Heat flow from polygons

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Abstract

We study the heat flow from an open, bounded set \( D \) in \( \mathbb{R}^2 \) with a polygonal boundary \( \partial D \). The initial condition is the indicator function of \( D \). A Dirichlet 0 boundary condition has been imposed on some but not all of the edges of \( \partial D \). We calculate the heat content of \( D \) in \( \mathbb{R}^2 \) at \( t \) up to an exponentially small remainder as \( t \downarrow 0 \).

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1 Introduction

Let \( D \) be an open, bounded set in \( \mathbb{R}^m \) with finite Lebesgue measure \( |D| \), and with boundary \( \partial D \). We consider the heat equation

\[
\Delta u = \frac{\partial u}{\partial t},
\]

and impose a Dirichlet 0 boundary condition on \( \partial D \). That is

\[
u(x; t) = 0, \quad x \in \partial D, \quad t > 0.
\]

We denote the (weak) solution corresponding to the initial datum

\[
limit_{t \downarrow 0} u(x; t) = 1, \quad x \in D,
\]

by \( u_D \). Then \( u_D(x; t) \) represents the temperature at \( x \in D \) at time \( t \) when \( D \) has initial temperature 1, and its boundary is kept at fixed temperature 0. The heat content of \( D \) at \( t \) is denoted by

\[
Q_D(t) = \int_D dx u_D(x; t).
\]

Both \( u_D \) and \( Q_D(t) \) have been the subject of a thorough investigation going back to the treatise by Carslaw and Jaeger, [9]. For a more recent account we refer to [12]. Many different versions and
extensions have already been considered. For example, the case where \( \partial D \) is smooth, and \( A \) is an open subset of \( \partial D \) on which a Neumann (insulating) boundary condition has been imposed, while the temperature \( 0 \) Dirichlet condition has been maintained on \( \partial D - A \). This Zaremba boundary condition for the heat equation has been considered in [4], for example. Even in the case where no boundary condition has been imposed on \( \partial D \), the corresponding heat content, denoted by \( H_D(t) \), has (if \( \partial D \) is smooth) an asymptotic series as \( t \downarrow 0 \) similar to the one for \( Q_D(t) \), see [3], for example.

In this paper we consider the heat flow out of \( D \) into \( \mathbb{R}^m \), where a Dirichlet \( 0 \) boundary condition has been imposed on a closed subset \( \partial D_- \subset \partial D \), and where no boundary condition has been imposed on \( \partial D_+ := \partial D - \partial D_- \). That is

\[
\Delta u = \frac{\partial u}{\partial t},
\]

with boundary condition

\[
u(x; t) = 0, \ x \in \partial D_-, \ t > 0.
\]

We denote the solution corresponding to the initial datum

\[
\lim_{t \downarrow 0} u(x; t) = 1_D(x), \text{ almost everywhere,}
\]

by \( u_{D,\partial D_-} \). Here \( 1_D \) is the indicator function of \( D \). Then \( u_{D,\partial D_-} \) is the weak solution of \([1.1],[1.2]\) and \([1.3]\), where \([1.2]\) holds at all regular points of \( \partial D_- \). The open set \( D \) looses heat via two mechanisms: (i) part of the boundary, \( \partial D_- \), is at fixed temperature \( 0 \) and cools the interior of \( D \); (ii) since the complement of \( D \) is at initial temperature \( 0 \), heat flows over the open part of the boundary, \( \partial D_+ \). The corresponding heat content is denoted by

\[
G_{D,\partial D_-}(t) = \int_D dx u_{D,\partial D_-}(x; t).
\]

Let \( A \) be a closed subset of \( \mathbb{R}^m \), and let \( p_{\mathbb{R}^m - A}(x, y; t), x \in \mathbb{R}^m - A, y \in \mathbb{R}^m - A, t > 0 \) be the heat kernel with a Dirichlet \( 0 \) boundary condition on \( A \). Then

\[
u_{D,\partial D_-}(x; t) = \int_D dy p_{\mathbb{R}^m - \partial D_-}(x, y; t).
\]

Let \( (B(s), s \geq 0, \mathbb{P}_x, x \in \mathbb{R}^m) \) be Brownian motion associated with \( \Delta \). Recall that \( p_{\mathbb{R}^m - \partial D_-} \) is the transition density for Brownian motion on \( \mathbb{R}^m \) with killing on \( \partial D_- \). If \( \tau_{\partial D_-} = \{ \inf s \geq 0 : B(s) \in \partial D_- \} \), then

\[
u_{D,\partial D_-}(x; t) = \mathbb{P}_x(\tau_{\partial D_-} \geq t, B(t) \in D),
\]

which jibes with \([1.4]\).

Since the Dirichlet heat kernel is monotone in its domain, and \( D \subset \mathbb{R}^m - \partial D_- \subset \mathbb{R}^m \), we have that

\[
0 \leq u_D(x; t) = u_{D,\partial D}(x; t) \leq u_{D,\partial D_-}(x; t) \leq u_{D,\partial}(x; t).
\]

It follows that

\[
Q_D(t) \leq G_{D,\partial D_-}(t) \leq H_D(t), \ t > 0.
\]

Using the spectral resolution for the Dirichlet heat kernel on \( \mathbb{R}^m - \partial D_- \) it is possible to show that all three heat contents in \([1.6]\) are strictly decreasing in \( t \). Moreover, \([1.3]\) implies that for \( 1 \leq p < \infty \),

\[
\lim_{t \downarrow 0} \| u_{D,\partial D_-}(.; t) - 1_D(\cdot) \|_{L^p(\mathbb{R}^m - \partial D_-)} = 0.
\]

Indeed, by monotonicity of the Dirichlet heat kernel,

\[
p_{\mathbb{R}^m - \partial D_-}(x, y; t) \leq (4\pi t)^{-m/2}e^{-|x-y|^2/(4t)}.
\]
Hence $0 < u_{D,\partial D_-}(x; t) \leq 1$, and $|u_{D,\partial D_-}(x; t) - 1| \leq 1$. Moreover,

$$\|u_{D,\partial D_-}(\cdot; t) - 1_D(\cdot)\|_{L^p(\mathbb{R}^n)} = \int_D dx|u_{D,\partial D_-}(x; t) - 1|^p + \int_{\mathbb{R}^n-D} dx u_{D,\partial D_-}(x; t)^p$$

$$\leq \int_D dx|u_{D,\partial D_-}(x; t) - 1| + \int_{\mathbb{R}^n-D} dx u_{D,\partial D_-}(x; t)$$

$$= \int_D dx|u_{D,\partial D_-}(x; t) - 1| + \int_{\mathbb{R}^n} dx u_{D,\partial D_-}(x; t) - \int_D dx u_{D,\partial D_-}(x; t)$$

$$\leq 2 \int_D dx|u_{D,\partial D_-}(x; t) - 1|,$$

where we use (1.5) to obtain the final inequality. The assertion follows by Lebesgue’s Dominated Convergence Theorem and (1.3).

