Notes on $A_{\infty}$-algebras, $A_{\infty}$-categories and non-commutative geometry. I

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1 Introduction

1.1 $A_\infty$-algebras as spaces

The notion of $A_\infty$-algebra introduced by Stasheff as well as the notion of $A_\infty$-category introduced by Fukaya has two different versions. First one is operadic:
an $A_\infty$-algebra is an algebra over the $A_\infty$-operad. Second one is geometric: an $A_\infty$-algebra is the same as a non-commutative formal graded manifold $X$ over a field $k$, having a marked $k$-point $pt$, and equipped with a vector field $d$ of degree +1 such that $d|_{pt} = 0$ and $[d, d] = 0$ (such vector fields are called homological). By definition the algebra of functions on the non-commutative formal pointed graded manifold is isomorphic to the algebra of formal series

$$\sum_{n \geq 0} \sum_{\sum i_j \in I} a_{i_1, \ldots, i_n} x_{i_1} \cdots x_{i_n} := \sum_M a_M x^M$$

of free graded variables $x_i, i \in I$ (the set $I$ can be infinite). Here $M = (i_1, \ldots, i_n), n \geq 0$ is a non-commutative multi-index, i.e. an element of the free monoid generated by $I$. Homological vector field makes the above graded algebra into a complex of vector spaces. The triple $(X, pt, d)$ is called a non-commutative formal pointed differential-graded (or simply dg-) manifold.

It is an interesting problem to make a dictionary from the pure algebraic language of $A_\infty$-algebras and $A_\infty$-categories to the language of non-commutative geometry. One purpose of these notes is to make few steps in this direction.

From the point of view of Grothendieck’s approach to the notion of “space”, our formal pointed manifolds are given by functors on graded associative Artin algebras commuting with finite projective limits. It is easy to see that such functors are represented by graded coalgebras. These coalgebras can be thought of as coalgebras of distributions on formal pointed manifolds. The above-mentioned algebras of formal power series are dual algebras to the coalgebras of distributions.

In the case of small $A_\infty$-categories one should modify the above definitions. Instead of one marked point one considers a closed subscheme of disjoint points in a formal graded manifold (they corresponds to objects of the category), and the homological vector field $d$ must be compatible with the embedding of this subscheme as well as with the projection onto it.

### 1.2 Some applications of the geometric language

Geometric approach to $A_\infty$-algebras and $A_\infty$-categories gives answers to several questions which are not transparent from the pure algebraic point of view. In particular one can obtain an explicit description of the $A_\infty$-structure on $A_\infty$-functors. In geometric language $A_\infty$-functors are interpreted as maps between non-commutative formal dg-manifolds commuting with homological vector fields. One can introduce a non-commutative formal dg-manifold of maps between two such spaces. Functors are just “commutative” points of the latter. As a result of the above considerations one can describe explicitly the $A_\infty$-structure on functors in terms of sums over certain sets of trees. In this paper we provide the answer in the case of $A_\infty$-categories with one object, i.e. in the case of $A_\infty$-algebras. The general case reflects the difference between

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1The $A_\infty$-operad can be realized as the operad of singular chains of the operad of intervals in the real line.

2We use “formal” non-commutative geometry in tensor categories, which is different from the non-commutative geometry in the sense of Alain Connes.

3By commutative points we mean points with values in commutative algebras.
Among other applications of our geometric language we mention an interpretation of the Hochschild chain complex of an $A_\infty$-algebra in terms of cyclic differential forms on the corresponding formal pointed dg-manifold (Section 7.1).

Geometric language simplifies some proofs as well. For example, Hochschild cohomology of an $A_\infty$-category $\mathcal{C}$ is isomorphic to $\text{Ext}^\bullet(\text{Id}_\mathcal{C}, \text{Id}_\mathcal{C})$ taken in the $A_\infty$-category of endofunctors $\mathcal{C} \to \mathcal{C}$. This result admits an easy proof, if one interprets Hochschild cochains as vector fields and functors as maps.

1.3 Contents of the paper

Present paper contains two parts out of the planned three (the last one is devoted to $A_\infty$-categories). In this paper we discuss $A_\infty$-algebras (=non-commutative formal pointed dg-manifolds with fixed affine coordinates). We have tried to develop in detail the dictionary between algebra and geometry and provide proofs in the cases when they are not straightforward.

Part I is devoted to the geometric description of $A_\infty$-algebras. We start with basics on formal graded affine schemes, then add homological vector fields to the story, thus arriving to the geometric definition of $A_\infty$-algebras as formal pointed dg-manifolds. Most of the material is well-known in algebraic language. We cannot completely avoid $A_\infty$-categories (subject of another planned paper). They appear in the form of categories of $A_\infty$-modules and $A_\infty$-bimodules, which we define using our geometric language.

Since in the $A_\infty$-world many notions are defined “up to quasi-isomorphism”, their geometric meaning is not obvious. As an example we mention the notion of weak unit. Basically, the latter means that the unit exists at the level of cohomology only. In Section 4 we discuss the relationship of weak units with the “differential-graded” version of the affine line.

We start Part II with the definition of the Hochschild complexes of $A_\infty$-algebras. As we have already mentioned, Hochschild cochain complex is interpreted in terms of graded vector fields on the non-commutative formal affine space. Dualizing, Hochschild chain complex is interpreted in terms of degree one cyclic differential forms. This interpretation is motivated by [Ko2]. It differs from the traditional picture (see e.g. [Co], [CST]) where one assigns to a Hochschild chain $a_0 \otimes a_1 \otimes \ldots \otimes a_n$ the differential form $a_0 da_1 \ldots da_n$. In our approach we interpret $a_i$ as the dual to an affine coordinate $x_i$, and the above expression is dual to the cyclic differential 1-form $x_1 \ldots x_n dx_0$. We also discuss graphical description of Hochschild chains, the differential, etc.

After that we discuss homologically smooth compact $A_\infty$-algebras. Those are analogs of smooth projective varieties in algebraic geometry. Indeed, the derived category $D^b(\mathcal{X})$ of coherent sheaves on a smooth projective variety $\mathcal{X}$ is another, purely algebraic approach to the $A_\infty$-structure on functors was suggested in [Lyu].

The idea to treat $\text{Ext}^\bullet(\text{Id}_\mathcal{C}, \text{Id}_\mathcal{C})$ as the tangent space to deformations of the derived category $D^b(\mathcal{C})$ goes back to A.Bondal.
$A_\infty$-equivalent to the category of perfect modules over a homologically smooth compact $A_\infty$-algebra (this can be obtained using the results of [BvB]). The latter contains as much information about the geometry of $X$ as the category $D^b(X)$ does. A good illustration of this idea is given by the non-commutative version of the Hodge theory presented in Section 9. It is a largely conjectural topic, which eventually should be incorporated in the theory of “non-commutative motives”. Encoding smooth proper varieties by homologically smooth compact $A_\infty$-algebras one can forget about the underlying commutative geometry and develop a theory of “non-commutative smooth projective varieties” in an abstract form.

Let us briefly explain how all that looks in the case of the Hodge theory. Let $(C_\bullet(A,A), b)$ be the Hochschild chain complex of a weakly unital homologically smooth compact $A_\infty$-algebra $A$. The corresponding negative cyclic complex $(C_\bullet(A,A)[[u]], b + uB)$ gives rise to a family of complexes over the formal affine line $A_1$ (shift of the grading reflects the fact that the variable $u$ has degree $+2$, cf. [Co], [CST]). We conjecture that the corresponding family of cohomology groups gives rise to a vector bundle over the formal line. The generic fiber of this vector bundle is isomorphic to the periodic cyclic homology of $A$, while the fiber over $u = 0$ is isomorphic to the Hochschild homology of $A$. If a compact homologically smooth $A_\infty$-algebra $A$ comes from a smooth projective variety as explained above, then the generic fiber is just the algebraic de Rham cohomology of the variety, while the fiber over $u = 0$ is the Hodge cohomology. In this case our conjecture becomes the classical theorem which claims degeneration of the spectral sequence Hodge-to-de Rham.

Last section of Part II is devoted to the relationship between moduli spaces of points on a cylinder and algebraic structures on the Hochschild complexes. In Section 11.3 we formulate a generalization of Deligne’s conjecture. Recall that Deligne’s conjecture says (see e.g. [KoSo1], [McS], [T]) that the Hochschild cochain complex of an $A_\infty$-algebra is an algebra over the operad of chains on the topological operad of little discs. In the conventional approach to non-commutative geometry Hochschild cochains correspond to polyvector fields, while Hochschild chains correspond to de Rham differential forms. One can contract a form with a polyvector field or take a Lie derivative of a form with respect to a polyvector field. This geometric point of view leads to a generalization of Deligne’s conjecture which includes Hochschild chains equipped with the structure of (homotopy) module over cochains, and to the “Cartan type” calculus which involves both chains and cochains (cf. [CST], [TaT1]). We unify both approaches under one roof formulating a theorem which says that the pair consisting of the Hochschild chain and Hochschild cochain complexes of the same $A_\infty$-algebra is an algebra over the colored operad of singular chains on configurations of discs on a cylinder with marked points on each of the boundary circles.

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6In a recent preprint [Kal], D.Kaledin claims the proof of our conjecture. He uses a different approach to the cyclic homology theory than we do.

7After our paper was finished we received the paper [TaT2] where the authors proved an equivalent result.
Sections 10.1.4 and 11.6 are devoted to $A_\infty$-algebras with scalar product. Geometrically this is the same as non-commutative formal symplectic manifolds. In Section 10.1.4 we also discuss a homological version of this notion and explain that it corresponds to the notion of Calabi-Yau structure on a manifold. In Section 11.6 we define an action of the PROP of singular chains of the topological PROP of smooth oriented 2-dimensional surfaces with boundaries on the Hochschild chain complex of an $A_\infty$-algebra with scalar product. If in addition the algebra $A$ is homologically smooth and the spectral sequence Hodge-to-de Rham degenerates, then the above action extends to the action of the PROP of singular chains on the topological PROP of stable 2-dimensional surfaces. This is essentially equivalent to a structure of 2-dimensional Cohomological TFT. More details and an application of this approach to Fukaya categories is given in [KaKoP]. We also mention that in Sections 11.4, 11.5 we propose an approach to the Gauss-Manin connection on periodic cyclic homology which is different from the one in [Get]. In particular we conjecture the existence of flat connection which is derived from a certain colored operad.

1.4 Generalization to $A_\infty$-categories

Let us say few words about the planned paper devoted to $A_\infty$-categories. The formalism of present paper admits a straightforward generalization to the case of $A_\infty$-categories. The latter should be viewed as non-commutative formal dg-manifolds with a closed marked subscheme of objects. One can hope for a translation of the theory of $A_\infty$-categories into our geometric language of non-commutative geometry. This goal is not fully achieved at present. Although some parts of the theory of $A_\infty$-categories admit nice interpretation in terms of non-commutative geometry, some other still wait for it. The latter includes e.g. triangulated $A_\infty$-categories. As a first step to the future geometric description of triangulated $A_\infty$-categories we propose to describe their theory from the point of view of $A_\infty$-functors from “elementary” categories to a given $A_\infty$-category (see a summary in [Ko5], [So1], [So2]). Those “elementary” categories are, roughly speaking, derived categories of representations of quivers with small number of vertices. Our approach has certain advantages over the traditional one. For example the complicated “octahedron axiom” admits a natural interpretation in terms of functors from the $A_\infty$-category associated with the quiver of the Dynkin diagram $A_2$ (there are six indecomposable objects in the category $D^b(A_2-mod)$ corresponding to six vertices of the octahedron). Interpreting such functors as morphisms of non-commutative formal dg-schemes one can hope to describe some parts of the theory of $A_\infty$-categories geometrically.

At the same time we have not been able to provide pure geometric proofs of all the results, thus relying on less flexible approach which uses differential-graded categories (see [Dr]).

In present and subsequent paper on the subject we mostly consider $A_\infty$-algebras and categories over a field of characteristic zero. This assumption

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8A different approach to this topic was developed by Kevin Costello, see [Cos]
simplifies many results, but also makes some other less general. We refer the reader to [Lyu], [LyuOv] for a theory over a ground ring instead of ground field (the approach of [Lyu],[LyuOv] is pure algebraic and different from ours). Most of the results of present paper are valid for an $A_{\infty}$-algebra $A$ over the unital commutative associative ring $k$, as long as the graded module $A$ is flat over $k$. More precisely, the results of Part I remain true except of the results of Section 3.2 (the minimal model theorem). In these two cases we assume that $k$ is a field of characteristic zero. Constructions of Part II work over a commutative ring $k$. The results of Section 10.1.4 are valid (and the conjectures are expected to be true) over a field of characteristic zero. Algebraic version of Hodge theory from Section 9 and the results of Section 7 are formulated for an $A_{\infty}$-algebra over the field of characteristic zero, although e.g. the Conjecture 9.1.5 is expected to be true for any $\mathbb{Z}$-flat $A_{\infty}$-algebra.

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Part I

$A_{\infty}$-algebras and non-commutative dg-manifolds

2 Coalgebras and non-commutative schemes

Geometric description of $A_{\infty}$-algebras will be given in terms of geometry of non-commutative ind-affine schemes in the tensor category of graded vector spaces (we will use $\mathbb{Z}$-grading or $\mathbb{Z}/2$-grading). In this section we are going to describe these ind-schemes as functors from finite-dimensional algebras to sets (cf. with the description of formal schemes in [Gr]). More precisely, such functors are represented by counital coalgebras. Corresponding geometric objects are called non-commutative thin schemes.

2.1 Coalgebras as functors

Let $k$ be a field, and $\mathcal{C}$ be a $k$-linear abelian symmetric monoidal category (we will call such categories tensor), which admits infinite sums and products (we refer to [DM] about all necessary terminology of tensor categories). Then we can do simple linear algebra in $\mathcal{C}$, in particular, speak about associative algebras or coassociative coalgebras. For the rest of the paper, unless we say otherwise, we will assume that either $\mathcal{C} = Vect_k^\mathbb{Z}$, which is the tensor category of $\mathbb{Z}$-graded vector spaces $V = \oplus_{n \in \mathbb{Z}} V_n$, or $\mathcal{C} = Vect_k^\mathbb{Z}/2$, which is the tensor
category of $\mathbb{Z}/2$-graded vector spaces (then $V = V_0 \oplus V_1$), or $\mathcal{C} = \text{Vect}_k$, which is the tensor category of $k$-vector spaces. There is a natural embedding (faithful tensor functor) from $\text{Vect}_k$ to the other two categories such that objects and morphisms are concentrated in degree zero. The unit object $1$ in all three examples can be identified with the field $k$ placed in degree zero. Since most of our definitions hold for more general tensor categories, we will abuse the notation and use $1$ and $k$ interchangeably.

Associativity morphisms in all these categories are identity maps, and commutativity morphisms are given by the Koszul rule of signs: $c(v_i \otimes v_j) = (-1)^{ij} v_j \otimes v_i$, where $v_n$ denotes an element of degree $n$.

We will denote by $\mathcal{C}_f$ the artinian category of finite-dimensional objects in $\mathcal{C}$ (i.e. objects of finite length). The category $\text{Alg}_{\mathcal{C}_f}$ of unital finite-dimensional algebras is closed with respect to finite projective limits. In particular, finite products and finite fiber products exist in $\text{Alg}_{\mathcal{C}_f}$. One has also the categories $\text{Coalg}_\mathcal{C}$ (resp. $\text{Coalg}_{\mathcal{C}_f}$) of coassociative counital (resp. coassociative counital finite-dimensional) coalgebras. In the case $\mathcal{C} = \text{Vect}_k$ we will also use the notation $\text{Alg}_k$, $\text{Alg}_f$, $\text{Coalg}_k$ and $\text{Coalg}_f$ for these categories. The category $\text{Coalg}_{\mathcal{C}_f} = \text{Alg}_{\mathcal{C}_f}^{op}$ admits finite inductive limits.

We will need few basic facts about coalgebras. The proofs are given in the Appendix.

**Theorem 2.1.1** Let $\mathcal{F} : \text{Alg}_{\mathcal{C}_f} \to \text{Sets}$ be a covariant functor commuting with finite projective limits. Then it is isomorphic to a functor of the type $A \mapsto \text{Hom}_{\text{Coalg}_{\mathcal{C}^{op}}}(A^*, B)$ for some counital coalgebra $B$. Moreover, the category of such functors is equivalent to the category of counital coalgebras.

**Proposition 2.1.2** If $B \in \text{Ob}(\text{Coalg}_\mathcal{C})$, then $B$ is a union of finite-dimensional counital coalgebras.

Objects of the category $\text{Coalg}_{\mathcal{C}_f} = \text{Alg}_{\mathcal{C}_f}^{op}$ can be interpreted as “very thin” non-commutative affine schemes (cf. with finite schemes in algebraic geometry). Proposition 2.1.2 implies that the category $\text{Coalg}_\mathcal{C}$ is naturally equivalent to the category of ind-objects in $\text{Coalg}_{\mathcal{C}_f}$.

For a counital coalgebra $B$ we denote by $\text{Spc}(B)$ (and call the “spectrum” of the coalgebra $B$) the corresponding functor on the category of finite-dimensional algebras. A functor isomorphic to $\text{Spc}(B)$ for some $B$ is called a non-commutative thin scheme. The category of non-commutative thin schemes is equivalent to the category of counital coalgebras. For a non-commutative scheme $X$ we denote by $B_X$ the corresponding coalgebra. We will call it the coalgebra of distributions on $X$. The algebra of functions on $X$ is by definition the dual algebra $\mathcal{O}(X) = B_X^* := \text{Hom}_{\mathcal{C}}(B, 1)$.

Non-commutative thin schemes form a full monoidal subcategory $\text{NAff}_{\mathcal{C}}^{th} \subset \text{Ind}(\text{NAff}_{\mathcal{C}})$ of the category of non-commutative ind-affine schemes (see Appendix). Tensor product in this monoidal category corresponds to the tensor product of coalgebras.
Example 2.1.3 Let $V \in \text{Ob}(\mathcal{C})$. Then $T(V) = \bigoplus_{n \geq 0} V^\otimes n$ carries a structure of counital cofree coalgebra in $\mathcal{C}$ with the coproduct $\Delta(v_0 \otimes \ldots \otimes v_n) = \sum_{0 \leq i \leq n} (v_0 \otimes \ldots \otimes v_i) \otimes (v_{i+1} \otimes \ldots \otimes v_n)$. Here $V^\otimes 0 = 1 = k$. The corresponding to $T(V)$ non-commutative thin scheme is called the non-commutative formal affine space $V_{\text{form}}$ (or formal neighborhood of zero in $V$).

Definition 2.1.4 A non-commutative formal manifold $X$ is a non-commutative thin scheme isomorphic to some $\text{Spec}(T(V))$ from the example above. The dimension of $X$ is defined as $\dim_k V$.

The algebra $\mathcal{O}(X)$ of functions on a non-commutative formal manifold $X$ of dimension $n$ is by definition the topological dual to the corresponding colagebra. Hence it isomorphic to the topological algebra $k\langle\langle x_1, \ldots, x_n \rangle\rangle$ of formal power series in free graded variables $x_1, \ldots, x_n$.

Let $X$ be a non-commutative formal manifold, and $pt : k \to B_X$ a $k$-point in $X$.

Definition 2.1.5 The pair $(X, pt)$ is called a non-commutative formal pointed manifold. If $\mathcal{C} = \text{Vect}_k^Z$ it will be called non-commutative formal pointed graded manifold. If $\mathcal{C} = \text{Vect}_k^{Z/2}$ it will be called non-commutative formal pointed supermanifold.

The notion of tangent space $T_{pt} X$ at the point $pt$ follows from the agreement than for $\text{Spec}(T(V))$ and the natural map $pt : k \to V^\otimes 0 = k$ we set $T_{pt}(\text{Spec}(T(V))) = V$. The above-mentioned coordinates $x_1, \ldots, x_n$ can be interpreted as “affine coordinates” on $V$. For more details see Section 3.1 and Appendix.

The following example is a generalization of the Example 2.1.3 from the quiver with one vertex to an arbitrary quiver.

Example 2.1.6 Let $I$ be a set and $B_I = \oplus_{i \in I} 1_i$ be the direct sum of trivial coalgebras. We denote by $\mathcal{O}(I)$ the dual topological algebra. The latter can be thought of as the algebra of functions on a discrete non-commutative thin scheme $I$.

A quiver $Q$ in $C$ with the set of vertices $I$ is given by a collection of objects $E_{ij} \in \mathcal{C}, i,j \in I$ called spaces of arrows from $i$ to $j$. The coalgebra of $Q$ is the coalgebra $B_Q$ generated by the $\mathcal{O}(I) - \mathcal{O}(I)$-bimodule $E_Q = \oplus_{i,j \in I} E_{ij}$, i.e. $B_Q \simeq \oplus_{n \geq 0} \oplus_{i_0, i_1, \ldots, i_n \in I} E_{i_0 i_1} \otimes \ldots \otimes E_{i_{n-1} i_n} : = \oplus_{n \geq 0} B_Q^n$, $B_Q^0 := B_I$. Elements of $B_Q^n$ are called trivial paths. Elements of $B_Q^n$ are called paths of the length $n$. Coproduct is given by the formula

$$\Delta(e_{i_0 i_1} \otimes \ldots \otimes e_{i_{n-1} i_n}) = \oplus_{0 \leq m \leq n} (e_{i_0 i_1} \otimes \ldots \otimes e_{i_{m-1} i_m}) \otimes (e_{i_m i_{m+1}} \otimes \ldots \otimes e_{i_{n-1} i_n}),$$

where for $m = 0$ (resp. $m = n$) we set $e_{i_{-1} i_0} = 1_{i_0}$ (resp. $e_{i_n i_{n+1}} = 1_{i_n}$).

In particular, $\Delta(i_1) = 1_i \otimes 1_i, i \in I$ and $\Delta(e_{ij}) = 1_i \otimes e_{ij} + e_{ij} \otimes 1_j$, where $e_{ij} \in E_{ij}$, and $1_m \in B_I$ corresponds to the image of $1 \in 1$ under the natural embedding into $\oplus_{m \in I} 1$. 

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The coalgebra $B_Q$ has a counit $\varepsilon$ such that $\varepsilon(1_i) = 1_i$, and $\varepsilon(x) = 0$ for $x \in B_Q^n, n \geq 1$.

**Example 2.1.7** (Generalized quivers). Here we replace $1_i$ by a unital simple algebra $A_i$ (e.g. $A_i = \text{Mat}(n_i, D_i)$, where $D_i$ is a division algebra). Then $E_{ij}$ are $A_i - \text{mod} - A_j$-bimodules. We leave as an exercise to the reader to write down the coproduct (for that one uses the tensor product of bimodules) and to check that we indeed obtain a coalgebra.

**Example 2.1.8** Let $I$ be a set. Then the coalgebra $B_I = \bigoplus_{i \in I} 1_i$ is a direct sum of trivial coalgebras, isomorphic to the unit object in $C$. This is a special case of Example 2.1.6. Notice that in general $B_Q$ is a $\mathcal{O}(I) - \mathcal{O}(I)$-bimodule.

