The resistance distance and Kirchhoff index on quadrilateral graph and pentagonal graph

Qun Liu a,c, Zhongzhi Zhang a,b

a. School of Computer Science, Fudan University, Shanghai 200433, China
b. Shanghai Key Laboratory of Intelligent Information Processing, Fudan University, Shanghai 200433, China
c. School of Mathematics and Statistics, Hexi University, Zhangye, Gansu, 734000, China.

Abstract

The quadrilateral graph \( Q(G) \) is obtained from \( G \) by replacing each edge in \( G \) with two parallel paths of length 1 and 3, whereas the pentagonal graph \( W(G) \) is obtained from \( G \) by replacing each edge in \( G \) with two parallel paths of length 1 and 4. In this paper, closed-form formulas of resistance distance and Kirchhoff index for quadrilateral graph and pentagonal graph are obtained whenever \( G \) is an arbitrary graph.

Keywords: Kirchhoff index, Resistance distance, Generalized inverse

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1 Introduction

All graphs considered in this paper are simple and undirected. Let \( G = (V(G), E(G)) \) be a graph with vertex set \( V(G) \) and edge set \( E(G) \). Let \( d_i \) be the degree of vertex \( i \) in \( G \) and \( D_G = \text{diag}(d_1, d_2, \cdots, d_{|V(G)|}) \) the diagonal matrix with all vertex degrees of \( G \) as its diagonal entries. For a graph \( G \), let \( A_G \) and \( B_G \) denote the adjacency matrix and vertex-edge incidence matrix of \( G \), respectively. The matrix \( L_G = D_G - A_G \) is called the Laplacian matrix of \( G \), where \( D_G \) is the diagonal matrix of vertex degrees of \( G \).

The resistance distance between vertices \( u \) and \( v \) of \( G \) was defined by Klein and Randić [1] to be the effective resistance between nodes \( u \) and \( v \) as computed with Ohm’s law when all the edges of \( G \) are considered to be unit resistors. The Kirchhoff index \( Kf(G) \) was defined in [1] as \( Kf(G) = \sum_{u<v} r_{uv} \), where \( r_{uv}(G) \) denotes the resistance distance between \( u \) and \( v \) in \( G \). Resistance distance are, in fact, intrinsic to the graph, with some nice purely mathematical interpretations and other interpretations. The Kirchhoff index was introduced in chemistry as a better alternative to other parameters used for discriminating different molecules with similar shapes and structures. See [1]. The resistance distance and the Kirchhoff index have attracted extensive attention due to its wide applications in physics, chemistry and others.
till now, many results on the resistance distance and the Kirchhoff index are obtained. See ([2], [4], [6] – [13]) and the references therein to know more.

Recently the quadrilateral graph and the pentagonal graph have attracted interesting study. In [3], the normalized Laplacian spectrum of the quadrilateral graph are obtained and calculate the multiplicative degree-Kirchhoff index and the spanning trees of the the quadrilateral graph. Let $G$ be a simple and connected graph with $n$ vertices and $m$ edges. Replacing each edge of $G$ with two parallel paths of lengths 1 and 3 and we call the resulting graph as quadrilateral graph. We denote by $N$ the total number of vertices and $E$ the total number of edges of $G$. It is clear that $E = 4m, N = n + 2m$. Based on the quadrilateral graph, we define the new graph. The pentagonal graph of the graph $G$ is obtained that replacing each edge of $G$ with two parallel paths of lengths 1 and 4. It is clear that $E = 5m, N = n + 3m$.

The rest of this paper is organized as follows. In Section 2, we list some lemmas and give some preliminary results which are used to prove our main results. In Section 3, we give the resistance distance and Kirchhoff index of quadrilateral graph $Q(G)$ whenever $G$ is an arbitrary graph. In Section 4, we give the resistance distance and Kirchhoff index of pentagonal graph $W(G)$ whenever $G$ is an arbitrary graph.

2 Preliminaries

The $\{1\}$-inverse of $M$ is a matrix $X$ such that $MXM = M$. If $M$ is singular, then it has infinite $\{1\}$-inverse [8]. For a square matrix $M$, the group inverse of $M$, denoted by $M^\#$, is the unique matrix $X$ such that $MXM = M, XMX = X$ and $MX = XM$. It is known that $M^\#$ exists if and only if $\text{rank}(M) = \text{rank}(M^2)$ ([8], [9]). If $M$ is real symmetric, then $M^\#$ exists and $M^\#$ is a symmetric $\{1\}$-inverse of $M$. Actually, $M^\#$ is equal to the Moore-Penrose inverse of $M$ since $M$ is symmetric [9].

It is known that resistance distances in a connected graph $G$ can be obtained from any $\{1\}$-inverse of $G$ ([2]). We use $M^{(1)}$ to denote any $\{1\}$-inverse of a matrix $M$, and let $(M)_{uv}$ denote the $(u, v)$-entry of $M$.

**Lemma 2.1** ([9]) Let $G$ be a connected graph. Then

$$r_{uv}(G) = (L_G^{(1)})_{uu} + (L_G^{(1)})_{uv} - (L_G^{(1)})_{vu} = (L_G^\#)_{uu} + (L_G^\#)_{vv} - 2(L_G^\#)_{uv}.$$ 

Let $1_n$ denotes the column vector of dimension $n$ with all the entries equal one. We will often use $1$ to denote an all-ones column vector if the dimension can be read from the context.

