ON GLOBAL SOLUTIONS IN ONE-DIMENSIONAL THERMOELASTICITY WITH SECOND SOUND IN THE HALF LINE

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Abstract. In this paper, we investigate the initial boundary value problem for one-dimensional thermoelasticity with second sound in the half line. By using delicate energy estimates, together with a special form of Helmholtz free energy, we are able to show the global solutions exist under the Dirichlet boundary condition if the initial data are sufficient small.

1. Introduction. In this paper, we investigate the global solvability of smooth small solutions to the one-dimensional thermoelasticity with second sound in the half line. The – essentially nonlinear – equations are described as follows:

\[
\begin{cases}
\omega_t = v_x, \\
v_t = S_x, \\
e_t = Sv_x - q_x, \\
\tau q_t + q + k\theta_x = 0,
\end{cases}
\]

(1)

where \(\omega, v, S, e, \eta\) denote strain, velocity, Piola-Kirchhoff stress tensor, internal energy, and entropy, respectively, and the constitutive equations are given by

\[
\psi = e - (\theta + T_0)\eta, \quad \psi_\omega = S, \quad \psi_\theta = -\eta,
\]

(2)

where \(\psi = \psi(\omega, \theta, q)\) is the Helmholtz free energy, and \(\theta, q, T_0\) denote the temperature difference, the heat flux and the reference temperature respectively. In view of Tarabek[26], see also [8] and [22], we assume the Helmholtz free energy have the following form:

\[
\psi(\omega, \theta, q) = \tilde{\psi}(\omega, \theta) + \frac{\tau}{2(\theta + T_0)\kappa} q^2.
\]

(3)

Here, for simplicity, we assume that the relaxation parameter \(\tau\) is a positive constant and the heat conductive coefficient \(\kappa\) is also a given constant. Note that the Clausius-Duhem inequality

\[
\psi_t + \eta\theta_t - S\omega_t + \frac{q\theta_x}{\theta + T_0} \leq 0
\]

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holds under this special choice of the Helmholtz free energy. Therefore, in view of (2) and (3), the original system (1) can be written as

\[
\begin{align*}
\omega_t - v_x &= 0, \\
v_t - a(\omega, \theta)\omega_x + b(\omega, \theta)\theta_x &= 0, \\
\bar{a}(\omega, \theta, q)\theta_t + b(\omega, \theta)v_x + c(\theta)q_x &= d(\omega, \theta, q)q_t, \\
\tau q_t + q + \kappa\theta_x &= 0
\end{align*}
\] (4)

where

\[
a(\omega, \theta) = \tilde{\psi}_\omega, \quad b(\omega, \theta) = -\tilde{\psi}_\omega, \quad \bar{a}(\omega, \theta, q) = -\tilde{\psi}_\theta, \\
c(\theta) &= \frac{1}{\theta + T_0}, \quad d(\omega, \theta, q) = \frac{1}{\theta + T_0}[(\theta + T_0)\psi_{\theta q} - \psi_q] = -\frac{2\tau}{(\theta + T_0)^2}\kappa q.
\] (5)

We consider the initial boundary value problem for the functions \( \omega, v, \theta, q : \Omega \times [0, \infty) \rightarrow \mathbb{R} \) with initial conditions

\[
(\omega(x, 0), v(x, 0), \theta(x, 0), q(x, 0)) = (\omega^0(x), v^0(x), \theta^0(x), q^0(x)), \quad (\omega^0, v^0, \theta^0, q^0) \quad (7)
\]

and Dirichlet boundary conditions (rigidly clamped, constant temperature)

\[
v|_{\partial\Omega} = \theta|_{\partial\Omega} = 0, \quad t \geq 0, \quad (8)
\]

where \( \Omega = (0, +\infty) \) and the initial data \( v_0, \theta_0 \) should be compatible with the boundary condition.

For the limit case \( \tau = 0 \), the equations (1) and (2) constitute the system of one-dimensional classical thermoelasticity, in which the relation between the heat flux and the temperature is governed by Fourier’s law,

\[
q = -k\theta_x. \quad (9)
\]

For this system, global existence of solutions for small data has been established for various domains, i.e., the whole space, the half space and the bounded interval, see [11, 19, 20, 12, 13, 14]. The formation of singularities for large data was considered by Dafermos and Hsiao [4], Hrusa and Messaoudi [10], and Hansen [6]. For more results in this respect, see the monograph [15] for details.

In case \( \tau > 0 \), Fourier’s law is replaced by Cattaneo’s law \( (4)_4 \) for heat conduction, and the corresponding system is called thermoelasticity with second sound reflecting the appearance of heat waves with finite propagation speed. For the Cauchy problem, Tarabek [26] has established a global existence theorem for small initial data, and the decay to an equilibrium. The local existence theorem, stated in [26] with a hint to the paper of Hughes, Nato and Marsden [9], was completely proved, and the decay rates for global solutions were given by Racke and Wang [23]. Recently, by using weighted energy estimates, A. Kasimov, R. Racke and B. Said-Houari [16] proved a global existence theorem for small data with improved decay rates. In [16], they first analyzed the decay property for linearized model by using the classical Shizuta-Kawashima condition [25] and then proved the decay estimates for both solutions and its derivatives of the nonlinear system. For initial boundary value problem, Racke [21] has proved the exponential stability and global existence on bounded domain, see [18] for the general case. In the case \( \Omega = \mathbb{R}^+ \), Hu [7] proved the global existence for small data under the traction free and constant temperature boundary condition, i.e. \( \omega|_{\partial\Omega} = \theta|_{\partial\Omega} = 0 \).

