An Inexact-Penalty Method for GNE Seeking in Games With Dynamic Agents

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Abstract—In this article, we consider a network of autonomous agents, whose outputs are actions in a game with coupled constraints. In such network scenarios, agents seek to minimize coupled cost functions using distributed information while satisfying the coupled constraints. Current methods consider a small class of multi-integrator agents using primal-dual methods. These methods can only ensure constraint satisfaction in steady state. In contrast, we propose an inexact penalty method using a barrier function for nonlinear agents with equilibrium-independent passive dynamics. We show that these dynamics converge to an \( \varepsilon \)-GNE while satisfying the constraints for all time, not just in steady state. We develop these dynamics in both full-information and partial-information settings. In the partial-information setting, dynamic estimates of the others’ actions are used to make decisions and are updated through local communication. Applications to optical networks, velocity synchronization of flexible robots, and wind farm optimization are provided.

Index Terms—Barrier methods, control systems, game theory, networks.

I. INTRODUCTION

GAME THEORY has become a widely used tool in the control of multiagent systems, having found many areas of application such as power control of communication networks [1] and formation control for robotics [2]. In a generalized game, the relevant equilibrium is the generalized Nash equilibrium, or GNE. At a GNE, each agent (player) is minimizing its own, coupled cost-function subject to the constraints, given that the other agents’ actions remain fixed. In real-world applications, each agent’s action in the game often corresponds to the output of a dynamical system, e.g., the position of a mobile robot in a sensor network [3] or the axial induction factor of a wind turbine in a wind farm [4]. In these applications, a GNE-seeking problem can be viewed as a steady-state set point regulation problem with reference determined by the a priori unknown GNE and actions updated in real time. Many GNE-seeking algorithms only consider static agents, i.e., agents who can instantaneously and arbitrarily update their actions from one iteration to the next. In addition, many assume that agents have full information about the others’ actions. These are restrictive assumptions in many control applications as information may be distributed among agents who must communicate with one another and agents may have inherent dynamics. Any GNE-seeking algorithm that is applied in these scenarios must deal with these two issues.

Recently, NE seeking partial-decision information has been considered for networks of dynamic agents (multiintegrator, LTI) [5], [6], but most existing results are restricted to games with decoupled constraints. Considering games with coupled constraints, results exist for integrator agents, e.g., [7] in discrete-time and [8], [9] in continuous-time and for multiintegrator agents in continuous-time [10]. Semidecentralized methods have also been considered [11]. However, existing methods, such as primal-dual methods, ensure that the coupled constraints are satisfied only in steady state. These algorithms are not applicable in real-world applications where the constraints must be satisfied for all time, e.g., sensor networks [3] or demand-side management in smart grids [12]. Recently, agents with nonlinear dynamics have been considered. Zhang et al. [13] considered nonlinear agents of relative-degree-one, subject to external disturbances playing an aggregative game with no constraints. Huang et al. [14] considered first-order nonlinear agents playing a quadratic game, also with no constraints. In both papers, the action space of each agent is restricted to be 1-D. Compared to both of these papers, we consider a different class of agents, allowing for the action set of each agent to be higher than 1-D. Notably, we do not assume a specific class of game outside of a standard monotonicity assumption, e.g., aggregative or quadratic. Also, we allow for the existence of coupled constraints, which we handle through the use of a log-barrier method.

Our work is related to the so-called optimal steady-state control problem, wherein an LTI system is regulated to an optimal set point, in steady state [15]. In contrast, our problem considers the regulation of a set of decoupled nonlinear systems to the (generalized) Nash equilibrium of the game given by coupled costs functions. Compared to [15], we must consider distributed feedbacks for nonlinear agents. In addition, we consider inequality constraints through the use of penalty methods which are not addressed in [15].

In this article, we investigate an inexact-penalty-based dynamics for GNE-seeking passive agents in networks. These dynamics can ensure that the coupled constraints are satisfied for
all time, not just in steady state. Moreover, this approach allows for the extension from the full-decision information setting to the partial-information one.

Penalty methods, using exact penalty functions, have been used in GNE-seeking algorithms [16], [17]. In our work, in order to enforce constraint satisfaction, the penalty function takes the form of a barrier function that prevents each agent’s action from exiting the interior of the constraint set. Related to our work, [18] considers NE seeking in potential games with virtual couplings used to satisfy connectivity constraints for all time. Compared to [18], we allow for arbitrary convex inequality constraints in nonpotential games. As far as we are aware, there do not exist any general GNE-seeking methods that can ensure constraint satisfaction for all time.

Contributions: Interior point methods are a widely used tool in convex optimization [19]. We propose using the log-barrier function on the coupled inequality constraints as our inexact penalty function. The GNE problem is then converted into an NE problem, whose costs go to infinity at the boundary of the constraint set. We consider the problem of regulating the outputs of a set of nonlinear agents with a class of equilibrium-independent passive (EIP) dynamics to the NE of the new penalized problem. The benefit of considering these agents is two-fold. First, we are able to capture versions of a variety of already-known NE seeking algorithms. Second, we are able to consider certain types of dynamic agents. For the full information case, a gradient-based feedback is used. In a partial information setting, we instead use a Laplacian-based feedback [20] for the case where agents have full knowledge of the constraint information.

In both of these cases, assuming that the initial conditions satisfy the constraints, we show that the resulting action trajectories also satisfy the constraints for all time. In addition, we extend these results to cases where the agents do not have full information about the constraints, but must communicate in order to get it, using a two-time scale approach.

A preliminary version of this work appeared in [21] concerning systems with LTI passive dynamics. Only cases with full information of the constraints are treated therein. In addition, a time-varying log-barrier method appears in [22]. Under stronger assumptions than those contained herein, this method can guarantee exact convergence to the variational GNE of the game. Herein, we consider nonlinear EIP agents and propose a fully distributed algorithm not contained in [21] or [22].

The rest of this article is organized as follows. In Section II, we provide the necessary background information on invariance, passivity, and graph theory. In Section III, we formulate the inexact penalized GNE problem. In Section IV, we provide a GNE-seeking strategy for agents’ with EIP dynamics under full information of the others’ actions. In Section V, these results are extended to the case of partial action information but full-knowledge of the constraint. Section VI considers a fully distributed algorithm based on a two-time scale approach. Sections VII–IX provide applications to optical networks, velocity synchronization, and wind farms, respectively. Finally, Section X concludes this article.

Notations: Let $\mathbb{R}$ and $\mathbb{R}_{\geq 0}$ denote the real numbers and non-negative real numbers, respectively. Given $x, y \in \mathbb{R}^n$, $x^\top y$ denotes their inner product. Let $\| \cdot \| : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ denote the Euclidean norm and $\| \cdot \| : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}_{\geq 0}$ its induced matrix norm. Given a set $X \subset \mathbb{R}^n$, $\| \cdot \|_X : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ denotes the Euclidean point-to-set distance and $\partial X$ denotes its boundary. Given a function $\phi : \mathbb{R}^n \rightarrow \mathbb{R}$ and a vector field $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $L_f \phi = \nabla \phi ^\top f$ is the Lie-derivative of $\phi$ along $f$.

II. BACKGROUND

A. Positive- and Output-Positive-Invariance

Consider a system with dynamics given by

\[
\dot{x} = f(x) \\
y = h(x)
\]

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is locally Lipschitz. The following are some standard results concerning positive-invariance of sets from, e.g., [23].

Definition 1: A set $\mathcal{X} \subset \mathbb{R}^n$ is called positively invariant for (1) if for all $x(0) \in \mathcal{X}$, $x(t) \in \mathcal{X}$ for all $t \geq 0$.

Definition 2: A set $\mathcal{Y} \subset \mathbb{R}^m$ is called output-positively invariant (1) if $y(0) \in \mathcal{Y}$ implies that $y(t) \in \mathcal{Y}$ for all $t \geq 0$.

Definition 3: The Bouligand tangent cone of the set $\mathcal{X}$ at $x$ is $T_\mathcal{X} (x) = \{ v \in \mathbb{R}^n : \liminf_{t \rightarrow 0^+} \frac{\|x + tv\| - \|x\|}{t} = 0 \}$.

Definition 4: (Definition 4.9 [23]) Let $\mathcal{O}$ be an open set. A set $S \subset \mathcal{O}$ is a practical set if

1) $S$ is defined by a finite set of inequalities

$$S = \{ x \in \mathbb{R}^n : h_k(x) \leq 0, k = 1, \ldots, r \}$$

where $h_k(x)$ are continuously differentiable functions defined on $\mathcal{O}$.

2) For all $x \in S$, there exists $z$ such that

$$h_k(x) + \nabla h_k(x)^\top z < 0 \quad \forall k$$

3) There exists a Lipschitz continuous vector field $f_0(x)$ such that for all $x \in \partial S$

$$L_{f_0} h_k < 0 \quad \forall k \text{ s.t. } h_k(x) = 0$$

Lemma 1: A closed set $\mathcal{X} \subset \mathbb{R}^n$ is positively invariant for (1) if and only if $f(x) \in T_\mathcal{X} (x)$ for all $x \in \mathcal{X}$.

Lemma 2: Let $S \subset \mathcal{O}$ be a practical set. Then for all $x \in \partial S$

$$T_S (x) = \{ z \in \mathbb{R}^n : L_z h_k(x) \leq 0 \quad \forall k \text{ s.t. } h_k(x) = 0 \}.$$

B. Passivity and Observability

The following is from [24]. Consider a system

\[
\Xi : \begin{cases} 
\dot{x} = f(x) + Gu \\
y = h(x)
\end{cases}
\]

where $G$ is full column-rank. Let $E_\Xi$ denote the set of assignable equilibria of (2), and given $\bar{x} \in E_\Xi$, let $\bar{u} = k_u (\bar{x}) := - (G^\top G)^{-1} G^\top f (\bar{x})$ and $\bar{y} = h(\bar{x})$ be the equilibrium input and output.

Definition 5: The system (2) is equilibrium-independent passive if, for every equilibrium $\bar{x}$, there exists a continuously
differentiable storage function $V^x : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that
\[
\nabla V^x(x) \top (f(x) + G u) \leq (y - \bar{y}) \top (u - \bar{u}) \tag{3}
\]
where $\bar{u}$ and $\bar{y}$ are the steady-state input and output at $\bar{x}$. A set of storage functions $\{V^x(x), \bar{x} \in E_\Sigma\}$ satisfying (3) is an EIP storage function family.

