Asymptotic and factorial expansions of Euler series truncation errors via exponential polynomials

Riccardo Borghi

Dipartimento di Elettronica Applicata
Università degli Studi “Roma Tre”, Rome, Italy
e-mail: borghi@uniroma3.it

Abstract
A detailed analysis of the remainder obtained by truncating the Euler series up to the nth-order term is presented. In particular, by using an approach recently proposed by Weniger, asymptotic expansions of the remainder, both in inverse powers and in inverse rising factorials of n, are obtained. It is found that the corresponding expanding coefficients are expressed, in closed form, in terms of exponential polynomials, well known in combinatorics, and in terms of associated Laguerre polynomials, respectively. A study of the divergence and/or of the convergence of the above expansions is also carried out for positive values of the Euler series argument.

1 Introduction
We consider the so-called Euler series (ES henceforth), defined as

\[ \mathcal{E}(z) = \sum_{m=0}^{\infty} (-1)^m \frac{z^m}{m!}, \]

where \(z\) is a nonnegative number (possibly complex). The series in Eq. (1) has a zero convergence radius and gives a coded representation of the function

\[ \mathcal{E}(z) = \exp(1/z) E_1 \left( \frac{1}{z} \right), \]

where \(E_1(x)\) denotes the exponential integral, defined by

\[ E_1(x) = \int_x^{\infty} \frac{\exp(-t)}{t} \, dt, \]

for \(|\arg x| < \pi\). The function in Eq. (2) is called the antilimit of the series in Eq. (1) and, according to Euler, can be viewed as the mathematical entity whose
expansion gives rise to the divergent series. Incidentally, the ES represents the paradigm for many other factorially divergent asymptotic inverse power series occurring in special function theory, or arising from large-order perturbation expansions in theoretical physics.

The resummation of factorially diverging series like the ES can be successfully operated through the use of different strategies. Among them, those based on nonlinear sequence transformations have proved, especially in recent times, to achieve the retrieving action in an effective way. The common feature of several types of sequence transformations is the following decomposition of the nth-order partial sum $s_n = \sum_{k=0}^{n} a_k$ of the starting series:

$$s_n = s + r_n,$$

where $s$ denotes the antilimit and $r_n = -\sum_{n+1}^{\infty} a_k$ the nth-order remainder. For the ES the decomposition in Eq. (4) can be straightforwardly derived by writing

$$s_n = \sum_{m=0}^{n} (-z)^m m!,$$

and by expanding the factorial as $m! = \int_{0}^{\infty} dt \, \exp(-t)t^m$, thus obtaining

$$s_n = \int_{0}^{\infty} dt \, \exp(-t) \sum_{m=0}^{n} (-zt)^m =$$

$$= \int_{0}^{\infty} dt \, \frac{\exp(-t)}{1 + zt} - (-z)^{n+1} \int_{0}^{\infty} dt \, \frac{\exp(-t)t^{n+1}}{1 + zt},$$

where use has been made of the explicit expression of the partial sum of the geometric series, i.e.,

$$\sum_{m=0}^{n} x^m = \frac{1 - x^{n+1}}{1 - x}.$$  

On comparing Eq. (6) with Eqs. (2) and (4), it is seen at once that

$$s = \mathcal{E}(z),$$

$$r_n = -(-z)^{n+1} \int_{0}^{\infty} dt \, \frac{\exp(-t)t^{n+1}}{1 + zt}.$$  

The development of nonlinear sequence transformations is connected to the theory of converging factors. According to it, the remainder $r_n$ is expressed as the product between the first term of the series not included in the

---

1 There is a nice quotation of a letter from Euler to Goldbach (1745) in Ref. [10], which reads: "Summa cuiusque seriei est valor expressionis illius finitae, ex cuius evolutione illa serie oritur," which may be translated from the Latin language as: "The sum of any given series is the value of the specific finite expression whose expansion gave rise to that same series."
partial sum, $a_{n+1}$, and a converging factor, say $\varphi_n$, chosen in such a way that the relationship

$$s_n = s + a_{n+1} \varphi_n,$$

(9)

is satisfied. In particular, from Eq. (8) it follows that, for the ES, an integral representation of the $n$th-order converging factor (or terminant) is (see Ref. [12], Ch. 21)

$$\varphi_n = -\frac{1}{(n+1)!} \int_0^{\infty} dt \frac{\exp(-t) t^{n+1}}{1 + zt}.$$  

