Muonic Hydrogen and the Third Zemach Moment

J. L. Friar

Theoretical Division, Los Alamos National Laboratory,
Los Alamos, New Mexico 87545

Ingo Sick

Dept. für Physik und Astronomie, Universität Basel, Basel, Switzerland

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We determine the third Zemach moment of hydrogen \((\langle r^3 \rangle_3)\) using only the world data on elastic electron-proton scattering. This moment dominates the \(O(Z\alpha)^5\) hadronic correction to the Lamb shift in muonic atoms. The resulting moment, \(\langle r^3 \rangle_3 = 2.71(13) \text{ fm}^3\), is somewhat larger than previously inferred values based on models. The contribution of that moment to the muonic hydrogen 2S level is \(-0.0247(12)\text{ meV}\).

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I. INTRODUCTION

Recent improvements in experimental techniques have led to the measurement\(^1\) of the 1S-2S interval in hydrogen to an unprecedented accuracy of 2 parts in \(10^{14}\). At the same time the QED corrections (expressed as powers of \(\alpha\), the fine-structure constant) to this interval have been calculated through \(O(\alpha^7)\), with an important subset of \(O(\alpha^8)\) terms determined in the past few years (see Ref.\(^2\) for a comprehensive review of the theory and Appendix A of \(^3\) for very recent developments). With these improvements the difference between experiment and QED theory (including recoil corrections) is dominated by hadronic size corrections, which affect the 1S-2S interval in the 10th\(^{\text{th}}\) significant figure. Uncalculated QED corrections probably affect the 12th significant figure\(^2\).

This situation creates both an opportunity and a problem. The opportunity is to use the high precision measurements involving S-states to determine the rms charge radius of the proton at levels of precision of roughly 1\(^{\%}\). Recent values obtained in this way are: \(\langle r^2 \rangle_p^{1/2} = 0.883(14)\text{ fm}\), \(0.891(18)\text{ fm}\), \(0.869(12)\text{ fm}\), and \(0.875(7)\text{ fm}\). These consistent results reflect slightly different theoretical and experimental input.

The murky history of experimental values for the proton charge radius obtained from elastic electron-proton scattering data has recently been clarified by the comprehensive analysis of Ref.\(^4\). That analysis of all of the world’s data separated the charge from the magnetic scattering, incorporated (significant) Coulomb corrections\(^5\), and carefully treated systematic (as well as random) uncertainties. The resulting value of \(\langle r^2 \rangle_p^{1/2} = 0.895(18)\text{ fm}\) is significantly higher than most older values, but is consistent with the atomic determinations. It is unlikely that new and relevant electron-scattering data will become available in the near future, and significant improvements over the 2\(^{\%}\) uncertainty are therefore unlikely during this period.

The problem mentioned above is that the extremely precise measurements of hydrogen spectral lines cannot be used to test QED at anything approaching the level of accuracy of the QED theory, unless tricks are used to combine measurements of at least two experiments in such a way that the shortest-range hadronic and QED processes cancel between the measurements. Although this complementary approach is a very useful and active field of research\(^6\), significant QED information has nevertheless been removed in the effort to eliminate the leading-order hadronic effects. Any alternative method that could provide a highly accurate value of the proton charge radius would resolve much of this problem. A possible method is the measurement of the 2S-2P Lamb shift in muonic hydrogen, and such an experiment at PSI\(^7\) plans to determine the proton radius.

The Lamb shift in electronic hydrogen is dominated by the (repulsive) radiative corrections on the electron line, which are much larger than the (attractive) vacuum polarization corrections on the photon line. The electron spends most of its time outside the polarization cloud induced in the electron Fermi sea. In the muonic-atom case the much smaller Bohr radius is within a significant portion of that cloud and the (electron) vacuum polarization dominates the QED corrections. The smaller radius also means that the hadronic size corrections are significantly more important, as well. The goal of the PSI experiment is to determine the proton charge radius, \(\langle r^2 \rangle_p^{1/2}\), to 1 part per thousand. This requires a theoretical accuracy significantly better than .008 meV, which is the uncertainty introduced by a part per thousand error in the proton radius. The necessary theoretical developments have been recently reviewed in Ref.\(^8\), \(^9\), and 11.

