Topological Strings and QCD in Two Dimensions

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Abstract. I present a new class of topological string theories, and discuss them in two dimensions as candidates for the string description of large-$N$ QCD. The starting point is a new class of topological sigma models, whose path integral is localized to the moduli space of harmonic maps from the worldsheet to the target. The Lagrangian is of fourth order in worldsheet derivatives. After gauging worldsheet diffeomorphisms in this “harmonic topological sigma model,” we obtain a topological string theory dominated by minimal-area maps. The bosonic part of this “topological rigid string” Lagrangian coincides with the Lagrangian proposed by Polyakov for the QCD string in higher dimensions.

1. Introduction

Strings undoubtedly represent one of the most interesting and universal patterns in Nature, emerging at vastly different scales and in different physical phenomena. The Regge phenomenology of hadrons and the structure of dual models were first indications that the adequate degrees of freedom for the theory of strong interactions at low energies are qualitatively those of a string theory. These ideas received a more specific support soon after QCD had been identified as the proper theory for strong interactions. Two

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basic techniques that reveal some form of string structure in QCD are the expansion in the number of colors, and the strong coupling expansion. Yet, even though the success of QCD in the ultraviolet has been enormous, any progress in the identification of the “stringy” fixed point that governs the theory in the infrared has been very limited (for recent reviews, see [1]).

In the meantime, strings have been recognized as playing an important, although still mostly enigmatic, rôle in quantization of gravity. Ever since, string theory of quantum gravity has been changing profoundly the basic concepts of space and time in quantum theory, and promises to do so even more after it is understood better as a second-quantized theory. Among the fascinating tentative results in this direction are the apparent lack of degrees of freedom in string theory at short distances, an extremely soft behavior of strings in super-Planckian collisions, and the existence of a minimal length/enlarged uncertainty principle, to mention just a few. The idea of a possible topological phase of quantum gravity (and string theory) at short distances also arose in this context. Without any doubt, many aspects of string theory have been elucidated in the context of quantum gravity.

At a more mundane level, however, one may return back to old unsolved problems, and start wondering whether the enormous progress accomplished in string theory during last decade could possibly lead to some new insight, and help in the quest for the string theory of QCD in the infrared. This program has been raised recently by Gross [2] and Gross and Taylor [3], in the context of the exactly solvable, pure-glue QCD in two dimensions. The idea is that the theory, despite having no local degrees of freedom, is rich enough in the content that it leads to a non-trivial string theory. It is exactly solvable on any compact spacetime manifold, thus supplying us with lots of information about the possible character of the string theory. The authors of [3] have been able to interpret the results of the $1/N$ expansion for 2D QCD in terms of simple rules for counting maps between surfaces (very similar results were obtained for Wilson loops on the plane some time ago by Kazakov and Kostov, see [4]). The next step is to try and find the worldsheet Lagrangian for the string theory, with the hope that this two dimensional QCD string Lagrangian could grasp some important aspects of the “stringy” fixed point that supposedly describes the real case of our interest, namely four dimensional QCD in the infrared. This represents an extremely interesting program for string theorists.

A successful identification of the worldsheet theory for two dimensional QCD could also have implications for string theory itself. Assuming that the worldsheet theory for the 2D QCD string is found, we can follow standard rules and construct its corresponding string field theory. This theory could be sufficiently complicated, but in this case –
unlike in the “fundamental” string theory of quantum gravity – we do know the correct spacetime description of the theory, this being given by the QCD Lagrangian itself! Thus, the 2D QCD string theory could even teach us something about the conundrums of string field theory for quantum gravity.

In this talk I present a new class of topological string theories, and discuss them in two dimensions as candidates for the worldsheet description of the QCD string theory. The presentation is based on unpublished results that will appear elsewhere [5].

In §2 I review very briefly some aspects of quantum Yang-Mills theory in two dimensions, its relation to topological Yang-Mills theory, and the interpretation of the $1/N$ expansion of the SU($N$) theory in terms of maps between surfaces as obtained by the authors of [2-4].

