4. Cohomological symbol for henselian discrete valuation fields of mixed characteristic

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4.1. Cohomological symbol map

Let $K$ be a field. If $m$ is prime to the characteristic of $K$, there exists an isomorphism

$$h_{1,K}: K^*/m \to H^1(K, \mu_m)$$

supplied by Kummer theory. Taking the cup product we get

$$(K^*/m)^q \to H^q(K, \mathbb{Z}/m(q))$$

and this factors through (by [T])

$$h_{q,K}: K_q(K)/m \to H^q(K, \mathbb{Z}/m(q)).$$

This is called the cohomological symbol or norm residue homomorphism.

Milnor–Bloch–Kato Conjecture. For every field $K$ and every positive integer $m$ which is prime to the characteristic of $K$ the homomorphism $h_{q,K}$ is an isomorphism.

This conjecture is shown to be true in the following cases:

(i) $K$ is an algebraic number field or a function field of one variable over a finite field and $q = 2$, by Tate [T].

(ii) Arbitrary $K$ and $q = 2$, by Merkur’ev and Suslin [MS1].

(iii) $q = 3$ and $m$ is a power of 2, by Rost [R], independently by Merkur’ev and Suslin [MS2].

(iv) $K$ is a henselian discrete valuation field of mixed characteristic $(0, p)$ and $m$ is a power of $p$, by Bloch and Kato [BK].

(v) $(K, q)$ arbitrary and $m$ is a power of 2, by Voevodsky [V].

For higher dimensional local fields theory Bloch–Kato’s theorem is very important and the aim of this text is to review its proof.

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Theorem (Bloch–Kato). Let $K$ be a henselian discrete valuation fields of mixed characteristic $(0, p)$ (i.e., the characteristic of $K$ is zero and that of the residue field of $K$ is $p > 0$), then

$$h_{q,K}: K_q(K)/p^n \rightarrow H^q(K, \mathbb{Z}/p^n(q))$$

is an isomorphism for all $n$.

Till the end of this section let $K$ be as above, $k = k_K$ the residue field of $K$.

4.2. Filtration on $K_q(K)$

Fix a prime element $\pi$ of $K$.

Definition.

$$U_m K_q(K) = \begin{cases} K_q(K), & m = 0 \\ \langle \{1 + \mathcal{M}_K^m \} \cdot K_{q-1}(K) \rangle, & m > 0 \end{cases}$$

Put $\text{gr}_m K_q(K) = U_m K_q(K)/U_{m+1} K_q(K)$.

Then we get an isomorphism by [FV, Ch. IX sect. 2]

$$K_q(k) \oplus K_{q-1}(k) \xrightarrow{\rho_0} \text{gr}_0 K_q(K)$$

$$\rho_0 \left( \{x_1, \ldots, x_q\}, \{y_1, \ldots, y_{q-1}\} \right) = \{\widetilde{x}_1, \ldots, \widetilde{x}_q\} + \{\widetilde{y}_1, \ldots, \widetilde{y}_{q-1}, \pi\}$$

where $\widetilde{x}$ is a lifting of $x$. This map $\rho_0$ depends on the choice of a prime element $\pi$ of $K$.

For $m \geq 1$ there is a surjection

$$\Omega_k^{q-1} \oplus \Omega_k^{q-2} \xrightarrow{\rho_m} \text{gr}_m K_q(K)$$

defined by

$$\left( \begin{array}{c} \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-2}}{y_{q-2}} \\ \frac{dy_1}{y_1} \end{array} \right) \mapsto \{1 + \pi^m \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_{q-1}\},$$

$$\left( \begin{array}{c} \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_{q-2}}{y_{q-2}} \\ 0, \frac{dy_1}{y_1} \end{array} \right) \mapsto \{1 + \pi^m \widetilde{x}, \widetilde{y}_1, \ldots, \widetilde{y}_{q-2}, \pi\}.$$

