QUASI-FOREST SIMPLICIAL COMPLEXES AND ALMOST COHEN-MACAULAY

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Abstract. In this paper we study the quasi-forest simplicial complexes and we define the concept of simplicial $k$-cycle (denoted by $S_k$) and simplicial $k$-point (denoted by $P_k$). We show that a simplicial complex $\Delta$ is quasi-forest if and only if it does not have any $P_k$ and any $S_k$ for $k \geq 3$. Furthermore we characterize almost Cohen-Macaulay quasi-forest simplicial complexes. In the end we show that the cycle graph $G = C_n$ is almost Cohen-Macaulay if and only if $n = 3, 4, 5, 6, 7, 8, 9, 11$.

Introduction

Throughout this paper, we assume that $R = K[x_1, \ldots, x_n]$ is the polynomial ring in $n$ variables over a field $K$ and $G$ is a simple graph (without loops and multiple edges) with vertex set $V(G) = \{x_1, \ldots, x_n\}$ and edge set $E(G)$. One associates to $G$ the edge ideal $I(G)$ of $R$ which is generated by all monomial $x_ix_j$ such that $\{x_i, x_j\} \in E(G)$. The independence complex and the clique complex of the graph $G$ are defined by $\text{Ind}(G) = \{A \subseteq V(G) | A$ is an independence set in $G\}$ and $\Delta(G) = \{B \subseteq V(G) | B$ is a clique of $G\}$, respectively. Note that an independent set of $G$ is a subset $A$ of $V(G)$ such that none of its elements are adjacent and a clique of $G$ is a subset $B$ of $V(G)$ such that $\{x_i, x_j\} \in E(G)$ for all $x_i, x_j \in B$ with $i \neq j$. It easy to see that $\Delta(G) = \text{Ind}(\overline{G})$, where $\overline{G}$ is the complement of $G$. Using the Stanley-Reisner correspondence, we can associate to $G$ the independent complex $\text{Ind}(G)$, where $I_{\text{Ind}(G)} = I(G)$. Hence the Stanley-Reisner ring of $\text{Ind}(G)$ is $R/I(G)$. The graph $G$ or the edge ideal $I(G)$ is called Cohen-Macaulay if $R/I(G)$ is Cohen-Macaulay. Cohen-Macaulay graphs were studied in [25, 6]. A complete classification of Cohen-Macaulay graphs does not exist. Also, a graph $G$ or the edge ideal $I(G)$ is called almost Cohen-Macaulay if $R/I(G)$ is almost Cohen-Macaulay. We say that $R/I(G)$ is almost Cohen-Macaulay when $\text{depth } R/I(G) \geq \dim R/I(G) - 1$. Almost Cohen-Macaulay rings have been studied in [13, 17, 18, 16, 2, 20, 21, 19].

Let $\Delta$ be a simplicial complex. A facet $F$ of $\Delta$ is called leaf, if there exists a facet $M$ of $\Delta$ with $F \neq M$ such that $N \cap F \subset M \cap F$ for all facet $N$ of $\Delta$ with $N \neq F$. If each subcomplex $\Gamma$ of $\Delta$ has a leaf, then $\Delta$ is called a forest. A simplicial complex $\Delta$ is called quasi-forest, if there exists an order $F_1, \ldots, F_r$ of the facets of $\Delta$ such that $F_i$ is a leaf of the simplicial complex $\langle F_1, \ldots, F_i \rangle$ for each $i = 1, \ldots, r$. A free vertex is a vertex which belongs to precisely one facet. It is known that each

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leaf has a free vertex. But the converse is not true in general. It is clear that every forest is a quasi-forest. We say that the graph \( G \) is quasi-forest when \( \text{Ind}(G) \) is a quasi-forest simplicial complex. The concept of quasi-forest has been studied in \cite{28, 8, 15, 10, 11}.

