Special entangled fermionic systems and exceptional symmetries

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Abstract

Special fermionic systems entered the realm of quantum chemistry in the seventies in the work of Borland and Dennis in the form of a toy model. This work was leading to a detailed study of the N-representability problem by Klyachko. The topic then has been reconsidered in the light of entanglement theory boiling down to the notion of entanglement polytopes. Recently building on certain properties of such special fermionic systems, a connection between the coupled cluster method and entanglement has been established. In this paper we show that precisely such a special class of systems also provides an interesting physical realization for structures related to the Lie algebras of exceptional groups. This result draws such exotic symmetry structures under the umbrella of entangled systems of physical relevance.

Keywords  Fermionic codes · Quantum entanglement

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1 Introduction

In recent developments of entanglement theory special fermionic systems have played an important role. The interest in studying such systems dates back to 1972, to the classical work of Borland and Dennis on the N-representability problem [1]. They were studying three fermion systems with six single particle states, and have shown that the eigenvalues of the one particle reduced density matrices are subject to special inequalities [1, 2]. It turned out that these inequalities for the one particle marginals hold if and only if these marginals are coming from a pure three fermionic state. Hence these inequalities provide a sufficient and necessary condition for the pure state

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3-representability of a one-particle density matrix. After many years as a first new result in this field in 2006 Klyachko solved the pure state \( N \)-representability problem [3, 4]. Thanks to this work it has turned out that the constraints for the eigenvalues of the one particle marginals define a convex polytope. Surprisingly the mathematics of such polytopes is inherently connected to quantum entanglement theory [5, 6]. These observations are encapsulated in by now well-known terms terms like: entanglement polytopes and generalized Pauli principle.

Interestingly the system originally studied by Borland and Dennis also turned out to be very useful in another branch of entanglement theory: entanglement classification. The special role of three fermion systems with six single particle states in entanglement theory was first observed in Ref. [7]. It has been pointed out that for this system the so called SLOCC entanglement classes [8] can explicitly be given thanks to special mathematical structures, called Freudenthal triple systems [9]. These structures are familiar to mathematicians from studies concerning exceptional groups [10]. Later further connections between other entangled systems and Freudenthal systems have been discovered and the underlying correspondence has been elaborated [11]. This correspondence has recently been used to shed some light on connections between entanglement theory and the coupled cluster method [12]. In the meantime such special entangled systems have also been identified as embedded ones lurking within the realm of fermionic entanglement theory [13–15]. However, in these studies the precise mathematical connection between special entangled systems and exceptional symmetries have never been explored. The aim of the present paper is to fill in this gap, and to provide the first modest step in the direction of understanding exceptional Lie algebras generating exceptional symmetries, in fermionic entanglement terms.

As is well-known a Lie algebra is a mathematical structure describing infinitesimal symmetry transformations of a physical system which is exhibiting continuous symmetry properties described by the corresponding Lie group. The basic building blocks of Lie algebras are the simple Lie algebras. Over the field of complex numbers \( \mathbb{C} \) the finite simple Lie algebras have been classified by Cartan, Killing and Dynkin [16]. The complete classification (neglecting exceptional isomorphisms) is given by the list of four infinite series \( A_n, n \geq 1 \), \( B_n, n \geq 2 \), \( C_n, n \geq 3 \), \( D_n, n \geq 4 \), \( n \in \mathbb{N} \), and five exceptional cases \( E_6, E_7, E_8, F_4, G_2 \) in terms of Dynkin diagrams [16]. Every simple complex Lie algebra has a unique real form whose corresponding centerless Lie-group is compact. These are also classified by Dynkin diagrams and usually in the same “ABCDEFGHIJKLMNOPQRSTUVWXYZ” manner. For a physicist these compact groups of the four infinite series \( A_n, B_n, C_n, D_n \) are connected to well-known matrix Lie groups \( SU(n+1) \), \( SO(2n+1) \), \( Sp(2n) \), \( SO(2n) \) respectively. All of these matrix groups are leaving invariant some extra structure of physical relevance. \( SU(n+1) \) is leaving invariant a Hermitian inner product and the associated norm of \( \mathbb{C}^{n+1} \) familiar from quantum theory. \( SO(2n+1) \) and \( SO(2n) \) being the rotation groups which are leaving invariant an ordinary scalar product and hence the associated length of a vector in \( \mathbb{R}^{2n+1} \) and \( \mathbb{R}^{2n} \). Finally \( Sp(2n) \) is leaving invariant a symplectic form (an alternating bilinear form) on \( \mathbb{R}^{2n} \) regarded as a phase space. \( \mathbb{R}^{2n} \) equipped with this structure is familiar as the symplectic geometry of classical Hamiltonian mechanics expressed in terms of Poisson brackets.
Then the question is: what kind of extra structures of physical relevance the exceptional symmetries of the corresponding groups are leaving invariant? Moreover, what kind of physical systems they are describing in a natural manner? There have been many possible answers to these question addressed in the physics literature. In this paper we would like to add an entanglement based physical system, which realizes its symmetries in an exceptional manner, to the list. We coin a term for such systems: special entangled embedded fermionic systems. We show that for them the extra structures are of the following kind. One of them is a well-known one: a symplectic structure which is already known from classical mechanics. The second one however, is quite unusual: a triple product. Indeed, multiplying two objects to obtain a third one is familiar to everyone, but in the case of a triple product one should multiply three object to get a fourth one! To cap all this we will see (Sect. 6.) that these structures are interrelated in an intricate manner. As one suspects these rather exotic structures are well-known to mathematicians [9, 17]. However, to physicists having familiarity with fermionic systems they are definitely not. This paper is aimed to take the first step in the direction of drawing such symmetries under the umbrella of entangled systems of physical relevance.

The organization of this paper is as follows. In Sect. 2 we summarize some background material needed for our considerations. Fermionic systems with an even number of modes \( N = 2k \) are defined here. Our special entangled fermionic systems will be ones with six modes \( (k = 3) \). In Sect 3, we describe them as spinors in either the positive or negative chirality representation. There are special transformations acting on such fermionic states. These are called local stochastic entanglement transformations associated with classical communication [8]. The orbits of such transformations will be called SLOCC entanglement classes. Such transformations are considered in Sect. 4. Since we are interested in embedded systems, here we also describe how qubit systems can naturally be embedded into our fermionic ones via applying the so called single occupancy representation.

In Sect. 5, we list our five special entangled embedded femionic systems. They will be the ones corresponding to the groups \( D_4, E_7, E_6, F_4, G_2 \), four of them\(^1\) being exceptional. The mathematical basis of this correspondence will be elaborated in Sect. 6. As the main result of this paper here it will be shown that these systems exhibit a symplectic triple system structure. In proving this we explicitely give the form of the triple product and the symplectic form, and show that these are satisfying the basic axioms [17] of such systems. Curiously in order to establish this result an interesting mapping between states an operators shows up. Physically this means that the entangled states and the entanglement transformations acting on them merge into a unified mathematical structure. As a byproduct of these results in Sect. 7 we introduce a dual state, and an entanglement invariant. These objects are serving as physically relevant quantities which enable an explicit entanglement type identification.

In Sect. 8, we realize that by adjoining an extra single qubit SLOCC subgroup and an extra complex degree of freedom one can construct nonlinear realizations for our groups \( D_4, E_7, E_6, F_4, G_2 \). These groups enlarge the SLOCC group substantially

\(^1\) For the largest exceptional group \( E_8 \) (the only one left out from the list of exceptional groups) we did not manage to find a correspondence with any simple fermionic system.
capable of incorporating transformations changing the value of the basic entangle-
ment measure hence the entanglement type. Though the basic ideas of this realization
are already available in the literature [18] we did not attempt here to give an explicit
construction of it and explore its physical consequences. We postpone such elabora-
tions for future work. The conclusions and some comments are left for Sect. 8. Here
we also speculate on the role of the largest exceptional group $E_8$ for which we did
not manage to give a simple fermionic entangled system interpretation. Hopefully in
future investigations we will be able to add this item to our list.

2 Fermionic systems with an even number of modes

In the following we need the formalism as developed in [19] for fermionic systems
with an even number of modes. Let $k \in \mathbb{N}$ and let $V$ be an $N = 2k$ dimensional
complex vector space and $V^*$ its dual. We regard $V = \mathbb{C}^N$ with $\{e_I\}, I = 1, 2, \ldots, N$
the canonical basis and $\{e^I\}$ the dual basis. Elements of $V$ will be called single particle
states. The number $N = 2k$ is called the number of modes. Equipped with a Hermitian
inner product $\langle \cdot | \cdot \rangle$ our vector space $(V, \langle \cdot | \cdot \rangle)$ can be regarded as the Hilbert space of
single particle states.

We also introduce the $2^N$ dimensional vector space

$$ \mathcal{V} \equiv V \oplus V^*. $$

An element of $\mathcal{V}$ is of the form $x = v + \alpha$ where $v$ is a vector and $\alpha$ is a linear
form. According to the method of second quantization to any element $x \in \mathcal{V}$ one
can associate a linear operator $O_x$ acting on the fermionic Fock space $\mathcal{F}_N$ which is
represented as a direct sum of $m$-particle subspaces $\mathcal{F}^{(m)}, m = 0, 1, \ldots, N$.

Explicitly, to the basis vectors of $\mathcal{V}$ one associates fermionic creation and annihi-
lation operators

$$ O_{e^I} \equiv p^I, \quad O_{e^J} \equiv n_J, \quad I, J = 1, 2, \ldots, N $$

satisfying the usual fermionic anticommutation relations

$$ p^I n_J + n_J p^I = \delta^I_J, \quad p^I p^J + p^J p^I = n_1 n_J + n_J n_1 = 0. $$

Since a Hermitian inner product on $\mathcal{V}$ is at our disposal one can regard $p^I$ as the
Hermitian conjugate of $n_I$. In this way one arrives at the usual operators familiar from
the literature: $f^I_j \equiv p^I$ and $f_j \equiv n_I$. However, for notational simplicity we refrain
from this notation and we rather use the $p^I$s and $n_I$s in the text.

