IMPROVED RESOLVENT BOUNDS FOR RADIAL POTENTIALS

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Abstract. We prove semiclassical resolvent estimates for the Schrödinger operator in \( \mathbb{R}^d \), \( d \geq 3 \), with real-valued radial potentials \( V \in L^\infty(\mathbb{R}^d) \). In particular, we show that if \( V(x) = O((x)^{-\delta}) \) with \( \delta > 2 \), then the resolvent bound is of the form \( e^{Ch^{-4/3}} \) with some constant \( C > 0 \). We also get resolvent bounds when \( 1 < \delta \leq 2 \). For slowly decaying \( \alpha \)-Hölder potentials we get better resolvent bounds of the form \( e^{Ch^{-4/3}(\alpha+\delta)} \).

Key words: Schrödinger operator, resolvent estimates, radial potentials.

1. Introduction and statement of results

The aim of this work is to improve the recent results in [4], [13], [15] concerning the semiclassical behavior of the resolvent of the Schrödinger operator

\[
P(h) = -h^2\Delta + V(x)
\]

where \( 0 < h \ll 1 \) is a semiclassical parameter, \( \Delta \) is the negative Laplacian in \( \mathbb{R}^d \), \( d \geq 3 \), and \( V \in L^\infty(\mathbb{R}^d) \) is a real-valued short-range potential satisfying the condition

\[
|V(x)| \leq C(|x| + 1)^{-\delta}
\]

where \( C > 0 \) and \( \delta > 1 \) are some constants. More precisely, we are interested in bounding the quantity

\[
g_{s}^{\pm}(h, \varepsilon) := \log \|(|x| + 1)^{-s}(P(h) - E \pm i\varepsilon)^{-1}(|x| + 1)^{-s}\|_{L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)}
\]

from above by an explicit function of \( h \), independent of \( \varepsilon \). Here \( 0 < \varepsilon < 1 \), \( s > 1/2 \) is independent of \( h \) and \( E > 0 \) is a fixed energy level independent of \( h \). When \( \delta > 2 \) it has been proved in [4] that

\[
g_{s}^{\pm}(h, \varepsilon) \leq Ch^{-4/3}\log(h^{-1}).
\]

The bound (1.2) was previously proved in [8] and [11] when \( d \geq 2 \) for compactly supported potentials, and in [12] when \( \delta > 3 \) and \( d \geq 3 \). It was also shown in [14] that (1.2) still holds for more general asymptotically Euclidean manifolds. In the present paper we show that, if \( d \geq 3 \), the logarithmic term in the righthand side of (1.2) can be removed for potentials \( V \) depending only on the radial variable \( r = |x| \). It is not clear, however, if it is possible to do so for general potentials. This remains an interesting, open problem. We also improve significantly the bound

\[
g_{s}^{\pm}(h, \varepsilon) \leq Ch^{-4/3}\log(h^{-1})^{1/\delta-1}
\]

proved in [13] for \( 1 < \delta \leq 3 \) and \( d \geq 3 \). More precisely, we have the following

Theorem 1.1. Let \( d \geq 3 \) and suppose that the potential \( V \) depends only on the radial variable. If \( V \) satisfies (1.1) with \( \delta > 2 \), then there exist constants \( C > 0 \) and \( h_0 > 0 \) independent of \( h \) and \( \varepsilon \) but depending on \( s \), \( E \), such that the bound

\[
g_{s}^{\pm}(h, \varepsilon) \leq Ch^{-4/3}
\]
holds for all $0 < h \leq h_0$. If $V$ satisfies (1.1) with $1 < \delta \leq 2$, then we have the bound
\begin{equation}
(1.5) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-\frac{2\delta}{2\delta-1}}(\log(h^{-1}))^{\frac{\delta+1}{2\delta-1}}.
\end{equation}
If the potential satisfies the condition
\begin{equation}
(1.6) \quad |V(r)| \leq C(r+1)^{-1}(\log(r+2))^{-\rho}
\end{equation}
with some constants $C > 0$ and $\rho > 1$, then we have the bound
\begin{equation}
(1.7) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-2}.
\end{equation}
Note that when $d = 1$ we have much better resolvent bounds. Indeed, it has been proved in [7] that for $V \in L^1(\mathbb{R})$ we have the bound
\begin{equation}
(1.8) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-1}.
\end{equation}
The bound (1.8) is proved in [15] (see also [5]) when $d \geq 3$ for slowly decaying Lipschitz potentials $V$ with respect to the radial variable $r$ and satisfying the conditions
\begin{equation}
(1.9) \quad V(x) \leq p(|x|)
\end{equation}
where $p(r) > 0$, $r \geq 0$, is a decreasing function such that $p(r) \to 0$ as $r \to \infty$, and
\begin{equation}
(1.10) \quad \partial_r V(x) \leq C_1(|x| + 1)^{-\delta_1}
\end{equation}
where $C_1 > 0$ and $\delta_1 > 0$ are some constants. When $d = 2$ the bound (1.8) is proved in [10] for potentials which are Lipschitz with respect to the space variable $x$. Under this condition, the bound (1.8) is extended in [15] to general exterior domains and all dimensions $d \geq 2$. Note that the bound (1.8) was first proved for smooth potentials in [2]. A high-frequency analog of (1.8) on Riemannian manifolds was proved in [1] and [3]. It was also showed in [6] that the bound (1.8) is optimal for smooth potentials.

The bound (1.2) has been recently improved in [4, 15] for Hölder potentials $V \in C^\alpha_3(\mathbb{R}^+)$ with respect to the radial variable and satisfying the condition (1.9) as well. Hereafter, given $0 < \alpha < 1$ and $\beta > 0$, the space $C^\alpha_3(\mathbb{R}^+)$ denotes the set of all Hölder functions $a$ such that
\[
\sup_{r' \geq 0: 0 < |r-r'| \leq 1} \frac{|a(r) - a(r')|}{|r-r'|^\alpha} \leq C(r+1)^{-\beta}, \quad \forall r \in \mathbb{R}^+,
\]
for some constant $C > 0$. Indeed, it is shown in [4] (with $\beta > 3$) and in [15] (with $\beta = 4$) that in this case we have the bound
\begin{equation}
(1.11) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-4/(\alpha+3)} \log(h^{-1}).
\end{equation}
In [15] the bound (1.11) is also extended to exterior domains and all dimensions $d \geq 2$ for potentials which are $\alpha$-Hölder with respect to the space variable. In the present paper we show that the bound (1.11) can be improved for radial potentials. We have the following

**Theorem 1.2.** Let $d \geq 3$ and suppose that the potential $V$ depends only on the radial variable and satisfies (1.3). If $V \in C^\alpha_3(\mathbb{R}^+)$, then there exist constants $C > 0$ and $h_0 > 0$ independent of $h$ and $\varepsilon$ but depending on $s$, $E$ and the function $p$, such that the bound
\begin{equation}
(1.12) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-4/(\alpha+3)}
\end{equation}
holds for all $0 < h \leq h_0$. If $V \in C^\alpha_3(\mathbb{R}^+)$ with $2 < \beta < 3$, then we have the bound
\begin{equation}
(1.13) \quad g_s^\pm(h, \varepsilon) \leq Ch^{-k}(\log(h^{-1}))^g,
\end{equation}
where
where
\[ k = \frac{2\alpha\beta - 6\alpha + 4}{2\alpha\beta - 5\alpha + 3}, \quad q = \frac{\alpha(3 - \beta)}{2\alpha\beta - 5\alpha + 3}. \]

If \( V \in C_\beta^3(\mathbb{R}^d) \) with \( 1 < \beta \leq 2 \), then the bound (1.1) holds with
\[ k = \frac{2\beta - 2\alpha}{2\beta - \alpha - 1}, \quad q = \frac{\beta + 1}{2\beta - \alpha - 1}. \]

The advantage of having a radial potential is that we can separate the variables. This allows to reduce our \( d \)-dimensional resolvent estimate to infinitely many one dimensional resolvent estimates depending on an additional parameter, which is the eigenvalues of the Laplace-Beltrami operator on the unit sphere (see Section 2). Therefore, we can use simpler Carleman estimates with better phases, and hence we can get better bounds. We expect that the bound (1.4) is optimal for \( L^\infty \) potentials, but to our best knowledge showing this remains an open problem. Note also that (1.4) is proved in [14] for a quite large class of asymptotically hyperbolic manifolds with \( L^\infty \) potentials decaying sufficiently fast.

