Gradient Estimates for a Class of Elliptic and Parabolic Equations on Riemannian Manifolds

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Abstract Let \((N, g)\) be a complete noncompact Riemannian manifold with Ricci curvature bounded from below. In this paper, we study the gradient estimates of positive solutions to a class of nonlinear elliptic equations

\[
\Delta u(x) + a(x)u(x) \log u(x) + b(x)u(x) = 0
\]
on \(N\) where \(a(x)\) is \(C^2\)-smooth while \(b(x)\) is \(C^1\) and its parabolic counterparts

\[
\left( \Delta - \frac{\partial}{\partial t} \right) u(x,t) + a(x,t)u(x,t) \log u(x,t) + b(x,t)u(x,t) = 0
\]
on \(N \times [0, \infty)\) where \(a(x,t)\) and \(b(x,t)\) are \(C^2\) with respect to \(x \in N\) while are \(C^1\) with respect to the time \(t\). In contrast with lots of similar results, here we do not assume the coefficients of equations are constant, so our results can be viewed as extensions to several classical estimates.

Keywords Gradient estimates, nonlinear equations, maximum principle

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1 Introduction

Recently, one pays attention to the study the following elliptic equation defined on a complete, noncompact Riemannian manifold \((N, g)\)

\[
\Delta u(x) + a(x)u(x) (\log u(x))^\alpha + b(x)u(x) = 0,
\]

where \(\alpha \in \mathbb{R}\) and \(a(x), b(x) \in C^2(N)\). The equation is linked with gradient Ricci solitons, for example, see [3, 5, 12, 21] for detailed explanations.
On the other hand, it is closely related to log-Sobolev constants of Riemannian manifolds (see [7]). Recall that, log-Sobolev constants $S_N$, associated to a closed Riemannian manifold $(N,g)$, are the smallest positive constants such that the logarithmic-Sobolev inequality

$$
\int_N u^2 \log u^2 dN \leq S_N \int_N |\nabla u|^2 dN
$$

for all smooth functions $u$ defined on $N$ with $\int_N u^2 dN = \text{Vol}(N)$. If $\psi$ is an extremal function which achieves the log-Sobolev constant and satisfies $\int_N \psi^2 dN = \text{Vol}(N)$, i.e.,

$$
S_N = \frac{\int_N |\nabla \psi|^2 dN}{\int_N \psi^2 \log \psi^2 dN} = \inf_{\phi \neq 0 \in C^1(N)} \frac{\int_N |\nabla \phi|^2 dN}{\int_N \phi^2 \log \phi^2 dN},
$$

by variation we know that $\psi$ satisfies the Euler–Lagrange equation

$$
\Delta \psi + S_N \psi \log \psi^2 = 0.
$$

In physics, the logarithmic Schrödinger equation is of the form

$$
i\epsilon \frac{\partial \Psi}{\partial t} = -\epsilon^2 \Delta \Psi + (W(x) + \omega)\Psi - \Psi \log |\Psi|^2, \quad \Psi : [0, \infty) \times \mathbb{R}^n \to \mathbb{C}, \quad n \geq 1,$$

which has also received considerable attention. It is well known that this class of equation has some important physical applications, such as quantum mechanics, quantum optics, open quantum systems, effective quantum gravity, transport and diffusion phenomena, theory of superfluidity and Bose–Einstein condensation (see [23] and the references therein). Standing waves solution, $\Psi$, for this logarithmic Schrödinger equation is related to the solutions of the following equation

$$
\epsilon^2 \Delta u + u \log u^2 - V(x)u = 0,
$$

where $V(x)$ is a real function on $\mathbb{R}^n$. Many mathematicians have also ever studied the existence and properties of solutions to such elliptic equation on a Euclidean space (see [1,18] and references therein).

In this paper, we mainly study the gradient estimates of the following nonlinear elliptic equation

$$
\Delta u(x) + a(x)u(x) \log u(x) + b(x)u(x) = 0, \quad (1.2)
$$

which is a special case of (1.1), and its parabolic counterpart

$$
\left( \Delta - \frac{\partial}{\partial t} \right) u(x,t) + a(x,t)u(x,t) \log u(x,t) + b(x,t)u(x,t) = 0 \quad (1.3)
$$

on a complete non-compact Riemannian manifold $N$. In (1.2), $a(x)$ is $C^2$-smooth while $b(x)$ is $C^1$ on $N$, and in (1.3), both $a(x,t)$ and $b(x,t)$ are $C^2$ with respect to $x \in N$ and are $C^1$ with respect to the time $t$. 
For the case $a < 0$ is a constant, $b = 0$, and $\alpha = 1$ in (1.1), that is,

$$\Delta u(x) + au(x) \log u(x) = 0 \quad \text{on } N,$$

(1.4)

Ma [12] studied the gradient estimates of the positive solutions to the above elliptic equation. Later, L. Chen and W. Chen [5] improved the estimate of [12] and extended it to the case $a > 0$.

The first study of the corresponding heat equation of (1.4) and related nonlinear heat equations can be traced back to Yang [21], and later by Huang and Ma [10], Qian [17], Cao et al. [3], Zhu and Li [22] and H. Dung and N. Dung [8], who derived various gradient estimates and Harnack estimates and noted the relation to gradient Ricci solitons.

Very recently, Peng [14] studied the equation (1.2) with constant coefficients and obtained the gradient estimates without assuming $u$ is bounded. Moreover, Peng et al. [16] studied (1.1) with $a \neq 0$ a constant and $\alpha \neq 1$ (also see [15]), and obtained some Li–Yau type gradient estimates on the positive solutions to these equations. More precisely, for the case $\alpha = \frac{k_1}{2k_2+1} \geq 2$, they improved the classical methods and employed some delicate analytic techniques to obtain a gradient bound of a positive solution to (1.1) which does not depend on such quantities as the bounds of the solution and the Laplacian of the distance function.

Its parabolic counterpart,

$$\left( \Delta - q(x,t) - \frac{\partial}{\partial t} \right) u(x,t) = au(x,t)(\log(u(x,t)))^\alpha,$$

where $q(x,t)$ is a $C^2$ function, $a$ and $\alpha$ ($\alpha \neq 0$ or 1) are constants, was also considered by some mathematicians (see [5,21,22]). Besides, Wu [19] and Yang and Zhang [20] paid attention to a similar nonlinear parabolic equation defined on certain smooth metric measure spaces.

We note that it seems that few mathematicians considered the above equations when $a(x)$ is a function defined on $N$ except for [13] and [9]. This stimulates us to study the gradient estimates for the case $a$ is not a constant. So, in the present paper we intend to extend the results on gradient estimates of the positive solutions to (1.2) or (1.3) with constant coefficients in [6,12,21] to the case (1.2) or (1.3) with variable coefficients. In particular, we can extend some results in [13] to the case $N$ is a noncompact manifold, $a(x,t)$ is not a constant function and $b(x,t)$ is a bounded real function.

It seems that the methods in [14] can not be used to deal with the variable coefficient equation here directly. However, by improving the techniques in [14] (also see [15,16]), we can also obtain some similar gradient estimates for the equation (1.2) with variable coefficients. As for (1.3), our basic ideas mainly come from [21], but we also need to adopt some methods and techniques from [15,16].
In order to state our main results, we need to introduce some notations first. Throughout this paper, let the symbol \((g)^+\) denote
\[
(g)^+ \equiv \sup_{x \in B_p(2R)} \max(g(x), 0),
\]
or
\[
[g]^+ \equiv \sup_{(x,t) \in B_p(2R) \times (0,\infty)} \max(g(x,t), 0).
\]
Now we are in the position to state the main results of this paper.