Since the Poincaré inequality is invalid for unbounded domains, the methods used for bounded domains cannot be carried over into unbounded case. This is the
main difficulty for the half line problem. The main point in [7] is that the special type of boundary condition allows one to do Fourier analysis which helps to get the decay rates of the linearized system and thus establish the well-posedness of the nonlinear system. However, for Dirichlet boundary problem, the Fourier analysis fails. We shall use other methods to solve this problem.

This paper is mainly motivated by Jiang’s paper [14] where he obtained the global existence of smooth solutions to the system of classical thermoelasticity under Dirichlet boundary condition in the half line by directly using energy estimates. We observe that for a special type of Helmholtz free energy, we can get the lower energy estimates of system (4) immediately, which makes it possible to do the higher order estimates. This type of Helmholtz free energy is physical reasonable for second sound (see [26]) and was also used in dealing with the Cauchy problem in [26]. Moreover, since the classical thermoelasticity have more dissipation than the second sound case, one must be careful to deal with the boundary terms which is quite different compared with that in [14]. We adapted the method used in [21] to overcome this difficulties and finally obtain the desired a priori estimates. We note that even though there are many results in classical thermoelasticity that can be extended to thermoelasticity with second sound, see [19, 21] and [23], it is not true, for example, for Timoshenko-type thermoelastic systems, where a system can be or remain exponentially stable under Fourier’s law, while it loses this property under Cattaneo’s law, see [5]. Therefore, our results are meaningful in this respect.

We note that a global defined smooth solution for thermoelasticity with second sound should not be expected for large data. Indeed, Hu and Racke [8] showed that for special choice of constitutive equations, the solution to the Cauchy problem will develop singularities in finite time if the initial data are large.

We now introduce some notations which will be frequently used throughout the paper. For a non-negative integer N, let

$$D^N u = (\partial_t^l \partial_x^m u; l + m = N), \quad \bar{D}^N u = (\partial_t^l \partial_x^m u; 0 \leq l + m \leq N).$$

We denote by $W^{m,p}(\Omega)$, $0 \leq m \leq \infty$, $1 \leq p \leq \infty$, the usual Sobolev space with the norm $\| \cdot \|_{W^{m,p}}$. For convenience, $H^m(\Omega)$ and $L^p(\Omega)$ stand for $W^{m,2}(\Omega)$ and $W^{0,p}(\Omega)$ with the norm $\| \cdot \|_{H^m}$ and $\| \cdot \|_{L^p}$, respectively. For $p = 2$, we denote the norm $\| \cdot \|_{L^2}$ by $\| \cdot \|$. Let $X$ be a Banach space. We denote by $L^p(\alpha, \beta; X)$ $(1 \leq p \leq \infty)$ and $\| \cdot \|_{L^p(\alpha, \beta; X)}$ the space of all measurable $p$-th power functions from $[\alpha, \beta]$ to $X$ and its norm, respectively.

For $U = (\omega, v, \theta, q)$ a function of $t$ and $x$, we denote

$$K(U) = \begin{pmatrix}
0 & -\partial_x & 0 & 0 \\
-a\partial_x & 0 & b\partial_x & 0 \\
0 & -\frac{b}{a}\partial_x & \frac{\kappa d}{a^2\tau}\partial_x & \frac{c}{a}\partial_x + \frac{d}{a\tau} \\
0 & 0 & \frac{\kappa}{\tau}\partial_x & \frac{1}{\tau}
\end{pmatrix} U.$$

In order to formulate the compatibility conditions for the initial data $U^0(x) = (\omega^0(x), v^0(x), \theta^0(x), q^0(x))$, we define $(\partial_t^k U(0, x))$ for $t \geq 0$ recursively by

$$(\partial_t^k U)(0, x) = \partial_t^{k-1}(-K(U)U)(0, x), \quad k \geq 1, \quad U(0, x) = U^0(x) = (\omega^0, v^0, \theta^0, q^0).$$

We shall suppose throughout the paper that:
Assumption 1.1. 1. \( \psi \in C^4 \) for \( |\omega|, |\theta|, |q| \leq \min\{1, \frac{T_0}{2}\} \).
2. There exists positive constants \( \gamma_0, \gamma_1 \) such that
\[
\gamma_0 \leq \psi \omega, |\psi \theta|, -\psi \eta \leq \gamma_1 \quad \text{for} \quad |\omega|, |\theta|, |q| \leq \min\{1, \frac{T_0}{2}\}.
\]
3. \( U_0^k \in H^{2-k}(\Omega) \) for \( k = 0, 1, 2 \) and \( v_0^k \mid_{\partial \Omega} = \theta_0^k \mid_{\partial \Omega} = 0 \) for \( k = 0, 1 \).

Here \( U_0^k \equiv (\omega_0^k(x), v_0^k(x), \theta_0^k(x), q_0^k(x)) := (\partial_k U)(0, x) \) for \( k \geq 0 \) and \( (\partial_k U)(0, x) \) are defined by (10).

Here we present the main theorem and will give a proof in the end of this paper.

