Unitary evolution for a quantum Kantowski–Sachs cosmology

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Abstract

It is shown that like the Bianchi I, V and IX models, a Kantowski–Sachs cosmological model also allows unitary evolution on quantization. It has also been shown that this unitarity is not at the expense of anisotropy. Non-unitarity, if there is any, cannot escape notice here, as the evolution is studied against a properly oriented time parameter fixed by the evolution of the fluid. Furthermore, we have constructed a wave packet by superposing different energy eigenstates, thereby establishing unitarity in a non-trivial way, which is a stronger result than an energy eigenstate trivially giving a time independent probability density. For α = 1, we have proved that the Hamiltonian admits a self-adjoint extension under a reasonable assumption, using a standard theorem. This consolidates the fact that the problem of non-unitarity cannot be a generic problem of anisotropy.

Keywords: quantum cosmology, unitarity, Kantowski–Sachs model, Schutz formalism

1. Introduction

The Universe as a whole, being a viable physical system, is expected to have an underlying quantum theory. The interaction that governs the Universe on a large scale is gravity, which as yet does not have a consistent and universally accepted quantum theory and thus one cannot really look at the quantum Universe as an application of quantum gravity. The alternative, an
attempt to bypass the conceptual difficulties of quantum gravity, is provided by the Wheeler–
deWitt equation [1–3], which is actually the starting point for quantizing a cosmological model,
and forms the subject of quantum cosmology. However, quantum cosmology by itself is not
actually free from conceptual problems as well as some practical difficulties. One continuous
problem is that of the choice of a properly oriented time parameter, as in relativistic theory time
is a coordinate and not a scalar parameter that is the same for all observers. This problem and the
possible resolutions in various ways constitute almost a branch of quantum cosmology [4–7].
Recently, the choice of an oriented time parameter depending on the evolution of a perfect fluid
constituting the matter distribution in the Universe has been in use with a fair amount of success.
The strategy actually had been given long ago by Lapchinskii and Rubakov [8] based on
Schutz’s formalism of writing the velocity vector of the fluid in terms of some thermodynamic
potentials [9, 10]. Other serious conceptual problems, that of the interpretation of the wave
function and that of the boundary conditions, are reviewed by Wiltshire [11], Halliwell [12] and
very recently by Pinto-Neto and Fabris [13].

The problem at a fundamental level is that the Copenhagen interpretation of probability is
found to be suspect in the standard Wheeler–DeWitt quantum cosmology. There are alter-
native attempts, such as an interpretation through the deBroglie-Bohmian perspective [13].
There was a recent attempt to provide a solution to this using a decoherent history formulation
of the Wheeler–DeWitt quantization scheme by Craig and Singh [14].

A nagging problem in quantum cosmology, which we are concerned with in this paper,
had been the fact that anisotropic models are found to be nonunitary and hence do not respect
the conservation of probability. As the collection of the 3-space Riemannian metric, the
Superspace, is too big to work in, the usual practice is to restrict the description in a truncated
version of it, namely a minisuperspace. The natural choice of the minisuperspace is indeed a
spatially homogeneous and isotropic FRW 3-space which is compatible with the presently
observed Universe. But anisotropy has its relevance. The formation of the present structure of
the Universe requires a small but finite anisotropy in the CMB temperature \( \Delta T \sim 10^{-5} \) and
the recent observations also reveal the same order of magnitude of anisotropy. The other
requirement is aesthetic; there is no reason, a priori, for the Universe to be isotropic. So a
correct version of a quantum theory should not lead to inconsistent physics, such as a
nonconservation of probability, even for the anisotropic models. It is interesting to note that
without the existence of a time parameter with the correct orientation, this nonconservation
of probability in quantum anisotropic models is generally obscure and may escape being
detected [15, 16]. This alleged pathology of non-unitarity often ascribed to the hyperbolicity
of the Hamiltonian of the anisotropic cosmologies [17].

