Development of Calculation Research Method for Exhaust System Main Elements Acoustic Characteristics

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Abstract: This paper presents a calculation method for researching acoustic characteristics of the exhaust system main elements in modern motor vehicles. The acoustic characteristics of the absorbing materials used in the muffler have been defined by method of calculation and experiment.

1. Introduction

In recent years, the requirements for the vibroacoustic (NVH) characteristics of motor vehicles have become more stringent. The output noise remains the main source of noise of the running engine, whereby the acoustic power of a not muffled exhaust noise reaches 100 W (up to 140 dBA) and is tens and hundreds of times superior to those of other elements and systems of the vehicle. The vibration frequency of an acoustic wave generated as a result of operation of an 8-cylinder 4-stroke internal combustion engine (ICE) with 650 rpm at idle will be 43.3 Hz, according to formula 1:

\[ f = \frac{n \cdot i}{60 \cdot \tau} = \frac{650 \cdot 8}{60 \cdot 2} = 43.3 \text{ Hz} \quad (1) \]

Where: \( n \) – engine crankshaft speed per minute; \( i \) – number of cylinders; \( \tau = 2 \) for four-stroke engines, \( \tau = 1 \) for two-stroke engines.

The wavelength at the frequency of 43.3 Hz at the exhaust gas temperature of 20°C is:

\[ \lambda = \frac{\vartheta}{f} = \frac{340 \cdot \frac{\text{m}}{\text{s}}}{43.3 \text{ Hz}} = 7.85 \text{ m} \quad (2) \]

The length of the whole exhaust system is 5.8 meters. The exhaust pipe cut falls on 3/4 of the wavelength at the frequency of 43.3 Hz. Respectively, it falls within the maximum sound pressure amplitude.

2. Main part

To solve this problem, a classical Helmholtz resonator, which is the most efficient within the narrow-band frequency spectrum, has been developed. First of all, the resonator quality factor depends on the chamber (cavity) volume. Figure 1 shows the classical Helmholtz resonator scheme:
In order to find the required value of transmission loss (TL) of the Helmholtz resonator the following formula was used:

\[ TL = 20 \log_{10} \left( \frac{1}{2} \left( 1 + \frac{\rho_0 c_0}{S_d Z_r} \right) \right) \]  

(3)

where

\[ Z_r = j \left( \omega \frac{\rho_0 l_n}{S_n} - \frac{1}{\omega} \frac{\rho_0 c_0^2}{V_c} \right) \]  

(4)

where \( \rho_0 = 1.2 \text{ kg/m}^3 \) is the air density at 20°C, \( c_0 = 343 \text{ m/s} \) - acoustic velocity at 20°C, \( S_d \) - longitudinal pipe area, \( Z_r \) - acoustic impedance, \( l_n \) - resonator's tube length, \( S_n \) - resonator's tube area, \( V_c \) - volume of Helmholtz resonator chamber, \( \omega \) - circular frequency

Parametric optimization was carried out for frequency adjustment and adjustment of the quality factor of the Helmholtz resonator. According to the results of the parametric optimization, the following geometric parameters of the resonator have been obtained: the tube length is 75 mm, the tube diameter is 22 mm, the tube thickness is 2 mm and the volume of the Helmholtz resonator chamber is 7452330.5 mm³. The results of the analytical calculation based on the results of the parametric optimization are shown in Figure 2.

The resonator is tuned to the frequency of 43.3 Hz (Figure 2). The transmission loss at this frequency is 22 dB.
For the exact mathematical calculation of the acoustic behavior of the main elements of the exhaust system, it is necessary to solve a diffractive problem for the wave equation describing the acoustic propagation in a gaseous medium filling this element under complex boundary conditions.

The whole exhaust system design can be divided acoustically into a limited set of elements having the same operating principle. These are reactive elements made in the form of chamber systems connected to each other and to the volume of the waveguide using pipes, gaps and holes. And also the dissipative elements containing different sound-absorbing materials. The whole system can be divided into a number of separate known acoustic elements, interconnected in a certain way. As a result of theoretical research for each element of this limited number of elements, a mathematical formula has been derived that describes its acoustic characteristics. The interaction of each of these elements with the neighboring one in the exhaust system is set through the input and output impedances \( Z \) (acoustic impedance), which is determined by the sound pressure \( P \) and the vibrational velocity of the particles of the medium \( V \). The connection of these parameters for each element is determined by the table (transfer matrix) \( T \), which fully characterizes the acoustic behavior of the element. In a matrix form this ratio can be written as follows:

\[
Z_{in} = T Z_{out}, \text{ where } Z_{in} = \begin{bmatrix} P_{out} \\ V_{out} \end{bmatrix} ; \quad T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}
\]  
(5)

or in a form of an equation system:

\[
P_{in} = T_{11} P_{out} + T_{12} V_{out}
\]  
(6)

\[
V_{in} = T_{21} P_{out} + T_{22} V_{out}
\]  
(7)

