ON A MOVE REDUCING THE GENUS OF A KNOT DIAGRAM

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Abstract. For a knot diagram we introduce an operation which does not increase the genus of the diagram and does not change its representing knot type. We also describe a condition for this operation to certainly decrease the genus. The proof involves the study of a relation between the genus of a virtual knot diagram and the genus of a knotoid diagram, the former of which has been introduced by Stoimenow, Tchernov and Vdovina, and the latter by Turaev recently. Our operation has a simple interpretation in terms of Gauss codes and hence can easily be computer-implemented.

1. Introduction

Seifert [3] gave an algorithm to construct from a diagram $D$ of a knot $K$ an orientable surface bounded by the knot (see Definition 2.1). We call the surface constructed by Seifert’s algorithm the canonical Seifert surface for $D$ and its genus the genus $g(D)$ of $D$. The canonical genus $g_c(K)$ of a knot $K$ is defined to be the minimal genus of all possible diagrams. It is an important knot invariant and extensively studied (see for example [4]). By definition the canonical genus of a knot $K$ gives an upper bound for the genus $g(K)$ of $K$, that is the minimum of genera of all possible Seifert surfaces for $K$.

In this paper, we introduce an operation, called the bridge-replacing move, for a knot diagram which does not change its representing knot type and does not increase the genus of the diagram (see Definition 3.9 and Theorem 3.11). A necessary and sufficient condition for the operation to actually decrease the genus of the diagram is given in Proposition 3.14.

Our move is derived from Turaev’s idea using the notion knotoid [6]. A knotoid diagram $D^\circ$ is a diagram which differs from usual knot diagrams in that the underlying curve is an immersed interval rather than an immersed circle (see Definition 2.15). Turaev has constructed in [6, §2.5] the canonical surface for a given knotoid diagram by using an analogous procedure to the usual Seifert algorithm (see Definition 2.16). The genus of a knotoid diagram is defined to be the genus of its canonical surface. In [6, §2.5] it has been suggested that the study of knotoid diagrams can be used to obtain a good estimate for the genus of a knot. For, if we have a diagram $D$ of a non-alternating knot $K$ which contains a consecutive sequence $B$ of $k$ over- or under-crossing segments ($k \geq 2$), then by removing $B$ from $D$ we can obtain the knotoid $D^\circ_B$ with genus $g(D^\circ_B) \geq g(K)$. In this context, we will prove in Proposition 3.5 that the inequality $g(D) \geq g(D^\circ_B)$ always holds, which has not been explicitly proven in [6, §2.5]. Moreover, the bridge-replacing move for $(D, B)$ (see Definition 3.9) proves to produce the knot diagram $D_B$ with the property $g(D_B) = g(D^\circ_B)$. This implies $g(D) \geq g(D_B)$, that is, the bridge-replacing move is indeed useful to obtain a better estimate for $g_c(K)$ than the one given by the initial diagram $D$.

For the proof, we will study in §3 a relation between the canonical surface associated to a knotoid diagram and the surface associated to a certain virtual knot diagram, which has been introduced by Stoimenow, Tchernov and Vdovina [5] through Gauss diagrams.

In §4, we discuss an example which suggests that the bridge-replacing move is actually useful for the determination of the genus of a knot.

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2. Preliminaries

2.1. Knots. Here we recall some well-known notions concerning classical knots in $\mathbb{R}^3$. A Seifert surface for a knot $K$ is a connected, oriented surface $\Sigma$ embedded in $\mathbb{R}^3$ whose boundary $\partial \Sigma$ coincides with the knot $K$. Recall Seifert’s algorithm which produces a Seifert surface $\Sigma_D$ for a knot $K$ from a given diagram $D = D_K$ of $K$:

**Definition 2.1.** (Seifert’s algorithm [3]). Given a diagram $D = D_K$ of a knot $K$, the canonical Seifert surface $\Sigma_D$ for $K$ is the surface obtained in the following way.

(i) Draw $D$ in $\mathbb{R}^2 \times \{0\}$ and orient $D$ in an arbitrary way.

(ii) Smooth all the crossings of $D$ as in Figure 1 to obtain a disjoint union of embedded circles in the plane (called Seifert circles).

(iii) Fill the Seifert circles with the disjoint disks in $\mathbb{R}^3$.

(iv) Take the half-twisted band sums along the original crossings (Figure 2) of $D$.

The following is obvious.

