Near-Optimality of Linear Recovery from Indirect Observations

Anatoli Juditsky *   Arkadi Nemirovski †

Contents
1 Introduction ........................................... 2
2 Situation and main result .............................. 3
  2.1 Situation and goal .................................. 3
  2.2 Preliminaries: Spectratopes ...................... 4
    2.2.1 Examples of spectratopes ..................... 6
    2.2.2 Upper-bounding quadratic form on a spectratope .................................. 8
  2.3 Building linear estimate ........................... 9
    2.3.1 Upper-bounding $\Phi_X(\cdot)$ .................. 10
    2.3.2 Upper-bounding $\Psi_\Pi(\cdot)$ .................. 10
    2.3.3 Putting things together: building linear estimate ............................. 12
  2.4 Near-optimality in Gaussian case .................. 12
  2.5 Illustration: covariance matrix estimation ......... 13
  2.6 Estimation from repeated observations .......... 18
3 Linear estimation in the case of uncertain-but-bounded noise ... 18
  3.1 Uncertain-but-bounded noise ..................... 19
    3.1.1 Building linear estimate ...................... 20
    3.1.2 Near-optimality ............................... 20
  3.2 Mixed noise ..................................... 20
4 Proofs .................................................. 21
  4.1 Technical lemma .................................. 21
  4.2 Proof of Proposition 2.1 .......................... 22
    4.2.1 Preliminaries: matrix concentration ............ 22
    4.2.2 Proving Proposition 2.1 ...................... 23
  4.3 Proof of Lemma 2.2 ................................ 24
  4.4 Proof of Proposition 2.3 .......................... 27
  4.5 Proof of Proposition 3.2 .......................... 33

* LJK, Université Grenoble Alpes, B.P. 53, 38041 Grenoble Cedex 9, France, anatoli.juditsky@imag.fr
† Georgia Institute of Technology, Atlanta, Georgia 30332, USA, nemirovs@isye.gatech.edu

The first author was supported by the CNRS-Mastodons project GARGANTUA, and the LabEx PERSYVAL-Lab (ANR-11-LABX-0025). Research of the second author was supported by NSF grants CCF-1523768 and CMMI-1262063.
A Conic duality

Abstract

We consider the problem of recovering linear image $Bx$ of a signal $x$ known to belong to a given convex compact set $\mathcal{X}$ from indirect observation $\omega = Ax + \xi$ of $x$ corrupted by random noise $\xi$ with finite covariance matrix. It is shown that under some assumptions on $\mathcal{X}$ (satisfied, e.g., when $\mathcal{X}$ is the intersection of $K$ concentric ellipsoids/elliptic cylinders, or the unit ball of the spectral norm in the space of matrices) and on the norm $\| \cdot \|$ used to measure the recovery error (satisfied, e.g., by $\| \cdot \|_p$-norms, $1 \leq p \leq 2$, on $\mathbb{R}^m$ and by the nuclear norm on the space of matrices), one can build, in a computationally efficient manner, a “presumably good” linear in observations estimate, and that in the case of zero mean Gaussian observation noise, this estimate is near-optimal among all (linear and nonlinear) estimates in terms of its worst-case, over $x \in \mathcal{X}$, expected $\| \cdot \|$-loss. These results form an essential extension of those in [14], where the assumptions on $\mathcal{X}$ were more restrictive, and the norm $\| \cdot \|$ was assumed to be the Euclidean one. In addition, we develop near-optimal estimates for the case of “uncertain-but-bounded” noise, where all we know about $\xi$ is that it is bounded in a given norm by a given $\sigma$. Same as in [14], our results impose no restrictions on $A$ and $B$.

1 Introduction

Broadly speaking, what follows contributes to a long line of research (see, e.g., [16, 17, 13, 33, 30, 26, 11, 10, 7] and references therein) aimed at building linear estimates of signals from noisy observations of linear images of these signals and analyzing performance of these estimates. More specifically, this paper is a follow-up to our paper [14]; similarly to the latter paper, we consider the estimation problem where one, given a “sensing matrix” $A \in \mathbb{R}^{m \times n}$ and an indirect noisy observation

$$\omega = Ax + \xi$$

of unknown deterministic “signal” $x$ known to belong to a given “signal set” $\mathcal{X}$, is interested to recover the linear image $Bx$ of the signal, where $B \in \mathbb{R}^{v \times n}$ is a given matrix. We assume that the observation noise $\xi$ is random with unknown (and perhaps depending on $x$) distribution belonging to some family $\mathcal{P}$ of Borel probability distributions on $\mathbb{R}^m$ associated with a given nonempty convex compact subset $\Pi$ of the set of positive definite $m \times m$ matrices, “associated” meaning that the covariance matrix $\text{Cov}[P] := \text{E}_{\xi \sim P} \{ \xi \xi^T \}$ of a distribution $P \in \mathcal{P}$ is $\preceq$-dominated by some matrix from $\Pi$:

$$P \in \mathcal{P} \Rightarrow \exists Q \in \Pi : \text{Cov}[P] \preceq Q.$$ (1)

We quantify a candidate estimate – a Borel function $\hat{x}(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^v$ – by its worst-case, under the circumstances, expected $\| \cdot \|$-error defined as

$$\text{Risk}_{\Pi,\| \cdot \|}[\hat{x} | \mathcal{X}] = \sup_{x \in \mathcal{X}, P \in \mathcal{P}} \text{E}_{\xi \sim P} \{ \| Bx - \hat{x}(Ax + \xi) \| \};$$

here $\| \cdot \|$ is a given norm on $\mathbb{R}^v$.

In the major part of the paper we assume that signal set $\mathcal{X}$ is a special type symmetric w.r.t. the origin convex compact set (a spectratope to be defined in Section 2.2), and require from the norm $\| \cdot \|$, conjugate to $\| \cdot \|$ to have a spectratope as the unit ball. This allows, e.g., for $\mathcal{X}$ to be the (bounded) intersection of finitely many centered at the origin ellipsoids/elliptic cylinders/$\| \cdot \|_p$-balls ($p \in [2, \infty]$), or the (bounded) solution set of a system of two-sided Linear Matrix Inequalities

$$\{ x \in \mathbb{R}^n : -L_k \preceq R_k[x] \preceq L_k, k \leq K \} \quad \quad \quad [R_k[x] : \text{linear in } x \text{ symmetric matrices}]$$

and for $\| \cdot \|$ to be $\| \cdot \|_p$-norm, $1 \leq p \leq 2$, on $\mathbb{R}^v$, or the nuclear norm on the space $\mathbb{R}^v = \mathbb{R}^{u \times v}$ of matrices.

Under these assumptions, we
• develop computationally efficient scheme for building “presumably good” linear estimates (i.e., estimates of the form $\hat{x}_H(\omega) = H^T\omega$) and for upper-bounding their risks (Proposition 2.2);

• demonstrate that in the case $\{N(0, Q) : Q \in \Pi\} \subset \mathcal{P}$, the above linear estimates are “near-optimal” (optimal up to logarithmic terms) among all estimates, linear and nonlinear alike (Proposition 2.3).

Progress as compared to [14] is as follows:

• in [14], we dealt with the case $\mathcal{P} = \{N(0, Q)\}$ of zero mean Gaussian observation noise with known covariance matrix, while now we allow for $\mathcal{P}$ to be a whatever family of probability distributions with covariance matrices $\succeq$-dominated by matrices from a given convex compact set $\Pi \subset \text{int} S^m_+$;

• our present results are applicable to an essentially wider family of signal sets than the ellitopes considered in [14]: every ellitope is a spectratope, but not vice versa. Say, the intersection of centered at the origin ellipsoids/elliptic cylinders/$\| \cdot \|_p$-balls, $p \in [2, \infty]$, is an ellitope, and the (bounded) solution set of a finite system of two-sided LMI’s is a spectratope, but not an ellitope;

• in [14], the only allowed norm $\| \cdot \|$ was $\| \cdot \|_2$, while now we allow for a much wider family of norms quantifying the recovery errors.

In addition to observations with random noise, in what follows we address also observations with “uncertain-but-bounded” noise, where $\xi$, instead of being random, is selected, perhaps in adversarial manner, from a given spectratope – the situation not considered in [14] at all.

Note that the outlined extensions of the results of [14] require advanced technical tools (“Noncommutative Khitnchine Inequality, see [31] and references therein).

The main body of the paper is organized as follows. We start with formulating our estimation problem (Section 2.1), introducing the family of spectratopes – the signal sets we intend to work with (Section 2.2), explain how to build in a computationally efficient fashion a “presumably good” linear estimate (Section 2.3) and establish near-optimality of this estimate (Section 2.4). In Section 3 we consider the case of uncertain-but-bounded observation noise, same as the situation when the observation noise contains both random and uncertain-but-bounded components. All technical proofs are relegated to Section 4. Appendix contains an “executive summary” of conic duality, which is one of our major working horses.

2 Situation and main result

2.1 Situation and goal

Given $\nu \times n$ matrix $B$, consider the problem of estimating linear image $Bx$ of unknown deterministic signal $x$ known to belong to a given set $X \subset \mathbb{R}^n$ via noisy observation

$$\omega = Ax + \xi$$

(2)

where $A$ is a given $m \times n$ matrix $A$ and $\xi$ is random observation noise. In typical signal processing applications, the distribution of noise is fixed and is part of the data of the estimation problem. In order to cover some applications (e.g., the one in Section 2.5), we allow for “ambiguous” noise distributions; all we know is that this distribution belongs to a family $\mathcal{P}$ of Borel probability distributions on $\mathbb{R}^m$ associated, in the sense of (1), with a given convex compact subset $\Pi$ of the interior of the cone $S^m_+$ of
positive semidefinite $m \times m$ matrices. Actual distribution of noise in (2) is somehow selected from $\mathcal{P}$ by nature (and may, e.g., depend on $x$).

In the sequel, for a Borel probability distribution $P$ on $\mathbb{R}^m$ we write $P \ll \Pi$ to express the fact that $\text{Cov}[P]$ is $\succeq$-dominated by a matrix from $\Pi$:

$$\{P \ll \Pi\} \iff \exists \Theta \in \Pi : \text{Cov}[P] \preceq \Theta.$$ 

From now on we make the following regularity assumption

**Assumption R:** All matrices from $\Pi$ are positive definite.

Given $\Pi$ and a norm $\| \cdot \|$ on $\mathbb{R}^\nu$, we quantify the risk of a candidate estimate – of a Borel function $\hat{x}(\cdot) : \mathbb{R}^m \to \mathbb{R}^\nu$ – by its $(\Pi, \| \cdot \|)$-risk on $X$ defined as

$$\text{Risk}_{\Pi, \| \cdot \|}[\hat{x}|X] = \sup_{x \in X, P \ll \Pi} \mathbb{E}_{\xi \sim P}\{\| \hat{x}(Ax + \xi) - Bx\|\}. \quad (3)$$

Our focus is on linear estimates – estimates of the form $\hat{x}_H(\omega) = H^T \omega$ given by $m \times \nu$ matrices $H$; our ultimate goal is to demonstrate that under some restrictions on the signal domain $X$, “presumably good” linear estimate yielded by an optimal solution to an efficiently solvable convex optimization problem is near-optimal in terms of its risk among all estimates, linear and nonlinear alike. Any result of this type should impose some restrictions on $X$ – it is well known that linear estimates are “heavily sub-optimal” on some simple signal domains (e.g., $\| \cdot \|_1$-ball). We start with describing the domains $X$ we intend to work with – spectratopes.

### 2.2 Preliminaries: Spectratopes

A **basic spectratope** is a set $X \subset \mathbb{R}^n$ given by basic spectratopic representation – representation of the form

$$X = \left\{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : R_k^2[x] \preceq t_k I_{d_k}, 1 \leq k \leq K \right\} \quad (4)$$

where

(S1) $R_k[x] = \sum_{i=1}^n x_i R^{ki}$ are symmetric $d_k \times d_k$ matrices linearly depending on $x \in \mathbb{R}^n$ (i.e., “matrix coefficients” $R^{ki}$ belong to $\mathbb{S}_{d_k}$)

(S2) $\mathcal{T} \subset \mathbb{R}^K_+$ is a monotonic set, meaning that $\mathcal{T}$ is a convex compact subset of $\mathbb{R}^K_+$ which contains a positive vector and is monotone:

$$0 \leq t' \leq t \in \mathcal{T} \Rightarrow t' \in \mathcal{T}.$$

(S3) Whenever $x \neq 0$, it holds $R_k[x] \neq 0$ for at least one $k \leq K$.

An immediate observation is as follows:

**Remark 2.1** By Schur Complement Lemma, the set (4) given by data satisfying (S1), (S2) can be represented as

$$X = \left\{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : \begin{bmatrix} t_k I_{d_k} & R_k[x] \\ R_k[x]^T & I_{d_k} \end{bmatrix} \succeq 0, k \leq K \right\}$$

By the latter representation, $X$ is nonempty, closed, convex, symmetric w.r.t. the origin and contains a neighbourhood of the origin. This set is bounded if and only if the data, in addition to (S1), (S2), satisfies (S3).
A spectratope $\mathcal{X} \subset \mathbb{R}^p$ is a set represented as linear image of a basic spectratope:

$$\mathcal{X} = \{ x \in \mathbb{R}^p : \exists (y \in \mathbb{R}^n, t \in \mathcal{T}) : x = Py, \ R^2_k[y] \preceq t_k I_{d_k}, 1 \leq k \leq K \}, \tag{5}$$

where $P$ is a $p \times n$ matrix, and $R_k[\cdot]$, $\mathcal{T}$ are as in $(S_1)$–$(S_3)$.

We associate with a basic spectratope $(4)$, $(S_1)$–$(S_3)$ the following entities:

1. The size

$$D = \sum_{k=1}^{K} d_k;$$

2. Linear mappings

$$Q \mapsto \mathcal{R}_k[Q] = \sum_{i,j} Q_{ij} R^{ki} R^{kj} : \mathbb{S}^n \to \mathbb{S}^d$$

As is immediately seen, we have

$$\mathcal{R}_k[xx^T] \equiv R^2_k[x], \tag{6}$$

implying that $\mathcal{R}_k[Q] \succeq 0$ whenever $Q \succeq 0$, whence $\mathcal{R}_k[\cdot]$ is $\succeq$-monotone:

$$Q' \succeq Q \Rightarrow \mathcal{R}_k[Q'] \succeq \mathcal{R}_k[Q]. \tag{7}$$

Besides this, if $\xi$ is a random vector taking values in $\mathbb{R}^n$ with covariance matrix $Q$, we have

$$E_{\xi} \{ R^2_k[\xi] \} = E_{\xi} \{ \mathcal{R}_k[\xi \xi^T] \} = \mathcal{R}_k[ E_{\xi} \{ \xi \xi^T \} ] = \mathcal{R}_k[Q], \tag{8}$$

where the first equality is given by $(6)$.

3. Linear mappings $\Lambda_k \mapsto \mathcal{R}_k^*[\Lambda_k] : \mathbb{S}^d \to \mathbb{S}^n$ given by

$$[\mathcal{R}_k^*[\Lambda_k]]_{ij} = \frac{1}{2} \mathrm{Tr}(\Lambda_k [R^{ki} R^{kj} + R^{kj} R^{ki}]), \ 1 \leq i,j \leq n. \tag{9}$$

It is immediately seen that $\mathcal{R}_k^*[\cdot]$ is the conjugate of $\mathcal{R}_k[\cdot]$: 

$$\forall (\Lambda_k \in \mathbb{S}^d, Q \in \mathbb{S}^n) : \langle \Lambda_k, \mathcal{R}_k[Q] \rangle = \mathrm{Tr}(\Lambda_k \mathcal{R}_k[Q]) = \mathrm{Tr}(\mathcal{R}_k^*[\Lambda_k]Q) = \langle \mathcal{R}_k^*[\Lambda_k], Q \rangle, \tag{10}$$

where $\langle A, B \rangle = \mathrm{Tr}(AB)$ is the Frobenius inner product of symmetric matrices. Besides this, we have$^1$

$$\Lambda_k \succeq 0 \Rightarrow \mathcal{R}_k^*[\Lambda_k] \succeq 0; \tag{11}$$

4. The linear space $\Lambda^K = \mathbb{S}^{d_1} \times \ldots \times \mathbb{S}^{d_K}$ of all ordered collections $\Lambda = \{ \Lambda_k \in \mathbb{S}^{d_k} \}_{k \leq K}$ along with the linear mapping

$$\Lambda \mapsto \lambda[\Lambda] := [\mathrm{Tr}(\Lambda_1); \ldots; \mathrm{Tr}(\Lambda_K)] : \Lambda^K \to \mathbb{R}^K.$$ 

Besides this, we associate with a monotonic set $\mathcal{T} \subset \mathbb{R}^K$

- its support function

$$\phi_{\mathcal{T}}(g) = \max_{t \in \mathcal{T}} g^T t,$$

which clearly is a convex positively homogeneous, of degree 1, nonnegative real-valued function on $\mathbb{R}^K$. Since $\mathcal{T}$ contains positive vectors, $\phi_\mathcal{T}$ is coercive on $\mathbb{R}^K_+$, meaning that $\phi_{\mathcal{T}}(\lambda^s) \to +\infty$ along every sequence $\{ \lambda^s \geq 0 \}$ such that $\| \lambda^s \| \to \infty$.