The main results of this paper concern the special case where $D$ is an open, bounded set in $\mathbb{R}^2$ with a polygonal boundary. Throughout we make the hypothesis that the vertices of $\partial D$ are the endpoints of exactly two edges, and that the collection of vertices $\mathcal{V} = \{V_1, V_2, \cdots\}$ is finite. We consider edges of two types: Dirichlet edges which include their endpoints, and open edges which include those vertices common to two open edges. The union of all Dirichlet edges, denoted by $\partial D_-$ as above, is a closed subset of $\mathbb{R}^2$, and we denote its length by $L(\partial D_-)$. The union of all open edges, denoted by $\partial D_+$, is a relatively open subset of $\partial D$. We denote its length by $L(\partial D_+)$. The length of $\partial D$ is given by

$$L(\partial D) = L(\partial D_-) + L(\partial D_+).$$

It was shown in [8] that if all edges are of Dirichlet type, then

$$Q_D(t) = |D| - \frac{2}{\pi^{1/2}} L(\partial D) t^{1/2} + \sum_{\gamma \in \mathcal{C}} c(\gamma) t + O(e^{-q_D/t}), \quad t \downarrow 0,$$

(1.8)

where $q_D > 0$ is a constant which depends on $D$ only, $c : (0, 2\pi] \rightarrow \mathbb{R}$ is defined as

$$c(\gamma) = \int_0^\infty d\theta \frac{4 \sinh((\pi - \gamma)\theta)}{(\sinh(\pi\theta))(\cosh(\gamma\theta))},$$

(1.9)

$\mathcal{C} = \{\gamma_1, \gamma_2, \cdots\}$ are the interior angles at the vertices $V_1, V_2, \ldots$, and $L(\partial D)$ is the total length of all Dirichlet edges.

On the other hand if all edges are of open type, that is $\partial D_- = \emptyset$, then it was shown in [6] that

$$H_D(t) = |D| - \frac{1}{\pi^{1/2}} L(\partial D) t^{1/2} + \sum_{\beta \in \mathcal{B}} b(\beta) t + O(e^{-h_D/t}), \quad t \downarrow 0,$$

(1.10)

where $h_D > 0$ is a constant which depends on $D$ only, $b : (0, 2\pi) \rightarrow \mathbb{R}$ is given by

$$b(\beta) = \begin{cases} \frac{1}{\pi} + \left(1 - \frac{\beta}{\pi}\right) \cot \beta, & \beta \in (0, \pi) \cup (\pi, 2\pi); \\ 0, & \beta = \pi, \end{cases}$$

$\mathcal{B} = \{\beta_1, \beta_2, \ldots\}$ are the interior angles at the vertices $V_1, V_2, \ldots$, and $L(\partial D)$ is the total length of all open edges.

The main result of this paper, Theorem 1.1 below, allows both open and Dirichlet edges. The collection of interior angles between two adjacent Dirichlet, respectively open, edges is denoted by $\mathcal{C}$, respectively $\mathcal{B}$. The collection of angles between an adjacent pair of open-Dirichlet edges (or Dirichlet-open edges) is denoted by $\mathcal{A}$ (see Figure 1).
Theorem 1.1 There exists a constant \( g_D > 0 \) depending on \( D \) only such that
\[
G_D(t) = |D| - \frac{1}{\pi^{1/2}} \left( 2L(\partial D_-) + L(\partial D_+) \right) t^{1/2} + \left( \sum_{\gamma \in C} c(\gamma) + \sum_{\beta \in B} b(\beta) + \sum_{\alpha \in A} a(\alpha) \right) t + O(e^{-g_D t}), \quad t \downarrow 0,
\]
where \( a : (0, 2\pi) \mapsto \mathbb{R} \) is given by
\[
a(\alpha) = -\frac{3}{4} + \frac{1}{4} \int_0^\infty d\theta \frac{4 \sinh^2((\pi - \frac{\alpha}{2})\theta) - \sinh^2((\pi - \alpha)\theta)}{(\sinh(\pi\theta/2))(\cosh(\pi\theta))}.
\]
(1.12)

Figure 1: An open set \( D \subset \mathbb{R}^2 \) with polygonal boundary: the Dirichlet, respectively open, edges are displayed as solid, respectively dashed, lines.

The main results of both [6] and [8] hold for more general polygons. For example, vertices with just one edge or more than two are allowed. If a vertex supports just one edge, then the corresponding angle equals \( 2\pi \) and will contribute \( c(2\pi) \) to the coefficient of \( t \) in (1.8). That edge counts double in the total length of Dirichlet edges. Indeed, that edge cools \( D \) at both sides. In general, the contribution from the angles to the coefficient \( t \) in (1.8) is additive. The Dirichlet condition on the edges implies this additivity. That does not hold true in the setting of open edges. If two wedges with angles, say \( \beta_1 \) and \( \beta_2 \), are supported by the same vertex, then there is an additional contribution to the coefficient of \( t \), depending on \( \beta_1, \beta_2 \) and the angle between these two wedges (see [6]). Furthermore, if a vertex supports just one edge, then the corresponding angle, and the corresponding edge contribute 0 to the coefficients of \( t \) and \( t^{1/2} \) respectively. Indeed, heat does not flow over this edge into \( \mathbb{R}^2 - D \). We shall not consider these cases, and we assume that each vertex supports precisely two edges.

The proof of Theorem 1.1 is based on a partition of \( D \) combined with model computations, as are the proofs of (1.8) and (1.10). The main computation is the one for circular sectors with radius \( R \) with opening angles \( \gamma, \beta, \alpha \) depending on whether one deals with a Dirichlet-Dirichlet wedge, an open-open wedge, or, as in this paper, a Dirichlet-open wedge. The geometry of the Dirichlet-open wedge is one edge on which a Dirichlet boundary condition has been imposed, and an open edge separated by angle \( \alpha \) (see Figure 2). Our main result for such a circular sector is the following.

Theorem 1.2 Let \( W_\alpha = \{(r, \phi) : r > 0, 0 < \phi < \alpha\} \) in polar coordinates, and let \( u_\alpha((r, \phi); t) \) be the solution of the heat equation with a Dirichlet boundary condition on the positive \( x_1 \) axis, and initial data \( 1_{W_\alpha} \). Then, in polar coordinates, we have that
\[
\int_0^R r \, dr \int_0^\alpha d\phi u_\alpha((r, \phi); t) = \frac{1}{2} \alpha R^2 - \frac{3}{\pi^{1/2}} R t^{1/2} + a(\alpha) t + 3 R t^{1/2} \int_1^\infty dv \int_0^1 d\zeta \frac{\zeta}{(1 - \zeta^2)^{1/2}} e^{-R^2 \zeta^2 v^2/(4t)} + O(e^{-m_\alpha R^2/(4t)}), \quad t \downarrow 0,
\]
(1.13)
Figure 2: A Dirichlet-open wedge with angle $\alpha$: the Dirichlet, respectively open, edge is displayed as a solid, respectively dashed, line.

where $m_\alpha > 0$ is a constant which depends on $\alpha$ only.

We recognise the various terms in the right-hand side as follows. The first term is the area of the circular sector with opening angle $\alpha$ and radius $R$. The second term combines the contributions from two edges of length $R$ with the contribution from the Dirichlet edge having a factor 2. The third term is the angle contribution. The fourth term represents the contribution from two cusps. See Section 3 for details.