**Example 2.1.9** Let $A$ be an associative unital algebra in $C$. It gives rise to the functor $F_A : \text{Coalg}_{C_{\text{fin}}} \to \text{Sets}$ such that $F_A(B) = \text{Hom}_{\text{Alg}}(A, B^*)$. This functor describes finite-dimensional representations of $A$. It commutes with finite direct limits, hence it is representable by a coalgebra. If $A = \mathcal{O}(X)$ is the algebra of regular functions on the affine scheme $X$, then in the case of algebraically closed field $k$ the coalgebra representing $F_A$ is isomorphic to $\bigoplus_{x \in X(k)} \mathcal{O}_{x, X}^*,$ where $\mathcal{O}_{x, X}^*$ denotes the topological dual to the completion of the local ring $\mathcal{O}_{x, X}$. If $X$ is smooth of dimension $n$, then each summand is isomorphic to the topological dual to the algebra of formal power series $k[[t_1, ..., t_n]]$. In other words, this coalgebra corresponds to the disjoint union of formal neighborhoods of all points of $X$.

**Remark 2.1.10** One can describe non-commutative thin schemes more precisely by using structure theorems about finite-dimensional algebras in $C$. For example, in the case $C = \text{Vect}_k$ any finite-dimensional algebra $A$ is isomorphic to a sum $A_0 \oplus r$, where $A_0$ is a finite sum of matrix algebras $\bigoplus \text{Mat}(n_i, D_i)$, $D_i$ are division algebras, and $r$ is the radical. In $\mathbb{Z}$-graded case a similar decomposition holds, with $A_0$ being a sum of algebras of the type $\text{End}(V_i) \otimes D_i$, where $V_i$ are some graded vector spaces and $D_i$ are division algebras of degree zero. In $\mathbb{Z}/2$-graded case the description is slightly more complicated. In particular $A_0$ can contain summands isomorphic to $(\text{End}(V_i) \otimes D_i) \otimes D_{\lambda}$, where $V_i$ and $D_i$ are $\mathbb{Z}/2$-graded analogs of the above-described objects, and $D_\lambda$ is a $1/1$-dimensional superalgebra isomorphic to $k[\xi]/(\xi^2 = \lambda), \deg \xi = 1, \lambda \in k^*/(k^*)^2$.

### 2.2 Smooth thin schemes

Recall that the notion of an ideal has meaning in any abelian tensor category. A 2-sided ideal $J$ is called nilpotent if the multiplication map $J^\otimes n \to J$ has zero image for a sufficiently large $n$.

**Definition 2.2.1** Counital coalgebra $B$ in a tensor category $C$ is called smooth if the corresponding functor $F_B : \text{Alg}_{C_{\text{fin}}} \to \text{Sets}, F_B(A) = \text{Hom}_{\text{Coalg}_C}(A^*, B)$ satisfies the following lifting property: for any 2-sided nilpotent ideal $J \subset A$ the map $F_B(A) \to F_B(A/J)$ induced by the natural projection $A \to A/J$ is surjective. Non-commutative thin scheme $X$ is called smooth if the corresponding counital coalgebra $B = B_X$ is smooth.
**Proposition 2.2.2** For any quiver $Q$ in $\mathcal{C}$ the corresponding coalgebra $B_Q$ is smooth.

**Proof.** First let us assume that the result holds for all finite quivers. We remark that if $A$ is finite-dimensional, and $Q$ is an infinite quiver then for any morphism $f : A^* \to B_Q$ we have: $f(A^*)$ belongs to the coalgebra of a finite sub-quiver of $Q$. Since the lifting property holds for the latter, the result follows.

In order to apply the above considerations we need to prove the Proposition for a finite quiver $Q$. For that let us choose a basis $\{e_{ij,\alpha}\}$ of each space of arrows $E_{ij}$. Then for a finite-dimensional algebra $A$ the set $F_{B_Q}(A)$ is isomorphic to the set $\{((\pi_i, x_{ij,\alpha}))_{i,j\in I}\}$, where $\pi_i \in A, \pi_i^2 = \pi_i, \pi_i \pi_j = \pi_j \pi_i$, if $i \neq j, \sum_{i \in I} \pi_i = 1_A$, and $x_{ij,\alpha} \in \pi_i A \pi_j$ satisfy the condition: there exists $N \geq 1$ such that $x_{ij_1,\alpha_1}...x_{ij_m,\alpha_m} = 0$ for all $m \geq N$. Let now $J \subset A$ be the nilpotent ideal from the definition of smooth coalgebra and $(\pi_i, x_{ij,\alpha})$ be elements of $A/J$ satisfying the above constraints. Our goal is to lift them to $A$. We can lift them to the projectors $\pi_i$ and elements $x_{ij,\alpha}$ for $A$ in such a way that the above constraints are satisfied except of the last one, which becomes an inclusion $x_{ij_1,\alpha_1}...x_{ij_m,\alpha_m} = 0$ for $m \geq N$. Since $J^\infty = 0$ in $A$ for some $n$ we see that $x_{ij_1,\alpha_1}...x_{ij_m,\alpha_m} = 0$ in $A$ for $m \geq nN$. This proves the result. ■

**Remark 2.2.3**

a) According to Cuntz and Quillen (see [CQ2]) a non-commutative algebra $R$ in $\mathbf{Vec}_k$ is called smooth if the functor $\mathbf{Alg}_k \to \mathbf{Sets}, F_R(A) = \text{Hom}_{\mathbf{Alg}_k}(R, A)$ satisfies the lifting property from the Definition 2.2.1 applied to all (not only finite-dimensional) algebras. We remark that if $R$ is smooth in the sense of Cuntz and Quillen then the coalgebra $R_{\text{dual}}$ representing the functor $\mathbf{Coal}_k \to \mathbf{Sets}, B \mapsto \text{Hom}_{\mathbf{Alg}_k}(R, B^*)$ is smooth. One can prove that any smooth coalgebra in $\mathbf{Vec}_k$ is isomorphic to a coalgebra of a generalized quiver (see Example 2.1.7).

b) Almost all examples of non-commutative smooth thin schemes considered in this paper are formal pointed manifolds, i.e. they are isomorphic to $\text{Spc}(T(V))$ for some $V \in \text{Ob}(\mathcal{C})$. It is natural to try to “globalize” our results to the case of non-commutative “smooth” schemes $X$ which satisfy the property that the completion of $X$ at a “commutative” point gives rise to a formal pointed manifold in our sense. An example of the space of maps is considered in the next subsection.

c) The tensor product of non-commutative smooth thin schemes is typically non-smooth, since it corresponds to the tensor product of coalgebras. Notice that the tensor product of colagebras is not a categorical product.

Let now $x$ be a $k$-point of a non-commutative smooth thin scheme $X$. By definition $x$ is a homomorphism of counital coalgebras $x : k \to B_X$ (here $k = 1$ is the trivial coalgebra corresponding to the unit object). The completion $\hat{X}$ of $X$ at $x$ is a formal pointed manifold which can be described such as follows. As a functor $F_{\hat{X}} : \mathbf{Alg}_k \to \mathbf{Sets}$ it assigns to a finite-dimensional algebra $A$ the set of such homomorphisms of counital coalgebras $f : A^* \to B_X$ which are compositions $A^* \to A_1^* \to B_X$, where $A_1^* \subset B_X$ is a conilpotent extension of
$A_1$ is a finite-dimensional unital nilpotent algebra such that the natural embedding $\mathbf{k} \to A_1^* \to B_X$ coincides with $x : \mathbf{k} \to B_X$).

Description of the coalgebra $B_{\tilde{X}_x}$ is given in the following Proposition.

Proposition 2.2.4 The formal neighborhood $\tilde{X}_x$ corresponds to the counital sub-coalgebra $B_{\tilde{X}_x} \subset B_X$ which is the preimage under the natural projection $B_X \to B_X/x(\mathbf{k})$ of the sub-coalgebra consisting of conilpotent elements in the non-counital coalgebra $B/x(\mathbf{k})$. Moreover, $\tilde{X}_x$ is universal for all morphisms from nilpotent extensions of $x$ to $X$.

We discuss in Appendix a more general construction of the completion along a non-commutative thin subscheme.

We leave as an exercise to the reader to prove the following result.

Proposition 2.2.5 Let $Q$ be a quiver and $pt_i \in X = X_{B_Q}$ corresponds to a vertex $i \in I$. Then the formal neighborhood $\tilde{X}_{pt_i}$ is a formal pointed manifold corresponding to the tensor coalgebra $T(E_{ii}) = \oplus_{n \geq 0} E_{ii}^{\otimes n}$, where $E_{ii}$ is the space of loops at $i$.

2.3 Inner Hom

Let $X, Y$ be non-commutative thin schemes, and $B_X, B_Y$ the corresponding coalgebras.

Theorem 2.3.1 The functor $\text{Alg}_{C^f} \to \text{Sets}$ such that

$$A \mapsto \text{Hom}_{\text{Coalg}_{C^f}}(A^* \otimes B_X, B_Y)$$

is representable.

The corresponding non-commutative thin scheme will be denoted by $\text{Maps}(X, Y)$.

Proof. It is easy to see that the functor under consideration commutes with finite projective limits. Hence it is of the type $A \mapsto \text{Hom}_{\text{Coalg}_{C^f}}(A^*, B)$, where $B$ is a counital coalgebra (Theorem 2.1.1). The corresponding non-commutative thin scheme is the desired $\text{Maps}(X, Y)$. ■

It follows from the definition that $\text{Maps}(X, Y) = \text{Hom}(X, Y)$, where the inner Hom is taken in the symmetric monoidal category of non-commutative thin schemes. By definition $\text{Hom}(X, Y)$ is a non-commutative thin scheme, which satisfies the following functorial isomorphism for any $Z \in \text{Ob}(\text{NAff}_{C^f}^\text{th})$:

$$\text{Hom}_{\text{NAff}_{C^f}^\text{th}}(Z, \text{Hom}(X, Y)) \cong \text{Hom}_{\text{NAff}_{C^f}^\text{th}}(Z \otimes X, Y).$$

Notice that the monoidal category $\text{NAff}_{C^f}$ of all non-commutative affine schemes does not have inner $\text{Hom}$’s even in the case $C = \text{Vect}_k$. If $C = \text{Vect}_k$ then one can define $\text{Hom}(X, Y)$ for $X = \text{Spec}(A)$, where $A$ is a finite-dimensional unital algebra and $Y$ is arbitrary. The situation is similar to the case of the conventional (i.e. ‘commutative’) algebraic geometry, where one can define an affine
scheme of maps from a scheme of finite length to an arbitrary affine scheme. On the other hand, one can show that the category of non-commutative ind-affine schemes admits inner Hom’s (the corresponding result for commutative ind-affine schemes is known.

**Remark 2.3.2** The non-commutative thin scheme \(\text{Maps}(X,Y)\) gives rise to a quiver, such that its vertices are \(k\)-points of \(\text{Maps}(X,Y)\). In other words, vertices correspond to homomorphisms \(B_X \to B_Y\) of the coalgebras of distributions. Taking the completion at a \(k\)-point we obtain a formal pointed manifold. More generally, one can take a completion along a subscheme of \(k\)-points, thus arriving to a non-commutative formal manifold with a marked closed subscheme (rather than one point). This construction can be used for the description of the \(A_\infty\)-structure on \(A_\infty\)-functors. We also remark that the space of arrows \(E_{ij}\) of a quiver is an example of the geometric notion of bitangent space at a pair of \(k\)-points \(i,j\). We plan to discuss this topic in another paper.

**Example 2.3.3** Let \(Q_1 = \{i_1\}\) and \(Q_2 = \{i_2\}\) be quivers with one vertex such that \(E_{i_1i_1} = V_1, E_{i_2i_2} = V_2\), \(\dim V_i < \infty, i = 1, 2\). Then \(B_{Q_i} = T(V_i), i = 1, 2\) and \(\text{Maps}(X_{B_{Q_1}}, X_{B_{Q_2}})\) corresponds to the quiver \(Q\) such that the set of vertices \(I_Q = \text{Hom}_{\text{CAlg}}(B_{Q_1}, B_{Q_2}) = \prod_{n \geq 1} \text{Hom}(V_1^\otimes n, V_2)\) and for any two vertices \(f, g \in I_Q\) the space of arrows is isomorphic to \(E_{f,g} = \prod_{n \geq 0} \text{Hom}(V_1^\otimes n, V_2)\).

**Definition 2.3.4** Homomorphism \(f : B_1 \to B_2\) of counital coalgebras is called a minimal conilpotent extension if it is an inclusion and the induced coproduct on the non-counital coalgebra \(B_2/f(B_1)\) is trivial.

Composition of minimal conilpotent extensions is simply called a conilpotent extension. Definition 2.2.1 can be reformulated in terms of finite-dimensional coalgebras. Namely, the coalgebra \(B\) is smooth if the functor \(C \mapsto \text{Hom}_{\text{CAlg}}(C, B)\) satisfies the lifting property with respect to conilpotent extensions of finite-dimensional counital coalgebras. The following result shows that we can drop the condition of finite-dimensionality.

**Proposition 2.3.5** If \(B\) is a smooth coalgebra then the functor \(\text{CAlg}_C \to \text{Sets}\) such that \(C \mapsto \text{Hom}_{\text{CAlg}}(C, B)\) satisfies the lifting property for conilpotent extensions.

**Proof.** Let \(f : B_1 \to B_2\) be a conilpotent extension, and \(g : B_1 \to B\) an arbitrary homomorphism of counital coalgebras. It can be thought of as homomorphism of \(f(B_1) \to B\). We need to show that \(g\) can be extended to \(B_2\). Let us consider the set of pairs \((C, g_C)\) such \(f(B_1) \subset C \subset B_2\) and \(g_C : C \to B\) defines an extension of counital coalgebras, which coincides with \(g\) on \(f(B_1)\). We apply Zorn lemma to the partially ordered set of such pairs and see that there exists a maximal element \((B_{max}, g_{max})\) in this set. We claim that \(B_{max} = B_2\).

Indeed, let \(x \in B_2 \setminus B_{max}\). Then there exists a finite-dimensional coalgebra \(B_x \subset B_2\) which contains \(x\). Clearly \(B_x\) is a conilpotent extension of \(f(B_1) \cap B_x\). Since \(B\) is smooth we can extend \(g_{max} : f(B_1) \cap B_x \to B\) to \(g_x : B_x \to B\)
and, finally to \(g_{x, \text{max}} : B_x + B_{\text{max}} \to B\). This contradicts to maximality of \((B_{\text{max}}, g_{\text{max}})\). Proposition is proved. ■

**Proposition 2.3.6** If \(X, Y\) are non-commutative thin schemes and \(Y\) is smooth then \(\text{Maps}(X, Y)\) is also smooth.

**Proof.** Let \(A \to A/J\) be a nilpotent extension of finite-dimensional unital algebras. Then \((A/J)^* \otimes B_X \to A^* \otimes B_X\) is a conilpotent extension of counital coalgebras. Since \(B_Y\) is smooth then the Proposition 2.3.5 implies that the induced map \(\text{Hom}_{\text{Coalg}_C}(A^* \otimes B_X, B_Y) \to \text{Hom}_{\text{Coalg}_C}(A/J)^* \otimes B_X, B_Y)\) is surjective. This concludes the proof. ■

Let us consider the case when \((X, pt_X)\) and \((Y, pt_Y)\) are non-commutative formal pointed manifolds in the category \(\mathcal{C} = \text{Vect}_k^Z\). One can describe “in coordinates” the non-commutative formal pointed manifold, which is the formal neighborhood of a \(k\)-point of \(\text{Maps}(X, Y)\). Namely, let \(X = \text{Spec}(B)\) and \(Y = \text{Spec}(C)\), and let \(f \in \text{Hom}_{\text{NAff}^\text{th}C}(X, Y)\) be a morphism preserving marked points. Then \(f\) gives rise to a \(k\)-point of \(Z = \text{Maps}(X, Y)\). Since \(\mathcal{O}(X)\) and \(\mathcal{O}(Y)\) are isomorphic to the topological algebras of formal power series in free graded variables, we can choose sets of free topological generators \((x_i)_{i \in I}\) and \((y_j)_{j \in J}\) for these algebras. Then we can write for the corresponding homomorphism of algebras \(f^* : \mathcal{O}(Y) \to \mathcal{O}(X)\):

\[
f^*(y_j) = \sum_M c_{j, M}^0 x^M,
\]

where \(c_{j, M}^0 \in k\) and \(M = (i_1, \ldots, i_n), i_s \in I\) is a non-commutative multi-index (all the coefficients depend on \(f\), hence a better notation should be \(c_{j, M}^0(f)\)). Notice that for \(M = 0\) one gets \(c_{j, 0}^0 = 0\) since \(f\) is a morphism of pointed schemes. Then we can consider an “infinitesimal deformation” \(f_{\text{def}}\) of \(f\)

\[
f_{\text{def}}(y_j) = \sum_M (c_{j, M}^0 + \delta c_{j, M}^0) x^M,
\]

where \(\delta c_{j, M}^0\) are new variables commuting with all \(x_i\). Then \(\delta c_{j, M}^0\) can be thought of as coordinates in the formal neighborhood of \(f\). More pedantically it can be spelled out such as follows. Let \(A = k \oplus m\) be a finite-dimensional graded unital algebra, where \(m\) is a graded nilpotent ideal of \(A\). Then an \(A\)-point of the formal neighborhood \(U_f\) of \(f\) is a morphism \(\phi \in \text{Hom}_{\text{NAff}^\text{th}C}(\text{Spec}(A) \otimes X, Y)\), such that it reduces to \(f\) modulo the nilpotent ideal \(m\). We have for the corresponding homomorphism of algebras:

\[
\phi^*(y_j) = \sum_M c_{j, M} x^M,
\]

where \(M\) is a non-commutative multi-index, \(c_{j, M} \in A\), and \(c_{j, M} \mapsto c_{j, 0} = c_{j, 0}^0\) under the natural homomorphism \(A \to k = A/m\). In particular \(c_{j, 0} \in m\). We can treat coefficients \(c_{j, M}\) as \(A\)-points of the formal neighborhood \(U_f\) of \(f \in \text{Maps}(X, Y)\).
Remark 2.3.7 The above definitions play an important role in the case when
the non-commutative smooth thin scheme $\text{Spc}(B_Q)$ is assigned to a (small) $A_\infty$-
category. Then the non-commutative smooth thin scheme $\text{Maps}(\text{Spc}(B_{Q_1}), \text{Spc}(B_{Q_2}))$
can be used for the description of the category of $A_\infty$-functors between $A_\infty$-
categories, and the formal neighborhood of a point in the space $\text{Maps}(\text{Spc}(B_{Q_1}), \text{Spc}(B_{Q_2}))$
corresponds to natural transformations between $A_\infty$-functors.

3 $A_\infty$-algebras

3.1 Main definitions

From now on assume that $\mathcal{C} = \text{Vect}_k$ unless we say otherwise. If $X$ is a thin
scheme then a vector field on $X$ is, by definition, a derivation of the coalgebra $B_X$. Since derivations can have different degrees, the vector fields form a graded Lie algebra $\text{Vect}(X)$.

Definition 3.1.1 A non-commutative thin differential-graded (dg for short) scheme is a pair $(X, d)$ where $X$ is a non-commutative thin scheme, and $d$ is a vector field on $X$ of degree $+1$ such that $[d, d] = 0$.

We will call the vector field $d$ homological vector field.

Let $X$ be a formal pointed manifold and $x_0$ be its unique $k$-point. Such a point corresponds to a homomorphism of counital coalgebras $k \to B_X$. We say that the vector field $d$ vanishes at $x_0$ if the corresponding derivation kills the image of $k$.

Definition 3.1.2 A non-commutative formal pointed dg-manifold is a pair $(X, x_0, d)$ such that $(X, x_0)$ is a non-commutative formal pointed graded manifold, and $d = d_X$ is a homological vector field on $X$ such that $d|_{x_0} = 0$.

Homological vector field $d$ has an infinite Taylor decomposition at $x_0$. More precisely, let $T_{x_0}X$ be the tangent space at $x_0$. It is canonically isomorphic to the graded vector space of primitive elements of the coalgebra $B_X$, i.e. the set of $a \in B_X$ such that $\Delta(a) = 1 \otimes a + a \otimes 1$ where $1 \in B_X$ is the image of $1 \in k$ under the homomorphism of coalgebras $x_0 : k \to B_X$ (see Appendix for the general definition of the tangent space). Then $d := d_X$ gives rise to a (non-canonically defined) collection of linear maps $d^{(a)}_X := m_n : T_{x_0}X \otimes^n \to T_{x_0}X[1]$, $a \geq 1$ called Taylor coefficients of $d$ which satisfy a system of quadratic relations arising from the condition $[d, d] = 0$. Indeed, our non-commutative formal pointed manifold is isomorphic to the formal neighborhood of zero in $T_{x_0}X$, hence the corresponding non-commutative thin scheme is isomorphic to the cofree tensor coalgebra $T(T_{x_0}X)$ generated by $T_{x_0}X$. Homological vector field $d$ is a derivation of a cofree coalgebra, hence it is uniquely determined by a sequence of linear maps $m_n$.

Definition 3.1.3 Non-unital $A_\infty$-algebra over $k$ is given by a non-commutative formal pointed dg-manifold $(X, x_0, d)$ together with an isomorphism of counital coalgebras $B_X \simeq T(T_{x_0}X)$.  

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As we already pointed out, a choice of isomorphism with the tensor coalgebra generated by the tangent space is a non-commutative analog of a choice of affine structure in the formal neighborhood of $x_0$.

From the above definitions one can recover the conventional definition of an $A_\infty$-algebra. We present it below for convenience of the reader.

**Definition 3.1.4** A structure of an $A_\infty$-algebra on $V \in \text{Ob}(\text{Vect}_k^\mathbb{Z})$ is given by a derivation $d$ of degree +1 of the non-counital cofree coalgebra $T_c(V[1]) = \oplus_{n \geq 1} V^\otimes n$ such that $[d, d] = 0$ in the differential-graded Lie algebra of coalgebra derivations.

Traditionally the Taylor coefficients of $d = m_1 + m_2 + \ldots$ are called (higher) multiplications for $V$. The pair $(V, m_1)$ is a complex of $k$-vector spaces called the tangent complex. If $X = \text{Spc}(T(V))$ then $V[1] = T_0 X$ and $m_1 = d^{(1)}_X$ is the first Taylor coefficient of the homological vector field $d_X$. The tangent cohomology groups $H^i(V, m_1)$ will be denoted by $H^i(V)$. Clearly $H^\bullet(V) = \oplus_{i \in \mathbb{Z}} H^i(V)$ is an associative (non-unital) algebra with the product induced by $m_2$.

An important class of $A_\infty$-algebras consists of unital (or strictly unital) and weakly unital (or homologically unital) ones. We are going to discuss the definition and the geometric meaning of unitality later.

Homomorphism of $A_\infty$-algebras can be described geometrically as a morphism of the corresponding non-commutative formal pointed dg-manifolds. In the algebraic form one recovers the following traditional definition.

