A sublinear query quantum algorithm for $s$-$t$ minimum cut on dense simple graphs

Simon Apers* Arinta Auza† Troy Lee‡

Abstract

An $s$-$t$ minimum cut in a graph corresponds to a minimum weight subset of edges whose removal disconnects vertices $s$ and $t$. Finding such a cut is a classic problem that is dual to that of finding a maximum flow from $s$ to $t$. In this work we describe a quantum algorithm for the minimum $s$-$t$ cut problem on undirected graphs. For an undirected graph with $n$ vertices, $m$ edges, and integral edge weights bounded by $W$, the algorithm computes with high probability the weight of a minimum $s$-$t$ cut after $\tilde{O}(\sqrt{mn^{5/6}W^{1/3}})$ queries to the adjacency list of $G$. For simple graphs this bound is always $\tilde{O}(n^{11/6})$, even in the dense case when $m = \Omega(n^2)$. In contrast, a randomized algorithm must make $\Omega(m)$ queries to the adjacency list of a simple graph $G$ even to decide whether $s$ and $t$ are connected.

1 Introduction

Let $G$ be a graph with $n$ vertices and $m$ edges, and let $s, t$ be two vertices of $G$. The problem of determining the maximum amount of flow $\lambda_{st}(G)$ that can be routed from $s$ to $t$ while respecting the capacity constraints of $G$ is one of the most fundamental problems in theoretical computer science, whose study goes back at least to the 1950s and the pioneering work of Ford and Fulkerson [FF56]. By the max-flow min-cut theorem, $\lambda_{st}(G)$ is equal to the minimum weight of a cut separating $s$ and $t$ in $G$. From a maximum $s$-$t$ flow one can compute a minimum $s$-$t$ cut in linear time, but no such reduction is known the other way around. The Goldberg-Rao [GR98] algorithm, with running time $O(\min\{\sqrt{m}, n^{2/3}\} m \log(n) \log(W))$, where $W$ is the largest weight of an edge, stood as the best bound for both problems for many years. In the past decade, however, beginning with work of Christiano, Kelner, Mądry, Spielman, and Teng [CKM+11] there has been tremendous progress in max-flow algorithms by incorporating techniques from continuous optimization [She13, Mąd13, KLOS14, Pen16, LS14, LS20, GLP21, vdBL21]. With a recent $m^{1+o(1)} \log W$ time max-flow algorithm by Chen, Kyng, Liu, Peng, Gutenberg, and Sachdeva [CKL+22], there are now almost-linear time randomized max-flow algorithms for graphs with polynomially bounded integral weights.

*IRIF, CNRS, Paris. Email: smgapers@gmail.com
†Centre for Quantum Software and Information, University of Technology Sydney.
‡Centre for Quantum Software and Information, University of Technology Sydney. Email: troyjlee@gmail.com
Given its fundamental nature, there has been a surprising lack of work on quantum algorithms for the exact maximum flow or minimum s-t cut problem. As far as we are aware, the only work on the quantum complexity of these problems is by Ambainis and Špalek [AS06], who gave a quantum algorithm for max flow in a directed graph with integral weights bounded by $W \leq n^{1/4}$ with running time $\tilde{O}(\min\{n^{7/6}\sqrt{mW^{1/3}}, m\sqrt{nW}\})$, given adjacency list access to $G$. This bound is completely subsumed by current classical randomized algorithms.

More recent work of Apers and de Wolf [AdW20] gives a $\tilde{O}(\sqrt{mn}/\varepsilon)$ time quantum algorithm for finding a $(1 + \varepsilon)$-approximation of the minimum s-t cut. This algorithm works by first constructing an $\varepsilon$-cut sparsifier $H$ of the input graph $G$ with a quantum algorithm in time $\tilde{O}(\sqrt{mn}/\varepsilon)$. An $\varepsilon$-cut sparsifier is a re-weighted subgraph of $G$ that has only $\tilde{O}(n/\varepsilon^2)$ edges but for which the weight of every cut agrees with that of $G$ up to a factor of $1 \pm \varepsilon$. One can then run a classical randomized s-t minimum cut algorithm on the sparse graph $H$ to find an approximate s-t minimum cut of $G$. The quantum sparsification algorithm also plays a key role in our work.

In this paper, we give a quantum algorithm that computes $\lambda_{st}(G)$ on dense simple graphs with fewer queries than is possible classically. More generally, for an undirected and integral weighted graph $G$ with maximum weight $W$, we give a quantum algorithm with adjacency list access to $G$ that with high probability outputs $\lambda_{st}(G)$ and makes $\tilde{O}(\sqrt{mn^{5/6}W^{1/3}})$ queries (see Theorem 15). Thus for unweighted graphs the number of queries is $\tilde{O}(n^{11/6})$ even for dense instances with $m \in \Omega(n^2)$ edges. On the other hand, it is easy to show that in the worst case a randomized algorithm requires $\Omega(m)$ queries to the adjacency list of a graph with $m$ edges even to decide whether $s$ and $t$ are connected (see Theorem 19). Our algorithm also works with adjacency matrix access to $G$, in which case the number of queries becomes $\tilde{O}(n^{11/6}W^{1/3})$ (see Corollary 16).

