NON-EXTENSIBILITY OF THE PAIR \{1, 3\} TO A
DIOPHANTINE QUINTUPLE IN \( \mathbb{Z}[\sqrt{-d}] \)

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Abstract. We show that the Diophantine pair \{1, 3\} can not be
extended to a Diophantine quintuple in the ring \( \mathbb{Z}[\sqrt{-2}] \). This
result completes the work of the first author and establishes non-
extensibility of the Diophantine pair \{1, 3\} to a Diophantine quin-
tuple in \( \mathbb{Z}[\sqrt{-d}] \) for all \( d \in \mathbb{N} \).

1. Introduction and results

Let \( R \) be a commutative ring with unity 1. The set \( \{a_1, a_2, \ldots, a_m\} \)
in \( R \) such that \( a_i \neq 0 \) for all \( i = 1, \ldots, m \), \( a_i \neq a_j \) and \( a_ia_j + 1 \) is a
square in \( R \) for all \( 1 \leq i < j \leq m \), is called a Diophantine \( m \)-tuple
in \( R \). The problem of constructing such sets was first studied by Diophantus
of Alexandria who found a set of four rationals \( \{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\} \) with the
given property. Fermat found a first Diophantine quadruple in integers - the set \( \{1, 3, 8, 120\} \). A Diophantine pair \( \{a, b\} \) in a ring \( R \), which
satisfies \( ab + 1 = r^2 \), can be extended to a Diophantine quadruple
in \( R \) by adding elements \( a + b + 2r \) and \( 4r(r + a)(r + b) \), provided
all four elements are nonzero and different. Hence, apart from some
exceptional cases, Diophantine quadruples in a ring \( R \) exist, but can
we obtain Diophantine \( m \)-tuples of size greater than 4?

The folklore conjecture is that there are no Diophantine quintuples
in integers. In 1969, Baker and Davenport [1] showed that the set \( \{1, 3, 8\} \) can not be extended to a Diophantine quintuple, which was
the first result supporting the conjecture. This result was first general-
ized by Dujella [4], who showed that the set \( \{k - 1, k + 1, 4k\} \),
with integer \( k \geq 2 \), can not be extended to a Diophantine quintuple in
\( \mathbb{Z} \). Dujella and Pethő [8] later showed that not even the Diophantine

2010 Mathematics Subject Classification. 11D09, 11R11.
pair $\{1, 3\}$ can be extended to a Diophantine quintuple in $\mathbb{Z}$. Greatest step towards proving the conjecture did Dujella [6] in 2004; he showed that there are no Diophantine sextuples in $\mathbb{Z}$ and that there are only finitely many Diophantine quintuples. In [7] it was proved that there are no Diophantine quintuples in the ring of polynomials with integers coefficients under assumption that not all elements are constant polynomials.

The size of Diophantine $m$-tuples can be greater than 4 in some rings. For instance, the set

$$\left\{\frac{11}{192}, \frac{35}{192}, \frac{155}{27}, \frac{512}{27}, \frac{1235}{16}, \frac{180873}{16}\right\}$$

is a Diophantine sextuple in $\mathbb{Q}$; it was found by Gibbs [10]. Furthermore, we can construct Diophantine quintuples in the ring $\mathbb{Z}[[\sqrt{d}]]$ for some values of $d$; for instance $\{1, 3, 8, 120, 1678\}$ is a Diophantine quintuple in $\mathbb{Z}[[\sqrt{201361}]]$. It is natural to start investigating the upper bound for the size of Diophantine $m$-tuples in $\mathbb{Z}[[\sqrt{d}]]$ by focusing on a problem of extensibility of Diophantine triples $\{k-1, k+1, 4k\}$ and Diophantine pair $\{1, 3\}$ to a Diophantine quintuple in $\mathbb{Z}[[\sqrt{d}]]$, since the problem in integers was approached similarly, see [8] and [4].

In [9] Franušić proved that the Diophantine pair $\{1, 3\}$ can not be extended to a Diophantine quintuple in $\mathbb{Z}[-\sqrt{d}]$ if $d$ is a positive integer and $d \neq 2$. The case $d = 2$ was also considered and it was shown that if $\{1, 3, c\}$ is a Diophantine triple in $\mathbb{Z}[\sqrt{-2}]$, then $c \in \{c_k, d_l\}$, where the sequences $(c_k)$ and $(d_l)$ are given by

$$(1) \quad c_k = \frac{1}{6}(2 + \sqrt{3})(7 + 4\sqrt{3})^k + (2 - \sqrt{3})(7 - 4\sqrt{3})^k - 4),$$

$$(2) \quad d_l = \frac{-1}{6}(7 + 4\sqrt{3})^l + (7 - 4\sqrt{3})^l + 4),$$

where $k \geq 1$ and $l \geq 0$. Sequences $(c_k)$ and $(d_l)$ are defined recursively as follows

$$(3) \quad c_0 = 0, \quad c_1 = 8, \quad c_{k+2} = 14c_{k+1} - c_k + 6;$$

$$(4) \quad d_0 = -1, \quad d_1 = -3, \quad d_{l+2} = 14d_{l+1} - d_l + 8.$$
It is known that \( \{1, 3, c_k, c_{k+1}\} \), with \( k \geq 1 \), is a Diophantine quadruple in \( \mathbb{Z} \), see [8], and hence also in \( \mathbb{Z}[\sqrt{-2}] \). The set \( \{1, 3, d_l, d_{l+1}\} \) is a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \) since

\[
d_l d_{l+1} + 1 = (c_l + 2)^2
d\tag{5}
\]

for every \( l \geq 0 \); this easily follows from identities (1) and (2). The set \( \{1, 3, c_k, d_l\} \) is not a Diophantine quadruple for \( k \geq 1 \) and \( l \geq 0 \) since \( 1 + c_k d_l \) is a negative odd number and hence it can not be a square in \( \mathbb{Z}[\sqrt{-2}] \). Therefore, if there is an extension of the Diophantine pair \( \{1, 3\} \) to a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \), then it is of the form \( \{1, 3, c_k, c_l\} \), with \( l > k \geq 1 \) or \( \{1, 3, d_k, d_l\} \), with \( l > k \geq 0 \). In the former case, the set can not be extended to a Diophantine quintuple in \( \mathbb{Z} \), see [8], wherefrom it easily follows that it can not be extended to a Diophantine quintuple in \( \mathbb{Z}[\sqrt{-2}] \). It remains to examine the latter case. We can formulate the following theorem.

**Theorem 1.1.** Let \( k \) be a nonnegative integer and \( d \) an integer. If the set \( \{1, 3, d_k, d_l\} \) is a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \), where \( d_k \) is given by (2), then \( d = d_{k-1} \) or \( d = d_{k+1} \).

