Some Physical Appearances of Vector Coherent States and CS Related to Degenerate Hamiltonians

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Abstract
In the spirit of some earlier work on the construction of vector coherent states over matrix domains, we compute here such states associated to some physical Hamiltonians. In particular, we construct vector coherent states of the Gazeau-Klauder type. As a related problem, we also suggest a way to handle degeneracies in the Hamiltonian for building coherent states. Specific physical Hamiltonians studied include a single photon mode interacting with a pair of fermions, a Hamiltonian involving a single boson and a single fermion, a charged particle in a three dimensional harmonic force field and the case of a two-dimensional electron placed in a constant magnetic field, orthogonal to the plane which contains the electron. In this last example, an interesting modular structure emerges for two underlying von Neumann algebras, related to opposite directions of the magnetic field. This leads to the existence of coherent states built out of KMS states for the system.

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I Introduction

In some earlier work [2, 17], a fairly systematic method has been introduced for constructing vector coherent states over various types of matrix domains. The construction included earlier types of vector coherent states, arising mainly in nuclear physical problems, under the additional assumption of the existence of a resolution of the identity. (A detailed discussion of this point, as well as an exhaustive reference to the earlier literature is given in [2]). In the present paper we apply the method developed in [2, 17] to construct vector coherent states arising from various physical Hamiltonians. The kind of coherent states we generate are thus vectorial generalizations of the Gazeau-Klauder type [9] of coherent states. Some of the Hamiltonians we consider have degenerate spectra and in order to deal with this situation, we attempt a second generalization of the Gazeau-Klauder formalism. There have been earlier attempts in the literature for handling degeneracies when constructing coherent states associated to Hamiltonians [8, 12]. The method we suggest here is somewhat different from the one suggested in [12] and radically different from that suggested in [8]. However, we feel that the present method is more economical in the introduction of additional parameters defining the coherent states – we only need one additional parameter. We also look at situations where the degeneracy is countably infinite. In this context, in the case of a two-dimensional electron placed in a constant magnetic field, orthogonal to the plane which contains the electron, we encounter a highly interesting modular algebraic structure generated by the observables of the problem, leading to the rather unexpected appearance of equilibrium statistical mechanical states of the well-known KMS type [11]. It is worth recalling that this model is quite an interesting one, since it is the building block for writing down the many-body Hamiltonian of the fractional quantum Hall effect, see [7] and references therein. It is well known that the eigenspectrum of the single electron Hamiltonian can be found explicitly, and that there exists an infinite degeneracy for each eigenvalue (the so-called Landau levels) [5].

The rest of this paper is organized as follows: In Section II we review the Gazeau-Klauder construction within the framework of reproducing kernel Hilbert spaces. This general framework is then used in Section III to construct vector coherent states of the Gazeau-Klauder type. We illustrate the method with a couple of physical examples. Section IV generalizes the treatment to Hamiltonians with
degeneracies. We treat the cases of finite and infinite degeneracies separately and illustrate the finite situation with a number of physical examples. In Section V we work out, in detail, a physical example in which infinite degeneracies occur. In this example we also observe the existence of a modular algebraic structure and the appearance of KMS states, familiar from equilibrium statistical mechanics. Finally, in the Appendix we collect together explicit computations of some of the more unfamiliar formulae in Sections IV.I and V.

II The Gazeau-Klauder scheme revisited

The Gazeau-Klauder scheme [9] is a method for constructing coherent states $|J, \gamma\rangle$, where $J \geq 0$ and $\gamma \in \mathbb{R}$, associated to physical Hamiltonians $H$, which have discrete non-degenerate spectra. The states have to satisfy the following properties:

- **Continuity**: the mapping $(J, \gamma) \mapsto |J, \gamma\rangle$ is continuous in some appropriate topology.

- **Resolution of the identity**: $\int |J, \gamma\rangle \langle J, \gamma| \, dm(J, \gamma) = I$, where $I$ is the identity in the Hilbert space and $dm$ is some appropriate measure;

- **Temporal stability**: $e^{-iHt}|J, \gamma\rangle = |J, \gamma + \omega t\rangle$, for some constant $\omega$;

- **Action identity**: $\langle J, \gamma|H|J, \gamma\rangle = \omega J$.

Their construction, which we shall review below, works if $H$ has no degenerate eigenstates and, furthermore, if the lowest eigenvalue is exactly zero. This second requirement can always be imposed for reasonable physical systems, since all physically relevant Hamiltonians $H$ must be bounded from below, in order to admit a ground state. This means that there exists a lowest eigenvalue $E_{\text{min}} > -\infty$, so that we can define a new Hamiltonian, $\tilde{H} = H - E_{\text{min}} I$, whose lowest eigenvalue is clearly zero. Furthermore $H$ and $\tilde{H}$ have exactly the same dynamical content, since they obey the same commutation relations with all the observables of the system. For such a Hamiltonian, in the Gazeau-Klauder scheme, one writes the eigenvalues as $E_n = \omega \epsilon_n$ by introducing a sequence of dimensionless quantities $\{\epsilon_n\}$ ordered as follows: $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \ldots$. Then, the Gazeau-Klauder coherent states are
defined as

\[ |J, \gamma \rangle := \mathcal{N}(J)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{J^{n/2} e^{-i\epsilon_n \gamma}}{\sqrt{\rho_n}} |n\rangle \]  

(2.1)

where \( \mathcal{N} \) is a normalization factor, which turns out to be dependent on \( J \) only, the \( |n\rangle \) are the eigenstates of \( H \) and the \( \rho_n \) are positive numbers, which are fixed by the requirement of the action identity to be \( \rho_n = \epsilon_1 \epsilon_2 \cdots \epsilon_n \).

In the rest of this section we recapitulate the Gazeau-Klauder construction, with the aim of putting the discussion in a somewhat more general context, which will also enable us to extend the construction to include vector coherent states and to cases where each energy level is (a) finitely degenerate and (b) infinitely degenerate.

The essential mathematical ingredient in the construction is a reproducing kernel Hilbert space. Although this concept is a familiar one, both in the physical and the mathematical literature, we summarize below some essential features, putting them in the context of the present discussion.

II.1 Some generalities

Recall that a reproducing kernel Hilbert space (see, for example, [3, 6, 13] for detailed discussions) \( \mathcal{H}_{\text{ker}} \), consists of functions \( f : X \rightarrow \mathbb{C} \) on some topological space \( X \), with the property that, for all \( x \in X \), the evaluation map \( E_x : \mathcal{H}_{\text{ker}} \rightarrow \mathbb{C}, \ E_x(f) = f(x), \) is continuous. Such a space may or may not be an \( L^2 \)-space or a subspace of an \( L^2 \)-space and its scalar product, which we denote by \( \langle \cdot | \cdot \rangle_{\text{ker}} \), may be given in more general ways. (Although the space \( \mathcal{H}_{\text{ker}} \) could be finite or infinite dimensional, we shall only be interested in the infinite dimensional case here.) The continuity of the evaluation map implies that for each \( x \in X \), there exists a vector \( \xi_x \in \mathcal{H}_{\text{ker}} \) such that

\[ f(x) = \langle \xi_x | f \rangle_{\text{ker}} , \quad \text{for any} \quad f \in \mathcal{H}_{\text{ker}} . \]  

(2.2)

The vectors \( \xi_x, \ x \in X \), are total in \( \mathcal{H}_{\text{ker}} \) (i.e., their linear span is dense in the space), as can be easily seen. Furthermore, they can be used to define the reproducing kernel, \( K : X \times X \rightarrow \mathbb{C} \), for this space:

\[ K(y, x) := \langle \xi_y | \xi_x \rangle_{\text{ker}} = \xi_x(y) , \]  

(2.3)
the second equality following from (2.2). If now \( \{\Psi_n\}_{n=0}^\infty \) is an orthonormal basis of \( \mathcal{H}_{\ker} \), then writing

\[
\xi_x = \sum_{n=0}^\infty \lambda_n(x) \Psi_n , \quad \lambda_n(x) = \langle \Psi_n | \xi_x \rangle_{\ker} = \overline{\Psi_n(x)} ,
\]

and taking account of (2.3), we get

\[
K(x, y) = \sum_{n=0}^\infty \Psi_n(x) \overline{\Psi_n(y)} .
\] (2.4)

It ought to be noted that the above equation is true for any orthonormal basis, so that the kernel \( K(x, y) \) is independent of the basis chosen to express it. An equivalent condition for the existence of a reproducing kernel is that there be an orthonormal basis for which,

\[
\sum_{n=0}^\infty |\Psi_n(x)|^2 < \infty , \quad \text{for all } x \in X . \tag{2.5}
\]

If we symbolically write the scalar product of \( \mathcal{H}_{\ker} \) as

\[
\langle f | g \rangle_{\ker} = \int_X f(x) g(x) \, d\mu(x) ,
\]

then using (2.2) and (2.3) we may also write

\[
\langle \xi_x | \xi_y \rangle_{\ker} = \int_X \overline{\xi_x(z)} \xi_y(z) \, d\mu(z) = \int_X \langle \xi_x | \xi_z \rangle_{\ker} \langle \xi_z | \xi_y \rangle_{\ker} \, d\mu(z) .
\]

Referring again to (2.3) and noting that the vectors \( \xi_x \) are total in \( \mathcal{H}_{\ker} \), the above equation may be re-expressed either as

\[
K(x, y) = \int_X K(x, z) K(z, y) \, d\mu(z) , \tag{2.6}
\]

or as

\[
\int_X |\xi_z\rangle \langle \xi_z| \, d\mu(z) = I_{\ker} , \tag{2.7}
\]

where \( I_{\ker} \) is the identity operator on \( \mathcal{H}_{\ker} \). Thus, these equations appear now as the well-known reproducing property for the kernel \( K(x, y) \) and the resolution of the identity generated by the vectors \( \xi_x \), respectively. Once more we emphasize that in general, equations (2.6) and (2.7) only have symbolic meaning. However, if in fact
\( \mathcal{H}_{\text{ker}} \) is an \( L^2 \)-space with respect to some real measure \( d\mu \) on \( X \) (or a subspace of such a space), then the above equations do make literal sense. In view of equations (2.6) and (2.7), we may call the vectors \( \xi_x \) the *coherent states* defined by the kernel \( K(x, y) \) and they in fact characterize the reproducing kernel Hilbert space \( \mathcal{H}_{\text{ker}} \). However, since \( \|\xi_x\|^2 = K(x, x) \), these states are generally not normalized. If \( K(x, x) \neq 0 \), we may define the normalized vectors \( \zeta_x = [K(x, x)]^{-\frac{1}{2}}\xi_x \), for which we would have the “resolution of the identity”

\[
\int_X |\zeta_z\rangle \langle \zeta_z| K(x, x) \, d\mu(z) = I_{\text{ker}}.
\]

Coherent states, of all types appearing in the physical literature, can be built by simply transporting the above structure to some other appropriate Hilbert space by a basis change. To see this, let \( \mathcal{H} \) be an abstract (separable, complex) Hilbert space and \( \{\phi_n\}_{n=0}^{\infty} \) an orthonormal basis of it. Define the unitary map, \( V : \mathcal{H}_{\text{ker}} \rightarrow \mathcal{H} \) by \( V|\Psi \rangle = |\phi_n\rangle \), \( n = 0, 1, 2, \ldots \). Then the vectors

\[
|\eta_x\rangle := V|\xi_x\rangle = \sum_{n=0}^{\infty} \Psi_n(x)|\phi_n\rangle,
\]

(2.8)
define (non-normalized) coherent states on \( \mathcal{H} \). They are associated to the same reproducing kernel as the \( \xi_x \) since,

\[
K(x, y) = \langle \eta_x|\eta_y\rangle_{\mathcal{H}} = \langle \xi_x|\xi_y\rangle_{\text{ker}}
\]

and satisfy a “resolution of the identity” similar to (2.7):

\[
\int_X |\eta_z\rangle \langle \eta_z| \, d\mu(z) = I_{\mathcal{H}},
\]

where again, this equation is to be generally interpreted in the sense of (2.6). Furthermore, for arbitrary \( \phi \in \mathcal{H} \), the function \( f(x) = \langle \eta_x|\phi\rangle_{\mathcal{H}} \) defines a vector in \( \mathcal{H}_{\text{ker}} \) and it is easy to see that the inverse of the isometry \( V \) is given by this relation, i.e., \( V^{-1}\phi(x) = \langle \eta_x|\phi\rangle_{\mathcal{H}} \). Usually, in the physical literature one works with the normalized vectors

\[
|x\rangle = [K(x, x)]^{-\frac{1}{2}}|\eta_x\rangle = [K(x, x)]^{-\frac{1}{2}}\sum_{n=0}^{\infty} \Psi_n(x)|\phi_n\rangle,
\]