Main theorem. Let Assumption 1.1 holds. Then there exists a number \( \varepsilon > 0 \) such that if \( \|C^0\|_{H^2}^2 + \|U_1^0\|_{H^1}^2 + \|U_2^0\|_{L^2}^2 \leq \varepsilon^2 \), then there exists a unique global solution
\[
U = (\omega, v, \theta, q) \text{ of } (4), (7), (8)
\]
with
\[
U \in C^1([0, +\infty) \times \Omega),
\]
\[
\bar{D}^2U \in C^0([0, +\infty), L^2(\Omega)), \quad q, \bar{D}^jU \in L^2([0, +\infty), L^2(\Omega))(1 \leq j \leq 2),
\]
and
\[
|\omega(t, x)|, |\theta(t, x)|, |q(t, x)| < \min\{1, \frac{T_0}{2}\} \quad \text{for} \quad t \geq 0, x \in \bar{\Omega}.
\]

Remark 1. We can slightly modify the method here to treat the other boundary conditions (e.g. the Neumann boundary condition, cf. [7]).

Remark 2. Our system is a partially dissipative quasi-linear hyperbolic system, see (14) below. We mention that for Cauchy problem of such systems, there are some results concerning the existence of global small smooth solutions, see [1, 3, 17] and the references cited therein. However, since we consider the half line problem with Dirichlet boundary condition, the methods used for Cauchy problem can not be applied directly in our case.

2. Local existence. The following theorem on the local existence of smooth solutions to (4), (7) and (8) can be established by using the local existence results of initial boundary value problem for hyperbolic systems (cf. [2, 7]).

Theorem 2.1. Let Assumption 1.1 holds. Assume that \( |\omega_0|, |\theta_0|, |q_0| < \min\{1, \frac{T_0}{2}\} \) for \( x \in \bar{\Omega} \). Then the initial-boundary value problem (4), (7) and (8) has a unique solution \((\omega, v, \theta, q)\) with
\[
U \in C^1([0, T) \times \bar{\Omega}),
\]
on a maximal time interval \( [0, T] \), for some \( 0 < T < +\infty \) such that
\[
\bar{D}^2U \in C^0([0, t], L^2(\Omega)), \quad q, \bar{D}^jU(1 \leq j \leq 2) \in L^2([0, t], L^2(\Omega)), \forall t \in [0, T),
\]
and
\[
|\omega(t, x)|, |\theta(t, x)|, |q(t, x)| < \min\{1, \frac{T_0}{2}\} \quad \forall t \in [0, T), x \in \bar{\Omega}. \tag{11}
\]
Furthermore, if
\[
\sup_{t \in [0, T)} \|\bar{D}^2U(t)\|^2 + \int_0^T (\|q\|^2 + \|\bar{D}^1U\|^2 + \|\bar{D}^2U\|^2)ds < +\infty \tag{12}
\]
and
\[
\sup_{t \in [0, T), x \in \bar{\Omega}} \{|\omega(t, x)|, |\theta(t, x)|, |q(t, x)|\} < \min\{1, \frac{T_0}{2}\}, \tag{13}
\]
then \( T = +\infty \).
Proof. We shall use the results from [2] to prove the local existence results. We first multiply (4), (4) by \( a, \frac{c}{\kappa} \) respectively and rewrite (4), (7) and (8) as follows

\[
\begin{cases}
A^0 U_t + A^1 U_x + DU = 0, \\
MU = 0, \quad \text{for} \ (t, x) \in ([0, T] \times \partial \Omega), \\
U^0 = U(0, x) \quad \text{for} \ x \in \Omega,
\end{cases}
\tag{14}
\]

where \( U = U(t, x) = (\omega(t, x), v(t, x), \theta(t, x), q(t, x)) \) and

\[
A^0 = \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & \tilde{a} & 0 & 0 \\
0 & 0 & \frac{cT}{\kappa} & 0
\end{pmatrix}, \quad A^1 = \begin{pmatrix}
0 & -a & 0 & 0 \\
-a & 0 & b & 0 \\
b & \kappa b & c & 0 \\
0 & 0 & c & 0
\end{pmatrix}, \quad D = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & \frac{d}{\kappa} & 0 \\
0 & 0 & 0 & \frac{\kappa}{\kappa}
\end{pmatrix},
\]

and \( T \) is an arbitrary but fixed constant.

We note that, by Assumption 1.1, \( A^0 \) is a positive definite matrix and \( A^1 \) is a symmetric matrix. Therefore, system (14) is a first order quasi-linear symmetric hyperbolic system. The local existence theorem for such systems has already been proved, see Theorem 11.2 in [2] for details. We only need to show that the conditions in Theorem 11.2 are satisfied for system (14). For reader’s convenience, we list them all and will check the corresponding conditions one by one for our case.

(CH) The matrices

\[
A(U, \xi) := \sum_{j=1}^{d} \xi_j A^j(U)
\]

are diagonalizable with real eigenvalues of constant multiplicities for \( (U, \xi) \in \mathcal{U} \times S^{d-1} \).

(NC) For all \((x, t) \in \partial \Omega \times [0, T]\) and for all \( U \in \mathcal{U} \) such that \( B(U)U = b(x, t) \), the matrix \( A(U, \nu(x)) \) is non-singular (where \( \nu(x) \) denotes the outward unit normal to \( \partial \Omega \) at point \( x \)).