**Lemma 2.2** ([5]) For any graph $G$, we have $L_G^\#1_n = 0$.

**Lemma 2.3** ([13]) Let

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be a nonsingular matrix. If $A$ and $D$ are nonsingular, then

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BS^{-1}CA^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix} = \begin{pmatrix} (A - BD^{-1}C)^{-1} & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix},$$
where $S = D - CA^{-1}B$.

For a square matrix $M$, let $tr(M)$ denote the trace of $M$.

**Lemma 2.4** Let $G$ be a connected graph on $n$ vertices. Then

$$Kf(G) = ntr(L_G^{(1)}) - 1^T L_G^{(1)} 1 = ntr(L_G^#).$$

**Lemma 2.7** Let

$$L = \begin{pmatrix} A & B \\ B^T & D \end{pmatrix}$$

be a symmetric block matrix. If $D$ is nonsingular, then

$$X = \begin{pmatrix} H^# & -H^#BD^{-1} \\ -D^{-1}B^TH^# & D^{-1} + D^{-1}B^TH^#BD^{-1} \end{pmatrix}$$

is a symmetric $\{1\}$-inverse of $L$, where $H = A - BD^{-1}B^T$.

### 3 The resistance distance and Kirchhoff index of quadrilateral graph

In this section, we focus on determining the resistance distance and Kirchhoff index of $Q(G)$ whenever $G$ is an arbitrary graph. Let $E_G = \{e_1, e_2, \ldots, e_m\}$. For each edge $e_i = u_iu_i \in E_G$, there exist two parallel paths of lengths 1 and 3 in $Q(G)$ corresponding to it, which are denoted by $u_iu_1u_iu_2u_i$ for $i = 1, 2, \ldots, m$. Let $V_1 = \{u_{11}, u_{21}, \ldots, u_{m1}\}$, $V_2 = \{u_{12}, u_{22}, \ldots, u_{m2}\}$. Then $V_{Q(G)} = V \cup V_1 \cup V_2$, where $V$ is the set of all the vertices inherited from $G$. Our main results in the following gives the explicit formula of the resistance distance and Kirchhoff index of $Q(G)$.

**Theorem 3.1** Let $G$ be a connected graph with $n$ vertices and $m$ edges and $Q(G)$ be its quadrilateral graph. Then we have the resistance distance and Kirchhoff index as follows:

(i) For any $i, j \in V(G)$, we have

$$r_{ij}(Q(G)) = \frac{3}{4}(L_G^#)_{ii} + \frac{3}{4}(L_G^#)_{jj} - \frac{3}{2}(L_G^#)_{ij} = \frac{3}{4}r_{ij}(G).$$

(ii) For any $i \in V$, $j \in V_1$ or $V_2$, $j \in V$, we have

$$r_{ij}(Q(G)) = \frac{3}{4}(L_G^#)_{ii} + \left[ \left( \frac{1}{2}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{2}{3}B_1 + \frac{1}{3}B_2 \right) \right]_{jj} - 2 \left[ L_G^# \left( \frac{1}{2}B_1 + \frac{1}{4}B_2 \right) \right]_{ij}.$$

(iii) For any $i \in V_1$, $j \in V_2$, we have

$$r_{ij}(Q(G)) = \frac{4}{3} + \left[ \left( \frac{1}{2}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{2}{3}B_1 + \frac{1}{3}B_2 \right) \right]_{ii} + \left[ \left( \frac{1}{4}B_1^T + \frac{1}{2}B_2^T \right) L_G^# \left( \frac{1}{3}B_1 + \frac{2}{3}B_2 \right) \right]_{jj}$$

$$- \left( \frac{1}{3}I_m + \left( \frac{1}{2}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{1}{3}B_1 + \frac{2}{3}B_2 \right) \right)_{ij}.$$
(iv) For any \(i, j \in V_1 \) or \(V_2\), we have

\[
\begin{align*}
    r_{ij}(Q(G)) &= 4 + \frac{1}{2} B_1^T + \frac{1}{4} B_2^T L_G^\#(\frac{2}{3} B_1 + \frac{1}{3} B_2)_{ii} + \frac{1}{4} B_2^T L_G^\#(\frac{2}{3} B_1 + \frac{1}{3} B_2)_{jj} \\
    &\quad - 2(\frac{1}{2} B_1^T + \frac{1}{4} B_2^T L_G^\#(\frac{2}{3} B_1 + \frac{1}{3} B_2)_{ij}.
\end{align*}
\]

(v)

\[
\begin{align*}
    K_f(Q(G)) &= (n + 2m) \left( \frac{3}{4n} K_f(G) + \frac{5}{12} \left( \text{tr}(B_1^T L_G^# B_1) + \text{tr}(B_1^T L_G^# B_2) \right) \\
    &\quad + \frac{1}{3} \left( \text{tr}(B_2^T L_G^# B_1) + \text{tr}(B_2^T L_G^# B_2) \right) \right) \\
    &\quad \quad - \frac{3}{4} \left( 1^T B_1^T L_G^# B_1 + 1^T B_2^T L_G^# B_2 \right) \\
    &\quad \quad + 1^T B_1^T L_G^# B_1 + 1^T B_2^T L_G^# B_1 - 2m,
\end{align*}
\]

where \(B_1 + B_2 = B, B_1 B_2^T + B_2^T B_1 = A_G\) and \(B_1 B_1^T + B_2 B_2^T = D_G\).