**Theorem 1.1.** Let \((N,g)\) be a complete non-compact Riemannian manifold of dimension \(n \geq 2\), \(p\) is a point on \(N\), and \(B_p(2R)\) is a geodesic ball of radius \(2R\) around \(p\) which does not intersect with \(\partial N\). Let \(u\) be positive smooth solutions to equation (1.2) and \(f = \log u\). Suppose that the Ricci curvature of \(N\) denoted by \(\text{Ric}_N\) satisfies \(\text{Ric}_N \geq -K\) on \(B_p(2R)\), where \(K\) is a nonnegative constant, \(|\nabla a|\) and \(|\nabla b|\) are bounded. We have

1. If \(a(x) \geq 2A_1 > 0\), where \(A_1\) is a positive constant, \(\Delta a(x)\) has a lower bound while \(b(x)\) has an upper bound on \(B_p(2R)\), then there exist constants \(C_1 > 0\, C_2 > 0\) and \(M_1\) which is a nonnegative constant depending on the bounds of \(a(x)\, b(x)\, \nabla a(x)\, \nabla b(x)\, K\) and the lower bound of \(\Delta a(x)\), such that the following local estimate holds on \(B_p(R)\):

\[
|\nabla f|^2 + (A_1 + a) f \leq n \left\{ 2B + 3n \left( \frac{a^2}{A_1^2} \right)^+ \frac{C_1^2}{R^2} + (3K + 3) \left( \frac{a}{A_1} \right)^+ \right\} 
+ n \left\{ \frac{8}{n} (M_1 - b)^+ + \left( \frac{\Delta a}{A_1} \right)^+ + 5(a)^+ \right\}. \tag{1.5}
\]

Here \(B = \frac{(n-1)(1+\sqrt{K})C_1^2+C_2}{R^2}\).

2. If \(A_3 \leq a \leq 2A_2 < 0\), where \(A_2\) and \(A_3\) are negative constants, \(\Delta a(x)\) and \(b(x)\) have upper bounds, then there exist constants \(C_1 > 0\, C_2 > 0\) and \(M_2\) which is a nonnegative constant depending on the bounds of \(a(x)\, b(x)\, \nabla a(x)\, \nabla b(x)\, K\) and the upper bound of \(\Delta a(x)\), such that the local estimate holds on \(B_p(R)\):

\[
|\nabla f|^2 + (A_2 + a) f \leq n \left\{ 2B + 3n \left( \frac{a^2}{A_2^2} \right)^+ \frac{C_1^2}{R^2} + (3K + 3) \left( \frac{a}{A_2} \right)^+ \right\} 
+ n \left\{ \frac{8}{n} (M_2 - b)^+ + \left( \frac{\Delta a}{A_2} \right)^+ + \left( -\frac{3a}{2} \right)^+ \right\}. \tag{1.6}
\]

Here \(B\) is the same as in the above.

Using the results of Theorem 1.1, we obtain the following estimates about the upper or lower bounds of the global solutions to (1.2).
Corollary 1.1. Let $N$ be a complete non-compact without boundary. Suppose $u$ is a global solution to equation (1.2) and in addition that $b \geq b_1$ and $|\Delta a| \leq a_1$ on $N$, where $b_1$ and $a_1 \geq 0$ are constants.

(1) If the conditions on $a(x)$ and $b(x)$ in (1) of Theorem 1.1 are satisfied on $N$, moreover, $0 < 2A_1 \leq a \leq A_4$ where $A_4$ is a positive constant on $N$, then
\[
u \leq e^{n\left(\frac{(K+1)A_4}{A_1^2} + \frac{8(M_1-b_1)}{3nA_1^2} + \frac{a_1}{3A_1^2} + \frac{5A_4}{3A_1}\right)}.\]

(2) If the conditions on $a(x)$ and $b(x)$ in (2) of Theorem 1.1 are satisfied on $N$, then
\[
u \geq e^{n\left(\frac{(K+1)A_3}{A_2^2} + \frac{8(M_2-b_1)}{3nA_2^2} - \frac{a_1}{3A_2^2} + \frac{A_3}{2A_2}\right)}.\]

For heat equation (1.3) we obtain the following Li–Yau type estimates:

Theorem 1.2. Let $(N, g)$ be a complete non-compact Riemannian manifold of dimension $n \geq 2$, $p$ is a point on $N$, and $B_p(2R)$ is a geodesic ball of radius $2R$ which does not intersect with $\partial N$. Let $u$ be positive smooth solutions to equation (1.3) on $N \times [0, \infty)$, $0 < u \leq D$ and $f = \log \frac{u}{D}$ for some positive constant $D$. Suppose that the Ricci curvature of $N$ denoted by $\text{Ric}_N$ satisfies $\text{Ric}_N \geq -K$ on $B_p(2R)$, where $K$ is a nonnegative constant, $a, b, |\nabla a|, |\nabla b|, |a_t|$ are bounded and $\Delta b$ has a lower bound on $B_p(2R) \times (0, \infty)$.

Then, there exist $C_1 > 0$, $C_2 > 0$ and $A$ which is a positive constant and strictly larger than $[a]^+$ on $B_p(2R) \times (0, \infty)$ and a constant $M$ depending on the bounds of $a, b, |\nabla a|, |\nabla b|, a_t$ and $\Delta b$ on $B_p(2R) \times (0, \infty)$, such that the following local estimate holds on $B_p(R) \times (0, \infty)$:
\[
|\nabla f|^2 + (A + a)f - 2f_t \leq \frac{4n}{t} + 4n \left(\frac{[a]^+}{t} + B + \frac{2[M - a \log D]^+}{R^2} + \frac{nC_1^2}{R^2} + \frac{2[M - a \log D]^+}{n}\right) + 4n \left(\frac{1}{2} [-2K - 2 - |\log D|][a]^+ + \frac{[\Delta a + a_t]^+}{4(A - [a]^+)}\right). \quad (1.7)
\]

Here $B = \frac{(n-1)(1+\sqrt{KR})C_1^2 + C_2}{R^2}$.

Remark 1.1. In fact, from the following proof of Theorem 1.2, we can easily see that, if $a(x, t)$ is a constant, we do not need to assume $u$ has an upper bound. Especially when $a(x, t) = 0$, it is even unnecessary to assume $b$ is bounded. Actually, under these special conditions, we can recover the results in [21] and the classical Li–Yau estimates in [11].

For applications, we apply the above results to a special logarithmic Schrödinger equation
\[
\Delta u + u \log u^2 + V(x)u = 0 \quad (1.8)
\]
defined on $\mathbb{R}^n$ where $V(x) \in C^2(\mathbb{R}^n)$ and study its global positive solutions. In this situation, (1.8) is equivalent to

$$\Delta u + 2u \log u + V(x)u = 0.$$  \hfill (1.9)

As a consequence, we have the following a priori estimates:

**Theorem 1.3.** Let $u$ be a positive solution to equation (1.9).

1. If $V(x)$, $|\nabla V(x)|$ are bounded, then by Theorem 1.1, $u$ must be bounded:

$$u \leq e^{\frac{1}{3}\left\{ 16n + 8 \sup_{\mathbb{R}^n} |V| + \frac{8}{3} \sup_{\mathbb{R}^n} |\nabla V|^2 - 8 V \right\}}.$$ \hfill (1.10)

Especially, when $V \geq 0$ is a constant, from (1.10) we know that the upper bound of $u$ is not related to $V$.

2. If $V(x)$, $\Delta V(x)$ and $|\nabla V(x)|$ are bounded, then by Theorem 1.2, $u$ must be bounded:

$$u \leq e^{\left\{ 2n + \sup_{\mathbb{R}^n} |\Delta V| + \sup_{\mathbb{R}^n} |\nabla V| + 2 \sup_{\mathbb{R}^n} |V| \right\}}.$$ \hfill (1.11)

Generally speaking, (1.10) is better than (1.11) since the former does not need $\Delta V$, but in some special situations, e.g., if $V \leq 0$ is a constant, (1.11) is more accurate than (1.10).

This paper is divided into four sections. Section 2 gives the proof of Theorem 1.1 and Corollary 1.1, Section 3 gives the proof of Theorem 1.2 and Section 4 gives the detailed discussions about Theorem 1.3.

## 2 Proof of Theorem 1.1 and Corollary 1.1

As usual, our main mathematical tool is the maximum principle, so the first step is to establish the following lemma.

