MIXED COMMUTING VARIETIES

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1. Introduction

1.1. Let \( k \) be an algebraically closed field of characteristic \( p \geq 0 \). Suppose \( \mathfrak{g} \) is a simple Lie algebra defined over \( k \). For each \( r \geq 2 \), let \( V_1, \ldots, V_r \) be irreducible closed subvarieties of a Lie algebra \( \mathfrak{g} \). Define

\[
C(V_1, \ldots, V_r) = \{(v_1, \ldots, v_r) \in V_1 \times \cdots \times V_r \mid [v_i, v_j] = 0, \ 1 \leq i \leq j \leq r\},
\]

a mixed commuting variety over \( V_1, \ldots, V_r \). Note that if \( V_1 = \cdots = V_r \), then this variety becomes \( C_r(V_1) \), the commuting variety of \( r \)-tuples. Mixed commuting varieties of two tuples were first introduced in [Vas 9.4], the \( r \)-tuple version was defined and studied in [N]. In particular, the author explicitly described the irreducible decomposition for any mixed commuting variety over \( \mathfrak{sl}_2 \) and its nullcone \( \mathcal{N} \). The result also implies that such varieties are mostly not Cohen-Macaulay or normal.

In general, mixed commuting varieties are still mysterious. Interesting questions include

1. What is the dimension of \( C_r(V_1, \ldots, V_r) \)?
2. What are the irreducible components of \( C_r(V_1, \ldots, V_r) \)?

The results in this paper were motivated by investigating the cohomology for Frobenius kernels of algebraic groups. The first connection between commuting varieties and support varieties in cohomology of Frobenius kernels was constructed by Suslin, Friedlander, and Bendel in their two papers [SFB1, SFB2]. To be precise, let \( G \) be an algebraic group defined over \( k \), and let \( G_r \) be the \( r \)-th Frobenius kernel of \( G \). Then there is a homeomorphism between the maximal ideal spectrum of the cohomology ring for \( G_r \) and the nilpotent commuting variety over the Lie algebra \( \mathfrak{g} = \text{Lie}(G) \) whenever the characteristic \( p \) is large enough. This variety is also the ambient space for support varieties of the \( r \)-th Frobenius kernels of \( G \). In the case \( r = 1 \), these support varieties for modules \( L(\lambda) \) and \( H^0(\lambda) \) are explicitly described by the work of Drupieski, Nakano, Parshall, and Vella [NPV], [DNP]. Sobaje uses these descriptions to compute the support varieties of \( L(\lambda) \) for higher \( r \) [So]. Our study on mixed commuting varieties is inspired from results in that paper [So Theorems 3.1, 3.2]. Most of our calculations about mixed commuting varieties are in the case \( \mathfrak{g} = \mathfrak{sl}_3 \). More explicitly, we first study the dimensions of the commuting varieties over \( \mathfrak{gl}_3, \mathfrak{sl}_3 \).

1We are aware of the same terminology arising in [Ya2 Section 4]. The definition of mixed commuting varieties in that paper is entirely different from ours.
These are not new results, however we use a new method to tackle the problem. Our approach does not depend on the irreducibility of \( C_r(\mathfrak{gl}_n) \). On the other hand, we investigate the dimension and reducibility of mixed commuting varieties over \( \mathcal{O}_{\text{sub}}, \mathcal{N} \) and \( \mathfrak{sl}_3 \). Our results indicates that these mixed commuting varieties are mostly not irreducible. Hence, they are rarely normal.

1.2. Main results. The paper is organized as follows. We first review terminology and notation in Section 2. Then in Section 3, we study the properties of \( C_r(z_{\text{sub}}) \) where \( z_{\text{sub}} \) is the centralizer of the subregular element corresponding to the partition \([n-1,1]\) in \( \mathfrak{gl}_n \). In particular, let \( \mathfrak{g} = \mathfrak{sl}_n \) and \( p \nmid n \). Then we prove that for every \( r \geq 1 \), \( C_r(z_{\text{sub}}) \) is a product of an affine space of dimension \((n-2)r\) with a determinantal variety generated by all \( 2 \times 2 \)-minors over a \((3 \times r)\)-matrix of indeterminants; hence it is normal and Cohen-Macaulay (cf. Theorem 5.2.1). Next, we apply our calculations to compute the dimensions of \( C_r(\mathfrak{sl}_3) \) and \( C_r(\mathfrak{gl}_3) \). Note that the latter variety was proved to be irreducible by Kirillov and Neretin [KN], see also [Gu]; hence the dimension easily follows. Our method does not depend on the irreducibility of this variety. We also emphasize the impact of the characteristic \( p \), i.e. \( p \nmid n \), on the commuting variety \( C_r(\mathfrak{sl}_3) \) (cf. Remark 5.2.3).