Now the direct sum structure of $\mathcal{F}_N$ in this formalism is made explicit as follows.
First we define the vacuum state $|\text{vac}\rangle \in \mathcal{F}_N$ by the property

$$ n_I |\text{vac}\rangle = 0, \quad I = 1, 2, \ldots, N. $$
Then $\mathcal{F}_N$ is a vector space of dimension $2^N$ which is spanned by the basis vectors created from the vacuum

$$|\text{vac}>, \quad p^{I_1} p^{I_2} \cdots p^{I_m} |\text{vac}>, \quad 1 \leq I_1 < I_2 < \cdots < I_m \leq N, \quad m = 1, \ldots N.$$  \hfill (5)

Hence

$$\mathcal{F}_N = \bigoplus_{m=0}^N \mathcal{F}^{(m)}. \hfill (6)$$

According to (6) an arbitrary state $|\varphi\rangle \in \mathcal{F}_N$ can be written in the form

$$|\varphi\rangle = \sum_{m=0}^N \sum_{I_1, I_2, \ldots, I_m = 1}^N \frac{1}{m!} \varphi^{(m)}_{I_1 I_2 \ldots I_m} p^{I_1} p^{I_2} \cdots p^{I_m} |\text{vac}>. \hfill (7)$$

Here the $m$th order totally antisymmetric tensors $\varphi^{(m)}_{I_1 I_2 \ldots I_m}$ encapsulate the complex amplitudes of the $m$-“particle” subspace and we introduced the new notation

$$p^{I_1} p^{I_2} \ldots p^{I_m} \equiv p^{I_1 I_2 \ldots I_m}. \hfill (8)$$

Before proceeding we should warn the reader that the formalism developed so far is not the canonical one familiar to physicists. This formalism is based on the work of Elie Cartan [20] and Claude Chevalley [21] called “The theory of spinors”, which has been introduced to the physics literature by Trautman, Budinich and others [22]. Though it had turned out to be extremely useful for studying fermionic entanglement [14, 15], our treatise is still not widespread among physicists working in the field of entanglement theory. We decided to use it due its mathematical convenience for exploring connections with exceptional groups our main concern here.

In order to orient the reader for connecting our setup to a one which should be well-known here is an alternative way of looking at the structure of $\mathcal{F}_N$. In this more familiar picture one describes the $2^N$ basis vectors of $\mathcal{F}_N$ as

$$|\kappa_1 \kappa_2 \ldots \kappa_N\rangle \equiv p_1^{\kappa_1} p_2^{\kappa_2} \cdots p_N^{\kappa_N} |\text{vac}>, \quad (\kappa_1, \kappa_2, \ldots, \kappa_N) \in \mathbb{Z}_2^{N} \hfill (9)$$

hence $|00 \ldots 0\rangle \equiv |\text{vac}>, \quad |10 \ldots 0\rangle \equiv p_1 |\text{vac}\rangle$ etc. Let us now define $2N$ Majorana fermion operators as follows

$$c_{2l-1} = p_l + n_l, \quad c_{2l} = i(p_l - n_l). \quad \hfill (10)$$

These operators are satisfying

$$c_\mu c_\nu + c_\nu c_\mu = 2\delta_{\mu\nu}, \quad c_\mu^\dagger = c_\mu, \quad \mu = 1, 2, \ldots 2N. \hfill (11)$$
We will refer to the number $2N$ as the number of Majorana modes.

One can represent the $c_μ$ via the $N$-fold tensor product\(^2\) of Pauli matrices (with identity matrices in the remaining slots implicit) as

\[
c_{2I-1} = \left( \prod_{J=1}^{I-1} \sigma_z^{(J)} \right) \sigma_x^{(I)} \equiv \prod_{J=1}^{I-1} Z_J X_I, \quad c_{2I} = \left( \prod_{J=1}^{I-1} \sigma_z^{(J)} \right) \sigma_y^{(I)} \equiv \prod_{J=1}^{I-1} Z_J Y_I.
\]

(12)

In this way we arrive at the familiar Jordan-Wigner representation of Majorana operators.

Define now the chirality operator as

\[
\Gamma = \prod_{I=1}^{N} (1 - 2p_I n_I) = (-i)^N \prod_{I=1}^{N} c_{2I-1} c_{2I}
\]

(13)

For $m$ even (odd) the states of Eq. (5) are eigenvectors of $\Gamma$ with eigenvalues $+1(-1)$. The corresponding eigensubspaces of $\Gamma$ will be denoted by $F_\pm$. One can also refer to their elements as spinors of positive or negative chirality [20–22].

Later it will be rewarding to introduce instead of the canonical labeling of single particle states as $1, 2, 3, 4, \ldots, 2k-1, 2k$, two other labelings. The first one will be called the occupied non-occupied labeling, the second the so called odd-even labeling. In the first case one defines a relabeling according to the formula

\[
(1, 2, \ldots, k, k+1, k+2, \ldots, 2k) \leftrightarrow (1, 2, \ldots, k, \bar{1}, \bar{2}, \ldots, \bar{k})
\]

(14)

in the case of the second one has

\[
(1, 2, 3, 4, \ldots, 2k-1, 2k) \leftrightarrow (1, \bar{1}, 2, \bar{2}, \ldots, k, \bar{k})
\]

(15)

The first labeling is useful for studying entanglement and the coupled cluster method [12]. In this context the $1, 2, 3, \ldots$ states can be regarded as the occupied and the $\bar{1}, \bar{2}, \bar{3}, \ldots$ states as the non occupied ones. The second one is important when considering qubit systems embedded into fermionic ones. It is well-known that $k$-qubit systems [15, 23–25] can be embedded into fermionic ones with $2k$ modes. In this case the pairs $j \bar{j}$ will be referring to the two possible states of the $j$th qubit in the computational basis. We will give a more detailed description of these embedding procedures in the next sections.

3 Special fermionic systems with six modes

The states of our special fermionic systems we will be interested in are represented by the state vectors of the spaces $F_\pm^6$. Hence in the following we will be exploring the

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\(^2\) In the following formula for the product the use of the $\otimes$ symbol is implicit.
special properties of fermionic systems with six modes (or twelve Majorana modes) with definite chirality \((k = 3, N = 6, 2N = 12)\). Explicitly

\[
|\varphi_+\rangle = \left( \eta \cdot 1 + \frac{1}{2!} y_{IJ} p_{IJ} + \frac{1}{4!} x_{IJKLMN} p_{KLMN} + \xi p_{123456} \right) |\text{vac}\rangle \in \mathcal{F}_6^+
\]

\[
|\varphi_-\rangle = \left( u_I p^I + \frac{1}{3!} z_{IJK} p_{IJK} + \frac{1}{5!} v^I x_{IJKLMN} p^{JKLMN} \right) |\text{vac}\rangle \in \mathcal{F}_6^-
\]

Notice that the dimension of \(\mathcal{F}_6\) is \(2^6 = 64\) and the fermionic Fock space splits into two 32 dimensional subspaces of positive (with representative \(|\varphi_+\rangle\)) and negative (with representative \(|\varphi_-\rangle\)) chirality. The 32 complex amplitudes of the representative cases split into two scalars \(\eta\) and \(\xi\), two antisymmetric tensors \(x_{IJ}\), \(y_{IJ}\) two vectors \(u_I\), \(v^I\), and a totally antisymmetric tensor of rank three \(z_{IJK}\) where \(I, J, K = 1, 2, \ldots 6\). Hence in the first case we have a split of dimensions \(32 = 1 + 15 + 15 + 1\), and in the second \(32 = 6 + 20 + 6\).

Note that there exists an intertwiner \(\Omega\) between the positive and negative chirality representations such that

\[
|\varphi_+\rangle = \Omega |\varphi_-\rangle, \quad \Omega = c_1 c_3 c_5 = (p^1 + n_1)(p^2 + n_2)(p^3 + n_3)
\]

This intertwiner relates the amplitudes via a bijection

\[
(\eta, y, x, \xi) \leftrightarrow (u, z, v)
\]

with explicit form [15]

\[
\begin{align*}
x_{IJ} &= \begin{pmatrix}
0 & u_3 & -u_2 & z_{123} & z_{131} & z_{112} \\
-u_3 & 0 & u_1 & z_{231} & z_{212} & z_{221} \\
u_2 & -u_1 & 0 & \psi_{23} & \psi_{21} & \psi_{22} \\
-z_1_{13} & -z_2_{13} & -z_3_{13} & 0 & v^3 & -v^2 \\
-z_1_{23} & -z_2_{23} & -z_3_{23} & -v^3 & 0 & v^1 \\
-z_1_{12} & -z_2_{12} & -z_3_{12} & v^2 & -v^1 & 0
\end{pmatrix}, & \xi = z_{123} \\
y_{IJ} &= \begin{pmatrix}
0 & v_3 & v_2 & -z_{123} & -z_{323} & -z_{323} \\
v_3 & 0 & v_1 & -z_{131} & -z_{311} & -z_{311} \\
v_2 & -v_1 & 0 & -z_{112} & -z_{212} & -z_{312} \\
\psi_{23} & \psi_{21} & \psi_{22} & u_5 & 0 & -u_7 \\
z_{123} & z_{321} & z_{112} & 0 & u_7 & u_7 \\
z_{223} & z_{321} & z_{312} & -u_7 & u_7 & 0
\end{pmatrix}, & \eta = -z_{123}
\end{align*}
\]

where for later convenience the left hand side is labelled \((x_{IJ}, y_{IJ})\) by \(I, J = 1, 2, \ldots 6\), and for the right hand side we applied the occupied-non occupied labeling of Eq.(14). Hence for example \(x^{25} = z_{231} = z_{264} = -z_{246}, y_{54} = u_5 = u_6\) etc.
One can also choose another intertwiner for connecting the positive and negative chirality representation. A one which is using the odd-even labeling is given by

$$|\varphi_-\rangle = \Omega |\varphi_+\rangle, \quad \Omega' = c_1 c_5 c_9 = (p^1 + n_1)(p^3 + n_3)(p^5 + n_5)$$

(22)

4 Entanglement transformations, and embedded qubits

Until this point we have been discussing states. However, focusing on states is merely half of the physically interesting story, since in order to connect our mathematical considerations to physical phenomena we also have to discuss observables, and symmetry transformations, or even a special class of such transformations which generate interesting dynamics. In order to do this we have to describe special subsets of operators as infinitesimal generators of transformations acting on the fermionic Fock space. We will be interested in a special class of such transformations. Such transformations will form a group. The orbits of these transformations are states that can be converted to each other. Such orbits form equivalence classes that we will call entanglement classes.