In this paper we define the concept of simplicial \( k \)-cycle (denoted by \( S_k \)) and simplicial \( k \)-point (denoted by \( P_k \)) and we give some examples. We study the quasi-forest simplicial complex and we prove that a simplicial complex \( \Delta \) is quasi-forest if and only if it does not have any \( P_k \) and any \( S_k \) for \( k \geq 3 \). Furthermore, we characterize almost Cohen-Macaulay quasi-forest simplicial complexes as a generalization of \cite{10} Proposition 2.3. In the end we prove that the cycle graph \( G = C_n \) is almost Cohen-Macaulay if and only if \( n = 3, 4, 5, 6, 7, 8, 9, 11 \). For any unexplained notion or terminology, we refer the reader to \cite{14, 26}. Several explicit examples were performed with help of the computer algebra systems Macaulay2 \cite{12}.

1. Preliminaries

In this section, we recall some definitions and known results which are used in this paper.

A simplicial complex \( \Delta \) on the vertex set \( V = \{x_1, \ldots, x_n\} \) is a collection of subsets of \( V \) such that (i) \( \{x_i\} \in \Delta \) for every \( 1 \leq i \leq n \), and (ii) if \( F \in \Delta \) and \( H \subseteq F \), then \( H \in \Delta \). Each element \( F \) of \( \Delta \) is called a face of \( \Delta \) and it is called an \( i \)-face when \( |F| = i + 1 \). The dimension of a face \( F \) is \( |F| - 1 \) and the dimension of \( \Delta \) is defined to be \( \dim \Delta = d - 1 \), where \( d = \max\{|F| \mid F \in \Delta\} \). A facet of \( \Delta \) is maximal face (with respect to inclusion). The set \( \mathcal{F}(\Delta) := \{F_1, \ldots, F_r\} \) is the set of all facets of \( \Delta \). A simplicial complex \( \Delta \) with the facets \( F_1, \ldots, F_r \) is denoted by \( \Delta = \langle F_1, \ldots, F_r \rangle \). The Stanley-Reisner ideal of \( \Delta \) is \( I_{\Delta} := \langle \prod_{x_i \in F} x_i \mid F \not\in \Delta \rangle \) and the quotient ring \( K[\Delta] = R/I_{\Delta} \) is the Stanley-Reisner ring of \( \Delta \) over a field \( K \) where \( R = K[x_1, \ldots, x_n] \). A simplicial complex \( \Delta \) is pure if every facet has the same cardinality. A simplicial complex \( \overline{\Delta} = \langle \overline{F} \mid F \in \mathcal{F}(\Delta) \rangle \) is the complement of \( \Delta \). The Alexander dual of \( \Delta \), denoted by \( \Delta^\vee \), is defined as \( \Delta^\vee = \{V \setminus F \mid F \not\in \Delta\} \). The subcomplex \( \Delta(i) = \{F \in \Delta \mid \dim F = i\} \) is called the pure \( i \)-skeleton of \( \Delta \). Terai \cite{22} proved that if \( I \) is a square-free monomial ideal of \( R \), then the Castelnuovo-Mumford regularity \( \text{reg}(I_{\Delta^\vee}) = \text{proj dim}(R/I_{\Delta}) \).

Herzog, Hibi and Zheng \cite{15} proved the following beautiful result:

**Proposition 1.1.** A simplicial complex \( \Delta \) is quasi-forest if and only if \( \text{proj dim } I(\overline{\Delta}) = 1 \).

Now, since \( I(\overline{\Delta}) = I_{\Delta^\vee} \) (see \cite{15} Lemma 1.2) and by using Terai’s result we conclude that a simplicial complex \( \Delta \) is quasi-forest if and only if \( \text{reg } I_{\Delta} = 2 \).

Recall that a simplicial complex is called flag, if all minimal nonfaces consist of two elements, equivalently, \( I_{\Delta} \) is generated by quadratic monomials. By \cite{15} Lemma 3.2, a quasi-forest simplicial complex is flag. A monomial ideal \( I \) generated in degree \( d \) has a linear resolution if and only if the Castelnuovo-Mumford regularity of \( I \) is \( \text{reg}(I) = d \) (see \cite{23} Lemma 49). Fröberg \cite{9} proved that the edge ideal \( I(G) \) has a
linear resolution if and only if $G$ is chordal. Recall that a graph $G$ is called chordal if each cycle of length $> 3$ has a chord.