(10)

The aim of the present work is to find expansions, both in inverse powers and in inverse rising factorials of $n$, of the converging factor of the ES in Eq. (1) in such a way that the $n$th-order remainder in Eq. (8) can be expressed through the forms

$$r_n = a_{n+1} \sum_{k=0}^{\infty} \frac{c_k}{(n+\alpha)^k},$$

(11)

and

$$r_n = a_{n+1} \sum_{k=0}^{\infty} \frac{d_k}{(n+\alpha)k},$$

(12)

where $\alpha$ denotes a positive parameter, $(\cdot)_k$ denotes the Pochhammer symbol, defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)},$$

(13)

and where the infinite sequences $\{c_k\}$ and $\{d_k\}$ are independent of $n$. In particular, the last prescription revealed crucial for the Levin [15] and the $\delta$ (or Weniger) [20] transformations, which are strictly related to Eqs. (11) and (12), respectively, to be derived. On the other hand, it should also be noted that different types of asymptotic expansions of the converging factors for the exponential integral function have already been found in the past [17, 5], for instance by using the Airey’s approach [2].

To find the closed-form expressions of the sequences $\{c_k\}$ and $\{d_k\}$ we shall use the approach recently introduced by Weniger in Ref. [22]. According to it, the starting point is the decomposition scheme in Eq. (4), from which it follows that the truncation error $r_n$ associated to the partial sum $s_n$ must satisfy the first-order difference equation

$$\Delta r_n = a_{n+1},$$

(14)

or

$$\frac{r_{n+1} - r_n}{a_{n+1}} = 1,$$

(15)

where $\Delta$ denotes the forward difference operator with respect to $n$, i.e., such that $\Delta f(n) = f(n+1) - f(n)$. In Ref. [22] such approach was used to reproduce

\footnote{For an extensive review about applications of the Weniger transformation, see for instance Refs. [10, 21, 6, 7] and references therein.}
the Euler-Maclaurin formula for the remainder of the Dirichlet series for the Riemann zeta function. In the case of the ES, the same approach led to the exact expression of the first few terms of expansions similar to those in Eqs. (11) and (12). In the present work we will show that the whole sequences \( \{c_k\} \) and \( \{d_k\} \) appearing in Eqs. (11) and (12), respectively, can be obtained, for the case \( \alpha = 1 \), through simple, analytical closed-form expressions. In particular, what we will find is that the \( c_k \)'s are expressed by exponential polynomials \([4, 9, 11]\), while the coefficients \( d_k \) turn out to be proportional to associated Laguerre polynomials \([1]\).

2 Theoretical Analysis

2.1 The Weniger approach for building up asymptotic expansions of truncation errors

As anticipated in the previous section, the approach proposed by Weniger in Ref. [22] will be pursued to express, in closed-form terms, the two sequences \( \{c_k\} \) and \( \{d_k\} \) in Eqs. (11) and (12), for \( \alpha = 1 \). We begin with the asymptotic inverse power series, needed for the Levin transformation. The inverse factorial expansion, related to the Weniger transformation, will be subsequently derived starting from the former. However, to obtain the asymptotic expansion of the converging factor as in Eq. (11), following the prescriptions given in Ref. [22], an intermediate step is necessary. It consists in replacing, in Eq. (15), the remainder \( r_n \) by the estimate, say \( r_n^{(m)} \), given by

\[
r_n^{(m)} \simeq a_n \sum_{k=0}^{m} \frac{\gamma_k}{(n + 1)^k} = (-z)^n n! \sum_{k=0}^{m} \frac{\gamma_k}{(n + 1)^k},
\] (16)

where \( \{\gamma_0, \gamma_1, \ldots, \gamma_m\} \) are unknowns quantities which must be independent of \( n \). It should be noted that in Eq. (16) the truncation error of the ES is represented as the last term included in the partial sum multiplied by a truncated inverse power series in \( n + 1 \). The correct converging factor will be then obtained starting from the knowledge of the \( \gamma_k \)'s. The key point in the approach of Ref. [22] consists in substituting from Eq. (16) into Eq. (15) and in requiring that the subsequent equation be satisfied up to the power \( n - m \), i.e., that