**Lemma 2.1.** Let $(N,g)$ satisfy the same conditions as in Theorem 1.1. Let $u$ be a positive smooth solution to (1.2), $w = \log u$, and $G = |\nabla w|^2 + (a + A)w + M$ where $A$ and $M$ are two constants to be determined later. Then, on $B_p(2R)$ the function $G$ satisfies

$$\Delta G \geq \frac{2G^2}{n} - 2\langle \nabla G, \nabla w \rangle + G \left\{ - \frac{4Aw}{n} + \frac{4(b - M)}{n} - 2K - 2a - 2 \right\} + \left\{ \frac{4(M - b)A}{n} + 2K + 2(A + a) + \Delta a \right\}w$$
$$+ \frac{2A^2w^2}{n} - A(A + a)w + \frac{2(b - M)^2}{n} + (A + a)(M - b)$$
$$+ (2 - 2A)M + 2KM - |\nabla a|^2 - |\nabla b|^2.$$ \hfill (2.1)
Proof. First, from the equation (1.2) and the definition of $G$, we derive the following two new equations

$$\Delta w = -G + Aw + M - b$$

(2.2)

and

$$|\nabla w|^2 = G - (A + a)w - M.$$  

(2.3)

Then, using the well-known Bochner formula, we have

$$\Delta G = \Delta |\nabla w|^2 + \Delta ((A + a)w)$$

$$= 2 \langle \nabla w, \nabla \Delta w \rangle + 2|D^2 w|^2 + 2Ric(\nabla w, \nabla w) + \Delta ((A + a)w).$$

(2.4)

The Cauchy–Schwarz inequality tells us that

$$2|D^2 w|^2 \geq \frac{2}{n}(\Delta w)^2.$$  

(2.5)

Next, we substitute (2.2), (2.3) and (2.5) into (2.4) to obtain

$$\Delta G \geq 2 \langle \nabla w, \nabla (-G + Aw + M - b) \rangle + \frac{2}{n}(-G + Aw + M - b)^2$$

$$- 2K(G - (A + a)w - M) + \Delta ((A + a)w).$$

(2.6)

Keeping (2.2) and (2.3) in mind and noting

$$\Delta ((A + a)w) = w\Delta a + (A + a)\Delta w + 2\langle \nabla w, \nabla a \rangle$$

$$= w\Delta a + (A + a)(-G + Aw + M - b) + 2\langle \nabla w, \nabla a \rangle,$$

(2.7)

we infer from (2.3), (2.6) and (2.7) the following

$$\Delta G \geq \frac{2G^2}{n} - 2\langle \nabla w, \nabla G \rangle + G\left\{- \frac{4Aw}{n} - 2a + \frac{4(b - M)}{n} - 2K \right\}$$

$$+ \frac{2A^2 w^2}{n} - A(A + a)w + \left\{ \frac{4(M - b)A}{n} + 2K(A + a) + \Delta a \right\}w$$

$$+ (-2AM + (A + a)(M - b) + 2\langle \nabla a, \nabla w \rangle)$$

$$+ \frac{2(b - M)^2}{n} - 2\langle \nabla b, \nabla w \rangle + 2KM.$$  

(2.8)

We can easily see that on $B_p(2R)$ the following inequalities hold:

$$2 \langle \nabla a, \nabla w \rangle \geq -2|\nabla a||\nabla w|$$

$$\geq -2\left( \frac{|\nabla a|^2}{2} + \frac{|\nabla w|^2}{2} \right) = -|\nabla a|^2 - |\nabla w|^2$$

$$= (A + a)w - G + M - |\nabla a|^2,$$

(2.9)

and similarly,

$$2 \langle \nabla b, \nabla w \rangle \geq (A + a)w - G + M - |\nabla b|^2.$$  

(2.10)

By substituting (2.9) and (2.10) into (2.8), we obtain (2.1). Hence we accomplish the proof. □
In order to apply the maximum principle, we need to use the cut-off function introduced by Li–Yau in [11]. Concretely, let $\psi(r)$ be a nonnegative $C^2$-smooth function on $\mathbb{R}^+ = [0, +\infty)$ such that $\psi(r) = 1$ for $r \leq 1$ and $\psi(r) = 0$ for $r \geq 2$. Moreover, there exist two positive constants $C_1$ and $C_2$ such that the derivatives of $\psi(r)$ satisfy the conditions as follows:

$$-C_1 \frac{\psi'(r)}{2} \leq \psi'(r) \leq 0 \quad \text{and} \quad -C_2 \leq \psi''(r). \quad (2.11)$$

Now, let $\phi(x) = \psi\left(\frac{d(x,p)}{R}\right)$ where $d(x,p)$ denotes the distance from $p$ to $x$ on $N$ and it is obvious that $\phi(x)$ is supported in $B_p(2R)$:

$$\phi|_{B_p(R)} = 1,$$

$$\phi|_{N \setminus B_p(2R)} = 0.$$

Furthermore, by Calabi’s trick in [2], we can assume without loss of generality that $\phi$ is smooth on $B_p(2R)$. Consequently, it follows from (2.11) and the Laplacian comparison theorem that

$$\frac{|\nabla \phi|^2}{\phi} \leq \frac{C_2}{R^2}, \quad (2.12)$$

$$\Delta \phi \geq -\frac{(n - 1)(1 + \sqrt{K}R)C_1^2 + C_2}{R^2}. \quad (2.13)$$

According to the definition of $\phi(x)$, we know that $\phi G(x)$ is also supported in $B_p(2R)$. Consequently, there exists a point $x_0$ in $B_p(2R) \setminus \partial B_p(2R)$ such that:

$$\sup_{x \in B_p(2R)} \phi G(x) = \phi G(x_0).$$

Hence, by maximum principle, we have

$$\nabla(\phi G)(x_0) = 0 \quad \text{and} \quad \Delta(\phi G)(x_0) \leq 0,$$

and these imply that at $x_0$

$$\phi \nabla G = -G \nabla \phi \quad \text{and} \quad \phi \Delta G + G \Delta \phi - 2G \frac{|\nabla \phi|^2}{\phi} \leq 0. \quad (2.14)$$

We can also assume without loss of generality that $\phi G(x_0) > 0$. Otherwise Theorem 1.1 is trivial. After a direct computation, (2.12), (2.13) and (2.14) yield that at $x_0$

$$BG \geq \phi \Delta G, \quad (2.15)$$

where

$$B = \frac{(n - 1)(1 + \sqrt{K}R)C_1^2 + C_2}{R^2}.$$
On the other hand, from (2.3) and (2.14), we also have

\[-\langle \nabla w, \nabla G \rangle = G \langle \nabla w, \nabla \phi \rangle \geq -G|\nabla \phi|(G - (A + a)w - M)^{\frac{1}{2}}.\]  

(2.16)

Eventually, by substituting (2.15) and (2.16) into (2.1), we obtain that there holds at \(x_0\)

\[BG \geq \phi \Delta G \]

\[\geq \frac{2\phi G^2}{n} - 2G(G - (A + a)w)^{\frac{1}{2}}|\nabla \phi| + \phi \left\{ \frac{2A^2w^2}{n} - A(A + a)w \right\} \]

\[+ \phi G \left\{ -\frac{4Aw}{n} + \frac{4(b - M)}{n} - 2K - 2a - 2 \right\} \]

\[+ \phi w \left\{ 4(M - b)A + (2K + 2)(A + a) + \Delta a \right\} \]

\[+ \phi \left\{ (A + a)(M - b) + (2 - 2A)M + \frac{2(b - M)^2}{n} \right\} \]

\[+ \phi \left\{ 2KM - |\nabla a|^2 - |\nabla b|^2 \right\}.\]  

(2.17)

Now, we are ready to give the complete proof of Theorem 1.1. In consideration of the pretty long whole proof, it will be divided into several parts. Since the following calculations are all considered at the point \(x_0\), for simplicity, we omit the \(x_0\).

