Nonlocal error bounds for piecewise affine mappings

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Abstract

The paper is devoted to a detailed analysis of nonlocal error bounds for nonconvex piecewise affine mappings. We both improve some existing results on error bounds for such mappings and present completely new necessary and/or sufficient conditions for a piecewise affine function to have an error bound on various types of bounded and unbounded sets. In particular, we show that any piecewise affine function has an error bound on an arbitrary bounded set and provide several types of easily verifiable sufficient conditions for such functions to have an error bound on unbounded sets. We also present general necessary and sufficient conditions for a piecewise affine map to have an error bound on a finite union of polyhedral sets (in particular, to have a global error bound), whose derivation reveals a structure of sublevel sets and recession functions of piecewise affine mappings.

1 Introduction

Piecewise affine mappings have been an object of active research for many years. General topological and order-theoretic properties of the set of piecewise affine (and locally piecewise affine) functions were studied in [1, 2]. Various representations of piecewise affine mappings, such as max-min, min-max, and DC (Difference-of-Convex functions) representations, were studied in [11, 12, 22], while algorithms for constructing such representations were developed in [3, 9, 16, 28]. The surjectivity property for piecewise affine maps was analysed in [24, 25], while the bijectivity of such mappings was studied in [18].

Error bound property is an important concept in variational analysis having multiple applications and attracting significant attention of researchers [4, 5, 7, 10, 17, 23]. Usually, one studies only local error bounds, since theorems on nonlocal/global error bound property for general nonlinear mappings involve conditions [4, 7] that are very hard to verify in particular cases. Nonetheless, in some specific cases (such as convex [3, 21, 30], piecewise convex [20], DC [19], and polynomial [29] cases) one can use a structure of the problem to obtain simple condition ensuring the nonlocal/global error bound property. In the case of piecewise affine mappings, such conditions can be expressed in terms of the so-called recession function of a piecewise affine map [13]. However, to the best of

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the author’s knowledge, nonlocal error bounds (that is, error bounds on various types of bounded and unbounded sets) for piecewise affine mappings, as well as conditions for such maps to have the global error bound property that do not involve the recession function, have not been properly studied before.

The main goal of this paper is to present a detailed analysis of nonlocal error bounds for nonconvex piecewise affine mapping on various types of sets. We aim at improving some existing results on this topic, as well as obtaining new necessary and/or sufficient conditions for a piecewise affine function to have a nonlocal error bound. To this end, we heavily utilise Gorokhovik-Zorko’s representation theorem for piecewise affine functions \[11, 12\] that allows one to better understand a structure of sublevel sets of such functions and conditions ensuring that they have nonlocal error bounds.

We prove that a piecewise affine function always has an error bound on an arbitrary bounded set and provided necessary and/or sufficient conditions for such function to have an error bound on various types of unbounded sets. In the unbounded case, we obtain simple and easily verifiable sufficient conditions for a piecewise affine mapping to have an error bound, as well as more theoretical necessary and sufficient conditions that might be less appealing for applications, but nonetheless reveal deep interrelations between some properties of piecewise affine functions, their sublevel sets, recession functions, and “flat parts” (that is, polyhedral sets on which a piecewise affine map is constant). We also apply all these results to an analysis of error bounds for systems of piecewise affine equality and inequality constraints.

It should be mentioned that apart from many completely new results, we present improved versions of Robinson’s \[26\] and Gowda’s \[13\] results on error bounds for piecewise affine maps. Robinson \[26\] showed that for a piecewise affine map \(F : \mathbb{R}^d \to \mathbb{R}^m\) there exists \(\rho > 0\) such that the function \(\|F(\cdot)\|\) has an error bound on the set \(V(\rho) = \{x : \|F(x)\| \leq \rho\}\). We improve this result by showing how \(\rho > 0\) from the Robinson’s theorem can be easily estimated (see Theorem \[2\]) and how the procedure for estimating \(\rho\) can in some cases be used to verify that the function \(\|F(\cdot)\|\) has a global error bound.

In turn, Gowda in \[13\] presented necessary and sufficient conditions for a piecewise affine mapping to have a global error bound in terms of the recession function of this map. We extend Gowda’s result to the case of error bounds on a finite union of polyhedral sets and, moreover, obtain new necessary and sufficient conditions for a piecewise affine map to have an error bound on a finite union of polyhedral sets (in particular, a global error bound) that are not based on the use of the recession function (Lemma \[5\]).

The paper is organised as follows. Some auxiliary definitions and Gorokhovik’s representation theorem for piecewise affine functions are collected in Section \[2\]. Section \[3\] is devoted to error bounds for real-valued nonconvex piecewise affine functions. Subsection \[3.1\] contains an improved version of the Robinson’s result \[26\] and an analysis of error bounds for piecewise affine functions on bounded sets. Several types of sufficient conditions for the existence of an error bound on unbounded sets are given in Subsection \[3.2\], while general necessary and sufficient conditions for a piecewise affine function to have an error bound on a finite union of polyhedral sets are studied in Subsection \[3.3\]. Finally, in Section \[4\] the results on error bounds for piecewise affine functions are applied to an analysis of error bounds for systems of piecewise affine equality and inequality constraints.
2 Piecewise affine functions

Let us recall some auxiliary definitions and results on piecewise affine functions that will be used throughout the article. We start with the definition of piecewise affine function [11, 12, 16].

Definition 1. A set $Q \subseteq \mathbb{R}^d$ is called polyhedral, if it is the intersection of a finite number of closed half-spaces. A finite family $\sigma = \{Q_1, \ldots, Q_s\}$ of polyhedral subsets $Q_i$ of $\mathbb{R}^d$ is called a polyhedral partition of $\mathbb{R}^d$, if

$$\bigcup_{i=1}^s Q_i = \mathbb{R}^d, \quad \text{int } Q_i \neq \emptyset, \quad \text{int } Q_i \cap \text{int } Q_j \neq \emptyset \quad \forall i, j \in \{1, \ldots, s\}, \; i \neq j,$$

where int $Q$ is the topological interior of a set $Q \subset \mathbb{R}^d$. Finally, a function $F : \mathbb{R}^d \to \mathbb{R}^m$ is called piecewise affine, if there exists a polyhedral partition $\sigma = \{Q_1, \ldots, Q_s\}$ of $\mathbb{R}^d$ and a collection of affine mappings $F_i : \mathbb{R}^d \to \mathbb{R}^m$, $F_i(x) = A_i x + b_i$ with $A_i \in \mathbb{R}^{m \times n}$ and $b_i \in \mathbb{R}^m$, such that $F(x) = F_i(x)$ for all $x \in Q_i$ and $i \in \{1, \ldots, s\}$.

Remark 1. As was noted in [11], the assumption that the sets $Q_i$ from the polyhedral partition have nonempty interiors is, in fact, redundant. It is sufficient to suppose that only relative interiors of the sets $Q_i$, $i \in \{1, \ldots, s\}$, are pairwise disjoint.

It is worth mentioning that the set of all piecewise affine mappings from $\mathbb{R}^d$ to $\mathbb{R}^m$ is closed under addition, multiplication by scalar, as well as coordinate-wise supremum and infimum of finite families of functions. Furthermore, this set is the smallest vector lattice (with respect to pointwise operations) containing all affine functions. Finally, the composition of piecewise affine mappings is also a piecewise affine map [11].

Apart from representations of piecewise affine functions in terms of polyhedral partitions, one often has to deal with various analytical representation of such functions. As was proved in [11,12], among various analytical representations of piecewise affine functions there always exist a natural DC (Difference-of-Convex functions) decomposition of such functions and a min-max/max-min representation that are especially convenient for theoretical analysis.

Theorem 1. Let $F : \mathbb{R}^d \to \mathbb{R}^m$ be a given map. The following assertions are equivalent:

1. $F$ is piecewise affine;

2. $F$ can be represented in the form

$$F(x) = \sup_{i \in I} G_i(x) + \inf_{j \in J} H_j(x) \quad \forall x \in \mathbb{R}^d$$

for some finite families of affine mappings $G_i : \mathbb{R}^d \to \mathbb{R}^m$, $i \in I := \{1, \ldots, \ell\}$, and $H_j : \mathbb{R}^d \to \mathbb{R}^m$, $j \in J := \{1, \ldots, s\}$, where the supremum and the infimum are taken with respect to the coordinate-wise partial order in $\mathbb{R}^m$;

3. $F$ can be represented in the form

$$F(x) = \inf_{i \in I} \sup_{j \in J(i)} G_{ij}(x) \quad \forall x \in \mathbb{R}^d$$
for some affine mappings $G_{ij}: \mathbb{R}^d \to \mathbb{R}^m$, $i \in I := \{1, \ldots, \ell\}$, and $j \in J(i) = \{1, \ldots, s(i)\}$.

4. $F$ can be represented in the form

$$F(x) = \sup_{i \in I} \inf_{j \in J(i)} G_{ij}(x) \quad \forall x \in \mathbb{R}^d$$

for some affine mappings $G_{ij}: \mathbb{R}^d \to \mathbb{R}^m$, $i \in I := \{1, \ldots, \ell\}$, and $j \in J(i) = \{1, \ldots, s(i)\}$.

Remark 2. Methods for constructing analytical representations of piecewise affine functions from their representations via polyhedral partitions were studied in [16, 28]. In turn, methods for constructing DC decompositions of piecewise affine functions (which can be used to construct max-min and min-max representations of such functions) from their arbitrary analytical representations were developed in [3] (see also [9]).

3 Nonlocal error bounds for piecewise affine maps

In this section we study error bounds for piecewise affine functions on various types of sets. Our main goal is to show that nonconvex piecewise affine functions have an error bound on any bounded set and provide necessary and/or sufficient conditions for such functions to have an error bound on an unbounded set.

