ASSESMENT OF SMARANDACHE CURVES IN THE NULL CONE $Q^2$

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Abstract. In this paper, we give Smarandache curves according to the asymptotic orthonormal frame in null cone $Q^2$. By using cone frame formulas, we present some characterizations of Smarandache curves and calculate cone Frenet invariants of these curves. Also, we illustrate these curves with an example.

1. Introduction

Human being were bewitched by curves and curved shapes long before they took into account them as mathematical objects. But the greatest effect in the research of curves was, of course, the discovery of the calculus. Geometry before calculus includes only the simplest curves.

In classical curve theory, the geometry of a curve in three-dimensions is actually characterized by Frenet vectors.

Smarandache geometry is a geometry which has at least one Smarandachely denied axiom [4]. An axiom is said to be Smarandachely denied, if it behaves in at least two different ways within the same space. Smarandache curve is defined as a regular curve whose position vector is composed by Frenet frame vectors of another regular curve. Smarandache curves in various ambient spaces have been classified in [1]-[14], [19], [21]-[32].

In this study, we define special Smarandache curves such as $x\alpha, xy, \alpha y$ and $x\alpha y$ Smarandache curves according to asymptotic orthonormal frame in the null cone $Q^2$ and we examine the curvature and the asymptotic orthonormal frame’s vectors of Smarandache curves. We also give an example related to these curves.

2. Preliminaries

Some basics of the curves in the null cone are provided from, [15]-[18]. Let $E^3_3$ be the 3-dimensional pseudo-Euclidean space with the

$$\tilde{g}(X,Y) = \langle X,Y \rangle = x_1y_1 + x_2y_2 - x_3y_3$$

for all $X = (x_1,x_2,x_3), Y = (y_1,y_2,y_3) \in E^3_3$. $E^3_3$ is a flat pseudo-Riemannian manifold of signature $(2,1)$.

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Let $M$ be a submanifold of $E^3_1$. If the pseudo-Riemannian metric $\bar{g}$ of $E^3_1$ induces a pseudo-Riemannian metric $g$ (respectively, a Riemannian metric, a degenerate quadratic form) on $M$, then $M$ is called a timelike (respectively, spacelike, degenerate) submanifold of $E^3_1$. Let $c$ be a fixed point in $E^3_1$. The pseudo-Riemannian lightlike cone (quadric cone) is defined by

$$Q^3_1(c) = \{ x \in E^3_1 : g(x - c, x - c) = 0 \},$$

where the point $c$ is called the center of $Q^3_1(c)$. When $c = 0$, we simply denote $Q^3_1(0)$ by $Q^2$ be and call it the null cone.

Let $E^3_1$ be 3-dimensional Minkowski space and $Q^2$ the lightlike cone in $E^3_1$. A vector $V \neq 0$ in $E^3_1$ is called spacelike, timelike or lightlike, if $\langle V, V \rangle > 0$, $\langle V, V \rangle < 0$ or $\langle V, V \rangle = 0$, respectively. The norm of a vector $x \in E^3_1$ is given by $\|x\| = \sqrt{\langle x, x \rangle}$.

We assume that curve $x : I \to Q^2 \subset E^3_1$ is a regular curve in $Q^2$ for $t \in I$. In the following, we always assume that the curve is regular.

A frame field $\{x, \alpha, y\}$ on $E^3_1$ is called an asymptotic orthonormal frame field, if

$$\langle x, x \rangle = \langle y, y \rangle = \langle x, \alpha \rangle = \langle y, \alpha \rangle = 0, \ \langle x, y \rangle = \langle \alpha, \alpha \rangle = 1.$$

Using $x'(s) = \alpha(s)$ we know that $\{x(s), \alpha(s), y(s)\}$ from an asymptotic orthonormal frame along the curve $x(s)$ and the cone frenet formulas of $x(s)$ are given by

$$x'(s) = \alpha(s)$$
$$\alpha'(s) = \kappa(s)x(s) - y(s)$$
$$y'(s) = -\kappa(s)\alpha(s)$$

where the function $\kappa(s)$ is called cone curvature function of the curve $x(s)$.