As before the independent simplicial complex of a graph $G$ is the clique complex of $G$ and vice versa. One can rephrase [15, Lemma 3.1] as follows:

**Theorem 1.2.** Let $G$ be a graph and $I_\Delta = I(G)$ be its edge ideal. Then

1. $\Delta = \text{Ind}(G)$;
2. $\overline{G} = \Delta(1)$;
3. $\Delta$ is quasi-forest if and only if $\overline{G}$ is chordal.

Now, one can conclude that the independence complex $\text{Ind}(G)$ is quasi-forest if and only if the edge ideal $I(G)$ has a linear resolution.

A Ferrers graph is a bipartite graph on two distinct vertex sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ such that if $x_iy_j$ is an edge of $G$, then so is $x_py_q$ for all $1 \leq p \leq i$ and $1 \leq q \leq j$.

Corso and Nagel [5, Theorem 4.2] proved the following result:

**Theorem 1.3.** Let $G$ be a bipartite graph without isolated vertices. Then its edge ideal has a 2-linear resolution if and only if $G$ is (up to a relabeling of the vertices) a Ferrers graph.

By the above theorem, one can conclude that if $G$ is a Ferrers graph, then $	ext{reg}(I(G)) = 2$. In particular, the independence complex $\text{Ind}(G)$ is quasi-forest.

Fröberg [9] proved that the following result:

**Theorem 1.4.** Let $\Delta$ be a $(d-1)$-dimensional quasi-forest. Then $f_{d-1} \leq n - d + 1$ with equality if and only if $K[\Delta]$ is Cohen-Macaulay, where $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$ is the $f$-vector of $\Delta$.

2. RELATIONS ON QUASI-FOREST SIMPLICIAL COMPLEX

We start this section by the following definition:

**Definition 2.1.** An induced $k$-cycle in a graph $G$ is a 3-cycle or a chordless cycle of length $k \geq 4$, we denote it by $C_k$. A simplicial $k$-cycle, denoted by $S_k$, in $\Delta$ is an induced $k$-cycle $C_k$ in $\Delta(1)$ such that no more than two vertices of $C_k$ are in the same facet of $\Delta$.

**Example 2.2.** (i) Let $\Delta$ be a simplicial complex which has the facets $\{x_1, x_2, x_3\}, \{x_2, x_4, x_5\}, \{x_3, x_4, x_5\}$. The cycle $x_2x_3, x_3x_4, x_4x_2$ is $S_3$ in $\Delta$. 
(ii) Let $\Gamma$ be a simplicial complex which has the facets\
\{x_1, x_2, x_3\}, \{x_2, x_4, x_6\}, \{x_3, x_4, x_5\}, \{x_2, x_3, x_4\}. Then $\Gamma$ does not have any $S_k$. 

**Definition 2.3.** Let $\Delta$ be a simplicial complex. For $k \geq 3$, we say that $\Delta$ has a simplicial $k$-point $P_k$ if there is a vertex $x_t \in V$ and a subset $\{x_{j_1}, \ldots, x_{j_k}\}$ of $V$ such that for each $1 \leq i \leq k$ we have $\{x_t, x_{j_1}, \ldots, \hat{x}_{j_i}, \ldots, x_{j_k}\} \in \Delta$ and $\{x_t, x_{j_1}, \ldots, x_{j_k}\} \notin \Delta$. Also, we say that $\Delta$ has a simplicial 2-point $P_2$ if and only if it contains an $S_3$.

**Example 2.4.** Let $\Theta$ be a simplicial complex as the following: 

Then $\Theta$ has $P_3$ since $x_4 \in V$ and $\{x_1, x_2, x_4\}$, $\{x_1, x_3, x_4\}$, $\{x_2, x_3, x_4\}$ are in $\Theta$ and $\{x_1, x_2, x_3, x_4\} \notin \Theta$. 