**Definition 3.1.5** A homomorphism of non-unital $A_\infty$-algebras (A$_\infty$-morphism for short) $(V, d_V) \to (W, d_W)$ is a homomorphism of differential-graded coalgebras $T_c(V[1]) \to T_c(W[1])$.

A homomorphism $f$ of non-unital $A_\infty$-algebras is determined by its Taylor coefficients $f_n : V^\otimes n \to W[1 - n], n \geq 1$ satisfying the system of equations

\[
\sum_{1 \leq i_1 < \ldots < i_n = n} (-1)^n m_i^W (f_{i_1}(a_1, \ldots, a_{i_1}), f_{i_2-i_1}(a_{i_1+1}, \ldots, a_{i_2}), \ldots, f_{i_n-i_{n-1}}(a_{n-i_{n-1}+1}, \ldots, a_n)) = \\
\sum_{s+r=n+1} \sum_{1 \leq j \leq s} (-1)^r f_s(a_1, \ldots, a_{j-1}, m_r^V(a_j, \ldots, a_{j+r-1}), a_{j+r}, \ldots, a_n).
\]

Here $\epsilon = r \sum_{1 \leq p \leq s-j} \text{deg}(a_p) + j - 1 + r(s-j)$, $\gamma_i = \sum_{1 \leq p \leq s-i} (i-p)(l_p - l_{p-1} - 1) + \sum_{1 \leq p \leq s-i} \nu(l_p) \sum_{l_{p-1}+1 \leq q \leq l_p} \text{deg}(a_q)$, where we use the notation $\nu(l_p) = \sum_{p+1 \leq m \leq s-i} (1 - l_m + l_{m-1})$, and set $l_0 = 0$.

**Remark 3.1.6** All the above definitions and results are valid for $\mathbb{Z}/2$-graded $A_\infty$-algebras as well. In this case we consider formal manifolds in the category $\text{Vect}_k^\mathbb{Z}/2$ of $\mathbb{Z}/2$-graded vector spaces. We will use the corresponding results without further comments. In this case one traditionally denotes by $\text{IIA}$ the $\mathbb{Z}/2$-graded vector space $A[1]$.

### 3.2 Minimal models of $A_\infty$-algebras

One can do simple differential geometry in the symmetric monoidal category of non-commutative formal pointed dg-manifolds. New phenomenon is the possibility to define some structures up to a quasi-isomorphism.
Definition 3.2.1 Let \( f : (X, d_X, x_0) \to (Y, d_Y, y_0) \) be a morphism of non-commutative formal pointed dg-manifolds. We say that \( f \) is a quasi-isomorphism if the induced morphism of the tangent complexes \( f_1 : (T_{x_0}X, d^{(1)}_X) \to (T_{y_0}Y, d^{(1)}_Y) \) is a quasi-isomorphism. We will use the same terminology for the corresponding \( A_\infty \)-algebras.

Definition 3.2.2 An \( A_\infty \)-algebra \( A \) (or the corresponding non-commutative formal pointed dg-manifold) is called minimal if \( m_1 = 0 \). It is called contractible if \( m_n = 0 \) for all \( n \geq 2 \) and \( H^\bullet(A, m_1) = 0 \).

The notion of minimality is coordinate independent, while the notion of contractibility is not.

It is not difficult to prove that any \( A_\infty \)-algebra \( A \) has a minimal model \( M_A \). Here we call \( M_A \) a minimal model if \( M_A \) is minimal and there exists a quasi-isomorphism \( M_A \to A \). The proof of the existence of a minimal model is similar to the one from [Ko1], [KoSo2]. The minimal model is unique up to an \( A_\infty \)-isomorphism. We are going to use the same terminology when talking about non-commutative formal pointed dg-manifolds. In particular, in geometric language a non-commutative formal pointed dg-manifold \( X \) is isomorphic to a categorical product (the latter corresponds to the completed free product \( X_m \times X_{lc} \), where \( X_m \) is minimal and \( X_{lc} \) is linear contractible. The above-mentioned quasi-isomorphism corresponds in the geometric language to the projection \( X \to X_m \). In what follows we will freely move from algebraic to geometric language and back at our convenience.

The following result (homological inverse function theorem) can be easily deduced from the above product decomposition.

Proposition 3.2.3 If \( f : A \to B \) is a quasi-isomorphism of \( A_\infty \)-algebras then there is a (non-canonical) quasi-isomorphism \( g : B \to A \) such that \( fg \) and \( gf \) induce identity maps on zero cohomologies \( H^0(B) \) and \( H^0(A) \) respectively.

3.3 Centralizer of an \( A_\infty \)-morphism

Let \( A \) and \( B \) be two \( A_\infty \)-algebras, and \( (X, d_X, x_0) \) and \( (Y, d_Y, y_0) \) be the corresponding non-commutative formal pointed dg-manifolds. Let \( f : A \to B \) be a morphism of \( A_\infty \)-algebras. Then the corresponding \( k \)-point \( f \in \text{Maps} \left( \text{Spc}(A), \text{Spc}(B) \right) \) gives rise to the formal pointed manifold \( U_f = \text{Maps}(X, Y)_f \) (completion at the point \( f \)). Functoriality of the construction of \( \text{Maps}(X, Y) \) gives rise to a homomorphism of graded Lie algebras of vector fields \( \text{Vect}(X) \oplus \text{Vect}(Y) \to \text{Vect}(\text{Maps}(X, Y)) \). Since \( [d_X, d_Y] = 0 \) on \( X \otimes Y \), we have a well-defined homological vector field \( d_Z \) on \( Z = \text{Maps}(X, Y) \). It corresponds to \( d_X \otimes 1_Y - 1_X \otimes d_Y \) under the above homomorphism. It is easy to see that \( d_Z|_f = 0 \) and in fact morphisms \( f : A \to B \) of \( A_\infty \)-algebras are exactly zeros of \( d_Z \). We are going to describe below the \( A_\infty \)-algebra \( \text{Centr}(f) \) (centralizer of \( f \)) which corresponds to the formal neighborhood \( U_f \) of the point \( f \in \text{Maps}(X, Y) \). We can write (see Section 3.3 for the notation)

\[
c_{j,M} = c^0_{j,M} + r_{j,M},
\]
where $c^0_{j,M} \in k$ and $r_{j,M}$ are formal non-commutative coordinates in the neighborhood of $f$. Then the $A_\infty$-algebra $Centr(f)$ has a basis $(r_{j,M})_{j,M}$ and the $A_\infty$-structure is defined by the restriction of the homological vector $d_Z$ to $U_f$.

As a $\mathbb{Z}$-graded vector space $Centr(f) = \prod_{n \geq 0} Hom_{Vect}(A^\otimes n, B)[-n]$. Let $\phi_1, ..., \phi_n \in Centr(f)$, and $a_1, ..., a_N \in A$. Then we have $m_n(\phi_1, ..., \phi_n)(a_1, ..., a_N) = I + R$. Here $I$ corresponds to the term $= 1_X \otimes 1_Y$ and is given by the following expression:

\[
\mathbf{I} = \sum_{\text{trees } T}^{+} m^B j + \sum_{\text{trees } T}^{-} \mathbf{a}
\]

Similarly $R$ corresponds to the term $d_X \otimes 1_Y$ and it is described by the following picture:
Here are some comments on the figure describing $I$:

1) We partition a sequence $(a_1, ..., a_N)$ into $l \geq n$ non-empty subsequences.
2) We mark $n$ of these subsequences counting from the left (the set can be empty).
3) We apply multilinear map $\phi_i, 1 \leq i \leq n$ to the $i$th marked group of elements $a_i$.
4) We apply Taylor coefficients of $f$ to the remaining subsequences.

Notice that the term $R$ appear only for $m_1$ (i.e. $n = 1$). For all subsequences we have $n \geq 1$.

From geometric point of view the term $I$ corresponds to the vector field $d_Y$, while the term $R$ corresponds to the vector field $d_X$.

**Proposition 3.3.1** Let $d_{\text{Centr}}(f)$ be the derivation corresponding to the image of $d_X \oplus d_Y$ in $\text{Maps}(X,Y)$. Then one has:

$$[d_{\text{Centr}}(f), d_{\text{Centr}}(f)] = 0.$$  

*Proof.* Straightforward computation . ■

**Remark 3.3.2** The $A_\infty$-algebra $\text{Centr}(f)$ and its generalization to the case of $A_\infty$-categories provide geometric description of the notion of natural transformation in the $A_\infty$-case (see [Lyu],[LyuOv] for a pure algebraic approach to this notion).

4 Non-commutative dg-line L and weak unit

4.1 Main definition

**Definition 4.1.1** An $A_\infty$-algebra is called unital (or strictly unital) if there exists an element $1 \in V$ of degree zero, such that $m_2(1,v) = m_2(v,1)$ and
The notion of strict unit depends on a choice of affine coordinates on $\text{Spc}(T(V))$, while the notion of weak unit is “coordinate free”. Moreover, one can show that a weakly unital $A_\infty$-algebra becomes strictly unital after an appropriate change of coordinates.

The category of unital or weakly unital $A_\infty$-algebras are defined in the natural way by the requirement that morphisms preserve the unit (or weak unit) structure.

In this section we are going to discuss a non-commutative dg-version of the odd 1-dimensional supervector space $A_0|_1$ and its relationship to weakly unital $A_\infty$-algebras. The results are valid for both $\mathbb{Z}$-graded and $\mathbb{Z}/2\mathbb{Z}$-graded $A_\infty$-algebras.

**Definition 4.1.2** Non-commutative formal dg-line $L$ is a non-commutative formal pointed dg-manifold corresponding to the one-dimensional $A_\infty$-algebra $A \simeq k$ such that $m_2 = \text{id}, m_n \neq 2 = 0$.

The algebra of functions $O(L)$ is isomorphic to the topological algebra of formal series $k(\langle \xi \rangle)$, where $\deg \xi = 1$. The differential is given by $\partial(\xi) = \xi^2$.

### 4.2 Adding a weak unit

Let $(X, d_X, x_0)$ be a non-commutative formal pointed dg-manifold corresponding to a non-unital $A_\infty$-algebra $A$. We would like to describe geometrically the procedure of adding a weak unit to $A$.

Let us consider the non-commutative formal pointed graded manifold $X_1 = L \times X$ corresponding to the free product of the corresponding coalgebras, i.e. to $B_L \ast B_X$. Clearly one can lift vector fields $d_X$ and $d_L := \partial/\partial \xi$ to $X_1$.

**Lemma 4.2.1** The vector field $d := d_{X_1} = d_X + ad(\xi) - \xi^2 \partial/\partial \xi$

satisfies the condition $[d, d] = 0$.

**Proof.** Straightforward check. ■

**Proposition 4.2.2** The $A_\infty$-algebra $A_1$ has a strict unit.

**Proof.** It follows from the formulas that $\xi$ appears in the expansion of $d_{X_1}$ in quadratic expressions only. Let $A_1$ be the $A_\infty$-algebras corresponding to $X_1$ and $1 \in T_{pt}X_1 = A_1[1]$ be the element of $A_1[1]$ dual to $\xi$ (it corresponds to the tangent vector $\partial/\partial \xi$). Thus we see that $m_2^{A_1}(1, a) = m_2^{A_1}(a, 1) = a, m_2^{A_1}(1, 1) = 1$ for any $a \in A$ and $m_n^{A_1}(a_1, ..., 1, ..., a_n) = 0$ for all $n \geq 2, a_1, ..., a_n \in A$. This completes the proof. ■

Notice that we have a canonical morphism of non-commutative formal pointed dg-manifolds $e : X \to X_1$ such that $e^*_{X_1} = \text{id}, e^*(\xi) = 0$. 

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Definition 4.2.3 Weak unit in $X$ is given by a morphism of non-commutative formal pointed dg-manifolds $p : X_1 \to X$ such that $p \circ e = \text{id}$.

It follows from the definition that if $X$ has a weak unit then the associative algebra $H^\bullet(\Lambda, m^A)$ is unital. Hence our geometric definition agrees with the pure algebraic one (explicit algebraic description of the notion of weak unit can be found e.g. in [FOOO], Section 20\footnote{V. Lyubashenko has informed us that the equivalence of two descriptions also follows from his results with Yu. Bespalov and O. Manzyuk.}.

5 Modules and bimodules

5.1 Modules and vector bundles

Recall that a topological vector space is called linearly compact if it is a projective limit of finite-dimensional vector spaces. The duality functor $V \mapsto V^*$ establishes an anti-equivalence between the category of vector spaces equipped with the discrete topology and the category of linearly compact vector spaces. All that can be extended in the obvious way to the category of graded vector spaces.

Let $X$ be a non-commutative thin scheme in $\text{Vect}_k^\mathbb{Z}$.

Definition 5.1.1 Linearly compact vector bundle $\mathcal{E}$ over $X$ is given by a linearly compact topologically free $\mathcal{O}(X)$-module $\Gamma(\mathcal{E})$, where $\mathcal{O}(X)$ is the algebra of function on $X$. Module $\Gamma(\mathcal{E})$ is called the module of sections of the linearly compact vector bundle $\mathcal{E}$.

Suppose that $(X, x_0)$ is formal pointed graded manifold. By definition, the fiber of $\mathcal{E}$ over $x_0$ is given by the quotient space $\mathcal{E}_{x_0} = \Gamma(\mathcal{E})/m_{x_0}\Gamma(\mathcal{E})$ where $m_{x_0} \subset \mathcal{O}(X)$ is the 2-sided maximal ideal of functions vanishing at $x_0$, and the bar means the closure.

Definition 5.1.2 A dg-vector bundle over a formal pointed dg-manifold $(X, d_X, x_0)$ is given by a linearly compact vector bundle $\mathcal{E}$ over $(X, x_0)$ such that the corresponding module $\Gamma(\mathcal{E})$ carries a differential $d_\mathcal{E} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})[1], d_\mathcal{E}^2 = 0$ so that $(\Gamma(\mathcal{E}), d_\mathcal{E})$ becomes a dg-module over the dg-algebra $(\mathcal{O}(X), d_X)$ and $d_\mathcal{E}$ vanishes on $\mathcal{E}_{x_0}$.

Definition 5.1.3 Let $A$ be a non-unital $A_\infty$-algebra. A left $A$-module $M$ is given by a dg-bundle $\mathcal{E}$ over the formal pointed dg-manifold $X = \text{Spc}(T(A[1]))$ together with an isomorphism of vector bundles $\Gamma(\mathcal{E}) \simeq \mathcal{O}(X) \hat{\otimes} M^*$ called a trivialization of $\mathcal{E}$.

Here $\hat{\otimes}$ denotes the topologically completed tensor product. Passing to dual spaces we obtain the following algebraic definition.
Definition 5.1.4 Let $A$ be an $A_{\infty}$-algebra and $M$ be a $\mathbb{Z}$-graded vector space. A structure of a left $A_{\infty}$-module on $M$ over $A$ (or simply a structure of a left $A$-module on $M$) is given by a differential $d_M$ of degree $+1$ on $T(A[1]) \otimes M$ which makes it into a dg-comodule over the dg-coalgebra $T(A[1])$.

The notion of right $A_{\infty}$-module is similar. Right $A$-module is the same as left $A^{op}$-module. Here $A^{op}$ is the opposite $A_{\infty}$-algebra, which coincides with $A$ as a $\mathbb{Z}$-graded vector space, and for the higher multiplications one has $m^{op}_n(a_1, ..., a_n) = (-1)^{n(n-1)/2}m_n(a_n, ..., a_1)$. The $A_{\infty}$-algebra $A$ carries the natural structures of the left and right $A$-modules. If we simply say “$A$-module” it will always mean “left $A$-module”.

Taking the Taylor series of $d_M$ we obtain a collection of $k$-linear maps (higher action morphisms) for any $n \geq 1$

$$m^M_n : A^{\otimes (n-1)} \otimes M \rightarrow M[2-n],$$

satisfying the compatibility conditions which can be written in exactly the same form as compatibility conditions for the higher products $m^A_n$ (see e.g. [Ke1]). All those conditions can be derived from just one property that the cofree $T_+(A[1])$-comodule $T_+(A[1], M) = \oplus_{n \geq 0} A[1]^{\otimes n} \otimes M$ carries a derivation $m^M = (m^M_n)_{n \geq 0}$ such that $[m^M, m^M] = 0$. In particular $(M, m^M_1)$ is a complex of vector spaces.

Definition 5.1.5 Let $A$ be a weakly unital $A_{\infty}$-algebra. An $A$-module $M$ is called weakly unital if the cohomology $H^\bullet(M, m^M_1)$ is a unital $H^\bullet(A)$-module.

It is easy to see that left $A_{\infty}$-modules over $A$ form a dg-category $A-mod$ with morphisms being homomorphisms of the corresponding comodules. As a graded vector space

$$\text{Hom}_{A-mod}(M, N) = \oplus_{n \geq 0} \text{Hom}_{\text{Vect}}(A[1]^{\otimes n} \otimes M, N).$$

It easy to see that $\text{Hom}_{A-mod}(M, N)$ is a complex.

If $M$ is a right $A$-module and $N$ is a left $A$-module then one has a naturally defined structure of a complex on $M \otimes_A N := \oplus_{n \geq 0} M \otimes A[1]^{\otimes n} \otimes N$. The differential is given by the formula:

$$d(x \otimes a_1 \otimes ... \otimes a_n \otimes y) = \sum \pm m^M_i(x \otimes a_1 \otimes ... \otimes a_i) \otimes a_{i+1} \otimes ... \otimes a_n \otimes y) +$$

$$\sum \pm x \otimes a_1 \otimes ... \otimes a_{i-1} \otimes m^A_i(a_i \otimes ... \otimes a_{i+k-1}) \otimes a_{i+k} \otimes ... \otimes a_n \otimes y +$$

$$\sum \pm x \otimes a_1 \otimes ... \otimes a_{i-1} \otimes m^N_j(a_i \otimes ... \otimes a_n \otimes y).$$

We call this complex the derived tensor product of $M$ and $N$.

For any $A_{\infty}$-algebras $A$ and $B$ we define an $A-B$-bimodule as a $\mathbb{Z}$-graded vector space $M$ together with linear maps

$$e^M_{n_1, n_2} : A[1]^{\otimes n_1} \otimes M \otimes B[1]^{\otimes n_2} \rightarrow M[1]$$
satisfying the natural compatibility conditions (see e.g. [Ke1]). If \( X \) and \( Y \) are formal pointed dg-manifolds corresponding to \( A \) and \( B \) respectively then an \( A - B \)-bimodule is the same as a dg-bundle \( E \) over \( X \otimes Y \) equipped with a homological vector field \( d_E \) which is a lift of the vector field \( d_X \otimes 1 + 1 \otimes d_Y \).

**Example 5.1.6** Let \( A = B = M \). We define a structure of diagonal bimodule on \( A \) by setting \( c_{n_1,n_2}^A = m_{n_1+n_2+1}^A \).

**Proposition 5.1.7** 1) To have a structure of an \( A_\infty \)-module on the complex \( M \) is the same as to have a homomorphism of \( A_\infty \)-algebras \( \phi : A \to \text{End}_K(M) \), where \( K \) is a category of complexes of \( \mathbb{k} \)-vector spaces.

2) To have a structure of an \( A - B \)-bimodule on a graded vector space \( M \) is the same as to have a structure of left \( A \)-module on \( M \) and to have a morphism of \( A_\infty \)-algebras \( \varphi_{A,B} : B^{\text{op}} \to \text{Hom}_{A-\text{mod}}(M,M) \).

Let \( A \) be an \( A_\infty \)-algebra, \( M \) be an \( A \)-module and \( \varphi := \varphi_{A,A} : A^{\text{op}} \to \text{Hom}_{A-\text{mod}}(M,M) \) be the corresponding morphism of \( A_\infty \)-algebras. Then the dg-algebra \( \text{Centr}(\varphi) \) is isomorphic to the dg-algebra \( \text{Hom}_{A-\text{mod}}(M,M) \).

If \( M \equiv_M M_B \) is an \( A - B \)-bimodule and \( N \equiv_B N_C \) is a \( B - C \)-bimodule then the complex \( AM_B \otimes_B BN_C \) carries an \( A - C \)-bimodule structure. It is called the tensor product of \( M \) and \( N \).

Let \( f : X \to Y \) be a homomorphism of formal pointed dg-manifolds corresponding to a homomorphism of \( A_\infty \)-algebras \( A \to B \). Recall that in Section 3.3 we constructed the formal neighborhood \( U_f \) of \( f \) in \( \text{Maps}(X,Y) \) and the \( A_\infty \)-algebra \( \text{Centr}(f) \). On the other hand, we have an \( A - \text{mod} - B \) bimodule structure on \( B \) induced by \( f \). Let us denote this bimodule by \( M(f) \). We leave the proof of the following result as an exercise to the reader. It will not be used in the paper.

**Proposition 5.1.8** If \( B \) is weakly unital then the dg-algebra \( \text{End}_{A-\text{mod}-B}(M(f)) \) is quasi-isomorphic to \( \text{Centr}(f) \).

\( A_\infty \)-bimodules will be used in Part III for the study of homologically smooth \( A_\infty \)-algebras. In the case of \( A_\infty \)-categories bimodules give rise to \( A_\infty \)-functors between the corresponding categories of modules. Tensor product of bimodules corresponds to the composition of \( A_\infty \)-functors.

### 5.2 On the tensor product of \( A_\infty \)-algebras

The tensor product of two dg-algebras \( A_1 \) and \( A_2 \) is a dg-algebra. For \( A_\infty \)-algebras there is no canonical simple formula for the \( A_\infty \)-structure on \( A_1 \otimes_k A_2 \) which generalizes the one in the dg-algebras case. Some complicated formulas were proposed in [SU]. They are not symmetric with respect to the permutation \( (A_1,A_2) \mapsto (A_2,A_1) \). One can give a definition of the dg-algebra which is quasi-isomorphic to the one from [SU] in the case when both \( A_1 \) and \( A_2 \) are weakly unital. Namely, we define the \( A_\infty \)-tensor product

\[
A_1^\prime \otimes^\prime A_2 = \text{End}_{A_1-\text{mod}-A_2}(A_1 \otimes A_2).
\]
Notice that it is a unital dg-algebra. One can show that the dg-category $A - mod - B$ is equivalent (as a dg-category) to $A_1 \otimes A_2^{op} - mod$.

6 Yoneda lemma

6.1 Explicit formulas for the product and differential on $Centr(f)$

Let $A$ be an $A_\infty$-algebra, and $B = \text{End}_K(A)$ be the dg-algebra of endomorphisms of $A$ in the category $K$ of complexes of $k$-vector spaces. Let $f = f_A : A \to B$ be the natural $A_\infty$-morphism coming from the left action of $A$ on itself. Notice that $B$ is always a unital dg-algebra, while $A$ can be non-unital. The aim of this Section is to discuss the relationship between $A$ and $Centr(f_A)$. This is the simplest case of the $A_\infty$-version of Yoneda lemma (the general case easily follows from this one. See also [Lyu],[LyuOv]).