Very recently it was shown that the problem of non-unitarity can actually be cured. For a
Bianchi type I model, either by a clever ordering of operators or by a suitable transformation
of coordinates, one can find a self-adjoint extension of the Hamiltonian [18]. In fact this is not
exclusive for a Bianchi I model where the spatial curvature is zero; a similar self-adjoint
extension is possible for anisotropic spaces with negative and positive spatial curvature given
by Bianchi V and IX models as well [19]. It deserves mention that both of these investigations
involve a properly oriented time parameter through the evolution of the fluid as mentioned
earlier. Of course, the construction of the wave packet and the evaluation of the norm depends
on the degree of difficulty of the integration involved, so one cannot work out what may be
called the ‘general’ situation, one actually has to depend on some specific equations of state
for the fluid present. But one counter example is good enough to disprove the fact that alleged
non-unitarity is a generic problem of anisotropic models, and one now has several non-trivial
examples. It deserves mention that even isotropic models actually can show a nonunitary
evolution unless the operator ordering is carefully chosen. Issues of unitarity of a quantized isotropic model in the presence of a scalar field have been discussed very recently by Almeida et al [20].

The motivation of this work is to check if unitary evolution is a possibility in a Kantowski–Sachs (KS) cosmology, which is qualitatively different from Bianchi I, V and IX models. The KS metric represents a homogeneous but anisotropic spacetime and has its own relevance in gravity in various ways, such as the fact that a KS metric is isometric to the metric for the interior of a spherically symmetric black hole. For a concise list of the relevance of a KS metric, we refer the reader to the recent work by Parisi et al [21]. A quantization scheme for a KS metric, although not often discussed, is not completely new. It has been discussed in the context of a non-commutative geometry by García-Compean et al [22], and very recently in the context of a loop quantum cosmology by Joe and Singh [23]. Quantizing a KS cosmology in a more standard version of gravity was discussed long ago by Conradi [24]. The latter includes a fluid, namely a pressure-less fluid in the quantization scheme, but does not indicate anything regarding unitarity.

We show that the Hamiltonian is indeed self-adjoint in the case of a Kantowski–Sachs model with a stiff fluid (i.e. fluid with an equation of state \( P = \rho \), where \( P \) is pressure and \( \rho \) is the density of the fluid) upon choosing a suitable weight factor while defining the norm. An explicit example of a unitary solution has also been obtained. The complete calculations are possible, however, only for a particular equation of state, that of a stiff fluid. For other equations of state, we cannot find any explicit solution owing to the computational difficulty. Nonetheless, we could show that even for such cases \( P \neq \rho \) the Hamiltonian admits a self-adjoint extension under reasonable assumption with a clever ordering of operators.

The general formalism and the Wheeler–deWitt equation for the KS model is given in section 2. Section 3 takes up the case of the particular example with a stiff fluid. In section 4, we arrive at the relevant Wheeler–deWitt equation for the general barotropic fluid and show how a suitable operator ordering can make the Hamiltonian self-adjoint extendible. Section 5 includes some discussions.

2. Quantization of a Kantowski–Sachs cosmological model

We start with the standard Einstein–Hilbert action

\[
\mathcal{A} = \int_M d^4x \sqrt{-g} R + 2 \int_{\partial M} \sqrt{h} h_{ab} K^{ab} + \int_M d^4x \sqrt{-g} P, \tag{1}
\]

where \( K^{ab} \) is the extrinsic curvature, \( h_{ab} \) the induced metric over the boundary \( \partial M \) of the four-dimensional spacetime manifold \( M \) and \( P \) is the fluid pressure, given by \( P = \alpha \rho \), where \( \rho \) is the density of the fluid and \( \alpha \) is a constant. The units are so chosen that \( 16\pi G = 1 \).

The Kantowski–Sachs metric is given by

\[
d^2 s^2 = -n^2 dt^2 + X^2(t) dr^2 + Y^2(t) \left[ d\theta^2 + \sin^2(\theta) d\phi^2 \right], \tag{2}
\]

where \( n(t) \) is a lapse function while \( X(t) \) and \( Y(t) \) are scale factors. In order to facilitate the process of quantization, we go over to the Misner representation via the transformation [25]

\[
X(t) = e^{\sqrt{T} \beta_0}, \tag{3}
\]

\[
Y(t) = e^{-\sqrt{T} (\beta_0 + \beta_1)}. \tag{4}
\]
This transformation allows us to write the metric (2) as
\[ \text{d}s^2 = -n^2 \text{d}t^2 + e^{2\sqrt{3}\beta x} \text{d}x^2 + e^{-2\sqrt{3}(\beta_+ + \beta_-)} [\text{d}\theta^2 + \sin^2(\theta) \text{d}\phi^2]. \] (5)

The new variables \( \beta_+ \) and \( \beta_- \) do not really have much of a new significance. But writing the metric in terms of exponentials results in a simplification in the Lagrangian by avoiding terms like \( \frac{\dot{X}}{X} \) and thus facilitates subsequent separation of variables. One should note that the number of independent variables is not affected in any way; \( (\beta_+, \beta_-) \) replace \( (X, Y) \).

