On a fractional $q$-differential inclusion on a time scale via endpoints and numerical calculations

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Abstract

By using an endpoint result for set-valued maps, we study the existence of solutions for a fractional $q$-differential inclusion with sum and integral boundary value conditions on the time scale $\mathbb{T}_{t_0} = \{t_0, t_0q, t_0q^2, \ldots\} \cup \{0\}$, where $t_0$ is a real number and $q \in (0, 1)$. We provide an example involving some graphs and algorithms via numerical calculations to illustrate our main result.

MSC: Primary 34A08; 34B16; secondary 39A13

Keywords: Endpoint; $q$-differential inclusions; Sum boundary conditions; Caputo $q$-derivative; Integral boundary conditions

1 Introduction

In 1910, Jackson started the subject of $q$-difference equations [1]. The fractional calculus provides a meaningful generalization for the classical integration and differentiation to any order. Also, quantum calculus is equivalent to traditional infinitesimal calculus without the notion of limits. Despite the long history of these two theories, the investigation of their properties has remained untouched until recent time. In last decades, some researchers investigated $q$-fractional difference equations [2–5]. Later, $q$-fractional boundary value problems considered by many researchers (see for examples, [6–11]).

Plasma is also known as the ionized state of the matter. In this state of matter, plasma contains components such as free electrons, ions neutrals, and dust. The multi-component plasmas deal with the partially or fully ionized state of plasma. It also fulfils the condition of quasi neutrality. The multi-component plasmas play a crucial role in plasma discharge and other industrial processing. The multi-component plasma has more than two components while the general plasma has ions and electrons. The wide range of plasmas itself give an opportunity to analyze such plasmas on distinct scales. There are two main areas of multi-component plasma: dusty plasma and the negative ions plasma. Both areas of multi-component plasma have a wide range of emerging applications in science and engineering. The time fractional Kersten–Krasil’shchik coupled KdV–mKdV nonlinear system and homogeneous two component time fractional coupled third order KdV systems are very important fractional nonlinear systems for describing the behavior of...
waves in multi-component plasma and elaborate various nonlinear phenomena in plasma physics. In the past decades many researchers are used to various techniques for solving fractional nonlinear partial differential equation and find approximate and exact solutions of the fractional evolution equations (for more details, see [12–26]) and different applications of fractional calculus (see for example [27–30]).

In 2013, Ahmad et al. studied the fractional inclusion problem \( cD^\beta [u](t) \in T(t,u(t)) \) with the integral boundary conditions \( u'(0) - c_i u'(\delta) = a_i \int_0^1 f_i(r,u(r)) \, dr \) for \( j = 0,1,2 \), where \( T \) is a multifunction [31]. In 2014, Ghorbanian et al. reviewed the fractional differential inclusion problems

\[
\begin{align*}
  cD^\gamma [z](t) &\in T_1(t,z(t),z'(t),z''(t)), \\
  cD^\delta [z](t) &\in T_2(t,z(t),cD^\beta [z](t),\ldots,cD^\delta [z](t)),
\end{align*}
\]

with some integral boundary value conditions

\[
\begin{align*}
  z(0) + az(1) &= \sum_{i=1}^m \int_{\alpha_i}^{\beta_i} f_i(r,z(r)) \, dr, \\
  cD^\xi [z](0) + cD^\sigma [z](\eta) + cD^\zeta [z](1) &= \int_0^1 f_1(r,z(r)) \, dr, \\
  cD^\rho [z](0) + cD^\phi [z](\eta) + cD^\psi [z](1) &= \int_0^1 f_2(r,z(r)) \, dr,
\end{align*}
\]

and \( z(0) + az(1) = \sum_{i=1}^m cD^\rho [z](\eta), \ Z(0) + bz'(1) = \sum_{i=1}^m cD^\phi [z](\eta), \) respectively, where \( t \in J, 2 < \sigma_1 < \beta_2 < 1, b < \eta, \zeta, \xi < 1, 1 < \beta < 2, \sigma_2 - \beta_i \geq 1 \) for \( 1 \leq i \leq n, a > \sum_{i=1}^m \rho_i \), \( b > \sum_{i=1}^m |\zeta_i - \eta_i|, n \in \mathbb{N}, T_1 : J \times \mathbb{R}^m \to P_{cp}(\mathbb{R}), T_2 : J \times \mathbb{R}^{m+1} \to P_{cp}(\mathbb{R}) \) are multifunctions, \( f_i : J \times \mathbb{R} \to \mathbb{R} \) are continuous functions for \( i = 0,1,2 \) and \( P_{cp}(\mathbb{R}) \) is the set of all compact subsets of \( \mathbb{R}^n [32] \).

In 2015, Agarwal et al. investigated the fractional derivative inclusions \( cD^\sigma [x](t) \in F_1(t,x(t)) \) and \( cD^\rho [x](t) \in F_2(t,x(t)) \) with the boundary value problems \( x(0) = a \int_0^1 \xi(\xi) \, d\xi, x(1) = b \int_0^1 \xi(\xi) \, d\xi \) and \( x(1) + x'(1) = \int_0^1 \eta(\xi) \, d\xi, x(0) = 0 \), respectively, where \( t \in J, \xi, \eta, \nu \in (0,1), \beta \in (1,2) \) with \( \beta - \beta_i > 1 \), \( a,b \in \mathbb{R}, cD^\sigma \) is the Caputo differentiation and \( F_1 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m, F_2 : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m \) are compact valued multifunction [33]. Also, Ntouyas et al. studied the fractional inclusion problem \( cD^\sigma_0 [u](t) \in F(t,u(t),u'(t)) \) with sum and integral boundary conditions \( u(0) + \sum_{i=1}^{m_i} b_i u_i(0) = 0, \gamma_1 u(\eta) + \gamma_2 \int_0^1 u(\tau) \, d\tau = 0 \) and \( \sum_{i=1}^{m_i} b_i u_i(1) + \gamma_3 \int_0^1 u(\tau) \, d\tau = 0, \) where \( cD^\sigma_0 \) denotes the Caputo fractional derivative of order \( \alpha, \) \( t \in [0,1], 2 < \alpha \leq 3, 1 < p_i \leq 2, (i = 1, \ldots,k; k \geq 1), 0 < \eta < 1, b_j (j = 1, \ldots,m; m \geq 1), \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R} \) and \( F : J \times \mathbb{R}^{m+1} \to \mathbb{P}(\mathbb{R}) \) is a compact-valued multifunction (citeNtouyasEtemad). In 2019, Samei et al. studied the hybrid Caputo–Hadamard fractional inclusion problem

\[
\begin{align*}
  cH^\alpha \int_a^t f(t,x(t)) \, dt \in &K(t,x(t)), \\
  x(1) = \mu(x), \quad x(\eta) = \eta(x),
\end{align*}
\]

where \( cH^\alpha \) denote the Caputo–Hadamard fractional derivative and Hadamard integral of order \( \alpha, \) respectively, \( t \in J = [1,e], n,m \in \mathbb{N}, 1 < \alpha \leq 2, \beta_i > 0 \) for \( i = 1,2,\ldots,n, \gamma_i > 0 \) for \( i = 1,2,\ldots,m, \) the functions \( f : J \times \mathbb{R}^{m+1} \to \mathbb{R}, g : J \times \mathbb{R}^{m+1} \to \mathbb{R} \setminus \{0\}, h_i : J \times \mathbb{R}^{m+1} \to \mathbb{R} \).
\( \mathbb{R} \rightarrow \mathbb{R} \) for \( i = 1, 2, \ldots, n \), functions \( \mu, \eta \) map \( C(f, \mathbb{R}) \) into \( \mathbb{R} \) and the multifunction \( K : \mathcal{J} \times \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \) satisfies certain conditions [35]. Also, Ntouyas et al. reviewed the multi-term nonlinear fractional \( q \)-integro-differential equation

\[
\begin{align*}
\mathcal{D}_q^\beta \{ \mathcal{D}_q^\alpha \{ \mathcal{D}_q^\beta \{ \mathcal{D}_q^\alpha \{ x(t) \} \} \} \} \in \mathcal{F}(t, k, \mathcal{I}_q^{\beta_1}[k], \ldots, \mathcal{I}_q^{\beta_m}[k]),
\end{align*}
\]

under some boundary conditions [36]. In 2019, Samei et al. discussed the fractional hybrid \( q \)-differential inclusions

\[
\mathcal{D}_q^\alpha \{ \mathcal{D}_q^\beta \{ \mathcal{D}_q^\beta \{ \mathcal{D}_q^\alpha \{ x(t) \} \} \} \} \in \mathcal{F}(t, k, \mathcal{I}_q^{\beta_1}[k], \ldots, \mathcal{I}_q^{\beta_m}[k]),
\]

with the boundary conditions \( k(0) = k_0 \) and \( k(1) = k_1 \), where \( 1 < \alpha \leq 2, q \in (0, 1), k_0, k_1 \in \mathbb{R}, \alpha_i > 0 \), for \( i = 1, 2, \ldots, n \), \( \beta_j > 0 \), for \( j = 1, 2, \ldots, m \), \( n, m, m \in \mathbb{N} \), \( \mathcal{D}_q^\alpha \) denotes Caputo type \( q \)-derivative of order \( \alpha \), \( \mathcal{I}_q^\beta \) denotes the Riemann–Liouville type \( q \)-integral of order \( \beta \), \( f : \mathcal{J} \times \mathbb{R}^n \rightarrow (0, \infty) \) is continuous and \( \mathcal{F} : \mathcal{J} \times \mathbb{R}^m \rightarrow \mathcal{P}(\mathbb{R}) \) is a multifunction [10].