**Lemma 2.2.** Given a knot diagram $D = D_K$ with $n$ crossings, the genus of the canonical Seifert surface $\Sigma_D$ of $D$ is given by the formula

$$g(\Sigma_D) = \frac{n - s_D + 1}{2},$$

where $s_D$ is the number of the Seifert circles obtained in the step (ii) in Definition 2.1.

According to Stoimenow, Tchernov and Vdovina [5], for the Gauss diagram of a knot diagram $D$ we can construct a surface, which turns out to be homeomorphic to the canonical Seifert surface $\Sigma_D$, as explained below.

**Definition 2.3.** A Gauss diagram is an oriented circle equipped with some number $n$ of signed, oriented chords each of which connects distinct two points on the circle (all the $2n$ points are distinct with each other).

To each knot diagram $D = D_K$ with $n$ crossings, we can assign the Gauss diagram $G_D$ with $n$ chords as follows:

(i) Connect the preimages of each crossing of $D$ by a chord.

(ii) Choose the orientation of each chord from the overpass branch to the underpass one.

(iii) Give to each chord the sign $+$ or $-$ depending on whether the corresponding crossing is positive (the left crossing in Figure 1) or negative, respectively.

**Example 2.4.** Figure 3 shows a knot diagram $D$ of the trefoil knot $3_1$ and its Gauss diagram. The endpoint $i$ of $G$ corresponds to the over-arc of the crossing $i$ of $D$.

**Definition 2.5.** Let $G$ be a Gauss diagram. The canonical surface $\Sigma_G$ of $G$ is the surface (with exactly one boundary component) obtained in the following way.

(i) Consider an annulus $A = S^1 \times [0, 1]$.

(ii) Take a band $B_c = [0, 1] \times [0, 1]$ for each chord $c$ of $G$. 
For each chord \( c \) and \( \epsilon \in \{0, 1\} \), glue \( \{\epsilon\} \times [0, 1] \subset B_c \) to \([x_\epsilon - \delta, x_\epsilon + \delta] \times \{0\} \subset \partial A\) (where \( x_0, x_1 \in S^1 = \mathbb{R}/\mathbb{Z} \) are the endpoints of \( c \), and \( \delta > 0 \) is a sufficiently small number) so that \( \Sigma_G' := A \cup \bigcup_c B_c \) is oriented.

(iv) Glue all the boundary components of \( \Sigma_G' \) but \( S^1 \times \{1\} \subset \partial A \) with disks to obtain an oriented surface \( \Sigma_G \) with one boundary component.

**Example 2.6.** Figure 4 shows \( \Sigma_G' \) for the Gauss diagram \( G \) of Figure 3. Attaching two disks along the boundary components of \( \Sigma_G' \) other than \( S^1 \times \{1\} \)(the outermost circle), we obtain the canonical surface \( \Sigma_G \) of \( G \), which is homeomorphic to the canonical Seifert surface of \( D \) shown in Figure 3 (as proven in Proposition 2.8).

**Remark 2.7.** The canonical surface for a Gauss diagram constructed by Stoimenow, Tchernov and Vdovina in [5] is a closed surface. Removing a disk (attached along \( S^1 \times \{1\} \)) from their surface, we obtain the surface defined in Definition 2.5.

**Proposition 2.8 ([5, Theorem 2.5]).** For a knot diagram \( D \), the surfaces \( \Sigma_D \) and \( \Sigma_G \) are homeomorphic to each other.

For the proof, we introduce the notion of a cycle in a Gauss diagram which corresponds to a Seifert circle in Seifert’s algorithm.

**Definition 2.9.** Let \( G \) be a Gauss diagram. A cycle in \( G \) is the cycle obtained by repeating the following steps: starting from some endpoint \( x \) of a chord \( c \),

(i) go to the other endpoint of \( c \) along \( c \),
(ii) go to the next endpoint \( y \) along the circle with respect to its orientation,
(iii) from \( y \), repeat (i) and (ii) above until coming back to \( x \) in the step (ii).

We will denote a cycle as a cyclic sequence \( \{i_1, i_2, \ldots, i_{2k}\} \) of endpoints which appear in the above steps (i) and (ii).

**Example 2.10.** The Gauss diagram \( G \) of Figure 3 has two cycles \( \{\bar{1}, 1, \bar{2}, 2, \bar{3}, 3\} \) and \( \{1, \bar{1}, 2, \bar{2}, 3, 3\} \).

