State estimation for dynamical system described by linear equation with uncertainty

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Abstract. In this paper we investigate a problem of state estimation for the dynamical system described by the linear operator equation with unknown parameters in Hilbert space. We present explicit expressions for linear minimax estimation and error provided that any pair of uncertain parameters belongs to the quadratic bounding set. As an application of the main result we present the solution of minimax estimation problem for the linear descriptor differential equation with constant matrices.

Key words. state estimation, minimax, linear equation, DAE, descriptor system.

Introduction

One of the major problems in applied mathematics is the problem of state estimation for dynamical systems described by the linear equations with uncertain parameters. This problem belongs to the so called “uncertain inverse problems”. In the strict sense the problem is described as follows: given some element (for instance measurements of the system output) $y$ from some functional space one needs to estimate the expression $\ell(\theta)$ provided that $\theta$ obey the equation $g(\theta) = 0$. This problem is non-trivial if there exists non-unique $\theta$ satisfying the equation $g(\theta) = 0$ and the equality $y = C(\theta)$ holds. In this case the estimation problem may be reformulated as follows: given $y = C(\theta), \theta \in \Theta, y \in Y$ one needs to find the estimation $\tilde{\ell}(\theta)$ of the expression $\ell(\theta)$ provided that $g(\theta) = 0$ and $C(\cdot), \ell(\cdot)$ are given functions. Note that this problem becomes trivial if the equation $y = C(\theta)$ has the unique solution $\hat{\theta}$. Really, in this case we just set $\tilde{\ell}(\hat{\theta}) := \ell(\hat{\theta})$.

The estimation problem is said to be linear if $\Theta, Y$ are linear spaces and $C(\cdot), \ell(\cdot)$ are linear mappings. It is the common case when $\Theta, Y$ are Banach spaces.

Classical theory of uncertain estimation problems [1, 2, 3, 4, 5, 6] works well when the linear mapping $L$ in $(*)$ has bounded inverse mapping. The solution of the linear uncertain estimation problem for linear equations with Noether operator was introduced in [7]. Note that the introduced approach is not suitable when $\dim(L) = \infty$. This is the case for linear differential-algebraic (DAE) equations.

The major contribution of this paper is an state estimation approach for uncertain equations with linear closed operator in Hilbert space. This approach is valid when $L$ has non-closed range. It generalizes the classical theory [1, 2, 3, 4] to the linear noncausal uncertain differential-algebraic equations. The application of presented method to the linear equations with Noether operator was described in [9].

Notation. Set $c(G, \cdot) = \sup\{(z, f) : f \in G\}$, let $\delta(G, \cdot)$ denotes the indicator function of $G$, set $\text{dom} f = \{x \in \mathcal{H} : f(x) < \infty\}$, $f^*(x^*) = \sup_x \{f^*(x, x) - f(x)\}$, $(L^*c)(u) = \inf \{c(G, z), L^*z = u\}$, $(fL)(x) = f(Lx)$, $(L^*c)(u) = \inf \{c(G, z), L^*z = u\}$, let $\text{cl} f = f^*$. 

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denotes the closure of $f$ in the Young-Fenchel sense, $\text{Arginf}_u f(u)$ denotes the set of minimum points of $f$, $P_{L^*}$ denotes the orthogonal projector onto $R(L^*)$, $\partial f(x)$ denotes the subdifferential of $f$ at $x$ and $(\cdot, \cdot)$ denotes the inner product in Hilbert space.

**Problem formulation and definitions.**

Suppose that $L \varphi \in \mathcal{G}$ and

$$y = H \varphi + \eta$$

(1)

The mappings $L, H$ and the set $\mathcal{G}$ are supposed to be given. The element $\eta$ is uncertain. Our aim is to solve the inverse problem: to construct the operator mapping the given $y$ into the estimation $\hat{\ell}(\varphi)$ of expression $\ell(\varphi)$ and to calculate the estimation error $\sigma$. Now let us introduce some definitions.

The operator $L : \mathcal{H} \mapsto \mathcal{F}$ is assumed to be closed. Its domain $\mathcal{D}(L)$ is supposed to be a dense subset of the Hilbert space $\mathcal{H}$, $H \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$.