Let $x : I \to Q^2 \subset E^3_1$ be a spacelike curve in $Q^2$ with an arc length parameter $s$. Then $x = x(s) = (x_1, x_2, x_3)$ can be written as

$$x(s) = \frac{f^{-1}}{2}(f^2 - 1, 2f, f^2 + 1)$$

for some non constant function $f(s)$ and $f_s = f'$.

3. SMARANDACHE CURVES IN THE NULL CONE $Q^2$

In this section, we define the Smarandache curves according to the asymptotic orthonormal frame in $Q^2$. Also, we obtain the asymptotic orthonormal frame and cone curvature function of the Smarandache partners lying on $Q^2$ using cone frenet formulas.

Smarandache curve $\gamma = \gamma(s^*(s))$ of the curve $x$ is a regular unit speed curve lying fully on $Q^2$. Let $\{x, \alpha, y\}$ and $\{\gamma, \alpha_\gamma, y_\gamma\}$ be the moving asymptotic orthonormal frames of $x$ and $\gamma$, respectively.

**Definition 1.** Let $x$ be unit speed spacelike curve lying on $Q^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$. Then, $x\alpha$—Smarandache curve of $x$ is defined by

$$\gamma_{x\alpha}(s^*) = \frac{c}{b}x(s) + \alpha(s),$$

where $c$ and $b$ are non constant functions.
where \( c, b \in \mathbb{R}_0^+ \).

**Theorem 1.** Let \( x \) be unit speed spacelike curve in \( Q^2 \) with the moving asymptotic orthonormal frame \( \{ x, \alpha, y \} \) and cone curvature \( \kappa(s) \) and let \( \gamma_{x\alpha} \) be \( x\alpha \)-smarandache curve with asymptotic orthonormal frame \( \{ \gamma_{x\alpha}, \alpha_{x\alpha}, y_{x\alpha} \} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{ \gamma_{x\alpha}, \alpha_{x\alpha}, y_{x\alpha} \} \) of the \( x\alpha \)-smarandache curve \( \gamma_{x\alpha} \) is given as

\[
\begin{bmatrix}
\gamma_{x\alpha} \\
\alpha_{x\alpha} \\
y_{x\alpha}
\end{bmatrix} = \begin{bmatrix}
c \\[ \psi \] \\
\frac{b}{c} \kappa \psi - \frac{b}{c} \psi \\
q_1 \\
q_2 \\
q_3
\end{bmatrix} \begin{bmatrix}
x \\
\alpha \\
y
\end{bmatrix}
\]

where

\[
\psi = \frac{c}{\sqrt{c^2 - 2b^2 \kappa(s)}},
\]

\[
\Upsilon_1 = \frac{b}{c} (\kappa' + \kappa) \psi \psi' + \frac{b^2}{c^2} \kappa (\psi')^2,
\]

\[
\Upsilon_2 = \frac{b}{c} \psi' (\psi' + 2 \frac{b}{c} \kappa \psi),
\]

\[
\Upsilon_3 = - \frac{b}{c} \psi (\psi' + \psi)
\]

and

\[
q_1 = -(\Upsilon_1 + \frac{c}{2b} (2\Upsilon_1 \Upsilon_3 + \Upsilon_2^2)) = -\Upsilon_1 + \frac{c}{b} \kappa_{x\alpha}(s^*),
\]

\[
q_2 = -(\Upsilon_2 + \frac{1}{2} (2\Upsilon_1 \Upsilon_3 + \Upsilon_2^2)) = -\Upsilon_2 + \kappa_{x\alpha}(s^*),
\]

\[
q_3 = -\Upsilon_3.
\]

ii) The cone curvature \( \kappa_{x\alpha}(s^*) \) of the curve \( \gamma_{x\alpha} \) is given by

\[
\kappa_{x\alpha}(s^*) = -\frac{1}{2} (2\Upsilon_1 \Upsilon_3 + \Upsilon_2^2).
\]

where

\[
s^* = \frac{1}{b} \int \sqrt{c^2 - 2b^2 \kappa(s)} ds.
\]

