 Strings with Negative Stiffness and Hyperfine Structure

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We propose a new string model by adding a higher-order gradient term to the rigid string, so that the stiffness can be positive or negative without loosing stability. In the large-$D$ approximation, the model has three phases, one of which with a new type of generalized “antiferromagnetic” orientational correlations. We find an infrared-stable fixed point describing world-sheets with vanishing tension and Hausdorff dimension $D_H = 2$. Crumpling is prevented by the new term which suppresses configurations with rapidly changing extrinsic curvature.

Moreover, it has been proposed [16] as a mechanism leading to smooth strings, with “long-range” orientational correlations, a fact confirmed by recent numerical simulations [17].

In this letter we report on further progress in this field by considering the simplest possible model in this class of actions, obtained by adding to the rigid string the next-order gradient term.

1. The new model is defined in euclidean space by the action

$$ S = \int d^2 \xi \sqrt{g} g^{ab} D_a x_\mu \left( T - s D^2 + \frac{1}{M^2} D^2 \right) D_b x_\mu, $$

where $D_a$ are covariant derivatives with respect to the induced metric $g_{ab} = \partial_\alpha x_\mu \partial_\beta x_\mu$ on the surface $x_\mu(\xi_0, \xi_1)$ and we have used units $c = 1$, $\hbar = 1$. Here, the bracket has to be considered as representing the first few terms in the expansion of the non-local interaction mediated by the original antisymmetric tensor. The first term provides a bare surface tension $2T$, while the second accounts for rigidity [8] with stiffness parameter $s$. The last term can be written (up to surface terms) as a combination of the fourth power and the square of the gradient of the extrinsic curvature matrices, with $M$ being a new mass scale. It thus suppresses world-sheet configurations with rapidly changing extrinsic curvature; due to its presence, the stiffness $s$ can be positive or negative, as is actually the case [8] when the stiffness is generated dynamically by a tensor field in four-dimensional space-time.

Note that the action (1) is not Weyl invariant, so that no conformal anomaly appears.

We analyze the model (1) in the large-$D$ approximation along the lines of Ref. [3]. To this end we introduce a Lagrange multiplier matrix $L^{ab}$ to enforce the constraint

$$ g_{ab} = \partial_\alpha x_\mu \partial_\beta x_\mu, $$

extending the action (1) to

$$ S + \int d^2 \xi \sqrt{g} L^{ab} \left( \partial_\alpha x_\mu \partial_\beta x_\mu - g_{ab} \right). $$

Then we parametrize the world-sheet in a Gauss map by

$$ x_\mu(\xi) = \left( \xi_0, \xi_1, \phi^i(\xi) \right), \quad (i = 2, \ldots, D - 1), $$

where $-\beta/2 \leq \xi_0 \leq \beta/2$, $-R/2 \leq \xi_1 \leq R/2$ and $\phi^i(\xi)$ describe the $(D-2)$ transverse fluctuations. With the usual homogeneity and isotropy ansatz $g_{ab} = \rho \delta_{ab}$, $L^{ab} = L g^{ab}$ of infinite surfaces ($\beta, R \to \infty$) at the saddle point, we obtain
\[ S = 2 \int d^2 \xi \left[ T + L(1 - \rho) \right] + \int d^2 \xi \, \partial_a \phi^i \, V(T, s, M, L, D^2) \, \partial_a \phi^i , \] (3)

where

\[ V(T, s, M, L, D^2) = T + L - s D^2 + \frac{1}{M^2} D^4 , \] (4)

Integrating over the transverse fluctuations, in the infinite area limit, we get the effective action

\[ S_{\text{eff}} = 2 A_{\text{ext}} \left[ T + L(1 - \rho) \right] + A_{\text{ext}} \frac{D - 2}{8 \pi^2} \int d^2 p \ln \left[ p^2 V(T, s, M, L, p) \right] , \] (5)

where \( A_{\text{ext}} = \beta R \) is the extrinsic, physical, space-time area. For large \( D \), the fluctuations of \( L \) and \( \rho \) are suppressed and these variables take their “classical values”, determined by the two saddle-point equations

\[ 0 = f(T, s, M, L) , \quad \rho = \frac{1}{f'(T, s, M, L)} , \] (6)

where the prime denotes a derivative with respect to \( L \) and the “saddle-function” \( f \) is defined by

\[ f(T, s, M, L) \equiv L - \frac{D - 2}{8 \pi} \int d^2 p \ln \left[ p^2 V(T, s, M, L, p) \right] . \] (7)

