A “BOUNDEDNESS IMPLIES CONVERGENCE” PRINCIPLE AND ITS APPLICATIONS TO COLLAPSING ESTIMATES IN KÄHLER GEOMETRY

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ABSTRACT. We establish a general “boundedness implies convergence” principle for a family of evolving Riemannian metrics. We then apply this principle to collapsing Calabi-Yau metrics and normalized Kähler-Ricci flows on torus fibered minimal models to obtain convergence results.

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1. Introduction

In differential geometry, we usually try to find “canonical” geometric objects, e.g. Einstein metrics, minimal surfaces, etc., through a deformation process. Starting with a given object, we deform it through a suitable geometric flow or continuity path to get the desired metric or submanifold. To obtain convergence, we need a priori bounds for the higher order covariant derivatives of the evolving functions, metrics or curvature tensors. In general, such bounds are invalid and singularities may form. However, if we have curvature bounds or non-collapsing condition, general theories from Riemannian geometry and PDE usually gave us a lot of information about the formation of singularities and in the end we still get good geometric results. On the other hand, there are some important cases in which we lose curvature bounds and the metrics may collapse to lower dimensional spaces. In such cases, a priori bounds play a more decisive role. It is still interesting to know whether certain tensor fields converge to some degenerate or singular one.

In this paper, we prove a simple but useful “Boundedness Implies Convergence” lemma, which says that in certain situation if we have the convergence of an evolving tensor field together with uniform
estimates of its covariant derivatives, then we automatically have the convergence of its covariant derivatives. In spirit, this is similar to the simple fact in calculus that if a smooth function on $\mathbb{R}^n$ has bounded partial derivatives of all orders and if the function converges to 0 at infinity, then all its partial derivatives converge to 0 at infinity. Indeed, if we have uniform equivalence of metrics, this follows easily from interpolation type inequalities. Instead, here we assume the existence of good cut-off functions. In applications, such cut-off functions usually arise as the pull back of cut-off functions from the base manifold of a fibration. To be precise, we have:

**Lemma 1.1** (The “Boundedness Implies Convergence” Principle). Let $X$ be an $n$-dimension Riemannian manifold (not necessarily to be compact or complete) and $U$ be an open subset. Let $\tilde{g}(t)$ be a family of Riemannian metrics on $X$, $t \in \mathbb{R}$ and let $\eta(t)$ be a family of smooth functions or general tensor fields on $X$, satisfying the following conditions:

(A) $\|\eta(t)\|_{C^0(U, \tilde{g}(t))} \leq h_0(t)$.

(B) $\|\eta(t)\|_{C^k(U, \tilde{g}(t))} \leq A_k$, for $k = 1, 2, \ldots$.

(C) For any compact subset $K \subset U$, there exists smooth cut-off function $\rho$ with compact support $\tilde{K} \subset U$ such that $0 \leq \rho \leq 1$, and $\rho \equiv 1$ in a neighborhood of $K$, satisfying

\[ |\nabla^2 \rho|_{\tilde{g}(t)} + |\Delta \rho|_{\tilde{g}(t)} \leq B_K. \tag{1.1} \]

on $\tilde{K} \times [0, \infty)$, for some constant $B_K$ independent of $t$ (but may depend on the geometry of $K$).

Then we have: For any compact subset $K \subset U$ the estimates

\[ \|\eta(t)\|_{C^k(K, \tilde{g}(t))} \leq h_{K,t}(t). \tag{1.2} \]

where $h_{K,t}(t)$ are positive functions which tend to zero as $t \to \infty$, depending on the constants $A_0$, $A_1$, $\ldots$, $A_{k+2}$, $B_K$ and the function $h(t)$.

**Remark 1.2.**

1. Lemma 1.1 also holds true for discrete sequences of metrics and tensors. This is easily seen from the proof in section 2.

2. In Condition (A) if $h_0(t)$ is of the form $Ce^{-ct}$ for some positive constants $C < \infty$ and $c > 0$, then the functions $h_{K,t}(t)$ can be chosen to be $C_{K,t}e^{-ckt}$ for some constants $C_{K,t}, c_{K,t}$, i.e. covariant derivatives of $\eta(t)$ also decay exponentially.

3. In Lemma 1.1 if we only have Condition (B) for $1 \leq k \leq N + 2$, then we still have the estimate (1.2) for $1 \leq k \leq N$.

Note that in Lemma 1.1 we do not require the metrics to be uniformly non-degenerate or have bounded curvature and do not require the metrics and tensors satisfy differential equations. So it
applies even when the metrics collapse to lower dimensional spaces without curvature bounds. In particular, we shall apply this principle to two collapsing problems in Kähler geometry. We hope this simple “BIC principle” may find other applications in geometric analysis, especially in collapsing problems.

The first application in this paper is on Calabi-Yau degenerations.

Given a Calabi-Yau manifold $X$ with a holomorphic fiber space structure (i.e. there is a holomorphic surjection $f$ onto a lower dimensional variety, which is a holomorphic submersion outside a subvariety), the Calabi-Yau theorem [38] assures the existence of unique Ricci-flat Kähler metrics on the total space in every Kähler classes. Now let the Kähler classes approach the pull back of a Kähler class from the base, then the volume of these Ricci-flat metrics go to zero. One would like to understand the asymptotic behaviors of these degenerating metrics.

This problem has been much studied in recent years, starting from the pioneering work of Gross-Wilson [10] on elliptic $K3$ surfaces fibered over the 2-sphere, to more recent works in general dimensions by Tosatti [27], Tosatti-Zhang [31, 32], Gross-Tosatti-Zhang [8, 9], Hein-Tosatti [13], Tosatti-Weinkove-Yang [34], Li [17], and elsewhere. From these works, we know that the Ricci-flat metrics collapse, in some sense, to the pullback of a canonical Kähler metric on the base, uniformly on compact sets away from the singular fibers.

In this general setting, a $C^0_{\text{loc}}$ estimate was proved by Tosatti-Weinkove-Yang in [34], and this estimate can be improved to $C^\infty_{\text{loc}}$ estimate when the smooth fibers are tori or finite étale quotients of tori by Gross-Tosatti-Zhang [8] and Hein-Tosatti [13]. Certain components of the first order derivatives were bounded by Tosatti [27] and Tosatti-Weinkove-Yang in [34]. An even stronger partial estimate was proved by Tosatti-Zhang in [31]: the restriction of $e^t\omega(t)$ to $f^{-1}(U)$ converges in the pointed $C^\infty$ Cheeger-Gromov topology to the product of a flat $C^\infty$ with a fiber equipped with Ricci-flat metric.

In [14], with a systematic use of iterated blow-up-and-contradiction type arguments, Hein-Tosatti substantially improved the estimate to $C^\infty_{\text{loc}}$ if the regular fibers are pairwise bi-holomorphic to each other. In the general fibration case, they can improve the $C^0$ convergence of [34] to $C^\infty$ convergence. In a more recent preprint [23], Song-Tian-Zhang proved the uniform diameter bound and the Gromov-Hausdorff convergence of this family of collapsing metrics.

In this paper, we derive from Hein-Tosatti’s $C^\infty_{\text{loc}}$ estimate the corresponding convergence results, and such estimates also imply the $C^\infty_{\text{loc}}$ asymptotic behavior of the curvature tensor.

To state our results, let $X$ be a compact Kähler $(n + m)$-manifold with $c_1(X) = 0$ in $H^2(X, \mathbb{R})$ (i.e. a Calabi-Yau manifold), and let $\omega_X$ be a Ricci-flat Kähler metric on $X$. Suppose that we have a holomorphic map $f : X \rightarrow Z$ with connected fibers, where $(Z, \omega_Z)$ is a compact Kähler manifold, with image $B = f(X) \subset Z$ an irreducible normal subvariety of $Z$ of dimension $m > 0$. Then the induced surjective
map \( f : X \to B \) is a fiber space, and if \( S' \subset B \) denotes the singular set of \( B \) together with the set of critical values of \( f \), and \( S = f^{-1}(S') \), then \( S' \) is a proper analytic subvariety of \( B \), \( S \) is a proper analytic subvariety of \( X \), and \( f : X \setminus S \to B \setminus S' \) is a submersion between smooth manifolds. The fibers \( X_b \) for \( b \in B \setminus S' \) are Calabi-Yau \( n \)-folds. Write \( \chi = f^* \omega_Z \), which is a smooth nonnegative \((1, 1)\) form on \( X \), and we will also write \( \chi \) for the restriction of \( \omega_Z \) to \( B \setminus S' \).

Let \( \omega_B \) be the current in \([\chi]\) that is smooth and positive on \( B \setminus S' \) satisfying

\[
\text{Ric}(\omega_B) = \omega_{WP},
\]

where \( \omega_{WP} \) is a smooth semipositive Weil-Petersson form on \( B \setminus S' \). Its construction will be briefly recalled in section 3.2. We also have the semi-Ricci flat metric \( \omega_{SRF} \) on \( X \setminus S \), such that for each \( b \in B \setminus S' \), \( \omega_{SRF|X_b} \) is the unique Ricci-flat metric on \( X_b \) in the Kähler class \([\omega_X|X_b]\). Define the reference metrics \( \tilde{\omega}_t \) on the regular part \( X \setminus S \) by

\[
\tilde{\omega}_t = \omega_B + e^{-t} \omega_{SRF}.
\] (1.3)

Let \( \omega(t) \) be the unique Ricci-flat metric in \([\tilde{\omega}_t]\) that is smooth and positive on \( X \setminus S \) for any \( k \), depending only on \( k \) and the domain \( X \setminus S \). Then Tosatti-Weinkove-Yang proved in [3,4] that \( \|\omega(t) - \omega_B\|_{C^0(K, \tilde{\omega}_t)} \to 0 \), as \( t \to \infty \) on any compact set \( K \subset X \setminus S \). Denote the \((1,3)\)-curvature tensor of a Kähler metric \( \omega \) (associated to the Riemannian metric \( g \)) by \( R^\omega \), i.e.

\[
R^\omega_{i\bar{j}k} = g^{l\bar{q}} R(\omega)_{i\bar{j}k\bar{q}}.
\] (1.4)

If all the regular fibers are bi-holomorphic to each other, then \( \omega_{SRF,b} \) is independent of the base regular point \( b \) (see Equation [5,8]), and at this time we define the Kähler metric \( \omega_Y \) on \( Y \) by

\[
\omega_Y = \omega_{SRF,b},
\]

for any \( b \in B \setminus S' \). Based on the higher estimates of Hein-Tosatti [14], we have:

**Theorem 1.3.** Assume all the regular fibers are bi-holomorphic to a fixed Calabi-Yau manifold \( Y \). Let \( U \subset B \setminus S' \) be an open set such that the fibration is holomorphically trivial over \( U \). Identify \( f^{-1}(U) \) with \( U \times Y \). Define another reference metrics \( \tilde{\omega}(t) \) on \( U \times Y \) by

\[
\tilde{\omega}(t) = \omega_B + e^{-t} \omega_Y.
\] (1.5)

Then for each compact set \( K \subset U \), for any \( k \in \mathbb{N} \), we have

\[
\|\omega(t) - \tilde{\omega}(t)\|_{C^0(U \times Y, \tilde{\omega}(t))} \leq h_{K,k}(t),
\] (1.6)

and

\[
\|R^\omega(\omega(t)) - R^\omega(\tilde{\omega}(t))\|_{C^0(\omega, \omega(t))} \leq h_{K,k}(t),
\] (1.7)

where \( h_{K,k}(t) \) are positive functions which tends to zero as \( t \to \infty \), depending only on \( k \) and the domain \( K \).
Note that since we do not have curvature bounds, we can not derive the decay estimate of $\|Rm(\omega(t)) - Rm(\tilde{\omega}(t))\|_{C^k}$ from (1.7).

Another special case is when the smooth fibers $X_b$ are all complex torus by a holomorphic free action of a finite group, but we allow the complex structure to change. By [8, 13, 31], we have $C^\infty$ estimates as well as local curvature bounds on $X \setminus S$. We can apply Lemma 1.1 to obtain:

**Theorem 1.4.** Assume that for some $b \in B \setminus S'$ the fiber $X_b = f^{-1}(b)$ is bi-holomorphic to a finite quotient of a torus. Let $K \subset X \setminus S$ be any compact subset. Then we have

$$\|\omega(t) - \tilde{\omega}_t\|_{C^k(K, \tilde{\omega}_t)} \leq h_{K,k}(t).$$

and

$$\|Rm(\omega(t)) - Rm(\tilde{\omega}_t)\|_{C^k(K, \tilde{\omega}_t)} \leq h_{K,k+2}(t).$$

where $h_{K,k}(t)$ are positive functions which tends to zero as $t \to \infty$, depending only on $k$ and the domain $K$. In particular, when $S = \emptyset$, the estimates are globally true and each $h_k(t)$ is of exponential fast decay.