2. Preliminaries

To get our resolvent bounds we will use the following

**Lemma 2.1.** Let \( s > 1/2 \) and suppose that for all functions \( f \in H^2(\mathbb{R}^d) \) such that
\[ (|x| + 1)^s(P(h) - E \pm i\varepsilon)f \in L^2(\mathbb{R}^d) \]
we have the estimate
\[ \|(|x| + 1)^{-s}f\|_{L^2(\mathbb{R}^d)} \leq M\|(|x| + 1)^s(P(h) - E \pm i\varepsilon)f\|_{L^2(\mathbb{R}^d)}^2 + M\varepsilon f\|_{L^2(\mathbb{R}^d)}^2 \]
with some \( M > 0 \) independent of \( \varepsilon \) and \( f \). Then we have the resolvent bound
\[ g^\pm_s(h,\varepsilon) \leq \log(M + 1). \]

**Proof.** Since the operator \( P(h) \) is symmetric, we have
\[ \varepsilon\|f\|_{L^2}^2 = \pm\operatorname{Im} \langle (P(h) - E \pm i\varepsilon)f, f \rangle_{L^2} \]
\[ \leq (2M)^{-1}\|(|x| + 1)^{-s}f\|_{L^2}^2 + \frac{M^2}{2}\|(|x| + 1)^s(P(h) - E \pm i\varepsilon)f\|_{L^2}^2 \]
which can be rewritten in the form
\[ M\varepsilon\|f\|_{L^2}^2 \leq \frac{1}{2}\|(|x| + 1)^{-s}f\|_{L^2}^2 + \frac{M^2}{2}\|(|x| + 1)^s(P(h) - E \pm i\varepsilon)f\|_{L^2}^2. \]
By (2.1) and (2.3) we get
\[ \|(|x| + 1)^{-s}f\|_{L^2} \leq (M + 1)\|(|x| + 1)^s(P(h) - E \pm i\varepsilon)f\|_{L^2} \]
which clearly implies (2.2). \( \square \)

We will now use that the potential is radial to reduce the estimate (2.1) to infinitely many similar estimates on \( \mathbb{R}^+ \). To this end we will write the operator \( P(h) \) in polar coordinates \((r, w) \in \mathbb{R}^+ \times S^{d-1}, r = |x|, w = x/|x| \) and we will use that \( L^2(\mathbb{R}^d) = L^2(\mathbb{R}^+ \times S^{d-1}, r^{d-1} dr dw) \). We have the identity
\[ r^{(d-1)/2} \Delta r^{-(d-1)/2} = \partial_r^2 + \frac{\tilde{\Delta}_w}{r^2} \]
where $\tilde{\Delta}_w = \Delta_w - \frac{1}{4}(d-1)(d-3)$ and $\Delta_w$ denotes the negative Laplace-Beltrami operator on $\mathbb{S}^{d-1}$. Set $v = r^{(d-1)/2} f$ and
\[ P^\pm(h) = r^{(d-1)/2}(P(h) - E \pm i\varepsilon)r^{-(d-1)/2}. \]
Using (2.5) we can write the operator $P^\pm(h)$ in the coordinates $(r, w)$ as follows
\[ P^\pm(h) = D_r^2 + \frac{\Delta_w}{r^2} + V(r) - E \pm i\varepsilon \]
where we have put $D_r = -ih\partial_r$ and $\Delta_w = -h^2\tilde{\Delta}_w$. Clearly, the estimate (2.1) can be rewritten in the form
\begin{equation}
\|(r+1)^{-s}v\|^2_{L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1})} \leq M\|(r+1)^sP^\pm(h)v\|^2_{L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1})} + M\varepsilon\|v\|^2_{L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1})}.
\end{equation}
Let $\lambda_j \geq 0$ be the eigenvalues of $-\Delta_w$ repeated with the multiplicities and let $e_j \in L^2(\mathbb{S}^{d-1})$ be the corresponding eigenfunctions. Set
\[ \nu = h\sqrt{\lambda_j + \frac{1}{4}(d-1)(d-3)} \]
and
\[ v_j(r) = \langle v(r, \cdot), e_j \rangle_{L^2(\mathbb{S}^{d-1})}, \]
\[ Q^\pm_{\nu}(h) = D_r^2 + \frac{\nu^2}{r^2} + V(r) - E \pm i\varepsilon. \]
Thus we can write
\[ v = \sum_j v_je_j, \]
\[ P^\pm(h)v = \sum_j Q^\pm_{\nu}(h)v_je_j, \]
so we have the identities
\begin{align*}
\|v\|^2_{L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1})} &= \sum_j \|v_j\|^2_{L^2(\mathbb{R}^+)}, \\
\|(r+1)^{-s}v\|^2_{L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1})} &= \sum_j \|(r+1)^{-s}v_j\|^2_{L^2(\mathbb{R}^+)}, \\
\|(r+1)^sP^\pm(h)v\|^2_{L^2(\mathbb{R}^+ \times \mathbb{S}^{d-1})} &= \sum_j \|(r+1)^sQ^\pm_{\nu}(h)v_j\|^2_{L^2(\mathbb{R}^+)}. 
\end{align*}
We have the following