**Proof of Theorem 1.1.** First, we give the proof of (1) in Theorem 1.1.

Since \(a \geq 2A_1 > 0\), \(|\nabla a|, |\nabla b|\) are bounded, \(\Delta a\) is bounded from below and \(b\) has an upper bound, there exists a nonnegative constant \(M_1\) such that on \(B_p(2R)\)

\[\frac{4(M_1 - b)A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \geq 0.\]

Therefore, letting \(A = A_1 > 0\) and \(M = M_1 \geq 0\) in (2.17), we obtain

\[BG \geq \frac{2\phi G^2}{n} - 2G(G - (A + a)w)^{\frac{1}{2}}|\nabla \phi| + \phi \left\{ \frac{2A^2w^2}{n} - A_1(A_1 + a)w \right\} \]

\[+ \phi G \left\{ -\frac{4A_1w}{n} + \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\} \]

\[+ \phi w \left\{ 4(M_1 - b)A_1 + (2K + 2)(A_1 + a) + \Delta a \right\}.\]  

(2.18)
Now, we need to consider the following three cases:

**Case 1.** The case \( w \geq \frac{n(A_1 + a)}{2A_1} > 0 \).

Since \( w \geq \frac{n(A_1 + a)}{2A_1} > 0 \), there hold

\[
\frac{2A_1^2w^2}{n} - A(A + a)w \geq 0 \quad \text{and} \quad G \geq G - (A_1 + a)w > 0.
\]

Then, from (2.18) we obtain

\[
BG \geq \frac{2\phi G^2}{n} - 2G^\frac{1}{2} \left| \nabla \phi \right| + \phi \left\{ -\frac{4A_1w}{n} + \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\},
\]

i.e.,

\[
B \geq \frac{2\phi G}{n} - 2G^\frac{1}{2} \left| \nabla \phi \right| + \phi \left\{ -\frac{4A_1w}{n} + \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}. \tag{2.19}
\]

Considering that \( w \leq \frac{G}{A_1 + a} \), we can rewrite (2.19) as

\[
B \geq \frac{2\phi G}{n} - 2G^\frac{1}{2} \left| \nabla \phi \right| - \frac{4A_1G\phi}{n(A_1 + a)} + \phi \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}. \tag{2.20}
\]

Since \( a \geq 2A_1 > 0 \), by Young’s inequality, there holds

\[
2G^\frac{1}{2} \left| \nabla \phi \right| \leq \frac{\phi G(3a - 5A_1)}{2n(A_1 + a)} + \frac{\left| \nabla \phi \right|^2 2n(A_1 + a)}{\phi(3a - 5A_1)}. \tag{2.21}
\]

Then, from (2.12) and substituting (2.21) into (2.20), we have

\[
B \geq \frac{\phi G}{2n} - \frac{2n(A_1 + a)C_1^2}{(3a - 5A_1)R^2} + \phi \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}. \tag{2.22}
\]

Also note that \( A_1 + a \leq \frac{3a}{2} \), then

\[
\frac{2n(A_1 + a)}{(3a - 5A_1)} \leq \frac{2n \frac{3a}{2}}{6A_1 - 5A_1} = \frac{3na}{A_1},
\]

so, from (2.22) we can derive

\[
B \geq \frac{\phi G}{2n} - \frac{3naC_1^2}{A_1R^2} + \phi \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}.
\]

Note that

\[
\sup_{B_p(R)} G \leq \sup_{B_p(2R)} \phi G = \phi G(x_0).
\]

Therefore, on \( B_p(R) \) we have

\[
G \leq 2n \left\{ B + \frac{3n(a)^+C_1^2}{A_1R^2} + 2K + 2(a)^+ + 2 + \frac{4(M_1 - b)^+}{n} \right\}. \tag{2.23}
\]
Case 2. The case $0 \leq w < \frac{(A_1 + a) n}{2A_1}$. For this case, $\frac{2A_1^2 w^2}{n} \geq 0$ and $w \leq \frac{G}{A_1 + a}$ by the definition of $G$, hence $A_1 (A_1 + a) w \leq A_1 G$. So we have

$$BG \geq \frac{2\phi G^2}{n} - 2G \frac{3}{2} |\nabla \phi| + \phi G \left\{ - \frac{4A_1 w}{n} + \frac{4(b - M_1)}{n} - 2K - 2a - 2 - A_1 \right\}. $$

Then, by the same discussions as in Case 1, it holds that on $B_p(R)$

$$G \leq 2n \left\{ B + \frac{3n(a) + C_1^2}{A_1 R^2} + 2K + 2(a)^+ + 2 + \frac{4(M_1 - b)^+}{n} + A_1 \right\}. \quad (2.24)$$

Case 3. The case $w < 0$. Since $w < 0$, it follows that $\frac{2A_1^2 w^2}{n} - A_1 (A_1 + a) w > 0$.

By (2.18) there holds

$$BG \geq \frac{2\phi G^2}{n} - 2(G - (A_1 + a) w)^{\frac{1}{2}} |\nabla \phi| + \phi G \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\} + \phi w \left\{ - \frac{4A_1 G}{n} + \frac{4(M_1 - b) A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \right\}. \quad (2.25)$$

It follows by Young’s inequality that

$$2(G - (A_1 + a) w)^{\frac{1}{2}} |\nabla \phi| \leq \frac{R |\nabla \phi|^2 G^\frac{3}{2}}{C_1 \phi^\frac{1}{2}} + \frac{C_1 \phi^\frac{1}{2} G^\frac{3}{2} (G - (A_1 + a) w)}{R}. $$

Then (2.25) can be written as

$$BG \geq \frac{2\phi G^2}{n} - \frac{R |\nabla \phi|^2 G^\frac{3}{2}}{C_1 \phi^\frac{1}{2}} - \frac{C_1 \phi^\frac{1}{2} G^\frac{3}{2}}{R} + \frac{C_1 \phi^\frac{1}{2} G^\frac{3}{2} (A_1 + a) w}{R} + \phi w \left\{ - \frac{4A_1 G}{n} + \frac{4(M_1 - b) A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \right\} + \phi G \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}. \quad (2.26)$$

Now, if

$$\frac{C_1 \phi^\frac{1}{2} G^\frac{3}{2} (A_1 + a) w}{R} + \phi w \left\{ - \frac{4A_1 G}{n} + \frac{4(M_1 - b) A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \right\} \geq 0,$$
then we can drop the above terms in (2.26) to get

\[ BG \geq \frac{2\phi G^2}{n} - \frac{R|\nabla \phi|^2 G^{\frac{3}{2}}}{C_1 \phi^{\frac{3}{2}}} - \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \phi G \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}, \]

i.e.,

\[ B \geq \frac{2\phi G}{n} - \frac{R|\nabla \phi|^2 G^{\frac{3}{2}}}{C_1 \phi^{\frac{3}{2}}} - \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \phi \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}. \]

Therefore, it follows

\[ B \geq \frac{2\phi G}{n} - \frac{C_1 (\phi G)^{\frac{1}{2}}}{R} - \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \phi \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}. \]

By (2.12), there holds

\[ B \geq \frac{2\phi G}{n} - \frac{C_1 (\phi G)^{\frac{1}{2}}}{R} - \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \phi \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}, \]

consequently,

\[ B \geq \frac{2\phi G}{n} - \frac{C_1 (\phi G)^{\frac{1}{2}}}{R} + \phi \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}. \]

Using Young’s inequality again leads to

\[ \frac{2C_1 (\phi G)^{\frac{1}{2}}}{R} \leq \frac{\phi G}{n} + \frac{nC_1^2}{R^2}. \]

Then we get

\[ B \geq \frac{\phi G}{n} - \frac{nC_1^2}{R^2} + \phi \left\{ \frac{4(b - M_1)}{n} - 2K - 2a - 2 \right\}. \]

Finally on \( B_p(R) \), there holds

\[ G \leq n \left\{ B + \frac{nC_1^2}{R^2} + \frac{4(M_1 - b)^+}{n} + 2K + 2(a)^+ + 2 \right\}. \quad (2.27) \]