Note that the condition $L \varphi \in \mathcal{G}$ is equal to the following

$$L \varphi = f,$$

(2)

where $f$ is uncertain and belongs to the given subset $\mathcal{G}$ of the Hilbert space $\mathcal{F}$. In the sequel $\eta$ is supposed to be a random $\mathcal{Y}$-valued vector with zero mean so that its correlation $R_\eta \in \mathcal{R}$, where $\mathcal{R}$ is some subset of $\mathcal{L}(\mathcal{Y}, \mathcal{Y})$. Also we deal with deterministic $\eta$ so that $(f, \eta) \in \mathcal{G}$, where $\mathcal{G}$ is some subset of $\mathcal{F} \times \mathcal{Y}$. Note that the realization of $y$ depends $\eta, H$ and $f$. Also it depends on elements of $N(L) = \{ \varphi \in \mathcal{D}(L) : L \varphi = 0 \}$ so that $y = H(\varphi_0 + \varphi) + \eta$, where $\varphi_0$ may be thought as inner noise in the state model [2].

Let $\ell(\varphi) = (\ell, \varphi), \hat{\ell}(\varphi) = (u, y) + c$. Since $L, H$ are not supposed to have a bounded inverse mappings the $\ell(\varphi)$ and $\hat{\ell}(\varphi)$ are not stable with a respect to small deviations in $f, \eta$. Also $f, \eta$ are supposed to be uncertain. Therefore we use the minimax design in order to construct the estimation.

**Definition 1.** The function $\hat{\ell}(\varphi) = (\hat{u}, \cdot) + \hat{c}$ is called the a priori minimax mean-squared estimation iff $\sigma(\ell, \hat{u}) = \text{inf}_{u, c} \sigma(\ell, u)$ where

$$\sigma(\ell, u) := \sup_{L \varphi \in \mathcal{G}, \eta \in \mathcal{R}} M(\ell(\varphi) - \ell(\varphi))^2$$

(3)

The number $\hat{\sigma}(\ell) = \sigma(\hat{\ell}, \hat{u})$ is said to be the minimax mean-squared error in the direction $\ell$.

On the contrast the a posteriori estimation describes the evolution of the central point of the system reachability set

$$(L \varphi, y - H \varphi) \in \mathcal{G}$$

consistent with measured output $y$ [10, 11, 12]. Note that the condition $(L \varphi, y - H \varphi) \in \mathcal{G}$ holds if $\|y\| < C$ for some real $C$. But it doesn’t hold in our assumptions if $\eta$ is random since $\|R_\eta\| < c$ doesn’t imply $\|y\| < C$ for realizations of $\eta$. Therefore $\eta$ is supposed to be deterministic.

**Definition 2.** The set

$$X_y = \{ \varphi \in \mathcal{D}(L) : (L \varphi, y - H \varphi) \in \mathcal{G} \}$$

is called an a posteriori set. The vector $\hat{\varphi}$ is said to be minimax a posteriori estimation of $\varphi$ in the direction $\ell$ iff

$$\hat{d}(\ell) := \inf_{\varphi \in X_y} \sup_{\psi \in X_y} |(\ell, \varphi) - (\ell, \psi)| = \sup_{\psi \in X_y} |(\ell, \varphi) - (\ell, \psi)|$$

The expression $\hat{d}(\ell)$ is called the minimax a posteriori error in the direction $\ell$.

In the sequel the minimax mean-squared a priori estimation (error) is referred as minimax estimation (error).

**Main results**

**Proposition 1.** Assume that $\mathcal{G}, \mathcal{R}$ are convex bounded closed subsets of $\mathcal{F}, \mathcal{L}(\mathcal{Y}, \mathcal{Y})$ respectively. For the given $\ell \in \mathcal{H}$ the minimax error $\hat{\sigma}(\ell)$ is finite iff

$$\ell - H^* u \in \text{dom cl}(L^* c) \cap (-1) \text{dom cl}(L^* c)$$

(4)

for some $u \in \mathcal{Y}$. Under this condition

$$\sigma(\ell, u) = \sup_{R_\eta \in \mathcal{R}} (R_\eta u, u) +$$

$$\frac{1}{4} \text{cl}(L^* c)(\ell - H^* u) + \text{cl}(L^* c)(-\ell + H^* u)]^2$$