The second application is on the normalized Kähler-Ricci flow on torus fibered minimal models.

Let $(X, \omega_0)$ be a compact Kähler $(n + m)$-manifold with semiample canonical bundle and Kodaira dimension $m$. Here we assume $m > 0, n > 0$. The sections of $K^\ell_X$, for $\ell$ large, give rise to a fiber space $f : X \to B$ called the Iitaka fibration of $X$, with $B$ a normal projective variety of dimension $m$ and the smooth fibers $X_b = f^{-1}(b), b \in B \setminus S'$ are all Calabi-Yau $n$-manifolds, diffeomorphic to each other. Let $\chi$ be the restriction of $\frac{1}{\ell} \omega_{FS}$ to $B$, as well as its pullback to $X$. This time we consider the solution $\omega = \omega(t)$ of the normalized Kähler-Ricci flow

$$\frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) - \omega, \quad \omega(0) = \omega_0.$$ 

which exists for all $t \geq 0$. Thanks to [19, 20, 21, 4, 34, 31, 6, 8] we have that the evolving metrics have uniformly bounded scalar curvature globally and collapse locally uniformly on $X \setminus S$ to a canonical Kähler metric on $B \setminus S'$, and moreover the rescaled metrics along the fibers $e^t \omega|_{X_b}$ converge in $C^\infty$ to a Ricci-flat metric on $X_b$. This is the collapsing phenomenon in the Kähler-Ricci flow case.

Now, assume the smooth fibers are the quotient of a complex torus by a holomorphic free action of a finite group, then we have smooth collapsing to the generalized Kähler-Einstein metrics defined by Song-Tian [20] on the regular part with respect to a fixed metric.

An immediate corollary of Lemma 1.1 is the smooth convergence of the solution and its curvatures.

**Theorem 1.5.** Let $(X^{n+m}, \omega_0)$ with $n > 0$ be a compact Kähler manifold with $K_X$ semiample and $\kappa(X) = m > 0$, and let $f : X \to B$ be the fibration as described above. Assume that for some $y \in B \setminus S'$
the fiber $X_b = f^{-1}(b)$ is bi-holomorphic to a finite quotient of a torus. Let $\omega(t), t \in [0, \infty)$ be the solution of the Kähler-Ricci flow \ref{eq:KRF} starting at $\omega_0$. Let $K \subset X \setminus S$ be any compact subset. Then we have

$$\|\omega(t) - \tilde{\omega}(t)\|_{C^k(K, \tilde{\omega}(t))} \leq h_{K,k}(t),$$ \tag{1.11}

where $\tilde{\omega}(t) = e^{-t}\omega_{SRF} + (1 - e^{-t})\omega_B$ with $\omega_B$ the Song-Tian’s generalized Kähler-Einstein metric current on $B$. (See section 4 for its definition) Moreover, we have the smooth convergence of the curvature tensors

$$\|\text{Rm}(\omega(t)) - \text{Rm}(\tilde{\omega}(t))\|_{C^k(K, \tilde{\omega}(t))} \leq h_{K,k}(t),$$ \tag{1.12}

$$\|\text{Ric}(\omega(t)) - \text{Ric}(\tilde{\omega}(t))\|_{C^k(K, \tilde{\omega}(t))} \leq h_{K,k}(t),$$ \tag{1.13}

$$\|\text{R}(\omega(t)) - \text{R}(\tilde{\omega}(t))\|_{C^k(K, \tilde{\omega}(t))} \leq h_{K,k}(t),$$ \tag{1.14}

where $h_{K,k}(t)$ are positive functions which tends to zero as $t \to \infty$, depending only on $k$ and the domain $K$. In particular, when $S = \emptyset$, the estimates are globally true and each $h_k(t)$ is of exponential fast decay.

Our next result is to exhibit the relation between the Ricci curvature and scalar curvature of the solution $\omega(t)$ and the generalized Kähler-Einstein metric $\omega_B$. We have

**Theorem 1.6.** Assume the same set-up as in Theorem 1.5. Let $K \subset X \setminus S$ be any compact subset. Then we have

$$\|\text{Ric}(\omega(t)) + \omega_B\|_{C^k(K, \tilde{\omega}(t))} \leq h_{K,k+2}(t),$$ \tag{1.15}

and the convergence of scalar curvature

$$\|\text{R}(\omega(t)) + m\|_{C^k(K, \tilde{\omega}(t))} \leq h_{K,k+2}(t),$$ \tag{1.16}

where $h_{K,k}(t)$ are positive functions which tends to zero as $t \to \infty$, depending only on $k$ and the domain $K$. In particular, when $S = \emptyset$, the estimate is globally true and each $h_k(t)$ is of exponential fast decay.

In \cite{15}, the first author showed that the scalar curvature converges to $-m$ in the $C^0_{loc}$ topology in the general fibration case. Here we improved the topology to $C^\infty_{loc}$ in the special case when the fibers are flat. We do expect that Theorem 1.6 holds for the general case.

This paper is arranged as follows: In section 2, we prove Lemma 1.1 by maximum principle. Then we apply it to Calabi-Yau degenerations in section 3 where Theorem 1.3, 1.4 are proved. Finally in section 4 we prove Theorem 1.5 and 1.6 for normalized Kähler-Ricci flow on minimal models whose regular fibers are all finite quotients of complex tori.

In this paper, we use the following notations and conventions.
We always denote by $h(t)$ a positive function which tends to zero as $t \to \infty$, and by $h_{k,k}(t)$ we mean that this function also depends on the domain $K$ and the order $k$. We allow these functions change from line to line.

When we compute on a product manifold $X = B \times Y$, we always use a product coordinate system and, we call $B$ the base space and the corresponding indices the base directions, and we call $Y$ the fiber space and the corresponding indices the fiber directions. We will denote any complex $(1,0)$ "base" $\mathbb{C}^m$ direction by a subscript $b$ and any complex $(1,0)$ "fiber" $Y$ direction by a subscript $f$.

By $\nabla^{k,g}$ we means all the possible covariant derivatives with respect to the metric $g$, including holomorphic and anti-holomorphic covariant derivatives when $g$ is a Kähler.

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2. Proof of the "Boundedness Implies Convergence" Principle

We use the maximum principle to prove Lemma [1.1]

Proof of Lemma [1.1] Let $K$ be any compact subset of $U$, and choose compact subsets $\hat{K} \subset \subset U$ and smooth cut-off function $\rho$ as in the Condition (C) of Lemma [1.1] Let $C_k$ denote constants which depend on $A_1, \ldots, A_{k+2}$ which may change from line to line.

The proof is by induction on the order $k$. The case $k = 0$ for Equation (1.2) is just the Condition (A). Suppose we have established (1.2) for $0, \ldots, k-1$ for $k \geq 1$. Now we prove the estimate (1.2) for $k$.

First, using induction hypothesis, we can find some positive function $h(t)$ converging to 0 such that

$$\left| \nabla^{k-1,\hat{g}(t)} \eta \right|_{\hat{g}(t)}^2 \leq h(t) \quad (2.1)$$

holds on $\hat{K}$.

Next, using Condition (B), we can compute for every $k \geq 0$ on $U$

$$\left( -\Delta_{\hat{g}(t)} \right) \left( \nabla^{k,\hat{g}(t)} \eta \right)_{\hat{g}(t)}^2 = -\tilde{g}(t)^{ab} \cdot \nabla_{a}^{\hat{g}(t)} \nabla_{b}^{\hat{g}(t)} \left( \left| \nabla^{k,\hat{g}(t)} \eta \right|_{\hat{g}(t)}^2 \right)$$

$$= -2 \left| \nabla^{k+1,\hat{g}(t)} \eta \right|_{\hat{g}(t)}^2 + \nabla^{k+2,\hat{g}(t)} \eta \star \nabla^{k,\hat{g}(t)} \eta \quad (2.2)$$

$$\leq -2 \left| \nabla^{k+1,\hat{g}(t)} \eta \right|_{\hat{g}(t)}^2 + C \cdot \left| \nabla^{k+2,\hat{g}(t)} \eta \right|_{\hat{g}(t)} \cdot \left| \nabla^{k,\hat{g}(t)} \eta \right|_{\hat{g}(t)}$$

$$\leq -2 \left| \nabla^{k+1,\hat{g}(t)} \eta \right|_{\hat{g}(t)}^2 + C \cdot \left| \nabla^{k,\hat{g}(t)} \eta \right|_{\hat{g}(t)}$$
where $C = C_k$ and * denotes the tensor contraction by the metric $\tilde{g}(t)$. Note that in the above inequalities we do not use Bochner-type formulas since we do not have curvature bounds. We use instead the assumption of higher order estimates. Applying (2.2) for $k - 1 \geq 0$, we have on $U$

$$\left( -\Delta_{\tilde{g}(t)} \right) \left( \left| \nabla^{k-1,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \right) \leq -2 \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} + C \cdot \left| \nabla^{k-1,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)}, \tag{2.3}$$

Also by (2.2) and Condition (B), we have on $U$

$$\left( -\Delta_{\tilde{g}(t)} \right) \left( \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} \right) = 2 \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \cdot \left( -\Delta_{\tilde{g}(t)} \right) \left( \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \right) - 2 \left| \nabla \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \right|^2_{\tilde{g}(t)} \leq 2 \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \cdot \left( -2 \left| \nabla^{k+1,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} + C \cdot \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \right) - 2 \left| \nabla \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \right|^2_{\tilde{g}(t)} \leq 2C \cdot \left| \nabla^{k,\tilde{g}(t)} \eta \right|^3_{\tilde{g}(t)} - 2 \left| \nabla \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \right|^2_{\tilde{g}(t)} \tag{2.4}$$

where $C = C_k$. Now, take the cut-off function $\rho$ into consideration. By (2.4) and Condition (B) we have

$$\left( -\Delta_{\tilde{g}(t)} \right) \rho^2 \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} = \rho^2 \left( -\Delta_{\tilde{g}(t)} \right) \left( \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} \right) + \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} \left( -\Delta_{\tilde{g}(t)} \right) \left( \rho^2 \right) - 2 \left| \nabla \rho^2, \nabla \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} \right|_{\tilde{g}(t)} \leq \rho^2 \left( C \cdot \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} - 2 \left| \nabla \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \right|^2_{\tilde{g}(t)} \right) + C \cdot \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} + C \cdot \rho \left| \nabla \rho \right|_{\tilde{g}(t)} \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \left| \nabla \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \right|_{\tilde{g}(t)} \leq C \cdot \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} + C \cdot \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} \leq C \cdot \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} \tag{2.5}$$

where $C = C_k$. Now, put (2.1), (2.3) and (2.5) together, we conclude that we can find some $h(t)$ such that on $\tilde{K}$

$$\left\{ \begin{align*}
\left| \nabla^{k-1,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} h(t)^{-1} & \leq 1, \\
\left( -\Delta_{\tilde{g}(t)} \right) \left( \left| \nabla^{k-1,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} h(t)^{-1} \right) & \leq -2 \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} h(t)^{-1} + 1, \\
\left( -\Delta_{\tilde{g}(t)} \right) \rho^2 \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} h(t)^{-1} & \leq C \cdot \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} h(t)^{-1},
\end{align*} \right. \tag{2.6}$$

where $C = C_k$. Define an auxiliary function

$$Q := \rho^2 \left| \nabla^{k,\tilde{g}(t)} \eta \right|^4_{\tilde{g}(t)} h(t)^{-1} + C \cdot \left| \nabla^{k-1,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} h(t)^{-1}.$$ 

Using (2.6), on $\tilde{K}$ we have

$$\left( -\Delta_{\tilde{g}(t)} \right) (Q) \leq - \left| \nabla^{k,\tilde{g}(t)} \eta \right|^2_{\tilde{g}(t)} h(t)^{-1} + C.$$
Now, at a given time \( t \geq 0 \), assume \( Q \) achieves its maximum in \( U \) at a point \( x_0 \in \bar{U} \). If \( x_0 \notin \text{Int} \hat{K} \), then \( \rho \equiv 0 \), \( \rho(t) \), implies that \( Q \) has a uniform upper bound \( C \), and we are done. Otherwise \( x_0 \in \text{Int} \hat{K} \), we have

\[
0 \leq \left(-\Delta_{\hat{g}(t)}\right)(Q)(x_0) \leq -\left|\nabla^{\hat{g}(t)}_{\hat{g}(t)}\eta\right|^2_{\hat{g}(t)}(x_0) h(t)^{-1} + C.
\]

which gives

\[
\left|\nabla^{\hat{g}(t)}_{\hat{g}(t)}\eta\right|^2_{\hat{g}(t)}(x_0) h(t)^{-1} \leq C.
\]

Then by Condition \((B)\) and \( \rho(t) \) we have on \( \hat{K} \)

\[
Q \leq Q(x_0) \leq A_k^2 \left|\nabla^{\hat{g}(t)}_{\hat{g}(t)}\eta\right|^2_{\hat{g}(t)}(x_0) h(t)^{-1} + C \cdot \left|\nabla^{k-1,\hat{g}(t)}_{\hat{g}(t)}\eta\right|^2_{\hat{g}(t)}(x_0) h(t)^{-1} \leq C.
\]

Since \( \rho \equiv 1 \) on \( K \), we obtain

\[
\left|\nabla^{k,\hat{g}(t)}_{\hat{g}(t)}\eta\right|^2_{\hat{g}(t)} \leq C h(t)^{\frac{k}{2}}
\]

on \( K \), where \( h(t) \) depends on the constants \( A_1, \ldots, A_{k+2}, B_K \) and the function \( h_0(t) \). This establishes \( (1.2) \) for \( k \) and hence completes the proof.

