QUANTUM BEHAVIOR IN ASYMMETRIC, WEYL-LIKE CARTAN GEOMETRIES *

J.E. Rankin†
Rankin Consulting, 527 Third Avenue, Ste. 298, New York, NY 10016, USA
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Abstract

This discussion examines recent developments in the theory of a Weyl-like, Cartan geometry with natural Schrödinger field behavior proposed previously. In that model, very nearly exactly a coupled Einstein-Maxwell- Schrödinger, classical field theory emerges from a gauge invariant, purely geometric action based solely on variations of the electromagnetic potentials and the metric. In spite of this, only slight differences appear between the resulting Schrödinger part, and the conventional theory of the Schrödinger field. Close examination of the differences reveals that most are general relativistic effects which are unobservable in flat spacetime, and which are estimated to interact significantly only via their gravitational fields, or on scales comparable with neutrino interaction cross sections. The only remaining difference is that the wavefunction obeying the conjugate wave equation is not always restricted to be exactly the complex conjugate of the primary wavefunction. Generalizations of the model lead naturally to spinlike phenomena, a possible new mechanism for a theory of rest mass, and spinor connections containing the form of an SU(2) potential.

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†E-mail address: (Internet) jrankin@panix.com
I. INTRODUCTION

In 1927, London published his famous paper on the quantum mechanical interpretation of Weyl’s classical geometry\(^1\). In the last 15 years, a flurry of new articles has appeared, reviving interest in intrinsic quantum properties of both Weyl geometries, and closely related geometric structures\(^2\)–\(^7\). In this particular discussion, I will elaborate on a few points mentioned in a recent paper on such a case, a purely field theoretic model cast in a Weyl-like Cartan geometry with intrinsic Schrödinger field behavior\(^7\). Indeed, this discussion should be viewed as an additional section of comments in that paper.

One reason this model is of interest is because it yields essentially a full, coupled Einstein-Maxwell-Schrödinger set of fields from a purely geometric action in which only the Weyl four vector and the metric are varied. Yet the Schrödinger field also emerges from the results, correctly obeying the Klein-Gordan equation and a charge conjugate, complementary equation, and producing the correct Schrödinger current and stress tensor as sources of the electromagnetic and gravitational fields. Additional terms do appear in the equations, but they will be noted to produce vanishing effects in the limit of a flat spacetime of special relativity. Thus, they are essentially general relativistic modifications to special relativistic Schrödinger theory. Addition of further geometric fields to the model produces spinlike phenomena, and terms which have the form of an SU(2) potential.

II. GENERAL RELATIVISTIC NATURE OF ADDITIONAL TERMS

In the earlier reference (\(\hat{\phi} = \pi\) case)\(^8\), the Schrödinger wave function \(\psi\) obeys the wave equation

\[
\frac{1}{\sqrt{-\hat{g}}}(\sqrt{-\hat{g}} \hat{g}^{\mu\nu} \psi_{,\mu} + 2\hat{g}^{\mu\nu} v_{,\mu} \psi_{,\nu}) + \frac{1}{\sqrt{-\hat{g}}}(\sqrt{-\hat{g}} \hat{g}^{\mu\nu} v_{,\nu} \psi_{,\mu} + \hat{g}^{\mu\nu} v_{,\nu} v_{,\mu} \psi) = -\frac{1}{6} (1 + \hat{R}) \psi
\]

(1)

where \(v_{,\mu}\) is the Weyl vector, and is proportional to the standard Maxwell potential through an imaginary coefficient. A conjugate wave equation (minus \(v_{,\mu}\)) governs the conjugate wavefunction \(\xi\). The formalism does not require \(\xi\) always to be the complex conjugate of \(\psi\). Except for that, the Schrödinger current and stress tensor contain the standard terms.\(^9\) However, the stress tensor also contains additional terms \(\hat{\beta}_{||\gamma} ||\gamma \hat{g}_{\mu\nu} - \hat{\beta}_{||\mu\nu} ||\nu\), where \(\hat{\beta} = \xi \psi\), and the || derivative is the covariant derivative using the Christoffel symbols formed from \(\hat{g}_{\mu\nu}\). These terms are in addition to manifestly general relativistic modifications, such as the \(\hat{R}\) term in Eq. (1).

