We here consider a generalization of the Klein-Gordon scalar wave equation which involves a single arbitrary function. The quantization may be viewed as allowing $\hbar$ to be a function of the momentum or wave vector rather than a constant. The generalized theory is most easily viewed in the wave vector space analog of the Lagrangian. We need no reference to spacetime. In the generalized theory the de Broglie relation between wave vector and momentum is generalized, as are the canonical commutation relations and the uncertainty principle. The generalized uncertainty principle obtained is the same as has been derived from string theory, or by a general consideration of gravitational effects during the quantum measurement process. The propagator of the scalar field is also generalized, and an illustrative example is given in which it factors into the usual propagator times a "propagator form factor."

I. INTRODUCTION

Early in the history of quantum mechanics de Broglie [1] suggested that a wave might be associated with a particle such as an electron, and that the wave vector \( \vec{k} \) be related to the momentum \( \vec{p} \) by

\[
\vec{k} = \frac{\vec{p}}{\hbar},
\]

or, in terms of the wavelength,

\[
\lambda = \frac{2\pi\hbar}{|\vec{p}|}.
\]

This ad hoc suggestion is widely considered as an important step leading to the Schroedinger equation and nonrelativistic quantum mechanics.

In this work we study a generalization of this relation. The generalization may be viewed as allowing $\hbar$ to be a function of the square of the four-momentum or wave vector; that is we take $\hbar \to \hbar f(p^2)$ or equivalently $\hbar \to \hbar f(k^2)$. In this paper we study this in the context of a generalized Klein-Gordon or scalar wave equation, which we study in both spacetime, where it may have an arbitrarily large number of derivatives, and also in wave vector space, where it contains the arbitrary function $f(k^2)$. We emphasize that the analysis is particularly simple and logically complete in wave vector space, and that spacetime need not be considered at all.

The generalized de Broglie relation leads to a modified canonical commutation relation between position and momentum, and that in turn leads to a generalized uncertainty principle, or GUP. The GUP is the same as that obtained in string theory by Veneziano and other authors [2]; it has also been obtained from a very general consideration of gravitational effects in the quantum measurement process [3]. Since we obtain the GUP here without reference to gravity we may speculate that there is some connection between the generalized de Broglie relation and gravity, or that gravity is somehow generated by intrinsic quantum effects [4].
Lastly we very briefly consider interaction of the scalar field with a given external scalar field in the context of wave vector space. Of course the propagator of the scalar field is generalized, and with one illustrative choice of the function $f(k^2)$, we show that the generalized propagator is equal to the usual propagator times a “propagator form factor.” The form factor has the form of the propagator of a virtual particle of Planck mass.

Some time ago Snyder [5] considered the idea of a quantized or “discretized” spacetime. His ideas are equivalent to our modified de Broglie relation, but his motivation was different from ours. Recently, from a somewhat different perspective, some of our results have been derived by Kempf and collaborators [6]. In particular, they obtain our results regarding commutators and uncertainty relations and construct a theory that is regular in the ultraviolet. Our philosophy and approach in the present work is different than theirs, and we treat interactions differently.

II. THE SCALAR WAVE EQUATION IN SPACETIME

We first recall briefly some simple properties of the scalar wave equation in spacetime. Throughout we use units in which $c = 1$ but take pains to retain $\hbar$ explicitly. The Lagrangian density for a noninteracting or free scalar field is

$$L = \hbar^2 \phi(x)_{,\mu} \phi(x)_{,\nu} \eta^{\mu\nu} - m^2 \phi(x)^* \phi(x), \quad \phi(x)_{,\alpha} \equiv \frac{\partial \phi(x)}{\partial x^\alpha},$$

(2.1)

where $\eta^{\mu\nu}$ is the Lorentz metric with signature (1,-1,-1,-1), and the scalar field and its conjugate are treated as independent variables. From this the Euler-Lagrange equations are

$$\hbar^2 \phi(x)_{,\mu} \eta^{\mu\nu} + m^2 \phi(x) = 0,$$

(2.2)

and similarly for the conjugate. As a solution we seek a plane wave of the form

$$\phi(x) = \phi_0 e^{-ik_\beta x^\beta}.$$  

(2.3)