From Theorem 1.1 we immediately obtain the following corollary.

**Corollary 1.2.** The Diophantine pair \( \{1, 3\} \) can not be extended to a Diophantine quintuple in \( \mathbb{Z}[\sqrt{-2}] \).

The organization of the paper is as follows. In Section 2 assuming \( k \) to be minimal integer for which Theorem 1.1 does not hold, we translate the assumption of Theorem 1.1 into system of Pellian equations from which recurrent sequences \( \nu_m^{(i)} \) and \( \omega_n^{(j)} \) are deduced, intersections of which give solutions to the system. In Section 3 we use a congruence method introduced by Dujella and Pethö [8] to determine the fundamental solutions of Pellian equations. In Section 4 we give a lower bound for \( m \) and \( n \) for which the sequences \( \nu_m^{(i)} \) and \( \omega_n^{(j)} \) intersect. In Section 5 we use a theorem of Bennett [3] to establish an upper bound for \( k \). Remaining cases are examined separately in Section 6 using linear forms in logarithms, Baker-Wüstholz theorem [2] and the Baker-Davenport method of reduction [1].
2. The system of Pellian equations

Let \( \{1, 3, d_k, d\} \) be a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \) where \( k \) is the minimal integer for which Theorem 1.1 does not hold. Assume \( k \geq 6 \). Clearly \( d = d_l \) for some \( l \geq 0 \). Since \( d + 1 \) and \( 3d + 1 \) are negative integers and \( d_kd + 1 \) is a positive integer, it follows that there exist \( x, y, z \in \mathbb{Z} \) such that

\[
\begin{align*}
    d + 1 &= -2x^2, \\
    3d + 1 &= -2y^2, \\
    d_kd + 1 &= z^2.
\end{align*}
\]

The system of equations (6) is equivalent to the following system of Pellian equations

\[
\begin{align*}
    z^2 + 2d_kx^2 &= 1 - d_k \\
    3z^2 + 2dky^2 &= 3 - d_k
\end{align*}
\]

where

\[
\begin{align*}
    d_k + 1 &= -2s_k^2, \\
    3d_k + 1 &= -2t_k^2,
\end{align*}
\]

for some \( s_k, t_k \in \mathbb{Z} \). Note that we may assume \( s_k, t_k \in \mathbb{N} \). Conditions (9) follow from the fact that \( \{1, 3, d_k\} \) is a Diophantine triple in \( \mathbb{Z}[\sqrt{-2}] \) and the fact that \( d_k + 1 \) and \( 3d_k + 1 \) are negative integers.

The following propositions describe the set of positive integer solutions of equations (7) and (8).

**Proposition 2.1.** There exist \( i_0 \in \mathbb{N} \) and \( z_0^{(i)}, x_0^{(i)} \in \mathbb{Z}, i = 1, 2, \ldots, i_0 \), such that \( (z_0^{(i)}, x_0^{(i)}) \) are solutions of the equation (7), which satisfy

\[
1 \leq z_0^{(i)} \leq \sqrt{-d_k(1-d_k)}, \quad 1 \leq |x_0^{(i)}| \leq \sqrt{\frac{1-d_k^2}{2d_k}},
\]

and such that for every solution \( (z, x) \in \mathbb{N} \times \mathbb{N} \) of the equation (7), there exists \( i \in \{1, 2, \ldots, i_0\} \) and an integer \( m \geq 0 \) such that

\[
z + x\sqrt{-2d_k} = \left(z_0^{(i)} + x_0^{(i)}\sqrt{-2d_k}\right) \left(-2d_k - 1 + 2s_k\sqrt{-2d_k}\right)^m.
\]

**Proof.** The fundamental solution of the related Pell’s equation \( z^2 + 2d_kx^2 = 1 \) is \( -2d_k - 1 + 2s_k\sqrt{-2d_k} \) since

\[
(-2d_k - 1)^2 + 2d_k \cdot (2s_k)^2 = 4d_k^2 + 4d_k + 1 - 4d_k(1 + d_k) = 1
\]
and $-2d_k - 1 > 2s_k^2 - 1 = -d_k - 2$, see \[11\] Theorem 105. Following arguments of Nagell \[11\] Theorem 108 we obtain that there are finitely many integer solutions \( (z_0^{(i)}, x_0^{(i)}) \), \( i = 1, 2, \ldots, i_0 \) of the equation (7) such that the following inequalities hold

\[
1 \leq \left| z_0^{(i)} \right| \leq \sqrt{-d_k(1 - d_k)}, \quad 0 \leq \left| x_0^{(i)} \right| \leq \sqrt{\frac{1 - d_k^2}{2d_k}},
\]

and such that if \( z + x\sqrt{-2d_k} \) is a solution of the equation (7) with \( z \) and \( x \) in \( \mathbb{Z} \), then

\[
z + x\sqrt{-2d_k} = \left( z_0^{(i)} + x_0^{(i)} \sqrt{-2d_k} \right) \left( -2d_k - 1 + 2s_k\sqrt{-2d_k} \right)^m
\]

for some \( m \in \mathbb{Z} \) and \( i \in \{1, 2, \ldots, i_0\} \). Hence

\[
z_0^{(i)} + x_0^{(i)} \sqrt{-2d_k} = \left( z + x\sqrt{-2d_k} \right) \left( -2d_k - 1 + 2s_k\sqrt{-2d_k} \right)^{-m},
\]

wherefrom it can be easily deduced that if \( z + x\sqrt{-2d_k} \) is a solution of the equation (7) with \( z \) and \( x \) in \( \mathbb{N} \), then \( z_0^{(i)} > 0 \). Hence

\[
1 \leq z_0^{(i)} \leq \sqrt{-d_k(1 - d_k)}
\]

for all \( i \in \{1, 2, \ldots, i_0\} \). If \( x_0^{(i)} = 0 \), we get a contradiction with the upper bound for \( z_0^{(i)} \), hence \( \left| x_0^{(i)} \right| \geq 1 \). To complete the proof it remains to show that \( m \geq 0 \). Assume to the contrary that \( m < 0 \). Then

\[
\left( -2d_k - 1 + 2s_k\sqrt{-2d_k} \right)^m = \alpha - \beta\sqrt{-2d_k}
\]

with \( \alpha, \beta \in \mathbb{N} \) and \( \alpha^2 + 2d_k\beta^2 = 1 \). Since

\[
z + x\sqrt{-2d_k} = \left( z_0^{(i)} + x_0^{(i)} \sqrt{-2d_k} \right) \left( \alpha - \beta\sqrt{-2d_k} \right),
\]

we have \( x = -z_0^{(i)}\beta + x_0^{(i)}\alpha \). By squaring \( x_0^{(i)}\alpha = x + z_0^{(i)}\beta \) and substituting \( \alpha^2 = 1 - 2d_k\beta^2 \) we get

\[
\left( x_0^{(i)} \right)^2 = \beta^2(1 - d_k) + x^2 + 2xz_0^{(i)}\beta > \beta^2(1 - d_k) \geq 1 - d_k > \frac{1 - d_k^2}{2d_k},
\]

since \( x, z_0^{(i)}, \beta \) and \( k \) are positive integers. This is in contradiction with the upper bound for \( x_0^{(i)} \).

