Bipolar Weighted Argumentation Graphs

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Abstract

This paper discusses the semantics of weighted argumentation graphs that are bipolar, i.e. contain both attacks and support graphs. The work builds on previous work by Amgoud, Ben-Naim et. al. [1, 2], which presents and compares several semantics for argumentation graphs that contain only supports or only attacks relationships, respectively.

1 Introduction

In [3] we presented a prototype of a system that enables users to explore arguments for a given topic. This involves these steps:

1. **Argument identification.** In the first step, arguments concerning a given topic are identified in a given text and attacking and supporting relationships between the propositions are established. The result is an argumentation graph. In the future we hope to use argumentation mining techniques to automate this step. At this time, this is done manually by marking up some text.

2. **Initial plausibility assessment.** The propositions (represented as the nodes of the argumentation graph) are assigned an initial plausibility based on Web searches.

3. **Acceptability degree calculation.** The acceptability degree of the propositions are calculated. This calculation depends on the plausibility assessment of the propositions and the attacks and supports between them.

4. **Recommendation.** Based on the acceptability degree calculation the system determines an answer and the argument for it.

5. **User Interaction.** The user is able to explore the argument graph and may manually change the plausibility rating of any proposition. The system recalculates the acceptability degrees and its recommendation.

In this paper we focus on the third step: the calculation of acceptability degrees of the propositions of the arguments. For the sake of this paper we treat the propositions as arguments in an argument network.\(^1\)

\(^1\) This is a significant simplification, since it only allows us to represent convergent arguments in the sense of [4]. Further, this approach does not consider that the strength of a support or attack may be weighted as well; see section [7].
Abstract argumentation has been extensively studied since Dung’s pioneering work on argumentation graphs featuring an attack relation between arguments. While the Dung framework uses a classical logic approach in the sense that arguments can be only true or false, the framework has been generalised to gradual (or rank-based, or weighted) argumentation graphs that assign real numbers as weights to arguments. These have been widely discussed, in particular in [6, 7, 8], and also for the bipolar case (involving both attack and support relations) [9]. However, these works do not considering initial weightings (also called initial plausibilities). Given that the result of the initial plausibility assessment provide a continuous initial weighting, the most relevant previous work is on the evaluation argumentation graphs with support relationships by Leila Amgoud and Jonathan Ben-Naim [1] and an unpublished work by the same authors and two other authors on the evaluation of argumentation graphs with attack relationships [2].

Since for our purposes we need to consider bipolar [10] argumentation graphs that contain both attack and support relationships, we need to generalise the results of [1] and [2]. The result is a novel acceptability semantics for weighted argumentation graphs that contain both attacks and supports between arguments. The outline of the paper follows the presentations in [1, 2]. In section 2 we introduce the basic notions. In section 4 we discuss the characteristics an acceptability semantics should have. These characteristics are characterised axiomatically. We discuss how these axioms relate to the axioms from [1, 2].

In section 5 we discuss a semantics that meets the characteristics from section 4 show that it converges, study some properties and derive a variant with weights in the interval (0, 1). In section 6 we compare our approach to [1, 2] in more detail. Since a naïve combination of the semantics [1, 2] to a bipolar one fails, we discuss two suitable modifications. Finally, in section 7 we discuss some limitations of our approach and future work.

The two main contributions of this work are as follows. Firstly, we generalise the axiomatic framework of [1, 2] in various directions (bipolarity, unboundedness, multi-graph characteristics), as well as strengthen it (partly much stronger characteristics, as well as new ones like Continuity). Secondly, we design an unbounded bipolar semantics for weighted argumentation graphs that meets the developed characteristics. This semantics meets the requirements that emerged when developing a prototype of a system that enables users to explore arguments for a given topic.

2 Basic Concepts

In [1, 2] argumentation graphs are represented as a set of weighted nodes, which represent the attacks, and a set of vertices, which represent an attack relationship or a support relationship, respectively. We choose an alternative representation. An argument graph consists of three elements: a vector of arguments \( A = a_1, \ldots, a_n \), a matrix \( G \) that determines the attack and support relationships between the arguments, and a weighting \( w \) of the arguments which provides initial plausibilities. More specifically, \( G \) is a square matrix of order \( n \), which elements are either \(-1, 0 \) or \( 1 \). Given a matrix \( G \), if the element \( g_{ij} = 1 \), then this is intended to represent that the argument \( a_j \) supports
the argument \( a_i \); if \( g_{ij} = -1 \), then argument \( a_j \) attacks \( a_i \); and if \( g_{ij} = 0 \), then \( a_j \) does neither support nor attack \( a_i \). The vector \( w \) assigns to each argument a real number to represent its initial plausibility. The larger the \( w(a) \) for some argument \( a \) is, the larger its initial plausibility.

Note that our approach deviates from the approach in [1, 2], where only values in the interval [0, 1] are considered. In this paper we decided to allow \( \mathbb{R} \) as the value space, since there is no a priori reason why weights should always be restricted to the interval [0, 1], and, thus, we aim to support the more general case. For example, we plan to use hit counts as initial plausibilities, which can be used directly without any normalisation. 0 is the neutral value (neither plausible nor implausible). Negative values denote implausibility (e.g. consider hit counts contradicting the argument). In section 4.3 we show that the support argumentation graphs in [1] ranging in the interval [0, 1] may be considered as a special instance of \( \text{wasa} \). Within this approach 0 is the neutral value, and, hence, implausibility cannot be expressed. In section 5.5 we show how our proposed semantics may be adopted to support initial plausibilities and acceptability degrees in the interval (0, 1), which enables a direct comparison to the semantics in [1, 2]. One difference, though, is that within this approach \( \frac{1}{2} \) is the neutral value and that a value in the interval \((\frac{1}{2}, 0)\) expresses implausibility.

**Definition 1 (Weighted Attack/Support Argumentation Graph)** A weighted attack/support argumentation graph \( \text{wasa} \) is a triple \( \mathcal{A} = (\mathcal{A}, G, w) \), where

- \( \mathcal{A} \) is a vector of size \( n \) (for some \( n \in \mathbb{N}^+ \)), where all components of \( \mathcal{A} \) are pairwise distinct.
- \( G = \{g_{ij}\} \) is a square matrix of order \( n \) where \( g_{ij} \in \{-1, 0, 1\} \),
- \( w \) is a vector in \( \mathbb{R}^n \).

If \( \mathcal{A} = (\mathcal{A}, G, w) \) is a wasa and \( \mathcal{A} \) is of size \( n \), then \( \mathcal{A} \) consists of \( n \) arguments.

**Example 1**

\[
\mathcal{A} = \left( \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_3 & a_1 & a_4 \\ a_3 & a_1 & a_2 & a_4 \\ a_4 & a_1 & a_2 & a_3 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 2 \\ 2 \\ 1 \end{pmatrix} \right)
\]

The wasa \( \mathcal{A} \) in Example 1 consists of four arguments, namely \( a_1, a_2, a_3, a_4 \). The second component of \( \mathcal{A} \) determines that \( a_1 \) and \( a_4 \) are neither attacked nor supported. \( a_1 \) and \( a_2 \) support both \( a_2 \) and \( a_3 \). \( a_2 \) is attacked by \( a_3 \) and \( a_4 \), and \( a_3 \) is attacked by \( a_4 \). The third component of \( \mathcal{A} \) assigns initial plausibilities to the arguments, namely the weights \( w(a_1) = 0.5, w(a_2) = 2, w(a_3) = 2 \) and \( w(a_4) = 1 \). Example 1 also contains a graphical representation of \( \mathcal{A} \), which represents support relationships as connections with an arrow head and attacks as connections with a round head.
A WASA is a representation of a set of arguments, their attack and support relationships, and the initial plausibility of the arguments. The question that this paper needs to address is: How do we calculate the acceptability of the arguments based on their initial plausibility and their relations? Following the terminology in [1, 2], an answer to this question is called an acceptability semantics:

**Definition 2 (Acceptability Semantics)** An acceptability semantics is a function $S$ transforming any WASA $A = \langle A, G, w \rangle$ into a vector $\text{Deg}_A^S$ in $\mathbb{R}^n$, where $n$ is the number of arguments in $A$. For any argument $a_i$ in $A$, $\text{Deg}_A^S(a_i)$ is called the acceptability degree of $a_i$.

Obviously, there are many possible acceptability semantics. Example 2 defines the acceptability semantics $S^G$ that may have been embraced by the Greek sophist Gorgias, who believed that knowledge and communication is impossible.

**Example 2 (Gorgias Semantics)** $S^G$ is the function such that for, any WASA $A$ that consists of $n$ arguments, $S^G(A)$ is a vector of size $n$ such that $S^G(A) = (0 \ldots 0)$. According the Georgias Semantics any argument is equally acceptable and unacceptable, thus, $\text{Deg}_A^{S^G}(a) = 0$ for any argument $a$ in any WASA $A$.

Most people would probably agree that $S^G$ does not provide us with a useful tool for analysing argumentations. However, it raises the questions what requirements a suitable acceptability semantics should meet. We will discuss this question in the section 4.

### 3 Notation and auxiliary definitions

Unless otherwise specified $A$ is a WASA such that $A = \langle A, G, w \rangle$. If $a_1, \ldots, a_n$ are the components of $A$ we denote by

- $\text{Att}_A(a_i)$ the set of all attackers of $a_i$ in $A$, that is $\text{Att}_A(a_i) = \{a_j \mid g_{ij} = -1\}$;
- $\text{Sup}_A(a_i)$ the set of all supporters of $a_i$ in $A$, that is $\text{Sup}_A(a_i) = \{a_j \mid g_{ij} = 1\}$;
- $\text{Back}_A(a_i)$ and $\text{Detr}_A(a_i)$ are the sets of the backers of $a_i$ and the set of the detractors of $a_i$. They are defined recursively as the set of all arguments that directly or indirectly support $a_i$ (e.g., attacking an attacker of $a_i$) and, respectively, the set of all arguments that directly or indirectly attack $a_i$ (e.g., supporting an attacker of $a_i$). Thus, $\text{Back}_A(a_i)$ and $\text{Detr}_A(a_i)$ are the minimal sets such that the following equations hold:

\[
\text{Back}_A(a_i) = \text{Sup}_A(a_i) \cup \{a_j \mid \exists x : a_j \in \text{Sup}_A(x) \land x \in \text{Back}_A(a_i)\} \\
\cup \{a_j \mid \exists x : a_j \in \text{Att}_A(x) \land x \in \text{Detr}_A(a_i)\}
\]

\[
\text{Detr}_A(a_i) = \text{Att}_A(a_i) \cup \{a_j \mid \exists x : a_j \in \text{Att}_A(x) \land x \in \text{Back}_A(a_i)\} \\
\cup \{a_j \mid \exists x : a_j \in \text{Sup}_A(x) \land x \in \text{Detr}_A(a_i)\}
\]
• **Parent** \(a_i\) is the \(i\)th matrix row \((g_{i1}, \ldots, g_{in})\) of \(G\). It contains the parents of \(a_i\) in the argument graph and hence combines the information of \(\text{Sup}_A(a_i)\) and \(\text{Att}_A(a_i)\) in one vector.

**Definition 3 (Influence)** Given a WASA \(A = (A, G, w)\) and a vector \(v \in \mathbb{R}^n\) (e.g. \(v\) could be \(w\)), the influence of \(v\) on \(a_i\) is defined as the number:

\[
\sum_{b \in \text{Sup}_A(a_i)} v(b) - \sum_{c \in \text{Att}_A(a_i)} v(c) = \text{Parent}_A(a_i) v
\]

The influence of \(v\) in general is computed as the vector of the individual influences:

\[
\begin{pmatrix}
\text{Parent}_A(a_1)v \\
\vdots \\
\text{Parent}_A(a_n)v
\end{pmatrix} = Gv
\]

Note that supporters and attackers cancel each other out when computing the influence. Moreover, support by an implausible argument (weighted negatively) behaves like an attack, an vice versa, an attack by an implausible argument behaves like a support. This is called reverse impact, see Characteristics 17 below.

**Definition 4 (Isomorphism)** Let \(A = (A, G, w)\) and \(A' = (A', G', w')\) be two WASA, such that:

\[
A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, A' = \begin{pmatrix} a'_1 \\ \vdots \\ a'_m \end{pmatrix}, G = \begin{pmatrix} g_{11} & \ldots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \ldots & g_{nn} \end{pmatrix}, G' = \begin{pmatrix} g'_{11} & \ldots & g'_{1m} \\ \vdots & \ddots & \vdots \\ g'_{m1} & \ldots & g'_{mm} \end{pmatrix}
\]

An isomorphism from \(A\) to \(A'\) is a bijective function \(f\) from \(A\) to \(A'\) such that the following holds, for any \(a_i, a_j\),

• \(w(a_i) = w'(f(a_i))\),

• if \(f(a_i) = a'_k\) and \(f(a_j) = a'_m\), then \(g_{ij} = g'_{km}\).