Here the wave vector $k_\beta$ is a set of four real parameters. Substitution of this into the wave equation (2.2) shows it is a solution if

$$\hbar^2 (k_\mu k_\nu \eta^{\mu\nu}) \equiv \hbar^2 k^2 = m^2.$$  

(2.4)

Special relativity tells us that the four-momentum squared of a particle is equal to its mass squared, $p^2 = m^2$. Comparing this with (2.4) we naturally assume the wave vector is related to the momentum by $k_\mu = p_\mu / \hbar$, which we refer to as the de Broglie relation. This is not the most general solution, as we will discuss in section IV. In particular the de Broglie relation implies the usual relation between the wavelength $\lambda = 2\pi / |\vec{k}|$ and the magnitude of the three-momentum $p = |\vec{p}|$

$$\lambda = \frac{2\pi \hbar}{p},$$

(2.5)

Since the wave vector may be identified with the derivative operator $i\partial / \partial x^\alpha$ this gives the standard operator expression for momentum in a position representation, and the standard commutation relation

$$p_\gamma = i\hbar \frac{\partial}{\partial x^\gamma}, \quad [x^\alpha, p_\beta] = -i\hbar.$$  

(2.6)

In a section IV we will modify and generalize the equations in this section.

III. THE SCALAR WAVE EQUATION IN WAVE VECTOR SPACE

It is interesting to express the action principle for fields in the space of wave vectors, without reference to spacetime. This is an elementary exercise in Fourier transforms, but the physical viewpoint is different, and may allow more flexibility and insight when seeking modifications of present theories. We use the scalar field to illustrate this. The action is defined as the integral of the Lagrangian in spacetime,

$$S = \int d^4 x L[\phi(x), \phi(x)_{,\mu}],$$

(3.1)
where the integral may be taken over all space, and time from an initial time, \( t_i \), to a final time, \( t_f \). (Dependence of the Lagrangian on the conjugate field is implicit.) In particular we may take \( t_i \to -\infty \) and \( t_f \to \infty \) so the integral is over all spacetime. If we Fourier transform the field in spacetime \((x)\) to wave vector space \((k)\) by

\[
\phi(x) = \int \frac{d^4k}{(2\pi)^4} \tilde{\phi}(k)e^{-ikx},
\]

then the action may be expressed in terms of an analog of the Lagrangian in wave vector space,

\[
S = \int \frac{d^4k}{(2\pi)^4} K[\phi(k), k_\mu \tilde{\phi}(k)], \quad K[\phi(k), k_\mu \tilde{\phi}(k)] = \hbar^2[k_\mu \tilde{\phi}(k)^*][k_\mu \phi(k)]\eta^{\mu\nu} - m^2 \phi^*(k)\phi(k).
\]

To obtain the appropriate dynamical equations in wave vector space we extremize the action with respect to variations of the fields. This can be done in two equivalent ways. In the first way we take the fields, \( \tilde{\phi}(k) \) and \( \tilde{\phi}(k)^* \), and the product of the wave vector and the fields, \( k_\mu \tilde{\phi}(k) \) and \( k_\mu \tilde{\phi}(k)^* \), as independent variables; this may appear somewhat artificial, but it is the analog of treating the field \( \phi(x) \) and its derivatives \( \phi(x)_\mu \) as independent variables in spacetime, and is useful for formal manipulations. Extremizing in this way we obtain

\[
\delta S = \int \frac{d^4k}{(2\pi)^4} \left[ \left( \frac{\partial K}{\partial \phi(k)} \delta \tilde{\phi}(k) + \frac{\partial K}{\partial (k_\mu \phi(k))} \delta (k_\mu \tilde{\phi}(k)) \right) + \left( \frac{\partial K}{\partial (k_\mu \phi^*(k))} \delta (k_\mu \phi(k)^*) + \frac{\partial K}{\partial (k_\mu \phi^*(k))} \delta (k_\mu \phi(k)^*) \right) \right] = 0.
\]

Since the variations of the fields are arbitrary, and since \( \delta (k_\mu \tilde{\phi}(k)) = k_\mu \delta \tilde{\phi}(k) \), this leads to an analog of the Euler-Lagrange equations

\[
\frac{\partial K}{\partial \phi(k)} + \frac{\partial K}{\partial (k_\mu \phi(k))} k_\mu = 0,
\]

and similarly for the conjugate.

