Linear relations, monodromy and Jordan cells of a circle valued map.

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Abstract

In this paper we review the definition of the monodromy of an angle valued map based on linear relations as proposed in [3]. This definition provides an alternative treatment of the Jordan cells, topological persistence invariants of a circle valued maps introduced in [2].

We give a new proof that homotopic angle valued maps have the same monodromy, hence the same Jordan cells, and we show that the monodromy is actually a homotopy invariant of a pair consisting of a compact ANR $X$ and a one dimensional integral cohomology class $\xi \in H^1(X; \mathbb{Z})$.

We describe an algorithm to calculate the monodromy for a simplicial angle valued map $f : X \to S^1$, $X$ a finite simplicial complex, providing a new algorithm for the calculation of the Jordan cells of the map $f$.

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1 Introduction

Let $X$ be a compact ANR, $\xi \in H^1(X; \mathbb{Z})$ and $\kappa$ a field with algebraic closure $\overline{\kappa}$.

The $r-$monodromy, $r \in \mathbb{Z}_{\geq 0}$, is a similarity (= conjugacy) class of linear isomorphisms $T^{(X,\xi)}(r) : V_r(X, \xi) \to V_r(X, \xi)$, cf definition 2.1. The Jordan decomposition of a square matrix permits to assign to each $T^{(X,\xi)}(r)$ the collection $\mathcal{J}_r(X; \xi)$ of pairs $(\lambda, k), \lambda \in \pi \setminus 0, k \in \mathbb{Z}_{\geq 1}$, referred to as Jordan cells in dimension $r$.

If $f : X \to S^1$ is a tame map as in [2] and $\xi_f$ the cohomology class defined by $f$ then the set $\mathcal{J}_r(X; \xi_f)$ coincides with the set of Jordan cells $\mathcal{J}_r(f)$ considered in [2] in relation with the topological persistence of the circle valued map $f$.

Recall that topological persistence for a real or circle valued map $f : X \to \mathbb{R}$ or $f : X \to S^1$ ($S^1 = \mathbb{R}/2\pi \mathbb{Z}$) analyses the changes in the homology of the levels $f^{-1}(\theta), \theta \in \mathbb{R}$ or $\mathbb{S}$. It records the detectability and the death of homology of the levels in terms of bar codes cf. [2], or [4]. In case of a circle valued map in addition to death and detectability there is an additional feature of interest to be recorded, the return of some homology classes of $f^{-1}(\theta)$, when the angle $\theta$ increases or decreases with $2\pi$. This feature is recorded as Jordan cells which were introduced in [2], and describe what the topologists refer to as the homological monodromy or simply the monodromy.

In [3] we have proposed an alternative definition for Jordan cells and for monodromy based on linear relations. For background on linear relations the reader can consult [11] or [3] section 8.

In this paper we review this definition, provide a new geometric proof of its homotopy invariance (without any reference to Novikov homology used in [3]) and propose a new algorithm for the calculation of $\mathcal{J}_r(f) = \mathcal{J}_r(X; \xi_f)$, for $X$ a finite simplicial complex and $f$ a simplicial map.

In the present approach the monodromy is first defined for a continuous map $f : X \to S^1$ and a weakly regular angle $\theta \in S^1$ (see the definitions in section 3). Note that not all compact ANR’s admit enough many angle valued maps with weakly regular angles cf [10]. Note also that for a simplicial map all angles are weakly regular.

Proposition 3.4 shows that the monodromy proposed is independent of the weakly regular angle, remains the same for maps which have weakly regular angles and are homotopic and does not change when one replaces the map by its composition with the projection $X \times K \to X$ when $K$ is an acyclic compact ANR. These facts ultimately show that the monodromy can be associated to a pair $(X, \xi \in H^1(X; \mathbb{Z}))$, $X$ any compact ANR, and the assignment is a homotopy invariant of the pair $(X, \xi)$. All these facts are established in section 3, based on elementary linear algebra of bar codes summarized in section 2. The algorithm for calculating $\mathcal{J}_r(f)$ for $f$ a simplicial angle valued map is discussed in section 4. This algorithm can be also used for the calculation of the Alexander polynomial of a knot and of some type of Reidemeister torsions, useful topological invariants. These topological invariants can be regarded as particular cases of monodromy or derived from monodromy.

In section 3 we notice that a generalization of the homological the monodromy discussed in this paper can be obtained when the singular homology $H_r$ is replaced by a vector space valued homotopy functor $F$ which is half exact in the sense of A. Dold cf [6]. This $F-$ monodromy is not investigated in this paper but it might deserve attention.

Acknowledgements:

The idea of describing the Jordan cells considered in [2] using linear relations belongs to Stefan Haller and was pursued in [3] not yet in print.

It is a pleasure to thank S. Ferry for help in relation with the Appendix 2. and for bringing to our attention the reference [10].

1 The reader unfamiliar with the notion of ANR should always think to the main examples, spaces homeomorphic to simplicial or CW complexes.
2 Linear relations

Fix a field \( K \) and let \( \bar{K} \) be its algebraic closure.

2.1 Generalities

Recall from [11]:

- A linear relation \( R : V_1 \rightarrow V_2 \) is a linear subspace \( R \subseteq V_1 \times V_2 \). One writes \( v_1Rv_2 \iff (v_1, v_2) \in R, v_1 \in V_1 \).

- Two linear relations \( R_1 : V_1 \rightarrow V_2 \) and \( R_2 : V_2 \rightarrow V_3 \) can be composed in an obvious way, \((v_1(R_2 \cdot R_1)v_3) \iff \exists v_2 \) such that \( v_1R_1v_2 \) and \( v_2R_2v_3 \). The diagonal \( \Delta \subset V \times V \) is playing the role of the identity.

- Given a linear relation \( R : V_1 \rightarrow V_2 \) denote by \( R^\dagger : V_2 \rightarrow V_1 \) the relation defined by the property \( v_2R^\dagger v_1 \iff v_1Rv_2 \). Clearly \( (R_1 \cdot R_2)^\dagger = R_2^\dagger \cdot R_1^\dagger \) and \( R^{\dagger \dagger} = R \).

The familiar category of finite dimensional vector spaces and linear maps can be extended to incorporate all linear relations as morphisms. The linear map \( f : V_1 \rightarrow V_2 \) can be interpreted as the relation "graph \( f \subset V_1 \times V_2 \)" denoted by \([R(f)]\), providing the embedding of the category of vector spaces and linear maps in the category of vector spaces and linear relations. This extended category remains abelian.

- The direct sums \( R' \oplus R'' : V'_1 \oplus V'_2 \rightarrow V''_1 \oplus V''_2 \) of two relations \( R' : V'_1 \rightarrow V'_2 \) and \( R'' : V''_1 \rightarrow V''_2 \) is defined in the obvious way, \((v'_1, v''_1)(R' \oplus R'')(v'_2, v''_2) \iff (v'_1 R' v'_2) \) and \((v''_1 R'' v''_2) \).

One says that:

- The relation \( R' : V' \rightarrow W' \) and \( R'' : V'' \rightarrow W'' \) are isomorphic or equivalent and one writes \([R' \equiv R'']\) if there exists the linear isomorphisms \( \alpha : V' \rightarrow V'' \) and \( \beta : V'' \rightarrow V' \) s.t. \( R'' \cdot R(\alpha) = R(\beta) \cdot R' \).

- The relation with the same source and target \( R' : V' \rightarrow V' \) and \( R'' : V'' \rightarrow V'' \) are similar and one writes \([R' \sim R'']\) if there exists the linear isomorphisms \( \alpha : V' \rightarrow V'' \) s.t. \( R'' \cdot R(\alpha) = R(\alpha) \cdot R' \).

Recall that two linear endomorphisms \( T : V \rightarrow V \) and \( T' : V' \rightarrow V' \) are called similar if there exists a linear isomorphism \( C : V \rightarrow V' \) s.t. \( C^{-1} \cdot T' \cdot C = T \). One writes \( T \sim T' \) if \( T \) and \( T' \) are similar and one denotes the similarity class of \( T : V \rightarrow V \) by \([T] \); so \( T \sim T' \) and \([T] = [T'] \) mean the same thing.

As in the case of linear maps one denotes the similarity class of the relation \( R : V \rightarrow V \) by \([R] \). Clearly when \( T : V \rightarrow V \) is a linear map both notations \([T] \) and \([R(T)] \) means the same thing.

There are two familiar ways to describe a linear relation \( R : V \rightarrow W \). They are equivalent.

1. Two linear maps \( V_1 \xrightarrow{\alpha} W \xleftarrow{\beta} V_2 \) provide the relation

\[ R(\alpha, \beta) \subset V_1 \times V_2 := \{(v_1, v_2) \mid \alpha(v_1) = \beta(v_2)\} \]

2. Two linear maps \( V_1 \xleftarrow{a} U \xrightarrow{b} V_2 \) provide the relation

\[ R < a, b > \subset V_1 \times V_2 := \{(v_1, v_2) \mid \exists u, a(u) = v_1, b(u) = v_2\} \]

In view of 1. the category of linear relation can be regarded as the category of \( K_2 \) representations where \( K_2 \) denotes the Kronecker quiver with two vertices \( a \) and \( b \) and two oriented arrows from \( a \) to \( b \), cf [1].

A linear relation \( R : V \rightarrow W \) gives rise to the following subspaces:

\[
\begin{align*}
\text{dom}(R) & := \{v \in V \mid \exists w \in W : vRw\} = pr_V(R) \\
\text{img}(R) & := \{w \in W \mid \exists v \in V : vRw\} = pr_W(R) \\
\text{ker}(R) & := \{v \in V \mid vR0\}\cong V \times 0 \cap R \\
\text{mul}(R) & := \{w \in W \mid 0Rw\}\cong 0 \times W \cap R \\
\end{align*}
\]

Here \( pr_V \) and \( pr_W \) denote the projections of \( V \times W \) on \( V \) and \( W \). We have
Observation 2.1

1. \( \ker(R) \subseteq \text{dom}(R) \subseteq V \) and \( W \supseteq \text{img}(R) \supseteq \text{mul}(R) \),

2. \( \ker(R^\dagger) = \text{img}(R) \) and \( \text{dom}(R^\dagger) = \text{img}(R) \),

3. \( \dim \text{dom}(R) + \dim \ker(R^\dagger) = \dim(R) = \dim \text{dom}(R^\dagger) + \dim \ker(R) \).

It is immediate, in view of the above definitions and observations that:

Lemma 2.2

1. A linear relation \( R : V \sim W \) is of the form \( R(f) \) for \( f : V \to W \) linear map iff \( \text{dom}R = V \) and \( \text{mul}R = 0 \).

2. A linear relation \( R : V \sim V \) is of the form \( R(T) \) for \( T : V \to V \) a linear isomorphism iff \( \text{dom}R = V \) and \( \ker R = 0 \).