3.1 Error bounds on bounded sets

For any function $f: \mathbb{R}^d \to \mathbb{R}$ denote by $S(f) = \{x \in \mathbb{R}^d \mid f(x) \leq 0\}$ the lower 0-level set of $f$, and let $[f]_+(x) = \max\{f(x), 0\}$. Recall that $f$ is said to have an error bound with constant $\tau > 0$ on a set $V \subset \mathbb{R}^d$, if

$$\tau \text{dist}(x, S(f)) \leq [f]_+(x) \quad \forall x \in V,$$

where $\text{dist}(x, K) = \inf\{y \in K \mid \|x - y\|\}$ is the distance between a point $x \in \mathbb{R}^d$ and a set $K \subset \mathbb{R}^d$, and $\|\cdot\|$ is a norm on $\mathbb{R}^n$. The supremum of all those $\tau$ for which inequality (1) holds true is denoted by $\tau(f, V)$ or simply $\tau(V)$, if a function $f$ is fixed. Finally, $f$ is said to have a global error bound, if there exists $\tau > 0$ such that inequality (1) holds true for $V = \mathbb{R}^d$.

Let $f: \mathbb{R}^d \to \mathbb{R}$ be a piecewise affine function. We start our analysis of error bounds for piecewise affine functions by proving a new improved version of the Robinson’s result on error bounds for piecewise affine maps from [26].

Theorem 2. Let $S(f) \neq \emptyset$ and $f$ have the form

$$f(x) = \min_{i \in I} \max_{j \in J(i)} (a_{ij} + \langle v_{ij}, x \rangle) \quad \forall x \in \mathbb{R}^d$$

for some $a_{ij} \in \mathbb{R}$, $v_{ij} \in \mathbb{R}^d$, $i \in I := \{1, \ldots, \ell\}$, and $j \in J(i) = \{1, \ldots, s(i)\}$. Let also

$$f_i(x) = \max_{j \in J(i)} (a_{ij} + \langle v_{ij}, x \rangle), \quad f_i^* = \inf_{x \in \mathbb{R}^d} f_i(x)$$

for all $i \in I$. Then the following statements hold true:
1. if \( f_i^* \leq 0 \) for all \( i \in I \), then \( f \) has a global error bound;

2. if there exists \( i \in I \) such that \( f_i^* > 0 \), then \( f \) has an error bound on the set \( V = \{ x \in \mathbb{R}^d \mid f(x) < \rho \} \) with \( \rho = \min \{ f_i^* \mid i \in I : f_i^* > 0 \} \).

**Proof.**  **Case I.** Suppose that \( f_i^* \leq 0 \) for all \( i \in I \). Note, at first, that each \( f_i \) has a global error bound with some constant \( \tau_i > 0 \). Indeed, if \( f_i^* < 0 \), then the set \( S(f_i) \) is obviously nonempty. If \( f_i^* = 0 \), then taking into account the fact that a bounded below piecewise affine function attains a global minimum (see, e.g. [9, Thm. 4.7]) one can conclude that the set \( S(f_i) \) is nonempty, since it contains a global minimizer of \( f_i \).

As is easily seen, the set \( S(f_i) \) coincides with the set of solutions of the following systems of linear inequalities:

\[
a_{ij} + \langle v_{ij}, x \rangle \leq 0, \quad j \in J(i).
\]

Therefore, by Hoffman’s theorem [15] there exists \( \tau_i > 0 \) such that

\[
\tau_i \text{ dist}(x, S(f_i)) \leq \max_{j \in J(i)} \max \{ 0, a_{ij} + \langle v_{ij}, x \rangle \} = [f_i(x)]_+ \quad \forall x \in \mathbb{R}^d,
\]

that is, \( f_i \) has a global error bound with constant \( \tau_i \).

From (2) and the definitions of \( f_i \) it obviously follows that \( S(f) = \bigcup_{i \in I} S(f_i) \).

Therefore, for any \( i \in I \) and \( x \in \mathbb{R}^d \) one has \( \text{dist}(x, S(f_i)) \geq \text{dist}(x, S(f)) \). Let us check that

\[
[f]_+(x) = \min_{i \in I} [f_i]_+(x) \quad \forall x \in \mathbb{R}^d. \tag{3}
\]

Then for any \( x \in \mathbb{R}^d \) one has

\[
[f]_+(x) = \min_{i \in I} [f_i]_+(x) \geq \min_{i \in I} \tau_i \text{ dist}(x, S(f_i)) \geq \tau \min_{i \in I} \text{ dist}(x, S(f_i)) = \tau \text{ dist}(x, S(f)),
\]

where \( \tau = \min_{i \in I} \tau_i > 0 \). In other words, \( f \) has a global error bound with constant \( \tau \).

Thus, it remains to prove equality (3). Fix any \( x \in \mathbb{R}^d \). For any \( i \in I \) one has \( f(x) \leq f_i(x) \), which implies that \( [f]_+(x) \leq [f_i]_+(x) \) and inequality \( [f]_+(x) \leq \min_{i \in I} [f_i]_+(x) \) holds true.

To prove the converse inequality, note that by definition there exists \( i \in I \) such that

\[
[f]_+(x) := \max \left\{ 0, \min_{i \in I} f_i(x) \right\} = \max \{ 0, f_i(x) \} = [f_i]_+(x).
\]

Therefore \( [f]_+(x) \geq \min_{i \in I} [f_i]_+(x) \), and the proof of the first case is complete.

**Case II.** Suppose now that \( f_i^* > 0 \) for some \( i \in I \). Introduce the index set \( I_0 = \{ i \in I \mid f_i^* \leq 0 \} \) and the function \( g(x) = \min_{i \in I_0} f_i(x) \). Note that \( f(x) = g(x) \) for any \( x \in V := \{ x \in \mathbb{R}^d \mid f(x) < \rho \} \) and \( S(f) = S(g) \). Moreover, by the first part of the proof \( g \) has a global error bound. Hence, as one can readily check, \( f \) has an error bound on \( V \).

As a simple corollary to the theorem above we can prove that positively homogeneous piecewise affine functions (such functions are sometimes called piecewise linear [11,12]) always have a global error bound.
Corollary 1. Let $f$ be positively homogeneous. Then it has a global error bound.

Proof. Since $f$ is positively homogeneous, by [12 Thm. 3.2] this function can be represented in the form (2) with $a_{ij} = 0$ for all $i$ and $j$. Therefore, for the functions $f_i$, defined as in Theorem 2, one has $f_i(0) = 0$, which implies that $f_i^* \leq 0$ for all $i \in I$. Consequently, $f$ has a global error bound by Theorem 2.

The two following simple examples demonstrate that in the case of non-positively homogeneous piecewise affine functions the value $\rho$ from Theorem 2 cannot be improved, but, at the same time, this theorem does not describe the largest set on which a piecewise affine function has an error bound.

Example 1. Let $d = 1$ and

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } x \in [0, 1], \\ 1, & \text{if } x \geq 1. \end{cases}$$

Clearly, one has $S(f) = (-\infty, 0]$ and

$$f(x) = \min \{f_1(x), f_2(x)\}, \quad f_1(x) = 1, \quad f_2(x) = \max \{0, x\}.$$ 

Therefore $f_1^* = 1, f_2^* = 0$, and by Theorem 2 $f$ has an error bound on the set $\{x \in \mathbb{R} \mid f(x) < 1\}$. Furthermore, as is easily seen, $f$ does not have an error bound on the set $\{x \in \mathbb{R} \mid f(x) < \rho\}$ for any $\rho > 1$.

Example 2. Let $d = 1$ and

$$f(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ x, & \text{if } x \in [0, 1], \\ 1, & \text{if } x \in [1, 2], \\ x - 1, & \text{if } x \geq 2. \end{cases}$$

Then one has

$$f(x) = \min \{f_1(x), f_2(x)\}, \quad f_1(x) = \max \{0, x\}, \quad f_2(x) = \max \{1, x - 1\}.$$ 

Consequently, $f_1^* = 0, f_2^* = 1$, and by Theorem 2 $f$ has an error bound on the set $\{x \in \mathbb{R} \mid f(x) < 1\}$. However, in actuality, $f$ has a global error bound and $\tau(\mathbb{R}) = 0.5$.

Denote by $U(x, r) = \{y \in \mathbb{R}^d \mid \|x - y\| < r\}$ the open ball with centre $x$ and radius $r > 0$. Recall that $f$ is said to have an error bound at a point $\overline{x} \in S(f)$, if there exist $\tau > 0$ and a neighbourhood $V$ of $\overline{x}$ for which inequality (1) holds true. Let us show that from Theorem 2 it follows that any piecewise affine function $g$ has a uniform error bound at every $\overline{x} \in S(g)$.

Proposition 1. For any $\overline{x} \in S(f)$ the function $f$ has an error bound at $\overline{x}$. Moreover, there exist $r > 0$ and $\tau_0 > 0$ such that $\tau(U(\overline{x}, r)) \geq \tau_0$ for all $\overline{x} \in S(f)$, provided $S(f) \neq \emptyset$. 

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Proof. By Theorem 1 the function $f$ can be represented in the min-max form (2). If, in the notation of Theorem 2, one has $f^*_i \leq 0$ for all $i \in I$, then by this theorem one can set $\tau_0 = \tau(\mathbb{R}^d)$ and choose any $r > 0$.

Therefore, suppose that $f^*_i > 0$ for some $i \in I$. Let $\rho > 0$ be defined as in Theorem 2 and denote $V = \{ x \in \mathbb{R}^d \mid f(x) < \rho \}$. Then $\tau(V) > 0$ by Theorem 2.

From the representation (2) it obviously follows that $f$ is globally Lipschitz continuous with constant

$$ L = \max_{i,j \in J(0)} \| v_{ij} \|_* , \quad \| x \|_* = \max \{ \langle x, y \rangle \mid y \in U(0,1) \} $$

Consequently, $U(\mathbb{R},r) \subseteq V$ for any $\mathbb{R} \in S(f)$ and $r = \rho/L$, which implies that for any such $\mathbb{R}$ the inequality $\tau(U(\mathbb{R},r)) \geq \tau(V)$ holds true. Thus, one can set $r = \rho/L$ and $\tau_0 = \tau(V)$. \qed

With the use of the previous proposition one can easily prove that a piecewise affine function has an error bound on any bounded set.

