Twist Deformation of the rank one Lie Superalgebra

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Abstract

The Drinfeld twist is applied to deform the rank one orthosymplectic Lie superalgebra $osp(1|2)$. The twist element is the same as for the $sl(2)$ Lie algebra due to the embedding of the $sl(2)$ into the superalgebra $osp(1|2)$. The $R$-matrix has the direct sum structure in the irreducible representations of $osp(1|2)$. The dual quantum group is defined using the FRT-formalism. It includes the Jordanian quantum group $SL_\xi(2)$ as subalgebra and Grassmann generators as well.

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1 The deformed algebra $osp_\xi(1|2)$

It is difficult to overestimate the role of the rank one Lie algebra $sl(2)$ in the theory of Lie groups and their applications. The corresponding role for Lie superalgebras is played by the orthosymplectic superalgebra $osp(1|2)$ with five generators $\{h, X_-, X_+, v_-, v_+\}$ and commutation relations (Lie super- or $Z_2$ graded-brackets):

$$[h, X_\pm] = \pm 2X_\pm, \quad [X_+, X_-] = h, \quad \text{(1)}$$

$$[h, v_\pm] = \pm v_\pm, \quad [v_+, v_-]_+ = -h/4, \quad \text{(2)}$$

$$[X_\pm, v_\pm] = 0, \quad [X_\pm, v_\mp] = v_\pm, \quad [v_\pm, v_\pm]_+ = \pm X_\pm/2. \quad \text{(3)}$$

The generators $h$ and $X_\pm$ are even (zero parity $p = 0$), while $v_\pm$ are odd, $p = 1$. As a Hopf superalgebra, the universal enveloping $\mathcal{U}(osp(1|2))$ of $osp(1|2)$ is generated, as $sl(2)$, just by three elements: it is sufficient to start from $\{h, v_-, v_+\}$ restricted by the relations (2) only, and define $X_\pm \equiv \pm 4v_\pm^2$.

The quantum deformation of $sl(2)$ can be considered as a "pivot" of the quantum group theory [1, 2], while the corresponding quantum superalgebra $osp_q(1|2)$ constructed in [3, 4, 5], is the corresponding analogue for the quantum supergroups. As a quasitriangular Hopf superalgebra $osp_q(1|2)$, analogously to the universal enveloping of $osp(1|2)$, is generated by three elements $\{h, v_-, v_+\}$ under the relations

$$[h, v_\pm] = \pm v_\pm, \quad [v_+, v_-] = -\frac{1}{4} (q^h - q^{-h})/(q - q^{-1}).$$

It is worthy to note that, while $sl(2)$ is embedded into $osp(1|2)$, such embedding does not exist for $sl_q(2)$ into $osp_q(1|2)$ because the coproduct of even elements $X_\pm \sim v_\pm^2$ includes also odd ones.

The aim of this paper is to construct and study the twist deformation [6] of $osp(1|2)$ that looks, in some sense, more natural than $osp_q(1|2)$ because it is consistent with this fundamental property of inclusion $sl(2) \subset osp(1|2)$ and it is generated by the same twist element of $sl(2)$.

The triangular Hopf algebra $sl_\xi(2)$ (cf. [4, 8, 9, 10, 11, 12], and Refs therein) is given by the extension of the twist deformation of the universal
enveloping of the Borel sub-algebra $B_- \equiv \{h, X_-\}$ to the whole $\mathcal{U}(sl(2))$. The twist element $\mathcal{F}$ is

$$\mathcal{F} = 1 + \xi h \otimes X_- + \frac{\xi^2}{2} h(h + 2) \otimes X_-^2 + \ldots$$

that can be written as

$$\mathcal{F} = (1 - 2\xi 1 \otimes X_-)^{-\frac{1}{2}(h \otimes 1)} = \exp\left(\frac{1}{2} h \otimes \sigma\right) \quad (4)$$

where $\sigma = -\ln (1 - 2\xi X_-)$.

