The evolution of a warped disc around a Kerr black hole

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ABSTRACT
We consider the evolution of a warped disc around a Kerr black hole, under conditions such that the warp propagates in a wavelike manner. This occurs when the dimensionless effective viscosity, $\alpha$, that damps the warp is less than the characteristic angular semi-thickness, $H/R$, of the disc. We adopt linearized equations that are valid for warps of sufficiently small amplitude in a Newtonian disc, but also account for the apsidal and nodal precession that occur in the Kerr metric. Through analytical and time-dependent studies, we confirm the results of Demianski & Ivanov, and of Ivanov & Illarionov, that such a disc takes on a characteristic warped shape. The inner part of the disc is not necessarily aligned with the equator of the hole, even in the presence of dissipation. We draw attention to the fact that this might have important implications for the directionality of jets emanating from discs around rotating black holes.

Key words: accretion, accretion discs – black hole physics – hydrodynamics – waves

1 INTRODUCTION
Accretion discs are found around black holes in the centres of active galaxies and also in galactic X-ray binary stars. These systems may form in such a way that the angular momentum vectors of the disc and of the black hole are not parallel. As a particle in a tilted orbit around a rotating black hole undergoes Lense–Thirring precession at a rate dependent on the radius of the orbit, so a tilted disc experiences a gravitomagnetic torque that tends to twist and warp the disc. If the disc is able to communicate a warping disturbance in a diffusive manner as a result of its pressure and viscosity, it may adopt a characteristic warped shape in which the plane of the disc undergoes a smooth transition from one plane to another in the vicinity of a certain radius (Bardeen & Petterson 1975; Kumar & Pringle 1985; Scheuer & Feiler 1996). At large radius the disc is essentially flat and its plane is determined by the total angular momentum vector of the disc. At small radius the disc is essentially flat and lies in the equatorial plane of the black hole. Over a longer time-scale the disc and hole tend towards mutual alignment (Scheuer & Feiler 1996; Natarajan & Pringle 1998).

However, there are circumstances in which a warping disturbance propagates in a wavelike, rather than diffusive, manner. This occurs in a Keplerian disc when the dimensionless viscosity parameter $\alpha$ (Shakura & Sunyaev 1973) is smaller than the angular semi-thickness $H/R$ of the disc (Papaloizou & Lin 1995; Wijers & Pringle 1999; see the discussion by Pringle 1999); in a non-Keplerian disc the propagation is also wavelike, although it is dispersive (Ogilvie 1999). The evolution of a disc around a rotating black hole under these conditions has received less attention. On the basis of analytical calculations, Demianski & Ivanov (1997) and Ivanov & Illarionov (1997) found that the disc can adopt a steady warped shape in which the tilt angle is an oscillatory function of radius. More recently, Nelson & Papaloizou (2000) conducted smoothed particle hydrodynamic simulations of accretion discs around rotating black holes under a variety of assumptions. They did not find the tilt oscillations, but found that the inner part of the disc was aligned with the equatorial plane of the hole.

In this paper we re-examine the shape of a warped disc around a Kerr black hole, under conditions such that the warp propagates in a wavelike manner. We explore the reasons for the existence of a steady wavelike solution, and discuss the expected wavelength, amplitude and phase of the oscillations in disc tilt. Using time-dependent calculations of the linearized equations, we study the process by which such a steady solution is established. Finally, we discuss the potentially important implications of the shape of the disc for the directionality of jets emanating from discs around rotating black holes.

2 WAVELIKE PROPAGATION OF WARPS
The propagation of warping disturbances of small amplitude in a nearly Keplerian and nearly inviscid disc has been discussed by Papaloizou & Lin (1995), Masset & Tagger (1996), Demianski & Ivanov (1997), Pringle (1999) and Lubow & Ogilvie (2000). These authors differ in respect of the notations and assumptions they adopt, but all agree that bend-
ing waves propagate approximately non-dispersively under these conditions, with a wave speed that is somewhat less than the average sound speed of the disc.