**Proof.** i) We assume that the curve \( x \) is a unit speed spacelike curve with the asymptotic orthonormal frame \( \{ x, \alpha, y \} \) and cone curvature \( \kappa \). Differentiating the equation \( 3.1 \) with respect to \( s \) and considering \( 2.1 \), we have

\[
\gamma'_{x\alpha}(s^*) = A \left( \alpha(s) + \frac{b}{c} \kappa x(s) - \frac{b}{c} y(s) \right),
\]

where

\[
\frac{ds^*}{ds} = \frac{1}{b} \sqrt{c^2 - 2b^2 \kappa(s)},
\]

\[
\psi = \frac{c}{\sqrt{c^2 - 2b^2 \kappa(s)b^2}}.
\]

It can be easily seen that the tangent vector \( \gamma'_{x\alpha}(s^*) = \alpha_{x\alpha}(s^*) \) is a unit spacelike vector.

Differentiating \( 3.7 \), we obtain equation as follows

\[
\gamma''_{x\alpha}(s^*) = \Upsilon_1 x(s) + \Upsilon_2 \alpha(s) + \Upsilon_3 y(s),
\]
where

\[ \Upsilon_1 = \frac{b}{c}(\kappa' + \kappa)\psi\psi' + \frac{b^2}{c^2}\kappa(\psi')^2, \]
\[ \Upsilon_2 = \frac{b}{c}\psi(\psi' + 2\frac{b}{c}\kappa\psi), \]
\[ \Upsilon_3 = -\frac{b}{c}\psi(\frac{b}{c}\psi' + \psi). \]

(3.11)

\[ y_{x\alpha}(s^*) = -\gamma''_{x\alpha} - \frac{1}{2}\langle \gamma''_{x\alpha}, \gamma''_{x\alpha} \rangle_{\gamma_{x\alpha}} \]

By the help of previous equation (3.11), we obtain

(3.12)

\[ y_{x\alpha}(s^*) = \varrho_1 x(s) + \varrho_2 \alpha(s) + \varrho_3 y(s), \]

where \( \varrho_1 = -(\Upsilon_1 + \frac{2b}{c^2}(2\Upsilon_1 \Upsilon_3 + \Upsilon_2^2)), \varrho_2 = -(\Upsilon_2 + \frac{1}{2}(2\Upsilon_1 \Upsilon_3 + \Upsilon_2^2)), \varrho_3 = -\Upsilon_3. \)

ii) The curvature \( \kappa_{\gamma_{x\alpha}}(s^*) \) of the \( \gamma_{x\alpha}(s^*) \) is explicitly obtained by

\[ \kappa_{\gamma_{x\alpha}}(s^*) = -\frac{1}{2}\langle \gamma''_{x\alpha}, \gamma''_{x\alpha} \rangle = -\frac{1}{2}(2\Upsilon_1 \Upsilon_3 + \Upsilon_2^2). \]

(3.13)

Thus, the theorem is proved. \( \Box \)

**Definition 2.** Let \( x \) be unit speed spacelike curve lying on \( Q^2 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, y\} \). Then, \( xy \)–smarandache curve of \( x \) is defined by

(3.14)

\[ \gamma_{xy}(s^*) = \frac{1}{\sqrt{2cb}}(cx(s) + by(s)), \]

where \( c, b \in \mathbb{R}_0^+ \).

**Theorem 2.** Let \( x \) be unit speed spacelike curve in \( Q^2 \) with the moving asymptotic orthonormal frame \( \{x, \alpha, y\} \) and cone curvature \( \kappa \) and let \( \gamma_{xy} \) be \( xy \)–smarandache curve with asymptotic orthonormal frame \( \{\gamma_{xy}, \alpha_{xy}, y_{xy}\} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{\gamma_{xy}, \alpha_{xy}, y_{xy}\} \) of the \( xy \)–smarandache curve \( \gamma_{xy} \) is given as

