LMI approach to global stability analysis of stochastic delayed Lotka-Volterra models

Krisztina Kiss\textsuperscript{a, *}, Éva Gyurkovics\textsuperscript{a}

\textsuperscript{a}Mathematical Institute, Budapest University of Technology and Economics, Budapest, Pf. 91, 1521, Hungary

Abstract
This paper is devoted to the stability analysis of an $n$ species Lotka-Volterra system with discrete and distributed delays. Stochastic perturbations to the parameters of the model are allowed. Sufficient conditions for the almost sure global asymptotic stability of the positive equilibrium are derived in terms of LMIs. The efficiency of the proposed method is illustrated by numerical examples.

Keywords: Stochastic differential delay equation; Global asymptotic stability; Discrete time-dependent delay; Distributed delay; LMI

1. Introduction
In the past decades, one of the most popular models in mathematical biology has been the Lotka-Volterra model that have been studied in a huge number of works. In particular, the books \cite{1}–\cite{2} are good references in this area. A large class of models is given by ordinary differential equations, but it is often more realistic to use delayed functional differential equations (FDEs) to describe such models (e.g. \cite{3}–\cite{6}). Consider the $n$-species Lotka-Volterra model of the form

\[
\dot{u}_i(t) = u_i(t) \left[ \rho_i - \sum_{j=1}^{n} a_{ij} u_j(t) - \sum_{j=1}^{n} a_{ij}^d u_j(t - \tau_{ij}(t)) - \sum_{j=1}^{n} a_{ij}^D \int_{t - \tau_{ij}}^{t} e^{\alpha_{ij} (\eta - t)} u_j(\eta) d\eta \right], \quad i = 1, \ldots, n \tag{1}
\]

consisting both discrete and distributed delays. Here $u_i(t)$, $(i = 1, \ldots, n)$ represent the population sizes, the parameters $a_{ij}, a_{ij}^d, a_{ij}^D$ are the so-called interaction coefficients, $\rho_i > 0$ is the $i$th intrinsic growth rate, while the values $\alpha_{ij} \geq 0$ play the role of the weighting parameters of the distributed delays. The discrete delays $\tau_{ij}$ are supposed to be differentiable functions of the time satisfying conditions

\[
0 < \tau_{ij}(t) \leq \tau_{ij}, \quad \dot{\tau}_{ij}(t) \leq \tau_{ij} \tag{2}
\]

with known constants $\tau_{ij}, \tau_{ij}^d$. Define $\tau = \max_{1 \leq i,j \leq n} \tau_{ij}$.

Let us suppose that $\hat{A}$ has a positive equilibrium state $u^* = (u_1^*, u_2^*, \ldots, u_n^*)^T \in \mathbb{R}_+^n$, i.e. $\hat{A}u^* = \rho$, where the notations $\hat{A} = A + A^d + A^D$, $A = (a_{ij})_{n \times n}$, $A^d = (a_{ij}^d)_{n \times n}$, $A^D = (\beta_{ij} a_{ij}^D)_{n \times n}$, $\rho = (\rho_1, \ldots, \rho_n)^T$, with

\[
\beta_{ij} = \int_{t - \tau_{ij}}^{t} e^{\alpha_{ij} (\eta - t)} d\eta = \begin{cases} \frac{1}{\alpha_{ij}} (1 - e^{-\alpha_{ij} \tau_{ij}}), & \text{if } \alpha_{ij} \neq 0, \\ \frac{1}{\tau_{ij}}, & \text{if } \alpha_{ij} = 0 \end{cases} \tag{3}
\]

have been used.

The population systems are almost always subjected to environmental noises (see e.g. \cite{1}, \cite{7}–\cite{10}). Similarly to \cite{8}, we will take into account random fluctuations, namely white noise, affecting on $\rho_i$, and depending on how much the current population sizes differ from the equilibrium state. Thus, we replace $\rho_i$ by $\rho_i + \sigma$.
We shall consider system (6) with the initial condition $u_t = 1_{ij}$. Then, for any initial function $u_t$, we have $u_t$ has global positive solution almost surely for any initial function $u_t$.

In order to write model (4) in a more compact form, we introduce some notations. Let $u^d (t)$, $u^D (t) \in \mathbb{R}^{n^2}$ be vectors having elements

$$u^d_k(t) = u_j(t - \tau_{ij}(t)), \quad u^D_k(t) = \int_{t-\tau_{ij}}^{t} e^{\alpha_{ij}(\eta-t)} u_j(\eta) d\eta,$$

for $k = (i-1)n + j$, $i, j = 1, \ldots, n$.

