Curvature perturbations from ekpyrotic collapse with multiple fields

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Abstract

A scale-invariant spectrum of isocurvature perturbations is generated during collapse in the ekpyrotic scaling solution in models where multiple fields have steep negative exponential potentials. The scale invariance of the spectrum is realized by a tachyonic instability in the isocurvature field. This instability drives the scaling solution to the late-time attractor that is the old ekpyrotic collapse dominated by a single field. We show that the transition from the scaling solution to the single-field-dominated ekpyrotic collapse automatically converts the initial isocurvature perturbations about the scaling solution to comoving curvature perturbations about the late-time attractor. The final amplitude of the comoving curvature perturbation is determined by the Hubble scale at the transition.

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1. Introduction

The existence of an almost scale-invariant spectrum of primordial curvature perturbations on large scales is one of the most important observations that any model for the early universe should explain. An inflationary expansion in the very early universe is most commonly assumed to achieve this, but it is important to consider whether there is any alternative model. In this paper, we focus on the ekpyrotic scenario as an alternative [1] (see also [2, 3]). In the old ekpyrotic scenario, the large-scale perturbations are supposed to be generated during a collapse driven by a single scalar field with a steep negative exponential potential. It was shown that the Newtonian potential acquires a scale-invariant spectrum, but the comoving curvature perturbation has a steep blue spectrum [4]. In this scenario we need a mechanism...
to convert contraction to expansion, and for a regular four-dimensional bounce the scale-invariant Newtonian potential is matched to the decaying mode in an expanding universe, and the growing mode of curvature perturbations acquires a steep blue spectrum [5, 6]. It has been suggested that this conclusion might be altered by allowing a singular matching between collapse and expansion [7], but the general rule that the comoving curvature perturbation remains constant still holds for adiabatic perturbations on large scales [8]. In a braneworld context, the conversion from contraction to expansion might be accomplished by a collision of two branes where one of extra dimensions disappears [9]. It was argued that the scale-invariant Newtonian potential can be transferred to the comoving curvature perturbations by this singular bounce [10]. However, without having a concrete theory to describe the singularity, it is difficult to have a definite conclusion on how perturbations pass through the singularity.

Recently, there has been some progress in generating a scale-invariant spectrum for curvature perturbations in the ekpyrotic scenario [11–13], by considering non-adiabatic perturbations which have been suggested previously by [15]. In this case, we require two or more fields. If these have steep exponential potentials then there exists a scaling solution where the energy densities of the fields grow at the same rate during collapse [16, 17]. The isocurvature perturbations then have a scale-invariant spectrum [16]. These isocurvature perturbations can be converted to curvature perturbations if there is a sharp turn in the trajectory in field space [11–13]. For example, a situation where one of the fields changes its direction in field space has been considered in [11], which corresponds to a time when a negative tension brane is reflected by a curvature singularity in the bulk, in the context of the heterotic $M$-theory. A regular bounce realized by a ghost condensate has been considered in [12, 13]. One of the fields exits the ekpyrotic phase and hits the transition to the ghost condensate phase that creates a sharp turn in the trajectory in field space and curvature perturbations can be generated [12]. It is still necessary to match curvature perturbations in a contracting phase to those in an expanding universe, but it is shown that the comoving curvature perturbation is conserved on large scales resulting in an almost scale-invariant spectrum observed today for a regular bounce like a ghost condensate model [12].

The isocurvature perturbations behave like $\delta s \propto H$ on large scales. As the Hubble parameter is rapidly increasing in a collapsing universe, this signals an instability. In fact, it is easy to see that we always require an instability of this form in order to generate a scale-invariant spectrum with a canonical scalar field during collapse\(^3\). As the amplitude of field perturbations at Hubble exit is of order $H$, we require the super-Hubble perturbations to grow at the same rate to maintain a scale-invariant spectrum. This instability in a phase space analysis has been studied in [19]. The multi-field scaling solution was shown to be a saddle point in field space and the late-time attractor is the old ekpyrotic collapse dominated by a single field. A tachyonic instability drives the scaling solution towards the late-time attractor (see also [11, 12]).