**Definition 6:** The system (2) is equilibrium-independent observable if, for every $\bar{x} \in E_\Sigma$ with associated equilibrium input/output vectors $\bar{u} = h_u(\bar{x})$ and $\bar{y} = h(\bar{x})$, no trajectory of $x = f(x) + G u$ can remain within the set $\{x \in \mathbb{R}^n : h(x) = \bar{y}\}$ other than the equilibrium trajectory $x(t) = \bar{x}$.

**Lemma 3:** If the system (2) is equilibrium-independent observable then for a given equilibrium I/O pair $(\bar{u}, \bar{y})$, there is exactly one $\bar{x} \in E_\Sigma$ satisfying $\bar{u} = h_u(\bar{x})$ and $\bar{y} = h(\bar{x})$.

### C. Graph Theory

The following is from [25]. An undirected graph $G = (\mathcal{I}, \mathcal{E})$ is a set of vertices, $\mathcal{I} = \{1, \ldots, N\}$, and edges, $\mathcal{E} \subset \mathcal{I} \times \mathcal{I}$. $(i, j) \in \mathcal{E}$ means that vertex $j$ can receive information from vertex $i$. Let $G$ be assumed to be undirected so that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$ for all $i, j \in \mathcal{I}$. The adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{N \times N}$ of $G$ is defined by $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. Since $G$ is undirected, $A = A^\top$. $G$ is connected if given any $i, j \in \mathcal{I}$, there is a path connecting them. Let $\mathcal{N}_i$ denote the neighbors of vertex $i$ and let $D = \text{diag}([\mathcal{N}_i])_{i \in \mathcal{I}}$. The Laplacian of $G$ is defined as $L = D - A$. $\lambda_2(L)$ denotes the second smallest eigenvalue of $L$, which is known as the algebraic connectivity of $G$.

### III. PROBLEM FORMULATION

In this work, we consider a set of $N$ agents (players) $\mathcal{I} = \{1, \ldots, N\}$ in a generalized game. Each agent controls its action $y_i \in \mathbb{R}^m$ and attempts to minimize a cost function $J_i$ subject to shared, (possibly) coupled inequality constraints $g(y) \leq 0, g : \mathbb{R}^m \to \mathbb{R}^p$, where $y = \text{col}(y_i)_{i \in \mathcal{I}} \in \mathbb{R}^m$, $m = \sum_{i \in \mathcal{I}} m_i$. This gives the following for each $i \in \mathcal{I}$:
\[
\min_{y_i} J_i(y_i, y_{-i}) \quad \text{s.t.} \quad g(y) \leq 0 \tag{4}
\]
where $y_{-i} = \text{col}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_N)$, all agents actions except for agent $i$’s.

**Definition 7:** A generalized Nash equilibrium (GNE) of (4) is a strategy profile $y^*$ satisfying
\[
y^*_i \in \text{arg min}_{z_i} J_i(z_i, y^*_{-i}) \text{ s.t. } g(z_i, y^*_{-i}) \leq 0 \quad \forall i \in \mathcal{I}.
\]

**Assumption 1:** The cost function of each agent $J_i(y_i, y_{-i})$ is convex and continuously differentiable in $y_i$, for each fixed $y_{-i}$, $g(y)$ is component-wise convex and $C^2$ in $y$, either all $g_i(y)$ are affine or at least one $g_i(y)$ is strictly convex, where $g_i(y)$ is the $i$th component of $g(y)$, and the feasible set $\Omega = \{y \in \mathbb{R}^m : g(y) \leq 0\}$ is nonempty, convex, compact, and satisfies Slater’s constraint qualification.

Let the stacked vector of partial gradients of all cost functions, called the pseudogradient, be denoted as $F(y) := \text{col} (\nabla_i J_i(y_i, y_{-i}))_{i \in \mathcal{I}} \tag{5}$
where $\nabla_i J_i(y_i, y_{-i}) = \partial_i J_i(y_i, y_{-i})^\top$.

A specific type of GNE is the so-called variational-GNE (vGNE). Under Assumption 1, from [26, Th. 4.8], $y^*$ is a vGNE if and only if there exist a dual variable $\lambda^* \in \mathbb{R}^p$ such that the KKT conditions hold
\[
F(y^*) + \sum_{\ell=1}^p \lambda^*_\ell \nabla \ell g_\ell(y^*) = 0
\]
\[
\lambda^*_\ell \geq 0
\]
\[
- g_\ell(y^*) \geq 0 \quad \forall \ell = 1, \ldots, p
\]
\[
- \lambda^*_\ell g_\ell(y^*) = 0. \tag{6}
\]
A vGNE has the interpretation of no price-discrimination among the agents, that is, all agents are penalized equally for constraint violation.

**Assumption 2:** The pseudogradient $F(y)$ is Lipschitz continuous, i.e., $\|F(y) - F(y')\| \leq \theta_1 \|y - y'\|$ for $\theta_1 > 0$, for all $y, y' \in \mathbb{R}^m$, and is either:
1) strictly monotone, i.e., $(y - y') \top (F(y) - F(y')) > 0$ for all $y \neq y' \in \mathbb{R}^m$;
2) strongly monotone, i.e., $(y - y') \top (F(y) - F(y')) \geq \mu \|y - y'\|^2$ for $\mu > 0$ and for all $y, y' \in \mathbb{R}^m$.

Assumptions 1 and 2 are standard assumptions that guarantee the existence and uniqueness of a vGNE and are commonly used to show convergence, see, e.g., [7] or [10]. Under Assumptions 1 and 2, by [27, Cor. 2.2.5 and Th. 2.3.3], the GNE problem (4) has a unique variational-GNE.

Our goal is to design a GNE-seeking algorithm in continuous-time that satisfies the constraint $g(y) \leq 0$ for all time. Inspired by interior-point methods, we consider solving problem (4) using an exact penalty function. Thus, consider transforming the GNE problem into an NE-seeking problem given by the following set of unconstrained programs:
\[
\min_{y_i} J_i(y_i, y_{-i}) + \phi(y) \quad \forall i \in \mathcal{I} \tag{7}
\]
where $\phi(y)$ is the so-called log-barrier function
\[
\phi(y) = \begin{cases} -\rho \sum_{\ell=1}^p \log(-g_\ell(y)) & g(y) < 0 \\ +\infty, & \text{else} \end{cases}
\]
where $\rho > 0$. Note that under Assumption 1, $\phi(y)$ is strictly convex and $C^2$ in $y$ on $\Omega = \{y \in \mathbb{R}^m : g(y) < 0\}$ and, thus, $\nabla \phi(y)$ is locally Lipschitz on $\Omega$.

**Remark 1:** Other barrier functions may be considered, such as polynomial-type functions
\[
\phi(y) = \begin{cases} -\rho \sum_{\ell=1}^p \frac{1}{g_\ell(y)} & g(y) < 0 \\ +\infty, & \text{else} \end{cases}
\]
where $\gamma > 0$. Convergence of the dynamics presented in Sections IV–VI holds for any barrier function that is strictly convex and $C^2$. In order to characterize the error generated by solutions
to the penalized problem, we assume for the remainder of the article that \( \phi(y) \) is the log-barrier function.

Under Assumptions 1 and 2, by [28, Cor. 4.3] and [29, Th. 3], the NE problem (7) has a unique solution \( y^* \). Moreover, \( y^* \) satisfies

\[
F(y^*) + \nabla \phi(y^*) = 0 \tag{9}
\]

where \( \nabla \phi \) is the gradient of \( \phi \). Following from [19, Sec. 11.2.2], since \( \phi(y) \) is the log-barrier function, (9) becomes:

\[
F(y^*) + \sum_{\ell=1}^{p} \frac{\rho}{-g_\ell(y^*)} \nabla g_\ell(y^*) = 0.
\]

By letting \( \lambda_\ell^*(\rho) := -\frac{\rho}{-g_\ell(y^*)} \), we get the following conditions:

\[
F(y^*) + \sum_{\ell=1}^{p} \lambda_\ell^*(\rho) \nabla g_\ell(y^*) = 0
\]

\[
\lambda_\ell^*(\rho) \geq 0
\]

\[
-\rho g_\ell(y^*) \geq 0 \quad \forall \ell = 1, \ldots, p
\]

\[
-\lambda_\ell^*(\rho) g_\ell(y^*) = \rho.
\]

Thus, the NE of (7) is an approximate vGNE of (4). Moreover, it is an \( \varepsilon \)-GNE of (4).

**Definition 8:** A generalized \( \varepsilon \)-Nash equilibrium (\( \varepsilon \)-GNE) of (4) is a strategy profile \( y^* \) satisfying

\[
\mathcal{J}_i(y_i^*, y_i^{\varepsilon}) \leq \inf_{y_i} \left\{ \mathcal{J}_i(y_i, y_i^{\varepsilon}) + \varepsilon : g(y_i, y_i^{\varepsilon}) \leq 0 \right\} \quad \forall i \in \mathcal{I}
\]

for \( \varepsilon > 0 \). When \( \varepsilon = 0 \), \( y^* \) is a GNE. Moreover, if the dual variables are the same for each agent, we call \( y^* \) an \( \varepsilon \)-vGNE.

**Lemma 4:** The NE of (7) is an \( \varepsilon \)-vGNE of (4) with \( \varepsilon = \rho p \).

**Proof:** Consider the dual function \( h_i \) of (4) for agent \( i \) evaluated at the NE of (7) \( y^* \) and \( \lambda^*(\rho) \) as above

\[
h_i(\lambda^*(\rho)) = \mathcal{J}_i(y_i^*, y_i^{\varepsilon}) + \sum_{\ell=1}^{p} \lambda_\ell^*(\rho) g_\ell(y^*)
\]

\[
= \mathcal{J}_i(y_i^*, y_i^{\varepsilon}) - \rho p p.
\]

By properties of the dual function in [19, Sec. 5.1.3]

\[
\mathcal{J}_i(y_i^*, y_i^{\varepsilon}) - \rho p p \leq \inf_{y_i} \left\{ \mathcal{J}_i(y_i, y_i^{\varepsilon}) : g(y_i, y_i^{\varepsilon}) \leq 0 \right\}.
\]

Therefore

\[
\mathcal{J}_i(y_i^*, y_i^{\varepsilon}) \leq \inf_{y_i} \left\{ \mathcal{J}_i(y_i, y_i^{\varepsilon}) + \rho p p : g(y_i, y_i^{\varepsilon}) \leq 0 \right\}.
\]

Since this holds for all \( i \in \mathcal{I} \), \( y^* \) is an \( \varepsilon \)-GNE of (4) with \( \varepsilon = \rho p \). Moreover, the dual variables, \( \lambda^*(\rho) \) are the same for all agents. Therefore, \( y^* \) is an \( \varepsilon \)-vGNE.

