UNIFIED VECTOR QUASIEQUILIBRIUM PROBLEMS VIA IMPROVEMENT SETS AND NONLINEAR SCALARIZATION WITH STABILITY ANALYSIS

HONG-ZHI WEI  
College of Mathematics and Statistics  
Chongqing University, Chongqing, 401331, China  

XIN ZUO  
Department of Basic, Yinchuan Energy College  
Yinchuan, 750105, China  

CHUN-RONG CHEN*  
College of Mathematics and Statistics  
Chongqing University, Chongqing, 401331, China  

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Abstract. This paper has two objectives. The first one is to propose a new vector quasiequilibrium problem where the ordering relation is defined via an improvement set $D$, and its weak version, also their Minty-type dual problems and the corresponding set-valued cases. These models provide unified frameworks to deal with well-known exact and approximate vector quasiequilibrium problems with vector-valued or set-valued mappings. The second one is to study solution stability in the sense of Hölder continuity of the unique solution to parametric unified (resp. weak) vector quasiequilibrium problems, by employing the Gerstewitz scalarization techniques. In particular, we deduce a new stability result for the typical vector optimization problem related with (resp. weak) $D$-optimality, by considering perturbations of both the objective function and the feasible set.

1. Introduction. Vector optimization problems can be found not only in mathematics but also in engineering, economics and so on, which have a large number of real life applications; see [22, 26, 13, 5, 25, 20, 31] and the references therein. Recently, a research line related with this field is to develop concepts and settings in order to unify different kinds of solution notions of these problems, like the efficiency, weak efficiency, proper efficiency and $\varepsilon$-efficiency (see [23] for more details). Among them, the notion of so-called improvement set provides a useful tool. Chicco et al. [14] introduced the concept of improvement set $E$ and a kind of optimality named as $E$-optimality in finite dimensional spaces, where the ordering relation...
of \(E\)-optimality is given by an improvement set \(E\), and \(E\)-optimality unifies some known concepts of exact and approximate solutions of vector optimization problems. Gutiérrez et al. [23] generalized the concepts of improvement set and \(E\)-optimality to a general topological vector space. Subsequent works about this aspect one can also refer to Zhao and Yang [39, 40], Zhao et al. [41], Oppezzi and Rossi [33], Lalitha and Chatterjee [28]. As mentioned in [23], \(E\)-optimality via improvement sets has been shown to be very suitable to deal in a unified way with well-known exact and approximate nondominated concepts. Moreover, it works with very general ordering sets (non-necessarily cones neither pointed, convex, solid nor closed sets) and so it can be used to deal with problems where the ordering relation is not a quasi order, which frequently appear in economics, and in linear spaces where the ordering set is not solid, such as the spaces \(\ell^p\) and \(L^p\), \(1 \leq p < \infty\), and their natural ordering cones.

In the development of vector optimization, the popular Gerstewitz and oriented distance scalarizations [18, 24] are proved to be powerful tools. Both scalarizing functions have many good properties and various applications (see [22, 26, 13, 31, 19, 38, 17]). Recently, Tammer and Zălinescu [37] had studied Lipschitz continuity properties of the Gerstewitz (Tammer) function \(\varphi_{D,k,0}\). Close works related with this topic one can also refer to Nam and Zălinescu [32], Durea and Strugariu [16]. These Lipschitz properties are very useful in qualitative or quantitative stability analysis, nonsmooth/variational analysis and so on (see [17, 37, 32, 16, 7, 8, 9]). However, they still have not received enough attention, unlike the oriented distance function introduced by Hiriart-Urruty [24, 38].

Vector equilibrium problems (VEPs, for short), also known as generalized Ky Fan inequalities recently, contain many important models as special cases, such as vector variational inequalities, vector complementarity problems and vector optimization problems (see, for example, [22, 26, 13, 5, 20] and the references therein). As a significant topic of stability analysis, Hölder or Lipschitz continuity of solutions to parametric VEPs is of considerable interest [7, 8, 9, 10, 1, 2, 3, 6, 29, 11, 30]. Scalarizations have been used as efficient methods to study stability (semicontinuity, Hölder continuity, etc.) of parametric VEPs [7, 8, 9, 10, 6, 30, 12, 21, 36, 34]. Very recently, Chen et al. [7] had established new stability results for Hölder continuity of the unique solution to a parametric generalized vector quasiequilibrium problem via nonlinear scalarization. For proving, the Gerstewitz (Tammer) nonlinear scalarization function \(\varphi_{D,k,0}\) as a fundamental tool plays key roles, especially, its globally Lipschitz property obtained by Tammer and Zălinescu [37]. Subsequently, by means of improvement sets, linear scalarization characterizations were established for vector equilibrium problems, and some continuity (both lower and upper semicontinuities) conclusions of parametric problems with respect to vector equilibria were deduced (see [10]).

Inspired by above observations, combining current developments and techniques on vector optimization, the aim of this paper is twofold.

First, we propose a new vector quasiequilibrium problem (UVQEP) where the ordering relation is given by an improvement set \(D\), and its weak version (UWVQEP), also their Minty-type dual problems and the corresponding set-valued cases. These models contain well-known exact and approximate vector quasiequilibrium problems with vector-valued or set-valued mappings as special ones, by taking different improvement sets. In particular, the vector optimization problem (VOP) defined via (resp. weak) \(D\)-optimality [23] is a special case of our general settings. So, our
models introduced herein can provide unified frameworks to deal with exact and approximate vector quasiequilibrium problems.

Second, we study solution stability in the sense of Hölder continuity of the unique solution to parametric UVQEP and UWVQEP. Without doubt, it is very important to know continuity properties of solutions for the perturbed UWVQEP technique solution to parametric UVQEP and UWVQEP. Without doubt, it is very important to know continuity properties of solutions for the perturbed UWVQEP technique solution to parametric UVQEP and UWVQEP.

Notations and Preliminaries. Throughout this paper, let $Y$ be a normed linear space, and let $C \subset Y$ be a proper closed convex cone with nonempty interior $\text{int} C$ (we consider that $0_Y \in C$). Let $D \subset Y$ be a nonempty proper (i.e., $D \neq Y$) set with nonempty interior $\text{int} D$ and $k^0 \in Y \setminus \{0_Y\}$.

Definition 2.1. [22, 18, 19] The nonlinear scalarization function $\varphi_{D,k^0} : Y \to \mathbb{R} \cup \{\pm \infty\}$ is defined by

$$
\varphi_{D,k^0}(y) := \inf \{ t \in \mathbb{R} \mid y \in tk^0 - D \}.
$$

In the special case of $D = C$ and $k^0 \in \text{int} C$, $\varphi_{C,k^0} : Y \to \mathbb{R}$ is a finite-valued, Lipschitz continuous, sublinear and convex function, and it is $C$-monotone (i.e., $y^2 - y^1 \in C \Rightarrow \varphi_{C,k^0}(y^1) \leq \varphi_{C,k^0}(y^2)$) and strictly $\text{int} C$-monotone (i.e., $y^2 - y^1 \in \text{int} C \Rightarrow \varphi_{C,k^0}(y^1) < \varphi_{C,k^0}(y^2)$) (see [22, 13, 17]).

Let $Y^*$ be the topological dual space of $Y$, and $C^*$ be the dual cone of $C$, i.e., $C^* := \{ \zeta \in Y^* \mid \langle \zeta, c \rangle \geq 0, \forall c \in C \}$. Let $v \in \text{int} C$ be a fixed point. The set

$$
C^v := \{ \zeta \in C^* \mid \langle \zeta, v \rangle = 1 \}
$$

is a weak$^*$-compact set of $Y^*$. Clearly, $C^v$ is a weak$^*$-compact base of $C^*$, i.e., $C^v$ is convex and weak$^*$-compact such that $0_{Y^*} \not\in C^v$ and $C^* = \bigcup_{t \geq 0} tc^v$. For every $y \in Y$, $\varphi_{C,v}(y) = \max_{\zeta \in C^v} \langle \zeta, y \rangle$ (see [8, 9]).

We shall use the following conditions proposed in [37] and [22], respectively:

**Assumption (P):** $D$ is proper closed, and satisfies the free-disposal assumption $D + C = D$;
Assumption $(P_w)$: $D$ is proper closed, and satisfies the weak free-disposal assumption $D + \text{int} C \subset D$.

It is clear that $(P) \Rightarrow (P_w)$, and the condition $D + C = D$ is equivalent to $D + C \subset D$. In addition, obviously, $D$ is not necessarily convex when assumption $(P)$ holds. For example, we may take $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and $D = \mathbb{R}^2_+ \backslash (0,1] \times [0,1]$.

In what follows, we use the standard notations: the closed ball with center $\bar{x} \in Y$ and radius $r \geq 0$ is denoted by $B(\bar{x};r)$; the distance from $\bar{y} \in Y$ to the set $\Theta \subset Y$ is denoted by $\text{dist}(\bar{y},\Theta) := \inf_{y \in \Theta} \| \bar{y} - y \|$ with $\text{dist}(\bar{y},\emptyset) = +\infty$.

The following properties of $\varphi$ come from [22, Theorem 2.3.1 and Proposition 2.3.4], [37, Theorem 3.1] and [7, Lemma 2.1].