The fluid sector can be expressed in terms of some thermodynamic variables following the Schutz formalism [9, 10], which has been subsequently developed by Lapchinskii and Rubakov [8] in context of quantum cosmology. This method was later used quite extensively [18, 19, 26–28]. In this formalism, the Lagrangian for the system can be extracted out of the action (1) as
\[ \mathcal{L} = \frac{6}{n} e^{-\sqrt{3}(\beta_+ + 2\beta_-)} \left[ \beta_+^2 - \beta_-^2 \right] + 2ne^{\sqrt{3}\beta_+} + \left[ n^{-\frac{\alpha}{2}} e^{\beta_+} \frac{\alpha}{(1 + \alpha)^{\frac{1}{2}}} (\epsilon + \theta S)^{\frac{1}{2}} + \frac{1}{2} e^{\frac{\alpha}{2}} \right], \] (6)

where an overhead dot implies a differentiation with respect to \( t \).

The metric as well as all other quantities are spatially homogeneous, and the integration of space yields a constant in (1) and is thus inconsequential as it can be absorbed in the right hand side zero of the variational principle. The quantities \( \theta, \epsilon \) and \( s \) are thermodynamic potentials that determine the velocity vector. The details are given in [8]. The last term within the square bracket is the contribution from the fluid sector whereas the rest denotes the contribution from the gravity sector. The corresponding Hamiltonian for the gravity sector for the metric (5) can be written as
\[ H_g = \frac{n}{24} e^{\sqrt{3}(\beta_+ + 2\beta_-)} \left[ -p_{\beta_+}^2 + p_{\beta_-}^2 - 48e^{-2\sqrt{3}\beta_+} \right], \] (7)

where the canonical momenta are defined in the usual way. We define the canonical momenta for the fluid sector as \( p_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} \) and \( p_s = \frac{\partial \mathcal{L}}{\partial \dot{s}} \) and the corresponding Hamiltonian is written as
\[ H_f = ne^{-3\alpha n} p_\theta^\alpha e^\epsilon. \] (8)

With the canonical transformation,
\[ T = -p_s \exp(-S)p_\theta^{-\alpha - 1}, \] (9)
\[ p_T = p_\theta^{\alpha + 1} \exp(S), \] (10)
\[ \epsilon' = \epsilon + (\alpha + 1) \frac{p_s}{p_\theta}, \] (11)
\[ p_T' = p_T, \] (12)
the Hamiltonian for the fluid sector becomes
\[ H_f = ne^{\alpha \sqrt{3}(\beta_+ + 2\beta_-)} p_T. \] (13)

Here \( T, \epsilon' \) replace \( S, \epsilon \) and \( p_T \) and \( p_T' \) are the momenta conjugate to the new variables \( T, \epsilon' \) respectively.
Now, combining (7) and (13), the super Hamiltonian is given by

\[ H = H_0 + H_f \]

\[ = \frac{ne^{\alpha_\beta(\beta_+ + 2\beta)}}{24} \left[ e^{\sqrt{3}(1-\alpha)(\beta_+ + 2\beta)} \left\{ -p_{\beta_+}^2 + p_{\beta_+}^2 - 48e^{-2\sqrt{3}\beta} \right\} + 24p_T \right]. \]  

(14)

Now, one can choose a gauge \( n = 24n_0e^{-\alpha_\beta(\beta_+ + 2\beta)} \) and vary (14) with respect to \( n_0 \) to obtain the Hamiltonian constraint as

\[ e^{\sqrt{3}(1-\alpha)(\beta_+ + 2\beta)} \left\{ -p_{\beta_+}^2 + p_{\beta_+}^2 - 48e^{-2\sqrt{3}\beta} \right\} + 24p_T = 0. \]  

(15)