It has been noted (see p.43 in [8]) that there are three closed form expressions for the heat kernel of a wedge with opening angle $\gamma$, see [10], [14], and [16]. The authors of [8] were unable to extract, for example, the angle contribution $c(\gamma)t$ featuring in (1.9). In the case at hand, there is a fourth explicit formula for the heat kernel of a wedge with opening angle $2\pi$ (see p.380 in [9]). However, we were unable to obtain a workable expression using that formula. D.B. Ray managed to compute the angle contribution of the trace of the Dirichlet heat semigroup for a polygon using the Laplace transform of the heat kernel for a wedge, expressed as a Kontorovich Lebedev transform (see the footnote on p.44 of [15]). This strategy has been successfully employed in both [7] and [8].

Unlike the integral for $c(\gamma)$ in (1.9) it is possible to evaluate the expression for $a(\alpha)$ in (1.12). To do so we write

$$\frac{1}{(\sinh(\pi t)/2)(\cos(\pi t/2))} = \frac{1}{\sinh^2(\pi \theta/2)} - \frac{2}{\cosh(\pi \theta)}$$

and compute the resulting four integrals using formulae 3.511.7 and 3.511.9 in [13]. The common range of convergence for these four integrals is $\pi < \alpha < 3\pi/2$. We find that

$$a(\alpha) = -\frac{3}{8} + \frac{3}{4\pi} - \frac{1}{8\cos \alpha} + \frac{1}{2\cos(\alpha/2)} + \left(\frac{7}{4} - \frac{3\alpha}{4\pi}\right) \frac{1}{\tan \alpha} + \left(\frac{1}{4} - \frac{\alpha}{4\pi}\right) \tan \alpha, \pi < \alpha < 3\pi/2.$$

Outside this interval we can use (1.12) to evaluate $a(\alpha)$. For example we have that

$$a(\pi/2) = -\frac{3}{8} + \frac{1}{\pi} + \frac{1}{2\sqrt{2}},$$

$$a(\pi) = -\frac{1}{4},$$

$$a(3\pi/2) = -\frac{3}{8} + \frac{1}{\pi} - \frac{1}{2\sqrt{2}}.$$

The value $a(\pi) = -\frac{1}{4}$ in (1.14) is of particular interest. Consider an open, bounded set $D$ in $\mathbb{R}^m$ with $C^\infty$ boundary $\partial D$. Let $\partial D_\pm$ be a closed subset of $\partial D$ with $C^\infty$ boundary $\Sigma$, and $\dim \Sigma = m - 2$. Let $u_{D,\partial D_-}$ be the solution of (1.1), (1.2), and (1.3). Then, provided an asymptotic series in half powers
of \( t \) exists, we have that

\[
G_{D, \partial D_-}(t) = |D| - \pi^{-1/2} \left( 2 \int_{\partial D_-} d\sigma + \int_{\partial D - \partial D_-} d\sigma \right) t^{1/2} + \left( \frac{1}{2} \int_{\partial D_-} d\sigma L_a(a) - \frac{1}{4} \text{vol}(\Sigma) \right) t + O(t^{3/2}), \quad t \downarrow 0, \tag{1.15}
\]

where \( d\sigma \) denotes the surface measure on \( \partial D \), \( L_a \) is the trace of the second fundamental form defined by the inward unit normal vector field of \( \partial D \) in \( D \), \( \text{vol}(\Sigma) \) is the \((m - 2)\)-dimensional volume of the boundary of \( \partial D_- \) in \( \partial D \), and \( a(\pi) \) is its coefficient. To see that (1.15) holds we note that the local geometry around \( \Sigma \) is as follows. Let \( P \) be a point of \( \Sigma \). Then straightening out the boundary of \( \partial D \) around \( P \) we obtain, locally, an \((m - 1)\)-dimensional hyper plane. The straightening out of \( \Sigma \) around \( P \) partitions this hyper plane into two hyper half-planes at angle \( \pi \). On one (closed) hyper half-plane we have a Dirichlet 0 boundary condition, and on the remaining open hyper half-plane we do not have boundary conditions. This is precisely the geometry of a Dirichlet-open wedge with angle \( \pi \) times \( \Sigma \). This then leads to the \( a(\pi)\text{vol}(\Sigma)t \) contribution in (1.15). The computation of the coefficient of \( t^{3/2} \) promises to be more complicated even in this special setting. One expects that there is an integral over \( \Sigma \) involving both the second fundamental form of \( \Sigma \) in \( D \) and the second fundamental form of \( \partial D \) in \( D \). Consequently, several special case calculations would be required. See also [4].

The proofs of Theorems 1.2 and 1.4 have been deferred to Sections 2 and 3 respectively.

2 Proof of Theorem 1.2

Proof of Theorem 1.2 Let \( p_{W, \alpha}(A_1, A_2; t) \) denote the Dirichlet heat kernel for the open set \( \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \notin [0, \infty)\} \). Let

\[
\hat{p}_{W, \alpha}(A_1, A_2; s) = \int_0^\infty ds \ e^{-st} p_{W, \alpha}(A_1, A_2; t)
\]

be the associated Green’s function. Then, by the footnote on p.44 in [15],

\[
\hat{p}_{W, \alpha}(A_1, A_2; s) = \frac{1}{\pi^2} \int_0^\infty d\theta K_{i\theta}(\sqrt{s} a_1) K_{i\theta}(\sqrt{s} a_2) \times \left( \cosh((\pi - |\alpha_1 - \alpha_2|)\theta) - \frac{\sinh(\pi\theta)}{\sinh(2\pi\theta)} (\cosh((2\pi - \alpha_1 - \alpha_2)\theta) + \cosh((\alpha_1 - \alpha_2)\theta)) \right),
\]

where \( A_i = (a_i, \alpha_i), i = 1, 2 \) in polar coordinates.

In what follows, we consider

\[
\int_0^R d\alpha_1 \int_0^\alpha d\alpha_2 \int_0^\alpha d\alpha_1 \int_0^\alpha d\alpha_2 \hat{p}_{W, \alpha}(A_1, A_2; s),
\]

and then take the inverse Laplace transform. A straightforward computation shows that

\[
\int_0^\alpha d\alpha_1 \int_0^\alpha d\alpha_2 \left( \cosh((-\alpha + |\alpha_1 - \alpha_2|)\theta) - \frac{\sinh(\pi\theta)}{\sinh(2\pi\theta)} (\cosh((2\pi - \alpha_1 - \alpha_2)\theta) + \cosh((\alpha_1 - \alpha_2)\theta)) \right)
= \frac{2\alpha}{\theta} \sinh(\pi\theta) + \frac{1}{2\theta^2 \cosh(\pi\theta)} (3 - 3 \cosh(2\pi\theta))
+ \frac{1}{2\theta^2 \cosh(\pi\theta)} (4 \cosh((2\pi - \alpha)\theta) - \cosh((2\pi - 2\alpha)\theta) - 3)
=: C_1 + C_2 + C_3,
\]

with obvious notation. Recall that by formulae 6.561.16, 8.332.3, and 6.794.2 in [13],

\[
\int_0^\infty da \ a K_{i\theta}(\sqrt{s} a) = \frac{\pi \theta}{2s \sinh(\pi\theta/2)},
\]
\[
\int_0^\infty d\theta \cosh(\pi \theta/2) K_{i\theta}(\sqrt{a} \alpha) = \frac{\pi}{2}, \quad a > 0.
\]  
(2.3)

