Thomsen’s D-theory and the K-theoretic Kronecker pairing

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September 8, 2020

We calculate Thomsen’s D-theory groups $D(\Sigma, B)$, where $\Sigma = C_0(\mathbb{R})$. Furthermore, we relate the pairing $K_0(A) \times E(A, B) \to K_0(B)$ to a similar pairing which is defined using D-theory.

1 Introduction

Connes and Higson [CH90] introduced the so-called E-theory groups $E(A, B)$ as the morphism sets in an additive category $E$ whose objects are the separable C*-algebras. There is a natural functor from the category of C*-algebras to the category $E$ which has the universal property that every stable, homotopy-invariant, and half-exact functor from the category of C*-algebras to an additive category factors through $E$. In particular, E-theory is closely related to Kasparov’s KK-theory [Kas80]. E-theory was successfully used by Higson and Kasparov [Hig00; HK97] to prove special cases of the Baum–Connes conjecture.

In [Tho03], Thomsen introduced D-theory, a discrete variant of Connes’s and Higson’s E-theory. Like E-theory, D-theory is a bifunctor from the category of separable C*-algebras to the category of abelian groups, and there exist products

\[
D(A, B) \times D(B, C) \to D(A, C),
\]

\[
D(A, B) \times E(B, C) \to D(A, C),
\]

\[
E(A, B) \times D(B, C) \to D(A, C),
\]

relating E-theory and D-theory.

It is a well-known calculation that the E-theory groups satisfy $E(\mathbb{C}, B) \cong K_0(B)$ for all C*-algebras $B$. However, there is, up to now, no concrete calculation of D-theory groups at all. Of course, one might hope to calculate $D(\mathbb{C}, B)$ in a similar way. Since D-theory (like E-theory) supports a suspension isomorphism $\Sigma: D(A, B) \cong D(\Sigma A, \Sigma B)$ for all separable C*-algebras $A$ and $B$ (where $\Sigma A = C_0(\mathbb{R}) \otimes A$ is the suspension of $A$), it suffices to calculate the groups $D(\Sigma, B)$ for $\Sigma = \Sigma \mathbb{C} = C_0(\mathbb{R})$. 

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Thus, the aim of the first part of this paper will be to prove the existence of a natural isomorphism

\[ D(\Sigma, B) \cong \prod_{n \in \mathbb{N}} K_0(B) \oplus \bigoplus_{n \in \mathbb{N}} K_0(B) \]  

where \( \Sigma = C_0(\mathbb{R}) \) is the suspension C*-algebra. The proof of (1) is similar, in structure, to the proof of the natural isomorphism \( E(C, B) \cong K_0(B) \). However, the technical details are slightly more complicated and involve the use of the concrete form of Cuntz’s periodicity map [Cun84]. Product-modulo-sum quotients as in the right hand of (1) have appeared before in various contexts, for example in the work of Blackadar and Kirchberg [BK97] on inductive limits of C*-algebras, of Hanke and Schick [HS06, HS07, HS08] or of Carrión and Dadarlat [CD18] on the maximal Baum–Connes assembly map.

There exists a natural pairing \( K_0(A) \times E(A, B) \cong E(C, A) \times E(A, B) \to E(C, B) \cong K_0(B) \), defined using the E-theory product [CH90]. By (1) there is also a pairing

\[ \prod_{n \in \mathbb{N}} K_0(A) \oplus \bigoplus_{n \in \mathbb{N}} K_0(A) \times E(A, B) \to D(\Sigma, A) \times E(A, B) \to D(\Sigma, B) \cong \prod_{n \in \mathbb{N}} K_0(B) \oplus \bigoplus_{n \in \mathbb{N}} K_0(B) \]

defined using Thomsen’s product. In the second part of this paper, we will show that this pairing agrees with the pairing \( \prod_{n \in \mathbb{N}} K_0(A) / \bigoplus_{n \in \mathbb{N}} K_0(A) \times E(A, B) \to \prod_{n \in \mathbb{N}} K_0(B) / \bigoplus_{n \in \mathbb{N}} K_0(B) \) induced by the E-theory product. This shows that it is possible to investigate the asymptotic behaviour of the pairing \( K_0(A) \times E(A, B) \to K_0(B) \) using Thomsen’s D-theory product.

This paper is based on part of the author’s doctoral dissertation at the Universität Augsburg. The author would like to thank his advisor Bernhard Hanke for his support and advice, and his helpful remarks on first version of this paper. The dissertation project was supported by a scholarship of the Studienstiftung des deutschen Volkes and by the TopMath program of the Elitenetzwerk Bayern.

2 E-theory and D-theory

We will define the D-theory groups, and in particular Thomsen’s products, in a way which differs slightly from [Tho03], and which is inspired by the definition of the corresponding objects for E-theory in [GHT00]. We will also review the definition of E-theory, as we will need it later on.

If \( X \) is a locally compact Hausdorff space and \( B \) is a C*-algebra, we denote by \( C_b(X; B) \) the C*-algebra of bounded continuous \( B \)-valued continuous functions on \( X \), and by \( C_0(X; B) \subset C_b(X; B) \) the ideal of functions vanishing at infinity. We are particularly interested in the special case \( X = \mathbb{P} = [0, \infty) \), where we abbreviate \( TB = C_b(P; B) \) and \( T_0B = C_0(P; B) \). Similarly, we write \( T_\delta B = C_b(N; B) \), \( T_\delta,0B = C_0(N; B) \).