In §3, a new class of topological sigma models is defined, as the “worldsheet matter” for the topological string theory that we are about to construct. The path integral of these topological sigma models is localized at the moduli space of harmonic maps from the worldsheet to the target (with respect to a fixed metric on both). To distinguish the theory from the traditional topological sigma model dominated by holomorphic maps, I refer to it as the “harmonic topological sigma model” henceforth. The bosonic part of the Lagrangian is of fourth order in worldsheet derivatives, and is reminiscent of the rigid string Lagrangian proposed by Polyakov [6] some time ago as a candidate for the QCD string in higher dimensions.

In §4 we gauge worldsheet diffeomorphisms in the harmonic topological sigma model. Instead of introducing a dynamical metric on the worldsheet, we will replace the fixed metric by the induced metric. This completes the analogy with the rigid string, and I refer to the theory as the “topological rigid string theory.” The instanton moduli space that dominates the path integral is the space of maps harmonic in their own induced metric, i.e. the moduli space of minimal-area maps.

In §5 I discuss possible variations on the theme of topological rigid strings in dimensions higher than two, in particular in four dimensions.

2. Yang-Mills Theory and Large-$N$ QCD in Two Dimensions

The recent revival of interest in two dimensional Yang-Mills theory has led to very interesting results. The quantum theory is exactly solvable on any Riemann surface, a fact that can be explained from different points of view. The original one, due to Migdal and recently Rusakov [7], uses a RG-invariant lattice formulation. Another point of view, developed by Witten [8], explains the exact solvability in the continuum theory in
terms of localization of the path integral in equivariant cohomology/topological Yang-Mills theory to a moduli space of classical configurations. Since we are interested in the continuum limit of the worldsheet theory for the QCD string, the results of Witten seem to be particularly relevant. In [8] it is shown that after some fermion fields are added to the Yang-Mills theory, the “physical” Yang-Mills Lagrangian can be treated as a BRST invariant observable in the underlying topological Yang-Mills theory. This correspondence then allows one to calculate the partition function of the “physical” theory as a correlation function in the underlying topological theory. In this picture, two dimensional Yang-Mills theory is so special because there is an underlying topological theory that actually governs the calculation of the partition function in the “physical” theory.

The partition function of the Yang-Mills theory on an arbitrary two dimensional surface $M$ of genus $G$ is given by

$$Z \equiv \int DA \mu e^{(1/\tilde{g}^2) \int_M d\mu Tr F_{\mu\nu}F^{\mu\nu}} = \sum_{R}(\text{dim } R)^2 - 2G - \tilde{g}^2 A C_2(R)/2,$$

(1)

where $d\mu$ is the Riemannian volume element and $A$ the area of a fixed metric $G_{\mu\nu}$ on $M$, the sum runs over all irreducible representations of the gauge group, and $C_2(R)$ is the second Casimir of $R$. For $SU(N)$ we can take the large-$N$ limit that keeps $\lambda \equiv \tilde{g}^2 N$ fixed, and expand the exact result (1) in the powers of $1/N$. In [2,3], Gross and Taylor obtained the following expression for the $1/N$ expansion of (1):

$$Z = \sum_{g \geq 0} \frac{1}{N^{2g-2}} \sum_{n,\tilde{n}} e^{-(n+\tilde{n})\lambda A/2} \sum_{i=0}^{2g-2-(n+\tilde{n})(2G-2)} (\lambda A)^i \omega^{n,\tilde{n},i}_{g,G},$$

(2)

where $\omega^{n,\tilde{n},i}_{g,G}$ are calculable coefficients independent of $A$. The expansion is reminiscent of partition functions of string theory, with $1/N$ being the string coupling constant, $\lambda$ the string tension, $g$ the worldsheet genus, and $n$ ($\tilde{n}$) the number of orientation-preserving (-reversing) sheets of the map from the worldsheet to the target.

The authors of [2,3] interpreted the coefficients of this expansion in terms of rules for counting specific maps between surfaces. Maps that are allowed to contribute to the sum are branched covers of the target, with collapsed handles and infinitesimal tubes that connect various sheets of the cover (see [2,3] for details). The string partition function can then be rewritten as an integral over the moduli space $\mathcal{M}$ of such coverings of $M$,

$$Z = \int_\mathcal{M} d\nu W(\nu),$$

(3)
with the location of the branch points, collapsed handles and connecting tubes serving as coordinates on $\mathcal{M}$. The density $W(\nu)$ to be integrated is given essentially by the symmetry factor of the cover (see [2,3]).