**Definition 5 (Union)** Let \(A = (A, G, w)\) and \(A' = (A', G', w')\) be two WASA such that \(A\) and \(A'\) do not share a component and

\[
A = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a'_1 \\ \vdots \\ a'_m \end{pmatrix}, A' = \begin{pmatrix} a'_1 \\ \vdots \\ a'_m \end{pmatrix}, G = \begin{pmatrix} g_{11} & \ldots & g_{1n} \\ \vdots & \ddots & \vdots \\ g_{n1} & \ldots & g_{nn} \\ g'_{11} & \ldots & g'_{1m} \\ \vdots & \ddots & \vdots \\ g'_{m1} & \ldots & g'_{mm} \end{pmatrix}, G' = \begin{pmatrix} g'_{11} & \ldots & g'_{1m} \\ \vdots & \ddots & \vdots \\ g'_{m1} & \ldots & g'_{mm} \end{pmatrix}
\]

\[
w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \\ w'_1 \\ \vdots \\ w'_m \end{pmatrix}, w' = \begin{pmatrix} w'_1 \\ \vdots \\ w'_m \end{pmatrix}
\]

The union \(A \oplus A' = (A^\uparrow, G^\uparrow, w^\uparrow)\) of \(A\) and \(A'\) is defined as follows:

\[
A^\uparrow = \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a'_1 \\ \vdots \\ a'_m \\ \vdots \\ a'_m \end{pmatrix}, G^\uparrow = \begin{pmatrix} g_{11} & \ldots & g_{1n} & 0 & \ldots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ g_{n1} & \ldots & g_{nn} & 0 & \ldots & 0 \\ 0 & \ldots & 0 & g'_{11} & \ldots & g'_{1m} \\ \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & g'_{m1} & \ldots & g'_{mm} \end{pmatrix}, w^\uparrow = \begin{pmatrix} w_1 \\ \vdots \\ w_n \\ w'_1 \\ \vdots \\ w'_m \end{pmatrix}
\]
To improve readability we will in the rest of the paper use a more compact notation, which does not list the individual components but refers to the matrices that are merged. (”0” represent a zero matrix of appropriate dimensions, that is $n \times m$ and $m \times n$, respectively.)

$$A^\dagger = (A^A) \ , \ G^\dagger = (G^G \ 0) \ , \ w^\dagger = (w')$$

4 Characteristics of Acceptability Semantics

There are many possible acceptability semantics that one may consider for bipolar argumentation graphs. As Example 2 illustrates, some of them are not useful. Thus, the question arises, which characteristics an acceptability semantics should have to be any good?

In [1, 2] the authors enumerate several desirable characteristics, which they state axiomatically. These characteristics are distinguished between mandatory and optional. In [2] eleven mandatory characteristics are discussed, [1] contains eleven similar characteristics and two additional ones (Monotony, Boundedness).

2Technically, the characteristics are all different, since argument graphs in [1] involve (only) support relationships and the argument graphs in [2] involve (only) attack relationships. Further, there are additional technical differences in their axiomatisations. However, for the purposes of this paper we disregard these differences.
In this section we discuss these characteristics and define them within our framework. Table 1 provides an overview over the mandatory characteristics in [1, 2] and maps them to the terminology that we use in this paper.

The definition of the mandatory characteristics in [1, 2] within our framework involves different kind of changes. First, the definitions in [1, 2] assume that weightings and acceptability degrees are within the interval $[0,1]$, whereas we allow arbitrary real numbers. Second, we need to account for the fact that a WASA may contain both attack and support relationships. Third, because of the way the characteristics were formulated in [1, 2] some of them allowed for unintended semantics. These three points will be detailed below. We formulated the axioms in a way that captures the intended characteristic in a more general way.

The definition of the characteristics depends on three parameters. The first one is a neutral acceptability degree $\text{Neutral}_S$. Attacks or supports by arguments with the neutral acceptability degree will have no effect. If not stated otherwise, we assume $\text{Neutral}_S = 0$. The other two parameters are the minimum and maximum acceptability degrees. They can be derived from the acceptability degree space as follows:

Let $S$ be an acceptability semantics. Its acceptability degree space $\text{ADS}_S = \{x \mid x = \text{Deg}_S^a(a) \text{ for some } \text{wasa } \mathbb{A} = \langle A, G, w \rangle \text{ and } a \in A \}$.

If there is some $x \in \text{ADS}_S$ such that $x \geq y$ for all $y \in \text{ADS}_S$, then $\text{ADS}_S$ is bounded from above and its maximum acceptability degree $\text{Max}_S = x$. Otherwise, $\text{Max}_S$ is undefined.

If there is some $x \in \text{ADS}_S$ such that $x \leq y$ for all $y \in \text{ADS}_S$, then $\text{ADS}_S$ bounded from below and its minimum acceptability degree $\text{Min}_S = x$. Otherwise, $\text{Min}_S$ is undefined.

The neutral value also plays a role in “neutralising” arguments as follows:

**Definition 6 (Isolation)** Let $\mathbb{A} = \langle A, G, w \rangle$ be a wasa such that $a_1, \ldots, a_n$ are the components of $A$. The isolation $\mathbb{A}|_{a_i}$ of $a_i$ in $\mathbb{A}$ ($1 \leq i \leq n$) is defined as follows

$$\mathbb{A}|_{a_i} = \left( A, \left( \begin{array}{cccc} g'_{11} & \cdots & g'_{1n} \\ \vdots & \ddots & \vdots \\ g'_{n1} & \cdots & g'_{nn} \end{array} \right), \left( \begin{array}{c} w'_1 \\ \vdots \\ w'_n \end{array} \right) \right)$$

where

$$
g'_{jk} = \text{Neutral}_S \text{ if } j = i \text{ or } k = i$$

$$g'_{jk} = g_{jk} \text{ otherwise}$$

$$w'_j = \text{Neutral}_S \text{ if } a_j = a_i$$

$$w'_j = w_j \text{ otherwise}$$

If $c_1, \ldots, c_n$ are arguments in $\mathbb{A}$, then $\mathbb{A}|_{c_1, \ldots, c_n}$ is defined as $(\ldots ((\mathbb{A}|_{c_1})|_{c_2}) \ldots)|_{c_n}$.

### 4.1 Mandatory Characteristics

**Anonymity** implies that the identity of an argument (or its internal structure) has no impact on an acceptability degree semantics. **Independence** requires that the acceptability degree of an argument is influenced only by arguments that are (directly or indirectly) connected to it. These definitions are, modulo trivial changes, identical with the corresponding definitions in [1, 2].

**Characteristic 1 (Anonymity)** A semantics $S$ satisfies Anonymity iff, for any two wasa $\mathbb{A} = \langle A, G, w \rangle$ and $\mathbb{A}' = \langle A', G', w' \rangle$ and for any isomorphism $f$ from $\mathbb{A}$ to $\mathbb{A}'$, the following property holds: for any $a$ in $\mathbb{A}$, $\text{Deg}_S^a(a) = \text{Deg}_{S'}^{f(a)}(f(a))$. 
Characteristic 2 (Independence) A semantics $S$ satisfies Independence iff, for any two WASA $\mathcal{A} = \langle \mathcal{A}, G, w \rangle$ and $\mathcal{A}' = \langle \mathcal{A}', G', w' \rangle$ such that $\mathcal{A}$ and $\mathcal{A}'$ do not share a component, the following property holds: for any $a$ in $\mathcal{A}$, $\text{Deg}^S_{\mathcal{A}}(a) = \text{Deg}^S_{\mathcal{A} \oplus \mathcal{A}'}(a)$.

Equivalence requires that if two arguments start out with the same initial plausibility and if they share the same degree of attack and support, they have the same acceptability degree. This is in the same spirit as Anonymity, the major difference is that Anonymity compares arguments across different WASA, while Equivalence is about the arguments within one WASA. The definition is a straightforward combination of the corresponding definitions in [1, 2].

Characteristic 3 (Equivalence) A semantics $S$ satisfies Equivalence iff, for any weighted argumentation graph $\mathcal{A} = \langle \mathcal{A}, G, w \rangle$ and for any $a, b$ in $\mathcal{A}$, if

- $w(a) = w(b)$,
- there exists a bijective function $f$ from $\text{Att}_{\mathcal{A}}(a)$ to $\text{Att}_{\mathcal{A}}(b)$ such that $\forall x \in \text{Att}_{\mathcal{A}}(a)$, $\text{Deg}^S_{\mathcal{A}}(x) = \text{Deg}^S_{\mathcal{A}}(f(x))$,
- there exists a bijective function $g$ from $\text{Sup}_{\mathcal{A}}(a)$ to $\text{Sup}_{\mathcal{A}}(b)$ such that $\forall x \in \text{Sup}_{\mathcal{A}}(a)$, $\text{Deg}^S_{\mathcal{A}}(x) = \text{Deg}^S_{\mathcal{A}}(g(x))$,

then $\text{Deg}^S_{\mathcal{A}}(a) = \text{Deg}^S_{\mathcal{A}}(b)$.

Directionality captures the idea that attack and support are directed relationships, that is, the attacker (supporter) influences the acceptability degree of the attacked (supported), but not vice versa. Thus, assume $\mathcal{A}$ is a WASA and one adds a new attack (or support) relationship from $a_i$ to $a_j$, then this should only affect the acceptability degree of $a_j$ and arguments that $a_j$ directly or indirectly attacks or supports. To put it in a different way, all arguments (other than $a_j$) that do not have $a_j$ as backer or detractor, should not be affected by adding the new attack (support, respectively) relationship and their acceptability degree should not change.

Characteristic 4 (Directionality) A semantics $S$ satisfies Directionality iff, for any two WASA $\mathcal{A} = \langle \mathcal{A}, G, w \rangle$ and $\mathcal{A}' = \langle \mathcal{A}', G', w \rangle$ the following holds: if $G$ and $G'$ are of order $n$ (for some $n \in \mathbb{N}^+$) and there exists $i, j \in \mathbb{N}^+$ such that

- $g_{ji} = 0$,
- $g'_{ji} \neq 0$,
- for any $k, m$: if $k \neq i$ or $m \neq j$, then $g_{mk} = g'_{mk}$,

then for all $x$ in $\mathcal{A}$: if $x \neq a_j$ and $a_j \notin \text{Back}_{\mathcal{A}}(x)$ and $a_j \notin \text{Detr}_{\mathcal{A}}(x)$, then $\text{Deg}^S_{\mathcal{A}}(x) = \text{Deg}^S_{\mathcal{A}'}(x)$.
Our definition of Directionality translates the corresponding definition in [2] into our our matrix-based approach. Both are more general than Non-dilution in [1].

Conservativity expresses that, given any lack of supports or attacks, the acceptability degree of an argument should be identical to its initial plausibility. It combines Minimality in [1] with Maximality in [2].

**Characteristic 5 (Conservativity)** A semantics \( S \) satisfies Conservativity iff for any wasa \( \mathcal{A} = \langle A, G, w \rangle \), for any argument \( a \) in \( A \), if \( \text{Att}_A(a) = \text{Sup}_A(a) = \emptyset \), then \( \text{Deg}_S^A(a) = w(a) \).

In [1] the Coherence axiom is explained as follows: “[...] the impact of support is proportional to the basic strength of its target”. The same characteristic is named ‘Proportionality’ in [2]. However, neither of the axiomatisations in [1] and [2] represents proportionality in its usual sense. They rather require that an increase in the weights leads to an increase of the acceptability degree. Since it is about monotony in the initial plausibility, we call it Initial Monotony (in contrast to Parent Monotony introduced later on).

**Characteristic 6 (Initial Monotony)** A semantics \( S \) satisfies Initial Monotony iff, for any wasa \( \mathcal{A} = \langle A, G, w \rangle \) and for any arguments \( a, b \) in \( A \), if

- \( \text{Parent}_A(a) = \text{Parent}_A(b) \), and
- \( w(a) > w(b) \),

then

\[
\text{Deg}_S^A(a) > \text{Deg}_S^A(b) \text{ or } \text{Deg}_S^A(a) = \text{Deg}_S^A(b) = \text{Max}_S \text{ or } \text{Deg}_S^A(a) = \text{Deg}_S^A(b) = \text{Min}_S.
\]

(Note that the equation \( \text{Deg}_S^A(a) = \text{Deg}_S^A(b) = \text{Max}_S \) is taken to be false if \( \text{Max}_S \) does not exist, and similarly for \( \text{Min}_S \).

The initial plausibilities and the acceptability degrees of arguments are expressed as real numbers. Numbers (much) greater than \( \text{Neutral}_S \) represent high plausibility and a high acceptability, respectively, of an argument. Numbers (much) less than \( \text{Neutral}_S \) represent high implausibility and a strong inadequateness, respectively. \( \text{Neutral}_S \) plays a special role as the middle ground. An initial plausibility of \( \text{Neutral}_S \) means that the argument is neither plausible nor implausible, and an acceptability degree of \( \text{Neutral}_S \) means that within the given wasa there is neither grounds for accepting nor for rejecting the argument.

**Neutrality** expresses that, given an argument \( a \) with an acceptability degree of \( \text{Neutral}_S \), one can remove all attack and support relationships that \( a \) is involved in, since \( a \) has no impact on the acceptability degrees of rest of the arguments. Together with Independence this implies that arguments with an acceptability degree of \( \text{Neutral}_S \) can be eliminated from a wasa without changing the acceptability degrees of the other arguments.
Characteristic 7 (Neutrality) A semantics $S$ satisfies Neutrality iff, for any WASA $A = (A, G, w)$ the following holds: if there is an argument $a$ in $A$ such that $\text{Deg}^S_{A}(a) = \text{Neutral}^S$, then $\text{Deg}^S_{A} = \text{Deg}^S_{A|a}$.

Our definition of Neutrality is wider applicable than the corresponding notions in [2] and [1]. Example 3 (with $\text{Neutral}^S = 0$) illustrates one difference to [1, 2]: for any semantics $S$ that exemplifies Conservativity and Neutrality, $\text{Deg}^S_{A}(a_2) = 1$. (Because of Conservativity $\text{Deg}^S_{A}(a_1) = 0$, hence neutrality implies that $\text{Deg}^S_{A}(a_2) = \text{Deg}^S_{A'}(a_2)$, and thus, by Conservativity, $\text{Deg}^S_{A'}(a_2) = 1$.) In [1, 2] the Neutrality and Minimality (or Maximality, respectively) would not entail that $\text{Deg}^S_{A}(a_2) = 1$, because Neutrality is defined in a way that only compares acceptability degrees within one argumentation graph.