Using (3) in (5) we get \( S_{\text{eff}} = 2(T + L) \) \( A_{\text{ext}} \) showing that \( T = 2(T + L) \) is the physical string tension.

The stability condition for the euclidean surfaces is that \( V(T, s, M, L, p) \) be positive for all \( p^2 \geq 0 \). However, we shall require the same condition also for \( p^2 \leq 0 \), so that strings propagating in Minkowski space-time are not affected by the propagating states of negative norm which plague rigid strings. The stability condition becomes thus \( \sqrt{T + L} \geq |s M|/2 \), which allows us to introduce the real variables \( R \) and \( I \) defined by

\[ R^2 \equiv \frac{M}{2} \sqrt{T + L} + \frac{s M^2}{4} , \quad I^2 \equiv \frac{M}{2} \sqrt{T + L} - \frac{s M^2}{4} . \] (8)

In terms of these, the kernel \( V \) can be written as

\[ M^2 V(T, s, M, L, p) = (R^2 + I^2)^2 + 2 (R^2 - I^2)^2 p^2 + p^4 . \] (9)

3. In order to distinguish the various phases of our model we compute two correlation functions. First, we consider the orientational correlation function \( g_{ab} (\xi - \xi') \equiv \langle \partial_a \phi^i (\xi) \partial_b \phi^i (\xi') \rangle \) for the normal components of tangent vectors to the world-sheet. From (5) this is given by

\[ g_{ab} (\xi - \xi') = \frac{\delta_{ab}}{8 \pi^2} \int d^2 p \, \frac{1}{V(T, s, M, L, p)} \, e^{i \sqrt{\rho} (\xi - \xi')} . \] (10)

In terms of \( R \) and \( I \), the Fourier components can be written as

\[ \frac{1}{V(T, s, M, L, p)} = \frac{M^2}{2 R I} \, \text{Im} \, \frac{1}{p^2 + (R - i I)^2} , \] (11)

from where we obtain the following exact result for the diagonal elements \( g \equiv g_{aa} \) of (10):

\[ 4 \pi g(d) = \frac{M^2}{2 R I} \, \text{Im} \, K_0 \left( (R - i I) \sqrt{d^2} \right) , \] (12)

where \( d \equiv |\xi - \xi'| \) and \( K_0 \) is a Bessel function of imaginary argument (13).

Secondly, we compute the scaling law of the distance \( d_E \) in embedding space between two points on the world-sheet when changing its projection \( d \) on the reference plane. The exact relation between the two lengths is

\[ d_E^2 = d^2 + \sum_i \langle (\phi^i (\xi) - \phi^i (\xi'))^2 \rangle . \] (13)

With a computation analogous to the one leading to (12) we obtain the following behaviour:

\[ d_E^2 = \begin{cases} \left( \frac{R^2 + I^2}{16 \pi T R I} \arctan(I/R) \right) \alpha d^2 , & d^2 \ll \frac{1}{\alpha} , \\ \frac{1}{4 \pi T} \ln \left( \alpha d^2 / 4 \right) + \text{const} , & \frac{1}{\alpha} \ll d^2 \ll \frac{1}{2 \pi T} , \\ d^2 , & d^2 \gg \frac{1}{2 \pi T} , \end{cases} \]

with \( \alpha \equiv (R^2 + I^2) \rho \).