In addition to $\lambda_{st}(G)$, our algorithm can output the corresponding bipartition of the vertex set. It does not, however, compute a maximum s-t flow, and finding a sublinear quantum query algorithm for this problem remains a tantalizing open question.

Our quantum algorithm follows an algorithm of Rubinstein, Schramm and Weinberg (RSW) [RSW18], which computes a minimum s-t cut in the cut query model. In the cut query model one can query a subset of vertices $S$ and receive as an answer the total weight of edges with exactly one endpoint in $S$. While designed for cut query complexity, we show the algorithm also has a surprisingly efficient implementation in the adjacency list and adjacency matrix models that is well-suited for quantum speedups. The crux of the algorithm is a procedure to transform the input graph $G$ into a sparser graph $G'$ while preserving minimum s-t cuts. We use two quantum tools to speed up this procedure. The first is to use the quantum algorithm from [AdW20] to find an $\varepsilon$-cut sparsifier of $G$ faster than a classical sparsification algorithm. The second is to use Grover’s algorithm to learn all the edges in the sparser graph $G'$. Once we explicitly know the graph $G'$ we can use a classical randomized algorithm to compute a minimum s-t cut in $G'$. This blueprint is similar to the global min cut algorithm from [AL21], which is also based on a (different) cut query

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1 The bound quoted in [AdW20, Claim 9] for finding a $(1 + \varepsilon)$-approximate minimum s-t cut is $\tilde{O}(\sqrt{mn}/\varepsilon + n/\varepsilon^5)$. However, plugging into the proof the improved randomized approximate s-t minimum cut algorithm of [ST18] with running time $\tilde{O}(m + \sqrt{mn}/\varepsilon)$ improves this to $\tilde{O}(\sqrt{mn}/\varepsilon)$.

2 A simple graph is an undirected and unweighted graph with no self loops and at most one edge between any pair of vertices.
algorithm from \textsc{[RSW18]}. There as well a cut sparsifier of \( G \) is used to identify edges that can be contracted while preserving (non-trivial) global minimum cuts, and then Grover search is applied to learn the edges in the sparser contracted graph.

The contracted graph in the RSW procedure is obtained by contracting edges that cannot participate in any minimum \( s \)-\( t \) cut. To get an intuition for the idea, suppose that \( G \) is a simple graph and first imagine that we compute a maximum \( s \)-\( t \) flow \( F \) in \( G \). If we let \( G_F \) be the graph whose edge weights are those of \( G \) minus the flow \( F \), then \( s \) and \( t \) become disconnected in \( G_F \)—otherwise we could route more flow from \( s \) to \( t \). Contracting all connected components of \( G_F \) then results in a sparser graph while preserving any edge that is part of a minimum \( s \)-\( t \) cut. This routine is also how one can compute a minimum \( s \)-\( t \) cut from a maximum \( s \)-\( t \) flow: the connected component of \( s \) in \( G_F \) gives one side of a vertex bipartition corresponding to a minimum \( s \)-\( t \) cut.

Of course we did not save anything in this example as we had to compute a maximum \( s \)-\( t \) flow in \( G \). What RSW actually do is to first obtain an \( \varepsilon \)-cut sparsifier \( H \) of \( G \). This is where we use the \( \tilde{O}(\sqrt{mn}/\varepsilon) \) time quantum sparsification algorithm from \textsc{[AdW20]}. As \( H \) has only \( \tilde{O}(n/\varepsilon^2) \) edges it is much cheaper to carry out the above plan on \( H \) instead: we compute a maximum flow \( F \) in \( H \) and consider the graph \( H_F \) whose weights are those of \( H \) minus the flow \( F \). As cuts of \( H \) only approximate those in \( G \) we cannot fully contract the connected components in \( H_F \) and hope to preserve all minimum \( s \)-\( t \) cut of \( G \). However, a key insight of RSW is that we can still safely contract induced subgraphs of \( H_F \) that are highly connected. Doing so results in the desired contraction \( G' \) of \( G \) that is relatively sparse—one can argue it has only \( O(\varepsilon n^2) \) edges—yet preserves all minimum \( s \)-\( t \) cuts. We can learn the \( O(\varepsilon n^2) \) edges of \( G' \) among the \( m \) edges of \( G \) using Grover search in time \( \tilde{O}(n\sqrt{m\varepsilon}) \). We then again run a max flow algorithm on the now explicitly known \( G' \) to compute \( \lambda_{st}(G) \). Balancing the terms \( \sqrt{mn}/\varepsilon \) from the sparsifier computation and \( n\sqrt{m\varepsilon} \) for learning the edges of \( G' \) leads to the choice \( \varepsilon = n^{-1/3} \), and \( \tilde{O}(\sqrt{mn}^{5/6}) \) queries for the quantum steps of the algorithm. All other steps are on explicitly learned graphs and require no queries.

\section{Preliminaries}

\subsection{Graph notation and background}

Let \( V \) be a finite set and \( V^{(2)} \) the set of all subsets of \( V \) of cardinality 2. We represent a weighted undirected graph as a pair \( G = (V, w) \) where \( w : V^{(2)} \to \mathbb{R} \) is a non-negative function. We let \( V(G) \) be the vertex set of \( G \) and \( E(G) = \{ e \in V^{(2)} : w(e) > 0 \} \) be the set of edges of \( G \). We extend the weight function to sets \( S \subseteq V^{(2)} \) by \( w(S) = \sum_{e \in S} w(e) \). We say that \( G \) is \textit{simple} if \( w : V^{(2)} \to \{0,1\} \) and in this case also denote \( G \) as \( G = (V,E) \), where \( E \) is the set of edges. For a subset \( X \subseteq V \) we write \( G[X] \) for the induced subgraph on \( X \).