We consider disturbances of a steady disc of surface density $\Sigma(R)$ and vertically integrated pressure $P(R)$ defined by

$$\Sigma = \int \rho \, dz, \quad P = \int p \, dz,$$

where $(R, \phi, z)$ are cylindrical polar coordinates. Let $\Omega(R)$, $\kappa(R)$ and $\Omega_z(R)$ be the orbital angular velocity, the epicyclic frequency and the vertical oscillation frequency associated with circular orbits at radius $R$ from the central object. After an integration by parts, and using the vertical hydrostatic equilibrium of the disc, we find

$$P = \int z \left( \frac{\partial p}{\partial z} \right) \, dz = \Omega_z^2 \int \rho z^2 \, dz = \Omega_z^2 \Sigma H^2,$$

where $H(R)$ is an effective density scale-height. In the case of a vertically isothermal disc, $H$ is equal to the usual Gaussian scale-height.

A simple way to describe a warped disc is as a continuous set of tilted circular rings, and to define a unit tilt vector $t(R,t)$ normal to the plane of the ring of radius $R$ at time $t$. When the disc is only slightly tilted out of the $xy$-plane, $l_z \approx 1$. As shown by Lubow & Ogilvie (2000), the equations governing warping disturbances of small amplitude in a nearly Keplerian and nearly inviscid disc may be written in the form

$$\Sigma R^2 \frac{\partial \Omega}{\partial t} = \frac{1}{R} \frac{\partial G}{\partial R} + T,$$

(3)

$$\frac{\partial G}{\partial t} - \left( \frac{\Omega^2 - \kappa^2}{2 \Omega} \right) l \times G + \alpha \Omega G = \frac{PR^2 \Omega}{4} \frac{\partial l}{\partial R}.$$  

(4)

Equation (3) is derived from the vertical component of the equation of motion, and can be understood as expressing the conservation of horizontal angular momentum: $2\pi G(R,t)$ is the horizontal internal torque in the disc, and $T(R,t)$ is the horizontal external torque, per unit area, acting on the disc. Equation (4) is derived from the horizontal components of the equation of motion, and determines the internal torque $G$. The internal torque is mediated by shearing epicyclic motions, driven by the radial pressure gradients that are set up when a stratified disc is warped. These motions undergo apsidal precession when $\kappa \neq \Omega$, and also viscous decay, here described by a dimensionless viscosity parameter $\alpha$.

The parameter $\alpha$ appears in equation (4) only to quantify the rate, $\alpha \Omega$, at which the shearing epicyclic motions associated with the warp decay. Although this term is derived from a viscous model, numerical simulations of the interaction of shearing epicyclic motions with three-dimensional magnetorotational turbulence (Torkelsson et al. 2000), and an analytical closure model of such turbulence (Ogilvie 2002), both suggest that small-amplitude shearing epicyclic motions decay exponentially in time. The measured rate is in fact through to be somewhat smaller than $\alpha \Omega$ if $\alpha$ is calibrated in the usual way with reference to the $R\phi$-component of the turbulent stress.

It is often more convenient to adopt a complex representation in which $W = l_x + il_y$ and $G = G_x + iG_y$. Then

$$\Sigma R^2 \left[ \frac{\partial W}{\partial t} - i \left( \frac{\Omega^2 - \kappa^2}{2 \Omega} \right) W \right] = \frac{1}{R} \frac{\partial G}{\partial R}.$$  

(5)

\[ \begin{align*}
\frac{\partial G}{\partial t} - i \left( \frac{\Omega^2 - \kappa^2}{2 \Omega} \right) G + \alpha \Omega G &= \frac{PR^2 \Omega}{4} \frac{\partial l}{\partial R}.
\end{align*} \]

(6)

Note that the external torque is related to the non-sphericity of the potential, and therefore to the difference between $\Omega^2$ and $\Omega_z^2$. In addition, $W(R,t)$ represents the tilt of the disc at each radius in the sense that $W(R,t) = \beta(R,t) \exp[i\gamma(R,t)]$, where $\beta$ is the amplitude of the local tilt and $\gamma$ is the azimuth (cf. Pringle 1996).

These equations are known to be valid when the quantities $W$ and $G$ are sufficiently small, and vary on a length-scale long compared to $H$ and on a time-scale long compared to $\Omega^{-1}$. In deriving them it is also assumed that the quantities $|1 - \Omega_z^2/\Omega^2|$, $|1 - \kappa^2/\Omega^2|$ and $\alpha$ are of order $H/R$ or smaller.