A natural set of infinitesimal transformations that immediately springs into ones mind is the set of transformations generated by self-adjoint operators i.e. observables. The exponentials of such operators generate unitary symmetry transformations. The canonical example of that kind is the Hamiltonian as an observable associated to energy. The Hamiltonian then, via exponentialization, gives rise to the operator of time evolution. More generally one can consider a local set of unitary operators, that are acting merely on a chosen subsystem of our fermionic one. In this case the orbits regarded as entanglement classes are certain local unitary orbits represented by particular states. For an example (of a special system wich is a subset of our system with six modes showing up in the previous section) of such a classification of entanglement classes see Ref. [25]. Such local unitary transformations, combined with classical manipulations, forms the LOCC transformations [26, 27] (local operations and classical communication).

However, in entanglement theory it has turned out that confining our attention to such subsets of transformations is too restrictive, and by relaxing the assumption of unitarity a more general set of transformations should be considered [8]. Such transformations are called SLOCC transformations (stochastic local operations and classical communication). In the fermionic context later even a further generalization has been introduced the GSLOCC transformations [15] (generalized SLOCC transformations).

Now the set of operators we will be interested in is of the form

$$s \equiv \frac{1}{2} A^J_I [p^I, n_J] + \frac{1}{2} B_{IJ} p^I p^J + \frac{1}{2} C^{IJ} n_I n_J. \quad \quad \quad \quad \quad \quad \quad \quad (23)$$

Here $A$, $B$, $C$ are $\mathcal{N} \times \mathcal{N}$ complex matrices with $B$ and $C$ skew-symmetric and for the repeated indices summation is understood. Our special choice is dictated by the convenience that via forming $\Lambda = e^s$ we obtain special group elements capable of representing GSLOCC transformations. Indeed, it can be shown [15] that $\Lambda =
$e^s \in \text{Spin}(2N, \mathbb{C})$ taken together with a nonzero complex number $\lambda$ is generating transformations of the form

$$|\psi\rangle \mapsto \lambda \Lambda |\psi\rangle, \quad (\lambda, \Lambda) \in \mathbb{C}^\times \times \text{Spin}(2N, \mathbb{C}), \quad |\psi\rangle \in \mathcal{F}_N$$

(24)

representing GSLOCC transformations.\(^3\) First we will be interested in the Spin($2N, \mathbb{C}$) subgroup represented by elements of the form $(1, \Lambda)$. In order to understand what type of transformations are showing up in this mathematical setting let us consider the particle number conserving subgroup of the generalized SLOCC group which is obtained by setting $B = C = 0$ in Eq.(23). Then we get

$$\Lambda = e^{-\text{Tr}A/2} e^{A_{1}^T p^I n_J}.$$  

(25)

The action of $\Lambda$ on a state of the $m$-particle subspace $\mathcal{F}^{(m)}$ of $\mathcal{F}_N$ is\(^1\)

$$\Lambda |\phi^{(m)}\rangle = \frac{1}{m!} \phi^{(m)}_{l_1...l_m} p^{l_1...l_m} |\text{vac}\rangle, \quad \phi^{(m)}_{l_1...l_m} = (\text{Det}A)^{-1/2} A_{j_1} I_1 \cdots A_{j_m} I_m \phi^{(m)}_{I_1...I_m}$$

(26)

where

$$A = e^A \in GL(N, \mathbb{C})$$

(27)

with $GL(N, \mathbb{C})$ being the group of invertible $N \times N$ complex matrices. This group is known to represent SLOCC transformations on $N$ state systems\(^8\).

Since we have chosen $N = 2k$ our formalism is also capable of describing embedded $k$-qubit systems\(^15\). In order to see this first we write the states of the $k$-fermion subspace $\mathcal{F}^{(k)}$ of $\mathcal{F}_{2k}$ as

$$|z\rangle \equiv \frac{1}{k!} z_{I_1 I_2 \ldots I_k} p^{l_1 l_2 \ldots l_k} |\text{vac}\rangle.$$  

(28)

Here for the special case $m = k$ we introduced a special notation $z_{I_1 I_2 \ldots I_k}$ for the amplitudes of $|z\rangle$. Notice that for the $k = 3$ case, our main concern here, we get back to the amplitudes $z_{IJK}$ showing up in Eq. (17). Such states are precisely the Fock space representatives of the three fermionic ones with six single particle states studied by Borland and Dennis in their classical paper\(^1\).

Under the subgroup of (25) the amplitudes of this state transform as

$$z_{J_1...J_k} \mapsto (\text{Det}A)^{-1/2} A_{j_1}^{I_1} \cdots A_{j_k}^{I_k} z_{I_1...I_k} \equiv S_{j_1}^{I_1} \cdots S_{j_k}^{I_k} z_{J_1...J_k},$$

(29)

where

\(^3\) The group Spin($2N, \mathbb{C}$) is the covering group of the rotation group $SO(2N, \mathbb{C})$. $\mathbb{C}^\times$ is the group of nonzero complex numbers. For SLOCC orbits one uses unnormalized states. $\mathbb{C}^\times$ accounts for the possibility of renormalizing states\(^8\).
\[ S_I^J \equiv (\text{Det} A)^{-\frac{1}{2}} A_I^J \in SL(2k, \mathbb{C}). \]  

(30)

Hence in this special case the (25) subgroup of transformations coming from the group \( \text{Spin}(4k, \mathbb{C}) \) with \( B_{I J} = C_{I J} = 0 \) will produce an \( SL(2k, \mathbb{C}) \) subgroup comprising SLOCC transformations with unit determinant.

Now we employ our odd-even labeling of Eq. (15).

\[ \{ e_1, e_T, e_2, e_T, \ldots, e_k, e_T \} \equiv \{ e_1, e_2, e_3, \ldots, e_{2k-1}, e_{2k} \}. \]  

(31)

Hence we have two sets of basis vectors \( \{ e_j \} \) and \( \{ e_j \} \) where \( j = 1, \ldots k \). From the set of \( \binom{2k}{k} \) basis vectors of \( \mathcal{F}(k) \) we choose a special subset containing merely \( 2^k \) elements as follows

\[ p_1 p_2^2 \cdots p_k^k |\text{vac}\rangle, \quad p_1 p_2^2 \cdots p_k^k |\text{vac}\rangle, \ldots, \quad p_1^T p_2^T \cdots p_k^T |\text{vac}\rangle, \quad p_1^T p_2^T \cdots p_k^T |\text{vac}\rangle, \]  

(32)

or in the notation of Eqs. (9) and (31),

\[ |1010 \ldots 10\rangle, \quad |1010 \ldots 01\rangle, \ldots, \quad |0101 \ldots 10\rangle, \quad |0101 \ldots 01\rangle \]  

(33)

These basis vectors are spanning the linear subspace \( \mathcal{K}_s \) of an embedded \( k \)-qubit system. We will refer to this embedding of \( k \)-qubits as the single occupancy representation [15, 25].

We give the explicit form of this embedding as follows. For the \( k \)-qubit state \( |\psi\rangle \in \mathbb{C}^{2^k} \) with amplitudes \( \psi_{00\ldots 0}, \psi_{00\ldots 1}, \ldots, \psi_{11\ldots 1} \) we associate an element \( |z_\psi\rangle \in \mathcal{K}_s \subset \mathcal{F}(k) \) via the mapping

\[ |\psi\rangle \mapsto |z_\psi\rangle = (\psi_{11\ldots 1} p_1 p_2 \cdots p_k + \psi_{11\ldots 0} p_1 p_2 \cdots p_k + \cdots + \psi_{00\ldots 0} p_1 p_2 \cdots p_k) |\text{vac}\rangle. \]  

(34)

Now the operators\(^4\)

\[ \bar{X}^{(j)} = p_j n_j + p_j n_j, \quad \bar{Y}^{(j)} = i(p_j n_j - p_j n_j), \quad \bar{Z}^{(j)} = p_j n_j - p_j n_j \]  

(35)

taken together with the triple \( \{ i\bar{X}^{(j)}, i\bar{Y}^{(j)}, i\bar{Z}^{(j)} \} \) form the \( 2 \times 2 \) representation for the generators of \( n \) copies of the group \( SL(2, \mathbb{C}) \). Clearly on the basis states (the computational basis for the \( j \)-th qubit)

\[ |\bar{0}\rangle_j \equiv \ldots p_j^T \ldots |\text{vac}\rangle = |\ldots 01 \ldots \rangle, \quad |\bar{1}\rangle_j \equiv \ldots p_j \ldots |\text{vac}\rangle = |\ldots 10 \ldots \rangle \]  

(36)

\(^4\) The overline on \( (\sigma_x, \sigma_y, \sigma_z) \equiv (X, Y, Z) \) is not referring to complex conjugation. It is merely indicating that the Pauli matrices obtained in this new way are different from the ones of Eq. (12).
these operators give rise to the usual Pauli spin matrices.

In order to see the physical meaning of our single occupancy representation [15] we express this unusual form of the Pauli operators in terms of the Majorana operators $c_{\mu, \mu} = 1, 2, \ldots 4k$ introduced in Eqs. (10)-(11). On $\mathcal{F}_{2k}$ we have

$$X^{(j)} = \frac{i}{2}(c_{4j-3}c_{4j} - c_{4j-2}c_{4j-1}),$$
$$Y^{(j)} = \frac{i}{2}(c_{4j-2}c_{4j} - c_{4j-1}c_{4j-3}),$$
$$Z^{(j)} = \frac{i}{2}(c_{4j-1}c_{4j} - c_{4j-3}c_{4j-2}).$$

Notice now that on $\mathcal{K}_s$ we also have $c_{4j-3}c_{4j} = -c_{4j-2}c_{4j-1}$ etc. hence on $\mathcal{K}_s$ one can alternatively choose

$$X^{(j)} = ic_{4j-3}c_{4j}, \quad Y^{(j)} = ic_{4j-2}c_{4j}, \quad Z^{(j)} = ic_{4j-1}c_{4j}, \quad j = 1, 2, \ldots k.$$  

(40)

This result [19] connects the single occupancy embedding [15, 25] of qubits into fermions to the influential work of Kitaev [28] on toric codes popular in possible implementations of quantum computing in solid state physics.