\[
\frac{r_{n+1}^{(m)} - r_n^{(m)}}{a_{n+1}} = 1 + \mathcal{O}(n^{-m-1}),
\] (17)

for \( n \to \infty \). After some algebra, it is obtained \([22]\)

\[
\frac{r_{n+1}^{(m)} - r_n^{(m)}}{(-1/x)^{n+1}(n + 1)!} = \frac{x}{n + 1} \sum_{k=0}^{m} \frac{\gamma_k}{(n + 1)^k} + \sum_{k=0}^{m} \frac{\gamma_k}{(n + 2)^k} = 1 + \mathcal{O}(n^{-m-1}),
\] (18)
where, for convenience, it has been set $x = 1/z$. Note that Eq. (18) is asymptotically equivalent to

$$\frac{x}{n} \sum_{k=0}^{m} \gamma_k \frac{n^k}{k!} + \sum_{k=0}^{m} \gamma_k \frac{n^k}{(n+1)^k} = \sum_{k=0}^{m} \delta_k,0 \frac{n^k}{k!} + O(n^{-m-1}),$$

which led to a linear system for the coefficients $\gamma_k$ that was explicitly solved by Weniger in Ref. [22] for $m = 4$.

We are now going to prove that such system can be solved in closed form for any values of $m$. To show this, we start by writing

$$\frac{x}{n} \sum_{k=0}^{m} \gamma_k \frac{n^k}{k!} + \sum_{k=0}^{m} \gamma_k \frac{n^k}{(n+1)^k} = \gamma_0 + \sum_{k=1}^{m+1} \frac{x \gamma_{k-1}}{n^k} + \sum_{k=1}^{m} \frac{\gamma_k}{(n+1)^k},$$

and we note that

$$\frac{1}{(n+1)^k} = \frac{1}{n^k} \frac{1}{(1+1/n)^k} = \frac{1}{n^k} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \frac{(k)}{j!} =$$

$$= (-1)^k \sum_{j=0}^{\infty} \frac{(-1)^j}{n^j} \frac{(k+j-1)!}{j!(k-1)!} = (-1)^k \sum_{j=0}^{\infty} \frac{(-1)^j}{n^j} \frac{(j+1)_{k-1}}{(k-1)!} =$$

$$= (-1)^k \sum_{j=0}^{\infty} \frac{(-1)^j}{n^j} \frac{(j-k+1)_{k-1}}{(k-1)!} = (-1)^k \sum_{j=0}^{\infty} \frac{(-1)^j}{n^j} \frac{(j-1)}{k-1}. $$

Furthermore, on substituting from Eq. (21) into Eq. (20), we have

$$\frac{x}{n} \sum_{k=0}^{m} \gamma_k \frac{n^k}{k!} + \sum_{k=0}^{m} \gamma_k \frac{n^k}{(n+1)^k} =$$

$$= \gamma_0 + \sum_{k=1}^{m+1} \frac{x \gamma_{k-1}}{n^k} + \sum_{k=1}^{m} \sum_{j=k}^{\infty} (-1)^{k+j} \frac{\gamma_k}{n^j} \frac{(j-1)}{k-1} =$$

$$= \gamma_0 + \sum_{k=1}^{m} \frac{x \gamma_{k-1}}{n^k} + \sum_{k=1}^{m} \sum_{j=k}^{\infty} (-1)^{k+j} \frac{\gamma_k}{n^j} \frac{(j-1)}{k-1}$$

$$+ \frac{x \gamma_m}{n^{m+1}} + \sum_{k=1}^{m} \sum_{j=m+1}^{\infty} (-1)^{k+j} \frac{\gamma_k}{n^j} \frac{(j-1)}{k-1} =$$

$$= \gamma_0 + \sum_{k=1}^{m} \frac{x \gamma_{k-1}}{n^k} + \sum_{k=1}^{m} \sum_{j=k}^{\infty} (-1)^{k+j} \frac{\gamma_k}{n^j} \frac{(j-1)}{k-1} + O(n^{-m-1}),$$
or, by interchanging the symbols $j$ and $k$,

$$
\frac{x}{n} \sum_{k=0}^{m} \frac{\gamma_k}{n^k} + \frac{1}{n} \sum_{k=0}^{m} \frac{\gamma_k}{(n+1)^k} = \gamma_0 + \sum_{k=1}^{m} \frac{1}{n^k} \left[ x \gamma_{k-1} + (-1)^k \sum_{j=1}^{k} (-1)^j \left( \frac{k-1}{j-1} \right) \gamma_j \right] + \mathcal{O}(n^{-m-1}).
$$

