Evidence for non-Gaussianity in the CMB

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ABSTRACT In a recent Letter we have shown how COBE-DMR maps may be used to disprove Gaussianity at a high confidence level. In this report we digress on a few issues closely related to this Letter. We present the general formalism for surveying non-Gaussianity employed. We present a few more tests for systematics. We wonder about the theoretical implications of our result.

KEYWORDS: Cosmic Microwave Background, Statistics

1 Introduction

It is hard to overemphasise the importance of Gaussianity in theories of structure formation. Under the assumption of Gaussianity calculations become much simpler, and only the power spectrum of fluctuations has to be computed. Furthermore it is thought that Gaussianity is a hallmark of the inflationary paradigm \cite{1}. A simple argument for this hinges on the fact that small perturbations during inflation satisfy an harmonic oscillator equation, at the relevant scales for structure formation. The ground state of an harmonic oscillator has a Gaussian wave function. Hence although an unperturbed background is the classical ground state, quantum theory forces the existence of “zero-level” fluctuations with Gaussian statistics. These quantum fluctuation are the seeds of structure, according to inflation, and one identifies quantum probability with the classical statistics of these macroscopic seeds. This simple argument linking inflation and Gaussian statistics is bypassed in non minimal models of inflation \cite{2}.

In a recent Letter \cite{3} we showed how Gaussianity could be disproved at a high confidence level, using COBE-DMR maps. Here we review that Letter and report on work in progress expanding \cite{3}. In Section 2 we describe a generalisation of the method employed in \cite{3}, which incorporates inter-\ell correlators. In Section 3 we review how the method was applied to the data in \cite{3}. In Section 4 we review the checks for systematics considered in \cite{3} and present a few extra tests.

We conclude with a few remarks on recent work \cite{4,5,6,7} which comments on our result.
2 The statistics

The statistics used in [3] are a subset of a general class of statistics, to be presented in [4]. They are inspired by [9, 10, 11]. The idea is to extract from a laboriously produced CMB map all the relevant information, with as little theoretical prejudice as possible. If we believe inflation is the answer, then one may simply reduce a map to a fit for a set of cosmological parameters. A less omniscient approach is to assume only Gaussianity, and concentrate on an unbiased estimate of the CMB power spectrum (see [12] for an example). The ultimate open mind would not assume Gaussianity, but try to extract from the map the whole set of correlation functions characterising the most general random process.

It is fair to keep one prejudice: statistical isotropy. The idea is therefore to extract from a map with \( N \) pixels \( N - 3 \) rotationally invariant independent quantities. We work in the spherical harmonic representation \( a_{\ell m} \). To construct a an \( n \)-linear invariant one takes the tensor product of \( n \) \( \Delta T_{\ell} \):

\[
(\Delta T_{\ell_1} \otimes \Delta T_{\ell_2} \otimes \cdots \otimes \Delta T_{\ell_n})(n) = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_n} a_{\ell_1 m_1} a_{\ell_2 m_2} \cdots a_{\ell_n m_n} Y_{\ell_1 m_1}(n) \otimes Y_{\ell_2 m_2}(n) \otimes \cdots \otimes Y_{\ell_n m_n}(n) \tag{1}
\]

One is interested in rotationally invariant quantities. These can be trivially obtained if one rewrites the tensor product in terms of the total angular momentum basis. The coefficient of the singlet will be the higher order invariant we are looking for.

