On the Black-Hole Conformal Field Theory Coupled to the Polyakov’s String Theory. A Non Perturbative Analysis

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ABSTRACT

We couple the 2D black-hole conformal field theory discovered by Witten to a $D-1$ dimensional Euclidean bosonic string. We demonstrate that the resulting planar (=zero genus) string susceptibility is real for any $0 \leq D \leq 4$.

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Witten [1] has constructed a modular invariant SL(2, \mathbb{R})/U(1) coset model as a gauged SL(2, \mathbb{R})-Wess-Zumino-Witten (WZW) theory which, for a large Kac-Moody level \( k \), describes a bosonic string propagating over a two dimensional black-hole target space-time. If we denote the two target space-time coordinates by \((r, \theta)\), one sees that for large \( r \) (radial component), the same \( r \) can be identified with the Liouville field and one gets the two dimensional space-time that appears in the standard \( c = 1 \) non-critical string theory.

However, for finite \( r \), the two theories are different. In fact the exact black-hole quantum field theory seems in some sense more ”fundamental”. Indeed, quoting Witten [1], ”the region \( k > \frac{9}{4} \), which corresponds to what is normally regarded as the forbidden region of \( c > 1 \) in Liouville theory, makes perfect sense for Euclidian black-hole.”

Therefore, replacing the standard Liouville theory with the Euclidian black-hole, one could get past the Liouville theory barrier at \( c = 1 \).

In this letter, continuing the line of Witten’s arguments starting from these observations, we would like to show that if one couples the conformal field theory (CFT) governing the Euclidean 2D black-hole to \( D - 1 \) massless scalar free fields described by a Polyakov’s action with the central charge \( c \) (before coupled to the black-hole), then one gets a well defined theory also in the physically interesting range \( c = D > 1 \). The original Witten’s theory corresponds to the case \( D = 1 \).

Our starting point is the result of Bershadsky-Kutasov [2] which shows that the Euclidian SL(2, \mathbb{R})/U(1) CFT with the action \( S_{\text{W}} \) in free field representation is identical quantum mechanically to a Liouville field \( \phi \) coupled to a \( c = 1 \) conformal field \( X^0 \), with a non standard cosmological constant term \( O(\mu) \). Indeed, here the cosmological constant \( \mu \) is not controlling the area (identity) operator \( \hat{A} = \sqrt{g} e^{\beta \phi} \) (the physical world-sheet metric being taken as \( g_{\mu\nu} = e^{\beta \phi} \hat{g}_{\mu\nu} \)), but rather the primary spinless operator \( \Phi \) of lower dimension \( \Delta_0 \) of the SL(2, \mathbb{R})/U(1) theory before the ”gravitational dressing” by \( \sqrt{g} e^{\beta \phi} \).
Explicitly the action $S_W$ can be rewritten as

$$S_W = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \hat{g}^{\alpha\beta} [\partial_\alpha \phi \partial_\beta \phi + \partial_\alpha X^a \partial_\beta X^a] - \frac{Q}{4} \int d^2z \sqrt{\hat{g}} \hat{R}^2 (\hat{g}) \phi + \mu \int d^2z \sqrt{\hat{g}} e^{\beta \phi} \Phi (\phi, X^a)$$

(1)

where

$$\Phi (\phi, X^a) = (a \partial \phi + ib \partial X^a)(a \bar{\partial} \phi + ib \bar{\partial} X^a)$$

(2)

and (a,b) are the two free parameters fixed by the conformal invariance, and whose explicit values are not relevant for our purpose.

The total action is then obtained by adding to (1) the action of reparametrization ghosts plus the Polyakov’s string action for $D - 1$ massless scalar fields

$$S_P = \frac{1}{8\pi} \int d^2z \sqrt{\hat{g}} \hat{g}^{\alpha\beta} \sum_{a=1}^{D-1} (\partial_\alpha X^a \partial_\beta X^a)$$

(3)

Returning to eq. (1), we see that the cosmological constant operator $\sqrt{\hat{g}} e^{\beta \phi} \Phi$ is a conformal field of dimension (1,1) under conformal transformation, if the coefficient $\beta$ satisfies the constraint equation

$$\Delta_0 - \frac{1}{2} \beta (Q + \beta) = 1$$

(4)

where $\Delta_0$ is the dimension of the primary spinless field $\Phi$, eq. (2).

In eq. (4), $Q$ is the background charge whose value is fixed by vanishing of the total charge.

$$c_\phi + c_{X^a} + c_{\{X^a\}} + c_{\text{gh}} = 0$$

(5)

where

$$\begin{cases} c_\phi = 1 + 3Q^2 \\ c_{X^a} = 1 \\ c_{\{X^a\}} = D - 1 \\ c_{\text{gh}} = -26 \end{cases}$$

(6)
Thus one finds from eq. (5) and (6) that

\[ Q = \sqrt{\frac{25 - D}{3}} \]  

(7)

Substituting this value for \( Q \) into eq. (4) we get

\[ \beta_{\pm}(\Delta_0) = -\frac{1}{2\sqrt{3}}(\sqrt{25 - D} \pm \sqrt{1 - D + 24\Delta_0}) \]  

(8)

We shall choose in what follows the plus sign in (8), i.e. \( \beta = \beta_+ \).