For a subset \( X \subseteq V \) we use the shorthand \( \ol{X} = V \setminus X \), and we say \( X \) is \textit{non-trivial} if \( \emptyset \neq X \subseteq V \). For disjoint sets \( X,Y \subseteq V \) we use \( E(X,Y) \) for the set of edges with one endpoint in \( X \) and one endpoint in \( Y \). For a non-trivial set \( X \), let \( \Delta_G(X) = \{ \{i,j\} \in E(G) : i \in X, j \in \ol{X} \} \) be the set of edges of \( G \) with one endpoint in \( X \) and one endpoint in \( \ol{X} \). A \textit{cut} of \( G \) is a set of the form \( \Delta_G(X) \) for some non-trivial set \( X \). We call \( X \) and \( \ol{X} \) the \textit{shores} of the cut \( \Delta_G(X) \). For
two distinguished vertices \( s, t \in V \) we call \( \Delta_G(X) \) an \( s-t \) cut if \( s \in X, t \notin X \). When we speak of the shore of an \( s-t \) cut we always refer to the shore containing \( s \). We let \( \lambda_{st}(G) \) be the minimum weight of an \( s-t \) cut of \( G \).

By the famous max-flow min-cut theorem [FF56], \( \lambda_{st}(G) \) is equal to the maximum amount of flow that can be routed from \( s \) to \( t \) in \( G \). While all the graphs \( G \) in this paper will be undirected, for flows it is most natural to think of an undirected graph \( G \) as a directed graph with each edge directed in both directions. Without loss of generality, an \( s-t \) flow \( F \) in \( G \) can be defined to use an edge in at most one direction. For \( e \in E(G) \) it will be convenient for us to let \( F(e) \) be the flow along edge \( e \) in the direction it is used by the flow \( F \).

We will need the following result about the total weight of a flow from Rubinstein, Schramm, and Weinberg [RSW18].

**Lemma 1** ([RSW18 Lemma 5.4]). Let \( G = (V, w) \) be a graph with integral weights from \([0, W]\) and let \( s, t \in V \). Let \( F \) be a non-circular \( s-t \) flow of value \( f \) in \( G \). Then the total weight of flow

\[
\sum_{e \in E(G)} F(e) \leq 10 \cdot n \sqrt{fW}.
\]

### 2.2 Quantum models and background

We will work with the two most standard quantum models for graph algorithms, the adjacency list and adjacency matrix models. We refer the reader to [AL21, Section 2.3] for definitions of these models.

Our quantum algorithm can be viewed as a classical algorithm with two quantum primitives. The first is a quantum algorithm to compute a sparsifier [AdW20] described in the next section, and the second is the following application of Grover’s algorithm.

**Theorem 2** (cf. [AL21, Theorem 13]). Given \( t, N \in \mathbb{N} \) with \( 1 \leq t \leq N \) and oracle access to \( x \in \{0, 1\}^N \), there is a quantum algorithm such that

- if \( |x| \leq t \) then the algorithm outputs \( x \) with certainty, and
- if \( |x| > t \) then the algorithm reports so with probability at least \( 9/10 \).

The algorithm makes \( O(\sqrt{tN}) \) queries to \( x \) and has time complexity \( O(\sqrt{tN} \log(N)) \).

### 2.3 Sparsifiers and strong components

A key component of our algorithm will be a cut sparsifier of a graph.

**Definition 3** (cut sparsifier). Let \( G = (V, w_G) \) be a weighted graph. An \( \varepsilon \)-cut sparsifier \( H = (V, w_H) \) of \( G \) is a reweighted subgraph of \( G \), i.e., \( w_H(e) > 0 \) only if \( w_G(e) > 0 \), satisfying

\[
(1 - \varepsilon)w_G(\Delta_G(X)) \leq w_H(\Delta_H(X)) \leq (1 + \varepsilon)w_G(\Delta_G(X))
\]

for every \( X \subseteq V \).
We will use a quantum algorithm to construct a cut sparsifier of a graph by Apers and de Wolf [AdW20].

**Theorem 4** ([AdW20 Theorem 1]). Let $G$ be a weighted and undirected graph with $n$ vertices and $m$ edges. There exists a quantum algorithm that outputs with high probability the explicit description of an $\varepsilon$-cut sparsifier $H$ of $G$ with $\tilde{O}(n/\varepsilon^2)$ edges after $\tilde{O}(\sqrt{mn}/\varepsilon)$ queries to the adjacency list model or $\tilde{O}(n^{3/2}/\varepsilon)$ queries in the adjacency matrix model. Moreover, if $G$ has integral weights then so does $H$, and if the largest weight of an edge of $G$ is $W$ then the largest weight of an edge of $H$ is $\tilde{O}(\varepsilon^2 nW)$.

