Breather solitons in highly nonlocal media

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Abstract

We investigate the breathing of optical spatial solitons in highly nonlocal media. We use a generalization of the Ehrenfest theorem (1990 Am. J. Phys. 58 742) leading to a fourth-order ordinary differential equation, the latter ruling the beam width evolution in propagation. In actual highly nonlocal materials, the original accessible soliton model by Snyder and Mitchell (1997 Science 276 1538) cannot accurately describe the dynamics of self-confined beams: the transverse size oscillations have a period which not only depends on power, but also on the initial width. Modeling the nonlinear response by a Poisson equation driven by the beam intensity we verify the theoretical results against numerical simulations.

Keywords: solitons, nonlocality, light localization, quantum harmonic oscillator

(Some figures may appear in colour only in the online journal)

1. Introduction

Since the invention of the laser, optics has played an important role in nonlinear physics. One of the most known phenomena in nonlinear optics is the all-optical Kerr effect or an intensity-dependent refractive index [1]. While in the simplest limit the change in refractive index depends on the local intensity value, in nonlocal media the nonlinear perturbation depends also on the intensity in neighbouring points. Nonlocality strongly affects light propagation, leading, for example, to the stabilization of fundamental bright (2+1)D spatial solitons [2, 3] higher-order and vector solitons [4–13] as well as spatio-temporal solitons [14], complex dynamics and long-range interactions of solitons [15–17] as well as between solitons and boundaries [18–20]. Optical nonlocality also entails the observation of fundamental phenomena, from soliton bistability [21] to spontaneous symmetry breaking [22], from turbulence to condensation [23], from irreversibility and shock waves [24–27] to gravity-like effects [28].

In general, even in the absence of losses, self-trapped beams in nonlocal media undergo variations in transverse size owing to a dynamic balance between self-focusing and diffractive spreading [29, 30]. Such behavior resembles the collective excitation phenomena in condensed matter, for example the collective modes in Bose–Einstein condensates where the center of mass or the condensate size in a harmonic trap undergoes oscillations [31, 32]. In optics, if the nonlinear index well depends only on the input power, nonlinear beam propagation can be described by a linear quantum harmonic oscillator and the breathing is purely periodic [33, 34]. In actual media, however, self-focusing also depends on the transverse profile of the beam [30]. It was numerically shown that soliton breathing remains periodic in a (1+1)D simplified model, connecting this result with the existence of a (quasi) parabolic potential well [35].

In this paper, we employ a generalization of the Ehrenfest theorem, the latter providing a set of ordinary differential equations ruling the moments evolution for a wave satisfying the Schrödinger equation [36]. We find that, when the wave is subject to a parabolic potential explicitly dependent on the propagation distance, the beam width dynamics is governed by a single fourth-order equation [37]. We then apply such equations to the investigation of spatial optical solitons in highly nonlocal media. In such a limit, as first demonstrated by Snyder and Mitchell [33] and later confirmed experimentally in nematic liquid crystals [5, 30] and thermo-optic media [16], the light-induced waveguide can be satisfactorily approximated with a parabola, allowing the usage of all the mathematical tools developed for the quantum harmonic oscillator [38]. This important result led to the coinage of the
term accessible solitons [33]. The original model for accessible solitons predicts a breathing period depending only on the input power. Here we demonstrate that—in real media showing a non-differentiable response function—both extrema and period of the oscillations strongly depend on the input beam width. More specifically, we show that the oscillations in the beam width are ruled by a Newton equation encompassing a conservative force. Thus an effective potential, the latter depending on both the input power and input width, can be introduced. This approach allows one to describe the beam dynamics with a semi-analytical formulation and rules out the presence of chaotic dynamics, at least in the highly nonlocal limit. The validity of the Snyder–Mitchell model in real materials was analysed in [39] with respect to the stationary states, whereas the dependence of the breathing period on the input beam width was briefly mentioned in [37]. Numerical simulations with reference to a nonlinear response modelled by a Poisson equation, the latter modelling of both nematic liquid crystals and thermo-optic materials [40], support our findings.