By using the main idea of this work, we investigate the fractional \( q \)-differential inclusion

\[
\begin{align}
\mathcal{D}_q^\sigma \{ \mathcal{D}_q^\alpha \{ \mathcal{D}_q^\beta \{ \mathcal{D}_q^\gamma \{ x(t) \} \} \} \} \in \mathcal{T}(t, k(t), k'(t), \mathcal{I}_q^{\alpha_1}[k](t), \ldots, \mathcal{I}_q^{\alpha_m}[k](t)),
\end{align}
\]

with sum and integral boundary value conditions

\[
\begin{align}
\begin{cases}
   k(0) + \sum k''(0) = 0, \\
   a_1k(t) + a_2\phi(1) = 0, \\
   \sum k'(1) + a_3\phi(1) = 0,
\end{cases}
\end{align}
\]

where \( 2 < \sigma \leq 3 \), \( t \in \mathcal{J} := [0, 1] \), \( \mathcal{D}_q^\sigma \) denotes the Caputo fractional \( q \)-derivative of order \( \sigma \), \( 1 < \zeta_i \leq 2 \) (\( i = 1, \ldots, m \)), \( 0 < \tau < 1 \), \( \Sigma = \sum_{j=1}^k c_j \) with \( c_j \in \mathbb{R} \), \( \phi : [0, \infty) \rightarrow [0, \infty) \) defined by \( \phi(r) = \int_0^r \phi(\tau) \, d\tau \) and \( \mathcal{T} : \mathcal{J} \times \mathbb{R}^{m+2} \rightarrow \mathcal{P} \) is the compact-valued multifunction.

This work is arranged as: In Sect. 2, we state some useful definitions and lemma on the fundamental concepts of \( q \)-fractional calculus and multifunctions. In Sect. 3, some main theorems on the solutions of fractional \( q \)-differential inclusion (1)–(2) are stated. Section 4 contains an illustrative example to show the validity and applicability of our results. The paper concludes with some interesting observations. In Sect. 5, conclusions are presented.

### 2 Essential preliminaries

Throughout this article, we shall apply the time scales calculus notation [37]. In fact, we consider the fractional \( q \)-calculus on the time scale \( \mathbb{T}_0 = \{ 0 \} \cup \{ t : t = t_0 q^n \} \), where \( n \geq 0 \), \( t_0 \in \mathbb{R} \) and \( q \in (0, 1) \). Let \( a \in \mathbb{R} \). Define \( [a]_q = \frac{a - q^a}{1 - q} \) [1]. The power function \( (x - y)^{\alpha} \) with \( n \in \mathbb{N}_0 \) is defined by \( (x - y)^{\alpha} = \prod_{k=0}^{\alpha-1} (x - y^k) \) for \( n \geq 1 \) and \( (x - y)^{(0)} = 1 \), where \( x \) and \( y \) are real numbers and \( \mathbb{N}_0 := \{ 0 \} \cup \mathbb{N} \) [2]. Also, for \( \alpha \in \mathbb{R} \) and \( a \neq 0 \), we have

\[
(x - y)^{\alpha} = x^\alpha \prod_{k=0}^{\infty} \frac{x - y^k}{x - y^{\alpha+k}}.
\]
Algorithm 1 The proposed method for calculating \((a - b)_q\)

```
1 function g = powerfunction(a, b, n, q)
2 %Power Gamma (a-b)^{(n)}
3 n=1
4 if n==0
5 \(p=1\)
6 else
7 for k=1:n-1
8 \(g=g*\{(a-b)*q^-k\}/(a-b*q^-\{\alpha+k\})\);
9 end
10 \(p=\alpha*\text{alpha} * \, ds\);
11 end
12 end
```

Algorithm 2 The proposed method for calculating \(I_q^\alpha(x)\)

```
1 function g = qGamma(q, x, n)
2 %q-Gamma Function
3 \(p=1,\)
4 for k=0:n
5 \(p=p*(1-q^-\{k+1\})/(1-q^-\{x+k\})\);
6 end
7 \(g=p/\{(1-q^-\{x\})*\, x\}\);
8 end
```

Algorithm 3 The proposed method for calculating \((D_q^\alpha f)(x)\)

```
1 function g = Dq(q, x, n, fun)
2 if x==0
3 \(q=\text{limit}\,\{(\text{fun}\,(x)\text{-}\text{fun}\,(q*x))/\{(1-q)*x\},x,0\};\)
4 else
5 \(q=(\text{fun}\,(x)\text{-}\text{fun}\,(q*x))/\{(1-q)*x\};\)
6 end
7 end
```

If \(y = 0\), then it is clear that \(x^{(\alpha)} = x^\alpha\) [7] (see the Algorithm 1). The \(q\)-Gamma function is given by \(I_q^\alpha(z) = (1 - q)^{\{(z-1)/(1-q)\}^{-1}}\), where \(z \in \mathbb{R} \setminus \{0, -1, -2, \ldots\} \) [1]. Note that, \(I_q^\alpha(z + 1) = \lfloor z \rfloor I_q^\alpha \lfloor z \rfloor\). The Algorithm 2 shows a pseudo-code description of the technique for estimating \(q\)-Gamma function of order \(n\). The \(q\)-derivative of function \(f\), is defined by \((D_q^\alpha f)(x) = \frac{\{(x^\alpha) - f(qx)\}}{(1-q)\, x}\) and \((D_q^\alpha f)(0) = \lim_{x \to 0}(D_q^\alpha f)(x)\) which is shown in Algorithm 3 [2, 3]. Furthermore, the higher order \(q\)-derivative of a function \(f\) is defined by \((D_q^{n\alpha} f)(x) = D_q(D_q^{n-1} f)(x)\) for \(n \geq 1\), where \((D_q^{n\alpha} f)(x) = f(x)\) [2, 3]. The \(q\)-integral of a function \(f\) is defined on \([0, b]\) by \(I_q^\alpha f(x) = \int_0^x f(s) \, dq^\alpha s = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k)\) for \(0 \leq x \leq b\), provided the series is absolutely converges [2, 3]. If \(x\) in \([0, T]\), then

\[
\int_x^T f(r) \, dq^\alpha r = I_q^\alpha f(T) - I_q^\alpha f(x) = (1-q) \sum_{k=0}^{\infty} q^k \left[ I_q^\alpha f(Tq^k) - \alpha f(xq^k) \right],
\]

whenever the series exists. In addition, we can interchange the order of double \(q\)-integral by \(\int_0^T \int_0^T h(r) \, dq^\alpha r \, dq^\alpha s = \int_0^T \int_0^T h(r) \, dq^\alpha s \, dq^\alpha r\) [38]. Actually the interchange of order is true, since

\[
\int_0^T \int_0^T h(s) \, dq^\alpha s \, dq^\alpha r = \int_0^T (t - qr)^{(\alpha-1)} h(r) \, dq^\alpha r
\]
Algorithm 4 The proposed method for calculating $I_q^n [x]$

```
function g = Iq.sigma(q, sigma, x, n, fun)
    p=0;
    for k=0:n
        s1=1;
        for i=0:k-1
            s1=s1*(1-q*(sigma+i));
        end
        s2=1;
        for i=0:k-1
            s2=s2*(1-q*(i+1));
        end
        p=p + q^k*s1*fun(x*q^k)/s2;
    end
    g=round({x*sigma}*{1-q*sigma}*p, 6);
end
```

$$
= t(1-q) \sum_{i=0}^{\infty} q^i h(q^i t)(t-q^{i+1} t) \\
= t^2(1-q)^2 \sum_{i=0}^{\infty} q^i h(q^i t) \left( \sum_{i=0}^{\infty} q^i \right).
$$

In addition the left side can be written as

$$
\int_0^t \int_0^r h(s) d_q s d_q r = t(1-q) \sum_{i=0}^{\infty} q^i \int_0^{qt_i} h(r) d_q r \\
= t^2(1-q)^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} q^{i+j} h(q^{i+j} t).
$$

(3)

The operator $I_q^n$ is given by $(I_q^n h)(x) = h(x)$ and $(I_q^n h)(x) = (I_q^n(I_q^{n-1} h))(x)$ for all $n \geq 1$ and $h \in C([0, T])$ [2, 3]. It has been proved that $(D_q(I_q^n h))(x) = h(x)$ and $(I_q^n(D_q h))(x) = h(x) - h(0)$ whenever $h$ is continuous at $x = 0$ [2, 3]. The fractional Riemann–Liouville type $q$-integral of the function $h$ on $J = (0, 1)$ for $\sigma \geq 0$ is defined by $I_q^{\sigma}[h](t) = h(t)$ and

$$
I_q^{\sigma}[h](t) = \frac{1}{\Gamma_q(\sigma)} \int_0^t (t - qr)^{(\sigma-1)} h(r) d_q r \\
= t^\sigma(1-q)^\sigma \sum_{k=0}^{\infty} q^k \frac{\prod_{i=1}^{k-1}(1-q^{\sigma+i}) \prod_{i=1}^{k-1}(1-q^{i+1})}{\prod_{i=1}^{k}(1-q^{\sigma+i})} h(tq^k)
$$

for $t \in J$ [39]. Also, the Caputo fractional $q$-derivative of a function $h$ is defined by

$$
cD_q^{\sigma}[h](t) = I_q^{\sigma-\sigma} \left[ D_q^{\sigma}[h](t) \right] = \frac{1}{\Gamma_q(\sigma)} \int_0^t (t - qr)^{-(\sigma-1)} D_q^{\sigma}[h](r) d_q r,
$$

where $t \in J$ and $\sigma > 0$ [39]. It has been proved that $I_q^{\sigma}[I_q^{\sigma}[h]](x) = I_q^{\sigma+\sigma}[h](x)$ and $D_q^{\sigma}[I_q^{\sigma}[h]](x) = h(x)$, where $\alpha, \beta \geq 0$ [39]. The Algorithm 4 shows pseudo-code $I_q^{\sigma}[h](x)$.