(Nb) The boundary matrix \( B(U) \) is of constant, maximal rank for all \((x, t) \in \partial \Omega \times [0, T] \) and \( U \in \mathcal{U} \) such that \( B(U)U = b(x, t) \) and

\[
\mathbb{R}^n = \ker B(U) \oplus E^s(A(U, \nu(x))),
\]

with \( \nu \) an outward normal vector to \( \partial \Omega \) and \( E^s(A(U, \nu)) \) is the stable subspace of the matrix \( A(U, \nu) \).

(UKL) For all \((U, x, \xi, z) \in \mathcal{M}_b(x, t) \times T^* \partial \Omega \times \mathbb{C} \) with \( \text{Re} z > 0 \) where

\[
\mathcal{M}_b(x, t) := \{ U \in \mathcal{U}; B(U)U = b(x, t) \},
\]

there exists \( C > 0 \) such that

\[
\|V\| \leq C \|B(U)V\| \quad \text{for all} \ V \in E_-(U, x, \xi, z),
\]

where \( E_-(U, x, \xi, z) \) is the stable subspace of

\[
\mathcal{A}(U, x, \xi, z) := A(U, \nu(x))^{-1}(zI_n + iA(U, \xi)),
\]

and \( \nu(x) \) denotes the outward unit normal to \( \partial \Omega \) at point \( x \); and the same is true for \( \text{Re} z = 0 \) once the subspace \( E_- \) has been extended by continuity.

In our case, the domain \( \Omega \) equals to \((0, +\infty)\) and the corresponding outward normal vector \( \nu(x)|_{x=0} = -1 \). The phase space \( \mathcal{U} = \mathbb{R}^4 \) and \( d = 1 \); the boundary
matrix $B(U)$ equals to $M$ and $\bar{b} = 0$. We say $W$ is a stable subspace of a matrix $A$ if $W$ is generated by eigenvectors associated with negative eigenvalues of matrix $B$. Clearly, the condition $(CH)$ is satisfied automatically for symmetric hyperbolic systems. All we need is to check that $(NC_b), (N_b)$ and $(UKL_b)$ are satisfied. Since

$$A(U, \nu) = (-1) \cdot A^1 = \begin{pmatrix} 0 & a & 0 & 0 \\ a & 0 & -b & 0 \\ 0 & -b & \frac{kd}{\tau} & -c \\ 0 & 0 & -c & 0 \end{pmatrix},$$

we get that $\det A(U, \nu) = a^2c^2 \neq 0$ and thus condition $(NC_b)$ holds.

Next, let’s check the condition

$$\mathbb{R}^4 = \ker M \oplus E^s(A(U, \nu)).$$

After some simple calculation, we know that

$$\ker M = \text{span}\{(1, 0, 0, 0), (0, 0, 0, 1)\}.$$ Let $\det(\lambda I - A(U, \nu))$ be zero, we have

$$\lambda^4 - \frac{kd}{\tau} \lambda^3 - (a^2 + b^2 + c^2) \lambda^2 + \frac{a^2kd}{\tau} \lambda + a^2c^2 = 0.$$ Suppose $\lambda_4 \leq \lambda_3 \leq \lambda_2 \leq \lambda_1$ are four real eigenvalues of the matrix $A(U, \nu)$. We know easily that

$$\lambda_1\lambda_2\lambda_3\lambda_4 = a^2c^2 > 0, (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)(\lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4 + \lambda_1\lambda_2\lambda_4 + \lambda_1\lambda_2\lambda_3) \leq 0$$ and

$$\lambda_1(\lambda_2 + \lambda_3 + \lambda_4) + \lambda_2(\lambda_3 + \lambda_4) + \lambda_3\lambda_4 = -(a^2 + b^2 + c^2) < 0.$$ This implies that $\lambda_4 \leq \lambda_3 < 0 < \lambda_2 \leq \lambda_1$. Let $v_i (i = 1, 2, 3, 4)$ be eigenvectors corresponding to $\lambda_i$, which means that

$$(\lambda_i I - A(U, \nu))v_i = 0.$$ For each $\lambda_i$, $v_i$ is uniquely determined by

$$v_i = k(1, \frac{\lambda_i}{a} \frac{a}{b} \frac{c}{ab} + \frac{c\lambda_i}{ab})$$ for any $k \in \mathbb{R}, k \neq 0.$ This show that all the four eigenvalues are different from each other, i.e., we have $\lambda_4 < \lambda_3 < 0 < \lambda_2 < \lambda_1$. So, $E^s(A(U, \nu)) = \text{span}\{v_3, v_4\}$. Since

$$\det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\lambda_3}{a} & \frac{a}{b} & -\frac{\lambda_2^2}{ab} & -\frac{ac\lambda_3}{ab} \\ \frac{\lambda_4}{a} & \frac{a}{b} & -\frac{\lambda_4^2}{ab} & -\frac{ac\lambda_4}{ab} \end{pmatrix} = (1 + \frac{\lambda_3\lambda_4}{a^2}) \frac{\lambda_3 - \lambda_4}{b} \neq 0,$$ we derive that

$$\mathbb{R}^4 = \ker M \oplus E^s(A(U, \nu)),$$ and the condition $(N_b)$ is satisfied.

