Equivariant holonomy of U(1)-bundles

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Abstract
We define the equivariant holonomy of an invariant connection on a principal U(1)-bundle. The properties of the ordinary holonomy are generalized to the equivariant setting. In particular, equivariant U(1)-bundles with connection are shown to be classified by its equivariant holonomy modulo isomorphisms. We also show that the equivariant holonomy can be used to obtain results about equivariant prequantization and anomaly cancellation.

Mathematics Subject Classification 2010: 53C29; 53D50, 55N91, 81Q70.
Key words and phrases: equivariant holonomy, equivariant prequantization.

Acknowledgments: Supported by Ministerio de Ciencia, Innovación y Universidades of Spain under grant PGC2018-098321-B-I00.

1 Introduction
When a group G acts on a principal bundle P → M, we can consider the G-equivariant version of almost any mathematical construction. For example we have G-equivariant homology and cohomology ([15]), G-equivariant characteristic classes ([6]), G-equivariant differential cohomology ([14],[18]), etc. However, in our study of anomaly cancellation in [11] we needed to consider the G-equivariant holonomy of a G-invariant connection on a U(1)-bundle. Although there is a natural definition of this concept, we have been unable to find a detailed study of it in the literature (and we are not the only ones to face this problem, e.g. see [3, page 8]). In [11] the G-equivariant holonomy is studied in the particular case of topologically trivial U(1)-bundles over a contractible space. The reason is that this is the case that appears in the study of gravitational anomaly cancellation (see Section 4 for more details).

In the present paper we study the G-equivariant holonomy of a G-invariant connection Ξ on an arbitrary G-equivariant U(1)-bundle p: P → M. We show that the basic properties of ordinary holonomy can be extended to G-equivariant
holonomy. In particular, we prove that the $G$-equivariant holonomy classifies $G$-equivariant $U(1)$-bundles with connection modulo isomorphisms. We give an application of the equivariant holonomy to study the existence of equivariant prequantization bundles. Finally we study the relation of equivariant holonomy with the study of anomaly cancellation.

To motivate our definition of equivariant holonomy, let us consider the case of a free and proper action of a discrete group $G$ on a $U(1)$-bundle $p: P \to M$. If $\Xi$ is a $G$-invariant connection on $P \to M$, then it projects onto a connection $\Xi$ on the quotient bundle $P/G \to M/G$. We want to compute the holonomy of $\Xi$ in terms of $\Xi$. A loop $\gamma$ on $M/G$ based at $[x] \in M/G$ can be represented by a curve $\gamma: [0,1] \to M$ such that $\gamma(0) = x$ and $\gamma(1) = \phi_M(x)$ for an element $\phi \in G$. If $y \in p^{-1}(x) \subset P$, we can consider the $\Xi$-horizontal lift $\gamma^y: I \to P$ with $\gamma^y(0) = y$. The problem is that $\gamma^y(1)$ and $\phi_P(y)$ are in different fibers, and hence we cannot compare them in order to define the ordinary holonomy. However, $\gamma^y(1)$ and $\phi_P(y)$ are in the same fiber, and hence they can be related. The $\phi$-equivariant holonomy $\text{Hol}_\phi^\Xi(\gamma) \in U(1)$ of $\gamma$ is characterized by the property $\gamma^y(1) = \phi_P(y) \cdot \text{Hol}_\phi^\Xi(\gamma)$. Moreover, we prove that we have $\text{Hol}_\phi^\Xi(\gamma) = \text{Hol}_{\phi P}^\Xi(\gamma)$.

For the action of an arbitrary Lie group $G$ on $P \to M$, we define the equivariant holonomy in a similar way. As it is well known, the curvature of a connection $\text{curv}(\Xi)$ measures the infinitesimal holonomy of $\Xi$. In the equivariant case, we have the equivariant curvature $\text{curv}_G(\Xi) = \text{curv}(\Xi) + \mu^\Xi (e.g. see [6])$, where $\mu^\Xi: g \to \mathbb{R}$ is called the momentum of $\Xi$. We show that $\mu^\Xi$ measures the infinitesimal variation of the equivariant holonomy $\text{Hol}_\phi^\Xi(\gamma)$ with respect to $\phi \in G$.

The results of this paper can be generalized in several ways. For example, the equivariant holonomy can be defined for connections on principal bundles with arbitrary structural group. We left the study of the general case for a separate paper.

Another possible extension is the following. In [8] the concept of differential character is introduced as an object similar to the holonomy of a connection. In the equivariant case, we can define an equivariant differential character (of second order) as a map $\chi$ which assigns a complex number $\chi(\phi, \gamma) \in U(1)$ to a pair $(\phi, \gamma)$, with $\phi \in G$ and $\gamma: [0,1] \to M$ such that $\gamma(1) = \phi_M(\gamma(0))$. This concept of equivariant differential character has been introduced in [10] in order to classify equivariant $U(1)$-bundles modulo isomorphisms. Furthermore, it can be seen that there are important geometrical objects that appear in a natural way as equivariant differential characters not necessarily associated to an invariant connection. One example is the definition of the Chern-Simons line bundle (see [10]) and another example is Witten’s formula for global anomalies (see Section 9).
2 Holonomy on \( U(1) \)-bundles

In this section we recall some classical results about the holonomy of connections on \( U(1) \)-bundles (e.g. see [5, 17] for more details). In the rest of the paper we show how to extend these results to the equivariant setting.

The interval \([0, 1]\) is denoted by \( I \). The space of loops on \( M \) is defined by \( \mathcal{C}(M) = \{ \gamma: I \to M \mid \gamma \text{ is piecewise } C^1 \text{ and } \gamma(1) = \gamma(0) \} \), and the space of loops based at \( x \) is defined by \( \mathcal{C}_x(M) = \{ \gamma \in \mathcal{C}(M) \mid \gamma(0) = x \} \). The holonomy can be defined for connections on principal bundles with arbitrary structural group (e.g. see [16]). Let \( K \) be a Lie group, \( p: P \to M \) a principal \( K \)-bundle, and let \( \Xi \) be a connection on \( P \to M \). If \( \gamma \in \mathcal{C}_x(M) \) and \( y \in p^{-1}(x) \), we have a \( \Xi \)-horizontal lift \( \Xi^y: I \to P \) with \( \Xi^y(0) = y \). Furthermore, if \( \gamma \in \mathcal{C}_x(M) \), then we have \( p(\Xi^y(1)) = p(\Xi^y(0)) = x \) and hence there exist \( \text{Hol}^{\Xi,y}(\gamma) \in K \) such that \( \Xi^y(1) = \Xi^y(0) \cdot \text{Hol}^{\Xi,y}(\gamma) \). If \( K \) is abelian then it can be seen that \( \text{Hol}^{\Xi,y}(\gamma) \) is independent of the element \( y \in p^{-1}(x) \) chosen and it is denoted simply by \( \text{Hol}^{\Xi}(\gamma) \).

Two curves \( \gamma_1 \) and \( \gamma_2 \) are said to differ by a reparametrization if there exists an orientation preserving homeomorphism \( \varphi: I \to I \) such that \( \varphi \) and \( \varphi^{-1} \) are piecewise \( C^1 \) and \( \gamma_2 = \gamma_1 \circ \varphi \). In this case we have \( \gamma_2^y(1) = \gamma_1^y(1) \) (e.g. see [16]). Hence if \( \gamma_1, \gamma_2 \in \mathcal{C}_x(M) \) differ by a reparametrization then \( \text{Hol}^{\Xi}(\gamma_1) = \text{Hol}^{\Xi}(\gamma_2) \).