Let $(\mathcal{E}, \rho)$ be a metric space. Denote by $\mathcal{P}(\mathcal{E})$ and $2^\mathcal{E}$ the class of all subsets and the class of all nonempty subsets of $\mathcal{E}$, respectively. Thus, $\mathcal{P}(\mathcal{E})$, $\mathcal{P}(bd(\mathcal{E}))$, $\mathcal{P}(cv(\mathcal{E}))$ and $\mathcal{P}(cp(\mathcal{E}))$ denote the classes of all closed, bounded, convex and compact subsets of $\mathcal{E}$, respectively. A mapping $T : \mathcal{E} \rightarrow 2^\mathcal{E}$ is called a multifunction on $\mathcal{E}$ and $e \in \mathcal{E}$ is called a fixed point of $T$.
whenever \( e \in T(e) \). An element \( e \in E \) is called an endpoint of a multifunction \( T : E \to 2^E \) whenever \( T(e) = \{ e \} \) [40]. Also, we say that \( T \) has an approximate endpoint property whenever \( \inf_{e \in E} \sup_{f \in T(e)} \rho(e, f) = 0 \) [40]. A function \( \psi : \mathbb{R} \to \mathbb{R} \) is called upper semi-continuous whenever \( \lim sup_{n \to \infty} \psi(r_n) \leq \psi(r) \) for all sequence \( \{r_n\}_{n \geq 1} \) with \( r_n \to r \).

A multifunction \( T : E \to \mathcal{P}_c(E) \) is said to be lower semi-continuous whenever for every open set \( O \) of \( E \), the set \( T^{-1}(O) := \{ z \in E : T(z) \cap O \neq \emptyset \} \) is open [41]. We say that \( T \) is upper semi-continuous whenever the set \( \{ z \in E : T(z) \subset O \} \) is open for any open set \( O \) of \( E \) [41]. Also, \( T : E \to \mathcal{P}_c(E) \) is called compact if \( T(B) \) is a compact set of \( E \) for any bounded subsets \( B \) of \( E \) [41]. A multifunction \( T : \bar{J} \to \mathcal{P}_c(\mathbb{R}) \) is said to be measurable whenever the function \( t \mapsto \rho(y, T(t)) = \inf \{ |y - z| : z \in T(t) \} \) is measurable for all \( y \in \mathbb{R} \), \( \bar{J} = [0,1] \) [41]. Define the Pompeiu–Hausdorff metric \( P_\rho : 2^E \times 2^E \to [0, \infty) \) by

\[
P_\rho(S, T) = \max \left\{ \sup_{s \in S} \rho(s, T), \sup_{t \in T} \rho(S, t) \right\},
\]

where \( \rho(s, t) = \inf_{e \in E} \rho(s; t) \). Then \( (\mathcal{P}_{b,d}(E), P_\rho) \) is a metric space and \( (\mathcal{P}_{c}(E), P_\rho) \) is a generalized metric space [41]. A multifunction \( T : E \to \mathcal{P}_d(E) \) is called a \( \lambda \)-contraction whenever there exists \( \lambda \in (0, 1) \) such that \( P_\rho(T(e_1), \Theta(e_2)) \leq \lambda \rho(e_1, e_2) \) for all \( e_1, e_2 \in E \) [42]. In 1970, Covitz and Nadler proved that each closed valued contractive multifunction on a complete metric space has a fixed point [42]. We say that \( T : \bar{J} \times \mathbb{R}^{2m} \to 2^\mathbb{R} \) is a Carathéodory multifunction whenever \( t \mapsto T(t, r_1, \ldots, r_{2m}) \) is measurable for all \( r_1, \ldots, r_{2m} \in \mathbb{R} \) and \( \{ T(t, r_1, \ldots, r_{2m}) \} \) is an upper semi-continuous map for almost all \( t \in \bar{J} \) [41, 43, 44]. Also, a Carathéodory multifunction \( T : \bar{J} \times \mathbb{R}^{2m} \to 2^\mathbb{R} \) is called \( L^1 \)-Carathéodory whenever for each \( \eta > 0 \) there exists \( \bar{r}_\eta \in L^1(\bar{J}, \mathbb{R}^m) \) such that

\[
\| T(t, r_1, \ldots, r_{2m}) \| = \sup \{ |k| : k \in T(t, r_1, \ldots, r_{2m}) \} \leq \bar{r}_\eta(t),
\]

for all \( |r_1|, \ldots, |r_{2m}| \leq \eta \) and for almost all \( t \in \bar{J} \) [41, 43, 44]. For each \( i \), define the space \( E_i = \{ k(t) : k(t), k'(t), \psi D_q^k[k(t)] \in \mathcal{A} \} \) endowed with the norm

\[
\| k \|_i = \sup_{t \in \bar{J}} |k(t)| + \sup_{t \in \bar{J}} |k'(t)| + \sup_{t \in \bar{J}} \| \psi D_q^k[k(t)] \|,
\]

where \( \mathcal{A} = C(\bar{J}, \mathbb{R}) \). Also, consider the product space \( E = E_1 \times \cdots \times E_m \) endowed with the norm \( \| (k_1, \ldots, k_m) \| = \sum_{i=1}^m \| k_i \|_i \). Then \( (E, \| \cdot \|) \) is a Banach space [45]. By using the idea of [31, 34, 46], define the set of the selections of \( S \), at \( k \) by

\[
S_{T,k} = \{ p \in L^1(\bar{J}) : p(t) \in T(t, k(t), k'(t), \psi D_q^k[k(t)], \ldots, \psi D_q^{m-1}[k(t)]) \}
\]

for all \( t \in \bar{J}, k = (k_1, \ldots, k_m) \in E \) and \( 1 \leq i \leq m \). One can check that \( S_{T,k} \neq \emptyset \) for all \( k \in E \) whenever \( \dim E < \infty \) [47]. For the proof of our main result we use the following endpoint fixed point theorem of Amini–Harandi [40].

**Lemma 1** ([40]) Let \( \psi : [0, \infty) \to [0, \infty) \) be an upper semi-continuous function such that \( \psi(s) < s \) and \( \liminf_{s \to \infty} (s - \psi(s)) > 0 \) for all \( s > 0 \), \( (E, \rho) \) a complete metric space and \( T : E \to \mathcal{P}_{c}(E) \) a multifunction such that \( P_\rho(T(e), T(f)) \leq \psi(\rho(e,f)) \) for all \( e, f \in E \). Then \( T \) has a unique endpoint if and only if \( T \) has approximate endpoint property.
3 Main results

Now, we are ready to provide our main results.

**Lemma 2** Let \( z(t) \in \mathcal{A} \) and \( \sigma \in (2,3) \). Then the unique solution of the fractional problem
\[
{^cD_q^\sigma} [k](t) = z(t)
\]
under boundary value conditions \( k(0) + \Sigma k'(0) = 0, a_1 k(\tau) + a_2 \varrho(1) = 0 \) and \( \Sigma k'(1) + a_3 \varrho(1) = 0 \) is given by
\[
k(t) = \mathcal{I}_q^\sigma [z](t) + \Delta a_1 A_1(t) \mathcal{I}_q^\sigma [z](\tau) + \Delta (a_2 A_1(t) + a_3 A_2(t)) \mathcal{I}_q^\sigma [\mathcal{I}_q^\sigma [z](s)](1) + \Delta A_2(t) \Sigma \mathcal{I}_q^{\sigma -1} [z](1),
\]
where \( \mathcal{I}_q^\sigma [\mathcal{I}_q^\sigma [z](s)](1) \) is defined by (3), \( \Sigma = \sum_{i=1}^k c_i \) with \( c_1, \ldots, c_k \in \mathbb{R}, \varrho : [0,\infty) \rightarrow [0,\infty) \) define by \( \varrho(v) = \int_0^v k(r) \, dr \),
\[
\Delta = \left[ \left( a_1 \tau + \frac{a_2}{2} \right) \left( 2 \Sigma + \frac{a_3}{3} - 2 a_3 \Sigma \right) + \left( \Sigma + \frac{a_3}{2} \right) \left( 2 \Sigma (a_1 + a_2) - \left( a_1 \tau^2 + \frac{a_2}{3} \right) \right) \right]^{-1} \neq 0
\]
and
\[
A_1(t) = \left( \Sigma + \frac{a_3}{2} \right) \left( t^2 - t - 2 \Sigma \right) + 2 a_3 t \Sigma,
A_2(t) = \left( a_1 \tau + \frac{a_2}{2} \right) \left( 2 \Sigma - t^2 \right) + t \left( a_1 \tau^2 + \frac{a_2}{3} - 2 (a_1 + a_2) \Sigma \right).
\]

**Proof** It is known that the solution of the fractional \( q \)-differential equation \( {^cD_q^\sigma} k(t) = z(t) \) is
\[
k(t) = \mathcal{I}_q^\sigma [z](t) + d_0 + d_1 t + d_2 t^2,
\]
where \( d_0, d_1 \) are real constants and \( t \in \mathcal{J} [48] \). Thus, we have \( k'(t) = \mathcal{I}_q^{\sigma -1} [z](t) + d_1 + 2 d_2 t \) and \( k''(t) = \mathcal{I}_q^{\sigma -2} [z](t) + 2 d_2 \). By using the boundary conditions, we obtain \( a_0 + 2 d_2 \Sigma = 0 \),
\[
(a_1 + a_2) d_0 + \left( a_1 \tau + \frac{a_2}{2} \right) d_1 + \left( a_1 \tau^2 + \frac{a_2}{3} \right) d_2 = -a_2 \mathcal{I}_q^\sigma \left[ \mathcal{I}_q^\sigma [z](s) \right](1) - a_1 \mathcal{I}_q^\sigma [z](\tau)
\]
and
\[
a_3 d_0 + \left( \Sigma + \frac{a_3}{2} \right) d_1 + \left( 2 \Sigma + \frac{a_3}{3} \right) d_2 = -a_3 \mathcal{I}_q^\sigma \left[ \mathcal{I}_q^\sigma [z](s) \right](1) - \Sigma \mathcal{I}_q^{\sigma -1} [z](1).
\]
Hence by an easy calculation, we get
\[
d_0 = 2 \Sigma \Delta \left[ - \left( \Sigma + \frac{a_3}{2} \right) (a_2 \mathcal{I}_q^\sigma \left[ \mathcal{I}_q^\sigma [z](s) \right](1) + a_1 \mathcal{I}_q^\sigma [z](\tau)) + \left( a_1 \tau + \frac{a_2}{2} \right) (a_3 \mathcal{I}_q^\sigma \left[ \mathcal{I}_q^\sigma [z](s) \right](1) + \Sigma \mathcal{I}_q^{\sigma -1} [z](1)) \right],
\]
\[d_1 = \Delta \left[ \left( 2a_1 \Sigma - \Sigma - \frac{a_3}{3} \right) (a_2 T_q^\sigma [T_q^\sigma [z](s)](1) + a_1 T_q^\sigma [z](\tau)) + \left( a_1 \tau^2 + \frac{a_2^2}{3} - 2\Sigma(a_1 + a_2) \right) (a_3 T_q^\sigma [T_q^\sigma [z](s)](1) + \Sigma T_q^{\sigma - 1}[z](1)) \right],
\]
\[d_2 = \Delta \left[ \left( \Sigma + \frac{a_2}{2} \right) (a_2 T_q^\sigma [T_q^\sigma [z](s)](1) + a_1 T_q^\sigma [z](\tau)) - \left( a_1 \tau^2 + \frac{a_2^2}{2} \right) (a_3 T_q^\sigma [T_q^\sigma [z](s)](1) + \Sigma T_q^{\sigma - 1}) \right].\]

Now, substituting the values of \(d_0\), \(d_1\) and \(d_2\) in (7), we obtain (4).

