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1. Introduction

According to data and statistics in a global summary of the Acquired immune deficiency syndrome (AIDS) epidemic from The World Health Organization (WHO), by the end of 2007 an estimated 33 million people worldwide were living with human immunodeficiency virus, HIV. That same year, some 2 million died of AIDS, and the number of people receiving antiretroviral therapy (ART) was reported in 2.990.000, while an estimated of about 9.700.000 the people needing ART. In other words, globally, less than one person in five at risk of HIV has access to basic HIV prevention services. The same study indicates a total 31% as the ART coverage at that same period (WHO, 2007).

Highly Active Antiretroviral Therapy (HAART) has demonstrated to be effective at slowing the progression of (HIV) infection to Acquired immune deficiency syndrome (AIDS) and, subsequently, to improve quality of life for infected people. However, if on the one hand the cocktail of drugs has been making possible to extend patient’s lives, on the other hand the many problems associated with it and its high cost, particularly to poor people are a clear indication that new approaches to address the situation are needed. Most efforts to control HIV replication has been focused on developing and optimizing antiretroviral therapies.

The immune system of human beings contains different types of cells that help protect the body from infections. One of these types of specialized cells are called Cluster of Differentiation Antigen 4 (CD4) or T-cells, by the fact that CD4 is a glycoprotein predominantly found on the surface of helper T cells. The Human Immunodeficiency Virus (HIV) is a retrovirus and therefore it needs cells from a host so that it can make copies of itself. The CD4 cells are receptors for HIV and they aid the virus to initiate its replication process by enabling it to enter into its host. HIV is essentially considered as an infection of the immune system in the sense that this virus infects and damages CD4 during the virus replication process. The more virus is produced by infected cells, the higher is the viral load and consequently, lower will be the number of functioning CD4 cells. When this number of uninfected cells declines below a critical value, the immune system is seriously deteriorated by HIV.
Definition 1. The equilibrium point $x = 0$ of (3) is such that $V(0) < 0$, $V(x) < 0$ for $x \neq 0$, and $V(x) \rightarrow 0$ as $t \rightarrow \infty$. Since $\Omega$, the domain of attraction, is a compact set (is closed and bounded since it is contained in $R^n$), there is $r > 0$ such that $\{ x \in D : \|x\| < r \} \subset B_x(\delta)$ and $V(\delta) < 0. \quad (6)$

This follows from (5) since $V$ and $\partial V/\partial x$ are negative, $V(x) < \beta$ for $\|x\| < \delta$, $\|x - \alpha\| < \beta$, and $\beta < \alpha$. Then, $x_1$ is a unique solution defined for all $t \geq 0$. The function $V(x)$ is continuous and $\partial V / \partial x$ is the system's equation. Hence, if $\|x\| < \delta$ can be chosen such that $\|x_1\| < \delta \leq \delta$, $\|x\| < \beta - \alpha$, there is $\|x\| < \beta\|x\|$, and $\|x\| < \beta < \alpha$. Moreover, if $\|x\| < \beta$, $\|x\| < \beta - \alpha$, there is $\|x\| < \beta\|x\|$, and $\|x\| < \beta < \alpha$. The equilibrium point $x = 0$ is stable. Moreover, if $\|x\| < \beta$, $\|x\| < \beta - \alpha$, there is $\|x\| < \beta\|x\|$, and $\|x\| < \beta < \alpha$. Then, $x = 0$ is asymptotically stable. If $\|x\| < \delta$, $\|x\| < \beta - \alpha$, there is $\|x\| < \beta\|x\|$, and $\|x\| < \beta < \alpha$. Then, $x = 0$ is unstable. If $\|x\| < \beta$, $\|x\| < \beta - \alpha$, there is $\|x\| < \beta\|x\|$, and $\|x\| < \beta < \alpha$. Then, $x = 0$ is unstable.
Then $B \subseteq \Omega \subseteq B^r$, and $x(0) \in B \delta \Rightarrow x(0) \in \Omega \beta \Rightarrow x(t) \in \Omega \beta \Rightarrow x(t) \in B$. This theorem does not mean that such a stability property cannot be established by a Lyapunov function candidate. The origin is stable if there is a continuously differentiable positive definite function $V(x) = x^t(x_0) \leq \parallel x \parallel^r \leq \parallel x \parallel < a$. By continuity of $V(x)$, there is $\parallel x \parallel < a$ such that $\parallel x \parallel \parallel \dot{V}(x) \parallel = \parallel x \parallel \leq \parallel V \parallel (0) \leq \parallel x \parallel$. Therefore, $\parallel x \parallel < a$ implies that the trajectory $\parallel x \parallel < a$ as $t \to \infty$. This shows that the equilibrium point $x(0)$ is stable. Moreover, if $c \parallel x \parallel \geq \parallel \dot{x} \parallel$, then $\parallel \dot{x} \parallel < c \parallel x \parallel$. If $c \parallel x \parallel < \parallel \dot{x} \parallel$, then $\parallel \dot{x} \parallel < c \parallel x \parallel$.