Now we consider the case \( K = U(1) \). If \( \gamma: I \to M \) is a curve, we define the inverse curve \( \Xi^y(t) = \text{Hol}^{\Xi}(\gamma)^{-1} \). Moreover, if \( \gamma_1 \) and \( \gamma_2 \) are curves with \( \gamma_1(1) = \gamma_2(0) \) we define \( \gamma_1 \ast \gamma_2: I \to \mathbb{R}, \gamma_1 \ast \gamma_2(t) = \gamma_1(2t) \) for \( t \in [0, 1/2] \) and \( \gamma_1 \ast \gamma_2(t) = \gamma_2(2t - 1) \) for \( t \in [1/2, 1] \). If \( \gamma_1, \gamma_2 \in \mathcal{C}_x(M) \) then we have \( \gamma_1 \ast \gamma_2 \in \mathcal{C}_x(M) \) and \( \text{Hol}^{\Xi}(\gamma_1 \ast \gamma_2) = \text{Hol}^{\Xi}(\gamma_1) \cdot \text{Hol}^{\Xi}(\gamma_2) \).

The connection is a form \( \Xi \in \Omega^1(P, \mathfrak{h}\mathbb{R}) \) and the curvature form \( \text{curv}(\Xi) \in \Omega^2(M) \) is defined by the property \( p^*(\text{curv}(\Xi)) = \frac{1}{2\pi} d\Xi \). As it is well known, for bundles with arbitrary group the curvature of \( \Xi \) measures the infinitesimal holonomy. For \( U(1) \)-bundles we have a more precise result that is a generalization of the classical Gauss-Bonnet Theorem

**Proposition 1** If \( \Sigma \subset M \) is a 2-dimensional submanifold with boundary \( \partial \Sigma = \bigcup_{i=1}^k \gamma_i \), with \( \gamma_i \in \mathcal{C}(M) \) then we have \( \prod_{i=1}^k \text{Hol}^{\Xi}(\gamma_i) = \exp(2\pi i \int_\Sigma \text{curv}(\Xi)) \).

We also have the following

**Proposition 2** If \( \Xi \) and \( \Xi' \) are connections on \( P \to M \), then we have \( \Xi' = \Xi - 2\pi i (p^* \rho) \) for a form \( \rho \in \Omega^1(M) \) and \( \text{Hol}^{\Xi'}(\gamma) = \text{Hol}^{\Xi}(\gamma) \cdot \exp(2\pi i \int_\gamma \rho) \) for any \( \gamma \in \mathcal{C}(M) \).

If \( p: P \to M, p': P' \to M \) are two principal \( U(1) \)-bundles, we write that \( P \simeq P' \) if there exists a \( U(1) \)-bundle isomorphism \( \Phi: P' \to P \) covering the identity map of \( M \). And \( P \to M \) is a trivial \( U(1) \)-bundle if \( P \simeq M \times U(1) \). A \( U(1) \)-bundle with connection is a pair \( (P, \Xi) \), where \( p: P \to M \) is a \( U(1) \)-bundle and \( \Xi \) is a connection on \( P \). We write that \( (P, \Xi) \simeq (P', \Xi') \) if there exists a
The holonomy can be used to classify $U(1)$-bundles modulo isomorphisms. Precisely, we recall the following classical result (e.g. see \[17\] Theorem 2.5.1)

**Theorem 3** If $(P, \Xi)$ and $(P', \Xi')$ are $U(1)$-bundles with connection over $M$, then $(P, \Xi) \simeq (P', \Xi')$ if and only if $\text{Hol}^\Xi(\gamma) = \text{Hol}^{\Xi'}(\gamma)$ for any $\gamma \in C(M)$.

A consequence of the preceding theorem and Proposition 3 is the following

**Proposition 4** Let $(P, \Xi)$ be a $U(1)$-bundle with connection over $M$. Then $P \to M$ is a trivial $U(1)$-bundle if and only if there exists a 1-form $\beta \in \Omega^1(M)$ such that $\text{Hol}^\Xi(\gamma) = \exp(2\pi i \int_\gamma \beta)$ for any $\gamma \in C(M)$.

A connection $\Xi$ is flat if $\text{curv}(\Xi) = 0$. In this case, it follows from Proposition 4 that $\text{Hol}^\Xi(\gamma)$ depends only on the homotopy class of $\gamma$. Hence the holonomy of a flat connection defines a homomorphism $\text{Hol}^\Xi: \pi_1(M) \to U(1)$.

If $\omega \in \Omega^2(M)$ is closed, then $\omega$ is prequantizable if there exist a $U(1)$-bundle with connection $(P, \Xi)$ such that $\text{curv}(\Xi) = \omega$. By a classical result of Weil and Kostant (e.g. see \[17\] Proposition 2.1.1), $\omega$ is prequantizable if and only if it is integral, i.e., if its de Rham cohomology class comes from an integral class under the natural map $H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R})$.

## 3 Equivariant holonomy

In this section we define the equivariant holonomy and the notations that are used in the rest of the paper. Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and let $M$ be a connected manifold. A $G$-equivariant $U(1)$-bundle is a principal $U(1)$-bundle $p: P \to M$ in which $G$ acts (on the left) by principal bundle automorphisms. If $\phi \in G$ and $y \in P$, we denote by $\phi_p(y)$ the action of $\phi$ on $y$. In a similar way, for $X \in \mathfrak{g}$ we denote by $X_P \in \mathfrak{X}(P)$ the corresponding vector field on $P$ defined by $X_P(x) = \left. \frac{d}{dt} \right|_{t=0} \exp(-tX)\gamma_0(x)$. For any $\phi \in G$ we define

$$
C^\phi(M) = \{ \gamma: I \to M \mid \gamma \text{ is piecewise } C^1 \text{ and } \gamma(1) = \phi_M(\gamma(0)) \},
$$

and $C^\phi_x(M) = \{ \gamma \in C^\phi(M) \mid \gamma(0) = x \}$. Note that if $e \in G$ is the identity element, then $C^\phi_x(M) = \mathcal{L}_x(M)$ is the space of loops based at $x$. If $\phi \in G$ and $\gamma \in C^\phi_x(M)$ then we define $\phi \circ \gamma \in C^{\phi \circ \gamma}(M)$ by $(\phi \circ \gamma)(t) = \phi_M(\gamma(t))$.

Let $\Xi$ be a $G$-invariant connection on a $G$-equivariant $U(1)$-bundle $P \to M$. If $\gamma \in C^\phi_x(M)$ and $y \in p^{-1}(x)$, we have a $\Xi$-horizontal lift $\tau^y: I \to P$ with $\tau^y(0) = y$. We have $p(\tau^y(1)) = p(\phi_p(y)) = \phi_M(x)$, and hence there exists $u \in U(1)$ such that $\tau^y(1) = (\phi_p(y)) \cdot u$. As $\tau^{uy}z = \tau^y \cdot z$ for $z \in U(1)$, it follows that $u$ does not depend on the $y \in p^{-1}(x)$ chosen and we denote it

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1In the definition of the fundamental vector field $X_P$ we follow the sign convention of \[15\] page 10
by $\text{Hol}_{\phi}^G(\gamma) \in U(1)$. Hence the equivariant holonomy is characterized by the property

$$\overline{\gamma^p}(1) = \phi_p(y) \cdot \text{Hol}_{\phi}^G(\gamma).$$

(1)

Note that if $\gamma \in C^\phi_x(M)$ is a loop on $M$, then $\text{Hol}_{\phi}^G(\gamma) = \text{Hol}_{\phi}^G(\gamma)$ is the ordinary holonomy of $\gamma$. Furthermore, if $\gamma, \gamma' \in C^\phi(M)$ differ by a reparametrization then we have $\text{Hol}_{\phi}^G(\gamma') = \text{Hol}_{\phi}^G(\gamma)$.