From definition (6), we can see that
\[
|A_1(t)| \leq 2 \left( |\Sigma| + \frac{|a_3|}{2} \right) (1 + |\Sigma|) + 2|a_3||\Sigma| := \o A_1,
\]
\[
|A_1'(t)| \leq (2|a_3| + 1)|\Sigma| + \frac{|a_3|}{2} := \o A_1',
\]
\[
|A_2(t)| \leq \left( |a_1| \tau + \frac{|a_2|}{2} \right) (2|\Sigma| + 1) + |a_1|\tau^2 + \frac{|a_2|}{3} + 2|a_1 + a_2||\Sigma| := \o A_2,
\]
\[
|A_2'(t)| \leq 2 \left( |a_1| \tau + \frac{|a_2|}{2} \right) + |a_1|\tau^2 + \frac{|a_2|}{3} + 2|a_1 + a_2||\Sigma| := \o A_2'.
\]

Also,
\[
cD_q^\sigma [A_1](t) = \left( \Sigma + \frac{a_3}{2} \right) \left[ -\frac{1}{\Gamma_q(2 - \xi)} t^{1-\xi} + \frac{2}{\Gamma_q(3 - \xi)} t^{2-\xi} \right] + \frac{2a_3 \Sigma \Gamma_q(2 - \xi)}{\Gamma_q(3 - \xi)} t^{1-\xi}
\]
and
\[
cD_q^\sigma [A_2](t) = \left( a_1 \tau + \frac{a_2}{2} \right) \left( -\frac{2}{\Gamma_q(3 - \xi)} t^{2-\xi} \right) + \left( a_1 \tau^2 + \frac{a_2}{3} + 2(a_1 + a_2) \Sigma \right) \frac{1}{\Gamma_q(2 - \xi)} t^{1-\xi},
\]
for \(t \in \mathcal{J}\) and \(i = 1, \ldots, m\). From this we obtain
\[
|cD_q^\sigma [A_1](t)| \leq \left( \Sigma + \frac{|a_3|}{2} \right) \left[ -\frac{1}{\Gamma_q(2 - \xi)} + \frac{2}{\Gamma_q(3 - \xi)} \right] + 2|a_3||\Sigma| \frac{1}{\Gamma_q(2 - \xi)} := \o A_1,
\]
\[
|cD_q^\sigma [A_2](t)| \leq \left( |a_1| \tau + \frac{|a_2|}{2} \right) \frac{2}{\Gamma_q(3 - \xi)} + \left( |a_1|\tau^2 + \frac{|a_2|}{3} + 2|a_1 + a_2||\Sigma| \right) \frac{1}{\Gamma_q(2 - \xi)} := \o A_2.
\]

**Definition 3** A function \((k_1, k_2, \ldots, k_m)\) belongs to \(\prod_{i=1}^m AC^i(\mathcal{J})\) is a solution for the fractional \(q\)-differential inclusion (1) whenever it satisfies the boundary value conditions and
there exists a function \((z_1, z_2, \ldots, z_m) \in \prod_{i=1}^{m} L^1(J)\) such that

\[
z_i(t) \in T(t, k(t), k'(t), cD^0_q [k](t), \ldots, cD^m_q [k](t))
\]

and

\[
k(t) = \mathcal{I}_q^\sigma [z](t) + \Delta a_1 A_1(t) T_q^\sigma [z](\tau) + \Delta (a_2 A_1(t) + a_3 A_2(t)) T_q^\sigma [z](\sigma) + \Delta A_2(t) \Sigma T_q^{\sigma - 1}[z](1),
\]

for \(t \in J\) and \(1 \leq i \leq m\), where \(A_1(t)\) and \(A_2(t)\) are defined in (6).

**Theorem 4** Suppose that \(T : \mathcal{J} \times \mathbb{R}^{mv} \to \mathcal{P}_{\mathfrak{q}_p}(\mathbb{R})\) is an integrable bounded multifunction such that \(t \mapsto T(t, x_1, x_2, v_1, \ldots, v_m)\) is measurable, \(\psi : [0, \infty) \to [0, \infty)\) is a nondecreasing upper semi-continuous mapping such that \(\liminf_{t \to \infty}(t - \psi(t)) > 0\) and \(\psi(t) < t\) for all \(t > 0\) and there exist continuous functions \(\gamma : J \to [0, \infty)\) such that

\[
P_p(T(k_1, k_2, \ldots, k_m), T(t, k'_1, k'_2, \ldots, k'_m)) \leq \mathcal{E} \gamma(t) \sum_{i=1}^{mv^2} |k_i - k'_i|,
\]

for all \(t \in J\), \((k_1, k_2, \ldots, k_m) \in \mathcal{E}, x_1, x_2\) and \(v_i \in \mathbb{R}\) for \(1 \leq i \leq m + 2\), where

\[
\mathcal{E} = \frac{1}{M_1 + M_2 + \sum_{i=1}^{m} M_3},
\]

\[
M_1 = \|\gamma\| \left[ \frac{1}{T_q(\sigma + 1)} + \frac{|a_1| \sigma \tau^\sigma |\Delta|}{T_q(\sigma + 1)} + \frac{(|a_2| A_1 + |a_3| A_2) \Delta}{T_q(\sigma + 2)} + \frac{A_2 \Delta}{T_q(\sigma)} \right],
\]

\[
M_2 = \|\gamma\| \left[ \frac{1}{T_q(\sigma - 1)} + \frac{|a_1| \sigma \tau^\sigma |\Delta|}{T_q(\sigma + 1)} + \frac{(|a_2| A_1 + |a_3| A_2) \Delta}{T_q(\sigma + 2)} + \frac{A_2 \Delta}{T_q(\sigma)} \right],
\]

\[
iM_3 = \|\gamma\| \left[ \frac{1}{T_q(\sigma - \zeta + 1)} + \frac{|a_1| \sigma \tau^\sigma |\Delta|}{T_q(\sigma + 1)} + \frac{(|a_2| A_1 + |a_3| A_2) \Delta}{T_q(\sigma + 2)} + \frac{A_2 \Delta}{T_q(\sigma)} \right],
\]

for \(i = 1, 2, \ldots, m\). Define the operator \(\Omega : \mathcal{E} \to 2^\mathcal{E}\) by

\[
\Omega(k) = \left\{ \begin{array}{ll}
p \in \mathcal{E} : \\
p(t) = \begin{cases} 
\mathcal{I}_q^\sigma [z](t) + a_1 \Delta A_1(t) \mathcal{I}_q^\sigma [z](\tau) + \Delta (a_2 A_1(t) + a_3 A_2(t)) \mathcal{I}_q^\sigma [z](\sigma) + \Delta A_2(t) \Sigma \mathcal{I}_q^{\sigma - 1}[z](1), \\
\end{cases}
\end{array} \right\}
\]

for \(z \in S_{\mathfrak{q}_p}\). If the multifunction \(\Omega\) has the approximate endpoint property, then the boundary value inclusion problem (1)–(2) has a solution.
Proof  First, we prove that $\Omega(k)$ is a closed subset of $\mathcal{P}(\mathcal{E})$ for all $k \in \mathcal{E}$. Since the multivalued map $t \mapsto \mathcal{T}(t, k(t), k'(t), \mathcal{D}^1 k(t), \ldots, \mathcal{D}^m k(t))$ is measurable and has closed values for all $k \in \mathcal{E}$, has measurable selection and so $S_{\mathcal{T}, k}$ is nonempty for all $k$. Assume that $k \in \mathcal{E}$ and $\{v_n\}_{n \geq 1}$ be a sequence in $\Omega(k)$ with $v_n \to v$. For every $n \geq 1$, choose $z_n \in S_{\mathcal{T}, k}$ such that

$$v_n(t) = \mathcal{T}^n_q [z_n](t) + \Delta a_1 A_1(t) \mathcal{T}^n_q [z_n](\tau) + \Delta a_2 A_2(t) \mathcal{T}^n_q [z_n](s)(1) + \Delta a_3 A_3(t) \mathcal{T}^n_q [z_n](1)$$

for all $t \in \mathcal{T}$. By compactness of $\mathcal{T}$, the sequence $\{z_n\}_{n \geq 1}$ has a subsequence which converges to some $z \in L^1(\mathcal{T})$. We denote this subsequence again by $\{z_n\}_{n \geq 1}$. One can easily check that $z \in S_{\mathcal{T}, k}$ and