Alternatively, in the second way, we treat the fields as the independent variables, however they occur in \( K \), and obtain

\[
\delta S = \int \frac{d^4k}{(2\pi)^4} \left[ \left( \frac{\partial K}{\partial \phi(k)} \right)_T \delta \phi(k) + \left( \frac{\partial K}{\partial (k_\mu \phi(k))} \right)_T \delta (k_\mu \phi(k)^*) \right] = 0,
\]

where the subscript \( T \) denotes a total derivative. Thus the analog of the Euler-Lagrange equations is

\[
\left( \frac{\partial K}{\partial \phi(k)} \right)_T = 0,
\]

and similarly for the conjugate. This is obviously the same as (3.3). For the scalar Lagrangian (3.3) we vary the action with respect to the conjugate field and obtain explicitly

\[
\left( \frac{\partial K}{\partial \phi(k)^*} \right)_T = (\hbar^2 k^2 - m^2) \tilde{\phi}(k) = 0,
\]

The solution to this may be expressed as

\[
\tilde{\phi}(k) = 2\pi \delta \left( k_0 - \sqrt{k^2 + m^2/\hbar^2} \right) \tilde{\phi}_+(\tilde{k}) + 2\pi \delta \left( k_0 + \sqrt{k^2 + m^2/\hbar^2} \right) \tilde{\phi}_-(\tilde{k}),
\]

where the \( \tilde{\phi}_+ \) and \( \tilde{\phi}_- \) are arbitrary functions of the three-vector momentum. This corresponds to a superposition of plane wave solutions of the form (2.3).

We may consider the variational principle in wave vector space, in terms of the function \( K \), as the fundamental basis of the theory, and thereby make no explicit reference to spacetime. This may have some conceptual or philosophical merit in that experiments in particle physics are generally scattering experiments and are more directly related to the momenta of particles rather than their position. It may also be possible that such a viewpoint might allow easier generalizations of the theory, as we will discuss in the next section.
IV. A GENERALIZATION OF THE THEORY

It has long been a supposed fact of life that the differential equations of physics are first or second order. This is well born out by experience in that classical mechanics, classical electromagnetism, general relativity, nonrelativistic quantum mechanics, relativistic quantum mechanics, and all the equations of the standard model of particles are at most of second order. But it may well be that we also have developed an unjustified bias in favor of second order equations due to mathematical convenience. As such it is particularly interesting to consider higher order equations with all their inherent dangers and difficulties with boundary conditions. Motivated thereby, we consider the following generalization of the scalar wave equation,\(^4\)

\[
h^2(\partial^2 - a_l^2\partial^4 + \ldots)\phi(x) + m^2\phi(x) = 0, \quad \partial^2 \equiv \eta^{\mu\nu} \frac{\partial^2}{\partial x^\mu \partial x^\nu}, \quad \partial^4 \equiv \partial^2\partial^2, \quad \text{etc.} \tag{4.1}
\]

Here \(l_p\) is a characteristic distance, presumably of the order of the Planck distance, and \(a\) is a dimensionless parameter of order 1, with either sign. The series represents any reasonable function with a power series expansion. This equation does not fit easily into the action formalism in spacetime since it may be of infinite order in derivatives. If the series is infinite this equation is, in a sense, nonlocal.