Definition.

$$k_q(K) = K_q(K)/p, h_q(K) = H^q(K, \mathbb{Z}/p(q)),$$

$$U_m k_q(K) = \text{im}(U_m K_q(K)) \text{ in } k_q(K), \quad U_m h_q(K) = h_{q,K}(U_m k_q(K)),$$

$$\text{gr}_m h_q(K) = U_m h_q(K)/U_{m+1} h_q(K).$$
Proposition. Denote \( \nu_q(k) = \ker(\Omega^1_k \to \Omega^q_k \otimes \Omega^{q-1}_k) \) where \( C^{-1} \) is the inverse Cartier operator:

\[
x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \mapsto x^p \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}.
\]

Put \( e' = pe/(p-1) \), where \( e = v_K(p) \).

(i) There exist isomorphisms \( \nu_q(k) \to k_q(k) \) for any \( q \); and the composite map denoted by \( \tilde{\rho}_0 \)

\[
\tilde{\rho}_0 : \nu_q(k) \oplus \nu_{q-1}(k) \to k_q(k) \oplus k_{q-1}(k) \to \gr_0 k_q(K)
\]

is also an isomorphism.

(ii) If \( 1 \leq m < e' \) and \( p \nmid m \), then \( \rho_m \) induces a surjection

\[
\tilde{\rho}_m : \Omega^{q-1}_k \to \gr_m k_q(K).
\]

(iii) If \( 1 \leq m < e' \) and \( p \mid m \), then \( \rho_m \) factors through

\[
\tilde{\rho}_m : \Omega^{q-1}_k \otimes \Omega^{q-2}_k \to \gr_m k_q(K)
\]

and \( \tilde{\rho}_m \) is a surjection. Here we denote \( Z^q_1 = \ker(d : \Omega^{q-2}_k \to \Omega^{q-1}_k) \).

(iv) If \( m = e' \in \mathbb{Z} \), then \( \rho_m \) factors through

\[
\tilde{\rho}_m : \Omega^{q-1}_k / (1 + a C) Z^q_1 \otimes \Omega^{q-2}_k / (1 + a C) Z^q_1 \to \gr_m k_q(K)
\]

and \( \tilde{\rho}_m \) is a surjection.

Here \( a \) is the residue class of \( p\pi^{-e} \), and \( C \) is the Cartier operator

\[
x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q} \mapsto x \frac{dy_1}{y_1} \wedge \cdots \wedge \frac{dy_q}{y_q}, \quad d\Omega^{q-1}_k \to 0.
\]

(v) If \( m > e' \), then \( \gr_m k_q(K) = 0 \).

Proof. (i) follows from Bloch–Gabber–Kato’s theorem (subsection 2.4). The other claims follow from calculations of symbols. \( \square \)

Definition. Denote the left hand side in the definition of \( \tilde{\rho}_m \) by \( G^q_m \). We denote the composite map \( G^q_m \xrightarrow{\tilde{\rho}_m} \gr_m k_q(K) \xrightarrow{h_{q,k}} \gr_m h^q(K) \) by \( \rho_m \); the latter is also surjective.
4.3

In this and next section we outline the proof of Bloch–Kato’s theorem.

4.3.1. Norm argument.

We may assume \( \zeta_p \in K \) to prove Bloch–Kato’s theorem. Indeed, \(|K(\zeta_p) : K|\) is a divisor of \( p - 1 \) and therefore is prime to \( p \). There exists a norm homomorphism \( N_{L/K} : K_q(L) \to K_q(K) \) (see [BT, Sect. 5]) such that the following diagram is commutative:

\[
\begin{array}{ccc}
K_q(K)/p^n & \xrightarrow{h_{q,K}} & K_q(L)/p^n & \xrightarrow{N_{L/K}} & K_q(K)/p^n \\
\downarrow h_{q,L} & & \downarrow h_{q,L} & & \downarrow h_{q,K} \\
H^q(K, \mathbb{Z}/p^n(q)) & \xrightarrow{\text{res}} & H^q(L, \mathbb{Z}/p^n(q)) & \xrightarrow{\text{cor}} & H^q(K, \mathbb{Z}/p^n(q))
\end{array}
\]