$$v_n(t) \to v(t) = \mathcal{T}^n_q [z](t) + \Delta a_1 A_1(t) \mathcal{T}^n_q [z](\tau) + \Delta a_2 A_2(t) \mathcal{T}^n_q [z](s)(1) + \Delta a_3 A_3(t) \mathcal{T}^n_q [z](1)$$

for all $t \in \mathcal{T}$. This shows that $v \in \Omega(k)$ and so $\Omega$ is closed-valued. On the other hand, $\Omega(k)$ is a bounded set for all $k \in \mathcal{E}$ because $\mathcal{T}$ is a compact multivalued map. Finally, we show that $P_v(\Omega(k), \Omega(l)) \leq \psi(||l - k||)$. Let $k, l \in \mathcal{E}$ and $f_1 \in \Omega(l)$. Choose $z_1 \in S_{\mathcal{T}, l}$ such that

$$f_1(t) = \mathcal{T}^n_q [z_1](t) + \Delta a_1 A_1(t) \mathcal{T}^n_q [z_1](\tau) + \Delta a_2 A_2(t) \mathcal{T}^n_q [z_1](s)(1) + \Delta a_3 A_3(t) \mathcal{T}^n_q [z_1](1)$$

for almost all $t \in \mathcal{T}$. Since

$$P_v(\mathcal{T}, k(t), k'(t), \mathcal{D}^1 k(t), \ldots, \mathcal{D}^m k(t)$$

$$- \mathcal{T}(t, l(t), l'(t), \mathcal{D}^1 l(t), \ldots, \mathcal{D}^m l(t))$$

$$\leq \mathcal{S} \mathcal{Y}(t) \psi \left( ||k(t) - l(t)|| + ||k'(t) - l'(t)|| + \sum_{i=1}^{m} ||\mathcal{D}^i k(t) - \mathcal{D}^i l(t)|| \right)$$

for $t \in \mathcal{T}$, there exist $v \in \mathcal{T}(t, k(t), k'(t), \mathcal{D}^1 k(t), \ldots, \mathcal{D}^m k(t))$ such that

$$|z_1(t) - v| \leq \mathcal{S} \mathcal{Y}(t) \psi \left( ||k(t) - l(t)|| + ||k'(t) - l'(t)|| + \sum_{i=1}^{m} ||\mathcal{D}^i k(t) - \mathcal{D}^i l(t)|| \right),$$

for $t \in \mathcal{T}$. Now, we consider the multivalued map $K : \mathcal{T} \to \mathcal{P}(\mathcal{R})$ which is given by

$$K(t) = \left\{ v \in \mathcal{R} : |z_1(t) - v| \leq \mathcal{S} \mathcal{Y}(t) \right.$$ 

$$\times \psi \left( ||k(t) - l(t)|| + ||k'(t) - l'(t)|| + \sum_{i=1}^{m} ||\mathcal{D}^i k(t) - \mathcal{D}^i l(t)|| \right) \right\}.$$
Since $z_1$ and
\[ \mathcal{E} \gamma(t) \psi \left( |k(t) - l(t)| + |k'(t) - l'(t)| + \sum_{i=1}^{m} |c \mathcal{D}_q^{i} [k](t) - c \mathcal{D}_q^{i} [l](t)| \right), \]
are measurable, the multifunction
\[ K(\cdot) \cap T \left( \cdot, k(\cdot), k'(\cdot), c \mathcal{D}_q^{i} [k](\cdot), \ldots, c \mathcal{D}_q^{\infty} [k](\cdot) \right) \]
is measurable. Now, choose $z_2(t) \in T(t, k(t), k'(t), c \mathcal{D}_q^{i} [k](t), \ldots, c \mathcal{D}_q^{\infty} [k](t))$ such that
\[ |z_1(t) - z_2(t)| \leq \mathcal{E} \gamma(t) \psi \left( |k(t) - l(t)| + |k'(t) - l'(t)| + \sum_{i=1}^{m} |c \mathcal{D}_q^{i} [k](t) - c \mathcal{D}_q^{i} [l](t)| \right), \]
for all $t \in \mathcal{J}$. Define the element $f_2 \in \Omega(k)$ by
\[ f_2(t) = \mathcal{T}_q'[z_2](t) + \Delta a_1 A_1(t) \mathcal{T}_q^\sigma[z_2](\tau) + \Delta (a_2 A_1(t) + a_3 A_2(t)) \mathcal{T}_q^\sigma[z_2](s)(1) + \Delta A_2(t) \Sigma \mathcal{T}_q^{\sigma-1}[z_2](1) \]
for all $t \in \mathcal{J}$. Let $\sup_{t \in \mathcal{J}} |\gamma(t)| = \| \gamma \|$. Thus, one can get
\[ |f_1(t) - f_2(t)| \leq \mathcal{T}_q^\sigma[|z_1 - z_2|](t) + \Delta a_1 A_1(t) \mathcal{T}_q^\sigma[|z_1 - z_2|](\tau) + \Delta (a_2 A_1(t) + a_3 A_2(t)) \mathcal{T}_q^\sigma[|z_1 - z_2|](s)(1) + \Delta A_2(t) \Sigma \mathcal{T}_q^{\sigma-1}[|z_1 - z_2|](1) \]
\[ \leq \mathcal{E} \| \gamma \| \psi \left( \| k - l \| \right) \left[ \frac{1}{\mathcal{F}_q(\sigma + 1)} + \frac{|a_1|}{|\Delta|} \frac{\tau^\sigma}{\mathcal{F}_q(\sigma + 1)} + \frac{|\Delta(\|a_2\| A_1 + \|a_3\| A_2)|}{\mathcal{F}_q(\sigma + 2)} + \frac{|\Delta| A_2 \Sigma}{\mathcal{F}_q(\sigma)} \right] \]
\[ = A_1 \mathcal{E} \psi \left( \| k - l \| \right). \]

On the other hand,
\[ |f'_1(t) - f'_2(t)| \leq \mathcal{E} \| \gamma \| \psi \left( \| k - l \| \right) \left[ \frac{1}{\mathcal{F}(\sigma - 1)} + \frac{|\Sigma| |a_1| |\Delta| |a_1| A_1^\prime}{\mathcal{F}_q(\sigma + 1)} \frac{\tau^\sigma}{\mathcal{F}_q(\sigma + 1)} + \frac{|\Delta(\|a_2\| A_1^\prime + \|a_3\| A_2^\prime)|}{\mathcal{F}_1(\sigma + 2)} + \frac{|\Delta| A_2^\prime \Sigma}{\mathcal{F}_1(\sigma)} \right] \]
\[ = A_2 \mathcal{E} \psi \left( \| k - l \| \right) \]
and, for each $i = 1, \ldots, k$, we have
\[ |c \mathcal{D}_q^{i} [f_1](t) - c \mathcal{D}_q^{i} [f_2](t)| \leq \mathcal{E} \| \gamma \| \psi \left( \| k - l \| \right) \times \left[ \frac{1}{\mathcal{F}_q(\sigma - \zeta_i + 1)} + |a_1| |\Delta| \frac{A_1^\prime \tau^\sigma}{\mathcal{F}_q(\sigma + 1)} \right]. \]
Algorithm 5 The proposed method for calculating $\int_{a}^{b} f(q(r)) \, dr$

```
function g = lq(q, x, n, fun)
  p = 1;
  for k = 0:n
    p = p + q^k * fun(x * q^k);
  end;
  g = x * (1 - q) * p;
end
```

\[ + \frac{|\Delta|(|A_2|A_1 + |A_3|A_2)}{\Gamma_q(\sigma + 2)} + \frac{|\Delta|A_2 \Sigma}{\Gamma_q(\sigma)} \]

\[ = \mathcal{X}_1 A_3 \psi (\| k - l \|). \]

Thus, we get

\[ \| f_1 - f_2 \| = \sup_{t \in J} |f_1(t) - f_2(t)| + \sup_{t \in J} |f_1'(t) - f_2'(t)| + \sum_{i=1}^{m} \sup_{t \in J} |^{\sigma}D_q^i [f_1(t) - f_2(t)]| \]

\[ \leq \mathcal{X}_1 \psi (\| k - l \|) (\mathcal{S}) = \psi (\| k - l \|). \]

Hence, $P_\rho (\Omega (k), \Omega (l)) \leq \psi (\| k - l \|)$ for all $k, l \in \mathcal{E}$. By using the hypothesis, the multifunction $\Omega$ has approximate endpoint property. Now by using Lemma 1, there exists $k^* \in \mathcal{E}$ such that $\Omega (k^*) = \{ k^* \}$. Consequently, $k^*$ is a solution for the $q$-inclusion problem (1)–(2).

4 An example by using the algorithms and some numerical calculations

Here, we give an example to illustrate our main result. In this way, we give a computational technique for checking the problem. We need to present a simplified analysis that is able to execute the values of the $q$-Gamma function. For this purpose, we provided a pseudo-code description of the method for calculation of the $q$-Gamma function of order $n$ in the Algorithms 2, 3, 4 and 5. The Algorithm 6 help us for numerical solving of the problem. Also, we provide some figures and numerical tables.