In the absence of viscosity, the dispersion relation associated with this system, for wavelike solutions with a rapidly varying phase factor $\exp(\int (\omega \, dt - ik \, dR)$, is

\[ \begin{align*}
\left[ \omega - \left( \frac{\Omega^2 - \kappa^2}{2 \Omega} \right) \right] \left[ \omega - \left( \frac{\Omega^2 - \kappa^2}{2 \Omega} \right) \right] &= \frac{1}{4} k^2 H^2 \Omega_z^2.
\end{align*} \]

(7)

This shows that, in an exactly Keplerian disc, the warp propagates as a non-dispersive wave with wave speed $\sqrt{H \Omega_z/2}$, which is equal to half the isothermal sound speed in the case of a vertically isothermal disc. The presence of nodal and/or apsidal precession introduces dispersion, and forbids wave propagation for frequencies between $(\Omega^2 - \kappa^2)/2\Omega$ and $(\Omega^2 - \kappa^2)/2\Omega$ (Fig. 1).

It is interesting to compare this dispersion relation with that derived by Lubow & Pringle (1993), which describes the full set of wave modes of a vertically isothermal disc. Ogilvie & Lubow (1999) slightly extended that work to allow for the possibility that $\Omega_z \neq \Omega$, and found that, for a disc undergoing isothermal perturbations ($\gamma = 1$),

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{dispersion_relation.png}
\caption{Dispersion relation for bending waves in the case of prograde nodal and apsidal precession with $\Omega_z/\Omega = 0.95$ and $\kappa/\Omega = 0.9$, according to equation (5) (solid lines) and equation (6) (dashed lines).}
\end{figure}
\[
\left( \frac{\omega^2 - \Omega^2}{\Omega^2} \right) - \left( \frac{\dot{\omega}^2}{\Omega^2} \right) k^2 H^2 = n, 
\]

where \( \dot{\omega} = \omega - m \Omega \), \( m \) is the azimuthal mode number and \( n \) is the vertical mode number. For the case \( m = 1, n = 0 \), which corresponds to a tilt or warp, we find

\[
\left[ \omega - \left( \frac{\Omega^2 - \Omega^2 + \omega^2}{2\Omega} \right) \right] \left[ \omega - \left( \frac{\Omega^2 - \kappa^2 + \omega^2}{2\Omega} \right) \right] = \frac{1}{4} k^2 H^2 \Omega^2 \left( 1 - \frac{\omega}{\Omega} \right)^2. 
\]

Unlike equation (7), this dispersion relation is valid when the quantities \( |1 - \Omega^2/\Omega^2| \), \( |1 - \kappa^2/\Omega^2| \) and \( kH \) are of order unity. The two equations clearly agree well in the limit of low frequency, \( |\omega/\Omega| \ll 1 \), and agree exactly when \( \omega = 0 \) (see Fig. 1).

The case \( \omega = 0 \) is of some interest, as it relates to warped discs that are independent of time. When both nodal and apsidal precession are present, and are either both prograde or both retrograde, the dispersion relation for \( \omega = 0 \) indicates that the spatial structure of the warp is oscillatory in character, its radial wavenumber being given by

\[
k^2 H^2 = \frac{(\Omega^2 - \Omega^2)(\Omega^2 - \kappa^2)}{\Omega^2 \Omega^2}. 
\]

An oscillatory solution of this kind may be interpreted as a bending wave having zero phase velocity but non-zero group velocity.

Conversely, when the nodal and apsidal precession are in opposite senses, a steady warp is spatially evanescent. This situation is qualitatively similar to the case when the viscosity is large and the warp satisfies a diffusion equation. It applies when the orbital, epicyclic and vertical frequencies are derived from an axisymmetric Newtonian gravitational potential that satisfies Laplace’s equation in the mid-plane of the disc,

\[
\kappa^2 - 2\Omega^2 + \Omega^2 = 0, 
\]

since this implies

\[
\text{sgn} (\Omega^2 - \Omega^2) = -\text{sgn} (\Omega^2 - \kappa^2). 
\]

However, this need not hold in the metric of a black hole.