**Definition 2.1.** The discrete asymptotic algebra [Tho03] over \( B \) is the C*-algebra \( A_\delta B = T_\delta B / T_\delta,0B \), and the asymptotic algebra [GHT00] over \( B \) is \( AB = TB / T_0B \).
Every *-homomorphism \( f : A \to B \) induces, by postcomposition, *-homomorphisms \( C_b(X; A) \to C_b(X; B) \) which restrict to *-homomorphisms \( C_0(X; A) \to C_0(X; B) \). In particular, \( f \) also induces *-homomorphisms \( A_δ A \to A_δ B \) and \( AA \to AB \). These definitions turn \( A \) and \( A_δ \) into functors, which are easily seen to be exact.

**Definition 2.2.** An asymptotic homomorphism between C*-algebras \( A \) and \( B \) is a *-homomorphism \( A \to IB \), where \( IB = C([0,1], B) \) is the C*-algebra of continuous \( B \)-valued functions on the unit interval \([0,1]\). The asymptotic homomorphisms \( \text{Aev}_0 \circ H \) and \( \text{Aev}_1 \circ H \) from \( A \to B \) are then called asymptotically homotopic. Here \( \text{ev}_\tau : IB \to B \) denotes the evaluation at \( \tau \in [0,1] \).

Similarly, a discrete asymptotic homotopy is a *-homomorphism \( H : A \to A_δ IB \), and again we call \( A_δ \text{ev}_0 \circ H \) and \( A_δ \text{ev}_1 \circ H \) asymptotically homotopic.

One can show [GHT00, Proposition 2.3] that asymptotic homotopy defines an equivalence relation on the sets of (discrete) asymptotic homomorphisms from \( A \) to \( B \).

**Definition 2.4.** We denote the set of asymptotic homotopy classes of asymptotic homomorphisms from \( A \) to \( B \) by \( [A,B] \), and the set of asymptotic homotopy classes of discrete asymptotic homomorphisms from \( A \) to \( B \) by \( [A,B]_δ \).

**Definition 2.5.** If \( A \) and \( B \) are separable C*-algebras, then we define

\[
E(A, B) = [ΣA ⊗ K, ΣB ⊗ K]
\]

and

\[
D(A, B) = [ΣA ⊗ K, Σ^2 B ⊗ K]_δ,
\]

where \( ΣA = \{ φ ∈ IA : φ(0) = φ(1) = 0 \} \cong C_0(\mathbb{R}) \otimes A \) is the suspension of \( A \), and where \( K = K(ℓ^2) \) is the C*-algebra of compact operators on a separable Hilbert space.

Note that the definitions of the E-theory and D-theory groups directly imply that there are stability isomorphisms \( E(A, B) \cong E(A \otimes K, B) \cong E(A, B \otimes K) \) and analogous stability isomorphisms for the D-theory groups.

The double suspension which appears in the definition of \( D(A, B) \) in front of the C*-algebra \( B \) has its origin in the following alternative viewpoint towards the discrete asymptotic algebra: The inclusion \( \mathbb{N} ⊂ P = [0, \infty) \) induces restriction maps \( T^B \to T^δ B \) and

\footnote{The separability assumption here is in order to make sure that we can define the products later on.}
These maps are clearly natural, so they induce a natural transformation \( A \to A_\delta \), which is clearly surjective. Thus, we may define another functor by considering the kernel of this natural transformation.

**Definition 2.6.** The sequentially trivial asymptotic algebra \( \Theta_0 \) over \( B \) is the \( \mathrm{C}^* \)-algebra \( A_0 B = \ker(AB \to A_\delta B) \). A sequentially trivial asymptotic homomorphism is a \( \ast \)-homomorphism \( A \to A_0 B \), and a sequentially trivial asymptotic homotopy is a \( \ast \)-homomorphism \( A \to A_0 IB \) as before. We denote by \( [A, B]_0 \) the set of asymptotic homotopy classes of sequentially trivial asymptotic homomorphisms.

There is a natural equivalence between the functors \( A_\delta \circ \Sigma \) and \( A_0 \) [cf. \( \Theta_0 \), Lemma 5.4]. Indeed, we may define maps \( \eta_B : T_\delta \Sigma B \to TB \) by

\[
\eta_B(\phi)(t) = \phi([t])(t - [t]).
\]

Then \( \eta_B(\phi) \in \Theta_0 B \) whenever \( \phi \in T_\delta,0 \Sigma B \), so there is an induced \( \ast \)-homomorphism \( \eta_B : A_\delta \Sigma B \to AB \).

**Lemma 2.7.** The \( \ast \)-homomorphism \( \eta_B : A_\delta \Sigma B \to AB \) is injective, and its image equals \( A_0 B \). Thus, \( \eta_B : A_\delta \Sigma B \to A_0 B \) is a natural \( \ast \)-isomorphism.

**Proof.** It suffices to prove that the rightmost column in the diagram

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
0 \rightarrow & T_\delta,0 \Sigma B & \rightarrow & T_\delta \Sigma B & \rightarrow & A_\delta \Sigma B & \rightarrow 0 \\
0 \rightarrow & T_0 B & \rightarrow & TB & \rightarrow & AB & \rightarrow 0 \\
0 \rightarrow & T_\delta,0 B & \rightarrow & T_\delta B & \rightarrow & A_\delta B & \rightarrow 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\]

is exact. Since the rows of the diagram are exact by definition, it suffices, by the Nine Lemma, to prove that the two left columns are exact, which is straightforward. \( \square \)

Hence, there are natural bijections \( [A, \Sigma B]_\delta \to [A, B]_0 \), and we may equally well define \( D(A, B) \) by

\[
D(A, B) = [\Sigma A \otimes K, \Sigma B \otimes K]_0.
\]

This description is most useful for constructing the products relating \( \mathrm{D} \)-theory and \( \mathrm{E} \)-theory. These products were introduced by Connes and Higson \( \Theta_0 \) for \( \mathrm{E} \)-theory, and
by Thomsen \cite{Tho03} for D-theory. The E-theory products were defined in a slightly different way by Guentner, Higson and Trout \cite{GHT00}. We will give definitions of the products which combine the methods of Thomsen and of Guentner, Higson and Trout. Indeed, we will use the following statement from \cite{GHT00} to simplify the definition given in \cite{Tho03} significantly.