On the other hand, if

\[ \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} (A_1 + a) w + \phi w \left\{ - \frac{4A_1 G}{n} + \frac{4(M_1 - b) A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \right\} \leq 0, \]
noting \( w < 0 \) then we know there holds

\[
\frac{C_1 \phi^\frac{1}{2} G^\frac{1}{2}(A_1 + a)}{R} + \phi \left\{ -\frac{4A_1 G}{n} + \frac{4(M_1 - b)A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \right\} \geq 0.
\]

By Young’s inequality we have

\[
\frac{C_1 \phi^\frac{1}{2} G^\frac{1}{2}(A_1 + a)}{R} \leq \frac{1}{2} \left( \frac{6A_1 \phi G}{n} + \frac{nC_1^2(A_1 + a)^2}{6A_1 R^2} \right),
\]

hence,

\[
\frac{A_1 \phi G}{n} \leq \frac{nC_1^2(A_1 + a)^2}{12A_1 R^2} + \phi \left\{ \frac{4(M_1 - b)A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \right\}.
\]

Since we have assumed that on \( B_p(2R) \)

\[
\frac{4(M_1 - b)A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \geq 0,
\]

on \( B_p(R) \) we have

\[
G \leq n\left\{ \frac{n(A_1 + (a)^+)^2 C_1^2}{12A_1^2 R^2} + \frac{4(M_1 - b)^+}{n} + \frac{(2K + 2)(A_1 + a)^+}{A_1} + \frac{(\Delta a)^+}{A_1} \right\}.
\]

(2.28)

By combining (2.23), (2.24), (2.27) and (2.28) and noting \( a \geq 2A_1 > 0 \), we complete the proof of (1.5).

Now, we turn to the proof of (2) of Theorem 1.1.

Since \( A_3 \leq a \leq 2A_2 < 0 \), \( |\nabla a|, |\nabla b| \) are bounded, \( \Delta a \) is bounded from above and \( b \) has an upper bound, there exists a nonnegative constant \( M_2 \) such that on \( B_p(2R) \)

\[
\frac{2(M_2 - b)^2}{n} + (A_2 + a)(M_2 - b) + (2 - 2A_2)M_2 + 2K M_2 - |\nabla a|^2 - |\nabla b|^2 \geq 0
\]

and

\[
\frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a \leq 0.
\]

Therefore, it follows from (2.17) that
\[ BG \geq \frac{2\phi G^2}{n} - 2(G - (A_2 + a)w)^{\frac{1}{2}}|\nabla \phi| + \phi \left( \frac{2A_2^2w^2}{n} - A_2(A_2 + a)w \right) \]

\[ + \phi G \left\{ -4A_2w + \frac{4(b - M_2)}{n} - 2K - 2a - 2 \right\} \]

\[ + \phi w \left\{ \frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a \right\}. \quad (2.29) \]

Now, we also need to discuss the following three situations case by case:

**Case 4.** The case \( w \leq 0 \).

Since \( w \leq 0 \), it is easy to see

\[ \phi \left( \frac{2A_2^2w^2}{n} - A_2(A_2 + a)w \right) \geq 0 \]

and

\[ \left\{ \frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a \right\} \phi w \geq 0. \]

Letting \( A = A_2 \), it follows from (2.29) that

\[ BG \geq \frac{2\phi G^2}{n} - 2(G - (A_2 + a)w)^{\frac{1}{2}}|\nabla \phi| + \phi G \left\{ -4A_2w + \frac{4(b - M_2)}{n} - 2K - 2 \right\}. \]

Noting also that

\[ 0 < A_2w \leq \frac{A_2G}{A_2 + a}, \]

we have

\[ BG \geq \frac{2\phi G^2}{n} - 2G^{\frac{1}{2}}|\nabla \phi| + \phi \left\{ -4A_2G \frac{n}{n(A_2 + a)} + \frac{4(b - M_2)}{n} - 2K - 2 \right\}, \]

i.e.,

\[ B \geq \frac{2\phi G}{n} - 2G^{\frac{1}{2}}|\nabla \phi| + \phi \left\{ -4A_2G \frac{n}{n(A_2 + a)} + \frac{4(b - M_2)}{n} - 2K - 2 \right\}. \]

Since \( A_2 + a < 0 \) and \( 3a - 5A_2 < 0 \), it follows from Young’s inequality that

\[ 2G^{\frac{1}{2}}|\nabla \phi| \leq \frac{\phi G(3a - 5A_2)}{2n(A_2 + a)} + \frac{|\nabla \phi|^22n(A_2 + a)}{\phi(3a - 5A_2)}. \]

Hence, in view of (2.12) we can infer from the above two inequalities that

\[ B \geq \frac{\phi G}{2n} - \frac{2n(A_2 + a)C_1^2}{(3a - 5A_2)R^2} + \phi \left\{ \frac{4(b - M_2)}{n} - 2K - 2 \right\}. \]
Obviously \( a \leq 2A_2 < 0 \) also implies that
\[
\frac{2n(A_2 + a)}{3a - 5A_2} \leq \frac{3na}{A_2}.
\]

Therefore, on \( B_p(R) \) we have
\[
G \leq 2n \left\{ B + \frac{3n(a)^+ C_1^2}{A_2 R^2} + 2K + 2 + \frac{4(M_2 - b)^+}{n} \right\}.
\]

(2.30)

**Case 5.** The case \( 0 \leq w \leq \frac{n(A_2 + a)}{2A_2} \).

In the present situation, from (2.29) we can derive
\[
BG \geq \frac{2\phi G^2}{n} - 2G(G - (A_2 + a)w)^{\frac{1}{2}}|\nabla \phi|
+ \phi G \left\{ - \frac{4A_2 w}{n} + \frac{4(b - M_2)}{n} - 2K - 2a - 2 \right\}
+ \phi w \left\{ \frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a - A_2(A_2 + a) \right\}.
\]

(2.31)

Note that \( (A_2 + a)w \leq 0 \), and by Young’s inequality we have
\[
2G(G - (A_2 + a)w)^{\frac{1}{2}}|\nabla \phi| \leq \frac{R|\nabla \phi|^2 G^{\frac{3}{2}}}{C_1 \phi^{\frac{1}{2}}} + \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}} (G - (A_2 + a)w)}{R}.
\]

Therefore, we can rewrite (2.31) as
\[
BG \geq \frac{2\phi G^2}{n} - \frac{R|\nabla \phi|^2 G^{\frac{3}{2}}}{C_1 \phi^{\frac{1}{2}}} - \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \phi G \left\{ \frac{4(b - M_2)}{n} - 2K - 2a - 2 \right\}
+ \phi w \left\{ \frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a - A_2(A_2 + a) \right\}
+ \phi w \left\{ \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{1}{2}} (A_2 + a)}{R} - \frac{4A_2 G}{n} \right\}.
\]

Hence, if
\[
\phi w \left\{ \frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a - A_2(A_2 + a) \right\}
+ \phi w \left\{ \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{1}{2}} (A_2 + a)}{R} - \frac{4A_2 G}{n} \right\} \geq 0,
\]

we know that there holds
\[ BG \geq \frac{2\phi G^2}{n} - \frac{R|\nabla \phi|^2 G^{\frac{3}{2}}}{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}} - \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \phi G \left\{ \frac{4(b - M_2)}{n} - 2K - 2a - 2 \right\}. \]

By (2.12) again and the above expression we can see that there holds
\[ BG \geq \frac{2\phi G^2}{n} - \frac{2C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \phi \left\{ \frac{4(b - M_2)}{n} - 2K - 2a - 2 \right\}, \]
i.e.,
\[ B \geq \frac{2\phi G^2}{n} - \frac{2C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}}}{R} + \phi \left\{ \frac{4(b - M_2)}{n} - 2K - 2a - 2 \right\}. \]

Now, by the same arguments as deducing (2.27) from (2.26) we can easily see that there holds that on \( B_p(R) \)
\[ G \leq n \left\{ B + \frac{nC_1^2}{R^2} + \frac{4(M_2 - b)^+}{n} + 2K + 2(a)^+ + 2 \right\}. \quad (2.32) \]