Let \( R : V \sim V \) be a linear relation. Define

1. \( D = \{ v \in V \mid \exists v_i \in V, i \in \mathbb{Z}, v_i R v_{i+1}, v_0 = v \} \). The relation \( R \) restricts to a relation \( R_D : D \sim D \)

2. \( K_+ := \{ v \in V \mid \exists v_i, i \in \mathbb{Z}_0, v_i R v_{i+1}, v_0 = v \} \)

3. \( K_+ := \{ v \in V \mid \exists v_i, i \in \mathbb{Z}_0, v_i R v_{i+1}, v_0 = v \} \)

4. \( V_{\text{reg}} := \frac{D_{D \cap (K_+ + K_-)}}{D_{D \cap (K_+ + K_-)}} \), \( \pi D \to \frac{D_{D \cap (K_+ + K_-)}}{D_{D \cap (K_+ + K_-)}} \) and \( I : D \to V \) the inclusion.

Consider the composition of relations

\[ R_D = R(I)^\dagger \cdot R \cdot R(I) \]

and

\[ R_{\text{reg}} := R(I) \cdot R_D \cdot R(I)^\dagger : V_{\text{reg}} \sim V_{\text{reg}}. \]

Proposition 2.3 (cf [3])

1. There exists a linear isomorphism \( T^R : V_{\text{reg}} \to V_{\text{reg}} \) such that \( R_{\text{reg}} = R(T^R) \).

2. If \( R : V \sim V \) and \( R' : V' \sim V' \) are similar relations, i.e. there exists an isomorphism of vector spaces \( \omega : V \to V' \) such that \( R' = R(\omega) \cdot R \cdot R(\omega)^{-1} \), then \( T^R \) and \( T^{R'} \) are similar linear isomorphisms (precisely \( T^{R'} = \omega \cdot T^R \cdot \omega^{-1} \) for some induced isomorphism \( \omega \)).

3. \( R_{\text{reg}}^{-1} = (R^\dagger)_{\text{reg}} \).

4. \( (R' \oplus R'')_{\text{reg}} = R'_{\text{reg}} \oplus R''_{\text{reg}} \).

5. Suppose \( R_i : V_i \sim V_{i+1}, i = 1, 2, \cdots k \) with \( V_1 = V_{k+1} \) then \( (R_1 \cdots R_{i-1} \cdots R_1 \cdots R_k \cdots R_{i-1} \cdots R_1)_{\text{reg}} \sim (R_k \cdots R_2 \cdots R_1)_{\text{reg}} \) where we continue to write \( R_{\text{reg}}' \sim R''_{\text{reg}} \) if \( T^{R'} \sim T^{R''} \).

In view of the definition of \( R_{\text{reg}} \) it is immediate that:

Observation 2.4

1. If \( \alpha, \beta : V \to W \) are two isomorphisms then \( T^{R(\alpha, \beta)} = \beta^{-1} \cdot \alpha \).

2. If \( f : V \to V \) is a linear map and \( V_0 \) is the generalized eigen-space of the eigenvalue 0 then \( f(V_0) \subset V_0 \), \( f \) induces \( \hat{f} : V/V_0 \to V/V_0 \) and \( T^{R(f)} \sim \hat{f} : V/V_0 \to V/V_0 \).
The following rather technical Proposition will be used in section 4.3, where an algorithm for the calculation of \( R(a, b)_{\text{reg}} \) will be presented as a part of an algorithm for the calculation of \( r - \) monodromy.

**Proposition 2.5**

1. Consider the diagram

\[
\begin{array}{c}
V \xrightarrow{\alpha} W \xleftarrow{\beta} V \\
\downarrow \quad \downarrow \quad \downarrow \\
V' \xrightarrow{\alpha'} W' \xleftarrow{\beta'} V'
\end{array}
\]

(1)

with \( W' \supseteq \text{img}\alpha \cap \text{img}\beta V' = \alpha^{-1}(W') \cap \beta^{-1}(W') \) and \( \alpha' \) and \( \beta' \) the restriction of \( \alpha \) and \( \beta \). Then \( R(\alpha, \beta)_{\text{reg}} = R(\alpha', \beta')_{\text{reg}} \)

2. Consider the diagram

\[
\begin{array}{c}
V \xrightarrow{\alpha} W \xleftarrow{\beta} V \\
\downarrow \quad \downarrow \quad \downarrow \\
V' \xrightarrow{\alpha'} W' \xleftarrow{\beta'} V'
\end{array}
\]

(2)

with both \( \alpha \) and \( \beta \) surjective. Define

\[
\begin{align*}
V' &= V/\ker\alpha, \ W' = W/\beta(\ker\alpha) \\
p : W \to W', \ p' : V \to V' & \text{ the canonical quotient maps} \\
\overline{\alpha} : V' \to W & \text{ induced from } \alpha, \ \alpha' = p \cdot \overline{\alpha} \\
\beta' & \text{ induced by passing to quotient from } \beta.
\end{align*}
\]

Then \( R(\alpha, \beta)_{\text{reg}} = R(\alpha', \beta')_{\text{reg}} \)

For the reader’s convenience the proofs of Propositions 2.3 and 2.5 are included in Appendix 1.

**2.2 Jordan cells, characteristic polynomial and its characteristic divisors**

Recall that a Jordan matrix is determined by a pair \((\lambda, k)\), \(\lambda \in \mathbb{K}\) and \(k\) a positive integer and when \(\lambda \neq 0\) is is called Jordan cell. It is defined by:

\[
T(\lambda; k) = \begin{pmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \lambda
\end{pmatrix}
\]

Any invertible square \(n \times n\) matrix is conjugated with a direct sum of Jordan cells (by Jordan decomposition theorem, cf [8]) with \(\lambda\) eigenvalue of the matrix. In different words any conjugacy class of linear isomorphism \(T : V \to V\) denoted by \([T]\) is determined by a unique collection of pairs = Jordan cells \(\mathcal{J}(T) \equiv \mathcal{J}([T])\). Any such collection determines and is determined by the collection of monic polynomials

\[
P^T(z) | P_1^T(z) | P_2^T(z) | \cdots | P_{n-1}^T(z)
\]
where \( P^T(z) = \det(zI - T) \) and \( P^T_1(z) \) is the greatest common divisor of all \((n - i) \times (n - i)\) minors of \( zI - T \) cf [8].

Note that the polynomials \( P^T(z)|P^T_1(z)|P^T_2(z)| \cdots \) do not involve the algebraic closure \( \bar{\kappa} \).

The precise relation between them and the elements of \( J([T]) \) is given in [8].

**Definition 2.6** The Jordan cells of the linear relation \( R: V \rightarrow V \) is the collection \( J([T^{reg}]) \).

### 3 Monodromy

In this section the homology of a space \( X \) is the singular homology with coefficients in a field \( \kappa \) fixed once for all and is denoted by \( H_r(X), r = 0, 1, 2, \ldots \).

An angle is a complex number \( \theta = e^{it} \in \mathbb{C}, t \in \mathbb{R} \) and the set of all angles is denoted by \( S^1 = \{ \theta = e^{it} \mid t \in \mathbb{R} \} \). The space of angles, \( S^1 \), is equipped with the distance

\[
d(\theta_2, \theta_2) = \inf \{ |t_2 - t_1| \mid e^{it_1} = \theta_1, e^{it_2} = \theta_2 \}.
\]

In this paper all real valued or angle valued maps \( f: X \rightarrow \mathbb{R} \) or \( f: X \rightarrow S^1 \) are proper continuous maps with \( X \) an ANR. The properness of \( f \) forces the space \( X \) to be locally compact in the first case and compact in the second.

- A value \( t \in \mathbb{R} \) or \( \theta \in S^1 \) is weakly regular if \( f^{-1}(t) \) or \( f^{-1}(\theta) \) is an ANR, hence a compact ANR[2]

- A map \( f \) whose set of weakly regular values is not empty is called good and a map with all values weakly regular is called weakly tame. If \( X \) is compact real valued map \( f \) is always good for trivial reasons.

- An ANR \( X \) whose set of good angle valued maps maps is dense in the space of all maps with the \( C^0 \) compact open topology is called a good ANR.

- An ANR with the property that the set of all weakly tame maps is dense in the set of all maps with \( C^0 \) fine topology is called very good. Clearly very good implies good. The tame maps considered in [2] are weakly tame with the domain= source a very good ANR.

There exist compact ANR’s (actually compact homological n-manifolds, cf [10]) with no co-dimension one subsets which are ANR’s, hence compact ANR’s which are not good ANR’s.

- The spaces homeomorphic to simplicial complexes, or finite dimensional topological manifolds, or Hilbert cube manifolds (see Appendix 2 for definitions) are all very good ANR’s. The first because any continuous map can be approximated by simplicial maps w.r. to a convenient subdivision, the last by the more subtle reasons explained in Appendix 2.

For this paper the concepts of good map, good ANR, very good ANR will be considered under the hypothesis that the space is compact.

As pointed out in introduction, the \( r-\)monodromy is first defined for good maps and involves an angle \( \theta \), a weakly regular value. It will be shown that the angle is irrelevant. It will be also shown that the \( r-\)monodromy depends only on the cohomology class \( \xi_f \) associated with the map.

Once some elementary properties are established, it will be shown that the \( r-\)monodromies can be associated to any angle valued map and provide homotopy invariant of any pair \( (X, \xi \in H^1(X; \mathbb{Z})) \) for \( X \) any compact ANR.

The following observations will be useful.

**Proposition 3.1**

---

\[\footnote{A compact ANR has the homotopy type of finite simplicial complex.} \]
1. Two maps \( f, g : X \to \mathbb{S}^1 \) with \( D(f, g) = \sup_{x \in X} d(f(x), g(x)) < \pi \) are homotopic by a canonical homotopy, the convex combination homotopy.

2. Suppose \( X \) is a good ANR \( f, g : X \to \mathbb{S}^1 \) are two maps which are homotopic and \( \epsilon > 0 \). There exists a finite collection of maps \( f_0, f_2, \cdots f_k, f_{k+1} \), such that:
   a) \( f_0 = f, f_{k+1} = g \),
   b) \( f_i \) are good maps for \( i = 1, 2, \cdots k \),
   c) \( D(f_i, f_{i+1}) < \epsilon \).

   Indeed if \( f \) and \( g \) are viewed as maps with values in \( \mathbb{C} \) then the map \( h_t(x) = \frac{tg(x) + (1-t)f(x)}{|tg(x) + (1-t)f(x)|} \) \( 0 \leq t \leq 1 \) provides the desired homotopy stated in item 1. The condition \( D(f(x), g(x)) < \pi \) insures that \( |tg(x) + (1-t)f(x)| \neq 0 \).

   Item 2. follows from the local contractibility of the space of maps when equipped with the distance \( D \).