Reference\(^10\) estimates that uncalculated QED diagrams are likely to be smaller than 0.002 meV (although Ref.\(^11\) points out that the light-by-light-scattering contribution has not yet been calculated). The largest uncertainties are from hadronic contributions. Most of the latter corrections fortunately are proportional (or nearly so) to the mean-square charge radius of the proton. All such terms can be simultaneously fit to the observed muonic Lamb shift together with the leading-order size correction.
in Eqn. (1) below. One contribution, however, is significantly different, and estimates of its size have shown considerable variation. We address this quantity in the next section.

II. ZEMACH MOMENTS

The primary hadronic size corrections are the Coulomb correction of $\mathcal{O}(Z\alpha)^2$, the one-loop correction of $\mathcal{O}(Z\alpha)^3$, the two-loop correction of $\mathcal{O}(Z\alpha)^6$, and the hadronic-size modifications of the radiative and vacuum polarization corrections. In addition there are hadronic vacuum polarization corrections and polarizability corrections (which are significant, but which we ignore).

The dominant and long-known[12] hadronic size correction arises from modifying the Coulomb potential with the hadronic charge distribution, $\rho_{ch}$. Higher-order contributions of this mechanism can be obtained as well, and these were calculated many years ago in the context of muonic atoms[13, 15]. The first three orders of corrections for the $m$th S-state can be written in the general form:

$$\Delta E_n = \frac{2\pi}{3} Z \alpha |\phi_n(0)|^2 \left( \langle r^2 \rangle - \frac{Z\alpha}{2} \langle r^3 \rangle_{(2)} \right) + (Z\alpha)^2 F_{REL} + (Z\alpha\mu)^2 F_{NR} + \cdots,$$

(1)

where $Z$ is the nuclear charge, $\langle r^m \rangle$ is the $m$th moment of the nuclear charge distribution (normalized to unit charge), $\mu$ is the reduced mass of the muon-nucleus system, $\phi_n(0)$ is the muon wave function at the origin, and the Zemach moment $\langle r^3 \rangle_{(2)}$ is defined by

$$\langle r^3 \rangle_{(2)} = \int d^3r r^3 \rho_{(2)}(r),$$

(2)

where the convoluted (Zemach) charge density is given by

$$\rho_{(2)}(r) = \int d^3z \rho_{ch}(z) \rho_{ch}(z - r).$$

(3)

The nonrelativistic term $F_{NR}$ is part of the Coulomb correction of relative order (to the leading-order term in Eqn. (1)) $(Z\alpha)^2 \mu^2 R^2$, where $R$ is a generic proton radius, while the corresponding relativistic correction is determined by $F_{REL}$, and is of relative order $(Z\alpha)^2$.

The finite-size radiative corrections are discussed in detail in Refs. 2, 10, 11 and should scale like $(r^2)_p$. The complete hadronic correction of $\mathcal{O}(Z\alpha)^6$ has never been worked out but the Coulomb approximation to it is small (see Refs. 2, 17, 11 and above) and this is likely to be adequate. The remaining term is the one-loop contribution of $\mathcal{O}(Z\alpha)^5$, discussed in Refs. 2, 11, 11, 12, 17, which we discuss next.