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**Proposition 2.5.** If \( \text{Ind}(G) \) is a quasi-forest simplicial complex. Then \( G \) does not contain an induced cycle \( C_k \) for all \( k \geq 5 \).

*Proof.* Suppose \( G \) contains an induced cycle \( C_k : x_{i_1}x_{i_2}, \ldots, x_{i_{k-1}}x_{i_k}, x_{i_k}x_{i_1}, k \geq 5 \).

Then we have two cases:

1. If \( k = 5 \), then we have a cycle \( x_{i_1}x_{i_2}, x_{i_1}x_{i_4}, x_{i_2}x_{i_3}, x_{i_2}x_{i_5}, x_{i_3}x_{i_4} \) in \( \overline{G} \). Hence by Theorem 1.2, \( \text{Ind}(G) \) is not quasi-forest and this is a contradiction.

2. If \( k \geq 6 \), then \( x_{i_1}x_{i_3} \) and \( x_{i_4}x_{i_5} \) are in \( G \). Hence \( \overline{G} \) contains a cycle \( C_4 \) and so \( \overline{G} \) is not chordal. By Theorem 1.2, \( \text{Ind}(G) \) is not quasi-forest and this is a contradiction.

This completes the proof. \( \square \)

From the above proposition one can deduce that every graph containing \( C_k \) \((k \geq 5)\) is not quasi-forest.

**Corollary 2.6.** If \( \text{Ind}(G) \) is quasi-forest and \( G \) does not have an induced 3-cycle and any isolated vertex, then \( G \) is a bipartite graph. In particular, \( G \) is a Ferrers graph.

*Proof.* By Proposition 2.5 and our hypothesis, \( G \) does not have any odd cycle. Thus \( G \) is a bipartite graph. \( \square \)

Let \( I \) be a monomial ideal, in the following we use \( \mathcal{G}(I) \) the unique minimal set of monomial generators of \( I \).

**Proposition 2.7.** A simplicial complex \( \Delta \) is flag if and only if it does not have any \( \mathcal{P}_k \) for \( k \geq 2 \).

*Proof.* Suppose that \( \Delta \) is not flag. Then there is a monomial \( x_{j_1} \ldots x_{j_k} \in \mathcal{G}(I_\Delta) \) for \( k \geq 3 \). It follows that \( \{x_{j_2}, x_{j_3}, \ldots, x_{j_k}\}, \{x_{j_1}, x_{j_3}, \ldots, x_{j_k}\}, \ldots, \{x_{j_1}, x_{j_2}, \ldots, x_{j_{k-1}}\} \) are faces of \( \Delta \) and \( \{x_{j_1}, \ldots, x_{j_k}\} \notin \Delta \). Hence \( \Delta \) has a \( \mathcal{P}_{k-1} \) for \( k \geq 3 \).

Conversely, assume \( \Delta \) has a \( \mathcal{P}_k \). If \( k = 2 \), then there are faces \( \{x_{j_1}, x_{j_2}\}, \{x_{j_2}, x_{j_3}\} \) and \( \{x_{j_1}, x_{j_3}\} \) of \( \Delta \) such that \( \{x_{j_1}, x_{j_2}, x_{j_3}\} \notin \Delta \). Hence \( x_{j_1}x_{j_2}x_{j_3} \notin \mathcal{G}(I_\Delta) \). If \( k \geq 3 \), then there is a vertex \( x_t \in V \) and \( \{x_{j_1}, \ldots, x_{j_k}\} \subset V \) such that \( \{x_t, x_{j_1}, \ldots, x_{j_k}\} \) are faces of \( \Delta \) for \( i = 1, \ldots, k \). If \( \{x_{j_1}, \ldots, x_{j_k}\} \notin \Delta \), then \( x_{j_1} \ldots x_{j_k} \notin \mathcal{G}(I_\Delta) \). Now if \( \{x_{j_1}, \ldots, x_{j_k}\} \in \Delta \), then \( x_t x_{j_1} \ldots x_{j_k} \notin \mathcal{G}(I_\Delta) \) since \( \{x_t, x_{j_1}, \ldots, x_{j_k}\} \notin \Delta \). This completes the proof. \( \square \)

**Remark 2.8.** Consider Examples 2.2 and 2.4, the simplicial complexes \( \Delta \) and \( \Theta \) are not flag since they contain an \( S_3 \) and a \( \mathcal{P}_3 \), respectively. However, the simplicial complex \( \Gamma \) is flag since \( \Gamma \) does not contain any \( \mathcal{P}_3 \).