Let the matrices $A^d, A^D \in \mathbb{R}^{n \times n^2}$ be defined by $A^d = \text{diag}\{a^d_1, \ldots, a^d_n\}$ and $A^D = \text{diag}\{a^D_1, \ldots, a^D_n\}$, where $a^d_i$ and $a^D_i$ are the $i$th row vectors of the matrices $A^d, A^D$, respectively. Further, let $g : \mathbb{R}^n \to \mathbb{R}^{n \times n}$ be a diagonal matrix with $g_{ii}(u) = \sum_{j=1}^{n} \sigma_{ij} u_j$, and zero otherwise. Then system (4) can be written as

$$du(t) = \text{diag}\{u_1(t), \ldots, u_n(t)\} \left\{ \left[ \rho - A u(t) - A^d u^d(t) - A^D u^D(t) \right] dt + g(u(t) - u^*) dw(t) \right\}.$$

We shall consider system (4) with the initial condition $u_t = \varphi_0(t)$, if $t \in [-\tau, 0]$ and $\varphi_0 \in C([-\tau, 0], \mathbb{R}_+^n)$. Our aim is to derive sufficient conditions for ensuring that

- equation (4) has global positive solution almost surely for any initial function $\varphi_0 \in C([-\tau, 0], \mathbb{R}_+^n)$;
- the equilibrium state of (4) is almost surely globally asymptotically stable in $\mathbb{R}_+^n$.

Based on some new developments in the field, a new Lyapunov-Krasovskii functional is introduced, and the stability condition is given in terms of linear matrix inequalities (LMIs). To the best of our knowledge, only variational system based local results have been given up to now by means of LMIs in the literature (see [4], and the references therein). The result obtained in this work demonstrates that LMI can be applied for investigating the stability behaviour of the nonlinear Lotka-Volterra equation.

2. Main results

We shall first formulate a condition under which system (4) has a unique global positive solution a.s.

**Theorem 1.** Assume that the discrete delay functions satisfy (2) and the condition

$$\tau^d := \max_{1 \le i, j \le n} \tau_{ij}^d < 1,$$

and the noise intensity matrix $\sigma = (\sigma_{ij})_{n \times n}$ has the property that $\sigma_{ii} > 0$ for $i = 1, \ldots, n$, and $\sigma_{ij} \ge 0$ for $i, j = 1, \ldots, n$. Then, for any initial function $\varphi_0 \in C([-\tau, 0], \mathbb{R}_+^n)$, equation (4) has a unique positive solution $u(t)$ on $[-\tau, \infty)$, and the solution remains in $\mathbb{R}_+^n$ with probability 1.

**Proof.** The proof follows the same line as [8] and [7], therefore the details are omitted to save space. \(\square\)

Now we turn to the problem of stochastic asymptotic stability of the equilibrium state of (4). To this end, shifting the origin to the equilibrium and applying the notation of $x(t) = u(t) - u^*$, (8) is transformed to

$$dx(t) = X^*(t) \left\{ \left[ -\tilde{A} x(t) - A^d \tilde{x}^d(t) - A^D \tilde{x}^D(t) \right] dt + g(x(t)) dw(t) \right\},$$
In order to formulate the stability condition, let us define the block entry matrices

\[ e_n \] are applied, \( B \in R^{n \times n^2} \) is a diagonal matrix with diagonal elements \( B_{kk} = \beta_{ij} \), for \( k = (i-1)n + j, \ i, j = 1, \ldots, n \) and the vectors \( x^D(t), x^D(t) \in R^{n^2} \) are defined analogously to \([3]\).

In order to formulate the stability condition, let us define the block entry matrices

\[
A = [A \, A^d, A^d, 0_{n \times n^2}] \in R^{n \times (n+3n^2)}, \quad e = [I_n, 0_{n \times n^2}, 0_{n \times n^2}, 0_{n \times n^2}] \in R^{n \times (n+3n^2)},
\]

\[
e_2 = [0_{n^2 \times n}, I_{n^2}, 0_{n^2}, 0_{n^2}] \in R^{n^2 \times (n+3n^2)},
\]

\[
e_4 = [0_{n^2 \times n}, 0_{n^2}, 0_{n^2}, I_{n^2}] \in R^{n^2 \times (n+3n^2)},
\]

and the diagonal matrices \( U^* = \text{diag}\{u^*_1, \ldots, u^*_n\} \in R^{n \times n} \), \( T, T^d, A^\alpha \in R^{n \times n^2} \) with diagonal entries

\[
T_{kk} = \tau_{ij}, \quad T^d_{kk} = 1 - \tau_{ij}, \quad A^\alpha_{kk} = \alpha_{ij}, \quad k = (i-1)n + j, \ i, j = 1, \ldots, n.
\]