These results show that the model has three possible phases. The first one is realized when there are no solutions to the saddle-point equations in the allowed range of parameters, which means that, for this choice of parameters, there exist no homogeneous, isotropic surfaces. This is realized, as we shall see, for \( R \) very small, when the spectrum of transverse fluctuations develops an instability at a finite value \( \rho = \sqrt{T^2 - R^2} \). In this phase (I) the surfaces will form inhomogeneous structures (14).

If a solution to the saddle-point equations exists, two situations can be realized. For large positive stiffness \( s \) we have \( R \gg I \), the asymptotic region is \( d \gg 1/R \sqrt{\rho} \) and \( I \) can be neglected. In this region we have

\[ g(d) \propto \frac{1}{\sqrt{R \sqrt{\rho}}} \, e^{-R \sqrt{\rho} d} , \] (14)

exhibiting short-range orientational order. For short distances \( d \ll 1/R \sqrt{\rho} \), world-sheets behave as two-dimensional objects. If \( \rho \) becomes large we have a region \( 1/R \sqrt{\rho} \ll d \ll 1/\sqrt{2 \pi T} \) in which \( d_E \) scales logarithmically with \( d \) and distances along the surface become large. The transition to this regime happens on the scale of the persistence length \( d_E^{1/2} = 1/\sqrt{T} \). Above this scale world-sheets are crumpled, with no orientational correlations (if the tension is not large enough to dominate over the entire surface, causing \( \rho \approx 1 \)). This phase (R)
corresponds to the behaviour of the familiar rigid strings [3].

For large negative stiffness $s$, in contrast, we have $I \gg R$, the asymptotic region is $d \gg 1/I\sqrt{\rho}$, and $R$ can be neglected in $K_0$ for $d \ll 1/R\sqrt{\rho}$. In this region we have

$$8g(d) = \frac{M^2}{2RI} J_0 (I\sqrt{\rho}d) ,$$

with $1/R\sqrt{\rho}$ playing the role of an infrared cutoff for the oscillations on the scale $1/I\sqrt{\rho}$ over which the Bessel function $J_0$ varies. In this case $d_E^0 = \sqrt{1/RI}$ is the scale on which world-sheets form oscillating structures characterized by the “long-range antiferromagnetic” orientational correlations [5]. Crumpling takes place only if $1/R\sqrt{\rho} \ll 1/\sqrt{2\pi T}$ and the corresponding persistence length is $d_E^p = (1/\sqrt{T}) \sqrt{(\pi I/2R) + \ln (I^2/4R^2)}$, which is much larger than $1/\sqrt{T}$. Otherwise, the oscillating superstructure, which we shall call the hyperfine structure of strings, goes over directly into the tension dominated regime.

Such world-sheets constitute a liquid version of the “egg carton” surfaces (LEC) of the biomembrane literature [20]. In this case $d_E$ scales logarithmically with $d$ for $1/I\sqrt{\rho} \ll d \ll 1/\sqrt{2\pi T}$, and $\rho$ is large not because of crumpling but because of the many “hills and valleys” on the liquid egg carton; indeed there are strong orientational correlations in this region. For very small $R$ one can thus obtain long strings with an oscillating superstructure, a mechanism proposed in [16]. For symmetry reasons, the transition from the rigid to the liquid egg carton phase occurs on the line $R = I$, where $s = 0$. On this line the spectrum of transverse fluctuations, $E(T, s, M, L, p) = p^2 V(T, s, M, L, p)$ acquires a roton-like minimum, as shown schematically in Fig. 1.

$$f(T, s, M, L) = L + \frac{1}{16\pi^3} (R^2 - I^2) \ln \frac{R^2 + I^2}{\Lambda^2} - \frac{1}{8\pi^3} RI \left( \frac{\pi}{2} + \arctan \frac{I^2 - R^2}{2R} \right) ,$$

where $\Lambda \equiv \mu \exp(2/\epsilon)$ and $\mu$ is a reference scale which must be introduced for dimensional reasons. The scale $\Lambda$ plays the role of an ultraviolet cutoff, diverging for $\epsilon \to 0$.