On the other hand, if
\[ \phi w \left\{ \frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a - A_2(A_2 + a) \right\} \]
\[ + \phi w \left\{ \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}} (A_2 + a)}{R} - 4A_2G \right\} \leq 0, \]

then, noting \( w \geq 0 \) we have
\[ \frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a - A_2(A_2 + a) \]
\[ + \frac{C_1 \phi^{\frac{1}{2}} G^{\frac{3}{2}} (A_2 + a)}{R} - 4A_2G \leq 0. \]

Since \( a < 0 \) and \( A_2 + a < 0 \), by multiplying \(-1\) and adopting almost the same discussions as those to obtain (2.28) in Case 3, on \( B_p(R) \) we have
\[ G \leq n \left\{ \frac{(A_2 + (a)^+)^2 C_1^2}{3A_2^2 R^2} + \frac{4(M_2 - b)^+}{n} \right. \]
\[ + \frac{(2K + 2)(A_2 + a)^+}{A_2} + \frac{(\Delta a)^+}{A_2} - (A_2 + a)^+ \right\}. \quad (2.33) \]

**Case 6.** The case \( w \geq \frac{(A_2 + a)n}{2A_2} \).

For this situation, \( A_2(A_2 + a)w < 0 \) and \( \frac{2A_2^2 w^2}{n} - A_2(A_2 + a)w \geq 0 \), then by (2.29), there holds
Gradient Estimates for a Class of Elliptic and Parabolic Equations

\[ BG \geq \frac{2\phi G^2}{n} - 2G(G - (A_2 + a)w)^{\frac{1}{2}}|\nabla \phi| + \phi G \left\{ -\frac{4A_2w}{n} + \frac{4(b - M_2)}{n} - 2K - 2a - 2 \right\} + \phi w \left\{ \frac{4(M_2 - b)A_2}{n} + (2K + 2)(A_2 + a) + \Delta a \right\}. \tag{2.34} \]

Notice that the right-hand sides of (2.34) and (2.31) only differ by \( A_2(A_2 + a)\phi w \), so all the discussions here will be almost the same as those in Case 5. So the results here are almost the same as those in Case 5. As a consequence, we have

\[ G \leq n \left\{ B + \frac{nC_1^2}{R^2} + \frac{4(M_2 - b)^+}{n} + 2K + 2(a)^+ + 2 \right\}, \]

which is similar to (2.32) or

\[ G \leq n \left\{ \frac{(A_2 + (a)^+)^2C_1^2}{3A_2^2R^2} + \frac{4(M_2 - b)^+}{n} + \frac{(2K + 2)(A_2 + a)^+}{A_2} + \frac{(\Delta a)^+}{A_2} \right\}, \]

which is similar to (2.33).

Eventually, since \( \frac{a}{A_2} \geq 2 \), then by combining (2.30), (2.32), (2.33) and the arguments in Case 6, we know that (1.6) holds. Thus we accomplish the proof of Theorem 1.1.

\[ \square \]

Next, we give the proof of Corollary 1.1.

**Proof of Corollary 1.1.** (1) Let \( R \to +\infty \). Then from (1.5), there holds

\[ (A_1 + a)f \leq n \left\{ (3K + 3) \left( \frac{A_4}{A_1} \right) + \frac{8}{n}(M_1 - b_1) + 5A_4 \right\} \]

by \(|\nabla f|^2 \geq 0\) and \( M_1 \geq 0 \). Then just using \( A_1 + a \geq 3A_1 \) we can see that the required estimate follows at once. The proof of (2) is just the same as (1). \( \square \)

### 3 Proof of Theorem 1.2

The main methods here are similar to those in the proof of Theorem 1.1. First of all, we need to establish the following lemma.

**Lemma 3.1.** Let \( u \leq D \) be a positive smooth solution of (1.3) for some positive constant \( D \). Let \( f = \log \frac{u}{h} \), \( F = t\{ |\nabla f|^2 + (A + a)f + 2(M + b) - 2f_t \} \) for some constants \( A, M \) to be determined later, and \( h = \frac{|f|^2}{F} \). Then, on \( B_p(2R) \times (0, \infty) \)
there holds

\[ \Delta F - F_t \geq t \left\{ (A - 2K - 2 - |\log D|)hF + \frac{(1 + ht)^2F^2}{2nt^2} - \frac{2(1 + ht)(M - a \log D)F}{nt} \right\} \]

\[ + tf \left\{ \frac{(1 + ht)(A - a)F}{nt} + \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \right\} \]

\[ + t\left\{ 2a(M + b) + 2a_t \log D + 2\Delta b + \frac{2(M - a \log D)^2}{n} \right\} - \frac{F}{t} - aF \]

\[ - t\{(A + a)(a \log D + b) + |\nabla b|^2 + (1 + |\log D|)|\nabla a|^2\} - 2\langle \nabla f, \nabla F \rangle. \]

(3.1)

**Proof.** By direct calculations, we derive the following identities

\[ |\nabla f|^2 = \frac{F}{t} - (A + a)f - 2(M + b) + 2f, \quad (3.2) \]

\[ \Delta f = f_t - af - a \log D - b - |\nabla f|^2 \]

and

\[ f_{tt} = \Delta f_t + af_t + 2\langle \nabla f, \nabla f_t \rangle + b_t + a_t f + a_t \log D. \quad (3.4) \]

By the definition of \( F \), (3.3) is equivalent to

\[ \Delta f = -\frac{F}{2t} + \frac{(A - a)f}{2} - \frac{|\nabla f|^2}{2} + M - a \log D. \quad (3.5) \]

By virtue of Bochner Formula, there holds

\[ \Delta F = t\left\{ \Delta |\nabla f|^2 + (A + a)\Delta f + f\Delta a + 2\langle \nabla a, \nabla f \rangle + 2\Delta b - 2\Delta f_t \right\} \]

\[ = t\left\{ 2\langle \nabla f, \nabla \Delta f \rangle + 2|D^2 f|^2 + 2\text{Ric}(\nabla f, \nabla f) \right\} \]

\[ + t\left\{ f\Delta a + (A + a)\Delta f + 2\langle \nabla a, \nabla f \rangle + 2\Delta b - 2\Delta f_t \right\}. \quad (3.6) \]

Next, substituting (3.2) into (3.5), we have

\[ \Delta f = -\frac{F}{t} + Af - f_t + 2M + b - a \log D. \]

Then we get

\[ \langle \nabla f, \nabla \Delta f \rangle = -\left( \nabla f, \frac{\nabla F}{t} \right) + A|\nabla f|^2 - \langle \nabla f, \nabla f_t \rangle + \langle \nabla f, \nabla b \rangle - \langle \nabla f, \nabla a \rangle \log D. \]

By Cauchy–Schwarz inequality, on \( B_p(2R) \) it holds that

\[ |D^2 f|^2 \geq \frac{(\Delta f)^2}{n}, \]
Gradient Estimates for a Class of Elliptic and Parabolic Equations

\[ 2 \langle \nabla f, \nabla b \rangle \geq -(|\nabla f|^2 + |\nabla b|^2) \]

and

\[ (2 - 2 \log D) \langle \nabla f, \nabla a \rangle \geq (-1 - |\log D|) (|\nabla f|^2 + |\nabla a|^2) . \]

Then, from (3.6) we derive

\[
\Delta F \geq -2 \langle \nabla f, \nabla F \rangle + t \left\{ (2A - 2K - 2 - |\log D|) |\nabla f|^2 + \frac{2(\Delta f)^2}{n} \right\} \\
+ t \left\{ f \Delta a + (A + a) \Delta f + 2\Delta b - 2\Delta f_t - 2 \langle \nabla f, \nabla f_t \rangle \right\} \\
- t \left\{ |\nabla b|^2 + (1 + |\log D|) |\nabla a|^2 \right\} .
\]

(3.7)

As for \( F_t \), first of all, we have:

\[ F_t = \frac{F}{t} + t \{ 2 \langle \nabla f, \nabla f_t \rangle + (A + a)f_t + a_t f + 2b_t - 2f_{tt} \} . \] (3.8)