3.1 Real valued maps

For \( f : X \to \mathbb{R} \) a real valued map and \( a \in \mathbb{R} \) denote by:

\[ X^f_a, \text{ the sub-level } X^f_a := f^{-1}((-\infty, a]); \] if \( a \) is weakly regular value then \( X^f_a := f^{-1}((-\infty, a]) \) is an ANR,

\[ X^f_a, \text{ the super-level } X^f_a := f^{-1}([a, \infty)); \] if \( a \) is weakly regular value then \( X^f_a := f^{-1}([a, \infty)) \) is an ANR.

For \( f : X \to \mathbb{R} \) and \( g : X \to \mathbb{R} \) maps as above and \( a < b \) s.t \( f^{-1}(a) \subset g^{-1}((-\infty, b) \) denote by

\[ X^f_{a,b} := X^f_a \cap X^g_b, \]

if \( b \) is a weakly regular value for \( g \) and \( a \) is weakly regular value for \( f \) then \( X^f_{a,b} \) is a compact ANR. This insures that \( H_r(g^{-1}(a)), H_r(f^{-1}(b)) \) and \( H_r(X^f_{a,b}) \) have finite dimension.

Denote by \( R^{f,g}_{a,b}(r) \) the linear relation defined by inclusion induced linear maps \( i_1(r) \) and \( i_2(r) \).

\[ H_r(g^{-1}(a)) \xrightarrow{i_{1}(r)} H_r(X^f_{a,b}) \xrightarrow{i_{2}(r)} H_r(f^{-1}(b)). \]

**Proposition 3.2** Let \( t_1 < t_2 < t_3 \). Suppose that \( t_1 \) is weakly regular for \( f \), \( t_2 \) is weakly regular for \( g \) and \( g^{-1}(t_2) \subset f^{-1}((t_1, t_3)) \). Then one has

\[ R^{g,f}_{t_2, t_3}(r) : R^{f,g}_{t_1, t_2} = R^{f,f}_{t_1, t_3}(r). \]

**Proof:** The verification is a consequence of the exactness of the following piece of of Meyer Vietoris sequence

\[ H_r(g^{-1}(t_2)) \xrightarrow{i'_{1} \oplus i'_{2}} H_r(X^f_{t_1, t_2}) \oplus H_r(X^g_{t_2, t_3}) \xrightarrow{i_1 - i_2} H_r(X^f_{t_1, t_3}) \]

whose linear maps involved in the sequence \( \text{(3)} \) and part of the commutative diagram below are induced by obvious inclusions.

\[ H_r(f^{-1}(t_3)) \xrightarrow{j_1} H_r(X^f_{t_1, t_3}) \xrightarrow{i_1} H_r(g^{-1}(t_2)) \xrightarrow{i_2} H_r(X^g_{t_2, t_3}) \xrightarrow{j_2} H_r(f^{-1}(t_3)) \]

\[ H_r(X^f_{t_1, t_3}) \]

\[ H_r(f^{-1}(t_3)) \xrightarrow{j_1} H_r(X^f_{t_1, t_3}) \xrightarrow{i_1} H_r(g^{-1}(t_2)) \xrightarrow{i_2} H_r(X^g_{t_2, t_3}) \xrightarrow{j_2} H_r(f^{-1}(t_3)) \]
Note that in \([3]\) and \([4]\) to lighten the writing, ”\(r\)” was dropped off the notation. We will continue to do so when possible.

Indeed for \(x \in H_r(f^{-1}(t_1))\) and \(y \in H_r(f^{-1}(t_3))\) the commutativity of the diagram \([3]\) implies that \(xR^{f,g}_{t_1,t_3}y\) iff \(i_1(j_1(x)) - i_2(j_2(y)) = 0\).

By the exactness of the sequence \([3]\) one has \(i_1(j_1(x)) - i_2(j_2(y)) = 0\) iff there exists \(u \in H_r(g^{-1}(t_2))\) such that \((i_1' \oplus i_2')(u) = (j_1(x), j_2(y))\). This happens iff \(xR^{f,g}_{t_1,t_2}u\) and \(uR^{g,f}_{t_2,t_3}y\), which means \(xR^{f,g}_{t_1,t_2}y\).

\[\square\]

### 3.2 Angle valued maps

Let \(f : X \to \mathbb{S}^1\) be an angle valued map. Let \(u \in H^1(S^1; \mathbb{Z}) \cong \mathbb{Z}\) be the generator defining the orientation of \(\mathbb{S}^1\). Here \(\mathbb{S}^1\) is regarded as an oriented one dimensional manifold. Let \(f^* : H^1(\mathbb{S}^1; \mathbb{Z}) \to H^1(X; \mathbb{Z})\) be the homomorphism induced \(f\) in integral cohomology and \(\xi_f = f^*(u) \in H^1(X; \mathbb{Z})\).

It is well known fact in homotopy theory that the assignment \(f \mapsto \xi_f\) establishes a bijective correspondence between the set of homotopy classes of continuous maps from \(X\) to \(\mathbb{S}^1\) and \(H^1(X; \mathbb{Z})\).

**The cut at \(\theta\) with respect to \(f\)**

For \(\theta \in \mathbb{S}^1\) a weakly regular value for \(f\) define the cut at \(\theta = e^{it}\), to be the space \(\overline{X}^f_{\theta}\), the two sided compactification of \(X \setminus f^{-1}(\theta)\) with sides \(f^{-1}(\theta)\). Precisely as a set \(\overline{X}^f_{\theta}\) is a disjoint union three parts, \(\overline{X}^f_{\theta} = f^{-1}(\theta)(1) \sqcup f^{-1}(\mathbb{S}^1 \setminus \theta) \sqcup f^{-1}(\theta)(2)\), with \(f^{-1}(\theta)(1)\) and \(f^{-1}(\theta)(2)\) two copies of \(f^{-1}(\theta)\).

The topology on \(\overline{X}^f_{\theta}\) is the only topology which makes \(\overline{X}^f_{\theta}\) compact and the map from \(\overline{X}^f_{\theta}\) to \(X\) defined by identity on each part continuous. The compact space \(\overline{X}^f_{\theta}\) is a compact ANR.

We have \(f^{-1}(\theta) \xrightarrow{i_1} \overline{X}^f_{\theta} \xrightarrow{i_2} f^{-1}(\theta)\) with \(i_1, i_2\) the obvious inclusions which induce in homology in dimension \(r\) the linear maps (between finite dimensional vector spaces)

\[
H_r(f^{-1}(\theta)) \xrightarrow{i_1(r)} \overline{H}_r(\overline{X}^f_{\theta}) \xrightarrow{i_2(r)} H_r(f^{-1}(\theta)).
\]

These linear maps define the linear relation \(R(i_1(r), i_2(r)) := R^f_{\theta}(r)\) and then the relation \((R^f_{\theta}(r))_{\text{reg}}\).

**Definition 3.3** The \(r\)– monodromy of \(f : X \to \mathbb{S}^1\) at \(\theta \in \mathbb{S}^1\), for \(\theta\) a weakly regular value, is the similarity class \([R^f_{\theta}(r)]_{\text{reg}}\) of the linear relation \((R^f_{\theta}(r))_{\text{reg}}\), equivalently the similarity class of the linear isomorphism

\[
T^{(R^f_{\theta}(r))_{\text{reg}}} : V_{\text{reg}}(R^f_{\theta}(r)) \to V_{\text{reg}}(R^f_{\theta}(r)).
\]

We will abbreviate the linear isomorphism \(T^{(R^f_{\theta}(r))_{\text{reg}}}\) to \(T^f_{\theta}(r)\) and denote the similarity class of the linear relation \((R^f_{\theta}(r))_{\text{reg}}\) by \([T^f_{\theta}(r)]\).

For a map \(f : X \to \mathbb{S}^1\) and \(K\) a compact ANR denote by

\[
\overline{f}_{K} : X \times K \to \mathbb{S}^1,
\]

the composition of \(f\) with the projection of \(X \times K\) on \(X\). Note that if \(\theta\) is a weakly regular value for \(f\) it remains a weakly regular value for \(\overline{f}_{K}\) and \((X \times K)_{\theta} = \overline{X}^f_{\theta} \times K\). Therefore in view of the Kunneth formula (expressing the homology of the product of two spaces) one has

\[
V_{\text{reg}}(R_{\theta}^{f \times K}(r)) = \bigoplus_l V_{\text{reg}}(R_{\theta}^f(r - l)) \otimes H_l(K)
\]

\[
T^f_{\theta}^{K}(r) = \bigoplus_l T^f_{\theta}(r - l) \otimes Id_{H_l(K)}
\]

(5)
Proposition 3.4

1. If $\theta_1$ and $\theta_2$ are two different weakly regular angles of $f$ then $[T_{\theta_1}^f (r)] = [T_{\theta_2}^f (r)]$.

2. If $f, g : X \to S^1$ are two maps with $\theta_1$ a weakly regular value for $f$ and $\theta_2$ a weakly regular value for $g$ and $D(f, g) < \pi$ then $[T_{\theta_1}^f (r)] = [T_{\theta_2}^g (r)]$.

3. If $f : X \to S^1$ and $g : Y \to S^1$ are two maps with $\theta_1$ weakly regular value for $f$ and $\theta_2$ weakly regular value for $g$ then $[T_{\theta_1}^f (r)] = [T_{\theta_2}^g (r)]$ iff $[T_{H_0}^g (r)] = [T_{\theta_2}^g (r)]$.

4. If $f : X \to S^1$ and $g : Y \to S^1$ are two maps with $\theta_1$ weakly regular value for $f$ and $\theta_2$ weakly regular value for $g$, and $\omega : X \to Y$ is a homeomorphisms such that $g \cdot \omega$ and $f$ are homotopic then $[T_{\theta_1}^f (r)] = [T_{\theta_2}^g (r)]$.

Proof:

Proof of 1.: For $X$ a compact ANR and $\xi \in H^1(X; \mathbb{Z})$ consider $\pi : \tilde{X} \to X$ an infinite cyclic cover associated to $\xi$.

Any map $f : X \to S^1$ such that $f^*(u) = \xi$, $u$ the canonical generator of $H^1(S^1)$, has lifts $\tilde{f} : \tilde{X} \to \mathbb{R}$, which make the diagram below a pull-back diagram

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{p} & S^1 \\
\downarrow f & & \downarrow f \\
\tilde{X} & \xrightarrow{\pi} & X
\end{array}
\] (8)

Here $p(t)$ is given by $p(t) = e^{it} \in S^1$.