where the left horizontal arrow of the top row is the natural map, and \( \text{res} \) (resp. \( \text{cor} \)) is the restriction (resp. the corestriction). The top row and the bottom row are both multiplication by \( |L : K| \), thus they are isomorphisms. Hence the bijectivity of \( h_{q,K} \) follows from the bijectivity of \( h_{q,L} \) and we may assume \( \zeta_p \in K \).

4.3.2. Tate’s argument.

To prove Bloch–Kato’s theorem we may assume that \( n = 1 \).

Indeed, consider the cohomological long exact sequence

\[
\cdots \to H^{q-1}(K, \mathbb{Z}/p(q)) \xrightarrow{\delta} H^q(K, \mathbb{Z}/p^{n-1}(q)) \xrightarrow{p} H^q(K, \mathbb{Z}/p^n(q)) \to \cdots
\]

which comes from the Bockstein sequence

\[
0 \to \mathbb{Z}/p^{n-1} \xrightarrow{p} \mathbb{Z}/p^n \xrightarrow{\text{mod } p} \mathbb{Z}/p \to 0.
\]

We may assume \( \zeta_p \in K \), so \( H^{q-1}(K, \mathbb{Z}/p(q)) \simeq h_{q-1}(K) \) and the following diagram is commutative (cf. [T, §2]):

\[
\begin{array}{ccc}
k_{q-1}(K) & \xrightarrow{\ast \zeta_p} & K_q(K)/p^{n-1} & \xrightarrow{p} & K_q(K)/p^n & \xrightarrow{\text{mod } p} & k_q(K) \\
\downarrow h_{q-1, K} & & \downarrow h_{q, K} & & \downarrow h_{q, K} & & \downarrow h_{q, K} \\
H^q(K, \mathbb{Z}/p^{n-1}(q)) & \xrightarrow{\cup \zeta_p} & H^q(K, \mathbb{Z}/p^n(q)) & \xrightarrow{p} & H^q(K, \mathbb{Z}/p^n(q)) & \xrightarrow{\text{mod } p} & h^q(K).
\end{array}
\]

The top row is exact except at \( K_q(K)/p^{n-1} \) and the bottom row is exact. By induction on \( n \), we have only to show the bijectivity of \( h_{q,K} : k_q(K) \to h^q(K) \) for all \( q \) in order to prove Bloch–Kato’s theorem.
4.4. Bloch–Kato’s Theorem

We review the proof of Bloch–Kato’s theorem in the following four steps.

I \( \bar{\rho}_m : \text{gr}_m k_q(K) \to \text{gr}_m h^q(K) \) is injective for \( 1 \leq m < e' \).

II \( \bar{\rho}_0 : \text{gr}_0 k_q(K) \to \text{gr}_0 h^q(K) \) is injective.

III \( h^q(K) = U_0 h^q(K) \) if \( k \) is separably closed.

IV \( h^q(K) = U_0 h^q(K) \) for general \( k \).

4.4.1. Step I.

Injectivity of \( \bar{\rho}_m \) is preserved by taking inductive limit of \( k_0 \). Thus we may assume \( k \) is finitely generated over \( \mathbb{P}_p \) of transcendence degree \( r < \infty \). We also assume \( \zeta_p \in K \). Then we get

\[
\text{gr}_{e'-r+2} h^r(K) = U_{e'-r+2} h^r(K) \neq 0.
\]

For instance, if \( r = 0 \), then \( K \) is a local field and \( U_{e'-r+2} h^r(K) = p \cdot \Br(K) = \mathbb{Z}/p \). If \( r \geq 1 \), one can use a cohomological residue to reduce to the case of \( r = 0 \). For more details see [K1, Sect. 1.4] and [K2, Sect. 3].