By combining (3.3), (3.4), (3.7), (3.8) and also noting that

\[ -aF = t \left\{ -a|\nabla f|^2 - (A + a)af - 2a(M + b) + 2af_t \right\}, \]

we can see that there holds

\[
\Delta F - F_t \geq -2 \langle \nabla f, \nabla F \rangle - \frac{F}{t} - aF + t \left\{ (A - 2K - 2 - |\log D|) |\nabla f|^2 \right\} \\
+ t \left\{ \frac{2(\Delta f)^2}{n} + f \Delta a + a_t f + 2a_t |\log D| + 2a(M + b) + 2\Delta b \right\} \\
- t \left\{ (A + a) (a \log D + b) + |\nabla b|^2 + (1 + |\log D|) |\nabla a|^2 \right\} .
\]

(3.9)

Next, substituting \( hF = |\nabla f|^2 \) into (3.5) and noting that \((\frac{A-a)f}{2})^2 \geq 0\), we deduce that

\[
(\Delta f)^2 = \left( -\frac{(1 + ht)F}{2t} + \frac{(A - a)f}{2} + (M - a \log D) \right)^2 \\
\geq \frac{(1 + ht)^2 F^2}{4t^2} + (M - a \log D)^2 - \frac{(1 + ht)(A - a)F f}{2t} \\
- \frac{(1 + ht)(M - a \log D)F f}{t} + (A - a)(M - a \log D) f .
\]

(3.10)

Then, just substituting (3.10) into (3.9) we obtain (3.1) at once.

Now we are going to give the proof of Theorem 1.2, and it is divided into two parts. Without loss of generality, we assume \( F > 0 \), since otherwise the result will be trivial.
Proof of Theorem 1.2. Since on $B_p(2R) \times (0, T]$ for any $T > 0$, $a, b, a_t, |\nabla a|$ and $|\nabla b|$ are bounded and $\Delta b$ has a lower bound, there exists a constant $M$ such that

$$
\begin{cases}
M + b \geq 0, \\
M - a \log D \geq 0, \\
 \frac{2(M - a \log D)^2}{n} + 2a(M + b) - (A + a)(a \log D + b) \\
+ 2a_t \log D + 2\Delta b - \{|\nabla b|^2 + (1 + |\log D|)|\nabla a|^2\} \geq 0.
\end{cases} 
$$

(3.11)

Consequently, let $A$ be a constant strictly larger than $[a]^+$. Then from (3.1) we obtain

$$
\Delta F - F_t \\
\geq t\left\{ (A - 2K - 2 - |\log D|)hF + \frac{(1 + ht)^2F^2}{2nt^2} - \frac{2(1 + ht)(M - a \log D)F}{nt} \right\} \\
+ tf\left\{ \frac{(1 + ht)(a - A)F}{nt} + \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \right\} \\
- 2\langle \nabla f, \nabla F \rangle - \frac{F}{t} - aF.
$$

(3.12)

Next, we consider the same cut-off function $\phi$ as in Section 2, and denote $\phi F$ by $\lambda$. Let $(x_0, t_0) \in B_p(2R) \times (0, T]$ be the maximum point of $\lambda$. Then

$$
\nabla \lambda(x_0, t_0) = 0, \quad \Delta \lambda(x_0, t_0) \leq 0, \quad \text{and} \quad F_t \geq 0.
$$

Then, by the same discussions as to obtain (2.15), we also have $BF \geq \phi \Delta F$ and $\phi \nabla F = -F \nabla \phi$ at $(x_0, t_0)$. From now on, all the discussions are considered at the point $(x_0, t_0)$. For simplicity, we still write $t_0$ as $t$ and omit $x_0$.

By the above discussions, (3.12) changes into

$$
BF \geq \phi \Delta F \\
\geq \phi t\left\{ (A - 2K - 2 - |\log D|)hF + \frac{(1 + ht)^2F^2}{2nt^2} - \frac{2(1 + ht)(M - a \log D)F}{nt} \right\} \\
+ \phi tf\left\{ \frac{(1 + ht)(a - A)F}{nt} + \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \right\} \\
- 2\phi \langle \nabla f, \nabla F \rangle - \frac{\phi F}{t} - a\phi F.
$$

(3.13)

Now, at $(x_0, t_0)$ we need to consider the following cases:

Case I. The case

$$
\left\{ \frac{(1 + ht)(a - A)F}{nt} + \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \right\} \leq 0.
$$
For the present situation, since \( f \leq 0 \), we have
\[
\frac{(1 + ht)(a - A)F}{nt} + \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \geq 0.
\]
Noting \( A > [a]^{+} \), we derive
\[
F \leq \frac{nt}{(A - a)(1 + ht)} \left( \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \right)
\leq \frac{nt}{A - a} \left( \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \right)
\leq \frac{nT[\Delta a + a_t]^{+} + 2T[M - a \log D]^{+}}{A - [a]^{+}}.
\]
(3.14)

In fact, if furthermore assume that
\[
\Delta a + a_t + \frac{2}{n}(A - a)(M - a \log D) \leq 0,
\]
then we have that there holds
\[
f \left\{ \frac{(1 + ht)(a - A)F}{nt} + \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \right\} \geq 0,
\]
and this becomes a special case of Case II. So, here we always assume
\[
\Delta a + a_t + \frac{2}{n}(A - a)(M - a \log D) > 0
\]
to ensure (3.14) makes sense.

Case II. The case
\[
f \left\{ \frac{(1 + ht)(a - A)F}{nt} + \Delta a + a_t + \frac{2(A - a)(M - a \log D)}{n} \right\} \geq 0.
\]

Now, from (3.13) we obtain
\[
BF \geq \phi \Delta F
\geq \phi t \left\{ (A - 2K - 2 - |\log D|)hF + \frac{(1 + ht)^2F^2}{2nt^2} - \frac{2(1 + ht)(M - a \log D)F}{nt} \right\}
- 2\phi \langle \nabla f, \nabla F \rangle - \frac{\phi F}{t} - a\phi F.
\]
(3.15)

By \( \phi \nabla F = -F \nabla \phi \), we deduce that
\[
-2\phi \langle \nabla f, \nabla F \rangle = 2 \langle \nabla f, \nabla \phi \rangle F.
\]
Then by (2.12) and \( hF = |\nabla f|^2 \), it turns out that
\[
2 \langle \nabla f, \nabla \phi \rangle F \geq -2|\nabla \phi||\nabla f|F
\]
\[
= -2 \left( \frac{|\nabla \phi|^2}{\phi} \right)^{\frac{1}{2}} \phi^{\frac{1}{2}} F |\nabla f|
\]
\[
\geq -\frac{2C_1^{\frac{1}{2}} F^{\frac{3}{2}} h^{\frac{1}{2}}}{R}.
\] (3.16)
Moreover, we have
\[
\begin{cases}
-a\phi F \geq -[a]^+ \phi F, \\
\phi t (A - 2K - 2 - |\log D|) hF \geq \phi t \{ -[-(A - 2K - 2 - |\log D|)]^+ \} hF.
\end{cases}
\] (3.17)
Combining (3.16) and (3.17), in view of (3.15) we deduce that
\[
BF \geq -\frac{2C_1^{\frac{1}{2}} F^{\frac{3}{2}} h^{\frac{1}{2}}}{R} - \frac{\phi F}{t} - [a]^+ \phi F \\
+ \phi t \{ -[-(A - 2K - 2 - |\log D|)]^+ \} hF \\
+ \phi t \left\{ \frac{(1 + ht)^2 F^2}{2nt^2} - \frac{2(1 + ht)(M - a \log D) F}{nt} \right\}.
\] (3.18)
Then, multiplying by \( \phi t \) on both sides of (3.18), and noting that on \( B_{2R}(p) \):
\( 0 < \phi \leq 1, \phi^2 \leq \phi \) and \( M > a \log D \), from (3.18) we deduce that
\[
Bt\lambda \geq -\frac{2C_1^{\frac{1}{2}} F^{\frac{3}{2}} h^{\frac{1}{2}}}{R} - \lambda - [a]^+ t\lambda - \lambda t^2 h [- (A - 2K - 2 - |\log D|)]^+ \\
+ \frac{(1 + ht)^2 \lambda^2}{2n} - \frac{2t(1 + ht)(M - a \log D) \lambda}{n},
\]
i.e.,
\[
Bt \geq -\frac{2C_1^{\frac{1}{2}} F^{\frac{3}{2}} h^{\frac{1}{2}}}{R} - 1 - [a]^+ t - t^2 h [- (A - 2K - 2 - |\log D|)]^+ \\
+ \frac{(1 + ht)^2 \lambda}{2n} - \frac{2t(1 + ht)(M - a \log D)}{n}.
\]
Hence, there holds
\[
\frac{(1 + ht)^2 \lambda}{2n} \leq Bt + \frac{2C_1^{\frac{1}{2}} F^{\frac{3}{2}} h^{\frac{1}{2}}}{R} + 1 + [a]^+ t + t^2 h [- (A - 2K - 2 - |\log D|)]^+ \\
+ \frac{2t(1 + ht)(M - a \log D)}{n}.
\]
By virtue of Young’s inequality we have
\[
\frac{2C_1^{\frac{1}{2}} F^{\frac{3}{2}} h^{\frac{1}{2}}}{R} \leq \frac{\lambda(1 + ht)^2}{4n} + \frac{4nC_1^2 h t^2}{R^2 (1 + ht)^2}.
\]
Consequently, we derive
\[
\frac{(1 + ht)^2}{4n} \lambda \leq Bt + 1 + [a]^+ t + \frac{4nC_1^2h^2}{R^2(1 + ht)^2} + t^2h \left[-(A - 2K - 2 - |\log D|)^+ \right] + \frac{2t(1 + ht)(M - a \log D)}{n}.
\]