**Proposition 1.** Let $k^0 \in \text{int} C$. If assumption $(P_w)$ holds, then $\varphi := \varphi_{D,k^0}$ is finite-valued and continuous. Moreover,

- (i) $\{ y \in Y \mid \varphi(y) \leq r \} = rk^0 - D$, $\forall r \in \mathbb{R}$;
- (ii) $\{ y \in Y \mid \varphi(y) < r \} = rk^0 - \text{int} D$, $\forall r \in \mathbb{R}$;
- (iii) $\{ y \in Y \mid \varphi(y) = r \} = rk^0 - \text{bd} D$, $\forall r \in \mathbb{R}$, where $\text{bd} D$ denotes the boundary of $D$;
- (iv) $\varphi(y + rk^0) = \varphi(y) + r$, $\forall y \in Y$, $\forall r \in \mathbb{R}$ (translation property);
- (v) $\varphi$ is convex if and only if $D$ is convex;
- (vi) $\forall y \in Y$, $\forall r > 0 : \varphi(ry) = r\varphi(y)$ if and only if $D$ is a cone;
- (vii) $\varphi$ is subadditive if and only if $D + D \subset D$.

Furthermore, suppose that assumption $(P)$ holds. Then $\varphi$ is $C$-monotone (i.e., $y^2 - y^1 \in C \Rightarrow \varphi(y^1) \leq \varphi(y^2)$) and

- (viii) $\varphi$ is (globally) Lipschitz on $Y$ with Lipschitz constant $L := \sup_{\xi \in C \setminus 0} \| \xi \| \in \left[ \frac{1}{\| k^0 \|}, +\infty \right]$, or equivalently,

$$L' := 1/\text{dist}(k^0, \text{bd} C) = \inf \left\{ \frac{1}{r} > 0 \mid B(k^0; r) \subset C \right\}.$$  

We give an example (see also [37, Example 2.1]) to show that when $k^0 \notin \text{int} C$, even if assumption $(P)$ holds, $\varphi := \varphi_{D,k^0}$ is not finite-valued, and so it is not (globally) Lipschitz on $Y$. Recall that $\inf \emptyset = +\infty$.

**Example 1.** Let $Y = \mathbb{R}^2$, $C = \mathbb{R}^2_+$, $k^0 = (1,0) \in \text{bd} C$ and $D = \mathbb{R}^2_+ \cup (\mathbb{R} \times [0,\infty)) \times [1, +\infty])$. Obviously, assumption $(P)$ is satisfied. Direct computations show that

$$\varphi(y) = \varphi(y_1, y_2) = \begin{cases} +\infty, & \text{if } y_2 \in [0, +\infty], \\ y_1, & \text{if } y_2 \in [-1,0], \\ -\infty, & \text{if } y_2 \in [-\infty, -1]. \end{cases}$$

Surely, $\varphi$ is not globally Lipschitz on $\mathbb{R}^2$.

We give another example to show that when $k^0 \in \text{int} C$, if the free-disposal assumption $D + C = D$ fails (i.e., assumption $(P)$ does not hold), then $\varphi := \varphi_{D,k^0}$ is not (globally) Lipschitz on $Y$.

**Example 2.** Let $Y = \mathbb{R}^2$, $C = \{ y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq y_1, y_2 \geq 0 \}$, $k^0 = (0, 1) \in \text{int} C$ and $D = (\mathbb{R} \times [0, +\infty)) \cup ([0, +\infty) \times (\mathbb{R} \times [0, +\infty))$. The free-disposal assumption $D + C = D$ does not hold. Direct computations show that

$$\varphi(y) = \varphi(y_1, y_2) = \begin{cases} +\infty, & \text{if } y_1 \in [1, +\infty], \\ y_2, & \text{if } y_1 \in [0,1], \\ y_2 - 1, & \text{if } y_1 \in [-\infty, 0]. \end{cases}$$
Obviously, $\varphi$ is not globally Lipschitz on $\mathbb{R}^2$.

**Remark 1.** If $C \subseteq \mathbb{R}^n$ with nonempty interior is a polyhedral convex cone described by $C := \{ y \in \mathbb{R}^n \mid a_i^T y := (a_i, y) \leq 0, i = 1, \cdots, m \}$, where $a_i \in \mathbb{R}^n$ and $a_i \neq 0 \iff$, then for given $k^0 \in \text{int}C$, $L = L' = \max_{1 \leq i \leq m} \left\{ -\frac{\|a_i\|}{a_i^T k^0} \right\}$. Particularly, if $C = \mathbb{R}^n_+$, then $L = L' = \max_{1 \leq i \leq n} \left\{ \frac{1}{k_i^0} \right\}$, where $k^0 := (k_1^0, \cdots, k_n^0) \in \text{int} \mathbb{R}^n_+$.

We now verify this claim. Obviously, $C$ is a closed convex cone. By the expression of $L'$, we wish to maximize $\tau > 0$ subject to the constraint $\mathbb{B}(k^0; \tau) = \{ k^0 + r \mid \|r\| \leq \tau \} \subseteq C$, i.e., $a_i^T x \leq 0, i = 1, \cdots, m$ for all $x \in \mathbb{B}(k^0; \tau)$. Therefore $\mathbb{B}(k^0; \tau) \subseteq C$ if and only if

$$\sup \{ a_i^T (k^0 + r) \mid \|r\| \leq \tau \} \leq 0.$$ 

Since $\sup \{ a_i^T r \mid \|r\| \leq \tau \} = \tau \|a_i\|$, we can rewrite above inequality as $a_i^T k^0 + \tau \|a_i\| \leq 0, i = 1, \cdots, m$. Thus, we have $\tau \leq -\frac{a_i^T k^0}{\|a_i\|}, i = 1, \cdots, m$. Because $k^0 \in \text{int}C$, $a_i^T k^0 < 0, i = 1, \cdots, m$. Whence, we obtain that

$$L' = 1/\max \left\{ \tau > 0 \mid \tau \leq -\frac{a_i^T k^0}{\|a_i\|}, i = 1, \cdots, m \right\} = 1/ \min_{1 \leq i \leq m} \left\{ -\frac{a_i^T k^0}{\|a_i\|} \right\} > 0.$$

We should mention another famous scalarizing function, the oriented distance function $\Delta_D(y) := \text{dist}(y, D) - \text{dist}(y, Y \setminus D)$ introduced by Hiriart-Urruty [24]. It is well known that this function has many good properties (see [38, Proposition 3.2]), especially, which is always 1-Lipschitz. While, it seems to be unsuitable in this paper, since an explanation is given in Remark 9.

Next, we recall a very useful concept named as improvement set, which was introduced by Chicco et al. [14] in finite dimensional spaces and was generalized in [23] to a linear topological space ordered via a convex cone.

**Definition 2.2.** [23, 14] A nonempty proper set $E \subseteq Y$ is said to be an improvement set with respect to $C$ if $0_Y \notin E$ and $E$ is free-disposal (i.e., $E + C = E$).

**Remark 2.** Some classical improvement sets with respect to $C$ include as: (i) $C \setminus \{0_Y\}$; (ii) $\text{int}C$; (iii) $Y \setminus (-C)$; (iv) $\kappa + C$, where $\kappa \in C \setminus \{0_Y\}$; (v) $\varepsilon E$, where $\varepsilon > 0$ and $E$ is an improvement set with respect to $C$; (vi) $\text{int} E$, where $E$ is a solid (i.e., $\text{int} E \neq \emptyset$) improvement set with respect to $C$. For more details and examples of improvement sets, one can refer to [23, 14, 39, 40, 41, 33, 28].

**Remark 3.** It is clear that if $D \subseteq Y$ is a closed improvement set with respect to $C$ and $k^0 \in \text{int} C$, then all properties of the function $\varphi := \varphi_{D, k^0}$ in Proposition 1 still hold.

3. **Unified Vector Quasiequilibria via Improvement Sets.** Throughout this paper, let $(X, d_X)$ be a metric linear space. Let $K : X \Rightarrow X$ be a set-valued mapping with nonempty values and $f : X \times X \rightarrow Y$ be a vector-valued mapping. From now on we assume that $D \subseteq Y$ is an improvement set with respect to $C$.

We consider the following unified vector quasiequilibrium problem (UVQEP) via improvement set $D$ of finding $\bar{x} \in K(\bar{x})$ such that

$$f(\bar{x}, y) \notin -D, \quad \forall y \in K(\bar{x}).$$

When $K(x) \equiv K$, a nonempty subset of $X$, the model (UVQEP) reduces to the unified vector equilibrium problem (UVEP) via improvement set $D$.

By taking special improvement sets $D$, the model (UVQEP) reduces to well-known exact and approximate vector quasiequilibrium problems, as it is showed in
Remark 4. If we take the improvement set $D$ as $C \setminus \{0_Y\}$ (resp. $Y \setminus (-C)$, int$C$), then the model (UVQEP) reduces to the classical (resp. strong, weak) vector quasiequilibrium problem in the literature. The main approximate versions for the (resp. strong, weak) vector quasiequilibrium problem can also be seen as particular cases of (UVQEP) by taking certain improvement set $D$ as $\epsilon e + C \setminus \{0_Y\}$ (resp. $\epsilon e + Y \setminus (-C)$, $\epsilon e + \text{int}C$), where $\epsilon \geq 0$ and $e \in \text{int}C$.

We also consider the following unified weak vector quasiequilibrium problem (UWVQEP) via improvement set $D$ of finding $\bar{x} \in K(\bar{x})$ such that

$$f(\bar{x}, y) \not\in \text{int}D, \quad \forall y \in K(\bar{x}).$$

When $K(x) \equiv K$, a nonempty subset of $X$, the model (UWVQEP) reduces to the unified weak vector equilibrium problem (UWVEP) via improvement set $D$.

Remark 5. The model (UVQEP) (resp. (UWVQEP)) contains many important problems as special cases, such as unified (resp. weak) vector variational inequalities, vector quasiequilibrium problems in the literature. The main approximate versions for the (resp. strong, weak) vector quasiequilibrium problem can also be seen as particular cases of (UVQEP) by taking certain improvement set $D$ as $\epsilon e + C \setminus \{0_Y\}$ (resp. $\epsilon e + Y \setminus (-C)$, $\epsilon e + \text{int}C$), where $\epsilon \geq 0$ and $e \in \text{int}C$.

