Klein-Dirac Quadric and Multidimensional Toda Lattice via Pseudo-positive Moment Problem

S. Albeverio
Institute of Applied Mathematics and HCM, IZKS, BiBoS
University of Bonn, Endenicherallee 60, 53115 Bonn
CERFIM (Locarno)
and
O. Kounchev
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad. G. Bonchev St. 8, 1113 Sofia
and IZKS-University of Bonn

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ABSTRACT. In 1974 Jürgen Moser has shown that the classical Moment Problem plays a fundamental role for the theory of completely integrable systems, by proving that the simplest case of the finite Toda lattice is described exhaustively in its terms. In particular, the Jacobi matrix defined by the Flaschka variables corresponds to the Schrödinger operator related to the Korteweg-de Vries operator.

In the present paper we use a recent breakthrough in the area of the multidimensional Moment Problem in order to develop a multidimensional generalization of the finite Toda lattice. Our construction is based on the notion of pseudo-positive measure and the notion of multidimensional Markov-Stieltjes transform naturally defined on the Klein-Dirac quadric. An important consequence of the properties of the Markov-Stieltjes transform is a new method for summation of multidimensional divergent series on the Klein-Dirac quadric.

Key words: Multidimensional Toda lattice, Klein-Dirac quadric, pseudo-positive moment problem, Markov-Stieltjes transform, Schrödinger operator, Jacobi matrices, multidimensional divergent series.

AMS Classification: Primary 37K40, 37K45; Secondary 35Q15, 37K10
1 Introduction

In the present paper we provide a novel multidimensional generalization of the finite Toda lattice which is governed by an operator which generalizes the Schrödinger one-dimensional operator. Our construction is based on a new notion of multidimensional Markov-Stieltjes transform naturally defined on the Klein-Dirac quadric. Our research is a direct generalization of the work of Jürgen Moser, [20], [21], [22], who applied the one-dimensional Moment Problem to study exhaustively the non-periodic finite Toda lattice.

Let us note that there has been a large amount of research in the area of multidimensional non-linear evolutions aimed at generalization of the Toda lattice and the KdV equation. These have led some researchers ([3]) to the conclusion that there is a "...lack of interesting non-linear evolutions associated to the Laplacian. A major open question is to find those operators and systems that do have associated, stable computable evolutions". Accordingly, one of the major objectives of the present research is to create models for such operators.

The structure of the paper is as follows: In section 2 we recall some basic facts from the classical Moment Problem and the finite Toda lattice. In section 3 we recall the (multidimensional) pseudo-positive Moment Problem by using the framework of the Klein-Dirac quadric. In section 4 we generalize the Nevanlinna class of analytic functions to holomorphic functions on the Klein-Dirac quadric, and we prove a generalization of the theorem of Riesz-Nevanlinna for encoding the Moment Problem by the means of function theory. As an interesting by-product we obtain a new method for summation of multidimensional divergent series on the Klein-Dirac quadric. In section 5 we define our generalization as the pseudo-positive Toda lattice and prove conditions for its existence, Theorem 21. In section 6 we describe a functional model for a generalization of the one-dimensional discrete Schrödinger (Jacobi matrix) operator. Finally, in section 7 we present a special class of pseudo-positive Toda lattices.

2 The classical finite Toda lattice and the Moment Problem

The Toda lattice is a system of unit masses, connected by nonlinear springs governed by an exponential restoring force. The equations of motion are derived from a Hamiltonian containing the displacements $x_j$ of the $j$-th mass and the corresponding momentum $y_j$. The Toda lattice is a finite-dimensional analog to the Kortevg-de Vries partial differential equations, cf. [6], [7].