3 THE STEADY SHAPE OF A WARPED DISC AROUND A BLACK HOLE

The orbital, epicyclic and vertical frequencies around a Kerr black hole satisfy (Kato 1990)

\[
\Omega^{-1} = \left( \frac{GM}{c^2} \right) \left( r^{3/2} + a \right). 
\]

\[
\Omega^2 - \kappa^2 = 6r^{-1} - 8a \alpha^{3/2} + 3a^2 r^{-2}, 
\]

\[
\Omega^2 - \Omega^2 = 4a \alpha^{-3/2} - 3a^2 r^{-2}, 
\]

where \( -1 < a < 1 \) is the dimensionless angular momentum parameter of the hole \( (a < 0 \) implying a retrograde disc) and \( r \) is the radius in units of \( GM/c^2 \). For a prograde disc \( (a > 0) \) we have the ordering \( \kappa < \Omega_+ < \Omega \) at any radius, corresponding to a situation as illustrated in Fig. 1.

![Figure 2. Ratio of the radial wavelength of the steady warp to the effective density scale-height, plotted against the distance from the black hole, expressed in units of \( GM/c^2 \). The four curves relate to Kerr black holes with \( a = 0.1 \) (top), \( a = 0.2 \), \( a = 0.4 \) and \( a = 1 \) (bottom). The inner radius plotted corresponds to the marginally stable circular orbit in each case.](attachment:image.png)
and the tilt tends to a constant value, \( W \) amplitude of \( R \) the disc, but is likely to do so slowly. At small \( x = \epsilon \) constant and \( h > \sigma > -\frac{1}{2} \), equation (18) in the absence of viscosity has a solution in Bessel functions, \( W = x^\nu \left[ C_1 J_\nu(x) + C_2 Y_\nu(x) \right] \), (21)

where

\[
\nu = \frac{h - \sigma}{h + \frac{3}{4}}, \quad x = \left( \frac{24 \kappa}{c^2} \right)^{1/2} \frac{r^{-(h+1/4)}}{h + \frac{1}{4}}.
\]

At large \( R \) \( x \ll 1 \) the second Bessel function is dominant and the tilt tends to a constant value,

\[
W \rightarrow W_\infty = -\frac{2\nu}{\pi} \Gamma(\nu) C_2.
\]

At small \( R \) \( x \gg 1 \) the Bessel functions are oscillatory and

\[
W \sim x^\nu \left( \frac{2}{\pi \nu} \right)^{1/2} \left[ C_1 \cos \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) + C_2 \sin \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \right].
\]

Note that the amplitude of \( W \propto x^{\nu - 1/2} \propto R^{1/8} (\Sigma H)^{-1/2} \) as quoted above. In the special case considered by Ivanov & Illarionov (1997), \( \sigma = \frac{1}{4} \) and \( h = \frac{1}{2} \), and the amplitude of \( W \propto R^{-1/8} \). A different special case occurs when \( \nu = \frac{1}{2} \), since then

\[
W = \left( \frac{2}{\pi} \right)^{1/2} \left( C_1 \sin x - C_2 \cos x \right)
\]

exactly.

While the constant \( C_2 \) is fixed by the tilt at large radius, \( C_1 \) is determined by the inner boundary condition. If this is such that the torque vanishes at the inner radius \( R = R_{\text{in}} \) (usually at the marginally stable circular orbit), then \( dW/dR = 0 \) there. In the special case \( \nu = \frac{1}{2} \), this determines the solution as

\[
W = W_\infty \frac{\cos(x_{\text{in}} - x)}{\cos x_{\text{in}}},
\]

provided that \( \cos x_{\text{in}} \neq 0 \). This solution can be understood as a standing wave consisting of a superposition of ingoing and outgoing bending waves (having zero phase velocity but non-zero group velocity). The condition of vanishing torque at the inner radius reflects the ingoing bending waves. Note that in this case the warp consists of pure tilt, without twisting (i.e. \( \gamma = \text{constant} \)).