The one-loop correction of $\mathcal{O}(Z\alpha)^5$ was considered by Pachucki[16], and was expressed in terms of the proton’s Dirac form factors, $F_1$ and $F_2$. He also showed in the limit that the proton mass becomes very large that this contribution approached a relatively simple expression involving the same limit of the Sachs electric form factor, $G_E(q^2)$. The expression

$$\langle r^3 \rangle_{(2)} = \frac{48}{\pi} \int_0^\infty dq (G_E^2(q^2) - 1 + q^2(r^2)_p/3)$$

(4)

can be verified by writing the form factor $G_E(q^2)$ as the Fourier transform of $\rho_{ch}(r)$ and repeatedly integrating by parts in Eqn. (4). Substituting Eqn. (4) for $\langle r^3 \rangle_{(2)}$ in Eqn. (1) verifies that Pachucki’s simplified expression is just the $\mathcal{O}(Z\alpha)^5$ term in Eqn. (1). The terms that vanish in this approximation can be considered as higher-order recoil corrections, as was noted in Refs. 2, 10. According to [2] the Coulomb approximation is good to within 10% for a simple model of the form factors.

The convoluted density $\rho_{(2)}(r)$ arises because each of the Coulomb photons is modified at short distances by the nuclear charge distribution. An alternative description is that the modified Coulomb potential changes the muon wave function at short distances, and this has an effect on all expectation values[10, 10, 10]. Thus the $\langle r^3 \rangle_{(2)}$ term above bears the same relationship to the full elastic one-loop contribution to the Lamb shift as the (traditional Zemach moment) $\langle r^3 \rangle_{(2)}$ term does to the full elastic one-loop contribution to the hyperfine structure. We also note that analytic results exist for $\langle r^3 \rangle_{(2)}$ for three simple charge distributions, including the dipole form factor case (viz., an exponential charge distribution, for which $\langle r^3 \rangle_{(2)} = 3\sqrt{3} r^2(2^3/16)$[13]). The necessity to resort to models of the proton form factor has lead to significant variations in the size expected for $\langle r^3 \rangle_{(2)}$. For a treatment using heavy-baryon effective field theory see Ref. 10.

III. CALCULATIONS

In Ref. 6 the world data on electron-proton scattering for momentum transfers $q \leq 4$ fm$^{-1}$ have been analyzed (for references to the data see 6). The electric and magnetic Sachs form factors $G_E(q^2)$ and $G_M(q^2)$ have been parameterized using a Continued Fraction (CF) expansion. It has been shown that this CF-expansion is more suitable than other parameterizations used in the past, and the contribution of the model dependence due to this choice has been evaluated. The longitudinal/transverse separation then is done during the global fit of the cross sections, an approach that is superior to the L/T-separations performed when determining $G_E$ and $G_M$ from individual data sets.

The fit cross sections have been calculated from $G_E$ and $G_M$ including, in second-order Born approximation 7, the Coulomb distortion of the electron waves; this correction, although neglected in almost all analyses in the past, is important at low $q$.

The data have been fitted using their random errors, and the error propagation treated via the error matrix.
The systematic uncertainties of the data have been taken into account by changing the data sets by the quoted error, refitting and adding all resulting changes quadratically, hereby obtaining a very conservative estimate of the systematic uncertainty of \( \langle r^3 \rangle_{(2)} \).

The result: \( \langle r^3 \rangle_{(2)} \) amounts to 2.71(13) fm\(^3\), where the error bar includes both random and systematic errors of the data, the latter dominating by far. For comparison we quote \( \langle r^3 \rangle_{(2)} \) for the “standard” dipole parameterization, which corresponds to an rms charge radius of 0.811 fm and produces \( \langle r^3 \rangle_{(2)} = 2.02 \) fm\(^3\). If the charge radius of the dipole model is scaled to 0.895 fm, the dipole result becomes 2.72 fm\(^3\), which is in very good agreement with the value we determined directly from the data.