For our considerations the value of this formalism is that we can now relate this action of the Pauli operators to a special subset of the infinitesimal fermionic $SL(2k, \mathbb{C})$ transformations acting on $\mathcal{F}(k)$ in the (29) form. Indeed, these are transformations, characterized by a $2k \times 2k$ matrix $S$ of the (30) form, that leave the subspace $\mathcal{K}_s$ invariant. Looking at Eq. (29) it is easy to see that such transformations in the (15) odd-even representation can be organized into a block diagonal matrix of the form

$$S = \text{diag}(S^{(1)}, S^{(2)}, S^{(3)}, \ldots S^{(k)}) \in SL(2k, \mathbb{C})$$

(41)

where the $2 \times 2$ blocks of $S^{(j)}$ of $S$ are of the form

$$S^{(j)} = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} \in SL(2, \mathbb{C}), \quad j = 1, 2, \ldots k$$

(42)

with the Pauli operators, in the Kitaev representation discussed above, being the generators of this group action. Now the transformation $|z_\psi \rangle \mapsto \Lambda|z_\psi \rangle$ gives rise to the one

$$\psi_{\alpha_1 \ldots \alpha_n} \mapsto S^{(1)}_{\alpha_1 \beta_1} \ldots S^{(k)}_{\alpha_k \beta_k} \psi_{\beta_1 \ldots \beta_k}, \quad S^{(j)} \in SL(2, \mathbb{C}), \quad j = 1, 2 \ldots k$$

(43)

which is just an $SL(2, \mathbb{C})^{\times k}$ action for $k$-qubits. Let us finally look at Eq.(24). Clearly after taking also into account transformations of the form $(\lambda, \Lambda)$ where $\lambda \in \mathbb{C}^\times \simeq GL(1, \mathbb{C})$ gives rise to the $GL(2, \mathbb{C})^{\times k}$ action which represents SLOCC.
transformations for $k$-qubit systems [8]. Hence we can conclude that in this single occupancy Kitaev representation GSLOCC transformations for $N = 2k$ fermionic systems incorporate also SLOCC transformations for $k$-qubit ones.

In the following we will be interested only in the $k = 3$ case. We will see that such systems are very special ones, namely in this case one can define a mapping between states and operators representing entanglement transformations. Our aim is to show that such a mapping gives rise to structures well-known from mathematical constructions of exceptional groups. In this manner we will be able to establish a nice connection between embedded entangled fermionic systems and exceptional symmetries.

5 Embedded systems of the $k = 3$ case

As a starting point let us consider a fermionic system represented by a state in the negative chirality (17) form. Our choice for the negative chirality representation is dictated by convenience. Using the intertwiner (18) one can convert mathematical data to the (16) positive chirality one as well. Though such a conversion via an intertwiner obviously will yield a very different physical interpretation, it will turn out that the mathematical structures related to exceptional groups can be revealed in any of such representations. Now on the one hand elaborations are more easy in the (16) representation, on the other hand as far as physics is concerned the structure of embedded systems is manifesting itself nicer in the (17) one. But this is not a problem since using the intertwiner the physically relevant systems based on (17) will be also exhibiting the mathematical structures we are interested in. In the following first we give a list of the physically relevant embedded systems. They will be the main actors of our further considerations. Then after presenting some motivation for the structures we are intending to study, in the next section we turn to their mathematical elaboration.

1. According to Eq. (17) our basic system is a fermionic one with six modes. The representative state containing 1, 3, 5 “particle” states is encapsulated in the $32 = 6 + 20 + 6$ complex amplitudes $(u_I, z_{IJK}, v_I)$. The GSLOCC group is acting on this space in the manner as shown in Eq.(24) where $(\lambda, \Lambda) \in GL(1, \mathbb{C}) \times Spin(12, \mathbb{C})$.

2. Three fermion states with six single particle states are naturally embedded into our system. The corresponding states are of the form $|z\rangle \equiv \frac{1}{3!} z_{IJK} p^{IJK} |\text{vac}\rangle$. (44)

As we have already discussed, the GSLOCC group action is represented on this subspace by the action of the SLOCC group in the (24) and (29) way where $(\lambda, \Lambda) \in GL(1, \mathbb{C}) \times SL(6, \mathbb{C})$. Explicitly, the amplitudes transform as

$$z_{IJK} \mapsto \lambda S_I S_J S_K z_{LMN}, \quad \lambda \in GL(1, \mathbb{C}), \quad S \in SL(6, \mathbb{C}). \quad (45)$$

3. Our six dimensional vector space $V$ of single particle states is equipped with a Hermitian form. One can put further structures on this space. A structure that has
also appeared in the literature [29] is a symplectic one, which should be familiar from classical mechanics. As is well-known this can be represented by an alternating bilinear form $\omega$, having the property $\omega(u, v) = -\omega(v, u)$, where $u, v \in V$. The group which is leaving invariant this form is the symplectic group.

$$\omega(Sv, Su) = \omega(u, v), \quad S \in Sp(6, \mathbb{C}).$$  \hspace{1cm} (46)

One can then also consider entanglement SLOCC transformations that also leave invariant $\omega$. In this picture two states can be regarded to be equivalent iff they can be mutually converted to each other by transformations representing physical manipulations that are leaving invariant the extra structure of the single particle space. However, it can be shown [30] that on the space $\mathcal{F}^{(3)}$ the group $Sp(6, \mathbb{C})$ acts in a reducible manner. The 20 dimensional space $\mathcal{F}^{(3)}$ splits into two irreducible blocks with the corresponding dimensional split being: $20 = 14 + 6$. The 14 dimensional invariant subspace is consisting of such three fermionic states for which the two independent $3 \times 3$ submatrices showing up in Eqs. (20)-(21) are symmetric. Hence for example in Eq. (20) we have $z_{131} = z_{232}$ etc. In this way instead of $2 \times 9$ components the two independent matrices are having merely $2 \times 6$ ones. Then after taking also into account the $z_{123}$ and $z_{123}$ amplitudes we are having a $14 = 1 + 6 + 6 + 1$ dimensional invariant subspace. We will call the fermionic states living in this irreducible subspace symplectic fermionic systems with six modes. Clearly such states are transforming under symplectic SLOCC transformations, namely the group $GL(1, \mathbb{C}) \times Sp(6, \mathbb{C})$ as

$$z_{IJK} \mapsto \lambda S_I^L S_J^M S_K^N z_{LMN}, \quad \lambda \in GL(1, \mathbb{C}), \quad S \in Sp(6, \mathbb{C}).$$  \hspace{1cm} (47)

4. One can also consider three qubit systems as embedded ones in the single occupancy representation. Here the 8 complex amplitudes $\psi_{\alpha\beta\gamma}$ of three qubits are embedded into the three fermion state by the $k = 3$ version of (34), i.e. the relevant $|\psi\rangle \mapsto |z_{\psi}\rangle$ mapping. The transformation rule for the amplitudes is then

$$\psi_{\alpha\beta\gamma} \mapsto \lambda S^{(1)}_{\alpha} S^{(2)}_{\beta} S^{(3)}_{\gamma} \psi'_{\alpha'\beta'\gamma'}, \quad S^{(j)} \in SL(2, \mathbb{C}), \quad j = 1, 2, 3$$  \hspace{1cm} (48)

with the group of SLOCC transformations being [8] $GL(1, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \simeq GL(2, \mathbb{C}) \times GL(2, \mathbb{C}) \times GL(2, \mathbb{C})$.

5. There is still one more embedded system that can naturally be included in our list. This is the case of three bosonic qubits. In this case the three qubits are indistinguishable hence instead of the 8 amplitudes we are having merely 4. Explicitly a bosonic three qubit state $|\psi_b\rangle$ is having the form

$$|\psi_b\rangle = \psi_0|000\rangle + \psi_1(|001\rangle + |010\rangle + |100\rangle)$$  \hspace{1cm} (49)

$$+ \psi_2(|110\rangle + |101\rangle + |011\rangle) + \psi_3|111\rangle.$$
group \((GL(2, \mathbb{C})^2)_{\text{diag}}\) i.e. the diagonal action of three copies of \(GL(2, \mathbb{C})\). Explicitly we have

\[
\psi_{\alpha\beta\gamma} \mapsto \lambda S_{\alpha} S'_{\beta} S'_{\gamma}\psi_{\alpha'}\beta'\gamma' \quad S \in SL(2, \mathbb{C}). \tag{50}
\]
i.e. due to indistinguishability now the same \(SL(2, \mathbb{C})\)s are acting on each qubit. Taking into account the arbitrary \(\lambda \in GL(1, \mathbb{C})\) degree of freedom we are left with the same \(GL(2, \mathbb{C})\)s acting on each qubit, i.e. the diagonal action.

The entanglement classes (the SLOCC classes) of these systems are orbits under the action of the corresponding groups. The classification problem of entanglement types in this case boils down to finding representative states from each class. For the systems 1.-5., the structure of these classes (with representatives) follows a similar pattern. For example in the three qubit case 4., we have the fully separable class \((|000\rangle\)), biseparable classes \((|011\rangle + |000\rangle\) and permutations of the qubits), the so-called W class \((|001\rangle + |010\rangle + |100\rangle\) and the famous GHZ class \((|000\rangle + |111\rangle\)). Hence we see that the number of nontrivial entanglement classes is four. This pattern survives for the remaining cases except for 5., where the biseparable class is missing.

For system 1., the classification of orbits under the group \(GL(1, \mathbb{C}) \times Spin(12, \mathbb{C})\) is also known as the classification of the corresponding spinors. This problem has been solved by Igusa [31]. The interpretation of the classification of spinors in terms of entanglement theory first appeared in [14, 15]. The classification problem for three fermions with six single particle states has been solved by Reichel [32] in 1907, his classification has been interpreted as classification of entanglement types in [7]. It has also been shown that the different entanglement types can be related to the structure of cluster operators in the coupled cluster method [12].