Remark 6. In fact, the model (UWVQEP) can also be viewed as a special case of the model (UVQEP), because int$D$ is an improvement set with respect to $D$ if $D$ is a solid improvement set with respect to $C$ (see [23, Proposition 2.4]).

By the way, we also introduce the following unified Minty-type (resp. weak) vector quasiequilibrium problem (UMQEP) (resp. (UMWVQEP)) via improvement set $D$ of finding $\bar{x} \in K(\bar{x})$ such that

$$f(y, \bar{x}) \not\in D, \quad \forall y \in K(\bar{x}).$$

When $K(x) \equiv K$, a subset of $X$, the models (UMQEP) and (UMWVQEP) reduce to the unified Minty-type (resp. weak) vector quasiequilibrium problem (UMVEP) and (UMWVEP) via improvement set $D$, respectively.

Let $F : X \times X \to Y$ be a set-valued mapping with nonempty values. We can extend the model (UVQEP) (resp. (UWQEP)) to the set-valued setting: Find $\bar{x} \in K(\bar{x})$ such that

$$F(\bar{x}, y) \cap (-D) = \emptyset, \quad \forall y \in K(\bar{x}).$$

The corresponding Minty-type problems with set-valued mappings: Find $\bar{x} \in K(\bar{x})$ such that

$$F(y, \bar{x}) \cap D = \emptyset, \quad \forall y \in K(\bar{x}).$$

In this paper, we will mainly focus on the unified model (UWVQEP) (resp. (UVQEP)), and discuss its stability analysis. It is very important to understand
behaviors of solutions for (UWVQEP) (resp. (UVQEP)) when the problem’s data vary. In other words, we need to know properties of solutions for the so-called parametric UWVQEP (resp. UVQEP) when parameters vary. Therefore, one of main topics is to investigate stability of solution mappings for parametric UWVQEP (resp. UVQEP). Herein, the solution stability investigation was devoted to Hölder continuity.

4. Hölder Continuity of the Unique Solution. From now on, let Λ and Ω be nonempty subsets of metric spaces. Let $K : X \times \Lambda \rightrightarrows X$ be a set-valued mapping with nonempty values and $f : X \times X \times \Omega \rightrightarrows Y$ be a vector-valued mapping.

For the parameters $\lambda \in \Lambda$ and $\mu \in \Omega$, we consider the following parametric UWVQEP (PUWVQEP) of finding $\bar{x} \in K(\bar{x}, \lambda)$ such that

$$f(\bar{x}, y, \mu) \not\in -\text{int} D, \quad \forall y \in K(\bar{x}, \lambda).$$

Let $E(\lambda) := \{x \in X \mid x \in K(x, \lambda)\}$. Let $S_u(\lambda, \mu)$ be the subset of $E(\lambda)$ of the solutions of (PUWVQEP). For the reference point $(\bar{\lambda}, \bar{\mu}) \in \Lambda \times \Omega$, we assume that $S_u(\lambda, \mu) \neq \emptyset$ for every $\lambda \in U(\bar{\lambda})$ and $\mu \in U(\bar{\mu})$, where $U(\nu)$ denotes some neighborhood of the reference point $\nu$. It follows from Proposition 4(ii) and Remark 3 that for any given $k^0 \in \text{int} C$ and closed improvement set $D$ with respect to $C$,

$$S_u(\lambda, \mu) = \{x \in E(\lambda) \mid \varphi_{D,k^0}(f(x, y, \mu)) \geq 0, \forall y \in K(x, \lambda)\}.$$

**Definition 4.1.** (Classical notion) A set-valued mapping $G : \Omega \rightrightarrows X$ is said to be $\ell, \alpha$-Hölder continuous at $\mu_0$, iff there is a neighborhood $U(\mu_0)$ of $\mu_0$ such that, $\forall \mu_1, \mu_2 \in U(\mu_0), G(\mu_1) \subset G(\mu_2) + \ell d^\alpha(\mu_1, \mu_2)B_X$, where $\ell \geq 0$ and $\alpha > 0$, and $B_X$ denotes the unit ball of $X$.

In particular, when $X$ is a normed linear space, the vector-valued mapping $g : \Omega \rightrightarrows X$ is said to be $\ell, \alpha$-Hölder continuous at $\mu_0$, iff $\|g(\mu_1) - g(\mu_2)\| \leq \ell d^\alpha(\mu_1, \mu_2)$, $\forall \mu_1, \mu_2 \in U(\mu_0)$.

**Definition 4.2.** [3] A set-valued mapping $G : X \times \Lambda \rightrightarrows X$ is called $(\ell_1, \alpha_1, \ell_2, \alpha_2)$-Hölder continuous at $(x_0, \lambda_0)$, iff there exist neighborhoods $U(x_0)$ of $x_0$ and $U(\lambda_0)$ of $\lambda_0$ such that, $\forall x_1, x_2 \in U(x_0)$, $\forall \lambda_1, \lambda_2 \in U(\lambda_0)$, $G(x_1, \lambda_1) \subset \{x \in X \mid \exists z \in G(x_2, \lambda_2), d_X(x, z) \leq \ell_1 d^\alpha_1(x_1, x_2) + \ell_2 d^\alpha_2(\lambda_1, \lambda_2)\}$, where $\ell_1, \ell_2 \geq 0$ and $\alpha_1, \alpha_2 > 0$.

Now we state and prove the following stability result.

**Theorem 4.3.** Suppose that $D \subset Y$ is a closed improvement set with respect to $C$ and the following conditions hold:

(i) $K(\cdot, \cdot)$ is $(\ell_1, \alpha_1, \ell_2, \alpha_2)$-Hölder continuous on $E(U(\bar{\lambda})) \times \{\bar{\lambda}\}$;

(ii) for any $x \in E(U(\bar{\lambda}))$, $f(x, \cdot, \cdot)$ is $n, \delta$-Hölder continuous on $E(U(\bar{\lambda}))$;

(iii) $f(x, y, \cdot)$ is $m, \gamma$-Hölder continuous at $\bar{\mu}$ with $\theta$-relative to $E(U(\bar{\lambda}))$, which is equivalent to $\forall \mu_1, \mu_2 \in U(\bar{\mu}), \forall x, y \in E(U(\bar{\lambda})), x \neq y$, it holds

$$\|f(x, y, \mu_1) - f(x, y, \mu_2)\| \leq \ell d^\alpha(x, y)d^\gamma(\mu_1, \mu_2);$$

(iv) $f(\cdot, y, \cdot)$ is $h, \beta$-Hölder-related weakly monotone with respect to $\varphi_{D,q}$, i.e., there exist $q \in \text{int} C$ and $h > 0$, $\beta > 0$, such that for every $x, y \in E(U(\bar{\lambda})): x \neq y$,

$$hd^\beta(x, y) \leq |\varphi_{D,q}(f(x, y, \mu))| + |\varphi_{D,q}(f(y, x, \mu))|, \quad \text{where } |a, b| := \inf_{t \in \mathbb{R}_+} |a - t| \text{ for } a \in \mathbb{R} \text{ and } \mathbb{R}_+ := [0, +\infty];$$
(v) \( \beta = \alpha_1 \delta > \theta, \ h > 2n \rho \ell_1^s \) where \( \rho := \sup_{s \in C^T} ||s|| \in \left[ \frac{1}{|q|}, +\infty \right] \) (or \( \rho = 1/\text{dist}(q, b\mathbb{C}) \)) is the Lipschitz constant of \( \varphi_{D,q} \) on \( Y \) and \( q \in \text{int}C \) is given in above condition.

Then for every \((\lambda, \mu) \in U(\bar{\lambda}) \times U(\bar{\mu})\), the solution of (PUWVQEP) is unique, \( x(\lambda, \mu) \), and this function satisfies the Hölder condition:

\[
\begin{align*}
d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) & \leq \left( \frac{2n \rho \ell_2^s}{h - 2n \rho \ell_1^s} \right)^{\frac{1}{\theta}} d_X^{s_\ell}(1, 1) \\
& + \left( \frac{m \rho}{h - 2n \rho \ell_1^s} \right)^{\frac{1}{\theta}} d^{s_\ell}(\mu_1, \mu_2).
\end{align*}
\]

Proof. Let \((\lambda_1, \mu_1), (\lambda_2, \mu_2) \in U(\bar{\lambda}) \times U(\bar{\mu})\). We prove the Hölder condition holds by the following three steps, based on the fact that \( d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq d_X(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2) + d_X(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2)) \). It follows from Proposition 1(ii) that \( S_w(\lambda, \mu) = \{x \in E(\lambda) | \varphi_{D,q}(f(x, y, \mu)) \geq 0, \forall y \in K(x, \lambda)\} \).