We can easily find plane wave solutions of the form \((2.3)\). These are solutions if

\[
-\hbar^2k^2(1 + a_l^2k^2 + \ldots) + m^2 \equiv -\hbar^2k^2f^2(k^2) + m^2 = 0. \tag{4.2}
\]

As before we note that special relativity tells us that \(p^2 = m^2\), so that

\[
\hbar^2k^2f^2(k^2) = p^2. \tag{4.3}
\]

Since \(f^2(k^2)\) is a scalar and the wave vector and the momentum are four-vectors the quotient theorem tell us that they are related by a second rank tensor,

\[
p_\mu = t_\mu^\alpha k_\alpha. \tag{4.4}
\]

Substitution of this into \((4.1)\) shows that the tensor \(t_\mu^\alpha\) must be a scalar multiple of an element of the Lorentz group, or

\[
t_\mu^\alpha = \hbar f(k^2)\Lambda_\mu^\alpha, \quad \Lambda_\mu^\alpha\eta^{\nu\beta}\Lambda_\nu^\beta = \eta^{\alpha\beta}. \tag{4.5}
\]

The simplest choice for \(\Lambda_\mu^\alpha\), and the only one that does not depend on some intrinsic parameter, is the identity \(\delta_\mu^\alpha\). We accordingly assume, in direct analogy with the assumption in section II, a generalized de Broglie relation

\[
p_\mu = \hbar f(k^2)k_\mu. \tag{4.6}
\]

This implies in particular that the wavelength of the particle is given by

\[
\lambda = \frac{2\pi\hbar}{p}f(k^2). \tag{4.7}
\]

Such a modification may not be easy to test experimentally since the wavelengths of particles in scattering experiments are not generally observed, and at sufficiently high energies may not even be observable in principle.

It is amusing to note that the above generalization of the theory may be viewed as allowing Planck’s constant to become a function of the wave vector squared, or equivalently the mass of the associated free particle. It is for this reason that we retain it explicitly as a bookkeeping tool in all equations.

Our consideration of the generalized scalar wave equation is not difficult, but is slightly cumbersome in the spacetime Lagrangian formalism. It amounts to making the substitution \(\hbar \to \hbar f(-\partial^2)\). In wave vector space however it is extremely simple and natural. We may view the generalization as multiplying the wave vector by \(f(k^2)\) or equivalently as allowing Planck’s constant to be a scalar function via \(\hbar \to \hbar f(k^2);\) in either view we substitute \(\hbar k_\mu \to \hbar k_\mu f(k^2)\), so the function \(K\) in \((3.3)\) is modified to

\[
K = \tilde{\phi}(k)^* [\hbar^2 f^2(k^2)k^2 - m^2] \tilde{\phi}(k). \tag{4.8}
\]

The wave equation is then

\[
[\hbar^2 f^2(k^2)k^2 - m^2] \tilde{\phi}(k) = 0, \tag{4.9}
\]
and the solution is
\[ \tilde{\phi}(k) = 2\pi\delta \left( k_0 - \sqrt{k^2 + m^2/\hbar^2 f^2} \right) \tilde{\phi}_+(\tilde{k}) + 2\pi\delta \left( k_0 + \sqrt{k^2 + m^2/\hbar^2 f^2} \right) \tilde{\phi}_-(\tilde{k}), \] (4.10)
which is the obvious generalization of (4.3).

One might expect that the generalized de Broglie relation (4.6) would lead to a change in the dispersion relation for the velocity of a massive free particle. It can however be easily seen that this is not the case. From (4.3) with \( p^2 = m^2 \) we have
\[ \hbar^2 f^2(k^2)k^2 = m^2 \Rightarrow k^2 = (k_0)^2 - \tilde{k}^2 = \text{constant}. \] (4.11)
The group or particle velocity is then given by
\[ v_g = \frac{dk_0}{d|\tilde{k}|} = \frac{|\tilde{k}|}{k_0} = \frac{|\tilde{p}|}{p^0}, \] (4.12)
which is independent of the function \( f(k^2) \).