In [19], we pointed out that the natural turning point in the field space trajectory due to the instability of the scaling solution might itself offer the possibility of converting the scale-invariant spectrum of isocurvature field perturbations into a scale-invariant spectrum of curvature perturbations. In this paper, we confirm this expectation by explicitly solving the evolution equations for perturbations in a two-field model. We find that the ratio between curvature perturbations and isocurvature perturbations at the final old ekpyrotic phase is solely

\(^3\) The only way to produce a scale-invariant spectrum without the presence of an instability seems to be due to non-canonical kinetic terms, as in the case of axion-type fields which can acquire scale-invariant perturbation spectra while remaining massless [18].
determined by the ratio of exponents of the two exponential potentials, and the amplitude is set by the Hubble rate at the transition time.

2. Homogeneous field dynamics

We first review the background dynamics of the fields. During the ekpyrotic collapse, the contraction of the universe is assumed to be described by a 4D Friedmann equation in the Einstein frame with scalar fields with negative exponential potentials:

\[ 3H^2 = V + \frac{1}{2} \dot{\phi}^2, \]  

where

\[ V = - \sum_i V_i e^{-c_i \phi_i}, \]  

and we take \( V_i > 0 \) and set \( 8\pi G \) equal to unity.

The authors of [11] found a scaling solution (previously studied in [16, 17]) in which both fields roll down their potential as the universe approaches a big crunch singularity. In this ekpyrotic scaling collapse, we find a power-law solution for the scale factor

\[ a \propto (-t)^p, \quad \text{where} \quad p = \sum_i \frac{2}{c_i^2} < \frac{1}{3}, \]  

where

\[ \frac{\dot{\phi}_i^2}{\dot{\phi}_j^2} = \frac{-V_i e^{-c_i \phi_i} - c_j^2}{-V_j e^{-c_j \phi_j}} = \frac{c_j^2}{c_i^2}. \]  

As we will see in the following section, it is possible to generate scale-invariant isocurvature perturbations around this background. However, the ekpyrotic scaling solution (4) is unstable.

In addition to the scaling solution we have fixed points corresponding to any one of the original fields \( \phi_i \) dominating the energy density where the other fields have negligible energy density. These correspond to the original ekpyrotic power-law solutions where

\[ a \propto (-t)^{p_i}, \quad \text{where} \quad p_i = \frac{2}{c_i^2}, \]  

for \( c_i^2 > 6 \). We find that any of these single-field-dominated solutions is a stable local attractor at late times during collapse.

In [19], the ekpyrotic scaling solution (4) was shown to be a saddle point in the phase space. We briefly review the phase space analysis. Introducing phase space variables [17, 20, 21]

\[ x_i = \frac{\dot{\phi}_i}{\sqrt{6} H}, \]  

\[ y_i = \sqrt{\frac{V_i}{6}} e^{-c_i \phi_i} \sqrt{3} H, \]  

the first-order evolution equations for the phase space variables are given by

\[ \frac{dx_i}{dN} = -3x_i \left( 1 - \sum_j x_j^2 \right) - c_i \left( \frac{3}{2} y_j^2 \right), \]
Figure 1. Left: numerical solutions for $x_1(N)$. The horizontal axis is $N = \log a$ and we take $c_1 = 40$ and $c_2 = 30$. The initial time is $N = 0.05$. Note that $N$ decreases towards the future in a collapsing universe. Right: the corresponding phase space trajectories in the $(x_1, x_2)$-plane.

$$\frac{dy_i}{dN} = y_i \left( 3 \sum x_j^2 - c_i \sqrt{\frac{3}{2}} x_i \right),$$

(9)

where $N = \log a$. The Friedmann equation gives a constraint

$$\sum_j x_j^2 - \sum_j y_j^2 = 1.$$

(10)

There are $(n + 2)$ fixed points of the system where $dx_i/dN = dy_i/dN = 0$:

- **A:** $\sum_j x_j^2 = 1, \quad y_j = 0$.

- **B_i:** $x_i = \frac{c_i}{\sqrt{6}}, \quad y_i = -\sqrt{\frac{c_i^2}{6} - 1}, \quad x_j = y_j = 0$ (for $j \neq i$).

- **B:** $x_j = \frac{1}{\sqrt{6} \sqrt{p}} c_j, \quad y_j = -\sqrt{\frac{2}{c_j p}} \left( \frac{1}{3p} - 1 \right)$.

