SOME RESULTS IN QUASITOPOLOGICAL HOMOTOPY GROUPS

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UDC 515.14

It is shown that the $n$th quasitopological homotopy group of a topological space is isomorphic to the $(n-1)$th quasitopological homotopy group of its loop space. By using this fact, we obtain some results about quasitopological homotopy groups. Finally, with the help of the long exact sequence of a based pair and a fibration in the qTop introduced by Brazas in 2013, we obtain some results in this field.

1. Introduction

Being endowed with the quotient topology induced by the natural surjective map

$$q: \Omega^n(X, x) \to \pi_n(X, x),$$

where $\Omega^n(X, x)$ is the $n$th loop space of $(X, x)$ with the compact-open topology, the familiar homotopy group $\pi_n(X, x)$ turns into a quasitopological group, which is called a quasitopological $n$th homotopy group of the pointed space $(X, x)$ and denoted by $\pi_n^{\text{qtop}}(X, x)$ (see [3–5, 10]).

It was claimed by Biss [3] that $\pi_1^{\text{qtop}}(X, x)$ is a topological group. However, Calcut and McCarthy [7] and Fabel [8] showed that there is a gap in the proof of Proposition 3.1 in [3]. The misstep in the proof was repeated by Ghane, et al. [10] to prove that $\pi_n^{\text{qtop}}(X, x)$ is a topological group [10] (Theorem 2.1) (see also [7]).

Calcut and McCarthy [7] showed that $\pi_1^{\text{qtop}}(X, x)$ is a homogeneous space and, more precisely, Brazas [5] mentioned that $\pi_1^{\text{qtop}}(X, x)$ is a quasitopological group in the sense of [1].

Calcut and McCarthy [7] also proved that, for a path connected and locally path connected space $X$, $\pi_1^{\text{qtop}}(X)$ is a discrete topological group if and only if $X$ is semilocally 1-connected (see also [5]). Pakdaman, et al. [12] showed that, for a locally $(n-1)$-connected space $X$, $\pi_n^{\text{qtop}}(X, x)$ is discrete if and only if $X$ is semilocally $n$-connected at $x$ (see also [10]). In addition, they proved that the quasitopological fundamental group of every small loop space is an indiscrete topological group. We recall that a loop in $X$ at $x$ is called small if it is homotopic to a loop in every neighborhood $U$ of $x$. Moreover, a topological space $X$ with nontrivial fundamental group is called a small loop space if every loop of $X$ is small.

In the present paper, we obtain some results for quasitopological homotopy groups. One of the main results of Section 2 is Theorem 2.1.

By using this fact, we can show that some properties of a space can be transferred to its loop space. In addition, we obtain several results for quasitopological homotopy groups. Moreover, we show that, for a fibration

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Published in Ukraïns'kyi Matematychnyi Zhurnal, Vol. 72, No. 1, pp. 1663–1668, December, 2020. Ukrainian DOI: 10.37863/umzh.v72i12.564. Original article submitted September 14, 2017.
$p : E \to X$ with fiber $F$, the induced map

$$f_* : \pi_n^{qtop}(B, b_0) \to \pi_{n-1}^{qtop}(F, \tilde{b}_0)$$

is continuous.

In his thesis [6], Brazas exhibited two long exact sequences of a based pair $(X, A)$ and a fibration $p : E \to X$ in the qTop. In Section 3, we use these sequences and obtain some results in this field.

2. Quasitopological Homotopy Groups

It is well known that, for a pointed topological space $(X, x)$, we have

$$\pi_n(X, x) \cong \pi_{n-k}(\Omega^k(X, x), e_x)$$

for all $n \geq 1$ and $1 \leq k \leq n - 1$. In this section, we extend this result to quasitopological homotopy groups and obtain some results for these groups. The following theorem is one of the main results of the present paper:

**Theorem 2.1.** Let $(X, x)$ be a pointed topological space. Then, for all $n \geq 1$ and $1 \leq k \leq n - 1$,

$$\pi_n^{qtop}(X, x) \cong \pi_{n-k}^{qtop}(\Omega^k(X, x), e_x),$$

where $e_x$ is a constant $k$-loop in $X$ at $x$.