As a graded vector space $Centr(f_A)$ is isomorphic to $\prod_{n \geq 0} \text{Hom}(A^\otimes(n+1), A)[-n]$.

Let us describe the product on $Centr(f)$ for $f = f_A$. Let $\phi, \psi$ be two homogeneous elements of $Centr(f)$. Then

$$(\phi \cdot \psi)(a_1, a_2, \ldots, a_N) = \pm \phi(a_1, \ldots, a_{p-1}, \psi(a_p, \ldots, a_N)).$$

Here $\psi$ acts on the last group of variables $a_p, \ldots, a_N$, and we use the Koszul sign convention for $A_\infty$-algebras in order to determine the sign.

Similarly one has the following formula for the differential (see Section 3.3):

$$(d\phi)(a_1, \ldots, a_N) = \sum \pm \phi(a_1, \ldots, a_s, m_i(a_{s+1}, \ldots, a_{s+i}), a_{s+i+1} \ldots, a_N) + \sum \pm m_i(a_1, \ldots, a_{s-1}, \phi(a_s, \ldots, a_j, \ldots, a_N)).$$

6.2 Yoneda homomorphism

If $M$ is an $A - B$-bimodule then one has a homomorphism of $A_\infty$-algebras $B^{op} \to Centr(\phi_{A,M})$ (see Propositions 5.1.7, 5.1.8). We would like to apply this general observation in the case of the diagonal bimodule structure on $A$. Explicitly, we have the $A_\infty$-morphism $A^{op} \to \text{End}_{mod - A}(A)$ or, equivalently, the collection of maps $A^\otimes m \to \text{Hom}(A^\otimes n, A)$. By conjugation it gives us a collection of maps

$A^\otimes m \otimes \text{Hom}(A^\otimes n, A) \to \text{Hom}(A^\otimes (m+n), A)$.

In this way we get a natural $A_\infty$-morphism $Yo : A^{op} \to Centr(f_A)$ called the Yoneda homomorphism.

Proposition 6.2.1 The $A_\infty$-algebra $A$ is weakly unital if and only if the Yoneda homomorphism is a quasi-isomorphism.
Proof. Since $\text{Centr}(f_A)$ is weakly unital, then $A$ must be weakly unital as long as Yoneda morphism is a quasi-isomorphism.

Let us prove the opposite statement. We assume that $A$ is weakly unital. It suffices to prove that the cone $\text{Cone}(Yo)$ of the Yoneda homomorphism has trivial cohomology. Thus we need to prove that the cone of the morphism of complexes

$$(A^{op}, m_1) \to (\oplus_{n \geq 1} \text{Hom}(A^{\otimes n}, A), m_1^{\text{Centr}(f_A)}).$$

is contractible. In order to see this, one considers the extended complex $A \oplus \text{Centr}(f_A)$. It has natural filtration arising from tensor powers of $A$. The corresponding spectral sequence collapses, which gives an explicit homotopy of the extended complex to the trivial one. This implies the desired quasi-isomorphism of $H^0(A^{op})$ and $H^0(\text{Centr}(f_A))$. ■

Remark 6.2.2 It looks like the construction of $\text{Centr}(f_A)$ is the first known canonical construction of a unital dg-algebra quasi-isomorphic to a given $A_{\infty}$-algebra (canonical but not functorial). This is true even in the case of strictly unital $A_{\infty}$-algebras. Standard construction via bar and cobar resolutions gives a non-unital dg-algebra.

Part II
Smoothness and compactness

7 Hochschild cochain and chain complexes of an $A_{\infty}$-algebra

7.1 Hochschild cochain complex

We change the notation for the homological vector field to $Q$, since the letter $d$ will be used for the differential. Let $((X, pt), Q)$ be a non-commutative formal pointed dg-manifold corresponding to a non-unital $A_{\infty}$-algebra $A$, and $\text{Vect}(X)$ the graded Lie algebra of vector fields on $X$ (i.e. continuous derivations of $\mathcal{O}(X)$).

We denote by $C^\bullet(A, A) := C^\bullet(X, X) := \text{Vect}(X)[-1]$ the Hochschild cochain complex of $A$. As a $\mathbb{Z}$-graded vector space

$$C^\bullet(A, A) = \prod_{n \geq 0} \text{Hom}_\mathbb{C}(A[1]^{\otimes n}, A).$$

The differential on $C^\bullet(A, A)$ is given by $[Q, \bullet]$. Algebraically, $C^\bullet(A, A)[1]$ is a DGLA of derivations of the coalgebra $T(A[1])$ (see Section 3).

We recall that the super version of the notion of formal dg-manifold was introduced by A. Schwarz under the name “Q-manifold”. Here letter $Q$ refers to the supercharge notation from Quantum Field Theory.
Theorem 7.1.1 Let $X$ be a non-commutative formal pointed dg-manifold and $C^\bullet(X, X)$ be the Hochschild cochain complex. Then one has the following quasi-isomorphism of complexes

$$C^\bullet(X, X)[1] \simeq T_{id_X}(Maps(X, X)),$$

where $T_{id_X}$ denotes the tangent complex at the identity map.

Proof. Notice that $Maps(Spec(k[\varepsilon]/(\varepsilon^2)) \otimes X, X)$ is the non-commutative dg ind-manifold of vector fields on $X$. The tangent space $T_{id_X}$ from the theorem is the DGLA of vector fields on $X$. It can be identified with the set of such $f \in Maps(Spec(k[\varepsilon]/(\varepsilon^2)) \otimes X, X)$ that $f|_{\{pt\} \otimes X} = id_X$. On the other hand the DGLA $C^\bullet(X, X)[1]$ is the DGLA of vector fields on $X$. The theorem follows. □

The Hochschild cochain complex admits a couple of other interpretations. We leave to the reader to check the equivalence of all of them. Namely:

a) $C^\bullet(A, A) \simeq Centr(id_A)$.

b) For a weakly unital $A_\infty$-algebra $A$ one has $C^\bullet(A, A) \simeq Hom_{A-mod-A}(A, A)$.

Both a) and b) are quasi-isomorphisms of complexes.

Remark 7.1.2 Interpretation of $C^\bullet(A, A)[1] = C^\bullet(X, X)[1]$ as vector fields gives a DGLA structure on this space. It is a Lie algebra of the “commutative” formal group in $Vect^F_k$, which is an abelianization of the non-commutative formal group of inner (in the sense of tensor categories) automorphisms $Aut(X) \subset Maps(X, X)$. Because of this non-commutative structure underlying the Hochschild cochain complex, it is natural to expect that $C^\bullet(A, A)[1]$ carries more structures than just DGLA. Indeed, Deligne’s conjecture (see e.g. [KoSo1] and the last section of this paper) claims that the DGLA algebra structure on $C^\bullet(A, A)[1]$ can be extended to a structure of an algebra over the operad of singular chains of the topological operad of little discs. Graded Lie algebra structure can be recovered from cells of highest dimension in the cell decomposition of the topological operad.

7.2 Hochschild chain complex

In this subsection we are going to construct a complex of $k$-vector spaces which is dual to the Hochschild chain complex of a non-unital $A_\infty$-algebra.

7.2.1 Cyclic differential forms of order zero

Let $(X, pt)$ be a non-commutative formal pointed manifold over $k$, and $O(X)$ the algebra of functions on $X$. For simplicity we will assume that $X$ is finite-dimensional, i.e. $dim_k T_{pt}X < \infty$. If $B = B_X$ is a counital coalgebra corresponding to $X$ (i.e. the coalgebra of distributions on $X$) then $O(X) \simeq B^*$. Let us choose affine coordinates $x_1, x_2, ..., x_n$ at the marked point $pt$. Then we have an isomorphism of $O(X)$ with the topological algebra $k\langle\langle x_1, ..., x_n\rangle\rangle$ of formal series in free graded variables $x_1, ..., x_n$.  

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We define the space of cyclic differential degree zero forms on $X$ as

$$\Omega^0_{cyc}(X) = \mathcal{O}(X)/[\mathcal{O}(X), \mathcal{O}(X)]_{top},$$

where $[\mathcal{O}(X), \mathcal{O}(X)]_{top}$ denotes the topological commutator (the closure of the algebraic commutator in the adic topology of the space of non-commutative formal power series).

Equivalently, we can start with the graded $k$-vector space $\Omega^0_{cyc,dual}(X)$ defined as the kernel of the composition $B \rightarrow B \otimes B \rightarrow \wedge^2 B$ (first map is the coproduct $\Delta : B \rightarrow B \otimes B$, while the second one is the natural projection to the skew-symmetric tensors). Then $\Omega^0_{cyc}(X) \simeq (\Omega^0_{cyc,dual}(X))^*$ (dual vector space).

### 7.2.2 Higher order cyclic differential forms

We start with the definition of the odd tangent bundle $T[1]X$. This is the dg-analog of the total space of the tangent supervector bundle to a supermanifold with the changed parity of fibers. It is more convenient to describe this formal manifold in terms of algebras rather than coalgebras. Namely, the algebra of functions $\mathcal{O}(T[1]X)$ is a unital topological algebra isomorphic to the algebra of formal power series $k\langle\langle x_i, dx_i \rangle\rangle, 1 \leq i \leq n$, where $\text{deg } dx_i = \text{deg } x_i + 1$ (we do not impose any commutativity relations between generators). More invariant description involves the odd line. Namely, let $t_1 := \text{Spc}(B_1)$, where $(B_1)^* = k\langle\langle \xi \rangle\rangle/(\xi^2), \text{deg } \xi = +1$. Then we define $T[1]X$ as the formal neighborhood in $\text{Maps}(t_1, X)$ of the point $p$ which is the composition of $pt$ with the trivial map of $t_1$ into the point $\text{Spc}(k)$.

**Definition 7.2.1**

a) The graded vector space

$$\mathcal{O}(T[1]X) = \Omega^*(X) = \prod_{m \geq 0} \Omega^m(X)$$

is called the space of de Rham differential forms on $X$.

b) The graded space

$$\Omega^0_{cyc}(T[1]X) = \prod_{m \geq 0} \Omega^m_{cyc}(X)$$

is called the space of cyclic differential forms on $X$.

In coordinate description the grading is given by the total number of factors $dx_i$. Clearly each space $\Omega^m_{cyc}(X), n \geq 0$ is dual to some vector space $\Omega^m_{cyc,dual}(X)$ equipped with the discrete topology (since this is true for $\Omega^0(T[1]X)$).

The de Rham differential on $\Omega^*(X)$ corresponds to the vector field $\partial/\partial \xi$. This can be seen from the description which uses the odd line, where the variable $\xi$ has the same meaning. Since $\Omega^0_{cyc}$ is given by the natural (i.e. functorial) construction, the de Rham differential descends to the subspace of cyclic differential forms. We will denote the former by $d_{DR}$ and the latter by $d_{cyc}$. 

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The space of cyclic 1-forms $\Omega^1_{cycl}(X)$ is a (topological) span of expressions $x_1x_2...x_1 \ dx_1, x_i \in \mathcal{O}(X)$. Equivalently, the space of cyclic 1-forms consists of expressions $\sum_{1 \leq i \leq n} f_i(x_1, ..., x_n) \ dx_i$ where $f_i \in \mathcal{k}\langle x_1, ..., x_n \rangle$.

There is a map $\varphi : \Omega^1_{cycl}(X) \rightarrow \mathcal{O}(X)_{red} := \mathcal{O}(X)/\mathcal{k}$, which is defined on $\Omega^1(X)$ by the formula $adb \mapsto [a, b]$, and then one checks that the induced map on the cyclic 1-forms is well-defined. This map does not have an analog in the commutative case.\[\]7.2.3 Non-commutative Cartan calculus

Let $X$ be a formal graded manifold over a field $k$. We denote by $g := gX$ the graded Lie algebra of continuous linear maps $\mathcal{O}(T[1]X) \rightarrow \mathcal{O}(T[1]X)$ generated by de Rham differential $d = d_{dR}$ and contraction maps $\iota_\xi, \xi \in Vect(X)$ which are defined by the formulas $\iota_\xi(f) = 0, \iota_\xi(df) = \xi(f)$ for all $f \in \mathcal{O}(T[1]X)$. Let us define the Lie derivative $\text{Lie}_\xi = [d, \iota_\xi]$, where the RHS is the graded commutator. Then one can easily checks the usual formulas of the Cartan calculus

$$[d, d] = 0, \text{Lie}_\xi = [d, \iota_\xi], [d, \text{Lie}_\xi] = 0,$$

$$[\text{Lie}_\xi, \iota_\eta] = \iota_{[\xi, \eta]}, [\text{Lie}_\xi, \text{Lie}_\eta] = \text{Lie}_{[\xi, \eta]}, [\iota_\xi, \iota_\eta] = 0,$$

for any $\xi, \eta \in Vect(X)$.

By naturality, the graded Lie algebra $gX$ acts on the space $\Omega^\bullet_{cycl}(X)$ as well as on the dual space $(\Omega^\bullet_{cycl}(X))^\ast$.

7.2.4 Differential on the Hochschild chain complex

Let $Q$ be a homological vector field on $(X, pt)$. Then $A = T_{pt}X[-1]$ is a non-unital $A_\infty$-algebra.

We define the dual Hochschild chain complex $(C_\bullet(A, A))^\ast$ as the graded vector space $\Omega^1_{cycl}(X)[2]$ endowed with the differential $\text{Lie}_Q$. Our terminology is explained by the observation that $\Omega^1_{cycl}(X)[2]$ is dual to the conventional Hochschild chain complex

$$C_\bullet(A, A) = \oplus_{n \geq 0} (A[1])^\otimes n \otimes A.$$

Notice that we use the cohomological grading on $C_\bullet(A, A)$, i.e. chains of degree $n$ in conventional (homological) grading have degree $-n$ in our grading. In our grading the differential has degree +1.

In coordinates the isomorphism identifies an element $f_i(x_1, ..., x_n) \otimes x_i \in (A[1]^\otimes n \otimes A)^\ast$ with the homogeneous element $f_i(x_1, ..., x_n) \ dx_i \in \Omega^1_{cycl}(X)$. Here $x_i \in (A[1])^\ast, 1 \leq i \leq n$ are “affine coordinates” on $X$.

The graded Lie algebra $Vect(X)$ of vector fields of all degrees acts on any functorially defined space, in particular, on all spaces $\Omega^i(X), \Omega^i_{cycl}(X)$, etc.

\[1\] V. Ginzburg pointed out that the geometric meaning of the map $\varphi$ as a “contraction with double derivation” was suggested in Section 5.4 of [G3].

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Then we have a differential of degree +1 on $\Omega^j_{cycl}(X)$ given by $b = \text{Lie}_Q$. There is an explicit formula for the differential $b$ on $C_\bullet(A,A)$ (cf. [T]):

$$b(a_0 \otimes \ldots \otimes a_n) = \sum \pm a_0 \otimes \ldots \otimes m_1(a_i \otimes \ldots \otimes a_j) \otimes \ldots \otimes a_n$$

$$+ \sum \pm m_1(a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_i) \otimes a_{i+1} \otimes \ldots \otimes a_{j-1}.$$  

It is convenient to depict a cyclic monomial $a_0 \otimes \ldots \otimes a_n$ in the following way. We draw a clockwise oriented circle with $n + 1$ points labeled from 0 to $n$ such that one point is marked. We assign the elements $a_0, a_1, \ldots, a_n$ to the points with the corresponding labels, putting $a_0$ at the marked point.

Then we can write $b = b_1 + b_2$ where $b_1$ is the sum (with appropriate signs) of the expressions depicted below:

Similarly, $b_2$ is the sum (with appropriate signs) of the expressions depicted
In both cases maps $m_i$ are applied to a consecutive cyclically ordered sequence of elements of $A$ assigned to the points on the top circle. The identity map is applied to the remaining elements. Marked point on the top circle is the position of the element of $a_0$. Marked point on the bottom circle depicts the first tensor factor of the corresponding summand of $b$. In both cases we start cyclic count of tensor factors clockwise from the marked point.

7.3 The case of strictly unital $A_\infty$-algebras

Let $A$ be a strictly unital $A_\infty$-algebra. There is a reduced Hochschild chain complex

$$C^\bullet_{\text{red}}(A, A) = \oplus_{n \geq 0} A \otimes ((A/k \cdot 1)[1])^\otimes n,$$

which is the quotient of $C^\bullet(A, A)$. Similarly there is a reduced Hochschild cochain complex

$$C^\bullet_{\text{red}}(A, A) = \prod_{n \geq 0} \text{Hom}_C((A/k \cdot 1)[1]^\otimes n, A),$$

which is a subcomplex of the Hochschild cochain complex $C^\bullet(A, A)$.

Also, $C^\bullet(A, A)$ carries also the “Connes’s differential” $B$ of degree $-1$ which is called sometimes “de Rham differential” and which is given by the formula (see [Co], [T])

$$B(a_0 \otimes \ldots \otimes a_n) = \sum_i \pm 1 \otimes a_i \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{i-1}, B^2 = 0, Bb + bB = 0.$$

Here is the graphical description of $B$, which will receive an explanation in the section devoted to generalized Deligne’s conjecture:
Let \( u \) be an independent variable of degree +2. It follows that for a strictly unital \( A_\infty \)-algebra \( A \) one has a differential \( b + u B \) of degree +1 on the graded vector space \( C_\bullet(A, A)[[u]] \) which makes the latter into a complex called \textit{negative cyclic complex} (see [Co], [T]). In fact \( b + u B \) is a differential on a smaller complex \( C_\bullet(A, A)[u] \). In the non-unital case one can use Cuntz-Quillen complex instead of the negative cyclic complex (see next subsection).

### 7.4 Non-unital case: Cuntz-Quillen complex

In this subsection we are going to present a formal dg-version of the mixed complex introduced by Cuntz and Quillen (see [CQ1]). In the previous subsection we introduced the Connes differential \( B \) in the case of strictly unital \( A_\infty \)-algebras. In the non-unital case the construction has to be modified. Let \( X = A[1]_{\text{form}} \) be the corresponding non-commutative formal pointed dg-manifold. The algebra of functions \( \mathcal{O}(X) \cong \prod_{n \geq 0} (A[1]_{\text{form}})^{\otimes n} \) is a complex with the differential \( \text{Lie}_{\mathcal{Q}} \).

**Proposition 7.4.1** If \( A \) is weakly unital then all non-zero cohomology of the complex \( \mathcal{O}(X) \) are trivial, and \( H^0(\mathcal{O}(X)) \cong k \).

**Proof.** Let us calculate the cohomology using the spectral sequence associated with the filtration \( \prod_{n \geq n_0} (A[1]_{\text{form}})^{\otimes n} \). The term \( E_1 \) of the spectral sequence is isomorphic to the complex \( \prod_{n \geq 0} ((H^*(A[1], m_1))^{\otimes n})^\bullet \) with the differential induced by the multiplication \( m_2^A \) on \( H^*(A, m_1^A) \). By assumption \( H^*(A, m_1^A) \) is a unital algebra, hence all cohomology groups of the complex vanish except for the zeroth one, which is isomorphic to \( k \). This concludes the proof. \( \blacksquare \).

It follows from the above Proposition 7.4.1 that the complex \( \mathcal{O}(X)/k \) is acyclic. We have the following two morphisms of complexes

\[
d_{\text{cycl}} : (\mathcal{O}(X)/k \cdot 1, \text{Lie}_Q) \to (\Omega^1_{\text{cycl}}(X), \text{Lie}_Q)
\]
and
\[ \varphi : (\Omega^1_{\text{cycl}}(X), \text{Lie}_Q) \rightarrow (\mathcal{O}(X)/k \cdot 1, \text{Lie}_Q). \]

Here \( d_{\text{cycl}} \) and \( \varphi \) were introduced in the Section 7.2. We have: \( \text{deg}(d_{\text{cycl}}) = +1, \text{deg}(\varphi) = -1, d_{\text{cycl}} \circ \varphi = 0, \varphi \circ d_{\text{cycl}} = 0. \)

Let us consider a modified Hochschild chain complex

\[ C^\bullet_{\text{mod}}(A, A) := (\Omega^1_{\text{cycl}}(X)[2])^* \oplus (\mathcal{O}(X)/k \cdot 1)^* \]

with the differential

\[ b = \begin{pmatrix} (\text{Lie}_Q)^* & \varphi^* \\ 0 & (\text{Lie}_Q)^* \end{pmatrix} \]

Let

\[ B = \begin{pmatrix} 0 & 0 \\ d_{\text{cycl}}^* & 0 \end{pmatrix} \]

be an endomorphism of \( C^\bullet_{\text{mod}}(A, A) \) of degree \(-1\). Then \( B^2 = 0 \). Let \( u \) be a formal variable of degree \(+2\). We define modified negative cyclic, periodic cyclic and cyclic chain complexes such as follows

\[ CC^{\mod}_{-1}(A) = (C^\bullet_{\text{mod}}(A, A)[[u]], b + uB), \]

\[ CP^{\mod}_{-1}(A) = (C^\bullet_{\text{mod}}(A, A)((u)), b + uB), \]

\[ CC_{\text{mod}}^\bullet(A) = (CP^{\mod}_{-1}(A)/CC^{\mod}_{-1}(A))[-2]. \]

The corresponding cohomology groups are called (modified) negative cyclic, periodic cyclic and cyclic homology respectively.

For unital dg-algebras these complexes are quasi-isomorphic to the standard ones. If \( \text{char} \ k = 0 \) and \( A \) is weakly unital then \( CC^{\mod}_{-1}(A) \) is quasi-isomorphic to the complex \( (\Omega^1_{\text{cycl}}(X), \text{Lie}_Q)^* \). Notice that the \( k[[u]] \)-module structure on the cohomology \( H^\bullet((\Omega^0_{\text{cycl}}(X), \text{Lie}_Q)^*) \) is not transparent from its definition.

8 Homologically smooth and compact \( A_\infty \)-algebras

From now on we will assume that all \( A_\infty \)-algebras are weakly unital unless we say otherwise.

8.1 Homological smoothness

Let \( A \) be an \( A_\infty \)-algebra over \( k \) and \( E_1, E_2, ..., E_n \) be a sequence of \( A \)-modules. Let us consider a sequence \((E_{\leq i})_{1 \leq i \leq n}\) of \( A \)-modules together with exact triangles

\[ E_i \rightarrow E_{\leq i} \rightarrow E_{i+1} \rightarrow E_i[1], \]

such that \( E_{\leq 1} = E_1 \).

We will call \( E_{\leq n} \) an extension of the sequence \( E_1, ..., E_n \).

Notice that we can speak in the same way about extensions in the category of \( A - A \)-bimodules.
Definition 8.1.1 1) A perfect $A$-module is the one which is quasi-isomorphic to a direct summand of an extension of a sequence of modules each of which is quasi-isomorphic to $A[n], n \in \mathbb{Z}$.

2) A perfect $A - A$-bimodule is the one which is quasi-isomorphic to a direct summand of an extension of a sequence consisting of bimodules each of which is quasi-isomorphic to $(A \otimes A)[n], n \in \mathbb{Z}$.