For \( 1 \leq m < e' \), consider the following diagram:

\[
\begin{array}{ccc}
G^q_m \times G^{r+2-q}_{e'-m} & \xrightarrow{\rho_m \times \rho_{e'-m}} & \text{gr}_m h^q(K) \oplus \text{gr}_{e'-m} h^{r+2-q}(K) \\
\downarrow \varphi_m & & \downarrow \text{cup product} \\
\Omega^q_k/d\Omega^q_k & \xrightarrow{\rho_e'} & \text{gr}_{e'} h^{r+2}(K)
\end{array}
\]

where \( \varphi_m \) is, if \( p \nmid m \), induced by the wedge product \( \Omega^{q-1}_k \times \Omega^{r+1-q}_k \to \Omega^q_k / d\Omega^{q-1}_k \), and if \( p \mid m \),

\[
\begin{array}{c}
\Omega^{q-1}_k / Z^{q-1}_{1,1} \oplus \Omega^{q-2}_k / Z^{q-2}_{1,1} \times \Omega^{r+1-q}_k / Z^{r+1-q}_{1,1} \oplus \Omega^{r-q}_k / Z^{r-q}_{1,1} \\
\varphi_m \mapsto \Omega^q_k / d\Omega^q_k
\end{array}
\]

\[
(x_1, x_2, y_1, y_2) \mapsto x_1 \wedge dy_2 + x_2 \wedge dy_1,
\]

and the first horizontal arrow of the bottom row is the projection

\[
\Omega^{q-1}_k / d\Omega^q_k \longrightarrow \Omega^q_k / (1 + a C) Z^{q-1}_1 = G^{r+2}_k
\]

since \( \Omega^{r+1}_k = 0 \) and \( d\Omega^q_k \subset (1 + a C) Z^{q-1}_1 \). The diagram is commutative, \( \Omega^q_k / d\Omega^q_k \) is a one-dimensional \( k^p \)-vector space and \( \varphi_m \) is a perfect pairing, the arrows in the bottom row are both surjective and \( \text{gr}_{e'} h^{r+2}(K) \neq 0 \), thus we get the injectivity of \( \bar{\rho}_m \).
4.4.2. Step II.

Let $K'$ be a henselian discrete valuation field such that $K \subset K'$, $e(K'|K) = 1$ and $k_{K'} = k(t)$ where $t$ is an indeterminate. Consider

$$\text{gr}_0 h_q(K) \xrightarrow{\cup l+\pi t} \text{gr}_1 h^{q+1}(K').$$

The right hand side is equal to $\Omega_{k(t)}^q$ by (I). Let $\psi$ be the composite

$$\nu_q(k) \oplus \nu_{q-1}(k) \xrightarrow{\bar{\rho}_0} \text{gr}_0 h_q(K) \xrightarrow{\cup l+\pi t} \text{gr}_1 h^{q+1}(K') \simeq \Omega_{k(t)}^q.$$

Then

$$\psi\left(\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q}, 0\right) = l\frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_q}{x_q},$$

$$\psi\left(0, \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{q-1}}{x_{q-1}}\right) = \pm dt \wedge \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dx_{q-1}}{x_{q-1}}.$$

Since $t$ is transcendental over $k$, $\psi$ is an injection and hence $\bar{\rho}_0$ is also an injection.