**Proof.** Consider the following commutative diagram:

$$
\begin{array}{ccc}
\Omega^n(X, x) & \xrightarrow{\phi} & \Omega^{n-k}(\Omega^k(X, x), e_x) \\
\downarrow q & & \downarrow q \\
\pi_n^{qtop}(X, x) & \xrightarrow{\phi_*} & \pi_{n-k}^{qtop}(\Omega^k(X, x), e_x),
\end{array}
$$

(1)

given by $\phi(f) = f^g$ is a homeomorphism with inverse $g \to g^\phi$ in a sense of [13]. Since $q$ is a quotient map, the homomorphism $\phi_*$ is an isomorphism between quasitopological homotopy groups.

The following result is a consequence of Theorem 2.1:

**Corollary 2.1.** Let $X$ be locally $(n - 1)$-connected. Then $X$ is semilocally $n$-connected at $x$ if and only if $\Omega^{n-1}(X, x)$ is semilocally simply connected at $e_x$, where $e_x$ is a constant loop in $X$ at $x$.

**Proof.** Since $X$ is locally $(n - 1)$-connected, by [12] (Theorem 6.7), $X$ is semilocally $n$-connected at $x$ if and only if $\pi_n^{qtop}(X, x)$ is discrete. By Theorem 2.1,

$$\pi_n^{qtop}(X, x) \cong \pi_1^{qtop}(\Omega^{n-1}(X, x), e_x).$$

Furthermore, $\pi_1^{qtop}(\Omega^{n-1}(X, x), e_x)$ is discrete if and only if $\Omega^{n-1}(X, x)$ is semilocally simply connected at $e_x$ by [12] (Theorem 6.7).
Note that the above-mentioned result was proved by H. Wada [17] (Remark) and the authors of [11] (Lemma 3.1) by using other methods.

**Corollary 2.2.** Let \((X, x) = \varprojlim (X_i, x_i)\) be the inverse limit of an inverse system \(\{(X_i, x_i), \varphi_{ij}\}_I\). Then, for all \(n \geq 1\) and \(1 \leq k \leq n - 1\),

\[
\pi_{n}^{\text{qtop}}(X, x) \cong \pi_{n-k}^{\text{qtop}}\left(\varprojlim \Omega^k(X_i, x_i), e_x\right).
\]

Virk [16] introduced a SG (small generated) subgroup of a fundamental group \(\pi_1(X, x)\) denoted by \(\pi_{1}^{SG}(X, x)\) as the subgroup generated by the following elements:

\[
[\alpha * \beta * \alpha^{-1}],
\]

where \(\alpha\) is a path in \(X\) with initial point \(x\) and \(\beta\) is a small loop in \(X\) at \(\alpha(1)\). Recall that a space \(X\) is said to be small generated if \(\pi_1(X, x) = \pi_1^{SG}(X, x)\). Moreover, a space \(X\) is said to be semilocally small generated if, for every \(x \in X\), there exists an open neighborhood \(U\) of \(x\) such that

\[
i_x\pi_1(U, x) \leq \pi_1^{SG}(X, x).
\]

Torabi, et al. [15] proved that if \(X\) is a small generated space, then \(\pi_1^{qtop}(X, x)\) is an indiscrete topological group and the quasitopological fundamental group of a semilocally small generated space is a topological group. By using Theorem 2.1, we obtain several results for quasitopological homotopy groups formulated in what follows:

**Corollary 2.3.** Let \(X\) be a topological space such that \(\Omega^{n-1}(X, x)\) is small generated. Then \(\pi_{n}^{qtop}(X, x)\) is an indiscrete topological group.

**Proof.** Since \(\Omega^{n-1}(X, x)\) is a small generated space, we conclude that \(\pi_{1}^{qtop}(\Omega^{n-1}(X, x), e_x)\) is an indiscrete topological group by Remark 2.11 in [15]. Therefore, the fact that

\[
\pi_{n}^{qtop}(X, x) \cong \pi_{1}^{qtop}(\Omega^{n-1}(X, x), e_x)
\]

implies that \(\pi_{n}^{qtop}(X, x)\) is an indiscrete topological group.

**Corollary 2.4.** Let \(X\) be a topological space such that \(\Omega^{n-1}(X, x)\) is a semilocally small generated space. Then \(\pi_{n}^{qtop}(X, x)\) is a topological group.

**Proof.** Since \(\Omega^{n-1}(X, x)\) is semilocally small generated, we conclude that \(\pi_{1}^{qtop}(\Omega^{n-1}(X, x), e_x)\) is a topological group by Theorem 4.1 in [15]. Therefore,

\[
\pi_{n}^{qtop}(X, x) \cong \pi_{1}^{qtop}(\Omega^{n-1}(X, x), e_x)
\]

implies that \(\pi_{n}^{qtop}(X, x)\) is a topological group.

