Variational Formulations of Flexible Ropes Dynamics Problems

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Abstract. Variational statements of the problems of the dynamics of flexible ropes with significant displacements, the shape of which is determined by constraints and load, are obtained. The position of the flexible rope axis is described by a vector of Cartesian coordinates. The stress state of the rope is characterized by the force in the rope. At present, the methods of calculating the ropes systems are based on a differential formulation. Variational formulations of such problems are proposed.

1. Introduction
The problem of the dynamics of flexible ropes is considered, in which the stress-strain state is described by the coordinates of the rope axis and the forces in the rope [1]. At present, numerical methods [2] based on the differential formulation of the problem are used to solve various problems of calculating cable systems. At the same time, in order to use modern universal numerical methods, for example, the finite element method, a variational statement of the problem is required.

To form the variational statement of the problem, one can apply a formal mathematical procedure, which is used [3] to obtain known variational principles of problems in the theory of elasticity. On the basis of this approach, in 1991 a mixed variational formulation of the flexible ropes equilibrium problem was developed [4].

The complexity of constructing the scheme of the finite element method is related to the geometric nonlinearity and the mixed nature of the problem. To solve a mixed problem, it is proposed to use the approach used in "mixed finite element method with the elimination of some unknowns at the element level" [5-7]. With this in mind, schemes of the finite element method were constructed [8, 9], in which the forces are calculated at the element level, and when solving the global system of algebraic equations, the coordinates of the axes of the ropes are calculated. And because of the geometric nonlinearity, an iterative procedure is used. In [10] examples of solving test problems are given. The variational statements of the problems of the dynamics of flexible ropes are given below.

2. Differential formulation
The axis position of the flexible ropes is described as vectors of its Cartesian coordinates $\mathbf{x} = \{x_1, x_2, x_3\}^T$. The stress state in the ropes is characterized by the force $T \ (T \geq 0)$. Coordinates along
the ropes axis are denoted by $s$. We assume one end of the ropes is fixed and the other is attached where the load is applied. The differential formulation of the equilibrium problem for flexible inextensible ropes is given, for example, in [1].

If the vector of coordinates $\mathbf{x}$ is a function of not only the coordinates along the axis of the rope $s$, but also of time, then the infinitesimal element selected near an arbitrary point of the rope will move with acceleration $\frac{\partial^2 \mathbf{x}}{\partial t^2}$. An inertial force will act on such an element, which in relation to a unit of rope length is equal to $\mu \frac{\partial^2 \mathbf{x}}{\partial t^2}$. In this case, $\mu ds$ is the mass of the element $ds$.

In accordance with the beginning of d'Alembert, the equations describing the movement of an element can be obtained by considering the conditions of its equilibrium, taking into account all forces acting on it, including the forces of inertia. As a result, the equations of the problem of the dynamics of a flexible rope will be

$$
\begin{aligned}
\frac{\partial}{\partial s} \left( T \frac{\partial \mathbf{x}}{\partial s} \right) + \mathbf{p} &= \mu \frac{\partial^2 \mathbf{x}}{\partial t^2}, \\
\left( \frac{\partial \mathbf{x}}{\partial s} \right)^T \frac{\partial \mathbf{x}}{\partial s} &= 1,
\end{aligned}
$$

$$
T \left. \frac{\partial \mathbf{x}}{\partial s} \right|_{s=l} = \mathbf{P}_t,
$$

$$
\mathbf{x}\big|_{s=0} = \mathbf{x}_0.
$$

Here

$\mathbf{p} = \{p_1, p_2, p_3\}^T$ – vector of distributed loads;

$\mathbf{x}_0 = \{x_{01}, x_{02}, x_{03}\}^T$ – specified vector coordinates of the fixed end of the ropes;

$\mathbf{P}_t = \{P_{t1}, P_{t2}, P_{t3}\}^T$ – the load on the free end of the ropes.

Equations (1) must be supplemented with the initial conditions

$$
t = 0, \quad \mathbf{x}(s, 0) = \mathbf{X}(s), \quad \mathbf{v}(s, 0) = \frac{\partial \mathbf{x}(s, 0)}{\partial t} = \mathbf{V}(s)
$$

We introduce the vector $\mathbf{r}$ of the momentum – the momentum in the form $\mathbf{r} = \mu \mathbf{v}$. Then equations (1), (2) take the form

$$
\begin{aligned}
\frac{\partial}{\partial s} \left( T \frac{\partial \mathbf{x}}{\partial s} \right) + \mathbf{p} &= \mu \frac{\partial \mathbf{x}}{\partial t}, \\
\left( \frac{\partial \mathbf{x}}{\partial s} \right)^T \frac{\partial \mathbf{x}}{\partial s} &= 1,
\end{aligned}
$$

$$
\mathbf{r} = \mu \frac{\partial \mathbf{x}}{\partial t},
$$

$$
T \left. \frac{\partial \mathbf{x}}{\partial s} \right|_{s=l} = \mathbf{P}_t,
$$

$$
\mathbf{x}\big|_{s=0} = \mathbf{x}_0,
$$

$$
t = 0, \quad \mathbf{x} = \mathbf{X}, \quad \mathbf{r} = \mu \mathbf{V} = \mathbf{r}_0, \quad \in l.
$$
3. Variational formulation