In applications, it is crucial to have Condition \((C)\). If there is a fibration structure, then we can find such cut-off functions by pulling back a cut-off function from the base manifold, as shown by the following Lemma:

**Lemma 2.1.** If \( f : \mathbb{C}^{m+n} \to B \) is a proper holomorphic submersion onto a ball in \( \mathbb{C}^m \), and if \( \tilde{\omega}(t) \) is of the form \( \omega_B + e^{-t} \omega_0 \), where \( \omega_B \) is a Kähler form on \( B \) and \( \omega_0 \) is real closed \((1, 1)\) form on \( X \) whose restriction to each fiber is positive, then for any compact subset \( K \subset X \), we can find cut-off function \( \rho \) satisfying Condition \((C)\) of Lemma 1.1.

**Proof.** Since \( K \) is compact, so is \( f(K) \subset B \). Then we can find a ball \( B_1 \subset B \) such that \( f(K) \subset B_1 \). Choose a cut-off function \( \rho_0 \in C_0^\infty(B) \) such that \( \text{supp} \rho_0 \subset B_1 \) and \( \rho_0|_{f(K)} \equiv 1 \). Then we can take \( \rho := f^* \rho_0 \) and \( \hat{K} := f^{-1}(B_1) \). Since

\[
\sqrt{-1} \partial \bar{\partial} \rho_0 \leq C \omega_B, \quad -C \omega_B \leq \sqrt{-1} \partial \bar{\partial} \rho_0 \leq C \omega_B,
\]

for some \( C > 0 \) on \( B \), we have on \( \hat{K} \)

\[
\left|\nabla \rho\right|^2_{\hat{g}(t)} = \sum_{i,j=1}^{m} \hat{g}(t)^{ij} \partial_i \rho \partial_j \rho \leq C,
\]

\[
\left|\Delta_{\hat{g}(t)} \rho\right| = \left|\text{tr}_{\hat{g}(t)} \sqrt{-1} \partial \bar{\partial} \rho\right| \leq C \text{tr}_{\hat{g}(t)} \omega_B \leq C
\]

with some constant \( C \) depending on the domain \( \hat{K} \). This verifies Condition \((C)\).

\( \square \)
3. Applications to collapsing Calabi-Yau metrics

3.1. Metric and curvature convergence on locally trivial Calabi-Yau fibrations. We recall basic definitions in the general fibration case. Let $X$ be a compact Kähler $(n + m)$-manifold with $c_1(X) = 0$ and let $\omega_X$ be a Ricci-flat Kähler metric on $X$. Let $f : X \to Z$ be the fibration map with $B = f(X)$. Write $\chi = f^* \omega_Z$, where $\omega_Z$ is a smooth Kähler form on $Z$. Then $\chi$ is a smooth nonnegative $(1, 1)$ form on $X$, and we will also write $\chi$ for the restriction of $\omega_Z$ to $B$. Note that $\int_{B\setminus S'} \chi^m$ is finite.

We define a semi Ricci-flat form $\omega_{SRF}$ on $X \setminus S$ in the usual way. Namely, for each $b \in B \setminus S'$ there is a smooth function $\rho_b$ on $X_b$ so that $\omega_X|_{X_b} + \sqrt{-1} \bar{\partial} \rho_b = \omega_{SRF,b}$ is Ricci-flat, normalized by $\int_{X_b} \rho_b (\omega_X|_{X_b})^n = 0$. As $b$ varies, this defines a smooth function $\rho$ on $X \setminus S$ and we define $\omega_{SRF} = \omega_X + \sqrt{-1} \bar{\partial} \rho$.

Let $F$ be the function on $X \setminus S$ given by

$$F = \frac{\omega_X^{n+m}}{(n+m) \omega_{SRF}^{n} \wedge \chi^m}.$$ 

It is easy to see that $F$ is constant along the fibers $X_b$, $b \in B \setminus S'$, so it descends to a smooth function, also denoted by $F$, on $B \setminus S'$. We see that $F$ satisfies $\int_{B \setminus S'} F \chi^m = \int_X \omega_X^{n+m} / (n+m) \int_X \omega_X^n$ (see [20 Section 3] and [27 Section 4]). Here note that $\int_X \omega_X^n$ is independent of $b \in B \setminus S'$.

Then [20 Section 3] shows that the Monge-Ampère equation

$$(\chi + \sqrt{-1} \bar{\partial} \bar{\partial} v)^m = \frac{(n+m)}{n} \frac{\int_X \omega_X^n \wedge \chi^m}{\int_X \omega_X^{n+m}} F \chi^m,$$  

has a unique solution $v$ which is a bounded $\chi$-plurisubharmonic function on $B$, smooth on $B \setminus S'$, with $\int_X v \omega_X^{n+m} = 0$, where here and henceforth we write $v$ for $\pi^* v$.

Define

$$\omega_B = \chi + \sqrt{-1} \bar{\partial} \bar{\partial} v,$$ 

for $v$ solving (3.1). Note that we have

$$\omega_{SRF} \wedge \omega_B^n = \frac{(n+m)}{n} \frac{\int_X \omega_X^n \wedge \chi^m}{\int_X \omega_X^{n+m}} F \omega_{SRF} \wedge \chi^m = \frac{\int_X \omega_X^n \wedge \chi^m}{\int_X \omega_X^{n+m}} \omega_X^{n+m}.$$  

Moreover, $\omega_B$ is a smooth Kähler metric on $B \setminus S'$, and satisfies

$$\text{Ric}(\omega_B) = \omega_{WP},$$ 

where $\omega_{WP}$ is the semipositive Weil-Petersson form on $B \setminus S'$, characterizing the change of complex structures of the fibers. If on a domain $U \subset B \setminus S'$ the bundle is holomorphically trivial, then we have $\omega_{WP} \equiv 0$ on $U$.

For $t \geq 0$, let $\omega_t$ be the global reference metrics defined by

$$\omega_t = \chi + e^{-t} \omega_X \in \alpha_t = [\chi] + e^{-t} [\omega_X],$$

where $\omega_X$ is the semipositive Weil-Petersson form on $WP \equiv\star WP$. 

We have $\omega_{SRF} \to \omega_{SRF}$ uniform over $B_{t_0}$ where $t_0 = \max \{ t \mid \omega_t \text{is smooth} \}$.

Moreover, $\omega_t$ converges to $\omega_{SRF}$ in the sense of $L^p$ for $p < \infty$. hence $\omega_t$ converges to $\omega_{SRF}$ in the sense of $C^{1,\alpha}$ for $\alpha < 1$.
and let $\omega(t) = \omega_r + \sqrt{-1} \partial \bar{\partial} \varphi(t)$ be the unique Ricci-flat Kähler metric on $X$ cohomologous to $\omega_r$, with the normalization $\int_X \varphi \omega_X^{n+m} = 0$. Then $\omega(t)$ solves the Calabi-Yau equation

$$
(\omega(t))^{n+m} = c_i e^{-m} \omega_X^{n+m},
$$

where $c_i$ is the constant given by

$$
c_i = \frac{\int_X e^{nt} \omega_t^{n+m}}{\int_X \omega_X^{n+m}} = \frac{1}{\int_X \omega_X^{n+m}} \sum_{k=0}^{m} \binom{n+m}{k} e^{-(m-k)t} \int_X \omega_X^{n+m-k} \wedge \chi^k = \binom{n+m}{n} \frac{\int_X \omega_X^n \wedge \chi^n}{\int_X \omega_X^{n+m}} + O(e^{-t}),
$$

which has a positive limit as $t \to \infty$.

Now, define the reference metrics $\tilde{\omega}_t$ on the regular part $X \setminus S$ by

$$
\tilde{\omega}_t = \omega_B + e^{-t} \omega_{SRF}.
$$

In [34], Tosatti-Weinkove-Yang proved the following $C^0$ convergence theorem:

**Theorem 3.1** (Tosatti-Weinkove-Yang [34]). Let $\omega = \omega(t) \in \alpha_t$ be Ricci-flat Kähler metrics on $X$ as described above. Then the following holds: For each compact set $K \subset X \setminus S$,

$$
\|\omega(t) - \tilde{\omega}_t\|_{C^0(K, \omega_t)} \to 0, \quad \text{as } t \to \infty.
$$

In particular, if $S = \emptyset$, then the convergence is global and exponentially fast.

In [14], Hein-Tosatti obtained higher-order estimate when the smooth fibers are pairwise bi-holomorphic:

**Theorem 3.2** (Hein-Tosatti [14]). Assume that all the fibers $X_b$ ($b \in U \subset B \setminus f(S)$) are bi-holomorphic to the same Calabi-Yau manifold $Y$. Over any small coordinate ball $U$ compactly contained in $B \setminus f(S)$, use [2] to trivialize $f$ holomorphically to a product $U \times Y \to U$. Define another Ricci-flat reference Kähler forms on $U \times Y$ by $\tilde{\omega}(t) = \omega_B + e^{-t} \omega_Y$. Then for any $k \in \mathbb{N}$, there exists a constant $C_{U,k}$ such that

$$
\|\omega(t)\|_{C^k(U \times Y, \tilde{\omega}(t))} \leq C_{U,k}.
$$

holds uniformly for all $t \in [0, \infty)$.

Here $\omega_Y$ is defined as follows. Let $f : X \to B$ be as in Theorem 3.2. By [2], $f$ is a holomorphic fiber bundle over $U$. Fix any small coordinate ball in $U$ over which this holomorphic fiber bundle is trivial. We may assume that $U$ is a ball in $\mathbb{C}^m$ and $f : U \times Y \to U$ is the projection, with $Y = X_b$ a compact Calabi-Yau manifold. Then we need to apply a gauge transformation. By [12, Prop. 3.1] (cf. [8, Prop. 3.1], [10, Lemma 4.1], [11, Claim 1, p.382], [31, p.2936–2937]), there is a unique Ricci-flat Kähler metric $\omega_Y$ on $Y$ such that, we can find a bi-holomorphism $T$ of $U \times Y$ (over $U$) such that

$$
T^* \omega_Y = S^* \omega_{\mathbb{C}^m} + \omega_Y \quad \text{for some } S \in \text{GL}(m, \mathbb{C}).
$$

Note that [12, Prop. 3.1] is stated with $U = \mathbb{C}^m$, but the proof applies also if $U$ is a ball in $\mathbb{C}^m$. Let us also note that $T$ takes the form $T(z, y) = (z, y + \sigma(z))$, where $\sigma(z)$ is a smooth function.
where $\sigma$ is a holomorphic function from $U$ to the space of $g_Y$-parallel $(1, 0)$-vector fields on $Y$, and where the addition $y + \sigma(z)$ has the same meaning as in [12, (1.1)].

We should note that for each $b \in U$ the metrics $\omega_{SRFB}$ and $\omega_Y$ are in the same Kähler class and both are Ricci-flat, so they are equal by the uniqueness part of Calabi-Yau theorem, i.e., we have

$$\omega_{SRFB} = \omega_Y,$$  \hfill (3.8)
on $Y = X_b$ for all $b \in U$.

Before we prove Theorem 1.3 we need some useful lemmas.

**Lemma 3.3.** Let $(Y, \omega_Y)$ be a Kähler manifold and $B$ the unit ball in $\mathbb{C}^m$ ($m \geq 1$). Let $\omega_1$, $\omega_2$ be any two Kähler metrics on $B$, and define two families of product metrics $\hat{\omega}(t)$ and $\tilde{\omega}(t)$ for $t \in [0, \infty)$ on $X = B \times Y$ as

$$\left\{ \begin{array}{l}
\hat{\omega}(t) = \omega_1 + e^t \omega_Y, \\
\tilde{\omega}(t) = \omega_2 + e^{-t} \omega_Y,
\end{array} \right. \hfill (3.9)$$

Then

$$\|\hat{\omega}(t)\|_{C^k(X, \hat{\omega}(t))} \leq C_k.$$  \hfill (3.10)

for all $k \geq 0$, where the constant $C_k$ depends only on $\omega_1$ and $\omega_2$, but independent of $t$.