**Proof.** The first part of the statement is directly from Theorem 1 of [AdW20]. The “moreover” part follows from an examination of Algorithm 1 of [AdW20]. There it can be seen that when an edge is added to the sparsifier it is always done so with a power of 4 times its original weight. The power of 4 is at most the number of times Algorithm 1 is run, which depends on the error parameter $\varepsilon$. To achieve an $\varepsilon$-cut sparsifier, Algorithm 1 is run $k = \log(n/\varepsilon^2)/2 + O(\log \log n)$ times, thus an edge of $G$ is placed in the sparsifier with weight at most its original weight times $4^k = \tilde{O}(\varepsilon^2 nW)$. \hfill $\square$

Apers and de Wolf actually show how to construct a spectral sparsifier. However, we will not need the stronger properties of a spectral sparsifier.

We will also need some definitions and results related to the minimum weight of cuts in induced subgraphs of $G$.

**Definition 5** ($k$-strong component). A graph $G$ is $k$-connected if the weight of each cut in $G$ is at least $k$. A $k$-strong component of $G$ is a maximal $k$-connected vertex induced subgraph of $G$. Individual vertices are defined to be $\infty$-strong components.

**Definition 6** (edge strength). The strength of an edge $e$, denoted $k_e$, is the maximum value of $k$ such that a $k$-strong component contains both endpoints of $e$. We say that $e$ is $k$-strong if its strength is $k$ or more, and $k$-weak otherwise.

Benczúr and Karger [BK15] show the following lemma about edge strengths.

**Lemma 7** (Lemma 4.11 [BK15]). Let $G = (V, w)$ be an $n$-vertex graph where the strength of edge $e$ is $k_e$. Then

$$\sum_e \frac{w(e)}{k_e} \leq n - 1.$$ 

**Definition 8** ($k$-strong partition). For a graph $G = (V, w)$, a $k$-strong partition of $G$ is a partition $\mathcal{P} = \{S_1, \ldots, S_t\}$ of $V$ such that each $G[S_i]$ is a $k$-strong component for $i = 1, \ldots, t$.

Benczúr and Karger give an algorithm to find a $k$-strong partition in Lemma 4.8 of [BK15]. If the weight of a minimum cut of $G$ is at least $k$ then $\{V\}$ provides a $k$-strong partition. Otherwise one finds all minimum cuts of $G$ and removes the union of all of their edges. All of these edges have edge strength less than $k$, and by Lemma 4.5 of [BK15] removing a $k$-weak edge does not affect the edge strength of a $k$-strong edge. One then applies the same procedure to the components created after the removal of these edges. While not particularly fast, this provides an algorithm to compute a $k$-strong partition.

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Remark 9. The first arXiv version of this paper claimed (in Corollary 10) a nearly-linear time algorithm to compute a $k$-strong partition. As pointed out by an ICALP 2022 reviewer, the proof of this had a fatal flaw, and we currently do not know a nearly-linear time algorithm to compute a $k$-strong partition. This is the bottleneck to making our quantum algorithm for $s$-$t$-mincut time efficient instead of just query efficient.

A key property about a $k$-strong partition that makes $s$-$t$-mincut algorithm of Rubinstein, Schramm, and Weinberg [RSW18, Algorithm 5.1] work efficiently is the following.

Theorem 10. For a graph $G = (V, w)$, let $\{S_1, \ldots, S_t\}$ be a $k$-strong partition. Then

$$\sum_{i,j \in [t]^2} w(S_i, S_j) \leq k(t-1).$$

Proof. Each $G[S_i]$ is a $k$-strong component, thus any edge with both endpoints in $S_i$ is $k$-strong. Also, any edge with endpoints in distinct sets $S_i, S_j$ is $k$-weak. By Lemma 4.6 of [BK15], contracting a $k$-strong edge does not change the strength of any $k$-weak edge. Consider the graph $G'$ formed by contracting each $S_i$. Every edge of $G'$ is $k$-weak. By [Lemma 7] the total edge weight of $G'$ is at most $k(t-1)$, which gives the lemma.

3 Main algorithm

Our algorithm is based on the following algorithm by Rubinstein, Schramm, and Weinberg [RSW18, Algorithm 5.1], who used it to give a randomized cut query algorithm for the minimum $s$-$t$ cut problem making $\tilde{O}(n^{5/3})$ cut queries.

\begin{algorithm}
\caption{Algorithm for $s$-$t$-mincut on a weighted graph $G$}
\begin{algorithmic}[1]
\STATE \textbf{Input:} Oracle access to a weighted graph $G = (V, w_G)$, vertices $s, t \in V$, and a parameter $0 < \varepsilon < 1/3$.
\STATE \textbf{Output:} The shore of a minimum $s$-$t$ cut in $G$ and the value $\lambda_{st}(G)$.
\STATE 1: Determine the largest edge weight $W$ in $G$.
\STATE 2: Compute an $\varepsilon$-cut sparsifier $H$ of $G$.
\STATE 3: Compute a maximum $s$-$t$ flow $F$ in $H$ and subtract $F$ from $H$. Denote the result as $H'$.
\STATE 4: Compute a $3\varepsilon nW$-strong partition $\mathcal{P} = \{A_1, \ldots, A_t\}$ of $H'$.
\STATE 5: Let $G'$ be the graph formed from $G$ by contracting the vertices in each $A_i$, and let $a, b \in \mathcal{P}$ be the sets containing $s, t$ respectively. Compute a minimum $a$-$b$ cut $\Delta_{G'}(X')$ in $G'$ and return $\lambda_{ab}(G')$ and the shore $X = \bigcup_{A \in X'} A$.
\end{algorithmic}
\end{algorithm}

For completeness we show correctness of the template, largely following the discussion of [RSW18]. In the next section we analyze the running time for a quantum algorithm that implements this algorithm with adjacency list access to $G$.