**Theorem 2.** Assume that the delay functions satisfy conditions \([2]\) and \([7]\). If there exist positive numbers \( p_i, q_{ij}, r_{ij}, s_{ij} \) for \( i, j = 1, \ldots, n \) such that diagonal matrices \( P \in R^{n \times n}, Q, R, S \in R^{n^2 \times n^2} \) with diagonal entries

\[
P_{ii} = p_i, \quad Q_{kk} = q_{ij}, \quad R_{kk} = r_{ij}, \quad S_{kk} = s_{ij}, \quad k = (i-1)n + j, \ i, j = 1, \ldots, n
\]

satisfy the LMI

\[
\Sigma = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 < 0,
\]

where

\[
\Sigma_1 = -\frac{1}{2} e_i^T P A + \frac{1}{2} A^T P e_i + \frac{1}{2} e_i^T A^T P U^* e_i,
\]

\[
\Sigma_2 = e_i^T Q_1 e_i - (e_2 + e_3)^T Q_d e_2 + e_3), \quad Q_1 = T^T Q T, \quad Q_d = T^d Q,
\]

\[
\Sigma_3 = e_i^T R_{ij} e_i - \left[ (e_3 + B e_5)^T, e_4^T \right] \left[ \begin{array}{cc} 4R_{ij} & -6R_{ij} \\ -6R_{ij} & 12R_{ij} + 4A_{ij} R \end{array} \right] \left[ (e_3 + B e_5) \right] + e_4,
\]

\[
\Sigma_4 = e_i^T S_{ij} e_4 + e_j^T S_{ij} e_1 - (e_2 + B e_5)^T S_{ij} (e_2 + B e_5) - 2e_4^T A_{ij} S_{ij} e_4,
\]

\[
R_{ij} = T^T T R_{ij}, \quad R_{ij} = T - R_{ij}, \quad S_{ij} = T^T S_{ij}, \quad S_{ij} = T - S_{ij},
\]

then the equilibrium state \( u^* \) of system \([4]\) is almost surely globally asymptotically stable.

**Proof.** Consider the Lyapunov-Krasovskii functional candidate

\[
V(t, x_t) = \sum_{i=1}^{4} \sum_{l=1}^{n} V_i(t, x_t) = \sum_{i=1}^{n} \sum_{l=2, i,j=1}^{n} V_{ij}^l(t, x_t),
\]

\[
V_i^l(x_t) = p_i \left( x_i - u^*_i \right) \text{l} \left( \frac{x_i + u^*_i}{u^*_i} \right), \quad V_{ij}^l(t, x_t) = r_{ij} \int_{i-1}^{t} (\eta - t + r_{ij}) e^{2a_{ij}(\eta-t)} x_i(\eta)^2 d\eta,
\]

\[
V_{ij}^2(t, x_t) = q_{ij} \int_{i-1}^{t} x_i(\eta)^2 d\eta, \quad V_{ij}^4(t, x_t) = s_{ij} \int_{i-1}^{t} \left( \int_{i-1}^{t} e^{\alpha_{ij}(\zeta-t)} x_j(\zeta) d\zeta \right)^2 d\eta.
\]

Fix an arbitrary \( \varphi_0 \in C([-\tau, 0], R^+) \) for \([6]\), and consider the solution of \([5]\) corresponding to the initial function \( \varphi_0(t) = \varphi_0(t) - u^* \), and compute the derivative of \( V \) applying the functional Ito’s formula given in Eqn. (3) of \([11]\). (Details and notations see in \([11]\).)
Since $V_1$ is independent of $t$, \( \frac{\partial V_1}{\partial x}(x) = \frac{p}{x+i_u^k x}, \) and $\frac{\partial^2 V_1}{\partial x^2}(x) = \frac{p u^*}{(x_i(t)+u^*_j x)^2}$, applying the above mentioned Itô’s formula in symbolic differential form to $V_1(x)$ yields
\[
\begin{align*}
dV_1(x(t)) &= \left[-x(t)^T P \left( \hat{A} x(t) + A^D \hat{x}^D(t) + A^D \tilde{x}_D(t) \right) + x(t)^T \sigma^T P \sigma x(t) \right] dt + G(x(t)) dw(t),
\end{align*}
\]
where $G(x) = [p_1 x_1 \Sigma_{i=1}^n \sigma_{ij} x_j, \ldots, p_n x_n \Sigma_{i=1}^n \sigma_{ij} x_j] \in \mathbb{R}^{1 \times n}$.