The structure of the $1/N$ expansion and the character of maps that contribute to the rules of [2-4] give us some indications about what we can expect from the QCD string theory. There are no terms in (2) with both $n$ and $\tilde{n}$ equal to zero, which indicates that folds of the maps are suppressed and the QCD string is sensitive to its “extrinsic curvature” in the target. This point of view is also supported by the fact that worldsheets of genus zero do not contribute to $Z$, and by the form of the terms exponential in the area. The fact that both orientation-preserving and orientation-reversing covers contribute indicates that the corresponding string theory is non-chiral.

Notice also that the string theoretical expression (2) for the QCD partition function allows us to take the limit $\lambda \to 0$. Although this limit is rather singular from the spacetime point of view, the large-$N$ expansion serves as a nice regulator. Indeed, when we take the $\lambda \to 0$ limit for fixed $n$ and $\tilde{n}$ in (2), we obtain finite results. From the point of view of string theory, this corresponds to the zero tension limit, and the only structure carried by the allowed maps are the so-called $\Omega$-points [3]. It seems very reasonable to expect that in the zero tension limit, the string theory of 2D QCD is topological.

3. Harmonic Topological Sigma Models

In the previous section we have seen that the coefficients of the large-$N$ expansion of the QCD partition function on an arbitrary spacetime surface $M$ has the structure of a sum over classes of maps from the worldsheet to the spacetime. The coefficients of this sum are exponentials of the area of the worldsheet that covers the spacetime times finite polynomials in the area. One of the first observations made by the authors of [3] is that the contribution to the leading term in this polynomial is essentially one from each homotopy sector that is allowed to contribute by Kneser’s formula. As a first step towards the worldsheet Lagrangian for QCD string theory in two dimensions, let us first try to construct a string theory whose partition function is essentially equal to one in each allowed homotopy sector of maps.

The field configurations of the theory are given solely by the maps from a worldsheet $\Sigma$ (of genus $g$) to the spacetime $M$ (of genus $G$),

$$\Phi : \Sigma \to M.$$  (4)
We will be using coordinates $X^\mu$ on the target, and $\sigma^a$ on the worldsheet. Before specializing to two target dimensions, we will work with targets of arbitrary real dimension $D$.

The theory that we are about to construct is a topological sigma model, with the topological gauge symmetry given by all possible deformations of $\Phi$. We do not choose any other fixed geometrical structure either on the worldsheet or in the target except the target metric $G_{\mu\nu}$. In particular, we do not assume a complex structure on $M$. (In fact, the theory will make perfect sense even for targets on which no complex structure exists.) The gauge-invariant Lagrangian is identically equal to zero, and we use the standard BRST technology to get a gauge-fixed version of the theory. First, we introduce the BRST charge that maps the fields to their corresponding ghosts,

$$[Q, X^\mu] = \psi^\mu, \quad \{Q, \psi^\mu\} = 0. \quad (5)$$

Then we choose a gauge fixing condition. Whereas the standard choice in the traditional topological sigma model is the holomorphicity of $\Phi$, here we will proceed in a different way. Inspired by the results of the $1/N$ expansion of 2D QCD as reviewed in §2, we are looking for a non-chiral topological sigma model, and we need a gauge fixing condition that prefers neither holomorphic nor anti-holomorphic maps.