Let us recall from [6] that for a quasitriangular Hopf algebra $A$ with an $R$-matrix $\mathcal{R}$ the twisted Hopf algebra $A_t$ has $R$-matrix $\mathcal{R}^{(F)}$ given by the twist transformation

$$\mathcal{R}^{(F)} = \mathcal{F}_{21} \mathcal{R} \mathcal{F}^{-1} \quad (5)$$

of the original $R$-matrix $\mathcal{R}$, where $\mathcal{F}_{21} = \mathcal{P} \mathcal{F} \mathcal{P}$, and $\mathcal{P}$ is the permutation map in $A \otimes A$. The algebraic sector of $A_t$ is not changed, and new coproduct is $\Delta_t = \mathcal{F} \Delta \mathcal{F}^{-1}$. The twist element satisfies the relations in $A \otimes A$ [6]

$$(\epsilon \otimes \text{id}) \mathcal{F} = (\text{id} \otimes \epsilon) \mathcal{F} = 1,$$

and in $A \otimes A \otimes A$

$$\mathcal{F}_{12} (\Delta \otimes \text{id}) \mathcal{F} = \mathcal{F}_{23} (\text{id} \otimes \Delta) \mathcal{F}.$$ 

According to this Drinfeld definition, the algebraic relations of eqs. (6) for the twisted $sl(2)$ are still the same, while the twisted coproduct $\Delta_t \equiv \mathcal{F} \Delta \mathcal{F}^{-1}$ is now on the generators

$$\Delta_t(h) = h \otimes e^\sigma + 1 \otimes h,$$

$$\Delta_t(X_-) = X_- \otimes 1 + 1 \otimes X_- - 2\xi X_- \otimes X_- = X_- \otimes e^{-\sigma} + 1 \otimes X_-,$$

$$\Delta_t(X_+) = X_+ \otimes e^\sigma + 1 \otimes X_+ - \xi h \otimes e^\sigma h + \frac{\xi}{2} h(h - 2) \otimes e^\sigma (1 - e^\sigma).$$

Let us stress that this twist of the whole $sl(2)$ is obtained due to the embedding $B_- \subset sl(2)$.

Thus, knowing that $B_- \subset sl(2) \subset osp(1|2)$, the procedure can be simply iterated to find $osp_\xi(1|2)$ (as well as the twisted deformations of all others nontrivial embeddings of $B_-$). It is an easy exercise, keeping in mind the
expression of $\mathcal{F}$ (eq. (4)), commutation relations (2), (3) and the primitive coproduct of $osp(1|2)$, to obtain:

$$
\begin{align*}
\Delta_t(h) &= h \otimes e^\sigma + 1 \otimes h, \\
\Delta_t(v_-) &= v_- \otimes e^{-\sigma/2} + 1 \otimes v_-, \\
\Delta_t(v_+) &= v_+ \otimes e^{\sigma/2} + 1 \otimes v_+ + \xi h \otimes v_- e^\sigma.
\end{align*}
$$

One can reproduce the coproducts of $X_\pm$ by squaring the coproducts of $v_\pm$, taking into account the $\mathbb{Z}_2$-grading of tensor product:

$$(x \otimes y) (u \otimes w) = (-1)^{p(x)p(y)(x u \otimes y w)},$$

and the commutation relations (2), (3).

The maps of counit $\epsilon$ and antipode $S$, necessary for a Hopf superalgebra definition, are

$$
\begin{align*}
\epsilon(h) &= \epsilon(v_-) = 0, \quad \epsilon(1) = 1, \\
S(h) &= -he^{-\sigma}, \quad S(v_-) = -v_- e^{\sigma/2}, \quad S(v_+) = -(v_+ - \xi hv_-) e^{-\sigma/2}.
\end{align*}
$$

We can thus arrive to the following \underline{Definition}. The Hopf superalgebra generated by three elements $\{h, v_-, v_+\}$ satisfying the relations (2), (3) and (4) is said to be the twist deformation of $U(osp(1|2))$ or $osp_\xi(1|2)$.

This is a triangular Hopf superalgebra ($R_{21}R = 1$) with universal $R$-matrix

$$
R = \exp(\frac{1}{2} \sigma \otimes h) \exp(-\frac{1}{2} h \otimes \sigma).
$$

The irreducible finite dimensional representations of $osp_\xi(1|2)$

$$
\rho_s : osp_\xi(1|2) \rightarrow End(W_s)
$$

are the same as for $osp(1|2)$, due to the unchanged algebraic relations (2). They are parametrized by the half-integer spin $s = 0, \frac{1}{2}, 1, \ldots$, have dimension $4s + 1$, and are decomposed into a direct sum of two irreps of the $sl(2)$ [4]: $W_s = V_s + V_{s-\frac{1}{2}}$. Hence, the $R$-matrix in the irreps of $osp_\xi(1|2)$ is a direct sum of four $R$-matrices of $sl_\xi(2)$. For the first non-trivial case $s = 1/2$ one gets

$$
R = (\rho_{\frac{1}{2}} \otimes \rho_{\frac{1}{2}}) R = R(\xi) + I_2 + I_2 + 1,
$$