3. Stiff fluid (\( \rho = P \))

As an example, we choose a stiff fluid given by \( \alpha = 1 \). This choice avoids the ordering ambiguity since \( e^{\sqrt{3}(1-\alpha)(\beta_+ + 2\beta)} = 1 \) for \( \alpha = 1 \). On quantizing the model, we have the Wheeler–deWitt equation,

\[ \left\{ \frac{\partial^2}{\partial \beta_+^2} + \frac{\partial^2}{\partial \beta_-^2} - 48e^{-2\sqrt{3}\beta} \right\} \psi = 24\frac{\partial \psi}{\partial T}, \]  

(16)

in units of \( \hbar = 1 \). The canonical momenta \( p_{\beta_+} \) and \( p_T \) are replaced by \( -i\frac{\partial}{\partial \beta_+} \) and \( -i\frac{\partial}{\partial T} \), respectively following the procedure elucidated in [17, 18, 28].

3.1. Probability density

With the separability ansatz \( \psi = e^{i\sqrt{3}k_0\beta_+}\phi(\beta_+)e^{-iET} \), (16) becomes

\[ \left\{ \frac{\partial^2}{\partial \beta_+^2} + 3k_+^2 - 48e^{-2\sqrt{3}\beta} \right\} \phi = 24E\phi, \]  

(17)

which has the solution

\[ \phi = K_0(4e^{-\sqrt{3}\beta}), \]  

(18)

where \( \nu \equiv \sqrt{k_+^2 - 8E} \). It deserves mention that this solution is similar to that obtained by Garcia-Compean et al [22]. The expression for \( \nu \) in this work is different due to the contribution from the fluid sector.

Now, we note that the \( \beta_+ \) sector admits a plane wave like solution. Hence, we might be tempted to form a Gaussian wavepacket superposing different plane wave solutions by doing an integration over \( k_+ \) as the variable. But this is actually not that simple. If we want to construct a wavepacket with definite energy \( E \), then varying \( k_+ \) would imply varying \( \nu = \sqrt{k_+^2 - 8E} \), hence we are forced to superpose \( K_0(4e^{-\sqrt{3}\beta}) \) as well, resulting in a complicated wavepacket, which is not Gaussian at all. This implies that if we intend to form a Gaussian wavepacket corresponding to the \( \beta_+ \) sector, it cannot be an energy eigenfunction, which motivates us to define a new parameter

\[ \sigma \equiv k_+^2 - 8E = \nu^2 \]  

(19)

and trade \((E, k_+)\) with \((\sigma, k_+)\) i.e the wave function can be thought of as dependent on \( k_+ \) and \( \sigma \) independently, and \( E \) becomes a parameter dependent on \( k_+ \) and \( \sigma \) through equation (19).
In terms of $\sigma$, the wave function for the $\beta_-$ sector reads

$$\phi = K_{i, \sigma}(4\chi),$$

where we have defined, for brevity, $\chi \equiv e^{-\sqrt{3}\beta_-}$.

Now we can superpose the plane wave like solutions for the $\beta_+$ sector, keeping $\sigma$ fixed. This, in turn, means the resulting wavepacket will no longer be an energy eigenstate, it is rather a superposition of energy eigenstates. The wavepacket is given by the following expression,

$$\Psi = \phi(\chi)e^{\frac{i}{\hbar} T} \psi(\beta_+),$$

where we define $\psi(\beta_+)$ as

$$\psi(\beta_+) = \int dk_+ e^{-(k_+ - k_{+0})^2} e^{i\sqrt{3}k_+\beta_+ - \frac{k_+^2}{\hbar}}.$$  

(21)

If we want to insist on superposition of negative energy eigenstates only, the integral over $k_+$ should be in the interval $(-\sqrt{\sigma}, \sqrt{\sigma})$. The requirement of a negative $E$ for the gravity sector stems from the demand that the energy for the fluid part is positive whereas the total energy is zero in view of the Hamiltonian constraint. A similar kind of consideration has been made in earlier work [18, 19].