**Lemma 2.2.** Let $s > 1/2$ and suppose that for all $\nu$ the estimates
\begin{equation}
\|(r+1)^{-s}u\|^2_{L^2(\mathbb{R}^+)} \leq M_\nu\|(r+1)^sQ^\pm_{\nu}(h)u\|^2_{L^2(\mathbb{R}^+)} + M\varepsilon\|D_ru\|^2_{L^2(\mathbb{R}^+)}
\end{equation}
hold for every $u \in H^2(\mathbb{R}^+)$ such that $u(0) = 0$ and $(r+1)^sQ^\pm_{\nu}(h)u \in L^2(\mathbb{R}^+)$, with $M_\nu > 0$ independent of $\varepsilon$ and $u$. Then the estimate (2.6) holds with
\[ M = (2 + E + \|V\|_{L^\infty}) \max_{\nu \in \text{spec} \Lambda_w} M_\nu. \]
Proof. We integrate by parts to obtain
\[ \text{Re} \int_0^\infty \mathcal{D}_r u \overline{u} dr = \int_0^\infty |\mathcal{D}_r u|^2 dr, \]
which leads to
\[ \text{Re} \int_0^\infty Q_\pm (h) u \overline{u} dr = \int_0^\infty |\mathcal{D}_r u|^2 dr + \nu^2 \int_0^\infty r^{-2} |u|^2 dr + \int_0^\infty (V(r) - E) |u|^2 dr \]
\[ \geq \int_0^\infty |\mathcal{D}_r u|^2 dr - (E + \|V\|_{L^\infty}) \int_0^\infty |u|^2 dr. \]
This implies
\[ (2.8) \quad \int_0^\infty |\mathcal{D}_r u|^2 dr \leq \int_0^\infty |Q_\pm (h) u|^2 dr + (1 + E + \|V\|_{L^\infty}) \int_0^\infty |u|^2 dr. \]
Combining (2.2) and (2.8) we get
\[ \|(r + 1)^{-s} u\|_{L^2(\mathbb{R}^+)} \leq 2M_\nu \|(r + 1)^s Q_\pm (h) u\|_{L^2(\mathbb{R}^+)} \]
\[ + (2 + E + \|V\|_{L^\infty}) M_\nu \varepsilon \|u\|_{L^2(\mathbb{R}^+)}^2. \]
Applying (2) with \( u = v_j \) and summing up all the inequalities clearly lead to (2.6) with the desired value of \( M \). \( \square \)

Thus we reduce our problem to proving estimates like (2.2) with as good bounds \( M_\nu \) as possible. This will be carried out in the next sections.

3. Bounding \( M \) for \( L^\infty \) potentials

We will first prove the following

Proposition 3.1. Let \( V \in L^1 \cap L^\infty \). Then the estimate (2.2) holds for all \( \nu \) with \( M_\nu = e^{C(\nu + 1)/h} \), where \( C > 0 \) is a constant independent of \( \nu \) and \( h \).

Proof. We will first consider the simplier case when \( \nu = 0 \). Note that this may happen only when \( d = 3 \). Set
\[ F(r) = E|u(r)|^2 + |\mathcal{D}_r u(r)|^2 \]
and observe that the first derivative of \( F \) satisfies the identity
\[ F'(r) = 2h^{-1} \text{Im} V u \overline{\mathcal{D}_r u} - 2h^{-1} \text{Im} Q_\pm (h) u \overline{\mathcal{D}_r u} \pm 2\varepsilon h^{-1} \text{Re} u \overline{\mathcal{D}_r u}. \]
We have
\[ -F'(r) \leq h^{-1} |V|(|u|^2 + |\mathcal{D}_r u|^2) + h^{-1} \gamma (r + 1)^{-2s} |\mathcal{D}_r u|^2 \]
\[ + h^{-1} \gamma^{-1} (r + 1)^{2s} |Q_\pm (h) u|^2 + \varepsilon h^{-1} (|u|^2 + |\mathcal{D}_r u|^2) \]
for any \( \gamma > 0 \). Let \( s > 1/2 \) and \( \eta = \frac{1}{2} \min\{1, E\} \). Set \( \mu = e^{\psi/h} \), where
\[ \psi(r) = \int_0^r \left( \eta^{-1} |V(\sigma)| + (\sigma + 1)^{-2s} \right) d\sigma \leq \eta^{-1} \|V\|_{L^1} + (2s - 1)^{-1}. \]
We have \( |V(r)| \leq \eta \psi'(r) \). Using this together with the identity \( \mu = h \mu'/\psi' \), we obtain
\[ -(\mu F)' = -\mu' F - \mu F' \leq -\mu' F + \mu \psi' (|u|^2 + |\mathcal{D}_r u|^2) + h^{-1} \gamma \mu (r + 1)^{-2s} |\mathcal{D}_r u|^2 \]
\[ + h^{-1} \gamma^{-1} \mu (r + 1)^{2s} |Q_\pm (h) u|^2 + \varepsilon h^{-1} \mu (|u|^2 + |\mathcal{D}_r u|^2) \]
\[ \leq -\frac{1}{2} \mu' F + h^{-1} \gamma \mu (r + 1)^{-2s} |\mathcal{D}_r u|^2 \]

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\[ + h^{-1} \gamma^{-1} \mu (r + 1)^{2s} |Q_0^\pm (h) u|^2 + \varepsilon h^{-1} \mu \left( |u|^2 + |\mathcal{D}_r u|^2 \right). \]

Integrating this inequality gives
\[ 0 \leq F(0) = - \int_0^\infty (\mu F)' \leq - \frac{1}{2} \int_0^\infty \mu' F + h^{-1} \gamma \max \mu \int_0^\infty (r + 1)^{-2s} |\mathcal{D}_r u|^2 + h^{-1} \gamma \max \mu \int_0^\infty (|u|^2 + |\mathcal{D}_r u|^2). \]

Using that \( \mu' \geq \psi'/h \geq h^{-1} (r + 1)^{-2s} \) we deduce from the above inequality
\[ \frac{1}{2} \int_0^\infty (r + 1)^{-2s} F \leq \frac{h}{2} \int_0^\infty \mu' F \leq \gamma \max \mu \int_0^\infty (r + 1)^{-2s} |\mathcal{D}_r u|^2 + \varepsilon \max \mu \int_0^\infty (|u|^2 + |\mathcal{D}_r u|^2). \]

Now we take \( \gamma = (3 \max \mu)^{-1} \) so that we can absorb the first term in the right-hand side of the above inequality. Thus we get the estimate
\[ \frac{1}{6} \int_0^\infty (r + 1)^{-2s} F \leq 3 (\max \mu)^2 \int_0^\infty (r + 1)^{2s} |Q_0^\pm (h) u|^2 + \varepsilon \max \mu \int_0^\infty (|u|^2 + |\mathcal{D}_r u|^2) \]

which clearly implies the desired bound because \( \max \mu \leq e^{C/h} \) with some constant \( C > 0 \).

Consider now the case \( \nu > 0 \). Then \( \nu \geq h \nu_0 \) with some constant \( \nu_0 > 0 \). Let \( \phi_j \in C^\infty(\mathbb{R}), \ j = 0, 1, 2, \) be real-valued functions such that \( 0 \leq \phi_j \leq 1, \phi'_j \geq 0, \phi_0(\sigma) = 0 \) for \( \sigma \leq 1/3, \phi_0(\sigma) = 1 \) for \( \sigma \geq 1/2, \phi_1(\sigma) = 0 \) for \( \sigma \leq 1, \phi_1(\sigma) = 1 \) for \( \sigma \geq 2, \phi_2(\sigma) = 0 \) for \( \sigma \leq 3, \phi_2(\sigma) = 1 \) for \( \sigma \geq 4 \). Set \( \kappa = 4\sqrt{1 + \mathcal{E} + ||V||_{L^\infty}} \) and \( V_\nu = \nu^2 \phi_0(r/\kappa) r^{-2} + V, u_1 = \phi_1(r/\kappa) u, u_2 = (1 - \phi_2)(r/\kappa) u \). Observe that
\[ (\nu^2 r^{-2} + V) u_1 = V_\nu u_1 \]
and \( V_\nu \in L^1 \) with norm \( ||V_\nu||_{L^1} = O(\nu + 1) \). As above, we are going to bound from below the first derivative of the function
\[ F_1(r) = E|u_1(r)|^2 + |\mathcal{D}_r u_1(r)|^2. \]