2. The Schrödinger equation in the Heisenberg picture

In the scalar approximation, in the harmonic regime and for small refractive index perturbations, the paraxial propagation of an optical wavepacket $\psi$ along $z$ is governed by

$$i \frac{\partial \psi}{\partial z} + \frac{1}{2\kappa_0n_0} \left( \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) + k_0 \Delta n(x, y, z) \psi = 0,$$  \hspace{1cm} (1)

where $k_0$ is the vacuum wave-number and $n_0$ is the refractive index of the unperturbed medium. The quantity $\Delta n$ is a space-dependent perturbation in the refractive index, of both linear or nonlinear origin. In a homogeneous nonlinear medium, the spatial dependence of $\Delta n$ is through $\psi$, i.e., $\Delta n[\psi(x, y, z)]$. Due to its formal equivalence to the Schrödinger equation [41], equation (1) states the equivalence between light propagation in space and temporal evolution of a quantum particle in a two-dimensional potential with $\hbar = 1$ and an effective mass $k_0n_0$; equation (1) can thus be analysed with the tools of quantum mechanics. We find (see appendix and [36] for details)

$$\frac{k_0n_0}{2} \frac{d^2 \langle x_j^2 \rangle}{dz^2} = \frac{\bar{p}_j}{k_0n_0} + k_0 \bar{\psi}_j \frac{\partial \Delta n}{\partial x_j},$$  \hspace{1cm} (2)

for the width of the wavepacket in the real space and

$$\frac{d^2 \bar{p}_j}{dz^2} = 2k_0 \left( \frac{\partial \Delta n}{\partial x_j} \right)^2 + k_0 \bar{\psi}_j \frac{d}{dz} \left( \frac{\partial \Delta n}{\partial x_j} \right) + k_0 \frac{d}{dz} \left( \frac{\partial \Delta n}{\partial x_j} \right) \bar{p}_j,$$  \hspace{1cm} (3)

for the width of the wavepacket expressed in the transformed space.

We want to stress that the results computed above hold valid even in the limit of a nonlinear response, that is, when the photonic potential $-k_0\Delta n$ depends on the wave profile $\psi$ itself. In fact, let us suppose that the solution of the Schrödinger equation (1) in the case of a nonlinear potential is known. Then, the nonlinear wave propagation is fully equivalent to a linear propagation in the presence of a field-dependent index well $\Delta n(x, y, z) = \Delta n[\psi(x, y, z)]$. Hence, the validity of equations (2) and (3) is extended to the nonlinear case. This type of a posteriori reasoning corresponds to the physical interpretation of a spatial soliton as the mode of its own self-written waveguide [42], and is routinely employed in nonlinear optics [7, 17, 18, 40].

3. Waves in a $z$-dependent parabolic potential

The beam trajectory obeys the Ehrenfest theorem, whereas the beam width can be obtained from two (generally vectorial) second-order ODEs with two unknowns: the width $\langle x_j^2 \rangle$ of its transverse profile and the width of its Fourier transform $\langle \bar{p}_j^2 \rangle$. The solution is not straightforward, as a complete knowledge of the profile $\psi(x, y, z)$ is needed to calculate the average refractive index well and its derivative [43]. In the general case, all momenta of $\psi$—i.e., $\langle x^n \rangle$ with $n \in \mathbb{N}$—are required to get the second momentum evolution with $z$. A substantial simplification applies when the index well is parabolic, that is, $\Delta n(x, y, z) = \frac{n_0^2}{2}(x^2 + y^2)$. In this case, the beam width is governed by the fourth-order ODE [37]

$$\frac{n_0}{2} \frac{d^4 \langle x_j^2 \rangle}{dz^4} - 2a \frac{d^2 \langle x_j^2 \rangle}{dz^2} - \frac{3a}{dz} \frac{d \langle x_j^2 \rangle}{dz} - \frac{d^2 a}{dz} \langle x_j^2 \rangle = 0,$$  \hspace{1cm} (4)