However, terms in the stress tensor must interact with their environment in some way in order to be observable. In that regard, even though \(\hat{\beta}_{||\gamma} ||\gamma \hat{g}_{\mu\nu} - \hat{\beta}_{||\mu\nu} ||\nu\) appear capable of carrying energy and momentum magnitudes that might be comparable to the conventional Schrödinger terms, these terms can only be detected insofar as their divergence is nonvanishing (allowing energy exchange with other terms), or via the gravitational field they produce. But the gravitational field is a manifestly general relativistic effect. And, the divergence of the terms is

\[
[\hat{\beta}_{||\gamma} ||\gamma \delta_{\mu} - \hat{\beta}_{||\mu} ||\nu]||\nu = \hat{R}_{\mu}^\nu \hat{\beta}_{\nu}
\]

(2)
Thus, the interaction of these terms with the rest of the stress tensor is also a manifestly general relativistic effect (more precisely, it is a higher order effect). The terms are therefore completely unobservable in the limit of the flat spacetime of special relativity.

But given that these terms are therefore general relativistic, when might they, or other general relativistic effects be important? Of course, they might contribute significantly to overall gravitational fields in any almost any configuration. But beyond that, examination of Eq. (2) suggests that whenever the magnitude of $\hat{R}_{\mu\nu}$ approaches or exceeds unity, these modifications to standard Schrödinger theory may become as significant as the usual kinetic energy and rest mass terms in the stress tensor.

To obtain some crude estimate of when such cases arise, assume the Reissner-Nordström metric\(^9\) gives at least reasonable magnitude estimates of $\hat{R}_{\mu\nu}$ in the vicinity of electrons, quarks, and other nearly pointlike, fundamental particles. Further assume that the natural charge magnitude associated with the problem is the electronic charge, and that the natural length scale (all quantities in these equations are kept dimensionless) is $\hbar/(\sqrt{\pi} m_0 c)$, where $m_0$ is the electron rest mass. This appears to be the natural rest mass to use here rather than the Planck mass because later extensions of this structure make it easy to generate masses larger than some reference mass, but hard to generate masses smaller than that mass.\(^7\) Thus, the natural reference mass to introduce is one of the very smallest, nonzero masses known, in this case, the electron rest mass.

Using the Reissner-Nordström solution then as a very rough guide, one finds that to lowest order, $\hat{R}_{\mu\nu} \sim 12\left[\left(Gm_0^2/\hbar c\right)e^2/(\hbar c)\right](1/r^4)$, and that this approaches unity for a radius $r_0 = 5.6 \cdot 10^{-23} \text{ cm.}$ While this is noticeably smaller than the current resolution of the pointlike nature of electrons or quarks,\(^8\), which is about $10^{-17} \text{ cm.}$, it is still huge compared to the Planck scale. Indeed, if $r_0$ is translated into a very crude estimate of a cross section\(^8\) by forming $\pi r_0^2$, this is found to be $\sim 10^{-44} \text{ cm.}^2$, a value actually comparable to the low end of observed neutrino cross sections.\(^9\) And, some additional general relativistic terms in the equations will also begin to be important under the same conditions.

**III. COMMENTS ON AN EXTENDED FORMALISM**

The previous paper also proposes enhancements to the proposed model via inclusion of a self dual antisymmetric part to the metric, $\hat{a}_{\mu\nu}$ (for spinlike phenomena), and a tracefree part to the torsion.\(^7\) For such cases, the results suggest that the effective rest mass in the wave equations is determined via $m_0^2 \sim -e^{-i\hat{\phi}}(1 + \frac{4}{3}\hat{\alpha})$, where $\hat{\phi}$ is a constant angle, and $\hat{\alpha} = \hat{a}_{\mu\nu}\hat{a}_{\mu\nu}$. This form illustrates why it would be difficult to produce rest masses smaller than some reference mass, since that would require $\hat{\alpha} \approx -4$. On the other hand, $\hat{\alpha}$ tends to be complex, which should mimic the effects of a complex scattering potential, creating wavefunction sources and sinks (but without violating conservation laws here). It remains to be seen if this can be controlled and used to advantage to require something like a real eigenvalue spectrum for $\hat{\alpha}$ by forbidding pathological behavior. If not, it may be necessary to simply constrain $\hat{\alpha}$ to be real.