(3.15)

\[ \begin{bmatrix} \gamma_{xy} \\ \alpha_{xy} \\ y_{xy} \end{bmatrix} = \frac{\sqrt{2c}}{2b} \begin{bmatrix} 0 & \sqrt{\frac{b}{c^2}} \\ 0 & 1 \\ \frac{b\kappa^2\sqrt{2c}}{(c-b\kappa)^2} & \frac{c\sqrt{2c}}{(c-b\kappa)^2} \end{bmatrix} \begin{bmatrix} x \\ \alpha \\ y \end{bmatrix}. \]

ii) The cone curvature \( \kappa_{\gamma_{xy}}(s^*) \) of the curve \( \gamma_{xy} \) is given by

(3.16)

\[ \kappa_{\gamma_{xy}}(s^*) = \frac{2bcx(s)}{(c-b\kappa(s))^2}, \]

where

(3.17)

\[ s^* = \frac{1}{\sqrt{2cb}} \int (c-b\kappa(s))ds. \]
Proof. i) We assume that the curve \( x \) is a unit speed spacelike curve with the asymptotic orthonormal frame \( \{ x, \alpha, y \} \) and cone curvature \( \kappa \). Differentiating the equation (3.14) with respect to \( s \) and considering (2.1), we have

\[
\gamma'_{xy}(s^*) \frac{ds^*}{ds} = \frac{1}{\sqrt{2cb}} (c - b\kappa(s)) \vec{\alpha}(s) .
\]

By considering (3.17), we get

\[
\gamma'_{xy}(s^*) = \alpha(s) = \alpha_{xy} .
\]

Here, it can be easily seen that the tangent vector \( \vec{\alpha}_{xy} \) is a unit spacelike vector.

\[
\gamma''_{xy}(s^*) \frac{ds^*}{ds} = \kappa x(s) - y(s) .
\]

By substituting (3.17) into (3.20) and making necessary calculations, we obtain

\[
\gamma''_{xy}(s^*) = \frac{\kappa \sqrt{2bc}}{(c - b\kappa)^2} \vec{x} - \frac{\sqrt{2bc}}{(c - b\kappa)^2} \vec{y} .
\]

By the help of equation \( y_{xy}(s^*) = -\gamma''_{xy} - \frac{1}{2} \langle \gamma''_{xy}, \gamma''_{xy} \rangle \gamma_{xy} \), we write

\[
y_{xy}(s^*) = \frac{b \sqrt{2bc} \kappa^2}{(c - b\kappa)^2} x(s) + \frac{c \sqrt{2bc}}{(c - b\kappa)^2} y(s) .
\]

ii) The curvature \( \kappa_{\gamma_{xy}}(s^*) \) of the \( \gamma_{xy}(s^*) \) is explicitly obtained by

\[
\kappa_{\gamma_{xy}}(s^*) = -\frac{1}{2} \langle \gamma''_{xy}, \gamma''_{xy} \rangle = \frac{2bck(s)}{(c - b\kappa(s))^2} .
\]

\[\square\]

Definition 3. Let \( x \) be unit speed spacelike curve lying on \( Q^2 \) with the moving asymptotic orthonormal frame \( \{ x, \alpha, y \} \). Then, \( \alpha y \)-smarandache curve of \( x \) is defined by

\[
\gamma_{\alpha y}(s^*) = \alpha(s) + \frac{b}{c} y(s) ,
\]

where \( c, b \in \mathbb{R}_0^+ \).

Theorem 3. Let \( x \) be unit speed spacelike curve in \( Q^2 \) with the moving asymptotic orthonormal frame \( \{ x, \alpha, y \} \) and cone curvature \( \kappa \) and let \( \gamma_{\alpha y} \) be \( \alpha y \)-smarandache curve with asymptotic orthonormal frame \( \{ \gamma_{\alpha y}, \alpha_{\alpha y}, y_{\alpha y} \} \). Then the following relations hold:

i) The asymptotic orthonormal frame \( \{ \gamma_{\alpha y}, \alpha_{\alpha y}, y_{\alpha y} \} \) of the \( \alpha y \)-smarandache curve \( \gamma_{\alpha y} \) is given as