In one dimensional case, $T^\ast \partial \Omega$ is trivial. So $A(U, \xi)$ in condition $UKL_b$ is zero. In this case, $E_-(U, x, \xi, z)$ is the stable subspace of $zA(U, \nu(x))^{-1}$. We know from matrix theory that if $\lambda$ is an eigenvalue of matrix $A$, then $\frac{1}{\lambda}$ is a eigenvalue of matrix $A^{-1}$. So, the eigenspace generated by eigenvectors with negative eigenvalues of matrix $A$ is exactly the same with eigenspace generated by eigenvectors with
negative eigenvalues of matrix $A^{-1}$. This means that the space $E_-(U, x, \xi, z)$ equals to the space $E^*(A(U, \nu))$ when $\text{Re} \nu > 0$. Since $\mathbb{R}^4 = \ker M \oplus E^*(A(U, \nu))$, $M$ restricted to $E^*$ is isomorphism, which means that there exists $C$ such that

$$||V|| \leq C||MV|| \quad \text{on } E^* = E_-.$$

Thus the condition $(\text{UKL}_b)$ is also satisfied. So far, we have proved all conditions needed in theorem 11.2. So, we can now apply theorem 11.2 to our systems and this complete our proof. \hfill \square

3. Proof of the global existence. Let

$$E_0 := ||U^0||_{H^2}^2 + ||U^0||_{H^1}^2 + ||U^0||^2,$$

$$E(t) := \sup_{s \in [0,t]} (\bar{D}^2U(s)) + \int_0^t (||q||^2 + ||D^1 U||^2 + ||D^2 U||^2)(s)ds.$$

We first show the following a priori estimates.

**Lemma 3.1.** Let $U = (\omega, v, \theta, q)$ be the local solution of (4), (7), (8) established in Theorem 2.1. Then we have

$$E(t) \leq \Gamma (E_0 + E^2(t)) \quad \forall t \in [0, T),$$

where $\Gamma > 0$ is a constant independent of $T$ and $E_0$.

**Proof.** Throughout the proof, $C$ will denote a generic positive constant which is independent of $T$ and $E_0$, not necessarily the same in any two spaces.

Firstly, by Sobolev imbedding theorem, we have the following inequality

$$\sup_{0 \leq s \leq t} \| (\omega, v, \theta, \omega_x, v_x, \theta_x, q_x, v, \theta, q) \|_{L^\infty} \leq CE^{2}(t) \quad \forall t \in [0, T).$$

We start with the following lower order energy estimates

$$||U(t)||^2 + \int_0^t ||q||^2 ds \leq CE_0, \quad \forall t \in [0, T).$$

In fact as in [26], the original system (1) implies the following equation

$$\partial_t (\psi - \hat{\psi} - \hat{S}\omega + \theta q + \frac{1}{2} q^2) + \frac{T_0}{\kappa (\theta + T_0)^2} q^2 = \partial_x ((S - \hat{S}) v - \theta \frac{q}{\theta + T_0})$$

holds where $\hat{S} = S(0, 0), \hat{\psi} = \psi(0, 0, 0)$. Therefore, by Taylor’s theorem and (3), we have

$$\psi(\omega, \theta, q) - \psi(\omega, 0, 0) - \theta \psi_\theta(\omega, \theta, q) = - \int_0^\theta \int_s^\theta \psi_{rr}(\omega, r, q) dr ds \geq \frac{1}{2} \min \{-\psi_{\theta \theta}\} \theta^2,$$

$$\psi(\omega, 0, q) - \psi(\omega, 0, 0) \geq \frac{T_0}{2\kappa T_0^2} q^2,$$

$$\psi(\omega, 0, 0) - \psi(\omega, 0, 0) \geq \frac{1}{2} \min \{\psi_{\omega \omega}\} \omega^2.$$
over $[0, t] \times \Omega$. Using the boundary conditions $v_t|_{\partial\Omega} = \theta_t|_{\partial\Omega} = 0$, we get

$$\int_\Omega (\omega_t^2 + v_t^2 + \hat{a} \theta_t^2 + \tau q_t^2) dx + \int_0^t \int_\Omega q_t^2 dx dt$$

$$\leq C(E_0 + E^{\frac{2}{3}}(t)) + \int_0^t \int_\Omega (d(\omega, \theta, q)q_t) \theta_t d x dt$$

$$\leq C(E_0 + E^{\frac{2}{3}}(t)) + \int_0^t \int_\Omega d(\omega, \theta, q)(-\frac{1}{\tau} q_t - \frac{\tau}{\tau} \theta_{xx}) \theta_t d x dt$$

$$\leq C(E_0 + E^{\frac{2}{3}}(t)), \quad (21)$$

where we have used the fact that $|d(\omega, \theta, q)| \leq C|q|$ by (6). Therefore, we have

$$\|U_t(t)\|^2 + \int_0^t \|q(s)\|^2 ds \leq C(E_0 + E^{\frac{2}{3}}(t)), \quad \forall t \in [0, T). \quad (22)$$

In view of (19), (22) and equations (4), we have

$$\|\omega_x(t)\|^2 + \|v_x(t)\|^2 + \|\theta_x(t)\|^2 + \|q_x(t)\|^2 + \int_0^t \|\theta_x\|^2 ds \leq C(E_0 + E^{\frac{2}{3}}(t)). \quad (23)$$

