SYMPLECTIC REDUCTION AND SYMMETRY ALGEBRA
IN BOUNDARY CHERN-SIMONS THEORY

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Abstract

We derive the Kac-Moody algebra and Virasoro algebra in Chern-Simons theory with boundary by using the symplectic reduction method and the Noether procedures.
**I. INTRODUCTION**

There has been vast interest in Chern-Simons theory [1] with boundary [2] in diverse areas of physics, especially in Chern-Simons gravity theory [3] because of a relation with black hole entropy [4,5]. Recently, Bañados et. al [6] further argued that the Kac-Moody and Virasoro algebras on the boundary of the black hole play a crucial role in understanding the statistical origin of the BTZ black hole entropy [7]. However, the derivation of these algebras was based on some assumptions about boundary charges. Also the relation with standard Noether procedures and constraint analysis were not clear.

In this Letter, we rederive these algebras by applying the symplectic method [8] and the Noether procedure. We shall identify the correct symplectic structure on the boundary, calculate the Poisson bracket between gauge and diffeomorphism charges, and obtain the central terms in the Kac-Moody and Virasoro algebras.

**II. SYMPLECTIC STRUCTURE**

Let us start from the Chern-Simons Lagrangian on the disc $D$

$$L = \frac{\kappa}{2\pi} \int_D d^2 x \epsilon^{\mu \nu \rho} \left\langle A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right\rangle,$$

where $\langle \cdots \rangle$ denotes trace. Up to a boundary term, (1) can be put into the canonical form with the Lagrangian

$$L = \frac{\kappa}{4\pi} \int_D d^2 x \epsilon^{ij} (A_i^a \dot{A}_j^a - A_0^a F^a_{ij}),$$

(Here, $\epsilon^{012} \equiv \epsilon^{12} \equiv 1$, $A_i = A_i^a t_a$, $F_{ij} = F^a_{ij} t_a$, $F^a_{ij} = \partial_i A_j^a - \partial_j A_i^a + f^{abc} A_i^b A_j^c$, and the group generators $t^a$ satisfy $[t^a, t^b] = f^{abc} t^c$, $\left\langle t^a t^b \right\rangle = -\frac{1}{2} \delta^{ab}$.) We shall take (2) as our starting point. Variation with respect to $A_0^a$ gives the Gauss’ law constraint

$$F^a_{ij} = 0.$$  

We adopt symplectic method and first solve the constraint explicitly [8]. The solution is the pure gauge
\[ A_i = g^{-1} \partial_i g. \] (4)

Substitution into the Lagrangian (2) gives

\[
L = -\frac{\kappa}{2\pi} \int_D d^2x \epsilon^{ij} \left \langle g^{-1} \partial_i g \partial_k (g^{-1} \partial_j g) \right \rangle \\
= \frac{\kappa}{2\pi} \int_D d^2x \epsilon^{ij} \left \langle \partial_i g^{-1} \partial_j g g^{-1} \dot{g} \right \rangle + \frac{\kappa}{2\pi} \oint_{\partial D} d\varphi \left \langle g^{-1} \partial_\varphi g g^{-1} \dot{g} \right \rangle \\
\equiv L_B + L_S, \tag{5}
\]

where \( \varphi \) denotes the angular coordinate on the boundary \( \partial D \) of disc \( D \), and \( L_B \) and \( L_S \) are the bulk and surface Lagrangians, respectively. Upon the parameterization of \( g \) locally, it can be shown that the \( L_B \) is also a surface term \[ 4].

Let us first compute the symplectic structure \[ 10]. Lagrangian (5) suggests the following canonical 1-form

\[
\Theta = \frac{\kappa}{2\pi} \int_D d^2x \epsilon^{ij} \left \langle \partial_i g^{-1} \partial_j g g^{-1} dg \right \rangle + \frac{\kappa}{2\pi} \oint_{\partial D} d\varphi \left \langle g^{-1} \partial_\varphi g g^{-1} dg \right \rangle, \tag{6}
\]

where \( dg \) is the functional exterior derivative of \( g \). Then, a straightforward computation gives

\[
\Omega = d\Theta = \frac{\kappa}{2\pi} \oint_{\partial D} d\varphi \left \langle d(g^{-1} \partial_\varphi g) \wedge g^{-1} dg \right \rangle. \tag{7}
\]

The above symplectic structure \[ 11\] yields the following Poisson bracket

\[
\{(g^{-1} \partial_\varphi g)_{AB}(\varphi), \ g_{CD}(\varphi')\} = \frac{\pi}{\kappa} \delta_{AB} g_{CB} \delta(\varphi - \varphi'), \tag{8}
\]

where indices \( A, B, \cdots \) denote the components of the matrix. To be more explicit, we parameterize the group element by some local coordinates \( \theta^a (a = 1, \cdots, \dim G) \): \( g \equiv g(\theta^a) \).