In Section 4, we investigate mixed commuting varieties over \( \mathcal{O}_{\text{sub}}, \mathcal{N} \), and \( \mathfrak{sl}_3 \). Our key ingredient is determinantal varieties over certain matrices. In particular, by analyzing the intersections \( z_{\text{sub}} \cap \mathcal{O}_{\text{sub}} \) and \( z_{\text{sub}} \cap \mathcal{N} \), we reduce to the problem of computing the dimension of the varieties generated by 2 by 2 minors of the following matrix of indeterminates

\[
\begin{pmatrix}
  x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} & x_{i+j+1} & \cdots & x_{i+j+m} \\
  0 & \cdots & 0 & y_1 & \cdots & y_j & y_{j+1} & \cdots & y_{j+m} \\
  0 & \cdots & 0 & 0 & \cdots & 0 & z_1 & \cdots & z_m
\end{pmatrix}.
\]

We can also generalize this result for any arbitrarily large matrix (cf. Theorem 4.3.2). This observation implies the dimension formula for mixed commuting varieties. Moreover, we are able to determine whether a mixed commuting variety is irreducible or not (cf. Theorems 4.4.1 and 4.4.3).

In the last section, we use results in the previous section to compute the dimension of support varieties of Frobenius kernels for a simple module \( L(\lambda) \) in a certain case. From results in [DNP] and [So], we point out a connection between support varieties and mixed commuting varieties (cf. Proposition 5.2.2). Then using an explicit calculation of support varieties of the first Frobenius kernels in [NPA] we are able to describe all support varieties of the \( r \)-th Frobenius kernels of \( G = SL_3 \). In particular, suppose that

\[
\lambda = \lambda_0 + \lambda_1 p + \cdots + \lambda_q p^q
\]

with \( \lambda_i \in X_1 \), which implies \( \lambda \in X^+ \). We compute the dimension of the support variety \( V_{G_r}(L(\lambda)) \) as follows:

\[
\dim V_{G_r}(L(\lambda)) = \begin{cases} 
2b_\lambda + 4 & \text{if } a_\lambda = 0, \\
2(a_\lambda + b_\lambda) + 3 & \text{if } a_\lambda = 1, \\
2(a_\lambda + b_\lambda) + 2 & \text{if } a_\lambda > 1.
\end{cases}
\]

where \( a_\lambda, b_\lambda \) are the number of singular weights and regular weights respectively in \( \{\lambda_1, \ldots, \lambda_0\} \). We further show that \( V_{G_r}(L(\lambda)) \) is always reducible unless every \( \lambda_i \) is regular (cf. Theorems 5.3.2 and 5.3.3).

2. Notation

2.1. Root systems and combinatorics. Let \( k \) be an algebraically closed field of characteristic \( p \). Let \( G \) be a simple, simply-connected algebraic group over \( k \), defined and split over the prime field \( \mathbb{F}_p \). Fix a maximal torus \( T \subset G \), also split over \( \mathbb{F}_p \), and let \( \Phi \) be the root system of \( T \) in \( G \). Fix a set \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \) of simple roots in \( \Phi \), and let \( \Phi^+ \) be the corresponding set of positive roots. Let \( B \subseteq G \) be the Borel subgroup of \( G \) containing \( T \) and corresponding to the set of negative
roots $\Phi^-$, and let $U \subseteq B$ be the unipotent radical of $B$. Set $\mathfrak{g} = \text{Lie}(G)$, the Lie algebra of $G$, $\mathfrak{b} = \text{Lie}(B)$, $\mathfrak{u} = \text{Lie}(U)$.

Let $X$ be the weight lattice of $\Phi$. Write $X^+$ for the set of dominant weights in $X$, and $X_\rho$ for the set of $p'$-restricted dominant weights in $X^+$. Given $\lambda \in X^+$, let $L(\lambda)$ be the simple rational $G$-module of highest weight $\lambda$. For each $r \geq 1$, let $F^r : G \to G$ be the $r$-th iterate of the Frobenius morphism of $G$. We call $G_r = \ker F_r$ the $r$-th Frobenius kernel of $G$.

2.2. Nilpotent orbits. Given a $G$-variety $V$ and a point $v$ of $V$, we denote by $O_v$ the $G$-orbit of $v$ (i.e., $O_v = G \cdot v$). For example, consider the nilpotent cone $\mathcal{N}$ of $\mathfrak{g}$ as a $G$-variety with the adjoint action. There are well-known orbits: $O_{\text{reg}} = G \cdot v_{\text{reg}}, O_{\text{subreg}} = G \cdot v_{\text{subreg}}$, (we abbreviate it by $O_{\text{sub}}$) and $O_{\text{min}} = G \cdot v_{\text{min}}$ where $v_{\text{reg}}, v_{\text{subreg}}, v_{\text{min}}$ are representatives for the regular, subregular, and minimal orbits. Denote by $z(v)$ the centralizer of $v$ in $\mathfrak{g}$. For convenience, we write $z_{\text{reg}}$ (or $z_{\text{sub}}$ and $z_{\text{min}}$) for the centralizers of $v_{\text{reg}}$ ($v_{\text{sub}}$ or $v_{\text{min}}$).