Equation (4) must be solved with initial conditions on the beam width $\xi_0 = \langle x_j^2 \rangle \mid_{z=0} = \langle x_j^2 \rangle \mid_{z=0}$, its initial variation $\xi_1 = \frac{d\langle x_j^2 \rangle}{dz} \mid_{z=0} = \frac{d\langle x_j^2 \rangle}{dz} \mid_{z=0}$ (vanishing in the presence of a flat phase profile), its convexity $\xi_2 = \frac{d^2\langle x_j^2 \rangle}{dz^2} \mid_{z=0} = \frac{d^2\langle x_j^2 \rangle}{dz^2} \mid_{z=0}$, as well as $\xi_3 = \frac{d^3\langle x_j^2 \rangle}{dz^3} \mid_{z=0} = \frac{d^3\langle x_j^2 \rangle}{dz^3} \mid_{z=0}$ (prime indicates derivative with respect to $z$). The quantity $\xi_2$ determines the initial diffraction of the beam, with spreading depending on both the intensity profile (i.e., $\langle x_j^2 \rangle$) and the phase distribution (i.e., $\langle \bar{p}_j^2 \rangle$). In free space (where $a = 0$ everywhere) equation (4) reduces to $\frac{d^4\langle x_j^2 \rangle}{dz^4} = 0$. Since for a real Gaussian beam of radial waist $w$ (i.e. $I(x, y) = I_0 e^{-2(x^2+y^2)/w^2}$), it is $\langle \bar{p}_j^2 \rangle = \frac{I_0}{w^2}$ consistently with diffraction we find $w^2(z) = w_0^2 + \frac{4\pi^2}{k_0n_0 w_0^2}$, where $w^2 = \frac{1}{4} \langle x_j^2 \rangle$ is valid whenever the intensity profile is Gaussian.
4. Self-trapped nonlinear waves in highly nonlocal media

Equation (4) is valid whenever the refractive index well is parabolic, in both linear ($\alpha(z)$ independent of excitation) and nonlinear ($\alpha(z)$ depending on the wavepacket profile and amplitude) regimes [44]. In the highly nonlocal limit the light-induced index well is much wider than the beam [33]; the photonic potential can be Taylor-expanded to the second-order and equation (4) accurately models light propagation.

We solve it in Kerr media (reductive index dependent on intensity $I = n_0 |\psi|^2/(2\varepsilon_0)$ with $\varepsilon_0$ the vacuum impedance) with reference to two common responses: Gaussian [33, 45] and diffusive-like [28]. The two responses differ for the Green function $G$ linking the beam intensity $I$ to the nonlinear perturbation $\Delta n = \int \int I(x',y')G(x-x',y-y')dx'dy'$, with $\int \int Gdx'dy' = 1$. Hereafter, for the sake of simplicity we refer to (2+1)D structures with cylindrical symmetry or (1+1)D geometries.

4.1. Ideal limit: differentiable Green function

When the Green function $G$ is twice differentiable in the origin, it is easy to obtain $\Delta n \approx (G_0 + 2G_2(x^2)y^2)P + G_2P(x^2 + y^2)$, with coefficients $G_m = \frac{1}{m!} \frac{\partial^m G}{\partial z^m}$ where $G_m$ is computed in the origin. This exactly matches the Snyder–Mitchell model, with a nonlinear response exclusively dependent on input power $P$ [33]. The term $(G_0 + 2G_2(x^2)y^2)P$ is $x$-independent, but it varies along $z$ through the local beam width $\sqrt{\int |\psi|^2 dx}$, yielding an overall power-dependent phase shift of the beam [46]. The nonlinear lens, modelled by the transverse term proportional to $(x^2 + y^2)$, is constant with $z$ because the strength $\alpha = 2G_2P$ of the quantum harmonic oscillator is invariant across the sample. For planar phase fronts at the input, the beam breathing along $z$ follows [33]

$$w^2 = \frac{w_0^2 + w_0^2(P)}{2w_0^2} + \frac{w_0^4 - w_0^4(P)}{2w_0^2} \cos \left( \frac{8|G_2|P}{n_0 z} \right),$$  

where $w_0(P) = \left( \frac{2}{mk_0^2 |G_2| P} \right)^{1/4}$ is the radial waist of the soliton at power $P$. According to (5), self-confined beams oscillate around $w_{sv} = \sqrt{\frac{w_0^2 + w_0^2(P)}{2w_0^2}}$ and are affected by both input power $P$ and width $w_0$. The breathing period is proportional to $1/\sqrt{P}$, whereas the breathing amplitude increases with the difference $|w_0 - w_s|$. Figure 1 compares the predictions of (5) with BPM (beam propagation method) simulations in (1+1)D (with power $P$ replaced by a power density $P$ in Wm⁻¹) taking a Gaussian response $(w_0 \sqrt{\pi}^{-1}e^{-x^2/n_0^2})$, confirming that the Snyder–Mitchell model is valid for differentiable responses $G$ in the highly nonlocal limit [46].