Notice that after applying the intertwiner one can embed our systems also into a state of the \((16)\) form. Then looking at the corresponding amplitudes in Eqs.(20)-(21) the following interesting pattern emerges. We always have a single vacuum state and a single six fermion state. For the two (labeled by the matrix \(y\)) and four fermion states (labeled by the matrix \(x\)) we have the following numbers for the complex amplitudes of the corresponding cases: \((1., 2., 3., 4., 5.) \leftrightarrow (15, 9, 6, 3, 1)\). One can notice that the \((15, 9, 6, 3, 1)\) numbers are the numbers of independent components of \(6 \times 6\) antisymmetric matrices (15), \(3 \times 3\) matrices (9), \(3 \times 3\) symmetric matrices (6), \(3 \times 3\) diagonal matrices (3), and \(3 \times 3\) diagonal matrices with the same elements in the diagonal.

Let us now consider \(3 \times 3\) Hermitian matrices with elements taken from \(\mathbb{R}, \mathbb{C}, \mathbb{H}\), i.e. the elements of these matrices are taken to real, complex or quaternionic. Using \(\overline{\psi}\) for the corresponding conjugation one can give the structure to the matrices \(x\) and \(y\) as \(3 \times 3\) matrices of the form

\[
x, y \leftrightarrow \begin{pmatrix}
\alpha & c & \overline{b} \\
\overline{c} & \beta & a \\
b & \overline{a} & \gamma
\end{pmatrix} \in \text{Herm}(3, F), \quad F = (\mathbb{R}, \mathbb{C}, \mathbb{H}) \tag{51}
\]
These matrices have $(15, 9, 6)$ real elements. Now we see that by using the complex extensions of these matrices, and using instead of $(\mathbb{R}, \mathbb{C}, \mathbb{H})$ the triple $(\mathbb{R} \times \mathbb{C}, \mathbb{C} \times \mathbb{C}, \mathbb{H} \times \mathbb{C})$ we obtain matrices with $(15, 9, 6)$ complex elements.

For example in the quaternionic case one can consider the correspondence

\[
\begin{pmatrix}
\alpha & c & \tilde{b} \\
\overline{\alpha} & \beta & a \\
\overline{c} & \beta & a
\end{pmatrix}
\leftrightarrow
\begin{pmatrix}
\alpha \epsilon & c \epsilon & \tilde{b} \epsilon \\
\overline{\alpha} \epsilon & \beta \epsilon & a \epsilon \\
\overline{c} \epsilon & \beta \epsilon & a \epsilon
\end{pmatrix}.
\]

Here on the left hand side $\alpha, \beta, \gamma \in \mathbb{C}$, $a, b, c \in \mathbb{H} \otimes \mathbb{C}$, and overline means quaternionic conjugation. On the right hand side these are $2 \times 2$ matrices. $\tilde{a} \equiv - \epsilon a^T \epsilon$ Here $\epsilon$ is the $2 \times 2$ matrix with components $\epsilon_{12} = - \epsilon_{21} = 1, \epsilon_{11} = \epsilon_{22} = 0$. One can then immediately verify that the $6 \times 6$ matrix on the right hand side is antisymmetric i.e. an element of $\text{Skew}(6, \mathbb{C})$ having 15 independent complex components. On the other hand on the left hand side we have a Hermitian $3 \times 3$ matrix with elements being biquaternions (i.e. complexified quaternions).

Hence we have the following sets of splits

\[
32 = 1 + 15 + 15 + 1, \quad 20 = 1 + 9 + 9 + 1, \quad 14 = 1 + 6 + 6 + 1, \quad (53)
\]

\[
8 = 1 + 3 + 3 + 1 \quad 4 = 1 + 1 + 1 + 1. \quad (54)
\]

Then in the (53) case we get the following objects

\[
\text{Sym}(3, \mathbb{C}), \quad \text{Mat}(3, \mathbb{C}), \quad \text{Skew}(6, \mathbb{C})
\]

all of them described according to the similar (51) pattern! The remaining cases of course also fit into this scheme, since these are merely arising as restriction to the diagonal components of the relevant matrices. These considerations clearly indicate that we should be able to describe our systems in a unified framework. This framework is of course well-known in the mathematics literature. It is related to the construction of the so called Freudenthal triple systems [9, 35] based on cubic Jordan algebras [34]. Note that the connection between entangled systems and the Freudenthal construction has already been studied in [11]. In the next section we would like to initiate a new way of looking at this construction based on embedded fermionic systems. In order to do this we have to introduce a quite unusual structure a triple product.

### 6 Special fermionic systems as symplectic triple systems

A symplectic triple system [17] is an $\mathcal{R}$ vector space over $\mathbb{C}$ equipped with a $\{\cdot, \cdot\} : \mathcal{R} \times \mathcal{R} \rightarrow \mathbb{C}$ symplectic form\(^5\) and a $[\cdot, \cdot, \cdot] : \mathcal{R} \times \mathcal{R} \times \mathcal{R} \rightarrow \mathcal{R}$ trilinear product, or triple product, such that the following set of axioms is fulfilled

\[
[x, y, z] = [y, x, z] \quad (56)
\]

\(^5\) This is an alternating bilinear form such that $\{x, y\} = -\{y, x\}, \quad x, y \in \mathcal{R}$. 

\[\text{Springer}\]
\[
\begin{align*}
[x, y, z] &= [x, z, y] + 2\{y, z\}x - \{z, x\}y - \{x, y\}z \\
[u, v, [x, y, z]] &= [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]]
\end{align*}
\]

(57)  (58)

One can show that as a result of (57)-(58) we also have the property

\[
\{[u, v, x], y\} + \{x, [u, v, y]\} = 0.
\]

(59)

In order to obtain some insight into this rather abstract structure let us observe that for a fixed pair of vectors \(u, v\) the map

\[
L_{uv} : \mathcal{R} \to \mathcal{R}, \quad x \mapsto [u, v, x]
\]

(60)
is a linear one. Then in terms of this map the first axiom says that

\[
L_{uv} = L_{vu},
\]

(61)

and the third

\[
[L_{uv}, [x, y, z]] = [L_{uv}x, y, z] + [x, L_{uv}y, z] + [x, y, L_{uv}z].
\]

(62)

This property shows that \(L_{uv}\) is a derivation, since its action resembles the action of the ordinary derivative on a triple product. The (59) property shows that this linear map preserves the symplectic structure

\[
\{L_{uv}x, y\} + \{x, L_{uv}y\} = 0.
\]

(63)

Clearly the second axiom of Eq.(57) of complicated appearance gives a very stringent constraint between the symplectic structure and the triple product.

As a main result of this paper now we prove that the fermionic states (16) and (17) and their embedded subsystems define a symplectic triple system. In order to prove this it is enough to use the (16) positive chirality states, since this result can be transferred to the (17) negative chirality one via the use of the (18) intertwiner. In the following we use the notation for the basic state of Eq. (16)

\[
|\varphi\rangle := |\varphi_+\rangle, \quad \mathcal{F} := \mathcal{F}^+_6, \quad \leftrightarrow (\eta, y_{IJ}, x^{IJ}, \xi)
\]

(64)

where for simplicity we have also reminded the reader the basic \(1 + 15 + 15 + 1\) split of the complex amplitudes, where \(y_{IJ}\) and \(x^{IJ}\) are \(6 \times 6\) antisymmetric matrices with complex elements.

Let us consider first an infinitesimal GSLOCC transformation \(G\) of the (23) form where in the formula (23) for convenience we take instead of the triple of \(3 \times 3\) matrices \((A, B, C)\) the one \((A^T, B^T, C^T)\). Hence our infinitesimal generator is of the form

\[
G = A^T_{IJ} p^J p^I n_I - \frac{1}{2} (\text{Tr} A) 1 - \frac{1}{2} B_{IJ} p^I p^J - \frac{1}{2} C^{IJ} n_I n_J
\]

(65)
Then for the amplitudes of a transformed state

$$|\varphi'\rangle = G|\varphi\rangle$$  \hfill (66)

we have

$$\eta' = (C, y) - \frac{1}{2}(\text{Tr}A)\eta$$ \hfill (67)

$$\xi' = -(B, x) + \frac{1}{2}(\text{Tr}A)\xi$$ \hfill (68)

$$x'_{IJ} = -2(B \times y)_{IJ} + \xi C^{IJ} + A^K_{I}x^K_{J} - A^K_{J}x^K_{I} + \frac{1}{2}(\text{Tr}A)x_{IJ}$$ \hfill (69)

$$y'_{IJ} = 2(C \times x)_{IJ} - \eta B_{IJ} + A^K_{I}y^K_{J} - A^K_{J}y^K_{I} - \frac{1}{2}(\text{Tr}A)y_{IJ}$$ \hfill (70)

where

$$(B \times y)_{IJ} = \frac{1}{8}\epsilon^{IJKLMN}B_{KL}y_{MN}, \quad (C \times x)_{IJ} = \frac{1}{8}\epsilon^{IJKLMN}C^{KL}x_{MN}$$ \hfill (71)

$$(C, y) = -\frac{1}{2}\text{Tr}(Cy) = \frac{1}{2}C^{IJ}y_{IJ}, \quad (B, x) = -\frac{1}{2}\text{Tr}(Bx) = \frac{1}{2}B_{IJ}x^{IJ}.$$ \hfill (72)

We also introduce the notation

$$x^{\sharp} := x \times x \quad y^{\sharp} := y \times y.$$ \hfill (73)

To every fermionic state $|\varphi\rangle$ let us now define an operator $\mathcal{K}_\varphi$ which is of the (65) form as follows

$$\mathcal{K}_\varphi := A_\varphi - \frac{1}{2}(\text{Tr}[A_\varphi]) \mathbf{1} - B_\varphi - C_\varphi$$ \hfill (74)

where

$$[A_\varphi]_{IJ} = 2x^K_{I}y_{KJ} + ((x, y) - \eta \xi) \delta^I_J$$ \hfill (75)

$$[B_\varphi]_{IJ} = 2(x \times x - \xi y)_{IJ}, \quad [C_\varphi]^{IJ} = 2(y \times y - \eta x)^{IJ}.$$ \hfill (76)

Using these expressions the explicit form of $\mathcal{K}_\varphi$ is

$$\frac{1}{2}\mathcal{K}_\varphi = (xy)^{IJ} p^I n_I + (\xi y - x^{\sharp})_{IJ} p^I p^J + (\eta x - y^{\sharp})^{IJ} n_I n_J$$ \hfill (77)

As we see this operator is quadratic in the amplitudes of the original state. This association of an operator to a state is a special case of a more general construction.
valid for $N = 2k$ with $k$ odd. This association is based on the so called moment map. For more information on the mathematical background of this map see Ref. [14].