(2.9)
It will later become apparent that the above coherent states coincide with \( |J, \gamma\rangle \) in (2.1) upon identifying \( \Psi_n(x) \) with \( J^n e^{i\epsilon_n \gamma}/\sqrt{\rho_n} \), \( \rho_n \) with \( \epsilon_1 \epsilon_2 \cdots \epsilon_n = \epsilon_n! \) and \( N(J) \) with \( K(x, x) \).
To summarize the preceding discussion, coherent states are linear superpositions of the elements of a basis in a Hilbert space, the components in the expansion being the values taken at a point by a set of vectors forming a basis in a reproducing kernel Hilbert space. Alternatively, referring to (2.5), we may identify the reproducing kernel Hilbert space $H_{\ker}$ with a subspace of $\ell^2$ generated by the infinite sequences, \( \{ \Psi_0(x), \Psi_1(x), \Psi_2(x), \ldots, \Psi_n(x), \ldots \} \), \( x \in X \). An associated family of coherent states is then simply given by the vectors, \( \{ \Psi_0(x), \Psi_1(x), \Psi_2(x), \ldots, \Psi_n(x), \ldots \} \), \( x \in X \), in this subspace. To see that this way of looking at coherent states does indeed include all the standard types of coherent states, let us assume that we are give a family of coherent states, \( |\lambda\rangle \), \( \lambda \in \Lambda \), on some Hilbert space $K$. The parameter space $\Lambda$ is assumed to be a topological space. Being coherent states means that the vectors either satisfy a resolution of the identity, 
\[
\int_\Lambda |\lambda\rangle\langle\lambda| \, dw(\lambda) = I_K ,
\]
with respect to some measure $dw$ defined on $\Lambda$, or else that the mapping $\phi \rightarrow f$, with $f(\lambda) = \langle \lambda|\phi \rangle$, where $\phi$ runs through $K$, is an isometry between $K$ and a reproducing kernel Hilbert space $K_{\ker}$ of functions on $\Lambda$. (In fact the first case implies the second.) In either case, if we choose an orthonormal basis $\{ \phi_n \}_{n=0}^\infty$ in $K$ and expand the coherent states in this basis,
\[
|\lambda\rangle = \sum_{n=0}^\infty f_n(\lambda)|\phi_n\rangle , \quad f_n(\lambda) = \langle \lambda|\phi_n \rangle ,
\]
then the functions $f_n$ are easily seen to form a basis for the Hilbert space $K_{\ker}$ with reproducing kernel $K(\lambda, \lambda') = \langle \lambda|\lambda' \rangle$.

The above considerations can also be generalized to the case where $H_{\ker}$ is a space of vector valued functions and the kernel $K(x,y)$ is matrix valued, yielding vector coherent states (see [2, 17]).

II.2 The Gazeau-Klauder situation

In the light of the preceding discussion, in order to develop a systematic method for generating coherent states and vector coherent states of the Gazeau-Klauder type, we begin by defining a Hilbert space, $\mathcal{H}_{\text{hs}}$, of functions $f : \mathbb{R} \rightarrow \mathbb{C}$, which is complete with respect to the scalar product
\[
\langle f | g \rangle_{\text{hs}} = \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^{T} f(\gamma)g(\gamma) \, d\gamma . \tag{2.10}
\]
The vectors $f_x, \ x \in \mathbb{R},$
\[ f_x(\gamma) = e^{i x \gamma}, \quad (2.11) \]
are of unit norm and for any two distinct numbers $x, x'$, the corresponding vectors $f_x$ and $f_{x'}$ are orthogonal. This also means that the space $\mathcal{F}_{ns}$ is non-separable. Although this space is not an $L^2$-space, by abuse of notation we shall still symbolically write the scalar product as
\[ \langle f | g \rangle_{ns} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(\gamma)g(\gamma) \, d\gamma := \int_{\mathbb{R}} f(\gamma)g(\gamma) \, d\mu(\gamma). \quad (2.12) \]
If $\{\epsilon_n\}_{n=0}^\infty$ is a sequence of numbers in $\mathbb{R}$ (we assume that $\epsilon_n \neq \epsilon_m$ if $n \neq m$), then the set of vectors
\[ f_n(\gamma) = e^{i \epsilon_n \gamma}, \quad n = 0, 1, 2, \ldots, \quad (2.13) \]
forms a countable orthonormal set and hence the closure of their linear span is a separable subspace of $\mathcal{F}_{ns}$. We denote this subspace by $\mathcal{F}_{ang}$ and it is such subspaces of $\mathcal{F}_{ns}$ that we shall use for constructing coherent states. The reason for the subscript will become clear presently. Suppose next, that the sequence $\{\epsilon_n\}_{n=0}^\infty$ is so chosen that the following conditions are satisfied,
\begin{enumerate}
  \item $\epsilon_0 = 0$ and the series
    \[ \sum_{n=0}^{\infty} \frac{J^n}{\epsilon_n!}, \quad J \in \mathbb{R}^+, \quad \epsilon_n! = \epsilon_1 \epsilon_2 \epsilon_3 \cdots \epsilon_n, \quad \epsilon_0! = 1, \]
    has a radius of convergence $L > 0$.
  \item There exists a measure $d\nu$ on $\mathbb{R}^+$ which solves the moment problem
    \[ \int_{0}^{L} J^n \, d\nu(J) = \epsilon_n!, \quad \int_{0}^{L} d\nu(J) = 1. \]
    Then the vectors $r_n, \ n = 0, 1, 2, \ldots,$ in $L^2((0, L), d\nu)$ defined by
    \[ r_n(J) = \frac{J^{\frac{n}{2}}}{\sqrt{\epsilon_n!}}, \quad (2.14) \]
    are of unit norm and span the space. Thus the vectors
    \[ \Psi_n = r_n \otimes f_n, \quad \Psi_n(J, \gamma) = \frac{J^{\frac{n}{2}} e^{i \epsilon_n \gamma}}{\sqrt{\epsilon_n!}}, \quad n = 0, 1, 2, 3, \ldots, \quad (2.15) \]
\end{enumerate}
form an orthonormal basis in the Hilbert space \( \mathcal{H}_{\text{ac-ang}} = L^2((0, L), d\nu) \otimes \mathcal{H}_{\text{ang}} \). Since the vectors \( \Psi_n \) satisfy the condition (analogous to (2.5)),

\[
\sum_{n=0}^{\infty} |\Psi_n(J, \gamma)|^2 = \sum_{n=0}^{\infty} J^n/n! := \mathcal{N}(J) < \infty ,
\]

(2.16)

for all \((J, \gamma) \in (0, L) \times \mathbb{R}^+\), the space \( \mathcal{H}_{\text{ac-ang}} \) is a reproducing kernel Hilbert space with kernel

\[
K(J, \gamma; J', \gamma') = \sum_{n=0}^{\infty} \Psi_n(J, \gamma) \overline{\Psi_n(J', \gamma')} = \sum_{n=0}^{\infty} \frac{(J J')^n}{n!} e^{i\epsilon_n(\gamma - \gamma')}. 
\]

(2.17)

By (2.3), the (non-normalized) coherent states, \( \xi_{J,\gamma} \), defined on \( \mathcal{H}_{\text{ac-ang}} \) and associated to this kernel are then:

\[
\xi_{J,\gamma}(J', \gamma') = K(J', \gamma'; J, \gamma) = \langle \xi_{J',\gamma'} | \xi_{J,\gamma} \rangle_{\text{ac-ang}} ,
\]

(2.18)

while for any \( \Psi \in \mathcal{H}_{\text{ac-ang}} \), we have the relation,

\[
\langle \xi_{J,\gamma} | \Psi \rangle_{\text{ac-ang}} = \Psi(J, \gamma).
\]

Adopting the notation of (2.12), we may also symbolically write a resolution of the identity as,

\[
\int_0^L \left[ \int_{-\infty}^{\infty} |\xi_{J,\gamma}| \langle \xi_{J,\gamma} | \Phi \rangle d\mu(\gamma) \right] d\nu(J) = I_{\text{ac-ang}} ,
\]

(2.19)

where \( I_{\text{ac-ang}} \) denotes the identity in \( \mathcal{H}_{\text{ac-ang}} \). The above equation is to be understood in the sense that for arbitrary \( \Phi, \Psi \in \mathcal{H}_{\text{ac-ang}} \),

\[
\int_0^L \left[ \int_{-\infty}^{\infty} \langle \Psi | \xi_{J,\gamma} \rangle \langle \xi_{J,\gamma} | \Phi \rangle d\mu(\gamma) \right] d\nu(J)
\]

\[
= \int_0^L \left[ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \overline{\Psi(J, \gamma)} \Phi(J, \gamma) \ d\gamma \right] d\nu(J) = \langle \Psi \ | \ \Phi \rangle .
\]

In the Gazeau-Klauder construction of coherent states, related to Hamiltonians with discrete spectra, one assumes that the Hamiltonian is given on some abstract Hilbert space \( \mathcal{H} \) in the orthonormal basis \( \{ \phi_n \}_{n=0}^{\infty} \) by

\[
H = \omega \sum_{n=0}^{\infty} \epsilon_n |\phi_n \rangle \langle \phi_n | , \quad \epsilon_0 = 0 ,
\]

(2.20)
where $\omega$ is a constant with the dimensions of energy (we take $\hbar = 1$). The variable $J$ is then generally identified with the classical action and $\gamma$ with the conjugate angle. It is this identification that prompted our choice of the subscripts for the Hilbert spaces $\mathcal{H}_{\text{ang}}$ and $\mathcal{H}_{\text{ac-ang}}$.

Following (2.8) we can now construct the non-normalized Gazeau-Klauder type coherent states in $\mathcal{H}$ using the vectors (2.15),

$$|\eta_{J,\gamma}\rangle = \sum_{n=0}^{\infty} \frac{J_n^2 e^{-i\epsilon_n \gamma}}{\sqrt{\epsilon_n!}} |\phi_n\rangle.$$  \hfill (2.21)

Once again, the map

$$W : \mathcal{H} \rightarrow \mathcal{H}_{\text{ac-ang}}, \quad (W\phi)(J, \gamma) = \langle \eta_{J,\gamma} | \phi \rangle_{\mathcal{H}},$$

is unitary. If instead, we use the normalized vectors,

$$|J,\gamma\rangle = \mathcal{N}(J)^{-\frac{1}{2}} |\eta_{J,\gamma}\rangle,$$  \hfill (2.22)

with $\mathcal{N}$ as in (2.16), the resolution of the identity becomes

$$\int_0^L \left[ \int_{\mathbb{R}} |J,\gamma\rangle \langle J,\gamma| \mathcal{N}(J) d\mu(\gamma) \right] d\nu(J) = I_{\mathcal{H}}. \hfill (2.23)$$

We also have the formal reconstruction formula,

$$|\phi\rangle = \int_0^L \left[ \int_{\mathbb{R}} \Phi(J,\gamma)|J,\gamma\rangle \mathcal{N}(J) d\mu(\gamma) \right] d\nu(J), \quad \Phi(J,\gamma) = \langle J,\gamma | \phi \rangle_{\mathcal{H}}, \hfill (2.24)$$

which easily follows from (2.23).

The Gazeau-Klauder coherent states are characterized by the temporal stability property,

$$e^{-iHt} |J,\gamma\rangle = |J,\gamma + \omega t\rangle,$$  \hfill (2.25)

and the action identity,

$$\langle J,\gamma | H | J,\gamma \rangle_{\mathcal{H}} = \omega J.$$  \hfill (2.26)

If for a given Hamiltonian, $\epsilon_0 \neq 0$, we will work with the new Hamiltonian $H' = H - \omega \epsilon_0 I$, and use $\epsilon_n = \epsilon_n - \epsilon_0$ to construct coherent states. Note that this amounts to simply shifting all the energy levels by a constant so as to bring the ground state energy to zero and moreover, the new Hamiltonian commutes with the old Hamiltonian. In this case,

$$e^{-iHt} |J,\gamma\rangle = e^{-iH't} e^{-i\omega t} |J,\gamma\rangle = e^{-i\omega t} |J,\gamma + \omega t\rangle$$

$$\langle J,\gamma | H | J,\gamma \rangle_{\mathcal{H}} = \langle J,\gamma | H' + \omega \epsilon_0 | J,\gamma \rangle_{\mathcal{H}} = J + \omega \epsilon_0.$$  \hfill (2.27)
III Vector coherent states of the Gazeau-Klauder type

Suppose now that the Hamiltonian \( H \) (acting on the Hilbert space \( \mathcal{H} \)) has a discrete positive spectrum and that the eigenvectors \( \phi_{jk}, \ j = 1, 2, 3, \ldots, N < \infty, \ k = 0, 1, 2, 3, \ldots, \infty, \) can be grouped into \( N \) families, each containing an infinite number of vectors. (Such a situation could arise, for example, through the lifting of an \( N \)-fold degeneracy in the energy spectrum, by an interaction. Therefore \( k \) labels the main energy levels while \( j \) labels the sublevels generated by, e.g., a small perturbation.) Furthermore, assume that the corresponding eigenvalues \( E_{jk} = \omega_{\gamma jk} \) satisfy \( \gamma_{j0} = 0, \ j = 1, 2, 3, \ldots, N, \) and for any \( j, \) \( \epsilon_{jk} \neq \epsilon_{j'\ell} \) if \( k \neq \ell \) and \( \forall j, j'. \)