Perfect $A$-modules form a full subcategory $\text{Perf}_A$ of the dg-category $A-\text{mod}$. Perfect $A - A$-bimodules form a full subcategory $\text{Perf}_{A-\text{mod}-A}$ of the category of $A - A$-bimodules.

Definition 8.1.2 We say that an $A_{\infty}$-algebra $A$ is homologically smooth if it is a perfect $A - A$-bimodule (equivalently, $A$ is a perfect module over the $A_{\infty}$-algebra $A_{\infty} \otimes A_{\infty}$).

Remark 8.1.3 An $A - B$-bimodule $M$ gives rise to a dg-functor $B - \text{mod} \to A - \text{mod}$ such that $V \mapsto M \otimes_B V$. The diagonal bimodule $A$ corresponds to the identity functor $\text{Id}_{A - \text{mod}} : A - \text{mod} \to A - \text{mod}$. The notion of homological smoothness can be generalized to the framework of $A_{\infty}$-categories. The corresponding notion of saturated $A_{\infty}$-category can be spelled out entirely in terms of the identity functor.

Let us list few examples of homologically smooth $A_{\infty}$-algebras.

Example 8.1.4 a) Algebra of functions on a smooth affine scheme.

b) $A = k[x_1, \ldots, x_n]_q$, which is the algebra of polynomials in variables $x_i, 1 \leq i \leq n$ subject to the relations $x_i x_j = q_{ij} x_j x_i$, where $q_{ij} \in k^*$ satisfy the properties $q_{ii} = 1, q_{ij} q_{ji} = 1$. More generally, all quadratic Koszul algebras, which are deformations of polynomial algebras are homologically smooth.

c) Algebras of regular functions on quantum groups (see [KorSo]).

d) Free algebras $k(x_1, \ldots, x_n)$.

e) Finite-dimensional associative algebras of finite homological dimension.

f) If $X$ is a smooth scheme over $k$ then the bounded derived category $D^b(\text{Perf}(X))$ of the category of perfect complexes (it is equivalent to $D^b(\text{Coh}(X))$) has a generator $P$ (see [BeB]). Then the dg-algebra $A = \text{End}(P)$ (here we understand endomorphisms in the “derived sense”, see [Ke2]) is a homologically smooth algebra.

Let us introduce an $A - A$-bimodule $A^! = \text{Hom}_{A - \text{mod}-A}(A, A \otimes A)$ (cf. [Gi2]). The structure of an $A - A$-bimodule is defined similarly to the case of associative algebras.

Proposition 8.1.5 If $A$ is homologically smooth then $A^!$ is a perfect $A - A$-bimodule.

\[\text{Sometimes \ Perfa is called a thick triangulated subcategory of } A - \text{mod \ generated by } A \text{ and denoted by } \langle A \rangle. \text{ In the case of } A - A\text{-bimodules we have a thick triangulated subcategory generated by } A \otimes A, \text{ which is denoted by } \langle A \otimes A \rangle.\]
Proof. We observe that $\text{Hom}_{C^{-\text{mod}}}(C, C)$ is a dg-algebra for any $A_{\infty}$-algebra $C$. The Yoneda embedding $C \to \text{Hom}_{C^{-\text{mod}}}(C, C)$ is a quasi-isomorphism of $A_{\infty}$-algebras. Let us apply this observation to $C = A \otimes A^{\text{op}}$. Then using the $A_{\infty}$-algebra $A^\omega \otimes A^{\text{op}}$ (see Section 5.2) we obtain a quasi-isomorphism of $A^{\omega}$-bimodules $\text{Hom}_{A^{\omega^{-\text{mod}}}}(A \otimes A, A \otimes A) \simeq A \otimes A$. By assumption $A$ is quasi-isomorphic (as an $A_{\infty}$-bimodule) to a direct summand in an extension of a sequence $(A \otimes A)[n_i]$ for $n_i \in \mathbb{Z}$. Hence $\text{Hom}_{A^{\omega^{-\text{mod}}}}(A \otimes A, A \otimes A)$ is quasi-isomorphic to a direct summand in an extension of a sequence $(A \otimes A)[m_i]$ for $m_i \in \mathbb{Z}$. The result follows. ■

**Definition 8.1.6** The bimodule $A^!$ is called the inverse dualizing bimodule.

The terminology is explained by an observation that if $A = \text{End}(P)$ where $P$ is a generator of of $\text{Perf}(X)$ (see Example 8.1.4 f)) then the bimodule $A^!$ corresponds to the functor $F \mapsto F \otimes K_X^{\omega}[\text{dim} X]$, where $K_X$ is the canonical class of $X$.\[13\]

**Remark 8.1.7** In [ToVa] the authors introduced a stronger notion of fibrant dg-algebra. Informally it corresponds to “non-commutative homologically smooth affine schemes of finite type”. In the compact case (see the next section) this notion coincides with the one of homologically smooth algebra.

### 8.2 Compact $A_{\infty}$-algebras

**Definition 8.2.1** We say that an $A_{\infty}$-algebra $A$ is compact if the cohomology $H^\bullet(A, m_1)$ is finite-dimensional.

**Example 8.2.2**

a) If $\dim_k A < \infty$ then $A$ is compact.

b) Let $X/k$ be a proper scheme of finite type. According to [BoB] there exists a compact dg-algebra $A$ such that $\text{Perf}_A$ is equivalent to $D^b(\text{Coh}(X))$.

c) If $Y \subset X$ is a proper subscheme (possibly singular) of a smooth scheme $X$ then the bounded derived category $D_Y^b(\text{Perf}(X))$ of the category of perfect complexes on $X$, which are supported on $Y$ has a generator $P$ such that $A = \text{End}(P)$ is compact. In general it is not homologically smooth for $Y \neq X$. More generally, one can replace $X$ by a formal smooth scheme containing $Y$, e.g. by the formal neighborhood of $Y$ in the ambient smooth scheme. In particular, for $Y = \{pt\} \subset X = \mathbb{A}^1$ and the generator $\mathcal{O}_Y$ of $D^b(\text{Perf}(X))$ the corresponding graded algebra is isomorphic to $k\langle \xi \rangle / (\xi^2)$, where $\deg \xi = 1$.

**Proposition 8.2.3** If $A$ is compact and homologically smooth then the Hochschild homology and cohomology of $A$ are finite-dimensional.

**Proof.** a) Let us start with Hochschild cohomology. We have an isomorphism of complexes $C^\bullet(A, A) \simeq \text{Hom}_{A^{-\text{mod}}_A}(A, A)$. Since $A$ is homologically smooth\[13\]We thank to Amnon Yekutieli for pointing out that the inverse dualizing module was first mentioned in the paper by Michel van den Bergh “Existence theorems for dualizing complexes over non-commutative graded and filtered rings”, J. Algebra, 195:2, 1997, 662-679.
the latter complex is quasi-isomorphic to a direct summand of an extension of the bimodule $\text{Hom}_{A-\text{mod}}(A \otimes A, A \otimes A)$. The latter complex in turn is quasi-isomorphic to $A \otimes A$ (see the proof of the Proposition rehomologically smoothness and perfectness). Since $A$ is compact, the complex $A \otimes A$ has finite-dimensional cohomology. Therefore any perfect $A-A$-bimodule enjoys the same property. We conclude that the Hochschild cohomology groups are finite-dimensional vector spaces.

b) Let us consider the case of Hochschild homology. With any $A-A$-bimodule $E$ we associate a complex of vector spaces $E^\bullet = \oplus_{n \geq 0} A[1]^{\otimes n} \otimes E$ (cf. [Gi2]). The differential on $E^\bullet$ is given by the same formulas as the Hochschild differential for $C_i(A, A)$ with the only change: we place an element $e \in E$ instead of an element of $A$ at the marked vertex (see Section [7]). Taking $E = A$ with the structure of the diagonal $A-A$-bimodule we obtain $A^\bullet = C_i(A, A)$. On the other hand, it is easy to see that the complex $(A \otimes A)^\bullet$ is quasi-isomorphic to $(A, m_1)$, since $(A \otimes A)^\bullet$ is the quotient of the canonical free resolution (bar resolution) for $A$ by a subcomplex $A$. The construction of $E^\bullet$ is functorial, hence $A^\bullet$ is quasi-isomorphic to a direct summand of an extension (in the category of complexes) of a shift of $(A \otimes A)^\bullet$, because $A$ is smooth. Since $A^\bullet = C_i(A, A)$ we see that the Hochschild homology $H_i(A, A)$ is isomorphic to a direct summand of the cohomology of an extension of a sequence of $k$-modules $(A[n_1], m_1)$. Since the vector space $H^\bullet(A, m_1)$ is finite-dimensional the result follows. ■

**Remark 8.2.4** For a homologically smooth compact $A_{\infty}$-algebra $A$ one has a quasi-isomorphism of complexes $C_i(A, A) \simeq \text{Hom}_{A-\text{mod}}(A^i, A)$. Also, the complex $\text{Hom}_{A-\text{mod}}(M^\bullet, N)$ is quasi-isomorphic to $(M \otimes_A N)^\bullet$ for two $A-A$-bimodules $M, N$, such that $M$ is perfect. Here $M^\bullet := \text{Hom}_{A-\text{mod}}(M, A \otimes A)$ Having this in mind one can offer a version of the above proof which uses the isomorphism

$$\text{Hom}_{A-\text{mod}}(A^i, A) \simeq \text{Hom}_{A-\text{mod}}(\text{Hom}_{A-\text{mod}}(A, A \otimes A), A).$$

Indeed, since $A$ is homologically smooth the bimodule $\text{Hom}_{A-\text{mod}}(A, A \otimes A)$ is quasi-isomorphic to a direct summand $P$ of an extension of a shift of $\text{Hom}_{A-\text{mod}}(A \otimes A, A \otimes A) \simeq A \otimes A$. Similarly, $\text{Hom}_{A-\text{mod}}(P, A)$ is quasi-isomorphic to a direct summand of an extension of a shift of $\text{Hom}_{A-\text{mod}}(A \otimes A, A \otimes A) \simeq A \otimes A$. Combining the above computations we see that the complex $C_i(A, A)$ is quasi-isomorphic to a direct summand of an extension of a shift of the complex $A \otimes A$. The latter has finite-dimensional cohomology, since $A$ enjoys this property.

Algebras of finite quivers are homologically smooth and compact. Besides of that there are two main sources of homologically smooth compact $\mathbb{Z}$-graded $A_{\infty}$-algebras.

**Example 8.2.5** a) Combining Examples [8.1.4](#) and [8.2.2](#) we see that if $X/k$ is a smooth proper variety then the bounded derived category $D^b(\text{Coh}(X))$
is equivalent to the category Perf$_A$ for a homologically smooth compact $A_\infty$-algebra $A/k$.

b) According to [Se] the derived category $D^b(F(X))$ of the Fukaya category $F(X)$ of a K3 surface $X$ is equivalent to Perf$_A$ for a homologically smooth compact $A_\infty$-algebra $A$. The latter is generated by Lagrangian spheres, which are vanishing cycles at the critical points for a fibration of $X$ over $\mathbb{CP}^1$. This result can be generalized to other Calabi-Yau manifolds.

In $\mathbb{Z}/2$-graded case examples of homologically smooth compact $A_\infty$-algebras come from Landau-Ginzburg models, i.e. from Fukaya-Seidel categories (see [Or], [R]) as well as from Fukaya categories of Fano varieties.

**Remark 8.2.6** Formal deformation theory of smooth compact $A_\infty$-algebras gives a finite-dimensional formal pointed (commutative) dg-manifold. The global moduli stack can be constructed using methods of [ToVa]). It can be thought of as a moduli stack of non-commutative smooth proper varieties.

9 Degeneration Hodge-to-de Rham

9.1 Main conjecture

Let us assume that $\text{char } k = 0$ and $A$ is a weakly unital $A_\infty$-algebra, which can be $\mathbb{Z}$-graded or $\mathbb{Z}/2$-graded.

For any $n \geq 0$ we define the truncated modified negative cyclic complex $C_{\bullet}^{\text{mod},(n)}(A, A) = C_{\bullet}^{\text{mod}}(A, A) \otimes k[u]/(u^n)$, where $\deg u = +2$. It is a complex with the differential $b + uB$. Its cohomology will be denoted by $H^\bullet(C_{\bullet}^{\text{mod},(n)}(A, A))$ and will be called truncated modified negative cyclic homology.

**Definition 9.1.1** We say that an $A_\infty$-algebra $A$ satisfies the degeneration property if for any $n \geq 1$ one has the following: the cohomology $H^\bullet(C_{\bullet}^{\text{mod},(n)}(A, A))$ is a flat $k[u]/(u^n)$-module.

**Conjecture 9.1.2** (Degeneration Hodge-to-de Rham). Let $A$ be a weakly unital compact homologically smooth $A_\infty$-algebra. Then it satisfies the degeneration property.

We call the Conjecture 9.1.2 the degeneration conjecture.

**Corollary 9.1.3** If $A$ satisfies the degeneration property then the modified negative cyclic homology coincides with $\lim_n H^\bullet(C_{\bullet}^{\text{mod},(n)}(A, A))$, and furthermore it is a flat $k[[u]]$-module.

Assuming the degeneration property for $A$ we see that there is a $\mathbb{Z}$-graded vector bundle $\xi_A$ over $\mathbb{A}^1_{\text{form},[-2]} = \text{Spf}(k[[u]])$ with the space of sections isomorphic to

$$\lim_n H^\bullet(C_{\bullet}^{\text{mod},(n)}(A, A)) = HC_{\bullet}^{\text{mod}}(A),$$
which is the modified negative cyclic homology of $A$. We will skip the word “modified” from the notation. The fiber of $\xi_A$ at $u = 0$ is isomorphic to the Hochschild homology $H^*_\text{mod}(A, A) := H_*(C_*(A, A))$.

Notice that $\mathbb{Z}$-graded $k((u))$-module $HP^\text{mod}_*(A)$ of periodic cyclic homology can be described in terms of just one $\mathbb{Z}/2$-graded vector space $HP^\text{mod}_{\text{even}}(A)$ and by odd numbers on $HP^\text{mod}_{\text{odd}}(A)$. The associated $\mathbb{Z}$-graded vector space coincides with $H_*(A, A)$.

**Remark 9.1.4** One can speak about degeneration property (modulo $u^n$) for $A_\infty$-algebras which are flat over unital commutative $k$-algebras. For example, let $R$ be an artinian local $k$-algebra with the maximal ideal $m$, and $A$ be a flat $R$-algebra such that $A/m$ is weakly unital, homologically smooth and compact. Then, assuming the degeneration property for $A/m$, one can easily see that it holds for $A$ as well. In particular, the negative cyclic homology of $A$ gives rise to a vector bundle over $\text{Spec}(R) \times A^\text{form}_{\text{form}}[-2]$, where the second factor means the formal graded scheme obtained from the formal completion of the affine line with the grading shifted by 2.

We can say few words in support of the degeneration conjecture. One is, of course, the classical Hodge-to-de Rham degeneration theorem (see Section 9.2 below). It is an interesting question to express the classical Hodge theory algebraically, in terms of a generator $E$ of the derived category of coherent sheaves and the corresponding $A_\infty$-algebra $A = R\text{Hom}(E, E)$. The degeneration conjecture also trivially holds for algebras of finite quivers without relations.

In classical algebraic geometry there are basically two approaches to the proof of degeneration conjecture. One is analytic and uses Kähler metric, Hodge decomposition, etc. Another one is pure algebraic and uses the technique of reduction to finite characteristic (see [DI]). Recently Kaledin (see [Kal]) suggested a proof of a version of the degeneration conjecture based on the reduction to finite characteristic.

Below we will formulate a conjecture which could lead to the definition of crystalline cohomology for $A_\infty$-algebras. Before doing that we remark that one can define homologically smooth and compact $A_\infty$-algebras over any commutative ring, in particular, over the ring of integers $\mathbb{Z}$. We assume that $A$ is a flat $\mathbb{Z}$-module.

**Conjecture 9.1.5** Suppose that $A$ is a weakly unital $A_\infty$-algebra over $\mathbb{Z}$, such that it is homologically smooth (but not necessarily compact). Truncated negative cyclic complexes $(C_*(A, A) \otimes \mathbb{Z}[[u, p]]/(u^n, p^m), b + uB)$ and $(C_*(A, A) \otimes \mathbb{Z}[[u, p]]/(u^n, p^m), b - puB)$ are quasi-isomorphic for all $n, m \geq 1$ and all prime numbers $p$. 

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If, in addition, \(A\) is compact then the homology of either of the above complexes is a flat module over \(Z[[u, p]]/(u^n, p^m)\).

If the Conjecture 9.1.5 holds then the degeneration conjecture, probably, can be deduced along the lines of [DI].

One can also make some conjectures about Hochschild complex of an arbitrary \(A_\infty\)-algebra, not assuming that it is compact or homologically smooth. More precisely, let \(A\) be a unital \(A_\infty\)-algebra over the ring of \(p\)-adic numbers \(Z_p\). We assume that \(A\) is topologically free \(Z_p\)-module. Let \(A_0 = A \otimes Z_p Z / p\) be the reduction modulo \(p\). Then we have the Hochschild complex \((C_\bullet(A_0, A_0), b)\) and the \(Z/2\)-graded complex \((C_\bullet(A_0, A_0), b + B)\).

**Conjecture 9.1.6** There is a natural isomorphism of \(Z/2\)-graded vector spaces over the field \(Z/p\):

\[ H^\bullet(C_\bullet(A_0, A_0), b) \simeq H^\bullet(C_\bullet(A_0, A_0), b + B). \]

There are similar isomorphisms for weakly unital and non-unital \(A_\infty\)-algebras, if one replaces \(C_\bullet(A_0, A_0)\) by \(C_\bullet^{\text{mod}}(A_0, A_0)\). Also one has similar isomorphisms for \(Z/2\)-graded \(A_\infty\)-algebras.

The Conjecture 9.1.6 presumably gives an isomorphism used in [DI], but does not imply the degeneration conjecture.

**Remark 9.1.7** One can formulate similar conjectures for saturated \(A_\infty\)-categories, which are categorical generalizations of homologically smooth compact \(A_\infty\)-algebras. This observation supports the idea of introducing the category \(NCMot\) of non-commutative pure motives. Objects of the latter are saturated \(A_\infty\)-categories over a field, and \(\text{Hom}_{NCMot}(C_1, C_2) = K_0(\text{Funct}(C_1, C_2)) \otimes \mathbb{Q}/\text{equiv}\). Here \(K_0\) means the \(K_0\)-group of the \(A_\infty\)-category of functors and equiv means numerical equivalence, i.e. the equivalence relation generated by the kernel of the Euler form \(\langle E, F \rangle := \chi(R\text{Hom}(E, F))\), where \(\chi\) is the Euler characteristic. Some of the properties of the category \(NCMot\) are discussed in [Ko4]. One can formulate non-commutative analogs of Weil and Beilinson conjectures for the category \(NCMot\).

### 9.2 Relationship with the classical Hodge theory

Let \(X\) be a quasi-projective scheme of finite type over a field \(k\) of characteristic zero. Then the category \(\text{Perf}(X)\) of perfect sheaves on \(X\) is equivalent to \(H^0(A - \text{mod})\), where \(A - \text{mod}\) is the category of \(A_\infty\)-modules over a dg-algebra \(A\). Let us recall a construction of \(A\). Consider a complex \(E\) of vector bundles which generates the bounded derived category \(D^b(\text{Perf}(X))\) (see [BvB]). Then \(A\) is quasi-isomorphic to \(R\text{Hom}(E, E)\). More explicitly, let us fix an affine covering \(X = \bigcup U_i\). Then the complex \(A := \oplus_{i_0, \ldots, i_n} \Gamma(U_{i_0} \cap \ldots \cap U_{i_n}, E^* \otimes E)[-n], n = \dim X\) computes \(R\text{Hom}(E, E)\) and carries a structure of dg-algebra. Different choices of \(A\) give rise to equivalent categories \(H^0(A - \text{mod})\) (derived Morita equivalence).
Properties of $X$ are encoded in the properties of $A$. In particular:

a) $X$ is smooth iff $A$ is homologically smooth;

b) $X$ is compact iff $A$ is compact.

Moreover, if $X$ is smooth then

$$H^\bullet(A, A) \simeq \bigoplus_{i,j \geq 0} H^i(X, \wedge^j T_X)[-(i + j)]$$

where $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset X \times X$.

Similarly

$$H^\bullet(A, A) \simeq \bigoplus_{i,j \geq 0} H^i(X, \wedge^j T^*_X)[j - i].$$

The RHS of the last formula is the Hodge cohomology of $X$. One can consider the hypercohomology $H^\bullet(X, \Omega^\bullet X)$ which is isomorphic to the generic fiber of the corresponding flat vector bundle over the formal line $A_\infty^1$. In order to make a connection with the “abstract” theory of the Section 9.1 we remark that $H^\bullet_{dR}(X)$ is isomorphic (as a $\mathbb{Z}/2$-graded vector space) to the periodic cyclic homology $HP^\bullet(A)$ while $H^\bullet(A, A)$ is isomorphic to $H_{Hodge}^\bullet(X)$.

10 $A_\infty$-algebras with scalar product

10.1 Main definitions

Let $(X, pt, Q)$ be a finite-dimensional formal pointed dg-manifold over a field $k$ of characteristic zero.

**Definition 10.1.1** A symplectic structure of degree $N \in \mathbb{Z}$ on $X$ is given by a cyclic closed 2-form $\omega$ of degree $N$ such that its restriction to the tangent space $T_{pt}X$ is non-degenerate.

One has the following non-commutative analog of the Darboux lemma.

**Proposition 10.1.2** Symplectic form $\omega$ has constant coefficients in some affine coordinates at the point $pt$.

**Proof.** Let us choose an affine structure at the marked point and write down $\omega = \omega_0 + \omega_1 + \omega_2 + \ldots$, where $\omega_l = \sum_{i,j} c_{ij}(x) dx_i \otimes dx_j$ and $c_{ij}(x)$ is homogeneous of degree $l$ (in particular, $\omega_0$ has constant coefficients). Next we observe that the following lemma holds.

**Lemma 10.1.3** Let $\omega = \omega_0 + r$, where $r = \omega_l + \omega_{l+1} + \ldots, l \geq 1$. Then there is a change of affine coordinates $x_i \mapsto x_i + O(x^{l+1})$ which transforms $\omega$ into $\omega_0 + \omega_{l+1} + \ldots$.
Lemma implies the Proposition, since we can make an infinite product of
the above changes of variables (it is a well-defined infinite series). The resulting
automorphism of the formal neighborhood of $x_0$ transforms $\omega$ into $\omega_0$.

Proof of the lemma. We have $d_{cycl}\omega_j = 0$ for all $j \geq l$ (see Section 7.2.2
for the notation). The change of variables is determined by a vector field $v = (v_1, \ldots, v_n)$ such that $v(x_0) = 0$. Namely, $x_i \mapsto x_i - v_i$, $1 \leq i \leq n$. Moreover, we will look for a vector field such that $v_i = O(x^{l+1})$ for all $i$.