One can generalize this construction further by noticing that even for a pair of states $(|\varphi_1\rangle, |\varphi_2\rangle)$ one can associate an operator of the above form. Indeed, in this case one can define

$$K_{\varphi_1\varphi_2} := A_{\varphi_1\varphi_2} - \frac{1}{2} \left( \text{Tr} \left[ A_{\varphi_1\varphi_2} \right] \right) I - B_{\varphi_1\varphi_2} - C_{\varphi_1\varphi_2}$$

(78)

where

$$A_{\varphi_1\varphi_2} = x_1 y_2 + x_2 y_1 + [\eta_1 \xi_2 + \eta_2 \xi_1] - \omega_{12}I$$

(79)

$$B_{\varphi_1\varphi_2} = 2x_1 \times x_2 - \xi_1 y_2 - \xi_2 y_1,$$

$$C_{\varphi_1\varphi_2} = 2y_1 \times y_2 - \eta_1 x_2 - \eta_2 x_1$$

(80)

$$2\omega_{12} = 3\eta_1 \xi_2 + 3\eta_2 \xi_2 - (x_1, y_2) - (x_2, y_1).$$

(81)

Here $I$ is the $6 \times 6$ identity matrix and the matrix indices are left implicit. One can also check that $\text{Tr} \left[ A_{\varphi_1\varphi_2} \right] = -2\omega_{12}$.

Notice that

$$K_{\varphi_1\varphi_2} = K_{\varphi_1\varphi_2}.$$ 

(82)

Moreover one also has

$$K_{\varphi_1\varphi_2} = K_{\varphi_2\varphi_1}.$$ 

(83)

This property reminds us of Eq. (61). Moreover, since $K_{\varphi_1\varphi_2}$ is a linear operator, motivated by Eq. (60) it is worth considering the expression

$$|\varphi'\rangle \equiv K_{\varphi_1\varphi_2} |\varphi_3\rangle$$

(84)

to define a new fermionic state associated to the three original ones $(\varphi_1, \varphi_2, \varphi_3)$.

This new $|\varphi'\rangle$ state has the (16) form where now the corresponding 32 amplitudes are of the form $(\eta', y', x', \xi')$. Similarly for the original states $|\varphi_\alpha\rangle$ we have $(\eta_\alpha, y_\alpha, x_\alpha, \xi_\alpha)$ ($\alpha = 1, 2, 3$). One can then calculate the explicit form of $(\eta', y', x', \xi')$ using the expressions (67)-(70) where now instead of the $A$, $B$ and $C$ matrices one has to use the (79)-(80) ones, and instead of $(\eta, y, x, \xi)$ the set $(\eta_3, y_3, x_3, \xi_3)$ is to be used. With this process we have associated to the triplet $(|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle)$ the new $|\varphi'\rangle$ fermionic state.

Now we conjecture that the

$$\binom{|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle}{\mapsto} |\varphi'\rangle = K_{\varphi_1\varphi_2} |\varphi_3\rangle.$$ 

(85)

map is a triple product. In order to check this let us denote the $6 \times 6$ matrices of Eqs.(79)-(80) by $A_{12} \equiv \left[ A_{\varphi_1\varphi_2} \right]$, $B_{12} \equiv \left[ B_{\varphi_1\varphi_2} \right]$ and $C_{12} \equiv \left[ C_{\varphi_1\varphi_2} \right]$. Then the explicit form of our conjectured product is
\[
\eta' = (C_{12}, y_3) + \omega_{12}\eta_3, \quad \xi' = -(B_{12}, x_3) - \omega_{12}\xi_3
\]  
(86)

\[
x' = -2B_{12} \times y_3 + C_{12}\xi_3 - A_{12}x_3 + (A_{12}x_3)^T - \omega_{12}x_3
\]  
(87)

\[
y' = 2C_{12} \times x_3 - B_{12}\eta_3 + y_3A_{12} - (y_3A_{12})^T + \omega_{12}y_3
\]  
(88)

Let us refer to this product as

\[
[\cdot, \cdot, \cdot]: \mathcal{F} \otimes \mathcal{F} \otimes \mathcal{F} \to \mathcal{F}, \quad (\varphi_1, \varphi_2, \varphi_3) \mapsto \mathcal{K}_{\varphi_1 \varphi_2 \varphi_3}.
\]  
(89)

Let us also define a symplectic form as

\[
\{\cdot, \cdot\}: \mathcal{F} \otimes \mathcal{F} \to \mathbb{C}, \quad (\varphi_1, \varphi_2) \mapsto \{\varphi_1, \varphi_2\} = -\frac{1}{2}(\varphi_1|\varphi_2)
\]  
(90)

where

\[
(\varphi_1|\varphi_2) = \eta_1\xi_2 - \eta_2\xi_1 + (x_1, y_2) - (x_2, y_1)
\]  
(91)

Now via a straightforward calculation one can check that with these definitions the axioms of Eqs.(56)-(58) are satisfied. Hence we have obtained the nice result that the complex 32 dimensional vector space \(\mathcal{F}\) of fermionic states equipped with the (90) symplectic form and the triple product (89), i.e. the mathematical object \((\mathcal{F}, \{\cdot, \cdot\}, [\cdot, \cdot, \cdot])\) is a symplectic triple system.

As we have already discussed using the (18) intertwiner one can convert our results obtained for the positive chirality representation to the negative chirality one which has a direct connection with the fermionic embedded systems related to physically relevant ones. For example if we are using the (17) negative chirality representation (now \(|\varphi\rangle = |\varphi_-\rangle\)) then instead of Eq. (78) for the components of \(\mathcal{K}_{\varphi}\) we obtain

\[
[A_{\varphi}]^I_J = 2v^I u_J - (\mathcal{K}_z)^I_J = v^K u_K \delta^I_J
\]
\[
[B_{\varphi}]_{IJ} = -2z_{1JK} v^K
\]
\[
[C_{\varphi}]^{IJ} = \frac{2}{3!} \varepsilon^{IL_1L_2L_3L_4L_5} z_{KL_1L_2L_3L_4L_5} u_N
\]  
(92)

where

\[
(\mathcal{K}_z)^I_J = \frac{1}{2!3!} \varepsilon^{IL_1L_2L_3L_4L_5} z_{KL_1L_2L_3L_4L_5}.
\]  
(93)

Notice now that the (90) symplectic form is based on the polarization of the quantity \(-\xi\eta - (x, y)\) hence by virtue of Eqs.(20)-(21) its new form can be rewritten as (now \(\mathcal{F} = \mathcal{F}_-\))

\[
\{\cdot, \cdot\}: \mathcal{F} \otimes \mathcal{F} \to \mathbb{C}, \quad (\varphi_1, \varphi_2) \mapsto \{\varphi_1, \varphi_2\} = \frac{1}{2}(v_1^I u_2I - v_2^I u_1I + \{z_1, z_2\})
\]  
(94)
where
\[ \{z_1, z_2\} := \frac{1}{3!3!} \epsilon^{IJKLMN} z_{1IK} z_{2LMN}. \] (95)

Clearly after going through the same steps as in the positive chirality case we obtain again a symplectic triple system.

However, since now we are in the realm of embedded fermionic systems we can also easily study symplectic triple subsystems corresponding to our physically relevant ones. Indeed, the triple products for these systems are just arising from the expressions above via doing suitable restrictions to the set of parameters: \((v_\alpha, u_\alpha, z_\alpha)\) where \(\alpha = 1, 2, 3\). For example in order to obtain a symplectic triple system for three fermion systems with six single particle states one just have to set \(u = v = 0\). In this case instead of \(K_\varphi\) one has to use \(K_\varepsilon\) and objects like \(K_{z_1z_2}|z_3]\rangle to create the relevant triple product. The symplectic form in this case is \(\frac{1}{2}\{z_1, z_2\}\). In this way we arrive at the nice result that all of our five embedded entangled systems discussed in Sect. 5 can be given the structure of a symplectic triple one. In the next section we elaborate further on the structure of such systems revealing their connection to exceptional groups.

### 7 Invariants, dual states and entanglement classes

Since symplectic triple systems have a general appearance in the following we refrain from the use of \(|\varphi\rangle\) for states and we simply refer to a state in the notation \(\varphi \in F\) where the notation \(F\) can be used for any of the positive or negative chirality representations. However, for historic reasons connected to the literature on exceptional groups we opted to use the positive chirality one.

Let us introduce the following notation
\[ \varphi_1 \varphi_2 \varphi_3 \equiv \frac{1}{3}[\varphi_1, \varphi_2, \varphi_3], \quad \varphi_1, \varphi_2, \varphi_3 \in F \] (96)
and the new quantity (a dual state cubic in the original amplitudes)
\[ \tilde{\varphi} \equiv \varphi \varphi \varphi = \frac{1}{3} K_\varphi \varphi \] (97)
(Recall Eq. (82) and (89).)

Then the new form of (57) is
\[ \varphi_1 \varphi_2 \varphi_3 = \varphi_1 \varphi_2 \varphi_3 + \frac{2}{3} \{\varphi_2, \varphi_3\} \varphi_1 - \frac{1}{3} \{\varphi_3, \varphi_1\} \varphi_2 - \frac{1}{3} \{\varphi_1, \varphi_2\} \varphi_3 \] (98)
with the remaining axioms left intact.