**Theorem 2.9.** A simplicial complex \( \Delta \) is quasi-forest if and only if it does not have any \( \mathcal{P}_k \) and any \( S_k \).

*Proof.* (\( \Rightarrow \)). Suppose \( \Delta \) has a \( \mathcal{P}_k \) such that \( k \geq 2 \). By Proposition 2.7 we obtain that \( \Delta \) is not flag. Hence \( \Delta \) is not quasi-forest and this is a contradiction. Suppose that \( \Delta \) contains an \( S_k \) for \( k \geq 4 \). Then \( C_k \) is in \( \Delta(1) \). Hence \( \overline{G} \) is not chordal. Hence by Theorem 1.2 it follows that \( \Delta \) is not quasi-forest and this is a contradiction. Therefore \( \Delta \) does not have any \( \mathcal{P}_k \) and any \( S_k \).
Suppose, by contrary, that $\Delta$ is not quasi-forest. Thus $\Delta$ is not flag or $\Delta(1)$ is not chordal. If $\Delta$ is not flag, then Proposition 2.7 implies that $\Delta$ has a $P_k$ for $k \geq 2$, a contradiction. If $\Delta(1)$ is not chordal, then $\Delta(1)$ contains $C_k, k \geq 4$. By assumption we may assume that at least three vertices of $C_k$ is a face in $\Delta$, say $F$. Then the 1-faces of $F$ are in $\Delta(1)$. It follows that $C_k$ has a chord and this is a contradiction. Therefore $\Delta$ is quasi-forest. □

Remark 2.10. The simplicial complexes $\Delta$ and $\Theta$ in Examples 2.2, 2.4 are not quasi-forest since they contain an $S_3$ and a $P_3$, respectively. The simplicial complex $\Gamma$ in Example 2.2 is quasi-forest since it does not have any $P_k$ and any $S_k$.

The following result immediately follows by Theorem 2.9.

Corollary 2.11. A simplicial complex $\Delta$ is forest if and only if any subcomplex $\Gamma$ of $\Delta$ does not have a $P_k$ and an $S_k$.

In Example 2.2 the simplicial complex $\Gamma$ is not forest since if we remove the facet $\{x_2, x_3, x_4\}$, then $\Gamma$ is equal to $\Delta$ and $\Delta$ contains an $S_3$.

Definition 2.12 ([28, Definition 2.15], [7, Definition 2.1]). Let $G$ be a graph. Two edges $xy$ and $zu$ form a gap in $G$ if $G$ does not have an edge with one endpoint in $xy$ and the other in $zu$. A graph without gaps is called gap-free. Equivalently, $G$ is gap-free if and only if $C_4$ is not an induced subgraph of $G$.

The following result immediately follows by Theorem 2.9.

Corollary 2.13. If $\Delta = \text{Ind}(G)$ is quasi-forest, then $G$ is a gap-free graph.

We recall the definition of a perfect graph and a Berge graph introduced in [3]. A graph $G$ is perfect if for every induced subgraph $H$, the chromatic number of $H$ equals the size of the largest complete subgraph of $H$, and $G$ is Berge if and only if neither it nor its complementary graph has an odd induced cycle of length at least five. Chudnovsky et al. in [3] proved that a graph is perfect if and only if it is Berge.

Corollary 2.14. If $\Delta = \text{Ind}(G)$ is a quasi-forest, then $G$ is a perfect graph.