(23)

Equation (23) together with Eq. (19), leads to the following linear system for the $\gamma_k$’s:

$$
\gamma_0 = 1,
\gamma_k = x \gamma_{k-1} + (-1)^k \sum_{j=1}^{k} (-1)^j \left( \frac{k-1}{j-1} \right) \gamma_j = 0.
$$

(24)

The solution of such system can be expressed in closed form simply by evaluating the quantity

$$
-x \sum_{k=1}^{n} \left( \frac{n-1}{k-1} \right) \gamma_{k-1},
$$

(25)

which, by taking Eq. (24) into account, takes on the form

$$
-x \sum_{k=1}^{n} \left( \frac{n-1}{k-1} \right) \gamma_{k-1} = \sum_{k=1}^{n} \sum_{j=1}^{k} (-1)^{k+j} \left( \frac{n-1}{k-1} \right) \left( \frac{k-1}{j-1} \right) \gamma_j = \sum_{j=1}^{n} (-1)^{2j} \gamma_j \delta_{n,j} = \gamma_n.
$$

(26)

where formula 4.2.4.45 of [18] has been used. It is worth exploring Eq. (26) in a deeper way. Actually, this equation already contains the closed-form expression of the $\gamma_n$ coefficients, as we shall see in a moment.

2.2 The asymptotic expansion of the truncation error and the exponential polynomials

Due to its importance, we rewrite Eq. (26) as

$$
\gamma_n = -x \sum_{k=1}^{n} \left( \frac{n-1}{k-1} \right) \gamma_{k-1}.
$$

(27)

It must be noted that there exist a whole class of functions satisfying the relation in Eq. (27). Such functions are called exponential, or Bell, polynomials [4], and
will be denoted $\phi_n(x)$. These polynomials are defined through the following generating function formula:

$$
\sum_{n=0}^{\infty} \phi_n(x) \frac{t^n}{n!} = \exp \{x [\exp(t) - 1]\},
$$

and have the explicit expansion

$$
\phi_n(x) = \sum_{k=0}^{n} S(n,k) x^k,
$$

where $S(n,k)$ denotes the Stirling number of the second kind, which is defined through

$$
z^n = (-1)^n \sum_{k=0}^{n} (-1)^k S(n,k) (z)_k.
$$

Exponential polynomials satisfy the recurrence relationship

$$
\phi_{n+1}(x) = x[\phi_n'(x) + \phi_n(x)],
$$

with $\phi_0(x) = 1$ and, more importantly, they fulfill the relation

$$
\phi_n(x) = x \sum_{k=1}^{n} \binom{n-1}{k-1} \phi_{k-1}(x),
$$

which, when compared to Eq. (27), shows that the coefficient $\gamma_k$ is proportional to $\phi_k(-x)$ and, by virtue of the initial condition $\gamma_0 = 1$, that

$$
\gamma_k = \phi_k(-x).
$$

Equation (33) represents one of the main results of the present work. According to it, we shall express the $n$th-order remainder of the ES through the following asymptotic series:

$$
r_n = a_n \sum_{k=0}^{\infty} \frac{\phi_k(-1/z)}{(n+1)^k},
$$

which, as pointed out at the beginning of Sec. 2.1, is not yet of the desired form given in Eq. (11). To find the correct sequence $\{c_k\}$ for $\alpha = 1$, it is sufficient to recast Eq. (34) as

$$
r_n = -\frac{a_{n+1}}{z(n+1)} \sum_{k=0}^{\infty} \frac{\gamma_k}{(n+1)^k} = a_{n+1} \sum_{k=1}^{\infty} \frac{-\phi_{k-1}(-1/z)z}{(n+1)^k},
$$

Here and in the following we are going to use, for the Bell polynomials, the notation given in the recent review by Boyadzhiev. Furthermore, note that the numerical evaluation of Bell polynomials $\phi_n(x)$ of arbitrary order is currently implemented, up to arbitrary precision, within the Mathematica platform through the command `BellB[n,x]`.
which, once compared to Eq. (11), leads to
\[
c_k = \begin{cases} 
0, & k = 0, \\
\frac{\phi_{k-1}(-1/z)}{z}, & k > 0.
\end{cases} \tag{36}
\]