**Theorem 3.** Let $S(f) \neq \emptyset$. Then $f$ has an error bound on every bounded subset of $\mathbb{R}^d$.

**Proof.** Fix any bounded set $V \subset \mathbb{R}^d$. Replacing $V$ with its closure, if necessary, one can suppose that $V$ is compact. We need to check that $f$ has an error bound on $V$.

Let $r > 0$ and $\tau_0 > 0$ be from Prp. 1. Note that the set $S(f)$ is closed, since piecewise affine functions are continuous. Therefore, the set $S(f) \cap V$ is compact, and one can find points $x_1, \ldots, x_N$ from this set such that

$$ S(f) \cap V \subset U := \bigcup_{k=1}^N U(x_k, r). $$

Denote $W = V \setminus U$. If $W = \emptyset$, then $V \subset U$, which, as is easily seen, implies that $f$ has an error bound on $V$ with $\tau(V) \geq \tau_0$. Therefore, suppose that $W \neq \emptyset$.

The set $W$ is obviously compact, which implies that the following values are correctly defined and finite:

$$ \alpha = \max_{x \in W} \text{dist}(x, S(f)), \quad \beta = \min_{x \in W} f(x). $$

By definition the sets $S(f)$ and $W$ do not intersect, which implies that $\alpha > 0$ and $\beta > 0$. Consequently, one has

$$ \frac{\beta}{\alpha} \text{dist}(x, S(f)) \leq f(x) = [f]_+(x) \quad \forall x \in W. $$

On the other hand, for any $x \in V \cap U$ the point $x$ belongs to some $U(x_i, r)$, which yields $\tau_0 \text{dist}(x, S(f)) \leq [f]_+(x)$. Therefore $f$ has an error bound on $V$ with constant $\tau = \min \{ \beta/\alpha, \tau_0 \} > 0$. \qed

**Remark 3.** From the theorem above it follows that $f$ has an error bound on any set $V \subset \mathbb{R}^d$ such that the set $V \setminus S(f)$ is bounded, even if both $S(f)$ and $V$ are unbounded (one simply has to apply the theorem to the set $V \setminus S(f)$).
3.2 Error bounds on unbounded sets: sufficient conditions

Now we turn to analysis of error bounds on unbounded sets. As Example 1 shows, a piecewise affine function might not have a global error bound. Therefore, below we study error bounds on an arbitrary unbounded set. First, we provide verifiable sufficient conditions for a piecewise affine function to have an error bound on an unbounded set that do not require any information about the set $S(f)$ and show when these conditions become necessary.

For any nonempty set $V \subseteq \mathbb{R}^d$ denote by $\text{cone} V$ the conic hull of $V$ (i.e., the smallest cone containing $V$), and by $\text{cl} V$ the closure of $V$. Recall also that the 

\textit{recession cone} $0^+ V$ of the set $V$ consists of all those vectors $z \in \mathbb{R}^d$ for which one can find $x \in V$ such that $x + \lambda z \in V$ for all $\lambda \geq 0$.

**Theorem 4.** Let $S(f)$ be nonempty and $V \subset \mathbb{R}^d$ be an unbounded set. Then for $f$ to have an error bound on $V$ it is sufficient that

$$\liminf_{\|x\| \to +\infty, x \in V} \frac{f(x)}{\|x\|} > 0. \quad (4)$$

Moreover, this condition becomes necessary, when $0^+ S(f) \cap \text{cl} \text{cone} V = \{0\}$.

**Proof.** **Sufficiency.** Suppose that condition (4) holds true. Then there exist $r > 0$ and $a > 0$ such that $f(x) \geq a\|x\| \forall x \in V \setminus U(0,r)$. Fix any $x_0 \in S(f)$. With the use of the inequality above one obtains that for any $x \in V \setminus U(0,R)$ with $R = \max\{r, \|x_0\|\}$ the following inequalities hold true:

$$\text{dist}(x, S(f)) \leq \|x - x_0\| \leq \|x_0\| + \|x\| \leq 2\|x\| \leq \frac{2}{a}f(x).$$

On the other hand, by Theorem 3 the function $f$ has an error bound on $U(0,R)$. Hence $f$ has an error bound on $V$ with constant $\tau = \max\{a/2, \tau(U(0,R))\}$.

**Necessity.** Suppose that $f$ has an error bound on $V$ with constant $\tau > 0$ and $0^+ S(f) \cap \text{cl} \text{cone} V = \{0\}$. Our aim is to show that under this assumption

$$\liminf_{\|x\| \to +\infty, x \in V} \frac{\text{dist}(x, S(f))}{\|x\|} > 0. \quad (5)$$

Then taking into account the fact that $[f(x)]_+ \geq \tau \text{dist}(x, S(f))$ by the definition of error bound one obtains the required result.

To prove inequality (5), recall that by Theorem 1 the function $f$ can be represented in the min-max form (2). Let the functions $f_i$ and the values $f^*_i$ be defined as in Theorem 2. Let also $I_0 = \{i \in I | f^*_i \leq 0\}$. Then with the use of (2) one gets that

$$S(f) = \bigcup_{i \in I_0} S(f_i), \quad (6)$$

Therefore, as is easily seen, for any $x \in \mathbb{R}^d$ the following equalities hold true:

$$\text{dist}(x, S(f)) = \inf_{y \in S(f)} \|x - y\| = \min_{i \in I_0} \inf_{y \in S(f_i)} \|x - y\| = \min_{i \in I_0} \text{dist}(x, S(f_i)).$$
Consequently, for the validity of inequality (5) it is sufficient to prove that
\[ \liminf_{\|x\| \to +\infty, x \in V} \frac{\text{dist}(x, S(f_i))}{\|x\|} > 0 \quad \forall i \in I_0. \] (7)

Let us prove this inequality.

Fix any \( i \in I_0 \). By the Motzkin theorem (see [24] Thms. 19.1 and 19.5) there exist a compact set \( K \) such that \( S(f_i) = K + 0^+ S(f_i) \), since \( S(f_i) \) is a polyhedral set. Therefore by the reverse triangle inequality for any \( x \in \mathbb{R}^d \) one has
\[ \text{dist}(x, S(f_i)) = \inf_{y \in K, z \in 0^+ S(f_i)} \|x - y - z\| \geq \inf_{z \in 0^+ S(f_i)} \|x - z\| - C = \text{dist}(x, 0^+ S(f_i)) - C \] (8)
where \( C = \sup_{y \in K} \|y\| \). From equality (8) it follows that \( 0^+ S(f_i) \subseteq 0^+ S(f) \), which thanks to our assumption implies that \( 0^+ S(f_i) \cap \text{cl cone} V = \{0\} \). Consequently, one has
\[ \beta := \min \left\{ \text{dist}(x, 0^+ S(f_i)) \mid x \in \text{cl cone} V, \|x\| = 1 \right\} > 0. \]

Since the recession cone \( 0^+ S(f_i) \) is a cone, the function \( x \mapsto \text{dist}(x, 0^+ S(f_i)) \) is positively homogeneous. Hence
\[ \text{dist}(x, 0^+ S(f_i)) \geq \beta \|x\| \quad \forall x \in \text{cl cone} V. \]
Combining this inequality with inequality (8) one finally obtains that
\[ \text{dist}(x, S(f_i)) \geq \beta \|x\| - C \quad \forall x \in \text{cl cone} V, \]
which obviously implies that condition (7) holds true. \( \square \)

In the case when \( V \) is a cone, one can provide somewhat less restrictive conditions for \( f \) to have an error bound on \( V \) than in the theorem above. Recall that a function \( h: \mathbb{R}^d \to \mathbb{R} \) is said to be coercive on an unbounded set \( V \), if \( h(x_n) \to +\infty \) for any sequence \( \{x_n\} \subset V \) such that \( \|x_n\| \to +\infty \) as \( n \to \infty \). In the case when \( V = \mathbb{R}^d \) we simply say that \( h \) is coercive.

**Theorem 5.** Let \( S(f) \) be nonempty and \( V \subseteq \mathbb{R}^d \) be a cone. Then for \( f \) to have an error bound on \( V \) it is sufficient that \( f \) is coercive on \( V \). Furthermore, this condition becomes necessary, when \( 0^+ S(f) \cap \text{cl cone} V = \{0\} \).

**Proof.** **Necessity.** Let \( f \) have an error bound on \( V \) and \( 0^+ S(f) \cap \text{cl cone} V = \{0\} \). Then by Theorem 4 inequality (4) holds true, which obviously implies that \( f \) is coercive on \( V \).

**Sufficiency.** Suppose now that that \( f \) is coercive on \( V \). Recall that \( f \) can be represented in the form (3) by virtue of Theorem 3. Let the functions \( f_i \) and values \( f_i^* \) be defined as in Theorem 2 and denote \( I_0 = \{i \in I \mid f_i^* \leq 0\} \). As was shown in the proof of Theorem 2 each function \( f_i, i \in I_0, \) has a global error bound with some constant \( \tau_i \).

Let us check that for any \( i \in I \setminus I_0 \) there exists \( \tau_i > 0 \) such that
\[ \tau_i \text{dist}(x, S(f)) \leq [f_i(x)]_+ \quad \forall x \in V. \] (9)
Then taking into account equality (3) and the fact that \( S(f) = \bigcup_{i \in I_0} S(f_i) \) one gets that
\[
[f(x)]_+ = \min_{i \in I} [f_i(x)]_+ \geq \min \left\{ \min_{i \in I_0} \text{dist}(x, S(f_i)), \min_{i \in I \setminus I_0} \text{dist}(x, S(f)) \right\} \\
\geq \tau \text{dist}(x, S(f))
\]
for \( \tau = \min_{i \in I} \tau_i \) and all \( x \in V \), that is, \( f \) has and error bound on \( V \).

Thus, it remains to prove inequality (9). Fix any \( i \in I \setminus I_0 \). Note that by the definition of \( I_0 \) one has \( f_i^* > 0 \), which implies that \( f_i(x) > 0 \), i.e. \( f_i(x) = [f_i]_+(x) \), for all \( x \in \mathbb{R}^d \).