4.2. Real case: singular Green function

Since actual highly nonlocal materials obey a diffusion-like equation with a Green function non-differentiable in the origin, this leads to discrepancies and quantitative inaccuracies when they are described by the original Snyder–Mitchell model [39, 46]. For instance, in thermo-optic materials or nematic liquid crystals in the perturbative regime [47], the nonlinear index well in the perturbation regime stems from a Poisson equation (we neglect the $\Delta n$ derivative along $z$ for simplicity)

$$\nabla^2 \Delta n + n_2 I = 0,$$

with the factor $n_2$ ($n_2 > 0$ for self-focusing) an equivalent nonlocal Kerr coefficient accounting for the ratio between the beam amplitude and the corresponding index perturbation. From (6), the nonlinear index well for an arbitrary beam profile can be Taylor expanded as $\Delta n \approx \Delta n_0 - \frac{n_2 I_0}{4} (x^2 + y^2)$, with $I_0$ denoting the intensity in the origin [30]. For a Gaussian beam it is $I_0 = 2P/(\pi w^2)$, thus

$$a = \frac{n_2 P}{\pi w^2} = \frac{n_2 P}{4\pi \langle x^2 \rangle}.$$

Equation (7) states that the strength of the self-induced quantum harmonic oscillator continuously changes along $z$, i.e., it depends on the evolution of the beam shape. The original Snyder–Mitchell model is recovered by neglecting variations of $\langle x^2 \rangle$ with $z$: light propagation is then governed by a linear quantum harmonic oscillator with a strength dictated by the input power $P$ [33]. Finally, we note that the
insertion of (7) into (4) provides a nonlinear equation in the beam width \( w \), and is generally harder to solve than in the linear case; for example, in [48] an analytic solution for the whole wavefunction was calculated for a periodic \( \lambda \).

Substitution of (7) in equation (4) provides

\[
\frac{d^2}{dz^2} \left( \frac{d^2}{dz^2} \langle x^2 \rangle \right) + \frac{n_2 P}{2 \pi n_0} \ln \langle x^2 \rangle = 0.
\]

Equation (8) shows that the fourth-order ODE equation (4) turns into a second-order ODE when the medium nonlinearity is governed by equation (7). Applying the proper boundary conditions we find

\[
\frac{d^2}{dz^2} \langle x^2 \rangle + \frac{n_2 P}{2 \pi n_0} \ln \langle x^2 \rangle + \frac{2}{k_0^2 n_0^2} \left( \frac{1}{w_s^2} - \frac{1}{w_0^2} \right) = 0,
\]

with the width of the shape-preserving soliton now given by

\[
w_s(P) = \left( \frac{4\pi}{n_0 n_2 k_0^2 P} \right)^{1/2}.
\]

As expected, in the linear regime \( \frac{d}{dz^2} \langle x^2 \rangle = \frac{2}{k_0^2 n_0^2} \) [43]; when \( w_0 = w_s(P) \) it is \( \frac{d}{dz^2} \langle x^2 \rangle = 0 \), i.e., a shape-preserving soliton is excited.

Equation (9) corresponds to the motion of a classical particle subject to a conservative force

\[
F = -\frac{n_2 P}{2 \pi n_0} \ln \langle x^2 \rangle + \frac{2}{k_0^2 n_0^2} \left( \frac{1}{w_s^2} - \frac{1}{w_0^2} \right).
\]

with the latter depending on both the normalized excitation \( n_2 P \) and the initial width \( w_0 \). The force \( F \) vanishes when \( \langle x^2 \rangle = \langle x^2 \rangle_{av} = \langle x^2 \rangle_0 \exp(w_s^2/w_0^2 - 1) \). Then equation (9)

\[
\frac{d^2}{dz^2} \langle x^2 \rangle + \frac{n_2 P}{2 \pi n_0} \ln \langle x^2 \rangle_{av} = 0,
\]

with an effective power-dependent potential acting on the beam width

\[
V(\langle x^2 \rangle, P, w_0) = \frac{n_2 P}{2 \pi n_0} \langle x^2 \rangle \left( \ln \langle x^2 \rangle_{av} - 1 \right).
\]