(5)
where
\[ R(L^*) \subset \text{dom} \text{ cl}(L^*c) \subset \overline{R(L^*)} \]
If \( \text{Arginf}_u \sigma(\ell, u) \neq \emptyset \), then \( \ell(\varphi) = (\hat{u}, y) + \hat{c} \), where
\[ \hat{u} \in \text{Arginf}_u \sigma(\ell, u) \]
and
\[ \hat{c} = \frac{1}{2}(\text{cl}(L^*c)(\ell - H^*\hat{u}) - \text{cl}(L^*c)(-\ell + H^*\hat{u})) \]

**Theorema 1.** Suppose that \( \mathcal{G}_1 \) is convex bounded closed balanced set and its 0 \( \in \text{int} \mathcal{G}_1 \). Also assume that
\[ \eta \in \{ \eta : M(\eta, \eta) \leq 1 \} \]
Then for the given \( \ell \in \mathcal{H} \) the minimax estimation \( \hat{\sigma}(\ell) \) is finite iff \( \ell - H^*u \in R(L^*) \) for some \( u \in \mathcal{Y} \). Under this condition there exists a unique minimax estimation \( \hat{u} \in \mathcal{U}_\ell \)
\[ \sigma(\ell, \hat{u}) = \min_u \sigma(\ell, u), \]
\[ \sigma(\ell, u) = (u, u) + \min_z \{ c^2(\mathcal{G}_1, z), L^*z = \ell - H^*u \} \] \hspace{1cm} (6)
If \( R(L), H(N(L)) \) are closed sets then \( \hat{u} \) is determined by the following conditions
\[ \hat{u} - Hp_0 \in H(\partial I_2(H^*\hat{u})), Lp_0 = 0, \]
\[ I_2(w) = \min_z \{ c^2(\mathcal{G}_1, z), L^*z = P_{L^*}(\ell - w) \} \] \hspace{1cm} (7)

**Corollary 1.** Let
\[ \mathcal{G}_1 = \{ f \in \mathcal{F} : (f, f) \leq 1 \}, \eta \in \{ \eta : M(\eta, \eta) \leq 1 \}, \]
and suppose that
1) \( R(L), H(N(L)) \) are closed sets;
2) \( R(T) = \{ [Lx, Hx], x \in \mathcal{D}(L) \} \) is closed set.
Then only for \( \ell \in R(L^*) + R(H^*) \) the unique minimax estimation \( \hat{u} \) is given by \( \hat{u} = H\hat{p} \), where \( \hat{p} \) is any solution of the equations
\[ L^*\hat{z} = \ell - H^*H\hat{p}, \]
\[ L\hat{p} = \hat{z} \] \hspace{1cm} (8)
The minimax error is given by the following expression
\[ \hat{\sigma}(\ell) = (\ell, \hat{p})^\frac{1}{2} \]

**Corollary 2.** Assume that linear mappings \( L : \mathcal{H} \mapsto \mathcal{F}, H \in \mathcal{L}(\mathcal{H}, \mathcal{Y}) \) obey 1) or 2) (Cor. 1). Then \( \mathcal{G} \) has a solution \( \hat{z} \in \mathcal{D}(L^*), \hat{p} \in \mathcal{D}(L) \) iff \( \ell = L^*\hat{z} + H^*u \) for some \( z \in \mathcal{D}(L^*), u \in \mathcal{Y} \).

**Corollary 3.** Under the conditions of Cor. 1 for any \( \ell \in R(L^*) + R(H^*) \) and some realization of \( y(\cdot) \) we have \( (\hat{u}, y) = (\ell, \hat{\varphi}) \), where \( \hat{\varphi} \) obeys the equation
\[ L^*\hat{q} = H^*(y - H\hat{\varphi}), \]
\[ L\hat{\varphi} = \hat{q} \] \hspace{1cm} (9)
Consider an a posteriori estimation.

