Gravity Probe C\textsubscript{lock} – Probing the gravitomagnetic field of the Earth by means of a clock experiment*

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Abstract

We outline a mission with the aim of directly detecting the gravitomagnetic field of the Earth. This mission is called Gravity Probe C. Gravity Probe C\textsubscript{lock} is based on a recently discovered and surprisingly large gravitomagnetic clock effect. The main idea is to compare the proper time of two standard clocks in direct and retrograde orbits around the Earth. After one orbit the proper time difference of two such clocks is predicted to be of the order of $2 \times 10^{-7}$ s. The conceptual difficulty to perform Gravity Probe C is expected to be comparable to that of the Gravity Probe B–mission.

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1 Introduction

Within our solar system relativistic gravity theories can be tested only in the weak-field limit. The simple reason for this is the absence of strongly gravitating objects in this part of our universe. It follows that relativistic gravity experiments in the solar system produce only small corrections to the Newtonian theory which are, typically, difficult to measure.

In order to describe relativistic gravity on macroscopic scales we usually use Einstein’s (classical) general relativity. It is well-known that in a weak field approximation the structure of general relativity becomes formally analogous to that of Maxwell’s special relativistic electrodynamics (see Ref. [1], for example). It is then possible to describe far fields, which are generated by isolated charge and current distributions, by means of multipole expansions. This circumstance is familiar from Maxwell’s theory where the corresponding multipole moments are divided into electric ones, due to electric charge distributions, and magnetic ones, due to electric current distributions. An analogous terminology is used in the linearized Einstein theory where gravitoelectric and gravitomagnetic fields are introduced according to whether they are due to mass distributions or mass current distributions, respectively. The standard tests of Einstein’s general relativity, like perihelion precession of Mercury, light deflection of the Sun, gravitational redshifts, and radar time delays, are explained by relativistic gravitoelectric corrections. However, they yield no direct information on relativistic gravitomagnetic corrections.

In general relativity there are also “classic” gravitomagnetic effects. They can be accounted for by what is commonly called the Lense–Thirring effect, i.e. the “dragging of inertial frames” by a spinning mass. The Lense–Thirring effect was discovered in 1918 [2] but turned out to be hard to verify: It was calculated by Schiff in 1960 [3] how to measure this effect by means of the precession of an orbiting gyroscope. To detect this precession is the main objective of the Gravity Probe B–mission which is awaited to be launched by NASA at the end of 1999: Highly sensitive gyroscopes will be carried in a drag free–satellite around the Earth and are expected to measure directly the gravitomagnetic field of the Earth [4]. A recent study indicates a verification of the Lense–Thirring effect by means of the numerical evaluation of data which were obtained from laser–ranging observations of the satellites LAGEOS and LAGEOS II [5]. Here the orbital planes of the satellites can be viewed as very large gyroscopes which are embedded in the gravitomagnetic field of the earth. Then the predicted Lense–Thirring precession of the orbital planes is analogous to that of an orbiting gyroscope, and this is what was numerically evaluated.

In this article we propose a mission, in the following called Gravity Probe C, which, similar to Gravity Probe B, is designed to directly measure the gravitomagnetic field of the Earth. We intend to make use of a remarkable gravitomagnetic clock effect that was recently pointed out by Cohen and Mashhoon [6]: Consider the difference in proper time of two standard clocks, following identical orbits around a rotating body in direct and

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1 This is true, at least, in the approximation used.
retrograde motion, respectively. After a proper azimuthal period this time difference turns out to be proportional to $J/Mc^2$, with $J$ the angular momentum and $M$ the mass of the rotating body. In the case of two standard clocks orbiting the Earth this time difference amounts to the order of $2 \times 10^{-7} \text{s}$. It seems to be not well-known that a gravitomagnetic clock effect of this order exists. Therefore we will give a complete derivation of it in the next Section 2. To actually measure this effect is the objective of Gravity Probe C. An experimental realization of Gravity Probe C is expected to be technically demanding. This will be discussed in Section 3.

2 A remarkable gravitomagnetic clock effect

The exterior spacetime of a system with mass $M$ and specific angular momentum $a = J/M$ can be described by the Kerr geometry. The Kerr geometry is an exact solution of the vacuum field equation of general relativity. In Boyer–Lindquist coordinates $(t, r, \vartheta, \varphi)$ the Kerr geometry takes the form

$$ds^2 = -dt^2 + \Sigma \left( \frac{1}{\Delta} dr^2 + d\vartheta^2 \right) + (r^2 + a^2) \sin^2 \vartheta \, d\varphi^2 + 2Mr \Sigma \left( dt - a \sin^2 \vartheta \, d\varphi \right)^2.$$  (1)

Here we introduced the standard abbreviations $\Sigma := r^2 + a^2 \cos^2 \vartheta$ and $\Delta := r^2 - 2Mr + a^2$. Except otherwise indicated we use units such that the gravitational constant $G$ and the velocity of light $c$ are set to unity, $G = c = 1$. The following derivation of the gravitomagnetic clock effect under consideration is based on Ref. [6].