To obtain a gauge fixing condition that satisfies these requirements, we pick an auxiliary fixed metric $h_{ab}$ on the worldsheet, define the Laplacian on the maps from the worldsheet to the target in the standard way:

$$\Delta X^\mu \equiv h^{ab} \nabla_a \partial_b X^\mu \equiv h^{ab} \left( \partial_a \partial_b X^\mu + \Gamma^\mu_{\sigma\rho} \partial_a X^\sigma \partial_b X^\rho - \Gamma^\sigma_{ac} \partial_c X^\mu \right), \quad (6)$$

and take the harmonicity condition

$$\Delta X^\mu = 0 \quad (7)$$

as the gauge fixing condition.\footnote{It is amusing to note here that the theory of harmonic maps between surfaces (see [9] for details) produces one of the shortest and most elegant analytic proofs of the (purely topological) Kneser formula that seems to be so important for the 2D QCD string.}

To implement the gauge fixing, we introduce the antighost/auxiliary BRST multiplet,

$$\{Q, \chi^\mu\} = B^\mu, \quad [Q, B^\mu] = 0. \quad (8)$$

The tensorial properties of the multiplet are determined by the properties of the gauge fixing condition (7), which in our case is a section of $\Phi^{-1}(TM)$. Note that both the
ghosts and the antighosts are (fermionic) sections of the same bundle over the worldsheet, an important fact that makes the theory very different from the traditional topological sigma model.

The choice of the gauge fixed Lagrangian that ensures its general covariance and yields the required gauge fixing condition is

$$\mathcal{L} = \int_{\Sigma} d^2 \sigma \sqrt{h} \left\{ Q, G_{\mu \nu} \chi^{\mu} \left( -\frac{1}{2} B^\nu + \Delta X^\nu + \frac{1}{2} \Gamma^\nu_{\sigma \rho} \chi^\sigma \psi^\rho \right) \right\}. \quad (9)$$

After performing the BRST commutator and integrating out the auxiliaries, we end up with the following form of the gauge fixed Lagrangian:

$$\mathcal{L} = \int_{\Sigma} d^2 \sigma \sqrt{h} \left( \frac{1}{2} G_{\mu \nu} \Delta X^\mu \Delta X^\nu - G_{\mu \nu} \chi^\mu \Delta \psi^\nu + R_{\mu \sigma \rho \nu} h^{ab} \partial_a X^\sigma \partial_b X^\rho \chi^\mu \psi^\nu \right. \quad (10)$$

$$\left. - \frac{1}{2} R_{\mu \nu} \chi^\mu \chi^\nu \right),$$

which is invariant under the on-shell BRST transformation given by

$$[Q, X^\mu] = \psi^\mu, \quad \{Q, \psi^\mu\} = 0, \quad \{Q, \chi^\mu\} = \Delta X^\mu + \Gamma^\mu_{\sigma \rho} \chi^\sigma \psi^\rho. \quad (11)$$

In (11) we have used the following definition of the covariant Laplacian on the ghosts,

$$\Delta \psi^\mu = h^{ab} \nabla_a \nabla_b \psi^\mu \equiv h^{ab} \left( \partial_a \partial_b \psi^\mu + 2 \Gamma^\mu_{\sigma \rho} \partial_a X^\sigma \partial_b \psi^\rho - \Gamma^c_{ab} \partial_c \psi^\mu \right) \quad (12)$$

$$+ \Gamma^\mu_{\sigma \rho} \Delta X^\sigma \psi^\rho + h^{ab} \partial_a X^\sigma \partial_b X^\rho \psi^\nu \left( R^\mu_{\rho \nu} + \partial_\rho \Gamma^\mu_{\sigma \rho} \right);$$

in this notation, the BRST transformation of $\Delta X^\mu$ is equal to $\Delta \psi^\mu$ plus additional terms,

$$[Q, \Delta X^\mu] = \Delta \psi^\mu - R^\mu_{\sigma \rho} h^{ab} \partial_a X^\sigma \partial_b X^\rho \psi^\lambda - \Gamma^\mu_{\sigma \rho} \Delta X^\sigma \psi^\rho. \quad (13)$$

This formula will be useful below.

The Lagrangian (11) is an exact BRST commutator, and the path integral of the theory is by standard arguments localized at the moduli space of instantons, which in our case are harmonic maps from $\Sigma$ to $M$. To distinguish the theory from the traditional topological sigma model, I refer to it as the “harmonic topological sigma model.”