V. GENERALIZED COMMUTATION RELATION

The generalized de Broglie relation (4.6) leads to a modification of the canonical position-momentum commutation relation of quantum mechanics. This may be obtained as follows. The commutation relation between the position and wave vector operators is evident, since the wave vector \( k_\alpha \) may be expressed as the derivative \( i\partial / \partial x^\alpha \) in the position representation, and thus
\[ [x^\mu, k_\alpha] = -i\delta_\alpha^\mu. \] (5.1)
Alternatively, in the wave vector representation, the position operator may be expressed as \( x^\mu = -i\partial / \partial k_\mu \). Then the commutator of \( x^\mu \) and \( p_\alpha \) can be written as
\[ [x^\mu, p_\alpha] = [-i \frac{\partial}{\partial k_\mu}, p_\alpha] = -i \frac{\partial p_\alpha}{\partial k_\mu}. \] (5.2)
We may calculate the indicated derivatives from (4.6) and obtain
\[ [x^\mu, p_\alpha] = -i\hbar \left[ f\delta_\alpha^\mu + 2f'k_\alpha k^\mu \right], \quad f' \equiv \frac{df}{dk^2}. \] (5.3)
To obtain this entirely in terms of momentum we define \( g(p^2) \equiv f(k^2) \), and use (4.3) and (4.6) to get
\[ [x^\mu, p_\alpha] = -i\hbar \left[ g\delta_\alpha^\mu + \frac{2g'p_\alpha p^\mu}{1 - 2g'p^2/g} \right], \quad g' \equiv \frac{dg}{dp^2}. \] (5.4)
As an explicit example suppose that to lowest order
\[ f = 1 + a\frac{p^2}{m_p^2} k^2. \] (5.5)
Then the commutation relation takes the simple form, to lowest order,
\[ [x^\mu, p_\alpha] = -i\hbar \left[ \delta_\alpha^\mu + \frac{a}{m_p^2} \left( p^2 \delta_\alpha^\mu + 2p_\alpha p^\mu \right) \right], \] (5.6)
where \( m_p = \hbar/l_p \).

We note parenthetically that the above results may also be obtained directly in the position representation by using
\[ p_\mu = i\hbar f\frac{\partial}{\partial x^\mu} \] (5.7)
and assuming a power series representation for \( f \), but the derivation is somewhat more tedious than the above.
VI. THE GENERALIZED UNCERTAINTY PRINCIPLE

The commutation relation obtained in the previous section leads immediately to a generalization of the Heisenberg uncertainty principle, variously called the extended uncertainty principle, or the gravitational uncertainty principle, or the generalized uncertainty principle (GUP). We recall that the uncertainty principle of quantum mechanics gives the product of the variances of two observables (or Hermitian operators) $A$ and $B$ in terms of their commutator,

$$ (\Delta A)^2(\Delta B)^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2 . $$

The variances are for a series of measurements of the observables and the expectation value is for whatever state is being measured. We apply this to the position and momentum operators in the $x$ direction and use the approximate relation (V) from the preceding section, with $\alpha = \mu = 1$, to obtain

$$ \Delta x \Delta p \geq \frac{\hbar}{2} \left( 1 + \frac{a m^2}{m_p^2} - \frac{2a}{m_p^2} \langle p_x^2 \rangle \right) . $$

(6.2)

Consider now an ordinary particle such as an electron with $m \ll m_p$ at rest, with momentum expectation equal to zero. If it is subjected to a position measurement localizing its position to a very small region its momentum spread will be very large. Then, since

$$ \Delta p_x^2 = \langle p_x^2 \rangle - \langle p_x \rangle^2 = \langle p_x^2 \rangle $$

(6.3)

equation (6.2) becomes

$$ \Delta x \Delta p \geq \frac{\hbar}{2} - \frac{a \hbar}{m_p^2} \Delta p^2, \quad \Delta x \geq \frac{\hbar}{2 \Delta p} - \frac{a \hbar \Delta p}{m_p^2} . $$

(6.4)

This is precisely the GUP, as usually obtained in string theory [2] or by consideration of gravitational effects in a quantum measurement process [3]. Moreover, for agreement with the usual GUP as discussed below, we choose $a \simeq -1$.