But when $\hat{a}_{\mu\nu} \neq 0$, previous results also suggest that it generates the tracefree part to the torsion (the vector $\hat{v}_\mu$ is 2/3 the trace of the torsion). Indeed, if the tracefree part of the torsion is constrained to be totally antisymmetric, the forms show a similarity to a theory of
torsion and spin developed by Hammond\textsuperscript{12}. Unfortunately, this constrained case is omitted in deference to the more general case in the earlier reference\textsuperscript{7}.

Another point associated with a tracefree part to the torsion (denoted by $\hat{Q}_{\nu\alpha}^\mu$) concerns the form of the spinor connection associated with the affine connection of the four space. In this case, the affine connection is presumed to have a complex, non-Christoffel (tensor) portion, so the spinor connection can be formed from either the regular four space connection, or from its complex conjugate. In practice, it appears necessary to mix both forms into some expressions in order to have consistent results. However, rather than elaborate on that here, simply assume that the spinor connection is defined via the usual relation\textsuperscript{13}

$$\hat{\Gamma}^A_{B\alpha} = -\frac{1}{2} \hat{g}_{\mu\nu} \sigma^\nu_{E' E} \left[ \sigma^\mu E' A \alpha + \sigma^\gamma E' A \hat{Q}_{\gamma\alpha} \right]$$  \hspace{1cm} (3)

While space will not permit much detail here, the above expression can be reworked by reexpressing $2\sigma^\nu_{E' B} \sigma^\gamma E' A$ as the sum of two terms, $\hat{g}^{\nu\gamma} \delta^A_B$ symmetric in $\nu$ and $\gamma$, and $S_{\nu\gamma}^A$ antisymmetric (and self dual) in the same index pair. Utilizing properties of self dual expressions\textsuperscript{5}, Eq. (3) becomes

$$\hat{\Gamma}^A_{B\alpha} = \{ \hat{A}^{\alpha}_{B\alpha} \} - \frac{1}{2} S_{\alpha}^B \gamma^A \left( \hat{v}_\gamma - i 3 \hat{W}_\gamma \right) + \frac{1}{2} S_{\mu\gamma}^A B \hat{Q}_{\mu\gamma\alpha}$$  \hspace{1cm} (4)

where $\hat{W}_\gamma$ is the dual of the totally antisymmetric part of $\hat{Q}_{\mu\gamma\alpha}$. Thus, a portion of the tracefree part of the torsion is seen to mix directly with the four vector, $\hat{v}_\mu$.

Furthermore, since the term $S_{\nu\gamma}^A$ is self dual in the tensor indices, the term $\frac{1}{2} S_{\mu\gamma}^A B \hat{Q}_{\mu\gamma\alpha}$ can be rewritten as $\frac{1}{4} S_{\mu\gamma}^A B \left( \hat{Q}_{\mu\gamma\alpha} + i \ast \hat{Q}_{\mu\gamma\alpha} \right)$ where the dual is taken on the first two indices of $\hat{Q}_{\mu\gamma\alpha}$. But this is now the inner product of two self dual forms, so that it can be expressed as the three space dot product of two complex three vectors (with additional free spinor and tensor indices in this case)\textsuperscript{5}. But the three vector corresponding to $S_{\nu\gamma}^A$ is just the Pauli matrices. The overall result then is that

$$\hat{\Gamma}^A_{B\alpha} = \{ \hat{A}^{\alpha}_{B\alpha} \} - \frac{1}{2} S_{\alpha}^B \gamma^A \left( \hat{v}_\gamma - i 3 \hat{W}_\gamma \right) + \sum_{\tau} \tau_n b_{n\alpha}$$  \hspace{1cm} (5)

where the $\tau_n$ are the Pauli matrices. But the last term now is in the usual form for an SU(2) potential\textsuperscript{13}. The fact that the preceding term is not in the standard form for the U(1) potential in electroweak theory does not appear so serious, since the purely U(1) case here is already the cleanest form of this model.

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