**Step 1:** We prove that, \( \forall x(\lambda_1, \mu_1) \in S_w(\lambda_1, \mu_1) \), \( \forall x(\lambda_1, \mu_2) \in S_w(\lambda_1, \mu_2) \),

\[
d_1 := d_X(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)) \leq \left( \frac{m \rho}{h - 2n \rho \ell_1^s} \right)^{\frac{1}{\theta}} d^{s_\ell}(\mu_1, \mu_2).
\]

If \( x(\lambda_1, \mu_1) = x(\lambda_1, \mu_2) \), then the estimate (2) holds trivially. So we suppose \( x(\lambda_1, \mu_1) \neq x(\lambda_1, \mu_2) \). Since \( x(\lambda_1, \mu_1) \in K(x(\lambda_1, \mu_1), \lambda_1) \) and \( x(\lambda_1, \mu_2) \in K(x(\lambda_1, \mu_2), \lambda_1) \), by the Hölder continuity of \( K(\cdot, \lambda_1) \), there exist \( x_1 \in K(x(\lambda_1, \mu_1), \lambda_1) \) and \( x_2 \in K(x(\lambda_1, \mu_2), \lambda_1) \) such that

\[
\begin{align*}
d_X(x(\lambda_1, \mu_1), x_1) & \leq \ell_1 d_X(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)) = \ell_1 d_X^{s_\ell}, \\
d_X(x(\lambda_1, \mu_2), x_2) & \leq \ell_1 d_X(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)) = \ell_1 d_X^{s_\ell}.
\end{align*}
\]

Because \( x(\lambda_1, \mu_1) \in S_w(\lambda_1, \mu_1) \) and \( x(\lambda_1, \mu_2) \in S_w(\lambda_1, \mu_2) \), we have

\[
\begin{align*}
\varphi_{D,q}(f(x(\lambda_1, \mu_1), x_1, \mu_1)) & \geq 0, \\
\varphi_{D,q}(f(x(\lambda_1, \mu_2), x_2, \mu_2)) & \geq 0.
\end{align*}
\]

By virtue of condition (iv) and (4), we get

\[
\begin{align*}
\varphi_X^{s_\ell}(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)) & = \varphi_X^{s_\ell}(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)) \\
& \leq |\varphi_{D,q}(f(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2), \mu_1)), \mathbb{R}_+| \\
& + |\varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_1), \mu_1)), \mathbb{R}_+| \\
& \leq |\varphi_{D,q}(f(x(\lambda_1, \mu_1), x(\lambda_1, \mu_1)) - \varphi_{D,q}(f(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2), \mu_1))| \\
& + |\varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_2), \mu_2)) - \varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_1), \mu_1))| \\
& \leq |\varphi_{D,q}(f(x(\lambda_1, \mu_1), x(\lambda_1, \mu_1)) - \varphi_{D,q}(f(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2), \mu_1))| \\
& + |\varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_2), \mu_2)) - \varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_1), \mu_2))| \\
& + |\varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_1), \mu_2)) - \varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_2), \mu_2))|.
\end{align*}
\]
From the Lipschitz property of $\varphi_{D,q}$ and conditions (ii) and (iii), together with (3), we can deduce

\[
\begin{align*}
hd_1^3 & \leq \rho \|f(x(\lambda_1, \mu_1), x_1, \mu_1) - f(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2), \mu_1))
+ \rho \|f(x(\lambda_1, \mu_1), x_2, \mu_2) - f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_1), \mu_2))
+ \rho \|f(x(\lambda_2, \mu_2), x(\lambda_1, \mu_1), \mu_2) - f(x(\lambda_1, \mu_2), x(\lambda_1, \mu_1), \mu_1))
\leq n\rho d_X^2(x_1, x(\lambda_1, \mu_2)) + n\rho d_X^2(x_2, x(\lambda_1, \mu_1)) + m\rho d_1^3 d^\gamma(\mu_1, \mu_2)
\leq 2n\rho d_1^3 d_1^3 + m\rho d_1^3 d^\gamma(\mu_1, \mu_2).
\end{align*}
\]

Whence, condition (v) yields that

\[
d_1^{3-\theta} \leq \frac{m\rho}{h - 2n\rho d_1^3} d^\gamma(\mu_1, \mu_2),
\]

and it shows that the estimate (2) holds.

**Step 2:** We further prove that, $\forall x(\lambda_1, \mu_2) \in S_w(\lambda_1, \mu_2), \forall x(\lambda_2, \mu_2) \in S_w(\lambda_2, \mu_2)$,

\[
d_2 := d_X(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2)) \leq \left( \frac{2n\rho d_2^3}{h - 2n\rho d_1^3} \right)^{\frac{1}{3}} d^{\frac{2\gamma}{3}}(\lambda_1, \lambda_2).
\]  

(5)

Let $x(\lambda_1, \mu_2) \neq x(\lambda_2, \mu_2)$ (the case of $x(\lambda_1, \mu_2) = x(\lambda_2, \mu_2)$ is trivial). As $x(\lambda_1, \mu_2) \in K(x(\lambda_1, \mu_2), \lambda_1)$ and $x(\lambda_2, \mu_2) \in K(x(\lambda_2, \mu_2), \lambda_2)$, it follows from condition (i) that there are $x'_1 \in K(x(\lambda_2, \mu_2), \lambda_1)$ and $x''_2 \in K(x(\lambda_1, \mu_2), \lambda_2)$ such that

\[
\begin{align*}
d_X(x(\lambda_1, \mu_2), x'_1) & \leq \ell_2 d^{\alpha_2}(\lambda_1, \lambda_2),
\{x(\lambda_2, \mu_2), x''_2) & \leq \ell_2 d^{\alpha_2}(\lambda_1, \lambda_2).
\end{align*}
\]

(6)

By the Hölder continuity of $K(\cdot, \cdot)$ again, there exist $x''_1 \in K(x(\lambda_1, \mu_2), \lambda_1)$ and $x''_2 \in K(x(\lambda_2, \mu_2), \lambda_2)$ such that

\[
\begin{align*}
d_X(x'_1, x''_1) & \leq \ell_1 d_X^2(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2)) = \ell_1 d_X^2, \\
d_X(x''_2, x''_2) & \leq \ell_1 d_X^2(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2)) = \ell_1 d_X^2.
\end{align*}
\]

(7)

Since $x(\lambda_1, \mu_2) \in S_w(\lambda_1, \mu_2)$ and $x(\lambda_2, \mu_2) \in S_w(\lambda_2, \mu_2)$, we obtain

\[
\begin{align*}
\varphi_{D,q}(f(x(\lambda_1, \mu_2), x'_1, \mu_2)) & \geq 0,
\varphi_{D,q}(f(x(\lambda_2, \mu_2), x''_2, \mu_2)) & \geq 0.
\end{align*}
\]

(8)

By virtue of condition (iv) and (8), we get

\[
\begin{align*}
hd_2^3 &= \hd_2^3(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2)) \\
&\leq |\varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2), \mu_2))| + |\varphi_{D,q}(f(x(\lambda_2, \mu_2), x(\lambda_1, \mu_2), \mu_2))| \\
&\leq |\varphi_{D,q}(f(x(\lambda_1, \mu_2), x''_2, \mu_2)) - \varphi_{D,q}(f(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2), \mu_2))| + |\varphi_{D,q}(f(x(\lambda_2, \mu_2), x''_2, \mu_2))| \\
&\leq |\varphi_{D,q}(f(x(\lambda_1, \mu_2), x''_1, \mu_2)) - \varphi_{D,q}(f(x(\lambda_1, \mu_2), x''_2, \mu_2))| + |\varphi_{D,q}(f(x(\lambda_2, \mu_2), x''_1, \mu_2))| + |\varphi_{D,q}(f(x(\lambda_2, \mu_2), x''_2, \mu_2))| + |\varphi_{D,q}(f(x(\lambda_1, \mu_2), x''_2, \mu_2)) - \varphi_{D,q}(f(x(\lambda_2, \mu_2), x''_2, \mu_2))|.
\end{align*}
\]
From the Lipschitz property of \( \varphi_{D,q} \) and condition (ii), together with (6) and (7), we can deduce

\[
hd_{2}^{\beta} \leq \rho \| f(x(\lambda_{1}, \mu_{2}), x'_{\mu}, \mu_{2}) - f(x(\lambda_{1}, \mu_{2}), x'_{\mu}, \mu_{2}) \|
\]

\[
+ \rho \| f(x(\lambda_{1}, \mu_{2}), x'_{\mu}, \mu_{2}) - f(x(\lambda_{2}, \mu_{2}), x'_{\mu}, \mu_{2}) \|
\]

\[
+ \rho \| f(x(\lambda_{2}, \mu_{2}), x'_{\mu}, \mu_{2}) - f(x(\lambda_{2}, \mu_{2}), x'_{\mu}, \mu_{2}) \|
\]

\[
\leq n \rho d_{X}(x_{1}, x'_{\mu}, x_{1}, x(\lambda_{2}, \mu_{2})) + n \rho d_{X}(x'_{\mu}, x_{1}, x(\lambda_{2}, \mu_{2}))
\]

\[
+ n \rho d_{X}(x'_{\mu}, x(\lambda_{1}, \mu_{2}))
\]

\[
\leq 2n \rho \ell_{1}^{\beta} d_{2} + 2n \rho \ell_{2}^{\beta} d_{2} \rho \ell_{2}^{\beta}(\lambda_{1}, \lambda_{2}).
\]

It follows from condition (v) that

\[
d_{2} \leq \frac{2n \rho \ell_{2}^{\beta}}{h - 2n \rho \ell_{2}^{\beta}} d_{2} \rho \ell_{2}^{\beta}(\lambda_{1}, \lambda_{2}).
\]

This implies that the estimate (5) holds.