The saddle-point function above is best studied by introducing the dimensionless couplings $t \equiv T/\Lambda^2$, $m \equiv M/\Lambda$ and $l \equiv L/\Lambda^2$. A numerical analysis of the solutions of the saddle-point equation $f(t, s, m, l) = 0$ leads to the phase diagram represented in Fig. 2 for $m = 100$.

![FIG. 1. The roton-like minimum in the spectrum of transverse fluctuations for $I > R$.](image1)

In order to fully establish the phase diagram of our model, we analyze the saddle-point function $f(T, s, M, L)$ in [8]. To this end we must prescribe a regularization for the ultraviolet divergent integral. We use dimensional regularization, computing the integral in $(2 - \epsilon)$ dimensions. For small $\epsilon$, this leads to

$$t = \frac{m^2 c^2}{2} \left( 1 + \sqrt{1 + \frac{4t}{m^2 c^2}} \right) ,$$

$$\rho = \left( 1 - \frac{mc}{2\sqrt{t} + t} \right)^{-1} ,$$

where $c \equiv 1/32\pi^2$. This shows that the point $(t^* = 0, s^* = 0, m^* = 0)$ constitutes an infrared-stable fixed-point with vanishing physical string tension $T$. This point is characterized by long-range correlations $g(d) = 2\pi^2/a$, with a constant $a$, and by the scaling law

![FIG. 2. The three phases of our model as a function of the parameters $s$ and $t$ for $m = 100$: an inhomogeneous (I) phase, a rigid (R) phase and the new liquid egg carton (LEC) phase.](image2)
\[ d_E^2 = \frac{\pi^2}{a} \rho^* \, d^2 \, , \quad \rho^* = \left(1 - \frac{1}{2\alpha}\right)^{-1} \, , \] (19)

which shows that the Hausdorff dimension of world sheets is \( D_H = 2 \). For \( s = 0 \), the constant \( a \) can be computed analytically:

\[ a^2 = \lim_{m \to 0} \frac{1 + (2t/m^2c^2) + \sqrt{1 + 4t/m^2c^2}}{2} \, , \] (20)

don which we recognize that \( 1 \leq \rho^* \leq 2 \).

At the infrared fixed-point we can remove the cutoff. The renormalization of the model is easily obtained by noting that the effective action for transverse fluctuations to quadratic order decouples from other modes and is identical with the second term in (3) with \( D^2 = \nabla^2 / \rho \) and \( \rho \) taking its saddle-point value. (see also [3]). From here we identify the physical tension, stiffness and mass as

\[ T = \Lambda^2(t + l) \, , \quad S = \frac{s}{\rho} \, , \quad M = \Lambda \, m \rho \, . \] (21)

For \( s = 0 \), we can compute analytically the corresponding \( \gamma \)-functions:

\[ \gamma_t = -\Lambda \frac{d}{d\Lambda} \ln t = 2 + O \left(\frac{t}{m^2}\right) \, , \] (22)
\[ \gamma_m = -\Lambda \frac{d}{d\Lambda} \ln m = 1 + O \left(\frac{t^2}{m^4}\right) \, . \] (23)

4. In summary, the vicinity of the infrared fixed-point defines a new theory of smooth strings for which the range of the orientational correlations in embedding space is always of the same order or bigger than the length scale \( 1/\sqrt{\rho} \) associated with the tension. The naive irrelevant term \( D^2/M^2 \) in [1] becomes relevant in the large-\( D \) approximation since it generates a string tension proportional to \( M^2 \) which takes over the control of the fluctuations after the orientational correlations die off. Note moreover, that it is exactly this new quartic term which guarantees that the spectrum \( p^2 V(T,s,M,L,p) \) has no other pole than \( p = 0 \), contrary to the rigid string, which necessarily has a ghost pole at \( p^2 = -T/s \).

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[1] For a review see e.g.: M. B. Green, J. H. Schwarz and E. Witten, Superstring Theory, Cambridge University Press, Cambridge (1987).

[2] For a review see e.g.: J. Polchinski, Strings and QCD, in Symposium on Black Holes, Wormholes, Membranes and Superstrings, H.A.R.C., Houston (1992).