The steady shape of the warp can be interpreted in terms of a resonant wave-launching process (e.g. Goldreich & Tremaine 1979). However, in the present case, the resonance is at \( R = \infty \) because the forcing, caused by the steady disc tilt at large radius, occurs at zero frequency. In addition, the region of maximum forcing, close to the black hole, is far removed from the resonance (cf. Lubow & Ogilvie 2000, where the resonance is at \( R = 0 \) and the forcing is concentrated in the outer part of the disc). As a consequence, the forcing effectively occurs at an intermediate radius, typically where the wave executes its outermost wavelength \( (kr \sim x \sim 1) \). The wave is launched as a trailing \( (k > 0) \) inwardly propagating wave, having group velocity less than or of order the sound speed (see also the discussion by Nelson & Papaloizou 2000).

4 TIME-DEPENDENT CALCULATIONS

We now demonstrate how these equations can be applied to the time-dependent behaviour of a disc in the neighbourhood of a rotating black hole. The simple application we have in mind here is the Newtonian approximation to Lense–Thirring precession around a rotating black hole. In this case, we can write approximately (Demianski & Ivanov 1997; Nelson and Papaloizou 2000; Section 3 above)

\[
\eta \equiv \frac{\kappa^2}{2 \Omega^2} - \frac{3}{2} \frac{R_\kappa}{R},
\]

and

\[
\zeta \equiv \frac{\Omega^2 - \Omega^2}{2 \Omega^2} = -\frac{a}{\sqrt{2}} \left( \frac{R_\kappa}{R} \right)^{3/2},
\]

where \( R_\kappa = 2GM/c^2 \) is the Schwarzschild radius.

In the numerical calculations we choose to solve for the quantities \( A(R, t) \) and \( D(R, t) \), which define the radial velocity perturbation and the enthalpy perturbation according to

\[
u_R = A z e^{i\phi}, \quad w' = D z e^{i\phi},
\]

where the real part is understood. These are simply related to \( W \) and \( G \) by \( W = -D' / R \Sigma H^2 \) and \( G = -2 \Sigma H^2 \Sigma R^2 / A^2 \). To be explicit, we use the equations (30) and (31) to solve for \( A \) and \( D \). Thus,

\[
\frac{\partial A}{\partial t} = - \frac{D}{R} - \frac{1}{2} \frac{\partial D}{\partial R} + i\eta A - i\Omega A,
\]

and

\[
\frac{\partial D}{\partial t} = - \frac{\zeta}{2} \left[ \frac{1}{\Sigma H^2 R^{3/2}} \frac{\partial}{\partial R} \left( \Sigma H^2 R^{1/2} A \right) \right] + i\Omega D.
\]

We consider a disc which is inclined to the spin axis of the hole at large radius. For ease of comparison with the analytic results we consider the particular case in which steady shape of the disc is described by Bessel functions of order \( \frac{1}{2} \), and so can be written in terms of trigonometric functions. Thus we assume the disc with inner radius \( R = R_{\text{in}} \) to have \( \Sigma = \Sigma_{\text{in}} (R/R_{\text{in}})^{-3/2}, \quad H = H_{\text{in}} (R/R_{\text{in}})^{3/4}, \quad \Omega = \Omega_{\text{in}} (R/R_{\text{in}})^{-3/2}, \) and thus (in the vertically isothermal case) a sound speed \( c_s = c_{s_{\text{in}}} (R/R_{\text{in}})^{-1/4} \). If we let

\[
\eta = \eta_{\text{in}} (R/R_{\text{in}})^{-1}, \quad \zeta = \zeta_{\text{in}} (R/R_{\text{in}})^{-3/2},
\]

then the steady solution of the equations, with \( \alpha = 0 \), is

\[
D = \beta_{\text{in}} R_{\text{in}}^{-2} \frac{\cos[\lambda (R_{\text{in}}^{-1} - R^{-1})]}{\cos[\lambda (R_{\text{in}}^{-1})]},
\]

and

\[
A = \frac{i \beta_{\text{in}}}{2 \eta_{\text{in}} \Omega_{\text{in}} R_{\text{in}}^{-3/2} \cos[\lambda (R_{\text{in}}^{-1} - R^{-1})]} \sin[\lambda (R_{\text{in}}^{-1} - R^{-1})],
\]

where
\( \lambda^2 \equiv \frac{4\eta_\infty c_\infty R_\infty^4}{H_\infty^2} \),

and \( \beta_\infty \) is the tilt angle of the disc as \( R \to \infty \). This solution is chosen to ensure that there is zero torque \( (A = 0) \) at the inner boundary.