Consider $\theta_1 = e^{it_1}, \theta_2 = e^{it_2} \in S^1$ two weakly regular values for $f$ with $t_2 - t_1 \leq \pi$ hence $t_1 < t_2 < t_1 + 2\pi < t_2 + 2\pi$. We apply the discussion in subsection 3.1 to the real valued map $\tilde{f} : \tilde{X} \to \mathbb{R}$ and note that

\[
R_{\theta_1}^f = R_{t_1, t_1+2\pi}^{\tilde{f}, \tilde{f}} = R_{t_2, t_1+2\pi}^{\tilde{f}, \tilde{f}} \cdot R_{t_1, t_2}^{\tilde{f}, \tilde{f}}
\]

and

\[
R_{\theta_2}^f = R_{t_2, t_2+2\pi}^{\tilde{f}, \tilde{f}} = R_{t_1+2\pi, t_2+2\pi}^{\tilde{f}, \tilde{f}} \cdot R_{t_1, t_2}^{\tilde{f}, \tilde{f}}.
\]

---

3 An infinite cyclic cover is a map $\pi : \tilde{X} \to X$ together with a free action $\mu : \mathbb{Z} \times \tilde{X} \to \tilde{X}$ which satisfies $\pi(\mu(n, x)) = \pi(x)$ such that the map induced by $\pi$ from $\tilde{X}/\mathbb{Z}$ to $X$ is a homeomorphism. The above covering is called associated to $\xi$ if any $\tilde{f} : \tilde{X} \to \mathbb{R}$ which satisfies $\tilde{f}(\mu(n, x)) = \tilde{f}(x) + 2\pi n$ induces a map from $X$ to $\mathbb{R}/2\pi \mathbb{Z} = S^1$ representing the cohomology class $\xi(f) = \xi$. Any two infinite cyclic cover $\pi_1 : \tilde{X}_1 \to X$ representing $\xi$ are isomorphic, namely there exists an homeomorphism $\omega : \tilde{X}_1 \to \tilde{X}_2$ which intertwines the free actions $\mu_1$ and $\mu_2$ and satisfies $\pi_2 \cdot \omega = \pi_1$.  

9
Using the linear isomorphisms induced by 1, the linear relations \( R_{t_1,t_2}(r) \) and \( R_{t_1+2\pi,t_2+2\pi}(r) \) can be identified to the linear relation \( R'(r) := R_{\theta_1}(r) : H_r(f^{-1}(\theta_1)) \to H_r(f^{-1}(\theta_2)) \) while \( R_{t_2,t_1+2\pi}(r) \) to the linear relation \( R''(r) = R_{\theta_2}(r) : H_r(f^{-1}(\theta_2)) \to H_r(f^{-1}(\theta_2)). \)

Therefore \( R_{\theta_1} = R' \cdot R'' \) and \( R_{\theta_2} = R' \cdot R'' \) which implies, in view of Proposition 2.3(5), imply that \( (R_{\theta_1})_{\text{reg}} \sim (R_{\theta_2})_{\text{reg}}. \)

**Proof of 2:** Let \( f, g : X \to S^1 \) be two continuous maps as in item 2. By Proposition 3.1(1) they are homotopic hence \( \xi_f = \xi_g. \) For any infinite cyclic cover \( \tilde{X} \to X \) associated with \( \xi = \xi_f = \xi_g \) both \( f \) and \( g \) have lifts \( \tilde{f} \) and \( \tilde{g} \) as indicated in the diagrams below.

\[
\begin{align*}
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{P} & S^1 \\
\downarrow f & & \downarrow f \\
\tilde{X} & \xrightarrow{\pi} & X
\end{array}
& \quad & \begin{array}{ccc}
\mathbb{R} & \xrightarrow{P} & S^1 \\
\downarrow \tilde{g} & & \downarrow g \\
\tilde{X} & \xrightarrow{\pi} & X
\end{array}
\end{align*}
\]

These lifts can be chosen to satisfy \( |\tilde{f}(x) - \tilde{g}(x)| < \epsilon \) and therefore \( g^{-1}(t_2) \subset \tilde{f}^{-1}(t_1, t_1 + 2\pi) \) and \( \tilde{f}^{-1}(t_1 + 2\pi) \subset \tilde{g}^{-1}(t_2, t_2 + 2\pi) \). We apply the considerations in subsection 3.1 to the real valued maps \( \tilde{f}, \tilde{g} : \tilde{X} \to \mathbb{R} \) and conclude that:

\[
R_{\theta_1} = R_{t_1,t_2}(r) = R_{t_1+2\pi,t_2}(r) \cdot R_{t_1+2\pi,t_2}.
\]

and

\[
R_{\theta_2} = R_{t_1,t_2+2\pi} = R_{t_1+2\pi,t_2+2\pi} \cdot R_{t_1+2\pi,t_2+2\pi}.
\]

Let \( R' := R_{t_1,t_2+2\pi} \) and \( R'' := R_{t_1+2\pi,t_2+2\pi} \). Then \( R_{\theta_1} = R' \cdot R'' \) and \( R_{\theta_2} = R' \cdot R'' \) which, by Proposition 2.3(5), imply that \( (R_{\theta_1})_{\text{reg}} \sim (R_{\theta_2})_{\text{reg}}. \)

**Proof of 3:**

Recall that for a linear isomorphism \( T : V \to V \) one denotes by \( J(T) \) the set of Jordan cells which is a similarity invariant.

First observe that if \( T_1 : V_1 \to V_1 \) and \( T_2 : V_2 \to V_2 \) are two linear isomorphism then \( J(T_1 \oplus T_2) = J(T_1) \cup J(T_2). \)

If so \( [T_1 \oplus T_2] = [T_1'] \cup [T_2'] \) hence \( J([T_1]) \cup J([T_2]) = J([T_1']) \cup J([T_2']) \) and \([T_1] = [T_1'] \) since \( J([T_1]) = J([T_1']) \) imply \( J([T_2]) = J([T_2']) \) hence \( [T_2] = [T_2'] \).

We apply this observation to \( T_1 = T_{\theta_1}(r - 1), T_1 = T_{\theta_2}(r - 1) \) and \( T_2 = T_{\theta_1}(r), T_2 = T_{\theta_2}(r) \). Then \( \odot \) implies item 3.

**Proof of 4:** In view of item 2. one has \( [T_{\theta_2}^\omega(r)] = [T_{\theta_1}^\omega(r)] \).

Since \( \omega \) induces a homeomorphism between \( \tilde{X}_{\theta_2} \) and \( \tilde{Y}_{\theta_2} \) then \( R_{\theta_2} \sim R_{\theta_2}^\omega \) which implies \( [T_{\theta_2}^\omega] = [T_{\theta_2}] \) which implies item 4..
with \([T^f(r)]\). Moreover, based on Proposition \[3.1\] if \(f\) and \(g\) are homotopic then \([T^f(r)] = [T^g(r)]\). Then for \(X\) a good ANR and \(\xi \in H^1(X; \mathbb{Z})\) one chooses \(f\), with \(\xi_f = \xi\), and one defines

\[T^{(X;\xi)}(r) := [T^f(r)].\]

In order to show that \([T^{(X;\xi)}(r)]\) can be extended to any compact ANR \(X\) and that it is a homotopy invariant of the pair \((X, \xi)\), one uses Proposition \[3.4\] (3) and (4) and the Stabilization Theorem below. This theorem is a consequence of remarkable topological results of Edwards and Chapman about Hilbert cube manifolds, cf [7]. An homological proof is also possible but requires more algebraic topology, cf [3].

**Theorem 3.5 Stabilization theorem (R. Edwards and T. Chapman)**

1. For any compact ANR there exists \(K\), a contractible compact ANR, such that \(X \times K\) is a very good compact ANR.

2. Given \(\omega : X \to Y\) a homotopy equivalence of compact ANR’s there exists a contractible compact ANR \(K\) such that \(\omega \times \text{id}_{K \times S^1} : X \times K \times S^1 \to Y \times K \times S^1\) is homotopic to a homeomorphism \(\omega' : X \times K \times S^1 \to Y \times K \times S^1\).

The statements above are rather straightforward consequences of Edwards and Chapman results however neither 1. nor 2., as formulated above, can be found in their work or in [7]. They can be derived from the mathematics presented in [7] as explained in Appendix 2. The compact ANR \(K\) claimed above is actually the Hilbert cube \(Q\), the product of countable many copies of the segment \([0, 1]\).

**Extension of \(r\)–monodromy to all pairs \((X, \xi)\)**

To any pair \((X, \xi)\), \(X\) compact ANR, \(\xi \in H^1(X; \mathbb{Z})\), for any \(r \in \mathbb{Z}_{\geq 0}\), one defines the \(r\)–monodromy by

\[T^{X;\xi}(r) := [T^{X \times K, \xi}(r)]\]

with \(\xi\) is the pull back of \(\xi\) by the projection of \(X \times K \to X\). In view of the equality [7] if \(X\) was already a good ANR then \([T^{X;\xi}(r)] = [T^{X \times K, \xi}(r)]\).

To verify the homotopy invariance consider \(f_i : X_i \to S^1\) representing the cohomology class \(\xi_i\). Since \(\omega^*(\xi_2) = \xi_1\) the composition \(f_2 \cdot \omega\) and \(f_1\) are homotopic and then in view of item 2. of Stabilization Theorem one has the homeomorphism \(\omega'\) homotopic to \(\omega \times \text{id}_{K \times S^1}\). Hence \((f_2)_{K \times S^1} \cdot \omega'\) is homotopic to \((f_1)_{K \times S^1}\). This, in view of Proposition \[3.4\] (4), implies that \([T^{(f_2)_{K \times S^1}}(r)] = [T^{(f_1)_{K \times S^1}}(r)]\) and in view of Proposition \[3.4\] (3) implies \([T^{(f_2)_{K \times S^1}}(r)] = [T^{(f_1)_{K \times S^1}}(r)]\), hence \([T^{(X;\xi_1)}(r)] = [T^{(X;\xi_2)}(r)]\).

As a summary of the considerations above one has the following Theorem.

**Theorem 3.6** To any pair \((X, \xi)\), and \(r = 0, 1, 2, \ldots\), \(X\) compact ANR, and \(\xi \in H^1(X; \mathbb{Z})\) one can associate the similarity class of linear isomorphisms \([T^{X;\xi}(r)]\) which is a homotopy invariant of the pair. When \(f : X \to S^1\) is a good map with \(\xi_f = \xi\) is the \(r\)–monodromy defined for a good map \(f\) and a weakly regular value.

The collection \(J_f(X; \xi)\) consisting of the pairs with multiplicity, \((\lambda, k)\), \(\lambda \in \pi, k \in \mathbb{Z}_{\geq 0}\), which determine the similarity class \([T^{(X;\xi)}(r)]\) as \([\oplus_{(\lambda, k) \in J_f(\xi)} T(\lambda, k)]\) is referred to as the Jordan cells of the \(r\)–monodromy.