Proof. From Proposition 2.5 and Theorem 2.9 it follows that $G$ is a Berge graph and so $G$ is a perfect graph. □

Note that the converse of Corollaries 2.13 and 2.14 are not true in general, for example, the graph $G$ of independence complex $\text{Ind}(G) = S_4$ is gap-free and perfect, however, $\text{Ind}(G)$ is not quasi-forest.

Following [27] a simplicial complex $\Delta$ is recursively defined to be vertex decomposable if it is either a simplex or else has some vertex $v$ so that

1. both $\Delta \setminus v$ and $\text{link}_\Delta v$ are vertex decomposable, and
2. no face of $\text{link}_\Delta v$ is a facet of $\Delta \setminus v$, where $\text{link}_\Delta v = \{F \in \Delta \mid F \cup \{v\} \in \Delta, v \notin F\}$.

Woodroofe in [27, Theorem 1] proved the following theorem:

Theorem 2.15. If $G$ is a graph with no chordless cycle of length other than 3 or 5, then $G$ is vertex decomposable (hence shellable and sequentially Cohen-Macaulay).
The following result immediately follows from Theorem 2.15 and Proposition 2.5.

**Corollary 2.16.** If $\Delta = \text{Ind}(G)$ is quasi-forest and $G$ does not contain an induced 4-cycle, then $\Delta$ is a vertex decomposable (so shellable and sequentially Cohen-Macaulay).

It can be noted from Corollary 2.11 that if $\Delta$ is a quasi-forest and contains at most three facets, then $\Delta$ is a forest.

**Corollary 2.17.** The simplicial complex $\text{Ind}(C_k)$ is quasi-forest if and only if $k = 3, 4$. In particular, $\text{Ind}(C_k)$ is a forest if and only if $\text{Ind}(C_k)$ is a quasi-forest.

**Proof.** If $k = 3, 4$, then it is immediately follows that $\text{Ind}(C_k)$ is quasi-forest. Conversely, we consider the following cases:

1. if $k = 3$, then $\deg(I_{\text{Ind}(C_3)}) = 2$ and $\text{Ind}(C_3) = \langle x_1, x_2, x_3 \rangle$. Thus it is clear that $\text{Ind}(C_3)$ is a forest;
2. if $k = 4$, then $\deg(I_{\Delta(C_4)}) = 2$ and $\text{Ind}(C_4) = \langle x_1x_3, x_2x_4 \rangle$. It therefore follows that $\text{Ind}(C_4)$ is a forest;
3. if $k \geq 5$, then by Proposition 2.5 we get the result.

\[ \square \]

3. The $h$-vector of Cohen-Macaulay quasi-forest

Let $I$ be a homogeneous ideal of $R$ with $\dim R/I = d$. The Hilbert series of $R/I$ is of the form $H_{R/I}(t) = (h_0 + h_1t + h_2t^2 + \ldots + h_st^s)/(1 - t)^d$, where each $h_i \in \mathbb{Z}$. The polynomial $h_{R/I}(t) = h_0 + h_1t + h_2t^2 + \ldots + h_st^s$ with $h_s \neq 0$ is called the $h$-polynomial of $R/I$ (see [14, Theorem 6.1.3]). The $a$-invariant is the degree of the Hilbert series $H_{R/I}(t)$, that is, the number $s - d$.

From [24, Corollary B.4.1] we have $a(K[\Delta]) \leq \deg(K[\Delta]) - \text{depth}(K[\Delta])$ with equality if $K[\Delta]$ is Cohen-Macaulay. Then we have the following inequality $\deg(h_{K[\Delta]}(t) - \deg(K[\Delta]) \leq \dim(K[\Delta]) - \text{depth}(K[\Delta])$. The equality holds if $K[\Delta]$ is Cohen-Macaulay or $K[\Delta]$ has a pure resolution (see [11, P. 153]). In particular, if $\Delta$ is a quasi-forest, then the equality holds.

**Proposition 3.1.** (Compare with [11, Proposition 2.3]) Let $\Delta$ be a $(d-1)$-dimensional simplicial quasi-forest with $h(\Delta) = (h_0, h_1, \ldots, h_d)$. Then $\Delta$ is Cohen-Macaulay if and only if $h_2, \ldots, h_d$ are zero.