We have $\text{Lie}_v(\omega) = d(i_v \omega_0) + d(i_v \tau)$. Since $d\omega_l = 0$ we have $\omega_l = d\alpha_{l+1}$ for some form $\alpha_{l+1} = O(x^{l+1})$ in the obvious notation. This follows from the formal Poincare lemma. Therefore in order to kill the term with $\omega_l$ we need to solve the equation $d\alpha_{l+1} = d(i_v \omega_l)$. It suffices to solve the equation $\alpha_{l+1} = i_v \omega_0$. Since $\omega_0$ is non-degenerate, there exists a unique vector field $v = O(x^{l+1})$ solving last equation. This proves the lemma. ■

**Definition 10.1.4** Let $(X, pt, Q, \omega)$ be a non-commutative formal pointed sym-
plectic dg-manifold. A scalar product of degree $N$ on the $A_{\infty}$-algebra $A = T_{pt}X[-1]$ is given by a choice of affine coordinates at $pt$ such that the $\omega$ becomes constant and gives rise to a non-degenerate bilinear form $A \otimes A \to k[-N]$.

**Remark 10.1.5** Notice that since $\text{Lie}_Q(\omega) = 0$ there exists a cyclic function
$S \in \Omega^0_{cycl}(X)$ such that $i_Q \omega = dS$ and $\{S, S\} = 0$ (here the Poisson bracket corresponds to the symplectic form $\omega$). It follows that the deformation theory of a non-unital $A_{\infty}$-algebra $A$ with the scalar product is controlled by the DGLA $\Omega^0_{cycl}(X)$ equipped with the differential $\{S, \cdot\}$.

We can restate the above definition in algebraic terms. Let $A$ be a finite-
dimensional $A_{\infty}$-algebra, which carries a non-degenerate symmetric bilinear form $(\cdot, \cdot)$ of degree $N$. This means that for any two elements $a, b \in A$ such that $\text{deg}(a) + \text{deg}(b) = N$ we are given a number $(a, b) \in k$ such that:

1) for any collection of elements $a_1, \ldots, a_{n+1} \in A$ the expression $(m_n(a_1, \ldots, a_n), a_{n+1})$
is cyclically symmetric in the graded sense (i.e. it satisfies the Koszul rule of
signs with respect to the cyclic permutation of arguments);

2) bilinear form $(\cdot, \cdot)$ is non-degenerate.

In this case we will say that $A$ is an $A_{\infty}$-algebra with the scalar product of
degree $N$.

### 10.2 Calabi-Yau structure

The above definition requires $A$ to be finite-dimensional. We can relax this con-
dition requesting instead that $A$ is compact. As a result we will arrive to a ho-
monological version of the notion of scalar product. More precisely, assume that $A$
is weakly unital compact $A_{\infty}$-algebra. Let $CC^{mod}(A) = (CC^{mod}(A, A)[u^{-1}], b + uB)$ be the cyclic complex of $A$. Let us choose a cohomology class $[\varphi] \in H^\bullet(CC^{mod}(A))^*$ of degree $N$. Since the complex $(A, m_1)$ is a subcomplex of $CC^{mod}(A, A) \subset CC^{mod}(A)$ we see that $[\varphi]$ defines a linear functional $Tr_{[\varphi]} : H^\bullet(A) \to k[-N]$. 40
Definition 10.2.1 We say that \([\varphi]\) is homologically non-degenerate if the bilinear form of degree \(N\) on \(H^\bullet(A)\) given by \((a, b) \mapsto Tr_{[\varphi]}(ab)\) is non-degenerate.

Notice that the above bilinear form defines a symmetric scalar product of degree \(N\) on \(H^\bullet(A)\).

Theorem 10.2.2 For a weakly unital compact \(A_\infty\)-algebra \(A\) a homologically non-degenerate cohomology class \([\varphi]\) gives rise to a class of isomorphisms of non-degenerate scalar products on a minimal model of \(A\).

Proof. The proof will consist of several steps. Since \(\text{char } k = 0\) the complex \((CC^\bullet_{\text{mod}}(A))^*\) is quasi-isomorphic to \((\Omega^0_{\text{cycl}}(X)/k, \text{Lie}_Q)\).

Lemma 10.2.3 Complex \((\Omega^2_{\text{cycl}}(X), \text{Lie}_Q)\) is quasi-isomorphic to the complex \((\Omega^0_{\text{cycl}}(X)/k, \text{Lie}_Q)\)^{14}

Proof. Notice that as a complex \((\Omega^2_{\text{cycl}}(X), \text{Lie}_Q)\) is isomorphic to the complex \(\Omega^1_{\text{cycl}}(X)/d_{\text{cycl}} \Omega^0_{\text{cycl}}(X)\). The latter is quasi-isomorphic to \([O(X), O(X)]_{\text{top}}\) via \(a db \mapsto [a, b]\) (recall that \([O(X), O(X)]_{\text{top}}\) denotes the topological closure of the commutator).

By definition \(\Omega^0_{\text{cycl}}(X) = O(X)/[O(X), O(X)]_{\text{top}}\). We know that \(O(X)/k\) is acyclic, hence \(\Omega^0_{\text{cycl}}(X)/k\) is quasi-isomorphic to \([O(X), O(X)]_{\text{top}}\). Hence the complex \((\Omega^2_{\text{cycl}}(X), \text{Lie}_Q)\) is quasi-isomorphic to \((\Omega^0_{\text{cycl}}(X)/k, \text{Lie}_Q)\). ■

As a corollary we obtain an isomorphism of cohomology groups \(H^\bullet(\Omega^2_{\text{cycl}}(X)) \cong H^\bullet(\Omega^0_{\text{cycl}}(X)/k)\). Having a non-degenerate cohomology class \([\varphi]\) \(\in H^\bullet(CC^\bullet_{\text{mod}}(A))^* \cong H^\bullet(\Omega^2_{\text{cycl}}(X), \text{Lie}_Q)\) as above, we can choose its representative \(\omega \in \Omega^2_{\text{cycl}}(X)\), \(\text{Lie}_Q \omega = 0\). Let us consider \(\omega(x_0)\). It can be described pure algebraically such as follows. Notice that there is a natural projection \(H^\bullet(\Omega^0_{\text{cycl}}(X)/k) \to (A/\langle [A, A]\rangle)^*\) which corresponds to the taking the first Taylor coefficient of the cyclic function. Then the above evaluation \(\omega(x_0)\) is the image of \(\varphi(x_0)\) under the natural map \((A/\langle [A, A]\rangle)^* \to (\text{Sym}^2(A))^*\) which assigns to a linear functional \(l\) the bilinear form \(l(ab)\).

We claim that the total map \(H^\bullet(\Omega^2_{\text{cycl}}(X)) \to (\text{Sym}^2(A))^*\) is the same as the evaluation at \(x_0\) of the closed cyclic 2-form. Equivalently, we claim that \(\omega(x_0)(a, b) = Tr_{\varphi}(ab)\). Indeed, if \(f \in \Omega^0_{\text{cycl}}(X)/k\) is the cyclic function corresponding to \(\omega\) then we can write \(f = \sum_i a_i x_i + O(x^2)\). Therefore \(\text{Lie}_Q(f) = \sum_{i,j} a_i c_{ij}^l [x_i, x_j] + O(x^3)\), where \(c_{ij}^l\) are structure constants of \(O(X)\). Dualizing we obtain the claim.

Proposition 10.2.4 Let \(\omega_1\) and \(\omega_2\) be two symplectic structures on the finite-dimensional formal pointed minimal dg-manifold \((X, pt, Q)\) such that \([\omega_1] = [\omega_2]\) in the cohomology of the complex \((\Omega^2_{\text{cycl}}(X), \text{Lie}_Q)\) consisting of closed cyclic 2-forms. Then there exists a change of coordinates at \(x_0\) preserving \(Q\) which transforms \(\omega_1\) into \(\omega_2\).

\(^{14}\)See also Proposition 5.5.1 from [Gi3].
Corollary 10.2.5 Let \((X, pt, Q)\) be a (possibly infinite-dimensional) formal pointed dg-manifold endowed with a (possibly degenerate) closed cyclic 2-form \(\omega\). Assume that the tangent cohomology \(H^0(T_{pt}X)\) is finite-dimensional and \(\omega\) induces a non-degenerate pairing on it. Then on the minimal model of \((X, pt, Q)\) we have a canonical isomorphism class of symplectic forms modulo the action of the group \(\text{Aut}(X, pt, Q)\).

Proof. Let \(M\) be a (finite-dimensional) minimal model of \(A\). Choosing a cohomology class \([\varphi]\) as above we obtain a non-degenerate bilinear form on \(M\), which is the restriction \(\omega(x_0)\) of a representative \(\omega \in \Omega^{2,cl}(X)\). By construction this scalar product depends on \(\omega\). We would like to show that in fact it depends on the cohomology class of \(\omega\), i.e. on \(\varphi\) only. This is the corollary of the following result.

Lemma 10.2.6 Let \(\omega_1 = \omega + \text{Lie}_Q(d\alpha)\). Then there exists a vector field \(v\) such that \(v(x_0) = 0\), \([v, Q] = 0\) and \(\text{Lie}_v(\omega) = \text{Lie}_Q(d\alpha)\).

Proof. As in the proof of Darboux lemma we need to find a vector field \(v\), satisfying the condition \(d\text{Lie}_v(\omega) = \text{Lie}_Q(d\alpha)\). Let \(\beta = \text{Lie}_Q(\alpha)\). Then \(d\beta = d\text{Lie}_Q(\alpha) = 0\). Since \(\omega\) is non-degenerate we can find \(v\) satisfying the conditions of the Proposition and such that \(d\text{Lie}_v(\omega) = \text{Lie}_Q(d\alpha)\). Using this \(v\) we can change affine coordinates transforming \(\omega + \text{Lie}_Q(d\alpha)\) back to \(\omega\). This concludes the proof of the Proposition and the Theorem 10.2.2.

Presumably the above construction is equivalent to the one given in [Ka]. We will sometimes call the cohomology class \([\varphi]\) a Calabi-Yau structure on \(A\) (or on the corresponding non-commutative formal pointed dg-manifold \(X\)). The following example illustrates the relation to geometry.

Example 10.2.7 Let \(X\) be a complex Calabi-Yau manifold of dimension \(n\). Then it carries a nowhere vanishing holomorphic \(n\)-form \(\text{vol}\). Let us fix a holomorphic vector bundle \(E\) and consider a dg-algebra \(A = \Omega^0(X, \text{End}(E))\) of Dolbeault \((0, p)\)-forms with values in \(\text{End}(E)\). This dg-algebra carries a linear functional \(a \mapsto \int_X \text{Tr}(a) \wedge \text{vol}\). One can check that this is a cyclic cocycle which defines a non-degenerate pairing on \(H^\bullet(A)\) in the way described above.

There is another approach to Calabi-Yau structures in the case when \(A\) is homologically smooth. Namely, we say that \(A\) carries a Calabi-Yau structure of dimension \(N\) if \(A^! \simeq A[N]\) (recall that \(A^!\) is the \(A - A\)-bimodule \(\text{Hom}_{A^{-\text{mod}} - A}(A, A \otimes A)\) introduced in Section 8.1). Then we expect the following conjecture to be true.

Conjecture 10.2.8 If \(A\) is a homologically smooth compact finite-dimensional \(A_\infty\)-algebra then the existence of a non-degenerate cohomology class \([\varphi]\) of degree \(\text{dim } A\) is equivalent to the condition \(A^! \simeq A[\text{dim } A]\).

If \(A\) is the dg-algebra of endomorphisms of a generator of \(D^b(\text{Coh}(X))\) \((X\) is Calabi-Yau) then the above conjecture holds trivially.
Finally, we would like to illustrate the relationship of the non-commutative symplectic geometry discussed above with the commutative symplectic geometry of certain spaces of representations\[15\]. More generally we would like to associate with \(X = \text{Spc}(T(A[1]))\) a collection of formal algebraic varieties, so that some “non-commutative” geometric structure on \(X\) becomes a collection of compatible “commutative” structures on formal manifolds \(\mathcal{M}(X, n):= \widetilde{\text{Rep}}_0(\mathcal{O}(X), \text{Mat}_n(\mathbb{k}))\), where \(\text{Mat}_n(\mathbb{k})\) is the associative algebra of \(n \times n\) matrices over \(\mathbb{k}\), \(\mathcal{O}(X)\) is the algebra of functions on \(X\) and \(\widetilde{\text{Rep}}_0(\ldots)\) means the formal completion at the trivial representation. In other words, we would like to define a collection of compatible geometric structure on “\(\text{Mat}_n(\mathbb{k})\)-points” of the formal manifold \(X\). In the case of symplectic structure this philosophy is illustrated by the following result.

**Theorem 10.2.9** Let \(X\) be a non-commutative formal symplectic manifold in \(\text{Vect}_\mathbb{k}\). Then it defines a collection of symplectic structures on all manifolds \(\mathcal{M}(X, n), n \geq 1\).

**Proof.** Let \(\mathcal{O}(X) = A, \mathcal{O}(\mathcal{M}(X, n)) = B\). Then we can choose isomorphisms \(A \simeq \mathbb{k}\langle x_1, \ldots, x_m \rangle\) and \(B \simeq \mathbb{k}\langle x_{1, \beta}, \ldots, x_{m, \beta} \rangle\), where \(1 \leq \alpha, \beta \leq n\). To any \(a \in A\) we can assign \(\tilde{a} \in B \otimes \text{Mat}_n(\mathbb{k})\) such that:

\[
\hat{x}_i = \sum_{\alpha, \beta} x_{i, \alpha}^{\alpha, \beta} \otimes e_{\alpha, \beta},
\]

where \(e_{\alpha, \beta}\) is the \(n \times n\) matrix with the only non-trivial element equal to 1 on the intersection of \(\alpha\)-th line and \(\beta\)-th column. The above formulas define an algebra homomorphism. Composing it with the map \(\text{id}_B \otimes \text{Tr}_{\text{Mat}_n(\mathbb{k})}\) we get a linear map \(\mathcal{O}_{\text{cycl}}(X) \to \mathcal{O}(\mathcal{M}(X, n))\). Indeed the closure of the commutator \([A, A]\) is mapped to zero. Similarly, we have a morphism of complexes \(\Omega^*_\text{cycl}(X) \to \Omega^*(\mathcal{M}(X, n))\), such that

\[
dx_i \mapsto \sum_{\alpha, \beta} dx_{i, \alpha}^{\alpha, \beta} e_{\alpha, \beta}.
\]

Clearly, continuous derivations of \(A\) (i.e. vector fields on \(X\)) give rise to the vector fields on \(\mathcal{M}(X, n)\).

Finally, one can see that a non-degenerate cyclic 2-form \(\omega\) is mapped to the tensor product of a non-degenerate 2-form on \(\mathcal{M}(X, n)\) and a nondegenerate 2-form \(\text{Tr}(XY)\) on \(\text{Mat}_n(\mathbb{k})\). Therefore a symplectic form on \(X\) gives rise to a symplectic form on \(\mathcal{M}(X, n), n \geq 1\).

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11 Hochschild complexes as algebras over operads and PROPs

Let \(A\) be a strictly unital \(A_\infty\)-algebra over a field \(\mathbb{k}\) of characteristic zero. In this section we are going to describe a colored dg-operad \(P\) such that the pair \[15\] It goes back to [Ko2], and since that time has been discussed in many papers, see e.g. [Gi2].
$(C^*(A,A), C_*(A,A))$ is an algebra over this operad. More precisely, we are going to describe $\mathbb{Z}$-graded $k$-vector spaces $A(n,m)$ and $B(n,m)$, $n, m \geq 0$ which are components of the colored operad such that $B(n,m) \neq 0$ for $m = 1$ only and $A(n,m) \neq 0$ for $m = 0$ only together with the colored operad structure and the action

a) $A(n,0) \otimes (C^*(A,A))^{\otimes n} \to C^*(A,A)$,

b) $B(n,1) \otimes (C^*(A,A))^{\otimes n} \otimes C_*(A,A) \to C_*(A,A)$.

Then, assuming that $A$ carries a non-degenerate scalar product, we are going to describe a PROP $R$ associated with moduli spaces of Riemannian surfaces and a structure of $R$-algebra on $C_*(A,A)$.

### 11.1 Configuration spaces of discs

We start with the spaces $A(n,0)$. They are chain complexes. The complex $A(n,0)$ coincides with the complex $M_n$ of the minimal operad $M = (M_n)_{n \geq 0}$ described in [KoSo1], Section 5. Without going into details which can be found in loc. cit. we recall main facts about the operad $M$. A basis of $M_n$ as a $k$-vector space is formed by $n$-labeled planar trees (such trees have internal vertices labeled by the set $\{1, \ldots, n\}$ as well as other internal vertices which are non-labeled and each has the valency at least 3).

We can depict $n$-labeled trees such as follows

![Labeled Trees](image)

Labeled vertices are depicted as circles with numbers inscribed, non-labeled vertices are depicted as black vertices. In this way we obtain a graded operad $M$ with the total degree of the basis element corresponding to a tree $T$ equal to

$$\text{deg}(T) = \sum_{v \in V_{\text{lab}}(T)} (1 - |v|) + \sum_{v \in V_{\text{nonl}}(T)} (3 - |v|)$$

where $V_{\text{lab}}(T)$ and $V_{\text{nonl}}(T)$ denote the sets of labeled and non-labeled vertices respectively, and $|v|$ is the valency of the vertex $v$, i.e. the cardinality of the set of edges attached to $v$. 
The notion of an *angle* between two edges incoming in a vertex is illustrated in the following figure (angles are marked by asterisks).

Operadic composition and the differential are described in [KoSo1], sections 5.2, 5.3. We borrow from there the following figure which illustrates the operadic composition of generators corresponding to labeled trees $T_1$ and $T_2$. 
Informally speaking, the operadic gluing of $T_2$ to $T_1$ at an internal vertex $v$ of $T_1$ is obtained by:

a) Removing from $T_1$ the vertex $v$ together with all incoming edges and vertices.

b) Gluing $T_2$ to $v$ (with the root vertex removed from $T_2$). Then

c) Inserting removed vertices and edges of $T_1$ in all angles between incoming edges to the new vertex $v_{\text{new}}$.

d) Taking the sum (with appropriate signs) over all possible inserting of edges in c).

The differential $d_M$ is a sum of the “local” differentials $d_v$, where $v$ runs through the set of all internal vertices. Each $d_v$ inserts a new edge into the set of edges attached to $v$. The following figure borrowed from [KoSo1] illustrates the difference between labeled (white) and non-labeled (black) vertices.

In this way we make $M$ into a dg-operad. It was proved in [KoSo1], that $M$ is quasi-isomorphic to the dg-operad $\text{Chains}(FM_2)$ of singular chains on the Fulton-Macpherson operad $FM_2$. The latter consists of the compactified moduli spaces of configurations of points in $\mathbb{R}^2$ (see e.g. [KoSo1], Section 7.2.
for a description). It was also proved in [KoSo1] that $C^\bullet(A, A)$ is an algebra over the operad $M$ (Deligne’s conjecture follows from this fact). The operad $FM_2$ is homotopy equivalent to the famous operad $C_2 = (C_2(n))_{n \geq 0}$ of 2-dimensional discs (little disc operad). Thus $C^\bullet(A, A)$ is an algebra (in the homotopy sense) over the operad $Chains(C_2)$.

11.2 Configurations of points on the cylinder

Let $\Sigma = S^1 \times [0, 1]$ denotes the standard cylinder.

Let us denote by $S(n)$ the set of isotopy classes of the following graphs $\Gamma \subset \Sigma$:

a) every graph $\Gamma$ is a forest (i.e. disjoint union of finitely many trees $\Gamma = \bigsqcup_i T_i$);

b) the set of vertices $V(\Gamma)$ is decomposed into the union $V_{\partial \Sigma} \sqcup V_{lab} \sqcup V_{nonl} \sqcup V_1$ of four sets with the following properties:

b1) the set $V_{\partial \Sigma}$ is the union $\{in\} \cup \{out\} \cup V_{out}$ of three sets of points which belong to the boundary $\partial \Sigma$ of the cylinder. The set $\{in\}$ consists of one marked point which belongs to the boundary circle $S^1 \times \{1\}$ while the set $\{out\}$ consists of one marked point which belongs to the boundary circle $S^1 \times \{0\}$. The set $V_{out}$ consists of a finitely many unlaabeled points on the boundary circle $S^1 \times \{0\}$;

b2) the set $V_{lab}$ consists of $n$ labeled points which belong to the surface $S^1 \times (0, 1)$ of the cylinder;

b3) the set $V_{nonl}$ consists of a finitely many non-labeled points which belong to the surface $S^1 \times (0, 1)$ of the cylinder;

b4) the set $V_1$ is either empty or consists of only one element denoted by $1 \in S^1 \times (0, 1)$ and called special vertex;

c) the following conditions on the valencies of vertices are imposed:

c1) the valency of the vertex $out$ is less or equal than 1;

c2) the valency of each vertex from the set $V_{\partial \Sigma} \setminus V_{out}$ is equal to 1;

c3) the valency of each vertex from $V_{lab}$ is at least 1;

c4) the valency of each vertex from $V_{nonl}$ is at least 3;

c5) if the set $V_1$ is non-empty then the valency of the special vertex is equal to 1. In this case the only outcoming edge connects $1$ with the vertex $out$.

d) Every tree $T_i$ from the forest $\Gamma$ has its root vertex in the set $V_{\partial \Sigma}$.

e) We orient each tree $T_i$ down to its root vertex.
Remark 11.2.1 Let us consider the configuration space $X_n$, $n \geq 0$ which consists of (modu

...
parametrized by elements of the set $S(n)$ described in the previous subsection.

Let us describe the action of a generator of $B(n, 1)$ on a pair $(\gamma_1 \otimes \ldots \otimes \gamma_n, \beta)$, where $\gamma_1 \otimes \ldots \otimes \gamma_n \in C^*(A, A)^{\otimes n}$ and $\beta = a_0 \otimes a_1 \otimes \ldots \otimes a_l \in C_l(A, A)$. We attach elements $a_0, a_1, \ldots, a_l$ to points on $S^1 \times \{1\}$ in a cyclic order, such that $a_0$ is attached to the point $in$. We attach $\gamma_i$ to the $i$th numbered point on the surface $S^1 \times (0, 1)$. Then we draw disjoint continuous segments (in all possible ways, considering pictures up to an isotopy) starting from each point marked by some element $a_i$ and oriented downstairs, with the requirements as above, with the only modification that we allow an arbitrary number of points on $S^1 \times \{1\}$.

We attach higher multiplications $m_j$ to all non-numbered vertices, so that $j$ is equal to the incoming valency of the vertex. Reading from the top to the bottom and composing $\gamma_i$ and $m_j$ we obtain (on the bottom circle) an element $b_0 \otimes \ldots \otimes b_m \in C_*(A, A)$ with $b_0$ attached to the vertex $out$. If the special vertex $1$ is present then we set $b_0 = 1$. This gives the desired action.

Composition of the operations in $B(n, 1)$ corresponds to the gluing of the cylinders such that the point $out$ of the top cylinder is identified with the point $in$ of the bottom cylinder. If after the gluing there is a line from the point marked $1$ on the top cylinder which does not end at the point $out$ of the bottom cylinder, we will declare such a composition to be equal to zero.