In the seminal papers by Jürgen Moser [20], [21], [22], the finite Toda lattice was considered with non-periodic boundary conditions, and a complete solution was given in terms of the classical Moment Problem. We provide a short outline of these results since we will use them essentially in our multidimensional generalization.
We introduce the Hamiltonian given by
\[ H = \frac{1}{2} \sum_{j=1}^{N} y_j^2 + \sum_{j=1}^{N-1} e^{x_j-x_{j+1}}, \] (1)

\( x_j \) and \( y_j \) being real variables corresponding to the mass displacements and their momenta. The corresponding equations of motion for all times \( t \in \mathbb{R} \) are
\[ x_j' = y_j, \quad j = 1, 2, ..., N \] (2a)
\[ y_j' = e^{x_{j-1}-x_j} - e^{x_j-x_{j+1}}, \quad j = 2, 3, ..., N-1 \]
\[ y_1' = -e^{x_1-x_2} \]
\[ y_N' = e^{x_{N-1}-x_N}, \]
(where \( ' \) denotes time derivative). Obviously, they are equivalent to
\[ x_j'' = e^{x_j-x_{j+1}} - e^{x_{j-1}-x_j}, \quad j = 1, 2, ..., N \]
\[ x_0 := -\infty, \quad x_{N+1} := +\infty, \]
so that \( e^{x_0-x_1} = 0 \) and \( e^{x_{N-1}-x_N} = 0 \). Following H. Flaschka [7], one introduces the new variables
\[ a_j = \frac{1}{2} e^{(x_j-x_{j+1})/2}, \quad j = 1, 2, ..., N-1 \] (3)
\[ b_j = -\frac{1}{2} y_j, \quad j = 1, 2, ..., N, \] (4)
\[ a_0 = 0, \quad a_N = 0, \] (5)

which satisfy the system
\[ a_j' = a_j (b_{j+1} - b_j), \quad j = 1, 2, ..., N-1 \] (6)
\[ b_j' = 2 (a_j^2 - a_{j-1}^2), \quad j = 1, 2, ..., N \] (7)
\[ a_0 = 0, \quad a_N = 0. \] (8)

Let us note that the map from the domain \( D_1 := \{(x, y) \in \mathbb{R}^{2N}\} \) to the domain \( D \subset \mathbb{R}^{2N-1} \) given by
\[ D := \{ a \in \mathbb{R}^{N-1}, b \in \mathbb{R}^N : a_j > 0, \quad \text{for } j = 1, ..., N-1 \} \]
is invertible only on equivalence classes in \( D_1 \) defined by the equivalence relation
\[ (x, y) \sim (\tilde{x}, \tilde{y}) \iff x_j - \tilde{x}_j \text{ is independent of } j, \] (9)
cf. [20], p. 471. We call these every such class configuration of \((x, y)\).

We have then for the Hamiltonian (1) the following representation in terms of the new variables:
\[ H = 4 \left\{ \sum_{j=1}^{N-1} a_j^2 + \frac{1}{2} \sum_{j=1}^{N} b_j^2 \right\}. \] (10)

The following result holds (see [20], Lemma in Section 2, and (2.5)).
**Proposition 1** Let us consider the system (6)–(7) with the requirement (8) replaced by the requirement that $a_0(t)$ and $a_N(t)$ belong to $L_2(\mathbb{R})$ as functions of $t \in \mathbb{R}$, i.e. satisfy
\[
\int_{-\infty}^{\infty} (a_0^2(t) + a_N^2(t)) \, dt < \infty.
\]
Then for every $j = 2, ..., N$ we have
\[
\int_{-\infty}^{\infty} (a_j^2(t) + a_{j-1}^2(t)) \, dt < \infty.
\]
Moreover, for any solution of (6)-(8) satisfying $a_j(t) > 0$, $j = 1, 2, ..., N-1$, we have
\[
a_j(t) \to 0 \quad \text{for } t \to \pm \infty \quad \text{and } j = 1, 2, ..., N-1,
\]
and
\[
b_j(t) \xrightarrow{t \to \infty} b_j(\infty)
\]
for some $b_j(\infty) \in \mathbb{R}$.