Remark 1. The origin is stable if there is a positively invariant set $\Omega$ that is bounded from below by zero, and bounded from above by $V(x) = x^t(x_0)$. The principle and the Barbashin-Krasovskii theorem.

Remark 2. The origin is asymptotically stable if there is a continuously differentiable positive definite function $V(x) = x^t(x_0) \leq \parallel x \parallel^r \leq \parallel x \parallel < a$. By continuity of $V(x)$, there is $\parallel x \parallel < a$ such that $\parallel x \parallel \parallel \dot{V}(x) \parallel = \parallel x \parallel \leq \parallel V \parallel (0) \leq \parallel x \parallel$. Therefore, $\parallel x \parallel < a$ implies that the trajectory $\parallel x \parallel < a$ as $t \to \infty$. This shows that the equilibrium point $x(0)$ is asymptotically stable. Moreover, if $c \parallel x \parallel \geq \parallel \dot{x} \parallel$, then $\parallel \dot{x} \parallel < c \parallel x \parallel$. If $c \parallel x \parallel < \parallel \dot{x} \parallel$, then $\parallel \dot{x} \parallel < c \parallel x \parallel$.
Consider a quadratic Lyapunov function candidate
\[ V(x) = x^T P x, \]
where \( P \) is a real symmetric positive definite matrix. The derivative of \( V \) along the trajectories of the linear system (8) is given by
\[ \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T (PA + A^T P) x = -x^T Q x. \]

Asymptotic stability of the origin can be also investigated using Lyapunov's equation, as it is shown on corollary 4.

**Corollary 4.** A matrix \( A \) is a stability matrix, that is, \( \text{Re}(\lambda_i) < 0 \) for all eigenvalues of \( A \), if and only if for any given positive definite symmetric matrix \( Q \) there exists a positive definite symmetric matrix \( P \) that satisfies the Lyapunov equation
\[ PA + A^T P = -Q. \]

If \( Q \) is positive definite, then the origin is asymptotically stable, that is, \( \text{Re}(\lambda_i) < 0 \) for all eigenvalues of \( A \). Here it follows the usual procedure of Lyapunov's method, where it chooses \( V(x) \) to be positive definite and then checks negative definiteness of \( \dot{V}(x) \).

**Remark 4.** If matrix \( A \) is a stability matrix, then \( P \) is a unique solution of (9).

The Lyapunov equation can be used to test whether or not a matrix \( A \) is a stability matrix, as an alternative to calculating the eigenvalues of \( A \). The existence of a Lyapunov function will allow us to draw conclusions about the system when the linear term \( Ax \) is perturbed, whether such perturbation is a linear perturbation in the coefficients of \( A \) or a nonlinear autonomous perturbation. The following lemma is known as Lyapunov's indirect method or Lyapunov first method.

**Lemma 2.** Let \( x = 0 \) be an equilibrium point for the nonlinear system
\[ \dot{x} = f(x), \]
where \( f: D \rightarrow \mathbb{R}^n \) is continuously differentiable and \( D \) is a neighbourhood of the origin. Let
\[ A = \delta f(x) \delta x \bigg|_{x=0}. \]

Then, the origin is asymptotically stable if \( \text{Re}(\lambda_i) < 0 \) for all eigenvalues of \( A \).