2.3 The factorial expansion

The transformation of an inverse power series to a factorial series can be accomplished with the help of the Stirling numbers of the first kind, say \( s(n,j) \), which are defined through
\[
(-1)^n (z)_n = \sum_{j=0}^{n} s(n,j) (-z)^j. \tag{37}
\]

In particular, the Stirling numbers of the first kind occur in the factorial series expansion of an inverse power, namely \cite{22}
\[
\frac{1}{\zeta^k} = (-1)^k \sum_{j=k}^{\infty} (-1)^j \frac{s(j-1,k-1)}{(\zeta)_j} = (-1)^k \sum_{j=1}^{\infty} (-1)^j \frac{s(j-1,k-1)}{(\zeta)_j}, \tag{38}
\]
valid for \( k \geq 1 \), where in the last passage use has been made of the fact that \( s(n,m) = 0 \) when \( m > n \). Accordingly, as pointed out in Ref. \cite{22}, given an asymptotic power series of the form \( \sum_{k=0}^{\infty} c_k / \zeta^k \), the following identity can be established:
\[
\sum_{k=0}^{\infty} \frac{c_k}{\zeta^k} = \sum_{k=0}^{\infty} \frac{d_k}{(\zeta)_k}, \tag{39}
\]
where
\[
d_k = \begin{cases} 
c_0, & k = 0, \\
(-1)^k \sum_{j=1}^{k} (-1)^j s(k-1,j-1) c_j, & k \geq 1.
\end{cases} \tag{40}
\]

Then, on substituting from Eq. (36) into Eq. (40), after some algebra it is found that the expanding coefficients \( \{d_k\} \) in Eq. (12) are given by
\[
d_k = \begin{cases} 
0, & k = 0, \\
\frac{\psi_{k-1}(-1/z)}{z}, & k > 0,
\end{cases} \tag{41}
\]
where the function \( \psi_k(x) \) is defined as
\[
\psi_k(x) = (-1)^k \sum_{j=0}^{k} (-1)^j s(k,j) \phi_j(x). \tag{42}
\]
On substituting from Eq. (41) into Eq. (12), we eventually obtain
\[ r_n = a_{n+1} \sum_{k=1}^{\infty} \frac{-\psi_{k-1}(-1/z)z}{(n+1)k}. \] (43)

In the next section it will be proved that \( \psi_n(x) \) is proportional to the associated Laguerre polynomial of orders \( n \) and -1 [1].

3 The \( \psi_n(x) \) polynomials

From the definition given in Eq. (42), on using Eq. (29) we have
\[
\psi_n(x) = (-1)^n \sum_{j=0}^{n} (-1)^j s(n,j) \sum_{k=0}^{j} S(j,k) x^k =
\]
\[
= (-1)^n \sum_{k=0}^{n} x^k \sum_{j=k}^{n} (-1)^j s(n,j) S(j,k) =
\] (44)
\[
= (-1)^n \sum_{k=0}^{n} x^k \sum_{j=0}^{k} (-1)^j s(n,j) S(j,k),
\]
where in the last passage use has been made of the fact that \( S(j,k) = 0 \) for \( k > j \). The expanding coefficients in Eq. (44) can be given a closed form. To show this, we first recall formula 24.1.4.C of Ref. [1], i.e.,
\[
S(j,k) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \ell^j,
\] (45)
which gives
\[
\sum_{j=0}^{n} (-1)^j s(n,j) S(j,k) = \frac{(-1)^k}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} \sum_{j=0}^{n} (-\ell)^j s(n,j) =
\] (46)
\[
= \frac{(-1)^{k+n}}{k!} \sum_{\ell=0}^{k} (-1)^\ell \binom{k}{\ell} (\ell)_n,
\]
where in the last passage the definition of the generating function of the Stirling numbers of the first kind, given in Eq. (37), has been used. Furthermore, on taking into account that
\[
(\ell)_n = n! \binom{\ell + n - 1}{n},
\] (47)
Eq. (46) becomes
\[ \sum_{j=0}^{n} (-1)^j s(n,j) S(j,k) = (-1)^{k+n} \frac{n!}{k!} \sum_{\ell=0}^{k} (-1)^\ell \left( \begin{array}{c} k \\ \ell \end{array} \right) \left( \frac{\ell + n - 1}{n} \right), \] (48)