Observe that from equality (2) and our assumption on coercivity of \( f \) on \( V \) it follows that \( f_i \) is coercive on \( V \) as well. Consequently, the function \( g_i(x) = \max_{j \in J(i)} |v_{ij}, x| \) is also coercive on \( V \), since
\[
g_i(x) \geq f_i(x) - a_0 \quad \forall x \in \mathbb{R}^d, \quad a_0 = \max_{j \in J(i)} |a_{ij}|.
\]

Let us check that
\[
\beta := \inf \{ g_i(x) \mid x \in V, \|x\| = 1 \} > 0.
\]

Indeed, arguing by reductio ad absurdum suppose that there exists a sequence \( \{x_n\} \subset V \) with \( \|x_n\| = 1 \) such that the sequence \( \beta_n := g_i(x_n) \) converges to zero. Define \( y_n = (1/\beta_n)x_n \). Clearly, \( \{y_n\} \subset V \), since \( V \) is a cone, and \( \|y_n\| \to +\infty \) as \( n \to \infty \). Furthermore, taking into account the fact that \( g_i \) is a positively homogeneous function one gets that
\[
g_i(y_n) = \frac{1}{\beta_n} g_i(x_n) = 1 \quad \forall n \in \mathbb{N},
\]
which contradicts the fact that \( g_i \) is coercive on \( V \). Therefore, \( \beta > 0 \) and \( g_i(x) \geq \beta \|x\| \) for all \( x \in V \), since \( V \) is a cone and the function \( g_i \) is positively homogeneous.

Thus, the following lower estimate of the function \( f_i = [f_i]_+ \) holds true:
\[
f_i(x) \geq g_i(x) - a_0 \geq \beta \|x\| - a_0 \quad \forall x \in V.
\]

Fix any \( x_0 \in S(f) \). Then for any \( x \in V \setminus U(0, R) \) with \( R = \max\{\|x_0\|, 2a_0/\beta\} \) one has
\[
f_i(x) \geq \beta \|x\| - a_0 = \frac{\beta}{2} \|x\| + \frac{\beta}{2} \|x\| - a_0 \geq \frac{\beta}{2} \|x\| \\
\geq \frac{\beta}{4} \|x\| + \frac{\beta}{4} \|x_0\| \geq \frac{\beta}{4} \|x - x_0\| \geq \frac{\beta}{4} \text{dist}(x, S(f)).
\]

In turn, for any \( x \in U(0, R) \) and for \( \gamma = f_i^*/(R + \|x_0\|) \) (recall that \( i \in I \setminus I_0 \), i.e. \( f_i^* > 0 \)) one has
\[
\gamma \text{dist}(x, S(f)) \leq \gamma \|x - x_0\| \leq \gamma (R + \|x_0\|) \leq f_i^* \leq f_i(x).
\]

Thus, inequality (9) is satisfied with \( \tau_i = \min\{\gamma, \beta/4\} \).

**Corollary 2.** Let \( S(f) \neq \emptyset \) and \( f \) be coercive. Then \( f \) has a global error bound.
Corollary 3. The function $f$ is coercive if and only if $\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} > 0$.

Proof. Replacing, if necessary, $f$ with $f - C$ for a sufficiently large $C$ one can suppose that the set $S(f)$ is nonempty. If $f$ is coercive, then by the previous corollary $f$ has a global error bound, which by Theorem 4 implies that $\liminf_{\|x\| \to \infty} \frac{f(x)}{\|x\|} > 0$. In turn, if this condition is satisfied, then $f$ is obviously coercive.

The following example demonstrates that when $V$ is not a cone, the coercivity of $f$ on $V$ does not guarantee that $f$ has an error bound on $V$. Furthermore, it also shows that $f$ can have an error bound on an unbounded set $V$, but not have error bound on the conic hull of $V$, which means that, roughly speaking, necessary conditions for $f$ to have an error bound on $V$ cannot be expressed in terms of the conic hull of $V$.

Example 3. Let $d = 2$, $f(x^{(1)}, x^{(2)}) = \min\{|x^{(1)}| + |x^{(2)}|, 1 + |x^{(1)}|\}$, and $\|\cdot\|$ be the Euclidean norm. Then $S(f) = \emptyset$. Define $V = \{x_n\}_{n \in \mathbb{N}}$, where $x_n = (n, n^2)$. Note that $f(x_n) = 1 + n$ for any $n \in \mathbb{N}$, that is, the function $f$ is coercive on $V$. However, for any $\tau > 0$ one has

$$f(x_n) = 1 + n < \tau n^2 < \tau \|x_n\| = \tau \text{dist}(x_n, S(f)) \quad \forall n > \frac{1 + \sqrt{1 + 4\tau}}{2\tau},$$

which implies that $f$ does not have an error bound on $V$.

Suppose now that $V = \{(0, 1)\} \cup \{(n, 0)\}_{n \in \mathbb{N}}$. As one can readily verify,

$$f(0, 1) = 1 = \text{dist}\left((0, 1), S(f)\right), \quad f(n, 0) = n = \text{dist}\left((n, 0), S(f)\right) \quad \forall n \in \mathbb{N},$$

that is, $f$ has an error bound on $V$. On the other hand, the ray $\{(0, s) | s \geq 0\}$ is obviously contained in the conic hull of $V$ and for any $\tau > 0$ one has

$$f(0, s) = 1 < \tau s = \tau \text{dist}\left((0, s), S(f)\right) \quad \forall s \geq \frac{1}{\tau},$$

Thus, $f$ does not have an error bound on the conic hull of $V$.

In the case when $V$ is a convex cone and a representation of $f$ of the form (2) is known, one can provide geometric necessary and sufficient conditions for $f$ to have an error bound on $V$.

For any convex cone $K \subseteq \mathbb{R}^d$ denote by $K^* = \{y \in \mathbb{R}^d | \langle y, x \rangle \leq 0 \ \forall x \in K\}$ the polar cone of $K$. Let also $\text{co} C$ be the convex hull of a set $C \subset \mathbb{R}^d$.

Theorem 6. Let $f$ have the form (2), and the functions $f_i$ and the values $f_i^*$ be defined as in Theorem 3. Let also the set $S(f)$ be nonempty and $V$ be a closed convex cone. Then for $f$ to have an error bound on $V$ it is sufficient that one of the two following equivalent conditions holds true:

1. $f_i$ is coercive on $V$ for any $i \in I$ such that $f_i^* > 0$;

2. $0 \in \text{int}(\text{co}\{v_{ij} | j \in J(i)\} + V^*)$ for any $i \in I$ such that $f_i^* > 0$.

Moreover, these conditions become necessary, when $0^+ S(f) \cap V = \{0\}$.
Proof. **Part 1.** Let us first verify that the two conditions in the formulation of the theorem are indeed equivalent.

1. \[\implies\] 2. Fix any \( i \in I \) such that \( f_i^* > 0 \). Arguing by reductio ad absurdum, suppose that \( 0 \notin \text{int}(\text{co}\{v_{ij} \mid j \in J(i)\} + V^*) \). Then by the separation theorem there exists a nonzero vector \( z \in \mathbb{R}^d \) such that

\[
\langle z, v \rangle \leq 0 \quad \forall v \in \text{co}\{v_{ij} \mid j \in J(i)\} + V^*.
\]

(10)

Note that \( z \in V^{**} = V \). Indeed, if \( z \notin V^{**} \), then one can find \( y \in V^* \) such that \( \langle z, y \rangle > 0 \). On the other hand, taking into account (10) and the fact that \( V^* \) is a cone one obtains that

\[
\langle z, v_{ij} + ty \rangle \leq 0 \quad \forall t > 0, \quad \forall j \in J(i),
\]

which obviously contradicts the inequality \( \langle z, y \rangle > 0 \).

From (10) it follows that \( \langle z, v_{ij} \rangle \leq 0 \) for all \( j \in J(i) \). Thus, we have found \( z \in V, z \neq 0 \), such that

\[
f_i(tz) = \max_{j \in J(i)} (a_{ij} + t\langle v_{ij}, z \rangle) \leq \max_{j \in J(i)} a_{ij} \quad \forall t \geq 0,
\]

which contradicts the fact that \( f_i \) is coercive on \( V \).

2. \[\implies\] 1. Fix any \( i \in I \) such that \( f_i^* > 0 \). By our assumption there exists \( r > 0 \) such that

\[
U(0, r) \subset \text{co}\{v_{ij} \mid j \in J(i)\} + V^*,
\]

which with the use of the definition of \( V^* \) yields

\[
r\|x\| \leq \max \left\{ \langle v, x \rangle \mid v \in \text{co}\{v_{ij} \mid j \in J(i)\} + V^* \right\} \leq \max_{j \in J(i)} \langle v_{ij}, x \rangle \quad \forall x \in V.
\]

Consequently, for any \( x \in V \) one has

\[
f_i(x) = \max_{j \in J(i)} (a_{ij} + \langle v_{ij}, x \rangle) \geq \max_{j \in J(i)} \langle v_{ij}, x \rangle - \max_{j \in J(i)} |a_{ij}| \geq r\|x\| - \max_{j \in J(i)} |a_{ij}|,
\]

which obviously implies that \( f_i \) is coercive on \( V \).

**Part 2.** Let us now prove the main statement of the theorem. Suppose that each function \( f_i \) with \( f_i^* > 0 \) is coercive on \( V \). Then, as was shown in the proof of Theorem 5 for any such function \( f_i \) there exists \( \tau_i > 0 \) such that

\[
\tau_i \text{dist}(x, S(f)) \leq f_i(x) \quad \forall x \in V.
\]

In turn, as was shown in the proof of Theorem 2 for any \( i \in I \) with \( f_i^* \leq 0 \) there exists \( \tau_i > 0 \) such that

\[
\tau_i \text{dist}(x, S(f_i)) \leq [f_i]+(x) \quad \forall x \in \mathbb{R}^d.
\]

Hence bearing in mind equality (2) and the fact that \( S(f) = \bigcup_{i \in I} S(f_i) \) one can easily check that \( f \) has an error bound on \( V \) with constant \( \tau = \min_{i \in I} \tau_i \).