**Proof** Let \(A_G, D_G\) and \(B_G\) be the adjacency matrix, degree matrix and incidence matrix of \(G\). With a suitable labeling for vertices of \(Q(G)\), the Laplacian matrix of \(Q(G)\) can be written as follows:

\[
    L_{QG} = \begin{pmatrix}
        2D_G - A_G & -B_1 & -B_2 \\
        -B_1^T & 2I_m & -I_m \\
        -B_2^T & -I_m & 2I_m
    \end{pmatrix},
\]

where \(B_1 + B_2 = B, B_1 B_2^T + B_2^T B_1 = A_G\) and \(B_1 B_1^T + B_2 B_2^T = D_G\).

Let \(A = 2D_G - A_G, B = (-B_1 - B_2), B^T = \begin{pmatrix} -B_1^T \\ -B_2^T \end{pmatrix}\) and \(D = \begin{pmatrix} 2I_m & -I_m \\ -I_m & 2I_m \end{pmatrix}\).

By Lemma 2.3, it is easily obtained that \(D^{-1} = \begin{pmatrix} \frac{2}{3}I_m & \frac{1}{3}I_m \\ \frac{1}{3}I_m & \frac{2}{3}I_m \end{pmatrix}\).

First we begin with the computation of \(\{1\}\)-inverse of \(Q(G)\).

By Lemma 2.7, we have

\[
    H = 2D_G - A_G - (-B_1 - B_2) \begin{pmatrix} \frac{2}{3}I_m & \frac{1}{3}I_m \\ \frac{1}{3}I_m & \frac{2}{3}I_m \end{pmatrix} \begin{pmatrix} -B_1^T \\ -B_2^T \end{pmatrix} \\
    = 2D_G - A_G - \frac{2}{3} \begin{pmatrix} -B_1 & -B_2 \end{pmatrix} \begin{pmatrix} \frac{2}{3}B_1 & \frac{1}{3}B_2 \\ \frac{1}{3}B_1 & \frac{2}{3}B_2 \end{pmatrix} \begin{pmatrix} -B_1^T \\ -B_2^T \end{pmatrix} \\
    = 2D_G - A_G - \frac{2}{3} (B_1 B_1^T + B_2 B_2^T) - \frac{1}{3} (B_1 B_2^T + B_2 B_1^T) \\
    = \frac{3}{4} L_G^#,
\]

so \(H^# = \frac{3}{4} L_G^#\).

According to Lemma 2.7, we calculate \(-H^# BD^{-1}\) and \(-D^{-1} B^T H^#\).

\[
    -H^# BD^{-1} = -\frac{3}{4} L_G^# \begin{pmatrix} -B_1 & -B_2 \end{pmatrix} \begin{pmatrix} \frac{2}{3}I_m & \frac{1}{3}I_m \\ \frac{1}{3}I_m & \frac{2}{3}I_m \end{pmatrix} \begin{pmatrix} -B_1^T \\ -B_2^T \end{pmatrix} \\
    = -\frac{3}{4} L_G^# \begin{pmatrix} -\frac{2}{3}B_1 & -\frac{1}{3}B_2 \\ -\frac{1}{3}B_1 & -\frac{2}{3}B_2 \end{pmatrix} \\
    = \begin{pmatrix} L_G^#(\frac{1}{2}B_1 + \frac{1}{4}B_2) & L_G^#(\frac{1}{4}B_1 + \frac{1}{2}B_2) \end{pmatrix}
\]

and

\[
    -D^{-1} B^T H^# = -(H^# BD^{-1})^T = \begin{pmatrix} (\frac{1}{2}B_1^T + \frac{1}{4}B_2^T) L_G^# \\ (\frac{1}{4}B_1^T + \frac{1}{2}B_2^T) L_G^# \end{pmatrix}.
\]
We are ready to compute $D^{-1}B^TH^#BD^{-1}$.

\[
D^{-1}B^TH^#BD^{-1} = \left( \begin{array}{ccc}
\frac{2}{3}I_m & \frac{1}{2}I_m & \frac{1}{2}I_m \\
\frac{1}{3}I_m & \frac{1}{2}I_m & \frac{1}{3}I_m \\
\frac{1}{3}I_m & \frac{1}{2}I_m & \frac{1}{3}I_m \\
\end{array} \right) \left( \begin{array}{ccc}
-B_1^T & 0 & 0 \\
0 & -B_2^T & 0 \\
0 & 0 & -B_2^T \\
\end{array} \right) \left( \begin{array}{ccc}
\frac{2}{3}I_m & \frac{1}{2}I_m & \frac{1}{2}I_m \\
\frac{1}{3}I_m & \frac{1}{2}I_m & \frac{1}{3}I_m \\
\frac{1}{3}I_m & \frac{1}{2}I_m & \frac{1}{3}I_m \\
\end{array} \right) \left( \begin{array}{ccc}
\frac{1}{2}B_1^T + \frac{1}{3}B_2^T) & L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2) & -2L^#_G(\frac{1}{2}B_1 + \frac{1}{2}B_2) \\
\frac{1}{2}B_1^T + \frac{1}{3}B_2^T & L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2) & -2L^#_G(\frac{1}{2}B_1 + \frac{1}{2}B_2) \\
\frac{1}{2}B_1^T + \frac{1}{3}B_2^T & L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2) & -2L^#_G(\frac{1}{2}B_1 + \frac{1}{2}B_2) \\
\end{array} \right).
\]