(12)

(13)

In this paper, we focus on the fixed points $B$ and $B_i$ assuming $c_i^2 > 6$ and $\sum c_j^{-2} < 1/6$. The linearized analysis shows that the multi-field scaling solution, $B$, always has one unstable mode. On the other hand, the single-field-dominated fixed points, $B_i$, are always stable.

From now on we concentrate our attention on the two-field case. Then we have three fixed points $B, B_1$ and $B_2$. It is interesting to note that in the $(x_1, x_2)$-plane, the fixed points $B, B_1$ and $B_2$ are connected by a straight line, which is given by

$$c_2 x_1 + c_1 x_2 = \frac{c_1 c_2}{\sqrt{6}}.$$

(14)

The eigenvector associated with the unstable mode around the scaling solution $B$ lies in the same direction as the line (14). Thus this is an attractor trajectory, which all solutions near $B$ approach. Figure 1 shows numerical solutions for the evolution of $x_1$. Initial positions in the phase space are perturbed away from $B$ along the line (14). The solutions go to $B_1$ or $B_2$, depending on the initial position in the phase space.

An important observation is that, as we follow phase space trajectories during the transition from the scaling solution $B$ to the attractor solutions $B_1$ or $B_2$, the solutions obey relation (14),
even far away from the saddle point, B. Using the Friedmann equation and the field equations, we can show that
\[
\left( H - \frac{\dot{\phi}_1}{c_1} - \frac{\dot{\phi}_2}{c_2} \right) + 3H \left( H - \frac{\dot{\phi}_1}{c_1} - \frac{\dot{\phi}_2}{c_2} \right) = 0, \quad (15)
\]
and hence
\[
H - \frac{\dot{\phi}_1}{c_1} - \frac{\dot{\phi}_2}{c_2} = \frac{C}{a^3}, \quad (16)
\]
where \( C \) is an integration constant. In terms of \( \phi_1 \) and \( \phi_2 \), equation (14) can be rewritten as
\[
\dot{\phi}_1 + \frac{\dot{\phi}_2}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}} = H, \quad (17)
\]
and hence we see that for trajectories starting from point B we have \( C = 0 \), which is the late-time attractor. Thus we see that equation (17) holds even during the transition caused by the tachyonic instability from point B to \( B_1 \) or \( B_2 \) and in the final single-field-dominated phase. This fact will be important when we study perturbations.

We will study the behaviour of perturbations during the transition in the following section.

3. Generation of quantum fluctuations

In this section, we consider inhomogeneous linear perturbations around the background solution. We consider the scalar field perturbations on spatially flat hypersurfaces. Then, the scalar field perturbations are given by [22–25]
\[
\delta \ddot{\phi} + 3H \delta \dot{\phi} + \frac{k^2}{a^2} \delta \phi - c_1^2 V_i \exp(-c_i \phi) \delta \phi_i - \sum_j \frac{1}{a^3} \left( \frac{a^3}{H} \phi_i \phi_j \right) \delta \phi_j = 0. \quad (18)
\]
We can decompose the perturbations into the instantaneous adiabatic and entropy field perturbations as follows [23]:
\[
\delta r = \frac{\dot{\phi}_1 \delta \phi_1 + \dot{\phi}_2 \delta \phi_2}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}, \quad \delta s = \frac{\dot{\phi}_2 \delta \phi_1 - \dot{\phi}_1 \delta \phi_2}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}. \quad (19)
\]
The adiabatic field perturbation \( \delta r \) is the component of the two-field perturbation along the direction of the background fields’ evolution while the entropy perturbation \( \delta s \) represents fluctuations orthogonal to the background classical trajectory. The adiabatic field perturbation leads to a perturbation in the comoving curvature perturbation:
\[
R_c = \frac{H \delta r}{\dot{r}}, \quad (20)
\]
whereas the entropy field perturbations correspond to isocurvature perturbations.