Upon integration over the said interval, we have

$$\psi = \frac{1 + t}{\sqrt{-8t + T}} \exp\left[\frac{-k_{+0}^2 T + 8\sqrt{3}k_{+0} \beta_+ + 6t\beta_+^2}{-8t + T}\right]$$

$$\times \left[g(\sqrt{\sigma}) - g(-\sqrt{\sigma})\right].$$

(23)

where $g(a)$ is given by

$$g(a) = Erf\left[\frac{1 + t}{4} \left(8k_{+0} + a(-8 - tT) + 4\sqrt{3}\beta_+\right)\right]$$

(24)

With the definition $\|\psi\| \equiv \int_{-\infty}^{\infty} d\beta_+ \psi^* \psi$, we have

$$||\psi|| = \frac{1}{\sqrt{3}} \int_{-\sqrt{\sigma}}^{\sqrt{\sigma}} dk_+ e^{-(k_+ - k_{+0})^2}$$

$$= \frac{1}{2\sqrt{3} \sqrt{2}} \left(Erf\left[\sqrt{2}\left(\sqrt{\sigma} + k_{+0}\right)\right] - Erf\left[\sqrt{2}\left(-\sqrt{\sigma} + k_{+0}\right)\right]\right).$$

(25)

We have specifically shown, for the $\beta_-$ sector, that the norm is finite and time independent. This happens non-trivially since we are not dealing with the energy eigenstate. We recall that for the energy eigenstate, the probability density function is trivially time independent. Now we turn our attention to the $\beta_+$ sector. We recast the equation (17) using the variables $\chi$ and $\sigma$:

$$\chi^2 \frac{d^2 \phi}{d\chi^2} + \phi \frac{d\phi}{d\chi} + \left(16\chi^2 - \sigma\right)\phi = 0.$$  

(26)

This equation can be rewritten in the standard self-adjoint form:

$$\frac{d}{d\chi} \left(\chi \frac{d\phi}{d\chi}\right) + \left(16\chi - \frac{\sigma}{\chi}\right)\phi = 0.$$  

(27)
with the definition of the inner product given by

$$\langle \phi_1 | \phi_2 \rangle \equiv \int_0^{\infty} d\chi \phi_1^* (\chi) \phi_2 (\chi).$$  \hspace{1cm} (28)

Hence, the Hamiltonian for the $\beta_-$ sector is self-adjoint as well, ensuring a unitary time evolution. The norm can explicitly be calculated and shown to be finite, since

$$\int_0^{\infty} d\chi \phi^* (\chi) \phi (\chi) = \frac{\pi \nu}{32 \sinh (\pi \nu)}$$

where $\phi (\chi) = K_{\nu} (4 \chi)$ and $\nu^2 = \sigma$.

The relevant plots of the probability density function at different time instants are given in figures 1 and 2. We note that $\phi$ is time independent, and hence the probability density due to the $\beta_-$ sector (we will call it $P_{\beta_-} \equiv \chi \phi (\chi) \phi^* (\chi)$) is time independent, while the probability density due to the $\beta_+$ sector, which we call $P_{\beta_+} = \psi (\beta_+) \psi^* (\beta_+)$, is time dependent (even though the norm is finite and time independent).

We note that the wave function dies out for large $\chi$, which translates to the fact that the probability of having $\beta_- = -\infty$ is zero. Similarly, the probability density dies out as $\beta_+$ approaches $-\infty$. This essentially means that the volume of the Universe, i.e, $e^{\sqrt{\lambda} (\beta_+ + 2 \beta_-)}$, does not hit singularity at any time. Similarly, one can argue that the wave function dies out for $\chi = 0$ which does imply that the probability of large $\beta_-$ is very small. But we note that $P_{\beta_+}$ is time dependent and the peak of $P_{\beta_+}$ does shift to the right with time. Hence, the probability of having large $\beta_+$ is not small if we give enough time for evolution. Hence, the volume can get arbitrarily large with time.

### 3.2. Anisotropy

The scalar $\tilde{\sigma}^2 = \frac{1}{2} \sigma_{\mu \nu} \sigma_{\mu \nu}$ found from shear tensor $\sigma_{\mu \nu}$, can be written for a diagonal metric as

$$\tilde{\sigma}^2 = \frac{1}{12n^2} \left[ \left( \frac{b_{11}}{b_{11}} - \frac{g_{22}}{g_{22}} \right)^2 + \left( \frac{g_{22}}{g_{22}} - \frac{g_{33}}{g_{33}} \right)^2 + \left( \frac{g_{33}}{g_{33}} - \frac{b_{11}}{b_{11}} \right)^2 \right]$$

which yields

$$\tilde{\sigma}^2 = \frac{1}{2} e^{-2 \sqrt{\lambda} (\beta_+ + 2 \beta_-)} \left( 2 p_{\beta_-} - p_{\beta_+} \right)^2 \geq 0,$$  \hspace{1cm} (30)

for the metric (3) where $p_{\beta_\pm}$ are conjugate variables to $\beta_\pm$.