We obtain by definition of \(C_1\), Fubini’s theorem, (2.2) and (2.3),
\[
\int_0^R da_1 a_1 \int_0^\infty da_2 \frac{1}{\pi^2} \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) K_{i\theta}(\sqrt{s}a_2) C_1
\]
\[
= \frac{2\alpha}{\pi}\int_0^R da_1 a_1 \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) \cosh(\pi \theta/2)
\]
\[
= \alpha R^2/2s.
\]

Throughout this paper we denote by \(L^{-1}\) the inverse Laplace transform. That is, if \(\hat{f}(s) = \int_0^\infty dt e^{-st} f(t)\) then \(L^{-1}\{\hat{f}\}(t) = f(t)\), at points of continuity of \(f\). So \(L^{-1}\{(2s)^{-1}R^2\}(t) = 2^{-1}\alpha R^2\), which is the first term in right-hand side of (1.13).

Furthermore, by Fubini’s theorem, (2.2), and the definition of \(C_2\) in (2.1), we find that
\[
\int_0^R da_1 a_1 \int_0^\infty da_2 \frac{1}{\pi^2} \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) K_{i\theta}(\sqrt{s}a_2) C_2
\]
\[
= -\frac{3}{\pi s} \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) \sinh(\pi \theta/2) = \frac{3}{\pi s} \int_0^{\pi/2} d\eta \int_0^\infty d\theta \cosh(\eta \theta) K_{i\theta}(\sqrt{s}a_1)
\]
\[
= \frac{3}{2s} \int_0^{\pi/2} d\eta e^{-a_1 \sqrt{s} \cos \eta}.
\]  
(2.5)

By (2.5), we obtain for the first term in the right-hand side of (2.4)
\[
-\frac{3}{\pi} \int_0^R da_1 a_1 \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) \sinh(\pi \theta/2)
\]
\[
= -\frac{3}{2s} \int_0^R da \int_0^{\pi/2} d\eta e^{-a \sqrt{s} \cos \eta}.
\]  
(2.6)

From the calculation in (2.14) of [8], we find that the inverse Laplace transform of the right-hand side of (2.6) is given by
\[
-\frac{3 R t^{1/2}}{\pi^{1/2}} + \frac{3 R t^{1/2}}{\pi^{1/2}} \int_1^\infty dv \frac{\zeta d\zeta}{\sqrt{1 - \zeta^2}} e^{-R^2 \zeta^2 t^2/4t}.
\]

For the second term in the right-hand side of (2.4), we have by Fubini’s theorem and (2.2) that
\[
-\frac{3}{\pi} \int_0^R da_1 a_1 \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) \frac{\sin(\pi \theta/2)}{\cosh(\pi \theta)}
\]
\[
= -\frac{3}{2s^2} \int_0^\infty d\theta \cosh(\pi \theta) + \frac{3}{\pi} \int_0^R da_1 a_1 \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) \frac{\sin(\pi \theta/2)}{\cosh(\pi \theta)}
\]
\[
= -\frac{3}{4s^2} + \frac{3}{\pi} \int_0^R da_1 a_1 \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) \frac{\sin(\pi \theta/2)}{\cosh(\pi \theta)}.
\]  
(2.7)
Taking the inverse Laplace transform of the first term in the right-hand side of (2.7) yields \(-\frac{3}{4}t\), which accounts for the \(-\frac{3}{4}\) term in (1.12).

To invert the Laplace transform of the second term in the right-hand side of (2.7), we use formula 3.547.4 of [13],

\[ K_{i\theta}(\sqrt{s}a) = \int_0^\infty dw (\cos(w\theta))e^{-\sqrt{s}a \cosh w}, \]  

(2.8)

and obtain, by Fubini’s theorem, that

\[
\frac{3}{\pi} \int_0^\infty da \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a) \frac{\sin(\pi\theta/2)}{\cosh(\pi\theta)}
= \frac{3}{\pi} \int_0^\infty da \int_0^\infty dw e^{-\sqrt{s}a \cosh w} \int_0^\infty \frac{d\theta}{\theta} (\cos(w\theta)) \frac{\sin(\pi\theta/2)}{\cosh(\pi\theta)}.
\]

By formula 5.6.3 in [11],

\[
L^{-1}\{s^{-1}e^{-\sqrt{s}a \cosh w}\}(t) = \text{Erfc} ((a \cosh w)/(4t)^{1/2}) = \frac{2}{\sqrt{\pi}} \int_{(a \cosh w)/(4t)^{1/2}}^\infty dr e^{-r^2}. \tag{2.9}
\]

Hence the inverse Laplace transform of the second term in the right-hand side of (2.7) is bounded in absolute value by

\[
\frac{3}{\pi} \int_0^\infty da \int_0^\infty dw \text{Erfc} ((a \cosh w)/(4t)^{1/2}) \int_0^\infty \frac{d\theta}{\theta} \frac{\sin(\pi\theta/2)}{\cosh(\pi\theta)}
\leq \frac{3}{\pi} \int_0^\infty da \int_0^\infty dw \text{Erfc} ((a \cosh w)/(4t)^{1/2}) \int_0^\infty \frac{d\theta}{\theta} \frac{\sin(\pi\theta/2)}{\cosh(\pi\theta)}
\leq \frac{3\log(1 + \sqrt{2})}{\pi} \int_0^\infty dw \int_R^\infty da \text{Erfc} ((a \cosh w)/(4t)^{1/2}), \tag{2.10}
\]

where we have used formula 4.114.2 in [13]. Since

\[
\text{Erfc}(z) := \frac{2}{\sqrt{\pi}} \int_z^\infty dr e^{-r^2} = \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} dr e^{-(z+r)^2} \leq \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} dr e^{-z^2-r^2} = e^{-z^2}, \tag{2.11}
\]

we obtain that the right-hand side of (2.10) is bounded from above by

\[
\frac{3\log(1 + \sqrt{2})}{\pi} \int_0^\infty dw \int_R^\infty da e^{-(a \cosh w)^2/(4t)} = \frac{6t\log(1 + \sqrt{2})}{\pi} \int_0^\infty \frac{dw}{(\cosh w)^2} e^{-(R \cosh w)^2/(4t)}
= O(te^{-R^2/(4t)}). \tag{2.12}
\]