$^1$note that when $\Lambda_k \succeq 0$ and $Q = xx^T$, the first quantity in $(10)$ is nonnegative by $(6)$, and therefore $(10)$ states that $x^T \mathcal{R}_k^*[\Lambda_k]x \geq 0$ for every $x$, implying $\mathcal{R}_k^*[\Lambda_k] \succeq 0.$
the conic hull
\[ K[T] = \text{cl}\{ [t; s] \in \mathbb{R}^{K+1} : s > 0, s^{-1}t \in T \} \]
which clearly is a regular (i.e., closed, convex, pointed and with a nonempty interior) cone in \( \mathbb{R}^{K+1} \) such that
\[ T = \{ t : [t; 1] \in K[T] \}. \]
Besides this, it is immediately seen that the cone \((K[T])^*\) dual to \(K[T]\) can be described as follows:
\[ (K[T])^* := \{ [g; r] \in \mathbb{R}^{K+1} : [g; r]^T [t; s] \geq 0 \forall [t; s] \in K[T] \} = \{ [g; r] \in \mathbb{R}^{k+1} : r \geq \phi_T(-g) \} . \]

2.2.1 Examples of spectratopes

Example 1: Ellitopes. An ellitope was defined in [14] as a set \( \mathcal{X} \subset \mathbb{R}^n \) representable as
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists (y \in \mathbb{R}^N, t \in T) : x = Py, y^T S_k y \leq t_k, k \leq K \} , \quad (12) \]
where \( S_k \succeq 0, \sum_K S_k \succ 0, \) and \( T \) satisfies \((S_2)\). Basic examples of ellitopes are:
- bounded intersections of centered at the origin ellipsoids/elliptic cylinders: whenever \( S_k \succeq 0 \) and \( \sum_K S_k \succ 0, \)
  \[ \bigcup_{k=1}^{K} \{ x \in \mathbb{R}^n : x^T S_k x \leq 1 \} = \{ x \in \mathbb{R}^n : \exists t \in T = [0, 1]^K : x^T S_k x \leq t_k, 1 \leq k \leq K \}. \]
- \( \| \cdot \|_p \)-balls, \( 2 \leq p \leq \infty \):
  \[ \{ x \in \mathbb{R}^n : \| x \|_p \leq 1 \} = \{ x \in \mathbb{R}^n : \exists t \in T := \{ t \geq 0, \| t \|_{p/2} \leq 1 \} : x^T S_k x := x_k^2 \leq t_k, k \leq n \}. \]
It is immediately seen that an ellitope \((12)\) is a spectratope as well. Indeed, let \( S_k = \sum_{j=1}^{r_k} s_{kj} s_{kj}^T, \) \( r_k = \text{Rank}(S_k), \) be a dyadic representation of the positive semidefinite matrix \( S_k, \) so that
\[ y^T S_k y = \sum_j (s_{kj} y)^2 \forall y, \]
and let
\[ \hat{T} = \{ \{ t_{kj} \geq 0, 1 \leq j \leq r_k, 1 \leq k \leq K \} : \exists t \in T : \sum_j t_{kj} \leq t_k, k \leq K, \ R_{kj} = s_{kj}^T y \in \mathbf{S}^1 = \mathbb{R} \}. \]
We clearly have
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists \{ t_{kj} \} \in \hat{T}, y) : x = Py, R_{kj}^2[y] \leq t_{kj} I_1 \forall k, j \} \]
and the right hand side is a valid spectratopic representation of \( \mathcal{X}. \) Note that the spectratopic size of \( \mathcal{X} \) is \( D = \sum_{k=1}^{K} r_k. \)
Example 2: “Matrix box.” Let $L$ be a positive definite $d \times d$ matrix. Then the “matrix box”

$$
\mathcal{X} = \{ x \in \mathbb{S}^d : -L \preceq x \preceq L \} = \{ x \in \mathbb{S}^d : -I_d \preceq L^{-1/2}XL^{-1/2} \preceq I_d \}
$$

is a basic spectratope (augment $R_1[.] := R[.]$ with $K = 1$, $T = [0, 1]$). As a result, a bounded set $\mathcal{X} \subset \mathbb{R}^n$ given by a system of “two-sided” Linear Matrix Inequalities, specifically,

$$
\mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : -\sqrt{t_k}L_k \preceq S_k[x] \preceq \sqrt{t_k}L_k, 1 \leq k \leq K \}
$$

where $S_k[x]$ are symmetric $d_k \times d_k$ matrices linearly depending on $x$, $L_k > 0$ and $\mathcal{T}$ satisfies $(S_2)$, is a basic spectratope:

$$
\mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : R_k^2[x] \preceq t_kI_{d_k}, 1 \leq k \leq K \} \quad \quad [R_k[x] = L_k^{-1/2}S_k[x]L_k^{-1/2}]
$$

Calculus of spectratopes. Spectratopes admit fully algorithmic “calculus” – nearly all basic operations with sets preserving convexity, symmetry w.r.t. the origin, and boundedness (these are “built-in” properties of spectratopes), as applied to spectratopes, yield spectratopes as well, and a spectratopic representation of the result of such an operation is readily given by spectratopic representations of the operands. The main calculus rules are as follows:

- **[finite intersections]** If

  $$
  \mathcal{X}_\ell = \{ x \in \mathbb{R}^n : \exists (y^\ell \in \mathbb{R}^{n_\ell}, t^\ell \in \mathcal{T}_\ell) : x = P_\ell y^\ell, R_{k\ell}^2[y^\ell] \preceq t^\ell_k I_{d_{k\ell}}, 1 \leq k \leq K_\ell \}, 1 \leq \ell \leq L,
  $$

  are spectratopes, so is $\mathcal{X} = \bigcap_{\ell \leq L} \mathcal{X}_\ell$. Indeed, let

  $$
  E = \{ [y = [y^1; \ldots; y^L] \in \mathbb{R}^{n_1} \times \ldots \times \mathbb{R}^{n_L} : P_1 y^1 = P_2 y^2 = \ldots = P_L y^L \}.
  $$

  When $E = \{ 0 \}$, we have $\mathcal{X} = \{ 0 \}$, so that $\mathcal{X}$ is a spectratope; when $E \neq \{ 0 \}$, we have

  $$
  \mathcal{X} = \{ x \in \mathbb{R}^n : \exists (y = [y^1; \ldots; y^L] \in E, t = [t^1; \ldots; t^L] \in \mathcal{T} := \mathcal{T}_1 \times \ldots \times \mathcal{T}_L) : x = P y := P_1 y^1, R_{k\ell}^2[y^\ell] \preceq t^\ell_k I_{d_{k\ell}}, 1 \leq k \leq K_\ell, 1 \leq \ell \leq L \};
  $$

  identifying $E$ and appropriate $\mathbb{R}^n$, we arrive at a valid spectratopic representation of $\mathcal{X}$.

- **[direct product]** If

  $$
  \mathcal{X}_\ell = \{ x^\ell \in \mathbb{R}^{n_\ell} : \exists (y^\ell \in \mathbb{R}^{n_\ell}, t^\ell \in \mathcal{T}_\ell) : x^\ell = P_\ell y^\ell, R_{k\ell}^2[y^\ell] \preceq t^\ell_k I_{d_{k\ell}}, 1 \leq k \leq K_\ell \}, 1 \leq \ell \leq L,
  $$

  are spectratopes, so is $\mathcal{X} = \mathcal{X}_1 \times \ldots \times \mathcal{X}_L$:

  $$
  \mathcal{X}_1 \times \ldots \times \mathcal{X}_L = \{ [x^1; \ldots; x^L] : \exists (y = [y^1; \ldots; y^L], t = [t^1; \ldots; t^L] \in \mathcal{T} := \mathcal{T}_1 \times \ldots \times \mathcal{T}_L) : x = P y := [P_1 y^1; \ldots; P_L y^L], R_{k\ell}^2[y^\ell] \preceq t^\ell_k I_{d_{k\ell}}, 1 \leq k \leq K_\ell, 1 \leq \ell \leq L \};
  $$

- **[linear image]** If

  $$
  \mathcal{X} = \{ x \in \mathbb{R}^n : \exists (y \in \mathbb{R}^n, t \in \mathcal{T}) : x = P y, R_k^2[y] \preceq t_k I_{d_k}, k \leq K \}
  $$

  is a spectratope and $S$ is a $\mu \times \nu$ matrix, the set $S\mathcal{X} = \{ z = Sx : x \in \mathcal{X} \}$ is a spectratope:

  $$
  S\mathcal{X} = \{ z \in \mathbb{R}^\mu : \exists (y \in \mathbb{R}^n, t \in \mathcal{T}) : z = SP y, R_k^2[y] \preceq t_k I_{d_k}, k \leq K \}.
  $$
• [inverse linear image under embedding] If
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists (y \in \mathbb{R}^m, t \in \mathcal{T}) : x = Py, R_k^2[y] \preceq t_k I_{d_k}, k \leq K \} \]
is a spectrotape, and $S$ is a $\nu \times \mu$ matrix with trivial kernel, the set $S^{-1} \mathcal{X} = \{ z : Sz \in \mathcal{X} \}$ is a spectrotape. Indeed, setting $E = \{ y \in \mathbb{R}^m : Py \in \text{Im} S \}$, we get a linear subspace of $\mathbb{R}^n$; if $E = \{ 0 \}$, $S^{-1} \mathcal{X} = \{ 0 \}$ is a spectrotape, otherwise we have
\[ S^{-1} \mathcal{X} = \{ z \in \mathbb{R}^m : \exists (y \in E, t \in \mathcal{T}) : z = Qy, R_k^2[y] \preceq t_k I_{d_k}, k \leq K \}, \]
where linear mapping $y \mapsto Qy : E \to \mathbb{R}^n$ is uniquely defined by the relation $Py = SQy$. When identifying $E$ with appropriate $\mathbb{R}^n$, we get a valid spectrotape representation of $S^{-1} \mathcal{X}$.

• [arithmetic sum] If $\mathcal{X}_\ell$, $\ell \leq L$, are spectrotapes in $\mathbb{R}^n$, so is the arithmetic sum $\mathcal{X} = \mathcal{X}_1 + \ldots + \mathcal{X}_L$ of $\mathcal{X}_\ell$. Indeed, $\mathcal{X}$ is the image of $\mathcal{X}_1 \times \ldots \times \mathcal{X}_L$ under the linear mapping $[x^1; \ldots; x^L] \mapsto x^1 + \ldots + x^L$, and taking direct products and linear images preserve spectrotapes.

### 2.2.2 Upper-bounding quadratic form on a spectrotape

The first nontrivial fact we are about to establish is that the maximum of an (indefinite) quadratic form over a spectrotape admits reasonably tight efficiently computable upper bound.

**Proposition 2.1** Let $C$ be a symmetric $p \times p$ matrix, let $\mathcal{X} \subset \mathbb{R}^p$ be given by spectrotape representation (5), let
\[ \text{Opt} = \max_{x \in \mathcal{X}} x^T C x \]
and
\[ \text{Opt}_* = \min_{\Lambda = \{ \Lambda_k \}_{k \leq K}} \{ \phi_T(\lambda[\Lambda]) : \Lambda_k \succeq 0, k \leq K, P^T C P \preceq \sum_k R_k^*[\Lambda_k] \} \]
where $\Lambda = \{ \Lambda_k \}_{k \leq K}$ is a spectrotape representation of $\mathcal{X}$.

Then (13) is solvable, and
\[ \text{Opt} \leq \text{Opt}_* \leq 2 \max[\ln(2D), 1]\text{Opt}, \]
where $D = \sum_k d_k$ is the size of the spectrotape $\mathcal{X}$.

To explain where the result of the proposition comes from, let us prove right now its easy part – the first inequality in (14); the remaining, essentially less trivial, part of the claim is proved in Section 4.2. Let $\Lambda$ be a feasible solution to the optimization problem in (13), and let $x \in \mathcal{X}$, so that $x = Py$ for some $y$ such that $R_k^2[y] \preceq t_k I_{d_k}$, $k \leq K$, for properly selected $t \in \mathcal{T}$. We have
\[ x^T C x = y^T [P^T C P] y \leq \sum_k y^T R_k^*[\Lambda_k] y = \sum_k \text{Tr}(R^*_k[\Lambda_k] yy^T) \leq \sum_k \text{Tr}(\Lambda_k R_k[yy^T]) \]
\[ \leq \sum_k \text{Tr}(\Lambda_K R_k^K[y]) \leq \sum_k \text{Tr}(\Lambda_k t_k I_{d_k}) = \sum_k t_k \text{Tr}(\Lambda_k) = \lambda^T[\Lambda] t \leq \phi_T(\lambda[\Lambda]), \]
where (a) is due to the fact that $\Lambda$ is feasible for the optimization problem in (13), (b) is by (10), (c) is by (6), (d) is due to $\Lambda_K \succeq 0$ and $R_k^2[y] \preceq t_k I_{d_k}$, and (e) is by the definition of $\phi_T$. The bottom line is that the value of the objective of the optimization problem in (13) at every feasible solution to this problem upper-bounds $\text{Opt}$, implying the first inequality in (14). Note that the derivation we have carried out is nothing but a minor modification of the standard semidefinite relaxation scheme.
Remark 2.2 Proposition 2.1 has some history. When \( \mathcal{X} \) is an intersection of centered at the origin ellipsoids/elliptic cylinders, it was established in [23]; matrix analogy of the latter result can be traced back to [24], see also [22]. The case when \( \mathcal{X} \) is a general-type ellitope (12) was considered in [14], with tightness guarantee slightly better than in (14), namely,

\[
\text{Opt} \leq \text{Opt}_s \leq 4 \ln(5K)\text{Opt}.
\]

Note that in the case of (12), Proposition 2.1 results in a worse than \( O(1) \ln(K) \) “nonoptimality factor” \( O(1) \ln(\sum_{k=1}^K \text{Rank}(S_k)) \). We remark that passing from ellitopes to spectratopes needs replacing elementary bounds on probabilities of large deviations used in [23, 14] by much more powerful tool – matrix concentration inequalities, see [31, 32] and references therein.

2.3 Building linear estimate

Our goal is to process the estimation problem posed in Section 2.1 in the case when \( \mathcal{X} \) is a spectratope, and in this context we can assume w.l.o.g. that the spectratope in question is basic. Indeed, when the “true” signal set is (4), and in this context we can assume w.l.o.g. that the spectratope in question is basic. Indeed, when the signal set satisfies (4). For this reason, we assume from now on that the signal set in question is the basic spectratope (4).

Our current goal is to build a “good” linear estimate. To this end observe that the (\( \Pi, \| \cdot \| \))-risk of the linear estimate \( \hat{x}_H(\omega) = H^T\omega, H \in \mathbb{R}^{m \times \nu} \), can be upper-bounded as follows:

\[
\text{Risk}_{\Pi,\| \cdot \|}[\hat{x}_H(\cdot)|\mathcal{X}] = \sup_{x \in \mathcal{X}, P \in \Pi} E_{\xi \sim P}\{\|H^T(Ax + \xi) - Bx\|\} \\
\leq \sup_{x \in \mathcal{X}} \|H^T Ax - Bx\| + \sup_{P \in \Pi} E_{\xi \sim P}\{\|H^T\xi\|\} \\
\leq \Phi_\mathcal{X}(B - H^T A) + \Psi_\Pi(H),
\]

where

\[
\Phi_\mathcal{X}(V) = \max_x \{\|Vx\| : x \in \mathcal{X}\}, \quad \Psi_\Pi(H) = \sup_{P \in \Pi} E_{\xi \sim P} \{\|H^T\xi\|\}.
\]

Note that while \( \Phi_\mathcal{X}(B - H^T A) \) and \( \Psi_\Pi(H) \) are convex functions of \( H \), these functions can be difficult to compute\(^2\). A matrix \( H \) of a “good” linear estimate \( \hat{x}_H \) which is also efficiently computable can be taken as a minimizer of the sum of efficiently computable convex upper bounds on \( \Phi_\mathcal{X} \) and \( \Psi_\Pi \). We make from now on the following

**Assumption A**: The unit ball \( B_s \) of the norm \( \| \cdot \| \), conjugate to the norm \( \| \cdot \| \) participating in the formulation of our estimation problem is a spectratope:

\[
B_s = \{z \in \mathbb{R}^q : \exists y \in \mathcal{Y} : z = My\}, \\
\mathcal{Y} := \{y \in \mathbb{R}^q : \exists r \in \mathcal{R} : S_T^2[y] \preceq r_t I_{t_t}, 1 \leq t \leq L\}, \quad (16)
\]

where the right hand side data are as required in a spectratopic representation.

Examples of norms satisfying Assumption A include \( \| \cdot \|_q \)-norms on \( \mathbb{R}^q \), \( 1 \leq q \leq 2 \) (conjugates of the norms \( \| \cdot \|_p \) with \( 1/p + 1/q = 1 \), see Example 1 in Section 2.2.1). Another example is nuclear norm \( \|V\|_{\text{S,1}} \) on the space \( \mathbb{R}^q \) of \( p \times q \) matrices – the sum of singular values of a matrix \( V \).

\(^2\)For instance, computing \( \Psi_\mathcal{X}(B - H^T A) \) reduces to maximizing the convex function \( \|(B - H^T A)x\| \) over \( x \in \mathcal{X} \), which is computationally intractable even when \( \mathcal{X} \) is as simple as the unit box, and \( \| \cdot \| \) is the Euclidean norm.
conjugate of the nuclear norm is the spectral norm \( \| \cdot \|_{\text{sh}, \infty} \) on \( \mathbb{R}^p = \mathbb{R}^{p \times q} \), and the unit ball of the latter norm is a spectrapoly:

\[
\{ X \in \mathbb{R}^{p \times q} : \| X \|_{\text{sh}, \infty} \leq 1 \} = \{ X : \exists t \in \mathcal{T} = [0, 1] : R^2[X] \leq tI_{p+q} \}, \quad R[X] = \frac{X^T}{X}.
\]

### 2.3.1 Upper-bounding \( \Phi_X(\cdot) \)

Assuming that Assumption A holds true, let us consider the direct product spectrapoly

\[
\mathcal{Z} := \mathcal{X} \times \mathcal{Y} = \{ [x; y] \in \mathbb{R}^n \times \mathbb{R}^q : \exists s = [t; r] \in \mathcal{T} \times \mathcal{R} : R^2_k[x] \leq t_kI_{d_k}, 1 \leq k \leq K, S^2_r[y] \leq r_lI_{f_l}, 1 \leq l \leq L \} = \{ w = [x; y] \in \mathbb{R}^n \times \mathbb{R}^q : \exists s = [t; r] \in \mathcal{S} \times \mathcal{T} : U^2_r[w] \leq s_iI_{h_i}, 1 \leq i \leq I = K + L \}
\]

with \( U_\ell[\cdot] \) readily given by \( R_k[\cdot] \) and \( S_\ell[\cdot] \). Given a \( \nu \times n \) matrix \( V \) and setting

\[
W[V] = \frac{1}{2} \begin{bmatrix} V^T M \end{bmatrix}
\]

it clearly holds

\[
\Phi_X(V) = \max_{x \in \mathcal{X}} \| Vx \| = \max_{x \in \mathcal{X}, y \in \mathcal{Y}, z \in \mathcal{B}} \begin{bmatrix} V^T V \end{bmatrix} = \max_{x \in \mathcal{X}, y \in \mathcal{Y}} \begin{bmatrix} V^T M \end{bmatrix} = \max_{w \in \mathcal{X}} w^T W[V] w.
\]

Applying Proposition 2.1, we arrive at the following result:

**Corollary 2.1** *In the just defined situation, the efficiently computable convex function*

\[
\overline{\Phi}_X(V) = \min_{\Lambda, Y} \left\{ \phi_T(\lambda[\Lambda]) + \phi_R(\lambda[Y]) : \Lambda = \{\Lambda_k \in \mathbb{S}^{d_k}_{+}\}_{k \leq K}, Y = \{Y_\ell \in \mathbb{S}_+^{f_\ell}\}_{\ell \leq L}, \sum_k R^*_k[\Lambda_k] \geq 0, \frac{1}{2} y^T M y + \frac{1}{2} \sum_{\ell \leq L} S^*_\ell[Y_\ell] \geq 0 \right\}
\]

*is a norm on \( \mathbb{R}^{p \times n} \), and this norm is a tight upper bound on \( \Psi_X(\cdot) \), namely,

\[
\forall V \in \mathbb{R}^{p \times n} : \Phi_X(V) \leq \overline{\Phi}_X(V) \leq 2 \max[\ln(2D), 1] \Phi_X(V), \quad D = \sum_k d_k + \sum_{\ell \leq L} f_\ell
\]

*(recall, that here)*

\[
R^*_k[\Lambda_k]_{ij} = \frac{1}{2} \text{Tr}(\Lambda_k R^k[ij] R^k[ij]^T), \quad R_k[x] = \sum_{l \leq L} x_i R^k[i], \quad S^*_\ell[Y_\ell]_{ij} = \frac{1}{2} \text{Tr}(Y_\ell S^\ell[ij] S^\ell[ij]^T), \quad S_\ell[y] = \sum_{l \leq L} y_i S^\ell[i],
\]

\[
\phi_T(\lambda) = \max_{t \in \mathcal{T}} \lambda^T t, \quad \phi_R(\lambda) = \max_{r \in \mathcal{R}} \lambda^T r, \text{ and } \lambda[\{\Xi_1, \ldots, \Xi_N\}] = [\text{Tr}(\Xi_1); \ldots; \text{Tr}(\Xi_N)].
\]

### 2.3.2 Upper-bounding \( \Psi_\Pi(\cdot) \)

We are about to present an efficiently computable upper bound on \( \Psi_\Pi \) capable to handle any norm obeying Assumption A. The underlying observation is as follows:
Lemma 2.1. Let $V$ be a $m \times n$ matrix, $Q \in S^n_{++}$, and $P$ be a probability distribution on $R^m$ with $\text{Cov}[P] \preceq Q$. Let, further, $\| \cdot \|$ be a norm on $R^r$ with the unit ball $B_*$ of the conjugate norm $\| \cdot \|_*$ given by (16). Finally, let $\Upsilon = \{\Upsilon_{\ell} \in S^M_{+}\}_{\ell \in L}$ and a matrix $\Theta \in S^m$ satisfy the constraint

$$\left[ \begin{array}{c} \Theta \\ \frac{1}{2} M^T V^T \\ \sum_{\ell} S_\ell^2 [\Upsilon_{\ell}] \end{array} \right] \succeq 0 \quad (19)$$

(for notation, see (16), (18)). Then

$$E_{\eta \sim P\{\|V^T \eta\|} \leq \text{Tr}(Q \Theta) + \phi_R(\lambda[\Upsilon]). \quad (20)$$

Proof is immediate. In the case of (19), we have

$$\|V^T \xi\| = \max_{\xi \in B_*} \|V^T \xi\| = \max_{y \in Y} \|y^T M^T V^T \xi\| \leq \max_{y \in Y} \|\xi^T \Theta \xi + \sum_{\ell} \text{Tr}(S_\ell^2 [\Upsilon_{\ell}] y y^T)\| \leq \max_{y \in Y} \|\xi^T \Theta \xi + \sum_{\ell} \text{Tr}(Y_{\ell} S_\ell^2 [y])\| \leq \xi^T \Theta \xi + \sum_{r \in R} \text{Tr}(Y_{\ell} ) \leq \xi^T \Theta \xi + \phi_R(\lambda[\Upsilon]). \quad (21)$$

Taking expectation of both sides of the resulting inequality w.r.t. distribution $P$ of $\xi$ and taking into account that $\text{Tr}(\text{Cov}[P]|\Theta) \leq \text{Tr}(Q \Theta)$ due to $\Theta \succeq 0$ (by (19)) and $\text{Cov}[P] \preceq Q$, we get (20).