**Step 3:** \( \forall x(\lambda_{1}, \mu_{1}) \in S_{w}(\lambda_{1}, \mu_{1}), \forall x(\lambda_{2}, \mu_{2}) \in S_{w}(\lambda_{2}, \mu_{2}), \)

\[
d_{X}(x(\lambda_{1}, \mu_{1}), x(\lambda_{2}, \mu_{2})) \leq d_{1} + d_{2}.
\]

Thus,

\[
d_{S}(S_{w}(\lambda_{1}, \mu_{1}), S_{w}(\lambda_{2}, \mu_{2}))
\]

\[
= \sup_{x(\lambda_{1}, \mu_{1}) \in S_{w}(\lambda_{1}, \mu_{1}), x(\lambda_{2}, \mu_{2}) \in S_{w}(\lambda_{2}, \mu_{2})} d_{X}(x(\lambda_{1}, \mu_{1}), x(\lambda_{2}, \mu_{2}))
\]

\[
\leq \left( \frac{2n \rho \ell_{2}^{\beta}}{h - 2n \rho \ell_{2}^{\beta}} \right)^{\beta} d_{2}^{\rho \ell_{2}^{\beta}}(\lambda_{1}, \lambda_{2}) + \left( \frac{m \rho}{h - 2n \rho \ell_{2}^{\beta}} \right)^{\beta} d_{2}^{\rho \ell_{2}^{\beta}}(\mu_{1}, \mu_{2}).
\]

Taking \( \lambda_{2} = \lambda_{1} \) and \( \mu_{2} = \mu_{1} \), we see the diameter \( d_{S} \) of \( S_{w}(\lambda_{1}, \mu_{1}) \) is 0, that is, this set is a singleton \( \{ x(\lambda_{1}, \mu_{1}) \} \). \( S_{w}(\lambda_{2}, \mu_{2}) \) is similar. Thus the problem (PUVVQEP) has a unique solution in a neighborhood of \( (\lambda, \mu) \) and the Hölder condition (1) is satisfied.

**Remark 7.** If \( E(U(\hat{\lambda})) \) in condition (iii) of Theorem 4.3 is bounded, then without loss of generality we can take \( \theta = 0 \) in condition (iii), since \( d_{X}(x, y) \leq w \) for some \( w > 0 \), \( \forall x, y \in E(U(\hat{\lambda})). \) Thus, the condition \( '\beta > \theta' \) in Theorem 4.3 can be omitted.

When \( K(x, \lambda) = K(\lambda) \), i.e., \( K \) does not depend on \( x \), the model (PUVVQEP) reduces to the parametric unified weak vector equilibrium problem (PUVVEP). Under this case, \( \ell_{1} = 0 \) and \( \alpha_{1} \) is arbitrary. So the condition \( '\beta = \alpha_{1} \ell_{1}, \ h > 2n \rho \ell_{1}^{\beta}' \) is satisfied. Further, if \( K(U(\hat{\lambda})) \) in condition (iii) is bounded, taking \( \theta = 0 \), then condition (v) of Theorem 4.3 holds automatically. Thus, the Hölder condition (1) becomes that

\[
d_{X}(x(\lambda_{1}, \mu_{1}), x(\lambda_{2}, \mu_{2})) \leq \left( \frac{2n \rho \ell_{1}^{\beta}}{h} \right)^{\beta} d_{2}^{\rho \ell_{1}^{\beta}}(\lambda_{1}, \lambda_{2}) + \left( \frac{m \rho}{h} \right)^{\beta} d_{2}^{\rho \ell_{1}^{\beta}}(\mu_{1}, \mu_{2}),
\]

where \( x(\lambda, \mu) \) stands for the unique solution of (PUVVEP).

Now we give two examples to explain that Theorem 4.3 holds when \( D \) is convex and \( D \) is nonconvex, respectively.
Example 3. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq |y_1|\}$ and $D = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq |y_1|\}$. Obviously, $D$ is a closed improvement set with respect to $C$ and $D$ is convex. Let $\Lambda \equiv \Omega = [1, 2]$, $K(x, \lambda) = [\lambda, 2]$ and $f(x, y, \lambda) = (\lambda(x - y), \lambda(y - x) - 1)$. Let $\lambda = \frac{2}{3}$ and $U(\lambda) = \Lambda$.

Then, condition (i) is fulfilled with $\ell_1 = 0$, $\ell_2 = \alpha_2 = 1$ and $\alpha_1$ is arbitrary; condition (ii) is satisfied with $n = 2\sqrt{2}$ and $\delta = 1$; condition (iii) holds with $m = \sqrt{2}$ and $\theta = \gamma = 1$. Now we check that condition (iv) holds with $h = \beta = 1$ and $q = (0, 1) \in \text{int} C$. Direct computations show that $\varphi_{D,q}(y) = \varphi_{D,q}(y_1, y_2) = y_2 + 1$ and $\rho = \sqrt{2}$. Hence, $\varphi_{D,q}(f(x, y, \lambda)) = \lambda(y - x)$ and $\varphi_{D,q}(f(y, x, \lambda)) = \lambda(x - y)$, and thus, for every $x, y \in E(U(\lambda)) = [1, 2] : x \neq y$, $hd^\lambda_X(x, y) = |x - y| \leq \lambda|x - y| = |\varphi_{D,q}(f(x, y, \lambda)), \varphi_{D,q}(f(y, x, \lambda))|$. Since $\ell_1 = 0$ and $\alpha_1$ is arbitrary, noting that Remark 7, we can take $\alpha_1 = 1$ and $\theta = 0$ to see that condition (v) is satisfied. Whence, all conditions of Theorem 4.3 hold, and Theorem 4.3 derives the Hölder continuity of the solution around $\bar{\lambda}$ (in fact, $S_\omega(\lambda) = \{\lambda\}$, $\forall \lambda \in \Lambda$).

Example 4. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq |y_1|\}$ and $D = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq -|y_1| + 1\}$. Obviously, $D$ is a closed improvement set with respect to $C$ and $D$ is nonconvex. Let $\Lambda \equiv \Omega = [1, 2]$, $K(x, \lambda) = [\lambda + 1, 4]$ and $f(x, y, \lambda) = (\lambda(x - y), \lambda(y - x) - 1)$. Let $\bar{\lambda} = \frac{2}{3}$ and $U(\bar{\lambda}) = \Lambda$. Similar to Example 3, it can be checked that conditions (i)-(iv) hold with $\ell_1 = 0$, $n = 2\sqrt{2}$, $m = \sqrt{2}$, $\ell_2 = \alpha_2 = \delta = \theta = \gamma = \beta = 1$, $h = 2$, $\alpha_1$ is arbitrary and $q = (0, 1) \in \text{int} C$. Direct computations show that $\varphi_{D,q}(y) = \varphi_{D,q}(y_1, y_2) = y_2 - |y_1| + 1$ and $\rho = \sqrt{2}$. Noting that Remark 7, we can take $\alpha_1 = 1$ and $\theta = 0$ to see that condition (v) is satisfied. Whence, all conditions of Theorem 4.3 hold, and Theorem 4.3 derives the Hölder continuity of the solution around $\bar{\lambda}$ (in fact, $S_\omega(\lambda) = \{\lambda + 1\}$, $\forall \lambda \in \Lambda$).

Proposition 2. Let $D \subset Y$ be a closed improvement set with respect to $C$. If there exist $q \in \text{int} C$ and $h > 0$, $\beta > 0$, such that for every $x, y \in E(U(\lambda)) : x \neq y$, $Q(x, y, \mu) + hd^\lambda_X(x, y)q \in -D$, where

\[
Q(x, y, \mu) := \begin{cases} 
 f(y, x, \mu), & \text{if } f(x, y, \mu) \notin -\text{int} D, \\
 f(x, y, \mu), & \text{if } f(x, y, \mu) \in -\text{int} D,
\end{cases}
\]

then $f(\cdot, \cdot, \cdot)$ is $h, \beta$-Hölder-related weakly monotone with respect to $\varphi_{D,q}$.

Proof. If $f(x, y, \mu) \notin -\text{int} D$, then $f(y, x, \mu) + hd^\lambda_X(x, y)q \in -D$. By Proposition 1(i), we have $\varphi_{D,q}(f(y, x, \mu)) \leq -hd^\lambda_X(x, y)$. Thus,

\[
hd^\lambda_X(x, y) \leq -\varphi_{D,q}(f(y, x, \mu)) \leq |\varphi_{D,q}(f(y, x, \mu))|, R_+ + |\varphi_{D,q}(f(y, x, \mu)), R_+|.
\]

If $f(x, y, \mu) \in -\text{int} D$, then $f(x, y, \mu) + hd^\lambda_X(x, y)q \in -D$. Similarly, we also get

\[
hd^\lambda_X(x, y) \leq -\varphi_{D,q}(f(x, y, \mu)) \leq |\varphi_{D,q}(f(x, y, \mu)), R_+ + |\varphi_{D,q}(f(x, y, \mu)), R_+|.
\]

This completes the proof. \qed

Remark 8. The converse of Proposition 2 is not true generally (see the following Example 5). Hence, Theorem 4.3 relaxes the corresponding assumption of monotonicity in [7, Theorem 3.1], and thus improves [7, Theorem 3.1] when $D$ is an improvement set with respect to $C$.

Example 5. Let $X = \mathbb{R}$, $Y = \mathbb{R}^2$, $C = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq |y_1|\}$ and $D = \{y = (y_1, y_2) \in \mathbb{R}^2 \mid y_2 \geq |y_1|\}$. Obviously, $D$ is a closed improvement set with respect to $C$. Let $\Lambda \equiv \Omega = [1, 2]$, $K(x, \lambda) = [\lambda, 2]$ and $f(x, y, \lambda) = (\lambda(x - y), -2)$. 

Let $\lambda = \frac{2}{q}$ and $U(\lambda) = A$. Now we check that the H"older-related weak monotonicity of $f$ holds with $h = 2$, $\beta = 1$ and $q = (0, 1) \in \text{int} C$. Direct computations show that $\varphi_{D,q}(y) = \varphi_{D,q}(y_2) = 0$. Hence, $\varphi_{D,q}(f(x, y, \lambda)) = \varphi_{D,q}(f(y, x, \lambda)) = -1$, and thus, for every $x, y \in E(U(\lambda)) = [1, 2] : x \neq y$, $\varphi_{D,q}(f(x, y, \lambda)) = 2|x - y| \leq 2 = |\varphi_{D,q}(f(x, y, \lambda)), \mathbb{R}^+| + |\varphi_{D,q}(f(y, x, \lambda)), \mathbb{R}^+|$. It is clear that $f(x, y, \lambda) \in \text{int} D$ holds for any $x, y \in E(U(\lambda)) = [1, 2] : x \neq y$. However, $Q(x, y, \lambda) + \varphi_{D,q}(f(y, x, \lambda))q = f(x, y, \lambda) + \varphi_{D,q}(f(x, y, \lambda))q = (\lambda(x - y), -2) + 2|x - y|(0, 1) = (\lambda(x - y), 2|x - y| - 2) \not\in -D$ when $\frac{1}{2} < |x - y| \leq 1$.