**Proof.** Let \( A \) be a stability matrix. Then, by corollary 4 it is known that for any positive definite symmetric matrix \( Q \), the solution \( P \) of the Lyapunov equation (9) is positive definite. Consider \( V(x) = x^T P x \), as a Lyapunov function candidate for the nonlinear system. The derivative of \( V \) along the trajectories of the system is given by
\[ \dot{V}(x) = x^T P \delta f(x) \delta x + \delta f^T(x) \delta x^T P x = x^T (PA + A^T P) x = -x^T Q x + 2 x^T P g(x). \]

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The first term on the right-hand side is negative definite, while the second term is indefinite. But the function $g$ satisfies

$$\forall \|x\| > D \text{ and } t = 0,$$

$$\|g(x)\| < k \delta \leq \frac{1}{2} \|x\|,$$

where $\delta$ can be chosen such that

$$\|g(x)\| + \|Qx\| < \|x\|,$$

Since $\|Qx\| \geq \lambda_{min} \|x\|$, we have $\lambda_{min} \geq 0$ for all $t$, and

$$\|g(x)\| < \|x\| - \epsilon \|x\|,$$

for all $t$, $\epsilon > 0$, and any $t \geq 0$. Hence, after majorize the right-hand side $\|\dot{x}\| < \|x\|$, for all $t$, $\epsilon > 0$, and any $t \geq 0$. Therefore, the stability behavior of the equilibrium point will be dependent on $\lambda_{min}$.
Definition 4. A scalar function \( w(r) \in \mathbb{R} \) is said to be positive definite, if it is continuous and decreasing with respect to \( r \), and for each \( s \), the function \( u(s) = \max \{0, w(r)\} \) is continuous with respect to \( r \) and \( u(s) \) is decreasing with respect to \( s \).

The equilibrium point \( x_0 \) is uniformly asymptotically stable if inequality (12) is satisfied for any initial state \( x(0) \), and \( x(t) \rightarrow x_0 \) as \( t \rightarrow \infty \).

The following corollary states some properties of positive definite functions.

Corollary 5. Let \( w(r) \in \mathbb{R} \) be a positive definite function. The following stability properties of the origin are given.

1. Uniformly stable, if there exists a positive definite function \( w(r) \in \mathbb{R} \) such that for each \( s \), \( u(s) = \max \{0, w(r)\} \) is continuous with respect to \( r \) and \( u(s) \) is decreasing with respect to \( s \).
2. Uniformly asymptotically stable, if there exist a positive definite and decreasing function \( u(s) \in \mathbb{R} \) and a positive constant \( r > 0 \) such that \( \|x(t)\| \leq r \cdot \|x(0)\| \) for all \( t > 0 \), where \( \|x(t)\| \) denotes the Euclidean norm of \( x(t) \).
3. Globally uniformly asymptotically stable, if inequality (12) is satisfied for any initial state \( x(0) \).
4. Exponentially stable, if inequality (12) is satisfied with \( \|x(t)\| \leq r \cdot \|x(0)\| \cdot e^{-\gamma t} \) for all \( t > 0 \), where \( \gamma > 0 \) is a positive constant.

The following corollary defines a scalar solution of a special equation.

Corollary 6. Consider the scalar differential equation

\[ \dot{y}(t) = ky(t) \quad \text{with the aid of an auxiliary scalar differential equation.} \]

The scalar function \( y(t) \) is radially unbounded if \( w(r) \) is continuous with respect to \( r \), and for each fixed \( r \), \( u(s) = \max \{0, w(r)\} \) is continuous with respect to \( r \) and \( u(s) \) is decreasing with respect to \( s \).

The equilibrium point \( x_0 \) is uniformly asymptotically stable if inequality (12) is satisfied for any initial state \( x(0) \), and \( x(t) \rightarrow x_0 \) as \( t \rightarrow \infty \).

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Exponential Equilibria and Uniform Boundedness of HIV Infection Model

Lyapunov stability theorems give sufficient conditions for stability, asymptotic stability, and equilibrium point. We can no longer study stability of the origin as an equilibrium point, nor actually constructing auxiliary functions that satisfy the conditions of the respective theorems. Theorem 1.

The following Lyapunov like theorem is useful to show uniform boundedness. Let \( V(t, x) \) be continuous in \( t \) and locally Lipschitz in \( x \). Let \( \rho \langle \rangle \) be continuous in \( t \) and locally Lipschitz in \( x \).