2.3. Basic algebraic geometry conventions. Let $R$ be a commutative Noetherian ring with identity. We use $R_{\text{red}}$ to denote the reduced ring $R/\sqrt{0}$ where $\sqrt{0}$ is the radical ideal of the trivial ideal 0, which consists of all nilpotent elements of $R$. If $V$ is a closed subvariety of an affine space $\mathbb{A}^n$, we denote by $I(V)$ the radical ideal of $k[\mathbb{A}^n] = k[x_1, \ldots, x_n]$ associated to this variety. Let $X$ be an affine variety. Then we always write $k[X]$ for the coordinate ring of $X$ which is the same as the ring of global sections $\mathcal{O}_X(X)$.

2.4. Commutative algebra. Consider an $m \times n$ matrix

$$X = \begin{pmatrix} x_{11} & \cdots & x_{1n} \\ \vdots & \ddots & \vdots \\ x_{m1} & \cdots & x_{mn} \end{pmatrix}$$

whose entries are independent indeterminates over the field $k$. Let $k[X]$ be the polynomial ring over all the indeterminates of $X$, and let $I_t(X)$ be the ideal in $k(X)$ generated by all $t$ by $t$ minors of $X$. For each $t \geq 1$, the ring

$$R_t(X) = \frac{k[X]}{I_t(X)}$$

is called a determinantal ring. The following is one of the nice properties of determinantal rings.

**Proposition 2.4.1.** [BV] For every $1 \leq t \leq \min(m, n)$, $R_t(X)$ is a reduced, Cohen-Macaulay, normal domain of dimension $(t - 1)(m + n - t + 1)$.

We denote by $D_t(X)$ the determinantal variety defined by $I_t(X)$.

3. Commuting varieties of centralizers

Let $\mathfrak{g} = \mathfrak{sl}_n$. It is easy to see that $C_r(z_{\text{reg}}) = z_{\text{reg}}^r$ for every $r \geq 1$. We study in this section the variety $C_r(z_{\text{sub}})$. Then we apply our calculations to compute the dimensions of $C_r(\mathfrak{sl}_3)$ and $C_r(\mathfrak{gl}_3)$ for each $r \geq 1$.

3.1. Nice properties of $C_r(z_{\text{sub}})$.

**Theorem 3.1.1.** For each $r \geq 1$, the variety $C_r(z_{\text{sub}})$ is irreducible, Cohen-Macaulay and normal. Moreover, we have

$$\dim C_r(z_{\text{sub}}) = \begin{cases} (n - 1)r + 2, & \text{if } p \nmid n, \\ nr + 1, & \text{otherwise.} \end{cases}$$
Proof. Without loss of generality, let \( v_{\text{sub}} \) be the Jordan matrix corresponding to the partition \([n-1,1]\). Then an element \( u \) of \( z_{\text{sub}} \) is of the form

\[
\begin{pmatrix}
  a_1 & 0 & 0 & \cdots & 0 \\
  a_2 & a_1 & 0 & \ddots & \\
  \vdots & \vdots & \ddots & \ddots & 0 \\
  a_{n-1} & \cdots & a_2 & a_1 & c \\
  b & 0 & \cdots & 0 & (1-n)a_1
\end{pmatrix}.
\]

By using the multiplication of matrices by blocks, we obtain for any pair \( u, u' \) in \( z_{\text{sub}} \)

\[
[u, u'] = \begin{pmatrix}
  0 & 0 & 0 & \cdots & 0 \\
  0 & 0 & 0 & \ddots & \\
  \vdots & \vdots & \ddots & \ddots & 0 \\
  cb' - bc' & \cdots & \cdots & 0 & n(a_1c' - a_1c) \\
  n(ba_1' - a_1b') & 0 & \cdots & 0 & 0
\end{pmatrix}.
\]

If \( p \) does not divide \( n \), the defining polynomials of the commutator are \( a_1c' - a_1c, cb' - bc' \), and \( ba_1' - b'a_1 \). This implies that the variety \( C_r(z_{\text{sub}}) \) is defined by the collection of polynomials \( \{x_iy_j - x_jy_i, y_iz_j - y_jz_i, x_iz_j - x_jz_i \mid 1 \leq i \leq j \leq r\} \) in \( k[x_1, y_1, z_1, t_{ij}] \mid 1 \leq i \leq r, 2 \leq j \leq n - 1 \}. So we can identify \( C_r(z_{\text{sub}}) \) with the determinantal variety \( D_2(X) \) where \( X \) is the matrix

\[
X = \begin{pmatrix}
  x_1 & x_2 & \cdots & x_r \\
  y_1 & y_2 & \cdots & y_r \\
  z_1 & z_2 & \cdots & z_r
\end{pmatrix}.
\]

This identification implies the following isomorphism of varieties

\[
C_r(z_{\text{sub}}) \cong D_2(X) \times k^{(n-2)r}.
\]

On the other hand, if \( p \) divides \( n \), then the commutator is defined by \( cb' - c'b \). This gives us the following

\[
C_r(z_{\text{sub}}) \cong D_2(Y) \times k^{(n-1)r}
\]

where \( Y \) is the matrix of indeterminates defined by

\[
Y = \begin{pmatrix}
  x_1 & x_2 & \cdots & x_r \\
  y_1 & y_2 & \cdots & y_r
\end{pmatrix}.
\]

Hence \( D_2(Y) \) is of dimension \( r + 1 \) and so we obtain \( \dim C_r(z_{\text{sub}}) = (n - 1)r + r + 1 = nr + 1 \).