**Remark 9.** Let $S(\lambda, \mu) := \{x \in E(\lambda) \mid f(x, y, \mu) \not\in -D, \forall y \in K(x, \lambda)\}$ be the solution set of parametric UVQEP (PUVQEP). It follows from Proposition 1(i) and Remark 3 that for any given $k^0 \in \text{int} C$ and closed improvement set $D$ with respect to $C$, $S(\lambda, \mu) = \{x \in E(\lambda) \mid f(x, y, \mu) > 0, \forall y \in K(x, \lambda)\}$. Obviously, a stability result for (PUVQEP) which is similar to that of Theorem 4.3 can also be obtained, just replacing $\mathbb{R}^+$ with $\mathbb{R}^+ := [0, +\infty]$ in condition (iv). Actually, a similar result for (PUVWQEP) has been shown in [7, Theorem 3.3] by using the oriented distance scalarization. However, we think the Gerstewitz (Tammer) scalarizing function $\varphi$ is more suitable than the oriented distance function $\Delta_D$ used in current framework via improvement sets. This is because if we employ the oriented distance function $\Delta_D$ herein, then the set $D$ in the model (PUVWQEP) or (PUVQEP) is not necessarily free-disposal (i.e., $D + C = D$).

**Remark 10.** All discussions on the models (PUVQEP) and (PUWVQEP) can be applied to parametric Minty-type dual problems (UMVQEP) and (UMWVQEP) by setting $f(x, y, \mu) := -f(y, x, \mu)$, respectively.

As a by-product of Theorem 4.3, in the sequel, we consider the case that $\beta = 2$, $\theta = 0$ and the mappings $K$ and $f$ are Lipschitz continuous (i.e., $\alpha_1 = \alpha_2 = \delta = \gamma = 1$). Note that under this quadratic analysis, $\beta \neq \alpha_1 \delta$ and the requirement of $h > 2n\rho_{f}^1$ is dropped. Hence, the following H"older condition is especially convenient, since H"older-related constants are independent each other (cf. Theorem 4.3).

**Proposition 3.** Assume that $D \subset Y$ is a closed improvement set with respect to $C$ and conditions (i)-(iv) in Theorem 4.3 are satisfied with $\alpha_1 = \alpha_2 = \delta = \gamma = 1$, $\theta = 0$ and $\beta = 2$. Then for any $x(\lambda_i, \mu_i) \in S_w(\lambda_i, \mu_i), i = 1, 2$, where $(\lambda_i, \mu_i) \in U(\lambda) \times U(\mu)$, it holds that

$$d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \frac{4n\rho_{f}^1}{h} + \left(\frac{2n\rho_{f}^2}{h}\right)^{\frac{1}{2}} d^2(\lambda_1, \lambda_2) + \left(\frac{m\rho}{h}\right)^{\frac{1}{2}} d^2(\mu_1, \mu_2). \quad (9)$$

**Proof.** Let $(\lambda_1, \mu_1), (\lambda_2, \mu_2) \in U(\lambda) \times U(\mu)$. We prove the H"older condition (9) holds by the following two steps, based on the fact that $d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq d_X(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)) + d_X(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2))$.

**Step 1:** We prove that, $\forall x(\lambda_1, \mu_1) \in S_w(\lambda_1, \mu_1)$, $\forall x(\lambda_1, \mu_2) \in S_w(\lambda_1, \mu_2)$,

$$d_1 := d_X(x(\lambda_1, \mu_1), x(\lambda_1, \mu_2)) \leq \frac{2n\rho_{f}^1}{h} + \left(\frac{m\rho}{h}\right)^{\frac{1}{2}} d^2(\mu_1, \mu_2). \quad (10)$$

If $x(\lambda_1, \mu_1) = x(\lambda_1, \mu_2)$, then the estimate (10) holds trivially. So we suppose $x(\lambda_1, \mu_1) \neq x(\lambda_1, \mu_2)$. Similar to Step 1 of the proof of Theorem 4.3, we can deduce
By solving the above quadratic inequality we get from Theorem 4.3 consequences for such special cases. In what follows, for given by $X$ continuous (bounded) linear mappings from $\mathbb{C}^n$ we obtain

$$d_1 \leq \frac{n\rho_1 + \sqrt{(n\rho_1)^2 + mp\delta}}{h}.$$

Applying to the last inequality the inequality $\sqrt{a^2 + b^2} \leq a + b$, for $a \geq 0$, $b \geq 0$, we obtain

$$d_1 \leq \frac{2n\rho_1}{h} + \left(\frac{mp}{h}\right)^{\frac{1}{2}} d_2 (\mu_1, \mu_2).$$

**Step 2:** We further prove that, $\forall x(\lambda_1, \mu_2) \in S_u(\lambda_1, \mu_2)$, $\forall x(\lambda_2, \mu_2) \in S_u(\lambda_2, \mu_2)$,

$$d_2 := d_X(x(\lambda_1, \mu_2), x(\lambda_2, \mu_2)) \leq \frac{2n\rho_1}{h} + \left(\frac{2n\rho_2}{h}\right)^{\frac{1}{2}} d_2 (\lambda_1, \lambda_2). \quad (11)$$

Let $x(\lambda_1, \mu_2) \neq x(\lambda_2, \mu_2)$ (the case of $x(\lambda_1, \mu_2) = x(\lambda_2, \mu_2)$ is trivial). Similar to Step 2 of the proof of Theorem 4.3, we can deduce that $hd_2^2 \leq 2n\rho_1d_2^{\gamma_1} + 2n\rho_2d_2^{\gamma_2}(\lambda_1, \lambda_2)$, that is

$$hd_2^2 - 2n\rho_1d_2 - 2n\rho_2d(\lambda_1, \lambda_2) \leq 0.$$

By the same analysis to Step 1, we can obtain that

$$d_2 \leq \frac{2n\rho_1}{h} + \left(\frac{2n\rho_2}{h}\right)^{\frac{1}{2}} d_2 (\lambda_1, \lambda_2).$$

Hence, the estimate (11) also holds. This completes the proof.

Remark that when $K(x, \lambda) = K(\lambda)$, i.e., $K$ does not depend on $x$, $\ell_1 = 0$. Thus, the Hölder condition (9) becomes that

$$d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \frac{2n\rho_2}{h} d_2^{\gamma}(\lambda_1, \lambda_2) + \left(\frac{mp}{h}\right)^{\frac{1}{2}} d_2^{\gamma}(\mu_1, \mu_2),$$

where $x(\lambda, \mu)$ is the unique solution of (PUWVEP).

Since the model (PUWQEP) (resp. (PUQEP)) contains many problems as special cases, including parametric vector quasivariational inequalities, vector quasi-optimization problems, vector quasiconcomplementarity problems, etc., we can derive from Theorem 4.3 consequences for such special cases. In what follows, for simplicity we discuss only corollaries for classical parametric vector variational inequalities and vector optimization problems as examples.

We now assume that $X$ is a normed linear space. Let $L(X, Y)$ be the space of all continuous (bounded) linear mappings from $X$ to $Y$. The value of a linear mapping $t \in L(X, Y)$ at $x \in X$ is denoted by $\langle t, x \rangle$. The norm $\| \cdot \|_L : L(X, Y) \to \mathbb{R}_+$ is given by $\| \Gamma \|_L := \sup_{x \neq 0} \frac{\| t(x) \|}{\| x \|}$ for all $\Gamma \in L(X, Y)$. Let $T : X \times \Omega \to L(X, Y)$ be a vector-valued mapping. If $f(x, y, \mu) = \langle T(x, \mu), y - x \rangle$ and $K(x, \lambda) = K(\lambda)$, then the model (PUWQEP) collapses to the parametric weak vector variational inequality (PWVVI) via improvement set $D$ of finding $\bar{x} \in K(\lambda)$ such that

$$\langle T(\bar{x}, \mu), y - \bar{x} \rangle \notin -\text{int}D, \quad \forall y \in K(\lambda).$$

Denote the solution set of (PWVVI) by $S_{V1}(\lambda, \mu) := \{ x \in K(\lambda) \mid \langle T(x, \mu), y - x \rangle \notin -\text{int}D, \forall y \in K(\lambda) \}$. For the reference point $(\lambda, \bar{\mu}) \in \Lambda \times \Omega$, we assume that $S_{V1}(\lambda, \mu) \neq \emptyset$ for every $\lambda \in U(\Lambda)$ and $\mu \in U(\bar{\mu})$. 