**Proof.** Since $\Delta$ is a quasi-forest, we have $\deg h_{K[\Delta]}(t) = \dim(K[\Delta]) - \text{depth}(K[\Delta]) + 1$. If $\Delta$ is Cohen-Macaulay, then $\deg h_{K[\Delta]}(t) = 1$. Therefore $h_2, \ldots, h_d$ are zero. Conversely, let $h_2, \ldots, h_d$ be zero. Then $1 = \deg h_{K[\Delta]}(t) = \dim(K[\Delta]) - \text{depth}(K[\Delta]) + 1$. Thus $\dim(K[\Delta]) = \text{depth}(K[\Delta])$ and so $\Delta$ is Cohen-Macaulay.

\[ \square \]

**Theorem 3.2.** Let $\Delta$ be a $(d-1)$-dimensional simplicial quasi-forest with $h(\Delta) = (h_0, h_1, \ldots, h_d)$. Then $\Delta$ is almost Cohen-Macaulay if and only if $h_2$ is non-positive, and $h_3, \ldots, h_d$ are zero.

**Proof.** Since $\Delta$ is a quasi-forest, we have $\deg h_{K[\Delta]}(t) = \dim(K[\Delta]) - \text{depth}(K[\Delta]) + 1$. If $\Delta$ is almost Cohen-Macaulay, then $\deg h_{K[\Delta]}(t) \leq 2$. Therefore $h_3, \ldots, h_d$ are...
zero. By [26, Theorem 6.7.6] \( h_0 = 1 \) and \( h_1 = f_0 - d = n - d \) which is not negative.

Now, we consider the following cases:

(i) : let \( h_1 = 0 \). Then \( f_0 = d \) and so \( h_2 \) is zero;

(ii) : let \( h_1 \) be positive. By Theorem 1.4 we have \( f_{d-1} \leq n - d + 1 \). Since

\[
    f_{d-1} = h_0 + h_1 + \cdots + h_d,
\]

it follows that \( h_0 + h_1 + \cdots + h_d \leq n - d + 1 \). Thus \( h_2 + \cdots + h_d \leq 0 \). It therefore follows that \( h_2 \leq 0 \).

Conversely, suppose that \( h_2 \leq 0 \) and \( h_3 = \cdots = h_d = 0 \). Since \( \dim(K[\Delta]) - \depth(K[\Delta]) + 1 = \deg h_{K[\Delta]}(t) \leq 2 \), it follows that \( \dim(K[\Delta]) - \depth(K[\Delta]) \leq 1 \).

Hence \( \Delta \) is almost Cohen-Macaulay, as required. \( \square \)

As before if \( K[\Delta] \) is Cohen-Macaulay, then \( a(K[\Delta]) = \reg(K[\Delta]) - \depth(K[\Delta]) \).

It is natural to ask whether if \( K[\Delta] \) is almost Cohen-Macaulay, then \( a(K[\Delta]) = \reg(K[\Delta]) - \depth(K[\Delta]) - 1 \) is it true in general. In the following we give a counter example for this question.

**Example 3.3.** Let \( n = 5 \) and \( I = (x_4 x_5, x_1 x_3 x_5, x_1 x_2 x_5, x_1 x_2 x_3 x_4) \) be an ideal of \( R \). Then by using Macaulay 2 we have \( \dim(R/I) = 3 \) and \( \depth(R/I) = 2 \) and so \( I \) is almost Cohen-Macaulay. On the other hand, \( \reg(R/I) = 3 \) and \( H_{R/I}(t) = \frac{1+2t+2t^2}{(1-t)^3} \).

Thus \( \reg(R/I) = -1 \) and \( \reg(R/I) - \depth(R/I) - 1 \neq a(R/I) \).