Their evolution equations are given by [23]
\[
\ddot{r} + 3H \dot{r} + \frac{k^2}{a^2} \dot{r} + \left[ V_{,rr} - \dot{\theta}^2 - \frac{1}{a^3} \left( \frac{a^3}{H} \right) \right] \delta r = 2\dot{\theta} \delta s + 2 \left[ \dot{\theta} - \left( \frac{V_{,r}}{r \dot{r} + H} \right) \dot{\theta} \right] \delta s, \quad (21)
\]
\[
\ddot{s} + 3H \dot{s} + \frac{k^2}{a^2} \delta s + (V_{,rs} - \dot{\theta}^2) \delta s = -2 \dot{\theta} \left[ \dot{r} \delta r - \left( \frac{r^3}{2H} \right) \dot{r} \right], \quad (22)
\]
where the angle \( \theta \) is defined as
\[
\cos \theta = \frac{\dot{\phi}_2}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}, \quad \sin \theta = \frac{\dot{\phi}_1}{\sqrt{\dot{\phi}_1^2 + \dot{\phi}_2^2}}. \quad (23)
\]
such that
\[ \dot{r} = (\cos \theta)\dot{\phi}_2 + (\sin \theta)\dot{\phi}_1, \]
\[ \dot{\theta} = -\frac{V_s}{r}, \]
and
\[ V_r = (\sin \theta)c_1 V_1 \exp(-c_1\phi_1) + (\cos \theta)c_2 V_2 \exp(-c_2\phi_2), \]
\[ V_s = (\cos \theta)c_1 V_1 \exp(-c_1\phi_1) - (\sin \theta)c_2 V_2 \exp(-c_2\phi_2), \]
\[ V_{rr} = -(\sin \theta)^2 c_1^2 V_1 \exp(-c_1\phi_1) - (\cos \theta)^2 c_2^2 V_2 \exp(-c_2\phi_2), \]
\[ V_{ss} = -(\sin \theta)^2 c_1^2 V_1 \exp(-c_1\phi_1) - (\cos \theta)^2 c_2^2 V_2 \exp(-c_2\phi_2). \]

For the multi-field scaling solution, \( B \), we have
\[ \theta = \arctan \frac{c_2}{c_1}, \]
and for the single-field scaling solutions we have
\[ \theta = \frac{\pi}{2} (B_1), \quad \theta = 0 (B_2). \]

Thus we have \( \theta = \text{constant} \) for the fixed points and the adiabatic and entropy fields are decoupled. This allows us to quantize the independent fluctuations in the two fields.

For the multi-field scaling solution \( B \), the spectrum of quantum fluctuations of the entropy field is given on large scales \( (k \ll aH) \) by
\[ P_{s\delta} \equiv \frac{k^3}{2\pi^2} |\delta s|^2 = C_v^2 \frac{k^2}{a^2} (-k\tau)^{1-2\nu}, \]
where \( \tau < 0 \) is conformal time, \( a(\tau) \propto (-\tau)^{p/(1-p)} \), and
\[ \nu^2 = \frac{9}{4} - \frac{3\epsilon}{(\epsilon - 1)^2}, \quad \epsilon \equiv -\dot{H}/H^2 = 1/p, \]
and \( C_v = 2^{\nu-3/2}\Gamma(\nu)/\pi^{3/2} \) [16]. The spectral tilt is given by
\[ \Delta n_{s\chi} \simeq \frac{2}{\nu}, \]
to leading order in a fast-roll expansion \( (\epsilon \gg 1) \) [11–13]. In this limit, the spectrum (32) can be written as
\[ P_{s\delta}^{\nu/2} = \epsilon \left| \frac{H}{2\pi} \right|. \]
Note that \( |H(\tau)| \) is rapidly increasing and thus \( \delta s \) is also growing on super-Hubble scales due to the tachyonic instability. This instability is essential in order to realize the scale invariance of the spectrum (35). The amplitude of field perturbations at the Hubble exit is of order \( H \) and thus we require the super-Hubble perturbations to grow at the same rate in order to maintain a scale-invariant spectrum.

Note that in the simplest model the spectrum is slightly blue [11–13]. However, any deviations from an exponential potential for adiabatic field can introduce the corrections to the spectral tilt and thus it becomes model dependent [11, 12].
The spectrum of quantum fluctuations in the adiabatic field about the scaling solution has the same power-law form on large scales

\[ P_{\delta r} = C_\mu \frac{k^2}{a^2} (-k \tau)^{1-2\mu}, \]  

where \( \mu \simeq 1/2 \) to leading order in \( 1/\epsilon \). Thus, the adiabatic field perturbations become constant in the large-scale limit and the spectral tilt is given by

\[ \Delta n_{\delta r} \simeq 2. \]  

(37)

Thus, we have

\[ \frac{P_{\delta s}}{P_{\delta r}} \propto (-k \tau)^2, \]  

(38)

and hence in what follows we can neglect the adiabatic field fluctuations in the large-scale limit.