Theorem 11. If every step of Algorithm 1 is performed correctly then the algorithm returns a minimum $s$-$t$ cut of $G$. 

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We first prove two claims from which the proof of Theorem 11 will easily follow.

**Claim 12.** Let \( G = (V, w_G) \) be an \( n \)-vertex weighted graph with largest edge weight \( W \) and \( s, t \in V \). For \( 0 < \varepsilon < 1/3 \), let \( H = (V, w_H) \) be an \( \varepsilon \)-cut sparsifier of \( G \) and let \( F \) be a maximum s-t flow in \( H \). Form the graph \( H' = (V, w_{H'}) \) where \( w_{H'}(e) = w_H(e) - F(e) \) for all \( e \in V^{(2)} \). Let \( \mathcal{P} \) be a \( 3\varepsilon nW \)-strong partition of \( H' \). Then for any minimum s-t cut \( \Delta_G(X) \) of \( G \) and all \( A \in \mathcal{P} \) it holds that either \( A \subseteq X \) or \( A \subseteq \overline{X} \).

**Proof.** Let \( f_G = \lambda_{st}(G) \) and \( f_H = \lambda_{st}(H) \). Let \( \Delta_G(X) \) be a minimum s-t cut of \( G \), i.e., such that \( s \in X, t \not\in X \) and \( w_G(\Delta_G(X)) = f_G \). We want to upper bound \( \tau = w_{H'}(\Delta_H(X)) \). By definition of \( H' \) and \( F \), \( w_H(\Delta_H(X)) = \tau + w_F(\Delta_F(X)) \). We must have \( w_F(\Delta_F(X)) \geq f_H \) since there is a flow of value \( f_h \) from \( s \) to \( t \) in \( H \). Thus \( w_H(\Delta_H(X)) \geq f_H + \tau \). As \( H \) is an \( \varepsilon \)-cut sparsifier of \( G \), \( w_H(\Delta_H(X)) \leq (1 + \varepsilon) f_G \) and so \( \tau \leq (1 + \varepsilon) f_G - f_H \).

If \( f_H \geq f_G \), then we immediately have \( \tau \leq \varepsilon f_H \leq \varepsilon(1 + \varepsilon)nW \) as \( f_H \leq (1 + \varepsilon)f_G < (1 + \varepsilon)nW \). In the case \( f_H < f_G \), we use the lower bound \( f_G \geq f_G/(1 - \varepsilon) \) to obtain \( \tau \leq \frac{2\varepsilon}{1-\varepsilon} f_G \leq \frac{2\varepsilon}{1-\varepsilon} nW \) as \( f_H < f_G < nW \). In either case we have \( \tau < 3\varepsilon nW \) as \( \varepsilon < 1/3 \).

Now let \( A \in \mathcal{P} \) and let \( B = A \cap X \). If \( B \) is nontrivial, i.e., \( \emptyset \neq B \subseteq A \), then there is a cut of \( H'[A] \) of weight less than \( 3\varepsilon nW \), a contradiction to the assumption that \( \mathcal{P} \) is a \( 3\varepsilon nW \)-strong partition of \( H' \). Thus \( B \) must be trivial and either \( A \subseteq X \) or \( A \subseteq \overline{X} \).

**Claim 13.** Let \( G, H, H' \) and \( \varepsilon \) be as in Claim 12. Let \( \mathcal{P} = \{A_1, \ldots, A_t\} \) be a \( 3\varepsilon nW \)-strong partition of \( H' \) and \( G' \) be the graph formed from \( G \) by contracting vertices in the same set of \( \mathcal{P} \). Let \( a, b \in \mathcal{P} \) be the sets containing \( s \) and \( t \) respectively. Then \( \lambda_{st}(G) = \lambda_{ab}(G') \). Moreover, if \( X' \) is the shore of a minimum a-b cut of \( G' \) then \( X = \bigcup_{A \in \mathcal{P}} A \) is the shore of a minimum s-t cut of \( G \).

**Proof.** Let \( G = (V, w_G) \) and note that by the definition of \( G' \) we have \( G' = (\mathcal{P}, w_{G'}) \) where \( w_{G'}(A_i, A_j) = w_G(E(A_i, A_j)) \) for \( i \neq j \).

Let us show that \( \lambda_{ab}(G') \leq \lambda_{st}(G) \). Let \( \Delta_G(X) \) be a minimum s-t cut in \( G \). For every \( A_i \in \mathcal{P} \) we either have \( A_i \subseteq X \) or \( A_i \subseteq \overline{X} \) by Claim 12. Note that in particular this means \( a \subseteq X \) and \( b \cap X = \emptyset \). From \( X \) we define a set \( X' \) where for every \( A_i \in \mathcal{P} \) we put \( A_i \in X' \) iff \( A_i \subseteq X \). Note that \( \Delta_G'(X') \) is an a-b cut of \( G' \) and \( w_{G'}(\Delta_G'(X')) = w_G(\Delta_G(X)) \), thus \( \lambda_{ab}(G') \leq \lambda_{st}(G) \).