Example 1 Consider the fractional $q$-differential inclusions

\[ ^{c}D_q^{\frac{14}{5}} [k](t) \in \left[ 0, \frac{3t \cos^2 k(t)}{20(1 + \cos^2 k(t))} + \frac{3t|k'(t)|}{20(1 + |k'(t)|)} \right. \]

\[ + \left. \frac{3t \sin(\frac{11}{5} [k](t))}{20(1 + \sin(\frac{11}{5} [k](t)))} + \frac{3t \exp((\frac{3}{5} [k](t)))}{20(1 + \exp((\frac{3}{5} [k](t))))) \right] + \frac{3t|^{c}D_q^{\frac{11}{12}} [k](t)|}{20(1 + |^{c}D_q^{\frac{11}{12}} [k](t)|)} + \frac{3t(\frac{3}{5} [k](t))^2}{20(1 + (\frac{3}{5} [k](t))^2)} \]

(11)
with the sum and integral boundary conditions

\[
\begin{align*}
{k(0) + \Sigma k'(0) &= 0,} \\
\frac{37}{100} k\left(\frac{11}{80}\right) + \frac{19}{100} \phi(1) &= 0, \\
\Sigma k'(1) + \frac{9}{100} \phi(1) &= 0,
\end{align*}
\] (12)

where \( t \in [0, 1] \) and \( \phi(v) = \int_0^v \sin(k(r)) \, dr \). Consider the set-valued map \( T : \bar{T} \times \mathbb{R}^6 \rightarrow \mathcal{P}_{cp}(\mathbb{R}) \) defined by

\[
T(t, k_1, k_2, k_3, k_4, k_5, k_6) = \left[ 0, \frac{3t \cos^2 k_1(t)}{20(1 + \cos^2 k_1(t))} + \frac{3t |k_2(t)|}{20(1 + |k_2(t)|)} + \frac{3t |k_3(t)|}{20(1 + |k_3(t)|)} + \frac{3t |k_4(t)|}{20(1 + |k_4(t)|)} + \frac{3t |k_5(t)|}{20(1 + |k_5(t)|)} + \frac{3t |k_6(t)|}{20(1 + |k_6(t)|)} \right]
\]

It is clear that \( \sigma = \frac{13}{5}, \xi_1 = \frac{11}{9}, \xi_2 = \frac{8}{5}, \xi_3 = \frac{7}{6}, \xi_4 = \frac{11}{8}, m = 4, a_1 = \frac{37}{100}, a_2 = \frac{19}{100}, a_3 = \frac{9}{100}, r = \frac{11}{80}, k = 5, \Sigma = \sum_{j=1}^5 c_j = \frac{1}{10} \) with \( c_1 = c_2 = c_3 = c_4 = c_5 = \frac{1}{50} \). In this case, we have \( \gamma : [0, 1] \rightarrow [0, \infty) \) by \( \gamma(t) = \frac{2}{3} t \) and \( \|\gamma\| = \frac{2}{3} \). Clearly, the function \( \gamma \) is nondecreasing upper semi-continuous on \([0, 1]\) such that \( \lim_{t \to \infty} (t - \psi(t)) > 0 \) and \( \psi(t) < t \) for all \( t > 0 \).

From (5), (9) and (10), we get

\[
\Delta = \left[ \left( a_1 t + \frac{a_2}{2} \right) \left( 2 \Sigma + \frac{a_3}{3} - 2 a_3 \Sigma \right) \\
+ \left( \Sigma + \frac{a_3}{2} \right) \left( 2 \Sigma (a_1 + a_3) - \left( a_1 t^2 + \frac{a_2}{3} \right) \right) \right]^{-1}
= \left[ \left( \frac{37}{100} \frac{11}{80} + \frac{19}{200} \right) \left( \frac{1}{5} + \frac{3}{100} - \frac{9}{50} \right) \right. \\
+ \left( \frac{1}{10} + \frac{9}{200} \right) \left( \frac{14}{125} \frac{37}{100} \left( \frac{11}{80} \right)^2 + \frac{19}{300} \right) \right]^{-1} = 27.0505 \] (13)

and

\[
\phi A_1 = 2 \left( |\Sigma| + \frac{|a_3|}{2} \right) \left( 1 + |\Sigma| \right) + 2|a_3| |\Sigma| \\
= 2 \left( |0.1| + \frac{9}{200} \right) \left( 1 + |0.1| \right) + \frac{9}{50} |0.1| = 0.3370,
\]

\[
\phi A'_1 = (2|a_3| + 1) |\Sigma| + \frac{|a_3|}{2} \\
= \left( \frac{2}{100} + 1 \right) |0.1| + \frac{9}{200} = 0.1630
\]

\[
\phi A_2 = \left( |a_1 t + \frac{a_2}{2} \right) \left( 2 |\Sigma| + 1 \right) + |a_1 t^2 + \frac{a_2}{3} + 2|a_1 + a_2| |\Sigma| \\
= 1.2 \left( \frac{407}{8000} \frac{19}{200} \right) + \frac{37}{100} \left( \frac{11}{80} \right)^2 + \frac{19}{300} + \frac{28}{25} |0.1| = 0.3574,
\]
Table 1 Some numerical results for $\mathcal{A}_1$ and $\mathcal{A}_2$ in Example 1 for $q = \frac{1}{10}, \frac{1}{7}, \frac{6}{7}$

| $n$ | $q = \frac{1}{10}$ | $q = \frac{1}{7}$ | $q = \frac{6}{7}$ |
|-----|-------------------|-------------------|-------------------|
|     | $\mathcal{A}_1$   | $\mathcal{A}_2$   | $\mathcal{A}_1$   | $\mathcal{A}_2$   |
| 1   | 0.4478            | 0.4678            | 0.4226            | 0.442             | 0.3201            | 0.3423            |
| 2   | 0.4479            | 0.4679            | 0.4353            | 0.4544            | 0.3365            | 0.3575            |
| 3   | 0.448             | 0.4679            | 0.4417            | 0.4606            | 0.3518            | 0.3722            |
| 4   | 0.448             | 0.4679            | 0.4449            | 0.4638            | 0.3654            | 0.3853            |
| 5   | 0.448             | 0.4679            | 0.4465            | 0.4654            | 0.3773            | 0.3969            |
| ... |                  |                  |                  |                  |                  |                  |
| 14  | 0.448             | 0.4679            | 0.4481            | 0.4669            | 0.4316            | 0.4501            |
| 15  | 0.448             | 0.4679            | 0.4482            | 0.4669            | 0.4342            | 0.4526            |
| 16  | 0.448             | 0.4679            | 0.4482            | 0.4669            | 0.4364            | 0.4548            |
| 17  | 0.448             | 0.4679            | 0.4482            | 0.4669            | 0.4383            | 0.4567            |
| ... |                  |                  |                  |                  |                  |                  |
| 49  | 0.448             | 0.4679            | 0.4482            | 0.4669            | 0.4496            | 0.4678            |
| 50  | 0.448             | 0.4679            | 0.4482            | 0.4669            | 0.4496            | 0.4679            |
| 51  | 0.448             | 0.4679            | 0.4482            | 0.4669            | 0.4496            | 0.4679            |
| 52  | 0.448             | 0.4679            | 0.4482            | 0.4669            | 0.4496            | 0.4679            |

Tables 1, 2, 3, 4 show that

\[
\begin{align*}
\mathcal{A}_1 & \approx 0.4480, 0.4482, 0.4496, \\
\mathcal{A}_2 & \approx 0.4679, 0.4669, 0.4679,
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}_1 & \approx 0.4109, 0.4260, 0.4001, \\
\mathcal{A}_2 & \approx 0.4260, 0.4128, 0.4111,
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}_1 & \approx 0.4500, 0.4508, 0.4521, \\
\mathcal{A}_2 & \approx 0.4704, 0.4703, 0.4712,
\end{align*}
\]

\[
\begin{align*}
\mathcal{A}_1 & \approx 0.4386, 0.4354, 0.4364, \\
\mathcal{A}_2 & \approx 0.4572, 0.4519, 0.4521,
\end{align*}
\]

for $q = \frac{1}{10}, \frac{1}{7}, \frac{6}{7}$, respectively. By using (8), we get

\[
M_1 = \| \gamma \| \left[ \frac{1}{\Gamma_q(\sigma + 1)} + \frac{|a_1|\tau}{\Gamma_q(\sigma + 1)}|\Delta| \right]
\]
| $n$ | $q = \frac{1}{10}$ | $q = \frac{1}{2}$ | $q = \frac{6}{7}$ |
|-----|----------------|----------------|----------------|
|     | $j_2 A_1$ | $j_2 A_2$ | $j_2 A_1$ | $j_2 A_2$ | $j_2 A_1$ | $j_2 A_2$ |
| 1   | 0.4111 | 0.4262 | 0.3941 | 0.4076 | 0.3713 | 0.3916 |
| 2   | 0.4109 | 0.426 | 0.397 | 0.4097 | 0.3663 | 0.3836 |
| 3   | 0.4109 | 0.426 | 0.3988 | 0.4112 | 0.3674 | 0.383 |
| ... | ... | ... | ... | ... | ... | ... |
| 8   | 0.4109 | 0.426 | 0.4007 | 0.4127 | 0.3827 | 0.3951 |
| 9   | 0.4109 | 0.426 | 0.426 | 0.4128 | 0.3871 | 0.3991 |
| 10  | 0.4109 | 0.426 | 0.4007 | 0.4128 | 0.389 | 0.4008 |
| ... | ... | ... | ... | ... | ... | ... |
| 43  | 0.4109 | 0.426 | 0.4007 | 0.4128 | 0.4 | 0.411 |
| 44  | 0.4109 | 0.426 | 0.4007 | 0.4128 | 0.4001 | 0.411 |
| 45  | 0.4109 | 0.426 | 0.4007 | 0.4128 | 0.4001 | 0.411 |
| 46  | 0.4109 | 0.426 | 0.4007 | 0.4128 | 0.4001 | 0.411 |
| 47  | 0.4109 | 0.426 | 0.4007 | 0.4128 | 0.4001 | 0.411 |