**IV. FULL INFORMATION GRADIENT FEEDBACK—NONLINEAR AGENTS**

We consider GNE-seeking for a class of passive nonlinear agents with dynamics of the form

\[
\mathcal{P}_i : \begin{cases} 
\dot{x}_i = f_i(x_i) + G_i u_i \\
y_i = G_i^T \nabla V_i(x_i)
\end{cases} \tag{10}
\]

where \( x_i \in \mathbb{R}^n \) is the state of agent \( i \), \( u_i \in \mathbb{R}^m \) is agent \( i \)'s input, and the output \( y_i \in \mathbb{R}^m \) of \( \mathcal{P}_i \) is the action of agent \( i \). Agent \( i \) seeks to regulate its action to the minimum of a coupled optimization problem in steady state. In this section, we assume that each agent \( i \) has full knowledge of all other agents’ actions, \( y_{-i} \), but no knowledge of their states \( x_j \) or cost functions \( J_j \) for \( j \neq i \). This is what we call the full-(decision) information case.

**Assumption 3:** \( \mathcal{P}_i \) satisfies the following:

1) \( f_i(x_i) \) is Lipschitz continuous;
2) (10) is equilibrium independent observable;
3) \( G_i \) is full column-rank;
4) there exists a map \( \pi_i : \mathbb{R}^m \rightarrow \mathbb{R}^n \) that solves the regulator equations for any \( \tilde{y}_i \in \mathbb{R}^m \),

\[
0 = f_i(\pi_i(\tilde{y}_i))
\]

\[
0 = G_i^T \nabla V_i(\pi_i(\tilde{y}_i)) - \tilde{y}_i
\]

i.e., any \( \tilde{y}_i \in \mathbb{R}^m \) is an equilibrium output of (10) with equilibrium input \( \bar{u}_i = 0 \);
5) \( V_i(x_i) \) is a strongly convex function with \( \nabla V_i(x_i) \) Lipschitz continuous and the mapping \( -f_i \circ \nabla V_i^{-1} \) is monotone.

**Remark 2:** Under Assumption 3, by [24, Corollary 3.6] in (10) is equilibrium-independent passive with storage function family \( \{ V_i(x_i), \bar{x}_i \in \mathcal{F}_i \} \), where

\[
V_i(x_i) := V_i(x_i) - V(x_i) - \nabla V_i(x_i)^T (x_i - \bar{x}_i) \tag{11}
\]

i.e., \( V_i(x_i) \) and (10) satisfy (3) for all \( \bar{x}_i \in \mathcal{F}_i \).

**Remark 3:** Assumption 3 can be seen as a generalization of the passivity assumption commonly used in (G)NE seeking problems. It captures integrator dynamics [20], the passive LTI agents in [21], and can be achieved through suitable feedback transformations and output choice, e.g., [6] or [10]. Examples of systems that meet Assumption 3 are the following.

1) Integrators

\[
\dot{y}_i = u_i. \tag{12}
\]

2) PI controllers in cascade with certain stable linear systems

\[
\dot{x}_i = \begin{bmatrix} -v_i I & k_i I \\ 0 & 0 \end{bmatrix} x_i + \begin{bmatrix} I \\ I \end{bmatrix} u_i
\]

\[
y_i = \begin{bmatrix} I \\ 0 \end{bmatrix} x_i \tag{13}
\]

where \( 0 < k_i < v_i \).

3) Nonlinear systems of the form

\[
\dot{x}_i = x_i^2 - x_i^3
\]

\[
M_i \dot{x}_i^2 = -\psi_i(x_i^2) - \eta_i(x_i^2 - x_i^3) + u_i
\]

\[
m_i \dot{x}_i^2 = \psi_i(x_i^2) + \eta_i(x_i^2 - x_i^3)
\]

\[
y_i = x_i^2 \tag{14}
\]

where \( x_i^1, x_i^2, x_i^3 \in \mathbb{R} \), \( \psi_i(\cdot) \) is Lipschitz, strongly monotone, and \( \psi_i(0) = 0 \) can be shown to satisfy Assumption 3 with \( V_i(x_i) = \int_0^{x_i^1} \psi_i(x_i^1) dx_i^1 + \frac{M_i}{2} (x_i^1)^2 + \frac{m_i}{2} (x_i^3)^2 \). This can model certain mechanical systems, such as flexible mobile robots.
By stacking the plant dynamics (10), we get
\[
\mathcal{P} : \begin{cases} 
\dot{x} = f(x) + Gu \\
y = G^T \nabla V(x)
\end{cases}
\]
(15)
where \( x = \text{col}(x_i)_{i \in I}, \ u = \text{col}(u_i)_{i \in I}, \ y = \text{col}(y_i)_{i \in I}, \ f(x) = \text{col}(f_i(x_i))_{i \in I}, \ G = \text{blkdiag}(G_i)_{i \in I}, \) and \( \nabla V = \text{col}(\nabla V_i(x_i))_{i \in I} \).

**Remark 4:** It can be easily verified that (15) satisfies the following:
1) \( f(x) \) is Lipschitz continuous;
2) (15) is equilibrium-independent observable;
3) \( G \) is full column-rank;
4) the map \( \pi : \mathbb{R}^m \to \mathbb{R}^n, \pi(y) = \text{col}(\pi_i(y_i))_{i \in I} \) solves the regulator equations for any \( \bar{y} \in \mathbb{R}^m \)
\[
0 = f(\pi(y)) \\
0 = G^T \nabla V(\pi(y)) - \bar{y}
\]
i.e., any \( \bar{y} \in \mathbb{R}^m \) is a steady-state output of (10) with steady-state input \( u = 0; \)
5) (15) is equilibrium independent observable with strongly convex storage function family
\[
V^x(x) := V(x) - V(\bar{x}) - \nabla V(\bar{x})^T(x - \bar{x}),
\]
where \( V(x) = \sum_{i \in I} V_i(x_i) \).

We consider solving GNE-seeking (4) for (15) by converting the problem into (7). In the full-(decision) information case, we consider a static partial-gradient feedback. Thus, agent \( i \) with \( \mathcal{P}_i \) (10) takes
\[
u_i = -\nabla_i \bar{J}_i(y) - \nabla_i \phi(y)
\]
(16)
where \( \nabla_i \phi(y) = \frac{\partial \phi(y)}{\partial y_i} \). For all agents, from \( \mathcal{P} \) (15) with \( u = -F(y) - \nabla \phi(y) \), this leads to an overall stacked dynamics
\[
\Sigma : \begin{cases} 
\dot{x} = f(x) - G(F(y) + \nabla \phi(y)) =: q(x) \\
y = G^T \nabla V(x).
\end{cases}
\]
(17)

**Remark 5:** In the absence of constraints (no penalty), the dynamics (17) capture a number of previously investigated NE seeking algorithms. If each \( \mathcal{P}_i \) is taken to be an integrator (12), then gradient-play
\[
\Sigma : \begin{cases} 
\dot{y} + F(y) = 0
\end{cases}
\]
is recovered [20]. If instead, each \( \mathcal{P}_i \) is governed by (13), with \( f_i(x_i) = v_i x_i \), then the dynamics become the following second-order method investigated in the optimization literature:
\[
\Sigma : \begin{cases} 
\dot{y} + \left( V + \frac{\partial F(y)}{\partial y} \right) \dot{y} + K F(y) = 0
\end{cases}
\]
where \( V = \text{blkdiag}(v_i I)_{i \in I} \) and \( K = \text{blkdiag}(\kappa_i I)_{i \in I} \). See, e.g., [30] and [31].

Next, we investigate the behavior of (17) and show that not only do the solutions converge to an equilibrium corresponding to the output being the NE, but that the solutions satisfy the output constraints for all time. First, we investigate the equilibria of (17).

**Lemma 5:** Under Assumptions 1, 2, and 3, \( x^* = \pi(y^*) \) is the unique equilibrium point of (17) with \( y^* \) as in (9), the NE of (7), and \( \varepsilon \text{-vGNE of (4)}. \)

**Proof:** See Appendix A.

In order to show that the constraints are satisfied for all time, we look at positive-invariance of sublevel sets of the form \( S = \{ x \in \mathbb{R}^n : V^x(x) - c \leq 0, \phi^x(x) - d \leq 0 \} \) for \( c, d > 0 \), where \( x^* = \pi(y^*) \) and \( \phi^x(x) := \phi(G^T \nabla V(x)) - \phi(G^T \nabla V(x^*)) - \nabla \phi(G^T \nabla V(x^*))^T (G^T \nabla V(x) - G^T \nabla V(x^*)), \) related to the Bregman divergence of \( \phi(y) \). The interior of these sets corresponds to the output satisfying the constraints. The following lemma shows that \( S \) is a practical set.

**Lemma 6:** Under Assumption 3, \( S = \{ x \in \mathbb{R}^n : V^x(x) - c \leq 0, \phi^x(x) - d \leq 0 \} \) is a compact, practical set for all \( c, d > 0 \).

**Proof:** See Appendix B.

**Remark 6:** With LTI agents as in [21], sets of the form \( S \) are convex with nonempty interior. Thus, practicality follows trivially. In potential games, as in [18], the sublevel sets of the potential function are positively invariant under gradient feedback, and thus, constraint satisfaction also follows trivially.

**Lemma 7:** Under Assumptions 1, 2(a), and 3, the set int \( \Omega = \{ y : g(y) < 0 \} \) is output-positively invariant for the dynamics (17). That is, for all \( x(0) \) such that \( y(0) \in \text{int} \Omega, \ y(t) \in \text{int} \Omega \) for all \( t \geq 0, \) i.e., the output constraints are satisfied for all time.