By contrast, for the single-field-dominated scaling solutions, both the adiabatic and entropy field perturbations are frozen on super-Hubble scales:

\[ \delta s, \delta r = \text{const.} \]  

(39)

These perturbations have a steep blue spectrum if they cross the horizon when the background solutions are described by the single-field-dominated solution [19].

4. Generation of curvature perturbations

Now let us consider the evolution of perturbations in a situation where the classical solution starts from near the saddle point, \( B \). As emphasized in [19] this requires an additional preceding mechanism that drives the classical background solution to the unstable saddle point throughout our observable part of the universe. In this paper, we will not discuss the mechanism required to bring the classical solution to the saddle point and we just assume that the classical solution stays near the saddle point for long enough to ensure that a scale-invariant spectrum of isocurvature perturbations is generated over the relevant scales for the observed large-scale structure of our universe.

Then, the initial conditions for the adiabatic and entropy field perturbations can be set from the amplitude of quantum fluctuation as described in the previous section:

\[ \delta r = 0, \quad \delta s = \epsilon \left| \frac{H}{2\pi} \right|, \]  

(40)

on sufficiently large scales and for \( 1/\epsilon \ll 1 \).

Unless the spatially homogeneous background solution is located exactly at the fixed point, the tachyonic instability drives the background solution away from the multi-field scaling solution, \( B \), to one of the single-field-dominated solutions, \( B_1 \) or \( B_2 \), depending on the initial conditions. During the transition \( \theta \) is not constant and the adiabatic and entropy field perturbations mix, so it is possible to generate perturbations in the adiabatic field, and hence comoving curvature perturbations (20), from initial fluctuations in the entropy field.

We can solve the evolution equations (21) and (22) numerically for any given classical background. Figure 2 shows the behaviour of \( \delta r \) and \( \delta s \). Due to the coupling between \( \delta r \) and \( \delta s \) during the transition, curvature perturbations are generated during the transition. On the other hand, \( \delta s \) shows a tachyonic instability according to equation (40) close to the scaling solution, but when the background solution goes to \( B_1 \), the entropy field perturbation becomes constant. The final amplitude of \( \delta r \) depends on when the transition from \( B \) to \( B_1 \) occurs, but,
Interestingly, the final ratio between \( \delta r \) and \( \delta s \) does not depend on the details of the transition. We find that the ratio is determined solely by the parameters \( c_1 \) and \( c_2 \) as

\[
\frac{\delta r}{\delta s} = \frac{c_1}{c_2}, \quad \text{at } B_1, \tag{41}
\]

\[
\frac{\delta r}{\delta s} = -\frac{c_2}{c_1}, \quad \text{at } B_2, \tag{42}
\]

as shown in figure 2. We will explain later why such a simple result is found.

The resulting curvature perturbation on a comoving hypersurface in the final single-field-dominated phase is thus given by

\[
\mathcal{R}_c = \frac{H}{\dot{\phi}_1} \delta r = \frac{1}{c_1} \delta r, \quad \text{at } B_1, \tag{43}
\]

\[
\mathcal{R}_c = \frac{H}{\dot{\phi}_2} \delta r = \frac{1}{c_2} \delta r, \quad \text{at } B_2. \tag{44}
\]

It is interesting to note that the equation for the entropy field perturbation (22) can be rewritten as [23]

\[
\ddot{\delta s} + 3H \dot{\delta s} + \left( \frac{k^2}{a^2} + V_{ss} + 3 \dot{\theta}^2 \right) \delta s = 4\frac{\dot{\theta}^2 k^2}{r a^2} \Psi, \tag{45}
\]

where \( \Psi \) is the curvature perturbations in the Newtonian gauge. The change of the curvature perturbations is determined by [23]

\[
\mathcal{R}_c = \frac{H k^2}{H a^2} \Psi + \frac{2H}{r} \dot{\theta} \delta s. \tag{46}
\]

In [11, 12], the curvature perturbation generated from the initial entropy perturbations is estimated by neglecting \( (k^2/a^2)\Psi \) on large scales, as can be done during slow-roll inflation [23]. However, this assumption is not necessarily justified in a collapsing universe where \( \Psi \) may grow rapidly on large scales. Even in a single-field inflation where \( \delta s = 0 \), there are cases where \( \mathcal{R}_c \) can change its amplitude on super-horizon scales [26]. In fact, in our case, we checked that the contribution from the \( \Psi \) terms in equations (45) and (46) cannot be neglected during the transition from scaling solution to the single-field-dominated solution.