Using same arguments we can prove the following proposition. \( \square \)
**Proposition 2.2.** There exists \( j_0 \in \mathbb{N} \) and \( z^{(j)}_1, y^{(j)}_1 \in \mathbb{Z}, j = 1, 2, \ldots, j_0 \), such that \( (z^{(j)}_1, y^{(j)}_1) \) are solutions of the equation (8), which satisfy

\[
1 \leq z^{(j)}_1 \leq \sqrt{-d_k (3 - d_k)}, \quad 1 \leq |y^{(j)}_1| \leq \sqrt{\frac{(3 - d_k)(1 + 3d_k)}{2d_k}},
\]

and such that for every solution \((z, y) \in \mathbb{N} \times \mathbb{N}\) of the equation (8), there exists \( j \in \{1, 2, \ldots, j_0\} \) and an integer \( n \geq 0 \) such that

\[
z \sqrt{3} + y \sqrt{-2d_k} = \left( z^{(j)}_1 \sqrt{3} + y^{(j)}_1 \sqrt{-2d_k} \right) \left( -6d_k - 1 + 2t_k \sqrt{-6d_k} \right)^n.
\]

□

Finitely many solutions that satisfy bounds given in Proposition 2.1 and Proposition 2.2 will be called fundamental solutions.

From Proposition 2.1 and Proposition 2.2 it follows that if \((z, x)\) is a solution in positive integers of the equation (7), then \( z = \nu^{(i)}_m \) for some \( m \geq 0 \) and \( i \in \{1, 2, \ldots, i_0\} \), where

\[
\nu^{(i)}_0 = z^{(i)}_0,
\]

\[
\nu^{(i)}_1 = (-2d_k - 1)z^{(i)}_0 - 4s_k d_k x^{(i)}_0,
\]

\[
\nu^{(i)}_{m+2} = (-4d_k - 2)\nu^{(i)}_{m+1} - \nu^{(i)}_m,
\]

and if \((z, y)\) is a solution in positive integers of the equation (8), then \( z = \omega^{(j)}_n \) for some \( n \geq 0 \) and \( j \in \{1, 2, \ldots, j_0\} \), where

\[
\omega^{(j)}_0 = z^{(j)}_1,
\]

\[
\omega^{(j)}_1 = (-6d_k - 1)z^{(j)}_1 - 4t_k d_k y^{(j)}_1,
\]

\[
\omega^{(j)}_{n+2} = (-12d_k - 2)\omega^{(j)}_{n+1} - \omega^{(j)}_n.
\]

Therefore, we are looking for the intersection of sequences \( \nu^{(i)}_m \) and \( \omega^{(j)}_n \).

3. **Congruence method**

Using the congruence method introduced by Dujella and Pethő [8] we determine the fundamental solutions of the equations (7) and (8).

**Lemma 3.1.**

\[
\nu^{(i)}_{2m} \equiv z^{(i)}_0 \pmod{2d_k}, \quad \nu^{(i)}_{2m+1} \equiv -z^{(i)}_0 \pmod{2d_k},
\]
\[ \omega_{2m}^{(j)} \equiv z_1^{(j)} \pmod{2d_k}, \quad \omega_{2m+1}^{(j)} \equiv -z_1^{(j)} \pmod{2d_k}, \]

for all \( m, n \geq 0, \ i \in \{1, 2, \ldots, i_0\}, \ j \in \{1, 2, \ldots, j_0\} \).

**Proof.** Easily follows by induction. \( \square \)

**Lemma 3.2.** If \( \nu_m^{(i)} = \omega_n^{(j)} \) for some \( m, n \geq 0, \ i \in \{1, 2, \ldots, i_0\}, \ j \in \{1, 2, \ldots, j_0\} \), then \( z_0^{(i)} = z_1^{(j)} \) or \( z_0^{(i)} + z_1^{(j)} = -2d_k \).

**Proof.** From Lemma 3.1 it follows that either \( z_0^{(i)} \equiv z_1^{(j)} \pmod{2d_k} \) or \( z_0^{(i)} \equiv -z_1^{(j)} \pmod{2d_k} \). In the latter case \( z_0^{(i)} + z_1^{(j)} \equiv 0 \pmod{2d_k} \). From Proposition 2.1 and Proposition 2.2 we get

\[
0 < z_0^{(i)} + z_1^{(j)} \leq \sqrt{-d_k(1 - d_k)} + \sqrt{-d_k(3 - d_k)}
\]

\[
< -d_k + 1 - d_k + 2 = -2d_k + 3,
\]

wherefrom it follows that \( z_0^{(i)} + z_1^{(j)} = -2d_k \). If \( z_0^{(i)} \equiv z_1^{(j)} \pmod{2d_k} \) and \( z_0^{(i)} > z_1^{(j)} \), then

\[
0 < z_0^{(i)} - z_1^{(j)} < z_0^{(i)} \leq \sqrt{-d_k(1 - d_k)} < -2d_k,
\]

contradiction. Analogously, if \( z_1^{(j)} > z_0^{(i)} \), then

\[
0 < z_1^{(j)} - z_0^{(i)} < z_1^{(j)} \leq \sqrt{-d_k(3 - d_k)} < -2d_k,
\]

contradiction. \( \square \)

**Lemma 3.3.**

\[
\nu_m^{(i)} \equiv (-1)^m \left( z_0^{(i)} + 2d_k m^2 z_0^{(i)} + 4d_k s_k mx_0^{(i)} \right) \pmod{8d_k^2}
\]

\[
\omega_n^{(j)} \equiv (-1)^n \left( z_1^{(j)} + 6d_k n^2 z_1^{(j)} + 4d_k t_k n y_1^{(j)} \right) \pmod{8d_k^2}
\]

for all \( m, n \geq 0, \ i \in \{1, 2, \ldots, i_0\}, \ j \in \{1, 2, \ldots, j_0\} \).

**Proof.** Easily follows by induction. \( \square \)

**Lemma 3.4.** If \( \nu_m^{(i)} = \omega_n^{(j)} \) for some \( m, n \geq 0, \ i \in \{1, 2, \ldots, i_0\}, \ j \in \{1, 2, \ldots, j_0\} \), then \( m \equiv n \pmod{2} \).