**An example**

\[4\]This means that for \((X_1, \xi_1)\), and \((X_2, \xi_2)\) pairs with \(X_i\), \(i = 1, 2\) compact ANRs, \(\xi_i \in H^1(X_i; \mathbb{Z})\), \(i = 1, 2\), the existence of a homotopy equivalence \(\omega : X_1 \to X_2\) satisfying \(\omega^*(\xi_2) = \xi_1\) implies \([T^{(X_1;\xi_1)}(r)] = [T^{(X_2;\xi_2)}(r)]\).
Consider the space $X$ obtained from $Y$ by identifying its right end $Y_1$ (a union of three circles) to the left end $Y_0$ (a union of three circles) following the map $\phi: Y_1 \to Y_0$ given by the matrix

$$
\begin{pmatrix}
3 & 3 & 0 \\
2 & 3 & -1 \\
1 & 2 & 3
\end{pmatrix}.
$$

The meaning of this matrix as a map is the following: Circle (1) is divided in 6 equal parts, circle (2) in 8 parts and and circle (3) in 4 parts; the first three parts of circle (1) wrap clockwise around circle (1) to cover it three times, the next 2 wrap clockwise around circle (2) to cover it twice and around circle three to cover it three times. Similarly circle (2) and (3) wrap over circles (1)(2) and (3) as indicated by the matrix. The first part of circle (2) wraps counterclockwise on circle (2).

The map $f: X \to S^1$ is induced by the projection of $Y$, on the interval $[0, 2\pi]$ which becomes $S^1$ when 0 and $2\pi$ are identified. This map has all values weakly regular.

In this example it is not hard to see that $J_0(f) = \{(1, 1)\}$, $J_1(f) = \{(\lambda = 2, k = 2)\}$ and $J_0(f) = \emptyset$ However this will be obvious applying the algorithm described in Section 4.

### 3.3 F-monodromy

For a field $\kappa$, instead of the homology vector space $H_*(X)$, one can consider a more general functor $F$, a so called Dold half-exact functor cf. [6]. Recall that this is a covariant functor defined from the category $Top_c$ of compact ANR’s and continuous maps (or any subcategory with the same homotopy category) to the category $\kappa - Vect$ of finite dimensional vector spaces and linear maps which satisfies the following properties:

1. $F$ is a homotopy functor, i.e. $F(f) = F(g)$ for any two homotopic maps $f$ and $g$,

2. $F$ satisfies the Meyer Vietoris property, precisely, if $A$ is a compact ANR with $A_1$ and $A_2$ closed subsets such that $A_1, A_2$ and $A_{12} = A_1 \cap A_2$ all ANR’s and $A = A_1 \cup A_2$ then the sequence

$$
F(A_{12}) \xrightarrow{i} F(A_1) \oplus F(A_2) \xrightarrow{j} F(A)
$$

with $i = F(i_1) \oplus F(i_2)$, $j = F(j_1) - F(j_2)$, $i_1, i_2$ the obvious inclusions of $A_{12}$ in $A_1$ and $A_2$ and $j_1, j_2$ the obvious inclusion of $A_1$ and $A_2$ in $A$ is exact.
An analogue of Propositions 3.2 and 3.4 hold for $F$ instead of $H_r$ since they are based only on the Meyer-Vietoris property.

The same constructions with the same arguments work of define the $F$ monodromy and as the similarity class $R^{(X,\xi)}(F)$. There are plenty of such functors and the $F$–monodromy might be a useful invariant.

3.4 Comments

Theorem 3.6 is Implicit in [3] (cf section 4 combined with with Theorem 8.14) based on the interpretation of the monodromy as the similarity class of the linear isomorphism induced by the generator of the group of deck transformations, on the vector space $\ker(H_rX) \rightarrow H^N_r(X,\xi)$. Here $\tilde{X}$ denotes the infinite cyclic cover of $X$ defined by $\xi$ and $H^N_r(X;\xi)$ denotes the Novikov homology of $(X,\xi)$.

In [3] it is shown that the Jordan cells $J_r(f)$ defined in [2] as invariants for persistence of the circle valued map $f$ are the same as the Jordan cell defined above. Since [3] is not yet in print, for the reader familiar with the notations in [2] section 5 we will provide a short explanations of this statement in Appendix 3.

The characteristic polynomial of $[T^{(X,\xi)}(1)]$ for the pair $(X;\xi)$ with $X = S^3 \setminus K$, $K$ an open tube around an embedded oriented circle (knot) and $\xi$ the canonical generator of $H^1(S^3 \setminus K) = \mathbb{Z}$ is exactly the Alexander polynomial of the knot.

The alternating product of the characteristic polynomials $P_r(z)$ of the monodromies $[T^{X,\xi}(r)]$

$$A(X;\xi)(z) = \prod P_{A_r}(z)(-1)^r,$$

calculates (essentially [1]) the Reidemeister torsion of $X$ equipped with the degree one representation of $\pi_1(X)$ defined by $\xi$, when interpreted as an homomorphism $\pi_1(X,x) \rightarrow GL_1(C)$, and the complex number $z \in \mathbb{C}$, when $z \neq 0$. This was pointed out first by J. Milnor and refined by V. Turaev, cf [12]. A precise formulation of this identification will be discussed elsewhere.

4 The calculation of Jordan cells of an angle valued map

4.1 Generalities

Recall

- A convex $k$–cell $\sigma$ in an affine space $\mathbb{R}^n$, $n \geq k$, is the convex hull of a finite collection of points $e_0, e_1, \cdots e_N$ called vertices, with the property that:
  - there are subsets with $(k+1)$–points linearly independent but no $(k+2)$–points linearly independent,
  - no vertex lies in the topological interior of the convex hull.

The topology of the cell is the one induced from the ambient affine space $\mathbb{R}^n$.

A $k$–simplex is a convex $k$–cell with exactly $k + 1$ vertices.

- A $k'$–face $\sigma'$ of $\sigma$, $k' < k$, is a convex $k'$ cell whose vertices is a subset of the set of vertices of $\sigma$.

One indicates that $\sigma'$ is a face of $\sigma$ by writing $\sigma' \prec \sigma$.

---

5 a precise formulation requires additional data which have to be explained.
A space homeomorphic to a convex $k$-cell is called simply a $k$-cell and the subset homeomorphic to a face continues to be called face.

- A finite cell complex $Y$ is a space together with a collection $\mathcal{Y}$ of compact subsets $\sigma \subset Y$, each homeomorphic with a convex cell, with the following properties:
  1. If a $k$-cell $\sigma$ is a member of the collection $\mathcal{Y}$ then any of its faces $\omega \prec \sigma$ is a member of the collection $\mathcal{Y}$.
  2. If $\sigma$ and $\sigma'$ are two cells members of the collection $\mathcal{Y}$ then their intersection is a union of cells and each cell of this union is face of both $\sigma$ and $\sigma'$.

The concept of sub complex $Y' \subset Y$ is obvious; the face of each cell of $Y'$ is a cell of $Y'$. A simplicial complex is a cell complex with all cells simplexes.

Denote by $Y_k$ the set of the $k$-cells in $Y$. Clearly $Y_0$ is the set of all vertices of the cells in $\mathcal{Y}$.

- If a cell $\sigma \in \mathcal{Y}$ is equipped with an orientation $o(\sigma)$ this orientation induces an orientation for any codimension one face $\sigma$ described by the rule: first the induced orientation, next the normal vector pointing inside give the orientation $o(\sigma)$.

If each cell $\sigma$ of a cell complex is equipped with an orientation $o(\sigma)$ one has the incidence function $I : \mathcal{Y} \times \mathcal{Y} \to \{0, +1, -1\}$ defined as follows:

$$I(\sigma, \tau) := \begin{cases} I(\tau, \sigma) = +1 & \text{if } \sigma \in \mathcal{Y}_k, \tau \in \mathcal{Y}_{k+1}, \sigma \prec \tau, o(\sigma)|_{\sigma'} = o(\sigma'), \\ I(\tau, \sigma) = -1 & \text{if } \sigma \in \mathcal{Y}_k, \tau \in \mathcal{Y}_{k+1}, \sigma \prec \tau, o(\sigma)|_{\sigma'} \neq o(\sigma'), \\ I(\tau, \sigma) = 0 & \text{if } \sigma \cap \sigma' = \emptyset \end{cases} \quad (10)$$

The incidence function determine the homology of $Y$ with coefficients in any field.

- Suppose that a total order "$\leq$" of the set $\mathcal{Y}$ of all cells of $Y$ is given and the total number of cells is $N$. The order is called good order if:
  1. $\sigma \prec \tau$ implies $\sigma < \tau$.

In this case the function $I(\cdot, \cdot, \cdot, \cdot)$ can be regarded as $N \times N$ upper triangular matrix (all entries on and below diagonal are 0) and is referred below as the incidence matrix of $Y$.

Suppose that inside $Y$ one has two disjoint sub complexes, $Y_1, Y_2 \subset Y$. In this case a good order for $\mathcal{Y}$ (compatible with $\mathcal{Y}_1$ and $\mathcal{Y}_2$) needs in addition to (1) above the following requirements satisfied:
  2. $\sigma_1 \in \mathcal{Y}_1$ and $\sigma_2 \in \mathcal{Y} \setminus (\mathcal{Y}_1 \cup \mathcal{Y}_2)$ imply $\sigma_1 < \sigma_2$, and
  3. $\sigma' \in \mathcal{Y}_i$ and $\sigma \in \mathcal{Y} \setminus \mathcal{Y}_i$ imply $\sigma' \prec \sigma$.

Note that:

1. Given a total order of the cells in $\mathcal{Y}$ a simple algorithm permits to change this order into a good total order.

   The algorithm the Ordering algorithm is based on the inspection of the $n-$th cell with respect with all previous cells. If the requirements (1)-(3) are not violated move to the $(n+1)-$cell. If at least one of the three requirements is violated, change the position of this cell, and implicitly of the preceding ones if the case, by moving the cell to the left until (1), (2), or (3) are no more violated.
2. With the requirements 1, 2, 3 of good order satisfied the incidence matrix of $Y$, $I(\cdots, \cdots)$, should have the form

$$
\begin{pmatrix}
A_1 & 0 & X \\
0 & A_2 & Y \\
0 & 0 & Z
\end{pmatrix}
$$

with $A_1 = I_1$, $A_2 = I_2$ the incidence matrices for $Y_1$ and for $Y_2$.

3. The persistence algorithm [5], [9] permits to calculate from the incidence matrix :

(a) first, a base for $H_r(Y_1)$, then a base for $H_r(Y_2)$, then a base for $H_r(Y)$,

(b) second, the $\dim H_r(Y) \times \dim H_r(Y_1)$ matrix $A$ and the $\dim H_r(Y) \times \dim H_r(Y_2)$ matrix $B$

representing the linear maps induced in homology by the inclusions of $Y_1$ and $Y_2$ in $Y$ in any dimension $r$.