**Proof:** We show that for each \( x(0) \) such that \( y(0) = G^T \nabla V(x(0)) \), there exists \( c, d > 0 \) such that \( x(0) \in S = \{ x \in \mathbb{R} : V^x(x) - c \leq 0 \} \cap \{ x \in \mathbb{R} : \phi^x(x) - d \leq 0 \} \) where \( x^* \) as in Lemma 5, and \( S \) is positively invariant.

First, consider the Lie derivatives of \( V^x(x) \) and \( \phi^x(x) \) on \( \partial S \) along the solutions of (17). There are two cases.
1) \( V^x(x) = c = \phi^x(x) \leq d \)
In this case, we consider the Lie derivative of \( V^x(x) \) along the solutions of (17). By equilibrium-independent passivity of (15), we have
\[
L_q V^x \leq -(y - y^*)^T u = -(y - y^*)^T (F(y) + \nabla \phi(y)).
\]
Using (9), \( F(y) \) monotone and \( \phi(y) \) convex, we have
\[
L_q V^x \leq -(y - y^*)^T (F(y) - F(y^*) + \nabla \phi(y) - \nabla \phi(y^*)) \leq 0.
\]
2) \( V^x(x) \leq c \) and \( \phi^x(x) = d \)
We take the Lie derivative of \( \phi^x(x) \) along the solutions of (17), giving
\[
L_q \phi^x = (\nabla \phi(y) - \nabla \phi(y^*))^T G^T \nabla^2 V(x)^T [f(x) - G(F(y) + \nabla \phi(y))] \\
= (\nabla \phi(y) - \nabla \phi(y^*))^T G^T \nabla^2 V(x)^T [f(x) - f(x^*)] \\
- G(F(y) - F(y^*) + \nabla \phi(y) - \nabla \phi(y^*)].
\]
By Assumption 3(c) and (e), \( G^T \nabla^2 V(x) G \succeq \kappa I \) for some \( \kappa > 0 \). Then, we can get
\[
L_q \phi^x \leq \|\nabla \phi(y) - \nabla \phi(y^*)\| \|G^T \nabla^2 V(x)\| \|f(x) - f(x^*)\| + \|G^T \nabla^2 V(x) G\|
\]
\[ \| F(y) - F(y^*) \| - \alpha \| \nabla \phi(y) - \nabla \phi(y^*) \| . \]

Since \( V^{x^*} < c \) and \( V^{x^*} \) strongly convex, we have \( \| x - x^* \| \leq c_1 \), for some \( c_1 \). Since \( f(x) \) and \( \nabla V(x) \) are Lipschitz continuous, we have \( \| f(x) - f(x^*) \| \leq \theta_{1} | x - x^* | \) and \( | G^{T} \nabla V(x) | \leq \theta_{3}, \) for some \( \theta_{3}, \theta_{4} \). Since \( f(x) \) is Lipschitz continuous, we have \( \| F(y) - F(y^*) \| \leq \theta_{1} \| y - y^* \| \leq \theta_{1} \theta_{4} \| x - x^* \| \leq c_1 \theta_{1} \theta_{4}. \)

Therefore,

\[ L_{0} \phi^{x^*} \leq \| \nabla \phi(y) - \nabla \phi(y^*) \| [c_1 \theta_{1} \theta_{4} + c_1 \theta_{1} \theta_{4}^{2} | G |] - \alpha [ \| \nabla \phi(y) - \nabla \phi(y^*) \| ] . \]

By strict-convexity of \( \phi^{x^*}(x) \), we have that \( \| \nabla \phi(y) - \nabla \phi(y^*) \| \geq \phi^{x^*}(x) \| y - y^* \| \geq c_2 \| y - y^* \| . \) If we take \( d \geq d_{c} := \frac{\alpha^{2} \theta_{4}}{c_{2} \theta_{1} \theta_{4}} \), then \( L_{0} V_{2} \leq 0. \)

Since \( S \) is a practical set by Lemma 6, by Lemma 2, \( q(x) \in T_{S}(x) \) for all \( x \in O_{S}. \) In the interior of \( S, q(x) \in T_{S}(x) = R^{n}. \) Therefore, by Lemma 1, \( S \) is positively invariant. For all initial conditions \( x(0) \) such that \( y(0) \in \Omega \), then \( e \geq V^{x}(x(0)) \) and \( d \geq d_{c}. \) Then \( x(t) \in S \) for all \( t \geq 0 \) and \( y(t) \in \Omega \) for all \( t \geq 0 \) since for all \( x \in S, y = G^{T} \nabla V(x) \in \Omega. \)

**Theorem 1:** Under Assumptions 1, 2(a), and 3, the equilibrium \( x^{*} = \pi(y^{*}) \) of (17), where \( y^{*} \) is the NE of (7) and \( v \)-vGNE of (4), is asymptotically stable. Moreover, if \( y(0) \in \Omega \), then the constraint \( y(t) \in \Omega \) is satisfied for all \( t \geq 0. \)

**Proof:** Take \( V^{x^*} \) as the storage function of \( P \), where \( x^{*} \) as in Lemma 5. From equilibrium-independent passivity of (15), the derivative along the solutions of (17) is

\[
\dot{V}^{x^*} \leq -(y - y^*)^{T} u = -(y - y^*)^{T} (F(y) + \nabla \phi(y)) \leq 0 \]

and by strict-monotonicity of \( F(y) \) and convexity of \( \phi(y) \), \( \dot{V} = 0 \) if and only if \( y = y^*. \) By equilibrium independent observability of (10), \( x \rightarrow x^{*} \) is asymptotically stable. By Lemma 7, if \( y(0) \in \Omega \), then \( y(t) \in \Omega \) for all \( t \geq 0. \)

**V. PARTIAL-INFORMATION GRADIENT FEEDBACK—NONLINEAR SYSTEMS**

Now, let us assume that each agent has only partial-information of the actions taken by the other players exchanged over an undirected, connected graph \( G_{c}. \) For now, we assume that each agent has enough knowledge of \( g(y) \) in order to be able to compute its partial-gradient of the penalty function exactly. This assumption is motivated by scenarios in which the agents could have this knowledge:

1) in the case of standard NE seeking, each agent’s constraints depend only on its own action, i.e., \( g(y) = col(g_{1}(y_{1}), \ldots, g_{M}(y_{M})) \);

2) if each agent’s constraint set depends only on the actions of its neighbors in the communication graph \( G_{c}. \) or

3) if the constraints are separable, i.e., \( g(y) = \sum_{j \in \mathcal{G}} g_{j}(y) \), then \( \nabla \phi(y) = \sum_{j \in \mathcal{G}} \frac{\partial \phi}{\partial y_{j}} \nabla g_{j}(y). \) Thus, the only information agent \( i \) needs from the others’ to compute its gradient is the current value of each constraint, \( g_{i}(y) \). In some cases, this may be measured from the environment or communicated by a network manager.

In Section VI, we relax this assumption.

For the individual actions, assume that each agent \( i \) maintains an estimate, \( y_{j}^{e} \), of the action of each agent \( j \) and uses these to evaluate the partial-gradient of its original cost function instead of the true actions. Let \( y_{i,j}^{e} := col(y_{i,1}, \ldots, y_{i,i-1}, y_{i,i+1}, \ldots, y_{i,N}) \) and \( y_{i}^{e} := col(y_{i,1}, \ldots, y_{i,i-1}, y_{i,i+1}, \ldots, y_{i,N}) \). Stacking the actions and estimates, we get \( y_{-i} := col(y_{1}, \ldots, y_{N}) \), the estimates only, and \( y := col(y_{1}^{e}, \ldots, y_{N}^{e}) \), the stacked actions and estimates. These actions and estimates are then exchanged over a communication graph, \( G_{c} \), with Laplacian \( L \), using a proportional consensus algorithm.

**Assumption 4:** The graph \( G_{c} \) is undirected and connected.

Let matrices \( R_{i}, S_{i} \) for action and estimates select be

\[
R_{i} := \begin{bmatrix} m_{i,m_{<i}} & I_{m_{i}} & 0_{m_{i},m_{>i}} \end{bmatrix} \quad S_{i} := \begin{bmatrix} I_{m_{<i}} & 0_{m_{<i},m_{i}} & 0_{m_{<i},m_{>i}} \end{bmatrix} \quad \text{(18)}
\]

with \( m_{<i} := \sum_{j < i, j \in \mathcal{G}} m_{j} \) and \( m_{>i} := \sum_{j > i, j \in \mathcal{G}} m_{j} \). Note that \( y_{i} = R_{i} y_{i}^{e} \) and \( y_{i}^{e} = R_{i} y_{i}^{e}. \)

Inspired by [20], instead of (16), we consider that each agent (10) uses the following dynamic feedback:

\[
y_{i}^{e} = -\delta S_{i} \sum_{j \in \mathcal{N}_{i}} (y_{j}^{e} - y_{i}^{e}) \quad u_{i} = -\nabla_{i} F_{i}(y_{i}, y_{i}^{e}) - \nabla_{i} \phi(y_{i}) - \delta R_{i} \sum_{j \in \mathcal{N}_{i}} (y_{j}^{e} - y_{i}^{e}) \quad \text{(19)}
\]

where \( \delta \) is a communication connectivity parameter and \( \nabla_{i} \phi(y_{i}) \) can be computed using the information available to each agent. Note that (19) has a gradient-play term (evaluated at estimates) and penalty term, as well as a dynamic and Laplacian-based estimate-consensus component \( y_{i}^{e} \), which, in steady state, should bring all \( y_{i}^{e} \) to consensus. We call the stacked vector of partial gradients evaluated at estimates \( F(y) := \col(\nabla_{i} F_{i}(y_{i}, y_{i}^{e}), j \in \mathcal{G}) \) the extended pseudogradient. Note that \( F(1 \otimes y) = F(y) \) for all \( y \in R^{m}. \) We will use the following results concerning the extended pseudogradient. Let \( L := L \otimes I. \)

**Lemma 8:** [10, Lemma 3] Let Assumption 2 hold. Then, the mapping \( F = \theta_{2}-\text{Lipschitz continuous for } \theta_{2} \in [\mu, \theta_{1}]. \)