**Proposition 5** If $P \to M$ is a $G$-equivariant principal $U(1)$-bundle, and $\Xi$ is a $G$-invariant connection on $P$, then for any $\phi, \phi' \in G$ and $x \in M$ we have

a) If $\gamma \in C^\phi(M)$ then $\phi \cdot \gamma \in C^{\phi \cdot \phi'^{-1}}(M)$ and $\text{Hol}_{\phi \cdot \phi'^{-1}}(\phi \cdot \gamma) = \text{Hol}_{\phi}^G(\gamma)$.

b) If $\gamma \in C^\phi(M)$ and $\gamma' \in C^\phi(M)$, then $\gamma \ast \gamma' \in C^{\phi}(M)$ and we have

$$\text{Hol}_{\phi}^G(\gamma \ast \gamma') = \text{Hol}_{\phi}^G(\gamma) \cdot \text{Hol}_{\phi'}^G(\gamma').$$

c) If $\gamma \in C^\phi(M)$ then $\overline{\gamma} \in C^{\phi^{-1}}(M)$ and $\text{Hol}_{\phi}^G(\overline{\gamma}) = \text{Hol}_{\phi}^G(\gamma)^{-1}$.

d) If $\gamma, \gamma' \in C^\phi(M)$ then $\gamma' \ast \overline{\gamma} \in C^\phi(M)$ and $\text{Hol}_{\phi}^G(\gamma' \ast \overline{\gamma}) = \text{Hol}_{\phi}^G(\gamma') \cdot \text{Hol}_{\phi}^G(\gamma^{-1})$.

e) If $\zeta : I \to M$ is a curve on $M$ such that $\zeta(0) = \gamma(0)$ then $\zeta \ast \gamma \ast (\phi \cdot \zeta) \in C^{\phi}(M)$ and $\text{Hol}_{\phi}^G(\zeta \ast \gamma \ast (\phi \cdot \zeta)) = \text{Hol}_{\phi}^G(\gamma)$.

Let $P \to M$ be another $G$-equivariant $U(1)$-bundle with connection and $\Phi : P' \to P$ be a $G$-equivariant $U(1)$-bundle morphism with covers $\Phi : M' \to M$. The connection $\Xi' = \Phi^* \Xi$ is $G$-invariant and we have $\text{Hol}_{\phi}^G(\gamma) = \text{Hol}_{\phi}^G(\Phi \circ \gamma)$ for any $\phi \in G$ and $\gamma \in C^\phi(M)$.

**Proof.** a) For any $\gamma \in C^\phi(M)$ and $y \in p^{-1}(\gamma(0))$ by equation (1) we have

$$\overline{\gamma^p}(1) = \phi_p(y) \cdot \text{Hol}_{\phi}^G(\gamma) \text{ and } \overline{\gamma^p}(1) = (\phi \cdot \phi' \cdot \phi'^{-1})(\phi \cdot \gamma).$$

Using that $\Xi$ is $G$-invariant we obtain $\phi \cdot \gamma^p = \phi \cdot \gamma^{p}(y)$, and hence

$$\phi \cdot \gamma^{p}(y) = (\phi \cdot \gamma^p(1)) \cdot (\phi \cdot \phi') \cdot \text{Hol}_{\phi}^G(\gamma) = (\phi \cdot \phi') \cdot \phi_p(y) \cdot \text{Hol}_{\phi}^G(\gamma).$$

We conclude that we have $\text{Hol}_{\phi \cdot \phi'^{-1}}(\phi \cdot \gamma) = \text{Hol}_{\phi}^G(\gamma)$.

b) If $\gamma \in C^\phi(M)$ and $y \in p^{-1}(\gamma(0))$ then we have

$$\overline{\gamma^p}(1) = \gamma^{p}(1) = \phi_p(y) \cdot \text{Hol}_{\phi}^G(\gamma) \text{ and } \text{Hol}_{\phi}^G(\gamma) = \text{Hol}_{\phi}^G(\gamma).$$

We conclude that we have $\text{Hol}_{\phi}^G(\gamma) = \text{Hol}_{\phi}^G(\gamma)$.

c) If $y \in p^{-1}(\gamma(0))$ we define $y' = \gamma^p(1)$ and we have

$$\overline{\gamma} \ast \gamma^p(1) = \gamma \ast \gamma^p(1) = (\phi \cdot \gamma)^{p}(1) \cdot \text{Hol}_{\phi}(\gamma) = (\phi \cdot \phi') \cdot \phi_p(y) \cdot \text{Hol}_{\phi}^G(\gamma) = (\phi \cdot \phi') \cdot \text{Hol}_{\phi}^G(\gamma).$$

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f) follows easily from the properties of parallel transport. □

We recall that the equivariant holonomy $\text{Hol}_{\gamma}^G(\gamma)$ depends on the curve $\gamma$, but it also depends on $\phi \in G$. For example, if $x \in M$, $\gamma_x$ is the constant curve with value $x$ and $G_x$ is the isotropy group of $x$, then by Proposition 3 b) we have a homomorphism $\chi^G_x: G_x \to U(1)$ defined by $\chi^G_x(\phi) = \text{Hol}_{\phi}^G(\gamma_x)$.

The following example shows that the equivariant holonomy can be computed geometrically. We will return to this example later.

**Example 1:** Let $S^2 \subset \mathbb{R}^3$ be the sphere and $g$ the metric induced by the euclidean metric of $\mathbb{R}^3$. The Levi-Civita connection $\nabla$ of $g$ is a connection on the orthonormal oriented frame bundle $P \to S^2$, that has structure group $SO(2) \simeq U(1)$. The action of $G = SO(3)$ on $S^2$ lifts in a natural way to an action on $P$. By Proposition 3 a) it is enough to study the case of a rotation $\phi$ of angle $\alpha$ around the $z$ axis. We can compute the $\phi$-equivariant holonomy in geometrical terms. The curvature of $\nabla$ coincides with the Euler form of $g$ and hence we have $\text{curv}(\nabla) = \frac{1}{2} \text{vol}_g$. If $\gamma$ is a loop on $S^2$, by Proposition 3 we have $\text{Hol}_{\gamma}^G(\gamma) = \exp(i \cdot \text{Area}(D))$, where $D \subset S^2$ satisfies $\gamma = \partial D$ (it exists because $S^2$ is simply connected).

If $x$ is a point in $S^2$ and $\alpha \in \mathbb{R}/\mathbb{Z}$, we define $\sigma_{x,\alpha} \in C^0_{x,\alpha}(M)$ by $\sigma_{x,\alpha}(t) = \phi(t \alpha)(x)$. If $x$ is a point in the equator, then $\sigma_{x,\alpha}$ is a geodesic. For any $y \in p^{-1}(x)$ we have $\sigma_{x,\alpha}'(1) = (\phi_\alpha)_P(y)$, and hence $\text{Hol}_{\phi_\alpha}^G(\sigma_{x,\alpha}) = 1$. If $\gamma \in C^0_x(M)$ then $\gamma * \sigma_{x,\alpha}$ is a loop on $S^2$, and by Proposition 3 d) we have

$$\text{Hol}_{\phi_\alpha}^G(\gamma) = \text{Hol}_{\phi_\alpha}^G(\gamma * \sigma_{x,\alpha}) \cdot \text{Hol}_{\phi_\alpha}^G(\sigma_{x,\alpha}) = \exp(i \cdot \text{Area}(D)),$$

where $\gamma * \sigma_{x,\alpha} = \partial D$.