We have the identity
\[ F'_1(r) = 2h^{-1} \text{Im} V_\nu u_1 \overline{\mathcal{D}_r u_1} - 2h^{-1} \text{Im} Q_0^\pm (h) u_1 \overline{\mathcal{D}_r u_1} + 2 \varepsilon h^{-1} \text{Re} u_1 \overline{\mathcal{D}_r u_1} \]
\[ = 2h^{-1} \text{Im} V_\nu u_1 \overline{\mathcal{D}_r u_1} - 2h^{-1} \text{Im} Q_0^\pm (h) u_1 \overline{\mathcal{D}_r u_1} + 2 \varepsilon h^{-1} \text{Re} u_1 \overline{\mathcal{D}_r u_1} + T(r), \]

where
\[ T(r) = -2h^{-1} \text{Im} [\mathcal{D}_r^2, \phi_1(r/\kappa)] u \overline{\mathcal{D}_r \phi_1(r/\kappa) u} \]
\[ = 2(\kappa \nu)^{-1} \phi_1' \phi_1'' |\mathcal{D}_r u|^2 + h(\kappa \nu)^{-2} (\phi_1' \phi_1'' + 2 \phi_1''') \text{Im} u \overline{\mathcal{D}_r u} + h^2 (\kappa \nu)^{-3} \phi_1' |\phi_1''|^2 \]
\[ \geq -O(h \nu^{-2}) (\phi_1' + |\phi_1''|) (|u|^2 + |\mathcal{D}_r u|^2). \]

Thus we obtain
\[ -F'_1(r) \leq -h^{-1} |V_\nu| (|u_1|^2 + |\mathcal{D}_r u_1|^2) + h^{-1} \gamma (r + 1)^{-2s} |\mathcal{D}_r u_1|^2 \]
\[ + h^{-1} \gamma^{-1} (r + 1)^{2s} |Q_0^\pm (h) u_2|^2 + \varepsilon O(h^{-1}) (|u|^2 + |\mathcal{D}_r u|^2) \]
\[ + O(h \nu^{-2}) (\phi_1' + |\phi_1''|) (|u|^2 + |\mathcal{D}_r u|^2) \]
for any \( \gamma > 0 \). Let \( 1/2 < s \leq 1 \), let \( \eta \) be as above and let \( \lambda \gg 1 \) be a large parameter independent of \( h \) and \( \nu \) to be fixed later on. Set \( \mu = e^{\psi/h} \) with
\[ \psi(r) = \int_0^r (\eta^{-1} |V_\nu(\sigma)| + \lambda(\sigma + 1)^{-2s}) d\sigma \leq \eta^{-1} ||V_\nu||_{L^1} + \lambda(2s - 1)^{-1} \lesssim \nu + 1. \]
Integrating by parts and absorbing the term involving the potential in the same way as above we arrive at the inequality
\[
\frac{1}{2} \int_0^\infty \psi \mu F_1 \lesssim \gamma \max \mu \int_0^\infty (r + 1)^{-2s} |D_r u_1|^2 
+ \gamma^{-1} \max \mu \int_0^\infty (r + 1)^{2s} |Q_\nu^\pm (h) u|^2 
+ \varepsilon \max \mu \int_0^\infty (|u|^2 + |D_r u|^2) 
+ h^2 \nu^{-2} \mu(2\kappa \nu) \int_{\kappa \nu}^{2\kappa \nu} (|u|^2 + |D_r u|^2).
\]
On the other hand, we have
\[
\int_0^\infty \psi \mu F_1 \geq \lambda \mu(3\kappa \nu) \int_{3\kappa \nu}^{4\kappa \nu} (r + 1)^{-2s} F_1 \geq \lambda(4\kappa \nu + 1)^{-2} \mu(3\kappa \nu) \int_{3\kappa \nu}^{4\kappa \nu} F_1.
\]
Combining both inequalities and using that \(\mu(3\kappa \nu) > \mu(2\kappa \nu) > 1\) yield
\[
\int_{3\kappa \nu}^{4\kappa \nu} F_1 \lesssim \gamma(\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{-2s} |D_r u_1|^2 
+ \gamma^{-1}(\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{2s} |Q_\nu^\pm (h) u|^2 
+ \varepsilon(\nu + 1)^2 \max \mu \int_0^\infty (|u|^2 + |D_r u|^2) 
+ \lambda^{-1} h^2 \nu^{-2} (\nu + 1)^2 \int_{\kappa \nu}^{2\kappa \nu} (|u|^2 + |D_r u|^2).
\]
Observe now that \(h^2 \nu^{-2} (\nu + 1)^2 \lesssim 1\) as long as \(\nu \geq h\nu_0\). Therefore, we can rewrite the above inequality in the form
\[
\int_{3\kappa \nu}^{4\kappa \nu} (|u|^2 + |D_r u|^2) \lesssim \gamma(\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{-2s} |D_r u_1|^2 
+ \gamma^{-1}(\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{2s} |Q_\nu^\pm (h) u|^2 
+ \varepsilon(\nu + 1)^2 \max \mu \int_0^\infty (|u|^2 + |D_r u|^2) 
+ \lambda^{-1} \int_{\kappa \nu}^{2\kappa \nu} (|u|^2 + |D_r u|^2).
\]
On the other hand, the choice of \(\kappa\) guarantees the inequality
\[
(\nu^2 r^{-2} + V(r) - E)|u_2|^2 \geq |u_2|^2.
\]
Therefore, integrating by parts we obtain
\[
\text{Re} \int_0^\infty Q_\nu^\pm (h) u_2 \overline{u_2} = \int_0^\infty |D_r u_2|^2 + \int_0^\infty (\nu^2 r^{-2} + V(r) - E)|u_2|^2 
\geq \int_0^\infty |D_r u_2|^2 + \int_0^\infty |u_2|^2.
\]
which implies
\[ \int_0^\infty (|u_2|^2 + |D_r u_2|^2) \leq \int_0^\infty |Q^\pm_{\nu'}(h) u_2|^2 \]
\[ \leq \int_0^\infty |Q^\pm_{\nu'}(h) u|^2 + \int_0^\infty |[D^\nu_r, \phi_2(r/\kappa \nu)] u|^2 \]
\[ \lesssim \int_0^\infty |Q^\pm_{\nu'}(h) u|^2 + \int_{3\kappa \nu}^{4\kappa \nu} (|u|^2 + |D_r u|^2) . \]

In particular, this inequality yields
\[ (3.2) \quad \int_{\kappa \nu}^{2\kappa \nu} (|u|^2 + |D_r u|^2) \lesssim \int_0^\infty |Q^\pm_{\nu'}(h) u|^2 + \int_{3\kappa \nu}^{4\kappa \nu} (|u|^2 + |D_r u|^2) . \]

Combining (3) and (3.2) and taking \( \lambda \) big enough, we get
\[ \int_{\kappa \nu}^{2\kappa \nu} + \int_{3\kappa \nu}^{4\kappa \nu} (|u|^2 + |D_r u|^2) \lesssim \gamma (\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{-2s} |D_r u_1|^2 \]
\[ + \gamma^{-1} (\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{2s} |Q^\pm_{\nu'}(h) u|^2 \]
\[ (3.3) \quad + \varepsilon (\nu + 1)^2 \max \mu \int_0^\infty (|u|^2 + |D_r u|^2) . \]