and, by using formula 4.2.5.26 of Ref. [18], it is obtained
\[ \sum_{j=0}^{n} (-1)^j s(n,j) S(j,k) = (-1)^n \left( \frac{n-1}{n} \right) \frac{n!}{k!} = (-1)^n \left( \frac{n}{k} \right) \frac{(n-1)!}{(k-1)!}. \] (49)

On substituting from Eq. (49) into Eq. (44), the polynomial expansion of \( \psi_n(x) \) turns out to be, for \( n > 0 \),
\[ \psi_n(x) = (n-1)! \sum_{k=1}^{n} \frac{x^k}{(k-1)!} \left( \frac{n}{k} \right), \] (50)
from which it follows that
\[ \psi_n(x) = n! L_n^{(-1)}(-x), \] (51)
where \( L_n^{(\alpha)}(\cdot) \) denotes the associated Laguerre polynomial of orders \( n \) and \( \alpha \). Finally, on substituting from Eq. (51) into Eq. (43), the factorial expansion of the \( n \)th-order remainder of the ES reads
\[ r_n = -a_{n+1} \sum_{k=0}^{\infty} \frac{k!}{(n+1)_{k+1}} L_k^{(-1)}(1/z)/z, \] (52)
which, together with Eq. (55), constitutes the main result of the present paper.

4 Discussions

4.1 Preliminaries

First of all, we note that the remainder \( r_n \) in Eq. (8) can be evaluated in closed form, and turns out to be given by
\[ r_n = -a_{n+1} \exp \left( \frac{1}{z} \right) \frac{1}{z} E_{n+2} \left( \frac{1}{z} \right), \] (53)
which, once compared with Eqs. (85) and (62), gives at once the following two expansions for the exponential integral function:
\[ E_{n+1}(x) = \sum_{k=0}^{\infty} \frac{\phi_k(-x) \exp(-x)}{n^{k+1}}, \] (54)
and
\[ E_{n+1}(x) = \sum_{k=0}^{\infty} \frac{k!}{(n)_{k+1}} L_k^{(-1)}(x) \exp(-x), \quad (55) \]
respectively. Moreover, Eqs. (54) and (55) can be cast, by using the connection between the exponential integral and the incomplete gamma functions [1], in the following form:
\[ \Gamma(-n, x) = x^{-n} \exp(-x) \sum_{k=0}^{\infty} \frac{1}{n^{k+1}} \phi_k(-x), \quad (56) \]
and
\[ \Gamma(-n, x) = x^{-n} \exp(-x) \sum_{k=0}^{\infty} \frac{k!}{(n)_{k+1}} L_k^{(-1)}(x), \quad (57) \]
respectively. In particular, the expansion in Eq. (56) displays a structure very similar to an asymptotic expansion recently found in Ref. [8] for the (lower) incomplete gamma function \( \gamma(\lambda, x) \) [1], with \( \lambda > 0 \).

4.2 Analysis of the convergence of the two series for positive \( x \)

It is worth studying the character of the series in Eqs. (54) and (55), for \( x > 0 \). As far as the series in Eq. (54) is concerned, we use the asymptotics, for \( k \gg 1 \), of the Bell polynomials, recently reviewed in Ref. [13]. In particular, for \( x > 0 \) we have [13]
\[ \exp(-x) \phi_k(-x) \approx k! \sqrt{\frac{2}{\pi k}} \exp \left\{ k \left[ \log \frac{\sin \varphi}{\varphi} - \frac{\sin \varphi}{\varphi} \cos \varphi \right] \right\} \left[ \left( \frac{\varphi}{\sin \varphi} - \cos \varphi \right)^2 + \sin^2 \varphi \right]^{1/4} \times \sin \left[ k \left( \pi - \varphi + \frac{\sin^2 \varphi}{\varphi} \right) + \eta(\varphi) \right], \quad (58) \]
where
\[ \eta(\varphi) = \frac{\pi}{2} + \frac{1}{2} \arccos \left[ \frac{1 - \varphi \cot \varphi}{(1 - \varphi \cot \varphi)^2 + \sin^2 \varphi} \right]^{1/4}, \quad (59) \]
and where \( \varphi \) is the solution of the equation
\[ \frac{x}{n} = \frac{\sin \varphi}{\varphi} \exp \left( \frac{\varphi}{\sin \varphi} \cos \varphi \right). \quad (60) \]
For any fixed value of \( x \), we are interested in estimating the behavior of \( \exp(-x) \phi_k(-x) \) for \( k \to \infty \). In such limit the solution of Eq. (60) tends to \( \pi^- \). In the same
limit, $\eta(\varphi) \to \pi/2$, so that the whole term $\sin \left[ k \left( \pi - \varphi + \frac{\sin^2 \varphi}{\varphi} \right) + \eta(\varphi) \right]$ can be replaced by 1. Secondly, in the limit $\varphi \to \pi^-$, we have