Let $P = (\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2)$, $Q = (\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2)$, $M = (\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2)$ and $N = (\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{2}{3}B_2)$. Based on Lemma 2.3 and 2.7, the following matrix

\[
N = \left( \begin{array}{ccc}
\frac{3}{4}L^#_G & L^#_G(\frac{1}{2}B_1 + \frac{1}{2}B_2) & L^#_G(\frac{1}{2}B_1 + \frac{1}{2}B_2) \\
L^#_G(\frac{1}{2}B_1 + \frac{1}{2}B_2) & \frac{1}{2}I_m + P & \frac{1}{2}I_m + Q \\
L^#_G(\frac{1}{2}B_1 + \frac{1}{2}B_2) & \frac{1}{2}I_m + M & \frac{1}{2}I_m + N \\
\end{array} \right) \tag{3.1}
\]

is a symmetric $\{1\}$-inverse of $L_{Q(G)}$.

For any $i, j \in V(G)$, by Lemma 2.1 and the Equation (3.1), we have

\[
br_{ij}(Q(G)) = (\frac{3}{4}L^#_G)_{ij} + \frac{3}{4}(L^#_G)_{jj} - \frac{3}{2}(L^#_G)_{ij} = \frac{3}{4}r_{ij}(G),
\]

as stated in $(i)$.

For any $i \in V$, $j \in V_1$ or $V_2$, $j \in V$, by Lemma 2.1 and the Equation (3.1), we have

\[
br_{ij}(Q(G)) = \frac{3}{4}(L^#_G)_{ij} + \left[ \frac{1}{2}I_m + (\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2) \right]_{ij} - 2\left[ L^#_G(\frac{1}{2}B_1 + \frac{1}{3}B_2) \right]_{ij},
\]

as stated in $(ii)$.

For any $i \in V_1$, $j \in V_2$, by Lemma 2.1 and the Equation (3.1), we have

\[
br_{ij}(Q(G)) = \frac{4}{3} + [\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2)]_{ij} + [(\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{2}{3}B_2)]_{jj} - \frac{1}{3}I_m + (\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{2}{3}B_2)]_{ij},
\]

as stated in $(iii)$.

For any $i, j \in V_1$ or $V_2$, by Lemma 2.1 and the Equation (3.1), we have

\[
br_{ij}(Q(G)) = \frac{4}{3} + [\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2)]_{ii} + [(\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2)]_{jj} - 2[(\frac{1}{2}B_1^T + \frac{1}{3}B_2^T)L^#_G(\frac{2}{3}B_1 + \frac{1}{3}B_2)]_{ij},
\]

as stated in $(iv)$.

By Lemma 2.4, we have

\[
Kf(Q(G)) = (n + 2m)tr(N) - 1^TN1
\]
\[ (n + 2m) \left( \frac{3}{4} tr(L_G^\#) + tr \left( \frac{1}{2} B_1^T + \frac{1}{4} B_2^T \right) L_G^\# \left( \frac{2}{3} B_1 + \frac{1}{3} B_2 \right) \right) + tr \left( \frac{1}{2} B_1^T + \frac{1}{2} B_2^T \right) L_G^\# \left( \frac{1}{3} B_1 + \frac{2}{3} B_2 \right) + 2tr \left( \frac{2}{3} I_m \right) = 1^T N_1 \]

Since \( L_G^\# 1 = 0 \), then

\[ 1^T N_1 1^T = 1^T [(\frac{1}{2} B_1^T + \frac{1}{4} B_2^T) L_G^\# (\frac{2}{3} B_1 + \frac{1}{3} B_2)] 1 + 1^T (\frac{1}{2} B_1^T + \frac{1}{4} B_2^T) L_G^\# (\frac{1}{3} B_1 + \frac{2}{3} B_2) 1 + 2 \cdot \frac{1}{3} 1^T 1 \\
+ 1^T (\frac{1}{4} B_1^T + \frac{1}{2} B_2^T) L_G^\# (\frac{2}{3} B_1 + \frac{1}{3} B_2) 1 + 1^T (\frac{1}{4} B_1^T + \frac{1}{2} B_2^T) L_G^\# (\frac{1}{3} B_1 + \frac{2}{3} B_2) 1 + 2 \cdot \frac{2}{3} 1^T 1 \\
= \frac{3}{4} 1^T B_1^T L_G^\# B_1 1 + \frac{3}{4} 1^T B_2^T L_G^\# B_2 1 + \frac{3}{4} 1^T B_1^T L_G^\# B_2 1 + \frac{3}{4} 1^T B_2^T L_G^\# B_1 1 + 2m. \]

Plugging the above equation into \( Kf(Q(G)) \), we obtain the required result in (v).