**Proof.** If \( m \) is even and \( n \) odd, then Lemma 3.1 and Lemma 3.2 imply \( z_0^{(i)} + z_1^{(j)} = -2d_k \). Lemma 3.3 implies

\[
z_0^{(i)} + 2d_k m^2 z_0^{(i)} + 4d_k s_k mx_0^{(i)} \equiv -z_1^{(j)} - 6d_k n^2 z_1^{(j)} - 4d_k t_k n y_1^{(j)} \pmod{8d_k^2},
\]
wherefrom, by substituting \( z_0^{(i)} + z_1^{(j)} = -2d_k \) and dividing by \( 2d_k \), we obtain

\[
-1 + m^2 z_0^{(i)} + 2s_k mx_0^{(i)} \equiv -3n^2 z_1^{(j)} - 2t_k ny_1^{(j)} \pmod{4d_k}.
\]

Since \( d_k \) is always odd, from (7) and (8) we get that \( z_0^{(i)} \) and \( z_1^{(j)} \) are even, hence the last congruence cannot hold. Indeed, on the left side is an odd integer and on the right side is an even integer, contradiction.

If \( m \) is odd and \( n \) even, contradiction can be obtained analogously. □

Therefore, the equations \( \nu_{2m}^{(i)} = \omega_{2n+1}^{(j)} \) and \( \nu_{2m+1}^{(i)} = \omega_{2n}^{(j)} \) have no solutions in integers \( m, n \geq 0, i \in \{1, 2, \ldots, i_0\}, j \in \{1, 2, \ldots, j_0\} \). It remains to examine the cases when \( m \) and \( n \) are both even or both odd. In each of those cases we have \( z_0^{(i)} = z_1^{(j)} \). Since

\[
\left( z_0^{(i)} \right)^2 - 1 = d_k \left( -2 \left( x_0^{(i)} \right)^2 - 1 \right),
\]

it follows that

\[
\delta := \frac{\left( z_0^{(i)} \right)^2 - 1}{d_k}
\]

is an integer. Furthermore,

\[
\delta + 1 = -2 \left( x_0^{(i)} \right)^2, \quad 3\delta + 1 = -2 \left( y_1^{(j)} \right)^2, \quad \delta d_k + 1 = \left( z_0^{(i)} \right)^2.
\]

Thus \( \delta \) satisfies system (6) and hence \( \delta = d_l \) for some \( l \geq 0 \). Moreover, \( \{1, 3, d_k, d_l\} \) is a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \) since \( d_l \neq d_k \).

Indeed, if \( d_l = d_k \) then

\[
d_k^2 + 1 = \left( z_0^{(i)} \right)^2,
\]

contradiction with \( d_k^2 \equiv 1 \pmod{4} \). In what follows we show that \( l = k - 1 \). Assume \( \delta > d_{k-1} \), that is \( l < k - 1 \). Then the triple \( \{1, 3, d_l\} \) can be extended to a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \) by \( d_k \), which differs from \( d_{l-1} \) and \( d_{l+1} \) since \( l - 1 < l + 1 < k \) by assumption; this contradicts the minimality of \( k \). Therefore \( l \geq k - 1 \). On the other hand, since

\[
\delta d_k + 1 = \left( z_0^{(i)} \right)^2 \leq -d_k(-d_k + 1),
\]
from Proposition [2.1] it follows that \( \delta = d_l > d_k - 1 \) and hence \( l \leq k \). Since \( d_l \neq d_k \) we have \( d_l = d_{k-1} \). Hence

\[
(z_0^{(i)})^2 = d_k d_{k-1} + 1.
\]

From (5) we obtain \( z_0^{(i)} = z_0 = c_{k-1} + 2 \). Furthermore, from (7), (8) and (9) we get \( |x_0^{(i)}| = s_{k-1} \) and \( |y_1^{(j)}| = t_{k-1} \). Moreover, from

\[
s_k = \frac{1}{2\sqrt{3}} \left( (2 + \sqrt{3})^k - (2 - \sqrt{3})^k \right),
\]

\[
t_k = \frac{1}{2} \left( (2 + \sqrt{3})^k + (2 - \sqrt{3})^k \right),
\]

we get

\[
(14) \quad 2s_k s_{k-1} = c_{k-1}, \quad 2t_k t_{k-1} = 3c_{k-1} + 4.
\]

This brings us to the important conclusion. If the system of Pellian equations (7) and (8) has a solution in positive integers, where \( k \) is the smallest integer for which Theorem 1.1 does not hold and under assumption \( k \geq 6 \), the fundamental solutions of Pellian equations (7) and (8) are \((z_0, x_0^{\pm})\) and \((z_1, y_1^{\pm})\) respectively, where

\[
(15) \quad z_0 = z_1 = 2(s_k s_{k-1} + 1),
\]

\[
(16) \quad x_0^{\pm} = \pm s_{k-1}, \quad y_1^{\pm} = \pm t_{k-1}.
\]

4. THE LOWER BOUND FOR m AND n

After plugging (15) and (16) into (III) and (III) and expanding we get

\[
\nu^\pm_m = \frac{1}{2} \left( 2(s_k s_{k-1} + 1) \pm s_{k-1} \sqrt{-2d_k} \right) \left( -2d_k - 1 + 2s_k \sqrt{-2d_k} \right)^m
\]

\[
+ \frac{1}{2} \left( 2(s_k s_{k-1} + 1) \mp s_{k-1} \sqrt{-2d_k} \right) \left( -2d_k - 1 - 2s_k \sqrt{-2d_k} \right)^m,
\]

and

\[
\omega^\pm_n = \frac{1}{2\sqrt{3}} \left( 2(s_k s_{k-1} + 1) \sqrt{3} \mp t_{k-1} \sqrt{-2d_k} \right) \left( -6d_k - 1 + 2t_k \sqrt{-6d_k} \right)^n
\]

\[
+ \frac{1}{2\sqrt{3}} \left( 2(s_k s_{k-1} + 1) \sqrt{3} \pm t_{k-1} \sqrt{-2d_k} \right) \left( -6d_k - 1 - 2t_k \sqrt{-6d_k} \right)^n,
\]
for \( m, n \geq 0 \). One intersection of these sequences is clearly

\[
\nu_0^\pm = \omega_0^\pm = 2(s_k s_{k-1} + 1),
\]

wherefrom it follows that the triple \( \{1, 3, d_k\} \) can be extended to a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \) by \( d_{k-1} \). Another intersection is \( \nu_1^- = \omega_1^- \). Indeed, (14) implies

\[
(17) \quad s_k s_{k-1} + 1 = \frac{1}{3}(t_k t_{k-1} + 1)
\]

and hence

\[
\omega_1^- = -2 - 12d_k - 2s_k s_{k-1} - 12d_k s_k s_{k-1} + 4d_k t_k t_{k-1} = -2 - 4d_k - 2s_k s_{k-1} = \nu_1^-.
\]