Note that when $P = \mathcal{N}(0, Q)$, the smallest possible upper bound on $E_{\eta \sim P\{\|V^T \eta\|}$ which can be extracted from Lemma 2.1 (this bound is efficiently computable) is tight, see Lemma 2.2 below.

An immediate consequence is

Corollary 2.2 Let

$$\Gamma(\Theta) = \max_{Q \in \Pi} \text{Tr}(Q \Theta) \quad (22)$$

and

$$\Psi_{\Pi}(H) = \min_{\{\Upsilon_{\ell} \}_{\ell \in L}, \Theta \in S^m} \left\{ \Gamma(\Theta) + \phi_R(\lambda[\Upsilon]) : \forall \ell, \left[ \begin{array}{c} \Theta \\ \frac{1}{2} M^T H^T \\ \sum_{\ell} S_\ell^2 [\Upsilon_{\ell}] \end{array} \right] \succeq 0 \right\} \quad (22)$$

Then $\Psi_{\Pi}(\cdot) : R^{m \times n} \rightarrow R$ is efficiently computable convex upper bound on $\Psi_{\Pi}(\cdot)$.

Indeed, given Lemma 2.1, the only non-evident part of the corollary is that $\Psi_{\Pi}(\cdot)$ is a well-defined real-valued function, which is readily given by Lemma 4.1, see Section 4.1.

Remark 2.3 When $\Upsilon = \{\Upsilon_{\ell} \}_{\ell \in L}$, $\Theta$ is a feasible solution to the right hand side problem in (22) and $s > 0$, the pair $\Upsilon' = \{s \Upsilon_{\ell} \}_{\ell \in L}$, $\Theta' = s^{-1} \Theta$ also is a feasible solution; since $\phi_R(\cdot)$ and $\Gamma(\cdot)$ are positive homogeneous of degree 1, we conclude that $\Psi_{\Pi}(\cdot)$ is in fact the infimum of the function

$$2 \sqrt{\Gamma(\Theta) \phi_R(\lambda[\Upsilon])} = \inf_{\delta > 0} \left[ s^{-1} \Gamma(\Theta) + s \phi_R(\lambda[\Upsilon]) \right]$$

over $\Upsilon, \Theta$ satisfying the constraints of the problem (22).

In every feasible solution $\Upsilon = \{\Upsilon_{\ell} \}_{\ell \in L}$, $\Theta$ to the problem (22) with $M[\Upsilon] := \sum_{\ell} S_\ell^2 [\Upsilon_{\ell}] > 0$, the pair $\Upsilon, \hat{\Theta} = \frac{1}{2} \text{HM} M^{-1}[\Upsilon] M^T H^T$ is feasible for the problem as well and $0 \preceq \hat{\Theta} \preceq \Theta$ (Schur Complement Lemma), so that $\Gamma(\hat{\Theta}) \leq \Gamma(\Theta)$. As a result,

$$\Psi_{\Pi}(H) = \inf_{\Upsilon} \left\{ \frac{1}{2} \Gamma(\text{HM} M^{-1}[\Upsilon] M^T H^T) + \phi_R(\lambda[\Upsilon]) : \Upsilon = \{\Upsilon_{\ell} \in S^M_{+}\}_{\ell \in L}, M[\Upsilon] > 0 \right\}. \quad (23)$$
Illustration. Consider the case when $\|u\| = \|u\|_p$ with $p \in [1, 2]$, and let us apply the just described scheme for upper-bounding $\Psi_{\Pi}$, assuming $\{Q\} \subset \Pi \subset \{S \in S^m_+ : S \leq Q\}$ for some given $Q > 0$, so that $\Gamma(\Theta) = \text{Tr}(Q\Theta)$, $\Theta \geq 0$. The unit ball of the norm conjugate to $\| \cdot \|$, that is, the norm $\| \cdot \|_q$, $q = \frac{p}{p-1} \in [2, \infty)$, is the basic spectratope (in fact, ellitope)

$$B_\ast = \{y \in \mathbb{R}^k : \exists r \in \mathcal{R} := \{\mathbb{R}^r_+ : \|r\|_{q/2} \leq 1\} : S^2[y] \leq r_\ell, 1 \leq \ell \leq L = l\}, \ \text{with} \ S_\ell[y] = y_\ell.$$

As a result, $\Psi$’s from Remark 2.3 are collections of $\nu$ positive semidefinite $1 \times 1$ matrices, and we can identify them with $\nu$-dimensional nonnegative vectors $\nu$, resulting in $\lambda[\Psi] = \nu$ and $\mathcal{M}[\Psi] = \text{Diag}\{\nu\}$. Besides this, for nonnegative $\nu$ we clearly have $\phi_R(\nu) = \|\nu\|_{p/(2-p)}$. The optimization problem in (23) now reads

$$\overline{\Psi}_{\Pi}(H) = \inf_{v \in \mathbb{R}^\nu} \left\{ \frac{1}{2} \text{Tr}(V \text{Diag}^{-1}\{\nu\}V^T) + \|v\|_{p/(2-p)} : v > 0 \right\} \quad [V = Q^{1/2}H]$$

After setting $\alpha_\ell = \|\text{Col}[V]\|_2$, (23) becomes

$$\overline{\Psi}_Q(H) = \inf_{\nu > 0} \left\{ \frac{1}{4} \sum_\ell \frac{a_\ell^2}{\nu_\ell} + \|v\|_{p/(2-p)} \right\}.$$

This results in $\overline{\Psi}_Q(H) = \|a_1; \ldots ; a_\mu\|_p$. Recalling what $\alpha_\ell$ and $V$ are, we end up with

$$\Psi_{\Pi}(H) \leq \overline{\Psi}_{\Pi}(H) := \left\| \left[ \|\text{Row}_1[H^TQ^{1/2}]\|_2 ; \ldots ; \|\text{Row}_\nu[H^TQ^{1/2}]\|_2 \right] \right\|_p.$$  

2.3.3 Putting things together: building linear estimate

An immediate summary of Corollaries 2.1, 2.2 is the following recipe for building “presumably good” linear estimate:

**Proposition 2.2** In the situation of Section 2.1 and under Assumption A, consider the convex optimization problem (for notation, see (18) and (21))

$$\text{Opt} = \min_{H, \Lambda, \Psi, \Theta} \left\{ \phi_T(\lambda[\Lambda]) + \phi_R(\lambda[\Psi]) + \phi_R(\lambda[\Psi']) + \Gamma(\Theta) : \right.$$  

$$\lambda = \{\Lambda_k \geq 0, k \leq K\}, \ \Psi = \{\Psi_\ell \geq 0, \ell \leq L\}, \ \Psi' = \{\Psi'_\ell \geq 0, \ell \leq L\},$$

$$\left. \left[ \begin{array}{c} \sum_k R_k[\Lambda_k] \\ \frac{1}{2}M^T[B - H^TA] \\ \frac{1}{2}M^T[H^T] \\ \frac{1}{2}M^T[A^T] \\ \frac{1}{2}M^T[H^T] \end{array} \right] \geq 0, \left[ \begin{array}{c} \frac{1}{2}B^T - A^T[M] \\ \frac{1}{2}M^T[A^T] \\ \frac{1}{2}M^T[H^T] \end{array} \right] \geq 0 \right\} \quad (24)$$

The problem is solvable, and the $H$-component $H_\ast$ of its optimal solution yields linear estimate $\hat{x}_{H_\ast}(\omega) = H^T \omega$ such that

$$\text{Risk}_{\Pi,\|\cdot\|} [\hat{x}(\cdot)|\chi] \leq \text{Opt}. \quad (25)$$

2.4 Near-optimality in Gaussian case

The risk of the linear estimate $\hat{x}_{H_\ast}(\cdot)$ constructed in (24), (25) can be compared to the minimax optimal risk of recovering $Bx$, $x \in \chi$, from observations corrupted by zero mean Gaussian noise with covariance matrix from II; formally, this minimax optimal risk is defined as

$$\text{RiskOpt}_{\Pi,\|\cdot\|} [\chi] = \sup_{Q \in \Pi} \inf_{\overline{x}(\cdot)} \left\{ \sup_{x \in \chi} E_{\xi \sim \mathcal{N}(0,Q)} \{ \|Bx - \overline{x}(Ax + \xi)\| \} \right\} \quad (26)$$

where the infimum is taken over all estimates.
Proposition 2.3 Under the premise and in the notation of Proposition 2.2, let
\[ M_2^2 = \max_W \left\{ \mathbb{E}_{\eta \sim \mathcal{N}(0, I_n)} ||BW^{1/2}\eta||^2 : W \in Q := \{ W \in S_n^+ : \exists t \in T : R_k[W] \leq t_kI_{d_k}, 1 \leq k \leq K \} \right\} \]  
we have
\[ \text{Risk}_{\Pi, \| \cdot \|}[\hat{x}_H|X] \leq \text{Opt} \leq C \sqrt{\ln(2F) \ln \left( \frac{2DM_2^2}{\text{Risk}_{\Pi, \| \cdot \|}|X|} \right)} \text{Risk}_{\Pi, \| \cdot \|}[X], \]  
where \( C \) is a positive absolute constant, and
\[ D = \sum d_k, \quad F = \sum f_\ell. \]

For the proof, see Section 4.4. The key component of the proof is the following important by its own right fact (for proof, see Section 4.3):

Lemma 2.2 Let \( Y \) be an \( N \times \nu \) matrix, let \( \| \cdot \| \) be a norm on \( \mathbb{R}^\nu \) such that the unit ball \( B^* \) of the conjugate norm is the spectratope (16), and let \( \zeta \sim \mathcal{N}(0, Q) \) for some positive semidefinite \( N \times N \) matrix \( Q \). Then the best upper bound on \( \psi_Q(Y) := \mathbb{E}||Y^T\zeta|| \) yielded by Lemma 2.1, that is, the optimal value \( \text{Opt}[Q] \) in the convex optimization problem (cf. (22))
\[ \text{Opt}[Q] = \min_{\Theta, \Upsilon} \left\{ \phi_R(\lambda[\Upsilon]) + \text{Tr}(\Theta \Upsilon) : \Upsilon = \{ \Upsilon_\ell \succeq 0, 1 \leq \ell \leq L \}, \Theta \in S_n^m, \right\} \]
(30)
(31)
(for notation, see Lemma 2.1 and (18)) satisfies the identity
\[ \forall (Q \succeq 0) : \text{Opt}[Q] = \text{opt}[Q] := \min_{G, \Upsilon} \left\{ \phi_R(\lambda[\Upsilon]) + \text{Tr}(G) : \Upsilon = \{ \Upsilon_\ell \succeq 0, \left[ \frac{G}{2M^T\gamma^*Y\gamma^*} + \frac{4YM}{\sum S_\ell^*[\Upsilon_\ell]} \right] \geq 0 \right\}, \]
and is a tight bound on \( \psi_Q(Y) \), namely,
\[ \psi_Q(Y) \leq \text{Opt}[Q] \leq \frac{4\sqrt{\ln \left( \frac{2\sqrt{3}F}{\sqrt{2} - e^{1/4}} \right)}}{\sqrt{2} - e^{1/4}} \psi_Q(Y) \leq 31 \sqrt{\ln(44F)} \psi_Q(Y), \]
(32)
where \( F = \sum f_\ell \) is the size of the spectratope (16).

2.5 Illustration: covariance matrix estimation

Suppose that we observe a sample
\[ \eta^T = \{ \eta_k = A\xi_k \}_{k \leq T} \]
where \( A \) is a given \( m \times n \) matrix, and \( \xi_1, ..., \xi_T \) are sampled, independently of each other, from zero mean Gaussian distribution with unknown covariance matrix \( \vartheta \) known to satisfy
\[ \gamma \vartheta_* \preceq \vartheta \preceq \vartheta_*, \]
where \( \gamma \geq 0 \) and \( \vartheta_* > 0 \) are given. Our goal is to recover \( \vartheta \), and the norm on \( S^n \) in which recovery error is measured satisfies Assumption A.
Processing the problem. We can process the just outlined problem as follows.

1. We represent the set \( \{ \vartheta \in S^p : \gamma \vartheta_* \leq \vartheta \leq \vartheta_* \} \) as the image of the matrix box
   \[
   \mathcal{V} = \{ v \in S^n : \|v\|_{\text{sn},\infty} \leq 1 \}
   \]
   under affine mapping, specifically, we set
   \[
   \vartheta_0 = \frac{1 + \gamma \vartheta_*}{2}, \quad \sigma = \frac{1 - \gamma}{2}
   \]
   and treat the matrix
   \[
   v = \sigma^{-1} \vartheta_*^{-1/2} (\vartheta - \vartheta_0) \vartheta_*^{-1/2} \quad \iff \vartheta = \vartheta_0 + \sigma \vartheta_*^{1/2} v \vartheta_*^{1/2}
   \]
as the signal underlying our observations. Note that our a priori information on \( \vartheta \) reduces to \( v \in \mathcal{V} \).

2. We pass from observations \( \eta_k \) to “lifted” observations \( \eta_k \eta_k^T \in S^m \), so that
   \[
   \mathbb{E}\{ \eta_k \eta_k^T \} = \mathbb{E}\{ A \xi_k \xi_k^T A^T \} = A \vartheta A^T = A \left( \vartheta_0 + \sigma A \vartheta_*^{1/2} v \vartheta_*^{1/2} \right) A^T,
   \]
   and treat as “actual” observations the matrices
   \[
   \omega_k = \eta_k \eta_k^T - A \vartheta_0 A^T.
   \]
   We have\(^3\)
   \[
   \omega_k = \mathcal{A} v + \zeta_k \text{ with } \mathcal{A} v = \sigma A \vartheta_*^{1/2} v \vartheta_*^{1/2} A^T \text{ and } \zeta_k = \eta_k \eta_k^T - A \vartheta[v] A^T. \tag{35}
   \]
   Observe that random matrices \( \zeta_1, \ldots, \zeta_T \) are i.i.d. with zero mean and covariance mapping \( \mathbb{Q}[v] \) (that of random matrix-valued variable \( \zeta = \eta \eta^T - \mathbb{E}\{ \eta \eta^T \}, \eta \sim \mathcal{N}(0, A \vartheta[v] A^T) \)).