Example 3

$A = a_1 \rightarrow a_2$

$A' = a_1 \rightarrow a_2$

Parent Monotony requires that, for any given argument $a$ in a WASA, if one weakens or removes attackers of $a$ or strengthens or adds supporters of $a$, then this leads to a stronger or equal acceptability degree of $a$. Our formalisation of Parent Monotony significantly generalises the notion of Monotony in [1].

Characteristic 8 (Parent Monotony) A semantics $S$ satisfies Parent Monotony iff, for any two WASA $A = (A, G, w)$ and $A' = (A', G', w')$ and any argument $a$ which is both in $A$ and in $A'$, if

1. $w(a) = w'(a)$
2. $\text{Att}_{A'}(a) \subseteq \text{Att}_{A}(a)$ and $\text{Sup}_{A}(a) \subseteq \text{Sup}_{A'}(a)$,
3. for all $x \in \text{Att}_{A'}(a)$, $\text{Deg}^S_{A'}(x) \leq \text{Deg}^S_{A}(x)$,
4. for all $x \in \text{Sup}_{A}(a)$, $\text{Deg}^S_{A}(x) \leq \text{Deg}^S_{A'}(x)$,

then $\text{Deg}^S_{A}(a) \leq \text{Deg}^S_{A'}(a)$.

Impact requires that adding a new supporting argument (with positive acceptability degree) strengthens (and thus has an impact on) the supported argument. Further, the opposite is true for adding a new attacking argument. Impact generalises Weakening and Counting in [2] and Strengthening and Counting in [1].

---

$^3$In [1] Strengthening requires that the acceptability degree of a supported argument is higher than its initial plausibility. Counting requires that any additional support increases the acceptability more. Impact covers both given that Conservativity entails that in the absence of attackers and supporters the acceptability degree of an argument is identical to its initial plausibility. The analog is true for Strengthening and Counting in [2].
Characteristic 9 (Impact) A semantics \( S \) satisfies Impact iff, for any WASA \( \mathbb{A} = \langle A, G, w \rangle \) and any arguments \( a, b \) in \( A \) such that \( \text{Deg}_{\mathbb{A}}^S(b) > \text{Neutral}_S 
:

- If
  1. \( b \in \text{Att}_{\mathbb{A}}(a) \) and
  2. \( b \notin \text{Back}_{\mathbb{A}}(a) \)

  then \( \text{Deg}_{\mathbb{A}}^S(a) < \text{Deg}_{\mathbb{A}'}^S(a) \) or \( \text{Deg}_{\mathbb{A}}^S(a) = \text{Deg}_{\mathbb{A}'}^S(b) = \text{Min}_S \),

- If
  1. \( b \in \text{Sup}_{\mathbb{A}}(a) \) and
  2. \( b \notin \text{Detr}_{\mathbb{A}}(a) \)

  then \( \text{Deg}_{\mathbb{A}'}^S(b)(a) < \text{Deg}_{\mathbb{A}}^S(a) \) or \( \text{Deg}_{\mathbb{A}}^S(a) = \text{Deg}_{\mathbb{A}'}^S(b)(a) = \text{Max}_S \).

Reinforcement requires that if an attacker of an argument is weakened or a supporter is strengthened, then, ceteris paribus, the acceptability degree of the argument increases. Dually, if an attacker of an argument is strengthened or a supporter is weakened, then, ceteris paribus, the acceptability degree of the argument decreases. Reinforcement generalises the corresponding axioms in [1, 2] by considering both attacks and supports and by comparing acceptability degrees of arguments in different \( \text{wasa} \). Impact and Reinforcement together correspond to a kind of ‘Strict Parent Monotony’.

Characteristic 10 (Reinforcement) A semantics \( S \) satisfies Reinforcement iff, for any two WASA \( \mathbb{A} = \langle A, G, w \rangle \) and \( \mathbb{A}' = \langle A', G', w' \rangle \) and argument \( a \) such that: \( a \) is both in \( A \) and \( A' \), and \( w(a) = w'(a) \), \( \text{Att}_{\mathbb{A}}(a) = \text{Att}_{\mathbb{A}'}(a) \), and \( \text{Sup}_{\mathbb{A}}(a) = \text{Sup}_{\mathbb{A}'}(a) \) the following holds:

1. If
   - for all \( x \in \text{Att}_{\mathbb{A}}(a) \), \( \text{Deg}_{\mathbb{A}'}^S(x) \leq \text{Deg}_{\mathbb{A}}^S(x) \),
   - for all \( x \in \text{Sup}_{\mathbb{A}}(a) \), \( \text{Deg}_{\mathbb{A}'}^S(x) \geq \text{Deg}_{\mathbb{A}}^S(x) \),
   - there is some \( b \) such that either
     - \( b \in \text{Att}_{\mathbb{A}}(a) \) and \( \text{Deg}_{\mathbb{A}'}^S(b) < \text{Deg}_{\mathbb{A}}^S(b) \) or
     - \( b \in \text{Sup}_{\mathbb{A}}(a) \) and \( \text{Deg}_{\mathbb{A}'}^S(b) > \text{Deg}_{\mathbb{A}}^S(b) \),

   then \( \text{Deg}_{\mathbb{A}}^S(a) > \text{Deg}_{\mathbb{A}'}^S(a) \) or \( \text{Deg}_{\mathbb{A}}^S(a) = \text{Deg}_{\mathbb{A}'}^S(a) = \text{Max}_S \).

2. If
   - for all \( x \in \text{Att}_{\mathbb{A}}(a) \), \( \text{Deg}_{\mathbb{A}'}^S(x) \geq \text{Deg}_{\mathbb{A}}^S(x) \),
   - for all \( x \in \text{Sup}_{\mathbb{A}}(a) \), \( \text{Deg}_{\mathbb{A}'}^S(x) \leq \text{Deg}_{\mathbb{A}}^S(x) \),
   - there is some \( b \) such that either
– \( b \in \text{Att}_A(a) \) and \( \text{Deg}_S^b(a) > \text{Deg}_S^b(a) \) or
– \( b \in \text{Sup}_A(a) \) and \( \text{Deg}_S^b(a) < \text{Deg}_S^b(a) \),

then \( \text{Deg}_S^b(a) < \text{Deg}_S^b(a) \) or \( \text{Deg}_S^b(a) = \text{Deg}_S^b(a) = \text{Min}_S \).

Strengthening Soundness in [1] and Weakening Soundness in [2] express that any difference between an initial plausibility and the acceptability degree of an argument is caused by some supporting (attacking, respectively) argument. We call this characteristic Causality.

**Characteristic 11 (Causality)** A semantics \( S \) satisfies Causality iff, for any WASA \( \mathbb{A} = \langle A, G, w \rangle \) and any argument \( a \) in \( A \), if \( \text{Deg}_S^a(a) \neq w(a) \), then there exists an argument \( b \) in \( A \) such that \( \text{Deg}_S^b(b) \neq \text{Neutral}_S \) and \( b \in \text{Att}_A(a) \cup \text{Sup}_A(a) \).

**Theorem 1** Conservativity and Neutrality together imply Causality.

The idea behind Boundedness in [1] is the following: if an argument \( a \) has an acceptability degree of 1 and \( b \)'s support is stronger than \( a \)'s support, then the acceptability degree of \( b \) is also 1. The name of this characteristic is somewhat of a misnomer, since its definition does not entail that the argument degree space is bounded in the usual mathematical sense. E.g., consider a semantics \( S \) such that \( ADS_S \) is the open interval \((0, 1)\). Since \( S \) assigns the acceptability degree of 1 to no argument, the condition for Boundedness in [1] is met trivially, but the open interval \((0, 1)\) is not 'bounded' in the usual mathematical sense of the word.

To avoid any possible confusion, we are going to define Boundedness in its traditional sense and rename the characteristic from [1] into Stickiness. Note that Stickiness presupposes that the maximum acceptability degree is 1. Thus, Stickiness is not true for arbitrary WASA. In section 4.3 we introduce Stickiness for a subset of WASA and show that it is entailed by the mandatory characteristics in this section; see page 16. Since Boundedness is not a mandatory characteristic we discuss it in section 4.2.

Up to this point all characteristics in this section are (more or less loosely) based on the mandatory characteristics that are discussed in [1, 2]. We add several new mandatory characteristics, namely Neutralisation, Continuity, and Interchangeability.

**Neutralisation** is concerned with the relationship between attacks and supports. If argument \( a_m \) is attacked by \( a_l \) and supported by \( a_k \) and the acceptability degrees of \( a_l \) and \( a_k \) are identical, then \( a_k \) and \( a_l \) neutralise each other (with respect to \( a_m \)). Hence, if one removes both the attack from \( a_l \) on \( a_m \) and the support of \( a_k \) for \( a_m \), then, ceteris paribus, the acceptability degrees of the arguments in the WASA remain unchanged.

**Characteristic 12 (Neutralisation)** A semantics \( S \) satisfies Neutralisation iff, for any WASA \( \mathbb{A} = \langle A, G, w \rangle \) and any components \( a_k, a_l, a_m \) in \( A \), if

- \( a_k \in \text{Att}_A(a_m) \),
- \( a_l \in \text{Sup}_A(a_m) \),
\[ \text{Deg}_{\text{SA}}(a_k) = \text{Deg}_{\text{SA}}(a_l), \]

- \( A' = \langle A, G', w \rangle \) and
\[
G' = \begin{pmatrix}
  g'_{11} & \cdots & g'_{1m} \\
  \vdots & \ddots & \vdots \\
  g'_{m1} & \cdots & g'_{mm}
\end{pmatrix}
\]
where
\[
\begin{cases}
  g'_{ij} = 0 & \text{if } j = m \text{ and } i = k \\
  g'_{ij} = 0 & \text{if } j = m \text{ and } i = l \\
  g'_{ij} = g_{ij} & \text{otherwise}
\end{cases}
\]

then \( \text{Deg}_{\text{SA}} = \text{Deg}_{\text{SA'}}. \)

Given a semantics \( S \) that meets Neutralisation and Conservativity, the argument \( a_2 \) in Example 4 has an acceptability degree \( \text{Deg}_{\text{SA}}(a_2) = w(a_2) = 3 \). Because the attack of \( a_1 \) and the support of \( a_3 \) neutralise each other.

**Example 4**

\[ a_1 \quad 4 \quad a_2 \quad 3 \quad a_3 \quad 4 \]

*Continuity* requires that the acceptability degree of an argument is a continuous function of the initial plausibility. The main motivation for adding this mandatory characteristic is to exclude semantics that show chaotic behaviour, where small differences in the initial plausibility leads widely divergent acceptability degrees.

**Characteristic 13 (Continuity)** A semantics \( S \) satisfies Continuity iff, for any sequence of WASAs \( A_n = \langle A, G, w_n \rangle \), if
\[
\lim_{n \to \infty} w_n = w
\]
then for \( A = \langle A, G, w \rangle \)
\[
\lim_{n \to \infty} \text{Deg}_{\text{SA}} = \text{Deg}_{\text{SA}}
\]

*Interchangeability* requires that arguments with the same acceptability degree may be substituted for each other in attacking and supporting relationships without affecting the acceptability degrees of the arguments in the WASA. In other words, for the purpose of calculating the acceptability degree of an argument \( a \) the identity of the supporting and attacking arguments is not important, only their acceptability degrees matter.

**Characteristic 14 (Interchangeability)** A semantics \( S \) satisfies Interchangeability iff, for any any WASA \( A = \langle A, G, w \rangle \) and \( A' = \langle A, G', w \rangle \), if
- \( a_i, a_j, a_k \) are in the vector \( A \),
- \( \text{Deg}_{\text{SA}}(a_j) = \text{Deg}_{\text{SA}}(a_k) \),
- \( G' = \begin{pmatrix}
  g'_{11} & \cdots & g'_{1m} \\
  \vdots & \ddots & \vdots \\
  g'_{m1} & \cdots & g'_{mm}
\end{pmatrix} \), where
\[
\begin{cases}
  g'_{ij} = g_{ik} & \text{if } i = j = k \\
  g'_{ik} = g_{ij} & \text{if } i = k \text{ and } j = l \\
  g'_{lm} = g_{lm} & \text{otherwise}
\end{cases}
\]
then $\text{Deg}^S_{A} = \text{Deg}^S_{A'}$.

Example 5 illustrates Interchangeability. The difference between $A$ and $A'$ is the direction of attack and support for $a_2$. Since $a_1$ and $a_4$ have the same acceptability degree, it follows from Interchangeability that $\text{Deg}^S_{A} = \text{Deg}^S_{A'}$.

Example 5

\[
\begin{array}{c}
a_1 \rightarrow a_2 \\
| \downarrow \| \downarrow \\
a_3 \rightarrow a_4
\end{array}
\quad
\begin{array}{c}
a_1 \rightarrow a_2 \\
| \downarrow \| \downarrow \\
a_3 \rightarrow a_4
\end{array}
\]

4.2 Optional Characteristics

As we pointed out in the discussion of Initial Monotony, the characteristic of Proportionality is defined in [2] in a non-standard way. Hence, the question arises whether we can introduce a notion of proportionality in its usual sense? The answer is that proportionality is most likely not a useful concept for an acceptability semantics, since the acceptability degree of an argument depends on two variables: namely, its initial plausibility and the wasa it is part of. However, instead we can consider Linearity. Given a set of arguments and attack and support relationships between them, Linearity requires that the acceptability degree of an argument is a linear function of its initial plausibility.