Finally, let $x$ be any point in $S^2$ and $\gamma \in C^0_x(M)$. We chose a curve $\zeta$ on $S^2$ joining $x$ and a point $x'$ in the equator. Then by Proposition 3 e) we have

$$\text{Hol}_{\phi_\alpha}^G(\gamma) = \text{Hol}_{\phi_\alpha}^G(\zeta * \gamma * (\phi_\alpha \cdot \zeta)) = \exp(i \cdot \text{Area}(D)),$$

where $\zeta * \gamma * (\phi_\alpha \cdot \zeta) * \sigma_{x',\alpha} = \partial D$.

In particular, if $x = (0,0,1)$ then for any $\alpha \in \mathbb{R}/\mathbb{Z}$ we have $\phi_\alpha \in G_x$ and $\chi^G_x: G_x \to U(1)$ is given by $\chi^G_x(\phi_\alpha) = \exp(-i\alpha)$.

### 4 Equivariant Curvature

First, we recall the definition of equivariant cohomology in the Cartan model (e.g. see [14]). Suppose that we have a left action of a connected Lie group $G$ on a manifold $M$. We denote by $\Omega^k(M)^G$ the space of $G$-invariant $k$-forms on $M$. Let $\Omega^*_G(M) = (\mathbf{S}^*(g^*) \otimes \Omega^*(M))^G$ be the space of $G$-invariant polynomials on $g$ with values in $\Omega^*(M)$, with the graduation $\text{deg}(\alpha) = 2k + r$ if $\alpha \in \mathbf{S}^k(g^*) \otimes \Omega^r(M)$. Let $D: \Omega^*_G(M) \to \Omega^{*+1}_G(M)$ be the Cartan differential, $(D\alpha)(X) = d(\alpha(X)) - i_{X_M} \alpha(X)$ for $X \in g$. On $\Omega^*_G(M)$ we have $D^2 = 0$, and the equivariant
cohomology (in the Cartan model) of \( \Omega^2(M) \) with respect of the action of \( G \) is defined as the cohomology of this complex.

Let \( \varpi \in \Omega^2_G(M) \) be a \( G \)-equivariant 2-form. Then we have \( \varpi = \omega + \mu \) where \( \omega \in \Omega^2(M) \) is \( G \)-invariant and \( \mu \in \text{Hom} (g, \Omega^0(M))^G \). We have \( D\omega = 0 \) if and only if \( d\omega = 0 \), and \( \iota_X \omega = d(\mu_X) \) for every \( X \in g \). Hence \( \mu \) is a comoment map for \( \omega \).

If \( \Xi \) is a \( G \)-equivariant connection on a principal \( U(1) \) bundle \( P \to M \) then \( \frac{i}{2\pi} D(\Xi) \) projects onto a closed \( G \)-equivariant 2-form \( \text{curv}_G(\Xi) \in \Omega^2_G(M) \) called the \( G \)-equivariant curvature of \( \Xi \). If \( X \in g \) then we have \( \text{curv}_G(\Xi)(X) = \text{curv}(\Xi) + \mu^X_\Xi \), where \( \mu^X_\Xi = -\frac{i}{2\pi} \Xi(X_P) \) is the momentum of \( \Xi \). If \( \Xi' \) is another \( G \)-invariant connection we have \( \Xi' = \Xi - 2\pi i (p^* \lambda) \) for a \( G \)-invariant \( \lambda \in \Omega^1(M)^G \). Then \( \text{curv}_G(\Xi') = \text{curv}_G(\Xi) + D\lambda \) and hence the equivariant cohomology class \([\text{curv}_G(\Xi)] \in H^2_G(M)\) does not depend on the \( G \)-invariant connection chosen.

The Maurer-Cartan form on \( U(1) \) is denoted by \( \vartheta = z^{-1} dz \), and \( \xi \in \mathfrak{x}(U(1)) \) is the \( U(1) \)-invariant vector field \( \xi(z) = iz \) such that \( \vartheta(\xi) = i \). We denote by \( \xi_P \in \mathfrak{x}(P) \) the vector field on \( P \) corresponding to \( \xi \).

For a topologically trivial bundle \( M \times U(1) \to M \) the action of \( G \) on \( M \times U(1) \) is determined by a map \( \theta: G \times M \to U(1) \) characterized by the property \( \phi_P(x, u) = (\phi(x), u \cdot \theta(x)) \). It satisfies the cocycle condition \( \theta_{\phi'}(x) = \theta_{\phi}(x) \cdot \theta_{\phi'}(\phi(x)) \). Conversely, any cocycle determines an action of \( G \) on \( M \times U(1) \) such that it is a \( G \)-equivariant \( U(1) \)-bundle. In this case the equivariant holonomy can be studied in terms of the cocycle \( \theta(\phi) \) (e.g. see [11]). For an arbitrary bundle, \( \theta_\phi \) is defined only in a local trivialization. If \( \Psi: \mathbb{R}^n \times U(1) \to P \) is a local trivialization covering a map \( \Psi: \mathbb{R}^n \to M \) we have

\[
\Psi^* \Xi = \vartheta - 2\pi i \Psi^* \rho^\Psi \quad (2)
\]

for a form \( \rho^\Psi \in \Omega^1(\text{Im} \Psi) \). Note that we have \( \text{curv}(\Xi) = d\rho^\Psi \) on \( \text{Im} \Psi \).

We denote by \( \alpha: G \times M \to M \) the map defining the action of \( G \) on \( M \), i.e., \( \alpha(\phi, x) = \phi_M(x) \). Given \( x \in M \), we can find neighborhoods \( \mathcal{U} \) and \( \mathcal{V} \) of \( x \) and \( e \) such that \( \mathcal{V} \times \mathcal{U} \subset \alpha^{-1}(\text{Im} \Psi) \). We define \( \theta^\Psi: \mathcal{V} \times \mathcal{U} \to U(1) \) by the property \( \text{pr}_2(\Psi^{-1}(\phi_P(y))) = \text{pr}_2(\Psi^{-1}(y)) \cdot \theta^\Psi(x) \) for any \( y \in p^{-1}(x) \) with \( x \in \mathcal{U} \) and \( \phi \in \mathcal{V} \), and where \( \text{pr}_2: \mathbb{R}^n \times U(1) \to U(1) \) denotes the projection. At the infinitesimal level, for any \( X \in g \) we have

\[
\Psi^{-1}_\xi(X_P) = \Psi^{-1}_\xi(X_M) + 2\pi a^\Psi_X \cdot \xi, \quad (3)
\]

where \( a^\Psi_X(x) = \frac{i}{2\pi} \frac{d}{dt}|_{t=0} \theta^\Psi_{\exp(tX)}(x) \). Using equations (2) and (3) we obtain the following

**Lemma 6**

a) The horizontal lift of \( \gamma: I \to \text{Im} \Psi \subset M \) is given by \( \overline{\gamma}^\Psi(z(0), u)(s) = \Psi(z(s), u \cdot \exp(2\pi i \int_0^s \rho^\Psi_{\gamma(t)}(\gamma(t)) dt)) \), where \( z(s) = \Psi^{-1}(\gamma(s)) \).

b) \( \mu^\Psi_X = a^\Psi_X - \rho^\Psi(X_M) \).