Using (3) we can rewrite the above inequalities as follows:
\[ \lambda \int_{5\kappa \nu/2}^\infty (r + 1)^{-2s} (E |u|^2 + |D_r u|^2) \]
\[ \leq \lambda \int_0^\infty (r + 1)^{-2s} F_1 \leq \int_0^\infty \psi' F_1 \leq \int_0^\infty \psi' \mu F_1 \]
\[ \lesssim \gamma \max \mu \int_0^\infty (r + 1)^{-2s} |D_r u_1|^2 \]
\[ + \gamma^{-1} \max \mu \int_0^\infty (r + 1)^{2s} |Q^\pm_{\nu'}(h) u|^2 \]
\[ + \varepsilon \max \mu \int_0^\infty (|u|^2 + |D_r u|^2) + \max \mu \int_{\kappa \nu}^{2\kappa \nu} (|u|^2 + |D_r u|^2) \]
\[ \lesssim \gamma (\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{-2s} (|u|^2 + |D_r u|^2) \]
\[ + \gamma^{-1} (\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{2s} |Q^\pm_{\nu'}(h) u|^2 \]
\[ + \varepsilon (\nu + 1)^2 \max \mu \int_0^\infty (|u|^2 + |D_r u|^2) \]

and
\[ \int_{\kappa \nu/2}^{5\kappa \nu/2} (r + 1)^{-2s} (|u|^2 + |D_r u|^2) \leq \int_0^\infty (|u_2|^2 + |D_r u_2|^2) \]
\[ \lesssim \int_0^\infty |Q^\pm_{\nu'}(h) u|^2 + \int_{3\kappa \nu}^{4\kappa \nu} (|u|^2 + |D_r u|^2) \]
\[ \lesssim \gamma (\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{-2s} (|u|^2 + |D_r u|^2) \]
\[ + \gamma^{-1}(\nu + 1)^2 \max \mu \int_0^\infty (r + 1)^{2s}|Q^\pm_{\nu}(h)u|^2 \]
\[ + \epsilon(\nu + 1)^2 \max \mu \int_0^\infty (|u|^2 + |D_r u|^2). \]

Hence
\[ \int_0^\infty (r + 1)^{-2s}(|u|^2 + |D_r u|^2) \lesssim \gamma(\nu + 1)^2(\max \mu)^2 \int_0^\infty (r + 1)^{-2s}(|u|^2 + |D_r u|^2) \]
\[ + \gamma^{-1}(\nu + 1)^2(\max \mu)^2 \int_0^\infty (r + 1)^{2s}|Q^\pm_{\nu}(h)u|^2 \]
\[ + \epsilon(\nu + 1)^2(\max \mu)^2 \int_0^\infty (|u|^2 + |D_r u|^2). \]

We now take \( \gamma \) such that \( \gamma(\nu + 1)^2(\max \mu)^2 = \gamma_0 \) with a sufficiently small constant \( \gamma_0 > 0 \), so that we can absorb the first term in the right-hand side of the above inequality. This leads to the estimate
\[ \int_0^\infty (r + 1)^{-2s}(|u|^2 + |D_r u|^2) \lesssim (\nu + 1)^4(\max \mu)^4 \int_0^\infty (r + 1)^{2s}|Q^\pm_{\nu}(h)u|^2 \]
\[ + \epsilon(\nu + 1)^2(\max \mu)^2 \int_0^\infty (|u|^2 + |D_r u|^2), \]
which together with the fact that \( \max \mu \leq e^{C(\nu+1)/h} \) imply the desired estimate for \( 1/2 < s \leq 1 \), and hence for all \( s > 1/2 \). \( \square \)

We will next show that for large \( \nu \) much better bounds for \( M_\nu \) are possible. To this end, set \( \tau = h^{-1/3}, \tau_1 = \tau \) if \( V \) satisfies (1.1) with \( \delta > 2 \), \( \tau = h^{-\frac{3\delta}{2\delta - 3}} \epsilon^{-\frac{2\delta}{2\delta - 3}}, \tau_1 = \epsilon^{-1}\tau \) if \( V \) satisfies (1.1) with \( 1 < \delta \leq 2 \), and \( \tau = h^{-1}, \tau_1 = \tau \) if \( V \) satisfies (1.6), where \( \epsilon = (\log(h^{-1}))^{-1} \).

**Proposition 3.2.** There exist constants \( C, c > 0 \) such that the estimate (2.2) holds for all \( \nu \geq c \tau \) with \( M_\nu = e^{C\tau_1/h} \).

**Proof.** Set \( \lambda = 1, \omega(r) = (r + 1)^{-\delta + 3} \) if \( V \) satisfies (1.1) with \( \delta > 2 \), \( \lambda \gg 1, \omega(r) = (r + 1)^{-1-\epsilon} \) if \( V \) satisfies (1.1) with \( 1 < \delta \leq 2 \), and \( \lambda \gg 1, \omega(r) = (r + 1)^{-1}(\log(r + 2))^{-\delta} \) if \( V \) satisfies (1.6). Set
\[
\varphi(r) = \lambda \tau \int_0^r \omega(\sigma) d\sigma \lesssim \tau_1.
\]
The parameter \( \lambda \) in the second and the third cases is independent of \( \nu \) and \( h \) and will be fixed later on. Introduce the operator
\[
Q^\pm_{\nu, \varphi}(h) = e^{\varphi/h}Q^\pm_{\nu}(h)e^{-\varphi/h}
\]
\[
= D^2_r + \frac{\nu^2}{r^2} + V(r) - \varphi'^2 + h\varphi'' + 2i\varphi'D_r - E \pm i\varepsilon.
\]
Consider the function
\[
F_\varphi(r) = (E - \nu^2r^{-2} + \varphi'^2)|u(r)|^2 + |D_r u(r)|^2.
\]
It is easy to see that its first derivative is given by
\[
F'_\varphi(r) = 2(\nu^2r^{-3} + \varphi'\varphi'')|u|^2 + 4h^{-1}\varphi'|D_r u|^2
\]
\[
+ 2h^{-1}\Im(V + h\varphi'')u\overline{D_r u} - 2h^{-1}\Im Q^\pm_{\nu, \varphi}(h)u\overline{D_r u} \pm 2\varepsilon h^{-1}\Re u\overline{D_r u}
\]
\[
\geq (2\nu^2r^{-3} - 2\varphi'|\varphi'|^2 - (\varphi')^{-1}|\varphi''|^2)|u|^2 - h^{-1}(\varphi')^{-1}V^2|u|^2 - \Phi(r),
\]
where
\[ \Phi = \gamma(r + 1)^{-2s} |D_r u|^2 + \gamma^{-1} h^{-2}(r + 1)^{2s} \left| Q_{r,\nu}(h) u \right|^2 + \epsilon h^{-1} (|u|^2 + |D_r u|^2) \]
and \( \gamma > 0 \) and \( s > 1/2 \) are arbitrary. It is easy to check that
\[ 2\varphi'|\varphi'| + (\varphi')^{-1} |\varphi''|^2 \leq c_0^2 h^2(r + 1)^{-3} \]
with some constant \( c_0 > 0 \). Therefore, for \( \nu \geq c_0 \tau \), we obtain
\[ (3.4) \quad F'_\nu(r) \geq \nu^2 r^{-3}|u|^2 - (\lambda h\tau)^{-1} \omega^{-1} V^2 |u|^2 - \Phi. \]
Let us see that (3.4) implies the inequality
\[ (3.5) \quad F'_\nu(r) \geq \frac{1}{2} \nu^2 r^{-3}|u|^2 - K \lambda^{-1} \omega |u|^2 - \Phi, \]
provided \( \nu \geq c \tau \) with some constant \( c > c_0 \), where \( K = 0 \) if \( V \) satisfies (1.1) with \( \delta > 2 \), \( K > 0 \) is a constant if \( V \) satisfies (1.6), and \( K = \bar{K} \epsilon, \bar{K} > 0 \) is a constant if \( V \) satisfies (1.1) with \( 1 < \delta \leq 2 \). Indeed, in the first case this follows from the inequality \( (h\tau)^{-1} \omega^{-1} V^2 \lesssim \tau^2 (r + 1)^{-3} \), while in the case when \( V \) satisfies (1.6) it follows from \( (h\tau)^{-1} \omega^{-1} V^2 \lesssim \omega \). Let now \( V \) satisfy (1.1) with \( 1 < \delta \leq 2 \). Then we have
\[ \omega^{-1} V^2 \lesssim (r + 1)^{-2\delta + 1 + \epsilon} \leq a^4 - 2\delta + \epsilon (r + 1)^{-3} + a^2 - 2\delta + 2\epsilon (r + 1)^{-1 - \epsilon} \]
for every \( a > 1 \). Take \( a \) such that
\[ (h\tau)^{-1} a^4 - 2\delta = \tau^2, \quad (h\tau)^{-1} a^2 - 2\delta = \epsilon. \]
In view of the choice of \( \tau \) we find that these equations are satisfied with \( a = h^{- \frac{1}{2\delta - 1}} \epsilon^{- \frac{3}{2\delta - 1}} \).
Thus we get the inequality
\[ (h\tau)^{-1} \omega^{-1} V^2 \lesssim \tau^2 (r + 1)^{-3} + \epsilon (r + 1)^{-1 - \epsilon}, \]
which clearly implies (3.5) in this case.