Denote by \( \mathcal{H}_j \) the subspace of \( \mathcal{H} \) spanned by the vectors \( \phi_{jk}, k = 0, 1, 2, \ldots, \infty, \) and by \( \mathbb{P}_j \) the projection operator onto this subspace. Then \( \mathcal{H} = \bigoplus_{j=1}^{N} \mathcal{H}_j, \) with \( \mathcal{H}_j = \omega \sum_{k=0}^{\infty} \epsilon_{jk} |\phi_{jk}\rangle \langle \phi_{jk}|, \) which leaves \( \mathcal{H}_j \) stable. We will give an example of such a decomposition in the first application below. In \( \mathcal{H}_j \) we define the coherent states,

\[
|J_j, \gamma_j\rangle = \mathcal{N}(J_j)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{J_j^k}{\sqrt{\epsilon_{j1}\epsilon_{j2} \cdots \epsilon_{jk}}} e^{-i\epsilon_{jk}\gamma_j} |\phi_{jk}\rangle . \tag{3.1}
\]

Here \( -\infty < \gamma_j < \infty \) and \( 0 \leq J_j < L_j = \lim_{k \to \infty} \epsilon_{jk}, \) and we assume that \( L_j > 0. \) The normalization factor \( \mathcal{N}(J_j) \) is chosen so that

\[
\langle J_j, \gamma_j | J_k, \gamma_k \rangle = \delta_{jk} , \tag{3.2}
\]

These states also satisfy

\[
e^{-iH_j t} |J_j, \gamma_j\rangle = |J_j, \gamma_j + \omega t\rangle , \quad \langle J_j, \gamma_j | H_k | J_k, \gamma_k \rangle = \omega J_j \delta_{jk} , \tag{3.3}
\]

and the “partial resolution of the identity”:

\[
\int_{0}^{L_j} \left[ \int_{\mathbb{R}} |J_j, \gamma_j\rangle \langle J_j, \gamma_j| N(J_j) \ d\mu(\gamma_j) \right] \ dv_j(J_j) = \mathbb{P}_j , \tag{3.4}
\]

where \( d\mu \) is as in (2.12) and the measure \( dv_j(J_j) \) is defined through the moment problem

\[
\int_{0}^{L_j} J^n \ dv_j(J) = \epsilon_{j1}\epsilon_{j2} \cdots \epsilon_{jn} , \quad \int_{0}^{L_j} dv_j(J) = 1 . \tag{3.5}
\]
Next, introducing the diagonal matrices,

\[
J = \text{diag} \left( J_1, J_2, \ldots, J_N \right), \quad \varepsilon_k = \text{diag} \left( \varepsilon_{1k}, \varepsilon_{2k}, \ldots, \varepsilon_{Nk} \right),
\]

\[
\gamma = \text{diag} \left( \gamma_1, \gamma_2, \ldots, \gamma_N \right), \quad \varepsilon_k! = \varepsilon_{1k} \varepsilon_{2k} \ldots \varepsilon_{k},
\]

(3.6)

and the vectors

\[
| \Phi_k; j \rangle = \left( \begin{array}{c} 0 \\ \vdots \\ |\phi_{jk}\rangle \\ \vdots \\ 0 \end{array} \right), \quad j = 1, 2, \ldots, N, \quad k = 0, 1, 2, \ldots,
\]

(3.7)

we may rewrite the vectors (3.1) as

\[
| J, \gamma; j \rangle := \mathcal{N}(J_j)^{-\frac{1}{2}} \sum_{k=0}^{\infty} [\varepsilon_k!]^{-\frac{1}{2}} J_j^k \exp[-i\varepsilon_k \gamma] | \Phi_k; j \rangle
\]

\[
= \left( \begin{array}{c} 0 \\ \vdots \\ |J_j, \gamma_j\rangle \\ \vdots \\ 0 \end{array} \right).
\]

(3.8)

We call these states vector coherent states for the Hamiltonian \( H \). Note that, in this representation, \( H \) is a diagonal operator, \( H = \text{diag} \left( H_1, H_2, \ldots, H_N \right) \), each \( H_j \) being an infinite diagonal matrix with eigenvalues \( \omega_{jk}, k = 0, 1, 2, \ldots \).

\[
e^{-iHt} | J, \gamma; j \rangle = | J, \gamma + \omega t d_j; j \rangle, \quad \langle J, \gamma; j | H | J, \gamma; j \rangle = \omega J_j,
\]

(3.9)

where \( d_j \) is the diagonal matrix with one in the \( jj \)-position and zeroes elsewhere. Furthermore, we have the resolution of the identity on \( \mathfrak{g} \):

\[
\sum_{j=0}^{N} \int_{0}^{L_N} \cdots \int_{0}^{L_1} \left[ \int_{\mathbb{R}^N} | J, \gamma; j \rangle \langle J, \gamma; j | \mathcal{N}(J_j) \, d\mu(\gamma) \right] d\nu(J) = I_{\mathfrak{g}},
\]

(3.10)

with

\[
d\nu(J) = d\nu(J_1) \, d\nu(J_2) \cdots d\nu(J_N), \quad d\mu(\gamma) = d\mu_1(\gamma_1) \, d\mu_2(\gamma_2) \cdots d\mu_N(\gamma_N).
\]
In view of the fact that (see also (3.2))

\[ \langle J, \gamma; j | J, \gamma; k \rangle = \delta_{jk} , \]  

(3.11)
a general vector coherent state for such a system may be written as a linear combination,

\[ |J, \gamma; j\rangle = \sum_{j=0}^{N} c_j |J, \gamma; j\rangle . \]

However, such a state would, in general, not be of the Gazeau-Klauder type, unless the levels \( \epsilon_{jk} , j = 1, 2, \ldots, N \), are degenerate for all \( k \). Associated to the vector coherent states (3.8) is the matrix-valued reproducing kernel, \( K(J, \gamma; J', \gamma') \), with matrix elements

\[ K(J, \gamma; J', \gamma')_{jk} = \langle J, \gamma; j | J', \gamma'; k \rangle . \]  

(3.12)

This kernel has the properties,

\[ K(J, \gamma; J, \gamma)_{jj} = \| |J, \gamma; j\rangle \|^2 > 0 , \quad K(J, \gamma; J', \gamma')_{jk} = \overline{K(J', \gamma'; J, \gamma)_{kj}} , \]

\[ \sum_{\ell=0}^{N} \int_{L_N}^{L_1} \cdots \int_{L_1} \int_{\mathbb{R}^N} K(J, \gamma; J'', \gamma'')_{j\ell} K(J'', \gamma''; J', \gamma')_{\ell k} N(J_{\ell}) d\mu(\gamma'') \]  

\[ d\nu(J'') = K(J, \gamma; J', \gamma')_{jk} . \]  

(3.13)

**III.1 Some examples**

Let us consider a model described by the following Hamiltonian,

\[ H = \omega a^\dagger a + \epsilon_1 c_1^\dagger c_1 + \epsilon_2 c_2^\dagger c_2 + (g_1 c_1^\dagger c_1 + g_2 c_2^\dagger c_2)(a + a^\dagger) \]  

(3.14)

where the following commutation rules hold:

\[ [a, a^\dagger] = \{c_1, c_1^\dagger\} = \{c_2, c_2^\dagger\} = I , \]  

(3.15)

and

\[ [a^\dagger, c_1^\dagger] = \{c_1, c_1\} = \{c_2, c_2\} = 0 , \]  

(3.16)

where \( a^\dagger \) stands for \( a \) or \( a^\dagger \), \( [A, B] = AB - BA \) and \( \{A, B\} = AB + BA \). This model, which describes an interaction between a single mode, \( (a, a^\dagger) \), of the radiation field with two Fermi type modes, has been analyzed quite recently in [14].
A convenient feature of the above hamiltonian is that its spectrum can be obtained explicitly, as well as its eigenvectors. In fact, considering the fermionic part, it is clear that all the eigenstates of $H$ must be of the following form:

$$\Phi = \varphi \otimes \Psi_{kl},$$

with $k, l = 0, 1$, and where $\Psi_{0,0}$ is the fermionic vacuum: $c_j \Psi_{00} = 0$, for $j = 1, 2$. The vector $\varphi$ has still to be determined, but it is clear that it cannot, in general, be proportional to $(a^\dagger)^n \varphi_0$, where $a \varphi_0 = 0$, since the interaction part of $H$ is not diagonal on these vectors. However it is a rather simple exercise to check that

$$H (\varphi \otimes \Psi_{00}) = \omega a^\dagger a (\varphi \otimes \Psi_{00})$$

$$H (\varphi \otimes \Psi_{10}) = (\omega a^\dagger a + \epsilon_1 + g_1(a + a^\dagger)) (\varphi \otimes \Psi_{10})$$

$$H (\varphi \otimes \Psi_{01}) = (\omega a^\dagger a + \epsilon_2 + g_2(a + a^\dagger)) (\varphi \otimes \Psi_{01})$$

$$H (\varphi \otimes \Psi_{11}) = (\omega a^\dagger a + \epsilon_1 + \epsilon_2 + (g_1 + g_2)(a + a^\dagger)) (\varphi \otimes \Psi_{11})$$

To proceed further, we observe that in each of the four cases above, $\phi$ is an eigenvector of an self-adjoint operator of the type,

$$B_{kl} = \omega A_{kl}^\dagger A_{kl} + (\epsilon_{kl} - \frac{g_{kl}^2}{\omega}) I, \quad A_{kl} = a + \frac{g_{kl}}{\omega}, \quad [A_{kl}, A_{kl}^\dagger] = 1, \quad k, l = 0, 1,$$

where,

$$\epsilon_{kl} = l\epsilon_1 + k\epsilon_2, \quad g_{kl} = lg_1 + kg_2, \quad l, k = 0, 1.$$

We know, however, that

$$A_{kl} = \exp\left[i\sqrt{2} \frac{g_{kl}}{\omega} P\right] a \exp\left[-i\sqrt{2} \frac{g_{kl}}{\omega} P\right], \quad \text{where} \quad P = \frac{a - a^\dagger}{i\sqrt{2}}.$$\n
Thus, the eigenvectors of $B_{kl}$ are,

$$|\Phi_{kl}^n\rangle = \exp\left[i\sqrt{2} \frac{g_{kl}}{\omega} P\right]|\Phi_0^{kl}\rangle = \left(\frac{A_{kl}^\dagger}{\sqrt{n!}}\right)^n|\Phi_0^{kl}\rangle,$$

where $|n\rangle = \frac{a^n}{\sqrt{n!}}|0\rangle$ are the eigenvectors of the usual number operator $N = a^\dagger a$.

The diagonalization of $H$ is now complete. Our results can be summarized as follows:

**eigenstates of $H$:** $\{\varphi_{kl}^n := \Phi_{kl}^n \otimes \Psi_{kl}, \text{ where } n = 0, 1, 2, \ldots, \text{ and } k, l = 0, 1\}$
eigenvalues of $H$: \{ $E_n^{kl}$ with $n = 0, 1, 2, \ldots$, and $k, l = 0, 1$ \},

where the relevant quantities are shown in the following table:

| $k,l$ | $E_n^{kl} = $ | $\Psi_{kl} =$ | $\Phi_n^{kl} =$ | where | and |
|-------|----------------|----------------|----------------|-------|-------|
| 0,0   | $\omega n$     | $\Psi_{00}$    | $(a^n)^{\frac{1}{\sqrt{n}}} \Phi_{00}$ | $a \Phi_{00}^{00} = 0$ | |
| 1,0   | $\omega n + \epsilon_1 - \frac{g_{kl}^2}{\omega}$ | $c^1_{2} \Psi_{00}$ | $(A^n_{kl})^{\frac{g_{kl}^2}{\sqrt{n}}} \Phi_{01}$ | $A_{10} \Phi_{01}^{00} = 0$ | $A_{10} = a + \frac{g_{kl}^2}{\omega}$ |
| 0,1   | $\omega n + \epsilon_2 - \frac{g_{kl}^2}{\omega}$ | $c^2_{2} \Psi_{00}$ | $(A^n_{kl})^{\frac{g_{kl}^2}{\sqrt{n}}} \Phi_{01}$ | $A_{01} \Phi_{01}^{00} = 0$ | $A_{01} = a + \frac{g_{kl}^2}{\omega}$ |
| 1,1   | $\omega n + \epsilon_1 + \epsilon_2 - \frac{(g_{11} + g_{22})^2}{\omega}$ | $c^1_{1} c^2_{2} \Psi_{00}$ | $(A^n_{kl})^{\frac{g_{kl}^2}{\sqrt{n}}} \Phi_{11}$ | $A_{11} \Phi_{11}^{00} = 0$ | $A_{11} = a + \frac{g_{kl}^2}{\omega}$ |