Table 2 shows that $M_1 \approx 0.5015, 0.1483, 0.0617$ and $M_2 \approx 0.5053, 0.2350, 0.1570$ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively (Fig. 1). By using (10), we obtain

$$i M_3 = \| \phi \| \left[ \frac{1}{T_{q}(\sigma - \zeta_1)} \right] + \frac{|a_1| \tau^\sigma |\Delta|}{T_{q}(\sigma + 1)} + \frac{|a_2| |A_1| \tau^\sigma |\Delta|}{T_{q}(\sigma + 2)} + \frac{|a_2| \tau^\sigma |\Delta|}{T_{q}(\sigma)} \left[ \right]$$

$$= \left[ \right]$$

Table 5 shows that $M_1 \approx 0.5015, 0.1483, 0.0617$ and $M_2 \approx 0.5053, 0.2350, 0.1570$ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively (Fig. 1). By using (10), we obtain
Table 3  Some numerical results for $3A_1$ and $3A_2$ in Example 1 for $q = \frac{1}{10}, \frac{1}{7}, \frac{6}{7}$

| $n$ | $q = \frac{1}{10}$ | $q = \frac{1}{7}$ | $q = \frac{6}{7}$ |
|-----|-----------------|-----------------|-----------------|
|     | $3A_1$          | $3A_2$          | $3A_1$          |
|     | $3A_2$          | $3A_1$          | $3A_2$          |
| 1   | 0.4498          | 0.4702          | 0.4224          | 0.4423          |
| 2   | 0.45            | 0.4703          | 0.4366          | 0.4562          | 0.3274          | 0.3484          |
| 3   | 0.45            | 0.4704          | 0.4437          | 0.4632          | 0.3447          | 0.3652          |
| 4   | 0.45            | 0.4704          | 0.4472          | 0.4667          | 0.3599          | 0.38            |
| ... | ...             | ...             | ...             | ...             | ...             | ...             |
| 11  | 0.45            | 0.4704          | 0.4508          | 0.4702          | 0.4208          | 0.4401          |
| 12  | 0.45            | 0.4704          | 0.4508          | 0.4703          | 0.4253          | 0.4446          |
| 13  | 0.45            | 0.4704          | 0.4508          | 0.4703          | 0.4291          | 0.4484          |
| 14  | 0.45            | 0.4704          | 0.4508          | 0.4703          | 0.4324          | 0.4517          |

Table 4  Some numerical results for $4A_1$ and $4A_2$ in Example 1 for $q = \frac{1}{10}, \frac{1}{7}, \frac{6}{7}$

| $n$ | $q = \frac{1}{10}$ | $q = \frac{1}{7}$ | $q = \frac{6}{7}$ |
|-----|-----------------|-----------------|-----------------|
|     | $4A_1$          | $4A_2$          | $4A_1$          |
|     | $4A_2$          | $4A_1$          | $4A_2$          |
| 1   | 0.4385          | 0.4571          | 0.4179          | 0.4355          |
| 2   | 0.4386          | 0.4572          | 0.4265          | 0.4435          | 0.3565          | 0.3771          |
| 3   | 0.4386          | 0.4572          | 0.4309          | 0.4477          | 0.366           | 0.3853          |
| ... | ...             | ...             | ...             | ...             | ...             | ...             |
| 7   | 0.4386          | 0.4572          | 0.4352          | 0.4517          | 0.3972          | 0.4143          |
| 8   | 0.4386          | 0.4572          | 0.4353          | 0.4518          | 0.4028          | 0.4196          |
| 9   | 0.4386          | 0.4572          | 0.4354          | 0.4519          | 0.4075          | 0.4242          |
| ... | ...             | ...             | ...             | ...             | ...             | ...             |
| 44  | 0.4386          | 0.4572          | 0.4354          | 0.4519          | 0.4364          | 0.452           |
| 45  | 0.4386          | 0.4572          | 0.4354          | 0.4519          | 0.4364          | 0.452           |
| 46  | 0.4386          | 0.4572          | 0.4354          | 0.4519          | 0.4364          | 0.452           |
| 47  | 0.4386          | 0.4572          | 0.4354          | 0.4519          | 0.4364          | 0.452           |

Table 5  Some numerical results for $M_1$ and $M_2$ in Example 1 for $q = \frac{1}{10}, \frac{1}{7}, \frac{6}{7}$

| $n$ | $q = \frac{1}{10}$ | $q = \frac{1}{7}$ | $q = \frac{6}{7}$ |
|-----|-----------------|-----------------|-----------------|
|     | $M_1$           | $M_2$           | $M_1$           |
|     | $M_2$           | $M_1$           | $M_2$           |
| 1   | 0.501           | 0.5047          | 0.117           | 0.1905          | 0.0053          | 0.0257          |
| 2   | 0.5015          | 0.5052          | 0.1321          | 0.2121          | 0.0087          | 0.0372          |
| 3   | 0.5015          | 0.5052          | 0.1401          | 0.2234          | 0.0125          | 0.0488          |
| ... | ...             | ...             | ...             | ...             | ...             | ...             |
| 13  | 0.5015          | 0.5053          | 0.1483          | 0.2349          | 0.0465          | 0.1278          |
| 14  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0485          | 0.1318          |
| 15  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0503          | 0.1352          |
| 16  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0518          | 0.1382          |
| 17  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0532          | 0.1408          |
| 18  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0543          | 0.1431          |
| ... | ...             | ...             | ...             | ...             | ...             | ...             |
| 48  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0617          | 0.1569          |
| 49  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0617          | 0.157           |
| 50  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0617          | 0.157           |
| 51  | 0.5015          | 0.5053          | 0.1483          | 0.235           | 0.0617          | 0.157           |
Table 6 shows that $M_1 \approx 0.6407, 0.2231, 0.1142$, $M_2 \approx 0.6016, 0.2237, 0.1273$, $M_3 \approx 0.6424, 0.2218, 0.1120$, and $M_4 \approx 0.6315, 0.2252, 0.1199$ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively (Fig. 2). Thus, we can easily get

$$P(\tau(t, k_1, k_2, k_3, k_4, k_5, k_6), T(t, k'_1, k'_2, k'_3, k'_4, k'_5, k'_6)) \leq \Xi(t) \psi \left( \sum_{i=1}^{6} |k_i - k'_i| \right),$$

for all $k_i, k'_i \in \mathbb{R}$ $(i = 1, 2, 3, 4, 5, 6)$, where

$$\Xi = [M_1 + M_2 + M_3 + M_4]^{-1}.$$

Table 7 shows that $\Xi \approx 0.2839, 0.3449, 0.3657$ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$, respectively (Fig. 3). Let $E_1 = \{k : k, \sigma, D_q^{11} [k] (t) \in \mathcal{A} \}$, $E_2 = \{k : k, \sigma, D_q^{6} [k] (t) \in \mathcal{A} \}$, $E_3 = \{k : k, \sigma, D_q^{7} [k] (t) \in \mathcal{A} \}$, $E_4 = \{k : k, \sigma, D_q^{11} [k] (t) \in \mathcal{A} \}$ and $\mathcal{E} = E_1 \times E_2 \times E_3 \times E_4$. Now, define the operator $\Omega : \mathcal{E} \rightarrow 2^\mathcal{E}$ by $\Omega(u) = \{p \in \mathcal{E} : \text{there exists } v \in S_{t, \mathcal{A}} \text{ such that } p(t) = v(t) \text{ for all } t \in \mathcal{J} \}$, where

$$v(t) = I_q^{\sigma} [z] (t) + \Delta a_1 A_1 (t) I_q^{\sigma} [z] (\tau)$$
Table 6 Some numerical results of $M_i$ ($i = 1, 2, 3, 4$) in Example 1 for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$

| $n$ | $1_{M_1}$ | $2_{M_1}$ | $3_{M_1}$ | $4_{M_1}$ |
|-----|-----------|-----------|-----------|-----------|
| 1   | 0.6398    | 0.6012    | 0.6415    | 0.6307    |
| 2   | 0.6406    | 0.6016    | 0.6423    | 0.6314    |
| 3   | 0.6407    | 0.6016    | 0.6424    | 0.6315    |
| 4   | 0.6407    | 0.6016    | 0.6424    | 0.6315    |
| 5   | 0.6407    | 0.6016    | 0.6424    | 0.6315    |
| 6   | 0.6407    | 0.6016    | 0.6424    | 0.6315    |

$q = \frac{1}{10}$

| $n$ | $1_{M_2}$ | $2_{M_2}$ | $3_{M_2}$ | $4_{M_2}$ |
|-----|-----------|-----------|-----------|-----------|
| 1   | 0.171     | 0.1776    | 0.169     | 0.1752    |
| 2   | 0.1959    | 0.1997    | 0.1942    | 0.1991    |
| 3   | 0.2092    | 0.2115    | 0.2077    | 0.2119    |
| 10  | 0.223     | 0.2236    | 0.2217    | 0.2251    |
| 11  | 0.2231    | 0.2236    | 0.2217    | 0.2252    |
| 12  | 0.2231    | 0.2236    | 0.2218    | 0.2252    |
| 13  | 0.2231    | 0.2237    | 0.2218    | 0.2252    |
| 14  | 0.2231    | 0.2237    | 0.2218    | 0.2252    |
| 15  | 0.2231    | 0.2237    | 0.2218    | 0.2252    |

$q = \frac{1}{2}$

| $n$ | $1_{M_3}$ | $2_{M_3}$ | $3_{M_3}$ | $4_{M_3}$ |
|-----|-----------|-----------|-----------|-----------|
| 1   | 0.0104    | 0.0159    | 0.0097    | 0.0124    |
| 2   | 0.0167    | 0.024     | 0.0158    | 0.0195    |
| 3   | 0.0238    | 0.0327    | 0.0227    | 0.0272    |
| 46  | 0.1141    | 0.1272    | 0.1119    | 0.1198    |
| 47  | 0.1141    | 0.1272    | 0.1119    | 0.1198    |
| 48  | 0.1141    | 0.1273    | 0.1119    | 0.1198    |
| 49  | 0.1141    | 0.1273    | 0.1119    | 0.1199    |
| 50  | 0.1142    | 0.1273    | 0.1119    | 0.1199    |
| 51  | 0.1142    | 0.1273    | 0.112     | 0.1199    |
| 52  | 0.1142    | 0.1273    | 0.112     | 0.1199    |
| 53  | 0.1142    | 0.1273    | 0.112     | 0.1199    |
| 54  | 0.1142    | 0.1273    | 0.112     | 0.1199    |