Let us now combine \(\tilde{\varphi}\) with the (90) symplectic form to create a polynomial which is quartic in the amplitudes of our fermionic state \(\varphi \in F\).
\[ D(\varphi) \equiv \{\varphi, \tilde{\varphi}\} \] (99)
Table 1  GSLOCC Entanglement classes

| Class | $D(\psi)$ | $\tilde{\varphi}$ | $K_\varphi$ | $\varphi$ | $z_\Omega\psi$ |
|-------|-----------|------------------|-------------|----------|----------------|
| I.    | $\neq 0$  | $\neq 0$         | $\neq 0$    | $(1 + p^{1234} + p^{3456} + p^{1256})|\text{vac}\rangle$ | $|111\rangle + |001\rangle + |010\rangle + |100\rangle$ |
| II.   | $0$       | $\neq 0$         | $\neq 0$    | $(1 + p^{1234} + p^{3456})|\text{vac}\rangle$ | $|111\rangle + |001\rangle + |100\rangle$ |
| III.  | $0$       | $0$              | $\neq 0$    | $(1 + p^{1234})|\text{vac}\rangle$ | $|111\rangle + |001\rangle$ |
| IV.   | $0$       | $0$              | $0$         | $|\text{vac}\rangle$ | $|111\rangle$ |
| V.    | $0$       | $0$              | $0$         | $0$      | $0$            |

It is useful to organize formally the $(\eta, y, x, \xi)$ quantities into a matrix \[10\] then in this notation one has

$$
\tilde{\varphi} = \begin{pmatrix}
\xi \\
y \\
x \\
\eta
\end{pmatrix} = 2 \begin{pmatrix}
-x\kappa - \text{Pf}(x) & -(\eta x^2 - 2x \times y^2) + \kappa y \\
(\xi y^2 - 2y \times x^2) & \eta \kappa + \text{Pf}(y)
\end{pmatrix}
$$

where

$$
\kappa \equiv \frac{1}{2}(\eta \xi - (x, y)).
$$

The explicit form of the quartic polynomial is

$$
D(\psi) = 4[\kappa^2 - (x^2, y^2) + \xi \text{Pf}(y) + \eta \text{Pf}(x)].
$$

where

$$
\text{Pf}(x) = \frac{1}{2^3 3!} \epsilon_{IJKLMN} x^{IJ} x^{KL} x^{MN}, \quad \text{Pf}(y) = \frac{1}{2^3 3!} \epsilon_{IJKLMN} y^{IJ} y^{KL} y^{MN}.
$$

is the Pfaffian of a $6 \times 6$ antisymmetric matrix.

One can then show \[14\] that

$$
D(g\psi) = D(\psi), \quad \{g\varphi_1, g\varphi_2\} = \{\varphi_1, \varphi_2\}, \quad g \in \text{Spin}(12, \mathbb{C}), \quad \varphi \in \mathcal{F}.
$$

This means that our polynomial and the symplectic form is invariant under the non-trivial part of the SLOCC group $GL(1, \mathbb{C}) \times \text{Spin}(12, \mathbb{C})$.

Now the SLOCC classification of entanglement types for fermionic states (i.e. the classification of spinors \[31\]) can be described in the following elegant manner. Suppose we are given a state $\varphi \in \mathcal{F}$. We read off the 32 complex amplitudes. Then we calculate the matrix of the form \[100\], the operator $K_\varphi$ of the form \[77\], and the quartic invariant of the form \[102\]. Then one has to check which of the following five cases holds (see Table 1.).

Having determined the entanglement class one can be sure that via a SLOCC transformation our state $\varphi$ can be transformed into some of the representatives showing up in the Table 1.
Notice that in the sixth column of Table 1, we have given the embedded qubit version of the unnormalized fermionic state. In arriving at this form we have used Eqs. (22) and (34). The I. class in this picture corresponds to the GHZ class of three qubits, since the representative state is the $(X \otimes X \otimes X)(H \otimes H \otimes H)$ (bit flip gates followed by Hadamard gates) transformed version (a SLOCC transformation) of the famous $^6$ GHZ state $|000\rangle + |111\rangle$. As far as the II. class is concerned its representative can be transformed by a bit flip gate $I \otimes X \otimes I$ to the usual representative of the so called $W$-class i.e. $|110\rangle + |101\rangle + |011\rangle$. The class III. representative is a biseparable state of the form $(|11\rangle + |00\rangle) \otimes |1\rangle$. Finally $|111\rangle$ is representing the totally separable class. For a discussion of these classes see the original paper of Dür et. al. [8]. For a discussion on the geometric meaning of the fact that the same entanglement patterns appear for all of our special embedded entangled systems see Ref. [29].

8 Connecting fermionic systems to exceptional symmetries

Let us summarize what we have found in this paper in connection with special embedded fermionic systems. We have five such systems, all of them embedded in the fermionic system with 6 modes or 12 Majorana modes. In the positive chirality representation the number of complex amplitudes of their representative quantum states splits according to the pattern: $1 + \sigma + \sigma + 1$ where $\sigma = 1, 3, 6, 9, 15$. The numbers $6, 9, 15$ are related to the complexifications of $3 \times 3$ Hermitian matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$. The remaining numbers are arising as restriction to diagonal matrices. Such matrices are forming Jordan algebras [34] $J_F$ with $F$ taken from the three possibilities $\mathbb{R}, \mathbb{C}, \mathbb{H}$. There is also a construction of systems based on two copies of such Jordan algebras [34] and two scalars answering the $1 + \sigma + \sigma + 1$ type split, called Freudenthal triple systems [9, 10, 35]. Such objects are featuring triple products and are used for constructions of exceptional groups. Since we have come accross these algebraic structures in connection with our fermionic systems, we are expecting that these structures can also be linked to exceptional groups in a natural manner.

According to Table 2. We observe that for four of our systems the number $2m + |k| + 3$ gives the dimensions of exceptional groups i.e. $|g|$. But we have to point out that there is an important caviat to this. Indeed, for the largest exceptional group $e_8$ the question mark shows that we did not manage to associate to this case any simple fermionic system. The reader familiar with Jordan algebras realizes that the last entry of our table should be associated with Jordan algebras coming from the complexification of octonion hermitian matrices, i.e. $J_O$. Since the multiplication of octonions is non associative it is not at all surprising that a simple fermionic system is not capable of reproducing this algebraic structure. In the last section of our conclusions we present some hints, how this case could also be incorporated into our formalism.

In closing this paper we will not present an explicit dictionary between the generators of the groups $g_2, so(8), f_4, e_6, e_7$ and fermionic structures. In the following we

---

$^6$ Here $X, Z$ are the $\sigma_x, \sigma_z$ Pauli matrices implementing bit flip and phase flip. $\sqrt{2}H = X + Z$ is the Hadamard gate implementing discrete Fourier transformation for qubits.
only would like to reveal two different ways how the exceptional symmetries could be realized on our fermionic systems.

First we would only like to indicate that based on Ref. [18] in principle one should be able to obtain a nonlinear realization of exceptional symmetries in a following straightforward manner. We merely sketch the basic ideas details will be intended to be explored in future work.

The idea is to take our basic 32 dimensional fermionic system denoted by \( \mathcal{F} \) and an extra copy of \( \mathbb{C} \). The direct sum of these spaces forms a 33 dimensional vector space. This will be the representation space for the exceptional group \( E_7 \). Define now \( 32 + 32 \) extra generators acting on \( \mathcal{R} \equiv \mathcal{F} \oplus \mathbb{C} \) in a convenient manner [18]. Also take an extra copy of the one qubit SLOCC subgroup \( SL(2, \mathbb{C}) \) with 3 generators \( E_\pm \) and \( H \) satisfying the well-known commutation relations

\[
[E_+, E_-] = H, \quad [H, E_\pm] = \pm 2E_\pm.
\] (105)

and specify their action on \( \mathcal{R} \). An interesting feature of this realization is the action of the generator \( F \) on an element of \( \mathcal{R} \) of the form \((\varphi, \zeta) \in \mathcal{F} \oplus \mathbb{C}\). We have [18]

\[
E_-(\zeta) = D(\varphi) + \zeta^2, \quad E_-(\varphi) = \tilde{\varphi} - \zeta \varphi.
\] (106)

These expressions show that the action of the generator \( E_- \) is featuring two of the basic quantities needed for the classification of entanglement types showing up in Table 1. They are the quartic invariant \( D(\varphi) \) and the dual state \( \tilde{\varphi} \). Hence transformations generated by \( E_- \) are clearly capable of changing the entanglement type of the state \( \varphi \).

Hence the \( E_- \) generator of the extra SLOCC-like transformation is representing some sort of global manipulation on our entangled system. It would be very interesting to explore the physical meaning of this transformation.

Table 2: Special fermionic entangled systems and Freudenthal systems. The SLOCC group of all systems has the form \( GL(1, \mathbb{C}) \times K \). Here \(|k|\) is dimension of the Lie algebra of \( K \). This is the number of nontrivial complex SLOCC parameters. \( m \) is the number of complex amplitudes of the corresponding entangled system written in the (53)-(54) form. In the 4.-7. rows of the table the cubic Jordan algebras with elements of the (51) form, underlying the Freudenthal construction are \( J_{\mathbb{R}}, J_{\mathbb{C}}, J_{\mathbb{H}} \) and \( J_{\mathbb{O}} \). These are based on the real numbers, complex numbers, quaternions and octonions. In the 2.-3. rows the corresponding Jordan algebras are obtained by restriction to diagonal elements. Notice that the number \( 2m + |k| + 3 \) gives the dimensions of the complex \( g \) Lie algebras \( g_2, so(8), f_4, e_6, e_7 \) and \( e_8 \)

| Type                           | \( K \)          | \(|k|\) | \( m = \dim \mathcal{F} \) | \( 2m + |k| + 3 \) |
|--------------------------------|------------------|--------|-----------------------------|-------------------|
| 3 bosonic qubits               | \( SL(2, \mathbb{C})^{3} \) | 3      | 1 + 1 + 1 + 1 + 1 + 1 + 1 | 14                |
| 3 qubits                       | \( SL(2, \mathbb{C})^{3} \) | 9      | 1 + 3 + 3 + 1 + 1 + 1 + 1 | 28                |
| 3 fermions 6 symplectic modes  | \( Sp(6, \mathbb{C}) \)  | 21     | 1 + 6 + 6 + 1 + 1 + 1 + 1 | 52                |
| 3 fermions 6 modes             | \( SL(6, \mathbb{C}) \)  | 35     | 1 + 9 + 9 + 1 + 1 + 1 + 1 | 78                |
| fermions 6 modes               | \( Spin(12, \mathbb{C}) \) | 66     | 1 + 15 + 15 + 1 + 1 + 1 + 1 | 133               |
| \?                             | \( E_7(\mathbb{C}) \)  | 133    | 1 + 27 + 27 + 1 + 1 + 1 + 1 | 248               |
Further observations on this structure reveal other interesting possibilities. Take for instance now the infinitesimal generators of the \( \text{Spin}(12, \mathbb{C}) \) part of the SLOCC group (66 generators) and also specify their action on \( R \). Then it is well-known [18, 35] that one can decompose the adjoint representation of the exceptional group \( E_7 \) which has 133 generators with respect to the subgroup of \( \text{Spin}(12, \mathbb{C}) \times D \) where \( D \) refers to the subgroup generated by \( H \) in the following way

\[
g_{-2} \oplus g_{-1} \oplus g_0 \oplus g_1 \oplus g_2 \leftrightarrow 1 + 32 \oplus (66 + 1) \oplus 32 \oplus 1 \quad (107)
\]