**Proof.** Denote by $f$ the projection $B \times Y \to B$, and compute under product coordinates. We prove by induction that:

$$\nabla^{k, \hat{\omega}(t)} \hat{g}(t) = f^*(\alpha_k),$$  \hfill (3.11)

where $\alpha_k = \nabla^{k, 1} g_2$ is a well-defined covariant tensor on the base space $B$. Indeed, since $\hat{\omega}(t)$ is a product metric, we have

$$\Gamma(\hat{g}(t))_{i\ell}^{k} = \begin{cases}
(g_1)^{k\ell} \nabla^E_{i}(g_1)_{p\ell}, & i, k, p \in b, \\
(g_Y)^{k\ell} \nabla^F_{i}(g_Y)_{p\ell}, & i, k, p \in f, \\
0, & \text{otherwise},
\end{cases} \hfill (3.12)$$

and similarly

$$\Gamma(\tilde{g}(t))_{i\ell}^{k} = \begin{cases}
(g_2)^{k\ell} \nabla^E_{i}(g_2)_{p\ell}, & i, k, p \in b, \\
(g_Y)^{k\ell} \nabla^F_{i}(g_Y)_{p\ell}, & i, k, p \in f, \\
0, & \text{otherwise}.
\end{cases} \hfill (3.13)$$

So $\hat{g}(t)^{k\ell} \nabla_i^{(\hat{\omega}(t))} \hat{g}(t)_{p\ell}$ is nonzero only if $i, k, p \in b$, and when $i, k, p \in b$, we have

$$\hat{g}(t)^{k\ell} \nabla_i^{(\hat{\omega}(t))} \hat{g}(t)_{p\ell} = (g_1)^{k\ell} \nabla^E_{i}(g_1)_{p\ell} - (g_2)^{k\ell} \nabla^E_{i}(g_2)_{p\ell} = f^* \left( (g_1)^{k\ell} \nabla^g_{i}(g_1)_{p\ell} \right),$$  \hfill (3.14)

which gives that

$$\nabla_i^{\hat{\omega}(t)} \hat{g}(t)_{k\ell} = f^* \left( \nabla^{g_2}_{i}(g_1)_{k\ell} \right).$$
This is also true for all directions, and hence establishes (3.11) for \( k = 1 \).

Now assume that we have (3.11) for \( 1, 2, \ldots, k - 1 \) with \( k \geq 2 \). Then we have

\[
\nabla^{k-1} \hat{g}(t) = f^*(\alpha_{k-1}),
\]

with \( \alpha_{k-1} = \nabla^{k-1, g_1}(g_1) \) which is a covariant tensor. Then \( f^*(\alpha_{k-1})_{i_1 \ldots i_{k+2}} \) is nonzero only if \( i_2, \ldots, i_{k+2} \) are all of \( b \) or \( \overline{b} \) directions. Now suppose \( i_1 \) is of the \( b \) or \( f \) directions, then we have

\[
\nabla^{\hat{g}(t)} f^*(\alpha_{k-1})_{i_1 \ldots i_{k+2}} = \nabla^{\hat{g}(t)} E_{i_1} f^*(\alpha_{k-1})_{i_2 \ldots i_{k+2}} - \sum_{2 \leq j \leq k+2, i_j = b} \Gamma(\hat{g}(t))_{i_1 i_j}^p f^*(\alpha_{k-1})_{i_2 \ldots i_{j-1} p i_{j+1} \ldots i_{k+2}}.
\]

If \( i_1 \) is of the \( f \) directions, then since \( f^*(\alpha_{k-1})_{i_1 \ldots i_{k+2}} \) is a function only depends on the base invariant, and hence \( \nabla^{\hat{g}(t)} f^*(\alpha_{k-1})_{i_1 \ldots i_{k+2}} = 0 \), and using (3.13) we have \( \Gamma(\hat{g}(t))_{i_1 i_j}^p = 0 \) since \( i_j \in b \). Hence this covariant derivative is nonzero only if \( i_1 \) is of the \( b \) directions, and the second terms is nonzero only if \( \beta \) is of the \( b \) directions. Using (3.13) again we obtain

\[
\nabla^{\hat{g}(t)} f^*(\alpha_{k-1})_{i_2 \ldots i_{k+2}} = f^*(\nabla^{\hat{g}(t)} E_{i_1} (\alpha_{k-1}))_{i_2 \ldots i_{k+2}}.
\]

This is also true for all directions. The case for \( i_1 \) of the \( \overline{b} \) or \( \overline{f} \) directions is similar. Using induction, we obtain (3.11).

Now, we have on \( X \) the estimate

\[
|\nabla^k \nabla^{\hat{g}(t)} g(t)|_{\hat{g}(t)} = |f^*(\alpha_k)|_{\hat{g}(t)} = |\alpha_k|_{\hat{g}(t)} \leq C_k,
\]

where the constant is independent the time \( t \). This completes the proof of the Lemma. \( \square \)

Next, we compare covariant derivatives of two Kähler metrics.

**Lemma 3.4.** Let \( X \) be a Kähler manifold. Let \( \hat{\omega}, \tilde{\omega} \) be any two Kähler metrics on \( X \) and \( \alpha \) be any tensor on \( X \). Then we have for any \( k \geq 1 \)

\[
\nabla^k \nabla^{\hat{g}(t)} \alpha = \sum_{j \geq 1, j_1 + \cdots + j_k = k, j_1 \geq 0} \nabla^j \nabla^{\hat{g}(t)} \beta \ast \cdots \ast \nabla^j \nabla^{\hat{g}(t)} \beta,
\]

(3.15)

where \( \beta \) means either the metric \( \hat{g} \) or the tensor \( \alpha \), and \( \ast \) denotes the tensor contraction by \( \hat{g} \).

**Proof.** We prove by induction on \( k \). We compute under any given coordinate system \( \{z_i\} \) around a given point.

First consider the \( k = 1 \) case. For example, if \( \alpha = \alpha_{k\bar{k}} \) is a two tensor, then we have

\[
\nabla_{\hat{g}}^2 \alpha_{k\bar{k}} = \nabla_{\hat{g}} \nabla_{\hat{g}} \alpha_{k\bar{k}} = (\nabla_{\hat{g}}^E \alpha_{k\bar{k}} - \nabla_{\hat{g}}^{\hat{g}} p \alpha_{p}) - (\nabla_{\hat{g}}^E \alpha_{k\bar{k}} - \nabla_{\hat{g}}^{\hat{g}} p \alpha_{p})
\]

\[
= - \nabla_{\hat{g}}^{\hat{g}} p \alpha_{p}
\]

which gives

\[
\nabla_{\hat{g}}^2 \alpha = \nabla_{\hat{g}} \nabla_{\hat{g}} + \nabla_{\hat{g}}^{\hat{g}} \alpha \ast \alpha.
\]
where $\ast$ is tensor contraction by $\tilde{g}$. It’s easy to see that the same argument holds for any tensor field $\alpha$, this proves (3.15) for $k = 1$.

Now assume that we have established (3.15) for $1, 2, \ldots, k - 1$ with $k \geq 2$. Then we have

$$\nabla^{k,\tilde{g}}\alpha = \nabla^{\tilde{g}}\left(\nabla^{k-1,\tilde{g}}\alpha\right) = \nabla^{\tilde{g}}\left(\nabla^{k-1,\tilde{g}}\alpha\right) + \nabla^{\tilde{g}}\ast \nabla^{k-1,\tilde{g}}\alpha = \left(\nabla^{\tilde{g}} + \nabla^{\tilde{g}}\ast\right)\left(\sum_{j \geq 1, i_1 + \cdots + i_j = k - 1, i_1, \ldots, i_j \geq 0} \nabla^{i_1,\tilde{g}}\beta \ast \cdots \ast \nabla^{i_j,\tilde{g}}\beta\right) = \sum_{j \geq 1, i_1 + \cdots + i_j = k, i_1, \ldots, i_j \geq 0} \nabla^{i_1,\tilde{g}}\beta \ast \cdots \ast \nabla^{i_j,\tilde{g}}\beta,$$

where $\beta$ still denotes either the metric $\tilde{g}$ or the tensor $\alpha$, and $\ast$ still denotes the tensor contraction by $\tilde{g}$. This completes the inductive step and establish this lemma. \hfill \Box

As a corollary, we can change the reference metric in Hein-Tosatti’s estimate:

**Corollary 3.5.** For all compact sets $K \subset B$ and all $k \in \mathbb{N}$, there exists a constant $C_{k,K}$ independent of $t$ such that for all $t \in [0, \infty)$ we have that

$$\|\omega(t)\|_{C^k(K \times Y, \tilde{\omega}(t))} \leq C_{k,K}. \quad (3.16)$$

**Proof.** Denote $\beta(t)$ the Kähler form $\omega(t)$ or $\check{\omega}(t)$, using Theorem 3.2 and Lemma 3.3 we have for any compact subset $K \subset B$

$$\|\beta(t)\|_{C^k(K \times Y, \check{\omega}(t))} \leq C_{k,K}.$$ 

Hence, using Lemma 3.4 with $\alpha = \omega(t)$, $\bar{\omega} = \check{\omega}(t)$ and $\check{\omega} = \check{\omega}(t)$, we have the following estimate on $K \times Y$ for $k \geq 1$

$$\left|\nabla^{k,\tilde{g}(t)}g(t)\right|_{\check{\omega}(t)} = \left|\sum_{j \geq 1, i_1 + \cdots + i_j = k, i_1, \ldots, i_j \geq 1} \nabla^{i_1,\tilde{g}(t)}\beta(t) \ast \cdots \ast \nabla^{i_j,\tilde{g}(t)}\beta(t)\right|_{\check{\omega}(t)} \leq C \cdot \sum_{j \geq 1, i_1 + \cdots + i_j = k, i_1, \ldots, i_j \geq 1} \left|\nabla^{i_1,\tilde{g}(t)}\beta(t)\right|_{\check{\omega}(t)} \cdots \left|\nabla^{i_j,\tilde{g}(t)}\beta(t)\right|_{\check{\omega}(t)} \leq C_{k,K}.$$ 

where $\ast$ denotes tensor contraction by $\tilde{g}(t)$. Here we have used the uniformly equivalent relations between $\omega(t)$, $\check{\omega}(t)$ and $\check{\omega}(t)$, and hence completes the proof of Corollary 3.5. \hfill \Box

**Remark 3.6.** Suppose we have two uniformly equivalent families of Kähler metrics $\omega(t)$ and $\check{\omega}(t)$, it doesn’t matter which metric we use to measure the norm. Also, assume we have any quantity of the...
form

\[ A_1 \ast A_2 \ast \cdots \ast A_k, \]

where each \( A_i \) is a tensor, and \( \ast \) is the tensor contraction given by \( \omega(t) \) or \( \tilde{\omega}(t) \), then by the uniformly equivalent relations between \( \omega(t) \) and \( \tilde{\omega}(t) \), we have

\[ |A_1 \ast A_2 \ast \cdots \ast A_k| \leq C \cdot |A_1|_{\omega(t)} \cdots \cdot |A_k|_{\tilde{\omega}(t)}. \]

for some uniform constant \( C \) independent of \( t \), since here we have only finitely many contractions (depending only on \( k \) and the degrees of the \( A_i \)'s). The case for three or more uniformly equivalent metrics is similar. We will use such principle to take norms throughout this paper.

The following lemma is a standard result in Kähler geometry, and follows easily by direct computations. So we omit the proof.

**Lemma 3.7.** Given \( X \) be a Kähler manifold, and \( \omega, \tilde{\omega} \) be any two Kähler forms on \( X \). Define the tensor \( \Psi \) on \( X \) by

\[ \Psi_{ip} := \Gamma(g)_{ip} - \Gamma(\tilde{g})_{ip} = g^{kl} \nabla_i \tilde{g}^k \nabla_p \tilde{g}. \]

Then we have

\[ R(\omega)_{ijkl} = \tilde{g}^{st} g_{kl} R(\tilde{\omega})_{isjt} - \nabla_i \tilde{g}^{st} g_{kl} \nabla_j \tilde{g} + g_{st} \Psi_{ik} \nabla_{jl} \tilde{\Psi}. \]  \hspace{1cm} (3.17)

In particular, we have

\[ R^s(\omega)_{ijkl} = R^s(\tilde{\omega})_{ijkl} - g^{tu} \nabla_i \tilde{g}^{st} g_{kl} \nabla_j \tilde{g} + g^{ts} g_{st} \Psi^i_{jk} \nabla_{lj} \tilde{\Psi}. \]  \hspace{1cm} (3.18)

This lemma shows that we can express the difference of the Rm curvature tensor of two Kähler metrics as the tensor contraction of the first covariant derivatives and second covariant derivatives.

Now we can prove Theorem 1.3.