\[
\begin{bmatrix}
\gamma_{\alpha y} \\
\alpha_{\alpha y} \\
y_{\alpha y}
\end{bmatrix} = \begin{bmatrix}
0 & 1 & \frac{b}{c} \\
\sqrt{\kappa^2 - 2c} & \sqrt{\kappa^2 - 2c} & \kappa \sqrt{\kappa^2 - 2c} \\
\frac{\kappa \sqrt{\kappa^2 - 2c}}{\omega_1} & \frac{\kappa \sqrt{\kappa^2 - 2c}}{\omega_2} & \frac{\kappa \sqrt{\kappa^2 - 2c}}{\omega_3}
\end{bmatrix} \begin{bmatrix}
x \\
\alpha \\
y
\end{bmatrix} ,
\]
where
\begin{align}
\zeta_1 &= \frac{ck'}{b^2 - 2c^2} \left( \frac{c - 2kb}{2\kappa} \right), \\
\zeta_2 &= \frac{ck'}{b^2 - 2c^2} \left( \frac{c - b - 1}{2\kappa} \right), \\
\zeta_3 &= \frac{ck'}{b^2 - 2c^2} \left( \frac{bk - c}{2\kappa^2} \right)
\end{align}

and
\begin{align}
\omega_1 &= -\zeta_1, \\
\omega_2 &= -(\zeta_2 + \frac{1}{2}(2\zeta_1\zeta_3 + \zeta_2^2)), \\
\omega_3 &= -(\zeta_3 + \frac{b}{2c}(2\zeta_1\zeta_3 + \zeta_2^2)).
\end{align}

ii) The cone curvature \( \kappa_{\gamma_{ay}}(s^*) \) of the curve \( \gamma_{ay} \) is given by
\begin{equation}
\kappa_{\gamma_{ay}}(s^*) = -\frac{c^2}{8(b^2 - 2c^2)^2} \left( \frac{\kappa'}{\kappa} \right)^2 \left( \frac{1}{\kappa} (c - 2kb)(bk - c) + (c - b - 1)^2 \right),
\end{equation}
where
\begin{equation}
s^* = \frac{\sqrt{b^2 - 2c^2}}{c} \int \sqrt{\kappa(s)} ds.
\end{equation}

Proof. i) Let the curve \( x \) be a unit speed spacelike curve with the asymptotic orthonormal frame \( \{x, \alpha, y\} \) and cone curvature \( \kappa \). Differentiating the equation (3.28) with respect to \( s \) and considering (2.1), we find
\begin{equation}
\gamma'_{ay}(s^*) \frac{ds^*}{ds} = \kappa x(s) - \frac{b}{c} \kappa \alpha(s) - y(s).
\end{equation}
This can be written as following
\begin{equation}
\alpha_{ay}(s^*) \frac{ds^*}{ds} = \kappa x(s) - \frac{b}{c} \kappa \alpha(s) - y(s),
\end{equation}
where
\begin{equation}
\frac{ds^*}{ds} = \frac{\sqrt{b^2 - 2c^2}}{c} \sqrt{\kappa(s)}.
\end{equation}
By substituting (3.30) into (3.29), we find
\begin{equation}
\alpha_{ay}(s^*) = \frac{c\sqrt{\kappa}}{\sqrt{b^2 - 2c^2}} \frac{\dot{x}}{\sqrt{\dot{y}}} - \frac{b\sqrt{\kappa}}{\sqrt{b^2 - 2c^2}} \frac{\dot{\alpha}}{\sqrt{\dot{y}}} - \frac{c\sqrt{\kappa}}{\kappa \sqrt{b^2 - 2c^2}} \frac{\dot{y}}{\dot{y}}.
\end{equation}
Differentiating (3.31) and using (3.30), we get
\begin{equation}
\gamma''_{ay}(s^*) = \zeta_1 x(s) + \zeta_2 \alpha(s) + \zeta_3 y(s),
\end{equation}
where \( \zeta_1 = \frac{ck'}{b^2 - 2c^2} \left( \frac{c - 2kb}{2\kappa} \right), \zeta_2 = \frac{ck'}{b^2 - 2c^2} \left( \frac{c - b - 1}{2\kappa} \right), \zeta_3 = \frac{ck'}{b^2 - 2c^2} \left( \frac{bk - c}{2\kappa^2} \right) \).
\begin{equation}
y_{ay}(s^*) = -\frac{\gamma''_{ay}}{2} - \frac{1}{2} \left\langle \gamma''_{ay}, \gamma''_{ay} \right\rangle \gamma_{ay}.
\end{equation}
By the help of equation (3.32), we obtain
\begin{equation}
y_{ay}(s^*) = \omega_1 x(s) + \omega_2 \alpha(s) + \omega_3 y(s),
\end{equation}
where \( \omega_1 = -\zeta_1, \omega_2 = -(\zeta_2 + \frac{1}{2}(2\zeta_1\zeta_3 + \zeta_2^2)), \omega_3 = -(\zeta_3 + \frac{b}{2c}(2\zeta_1\zeta_3 + \zeta_2^2)) \).
ii) The curvature $\kappa_{\gamma_{xy}}(s^*)$ of the $\gamma_{xy}(s^*)$ is explicitly obtained by
\[
\kappa_{\gamma_{xy}}(s^*) = -\frac{c^2}{8(b^2 - 2c^2)^2} \left( \frac{\kappa'}{\kappa} \right)^2 \left( \frac{(c - 2cb)(bk - c)}{\kappa} + (c - b - 1)^2 \right).
\]