**Lemma 2.11.** Given a Gauss diagram \( G \) with \( n \) chords, the genus of the canonical surface \( \Sigma_G \) of \( G \) is given by the formula

\[
gg(\Sigma_G) = \frac{n - s_G + 1}{2},
\]

where \( s_G \) is the number of the cycles in \( G \).
Proof. This formula is a consequence of the following facts: a cycle corresponds to a boundary component of $Σ_G$ to which we glue the boundary of a disk in the final step (iv) in Definition 2.5. Thus the number of disks we glue in the step (iv) in Definition 2.5 is equal to the number of cycles in $G$. \hfill \square

Proof of Proposition 2.8. It is easy to observe that
- the number of the crossings of $D$ and that of the chords in $G_D$ are the same,
- $s_D = s_{G_D}$ (under the notations in Lemmas 2.2, 2.11).

These facts and formulas (1), (2) imply that the genus of $Σ_D$ is equal to that of $Σ_{G_D}$. This completes the proof, since both $Σ_D$ and $Σ_{G_D}$ have exactly one boundary component. \hfill \square

Remark 2.12. The formula (1) suggests that, to construct a knot diagram of smaller genus from a given knot diagram, we need an operation which decreases the number of the crossings or increases the number of Seifert circles. The formula (2) gives a similar suggestion for Gauss diagrams.

2.2. Virtual knots and the Gauss diagrams.

Definition 2.13. A virtual knot diagram is a generic immersion $S^1 \rightarrow \mathbb{R}^2$ with only transverse double points as its singularities, some of which are endowed with over- or under-crossing data but others are not (see Figure 6). A crossing endowed with over- or under-crossing information is called a real crossing, and one without such information is called a virtual crossing.

Seifert’s algorithm cannot be applied to a virtual knot diagram as it is. But the Gauss diagram $G_D^*$ of a virtual knot diagram $D^*$ can still be defined in the same way as explained after Definition 2.3, except that no chords are assigned to virtual crossings (see [5, §2.1]). Therefore in view of Proposition 2.8, the following definition would be natural.

Definition 2.14 ([5]). Let $D^*$ be a virtual knot diagram. The canonical surface $Σ_{G_D}^*$ of $D^*$ is defined to be the canonical surface $Σ_{G_D}^*$ of the Gauss diagram $G_D^*$ of $D^*$. The genus $g(D^*)$ of $D^*$ is defined to be the genus of the canonical surface $Σ_{G_D}^*$ of $D^*$.

2.3. Turaev’s Knotoids.

Definition 2.15 ([6]). A knotoid diagram is a generic immersion $f : [0, 1] \rightarrow \mathbb{R}^2$ which has only transverse double points (endowed with over- or under-crossing data) as its singularities. The endpoints $f(0)$ and $f(1)$ are called respectively the leg and the head of the knotoid diagram.

See the central diagram in Figure 6 for an example of a knotoid diagram. In the diagram, $P$ is the leg and $Q$ is the head. In [6] a surface is produced from a knotoid diagram $D^*$, in an analogous way to Seifert algorithm for knot diagrams.

Definition 2.16 ([6]). The canonical surface $Σ_{G_D}$ of a knotoid diagram $D^*$ is the surface obtained as follows. First, draw $D^*$ in $\mathbb{R}^2 \times \{0\} \subset \mathbb{R}^3$. Then:
(i) Orient $D^*$ in an arbitrary way.
(ii) Smooth all the crossings of $D^*$ as in Figure 1 to obtain a disjoint union of embedded circles in the plane (also called Seifert circles) and an embedded interval (denoted by $J$). We call $J$ the Seifert interval. $J$ has the same endpoints $x, y$ as $D^*$.
(iii) Fill the Seifert circles with the disjoint disks lying above $\mathbb{R}^2 \times \{0\}$, and take a band $J \times [0, 1]$ lying below $\mathbb{R}^2 \times \{0\}$ and meeting $\mathbb{R}^2 \times \{0\}$ along $J \times \{0\} \approx J$.
(iv) Take the half-twisted band sums along the original crossings (Figure 2) of $D^*$. The resulting surface $Σ_{G_D}$ is the canonical surface of $D^*$.

The boundary of $Σ_{G_D}$ is the union of $D^*$ with $J \times \{1\} \cup \{x, y\} \times [0, 1]$.

Definition 2.17. The genus $g(D^*)$ of a knotoid diagram $D^*$ is defined to be the genus of the canonical surface $Σ_{G_D}$ of $D^*$.
3. The bridge-replacing move

The main operation of this paper, the bridge-replacing move, is introduced in this section. It is clear by definition that this operation does not change the representing knot types. The observation given in Remark 2.12 is a key in proving that this operation does not increase, and sometimes certainly decreases, the genus of a knot diagram.