The Lagrangian of the harmonic topological sigma model resembles closely the Lagrangian of topological mechanics. First of all, since both the ghost field $\psi^\mu$ and the antighost $\chi^\mu$ are sections of the same bundle, their zero modes (in a fixed harmonic background $X_0^\mu$) satisfy identical equations,

$$\Delta \psi^\mu - R^\mu_{\sigma \rho} h^{ab} \partial_a X_0^\sigma \partial_b X_0^\rho \psi^\nu = 0. \quad (14)$$
(and similarly for $\chi^\mu$), and the ghost number anomaly in the path integral is identically zero.

The absence of quantum anomaly in the ghost number conservation has important implications for the structure of correlation functions. Topological sigma models have typically two classes of observables. The de Rham cohomology ring of the target produces one of them (with the degree of the cohomology class being the ghost number of the observable). If the first homotopy group is non-trivial, there is another class of observables, corresponding to the vacua of the winding sectors (cf. [10]). Assuming that there are no point-like observables of negative ghost numbers, every correlation function that contains at least one observable of non-zero ghost number is zero by the ghost number conservation in the quantum theory.

Given a fixed harmonic map $\Phi_0 : \Sigma \to M$, we can analyze the local structure of the moduli space of harmonic maps around $\Phi_0$ as follows. Consider a small deformation $\delta X^\mu$ of $X^\mu$. The deformed map is harmonic (to lowest order) if $\delta X^\mu$ satisfies the linearized harmonicity condition,

$$\Delta \delta X^\mu - R^\mu_{\sigma \rho \nu} h^{ab} \partial_a X^\sigma_0 \partial_b X^\rho \delta X^\nu = 0. \tag{15}$$

This equation, also known in the theory of harmonic maps as the Jacobi equation, is identical to Eqn. (14) for the zero modes of $\psi^\mu$. The (integrable) zero modes of $\psi^\mu$ are thus tangent to the moduli space of harmonic maps, and the number of linear independent integrable solutions to the Jacobi equation measures the dimension of the moduli space.

Except for the higher order form of the kinetic terms, the only apparent difference between the Lagrangian of the harmonic sigma model and the Lagrangian of topological mechanics is the existence of the curvature-dependent two-fermi term in (10). One might na"ively expect that this term will join the curvature-dependent four-fermi term in saturating the fermionic integral over the zero modes. The BRST transformation of $\Delta X^\mu$, Eqn. (13), indicates that this is not the case: In the path integral, the curvature-dependent two-fermi term gets absorbed into the kinetic term of the fermi fields, and leads to the determinant cancelation in the (exact) Gaussian approximation. Hence, the four-fermi term remains the only one that can be used to saturate the ghost/antighost zero modes in the path integral.

The structure of the partition function of the harmonic sigma model is thus very similar to that of topological mechanics with the same target manifold: In the homotopically trivial sector, harmonic maps are constant maps, and the moduli space is isomorphic to the target. The partition function is equal to the Euler character of the
target manifold. Non-trivial homotopy sectors provide interesting stringy corrections to this result, but what is always computed is the Euler character of the moduli space of harmonic maps in the corresponding homotopy class.

The fact that the path integral gives the Euler character of the moduli space of harmonic maps can be proven rigorously, following Atiyah and Jeffrey [11]. The path integral of a given topological field theory computes a regularized Euler number of a specific vector bundle $\mathcal{V}$ over the infinite dimensional manifold $\mathcal{A}$ of all field configurations. In our case, $\mathcal{A}$ contains all maps from $\Sigma$ to $M$, and $\mathcal{V}$ is the bundle whose typical section is the left hand side of the gauge fixing condition, i.e. $\mathcal{V}$ is the tangent bundle to $\mathcal{A}$. The regularized Euler number of this infinite dimensional bundle is defined as the Euler number of a restriction of $\mathcal{V}$ to the zero locus of $\Delta X^\mu$, and coincides with the Euler character of the moduli space of harmonic maps.