We next show the general fact that contraction cannot decrease the minimum s-t cut value. For any set \( X' \subseteq \mathcal{P} \) we can define a set \( X \subseteq V \) by \( X = \bigcup_{A \in \mathcal{P}} A \). Further \( w_{G'}(\Delta_G'(X')) = w_G(\Delta_G(X)) \) by the definition of \( w_{G'} \). Thus \( \lambda_{ab}(G') \geq \lambda_{st}(G) \).

This establishes \( \lambda_{st}(G) = \lambda_{ab}(G') \). To see the “moreover” part of the claim, let \( X' \) be a minimum a-b cut of \( G' \). We have just shown that \( w_{G'}(\Delta_G'(X')) = w_G(\Delta_G(X)) \) for \( X = \bigcup_{A \in \mathcal{P}} A \), thus \( X \) will be a minimum s-t cut of \( G \).

**Proof of Theorem 11** If the steps of the algorithm are implemented correctly then \( H, H' \) satisfy the hypotheses of Claim 12. We can therefore invoke Claim 12 and Claim 13 to conclude that \( \lambda_{st}(G) = \lambda_{ab}(G') \) and that \( X \) will be the shore of a minimum s-t cut in \( G \).
3.1 Implementation by a quantum algorithm

We now discuss the implementation of Theorem 15 by a quantum algorithm with adjacency list access to $G$. To upper bound the quantum query complexity it is important to have an upper bound on the number of edges in the contracted graph $G'$ formed in step 5 of Algorithm 1. We do this by means of the following claim.

Claim 14. Let $G, H, H'$ and $\varepsilon$ be as in Claim 12 with the following additional conditions:

1. The largest edge weight of $G$ is $W \geq 1$.
2. $H$ has integral weights and the largest edge weight is $W_H \in O(\varepsilon^2 n W)$.

Let $\mathcal{P} = \{A_1, \ldots, A_t\}$ be a $3\varepsilon n W$-strong partition of $H'$. Then $\sum_{i=1}^{t} w_G(\Delta_G(A_i)) = O(\varepsilon n^2 W)$.

Proof. As $H$ is an $\varepsilon$-cut sparsifier of $G$ we have

$$w_G(\Delta_G(A_i)) \leq \frac{w_H(\Delta_H(A_i))}{1 - \varepsilon} = \frac{w_{H'}(\Delta_{H'}(A_i)) + w_F(\Delta_F(A_i))}{1 - \varepsilon}.$$

By Theorem 10, we have $\sum_{i=1}^{t} w_{H'}(\Delta_{H'}(A_i)) = O(\varepsilon n^2 W)$. By Lemma 1 we have that

$$\sum_{i=1}^{t} w_F(\Delta_F(A_i)) \leq 20n \sqrt{\lambda_{st}(H) W_H},$$

using that $\lambda_{st}(H) \leq (1 + \varepsilon)n W$ and $W_H \in O(\varepsilon^2 n W)$. Thus overall $\sum_{i=1}^{t} w_G(\Delta_G(A_i)) = O(\varepsilon n^2 W)$. \qed

Theorem 15. Let $G$ be a graph with $n$ vertices and $m$ edges where all edge weights are integral and the largest weight of an edge is $W$. Given adjacency list access to $G$, there is a quantum algorithm that with high probability computes $\lambda_{st}(G)$ and the shore of a minimum $s$-$t$ cut of $G$ after $\tilde{O}(\sqrt{mn^{5/6} W^{1/3}})$ queries.

Proof. We follow the algorithm given in Algorithm 1 with the choice $\varepsilon = (n W)^{-1/3}$.

Line 1 We can find the maximum weight $W$ of an edge with high probability after $\tilde{O}(\sqrt{m})$ queries to the graph using the quantum maximum finding routine of Dürr and Høyer [DH96].

Line 2 We run quantum algorithm of Apers and de Wolf [AdW20] given in Theorem 4 to find an $\varepsilon$-cut sparsifier $H$ of $G$ with $\tilde{O}(n/\varepsilon^2)$ edges. As all edge weights of $G$ are integral and the largest weight of an edge is $W$, by the “moreover” part of this theorem the edge weights of $H$ are integral and the largest weight of an edge of $H$ is $O(\varepsilon^2 n W)$. This step succeeds with high probability and takes $\tilde{O}(\sqrt{mn}/\varepsilon)$ queries.
For this step we can run a classical maximum $s$-$t$ flow algorithm to compute a max flow $F$ in $H$. As the graph is explicitly known, this step requires no queries.

After step 3 the graph $H'$ is explicitly known. Thus we can compute a $3\varepsilon n$-strong partition of $H'$ classically with no queries.