Definition 3.1. An over-bridge (resp. under-bridge) of length $k$ of a knot diagram $D$ is a consecutive sequence of $k$ over-crossing (resp. under-crossing) segments of $D$ (see the leftmost diagram in Figure 6).

Notation 3.2. Let $G$ be any Gauss diagram and $G'$ be an over- or under-bridge of $G$. We denote by $D^n_G$ the knotoid diagram obtained by removing (the interior of) $B$ from $D$. We denote by $D^n_G$ the virtual knot diagram obtained from $D$ by turning each crossing along $B$ into a virtual crossing. See Figure 6.

Proposition 3.3. Under Notation 3.2, we have

$$g(D^n_G) = g(D^n_G).$$

Proof. If we denote by $\delta$ the number of disks that were glued in the final step (the step (iv) in Definition 2.5) of the construction of $\Sigma_D$ and by $\gamma$ the number of chords in the Gauss diagram $G_D$, then the Euler characteristic $\chi(\Sigma_D)$ of $D^n_G$ is equal to $\delta - \gamma$. We easily see that $\delta$ equals the number of Seifert circles in Turaev’s construction of the Seifert surface $\Sigma_D$ and that $\gamma$ equals the number of crossings of $D^n_G$. Thus it is clear that $\Sigma_D$ has the same Euler characteristic as $\Sigma_D$ and hence is homeomorphic to $\Sigma_D$.

Remark 3.4. Given a knotoid diagram $D^\circ$, we can construct a virtual knot diagram by connecting the two endpoints of $D^\circ$ with an arbitrary path along which only virtual crossings occur; then it is a natural idea to define the Gauss diagram of $D^\circ$ to be the Gauss diagram of such a virtual knot diagram. According to Proposition 3.3, it turns out that we can alternatively define the genus of a knotoid diagram (Definition 2.17) to be the genus of its Gauss diagram (just as Stoimenow, Tchernov and Vdovina [5] define the genus of a virtual knot diagram).

Proposition 3.5. Under Notation 3.2, we have

$$g(D) \geq g(D^n_G).$$

Proof. By Proposition 3.3, it suffices to compare the genera of the Gauss diagrams $G_D$ and of $G_D$. Note that $G_D$ is obtained by removing from $G_D$ the chords corresponding to the crossings along $B$. In Lemma 3.6, we will prove that the genus of any Gauss diagram does not increase after a removal of a chord. This implies the result.

Lemma 3.6. Let $G$ be any Gauss diagram and $G'$ be the Gauss diagram obtained by removing from $G$ a chord $c$. Then we have $g(G) \geq g(G')$.

In more detail, if $c$ appears twice in a single cycle of $G$ (as Case (i) in Figure 5), then we have $g(G) = g(G') + 1$ and hence $g(G) > g(G')$. Otherwise we have $g(G) = g(G')$.

Proof. In order to compare the genera of $G$ and of $G'$, we need only to know the change of the numbers of their chords and cycles in constructing $G'$ from $G$ by removing the chord $c$ (see the formula (2) in Lemma 2.11 and Remark 2.12).

In the case where $c$ appears twice in a single cycle (say $\alpha$) of $G$, this $\alpha$ is of the form

$$\alpha = \{w_1, a, b, w_2, b, a\}$$

for $a, b \in \partial c$ and some words $w_1, w_2$ in the endpoints. Let $f(w)$ (resp. $l(w)$) be the first (resp. last) endpoint contained in $w$. Then the points $a, b, f(w_1)$ and $l(w_i)$ ($i = 1, 2$) are located on the circle as Case (i) in Figure 5. After the chord $c$ is removed (then the points $a$ and $b$ are also removed), the cycle $\alpha$ splits into two cycles $\{w_1\}$ and $\{w_2\}$, because $f(w_1)$
(resp. \( f(w_2) \)) appears just after \( l(w_1) \) (resp. \( l(w_2) \)). Thus the number of cycles increases by one. Since the number of chords decreases by one after \( c \) is removed, we have \( g(G) = g(G') + 1 \) by the formula (2).