The possibility to describe the dynamics of the beam width, \( w \), as a classical particle subject to a conservative force means that the system is integrable. Integrability rules out the possibility of chaotic dynamics, at least in the highly nonlocal limit. This is a quite counter-intuitive result if we look at the initial equation (4). In fact, equation (4) would suggest that light self-changes its own breathing period through the nonlocal nonlinearity. Thus, a continuous generation of new oscillation frequencies along \( z \) would be expected, eventually leading to a strongly aperiodic behavior, or even chaos. This is what occurs in [49], where a linear index well \( \Delta n \), periodic along \( z \), yields wave chaos in the linear regime. Physical intuition would lead to predict an even stronger chaotic behavior in the nonlinear case due to the additional degree of freedom: conversely and quite surprisingly our model rules out chaos. In section 4.3 we check the validity of this conclusion via numerical simulations of the complete model, i.e., without the highly nonlocal approximation.

Figure 2(a) illustrates the potential, \( V \), asymmetric with respect to the local minimum \( \langle x^2 \rangle_{av} \), and therefore sustaining non-sinusoidal oscillations of the momentum \( \langle x^2 \rangle \). Such dynamics is confirmed by direct numerical integration of equation (9), as plotted in figure 2(b). Integrating the energy conservation law over one half-period yields the breathing
period

$$\Lambda = 2 \int_{x_0}^{\langle x^2 \rangle} \frac{d\langle x^2 \rangle}{\sqrt{2[V(\langle x^2 \rangle) - V(\langle x^2 \rangle)]}},$$

where $\langle x^2 \rangle_F$ is the extremum opposite to the initial value during one single oscillation. Results from (14) are graphed in figure 2(c) together with the direct numerical integration of equation (9) (corresponding to the beam width graphed in figure 2(b)): the match is perfect within the numerical accuracy. At variance with equation (5), the oscillation period $\Lambda$ depends not only on input power $P$ but also on input width $w_0$. In particular, $\Lambda$ has a local minimum

$$\Lambda_{\text{min}} = \sqrt{\frac{4\omega P w_0^3}{n_2 P}} = \frac{4\omega P w_0^3}{n_2 P},$$

when the input beam matches the shape-preserving soliton, i.e., $w_0 = w_5$, where small departures from $w_5$ cause the breathing to approximately follow equation (5) as the strength $\alpha$ of the quantum harmonic oscillator (7) undergoes small variations along $z$ [30]. For input beams narrower than the exact soliton ($w_0 < w_5(P)$) the period increases sharply due to the diffraction limit (beam size comparable with wavelength); when $w_0 > w_5(P)$ the period grows linearly with $w_0$. The location $w_{0,\nu} = 2\sqrt{\langle x^2 \rangle_{\nu}}$ of the minimum effective potential follows a trend similar to $\Lambda$ versus $w_0$ (inset of figure 2(c)), asymptotically tending to a straight line with slope $w_0/\sqrt{\sigma}$ for $w_0 \gg w_5$. Finally, figure 2(d) plots the maximum and minimum beam widths versus $w_0$ for $w_0 < w_3$ the initial width $w_0$ is the minimum (diffraction overcoming self-focusing at the input); conversely, when $w_0 > w_5$, $w_0$ is the maximum (self-focusing dominating over diffraction at the input).

4.3. Full numerical simulations in a Poisson material

To check our predictions we integrated equations (1) and (6) in a radially symmetric geometry, using a standard BPM in log-polar coordinates [37, 39]. The results for a given input power corresponding to a soliton of width $w_3 \approx 3 \mu m$ are summarized in figure 3. Given that the intensity profile overlaps with higher polynomial terms of the self-induced potential [39, 50], the soliton existence condition (10) now becomes:

$$w_5(P) = \left( \frac{s_\infty}{\omega P_0^2 n^2 P} \right)^{1/2},$$

i.e., the width of the actual (i.e., solution of equations (1) and (6)) shape-preserving soliton is $\sqrt{2}$ times larger than the theoretical prediction (10) for a fixed $n_2 P$, or alternatively, for a fixed $w_5$ the normalized input power $n_2 P$ has to be doubled with respect to the theoretical value provided by (10). Therefore, the normalized power $n_2 P$ corresponding to the shape-preserving soliton of equations (1) and (6) featuring $w_3 = 3 \mu m$ is $n_2 P/(4\pi) \approx 0.006$. We will use this value for $n_2 P$ in all the simulations reported hereafter. The intensity evolution shows a periodic to aperiodic transition for varying input widths. The case $w_0 = 3 \mu m$ does not exactly excite a shape-preserving soliton because in real Poisson media, the exact soliton profile slightly differs from a Gaussian profile [39, 50].