In order to compute \(C_3\), we extend the integral with respect to \(a_1\) to the interval [0, \(\infty\)], and obtain, via Fubini’s theorem and (2.2), that

\[
\int_0^\infty da_1 a_1 \int_0^\infty da_2 a_2 \frac{1}{\pi^2} \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) K_{i\theta}(\sqrt{s}a_2) C_3
= \int_0^\infty da_1 a_1 \int_0^\infty da_2 a_2 \frac{1}{\pi^2} \int_0^\infty d\theta K_{i\theta}(\sqrt{s}a_1) K_{i\theta}(\sqrt{s}a_2)
\times \frac{1}{2\theta^2 \cosh(\pi\theta)} (4 \cosh((2\pi - \alpha)\theta) - \cosh((2\pi - 2\alpha)\theta) - 3)
= \frac{1}{8s^2} \int_0^\infty d\theta \frac{4 \cosh((2\pi - \alpha)\theta) - \cosh((2\pi - 2\alpha)\theta) - 3}{\cosh(\pi\theta)) (\sinh^2(\pi\theta/2))}
= \frac{1}{4s^2} \int_0^\infty d\theta \frac{4 \sinh^2((\pi - \frac{\alpha}{2})\theta) - \sinh^2((\pi - \alpha)\theta)}{\sinh^2(\pi\theta/2)) (\cosh(\pi\theta))}.
\]
Inverting the Laplace transform yields a contribution \((\frac{1}{4} + a(\alpha))t\), where \(a(\alpha)\) is as defined in (1.12). This, together with the statement below (2.7) gives the contribution \(a(\alpha)t\) in (1.13). It remains to bound the inverse Laplace transform of
\[
\int_R^\infty da_1 a_1 \int_0^\infty da_2 a_2 \frac{1}{\pi^2} \int_0^\infty d\theta K_i(\sqrt{s}a_1) K_i(\sqrt{s}a_2) \times \frac{1}{2\theta^2 \cosh(\pi \theta)} (4 \cosh((2\pi - \alpha)\theta) - \cosh((2\pi - 2\alpha)\theta) - 3)
\]
\[
= \frac{1}{s} \int_R^\infty da \int_0^\infty d\theta K_i(\sqrt{s}a) \frac{4\cosh((2\pi - \alpha)\theta) - \cosh((2\pi - 2\alpha)\theta) - 3}{4\pi\theta \sinh(\pi \theta/2) \cosh(\pi \theta)}.
\]
(2.13)

We first consider the case \(\pi/2 < \alpha < 7\pi/4\), and we proceed as above. We use (2.8), and invert the Laplace transform of \(s^{-1}e^{-\sqrt{s}a}\cosh w\) as in (2.9). This gives that the inverse Laplace transform of (2.13) equals
\[
\int_R^\infty da \int_0^\infty dw \operatorname{Erfc}((a \cosh w)/(4t)^{1/2}) \int_0^\infty d\theta \frac{4\cosh((2\pi - \alpha)\theta) - \cosh((2\pi - 2\alpha)\theta) - 3}{4\pi\theta \sinh(\pi \theta/2) \cosh(\pi \theta)}
\]
(2.14)

Using \(|\cos(w\theta)| \leq 1\), we find that the absolute value of the expression under (2.14) is bounded from above by
\[
\int_R^\infty da \int_0^\infty dw \operatorname{Erfc}((a \cosh w)/(4t)^{1/2}) \int_0^\infty d\theta \frac{4\cosh((2\pi - \alpha)\theta) - \cosh((2\pi - 2\alpha)\theta) - 3}{4\pi\theta \sinh(\pi \theta/2) \cosh(\pi \theta)} = O(te^{-R^2/(4t)}),
\]
(2.15)

where, as before, we have used (2.11), and argued similarly to (2.12). We note that the integrals with respect to \(\theta\) in (2.14) and (2.15) converge for \(\pi/2 < \alpha < 7\pi/4\).

We next consider the case \(7\pi/4 < \alpha < 2\pi\). We write the right-hand side of (2.13) as the sum of two terms, say \(D_1(s) + D_2(s)\), where
\[
D_1(s) = \frac{1}{s} \int_R^\infty da \int_0^\infty d\theta K_i(\sqrt{s}a) \frac{\cosh((2\pi - \alpha)\theta) - 1}{(\pi \theta \sinh(\pi \theta/2)) \cosh(\pi \theta)},
\]
(2.16)

and
\[
D_2(s) = \frac{1}{s} \int_R^\infty da \int_0^\infty d\theta K_i(\sqrt{s}a) \frac{1 - \cosh((2\pi - 2\alpha)\theta)}{4\pi\theta \sinh(\pi \theta/2) \cosh(\pi \theta)}.
\]
(2.17)

Using (2.8), (2.9) gives that
\[
|L^{-1}\{D_1\}(t)| = \int_R^\infty da \int_0^\infty dw \operatorname{Erfc}((a \cosh w)/(4t)^{1/2}) \int_0^\infty d\theta \frac{\cosh((2\pi - \alpha)\theta) - 1}{(\pi \theta \sinh(\pi \theta/2)) \cosh(\pi \theta)}
\]
\[
\leq \int_R^\infty da \int_0^\infty dw \operatorname{Erfc}((a \cosh w)/(4t)^{1/2}) \int_0^\infty d\theta \frac{\cosh((2\pi - 2\alpha)\theta) - 1}{4\pi\theta \sinh(\pi \theta/2) \cosh(\pi \theta)} = O(te^{-R^2/(4t)}),
\]
(2.18)

where we have used (2.11), and argued similarly to (2.12). The integral with respect to \(\theta\) in (2.18) converges for \(\alpha \in (\pi/2, 2\pi) \supset (7\pi/4, 2\pi)\). To invert \(D_2(s)\) we rewrite the integrand as follows. For \(\epsilon \in \mathbb{R}\),
\[
\frac{1 - \cosh((2\pi - 2\alpha)\theta)}{4\pi\theta \sinh(\pi \theta/2) \cosh(\pi \theta)} = \frac{4 \cosh(\pi \theta/2) - 2 \cosh((2\alpha - \frac{3\pi}{2})\theta) - 2 \cosh((2\alpha - \frac{5\pi}{2})\theta)}{4\pi\theta \sinh(2\pi \theta)}
\]
\[
= \frac{4 \cosh(\pi \theta/2) - 2 \cosh((2\alpha - \frac{3\pi}{2})\theta) - 2 \cosh((2\alpha - \frac{5\pi}{2})\theta) + 2 \cosh((2\pi + \epsilon)\theta) - 2 \cosh((2\pi - \epsilon)\theta)}{4\pi\theta \sinh(2\pi \theta)}
\]
\[- \frac{1}{\pi \theta} \sinh(\epsilon \theta).
\]
We choose $2\alpha - \frac{3\pi}{2} = 2\pi + \epsilon$. This gives that $\epsilon = 2\alpha - \frac{\pi}{2}$, and

$$1 - \cosh((2\pi - 2\alpha)\theta) = \frac{2 \cosh(\pi\theta/2) - \cosh((2\alpha - \frac{5\pi}{2})\theta) - \cosh((\frac{11\pi}{2} - 2\alpha)\theta)}{2\pi \sinh(2\pi\theta)} - \frac{1}{\pi\theta} \sinh((4\alpha - 7\pi)\theta/2). \tag{2.19}$$

The first term in the right-hand side of (2.19) is integrable, and, analogously to the above, we proceed with (2.8), (2.9), and (2.11). This gives a remainder $O(te^{-R^2/(4t)})$.