### 4.1 Numerical results

The numerical method is to take \( A \) to be defined at the \( N \) logarithmically distributed grid points (typically \( N = 1001 \) or 2002), and to take \( D \) to be defined at the half-grid points. We take \( A \) and \( D \) to be defined both at full time points and at the intermediate half-time points. \( A \) is updated from: (a) \( D \) at the intermediate time-point using straightforward numerical differencing for the radial derivative, (b) \( A \) at the intermediate time-point for the precession term (leapfrog), and (c) \( A \) at the previous full time-point for the dissipative term (Euler method). Similarly \( D \) is updated from: (a) \( A \) at the intermediate time-point using straightforward numerical differencing for the radial derivative, and (b) \( D \) at the intermediate time-point for the precession term (leapfrog).

Boundary conditions are required only for \( A \), and we take \( A = 0 \) at the boundaries, which corresponds to a zero torque condition at the inner and outer disc radii.

For numerical convenience, we take units in which \( R_\infty = 1 \), \( \beta_\infty = 1 \), and \( \Omega_\infty = 1 \). We then choose the other parameters to produce a Newtonian approximation to a rotating black hole for which the angular momentum parameter is \( \alpha = \frac{4}{3}(4 - \sqrt{10}) \approx 0.5585 \) and for which the marginally stable circular orbit is at \( R_\infty = 2R_a \). This implies that \( \eta_\infty = -\frac{4}{3} \) and \( c_\infty = -\frac{2}{3}(4 - \sqrt{10}) \approx -0.1396 \). We then choose \( H_\infty/R_\infty = [(4 - \sqrt{10})/8\pi^2]^{1/2} \approx 0.1030 \) to ensure that \( \lambda = 2\pi R_a \). This gives a steady solution (with viscosity zero, i.e. \( \alpha = 0 \)) of the form:

\[
D = R^{-2} \cos[2\pi(1 - R^{-1})].
\]

This solution has the property that the tilt angle \( \beta(R) \equiv |DR^2| \) is unity at \( R = 1 \), at \( R = 2 \), and as \( R \to \infty \), and is zero at \( R = 4 \) and at \( R = 4/3 \). As a check, this solution was tested numerically by using it as initial input on a grid from \( R_{in} = 1 \) to \( R_{out} = 30 \), using \( N = 1001 \) logarithmically spaced grid points. No evolution was detected.

We then investigated time-dependent behaviour, by considering the effect of taking a disc which is initially aligned with the spin axis of the hole, and tilting the outer parts. Thus as an initial condition we take \( DR^2 = 0 \) for \( R \leq 18 \), \( DR^2 = \frac{1}{2}(1 + \sin[\pi(R - 20)/4]) \) for \( 18 \leq R \leq 22 \), and \( DR^2 = 1 \) for \( R \geq 22 \). To consider a reasonable physical situation we add a small amount of dissipation and take \( \alpha = 0.05 \). We take \( A = 0 \) initially, which satisfies the inner and outer boundary conditions and also corresponds to the correct solution for a disc with uniform tilt around a Keplerian point mass. Thus the initial condition corresponds to a discontinuity in the disc tilt at radius \( R = 20 \) smoothed over a distance of \( \Delta R = 4 \). We integrated the equations for a time of \( t = 20,000 \) using a grid with \( N = 2002 \) extending from \( R = 1 \) to \( R = 900 \).

In Fig. 3 we show the initial evolution up to time \( t = 4,000 \). As expected the discontinuity in the tilt initially propagates inwards and outwards in a wavelike fashion. The outward propagation continues throughout the calculation.

**Figure 3.** The tilt of the disc is shown as a function of radius at 11 equally spaced times \( t = 0, 400, 800, \ldots, 4,000 \). The tilt at the final time, \( t = 4,000 \), shown in bold, shows that the steady state solution has already been established at radii \( R \lesssim 8 \).