Next consider the case where such a cycle as above does not exist. Then there are exactly two cycles (say \( \beta = \{w_3, a, b\} \) and \( \gamma = \{w_4, b, a\} \) in each of which \( c \) appears exactly once. Then \( \beta \) and \( \gamma \) are located as Case (ii) in Figure 5. Thus after the removal of \( c \), the cycle \( \beta \) is unified with \( \gamma \) by a band-sum along \( c \), and the result is a single cycle \( \{w_3, w_4\} \). Thus the number of cycles decreases by one. Thus the formula (2) implies that \( g(G) = g(G') \). \( \square \)

**Remark 3.7.** In [6, §1], it reads that “The study of knotoid diagrams leads to an elementary but possibly useful improvement of the standard Seifert estimate from above for the genus of knot.” However, it has not been rigorously proven in [6] that a knotoid diagram always gives an estimate not worse than “the usual Seifert estimate” with respect to the genus. Propositions 3.5 and 3.3, which deduce the inequality \( g(D) \geq g(D') \), implement it. Notice that introducing the notion of the Gauss diagram of a knotoid has been the main key in the proof (see Remark 3.4).

**Remark 3.8.** We do not need to consider the removal of any “sub-bridge” of \( B \) since, by Lemma 3.6, the removal of the whole \( B \) always gives a better estimate of the genus than any removal of a sub-bridge.

Now we introduce the bridge-replacing move for a knot diagram which does not increase (Theorem 3.11), and in some cases certainly decrease (Proposition 3.13), the genus of the diagram (cf. [6, §2.5]).

**Definition 3.9** (the bridge-replacing move). Let \( D \) be a diagram of a knot \( K \) with an over-bridge (resp. under-bridge) \( B \). Then we define an operation, called the bridge-replacing move for \((D, B)\), which replaces the over-bridge (resp. under-bridge) \( B \) into another over-bridge (resp. under-bridge) \( \tilde{B} \), as follows.

Take an orientation on the diagram \( D \) and consider the (oriented) knotoid diagram \( D'_B \) by removing \( B \) from \( D \) (see Figure 6). Let \( J \) be the oriented Seifert interval of \( D'_B \), obtained by the smoothing process (the step (ii) in Definition 2.16), from the leg \( P \) to the head \( Q \) (see Figure 7) in \( D'_B \).

Now the new bridge \( \tilde{B} \) from \( Q \) to \( P \) is constructed just along \( J \) so that it goes on the right (resp. left) side of \( J \) if at all possible, and is allowed to overpass (resp. underpass) only the arcs of which \( J \) consists. Thus the union \( D'_B \cup \tilde{B} \) provides a new knot diagram, which we denote by \( \tilde{D}_B \) (see Figure 8). Clearly \( \tilde{D}_B \) represents the same knot type \( K \) as \( D \) does.

**Remark 3.10.** In Definition 3.9, in fact, it is not so important which side of \( J \) the new bridge goes on (although it is essential that the new bridge does not cause crossings outside \( J \)). We have adopted the convention here so that we can treat over-bridges and under-bridges symmetrically. This systematic treatment will be convenient in Algorithm A.6.
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Figure 6. A diagram $D$ of $8_{20}$, a knotoid diagram $D_B$ obtained by removing an over-bridge $B = (45)$ from $D$, and a diagram $D_B^*$ of the virtual knot obtained by turning the crossings 4, 5 along $B$ into virtual crossings. The dotted arrow in $D$ is a bypass for $B$ (see Proposition 3.14).

Figure 7. The diagram obtained by smoothing all the crossings of $D_B^*$. The thickened curve is the Seifert interval $J$. The dotted curve will be a guide for the new bridge $\tilde{B}$ in the new diagram $\tilde{D}$.

Figure 8. The diagram $\tilde{D}$ of $8_{20}$ obtained by replacing $B$ with $\tilde{B}$

Regarding the above bridge-replacing move, our main claim is the following.

**Theorem 3.11.** Let $D$ be a knot diagram with an over- or under-bridge $B$ and $\tilde{D}_B$ be the knot diagram obtained by the bridge-replacing move for $(D, B)$. Then we have

$$g(D) \geq g(\tilde{D}_B).$$

To prove Theorem 3.11, we just need to combine Lemma 3.12 below with Propositions 3.3 and 3.5.
Lemma 3.12. Let $D$ be a knot diagram with an over- or under-bridge $B$. Let $D_B$ be the knot diagram obtained by the bridge-replacing move for $(D, B)$ and $D^*_B$ be the knotoid diagram obtained by removing from $D$ the interior of $B$. Then we have
\[ g(D_B) = g(D^*_B). \]

Proof. We use the notations in Notation 3.2 and Definition 3.9.