If \( \gamma \in C^2_\mathcal{P}(\text{Im} \Psi) \) with \( x \in \mathcal{U} \) and \( \phi \in \mathcal{V} \), then using Lemma 6 a) we obtain
By Proposition 7 the curvature of the connection measures the infinitesimal holonomy. In a similar way the second term of the equivariant curvature, the momentum $\mu^\xi_\phi$, measures the infinitesimal variation of the equivariant holonomy with respect to $\phi \in G$.

**Proposition 7** Let $\varphi: (t_0 - \varepsilon, t_0 + \varepsilon) \to G$ be a curve on $G$ with $\varphi_{t_0} = e$ and $x \in M$. If $X = \dot{\varphi}_{t_0} \in \mathfrak{g}$, then $\sigma_{x,t}(s) = (\varphi_{t_0 + s(t-t_0)})_M(x)$, and we have

$$\frac{d}{dt}|_{t=t_0} \text{Hol}_{\varphi_t}(\sigma_{x,t}) = 2\pi i \mu^\xi_\phi(x).$$

**Proof.** We chose a local trivialization $\Psi: \mathbb{R}^n \times U(1) \to P$ and neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $x$ and $e$ such that $\mathcal{V} \times \mathcal{U} \subset \alpha^{-1}(\text{Im}\, \Psi)$. For $t$ close to $t_0$ we can use equation (4) and we have

$$\text{Hol}_{\varphi_t}(\sigma_{x,t}) = \theta^\Psi_{\varphi_t}(x)^{-1} \cdot \exp(2\pi i \int_{\sigma_{x,t}} \rho^\Psi).$$

Furthermore, we have

$$\int_{\sigma_{x,t}} \rho^\Psi = \int_{t_0}^1 \rho^\Psi(\dot{\varphi}_{t_0 + s(t-t_0)})_M(x)(t-t_0)ds = \int_{t_0}^1 \rho^\Psi(\varphi_{t_0 + s(t-t_0)})_M(x)(\varphi_{t_0 + s(t-t_0)}(x))ds.$$

By taking the derivative and using equation (4) we obtain

$$\frac{d}{dt}|_{t=t_0} \text{Hol}_{\varphi_t}(\sigma_{x,t}) = \frac{d}{dt}|_{t=t_0} \theta^\Psi_{\varphi_t}(x)^{-1} + \frac{d}{dt}|_{t=t_0} \exp(2\pi i \int_{\sigma_{x,t}} \rho^\Psi)$$

$$= 2\pi i a^\Psi_\phi(x) - 2\pi \rho^\Psi(X_M(x)) = 2\pi i \mu^\xi_\phi(x).$$

**Proposition 8** For any $X \in \mathfrak{g}$ and $x \in M$ we define $\tau_{x,X}(s) = \exp(sX)_M(x)$. Then $\tau_{x,X} \in C_x^{\text{exp}(X)}(M)$ and we have $\text{Hol}_{\exp(X)}(\tau_{x,X}) = \exp(2\pi i \mu^\xi_\phi(x)).$

**Proof.** We define $\sigma_t(s) = \exp(stX)_M \cdot x$ and we have $\sigma_t \in C_x^{\text{exp}(tx)}(M)$. By Proposition 7 we have $\frac{d}{dt}|_{t=0} \text{Hol}_{\exp(tx)}(\sigma_t) = 2\pi i \mu^\xi_\phi(x)$.

The curves $\sigma_{t+s}$ and $\sigma_t \cdot \exp(tx) \cdot \sigma_s$ differ by a reparametrization, and by Proposition 5 a) and b) we conclude that $\text{Hol}_{\exp((t+s)X)}(\sigma_{t+s}) = \text{Hol}_{\exp(tx)}(\sigma_t) \cdot \text{Hol}_{\exp(sX)}(\sigma_s)$.

By taking the derivative we obtain

$$\frac{d}{dt}|_{t=0} \text{Hol}_{\exp(tx)}(\sigma_t) = \frac{d}{dt}|_{s=0} \text{Hol}_{\exp((t+s)X)}(\sigma_{t+s})$$

$$= \text{Hol}_{\exp(tx)}(\sigma_t) \cdot \frac{d}{ds}|_{s=0} \text{Hol}_{\exp(sX)}(\sigma_s)$$

$$= 2\pi i \mu^\xi_\phi(x) \cdot \text{Hol}_{\exp(tx)}(\sigma_t).$$
The result follows by solving the differential equation and using that \( \tau_{x,\xi} = \sigma_1 \) and that \( \text{Hol}^\Xi_{\exp(b,\xi)}(\sigma_0) = 1 \).

**Example 1 (continuation):** We consider again example 1. We recall that the symplectic form \( \text{vol}_g \in \Omega^2(S^2)^{SO(3)} \) admits a canonical comoment map. For example it can be obtained by considering \( S^2 \) as a coadjoint orbit of \( SO(3) \) (e.g. see [1, §4.6.1]). The map \( \tilde{v}: \mathfrak{so}(3) \rightarrow \mathbb{R}^3 \) that assigns to \( X = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} \in \mathfrak{so}(3) \) the vector \( \tilde{v}_X = (c, -b, a) \) determines a Lie algebra isomorphism between \( (\mathfrak{so}(3), \left[ , \right]) \) and \( (\mathbb{R}^3, \times) \). Note that \( X \in \mathfrak{so}(3) \) is an infinitesimal generator of rotations around the axis determined by \( \tilde{v}_X \). We define \( h: \mathfrak{so}(3) \rightarrow \Omega^0(S^2) \) by \( h_X(x) = \langle \tilde{v}_X, x \rangle \) for \( x \in S^2 \) and \( h \) is a comoment map for \( \text{vol}_g \). Furthermore, the difference of two comoment maps for \( \text{vol}_g \) determines an element of \( H^1(\mathfrak{so}(3)) = 0 \), and hence the comoment map is unique. We conclude from this result that we have \( \mu^\Xi = \frac{1}{2\pi} h \) and \( \text{curv}^\Xi_{SO(3)} = \frac{1}{2\pi} \text{vol}_g + \frac{1}{2\pi} h \).

Let \( X \in \mathfrak{so}(3) \) be the infinitesimal generator of rotations \( \phi_\alpha \) around the \( z \)-axis, i.e.,

\[
X = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]

By Proposition 8 we have \( \text{Hol}^\Xi_{\phi_\alpha}(\tau_{x,\alpha X}) = \exp(i\alpha h(X)) \), a result that can be easily seen to coincide with the previous result using the formula for the area of a sector of the sphere.

## 5 Equivariant holonomy and quotient

In this section we assume that the actions of \( G \) on \( P \) and \( M \) are free, and that \( P \stackrel{\varrho_P}{\rightarrow} P/G \) and \( M \stackrel{\varrho_M}{\rightarrow} M/G \) are (left) principal \( G \)-bundles. This happens for example if the action is free and proper (e.g. see [1, page 264]). Then we have a well defined quotient bundle \( P/G \rightarrow M/G \), and the diagram

\[
P \xrightarrow{\varrho_P} P/G \\
\downarrow \\
M \xrightarrow{\varrho_M} M/G
\]

Let \( \Xi \) be a \( G \)-invariant connection on \( P \rightarrow M \). Then \( \Xi \) is \( \varrho_G \)-projectable onto a connection \( \underline{\Xi} \) on \( P/G \rightarrow M/G \) if and only if it is \( G \)-basic, and this is equivalent to \( \mu^\Xi = 0 \). The next proposition shows that in this case the holonomy of the connection \( \underline{\Xi} \) can be computed in terms of the \( G \)-equivariant holonomy of \( \Xi \).