Character 15 (Linearity) A semantics $S$ satisfies Linearity iff, for any wasa $A = \langle A, G, w \rangle$ and for any argument $a$ in $A$, there are constants $c_1$ and $c_2$ such that for all $w'$ that agree with $w$ except possibly on $a$,

\[
\text{Deg}^S_{(A, G, w')}(a) = c_1 + c_2 w'(a)
\]

In contrast to the other characteristics that we discussed in this section, we do not consider Linearity a mandatory characteristic of an acceptability semantics. The same is true for Boundedness and Reverse Impact.

A semantics $S$ is bounded if its acceptability degree space $ADS_S$ has a maximum and a minimum element.

Character 16 (Boundedness) A semantics $S$ satisfies Boundedness iff $ADS_S$ is bounded from above and bounded from below.

Reverse impact means that the effect of an attack relation can be supporting or vice versa. For example, assume that $b$ is a discredited argument, which is strongly rejected by the audience. If $b$ attacks the argument $a$, then $a$ may actually be considered more acceptable by the audience because of the attack.

Character 17 (Reverse impact) A semantics $S$ satisfies Reverse impact iff, for any wasa $A = \langle A, G, w \rangle$ and any argument $a$ in $A$ there is some argument $b$ in $A$ such that:
• If
  1. \( b \in \text{Att}_A(a) \) and
  2. \( b \notin \text{Back}_A(a) \)

  then \( \text{Deg}_{S_A}(a) > \text{Deg}_{S_A}(b) \) or \( \text{Deg}_{S_A}(a) = \text{Max}_S \) or \( \text{Deg}_{S_A}(b) = \text{Max}_S \) or \( \text{Deg}_{S_A}(a) = \text{Deg}_{S_A}(b) = \text{Min}_S \).

• If
  1. \( b \in \text{Sup}_A(a) \) and
  2. \( b \notin \text{Detr}_A(a) \),

  then \( \text{Deg}_{S_{A|b}}(a) > \text{Deg}_{S_A}(a) \) or \( \text{Deg}_{S_A}(a) = \text{Deg}_{S_{A|b}}(b) = \text{Max}_S \) or \( \text{Deg}_{S_A}(a) = \text{Deg}_{S_{A|b}}(b) = \text{Min}_S \).

4.3 WASA vs. Support Argumentation Graphs

In the discussion of Characteristic 7, we explained our motivation for deviating from definitions of the characteristics within [1, 2] by defining these characteristics by comparison across different WASA. We claimed that a cross-graph comparison is applicable to a wider range of examples, but we have not discussed the relationship between the approaches in more detail. In this section we show that our approach is more general. For this purpose we will focus on the definitions of support argumentation graphs in [1]. In a first step we show how the basic definitions in [1] can be represented within our framework by definitions 7 and 8. In a second step we show how that the mandatory axioms in [1] are entailed by our mandatory characteristics. A third step is relegated to section 6.1 below, namely the proof that all our mandatory axioms hold for the aggregation-based semantics in [1].

We first restrict the general setting to that in [1]:

Definition 7 (Bounded Weighted Support Argumentation Graph)
A \([0,1]\)-Bounded Weighted Support Argumentation Graph (BWSA) is a WASA \( A = \langle A, G, w \rangle \) such that

- \( g_{ij} \in \{0,1\} \), for any component \( g_{ij} \) of \( G \),
- \( w \) is a vector in the interval \([0,1]\).

Any support argumentation graph in [1] is a BWSA. Since there are no attacking relationships within BWSA, it follows that, for any argument \( a \) in an BWSA \( A \), \( \text{Att}_A(a) = \text{Detr}_A(a) = \emptyset \).

We use BWSA to define a corresponding semantic notion in Definition 8:

Definition 8 (Bounded Acceptability Semantics) A \([0,1]\)-bounded acceptability semantics \( S \) is an acceptability semantics \( S \) such that \( \text{Deg}_{S_{A|b}}(a) \in [0,1] \), for any BWSA \( A = \langle A, G, w \rangle \) and any argument \( a \in A \).
Note that any semantics for support argumentation graphs in [1] corresponds to a bounded acceptability semantics that is restricted to BWSA.

In the following we show that the mandatory axioms in [1] for bounded acceptability semantics are entailed by the mandatory characteristics for acceptability semantics in section 4.1. In some cases this is quite trivial, since the characteristics in section 4.1 are based on [1]. For this reason we consider only two examples.

The Dummy axiom in [1] corresponds to our notion of Neutrality and is defined by comparing the acceptability degrees within one argument graph.

**Characteristic 18 (Dummy)** A bounded acceptability semantics $S$ satisfies Dummy iff, for any BWSA and $a, b$ in $A$ such that

- $w(a) = w(b)$,
- $\text{Sup}_A(a) = \text{Sup}_A(b) \cup \{x\}$, such that $\text{Deg}_S^A(x) = \text{Neutral}_S$,

then $\text{Deg}_S^A(a) = \text{Deg}_S^A(b)$.

**Theorem 2** Any bounded acceptability semantics $S$ that satisfies Neutrality and Parent Monotony, satisfies Dummy.

Note that boundedness or the restriction to $[0,1]$ is not needed for Thm. 2 all that is needed is that there are no attacks.

**Stickiness** (which is called boundedness in [1]) expresses that if argument $b$ has an acceptability degree of 1 and argument $a$ has some stronger support than $a$ has also an acceptability degree of 1.

**Characteristic 19 (Stickiness)** A bounded acceptability semantics $S$ satisfies Stickiness iff, for any BWSA $A = \langle A, G, w \rangle$ and $a, b$ in $A$ such that

- $w(a) = w(b)$,
- $\text{Sup}_A(a) \setminus \text{Sup}_A(b) = \{x\}$,
- $\text{Sup}_A(b) \setminus \text{Sup}_A(a) = \{y\}$,
- $\text{Deg}_S^A(x) > \text{Deg}_S^A(y)$,

if $\text{Deg}_S^A(b) = 1$, then $\text{Deg}_S^A(a) = 1$.

**Theorem 3** Any bounded acceptability semantics $S$ that satisfies Parent Monotony, Independence, Interchangeability, Anonymity and Conservativity, satisfies Stickiness.

Analog theorems may be formulated for the other axioms in [1]:

**Theorem 4** Any bounded acceptability semantics $S$ that satisfies the mandatory characteristics in [4.1] satisfies Anonymity, Independence, Non-Dilution, Monotony, Equivalence, Dummy, Minimality, Strengthening, Strengthening Soundness, Coherence, Counting, Boundedness, and Reinforcement as defined in [1].

Thm. 4 together with Thm. 12 below, shows that our axiomatic approach is more general than that in [1].
5 Direct Aggregation Semantics

The main intuition behind the aggregation semantics is that the strength of an argument in an argumentation is based on its initial plausibility and it is strengthened by supports and weakened by attacks. If we understand "strengthening" as addition and "weakening" as subtraction, we get to a model where the acceptability degree of an argument is calculated by adding to its initial plausibility the sum of the acceptability degrees of its supporters and subtracting the sum of the acceptability degree of its attackers.

One additional intuition behind the semantics is that the influence of arguments is the strongest on arguments that they directly attack or support and increasingly weaker on arguments that are only indirectly connected to them.

Example 6

\[ a_3 \rightarrow a_2 \rightarrow a_1 \]
\[ a'_3 \rightarrow a'_2 \rightarrow a'_1 \]

The difference can be illustrated by Example 6: both \( a_1 \) and \( a'_1 \) are supported by 3 different arguments. However, while \( a_4 \) provides only indirect support for \( a_1 \), \( a'_4 \) supports \( a'_1 \) directly. For this reason, the acceptability degree of \( a'_1 \) should be larger than the acceptability degree of \( a_1 \). We achieve this effect by introducing a so-called "dampening factor" \( d \geq 1 \) that mitigates the effect of arguments along the paths of a WASA.

5.1 Definition of Direct Aggregation Semantics

These two intuitions are formalised by Definitions 9 and 11.

**Definition 9** Let \( \mathcal{A} = (A, G, w) \) be a WASA. Let the damping factor \( d \) be such that \( d \geq 1 \). For \( i \in \mathbb{N} \), let \( f_{i}^{\text{dir}} \) be a function \(^4\) from \( A \) to \( \mathbb{R} \) such that for any \( a \in A \), for \( i \in \{0, 1, 2, \ldots\} \), if \( i = 0 \), then \( f_{i}^{\text{dir}}(a) = w(a) \), otherwise

\[
\begin{align*}
  f_{i}^{\text{dir}}(a) &= w(a) + \frac{1}{d} \times \left( \sum_{b \in \text{Sup}_{A}(a)} f_{i-1}^{\text{dir}}(b) - \sum_{c \in \text{Att}_{A}(a)} f_{i-1}^{\text{dir}}(c) \right) \\
  \text{or shorter in matrix notation} \\
  f_{i}^{\text{dir}} &= w + \frac{1}{d} G f_{i-1}^{\text{dir}}
\end{align*}
\]

**Definition 10** We call \( G \) the incidence matrix and

\[
P_{r}^{G,d} = \sum_{i=0}^{\infty} \left( \frac{1}{d} G \right)^{i}
\]

\(^4 f^{\text{dir}} \) also depends on \( G \) and \( d \), which we omit here for readability. Since \( f^{\text{dir}} \) is only used locally in this definition and the next one, \( G \) and \( d \) are clear from context.
the propagation matrix.

**Definition 11** The direct aggregation semantics is a function \( s^d \) transforming any WASA \( \mathbb{A} = (A, G, w) \) into a weighting on \( A \) such that for any \( a \in A \)

\[
\text{Deg}^\text{dir}_{\mathbb{A}, d}(a) = \lim_{i \to \infty} f^\text{dir}_i(a)
\]

that is

\[
\text{Deg}^\text{dir}_{\mathbb{A}, d} = \lim_{i \to \infty} f^\text{dir}_i = \sum_{i=0}^{\infty} \left( \frac{1}{d} \right)^i w
\]

or short

\[
\text{Deg}^\text{dir}_{\mathbb{A}, d} = \Pr^{G,d}_{w}
\]

(degree = propagation matrix \( \times \) initial plausibility)

This means that the degree can be computed from the initial plausibility by a fixed linear transformation given by the propagation matrix.

Let inddegree\((G)\) be the maximal indegree, i.e. the maximum number of edges leading into an argument in \( \mathbb{A} \). For \( d \leq \text{indegree}(G) \), the direct aggregation semantics is not well-defined in general. Example 7 provides an example where \( f^\text{dir} \) does not converge.

**Example 7** Assume \( d = \text{indegree}(G) \) and consider an argument with an initial plausibility 1 that attacks itself. In this case \( d = \text{indegree}(G) = 1 \) and

\[\mathbb{A} = \langle (a), (-1), (1) \rangle \quad \text{or, graphically:} \quad \mathbb{A} = \bigcirc_{1}\]

In this case \( f^\text{dir}_{2i}(a) = 1 \) and \( f^\text{dir}_{2i+1}(a) = 0 \).

**Theorem 5** For \( d > \text{indegree}(G) \), the direct aggregation semantics is well-defined and converges to \( \left( I - \frac{1}{d} G \right)^{-1} w \).

For the proof, we need the following from [11], Corollary 5.6.16:

**Fact 1** If \( \| \cdot \| \) is a matrix norm, and if \( \| A \| < 1 \), then \( I - A \) is invertible and

\[
(I - A)^{-1} = \sum_{i=0}^{\infty} A^i
\]

**Proof (of Fact 1)**: \( \sum_{i=0}^{\infty} A^i \) converges because its norm does. But then \( (I - A)(\sum_{i=0}^{\infty} A^i) = \sum_{i=0}^{\infty} A^i - \sum_{i=1}^{\infty} A^i = A^0 = I \).

**Proof (of Thm. 5)**: We need to show that

\[
\Pr^{G,d}_{w} = \sum_{i=0}^{\infty} \left( \frac{1}{d} G \right)^i
\]
converges. Note that the maximum row sum norm $||G||_\infty$ defined by

$$||G||_\infty = \max_{i=1,...,n} \sum_{j=1,...,n} |g_{ij}|$$

coincides with the maximal indegree, i.e. $||G||_\infty = \text{indegree}(G)$. Hence, we have

$$||\frac{1}{d}G||_\infty \leq \frac{\text{indegree}(G)}{d} < 1$$

By Fact 1, this implies that $I - \frac{1}{d}G$ is invertible and $Pr^{G,d} = (I - \frac{1}{d}G)^{-1}$, thus

$$\text{Deg}_{\text{dir}}^A = Pr^{G,d}w = \sum_{i=0}^{\infty} (\frac{1}{d}G)^iw = (I - \frac{1}{d}G)^{-1}w$$

5.2 Application of Direct Aggregation Semantics

To illustrate the use of the direct aggregation semantics we revisit an example from the literature [12, 13]. Assume it is the last weekend of the football season and there is a close title race between Liverpool and Manchester United. Liverpool will win the title if it either wins its last game or Manchester does not win its last game. The question is: Will Liverpool win the Premiere League? (lpl). There are two arguments supporting Liverpool’s title ambitions: Liverpool will win, because Manchester United will not win its last game against Manchester City (mnw). And Liverpool will win the title, because it will win its match against Arsenal (wlm). However, there is a counterargument: Liverpool will not win against Arsenal, because Liverpool’s best player is injured (bpi).