4 The resistance distance and Kirchhoff index of pentagonal graph

In this section, we focus on determining the resistance distance and Kirchhoff index of \( W(G) \) whenever \( G \) be an arbitrary graph. Let \( E_G = \{e_1, e_2, \ldots, e_m\} \). For each edge \( e_i = u_i v_i \in E_G \), there exist two parallel paths of lengths 1 and 4 in \( W(G) \) corresponding to it, which are denoted by \( u_i v_i \) and \( u_i u_1 u_2 u_3 v_i \) for \( i = 1, 2, \ldots, m \). Let \( V_1 = \{u_1, u_2, \ldots, u_m\}, V_2 = \{u_2, u_2, \ldots, u_m2\}, V_3 = \{u_3, u_23, \ldots, u_m3\}. \) Then \( V_{W(G)} = V \cup V_1 \cup V_2 \cup V_3 \), where \( V \) is the set of all the vertices inherited from \( G \). Our main results in the following gives the explicit formula of the resistance distance and Kirchhoff index of \( W(G) \).

**Theorem 4.1** Let \( G \) be a connected graph with \( n \) vertices and \( m \) edges and \( W(G) \) be its pentagonal graph of \( G \). Then we have the resistance distance and Kirchhoff index as follows:

(i) For any \( i, j \in V(G) \), we have

\[ r_{ij}(W(G)) = \frac{4}{5} (L_G^\#)_{ii} + \frac{4}{5} (L_G^\#)_{jj} - \frac{8}{5} (L_G^\#)_{ij} = \frac{4}{5} r_{ij}(G). \]

(ii) For any \( i \in V, j \in V_1 \), we have

\[ r_{ij}(W(G)) = \frac{4}{5} (L_G^\#)_{ii} + \left[ \frac{3}{4} I_m + \left( \frac{3}{5} B_1^T + \frac{1}{5} B_2^T \right) L_G^\# \left( \frac{3}{4} B_1 + \frac{1}{4} B_2 \right) \right]_{jj} - 2 \left[ L_G^\# \left( \frac{3}{5} B_1 + \frac{1}{5} B_2 \right) \right]_{ij}. \]
(iii) For any \( i \in V, j \in V_2 \), we have
\[
\begin{align*}
  r_{ij}(W(G)) &= \frac{4}{5} (L^G)_{ij} + \left[ I_m + \left( \frac{3}{5} B_1^T + \frac{1}{5} B_2^T \right) L^G (\frac{3}{4} B_1 + \frac{1}{4} B_2) \right]_{ij} \\
  &\quad - 2 \left[ L^G (\frac{2}{5} B_1 + \frac{2}{5} B_2) \right]_{ij} .
\end{align*}
\]

(iv) For any \( i \in V, j \in V_3 \), we have
\[
\begin{align*}
  r_{ij}(W(G)) &= \frac{4}{5} (L^G)_{ij} + \left[ \frac{1}{5} B_1^T + \frac{3}{5} B_2^T \right] L^G (\frac{1}{2} B_1 + \frac{1}{2} B_2)_{ij} \\
  &\quad - 2 \left[ L^G (\frac{1}{5} B_1 + \frac{3}{5} B_2) \right]_{ij} .
\end{align*}
\]

(v) For any \( i \in V_1, j \in V_2 \), we have
\[
\begin{align*}
  r_{ij}(W(G)) &= \frac{5}{4} + \left[ \frac{3}{5} B_1^T + \frac{1}{5} B_2^T \right] L^G (\frac{3}{4} B_1 + \frac{1}{4} B_2)_{ii} + \left[ \frac{2}{5} B_1^T + \frac{2}{5} B_2^T \right] L^G (\frac{1}{2} B_1 + \frac{1}{2} B_2)_{jj} \\
  &\quad - \frac{1}{2} I_m + \left[ \frac{3}{5} B_1^T + \frac{1}{5} B_2^T \right] L^G (\frac{1}{4} B_1 + \frac{3}{4} B_2)_{ij} .
\end{align*}
\]

(vi) For any \( i \in V_1, j \in V_3 \), we have
\[
\begin{align*}
  r_{ij}(W(G)) &= \frac{3}{2} + \left[ \frac{3}{5} B_1^T + \frac{1}{5} B_2^T \right] L^G (\frac{3}{4} B_1 + \frac{1}{4} B_2)_{ii} + \left[ \frac{1}{5} B_1^T + \frac{3}{5} B_2^T \right] L^G (\frac{1}{4} B_1 + \frac{3}{4} B_2)_{jj} \\
  &\quad - \frac{1}{4} I_m + \left[ \frac{3}{5} B_1^T + \frac{1}{5} B_2^T \right] L^G (\frac{1}{4} B_1 + \frac{3}{4} B_2)_{ij} .
\end{align*}
\]