Fabel [8] proved that \(\pi_1^{qtop}(HE, x)\) is not a topological group. By analyzing the proof of this result, it is possible to conclude that if \(\pi_1(X, x)\) is an Abelian group, then \(\pi_{1}^{qtop}(X, x)\) is a topological group. Fabel [9] also showed that, for any \(n \geq 2\), there exists a compact and path connected metric space \(X\) such that \(\pi_{n}^{qtop}(X, x)\) is not a topological group. In the following example, we show that there is a metric space \(Y\) with Abelian fundamental group such that \(\pi_{1}^{qtop}(Y, y)\) is not a topological group.
Example 2.1. Let \( n \geq 1 \) and let \( X \) be a compact and path connected metric space introduced in [9] such that \( \pi_n^{qtop}(X, x) \) is not a topological group. By Theorem 2.1 \( \pi_1^{qtop}(\Omega^n(X, x), e_x) \) is not a topological group. Since, for every \( n \geq 2 \), \( \pi_n(X, x) \) is an Abelian group, there exists a metric space \( Y = \Omega^{n-1}(X, x) \) with Abelian fundamental group such that \( \pi_1^{qtop}(Y, y) \) is not a topological group.

In [4] (Proposition 3.25), it was proved that the quasitopological fundamental groups of shape-injective spaces are Hausdorff. By Theorem 2.1, we have the following result:

Corollary 2.5. Let \( X \) be a topological space such that \( \Omega^{n-1}(X, x) \) is a shape-injective space. Then \( \pi_n^{qtop}(X, x) \) is Hausdorff.

Proposition 2.1 [15]. For a pointed topological space \( (X, x) \), if \( \{[e_x]\} \) is closed (or, equivalently, if the topology of \( \pi_1^{qtop}(X, x) \) is \( T_0 \)), then \( X \) is homotopically Hausdorff.

We now generalize the above-mentioned proposition as follows:

Corollary 2.6. Let \( X \) be a topological space such that \( \Omega^{n-1}(X, x) \) is a shape-injective space. Then \( X \) is \( n \)-homotopically Hausdorff.

**Proof.** By Theorem 2.1, since \( \pi_n^{qtop}(X, x) \) is \( T_0 \), we conclude that \( \pi_1^{qtop}(\Omega^{n-1}(X, x), e_x) \) is \( T_0 \). Therefore, by the previous proposition, \( \Omega^{n-1}(X, x) \) is homotopically Hausdorff, which implies that \( X \) is \( n \)-homotopically Hausdorff by Lemma 3.5 in [11].

**Theorem 2.2.** Let \( (B, b_0) \) be a pointed space and let \( p : E \to B \) be a fibration with fiber \( F \). Consider its mapping fiber

\[
Mp = \{(e, \omega) \in E \times B^1 : \omega(0) = b_0 \text{ and } \omega(1) = p(e)\}.
\]

If \( \tilde{b}_0 \in p^{-1}(b_0) \), then the injection map \( k : \Omega(B, b_0) \to Mp \) given by \( k(\omega) = (\tilde{b}_0, \omega) \) induces a homomorphism \( f_* : \pi_n(B, b_0) \to \pi_{n-1}(F, \tilde{b}_0) \) [13].

**Proof.** The proof follows from Corollary 2.5 and Proposition 2.2. Let \( (B, b_0) \) be a pointed space and let \( p : E \to B \) be a fibration. If \( \tilde{b}_0 \in p^{-1}(b_0) \), then \( f_* : \pi_n^{qtop}(B, b_0) \to \pi_{n-1}^{qtop}(F, \tilde{b}_0) \) is continuous for all \( n \geq 1 \).