3. Let us \( \succeq \)-upper-bound the covariance mapping of \( \zeta \). Observe that \( \mathbb{Q}[v] \) is a symmetric linear mapping of \( S^m \) into itself given by
   \[
   \langle h, \mathbb{Q}[v] h \rangle = \mathbb{E}\{ \langle h, \zeta \rangle^2 \} = \mathbb{E}\{ \langle h, \eta \eta^T \rangle^2 \} - \langle h, \mathbb{E}\{ \eta \eta^T \} \rangle^2, \quad h \in S^m,
   \]
   and for all \( h \in S^m \)
   \[
   \langle h, \mathbb{Q}[v] h \rangle \leq \mathbb{E}\{ \langle h, \eta \eta^T \rangle^2 \} = \mathbb{E}_{\zeta \sim \mathcal{N}(0, \mathbb{Q}[v])}\{ \text{Tr}^2 \left( h A \xi \xi^T A^T \right) \}
   = \mathbb{E}_{\chi \sim \mathcal{N}(0, I_n)}\{ \text{Tr}^2 \left( h A \vartheta^{1/2} \chi \vartheta^{1/2} \chi^T A^T \right) \}.
   \]
   Setting \( \mathcal{H}(h) = A^T h A \) and denoting \( \vartheta = \vartheta[v] \), so that \( 0 \leq \vartheta \leq \vartheta_* \), we therefore get
   \[
   \langle h, \mathbb{Q}[v] h \rangle \leq \mathbb{E}_{\chi \sim \mathcal{N}(0, I_n)}\{ \text{Tr}^2 (\chi^T \vartheta^{1/2} \mathcal{H}(h) \vartheta^{1/2} \chi) \}.
   \]

\(^3\)In our current considerations, we need to operate with linear mappings acting from \( S^p \) to \( S^q \). We treat \( S^k \) as Euclidean space equipped with the Frobenius inner product \( (u, v) = \text{Tr}(uv) \) and denote linear mappings from \( S^p \) into \( S^q \) by capital calligraphic letters, like \( \mathcal{A} \), \( \mathbb{Q} \), etc. Thus, \( \mathcal{A} \) in (35) denotes the linear mapping which, on a closest inspection, maps matrix \( v \in S^n \) into the matrix \( \mathcal{A} v = A \vartheta[v] - \vartheta[0] A^T \).
We have $\theta^{1/2}H(h)\theta^{1/2} = U\text{Diag}\{\lambda\}U^T$ with orthogonal $U$; setting $\bar{\chi} = U^T\chi \sim \mathcal{N}(0,I_n)$, we further have

$$
E_{\chi \sim \mathcal{N}(0,I_n)}\{\text{Tr}^2(\chi^T\theta^{1/2}H(h)\theta^{1/2}\chi)\} = E_{\bar{\chi} \sim \mathcal{N}(0,I_n)}\{\sum_{i \neq j} \lambda_i \lambda_j \text{Tr}^2(\bar{\chi}_i^2 \bar{\chi}_j^2)\} = \sum_{i \neq j} \lambda_i \lambda_j \text{Tr}^2(\bar{\chi}_i^2 \bar{\chi}_j^2) + \sum_i \lambda_i^4 E_{\bar{\chi} \sim \mathcal{N}(0,I_n)}\{\bar{\chi}_i^4\}
$$

$$
= \sum_{i \neq j} \lambda_i \lambda_j + 3\sum_i \lambda_i^2 = \sum_i \lambda_i^2 = \text{Tr}^2(\theta^{1/2}H(h)\theta^{1/2}) + 2\text{Tr}(\theta^{1/2}H(h)\theta H(h)\theta^{1/2})
$$

$$
= \text{Tr}^2(\theta H(h)) + 2\text{Tr}(\theta H(h)\theta H(h)) = \text{Tr}^2(\theta \theta^{-1/2}[\theta^{1/2}H(h)\theta^{1/2}]\theta^{-1/2}) + 2\text{Tr}(\theta H(h)\theta H(h))
$$

$$
\leq \text{Tr}(\theta^{-1/2}\theta^{1/2}) \text{Tr}(\theta^{-1/2}H(h)\theta^{1/2})^2 + 2\text{Tr}(\theta H(h)\theta H(h))
$$

$$
\leq (n+2)\text{Tr}(\theta^{-1/2}H(h)\theta^{1/2}) \text{Tr}(\theta^{-1/2}H(h)\theta^{1/2}) + 2\text{Tr}(\theta H(h)\theta H(h)) [\text{since } 0 \leq \theta \leq \vartheta^4].
$$

We conclude that

$$
\forall v \in \mathcal{V} : Q[v] \leq Q, \quad \langle e, Qh \rangle = (n+2)\text{Tr}(\vartheta^4 A^T h A \vartheta^4 e A), \quad e, h \in S^m. \quad (36)
$$

4. To continue, we need to set some additional notation to be used when operating with Euclidean spaces $S^p$, $p = 1, 2, ...$

- We denote $\bar{p} = \frac{p(p+1)}{2} = \text{dim} S^p$, $\mathcal{I}_p = \{(i,j) : 1 \leq i \leq j \leq p\}$, and for $(i,j) \in \mathcal{I}_p$, set

$$
eij_p = \begin{cases} e_i e_j^T, & i = j \\ \frac{e_i e_j^T + e_j e_i^T}{\sqrt{2}}, & i < j \end{cases},
$$

where $e_i$ are the standard orthonormal bases in $\mathbb{R}^p$. Note that $\{eij_p : (i,j) \in \mathcal{I}_p\}$ is the standard orthonormal basis in $S^p$. Given $v \in S^p$, we denote by $x^p(v)$ the vector of coordinates of $v$ in this basis:

$$
x^p_{ij}(v) = \text{Tr}(veij_p) = \left\{ \begin{array}{ll} v_{ij}, & i = j \\ \sqrt{2}v_{ij}, & i < j \end{array} \right., \quad (i,j) \in \mathcal{I}_p.
$$

Similarly, for $x \in \mathbb{R}^\bar{p}$, we index the entries in $x$ by pairs $(i,j) \in \mathcal{I}_p$, and set $x^p(x) = \sum_{(i,j) \in \mathcal{I}_p} x_{ij}eij_p$, so that $v \mapsto x^p(v)$ and $x \mapsto x^p(x)$ are inverse to each other linear norm-preserving maps identifying the Euclidean spaces $S^p$ and $\mathbb{R}^\bar{p}$ (recall that the inner products on these spaces are, respectively, the Frobenius and the standard one).

- Recall that $\mathcal{V}$ is the matrix box $\{v \in S^n : v^2 \leq I_n\} = \{v \in S^n : \exists t \in \mathcal{T} := [0,1] : v^2 \leq tI_n\}$. We denote by $\mathcal{X}$ the image of $\mathcal{V}$ under the mapping $x^m$:

$$
\mathcal{X} = \{x \in \mathbb{R}^\bar{p} : \exists t \in \mathcal{T} : R^2|x| \leq tI_n\}, \quad R[x] = \sum_{(i,j) \in \mathcal{I}_n} x_{ij}eij_p, \quad \bar{n} = \frac{1}{2}n(n+1).
$$

Note that $\mathcal{X}$ is a basic spectratope of size $n$.

Now we can assume that the signal underlying our observations is $x \in \mathcal{X}$, and the observations themselves are

$$
w_k = x^m(\omega_k) = x^m(\mathcal{A}v^n(x)) + z_k, \quad z_k = x^m(\zeta_k).
$$

\footnote{Indeed, with $g := H(h)$, we have $\text{Tr}(\theta g \theta g^T) = \text{Tr}(\theta^{1/2}g \theta^{1/2}g^T) \leq \text{Tr}(\theta^{1/2}) \text{Tr}(g^T g) = \text{Tr}(\theta^{1/2}g^T g\theta^{1/2}) \leq \text{Tr}(\theta^{1/2}g^T g\theta^{1/2}) = \text{Tr}(\theta_g^T \theta_g^T)$ and $\text{Tr}(\theta^{1/2}g^T g\theta^{1/2}) \leq n$.}
Note that $z_k \in \mathbb{R}^n$, $1 \leq k \leq T$, are zero mean i.i.d. random vectors with covariance matrix $Q[x]$ satisfying, in view of (36), the relation

$$Q[x] \preceq Q,$$

where $Q_{ij,k\ell} = (n + 2)\text{Tr}(\vartheta^* A^T e^{\text{ij}}_e A \vartheta^* A^T e^{k\ell}_e A)$, $(i,j) \in \mathcal{I}_m$, $(k,\ell) \in \mathcal{I}_m$.

Our goal is to estimate $\vartheta[v] - \vartheta[0]$, or, what is the same, to recover $\overline{B}_x := X^m(\vartheta[v](x) - \vartheta[0])$.

We assume that the norm in which the estimation error is measured is “transferred” from $\mathbb{S}^m$ to $\mathbb{R}^m$; we denote the resulting norm on $\mathbb{R}^m$ by $\| \cdot \|$ and assume that the unit ball $B_*$ of the conjugate norm $\| \cdot \|_*$ is given by spectratopic representation:

$$\{u \in \mathbb{R}^m : \|u\|_* \leq 1\} = \{u \in \mathbb{R}^m : \exists y \in \mathcal{Y} : u = My\},$$

$$\mathcal{Y} := \{y \in \mathbb{R}^m : \exists r \in \mathcal{R} : S^2[y] \preceq r I_L, 1 \leq \ell \leq L\}. \quad (37)$$

The formulated description of the estimation problem fit the premises of Proposition 2.2, specifically:

- the signal $x$ underlying our observation $w^T = [w_1; \ldots; w_T]$ is known to belong to basic spectratope $\mathcal{X} \in \mathbb{R}^n$, and the observation itself is of the form

$$w^T = \mathcal{A}^T x + z^T, \quad \mathcal{A}^T = [\mathcal{A}_1 \ldots \mathcal{A}_T], \quad z^T = [z_1; \ldots; z_T];$$

- the noise $z^T$ is zero mean, and its covariance matrix is $\preceq Q_T := \text{Diag}\{Q, \ldots, Q\}$, which allows to set $\Pi = \{Q_T\}$;

- our goal is to recover $\overline{B}_x$, and the norm $\| \cdot \|$ in which the recovery error is measured satisfies (37).

Proposition 2.2 supplies the linear estimate

$$\hat{x}(w^T) = \sum_{k=1}^T H^T_{x,k} w_k,$$

of $\overline{B}_x$ with $H_* = [H_{x,1}; \ldots; H_{x,T}]$ stemming from the optimal solution to the convex optimization problem

$$\text{Opt} = \min_{H=[H_1;\ldots;H_T],\Lambda,\mathcal{Y}} \left\{ \text{Tr}(\Lambda) + \phi_\mathcal{R}(\lambda[\mathcal{Y}]) + \overline{\Psi}_{(Q_T)}(H_1,\ldots,H_T) : \right\} \quad \begin{array}{l}
\Lambda \in \mathbb{S}^n_{++}, \mathcal{Y} = \{Y_{\ell} \succeq 0, \ell \leq L\}, \\
\frac{M^T [B - \sum_k H_k] M}{\sum_{\ell} S^2_{\ell}[Y_{\ell}]} \succeq 0, \end{array} \quad (38),$$

where

$$\mathcal{R}^*[\Lambda] \in \mathbb{S}^n : (\mathcal{R}^*[\Lambda])_{ij,k\ell} = \text{Tr}(\Lambda e^{ij}_e e^{k\ell}_e), (i,j) \in \mathcal{I}_n, (k,\ell) \in \mathcal{I}_n,$$

and, cf. (22),

$$\overline{\Psi}_{(Q_T)}(H_1,\ldots,H_T) = \min_{\mathcal{Y}',\Theta} \left\{ \text{Tr}(Q_T \Theta) + \phi_\mathcal{R}(\lambda[\mathcal{Y}']) : \Theta \in \mathbb{S}^{mT}, \mathcal{Y}' = \{Y_{\ell} \succeq 0, \ell \leq L\}, \right\} \begin{array}{l}
\Theta \\
\frac{\Theta}{\frac{1}{2}[H_1 M; \ldots; H_T M]} \succeq 0 \end{array},$$

16
5. Evidently, the function \( \Psi_{\{Q_T\}}(H_1, \ldots, H_T) \) remains intact when permuting \( H_1, \ldots, H_T \); with this in mind, it is clear that permuting \( H_1, \ldots, H_T \) and keeping intact \( \Lambda \) and \( \Upsilon \) is a symmetry of (38) – such a transformation maps feasible set onto itself and preserves the value of the objective. Since (38) is convex and solvable, it follows that there exists an optimal solution to the problem with \( H_1 = \ldots = H_T = H \). On the other hand,

\[
\Psi_{\{Q_T\}}(H, \ldots, H) = \min_{Y', \Theta} \left\{ \text{Tr}(Q_T \Theta) + \phi_R(\lambda[Y']) : \Theta \in S^{mT}, Y' = \{Y'_\ell \geq 0, \ell \leq L\} \right\} = \inf_{Y', \Theta} \left\{ \text{Tr}(Q_T \Theta) + \phi_R(\lambda[Y']) : \Theta \in S^{mT}, Y' = \{Y'_\ell > 0, \ell \leq L\}, \Theta \geq \frac{1}{2}[H_M; \ldots; H_M] \left[ \sum_\ell S^*_\ell[Y'_\ell] \right]^{-1} [H_M; \ldots; H_M]^T \right\}
\]

\[
= \inf_{Y'} \left\{ \phi_R(\lambda[Y']) + \frac{T}{4} \text{Tr} \left( QHM \left[ \sum_\ell S^*_\ell[Y'_\ell] \right]^{-1} M^T H^T \right) : Y' = \{Y'_\ell > 0, \ell \leq L\} \right\}
\]

\[
= \min_{Y', G} \left\{ \text{Tr}(QG) + \phi_R(\lambda[Y']) : G \in S^{mT}, Y' = \{Y'_\ell \geq 0, \ell \leq L\}, \begin{bmatrix} G & \frac{1}{2}H_M \frac{1}{2}M^T H^T \\ \frac{1}{2}M^T \frac{1}{2}H \end{bmatrix} \left[ \sum_\ell S^*_\ell[Y'_\ell] \right] \geq 0 \right\}
\]

(39)

(we have used Schur Complement Lemma combined with the fact that \( \sum_\ell S^*_\ell[Y'_\ell] \geq 0 \) whenever \( Y'_\ell > 0 \) for all \( \ell \), see Lemma 4.1).

In view of the above observations, when replacing variables \( H \) and \( G \) with \( \overline{H} = TH \) and \( \overline{G} = T^2G \), respectively, problem (38), (39) becomes

\[
\text{Opt} = \min_{\overline{H}, \overline{G}, \Lambda, \Upsilon} \left\{ \text{Tr}(\Lambda) + \phi_R(\lambda[\Upsilon]) + \phi_R(\lambda[\Upsilon]) + \frac{1}{T} \text{Tr}(Q\overline{G}) : \Lambda \in S^+_n, \Upsilon = \{Y'_\ell \geq 0, \ell \leq L\}, \Upsilon' = \{Y'_\ell \geq 0, \ell \leq L\}, \begin{bmatrix} R^*[\Lambda] & \frac{1}{2}M^T B - A B^T \end{bmatrix} \begin{bmatrix} \frac{1}{2}M^T [B - \overline{H} - \overline{H}] M \end{bmatrix} \begin{bmatrix} \frac{1}{2}M^T [B - \overline{H} - \overline{H}] M \end{bmatrix} \left[ \sum_\ell S^*_\ell[Y'_\ell] \right] \geq 0 \right\},
\]

(40)

and the estimate

\[
\hat{x}(\omega^T) = \frac{1}{T} \overline{H}^T \sum_{k=1}^T w_k
\]

stemming from an optimal solution to (40) satisfies

\[
\text{Risk}_{\Pi, \|\cdot\|}[\hat{x}] \leq \text{Opt},
\]

where \( \Pi = \{Q_T\} \).
2.6 Estimation from repeated observations

Consider the special case of the situation considered in Section 2.1, where observation \( \omega \) in (2) is a \( T \)-element sample: \( \overline{\omega} = [\bar{\omega}_1; ...; \bar{\omega}_T] \) with components

\[
\bar{\omega}_t = \bar{A}x + \xi_t, \quad t = 1, ..., T
\]

and \( \xi \) are i.i.d. observation noises with zero mean distribution \( \bar{P} \) satisfying \( \bar{P} \ll \bar{\Pi} \) for some convex compact set \( \bar{\Pi} \subset \text{int} S_{\bar{m}}^T \). In other words, we are in the situation where

\[
A = [\bar{A}; ...; \bar{A}]^T \in \mathbb{R}^{m \times n} \text{ for some } \bar{A} \in \mathbb{R}^{\bar{m} \times n} \text{ and } m = \bar{T} \bar{m},
\]

\[
\Pi = \{ Q = \text{Diag}\{ \bar{Q}, ..., \bar{Q} \}^T, \bar{Q} \in \bar{\Pi} \}.
\]

The same argument as used in item 5 of Section 2.5 justifies the following

**Proposition 2.4** In the situation in question and under Assumption A, the linear estimate of \( Bx \) yielded by an optimal solution to problem (24) can be found as follows. We consider the convex optimization problem

\[
\text{Opt} = \min_{R, \Lambda, \Upsilon, \Theta} \left\{ \phi_T(\lambda[\Lambda]) + \phi_R(\lambda[\Upsilon]) + \phi_R(\lambda[\Upsilon']) + \frac{1}{T} \Upsilon(\Theta) : \right.
\]

\[
\begin{align*}
\Lambda &= \{ \Lambda_k \succeq 0, k \leq K \}, \quad \Upsilon = \{ \Upsilon_\ell \geq 0, \ell \leq L \}, \quad \Upsilon' = \{ \Upsilon'_\ell \geq 0, \ell \leq L \}, \\
&\left[ \frac{1}{2} M^T [B - H^T A] \right] \sum_k R_k[\Lambda_k] \geq 0, \\
&\left[ \frac{1}{2} M^T H^T \right] \sum_\ell S_\ell[\Upsilon'_\ell] \geq 0,
\end{align*}
\]

where

\[
\Upsilon(\Theta) = \max_{\bar{Q} \in \bar{\Pi}} \text{Tr}(\bar{Q}\Theta).
\]

The problem is solvable, and the estimate in question is yielded by the \( \bar{H} \)-component \( \bar{x}_* \) of the optimal solution according to

\[
\bar{x}(\overline{\omega}_1; ...; \overline{\omega}_T) = \frac{1}{T} \bar{H}^T \sum_{t=1}^T \bar{\omega}_t.
\]

The provided by Proposition 2.2 upper bound on the risk \( \text{Risk}_{\|x\|}^{\text{Opt}}(\cdot)[X] \) of this estimate is \( \text{Opt} \).

The advantage of this result as compared to what is stated under the circumstances by Proposition 2.2 is that the sizes of optimization problem (41) are independent of \( T \).

3 Linear estimation in the case of uncertain-but-bounded noise

So far, the main subject of our interest was recovering (linear images of) signals via indirect observations of these signals corrupted by random noise. In this section, we focus on alternative observation schemes – those with “uncertain-but-bounded” and with “mixed” noise.
3.1 Uncertain-but-bounded noise

Consider recovering problem where one, given observation
\[ \omega = Ax + \eta \] (41)
of unknown signal \( x \) known to belong to a given signal set \( \mathcal{X} \), wants to recover linear image \( Bx \) of \( x \). Here \( A \) and \( B \) are given \( m \times n \) and \( \nu \times n \) matrices. The situation looks exactly as before; the difference with our previous considerations is that now we do not assume the observation noise to be random; all we assume about \( \eta \) is that it belongs to a given compact set \( \mathcal{H} \) (“uncertain-but-bounded observation noise”). In the situation in question, a natural definition of the risk on \( \mathcal{X} \) of a candidate estimate \( \omega \mapsto \hat{x}(\omega) \) is
\[ \text{Risk}_{\mathcal{H}}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}, \eta \in \mathcal{H}} \| Bx - \hat{x}(Ax + \eta) \| \] (42)
(“\( \mathcal{H} \)-risk”).

We are about to prove that when \( \mathcal{X} \) and \( \mathcal{H} \) are spectratopes, and the unit ball of the norm \( \| \cdot \|_* \) conjugate to \( \| \cdot \| \) is a basic spectratope, an efficiently computable linear estimate is near-optimal in terms of its \( \mathcal{H} \)-risk.

Our initial observation is that the situation in question reduces straightforwardly to the one where there is no observation noise at all. Indeed, let \( \mathcal{Y} = \mathcal{X} \times \mathcal{H} \); then \( \mathcal{Y} \) is a spectratope, and we lose nothing when assuming that the signal underlying observation \( \omega \) is \( y = [x; \eta] \in \mathcal{Y} \):
\[ \omega = Ax + \eta = \bar{A}y, \quad \bar{A} = [A, I_m], \]
while the entity to be recovered is
\[ Bx = \bar{B}y, \quad \bar{B} = [B, 0_{\nu \times m}]. \]

With these conventions, the \( \mathcal{H} \)-risk of a candidate estimate \( \hat{x}(\cdot) : \mathbb{R}^m \to \mathbb{R}^\nu \) becomes the quantity
\[ \text{Risk}_{\| \cdot \|}[\hat{x}|\mathcal{X} \times \mathcal{H}] = \sup_{y = [x; \eta] \in \mathcal{X} \times \mathcal{H}} \| \bar{B}y - \hat{x}(\bar{A}y) \|, \]
that is, we indeed arrive at the situation where the observation noise is identically zero.