The quadripole (four-pole) method applied in the electroacoustics equivalent circuit theory is used in calculation of the entire system. Acoustic calculation is replaced with calculation of equivalent circuits, the characteristics of which are defined by well-known transfer matrices [1, 2, 3, 4].

This method is successfully used in calculation of simple elements of the exhaust system. As the design becomes more complicated, the calculations' accuracy falls dramatically, because it is difficult to build an equivalent circuit fully describing the operation of the elements and the exhaust system on the whole, it is difficult to derive an accurate analytical formula for a separate specific element, it is difficult to gather sufficient experimental statistics for deriving an empirical formula. In case of calculation using this method, it is difficult to consider the effect of temperature and gas flow on the acoustic parameters.

The finite-element modelling method has none of these disadvantages. It allows to accurately simulate acoustic behaviour of an exhaust system element of any complexity and to investigate the effect of the exhaust system geometric parameters and of such properties of the gas medium as density, temperature and flow rate on its acoustic parameters.

The finite-element modelling method is based on representation of the exhaust system element being investigated in the form of a gas volume filling this element and divided into a number of discrete volumes – finite elements approximating the geometric form of the whole element.

The calculation of transmission losses is performed based on the finite-element solution to the Helmholtz wave equation:

\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} + \frac{\omega}{c} p = 0
\]  
(8)

with the corresponding boundary conditions at the calculated volume boundaries, where \( p \) is vibration velocity potential, \( \omega \) is cyclic frequency, \( c \) is acoustic velocity in the medium.

The channel modes which were located at the end face of the muffler cylindrical surface "input" were selected as a source of sound waves. Any acoustic wave which propagates inside the channel is divided into incident and reflected waves.

\[
p = p^+ + p^-
\]  
(9)
The peculiarity of channel modes is as follows: only a plane wave propagates at low frequencies inside the channel, and at higher frequencies, the acoustic field may be more complex, but in any case, the pressure may be considered a mathematical superposition of channel modes, the formulas of which are shown below:

\[
p^+ = \sum_{\text{propagating mode}} a^+ \psi^+_i \tag{10}
\]

\[
p^- = \sum_{\text{propagating mode}} a^- \psi^-_i \tag{11}
\]

where \(a\) is a contribution factor, \(\psi\) is a mode form.

The boundary conditions for cylindrical surface reflecting walls are given as an example (see Fig. 3 and Formulas 4, 5).

\[
\frac{\partial p}{\partial x} \bigg|_{x=0} = \frac{\partial p}{\partial x} \bigg|_{x=r} = 0 \tag{12}
\]

\[
\frac{\partial p}{\partial y} \bigg|_{y=0} = \frac{\partial p}{\partial y} \bigg|_{y=r} = 0 \tag{13}
\]

**Figure 3.** Boundary conditions: reflecting wall of muffler model cylindrical surface.

Solution to the Helmholtz equation for round channel modes

\[
p(x, y, z) = A \cos(k_x x) \cos(k_y y)e^{ik_z z} \tag{14}
\]

true as long as the following ratio is maintained:

\[
k_x^2 + k_y^2 + k_z^2 = k^2 \tag{15}
\]

The following conditions shall be met for defining the boundary conditions of the channel modes with reflecting walls:

\[
k_x = m \frac{\pi}{r}, \quad k_y = n \frac{\pi}{r} \tag{16}
\]

From equation 7 taking into account equation 8, we define

\[
k_z = \sqrt{k^2 - \frac{\pi^2}{r^2} m^2 - \frac{\pi^2}{r^2} n^2} \tag{17}
\]

In case of excitation by a channel mode, the pressure profile in the cross-section is defined by Bessel functions and represents a set of waves. \(M\) and \(n\) are wave numbers showing how many half-waves fit along the radius of the cross-section.
Also, the red color shows (Figure 3) the position of the dissipative element (basalt fiber) in the muffler, the acoustic characteristics of which were determined by the calculation and experiment method.