Suppose now that \( f \) has an error bound on \( V \) and \( 0^+ S(f) \cap V = \{0\} \). Then by Theorem 2 the function \( f \) is coercive on \( V \), which due to the representation (2) and the definition of \( f_i \) obviously implies that each function \( f_i \) is coercive on \( V \).
3.3 Error bounds on unbounded sets: general conditions

Let us finally provide general necessary and sufficient conditions for \( f \) to have an error bound on an unbounded set (in particular, a global error bound) that extend the results of Gowda [13] on global error bounds for piecewise affine mappings.

One might be tempted to say that for \( f \) to have a global error bound it is necessary that all “flat pieces” of \( f \), on which \( f \) is positive (i.e. all polyhedral sets on which \( f \) is constant and positive), are bounded. However, this is not the case.

Example 4. Let \( d = 2 \), the space \( \mathbb{R}^2 \) be endowed with the Euclidean norm, and

\[
f(x) = \begin{cases} 
0, & \text{if } x^{(1)} \leq 0, \\
x^{(1)}, & \text{if } 0 \leq x^{(1)} \leq 1, \\
1, & \text{if } 1 \leq x^{(1)} \leq 2, \\
x^{(1)} - 1, & \text{if } x^{(1)} \geq 2.
\end{cases}
\]

Then \( S(f) = \{ x \in \mathbb{R}^2 \mid x^{(1)} \leq 0 \} \) and \( \text{dist}(x, S(f)) = [x^{(1)}]_+ \) for all \( x \in \mathbb{R}^2 \). Hence, as is easily seen,

\[
\frac{1}{2} \text{dist}(x, S(f)) \leq f(x) \quad \forall x \in \mathbb{R}^2,
\]

that is, \( f \) has a global error bound, despite the fact that \( f \) is constant and positive on the unbounded set \( \{ x \in \mathbb{R}^2 \mid 1 \leq x^{(1)} \leq 2 \} \).

Our aim is to show that the existence of a global error bound for \( f \) is completely defined by the location of directions along which \( f \) is constant and positive with respect to the set \( S(f) \). Namely, \( f \) has a global error bound if and only if such directions are, in a sense, parallel to the set \( S(f) \). To conveniently formulate this condition, we will use the recession function of the function \([f]_+\), being inspired by the approach of Gowda [13].

Recall that the recession function \( g^\infty \) of a piecewise affine map \( g: \mathbb{R}^d \to \mathbb{R} \) is defined as

\[
g^\infty(x) = \lim_{\lambda \to +\infty} \frac{g(\lambda x)}{\lambda}
\]

Note that if the function \( g \) has the form

\[
g(x) = \min_{i \in I} g_i(x), \quad g_i(x) = \max_{j \in J(i)} (a_{ij} + \langle v_{ij}, x \rangle)
\]

for some finite index sets \( I \) and \( J(i) \), then

\[
g^\infty(x) = \min_{i \in I} \max_{j \in J(i)} \langle v_{ij}, x \rangle \quad \forall x \in \mathbb{R}^d,
\]

that is, \( g^\infty \) is a positively homogeneous piecewise affine function. Since any piecewise affine function \( g \) can be represented in the form (11) by Theorem 1, one can conclude that the recession function of a piecewise affine function is correctly defined and is a positively homogeneous piecewise affine function.

Now we can formulate necessary and sufficient conditions for \( f \) to have an error bound on a finite union of polyhedral sets (e.g. to have a global error bound).
Theorem 7. Let $S(f)$ be nonempty and $V \subseteq \mathbb{R}^d$ be a finite union of polyhedral sets. Then for $f$ to have an error bound on $V$ it is necessary and sufficient that

$$S([f]_\infty^\infty) \cap 0^+ V \subseteq 0^+ S(f).$$

We divide the proof of this theorem into a series of lemmas. First, we reformulate the statement of the theorem in terms of the convex piecewise affine functions $f_i(x) = \max_{j \in J(i)} (a_{ij} + \langle v_{ij}, x \rangle)$ from the min-max representation $f = \min_{i \in I} f_i$ of the function $f$ given in Theorem 1.

Lemma 1. Let the assumptions of Theorem 7 hold true, the function $f$ have the min-max form (2), and the functions $f_i$ and the values $f_i^*$ be defined as in Theorem 2. Then $f$ has an error bound on $V$ if and only if for any $i \in I$ with $f_i^* > 0$ there exists $\tau_i > 0$ such that

$$f_i(x) \geq \tau_i \text{dist}(x, S(f)) \quad \forall x \in V. \quad (13)$$

Proof. By equality (2) and the definition of error bound, the function $f$ has an error bound on $V$ if and only if there exists $\tau > 0$ such that

$$f(x) = \min_{i \in I} f_i(x) \geq \tau \text{dist}(x, S(f)) \quad \forall x \in V.$$

Therefore, $f$ has an error bound on $V$ if and only if for each $i \in I$ one can find $\tau_i > 0$ for which inequality (13) holds true. It remains to note that, as was shown in the proof of Theorem 2 for any $i \in I$ with $f_i^* \leq 0$ there exists $\tau_i > 0$ such that

$$f_i(x) \geq \tau_i \text{dist}(x, S(f_i)) \quad \forall x \in \mathbb{R}^d,$$

which with the use of the obvious inclusion $S(f_i) \subseteq S(f)$ (see (2)) implies that inequality (13) is always satisfied for any $i \in I$ with $f_i^* \leq 0$. \qed

The second step is to reformulate condition (13) in geometric terms involving recession cones of some sets. To this end, we need to prove three auxiliary result on the distance to a finite union of polyhedral sets.

Lemma 2. Let $X \subseteq \mathbb{R}^d$ be a finite union of polyhedral sets and $x, z \in \mathbb{R}^d$ be fixed. Then the function $\lambda \mapsto \text{dist}(x + \lambda z, X)$ is bounded on $[0, +\infty)$ if and only if $z \in 0^+ X$.

Proof. Fix any $x, z \in \mathbb{R}^d$. If $z \notin 0^+ X$, then there exists $x_0 \in \mathbb{R}^d$ such that $x_0 + \lambda z \notin X$ for all $\lambda \geq 0$. Therefore

$$\text{dist}(x + \lambda z, X) \leq \|x + \lambda z - (x_0 + \lambda z)\| = \|x - x_0\| \quad \forall \lambda \geq 0,$$

i.e. the function $\lambda \mapsto \text{dist}(x + \lambda z, X)$ is bounded on $[0, +\infty)$.

Conversely, let the distance function be bounded. Arguing by reductio ad absurdum, suppose that $z \notin 0^+ X$. By our assumption $X$ is the union of some polyhedral sets $X_1, \ldots, X_\ell \subset \mathbb{R}^d$. As is easily seen,

$$\text{dist}(y, X) = \min \left\{ \text{dist}(y, X_1), \ldots, \text{dist}(y, X_\ell) \right\} \quad \forall y \in \mathbb{R}^d.$$

Therefore, if we prove that each function $\lambda \mapsto \text{dist}(x + \lambda z, X_\ell)$ is unbounded, we will get an obvious contradiction.
Fix any \( i \in I = \{1, \ldots, \ell\} \). Clearly, \( 0^+X_i \subseteq 0^+X \) for any \( i \in I \), which implies that \( z \notin 0^+X_i \). By the Motzkin theorem \cite[Thms. 19.1 and 19.5]{Motzkin} one has \( X_i = X_i + 0^+X_i \) for some polytope \( P_i \subseteq \mathbb{R}^d \), since \( X_i \) is a polyhedral set. Therefore, by the reverse triangle inequality for any \( \lambda \geq 0 \) one has

\[
\text{dist}(x + \lambda z, X_i) = \inf \left\{ \|x + \lambda z - (y_1 + y_2)\| \mid y_1 \in K_i, y_2 \in 0^+X_i \right\}
\geq \inf \left\{ \|\lambda z - y_2\| - \|x\| - \|y_1\| \mid y_1 \in K_i, y_2 \in 0^+X_i \right\}
\geq \text{dist}(\lambda z, 0^+X_i) - \|x\| - \max_{y \in K_i} \|w\|.
\]

Since \( 0^+X_i \) is a cone, the function \( \text{dist}(\cdot, 0^+X_i) \) is obviously positively homogeneous. Hence

\[
\text{dist}(x + \lambda z, X_i) \geq \lambda r - \|x\| - \max_{y \in K_i} \|w\| \quad \forall \lambda \geq 0,
\]

where \( r = \text{dist}(z, 0^+X_i) > 0 \) (recall that \( z \notin 0^+X_i \)). Thus, for any \( i \in I \) one has \( \text{dist}(x + \lambda z, X_i) \to +\infty \), which contradicts our assumption.

**Lemma 3.** Let \( Y, V \subseteq \mathbb{R}^d \) be polyhedral convex cones. Then there exists \( C > 0 \) such that

\[
C \text{dist}(x, Y) \geq \text{dist}(x, Y \cap V) \quad \forall x \in V. \tag{14}
\]

**Proof.** We prove the lemma in the case when \( \|\cdot\| \) is the Euclidean norm. Clearly, the validity of the lemma in the general case follows directly from its validity in the Euclidean case.