Let us now consider a topological colored operad $C_{2}^{col} = (C_{2}^{col}(n, m))_{n, m \geq 0}$ with two colors such that $C_{2}^{col}(n, m) \neq \emptyset$ only if $m = 0, 1$, and

a) In the case $m = 0$ it is the little disc operad.

b) In the case $m = 1$ $C_{2}^{col}(n, 1)$ is the moduli space (modulo rotations) of the configurations of $n \geq 1$ discs on the cylinder $S^1 \times [0, h]$ for $h \geq 0$, and two marked points on the boundary of the cylinder. We also add the degenerate circle of

Technically we should resize vertically the glued cylinder, so that it is identified with $\Sigma = S^1 \times [0, 1]$. Alternatively we can introduce an additional parameter $h \in \mathbb{R}_{>0}$ and consider cylinders $\Sigma_h = S^1 \times [0, h]$. Then the composition of the operations depicted on $\Sigma_{h_1}$ and $\Sigma_{h_2}$ amounts to the operation depicted on $\Sigma_{h_1 + h_2}$. We adopt this convention below when speaking about the colored operad. In that case it makes more sense since the case $h = 0$ has to be included.
configurations \( n = 0, h = 0 \). The topological space \( C_{2}^{col}(n, 1) \) is homotopically equivalent to the configuration space \( X_n \) described in the previous subsection.

Let \( \text{Chains}(C_{2}^{col}) \) be the colored operad of singular chains on \( C_{2}^{col} \). Then, similarly to [KoSo1], Section 7, one proves (using the explicit action of the colored operad \( P = (A(n, m), B(n, m))_{n,m\geq 0} \) described above) the following result.

**Theorem 11.3.1** Let \( A \) be a unital \( A_\infty \)-algebra. Then the pair \( (C^*(A, A), C^*(A, A)) \) is an algebra over the colored operad \( \text{Chains}(C_{2}^{col}) \) (which is quasi-isomorphic to \( P \)) such that for \( h = 0, n = 0 \) and coinciding points \( \text{in} = \text{out} \), the corresponding operation is the identity.

**Remark 11.3.2** The above Theorem 11.3.1 generalizes Deligne’s conjecture (see e.g. [KoSo1]). It is related to the abstract calculus associated with \( A \) (see [T], [TaT]). The reader also notices that for \( h = 0, n = 0 \) we have the moduli space of two points on the circle. It is homeomorphic to \( S^1 \). Thus we have an action of \( S^1 \) on \( C^*(A, A) \). This action gives rise to the Connes differential \( B \).

Similarly to the case of little disc operad, one can prove the following result.

**Proposition 11.3.3** The colored operad \( C_{2}^{col} \) is formal, i.e. it is quasi-isomorphic to its homology colored operad.

If \( A \) is non-unital we can consider the direct sum \( A_1 = A \oplus k \) and make it into a unital \( A_\infty \)-algebra. The reduced Hochschild chain complex of \( A_1 \) is defined as \( C^{\text{red}}_*(A_1, A_1) = \oplus_{n\geq 0} A_1 \otimes ((A_1/k)[1])^n \) with the same differential as in the unital case. One defines the reduced Hochschild cochain complex \( C^{\text{red}}_*(A_1, A_1) \) similarly. We define the modified Hochschild chain complex \( C^{\text{mod}}_*(A, A) \) from the following isomorphism of complexes \( C^{\text{red}}_*(A_1, A_1) \cong C^{\text{mod}}_*(A, A) \oplus k \). Similarly, we define the modified Hochschild cochain complex from the decomposition \( C^{\text{red}}_*(A_1, A_1) \cong C^{\text{mod}}_*(A, A) \oplus k \). Then, similarly to the Theorem 11.3.1 one proves the following result.

**Proposition 11.3.4** The pair \( (C^{\text{mod}}_*(A, A), C^{\text{mod}}_*(A, A)) \) is an algebra over the colored operad which is an extension of \( \text{Chains}(C_{2}^{col}) \) by null-ary operations on Hochschild chain and cochain complexes, which correspond to the unit in \( A \), and such that for \( h = 0, n = 0 \) and coinciding points \( \text{in} = \text{out} \), the corresponding operation is the identity.

### 11.4 Remark about Gauss-Manin connection

Let \( R = k[[t_1, ..., t_n]] \) be the algebra of formal series, and \( A \) be an \( R \)-flat \( A_\infty \)-algebra. Then the (modified) negative cyclic complex \( C_{\text{Cycl}}^-(A) = (C^*(A, A)[[u]], b + uB) \) is an \( R[[u]] \)-module. It follows from the existense of the Gauss-Manin connection (see [Get]) that the cohomology \( HC_{\text{Cycl}}^-(A) \) of this complex is in fact a module over the ring

\[
D_R(A) := k[[t_1, ..., t_n, u]][u\partial/\partial t_1, ..., u\partial/\partial t_n].
\]
Ineed, if $\nabla$ is the Gauss-Manin connection from [Get] then $u \partial / \partial t_i$ acts on the cohomology as $u \nabla_{\partial / \partial t_i}$, $1 \leq i \leq n$.

The conjecture below give another interpretation of this module structure. Let $g = C^*(A, A)[1]$ be the DGLA associated with the Hochschild cochain complex, and $M := C^*_{-\text{mod}}(A)$. We define a DGLA $\hat{g}$ which is the crossproduct $(g \otimes k(\xi)) \rtimes k(\partial / \partial \xi)$, where $\deg \xi = +1$.

**Conjecture 11.4.1** There is a structure of an $L_\infty$-module on $M$ over $\hat{g}$ which extends the natural structure of a $g$-module and such that $\partial / \partial \xi$ acts as Connes differential $B$. Moreover this $\hat{g}$-module structure should follow from the algebra structure over the colored operad $P$ described in Section 11.3.

It looks plausible that Theorem 11.3.1 gives formulas for the Gauss-Manin connection from [Get].

### 11.5 Flat connections and the colored operad

We start with $\mathbb{Z}$-graded case. Let us interpret the $\mathbb{Z}$-graded formal scheme $\text{Spf}(k[[u]])$ as even formal line equipped with the $G_m$-action $u \mapsto \lambda^2 u$. The space $HC_{-\text{mod}}(A)$ can be interpreted as a space of sections of a $G_m$-equivariant vector bundle $\xi_A$ over the formal line $\text{Spf}(k[[u]])$ corresponding to the $k[[u]]$-flat module $\lim_{\leftarrow n} H^\bullet(C_n^*(A, A))$. The action of $G_m$ identifies fibers of this vector bundle over $u \neq 0$. Thus we have a natural flat connection $\nabla$ on the restriction of $\xi_A$ to the complement of the point 0. This connection has pole of order one at $u = 0$.

We are going to propose a different construction of the connection $\nabla$ which works also in $\mathbb{Z}/2$-graded case. This connection will have in general a pole of degree two at $u = 0$. In particular we will have the following result.

**Proposition 11.5.1** The space of sections of the vector bundle $\xi_A$ can be endowed with a structure of a $k[[u]][[u^2 \partial / \partial u]]$-module.

In fact we are going to give an explicit construction of the connection, which is based on the action of the colored dg-operad $P$ discussed in Section 11.3. More precisely, we will consider a certain extension $P^{new}$ of $P$. Before presenting an explicit formula, we will make few comments.

1. For any $\mathbb{Z}/2$-graded $A_\infty$-algebra $A$ one can define canonically a 1-parameter family of $A_\infty$-algebras $A_\lambda$, $\lambda \in G_m$, such that $A_\lambda = A$ as a $\mathbb{Z}/2$-graded vector space and $m_n^{A_\lambda} = \lambda m_n^A$.

2. For simplicity we will assume that $A$ is strictly unital. Otherwise we will work with the pair $(C^*_{-\text{mod}}(A, A), C^*_{-\text{mod}}(A, A))$ of modified Hochschild complexes.

3. Let us consider an extension $P^{new}$ of the dg-operad $P$ allowing any non-zero valency for a non-labeled (black) vertex (recall that in the definition of $P$ we required that such a valency was at least three). All the formulas remain the same. But the dg-operad $P^{new}$ is no longer formal. It contains a dg-suboperad generated by trees with all vertices being non-labeled. Action of this suboperad $P^{new}_{nonl}$ is responsible for the flat connection discussed below.
4. In addition to the connection along the variable \( u \) one has the Gauss-Manin connection which acts along the fibers of \( \xi_A \) (see Section 11.4). Probably one can write down an explicit formula for this connection using the action of the colored operad \( P^{\text{new}} \). In what follows are going to describe a connection which presumably coincides with the Gauss-Manin connection.

Let us now consider a dg-algebra \( k[B, \gamma_0, \gamma_2] \) which is generated by the following operations of the colored dg-operad \( P^{\text{new}} \):

a) Connes differential \( B \) of degree \(-1\). It can be depicted such as follows (cf. Section 7.3):

\[
\begin{array}{c}
\text{in} \\
B = \\
\text{out}
\end{array}
\]

b) Generator \( \gamma_2 \) of degree 2, corresponding to the following figure:

\[
\begin{array}{c}
\text{in} \\
\gamma_2 = \\
\text{out}
\end{array}
\]

c) Generator \( \gamma_0 \) of degree 0, where \( 2\gamma_0 \) is depicted below:
Proposition 11.5.2 The following identities hold in $P_{\text{new}}$:

\[ B^2 = dB = d\gamma_2 = 0, d\gamma_0 = [B, \gamma_2], \]

\[ B\gamma_0 + \gamma_0 B := [B, \gamma_0]_+ = -B. \]

Here by $d$ we denote the Hochschild chain differential (previously it was denoted by $b$).

Proof. Let us prove that $[B, \gamma_0] = -B$, leaving the rest as an exercise to the reader. One has the following identities for the compositions of operations in $P_{\text{new}}$: $B\gamma_0 = 0$, $\gamma_0 B = B$. Let us check, for example, the last identity. Let us denote by $W$ the first summand on the figure defining $2\gamma_0$. Then $\gamma_0 B = \frac{1}{2}WB$.

The latter can be depicted in the following way:

\[ \frac{1}{2}B = \]

\[ \frac{1}{2}W = \]

It is easy to see that this equals to $2 \cdot \frac{1}{2}B = B$. ■

Corollary 11.5.3 Hochschild chain complex $C_\bullet(A, A)$ is a dg-module over the dg-algebra $k[B, \gamma_0, \gamma_2]$.

Let us consider the truncated negative cyclic complex ($C_\bullet(A, A)[[u]]/(u^n)$, $d_u = d + uB$). We introduce a $k$-linear map $\nabla$ of $C_\bullet(A, A)[[u]]/(u^n)$ into itself such that $\nabla u^2 \partial/\partial u = u^2 \partial/\partial u - \gamma_2 + u\gamma_0$. Then we have:
a) \[ \nabla_{u^2 \partial / \partial u} d_u = 0; \]

b) \[ \nabla_{u^2 \partial / \partial u} u = u^2. \]

Let us denote by \( V \) the unital dg-algebra generated by \( \nabla_{u^2 \partial / \partial u} \) and \( u \), subject to the relations a), b) and the relation \( u^n = 0 \). From a) and b) one deduces the following result.

**Proposition 11.5.4** The complex \( (C_\bullet(A, A)[[u]]/(u^n), d_u = d + uB) \) is a \( V \)-module. Moreover, assuming the degeneration conjecture, we see that the operator \( \nabla_{u^2 \partial / \partial u} \) defines a flat connection on the cohomology bundle corresponding to the module

\[ H^\bullet(C_\bullet(A, A)[[u]]/(u^n), d_u) \]

such that the connection has the only singularity at \( u = 0 \) which is a pole of second order.

Taking the inverse limit over \( n \) we see that \( H^\bullet(C_\bullet(A, A)[[u]]/(u^n), d_u) \) gives rise to a vector bundle over \( A_{form}[−2] \) which carries a flat connection with the second order pole at \( u = 0 \).

**Remark 11.5.5** In the above discussion one should notice a difference between \( \mathbb{Z} \)-graded and \( \mathbb{Z}/2 \)-graded \( A_\infty \)-algebras. It follows from the explicit formula for the connection \( \nabla \) that the coefficient in front of the second order pole is represented by multiplication by the cocyle \( (m_n)_{n \geq 1} \in C^\bullet(A, A) \). Taking the cohomology we see that the coefficient is trivial in \( \mathbb{Z} \)-graded case (because of the invariance with respect to the group action \( m_n \mapsto \lambda m_n \)), but nontrivial in \( \mathbb{Z}/2 \)-graded case. Therefore the order of the pole of \( \nabla \) is equal to one for \( \mathbb{Z} \)-graded \( A_\infty \)-algebras and is equal to two for \( \mathbb{Z}/2 \)-graded \( A_\infty \)-algebras. We see that in \( \mathbb{Z} \)-graded case the connection along the variable \( u \) comes from the action of the group \( \mathbb{G}_m \) on higher products \( m_n \), while in \( \mathbb{Z}/2 \)-graded case it is more complicated.

### 11.6 PROP of marked Riemann surfaces and TQFT associated with a Calabi-Yau category

Here we will describe a PROP naturally acting on the Hochschild complexes of a finite-dimensional \( A_\infty \)-algebra with the scalar product of degree \( N \).

Since we have a quasi-isomorphism of complexes

\[ C^\bullet(A, A) \simeq (C_\bullet(A, A))^\bullet[−N] \]

it suffices to consider the chain complex only.

We will assume that \( A \) is either \( \mathbb{Z} \)-graded (then \( N \) is an integer) or \( \mathbb{Z}/2 \)-graded (then \( N \in \mathbb{Z}/2 \)). We will present the results for non-unital \( A_\infty \)-algebras. In this case we will consider the modified Hochschild chain complex

\[ C^\bullet_{mod} = \bigoplus_{n \geq 0} A \otimes (A[1])^\otimes n \bigoplus \bigoplus_{n \geq 1} (A[1])^\otimes n, \]

equipped with the Hochschild chain differential (see Section 7.4).
Our construction is summarized in i)-ii) below.

i) Let us consider the topological PROP \( \mathcal{M} = (\mathcal{M}(n, m))_{n, m \geq 0} \) consisting of moduli spaces of metrics on compact oriented surfaces with boundary consisting of \( n + m \) circles and some additional marking (see precise definition below).

ii) Let \( \text{Chains}(\mathcal{M}) \) be the corresponding PROP of singular chains. Then there is a structure of a \( \text{Chains}(\mathcal{M}) \)-algebra on \( C^\text{mod}_\bullet(A, A) \), which is encoded in a collection of morphisms of complexes

\[
\text{Chains}(\mathcal{M}(n, m)) \otimes C^\text{mod}_\bullet(A, A) \otimes n \to (C^\text{mod}_\bullet(A, A)) \otimes m.
\]

In addition one has the following:

iii) If \( A \) is homologically smooth and satisfies the degeneration property then the structure of \( \text{Chains}(\mathcal{M}) \)-algebra extends to a structure of a \( \text{Chains}(\mathcal{M}) \)-algebra, where \( \mathcal{M} \) is the topological PROP of stable compactifications of \( \mathcal{M}(n, m) \).

**Definition 11.6.1** An element of \( \mathcal{M}(n, m) \) is an isomorphism class of triples \((\Sigma, h, \text{mark})\) where \( \Sigma \) is a compact oriented surface (not necessarily connected) with metric \( h \), and \( \text{mark} \) is an orientation preserving isometry between a neighborhood of \( \partial \Sigma \) and the disjoint union of \( n + m \) flat semiannuli \( \sqcup_{1 \leq i \leq n}[S^1 \times [0, \varepsilon)] \sqcup \sqcup_{1 \leq i \leq m}([S^1 \times [-\varepsilon, 0]) \), where \( \varepsilon \) is a sufficiently small positive number. We will call \( n \) circles “inputs” and the rest \( m \) circles “outputs”. We will assume that each connected component of \( \Sigma \) has at least one input, and there are no discs among the connected components. Also we will add \( \Sigma = S^1 \) to \( \mathcal{M}(1, 1) \) as the identity morphism. It can be thought of as the limit of cylinders \( S^1 \times [0, \varepsilon] \) as \( \varepsilon \to 0 \).

The composition is given by the natural gluing of surfaces.

Let us describe a construction of the action of \( \text{Chains}(\mathcal{M}) \) on the Hochschild chain complex. In fact, instead of \( \text{Chains}(\mathcal{M}) \) we will consider a quasi-isomorphic dg-PROP \( R = R(n, m)_{n, m \geq 0} \) generated by ribbon graphs with additional data. In what follows we will omit some technical details in the definition of the PROP \( R \). They can be recovered in a more or less straightforward way.

It is well-known (and can be proved with the help of Strebel differentials) that \( \mathcal{M}(n, m) \) admits a stratification with strata parametrized by graphs described below. More precisely, we consider the following class of graphs:

1) Each graph \( \Gamma \) is a (not necessarily connected) ribbon graph (i.e. we are given a cyclic order on the set \( \text{Star}(v) \) of edges attached to a vertex \( v \) of \( \Gamma \)). It is well-known that replacing an edge of a ribbon graph by a thin stripe (thus getting a “fat graph”) and gluing stripes in the cyclic order one gets a Riemann surface with the boundary.

2) The set \( V(\Gamma) \) of vertices of \( \Gamma \) is the union of three sets: \( V(\Gamma) = V_{\text{in}}(\Gamma) \cup V_{\text{middle}}(\Gamma) \cup V_{\text{out}}(\Gamma) \). Here \( V_{\text{in}}(\Gamma) \) consists of \( n \) numbered vertices \( \text{in}_1, ..., \text{in}_n \) of the valency 1 (the corresponding outcoming edges are called tails), \( V_{\text{middle}}(\Gamma) \) consists of vertices of the valency greater or equal than 3, and \( V_{\text{out}}(\Gamma) \) consists of \( m \) numbered vertices \( \text{out}_1, ..., \text{out}_m \) of valency greater or equal than 1.

3) We assume that the Riemann surface corresponding to \( \Gamma \) has \( n \) connected boundary components each of which has exactly one input vertex.
4) For every vertex $out_j \in V_{out}(\Gamma), 1 \leq j \leq m$ we mark either an incoming edge or a pair of adjacent edges (we call such a pair of edges a corner).

More pedantically, let $E(\Gamma)$ denotes the set of edges of $\Gamma$ and $E^{or}(\Gamma)$ denotes the set of pairs $(e, or)$ where $e \in E(\Gamma)$ and $or$ is one of two possible orientations of $e$. There is an obvious map $E^{or}(\Gamma) \rightarrow V(\Gamma) \times V(\Gamma)$ which assigns to an oriented edge the pair of its endpoint vertices: source and target. The free involution $\sigma$ acting on $E^{or}(\Gamma)$ (change of orientation) corresponds to the permutation map on $V(\Gamma) \times V(\Gamma)$. Cyclic order on each $Star(v)$ means that there is a bijection $\rho : E^{or}(\Gamma) \rightarrow E^{or}(\Gamma)$ such that orbits of iterations $\rho^n, n \geq 1$ are elements of $Star(v)$ for some $v \in V(\Gamma)$. In particular, the corner is given either by a pair of coinciding edges $(e, e)$ such that $\rho(e) = e$ or by a pair edges $e, e' \in Star(v)$ such that $\rho(e) = e'$. Let us define a face as an orbit of $\rho \circ \sigma$. Then faces are oriented closed paths. It follows from the condition 2) that each face contains exactly one edge outcoming from some $in_i$.

We depict below two graphs in the case $g = 0, n = 2, m = 0$. 
Here is a picture illustrating the notion of face:

Two faces: one contains $\text{in}_1$, another contains $\text{in}_2$. 

\[ \text{deg } \Gamma = 0 \]

\[ \text{deg } \Gamma = -1 \]
Remark 11.6.2 The above data (i.e. a ribbon graph with numerations of in and out vertices) have no automorphisms. Thus we can identify $\Gamma$ with its isomorphism class.

The functional $(m_n(a_1, ..., a_n), a_{n+1})$ is depicted such as follows.

![Diagram](image1)

We define the degree of $\Gamma$ by the formula

$$\deg \Gamma = \sum_{v \in V_{\text{middle}}(\Gamma)} (3 - |v|) + \sum_{v \in V_{\text{out}}(\Gamma)} (1 - |v|) + \sum_{v \in V_{\text{out}}(\Gamma)} \epsilon_v - N\chi(\Gamma),$$

where $\epsilon_v = -1$, if $v$ contains a marked corner and $\epsilon_v = 0$ otherwise. Here $\chi(\Gamma) = |V(\Gamma)| - |E(\Gamma)|$ denotes the Euler characteristic of $\Gamma$.

Definition 11.6.3 We define $R(n, m)$ as a graded vector space which is a direct sum $\oplus_{\Gamma} \psi_{\Gamma}$ of 1-dimensional graded vector spaces generated by graphs $\Gamma$ as above, each summand has degree $\deg \Gamma$.

One can see that $\psi_{\Gamma}$ is naturally identified with the tensor product of 1-dimensional vector spaces (determinants) corresponding to vertices of $\Gamma$.

Now, having a graph $\Gamma$ which satisfies conditions 1)-3) above, and Hochschild chains $\gamma_1, ..., \gamma_n \in C^{\text{mod}}_{\bullet}(A, A)$ we would like to define an element of $C^{\text{mod}}_{\bullet}(A, A)^{\otimes m}$. Roughly speaking we are going to assign the above $n$ elements of the Hochschild complex to $n$ faces corresponding to vertices $i_n, 1 \leq i \leq n$, then assign tensors corresponding to higher products $m_i$ to internal vertices $v \in V_{\text{middle}}(\Gamma)$, then using the convolution operation on tensors given by the scalar product on $A$ to read off the resulting tensor from $out_j, 1 \leq j \leq m$. More precise algorithm is described below.

a) We decompose the modified Hochschild complex such as follows:

$$C^{\text{mod}}_{\bullet}(A, A) = \oplus_{l \geq 0, \epsilon \in \{0, 1\}} C^{\text{mod}}_{l, \epsilon}(A, A),$$
where $C_{l,\epsilon}^{mod}(A, A) = A \otimes (A[1])^\otimes l$ and $C_{l,\epsilon}^{mod}(A, A) = k \otimes (A[1])^\otimes l$ according to the definition of modified Hochschild chain complex. For any choice of $l_i \geq 0, \epsilon_i \in \{0, 1\}, 1 \leq i \leq n$ we are going to construct a linear map of degree zero

$$f_\Gamma : \psi_\Gamma \otimes C_{l_1,\epsilon_1}(A, A) \otimes \ldots \otimes C_{l_n,\epsilon_1}(A, A) \rightarrow (C_{\bullet}^{mod}(A, A))^\otimes m.$$ 

The result will be a sum $f_\Gamma = \sum_{\Gamma'} f_{\Gamma'}$ of certain maps. The description of the collection of graphs $\Gamma'$ is given below.

b) Each new graph $\Gamma'$ is obtained from $\Gamma$ by adding new edges. More precisely one has $V(\Gamma') = V(\Gamma)$ and for each vertex $in_i \in V_{in}(\Gamma)$ we add $l_i$ new outcoming edges. Then the valency of $in_i$ becomes $l_i + 1$.