**Proof.** We consider the following commutative diagram:

\[
\begin{array}{ccc}
\Omega^{n-1}(\Omega(B, b_0), e_{b_0}) & \xrightarrow{k_2} & \Omega^{n-1}(Mp, *) \\
\downarrow q & & \downarrow q \\
\pi^{qtop}_{n-1}(\Omega(B, b_0), e_{b_0}) & \xrightarrow{k_*} & \pi^{qtop}_{n-1}(Mp, *)
\end{array}
\]

where \( q \) is a quotient map and \( k_2 \) is the induced map of \( k : \Omega(B, b_0) \to Mp \) by the functor \( \Omega^{n-1} \). Since \( k_2 \) is continuous and \( q \) is a quotient map, \( k_* : \pi^{qtop}_{n-1}(\Omega(B, b_0), e_{b_0}) \to \pi^{qtop}_{n-1}(F, \tilde{b}_0) \) is continuous. By Theorem 2.1, \( \pi^{qtop}_{n-1}(\Omega(B, b_0), e_{b_0}) \) is isomorphic to \( \pi^{qtop}_n(B, b_0) \). Therefore, \( f_* : \pi^{qtop}_n(B, b_0) \to \pi^{qtop}_{n-1}(F, \tilde{b}_0) \) is continuous.
3. Long Exact Sequence of $\pi_n^{qtop}(X)$

Brazas [6] (Theorem 2.49) proved that, for every based pair $(X, A)$ with the inclusion $i: A \to X$, there exists the following long exact sequence in the category of quasitopological groups:

$$\ldots \to \pi_{n+1}^{qtop}(A) \to \pi_{n+1}^{qtop}(X) \to \pi_{n+1}^{qtop}(X, A) \to \pi_n^{qtop}(A) \to \pi_n^{qtop}(X) \to \pi_n^{qtop}(X, A) \to \pi_n^{qtop}(A) \to \pi_n^{qtop}(X) \to \ldots$$

Brazas also showed [6] (Proposition 2.20) that, for every fibration $p: E \to B$ of path-connected spaces with fiber $F$, there exists the following long exact sequence in the category of quasitopological groups:

$$\ldots \to \pi_n^{qtop}(E) \to \pi_n^{qtop}(B) \to \pi_{n-1}^{qtop}(F) \to \pi_n^{qtop}(E) \to \pi_n^{qtop}(B) \to \pi_n^{qtop}(F) \to \pi_n^{qtop}(E) \to \pi_n^{qtop}(B). \tag{3}$$

In what follows, we obtain some results and examples by using these exact sequences.

**Example 3.1.** Consider a pointed pair $(HA, HE)$, where $HA$ is the harmonic archipelago and $HE$ is the hawaiian earring. Then, by [6] (Theorem 2.49), there exists a long exact sequence in $qTop$:

$$\ldots \to \pi_{n+1}^{qtop}(HE) \to \pi_{n+1}^{qtop}(HA) \to \pi_{n+1}^{qtop}(HA, HE) \to \pi_n^{qtop}(HE)$$

$$\ldots \to \pi_1^{qtop}(HA) \to \pi_1^{qtop}(HA, HE) \to \pi_0^{qtop}(HE) \to \pi_0^{qtop}(HA, HE) \to \pi_0^{qtop}(HE) \to \pi_0^{qtop}(HA).$$

Recall that a short exact sequence $E: 0 \to H \xrightarrow{i} X \xrightarrow{\pi} G \to 0$ of topological Abelian groups is called an extension of topological groups if both $i$ and $\pi$ are continuous and open homomorphisms when considered as maps onto their images. In addition, the extension $E$ is called split if and only if it is equivalent to the trivial extension $E_0: 0 \to H \xrightarrow{i_0} H \times G \xrightarrow{\pi_0} G \to 0$ (see [2]).

**Theorem 3.1** ([2], Theorem 1.2). Let $E: 0 \to H \xrightarrow{i} X \xrightarrow{\pi} G \to 0$ be an extension of topological Abelian groups. Then the following assertions are equivalent:

1. $E$ splits.
2. There exists a right inverse for $\pi$.
3. There exists a left inverse for $i$.

The above-mentioned results remain true for the quasitopological groups.

**Proposition 3.1.** If $r: X \to A$ is a retraction, then there are isomorphisms in quasitopological groups; for all $n \geq 2$,

$$\pi_n^{qtop}(X) \cong \pi_n^{qtop}(A) \times \pi_n^{qtop}(X, A).$$

**Proof.** Consider the pointed pair $(X, A)$. By [6] (Theorem 2.49), there exists a long exact sequence

$$\ldots \to \pi_{n+1}^{qtop}(X) \to \pi_{n+1}^{qtop}(X, A) \to \pi_n^{qtop}(A) \xrightarrow{i_*} \pi_n^{qtop}(X) \to \pi_n^{qtop}(X, A) \to \ldots$$
Since $r$ is a retraction and $i_*$ is an injection, there is a short exact sequence
\[ 0 \to \pi_n^{qtop}(A) \xrightarrow{i_*} \pi_n^{qtop}(X) \to \pi_n^{qtop}(X, A) \to 0. \]