To avoid messy notation, let us assume that the outlined reduction has been carried out in advance, so that

The problem of interest is to recover the linear image \( Bx \in \mathbb{R}^\nu \) of an unknown signal \( x \) known to belong to a given spectratope \( \mathcal{X} \) from noiseless observation
\[ \omega = Ax \in \mathbb{R}^m, \]
and the risk of a candidate estimate is defined as
\[ \text{Risk}_{\| \cdot \|}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \| Bx - \hat{x}(Ax) \|, \]
where \( \| \cdot \| \) is a given norm with a basic spectratope as the unit ball \( B_* \) of the conjugate norm. By our standard argument,

We lose nothing when assuming that the spectratope \( \mathcal{X} \) is basic as well, so that
\[ \mathcal{X} = \{ x \in \mathbb{R}^n : \exists t \in \mathcal{T} : R_k^2[x] \leq t_k I_d, k \leq K \}, \]
\[ B_* := \{ u \in \mathbb{R}^\nu : \| u \|_* \leq 1 \} = \{ u \in \mathbb{R}^\nu : \exists r \in \mathcal{R} : S^2_\ell[u] \leq r \ell I_{\ell}, \ell \leq L \} \] (43)
with the standard restrictions on \( \mathcal{T}, \mathcal{R} \) and \( R_k[\cdot], S_\ell[\cdot] \).
3.1.1 Building linear estimate

Let us build a seemingly good linear estimate. For a linear estimate \( \hat{x}_H(\omega) = H^T\omega \), we have

\[
\text{Risk}_{\|\cdot\|}[\hat{x}_H;\mathcal{X}] = \max_{x \in \mathcal{X}} \| (B - H^T A) x \| = \max_{[u;x] \in B \times \mathcal{X}} [u;x]^T \begin{bmatrix} \frac{1}{2} (B - H^T A) \end{bmatrix} [u;x].
\]

Applying Proposition 2.1, we arrive at the following

**Proposition 3.1**

In the situation of this section, consider the convex optimization problem

\[
\text{Opt}_\# = \min_{H,Y=\{Y_\ell\},\Lambda=\{\Lambda_k\}} \left\{ \phi_R(\lambda[Y]) + \phi_T(\lambda[\Lambda]) : \lambda_\ell \geq 0, \ \lambda_k \geq 0, \ \forall (\ell,k) \right\},
\]

where \( R^*_{\ell}[\cdot], S^*_{\ell}[\cdot] \) are induced by \( R_\ell[\cdot], S_\ell[\cdot] \), respectively, as explained in Section 2.2. The problem is solvable, and the risk of the linear estimate \( \hat{x}_H(\cdot) \) yielded by the \( H \)-component of an optimal solution does not exceed \( \text{Opt}_\# \).

3.1.2 Near-optimality

**Proposition 3.2**

The linear estimate \( \hat{x}_H \), yielded by Proposition 3.1 is near-optimal in terms of its risk:

\[
\text{Risk}_{\|\cdot\|}[\hat{x}_H;\mathcal{X}] \leq \text{Opt}_\# \leq 2 \ln(2D) \text{Risk}_{\text{opt}}[\mathcal{X}], \quad D = \sum_k d_k + \sum_\ell f_\ell,
\]

where \( \text{Risk}_{\text{opt}}[\mathcal{X}] \) is the minimax optimal risk:

\[
\text{Risk}_{\text{opt}}[\mathcal{X}] = \inf_{\hat{x}} \text{Risk}_{\|\cdot\|}[\hat{x};\mathcal{X}],
\]

where \( \inf \) is taken w.r.t. all possible estimates.

For proof, see Section 4.5.

3.2 Mixed noise

So far, we have considered separately the cases of random and uncertain-but-bounded observation noises in (2). Note that both these observation schemes are covered by the following “mixed” scheme:

\[
\omega = Ax + \xi + \eta,
\]

where, as above, \( A \) is a given \( m \times n \) matrix, \( x \) is an unknown deterministic signal known to belong to a given signal set \( \mathcal{X} \), \( \xi \) is random noise with distribution known to belong to a family \( \mathcal{P} \) of Borel probability distributions on \( \mathbb{R}^m \) satisfying (1) for a given convex compact set \( \Pi \subset \text{int} S^m_+ \), and \( \eta \) is “uncertain-but-bounded” observation error known to belong to a given set \( \mathcal{H} \). As before, our goal is to recover \( Bx \in \mathbb{R}^\nu \) via observation \( \omega \). In our present situation, given a norm \( \| \cdot \| \) on \( \mathbb{R}^\nu \), we can quantify the performance of a candidate estimate \( \omega \mapsto \hat{x}(\omega) : \mathbb{R}^m \to \mathbb{R}^\nu \) by its risk

\[
\text{Risk}_{\|\cdot\|}[\mathcal{H},\|\cdot\|][\hat{x};\mathcal{X}] = \sup_{x \in \mathcal{X}} \mathbf{E}_{\xi \sim P, \eta \in \mathcal{H}} \{ \| Bx - \hat{x}(Ax + \xi + \eta) \| \}. 
\]
Observe that the estimation problem associated with “mixed” observation scheme straightforwardly reduces to similar problem for random observation scheme, by the same trick we have used in Section 3 to eliminate observation noise at all. Indeed, let us treat \( x^+ = [x; \eta] \in X^+ := X \times H \) as the new signal/signal set underlying our observation, and set \( Ax^+ = Ax + \eta, Bx^+ = Bx \), where \( x^+ = [x; \eta] \). With these conventions, the “mixed” observation scheme reduces to

\[
\omega = \tilde{A}x^+ + \xi,
\]

and for every candidate estimate \( \tilde{x}(\cdot) \) it clearly holds

\[
\text{Risk}_{\Pi, H, \|\cdot\|}\{\tilde{x} | X\} = \text{Risk}_{\Pi, H, \|\cdot\|}\{\tilde{x} | X^+\},
\]

and we arrive at the situation of Section 2. Assuming that \( X \) and \( H \) are spectratopes, so is \( X^+ \), meaning that all results of Section 2 on building presumably good linear estimates and their near-optimality are applicable to our present setup.

An immediate question is: given the reduction we have described, what is the reason for considera-

\[ 4 \text{ Proofs} \]

\subsection{4.1 Technical lemma}

In the sequel, we frequently use the following technical fact:

\[ \text{Lemma 4.1} \quad \text{Given basic spectratope (4) and a positive definite } n \times n \text{ matrix } Q \text{ and setting } \Lambda_k = R_k[Q], \]

\[ \text{we get a collection of positive semidefinite matrices, and } \sum_k R_k^*[\Lambda_k] \text{ is positive definite. As a corollary, whenever } M_k, k \leq K, \text{ are positive definite matrices, the matrix } \sum_k R_k^*[M_k] \text{ is positive definite. In addition, the set } \]

\[ Q = \{Q \in S^n : Q \geq 0, \exists t \in T : R_k[Q] \preceq t_k I_{d_k}, k \leq K\} \]

\[ \text{is nonempty convex compact set containing a neighbourhood of the origin.} \]

\[ \text{Proof.} \quad \text{Let us prove the first claim, Assuming the opposite, we would be able to find a nonzero vector } \]

\[ y \text{ such that } \sum_k y^T R_k^*[\Lambda_k] y \leq 0, \text{ whence } \]

\[ 0 \geq \sum_k y^T R_k^*[\Lambda_k] y = \sum_k \text{Tr}(R_k^*[\Lambda_k][yy^T]) = \sum_k \text{Tr}(\Lambda_k R_k[yy^T]) \]

(we have used (10), (6)). Since \( \Lambda_k = R_k[Q] \geq 0 \) due to \( Q \geq 0 \), see (7), it follows that \( \text{Tr}(\Lambda_k R_k[yy^T]) = 0 \) for all \( k \). Now, the linear mapping \( R_k[\cdot] \) is \( \gamma \)-monotone, and \( Q \) is positive definite, implying that \( Q \succeq r_k yy^T \) for some \( r_k > 0 \), whence \( \Lambda_k \succeq r_k R_k[yy^T] \), and therefore \( \text{Tr}(\Lambda_k R_k[yy^T]) = 0 \) implies \( \text{Tr}(R_k[yy^T]) = 0 \), that is, \( R_k[yy^T] = R_k^*[y] = 0 \). Since \( R_k[\cdot] \) takes values in \( S^{d_k} \), we get \( R_k[y] = 0 \) for all \( k \), which is impossible due to \( y \neq 0 \) and property (S3), see Section 2.2.

The second claim is an immediate consequence of the first one. Indeed, when \( M_k \) are positive definite, we can find \( \gamma > 0 \) such that \( \Lambda_k \preceq \gamma M_k \) for all \( k \leq K \); invoking (11), we conclude that \( R_k^*[\Lambda_k] \preceq \gamma R_k^*[M_k] \), whence \( \sum_k R_k^*[M_k] \) is positive definite along with \( \sum_k R_k^*[\Lambda_k] \).
Finally, the only nonevident component in the third claim of Lemma is that $Q$ is bounded. To see that it is the case, let us fix a collection \( \{ M_k \} \) of positive definite matrices $M_k \in S^{d_k}$, and let us set $M = \sum_k R_k^T M_k$, so that $M \succ 0$ by already proved part of Lemma. For $Q \in Q$, we have $R_k[Q] \preceq t_k I_{d_k}$, $k \leq K$, for the properly selected $t \in \mathcal{T}$, so that
\[
\text{Tr}(QM) = \sum_k \text{Tr}(Q R_k^T M_k) = \sum_k \text{Tr}(R_k[Q] M_k) \leq \sum_k t_k \text{Tr}(M_k)
\]
(we have used (10)), and the concluding quantity does not exceed properly selected $C < \infty$ (since $\mathcal{T}$ is compact). Thus, $Q \subset \{ Q : Q \succeq 0, \text{Tr}(QM) \leq C \}$, whence $Q$ is bounded due to $M \succ 0$. \hfill \Box

4.2 Proof of Proposition 2.1

4.2.1 Preliminaries: matrix concentration

We are about to use the following deep matrix concentration result, see [32, Theorem 4.6.1]:

**Theorem 4.1** Let $Q_i \in S^n$, $1 \leq i \leq I$, and let $\xi_i$, $i = 1, ..., I$, be independent Rademacher ($\pm 1$ with probabilities 1/2) or $\mathcal{N}(0, 1)$ random variables. Then for all $t \geq 0$ one has
\[
\text{Prob} \left\{ \left\| \sum_{i=1}^I \xi_i Q_i \right\| \geq t \right\} \leq 2n \exp \left\{ -\frac{t^2}{2v_Q} \right\}
\]
where $\| \cdot \|$ is the spectral norm, and $v_Q = \left\| \sum_{i=1}^I Q_i^2 \right\|$.

We need the following immediate consequence of Theorem:

**Lemma 4.2** Given spectratope (4), let $Q \in S^n_+$ be such that
\[
R_k[Q] \preceq \rho t_k I_{d_k}, \ 1 \leq k \leq K,
\]
for some $t \in \mathcal{T}$ and some $\rho \in (0, 1]$. Then
\[
\text{Prob}_{\xi \sim \mathcal{N}(0, Q)} \{ \xi \notin X \} \leq \min \left\{ 2De^{-\frac{1}{2D}}, 1 \right\}, \quad D := \sum_{k=1}^K d_k.
\]

**Proof.** When setting $\xi = Q^{1/2} \eta$, $\eta \sim \mathcal{N}(0, I_n)$, we have
\[
R_k[\xi] = R_k[Q^{1/2} \eta] =: \sum_{i=1}^n \eta_i \tilde{R}_k^{ki} = \tilde{R}_k[\eta]
\]
with
\[
\sum_k \| \tilde{R}_k^{ki} \|^2 = E_{\eta \sim \mathcal{N}(0, I_n)} \{ R_k^2[\eta] \} = E_{\xi \sim \mathcal{N}(0, Q)} \{ R_k^2[\xi] \} = R_k[Q] \preceq \rho t_k I_{d_k}
\]
due to (8). Hence, by Theorem 4.1
\[
\text{Prob}_{\xi \sim \mathcal{N}(0, Q)} \{ \| R_k[\xi] \|^2 \geq t_k \} = \text{Prob}_{\eta \sim \mathcal{N}(0, I_n)} \{ \| \tilde{R}_k[\eta] \|^2 \geq t_k \} \leq 2d_k e^{-\frac{t_k}{2D}}.
\]
We conclude that
\[
\text{Prob}_{\xi \sim \mathcal{N}(0, Q)} \{ \xi \notin X \} \leq \text{Prob}_{\xi \sim \mathcal{N}(0, Q)} \{ \exists k : \| R_k[\xi] \|^2 > t_k \} \leq 2De^{-\frac{1}{2D}}. \hfill \Box
\]
4.2.2 Proving Proposition 2.1

1°. Under the premise of Proposition 2.1, let us set \( C = P^T CP \), and consider the conic problem

\[
\text{Opt}_# = \max_{Q, t} \left\{ \text{Tr}(\tilde{C}Q) : Q \succeq 0, \mathcal{R}_k[Q] \leq t_k I_{d_k} \forall k, [t; 1] \in \mathcal{K}[T] \right\}.
\]

(48)

Since \( T \) contains positive vectors, this problem is strictly feasible. Besides this, the feasible set of the problem is bounded by Lemma 4.1 and since \( T \) is compact. Thus, problem (48) is strictly feasible with bounded feasible set and thus is solvable along with its conic dual, both problems sharing a common optimal value (Conic Duality Theorem, see Appendix A):

\[
\text{Opt}_# = \min_{\Lambda \in \{\Lambda_k\}_{k \leq K} \mid g; L} \left\{ s : \text{Tr}\left(\sum_k \mathcal{R}_k^*[\Lambda_k] - L\right) - \sum_k [\text{Tr}(\Lambda_k) + g] t_k = \text{Tr}(\tilde{C}Q) \forall (Q, t), \right\}
\]

[recall that the cone dual to \( \mathcal{K}[T] \) is \( \{[g; s] : s \geq \phi_T(-g)\} \)]

\[
= \min_{\Lambda \mid g; L} \left\{ s : \sum_k \mathcal{R}_k^*[\Lambda_k] - L = \tilde{C}, g = -\lambda[\Lambda], \right\}
\]

\[
= \min_{\Lambda} \left\{ \phi_T(\lambda[\Lambda]) : \sum_k \mathcal{R}_k^*[\Lambda_k] \succeq \tilde{C}, \lambda \geq 0 \forall k \right\} = \text{Opt}_*.
\]

We see that (13) is solvable along with conic dual to problem (48), and

\[
\text{Opt}_# = \text{Opt}_*.
\]

2°. Problem (48), as we already know, is solvable; let \( Q_*, t^* \) be an optimal solution to the problem. Next, let us set \( R_* = Q_*^{1/2}, \tilde{C} = R_* \tilde{C} R_* \), and let \( \tilde{C} = UDU^T \) be the eigenvalue decomposition of \( \tilde{C} \), so that the matrix \( D = U^T R_* \tilde{C} R_* U \) is diagonal, and the trace of this matrix is \( \text{Tr}(R_* \tilde{C} R_*) = \text{Tr}(\tilde{C} Q_*) = \text{Opt}_# = \text{Opt}_* \). Now let \( V = R_* U \), and let \( \xi \sim \mathcal{R}, \) i.e. \( \eta \) is \( n \)-dimensional random Rademacher vector (with independent entries taking values \( \pm 1 \) with probabilities 1/2). We have

\[
\xi^T \tilde{C} \xi = \eta^T [V^T \tilde{C} V] \eta = \eta^T [U^T R_* \tilde{C} R_* U] \eta = \eta^T D \eta \equiv \text{Tr}(D) = \text{Opt}_*.
\]

(49)

(recall that \( D \) is diagonal) and

\[
\mathbf{E}_\xi \{\xi^T\} = \mathbf{E}_\eta \{V \eta^T V^T\} = V V^T = R_* U U^T R_* = R_*^2 = Q_*.
\]

From the latter relation,

\[
\mathbf{E}_\xi \{R_k^T \xi \} = \mathbf{E}_\xi \{\mathcal{R}_k[\xi^T]\} = \mathcal{R}_k[\mathbf{E}_\xi \{\xi^T\}] = \mathcal{R}_k[Q_*] \succeq t_k^* I_{d_k}, 1 \leq k \leq K.
\]

(50)

On the other hand, with properly selected symmetric matrices \( \tilde{R}^{kj} \) we have

\[
R_k[V y] = \sum_i \tilde{R}^{ki} y_i
\]

identically in \( y \in \mathbb{R}^n \), whence

\[
\mathbf{E}_\xi \{R_k^T \xi \} = \mathbf{E}_\eta \{\tilde{R}^{kj} \eta \} = \mathbf{E}_\eta \left\{ \left[ \sum_i \eta_i \tilde{R}^{ki} \right]^2 \right\} = \sum_{i,j} \mathbf{E}_\eta (\eta_i \eta_j) \tilde{R}^{ki} \tilde{R}^{kj} = \sum_i |\tilde{R}^{ki}|^2.
\]

This combines with (50) to imply that

\[
\sum_i |\tilde{R}^{ki}|^2 \leq t_k^* I_{d_k}, 1 \leq k \leq K.
\]

(51)
3°. Let us fix \( k \leq K \). Applying Theorem 4.1, we derive from (51) that
\[
\text{Prob}_{\eta \sim \mathcal{R}} \{ \| \bar{R}_k[\eta] \|^2 > t_k^* / \rho \} < 2d_k e^{-\frac{t_k^*}{\rho}},
\]
and recalling the relation between \( \xi \) and \( \eta \), we arrive at
\[
\text{Prob} \{ \xi : \| R_k[\xi] \|^2 > t_k^* / \rho \} < 2d_k e^{-\frac{t_k^*}{\rho}} \quad \forall \rho \in (0, 1).
\] (52)
Note that when \( t_k^* = 0 \) (51) implies \( \bar{R}_k^i = 0 \) for all \( i \), so that \( R_k[\xi] = \bar{R}_k[\eta] = 0 \), and (52) also holds for those \( k \).

Now let us set \( \rho = \frac{1}{2 \max \{ \ln(2 \ln (M)), 1 \}} \). For this \( \rho \), the sum over \( k \leq K \) of the right hand sides in inequalities (52) is \( \leq 1 \), implying that there exists a realization \( \bar{\xi} \) of \( \xi \) such that
\[
\| R_k[\bar{\xi}] \|^2 \leq t_k^* / \rho, \quad \forall k,
\]
or, equivalently,
\[
\bar{x} := \rho^{1/2} \bar{P} \bar{\xi} \in \mathcal{X},
\]
implying that
\[
\text{Opt} \geq \bar{x}^T \bar{C} \bar{x} = \rho \xi^T \bar{C} \xi = \rho \text{Opt}_*,
\]
(the concluding equality is due to (49)), and we arrive at the right inequality in (14).

4.3 Proof of Lemma 2.2

1°. Let us verify (31). When \( Q \succ 0 \), passing from variables \((\Theta, \Upsilon)\) in problem (30) to the variables \((G = Q^{1/2} \Theta Q^{1/2}, \Upsilon)\), the problem becomes exactly the optimization problem in (31), implying that \( \text{Opt}[Q] = \text{Opt}[\bar{Q}] \) when \( Q \succ 0 \). As it is easily seen, both sides in this equality are continuous in \( Q \succeq 0 \), and (31) follows.