Corollary 1. Suppose that $D \subset Y$ is a closed improvement set with respect to $C$ and the following conditions hold:

(i) $K(\cdot)$ is $t_\alpha$-Hölder continuous at $\bar{\lambda}$;
(ii) $\|T(x, \mu)\|_L \leq \vartheta$ for some $\vartheta > 0$, $\forall x \in K(U(\bar{\lambda}))$;
(iii) $T(x, \cdot)$ is $m_\gamma$-Hölder continuous at $\bar{\mu}$, i.e., $\forall x \in K(U(\bar{\lambda}))$, $\forall \mu_1, \mu_2 \in U(\bar{\mu})$,

$$\|T(x, \mu_1) - T(x, \mu_2)\|_L \leq \tilde{m}d^\gamma(\mu_1, \mu_2);$$

(iv) $T(\cdot, \mu)$ is $h, \beta$-Hölder-related weakly monotone with respect to $\varphi_{D,q}$, i.e., there exist $q \in \text{int} C$ and $h > 0$, $\beta > 0$, such that for every $x, y \in K(U(\bar{\lambda})) : x \neq y,$

$$h\|x - y\|^\beta \leq |\varphi_{D,q}(T(x, \mu), y - x))| + |\varphi_{D,q}(T(y, \mu), x - y))|, \mathbb{R}_+;$$

(v) $K(U(\bar{\lambda}))$ is bounded and let $\varrho := \sup_{x,y \in K(U(\bar{\lambda}))}\|x - y\| < +\infty$ (resp. $\beta > 1$).

Then for every $(\lambda, \mu) \in U(\bar{\lambda}) \times U(\bar{\mu})$, the solution of (PWVVI) is unique, $x(\lambda, \mu)$, and this function satisfies the Hölder condition:

$$d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \left(\frac{2\varrho L}{h}\right)^\frac{1}{\beta} d^\frac{\alpha}{\beta}(\lambda_1, \lambda_2) + \left(\frac{m\varrho}{h}\right)^\frac{1}{\beta} d^\frac{\gamma}{\beta}(\mu_1, \mu_2),$$

$$\text{resp.} d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \left(\frac{2\varrho L}{h}\right)^\frac{1}{\beta} d^\frac{\alpha}{\beta}(\lambda_1, \lambda_2) + \left(\frac{m\varrho}{h}\right)^\frac{1}{\beta} d^\frac{\gamma}{\beta}(\mu_1, \mu_2).$$

Proof. To apply Theorem 4.3, combining with Remark 7, we only need to check the Hölder continuity of $f$. First, $\forall x \in K(U(\bar{\lambda}))$, $\forall y_1, y_2 \in K(U(\bar{\lambda}))$,

$$\|f(x, y_1, \mu) - f(x, y_2, \mu)\| = \|T(x, \mu), y_1 - y_2\| \leq \|T(x, \mu)\|_L\|y_1 - y_2\| \leq \vartheta\|y_1 - y_2\|. $$

Second, $\forall x, y \in K(U(\bar{\lambda}))$, $\forall \mu_1, \mu_2 \in U(\bar{\mu})$,

$$\|f(x, y, \mu_1) - f(x, y, \mu_2)\| = \|T(x, \mu_1) - T(x, \mu_2), y - x\| \leq \|T(x, \mu_1) - T(x, \mu_2)\|_L\|y - x\| \leq \tilde{m}d^\gamma(\mu_1, \mu_2)\|y - x\| \leq \tilde{m}\varrho d^\gamma(\mu_1, \mu_2).$$

Whence, we see that Hölder constants of Theorem 4.3 are fulfilled with $\ell_1 = 0$, $\alpha_1 = \beta_1$ is arbitrary, $\ell_2 = \ell$, $\alpha_2 = \alpha$, $m = \tilde{m}\varrho$, $\vartheta = 0$, $n = \vartheta$ and $\delta = 1$ (resp. $\ell_1 = 0$, $\alpha_1$ is arbitrary, $\ell_2 = \ell$, $\alpha_2 = \alpha$, $m = \tilde{m}$, $\vartheta = 1$, $n = \vartheta$ and $\delta = 1$).

Let $g : X \times \Omega \to Y$ be a vector-valued mapping. If $f(x, y, \mu) = g(y, \mu) - g(x, \mu)$ and $K(x, \lambda) = K(\lambda)$, then the model (PUWVQEP) collapses to the parametric vector optimization problem (PVOP) via improvement set $D$ of finding $\bar{x} \in K(\lambda)$ such that

$$g(y, \mu) - g(\bar{x}, \mu) \notin -\text{int} D, \quad \forall y \in K(\lambda).$$

This is equivalent to consider the parametric problem of (VOP): Min$\{g(x, \mu) \mid x \in K(\lambda)\}$, where “Min” means to finding weak $D$-optimal solution $\bar{x} \in K(\lambda)$ of problem (PVOP), namely, finding $\bar{x} \in K(\lambda)$ such that

$$g(K(\lambda), \mu) - g(\bar{x}, \mu)) \cap (-\text{int} D) = \emptyset.$$
Corollary 2. Suppose that \( D \subset Y \) is a closed improvement set with respect to \( C \) and the following conditions hold:

(i) \( K(\cdot) \) is \( \ell.\alpha \)-Hölder continuous at \( \bar{\lambda} \);

(ii) \( g(\cdot, \mu) \) is \( n.\delta \)-Hölder continuous on \( K(U(\bar{\lambda})) \);

(iii) for any \( x \in K(U(\bar{\lambda})) \), \( g(x, \cdot) \) is \( m.\gamma \)-Hölder continuous at \( \bar{\mu} \);

(iv) \( g(\cdot, \mu) \) is \( h.\beta \)-Hölder-related weakly monotone with respect to \( \varphi_{D,q} \), i.e., there exists \( q \in \text{int}C \) and \( h > 0 \), \( \beta > 0 \), such that for every \( x, y \in K(U(\bar{\lambda})) : x \neq y \),

\[
h\|x - y\|^\beta \leq |\varphi_{D,q}(g(y, \mu) - g(x, \mu)), \mathbb{R}_+| + |\varphi_{D,q}(g(x, \mu) - g(y, \mu)), \mathbb{R}_+|.
\]

Then for every \( (\lambda, \mu) \in U(\bar{\lambda}) \times U(\bar{\mu}) \), the weak \( D \)-optimal solution of \( (PVOP) \) is unique, \( x(\lambda, \mu) \), and this function satisfies the Hölder condition:

\[
dx(\lambda_1, \mu_1), (\lambda_2, \mu_2) \leq \left( \frac{2n\rho\delta}{h} \right)^{\frac{1}{\alpha}} \frac{\alpha^\beta}{\mu} (\lambda_1, \lambda_2) + \left( \frac{2m\rho}{h} \right)^{\frac{1}{\gamma}} \frac{\gamma^{\beta}}{\mu} (\mu_1, \mu_2).
\]

Proof. To apply Theorem 4.3, combining with Remark 7, we only need to check the Hölder continuity of \( f \). First, \( \forall x \in K(U(\bar{\lambda})) \), \( \forall y_1, y_2 \in K(U(\bar{\lambda})) \),

\[
\|f(x, y_1, \mu) - f(x, y_2, \mu)\| = \|g(y_1, \mu) - g(y_2, \mu)\| \leq n\|y_1 - y_2\|^\delta.
\]

Second, \( \forall x, y \in K(U(\bar{\lambda})) \), \( \forall \mu_1, \mu_2 \in U(\bar{\mu}) \),

\[
\|f(x, y, \mu_1) - f(x, y, \mu_2)\| = \|g(y, \mu_1) - g(y, \mu_2) + g(x, \mu_2) - g(x, \mu_1)\| \\
\leq \|g(y, \mu_1) - g(y, \mu_2)\| + \|g(x, \mu_1) - g(x, \mu_2)\| \\
\leq \tilde{m}d^\gamma(\mu_1, \mu_2) + \tilde{m}d^\gamma(\mu_1, \mu_2) \\
= 2\tilde{m}d^\gamma(\mu_1, \mu_2).
\]

Whence, we see that Hölder constants of Theorem 4.3 are fulfilled with \( \ell_1 = 0, \alpha_1 \) is arbitrary, \( \ell_2 = \ell, \alpha_2 = \alpha, m = 2\tilde{m} \) and \( \theta = 0 \).

\[\square\]

Remark 11. Since Gutiérrez et al. [23] have introduced the concept of the \( D \)-optimal (resp. weak \( D \)-optimal) solution for vector optimization problem (VOP): \( \text{Min}\{g(x) \mid x \in K\} \), where \( g : X \to \mathbb{Y} \) and \( K \subset X \) is a nonempty feasible set, the stability analysis for \( D \)-optimal (resp. weak \( D \)-optimal) solutions has not received enough attention (cf. [33, 28]). In this paper, we establish a new result for Hölder continuity of the unique solution to the parametric VOP: \( \text{Min}\{g(x, \mu) \mid x \in K(\lambda)\} \), by considering perturbations of the objective mapping \( g \) and the constraint set \( K \). To the best of our knowledge, it seems to be the first attempt in this subject.

At the end of this section, we make some discussions on the models (PUVQEP) and (PUWQEP) with set-valued mappings. Let \( F : X \times X \times \Omega \rightrightarrows \mathbb{Y} \) be a set-valued mapping with nonempty values. Let

\[
\bar{S}(\lambda, \mu) := \{ x \in E(\lambda) \mid F(x, y, \mu) \cap (-D) = \emptyset, \forall y \in K(x, \lambda) \}
\]

(resp. \( \bar{S}_w(\lambda, \mu) := \{ x \in E(\lambda) \mid F(x, y, \mu) \cap (-\text{int}D) = \emptyset, \forall y \in K(x, \lambda) \} \)

be the corresponding solution set of (PUVQEP) (resp. (PUWQEP)) with the set-valued mapping \( F \), but not the vector-valued mapping \( f \). For the reference point \( (\lambda, \mu) \in \Omega \times \Omega \), we assume that \( \bar{S}(\lambda, \mu) \neq \emptyset \) (resp. \( \bar{S}_w(\lambda, \mu) \neq \emptyset \)) for every \( \lambda \in U(\bar{\lambda}) \) and \( \mu \in U(\bar{\mu}) \). It follows from Proposition 1(i)-(ii) and Remark 3 that for any given \( k^0 \in \text{int}C \) and closed improvement set \( D \) with respect to \( C \),

\[
\bar{S}(\lambda, \mu) \subset \bar{S}_w(\lambda, \mu) = \left\{ x \in E(\lambda) \mid \inf_{z \in F(x, y, \mu)} \varphi_{D,k^0}(z) \geq 0, \forall y \in K(x, \lambda) \right\}.
\]
Lemma 4.4. Let $\psi(x,y,\mu) := \inf_{z \in F(x,y,\mu)} \varphi_{D,k^0}(z)$, where $k^0 \in \text{int}C$ and $D \subset Y$ is a closed improvement set with respect to $C$.