This situation may be represented similarly as in [12]:

Example 8

$$\mathbb{A} = \text{mnw} \xrightarrow{8} \text{lpl} \xleftarrow{0} \text{wlm} \xleftarrow{5} \text{bpi} \xleftarrow{2}$$

However, since WASA are weighted, they allow us to not just represent the relationships between the arguments, but also the initial plausibility of the arguments involved. Since the question whether Liverpool will win is the topic under discussion we assign it an initial value of 0 (the neutral value). Let us assume that we polled a panel of eight experts on the plausibility of the arguments. 5 experts believe Liverpool will win its game. Let’s further assume that 3 experts believe that Manchester City will win and 5 experts believe in a draw in the Manchester derby. Hence, “Manchester will not win” (mnw) is assigned a value of 8. Further, only 2 experts believe that the loss of Liverpool’s star player will have a significant impact on the game.

Let’s assume a dampening factor of 2 for the remainder of this subsection. In this case it follows that in Example 8 $\text{Deg}^{\text{dir}}_A(bpi) = 2$, $\text{Deg}^{\text{dir}}_A(wlm) = 4$, $\text{Deg}^{\text{dir}}_A(mnw) = 8$, $\text{Deg}^{\text{dir}}_A(lpl) = 6$. Thus, given the argumentation $\mathbb{A}$ in Example 8 Liverpool fans should be quite optimistic.
This can be seen in more detail as follows. With \( A = \begin{pmatrix} mnw \\
pl \\
wlm \\
bpi \\
mcw \end{pmatrix} \), the matrix of the graph is

\[
G = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

and using damping factor \( d = 2 \), the propagation matrix is

\[
Pr^{G,2}_G = (I - \frac{1}{2}G)^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0.5 & 0 & -0.25 \\
0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

The computed acceptability degrees are \( Pr^{G,2}_G (\begin{pmatrix} 8 \\
9 \\
5 \\
2 \end{pmatrix}) = \begin{pmatrix} 8 \\
6 \\
4 \\
2 \end{pmatrix} \).

Note that in the example the first argument (mnw) may be considered as a summary of two different arguments: an argument that Manchester City wins (mcw) or that the Manchester derby is a draw (mdd). If we split these arguments up, we get the wasa \( A' \) in Example 9.

Example 9

\[
A' = \begin{array}{c}
\text{mdd} \\
\text{lpl} \\
\text{wlm} \\
\text{bpi} \\
\text{mcw}
\end{array}
\]

With \( A = \begin{pmatrix} mdd \\
lpl \\
wlm \\
bpi \\
mcw \end{pmatrix} \), the matrix of the graph is

\[
G' = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

and using damping factor \( d = 2 \), the propagation matrix is

\[
Pr^{G',2}_G = (I - \frac{1}{2}G')^{-1} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0.5 & 0 & -0.5 & 0.5 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

The computed acceptability degrees are \( Pr^{G',2}_G (\begin{pmatrix} 5 \\
9 \\
2 \\
3 \\
0 \end{pmatrix}) = \begin{pmatrix} 8 \\
6 \\
4 \\
2 \\
3 \end{pmatrix} \).

Because the effects of supporting arguments in the direct aggregation semantics is additive, it does not matter whether one chooses the representation in Example 8 or in Example 9. The acceptability degree of “Liverpool wins the Premiere League” (lpl) does not change.

A third alternative of representing the situation is to consider Manchester United’s perspective (see Example 10). “Manchester United wins the Premiere League” (mpl) is under the attack by (mdd), (mcw), and (wlm). As result, \( \text{Deg}_{A''}^{\text{dir}} (mpl) = -6 \).
Example 10

Another example concern two pupils Alice and Bob that accuse each other of lying about a certain circumstance. The teachers Miller and Smith support the views of Alice and Bob, respectively, because they know their pupils well. Of course, both pupils also accuse the other teacher to be wrong, while the teachers are wise enough not to attack each others’ views, but even support them because they have known each other for a long time. From the files, the director derives some judgement about credibilities:

Example 11

With $A = \begin{pmatrix} Miller & Smith \\ Alice & Bob \end{pmatrix}$, the matrix of the graph is

$G = \begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{pmatrix}$

and using damping factor $d = 3$, the propagation matrix is

$Pr^{G,3} = (I - \frac{1}{3}G)^{-1} = \frac{1}{21} \begin{pmatrix} 23 & 5 & 1 & -8 \\ 5 & 23 & 1 & -11 \\ -1 & -8 & 25 & -11 \\ -1 & 8 & -11 & 25 \end{pmatrix}$

The computed acceptability degrees are $Pr^{G,3} \left( \begin{pmatrix} 6 \\ 4 \\ 1.5 \end{pmatrix} \right) = \left( \begin{pmatrix} 5.7 \\ 5.5 \\ 2.5 \end{pmatrix} \right)$. This means that Alice’s view is valued equally with Bob’s, even though her initial assessment has been slightly worse than Bob’s. Also note that the all values have increased, because support has been in general stronger than attack.

5.3 Properties of Direct Aggregation Semantics

Theorem 6 $D = \deg_{A,d}^{dir}$ is the unique solution of the equation

$D = w + \frac{1}{d}GD$

(degree = initial plausibility + damped influence)

21
Proof: The equation can be rewritten as \((I - \frac{1}{d}G)D = w\), which shows that 
\[D = (I - \frac{1}{d}G)^{-1}w = \text{Deg}_{\text{dir}}^{\text{A},d}\] 

Moreover, if \(D = w + \frac{1}{d}GD\), by successively unfolding, we get
\[D = \sum_{i=0}^{\infty} (\frac{1}{d}G)^i w = \text{Deg}_{\text{dir}}^{\text{A},d}\]

A node \(a\) in an argumentation graph is called circular, if there is a non-empty path of parent relations (which can be attack or support relations) that starts and ends in \(a\). A node is called hereditarily circular, if one of its backers or detractors is circular.

Theorem 7 Convergence as in Thm. \(\square\) holds already if \(d\) is greater than the maximum indegree for the subgraph induced by the hereditarily circular nodes; or \(d > 0\) if there are no hereditarily circular nodes, i.e. if the argumentation graph is acyclic.

Proof: If \(a_i\) is not hereditarily circular and \(k \geq n\), the \(i\)-th row of \(G^k\) consists of zeros only. Hence, for convergence, it suffices to consider the hereditarily circular nodes only. \(\square\)

In the sequel, we will examine direct aggregation semantics w.r.t. the characteristics introduced in Sect. 4. Generally, we have two options of choosing the damping factor \(d\):

1. \(d\) is chosen such that \(d > \text{indegree}(G)\). To be definite, let us chose \(d = \text{indegree}(G) + 1\). This means that \(d\) depends on the argument matrix (graph).

2. \(d\) is chosen globally \(d\), independently of the matrix (graph). This means that for each \(d\), we get a separate semantics, which is guaranteed to be defined only for graphs \(G\) with \(\text{indegree}(G) < d\).

We will indicate these cases.

Example 12 If \(d = \text{indegree}(G) + 1\) (or generally, \(d\) does depend on \(\text{indegree}(G)\)), direct aggregation semantics does not satisfy Independence. Consider \(A = ((a), (1), (1))\) and 
\[A' = ((\frac{a}{b}, c), (\frac{0}{0}, 0, 0, 0), (\frac{1}{1})).\]

Then \(\text{Deg}_{\text{dir}}^{\text{A},2} = (\frac{2}{2}), \text{but Deg}_{\text{dir}}^{\text{A}',3} = (\frac{3}{3} \frac{3}{3} \frac{3}{3})^T\).

Theorem 8 If \(d\) is chosen globally (which implies that only wasas of maximal indegree \(< d\) are considered), direct aggregation semantics satisfies Independence.

Proof: \(\text{Deg}_{\text{dir}}^{\text{A},d}(a) = (\sum_{i=0}^{\infty} (\frac{1}{d}G)^i w)(a) = (\sum_{i=0}^{\infty} (\frac{1}{d}G)^i w)(a) = \text{Deg}_{\text{dir}}^{\text{A},d}(a).\) \(\square\)

Our semantics enjoys all other desirable properties (the proof has been relegated to the appendix):

Theorem 9 Direct aggregation semantics satisfies Anonymity, Equivalence, Directionality, Conservativity, Initial Monotony, Neutrality, Parent Monotony, Impact, Reinforcement, Causality, Neutralisation and Continuity, Linearity and Reverse impact.

\(^5\)Given a vector \(v\), we use the notation \(v(a_j)\) for the value \(v_j\). This is particularly convenient if we do not have the index \(j\) at hand.
5.4 Behaviour of the Propagation Matrix

We will now examine the behaviour of direct aggregation semantics for specific cases. As noted above, the degree is computed from the initial plausibility by multiplication with the propagation matrix. Hence, it suffices to examine propagation matrices, which are independent of the initial plausibility. Indeed, propagation matrices can be computed easily by matrix inversion, see Thm. 5. We list propagation matrices for a number of small argument graphs, in terms of the damping factor \( d \). We begin with acyclic graphs.

An isolated argument will be not affected at all:

\[
G \quad P_{d}^{G,a}
\]

Edges in the argument graph propagate (positive or negative) influence with strength \( \frac{1}{d} \):

\[
G \quad P_{d}^{G,a} \quad G \quad P_{d}^{G,a}
\]

\[
a \rightarrow b \quad \left( \begin{array}{cc} 1 & 0 \\ \frac{1}{d} & 1 \end{array} \right) \quad a \rightarrow b \quad \left( \begin{array}{cc} 1 & 0 \\ -\frac{1}{d} & 1 \end{array} \right)
\]

Influence of arguments can also propagated along \( k \) edges, resulting in a factor \( \frac{1}{d^{k}} \).

For \( k = 2 \), we get:

\[
G \quad P_{d}^{G,a} \quad G \quad P_{d}^{G,a}
\]

\[
a \rightarrow b \rightarrow c \quad \frac{1}{d^{2}} \left( \begin{array}{ccc} d^{2} & 0 & 0 \\ 0 & d^{2} & 0 \\ 1 & d & d^{2} \end{array} \right) \quad a \rightarrow b \rightarrow c \quad \frac{1}{d^{2}} \left( \begin{array}{ccc} d^{2} & 0 & 0 \\ -d & d^{2} & 0 \\ 1 & -d & d^{2} \end{array} \right)
\]

Now to cyclic graphs. The simplest case are self-support and self-attack, resulting respectively in a slight strengthening and weakening of the weight:

\[
G \quad P_{d}^{G,a} \quad G \quad P_{d}^{G,a}
\]

\[
a \rightarrow b \quad \left( \frac{d}{\pi^{2}} \right) \quad a \rightarrow b \quad \left( \frac{4}{\pi^{2}} \right)
\]

Since a chain of two attacks is a support, both mutual attack and support lead to a slight self-support as well, while an attack-support pair (called a vicious circle in [14]) leads to a slight self-attack of both arguments. We also show the combination with
explicit self-supports and self-attacks:

\[
G \quad P_{rG,d} \quad G \quad P_{rG,d}
\]

\[
a \overset{\rightarrow}{\leftarrow} b \quad \frac{1}{d^2-1} \left( \begin{array}{ccc}
d & d & d \\
d & d & d^2 \\
d & d^2 & d^2 \\
\end{array} \right) \quad a \overset{\rightarrow}{\leftarrow} b \quad \frac{1}{d^2-1} \left( \begin{array}{ccc}
d & d & d^2 \\
d & d & d^2 \\
-2 & -2 & -2 \\
\end{array} \right)
\]

\[
a \overset{\rightarrow}{\leftarrow} b \quad \frac{1}{d^2+1} \left( \begin{array}{ccc}
d & d & d \\
d & d & d^2 \\
d & d^2 & d^2 \\
\end{array} \right) \quad a \overset{\rightarrow}{\leftarrow} b \quad \frac{1}{d^2+1} \left( \begin{array}{ccc}
d & d & d^2 \\
d & d & d^2 \\
d & d & d^2 \\
\end{array} \right)
\]

For all arguments in a cycle of length three, we get a slight self-support (left hand-side, non-vicious circles) or self-attack of the argument (right hand-side, vicious circle in the sense of \[14\]):

\[
G \quad P_{rG,d} \quad G \quad P_{rG,d}
\]

\[
a \overset{\rightarrow}{\leftarrow} b \quad \frac{1}{d^3-1} \left( \begin{array}{ccc}
d & d & d \\
d & d & d^2 \\
d & d & d^3 \\
\end{array} \right) \quad a \overset{\rightarrow}{\leftarrow} b \quad \frac{1}{d^3+1} \left( \begin{array}{ccc}
d & d & d^2 \\
-2 & -2 & -2 \\
2 & 2 & 2 \\
\end{array} \right)
\]

Finally, we show various cycles of length four, all of which show a slight self-support. The numbers 2 and -2 in the last three examples are caused by the fact that between
diagonally opposite arguments, there are always two distinct paths:

\[
\begin{align*}
 G & \quad Pr_{G,d} \\
 a \rightarrow b & \quad \frac{1}{d^4-1} \begin{pmatrix} d^4 & d & d^3 & d^2 \\ d & d^3 & d^4 & d \\ d^2 & d^4 & d & d^3 \\
\end{pmatrix} & \quad G & \quad Pr_{G,d} \\
 a \leftarrow b & \quad \frac{1}{d^4-1} \begin{pmatrix} -d^4 & -d & -d^3 & -d^2 \\ -d & -d^3 & -d^4 & -d \\ -d^2 & -d^4 & -d^3 & -d^2 \\
\end{pmatrix} \\

d \leftarrow c & \\

d \rightarrow b & \\

d \rightarrow c & \\

d \leftarrow c & \\

d \rightarrow b & \\

d \rightarrow c & \\
\end{align*}
\]

5.5 Sigmoid Direct Aggregation Semantics

The direct aggregation semantics allows for the real numbers as values for the initial plausibility and for the acceptability degree. As discussed on page 3, this approach deviates from other approaches like [1, 2], where only a subset of \( \mathbb{R} \) is considered. In this subsection we illustrate how the real-valued direct aggregation semantics can be cast into the more traditional framework of weighting in \([0, 1]\). Actually, for technical reasons, we restrict ourselves to the interval \((0, 1)\). A similar approach could be chosen for other intervals.