Let us then assume that the by now classical ansatz of Ref.[4] can be applied to our present situation. The calculation of the string susceptibility \( \gamma_h \) for power-like scaling of the total partition function \( Z(\mu) \) on a closed compact surface of genus \( h \) follows in our case from the constant shift in the Liouville field \( \phi \) (cf. Ref.[4])

\[ \phi \rightarrow \phi + \frac{\ln(\mu)}{\beta_+ (\Delta_0)} \]  

(9)

where \( \beta_+ (\Delta_0) \) has replaced the usual \( \beta_+ (0) \).

In this manner, one obtains an “effective” string susceptibility as

\[ Z(\mu) \sim \mu^{\gamma_h^{\text{eff}} - 3} \]

\[ \gamma_h^{\text{eff}} = \frac{\chi(h)Q}{2\beta_+(\Delta_0)} + 2 \]

(10)

and, in particular for \( h = 0 \)

\[ \gamma_0^{\text{eff}} = \frac{1}{12} \frac{D - 1 - 24\Delta_0 + \sqrt{(25 - D)(1 - D + 24\Delta_0)}}{1 - \Delta_0} \]

(11)

Here \( \chi(h) = 2 - 2h \) is the Euler characteristic.

It is also possible to recast these discussions in terms of the U(1) gauged \( SL_k(2, \mathbb{R}) \) Kac-Moody algebra.
The level $k$ is now renormalized according to the relation (5), where $c_{\phi} = 2 + \frac{6}{k-2}$. One gets
\[ (2 + \frac{6}{k-2}) + (D - 1) - 26 = 0 \]
from which it follows
\[ k = 2 + \frac{6}{24 - D} \]
where we write $\tilde{D}$ for $D - 1$. The Kac-Moody (KM) conformal dimension $\Delta_{KM}$ of a primary coset operator $\Phi_{l,m,n}$ of the Euclidean black-hole CFT characterized by the integer $m, n$ ($m(n)$ is to be interpreted as the discrete momentum (winding number) of $X^0$) and by the $\text{SL}(2, \mathbb{R})$ isospin $l$, is given in terms of the principal discrete series of the unitary representation of $\hat{S}_L^k(2, \mathbb{R})$ [5].
\[ \Delta_{l,m,n} \equiv \Delta_{KM}(\Phi_{l,m,n}) = -\frac{l(l+1)}{k-2} + \frac{(m+nk)^2}{4k} \]  
(12)
The fusion algebra implies the constraints
\[
\begin{cases} 
|n - k| \geq |m| \\
2l = r - \frac{1}{2}|nk| + \frac{1}{2}|m| < -\frac{1}{2}
\end{cases}
\]  
(13)
where $r$ runs over the non-negative integers.

In our scenario, the gravitational dressing of a (spinless) primary field $v$ of bare dimension $\Delta_B$, is realized by the operator $\Phi_{l,m,n}$ with $m = n = 0$, i.e. without the $S^1$-compactified field $X^0$. In other words, $\Phi_l \equiv \Phi_{l,0,0}$ mimics the Liouville field operator $\exp(\beta \phi)$. Thus the constraint $\Delta_B + \Delta_l = 1$, where $\Delta_l \equiv \Delta_{l,0,0}$, gives
\[
\begin{cases} 
\Delta_B(v) = 1 - \Delta_l = 1 + \frac{l(l+1)}{k-2} \\
l = -1, -\frac{3}{2}, ... 
\end{cases}
\]  
(14)
Thus the lowest bare dimension $\Delta_0 = \Delta_{B,mim}(v)$ is reached by eq. (14) at $l = -1$, namely $\Delta_0 = 1$. Applied to the cosmological term operator $e^{\beta \phi} \Phi$ in eq. (1) (i.e. $v \equiv \Phi$), above general argument implies that $\Delta_0(\Phi) = 1$. 

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Such a result is in agreement with the free field realization of the $\text{SL}(2,\mathbb{R})/\text{U}(1)$ coset theory. Indeed the operator $\Phi$ defined by eq. (2) may be rewritten as [2]

$$\Phi \sim J_\zeta^{tot} J_\zeta^{tot}$$

(15)

where $J_\zeta^{tot}$ is the total axial U(1) current.

But $J_\zeta^{tot}(J_\zeta^{tot})$ is a primary field of dimension $(1,0)$ ($(0,1)$), and hence $\Phi$ is a primary (spinless) field of dimension

$$\Delta_0 = \overline{\Delta}_0 = 1$$

As a consequence of this fact, we see that the planar (i.e. zero genus) effective string susceptibility $\gamma_{eff}^{0} \equiv \gamma_{h=0}^{eff}$ given by eq. (10) has the value

$$\gamma_{0}^{eff} = \frac{Q}{\beta_{+}(1)} + 2 = -\frac{Q}{Q} + 2 = 1$$

(16)

since $\chi(h = 0) = 2$ and $\beta_{+}(1) = -\sqrt{\frac{25 + D}{3}} = -Q$.

This last result does not depend on $D$ and, if $h \neq 0$, is universal, i.e. it is a function only of the genus of the string world-sheet.

It is to be understood that at the level of the partition function, $\gamma \equiv \gamma_{0}^{eff} > 0$ is the only quantity which indicates the presence of $c > 1$ matter coupled to world-sheet, and there is numerical evidence that for $c > 1$ one has that $\gamma > 1$ [6].
ACKNOWLEDGEMENT

One of the authors (M.M.) have benefited from very useful discussions with M. Bianchi.

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