**Proposition 9** If \( \mu^\Xi = 0 \) then \( \Xi \) projects onto a connection \( \underline{\Xi} \in \Omega^1(P/G, i\mathbb{R}) \). If \( \gamma \in C^\infty(M) \) then \( \underline{\gamma} = \varrho_G \circ \gamma \) is a loop on \( M/G \) and we have \( \text{Hol}_{\underline{\phi}}^\underline{\Xi}(\gamma) = \text{Hol}^\Xi_{\phi}(\gamma) \).
**Proof.** It follows from the fact that for any \( y \in p^{-1}(\gamma(0)) \) the curve \( \hat{q}_G(\gamma' y) \) is a \( \Xi \)-horizontal lift of \( \gamma \). ■

We note that the case commented in the Introduction of a proper action of a discrete group \( G \), is a particular case of the preceding proposition.

In the case in which \( \mu^\Xi \neq 0 \), the connection \( \Xi \) is not projectable to the quotient bundle. However, it is possible to obtain a connection on the quotient using a connection \( \Theta \) on the bundle \( M \rtimes^q G/M \). In more detail, we have the following result (see [7]).

**Proposition 10** If \( \Xi \) is a \( G \)-invariant connection on \( p: P \to M \) and \( \Theta \) a connection on the (left) \( G \)-principal bundle \( q_G: M \to M/G \), then we define the \( \mathbb{R} \)-valued 1-form \( \Xi(\Theta)(\xi) = \Xi((\Theta(p_\ast \xi))_p), \xi \in TP \). Then \( \Xi - \Xi(\Theta) \) is projectable to \( P/G \) and the projection is a connection form \( \Xi_\Theta \) on \( P/G \to M/G \).

The following result computes the holonomy of \( \Xi_\Theta \) in terms of the \( G \)-equivariant holonomy of \( \Xi \) and the holonomy of \( \Theta \).

**Proposition 11** If \( \gamma \) is a loop on \( M/G \) and \( \hat{\gamma} \) is a \( \Theta \)-horizontal lift of \( \gamma \), then \( \hat{\gamma} \in C^{\text{Hol}^\Theta,\gamma(0)}(\gamma)(M) \) and \( \text{Hol}^{\Xi_\Theta}(\gamma) = \text{Hol}^{\Xi,\gamma(0)}(\gamma)(\hat{\gamma}) \).

**Proof.** By the definition of the holonomy of \( \Theta \) we have \( \hat{\gamma}(1) = \hat{\gamma}(0) \cdot \text{Hol}^{\Theta,\gamma(0)}(\gamma) \), and hence \( \hat{\gamma} \in C^{\text{Hol}^\Theta,\gamma(0)}(\gamma)(M) \). For any \( y \in p^{-1}(\hat{\gamma}(0)) \) we have \( \hat{\gamma}'(1) = \hat{\gamma}'(0) \cdot \text{Hol}^{\Xi_\Theta,\gamma(0)}(\gamma)(\hat{\gamma}) \).

The curve \( \hat{q}_G \circ \gamma' \) is a \( \Xi_\Theta \)-horizontal lift of \( \gamma \) and \( \hat{q}_G \circ \gamma'(1) = (\hat{q}_G \circ \gamma'(0)) \cdot \text{Hol}^{\Xi_\Theta,\gamma(0)}(\gamma)(\hat{\gamma}) \). Hence \( \text{Hol}^{\Xi_\Theta}(\gamma) = \text{Hol}^{\Xi,\gamma(0)}(\gamma)(\hat{\gamma}) \). ■

6 G-Flat connections

A \( G \)-equivariant connection \( \Xi \) is \( G \)-flat if \( \text{curv}_G(\Xi) = 0 \), i.e., if \( \text{curv}(\Xi) = 0 \) and \( \mu^\Xi = 0 \). As commented in Section 2 for flat connections the holonomy determines a homomorphism \( \pi_1(M) \to U(1) \). For \( G \)-flat connections we have a similar result, but with the \( G \)-equivariant fundamental group in place of \( \pi_1(M) \).

We define \( C^G_x(M) = \bigcup_{\phi \in G} C^G_{\phi}(M) \). The \( G \)-equivariant fundamental group can be defined by \( \pi_{1,G}(M)_x = C^G_x(M)/\sim_G \) where \( \sim_G \) is the equivalence relation \( \sim \) if there exist a curve \( \phi: I \to G \), and a continuous map \( h: I \times I \to \mathbb{R}, (t,s) \mapsto h_t(s) \) such that \( h_t \in C^G_{\phi}(M) \) for any \( t \in I \) and \( \varphi_0 = \phi, \varphi_1 = \phi', h_0 = \gamma, h_1 = \gamma' \). We define the product in \( \pi_{1,G}(M)_x \) by \([\phi, \gamma] \ast [\phi', \gamma'] = [(\phi \cdot \phi'), \gamma \ast (\phi \cdot \gamma')] \). As usual, the groups \( \pi_{1,G}(M)_x \) for different points \( x \) are isomorphic, and hence we suppress the point \( x \) in the notation.

**Lemma 12** If \( \mu^\Xi = 0 \) then for any curve \( \varphi: I \to G \) and \( x \in M \) we have \( \text{Hol}^\Xi(\varphi) = 1 \), where \( \varphi = \varphi_1 \cdot \varphi_0^{-1} \) and \( \gamma(s) = (\varphi_s)_M(x) \).

**Proof.** We define \( \gamma_t(s) = \gamma(st) \) and \( k(t) = \text{Hol}_{\varphi_t, \varphi_0^{-1}}^\Xi(\gamma_t) \). The result follows if we prove that \( \frac{dk}{dt} = 0 \). We fix \( t_0 \in I \) and we define \( \gamma_{t_0,t}(s) = \gamma(t_0 + s(t - t_0)) \).
The curves $\gamma_t$ and $\gamma_{t_0} \ast \gamma_{t_0,t}$ differ by a reparametrization and hence $k(t) = \text{Hol}_\Xi^{\varphi_t \varphi_0^{-1}}(\gamma_{t_0} \ast \gamma_{t_0,t}) = \text{Hol}_\Xi^{\varphi_t \varphi_0^{-1}}(\gamma_{t_0}) \cdot \text{Hol}_\Xi^{\varphi_t \varphi_0^{-1}}(\gamma_{t_0,t})$. Furthermore we have $\frac{dk}{dt}(t_0) = \frac{d}{dt} \big|_{t=t_0} k(t_0 + t) = \text{Hol}_\Xi^{\varphi_t \varphi_0^{-1}}(\gamma_{t_0}) \cdot \frac{d}{dt} \big|_{t=t_0} \text{Hol}_\Xi^{\varphi_t \varphi_0^{-1}}(\gamma_{t_0,t})$.

Using Proposition 5 a) we obtain $\text{Hol}_\Xi^{\varphi_t \varphi_0^{-1}}(\gamma_{t_0,t}) = \text{Hol}_\Xi^{\varphi_0^{-1}}(\varphi_t \varphi_0^{-1} \cdot \gamma_{t_0,t})$. The result follows by applying Proposition 7 to the curve $\varphi_t \varphi_0^{-1} \cdot \gamma$.

**Proposition 13** Let $\Xi$ be a $G$-flat connection on $P \to M$. If $(\phi, \gamma) \sim_G (\phi', \gamma')$ then $\text{Hol}_\Xi^{\phi}(\gamma) = \text{Hol}_\Xi^{\phi'}(\gamma')$. Hence the $G$-equivariant holonomy induces a group homomorphism $\text{Hol}_\Xi^G : \pi_1(G) \to U(1)$.