### 4. Models with other equations of state ($0 < \alpha < 1$)

In this section, we turn our attention to $\alpha \neq 1$. For $\alpha = 1$ we rewrite (15) with the following operator ordering.

$$\left[ \frac{\sqrt{\lambda}}{2} (1-\alpha) \beta_+ \beta_- p_{\beta_+} \right] \left[ \frac{\sqrt{\lambda}}{2} (1-\alpha) \beta_+ \beta_- p_{\beta_+} \right] - e^{\sqrt{\lambda} (1-\alpha)(\beta_+ + 2 \beta_-)} \left( p_{\beta_+} e\sqrt{\lambda} (1-\alpha) \beta_+ p_{\beta_+} \right. - 48 e^{-2\sqrt{\lambda} \beta_-} e\sqrt{\lambda} (1-\alpha)(\beta_+ + 2 \beta_-) = 0. \hspace{1cm} (31)$$

The guiding principle behind choosing such operator ordering is to ascertain the self-adjoint extension, which is explained later in this section.
By replacing the momenta by the corresponding operators as described before, we find the Wheeler–DeWitt equation as

$$\left[ \frac{-\text{e}^{\frac{\sqrt{3}}{2}(1-\alpha)(\beta_+ + 4\beta_-)}}{2} \frac{\partial}{\partial \beta_+} - \frac{\text{e}^\frac{\sqrt{3}}{2}(1-\alpha)\beta_-}{2} \frac{\partial}{\partial \beta_-} + \text{e}^{\frac{\sqrt{3}}{2}(1-\alpha)(\beta_+ + 2\beta_-)} \right] \Psi = 24i\hbar \frac{\partial \Psi}{\partial T}. $$

We now effect a transformation of variables as

$$\chi_+ \equiv \text{e}^{\frac{\sqrt{3}}{2}(1-\alpha)\beta_+},$$

$$\chi_- \equiv \text{e}^{-\frac{\sqrt{3}}{2}(1-\alpha)\beta_-}.$$  

and use separability ansatz $\Psi = \phi(\chi_+, \chi_-) \text{e}^{-i\mathcal{H}T}$ to obtain

$$\mathcal{H}_\phi \phi = -\frac{1}{\chi_-^2} \frac{\partial^2 \phi}{\partial \chi_-^2} + \frac{1}{\chi_+^2} \frac{\partial^2 \phi}{\partial \chi_+^2} - 48\chi_+^2 \frac{\partial}{\partial \chi_+} - 24E \phi. $$

This equation is not apparently separable in $\chi_\pm$ and thus one cannot investigate the behavior analytically. But under a reasonable assumption, we can show that the Hamiltonian given by (35) actually admits a self-adjoint extension. The assumption is that the Hamiltonian is a symmetric operator, by which we mean that the solution (if it admits any) obeys following conditions

$$\left[ 0 \frac{\partial \phi}{\partial \chi_\pm} - \phi \frac{\partial \phi}{\partial \chi_\pm} \right]_0 = 0.$$

To facilitate a clear view, we can think of this assumption in the following way. Imagine that we obtain the solution to equation (35) and restrict the solution space so as to satisfy equation (36). Now what we call a reasonable assumption is that this restricted solution space is not null. This is actually quite natural in the context of quantum mechanics, and we assume similar conditions while we work on a free particle on a half-line or the whole real line. In case of the whole line, 0 in the lower limit of equation (36) is replaced by $-\infty$. The condition is reasonable in the present context, since, for example, any normalizable solution that vanishes at $\beta_\pm = \pm \infty$ will satisfy the condition.

Now we can use Von Neumann’s theorem [29] asserting that a symmetric operator $\hat{A}$ defined on domain $\mathcal{D}$ has an equal deficiency index, if there exists a norm preserving the anti-unitary conjugation map $C : \mathcal{D} \to \mathcal{D}$ such that $[\hat{A}, C] = 0$, which, in turn, shows that $A$ admits self-adjoint extension.
It is easy to see that all the conditions for employing Neumann’s theorem are satisfied.

i. Complex conjugation map, \( C : \mathcal{H}_i \rightarrow \mathcal{H}_i \), between Hilbert space \( \mathcal{H}_i \), takes \( \phi \) to \( \phi^* \), which also belongs to the Hilbert space \( \mathcal{H}_i \).

ii. \( C \) is norm preserving, since whatever the definition of the norm, it involves \( \phi \phi^* \), and hence does not change under \( C \).

iii. \( C \) is anti-unitary and we have \( C \mathcal{H}_i = \mathcal{H}_i C \).