We start analyzing the wavepacket behavior when close to the input, i.e., for short propagation lengths. In the interval $2.5 \mu m < w_0 < 5 \mu m$ the excitation is close enough to the soliton state (i.e., $w_3 = 3 \mu m$ for the chosen power) and the self-trapped beam oscillates quasi-periodically (see figures 3 and 4(a)). Figure 4(b) shows the first oscillation period, the latter being computed doubling the position of the first local extremum in the beam width versus $z$. In agreement with the
theory, the oscillation period depends on $w_0$, where $\Lambda$ is shorter when $w_0 \approx w_S(P)$. The numerical results quite closely resemble the predictions from equation (14), with quantitative discrepancies arising when the input beam is much wider than the soliton (see figure 4(b)). As visible in figure 4(a), when the propagation distances are longer, the wavepacket evolution departs from the theory. Also, when the difference $|w_0 - w_S|$ is small, the oscillation period slightly varies along $z$; conversely, for very narrow ($w_0 \ll w_S(P)$) and very wide ($w_0 \gg w_S(P)$) inputs, the oscillations become markedly aperiodic. The discrepancy can be ascribed to two main causes, (i) the effective shape of the self-induced index well is not perfectly parabolic, as discussed above; (ii) the beam shape strongly departs from Gaussian due to the nonlinear interaction between a large number of modes; this in turn, breaks the validity of equation (7) and leads to the relationship between $a$ and $\langle x^2 \rangle$ requiring a more involved approach.

The general trends with $w_0$ can be confirmed by computing light propagation over longer distances. The results in figure 4(c) show soliton breathing over a propagation length of 5 mm. First, the evolution smoothly changes with $w_0$, ruling out the presence of chaotic dynamics [37, 49]. Second, the yellow portions in figure 4(c) (bottom and top) correspond to strongly aperiodic dynamics. Between them, in the center of the panel, the dynamics is quasi-periodic with a comb-like structure where each tooth is tilted towards the left (smaller $z$), showing that the oscillation period changes and tends to a minimum when $w_0 = w_S$. Consistently with the theory, the oscillation amplitude is proportional to $|w_0 - w_S|$. In addition, the oscillation amplitude unexpectedly drops with $z$ due to an effective dissipation (in the framework of the effective potential defined via equation (13)) arising from the nonlinear interaction between the modes of the structure, as modeled by the higher-order polynomial terms in the light induced index well.

Next we study beam breathing in the frequency domain. To carry out this analysis we use a wavelet transform, as the evolution is not periodic and extends over a finite domain. Wavelets allow to address the temporal fluctuations in the spectrum of a signal. Such goal is achieved by using a basis function, named the mother wavelet. Local components of the spectrum are found compressing or dilating the mother wavelet, the compression factor used to determine at which scale we are analyzing the signal. Here we choose the wavelet transform Daubechies db6 [51]. To avoid artifacts due to the boundaries, we limit our analysis in the interval 1 mm $< z < 4$ mm. For both $w_0 = 3 \mu$m (figure 4(d)) and $w_0 = 4 \mu$m (figure 4(e)), the peak of the wavelet transform does not move on the frequency axis with $z$. The wavelet transform is also strongly localized on the scale axis,
demonstrating that no diffusion effects occur in the frequency domain. The absolute value of the transform decreases with $z$, in line with the emerging dissipative mechanism described above. Comparing the two cases, it is evident that the spectral components are higher when $w_0 = 4 \mu m$ due to a larger oscillation amplitude. Noteworthy, for $w_0 = 3 \mu m$ the higher frequency components are much more relevant than for $w_0 = 4 \mu m$. In fact, in the former case, the oscillation around the average value is heavily affected by the non-Gaussian profile of the soliton [50], a contribution neglected in deriving equations (9) and (13). This is evident in figure 4(f) showing the average period in the range $1 \text{ mm} < z < 4 \text{ mm}$ is computed from the wavelet transform. The shape is very close to figure 4(b), except near $w_0 = w_S$ where a spurious peak appears. Physically, close to the input, the action of the higher-order modes can be neglected; for long distances, their effect accumulates and can no longer be neglected.