$q = \frac{6}{7}$

| $n$ | $1_{M_4}$ | $2_{M_4}$ | $3_{M_4}$ | $4_{M_4}$ |
|-----|-----------|-----------|-----------|-----------|
| 1   | 0.0104    | 0.0159    | 0.0097    | 0.0124    |
| 2   | 0.0167    | 0.024     | 0.0158    | 0.0195    |
| 3   | 0.0238    | 0.0327    | 0.0227    | 0.0272    |
| 46  | 0.1141    | 0.1272    | 0.1119    | 0.1198    |
| 47  | 0.1141    | 0.1272    | 0.1119    | 0.1198    |
| 48  | 0.1141    | 0.1273    | 0.1119    | 0.1198    |
| 49  | 0.1141    | 0.1273    | 0.1119    | 0.1199    |
| 50  | 0.1142    | 0.1273    | 0.1119    | 0.1199    |
| 51  | 0.1142    | 0.1273    | 0.112     | 0.1199    |
| 52  | 0.1142    | 0.1273    | 0.112     | 0.1199    |
| 53  | 0.1142    | 0.1273    | 0.112     | 0.1199    |
| 54  | 0.1142    | 0.1273    | 0.112     | 0.1199    |

$+ \Delta (a_2 A_1(t) + a_3 A_2(t)) \int_0^t \left[ \int_0^s [z(s)]^2(1) + \Delta A_2(t) \Sigma I_q^{\alpha-1}[z](1) \right]$ 

$= I_q^{\alpha} [z](t) + 10.0087 A_1(t) I_q^{\alpha} \left[ \int_0^s [z(s)]^2(1) \right]^{11} \left( \frac{80}{11} \right)$ 

$+ \left( 5.1396 A_1(t) + 2.4345 A_2(t) \right) I_q^{\frac{14}{11}} \left[ \int_0^s [z(s)]^2(1) \right]^{14} \int_q^0 \left[ I_q^{\alpha} [z](s) \right](1)$ 

$+ 2.70505 A_2(t) I_q^{\frac{9}{11}} [z](1)$.

Here, $I_q^{\alpha} \left[ \int_0^s [z(s)]^2(1) \right]^{11} = \int_0^1 \int_0^r \left( \frac{\alpha-q}{r_q^{\alpha-1}} \right)^{11} z(s) ds dr$.

$A_1(t) = \left( \Sigma + \frac{a_3}{2} \right) (t^2 - t - 2 \Sigma) + 2a_3 \Sigma = 0.145t^2 - 0.127t + 0.029,$

$A_2(t) = \left( a_1 \tau + \frac{a_2}{2} \right) (2 \Sigma - t^2) + \tau \left( a_1 \tau^2 + \frac{a_2}{3} - 2(a_1 + a_2) \Sigma \right)$

$= -0.145875t^2 + 0.041671t + 0.029175.$

The Table 7 shows the values of $\varepsilon \approx 0.2839, 0.3449, 0.3657$ for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$. One can easily check that $\inf_{k \in \bar{E}} \sup_{l \in \Omega(k)} \|k - l\| = 0$. Thus, the operator $\Omega$ has the approximate
Figure 2 Numerical results of $\mathcal{M}_i$, ($i = 1, 2, 3, 4$) where $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$ in Example 1

Table 7 Numerical results of $\mathcal{M}_i$ in Example 1 for $q = \frac{1}{10}, \frac{1}{2}, \frac{6}{7}$

| $n$ | $q = \frac{1}{10}$ | $q = \frac{1}{2}$ | $q = \frac{6}{7}$ |
|-----|--------------------|--------------------|-------------------|
| 1   | 0.2842             | 0.3545             | 0.393             |
| 2   | 0.2839             | 0.3496             | 0.3904            |
| 3   | 0.2839             | 0.3473             | 0.388             |
| ... | ...                | ...                | ...               |
| 8   | 0.2839             | 0.345              | 0.3778            |
| 9   | 0.2839             | 0.3449             | 0.3763            |
| 10  | 0.2839             | 0.3449             | 0.3749            |
| 11  | 0.2839             | 0.3449             | 0.3737            |
| ... | ...                | ...                | ...               |
| 38  | 0.2839             | 0.3449             | 0.3658            |
| 39  | 0.2839             | 0.3449             | 0.3657            |
| 40  | 0.2839             | 0.3449             | 0.3657            |
| 41  | 0.2839             | 0.3449             | 0.3657            |

endpoint property. Now, by using Theorem 4, we get the fractional $q$-differential inclusion problem (11) has a solution.

5 Conclusion
Most natural phenomena could be modeled by different types of fractional differential equations and inclusions. Recently some physicists have been studying the role of fractional calculus in better describing of physical phenomena. They have found that by using the $q$-fractional they can provide a better description by some physical notions. Thus, we should investigate distinct fractional differential equations and inclusions to increase our
Algorithm 6 The proposed method for calculating the problem (1)–(2)

```
function [IA1, IA2, M1, M2, IM3, XI] = funcproblem(q, sigma, zeta, m, k, ...
  a, Sigma, tau, normgamma)
  [eq, yq] = size(q);
  [zeta, yzeta] = size(zeta);
  Delta = (a(1) + tau + a(2)/2) * (2 * Sigma + a(3)/3 - 2 * ...
    a(3) * Sigma + (Sigma + a(3)/2) * (2 * Sigma + ...
    (a(1) + tau/2 + a(2)/3))) 
  A01 = 2 * (abs(Sigma) + abs(a(3))/2) * (1 + abs(Sigma)) + 2 * ...
    abs(a(3)) + abs(Sigma); 
  Aprim01 = (2 * abs(a(3)) + 1) * abs(Sigma) + abs(a(3))/2;
  A02 = (abs(a(1)) * tau + abs(a(2))/2) * (2 * abs(Sigma) + 1) + ...
    abs(a(1)) * tau*2 + abs(a(2))/3 + 2 * abs(a(1)) + a(2) + ...
    abs(Sigma); 
  Aprim02 = 2 * (abs(a(1)) + tau + abs(a(2))/2) + ...
    abs(a(1)) * tau*2 + abs(a(2))/3 + 2 * abs(a(1)) + a(2) + ...
    abs(Sigma); 
  for n=1:k
    gamzeta(n, 1) = n;
    M1(n, 1) = n;
    M2(n, 1) = n;
    IM3(n, 1) = n;
    IA1(n, 1) = n;
    IA2(n, 1) = n;
    Xi(n, 1) = n;
  end;
  column = 2;
  for s = 1:qy
    for n=1:k
      s00 = qGamma(q(s), sigma, n);
      s01 = qGamma(q(s), sigma - 1, n);
      s02 = qGamma(q(s), sigma + 1, n);
      s03 = qGamma(q(s), sigma + 2, n);
      M1(n, column) = normgamma(1/s00 + ...
        abs(a(1)) * A01 + tau*sigma + ...
        abs(Delta)/s01 + (abs(a(2)) + A01 + ...
        abs(a(3)) + A02) * abs(Delta)/s02 + A02 + ...
        abs(Delta) + abs(Sigma)/s01);
      M2(n, column) = normgamma(1/s03 + ...
        abs(a(1)) * Aprim01 + tau*sigma + ...
        abs(Delta)/s01 + (abs(a(2)) + Aprim01 + ...
        abs(a(3)) + Aprim02) * abs(Delta)/s02 + ...
        Aprim02 + abs(Delta) + abs(Sigma)/s00);
    end;
    column = column + 1;
  end;
  column = 2;
  for i = 1:m
    for n=1:k
      s00 = qGamma(q(s), sigma, n);
      s01 = qGamma(q(s), sigma - 1, n);
      s02 = qGamma(q(s), sigma + 1, n);
      s03 = qGamma(q(s), sigma + 2, n);
      zeta = qGamma(q(s), zeta - zeta(1) + 1, n);
      s1 = qGamma(q(s), zeta - zeta(1), n);
      s2 = qGamma(q(s), zeta + zeta(1), n);
      IA1(n, column) = abs(Sigma) + ...
        abs(a(3)/2) + (1/s1 - 2/s2) + ...
        2 * abs(a(1)) + abs(Sigma)/s1;
      IA2(n, column) = (abs(a(1)) + tau + ...
        abs(a(2)/2) + 2/s2 + ...
        abs(a(1)) * tau*2 + abs(a(2))/3 + ...
        2 * abs(a(1)) + ...
        (s1) + abs(Sigma))/s1;
      IM3(n, column) = normgamma(1/s00 + ...
        abs(a(1)) + IA1(n, ...
        column) + tau*sigma + abs(Delta)/s01 + ...
        abs(a(2)) + IA1(n, column) + ...
        abs(a(3)) + IA2(n, column) + ...
        abs(Delta)/s02 = IA2(n, column) + ...
        abs(Delta) + abs(Sigma)/s00);
    end;
    column = column + 1;
```
Algorithm 6 (Continued)

```matlab
45     end;
46     end;
47     column=2;
48     for s=1:nyq
49         for n=1:x;
50             mm=0;
51             for c=2:3:4*nyq
52                 mm=mm + 1*M3(n, c);
53             end;
54             Xi(n, column) = 1/(M1(n, column) + M2(n, ...
55                 column) + mm);
56         end;
57     column=column+1;
58     end;
```

**Figure 3** Numerical results of $X_i$ where $q = \frac{1}{10}, \frac{1}{7}, \frac{6}{7}$ in Example 1

ability for exact modelings of more phenomena. It is better we concentrate on numerical methods for reviewing of inclusion problems. In this work, by using an endpoint result for set-valued maps, we study a fractional q-differential inclusion problem with sum and integral boundary value conditions on a time scale. We provide an example involving some graphs and algorithms via numerical calculations to illustrate our main result.

Acknowledgements
The first author was supported by Bu-Ali Sina University. The second author was supported by Azarbaijan Shahid Madani University. The authors express their gratitude to the unknown referees for their helpful suggestions, which improved the final version of this paper.

Funding
Not applicable.

Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Ethics approval and consent to participate
Not applicable.

Competing interests
The authors declare that they have no competing interests.
Consent for publication
Not applicable.

Authors’ contributions
The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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Publisher’s Note
Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Received: 7 April 2020 Accepted: 24 August 2020 Published online: 03 September 2020

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