Now, we estimate the term $\int_0^T \|u_x(s)\|^2 ds$. Multiplying (4) by $\omega_x$, integrating over $\Omega \times [0, t]$, using (19), (23), equation (4), Assumption 1.1 and the Cauchy-Schwarz inequality, we get

$$\int_0^t \int_\Omega a(\omega, \theta) \omega_x^2 dx dt$$

$$= \int_0^t \int_\Omega v \omega_x dx dt + \int_0^t \int_\Omega b(\omega, \theta) \theta_x \omega_x dx dt$$

$$= \int_0^t \frac{d}{dt} \int_\Omega v \omega_x dx dt - \int_0^t \int_\Omega v \omega_{xt} dx dt + \int_0^t \int_\Omega b(\omega, \theta) \theta_x \omega_x dx dt$$

$$\leq \int_0^t \int_\Omega v_x^2 dx dt + \int_0^t \int_\Omega \frac{1}{2} a(\omega, \theta) \omega_x^2 dx dt + C(E_0 + E^{\frac{2}{3}}(t)).$$

Thus, by Assumption 1.1 and (11), we have

$$\int_0^T \|\omega_x\|^2 ds \leq C \int_0^T \|v_x\|^2 ds + C(E_0 + E^{\frac{2}{3}}(t)). \quad (24)$$

Now, multiplying (4) by $q_x$ and integrating with respect to $(t, x)$, we have

$$\int_0^t \int_\Omega c(\theta) q_x^2 dx dt$$

$$= - \int_0^t \int_\Omega \hat{a} \theta \omega_x dx dt - \int_0^t \int_\Omega b(\omega, \theta) v_x q_x dx dt - \int_0^t \int_\Omega d(\omega, \theta, q) q_x q_x dx dt,$$

$$= - \int_0^t \frac{d}{dt} \int_\Omega \hat{a} \theta q_x dx dt + \int_0^t \int_\Omega (\hat{a})_t \theta q_x dx dt$$

$$+ \int_0^t \int_\Omega \hat{a} \theta q_{xt} dx dt - \int_0^t \int_\Omega b(\omega, \theta) v_x q_x dx dt - \int_0^t \int_\Omega d(\omega, \theta, q) q_x q_x dx dt$$

$$:= I_1 + I_2 + I_3 + I_4 + I_5.$$
By using (6), (19), (23), the definition of \(E(t)\) and recalling the boundary condition \(\theta|_{\partial\Omega} = 0\), we can easily check that

\[
I_1 + I_2 + I_3 + I_5 \leq C(E_0 + E^{\frac{3}{2}}(t)).
\] (25)

Now, we intend to estimate the term \(I_4\). Integrating by part with respect to \(x\), using the same technique before, we get

\[
I_4 = -\int_0^t \int_\Omega b(\omega, \theta) v_x q_x dx dt
\]

\[
= -\int_0^t \int_\Omega (b(\omega, \theta) v_x q_x) (s, 0) ds + \int_0^t \int_\Omega (b(\omega, \theta) v_x) q_x dx dt
\]

\[
= -\int_0^t \int_\Omega (b v_x q_x) (s, 0) ds + \int_0^t \int_\Omega b_x v_x q_x dx dt
\]

\[
+ \int_0^t \frac{d}{dt} \int_\Omega b_\omega q_x - \int_0^t \int_\Omega b_\omega q_x dx dt - \int_0^t \int_\Omega b_\omega q_x dx dt
\]

\[
\leq -\int_0^t (b v_x q_x) (s, 0) ds + C(E_0 + E^{\frac{3}{2}}(t)) + \varepsilon \int_0^t \|\omega_x\|^2 ds.
\]

In term of Sobolev imbedding theorem \(W^{1,1}(\Omega) \hookrightarrow L^\infty\), we have

\[
\sup_{x \in \Omega} |q(x, t)|^2 \leq C(\|q\|^2 + \|q\|\|q_x\|).
\]

Thus

\[
\int_0^t (b v_x q_x) (s, 0) ds \leq \varepsilon \int_0^t v_x^2 (s, 0) ds + \frac{C}{\varepsilon} \int_0^t \|q\|^2 \|q_x\| ds
\]

\[
\leq \varepsilon \int_0^t v_x^2 (s, 0) ds + \frac{1}{2} \int_0^t \int_\Omega c(\theta) q_x^2 dx dt + C(\varepsilon) E_0.
\]

Combining the above calculations, we have

\[
\int_0^t \|q_x\|^2 ds \leq \varepsilon \int_0^t v_x^2 (s, 0) ds + \varepsilon \int_0^t \|\omega_x\|^2 ds + C(E_0 + E^{\frac{3}{2}}(t)).
\] (26)

Next we employ the technique of [24], also see [14] and [21], in dealing with the boundary term on the righthand side of (26). Note that, the methods used in [14] can not be used here since the estimates for \(\theta_{xt}\) are not expected. However, we shall use the techniques developed for the second sound case, see [21]. Differentiating (4)_2 with respect to \(t\), multiplying \(v_x^2\) and integrating, we have

\[
\int_0^t \int_\Omega v_t v_x dx dt - \int_0^t \int_\Omega (a(\omega, \theta) \partial_x (v_x^2)) + a_t \omega x v_x dx dt + \int_0^t \int_\Omega (b(\omega, \theta) \partial_x) v_x dx dt = 0.
\] (27)