(vii) For any \( i \in V_2, j \in V_3 \), we have
\[
\begin{align*}
  r_{ij}(W(G)) &= \frac{7}{4} + \left[ \frac{2}{5} B_1^T + \frac{2}{5} B_2^T \right] L^G (\frac{1}{2} B_1 + \frac{1}{2} B_2)_{ii} + \left[ \frac{1}{5} B_1^T + \frac{3}{5} B_2^T \right] L^G (\frac{1}{4} B_1 + \frac{3}{4} B_2)_{jj} \\
  &\quad - \frac{1}{2} I_m + \left[ \frac{1}{4} B_1^T + \frac{1}{4} B_2^T \right] L^G (\frac{1}{4} B_1 + \frac{3}{4} B_2)_{ij} .
\end{align*}
\]

(viii) \( Kf(W(G)) \)
\[
\begin{align*}
  &= (n + 3m) \left( \frac{4}{5n} Kf(G) + \frac{61}{100} (tr(B_1^T L^G B_1) + tr(B_2^T L^G B_2)) \\
  &\quad + \frac{1}{2} \left( tr(B_1^T L^G B_2) + tr(B_1^T L^G B_2) \right) + \frac{5m}{2} \right) - \frac{141}{80} 1^T B_1^T L^G B_1 - \frac{131}{80} 1^T B_2^T L^G B_2 \\
  &\quad - \frac{133}{80} 1^T B_2^T L^G B_1 - \frac{127}{80} 1^T B_2^T L^G B_2 - 5m ,
\end{align*}
\]

where \( B_1 + B_2 = B, B_1 B_2^T + B_2 B_1 = A_G \) and \( B_1 B_1^T + B_2 B_2^T = D_G \).

**Proof** Let \( A_G, D_G \) and \( B_G \) be the adjacency matrix, degree matrix and incidence matrix of \( G \). With a suitable labeling for vertices of \( W(G) \), the Laplacian matrix of \( W(G) \) can be written as follows:
\[
L_{W(G)} = \begin{pmatrix}
  2D_G - A_G & -B_1 & 0 & -B_2 \\
  -B_1^T & 2I_m & -I_m & 0 \\
  0 & -I_m & 2I_m & -I_m \\
  -B_2^T & 0 & -I_m & 2I_m
\end{pmatrix},
\]
where \( B_1 + B_2 = B, B_1 B_2^T + B_2^T B_1 = A_G \) and \( B_1 B_1^T + B_2 B_2^T = D_G \).

Let \( A = 2D_G - A_G, B = (-B_1 \ 0 \ -B_2) \), \( B^T = \begin{pmatrix} -B_1^T \\ 0 \\ -B_2^T \end{pmatrix} \) and \( D = \begin{pmatrix} 2I_m & -I_m & 0 \\ -I_m & 2I_m & -I_m \\ 0 & -I_m & 2I_m \end{pmatrix} \).

By Lemma 2.3, we have \( D^{-1} = \begin{pmatrix} \frac{3}{4}I_m & \frac{1}{4}I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & I_m & -\frac{1}{4}I_m \\ \frac{1}{4}I_m & \frac{1}{4}I_m & \frac{3}{4}I_m \end{pmatrix} \).

First we begin with the computation of \( \{1\}\)-inverse of \( W(G) \).

By Lemma 2.7, we have
\[
H = 2D_G - A_G - (-B_1 \ 0 \ -B_2) \begin{pmatrix} \frac{3}{4}I_m & \frac{1}{4}I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & \frac{1}{4}I_m & \frac{3}{4}I_m \end{pmatrix} \begin{pmatrix} -B_1^T \\ 0 \\ -B_2^T \end{pmatrix} = 2D_G - A_G - \left( -\frac{3}{4}B_1 \ - \frac{1}{4}B_2 \right) \left( -\frac{1}{4}B_1 \ - \frac{1}{4}B_2 \right)
\]
\[
= 2D_G - A_G - \left( -\frac{3}{4}B_1 \ - \frac{1}{4}B_2 \right) \left( -\frac{1}{4}B_1 \ - \frac{1}{4}B_2 \right)
\]
\[
= 2D_G - A_G - \frac{3}{4}(B_1 B_1^T + B_2 B_2^T) - \frac{1}{4}(B_1 B_1^T + B_2 B_2^T)
\]
\[
= \frac{5}{4}B_G^#.
\]
so \( H^# = \frac{4}{5}L_G^# \).

According to Lemma 2.7, we calculate \(-H^#BD^{-1}\) and \(-D^{-1}B^TH^#\).

\[
-H^#BD^{-1} = -\frac{4}{5}L_G^# \left( -B_1 \ 0 \ -B_2 \right) \begin{pmatrix} \frac{3}{4}I_m & \frac{1}{4}I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & \frac{1}{4}I_m & \frac{3}{4}I_m \end{pmatrix} = \frac{4}{5}L_G^# \left( \frac{3}{4}B_1 + \frac{1}{4}B_2 \right) \frac{1}{4}B_1 + \frac{1}{4}B_2 \frac{1}{4}B_1 + \frac{3}{4}B_2
\]

and
\[
-D^{-1}B^TH^# = -\left( -H^#BD^{-1} \right)^T = \begin{pmatrix} \frac{1}{4}B_1^T + \frac{1}{4}B_2^T \ L_G^# \left( \frac{1}{4}B_1 + \frac{1}{2}B_2 \right) \\ \frac{1}{4}B_1^T + \frac{1}{4}B_2^T \ L_G^# \left( \frac{1}{4}B_1 + \frac{1}{2}B_2 \right) \\ \frac{1}{4}B_1^T + \frac{1}{4}B_2^T \ L_G^# \left( \frac{1}{4}B_1 + \frac{1}{2}B_2 \right) \end{pmatrix}.
\]