Integrating (3.5) from 0 to \( \infty \) and using that \( F_\nu(0) = |D_r u(0)|^2 \geq 0 \), we get
\[ (3.6) \quad \frac{1}{2} \int_0^\infty \nu^2 r^{-3}|u|^2 \leq K \lambda^{-1} \int_0^\infty \omega |u|^2 + \int_0^\infty \Phi. \]
On the other hand, integrating (3.5) from \( r \) to \( \infty \) yields
\[ (3.7) \quad F'_\nu(r) = - \int_r^\infty F'_\nu(\sigma) d\sigma \leq K \lambda^{-1} \int_0^\infty \omega |u|^2 + \int_0^\infty \Phi. \]
Set \( \bar{\omega} = (r + 1)^{-2s} \) if \( V \) satisfies (1.1) with \( \delta > 2 \), and \( \bar{\omega} = \omega \) otherwise. Clearly, \( \| \bar{\omega} \|_{L^1} \lesssim \epsilon^{-1} \) if \( V \) satisfies (1.1) with \( 1 < \delta \leq 2 \) and \( \| \bar{\omega} \|_{L^1} \lesssim 1 \) in the other two cases. Therefore, multiplying (3.7) by \( \bar{\omega} \) and integrating from 0 to \( \infty \) lead to
\[ (3.8) \quad \int_0^\infty \bar{\omega} F'_\nu \lesssim \ell_1 \lambda^{-1} \int_0^\infty \omega |u|^2 + (1 + \ell_2 \epsilon^{-1}) \int_0^\infty \Phi, \]
where \( \ell_1 = \ell_2 = 0 \) if \( V \) satisfies (1.1) with \( \delta > 2 \), \( \ell_1 = \ell_2 = 1 \) if \( V \) satisfies (1.1) with \( 1 < \delta \leq 2 \), and \( \ell_1 = 1, \ell_2 = 0 \) if \( V \) satisfies (1.6). By (3.6) and (3.8), using that \( \bar{\omega} \lesssim C r^{-1} \), we obtain
\[ \int_0^\infty \bar{\omega} \left( E |u|^2 + |D_r u|^2 \right) \leq \int_0^\infty \bar{\omega} F'_\nu + C \int_0^\infty \nu^2 r^{-3}|u|^2 \]
\[ \lesssim \ell_1 \lambda^{-1} \int_0^\infty \bar{\omega} |u|^2 + (1 + \ell_2 \epsilon^{-1}) \int_0^\infty \Phi. \]
When $\ell_1 = 1$ we take $\lambda$ large enough in order to absorbe the first term in the right-hand side of the above inequality. Thus we get
\[
(3.9) \quad \int_0^\infty \tilde{\omega} \left( |u|^2 + |D_r u|^2 \right) \lesssim (1 + \ell_2 \epsilon^{-1}) \int_0^\infty \Phi.
\]

Since $\tilde{\omega} \geq (r + 1)^{-2s}$, we deduce from (3.9)
\[
\int_0^\infty (r + 1)^{-2s} \left( |u|^2 + |D_r u|^2 \right) \lesssim \gamma (1 + \ell_2 \epsilon^{-1}) \int_0^\infty (r + 1)^{-2s} |D_r u|^2 + \gamma^{-1} \epsilon^{-2} \int_0^\infty (r + 1)^{2s} |Q_{r,\nu}^\pm(h)u|^2 + \epsilon \gamma h^{-1}(1 + \ell_2 \epsilon^{-1}) \int_0^\infty (|u|^2 + |D_r u|^2).
\]

We now take $\gamma$ such that $\gamma (1 + \ell_2 \epsilon^{-1}) = \gamma_0$, where $\gamma_0$ is a sufficiently small constant. Thus we can absorbe the first term in the right-hand side of the above inequality to obtain
\[
\int_0^\infty (r + 1)^{-2s} |u|^2 \lesssim h^{-2} (1 + \ell_2 \epsilon^{-1})^2 \int_0^\infty (r + 1)^{2s} |Q_{r,\nu}^\pm(h)u|^2 + \epsilon \gamma h^{-1}(1 + \ell_2 \epsilon^{-1}) \int_0^\infty (|u|^2 + |D_r u|^2).
\]

We apply (3) with $u$ replaced by $e^{\varphi/h} u$. Thus we get the Carleman estimate
\[
\int_0^\infty (r + 1)^{-2s} e^{2\varphi/h} |u|^2 \lesssim h^{-2} (1 + \ell_2 \epsilon^{-1})^2 \int_0^\infty (r + 1)^{2s} e^{2\varphi/h} |Q_{r,\nu}^\pm(h)u|^2 + \epsilon \gamma h^{-1}(1 + \ell_2 \epsilon^{-1}) \tau^2 \int_0^\infty e^{2\varphi/h} (|u|^2 + |D_r u|^2).
\]

Since $1 \leq e^{2\varphi/h} \leq e^{\tilde{C} \tau_1/h}$ with some constant $\tilde{C} > 0$, (3) implies
\[
\int_0^\infty (r + 1)^{-2s} |u|^2 \lesssim h^{-2} (1 + \ell_2 \epsilon^{-1})^2 e^{\tilde{C} \tau_1/h} \int_0^\infty (r + 1)^{2s} |Q_{r,\nu}^\pm(h)u|^2 + \epsilon \gamma h^{-1}(1 + \ell_2 \epsilon^{-1}) \tau^2 e^{\tilde{C} \tau_1/h} \int_0^\infty (|u|^2 + |D_r u|^2),
\]
which gives the desired bound for $M_\nu$.