To illustrate the technique we shall work out a few examples. Let us first construct the rotationally invariant bilinear. For this we take the tensor product \( \Delta T_{\ell_1} \otimes \Delta T_{\ell_2} \):

\[
(\Delta T_{\ell_1} \otimes \Delta T_{\ell_2})(n) = \sum_{m_1 m_2} a_{\ell_1 m_1} a_{\ell_2 m_2} Y_{\ell_1 m_1}(n) \otimes Y_{\ell_2 m_2}(n) \tag{2}
\]

One can now use the angular momentum addition formulas to find the the coefficient of the singlet. Defining

\[
Y_{\ell_1 m_1}(n) \otimes Y_{\ell_2 m_2}(n) = \sum_{LM} \langle LM | \ell \ell m_1 m_2 \rangle Y_{LM}(n) \tag{3}
\]

we want the \( L = 0 \) term. This implies that \( \ell_1 = \ell_2 = \ell \) and from the condition \( M = 0 \) we have \( m_1 + m_2 = 0 \). So the coefficient of the singlet is

\[
I_{\ell}^2 = \sum_{m} a_{\ell m} a_{\ell - m} \frac{(-1)^{\ell - m}}{\sqrt{2 \ell + 1}} \tag{4}
\]

Up to normalisation this is the simplest quadratic estimator of the power spectrum.

The next simplest case is the cubic invariant. Let us first restrict ourselves to one “ring”, i.e. fixed \( \ell \):

\[
(\Delta T_{\ell} \otimes \Delta T_{\ell} \otimes \Delta T_{\ell})(n) \]
\[
\sum m_1 m_2 m_3 \langle \ell - m_3 | \ell \ell | m_1 m_2 \rangle \delta(m_1 + m_2 + m_3, 0)
\]

From the symmetry properties of the Wigner 3J coefficients we immediately see that for \( \ell \) odd this quantity is identically zero.

We are also interested in relating power between \( \ell \)'s. Consider then \((\Delta T_{\ell-1} \otimes \Delta T_\ell \otimes \Delta T_{\ell+1})(n)\). From the same manipulations one finds

\[
J_{\ell}^3 = \sum_{m_1 m_2 m_3} a_{\ell-1} a_{\ell+1} a_{m_3} \langle \ell - m_3 | \ell + 1 - 1 | m_1 m_2 \rangle \langle 00 | \ell \ell - m_3 m_3 \rangle
\]

\[
= \frac{1}{\sqrt{(2\ell + 1)}} (-1)^\ell \sum_{m_1 m_2 m_3} a_{\ell-1} a_{\ell+1} a_{m_3} \left( \begin{array}{ccc} \ell & \ell & \ell \\ m_1 & m_2 & m_3 \end{array} \right) \delta(m_1 + m_2 + m_3, 0)
\]

This procedure can be used to find all the invariants at any order, within a given multipole \((I^\ell)\) and relating different multipoles \((J^\ell)\). These may then be divided by the appropriate powers of the \(C_k\) in order to make them dimensionless, and suitably normalised, as was done with \(I_3^\ell\) in [3].

This method should produce the full set of independent invariant quantities in a set of \(a_m^\ell\). There should be \(2\ell - 2\) such quantities, for each \(\ell\), plus 3 inter-\(\ell\) invariants, for each pair of \(\ell\)'s. The power spectrum measures how much power there is on a given scale \(\ell\). The \(I^\ell\) describe how the power is divided between the various \(m\) modes, for a given \(\ell\). This encodes preferences for shapes within a given multipole. The \(J^\ell\) measure correlation between the orientations of preferred shapes in adjacent multipoles.

## 3 DMR bispectrum

We now summarise the application of the simplest of these statistics to DMR. We will be testing the inverse noise variance weighted, average maps of the 53A,
Fig. 1 The vertical thick dashed line represents the value of the observed $I_3^3$. The solid line is the probability distribution function of $I_3^3$ for a Gaussian sky with extended galactic cut and DMR noise.

53B, 90A and 90B COBE-DMR channels, with monopole and dipole removed, at resolution 6, in ecliptic pixelization. We use the extended galactic cut of [13], and [14] to remove most of the emission from the plane of the Galaxy. We apply our statistics to the DMR maps before and after correction for the plausible diffuse foreground emission outside the galactic plane as described in [13] and [14]. To estimate the $I_3^3$s we set the value of the pixels within the galactic cut to 0 and the average temperature of the cut map to zero. We then integrate the map multiplied with spherical harmonics to obtain the estimates of the $a_{\ell m}$s and compute the $I_3^3$ from these.