Although the physical meaning of the instantaneous adiabatic and entropy field perturbations is clear, the dynamics of the perturbations during the transition are rather
complicated in this basis. We find it is much easier to work in terms of new variables [19]
\[ \varphi = \frac{c_2 \phi_1 + c_1 \phi_2}{\sqrt{c_1^2 + c_2^2}}, \quad \chi = \frac{c_1 \phi_1 - c_2 \phi_2}{\sqrt{c_1^2 + c_2^2}}, \] (47)
corresponding to a fixed rotation in field space. The potential equation (2) can then be simply
rewritten as [16, 19, 30]
\[ V = -U(\chi) e^{-c\varphi}, \] (48)
where
\[ \frac{1}{c^2} = \sum_i \frac{1}{c_i^2}, \] (49)
and the potential for the orthogonal field is given by
\[ -U(\chi) = -V_1 e^{-\frac{c_1}{c_2} c\chi} - V_2 e^{\frac{c_2}{c_1} c\chi}, \] (50)
which has a maximum at
\[ \chi = \chi_0 = \frac{1}{\sqrt{c_1^2 + c_2^2}} \ln \left( \frac{c_1^2 V_1}{c_2^2 V_2} \right). \] (51)
The multi-field scaling solution corresponds to \( \chi = \chi_0 \), while \( \varphi \) is rolling down the
exponential potential. The potential for \( \chi \) has a negative mass squared around \( \chi = \chi_0 \), and
thus \( \chi \) represents the instability direction. If the initial condition for \( \chi \) is slightly different
from \( \chi_0 \) or \( \dot{\chi} \) is not zero, then \( \chi \) starts rolling down the potential and the solution approaches
a single-field-dominated solution.
Note that perturbations \( \delta \varphi \) and \( \delta \chi \) coincide with the instantaneous adiabatic and entropy
field perturbations, respectively, defined in equation (19), at the scaling solution. B. Thus, we
use the initial perturbations (40) due to vacuum fluctuations about the scaling solution
previously calculated. However as we follow the evolution away from this saddle point we
no longer identify \( \varphi \) and \( \chi \) with the adiabatic and entropy perturbations. Nevertheless, the
dynamics of perturbations turns out to be much simpler using these fields.
In terms of \( \varphi \) and \( \chi \) the equations for perturbations are given by
\[ \delta \ddot{\varphi} + 3H \dot{\delta \varphi} + \frac{k^2}{a^2} \delta \varphi + M_{\varphi \varphi} \delta \varphi + M_{\varphi \chi} \delta \chi = 0, \] (52)
\[ \delta \ddot{\chi} + 3H \dot{\delta \chi} + \frac{k^2}{a^2} \delta \chi + M_{\chi \chi} \delta \chi + M_{\varphi \chi} \delta \varphi = 0, \] (53)
where
\[ M_{\varphi \varphi} = V_{\varphi \varphi} - \frac{1}{a^3} \left( \frac{a^3}{H} \dot{\varphi}^2 \right), \] (54)
\[ M_{\varphi \chi} = V_{\varphi \chi} - \frac{1}{a^3} \left( \frac{a^3}{H} \dot{\varphi} \dot{\chi} \right), \] (55)
\[ M_{\chi \chi} = V_{\chi \chi} - \frac{1}{a^3} \left( \frac{a^3}{H} \dot{\chi}^2 \right). \] (56)
A key observation is that the phase space trajectory of the background fields during the
transition, equation (17), can be rewritten as
\[ \frac{\dot{\varphi}}{H} = c. \] (57)
Thus even away from the multi-field scaling solution, $\phi$ obeys a scaling relation. We can then show that two of the effective mass terms become

\[ M_{\phi\phi} = V_{\phi\phi} + cV_{\phi} = 0, \]
\[ M_{\phi\chi} = V_{\phi\chi} + cV_{\chi} = 0, \]

independently of the form of $U(\chi)$ since $V \propto \exp(-c\phi)$.