More pedantically, for every $i, 1 \leq i \leq n$ we have constructed a map from the set $\{1, ..., l_i\}$ to a cyclically ordered set which is an orbit of $\rho \circ \sigma$ with removed the tail edge oucoming from $in_i$. Cyclic order on the edges of $\Gamma'$ is induced by the cyclic order at every vertex and the cyclic order on the path forming the face corresponding to $in_i$. 

![Diagram](corner)
c) We assign $\gamma_i \in C_{l_i, \varepsilon_i}$ to $i_n_i$. We depict $\gamma_i$ as a “wheel” representing the Hochschild cocycle. It is formed by the endpoints of the $l_i + 1$ edges outcoming from $i_n_i \in V(\Gamma')$ and taken in the cyclic order of the corresponding face. If $\varepsilon_i = 1$ then (up to a scalar) $\gamma_i = 1 \otimes a_1 \otimes \ldots \otimes a_{l_i}$, and we require that the tensor factor 1 corresponds to zero in the cyclic order.

\[
\text{in}_i \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad 1
\]

\[\gamma_i\]

\[i_n_i\]

\[a_1 \quad a_2 \quad a_3 \quad a_4 \]

\[\text{out}_j\]

\[\text{middle}(\Gamma')\]

\[|v| = 3\]

\[1 \leq j \leq m\]

\[\text{out}_j\]

\[\text{marked edge for}\]

d) We remove from considerations graphs $\Gamma$ which do not obey the following property after the step c):

the edge corresponding to the unit $1 \in \mathbb{k}$ (see step c)) is of the type $(i_n_i, v)$ where either $v \in V_{\text{middle}}(\Gamma')$ and $|v| = 3$ or $v = \text{out}_j$ for some $1 \leq j \leq m$ and the edge $(i_n_i, \text{out}_j)$ was the marked edge for $\text{out}_j$.

Let us call unit edge the one which satisfies one of the above properties. We define a new graph $\Gamma''$ which is obtained from $\Gamma$ by removing unit edges.

e) Each vertex now has the valency $|v| \geq 2$. We attach to every such vertex either:

the tensor $c \in A \otimes A$ (inverse to the scalar product), if $|v| = 2$,

or

the tensor $(m_{|v|-1}(a_1, \ldots, a_{|v|-1}), a_{|v|})$ if $|v| \geq 3$. The latter can be identified with the element of $A^{|v|}$ (here we use the non-degenerate scalar product on $A$).

Let us illustrate this construction.
f) Let us contract indices of tensors corresponding to \( V_{in}(\Gamma'') \cup V_{middle}(\Gamma'') \) (see c), e)) along the edges of \( \Gamma'' \) using the scalar product on \( A \). The result will be an element \( a_{out} \) of the tensor product \( \otimes_{1 \leq j \leq m} A^{\text{Start}_{\Gamma''}(\text{out}_j)} \).

g) Last thing we need to do is to interpret the element \( a_{out} \) as an element of \( C^\text{mod}(A, A) \). There are three cases.

Case 1. When we constructed \( \Gamma'' \) there was a unit edge incoming to some \( \text{out}_j \). Then we reconstruct back the removed edge, attach \( 1 \in k \) to it, and interpret the resulting tensor as an element of \( C^\text{mod}_{|\text{out}_j|, \varepsilon_j = 1}(A, A) \).

Case 2. There was no removed unit edge incoming to \( \text{out}_j \) and we had a marked edge (not a marked corner) at the vertex \( \text{out}_j \). Then we have an honest element of \( C^\text{mod}_{|\text{out}_j|, \varepsilon_j = 0}(A, A) \).

Case 3. Same as in Case 2, but there was a marked corner at \( \text{out}_j \in V_{out}(\Gamma) \). We have added and removed new edges when constructed \( \Gamma'' \). Therefore the marked corner gives rise to a new set of marked corners at \( \text{out}_j \) considered as a vertex of \( \Gamma'' \). Inside every such a corner we insert a new edge, attach the element \( 1 \in k \) to it and take the sum over all the corners. In this way we obtain an element of \( C^\text{mod}_{|\text{out}_j|, \varepsilon_j = 1}(A, A) \). This procedure is depicted below.
$e_1$ and $e_2$ are new edges.

Three new corners with new unit edges

This concludes the construction of $f_{\Gamma}$. Notice that $R$ is a dg-PROP with the differential given by the insertion of a new edge between two vertices from $V_{\text{middle}}(\Gamma)$.

Proof of the following Proposition is tedious and technical. We will omit details here.

**Proposition 11.6.4** The above construction gives rise to a structure of a $R$-algebra on $C_{\text{mod}}^{\bullet}(A, A)$.

**Remark 11.6.5** The above construction did not use homological smoothness of $A$.

Finally we would like to say few words about an extension of the $R$-action to the $\text{Chains}(\overline{M})$-action.

If we assume the degeneration property for $A$, then the action of the PROP $R$ can be extended to the action of the PROP $\text{Chains}(\overline{M})$ of singular chains of the topological PROP of stable degenerations of $M_{g,n,m}^{\text{marked}}$. In order to see this, one introduces the PROP $D$ freely generated by $R(2,0)$ and $R(1,1)$, i.e. by singular
chains on the moduli space of cylinders with two inputs and zero outputs (they correspond to the scalar product on \(C_\bullet(A,A)\)) and by cylinders with one input and one output (they correspond to morphisms \(C_\bullet(A,A) \to C_\bullet(A,A)\)). In fact the (non-symmetric) bilinear form \(h : H_\bullet(A,A) \otimes H_\bullet(A,A) \to k\) does exist for any compact \(A_\infty\)-algebra \(A\). It is described by the graph of degree zero on the corresponding figure above. This is a generalization of the bilinear form \((a,b) \in A/[A,A] \otimes A/[A,A] \mapsto \text{Tr}(axb) \in k\).

It seems plausible that homological smoothness implies that \(h\) is non-degenerate. This allows us to extend the action of the dg sub-PROP \(D \subset R\) to the action of the dg PROP \(D' \subset R\) which contains also \(R(0,2)\) (i.e. the inverse to the above bilinear form). If we assume the degeneration property, then we can “shrink” the action of the homologically non-trivial circle of the cylinders, since the rotation around this circle corresponds to the differential \(B\). Thus \(D'\) is quasi-isomorphic to the dg-PROP of chains on the one-dimensional retracts of the above cylinders, where the retraction contracts the circles. Let us denote the dg-PROP generated by singular chains on the retraction by \(D''\). Thus, assuming the degeneration property, we see that the free product dg-PROP \(R' = R \ast_D D''\) acts on \(C_\bullet^{\text{mod}}(A,A)\). One can show that \(R'\) is quasi-isomorphic to the dg-PROP of chains on the topological PROP \(\mathcal{M}^{\text{marked}}_{g,n,m}\) of stable compactifications of the surfaces from \(M^{\text{marked}}_{g,n,m}\).

Remark 11.6.6 a) The above construction is a generalization of the construction from [Ko3], which assigns cohomology classes of \(M_{g,n}\) to a finite-dimensional \(A_\infty\)-algebra with scalar product (trivalent graphs were used in [Ko3]).

b) Different approach to the action of the PROP \(R\) was suggested in [Cos]. The above Proposition 11.6.4 gives rise to a structure of Topological Field Theory associated with a non-unital \(A_\infty\)-algebra with scalar product (or more generally with a Calabi-Yau category). If the degeneration property holds for \(A\) then one can define a Cohomological Field Theory in the sense of [KoM].

c) Homological smoothness of \(A\) is closely related to the existence of a non-commutative analog of the Chern class of the diagonal \(\Delta \subset X \times X\) of a projective scheme \(X\). This Chern class gives rise to the inverse to the scalar product on \(A\). Generalization of this observation plays an important role in the case of \(A_\infty\)-categories.

12 Appendix

12.1 Non-commutative schemes and ind-schemes

Let \(\mathcal{C}\) be an abelian \(k\)-linear tensor category. To simplify formulas we will assume that it is strict (see [McL]). We will also assume that \(\mathcal{C}\) admits infinite sums. To simplify the exposition we will assume below as we did in most cases in the main body of the paper that \(\mathcal{C} = \text{Vect}_k^\mathbb{Z}\).
Definition 12.1.1 The category of non-commutative affine $k$-schemes in $C$ (notation $NAff_C$) is defined as the category opposite to the category of associative unital $k$-algebras in $C$.

The non-commutative scheme corresponding to the algebra $A$ is denoted by $\text{Spec}(A)$. Conversely, if $X$ is a non-commutative affine scheme then the corresponding algebra is called the algebra of regular functions on $X$ and is denoted by $O(X)$. By analogy with commutative case we call a morphism $f : X \to Y$ a closed embedding if the corresponding homomorphism $f^* : O(Y) \to O(X)$ is an epimorphism.

Let us recall some terminology related to ind-objects (see for ex. [Gr], [AM], [KSch]). For a covariant functor $\phi : I \to A$ from a small filtering category $I$ (called filtrant in [KSch]) there is a notion of an inductive limit $\lim_{\to} \phi \in \hat{A}$ and a projective limit $\lim_{\leftarrow} \phi \in \hat{A}$, where $\hat{A}$ is the category of functors $A \to \text{Sets}$. By definition $\lim_{\to} \phi(X) = \lim_{\to} \text{Hom}_A(X, \phi(i))$ and $\lim_{\leftarrow} \phi(X) = \lim_{\leftarrow} \text{Hom}_A(\phi(i), X)$. All inductive limits form a full subcategory $\text{Ind}(A) \subset \hat{A}$ of ind-objects in $A$. Similarly all projective limits form a full subcategory $\text{Pro}(A) \subset \hat{A}$ of pro-objects in $A$.

Definition 12.1.2 Let $I$ be a small filtering category, and $F : I \to NAff_C$ a covariant functor. We say that $\lim_{\to} F$ is a non-commutative ind-affine scheme if for a morphism $i \to j$ in $I$ the corresponding morphism $F(i) \to F(j)$ is a closed embedding.

In other words a non-commutative ind-affine scheme $X$ is an object of $\text{Ind}(NAff_C)$, corresponding to the projective limit $\lim_{\to} A_\alpha, \alpha \in I$, where each $A_\alpha$ is a unital associative algebra in $C$, and for a morphism $\alpha \to \beta$ in $I$ the corresponding homomorphism $A_\beta \to A_\alpha$ is a surjective homomorphism of unital algebras (i.e. there exists an exact sequence $0 \to ? \to A_\beta \to A_\alpha \to 0$).

Remark 12.1.3 Not all categorical epimorphisms of algebras are surjective homomorphisms (although the converse is true). Nevertheless one can define closed embeddings of affine schemes for an arbitrary abelian $k$-linear category, observing that a surjective homomorphism of algebras $f : A \to B$ is characterized categorically by the condition that $B$ is the cokernel of the pair of the natural projections $f_{1,2} : A \times_B A \to A$ defined by $f$.

Morphisms between non-commutative ind-affine schemes are defined as morphisms between the corresponding projective systems of unital algebras. Thus we have

$$\text{Hom}_{NAff_C}(\lim_{I} X_i, \lim_{J} Y_j) = \lim_{I} \lim_{J} \text{Hom}_{NAff_C}(X_i, Y_j).$$

Let us recall that an algebra $M \in \text{Ob}(C)$ is called nilpotent if the natural morphism $M^\otimes n \to M$ is zero for all sufficiently large $n$. 

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Definition 12.1.4 A non-commutative ind-affine scheme $\hat{X}$ is called formal if it can be represented as $\hat{X} = \lim_{\to} \text{Spec}(A_i)$, where $(A_i)_{i \in I}$ is a projective system of associative unital algebras in $\mathcal{C}$ such that the homomorphisms $A_i \to A_j$ are surjective and have nilpotent kernels for all morphisms $j \to i$ in $I$.

Let us consider few examples in the case when $\mathcal{C} = \text{Vect}_k \subset \text{Vect}_k^\mathbb{Z}$.

Example 12.1.5 In order to define the non-commutative formal affine line $\hat{A}^1_{\text{NC}}$, it suffices to define $\text{Hom}(\text{Spec}(A), \hat{A}^1_{\text{NC}})$ for any associative unital algebra $A$. We define $\text{Hom}_{\text{NAff}}(\text{Spec}(A), \hat{A}^1_{\text{NC}}) = \lim_{\to} \text{Hom}_{\text{Alg}}(k[t]/(t^n), A)$. Then the set of $A$-points of the non-commutative formal affine line consists of all nilpotent elements of $A$.

Example 12.1.6 For an arbitrary set $I$ the non-commutative formal affine space $\hat{A}^I_{\text{NC}}$ corresponds, by definition, to the topological free algebra $k\langle \langle t_i \rangle \rangle_{i \in I}$. If $A$ is a unital $k$-algebra then any homomorphism $k\langle \langle t_i \rangle \rangle_{i \in I} \to A$ maps almost all $t_i$ to zero, and the remaining generators are mapped into nilpotent elements of $A$. In particular, if $I = \mathbb{N} = \{1, 2, \ldots\}$ then $\hat{A}^\mathbb{N}_{\text{NC}} = \lim_{\to} \text{Spec}(k\langle \langle t_1, \ldots, t_n \rangle \rangle/(t_1, \ldots, t_n)^m)$, where $(t_1, \ldots, t_n)$ denotes the two-sided ideal generated by $t_i, 1 \leq i \leq n$, and the limit is taken over all $n, m \to \infty$.

By definition, a closed subscheme $Y$ of a scheme $X$ is defined by a 2-sided ideal $J \subset \mathcal{O}(X)$. Then $\mathcal{O}(Y) = \mathcal{O}(X)/J$. If $Y \subset X$ is defined by a 2-sided ideal $J \subset \mathcal{O}(X)$, then the completion of $X$ along $Y$ is a formal scheme corresponding to the projective limit of algebras $\lim_{\to n} \mathcal{O}(X)/J^n$. This formal scheme will be denoted by $\hat{X}_Y$ or by $\text{Spf} (\mathcal{O}(X)/J)$.

Non-commutative affine schemes over a given field $k$ form a symmetric monoidal category. The tensor structure is given by the ordinary tensor product of unital algebras. The corresponding tensor product of non-commutative affine schemes will be denoted by $X \otimes Y$. It is not a categorical product, differently from the case of commutative affine schemes, where the tensor product of algebras corresponds to the Cartesian product $X \times Y$. For non-commutative affine schemes the analog of the Cartesian product is the free product of algebras.

Let $A, B$ be free algebras. Then $\text{Spec}(A)$ and $\text{Spec}(B)$ are non-commutative manifolds. Since the tensor product $A \otimes B$ in general is not a smooth algebra, the non-commutative affine scheme $\text{Spec}(A \otimes B)$ in general is not a manifold.

Let $X$ be a non-commutative ind-affine scheme in $\mathcal{C}$. A closed $k$-point $x \in X$ is by definition a homomorphism of $\mathcal{O}(X)$ to the tensor algebra generated by the unit object $1$. Let $m_x$ be the kernel of this homomorphism. We define the tangent space $T_x X$ in the usual way as $(m_x/m_x^2)^* \in \text{Ob}(\mathcal{C})$. Here $m_x^2$ is the image of the multiplication map $m_x^\otimes 2 \to m_x$.

A non-commutative ind-affine scheme with a marked closed k-point will be called pointed. There is a natural generalization of this notion to the case of many points. Let $Y \subset X$ be a closed subscheme of disjoint closed $k$-points (it corresponds to the algebra homomorphism $\mathcal{O}(X) \to 1 \oplus 1 \oplus \ldots$). Then $\hat{X}_Y$ is a formal manifold. A pair $(\hat{X}_Y, Y)$ (often abbreviated by $\hat{X}_Y$) will be
called (non-commutative) formal manifold with marked points. If \(Y\) consists of one such point then \((\hat{X}_Y, Y)\) will be called a (non-commutative) formal pointed manifold.

### 12.2 Proof of Theorem 2.1.1

In the category \(\text{Alg}_{C^f}\) every pair of morphisms has a kernel. Since the functor \(F\) is left exact and the category \(\text{Alg}_{C^f}\) is artinian, it follows from [Gr], Sect. 3.1 that \(F\) is strictly pro-representable. This means that there exists a projective system of finite-dimensional algebras \((A_i)_{i \in I}\) such that, for any morphism \(i \to j\) the corresponding morphism \(A_j \to A_i\) is a categorical epimorphism, and for any \(A \in \text{Ob}(\text{Alg}_{C^f})\) one has

\[
F(A) = \lim_{\rightarrow} \text{Hom}_{\text{Alg}_{C^f}}(A_i, A).
\]

Equivalently,

\[
F(A) = \lim_{\rightarrow} \text{Hom}_{\text{Coalg}_{C^f}}(A_i^*, A^*),
\]

where \((A_i^*)_{i \in I}\) is an inductive system of finite-dimensional coalgebras and for any morphism \(i \to j\) the corresponding homomorphism of coalgebras \(f_{ij}: A_j^* \to A_i^*\) is surjective.

All what we need is to replace the projective system of algebras \((A_i)_{i \in I}\) by another projective system of algebras \((\overline{A}_i)_{i \in I}\) such that

a) functors “\(\lim\)”\(h_{\overline{A}_i}\) and “\(\lim\)”\(h_{\overline{A}_i}\) are isomorphic (here \(h_X\) is the functor defined by the formula \(h_X(Y) = \text{Hom}(X,Y)\));

b) for any morphism \(i \to j\) the corresponding homomorphism of algebras \(\overline{f}_{ij}: \overline{A}_j \to \overline{A}_i\) is surjective.

Let us define \(\overline{A}_i = \bigcap_{i \to j} \text{Im}(f_{ij})\), where \(\text{Im}(f_{ij})\) is the image of the homomorphism \(f_{ij}: A_j \to A_i\) corresponding to the morphism \(i \to j\) in \(I\). In order to prove a) it suffices to show that for any unital algebra \(B\) in \(\mathcal{C}^f\) the natural map of sets

\[
\lim_{\rightarrow} \text{Hom}_{\mathcal{C}^f}(A_i, B) \to \lim_{\rightarrow} \text{Hom}_{\mathcal{C}^f}(\overline{A}_i, B)
\]

(the restriction map) is well-defined and bijective.

The set \(\lim_{\rightarrow} \text{Hom}_{\mathcal{C}^f}(A_i, B)\) is isomorphic to \((\bigcup_i \text{Hom}_{\mathcal{C}^f}(A_i, B))/\text{equiv}\), where two maps \(f_i: A_i \to B\) and \(f_j: A_j \to B\) such that \(i \to j\) are equivalent if \(f_i f_{ij} = f_j\). Since \(\mathcal{C}^f\) is an artinian category, we conclude that there exists \(A_m\) such that \(f_{im}(A_m) = \overline{A}_i\), \(f_{jm}(A_m) = \overline{A}_j\). From this observation one easily deduces that \(f_{ij}(\overline{A}_j) = \overline{A}_i\). It follows that the morphism of functors in a) is well-defined, and b) holds. The proof that morphisms of functors bijectively correspond to homomorphisms of coalgebras is similar. This completes the proof of the theorem. ■

### 12.3 Proof of Proposition 2.1.2

The result follows from the fact that any \(x \in B\) belongs to a finite-dimensional subcoalgebra \(B_x \subset B\), and if \(B\) was counital then \(B_x\) would be also counital.
Let us describe how to construct $B_x$. Let $\Delta$ be the coproduct in $B$. Then one can write

$$\Delta(x) = \sum_i a_i \otimes b_i,$$

where $a_i$ (resp. $b_i$) are linearly independent elements of $B$.

It follows from the coassociativity of $\Delta$ that

$$\sum_i \Delta(a_i) \otimes b_i = \sum_i a_i \otimes \Delta(b_i).$$

Therefore one can find constants $c_{ij} \in k$ such that

$$\Delta(a_i) = \sum_j a_j \otimes c_{ij},$$

and

$$\Delta(b_i) = \sum_j c_{ji} \otimes b_j.$$

Applying $\Delta \otimes id$ to the last equality and using the coassociativity condition again we get

$$\Delta(c_{ji}) = \sum_n c_{jn} \otimes c_{ni}.$$

Let $B_x$ be the vector space spanned by $x$ and all elements $a_i, b_i, c_{ij}$. Then $B_x$ is the desired subcoalgebra.

12.4 Formal completion along a subscheme

Here we present a construction which generalizes the definition of a formal neighborhood of a $k$-point of a non-commutative smooth thin scheme.

Let $X = \text{Spec}(B_X)$ be such a scheme and $f : X \to Y = \text{Spec}(B_Y)$ be a closed embedding, i.e. the corresponding homomorphism of coalgebras $B_X \to B_Y$ is injective. We start with the category $\mathcal{N}_X$ of nilpotent extensions of $X$, i.e. homomorphisms $\phi : X \to U$, where $U = \text{Spec}(D)$ is a non-commutative thin scheme, such that the quotient $D/f(B_X)$ (which is always a non-counital coalgebra) is locally conilpotent. We recall that the local conilpotency means that for any $a \in D/f(B_X)$ there exists $n \geq 2$ such that $\Delta^{(n)}(a) = 0$, where $\Delta^{(n)}$ is the $n$-th iterated coproduct $\Delta$. If $(X, \phi_1, U_1)$ and $(X, \phi_2, U_2)$ are two nilpotent extensions of $X$ then a morphism between them is a morphism of non-commutative thin schemes $t : U_1 \to U_2$, such that $t\phi_1 = \phi_2$. In particular, $\mathcal{N}_X$ is a subcategory of the naturally defined category of non-commutative relative thin schemes.

Let us consider the functor $G_f : \mathcal{N}_X^{op} \to \text{Sets}$ such that $G(X, \phi, U)$ is the set of all morphisms $\psi : U \to Y$ such that $\psi \phi = f$.

**Proposition 12.4.1** Functor $G_f$ is represented by a triple $(X, \pi, \hat{Y}_X)$ where the non-commutative thin scheme denoted by $\hat{Y}_X$ is called the formal neighborhood of $f(X)$ in $Y$ (or the completion of $Y$ along $f(X)$).
Proof. Let \(B_f \subset B_X\) be the counital subcoalgebra which is the pre-image of the (non-counital) subcoalgebra in \(B_Y/f(B_X)\) consisting of locally conilpotent elements. Notice that \(f(B_X) \subset B_f\). It is easy to see that taking \(\widetilde{Y}_X := \text{Spc}(B_f)\) we obtain the triple which represents the functor \(G_f\). ■

Notice that \(\widetilde{Y}_X \to Y\) is a closed embedding of non-commutative thin schemes.

**Proposition 12.4.2** If \(Y\) is smooth then \(\widetilde{Y}_X\) is smooth and \(\widetilde{Y}_X \cong \tilde{Y}_X\).

Proof. Follows immediately from the explicit description of the coalgebra \(B_f\) given in the proof of the Proposition 12.4.1. ■

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