Since the new bridge $\tilde{B}$ passes only the arcs which configure the Seifert interval $J$, the Seifert algorithm for the new knot diagram $D_B = D^*_B \cup \tilde{B}$ provides the same result that the Seifert algorithm for $D^*_B$ does, except in the part with which $J$ is involved.

Assume that the new bridge $\tilde{B}$ has length $k$, that is, $\tilde{B}$ passes $k$ times the arcs which configure $J$. Then, in applying the Seifert algorithm for the knot diagram $D_B$, considering the orientations of $J$ and of $\tilde{B}$, we have $(k + 1)$ new Seifert circles (instead of $J$) in the part with which $J$ is involved. This implies that $g(D_B)$ is equal to $g(D^*_B)$. \hfill \Box

Next we study the condition for the bridge-replacing move to certain decrease the genus.

An over-bridge (resp. under-bridge) of a Gauss diagram $G$ of length $k$ is a consecutive sequence of endpoints $v_0, \ldots, v_k+1$ on the oriented circle of $G$ such that $v_1, \ldots, v_k$ are the initial (resp. terminal) points of the chords of $G$.

Proposition 3.13. Let $G$ be a Gauss diagram with an over- or under-bridge $v_0, \ldots, v_{k+1}$, and let $G'$ be the Gauss diagram obtained by removing all the chords with $v_i \in \partial c$ for some $1 \leq i \leq k$. Then $g(G) > g(G')$ if and only if there is at least one cycle $\alpha$ of $G$ of the form
\[ \alpha = \{ \ldots v_i v_{i+1} \ldots v_j v_{j+1} \ldots \} \]
for some $0 \leq i \neq j \leq k$.

Proof. This is a consequence of Lemma 3.6. If such a cycle $\alpha$ as above exists, then Case (i) in Figure 5 happens for at least one chord $c$ having its endpoint between $v_i v_{i+1}$ and $v_j v_{j+1}$, and hence the genus certainly decreases. Conversely, suppose any two of the portions $v_i v_{i+1}$, $(0 \leq i \leq k)$ do not belong to the same cycle in $G$. Then any chord $c$ with $v_i \in \partial c$ is placed as Case (ii) in Figure 5. Thus any removal of them does not decrease the genus. \hfill \Box

In the case of knot diagrams, the condition in Proposition 3.13 can be stated as follows.

Proposition 3.14. Let $D$ be a diagram of a knot $K$ with an over- or under-bridge $B$. Then the bridge-replacing move for $B$ certainly decreases the genus if and only if there exists a collection of oriented arcs in $D$ which constitute, after the smoothing process (step (ii) in Seifert’s algorithm, Definition 2.1), a (well-oriented) path connecting two crossings included in $B$. We call such a collection of arcs a bypass.

For example in the leftmost diagram of Figure 6, the existence of the bypass (dotted) for the over-bridge $B$ (thickened) ensures that the bridge-replacing move for $(D, B)$ decrease the genus of the diagram. In fact, the diagram shown in Figure 8 obtained from that in Figure 6 by the bridge-replacing move for $B$ detects the genus 2 of $8_{20}$.

4. An example

The knot $16_{686716}$, which is the 306917th non-alternating 16 crossing knot in the Hoste-Thistlethwaite table [1], is referred to in Stoimenow’s recent paper [4, §10.3] as a knot whose (canonical) genus has not been determined (to be whether 2 or 3) yet. Here we demonstrate an approach to this problem using bridge-replacing moves. Indeed its genus turns out to be 2 (see Remark 4.1).

In the knot diagram of $16_{686716}$ shown in Figure 9, that is drawn by Knotscape [1] and has genus 5, we can find three over-bridges (thickened) each of which has a favorable bypass (see Proposition 3.14). In fact we can check that the bridge-replacing move for
Figure 9. The knot 16_{686716}. This diagram has Dowker-Thistlethwaite code (DT code) 4 10 -26 -22 -18 2 20 -26 -32 -28 14 30 -6 -12 -8 24

Figure 10. A diagram with genus 5

Figure 11. A minimal crossing diagram with genus 3 of 16_{686716}. Its DT code is -12 26 22 -14 28 -2 -20 30 -24 8 -32 -16 4 10 18 -6

one of the three bridges produces a new diagram with genus 4. In this case, however, such a move for one bridge destroys the bypasses of the other two bridges and apparently we cannot perform further bridge-replacing moves. To avoid this we precook the diagram into the diagram shown in Figure 10. Then we can perform for the diagram in Figure 10 bridge-replacing moves twice successively, first with respect to the bridge A and second with respect to B, so that we obtain a diagram with genus 3, but with 20 crossings, of the knot 16_{686716} (the number of the crossings of this diagram can be pared down to 18 by the second Reidemeister move).