**Proposition 3.4.** (Compare with [4, Corollary 2.8]) Let \( G \) be a Ferrers graph on two distinct vertex sets \( V_1 = \{x_1, \ldots, x_n\} \) and \( V_2 = \{y_1, \ldots, y_n\} \) and \( I = I(G) \) be the edge ideal in the ring \( S = K[x_1, \ldots, x_n, y_1, \ldots, y_n] \). Then \( I \) is Cohen-Macaulay if and only if the Hilbert series \( H_{S/I}(t) = \frac{1+nt}{(1-t)^n} \).

**Proof.** Let \( I \) be Cohen-Macaulay. Then by Proposition 3.1 \( h_2, \ldots, h_d \) are zero and \( d = \dim R/I = n \). Since \( h_1 = f_0 - d = 2n - n = n \), it follows that \( H_{S/I}(t) = \frac{1+nt}{(1-t)^n} \).

Conversely, suppose \( H_{S/I}(t) = \frac{1+nt}{(1-t)^n} \). Since \( I \) has a linear resolution, we have \( \deg h_{S/I}(t) = \reg(S/I) = \depth(S/I) \) and \( \reg(S/I) = 1 \). By hypothesis \( \deg h_{S/I}(t) = 1 \) and it therefore follows \( \dim(S/I) = \depth(S/I) \) Thus \( I \) is Cohen-Macaulay, as required. \( \square \)

The following result proved by Cimpoeas in [4, Proposition 1.3]:

**Proposition 3.5.** Let \( G = C_n \) be a cycle graph and \( I = I(G) \). Then \( \depth R/I = \left[ \frac{n-1}{3} \right] \).

**Theorem 3.6.** Let \( G = C_n \) be a cycle graph and \( I = I(G) \). Then \( I \) is almost Cohen-Macaulay if and only if \( n = 3, 4, 5, 6, 7, 8, 9, 11 \).

**Proof.** \((\Longleftarrow)\). By using Macaulay 2 one can easily obtain the result.

\((\Longrightarrow)\). Let \( J = I(P_n) \), where \( P_n \) is a path of length \( n - 1 \) with \( n - 1 \) edges \( x_i x_{i+1} \) such that \( 1 \leq i \leq n - 1 \). Now by induction on \( n \), we prove that \( \dim(R/J) = \left[ \frac{n}{3} \right] \).

If \( n = 3 \), then by using Macaulay 2 there is nothing to prove. Suppose \( n \geq 4 \) and the result has been proved for smaller values of \( n \). Consider the exact sequence

\[
0 \rightarrow R/(J : x_n) \xrightarrow{x_n} R/J \rightarrow R/(J, x_n) \rightarrow 0. \quad (*)
\]
Since \((J : x_n) = (I(P_{n-2}), x_{n-1})\), it follows that \(\dim R/(J : x_n) = \dim R'[x_n]/I(P_{n-2}) = \dim R'/I(P_{n-2}) + 1\), where \(R' = K[x_1, \ldots, x_{n-2}]\). Now, by induction hypothesis, we obtain \(\dim R/(J : x_n) = \lfloor \frac{n}{2} \rfloor + 1 = \lfloor \frac{n}{2} \rfloor\). Similarly, \((J, x_n) = (I(P_{n-1}), x_n)\) and again by using induction hypothesis \(\dim R/(J, x_n) = \dim K[x_1, \ldots, x_{n-1}]/(I(P_{n-1})) = \lfloor \frac{n-1}{2} \rfloor\). Since \(\dim R/J = \max \{\dim R/(J : x_n), \dim R/(J, x_n)\}\), it follows that \(\dim R/J = \lfloor \frac{n}{2} \rfloor\). Since \((I : x_n) = (I(P_{n-3}), x_{n-1}, x_1)\), similarly by using \(I\) in stead of \(J\) in the exact sequence (*) , we get \(\dim R/I = \lfloor \frac{n}{3} \rfloor\). Also, by Proposition 3.5 we have depth \(R/I = \lfloor \frac{n-1}{3} \rfloor\). Hence by comparing depth \(R/I\) and \(\dim R/I\) we conclude that \(n = 3, 4, 5, 6, 7, 8, 9, 11\). □

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