The difference between the equations of dynamics and statics consists in the presence of the vector of inertia forces \( \frac{\partial \mathbf{r}}{\partial t} \), which is included in the first equation (3) similarly to the vector of distributed loads \( \mathbf{p} \), and an additional equation that connects the vector of momentum \( \mathbf{r} \) with the vector of coordinates \( \mathbf{x} \). We give arbitrary variations \( \delta T, \delta \mathbf{r} \) and geometrically possible variation \( \delta \mathbf{x} \), then we can rewrite the variational equation [8, 9] for the problems of the dynamics of a flexible rope as follows:

\[
\frac{1}{2} \int_0^t \delta T \left[ \left( \frac{\partial \mathbf{x}}{\partial s} \right)^T \left( \frac{\partial \mathbf{x}}{\partial s} \right) - 1 \right] ds + \int_0^t \left( \frac{\partial \mathbf{x}}{\partial s} \right)^T \mathbf{T} \frac{\partial \mathbf{x}}{\partial s} ds - \int_0^t (\delta \mathbf{x})^T \left( \mathbf{p} - \frac{\partial \mathbf{r}}{\partial t} \right) ds + \frac{1}{\mu} \int_0^t \left( \mathbf{r} - \mu \frac{\partial \mathbf{x}}{\partial t} \right) ds - (\delta \mathbf{x})^T \mathbf{P} \Bigg|_{t=0} = 0
\]

The variational equation (4) differs from the variational equation of a flexible rope in statics in the form of the third and the presence of the fourth integrals. In (4), the third integral contains \( \mathbf{p} - \mathbf{r} \) instead of \( \mathbf{p} \), which is explained by the nature of the first equation in (3). The fourth integral in (4) is completely due to the third equation in (3).

It is easy to verify that if the \( T \) and vectors \( \mathbf{x}, \mathbf{r} \) satisfy the integral identity (4) for any \( \delta T, \delta \mathbf{r} \) geometrically possible variation \( \delta \mathbf{x} \). They satisfy equations and static boundary condition. In this case, the initial conditions may not be met.

4. Equations of the dynamic problem of flexible rope in convolutions

Difficulties in the formulation of the variational problem associated with the appearance of initial conditions in the dynamic theory can be overcome by using the operation of transformation of functions known in the analysis, which is called convolution. It allows one to include the initial conditions directly into the equations of motion and arrive at a boundary value problem for which variational formulations are of the usual nature.

The convolution of two vector functions \( \mathbf{a}(s,t) \) and \( \mathbf{b}(s,t), 0 \leq t < \infty \), can be written in the form

\[
\mathbf{a} * \mathbf{b} = \int_0^t (\mathbf{a}(s, t - \tau))^T \mathbf{b}(s, \tau) d\tau.
\]

When deriving the variational statements in [4, 8, 9], the basic integral formula was used in the form

\[
\int_0^t \left( \frac{d\mathbf{a}}{ds} \right)^T \mathbf{b} ds = \mathbf{a}^T \mathbf{b} \bigg|_0^t - \int_0^t \mathbf{a}^T \frac{d\mathbf{b}}{ds} ds.
\]

By the definition of convolution

\[
\int_0^t \frac{\partial \mathbf{a}}{\partial s} * \mathbf{b} ds = \int_0^t \left[ \frac{\partial \mathbf{a}}{\partial s} \mathbf{a}(t - \tau) \right]^T \mathbf{b}(\tau) d\tau ds = \int_0^t \left[ \int_0^\tau \left( \frac{\partial \mathbf{a}}{\partial s} \right)^T \mathbf{b} ds \right] d\tau.
\]

Using (7), we transform formula (6) to the form

\[
\int_0^t \frac{\partial \mathbf{a}}{\partial s} * \mathbf{b} ds = \mathbf{a} * \mathbf{b} \bigg|_0^t - \int_0^t \mathbf{a} * \frac{\partial \mathbf{b}}{\partial s} ds
\]

We introduce the function \( g(t) = t, \ 0 \leq t < \infty \) and the vector function

\[
\mathbf{f} = g * \mathbf{p} + \mu (\mathbf{iV} + \mathbf{X}), \quad \in I, \quad [0, \infty)
\]

Then the first equation in (1) can be written as
\[ g \frac{\partial}{\partial s} \left( T \frac{\partial \mathbf{x}}{\partial s} \right) + \mathbf{f} = \mu \mathbf{x}, \quad \in l, \quad [0, \infty). \] (10)

5. Variational equation in convolutions for dynamics problems

By analogy with (10), we rewrite the system of equations (1)

\[
\begin{align*}
g \frac{\partial}{\partial s} \left( T \frac{\partial \mathbf{x}}{\partial s} \right) + \mathbf{f} &= \mu \mathbf{x}, \\
g \left( \frac{\partial \mathbf{x}}{\partial s} \right)^T \frac{\partial \mathbf{x}}{\partial s} - 1 &= 0, \\
g \left( T \frac{\partial \mathbf{x}}{\partial s} \right) - \mathbf{P}_l &= 0.
\end{align*}
\]
(11)

Then we can write down a variational equation similar to (4)

\[
\frac{1}{2} \int_0^l g \left( \frac{\partial \mathbf{x}}{\partial s} \right)^T \left( \frac{\partial \mathbf{x}}{\partial s} \right) \delta T ds + \int_0^l g \left( \frac{\partial \mathbf{x}}{\partial s} \right)^T T \frac{\partial \mathbf{x}}{\partial s} + \mathbf{f} - \mu \mathbf{x} \right) \delta \mathbf{x} ds = 0
\]
(12)

6. Conclusion

The most significant difference of the obtained dynamic variational equation (4) from the variational equations for static problems is that it does not correspond to the complete system of initial equations (3). The system of equations (3) contains initial conditions, while they are not included in the dynamic variational equation. In static problems, there was a complete correspondence of the variational equations to the original equations in differential form and boundary conditions.

The variational equation in convolutions (12) differs from equation (4) in that the equations, the static boundary condition, and the initial conditions are satisfied.

7. References

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