VII. SPECULATION ON GRAVITY

The GUP is a well-known result of string theory [2], and is related to the interesting physical notions of duality and a minimum length [8]. It can be obtained in considerable generality using only basic concepts of quantum mechanics and gravity as is done in references [3] and [9]. A very brief derivation using dimensional arguments may be made as follows: consider a photon being used to measure the position of an electron, as in the classic discussion of Heisenberg [10]. In addition to the usual quantum mechanical position uncertainty, $\Delta x \simeq \hbar/\Delta p$, there must be an uncertainty due to the gravitational field of the photon. The extra uncertainty should accordingly be linear in the gravitational constant $G$ and in the energy or momentum of the photon. Moreover, The momentum of the photon should be of the order the uncertainty in the electron momentum, so the extra term should be proportional to $G \Delta p$. Thus the GUP must be, on dimensional grounds,

$$ \Delta x \simeq \frac{\hbar}{\Delta p} + \frac{\hbar}{m_p^2} \Delta p . $$

(7.1)

This is in qualitative agreement with (6.4).

In the derivation of the GUP in section VI we made no mention of gravity, but have obtained the result form the generalized de Broglie relation and the consequent commutation relation. We are led to speculate that the generalization may in some way generate gravity. In this view gravity may not be a fundamental force, but instead, it may be generated by quantum effects. This is not unreasonable physically: general relativistic gravity is characterized by its breaking of translational invariance in spacetime, and in a similar way a nonconstant function $f(k^2)$ breaks the translational invariance of the action in wave vector space.

A suggestion that is similar in spirit was made long ago by Sakharov [4], who speculated that gravity may not be fundamental, but induced by the residual effects of fundamental quantum fields on the vacuum, with the Lagrangian playing the role an elastic stress. Such a view of gravity has the merit of providing some understanding of why gravity is so weak. In the Sakharov view gravity is weak because the Planck scale provides an energy momentum cutoff that is very large. In the present view gravity is weak because the relation between momentum and wave number differs from linear only for very large momenta.
VIII. INTERACTIONS

Although the main parts of this paper deal with a free scalar field, it should be emphasized that interactions are easily accommodated in the context of wave vector space. As a simple example of this we consider an interaction in the spacetime Lagrangian of the form

$$L_I = g \phi^* \phi \Omega(x),$$  \hspace{1cm} (8.1)

where \( \Omega \) is a given external field and \( g \) is a coupling constant. The Lagrangian function in wave vector space is then modified from (4.8) to

$$K = \tilde{\phi}(k)^* \left[ \frac{\hbar^2 f^2(k^2)}{k^2} - m^2 \right] \tilde{\phi}(k) - g \int \frac{d^4l}{(2\pi)^4} \tilde{\phi}(k + l)^* \tilde{\phi}(k) \tilde{\Omega}(l).$$  \hspace{1cm} (8.2)

That is, the point interaction term becomes a convolution, so the interaction is nonlocal in wave vector space. The extremum condition, (3.5) or (3.7), then is an integral equation

$$\left[ \frac{\hbar^2 f^2(k^2)}{k^2} - m^2 \right] \tilde{\phi}(k) = g \int \frac{d^4l}{(2\pi)^4} \tilde{\phi}(k - l) \tilde{\Omega}(l),$$  \hspace{1cm} (8.3)

where we have made a shift of \( l \) in the wave vector space. This is easily solved by a perturbation expansion of \( \tilde{\phi}(k) \) of the form

$$\tilde{\phi}(k) = \tilde{\phi}_0(k) + g \tilde{\phi}_1(k) + \ldots = \sum_n g^n \tilde{\phi}_n(k).$$  \hspace{1cm} (8.4)

We substitute this into (8.3) and get a set of iterative equations

$$\left[ \frac{\hbar^2 f^2(k^2)}{k^2} - m^2 \right] \tilde{\phi}_n(k) = g \int \frac{d^4l}{(2\pi)^4} \tilde{\phi}_{n-1}(k - l) \tilde{\Omega}(l).$$  \hspace{1cm} (8.5)