The cut of a simplex

Let $\sigma$ be a $k$–dimensional simplex with vertices $e_0, e_1, \cdots e_k$, i.e. a convex $k$–cell generated by $(k+1)$ linearly independent points located in some vector space. Let $f : \sigma \to \mathbb{R}$ be a linear map determined by the values of $f(e_i)$ by the formula $f(\sum_i t_i e_i) = \sum_i t_i f(e_i)$ $t_i \geq 0$, $\sum t_i = 1$ and let $t \in \mathbb{R}$. Suppose that $
abla f e_i > t$ and inf $f e_i < t$.

The map $f$ and the number $t$ determine two $k$–convex cells $\sigma_+, \sigma_-$ and a $(k - 1)$–convex cell $\sigma'$:

$$
\sigma_+ = f^{-1}([t, \infty)) \cap \sigma \\
\sigma_- = f^{-1}((-\infty, t]) \cap \sigma \\
\sigma' = f^{-1}(t) \cap \sigma.
$$

An orientation $o(\sigma)$ on $\sigma$ provides orientations $o(\sigma_+, o(\sigma_-)$ on $\sigma_+, \sigma_-$ and induces an orientation $o'(\sigma')$ on $\sigma'$, precisely the unique orientation which followed by grad $f$ is consistent with the orientation of $o(\sigma)$. Then $I(\sigma_+, \sigma') = \pm 1$.

Recall that the map $f : X \to \mathbb{S}^1 \subset \mathbb{C}$ is simplicial if the restriction of $\ln f$ to any simplex $\sigma$ is linear as considered above.

4.2 The algorithm

The algorithm we propose inputs a simplicial complex $X$, a simplicial map $f$ and an angle $\theta$ different from the values of $f$ on vertices and outputs in STEP 1 two $m \times n$ matrices $A_r$ and $B_r$ with $m$, the number of rows, equal to the dimension of $H_r(\mathbb{X}_\theta^f)$ and $n$, the number of columns, equal to the dimension of $H_r(f^{-1}(\theta))$. The matrices represent the linear maps induced in homology by the two inclusions of $f^{-1}(\theta) = Y_1 = Y_2$ in $\mathbb{X}_\theta^f$. In STEP 2 one obtains from the matrices $A_r$ and $B_r$ the invertible square matrices $A'_r$ and $B'_r$ such that $(B'_r)^{-1} A'_r$ represents the $r$–monodromy and in STEP 3 one derives from $(B'_r)^{-1} A'_r$ the Jordan cells $J_r(f)$.

STEP 1.

The simplicial set $X$ is recorded by :

– the set of vertices with an arbitrary chosen total order,
– a specification of the subsets which define the collection $X$ of simplices.

\[\text{in view of 1-connectivity of each simplex } \ln f \text{ has continuous univalent determination when the value on one vertex of the simplex is specified}\]
Implicit in this data is an orientation $o(\sigma)$ of each simplex, orientation provided by the relative ordering of the vertices of each simplex, and therefore the incidence number $\mathbb{I}(\sigma', \sigma)$ of any two simplexes $\sigma'$ and $\sigma$. (Implicit is also a total order of the simplexes of $\mathcal{X}$ provided by the lexicographic order induced from the order of the vertices.)

The simplicial map $f$ is indicated by
- the sequence of $N_0$ (the number of vertices) different angles, the values of $f$ on vertices.
- the formulae (12).

The map $f$ and the angle $\theta$ provide a decomposition of the set $\mathcal{X}$ as $\mathcal{X}' \sqcup \mathcal{X}''$ with $\mathcal{X'} := \{ \sigma \in \mathcal{X} \mid \sigma \cap f^{-1}(\theta) \neq \emptyset \}$ and $\mathcal{X}'' := \mathcal{X} \setminus \mathcal{X}'$.

From these data we can derive:
- first, the collections $\mathcal{Y}$ with the sub collections $\mathcal{Y}(1)$ and $\mathcal{Y}(2)$ of the cells of the complex $Y = \mathbb{X}_\partial^f$ and the sub complexes $Y_1 = f^{-1}(\theta)$ and $Y_2 = f^{-1}(\theta)$,
- second, the incidence function on $\mathcal{Y} \times \mathcal{Y}$,
- third, a good order for the elements of $\mathcal{Y}$.

These all lead to the incidence matrix $\mathbb{I}(Y)$.

**Description of the cells of $\mathcal{Y}$**: Each oriented simplex $\sigma$ in $\mathcal{X}''$ provides a unique oriented cell $\sigma$ in $\mathcal{Y}$. Each oriented $k-$simplex $\sigma$ in $\mathcal{X}'$ provides two oriented $k-$cells $\sigma_+$ and $\sigma_-$ and two oriented $(k-1)-$cells $\sigma'(1)$ and $\sigma'(2)$, copies of the oriented cell $\sigma'$. So the cells of $\mathcal{Y}$ are of five types

\[
\begin{align*}
\mathcal{Y}_k'^{(1)} &= \mathcal{X}'_{k+1}, \\
\mathcal{Y}_k'^{(2)} &= \mathcal{X}'_{k+1}, \\
\mathcal{Y}_k'^{-} &= \mathcal{X}'_k, \\
\mathcal{Y}_k'^{+} &= \mathcal{X}'_k, \\
\mathcal{Y}_k'' &= \mathcal{X}''_k.
\end{align*}
\]

Note that $\mathcal{Y}_k'^{+}$ and $\mathcal{Y}_k'^{-}$ are two copies of the same set $\mathcal{X}'_k$ and $\mathcal{Y}_k'^{(1)}$ and $\mathcal{Y}_k'^{(2)}$ are in bijective correspondence with the set $\mathcal{X}'_{k+1}$.

Inside the cell complex $Y$ we have two sub complexes $Y_1$ and $Y_2$ whose cells are: $(\mathcal{Y}_1)_k = \mathcal{Y}_k'^{(1)}$, $(\mathcal{Y}_2)_k = \mathcal{Y}_k'^{(2)}$.

**Incidence of cells of $\mathcal{Y}$**: The incidence of two cells in the same group (one of the five types) are the same as the incidence of the corresponding simplexes. The incidence of two cells one in $\mathcal{Y}_1$ the other in $\mathcal{Y}_2$ or one in the group $\mathcal{Y}'(i)$, $i = 1, 2$ the other in the group $\mathcal{Y}''$ is always zero. The rest of incidences are provided by the formulae (12).

**The good order**: Start with a good order of $\mathcal{Y}_1$ followed by $Y_2$ with the same order (translated by the number of the elements of $\mathcal{Y}_1$) followed by the remaining elements of $\mathcal{Y}$. Without changing the order in the collection $\mathcal{Y}_1 \sqcup \mathcal{Y}_2$, since no violation of the requirements 1, 2, 3, appear, we can realize a good order for the entire collection $\mathcal{Y}$ with all remaining cells being preceded by the cells of $\mathcal{Y}_1 \sqcup \mathcal{Y}_2$. Simply we apply the forcing algorithm to obtain this good order.

As a result we have the incidence matrix $\mathbb{I}(Y)$ which is of the form

\[
\begin{pmatrix}
\mathbb{I} & 0 & X \\
0 & \mathbb{I} & Y \\
0 & 0 & Z
\end{pmatrix}
\]

with $\mathbb{I}$ the incidence matrix of $Y_1$ and $Y_2$.

Running the persistence algorithm one obtains the matrices representing $A_r : H_r(\mathcal{Y}_1) \to H_r(\mathcal{Y})$ and $B_r : H_r(\mathcal{Y}_2) \to H_r(\mathcal{Y})$ as follows.

We run the persistence algorithm on the incidence matrix $A$ to compute a base for of the homology of $H_r(Y_1) = H_r(Y_2)$. We continue the procedure by adding columns and rows to the matrix to obtain a base
of $H_r(Y)$. It is straightforward to compute a matrix representation for the the inclusion induced linear maps $H_r(Y_1) \rightarrow H_r(Y), i = 1, 2$.

**STEP 2.** One uses the algebraic algorithm to pass from $A_r, B_r$ to the invertible matrices $A'_r, B'_r$ and then to $(B'_r)^{-1} \cdot (A'_r)$, described in the next subsection.

**STEP 3.** One uses the standard algorithms to put the matrix $(B'_r)^{-1} \cdot A'_r$ in Jordan diagonal form (i.e. as block diagonal matrix with Jordan blocks on diagonal).

### 4.3 An algorithm for the calculation of $R(A, B)_{\text{reg}}$

The algorithm presented below inputs two $m \times n$ matrices $(A, B)$ defining a linear relation $R(A, B)$ and outputs two $k \times k, k \leq \inf\{m, n\}$, invertible matrices $(A', B')$ such that $R(A, B)_{\text{reg}} \sim R(A', B')_{\text{reg}}$. It is based on three modifications $T_1, T_2, T_3$ described below. The simplest way to perform these modification is to use familiar procedures of bringing a matrix to row or column echelon form (REF) or (CEF) explained below, but much less is actually needed as the reader will see.

**Modification $T_1(A, B) = (A', B')$:**

Produces the invertible $m \times m$ matrix $C$ and the invertible $n \times n$ matrix $D$ so that

$$CAD = \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad CBD = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix}.$$

Precisely, one constructs first $C$ which puts $A$ in REF (reduced row echelon form) such that $CA = \begin{pmatrix} A_1 \\ 0 \end{pmatrix}$ and makes $CB = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$.

Second, one constructs $D$ which puts $B_2$ in CEF (column echelon form). Precisely, $B_2D = \begin{pmatrix} B_{21} & 0 \end{pmatrix}$.

Clearly $CAD$ and $CBD$ are as wanted above.

Take $A' = A_{12}, B' = B_{12}$.

In view of Proposition 2.5(1) one has $R(A, B)_{\text{reg}} = R(A', B')_{\text{reg}}$

**Modification $T_2(A, B) = (A', B')$:**

Produces the invertible $m \times m$ matrix $C$ and the invertible $n \times n$ matrix $D$ so that

$$CAD = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \quad \text{and} \quad CBD = \begin{pmatrix} B_{11} & B_{12} \\ 0 & 0 \end{pmatrix}.$$

Precisely, one constructs $C$ which puts $B$ in REF (row echelon form) such that $CB = \begin{pmatrix} B_1 \\ 0 \end{pmatrix}$ and makes $CA = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$.

Then one constructs $D$ which puts $A_2$ in RCEF (reduced column echelon form), precisely $A_2D = \begin{pmatrix} A_{21} & 0 \end{pmatrix}$.

Take $A' = A_{12}, B' = B_{12}$.

Clearly $CAD$ and $CBD$ are as wanted above.

In view of Proposition 2.5(1) one has $R(A, B)_{\text{reg}} = R(A', B')_{\text{reg}}$.

Note that if $A$ was surjective then $A'$ remains surjective.

**Modification $T_3(A, B) = (A', B')$:**

Produces the invertible $n \times n$ matrix $D$ and the $m \times m$ invertible matrix $C$ so that

$$CAD = \begin{pmatrix} A_{11} \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad CBD = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix}.$$

Precisely, one constructs $D$ which puts $A$ in CEF (reduced row echelon form) i.e. $AD = \begin{pmatrix} A_1 \\ 0 \end{pmatrix}$ and makes $BD = \begin{pmatrix} B_1 & B_2 \end{pmatrix}$.