It remains to invert the contribution coming from the second term in the right-hand side of (2.19). We recall (2.18) in [8]. That is, for $-\frac{\pi}{2} < \beta < \frac{\pi}{2}$ we have, by Fubini’s theorem, 6.795.1 in [13], and 5.6.3 in [11], that

$$L^{-1}\left\{ \int_{\mathbb{R}} da a \int_{0}^{\infty} \frac{d\theta}{\theta s} K_{i\theta}(\sqrt{s}a) \sinh(\beta\theta) \right\}(t) = L^{-1}\left\{ \int_{\mathbb{R}} da a \int_{0}^{\infty} d\eta J_{\beta}(\sqrt{2s}a) \right\}(t)$$

$$= L^{-1}\left\{ \int_{\mathbb{R}} da a \int_{0}^{\infty} d\eta e^{-a\sqrt{2\cos \eta}} \right\}(t) = \pi \int_{\mathbb{R}} da a \int_{0}^{\pi} d\eta \text{Erfc} \left( \frac{a \cos \eta}{(4\theta)^{1/2}} \right)$$

$$\leq \frac{\pi \beta}{2} \int_{\mathbb{R}} da a \text{Erfc} \left( \frac{a \cos \beta}{(4\theta)^{1/2}} \right) = O(te^{-R^2/(4\theta^2)}) \tag{2.20},$$

where we have used once more (2.11).

For $7\pi/4 < \alpha < 2\pi$ we have that $2\alpha - \frac{3\pi}{2} \in (0, \pi/2)$. Hence the second term in the right-hand side of (2.19) gives a contribution $O(te^{-R^2/(\sin(2\alpha)^2)/(4t)})$.

For $\alpha = \frac{7\pi}{4}$ we have that

$$\int_{0}^{\infty} \frac{d\theta}{\theta s} \sinh((2\pi - \alpha)\theta) - 1 \left( \frac{\pi\theta \sinh(\pi\theta/2)}{\cosh(\pi\theta)} \right) < \infty.$$

Hence the inverse Laplace transform of $D_1$ is $O(te^{-R^2/(4\theta^2)})$. For $\alpha = 7\pi/4$ we rewrite (2.17) as

$$D_2 = \frac{1}{s} \int_{\mathbb{R}} da a \int_{0}^{\infty} \frac{d\theta}{\theta s} K_{i\theta}(\sqrt{s}a) \frac{\cosh(\pi\theta/2) - \frac{1}{2} \cos(\pi\theta) - \frac{1}{2} \cosh(2\pi\theta)}{\pi\theta \sinh(2\pi\theta)}$$

$$= \frac{1}{s} \int_{\mathbb{R}} da a \int_{0}^{\infty} \frac{d\theta}{\theta s} K_{i\theta}(\sqrt{s}a) \left\{ \frac{\cosh(\pi\theta/2) - \frac{1}{2} \cos(\pi\theta) - \frac{1}{2} \cosh(2\pi\theta)}{\pi\theta \sinh(2\pi\theta)} - \tan(\pi\theta) \frac{2}{2\pi\theta} \right\}$$

Since

$$\int_{0}^{\infty} \frac{d\theta}{\theta s} \left| \frac{\cosh(\pi\theta/2) - \frac{1}{2} \cos(\pi\theta) - \frac{1}{2}}{\pi\theta \sinh(2\pi\theta)} \right| < \infty,$$

we have that this part gives again a contribution $O(te^{-R^2/(4\theta^2)})$. By 4.116.2 in [13], and 5.6.3 in [11] (see (2.20) in [8]), we have for $\beta > 0$ that

$$L^{-1}\left\{ \frac{1}{\pi s} \int_{\mathbb{R}} da a \int_{0}^{\infty} \frac{d\theta}{\theta} (\tan(\beta\theta)) K_{i\theta}(\sqrt{s}a) \right\}(t)$$

$$= L^{-1}\left\{ \frac{1}{\pi s} \int_{\mathbb{R}} da a \int_{0}^{\infty} dw e^{-a\sqrt{2\cosh w}} \int_{0}^{\infty} \frac{d\theta}{\theta} (\tan(\beta\theta)) \cos(w\theta) \right\}(t)$$

$$= L^{-1}\left\{ \frac{1}{\pi s} \int_{\mathbb{R}} da a \int_{0}^{\infty} dw e^{-a\sqrt{2\cosh w}} \log(\cot(\pi w/(4\beta))) \right\}(t)$$

$$= \frac{1}{\pi} \int_{\mathbb{R}} da a \int_{0}^{\infty} dw \text{Erfc} \left( \frac{a \cos w}{(4\beta)^{1/2}} \right) \log(\cot(\pi w/(4\beta))). \tag{2.21}$$
By (2.11) we obtain that (2.21) is bounded from above by
\[ 2\pi^{-1}te^{-R^2/(4t)} \int_0^\infty dw \frac{\log(cosh(\pi w/(4\beta)))}{(cosh w)^2} = O(te^{-R^2/(4t)}). \]

In particular, for \( \beta = \pi \), the term \( -\frac{\tanh(\pi \theta)}{2\pi \theta} \) contributes a remainder \( O(te^{-R^2/(4t)}) \).

For \( \alpha = \pi/2 \) we have that
\[ \int_0^\infty \left| \frac{1 - \cosh((2\pi - \alpha)\theta)}{4\pi\theta(\sinh(\pi\theta/2))(\cosh(\pi\theta))} \right| < \infty. \]

Hence the inverse Laplace transform of \( D_2 \) is, for \( \alpha = \pi/2, O(te^{-R^2/(4t)}) \). On the other hand, for \( \alpha = \pi/2 \) the integrand in (2.16) equals the integrand of (2.17) for \( \alpha = 7\pi/4 \) up to a factor of \( -\frac{1}{4} \).

Hence the inverse Laplace transform of \( D_1 \) is also \( O(te^{-R^2/(4t)}) \).

For \( \pi/4 < \alpha < \pi/2 \) we have that
\[ \int_0^\infty \left| \frac{1 - \cosh((2\pi - 2\alpha)\theta)}{4\pi\theta(\sinh(\pi\theta/2))(\cosh(\pi\theta))} \right| < \infty. \]

Hence the inverse Laplace transform of \( D_2 \) is, for \( \pi/4 < \alpha < \pi/2, O(te^{-R^2/(4t)}) \). Similarly to the above, we rewrite the hyperbolic part of the integrand in (2.16) as follows.

\[
\begin{align*}
cosh((2\pi - \alpha)\theta) - 1 & \quad (\pi\theta \sinh(\pi\theta/2)) \cosh(\pi\theta) \\
& = 2 \cosh((\frac{3\pi}{2} - \alpha)\theta) + 2 \cosh((\frac{3\pi}{2} + \alpha)\theta) - 4 \cosh(\pi\theta/2) \\
& = 2 \cosh((\frac{3\pi}{2} - \alpha)\theta) + 2 \cosh((\frac{3\pi}{2} + \alpha)\theta) - 4 \cosh(\pi\theta/2) - 2 \cosh((2\pi + \epsilon)\theta) - 2 \cosh((2\pi - \epsilon)\theta) \\
& + \frac{4 \sinh(\epsilon\theta)}{\pi\theta}.
\end{align*}
\]

We subsequently choose \( \epsilon = \frac{\pi}{2} - \alpha \). With this choice of \( \epsilon \), the absolute value of the first term in the right-hand side of (2.22) is integrable with respect to \( \theta \) on \( \mathbb{R}^+ \). Hence this term contributes \( O(te^{-R^2/(4t)}) \) to the inverse Laplace transform of the corresponding integral in (2.16). Moreover, since \( \epsilon \in (0, \pi/2) \) for this case, we have by (2.20) that this term contributes \( O(te^{-R^2(\sin^2(\alpha)/(4t))}) \) to the inverse Laplace transform of the corresponding integral in (2.16).