**Lemma 9:** [7, Lemma 3] Consider that Assumptions 1, 2, and 4 hold and let

\[
\Psi = \begin{bmatrix} \frac{1}{N} \sum_{i} y_{i}^{e} - \theta_{1} m_{>i} \delta \lambda \delta (\bar{L}) - \theta_{2} \end{bmatrix} .
\]

Then, \( \Psi > 0 \) for any \( \delta > \delta_{\text{min}}, \delta_{\text{min}} = \frac{1}{\theta_{2} m_{>i}} \left( \left( \theta_{1} + \theta_{2} \right)^{2} + 4 \mu \right), \)

and for any \( y \) and \( y' \) such that \( y' = 1 \otimes y \) for some \( y' \)

\[
(y - y')^{T} (R^{T} (F(y) - F(y')) + \delta L (y - y')) \geq \mu |y - y'|^{2}
\]

where \( \mu = \min(\Psi), \) i.e., \( F \) is restricted strongly monotone.
From (10) and (19), this gives overall stacked dynamics of

\[
\begin{align*}
\dot{y}_- &= -\delta SLy =: f_1(x) \\
\Sigma : \dot{x} &= f(x) - G(y) + \nabla \phi(y) + \delta RLy =: f_2(x) \\
y &= \nabla^T V(x)
\end{align*}
\]

(20)

where \( x = \text{col}(y_-, x) \), \( R = \text{blkdiag}(R_i)_{i \in I} \) and \( S = \text{blkdiag}(S_i)_{i \in I} \). Let \( f(x) := \text{col}(f_1(x), f_2(x)) \). The unique equilibrium point of (20) is \((y_-, x^*) = (\Sigma_1 N \otimes y^*, \pi(x^*))\), where \( y^* \) is the NE of (7) and \( \epsilon \)-vGNE of (4). We denote \( \bar{y} = 1_N \otimes y^* \).

**Lemma 10:** Under Assumptions 1, 2(b), 3, and 4, \((y_-, x) = (\Sigma_1 N \otimes y^*, \pi(x^*))\) is the unique equilibrium of (20).

**Proof:** See Appendix C. ■

We show output-positive invariance but showing positive invariance of the sublevel sets of two functions

\[
V^S(x) := \frac{1}{2} \|y_--\bar{y}\|^2 + V^r(x) \\
\phi^S(x) := \phi^r(x).
\]

First, we show that the intersection of their sublevel sets forms a practical set.

**Lemma 11:** Under Assumption 3, \( S = \{ x \in \mathbb{R}^{M-m+n} : V^S(x) - c \leq 0, \phi^S(x) - d \leq 0 \} \) is a compact, practical set for all \( c, d > 0 \).

**Proof:** By letting \( G = \text{col}(0, G) \), we have that \( y = \nabla^T V(x) \). In addition, since \( V(x) \) is strongly-convex in \( x \), the proof follows almost identically to the proof of Lemma 6, replacing \( x \) with \( G \) with \( G, V^r(x) \) with \( \phi^r(x) \) with \( \phi^S(x) \) and is thus omitted for the sake of brevity. ■

**Lemma 12:** Under Assumptions 1, 2(b), 3, and 4, if \( \delta > \delta_{\text{min}} \), then for all \( x \in \Omega \), \( y(t) \in \Omega \) for all \( t \geq 0 \), i.e., the output constraints are satisfied for all time.

**Proof:** We show that for all \( c > 0 \), there exists \( d_c \) such that for all \( d \geq d_c \), \( \phi^S(x) \) is positively invariant for (20). Then, for all \( (y^*(y), x(t)) \) such that \( y(t) \in \Omega \), we have that \( (y^*(t), x(t)) \in \Sigma_1^d \) for some \( c, d > 0 \) and \( y(t) \in \Omega \) for all \( t \geq 0 \). The proof is similar to that of Lemma 7, however, the fact that \( F \) is not monotone complicates the analysis.

First, consider the Lie derivatives of \( V^S(x) \) and \( \phi^S(x) \) along the solutions of (17). There are two cases to check.

1) \( V^S(x) = c \) and \( \phi^S(x) \leq d \)

In this case, we consider the Lie derivative of \( V \) along the solutions of (20). By equilibrium-independent passivity of (15), we get

\[
L_f V^S \leq -\delta(y_--\bar{y})^T SLy - (y-y^*)^T u
\]

\[
\leq -\delta(y_--\bar{y})^T SLy
\]

\[
-(y-y^*)^T (F(y) + \nabla \phi(y) + \delta RLy).
\]

Since \( F(y) = F(y^*), F(y^*) + \nabla \phi(y^*) = 0 \) and \( L \bar{y} = 0 \), we have

\[
L_f V^S \leq -\delta(y_--\bar{y})^T SLy - (y-y^*)^T (F(y) - F(\bar{y}) + \nabla \phi(y) - \nabla \phi(y^*))
\]

\[
-\delta(y-y^*)^T RLy - (y-y^*)^T RLy.
\]

By monotonicity of \( \nabla \phi(y) \) and that \( RLy \) and \( \delta RLy \) are positively invariant for all \( x \geq d \) and \( \phi(x) = d \), we have that \( L_f V^S \leq 0 \).

If \( \delta > \delta_{\text{min}} \), then by Lemma 9, we have that \( L_f V^S \leq 0 \).

In the second case, we take the Lie derivative of \( \phi^S(x) \) along the solutions of (17), giving

\[
L_f \phi^S = (\nabla \phi(y) - \nabla \phi(y^*))^T G^T \nabla^2 V(x)f(x)
\]

\[
- G(F(y) + \nabla \phi(y) + \delta RLy).
\]

By Assumption 3(c) and (e), we have that \( \nabla^2 V(x) \geq \alpha I \) for some \( \alpha > 0 \). Then, we get

\[
L_f \phi^S = (\nabla \phi(y) - \nabla \phi(y^*))^T G^T \nabla^2 V(x)f(x) - f(x^*)
\]

\[
- G(F(y) + \nabla \phi(y) + \delta RLy). \]

Using similar arguments to those in the proof of Lemma 7, there exists \( d_c \) such that for all \( d \geq d_c \), \( L_f \phi^S \leq 0 \).

Then, by Lemmas 2 and 11 for all \( x \in \partial S, f(x) \in \Sigma_3(x) \) and for \( x \in \Omega \), \( f(x) \in \mathbb{R}^{M-m+n} = \Sigma_3(x) \). Therefore, \( S \) is positively invariant by Lemma 1. For all \( x(0) \) such that \( y(0) \in \Omega \), take \( c > V(y_-, x(0)) \) and \( d \geq d_c \). Then \( x(t) \in S \) for all \( t \geq 0 \) and \( y(t) \in \Omega \) for all \( t \geq 0 \).

**Theorem 2:** Under Assumptions 1, 2(b), 3, and 4, if \( \delta > \delta_{\text{min}} \), then the equilibrium \((y_-, x^*) = (\Sigma_1 N \otimes y^*, \pi(x^*))\) of (20), where \( y^* \) is the NE of (7) and \( \epsilon \)-vGNE of (4), is asymptotically stable. Moreover, if \( y(t) \in \Omega \), then the constraint \( y(t) \in \Omega \) is satisfied for all \( t \geq 0 \).

**Proof:** Consider the Lyapunov candidate function \( V = \frac{1}{2}\|y_--\bar{y}\|^2 + V^r(x) \), where \( x^* = \pi(x^*) \) as in Lemma 10. Taking the derivative along solutions of (20) yields, as in (21)

\[
\dot{V} = -\delta(y-y^*)^T L(y-y^*) - (y-y^*)^T RL(y-y^*).
\]

By Lemma 9, if \( \delta > \delta_{\text{min}} \), \( V \leq 0 \) and \( \dot{V} = 0 \) if and only if \( y = y^* \). By equilibrium independent observability, \((y_-, x) = (y^*, x^*)\) is asymptotically stable. By Lemma 12, if \( y(t) \in \Omega \), \( y(t) \in \Omega \) for all \( t \geq 0 \).

**VI. FULLY DISTRIBUTED CONSTRAINT INFORMATION**

Next, we consider a partial-information feedback with fully distributed constraint information. Accordingly, each agent maintains estimates of all other agents’ actions, which are exchanged over a communication graph in the same manner as (19), and uses these and an auxiliary variable in place of its own action to compute the gradient of the penalty function.
In order to maintain constraint satisfaction, the communication and action updates occur on two different time-scales, with the communication and estimate updates happening on a fast time scale. This way, the estimates will be brought to consensus at the true values before the actions can be updated.

We consider the following feedback law for dynamics (10):
\[
\begin{align*}
\dot{z}_i &= y_i - z_i \\
\dot{y}^i_i &= -S_i \sum_{j \in N_i} (y^j - y^i) \\
u_i &= -k \left[ \nabla_i f^i(y_i, y^i_i) + \nabla_i \phi(z_i, y^i_i) \right]
\end{align*}
\]
where \( k > 0 \) is a parameter to be chosen to guarantee exponential stability on the slow time-scale, and \( \epsilon > 0 \) is a parameter chosen small enough so that the communication may happen quickly relative to action updates. Stacking these together yields
\[
\Sigma : \begin{align*}
\dot{z} &= y - z \\
\dot{y} &= -SLS^\top y - SLR^\top y \\
\dot{x} &= f(x) - kG(F(y, y) + \psi(z, y)) \\
\dot{y} &= G^\top \nabla V(x)
\end{align*}
\]
where \( z = \text{col}(z_i), i \in \mathcal{I} \) and \( \psi(z, y) = \text{col}(\nabla_i \phi(z_i, y^i_i)) \), \( i \in \mathcal{I} \).

In this section, we replace Assumption 3(e) with a slightly different passivity assumption.

**Assumption 5:** For each \( P_i \), for every \( \bar{x}, V^i_i \) satisfies
\[
\omega_i \| x_i - \bar{x}_i \|^2 \leq V^i_i \leq \bar{\omega}_i \| x_i - \bar{x}_i \|^2
\]
\[
(\nabla V^i_i)^T f_i(x_i) + \bar{G}_i \bar{u}_i \leq -\alpha_i \| x_i - \bar{x}_i \|^2 + \beta_i \| y_i - \bar{y}_i \|^2 + (y_i - \bar{y}_i)^T (u_i - \bar{u}_i)
\]
for \( \alpha_i, \beta_i, \omega_i, \bar{\omega}_i > 0 \).