2°. Let us set \( \zeta = Q^{1/2} \eta \) with \( \eta \sim \mathcal{N}(0, I_N) \) and \( Z = Q^{1/2} Y \). All we need to complete the proof of Lemma 2.2 is to show that the quantity
\[
[\text{Opt}[Q] =] \quad \text{Opt} := \min_{\Theta, \Upsilon = \{ \Upsilon_\ell, \ell \leq L \}} \left\{ \phi_R(\lambda[\Upsilon]) + \text{Tr}(\Theta) : \Upsilon_\ell \succeq 0, \left[ \begin{array}{c} \Theta \\ \frac{1}{2} M^T Z^T \\ \frac{1}{2} M Z \end{array} \right] \succeq 0 \right\}
\] (53)
satisfies
\[
\text{Opt} \leq \frac{4}{\sqrt{2}} \ln \left( \frac{2^{2/4}}{\sqrt{2} - e^{-1/4}} \right) \psi_I(Z), \quad \psi_I(Z) = E_{\eta \sim \mathcal{N}(0, I_N)} \{ \| Z^T \eta \| \}. \quad \text{(54)}
\]

3°. Let us represent \text{Opt} as the optimal value of a conic problem. Setting
\[
\mathbf{K} = \mathbf{K}[\mathcal{R}] = \text{cl}\{ [r; s] : s > 0, r / s \in \mathcal{R} \},
\]
we ensure that
\[
\mathcal{R} = \{ r : [r; 1] \in \mathbf{K} \}, \quad \mathbf{K}_* = \{ [g; s] : s \geq \phi_R(-g) \},
\]
where \( \mathbf{K}_* \) is the cone dual to \( \mathbf{K} \). Consequently, (53) reads
\[
\text{Opt} = \min_{\Theta, \Upsilon, \theta} \left\{ \theta + \text{Tr}(\Theta) : \begin{array}{c} \Upsilon_\ell \succeq 0, 1 \leq \ell \leq L \\ \Theta \\ \frac{1}{2} M^T Z^T \\ \frac{1}{2} M Z \end{array} \succeq 0 \right\}. \quad \text{(P)}
\]
Now let us prove that there exists matrix \( W \in S^q_+ \) and \( r \in \mathbb{R} \) such that
\[
S_\ell[W] \preceq r_\ell I_{f_\ell}, \ell \leq L,
\]
and
\[
\text{Opt} \leq \sum_i \sigma_i(ZMW^{1/2}), \tag{56}
\]
where \( \sigma_1(\cdot) \geq \sigma_2(\cdot) \geq ... \) are singular values.

To get the announced result, let us pass from problem \((P)\) to its conic dual. Applying Lemma 4.1 we conclude that \((P)\) is strictly feasible; in addition, \((P)\) clearly is bounded, so that the dual to \((P)\) problem \((D)\) is solvable with optimal value \(\text{Opt}\). Let us build \((D)\).

Denoting by \( \Lambda_\ell \succeq 0, 1 \leq \ell \leq L, \) \[
\begin{bmatrix}
G - R^T W & -R \\
-R & W
\end{bmatrix} \succeq 0,
\]
\([r; \tau] \in K\) the Lagrange multipliers for the respective constraints in \((P)\), and aggregating these constraints, the multipliers being the aggregation weights, we arrive at the following aggregated constraint:
\[
\text{Tr}(\Theta G) + \text{Tr}(W \sum_\ell S_\ell^*[\Theta_\ell]) + \sum_\ell \text{Tr}(\Lambda_\ell \Theta_\ell) - \sum_\ell r_\ell \text{Tr}(\Theta_\ell) + \theta \tau \geq \text{Tr}(ZMR^T).
\]

To get the dual problem, we impose on the Lagrange multipliers, in addition to the initial conic constraints like \( \Lambda_\ell \succeq 0, 1 \leq \ell \leq L, \) the restriction that the left hand side in the aggregated constraint, identically in \( \Theta, \Theta_\ell \) and \( \theta \), is equal to the objective of \((P)\), that is,
\[
G = I, S_\ell[W] + \Lambda_\ell - r_\ell I_{f_\ell} = 0, 1 \leq \ell \leq L, \tau = 1,
\]
and maximize, under the resulting restrictions, the right-hand side of the aggregated constraint. After immediate simplifications, we arrive at
\[
\text{Opt} = \max_{W, R, r} \left\{ \text{Tr}(ZMR^T) : W \succeq R^T R, r \in \mathbb{R}, S_\ell[W] \preceq r_\ell I_{f_\ell}, 1 \leq \ell \leq L \right\}
\]
(note that \( r \in \mathbb{R} \) is equivalent to \([r; 1] \in K\), and \( W \succeq R^T R \) is the same as \( \begin{bmatrix} I & -R \\
-R & W \end{bmatrix} \succeq 0 \)). Now, to say that \( R^T R \preceq W \) is exactly the same as to say that \( R = SW^{1/2} \) with the spectral norm \( \|S\|_{\text{sh}, \infty} \) of \( S \) not exceeding 1, so that
\[
\text{Opt} = \max_{W, S, r} \left\{ \text{Tr}(ZMW^{1/2}) : W \succeq 0, \|S\|_{\text{sh}, \infty} \leq 1, r \in \mathbb{R}, S_\ell[W] \preceq r_\ell I_{f_\ell}, 1 \leq \ell \leq L \right\}
\]
and we can immediately eliminate the \( S \)-variable, using the well-known fact that for every \( p \times q \) matrix \( J \), it holds
\[
\max_{S \in \mathbb{R}^{p \times q}, \|S\|_{\text{sh}, \infty} \leq 1} \text{Tr}(JST^T) = \|J\|_{\text{sh}, 1},
\]
where \( \|J\|_{\text{sh}, 1} \) is the nuclear norm (the sum of singular values) of \( J \). We arrive at
\[
\text{Opt} = \max_{W, r} \left\{ \|ZMW^{1/2}\|_{\text{sh}, 1} : r \in \mathbb{R}, W \succeq 0, S_\ell[W] \preceq r_\ell I_{d_\ell}, 1 \leq \ell \leq L \right\}.
\]
The resulting problem clearly is solvable, and its optimal solution \( W \) ensures the target relations (55), (56).
5°. Given $W$ satisfying (55), (56), let $UJV = W^{1/2}M^T Z^T$ be the singular value decomposition of $W^{1/2}M^T Z^T$, so that $U$ and $V$ are, respectively, $q \times q$ and $V$ is $N \times N$ orthogonal matrices, $J$ is $q \times N$ matrix with diagonal $\sigma = [\sigma_1; ...; \sigma_q]$, $p = \min[q,N]$, and zero off-diagonal entries; the diagonal entries $\sigma_i$, $1 \leq i \leq p$ are the singular values of $W^{1/2}M^T Z^T$, or, which is the same, of $ZMW^{1/2}$, so we have

$$\sum_i \sigma_i \geq \text{Opt.}$$

(57)

Now consider the following construction. Let $\eta \sim \mathcal{N}(0,I_N)$, we denote by $v$ the vector comprised of the first $p$ entries in $V\eta$; note that $\nu \sim \mathcal{N}(0,I_p)$, since $V$ is orthogonal. We then augment, if necessary, $v$ by $q-p$ independent $\mathcal{N}(0,1)$ random variables to obtain a $q$-dimensional normal vector $v' \sim \mathcal{N}(0,I_q)$, and set $\chi = U^Tv'$; because $U$ is orthogonal we also have $\chi \sim \mathcal{N}(0,I_q)$. Observe that

$$\chi^TW^{1/2}M^TZ^T\eta = \chi^TUJV\eta = [v']^TJu = \sum_{i=1}^p \sigma_i v_i^2.$$  

(58)

To continue we need the following simple observations.

1. One has

$$\alpha := \text{Prob}\left\{ \sum_{i=1}^p \sigma_i v_i^2 < \frac{1}{2} \sum_{i=1}^p \sigma_i \right\} \leq \frac{e^{1/4}}{\sqrt{2}} [ < 1].$$

(59)

The claim is evident when $\sigma := \sum_i \sigma_i = 0$. Now let $\sigma > 0$, and let us apply the Cramer bounding scheme. Namely, given $\gamma > 0$, consider the random variable

$$\omega = \exp\left\{ \frac{1}{2} \gamma \sum_i \sigma_i - \gamma \sum_i \sigma_i v_i^2 \right\}.$$ 

Note that $\omega > 0$ a.s., and is $> 1$ when $\sum_{i=1}^p \sigma_i v_i^2 < \frac{1}{2} \sum_{i=1}^p \sigma_i$, so that $\alpha \leq \mathbb{E}\{\omega\}$, or, equivalently, thanks to $v \sim \mathcal{N}(0,I_p)$,

$$\ln(\alpha) \leq \ln(\mathbb{E}\{\omega\}) = \frac{1}{2} \gamma \sum_i \sigma_i + \sum_i \ln(\mathbb{E}\{\exp\{-\gamma \sigma_i v_i^2\}\}) \leq \frac{1}{2} \left[ \gamma \sigma - \sum_i \ln(1 + 2 \gamma \sigma_i) \right].$$

Function $-\sum_i \ln(1 + 2 \gamma \sigma_i)$ is convex in $[\sigma_1; ...; \sigma_p] \geq 0$, therefore, its maximum over the simplex $\{\sigma_i \geq 0, i \leq p, \sum_i \sigma_i = \sigma\}$ is attained at a vertex, and we get

$$\ln(\alpha) \leq \frac{1}{2} [ \gamma \sigma - \ln(1 + 2 \gamma \sigma) ].$$

Minimizing the right hand side in $\gamma > 0$, we arrive at (59).

2. Whenever $\nu \geq 1$, one has

$$\text{Prob}\{\|MW^{1/2}\chi\|_2 > \nu\} \leq 2F \exp\{-\nu^2/2\},$$ 

(60)

with $F$ given by (29).

Indeed, setting $\rho = 1/\nu^2 \leq 1$ and $\omega = \sqrt{\rho}W^{1/2}\chi$, we get $\omega \sim \mathcal{N}(0,\rho W)$. Let us apply Lemma 4.2 to $Q = \rho W$ and to $\mathcal{R}$ in the role of $\mathcal{T}$, $L$ in the role of $K$, and $S_r[\cdot]$ in the role of $\mathcal{R}_k[\cdot]$. Denoting

$$\mathcal{Y} := \{ y : \exists r \in \mathcal{R} : S_r^2[y] \leq r I_L, \ell \leq L \},$$

26
we have $S_t[Q] = \rho S_t[W] \leq \rho r I_{\ell^t}, \ell \leq L$, with $r \in R$ (see (55)), so we are under the premise of Lemma 4.2. Applying the lemma, we conclude that

$$\text{Prob} \{ \chi : \chi^{-1} W^{1/2} \chi \notin V \} \leq 2 F \exp\{-1/(2\rho)\} = 2 F \exp\{-\chi^2/2\}.$$  

Recalling that $B_m = M \gamma$, we see that $\text{Prob}\{ \chi : \chi^{-1} MW^{1/2} \chi \notin B_m \}$ is indeed upper-bounded by the right hand size of (60), and (60) follows.

3. For $\chi \geq 1$, let

$$E_{\chi} = \left\{ (\chi, \eta) : \|MW^{1/2} \chi\|_s \leq \chi, \sum_i \sigma_i v_i^2 \geq \frac{1}{2} \sum_i \sigma_i \geq \frac{1}{2} \text{Opt} \right\}.$$ 

Then one has

$$\text{Prob}\{E_{\chi}\} \geq \beta(\chi) := 1 - \frac{e^{1/4}}{\sqrt{2}} - 2 F \exp\{-\chi^2/2\}. \quad (61)$$

Indeed, relation (61) follows from (59), (60) due to the union bound.

When $(\chi, \eta) \in E_{\chi}$, we have

$$\chi \|Z^T \eta\| \geq \|MW^{1/2} \chi\|_s \|Z^T \eta\| \geq \chi W^{1/2} M^T Z^T \eta = \sum_i \sigma_i v_i^2 \geq \frac{1}{2} \sum_i \sigma_i \geq \frac{1}{2} \text{Opt},$$

(we have used (58) and (57)), so that whenever $(\chi, \eta) \in E_{\chi}$ one has $\|Z^T \eta\| \geq \frac{1}{2} \text{Opt}$. Hence, finally,

$$2E_{\eta \sim N(0, I_N)} \{|\|Z^T \eta\|\} \geq \text{Prob}\{(\chi, \eta) \in E_{\chi}\} \chi^{-1} \text{Opt} \geq \left[ 1 - \frac{e^{1/4}}{\sqrt{2}} - 2 F \exp\{-\chi^2/2\} \right] \chi^{-1} \text{Opt},$$

and we arrive at (54) when specifying $\chi$ as

$$\chi = \sqrt{\frac{2 \ln \left( \frac{4 \sqrt{2} F}{\sqrt{2} - e^{1/4}} \right)}{}}. \quad \Box$$

### 4.4 Proof of Proposition 2.3

1. Let

$$\Phi(H, \Lambda, \Upsilon, \Upsilon', \Theta; Q) = \phi_T(\lambda[\Lambda]) + \phi_R(\lambda[\Upsilon]) + \phi_R(\lambda[\Upsilon']) + \text{Tr}(Q \Theta) : M \times \Pi \to \mathbb{R},$$

$$\mathcal{M} = \left\{ (H, \Lambda, \Upsilon, \Upsilon', \Theta) : \begin{array}{l}
\Lambda = \{ \Lambda_k \geq 0, k \leq K \}, \ Upsilon = \{ \Upsilon_\ell \geq 0, \ell \leq L \}, \ Upsilon' = \{ \Upsilon'_\ell \geq 0, \ell \leq L \}
\sum_k R_k^* [\Lambda_k] & \left[ \frac{1}{2} B^T - A^T H \right] M \right] \leq 0
\sum_k \Upsilon_\ell S_k^* [\Upsilon_\ell] & \leq 0
\sum_k \Upsilon'_\ell S_k^* [\Upsilon'_\ell] & \leq 0
\end{array} \right\} \quad (62)$$

Looking at (24), we conclude immediately that the optimal value $\text{Opt}$ in (24) is nothing but

$$\text{Opt} = \min_{(H, \Lambda, \Upsilon, \Upsilon', \Theta) \in \mathcal{M}} \left[ \Phi(H, \Lambda, \Upsilon, \Upsilon', \Theta) := \max_{Q \in \Pi} \Phi(H, \Lambda, \Upsilon, \Upsilon', \Theta; Q) \right]. \quad (63)$$

Note that the sets $\mathcal{M}$ and $\Pi$ are closed and convex, $\Pi$ is compact, and $\Phi$ is a continuous convex-concave function on $\mathcal{M} \times \Pi$. In view of these observations, Assumption R combines with Sion-Kakutani
Theorem to imply that $\Phi$ possesses saddle point $(H_*, \Lambda_*, \Upsilon_*, \Upsilon'_*, \Theta; Q_*)$ (min in $(H, \Lambda, \Upsilon, \Upsilon', \Theta)$, max in $Q$) on $\mathcal{M} \times \Pi$, whence $\text{Opt}$ is the saddle point value of $\Phi$ by (63). We conclude that for properly selected $Q_* \in \Pi$ it holds

$$\text{Opt} = \min_{(H, \Lambda, \Upsilon, \Upsilon', \Theta) \in \mathcal{M}} \Phi(H, \Lambda, \Upsilon, \Upsilon', \Theta; Q_*)$$

$$= \min_{H, \Lambda, \Upsilon, \Upsilon', \Theta} \left\{ \phi_T(\lambda[\Lambda]) + \phi_R(\lambda[\Upsilon]) + \phi_R(\lambda[\Upsilon']) + \text{Tr}(Q_* \Theta) : \right.$$

$$\lambda = \{\Lambda_k \geq 0, k \leq K\}, \quad \Upsilon = \{\Upsilon_\ell \geq 0, \ell \leq L\}, \quad \Upsilon' = \{\Upsilon'_\ell \geq 0, \ell \leq L\}$$

$$\left. \begin{bmatrix} \sum_k R^*_k[\Lambda_k] & \frac{1}{2}[B^T - A^T H]M \\ \frac{1}{2}M^T [B^T - A^T H]M & \sum_{\ell} S^*_\ell[\Upsilon_\ell] \end{bmatrix} \succeq 0, \right\}$$

$$= \min_{H, \Lambda, \Upsilon, \Upsilon', G} \left\{ \phi_T(\lambda[\Lambda]) + \phi_R(\lambda[\Upsilon]) + \phi_R(\lambda[\Upsilon']) + \text{Tr}(G) : \right.$$

$$\lambda = \{\Lambda_k \geq 0, k \leq K\}, \quad \Upsilon = \{\Upsilon_\ell \geq 0, \ell \leq L\}, \quad \Upsilon' = \{\Upsilon'_\ell \geq 0, \ell \leq L\}$$

$$\left. \begin{bmatrix} \sum_k R^*_k[\Lambda_k] & \frac{1}{2}[B^T - A^T H]M \\ \frac{1}{2}M^T [B^T - A^T H]M & \sum_{\ell} S^*_\ell[\Upsilon_\ell] \end{bmatrix} \succeq 0, \right\}$$

$$\Psi(H) := \min_{G, \Upsilon'} \left\{ \phi_R(\lambda[\Upsilon']) + \text{Tr}(G) : \Upsilon' = \{\Upsilon'_\ell \geq 0, \ell \leq L\}, \right.$$

$$\left. \begin{bmatrix} \sum_{\ell} S^*_\ell[\Upsilon'_\ell] \end{bmatrix} \succeq 0, \right\}$$

where $\text{Opt}$ is given by (24), and the equalities are due to (30) and (31).

$\blacksquare$. From now on we assume that the observation noise $\xi$ in observation (2) is $\xi \sim \mathcal{N}(0, Q_*)$. Besides this, we assume that $B \neq 0$, since otherwise the conclusion of Proposition 2.3 is evident.