We are ready to compute the \( D^{-1}B^TH^#BD^{-1} \).

\[
D^{-1}B^TH^#BD^{-1} = \begin{pmatrix} \frac{3}{4}I_m & \frac{1}{4}I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & \frac{1}{4}I_m & \frac{3}{4}I_m \end{pmatrix} \begin{pmatrix} -B_1^T \\ 0 \\ -B_2^T \end{pmatrix} \begin{pmatrix} \frac{3}{4}I_m & \frac{1}{4}I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & I_m & \frac{1}{4}I_m \\ \frac{1}{4}I_m & \frac{1}{4}I_m & \frac{3}{4}I_m \end{pmatrix} = \begin{pmatrix} \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \ L_G^# \left( \frac{3}{4}B_1 + \frac{1}{4}B_2 \right) \\ \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \ L_G^# \left( \frac{3}{4}B_1 + \frac{1}{4}B_2 \right) \\ \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \ L_G^# \left( \frac{3}{4}B_1 + \frac{1}{4}B_2 \right) \end{pmatrix}.
\]

Let
\[
P_1 = \left( \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{3}{4}B_1 + \frac{1}{4}B_2 \right), \ P_2 = \left( \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{1}{4}B_1 + \frac{1}{4}B_2 \right), \ P_3 = \left( \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{1}{4}B_1 + \frac{3}{4}B_2 \right), \ Q_1 = \left( \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{3}{4}B_1 + \frac{1}{4}B_2 \right), \ Q_2 = \left( \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{1}{4}B_1 + \frac{3}{4}B_2 \right), \ Q_3 = \left( \frac{3}{4}B_1^T + \frac{1}{4}B_2^T \right) L_G^# \left( \frac{1}{4}B_1 + \frac{3}{4}B_2 \right).
\]
Based on Lemma 2.3 and 2.7, the following matrix

$$
N = \begin{pmatrix}
\frac{4}{5}L_G^# & L_G^#(\frac{3}{5}B_1 + \frac{1}{5}B_2) & L_G^#(\frac{3}{5}B_1 + \frac{3}{5}B_2) \\
(\frac{3}{5}B_1^T + \frac{1}{5}B_2^T)L_G^# & \frac{3}{4}I_m + P_1 & \frac{1}{4}I_m + P_2 \\
(\frac{2}{5}B_1^T + \frac{2}{5}B_2^T)L_G^# & \frac{1}{2}I_m + Q_1 & \frac{1}{2}I_m + Q_2 \\
(\frac{1}{5}B_1^T + \frac{3}{5}B_2^T)L_G^# & \frac{1}{4}I_m + R_1 & \frac{1}{2}I_m + R_2 \\
\end{pmatrix}
$$

(4.2)
is a symmetric \{1\}-inverse of \(L\).

For any \(i, j \in V(G)\), by Lemma 2.1 and the Equation (4.2), we have

$$r_{ij}(W(G)) = \frac{4}{5}(L_G^#)_{ij} + \frac{4}{5}(L_G^#)_{jj} - \frac{8}{5}(L_G^#)_{ij} = \frac{4}{5}r_{ij}(G),$$
as stated in (i).

For any \(i \in V, j \in V_1\), by Lemma 2.1 and the Equation (4.2), we have

$$r_{ij}(W(G)) = \frac{4}{5}(L_G^#)_{ij} + \left[\frac{3}{4}I_m + \left(\frac{3}{5}B_1^T + \frac{1}{5}B_2^T\right)L_G^#(\frac{3}{4}B_1 + \frac{1}{4}B_2)\right]_{jj}
-2\left[L_G^#(\frac{3}{5}B_1 + \frac{1}{5}B_2)\right]_{ij},$$
as stated in (ii).

For any \(i \in V, j \in V_2\), by Lemma 2.1 and the Equation (4.2), we have

$$r_{ij}(W(G)) = \frac{4}{5}(L_G^#)_{ij} + \left[I_m + \left(\frac{3}{5}B_1^T + \frac{1}{5}B_2^T\right)L_G^#(\frac{3}{4}B_1 + \frac{1}{4}B_2)\right]_{jj}
-2\left[L_G^#(\frac{2}{5}B_1 + \frac{2}{5}B_2)\right]_{ij},$$
as stated in (iii).

For any \(i \in V, j \in V_3\), by Lemma 2.1 and the Equation (4.2), we have

$$r_{ij}(W(G)) = \frac{4}{5}(L_G^#)_{ij} + \left[\left(\frac{1}{5}B_1^T + \frac{3}{5}B_2^T\right)L_G^#(\frac{1}{4}B_1 + \frac{3}{4}B_2)\right]_{jj}
-2\left[L_G^#(\frac{1}{5}B_1 + \frac{3}{5}B_2)\right]_{ij},$$
as stated in (iv).