From (3.20) it is also clear that the vectors $\Phi_{00}^{kl}$ are just the well known canonical coherent states $|$z$>$, with $z = -\frac{g_{kl}}{\omega}$. Thus, in the position space representation these vectors are shifted Gaussians,

$$\Phi_{00}^{kl}(x) \simeq e^{-\frac{1}{2}(x+\sqrt{2}g_{kl})}, \quad k = 0, 1.$$  

In order to build Gazeau-Klauder type of coherent states for this Hamiltonian, we see now that it breaks up into four orthogonal parts:

$$H = \oplus_{k,l=0,1} H_{kl}, \text{ where } H_{kl} = \sum_{n=0}^{\infty} E_n^{kl} |\varphi_n^{kl}\rangle \langle \varphi_n^{kl}|. \quad (3.21)$$

Since the lowest eigenvalue $E_0^{kl}$, for the component Hamiltonian $H_{kl}$, is zero only for $k = l = 0$, we work with $H' = \oplus_{k,l=0,1} H'_{kl}$, where $H'_{kl} = \sum_{n=0}^{\infty} (E_n^{kl} - E_0^{kl}) |\varphi_n^{kl}\rangle \langle \varphi_n^{kl}|$. But $E_n^{kl} - E_0^{kl} = \omega n$. (Note that $H$ and $H'$ commute.) Thus, the vector coherent states of the present model are 4-component vectors, involving the standard canonical coherent states, $|$z$kl$>, $k, l = 0, 1$, $z_{kl} \in \mathbb{C}$, built on the bosonic vacuum state $\Phi_{00}^{kl}$. Thus, introducing the diagonal matrix $3 = \text{diag}(z_{00}, z_{10}, z_{01}, z_{11})$, we can write the vectors (3.21) for the present case as

$$|3; kl\rangle = |z_{kl}\rangle |\Psi_{kl}\rangle = e^{-\frac{|z_{kl}|^2}{2}} \sum_{n=0}^{\infty} \frac{3^n}{\sqrt{n!}} |\Psi_{kl}\rangle |\Phi_n^{kl}\rangle, \quad j, k = 1, 2, \quad (3.22)$$

where in the present representation, the vectors $\Psi_{kl}$ form the canonical basis of $\mathbb{C}^4$:

$$\Psi_{00} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_{10} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Psi_{01} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \Psi_{11} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$
These then are the Gazeau-Klauder type vector coherent states for the Hamiltonian (3.14). Equations (3.9) and (3.10) have obvious transcriptions for these states.

One could also consider the following variant of the Hamiltonian (3.14):

$$ H = \omega a^+a + \epsilon_1 c_1^+c_1 + \epsilon_2 c_2^+c_2 + \sum_{i,j=1}^{2} g_{ij} c_i^+ c_j (a + a^+) $$

(3.23)

where the same commutation rules (3.15) and (3.16) are assumed and

$$ g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} $$

is a $2 \times 2$ hermitian matrix, $g = g^T$. Let $V$ be the unitary matrix which diagonalizes $g$:

$$ VgV^{-1} = g_d := \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix}, $$

so that, defining

$$ d = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = Vc = V \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, $$

and $d^\dagger = c^\dagger V^\dagger = (d_1^\dagger, d_2^\dagger)$, the operators $d_j$ again obey the same anticommutation relations as the $c_j$. Also, $\sum_{i,j=1}^{2} g_{ij} c_i^+ c_j = g_1 d_1^\dagger d_1 + g_2 d_2^\dagger d_2$. However, if $\epsilon_1 \neq \epsilon_2$, this change of variables would make the free fermionic Hamiltonian $\epsilon_1 c_1^+ c_1 + \epsilon_2 c_2^+ c_2$ no longer diagonal, while if $\epsilon_1 = \epsilon_2 = \epsilon$ we get

$$ H = \omega a^+ a + \epsilon d_1^\dagger d_1 + \epsilon d_2^\dagger d_2 + (g_1 d_1^\dagger d_1 + g_2 d_2^\dagger d_2)(a + a^\dagger), $$

for which the entire analysis performed above can be repeated.

**Remark** A possible method for describing a non-degenerate two-level atom (i.e., $\epsilon_1 \neq \epsilon_2$), which is the one considered in [14], can be obtained by adapting the previous procedure as follows: we consider a fictitious three-level atom interacting with the radiation field in the following way:

$$ H = \omega a^+ a + \epsilon (c_1^+ c_1 + c_2^+ c_2 + c_3^+ c_3) + \sum_{i,j=1}^{3} g_{ij} c_i^+ c_j (a + a^\dagger), $$

where now $\{g_{ij}\}$ is a $3 \times 3$ hermitian matrix. We recover a two-level system by considering a subspace of the complete Hilbert space spanned by the vectors $\Psi_{kl} \otimes$
Φ_{kl}^n$, where the Φ_{kl}^n are constructed by trivially extending the foregoing procedure. Next we take Ψ_{00}^0 = Ψ_0 to be the ground state of c_j, j = 1, 2, 3 and set Ψ_{10} = c_1^\dagger Ψ_0, Ψ_{01} = c_2^\dagger c_3 Ψ_0 and Ψ_{11} = c_1^\dagger c_2^\dagger c_3 Ψ_0. (The interpretation is clear: Ψ_0 corresponds to both levels of our atom being empty, while Ψ_{10}, Ψ_{01} and Ψ_{11} correspond respectively to the first, second and both levels being occupied.)

If it is now possible to ensure that the resulting energy spectrum E_{kl}^n, n = 0, 1, 2, ..., k, l = 0, 1, has no degeneracies, we could build Grazeau-Klauder type coherent states for this system. On the other hand, it is easily verified that degeneracy will be avoided if the physical constants of the model satisfy the following inequalities:

$$0 < \epsilon_1 - \frac{g_1^2}{\omega} < \epsilon_2 - \frac{g_2^2}{\omega} < \epsilon_1 + \epsilon_2 - \frac{g_1^2 + g_2^2}{\omega} < \omega$$

In this case we put $E_0 = E_{00}^0 = 0, E_1 = E_{00}^1 = \epsilon_1 - \frac{g_1^2}{\omega}, E_2 = E_{01}^0 = \epsilon_2 - \frac{g_2^2}{\omega}, E_3 = E_{11}^0 = \epsilon_1 + \epsilon_2 - \frac{g_1^2 + g_2^2}{\omega}, E_4 = E_{00}^0 = \omega$, and so on and write, for the corresponding eigenstates $\varphi_0 = \varphi_{00}^0, \varphi_1 = \varphi_{10}^0, \varphi_2 = \varphi_{01}^0, \varphi_3 = \varphi_{11}^0, \varphi_4 = \varphi_{10}^0$, and so on. Finally, defining $\epsilon_n = \frac{E_n}{\omega}$, we recover a sequence of quantities satisfying the inequalities $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < ...$, as required in [9]. Thus we obtain the coherent states $|J, \gamma\rangle = N(J)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{e^{-\epsilon_n}}{\sqrt{\epsilon_n}} \varphi_n$, with all the required properties.

**IV Hamiltonians with degeneracies**

Here we extend the preceding construction to the situation in which some (or perhaps all) of the eigenvalues of the given Hamiltonian have degeneracies. We will consider two situations: first, where all the degeneracies are finite and second, where they are all countably infinite. In the first case, we will show that a natural way to recover all the required properties of the Gazeau-Klauder type coherent states, such as the resolution of the identity, temporal stability and the action identity, among others, is to introduce a third parameter into the definition of the coherent states, replacing $|J, \gamma\rangle$ by $|J, \gamma, \theta\rangle$. The extension we are proposing is somewhat different from that suggested in [8, 12], since it only involves one extra parameter. Moreover, as we will demonstrate, our method can also be adapted to the case of infinite degeneracies.
IV.1 Finite degeneracies

Let us now consider a Hamiltonian $H$, the eigenvalues of which are all discrete with the lowest eigenvalue being again zero. Assume that the $n$-th level, $E_n = \omega \epsilon_n$, has a degeneracy $d(n)$, in general different from 1. We assume $d(n) < \infty$, for all $n$. Denote by $|n,j\rangle$, $n = 0, 1, 2, \ldots$, $j = 1, 2, \ldots, d(n)$, the eigenvectors of the Hamiltonian $H$ so that $H|n,j\rangle = E_n|n,j\rangle$, with $n$ labelling the level and $j$ counting the degeneracy. As usual we introduce the dimensionless quantity $\epsilon_n$ and again, without loss of generality, arrange them in the sequence $0 = \epsilon_0 < \epsilon_1 < \epsilon_2 < \ldots$. This means that the hamiltonian is

$$H = \omega \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \epsilon_n |n,j\rangle \langle n,j|.$$  

We next introduce the parameter $\theta \in [0, 2\pi)$ and define

$$|J,\gamma,\theta\rangle := \mathcal{N}(J)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{J} \frac{J^{n/2} e^{-i \epsilon_n \gamma} e^{-ij\theta}}{\sqrt{\rho_n}} |n,j\rangle,$$  

with $J$ and $\gamma$ as before. We now prove that, for appropriate choice of $\rho_n$, these states satisfy the following properties, which naturally generalize the analogous ones stated at the beginning of Section II:

- **Continuity:** if $(J, \gamma, \theta) \rightarrow (J', \gamma', \theta')$ then $|J, \gamma, \mu\rangle \rightarrow |J', \gamma', \mu'\rangle$;

- **Resolution of the identity:** $\int |J, \gamma, \theta\rangle \langle J, \gamma, \theta| \, dm(J, \gamma, \theta) = I$, for some appropriately chosen measure $dm$;

- **Temporal stability:** $e^{-iHt}|J, \gamma, \theta\rangle = |J, \gamma + \omega t, \theta\rangle$, for some constant $\omega$;

- **Action identity:** $\langle J, \gamma, \theta | H | J, \gamma, \theta\rangle = \omega J$.

Indeed, continuity follows automatically from the definition itself. As for normalization, we observe that

$$\langle J, \gamma, \theta | J, \gamma, \theta\rangle = \mathcal{N}(J)^{-1} \sum_{n,m=0}^{\infty} \sum_{j=1}^{d(n)} \sum_{l=1}^{d(m)} \frac{J^{n/2+m/2} e^{-i(\epsilon_n-\epsilon_m)\gamma} e^{-ij-l\theta}}{\sqrt{\rho_n \rho_m}} \langle m, l|n, j\rangle$$

$$= \mathcal{N}(J)^{-1} \sum_{n=0}^{\infty} \sum_{j=1}^{J} \frac{J^n}{\rho_n} = \mathcal{N}(J)^{-1} \sum_{n=0}^{\infty} \frac{J^n d(n)}{\rho_n},$$

from which we conclude that $\langle J, \gamma, \theta | J, \gamma, \theta\rangle = 1$ if and only if

$$\mathcal{N}(J) = \sum_{n=0}^{\infty} \frac{J^n d(n)}{\rho_n}.$$  

(4.2)
Of course, this is a power series in $J$ and we assume that it has a radius of convergence $L > 0$.

The proof of temporal stability is easy:

$$e^{-iHt}|J, \gamma, \mu \rangle = e^{-iHt}N(J)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \frac{J^n e^{-i\epsilon_n \gamma} e^{-ij\mu}}{\sqrt{\rho_n}} |n, j\rangle$$

$$= N(J)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \frac{J^n e^{-i\epsilon_n \gamma} e^{-ij\mu}}{\sqrt{\rho_n}} e^{-i\omega_n t}|n, j\rangle$$

$$= |J, \gamma + \omega t, \mu \rangle .$$

In order for the action identity to be satisfied, we need a condition on the $\rho_n$. Since $\epsilon_0 = 0$, we get

$$\langle J, \gamma, \theta | H | J, \gamma, \theta \rangle = \omega J \left[ N(J)^{-1} \sum_{n=1}^{\infty} \frac{\epsilon_n J^n d(n)}{\rho_n} \right] .$$

Thus, in order for the action identity to hold the expression within the square brackets must equal one. This can be achieved if we require that

$$\frac{\epsilon_n d(n)}{\rho_n} = \frac{d(n-1)}{\rho_{n-1}} , \quad n = 1, 2, 3, \ldots ,$$

for then

$$\rho_n = \epsilon_n \frac{d(n)}{d(n-1)} \rho_{n-1} = \ldots = \epsilon_n ! \frac{d(n)}{d(0)} \rho_0 , \quad \text{by iteration} .$$

We choose $\rho_0 = d(0)$ so that

$$\rho_n = \epsilon_n ! d(n) , \quad n = 0, 1, 2, \ldots , \quad \text{and} \quad N(J) = \sum_{n=0}^{\infty} \frac{J^n}{\epsilon_n !} . \quad (4.3)$$

Thus the coherent states (4.1) become

$$|J, \gamma, \theta \rangle := N(J)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \frac{J^n e^{-i\epsilon_n \gamma} e^{-ij\theta}}{\sqrt{\epsilon_n ! d(n)}} |n, j\rangle . \quad (4.4)$$

It remains only to determine the measure $dm$ in order for the resolution of the identity to be satisfied. Proceeding as in Section (4) and assuming that the measure $d\nu$ solves the moment problem

$$\int_0^L J^n d\nu(J) = \epsilon_n ! d(n) , \quad n = 0, 1, 2, \ldots , \quad (4.5)$$
we take
\[ dm(J, \gamma, \theta) = \frac{\mathcal{N}(J)}{2\pi} \, d\nu(J) \, d\mu(\gamma) \, d\theta, \quad (4.6) \]
where \( d\mu \) is the symbolic measure defined in (2.12). Then, we prove exactly as in Section II the identity (see (2.23))
\[ \frac{1}{2\pi} \int_0^L \left\{ \int_0^{2\pi} \int_\mathbb{R} |\langle J, \gamma, \theta | N(J) | \rangle \, d\mu(\gamma) \right\} d\theta \right\} d\nu(J) = I_H. \quad (4.7) \]

**Remark:** If \( d(n) = 1 \) for all \( n \), the above coherent states coincide, apart from an inessential overall phase \( e^{-i\theta} \), with the usual Gazeau-Klauder coherent states (2.1). However, when the Hamiltonian \( H \) has a non-trivial degeneracy, it is interesting to notice the presence of \( d(n) \) in the denominator of the expression for the coherent states in (4.4), which implies that the radius of convergence \( L \) depends not only on the eigenvalues of the Hamiltonian but also on their degeneracies. Similarly, the measure \( d\nu \), solving the moment problem (4.5) and appearing in the resolution of the identity, depends on the degeneracy.