Hence, the theorem goes through and we have self-adjoint extension and thereby a unitary evolution for Kantowski–Sachs cosmology. The theorem can be understood in following manner as well. Since, \( \phi_+ \) satisfies

\[
H_x \phi_+ = i \phi_+ ,
\]

if and only if

\[
H_x \phi_- = -i \phi_- ,
\]

where \( \phi_- = \phi_+^* = C\phi_+ \), the map \( C \) induces a one-to-one map between two spaces whose dimensions are actually named the Deficiency Index and thereby make them equal, and we know that if a symmetric operator has equal deficiency indices, it does admit a self-adjoint extension. A detailed and rigorous proof can be found in [29]. It deserves mentioning that the extension may modify the boundary condition (36) by making it more strict (i.e., the modified condition will imply equation (36), not necessarily implied by equation (36)).

The role of operator ordering is crucial in the sense that with such an ordering we have the kinetic term \( \frac{\partial^2 \phi_+}{\partial \chi_+^2} \) multiplied with \( \chi_+^2 \). Hence, the condition (36) for its being symmetric is the same as the condition for a standard Laplacian since the derivative with respect to the \( \chi_+ \) term is multiplied with \( \chi_- \) and vice versa.

5. Discussion

It has been shown that like anisotropic cosmological models with constant spatial curvature, such as Bianchi I, V and IX, the Kantowski–Sachs cosmology can also be quantized where the problem of non-unitarity can be successfully eradicated. The trick is to figure out the correct weight factor while defining the norm and the inner product. This has been explicitly demonstrated for a stiff fluid given by \( P = \rho \), i.e., for \( \alpha = 1 \). We have calculated the shear scalar \( \tilde{\sigma}^2 \) which is a positive definite quantity indicating that this unitarity restoration is not at the cost of anisotropy itself. In the case of a Bianchi I quantum cosmology, it had already
been shown that the unitarity is not purchased at the expense of anisotropy [30]. For $\alpha = 1$, however, we could not solve the equation due to inseparability but we have shown that the Hamiltonian admits a self-adjoint extension (with a suitable operator ordering). Thus the model can actually have a unitary evolution for any ideal barotropic ($P = \alpha \rho$) fluid.

One explicit example, namely that with a stiff fluid, along with the implicit proof for non-stiff fluids is good enough to disprove the myth that anisotropic quantum cosmology necessarily involves non-unitarity. Here we have shown that the Hamiltonian is indeed self-adjoint for $\alpha = 1$, thereby has to admit a unitary evolution. Furthermore, we have constructed an explicit wave packet showing unitary evolution explicitly. This restoration of unitarity is non-trivial in nature since the wave packet leads to a time dependent probability density, which integrates to a time independent finite quantity, identified as the norm, leading to a conservation of probability. This scenario is in complete contrast with having energy eigenstates which trivially yields a time independent probability density and thereby a time independent norm irrespective of the self-adjointness of the underlying Hamiltonian. Energy eigenstates, thus, may not actually serve as a signature of unitarity.

For $\alpha \neq 1$, the proof is implicit in nature and it does not shed light on how to obtain a solution or how to do a self-adjoint extension. Yet it is good enough to consolidate the idea that non-unitarity is not generic to anisotropic models, and it can be cured with suitable operator ordering or introducing a correct weight factor.

A strong motivation behind quantizing any cosmological model is to look for a singularity free model. For example, Ashtekar et al [31] showed that a spatially flat FRW cosmology with a scalar field indeed has a bounce from a nonzero minimum for the volume in loop quantum cosmology (LQC). Singh also showed that any singularity of a zero or diverging proper volume or the singularity of an infinite curvature can possibly be alleviated in the realm of LQC [32]. In fact the bounce from a minimum volume could be quite generic in LQC, at least in isotropic models [33]. There is a comprehensive recent review of the potential of LQC regarding the resolution of singularity (and also many other developments of the subject) by Ashtekar and Singh [34]. It is interesting to note that the stiff fluid model in Kantowski–Sachs spacetime, as presented here, can actually give rise to a singularity free cosmology at the quantum level. This is evident from the discussion on the probability of the proper volume. This stems from the fact that scale factors have some intrinsic quantum fluctuation around its average value and the wave function goes to zero whenever the scale factor hits 0. It may be interesting to investigate whether the results obtained in this work in the standard Wheeler–DeWitt scheme could be generalized to LQC.