We have used Monte Carlo simulations to find the distribution of the estimators $I_3^3$ as applied to Gaussian maps subject to DMR noise and galactic cut (see Fig.1). These distributions are very non-Gaussian. In principle this would complete the theoretical work required for converting the observed $I_3^3$ (which we also plot in Fig.1) into a statistical statement on Gaussianity, but we proceed further by defining a new “goodness of fit” statistic as follows.

We constructed a tool similar to the $\chi^2$ (often used for comparing predicted and observed $C_\ell$ spectra) but adapted to the non-Gaussian distributions $P(I_3^3)$:

$$X^2 = \frac{1}{N} \sum_\ell X_\ell^2 = \frac{1}{N} \sum_\ell (-2 \log P_\ell(I_3^3) + \beta_\ell),$$

(6)
where the constants $\beta_\ell$ are defined so that for each term of the sum $\langle X^2_\ell \rangle = 1$. The definition reduces to the usual $X^2$ for Gaussian $P_\ell$.

This is a suitable definition. As an illustration let us first approximate the distributions of the $I^3_\ell$ by $P(I^3_\ell) = 2(1 - I^3_\ell)$ — a good approximation for $\ell$ around 10. Then $X^2 = -2 \log(1 - I^3_\ell)$. Like the standard $X^2$ one has $0 < X^2 \ll 1$ for observations close to the peak of the distribution, here at $I^3_\ell = 0$. Indeed $X^2(0) = 0$. However the peak of $P(I^3_\ell)$ is far from its average, and so the standard $X^2$ would produce $X^2 = 0$ at the wrong observation. For observations far from the peak of the distribution (but subject to the constraint $I^3_\ell \leq 1$) $X^2$ goes to infinity. In contrast the standard $X^2$ would always remain finite.

The proposed $X^2$ therefore does for these non-Gaussian distributions what the usual $X^2$ does for normal distributions. We build a $X^2$ for the COBE-DMR data by means of Monte Carlo simulations. We proceed as follows. First we compute the distributions $P(I^3_\ell)$, for $\ell = 2, \ldots, 18$, for a Gaussian process as measured subject to our galactic cut, and pixel noises. These $P(I^3_\ell)$ were inferred from 25000 realizations (see Fig.1). From these distributions we then build the $X^2$ as defined above, taking special care with the numerical evaluation of the constants $\beta_\ell$. We call this function $X^2_{COBE}$. We then find its distribution $F(X^2_{COBE})$ from 10000 random realizations. This is very well approximated by a $\chi^2$ distribution with 12 degrees of freedom. If all $P(I^3_\ell)$ were as in the analytical fit above, we could conclude that we successfully measured an effective number of useful invariants equal to 6. This is less than the number of invariants we actually measured (10) and this is simply due to anisotropic noise and galactic cut. However, had we used a standard $\chi^2$ statistics the effective number of useful invariants would be only 3.

We then compute $X^2_{COBE}$ with the actual observations and find $X^2_{COBE} = 1.81$. One can compute $P(X^2_{COBE} < 1.7) = 0.98$. Hence, it would appear that we can reject Gaussianity at the 98% confidence level.

4 Systematics

Given the nature of this result checking for systematics has been the central aspect of our work. We checked a large number of effects, related to foreground emissions, pixelization effects, spurious offsets induced by the galactic cut, the cut itself, the underlying power spectrum, etc, etc. These checks are reported in [3, 4].

It is important to stress that the confidence level quoted above is a lower limit. It corresponds to the worse case obtained, within the systematic space surveyed. It assumes a conspiracy theory with all systematics lined up so as to take the blame for the observed non-Gaussianity. If one does not consider this worst case scenario, but takes other data sets differently treated for systematics, in most cases we obtained a confidence level for rejection in excess of 99.5%.