**Proof.** Let $\varphi : I \to G$ and $h : I \times I \to \mathbb{R}, (t,s) \mapsto h(t,s)$ such that $h_t \in C^G_c(M)$ for any $t \in I$ and $\varphi_0 = \phi, \varphi'_1 = \phi', h_0 = \gamma, h_1 = \gamma'$. We define $\sigma(s) = (\varphi_s)_M(x)$ and we have $\sigma \in C^0(M)$ and $\partial h = \gamma \ast * \gamma'$. As $\Xi$ is flat we have $\text{Hol}_\Xi^{\phi}(\gamma \ast * \gamma') = \exp(2\pi i \int_{M} h(\text{curv}(\Xi))) = 1$. But by Proposition 5 we also have $\text{Hol}_\Xi^{\phi}(\gamma \ast * \gamma') = \text{Hol}_\Xi^{\phi}(\gamma) \cdot \text{Hol}_\Xi^{\phi'(\gamma')} - 1$. The result follows because $\text{Hol}_\Xi^{\phi \ast * \gamma^{-1}}(\sigma) = 1$ by Lemma 12.

The projection $(\phi, \gamma) \mapsto \phi$ induces an epimorphisms $\pi_{1,G}(M) \longrightarrow \pi_0(G)$. In a similar way, the inclusion map $C_\ast(M) \to C^0_G(M)$ induces a homomorphism $\pi_1(M) \to \pi_{1,G}(M)$. It can be seen that we have an exact sequence

$$\pi_1(M) \to \pi_{1,G}(M) \to \pi_0(G) \to 1.$$

**Remark 14** It is possible to give another interpretation of this result in terms of the Borel model of equivariant cohomology. If $EG \to BG$ is a universal bundle for $G$ then we define the homotopy quotient $M_G = (M \times EG)/G$. Then $p_1^*\Xi$ is a $G$-flat connection on $P \times EG \to M \times EG$ and it projects onto a flat connection $\Xi_G$ on $P_G \to M_G$. As $\Xi_G$ is flat, its holonomy defines a homomorphism $\pi_1(M_G) \to U(1)$ that corresponds to the one defined in Proposition 13. It can be seen that we have isomorphisms $\pi_{1,G}(M) \simeq \pi_{1,G}(M \times EG) \simeq \pi_1(M_G)$. Furthermore, the homotopy exact sequence induces the following exact sequence $\pi_1(G) \to \pi_1(M) \to \pi_1(M_G) \to \pi_0(G) \to 1$. We prefer to work with $\pi_{1,G}(M)$ in place of $\pi_1(M)$ because it is defined in terms of curves on $M$ and $G$, and hence it can be related directly with the equivariant holonomy.

### 7 Classification of equivariant $U(1)$-bundles by their equivariant holonomy

In this section we obtain the equivariant versions of the results of Section 2. If $p : P \to M, p' : P' \to M$ are two $G$-equivariant $U(1)$-bundles then we write that $P \simeq_G P'$ if there exists a $G$-equivariant $U(1)$-bundle isomorphism $\Phi : P' \to P$.

We recall that it is equivalent to define the fundamental group $\pi_1(M)$ by using continuous or differentiable curves (e.g. see [3] 17.8.1).
covering the identity map of $M$. We say that $P$ is a trivial $G$-equivariant $U(1)$-bundle if $P \simeq_G M \times U(1)$ for an action of $G$ on $M$ and where $G$ acts trivially on $U(1)$. A $G$-equivariant $U(1)$-bundle with connection is a pair $(P, \Xi)$, where $\rho: P \to M$ is a $G$-equivariant $U(1)$-bundle and $\Xi$ is a $G$-invariant connection on $P$. We write that $(P, \Xi) \simeq_G (P', \Xi')$ if there exists a $G$-equivariant $U(1)$-bundle isomorphism $\Phi: P' \to P$ covering the identity map of $M$ such that $\Phi^* \Xi = \Xi'$.

**Theorem 15** If $(P, \Xi)$ and $(P', \Xi')$ are $G$-equivariant $U(1)$-bundles with connection over $M$, then $(P, \Xi) \simeq_G (P', \Xi')$ if and only if $\text{Hol}^\Xi_\phi(\gamma) = \text{Hol}^{\Xi'}_\phi(\gamma)$ for any $\phi \in G$, and $\gamma \in C^0(M)$.

**Proof.** That equivalent bundles have the same equivariant holonomy follows from Proposition 5 f). We prove the converse. If $\text{Hol}^\Xi_\phi(\gamma) = \text{Hol}^{\Xi'}_\phi(\gamma)$ for any $\phi \in G$, and $\gamma \in C^0(M)$, in particular we have $\text{Hol}^\Xi(\gamma) = \text{Hol}^{\Xi'}(\gamma)$ for any $\gamma \in C^0(M) = C(M)$ and by Theorem 8 there exists a a $U(1)$-bundle isomorphism $\Phi: P \to P'$ covering the identity map of $M$ such that $\Phi^* (\Xi') = \Xi$. We prove that $\Phi$ is $G$-equivariant. If $\Phi(y)$ is the $\Xi$-horizontal lift of $\gamma \in C^0(M)$ starting at $y$, then $\Phi \circ \Phi(y)$ coincides with the $\Xi'$-horizontal lift $\Phi(y)$ of $\gamma$. By applying equation 1 we obtain

$$\Phi(\phi_P(y)) = \Phi(\Phi(y)) \cdot \text{Hol}^\Xi_\gamma(\gamma)^{-1} = \Phi(\Phi(y)) \cdot \text{Hol}^{\Xi'}_\gamma(\gamma)^{-1} = \Phi(y).$$

**Theorem 17** Let $(P, \Xi)$ be a $G$-equivariant $U(1)$-bundle with connection over $M$. Then $P \to M$ is a trivial $G$-equivariant $U(1)$-bundle if and only if there exists a $G$-invariant 1-form $\beta \in \Omega^1(M)^G$ such that $\text{Hol}^\Xi_\phi(\gamma) = \int_\gamma \beta$ for any $\phi \in G$ and $\gamma \in C^0(M)$.

**Proof.** If $P \to M$ is a trivial $G$-equivariant $U(1)$-bundle, we can choose a global trivialization $\Psi: M \times U(1) \to P$ with $\theta^\Psi(x) = 1$ and hence $\text{Hol}^\Xi_\phi(\gamma) = \exp(2\pi i \int_\gamma \rho^\Psi)$ and we can take $\beta = \rho^\Psi$.

Conversely, if $\text{Hol}^\Xi_\phi(\gamma) = \exp(2\pi i \int_\gamma \beta)$, then we can define a new $G$-invariant connection $\Xi' = \Xi + 2\pi i \beta$ and by Proposition 16 we have $\text{Hol}^{\Xi'}_\phi(\gamma) = 1 = \text{Hol}^\Xi_\phi(\gamma)$ for any $\phi \in G$ and $\gamma \in C^0(M)$, and by Theorem 15 we have $(P, \Xi') \simeq (P \times U(1), \theta)$. In particular $P \simeq_G M \times U(1)$. ■
8 Equivariant prequantization and equivariant holonomy

A $D$-closed equivariant 2-form $\varpi = \omega + \mu \in \Omega^2_G(M)$ is $G$-equivariant prequantizable if there exist a $G$-equivariant $U(1)$-bundle with connection $(P, \Xi)$ such that curv$_G(\Xi) = \varpi$. By Weil-Kostant theorem (e.g. see [17]) a necessary condition is that $\omega$ should be integral, but it is known that there could be additional obstructions (e.g. see [20]). Necessary conditions for equivariant prequantizability follow from Proposition 8. If $(P, \Xi)$ is a $G$-equivariant prequantization of $\varpi$ and $X \in g$ satisfies $\exp(X) = e$, then by Proposition 8 we have $\exp(2\pi i \mu_Y(x)) = \text{Hol}_\Xi(\tau_{x,Y})$ for any $x \in M$, where $\tau_{x,Y}(t) = \exp(tX)_M(x)$ is the curve defined in Proposition 8. If this condition is not satisfied, then $\varpi$ is not $G$-equivariant prequantizable. We apply this condition to Example 1.