**Figure 4.** The tilt of the disc is shown as a function of radius for the inner disc regions at 11 equally spaced times \( t = 0, 400, 800, \ldots, 4,000 \). The tilt at the final time, \( t = 4,000 \), shown in bold, shows that the steady state solution has already been established at radii \( R \lesssim 8 \).
The inward propagation of the tilt interacts with the precessional effects at the hole and by the time $t = 4,000$ the steady state shape of the disc close to the hole is essentially established (Fig. 4). It takes the form of a steady warp, together with a slight twist (caused by the small amount of viscosity) as predicted by Ivanov & Illarionov (1997). The interaction of the inwardly propagating warp with the inner disc regions generates a reflected wave which propagates outwards. Thus towards the end of the computation ($t = 20,000$), shown in Fig. 5, at which time the initial outwardly propagating warp wave is approaching the outer edge of the grid, followed closely by the reflection of the initially inwardly propagating warp wave, the steady warped disc solution has been established over about half the grid.

5 DISCUSSION

We have considered the time-dependent evolution of a warped centrifugally supported disc in the low-viscosity limit ($\alpha \lesssim H/R$) in which warp propagation is wavelike rather than diffusive. We note that this is relevant to discs in which the viscosity is low, such as protoplanetary discs, and/or in which the disc thickness is large, such as non-radiative or high-luminosity discs around black holes. We make use of the linearized perturbation equations, which require that the tilt angle of the warp be sufficiently small.

As an illustration we apply the resulting equations to a Newtonian approximation of a moderately thin ($H/R \approx 0.1$), low viscosity ($\alpha = 0.05 \lesssim H/R$) disc around a Kerr black hole with angular momentum parameter $a \approx 0.5$. The disc in this case is subject to both apsidal and nodal precession, and the evolution equations we derive are similar to those derived by Demianski & Ivanov (1997) in the limit of low viscosity. Ivanov & Illarionov (1997) demonstrated that these equations give rise to a steady state in which the disc takes up a characteristic warped shape with the precession induced by the potential being balanced by wavelike stresses in the disc. We confirm this result, and we use our time-dependent equations to demonstrate how such a steady state is set up, starting from the initial condition of a disc whose inner parts are aligned with the spin axis of the hole, and whose outer parts are tilted at a fixed inclination. We show that, once the outer tilt has propagated (in a wavelike fashion) to the central regions of the disc, the steady-state configuration becomes rapidly established. The adjustment of disc shape at the inner radii is propagated outwards through the disc at the usual local wave propagation speed.

Another possible approach to describing the steady shape of a warped disc around a black hole is based on the work of Ogilvie (1999), who developed a non-linear theory of bending waves in accretion discs. The derivation is based on certain ordering assumptions that eliminate some of the time-derivatives from the problem. In the time-dependent case, this theory is applicable to the generic, ‘non-resonant’ case and not to the special case of discs that are both nearly Keplerian and nearly inviscid. However, in the case of steady warps, the theory of Ogilvie (1999) is applicable to all regimes and has the advantages of being fully non-linear, and of including the effect of the accretion flow. We do not pursue this idea here, but note that for an inviscid disc with a steady warp, the theory implies that

$$0 = \frac{1}{R} \frac{dG}{dR} + T,$$

where

$$G = \frac{PR^3\Omega^2}{2(\Omega^2 - \kappa^2)} t \times \frac{d\ell}{dR} \times \left[ 1 + \frac{(6 + \gamma)}{2(3 - \gamma)} \frac{\Omega^2}{\Omega^2 - \kappa^2} |\psi|^2 + O(|\psi|^3) \right],$$

and $\psi$ with the time-derivatives set to zero. The theory indicates that non-linear effects become noticeable when $|RdW/dR|^2$ is comparable to $(\Omega^2 - \kappa^2)/\Omega^2$. This can be expected to occur in the inner part of the disc unless the tilt at large radius is sufficiently small, i.e. $\beta_\infty \ll 1$. The effect of the non-linearity is likely to be to increase the wavelength of the warp and restrict its amplitude.