**Remark 7:** Assumption 5 is related to strict-passivity and output-to-state stability. It guarantees that \( x_i \) can rendered exponentially stable by a suitable static output feedback. Systems (12) and (13), which satisfy Assumption 3, also satisfy Assumption 5 for any \( \beta_i > 0 \). In addition, consider the following dynamics:
\[
\begin{align*}
\dot{x}_i &= -f_i(x_i) + u_i \\
y_i &= x_i
\end{align*}
\]
where \( x_i \in \mathbb{R}^n \) and \( f_i(x_i) \) is \( \eta_i \)-strongly monotone and Lipschitz continuous. Clearly, (24) does not meet the requirements of Assumption 3(d). However, consider augmenting (24) with preliminary PI compensation to yield
\[
\begin{align*}
\dot{\eta}_i &= u_i \\
\dot{x}_i &= -f_i(x_i) + \kappa_i \eta_i + u_i \\
y_i &= x_i.
\end{align*}
\]
It can be shown that the augmented dynamics (25) meet Assumptions 3(a)-(d) and 5 if \( 0 < \kappa_i < \eta_i \).

**Assumption 6:** For every \( P_i \) such that \( f_i(x_i) \) is \( C^2 \) in \( x_i \) and \( V(x) \) is \( C^3 \). In addition, \( F(y) \) is \( C^2 \) in its arguments.

**Lemma 13:** Under Assumptions 1 and 2(b) through 6, for every \( x(0) \in \mathbb{R}^n \) such that \( y(0) \) is \( C^3 \) in \( y(0) \) and for every \( (z(0), y_-(0)) \) such that \( y_-(0) \) is \( C^0 \) in \( \Omega \), there exists \( \gamma > 0 \) such that \( R^{\gamma}(0) \) is a compact subset of \( \Omega \), \( y(t) \in \int \Omega \), and for every \( (z(0), y_-(0)) \) such that \( y_-(0) \) is \( C^0 \) in \( \Omega \),
\[
\frac{dy}{d\tau} = -kG^\top \nabla V(x) = -kG \nabla G^\top \nabla V(x) + \frac{\partial^2 \psi}{\partial y^2} G^\top \nabla V(x).
\]

Under Assumption 6, \( \frac{dy}{d\tau} = \frac{\partial f(y, \mathcal{N} \otimes y, x)}{dx} \) is Lipschitz continuous on any compact subset of \( \Omega \).
Therefore, by [32, Th. 11.2], for each $k \geq k^*$ and for each compact subset $\tilde{\Omega} \subset \Omega$, there exists $\tilde{\epsilon} > 0$ such that for all $x(0) \in \tilde{\Omega}$ and for all $0 < \epsilon < \tilde{\epsilon}$ (23) has a unique solution $x(t)$ and $(z(t), y_-(t))$ and for some $\kappa_1 > 0$

$$\|x(t) - \hat{x}(t)\| \leq \epsilon \kappa_1$$

where $\hat{x}(t)$ is the solution to the reduced system. By Lemma 7, $\Omega$ is positively invariant for the reduced system. Consider $\kappa_x := \inf_{x \in \tilde{\Omega}} \|\hat{x}(t)\|_{x<0}$ and let $\epsilon^* = \min\{\hat{\epsilon}, \frac{\kappa_x}{\kappa_1}\}$. Then for all $0 < \epsilon < \epsilon^*$ we have that $\|x(t) - \hat{x}(t)\| < \kappa_x$ and thus $x(t) \in \Omega_x$ and $y(t) \in \Omega$ for all $t \geq 0$.

**Theorem 3:** Under Assumptions 1 and 2(b) through 6, there exists $k^* > 0$ and $\epsilon^* > 0$ such that for all $k \geq k^*$ and $0 < \epsilon < \epsilon^*$, the equilibrium $(\tilde{z}, \tilde{y}_-, x^*) = (y^*, S1_N \otimes y^*, \pi(y^*))$ of (23), where $y^*$ is the NE of (7) and $\vGNE$ of (4), is exponentially stable. Moreover, for all $x(0)$ such that $y(0) \in \Omega$, for all $(z(0), y_-(0)) \in R_g^{x(0)}$, the constraint $y(t) \in \Omega$ is satisfied for all $t \geq 0$.

**Proof:** Following from the proof of Lemma 13, if $k \geq k^* := 2$, then $x = x^*$ is exponentially stable for the reduced system, (27). $(\tilde{z}, \tilde{y}_-) = (0, 0)$ is exponentially stable for the boundary-layer system, (26). Therefore, by [32, Th. 11.4], there exists $\hat{\epsilon} > 0$ such that for all $0 < \epsilon < \hat{\epsilon}$, $(\tilde{z}, \tilde{y}_-, x^*) = (y^*, S1_N \otimes y^*, x^*)$ is an exponentially stable equilibrium of (23).

Moreover, by Lemma 13, for all $(z(0), y_-(0), x(0))$ such that $y(0) \in \Omega$ and $(z(0), y_-(0)) \in R_g^{x(0)}$, there exists $\epsilon^* > 0$ such that the constraint $y(t) \in \Omega$ is satisfied for all $t \geq 0$. Then take $\epsilon^* = \min\{\hat{\epsilon}, \epsilon^*\}$. 

**VII. OSNR Example**

Consider an optical-signal-to-noise ratio (OSNR) model for wave division multiplexing links with ten channels. Each channel chooses its transmission power $y_i$ in order to maximize its signal-to-noise ratio. The link is assumed to have a maximum transmission power $P_0$. This leads to a game given by the following set of optimization problems:

$$\min_{y_i} \quad a_i y_i - b_i \ln \left(1 + c_i \frac{y_i}{n_i^0 + \sum_{j \neq i} \Gamma_{ij} y_j}\right)$$

$$\text{s.t} \quad 0 \leq y_i, \quad \sum_{j \neq i} y_j \leq P_0$$

where $a_i > 0$ is a pricing parameter, $b_i > 0$, $\Gamma = [\Gamma_{ij}]$ is the link system matrix and $n_i^0$ is the channel noise power. We use the following parameters in the simulation: $P_0 = 2.5 \text{ mW}$, $a_i = 0.001$, $c_i = 1$, and $b = (1, 3, 2, 1, 3, 3, 2, 2, 1, 1)$. For the remaining parameters, see [6] and the references therein. In this example, the agents are static. However, we may choose their plant dynamics $P_i$ to meet the necessary assumptions. For a simple choice, we choose integrator dynamics for each agent (12). We consider that each autonomous agent uses a fully distributed partial-information gradient-play scheme, with agent dynamics given by (12) and (22). In order to get the action information, action estimates are communicated over graph $G_c$ (see Fig. 1).

Fig. 2(a) shows the transmission power $y_i$ for each agent over time using $v_i = 1$, $\kappa_i = 1/2$, $\rho = 0.1$, $k = 1$, and $\epsilon = 10^{-5}$. Each agent always has positive power and the total power usage on the link is less than the maximum (see Fig. 3), meaning the constraints are satisfied for all time. To compare our algorithm with current methods, we simulate a full information primal-dual algorithm with centralized dual variable, i.e., the dual-variable is maintained and updated by a central agent, and passed to the other agents. Fig. 2(b) shows that the convergence time can be much slower depending on the initial condition of the dual variable, note the difference in time-scale between Fig. 2(a) and (b). This is due to the fact that the actions are not constrained to satisfy the constraints for all time and overshoot the correct
values. Table I shows that at the NE each agent is at most 0.11\% suboptimal, with some agent outperforming their costs at the true vGNE value. This is much smaller than the theoretical upper bound, shown in the right-most column, likely due to the fact that out of the 11 constraints, only one is active at the vGNE.

**VIII. VELOCITY SYNCHRONIZATION**

Next, we consider a velocity synchronization problem for a group of flexible mobile robots. Consider a group of five flexible mobile robots moving in a line. Each robot is modeled as two masses connected by a nonlinear spring, with force $\psi_i(\cdot)$ and damper with coefficient $\eta_i$. Let $d_i$ denote robot $i$’s position, mass $M_i$, and $\gamma_i$ the position of its appendage, mass $m_i$. A force $u_i$ is applied to mass $M_i$. Letting $x_1^i = d_i - \gamma_i$, $x_2^i = d_i$ and $x_3^i = \gamma_i$, the dynamics of robot $i$ are given by

\[
P_i : \begin{cases} 
x_1^i = x_2^i - x_3^i \\
M_i x_2^i = -\psi_i(x_1^i) - \eta_i(x_2^i - x_3^i) + u_i \\
m_i x_3^i = \psi_i(x_1^i) + \eta_i(x_2^i - x_3^i) 
\end{cases} \quad (30)
\]

From Remark 3 (iii), if $\psi_i$ is strongly monotone and Lipschitz continuous and $\psi_i(0) = 0$, (30) satisfies Assumption 3.

This problem can be cast as in our framework as a game given by the following set of leader-follower problems:

$$\min_{y_i} (y_i - y_{i-1})^2$$

s.t. $(y_i - y_{i-1})^2 \leq \delta_i^{2,\text{max}}$ \quad (31)

where $y_0 := v_0$, the reference velocity for the leader.

We simulate a group of five robots using the full information feedback (16) with $v_0 = 3$, $d_{i,\text{max}} = d^{\text{max}} = 3$, $M_i = m_i = 1$, $\eta_i = 1$, $\psi_i(x_1^i) = -x_1^i - \text{atan}(x_1^i)$ and $\rho = 0.1$. Fig. 4 shows that the robots synchronize to $v_0 = 3$. Notably, in this example, the vGNE and NE values are the same for any choice of $\rho$. Fig. 5 shows that the velocity difference between neighbors never exceeds $d^{\text{max}}$.