Let $\mathcal{P} = \{A_1, \ldots, A_t\}$ be the $3\varepsilon n$-strong partition of $H'$ computed in the previous step. The graph $G'$ is formed by contracting vertices in the same set of this partition. By Claim 14 the graph $G'$ has $O(\varepsilon n^2 W)$ edges, assuming all previous steps have succeeded. Consider the concatenation of all the lists in the adjacency list representation of $G$. This gives a vector of dimension $2m$ and defines an ordering of edges of $G$ where every edge appears twice. Define a string $x \in \{0, 1\}^{2m}$ using the same ordering where $x(e) = 1$ if the endpoints of $e$ are in distinct sets of the partition $\mathcal{P}$ and $x(e) = 0$ otherwise. A query to a bit of $x$ can be answered with a single query to the adjacency list of $G$. By Theorem 2 via Grover search in time $\tilde{O}(n\sqrt{mn/\varepsilon})$ with high probability we can determine if $x$ has at most $O(\varepsilon n^2 W)$ ones, and if so learn the positions of all these ones. If the number of ones in $x$ is larger than $O(\varepsilon n^2 W)$ then we abort. With high probability we do not abort and once we learn the positions of all the ones in $x$ we can then classically learn the weight of the corresponding edges in $G$ with $O(\varepsilon n^2 W)$ more queries. Then we have explicitly learned the graph $G'$.

Once we have an explicit description of $G'$ we can run a classical max flow algorithm to compute a maximum $a$-$b$ flow $F'$ in $G'$, where $a, b \in \mathcal{P}$ are the sets of the partition containing $s, t$ respectively. This takes no queries. The flow of $F'$ gives $\lambda_{ab}(G')$. To also compute the shore of a minimum $a$-$b$ cut of $G'$ we consider the residual graph of the flow $F'$ and compute the connected component containing $a$. This can also be done classically with no queries.

**Correctness and total running time** Each step individually succeeds at least with high probability and so with high probability all steps will be correct. We can thus invoke Theorem 11 to conclude that the algorithm is correct with high probability.

Queries are only made on Lines 12 and 5 and the number of queries made on these lines is $\tilde{O}(\sqrt{m}), \tilde{O}(\sqrt{mn/\varepsilon})$ and $\tilde{O}(n\sqrt{m\varepsilon W})$. The term $\tilde{O}(\sqrt{m})$ is low order, and the choice $\varepsilon = (nW)^{-1/3}$ makes the remaining two terms $\tilde{O}(n^{5/6}W^{1/3})$. Thus the total number of queries is $\tilde{O}(\sqrt{mn^{5/6}W^{1/3}})$.

We have the following result for the adjacency matrix model.

**Corollary 16.** Let $G$ be a graph with $n$ vertices and $m$ edges where all edge weights are integral and the largest weight of an edge is $W$. Given vertices $s, t$ and oracle access to the adjacency matrix of $G$, there is a quantum algorithm that computes $\lambda_{st}(G)$ and the shore of a minimum $s$-$t$ cut of $G$ with high probability after $\tilde{O}(n^{11/6}W^{1/3})$ queries.

**Proof.** The result is trivial if $W \geq \sqrt{n}$, so we just need to consider the case $W < \sqrt{n}$. We follow the same algorithm Algorithm 1 again choosing $\varepsilon = (nW)^{-1/3}$. 

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Line 1 can be done with $\tilde{O}(n)$ queries using the quantum maximum finding routine of Dürr and Høyer [DH96]. In Line 2, we run the adjacency matrix version of the sparsifier algorithm from [AdW20] (quoted in Theorem 4), which takes $\tilde{O}(n^{3/2}/\varepsilon) = \tilde{O}(n^{11/6}W^{1/3})$ queries. In Line 5, we find the $\tilde{O}(\varepsilon n^2W)$ edges of $G'$ by applying Theorem 2 over the $O(n^2)$ entries of the adjacency matrix, rather than the $O(m)$ entries of the adjacency list. This takes $\tilde{O}(n^2\sqrt{\varepsilon W}) = \tilde{O}(n^{11/6}W^{1/3})$ queries.

The remaining steps are performed classically on graphs explicitly constructed by the algorithm and require no queries.

4 Randomized lower bound

In this section we show that a randomized algorithm that computes $\lambda_{st}(G)$ with success probability at least $9/10$ on simple graphs $G$ with $m$ edges must make $\Omega(m)$ queries to the adjacency list of $G$ in the worst case. This holds true even for just determining if $s$ and $t$ are connected in $G$, which we call the USTCON problem.

**Definition 17 (USTCON).** In the USTCON problem one is given adjacency list access to an undirected graph $G = (V, E)$ and two distinguished vertices $s, t \in V$. The problem is to determine if $s$ and $t$ are connected in $G$.

To show a lower bound on USTCON we will use the classical adversary method for randomized query complexity [Aar06, LM08, AKP+21], a randomized analog of the quantum adversary method [Amb02]. We will just need a simple unweighted version of the method, adapted from [Aar06] Theorem 4.3.

**Lemma 18 (cf. [Aar06, Theorem 4.3]).** For a finite set $\Sigma$ and $S \subseteq \Sigma^n$, let $f : S \to \{0, 1\}$. Let $X \subseteq f^{-1}(0)$ and $Y \subseteq f^{-1}(1)$. Let $R \subseteq X \times Y$ be such that for every $x \in X$ there are at least $t$ different $y \in Y$ such that $(x, y) \in R$ and for every $x \in X$ and $k \in [n]$ there are at most $\ell$ different $y \in Y$ such that $(x, y) \in R$ and $x_k \neq y_k$. Then any randomized algorithm that computes $f$ with success probability $9/10$ must make $\Omega(t/\ell)$ queries.