Other results immediately follow from Proposition \[2.4.1\] \( \square \)

**Remark 3.1.2.** This result can not be generalized to the centralizer of an arbitrary nilpotent element. In particular, there is a nilpotent element \( e \) in \( g \) such that \( C_2(z(e)) \) is reducible (hence so is \( C_r(z(e)) \)) \[Ya1\].

3.2. An application. Suppose \( g \) is an arbitrary simple Lie algebra. We first study a connection between the dimension of the commuting variety \( C_r(g) \) and that of a certain mixed commuting variety.

Define a mixed commuting variety by

\[
C(\mathcal{N}, g^{r-1}) = \{(v_1, \ldots, v_r) \mid v_1 \in \mathcal{N}, (v_2, \ldots, v_r) \in g^{r-1}, \ [v_i, v_j] = 0 \text{ with } 1 \leq i \leq j \leq r\},
\]

which is a subvariety of \( C_r(g) \) with nilpotency condition for the first factor. Note that

\( \dim C_r(g) \leq \dim C(\mathcal{N}, g^{r-1}) + \rank(g) \). \hspace{1cm} (1)

Hence, knowing the dimension of \( C(\mathcal{N}, g^{r-1}) \) allows one to obtain an upper bound on \( \dim C_r(g) \). Note also that in the case \( r = 2 \), Baranovsky used this variety to compute the dimension of \( C_2(\mathcal{N}) \).
Assume in this subsection that \( p \neq 3 \). We aim to compute the dimensions of \( C_r(\mathfrak{sl}_3) \) and \( C_r(\mathfrak{gl}_3) \). We begin with a lemma.

**Theorem 3.2.1.** For each \( r \geq 2 \), we have \( \dim \mathcal{N}_r(\mathfrak{sl}_3) = 2r + 4 \) and \( \dim C_r(\mathfrak{sl}_3) = 2r + 6 \).

**Proof.** Note that \( C_r(\mathfrak{sl}_3) \) contains a component \( G \cdot t^r \), where \( t \) is a Cartan subalgebra of \( \mathfrak{sl}_3 \). It is easy to see that the dimension of this component is \( 2r + 6 \). So it suffices to show that \( \dim C_r(\mathfrak{sl}_3) \leq 2r + 6 \). We proceed by induction and assume that \( \dim C_{r-1}(\mathfrak{sl}_3) \leq 2(r-1) + 6 = 2r + 4 \). As \( \mathcal{N} \) contains three nilpotent orbits: \( \mathcal{O}_{\text{reg}}, \mathcal{O}_{\text{sub}}, \) and \( 0 \), we have the following decomposition

\[
\dim C_r(\mathfrak{sl}_3) = \dim C_1(\mathfrak{sl}_3) + \dim C_{r-1}(\mathfrak{sl}_3) + \dim C_{r-2}(\mathfrak{sl}_3).
\]

where \( \dim C_{r-1}(\mathfrak{sl}_3) = 2(r-1) + 6 \) by Theorem 3.1.1. So the dimension of \( C(\mathcal{N}, \mathfrak{sl}_3^j) \) is \( 2r + 4 \), which completes our proof.

**Corollary 3.2.2.** For each \( r \geq 2 \), we have \( \dim C_r(\mathfrak{gl}_3) = 3r + 6 \).

**Proof.** It follows from Theorem 3.2.1 and [N] Theorem 4.2.1. \( \square \)

**Remark 3.2.3.** Note first that our computation does not rely on the irreducibility of \( C_r(\mathfrak{gl}_3) \) for each \( r \geq 1 \). In the case \( p = 3 \), Remark 3.1.2 shows that \( C_r(\mathfrak{sl}_3) \) is of dimension at least \( 3r + 2 \). This shows that \( C_r(\mathfrak{sl}_3) \) is reducible when \( r > 4 \).

## 4. Mixed Commuting Varieties

We apply in this section a new technique to compute the dimension of mixed commuting varieties over various closed sets in \( \mathfrak{sl}_3 \). Our calculations are based on the dimension for a certain class of varieties defined by minors of a matrix of indeterminates.

To begin we set

\[
C_{i,j,m} = C(\underbrace{\mathcal{O}_{\text{sub}}, \ldots, \mathcal{O}_{\text{sub}}}_{i \text{ times}}), \underbrace{\mathcal{N}, \ldots, \mathcal{N}}_{j \text{ times}}, \underbrace{\mathfrak{sl}_3, \ldots, \mathfrak{sl}_3}_{m \text{ times}}).
\]

Our goal is to compute the dimension of \( C_{i,j,m} \) for every set of non-negative integers \( i, j, m \).