Flaschka [6], [7] has found the following Lax representation
\[
L' = BL - LB,
\]
where we have the $N \times N$ (symmetric) Jacobi matrix
\[
L = \begin{pmatrix}
b_1 & a_1 & 0 \\
-a_1 & b_2 & \\
0 & b_{N-1} & a_{N-1} \\
& a_{N-1} & b_N
\end{pmatrix}
\]
(11)
and
\[
B = \begin{pmatrix}
0 & a_1 & 0 \\
-a_1 & 0 & a_{N-1} \\
0 & -a_{N-1} & 0
\end{pmatrix}.
\]
(12)

**Remark 2** The matrix $L$ is the analog to the Schrödinger operator appearing in the Inverse Scattering method for solving the KdV equation.

Moreover, he found an orthogonal matrix $U$ which satisfies
\[
U' = BU
\]
\[
U(0) = I
\]
for which one has
\[
(U^{-1}LU)' = 0
\]
which implies  
\[ U^{-1}LU = L(0). \]

The latter equality implies that the eigenvalues \((\lambda_j)_{j=1}^N\) of the symmetric matrix \(L\), which are real and distinct, do not depend on \(t\). Thus the eigenvalues (and symmetric functions of them) are first integrals of the Toda flow  
\[ (a_j(0), b_j(0)) \longrightarrow (a_j(t), b_j(t)) \quad \text{for} \quad j = 1, 2, \ldots, N \]

associated with (6)-(8). In particular, for every integer \(m \geq 0\), the trace \(\text{tr}(L^m)\) is a symmetric function of the eigenvalues. We see that, in particular,  
\[ \text{tr}(L^2) = \sum_{j=1}^N \lambda_j^2 = \frac{1}{2}H = 2 \left\{ \sum_{j=1}^{N-1} a_j^2 + \frac{1}{2} \sum_{j=1}^N b_j^2 \right\}. \tag{13} \]

Further Flaschka uses the resolvent matrix  
\[ R(\lambda) = (\lambda I - L)^{-1} \quad \text{for} \quad \lambda \notin \sigma(L), \]
with \(\sigma(L)\) denoting the spectrum of \(L\).

We define the important function \(f(\lambda, t)\) by putting  
\[ R_{N,N}(\lambda, t) = \langle R(\lambda) e_N, e_N \rangle =: f(\lambda, t), \quad \lambda \notin \sigma(L), \quad t \in \mathbb{R}. \]

Here \(e_N\) denotes the \(N\)–vector \((0, 0, \ldots, 0, 1)\). A simple argument [20] shows that  
\[ f(\lambda, t) = \sum_{j=1}^N \frac{r_j^2(t)}{\lambda - \lambda_j(t)}, \quad \lambda \notin \sigma(L), \quad t \in \mathbb{R}, \tag{14} \]

for some appropriate real functions \(r_j(t)\) and \(\lambda_j(t)\). The function \(f\) satisfies  
\[ \lambda f(\lambda, t) \longrightarrow 1 \quad \text{for} \quad |\lambda| \longrightarrow \infty \]

which implies  
\[ \sum_{j=1}^N r_j^2(t) = 1, \quad t \in \mathbb{R}. \]

These \(\lambda_j, r_j\) are the new variables which show the direct relation to the Moment Problem.

Let us remark that a classical result of Stieltjes shows that the function \(f(\lambda, t)\) has the pointwise convergent continued fraction representation:
\[ f(\lambda, t) = \frac{1}{\lambda - b_N(t)} \sum_{n=0}^{\infty} \frac{a_n(t)}{\lambda - b_{n+1}(t)} = \frac{Q_N(\lambda, t)}{P_N(\lambda, t)}, \tag{15} \]

in terms of the variables \(a_j, b_j\) given by (6)-(8), cf. [1], [23], [26], [30].
The matrix $L$ contains the 3-term recurrence relations which generate the polynomials $P_N(\lambda, t)$ which are orthogonal with respect to the measure $\mu$. Let $Q_N(\lambda, t)$ be the second kind orthogonal polynomials which are obtained by setting