By definition 6, it is necessary to prove that the origin is uniformly asymptotically stable for all \( \mu \) and \( b \), and for every \( \alpha \), \( \beta \), and \( \gamma \), the perturbation terms \( h(t, x) \) are indeed necessary (Hahn, 1967; Krasovskii, 1967). The converse theorems are proved by so on. They do not say whether the given conditions are also necessary. There are converse theorems which state that if an equilibrium is uniformly asymptotically stable, then the conditions of the theorem are satisfied. This is useful in practice because it is often easier to verify the conditions than to construct auxiliary functions.

If we assume that \( V(t, x) \) is positive definite and decreasing, then there exists a positive definite and decreasing function \( u(t, x) \) such that

\[
\frac{\partial}{\partial t} V(t, x) + \frac{\partial}{\partial x} V(t, x) \leq r(t, x) - s(t, x)
\]

where \( r(t, x) \) and \( s(t, x) \) are positive definite functions.

The solution of

\[
\frac{\partial}{\partial t} \rho(t, x) + \frac{\partial}{\partial x} \rho(t, x) \leq \mu(t, x) - \beta(t, x)
\]

is said globally uniformly bounded if (14) holds for arbitrarily large \( \rho(t, x) \).

Finally, we have shown that the solution of the perturbed non-autonomous system (2), may not be an equilibrium point. We can no longer study stability of the origin as an equilibrium point, nor actually constructing auxiliary functions that satisfy the conditions of the respective theorems.
Let $\Omega \subset \mathbb{R}^n \times [0, T]$ and $\rho \in C^1([0, T], \mathbb{R})$.

It is clear that for all $t \geq 0$, $w(\|x\|) \leq w(\rho)$ must hold.

Therefore, any solution starting in $\Omega$ for all $t \in [0, T]$.

Furthermore, for any $t \in [0, T]$, $\Omega$ never enters $\rho$.

This time $\Omega$ is decreasing. Therefore, the solution starting at $0$ for all $t \in [0, T]$.

Now, it will assume that $\Omega(\|x\|)$.

Then, it will assume that $\|x\| = \rho$.

Finally, for any $t \in [0, T]$, the solution

satisfy the auxiliary autonomous first order differential equation.

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Finally, for any $t \in [0, T]$, the solution
Subsequently, the function $u$ given by
\begin{equation}
\|u\| < c
\end{equation}
uniformly asymptotically stable. The exponentially decaying for
\begin{equation}
(12)
\end{equation}
is satisfied for all
\begin{equation}
\|x\| < \rho
\end{equation}
\begin{equation}
\{\|x\| < \rho, \|w\| < r, \|\sigma\| < \zeta\}
\end{equation}
Further, for scalar function $\|x\| < \rho$, sometimes, it is possible to combine inequalities (17) and (18) in one inequality
\begin{equation}
\|x\| < \rho
\end{equation}
Now, let us illustrate how Theorem 1 is used in the analysis of the perturbed system (2), when
\begin{equation}
\|x\| < \rho
\end{equation}
Hence, the positive definite and decreasing function $\sigma$ is given by
\begin{equation}
\|x\| < \rho
\end{equation}
Hence, the origin of the nominal system is exponentially stable and the system has a uniform bounded solution.
Exponential Equilibria and Uniform Boundedness of HIV Infection Model

Table 1. Parameters for HIV model

| Parameter Description                              | Value/units       |
|---------------------------------------------------|-------------------|
| Source term for $s$ uninfected cells              | $10^{-3}$ per day  |
| Death rate of $d$ $T$ uninfected cells            | 0.02 per day        |
| Infection rate of $\beta$ free virus particles    | $2.4 \times 10^{-5}$ mm$^{-3}$ per day |
| Death rate of $\delta$ infected cell              | 0.24 per day        |
| Rate of virions $p$ produced per $100$ per day    | $c$ free particle |
| $T$ Time                                          | days              |

3.1 Linearisation on infected and uninfected equilibrium point

On reference (Santos & Middleton, 2008), both equilibrium points (21) and (22) were studied. The uninfected state (21), see parameter values on Table 1, is an unstable equilibrium, where even a small perturbation (e.g. introduction of HIV virus to system's dynamic) leads to divergence. For infected state in (22), it is concluded that the infected equilibrium is locally stable for the parameter values given on Table 1. The qualitative behavior of a nonlinear system near an equilibrium point can be determined via linearisation (Khalil, 2002). The system can be linearised by computing the Jacobian which for (20) is given by