**Definition 4.** Let $x$ be unit speed spacelike curve lying on $Q^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$. Then, $x\gamma - \text{smarandache}$ curve of $x$ is defined by
\[
\gamma_{x\gamma}(s^*) = \frac{1}{\sqrt{2cc^* + b^2}}(cx(s) + ba(s) + c^*y(s)),
\]
where $c, c^*, b \in \mathbb{R}_0^+$. 

**Theorem 4.** Let $x$ be unit speed spacelike curve in $Q^2$ with the moving asymptotic orthonormal frame $\{x, \alpha, y\}$ and cone curvature $\kappa$ and let $\gamma_{x\gamma}$ be $x \gamma - \text{smarandache}$ curve with asymptotic orthonormal frame $\{\gamma_{x\gamma}, \alpha_{x\gamma}, y_{x\gamma}\}$. Then the following relations hold:

i) The asymptotic orthonormal frame $\{\gamma_{x\gamma}, \alpha_{x\gamma}, y_{x\gamma}\}$ of the $x \gamma - \text{smarandache}$ curve $\gamma_{x\gamma}$ is given as
\[
\begin{bmatrix}
\gamma_{x\gamma} \\
\alpha_{x\gamma} \\
y_{x\gamma}
\end{bmatrix} = \begin{bmatrix} c & b & c^* \\ \sqrt{2cc^* + b^2} & \sqrt{2cc^* + b^2} & \sqrt{2cc^* + b^2} \\ \rho_1 & \rho_2 & \rho_3 \\ \sigma_1 & \sigma_2 & \sigma_3 \\ x \\ \alpha \\ y
\end{bmatrix}
\]
where
\[
\eta = \sqrt{(c - c^*\kappa(s))^2 - 2b^2\kappa(s)}
\]
\[
\rho_1 = \frac{bk(s)}{\eta}, \rho_2 = \frac{c - c^*\kappa(s)}{\eta}, \rho_3 = -\frac{b}{\eta},
\]
\[
\xi_1 = (\rho_1 + \rho_2\kappa), \xi_2 = \rho_2' + \rho_1 + \rho_2\kappa, \xi_3 = -\rho_3 - \rho_2
\]
and
\[
\sigma_1 = -\xi_1 - \frac{c}{2\sqrt{2cc^* + b^2}}(2\xi_1\xi_3 + \xi_2^2),
\]
\[
\sigma_2 = -\xi_2 - \frac{b}{2\sqrt{2cc^* + b^2}}(2\xi_1\xi_3 + \xi_2^2),
\]
\[
\sigma_3 = -\xi_3 - \frac{c^*}{2\sqrt{2cc^* + b^2}}(2\xi_1\xi_3 + \xi_2^2).
\]