We next consider the case \( \alpha = \pi/4 \). Then \( D_2 \) for \( \pi/4 \) equals \( D_2 \) for \( 7\pi/4 \), we immediately conclude that this term is \( O(te^{-R^2/(4t)}) \). We rewrite the hyperbolic part of the integrand as follows.

\[
\begin{align*}
cosh(7\pi\theta/4) - 1 & \quad (\pi\theta \sinh(\pi\theta/2))(\cosh(\pi\theta)) \\
& = 2 \cosh(9\pi\theta/4) + 2 \cosh(5\pi\theta/4) - 2 \cosh((2\pi + \epsilon)\theta) + 2 \cosh((2\pi - \epsilon)\theta) - 4 \cosh(\pi\theta/2) + \frac{4 \sinh(\epsilon\theta)}{\pi\theta}.
\end{align*}
\]

We subsequently choose \( \epsilon = \frac{\pi}{2} \) and observe that the absolute value of the first term in the right-hand side of (2.23) is integrable. This then yields that the corresponding Laplace transform is \( O(te^{-R^2/(8t)}) \). The second term has been inverted in (2.20). Choosing \( \beta = \pi/4 \) gives a remainder \( O(te^{-R^2/(8t)}) \).

We finally consider the case \( 0 < \alpha < \pi/4 \). The contribution from \( D_1 \) to the inverse Laplace transform can be estimated by (2.22), and the lines below, since \( \epsilon \in (0, \pi/2) \) for this case too. Hence we obtain a remainder \( O(te^{-R^2(\sin^2(\alpha)/(4t))}) \). The contribution from \( D_2 \) to the inverse Laplace transform
follows by a minor modification of (2.22). We have that
\[
1 - \cosh((2\pi - 2\alpha)\theta)
\]
\[
\frac{(4\pi \sinh(\pi\theta/2))\cosh(\pi\theta)}{\pi\theta \sinh(2\pi\theta)}
\]
\[
= \frac{\cosh(\pi\theta/2) - \frac{1}{2} \cosh((\frac{3\pi}{2} - 2\alpha)\theta) - \frac{1}{2} \cosh((\frac{3\pi}{2} + 2\alpha)\theta) + \frac{1}{2} \cosh((2\pi + \epsilon)\theta) - \frac{1}{2} \cosh((2\pi - \epsilon)\theta)}{\pi\theta \sinh(2\pi\theta)}
\]
\[
- \frac{\sinh(\epsilon\theta)}{\pi\theta}.
\]
(2.24)

We choose \(\epsilon = \frac{\pi}{2} - 2\alpha \in (0, \pi/2)\), and obtain the remainder \(O(te^{-R^2\sin^2(2\alpha)/(4t)})\) from the corresponding integral in (2.17) by (2.20). The first term in the right-hand side of (2.24) gives \(O(te^{-R^2/(4t)})\).

\section{Proof of Theorem 1.1}
Kac’s principle of not feeling the boundary asserts that the solution of the heat equation with initial datum \(1_D\), where \(D\) is an open set in \(\mathbb{R}^m\), is equal to 1 on the interior of \(D\) up to an exponentially small remainder, as \(t \downarrow 0\). Kac formulated his principle in the case where a Dirichlet 0 boundary condition is imposed on all of \(\partial D\), that is \(\partial D = \emptyset\). It has been shown that it also holds if no boundary condition is imposed on \(\partial D\), that is \(\partial D = \emptyset\). See, for example, Proposition 9(i) in [1]. We have the following lemma.

\textbf{Lemma 3.1} If \(D\) is an open set in \(\mathbb{R}^2\), and if \(\partial D_+\) is a closed subset of \(\partial D\), then
\[
1 \geq \int_D dy \, p_{\mathbb{R}^2-\partial D_+}(x, y; t) \geq \int_D dy \, p_D(x, y; t) \geq 1 - 2e^{-d(x, \partial D)^2/(4t)}.
\]
(3.1)

\textbf{Proof.} Since the Dirichlet heat kernel is monotone in the domain, and since \(D \subset \mathbb{R}^m - \partial D_-\),
\[
p_{\mathbb{R}^m-\partial D_-}(x, y; t) \geq p_D(x, y; t) > 0, \, x \in D, \, y \in D, \, t > 0.
\]

Hence \(u_{D, \partial D_+}(x, t) \geq \int_D dy \, p_D(x, y; t)\). The latter integral has been bounded from below in Lemma 4 of [8]. Taking \(m = 2\) in the first line of (3.2) in that paper we find (3.1). The upper bound in (3.1) follows as \(p_{\mathbb{R}^m-\partial D_-}(x, y; t) \leq p_{\mathbb{R}^m}(x, y; t)\) and \(\int_{\mathbb{R}^m} dy \, p_{\mathbb{R}^m}(x, y; t) = 1\).

We now consider the following partition of the set \(D\) (see [7, 8, 4]). At each vertex of \(\partial D\) with angle \(\theta\), we consider the circular sector of radius \(R > 0\) and angle \(\theta\) that is contained in \(D\). For \(\delta > 0\) (to be specified later), we consider the set of points in \(D\) that are at distance less than \(\delta\) from \(\partial D\) and that are not contained in the union of the circular sectors. This set consists of a union of rectangles and cusps of height \(\delta\). Each sector has two neighboring cusps. In the partition of \(D\), cusp contributions of two types feature. That is, those cusps adjacent to a half-space with a Dirichlet 0 boundary condition on its boundary, and those cusps adjacent to a half-space with an open boundary (see Figure 3). Cusps of the latter type feature in [6], and those of the former type feature in [8].

Let \(H = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}\), and let
\[
E(\delta, R) = \{x \in \mathbb{R}^2 : x_1 < R, |x| > R, d(x, \partial H) < \delta\}
\]
be the cusp of height \(\delta\) adjacent to \(H\).

We first consider the case where a Dirichlet 0 boundary condition has been imposed on \(\partial H\).

\textbf{Lemma 3.2} If \(\delta < R\) then
\[
\int_{E(\delta, R)} dx \, u_H(x; t) = |E(\delta, R)| - \frac{2Rt^{1/2}}{\pi^{1/2}} \int_1^\infty dw \int_0^1 \frac{vdv}{(1 - v^2)^{1/2}} e^{-R^2v^2/(4t)} + O(t^{1/2}e^{-\delta^2/(4t)}).
\]
Figure 3: A circular sector contained in a Dirichlet-open wedge with angle $\alpha$ and its neighbouring cusps (the Dirichlet, respectively open, edge is displayed as a solid, respectively dashed, line).