Thus on large scales $\delta\phi$ is constant. As we take $\delta\phi = 0$ as our initial condition, this remains so even during the transition and in the final single-field-dominated phase. Thus from equation (47) we have a relation between $\delta\phi_1$ and $\delta\phi_2$,

\[ c_2\delta\phi_1 + c_1\delta\phi_2 = 0, \]

and $\delta\phi_1$ and $\delta\phi_2$ are given by

\[ \delta\phi_1 = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} \delta\chi, \]
\[ \delta\phi_2 = -\frac{c_2}{\sqrt{c_1^2 + c_2^2}} \delta\chi. \]

In the single-field-dominated solutions, $\phi$ is no longer the adiabatic field. The adiabatic field is simply $\phi_1$ (or $\phi_2$) in $B_1$ (or $B_2$) and the entropy field is $-\phi_2$ (or $\phi_1$) in $B_1$ (or $B_2$). Thus at the late-time attractor we have

\[ \delta r = \delta\phi_1, \quad \delta s = -\delta\phi_2, \quad \text{at } B_1, \]
\[ \delta r = \delta\phi_2, \quad \delta s = \delta\phi_1, \quad \text{at } B_2. \]

Thus, the ratio between $\delta r$ and $\delta s$ is determined by the ratio between $\delta\phi_1$ and $\delta\phi_2$ and we can easily find the ratio $\delta r/\delta s$ as given in equations (41) and (42).

The amplitude of the curvature perturbations can be estimated from $\delta\chi$. In the initial stage, $\delta\chi$ coincides with the entropy field perturbations $\delta s$, and thus its initial amplitude on super-Hubble scales is given by

\[ \delta\chi = \frac{c^2}{2} \left| \frac{H}{2\pi} \right|, \]

where we used $\epsilon = c^2/2$ in equation (40) and $c^2$ is given by equation (49). After the transition, $\delta\chi$ becomes massless and the amplitude becomes frozen on large scales. In the single-field-dominated solutions, the comoving curvature perturbation $R_c$ is given by

\[ |R_c| = \frac{1}{\sqrt{c_1^2 + c_2^2}} \delta\chi. \]

Assuming the transition occurs suddenly, the final amplitude of the comoving curvature perturbation is given by

\[ |R_c| = \frac{c^2}{2\sqrt{c_1^2 + c_2^2}} \left| \frac{H}{2\pi} \right| T, \]

where the subscript $T$ means that the quantity is evaluated at the transition time. On the other hand, the amplitude of the entropy field perturbation is given by
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Figure 3. Left: solutions for $\delta \chi(N)$. We used the same parameters as figures 1 and 2. Right: solutions for $\log|H|$ with three different background solutions. We also show $\log|H_T|$ that is determined from the numerical solutions for $\delta \chi$.

\[ \delta s = \frac{c_2 c^2}{2 \sqrt{c_1^2 + c_2^2}} \left| \frac{H}{2\pi} \right|_T, \quad \text{at} \quad B_1, \]  

\[ \delta s = \frac{c_1 c^2}{2 \sqrt{c_1^2 + c_2^2}} \left| \frac{H}{2\pi} \right|_T, \quad \text{at} \quad B_2. \]

From the numerical solutions for $\delta r$ and $\delta s$, we can reconstruct $H_T$. We confirm that $H_T$ constructed in this way agrees with the Hubble scale at the transition as shown in figure 3.

5. Conclusion

In this paper, we have studied the generation of curvature perturbations during an ekpyrotic collapse with multiple fields. We must assume that the classical background solution starts from a state very close to a saddle point in the phase space that corresponds to the multi-field ekpyrotic scaling solution. If the solution stays at this fixed point for long enough, a scale-invariant spectrum of isocurvature perturbations is generated over the range of scales that is relevant for the large-scale structure in the universe. Even if the background solution deviates only slightly from the multi-field scaling solution initially, the tachyonic instability eventually drives the solution to the old ekpyrotic collapse dominated by a single field. During this transition, the initial isocurvature field perturbations generate a scale-invariant spectrum of comoving curvature perturbations.