Therefrom it follows that the triple \( \{1, 3, d_k\} \) can be extended to a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \) by \( d_{k+1} \). Using (17) we can write \( \omega_n^\pm \) as follows

\[
\omega_n^\pm = \frac{1}{6} \left( 2(t_k t_{k-1} + 1) \pm t_{k-1} \sqrt{-6d_k} \right) \left( -6d_k - 1 + 2t_k \sqrt{-6d_k} \right)^n
\]

\[
+ \frac{1}{6} \left( 2(t_k t_{k-1} + 1) \mp t_{k-1} \sqrt{-6d_k} \right) \left( -6d_k - 1 - 2t_k \sqrt{-6d_k} \right)^n.
\]

Since

\[
2(s_k s_{k-1} + 1) - s_{k-1} \sqrt{-2d_k} = 2 - \frac{\sqrt{-2d_k - 2}}{\sqrt{-2d_k} - \sqrt{-2d_k}} + \sqrt{-2d_k} > 2 - \frac{\sqrt{-2d_k - 2}}{\sqrt{-2d_k} - \sqrt{-2d_k}} > 1,
\]

it follows that

\[
\nu_m^+ \geq \nu_m^- > \frac{1}{2} \left( -2d_k - 1 + 2s_k \sqrt{-2d_k} \right)^m.
\]

Furthermore,

\[
\omega_n^- \leq \omega_n^+ < \frac{1}{2} \left( -6d_k - 1 + 2t_k \sqrt{-6d_k} \right)^{n+1},
\]

since

\[
2(t_k t_{k-1} + 1) - t_{k-1} \sqrt{-6d_k} < \left( -6d_k - 1 - 2t_k \sqrt{-6d_k} \right)^{n+1}.
\]
and
\[
\frac{1}{3} \left(2(t_k t_{k-1} + 1) + t_{k-1} \sqrt{-6d_k + 1}\right) < -6d_k - 1 + 2t_k \sqrt{-6d_k},
\]
which can be easily verified using (9). Therefore, if one of the equations
\(\nu_m^\pm = \omega_n^\pm\) has solutions, then
\[
\frac{1}{2} \left(-2d_k - 1 + 2s_k \sqrt{-2d_k}\right)^m < \frac{1}{2} \left(-6d_k - 1 + 2t_k \sqrt{-6d_k}\right)^{n+1},
\]
wherefrom
\[
\frac{m}{n+1} < \frac{\log (-6d_k - 1 + 2t_k \sqrt{-6d_k})}{\log (-2d_k - 1 + 2s_k \sqrt{-2d_k})}.
\]
The expression on the right side of the inequality decreases when \(k\) increases. Since \(k \geq 6\) it follows that
\[
\frac{m}{n+1} < 1.072.
\]
We may assume \(n \geq 2\). Indeed for \(n = 1\) we have \(m \leq 2\) and since \(m\) and \(n\) are both even or both odd it follows that the only possibility is \(m = 1\). We have already established the intersection \(\nu_1^- = \omega_1^-\) and it can be easily verified that \(\nu_1^+ \neq \omega_1^+\) and \(\nu_1^- \neq \omega_1^+\). Now it can be easily deduced that \(m < n\sqrt{3}\). Hence, if the sequences \((\nu_m^\pm)\) and \((\omega_n^\pm)\) have any intersections besides two already established ones, then \(n \geq 2\), \(m\) and \(n\) are of the same parity and \(m < n\sqrt{3}\). We further on assume these conditions.

**Proposition 4.1.** Let \(n \geq 2\). If one of the equations \(\nu_m^\pm = \omega_n^\pm\) has solutions then
\[
m \geq n \geq \frac{2}{3} \sqrt[4]{-d_k}.
\]

**Proof.** If \(m < n\), then \(m \leq n - 2\), since \(m\) and \(n\) are of the same parity. From (10) and (11) using (14) one easily finds \(\nu_0^+ < \omega_2^-\). It can be shown by induction that \(\nu_m^+ < \omega_{m+2}^-\) for \(m \geq 0\). Indeed, sequences \((\nu_m^\pm)\) and \((\omega_n^\pm)\) are strictly increasing positive sequences, which can be easily checked by induction after plugging (15) and (16) into (10) and (11). Hence
\[
\nu_{m+1}^+ < (-4d_k - 2)\nu_m^+, \quad \omega_{m+3}^- > (-12d_k - 3)\omega_{m+2}^-.
\]
Then clearly $\nu_m^+ < \omega_{m+2}$ implies $\nu_{m+1}^+ < \omega_{m+3}^-$, which completes the proof by induction. Since
\[ \nu_m^- \leq \nu_m^+ < \omega_{m+2}^- \leq \omega_{m+2}^+, \]
it follows that if one of the equations $\nu_m^+ = \omega_n^+$ has solutions, then $m + 2 > n$, a contradiction. Hence $m \geq n$. For the second part of the statement assume to the contrary that $n < \frac{2}{3} \sqrt{-d_k}$. Let us show how we can reach a contradiction in the case $\nu_m^+ = \omega_n^+$. Other three case can be similarly resolved.

Since $m$ and $n$ are of the same parity, Lemma 3.3 implies that if $\nu_m^+ = \omega_n^+$, then
\[ (c_{k-1} + 2)(m^2 - 3n^2 + m - 3n) \equiv 2(m - n) \pmod{-4d_k}, \]and since (5) implies $(c_{k-1} + 2)^2 \equiv 1 \pmod{d_k}$, we obtain
\[ (m^2 - 3n^2 + m - 3n)^2 \equiv 4(m - n)^2 \pmod{d_k}. \]Moreover
\[ (m^2 - 3n^2 + m - 3n)^2 \equiv 4(m - n)^2 \pmod{4d_k} \]since $(4, d_k) = 1$ and both sides of the congruence relation are divisible by 4, since $m$ and $n$ are of the same parity. Under assumption $n < \frac{2}{3} \sqrt{-d_k}$ one easily sees that the expressions on both sides of the congruence relation (19) are strictly smaller than $-4d_k$. Indeed,
\[ 0 \leq 2(m - n) \leq 2n \left( \sqrt{3} - 1 \right) < 2 \left( \sqrt{3} - 1 \right) \frac{2}{3} \sqrt{-d_k} < \sqrt{-4d_k} \]
and
\[ 0 < -m^2 + 3n^2 - m + 3n \leq 2n^2 + 2n \leq 3n^2 < \frac{12}{9} \sqrt{-d_k} < \sqrt{-4d_k}. \]Therefore $-m^2 + 3n^2 - m + 3n = 2(m - n)$, wherefrom clearly $m \neq n$, so $m > n$. From (18) we obtain
\[ -(c_{k-1} + 2) \cdot 2(m - n) \equiv 2(m - n) \pmod{4d_k}, \]wherefrom
\[ -2s_k s_{k-1}(m - n) \equiv 3(m - n) \pmod{2d_k}. \]
Since (9) implies $-2s_k^2 \equiv 1 \pmod{d_k}$, by multiplying both sides of the previous equation by $s_k$ we obtain

$$s_{k-1}(m-n) \equiv 3s_k(m-n) \pmod{d_k},$$

and since $2 \mid m-n$ and $(d_k, 2) = 1$, it follows that

(20) \quad \quad (m-n)(3s_k - s_{k-1}) \equiv 0 \pmod{2d_k}.