**Lemma 2.8** (\cite{GHT00}, Claim 2.18). Consider a \(C^*-\)subalgebra \(E \subset T^2B\) which is separable. Then there exists a piecewise linear invertible continuous function \(r_0: P \rightarrow P\) with \(\lim_{t \to \infty} r_0(t) = \infty\), such that

\[
\lim_{t \to \infty} \sup_{r \geq r_0(t)} \| F(t)(r) \| \leq \| \pi \circ F \|_{A^2B}
\]

for all \(F \in E\), where \(\pi: TB \rightarrow AB\) is the projection.\(^3\) Any such function \(r_0\) will be called an admissible reparametrization for \(E \subset T^2B\).

Now the key idea in the definition of the products is the following: Suppose that \(f: A \rightarrow AB\) and \(g: B \rightarrow AC\) are asymptotic homomorphisms. Then we may consider \(Ag \circ f: A \rightarrow A^2C\). Thus, if we had a natural map \(\Phi: A^2C \rightarrow AC\) then we could use this map to define the product \(g \cdot f = \Phi \circ Ag \circ f: A \rightarrow AC\).

Using Lemma 2.8, this idea can be made rigorous provided that \(A\) is separable. Indeed, suppose that \(E_0 \subset A^2C\) is a separable \(C^*\)-subalgebra. Let \(\text{asc}_C: T^2C \rightarrow A^2C\) be the natural projection. Then we can choose a separable \(C^*\)-subalgebra \(E \subset T^2C\) with \(E_0 \subset \text{asc}_C(E)\), and an admissible reparametrization \(r_0: P \rightarrow P\) for \(E\). We define *-homomorphisms \(\Phi, \hat{\Phi}: E_0 \rightarrow AC\) by

\[
\Phi([\pi \circ F]) = [t \mapsto F(t)(r_0(t))],
\]

\[
\hat{\Phi}([\pi \circ F]) = [t \mapsto F(r_0^{-1}(t))(t)]
\]

for \(F \in E\) with \(\text{asc}_C(F) = [\pi \circ F] \in E_0\). Lemma 2.8 makes sure that \(\Phi\) and \(\hat{\Phi}\) are well-defined *-homomorphisms which, however, certainly do depend on the choices of \(E_0\) and \(r_0\).

Now suppose that \(f: A \rightarrow AB\) and \(g: B \rightarrow AC\) are asymptotic homomorphisms. If \(A\) is separable, then also \(Ag(f(A)) \subset A^2C\) is separable. Choose a separable \(C^*\)-subalgebra \(E \subset T^2C\) with \(Ag(f(A)) \subset \text{asc}_C(E)\), and fix an admissible reparametrization \(r_0: P \rightarrow P\) for \(E\). Let \(\Phi: Ag(f(A)) \rightarrow AC\) be a as constructed above. Then we define the asymptotic composition of \(f\) and \(g\) to be \(g \cdot f = \Phi \circ Ag \circ f: A \rightarrow AC\). This construction gives a product \([A, B] \times [B, C] \rightarrow [A, C]\), \(([f], [g]) \mapsto [g \cdot f]\), and hence a product \(E(A, B) \times E(B, C) \rightarrow E(A, C)\) if \(A\) is separable. It turns out that the choices which went into the definition of \(\Phi\) do not change this product, and that one could equally well replace \(\Phi\) by \(\hat{\Phi}\).

Now the situation is very similar when one considers sequentially trivial asymptotic homomorphisms: If \(f: A \rightarrow A_0B\) is sequentially trivial and \(g: B \rightarrow AC\) is as above,

\(^3\)Hence \(\pi \circ F: P \rightarrow AB\) represents an element of \(A^2B\).
then we may choose $E \subset T^2C$ with $A\sigma(f(A)) \subset as_C(E)$ in such a way that every $F \in E$ satisfies $F(n)(t) = 0$ for all $n \in \mathbb{N}$ and $t \in P$. Then $\Phi \circ A\sigma \circ f: A \to A\sigma_0C$ is sequentially trivial as well. Similarly, if $f: A \to AB$ is arbitrary and $g: B \to A\sigma_0C$ is sequentially trivial then we may choose $E$ such that $F \in E$ satisfies $F(t)(n) = 0$ for all $n \in \mathbb{N}$ and $t \in P$. In this case, we may put $g \cdot f = \Phi \circ A\sigma \circ f$. Thus, we have defined products $[A, B]_{0} \times [B, C] \to [A, C]_{0}$ and $[A, B] \times [B, C]_{0} \to [A, C]_{0}$. Using either of the natural maps $[A, B]_{0} \to [A, B]$ or $[B, C]_{0} \to [B, C]$, we also obtain a product $[A, B]_{0} \times [B, C]_{0} \to [A, C]_{0}$. All of these products are compatible with the maps $[\cdot, \cdot]_{0} \to [\cdot, \cdot]$, and they satisfy all possible kinds of associativity laws [cf. Tho03, Section 3].