5. Conclusions

In conclusion, we used a generalization of the Ehrenfest theorem to derive a general equation ruling the nonlinear evolution of the beam width in a parabolic index well [37]. Applying this model to light propagation in highly nonlocal media, we investigated how soliton breathing deforms in real materials (both qualitatively and quantitatively) from the ideal Snyder–Mitchell law. In particular, we showed that the beam width dynamics can be modelled as a classical particle subject to a potential which depends on the power and the width of the input beam. Thus, although the beam itself introduces a longitudinal change in the index well [49], remarkably no aperiodic or chaotic evolution [37] takes place within the validity of our model. Numerical simulations verify that the breathing period depends on the width of the input beam and confirm the absence of chaos. Moreover, the simulations indicate the emergence of novel and intriguing effects due to the nonlinear interaction of several modes, assessing the suitability of nonlocal nonlinear optics for the investigation of many-body physics [52, 53].

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Appendix. Generalization of the Ehrenfest theorem

In the Heisenberg picture, an operator $\hat{A}$ evolves in space $z$ (or time in quantum mechanics) according to [38]

$$\frac{d\hat{A}}{dz} = i[\hat{A}, \hat{H}] + \frac{\partial \hat{A}}{\partial z},$$

(A.1)

where $\hat{H} = \hat{p}^2 + \hat{r}^2 + k_0 \hat{n}$ is the effective Hamiltonian operator and the square brackets indicate the commutator $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. In the definition of the effective Hamiltonian $\hat{H}$, the quadratic term in the operator $\hat{p}_j = -i\partial / \partial \eta_j (\eta = x, y)$ and the term $-k_0 \hat{n}$ correspond to the effective kinetic energy and the photonic potential, respectively. Applying equation (A.1) to the spatial operator $\hat{x}$ and the momentum operator $\hat{p}$ leads to the Ehrenfest’s theorem [38]

$$n_0 \frac{d^2 \hat{x}_j}{dz^2} = -\frac{\partial \hat{n}}{\partial \eta_j} (j = 1, 2),$$

(A.2)

with $x_1 = x$ and $x_2 = y$. The Ehrenfest theorem has been successfully applied in optics to derive the trajectory of finite-size beams [54] and the interaction of multiple filaments [17]. Here, we aim to extend it and derive an ODE governing the beam width in the general case when the photonic potential is $z$-dependent. The beam width is related with the operator $\hat{x} = x^2$, Equation (A.1) then yields [36]

$$k_0 n_0 \frac{d\hat{x}_j}{dz} = \hat{p}_j \hat{x}_j + \hat{x}_j \hat{p}_j.$$

(A.3)

A further derivation with respect to the propagation distance $z$ of (A.3) provides equation (2) in the main text.

Similarly, taking $\hat{A} = \hat{p}^2$ we find for the momentum [36]

$$\frac{d\hat{p}^2}{dz} = k_0 \left( \frac{\partial \hat{n}}{\partial \eta_j} \hat{p}_j + \hat{p}_j \frac{\partial \hat{n}}{\partial \eta_j} \right),$$

(A.4)

Equation (3) in the main text can then be found by deriving (A.4) with respect to $z$. We stress that equation (3) is more general than equation (11) in [36]. In fact, we also account for explicit dependencies on $z$ of the photonic potential $-k_0 \hat{n}$, as in the case of highly nonlocal solitons (see equation (7)), neglecting this would yield to a wrong expression for equation (4) in the main text.

The advantage of the Heisenberg picture is that the generic bra $\langle \psi |$ and ket $| \psi \rangle$ are stationary [38] (invariant with $z$ in optics), thus all the equations dealing with operators hold valid for the average values $\langle \hat{A} \rangle_\psi = \langle \psi | \hat{A} | \psi \rangle \left( \int |\psi|^2 d\eta d\psi \right)^{-1}$, as well. For conciseness, in the main text we omitted the subscript $\psi$ when referring to average quantities related with the wave $\psi$.

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