To constrain our semantics to \((0, 1)\), we need a sigmoid function, that is, a bijection \( \sigma : \mathbb{R} \rightarrow (0, 1) \) that is continuous and strictly increasing such that \( \sigma(0) = 0 \). For example, the logistic function

\[
\sigma(x) = \frac{1}{1 + e^{-x}}
\]

the suitably normalised arcus tangens function

\[
\sigma(x) = \frac{\arctan(x)}{\pi} + \frac{1}{2}
\]

or the simple fraction

\[
\sigma(x) = \frac{1 + |x| + x}{2(1 + |x|)}
\]

will do.
Being such prepared, we now can define

**Definition 12** Let $\mathbb{A} = \langle \mathbb{A}, G, w \rangle$ be a WASA such that $w : \mathbb{A} \rightarrow (0, 1)$. Let the damping factor $d$ be such that $d \geq 1$. The sigmoid direct aggregation semantics is defined as

$$D_{\text{deg}_\mathbb{A}}^{\text{sd}, d} = \sigma \left( P_{T} G_{d} \sigma^{-1}(w) \right)$$

We have a fixed-point theorem similar to Thm. 6:

**Theorem 10** $D = D_{\text{deg}_\mathbb{A}}^{\text{sd}, d}$ is the unique solution of the equation

$$D = \sigma^{-1}(w) + \frac{1}{d} G_{d} \sigma^{-1}(D)$$

**Proof:** Rewrite the equation to

$$\sigma^{-1}(D) = \sigma^{-1}(w) + \frac{1}{d} G_{d} \sigma^{-1}(D) \quad (1)$$

and use Thm. 6

In the appendix, we prove the desirable properties of our semantics, with the neutral value taken to be $\frac{1}{2}$ instead of 0:

**Theorem 11** Using $\text{Neutral}_{\mathbb{A}} = \frac{1}{2}$, sigmoid direct aggregation semantics satisfies Anonymity, Equivalence, Directionality, Conservativity, Initial Monotony, Parent Monotony, Neutrality, Impact, Reinforcement, Causality, Neutralisation, Continuity and Reverse impact. Independence is satisfied if $d$ is globally fixed. Since sigmoid functions are non-linear, Linearity cannot be satisfied; instead only the weaker Initial Monotony holds.
It is straightforward to modify sigmoid direct aggregation semantics such that it works with the interval \((-1, 1]\) instead of \((0, 1)\). This might even be considered as more natural, because then 0 is the neutral value, and not \(\frac{1}{2}\). We here have chosen \((0, 1)\) because this interval is used more frequently in the literature\(^7\) and 0 and 1 can roughly be thought of as false and true.

6 Comparison to previous work

Generalising the classical work by Dung\(^5\) to rank-based argumentation, Amgoud et al.\(^2\) have introduced weighted argumentation graphs. Moreover, they restrict weightings to \([0, 1]\). Thus they consider a non-empty finite set \(A\) of arguments, a weighting \(w : A \to [0, 1]\), and an attack relation \(R_a \subseteq A \times A\). Amgoud et al. have also considered support argumentation graphs\(^1\), which are similar, except that \(R_a\) is replaced by a support relation \(R_s \subseteq A \times A\).

It is straightforward to organise \(A\) into a vector. Then \(R_a\) and \(R_s\) can be organised as incidence matrices \(G_a\) and \(G_s\). A combined attack/support graph then leads to a \(\text{wasa}\) in our sense by setting \(G := G_s - G_a\).

6.1 Recursive Sigmoid Aggregation Semantics

Both the h-categorizer semantics of \(^2\) (for attack relations) and the aggregation based semantics of \(^1\) (for support relations) work with a summation of the attacks and supports respectively. They can be combined into one semantics for \(\text{wasa}\) as follows (note that weights are restricted to \([0, 1]\)):

\[
\text{Definition 13 (Recursive Sigmoid Aggregation Function) Let } A = (A, G, w) \text{ be a } \text{wasa} \text{ such that } w : A \to [0, 1]. \text{ For } i \in \mathbb{N}, \text{ the recursive sigmoid aggregation function } f_{rsig}^i \text{ from } A \text{ to } [0, 1] \text{ is defined as follows: for any } a \in A, \text{ for } i \in \{0, 1, 2, \ldots\}, \text{ if } i = 0, \text{ then } f_{rsig}^i(a) = w(a), \text{ otherwise }
\]

\[
f_{rsig}^i(a) = \begin{cases} 
\frac{w(a)}{1 - s^i(a)} & \text{if } s^i(a) \leq 0 \\
\frac{w(a) + s^i(a)}{1 + s^i(a)} & \text{if } s^i(a) \geq 0
\end{cases}
\]

where, for any \(a \in A\):

\[
s^i(a) = \sum_{b \in \text{Sup}_a} f_{rsig}^{i-1}(b) - \sum_{b \in \text{Att}_a} f_{rsig}^{i-1}(b)
\]

\[
\text{Definition 14 (Recursive Sigmoid Aggregation Semantics) The recursive sigmoid aggregation semantics is a function } rsig \text{ transforming any } \text{wasa } A = (A, G, w) \text{ into a weighting on } A \text{ such that for any } a \in A
\]

\[
\text{Deg}_{rsig}^A(a) = \lim_{i \to \infty} f_{rsig}^i(a)
\]

\(^6\)Also other intervals are possible, but seem less natural.

\(^7\)More precisely, the literature used the closed interval \([0, 1]\).
Example 13 The function $f_i^{rsig}$ does not converge in general. Consider the WASA

```
  3 4
 a ---- b 1/4
  3 4
c ---- d 1/4
```

Then $f_{2i}^{rsig} = (\frac{3}{4}, \frac{3}{4}, \frac{3}{4})^T$ and $f_{2i+1}^{rsig} = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})^T$.

This shows that a naïve combination of the semantics of attacks from [2] with the semantics of supports from [1] is not possible.

In the next two subsections, we will study two modifications of this semantics that hopefully will lead to convergence.

However, as shown in [1], $f_i^{rsig}$ does converge when graphs are restricted to support relations only. This is called aggregation-based semantics in [1].

**Theorem 12** Aggregation-based semantics [1], which is Deg$_A^{rsig}$ restricted to graphs with only support relations, satisfies all of our mandatory characteristics.

In [1], two more semantics are defined. One is top-based semantics. It differs from aggregation-based semantics in that multiple supports for a given argument are not summed up, but only the maximum support is considered. Hence, the number of supported is ignored, only their quality matters. By contrast, reward-based semantics “favours the number of supporters over their quality” [1]. This is achieved by defining $f_i$ as, in binary representation, $0.111\ldots1$ ($m$ ones, where $m$ is the number of supporters), plus the strength of the supporters which is normalised in a way such that it has effect only in the subsequent bits. Both semantics fulfil specialised optional axioms. The principles of these semantics can easily be carried over to our (sigmoid) direct aggregation semantics. However, these semantics are not so interesting for our use case. Moreover, top-based semantics does not even satisfy all the axioms which are said to be mandatory in [1]. Therefore, we refrain from developing these semantics in detail here.

### 6.2 Recursive Damped Aggregation Semantics

Our first attempt to modify the sigmoid aggregation semantics in order to make it convergent removes the sigmoid character of the functions and uses linear functions instead. Convergence is ensured by dividing $s^i$ by a damping factor $d >$ indegree$(G)$, such that its components range between -1 and 1. This allows us to get rid of $s^i$ in the denominator and use functions linear in $s^i$ instead of a sigmoid function.

**Definition 15 (Recursive damped aggregation function)** Let $\mathcal{A} = \langle \mathcal{A}, G, w \rangle$ be a WASA such that $w : \mathcal{A} \rightarrow [0,1]$. Let the damping factor $d$ be such that $d \geq 1$. For $i \in \mathbb{N}$, the recursive damped aggregation function $f_{id}^{\mathcal{A}}$ from $\mathcal{A}$ to $[0,1]$ is defined as follows: for any $a \in \mathcal{A}$, for $i \in \{0, 1, 2, \ldots\}$, if $i = 0$, then $f_{0d}^{\mathcal{A}}(a) = w(a)$, otherwise
\[
\begin{align*}
f^{rd}_i(a) = \begin{cases} 
  w(a)(1 + s^i(a)) & \text{if } s^i(a) \leq 0 \\
  w(a) + (1 - w(a))s^i(a) & \text{if } s^i(a) \geq 0 
\end{cases}
\end{align*}
\]
where, for any \( a \in \mathcal{A} \):
\[
s^i(a) = \frac{1}{d} \times \left( \sum_{b \in \text{Sup}_A(a)} f^{rd}_{i-1}(b) - \sum_{b \in \text{Att}_A(a)} f^{rd}_{i-1}(b) \right)
\]
An short matrix notation is
\[
f^{rd}_i = w + \text{Diag}(p(w, s^i))s^i
\]
where \( s^i = \frac{1}{d}Gf^{rd}_{i-1} \)
\[
p(w, s)(x) = \begin{cases} 
  w(x) & \text{if } s(x) < 0 \\
  1 - w(x) & \text{if } s(x) \geq 0 
\end{cases}
\]
Here, \( \text{Diag} \) uses a vector to fill the diagonal of a matrix, which is otherwise zero.

**Conjecture 1** For \( d > \text{indegree}(G) \), \( \lim_{i \to \infty} f^{rd}_i \) converges.

### 6.3 Damped Dogged Semantics

Our second modification of the sigmoid aggregation semantics that shall reach convergence keeps the sigmoid character but avoids the case distinction of the previous subsections.

**Definition 16 (Dogged Function)** Let \( \sigma \) be any sigmoid function, see Sect. 5.5 for examples. Let \( \mathcal{A} = \langle \mathcal{A}, G, w \rangle \) be a WASA such that \( w : \mathcal{A} \to [0, 1] \). Let the damping factor \( d \) be such that \( d \geq 1 \). The dogged function \( f^{\sigma}_i \) from \( \mathcal{A} \) to \([0, 1]\) is defined as follows: for any \( a \in \mathcal{A} \), for \( i \in \{0, 1, 2, \ldots \} \), if \( i = 0 \), then \( f^{\sigma}_0(a) = w(a) \), otherwise
\[
f^{\sigma}_i(a) = \begin{cases} 
  1 & \text{if } w(a) = 1 \\
  \sigma(s^i(a) + \sigma^{-1}(w(a))) & \text{if } 0 < w(a) < 1 \\
  0 & \text{if } w(a) = 0 
\end{cases}
\]
where, for any \( a \in \mathcal{A} \):
\[
s^i(a) = \frac{1}{d} \times \left( \sum_{b \in \text{Sup}_A(a)} f^{\sigma}_{i-1}(b) - \sum_{b \in \text{Att}_A(a)} f^{\sigma}_{i-1}(b) \right)
\]
Example 14 For $d = 1$ and $\sigma_1(x) = \frac{1}{1+e^{-x}}$ or $\sigma_2(x) = \frac{\arctan(x)}{\pi} + \frac{1}{2}$ or $\sigma_3(x) = \frac{1+|x|+x}{2(1+|x|)}$, $\lim_{i \to \infty} f_i^{\sigma_j}$ does not converge in general. Consider the following graph

with initial plausibility $0.85$ for every node. For large enough $i$, we have

$$f_{2n}^{\sigma_1} \approx (0.386435, 0.529751, 0.357394, 0.236454, 0.236454)^T$$
$$f_{2n+1}^{\sigma_1} \approx (0.621398, 0.705838, 0.585527, 0.497027, 0.497027, 0.4970277)^T$$

$\sigma_2$ and $\sigma_3$ exhibit a similar behaviour (with the same graph and initial plausibilities).

Conjecture 2 For $d > \text{indegree}(G)$, $\lim_{i \to \infty} f_i^{\sigma}$ converges.

7 Conclusion and Future work

We have shown that bipolar argumentation graphs can be equipped with a weighting (ranked-based) semantics, both for weights ranging over real numbers as well as for weights in the range $(0, 1)$. The neutral value is 0 in the former case and $\frac{1}{2}$ in the latter case. Both semantics fulfil suitable characteristics. E.g. the computed acceptability degree of an argument is monotonic both in the initial plausibility and in the set of supporting arguments. These characteristics have been taken from the literature and suitably generalised and strengthened.

The comparison of our semantics to related work in the literature (see also Table 1) naturally lead to further (recursively defined) bipolar semantics (see section 6), the convergence of which is still open. Note that these semantics are defined over $[0, 1]$ but still use 0 as the neutral value. We think that it is conceptually more convincing to use $\frac{1}{2}$ as neutral value for bipolar semantics with weights in $[0, 1]$, as we did for Sigmoid direct aggregation semantics in section 5.5.