There are two natural ways to define a group structure on the sets $E(A, B)$ and $D(A, B)$. The first one, which is employed in GHT03 and Tho03, goes as follows: Choose a unitary isomorphism $V: \ell^2 \to \ell^2 \oplus \ell^2$. Then conjugation with $V$ induces a C*-algebra isomorphism $K(\ell^2 \oplus \ell^2) \to K(\ell^2)$, which is independent of the choice of $V$ up to homotopy. We may therefore define a product on the set $[A, \Sigma B \otimes K]$ by the composition

$$[A, \Sigma B \otimes K] \times [A, \Sigma B \otimes K] \cong [A, \Sigma B \otimes (K \oplus K)] \to [A, \Sigma B \otimes K(\ell^2 \oplus \ell^2)] \cong [A, \Sigma B \otimes K].$$

Another way would be to employ the natural map $\Sigma \otimes \Sigma \to \Sigma$ given by concatenation, where $\Sigma \cong C_0(0, 1)$. It can be shown that these two products on $[A, \Sigma B \otimes K]$ agree and define an abelian group structure on $[A, \Sigma B \otimes K]$. In an entirely analogous fashion, also $[A, \Sigma B \otimes K]_{0}$ and $[A, \Sigma B \otimes K]_{0}$ (and hence in particular $D(A, B)$ and $E(A, B)$) are abelian groups.

One can show that the asymptotic products described above define group homomorphisms $D(A, B) \times D(B, C) \to D(A, C)$, $D(A, B) \times E(B, C) \to D(A, C)$, $E(A, B) \times D(B, C) \to D(A, C)$, and $E(A, B) \times E(B, C) \to E(A, C)$.

3 Calculation of $D(\Sigma, B)$

In this section, we will give a calculation of the group $D(\Sigma, B)$ for any separable C*-algebra $B$. This calculation bears some similarities with the well-known natural isomorphism $E(C, B) \cong K_0(B)$, which we will review first.

Let $B$ be an arbitrary C*-algebra, and consider a unitary $u \in M_n(B_+)$ with $u - 1 \in M_n(B)_0^3$. As usual, we denote the set of such unitaries by $U_n^+(B) \subset M_n(B_+)$. In particular, $u$ represents an element $[u] \in K_1(B)$. We identify the C*-algebra $\Sigma = C_0(0, 1)$ with the C*-algebra of all functions $\phi \in C(S^1)$ with $\phi(1) = 0$. Then there exists a unique *-homomorphism $g_u: \Sigma \to M_n(B) \subset B \otimes K$ with $g_u(\omega) = u - 1$ where $\omega: S^1 \to \mathbb{C}$ is given by $\omega(z) = z - 1$. We define a map

$$g^B: K_1(B) \to [\Sigma, B \otimes K]$$

Here $B_+$ is the unitization of the C*-algebra $B$. 


by $[u] \mapsto [\kappa_{B \otimes K} \circ g_u]$ where $\kappa_{B \otimes K}: B \otimes K \to A(B \otimes K)$ is given by $\kappa_{B \otimes K}(x) = [t \mapsto x]$. The following statement is well-known:

**Proposition 3.1** ([Ros82, Theorem 4.1] and [GHT00, Proposition 2.19]). For every $C^*$-algebra $B$, the map $g^B_\Sigma: K_1(\Sigma B) \to [\Sigma, \Sigma B \otimes K]$ is an isomorphism of groups. □

One can show easily that the inclusion $\Sigma \to \Sigma \otimes K$ induces an isomorphism $E(\mathbb{C}, B) = [\Sigma \otimes K, \Sigma B \otimes K] \to [\Sigma, \Sigma B \otimes K]$. Together with Proposition 3.1, this establishes the isomorphism $K_0(B) \cong K_1(\Sigma B) \cong E(\mathbb{C}, B)$ for arbitrary $C^*$-algebras $B$.

Now let us turn to the calculation of $D(\Sigma, B)$, which has not appeared in the literature so far. Let again $B$ be a $C^*$-algebra, and let $(u_n)_{n \in \mathbb{N}}$ be a sequence in $\bigcup_{k \in \mathbb{N}} U^+_k(B)$, so that each $u_n$ represents an element of $K_1(B)$. The map $\phi: \mathbb{N} \to B \otimes K$, which is defined by $\phi(n) = u_n - 1$, determines an element $[\phi] \in A_\delta(B \otimes K)$ such that $[\phi] + 1 \in A_\delta(B \otimes K)_+$ is unitary. Hence there exists a unique discrete asymptotic homomorphism $\tilde{g}_{(u_n)}: \Sigma \to A_\delta(B \otimes K)$ such that $\tilde{g}_{(u_n)}(\omega) = [\phi] = [n \mapsto u_n - 1]$. The map

$$g^B_\delta : \prod_{n \in \mathbb{N}} K_1(B) \to [\Sigma, B \otimes K]_\delta,$$

$$([u_n])_{n \in \mathbb{N}} \mapsto [\tilde{g}_{(u_n)}],$$

is well-defined: Indeed if we are given continuous paths $(u_n^\tau)_{\tau \in [0, 1]}$ in $U^+_k(B)$ then the same construction as above yields a discrete asymptotic homotopy $H: \Sigma = A_\delta(B \otimes K)$ with $H(\omega) = [n \mapsto (\tau \mapsto u_n^\tau - 1)]$, and $H$ is a discrete asymptotic homotopy connecting $\tilde{g}_{(u_n^0)}$ and $\tilde{g}_{(u_n^1)}$. The key step in calculating $D(\Sigma, B)$ is the following analogue of Proposition 3.1.