### 4.1. Note that \( \dim C_{i,j,m} \) is known in the following cases:

- If \( i = j = 0 \) then \( \dim C_{i,j,m} = \dim C_m(\mathfrak{sl}_3) = 2m + 6 \),
- If \( i = m = 0 \) then \( \dim C_{i,j,m} = \dim C_j(\mathcal{N}) = 2j + 4 \),
- If \( j = m = 0 \) then \( \dim C_{i,j,m} = \dim C_i(\mathcal{O}_{\text{sub}}) = 2i + 2 \)

by Theorem 3.2.1 and [N] Theorems 7.1.2 and 7.2.3. When \( i = 0 \), the dimension of \( C_{i,j,m} \) can be easily computed as follows.

**Proposition 4.1.1.** For \( j, m \geq 1 \), we have \( \dim C_{0,j,m} = 2(j + m) + 4 \). Consequently, the variety \( C_{0,j,m} \) is never irreducible.

**Proof.** Observe that

\[
\dim C_{j+m}(\mathcal{N}) \leq \dim C_{0,j,m} \leq \dim C(\mathcal{N}, \mathfrak{sl}_3^{j+m-1}).
\]

From earlier we have

\[
2(j + m) + 4 \leq \dim C_{0,j,m} \leq 2(j + m) + 4.
\]
This gives us the dimension of $C_{0,j,m}$. It indicates that $C_{j+m}(\mathcal{N})$ is a proper irreducible component of $C_{0,j,m}$. Hence, the reducibility of $C_{0,j,m}$ is proved. \hfill\Box

4.2. Fix $v_{\text{sub}}$, the canonical Jordan block matrix corresponding to the partition $[2,1]$ of 3. Then the centralizer of $v_{\text{sub}}$ in $\mathfrak{sl}_3$ is

$$\begin{bmatrix} x & 0 & 0 \\ y & x & t \\ z & 0 & -2x \end{bmatrix} \begin{bmatrix} x, y, z, t \in k \end{bmatrix}.$$  

We recall results in [N] on the intersections of $z_{\text{sub}}$ with $\overline{O_{\text{sub}}}$ or $\mathcal{N}$ respectively which play important roles in our calculations.

**Proposition 4.2.1.** [N, Lemma 7.2.2]** There are identities**

$$z_{\text{sub}} \cap \mathcal{N} = \begin{bmatrix} 0 & 0 & 0 \\ y & 0 & t \\ z & 0 & 0 \end{bmatrix} \begin{bmatrix} y, z, t \in k \end{bmatrix},$$

$$z_{\text{sub}} \cap \overline{O_{\text{sub}}} = \begin{bmatrix} 0 & 0 & 0 \\ y & 0 & 0 \\ z & 0 & 0 \end{bmatrix} \begin{bmatrix} y, z \in k \end{bmatrix} \cup \begin{bmatrix} 0 & 0 & 0 \\ y & 0 & t \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y, t \in k \end{bmatrix} =: V_1 \cup V_2.$$

Moreover, if $u, v \in V_1 \cup V_2$ then

$$[u, v] = 0 \iff u, v \in V_1 \text{ or } u, v \in V_2.$$

4.3. **Some results on determinantal varieties.** Before investigating the dimension of the mixed commuting variety $C_{i,j,m}$, we need to prove some results related to dimensions of determinantal varieties.

**Theorem 4.3.1.** Let $X_{i,j,m}$ be the matrix of indeterminates

$$\begin{bmatrix} x_1 & \cdots & x_i & x_{i+1} & \cdots & x_{i+j} & x_{i+j+1} & \cdots & x_{i+j+m} \\ 0 & \cdots & 0 & y_1 & \cdots & y_j & y_{j+1} & \cdots & y_{j+m} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & z_1 & \cdots & z_m \end{bmatrix}.$$  

Then $\dim V(I_2(X_{i,j,m})) = \max\{m + 2, j + m + 1, i + j + m\}$.

**Proof.** Let

$$A = \{x_1, \ldots, x_i\} \quad A' = \{y_1, \ldots, y_{j+m}, z_1, \ldots, z_m\}$$

$$B = \{y_1, \ldots, y_j\} \quad B' = \{x_1, \ldots, x_i, z_1, \ldots, z_m\},$$

$$C = \{z_1, \ldots, z_m\} \quad C' = \{x_1, \ldots, x_{i+j}, y_1, \ldots, y_j\}.$$  

Then we define $XY = \{xy \mid x \in X, y \in Y\}$. It is observed that the sets $AA', BB'$, and $CC'$ are in $I_2(X_{i,j,m})$. These sets of monomials give us the following decomposition

$$V(I_2(X_{i,j,m})) = V(I_2(X_{i+j+m,0,0})) \cup V(I_2(X_{0,j+m,0})) \cup V(I_2(X_{0,0,m})).$$