ii) The cone curvature $\kappa_{\gamma_{x\gamma}}(s^*)$ of the curve $\gamma_{x\gamma}$ is given by
\[
\kappa_{\gamma_{x\gamma}}(s^*) = \left( \frac{b}{\eta} \frac{c - c^*\kappa}{\eta} \right) \left( \frac{b}{\eta} \right)' + \frac{c - c^*\kappa}{\eta} \left( \frac{b}{\eta} \right)'
\]
\[
-\frac{1}{2} \left( \frac{c - c^*\kappa}{\eta} \right)'.
\]
where
\[
s^* = \frac{1}{\sqrt{2cc^* + b^2}} \int \sqrt{(c - c^*\kappa(s))^2 - 2b^2\kappa(s)} ds, \ b, c, c^* \in \mathbb{R}_0^+.
\]
Proof. i) Differentiating the equation (3.34) with respect to \( s \) and considering (2.1), we find

\[
\gamma_{xay}(s^*) \frac{ds^*}{ds} = \frac{1}{\sqrt{2cc^* + b^2}} \left( b\gamma x(s) + (c - c^*\kappa)\alpha(s) - b\gamma y(s) \right).
\]

This can be written as follows

\[
\alpha_{xay}(s^*) = \frac{b\gamma}{\eta} x(s) + \frac{c - c^*\kappa}{\eta} \alpha(s) - \frac{b}{\eta} y(s),
\]
or

\[
\alpha_{xay}(s^*) = \rho_1 x(s) + \rho_2 \alpha(s) - \rho_3 y(s).
\]

where

\[
\frac{ds^*}{ds} = \frac{1}{\sqrt{2cc^* + b^2}} \sqrt{(c - c^*\kappa)^2 - 2b^2\kappa}.
\]

Differentiating (3.32) and using (3.43), we get

\[
\gamma^{''}_{xay}(s^*) = \xi_1 x(s) + \xi_2 \alpha(s) + \xi_3 y(s),
\]

where \( \xi_1 = (\rho_1' + \rho_2\kappa) \), \( \xi_2 = \rho_2' + \rho_1 + \rho_3\kappa \), \( \xi_3 = -\rho_3' - \rho_2 \).

By the help of equation (3.44), we obtain

\[
y_{xay}(s^*) = -\gamma^{''}_{xay} - \frac{1}{2} \langle \gamma^{''}_{xay}, \gamma^{''}_{xay} \rangle \gamma_{xay}.
\]

ii) From \( \kappa_{xay}(s^*) = -\frac{1}{2} \langle \gamma^{''}_{xay}, \gamma^{''}_{xay} \rangle \), we have

\[
\kappa_{xay}(s^*) = \left( b \left( \frac{\kappa}{\eta} \right)' + \frac{c - c^*\kappa}{\eta} \right) \left( \left( \frac{b}{\eta} \right)' + \frac{c - c^*\kappa}{\eta} \right) - \frac{1}{2} \left( \frac{c - c^*\kappa}{\eta} \right)'.
\]

\[ \square \]

Theorem 5. Let \( x : I \to \mathbb{Q}^2 \subset E^3_1 \) be a spacelike curve in \( \mathbb{Q}^2 \) as follows

\[
x(s) = \frac{f_s^{-1}}{2} (f^2 - 1, 2f, f^2 + 1),
\]

for some non constant function \( f(s) \). Then we can write the following statements:

- If \( x \) is a \( x\alpha- \) smarandache curve, then the \( x\alpha- \) smarandache curve \( \gamma_{x\alpha} \) can be written as

\[
\gamma_{x\alpha}(s^*) = \left( \frac{c}{b} - f_s^{-1} f_s \right) x(s) + (f, 1, f).
\]

- If \( x \) is a \( xy- \) smarandache curve, then the \( xy- \) smarandache curve \( \gamma_{xy} \) can be written as

\[
\gamma_{xy}(s^*) = \frac{1}{\sqrt{2bc}} \left( c - \frac{1}{2} f_s^{-2} f_s^2 \right) x(s) + f_s^{-1} f_s (f, 1, f) - f_s (1, 0, 1).
\]
If $x$ is a $\alpha y$– smarandache curve, then the $\alpha y$– smarandache curve $\gamma_{\alpha y}$ can be written as
\begin{equation}
\gamma_{\alpha y}(s^*) = \left( -f_s^{-1} f_{ss} - \frac{b}{2c} f_s^{-2} f_{ss} \right) x(s) + \left( 1 + \frac{b}{c} f_s^{-1} f_{ss} \right) (f, 1, f) - \frac{b}{c} f_s (1.0, 1).
\end{equation}