Choose some \( C > 0 \) and introduce the functions

\[
p_C(x) = C \text{dist}(x, Y), \quad q(x) = \text{dist}(x, Y \cap V) \quad \forall x \in \mathbb{R}^d.
\]

Since \( Y \) and \( V \) are convex cones, the functions \( p_C \) and \( q \) are sublinear. Recall that a sublinear function is equal to the supports function of its subdifferential at the origin (see, e.g. \cite[Thm. V.3.1.1]{PolyhedralConvex}), that is

\[
p_C(x) = \max_{v \in \partial p_C(0)} \langle v, x \rangle, \quad q(x) = \max_{w \in \partial q(0)} \langle w, x \rangle \quad \forall x \in \mathbb{R}^d.
\]

Therefore, inequality \( \text{(14)} \) holds true if and only if

\[
\max_{v \in \partial p_C(0) - w} \langle v, x \rangle \geq 0 \quad \forall x \in V \quad \forall w \in \partial q(0).
\]

With the use of the separation theorem one can readily check that this inequality is satisfied if and only if \( (\partial p_C(0) - w) \cap (-V^*) \neq \emptyset \) for any \( w \in \partial q(0) \) or, equivalently, if and only if

\[
0 \in \partial p_C(0) - w + V^* \quad \forall w \in \partial q(0).
\]

The inclusion above can be rewritten as

\[
\partial q(0) \subseteq \partial p_C(0) + V^*. \tag{15}
\]

Thus, inequality \( \text{(14)} \) is satisfied for some \( C > 0 \) if and only if inclusion \( \text{(15)} \) is satisfied for the same \( C > 0 \). Let us prove that this inclusion holds true, provided \( C > 0 \) is large enough.
Indeed, note that by [6] Example 2.130 one has
\[ \partial p_C(0) = \left\{ v \in Y^* \parallel v \parallel \leq C \right\}, \quad \partial q(0) = \left\{ w \in (Y \cap V)^* \parallel w \parallel \leq 1 \right\}. \]
Moreover, by [27] Crlr. 16.4.2 (see also [27] Thm. 16.4 and Thm. 20.1) one has
\( (Y \cap V)^* = Y^* + V^* \), since both \( Y \) and \( V \) are polyhedral cones. Thus, inclusion [13] can be rewritten as
\[ \left\{ w \in Y^* + V^* \parallel w \parallel \leq 1 \right\} \subseteq \left\{ v \in Y^* \parallel v \parallel \leq C \right\} + V^*. \]
Consequently, it is sufficient to prove that for any \( w \in Y^* + V^* \) with \( \parallel w \parallel \leq 1 \) one can find \( v_1 \in Y^* \) with \( \parallel v_1 \parallel \leq C \) and \( v_2 \in V^* \) such that \( w = v_1 + v_2 \). Note that if \( Y^* = V^* = \{0\} \), then this result is obvious. Therefore, one can suppose that \( Y^* + V^* \neq \{0\} \).
To prove the existence of the required \( C > 0 \), note that the cones \( Y^* \) and \( V^* \) are polyhedral by [27] Crlr. 19.2.2, since the cones \( Y \) and \( V \) are polyhedral. Therefore by [27] Thm. 19.1 both cones \( Y^* \) and \( V^* \) are finitely generated, that is, they are the convex conic hulls of some vectors \( y_1, \ldots, y_n \in Y^* \) and \( z_1, \ldots, z_k \in V^* \).
Let \( \mathcal{M} \) be the collection of all nonempty subsets \( M \) of the set \( \{1, \ldots, n\} \times \{1, \ldots, k\} \) such that the vectors \( y_i + z_j, (i, j) \in M \), are linearly independent. Note that the set \( \mathcal{M} \) is nonempty, since, as was noted above, one can assume that \( Y^* + V^* \neq \{0\} \) and the cone \( Y^* + V^* \) is obviously the convex conic hull of the vectors \( y_i + z_j, i \in \{1, \ldots, n\}, j \in \{1, \ldots, k\} \).
For any \( M \in \mathcal{M} \) introduce linear subspace \( \mathcal{E}_M = \text{span}\{y_i + z_j \mid (i, j) \in M\} \).
By definition, for any \( x \in \mathcal{E}_M \) there exist unique \( \alpha(i, j) \in \mathbb{R}, (i, j) \in M \), such that \( x = \sum_{(i, j) \in M} \alpha(i, j)(y_i + z_j) \). Denote by \( \parallel x \parallel_M = \sum_{(i, j) \in M} |\alpha(i, j)| \). Clearly, \( \parallel \parallel_M \) is a norm on \( \mathcal{E}_M \). Therefore, it is equivalent to the Euclidean norm, which, in particular, implies that there exists \( C_M > 0 \) such that \( \parallel x \parallel_M \leq C_M \parallel x \parallel \) for all \( x \in \mathcal{E}_M \).
Now, fix any \( w \in Y^* + V^* \) with \( \parallel w \parallel \leq 1 \). Since the cone \( Y^* + V^* \) is the convex conic hull of the vectors \( \{y_i + z_j\} \), by the version of Carathéodory’s theorem for convex cones [27] Crlr. 17.1.2 there exists \( M \in \mathcal{M} \) such that \( w \) can be represented as the convex conic combination of the vectors \( \{y_i + z_j\}, (i, j) \in M \), that is, one can find \( \alpha_{ij} \geq 0, (i, j) \in M \), such that \( w = \sum_{(i, j) \in M} \alpha_{ij}(y_i + z_j) \).
Moreover, one has \( \sum_{(i, j) \in M} \alpha_{ij} \leq C_M \).
Define
\[ v_1 = \sum_{(i, j) \in M} \alpha_{ij}y_i, \quad v_2 = \sum_{(i, j) \in M} \alpha_{ij}z_j. \]
Then \( v_1 \in Y^*, v_2 \in V^*, \) and \( w = v_1 + v_2 \). Moreover, one has
\[ \parallel v_1 \parallel \leq \sum_{(i, j) \in M} \alpha_{ij} \parallel y_i \parallel \leq C_M \max \left\{ \parallel y_1 \parallel, \ldots, \parallel y_n \parallel \right\}. \]
Thus, we have proved that for any \( w \in Y^* + V^* \) with \( \parallel w \parallel \leq 1 \) one can find \( v_1 \in Y^* \) and \( v_2 \in V^* \) such that \( w = v_1 + v_2 \) and \( \parallel v_1 \parallel \leq C \), where
\[ C = \left( \max_{M \in \mathcal{M}} C_M \right) \max \left\{ \parallel y_1 \parallel, \ldots, \parallel y_n \parallel \right\}. \]
Note that \( C < +\infty \), since the collection \( \mathcal{M} \) consists of a finite number of sets. \( \square \)

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As the following example shows, the previous lemma does not hold true in the case when either of the cones \( Y \) and \( V \) is not polyhedral.

**Example 5.** Let \( d = 3, Y = \{ x \in \mathbb{R}^3 \mid x^{(1)} = 0 \} \), and \( V \) be the convex conic hull of the disc

\[
D = \left\{ x \in \mathbb{R}^3 \mid (x^{(1)} - 1)^2 + (x^{(2)})^2 \leq 1, \ x^{(3)} = 1 \right\}.
\]

Note that the cone \( V \) is not polyhedral in this case.

Let \( \| \cdot \| \) be the Euclidean norm. Then \( Y \cap V = \{ x \in \mathbb{R}^3 \mid x^{(1)} = x^{(2)} = 0 \} \)
and, as is easily seen,

\[
\text{dist}(x, Y \cap V) = \sqrt{(x^{(1)})^2 + (x^{(2)})^2}, \quad \text{dist}(x, Y) = |x^{(1)}| \quad \forall x \in V.
\]

Observe that for \( x(t) = (1 + \sin t, \cos t, 1) \in V \) the inequality

\[
\text{dist}(x(t), Y \cap V) \leq C \text{dist}(x(t), Y) \quad \forall t \in \mathbb{R},
\]

is not satisfied for any \( C > 0 \), since, as is easily seen,

\[
\text{dist}(x(t), Y \cap V) = 2(1 + \sin t) > C^2(1 + \sin t)^2 = \text{dist}(x(t), Y)^2
\]

\[
\forall t \in \left( \frac{3\pi}{2} - \varepsilon, \frac{3\pi}{2} \right) \cup \left( \frac{3\pi}{2}, \frac{3\pi}{2} + \varepsilon \right)
\]

for any sufficiently small \( \varepsilon > 0 \), depending on \( C \). Thus, Lemma 3 does not hold true when the cone \( V \) is not polyhedral. Switching \( Y \) and \( V \) one can check that this lemma does not hold true in the case when the cone \( Y \) is not polyhedral either.

Now we are ready to prove the main result on the distance to a finite union of polyhedral sets that is the key part of the proof of Theorem 7.

**Lemma 4.** Let \( X, Y, V \subseteq \mathbb{R}^d \) be finite unions of polyhedral sets. Then the inequality

\[
\text{dist}(x, X) \leq C \text{dist}(x, Y) + \theta \quad \forall x \in V
\]

is satisfied for some \( C > 0 \) and \( \theta \geq 0 \) if and only if \( 0^+(Y \cap V) \subseteq 0^+X \).

**Proof.** Let inequality (16) hold true for some \( C > 0 \) and \( \theta \geq 0 \). Arguing by reductio ad absurdum, suppose that there exists \( z \in 0^+(Y \cap V) \) such that \( x \notin 0^+X \). Then by the definition of the recession cone \( x_0 + \lambda z \in Y \cap V \) for some \( x_0 \in \mathbb{R}^d \) and all \( \lambda \geq 0 \), which yields \( \text{dist}(x_0 + \lambda z, Y) = 0 \) for all \( \lambda \geq 0 \). On the other hand, by Lemma 2 one has \( \text{dist}(x_0 + \lambda z, X) \to +\infty \) as \( \lambda \to +\infty \), which obviously contradicts inequality (16).

Suppose now that \( 0^+(Y \cap V) \subseteq 0^+X \). We will prove inequality (16) by reducing the proof to the particular case when \( Y \) and \( V \) are polyhedral cones.