$$\frac{1}{\left[ \left( \frac{\varphi}{\sin \varphi} - \cos \varphi \right)^2 + \sin^2 \varphi \right]^{1/4}} \approx \left( \frac{\sin \varphi}{\varphi} \right)^{1/2}, \quad (61)$$

and

$$\exp \left\{ k \left[ \log \frac{\sin \varphi}{\varphi} - \frac{\sin \varphi \cos \varphi}{\varphi} \right] \right\} \approx \left( \frac{\sin \varphi}{\varphi} \right)^k \exp \left( k \frac{\sin \varphi}{\varphi} \right), \quad (62)$$

so that

$$\exp(-x) \phi_k(-x) \approx k! \sqrt{\frac{2}{\pi k}} \left( \frac{\sin \varphi}{\varphi} \right)^{k+1/2} \exp \left( k \frac{\sin \varphi}{\varphi} \right), \quad (63)$$

which, once inserted into Eq. (54), shows that the asymptotic series displays a factorial divergence.

We now prove that the factorial series in the r.h.s. of Eq. (55) is, for $x > 0$, convergent. This can be done by noting that, for $n \geq 1$,

$$\frac{k!}{(n)_{k+1}} = \frac{1}{k+1} \frac{(k+1)!}{(n)_{k+1}} = \frac{1}{k+1} \frac{1 \times 2 \times \ldots \times (k+1)}{n \times (n+1) \times \ldots \times (n+k)} \leq \frac{1}{k+1}, \quad (64)$$

which, by taking into account the asymptotics of $L_k^{(-1)}(x)$ for large $k$, namely,

$$L_k^{(-1)}(x) \exp(-x) \approx \frac{1}{\sqrt{\pi}} \exp(-x/2) x^{1/4} k^{-3/4} \cos \left( 2\sqrt{kx} + \frac{\pi}{4} \right), \quad (65)$$

leads, for sufficiently high values of $k$, to

$$\left| \frac{k! L_k^{(-1)}(x) \exp(-x)}{(n)_{k+1}} \right| \leq \frac{1}{k+1} \left| L_k^{(-1)}(x) \exp(-x) \right|$$

$$\leq \frac{1}{k+1} \exp(-x/2) x^{1/4} \frac{1}{\sqrt{\pi}} \frac{1}{k^{3/4}} < \exp(-x/2) x^{1/4} \frac{1}{\sqrt{\pi}} \frac{1}{k^{7/4}}, \quad (66)$$

that proves the absolute convergence of the factorial series.

### 4.3 Some remarks for negative values of the ES argument

If $z < 0$, the ES in Eq. (1) becomes a nonalternating, divergent asymptotic power series in $z$, which has been used in the literature as a paradigmatic example of a series that cannot be resummed by the Levin and Weniger transformations [14]. The reason for such inability in the resummation process is strictly related to the
5 Conclusions

The understanding of the retrieving action, as well as the development of new types of nonlinear sequence transformations aimed at resumming different classed of divergent series requires the large index asymptotics of the corresponding truncation errors to be investigated. A general approach for achieving such task has recently been proposed in Ref. [22]. In the present paper we showed that, for the (factorially divergent) Euler series, such approach allows the nth-order remainder to be represented via asymptotic and factorial expansions involving exponential and associated Laguerre polynomials, respectively. The convergence of the above expansions has also been investigated.