By Proposition 2.4.1 we have

$$\dim V(I_2(X_{i+j+m,0,0})) = i + j + m,$$

$$\dim V(I_2(X_{0,j+m,0})) = j + m + 1,$$

$$\dim V(I_2(X_{0,0,m})) = m + 2.$$  

Hence, the result follows. \hfill\Box
This computation can be generalized to calculate the dimension of the determinantal variety $W$ defined by $2$ by $2$ minors of the matrix

\[
\begin{pmatrix}
  x_{11} & \cdots & x_{1,a_1} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & x_{1,a_m} \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  x_{b_1,1} & \cdots & x_{b_1,a_1} & \cdots & x_{b_1,a_1+1} & \cdots & \cdots & \cdots & \cdots & x_{b_1,a_m} \\
  0 & \cdots & 0 & x_{b_1+1,a_1+1} & \cdots & \cdots & \cdots & \cdots & \cdots & x_{b_1,a_m-a_1} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & 0 & \cdots & 0 & x_{b_2+1,a_2+1} & \cdots & \cdots & x_{b_2+1,a_m-a_1-a_2} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  0 & \cdots & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & x_{b_n,a_m-a_1-1} \\
  \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
  \vdots & \ddots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\end{pmatrix}
\]

Denote this matrix by $X(a_1, \ldots, a_m, b_1, \ldots, b_n)$. Then we can decompose $W$ into the union of determinantal varieties defined by $2$ by $2$ minors of the following matrices:

\[
X(a_1, \ldots, a_m, b_1, 0, \ldots, 0), \; X(0, a_2, \ldots, a_m, b_1, b_2, 0, \ldots, 0), \; \ldots, \; X(0, \ldots, 0, a_m, b_1, \ldots, b_n).
\]

Hence, we obtain the following result.

**Theorem 4.3.2.** Given $a_1 < a_2 < \cdots < a_m$, $b_1 < b_2 < \cdots < b_n$, and the matrix $W$ defined as above. Then

\[
\dim V(I_2(W)) = \max\{a_m + b_1 + 1, a_m - a_1 + b_2 + 1, \ldots, a_m - \sum_{i=1}^{m-1} a_i + b_n + 1\}.
\]

It would be nice if one can generalize this result for larger minors of $W$.

### 4.4. Main Theorem

We can now compute the dimension of the mixed commuting variety $C_{i,j,m}$. If $i = 0$, then the answer is obtained from Proposition 4.1.1. So we assume that $i \geq 1$.

**Theorem 4.4.1.** For each $i \geq 1$, let $N_{i,j,m} = \max\{m + 2, j + m + 1, i + j + m\}$. Then we have

\[
\dim C_{i,j,m} = N_{i-1,j,m} + i + j + m + 3.
\]

**Proof.** Let first consider the decomposition

\[ (3) \]

\[
C_{i,j,m} = G \cdot (t_{\text{sub}}, D_{i-1,j,m}) \cup 0 \times C_{i-1,j,m}
\]

where $D_{i-1,j,m} = C(z_{\text{sub}} \cap C_{\text{sub}}, \ldots, z_{\text{sub}} \cap C_{\text{sub}}, z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}, z_{\text{sub}}, \ldots, z_{\text{sub}})$. By Proposition 4.2.1 we can further decompose

\[
D_{i-1,j,m} = C(V_1 \cup V_2, \ldots, V_1 \cup V_2, z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}, z_{\text{sub}}, \ldots, z_{\text{sub}})
\]

\[
= C(V_1, \ldots, V_1, z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}, z_{\text{sub}}, \ldots, z_{\text{sub}}) \cup
\]

\[
C(V_2, \ldots, V_2, z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}, z_{\text{sub}}, \ldots, z_{\text{sub}}).
\]

Analyzing the commutators of $V_1$ or $V_2$ with $z_{\text{sub}} \cap \mathcal{N}$ and $z_{\text{sub}}$, we have the following identities

\[
C(V_1, \ldots, V_1, z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}, z_{\text{sub}}, \ldots, z_{\text{sub}}) = V(I) \times k^{i+j+m-1}
\]

\[
C(V_2, \ldots, V_2, z_{\text{sub}} \cap \mathcal{N}, \ldots, z_{\text{sub}} \cap \mathcal{N}, z_{\text{sub}}, \ldots, z_{\text{sub}}) = V(J) \times k^{i+j+m-1}
\]
where the affine space \( k^{i+j+m-1} \) is from the freeness of \( y_1, \ldots, y_{i+j+m-1} \), \( I \) and \( J \) are respectively the ideals generated by \( 2 \times 2 \) minors of the following matrices

\[
\begin{pmatrix}
z_1 & \cdots & z_{i-1} & z_i & \cdots & z_{i+j-1} & z_{i+j} & \cdots & z_{i+j+m-1} \\
0 & \cdots & 0 & t_1 & \cdots & t_j & t_{j+1} & \cdots & t_{j+m} \\
0 & \cdots & 0 & 0 & \cdots & 0 & x_1 & \cdots & x_m
\end{pmatrix}
\]

Theorem 4.3.1 gives us

\[
\dim V(I) = \dim V(J) = N_{i-1,j,m}
\]

so that

\[
\dim D_{i-1,j,m} = i + j + m - 1 + N_{i-1,j,m}.
\]

Hence we have

\[
\dim G \cdot (v_{\text{sub}}, D_{i-1,j,m}) = 4 + \dim D_{i-1,j,m} = i + j + m + 3 + N_{i-1,j,m}.
\]

It remains to prove that \( \dim 0 \times C_{i-1,j,m} \leq i + j + m + 3 + N_{i-1,j,m} \). Indeed, if \( i = 1 \), then it equals to \( 2(j + m) + 4 \) (by Proposition 4.1.1) which is \( \leq i + j + m + 3 + N_{i-1,j,m} \) since \( N_{i-1,j,m} \geq j + m + 1 \).