$$Q_N(\tau, t) = \int \frac{P_N(\tau, t) - P_N(\lambda, t)}{\tau - \lambda} \, d\mu(\lambda, t),$$

where $d\mu(\lambda, t)$ is the $t$-dependent measure in $\lambda$ on $\mathbb{R}$ defined by

$$d\mu(\lambda, t) := \sum_{j=1}^{N} r_j^2(t) \delta(\lambda - \lambda_j(t)),$$

with $\delta$ being the Dirac function concentrated at the origin. $d\mu(\lambda, t)$ is in fact the spectral measure for the Jacobi matrix $L$.

From (14) and (16) we have

$$f(\lambda, t) = \sum_{j=1}^{N} r_j^2(t) = \int \frac{d\mu(\tau, t)}{\lambda - \tau}, \quad \lambda \notin \sigma(L), \ t \in \mathbb{R}.$$

The main observation by J. Moser is contained in the following Proposition (see [20], p. 480): it expresses the dynamics of the variables $r_j$ and $\lambda_j$ as a result of the dynamics of the variables $a_j$ and $b_j$.

**Proposition 3** The variables $r_j(t), \lambda_j(t)$ associated with the solutions of the classical finite Toda lattice given by (6)-(7) satisfy the linear equations

$$\lambda_j'(t) = 0, \quad r_j'(t) = -\lambda_j(t) r_j(t), \quad j = 1, 2, ..., N.$$

The solution $(\lambda_j, r_j), j = 1, 2, ..., N$ is given by

$$\lambda_j(t) = \lambda_j(0) := \lambda_j, \quad \lambda_j = \lambda_j(0) := \lambda_j$$

$$r_j^2(t) = \frac{r_j^2(0) e^{-2\lambda_j t}}{\sum_{j=1}^{N} r_j^2(0) e^{-2\lambda_j t}}, \quad t \in \mathbb{R}. \quad (18)$$

For the details of the above representations we refer to the excellent exposition on the Classical Moment Problem in [26], and for the finite Toda lattice to [27], [30].

**3 Klein-Dirac quadric and pseudo-positive Moment Problem**

A central role in our further consideration will be played by the Klein-Dirac quadric which generalizes the complex plane $\mathbb{C}$. It is defined by

$$\text{KDG} := \{ \zeta \theta : \zeta \in \mathbb{C}, \ \theta \in S^{n-1} \} = (\mathbb{C} \times S^{n-1})_{\mathbb{Z}_2}.$$
For \( n = 3 \) the set KDQ has been introduced by Klein in his famous treatise. The notation \( \zeta \theta \) denotes the identification of the points \((\zeta, \theta)\) and \((-\zeta, -\theta)\) in \( \mathbb{C} \times S^{n-1} \) which is the antipodal identification \( \mathbb{Z}_2 \) of points in \( \mathbb{C} \times S^{n-1} \) expressed by the notation \((\mathbb{C} \times S^{n-1})_{\mathbb{Z}_2}\). The unit ball in KDQ is given by \( B_1 := \{ \zeta \theta : \zeta \in \mathbb{C}, |\zeta| < 1, \theta \in S^{n-1} \} \); respectively we define the ball of radius \( R \) as \( B_R := R \cdot B_1 = \{ R \zeta \theta : \zeta \theta \in B_1 \} \).

The topological boundary \( \partial (B_1) \) is the \textbf{compactified Minkowski space-time} \( (S^1 \times S^{n-1})_{\mathbb{Z}_2} = \{ \zeta \theta : \zeta \in S^1, \theta \in S^{n-1} \} \), where again \( \zeta \theta \) denotes the identification of \((\zeta, \theta)\) and \((-\zeta, -\theta)\) in \( S^1 \times S^{n-1} \).