**Proof.** We include its short proof. See also (4.7) in [8]. We have that

$$u_H(x; t) = \frac{1}{(\pi t)^{1/2}} \int_0^{\pi/2} dq e^{-q^2/(4t)}.$$  

Since the length of the line segment in $E(\delta, R)$ parallel to the $x_1$ axis equals $R - (R^2 - x_2^2)^{1/2}$, we have that

$$\int_{E(\delta, R)} dx u_H(x; t) = \frac{1}{(\pi t)^{1/2}} \int_0^\delta dx_2 (R - (R^2 - x_2^2)^{1/2}) \int_0^{x_2} dq e^{-q^2/(4t)}$$

$$= |E(\delta, R)| - \frac{1}{(\pi t)^{1/2}} \int_0^\delta dx_2 (R - (R^2 - x_2^2)^{1/2}) \int_{x_2}^\infty dq e^{-q^2/(4t)}$$

$$= |E(\delta, R)| - \frac{1}{(\pi t)^{1/2}} \int_0^\delta dx_2 (R - (R^2 - x_2^2)^{1/2}) x_2 \int_1^\infty dw e^{-w^2 x_2^2/(4t)}$$

$$= |E(\delta, R)| - \frac{2t^{1/2}}{\pi^{1/2}} \int_1^\infty \frac{dw}{w^2} \int_0^\delta \frac{x_2 dx_2}{(R^2 - x_2^2)^{1/2}} e^{-w^2 x_2^2/(4t)}$$

$$+ \frac{2t^{1/2}}{\pi^{1/2}} (R - (R^2 - \delta^2)^{1/2}) \int_1^\infty \frac{dw}{w^2} e^{-w^2 \delta^2/(4t)}.$$  

Both the third and fourth terms in the right-hand side of (3.3) are $O(t^{1/2} e^{-\delta^2/(4t)})$.  

For the cusp adjacent to a half-space with an open boundary we have the following.

**Lemma 3.3** If $\delta < R$ then

$$\int_{E(\delta, R)} dx u_{H, \delta}(x; t) = |E(\delta, R)| - \frac{R t^{1/2}}{\pi^{1/2}} \int_1^\infty \frac{dw}{w^2} \int_0^1 \frac{vdv}{(1 - v^2)^{1/2}} e^{-R^2 v^2 w^2/(4t)} + O(t^{1/2} e^{-\delta^2/(4t)}).$$  

**Proof.**

$$u_{H, \delta}(x; t) = 1 - \frac{1}{(4\pi t)^{1/2}} \int_{x_2}^\infty dq e^{-q^2/(4t)}.$$  

(3.5)
Comparing (3.5) with
\[ u_H(x; t) = 1 - \frac{1}{(\pi t)^{1/2}} \int_{x_2}^{\infty} dq e^{-q^2/(4t)}, \]
we see that the second, third and fourth terms in the right-hand side of (3.3) are weighted with a factor \( \frac{1}{2} \) in the computation of the integral in the left-hand side of (3.4). This then gives (3.4).

\[ \text{Lemma 3.4} \]
If \( S = \{(x_1, x_2) \in \mathbb{R}^2 : 0 < x_1 < L, 0 < x_2 < \delta \} \), then
\[ \int_S dx \ u_H(x; t) = |S| - \frac{2Lt^{1/2}}{\pi^{1/2}} + O(t^{1/2}e^{-\delta^2/(4t)}), \] (3.6)
and
\[ \int_S dx \ u_H(\theta; x; t) = |S| - \frac{Lt^{1/2}}{\pi^{1/2}} + O(t^{1/2}e^{-\delta^2/(4t)}). \] (3.7)

\[ \text{Proof.} \]
\[ \int_S dx \ u_H(x; t) = \int_0^L dx_1 \int_0^{\delta} dx_2 \left(1 - \frac{1}{(\pi t)^{1/2}} \int_{x_2}^{\infty} dq e^{-q^2/(4t)}\right) \]
\[ = |S| - \frac{2Lt^{1/2}}{\pi^{1/2}} + \int_\delta^{\infty} dx_2 \frac{L}{(\pi t)^{1/2}} \int_{x_2}^{\infty} dq e^{-q^2/(4t)} \]
\[ = |S| - \frac{2Lt^{1/2}}{\pi^{1/2}} + O(t^{1/2}e^{-\delta^2/(4t)}). \]

This proves (3.6). The observation concluding the proof of Lemma 3.3 immediately implies (3.7).

\[ \text{Proof of Theorem 1.1} \]
Similarly to the strategies of the proofs in [7, 8, 6], it remains to add up the contributions described in the model computations above and to apply Lemma 3.1 to the compensating terms.

We first choose \( R \) and \( \delta \) appropriately in the partition of \( D \). Let \( v \) be an arbitrary vertex of the polygonal boundary, and let \( e_v \) denote the union of the two edges of \( \partial D \) adjacent to \( v \). We choose
\[ R = \frac{1}{2} \inf \{d(v, y) : y \in \partial D - e_v\}. \]

This choice of \( R \) guarantees that all circular sectors are non-overlapping. Moreover the distance from any point in a circular sector with vertex \( v \) and angle \( \theta \) to \( W_\theta - D \) is at least \( R \). By Lemma 3.1 we have that the model computations for the sectors with angles in \( A, B, C \) give the appropriate contributions to \( G_D(t) \) in (1.11) up to an additive constant which is bounded in absolute value by \( 2|D|e^{-R^2/(4t)} \).

Next we choose \( \delta \) sufficiently small to ensure that the cusps are pairwise disjoint. We define \( \epsilon \) to be the smallest interior angle of the boundary \( \partial D \):
\[ \epsilon = \min \{A \cup B \cup C\}. \]

It is straightforward to check that
\[ \delta = \frac{R}{2} \sin(\epsilon/2) \]
satisfies the aforementioned conditions. For this choice of \( \delta \), there is a pair of cusps whose boundaries intersect in a point. The distance between the cusp and \( H - D \) is larger than \( 2\delta = R\sin(\epsilon/2) \) (if we consider the cusp corresponding to the sector with angle \( \epsilon \)). By Lemma 3.1 we have that the model computations in Lemma 3.2 and 3.3 give the appropriate contributions to \( G_D(t) \) up to an additive constant which is bounded in absolute value by \( 2|D|e^{-R^2(\sin(\epsilon/2))^2/(4t)} \).

Next we consider the contribution of the subset of \( D \) which is within distance \( \delta \) of \( \partial D \), and which is not contained in any of the radial sectors and their corresponding cusps. This subset is a collection
of disjoint rectangles supported either by a Dirichlet or an open edge respectively. Each such rectangle has at least distance \( \delta \) to any of the other edges. We conclude that, by Lemma 3.4 and Lemma 3.1, they give the relevant contributions to \( G_D(t) \) up to an additive constant which is bounded in absolute value by \( 2|D|e^{-R^2\left(\sin(\epsilon/2)^2\right)/16t}\).

The remaining subset of \( D \) which is not contained in a sector, cusp or rectangle has distance \( \delta \) to the boundary, and so contributes its measure up to an additive constant which is bounded in absolute value by \( 2|D|e^{-R^2\left(\sin(\epsilon/2)^2\right)/16t} \), by Lemma 3.1. All remainders above and in the proof of Theorem 1.2 are of the form \( O(t^\zeta e^{-\eta/t}) \), \( \zeta \geq 0, \eta > 0 \). This gives the remainder in (1.11).

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