The determination of the acoustic characteristics of the dissipative element (basalt fiber) used in the muffler was performed using the method of four microphones. A material sample of the d thickness is installed in the pipe between two pairs of microphones. On one end of the pipe, S speaker is placed, which generates sound waves in the pipes, and on the other end, there is L load.

Figure 4. Test bench layout for determining acoustic parameters of dissipative elements.

The measurements and determination of coefficients of the dissipative element sample transfer matrix were carried out using the technique described in TL Transmission Losses Determination section. The difference is in the fact that for such symmetrical system, "sound-absorbing material" dissipative element, the following ratios occur between the transfer matrix coefficients:

\[ T_{11} = T_{22} \]  
\[ T_{11} T_{22} - T_{12} T_{21} = 1 \]  

Taking into account these ratios, the coefficients may be determined according to the measurement results with only one load according to the following formulas:

\[ T_{11} = T_{22} = \frac{P_d V_d + P_u V_u}{P_d V_u + P_u V_d} \]  
\[ T_{12} = \frac{P_u^2 - P_d^2}{P_d V_u + P_u V_d} \]  
\[ T_{21} = \frac{V_u^2 - V_d^2}{P_d V_u + P_u V_d} \]

On the other hand, for the sound-absorbing material (SAM) sample, the transfer matrix coefficients are given by:

\[ \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} = \begin{bmatrix} \cos kd & (j/W) \sin kd \\ jW \sin kd & \cos kd \end{bmatrix} \]

where k, W are the constant of distribution and wave drag of the dissipative element respectively.
The obtained coefficients of the SAM transfer matrices were further used in the final procedure on determining acoustic parameters in accordance with the developed calculation program [5]. The results of the SAM wave parameters calculations are given in Figures 5 and 6.

**Figure 5.** Comprehensive density averaged by the SAM volume.

**Figure 6.** Comprehensive acoustic velocity averaged by the SAM volume.
Thus, we obtained the SAM parameters, which were used in the muffler calculation. The calculation results for the muffler transmission loss are shown in Figure 7.

![Figure 7. Calculation results for the muffler transmission loss.](image1)

To check the accuracy of the calculation using the finite-element model, the muffler bench tests were conducted. Under the bench conditions, within the operating frequency range, the TL transmission loss parameter is determined and then compared with the calculated values.

The experimental method of the TL determination is based on the sound pressure measurement in several points at the input and output of the muffler under investigation, calculation division into the incident and reflected wave, subsequent calculation of sound wave power at the input $W_{in}$ and output $W_{out}$ of the muffler (Figure 8) and the TL determination using the formula:

$$\text{TL} = 10 \log \left( \frac{W_{in}}{W_{out}} \right)$$  \hspace{1cm} (24)

![Figure 8. Exhaust system test setup.](image2)

The exhaust system element being tested (muffler) is located between the source of the sound waves on one end of the pipe and the sound-absorbing structure on the other end of the pipe. The input measurement microphones unit is installed between the acoustic radiator and the sample being tested, while the output measurement microphones unit is installed between the sample being tested and the sound absorber. During the testing, both amplitude and phase relations between the measured signals are measured in the form of a complex transmission (transfer) function [5].
Having determined the sound-pressure spectra at the positions of microphones P1 and P2 at the input of the exhaust system element, P3 and P4 at the output of the exhaust system element as well as transmission functions H21=P2/P1, H31=P3/P1, H41=P4/P1, one can find the sound-pressure and oscillation (particle) velocity spectra before the exhaust system element P_u and V_u and after the exhaust system element P_d and V_d:

\[
P_u = P_1 \frac{H_{21} \sin[k_0(l_{12}+l_2)] - \sin(k_0l_2)}{\sin(k_0l_{12})} = P_1 \tilde{P}_u
\]
\[
P_d = P_1 \frac{H_{31} \sin[k_0(l_{34}+l_1)] - H_{41} \sin(k_0l_1)}{\sin(k_0l_{34})} = P_1 \tilde{P}_u
\]
\[
V_u = jP_1 \frac{H_{21} \cos[k_0(l_{12}+l_2)] - \cos(k_0l_2)}{W_0 \sin(k_0l_{12})} = jP_1 \tilde{V}_u
\]
\[
V_d = jP_1 \frac{H_{41} \cos(k_0l_1) - H_{31} \cos[k_0(l_{34}+l_1)]}{W_0 \sin(k_0l_{34})} = jP_1 \tilde{V}_u
\]

where \(k_0 = \frac{\omega}{c_0}\) is the wave number; \(\omega\) is the cyclic frequency; \(c_0\) is the air acoustic velocity; \(W_0 = \rho_0 c_0\) is the air wave drag; \(\rho_0\) is the air density, \(l_{12}\) is the distance between the microphones at the element input; \(l_2\) is the distance between the input microphones and the element; \(l_{34}\) is the distance between the microphones at the element output; \(l_1\) is the distance from the element to the output microphones.