Next one can verify that $\nabla_x V_1(t, x_t) = 0$, and $\nabla V_1(t, x_t)$ can be computed by taking the time derivative, if $\ell = 2, 3, 4$. Therefore $\mathcal{L} V_2(t, x_t)$ can be estimated as follows:
\[
\mathcal{L} V_2(t, x_t) \leq \sum_{j=1}^n \left( \frac{n}{2} \right) (1 - \tau_j) q_{ij} x_j (t - \tau_j(t))^2 = x(t)^T Q_1 x(t) - x^d(t)^T Q_d x^d(t),
\]
where $Q_1$ and $Q_d$ are given in (14), and (2) has been taken into account.

Let us compute now $\mathcal{L} V_3^j(t, x_t)$:
\[
\begin{align*}
\mathcal{L} V_3^j(t, x_t) &= r_{ij} \tau_j x_j (t)^2 - r_{ij} \int_{t - \tau_{ij}}^t e^{2\alpha_{ij}(n-t)} x_j(\eta)^2 d\eta - 2\alpha_{ij} r_{ij} \int_{t - \tau_{ij}}^t (\eta - t + \tau_{ij}) e^{2\alpha_{ij}(\eta-t)} x_j(t)^2 d\eta.
\end{align*}
\]

The first integral term can be estimated by the Wirtinger inequality (the applied form see in [12]), while the second integral term can be estimated by the double-integral Jensen inequality (see e.g. [13]). Thus, we obtain with $k = (i-1)n + j$, $i, j = 1, \ldots, n$ that
\[
\begin{align*}
\mathcal{L} V_3^j(t, x_t) &\leq r_{ij} \tau_j x_j (t)^2 - r_{ij} \tau_{ij} \left( (x^D_k(t))^2 + 3(x^D_k(t) - 2z^D_k(t))^2 \right) - 4\alpha_{ij} r_{ij} (z^D_k(t))^2 \\
&= r_{ij} \tau_j x_j (t)^2 - r_{ij} \tau_{ij} \left( 4(x^D_k(t))^2 - 12x^D_k(t)z^D_k(t) + 12(z^D_k(t))^2 \right) - 4\alpha_{ij} r_{ij} (z^D_k(t))^2.
\end{align*}
\]
where the vector $z^D(t) \in \mathbb{R}^n$ is defined with the elements
\[
z^D_k(t) = \frac{1}{\tau_{ij}} \int_{t - \tau_{ij}}^t \int_{\eta} e^{\alpha_{ij}(n-t)} x_j(\zeta)d\zeta d\eta.
\]

Summing up and applying (17), we obtain that
\[
\mathcal{L} V_3(t, x_t) \leq x(t)^T \mathcal{R}_1 x(t) - \left[ x^D(t)^T, z^D(t)^T \right] \begin{bmatrix} 4\mathcal{R}_{1} & -6\mathcal{R}_{1,2} \\ -6\mathcal{R}_{2,1} & 12\mathcal{R}_{2} + 4\mathcal{A}_0 \mathcal{R} \end{bmatrix} \begin{bmatrix} x^D(t) \\ z^D(t) \end{bmatrix}.
\]

Computing $\mathcal{L} V_4^j(t, x_t)$, and estimating the last term by Jensen’s inequality yields
\[
\begin{align*}
\mathcal{L} V_4^j(t, x_t) &= -\frac{s_{ij}}{\tau_{ij}} \int_{t - \tau_{ij}}^t e^{\alpha_{ij}(n-t)} x_j(\zeta)d\zeta + x_j(t) \frac{2s_{ij}}{\tau_{ij}} \int_{t - \tau_{ij}}^t \int_{\eta} e^{\alpha_{ij}(n-t)} x_j(\zeta)d\zeta d\eta \\
&= -\frac{2\alpha_{ij} s_{ij}}{\tau_{ij}} \int_{t - \tau_{ij}}^t \int_{\eta} e^{\alpha_{ij}(n-t)} x_j(\zeta)d\zeta d\eta \leq -\frac{s_{ij}}{\tau_{ij}} (x^D_k(t))^2 + 2s_{ij} z^D_k(t)x_j(t) - 2s_{ij} \alpha_{ij} (z^D_k(t))^2.
\end{align*}
\]

Summing up and applying (17), we obtain that
\[
\mathcal{L} V_4(t, x_t) \leq -x^D(t)^T \mathcal{S}_2 x^D(t) + x(t)^T \mathcal{S}_1 z^D(t) + z^D(t)^T \mathcal{S}_1 x(t) - 2z^D(t)^T \mathcal{A}_0 z^D(t).
\]