Future work should consider the questions whether a bipolar semantics for compact intervals like $[0, 1]$ or $\mathbb{R} \cup \{-\infty, \infty\}$ are possible. One way would be to prove the bipolar recursive semantics developed in in section 6 to be convergent. Another way would be to extend sigmoid direct aggregation semantics (see section 5.5) from $(0, 1)$ to $[0, 1]$. An obvious solution would define $\sigma(-\infty) = 0$ and $\sigma(\infty) = 1$. However, then a major difficulty is the development of a suitable arithmetics for the extended real line that keeps Thm. 9 true.

Another future direction is to equip attack and support relations with weights, e.g. in the interval $[-1, 1]$. See [15, 16] for work in this direction, but in a different context: only attacks are equipped with weights, not the arguments themselves.
| Semantics                  | weight range | neutral value | convergent | bounded | reverse impact |
|----------------------------|--------------|---------------|------------|---------|----------------|
| Direct aggregation         | $\mathbb{R}$ | 0             | yes        | no      | yes            |
| Sigmoid direct aggregation | $(0, 1)$     | $\frac{1}{2}$ | yes        | no      | yes            |
| Recursive sigmoid          | $[0, 1]$     | 0             | no         | yes     | no             |
| Recursive damped aggregation | $[0, 1]$   | 0             | ?          | yes     | no             |
| Damped dogged              | $[0, 1]$     | 0             | ?          | yes     | no             |

Figure 1: Overview of the different semantics.

Also, the study of characteristics leaves some open questions. For example, is it possible to generalise Counting in a way that one does not consider exactly the same set of attackers, but a set of comparable attackers?

Also, we would like to use our framework to define a semantics for the Argument Interchange Format (AIF, [17]) that is simpler and more direct than the one given in the literature [18].

Finally, large argumentation graphs will benefit from a modular design; e.g. in [19] they are often divided into subgraphs, e.g. by drawing boxes around some groups of arguments. The characteristics of our semantics suggest that modularity can be obtained by substituting suitable subgraphs with discrete graphs whose arguments are initially weighted with their degrees in the original graph.

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Omitted proofs

**Proof of Thm. 1.** Let $\mathcal{A} = \langle A, G, w \rangle$ be a wasa such that $a \in A$. Further assume that there is no $b \in A$ such that $\deg_{\mathcal{A}}^S(b) \neq \text{Neutral}_S$ and $b \in \att_{\mathcal{A}}(a) \cup \sup_{\mathcal{A}}(a)$. Let $c_1, \ldots, c_n$ be an enumeration of all arguments in $\att_{\mathcal{A}}(a) \cup \sup_{\mathcal{A}}(a)$. Neutrality entails $\deg_{\mathcal{A}|c_1,\ldots,c_n}^S = \deg_{\mathcal{A}}^S$. Since $\att_{\mathcal{A}|c_1,\ldots,c_n}(a) = \sup_{\mathcal{A}|c_1,\ldots,c_n}(a) = \emptyset$, Conservativity entails $\deg_{\mathcal{A}|c_1,\ldots,c_n}^S(a) = w(a)$. Thus, $\deg_{\mathcal{A}}^S(a) = w(a)$. □

**Proof of Thm. 2.** Assume a wasa $\mathcal{A} = \langle A, G, w \rangle$ such that $w(a) = w(b), \sup_{\mathcal{A}}(a) = \sup_{\mathcal{A}}(b) \cup \{x\}$ and $\deg_{\mathcal{A}}^S(x) = \text{Neutral}_S$. Note that $\att_{\mathcal{A}|x}(a) = \att_{\mathcal{A}|x}(b), \sup_{\mathcal{A}|x}(a) = \sup_{\mathcal{A}|x}(b)$. By definition $\att_{\mathcal{A}}(a) = \att_{\mathcal{A}|x}(a) = \att_{\mathcal{A}}(b) = \att_{\mathcal{A}|x}(b) = \emptyset$. Thus, Parent Monotony entails $\deg_{\mathcal{A}|x}^S(a) = \deg_{\mathcal{A}|x}^S(b)$. Because of Neutrality $\deg_{\mathcal{A}}^S(a) = \deg_{\mathcal{A}}^S(b)$ and $\deg_{\mathcal{A}|x}^S(a) = \deg_{\mathcal{A}|x}^S(b)$. Therefore, $\deg_{\mathcal{A}}^S(a) = \deg_{\mathcal{A}}^S(b)$. □

**Proof of Thm. 3.** Assume a wasa $\mathcal{A} = \langle A, G, w \rangle$ such that $w(a) = w(b), \sup_{\mathcal{A}}(a) \sup_{\mathcal{A}}(b) = \{x\}$, $\sup_{\mathcal{A}}(b) \sup_{\mathcal{A}}(a) = \{y\}, \deg_{\mathcal{A}}^S(x) > \deg_{\mathcal{A}}^S(y)$, and $\deg_{\mathcal{A}}^S(b) = 1$. Let $\sup_{\mathcal{A}}(a) \{x\} = \sup_{\mathcal{A}}(b) \{y\} = \{c_1, \ldots, c_n\}$. Let $\mathcal{A}' = \{d_0, \ldots, d_n, c_0, \ldots, c_n\}$ be a set of arguments that
Thus, Parent Monotony entails

$$\text{Deg}_{\mathcal{A}^t}(z) = \text{Deg}_{\mathcal{A}}(z).$$

Now we apply Interchangeability $2n + 2$ times in order to disconnect $a$ and $b$ from $x, c_1, \ldots, c_n$ and $y, z, w$, respectively. In the resulting $\text{bwsA} \mathcal{A}^t$ it holds that $\text{Sup}_{\mathcal{A}^t}(a) = \{d_0, d_1, \ldots, d_n\}$ and $\text{Sup}_{\mathcal{A}^t}(b) = \{e_0, e_1, \ldots, e_n\}$, while $\text{Deg}_{\mathcal{A}^t}(a) = \text{Deg}_{\mathcal{A}^t}(b)$.

Now consider the $\text{bwsA} \mathcal{A}'$ that is the result of removing all outgoing arcs from $a$ and $b$ in $\mathcal{A}^t$. $\mathcal{A}'$ consists of three disconnected subgraphs: (i) $a$ and its supporters $d_0, d_1, \ldots, d_n$; (ii) $b$ and its supporters $e_0, e_1, \ldots, e_n$; and (iii) the remainder of $\mathcal{A}$ without $a, b$. Since there is no argument $z \in \mathcal{A}$ such that $z \in \text{Back}_{\mathcal{A}^t}(a)$ or $z \in \text{Detr}_{\mathcal{A}^t}(a)$, it follows from Directionality that $\text{Deg}_{\mathcal{A}^t}(a) = \text{Deg}_{\mathcal{A}^t}(b)$. For the same reason $\text{Deg}_{\mathcal{A}^t}(b) = \text{Deg}_{\mathcal{A}^t}(b)$.

$\mathcal{A}^a = \langle \mathcal{A}^a, G^a, w^a \rangle$ and $\mathcal{A}^b = \langle \mathcal{A}^b, G^b, w^b \rangle$ are defined to match two of the subgraphs in $\mathcal{A}'$. Note that by assumption $w(a) = w(b)$. Further, $G^a = G^b$, since both graphs consists just of one node that is supported by $n + 1$ other nodes.

$$\mathcal{A}^a = \left( \begin{array}{c} a \\ d_0 \\ d_1 \\ \vdots \\ d_n \\ e_n \end{array} \right), \quad G^a = \left( \begin{array}{ccccc} 0 & 1 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{array} \right), \quad w^a = \left( \begin{array}{c} w(a) \\ \text{Deg}_g^a(x) \\ \text{Deg}_g^a(c_1) \\ \vdots \\ \text{Deg}_g^a(c_n) \end{array} \right)$$

$$\mathcal{A}^b = \left( \begin{array}{c} b \\ c_0 \\ c_1 \\ \vdots \\ c_n \end{array} \right), \quad G^b = G^a, \quad w^b = \left( \begin{array}{c} w(a) \\ \text{Deg}_g^a(y) \\ \text{Deg}_g^a(c_1) \\ \vdots \\ \text{Deg}_g^a(c_n) \end{array} \right)$$

Because of Independence and Anonymity it follows $\text{Deg}_{\mathcal{A}^t}(a) = \text{Deg}_{\mathcal{A}^t}(a)$ and $\text{Deg}_{\mathcal{A}^t}(b) = \text{Deg}_{\mathcal{A}^t}(b)$. In a last step we rename the arguments in $\mathcal{A}^b$ and define $\mathcal{A}^r$ such that $\mathcal{A}^a = \mathcal{A}^a, G^r = G^a, w^r = w^b$.

Note that $w^a(a) = w^r(a), \text{Att}_{\mathcal{A}^r}(a) = \text{Att}_{\mathcal{A}^a}(a) = \emptyset$, and $\text{Sup}_{\mathcal{A}^r}(a) = \text{Sup}_{\mathcal{A}^a}(a)$. Further, since by assumption $\text{Deg}_{\mathcal{A}^a}^y < \text{Deg}_{\mathcal{A}^a}^x$, it follows, for any argument $z \in \text{Sup}_{\mathcal{A}^a}(a)$, that $w^r(z) < w^a(z)$, and thus, $\text{Deg}_{\mathcal{A}^r}^z \leq \text{Deg}_{\mathcal{A}^a}^z$ (because of Conservativity). Hence, Parent Monotony entails $\text{Deg}_{\mathcal{A}^r}(a) \leq \text{Deg}_{\mathcal{A}^a}(a)$. Anonymity gives us that $\text{Deg}_{\mathcal{A}^a}^y = \text{Deg}_{\mathcal{A}^a}^x$. Thus, $\text{Deg}_{\mathcal{A}^r}(a) = \text{Deg}_{\mathcal{A}^a}(a) = \text{Deg}_{\mathcal{A}^a}(a) = \text{Deg}_{\mathcal{A}^a}(a) = \text{Deg}_{\mathcal{A}^a}(a) = \text{Deg}_{\mathcal{A}^a}(a) = \text{Deg}_{\mathcal{A}^a}(a)$. Therefore, $1 \leq \text{Deg}_{\mathcal{A}^a}(a)$.
Proof of Thm. 4. Anonymity, Independence, Equivalence: clear, since our definition (when restricted to graphs with supports relations only) is the same definition as in [1].

Non-Dilution: is easily seen to be a special case of Directionality.

Monotony and Counting and Reinforcement: this follows respectively from Parent Monotony and Impact and Reinforcement, using Independence, Interchangeability, Anonymity and Conservativity, as in the proof of Thm. 3 in order to switch from a single-graph property to a two-graph property.

Dummy: see Thm. 2.

Minimality: same as our Conservativity.

Strengthening: given arguments $a, b$ with $w(a) < 1$ and $b \in \text{Sup}_A(a)$, we need to show that $\deg_{A}^S(a) > w(a)$. Now let $\text{Sup}_A(a) = \{b_1, \ldots, b_n\}$. Without loss of generality, we can assume that $a$ is not among the $\{b_1, \ldots, b_n\}$. If not, use Independence and Interchangeability as in the proof of Thm. 3 in order to replace $a$ in $\text{Sup}_A(a)$ by an equivalently weighted node. Now by Conservativity, $\deg_{A|b_1,\ldots,b_n}^S(a) = w(a)$. Repeated application of Impact (noting that there are no attacks, hence no detractors) gives $\deg_{A}^S(a) > w(a)$, unless $w(a) = 1$ (which however we have excluded).

Strengthening Soundness: this follows easily from Causality.

Coherence: in case of $\deg_{A}^S(b) > 0$, this follows easily from Initial Monotony. If $\deg_{A}^S(b) = 0$, use Conservativity and Parent Monotony to show that $\deg_{A}^S(a) > 0$.

Boundedness: we have called this stickiness (in order to distinguish it from our boundedness), and stickiness is proven in Thm. 3.

□

Proof of Thm. 9. Anonymity: The isomorphism $f$ can be organised into a permutation matrix $P$ such that $w' = Pw$ and $G' = PGP^{-1}$. These equations transform $\deg_{A}^{\text{dir}} = w + \frac{1}{\alpha}G\deg_{A}^{\text{dir}}$ (holding by Thm. 3) into $P\deg_{A}^{\text{dir}} = w' + \frac{1}{\alpha}G'P\deg_{A}^{\text{dir}}$. Hence, again by by Thm. 3, $P\deg_{A}^{\text{dir}} = \deg_{A}^{\text{dir}}$.

Equivalence: Since the bijective functions $f$ and $g$ must have disjoint domains and images, they can be combined into a permutation matrix $P$ that behaves as identity outside the union of $f$’s and $g$’s domains. The assumptions then can be written more compactly as

$$w(a) = w(b)$$
$$P\deg_{A}^{\text{dir}} = \deg_{A}^{\text{dir}}$$
$$\text{Parent}_A(a)P = \text{Parent}_A(b)$$

Now by Thm. 3, $\deg_{A}^{\text{dir}}(b) = w(b) + \frac{1}{\alpha}G\deg_{A}^{\text{dir}}(b)$. This is $w(b) + \frac{1}{\alpha}\text{Parent}_A(b)\deg_{A}^{\text{dir}}$, which by (2) and (4) is $w(a) + \frac{1}{\alpha}\text{Parent}_A(a)P\deg_{A}^{\text{dir}}$. By (3), we arrive at $w(a) + \frac{1}{\alpha}\text{Parent}_A(a)\deg_{A}^{\text{dir}}$, which is $w(a) + \frac{1}{\alpha}G\deg_{A}^{\text{dir}}(a)$, and again by Thm. 3, this is $\deg_{A}^{\text{dir}}(a)$.