Let us define \[ 11\]

\[
g^{-1}(\varphi) \frac{\partial g(\varphi)}{\partial \theta^a(\varphi')} = C_{a}^{b}(\varphi) t_{b} \delta(\varphi - \varphi'). \tag{9}
\]

Then

\[
\Omega = \frac{1}{2} \oint_{\partial D} d\varphi \oint_{\partial D} d\varphi' \omega_{ab}(\varphi, \varphi') d\theta^a(\varphi) \wedge d\theta^b(\varphi'), \tag{10}
\]

\[ 2 \]
with
\[
\omega_{ab}(\varphi, \varphi') = -\frac{\kappa}{4\pi} \left[ \frac{\partial A^c_\varphi(\varphi)}{\partial \theta^a(\varphi')} C^e_b(\varphi) - \frac{\partial A^c_\varphi(\varphi)}{\partial \theta^b(\varphi')} C^e_a(\varphi) \right],
\] (11)

where \( g^{-1} \partial_\varphi g = A^a_\varphi(\theta)t_a \). In general, this 2-form is degenerate, and does not possess an inverse. Further reduction must occur from \( G \) to one of its coadjoint orbit \( G/H \). For our purpose, we just assume the symplectic reduction has been performed and denote the coordinates on \( G/H \) by \( \bar{\theta}^\alpha \) (\( \alpha = 1, \ldots, \dim G/H = \text{even} \)). Then the 2-form (11) descends onto symplectic 2-form \( \omega_{\alpha\beta}(\varphi, \varphi') \) on \( G/H \), and the Poisson bracket is defined by
\[
\{ \bar{\theta}^\alpha(\varphi), \bar{\theta}^\beta(\varphi') \} = \omega^{\beta\alpha}(\varphi', \varphi),
\] (12)

where \( \omega^{\alpha\beta} \) is the inverse of \( \omega_{\alpha\beta} \). Fortunately, we do not need an explicit expression in terms of local coordinates neither for \( \omega^{\alpha\beta} \) nor for \( C^a_{\alpha\beta} \) of (9). Starting from the reduced expression of (7) on \( G/H \), we find
\[
\{ A^a_\varphi(\bar{\theta}(\varphi)), g(\bar{\theta}(\varphi')) \} = -\frac{2\pi}{\kappa} \delta(\varphi - \varphi') g(\varphi') t^a.
\] (13)

With this we calculate the Poisson bracket for \( A^a_\varphi \):
\[
\{ A^a_\varphi(\varphi), A^b_\varphi(\varphi') \} = \frac{2\pi}{\kappa} f^{acb} A^c_\varphi(\varphi) \delta(\varphi - \varphi') + \frac{2\pi}{\kappa} \partial_\varphi \delta(\varphi - \varphi') \delta^{ab}
\]
\[
= \frac{2\pi}{\kappa} (D_\varphi \delta(\varphi - \varphi'))^{ab},
\] (14)

which is the Kac-Moody algebra in density form. (\( D_\varphi \) is the \( \varphi \)-th component of the covariant derivative \( D^a_{\phi} = \delta^{ab} \partial_i + f^{acb} A^c_i \).)

**III. KAC-MOODY ALGEBRA OF GAUGE TRANSFORMATION**

We now consider the gauge transformation generated by \( \delta g = g\lambda \). From (4), we obtain
\[
\delta L = \frac{d}{dt} \left[ \frac{\kappa}{2\pi} \int_D d^2x e^{ij} \left\langle g^{-1} \partial_j gg^{-1} \partial_i g \lambda \right\rangle - \frac{\kappa}{2\pi} \int_{\partial D} d\varphi \left\langle g^{-1} \partial_\varphi g \lambda \right\rangle \right] \\
\equiv \frac{dX}{dt}. \tag{15}
\]
Then the Noether charge associated with this gauge transformation is given by

\[ Q(\lambda) = \left\langle \frac{\partial L}{\partial \dot{g}} \delta g \right\rangle - X \]

\[ = \frac{\kappa}{\pi} \oint_{\partial D} d\varphi \left\langle g^{-1} \partial_\varphi g \lambda \right\rangle = -\frac{\kappa}{2\pi} \oint_{\partial D} d\varphi A^a_\varphi \lambda^a. \]  

(16)

Using (14), we find that \( Q(\lambda) \) satisfies the Kac-Moody algebra:

\[ \{Q(\lambda), Q(\eta)\} = Q([\lambda, \eta]) - \frac{\kappa}{\pi} \oint_{\partial D} d\varphi \left\langle \lambda \partial_\varphi \eta \right\rangle, \]

(17)

where \([\lambda, \eta]^a = f^{abc} \lambda^b \eta^c\).