Let an extended variable $\xi(t) = (x(t)^T, \hat{x}^d(t)^T, \hat{x}^D(t)^T, z^D(t)^T)^T \in \mathbb{R}^{n + 3n^2}$ be introduced. Then, applying (19)-(20) one can check with a straightforward computation that
\[
dV(t, x_t) \leq \xi(t)^T \left( \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 \right) \xi(t) dt + G(x(t)) dw(t),
\]
where $\Sigma_{\ell}, \ell = 1, \ldots, 4$ are give by (13)-(17). Therefore, the statement of the theorem follows from (12) based on the stability theory of stochastic differential equations ([1], [14]).

**Remark 1.** Observe that it follows from (12) and (27) that $\text{EV}(T, x_T) \leq \text{EV}(0, x_0)$ for any $T > 0$. Therefore the conditions of Theorem 2 yield as an alternative for the existence of global positive solutions of (10) for any initial function $\varphi_0 \in C([-\tau, 0], \mathbb{R}_+^n)$ with probability 1. (Cf. with Theorem 2.1 and Theorem 2.3 of [14].) This alternative is useful in such cases, when the conditions $\sigma_{ii} > 0, \forall i$, are not satisfied (see Example 2 below).

### 3. Numerical examples

**Example 1.** Consider a 3 species Lotka-Volterra model with the data

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0.5 & 2.5 & 0.5 \\ 0 & 1 & 2.5 \end{bmatrix}, \quad A^d = \begin{bmatrix} 0.5 & 0.2 & 0.1 \\ 0.1 & 0.6 & 0 \\ 0.4 & 0.1 & 0.8 \end{bmatrix}, \quad A^D = \lambda_1 \begin{bmatrix} 0.4 & 0.5 & 0 \\ 0.2 & 1 & 0.1 \\ 0.1 & 0.1 & 0.5 \end{bmatrix},$$

$$\tau = \begin{bmatrix} 0.9 & 0.5 & 0.05 \\ 0.4 & 1 & 0.05 \\ 0.05 & 0.1 & 0.5 \end{bmatrix}, \quad \tau^d = \tau_d \begin{bmatrix} 1 & 0.8 & 0.5 \\ 0.6 & 0.7 & 0.4 \\ 0.4 & 0.3 & 0.5 \end{bmatrix}, \quad \sigma = \lambda_2 \begin{bmatrix} 0.2 & 0.05 & 0 \\ 0.15 & 0.1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

and $\alpha_{ij} = 2$. The results obtained by Theorem 2 for different values of the parameters $\lambda_1$, $\lambda_2$, $\tau$ and $\tau_d$ are given in Table 1. We note that, in the case of $A^D = 0$, the LMI of Theorem 2 is independent of the delay upper bound, but it depends only on the delay derivative upper bound. Thus, if it has a feasible solution for some $\tau_d$, then it has a feasible solution for the same $\tau_d$ and arbitrary $\tau$. For the value $\tau_d = 0.6515$ in the last column of Table 1, LMI (12) has a feasible solution, if $A^D = 0$ and $\tau = 100$, but it turns to be infeasible for any $\tau_d$, if $\tau_d$ is slightly increased.

**Example 2.** Consider the two species model of [10], where no delays are taken into account, i.e. $A^d = A^D = 0$, and let

$$A_1 = \begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}, \quad \rho^1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix}, \quad \rho^2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}, \quad \sigma = \begin{bmatrix} 0 & \sigma_1 \\ \sigma_2 & 0 \end{bmatrix}.$$ 

If we take $A = A_1, \rho = \rho^1, \sigma_1 = 1.5, \sigma_2 = 2$, as in [10], and take formally the delay parameters as $\tau = 10^{-5} I_2$, $\tau^d = 0, \alpha_{ij} = 0$, then the LMI of Theorem 2 is feasible. If we change the data to $A = A_2, \rho = \rho^2, \sigma_1 = \sqrt{2}, \sigma_2 = \sqrt{2}$, then the LMI of Theorem 2 is feasible, while assumption (H1) of [10] fails. Simulation supports the stability of the equilibrium $u^* = [1, 1]^T$. This suggest that Theorem 2 may lead to less conservative result, than some previously published stability conditions.

### 4. Acknowledgements

The research reported in this paper has been supported by the National Research, Development and Innovation Fund (TUDFO/51757/2019-ITM, Thematic Excellence Program).
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