Since \( D \) itself is a part of the extra \( SL(2, \mathbb{C}) \) group, then the above decomposition of the adjoint of \( E_7 \) corresponds to a more general one with respect to the group \( \text{Spin}(12, \mathbb{C}) \times SL(2, \mathbb{C}) \) of the form

\[
133 \leftrightarrow (66, 1) \oplus (1, 3) \oplus (32, 2), \quad \text{Spin}(12, \mathbb{C}) \times SL(2, \mathbb{C}) \subset E_7. \quad (108)
\]

This gives the mathematical meaning of the formula \( 2m + |g| + 3 \) giving the dimensions of the exceptional groups we have found in Table 2.

Indeed, similar decompositions also hold for the remaining groups \( E_6, F_4, D_4 \) and \( G_2 \). The corresponding decompositions are of the form

\[
egin{align*}
78 & \leftrightarrow (35, 1) \oplus (1, 3) \oplus (20, 2), \quad \text{SL}(6, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \subset E_6. \quad (109) \\
52 & \leftrightarrow (21, 1) \oplus (1, 3) \oplus (14, 2), \quad \text{Sp}(6, \mathbb{C}) \times \text{SL}(2, \mathbb{C}) \subset F_4. \quad (110) \\
28 & \leftrightarrow (9, 1) \oplus (1, 3) \oplus (8, 2), \quad \text{SL}(2, \mathbb{C})^{\times 3} \times \text{SL}(2, \mathbb{C}) \subset D_4. \quad (111) \\
14 & \leftrightarrow (3, 1) \oplus (1, 3) \oplus (4, 2), \quad \text{SL}(2, \mathbb{C})_{\text{diag}}^{\times 3} \times \text{SL}(2, \mathbb{C}) \subset G_2. \quad (112)
\end{align*}
\]

Moreover, though in this paper we did not manage to find a fermionic realization of the largest exceptional group a similar decomposition holds even in this case

\[
248 \leftrightarrow (133, 1) \oplus (1, 3) \oplus (56, 2), \quad E_7(\mathbb{C}) \times \text{SL}(2, \mathbb{C}) \subset E_8. \quad (113)
\]

These results give rise to a second way of realizing exceptional symmetries. First notice that the common feature of these decompositions is the appearance of a new copy of \( \text{SL}(2, \mathbb{C}) \) which also suggests a new role for our exceptional symmetries in a new type of system consisting of embedded fermionic ones and an extra qubit. The states of such physical systems are represented by elements of a vector space of the form \( F \otimes \mathbb{C}^2 \).

As an example have a look at Eq. (111). Clearly the left hand side of \( \text{SL}(2, \mathbb{C})^{\times 3} \times \text{SL}(2, \mathbb{C}) \subset D_4 \) shows that in this case \( F \otimes \mathbb{C}^2 \) contains the four-qubit state space on which the four qubit SLOCC subgroup \( \text{SL}(2, \mathbb{C})^{\times 4} \) acts. Since \( D_4 \simeq SO(8, \mathbb{C}) \) in this case the (111) decomposition suggests that the four qubit states space taken togetheer with the corresponding SLOCC transformations can alternatively be described via studying the adjoint action of the \( D_4 \) Lie algebra \( \mathfrak{so}(8) \) on itself. Indeed, one has the Cartan decomposition

\[
\mathfrak{so}(8) = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{h} = \mathfrak{so}(4) \oplus \mathfrak{so}(4) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \quad (114)
\]
Here $m$ is just the vector space $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ on which the four qubit SLOCC group acts. The new way of looking at this action is effected by studying the commutator $[\mathfrak{h}, m] \subset m$, which is indeed given by the infinitesimal form of the adjoint action. Geometrically this means that the 16 dimensional four qubit state space in this picture is arising as the symmetric space $SO(8, \mathbb{C})/SO(4, \mathbb{C}) \times SO(4, \mathbb{C})$, where the SLOCC subgroup is $SO(4, \mathbb{C}) \times SO(4, \mathbb{C}) \simeq SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$. Hence the new form of the (111) decomposition is

$$28 \leftrightarrow (3, 1, 1, 1) \oplus (1, 3, 1, 1) \oplus (1, 1, 3, 1) \oplus (1, 1, 1, 3) \oplus (2, 2, 2, 2).$$

(115)

Clearly this

$$\mathfrak{g} = \mathfrak{h} \oplus m$$

(116)
type reinterpretation for the (108)-(112) decompositions of the Lie algebras $e_7, e_6, f_4, so_8, g_2$ can always be given. Hence in this picture one can conclude that exceptional symmetries can be realized in the form of $G$ Lie groups acting on entangled systems with state spaces of the form $\mathcal{F} \otimes \mathbb{C}^2$ that can alternatively be written as symmetric spaces of the form $G/H$ where $GL(1, \mathbb{C}) \times H$ is the group action of SLOCC transformations. Here $H = K \times SL(2, \mathbb{C})$ where $K$ is taken from the possible groups showing up in Table 2. The $[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$ type commutators are explicitly given by the $H = K \times SL(2, \mathbb{C})$ SLOCC group structure. The $[\mathfrak{h}, m] \subset m$ ones on the other hand boil down to the SLOCC action on $\mathcal{F}$ already discussed in Sect. 3 and the usual $SL(2, \mathbb{C})$ action on the special qubit state space $\mathbb{C}^2$. In order to explicitly build up the commutators of the corresponding exceptional Lie algebras, the hard part is to construct the commutators of the form $[m, m] \subset \mathfrak{h}$. Since $G/H$ is a symmetric space no terms belonging to $m$ appear on the right hand side. We are intending to elaborate on such a construction in a separate work.

9 Conclusions

In this paper we have been considering five special entangled quantum systems. They were embedded ones in the sense that they can be obtained by restriction from a basic one. This basic system was a fermionic one with six modes or twelve Majorana modes. The embedded systems contained in this system were: three fermions with six modes, three fermions with six symplectic modes, three qubits, and three bosonic qubits. These systems are called special because of their special symmetry properties. We observed that their symmetry properties are realized by simple Lie groups of exceptional type. We have established that four from our five systems is connected to the exceptional groups $E_6, E_7, F_4$ and $G_2$. It is well-known in the mathematical literature that the exceptional symmetries are connected to the appearance of special structures left invariant by the correponding exceptional transformations. These structures are a symplectic form and a triple product connected to each other in a peculiar manner forming a symplectic triple system. As the main result of the paper we verified by a detailed explicit construction that our special embedded fermionic systems are form-
ing such triple systems. We also suggested two possible ways how exceptional group
actions should be realized on our embedded systems. Our results draw exceptional
symmetry structures under the umbrella of entangled systems of physical relevance.

We must point out however, that we did not manage to give a fermionic entan-
glement interpretation of the largest exceptional group $E_8$. However, the results of
Sect. 8. clearly show that this system should be considered as a basic one capable of
incorporating all of the fermionic systems studied in this paper. Therefore the problem
of finding the physical interpretation of the $E_8$ symplectic triple system is of basic
importance. Clearly the construction of a fermionic Fock space representation of this
$E_8$ case is not at all obvious. One solid piece of evidence is that if we use two copies
of embedded fermionic systems one in the single and the other in the so called double
occupancy representation [15] of qubits there is a fermionic code [19] featuring oper-
ators related to the Lie algebra $\mathfrak{e}_8$. However, it is not at all clear how such constructs
should be adjusted in a manner to be able to form a fermionic system containing our
basic fermionic ones with six modes via a symplectic triple system structure.

There are two interesting possibilities left open by this paper. One of them is the
explicit construction of exceptional symmetry generators acting on $F \oplus \mathbb{C}$, and
the other is connected to $F \otimes \mathbb{C}^2$.

In the first case what we have is just a fermionic entangled state $\varphi \in F$ and an extra
variable $\zeta \in \mathbb{C}$. $\zeta$ is related to the change in the quantities $D(\varphi)$ and $\bar{\varphi}$ characterizing
the entanglement type. This change is effected via the action of a special generator
$E_-$ of the exceptional symmetry. See Eq. (106). Hence this exceptional symmetry
generator is connected to global, i.e. entanglement type changing transformations. This is to be contrasted with the local nature of SLOCC transformations. This example
already shows that in order to reveal the physical meaning of the exceptional symmetry
transformations the explicit form of its group action on $(F, \zeta)$ is of utmost importance.

In the second case we have a fermionic system and an extra qubit. Hence in this
case we have a new type of entangled system with properties encapsulated in the $m$
part of the exceptional Lie algebra. On the other hand, see Eq. (114), the $\mathfrak{h}$ part is encap-
sculating the algebra of SLOCC entanglement transformations. However for realizing
the exceptional symmetry transformations in the adjoint manner briefly discussed in
Sect. 8, we are faced with the following problem. What is the physical meaning of com-
mutators of the form $[m, m] \subset \mathfrak{h}$ that have the strange interpretation of commutators
of entangled states giving rise to SLOCC transformations?

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