**IX. WIND FARM OPTIMIZATION**

Finally, we consider a wind farm power optimization example, as considered in [4] with an extremum seeking method. Here, we apply our model based approach. Each wind turbine varies its axial induction factor (AIL), $y_i$ to try to maximize the total power production of the wind farm. Each wind turbine has diameter $D$. The power produced by agent $i$ is $P_i(y) = \frac{1}{2} \zeta A \rho (y_i) V_i (y)^3$, where $\zeta$ is the air density, $A$ is the turbine blade area, $P_i(y) = 4y_i(1-y_i)^2$ is the power efficiency coefficient, and $V_i$ is the average wind speed experience by agent $i$. From [33], $V_i(y) = U_\infty (1 - 2 \sqrt{\sum_{j \in I} (y_j c_{ij})^2})$, where $U_\infty$ is the ambient wind speed and $c_{ij}$ is the wake interaction coefficient between $j$ and $i$ which depends on the respective positions and orientation of $i$ and $j$ as well as a wake slope coefficient $k_w$, see [33].
Following from [4], we assume that each agent is modeled by the following linear system, which accounts for the delay between choosing an AIL and the realization of the value:

\[
\dot{y}_i = -\frac{1}{\tau} (y_i - u_i)
\]

where \(\tau > 0\) is the AIL time-constant. As in Remark 3 (iii), we can augment these dynamics with precompensation by a PI controller to meet Assumption 3. The final dynamics are

\[
P_i : \begin{cases} 
\dot{\eta}_i = u_i \\
\dot{y}_i = -\frac{1}{\tau} (y_i - \kappa_i \eta - \tau u_i) 
\end{cases}
\]  

(32)

We consider \(N = 25\) wind turbines in a \(5 \times 5\) grid. Each turbine communicates with its nearest neighbors in the grid, giving rise to graph, \(G_c\). The topology of the graph and the positions of each wind turbine are shown in Fig. 6. As in [4], we impose a coupling constraint between neighboring agents in \(G_c\) to ensure similar levels of mechanical stress on each turbine. We require \(|y_i - y_j| \leq b\) for \(j \in N_i\).

Two types of problems can be considered. The first is the so-called greedy solution, wherein each wind turbine seeks to maximize its own power output, with no regard for the total power output of the farm, expressed in the following set of optimization problems:

\[
\min_{y_i} \quad -\frac{1}{2} \zeta AC_P(y_i) V_i(y)^3 \\
\text{s.t} \quad y_{\text{min}} \leq y_i \leq y_{\text{max}}, \quad (y_i - y_j)^2 \leq b^2 \quad \forall j \in N_i 
\]

(33)

which, assuming \(y_{\text{min}} \leq 1/3 \leq y_{\text{max}}\), has known solution \(y_i = 1/3\) for all \(i\) regardless of the wind speed or wake interaction, [33]. Alternatively, the power for the whole wind farm can be maximized. This can be cast as a potential game, where each agent’s cost corresponds to the total power production

\[
\min_{y_i} \quad -\frac{1}{2} \zeta A \sum_{i \in I} C_P(y_i) V_i(y)^3 \\
\text{s.t} \quad y_{\text{min}} \leq y_i \leq y_{\text{max}}, \quad (y_i - y_j)^2 \leq b^2 \quad \forall j \in N_i.
\]

(34)

We refer to this as the distributed optimization or DOPT case. In this case, the vGNE of the game corresponds to the global optimum of the problem.

Since the agents’ constraints depend only on their neighbors in \(G_c\), but the costs depend on potentially all other agents, we consider using the partial information feedback given by (19) with \(\rho = 10^{-4}\). We simulate both the greedy and DOPT cases with \(\kappa_i = 0.5, \quad \zeta = 1.225, \quad D = 80, \quad U_\infty = 8, \quad y_{\text{max}} = 0.5, \quad y_{\text{min}} = 0, \quad b = 0.03\) and wake coefficient \(k_w = 0.04\). Since the gradients of (33) and (34) are large, we scale them by \(10^{-7}\). Each simulation is run for 1000 s. The wind direction is initially from the East, but switches to being from the North at \(t = 500\) s. Fig. 7 shows the total power production for both algorithms. Due to the low wake interaction, the greedy algorithm performs almost as well as the DOPT algorithm when the wind is coming from the North. However, it only produces 83\% as much power as the DOPT when the wind is out of the East (see Fig. 8).

X. CONCLUSION

In this article, we presented a novel approach to solving the GNE problem using an inexact penalty function on the inequality constraints that converts the GNE-seeking problem into an NE seeking problem. We then consider full- and partial-information gradient-play feedbacks for dynamic agents with passive dynamics. We prove both convergence to the NE and constraint satisfaction for all time.
APPENDIX

A. Proof of Lemma 5

By Assumptions 1 and 2, \( y^* \) is the unique point such that \( F(y^*) + \nabla \phi(y^*) = 0 \). Let \( x^* = \pi(y^*) \). By Assumption 3 (see Remark 4) and (9)

\[
f(\pi(y^*)) - G(F(y^*) + \nabla \phi(y^*)) = 0.
\]

Therefore, \( x^* = \pi(y^*) \) is an equilibrium point of (3). Now suppose there is another equilibrium \( \bar{x} \neq x^* \), we have

\[
0 = f(\bar{x}) - G(F(\bar{y}) + \nabla \bar{y}(\bar{y}))
\]

\[
\bar{y} = G^T \nabla V(\bar{x}).
\]

Let \( w = \nabla V(\bar{x}) \). By strong-monotonicity and Lipschitz continuity of \( \nabla V, \nabla V \) is invertible. Thus, we have

\[
f(\nabla V^{-1}(w)) = G(F(G^T w) + \nabla \phi(G^T w)).
\]

Consider the equilibrium \( x^* = \pi(y^*) \), and let \( w^* = \nabla V(x^*) \). From (35), we get

\[
f(\nabla V^{-1}(w)) - f(\nabla V^{-1}(w^*)) = G[F(G^T w) + \nabla \phi(G^T w) - F(G^T w^*) - \nabla \phi(G^T w^*)].
\]

Left multiplying by \((w - w^*)' \) gives

\[
(w - w^*)'[f(\nabla V^{-1}(w)) - f(\nabla V^{-1}(w^*))] = (w - w^*)' G[F(G^T w) + \nabla \phi(G^T w) - F(G^T w^*) - \nabla \phi(G^T w^*)].
\]

By monotonicity of \( F + \nabla \phi \) and of \( -f \circ \nabla V^{-1} \), we have

\[
0 \geq (w - w^*)' f(\nabla V^{-1}(w)) - f(\nabla V^{-1}(w^*)) = (w - w^*)' G[F(G^T w) + \nabla \phi(G^T w) - F(G^T w^*) - \nabla \phi(G^T w^*)] = (w - w^*)' G[G^T w - G^T w^* - \nabla \phi(G^T w)] \geq 0.
\]

Then, by strict-monotonicity of \( F, G^T w = G^T w^* \), implying \( \bar{y} = y^* \), NE of (7) and \( \varepsilon \)-vGNE of (4) by Lemma 4. Since, \( \bar{x} \neq x^* \), with \( \bar{y} = y^* \), the system is not equilibrium-independent observable by Lemma 3, contradicting Assumption 3(b) [see Remark 4(b)]. Thus, \( \bar{x} = x^* \). Therefore, \( x^* = \pi(y^*) \) is the unique equilibrium of (17).

B. Proof of Lemma 6

First, we show that \( S \subset \Omega_x := \{ x \in \mathbb{R}^n : g(G^T \nabla V(x)) < 0 \} \) is compact. Note that \( S = \{ x \in \mathbb{R} : V^z(x) - c \leq 0 \} \cap \{ x \in \mathbb{R} : \phi^z(x) - d \leq 0 \} \). Which is closed, since both functions are continuous. Since, the first set is compact, the intersection of the two is also compact.

Next, cf., Definition 4, Condition 2, we need to show that for all \( x \in S \), there exists \( z \in \mathbb{R}^n \) such that \( V^z(x) - c + \nabla \phi^z(x)' z < 0 \) and \( \phi^z(x) - d + \nabla \phi^z(x)' z < 0 \). Now, consider \( x \in S \). There are four cases to be considered.

1) \( V^z(x) - c < 0, \phi^z(x) - d < 0 \)

In this case, the inequalities hold trivially for \( z = 0 \).

2) \( V^z(x) - c < 0, \phi^z(x) - d = 0 \)

Let \( z = -G_\eta \), where \( \eta = (\nabla \phi(G^T \nabla V(x)) - \nabla \phi(G^T \nabla V(x^*))) \). Then

\[
V^z(x) - c + \nabla \phi^z(x)' z = V^z(x) - c - (\nabla \nabla V(x) - \nabla \nabla V(x^*))' G_\eta < 0
\]

by strict-convexity of \( \phi \) and the fact that \( G^T \nabla V(x) \neq G^T \nabla V(x^*) \) for all \( z \) such that \( \phi^z(x) = d \). In addition

\[
\phi^z(x) - d + \nabla \phi^z(x)' z = -\eta^T G^T \nabla V(x) G_\eta < 0
\]

by convexity of \( \phi(y) \), strong-convexity of \( V(x) \), full-column rank of \( G \) and since \( G^T \nabla V(x) \neq G^T \nabla V(x^*) \) when \( \phi^z(x) > 0 \).

3) \( V^z(x) - c = 0, \phi^z(G^T \nabla V(x)) - d < 0 \)

Let \( z = -a(\nabla \nabla V(x) - \nabla \nabla V(x^*)) \), where \( 0 < a < [\nabla \phi^z(x)'] G^T (\nabla \nabla V(x) - \nabla \nabla V(x^*)) \). This gives

\[
V^z(x) - c + \nabla \phi^z(x)' z = -a(\nabla \nabla V(x) - \nabla \nabla V(x^*))' (\nabla \nabla V(x) - \nabla \nabla V(x^*)) < 0
\]

by strong-convexity of \( V(x) \). Furthermore

\[
\phi^z(x) - d + a \nabla \phi^z(x)' (\nabla \nabla V(x) - \nabla \nabla V(x^*)) \leq \phi^z(x) - d + a \nabla \phi^z(x)' (\nabla \nabla V(x) - \nabla \nabla V(x^*)) < 0.
\]

4) \( V^z(x) - c = 0, \phi^z(G^T \nabla V(x)) - d = 0 \)

Let \( z = 0 \) as in Case 2. Then \( V^z(x) - c + \nabla \phi^z(x)' z < 0 \) and \( \phi^z(x) - d + \nabla \phi^z(x)' z < 0 \) as in Case 2.