**IV. VIRASORO ALGEBRA OF DIFFEOMORPHISM**

In the derivation of the Kac-Moody algebra (14), the existence of the central term does not depend on what boundary condition one chooses for \( \lambda \)'s. The \( \lambda \)'s just have to be non-constant and single-valued functions on the boundary. However, in the derivation of the Virasoro algebra, the existence of central term depends crucially on the boundary condition one imposes. In the computation of the Virasoro charge, constraints can be imposed from the beginning as in the Kac-Moody case, or after the Noether procedure has been applied. We examine both cases here, because they give different boundary conditions in general, even though the final expressions of the charges are the same. Let us start with the Lagrangian (2) and study the response of \( L \) to a spatial and time-independent diffeomorphism (Diff):

\[ \delta_f x^\mu = -\delta_1^\mu f^i, \]

\[ \delta_f A_i^a = f^j \partial_j A_i^a + (\partial_i f^j) A_j^a, \]

\[ \delta_f A_0^a = f^j \partial_j A_0^a. \]  

(18)

Under (18), we find

\[ \delta_f L = \frac{\kappa}{4\pi} \int_D d^2x \epsilon^{ij} \partial_k [f^k A_\varphi A_j^a - f^k A^a_\varphi f_{ij}] \]

\[ = \frac{\kappa}{4\pi} \oint_{\partial D} d\varphi f^r (A^a_\varphi A^a_\varphi - A^a_\varphi A^a_\varphi - A^a_0 \epsilon^{ij} f_{ij}). \]

(19)
Now, we have two possible boundary conditions in order that there be Diff invariance, i.e.,
\[ \delta f = \frac{d}{dt} X \]: (a). \( f^r|_{\partial D} = 0 \), (b). \( A^a_\varphi|_{\partial D} = \text{constant}, \partial_r A^a_\varphi|_{\partial D} = 0, \) and \( A^a_0|_{\partial D} \propto A^a_\varphi|_{\partial D} \).

(a). This is the simpler boundary condition for the Diff invariance with \( X = 0 \) and results in Diff only along the circle (\( \partial D \)). The Noether charge for this Diff becomes
\[
Q(f) = \frac{\partial L}{\partial A^a_i} \delta f A^a_i - X,
\]
\[
= \frac{\kappa}{4\pi} \int_D d^2 x f^k A^a_k \epsilon^{ij} F^a_{ij} - \frac{\kappa}{4\pi} \oint_{\partial D} d\varphi f^c A^a_\varphi A^a_\varphi, \tag{20}
\]
and, imposing the \( F^{a}_{ij} = 0 \), one is left with
\[
Q(f) = -\frac{\kappa}{4\pi} \oint_{\partial D} d\varphi f^c A^a_\varphi A^a_\varphi, \tag{21}
\]
where \( A_\varphi = g^{-1} \partial_\varphi g \). Using the previous Poisson bracket (14), we find
\[
\{Q(f), Q(g)\} = Q([f, g]), \tag{22}
\]
where \([f, g] \equiv g \partial_\varphi f - f \partial_\varphi g\). This is the Virasoro algebra without central term, called Witt algebra. So, if we restrict the Diff along the circle (\( \partial D \)) there is no central term classically.

The central term will arise only as a quantum mechanical effect of normal ordering.