What we first want to calculate is the proper time as shown by a standard clock which follows a geodesic in the Kerr geometry. To keep things simple we specialize on a circular, geodesic orbit, i.e., we put $r = \text{const}$ and $\vartheta = \frac{\pi}{2}$. The geodesic equation which interrelates the remaining coordinates $t$ and $\varphi$ is given by

$$\frac{d^2 r}{d\tau^2} + \Gamma_{ij}^r \frac{dx^i}{d\tau} \frac{dx^j}{d\tau} = 0.$$  (2)

(We note the use of the Einstein summation convention which involves the indices $i$ and $j$.) The Christoffel symbols $\Gamma_{ij}^r$ are determined from the metric (1). The geodesic equation (2) turns out to be

$$dt^2 - 2a \, d\varphi \, dt + \left( a^2 - \frac{r^3}{M} \right) \, d\varphi^2 = 0.$$  (3)

This quadratic equation can be solved immediately. The result is

$$\frac{dt}{d\varphi} = a \pm \left( \frac{r^3}{M} \right)^{\frac{1}{2}} =: a \pm \frac{1}{\omega_0}.$$  (4)

Here we introduced the symbol $\omega_0$ to denote the Kepler (and Schwarzschild!) result

$$\frac{d\varphi}{dt} = \pm \left( \frac{M}{r^3} \right)^{\frac{1}{2}} =: \omega_0.$$  (5)
The square of a proper time interval \( d\tau^2 \) is given, up to a factor \( c^2 \) and a sign, by the line element (1). For a circular, equatorial orbit we find from (1) the relation

\[
\left( \frac{d\tau}{d\varphi} \right)^2 = \left( 1 - \frac{2M}{r} \right) \left( \frac{dt}{d\varphi} \right)^2 + \frac{4Ma}{r} \frac{dt}{d\varphi} - r^2 - a^2 \left( 1 + \frac{2M}{r} \right).
\]  

(6)

We substitute (4) and obtain

\[
\left( \frac{d\tau}{d\varphi} \right)^2 = \frac{1}{\omega_0^2} \left( 1 - \frac{3M}{r} \pm 2a\omega_0 \right),
\]  

(7)

or

\[
\frac{d\tau}{d\varphi} = \pm \frac{1}{\omega_0} \sqrt{1 - \frac{3M}{r} \pm 2a\omega_0}.
\]  

(8)

Plus and minus sign apply to direct and retrograde orbits, respectively. For one closed orbit (\( \varphi \rightarrow \varphi + 2\pi \)) it is now trivial to obtain the integrated proper time as

\[
\tau_{\pm} = \frac{2\pi}{\omega_0} \sqrt{1 - \frac{3M}{r} \pm 2a\omega_0}.
\]  

(9)

This result is exact.

We are interested in the proper time difference \( \tau_+ - \tau_- \). We Taylor-expand the square root of (8) in powers of \( a \) and obtain to first order

\[
\tau_+ - \tau_- \approx 4\pi a = 4\pi \frac{J}{M},
\]  

(10)

or, in natural units,

\[
\tau_+ - \tau_- = 4\pi \frac{J}{Mc^2} + O(c^{-4}) + \ldots.
\]  

(11)

This formula manifests the gravitomagnetic clock effect we are interested in. It is interesting to note that, in this approximation, the difference \( \tau_+ - \tau_- \) is independent of the radius \( r \) of the orbit and the gravitational constant \( G \).