**Theorem 19.** A randomized algorithm that solves USTCON with success probability at least $9/10$ on graphs with $m$ edges makes $\Omega(m)$ queries.

**Proof.** First we show the lower bound in the dense case where $m = \Omega(n^2)$ using Lemma 18. We construct sets of negative instances $X$ and positive instances $Y$. Suppose that $n$ is even. Excluding $s$ and $t$ we partition the remaining $n - 2$ vertices into two sets $A$ and $B$ each of size $(n - 2)/2$. In all instances, $s$ will be connected to all vertices in $A$ and $t$ will be connected to all vertices in $B$. $X$ will consist of a single negative instance $G$ where we additionally put a clique on the vertices in $A$ and a clique on the vertices in $B$ and no further edges. Thus $s$ and $t$ will not be connected in $G$. Note that apart from $s$ and $t$, every vertex has degree $(n - 2)/2$ in $G$.

For $Y$ we create a family of $2\binom{n-2}{2}$ positive instances. A positive instance will be labeled by $a = \{a_1, a_2\} \in A^{(2)}$, $b = \{b_1, b_2\} \in B^{(2)}$ and a bit $c \in \{0, 1\}$. Associated to $a, b, c$ we construct a graph $G_{a,b,c}$ as follows. Take the graph $G$ and remove the edges $a$ and $b$. If $c = 0$ then add
edges \( \{\min\{a_1, a_2\}, \min\{b_1, b_2\}\} \) and \( \{\max\{a_1, a_2\}, \max\{b_1, b_2\}\} \) to form \( G_{a,b,c} \). Otherwise if \( c = 1 \) then add edges \( \{\min\{a_1, a_2\}, \max\{b_1, b_2\}\} \) and \( \{\max\{a_1, a_2\}, \min\{b_1, b_2\}\} \) to form \( G_{a,b,c} \).

In both cases all vertices in \( A \cup B \) have degree \( (n - 2)/2 \) in \( G_{a,b,c} \) and there will be two edges connecting \( A \) and \( B \) so \( \lambda_{st}(G_{a,b,c}) = 2 \). This completes the description of the instances.

The degree sequences of all instances are the same, thus we may assume this is known to the algorithm and the algorithm does not need to make any degree queries. We thus focus on queries to the name of the \( i \)th neighbor of a vertex \( v \). For this it is important to specify the ordering of vertices given in the adjacency lists. For all vertices we will use the same ordering in their list. Let \( k = (n - 2)/2 \) and label the vertices of \( A \) as \( a_1, a_2, \ldots, a_k \) and the vertices of \( B \) as \( b_1, b_2, \ldots, b_k \). We use the ordering \( s < t < a_1 < b_1 < \cdots < a_k < b_k \).

We are now ready to show the lower bound using [Lemma 18]. We put \( G \) in relation with all \( 2^{n-2}\choose 2 \) of the \( G_{a,b,c} \). Now consider how many of the \( G_{a,b,c} \) differ from \( G \) in a specific location of the adjacency list, specifically consider the \( i \)th neighbor of a vertex \( a_j \in A \). In \( G \) the \( i \)th neighbor of \( a_j \) is

1. \( s \) if \( i = 1 \).
2. \( a_{i-1} \) if \( i \leq j \).
3. \( a_i \) if \( i > j \).

The \( i \)th neighbor of \( a_j \) in \( G_{a,b,c} \) will be the same unless

1. \( i \leq j \) and \( a = \{a_{i-1}, a_j\} \).
2. \( i > j \) and \( a = \{a_j, a_i\} \).

In either case, the number of \( a, b, c \) for which \( G_{a,b,c} \) differs from \( G \) on the name of the \( i \)th neighbor of \( a_j \) is \( 2^{n-2}\choose 2 \), as one only has free choice of the edge \( b \) and the bit \( c \). A similar argument holds when considering the \( i \)th neighbor of a vertex \( b_j \in B \). Thus for any position of the adjacency list the number of \( G_{a,b,c} \) that differ from \( G \) is at most \( 2^{n-2}\choose 2 \) and by [Lemma 18] we obtain a lower bound of \( (n-2)! = \Omega(m) \), proving the theorem in the dense case.

Finally, let us treat the general case of graphs with at most \( m \) edges. We choose the largest integer \( p \) such that \( 2(p + \binom{p}{2}) \leq m \) and then take disjoint sets of vertices \( A \) and \( B \) both of size \( p \). We then repeat the construction from the dense case on \( s, t, A, B \). The lower bound will be \( \binom{p}{2} = \Omega(m) \) as desired.

\[ \square \]

**Acknowledgements**

We would like to thank an anonymous reviewer from ICALP 2022 for pointing out errors in a previous version of this paper. Troy Lee is supported in part by the Australian Research Council Grant No: DP200100950. Simon Apers is partially supported by French projects EPIQ (ANR-22-PETQ-0007), QUDATA (ANR-18-CE47-0010) and QUOPS (ANR22-CE47-0003-01), and EU project QOPT (QuantERA ERA-NET Cofund 2022-25)
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