Using the boundary condition \(v_t|_{\partial\Omega} = 0\) and the fact that \(v_t v_x = \partial_t (v_x v_x) - \partial_x (v_x^2)\), we have

\[
\int_0^t a(\omega, \theta) (s, 0) v_x^2 (s, 0) ds + \int_0^t \int_\Omega (b(\omega, \theta) \theta_{xt} v_x dx dt \leq C(E_0 + E^{\frac{3}{2}}(t)).
\] (28)
Next, multiplying \((4)_3\) by \(\theta_{xt}\) and integrating with respect to \((t, x)\), we obtain

\[
- \int_0^t \int_\Omega b v_x \theta_{xt} dxdt \\
= \int_0^t \int_\Omega \tilde{a} \theta_t \theta_{xt} dxdt + \int_0^t \int_\Omega c(\theta)q_x \theta_{xt} dxdt - \int_0^t \int_\Omega d q_t \theta_{xt} dxdt \\
= - \int_0^t \int_\Omega \left( \frac{1}{2} \tilde{a}_x \theta_t^2 \right) + \int_0^t \frac{d}{dt} \int_\Omega (c(\theta)q_x \theta_x) dxdt \\
- \int_0^t \int_\Omega (c(\theta)q_x \theta_x - d q_t \theta_x) dxdt - \int_0^t \int_\Omega (c(\theta)q_{xt} \theta_x - d q_{xt} \theta_x) dxdt \\
\leq C(E_0 + E^2(t)) + \int_0^t \int_\Omega \frac{1}{t} q_x + \frac{\kappa}{t} \theta_{xx} \theta_x dxdt - \int_0^t \int_\Omega d \left( \frac{1}{t} q_t + \frac{\kappa}{t} \theta_{xt} \right) \theta_x dxdt \\
\leq C(E_0 + E^2(t)) + \int_0^t \| q_x \|^2 ds - \int_0^t \theta_x^2 (s, 0) ds. \quad (29)
\]

Therefore, combining \((28)\) and \((29)\), we get

\[
\int_0^t \int_\Omega v_x^2 (s, 0) ds + \int_0^t \int_\Omega \theta_x^2 (s, 0) ds \leq \int_0^t \| q_x \|^2 ds + C(E_0 + E^2(t)). \quad (30)
\]

This implies, with the help of \((26)\), that

\[
\int_0^t \| q_x \|^2 ds \leq \varepsilon \int_0^t \| \omega_x \|^2 + C(E_0 + E^2(t)). \quad (31)
\]

Now, we intend to estimate the term \(\int_0^t \| v_x \|^2 ds\). Multiply \((4)_3\) by \(\frac{1}{\vartheta(\omega \theta)} v_x\), integrating with respect to \((t, x)\), and using again \((6)\), \((19)\) and \((23)\), we get

\[
\int_0^t \int_\Omega v_x^2 dxdt = - \int_0^t \int_\Omega \tilde{a} \theta_{xt} v_x dxdt - \int_0^t \int_\Omega \frac{c}{b} q_x v_x dxdt + \int_0^t \int_\Omega \frac{d}{b} q_t v_x dxdt \\
= - \int_0^t \int_\Omega \frac{d}{dt} \left( \frac{\tilde{a}}{b} \theta_{xt} v_x dxdt + \int_0^t \int_\Omega \left( \frac{\tilde{a}}{b} \right) \theta v_x dxdt + \int_0^t \int_\Omega \left( \frac{\tilde{a}}{b} \right) \theta_{xt} v_x dxdt \\
+ \frac{1}{2} \int_0^t \int_\Omega \tilde{a}_x v_x^2 dxdt + C \int_0^t \int_\Omega q_x^2 dxdt + \int_0^t \int_\Omega \frac{d}{b} q_t v_x dxdt \right) \\
\leq C(E_0 + E^2(t)) + \int_0^t \int_\Omega \frac{\tilde{a}}{b} \theta_{xt} v_x dxdt \\
+ \frac{1}{2} \int_0^t \int_\Omega \tilde{a}_x v_x^2 dxdt + C \int_0^t \int_\Omega q_x^2 dxdt \quad (32)
\]

The term \(\int_0^t \int_\Omega \frac{\tilde{a}}{b} \theta_{xt} v_x dxdt\) can be estimated as follows. Differentiating \((4)_2\) with respect to \(x\), multiply by \(\frac{\tilde{a}}{b} \theta_{xt}\), integrating, using \((19)\) and \((23)\), we get

\[
\int_0^t \int_\Omega \frac{\tilde{a}}{b} \theta_{xt} v_x dxdt = \int_0^t \int_\Omega \frac{\tilde{a}}{b} (a \omega_x)_x \theta dxdt - \int_0^t \int_\Omega (b \theta_x)_x \frac{\tilde{a}}{b} \theta dxdt \\
= \int_0^t \int_\Omega \left( \frac{\tilde{a}}{b} \right) a \omega_x \theta dxdt - \int_0^t \int_\Omega \frac{\tilde{a}}{b} \omega_x a \theta dxdt - \int_0^t \int_\Omega \frac{\tilde{a}}{b} \omega_x b \theta dxdt \\
\leq C(E_0 + E^2(t)) + \varepsilon \int_0^t \| \omega_x \|^2 dxdt. \quad (33)
\]
Combining (32) and (33), we get
\[ \int_0^t \int_\Omega v^2_x \, dx \, dt \leq C \int_0^t \int_\Omega q^2_x \, dx \, dt + \varepsilon \int_0^t \| \omega \|^2 \, ds + C(E_0 + E^\frac{2}{3}(t)). \] (34)
The above estimate combining (31) and (24) implies
\[ \int_0^t \left( \| \omega \|^2 + \| v \|^2 + \| q \|^2 \right) \, dx \, dt \leq C(E_0 + E^\frac{2}{3}(t)). \] (35)
Then, it follows from the equations (4) and the above estimates that
\[ \int_0^t \left( \| \omega_t \|^2 + \| \theta_t \|^2 + \| v_t \|^2 + \| q_t \|^2 \right) \, ds \leq C(E_0 + E^\frac{2}{3}(t)). \] (36)