In case of orbits around the Earth we put \( M \approx 6 \times 10^{24} \) kg, \( J \approx 10^{34} \) kg m\(^2\) s\(^{-1}\), and find

\[
\tau_+ - \tau_- \approx 2 \times 10^{-7} \text{ s}.
\]  

(12)

This time difference is surprisingly large. Time shifts due to the gravitomagnetic field of the Earth are widely believed to be of much lower order. Indeed, suppose the time difference of a direct and retrograde moving clock is taken at a fixed time, say, after one Kepler period \( T_0 = 2\pi/\omega_0 \). Then it is straightforward to show that for a circular equatorial orbit,

\[
\tau_+ - \tau_- = 12\pi \frac{GJ}{c^5 r} + O(c^{-6}) + \ldots.
\]  

(13)

\(^2\)This observation is discussed in more detail in Ref. [7].
As an example, we set \( r \approx 7000 \text{ km} \) and obtain

\[
\tau_+ - \tau_- \approx 3 \times 10^{-16} \text{s}. \tag{14}
\]

This is nine orders of magnitude smaller than the time difference (12). We recall that the result (12) presupposes that the time difference of the two clocks is taken with respect to a fixed angle \( \varphi \) (i.e., after each clock has covered an azimuthal interval of \( 2\pi \)) and not with respect to a fixed time. It is very important to note this conceptual difference in order to avoid confusion.

The gravitomagnetic clock effect of this section can be generalized to circular orbits with a nonvanishing inclination to the equatorial plane, i.e., to circular orbits which are not restricted to \( \vartheta = \frac{\pi}{2} \) \cite{7, 8}. It turns out that with increasing inclination the time difference \( \tau_+ - \tau_- \) becomes smaller and finally vanishes in case of a polar orbit. Analytic calculations of \( \tau_+ - \tau_- \) for an arbitrary (eccentric) orbit have not been conducted, yet. It is expected that the corresponding geodesic equations can only be solved by means of numerical integration.

3 Realization of Gravity Probe C

In principle, it is a trivial task to measure a time difference of \( 2 \times 10^{-7} \text{s} \) with today’s technology. However, an experimental verification of the gravitomagnetic clock effect does not only require the measurement of the time difference between two well-defined events up to an accuracy of \( 2 \times 10^{-7} \text{s} \). It is, in fact, the proper time along the direct and retrograde orbits that is used as a clock. Therefore, it is not sufficient to send two highly accurate and stable clocks into space and let them orbit in opposite directions. The orbits themselves have to be highly accurate and stable as well. We recall that in order to obtain the time difference (12) we have to subtract two periods, each of which represents the sum of a Kepler period and a much smaller relativistic correction. Under the assumption of identical orbits, the Kepler periods and their gravitoelectric corrections cancel upon subtraction while the gravitomagnetic contributions add up, yielding the actual clock effect under consideration here. Disturbances of the orbits will in general change the Kepler periods of the orbiting clocks. It follows that in this case the Kepler periods will not exactly cancel but may exhibit a significant time difference.

For an (equatorial and circular) orbit of Keplerian period \( T_0 \), the relative gravitomagnetic variation of the orbital period is given by

\[
\frac{\tau_+ - \tau_-}{T_0} = \frac{2J}{Mc^2} \left( \frac{GM}{r^3} \right)^{1/2}. \tag{15}
\]

In case of a near-Earth orbit this expression amounts to about \( 4 \times 10^{-11} \). It was noted in \cite{6} that this expression is exactly the same as the the relative gravitomagnetic precession angle of a Gravity Probe B–gyroscope: The precession angle \( \Omega_P \) (per \( 2\pi \) radians) of a
gyroscope after one orbit is predicted to be

\[ \Omega_P = \frac{2 G J}{c^2 r^3}, \]  

(16)
such that

\[ \frac{\Omega_P}{\omega_0} = \frac{2 G J}{c^2 r^3} \left( \frac{G M}{r^3} \right)^{-1/2}. \]  

(17)

This is identical to equation (15). Therefore it is expected that the difficulty of performing the Gravity Probe C mission is essentially equivalent to that of Gravity Probe B.

### 3.1 Mission design

Gravity Probe C is simple from a conceptual point of view. To perform Gravity Probe C requires to collect precise data from direct and retrograde orbits around the Earth. It follows from the previous discussion of the gravitomagnetic clock effect that these orbits should be as circular and as equatorial as possible. In order to collect the data required one could put atomic clocks on board of satellites and send them on specific orbits.

One should keep in mind that future space missions are expected to carry highly stable and accurate clocks [9]. Also there already exist satellite–based measuring–devices, like the Global Positioning System (GPS), which make it possible to determine precisely the positions of orbiting clocks in space. Therefore, Gravity Probe C could rely, in principle, on data which are collected in the context of other space missions.

In order to become more specific and to obtain some error budget we now assume a realization of Gravity Probe C in the form of atomic clocks on board of satellites and in direct and retrograde motion around the Earth. We divide the error sources of such an experiment in two groups, namely

(i) errors due to the tracking of the actual orbits, and

(ii) deviations from idealized orbits due to

- mass multipole moments of the Earth
- radiation pressure
- gravitational influence of the Moon, the Sun, and other planets
- other systematic errors (e.g., atmospheric disturbances).