Indeed, by our assumption

\[
X = \bigcup_{i=1}^t X_i, \quad Y = \bigcup_{j=1}^s Y_j, \quad V = \bigcup_{k=1}^t V_k
\]

(17)
for some \( \ell, s, t \in \mathbb{N} \) and some polyhedral sets \( X_i, Y_j, V_k \subset \mathbb{R}^d \). As is easily seen, one has
\[
0^+ X = \bigcup_{i=1}^{\ell} 0^+ X_i, \quad 0^+ (Y \cap V) = \bigcup_{j=1}^{s} \bigcup_{k=1}^{t} 0^+ (Y_j \cap V_k)
\]
and, furthermore, \( 0^+ (Y_j \cap V_k) = 0^+ Y_j \cap 0^+ V_k \) (see, e.g., [27, Crlr. 8.3.3]). Thus, for any \( j \in J_Y := \{1, \ldots, s\} \) and \( k \in K_V := \{1, \ldots, t\} \) one has
\[
0^+ Y_j \cap 0^+ V_k \subseteq 0^+ X,
\]
which obviously implies that
\[
\text{dist}(x, 0^+ Y_j \cap 0^+ V_k) \geq \text{dist}(x, 0^+ X) \quad \forall x \in \mathbb{R}^d.
\] (18)

Note that the recession cones \( 0^+ Y_j \) and \( 0^+ V_k \) are polyhedral by [27, Thm. 19.5], since the sets \( Y_j \) and \( V_k \) are polyhedral. Therefore, by Lemma 3 for any \( j \in J_Y \) and \( k \in K_V \) there exists \( C_{jk} > 0 \) such that
\[
C_{jk} \text{dist}(x, 0^+ Y_j) \geq \text{dist}(x, 0^+ Y_j \cap 0^+ V_k) \quad \forall x \in 0^+ V_k.
\]
Hence with the use of (18) one obtains that for any \( j \in J_Y \) the following inequality holds true:
\[
\left( \max_{k \in K_V} C_{jk} \right) \text{dist}(x, 0^+ Y_j) \geq \text{dist}(x, 0^+ X) \quad \forall x \in 0^+ V = \bigcup_{k=1}^{t} 0^+ V_k.
\]
Finally, taking into account the obvious equality
\[
\text{dist}(x, 0^+ Y) = \text{dist} \left( x, \bigcup_{j=1}^{s} 0^+ Y_j \right) = \min_{j \in J_Y} \text{dist}(x, 0^+ Y_j) \quad \forall x \in \mathbb{R}^d
\]
one gets that
\[
C \text{dist}(x, 0^+ Y) \geq \text{dist}(x, 0^+ X) \quad \forall x \in 0^+ V,
\] (19)
where \( C = \max\{C_{jk} \; | \; j \in J_Y, \; k \in K_V\} \).

By the Motzkin theorem [27, Thms. 19.1 and 19.5] for each \( i \in I_X := \{1, \ldots, \ell\} \), \( j \in J_Y \), and \( k \in K_V \) there exist polytopes \( P(X_i), P(Y_j), P(V_k) \subset \mathbb{R}^d \) such that
\[
X_i = P(X_i) + 0^+ X_i, \quad Y_j = P(Y_j) + 0^+ Y_j, \quad V_k = P(V_k) + 0^+ V_k,
\]
since the sets \( X_i, Y_j, \) and \( V_k \) are polyhedral by our assumptions. Define
\[
\theta_1 = \max \left\{ \max_{i \in I_X} \max_{v \in P(X_i)} \|v\|, \max_{j \in J_Y} \max_{w \in P(Y_j)} \|w\| \right\}.
\]
As is easily seen, by the reverse triangle inequality for any \( x \in \mathbb{R}^d \) one has
\[
| \text{dist}(x, X_i) - \text{dist}(x, 0^+ X_i) | \leq \theta_1, \quad | \text{dist}(x, Y_j) - \text{dist}(x, 0^+ Y_j) | \leq \theta_1
\] (20)
for any \( i \in I_X \) and \( j \in J_Y \). Put also \( \theta_2 = \max \{ \|x\| \; | \; x \in P(V_k), k \in K_V \} \) and
\[
\theta_3 = \max \left\{ \max_{i \in I_X, k \in K_V} \text{dist}(x, 0^+ X_i), \max_{j \in J_Y, k \in K_V} \max_{x \in P(V_k)} \text{dist}(x, 0^+ Y_j) \right\},
\]
Clearly, $\theta_3 < +\infty$, since the sets $P(V_k)$ are compact and the corresponding distance functions are continuous.

Choose any $k \in K_V$ and $x \in V_k$. Then $x = x_1 + x_2$ for some $x_1 \in P(V_k)$ and $x_2 \in 0^+ V_k$. Applying Hoffman’s Theorem in precisely the same way as in the proof of Theorem 2 one gets that the function $f$.

Applying inequalities (19) and (20), and the fact that the distance to a convex cone is a sublinear function one gets that

$$\text{dist}(x, X) = \min_{i \in I_X} \text{dist}(x, X_i) \leq \theta_1 + \min_{i \in I_X} \text{dist}(x, 0^+ X_i)$$

$$\leq \theta_1 + \theta_3 + \min_{i \in I_X} \text{dist}(x_2, 0^+ X_i) = \theta_1 + \theta_3 + \text{dist}(x_2, 0^+ X)$$

$$\leq \theta_1 + \theta_3 + C \max_{j \in J_V} \text{dist}(x_2, 0^+ Y_j)$$

$$\leq \theta_1 + C \theta_2 + \theta_3 + C \min_{j \in J_V} \text{dist}(x, 0^+ Y_j)$$

$$\leq 2 \theta_1 + C \theta_2 + \theta_3 + C \min_{j \in J_V} \text{dist}(x, Y_j) = 2 \theta_1 + C \theta_2 + \theta_3 + C \text{dist}(x, Y).$$

Since $k \in K_V$ and $x \in V_k$ were chosen arbitrary, one can conclude that inequality (16) holds true with $\theta = 2 \theta_1 + C \theta_2 + \theta_3$. \hfill $\Box$

Lemmas 2 and 4 enable us to reformulate condition (13) from Lemma 1 in geometric terms.

**Lemma 5.** Under the assumptions of Lemma 4 the function $f$ has an error bound on $V$ if and only if for any $i \in I$ such that $f_i^* > 0$ one has

$$0^+(S(f_i - f_i^*) \cap V) \subseteq 0^+ S(f).$$

**Proof.** Fix any $i \in I$ such that $f_i^* > 0$. Our aim is to show that the inequality $f_i(x) \geq \tau_i \text{dist}(x, S(f))$ holds true for all $x \in V$ if and only if the inclusion $0^+(S(f_i - f_i^*) \cap V) \subseteq 0^+ S(f)$ hold true. Then applying Lemma 4 one obtains the required result.

Let the inequality $f_i(x) \geq \tau_i \text{dist}(x, S(f))$ be satisfied for some $\tau_i > 0$. Arguing by reductio ad absurdum, suppose that there exists $z \in 0^+(S(f_i - f_i^*) \cap V)$ such that $z \notin 0^+ S(f)$. By the definition of the recession cone one can find $x_0 \in S(f_i - f_i^*) \cap V$ such that $x_0 + \lambda z \notin S(f_i - f_i^*) \cap V$ for all $\lambda \geq 0$. Hence, in particular, $f_i(x_0 + \lambda z) = f_i^*$ for all $\lambda \geq 0$. In turn, from Lemma 2 and the condition $z \notin 0^+ S(f)$ it follows that $\text{dist}(x_0 + \lambda z, S(f)) \to +\infty$ as $\lambda \to +\infty$, which contradicts the fact that by our assumption

$$f_i^* = f_i(x_0 + \lambda z) \geq \tau_i \text{dist}(x, S(f)) \quad \forall \lambda \geq 0.$$

Thus, $0^+(S(f_i - f_i^*) \cap V) \subseteq 0^+ S(f)$.

Conversely, suppose that $0^+(S(f_i - f_i^*) \cap V) \subseteq 0^+ S(f)$. Then by Lemma 4 there exist $C > 0$ and $\theta > 0$ such that

$$\text{dist}(x, S(f)) \leq \theta + C \text{dist}(x, S(f_i - f_i^*)) \quad \forall x \in V. \quad (21)$$

Applying Hoffman’s Theorem in precisely the same way as in the proof of Theorem 2 one gets that the function $f_i - f_i^*$ has a global error bound with some constant $\tau_i > 0$, that is,

$$f_i(x) \geq f_i^* + \tau_i \text{dist}(x, S(f_i - f_i^*)) \quad \forall x \in \mathbb{R}^d.$$
Recall that \( f^*_i > 0 \) by our assumption. Consequently, decreasing \( \tau_i > 0 \), if necessary, one can suppose that \( f^*_i / \tau_i \geq \theta / C \), which with the use of (23) implies that

\[
f_i(x) \geq \tau_i \left( \frac{f^*_i}{\tau_i} + \text{dist}(x, S(f_i - f^*_i)) \right) \geq \frac{\tau_i}{C} \text{dist}(x, S(f)) \quad \forall x \in V.
\]

Thus, the proof is complete.

The last step of the proof of Theorem 7 consists in establishing a direct connection between the sets \( S(f_i - f^*_i) \) and the 0-sublevel set \( S([f]_+^\infty) \) of the recession function of \([f]_+^\infty\).

**Lemma 6.** Under the assumptions of Lemma 1 the following equality holds true:

\[
S([f]_+^\infty) = 0^+ S(f) \cup \left( \bigcup_{i \in I: f^*_i > 0} 0^+ S(f_i - f^*_i) \right).
\]

**Proof.** Observe that for any \( x \in \mathbb{R}^d \) one has

\[
[f]_+^\infty = \lim_{\lambda \to +\infty} \max \left\{ 0, \min_{i \in I} \frac{f_i(\lambda x)}{\lambda} \right\} = \max \left\{ 0, \min_{i \in I} f_i^\infty(x) \right\}.
\]

Therefore

\[
S([f]_+^\infty) = \bigcup_{i \in I} S(f_i^\infty).
\]

Recall that by our assumption \( f_i(x) = \max_{j \in J(i)} (a_{ij} + \langle v_{ij}, x \rangle) \). Consequently, for any \( i \in I \) one has

\[
f_i^\infty(x) = \max_{j \in J(i)} \langle v_{ij}, x \rangle, \quad S(f_i^\infty) = \left\{ x \in \mathbb{R}^d \mid \langle v_{ij}, x \rangle \leq 0 \quad \forall j \in J(i) \right\}.
\]