For any \(i \in V_1, j \in V_2\), by Lemma 2.1 and the Equation (3.1), we have

$$r_{ij}(W(G)) = \frac{5}{4} + \left[\left(\frac{3}{5}B_1^T + \frac{1}{5}B_2^T\right)L_G^#(\frac{3}{4}B_1 + \frac{1}{4}B_2)\right]_{ii}
+ \left[\left(\frac{2}{5}B_1^T + \frac{2}{5}B_2^T\right)L_G^#(\frac{1}{2}B_1 + \frac{1}{2}B_2)\right]_{jj}
-\left[\frac{1}{2}I_m + \left(\frac{3}{5}B_1^T + \frac{1}{5}B_2^T\right)L_G^#(\frac{1}{2}B_1 + \frac{1}{2}B_2)\right]_{ij},$$
as stated in (v).
For any \( i \in V_1, j \in V_3 \), by Lemma 2.1 and the Equation (4.2), we have

\[
\begin{align*}
    r_{ij}(W(G)) &= \frac{3}{2} + \left[ \left( \frac{3}{5} B_1^T + \frac{1}{5} B_2^T \right) L_G^# \left( \frac{3}{4} B_1 + \frac{1}{4} B_2 \right) \right]_{ii} + \left[ \left( \frac{1}{5} B_1^T + \frac{3}{5} B_2^T \right) L_G^# \left( \frac{1}{4} B_1 + \frac{3}{4} B_2 \right) \right]_{jj} \\
    &\quad - \left[ \frac{1}{4} I_m + \left( \frac{3}{5} B_1^T + \frac{1}{5} B_2^T \right) L_G^# \left( \frac{1}{4} B_1 + \frac{3}{4} B_2 \right) \right]_{ij},
\end{align*}
\]

as stated in (vi).

For any \( i \in V_2, j \in V_3 \), by Lemma 2.1 and the Equation (4.2), we have

\[
\begin{align*}
    r_{ij}(W(G)) &= \frac{7}{4} + \left[ \left( \frac{2}{5} B_1^T + \frac{2}{5} B_2^T \right) L_G^# \left( \frac{1}{2} B_1 + \frac{1}{2} B_2 \right) \right]_{ii} + \left[ \left( \frac{1}{5} B_1^T + \frac{3}{5} B_2^T \right) L_G^# \left( \frac{1}{4} B_1 + \frac{3}{4} B_2 \right) \right]_{jj} \\
    &\quad - \left[ \frac{1}{4} I_m + \left( \frac{1}{4} B_1^T + \frac{1}{4} B_2^T \right) L_G^# \left( \frac{1}{4} B_1 + \frac{3}{4} B_2 \right) \right]_{ij},
\end{align*}
\]

as stated in (vii).

By Lemma 2.4, we have

\[
\begin{align*}
    Kf(W(G)) &= (n + 3m)tr(N) - 1^T N 1^T \\
    &= (n + 3m) \left( \frac{4}{5} tr(I_G^#) + tr(\frac{3}{5} B_1^T + \frac{1}{5} B_2^T) L_G^# \left( \frac{3}{4} B_1 + \frac{1}{4} B_2 \right) + tr(\frac{1}{5} B_1^T + \frac{3}{5} B_2^T) L_G^# \left( \frac{1}{4} B_1 + \frac{3}{4} B_2 \right) + \frac{5m}{2} \right) - 1^T N 1 \\
    &= (n + 3m) \left( \frac{4}{5} Kf(G) + \frac{61}{100} \left( tr(B_1^T L_G^# B_1) + tr(B_2^T L_G^# B_2) \right) \right) \\
    &\quad + \frac{1}{2} \left( tr(B_1^T L_G^# B_2) + tr(B_2^T L_G^# B_2) \right) + \frac{5m}{2} - 1^T N 1.
\end{align*}
\]

Since \( L_G^# 1 = 0 \), then

\[
1^T N 1^T = 1^T P_1 + 1^T P_2 + 1^T P_3 + 1^T Q_1 + 1^T Q_2 + 1^T Q_3 + 1^T R_1 + 1^T R_2 + 1^T R_3 \\
\quad + 2 \times 1^T \frac{3}{4} I_m + 4 \times 1^T \frac{1}{2} I_m + 2 \times 1^T \frac{1}{4} I_m + \times 1^T I_m \\
= \frac{141}{80} 1^T B_1^T L_G^# B_1 + \frac{131}{80} 1^T B_1^T L_G^# B_2 + \frac{133}{80} 1^T B_2^T L_G^# B_1 + \frac{127}{80} 1^T B_2^T L_G^# B_2 + 5m
\]

Plugging the above equation into \( Kf(W(G)) \), we obtain the required result in (viii).

5 Conclusion

In this paper using the Laplacian generalized inverse approach, we obtained the resistance distance and Kirchhoff indices of the quadrilateral graph and the pentagonal graph whenever \( G \) is an arbitrary graph. We can obtain the resistance distance and Kirchhoff index of the quadrilateral graph and the pentagonal graph in terms of the resistance distance and kirchhoff index of the factor graph \( G \).

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