The sound-pressure and oscillation (particle) velocity spectra on both sides of the exhaust system element are related according to the following matrix equation:

\[
\begin{bmatrix}
P_u \\
V_u
\end{bmatrix}_{x=0} =
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
P_d \\
V_d
\end{bmatrix}_{x=d}
\]

\[(29)\]

where \(T\) is the transmission matrix of the exhaust system element under study.

This matrix equation represents a system of two equations with four unknown coefficients of the transmission matrix of the exhaust system element \(T_{11}, T_{12}, T_{21}, T_{22}\). Two more equations, which can be obtained by additional similar measurements with the second load on the measurement pipe output, thus implementing the two-load method, are necessary to find these coefficients.

Thus, we have:

\[
\begin{bmatrix}
P_u \\
V_u
\end{bmatrix}_{x=0} =
\begin{bmatrix}
T_{11} & T_{12} \\
T_{21} & T_{22}
\end{bmatrix}
\begin{bmatrix}
P_d \\
V_d
\end{bmatrix}_{x=d}, \quad i=1,2
\]

\[(30)\]

Here, the value of index \(i\) corresponds to the load number at the measurement pipe output. The solution of this equation system gives the following expressions for the transmission matrix coefficients:

\[
T_{11} = \frac{P_{u1}V_{d1} - P_{u2}V_{d2}}{P_{d1}V_{d2} - P_{d2}V_{d1}}, \quad T_{12} = \frac{P_{u1}P_{d2} - P_{u2}P_{d1}}{P_{d1}V_{d2} - P_{d2}V_{d1}}
\]

\[(31)\]
The TL transmission loss parameter is defined by the following expression:

\[ T_{21} = \frac{V_{u1} V_{d2} - V_{u2} V_{d1}}{P_{d1} V_{d2} - P_{d2} V_{d1}}; \quad T_{22} = \frac{P_{d1} V_{u2} - P_{d2} V_{u1}}{P_{d1} V_{d2} - P_{d2} V_{d1}} \]  

\[ TL = 20 \cdot \log \left( \frac{1}{2} \left| \frac{T_{11} + T_{12} + W_0 \cdot T_{21} + T_{22}}{W_0} \right| \right) \]  

The bench basic diagram and structural setup are shown in Figures 9 and 10.

The bench consists of five series-connected units, the middle one of which is the exhaust system element being tested. The input and output microphone measuring units, which are also connected through the adapter flanges with gaskets to the acoustic radiator at the input and the acoustic absorber at the output, are attached to the input and output of the test sample through adapter flanges with gaskets.

The microphone measuring units are hollow metal cylinders, each of which has 3 holes for measuring microphones with the diameter of 1/2 inch. The positions of the measuring microphones are indicated sequentially as No. 1, 2 and 3 for the input measuring unit and No. 4, 5 and 6 for the output measuring unit. The microphones are installed in the operational position on the measuring units using holders made in the form of glasses. The holders set the position of the microphone measuring...
membrane at the level of the inner surface of the measuring unit, and also provide reliable and stable fixation, electrical, vibration and acoustic isolation of the microphone housing from the bench.

As an acoustic radiator, a dynamic head (speaker) with a passband of 50 … 10 000 Hz is used, mounted in a soundproof column, connected with its radiating hole to the inlet flange of the input measuring microphones unit. Basalt fiber placed in a box is used as an acoustic absorber. The inlet hole of the absorber is connected to the output flange of the output measuring microphones unit. According to the test procedure, it is necessary to carry out measurements with two versions of absorbers having different absorption coefficients. As a second acoustic absorber, an additional disk in the form of acoustic foam is used. The absorption coefficients of the two loads are shown in Figure 2.6.

Figure 11. Experimental research results for absorbers. Absorption coefficients of the two loads: Z1- basalt fiber, Z2- acoustic foam.