Acknowledgments

I am indebted to Ernst Joachim Weniger for giving me very useful suggestions. I also thank both reviewers for their remarks and Turi Maria Spinozzi for his...
help during the preparation of the manuscript.

References

[1] M. Abramowitz, I. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1972).

[2] J. R. Airey, “The “converging factor” in asymptotic series and the calculation of Bessel, Laguerre and other functions,” Phil. Mag. 24 (1937) 521-552.

[3] G. A. Baker, Jr., P. Graves-Morris, *Padé Approximants*, 2nd Edition, Cambridge U. P., Cambridge, 1996.

[4] E.T. Bell, “Exponential polynomials,” Ann. of Math. 35 (1934) 258-277.

[5] L. Berg, “On the estimation of the remainder term in the asymptotic expansion of the exponential integral,” Computation 18, (1977) 361-363 (in german).

[6] R. Borghi, “Joint use of the Weniger transformation and hyperasymptotics for accurate asymptotic evaluations of a class of saddle-point integrals,” Phys. Rev. E 78 (2008) 026703-1 - 026703-11.

[7] R. Borghi, “Joint use of the Weniger transformation and hyperasymptotics for accurate asymptotic evaluations of a class of saddle-point integrals. II. Higher-order transformations,” Phys. Rev. E 80 (2009) 016704-1 - 016704-15.

[8] K. N. Boyadzhiev, “A series transformation formula and related polynomials,” Int. J. Math. Math Sci., 2005 (2005), 3849-3866.

[9] K. N. Boyadzhiev, “Exponential polynomials, Stirling numbers, and evaluation of some Gamma integrals,” Abstract and Appl. Analysis, (2009) 68672-1 - 68672-18. doi:10.1155/2009/168672

[10] E. Caliceti, M. Meyer-Hermann, P. Ribeca, A. Surzhykov, U. D. Jentschura, “From Useful Algorithms for Slowly Convergent Series to Physical Predictions Based on Divergent Perturbative Expansions,” Phys. Rep. 446 (2007) 1-96. arXiv:0707.1596v1.

[11] L. Comtet, *Advanced Combinatorics* (Reidel, Dordrecht, 1974).

[12] R.B. Dingle, *Asymptotic expansions: Their derivation and interpretation* (Academic Press, London, 1973).

[13] D. Dominici, “Asymptotic analysis of the Bell polynomials by the ray method,” J. Comput. Appl. Math. 233 (2009) 708-718.
[14] D. Jentschura, “Resummation of nonalternating divergent perturbative expansions,” Phys. Rev. D 62 (2000) 076001-.

[15] D. Levin, “Development of non-linear transformations for improving convergence of sequences,” Int. J. Comput. Math. B 3 (1973) 371-388.

[16] J.C.P. Miller, “A method for the determination of converging factors, applied to the asymptotic expansions for the parabolic cylinder function”, Proc. Cambridge Phil. Soc. 48 (1952) 243-254.

[17] W. Neuhaus und S. Schottlaender, “The development of Airey’s converging factors of the exponential integral to a representation with remainder term,” Computing 15 (1975) 41-52 (in german).

[18] A. P. Prudnikov, Yu. A. Brychkov, O. I. Marichev, Integrals and Series. Vol. I (Gordon Breach Publisher, New York, 1986).

[19] H. Stahl, “Spurious poles in Pad approximation,” J. Comput. Appl. Math. 99 (1998) 511 527.

[20] E. J. Weniger, “Nonlinear sequence transformations for the acceleration of convergence and the summation of divergent series,” Comput. Phys. Rep. 10 (1989) 189-371. Los Alamos Preprint math-ph/0306302 [http://arXiv.org].

[21] E. J. Weniger “Mathematical properties of a new Levin-type sequence transformation introduced by Čížek, Zamastil, and Skála. I. Algebraic theory,” J. Math. Phys. 45 (2004) 1209-1246.

[22] E.J. Weniger, Asymptotic approximations to truncation errors of series representations for special functions, in Iske, A., Levesley, J. (Editors) [2007], Algorithms for Approximation (Springer-Verlag, Berlin), 331 - 348. Proceedings of the Conference “Algorithms for Approximation V”, University College Chester, UK, 16th - 22nd July 2005. Los Alamos Preprint math.CA/0511074 [http://arXiv.org].