(a) If for each $x, y \in E(U(\bar{\lambda}))$, $F(x,y,\cdot)$ is $m,\gamma$-Hölder continuous at $\bar{\mu}$, then the function $\psi(x,y,\cdot)$ is $Lm,\gamma$-Hölder continuous at $\bar{\mu}$, where

$$L := \sup_{z \in C^0} ||z||,$$

or equivalently, $L = 1/\text{dist}(k^0, \text{bd}C)$ is the Lipschitz constant of $\varphi_{D,k^0}$ on $Y$.

(b) If for each $x \in E(U(\bar{\lambda}))$ and $\mu \in U(\bar{\mu})$, $F(x,\cdot,\mu)$ is $n,\delta$-Hölder continuous on $E(U(\bar{\lambda}))$, then the function $\psi(\cdot,\cdot,\mu)$ is $Ln,\delta$-Hölder continuous on $E(U(\bar{\lambda}))$.

Proof. (a) Let $x, y \in E(U(\bar{\lambda}))$. The $m,\gamma$-Hölder continuity of $F(x,y,\cdot)$ at $\bar{\mu}$ implies that for all $\mu_1, \mu_2 \in U(\bar{\mu})$, $F(x,y,\mu_1) \subset F(x,y,\mu_2) + \text{md}^\gamma(\mu_1, \mu_2)B_Y$. Hence, for any $z_1 \in F(x,y,\mu_1)$, there exist $z_2 \in F(x,y,\mu_2)$ and $b \in B_Y$ such that $z_1 = z_2 + \text{md}^\gamma(\mu_1, \mu_2)b$. By Proposition 1(viii), we get

$$|\varphi_{D,k^0}(z_1) - \varphi_{D,k^0}(z_2)| \leq L \cdot |z_1 - z_2| = L \text{md}^\gamma(\mu_1, \mu_2) \cdot |b| \leq L \text{md}^\gamma(\mu_1, \mu_2).$$

Thus, we have that $-L \text{md}^\gamma(\mu_1, \mu_2) \leq \varphi_{D,k^0}(z_1) - \varphi_{D,k^0}(z_2)$. Since $z_1 \in F(x,y,\mu_1)$ is arbitrary and $\varphi_{D,k^0}(z_2) \geq \psi(x,y,\mu_2)$, it yields that

$$-L \text{md}^\gamma(\mu_1, \mu_2) \leq \psi(x,y,\mu_1) - \psi(x,y,\mu_2).$$

By the symmetry between $\mu_1$ and $\mu_2$, we also have that

$$-L \text{md}^\gamma(\mu_1, \mu_2) \leq \psi(x,y,\mu_2) - \psi(x,y,\mu_1).$$

The above two inequalities together show that

$$|\psi(x,y,\mu_1) - \psi(x,y,\mu_2)| \leq L \text{md}^\gamma(\mu_1, \mu_2), \quad \forall \mu_1, \mu_2 \in U(\bar{\mu}).$$

Therefore, $\psi(x,y,\cdot)$ is $Lm,\gamma$-Hölder continuous at $\bar{\mu}$.

(b) The proof is similar to that of part (a).

Based on Lemma 4.4, the following stability result can be proved similarly as Theorem 4.3, just mainly replacing the role of $\varphi_{D,q} \circ f$ with $\psi$.

Theorem 4.5. Suppose that $D \subset Y$ is a closed improvement set with respect to $C$ and the following conditions hold:

(i) $K(\cdot, \cdot, \cdot)$ is $(\ell_1, \alpha_1, \ell_2, \alpha_2)$-Hölder continuous on $E(U(\bar{\lambda})) \times \{\lambda\}$;

(ii) for any $x \in E(U(\bar{\lambda}))$, $F(x,\cdot,\cdot,\cdot)$ is $n,\delta$-Hölder continuous on $E(U(\bar{\lambda}))$;

(iii) for any $x, y \in E(U(\bar{\lambda}))$, $F(x,y,\cdot)$ is $m,\gamma$-Hölder continuous at $\bar{\mu}$;

(iv) $F(\cdot,\cdot)$ is $h,\beta$-Hölder-related weakly monotone with respect to $\varphi_{D,q}$, i.e., there exist $q \in \text{int}C$ and $h > 0$, $\beta > 0$, such that for every $x, y \in E(U(\bar{\lambda}))$:

$$x \neq y, \quad \text{hd}^\beta_X(x,y) \leq |\psi(x,y,\mu), R_+| + |\psi(y,x,\mu), R_+|, \quad \text{where} \quad \psi(x,y,\mu) := \inf_{z \in F(x,y,\mu)} \varphi_{D,q}(z);$$

(v) $\beta = n, \delta, h > 2n, \rho$, $\rho := \sup_{s \in C^0} ||s|| \in \left[\frac{1}{\text{dist}(q, \text{bd}C)}, +\infty\right]$ (or $\rho = 1/\text{dist}(q, \text{bd}C)$) is the Lipschitz constant of $\varphi_{D,q}$ on $Y$ and $q \in \text{int}C$ is given in above condition.

Then for every $(\lambda, \mu) \in U(\bar{\lambda}) \times U(\bar{\mu})$, the solution set $\tilde{S}_w(\lambda, \mu)$ is a singleton $\{x(\lambda, \mu)\}$ (of course $S(\lambda, \mu)$), and this function $x(\lambda, \mu)$ satisfies the Hölder condition:

$$d_X(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \left(\frac{2n, \rho, \beta}{h - 2n, \rho, \beta}\right)^\frac{1}{n, \rho, \beta} d^{\frac{m, \rho}{\rho}}(\lambda_1, \lambda_2) + \left(\frac{m, \rho}{h - 2n, \rho, \beta}\right)^\frac{1}{n, \rho, \beta} d^{\frac{m, \rho}{\rho}}(\mu_1, \mu_2).$$

Combining Theorem 4.5 and Proposition 3, we have the following result.
Proposition 4. Assume that $D \subset Y$ is a closed improvement set with respect to $C$ and conditions (i)-(iv) in Theorem 4.5 are satisfied with $\alpha_1 = \alpha_2 = \delta = \gamma = 1$ and $\beta = 2$. Then for any $x(\lambda_i, \mu_i) \in \tilde{S}_w(\lambda_i, \mu_i), \ i = 1, 2,$ where $(\lambda_i, \mu_i) \in U(\tilde{\lambda}) \times U(\tilde{\mu})$, it holds that
\[
d_{X}(x(\lambda_1, \mu_1), x(\lambda_2, \mu_2)) \leq \frac{4n \rho \ell_1}{h} + \left(\frac{2n \rho \ell_2}{h}\right) \frac{1}{2} d^{\frac{1}{2}}(\lambda_1, \lambda_2) + \left(\frac{m \rho}{h}\right) \frac{1}{2} d^{\frac{1}{2}}(\mu_1, \mu_2).
\]

Obviously, Theorem 4.5 and Proposition 4 can be applied to parametric Minty-type dual problems with set-valued mappings by setting $F(x, y, \mu) := -F(y, x, \mu)$.

5. Conclusions. In this paper, motivated by the recent development of improvement sets in vector optimization (see [23, 14, 39, 40, 41, 33, 28]), we have introduced a new vector quasiequilibrium problem (UVQEP) via improvement set $D$ and its weak version (UWVQEP), also their Minty-type dual problems and the corresponding set-valued cases. By taking different improvement sets, these models reduce to well-known exact and approximate vector quasiequilibrium problems with vector-valued or set-valued mappings in the literature. Therefore, these kinds of models provide unified frameworks to deal with exact and approximate vector quasiequilibria. Especially, the typical vector optimization problem (VOP) defined via (resp. weak) $D$-optimality [23] is a special case of our general settings.

As it happens with all new models, there are many issues to analyze and study, such as solution existence, stability and sensitivity analysis, etc. Naturally, it is very important to understand behaviors of solutions for (UVQEP) (resp. (UWVQEP)) when the problem’s data vary. In other words, we need to know properties of solutions for the parametric UVQEP (resp. UWVQEP) when parameters vary. Therefore, in this work we have mainly investigated solution stability in the sense of Hölder continuity for parametric UVQEP and UWQEP. To this aim, the techniques of Gerstewitz (Tammer) nonlinear scalarization [37, 7] have been fully applied, and several Hölder continuity results of the unique solution to the models (PUVQEP) and (PUWQEP) have been established, especially including the $D$-optimal (resp. weak $D$-optimal) solution for parametric (VOP).

The Gerstewitz (Tammer) scalarizing function $\varphi$ seems to be good at dealing with solution stability analysis of vector equilibrium and quasiequilibrium problems via improvement sets (cf. [7]). In particular, its globally Lipschitz property has a natural link with the improvement set (cf. [37, 7]). Whence, it needs more attention on how to establish other stability results of single-valued and set-valued solution mappings to parametric vector (quasi)equilibrium problems via improvement sets proposed in this paper, by employing Gerstewitz (Tammer) scalarization approaches.

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E-mail address: hongzhiwei10922163.com
E-mail address: zuoxin1991@126.com
E-mail address: chencr1981@163.com