**Proposition 3.2.** For every $C^*$-algebra $B$, the map $g^B_\Sigma: \prod_{n \in \mathbb{N}} K_1(\Sigma B) \to [\Sigma, \Sigma B \otimes K]_\delta$ is a surjective group homomorphism with

$$\ker g^B_\delta = \bigoplus_{n \in \mathbb{N}} K_1(\Sigma B),$$

where $\bigoplus_{n \in \mathbb{N}} K_1(\Sigma B) \subset \prod_{n \in \mathbb{N}} K_1(\Sigma B)$ is the subgroup consisting of all sequences $([u_n])_{n \in \mathbb{N}}$ which vanish eventually.

**Proof.** We begin by proving that $g^B_\Sigma$ is surjective. Thus, we consider an arbitrary element $[h] \in [\Sigma, \Sigma B \otimes K]_\delta$ which is represented by a discrete asymptotic homomorphism $h: \Sigma \to A_\delta(\Sigma B \otimes K)$. We may write $h(\omega) = [G]$ for a map $G: \mathbb{N} \to \Sigma B \otimes K$. We may replace each $G(n)$ by an element of $\bigcup_{k \in \mathbb{N}} M_k(\Sigma B) \subset \Sigma B \otimes K$ which is $n^{-1}$-close to $G(n)$, without altering $[G] \in A_\delta(\Sigma B \otimes K)$. Thus, we may assume that $G(n) \in \bigcup_{k \in \mathbb{N}} M_k(\Sigma B)$ for all $n \in \mathbb{N}$.

We will show next that there exists a map $U: \mathbb{N} \to \Sigma B \otimes K$ such that $U(n) \in \bigcup_{k \in \mathbb{N}} U^+_k(\Sigma B)$ for all $n \in \mathbb{N}$, and such that $[G] = [U - 1] \in A_\delta(\Sigma B \otimes K)$. Indeed, $[G + 1] \in A_\delta(\Sigma B \otimes K)$ must be unitary, so that

$$\lim_{n \to \infty} (G(n) + 1)^*(G(n) + 1) = 1.$$
We put $F(n) = G(n) + 1$. Thus, $F(n)F(n)$ is invertible if $n$ is sufficiently large. Without loss of generality, $F(n)^*F(n)$ is invertible for all $n \in \mathbb{N}$. Now we put $U(n) = F(n)(F(n)^*F(n))^{-1/2}$. It is straightforward to see that indeed $[G] = [F - 1] = [U - 1] \in A_{\delta}(\Sigma B \otimes K)$ and that each $U(n)$ is contained in $\bigcup_{k \in \mathbb{N}} U_k^+(\Sigma B)$. We have $g_\delta^\Sigma B([U(n)])_{n \in \mathbb{N}} = [\tilde{g}(U(n))]$ where $\tilde{g}(U(n)) : \Sigma \to A_{\delta}(\Sigma B \otimes K)$ is determined by the property

$\tilde{g}(U(n))(\omega) = [n \mapsto U(n) - 1] = [U - 1] = [G] = h(\omega)$.

Hence, $g_\delta^\Sigma B([U(n)])_{n \in \mathbb{N}} = [\tilde{g}(U(n))] = [h]$ and $g_\delta^\Sigma B$ is surjective.

Next suppose that $(u_n)_{n \in \mathbb{N}}$ is a sequence of unitaries in $\bigcup_{k \in \mathbb{N}} U_k^+(\Sigma B)$ such that $g_\delta^\Sigma B((U(n)))_{n \in \mathbb{N}} = 0$. Thus, there exists a discrete asymptotic homotopy $H : \Sigma \to A_{\delta}I(\Sigma B \otimes K)$ with $Aev_0 \circ H = \tilde{g}(u_n)$ and $Aev_1 \circ H = 0$. As above, we may write $H(\omega) = [U - 1]$ where $U : \mathbb{N} \to I(\Sigma B \otimes K)_+$ is a unitary-valued map with $U(n) - 1 \in I(\Sigma B \otimes K)$ for all $n \in \mathbb{N}$. By assumption, $\lim_{n \to \infty} ||U(n)(0) - u_n|| = \lim_{n \to \infty} \|U(n)(1) - 1\| = 0$. A standard argument shows that therefore $[u_n] = [U(n)(0)] = [U(n)(1)] = [1] = 0 \in K_1(\Sigma B)$ if $n$ is sufficiently large. Thus, $(([u_n])_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} K_1(\Sigma B)$.

On the other hand, if $(v_n)_{n \in \mathbb{N}} \in \bigoplus_{n \in \mathbb{N}} K_1(\Sigma B)$ then $\tilde{g}(u_n)(\omega) = [n \mapsto u_n - 1] = [n \mapsto 0]$ so that $g_\delta^\Sigma B([u_n])_{n \in \mathbb{N}} = 0$. This completes the calculation of ker $g_\delta^\Sigma B$.

Finally, it remains to prove that $g_\delta^\Sigma B$ is additive. Thus, let $(u_n)_{n \in \mathbb{N}}$ and $(v_n)_{n \in \mathbb{N}}$ be two sequences in $\bigcup_{k \in \mathbb{N}} U_k^+(\Sigma B)$. It is a well-known fact that $[u_n] + [v_n] = [u_nv_n] = [u_n*v_n] \in K_1(\Sigma B)$, where $u_n*v_n$ is the concatenation of $u_n$ and $v_n$, viewed as elements of $\Sigma(M_k(B)_+)$. In particular, $g_\delta^\Sigma B((u_n)_{n \in \mathbb{N}}) = [\tilde{g}(u_n) + \tilde{g}(v_n)]$, and by definition of the group structure on $[\Sigma, \Sigma B \otimes K]_\delta$ we obtain that indeed $[\tilde{g}(u_n*v_n)] = [\tilde{g}(u_n)] + [\tilde{g}(v_n)] = g_\delta^\Sigma B([u_n])_{n \in \mathbb{N}} + g_\delta^\Sigma B([v_n])_{n \in \mathbb{N}}$.