$\blacksquare$. Let $W$ be a positive semidefinite $n \times n$ matrix, let $\eta \sim \mathcal{N}(0, W)$ be random signal, and let $\xi \sim \mathcal{N}(0, Q_*)$ be independent of $\eta$; vectors $(\eta, \xi)$ induce random vector

$$\omega = A\eta + \xi \sim \mathcal{N}(0, AW^T + Q_*)$$

Now, consider the problem where given $\omega$ we are interested to recover $B\eta$, and the Bayesian risk of a candidate estimate $\hat{x}(\cdot)$ is quantified by $\mathbf{E}_{\eta, \xi}\{\|B\eta - \hat{x}(A\eta + \xi)\|\}$. Let us set

$$\varrho[W] = \inf_{\hat{x}(\cdot)} \mathbf{E}_{\eta, \xi}\{\|B\eta - \hat{x}(A\eta + \xi)\|}\}. \quad (65)$$

Our first observation is that $\varrho[W]$ is “nearly attainable” with a linear estimate. Indeed, let $P$ be the joint distribution of the Gaussian vector $[\omega; B\eta]$, $Q$ be the marginal distribution of $\omega$, and let $R_\omega$ stand for the conditional, given $\omega$, distribution of $B\eta$. Since $P$ is zero mean Gaussian, the conditional expectation $\mathbf{E}_{\omega;B\eta}\{B\eta\}$ of $B\eta$ given $\omega$ is linear in $\omega$: $\mathbf{E}_{\omega;B\eta}\{B\eta\} = \hat{H}^T \omega$ for some $\hat{H}$ depending on $W$ only. Given an estimate $\hat{x}(\cdot)$, its Bayesian risk satisfies

$$\varrho = \mathbf{E}_{\eta, \omega}\{\|B\eta - \hat{x}(\omega)\|\} \geq \mathbf{E}_{\omega}\{\|B\eta - \hat{x}(\omega)\|\} \geq \mathbf{E}_{\omega}\{\|\mathbf{E}_{\omega;B\eta}\{B\eta\} - \hat{x}(\omega)\|\}$$

28
by the Jensen inequality. Hence
\[
\mathbf{E}_{\eta, \xi} \{ \| B \eta - \bar{H}^T (A \eta + \xi) \| \} = \mathbf{E}_{\eta, \omega} \{ \| B \eta - \bar{H}^T \omega \| \} = \mathbf{E}_{\omega} \{ \mathbf{E}_{\eta} \{ \| B \eta - \bar{H}^T \omega \| \} \} \\
\leq \mathbf{E}_{\omega} \{ \mathbf{E}_{\eta} \{ \| B \eta - \hat{x}(\omega) \| + \| \hat{x}(\omega) - \bar{H}^T \omega \| \} \} \\
= \mathbf{E}_{\eta, \omega} \{ \| B \eta - \hat{x}(\omega) \| \} + \mathbf{E}_{\omega} \{ \| \bar{H}^T \omega - \hat{x}(\omega) \| \} \leq 2 \varrho,
\]
and thus
\[
\mathbf{E}_{\eta, \xi} \{ \| B \eta - \bar{H}^T (A \eta + \xi) \| \} = \mathbf{E}_{\eta, \omega} \{ \| B \eta - \bar{H}^T \omega \| \} \leq 2 \varrho. \tag{66}
\]
Further, by convexity of the norm, we have
\[
\| \bar{H}^T \omega \| = \| \bar{H}^T \xi + [B - \bar{H}^T A] \eta \| \geq \| \bar{H}^T \xi \| + v(\xi) [B - \bar{H}^T A] \eta,
\]
where \( v(\xi) \) satisfies the inequality
\[
satisfies the inequality \tag{67}
\]
so that
\[
2 \varrho \geq \mathbf{E}_{\eta, \xi} \{ \| \bar{H}^T \xi + [B - \bar{H}^T A] \eta \| \} \geq \mathbf{E}_{\eta, \xi} \{ \| \bar{H}^T \xi \| + v(\xi) [B - \bar{H}^T A] \eta \} \geq \mathbf{E}_{\xi} \{ \| \bar{H}^T \xi \| \}. \tag{68}
\]
In relations (67) and (68), \( \bar{H} \) depends solely on \( W \), and \( \varrho \) can be made arbitrarily close to \( \varrho(W) \), and we arrive at the following

**Lemma 4.3** Let \( W \) be a positive semidefinite \( n \times n \) matrix. Then the risk \( \varrho(W) \) defined by (65) satisfies the inequality
\[
\varrho(W) \geq \frac{\varrho}{4} \inf_{H \in \mathbb{R}^{m \times n}} \left( \mathbf{E}_{\eta \sim \mathcal{N}(0, W)} \{ \| [B - H^T A] \eta \| \} + \mathbf{E}_{\xi \sim \mathcal{N}(0, Q)} \{ \| H^T \xi \| \} \right). \tag{69}
\]

**4°. Lemma 4.3** combines with Lemma 2.2 to imply the following result:

**Lemma 4.4** Let \( W \) be a positive semidefinite \( n \times n \) matrix. Then the risk \( \varrho(W) \) defined by (65) satisfies the inequality
\[
\varrho(W) \geq (4 \kappa[F])^{-1} \inf_{\gamma \in \{ \gamma, \| \gamma \| \leq L \}, G, H} \left\{ \text{Tr} (WG) + \phi_R(\lambda[\gamma]) + \overline{\Psi}(H) : \gamma \| \gamma \| \geq \frac{\gamma}{4} \right\} \leq 0, \tag{70}
\]
where \( \overline{\Psi}(H) \) is given by (64) and \( \kappa[F] = \frac{4}{\sqrt{2 - e^{1/F}}} \left( \ln \left( \frac{\sqrt{2} \sqrt{F}}{\sqrt{2} - e^{1/F}} \right) \right) \).

**Proof.** Let \( H \) be \( m \times \nu \) matrix. Applying Lemma 2.2 to \( N = m, Y = H, Q = Q, \) we get
\[
\mathbf{E}_{\xi \sim \mathcal{N}(0, Q)} \{ \| H^T \xi \| \} \geq \kappa^{-1[F]} \overline{\Psi}(H). \tag{71}
\]
Applying Lemma 2.2 to \( N = n, Y = (B - H^T A)^T, Q = W, \) we get
\[
\kappa[F] \mathbf{E}_{\eta \sim \mathcal{N}(0, W)} \{ \| [B - H^T A] \eta \| \} \geq \min_{\gamma \in \{ \gamma, \| \gamma \| \leq L \}, G} \left\{ \phi_R(\lambda[\gamma]) + \text{Tr}(WG) : \gamma \| \gamma \| \geq \frac{\gamma}{4} \right\} \leq 0.
\]
The resulting inequality combines with (69) and (71) to imply (70). \[\Box\]
5°. For $0 < \varkappa \leq 1$, let us set

(a) $\mathcal{W}_\varkappa = \{W \in \mathbf{S}_+^n : \exists t \in \mathcal{T} : R_k[W] \leq \varkappa t_k I_{d_k}, 1 \leq k \leq K\}$,

(b) $\mathcal{Z} = \left\{ (\Upsilon = \{\Upsilon_\ell, \ell \leq L\}, G, H) : \left[ G + \frac{1}{2} M^T [B - H^T A] \left[ \frac{1}{2} |B^T - A^T H| M \right] \right]_{\ell} \geq 0 \right\}$ \quad (72)

Note that $\mathcal{W}_\varkappa$ is a nonempty convex compact (by Lemma 4.1) set such that $\mathcal{W}_\varkappa = \varkappa \mathcal{W}_1$, and $\mathcal{Z}$ is a nonempty closed convex set. Consider the parametric saddle point problem

$$\text{Opt}(\varkappa) = \max_{W \in \mathcal{W}_\varkappa} \min_{(\Upsilon, G, H) \in \mathcal{Z}} [E(W; \Upsilon, G, H) := \text{Tr}(WG) + \phi_R(\lambda[\Upsilon]) + \Psi(H)].$$ \quad (73)

This problem is convex-concave; utilizing the fact that $\mathcal{W}_\varkappa$ is compact and contains positive definite matrices, it is immediately seen that the Sion-Kakutani theorem ensures the existence of a saddle point whenever $\varkappa \in (0, 1]$. We claim that

$$0 < \varkappa \leq 1 \Rightarrow \text{Opt}(\varkappa) \geq \sqrt{\varkappa} \text{Opt}(1).$$ \quad (74)

Indeed, $\mathcal{Z}$ is invariant w.r.t. scalings

$$(\Upsilon = \{\Upsilon_\ell, \ell \leq L\}, G, H) \mapsto (\theta \Upsilon := \{\theta \Upsilon_\ell, \ell \leq L\}, \theta^{-1} G, H), \quad [\theta > 0].$$

When taking into account that $\phi_R(\lambda[\Upsilon]) = \theta \phi_R(\lambda[\Upsilon])$, we get

$$E(W) := \min_{(\Upsilon, G, H) \in \mathcal{Z}} E(W; \Upsilon, G, H) = \min_{(\Upsilon, G, H) \in \mathcal{Z} \theta > 0} E(W; \theta \Upsilon, \theta^{-1} G, H) = \min_{(\Upsilon, G, H) \in \mathcal{Z}} \left[ 2 \sqrt{\text{Tr}(WG) \phi_R(\lambda[\Upsilon])} + \Psi(H) \right].$$

Because $\Psi$ is nonnegative we conclude that whenever $W \succeq 0$ and $\varkappa \in (0, 1]$, one has

$$E(\varkappa W) \geq \sqrt{\varkappa} E(W),$$

which combines with $\mathcal{W}_\varkappa = \varkappa \mathcal{W}_1$ to imply that

$$\text{Opt}(\varkappa) = \max_{W \in \mathcal{W}_\varkappa} E(W) = \max_{W \in \mathcal{W}_1} E(\varkappa W) \geq \sqrt{\varkappa} \max_{W \in \mathcal{W}_1} E(W) = \sqrt{\varkappa} \text{Opt}(1),$$

and (74) follows.

6°. We claim that

$$\text{Opt}(1) = \text{Opt},$$ \quad (75)

where Opt is given by (24) (and, as we have seen, by (64) as well). Note that (75) combines with (74) to imply that

$$0 < \varkappa \leq 1 \Rightarrow \text{Opt}(\varkappa) \geq \sqrt{\varkappa} \text{Opt}. \quad (76)$$

Verification of (75) is given by the following computation. By Sion-Kakutani Theorem,

$$\text{Opt}(1) = \max_{W \in \mathcal{W}_1} \min_{(\Upsilon, G, H) \in \mathcal{Z}} [\text{Tr}(WG) + \phi_R(\lambda[\Upsilon]) + \Psi(H)]$$

$$= \min_{(\Upsilon, G, H) \in \mathcal{Z}} \max_{W \in \mathcal{W}_1} [\text{Tr}(WG) + \phi_R(\lambda[\Upsilon]) + \Psi(H)]$$

$$= \min_{(\Upsilon, G, H) \in \mathcal{Z}} \left[ \Psi(H) + \phi_R(\lambda[\Upsilon]) + \max_W \{\text{Tr}(GW) : W \succeq 0, \exists t \in \mathcal{T} : R_k[W] \leq t_k I_{d_k}, k \leq K\} \right]$$

$$= \min_{(\Upsilon, G, H) \in \mathcal{Z}} \left[ \Psi(H) + \phi_R(\lambda[\Upsilon]) + \max_W \{\text{Tr}(GW) : W \succeq 0, [t; 1] \in K[\mathcal{T}], R_k[W] \leq t_k I_{d_k}, k \leq K\} \right]$$
Now, using Conic Duality combined with the fact that \((K[T])_\ast = \{ [g; s] : s \geq \phi_T(-g) \}\) we obtain

\[
\text{Opt}(1) = \min_{Y, G, H, \Lambda} \left[ \mathbf{W}(H) + \phi_R(\lambda[Y]) + \phi_T(\lambda[\Lambda]) : \begin{array}{l}
Y = \{ Y_\ell \geq 0, \ell \leq \ell \}, \Lambda = \{ \Lambda_k \geq 0, k \leq K \}, \\
G \geq \sum_k \mathbb{R}_+^+[\Lambda_k], \\
\frac{1}{2} M^T [B - H^T A] \sum_i S_i^T [Y_i] \geq 0
\end{array} \right]
\]

\[
= \min_{Y, \Lambda} \left[ \mathbf{W}(H) + \phi_R(\lambda[Y]) + \phi_T(\lambda[\Lambda]) : \begin{array}{l}
Y = \{ Y_\ell \geq 0, \ell \leq \ell \}, \Lambda = \{ \Lambda_k \geq 0, k \leq K \}, \\
\frac{1}{2} M^T [B - H^T A] \sum_i S_i^T [Y_i] \geq 0
\end{array} \right]
\]

\[
= \text{Opt} \ [\text{see (64)}].
\]

Now we can complete the proof.

\section{Case 1.}

Let us set

\[
\varrho_* = \inf \mathbf{Risk}[\hat{x}|\mathcal{X}], \quad \mathbf{Risk}[\hat{x}|\mathcal{X}] = \sup_{x \in \mathcal{X}} \mathbb{E}_{\xi \sim \mathcal{N}(0, Q_x)} \{ \| Bx - \hat{x} (Ax + \xi) \| \},
\]

where \(\inf\) is taken over all estimates. It is immediately seen that \(\varrho_* > 0\) due to \(Q_\ast > 0\) (recall that \(Q_\ast \in \Pi\) and invoke Assumption \(R\)) combined with \(B \neq 0\) and \(0 \in \text{int} \mathcal{X}\). Consequently, there is an estimate \(\hat{x}(\cdot)\) such that \(\mathbf{Risk}[\hat{x}|\mathcal{X}] \leq \frac{3}{4} \varrho_*\). Further, when \(x \in \mathcal{X} \setminus \{0\}\), we have \(W := xx^T \in \mathbb{Q}\), see (27) and (6), and \(W^{1/2} = W/\|x\|_2\), whence for \(M_\ast\) as defined in (27) we have

\[
M_\ast^2 \geq \mathbb{E}_{\eta \sim \mathcal{N}(0, 1)_\ast} \{ \| BW^{1/2} \eta \|^2 \} = \|x\|_2^{-2} \|Bx\|^2 \mathbb{E}_{\eta \sim \mathcal{N}(0, 1)} \{ (x^T \eta)^2 \} = \|Bx\|^2,
\]

and we arrive at

\[
x \in \mathcal{X} \Rightarrow \|Bx\| \leq M_\ast. \quad (78)
\]

Now let us convert the estimate \(\hat{x}\) into the estimate \(\hat{x}\) defined as follows: \(\hat{x}(\omega)\) is the \(\| \cdot \|\)-closest to \(\hat{x}(\omega)\) point of the set \(B_{M_\ast} = \{ u : \|u\| \leq M_\ast \}\). When \(x \in \mathcal{X}\), we have \(Bx \in B_{M_\ast}\) by (78), and because, by construction, \(\hat{x}\) is the closest to \(\tilde{x}\) point of \(B_{M_\ast}\), we have also \(\| \hat{x}(\omega) - \hat{x}(\omega)\| \leq \|Bx - \hat{x}(\omega)\|\) for all \(\omega\).

Thus,

\[
x \in \mathcal{X} \Rightarrow \|Bx - \hat{x}(\omega)\| \leq \|Bx - \hat{x}(\omega)\| + \|\hat{x}(\omega) - \hat{x}(\omega)\| \leq 2\|Bx - \hat{x}(\omega)\|.
\]

We conclude that \(\|\hat{x}(\omega)\| \leq M_\ast \forall \omega\), and

\[
\mathbf{Risk}[\hat{x}|\mathcal{X}] \leq 2\mathbf{Risk}[\hat{x}|\mathcal{X}] \leq 3\varrho_* \quad (79)
\]

\section{Case 2.}

For \(\varepsilon \in (0, 1]\), let \(W_\varepsilon\) be the \(W\)-component of a saddle point solution to the saddle point problem (73). Then, by (76),

\[
\sqrt{\varepsilon} \text{Opt} \leq \text{Opt}(\varepsilon) = \min_{(Y, G, H) \in \mathcal{Z}} \left\{ \text{Tr}(W_\varepsilon G) + \phi_R(\lambda[Y]) + \mathbf{W}(H) \right\}
\]

\[
= \min_{(Y, G, H)} \left\{ \text{Tr}(W_\varepsilon G) + \phi_R(\lambda[Y]) + \mathbf{W}(H) : Y_\ell \geq 0 \forall \ell, \left[ \begin{array}{c}
G \\
\frac{1}{2} M^T [B - H^T A] \sum_i S_i^T [Y_i]
\end{array} \right] \geq 0 \right\}
\]

\[
\leq 4\kappa[F]\varrho(W_\varepsilon) \quad (80)
\]

(we have used (72.6) and (70); recall that \(\varrho[\cdot]\) is given by (65)). On the other hand, when applying Lemma 4.2 to \(Q = W_\varepsilon\) we obtain, in view of relations \(0 < \varepsilon \leq 1\), \(W_\varepsilon \in W_{\varepsilon}\),

\[
\delta(\varepsilon) := \text{Prob}_{\eta \sim \mathcal{N}(0, 1)} \{ W_{\varepsilon}^{1/2} \eta \notin \mathcal{X} \} \leq 2D \exp\{-(2\varepsilon)^{-1}\},
\]

(81)
with $D$ given by (29). Setting
\[ \mathcal{E}_x = \{ \zeta : W^{1/2}_x \zeta \in \mathcal{X} \}, \quad \mathcal{E}_x^c = \mathbb{R}^n \setminus \mathcal{E}_x, \quad \Sigma = \text{Diag}\{ I_n, Q_x \}, \]
we have by definition of the risk $\varrho(W_x)$
\[
\varrho[W_x] \leq \mathbf{E}_{(\eta, \xi) \sim \mathcal{N}(0, \Sigma)} \{ \| BW^{1/2}_x \eta - \hat{x}(AW^{1/2}_x \eta + \xi) \| \}
= \mathbf{E}_{\eta \sim \mathcal{N}(0, I_n)} \left\{ \mathbf{E}_{\xi \sim \mathcal{N}(0, Q_x)} \{ \| BW^{1/2}_x \eta - \hat{x}(AW^{1/2}_x \eta + \xi) \| \} \right\}
= \mathbf{E}_{\eta \sim \mathcal{N}(0, I_n)} \left\{ \mathbf{E}_{\xi \sim \mathcal{N}(0, Q_x)} \{ \| BW^{1/2}_x \eta - \hat{x}(AW^{1/2}_x \eta + \xi) \| \} 1\{ \eta \in \mathcal{E}_x \} \right\}
+ \mathbf{E}_{\eta \sim \mathcal{N}(0, I_n)} \left\{ \mathbf{E}_{\xi \sim \mathcal{N}(0, Q_x)} \{ \| BW^{1/2}_x \eta - \hat{x}(AW^{1/2}_x \eta + \xi) \| \} 1\{ \eta \in \mathcal{E}_x^c \} \right\}
\leq \text{Risk}[\hat{x}|\mathcal{X}] + \mathbf{E}_{\eta \sim \mathcal{N}(0, I_n)} \left\{ (\| BW^{1/2}_x \eta \| + M_x) 1\{ \eta \in \mathcal{E}_x^c \} \right\} \quad [\text{since } \| \hat{x}(\cdot) \| \leq M_x]
\leq 3\varrho_x + M_x \delta(\mathcal{X}) + \mathbf{E}_{\eta \sim \mathcal{N}(0, I_n)} \left\{ \| BW^{1/2}_x \eta \| 1\{ \eta \in \mathcal{E}_x^c \} \right\} \quad [\text{we have used (79)}].