(b). From \( A^a_\varphi|_{\partial D} = \text{constant}, \partial_r A^a_\varphi|_{\partial D} = 0, \) and \( A^a_0|_{\partial D} \propto A^a_\varphi|_{\partial D}, \) (19) becomes \( \frac{d}{dt} \) with \( X = \frac{\kappa}{4\pi} \oint_{\partial D} d\varphi f^r A^a_r A^a_\varphi \). The Noether charge becomes
\[
Q(f) = \frac{\partial L}{\partial A^a_i} \delta f A^a_i - X,
\]
\[
= \frac{\kappa}{4\pi} \int_D d^2 x f^k A^a_k \epsilon^{ij} F^a_{ij} - \frac{\kappa}{4\pi} \oint_{\partial D} d\varphi (2 f^r A^a_r A^a_\varphi + f^c A^a_\varphi A^a_\varphi), \tag{23}
\]
and, imposing \( F^{a}_{ij} = 0 \), one is left with
\[
Q(f) = -\frac{\kappa}{4\pi} \oint_{\partial D} d\varphi (2 f^r A^a_r A^a_\varphi + f^c A^a_\varphi A^a_\varphi), \tag{24}
\]
where \( A_\varphi = g^{-1} \partial_\varphi g \). Using the Poisson bracket (14) and treating \( A^a_r|_{\partial D} \) as a c-number, we find
\[
\{Q(f), Q(g)\} = Q([f, g]) + \frac{\kappa}{2\pi} A^a_r A^a_i \oint_{\partial D} d\varphi f^r \partial_\varphi g^r. \tag{25}
\]
In general, this algebra does not satisfy the Jacobi identity and so the Noether charge \(Q(f)\) as a symmetry generator cannot be accepted. Therefore, the only way to avoid this undesirable situation is to consider the subset of transformation with particular \(f^r|_{\partial D} \propto \partial_\varphi f^\varphi|_{\partial D}\) and \(g^r|_{\partial D} \propto \partial_\varphi g^\varphi|_{\partial D}\) such that only the third order derivatives appear in the central term and hence (23) satisfies the Jacobi identity. A constant term \(\propto \oint_{\partial D} d\varphi f^\varphi A^a_r A^a_r\) can be added to \(Q(f)\) in (23) to obtain the first order derivative term in the center. Then, (23) becomes the standard form of the Virasoro algebra with central term by proper normalization of the proportionality constant [6]. So, in contrast to the \(\text{Diff}\) along the circle \((\partial D), \text{Diff}\) which deforms across the boundary has the central term even classically.

Now, let us recompute the charges after \(F^a_{ij} = 0\) was imposed from the beginning, and compare with previous results. Using \(F^a_{ij} = 0\), one can show that the second equation of (18) becomes [12]

\[
\delta f A^a_i = D_i(f^i A_i)^a. \tag{26}
\]

Then by substituting this into the variation of action (2) without the \(F^a_{ij}\) term, we find

\[
\delta f L = \frac{d}{dt} \left[ \frac{\kappa}{2\pi} \int_D d^2x f^{ij} A^a_i D_j(f^k A_k)^a \right] + \frac{\kappa}{2\pi} \oint_{\partial D} d\varphi A^a_i \dot{A}^a_i. \tag{27}
\]

For \(\text{Diff}\) invariance, we demand the second term to be a total time derivative. Again, we have two boundary conditions: (a). \(f^r|_{\partial D} = 0\), and (b). \(\dot{A}^a_r|_{\partial D} = 0\).

(a). When \(f^r|_{\partial D} = 0\), we find the second term of (27) reduces to

\[
\frac{d}{dt} \left[ \frac{\kappa}{4\pi} \int_D d^2x f^\varphi A^a_\varphi A^a_\varphi \right], \tag{28}
\]

and Noether charge is precisely the expression given in (21).

(b). When \(f^r|_{\partial D} \neq 0\), the second term can again be combined into a total derivative term by demanding that \(\dot{A}^a_r|_{\partial D} = 0\). Then, we have the charge precisely given in (24), and again one finds the Virasoro algebra with central term. Note that unlike the derivation of (24), we need not to impose the extra conditions \(\partial_\varphi A^a_\varphi|_{\partial D} = 0\), and \(A^a_0|_{\partial D} \propto A^a_\varphi|_{\partial D}\).
V. CONCLUSION

In summary, Kac-Moody and Virasoro algebra of Chern-Simons theory were derived by using the boundary symplectic structure which emerges as a consequence of symplectic reduction, and by applying the standard Noether procedures. The merit of our approach is that no assumptions about boundary charges are needed, and the Kac-Moody algebra in its standard current density form (14) is obtained. It remains to be seen whether the Dirac’s method yields the same results. It would be also interesting to extend the symplectic method to the supersymmetric case and to the higher dimensional Chern-Simons theories.

Note added.-After completing this work, we became aware of Ref. [13] in which the Kac-Moody algebra in Yang-Mills theory with Chern-Simons term was derived and generalized to higher dimensions. We thank J. Mickelsson for informing us about his work.

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