Used equipment: Brüel & Kjær spectrum analyzer 3560C (with 3109 module) consisting of two signal generators capable of generating both a sinusoidal signal in the range from 2 Hz to 25.6 kHz, and white or pink noise, a measuring amplifier and a spectrum analyzer; Brüel & Kjær integrated microphones 4191C-001 with the frequency range of 3,15 Hz - 40 kHz (measured level 20 - 162 dB). Figure 12 shows the installation of the muffler on the bench.
The target TL parameter is determined based on measurement results using the calculations that include the air acoustic velocity and air density. Since these parameters are sensitive to temperature and atmospheric pressure, it is necessary to measure the specified values.

The microphone signals are transmitted through the amplifiers and commutator to the spectrum analyzer that determines the transfer functions between the signals. If all six microphones are used at the same time, a hardware multiplexer shall be used as a commutator, providing simultaneous measurement of all transfer functions in a complex form. However, to determine the target TL parameter, transfer functions are used, which are measured pairwise, and all the measurements shall be performed with only one pair of microphones. In this case, the microphones are one-by-one installed in all the necessary pairs of measuring positions, and the remaining holes during each measurement are closed with plugs imitating the measuring microphones.

The upper limit of the frequency range of the bench $F_u$ is selected from the condition of ensuring the propagation of plane waves in the pipe and preventing the occurrence of transverse oscillations of higher modes. For cylindrical pipes with an inner channel diameter $d$, this condition is formalized by ratio $d < 0.58 \lambda u$, where $\lambda u = c / F_u$ is the wavelength on the upper boundary frequency $F_u$, $c$ is the acoustic velocity in the medium. For a pipe with the diameter of 60 mm, we obtain upper frequency limit $F_u = 3286$ Hz. On the other hand, the $s$ distance between the microphones along the longitudinal axis of the pipe has an effect on $\lambda u$ and is chosen from $s < 0.45 \lambda u$ ratio, which also imposes a constraint on the frequency upper limit: $F_u < 0.45 c / s$.

The lower limit of the bench operating frequency also depends on the distance between the microphones $s$: to obtain the measurement error of no more than 1%, the distance between the microphones $s$ shall not exceed 5% of the sound wave length $\lambda d$, i.e. $F_d = 0.05 c / s$.

To extend the operating frequency range of the bench, the measurement microphones units are designed & built to allow measurement with three variants of the distance between the microphones: 30 mm, 70 mm and 100 mm. The tests for each sample are performed twice: for the low-frequency range of 170-1000 Hz, when the measurement microphones are installed at the distance of 100 mm, and for the high-frequency range of 1000-3300 Hz with the microphones installed at the distance of 30 mm. The measurement result processing and all calculations shall be performed for each frequency range, and the final result shall be formed by consolidation of the obtained results for the common frequency range of 170 - 3300 Hz.

As, when tested, only transmission functions between measuring channels 1 and 2 are measured, there is no need for absolute calibration of each measuring path. For amplitude-phase calibration and
definition of the complex amplitude-phase calibration factor, double measurement of the transmission function with mutual change of microphone positions in mounting seats of the test bench shall be used. The calibration factor defined this way will take into account differences in transmission of signal amplitude and phases for all measuring paths of channels 1 and 2. All values being measured shall be defined as a complex value and the calibration factor is as follows:

\[ H_c = |H_c| \times e^{i\phi} \]  

(34)

To obtain reliable values of the transmission function for the measured acoustic signals, when measuring, it is necessary to obtain the maximum value of the coherence function by selection of the optimum ratio between the acoustic signal level and amplification coefficient of the measuring paths. The following condition may be considered as acceptable:

\[ 0.95 \leq \text{Coherence Function} \leq 1.0 \]  

(35)

The exhaust system main element (muffler) acoustic characteristic calculation and experimental research result is given in figure 13:

\[ \text{Figure 13. Results of calculation and experimental research of the muffler by transmission loss parameter.} \]

At frequencies up to 200 Hz, there is a "diffusion" of experimental results connected with "cutting" (roll-off) of the unpowered installation speaker characteristics at low frequencies.

At frequencies from 1600 Hz, there is a deviation from calculated values connected with influence of natural frequencies and oscillation forms of housing parts and partitions of the muffler.

3. Conclusions

1. The developed calculation method allows defining acoustic characteristics of the main elements of the exhaust system;

2. The boundary conditions of the developed method, in particular, characteristics of basalt fiber were defined by calculation and experimental method;

3. The developed calculation method was validated with experiment;

4. For more exact calculation, it is necessary to consider natural frequencies and oscillation forms of housing parts of the exhaust system element.
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