For \( i > 1 \), we have by induction that

\[
\dim C_{i-1,j,m} = i - 1 + j + m + 3 + N_{i-2,j,m} \leq i + j + m + 3 + N_{i-1,j,m}.
\]

Finally, we have shown that \( \dim C_{i,j,m} = i + j + m + 3 + N_{i-1,j,m} \) as desired.

Next we show that mixed commuting varieties are usually not irreducible. We start with a lemma.

**Lemma 4.4.2.** For each \( i \geq 2 \), the variety \( C_{i,0,0} = C_i(\overline{\mathfrak{O}_{\text{sub}}} \mathfrak{a}) \) is reducible of dimension \( 2r + 2 \).

**Proof.** This is just a corollary of [N] Theorem 7.2.3]. In particular, we have the following irreducible decomposition

\[
C_{i,0,0} = G \cdot (v_{\text{sub}}, V_1, \ldots, V_1) \cup G \cdot (v_{\text{sub}}, V_2, \ldots, V_2)
\]

where \( V_1 \) and \( V_2 \) are defined in Proposition 4.2.1. \( \square \)

**Theorem 4.4.3.** For each \( i, j, m \geq 0 \), the mixed commuting variety \( C_{i,j,m} \) is irreducible if and only if \( i, j, m \) satisfy one of the following conditions:

1. \( i = j = 0 \),
2. \( i = m = 0 \),
3. \( i = 1, j = m = 0 \).

**Proof.** The conditions (1), (2), and (3) in the theorem are equivalent to the cases in which the variety \( C_{i,j,m} \) is either \( C_m(\mathfrak{sl}_3) \), or \( C_j(\mathcal{N}) \), or \( \overline{\mathfrak{O}_{\text{sub}}} \mathfrak{a} \). It is known that these varieties are irreducible. Indeed, the variety \( C_j(\mathcal{N}) \) is irreducible by Theorem 7.1.2 in [N], and the variety \( C_m(\mathfrak{sl}_3) \) is irreducible by Theorem 4.2.1 in [N] and the fact that \( C_m(\mathfrak{gl}_3) \) is irreducible [KN, Gn]. From the decomposition (3) in Theorem 4.4.1 we have

\[
C_{i,j,m} = G \cdot (v_{\text{sub}}, D_{i-1,j,m}) \cup 0 \times C_{i-1,j,m}
\]

where

\[
G \cdot (v_{\text{sub}}, D_{i-1,j,m}) = G \cdot (v_{\text{sub}}, D_{i-1,j,m}) \cup \{0\}.
\]

So if \( i \geq 1 \) and \( j \neq 0 \) or \( m \neq 0 \), we always have \( C_{i,j,m} \) is reducible. The case \( i > 1 \) and \( j = m = 0 \) was proved in the previous lemma. On the other hand, Proposition 4.1.1 shows that \( C_{i,j,m} \) is reducible in the case \( i = 0 \) and \( j, m \geq 1 \). \( \square \)

**Remark 4.4.4.** This result also indicates that mixed commuting varieties are rarely normal as all irreducible components contain the origin. In other words, if a mixed commuting variety is reducible, it is not normal. This behavior is analogous with that of mixed commuting varieties over \( \mathfrak{sl}_2 \) and its nullcone in [N, Proposition 6.1.1].
5. Applications to support varieties for Frobenius kernels

5.1. Support varieties. Let $G$ be a simple algebraic group defined over $k$ (we assume that $p \geq 3$ in this section). For each $r \geq 1$, let

$$H^\bullet(G_r, k) = \bigoplus_{i \geq 0} H^i(G_r, k), \quad H^{2\bullet}(G_r, k) = \bigoplus_{i \geq 0} H^{2i}(G_r, k).$$

Under the cup product, $H^{2\bullet}(G_r, k)$ is a commutative ring. Given a finite dimensional $G$-module $M$, we consider $\text{Ext}^\bullet_G(M, M)$ as a $H^{2\bullet}(G_r, k)$-module with the action induced by the cup product. Then the support variety of $M$, denoted by $V_{G_r}(M)$, is the variety of the annihilator of $\text{Ext}^\bullet_G(M, M)$ in the ring $H^{2\bullet}(G_r, k)$. Note that

$$V_{G_r}(k) = \text{Spec } H^{2\bullet}(G_r, k)_{\text{red}} = C_r(N)$$

when $p \geq h$ [SFB1] Theorem 5.2, [CLN] 1.1.

5.2. Connection to mixed commuting varieties. For sufficiently large values of $p$, one can compute the support variety of the simple $G$-module $L(\lambda)$ as follows.