It follows from Propositions 3.1 and 3.2 together with Lemma 2.2 that $M \leq e^{C \tau_1/h}$ with some constant $C > 0$, which, in view of Lemma 2.1, implies Theorem 1.1.

4. **Bounding $M$ for H"older potentials**

Let $\rho \in C_0^\infty([0,1])$, $\rho \geq 0$, be a real-valued function independent of $\nu$ and $h$ such that $\int_0^\infty \rho(\sigma) d\sigma = 1$. If $V \in C_0^\infty(\mathbb{R}^+)$, we can approximate it by the function
\[
V_\theta(r) = \theta^{-1} \int_0^\infty \rho((r - r')/\theta) V(r') dr' = \int_0^\infty \rho(\sigma)V(r + \theta \sigma) d\sigma
\]
where $0 < \theta \ll 1$ will be chosen later on. We have
\[
|V(r) - V_\theta(r)| \leq \int_0^\infty \rho(\sigma)|V(r + \theta \sigma) - V(r)| d\sigma.
\]
Proposition 4.1. Clearly, \( V \) is \( C^1 \) with respect to the variable \( r \) and its first derivative \( V' \) is given by
\[
V'(r) = \theta^{-2} \int_0^\infty \rho' (\theta r) V(\theta r) dr
\]
where we have used that \( \int_0^\infty \rho' (\theta) d\theta = 0 \). Hence
\[
|V'(r)| \lesssim \theta^{1-\alpha} (r+1)^{-\beta} \int_0^\infty \sigma^\alpha |\rho'(\sigma)| d\sigma \lesssim \theta^{1-\alpha} (r+1)^{-\beta}.
\]
Using the above inequalities we will prove the following

**Proposition 4.1.** Let \( V \in L^\infty \) satisfy (1.9). Suppose in addition that \( V \in C^\alpha (\mathbb{R}^+) \) with \( \beta > 1 \). Then the estimate (2.2) holds for all \( \nu \) with \( M_\nu = e^{C(\nu+1)/h} \), where \( C > 0 \) is a constant independent of \( \nu \) and \( h \).

**Proof.** We will modify the proof of Proposition 3.1 to avoid using that \( V \in L^1 \). Instead, we will use that \( V - V_\theta \) and \( V'_\theta \) belong to \( L^1 \). We will apply the above inequalities with \( \theta \) independent of \( h \) and \( \nu \). Set \( \varphi(r) = \lambda (1 - (r+1)^{-1}) \), where \( \lambda \gg 1 \) is independent of \( h \) and \( \nu \). Clearly, \( \varphi'(r) = \lambda (r+1)^{-2} \). Using (4.3) with \( \gamma = E/2 \) and \( N = 4 \), we obtain
\[
E + \varphi^2 - V_\theta \geq \frac{E}{2} + (\lambda^2 - C_\gamma A) (r+1)^{-4} \geq \frac{E}{2}
\]
provided \( \lambda \) is large enough. We set
\[
F(r) = (E + \varphi^2 - V_\theta)|u(r)|^2 + |D_r u(r)|^2
\]
when \( \nu = 0 \), and
\[
F_1(r) = (E + \varphi^2 - V_\theta)|u_1(r)|^2 + |D_r u_1(r)|^2
\]
when \( \nu > 0 \). By (4.5),
\[
\frac{E}{2} |u(r)|^2 + |D_r u(r)|^2 \leq F(r).
\]
Clearly, (4.6) holds with \( u \) and \( F \) replaced by \( u_1 \) and \( F_1 \). As in the previous section, the first derivative of \( F \) is given by
\[
F'(r) = (2 \varphi' \varphi'' - V'_\theta)|u|^2 + 4h^{-1} \varphi'|D_r u|^2
+ 2h^{-1} \text{Im} (V - V_\theta + h \varphi'') u D_r u - 2h^{-1} \text{Im} Q_{0, \varphi}(h) u D_r u \pm 2 \epsilon h^{-1} \text{Re} u D_r u.
\]
Hence
\[
-F'(r) \leq h^{-1} W_\theta (|u|^2 + |D_r u|^2) + h^{-1} \gamma (r+1)^{-2s} |D_r u|^2
+ h^{-1} \gamma^{-1} (r+1)^{2s} |Q_{0, \varphi}(h) u|^2 + \epsilon h^{-1} (|u|^2 + |D_r u|^2)
\]

If in addition \( V \) satisfies (1.9), we obtain from (4)
\[
V_\theta(r) \leq p(r) + O((r+1)^{-\beta}).
\]
Since \( p \) is a decreasing function tending to zero, it follows from (4.2) that given any \( \gamma, N > 0 \) there is a constant \( C_{\gamma, N} > 0 \) such that
\[
V_\theta(r) \leq C_{\gamma, N} (r+1)^{-N} + \gamma.
\]
for any $\gamma > 0$, where
\[ W_0 = |V - V_0| + |\varphi''| + 2|\varphi'| \in L^1. \]

Similarly
\[ -F'(r) \leq h^{-1} W_\nu \left( |u|^2 + |D_r u|^2 \right) + h^{-1} \gamma (r + 1) - 2s |D_r u|^2 \]
\[ + h^{-1} \gamma^{-1} (r + 1)^{2s} |Q_{\nu, \varphi}^\pm (h) u|^2 + \varepsilon O \left( h^{-1} (|u|^2 + |D_r u|^2) \right) \]

for any $\gamma > 0$, where
\[ W_\nu = \nu^2 r^{-2} \phi(r/\kappa\nu) + W_0 \in L^1. \]

Clearly, $\|W_\nu\|_{L^1} = O(\nu + 1)$. Now, arguing in the same way as in the proof of Proposition 3.1 and using the above inequalities, we obtain the estimate
\[ \int_0^\infty (r + 1)^{-2s} |u|^2 \leq e^{C(\nu + 1)/h} \int_0^\infty (r + 1)^{2s} |Q_{\nu, \varphi}^\pm (h) u|^2 + \varepsilon e^{C(\nu + 1)/h} \int_0^\infty (|u|^2 + |D_r u|^2) \]

with some constant $C > 0$. Applying this inequality with $u$ replaced by $e^{\varphi/h} u$, we get
\[ \int_0^\infty (r + 1)^{-2s} e^{2\varphi/h} |u|^2 \]
\[ \leq e^{C(\nu + 1)/h} \int_0^\infty (r + 1)^{2s} e^{2\varphi/h} |Q_{\nu, \varphi}^\pm (h) u|^2 + \varepsilon e^{C(\nu + 1)/h} \int_0^\infty e^{2\varphi/h} (|u|^2 + |D_r u|^2) \]

which implies
\[ \int_0^\infty (r + 1)^{-2s} |u|^2 \leq e^{C(\nu + 1)/h} \int_0^\infty (r + 1)^{2s} |Q_{\nu, \varphi}^\pm (h) u|^2 + \varepsilon e^{C(\nu + 1)/h} \int_0^\infty (|u|^2 + |D_r u|^2) \]

with a new constant $C > 0$, which is the desired estimate. \[ \square \]

Let $V \in C^2_\beta$ with $0 < \alpha < 1$ and $1 < \beta \leq 3$. Set $\tau = h^{-k_0} \epsilon^{-q_0}$, $\tau_1 = h^{-k_0} \epsilon^{-q}$, where
\[ k_0 = \frac{1 - \alpha}{\alpha + 3}, \quad q = q_0 = 0, \]
if $\beta = 3$,
\[ k_0 = \frac{1 - \alpha}{2\alpha \beta - 5\alpha + 3}, \quad q = q_0 = \frac{\alpha(3 - \beta)}{2\alpha \beta - 5\alpha + 3}, \]
if $2 < \beta < 3$,
\[ k_0 = \frac{1 - \alpha}{2\beta - \alpha - 1}, \quad q = q_0 = \frac{\alpha - \beta + 2}{2\beta - \alpha - 1}, \quad q = q_0 + 1, \]
if $1 < \beta \leq 2$. Clearly, to get Theorem 1.2 it suffices to prove the following

**Proposition 4.2.** There exist constants $C, c > 0$ such that the estimate (2.2) holds for all $\nu \geq \epsilon \tau$ with $M_\nu = e^{C\tau_1/h}$.