**Example 1:** Consider the following simple example, consisting of a single boson and a single fermion: \( H = \omega(a^\dagger a + c^\dagger c) \), where \([a, a^\dagger] = \{c, c^\dagger\} = I \) and \([a^2, c^2] = 0 \), \( x^a \) being \( x \) or \( x^\dagger \). Introducing the vacuum \( \Phi_0 \) of \( a \), and \( \Psi_0 \) of \( c \) and taking, as usual \( \Phi_n = \frac{(a^\dagger)^n}{\sqrt{n!}} \Phi_0, \ n = 0, 1, 2, \ldots \), and \( \Psi_j = (c^\dagger)^j \Psi_0, \ j = 0, 1 \), we can write the eigenvectors of \( H \) as \( \varphi_{n,j} = \Phi_n \otimes \Psi_j \) if \( n = j = 0 \), and \( \varphi_{n,j} = \Phi_{n-j} \otimes \Psi_j \), if \( n = 1, 2, 3, \ldots \), and \( j = 0, 1 \). The corresponding eigenvalues are \( E_{n,j} = n\omega \), so that they turn out to be degenerate in \( j \). In particular we have \( d(0) = 1 \) and \( d(n) = 2 \) for all \( n \geq 1 \). The normalization can be computed using (1.2), and we get
\[ \mathcal{N}(J) = \sum_{n=0}^{\infty} \frac{J^n d(n)}{\rho_n} = 1 + 2 \sum_{n=1}^{\infty} \frac{J^n}{\rho_n} = 1 + \sum_{n=1}^{\infty} \frac{J^n}{n!} = e^J. \]
Definition (4.1) yields therefore,
\[ |J, \gamma, \theta\rangle := e^{-J/2} \left[ e^{-i\theta} |\varphi_{00}\rangle + \sum_{n=1}^{\infty} \sum_{j=1}^{2} \frac{J^{n/2} e^{-in\gamma} e^{-ij\theta}}{\sqrt{2n!}} |\varphi_{nj}\rangle \right]. \quad (4.8) \]
Actually, this time we can restrict the variable \( \gamma \) to the interval \([0, 2\pi)\) and use the measure \( d\mu(\gamma) = \frac{1}{2\pi} \ d\gamma \) in (4.3) instead of the one in (2.12). Furthermore,
$0 \leq J < \infty$ and the measure $d\nu(J)$ has to solve the moment problem

$$
\int_0^\infty J^n \, d\nu(J) = \begin{cases} 
1, & \text{if } n = 0, \\
2n!, & \text{if } n \geq 1.
\end{cases}
$$

It is then easily seen that $d\nu(J) = [2e^{-J} - \delta(J)] \, dJ$. Thus, writing

$$
dm(J, \gamma, \theta) = \frac{e^J}{4\pi^2} [2e^{-J} - \delta(J)] \, dJ \, d\gamma \, d\theta,
$$

we can prove the resolution of the identity,

$$
\int |J, \gamma, \theta\rangle \langle J, \gamma, \theta| \, dm(J, \gamma, \theta) = I.
$$

Finally, introducing the complex variable $z = re^{-i\gamma} = J^\frac{1}{2}e^{-i\gamma}$, $z \in \mathbb{C}$, we can rewrite (4.8) as

$$
|z, \theta\rangle = e^{-|z|^2} \left[ e^{-i\theta} |\varphi_{00}\rangle + \sum_{n=1}^{\infty} \sum_{j=1}^{2} \frac{z^n}{\sqrt{2n!}} e^{-ij\theta} |\varphi_{nj}\rangle \right].
$$

(4.9)

**Example 2:** As a second example consider a particle of mass $m$ constrained to move on the $xy$-plane and subject to the force $\vec{F} = (-kx - by, -ky - bx, 0)$, derivable from the potential $V(x, y) = \frac{1}{2}k(x^2 + y^2) + bxy$. In the rotated coordinates $\xi_\pm = \frac{1}{\sqrt{2}}(x \pm y)$, this potential assumes the form $V(\xi_+, \xi_-) = \frac{1}{2}m(\omega_+^2 \xi_+^2 + \omega_-^2 \xi_-^2)$, where $\omega_{\pm}^2 = \frac{1}{m}(k \pm b)$. The Hamiltonian looks like a 2-dimensional harmonic oscillator since, in an obvious notation, we also have $p_x^2 + p_y^2 = p_+^2 + p_-^2$. Introducing finally the creation and annihilation operators for the $\pm$ modes and adding an inessential constant we get $H = \omega_+ a_+^+ a_+ + \omega_- a_-^+ a_-$. The eigenvalues are therefore $E_{n_+, n_-} = \omega_+ n_+ + \omega_- n_-$ and the corresponding eigenstates are $|\varphi_{n_+, n_-}\rangle$, where $a_- \varphi_{00} = a_+ \varphi_{00} = 0$. Let us now take, as a concrete example, $b = \frac{3k}{5}$. Then the eigenvalues can be written as $E_{n_+, n_-} = \omega_- (2n_+ + n_-)$ and the degeneracy can be simply deduced: we notice that the spectrum is $\omega_- n$, $n = 2n_+ + n_- = 0, 1, 2, \ldots$, and $d(2n) = d(2n + 1) = n + 1$. Therefore, since $\rho_{2n} = (2n)!/(n+1)!$ and $\rho_{2n+1} = (2n+1)!/(n+1)!$, we may write

$$
|J, \gamma, \theta\rangle = e^{-\frac{J^2}{2}} \sum_{l=0}^{\infty} \sum_{j=1}^{l+1} \frac{J^l e^{-2i\gamma} e^{-i\theta j}}{\sqrt{(2l)!}(l+1)!} \left[ \Psi_{2l,j} \right] + \frac{\sqrt{J} e^{-i\gamma}}{\sqrt{2l+1}} |\Psi_{2l+1,j}\rangle,
$$

(4.10)
where we have introduced the states $\Psi_{n,j}$, $n = 0, 1, 2, \ldots$, and $j = 1, 2, \ldots, d(n)$, in order to keep track of the degeneracy of $H$. It is trivial to check that these states display temporal stability and the action identity, while it does not seem to be an easy task to find an explicit expression for a measure with respect to which a resolution of the identity would be satisfied. However, as we will discuss in the Appendix, it is possible to find weight functions, which are not necessarily everywhere positive, with respect to which a resolution of the identity could be defined in a weak sense.

**Example 3:** Let us consider now a particle of mass $m$ and electric charge $e$, subject to a three-dimensional harmonic force $\vec{F} = -k(x, y, z)$ and placed in a uniform magnetic field, oriented along the $z$-axis and given by the vector potential $\vec{A} = B \frac{2}{2} (-y, x, 0)$. The Hamiltonian

$$H = \frac{1}{2m}(p_x + \frac{eB}{2} y)^2 + \frac{1}{2m}(p_y - \frac{eB}{2} x)^2 + \frac{1}{2} p_z^2 + \frac{1}{2} k(x^2 + y^2 + z^2),$$

can be rewritten as

$$H = N_+ (\tilde{\omega} + \Omega) + N_- (\tilde{\omega} - \Omega) + N_z \omega,$$

where we have introduced

$$\Omega = \frac{eB}{2m}, \quad \omega^2 = \frac{k}{m}, \quad \tilde{\omega}^2 = (\Omega^2 + \omega^2),$$

$$a_u = \frac{1}{\sqrt{2}} \left( \sqrt{m \omega} u \pm \frac{i}{\sqrt{m \omega}} p_u \right), \quad u = x, y, \quad a_z = \frac{1}{\sqrt{2}} \left( \sqrt{m \omega} z + \frac{i}{\sqrt{m \omega}} p_z \right),$$

$$a_{\pm} = \frac{a_x \pm ia_y}{\sqrt{2}}, \quad N_{\pm} = a_{\pm}^\dagger a_{\pm}, \quad N_z = a_z^\dagger a_z.$$

The eigenvalues and the eigenstates of $H$ are easily found to be

$$E_{n_+, n_-, n_z} = n_+ (\tilde{\omega} + \Omega) + n_- (\tilde{\omega} - \Omega) + n_z \omega,$$

$$\varphi_{n_+, n_-, n_z} = \frac{(a_+^\dagger)^{n_+} (a_-^\dagger)^{n_-} (a_z^\dagger)^{n_z}}{\sqrt{n_+! n_-! n_z!}} \varphi_{000},$$

where $a_- \varphi_{000} = a_+ \varphi_{000} = a_z \varphi_{000} = 0$. In order to simplify the computation of the degeneracy of this Hamiltonian we assume that $\Omega \ll \omega$. In this approximation $H$ can be written as $H \simeq \omega (N_+ + N_- + N_z)$, which means that the eigenvalues really depend only on $n = n_+ + n_- + n_z$. As in the previous examples we can
introduce the eigenvalues $E_n = \omega n$ while the degeneracy of the $n$-th energy level is $d(n) = \sum_{k=1}^{n+1} k = \frac{1}{2}(n+1)(n+2)$. If we denote the corresponding eigenstates by $\Psi_{nj}, n = 0, 1, 2, \ldots, j = 1, 2, \ldots, d(n)$, we find

$$|J, \gamma, \theta\rangle := \sqrt{2} e^{-\frac{J}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} \left( J_n/2 e^{-i\gamma} e^{-ij\theta} \right) \left| \Psi_{nj} \right\rangle .$$

(4.11)

Once again, in this case we may introduce the complex variable $z = re^{-i\gamma} = J^{\frac{1}{2}} e^{-i\gamma}, z \in \mathbb{C}$, and write these coherent states as

$$|z, \theta\rangle = \sqrt{2} e^{-\frac{|z|^2}{2}} \sum_{n=0}^{\infty} \sum_{j=1}^{d(n)} z^n e^{-ij\theta} \left| \Psi_{nj} \right\rangle .$$

(4.12)

In this case the resolution of the identity takes the form,

$$\frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} |z, \theta\rangle \langle z, \theta| r^5 d\gamma d\theta dr = I .$$

(4.13)

It is trivial to check that all the other stated properties are satisfied as well. We ought to mention here that coherent states for this Hamiltonian have been constructed before in [10]. However the treatment there is somewhat different, in that the authors obtain multidimensional coherent states which allow them to study the Berezin-Lieb inequalities for the associated thermodynamic potential.