Note also that for any \( i \in I \) such that \( f^*_i > 0 \) one has

\[
0^+ S(f_i - f^*_i) = 0^+ \left\{ x \in \mathbb{R}^d \mid a_{ij} + \langle v_{ij}, x \rangle \leq f^*_i \quad \forall j \in J(i) \right\} = \left\{ x \in \mathbb{R}^d \mid \langle v_{ij}, x \rangle \leq 0 \quad \forall j \in J(i) \right\},
\]

while for any \( i \in I \) with \( f^*_i \leq 0 \) one has

\[
0^+ S(f_i) = 0^+ \left\{ x \in \mathbb{R}^d \mid a_{ij} + \langle v_{ij}, x \rangle \leq 0 \quad \forall j \in J(i) \right\} = \left\{ x \in \mathbb{R}^d \mid \langle v_{ij}, x \rangle \leq 0 \quad \forall j \in J(i) \right\}.
\]

Combining the equalities above with (23) one gets that

\[
S([f]_+^\infty) = \left( \bigcup_{i \in I: f^*_i \leq 0} 0^+ S(f_i) \right) \cup \left( \bigcup_{i \in I: f^*_i > 0} 0^+ S(f_i - f^*_i) \right).
\]

Hence with the use of the equality

\[
S(f) = \bigcup_{i \in I: f^*_i \leq 0} S(f_i)
\]

and the corresponding equality for the recession cones we arrive at the required result. \( \square \)
Remark 4. It is worth mentioning that from the proof of the previous lemma it follows that for a piecewise affine function \( f \) of the form
\[
f(x) = \min_{i \in I} \max_{j \in J(i)} (a_{ij} + \langle v_{ij}, x \rangle),
\]
the following equality holds true:
\[
S([f]_{\infty}^\infty) = \bigcup_{i \in I} \left\{ x \in \mathbb{R}^d \mid \langle v_{ij}, x \rangle \leq 0 \ \forall j \in J(i) \right\}
\]
(this fact also follows directly from equality (12)). Thus, one can easily construct the 0-sublevel set of the recession function \([f]_{\infty}^\infty\), if a min-max representation of the function \( f \) is known.

Finally, it remains to combine all the lemmas above into a coherent proof of Theorem 7.

Lemma 7. Theorem 7 holds true.

Proof. By Theorem 1 the function \( f \) can be represented in the following form:
\[
f(x) = \min_{i \in I} f_i(x), \quad f_i(x) = \max_{j \in J(i)} (a_{ij} + \langle v_{ij}, x \rangle), \quad \forall x \in \mathbb{R}^d.
\]
Denote \( f_i^* = \inf_{x \in \mathbb{R}^d} f_i(x) \).

If the function \( f \) has an error bound on \( V \), then by Lemma 5 the inclusion
\[
0^+(S(f_i - f_i^*) \cap V) \subseteq 0^+ S(f) \text{ holds true for any } i \in I \text{ such that } f_i^* > 0.
\]
Hence with the use of Lemma 6 and the fact that
\[
0^+(X \cap V) = 0^+ X \cap 0^+ V
\]
for any set \( X \subset \mathbb{R}^d \) that is a finite union of closed convex sets (see, e.g. [27, Crlr. 8.3.3]) one obtains that
\[
S([f]_{\infty}^\infty) \cap 0^+ V = \left( 0^+ S(f) \cap 0^+ V \right) \cup \left( \bigcup_{i \in I : f_i^* > 0} 0^+ S(f_i - f_i^*) \cap 0^+ V \right) \subseteq 0^+ S(f) \cup \left( \bigcup_{i \in I : f_i^* > 0} 0^+(S(f_i - f_i^*) \cap V) \right) \subseteq 0^+ S(f).
\]
Let us prove the converse statement. Suppose that
\[
S([f]_{\infty}^\infty) \cap 0^+ V \subseteq 0^+ S(f).
\]
Then by equality (24) and Lemma 6 for any \( i \in I \) such that \( f_i^* > 0 \) one has
\[
0^+(S(f_i - f_i^*) \cap V) = 0^+ S(f_i - f_i^*) \cap 0^+ V \subseteq S([f]_{\infty}^\infty) \cap 0^+ V \subseteq 0^+ S(f),
\]
which by Lemma 5 implies that \( f \) has an error bound on \( V \).

Let us illustrate Theorem 7 and some of the lemmas above by applying them to two simple examples.
Example 6. Let $d = 2$ and $f$ be defined as in Example 4, that is,

$$f(x) = \begin{cases} 
0, & \text{if } x^{(1)} \leq 0, \\
x^{(1)}, & \text{if } 0 \leq x^{(1)} \leq 1, \\
1, & \text{if } 1 \leq x^{(1)} \leq 2, \\
x^{(1)} - 1, & \text{if } x^{(1)} \geq 2.
\end{cases}$$

Then

$$[f]^\infty_+ (x) = \begin{cases} 
0, & \text{if } x^{(1)} \leq 0, \\
x^{(1)}, & \text{if } x^{(1)} > 0 = \max \{0, x^{(1)}\}.
\end{cases}$$

Therefore $S(f) = S([f]^\infty) = \{x \in \mathbb{R}^2 \mid x^{(1)} \leq 0\}$ and $f$ has a global error bound by Theorem 4.

Let us now apply Lemma 5. Observe that $f(x) = \min \{f_1(x), f_2(x)\}$ with

$$f_1(x) = \max \{0, x^{(1)}\}, \quad f_2(x) = \max \{1 + x^{(2)}, 1 - x^{(2)}\}.$$ 

Clearly, $f_1^* = 0$ and $f_2^* = 1$. Moreover, one has

$$S(f_2 - f_2^*) = \{x \in \mathbb{R}^2 \mid x^{(1)} \leq 2\}.$$ 

Therefore $0^+ S(f_2 - f_2^*) = 0^+ S(f)$ and one can conclude that $f$ has a global error bound by Lemma 5.

Example 7. Let now $f(x) = \min \{f_1(x), f_2(x)\}$, where

$$f_1(x) = \max \{0, x^{(1)}\}, \quad f_2(x) = \max \{1 + x^{(2)}, 1 - x^{(2)}\}.$$ 

Let us apply Lemma 5 first. Indeed, one has $f_1^* = 0$, and $f_2^* = 1$, and

$$S(f) = 0^+ S(f) = \{x \in \mathbb{R}^2 \mid x^{(1)} \leq 0\},$$

$$S(f_2 - f_2^*) = 0^+ S(f_2 - f_2^*) = \{x \in \mathbb{R}^2 \mid x^{(2)} = 0\}.$$ 

Consequently, by Lemma 5 the function $f$ does not have a global error bound, but has an error bound on any finite union of polyhedral sets $V \subset \mathbb{R}^2$ having bounded intersection with the ray $\{x \in \mathbb{R}^2 \mid x^{(1)} \geq 0, x^{(2)} = 0\}$.

Let us now apply Theorem 7. As is easy to check, one has

$$[f]^\infty_+ = \min \{|x^{(1)}|, |x^{(2)}|\}.$$ 

(see (12)). Therefore

$$S([f]^\infty) = \left\{x \in \mathbb{R}^2 \mid x^{(1)} \leq 0 \text{ or } x^{(2)} = 0 \right\}.$$ 

Note that $0^+ S([f]^\infty)$ is not contained in $0^+ S(f)$. Consequently, by Theorem 7 the function $f$ does not have a global error bound, but has an error one any finite union of polyhedral sets $V \subset \mathbb{R}^2$ having bounded intersection with the ray $\{x \in \mathbb{R}^2 \mid x^{(1)} \geq 0, x^{(2)} = 0\}$. 

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4 Error bounds for systems of piecewise affine equalities and inequalities

To conveniently summarize the main results of this article, let us apply them to obtain a straightforward extension of Hoffman’s theorem [15] to the case of systems of piecewise affine equalities and inequalities.

Let $F: \mathbb{R}^d \to \mathbb{R}^m$ and $G: \mathbb{R}^d \to \mathbb{R}^\ell$ be piecewise affine maps, and denote by

$$\Omega = \{x \in \mathbb{R}^d \mid F(x) = 0, G(x) \leq 0\}$$

the solution set of the system $F(x) = 0, G(x) \leq 0$. Here the inequality is understood coordinate-wise.

**Theorem 8.** Let $\Omega \neq \emptyset$, $V \subset \mathbb{R}^d$ be a given set, and $\| \cdot \|$ be an arbitrary norm on $\mathbb{R}^m$. Then the inequality

$$\tau \operatorname{dist}(x, \Omega) \leq \varphi(x) := \|F(x)\| + \ell \sum_{i=1}^\ell \max\{G_i(x), 0\} \quad \forall x \in V. \quad (25)$$

is satisfied for some $\tau > 0$, provided one of the following five conditions holds true:

1. $V$ is bounded;
2. $V$ is unbounded and
   $$\liminf_{\|x\| \to +\infty, x \in V} \frac{\varphi(x)}{\|x\|} > 0;$$
3. $V$ is a cone and $\varphi$ is coercive on $V$;
4. $V = \mathbb{R}^d$ and both $F$ and $G$ are positively homogeneous;
5. $V$ is a finite union of polyhedral sets and $S(\varphi^\infty) \cap 0^+ V \subseteq 0^+ \Omega$.

Furthermore, the last condition is necessary for inequality (25) to hold true in the general case, while the second and third conditions are necessary for this inequality to hold true in the case $0^+ \Omega \cap \operatorname{cl} V = \{0\}$.

**Proof.** Let $\| \cdot \|_\infty$ be the $\ell_\infty$ norm on $\mathbb{R}^d$. Introduce the function

$$f(x) = \|F(x)\|_\infty + \sum_{i=1}^\ell \max\{G_i(x), 0\} \quad \forall x \in \mathbb{R}^d.$$ 

This function is obviously piecewise affine, $S(f) = \Omega$, and $[f]_+ = f$. Note also that the function

$$\varphi^\infty(x) = \lim_{\lambda \to +\infty} \frac{\varphi(\lambda x)}{\lambda} = \|F^\infty(x)\| + \sum_{i=1}^\ell \max\{G_i^\infty(x), 0\}$$

is correctly defined and $S(\varphi^\infty) = S(f^\infty)$. Hence applying the results of the previous section, and taking into account the fact that the norms $\| \cdot \|$ and $\| \cdot \|_\infty$ are equivalent, one obtains the required result.
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