If $x$ is a $x\alpha y$– smarandache curve, then the $x\alpha y$– smarandache curve $\gamma_{x\alpha y}$ can be written as
\begin{equation}
\gamma_{x\alpha y}(s^*) = \frac{1}{\sqrt{2cc^* + b^2}} \left( (c - bf_s^{-1} f_{ss} - \frac{c^*}{c} f_s^{-2} f_{ss}) x(s) + (b + c^* f_s^{-1} f_{ss}) (f, 1, f) - f_s (1.0, 1) \right),
\end{equation}
where $c, b, c^* \in \mathbb{R}^+_0$.

Proof. It is obvious from (3.1), (3.14), (3.23), (3.34) and (3.46). \hfill \square

We can give the following example to hold special Smarandache curves in the null cone $Q^2$, $x\alpha$, $xy$, $\alpha y$, and $x\alpha y$ special smarandache curves of $x$ curves are given in Figure 1 A, C, E, G, I, respectively. These figures rotated in three dimensions are also given in Figure 1 B, D, F, H, J, respectively.

**Example 1.** The curve
\[ x(s) = \left( \frac{\cosh s}{2} - \frac{1}{\cosh s} \right) \tanh s, \frac{\cosh s}{2} \]
is spacelike in $Q^2$ with arc length parameter $s$. Also, the shape of the $x$ curve is given as follows:

Then we can write the smarandache curves of the $x$-curve as follows:

i) $x\alpha$– smarandache curve $\gamma_{x\alpha}$ is given by
\[ \gamma_{x\alpha}(s) = \left( \frac{d \left( \frac{\cosh s}{2} - \frac{1}{\cosh s} \right) + \sinh s}{d \tanh s + \frac{1}{\cosh s}}, \frac{d \cosh s}{2} + \frac{1}{\cosh s} \right) \]

ii) $xy$– smarandache curve $\gamma_{xy}$ is given by
\[ \gamma_{xy}(s) = \left( \frac{m \cosh s - n \tanh s \sinh s - d \left( \frac{1}{\cosh s} \right)}{c \tanh s \left( e - \tanh^2 s \right)}, \frac{d \left( c - \tanh^2 s \right) \cosh s}{2} - \frac{1}{\cosh s} \right) \]

iii) $\alpha y$– smarandache curve $\gamma_{\alpha y}$ is given by
\[ \gamma_{\alpha y}(s) = \left( \frac{(\cosh s - \frac{1}{\cosh s}) (1 - e \tanh s) \tanh s + \sinh s (1 + d \tanh s)}{-d \cosh s, - \tanh^2 s (1 + d \tanh s) + d \tanh s, \sinh s}{\sqrt{2}}, -d \cosh s - e \sinh s \tanh s} \]

iv) $x\alpha y$– smarandache curve $\gamma_{x\alpha y}$ is given by
\[ \gamma_{x\alpha y}(s) = \left( \frac{m \sinh s - d \cosh s \sinh s - \left( \frac{c}{\cosh s} - \frac{b}{\cosh s} \right) \tanh^2 s}{m \cosh s \sinh s + \tanh s \sinh s, m + n \tanh s + k \tanh^2 s + l \tanh^3 s, m \cosh s + d \sinh s + c \tanh s \sinh s,} \right) \]

where $m, n, d, c, k, l, c \in \mathbb{R}^+_0$. 
(a) The curve $x$

(b) The rotated surface of curve $x$

(c) $x\alpha$–Smarandache curve

(d) $x\alpha$–Smarandache surface.

(e) $xy$–Smarandache curve

(f) $xy$–Smarandache surface.

(g) $\alpha y$–Smarandache curve.

(h) $\alpha y$–Smarandache surface.

(i) $x\alpha\beta$–Smarandache curve.

(j) $x\alpha\beta$–Smarandache surface.

Figure 1
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