\section*{Remark 4} The Klein-Dirac quadric has appeared in the works of P. Dirac while he extended the equations of physics from the Minkowski space-time to a 4-dimensional conformal space, in order "to bring a greater symmetry of the equations into evidence", \[4\], \[5\]. Apparently, the term Klein-Dirac space has been coined in Conformal Field Theory by Ivan Todorov, cf. \[24\], and references therein. The Klein-Dirac quadric appears in a natural way as a space where the solutions of elliptic equations (polyharmonic functions) are extended and enjoy a generalized Cauchy type formula, cf. \[2\], \[17\], \[18\].

In the previous section we saw the key role of the function

\[ f(\lambda) = \int \frac{d\mu(\tau)}{\lambda - \tau} \quad \text{for} \; \lambda \notin \sigma(L) \] (19)

which is the Stieltjes transform (sometimes also called Markov transform, cf. \[23\]) of the measure \( \mu \) defined by \( \[16\] \) and is obtained from the resolvent \( R(\lambda) = (\lambda I - L)^{-1} \) of the Jacobi operator \( L \), \( \lambda \notin \sigma(L) \).

A main achievement of the multidimensional pseudo-positive Moment Problem introduced in \[13\], \[12\] is that it has at its disposal analogs to all notions available in the one-dimensional case. In particular, we have a generalization of the Stieltjes transform (called here Stieltjes-Markov transform) which is naturally defined on the Klein-Dirac quadric. It has been introduced and studied in \[16\], \[15\], \[14\]. We will explain briefly this notion here.

First, a crucial point of the pseudo-positive Moment Problem is the remarkable representation of multivariate polynomials which is sometimes associated with the names of Gauss and Almansi (cf. \[2\], \[28\]):

\[ P(x) = \sum_{k,\ell} p_{k,\ell} (r^2) r^k Y_{k,\ell}(\theta), \quad \text{for} \; x \in \mathbb{R}^n, \; r = |x|, \; x = r\theta, \] (20)

for \( n \in \mathbb{N} \) with \( n \geq 2 \). Here, for every fixed integer \( k \geq 0 \), the set

\[ \{ Y_{k,\ell}(\theta) : \ell = 1, 2, ..., d_k \}, \] (21)
with
\[ d_k = \frac{(2k + n - 2)(n + k - 3)!}{(n - 2)k!} \quad \text{for } k \geq 0, \]
is a basis of the space \( H_k ({\mathbb S}^{n-1}) \) of spherical harmonics on the unit sphere \( {\mathbb S}^{n-1} \) which are homogeneous of degree \( k \), and \( p_{k,\ell} \) are polynomials. It exhibits the representation of a multivariate polynomial \( P \) through the one-dimensional polynomials \( p_{k,\ell} \), having control on the degree: namely, if \( \Delta^N P(x) = 0 \), with \( \Delta \) being the Laplacian in \( \mathbb{R}^n \), then \( \deg p_{k,\ell} \leq N - 1 \). This representation is alternative to the standard
\[
P(x) = \sum_{\alpha \in \mathbb{Z}_+^n} a_{\alpha} x^\alpha,
\]
where \( \alpha = (\alpha_1, \ldots, \alpha_n) \), \( \alpha_i \in \mathbb{N}_0 \), is a multi-index, \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \). It is based on the fact that the polynomials
\[
r^{k+2j} Y_{k,\ell}(\theta) = |x|^{2j} Y_{k,\ell}(x) \quad \text{for } j \geq 0, \text{ and all indices } (k, \ell),
\]
with \( k \geq 0 \) and \( \ell = 1, 2, \ldots, d_k \), are an alternative basis of the space of all polynomials in \( \mathbb{R}^n \). It provides also an alternative representation for the Moment Problem by considering the moments
\[
c_{k,\ell,j} := \int_{\mathbb{R}^n} r^{k+2j} Y_{k,\ell}(\theta) d\mu(x) = \int_{\mathbb{R}^n} |x|^{2j} Y_{k,\ell}(x) d\mu(x) \quad \text{for } k \geq 0, \ell = 1, 2, \ldots, d_k, j \geq 0,
\]
instead of the usual approach where one considers the Moment Problem in the form
\[
d_\alpha := \int x^\alpha d\mu(x), \quad \text{for } \alpha \in \mathbb{N}_0.
\]