On the other hand, from

$$0 < m - n < n \left( \sqrt{3} - 1 \right) < \left( \sqrt{3} - 1 \right) \frac{2}{3} \sqrt{-d_k} < 0.49 \cdot \sqrt{-d_k}$$

and

$$0 < 3s_k - s_{k-1} \leq 3s_k = 3 \cdot \frac{-d_k - 1}{2} < 3 \cdot \frac{-d_k}{2}$$

it follows that

$$0 < (m - n)(3s_k - s_{k-1}) < 1.04 \cdot 4\sqrt{-d_k} < -2d_k.$$

Therefore, we have a contradiction with (20). Completely analogously a contradiction can be obtained in other three cases, i.e. when $\nu_m^+ = \omega_n^-$, $\nu_m^- = \omega_n^+$ and $\nu_m^- = \omega_n^-$. □

5. APPLICATION OF BENNETT’S THEOREM

Lemma 5.1. Let

$$\theta_1 = \sqrt{1 + \frac{1}{d_k}}, \quad \theta_2 = \sqrt{1 + \frac{1}{3d_k}}$$

and let $(x, y, z)$ be a solution in positive integers of the system of Pellian equations (7) and (8). Then

$$\max \{ \left| \theta_1 - \frac{6s_kx}{3z} \right|, \left| \theta_2 - \frac{2t_ky}{3z} \right| \} < (1 - d_k)z^{-2}.$$

Proof. Clearly $\theta_1 = \frac{2s_k}{\sqrt{-2d_k}}$ and $\theta_2 = \frac{2t_k}{\sqrt{-6d_k}}$. Hence,

$$\left| \theta_1 - \frac{6s_kx}{3z} \right| = \left| \frac{2s_k}{\sqrt{-2d_k}} - \frac{2s_kx}{z} \right| = 2s_k \left| \frac{z - x\sqrt{-2d_k}}{\sqrt{-2d_k}} \right|$$

$$= \frac{2s_k}{z\sqrt{-2d_k}} \cdot \frac{1 - d_k}{z + x\sqrt{-2d_k}} < \frac{2s_k(1 - d_k)}{\sqrt{-2d_k}} \cdot z^{-2}$$

$$< (1 - d_k) \cdot z^{-2}. $$
and

\[
\left| \theta_2 - \frac{2t_k y}{3z} \right| = \left| \frac{2t_k}{\sqrt{6d_k}} - \frac{2t_k y}{3z} \right| = \left| \frac{z\sqrt{3} - y\sqrt{-2d_k}}{z\sqrt{-2d_k} \sqrt{3}} \right|
\]

\[
= \frac{2t_k}{3z\sqrt{-2d_k}} \cdot \frac{3 - d_k}{z\sqrt{3} + y\sqrt{-2d_k}} < \frac{2t_k(3 - d_k)}{3\sqrt{-6d_k}} \cdot z^{-2} < \frac{3 - d_k}{3} \cdot z^{-2} < (1 - d_k) \cdot z^{-2}.
\]

\[\Box\]

In order to establish the lower bound for the expression in Lemma 5.1 we use the following result of Bennett [3] on simultaneous rational approximations of square roots of rationals which are close to 1.

**Theorem 5.2.** If \(a_i, p_i, q, N\) are integers for \(0 \leq i \leq 2\) with \(a_0 < a_1 < a_2, a_j = 0\) for some \(0 \leq j \leq 2, q\) nonzero and \(N > M^9\) where

\[
M = \max\{|a_i| : 0 \leq i \leq 2\},
\]

then we have

\[
\max_{0 \leq i \leq 2} \left\{ \left| \sqrt{1 + \frac{a_i}{N}} - \frac{p_i}{q} \right| \right\} > (130N\gamma)^{-1}q^{-\lambda}
\]

where

\[
\lambda = 1 + \frac{\log(33N\gamma)}{\log \left( 1.7N^2 \prod_{0 \leq i < j \leq 2} (a_i - a_j)^{-2} \right)}
\]

and

\[
\gamma = \begin{cases} 
\frac{(a_2 - a_0)^2(a_2 - a_1)^2}{(a_2 - a_0)^2(a_1 - a_0)^2}, & a_2 - a_1 \geq a_1 - a_0 \\
\frac{2a_2 - 4a_0 - a_1}{a_1 + a_2 - 2a_0}, & a_2 - a_1 < a_1 - a_0.
\end{cases}
\]

We can apply Theorem 5.2 with

\[
N = -3d_k, \quad a_0 = -3, \quad a_1 = -1, \quad a_2 = 0,
\]

\[
M = 3, \quad q = 3z, \quad p_1 = 6s_k x, \quad p_2 = 2t_k y,
\]

since \(N = -3d_k > 3^9\) for \(k \geq 6\). So,

\[
\max \left\{ \left| \theta_1 - \frac{6s_k x}{3z} \right|, \left| \theta_2 - \frac{2t_k y}{3z} \right| \right\} > (130 \cdot (-3d_k)\gamma)^{-1} \cdot (3z)^{-\lambda},
\]
where
\[ \gamma = \frac{36}{5}, \quad \lambda = 1 + \frac{\log (-99d_k \cdot \frac{36}{5})}{\log (1.7 \cdot 9d_k^2 \cdot \frac{1}{36})}. \]

From Lemma 5.1 we get
\[ z^{-\lambda + 2} < (1 - d_k) \left( 130 \cdot (-3d_k) \cdot \frac{36}{5} \right) \cdot 3^\lambda. \]

Since \( \lambda < 2 \) and \( -d_k(1 - d_k) < 1.000000821d_k^2 \) for \( k \geq 6 \), it follows that \( z^{-\lambda + 2} < 25272.03d_k^2 \) and hence
\[ (-\lambda + 2) \log z < \log (25272.03d_k^2). \]