Example 1 (continuation): We consider again the case of $S^2 \subset \mathbb{R}^3$ and the action of $G = SO(3)$. We have seen that $\varpi = \frac{1}{2\pi} \text{vol}_g + \frac{1}{2\pi} h \in \Omega^2_{SO(3)}(S^2)$ is $SO(3)$-equivariant prequantizable. The form $\frac{1}{2\pi} \text{vol}_g$ is integral, and hence it is prequantizable. However, we have the following result

Proposition 18 The form $\varpi' = \frac{1}{2\pi} \text{vol}_g + \frac{1}{2\pi} h \in \Omega^2_{SO(3)}(S^2)$ is not $SO(3)$-equivariant prequantizable.

Proof. We consider the point $x = (0, 0, 1)$, and the vector $Y = -2\pi X \in so(3)$, where $X$ is the infinitesimal generator of rotations around the $z$-axis of equation (5). If $\varpi' = \omega' + \mu'$, we have $\exp(Y) = e, \mu'_Y(x) = \frac{1}{2\pi} h_Y(x) = \frac{1}{2}$ and the curve $\tau_{x,Y}$ is a constant curve with value $x$. Hence if $(P, \Xi)$ is any $U(1)$-bundle with connection such that curv$(\Xi) = \omega'$ we have $1 = \text{Hol}_\Xi(\tau_{x,Y}) \neq \exp(2\pi i \mu'_Y(x)) = \exp(\pi i) = -1$. ■

9 Anomalies and equivariant holonomy

In this section we study the application of the results of the present paper to the study of anomalies in quantum field theory. As commented in the Introduction this was our original motivation to study equivariant holonomy. Let $\mathcal{M}_M$ be the space of Riemannian metrics on $M$ and $\mathcal{D}_M$ the group of orientation preserving diffeomorphisms. Anomalies appear when a classical symmetry of a theory is broken at the quantum level. This happens for example in theories with chiral Dirac operators, where the path integral $Z \in \Omega^0(\mathcal{M}_M)$ (defined as a regularized determinant) fails to be $\mathcal{D}_M$-invariant. In this case, for $g \in \mathcal{M}_M$ and $\phi \in \mathcal{D}_M$ we have $Z(\phi(g)) = Z(g) \cdot \theta(\phi, g)$, where $\theta: \mathcal{D}_M \times \mathcal{M}_M \rightarrow U(1)$ satisfies the cocycle condition. Hence $\theta$ defines a $\mathcal{D}_M$-equivariant $U(1)$-bundle $P \rightarrow \mathcal{M}_M$. This bundle is called the anomaly bundle and admits a $\mathcal{D}_M$-invariant connection $\Xi$ (e.g. see [13]). If the gravitational anomaly cancels, the anomaly bundle admits a $\mathcal{D}_M$-invariant section (e.g. see [11]) and $P \rightarrow \mathcal{M}_M$ is a trivial $\mathcal{D}_M$-equivariant $U(1)$-bundle. A topological obstruction for anomaly cancellation
can be obtained by considering the quotient bundle $P/\mathcal{D}_M \rightarrow \mathcal{M}_M/\mathcal{D}_M$. If this quotient bundle is non-trivial, then the anomaly cannot be cancelled. Hence the Chern class of $P/\mathcal{D}_M$ represents an obstruction for anomaly cancellation.

This allows us to interpret the gravitational anomaly as a cohomology class on $\mathcal{M}_M/\mathcal{D}_M$. Furthermore, the principal $\mathcal{D}_M$-bundle $\mathcal{M}_M \rightarrow \mathcal{M}_M/\mathcal{D}_M$ admits a natural connection $\Theta$. By Proposition 13, the connections $\Xi$ and $\Theta$ determine a connection $\Xi_\Theta$ on $P/\mathcal{D}_M \rightarrow \mathcal{M}_M/\mathcal{D}_M$. The curvature of $\Xi_\Theta$ can be computed using the Atiyah-Singer theorem. Moreover, in [21] Witten introduces a formula that measures the variation of the path integral $Z$ along a curve $\gamma: I \rightarrow \mathcal{M}_M$ such that $\gamma(1) = \phi(\gamma(0))$, i.e., he defines a map $w: C^0(\mathcal{M}_M) \rightarrow U(1)$. Later Witten’s formula was interpreted (e.g. see [13]) as a computation of the holonomy of the connection $\Xi_\Theta$. By Proposition 11 $w$ can also be considered as a computation of the $\mathcal{D}_M$-equivariant holonomy $\Xi$, and this result is applied in [12]. Furthermore, Theorem 17 provides necessary and sufficient conditions for anomaly cancellation.

If the equivariant curvature of $\Xi$ vanishes (this happens for example if $\dim M \neq 2 \mod 4$), then $\Xi$ is a $\mathcal{D}_M$-flat connection and by the results of Section 6 the equivariant holonomy of $\Xi$ defines a homomorphism $w: \pi_{1,G}(\mathcal{M}_M) \rightarrow U(1)$. As $\mathcal{M}_M$ is simply connected we conclude from the exact sequence (6) that $\pi_{1,G}(\mathcal{M}_M) \simeq \pi_0(\mathcal{D}_M)$ is the mapping class group of $M$. Hence we obtain a homomorphism $w: \pi_0(\mathcal{D}_M) \rightarrow U(1)$, that in quantum field theory is called a global gravitational anomaly.

The preceding geometrical interpretation of anomalies is insufficient from the physical point of view due to the problem of locality (e.g. see [2], [11]). In quantum field theory, the gravitational anomalies can be cancelled only using local terms, i.e. terms obtained by integration over $M$ of forms depending on the metric and its derivatives. Geometrically this implies that to cancel gravitational anomalies the existence of an especial type of $\mathcal{D}_M$-equivariant section of the anomaly bundle $P \rightarrow \mathcal{M}_M$ is necessary (see [11]). The principal problem in the study of locality in anomaly cancellation is that the connection $\Theta$ contains non-local terms and hence it is difficult to deal with the locality problem using the quotient bundle $P/\mathcal{D}_M \rightarrow \mathcal{M}_M/\mathcal{D}_M$ and the connection $\Xi_\Theta$. However, the $\mathcal{D}_M$-equivariant curvature and holonomy of $\Xi$ have local expressions (see [9], [11], [12]), and Theorem 17 can be extended to characterize gravitational anomaly cancellation in a way compatible with locality (see [12, Proposition 20]).

Finally we note that in the case of gravitational anomalies the space of fields $\mathcal{M}_M$ is contractible and in this case the study of equivariant holonomy can be simplified (e.g. see [11]). However, if we consider other theories (for example sigma models or string theory), the space of fields can be not contractible. In those cases, the study of anomaly cancellation requires the equivariant holonomy of connections on arbitrary bundles that we consider in this paper.

Footnote 3: In order to have a well defined quotient manifold it is necessary to restrict the group $\mathcal{D}_M$ to a subgroup acting freely on $\mathcal{M}_M$, but we omit this point here. Furthermore, one of the advantages of working with equivariant holonomy is that this restriction is unnecessary.
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