$$
A = \frac{\partial f}{\partial x} x = \begin{bmatrix} T & \tilde{T} & V \\ -\left(dT + \beta V\right) & 0 & -\beta T \\ \beta V - \delta \beta T & 0 & p - c \end{bmatrix},
$$

where $A$ is a stability matrix for the evaluation of infected state given in (22) that leads to the characteristic polynomial

$$
|\lambda I - A| = \lambda^3 + a_1 \lambda^2 + a_2 \lambda + a_3
$$

where $a_1 = (c + \delta + \beta s p \delta c)$, $a_2 = \beta sp \delta c (c + \delta)$, and $a_3 = (\beta sp - \delta cd T)$. From (24), the Hurwitz stability conditions are $a_k > 0$, for $k = 1, 2, 3$ and $a_1 a_2 - a_3 > 0$. This stability domain is very conservative, because of the local behavior about the equilibrium point in (22). In next subsection, for equilibrium point in (22), we need to probe the exponentially uniform stable property and describe boundedness of the region of attraction.

3.2 Non autonomous perturbation analysis

Now, it is necessary to study the dynamics for the perturbation and to determine the extent of stability region to know how large a perturbation from the equilibrium can be allowed and it
Consider the following change of variables in (20):

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \delta(t) = \begin{bmatrix} 0 \\ 0 \\ d(t) \end{bmatrix}, \quad \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]

The linearisation about the origin

\[ \dot{x} = Ax, \quad x(0) = x_0, \]

is bounded disturbance that satisfies

\[ x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad \beta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \delta(t) = \begin{bmatrix} 0 \\ 0 \\ d(t) \end{bmatrix}, \quad \lambda = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \theta = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \]

The converse theorem of Lyapunov is based on linearisation about the origin, theorem supposes that matrix

\[ A \]

is Hurwitz when

\[ 0 \]

is unknown,

\[ d \]

evaluating its derivative along the trajectories of the nominal system (27) such that

\[ V(x) \leq \beta \parallel x \parallel \lambda, \quad \parallel \dot{V} \parallel \leq c \parallel x \parallel \beta. \]

The HIV model equations (20) can be rewritten as a perturbed model

\[ \dot{x} = P \delta(t) - x + \lambda \theta(t), \]

where

\[ \parallel \dot{V} \parallel \leq c \parallel x \parallel \beta. \]

By solving the Lyapunov equation

\[ PA + AP + Q = 0, \]

the stability analysis of matrix

\[ A \]

in (27) is described by matrix

\[ V(x) \leq \beta \parallel x \parallel \lambda, \quad \parallel \dot{V} \parallel \leq c \parallel x \parallel \beta. \]

Suppose the perturbation

\[ \parallel x \parallel \leq \parallel x_0 \parallel \delta(t), \quad \parallel \dot{x} \parallel \leq \parallel x_0 \parallel \delta(t). \]

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It is possible to find the unique solution with $\theta$, $\zeta$, $M$, and $s$

\[
\theta \zeta \quad M \quad s
\]

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Securing Connection...

or

for being a positive definite scalar function $P$

The candidate Lyapunov function $V$

Remark 6.

taking the parameter values in Table 1 the matrix $P$

where $f$

A candidate Lyapunov function $V$

Then, the function $\lambda$

$P$

$V$

$\min$

$\partial$

$V$

$\partial$

$x$

$V$

$T$

$\lambda$

$P$

$Ax$

$g$

$1$. $\lambda$

$\min$

$\partial$

$V$

$x$

$\partial$

$x$

$\partial$

$x$

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$V$

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where \( M = \beta \times 10^3 \frac{2}{(1 - \zeta)} \) > 0, 0 < \( \theta < 1, 0 < \zeta < 1 \).

Therefore, the function \( \dot{V}(x) \) is negative definite inside the ball \( \|x\| < r \sqrt{\frac{c_1}{c_2}} \). The ball defines the region of attraction for the solution, when condition (32) is satisfied. It is concluded that the origin \( x = 0 \) is exponentially uniform stable and the solution for system (20) is uniformly bounded in the large for disturbances that satisfy \( |d(t)| \leq \delta, \) for all \( t \geq 0 \).

3.3 Simulation of trajectories and the region of attraction

For the bound given in (32), the following simulations are shown, with initial condition \( T(0) = 520, \tilde{T}(0) = 0, V(0) = 1 \). The phase space for system (20) is depicted in Figure 1 for parameter values \( s = 10, \beta = 2.4 \times 10^{-5} \).