Note that Proposition 3.2 together with Bott periodicity for K-theory and for D-theory and the stability isomorphism for D-theory, yields a chain of isomorphisms

$$\frac{\prod_{n \in \mathbb{N}} K_0(B)}{\bigoplus_{n \in \mathbb{N}} K_0(B)} \cong \frac{\prod_{n \in \mathbb{N}} K_1(\Sigma^3 B)}{\bigoplus_{n \in \mathbb{N}} K_1(\Sigma^3 B)} \cong [\Sigma, \Sigma^3 B \otimes K]_\delta$$

$$\cong [\Sigma^2 \otimes K, \Sigma B \otimes K]_\delta$$

$$= D(\Sigma, \Sigma B \otimes K) \cong D(\Sigma, B).$$

However, we will give a more succinct description of this isomorphism next. Suppose for the moment that $B$ is a unital C*-algebra. Then we define

$$\Psi_B : \prod_{n \in \mathbb{N}} K_0(B) \to D(\Sigma, B)$$

as follows: We write an element of $\prod_{n \in \mathbb{N}} K_0(B)$ as $([p_n])_{n \in \mathbb{N}}$ where each $p_n$ is a projection in $M_{\infty}(B) = \bigcup_{k \in \mathbb{N}} M_k(B)$. We consider the discrete asymptotic homomorphism $f_{(p_n)} : \mathbb{C} \to A_{\delta}(B \otimes K)$ which is determined by $f_{(p_n)}(1) = [n \mapsto p_n]$. Now we define
Let $\Psi_B$ be the prescription $\Psi_B([p_n])_{n \in \mathbb{N}} = [\Sigma^2 f(p_n) \otimes \mathrm{id}_K] \in [\Sigma^2 \otimes K, \Sigma^2 B \otimes K \otimes K]_\delta = D(\Sigma, B \otimes K) \cong D(\Sigma, B)$. Of course,

$$\Sigma^2 f(p_n) \otimes \mathrm{id}_K(\phi \otimes \psi \otimes T) = [n \mapsto \phi \otimes \psi \otimes p_n \otimes T]$$

for all $\phi, \psi \in \Sigma$ and $T \in K$. If $B$ is non-unital we define $\Psi_B$ by requiring that the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \prod_{n \in \mathbb{N}} K_0(B) & \longrightarrow & \prod_{n \in \mathbb{N}} K_0(B^+) & \longrightarrow & \prod_{n \in \mathbb{N}} K_0(C) & \longrightarrow & 0 \\
\Psi_{B^+} & \downarrow & \Psi_{B^+} & & & \downarrow & \Psi_c & & \\
0 & \longrightarrow & D(\Sigma, B) & \longrightarrow & D(\Sigma, B^+) & \longrightarrow & D(\Sigma, C) & \longrightarrow & 0
\end{array}
$$

with exact rows commutes.

**Theorem 3.3.** For any $C^*$-algebra $B$, the map $\Psi_B : \prod_{n \in \mathbb{N}} K_0(B) \rightarrow D(\Sigma, B)$ is a natural surjective group homomorphism with $\ker \Psi_B = \bigoplus_{n \in \mathbb{N}} K_0(B)$.

**Proof.** It is clear that $\Psi_B$ is natural in $B$. The proof consists of several parts: First we will assume that $B$ is unital in order to give another description of the map $\Psi_B$. By naturality, we can extend this description to non-unital $C^*$-algebras. Secondly we will use this alternative description for the $C^*$-algebra $\Sigma^2 B$ in order to prove the statement of the theorem for double suspensions $\Sigma^2 B$. Finally we will use the concrete description of Cuntz’s version of Bott periodicity [Cun84] to reduce the general case to the case of double suspensions.

We define a map $\tilde{\Psi}_B : \prod_{n \in \mathbb{N}} K_0(B) \rightarrow D(\Sigma, B \otimes K)$ to be the composition

$$
\prod_{n \in \mathbb{N}} K_0(B) \overset{\beta}{\longrightarrow} \prod_{n \in \mathbb{N}} K_1(\Sigma B) \overset{g_B}{\longrightarrow} [\Sigma, \Sigma B \otimes K] \overset{\Sigma \otimes \mathrm{id}_K}{\longrightarrow} [\Sigma^2 \otimes K, \Sigma^2 B \otimes K \otimes K]_\delta = D(\Sigma, B \otimes K) \cong D(\Sigma, B),
$$

where $\beta$ is the Bott periodicity isomorphism. It is clear that $\tilde{\Psi}_B$ is natural in $B$. We will prove that $\Psi_B = \tilde{\Psi}_B$. By naturality, it suffices to prove this for unital $C^*$-algebras $B$.

Thus, let $B$ be a unital $C^*$-algebra, and let $p_n \in \bigcup_{k \in \mathbb{N}} M_k(B)$ be a sequence of projections, representing an element $([p_n])_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} K_0(B)$. By the description of the Bott periodicity isomorphism $K_0(B) \rightarrow K_1(\Sigma B)$, we have $\beta([p_n])_{n \in \mathbb{N}} = ([\omega \otimes p_n + 1])_{n \in \mathbb{N}}$. In particular, $g_B \circ \beta([p_n])_{n \in \mathbb{N}} = [\tilde{g}]$ where $\tilde{g} : \Sigma \rightarrow A_\delta(\Sigma B \otimes K)$ is such that $\tilde{g}(\omega) = [n \mapsto \omega \otimes p_n]$. Of course, this implies that $\tilde{g}(\psi) = [n \mapsto \psi \otimes p_n]$ for all $\psi \in \Sigma$. In particular, $\Sigma g \otimes \mathrm{id}_K : \Sigma^2 \otimes K \rightarrow A_\delta(\Sigma^2 B \otimes K \otimes K)$ is such that $\Sigma g \otimes \mathrm{id}_K(\phi \otimes \psi \otimes T) = [n \mapsto \phi \otimes \psi \otimes p_n \otimes T]$ for all $\phi, \psi \in \Sigma$ and $T \in K$. Therefore, $\Sigma g \otimes \mathrm{id}_K = \Sigma^2 f(p_n) \otimes \mathrm{id}_K$, which implies that indeed $\Psi_B = \tilde{\Psi}_B$. 