### IV.2 Infinite degeneracies

We are now in a position to construct coherent states for Hamiltonians with infinite degeneracies. Let $\tilde{\mathfrak{H}}$ be an abstract Hilbert space and $\{\phi_{k\ell}\}_{k,\ell=0}^{\infty}$ an orthonormal basis in it:

$$\langle \phi_{k\ell} | \phi_{k'\ell'} \rangle = \delta_{kk'} \delta_{\ell\ell'} .$$

Using these and the basis vectors $\Psi_n$ (see (2.15)) of $\mathfrak{H}_{\text{ac-ang}}$ we now build several families of coherent states on $\tilde{\mathfrak{H}}$

1. **Vector coherent states VCS1**

   These are infinite component vector coherent states,

   $$|J, \gamma; J', \gamma'; \ell\rangle^1 = \frac{\Psi_{J', \gamma'}(J', \gamma')}{{\mathcal{N}}(J')^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{\Psi_n(J, \gamma)^{\frac{1}{2}}}{\mathcal{N}(J)^{\frac{1}{2}}} |\phi_{n\ell}\rangle$$

   $$= \frac{J^{\ell}}{\mathcal{N}(J')^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{J^{n}}{\mathcal{N}(J)^{\frac{1}{2}}} e^{-i\ell \gamma'} |\phi_{n\ell}\rangle ,$$

   (4.14)
with components $\ell = 0, 1, 2, \ldots$. These vectors satisfy the normalization
\[
\sum_{\ell=0}^{\infty} \langle J, \gamma; J', \gamma'; \ell | J, \gamma; J', \gamma'; \ell \rangle = 1 ,
\]
(note that according to our present convention, the individual vectors are not normalized) and the resolution of identity condition,
\[
\sum_{\ell=0}^{\infty} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} \left[ \int_0^L \int_0^L \langle J, \gamma; J', \gamma'; \ell \rangle \langle J, \gamma; J', \gamma'; \ell \rangle \right] d\mu(\gamma) d\mu(\gamma') \mathcal{N}(J) \mathcal{N}(J') \, dv(J) \, dv(J') = I_{\tilde{\mathcal{H}}} .
\]
Consider now the Hamiltonian
\[
H_1 = \sum_{n, \ell=0}^{\infty} \omega \epsilon_n \langle \phi_{n\ell} | \phi_{n\ell} \rangle = \omega A_1^\dagger A_1 ,
\]
where $A_1, A_1^\dagger$ are the operators
\[
A_1 \phi_{n\ell} = \sqrt{\epsilon_n} \phi_{n-1 \ell} \, , \quad A_1^\dagger \phi_{n\ell} = \sqrt{\epsilon_{n+1}} \phi_{n+1 \ell} .
\]
Each level $\omega \epsilon_n$ of this Hamiltonian is infinitely degenerate, with $\ell$ counting the degeneracy. Thus the states (4.14) are Gazeau-Klauder type vector coherent states for this Hamiltonian. Indeed, they satisfy the time stability condition,
\[
e^{-iH_1 t} | J, \gamma; J', \gamma'; \ell \rangle ^1 = | J, \gamma + \omega t; J', \gamma'; \ell \rangle ^1 ,
\]
and an action identity, which we could write either as
\[
\frac{\langle J, \gamma; J', \gamma'; \ell | H_1 | J, \gamma; J', \gamma'; \ell \rangle ^1}{\| J, \gamma; J', \gamma'; \ell \rangle ^1} = \omega J ,
\]
or as
\[
\sum_{\ell=0}^{\infty} \langle J, \gamma; J', \gamma'; \ell | H_1 | J, \gamma; J', \gamma'; \ell \rangle ^1 = \omega J ,
\]
where we have summed over the degenerate levels.

Note that we could just as well have constructed vector coherent states in this example, using an orthonormal basis $\{ \Psi_n \}_{n=0}^{\infty}$ in an arbitrary reproducing kernel Hilbert space $H_{\text{ker}}$:
\[
| J, \gamma; x; \ell \rangle = \frac{\Psi_\ell(x)}{[K(x, x)\mathcal{N}(J')]^{\frac{1}{2}}} \sum_{n=0}^{\infty} \frac{J^2_{\ell} e^{-i\epsilon_n \gamma}}{[\epsilon_\ell \gamma^2]^{2}} | \phi_{n\ell} \rangle ,
\]
with \( K(x, x) \) as in (2.5) and the degeneracies would again be handled as before. However, the special choice made in (4.14) enables us to write down the related family of vector coherent states, appearing in (4.22) below, which are the coherent states of a second Hamiltonian, acting on the degeneracy levels.

(2) Vector coherent states \textbf{VCS2}

These are a second set of similar vector coherent states

\[
|J, \gamma; J', \gamma'; n\rangle^2 = \frac{\Psi_n(J, \gamma)}{[N(J)N(J')]^{\frac{1}{2}}} \sum_{\ell=0}^{\infty} \Psi_{\ell}(J', \gamma') |\phi_{n\ell}\rangle
\]

\[
= \frac{J^n}{2} e^{-i n \gamma} \frac{J'^{\ell}}{2} e^{i \ell \gamma'} \sum_{\ell=0}^{\infty} \left[ \epsilon_{\ell}! \epsilon_n! \right]^{\frac{1}{2}} |\phi_{n\ell}\rangle ,
\]

(4.22)

with components \( n = 0, 1, 2, \ldots \). Defining a second Hamiltonian,

\[
H_2 = \sum_{n, \ell=0}^{\infty} \omega \epsilon_{\ell} |\phi_{n\ell}\rangle \langle \phi_{n\ell}| = \omega A_2 A_2^\dagger ,
\]

(4.23)

where \( A_2, A_2^\dagger \) are the operators

\[
A_2 |\phi_{n\ell}\rangle = \sqrt{\epsilon_{\ell}} |\phi_{n\ell-1}\rangle , \quad A_2^\dagger |\phi_{n\ell}\rangle = \sqrt{\epsilon_{\ell+1}} |\phi_{n\ell+1}\rangle ,
\]

(4.24)

we see that the states (4.22) are Kazeau-Klauder type coherent states for this Hamiltonian. The two Hamiltonians \( H_1 \) and \( H_2 \) commute and, in fact, \( H_2 \) lifts the degeneracy of \( H_1 \) and vice versa.

Finally, we can define a third set of coherent states as below.

(3) ‘Bi-coherent states’ \textbf{BCS}

These are basically the summed-over versions of the previous two

\[
|J, \gamma; J', \gamma'; n\rangle^{\text{BCS}} = \frac{1}{[N(J)N(J')]^{\frac{1}{2}}} \sum_{n, \ell=0}^{\infty} \frac{\Psi_n(J, \gamma)\Psi_{\ell}(J', \gamma')}{\Psi_n(J, \gamma)\Psi_{\ell}(J', \gamma')} |\phi_{n\ell}\rangle
\]

\[
= \frac{1}{[N(J)N(J')]^{\frac{1}{2}}} \sum_{n, \ell=0}^{\infty} \frac{J^n J'^{\ell}}{\epsilon_{\ell}! \epsilon_n!} e^{-i (\epsilon_{\ell} \gamma - \epsilon_n \gamma')} \left[ \epsilon_{\ell}! \epsilon_n! \right]^{\frac{1}{2}} |\phi_{n\ell}\rangle ,
\]

(4.25)
which can be considered as being the multidimensional coherent states (see III) of the Hamiltonian

\[ H = H_1 - H_2 = \sum_{n,\ell=0}^{\infty} \omega(\epsilon_n - \epsilon_\ell) |\phi_{n\ell}\rangle \langle \phi_{n\ell}| = \omega [A_1^\dagger A_1 - A_2^\dagger A_2] . \]  

(4.26)

These coherent states are normalized to unity; they satisfy the resolution of the identity,

\[ \int_{R^+} \int_{R^+} \left[ \int_0^L \int_0^L |J, \gamma; J', \gamma'|_{BCS} \right] \times d\mu(\gamma) \, d\mu(\gamma') \] 

\[ N(J)N(J') \, d\nu(J) \, d\nu(J') = I_{\tilde{H}} , \]  

(4.27)

temporal stability condition,

\[ e^{-iHt} |J, \gamma; J', \gamma'|_{BCS} = |J, \gamma + \omega t; J', \gamma' + \omega t\rangle_{BCS} , \]  

(4.28)

and the action identity,

\[ \langle J, \gamma; J', \gamma'|H|J, \gamma; J', \gamma'|_{BCS} = \omega(J - J') . \]  

(4.29)

A physical example of a Hamiltonian admitting such infinite degeneracies is worked out in the following section.

VElectron in a magnetic field

A single electron of unit charge, placed in the xy-plane and subjected to a constant magnetic field, pointing along the negative z-direction, has the classical Hamiltonian

\[ H_{elec} = \frac{1}{2} (\vec{p} + \vec{A})^2 = \frac{1}{2} \left( p_x + \frac{y}{2} \right)^2 + \frac{1}{2} \left( p_y - \frac{x}{2} \right)^2 , \]  

(5.1)

where we have chosen the magnetic vector potential to be \( \vec{A} = \frac{1}{2}(y, -x, 0) \), using the convenient units introduced in [7]. On \( \tilde{H} = L^2(R^2, dxdy) \) we introduce the quantized observables,

\[ p_x + \frac{y}{2} \longrightarrow Q_1 = -i \frac{\partial}{\partial x} + \frac{y}{2} , \quad p_y - \frac{x}{2} \longrightarrow P_1 = -i \frac{\partial}{\partial y} - \frac{x}{2} , \]  

(5.2)
which satisfy \([Q_1, P_1] = i\mathcal{H}\) and in terms of which the quantum Hamiltonian, corresponding to \(H_{\text{elec}}\) becomes

\[
H_1 = \frac{1}{2} (P_1^2 + Q_1^2) .
\]  

(5.3)

This is just the oscillator Hamiltonian in one dimension, with eigenvalues \(E_n = \omega(n + \frac{1}{2})\), \(n = 0, 1, 2, \ldots \infty\). Each level is infinitely degenerate, and we will denote the corresponding normalized eigenvectors by \(\Psi_{n\ell}\), \(\ell = 0, 1, 2, \ldots, \infty\). If the magnetic field were aligned along the positive \(z\)-axis (with \(\vec{A} = \frac{1}{2}(-y, x, 0)\)), the corresponding quantum Hamiltonian would have been

\[
H_2 = \frac{1}{2} (P_2^2 + Q_2^2) .
\]  

(5.4)

with

\[
Q_2 = -i \frac{\partial}{\partial y} + \frac{x}{2}, \quad P_2 = -i \frac{\partial}{\partial x} - \frac{y}{2} ,
\]

(5.5)

and \([Q_2, P_2] = i\mathcal{H}\). The two sets of operators \(\{Q_i, P_i\}, i = 1, 2\), mutually commute:

\[
[Q_1, Q_2] = [Q_1, P_2] = [P_1, Q_2] = [P_1, P_2] = 0 .
\]  

(5.6)

(Note that at the classical level, the transformation \((x, y, px, py) \rightarrow (x' = px + \frac{y}{2}, y' = py + \frac{p_x}{2}, p_{x'} = py - \frac{p_y}{2}, p_{y'} = p_y - \frac{p_x}{2})\) is canonical, i.e., \(dx \wedge dp_x + dy \wedge dp_y = dx' \wedge dp_{x'} + dy' \wedge dp_{y'}\)). Thus, \([H_1, H_2] = 0\) and the eigenvectors \(\Psi_{n\ell}\) of \(H_1\) can be so chosen that they are also the eigenvectors of \(H_2\) in the manner

\[
H_1 \Psi_{n\ell} = \omega(n + \frac{1}{2}) \Psi_{n\ell} , \quad H_2 \Psi_{n\ell} = \omega(\ell + \frac{1}{2}) \Psi_{n\ell} ,
\]

(5.7)

so that \(H_2\) lifts the degeneracy of \(H_1\) and vice versa. We shall assume that this has been done.

While we shall follow the technique outlined in the previous section to construct vector coherent states for the above two Hamiltonians, we shall first analyze the algebraic structures generated by these operators, to get a deeper insight into the nature of the resulting coherent states. In the process we shall display some von Neumann algebraic properties, the appearance of KMS states and a certain modular structure carried by the above model. Details of the mathematical theory underlying these structures may be found in [3, 4, 11, 15, 16]. On \(\mathcal{F}_1 = L^2(\mathbb{R})\) let \(Q\) and \(P\) be the usual position and momentum operators in the Schrödinger representation. Denote
by $\mathcal{B}_2(\mathfrak{H}) \simeq \mathfrak{H} \otimes \overline{\mathfrak{H}}$ the space of Hilbert-Schmidt operators on $\mathfrak{H}$. This is again a Hilbert space, with the scalar product $\langle X \mid Y \rangle_2 = \text{Tr}[X^*Y]$. Let $\{\phi_n\}_{n=0}^{\infty}$ be the orthonormal basis of $\mathfrak{H}$ consisting of the eigenvectors of the oscillator Hamiltonian $H_{\text{osc}} = \frac{1}{2}(P^2 + Q^2)$, i.e., $H_{\text{osc}}\phi_n = \omega(n + \frac{1}{2})\phi_n$, $n = 0, 1, 2, \ldots$. Then,

$$\phi_{n\ell} := |\phi_n\rangle\langle\phi_\ell|, \quad n, \ell = 0, 1, 2, \ldots, \infty,$$

(5.8)
is an orthonormal basis for $\mathcal{B}_2(\mathfrak{H})$. On $\mathfrak{H}$ define the unitary operators,

$$U(x, y) = e^{-i(xQ + yP)}, \quad (U(x, y)\phi)(\xi) = e^{-ix(\xi - \frac{y}{2})}\phi(\xi - y), \quad (x, y) \in \mathbb{R}^2, \quad \phi \in \mathfrak{H}.$$  

(5.9)

Then, it is well known (see, for example, [3]) that the map,

$$\mathcal{W} : \mathcal{B}_2(\mathfrak{H}) \longrightarrow L^2(\mathbb{R}^2, dx
dy) = \tilde{\mathfrak{H}}, \quad (\mathcal{W}X)(x, y) = \frac{1}{(2\pi)^{\frac{1}{2}}}\text{Tr}[U(x, y)^*X],$$

(5.10)
is unitary. Next, if $A$ and $B$ are two operators on $\mathfrak{H}$, we define by $A \lor B$ the operator

$$A \lor B(X) = AXB^*, \quad X \in \mathcal{B}_2(\mathfrak{H}).$$

For a large class of operators $A, B$ (in particular when $A$ and $B$ are both bounded operators), $A \lor B$ defines a linear operator on $\mathcal{B}_2(\mathfrak{H})$. Then straightforward computations (as shown in the Appendix) yield,

$$\mathcal{W}\begin{pmatrix} Q \lor I_{\mathfrak{H}} \\ P \lor I_{\mathfrak{H}} \end{pmatrix}\mathcal{W}^{-1} = \begin{pmatrix} Q_1 \\ P_1 \end{pmatrix}, \quad \mathcal{W}\begin{pmatrix} I_{\mathfrak{H}} \lor \overline{Q} \\ I_{\mathfrak{H}} \lor P \end{pmatrix}\mathcal{W}^{-1} = \begin{pmatrix} P_2 \\ P_2 \end{pmatrix},$$

(5.11)

and

$$\mathcal{W}\begin{pmatrix} H_{\text{osc}} \lor I_{\mathfrak{H}} \\ I_{\mathfrak{H}} \lor H_{\text{osc}} \end{pmatrix}\mathcal{W}^{-1} = \begin{pmatrix} H_1 \\ H_2 \end{pmatrix}, \quad \mathcal{W}\phi_{n\ell} = \Psi_{n\ell},$$

(5.12)

where the $\phi_{n\ell}$ are the basis vectors defined in (5.8) and the $\Psi_{n\ell}$ are the normalized eigenvectors defined in (5.7). This also means that these latter vectors form a basis of $L^2(\mathbb{R}^2, dx
dy)$.