First, we studied the perturbations by decomposing them into the instantaneous adiabatic and the entropy field perturbations. These fields are decoupled at the fixed points. The adiabatic field perturbations are effectively massless around both the multi-field scaling solutions and the single-field-dominated solution, so become constant on large scales. On the other hand, the entropy field perturbations have a tachyonic mass around the multi-field scaling solution, where they grow like $\delta s \propto H$ on large scales, but they are effectively massless around the single-field-dominated solution. We set initial conditions for these perturbations from quantum fluctuations about the multi-field scaling solution. As the adiabatic field perturbations have a blue spectrum they can be neglected compared with the entropy field perturbations on large scales.
scales. However, during the transition to the single-field attractor, adiabatic field perturbations are generated from the entropy field perturbations.

It turns out to be more convenient to use new fields $\varphi$ and $\chi$ defined in equations (47) to follow the perturbations through the transition. These fields coincide with the adiabatic and entropy fields around the multi-field scaling solution but not during the transition or at the final single-field-dominated phase. In terms of $\varphi$ and $\chi$, the potential is given by a product of the potential for $\chi$, $U(\chi)$, and an exponential potential for $\varphi$. $U(\chi)$ has an extremum which corresponds to the multi-field scaling solution. A crucial observation is that even during the transition and in the final phase $\varphi$ satisfies the scaling relation equation (57). We can then show that the field perturbations for $\varphi$ and $\chi$ are always decoupled. Since $\delta \varphi = 0$ on large scales during the ekpyrotic scaling solution, this remains so. This determines the ratio between the adiabatic and entropy field perturbations at the final phase. On the other hand, $\delta \chi$ grows during the scaling solution and then becomes constant during the single-field-dominated solution.

Our final results are equations (67), (68) and (69) for the amplitude of comoving curvature and isocurvature field perturbations during the single-field ekpyrotic collapse phase. The amplitude of the comoving curvature perturbation is determined by the Hubble scale at the transition.

We still need to convert the ekpyrotic collapse to expansion (see, for instance, [12, 13]) and see how this curvature perturbation is matched to that in an expanding universe. For a regular bounce the comoving curvature perturbation is conserved for adiabatic perturbations and thus equation (67) is directly related to the amplitude of the observed primordial density perturbation. If the radiation and matter content in today’s universe comes solely from the single field that dominates the final ekpyrotic phase, we will have no isocurvature perturbations in an expanding universe.

It is interesting to compare this multi-field model with a single-field model that gives a scale-invariant spectrum of comoving curvature perturbations during collapse [14, 27, 28]. The single-field model requires the correct exponent for a relatively flat and positive exponential potential in order to obtain $a \propto |t|^{2/3}$, whereas the ekpyrotic model only requires sufficiently steep, negative exponential potentials to obtain $a \propto |t|^{1/\epsilon}$ with $\epsilon \gg 1$. On the other hand, both models require fine-tuned initial conditions as it is the existence of an instability that gives rise to the scale-invariant perturbation spectrum during collapse. In both models, the amplitude of tensor perturbations is determined by the Hubble scale when the perturbations leave the horizon as tensor perturbations are then frozen on super-horizon scales. In the single-field model, the tensor metric perturbations thus acquire the same almost scale-invariant spectrum [29] as the comoving curvature perturbation, with a similar amplitude, giving rise to a dangerously large tensor–scalar ratio which severely constrains the model [28]. But in the ekpyrotic scaling solution the tensor perturbations have a steep blue spectrum and are completely negligible at scales relevant for the cosmic microwave anisotropies.

In summary, a simple ekpyrotic model with two steep, negative exponential potentials is capable of generating a scale-invariant spectrum of comoving curvature perturbations. A key ingredient is the instability of the multi-field scaling solution [11, 12, 16, 19]. This instability generates a scale-invariant spectrum of isocurvature field perturbations from vacuum fluctuations about the scaling solution and converts them to a scale-invariant spectrum of comoving curvature perturbations. We should emphasize that this conversion occurs automatically due to the dynamics of the fields in our simple model and does not require any change in the shape of the potential or any additional dynamics [11–13].

Thus, ekpyrotic collapse with multiple fields can generate a scale-invariant spectrum for curvature perturbations in several different ways (see also [31] for a different idea). In all these models, we need some preceding phase that initially drives the classical background solution
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This problem cannot be solved within the simplest model with multiple exponential potentials that we considered in this paper and we would need to appeal to a more ambitious framework for the model such as the cyclic scenario [32, 33] to address this problem.

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