**Proposition 2.** Let \( \mathcal{G} \) be a convex closed bounded subset of \( \mathcal{Y} \times \mathcal{F} \). The minimax a posteriori error in the direction \( \ell \in \text{dom} c(\mathcal{X}_y, \cdot) \cap (-1)\text{dom} c(\mathcal{X}_y, \cdot) \) and
\[ R(L^*) + R(H^*) \subset \text{dom} c(\mathcal{X}_y, \cdot) \cap (-1)\text{dom} c(\mathcal{X}_y, \cdot) \subset R(L^*) + R(H^*) \] \hspace{1cm} (10)
Under this condition
\[ (\ell, \hat{\varphi}) = \frac{1}{2}(c(\mathcal{X}_y, \ell) - c(\mathcal{X}_y, -\ell)), \]
\[ \hat{d}(\ell) = \frac{1}{2}(c(\mathcal{X}_y, \ell) + c(\mathcal{X}_y, -\ell)) \] \hspace{1cm} (11)

**Theorema 2.** Let
\[ \mathcal{G} = \{ (f, \eta) : \|f\|^2 + \|\eta\|^2 \leq 1 \}, \]
and assume that one of the conditions of Cor. 1 is fulfilled. The minimax a posteriori estimation \( \hat{\varphi} \) obeys the equation
\[ L^*\hat{q} = H^*(y - H\hat{\varphi}), \]
\[ L\hat{\varphi} = \hat{q} \] \hspace{1cm} (12)
 iff \( \ell \in R(L^*) + R(H^*) \). The estimation error is given by
\[ \hat{d}(\ell) = (1 - (y, y - H\hat{\varphi}))^\frac{1}{2} \hat{\sigma}(\ell) \] \hspace{1cm} (13)

**Corollary 4.** Assume that the conditions of Theorem 2 are fulfilled and \( \ell(\hat{\varphi}) = (\ell, \hat{\varphi}) \) for any \( \ell \), where \( \hat{\varphi} \) obeys (12). Then \( \hat{\varphi} \) gives the minimax a posteriori estimation of \( \varphi \) so that
\[ \inf_{\varphi \in \mathcal{X}_y} \sup_{x \in \mathcal{X}_y} \|\varphi - x\| = \sup_{x \in \mathcal{X}_y} \|\varphi - x\| = (1 - (y, y - H\hat{\varphi}))^\frac{1}{2} \max \hat{\sigma}(\ell) \]
Now we shall apply Cor. 4 to the linear uncertain estimation problem for differential-algebraic equation. Taking into account that any DAE with constant matrices has the SVD canonical form, we assume without loss of generality that

\[ F = \begin{pmatrix} \mathbf{0} & \mathbf{1} \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \]

**Proposition 3.** Let \( t \mapsto x(t) \in \mathbb{R}^n \) be the solution of

\[
\frac{d}{dt} F x(t) - C x(t) = f(t), \quad F x(t_0) = 0,
\]

and set

\[
\mathcal{G} = \{ (f, \eta) : \int_{t_0}^T (\|f(t)\|^2 + \|\eta(t)\|^2) dt \leq 1 \}
\]

Then the minimax a posteriori estimation of the function \( x(\cdot) \) is given by \( \hat{x}(\cdot) \), where \( \hat{x}(t) = [x_1(t), x_2(t)] \),

\[
\dot{x}_1(t) = (C_1 - C_2 (E + C_4 C_4^{-1} C_4') x_1(t) + (C_2 (E + C_4 C_4^{-1} C_4' + E) q_1(t) + C_2 (E + C_4 C_4^{-1}) y_2(t),
\]

\[
\dot{q}_1(t) = (-C_1' + C_3 C_4 (E + C_4 C_4^{-1} C_4') q_1(t) + C_3 C_4 (E + C_4 C_4^{-1}) y_2(t) - y_1(t) + (C_3' (E - C_4 (E + C_4 C_4^{-1} C_4') C_3 + E) x_1(t),
\]

\[
q_1(T) = 0, x_1(t_0) = 0,
\]

\[
x_2(t) = -(E + C_4 C_4^{-1} C_4' C_3 x_1(t) + (E + C_4 C_4^{-1} (C_3 q_1(t) + y_2(t),
\]

\[
q_2(t) = -(E - C_4 (E + C_4 C_4^{-1} C_4') C_3 x_1(t) - C_4 (E + C_4 C_4^{-1}) (C_3 q_1(t) + y_2(t))
\]

provided that \( y(t) = x(t) + \eta(t), t_0 \leq t \leq T \). The minimax error is given by

\[
\sup_{x_\eta} \| x - \hat{x} \| = (1 - \int_{t_0}^T (y, y - \hat{x}) dt)^{\frac{1}{2}} \max_{\|\ell\| = 1} (\int_{t_0}^T (\ell, p) dt)^{\frac{1}{2}}
\]

where \( p(\cdot) \) obeys (14) with \( y(t) = \ell(t) \).

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