**Proof of Theorem 1.3.** We still just need to verify the Conditions (A) – (C) of Lemma 1.1 with

\[ \eta(t) = \omega(t) - \tilde{\omega}(t). \]

We have already trivialize \( f \) holomorphically to a product \( U \times Y \rightarrow U \). Let \( K \subset \subset U \) be any compact subset.

Condition (C) follows from Lemma 2.1.

Condition (B): Replacing \( U \) by a slightly smaller subset. With Theorem 3.2 at hand, Corollary 3.3 implies the estimate

\[ \|\omega(t)\|_{C^k(U \times Y, \tilde{\omega}(t))} \leq C_{U,k} \]

for any \( k \geq 0 \).

Condition (A): Replacing \( U \) by a smaller subset, we only need to verify the condition

\[ \|\omega(t) - \tilde{\omega}(t)\|_{C^0(U \times Y, \tilde{\omega}(t))} \leq h_0(t). \]
From the result of Tosatti-Weinkove-Yang, say Equation (3.6) of Theorem 3.1, we have

\[ \|\omega(t) - \bar{\omega}_t\|^2_{C^0(U \times Y, \bar{\omega}_t)} \leq h(t). \]

where \( \bar{\omega}_t = \omega_B + e^{-t}\omega_{\text{SRF}} \). Choosing product coordinates, say \( \{z_\alpha, 1 \leq \alpha \leq m; y_i, 1 \leq i \leq n\} \) with \( z_\alpha \) being base coordinates and \( y_i \) being fiber coordinates. Then the above estimate implies on \( U \times Y \)

\[
\begin{align*}
&\left|g(t)_{\alpha\beta} - (g_B)_{\alpha\beta}\right|^2 \leq h(t), \quad \alpha, \beta \in b, \\
dotprod{e'}{\left|g(t)_{ij} - (g_{\text{SRF}})_{ij}\right|^2} &\leq h(t), \quad \alpha \in b, \quad j \in f, \\
dotprod{e^2}{\left|g(t)_{ij} - (g_{\text{SRF}})_{ij}\right|^2} &\leq h(t), \quad i, j \in f.
\end{align*}
\]

The first inequality of (3.19) implies that

\[ \left|g(t)_{\alpha\beta} - (g_B)_{\alpha\beta}\right|^2 \leq 2h(t) + 2\left|e^{-t}(g_B)_{\alpha\beta} - e^{-t}(g_{\text{SRF}})_{\alpha\beta}\right|^2 \leq h(t), \quad \alpha, \beta \in b, \]

and the second inequality of (3.19) implies

\[ \cdotdotprod{e'}{\left|g(t)_{ij} - (g_{\text{SRF}})_{ij}\right|^2} \leq 2h(t) + \cdotdotprod{e^{-t}}{(g_{\text{SRF}})_{ij}}^2 \leq h(t), \quad \alpha \in b, \quad j \in f. \]

Since \( \omega_{\text{SRF}} = \omega_Y \) for any \( b \in U \), we have \( (g_{\text{SRF}})_{ij} = (g_Y)_{ij} \) with \( i, j \in f \) under the product coordinates, hence the third inequality of (3.19) implies

\[ \cdotdotprod{e^2}{\left|g(t)_{ij} - (g_Y)_{ij}\right|^2} \leq h(t), \quad i, j \in f. \]

So we conclude that on \( U \times Y \) under product coordinates

\[
\begin{align*}
&\left|g(t)_{\alpha\beta} - (g_B)_{\alpha\beta}\right|^2 \leq h(t), \quad \alpha, \beta \in b, \\
\cdotdotprod{e'}{\left|g(t)_{ij}\right|^2} &\leq h(t), \quad \alpha \in b, \quad j \in f, \\
\cdotdotprod{e^2}{\left|g(t)_{ij} - (g_Y)_{ij}\right|^2} &\leq h(t), \quad i, j \in f.
\end{align*}
\]

This implies that

\[ \|\omega(t) - \bar{\omega}(t)\|^2_{C^0(U \times Y, \bar{\omega}(t))} \leq h(t). \]

This verifies Condition (A) since we already have local uniform equivalence between \( \omega(t) \) and \( \bar{\omega}(t) \).

Now applying Lemma 1.1, we get the desired estimate (1.6).

It remains to prove curvature convergence estimates (1.7). Applying Lemma 3.7 with \( \omega = \omega(t) \) and \( \bar{\omega} = \bar{\omega}(t) \), and define the tensor

\[ \Psi(t)_{jlp}^k := \Gamma(g(t))_{jlp}^k - \Gamma(g_{\text{SRF}})_{jlp}^k = g(t)^{jk} \nabla_{l} g_{i}^{(t)} g(t)_{pl}, \]

we get

\[
R^i(\omega(t))_{jlp}^k - R^i(\bar{\omega}(t))_{jlp}^k = g(t)^{jk} \nabla_{l} \nabla_{j} g_{i}^{(t)} g(t)_{k\nu} + g(t)^{jk} \nabla_{l} g(t)_{i}^{\nu} \Psi(t)_{jlp}^k \Psi(t)_{jnu} \\
= g(t)^{jk} \nabla_{l} \nabla_{j} g_{i}^{(t)} g(t)_{k\nu} + g(t)^{jk} g(t)_{i}^{\nu} \nabla_{l} g(t)_{k\nu} \nabla_{j} g_{i}^{(t)} g(t)_{p\theta}. \]

Then by induction we can show that for all \( k \geq 0 \)
\[
\nabla^{k,\tilde{g}(t)} \left[ R^\tilde{g}(\omega(t)) - R^\tilde{g}(\tilde{\omega}(t)) \right] = \sum_{j_1, \ldots, j_k \geq 1} \nabla^{j_1,\tilde{g}(t)} g(t) \ast \cdots \ast \nabla^{j_k,\tilde{g}(t)} g(t). \tag{3.22}
\]
where * is the tensor contraction given by \( g(t) \). In fact, \( k = 0 \) case already follows from \((3.18)\). Suppose we have \((3.22)\) for \( 0, \ldots, k-1 \) for \( k \geq 1 \). Then for \( k \) we have
\[
\nabla^{k,\tilde{g}(t)} \left[ R^\tilde{g}(\omega(t)) - R^\tilde{g}(\tilde{\omega}(t)) \right] = \nabla^{\tilde{g}(t)} \left\{ \sum_{j_1, \ldots, j_k \geq 1} \nabla^{j_1,\tilde{g}(t)} g(t) \ast \cdots \ast \nabla^{j_k,\tilde{g}(t)} g(t) \right\}
= \sum_{j_1, \ldots, j_k \geq 1} \nabla^{j_1,\tilde{g}(t)} g(t) \ast \cdots \ast \nabla^{j_k,\tilde{g}(t)} g(t).
\]
This proves \((3.22)\) for \( k \).

Now, taking norms with respect to \( \tilde{g}(t) \) and using the equivalence of \( \omega(t) \) and \( \tilde{\omega}(t) \) we have on \( K \times Y \)
\[
\left\| \nabla^{k,\tilde{g}(t)} \left[ R^\tilde{g}(\omega(t)) - R^\tilde{g}(\tilde{\omega}(t)) \right] \right\|_{\tilde{\omega}(t)} \leq C_k \cdot \sum_{j_1, \ldots, j_k \geq 1} \left\| \nabla^{j_1,\tilde{g}(t)} g(t) \right\|_{\tilde{\omega}(t)} \cdots \left\| \nabla^{j_k,\tilde{g}(t)} g(t) \right\|_{\tilde{\omega}(t)} \leq h_{K,k}(t).
\]
So we get \((1.7)\), and this completes the proof.

\[\square\]

3.2. Metric and curvature convergence on torus-fibered Calabi-Yau manifolds. Now we consider the case when the smooth fibers \( X_b \) are finite quotients of complex tori, but we allow the complex structure to change. We denote the semi-Ricci flat form \( \omega_{\text{SRF}} \) by \( \omega_{\text{SF}} \) now, since in this case, its restriction to each smooth fiber is actually flat. We have the following higher-order estimates.

**Lemma 3.8** ([8] [13] [31]). For all compact sets \( K \subset X \setminus S \) and all \( k \geq 0 \), there exists constants \( C_{K,k} \) independent of \( t \) such that for all \( t \in [0, \infty) \) we have that
\[
\|\omega(t)\|_{C^k(K,\tilde{\omega}_t)} \leq C_{K,k}. \tag{3.23}
\]
and the curvature bound
\[
\|\text{Rm}(\omega(t))\|_{C^k(K,\tilde{\omega}_t)} \leq C_{K,k}. \tag{3.24}
\]

Now, we can prove Theorem 1.4

**Proof of Theorem 1.4** For all compact sets \( K \subset X \setminus S \), we choose disks \( B_1 \subset B_2 \subset B \setminus S' \) such that \( K \subset f^{-1}(B_1) \). Set \( U = f^{-1}(B_2) \). Then, as the proof of Theorem [1.3] with the help of Theorem [3.1] and Lemma [3.8] we can similarly verify the Conditions (A) – (C) of Lemma [1.1] with
\[
\eta(t) = \omega(t) - \tilde{\omega}_t.
\]
So we immediately get the higher-order convergence
\[
\|\omega(t) - \tilde{\omega}_t\|_{C^k(K,\tilde{\omega}_t)} \leq h_{K,k}(t),
\]
where $h_{K,k}(t)$ are positive functions which tends to zero as $t \to \infty$, depending only on $k$ and the domain $K$. It remains to show that

$$\|\text{Rm}(\omega(t)) - \text{Rm}(\tilde{\omega}_t)\|_{C^k(K,\tilde{\omega}_t)} \leq h_{K,k}(t).$$

Applying Lemma[3.7] with $\omega = \omega(t)$ and $\tilde{\omega} = \tilde{\omega}_t$, we get

$$R(\omega(t))_{ijkl} = (\tilde{g}_i)^{\tilde{s}}(t)_{k\tilde{l}} R(\tilde{\omega}_t)_{ij\tilde{s}} - \nabla_i^\tilde{R} \nabla_j^\tilde{g}(t)_{k\tilde{l}} + g(t)_{u\tilde{v}} ^\tilde{\psi}(t)^{u}_{ik} \nabla^\tilde{v}_{jl}. \quad (3.25)$$

which gives

$$R(\tilde{\omega}_t)_{ijkl} = g(t)_{v^\tilde{s}(\tilde{g}_i)_{k\tilde{l}}} R(\omega(t))_{ij\tilde{s}} + g(t)_{v^\tilde{s}(\tilde{g}_i)_{k\tilde{l}}} \nabla_i^\tilde{g}(t)_{j\tilde{s}} - g(t)^{v}_{l} (\tilde{g}_i)_{k\tilde{l}} g(t)_{u\tilde{v}} ^\tilde{\psi}(t)^{u}_{ik} \nabla^\tilde{v}_{jl}. \quad (3.26)$$

This is equivalent to

$$R(\tilde{\omega}_t)_{ijkl} = g(t)^{v}_{l} (\tilde{g}_i)_{k\tilde{l}} R(\omega(t))_{ij\tilde{s}} + g(t)_{v^\tilde{s}(\tilde{g}_i)_{k\tilde{l}}} \nabla_i^\tilde{g}(t)_{j\tilde{s}} - g(t)^{v}_{l} (\tilde{g}_i)_{k\tilde{l}} g(t)_{u\tilde{v}} ^\tilde{\psi}(t)^{u}_{ik} \nabla^\tilde{v}_{jl}. \quad (3.27)$$

Hence we have

$$R(\tilde{\omega}_t)_{ijkl} - R(\omega(t))_{ijkl} =g(t)^{v}_{l} (\tilde{g}_i)_{k\tilde{l}} - g(t)^{v}_{l} (\tilde{g}_i)_{k\tilde{l}} R(\omega(t))_{ij\tilde{s}} + g(t)_{v^\tilde{s}(\tilde{g}_i)_{k\tilde{l}}} \nabla_i^\tilde{g}(t)_{j\tilde{s}} - g(t)^{v}_{l} (\tilde{g}_i)_{k\tilde{l}} g(t)_{u\tilde{v}} ^\tilde{\psi}(t)^{u}_{ik} \nabla^\tilde{v}_{jl}. \quad (3.28)$$