In the sequel we shall also need the thermal equilibrium state, at inverse temperature $\beta$, corresponding to the Hamiltonian $H_{\text{osc}}$. This is the density matrix,

$$\rho_\beta = \frac{e^{-\beta H_{\text{osc}}}}{\text{Tr}[e^{-\beta H_{\text{osc}}}]} = (1 - e^{-\omega_\beta})\sum_{n=0}^{\infty}e^{-n\omega_\beta}|\phi_n\rangle\langle\phi_n|. \quad (5.13)$$
On $\widetilde{\mathcal{H}}$, for each $(x, y) \in \mathbb{R}^2$, define the operators

$$U_1(x, y) = \mathcal{W} [U(x, y) \vee I_{\mathcal{S}}] \mathcal{W}^{-1}, \quad U_2(x, y) = \mathcal{W} [I_{\mathcal{S}} \vee U(x, y)^*] \mathcal{W}^{-1},$$

(5.14)

and let $\mathfrak{A}_i, i = 1, 2$, be the von-Neumann algebra (see, e.g. [15]) generated by the unitary operators $\{U_i(x, y) \mid (x, y) \in \mathbb{R}^2\}$. Then using the unitary map $\mathcal{W}$, the following modular structure can easily be inferred for the pair of von Neumann algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ (for details on modular structures see [16] and for the particular type of algebras appearing here, see [3, 4]).

(1) The algebra $\mathfrak{A}_1$ is the commutant of the algebra $\mathfrak{A}_2$ and vice versa and $\mathfrak{A}_1 \cap \mathfrak{A}_2 = \mathbb{C} I_{\mathcal{S}}$.

(2) If $\{\lambda_n\}_{n=0}^\infty$ is a sequence of non-zero positive numbers such that $\sum_{n=0}^\infty \lambda_n = 1$, then the vector $\Phi = \sum_{n=0}^\infty \lambda_n^2 |\Psi_{nn}\rangle$ is cyclic and separating for $\mathfrak{A}_1$. In particular, we shall work with the vector $\Phi = \Phi_\beta$, for which the $\lambda_n$ correspond to the thermal state $\rho_\beta$ in (5.13):

$$\Phi_\beta = [1 - e^{-\omega\beta}]^{1/2} \sum_{n=0}^\infty e^{-n\omega\beta} |\Psi_{nn}\rangle, \quad \text{i.e.,} \quad \lambda_n = (1 - e^{-\omega\beta}) e^{-n\omega\beta}. \quad (5.15)$$

(3) The map

$$S_\beta : \widetilde{\mathcal{H}} \longrightarrow \widetilde{\mathcal{H}}, \quad S_\beta[U_1(x, y)\Phi_\beta] = U_1(x, y)^*\Phi_\beta,$$

(5.16)

is closable and has the polar decomposition,

$$S_\beta = J_\beta \Delta_\beta^{1/2}, \quad (5.17)$$

where $J_\beta$ is the antiunitary operator:

$$J_\beta |\Psi_{n\ell}\rangle = |\Psi_{\ell n}\rangle, \quad J_\beta^2 = I_{\mathcal{S}}, \quad J_\beta \Phi_\beta = \Phi_\beta, \quad (5.18)$$

so that $J_\beta \mathfrak{A}_1 J_\beta = \mathfrak{A}_2$, and $\Delta_\beta$ is the self-adjoint operator,

$$\Delta_\beta = \sum_{n, \ell=0}^\infty \frac{\lambda_n}{\lambda_\ell} |\Psi_{n\ell}\rangle \langle \Psi_{n\ell}| = e^{-\beta H} \quad \text{where} \quad H = H_1 - H_2, \quad (5.19)$$

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the Hamiltonians $H_1$ and $H_2$ being as in (5.7). (We reproduce the derivation of (5.17)-(5.19) in the Appendix). The operator $\Delta_\beta$ defines a one parameter group of evolution, $t \mapsto \alpha_\beta(t)$ on the algebra $\mathfrak{A}_1$:

$$
\alpha_\beta(t)[A] = \Delta_\beta^{-it} A \Delta_\beta^{-it} = e^{itH} A e^{-itH} = e^{itH_1} A e^{-itH_1}, \quad A \in \mathfrak{A}_1.
$$

(5.20)

(4) The state $\varphi_\beta$, defined on the algebra $\mathfrak{A}_1$ by the vector $\Phi_\beta$:

$$
\langle \varphi_\beta ; A \rangle = \langle \Phi_\beta | A \Phi_\beta \rangle, \quad A \in \mathfrak{A}_1,
$$

is a faithful normal vector state which is invariant under the evolution $\alpha_\beta$:

$$
\langle \varphi_\beta ; \alpha_\beta(t)[A] \rangle = \langle \varphi_\beta ; A \rangle.
$$

(5.22)

Furthermore, $\varphi_\beta$ is a KMS state [11, 16] in the following sense: for $A, B \in \mathfrak{A}_1$, define the function $F_{A,B}$ of the real variable $t$,

$$
F_{A,B}(t) = \langle \varphi_\beta ; A\alpha_\beta(t)[B] \rangle.
$$

(5.23)

Then this function has an analytic extension to the open strip $\{ z = t + iv \mid 0 < v < \beta \}$ and furthermore,

$$
F_{A,B}(t + i\beta) = \langle \varphi_\beta ; \alpha_\beta(t)[B]A \rangle.
$$

(5.24)

Going back now to the problem of constructing coherent states for this system, we can immediately write down three types of states, in analogy with (4.14), (4.22) and (4.25).

(1) Vector coherent states of the Hamiltonian $H_1 - \omega I_{\tilde{\mathfrak{H}}}$

These are the states on $\tilde{\mathfrak{H}} = L^2(\mathbb{R}^2, dx dy)$,

$$
|z, z'; \ell\rangle^1 = e^{-\frac{|z|^2+|z'|^2}{2}} z^\ell \sum_{n=0}^\infty \frac{z^n}{\sqrt{n!} \ell!} |\Psi_{n\ell}\rangle, \quad \ell = 0, 1, 2, \ldots, \infty.
$$

(5.25)

They are obtained by replacing $J^\theta e^{-i\epsilon_n \gamma}$ by $z^n = r^n e^{i\epsilon_n \theta}$ and $J^\theta e^{-i\epsilon_n \gamma'}$ by $z'^\ell = r'^\ell e^{i\epsilon_n \theta'}$ in (4.44), with $z, z' \in \mathbb{C}$. The resolution of the identity now takes the form:

$$
\frac{1}{(2\pi)^2} \sum_{\ell=0}^\infty \int_{\mathbb{C} \times \mathbb{C}} |z, z'; \ell\rangle^1 \langle z, z'; \ell| dx \, dx' \, dy \, dy' = I_{\tilde{\mathfrak{H}}},
$$

(5.26)
where \( z = \frac{1}{\sqrt{2}}(y - ix) \) and \( z' = \frac{1}{\sqrt{2}}(y' - ix') \). Let us introduce the operators,

\[
A_\ell = \frac{1}{\sqrt{2}}(Q_\ell + iP_\ell), \quad A_\ell^\dagger = \frac{1}{\sqrt{2}}(Q_\ell - iP_\ell), \quad H_\ell = A_\ell^\dagger A_\ell + \frac{\omega}{2}, \quad \ell = 1, 2.
\]

(5.27)

Then, it is not hard to see that,

\[
U_1(z) := U_1(x, y) = e^{zA_1^\dagger - \overline{z}A_1} = e^{-\frac{|z|^2}{2}} e^{zA_1^\dagger} e^{-\overline{z}A_1}.
\]

(5.28)

Also, since

\[
A_1|\Psi_{n\ell}\rangle = \sqrt{n}|\Psi_{n-1\ell}\rangle, \quad A_1^\dagger |\Psi_{n\ell}\rangle = \sqrt{n+1}|\Psi_{n+1\ell}\rangle,
\]

it easily follows that,

\[
|z, \overline{z}; \ell\rangle^1 = e^{-\frac{|z'|^2}{2}} \frac{z^\ell}{\sqrt{\ell!}} U_1(z)|\Psi_{0\ell}\rangle.
\]

(5.29)

(2) Vector coherent states of the Hamiltonian \( H_2 - \frac{\omega}{2} I_\beta \)

Following (4.22), we have the analogous set of vector coherent states

\[
|z, \overline{z}; n\rangle^2 = e^{-\frac{|z|^2 + |z'|^2}{2}} z^n \sum_{\ell=0}^{\infty} \frac{z^\ell}{\sqrt{n! \ell!}} |\Psi_{n\ell}\rangle
= e^{-\frac{|z|^2}{2}} \frac{z^n}{\sqrt{n!}} U_2(z')|\Psi_{00}\rangle, \quad n = 0, 1, 2, \ldots, \infty.
\]

(5.30)

which satisfy a resolution of the identity similar to (5.20).

(3) Coherent states of the Hamiltonian \( H = H_1 - H_2 \)

These are the “bi-coherent states”, analogous to (4.23),

\[
|z, \overline{z}\rangle^{BCS} = e^{-\frac{|z|^2 |z'|^2}{2}} \sum_{n, \ell=0}^{\infty} \frac{z^n \overline{z}^\ell}{\sqrt{n! \ell!}} |\Psi_{n\ell}\rangle = U_1(z)U_2(z')|\Psi_{00}\rangle.
\]

(5.31)

(4) Coherent states built from the thermal equilibrium state.

As yet another example related to this system, we build coherent states, starting with the thermal state \( \Phi_\beta \) (see (5.15) and (5.21)). We define these states as

\[
|z, \overline{z}, \beta\rangle^{KMS} = U_1(z)|\Phi_\beta\rangle = e^{zA_1^\dagger - \overline{z}A_1}|\Phi_\beta\rangle.
\]

(5.32)
In view of the fact that for any normalized vector \( \phi \in \mathcal{H} \), the vectors \( U(z)\phi, \ z \in \mathbb{C} \), where \( U(z) := U(x, y) \) (see (5.10)), satisfy

\[
\frac{1}{2\pi} \int_{\mathbb{C}} |U(z)\phi\rangle\langle U(z)\phi| \ dx \ dy = I_{\mathcal{H}},
\]

we deduce, using the isometry \( W \) in (5.10) that the coherent states (5.32) satisfy the resolution of the identity condition

\[
\frac{1}{2\pi} \int_{\mathbb{C}} |z, z, \beta\rangle_{\text{KMS}} \langle z, z, \beta| \ dx \ dy = I_{\tilde{\mathcal{H}}}. \tag{5.33}
\]

Also, since

\[
U_1(z)|\Psi_{nn}\rangle = \frac{1}{\sqrt{n!}} (A_1^1 - zI_{\tilde{\mathcal{H}}})^n U_1(z)|\Psi_{0n}\rangle = \frac{1}{\sqrt{n!}} \left( \frac{\partial}{\partial z} - \frac{z}{2} I_{\tilde{\mathcal{H}}} \right)^n U_1(z)|\Psi_{0n}\rangle,
\]

which follows from the fact that

\[
(A_1^1)^n|\Psi_{0n}\rangle = \sqrt{n!}|\Psi_{nn}\rangle, \quad \text{and} \quad U_1(z)|\Psi_{0n}\rangle = e^{-\frac{|z|^2}{2}} \sum_{k=0}^\infty \frac{(zA_1^1)^k}{k!}|\Psi_{0n}\rangle,
\]

we may rewrite (5.32) as

\[
|z, z, \beta\rangle_{\text{KMS}} = \left[ 1 - e^{-\omega\beta} \right]^{\frac{1}{2}} \sum_{n=0}^\infty \sqrt{n!} e^{-\frac{n\omega^2}{2}} \left( \frac{\partial}{\partial z} - \frac{z}{2} \right)^n |z; n\rangle, \tag{5.35}
\]

where we have set

\[
|z; n\rangle = U_1(z)|\Psi_{0n}\rangle.
\]