As the zeroth order or unperturbed solution we naturally choose an eigenstate of the wave vector \( k_i \),

$$\tilde{\phi}_0(k) = (2\pi)^4 \delta(k - k_i),$$  \hspace{1cm} (8.6)

and obtain the series solution

$$\tilde{\phi}(k) = (2\pi)^4 \delta(k - k_i) + g \frac{1}{\hbar^2 f^2(k^2) - m^2} \tilde{\Omega}(k - k_i) +$$

$$g^2 \frac{1}{\hbar^2 f^2(k^2) - m^2} \int \frac{d^4l}{(2\pi)^4} \tilde{\Omega}(l) - \frac{1}{\hbar^2 f^2[(k - l)^2] - m^2} \tilde{\Omega}(k - k_i - l) + \ldots$$ \hspace{1cm} (8.7)

This we recognize as the usual Feynman expansion, but with a modified propagator given by

$$D(k^2) = \frac{1}{\hbar^2 f^2(k^2) - m^2}.$$ \hspace{1cm} (8.8)

As an illustrative example let us take the function \( f \) to be linear, as in (5.5), but expressed in terms of the Planck mass instead of the Planck length:

$$f(k^2) = 1 - l_p^2 k^2 = 1 - \frac{\hbar^2}{m_p^2} k^2.$$ \hspace{1cm} (8.9)

We note parenthetically that the momentum of the particle will become zero when \( k^2 = m_p^2 / \hbar^2 \). Massless particles could have a null wave vector or a wave vector with magnitude \( m_p / \hbar \), the inverse of the Planck length! Combining equations (8.8) and (8.9), we obtain the propagator

$$D(k^2) = \frac{1}{\hbar^2 k^2 - \hbar^4 k^4 / m_p^2 - m^2}.$$ \hspace{1cm} (8.10)

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It has poles at
\[ \hbar^2 k^2 = \frac{m_p^2}{2} \left( 1 \pm \sqrt{1 + 4m^2/m_p^2} \right) \simeq \begin{cases} m^2 & \text{for } -m^2, \\ m_p^2 & \text{for } +, \end{cases} \]
(8.11)
where the approximation is valid for \( m^2 \ll m_p^2 \). In this approximation the propagator factors to
\[ D(k^2) = \frac{1}{\hbar^2 k^2 - \hbar^4 k^4/m_p^2 - m^2} \simeq \left( \frac{1}{1 - \hbar^2 k^2/m_p^2} \right) \left( \frac{1}{\hbar^2 k^2 - m^2} \right). \]
(8.12)
That is, a form factor factor appears as multiplying the usual propagator. Such a form factor is usually associated with a modification of the vertex in quantum field theories, but here it appears as a result of the modified de Broglie relation.

Further discussion of quantum field theories and perturbation theory, in particular gauge theories, will follow in another paper.

**IX. SUMMARY**

We have generalized the de Broglie relation (1.1) to the four-vector relation (4.6), which involves an arbitrary scalar function. Thus the wavelength of a particle of a given momentum depends on its invariant mass. This was done in the context of the Klein-Gordon or scalar wave equation. In spacetime the generalized equation contains an arbitrarily large number of derivatives; in a formalism involving a Lagrangian type function in wave vector space the derivation is simpler and more natural. We emphasize that the wave vector space formalism is more closely related to actual laboratory experiments in high energy scattering, and that spacetime on a small scale need not even exist in this point of view.

As a result of the generalized de Broglie relation the canonical commutation relations are modified, and lead to a generalized uncertainty principle (GUP). Since the GUP may otherwise be obtained from a consideration of gravitational effects in the quantum measurement process the question naturally arises as to whether gravity may in some way be related to, or generated by, quantum effects, and not be an independent fundamental force.

Interactions are easily accomodated in the wave vector space formalism. A Feynman type expansion for the scalar field with a modified propagator was given as an example. Further studies will deal with perturbative field theories, especially with gauge theories such as QED.

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