Proposition 5.2.1. [So] Theorem 3.2 Let $G$ be a classical simple algebraic group. Suppose that $p > hc$. Let $\lambda$ be a weight with $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_q$, $\lambda_i \in X_1(T)$. Then for each $r \geq 1$, we have

$$V_{G_r}(L(\lambda)) = \{(\beta_0, \ldots, \beta_{r-1}) \in C_r(N) \mid \beta_i \in V_{G_1}(L(\lambda_{r-i-1}))\}.$$

Here the number $c$ is an integer defined for every type of $G$ as in [So] Section 3. Recall that if $G$ is of type $A_n$, then $c = \left(\frac{4n+1}{4}\right)^2$. In the case that the Lusztig character formula holds for all restricted dominant weights, a result of Drupieski, Nakano, and Parshall [DNP] Theorem 4.1 can be combined to give an explicit description for $V_{G_r}(L(\lambda))$.

Proposition 5.2.2. Let $G$ be a classical simple algebraic group. Suppose that $p > hc$ and assume that the Lusztig character formula\(^2\) holds for all restricted dominant weights. Then for $\lambda \in X^+$ with $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_q$, $\lambda_i \in X_1(T)$,

$$V_{G_r}(L(\lambda)) = \{(\beta_0, \ldots, \beta_{r-1}) \in C_r(N) \mid \beta_i \in G \cdot u_{J_i} \}$$

where $J_i \subset \Pi$ such that $w(\Phi_{\lambda_i}) = \Phi_J$ for each $J_i$.

This result shows that $V_{G_r}(L(\lambda))$ is a mixed commuting variety where every component of an $r$-tuple is in a closed subvariety of the nilpotent cone $N$. So it is potential to be reducible for most of the cases.

5.3. Rank 2 case. We are going to use our calculations in previous sections to investigate the support variety $V_{G_r}(L(\lambda))$ when $G$ is of type $A_2$. We first need some notation.

Let $\lambda$ be a weight in $X$ and set

$$\Phi_{\lambda,p} = \{\beta \in \Phi \mid (\lambda + p, \beta^\vee) \in p\mathbb{Z}\}.$$

Suppose that the prime $p$ is good for $G$. We recall that $\lambda$ is called $p$-regular if $\Phi_{\lambda,p} = \emptyset$, otherwise it’s called $p$-singular. The following result classifies the behaviors of support varieties for $L(\lambda)$ up to its regularity in the case of simple algebraic groups of rank 2.

Proposition 5.3.1. [NPV] Corollary 6.6.1] Let $G$ be a simple algebraic group of rank 2 and $p$ good.

(a) If $\lambda$ is a $p$-singular weight, then $V_{G_1}(L(\lambda)) = \overline{\mathcal{O}}_{\text{sub}}$.

(b) If $\lambda$ is a $p$-regular weight, then $V_{G_1}(L(\lambda)) = V_{G_1}(k) = N$.

\(^2\)See formula (4.0.1) in [DNP]

\(^3\)Let $\lambda = w \cdot \lambda^-, \lambda^- \in N_\lambda, w \in W_p$ and $w$ is minimal dominant for $\lambda^-$. 
Now suppose that $G$ is of type $A_2$ and $p \geq 7$ (i.e., $p > hc$). Let $\lambda$ be a weight such that $
abla = \lambda_0 + p\lambda_1 + \cdots + p^r\lambda_q$ with $\lambda_i \in X_1(T)$. Let $a_{\lambda_r}, b_{\lambda_r}$ be the number of singular weights and regular weights respectively in \{\lambda_0, \ldots, \lambda_{r-1}\}. Then by the propositions above, we have for each $r \geq 1$

\[ V_{G_r}(L(\lambda)) = C(O_{\text{sub}}(\lambda_{0}), \ldots, O_{\text{sub}}(\lambda_{r-1}, N_r, \ldots, N_r)) = C_{a_{\lambda_r}, b_{\lambda_r}, 0}. \]

This is a special case of mixed commuting varieties appearing in previous sections. Hence we know the dimensional and irreducible behaviors of the support variety $V_{G_r}(L(\lambda))$.

**Theorem 5.3.2.** Let $G$ be of type $A_2$ and $p \geq 6$. Suppose $\lambda$ is a weight in $X^+$. Then for each $r \geq 1$, the support variety $V_{G_r}(L(\lambda))$ can be identified with the mixed commuting variety $C_{a_{\lambda_r}, b_{\lambda_r}, 0}$. Furthermore,

\[
\dim V_{G_r}(L(\lambda)) = \begin{cases} 
2b_{\lambda_r} + 4 & \text{if } a_{\lambda_r} = 0, \\
2(a_{\lambda_r} + b_{\lambda_r}) + 3 & \text{if } a_{\lambda_r} = 1, \\
2(a_{\lambda_r} + b_{\lambda_r}) + 2 & \text{if } a_{\lambda_r} > 1.
\end{cases}
\]

**Theorem 5.3.3.** Under the same assumption as in the previous theorem, for each $r \geq 2$, the support variety $V_{G_r}(L(\lambda))$ is irreducible if and only if $a_{\lambda_r} = 0$. In other words, $V_{G_r}(L(\lambda))$ is irreducible if and only if there is no singular weight in the decomposition of $\lambda$.

**Proof.** It immediately follows from Theorem 5.3.2 and Theorem 4.4.3. \qed

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