Harmonic topological sigma models in two dimensions

Let us now consider, as an example, the harmonic topological sigma model with a two dimensional target $M$ of genus $G$. We know that the path integral of the theory is localized at the moduli space of the harmonic maps from $\Sigma$ to $M$, and we would like to check explicitly that it computes the Euler character of the moduli space. In the case of maps between surfaces, harmonic maps have been analyzed thoroughly in the mathematical literature (see [9] and references therein), and many important results have been obtained. Consider for instance an arbitrary homotopy class of maps between surfaces with both $g$ and $G$ greater than one, and with non-zero degree. Then there is a theorem [9] claiming that in such a homotopy class, there is exactly one harmonic map! The moduli space thus consists of only one point, there are no zero modes of the fermi fields, and the path integral in this homotopy class gives exactly one. Analogously, in the homotopically trivial sector, the only harmonic maps are the constant maps, and the moduli space coincides with $M$. There are two zero modes for both $\psi^\mu$ and $\chi^\mu$, corresponding to translations of the image of $\Phi$ in $M$. These zero modes bring down the curvature in the path integral, which is then equal to the Euler character of the target.

The path integral is similarly simple in the general case with any two dimensional target (see [5]). The moduli space of harmonic maps between surfaces with fixed metrics in a given homotopy class is always non-empty if $G \geq 1$ [9]; the only case when there are no harmonic maps in some homotopy classes corresponds to $G$ equal to zero, i.e. to the spherical target.

Up to this point we have studied the topological sigma model as a matter system on the worldsheet. To get a string theory instead, we have to take care of worldsheet
diffeomorphisms. The minimal way of doing so is to take the partition function of the harmonic topological sigma model, and sum over all possible homotopy classes. The sum is naïvely infinite, and we have to factorize by the group of global diffeomorphisms of the worldsheet. The resulting theory is finite, but it is still far from describing the QCD string in two dimensions.

4. Topological Rigid Strings

In the standard way of gauging worldsheet diffeomorphisms in a topological matter system, one introduces a topological gravity multiplet and couples it to the matter system. Here we will proceed in a different way, motivated mainly by the observation that QCD string theory is probably very different from traditional string theory. In particular, the QCD string can be sensitive to the extrinsic geometry, and the worldsheet quantization of the string theory could use a spacetime cutoff instead of the worldsheet cutoff, as e.g. in the rigid string theory. The naïve dimension of \( X^\mu \) in such a theory is one instead of zero, and the quantum properties of the string are governed by a fixed point that is substantially different from the one usually employed in traditional critical string theory. Unfortunately, such a string theory is very hard to quantize explicitly, and little is known about its properties.

Here we will see how to avoid these shortcomings in a topological version of such a string theory. Indeed, we have noticed above that the bosonic part of the Lagrangian for the harmonic topological sigma model is reminiscent of the rigid string theory studied by Polyakov and others some time ago. The basic discrepancy is that we have used a fixed fiducial metric on the worldsheet (which does not violate the naïve background independence of the theory by virtue of the topological symmetry), whereas the rigid string theory uses the induced metric. We will use this observation here to gauge worldsheet diffeomorphisms as follows. The BRST multiplet combines the topological symmetry of the harmonic topological sigma model with the worldsheet diffeomorphism symmetry,

\[
\begin{align*}
\{Q, X^\mu \} &= \psi^\mu + c^a \partial_a X^\mu, \\
\{Q, \psi^\mu \} &= c^a \partial_a \psi^\mu + \phi^a \partial_a X^\mu, \\
\{Q, c^a \} &= c^b \partial_b c^a - \phi^a, \\
[Q, \phi^a] &= c^b \partial_b \phi^a - \phi^b \partial_b c^a. 
\end{align*}
\]

Here \( c^a \) is the diffeomorphism ghost, which can be omitted from the (equivariant) theory, under the assumption that we deal exclusively with diffeomorphism invariant objects.
on which the reduced BRST charge is still nilpotent. We have also introduced the bosonic ghost-for-ghost $\phi^a$ of ghost number two, which takes care of the overcounting of worldsheet diffeomorphisms in the BRST symmetry.