Since
\[ \frac{1}{2 - \lambda} = \frac{1}{1 - \frac{\log (-99d_k \cdot \frac{36}{5})}{\log (1.7 \cdot 9d_k^2 \cdot \frac{1}{36})}} \leq \frac{\log (0.425d_k^2)}{\log(-0.00059d_k)} \]
we have
\[ \log z < \frac{\log (25272.03d_k^2) \log (0.425d_k^2)}{\log(-0.00059d_k)}. \]

Furthermore, since \( z = \nu_m \) for some \( m \geq 0 \), it follows that
\[ z > \frac{1}{2} \left( -2d_k - 1 + 2s_k \sqrt{-2d_k} \right)^m. \]

Since \( 2s_k \sqrt{-2d_k} > -2d_k - 2 \) for \( k \geq 0 \) it follows that
\[ z > \frac{1}{2} \left( -4d_k - 3 \right)^m. \]

From \( (-4d_k - 3)^{-1} < \frac{1}{2} \) for \( k \geq 1 \), we get \( z > (-4d_k - 3)^{m-1} \). Therefore,
\[ \log z > (m - 1) \log(-4d_k - 3), \]
and since \( m \geq n \geq \frac{2}{3} \cdot \sqrt{-d_k} \), it follows that \( m - 1 > 0.5 \cdot \sqrt{-d_k} \) and hence
\[ \log z > 0.5 \cdot \sqrt{-d_k} \cdot \log(-4d_k - 3). \]

Using (21) we obtain
\[ 4 \sqrt{-d_k} < \frac{\log (25272.03d_k^2) \log (0.425d_k^2)}{0.5 \cdot \log(-0.00059d_k) \log(-4d_k - 3)}. \]
The expression on the right side of the inequality decreases when \( k \) increases, and hence by substituting \( k = 6 \) we obtain

\[ 4\sqrt{-d_k} < 20.477 \]

and finally

\[ -d_k < 175817. \]

This implies \( k \leq 5 \), which contradicts the assumption \( k \geq 6 \). Therefore, the minimal integer \( k \) for which Theorem 1.1 does not hold, if such exists, is smaller than 6.

6. Small cases

To complete the proof it remains to show that Theorem 1.1 holds also for \( 0 \leq k \leq 5 \). In each case we have to solve a system of Pellian equations where one of the equations is always the Pell’s equation

\[ y^2 - 3x^2 = 1 \]

and the second one is as follows

- if \( k = 0 \) \( z^2 - 2x^2 = 2 \),
- if \( k = 1 \) \( z^2 - 6x^2 = 4 \),
- if \( k = 2 \) \( z^2 - 22y^2 = 12 \),
- if \( k = 3 \) \( z^2 - 902x^2 = 452 \),
- if \( k = 4 \) \( z^2 - 4182y^2 = 2092 \),
- if \( k = 5 \) \( z^2 - 58242y^2 = 29122 \).

All the solutions in positive integers of \( y^2 - 3x^2 = 1 \) are given by \((x, y) = (x'_m, y'_m)\), where

\[ x'_m = \frac{1}{2\sqrt{3}} \left( (2 + \sqrt{3})^m - (2 - \sqrt{3})^m \right) \],

\[ y'_m = \frac{1}{2} \left( (2 + \sqrt{3})^m + (2 - \sqrt{3})^m \right) \]

and \( m \geq 0 \). Likewise, we can find a sequence of solutions for any of the equations listed above. The above systems can be reduced to finding
the intersections of \((x'_m)\) and following sequences:

\[
k = 0 : \quad x_n = \frac{1 + \sqrt{2}}{2} (3 + 2\sqrt{2})^n + \frac{1 - \sqrt{2}}{2} (3 - 2\sqrt{2})^n,
\]

\[
k = 1 : \quad x_n = \frac{1}{\sqrt{6}} (5 + 2\sqrt{6})^n - \frac{1}{\sqrt{6}} (5 - 2\sqrt{6})^n,
\]

\[
k = 3 : \quad x_n^\pm = \pm \frac{61 + 2\sqrt{902}}{\sqrt{902}} (901 \pm 30\sqrt{902})^n,
\]

\[
\quad \mp \frac{61 - 2\sqrt{902}}{\sqrt{902}} (901 \mp 30\sqrt{902})^n,
\]

that is to finding the intersections of \((y'_m)\) and following sequences:

\[
k = 2 : \quad y_n^\pm = \pm \frac{5 + \sqrt{22}}{\sqrt{22}} (197 \pm 42\sqrt{22})^n \mp \frac{5 - \sqrt{22}}{\sqrt{22}} (197 \mp 42\sqrt{22})^n,
\]

\[
k = 4 : \quad y_n^\pm = \pm \frac{841 + 13\sqrt{4182}}{\sqrt{4182}} (37637 \pm 582\sqrt{4182})^n \mp \\
\quad \mp \frac{841 - 13\sqrt{4182}}{\sqrt{4182}} (37637 \mp 582\sqrt{4182})^n,
\]

\[
k = 5 : \quad y_n^\pm = \pm \frac{23419 + 97\sqrt{58241}}{2\sqrt{58241}} (524177 \pm 2172\sqrt{58241})^n \mp \\
\quad \mp \frac{23419 - 97\sqrt{58241}}{2\sqrt{58241}} (524177 \mp 2172\sqrt{58241})^n,
\]

with \(n \geq 0\). In what follows, we will briefly resolve the case \(k = 1\), so to demonstrate a method based on Baker’s theory on linear forms in logarithms.

If \(k = 1\) the problem reduces to finding the intersection of sequences

\[
x'_m = \frac{1}{2\sqrt{3}} \left( (2 + \sqrt{3})^m - (2 - \sqrt{3})^m \right)
\]

\[
x_n = \frac{1}{\sqrt{6}} \left( (5 + 2\sqrt{6})^n - (5 - 2\sqrt{6})^n \right)
\]

Clearly \(x'_0 = x_0 = 0\) and \(x'_1 = x_1 = 4\). We have to show that there are no other intersections. Assume \(m, n \geq 3\) and \(x'_m = x_n\). Setting

\[
P = \frac{1}{2\sqrt{3}} (2 + \sqrt{3})^m, \quad Q = \frac{1}{\sqrt{6}} (5 + 2\sqrt{6})^n,
\]
we have
\[ P - \frac{1}{12}P^{-1} = Q - \frac{1}{6}Q^{-1}. \]

Since
\[ Q - P = \frac{1}{6}Q^{-1} - \frac{1}{12}P^{-1} > \frac{1}{6}(Q^{-1} - P^{-1}) = \frac{1}{6}P^{-1}Q^{-1}(P - Q), \]
we have \( Q > P \). Furthermore, from
\[ \frac{Q - P}{Q} = \frac{1}{6}Q^{-1}P^{-1} - \frac{1}{12}P^{-2} < \frac{1}{6}Q^{-1}P^{-1} + \frac{1}{12}P^{-2} < 0.25P^{-2} \]
we get
\[
0 < \log \frac{Q}{P} = -\log \left( 1 - \frac{Q - P}{Q} \right) < \frac{Q - P}{Q} + \left( \frac{Q - P}{Q} \right)^2 < \frac{1}{4}P^{-2} + \frac{1}{16}P^{-4} < 0.32P^{-2} < e^{-m}. \]

The expression \( \log \frac{Q}{P} \) can be written as a linear form in three logarithms in algebraic integers. Indeed
\[
\Lambda := \log \frac{Q}{P} = -m \log \alpha_1 + n \log \alpha_2 + \log \alpha_3,
\]
with \( \alpha_1 = 2 + \sqrt{3}, \alpha_2 = 5 + 2\sqrt{6} \) and \( \alpha_3 = \sqrt{2} \). Then \( 0 < \Lambda < e^{-m} \).