Moreover, this sequence is an extension. Indeed, the maps $i_*$ and $\pi_*$ are continuous and open homomorphisms when considered as maps onto their images. Therefore,
\[ \pi_n^{qtop}(X) \cong \pi_n^{qtop}(A) \times \pi_n^{qtop}(X, A). \]

**Proposition 3.2.** Let $B \subseteq A \subseteq X$ be pointed spaces. Then there is a long exact sequence of the triple $(X, A, B)$ in the $qTop$:
\[ \cdots \to \pi_{n+1}^{qtop}(X, A) \to \pi_n^{qtop}(A, B) \to \pi_n^{qtop}(X, B) \to \pi_n^{qtop}(X, A) \to \pi_{n-1}^{qtop}(A, B) \to \cdots. \]

**Proof.** Consider the following commutative diagram and chase a long diagram as follows:

\[
\begin{array}{ccccccccc}
\pi_n^{qtop}(A) & \to & \pi_n^{qtop}(X) \\
\downarrow & & \downarrow \\
\pi_n^{qtop}(A, B) & \to & \pi_n^{qtop}(X, B) & \to & \pi_n^{qtop}(X, A) \\
\downarrow & & \downarrow d & & \downarrow i_* \\
\pi_{n-1}^{qtop}(B) & \to & \pi_{n-1}^{qtop}(A) & \to & \pi_{n-1}^{qtop}(X) \\
\downarrow & & \downarrow i_* & & \\
\pi_{n-1}^{qtop}(A, B) & \to & \pi_{n-1}^{qtop}(X, B). \\
\end{array}
\]

The following results are immediate consequences of sequence (3).

**Corollary 3.1.** If $p : E \to B$ is a fibration with contractible $E$, then $f_* : \pi_n^{qtop}(B, b_0) \to \pi_n^{qtop}(F, \tilde{b}_0)$ is an isomorphism in quasitopological groups for all $n \geq 2$ and $f_* : \pi_1^{qtop}(B, b_0) \to \pi_0^{qtop}(F)$ is an isomorphism in the Set.

**Corollary 3.2.** Let $(X, x)$ be a pointed topological space. Then
\[ \pi_n^{qtop}(X, x) \cong \pi_n^{qtop}(\Omega(X, x), e_x) \]
in quasitopological groups for all $n \geq 2$, where $e_x$ is a constant loop in $X$ at $x$, and
\[ \pi_1^{qtop}(X, x) \cong \pi_0^{qtop}(\Omega(X, x)) \]
in the Set.

**Proof.** By Proposition 4.3 in [14], the map $p : PX \to X$ is a fibration with the fiber $\Omega(X, x)$, where
\[ PX = (X, x)^{I, 0}. \]
By [6] (Proposition 2.20), the sequence

$$\cdots \longrightarrow \pi_n^{\text{qtop}}(PX, e_x) \longrightarrow \pi_n^{\text{qtop}}(X, x) \longrightarrow \pi_{n-1}^{\text{qtop}}(\Omega(X, x), e_x) \longrightarrow \pi_{n-1}^{\text{qtop}}(PX, e_x)$$

$$\cdots \longrightarrow \pi_1^{\text{qtop}}(X, x) \longrightarrow \pi_0^{\text{qtop}}(\Omega(X, x)) \longrightarrow \pi_0^{\text{qtop}}(PX) \longrightarrow \pi_0^{\text{qtop}}(X)$$

is exact in the qTop. By [14] (Proposition 4.4), $(PX, e_x)$ is contractible and, therefore, the required result follows from Corollary 3.1.

**Corollary 3.3.** If $p: \tilde{X} \longrightarrow X$ is a covering projection, then, for all $n \geq 2$,

$$\pi_n^{\text{qtop}}(\tilde{X}) \cong \pi_n^{\text{qtop}}(X)$$

in the quasitopological groups and $\pi_1^{\text{qtop}}(\tilde{X})$ can be embedded in $\pi_1^{\text{qtop}}(X)$.

**Proof.** This result follows from sequence (3) and the fact that the fiber $F$ of the covering projection $p$ is discrete and, therefore, $\pi_n^{\text{qtop}}(F)$ is trivial for all $n \geq 1$.

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