Taking norms with respect to $\tilde{\omega}_t$ and applying (1.8) and Lemma[3.8], we have that on $U$

$$\|\text{Rm}(\tilde{\omega}_t) - \text{Rm}(\omega(t))\|_{\tilde{\omega}_t} \leq C \cdot \|\tilde{\omega}_t - \omega(t)\|_{\tilde{\omega}_t} \cdot \|\text{Rm}(\omega(t))\|_{\tilde{\omega}_t} + C \cdot \|\nabla^2 \tilde{g} ; g(t)\|_{\tilde{\omega}_t} + C \cdot \|\nabla^2 \tilde{g} ; g(t)\|_{\tilde{\omega}_t},$$

$$\leq C \cdot h_K(t) \cdot C_K + C \cdot h_K(t) + C \cdot h_K(t)^2$$

$$\leq h_K(t).$$

Also from (3.27), by induction we can easily obtain that for all $k \geq 0$

$$\nabla^{k,\tilde{g}_i} \text{Rm}(\tilde{\omega}_t) = \sum_{j_1 \geq 1; j_1 + \cdots + j_1 = k, l_1, \ldots, j_1 \geq 0} \nabla^{l_1,\tilde{g}_i} g(t) \ast \cdots \ast \nabla^{l_{k-1},\tilde{g}_i} g(t) \ast \nabla^{l_{k},\tilde{g}_i} \text{Rm}(\omega(t)), \quad (3.29)$$

where $\ast$ denotes tensor contraction by $g(t)$ or $\tilde{g}_i$. This implies that on $U$

$$\|\nabla^{k,\tilde{g}_i} \text{Rm}(\tilde{\omega}_t)\|_{\tilde{\omega}_t} \leq C \cdot \sum_{j_1 \geq 1; j_1 + \cdots + j_1 = k, l_1, \ldots, j_1 \geq 0} \|\nabla^{l_1,\tilde{g}_i} g(t)\|_{\tilde{\omega}_t} \cdots \|\nabla^{l_{k-1},\tilde{g}_i} g(t)\|_{\omega(t)} \cdot \|\nabla^{l_{k},\tilde{g}_i} \text{Rm}(\omega(t))\|_{\omega(t)} \leq C_K K_k.$$

Hence if we set

$$\tilde{\eta}(t) = \text{Rm}(\tilde{\omega}_t) - \text{Rm}(\omega(t)),$$
then we have
\[
\begin{align*}
\|\tilde{\eta}(t)\|_{C^0(U,\tilde{\omega}_t)} &= \|Rm(\tilde{\omega}_t) - Rm(\omega(t))\|_{C^0(U,\tilde{\omega}_t)} \leq h_K(t), \\
\|\tilde{g}(t)\|_{C^0(U,\tilde{\omega}_t)} &= \|Rm(\tilde{\omega}_t) - Rm(\omega(t))\|_{C^0(U,\tilde{\omega}_t)} \leq C_{K,k},
\end{align*}
\] (3.30)

Hence Conditions (A) and (B) of Lemma 1.1 are satisfied for \(\eta(t) = \tilde{\eta}(t)\) and \(\tilde{g}(t) = \tilde{g}_t\). Hence from Lemma 1.1 we conclude the local convergence
\[\|Rm(\omega(t)) - Rm(\omega_t)\|_{C^0(K,\tilde{\omega}_t)} \leq h_{K,k}(t).\]

This establish (1.9). \( \Box \)

4. Applications to normalized Kähler-Ricci flow on torus-fibered minimal models

Let \((X^{m+n},\omega_0)\) be a compact Kähler manifold with semi-ample canonical bundle \(K_X\) and assume its Kodaira dimension to be \(0 < m := \text{Kod}(X) < m + n\). Then the pluricanonical system \(|\ell K_X|\) for sufficiently large \(\ell \in \mathbb{Z}^+\) induces the so called “Iitaka fibration” map
\[f : X \to B \subset \mathbb{CP}^N := \mathbb{P}H^0(X,K_X^{\otimes \ell}),\] (4.1)

where \(B\) is the canonical model of \(X\) with \(\text{dim} B = m\). Let \(S'\) be the singular set of \(B\) together with the set of critical values of \(f\), and we define \(S = f^{-1}(S') \subset X\).

From (4.1), we have \(f^*O(1) = K_X^{\otimes \ell}\), hence if we let \(\chi = \frac{1}{\ell}\omega_{\text{FS}}\) on \(\mathbb{P}H^0(X,K_X^{\otimes \ell})\), we have that \(f^*\chi\) (denoted by \(\chi\) afterwards) is a smooth semi-positive representative of \(\omega\). Here, \(\omega_{\text{FS}}\) denotes the Fubini-Study metric.

Given the Kähler metric \(\omega_0\) on \(X\), since \(X_b := f^{-1}(b)\) is a Calabi-Yau manifold for each \(b \in B\setminus S'\), there exists a unique smooth function \(\rho_b\) on \(X_b\) with \(\int_{X_b} \rho_b^m \omega_0^n = 0\), such that \(\omega_0|_{X_b} + \sqrt{-1} \partial \bar{\partial} \rho_b := \omega_b\) is the unique Ricci-flat Kähler metric on \(X_b\) in the class \([\omega_0|_{X_b}]\). Moreover, \(\rho_b\) depends smoothly on \(b\), and so define a global smooth function on \(X\setminus S\). We define
\[\omega_{\text{SRF}} = \omega_0 + \sqrt{-1} \partial \bar{\partial} \rho,\]
which is a closed real \((1,1)\)-form on \(X\setminus S\), called the “semi-Ricci flat metric”.

Let \(\Omega\) be the smooth volume form on \(X\) with
\[\sqrt{-1} \partial \bar{\partial} \log \Omega = \chi, \quad \int_X \Omega = \binom{m+n}{m} \int_X \omega_0^n \wedge \chi^m.\] (4.2)

Define a function \(F\) on \(X\setminus S\) by
\[F := \frac{\Omega}{\binom{m+n}{m} \chi^m \wedge \omega_{\text{SRF}}^n},\] (4.3)
then \(F\) is constant along the fiber \(X_b\), \(b \in B \setminus S'\), so it descends to a smooth function on \(B \setminus S'\). Then [20] showed that the Monge-Ampère equation
\[(\chi + \sqrt{-1} \partial \bar{\partial} \nu)^m = Fe^n \chi^m,\] (4.4)
has a unique solution $v \in \text{PSH}(\chi) \cap C^0(B) \cap C^\infty(B \setminus S')$. Define
\[ \omega_B = \chi + \sqrt{-1} \partial \bar{\partial} v, \]
which is a smooth Kähler metric on $B \setminus S'$, satisfying the twisted Kähler-Einstein equation
\[ \text{Ric}(\omega_B) = -\omega_B + \omega_{\text{WP}}, \]
where $\omega_{\text{WP}}$ is the smooth Weil-Petersson form on $B \setminus S'$.

Now let $\omega = \omega(t)$ be the solution of the normalized Kähler-Ricci flow
\[ \frac{\partial}{\partial t} \omega = -\text{Ric}(\omega) - \omega, \quad \omega(0) = \omega_0, \tag{4.5} \]
whose solution exists for all time. Define the global reference metrics
\[ \hat{\omega}(t) = e^{-t} \omega_0 + (1 - e^{-t}) \chi, \]
then it is Kähler for all $t \geq 0$, and we can write $\omega(t) = \hat{\omega}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t)$. Then the Kähler-Ricci flow \eqref{4.5} is equivalent to the parabolic Monge-Ampère equation
\[ \frac{\partial}{\partial t} \varphi = \log \frac{e^{[n-m]t} \left( \hat{\omega}(t) + \sqrt{-1} \partial \bar{\partial} \varphi(t) \right)^n}{\Omega} - \varphi, \quad \varphi(0) = 0. \tag{4.6} \]

We denote by $T_0(t) = \text{tr}_{\hat{\omega}(t)} \omega_B$ and $u = \varphi + \varphi - v$ on $X \setminus S$. Define on $X \setminus S$ the reference metrics
\[ \hat{\omega}(t) = e^{-t} \omega_{\text{SRF}} + (1 - e^{-t}) \omega_B. \]

We always set $K = f^{-1}(K')$ where $K' \subset B \setminus S'$ is a compact subset. Then we can choose some open subset $U' \subset B \setminus S'$ such that $K' \subset U'$. Set $U = f^{-1}(U')$, then $K \subset U \subset X \setminus S$. First, we have the following lemma in the general fibration case. See \cite{20, 21, 4, 34, 30, 15}.

**Lemma 4.1.** There exist some constant $C = C(K) > 0$ and positive functions $h(t)$ which tends to zero as $t \to \infty$, depending on the domain $K$, such that

1. $C^{-1} \hat{\omega}(t) \leq \omega(t) \leq C \hat{\omega}(t)$, on $K \times [0, \infty)$.
2. $|\varphi - v| + |\dot{\varphi} + \varphi - v| \leq h(t)$, on $K \times [0, \infty)$.
3. $||\omega(t) - \hat{\omega}(t)||_{C^0(K, \omega(t))} \leq h(t)$.
4. $|T_0 - m| + ||\omega_B||_{\hat{\omega}(t)}^2 - m| \leq h(t)$, on $K \times [0, \infty)$.
5. There exists a uniform constant $C_0 > 0$ such that
   \[ |R| \leq C_0, \quad \text{on} \quad X \times [0, \infty). \]
6. Along the normalized Kähler-Ricci flow \eqref{4.5}, we have on $X \setminus S$
   \[ (\bar{\partial}_t - \Delta) u = \text{tr}_{\hat{\omega}(t)} \omega_B - m. \tag{4.7} \]
(7) We have
\[ |\nabla u|^2 \leq h(t), \text{ on } K \times [0, \infty). \] (4.8)

(8) \[ |R(t) + m| \leq h(t), \text{ on } K \times [0, \infty). \]

Especially, if \( S = \emptyset \), then all of the above estimates hold with \( K \) replaced by \( X \) and \( h(t) \) replaced by \( Ce^{-\eta t} \) for some constants \( \eta, C > 0 \) depending on \((X, \omega_0)\).

From now on, assume the smooth fibers are the quotients of complex tori by holomorphic free action of a finite group. In this case, the semi-Ricci flat metric \( \omega_{SRF} \) we constructed above is actually flat when restricted to any smooth fiber \( X_b, b \in B \setminus S' \), and we denote \( \omega_{SRF} \) by \( \omega_{SF} \) to indicate such semi-flat property. We have the following estimates.

**Lemma 4.2 ([4, 5, 8, 13, 31]).** For all compact sets \( K \subset X \setminus S \) and all \( k \geq 0 \), there exists constants \( C_{K,k} \) independent of \( t \) such that for all \( t \in [0, \infty) \) we have the higher-order derivatives bound
\[ \|\omega(t)\|_{C^k(K,\tilde{\omega}(t))} \leq C_{K,k}. \] (4.9)
and the curvatures bound
\[ \|Rm(\omega(t))\|_{C^k(K,\tilde{\omega}(t))} \leq C_{K,k}. \] (4.10)
\[ \|Ric(\omega(t))\|_{C^k(K,\tilde{\omega}(t))} \leq C_{K,k}. \] (4.11)
\[ \|R(\omega(t))\|_{C^k(K,\tilde{\omega}(t))} \leq C_{K,k}. \] (4.12)

Now we can prove Theorem 1.5.

**Proof of Theorem 1.5.** For all compact sets \( K \subset X \setminus S \), we choose disks \( B_1 \subset \subset B_2 \subset \subset B \setminus S' \) such that \( K \subset f^{-1}(B_1) \). Set \( U = f^{-1}(B_2) \). Then, as the proof of Theorem 1.4, with the help of Lemma 4.1 and Lemma 4.2, we can similarly verify the Conditions (A) – (C) of Lemma 1.1 with \( \eta(t) = \omega(t) - \tilde{\omega}(t) \).