Now, we can apply the famous result of Baker and Wüstholz [2].

**Lemma 6.1.** If \( \Lambda = b_1\alpha_1 + \cdots + b_l\alpha_l \neq 0 \), where \( \alpha_1, \ldots, \alpha_l \) are algebraic integers and \( b_1, \ldots, b_l \) are rational integers, then
\[
\log |\Lambda| \geq -18(l + 1)l^{l+1}(32d)^{l+2}h'(\alpha_1) \cdots h'(\alpha_l) \log(2ld) \log B,
\]
where \( B = \max\{|\alpha_1|, \ldots, |\alpha_l|\} \), \( d \) is the degree of the number field generated by \( \alpha_1, \ldots, \alpha_l \) over \( \mathbb{Q} \),
\[
h'(\alpha) = \frac{1}{d} \max\{h(\alpha), |\log \alpha|, 1\}
\]
and \( h(\alpha) \) denotes the logarithmic Weil height of \( \alpha \).

In our case \( l = 3, \ d = 4, \ B = m, \ \alpha_1 = 2 + \sqrt{3}, \ \alpha_2 = 5 + 2\sqrt{6} \) and \( \alpha_3 = \sqrt{2} \). From Lemma 6.1 and from \( \Lambda < e^{-m} \) we obtain
\[
m \leq 2 \cdot 10^{14} \log m.
\]
Since the previous inequality does not hold for \( m \geq M = 10^{16} \), we conclude that if there is a solution of \( x'_m = x_n \) then \( n \leq m < M = 10^{16} \). This upper bound can be reduced by using the following lemma, which was originally introduced in [1].

**Lemma 6.2** ([5], Lemma 4a). Let \( \theta, \beta, \alpha, a \) be a positive real numbers and let \( M \) be a positive integer. Let \( p/q \) be a convergent of the continued fraction expansion of \( \theta \) such that \( q > 6M \). If \( \varepsilon = \|\beta q\| - M \cdot \|\theta q\| > 0 \), where \( \| \cdot \| \) denotes the distance from the nearest integer, then the inequality

\[
|m\theta - n + \beta| < \alpha a^{-m},
\]

has no integer solutions \( m \) and \( n \) such that \( \log(\alpha q/\varepsilon)/\log a \leq m \leq M \).

After we apply Lemma 6.2 with \( \theta = \log \alpha_1/\log \alpha_2, \beta = \log \alpha_3/\log \alpha_2, \alpha = 1/\log \alpha_2, M = 10^{16}, \) and \( a = e \), we obtain a new upper bound \( M = 38 \) and by another application of Lemma 6.2 we obtain \( M = 7 \).

By examining all the possibilities, we prove that the only solutions of \( x'_m = x_n \) are \( x'_0 = x_0 = 0 \) and \( x'_2 = x_1 = 4 \).

All the other cases can be treated similarly. We get these explicit results.

\[
\begin{align*}
    k = 0 : & \quad x_0 = x'_1 = 1 \\
    k = 1 : & \quad x_0 = x'_0 = 0, \quad x_1 = x'_2 = 4 \\
    k = 2 : & \quad y^+_0 = y'_1 = 2, \quad y^-_1 = y'_3 = 26 \\
    k = 3 : & \quad x^+_0 = x'_2 = 4, \quad x^-_1 = x'_4 = 56 \\
    k = 4 : & \quad y^+_0 = y'_3 = 26, \quad y^-_1 = y'_5 = 362 \\
    k = 5 : & \quad y^+_0 = y'_4 = 97, \quad y^-_1 = y'_6 = 1351.
\end{align*}
\]

These can be interpreted in terms of Theorem 1.1. So, if \( 0 \leq k \leq 5 \) and the set \( \{1, 3, d_k, d\} \) is a Diophantine quadruple in \( \mathbb{Z}[\sqrt{-2}] \), then \( d = d_{k-1} \) or \( d = d_{k+1} \), which completes the proof of Theorem 1.1.

**Acknowledgements.** Zrinka Franušić was supported by the Ministry of Science, Education and Sports, Republic of Croatia, grant 037-0372781-2821. Dijana Kreso was supported by the Austrian Science Fund (FWF): W1230-N13 and NAWI Graz.
References

1. A. Baker and H. Davenport, *The equations* $3x^2 - 2 = y^2$ and $8x^2 - 7 = z^2$, Quart. J. Math. Oxford 20 (1969), 129–137.
2. A. Baker and G. Wüstholz, *Logarithmic forms and group varieties*, J. Reine Angew. Math. 442 (1993), 19–62.
3. M.A. Bennett, *On the number of solutions of simultaneous Pell equations*, J. Reine Angew. Math. 498 (1998), 173–200.
4. A. Dujella, *The problem of the extension of a parametric family of Diophantine triples*, Publ. Math. Debrecen, 51 (1997), 311–322.
5. ______, *A proof of the Hoggatt-Bergum conjecture*, Proc. Amer. Math. Soc. 127 (1999), 1999–2005.
6. ______, *There are only finitely many Diophantine quintuples*, J. Reine Angew. Math. 566 (2004), 183–214.
7. A. Dujella and C. Fuchs, *Complete solution of the polynomial version of a problem of Diophantus*, J. Number Theory 106 (2004), 326–344.
8. A. Dujella and A. Pethő, *A generalization of a theorem of Baker and Davenport*, Quart. J. Math. Oxford 49 (1998), 291–306.
9. Z. Franušić, *On the extension of the Diophantine pair* $\{1, 3\}$ in $\mathbb{Z}[\sqrt{d}]$, Journal of Integer Sequences 13 (2010), Article 10.9.6.
10. P. Gibbs, *Some rational Diophantine sextuples*, Glas. Mat. 41 (2006), 195–203.
11. T. Nagell, *Introduction to Number Theory*, Chelsea, 1981.

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