Next, to show condition (3) in Definition 4, consider the vector field \( f_0(x) = (\phi^z(x) - d)(\nabla \nabla V(x) - \nabla \nabla V(x^*)) - G_\eta \), where \( \eta = (\nabla \phi(G^T \nabla V(x)) - \nabla \phi(G^T \nabla V(x^*))) \) and \( a \in \mathbb{R} \), which is Lipschitz continuous on \( S \) since \( S \subset \Omega_x \) is compact and \( f_0 \) is \( G^1 \) on \( \Omega_x \). Here, we need to check three cases.

1) \( V^z(x) - c < 0, \phi^z(x) - d = 0 \)

\[
L_{f_0} \phi^z(x) = -\eta^T G^T \nabla^2 V(x) G_\eta < 0
\]

by \( \phi(x) \) strictly convex, \( V(x) \) strongly convex, rank of \( G \) and that \( G^T \nabla V(x) \neq G^T \nabla V(x^*) \) when \( \phi^z(x) > 0 \).

2) \( V^z(x) - c = 0, \phi^z(x) - d < 0 \)

\[
L_{f_0} V^z(x) = (\phi^z(x) - d)(\nabla \nabla V(x) - \nabla \nabla V(x^*))^2 - (\nabla \nabla V(x) - \nabla \nabla V(x^*))' G_\eta < 0
\]

by strong-convexity of \( V(x) \), \( \phi^z(x) < d \) and convexity of \( \phi(y) \).

3) \( V^z(x) - c = 0, \phi^z(x) - d = 0 \)

We have \( L_{f_0} \phi^z(x) < 0 \) as in (36). Furthermore

\[
L_{f_0} V^z(x) = -(\nabla \nabla V(x) - \nabla \nabla V(x^*))' G_\eta < 0
\]

by \( \phi(y) \) strictly convex and \( G^T \nabla V(x) \neq G^T \nabla V(x^*) \) when \( \phi^z(x) > 0 \).

Therefore, by Definition 4, we have that \( S \) is a practical set.
C. Proof of Lemma 10

First, we show that \((S_1 \otimes y^*, \pi(x^*))\) is an equilibrium point of (17). At \((S_1 \otimes y^*, \pi(x^*))\), \(y = 1 \otimes y^*\). Thus, using the fact that \(F(1 \otimes y) = F(y)\) for any \(y\), (20) becomes

\[
y = -S_1 y \otimes y = 0
\]

\[
\dot{x} = f(\pi(x^*)) - G(F(y^*) + \nabla \phi(y^*) + RL_1(1 \otimes y^*)) = 0.
\]

Therefore, \((S_1 \otimes y^*, \pi(x^*))\) is an equilibrium point of (20). Now, suppose there is another equilibrium point \((y^*, \pi(x^*))\). From (20)

\[
-S_1(S^T y - R^T y) = 0
\]

(37)

\[
f(\pi(x^*)) - G(F(y^*) + \nabla \phi(y^*) + RLy) = 0.
\]

(38)

From (37), \(y = 1 \otimes y^*\). Using \(F(1 \otimes y) = F(y)\), (38) becomes \(f(\pi(x^*)) - G(F(y^*) + \nabla \phi(y^*)) = 0\). Following the proof of Lemma 5, \(\pi(x^*)\) and \(y^*\).

REFERENCES

[1] T. Alpcan and T. Başar, Distributed Algorithms for Nash Equilibria of Flow Control Games. Cambridge, MA, USA: Birkhäuser, 2005.

[2] W. Lin, Z. Qu, and M. A. Samaan, “Distributed game strategy design with application to multi-agent formation control,” in Proc. IEEE 53rd Conf. Decis. Control, 2014, pp. 433–438.

[3] M. S. Stanković, K. H. Johansson, and D. M. Stipanović, “Distributed seeking of Nash equilibria with application to mobile sensor networks,” IEEE Trans. Autom. Control, vol. 57, no. 4, pp. 904–919, Apr. 2012.

[4] S. Kršalević and S. Grammatico, “Learning generalized Nash equilibria in multi-agent dynamical systems via extremum seeking control,” Automatica, vol. 133, 2021, Art. no. 109846.

[5] M. Guo and C. De Persis, “Linear quadratic network games with dynamic players: Stabilization and output convergence to Nash equilibrium,” Automatica, vol. 130, 2021, Art. no. 109711.

[6] A. R. Romano and L. Pavel, “Dynamic NE seeking for multi-integrator networked agents with disturbance rejection,” IEEE Trans. Control Netw. Syst., vol. 7, no. 1, pp. 129–139, Mar. 2020.

[7] L. Pavel, “Distributed GNE-seeking under partial-decision information over networks via a doubly-augmented operator splitting approach,” IEEE Trans. Autom. Control, vol. 65, no. 4, pp. 1584–1597, Apr. 2020.

[8] K. Lu, G. Jing, and L. Wang, “Distributed algorithms for searching generalized Nash equilibrium of noncooperative games,” IEEE Trans. Cybern., vol. 49, no. 6, pp. 2362–2371, Jun. 2019.

[9] Y. Zou, B. Huang, Z. Meng, and W. Ren, “Continuous-time distributed Nash equilibrium seeking algorithms for non-cooperative constrained games,” Automatica, vol. 127, Art. no. 109535.

[10] M. Bianchi and S. Grammatico, “Continuous-time fully distributed generalized Nash equilibrium seeking for multi-integrator agents,” Automatica, vol. 129, 2021, Art. no. 109660.

[11] C. De Persis and S. Grammatico, “Continuous-time integral dynamics for a class of aggregative games with coupling constraints,” IEEE Trans. Autom. Control, vol. 65, no. 5, pp. 2171–2176, May 2020.

[12] A.-H. Mohsenian-Rad, V. W. Wong, J. Jatskevich, R. Schober, and A. Leon-Garcia, “Autonomous demand-side management based on game-theoretic energy consumption scheduling for the future smart grid,” IEEE Trans. Smart Grid, vol. 1, no. 3, pp. 320–331, Dec. 2010.

[13] Y. Zhang, S. Liang, X. Wang, and H. Ji, “Distributed Nash equilibrium seeking for aggregative games with nonlinear dynamics under external disturbances,” IEEE Trans. Cybern., vol. 50, no. 12, pp. 4876–4885, Dec. 2020.

[14] B. Huang, Y. Zou, and Z. Meng, “Distributed-observer-based Nash equilibrium seeking algorithm for quadratic games with nonlinear dynamics,” IEEE Trans. Syst. Man, Cybern. Syst., vol. 51, no. 11, pp. 7260–7268, Nov. 2021.

[15] L. S. P. Lawrence, J. W. Simpson-Porco, and E. Mallada, “Linear-convex optimal steady-state control,” IEEE Trans. Autom. Control, vol. 66, no. 11, pp. 5377–5384, Nov. 2021.

[16] F. Facchinei and C. Kanzow, “Penalty methods for the solution of generalized Nash equilibrium problems,” SIAM J. Optim., vol. 20, no. 5, pp. 2228–2253, 2010.

[17] M. Fukushima, “Restricted generalized Nash equilibria and controlled penalty algorithm,” Comput. Manage. Sci., vol. 8, pp. 201–218, 2011.

[18] F. Fabiani and A. Calti, “Nash equilibrium seeking in potential games with double-integrator agents,” in Proc. 18th Eur. Conf. Comput. Vis., 2019, pp. 548–553.

[19] S. Boyd and L. Vandenberghe, Convex Optimization. Cambridge, U.K.: Cambridge Univ. Press, 2004.

[20] D. Gadžov and L. Pavel, “A passivity-based approach to Nash equilibrium seeking over networks,” IEEE Trans. Autom. Control, vol. 64, no. 3, pp. 1077–1092, Mar. 2019.

[21] A. R. Romano and L. Pavel, “GNE seeking in games with passive dynamic agents via inexact-penalty methods,” in Proc. 59th CDC, 2020, pp. 500–505.

[22] A. R. Romano and L. Pavel, “Exact convergence to GNE using penalty methods,” in Proc. Euor. Conf. Control, 2022, pp. 2285–2290.

[23] F. Blanchini and S. Miani, Set-Theoretic Methods in Control, 2nd ed. Cambridge, MA, USA: Birkhäuser, 2015.

[24] J. W. Simpson-Porco, “Equilibrium-independent dissipativity with quadratic supply rates,” IEEE Trans. Autom. Control, vol. 64, no. 4, pp. 1440–1455, Apr. 2019.

[25] C. Godsil and G. Royle, Algebraic Graph Theory. (Graduate Texts in Mathematics Series), Berlin, Germany: Springer, 2001.

[26] F. Facchinei and C. Kanzow, “Generalized Nash equilibrium problems,” Ann. Oper. Res., vol. 175, pp. 177–211, 2010.

[27] F. Facchinei and J. S. Pang. Finite-Dimensional Variational Inequalities and Complementarity Problems. Berlin, Germany: Springer, 2007.

[28] T. Basar and G. J. Olsder, Dynamic Noncooperative Game Theory, 2nd ed. (Classics Applied Mathematics Series), Philadelphia, PA, USA: SIAM, 1999.

[29] G. Scutari, F. Facchinei, J. S. Pang, and D. P. Palomar, “Real and complex monotone communication games,” IEEE Trans. Inf. Theory, vol. 60, no. 7, pp. 400–409, Jul. 2014.

[30] H. Attouch, J. Peypouquet, and P. Redont, “A dynamical approach to an inertial forward-backward algorithm for convex minimization,” SIAM J. Optim., vol. 24, no. 1, pp. 232–256, 2014.

[31] R. I. Boţ and E. R. Csetnek, “A second-order dynamical system with Hessian-driven damping and penalty term associated to variational inequalities,” Optimization, vol. 68, no. 7, pp. 1265–1277, 2019.

[32] H. K. Khalil, Nonlinear Systems, 3rd ed. Philadelphia, PA, USA: SIAM, 2002.

[33] J. R. Marden, S. D. Ruben, and L. Y. Pao, “A model-free approach to wind farm control using game theoretic methods,” IEEE Trans. Control Syst. Technol., vol. 21, no. 4, pp. 1207–1214, Jul. 2013.

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