The gauge fixing condition in the bosonic sector remains formally the same as in the harmonic topological sigma model,

$$\sqrt{h} \Delta X^\mu = 0,$$

but now we have not introduced any fiducial metric on the worldsheet. Instead, we identify the metric $h_{ab}$ with the induced metric,

$$h_{ab} = \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}.$$  \hspace{1cm} (18)

Consequently, the Laplacian that enters the gauge fixing condition (17) is now given by

$$\Delta X^\mu \equiv h^{ab} \left( \delta^\mu_\nu - h^{cd} G_{\nu\lambda} \partial_c X^\mu \partial_d X^\lambda \right) \left( \partial_a \partial_b X^\nu + \Gamma^\nu_{\sigma\rho} \partial_a X^\sigma \partial_b X^\rho \right).$$ \hspace{1cm} (19)

One may wonder whether the naïve counting of gauge fixing conditions agrees with the number expected in a diffeomorphism invariant theory. While in the harmonic topological sigma model in $D$ target dimensions the harmonicity condition gives $D$ gauge fixing conditions (as it should), here the requirement that $h_{ab}$ be the induced metric effectively reduces the number of independent components of $\Delta X^\mu$ by two. Indeed, assuming that the induced metric is smooth and non-degenerate, $\Delta X^\mu$ coincides with the trace of the second fundamental form of the map $\Phi : \Sigma \rightarrow M$, and it is easy to show that $\Delta X^\mu$ of (19) is normal to the image $\Phi(\Sigma)$ of $\Sigma$ in $M$,

$$G_{\mu\nu} \Delta X^\mu \partial_a X^\nu = 0.$$ \hspace{1cm} (20)

In $D$ target dimensions, the number of independent components in (17) is thus $D - 2$, which leaves exactly two unfixed coordinates for worldsheet diffeomorphism symmetry. In two dimensions, we seem to have no condition at all. What makes the theory non-trivial, however, is the fact that the requirements imposed on the induced metric that lead to (20) are generically impossible to satisfy everywhere on the worldsheet. In particular, the gauge fixing condition forbids the existence of folds on the map from $\Sigma$ to $M$.

The gauge fixing procedure can be easily completed at higher ghost numbers, and we end up with a topological string Lagrangian of the following form:

$$\mathcal{L} = \int_{\Sigma} \sqrt{h} \left( G_{\mu\nu} \Delta X^\mu \Delta X^\nu + \text{ghost terms} \right).$$ \hspace{1cm} (21)
Unlike in the harmonic topological sigma model, $h_{ab}$ in (21) is now the induced metric. The exact form of the ghost terms is really not so important for our purposes here, and can be found elsewhere [5]. Indeed, since the theory has been constructed as a topological string theory, we do not need to know the exact form of the Lagrangian in order to compute the partition function. In topological theories, we know a priori what the path integral is computing, and we can frequently do the calculation without invoking the path integral.

In the case at hand, the Lagrangian (21) defines a topological version of the rigid string theory. The ghost-independent part of the Lagrangian coincides with the rigid string theory Lagrangian of [6], and it is natural to refer to the theory as the “topological rigid string.” Its path integral is localized at the moduli space of solutions to the gauge fixing solution $\sqrt{h} \Delta X^\mu = 0$. This means that the maps dominating the path integral are harmonic in their own induced metric, i.e. the path integral is localized at the moduli space of minimal-area maps, counted up to worldsheet diffeomorphisms!

In the topological sigma model of §3, we knew a priori that the path integral computed the Euler number of the moduli space of harmonic maps, and we confirmed this fact by a direct calculation for the simplest case of two dimensional targets. In the topological rigid string theory, similarly, we know that the path integral gives an (equivariant) Euler number of the moduli space of minimal maps. The moduli spaces of minimal maps can be conveniently parametrized if we adopt the spacetime point of view. Indeed, the minimal-area condition forbids folds of the maps, and only allows branch points (and possibly collapsed handles). The location of branch points in the spacetime then serves as a natural set of coordinates on the moduli space of minimal-area maps. The branch points are indistinguishable, as a result of the worldsheet diffeomorphism symmetry. Consequently, the moduli spaces that dominate the path integral of the topological rigid string are quite similar to the spaces of maps that contribute to the $1/N$ expansion of 2D QCD. More details about this correspondence can be found in [5].