Furthermore, using the fact that

\[
(A_2^1)^n|\Psi_{n0}\rangle = \sqrt{n!}|\Psi_{nn}\rangle,
\]

we may also write

\[
|z, z, \beta\rangle_{\text{KMS}} = \left[ 1 - e^{-\omega\beta} \right]^{\frac{1}{2}} \sum_{n=0}^\infty e^{-\frac{n\omega^2}{2}} \left( \frac{\partial}{\partial z} - \frac{z}{2} \right)^n A_2^n|z; 0\rangle, \tag{5.36}
\]

It ought to be pointed out that the coherent states (5.32) are not of the Gazeau-Klauder type. States of the type

\[
\left( \frac{\partial}{\partial z} - \frac{z}{2} \right)^n |z; n\rangle = (A_1^1 - zI_{\tilde{\mathcal{H}}})^n|z; n\rangle,
\]

32
are finite linear combinations of photon-added coherent states (see [1]), which have been studied extensively in the optical literature. Note that
\[ \langle z; n | z'; m \rangle = e^{-|z|^2} e^{-|z'|^2} \delta_{nm}. \] (5.37)

Finally, note that since \( U_1(x, y)^* = U_1(-x, -y) \), using (5.16) we can get another family of coherent states built on the thermal state \( \Phi_\beta \):
\[ S_\beta |z, \beta\rangle_{\text{KMS}} = | -z, \beta\rangle_{\text{KMS}}. \]

Obviously, these also satisfy the same resolution of the identity as (5.33).

We shall consider in more detail the relationship between the above algebraic structure and the different kinds of coherent states discussed here, as well as their use in the analysis of the quantum Hall effect, in a subsequent paper.

VI Appendix

We work out here some of the results quoted in the last two sections.

VI.1 The measure in Example 2 of Section [IV.1]

The proof of the existence of the measure in Example 2 of Section [IV.1] will be considered as a particular case of a more general situation.

We are looking for a “density” \( f(x) \) such that, given a sequence of numbers \( \rho_n \), the following equation holds:
\[ \int_0^\infty f(x)x^ndx = \rho_n, \quad n = 0, 1, 2, \ldots \]

It is convenient to introduce a new function \( \tilde{f}(x) \) as \( f(x) = e^{-x} \tilde{f}(x) \) and restate the problem as follows: we are looking for a function \( \tilde{f}(x) \) such that
\[ \int_0^\infty \tilde{f}(x)x^ne^{-x}dx = \rho_n, \quad n = 0, 1, 2, \ldots. \] (6.1)

As is well known, the orthonormalization procedure in \( L^2(\mathbb{R}^+, e^{-x}dx) \) for \( x^n \) produces the Laguerre polynomials:
\[ x^n \rightarrow L_n(x) = \sum_{k=0}^n \binom{n}{n-k} \frac{(-1)^k}{k!} x^k, \] (6.2)
and $< L_n | L_l > = \delta_{nl}$, where the scalar product is, of course, the one in $L^2(\mathbb{R}^+, e^{-x}dx)$. If we consider the linear combination of (6.1) with the coefficients given in (6.2) we get

$$\int_0^\infty \tilde{f}(x)L_n(x)(e^{-x}dx) = \sum_{k=0}^{n} \binom{n}{n-k} \frac{(-1)^k}{k!} \rho_k =: d_n \quad (6.3)$$

It is clear then that we have to take $\tilde{f}(x) = \sum_{n=0}^\infty d_n L_n(x)$, provided this sum converges and consequently, the required “density” is $f(x) = e^{-x} (\sum_{n=0}^\infty d_n L_n(x))$. Note however, that this function is not everywhere positive.

We can say more on the coefficients $d_n$ by recalling that $\rho_{2n} = (2n)! (n+1)$ and $\rho_{2n+1} = (2n+1)! (n+1)$. It is an easy exercise to check that

$$d_n = \sum_{l=0}^{[n/2]} \binom{n}{n-2l} (l+1) - \sum_{l=0}^{[n-1]/2} \binom{n}{n-(2l+1)} (l+1),$$

where $[r]$ stands for the integer part of the rational number $r$. This implies that $d_1 = 0$ and $d_n = 2^{n-2}$ for all $n \geq 2$, so that $\tilde{f}(x)$ cannot be a square-integrable function. However, if we consider the sequence $\{\tilde{f}_N | N \in \mathbb{N}\}$, where $\tilde{f}_N(x) = \sum_{n=0}^N d_n L_n(x)$, it is possible to show that it converges with respect to a certain family of test functions. For that we define

$$D_b = \left\{ f \in D([0,1]) \bigg| \int_0^1 \left| \frac{d^k}{dx^k} f(x) \right| dx \leq 1, \forall k = 0, 1, 2, \ldots \right\}. \quad (6.4)$$

This is a non empty subset of $D([0,1])$. We can check that

$$I_{NM} := \int_0^\infty (\tilde{f}_N(x) - \tilde{f}_M(x))\varphi(x)dx \to 0, \quad (6.5)$$

as $N, M \to \infty$ for all $\varphi \in D_b$. This follows from the fact that

$$L_n(x) = \frac{1}{n!} e^x \frac{d^n}{dx^n} (e^{-x}x^n)$$

and from the properties of $D_b$. Thus, using integration by parts:

$$|I_{NM}| \leq \sum_{n=M+1}^N \frac{|d_n|}{n!} \int_0^1 \left| x^n \left( 1 + \frac{d}{dx} \right)^n \varphi(x) \right| dx \leq \sum_{n=M+1}^N \frac{2^{n-2} \cdot 2^n}{n!} \to 0,$$

as $N, M \to \infty$.

It may be worth remarking that the set $D_b$ could be replaced by some larger set without affecting the final result. However, the estimates above would have been harder to obtain. Thus, since such a stronger result would not be very relevant in the present context, we will not consider this generalization here.
VI.2 Proof of (5.11)

We only demonstrate the first two relations in (5.11), since the other two follow in an entirely analogous manner. Moreover, (5.12) is a direct consequence of (5.11). Consider $X \in B_2(H)$ of the type $X = |\psi\rangle\langle\phi|$, such that both $\phi$ and $\psi$ are in the domains of the operators $Q$ and $P$, are differentiable and vanish at infinity. Then, 

$$(WX)(x, y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \text{Tr} [U(x, y)^* X] = \frac{1}{(2\pi)^{\frac{1}{2}}} \langle U(x, y) \psi | \phi \rangle_s$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ix(\xi - \frac{y}{2})} \psi(\xi - y) \phi(\xi) \, d\xi.$$ 

Thus, 

$$(WQ \lor I_s(X))(x, y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \langle U(x, y) \psi | Q\phi \rangle_s$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ix(\xi - \frac{y}{2})} \psi(\xi - y) \xi \phi(\xi) \, d\xi$$

$$= \left(-i \frac{\partial}{\partial x} + \frac{y}{2}\right) \left[ \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ix(\xi - \frac{y}{2})} \psi(\xi - y) \phi(\xi) \, d\xi \right],$$

implying 

$$(WQ \lor I_s(X))(x, y) = \left(-i \frac{\partial}{\partial x} + \frac{y}{2}\right) (WX)(x, y).$$

Extending by linearity on appropriate domains, we get 

$$WQ \lor I_s W^{-1} = -i \frac{\partial}{\partial x} + \frac{y}{2} = Q_1.$$

Next, 

$$(WP \lor I_s(X))(x, y) = \frac{1}{(2\pi)^{\frac{1}{2}}} \langle U(x, y) \psi | P\phi \rangle_s$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{\mathbb{R}} e^{ix(\xi - \frac{y}{2})} \psi(\xi - y) \left(-i \frac{\partial}{\partial \xi}\right) \phi(\xi) \, d\xi.$$ 

Now, 

$$-i \frac{\partial}{\partial \xi} \left[ e^{ix(\xi - \frac{y}{2})} \psi(\xi - y) \phi(\xi) \right] = x e^{ix(\xi - \frac{y}{2})} \psi(\xi - y) \phi(\xi)$$

$$+ e^{ix(\xi - \frac{y}{2})} \left(-i \frac{\partial}{\partial \xi}\right) \psi(\xi - y) \phi(\xi)$$

$$+ e^{ix(\xi - \frac{y}{2})} \psi(\xi - y) \left(-i \frac{\partial}{\partial \xi}\right) \phi(\xi).$$
Integrating both sides of this equation with respect to $\xi$ from $-\infty$ to $\infty$ and noting that $\psi(\xi), \phi(\xi) \to 0$ as $\xi \to \pm\infty$, and $\frac{\partial}{\partial \xi} \psi(\xi - y) = -\frac{\partial}{\partial y} \psi(\xi - y)$, we get,

$$0 = \frac{x}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix(\xi - \frac{y}{2})} \psi(\xi - y) \phi(\xi) + \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix(\xi - \frac{y}{2})} \left( i \frac{\partial}{\partial y} \right) \psi(\xi - y) \phi(\xi)$$

$$+ \frac{1}{(2\pi)^{1/2}} \int_{\mathbb{R}} e^{ix(\xi - \frac{y}{2})} \frac{\partial}{\partial \xi} \psi(\xi - y) \phi(\xi).$$

Thus,

$$(WP \lor I_5(X))(x, y) = \left( -i \frac{\partial}{\partial y} - \frac{x}{2} \right) (WX)(x, y),$$

and again, extending by linearity on appropriate domains we get

$$WP \lor I_5W^{-1} = -i \frac{\partial}{\partial x} + \frac{y}{2} = P_1.$$

### VI.3 Proof of (5.17)-(5.19)

Since the vectors $\Psi_{jk}, j, k = 0, 1, 2, \ldots, \infty$, form a basis of $\tilde{\mathcal{H}} (= L^2(\mathbb{R}^2, dxdy))$, we may write

$$U_1(x, y)\Phi_\beta = \sum_{i=0}^{\infty} \lambda_i^{1/2} U_1(x, y) \Psi_{ii} = \sum_{i, j, k=0}^{\infty} \lambda_i^{1/2} \langle \Psi_{jk} \mid U_1(x, y) \Psi_{ii} \rangle \tilde{\mathcal{H}} \Psi_{jk}.$$

Now, using the isometry $\mathcal{W}\phi_{jk} = \mathcal{W}(|\phi_j\rangle \langle \psi_k|) = \Psi_{jk}$ (see (5.8) and (5.12)), the first relation in (5.14) and the fact that the vectors $\phi_i, i = 0, 1, 2, \ldots, \infty$, form an orthonormal basis of $\tilde{\mathcal{H}}$, we obtain

$$\langle \Psi_{jk} \mid U_1(x, y) \Psi_{ii} \rangle \tilde{\mathcal{H}} = \text{Tr} \left[ |\phi_k\rangle \langle \phi_j| U(x, y) |\phi_i\rangle \langle \phi_i| \right] = \langle \phi_j \mid U(x, y) \phi_i \rangle \delta_{ik}$$

$$= (2\pi)^{1/2} \tilde{\mathcal{H}} \psi_{ji}(x, y) \delta_{ik}.$$

Thus,

$$U_1(x, y)\Phi_\beta = (2\pi)^{1/2} \sum_{i, j=0}^{\infty} \lambda_i^{1/2} \tilde{\mathcal{H}} \psi_{ji}(x, y) \Psi_{ji}. \quad (6.6)$$

Similarly,

$$U_1(x, y)^*\Phi_\beta = (2\pi)^{1/2} \sum_{i, j=0}^{\infty} \lambda_i^{1/2} \tilde{\mathcal{H}} \psi_{ij}(x, y) \Psi_{ji} = (2\pi)^{1/2} \sum_{i, j=0}^{\infty} \lambda_j^{1/2} \tilde{\mathcal{H}} \psi_{ji}(x, y) \Psi_{ij}. \quad (6.7)$$
Next, applying the operator $S_\beta$ to both sides of (6.6) and taking account of the fact that this operator is antilinear, we get

$$S_\beta [U_1(x,y)\Phi_\beta] = U_1(x,y)^*\Phi_\beta = (2\pi)^{\frac{1}{2}} \sum_{i,j=0}^{\infty} \lambda_i^\frac{1}{2} \Psi_{ji}(x,y) S_\beta \Psi_{ji}$$

Comparing this equation with (6.7) we immediately see that

$$S_\beta \Psi_{ji} = \left[ \frac{\lambda_j}{\lambda_i} \right]^{\frac{1}{2}} \Psi_{ij} ,$$

from which (5.17)-(5.19) follow directly.

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