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Now we consider a double suspension \( \Sigma^2 B \). In this case, \( \Psi_B \) can also be written as the composition

\[
\prod_{n \in \mathbb{N}} K_0(\Sigma^2 B) \xrightarrow{\beta} \prod_{n \in \mathbb{N}} K_1(\Sigma^3 B) \xrightarrow{g_{\Sigma^3 B}} [\Sigma, \Sigma^3 B \otimes K] \cong [\Sigma, \Sigma^2 B \otimes K]_0
\]

\[
\xrightarrow{- \otimes \text{id}_K} [\Sigma \otimes K, \Sigma^2 B \otimes K \otimes K]_0 = D(\Sigma, \Sigma B \otimes K)
\]

\[
\xrightarrow{\Sigma} D(\Sigma, \Sigma^2 B \otimes K) \cong D(\Sigma, \Sigma^2 B)
\]

where all maps in this composition are isomorphisms, except for \( g_{\Sigma^3 B} \) which is surjective with kernel equal to \( \bigoplus_{n \in \mathbb{N}} K_1(\Sigma^3 B) \) by Proposition 3.2. Since \( \beta^{-1}(\bigoplus_{n \in \mathbb{N}} K_1(\Sigma^3 B)) = \bigoplus_{n \in \mathbb{N}} K_0(\Sigma^2 B) \), this implies the claim of the theorem for \( \Sigma^2 B \).

In the case of general \( B \), we consider the diagram

\[
\begin{array}{ccccccc}
0 & \longrightarrow & \bigoplus_{n \in \mathbb{N}} K_0(\Sigma^2 B) & \longrightarrow & \prod_{n \in \mathbb{N}} K_0(\Sigma^2 B) & \xrightarrow{\Psi_{\Sigma^2 B}} & D(\Sigma, \Sigma^2 B) & \longrightarrow & 0 \\
\uparrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\
0 & \longrightarrow & \bigoplus_{n \in \mathbb{N}} K_0(B) & \longrightarrow & \prod_{n \in \mathbb{N}} K_0(B) & \xrightarrow{\Psi_B} & D(\Sigma, B) & \longrightarrow & 0
\end{array}
\]

where the vertical arrows are the periodicity isomorphisms coming from Cuntz’s [Cun84] version of Bott periodicity. Recall that these periodicity isomorphisms are the index maps associated to a certain short exact sequence of C*-algebras. By construction of these index maps, the diagram above commutes because the horizontal maps are given by natural transformations. We have already seen that the top row in the diagram is exact, so the bottom row must be exact as well. \( \square \)

### 4 The K-theoretical Kronecker pairing and D-theory

As mentioned in the introduction, the calculation

\[
D(\Sigma, A) \cong \frac{\prod_{n \in \mathbb{N}} K_0(A)}{\bigoplus_{n \in \mathbb{N}} K_0(A)}
\]

implies that there are two different ways for defining the product of an element of \( \prod_{n \in \mathbb{N}} K_0(A) / \bigoplus_{n \in \mathbb{N}} K_0(A) \) with an E-theory class in \( E(A, B) \), yielding an element of \( \prod_{n \in \mathbb{N}} K_0(B) / \bigoplus_{n \in \mathbb{N}} K_0(B) \). The following result states that these two products are in fact equal.

**Theorem 4.1.** Let \( A \) and \( B \) be C*-algebras and fix \( \eta \in E(A, B) \). Then the compositions

\[
\frac{\prod_{n \in \mathbb{N}} K_0(A)}{\bigoplus_{n \in \mathbb{N}} K_0(A)} \cong \frac{\prod_{n \in \mathbb{N}} E(\mathbb{C}, A)}{\bigoplus_{n \in \mathbb{N}} E(\mathbb{C}, A)} \rightarrow \frac{\prod_{n \in \mathbb{N}} E(\mathbb{C}, B)}{\bigoplus_{n \in \mathbb{N}} E(\mathbb{C}, B)} \cong \frac{\prod_{n \in \mathbb{N}} K_0(B)}{\bigoplus_{n \in \mathbb{N}} K_0(B)}
\]
and
\[
\prod_{n \in \mathbb{N}} K_0(A) \cong D(\mathcal{S}, A) \rightarrow D(\mathcal{S}, B) \cong \prod_{n \in \mathbb{N}} K_0(B),
\]
which are given by the respective composition product with \(\eta\).

Before we step into the proof of the theorem, we state a lemma that we will need in the course of the proof.