Checking for systematics is always open ended, and in these Proceedings we merely present one new check we have recently performed. It concerns the so-called systematic templates. The procedure leading from the time series to a CMB map
produces, as well as the map, an estimate of the systematic effects possibly plaguing the final product. These “systematic templates” for DMR are well documented ([17]), and display strongly non-Gaussian structures, tracing the DMR scanning patterns.

The systematic templates have negligible power. The worst effect in the worst channel has a rms of about 6 $\mu$K at the 95% confidence level. It is unlikely that these effects could corrupt power spectrum estimates. Nonetheless it is well known that a non-Gaussian pattern with negligible power may visually stand out over a Gaussian map with much larger power. Similarly it could happen that these systematics, while irrelevant for the purpose of power spectrum estimation, could be responsible for the observed non-Gaussian bispectra, derived from DMR maps.

In order to address this problem, we subjected the systematic templates to two tests. Firstly we computed the $I_3^\ell$ spectra for the templates. The resulting $I_3^\ell$ are well outside the Gaussian prediction, but they do not correlate with the DMR observed $I_3^\ell$. Secondly, we added or subtracted these templates enhanced by a factor of up to 4 to DMR maps. The effect on the $I_3^\ell$ spectrum was always found to be negligible. This shows that the systematic effects documented in [17] have not only negligible power, but also negligible effect on the higher order statistics which we have studied.

To be more specific we have applied the above tests to templates for systematics in 53A, 53B, 90A, and 90B, separately. This is the sensible thing to do, given that the templates are highly correlated from pixel to pixel ([17]). We have considered the effect of instrument susceptibility to the Earth magnetic field; any unknown effects at the spacecraft spin period; errors in the calibration associated with long-term
drifts, and calibration errors at the orbit and spin frequency; errors due to incorrect removal of the COBE Doppler and Earth Doppler signals; errors in correcting for emissions from the Earth, and eclipse effects; artifacts due to uncertainty in the correction for the correlation created by the low-pass filter on the lock-in amplifiers (LIA) on each radiometer; errors due to emissions from the moon, and the planets. Presumably what is usually mentioned as De-striping goes under the removal of artifacts in the calibration. DMR did not seem to have a serious striping problem, but the problem was addressed none the less.

In Fig.2 we plot the result of adding templates associated with effect of the magnetic field of the Earth. The instruments’ magnetic susceptibility, and the emissions from the Earth have by far the strongest effects. The effect on the $I^3_6$ spectrum is always negligible, but when present occurs at scales around $\ell = 6$.

5 Cautionary remarks and a digression

By now two other groups have reported results similar to ours, albeit making use of different methods [6, 5]. According to skeptics, this may merely reflect a change in the psychological prior, triggered by our work. More seriously one should remember that the work performed by us and by these groups makes use of the same data set. Therefore this work provides an independent confirmation of our analysis of the DMR maps, but not an independent confirmation of the result itself. In particular we feel that the issue of systematics, and foreground contamination, will only be clarified further when an independent all sky data set becomes available.

If indeed our result is due to cosmic emission, then a number of fascinating theoretical issues are raised (see [7]). Clearly the minimal inflationary models cannot be right. On the other hand it is not obvious that the main competitor to inflation, topological defects, could explain this type of non-Gaussianity. Topological defects are non-Gaussian, but in ways which are often more subtle than commonly thought. Computing with defects is prohibitively expensive, and predicting a set of $I^3_6$ distributions in defect models is well beyond current computer technology.

An interesting possibility was recently proposed by Peebles [8]. This is an isocurvature model in which the underlying fluctuations are not a Gaussian random field, but the square of a Gaussian random field. The model is based on non minimal inflation, but produces fluctuations radically different from minimal inflationary fluctuations. The main advantage of this model over defects is that, while not trivial, it is easy enough to compute with it. In particular it is feasible, and topical, to repeat the exercise we have performed in [3] using Peebles theory. This model could well produce a better fit to the DMR bispectrum than Gaussian theories.

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