Now we can interpret the bridge-replacing move in terms of Gauss codes, of which we give a detailed account in Appendix A. By using the interpretation, we develop a small Python program which, for an inputted Gauss code, performs all possible bridge-replacing moves (and reduces excessive crossings in pairs by the second Reidemeister moves). With
the aid of the program (and with some heuristic procedures), we have found the diagram shown in Figure 11 of the knot \(16_{686716}\), that has genus 3 and minimal (that is, 16) crossings.

**Remark 4.1.** Inspired by Figures 10 and 11, Mikami Hirasawa found Seifert surfaces of genus 2 for the knot \(16_{686716}\) (by probing for “compression disks”). Figure 12 depicts one of such (possibly non-canonical) Seifert surfaces with genus 2. By an estimate via polynomial invariants, these surfaces turn out to attain the genus 2 of the knot. We do not know, however, if the canonical genus of the knot can go down to 2. Namely, we do not know whether any of these Seifert surfaces for \(16_{686716}\) with genus 2 can be realized as a canonical one.

### Appendix A. The bridge-replacing move in terms of Gauss codes

We give an interpretation of the bridge-replacing move, introduced in §3, in terms of Gauss codes. We will follow the convention of [2] for Gauss codes.

**Definition A.1.** A unit is a word of length three of the form “αβγ,” where

- “α” is either the characters O or U (which indicate “over” or “under” information),
- “β” is a natural number (the label of a crossing),
- “γ” is the sign + or − (the sign of a crossing).

A Gauss code of length \(n\) is a cyclic sequence of \(2n\) units \(X_1 \ldots X_{2n}\) such that:

(i) It includes \(n\) distinct natural numbers and each number appears exactly twice.

(ii) If the number \(k\) (which appears twice) appears in some unit including the character U (resp. O), then \(k\) must appear in the other unit including O (resp. U).

(iii) For each number \(k\), the two units including \(k\) have the same sign.

To an oriented knot diagram \(D = D_k\) with \(n\) crossings, we can associate the Gauss code \(X_D = X_1 \ldots X_{2n}\) of length \(n\) as follows: label all the crossings as 1, 2, \ldots, \(n\), and go along \(D\) according to the orientation (starting from an arbitrary point \(b\)). We will meet each crossing exactly twice before we come back to \(b\). The \(i\)-th unit \(X_i\) consists of

- the character O or U depending on whether we overpass or underpass the \(i\)-th crossing respectively,
- the natural number \(k\) taken from the label of the \(i\)-th crossing, and
- the sign + or − depending on whether the \(i\)-th crossing is positive or negative respectively.

The following definition is motivated by the notion of cycles in Gauss diagrams (Definition 2.9).

**Definition A.2.** Let \(C = X_1 \ldots X_{2n}\) be a Gauss code. A cycle in \(C\) is a cyclic sequence of units in \(C\) obtained in the following steps: starting from some unit \(X_i\),

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This should be deduced also from a calculation of the knot Floer homology.
(i) record the unit \(X_i\) that contains the same natural number as \(X_i\),
(ii) record the next unit \(X_{i+1}\) to \(X_i\),
(iii) from \(X_i\), repeat (i) and (ii) above until \(X_i\) appears again in the step (ii).
We will denote a cycle as a cyclic sequence \(X_i, X_{i+1}, \ldots, X_{i_n}\) of units which appear in the
above steps (i) and (ii).

Example A.3. The Gauss code of the knot diagram of 3\(i\) shown in Figure 3 is
\[ C = 01 - U2 - 03 - U1 - 02 - \text{U3}- \cdot \cdot \cdot \]
This code has two cycles 01 - U1 - 02 - U2 - 03 - U3- and U1 - 01 - U2 - 02 - U3 - 03- (compare them with cycles of \(G\) given in Example 2.10).

Remark A.4. A Gauss code for a virtual knot diagram can also be defined in the same way
as explained after Definition A.1, except that no units are assigned to virtual crossings.

Definition A.5. An over-bridge (resp. under-bridge) \(B\) in a Gauss code \(C = X_1 \ldots X_{2n}\) is a
subsequence \(B = X_i X_{i+1} \ldots X_j\) with all \(X_p\) \((i \leq p \leq j)\) including the character 0 (resp. \(U\)).