We conclude that
\[
\varrho[W_x] \leq 3\varrho_x + M_x \delta(\mathcal{X}) + \left[ \mathbf{E}_{\eta \sim \mathcal{N}(0, I_n)} \{ \| BW^{1/2}_x \eta \| \} \right]^{1/2} \left[ \text{Prob}_{\eta \sim \mathcal{N}(0, I_n)} \{ \eta \in \mathcal{E}_x^c \} \right]^{1/2}
\leq 3\varrho_x + M_x \delta(\mathcal{X}) + \sqrt{\varrho(\mathcal{X})} \quad [\text{by (27); note that } W_x \in Q \text{ due to } \kappa \leq 1]
\leq 3\varrho_x + 2M_x \sqrt{\varrho(\mathcal{X})} \quad [\text{since } \delta(\mathcal{X}) \leq 1]
\leq 3\varrho_x + 2M_x \sqrt{2D \exp \left( -\frac{\kappa}{4} \right)} \quad [\text{we have used (81)}].
\]

The bottom line here is that
\[
0 < \kappa \leq 1 \Rightarrow \varrho[W_x] \leq 3\varrho_x + 2M_x \sqrt{2D \exp \left( -\frac{1}{4} \right)}.
\]  

(82)

Observe that $\varrho_x \leq M_x$, since due to (78), for the trivial – identically zero – estimate $\hat{x}(\cdot)$ of $Bx$ one has $\text{Risk}[\hat{x}|\mathcal{X}] \leq M_x$. It follows that setting
\[
\tilde{\kappa} = \frac{1}{4 \ln \left( \frac{2M_x \sqrt{2D}}{\varrho_x} \right)}
\]
we ensure that $\tilde{\kappa} \in (0, 1]$, whence, by (82),
\[
\varrho[W_x] \leq 4\varrho_x.
\]

This combines with (80) to imply that
\[
\sqrt{\tilde{\kappa}} \text{Opt} \leq 4\kappa[F] \varrho[W_x] \leq 16\kappa[F] \varrho_x,
\]
whence finally
\[
\text{Opt} \leq \frac{16\kappa[F]}{\sqrt{\tilde{\kappa}}} \varrho_x \leq \frac{64\sqrt{2}}{\sqrt{2 - e^{1/4}}} \left[ \ln \left( \frac{4\sqrt{2}F}{\sqrt{2 - e^{1/4}}} \right) \ln \left( \frac{8M^2D}{\varrho_x^2} \right) \right] \varrho_x.
\]

Noting that by definition of $\varrho_x$ and $\text{RiskOpt}_{\Pi, \| \cdot \|}[\mathcal{X}]$ we have $\varrho_x \leq \text{RiskOpt}_{\Pi, \| \cdot \|}[\mathcal{X}] \leq M_x$ (the concluding $\leq$ is due to $\| Bx \| \leq M_x$ for $x \in \mathcal{X}$), we arrive at (28). \qed

32
4.5 Proof of Proposition 3.2

1°. Observe that setting

$$\varrho = \max_x \{ \| Bx \| : x \in \mathcal{X}, Ax = 0 \}, \quad (83)$$

we ensure that

$$\text{Risk}_{\text{opt}}[\mathcal{X}] \geq \varrho. \quad (84)$$

Indeed, let $\bar{x}$ be an optimal solution to the (clearly solvable) optimization problem in (83). Then observation $\omega = 0$ can be obtained from both the signals $x = \bar{x}$ and $x = -\bar{x}$, and therefore the risk of any (deterministic) recovery routine is at least $\| B\bar{x} \| = \varrho$, as claimed.

2°. It may happen that $\text{Ker} A = \{0\}$. In this case the situation is trivial: specifying $A^\dagger$ as a partial inverse to $A$: $A^\dagger A = I_n$ and setting $H^T = BA^\dagger$ (so that $B - H^T A = 0$), $\Upsilon_\ell = 0_{f \times f}$, $\ell \leq L$, $\Lambda_k = 0_{d_k \times d_k}$, $k \leq K$, we get a feasible solution to the optimization problem in (44) with zero value of the objective, implying that $\text{Opt}_\# = 0$; consequently, the linear estimate induced by an optimal solution to the problem is with zero risk, and the conclusion of Proposition 3.2 is clearly true. with this in mind, we assume from now on that $\text{Ker} A \neq \{0\}$. Denoting $k = \dim \text{Ker} A$, we can build an $n \times k$ matrix $E$ of rank $k$ such that $\text{Ker} A$ is the image space of $E$.

3°. Setting

$$Z := \{ z \in \mathbb{R}^k : E x \in \mathcal{X} \} = \{ z \in \mathbb{R}^k : \exists (t \in T) : \bar{R}_k^2 [z] \leq t_k I_{d_k}, k \leq K \},$$

$$C = \left[ \frac{1}{2} E^T B^T \right] \frac{\frac{1}{2} BE}{\| z \|_2} \right],$$

note that when $z$ runs through the spectratope $Z$, $Ez$ runs exactly through the entire set $\{ x \in \mathcal{X} : Ax = 0 \}$. with this in mind, invoking Proposition 2.1, we arrive at

$$\varrho = \max_{g \| g \|_{\ell_1} \leq 1} \max_{z \in Z} g^T B E z = \max_{[u; z] \in \mathbb{R}^{k} \times \mathcal{Z}} [u; z]^T C [u; z]$$

$$\leq \text{Opt} := \min_{\Upsilon = \{ \Upsilon_\ell, \ell \leq L \}} \left\{ \phi_R(\lambda[\Upsilon]) + \phi_T(\lambda[\Lambda]) : \Upsilon_\ell \leq 0, \Lambda_k \leq 0, \forall (\ell, k) \right\}$$

$$\left[ \frac{\sum_{\ell} S_{\ell}^*[\Upsilon_\ell]}{\frac{1}{2} E^T B^T} \right] \left[ \frac{\frac{1}{2} BE}{E^T [\sum_k \mathcal{R}_k^*[\Lambda_k]] E} \right] \succeq 0 \right \} \quad (85)$$

(we have used the straightforward identity $\mathcal{R}_k^*[\Lambda_k] = E^T \mathcal{R}_k^*[\Lambda_k] E$). By the same Proposition 2.1, the optimization problem in (85) specifying Opt is solvable, and

$$\text{Opt} \leq 2 \ln(2D) \varrho, \quad D = \sum_k d_k + \sum_{\ell} f_{\ell}. \quad (86)$$

4°. Let $\tilde{\Upsilon} = \{ \tilde{\Upsilon}_\ell \}, \tilde{\Lambda} = \{ \tilde{\Lambda}_k \}$ be an optimal solution to the optimization problem specifying Opt, see (85), and let

$$\Upsilon = \sum_{\ell} S_{\ell}^*[\tilde{\Upsilon}_\ell], \quad \Lambda = \sum_k \mathcal{R}_k^*[\tilde{\Lambda}_k],$$

so that

$$\text{Opt} = \phi_R(\lambda[\Upsilon]) + \phi_T(\lambda[\Lambda]) \& \left[ \frac{\Upsilon}{\frac{1}{2} E^T B^T} \right] \left[ \frac{\frac{1}{2} BE}{E^T \Lambda E} \right] \succeq 0. \quad (87)$$
We claim that for properly selected $m \times \nu$ matrix $H$ it holds
\[
\begin{bmatrix}
\Upsilon & \frac{1}{2}(B - H^T A) \\
\frac{1}{2}(B - H^T A)^T & \Lambda
\end{bmatrix} \succeq 0.
\] \tag{88}

This claim implies the conclusion of Proposition 3.2: by the claim, we have $\text{Opt}_\# \leq \text{Opt}$, which combines with (86) and (84) to imply (45).

In order to justify the claim, assume that it fails to be true, and let us lead this assumption to contradiction.

40. a. Consider the semidefinite program
\[
\tau_* = \min_{\tau, H} \left\{ \tau : \begin{bmatrix}
\Upsilon & \frac{1}{2}(B - H^T A) \\
\frac{1}{2}(B - H^T A)^T & \Lambda
\end{bmatrix} + \tau I_{\nu+n} \succeq 0 \right\}. \tag{89}
\]

The problem clearly is strictly feasible, and the value of the objective at every feasible solution is positive; in addition, the objective clearly is coercive on the feasible domain, so that the problem is solvable with positive optimal value.

40. b. As we have seen, (89) is a strictly feasible solvable problem with positive optimal value $\tau_*$, so that the problem dual to (89) is solvable with positive optimal value. Let us build the dual problem.
Denoting by \[
\begin{bmatrix}
U & V \\
V^T & W
\end{bmatrix} \succeq 0 \]
the Lagrange multipliers for the semidefinite constraint in (89) and taking inner product of the left hand side of the constraint with the multiplier, we get the aggregated constraint
\[
\text{Tr}(U \Upsilon) + \text{Tr}(W \Lambda) + \tau \text{Tr}(U) + \text{Tr}(W) + \text{Tr}((B - H^T A)V^T) \geq 0; \tag{90}
\]
the equality constraints of the dual should make the homogeneous in $\tau, H$ part of the left hand side in the aggregated constraint identically equal to $\tau$, which amounts to
\[
\text{Tr}(U) + \text{Tr}(W) = 1, VA^T = 0,
\]
and the aggregated constraint now reads
\[
\tau \geq - \left[ \text{Tr}(U \Upsilon) + \text{Tr}(W \Lambda) + \text{Tr}(BV^T) \right].
\]

The dual problem is to maximize the right hand side of the latter constraint over Lagrange multiplier \[
\begin{bmatrix}
U & V \\
V^T & W
\end{bmatrix} \succeq 0 \text{ satisfying } AV^T = 0, \text{ and its optimal value is } \tau_* > 0, \text{ that is, there exists}
\]
\[
\begin{bmatrix}
\bar{U} & \bar{V} \\
\bar{V}^T & \bar{W}
\end{bmatrix} \succeq 0 \text{ such that } AV^T = 0 \text{ and}
\]
\[
\text{Tr}(\bar{U} \Upsilon) + \text{Tr}(\bar{W} \Lambda) + \text{Tr}(B\bar{V}^T) < 0; \tag{90}
\]
adding to $\bar{U}$ a small positive multiple of the unit matrix, we can assume, in addition, that $\bar{U} \succ 0$. Now, the relation $A\bar{V}^T = 0$ combines with the definition of $E$ to imply that $\bar{V}^T = EF$ for properly selected matrix $F$, so that
\[
\begin{bmatrix}
\bar{U} & EF \\
EF & F^T E^T
\end{bmatrix} \succeq 0,
\]
whence, by Schur Complement Lemma,
\[
\bar{W} \succeq EF \bar{U}^{-1} F^T F^T,
\]

34
and therefore (90) combines with $\Lambda \succeq 0$ to imply that

$$0 > \text{Tr}(\bar{U} \Upsilon) + \text{Tr}(\bar{W} \Lambda) + \text{Tr}(B \bar{V}^T) = \text{Tr}(\bar{U} \Upsilon) + \text{Tr}(\bar{W} \Lambda) + \text{Tr}(BEF)$$

$$\geq \text{Tr}(\bar{U} \Upsilon) + \text{Tr}(EFU^{-1} F^T E \Lambda) + \text{Tr}(BEF) = \text{Tr}\left(\begin{bmatrix} \Upsilon & \frac{1}{2} B E \\ \frac{1}{2} E^T B^T & \frac{1}{2} E^T \Lambda E \end{bmatrix}\begin{bmatrix} \bar{U} & F^T \\ F & FU^{-1} F^T \end{bmatrix}\right)$$

Both matrix factors in the concluding the chain $\text{Tr}(\cdot)$ are positive semidefinite (first – by (87), and the second - by the Schur Complement Lemma); consequently, the concluding quantity in the chain is nonnegative, which is impossible. We have arrived at a desired contradiction. □

References

[1] B. F. Arnold and P. Stahlecker. Another view of the Kuks–Olman estimator. *Journal of statistical planning and inference*, 89(1):169–174, 2000.

[2] A. Ben-Tal, L. El Ghaoui, and A. Nemirovski. *Robust optimization*. Princeton University Press, 2009.

[3] A. Ben-Tal and A. Nemirovski. *Lectures on modern convex optimization: analysis, algorithms, and engineering applications*, volume 2. SIAM, 2001.

[4] S. P. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan. *Linear matrix inequalities in system and control theory*, volume 15. SIAM, 1994.

[5] A. Buchholz. Operator Khintchine inequality in the non-commutative probability. *Mathematische Annalen* 391:1–16, 2001.

[6] N. Christopeit and K. Helmes. Linear minimax estimation with ellipsoidal constraints. *Acta Applicandae Mathematica*, 43(1):3–15, 1996.

[7] D. L. Donoho. Statistical estimation and optimal recovery. *The Annals of Statistics*, pages 238–270, 1994.

[8] D. L. Donoho, R. C. Liu, and B. MacGibbon. Minimax risk over hyperrectangles, and implications. *The Annals of Statistics*, pages 1416–1437, 1990.

[9] H. Drygas. Spectral methods in linear minimax estimation. *Acta Applicandae Mathematica*, 43(1):17–42, 1996.

[10] S. Efromovich. *Nonparametric curve estimation: methods, theory, and applications*. Springer Science & Business Media, 2008.

[11] S. Efromovich and M. Pinsker. Sharp-optimal and adaptive estimation for heteroscedastic nonparametric regression. *Statistica Sinica*, pages 925–942, 1996.

[12] Y. K. Golubev, B. Y. Levit, and A. B. Tsybakov. Asymptotically efficient estimation of analytic functions in gaussian noise. *Bernoulli*, pages 167–181, 1996.

[13] I. Ibragimov and R. Khasminskii. Theory of statistic estimation: Asymptotic theory, 1981.

[14] A. Juditsky and A. Nemirovski. A Near-Optimality of Linear Recovery in Gaussian Observation Scheme under $\|\cdot\|^2_2$-Loss. Submitted (April 2016) to *Annals of Statistics*. 35
[15] A. Juditsky and A. Nemirovski, A. Supplement to “Near-Optimality of Linear Recovery in Gaussian Observation Scheme under \( \| \cdot \|_2^2 \)-Loss.” DOI: COMPLETED BY THE TYPESETTER Submitted (April 2016) to *Annals of Statistics*.

[16] J. Kuks and W. Olman. Minimax linear estimation of regression coefficients (i). *Izvestija Akademija Nauk Estonskoj SSR*, 20:480–482, 1971.

[17] J. Kuks and W. Olman. Minimax linear estimation of regression coefficients (ii). *Izvestija Akademija Nauk Estonskoj SSR*, 21:66–72, 1972.

[18] B. Laurent and P. Massart. Adaptive estimation of a quadratic functional by model selection. *Annals of Statistics*, pages 1302–1338, 2000.

[19] O. V. Lepskii. Asymptotically minimax adaptive estimation. I. Upper bounds. Optimally adaptive estimates. *Teoriya Veroyatnostei i ee Primeneniya*, 36(4):645–659, 1991.

[20] R. Liptser and A. Shiryayev. *Statistics of random processes*. New York, Springer, 1977.

[21] F. Lust-Piquard. Inégalités de Khintchine dans \( C_p \) (1 < \( p \) < \( \infty \)). *Comptes Rendus de l’Académie des Sciences de Paris, Série I* 393(7):289–292, 1986.

[22] A. Man-Cho So. Moment inequalities for sums of random matrices and their applications in Optimization. *Mathematical Programming*, 130(1):125-151, 2011.

[23] A. Nemirovski, C. Roos, and T. Terlaky. On maximization of quadratic form over intersection of ellipsoids with common center. *Mathematical Programming*, 86(3):463–473, 1999.

[24] A. Nemirovski. Sums of random symmetric matrices and quadratic optimization under orthogonality constraints. *Mathematical Programming*, 109(2-3): 283–317, 2007.

[25] J. Pilz. Minimax linear regression estimation with symmetric parameter restrictions. *Journal of Statistical Planning and Inference*, 13:297–318, 1986.

[26] M. Pinsker. Optimal filtration of square-integrable signals in gaussian noise. *Prob. Info. Transmission*, 16(2):120–133, 1980.

[27] G. Pisier. Non-commutative vector valued \( L_p \) spaces and completely \( p \)-summing maps. *Astérisque* 247, 1998.

[28] C. R. Rao. *Linear statistical inference and its applications*, volume 22. John Wiley & Sons, 1973.

[29] C. R. Rao. Estimation of parameters in a linear model. *The Annals of Statistics*, pages 1023–1037, 1976.

[30] A. B. Tsybakov. *Introduction to Nonparametric Estimation*. Springer Series in Statistics, Springer 2008.

[31] J. A. Tropp. The random paving property for uniformly bounded matrices. *Studia Mathematica* 185:67–82, 2008.

[32] J. A. Tropp et al. An introduction to matrix concentration inequalities. *Foundations and Trends® in Machine Learning*, 8(1-2):1–230, 2015.

[33] L. Wasserman. *All of nonparametric statistics*. Springer Science & Business Media, 2006.
A Conic duality

A conic problem is an optimization problem of the form
\[ \text{Opt}(P) = \max_x \{ c^T x : A_i x - b_i \in K_i, i = 1, ..., m, Px = p \} \]

where \( K_i \) are regular (i.e., closed, convex, pointed and with a nonempty interior) cones in Euclidean spaces \( E_i \). Conic dual of \( (P) \) “is responsible” for upper-bounding the optimal value in \( (P) \) and is built as follows: selecting somehow Lagrange multipliers \( \lambda_i \) for the conic constraints \( A_i x - b_i \in K_i \) in the cones dual to \( K_i \):
\[ \lambda_i \in K_i^* := \{ \lambda : \langle \lambda, y \rangle \geq 0 \forall y \in K_i \}, \]
and a Lagrange multiplier \( \mu \in \mathbb{R}^{\dim p} \) for the equality constraints, every feasible solution \( x \) to \( (P) \) satisfies the linear inequalities \( \langle \lambda_i, A_i x \rangle \geq \langle \lambda_i, b_i \rangle, i \leq m \), same as the inequality \( \mu^T Px \geq \mu^T p \), and thus satisfies the aggregated inequality
\[ \sum_i \langle \lambda_i, A_i x \rangle + \mu^T Px \geq \sum_i \langle \lambda_i, b_i \rangle + \mu^T p. \]

If the left hand side of this inequality is, identically in \( x \), equal to \(-c^T x\) (or, which is the same, \(-c = \sum_i A_i^* \lambda_i + P^T \mu\), where \( A_i^* \) is the conjugate of \( A_i \)), the inequality produces an upper bound \(-\langle \lambda_i, b_i \rangle - p^T \mu\) on \( \text{Opt}(P) \). The dual problem
\[ \text{Opt}(D) = \min_{\lambda_1, ..., \lambda_m, \mu} \left\{ \sum_i \langle \lambda_i, b_i \rangle - p^T \mu : \lambda_i \in K_i^*, i \leq m, \sum_i A_i^* \lambda_i + P^T \mu = -c \right\} \]
is the problem of minimizing this upper bound. Note that \( (D) \) is a conic problem along with \( (P) \) – it is a problem of optimizing a linear objective under a bunch of linear equality constraints and conic inclusions of the form “affine function of the decision vector should belong to a given regular cone.” Conic Duality Theorem (see, e.g., [3]) states that when one of the problems \( (P) \), \( (D) \) is bounded\(^5\) and strictly feasible, then the other problem in the pair is solvable, and \( \text{Opt}(P) = \text{Opt}(D) \). In this context, strict feasibility means that there exists a feasible solution for which all conic inclusions are satisfied strictly, that is, the left hand side of the inclusion belongs to the interior of the right hand side cone.

---

\(^5\)for a maximization (minimization) problem, boundedness means that the objective is bounded from above (resp., from below) on the feasible set.