Directionality: We prove the following lemma

$$\forall y \in \text{Back}_A(x) \cup \text{Detr}_A(x) \cup \{x\}, \ G^i w(y) = G^i w(y)$$
by induction over \( i \). For \( i = 0 \), both sides of the equation are \( w(y) \). Now let us prove the statement for \( i + 1 \): \( G^{i+1}w(y) = GG^iw(y) = Parent_{A'}(y)G^iw \). Since all for \( y' \in Parent_{A'}(y), \) we have \( y' \in Back_{A}(x) \cup Detr_{A}(x) \cup \{ x \} \), by induction hypothesis, \( G^{i}w(y') = G^{i}w(y') \) for such \( y' \), and hence \( Parent_{A}(y)G^iw = Parent_{A}(y)G^{i+1}w \). Now by the assumption, \( y \neq a_j \), and hence \( Parent_{A}(y) = Parent_{A'}(y) \). Thus \( Parent_{A}(y)G^iw = Parent_{A}(y)G^{i+1}w = Parent_{A}(y)G^{i+1}w(y) = G^{i+1}w(y) \). Hence altogether, \( G^{i+1}w(y) = G^{i+1}w(y) \), and the lemma is proved. Since the lemma applies in particular to \( y = x \), we get \( \text{Deg}_{A',d}(x) = \sum_{i=0}^{\infty} (\frac{1}{G})^i w(x) = \sum_{i=0}^{\infty} (\frac{1}{G})^i w(x) = \text{Deg}_{A,d}(x) \).

Conservativity: By Thm. 6, \( \text{Deg}_{A,d}(a) = w(a) + \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \). By the assumption, \( \text{Parent}_{A}(a) = (0...0) \), so \( \text{Deg}_{A,d}(a) = w(a) \).

Initial Monotony: Using the assumptions and Thm. 6, we have \( \text{Deg}_{A,d}(a) = w(a) + \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \).\( \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \).\( \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \).

Neutrality: By Thm. 6, \( \text{Deg}_{A,d}(w) = w + \frac{1}{G} \text{Deg}_{A,d}(w) \). Since \( \text{Deg}_{A,d}(a) = 0 \), this is \( w + \frac{1}{G} \text{Deg}_{A,d}(a) \), where \( G \) is \( G \) with the columns for \( a \) replaced by zeros. Replacing \( (G) \) also the row for \( a \) with zeros leads to \( G' \), where \( A|_{b} = \langle A', G', w' \rangle \). Therefore, \( w + \frac{1}{G} \text{Deg}_{A,d}(a) \) and \( w' + \frac{1}{G} \text{Deg}_{A,d}(a) \) are equal except possibly for \( a \). However, on \( a \) they both are 0. Thus \( \text{Deg}_{A,d} = \text{Deg}_{A,d} \).

Parent Monotony: We have

\[
\begin{array}{l}
\text{Deg}_{A',d}(a) \\
\text{Thm. 6} \quad w(a) + \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \quad \square \\
\text{Thm. 6} \quad w'(a) + \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \quad \square \\
\text{Thm. 6} \quad w'(a) + \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \quad \square \\
\text{Thm. 6} \quad w'(a) + \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \quad \square \\
\text{Thm. 6} \quad w'(a) + \frac{1}{G} \text{Parent}_{A}(a) \text{Deg}_{A,d}(a) \quad \square \\
\end{array}
\]

Impact: We only show the first half, the second half being dual. Let \( A|_{b} = \langle A', G', w' \rangle \).

We show the following lemma:

\[
\begin{align*}
x \neq b \land b \notin \text{Back}_{A}(x) & \Rightarrow \text{Deg}_{A,d}(x) \leq \text{Deg}_{A|_{b},d}(x) \\
b \notin \text{Detr}_{A}(x) & \Rightarrow \text{Deg}_{A,d}(x) \geq \text{Deg}_{A|_{b},d}(x)
\end{align*}
\]

This follows from

\[
\begin{align*}
x \neq b \land b \notin \text{Back}_{A}(x) & \Rightarrow G^i w(x) \leq G^i w'(x) \\
b \notin \text{Detr}_{A}(x) & \Rightarrow G^i w(x) \geq G^i w'(x)
\end{align*}
\]

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which we show simultaneously by induction over \( i \). The induction base \( i = 0 \) follows since \( w \) and \( w' \) only differ in that \( w'(b) = 0 \). Concerning \( 5 \) for \( i+1 \), \( G^{i+1}w(x) = GG'^{i}w(x) = Parent_{\Lambda}(x)G^{i}w = Parent_{\Lambda}(x)G'^{i}w' (\ast) Parent_{\Lambda|b}(x)G'^{i}w' = G'^{i}w'(x) = G'^{i+1}w'(x) \). Now for \( (\ast) \), we need to show

- \( y \in Sup_{\Lambda}(x) \) implies \( G^{i}w(y) \leq G'^{i}w(y) \). This follows from the induction hypothesis since \( y \neq b \land b \notin Back_{\Lambda}(y) \). The latter follows since \( y = b \) as well as \( b \in Back_{\Lambda}(y) \) both would imply \( b \in Back_{\Lambda}(x) \).

- \( y \in Att_{\Lambda}(x) \) implies \( G^{i}w(y) \geq G'^{i}w(y) \). This follows from the induction hypothesis since \( b \notin Detr_{\Lambda}(y) \). The latter follows since \( b \in Detr_{\Lambda}(y) \) would imply \( b \in Back_{\Lambda}(x) \).

\( (\ast\ast) \) holds because \( Parent_{\Lambda}(x) \) and \( Parent_{\Lambda|b}(x) \) can only differ in the column for \( b \) — but \( G'^{i}w \) has a zero row for \( b \).

\( 6 \) is shown similarly.

Finally, \( \text{Deg}_{G,a,d}^{\text{dir}}(a) = Parent_{\Lambda}(a)\text{Deg}_{G,a,d}^{\text{dir}} (\ast\ast) Parent_{\Lambda|b}(a)\text{Deg}_{G,a,d}^{\text{dir}} = \text{Deg}_{G,a,d}^{\text{dir}}(a) \). \( (\ast\ast\ast) \) holds because

- for \( y \in Sup_{\Lambda}(a) \), we have \( y \neq b \) (because \( b \) attacks \( a \)) and \( b \notin Back_{\Lambda}(y) \). By the lemma, \( \text{Deg}_{G,a,d}^{\text{dir}}(y) \leq \text{Deg}_{G,a,d}^{\text{dir}}(y) \).

- for \( y \in Att_{\Lambda}(a) \), we have \( b \notin Detr_{\Lambda}(y) \). By the lemma, \( \text{Deg}_{G,a,d}^{\text{dir}}(y) \leq \text{Deg}_{G,a,d}^{\text{dir}}(y) \).

Since \( Parent_{\Lambda|b}(a) \) is \( Parent_{\Lambda}(a) \) with the column for \( b \) set to 0, but \( Parent_{\Lambda}(a)(b) < 0 \) and \( \text{Deg}_{G,a,d}^{\text{dir}}(b) > 0 \), \( (\ast\ast\ast) \) follows.

Reinforcement: Concerning 1, the assumptions imply \( Parent_{\Lambda}(a)\text{Deg}_{G,a,d}^{\text{dir}} > Parent_{\Lambda'}(a)\text{Deg}_{G,a,d}^{\text{dir}} \). Then \( \text{Deg}_{G,a,d}^{\text{dir}}(a) = Parent_{\Lambda}(a)\text{Deg}_{G,a,d}^{\text{dir}} > Parent_{\Lambda}(a)\text{Deg}_{G,a,d}^{\text{dir}} (\ast) Parent_{\Lambda}(a)\text{Deg}_{G,a,d}^{\text{dir}} = \text{Deg}_{G,a,d}^{\text{dir}}(a) \), where \( (\ast) \) holds because the attackers and supports of \( a \) agree for \( \Lambda \) and \( \Lambda' \).

The proof of 2 is analogous.

Causality: follows from Thm. \( 1 \).

Neutralisation: Using Thm. \( 6 \), we get \( \text{Deg}_{G,a,d}^{\text{dir}}(b) = w + \frac{1}{3} G\text{Deg}_{G,a,d}^{\text{dir}} (\ast) w + \frac{1}{3} G'\text{Deg}_{G,a,d}^{\text{dir}} \), where \( (\ast) \) follows from the assumptions. By the uniqueness part of Thm. \( 6 \), \( \text{Deg}_{G,a,d}^{\text{dir}} = \text{Deg}_{G,a,d}^{\text{dir}}' \).

Continuity: Matrix multiplication is continuous.

Linearity: Let \( i \) be such that \( a = a_i \). Then \( c_1 = \sum_{1 \leq j \leq n, j \neq i} P_{ij}^{G,d}w_j \) and \( c_2 = P_{ii}^{G,d} \).

Interchangeability: since \( a_j \) and \( a_k \) have the same degree, the equation in Thm. \( 6 \) is not affected by the interchange.

Reverse impact: choose \( \text{Deg}_{G,a,d}^{\text{dir}}(b) = -1 \) and proceed similar to the proof of Impact. \( \square \)
Proof of Thm. 11: The proof parallels that of Thm. 9, using Equation 1 instead of Thm. 6 and noting that both $\sigma$ and $\sigma^{-1}$ are inverses of each other, and each of them is strictly monotone, continuous and commutes with permutation matrix multiplication. For Neutrality and Impact use that $\sigma(0) = \frac{1}{2}$. □

In the proof of Thm. 12 below, we essentially rely on the following fixed-point property, which is Theorem 9 in [1]:

**Theorem 13**

$$\text{Deg}_{\mathcal{A}}^{\text{rsig}}(a) = w(a) + (1 - w(a)) \frac{\sum_{b \in \text{Sup}_{\mathcal{A}}(a)} \text{Deg}_{\mathcal{A}}^{\text{rsig}}(b)}{1 + \sum_{b \in \text{Sup}_{\mathcal{A}}(a)} \text{Deg}_{\mathcal{A}}^{\text{rsig}}(b)}$$

or recast in matrix notation

$$\text{Deg}_{\mathcal{A}}^{\text{rsig}} = w + (I - \text{Diag}(w)) f(G \text{Deg}_{\mathcal{A}}^{\text{rsig}})$$

where $f(x) = x^{1+} = x + \frac{x}{1-x}$ is applied point-wise to a vector, and Diag uses a vector to fill the diagonal of a matrix, which is otherwise zero.

Proof of Thm. 12: Anonymity, Independence, Equivalence: clear, since our definition (when restricted to graphs with supports relations only) is the same definition as in [1]. The same holds for Conservativity, which however is called Minimality in [1].

Neutrality, Initial Monotony, Parent Monotony, Reinforcement, Interchangeability, Neutralisation: analogous to the proof of Thm. 9, where now Thm. 13 plays the role of Thm. 6, and noting that the function $f$ is strictly monotonic. Neutralisation additionally needs uniqueness for Thm. 13, which can be shown using the convergence proof.

Directionality: Since there are no attack relations, we can ignore detractors. The proof is analogous to the proof of Thm. 9, we still detail it here to indicate the necessary modifications. We prove the following lemma (where we need to make $\mathcal{A}$ explicit as parameter of $f^{\text{rsig}}$):

$$\forall y \in \text{Back}_{\mathcal{A}}(x) \cup \{x\}, f^{\text{rsig}}_{\mathcal{A},i}(y) = f^{\text{rsig}}_{\mathcal{A}',i}(y)$$

by induction over $i$. For $i = 0$, both sides of the equation are $w(y)$. Now let us prove the statement for $i + 1$: Since all for $y' \in \text{Parent}_{\mathcal{A}}(y)$, we have $y' \in \text{Back}_{\mathcal{A}}(x) \cup \{x\}$, by induction hypothesis, $f^{\text{rsig}}_{\mathcal{A},i}(y') = f^{\text{rsig}}_{\mathcal{A}',i}(y')$ for such $y'$, and hence $\text{Parent}_{\mathcal{A}}(y)f^{\text{rsig}}_{\mathcal{A},i} = \text{Parent}_{\mathcal{A}}(y)f^{\text{rsig}}_{\mathcal{A}',i}$. Now by the assumption, $y \neq a_j$, and hence $\text{Parent}_{\mathcal{A}}(y) = \text{Parent}_{\mathcal{A}}(y)$. Thus $\text{Parent}_{\mathcal{A}}(y)f^{\text{rsig}}_{\mathcal{A},i} = \text{Parent}_{\mathcal{A}}(y)f^{\text{rsig}}_{\mathcal{A}',i}$. Hence altogether, $f^{\text{rsig}}_{\mathcal{A},i+1}(y) = w(y) + (1 - w(y))f(\text{Parent}_{\mathcal{A}}(y)f^{\text{rsig}}_{\mathcal{A},i}(y)) = w(y) + (1 - w(y))f(\text{Parent}_{\mathcal{A}}(y)f^{\text{rsig}}_{\mathcal{A}',i}(y)) = f^{\text{rsig}}_{\mathcal{A}',i+1}(y)$, and the lemma is proved. Since the lemma applies in particular to $y = x$, we get $\text{Deg}_{\mathcal{A}}^{\text{rsig}}(x) = \lim_{i \to \infty} f^{\text{rsig}}_{\mathcal{A},i}(x) = \lim_{i \to \infty} f^{\text{rsig}}_{\mathcal{A}',i}(y) = \text{Deg}_{\mathcal{A}'}^{\text{rsig}}(x)$.

Impact: Modify the proof of Impact in Thm. 9 in the same way as we did for Directionality above.

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Causality: follows from Thm. 1.

Continuity: all the operations involved in the definition of $\text{Deg}_A^{\text{r,sig}}$ are continuous. □