### 3.2 Errors due to the tracking of orbits

The tracking of the actual orbits requires the measurement of distances and angles. (We note in passing that the angle \( \varphi \) has to be measured with respect to a fixed star.) The position of a satellite along an orbit can be determined to a few centimeters using a Global Positioning System; therefore, the temporal uncertainty that a near-Earth satellite has actually returned to the same azimuthal position in space can be roughly
estimated to be $\delta \tau \sim \delta r/v \sim 10^{-6}$ s. Here $\delta r \sim 1$ cm is the position uncertainty along track and $v$ is the orbital speed of the satellite. The gravitomagnetic clock effect, however, involves a definite temporal deviation of $10^{-7}$ s. A simple but more detailed error estimate, based on the formulas provided in Section 2, shows that, to first order in uncertainties $\Delta \varphi$ and $\Delta r$,

$$
\tau_+ - \tau_- = 4\pi \frac{J}{Mc^2} + \sqrt{\frac{r^3}{GM}} \Delta \varphi + 3\pi \sqrt{\frac{r}{GM}} \Delta r.
$$

(18)

From this it is easy to see that one should be able to measure the orbital radius up to an accuracy of the order of $10^{-4}$ m and to determine angles up to an accuracy of $10^{-10}$ rad in order to keep the errors due to the measurement smaller than the clock effect after one orbit. These requirements are about one order of magnitude higher than what can be achieved with today’s technology. However, one should keep in mind that the clock effect is cumulative – just like the precession angle of a GP-B gyroscope – and hence many orbital periods can be used for a measurement of the gravitomagnetic effect; that is, the statistical tracking errors could be overcome if one were able to perform many single measurements.

### 3.3 Systematic errors

The systematic errors of the second group of error sources have a more serious influence on the gravitomagnetic clock effect. In order to calculate the influence of such perturbative accelerations on the Kepler period one has to focus on sections of orbits rather than on complete closed orbits. This is because under favorable conditions (e.g., an almost constant perturbative acceleration) different temporal deviations can cancel each other when summed over a closed orbit. It is possible to show from Newtonian mechanics that perturbative accelerations should be kept below $10^{-11}$ g in order for the clock effect to become measurable.

#### 3.3.1 Influence of multipole moments of the Earth

Perturbative accelerations due to multipole moments of the Earth are of the order of $10^{-3}$ g. That is, the orbit of a satellite under the influence of the nonspherical and inhomogeneous form of the Earth resembles something like a bumpy road. However, we do know the gravitational field of the Earth up to an accuracy of $10^{-9}$ g – $10^{-10}$ g. NASA’s already approved gravity mapping mission GRACE is expected to push this accuracy higher by about two orders of magnitude. This would then make it possible, in principle, to correct for the influence of the multipole moments of the Earth on a gravitomagnetic clock experiment.
3.3.2 Influence of radiation pressure

The radiation pressure of the Sun causes perturbative accelerations of the order of $10^{-8}$ g. Using drag–free satellite techniques, this disturbance can be downsized by two orders of magnitude to $10^{-10}$ g. To keep this error source under control one thus has to be able to determine the solar pressure accurately enough such that corrective calculations can be performed. Alternatively, one must wait for drag–free satellites that can perform at least one order of magnitude better than current technology.

3.3.3 Influence of the gravitational fields of the Sun and other planets

The gravitational fields of the Moon and the Sun cause relative accelerations between the Earth and the orbiting clocks. The amplitudes of these accelerations are of the order of $10^{-7}$ g (Moon) and $10^{-8}$ g (Sun). The influence of the other planets of the solar system plays only a minor role. For Jupiter, e.g., we obtain an influence of the order of $10^{-12}$ g. The positions of the Moon and the Sun are known with much higher accuracy than is needed to determine their gravitational field at the level of $10^{-11}$ g; therefore, in principle, the influence of the Moon and the Sun can be properly taken into account.

3.4 Concluding remarks

It is clear from the considerations of this section that Gravity Probe C requires high precision measurements. Overall, the accuracy required cannot be reached with today’s technology; however, what is missing is a factor of the order of 10. As technology advances in time, a gravitomagnetic clock experiment might become realizable soon. With higher accuracy available the data required for Gravity Probe C might be obtained as a by–product of other space missions.

Gravity Probe C will be kept in mind. This is also true for the gravitomagnetic clock effect of this paper which might also become important in the context of other tests of general relativity.

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3One should remember that it took more than 35 years to realize the Gravity Probe B mission.
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