What we have considered thus far is a topological string theory with zero tension. Its partition function is naturally independent of the area of the target. To obtain a string theory with non-zero tension, we would like to add to the Lagrangian a term that behaves as

$$\delta \mathcal{L} = \lambda \int_\Sigma d^2 \sigma \sqrt{h}.$$  \hspace{1cm} (22)

When evaluated at the maps that dominate the path integral of the topological rigid string theory, i.e. at the minimal-area maps, this term gives

$$\delta \mathcal{L}(\Phi) = \lambda (n + \tilde{n}) A,$$  \hspace{1cm} (23)
where \( n, \tilde{n} \) are determined by the homotopy class of the minimal map \( \Phi \) and represent the number of orientation-preserving and orientation-reversing sheets of the worldsheet over the spacetime, and \( A \) is the area of the target. If we add (22) to the Lagrangian of the topological rigid string and then evaluate the deformed Lagrangian at the moduli space, we recover exactly the same exponential dependence on the area as observed in 2D QCD.

Of course, this cannot be the whole story. The naïve area term (22) does not respect the topological BRST symmetry of the rigid string theory and cannot be added to the Lagrangian without spoiling its BRST invariance. We can, however, make the induced area BRST-invariant by adding ghost-dependent terms to (22):

\[
\delta \mathcal{L}' = \lambda \int_{\Sigma} d^2 \sigma \left( \sqrt{h} + \text{ghost terms} \right), \quad [Q, \delta \mathcal{L}'] = 0.
\] (24)

This term can be added to the topological rigid string Lagrangian. In the partition function, the ghost corrections to the area term do not change the form of the terms that are exponential in the area. They do affect, however, the integral over the moduli space of minimal-area maps. Indeed, in the path integral of the deformed theory, the ghost corrections to (22) will enter the integral over the fermionic zero modes, thus changing the integration over the bosonic zero modes. In the partition function, this will produce a polynomial dependence on the target area in each homotopy sector of maps from the worldsheet to the target.

This procedure is reminiscent of the mechanism that connects the “physical” Yang-Mills theory in the target to a topological version thereof (cf. §2). It would be very interesting to see whether a similar mechanism can be involved in the worldsheet theory of the 2D QCD string as well.

5. Higher Dimensions

In previous paragraphs, we have presented some evidence in favor of the rigid string picture of QCD, in the exactly solvable case of the two dimensional, pure-glue theory on compact surfaces. We have observed some striking similarities between a topological rigid string theory and the results of the large-\( N \) expansion in QCD as obtained in [2-4]. Here I comment on possible extensions of the results to higher dimensions.

Although both the harmonic topological sigma model and the topological rigid string theory can be in principle constructed in arbitrary real dimension, the four-dimensional case is unique. Let us first rewrite the rigid string Lagrangian

\[ \mathcal{L} = \int_{\Sigma} d^2 \sigma \sqrt{h} G_{\mu\nu} \Delta X^\mu \Delta X^\nu \] (25)
in an equivalent form,

\[ \mathcal{L} = \int_{\Sigma} d^2 \sigma \sqrt{h} h^{ab} \nabla_a t^{\mu \nu} \nabla_b t^{\sigma \rho} G_{\mu \sigma} G_{\nu \rho}, \]  

with \( t^{\mu \nu} \) defined by

\[ t^{\mu \nu} = \frac{\epsilon^{ab}}{\sqrt{h}} \partial_a X^\mu \partial_b X^\nu. \]

In four dimensions, the rigid string theory described by (26) is known to have instantons, with the instanton number being the self-intersection number \( s(\Phi) \) of the worldsheet in the target. It is easy to show that

\[ s(\Phi) = \int_{\Sigma} d^2 \sigma \epsilon^{ab} \nabla_a t^{\mu \nu} \nabla_b t^{\sigma \rho} \epsilon_{\mu \nu \sigma \rho}. \]

In the string picture of 4D QCD, \( s(\Phi) \) is believed to correspond to the \( \theta \) angle of the spacetime Yang-Mills theory [12]. It is possible to write down a Lagrangian for a topological rigid string theory in four dimensions, such that its path integral is localized at the moduli space of instantons with the instanton number given by (28). One can naturally ask whether such a topological rigid string theory would correspond to topological Yang-Mills theory in four dimensions [13], i.e. to the SU(\( N \)) Donaldson theory in \( 1/N \) expansion. This question should be much easier to answer that its full-fledged physical counterpart.

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