It turns out that there holds
\[
\lambda \leq \frac{4n}{(1 + ht)^2} \left\{ Bt + 1 + [a]^+ t + \frac{4nC_1^2h^2}{R^2(1 + ht)^2} + t^2h \left[-(A - 2K - 2 - |\log D|)^+ \right] + \frac{2t(1 + ht)(M - a \log D)}{n} \right\} \tag{3.19}
\]

Moreover, it is well known that the following claim holds: If \( x \geq 0 \), then, for integer \( n \geq 1 \) there holds that
\[(1 + x)^n \geq 1 + nx \geq nx.\]

Then, by virtue of the claim we have
\[
\frac{ht^2}{(1 + ht)^4} \leq \frac{t^2h}{4ht} = \frac{t}{4}, \tag{3.21}
\]
\[
\frac{ht^2}{(1 + ht)^2} \leq \frac{t^2h}{2ht} = \frac{t}{2}, \tag{3.22}
\]
\[
\frac{1 + ht}{(1 + ht)^2} \leq \frac{1}{1 + ht} \leq 1. \tag{3.23}
\]

Combining (3.19) to (3.23), on \( B_p(2R) \times (0, T] \) we derive
\[
\lambda \leq 4n \left\{ Bt + 1 + [a]^+ t + \frac{4nC_1^2t}{R^2} + \frac{t}{2} \left[-(A - 2K - 2 - |\log D|)^+ \right] \right\} + 4n \left\{ \frac{2t [M - a \log D]^+}{n} \right\}
\]
\[
\leq 4n \left\{ BT + 1 + [a]^+ T + \frac{4nC_1^2T}{R^2} + \frac{T}{2} \left[-(A - 2K - 2 - |\log D|)^+ \right] \right\} + 4n \left\{ \frac{2T [M - a \log D]^+}{n} \right\} \tag{3.24}
\]

Ultimately, it is obvious that the following expressions hold:
\[
F|_{B_p(r)} = \phi F|_{B_p(r)} = \lambda|_{B_p(r)} \leq \lambda(x_0, t_0). \tag{3.25}
\]

Therefore combining (3.14) and (3.24) and also noting that \( T > 0 \) is arbitrary, we obtain (1.7). Then we accomplish the proof of Theorem 1.2. \( \square \)
Remark 3.1. Now we can give some explanations about the previous Remark 1.1. If \( a \) is a constant, e.g. \( a \equiv A \), then from (3.13) we know that the arguments on Case 1 are completely unnecessary, so we do not have to assume \( u \) has an upper bound by letting \( D \equiv 1 \). Especially, when \( a \equiv 0 \), from (3.11) we know that it is even unnecessary to assume \( b \) is bounded, but in this situation, \( b \) will appear on the left side of (1.7) since \( M + b \geq 0 \) may not be true.

4 Discussions About Theorem 1.3

In this section, we apply our results in Theorems 1.1 and 1.2 to logarithmic Schrödinger equation (1.8). To this end, we begin to prove Theorem 1.3.

Proof of Theorem 1.3. (1) From the beginning of proof of (1) of Theorem 1.1, we know that all our requirements about \( M_1 \) are

\[
(A_1 + a)(M_1 - b) + (2 - 2A_1)M_1 + \frac{2(b - M_1)^2}{n} + 2KM_1 - \nabla a^2 - \nabla b^2 \geq 0
\]

and

\[
\frac{4(M_1 - b)A_1}{n} + (2K + 2)(A_1 + a) + \Delta a \geq 0.
\]

For (1.9), \( a \equiv 2, \, A_1 = 1, \, K = 0 \) and \( b(x) = V(x) \), then the above requirements become

\[
\begin{cases}
3(M_1 - V) + \frac{2(M_1 - V)^2}{n} - |\nabla V|^2 \geq 0, \\
\frac{4(M_1 - b)A_1}{n} + 6 \geq 0.
\end{cases}
\]

Specifically, we can take

\[
\begin{cases}
M_1 - V \geq \frac{1}{3} \sup_{\mathbb{R}^n} |\nabla V|^2, \\
M_1 - V \geq -\frac{3n}{2}.
\end{cases}
\]

Then we may choose \( M_1 = \sup_{\mathbb{R}^n} |V| + \frac{1}{3} \sup_{\mathbb{R}^n} |\nabla V|^2 \). By Theorem 1.1 (1), letting \( R \to +\infty \), we obtain

\[
3f \leq 16n + 8(M_1 - V)^+ = 16n + 8 \sup_{\mathbb{R}^n} |V| + \frac{8}{3} \sup_{\mathbb{R}^n} |\nabla V|^2 - 8V,
\]

thus

\[
u \leq e^{\frac{1}{3} \left( 16n + 8 \sup_{\mathbb{R}^n} |V| + \frac{8}{3} \sup_{\mathbb{R}^n} |\nabla V|^2 - 8V \right)}, \tag{4.1}
\]

i.e., if \( V(x), \, |\nabla V(x)| \) is bounded, then the positive solutions to equation (1.8) must be bounded. Especially, when \( V \geq 0 \) is a constant, from (4.1) we know that the upper bound of \( u \) is not related to \( V \).
(2) On the other hand, we can also apply Theorem 1.2 to this equation. In this situation, we may assume $u_t = 0$, and then we can let $T \to \infty$ and $R \to +\infty$ in (1.7). By Remark 3.1, if we take $A = a = 2$, $D = 1$, $b(x) = V(x)$, $K = 0$, then

$$M = \frac{1}{2} \sup_{\mathbb{R}^n} |\Delta V| + \frac{1}{4} \sup_{\mathbb{R}^n} |\nabla V| + \sup_{\mathbb{R}^n} |V|$$

ensures (3.11) to be nonnegative and $M + b \geq 0$, $M \geq 0$. By (1.7), we have

$$4f \leq 4n \left\{ 2 + \frac{2}{n} \left( \frac{1}{2} \sup_{\mathbb{R}^n} |\Delta V| + \frac{1}{4} \sup_{\mathbb{R}^n} |\nabla V| + \sup_{\mathbb{R}^n} |V| \right) \right\},$$

i.e.,

$$u \leq e^{\left\{ 2n + \sup_{\mathbb{R}^n} |\Delta V| + \frac{1}{2} \sup_{\mathbb{R}^n} |\nabla V| + 2 \sup_{\mathbb{R}^n} |V| \right\}}. \quad (4.2)$$

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