References

[1] DeWitt B S 1967 Phys. Rev. 160 1113
[2] Wheeler J A 1968 Superspace and the nature of quantum geometrodynamics Batelle Recontres: 1967 Lectures in Mathematics and Physics ed C M DeWitt and J A Wheeler (New York: Benjamin)
[3] Misner C W 1969 Phys. Rev. 186 1319
[4] Kuchar K V 1991 Conceptual Problems in Quantum Gravity ed A Ashtekar and J Stachel (Boston: Birkhause)
[5] Isham C J 1993 Integrable Systems, Quantum Groups and Quantum Field Theory ed L A Ibort and M A Rodriguez (Dordrecht: Kluwer)
[6] Rovelli C 2011 Found Phys. 41 1475
[7] Anderson E 2012 Classical and Quantum Gravity: Theory, Analysis and Applications ed V R Frignanni (New York: Nova) chap 4; arXiv:gr-qc/1009.2157
[8] Lapchinskii V G and Rubakov V A 1977 Theor. Math. Phys. 33 1076
[9] Schutz B F 1970 Phys. Rev. D 2 2762
[10] Schutz B F 1971 Phys. Rev. D 4 3559
[11] Wiltshire D L 1995 An introduction to quantum cosmology Cosmology: The Physics of the Universe ed B Robson, N Visvanathan and W S Woolcock (Singapore: World Scientific) pp 473–531
[12] Halliwell J J 1991 Quantum Cosmology and Baby Universes ed S Coleman, J B Hartle, T Piran and S Weinberg (Singapore: World Scientific)
[13] Pinto-Neto N and Fabris J C 2013 Class. Quant. Grav. 30 143001
[14] Craig D A and Singh P 2010 Phys. Rev. D 82 123526
[15] Lidsey J E 1995 Phys. Lett. B 352 207
[16] Pinto-Neto N, Velasco A F and Collistete R Jr 2000 Phys. Lett. A 277 194
[17] Alvarenga F G, Batista A B, Fabris J C, Lemos N A and Goncales S V B 2003 Gen. Relativ. Gravit. 35 1639
[18] Pal S and Banerjee N 2014 Phys. Rev. D 90 104001
[19] Pal S and Banerjee N 2014 arXiv:1411.1167 [gr-qc]

Pal S and Banerjee N 2015 Phys. Rev. D 91 044042
[20] Almeida C R, Batista A B, Fabris J C and Moniz P R L V Gravit. Cosmol. 21 191
[21] Parisi L, Radicella N and Vilasi G Phys. Rev. D 91 063533
[22] Garcia-Compean H, Obregon O and Ramirez C 2002 Phys. Rev. Lett. 88 161301
[23] Joe A and Singh P 2015 Class. Quant. Grav. 32 015009
[24] Conradi H-D 1995 Class. Quant. Grav. 12 2423
[25] Bastos C, Bertolami O, Costa Dias N and Prata J N 2008 Phys. Rev. D 78 023516
[26] Alvarenga F G and Lemos N A 1998 Gen. Relativ. Gravit. 30 681
[27] Alvarenga F G, Fabris J C, Lemos N A and Monerat G A 2002 Gen. Relativ. Gravit. 34 651
[28] Majumder B and Banerjee N 2013 Gen. Relativ. Gravit. 45 1
[29] Reed M and Simon B 1975 Methods of Modern Mathematical Physics vol 2, 2nd edn (San Diego, CA: Academic)
[30] Pal S 2015 arXiv:1504.02912
[31] Ashtekar A, Corichi A and Singh P 2008 Phys. Rev. D 77 024046
[32] Singh P 2009 Class. Quantum Grav. 26 125005
[33] Date G and Hossain G M 2005 Phys. Rev. Lett. 94 011302
[34] Ashtekar A and Singh P 2011 Class. Quantum Grav. 28 213001