Therefore, combining (23), (35) and (36), we obtain
\[ \sup_{0 \leq s \leq t} \| D^1 U(s) \|^2 + \int_0^t \| D^1 U(s) \|^2 \, ds \leq C(E_0 + E^\frac{2}{3}(t)) \quad \forall t \in [0, T). \] (37)

To estimate higher derivatives of \( U \), we differentiate (4) with respect to \( t \) twice and multiply by \( \kappa \omega_{tt}, \kappa v_{tt}, \kappa \theta_{tt}, c_q t_t \), respectively, integrating by parts with respect to \( x \), we get
\[ \| U_{tt} \|^2 + \int_0^t \| q_{tt} \|^2 \, ds \leq C(E_0 + E^\frac{2}{3}(t)). \] (38)

Note that the above procedure (also in what follows) is formal because the regularity of the local solution is insufficient to validate several of the steps. However, rigorous derivations of the desired energy inequality can be achieved easily by using difference quotients and taking limits, or by using mollifiers.

It follows from (4), (19) and (37) that
\[ \| (\omega_{xt}, v_{xt}, \theta_{xt}, q_{xt}, \omega_{xx}, v_{xx}, \theta_{xx}, q_{xx}) \|^2 \leq C(E_0 + E^\frac{2}{3}(t)). \] (39)

With the help of (19) and (37)-(39), similarly arguments to those used for (31), we have
\[ \int_0^t \| q_{xt} \|^2 \, ds \leq \varepsilon \int_0^t \| \omega_{xt} \|^2 \, ds + C(E_0 + E^\frac{2}{3}(t)). \] (40)

for any \( t \in [0, T) \) and \( 0 < \varepsilon < 1 \). Now, using (19), (37)-(39), following the same argument to those used for (24) and (34), we can obtain
\[ \int_0^t \left( \| v_{tx} \|^2 + \| \omega_{tx} \|^2 \right) \, ds \leq C(E_0 + E^\frac{2}{3}(t)). \] (41)

Thus, we have
\[ \int_0^t \| q_{xt} \|^2 \, ds \leq C(E_0 + E^\frac{2}{3}(t)). \] (42)

Now, differentiating the equation (4) and using (19) and (37)-(42), we conclude that
\[ \| D^2 U(t) \|^2 + \int_0^t \| D^2 U(t) \|^2 \, ds \leq C(E_0 + E^\frac{2}{3}(t)), \quad \forall t \in [0, T). \] (43)

which combined with (19), (37) yields (17). This completes the proof.

Now we show the Main Theorem.

**Proof of the Main Theorem.** From Lemma 3.1, we have the following *a priori* estimate for the local solutions of (4)
\[ E(t) \leq \Gamma(E_0 + E^\frac{2}{3}(t)). \]
The Sobolev imbedding theorems imply
\[ \| \omega \|_{L^\infty}^2 + \| \theta \|_{L^\infty}^2 + \| q \|_{L^\infty}^2 \leq \lambda E(t), \quad \text{for } t \in [0, T), \] (44)
where the constant \( \lambda \) is independent of \( T \) and \( E_0 \). From the definition of \( E(t) \), we know that
\[ E(0) \leq \mu E_0, \]
where \( \mu \) is a constant independent of \( T \) and \( E_0 \). Now we choose \( \delta > 0 \) such that
\[ \sqrt{\lambda \delta} < \min \{1, \frac{T_0}{2}\}, \quad \Gamma \delta^{\frac{1}{2}} \leq \frac{1}{4}. \]
Now, we take \( \varepsilon > 0 \) such that
\[ \Gamma \varepsilon^2 \leq \frac{\delta}{4}, \quad \mu \varepsilon^2 \leq \frac{\delta}{2}. \]
Assume that \( E_0 = \| U^0 \|_{H^2}^2 + \| U_0^0 \|_{H_1}^2 + \| U_0^0 \|_2^2 \leq \varepsilon^2 \) holds with the above choice of \( \varepsilon \). Now we assume for the moment that \( E(t) \leq \delta \), we get
\[ E(t) \leq \Gamma (E_0 + \varepsilon^2(t)) \leq \frac{\delta}{4} + \Gamma \delta^{\frac{1}{2}} \leq \frac{\delta}{2}. \]
Since \( E(0) \leq \mu \varepsilon^2 \leq \frac{\delta}{2} \), we conclude that
\[ E(t) \leq \frac{\delta}{2} < (\min \{1, \frac{T_0}{2}\})^2/2\lambda, \quad \forall t \in [0, T). \] (45)
Furthermore, from (44) and (45), we get that
\[ \| \omega \|_{L^\infty}^2 + \| \theta \|_{L^\infty}^2 + \| q \|_{L^\infty}^2 \leq \frac{1}{2} \min \{1, \frac{T_0}{2}\}^2 \]
for all \( t \in [0, T) \). In view of (45) and (46), we see that (12) and (13) in Theorem 2.1 holds and consequently \( T = +\infty \). This complete the main theorem.

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