3
where $I_2$ are $2 \times 2$ unit matrices, and $R(\xi)$ is the Jordanian solution to the Yang-Baxter equation (cf. [7])

$$R(\xi) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
-\xi & 1 & 0 & 0 \\
\xi & 0 & 1 & 0 \\
\xi^2 & -\xi & \xi & 1
\end{pmatrix}.$$  \hspace{1cm} (10)

The twist parameter can be scaled: $\xi \rightarrow \exp(2u)\xi$ by the similarity transformation with the element $\exp(-uh)$. The basis of the irreps tensor product decomposition will include deformed Clebsch-Gordan coefficients, expressed as linear combinations of the usual ones and the matrix elements of the twist $F$ \[14\]. This is reflected in the spectral decomposition of the $R$-matrix itself in the tensor product $W_s \otimes W_l$

$$\hat{R}^{s,l} = F^{s,l} \left( \sum_{j=|s-l|}^{s+l} (\pm P^j) \right) (F^{s,l})^{-1},$$

where $P^j$ are projectors onto irreducible representations of $osp(1|2)$.

### 2 Quantum supergroup $OSp_\xi(1|2)$

The self-dual character of the twisted Borel subalgebra $(B_-)_\xi$ was pointed out in \[8\]. This is obvious in terms of the generators $\{h, \sigma\} \in (B_-)_\xi$ and the generators $\{s, p\} \in (B_-)_{\xi}$ of the dual, with the only non-trivial evaluations $\langle h, s \rangle = 2, \langle \sigma, p \rangle = 2 \hspace{1cm} [8, 9]:$

$$[h, \sigma] = 2(1 - e^\sigma), \quad [p, s] = 2(1 - e^s),$$

$$\Delta(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma, \quad \Delta(s) = s \otimes 1 + 1 \otimes s,$$

$$\Delta(h) = h \otimes e^\sigma + 1 \otimes h, \quad \Delta(p) = p \otimes e^s + 1 \otimes p,$$

$$\epsilon(h) = \epsilon(\sigma) = 0, \quad \epsilon(s) = \epsilon(p) = 0,$$

$$S(h) = -he^{-\sigma}, \quad S(\sigma) = -\sigma, \quad S(p) = -pe^{-s}, \quad S(s) = -s.$$  

The situation is different for the twisted Hopf super-subalgebra $(sB_-)_\xi$. The latter is generated by two elements $\{h, v_-\}$ as $(B_-)_\xi$. However, due to the $Z_2$-grading its basis as a linear space consists of even $\sigma^m h^n$ and odd $\sigma^m v_- h^n$ elements ($\sigma = -\ln (1 + 8 \xi v_-^2)$).
Proposition. The dual \((sB_-)'_\xi\) of the twisted Hopf superalgebra \((sB_-)_\xi\) is generated by three elements \(\{\nu, \eta, x\}\) satisfying the relations

\[
\begin{align*}
[\nu, \eta] &= 0, \quad [\nu, x] = \frac{1}{2}(1 - e^{-2\nu}), \quad [x, \eta] = \frac{1}{2} \eta, \quad \eta^2 = 0, \quad (11) \\
\Delta(\nu) &= \nu \otimes 1 + 1 \otimes \nu, \quad \Delta(\eta) = \eta \otimes 1 + e^{-\nu} \otimes \eta, \\
\Delta(x) &= x \otimes 1 + e^{-2\nu} \otimes x + \frac{1}{8\xi} e^{-\nu} \eta \otimes \eta, \\
\epsilon(x) &= \epsilon(\eta) = \epsilon(\nu) = 0, \\
S(\eta) &= -\eta e^\nu, \quad S(\nu) = -\nu, \quad S(x) = -x e^{2\nu}.
\end{align*}
\]

One can check this by a straightforward calculation of evaluating the dual basis \(x_k \eta^l \delta^m\) of \((sB_-)'_\xi\) and \(\sigma^m \nu^k \eta^n\) of \((sB_-)_\xi\), \(k, l, m, n = 0, 1, 2, \ldots; \delta = 0, 1\) with the only non-zero evaluations among the generators: \(\langle h, \nu \rangle = 1, \langle v_-, \eta \rangle = 1, \langle \sigma, x \rangle = 1\). We shall prove it below by a reduction from the quantum supergroup \(OSp_\xi(1|2)\). The universal \(T\)-matrix (bicharacter) is given in term of these basis by a product of three exponents

\[
T = \exp(\sigma \otimes x) \exp(v_- \otimes \eta) \exp(h \otimes \nu).
\]

It is interesting to point out that starting from a Hopf superalgebra without nilpotent elements we were forced to introduce Grassmann variables \(\eta\) in the dual superalgebra.