4.4.3. Step III.

Denote $sh^q(K) = U_0 h^q(K)$ (the letter $s$ means the symbolic part) and put

$$C(K) = h^q(K)/sh^q(K).$$

Assume $q \geq 2$. The purpose of this step is to show $C(K) = 0$. Let $\bar{K}$ be a henselian discrete valuation field with algebraically closed residue field $k_{\bar{K}}$ such that $K \subset \bar{K}$, $k \subset k_{\bar{K}}$, and the valuation of $\bar{K}$ is the induced valuation from $\bar{K}$. By Lang [L], $\bar{K}$ is a $C_1$-field in the terminology of [S]. This means that the cohomological dimension of $\bar{K}$ is one, hence $C(\bar{K}) = 0$. If the restriction $C(K) \rightarrow C(\bar{K})$ is injective then we get $C(K) = 0$. To prove this, we only have to show the injectivity of the restriction $C(K) \rightarrow C(L)$ for any $L = K(b^{1/p})$ such that $b \in \mathcal{O}_{\bar{K}}^*$ and $\bar{b} \notin k_{\bar{K}}^*$.

We need the following lemmas.

**Lemma 1.** Let $K$ and $L$ be as above. Let $G = \text{Gal}(L/K)$ and let $sh^q(L)^G$ (resp. $sh^q(L)_G$) be $G$-invariants (resp. $G$-coinvariants). Then

(i) $sh^q(K) \xrightarrow{\text{res}} sh^q(L)^G \xrightarrow{\text{cor}} sh^q(K)$ is exact.

(ii) $sh^q(K) \xrightarrow{\text{res}} sh^q(L)_G \xrightarrow{\text{cor}} sh^q(K)$ is exact.

**Proof.** A nontrivial calculation with symbols, for more details see ([BK, Prop. 5.4]). \qed

**Lemma 2.** Let $K$ and $L$ be as above. The following conditions are equivalent:

(i) $h^{q-1}(K) \xrightarrow{\text{res}} h^{q-1}(L)_G \xrightarrow{\text{cor}} h^{q-1}(K)$ is exact.

(ii) $h^{q-1}(K) \xrightarrow{\cup_{h}} h^{q}(K) \xrightarrow{\text{res}} h^{q}(L)$ is exact.

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Proof. This is a property of the cup product of Galois cohomologies for $L/K$. For more details see [BK, Lemma 3.2].

By induction on $q$ we assume $\text{sh}^{q-1}(K) = h^{q-1}(K)$. Consider the following diagram with exact rows:

\[
\begin{array}{cccccccc}
\text{h}^{q-1}(K) & \cup b & \downarrow & & \downarrow \text{res} & & \downarrow \text{res} & & \downarrow \\
0 & \rightarrow & \text{sh}^{q}(K) & \rightarrow & \text{h}^{q}(K) & \rightarrow & C(K) & \rightarrow & 0 \\
& \text{res} & \downarrow & & \text{res} & \downarrow & \text{res} & \downarrow & \\
0 & \rightarrow & \text{sh}^{q}(L)^G & \rightarrow & \text{h}^{q}(L)^G & \rightarrow & C(L)^G & \rightarrow & \\
& \text{cor} & \downarrow & & \text{cor} & \downarrow & & \\
0 & \rightarrow & \text{sh}^{q}(K) & \rightarrow & \text{h}^{q}(K). & & & &
\end{array}
\]

By Lemma 1 (i) the left column is exact. Furthermore, due to the exactness of the sequence of Lemma 1 (ii) and the inductive assumption we have an exact sequence

\[h^{q-1}(K) \xrightarrow{\text{res}} h^{q-1}(L)^G \rightarrow h^{q-1}(K).\]

So by Lemma 2

\[h^{q-1}(K) \xrightarrow{\cup b} h^{q}(K) \xrightarrow{\text{res}} h^{q}(L)\]

is exact. Thus, the upper half of the middle column is exact. Note that the lower half of the middle column is at least a complex because the composite map $\text{cor} \circ \text{res}$ is equal to multiplication by $[L : K] = p$. Chasing the diagram, one can deduce that all elements of the kernel of $C(K) \rightarrow C(L)^G$ come from $h^{q-1}(K)$ of the top group of the middle column. Now $h^{q-1}(K) = \text{sh}^{q-1}(K)$, and the image of

\[\text{sh}^{q-1}(K) \xrightarrow{\cup b} h^{q}(K)\]

is also included in the symbolic part $\text{sh}^{q}(K)$ in $h^{q}(K)$. Hence $C(K) \rightarrow C(L)^G$ is an injection. The claim is proved.