Then one constructs $C$ to put $B_2$ in REF precisely,
$CB_2 = \begin{pmatrix} B_{21} \\ 0 \end{pmatrix}$.

Take $A' = A_{21}, B' = B_{21}$.

Clearly $CAD$ and $CBD$ are as wanted above.

In view of Proposition 2.5 (2) one has $R(A, B)_{\text{reg}} = R(A', B')_{\text{reg}}$.

Note that if both $A$ and $B$ were surjective then $A'$ and $B'$ remain surjective.

Here is how the algorithm works.

- (I) Inspect $A$
  - if surjective move to (II)
  - else:
    - apply $T_1$ and obtain $A'$ and $B'$.
    - make $A = A'$ and $B = B'$ and
    - go to (I)

- (II) Inspect $B$
  - if surjective move to (III)
  - else:
    - apply $T_2$ and obtain $A'$ and $B'$.
    - make $A = A'$ and $B = B'$ and
    - go to (II)

(Note that if $A$ was surjective by applying $T_2$, $A'$ remains surjective.)

- (III) Inspect $A$
  - if injective go to (IV).
  - else
    - apply $T_3$ and obtain $A'$ and $B'$.
    - make $A = A'$ and $B = B'$ and
    - go to (III)

- (IV) Calculate $B^{-1} \cdot A$.
  (Note that if $A$ and $B$ were surjective by applying $T_3$, $A'$ remains surjective.)

**Echelon form for $n \times m$ matrices**

Let $\kappa$ be a field.

An $m \times n$ matrix with coefficient in the field $\kappa$ is is a table with $m$ rows and $m$ columns

$$M = \begin{pmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\
  a_{2,1} & a_{2,2} & a_{2,3} & \cdots & a_{2,n} \\
  a_{3,1} & a_{3,2} & a_{3,3} & \cdots & a_{3,n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  a_{m,1} & a_{m,2} & a_{m,3} & \cdots & a_{m,n}
\end{pmatrix}.$$

A row or column is zero-row or zero-column if all entries are zero and its leading entry is the first nonzero entry.
Definition 4.1

1. The matrix \( M \) is in row echelon form, \( = \text{REF} \), if the following hold:

(a) All zero rows are below nonzero ones.
(b) For each row the leading entry is to the right of the leading entry of the previous row.

The matrix below is in row echelon form

\[
M = \begin{pmatrix}
0 & 0 & m & x & x & 0 & x \\
0 & 0 & 0 & n & x & x & x \\
0 & 0 & 0 & 0 & 0 & p & x \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

with \( m, n, p \not= 0 \) and \( x \) unspecified element in \( \kappa \).

2. The matrix \( M \) is in column echelon form, \( = \text{CEF} \), iff the transposed matrix \( M^t \) is in REF i.e. the following hold:

(a) All zero columns succeed nonzero ones.
(b) For each column the leading entry is below of the leading entry of the previous column.

The matrix below is in reduced row echelon form

\[
M = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
m & 0 & 0 & 0 & 0 \\
x & n & 0 & 0 & 0 \\
x & x & 0 & 0 & 0 \\
x & x & 0 & 0 & 0 \\
x & x & p & 0 & 0 \\
\end{pmatrix}
\]

with \( m, n, p \not= 0 \) and \( x \) unspecified element in \( \kappa \).

Proposition 4.2

1. For any \( (m \times n) \) matrix \( M \) one can produce an invertible \( n \times n \) matrix \( C \) such that the composition \( CM \) is in REF.

2. For any \( (m \times n) \) matrix matrix \( M \) one can produce an invertible \( m \times m \) matrix \( D \) such that the composition \( MD \) in in CEF.

The construction of \( C \) is based on ”Gauss elimination” procedure consisting in operation of ”permuting rows , multiplying rows with a nonzero element in \( \kappa \) and replacing a row by itself plus a multiple of an other row, each such operation is realizable by left multiplication by elementary matrix or permutation matrix cf [8].

The construction of \( D \) is done by : transpose, then apply the construction of \( C \), then transpose again.

Note All basic softwares which carry linear algebra packages contain sub packages which input a matrix and output its (reduced) row/column echelon form as well as the matrix \( C \) or \( D \) or outputs a Jordan form for a square matrix (at least in case \( \kappa = \mathbb{C} \).
4.4 An example

We illustrate Step 2 of the algorithm with $A$ and $B$ derived for $r = 1$ in the example in Section 3; In this case it is easy to see that for the cut at the angle $\theta = e^0$ one has $H_1(f^{-1}(\theta)) = \kappa^3$, $H_1(X_0^\theta) = \kappa^4$ and the matrices $A_1$ and $B_1$ are equal to

$$A = \begin{pmatrix} 3 & 3 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 3 \\ 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$  \hfill (14)

Proceed with the algorithm:

Inspect $A$, since not surjective apply $T_1$ and find $C = Id$ and $D = Id$. Then

$$A' = \begin{pmatrix} 3 & 3 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ \hfill (15)

Update

$$A = \begin{pmatrix} 3 & 3 & 0 \\ 2 & 3 & -1 \\ 1 & 2 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$ \hfill (16)

Since $A$ is surjective regard $B$. Since $B$ is not surjective apply $T_2$ and find

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$  

Then $CAD = \begin{pmatrix} 2 & 1 & -1 \\ 1 & 1 & 3 \\ 3 & 0 & 0 \end{pmatrix}$ and $CBD = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

$$A' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \quad B' = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$ \hfill (17)

Both $A'$ and $B'$ are invertible, so consider

$$B^{-1} \cdot A = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}$$

Hence $J([R_{reg}(A, B)]) = \{(2, 2)\}$.

5 Appendices

5.1 Appendix 1

For the proof of Propositions 2.3 and 2.5 one needs the following observation.

**Observation 5.1**

i). $x \in D$ iff there exists $x_i \in V$, $i \in \mathbb{Z}$ with $x_i R x_{i+1}$, $x_0 = x$.

ii). $y \in K_+ + K_-$ iff there exists a nonnegative integer $k$. 

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– the sequences \( x_1^+, x_2^+, \ldots, x_k^+ \) all elements in \( V \),
– the sequence \( x_1^-, x_0^-, x_{-1}^-, \ldots, x_{-k}^- \) all elements in \( V \),
such that:
1. \( y = x_1^+ + x_1^- \),
2. \( x_1^+ R x_2^+ R \cdots x_k^+ R 0 \),
3. \( 0 R x_{-k}^- R x_{-(k-1)}^- R \cdots x_0^- R x_1^- \).

**Proof of Proposition 2.3** (cf. [3])

To establish item 1., one uses Lemma 2.2 (2) applied to the relation \( R_{reg} \). Clearly in view of the surjectivity of \( \pi : D \to V_{reg} \) and Observation 5.1 (i) one has \( \text{dom} R_{reg} = V_{reg} \), so it remains to check that \( \ker(R_{reg}) = 0 \).

To verify this we start with
\( x \in D \text{ s.t. } x_R y, y \in (K_+ + K_-) \) and want to check that \( x \in D \cap (K_+ + K_-) \).

One uses Observation 5.1 and one produces the elements \( x_i \in V, i \in \mathbb{Z}, \; x_1^+, x_2^+, \ldots, x_k^+ \in V \) and \( x_1^-, x_0^-, x_{-1}^-, \ldots, x_{-k}^- \in V \) with the properties stated. One observes that:
1. \( x_0^- \in K_- \),
2. \( (x - x_0^-) \in D \cap K_+ \) since
\[
\cdots R x_{-(k-1)}^- R(x_{-k}^- - x_{-k}^-) R \cdots R(x_0^- - x_0^-) R((y - x_0^-) = x_1^+) R x_2^+ \cdots R x_k^+ R 0
\]
and therefore \( x_0^- = -x + x_0^- \in D \), hence
3. \( x_0^- \in D \cap K_- \).

Combining (2.) and (3.) above one obtains \( x = x - x_0^+ + x_0^- \in (D \cap K_+) + (D \cap K_-) \subseteq D \cap (K_+ + K_-) \).

Items 2., 3., and 4. (in Proposition 2.3) are straightforward.

To verify item 5., it suffices to check the equality for \( k = 2 \), which holds in view of Observation 5.1 (i).

**Proof of Proposition 2.5**

Item 1. follows by observing that \( D \) and \( D \cap (K_+ + K_-) \) for both \( R(\alpha, \beta) \) and \( R(\alpha', \beta') \) are actually the same.

To check this consider the sequences
\[
\cdots \longrightarrow w_{-1} \xleftarrow{\beta} v_{-1} \xleftarrow{\alpha} w_0 \xleftarrow{\beta} v_0 = (v_0^- + v_0^+) \longrightarrow w_1 \xleftarrow{\beta} v_1 \longrightarrow w_2 \xleftarrow{\beta} v_2 \xrightarrow{\alpha} \cdots
\]
\[
v_0^- \longrightarrow w_1^- \longleftarrow v_1^+ \longrightarrow w_2^+ \longleftarrow \cdots \longrightarrow w_k^- \longleftarrow v_k^+ \longrightarrow w_{k+1}^+ \longrightarrow 0
\]
\[
0 \longrightarrow w_{-(k)}^- \longleftarrow v_{-k}^- \longrightarrow w_{-(k-1)}^- \longleftarrow \cdots \longrightarrow v_{-1}^- \longrightarrow w_{-0}^- \longleftarrow v_0^-
\]

Indeed, by Observation 5.1 (i), \( v_0 \in D \) implies the existence of the first sequence above, which implies that \( v_1 \in \bar{V}' \) and \( w_1 \in \bar{W}' \), which guarantees that \( D = D' \).

If \( v_0 \in D \cap (K_+ + K_-) \) all the three sequences above exist, which imply that \( v_0 - v_0^- = v_0^+ \in D \cap K_+ \subseteq D' \cap (K_+ + K_+) \) and similarly that \( v_0 - v_0^- = v_0^- \in D \cap K_- \subseteq D' \cap (K_+ + K_-) \), and therefore \( v_0 = v - v_0^+ + v - v_0^- = v_0 \in D' \cap (K_+ + K_) \).
To check item 2. observe that the diagram (2) (in Section 2) induces the linear map \( \pi : D/D \cap (K_+ + K_-) \to D'/D' \cap (K'_+ + K'_-) \). This map is obviously surjective since both pairs \( \alpha, \beta \) and \( \alpha', \beta' \) being surjective make \( V = D \) and \( V' = D' \). To check that is injective we will verify that \( p'^{-1}(K'_+) \subset K_+ \).