The dual of the twisted Hopf superalgebra \(osp_\xi(1|2)\) can be introduced using a \(Z_2\)-graded version of the FRT-formalism \[3\], because the \(R\)-matrix in the fundamental representation is known \[3\]. The \(T\)-matrix of generators of quantum supergroup \(OSp_\xi(1|2)\) in this representation has dimension \(3 \times 3\). There are two convenient basis in this irrep as \(C^3\): i) with grading (0, 1, 0) and ii) with grading (0, 0, 1). The odd generators \(v_-, v_+\) of \(osp(1|2)\) are lower and upper triangular in the former basis, while the latter one is more convenient to write \(T\) in a block matrix form. These forms are

\[
T = \begin{pmatrix} a & \alpha & b \\ \gamma & g & \beta \\ c & \delta & d \end{pmatrix}, \quad \begin{pmatrix} T \psi \\ \omega \end{pmatrix}, \quad (12)
\]

where \(T\) is \(2 \times 2\) matrix of the even generators \(\{a, b, c, d\}\), while \(\psi\) and \(\omega\) are two component column \((\alpha, \beta)^t\) and row \((\gamma, \beta)\) vectors of odd elements.
The $3 \times 3$ matrix $T$ of the $OSp(1|2)$ generators satisfies the FRT-relation
\[
RT_1T_2 = T_2T_1R
\]
with $\mathbb{Z}_2$-graded tensor product and $9 \times 9$ $R$-matrix $R$ \((9)\). From the block-diagonal form of $R$ \((9)\) it follows for $2 \times 2$ matrix $T$
\[
R(\xi)T_1T_2 = T_2T_1R(\xi) .
\]
Hence, one reproduces the algebraic sector (commutation relations) of the twisted quantum group $SL_\xi(2)$ for the generators \{$a,b,c,d$\} \([7]\). For the other blocks of different dimension we get from (13) \((9)\)
\[
R(\xi)T_1\psi_2 = \psi_2T_1 , \quad gT = Tg , \quad \omega_1T_2 = T_2\omega_1R(\xi) , \quad \omega_1\psi_2 = -\psi_2\omega_1 , \quad \omega_1\omega_2 = -\omega_2\omega_1R(\xi) , \quad R(\xi)\psi_1\psi_2 = -\psi_2\psi_1 .
\]
From the relations (14) - (17) one gets centrality of the following elements:
\[
det_\xi T = a(d - \xi b) - cb , \quad g = \omega T^{-1} \psi .
\]
Coproduct, counit and antipode are given by the standard expressions of the FRT-formalism \([2]\)
\[
\Delta(T) = T \otimes T , \quad \epsilon(T) = I_3 , \quad S(T) = T^{-1} .
\]
The inverse of $T$ is expressed in terms of the generators (12) provided invertability of $\det_\xi T$, and $(g - \omega T^{-1} \psi)$
\[
T^{-1} = \begin{pmatrix} I_2 & -T^{-1}\psi \\ 0 & 1 \end{pmatrix} \begin{pmatrix} T^{-1} & 0 \\ 0 & (g - \theta)^{-1} \end{pmatrix} \begin{pmatrix} I_2 & 0 \\ -\omega T^{-1} & 1 \end{pmatrix} .
\]
Thus we arrive to the following
\textbf{Definition.} The dual to the Hopf superalgebra $osp(1|2)$ generated by the entries of $T$ (12) subject to the relations (14) - (18) is said to be the quantum supergroup $OSp_\xi(1|2)$.

Another way to define this $OSp_\xi(1|2)$ is to use the twist element $\mathcal{F}$ as the pseudodifferential operator on the Lie supergroup $OSp(1|2)$, and redefine super-commutative product of functions on this supergroup.

The reduction or Hopf superalgebra homomorphism, of $OSp_\xi(1|2)$ to $(sB_-)\xi$ is given by:
\[
b = \alpha = \beta = 0 , \quad g = 1 , \quad a = d^{-1} = \exp(\nu) , \quad \gamma a^{-1} = \delta = \frac{1}{2} \eta , \quad c = 2\xi xa .
\]
3 Conclusion

Using embedding of the Lie algebra $sl(2)$ into the rank one orthosymplectic superalgebra the latter one was deformed by the twist element $\mathcal{F} \in \mathcal{U}(sl(2))^{\otimes 2}$. Although the deformed Lie superalgebra is finite dimensional it can be used for further deformation of infinite dimensional Hopf superalgebras (e.g. super-Yangians) and integrable models [14]. There are also possibilities for different contractions. The work in this direction is in progress.

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