4.4.4. Step IV.

We use the Hochschild–Serre spectral sequence

\[H^r(G, \text{h}^q(K_{ur})) \Longrightarrow h^{q+r}(K).\]

For any $q$,

\[\Omega^q_{k_{sep}} \simeq \Omega^q_k \otimes_k k_{sep}, \quad Z_1 \Omega^q_{k_{sep}} \simeq Z_1 \Omega^q_k \otimes_{k'} (k_{sep})^p.\]
Thus, \( \text{gr}_m h^q(K_{ur}) \simeq \text{gr}_m h^q(K) \otimes_{k^p} (k^{\text{sep}})^p \) for \( 1 \leq m < e' \). This is a direct sum of copies of \( k^{\text{sep}} \), hence we have

\[
\begin{align*}
H^0(G_k, U_1 h^q(K_{ur})) & \simeq U_1 h^q(K)/U_{e'} h^q(K), \\
H^r(G_k, U_1 h^q(K_{ur})) & = 0
\end{align*}
\]

for \( r \geq 1 \) because \( H^r(G_k, k^{\text{sep}}) = 0 \) for \( r \geq 1 \). Furthermore, taking cohomologies of the following two exact sequences

\[
0 \longrightarrow U_1 h^q(K_{ur}) \longrightarrow h^q(K_{ur}) \longrightarrow \nu_{k^p}^q \oplus \nu_{k^p}^{q-1} \longrightarrow 0,
\]

\[
0 \longrightarrow \nu_{k^p}^q \longrightarrow Z_1 \Omega_{k^p}^q \longrightarrow \Omega_{k^p}^{q-1} \Omega_{k^p}^q \longrightarrow 0,
\]

we have

\[
\begin{align*}
H^0(G_k, h^q(K_{ur})) & \simeq sh^q(K)/U_{e'} h^q(K) \simeq k^q(K)/U_{e'} k^q(K), \\
H^1(G_k, h^q(K_{ur})) & \simeq H^1(G_k, \nu_{k^p}^q \oplus \nu_{k^p}^{q-1}) \\
& \simeq (\Omega_{k^p}^q/(1 - C)Z_1 \Omega_{k^p}^q) \oplus (\Omega_{k^p}^{q-1}/(1 - C)Z_1 \Omega_{k^p}^{q-1}), \\
H^r(G_k, h^q(K_{ur})) & = 0
\end{align*}
\]

for \( r \geq 2 \), since the cohomological \( p \)-dimension of \( G_k \) is less than or equal to one (cf. [S, II-2.2]). By the above spectral sequence, we have the following exact sequence

\[
0 \longrightarrow (\Omega_{k^p}^{q-1}/(1 - C)Z_1^{q-1}) \oplus (\Omega_{k^p}^{q-2}/(1 - C)Z_1^{q-2}) \longrightarrow h^q(K) \\
\longrightarrow k^q(K)/U_{e'} k^q(K) \longrightarrow 0.
\]

Multiplication by the residue class of \((1 - \zeta_p)^{p}/\pi^e'\) gives an isomorphism

\[
(\Omega_{k^p}^{q-1}/(1 - C)Z_1^{q-1}) \oplus (\Omega_{k^p}^{q-2}/(1 - C)Z_1^{q-2}) \\
\longrightarrow (\Omega_{k^p}^{q-1}/(1 + a C)Z_1^{q-1}) \oplus (\Omega_{k^p}^{q-2}/(1 + a C)Z_1^{q-2}) \simeq \text{gr}_{e'} k_{q}(K),
\]

hence we get \( h^q(K) \simeq k_{q}(K) \).

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