We should perhaps add a comment on the integral in E\num{4}, which seems to indicate that, as a consequence of the \( 1/\omega^4 \)-factor, the \( \langle r^3 \rangle_{(2)} \) depends on data at extremely small \( q \). It must be noted, however, that the \( G_E(0) \) term cancels the “1” at \( q \sim 0 \), and the \( q^2 \langle r^2 \rangle_p/3 \) term cancels the first, \( q^2 \)-dependent term in a power series expansion of \( G_E(q) \). Sensitivity studies have shown that the main contribution to the integral comes from the region \( q \sim 1.1 \pm 0.5 \) fm\(^{-1}\) where the data base for electron-proton scattering is very good.

We have also looked at the effect of two-photon exchange (beyond Coulomb distortion), which recently has been studied \[20\] in connection with differences between values of \( G_E \) extracted from Rosenbluth-separations of cross sections and polarization transfer measurements. These corrections turn out to have a very minor effect; the value of \( \langle r^3 \rangle_{(2)} \) is increased by 0.02 fm\(^3\), i.e. by only a small fraction of the error bar.

IV. CONCLUSIONS

We have calculated the third Zemach moment for the charge distribution of hydrogen from the world’s electron-proton scattering data. That moment is \( \langle r^3 \rangle_{(2)} = 2.71(13) \) fm\(^3\), which contributes \(-0.0247(12)\) meV to the 2S state of muonic hydrogen. For comparison the result for the “standard” dipole model is an energy shift of \(-0.0185\) meV. Our result is model independent, and removes the concerns of Ref.\[11\], who noted that two simple models of the charge distribution with the same charge radius produced differences of 0.002 meV to the 2S energy shift. Our energy shift is somewhat larger than most recent values and significantly larger than a few. Our calculation removes most of the uncertainty from the contribution of the \( O(\alpha/\gamma)^5 \) (elastic) finite-size term.

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[1] M. Niering et al. Phys. Rev. Lett., 84:5496, 2000.
[2] M.L. Eides, H. Grotch, and V.A. Shelyuto. Phys. Rep., 342:63, 2001.
[3] P.J. Mohr and B.N. Taylor. Rev. Mod. Phys., 77:1 (2005).
[4] K. Melnikov and T. van Ritbergen. Phys. Rev. Lett., 84:1673, 2000.
[5] K. Pachucki. Phys. Rev. A, 63:042503, 2001.
[6] I. Sick. Phys. Lett. B, 576:62, 2003.
[7] I. Sick and D. Trautmann. Phys. Lett. B, 375:16, 1996.
[8] S.G. Karshenboim and V.G. Ivanov. Phys. Lett. B, 524:259, 2002.
[9] D. Taqqu et al. Hyperfine Int., 119:311, 1999.
[10] K. Pachucki. Phys. Rev. A, 60:3593, 1999.
[11] E. Borie. Phys. Rev. A, 71:032508, 2005.
[12] R. Karplus, A. Klein, and J. Schwinger. Phys. Rev., 86:288, 1952.
[13] J.L. Friar. Ann. Phys. (N.Y.), 122:151, 1979. There are typographical errors in Eqns. (24b) (\( d^3 s \) replaces \( d s \)) and (25c) (\( -\frac{1}{a^3} \) replaces \( +\frac{1}{a^3} \)) defining the nonrelativistic contributions, which are important for muonic atoms. The numerical results in the paper are correct.
[14] L.A. Borisoglebsky and E.E. Trofimenko. Phys. Lett. B, 81:175, 1979. E.E. Trofimenko, Phys. Lett. A 73:383, 1979.
[15] C. Zemach. Phys. Rev., 104:1771, 1956.
[16] K. Pachucki. Phys. Rev. A, 53:2092, 1996.
[17] A.P. Martynenko and R.N. Faustov. Phys. At. Nucl., 63:845, 2000.
[18] J.L. Friar. Z. Phys. A, 292:1, 1979. (E) 303:84 1981.
[19] A. Pineda. Phys. Rev. C, 71:065205, 2005.
[20] P.G. Blunden, W. Melnitchouk, and J.A. Tjon. Phys. Rev. Lett., 91:142304, 2003.