*Proof.* Set $\omega(r) = (r + 1)^{-2\beta + 3}$ if $2 < \beta \leq 3$, $\omega(r) = (r + 1)^{-1-\epsilon}$ if $1 < \beta \leq 2$ and
\[ \varphi(r) = \lambda \tau \int_0^r \omega(\sigma)d\sigma \lesssim \tau_1, \]
where $\lambda \gg 1$ is independent of $\nu$ and $h$. Consider the function
\[ F_\varphi(r) = (E - \nu^2 r^{-2} + \varphi^2 - V_\varphi)|u(r)|^2 + |D_r u(r)|^2. \]
Clearly, we can still arrange the inequality (4.5). Therefore, we have
\[ E \frac{1}{2} |u(r)|^2 + |D_r u(r)|^2 \leq F_\varphi(r) + \nu^2 r^{-2} |u(r)|^2. \]

(4.7)
In the same way as in the proof of Proposition 3.2 we can obtain the following analog of the inequality (3.1)

\[ F'_\varphi(r) \geq \nu^2 r^{-3} |u|^2 - W|u|^2 - \Phi, \]

for \( \nu \geq c_0 \tau \), where

\[ W = |V'_\varphi| + (\lambda h\tau)^{-1}\omega^{-1}|V - V'_\varphi|^2. \]

By (4.8) and (4.4), we have

\[ W \lesssim \theta^{-1+\alpha}(r+1)^{-\beta} + (\lambda h\tau)^{-1}\theta^{2\alpha}\omega^{-1}(r+1)^{-2\beta}. \]

Set \( \ell = 0, \bar{\omega} = (r+1)^{-2\beta} \) if \( \beta = 3 \) and \( \ell = 1, \bar{\omega} = (r+1)^{-1-\epsilon} \) if \( 1 < \beta < 3 \). We will show that (4.8) and (4.9) imply

\[ F'_\varphi(r) \geq \frac{1}{2} \nu^2 r^{-3} |u|^2 - \mathcal{O}(\lambda^{-1}) \ell \bar{\omega}|u|^2 - \Phi \]

for \( \nu \geq c\tau \) with some constant \( c \gg 1 \). If \( \beta = 3 \) we require that \( \tau \) and \( \theta \) satisfy the relations

\[ \tau^2 = \theta^{-1+\alpha} = (h\tau)^{-1}\theta^{2\alpha}, \]

which provides the desired value of \( \tau \). Since in this case \( \omega^{-1}(r+1)^{-2\beta} = (r+1)^{-3} \), we get (4.9) from (4.8) and (4.9). Let \( 2 < \beta < 3 \). Then we have the inequality

\[ (r+1)^{-\beta} \leq (b_0)^{3-\beta}(r+1)^{-3} + (b_0)^{1+\epsilon-\beta}(r+1)^{-1-\epsilon} \]

for every \( b, b_0 > 1 \), provided \( \epsilon \ll \beta - 1 \). We choose \( b_0 \) such that \( b_0^{1-\beta} = \lambda^{-1} \). We also let

\[ \tau^2 = b^{3-\beta}\theta^{-1+\alpha} = (h\tau)^{-1}\theta^{2\alpha}, \quad \theta^{-1+\alpha}b^{\beta+1} = \epsilon. \]

We are looking for solutions of these equations of the form \( \tau = h^{-k_0}\epsilon^{-q_0}, \theta^{-1} = h^{-k_1}\epsilon^{-q_1}, b = h^{-k_2}\epsilon^{-q_2} \). Thus the above equations take the form

\[
\begin{cases}
2k_0 = (3 - \beta)k_2 + (1 - \alpha)k_1 = 1 - k_0 - 2\alpha k_1, \\
(1 - \alpha)k_1 - (\beta - 1)k_2 = 0, \\
2q_0 = (3 - \beta)q_2 + (1 - \alpha)q_1 = -q_0 - 2\alpha q_1, \\
(1 - \alpha)q_1 - (\beta - 1)q_2 = -1.
\end{cases}
\]

Solving this linear system we find the desired values of \( k_0 \) and \( q_0 \). With this choice, by (4.9) we get

\[ W \lesssim \tau^2(r+1)^{-3} + \lambda^{-1} \epsilon(r+1)^{-1-\epsilon}, \]

which together with (4.8) imply (4.10).

Let \( 1 < \beta < 2 \). Then (4.11) still holds. We also have the inequality

\[ \omega^{-1}(r+1)^{-2\beta} = (r+1)^{-2\beta+1+\epsilon} \leq a^{4-2\beta+1+\epsilon}(r+1)^{-3} + a^{2-2\beta+2\epsilon}(r+1)^{-1-\epsilon} \]

for every \( a > 1 \). We let the parameters \( a \) and \( b \) satisfy the relations

\[ \tau^2 = b^{3-\beta}\theta^{-1+\alpha} = (h\tau)^{-1}\theta^{2\alpha} a^{4-2\beta}, \quad \theta^{-1+\alpha}b^{\beta+1} = (h\tau)^{-1}\theta^{2\alpha} a^{2-2\beta} = \epsilon. \]

As above, we are looking for solutions of these equations of the form \( \tau = h^{-k_0}\epsilon^{-q_0}, \theta^{-1} = h^{-k_1}\epsilon^{-q_1}, b = h^{-k_2}\epsilon^{-q_2} \) and \( a = h^{-k_3}\epsilon^{-q_3} \). Thus the above equations take the form

\[
\begin{cases}
2k_0 = (3 - \beta)k_2 + (1 - \alpha)k_1 = 1 - k_0 - 2\alpha k_1 + 2(2 - \beta)k_3, \\
(1 - \alpha)k_1 - (\beta - 1)k_2 = 1 - k_0 - 2\alpha k_1 - 2(\beta - 1)k_3 = 0, \\
2q_0 = (3 - \beta)q_2 + (1 - \alpha)q_1 = -q_0 - 2\alpha q_1 + 2(2 - \beta)q_3, \\
(1 - \alpha)q_1 - (\beta - 1)q_2 = -q_0 - 2\alpha q_1 - 2(\beta - 1)q_3 = -1.
\end{cases}
\]
Solving this linear system we find the desired values of \( k_0 \) and \( q_0 \). Thus we conclude that (4.13) still holds in this case, and hence (4.10) follows. On the other hand, (4.7) and (4.10) imply the estimate (2.2) in the same way as in the proof of Proposition 3.2. \( \square \)

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