Now we describe the bridge-replacing move, only for an over-bridge. For an under-bridge, we need only to interchange the letters 0 and \(U\) appearing in Algorithm A.6.

Algorithm A.6 (the bridge-replacing move in terms of Gauss codes). Given a Gauss code \(C\) and an over-bridge \(B\) in \(C\), we obtain the Gauss code \(\tilde{C}\) by the following algorithm. Let \(n\) be the maximal natural number used in the code \(C\).

(i) Starting from the unit just before \(B\), search \(C\) (cyclically) leftward and find the first
unit \(X\) which does not contain any number appearing in \(B\).
(ii) Remove all the unit which contain the numbers appearing in \(B\), so that we obtain the
new code \(C'\). Note that \(C'\) contains the unit \(X\).
(iii) Let \(c\) be the cycle of \(C'\) starting from the unit \(X\).
(iv) Starting from \(X\), search \(c\) (cyclically) leftward and find all the sequences as 0\(m\)+0\(m\)+
or as 0\(m\)-0\(m\)- in \(c\), where \(m\) is an arbitrary natural number; record these natural numbers
\(a_1, a_2, \ldots, a_k\) in order (leftward from \(X\)).
(v) For \(C'\), insert the sequence \(0(n+1)-0(n+2)+0(n+3)-0(n+4)+ \cdots 0(n+2k)+\) just af-
fer the unit \(X\) in \(C'\). Furthermore,
- if \(0a_j+ua_j+\) appears in \(c\):
  \[ \text{insert } U(n+2j-1) - \text{just after } ua_j+ \text{ and insert } U(n+2j)+ \text{ just before } 0a_j+. \]
- if \(0a_j-ua_j-\) appears in \(c\):
  \[ \text{insert } U(n+2j-1) - \text{just after } 0a_j- \text{ and insert } U(n+2j)+ \text{ just before } ua_j-. \]
Then we denote the resultant code by \(\tilde{C}\).

We can easily check that if \(D\) is a knot diagram and its Gauss code is \(C\), then \(\tilde{C}\) obtained
by Algorithm A.6 is nothing but the Gauss code of the diagram obtained by performing the
bridge-replacing move for \((D, B)\), abusing the notation \(B\) also for the bridge in the Gauss
diagram corresponding to \(B \subset C\).

Example A.7. Let \(D\) be the knot diagram of 8\(20\) as shown in Figure 6. Its Gauss code
\(C = C_D\) is
\[ 01+U2-U3+04+05-U1+U6-07-U8-05-02-06-U7-03+U4+08- \cdot \cdot \cdot \]
Consider an over-bridge \(B = 04+05-\).
(i) We find the unit \(X = U3+\) just before \(B\), which does not contain neither 4 nor 5.
(ii) Removing all the units containing 4 or 5, we obtain a code
\[ C' = 01+U2-U3+U1+U6-07-U8-02-06-U7-03+08- \]
which indeed contains \(X = U3+\). Here \(C'\) corresponds to the knotoid diagram \(D'\) in
Figure 6. The crossing labeled 3 is the first one which we meet after \(Q\) along the
knotoid diagram \(D'\).
(iii) The cycle \(c\) of \(C'\) starting from \(X = U3+\) is
This corresponds to the guide of the new bridge $\tilde{B}$ (Figure 7).

(iv) In the cycle $c$, we find sequences

$$U_2-O_2-, O_3+U_3+$$

(notice that the codes are regarded as cyclic sequences). We put $a_1=3$, $a_2=2$ (leftward order from $X$).

(v) Since $n=8$ and $k=2$, we insert the sequence

$$09-01\theta+011-012+$$

just after $X=U_3+$. Moreover

- since $0a_1+Ua_1+ (a_1=3)$ appears in $c$, insert $U9-$ just after $U3+$ and insert $U1\theta+$ just before $O3+$,
- since $Ua_2-Oa_2- (a_2=2)$ appears in $c$, insert $U11-$ just after $O2-$ and insert $U12+$ just before $U2-$.

These units correspond to the new crossings which are produced after the bridge $B$ is replaced by the new bridge $\tilde{B}$.

In this way we obtain the new code

$$01+U12+U2-U3+U9-09-01\theta+011-012+U1+U6-07-U8-02-U11-06-U7-U1\theta+O3+O8-$$

which represents the diagram (Figure 8) obtained by performing the bridge-replacing move to $D$.

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