Other authors have cautioned against taking seriously the possibility of the disc warp having a substantial amplitude in the centre of the disc. Nelson & Papaloizou (2000) did not find such warps in their non-linear SPH computations and attribute this failure to a presumption that "non-linear effects lead to the damping of short wavelength [warps], and thus cause the alignment of inner disc regions in which the tilt amplitude would otherwise change rapidly on small length-scales". Although this may be the case, we note that for the case we consider, with $H/R \sim 0.1$, the radial length-scales of the warps are generally at least of order the local radius, $R$, (Fig. 2) and are not small compared to the local disc scale $H$. We note further that once the radial length-scale of the warp variation becomes short, the dispersive nature of the warp waves is likely to become significant.
and may delay the onset of non-linearity of the waves. In addition, in their original paper, Ivanov & Illarionov (1997) point out that although the formal computations predict a disc warp at inner radii, such predictions might be nullified if the warp itself gives rise to turbulence within the disc, and so a high local effective viscosity. They suggest that the oscillatory vertical shear within the disc caused by the warp (Papaloizou & Pringle 1983) might become shear unstable. Gammie, Goodman & Ogilvie (2000) have investigated this possibility, and show that instability occurs through a parametric effect and is likely to set in when $|A| \gtrsim 30\alpha \Omega$.

We have not discussed the interesting question of the net torque between the disc and the black hole, or the timescale for mutual alignment, under conditions such that the warp propagates in a wavelike manner. The torque may be considered to be exerted in launching the steady train of inwardly propagating bending waves, which carries a certain flux of angular momentum. Unlike the case of resonantly launched waves in problems of discs subject to periodic tidal forcing (Goldreich & Tremaine 1979), the torque cannot be simply expressed in terms of the properties of the disc in the neighbourhood of a certain radius. Furthermore, if the viscosity is small enough that the waves reach the inner radius and reflect from it, the torque that would cause a mutual alignment is partially or completely cancelled. For example, in the case $\nu = \frac{1}{2}$ leading to the solution (24) in the absence of viscosity, the integrated horizontal torque is given in terms of an integral over the disc of the relevant components of the local torque $T(R, t)$. Writing $T = T_x + iT_y$, the total horizontal torque is

$$2\pi \int_{R_{in}}^{\infty} R T dR \propto \frac{iW_\infty}{\cos x_{in}} \int_{0}^{x_{in}} \cos(x_{in} - x) \, dx \propto iW_\infty \tan x_{in}. \quad (39)$$

Rather than causing a mutual alignment, this torque causes a slow mutual precession of the disc and black hole. The direction and magnitude of the precession depend sensitively on the parameters of the disc.

Finally, we note that if the outer disc and the black hole are misaligned by more than $90^\circ$ so that the disc can be considered retrograde, the steady wavelike solution is replaced by an evanescent solution. This occurs because the nodal and apsidal precession are in opposite senses (see equations 13, 14 and 15). In that case the steady shape of the disc may be expected to be qualitatively similar to that envisaged by Bardeen & Petterson (1975), with the inner disc aligned with the equator of the black hole but rotating in the opposite sense. We illustrate this in Fig. 6.

6 CONCLUSION

The result that, in low-viscosity discs around a Kerr black hole, the inner parts of the disc are not necessarily aligned with the black hole, as found by Ivanov & Illarionov (1997), is a general one. Indeed, depending on the disc properties, it is possible for the inner disc to be tilted at a greater angle to the hole than the outer parts of the disc. In addition, because the inner disc shape depends sensitively on the radial dependence of disc properties (such as surface density and disc thickness), a change (for example) in the accretion rate can give rise to a change in the inner disc warp, even without changing the tilt of the outer disc. These results contrast with the usual finding (e.g. Nelson & Papaloizou 2000) and/or assumption (e.g. Natarajan & Pringle 1998) that the inner regions of the disc align with the equator of the hole. In the discussion above, we have noted above that confirmation of these results awaits a proper calculation using full general relativity, as well as an assessment of possible non-linear, dispersive and parametric effects. Nevertheless, it is evident that the results presented here could have important implications for the directions in which jets might emanate from accreting spinning black holes. Indeed, if the region responsible for direction of jet collimation is at several radii from the hole, which is likely to be the case for relativistic jets, then the Newtonian approximations applied above may be adequate to confirm the effect, even if the very inner regions of the disc are indeed aligned by the fully relativistic effects close to the hole. A lack of correlation between the inner and outer disc tilts could provide one explanation of the finding by Kinney et al. (2000) that the directions of jets from low luminosity AGN appear to be uncorrelated with the disc plane of the host spiral galaxies.

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