Hence from Lemma 1.1 we have the convergence estimate
\[ \|\omega(t) - \tilde{\omega}(t)\|_{C^k(K,\tilde{\omega}(t))} \leq h_{K,k}(t). \]

Also, with Lemma 3.8 being replaced by Lemma 4.1, the same argument as in the proof of Theorem 1.4 gives the convergence estimates
\[ \|Rm(\omega(t)) - Rm(\tilde{\omega}(t))\|_{C^k(K,\tilde{\omega}(t))} \leq h_{K,k}(t). \] (4.13)

Next we consider the Ricci curvature. First from (4.13) and Lemma 4.2, we have
\[ \|Rm(\tilde{\omega}(t))\|_{C^k(U,\tilde{\omega}(t))} \leq C_{K,k}. \] (4.14)
and hence naturally
\[ \|\text{Ric}(\bar{w}(t))\|_{C^k(U, \bar{w}(t))} \leq C_{K,k}. \]  
(4.15)
Combining (4.15) and Lemma 4.2 we obtain
\[ \|\text{Ric}(\omega(t))\|_{C^k(U, \bar{w}(t))} + \|\text{Ric}(\bar{w}(t))\|_{C^k(U, \bar{w}(t))} \leq C_{K,k}. \]  
(4.16)
By definition, we have
\[ R(\omega(t))_{ij} - R(\bar{w}(t))_{ij} = g(t)^{kl} R(\omega(t))_{ijkl} - \bar{g}(t)^{kl} R(\bar{w}(t))_{ijkl} \]
(4.17)
Taking norms with respect to \( \bar{w}(t) \) gives that on \( U \)
\[ |\text{Ric}(\omega(t)) - \text{Ric}(\bar{w}(t))|_{\bar{w}(t)} \leq C \cdot |\bar{w}(t)^{-1} - \omega(t)^{-1}|_{\bar{w}(t)} \cdot |\text{Rm}(\omega(t))|_{\bar{w}(t)} + C \cdot |\text{Rm}(\omega(t)) - \text{Rm}(\bar{w}(t))|_{\bar{w}(t)} \]
\[ \leq C \cdot h_K(t) \cdot C_K + C \cdot h_K(t) \]
\[ \leq h_K(t). \]
Hence the Conditions (A) – (C) of Lemma 1.1 with
\[ \eta(t) = \text{Ric}(\omega(t)) - \text{Ric}(\bar{w}(t)) \]
are satisfied, and we conclude from Lemma 1.1 the convergence estimates
\[ \|\text{Ric}(\omega(t)) - \text{Ric}(\bar{w}(t))\|_{C^k(K, \bar{w}(t))} \leq h_K(t). \]  
(4.18)
Finally, for the scalar curvature, from Equation (4.16) and Lemma 4.2 we naturally have
\[ \|R(\omega(t))\|_{C^k(U)} + \|R(\bar{w}(t))\|_{C^k(U)} \leq C_{K,k}. \]  
(4.19)
By definition we have
\[ R(\omega(t)) - R(\bar{w}(t)) = g(t)^{kl} R(\omega(t))_{kl} - \bar{g}(t)^{kl} R(\bar{w}(t))_{kl} \]
(4.20)
As before, we have on \( U \)
\[ |R(\omega(t)) - R(\bar{w}(t))| \leq C \cdot |\bar{w}(t)^{-1} - \omega(t)^{-1}|_{\bar{w}(t)} \cdot |\text{Ric}(\omega(t))|_{\bar{w}(t)} + C \cdot |\text{Ric}(\omega(t)) - \text{Ric}(\bar{w}(t))|_{\bar{w}(t)} \]
\[ \leq C \cdot h_K(t) \cdot C_K + C \cdot h_K(t) \leq h_K(t). \]
Hence the Conditions (A) – (C) of Lemma 1 with
\[ \eta(t) = R(\omega(t)) - R(\tilde{\omega}(t)) \]
are satisfied, and we conclude from Lemma 1 that
\[ \| R(\omega(t)) - R(\tilde{\omega}(t)) \|_{C^k(K, \tilde{\omega}(t))} \leq h_{K,k}(t). \]

This completes the proof of Theorem 1.5.

Finally, we prove Theorem 1.6. First, we need the following Proposition.

**Proposition 4.3.** For all compact sets \( U \subset X \setminus S \) and all \( k \geq 0 \), there exists constants \( C_{U,k} \) independent of \( t \) such that for all \( t \in [0, \infty) \) we have that
\[ \| \omega_B \|_{C^k(U, \tilde{\omega}(t))} \leq C_{U,k}. \] (4.21)

**Proof.** We adopt the notations of [30, Theorem 5.24, p363-368]. We just need to consider the case when \( X_b \) is in fact bi-holomorphic to a torus for some \( b \in B \setminus S' \). Then we have
\[ \lambda^*_i p^* f^* \omega_B = p^* f^* \omega_B, \]
and
\[ \lambda^*_i p^* \omega_{SF} = \lambda^*_i \sqrt{-1} \partial \bar{\partial} \eta = \sqrt{-1} \partial \bar{\partial} (\eta \circ \lambda_i) = e^t \sqrt{-1} \partial \bar{\partial} \eta = e^t p^* \omega_{SF}. \]

Hence we have
\[ \lambda^*_i p^* (\tilde{\omega}(t)) = \lambda^*_i p^* ((1 - e^{-t}) f^* \omega_B + e^{-t} \omega_{SF}) = (1 - e^{-t}) p^* f^* \omega_B + p^* \omega_{SF} =: \tilde{\omega}(t). \]

Now we may assume that \( t \geq 1 \) so that the factor \((1 - e^{-t}) \in (\frac{1}{2}, 1)\) would be a harmless factor. Then we have for every compact set \( K \subset B' \times \mathbb{C}^{n-m} \) (here \( B' \) is the \( B \) in the notations of [30, Theorem 5.24, p363-368], which is a small ball in the regular part of the base space \( B \) here) there are constants \( C_{K,k} \) such that
\[ \| p^* f^* \omega_B \|_{C^k(K, \tilde{\omega}(t))} \leq C_{K,k}, \]
for all \( t \geq 1 \). We can rewrite this as
\[ \| \lambda^*_i p^* f^* \omega_B \|_{C^k(K, \lambda^*_i p^*(\tilde{\omega}(t)))} \leq C_{K,k}. \]

Given any open subset \( U \subset X \setminus S \), if \( K' \subset U \subset X \setminus S \) is a compact set which is small enough so that \( K = p^{-1}(K') \subset B' \times \mathbb{C}^{n-m} \) is compact and \( p \) is a bi-holomorphism on \( K \), (note that such compact sets \( K' \) cover \( U \)) then we have
\[ \| f^* \omega_B \|_{C^k(K', \tilde{\omega}(t))} = \| p^* f^* \omega_B \|_{C^k(K, p^* f^*(\tilde{\omega}(t)))} = \| \lambda^*_i p^* f^* \omega_B \|_{C^k(\lambda^*_i(K), \lambda^*_ip^*(\tilde{\omega}(t)))}, \]
where $\lambda_{1/t}$ is the inverse map of $\lambda_t$. But the compact sets $\lambda_{1/t}(K) \ (t \geq 1)$ are all contained in a fixed compact set of $B' \times \mathbb{C}^{n-m}$, hence we have

$$
\|f^*\omega_B\|_{C^1(K',\tilde{\omega}(t))} = \|\lambda_{t}^* p^* f^* \omega_B\|_{C^1(\lambda_{1/t}(K),\lambda_{1/t}^* p^* (\tilde{\omega}(t)))} \leq C_{K',k},
$$

and so by a covering argument we easily obtain

$$
\|\omega_B\|_{C^1(U,\tilde{\omega}(t))} \leq C_{U,k}.
$$

This finishes the proof of Proposition 4.3.

Remark 4.4. The same conclusion of Proposition 4.3 holds with $\omega_B$ being replaced by any other fixed Kähler metric on the regular part of the base space $B$.

Now we can prove Theorem 1.6.

Proof of Theorem 1.6. Along the normalized Kähler-Ricci flow (4.5), we have on $X \setminus S \times [0, \infty)$

$$
\text{Ric}(\omega(t)) = -\sqrt{-1} \partial \bar{\partial} (\phi + \varphi) - \chi
$$

$$
= -\sqrt{-1} \partial \bar{\partial} (\phi + \varphi - v) - (\chi + \sqrt{-1} \partial \bar{\partial} v) \tag{4.22}$$

$$
= -\sqrt{-1} \partial \bar{\partial} u - \omega_B.
$$

We define $\eta(t)$ to be the $(1, 1)$ form

$$
\eta(t) = \text{Ric}(\omega(t)) + \omega_B = -\sqrt{-1} \partial \bar{\partial} u.
$$

Combining Lemma 4.2 with Proposition 4.3 we have

$$
\|\eta(t)\|_{C^1(K,\tilde{\omega}(t))} \leq \|\text{Ric}(\omega(t))\|_{C^1(K,\tilde{\omega}(t))} + \|\omega_B\|_{C^1(K,\tilde{\omega}(t))} \leq C_{K,k}, \tag{4.23}
$$

for any compact subset $K \subset X \setminus S$.

We need to prove that

$$
\|\eta(t)\|_{C^0(K,\tilde{\omega}(t))} \leq h_K(t), \tag{4.24}
$$

for any compact subset $K \subset X \setminus S$. To this end, we choose disks $B_1 \subset B_2 \subset B \setminus S'$ such that $K \subset f^{-1}(B_1)$. Set $U = f^{-1}(B_2)$. According to Lemma 4.1 we have the estimate

$$
|\nabla u|_{\tilde{\omega}(t)}^2 \leq h_K(t)
$$
Hence we conclude that

\[
(-\Delta_{\tilde{g}(t)}) \left( |\nabla u|_{\tilde{g}(t)}^2 \right)
\]

\[
= -2 \left| \nabla^2 \tilde{g}(t) u \right|_{\tilde{g}(t)}^2 - \tilde{g}(t)^{ab} \tilde{g}(t)^{ij} \cdot \nabla_{a} \nabla_{b} \nabla_{i} \nabla_{j} u \nabla_{k} \nabla_{l} u - \tilde{g}(t)^{ab} \tilde{g}(t)^{ij} \cdot \nabla_{i} \nabla_{j} u \nabla_{a} \nabla_{b} \nabla_{k} \nabla_{l} u
\]

\[
= -2 \left| \nabla^2 \tilde{g}(t) u \right|_{\tilde{g}(t)}^2 - \tilde{g}(t)^{ab} \tilde{g}(t)^{ij} \cdot \nabla_{a} \nabla_{b} \nabla_{i} \nabla_{j} u \\
- \tilde{g}(t)^{ab} \tilde{g}(t)^{ij} \cdot \nabla_{i} \nabla_{j} u \left\{ \nabla_{a} \nabla_{b} \nabla_{i} \nabla_{j} u + \text{Rm}(\tilde{\omega}(t)) \delta_{jab} \nabla_{k} \nabla_{l} u \right\}
\]

\[
= -2 \left| \nabla^2 \tilde{g}(t) u \right|_{\tilde{g}(t)}^2 + \tilde{g}(t)^{ab} \tilde{g}(t)^{ij} \cdot \nabla_{a} \nabla_{b} \nabla_{i} \nabla_{j} u - \tilde{g}(t)^{ab} \tilde{g}(t)^{ij} \cdot \nabla_{i} \nabla_{j} u \left\{ \nabla_{a} \nabla_{b} \eta_{ij} \right\} + \text{Rm}(\tilde{\omega}(t)) \delta_{jab} \nabla_{k} \nabla_{l} u
\]

\[
\leq -2 \left| \nabla^2 \tilde{g}(t) u \right|_{\tilde{g}(t)}^2 + C \cdot \left| \nabla \eta \right|_{\tilde{g}(t)} \cdot \left| \nabla^2 \tilde{g}(t) u \right|_{\tilde{g}(t)} + C \cdot |\text{Rm}(\tilde{\omega}(t))|_{\tilde{g}(t)} \cdot \left| \nabla^2 \tilde{g}(t) u \right|_{\tilde{g}(t)}
\]

\[
\leq -2 \left| \nabla^2 \tilde{g}(t) u \right|_{\tilde{g}(t)}^2 + C \cdot \left| \nabla \eta \right|_{\tilde{g}(t)} \cdot \left| \nabla \eta \right|_{\tilde{g}(t)}
\]

where * denotes tensor contraction by $\tilde{g}(t)$. Also, we naturally have

\[
\left| \nabla^2 \tilde{g}(t) u \right|_{\tilde{g}(t)}^2 \geq |\eta|_{\tilde{g}(t)}^2.
\]

Hence we conclude that

\[
(-\Delta_{\tilde{g}(t)}) \left( |\nabla u|_{\tilde{g}(t)}^2 \right) \leq -2 |\eta|_{\tilde{g}(t)}^2 + h(t).
\]

(4.25)

Next, using (4.23) we have

\[
(-\Delta_{\tilde{g}(t)}) \left( |\eta|_{\tilde{g}(t)}^2 \right)
\]

\[
= -2 \left| \nabla \eta \right|_{\tilde{g}(t)}^2 + \nabla^2 \tilde{g}(t) \eta \cdot \eta
\]

\[
\leq C \cdot \left| \nabla \eta \right|_{\tilde{g}(t)} \cdot |\eta|_{\tilde{g}(t)}
\]

\[
\leq C \cdot |\eta|_{\tilde{g}(t)}
\]

(4.26)

where * denotes tensor contraction by $\tilde{g}(t)$. Hence using (4.23) we can further compute on $U$

\[
(-\Delta_{\tilde{g}(t)}) \left( |\eta|_{\tilde{g}(t)}^4 \right)
\]

\[
\leq 2 |\eta|_{\tilde{g}(t)}^4 \cdot (-\Delta_{\tilde{g}(t)}) \left( |\eta|_{\tilde{g}(t)}^2 \right) - 2 \left| \nabla \eta \right|_{\tilde{g}(t)}^2
\]

\[
\leq C \cdot |\eta|_{\tilde{g}(t)}^3 - 2 \left| \nabla \eta \right|_{\tilde{g}(t)}^2
\]