Fig. 1. Phase space for:
\( s = 10, \beta = 2.4 \times 10^{-5} \) with three initial conditions (order from above to bottom):

a) \( T(0) = 180.2, \tilde{T}(0) = 77.6, V(0) = 3305 \);

b) \( T(0) = 180.2, \tilde{T}(0) = 0, V(0) = 500 \);

c) \( T(0) = 180.2, \tilde{T}(0) = 0, V(0) = 1 \).

The result describing the region of attraction is useful for the clinical personal studying HIV behavior, since it allows to predict the infection development and then choose treatment options. This model does not describe the infection behavior when AIDS has already developed. The region of attraction describes the zone for which, given any initial state condition within it, its future dynamics will be particularly slow, i.e. exponentially uniform. In the positive sector (\( x_i > 0, i = 1, 2, 3 \)) of the trajectory space, the solutions will be exponentially uniform, but two different types of conditions are analysed: those starting outside and those starting inside of the domain. The latter are from an invariant set. In Fig. 2, the trajectory corresponds to initial condition given in the paper (Barao & Lemos, 2007). That trajectory belongs to solution with fast dynamics that becomes slow as soon as it traverses the invariant set.

One point which is closer to conditions found in reality is the point \( (180.2, 0, 1) \), which is depicted in Fig. 1, c). Here, the viral load is small, but the number of uninfected cells is zero. This makes us think of an HIV-infected patient who is not having a large viral load. This information may be useful to configure control law that locate the starting condition at a point such that no large viral load is generated. It is important to keep in mind that the attraction...
Fig. 2. Region of attraction for:

\[ T(0) = 520, \tilde{T}(0) = 0, V(0) = 1. \]

Fig. 3. Time response for three initial conditions: a) 

\[ T(0) = 180.2, \tilde{T}(0) = 77.6, V(0) = 3305; \]

b) 

\[ T(0) = 180.2, \tilde{T}(0) = 0, V(0) = 500; \]

c) 

\[ T(0) = 180.2, \tilde{T}(0) = 0, V(0) = 1. \]

The aim of plotting the trajectories generated from different initial conditions is to depict the dynamics generated by the three state HIV model. Remember that the perturbation term \( d(t) \) is constant.

4. Conclusions

In the paper of (Barao & Lemos, 2007), the study is made for 3rd order ODE, which focus on the analysis of eigenvalues resulting from linearisation around the equilibrium points (Santos 235). The proposed change in the values causes a shift in the response for the viral load for variable \( V \), without modifying the time response for \( T \) and \( \tilde{T} \), see Fig. 3. Also, the possibility of the initial condition \((180.2, 0, 500)\) is worth considering in Fig. 1,b) and Fig. 3. That represents the beginning of an infection with a high viral load. This generates a more benign transient concerning the viral load dynamics.
The disadvantage of linearizing about the equilibrium point when it is not the origin, is that the non-linearities of the system are not taken into account. Lyapunov converse analysis allows to obtain bounds on the phase space so that the exponential stability of the equilibrium point at the origin is guaranteed. This means that the system trajectory describes an exponentially uniform trajectory as it approaches to stable equilibrium point. It can be seen that, there are initial conditions which are not within the given sets but their respective trajectories eventually reach the stable equilibrium point. This dynamic characteristic is studied for the kind of nonlinear system which is studied in this chapter. It must be emphasized that the region of attraction will always determined by the initial conditions and the parameter values. It is also interesting in the future, to study the repulsion region, that means, the region which corresponds to the unstable equilibrium region. Both regions, attraction and repulsion are located in a manifold. The closer the initial condition is to the manifold in which the equilibrium point is located, the less stressful will the patient suffer from the dynamics. That is the main reason to justify the search for manifolds where the uniformly exponentially stable trajectories are found.

5. References

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Like any other book on the subject of HIV/AIDS, this book is not a substitute or exhausting the subject in question. It aims at complementing what is already in circulation and adds value to clarification of certain concepts to create more room for reasoning and being part of the solution to this global pandemic. It is further expected to complement a wide range of studies done on this subject, and provide a platform for the more updated information on this subject. It is the hope of the authors that the book will provide the readers with more knowledge and skills to do more to reduce HIV transmission and improve the quality of life of those that are infected or affected by HIV/AIDS.
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