For this purpose consider diagram (2) with \( \alpha' \) and \( \beta' \) as described and note:

**Lemma 5.2** If \( w \in W, w' \in W' \cap V' \) such that \( p(w) = w' \) and \( \beta'(v') = w' \) then there exists \( v \in V \) such that \( \beta(v) = w \) and \( p'(v) = v' \).

**Proof:** We first choose \( v \) with the property \( p'(v) = v' \), observe that \( p(w - \beta(v)) = 0 \), hence in view of the definition of the diagram \( w - \beta(v) = \beta(u), u \in \ker \alpha \) and correct finally take \( v = v - u \). q.e.d.

With Lemma 5.2 established observe that given a sequence \( v_0', v_1', \ldots v_k' \in V' \) and \( v_0 \in V \) with the property that
\[
\alpha'(v_{i-1}') = \beta'(v_i'), 1 \leq i \leq k
\]
\[
p(v_0) = v_0'
\]
(18)

one can produce \( v_1, v_2, \ldots v_k \in V \) such that
\[
\alpha(v_{i-1}) = \beta(v_i)
\]
\[
p(v_i) = v_i'.
\]
(19)

Indeed suppose inductively that \( v_1, v_2, \ldots v_i, i \leq r \) satisfying properties (19) are produced. Apply the remark to \( w = \alpha(v_i), w' = \alpha'(v_i') \) and \( v' = v_{i+1}' \) and construct \( v_{i+1} \).

To conclude \( p'^{-1}(K'_+) \subset K_+ \) we choose the sequence \( \{v_i'\} \) to have \( \alpha(v_k') = 0 \) which means that \( v_0' \in K'_+ \), then \( v_k \in \ker \alpha \) which means that \( v_0 \in K_+ \).

To conclude \( p'^{-1}(K'_-) \subset K_- \) choose a sequence \( \{v_i'\} \) have \( v' = v_k' \in \in K'_- \) for some \( k \) and \( v_0' = 0 \) and \( v_0 = 0 \). Then \( v_k \in K_- \), hence \( p'^{-1}(K'_-) \subset K_- \), which implies that \( \pi \) is also injective. q.e.d.

### 5.2 Appendix 2.

Recall that:

- The Hilbert cube \( Q \) is the infinite product \( Q = \prod_{i \in \mathbb{Z}_{\geq 0}} I_i = I^\infty \) with \( I_i = I = [0, 1] \). The topology of \( Q \) is given by the metric \( d(u, v) = \sum_i |u_i - v_i|/2^i \) with \( u = \{u_i \in I, i \in \mathbb{Z}_{\geq 0}\} \) and \( v = \{v_i \in I, i \in \mathbb{Z}_{\geq 0}\} \).

- The space \( Q \) is a compact ANR and so is \( X \times Q \) for any \( X \) compact ANR.

For any \( n \), positive integer, write \( Q = I^n \times Q'_n \), and denote by:
\[
\pi_n : Q \to I^n \text{ the first factor projection and } \pi_n^X : X \times Q \to X \times I^n \text{ the product } \pi_n^X = id_X \times \pi_x.
\]

For \( F : X \times Q \to \mathbb{R} \) let \( F_n \) be the restriction of \( F \) to \( X \times I^n \) and \( \overline{F}_n := F_n \cdot \pi_n^X \).

For \( f : X \to \mathbb{R} \) denote by \( \overline{f} := f \cdot \pi_X \) where \( \pi_X : X \times Q \to X \) is the canonical projection on \( X \).

In view of the definition of the metric on \( Q \) observe that:

**Observation 5.3**

1. If \( f : X \to \mathbb{R} \) is a tame map so is \( \overline{f} \).

2. The sequence of maps \( \overline{F}_n \) is uniformly convergent to the map \( F \).
Recall that a compact Hilbert cube manifold is a compact Hausdorff space locally homeomorphic to the Hilbert cube. The following are two results about Hilbert cube manifolds whose proof can be found in [7].

**Theorem 5.4**

1. (R Edwards) If $X$ is a compact ANR then $X \times Q$ is a Hilbert cube manifold.

2. (T Chapman) If $\omega: X \to Y$ is a homotopy of equivalence between two finite simplicial complexes with Whitehead torsion $\tau(\omega) = 0$ then there exists a homeomorphism $\omega': X \times Q \to Y \times Q$ such that $\omega'$ and $\omega \times \text{id}_Q$ are homotopic.

3. (folklore) If $\omega$ is a homotopy equivalence between two finite dimensional complexes then $\omega \times \text{id}_Q$ has the Whitehead torsion $\tau(\omega \times \text{id}_Q) = 0$.

**Proof of Stabilization theorem:**

Items 1. and 2. in Stabilization theorem follow from item 1. respectively item 2. combined with item 3. in Theorem 5.4. Recall for the non expert reader that for a homotopy equivalence $f: X \to Y$ two compact connected ANR’s one can associate an element $\tau(f) \in Wh(\pi_1(X,x))$ which measures the obstruction to $f$ to be a simple homotopy equivalence in the sense of J.H. Whitehead. Here $Wh(\Gamma)$ denotes the Whitehead group of $\Gamma$, which is an abelian group associated with a discrete group $\Gamma$, cf [13]. Actually $\tau(f)$, known as Whitehead torsion was defined in case $X \to Y$ are cell complexes, cf [13], however based on the theory of Hilbert manifold it was extended to all compact ANR’s, cf [14]. It is also known [13] that if $K$ is a finite cell complex (actually a compact ANR) with $\chi(K) = 0$ then $\tau(f \times \text{id}_K) = 0$.

One has also the following result whose proof was provided by S. Ferry:

**Proposition 5.5** A compact Hilbert cube manifold is a ”very good ANR”.

**Proof:** Let $M$ be a Hilbert cube manifold and $F: M \to \mathbb{R}$ a continuous map. We want to show that for $\epsilon > 0$ one can produce a tame map $P: M \to \mathbb{R}$ such that $|F(\pi) - P(\pi)| < \epsilon$ for any $\pi \in M$. For this purpose write $M = K \times Q$, $K$ a finite simplicial complex, cf [7] section 11.

It suffices to produce an $n$ and a simplicial map $p: K \times I^n \to \mathbb{R}$ such that $|F - p \cdot \pi_n^X| < \epsilon$.

The continuity of $F$ and the compacity of $M$ insure the existence of $\delta > 0$ such that $|\pi - \tau| < \delta$ implies $|F(\tau) - F(\pi)| < \epsilon/2$.

Choose $n$ such that $|\pi - (\pi_n^X(\tau),0)| < \delta$, $\tau \in K \times Q$ (here $(\pi_n^X(\tau),0) \in (K \times I^n) \times Q'_n = Q$) and denote by $F_n$ the restriction of $F$ to $K \times I^n$.

Choose $p: K \times I^n \to \mathbb{R}$ a simplicial map with $|p(x) - F_n(x)| < \epsilon/2$, $x \in K \times I^n$, and take $P = p \cdot \pi_n^X$.

Since $p$ is tame so is $P$.

Clearly then $|F(\pi) - p \cdot \pi_n^X(\tau)| \leq |F(\pi) - F_n \cdot \pi_n^X(\tau)| + |F_n \cdot \pi_n^X(\tau) - p \cdot \pi_n^X(\tau)| < \epsilon$.

### 5.3 Appendix 3

Recall from [2] or [3] the following notation:

- The oriented graph $G_{2m}$ has vertices $x_1, x_2, \ldots, x_{2m}$ and the oriented edges $a_i : x_{2i-1} \to x_{2i}, b_i : x_{2i+1} \to x_{2i}$ with $x_{2i+1} = x_1, i = 1, \ldots, m$ and

- A $G_{2m}$—representation $\rho$ is given by a collection of linear maps $\alpha_i : V_{2i-1} \to V_{2i}, \beta_i : V_{2i+1} \to V_{2i}$ with $V_i$ vector space corresponding to the vertex $x_i$, and the linear map $\alpha_i$ resp. $\beta_i$ corresponding to the arrow $a_i$ resp. $b_i$.

To such representation $\rho$ one associates the linear relation $R(\rho): V_1 \sim V_1 = R^\dagger(\beta_m) \cdot R(\alpha_m) \cdots R^\dagger(\beta_1) \cdot R(\alpha_1)$ and one denotes by $\mathcal{J}(\rho) := \mathcal{J}([R(\rho)_{\text{reg}}])$. 

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In view of the definitions it is immediate that:

**Observation 5.6**

1. \( R(\rho \oplus \rho') = R(\rho) \oplus R(\rho') \) and therefore \( J(\rho) \sqcup J(\rho') \),

2. \( J(\rho') = \emptyset \),

3. \( J(\rho(\lambda, k)) = \{ (\lambda, k) \} \).

We use the notation in [2].

For \( f : X \to S^1 \) a tame map in the sense of [2] with \( m \) critical angles \( 0 < s_1 < s_2 < \cdots s_m \leq 2\pi \) and \( t_1, t_2, \ldots t_m \) regular values such that \( 0 < t_1 < s_1 < t_2 < \cdots s_{m-1} < t_m < s_m \) let \( \tilde{f} \) be the infinite cyclic cover of the tame map \( f : X \to S^1 \).

Observe that \( V_{2i} = H_r(f^{-1}(s_i)) = H_r(\tilde{X}_{t_i, t_{i+1}}) \) and therefore the relation \( R_{\tilde{t}_i, \tilde{t}_{i+1}}^{\tilde{f}, \tilde{f}}(r) \) is

\[
R_{\tilde{t}_i, \tilde{t}_{i+1}}^{\tilde{f}, \tilde{f}}(r) = R(\alpha_i^r, \beta_i^r) = R(\beta_i^r) \cdot R(\alpha_i^r)
\]

with \( V_{2i-1} = H_r(\tilde{f}^{-1}(t_i), V_{2i} = H_r(f^{-1}(s_i)) \), and \( \alpha_i^r, \beta_i^r \) the linear maps induced in homology by the continuous maps \( a_i \) and \( b_i \), cf [2] Section 4.

Then the composition

\[
R_{\tilde{t}_{m-2}, \tilde{t}_1+2\pi}^{\tilde{f}, \tilde{f}}(r) \cdot R_{\tilde{t}_{m-1}, \tilde{t}_m}^{\tilde{f}, \tilde{f}}(r) \cdots R_{\tilde{t}_2, \tilde{t}_3}^{\tilde{f}, \tilde{f}}(r) \cdot R_{\tilde{t}_1, \tilde{t}_2}^{\tilde{f}, \tilde{f}}(r)
\]

identifies to \( R_{\tilde{t}_1}^{\tilde{f}}(r) \). Therefore the Jordan cells \( (R_{\tilde{t}_1}^{\tilde{f}})_{\text{reg}}(r) \) are the Jordan cells of the \( r \)–monodromy

\( T^{(X, \xi)}(r) \) and by Observation 5.6 they are the Jordan cells defined in [2].

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