(4.27)
Choose cut-off function \( \rho \) as in the proof of Theorem 1.3 and use again (4.23) we can compute on \( U \)

\[
(-\Delta \omega(t)) \left( \rho^2 |\eta^4_{\omega(t)}| \right)
\]

\[
= \rho^2 (-\Delta \omega(t)) \left( |\eta^4_{\omega(t)}| + |\eta^4_{\omega(t)}| (-\Delta \omega(t)) \left( \rho^2 \right) - 2 \text{Re} \left\{ \langle \nabla \rho^2, \nabla |\eta^4_{\omega(t)}| \rangle_{\omega(t)} \right\} \right)
\]

\[
\leq \rho^2 \left( C \cdot |\eta^2_{\omega(t)}| - 2 |\nabla |\eta^2_{\omega(t)}|_{\omega(t)}^2 \right) + C \cdot |\eta^4_{\omega(t)}| + C \cdot \rho |\nabla \rho|_{\omega(t)} |\eta^2_{\omega(t)}| \nabla |\eta^2_{\omega(t)}|_{\omega(t)}
\]

\[
\leq -2 \rho^2 |\nabla |\eta^2_{\omega(t)}|_{\omega(t)}^2 + C \cdot |\eta^2_{\omega(t)}| + C \cdot \left( \rho |\nabla |\eta^2_{\omega(t)}|_{\omega(t)} \right) |\eta^2_{\omega(t)}|
\]

\[
\leq -2 \rho^2 |\nabla |\eta^2_{\omega(t)}|_{\omega(t)}^2 + C \cdot |\eta^2_{\omega(t)}| + \rho^2 |\nabla |\eta^2_{\omega(t)}|_{\omega(t)}^2 + C \cdot |\eta^4_{\omega(t)}|
\]

\[
\leq C \cdot |\eta^2_{\omega(t)}|.
\]

Now we conclude that we can find some \( h(t) \) such that on \( U \)

\[
\begin{align*}
&\left| \nabla u^2_{\omega(t)} h(t)^{-1} \right| \leq 1, \\
&(-\Delta \omega(t)) \left( |\nabla u^2_{\omega(t)} h(t)^{-1} \right) \leq -2 |\eta^2_{\omega(t)}| h(t)^{-1} + 1, \\
&(-\Delta \omega(t)) \left( \rho^2 |\eta^4_{\omega(t)}| h(t)^{-1} \right) \leq C \cdot |\eta^2_{\omega(t)}| h(t)^{-1}.
\end{align*}
\]

(4.28)

Set

\[
Q := \rho^2 |\eta^4_{\omega(t)}| h(t)^{-1} + C \cdot |\nabla u^2_{\omega(t)} h(t)^{-1}|
\]

Using (4.28), on \( U \times [0, \infty) \) we have

\[
(-\Delta \omega(t)) (Q) \leq - |\eta^2_{\omega(t)}| h(t)^{-1} + C.
\]

Now, at a given time \( t \), assume \( Q \) achieves it’s maximum at point \( x_0 \). If \( x_0 \in \partial U \), where \( \rho \equiv 0 \), using (4.28), \( Q \) has an upper bound \( C \) at time \( t \), and we are done. Otherwise \( x_0 \in U \) and by maximum principle, we have

\[
0 \leq (-\Delta \omega(t)) (Q)(x_0) \leq - |\eta^2_{\omega(t)}| (x_0) h(t)^{-1} + C,
\]

which gives

\[
|\eta^2_{\omega(t)}| (x_0) h(t)^{-1} \leq C.
\]

Then by (4.23) and (2.6) we have on \( U \)

\[
Q \leq Q(x_0) \leq C^2_K |\eta^2_{\omega(t)}| (x_0) h(t)^{-1} + C \cdot |\nabla u^2_{\omega(t)}(x_0) h(t)^{-1} | \leq C.
\]

Since \( \rho \equiv 1 \) on \( K \), we obtain the estimate

\[
|\eta^2_{\omega(t)}| \leq Ch(t)^{\frac{1}{2}}
\]

on \( K \). Hence we conclude

\[
\begin{align*}
\|\eta(t)\|_{C^0(U, \omega(t))} & \leq h_K(t), \\
\|\eta(t)\|_{C^0(U, \omega(t))} & \leq C_{K,k}.
\end{align*}
\]

(4.29)
Now Conditions (A) and (B) of Lemma 1.1 are satisfied for $\eta(t)$. Hence from Lemma 1.1 we conclude the local convergence

$$\|\text{Ric}(\omega(t)) + \omega_B\|_{C^k(K,\bar{\omega}(t))} \leq h_{K,k}(t). \quad (4.30)$$

Finally, for the scalar curvature, Lemma 4.2 gives that

$$\|R(\omega(t))\|_{C^0(U,\bar{\omega}(t))} \leq C_{K,k}(t).$$

The estimate (4.30) or the main result of [15] implies that

$$\|R(\omega(t)) + m\|_{C^0(U)} \leq h(t).$$

Hence set

$$\bar{\eta}(t) = R(\omega(t)) + m$$

and apply Lemma 1.1 we conclude the local convergence

$$\|R(\omega(t)) + m\|_{C^k(K,\bar{\omega}_t)} \leq h_{K,k}(t).$$

This establish (1.16) and hence completes the proof of Theorem 1.6. $\square$

References

[1] T. Aubin, $\text{É}\text{q}u\text{ë}t\text{ë}n\text{s du type Monge-Ampère sur les variétés kähleriennes compactes}$, C. R. Acad. Sci. Paris Sér. A-B 283 (1976), no. 3, Aiii, A119–A121.
[2] W. Fischer, H. Grauert, $\text{Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten}$, Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II (1965), 89–94.
[3] F.T.-H. Fong, Y.S. Zhang, $\text{Local curvature estimates of long-time solutions to the Kähler-Ricci flow}$, preprint, arXiv:1903.05939.
[4] F.T.-H. Fong, Z. Zhang, $\text{The collapsing rate of the Kähler-Ricci flow with regular infinite time singularity}$, J. reine angew. Math. 703 (2015), 95–113.
[5] M. Gill, $\text{Convergence of the parabolic complex Monge-Ampère equation on compact Hermitian manifolds}$, Comm. Anal. Geom. 19 (2011), no. 2, 277–303.
[6] M. Gill, $\text{Collapsing of products along the Kähler-Ricci flow}$, Trans. Amer. Math. Soc. 366 (2014), no. 7, 3907–3924.
[7] B. Greene, A. Shapere, C. Vafa, S.-T. Yau, $\text{Stringy cosmic strings and noncompact Calabi-Yau manifolds}$, Nuclear Phys. B 337 (1990), no. 1, 1–36.
[8] M. Gross, V. Tosatti, Y. Zhang, $\text{Collapsing of abelian fibered Calabi-Yau manifolds}$, Duke Math. J. 162 (2013), no. 3, 517–551.
[9] M. Gross, V. Tosatti, Y. Zhang, $\text{Gromov-Hausdorff collapsing of Calabi-Yau manifolds}$, Comm. Anal. Geom. 24 (2016), no. 1, 93–113.
[10] M. Gross, P.M.H. Wilson, $\text{Large complex structure limits of K3 surfaces}$, J. Differential Geom. 55 (2000), no. 3, 475–546.
[11] H.-J. Hein, $\text{Gravitational instantons from rational elliptic surfaces}$, J. Amer. Math. Soc. 25 (2012), no. 2, 355–393.
[12] H.-J. Hein, $\text{A Liouville theorem for the complex Monge-Ampère equation on product manifolds}$, to appear in Comm. Pure Appl. Math.
[13] H.-J. Hein, V. Tosatti, Remarks on the collapsing of torus fibered Calabi-Yau manifolds, Bull. Lond. Math. Soc. 47 (2015), no. 6, 1021–1027.
[14] H.-J. Hein, V. Tosatti, Higher-order estimates for collapsing Calabi-Yau metrics, preprint, arXiv:1803.06697.
[15] W. Jian, Convergence of scalar curvature of Kähler-Ricci flow on manifolds of positive Kodaira dimension, preprint, arXiv:1805.07884.
[16] W. Jian, Y. Shi, and J. Song, A remark on constant scalar curvature Kähler metrics on minimal models, arXiv:1805.06863v1, to appear at Proceedings of A.M.S.
[17] Y. Li, On collapsing Calabi-Yau fibrations, preprint, arXiv:1706.10250.
[18] C. Li, J. Li, X. Zhang, A mean value formula and a Liouville theorem for the complex Monge-Ampère equation, preprint, arXiv:1709.05754.
[19] J. Song, G. Tian, The Kähler-Ricci flow on surfaces of positive Kodaira dimension, Invent. Math. 170 (2007), no. 3, 609–653.
[20] J. Song, G. Tian, Canonical measures and Kähler-Ricci flow, J. Amer. Math. Soc. 25 (2012), no. 2, 303–353.
[21] J. Song, G. Tian, Bounding scalar curvature for global solutions of the Kähler-Ricci flow, Amer. J. Math. 138 (2016), no. 3, 683–695.
[22] J. Song, G. Tian, The Kähler-Ricci flow through singularities, Invent. Math. 207 (2017), 519-595.
[23] J. Song, G. Tian and Z. Zhang, Collapsing behavior of Ricci-flat Kähler metrics and long time solutions of the Kähler-Ricci flow, preprint, arXiv:1904.08345.
[24] J. Song, B. Weinkove, Contracting exceptional divisors by the Kähler-Ricci flow, Duke Math. J. 162 (2013), no. 2, 367415.
[25] J. Song, B. Weinkove, Contracting exceptional divisors by the Kähler-Ricci flow II, Proc. Lond. Math. Soc. (3) 108 (2014), no. 6, 15291561.
[26] J. Song, B. Weinkove, Introduction to the Kähler-Ricci flow, Chapter 3 of ‘Introduction to the Kähler-Ricci flow’, eds S. Boucksom, P. Eyssidieux, V. Guedj, Lecture Notes Math. 2086, Springer 2013.
[27] V. Tosatti, Adiabatic limits of Ricci-flat Kähler metrics, J. Differential Geom. 84 (2010), no. 2, 427–453.
[28] V. Tosatti, Degenerations of Calabi-Yau metrics, in Geometry and Physics in Cracow, Acta Phys. Polon. B Proc. Suppl. 4 (2011), no. 3, 495–505.
[29] V. Tosatti, Calabi-Yau manifolds and their degenerations, Ann. N.Y. Acad. Sci. 1260 (2012), 8–13.
[30] V. Tosatti, KAWA lecture notes on the Kähler-Ricci flow, to appear in Ann. Fac. Sci. Toulouse Math.
[31] V. Tosatti, Y. Zhang, Infinite time singularities of the Kähler-Ricci flow, Geom. Topol. 19 (2015), no. 5, 2925–2948.
[32] V. Tosatti, Y. Zhang, Collapsing hyperkähler manifolds, preprint, arXiv:1705.03299.
[33] V. Tosatti, Y. Zhang, Triviality of fibered Calabi-Yau manifolds without singular fibers, Math. Res. Lett. 21 (2014), no.4, 905–918.
[34] V. Tosatti, B. Weinkove, and X.K. Yang, The Kähler-Ricci flow, Ricci-flat metrics and collapsing limits, arXiv:1408.0161, to appear in Amer. J. Math.
[35] G. Tian, Z.L. Zhang, Convergence of Kähler-Ricci flow on lower-dimensional algebraic manifolds of general type, Int. Math. Res. Not. IMRN 2016, no. 21, 6493-6511.
[36] G. Tian, Z.L. Zhang, Relative volume comparison of Ricci flow and its applications, arXiv:1802.09506.
[37] G. Tian, Z. Zhang, On the Kähler-Ricci flow on projective manifolds of general type, Chinese Ann. Math. Ser. B 27 (2006), no. 2, 179–192.
[38] S. T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.*, Comm. Pure Appl. Math. 31 (1978), no. 3, 339-411.

[39] S.-T. Yau, *A general Schwarz lemma for Kähler manifolds*, Amer. J. Math. 100 (1978), no. 1, 197–203.

[40] Y. Zhang, *Infinite-time singularity type of the Kähler-Ricci flow*, J. Geom. Anal. (2017), https://doi.org/10.1007/s12220-017-9949-2.

[41] Y. Zhang, *Collapsing limits of the Kähler-Ricci flow and the continuity method*, Math. Ann. (2018), https://doi.org/10.1007/s00208-018-1676-x.

[42] Y. Zhang, *Infinite-time singularity type of the Kähler-Ricci flow II*, Math. Res. Lett. (2019) (to appear), arXiv:1809.01305

[43] Z. Zhang, *Scalar curvature bound for Kähler-Ricci flows over minimal manifolds of general type*, Int. Math. Res. Not. 2009; doi: 1093/imrn/rnp073

[44] Z. Zhang, *Scalar curvature behavior for finite time singularity of Kähler-Ricci flow*, Michigan Math. J. 59 (2010), no. 2, 419–433

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