**Lemma 4.2.** Let \(B\) be a C*-algebra, and fix \(\tau \in [0, 1]\). Assume that \(\check{E} \subset \mathcal{A}^2IB\) is separable, and put \(E = \mathcal{A}^2ev(\check{E}) \subset \mathcal{A}^2B\). Let \(E \subset \mathcal{T}IB\) be separable with \(as_B(E) = \check{E}\), and put \(E_\tau = \mathcal{T}ev(\tau) \subset \mathcal{T}B\) (so that in particular \(as_B(E_\tau) = \check{E}\)). Let \(r_0 : P \rightarrow P\) be a reparametrization which is admissible for both \(E\) and \(E_\tau\). Then the diagram

\[
\begin{array}{ccc}
\check{E} & \xrightarrow{\Phi} & \mathcal{A}IB \\
\mathcal{A}^2ev & \downarrow & \mathcal{A}ev \\
\check{E}_\tau & \xrightarrow{\Phi} & \mathcal{A}B
\end{array}
\]

commutes if we use \(r_0\) to define both horizontal maps \(\Phi\).

**Proof.** Let \(F \in E\) be arbitrary. Then \(\Phi[\pi \circ F] = [t \mapsto F(t)(r_0(t))] \in \mathcal{A}IB\) and hence
\[
\mathcal{A}ev_\tau \circ \Phi[\pi \circ F] = [t \mapsto F(t)(r_0(t))(\tau)] \in \mathcal{A}B.
\]
On the other hand, \(\mathcal{A}^2ev_\tau[\pi \circ F] = [\pi \circ F_\tau]\) where \(F_\tau = \mathcal{T}ev_\tau(F) \in E_\tau\). Therefore,
\[
\Phi \circ \mathcal{A}^2ev_\tau[\pi \circ F] = \Phi[\pi \circ F_\tau] = [t \mapsto F_\tau(t)(r_0(t))] = [t \mapsto F(t)(r_0(t))(\tau)] = \mathcal{A}ev_\tau \circ \Phi[\pi \circ F]
\]
as claimed. \(\Box\)

**Proof of Theorem 4.1.** By a naturality argument, we may assume without loss of generality that \(A\) and \(B\) are unital. Let \((p_n)_{n \in \mathbb{N}}\) be a sequence of projections in \(A \otimes \mathcal{K}\), and represent \(\eta\) by an asymptotic homomorphism \(f : A \rightarrow AB\). Then the image of \([([p_n])_{n \in \mathbb{N}}] \in \prod_{n \in \mathbb{N}} K_0(A) / \bigoplus_{n \in \mathbb{N}} K_0(A)\) under the first composition is represented by a family \((\check{p}_n)_{n \in \mathbb{N}}\) of projections in \(B \otimes \mathcal{K}\) which have the property that the asymptotic homotopy classes of the asymptotic homomorphisms \(\phi \mapsto [t \mapsto \phi \otimes \check{p}_n]\) and \(\phi \mapsto f(\phi \otimes p_n)\) agree. Let \(H_n : \Sigma \rightarrow \mathcal{A}I(\mathcal{S}B \otimes \mathcal{K})\) be asymptotic homotopies with \(\mathcal{A}ev_0 \circ H_n(\phi) = f(\phi \otimes p_n)\) and \(\mathcal{A}ev_1 \circ H_n(\phi) = [t \mapsto \phi \otimes \check{p}_n]\) for all \(\phi \in \Sigma\).

We have to prove that the second composition in the statement of the theorem maps \([([p_n])_{n \in \mathbb{N}}]\) to \([([\check{p}_n])_{n \in \mathbb{N}}]\) as well. It follows from the description of \(\Psi_A\) in Theorem 3.3 that under the identification
\[
\prod_{n \in \mathbb{N}} K_0(A) / \bigoplus_{n \in \mathbb{N}} K_0(A) \cong [\Sigma^2, \Sigma^2 A \otimes \mathcal{K}]_0 \cong [\Sigma^2, \Sigma A \otimes \mathcal{K}]_0,
\]
the second composition maps \(\mathcal{A}ev_\tau \circ \Phi[\pi \circ F]\). \(\Box\)
where the right-hand side is defined using the composition product
Analogously, \([(\tilde{\varphi})_n]_{n \in \mathbb{N}}\) is identified with the class of the sequentially trivial homomorphism \(g: \Sigma^2 \to A_0(\Sigma \mathcal{B} \otimes \mathcal{K})\) which is given by
Thus, it is possible to use the D-theory product to calculate Kronecker pairings asymptotically homogeneous. For appropriate choices in the respective definitions of \(\Phi\) we have
Now suppose that \((\xi, \nu)_{n \in \mathbb{N}}\) is a sequence in \(K_{\ell}B \otimes A\). This sequence then defines a class in \(D(\Sigma, \Sigma^{\ell+1}A)\). (Note that \(K_{\ell}(\Sigma^{\ell+1}A)\) is a K-theory class, we can define the Kronecker pairing of \(\eta\) and \(\xi\) to be
Thus, it is possible to use the D-theory product to calculate Kronecker pairings asymptotically homogeneous. 

\[\langle \eta, \xi \rangle = (\Sigma^{\ell}\eta \otimes \text{id}_A) \bullet \xi \in E(\Sigma, \Sigma^{\ell+1}A) \cong K_{\ell+1}(A).\]

Corollary 4.3. In this situation,
\[\Psi_{\Sigma^{\ell+1}A}[\langle \eta, \Sigma_{n \in \mathbb{N}}\rangle] = (\Sigma^\ell \eta \otimes \text{id}_A) \bullet \Psi_{\Sigma^\ell(B \otimes A)}[\langle \xi_{n \in \mathbb{N}}\rangle] \in D(\Sigma, \Sigma^{\ell+1}A),\]
where the right-hand side is defined using the composition product \(E(\Sigma^{\ell}(B \otimes A), \Sigma^{\ell+1}A)\times D(\Sigma, \Sigma^{\ell}(B \otimes A)) \to D(\Sigma, \Sigma^{\ell+1}A)\).
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