Further, we consider the \textbf{Hua-Aronszajn kernel} defined by (cf. the details in [16], [15])
\[
K(\zeta \theta; x) := \frac{\zeta^{n-1}}{r(\zeta \theta - x)^n}, \quad \zeta \in \mathbb{C}, \theta \in {\mathbb S}^{n-1}, x \in \mathbb{C}^n, r = |x|,
\]
where \( n \in \mathbb{N} \) and \( n \geq 2 \). Here we use the following notation of Aronszajn [2]
\[
r(\zeta \theta - x)^n := \left( \sqrt{\zeta^2 - 2\zeta \langle \theta, x \rangle + |x|^2} \right)^n,
\]
where an appropriate root is chosen, \( \langle \cdot, \cdot \rangle \) denoting the scalar product in \( \mathbb{C}^n \) and \(|\cdot|\) the corresponding norm. We notice that the zeros \( \zeta^2 - 2\zeta \langle \theta, x \rangle + |x|^2 = 0 \) are given by
\[
\zeta_{1,2}(\theta, x) = \langle \theta, x \rangle \pm i \sqrt{|x|^2 - |\langle \theta, x \rangle|^2}.
\]
One has obviously
\[
|\zeta_{1,2}| = |x|.
\]
Let us consider the above kernel $K$ for arguments $(\zeta; x)$ satisfying

$$\zeta \neq \zeta_{1,2}(\theta, x).$$

For us the following representation of the Hua-Aronszajn kernel will be important (cf. e.g. formula (18) in [15]):

$$K(\zeta; x) = \zeta^{n-1} r(\zeta-x)^n = \frac{\zeta}{\zeta^2 - |x|^2} \sum_{k,\ell} \zeta^{-k} Y_{k,\ell}(\theta) Y_{k,\ell}(x),$$  \hspace{1cm} (25)

where $\zeta \in \mathbb{C}$, $\theta \in S^{n-1}$ and $x \in \mathbb{R}^n$. It is obtained through the representation of $K$ as a product of the Cauchy kernel and the “complexified Poisson kernel” $\zeta^{n-2}(\zeta^2 - x^2)$ by the representation:

$$\frac{\zeta^{n-1}}{\zeta^2 - x^2} r(\zeta-x)^n = \frac{\zeta^{n-2}(\zeta^2 - x^2)}{r(\zeta-x)^n}.$$

The expansion in spherical harmonics follows directly from the expansion in spherical harmonics of the “complexified Poisson kernel” (see [15], [29].

One may prove now easily the following analog and generalization to the Cauchy formula:

**Proposition 5** For every polynomial $P(x)$, $x \in \mathbb{C}^n$, the following representation holds:

$$P(x) = \frac{1}{2\pi i} \int_{S^1} \int_{S^{n-1}} \frac{\zeta^{n-1}}{r(\zeta-x)^n} P(\zeta \theta) d\zeta d\theta \text{ for } x \in \mathbb{C}^n.$$

**Proof.** The proof follows directly from the Almansi representation (20), (25), and the classical Cauchy formula, cf. [10], [2].

**Definition 6** We say that the function $f(\zeta, \theta)$ is KDQ-holomorphic on the compact ball $B_R \subset \text{KDQ}$, if it is representable by the series

$$f(\zeta \theta) = \sum_{k,\ell} f_{k,\ell}(\zeta^2) \zeta^k Y_{k,\ell}(\theta), \text{ for } \zeta \theta \in D,$$  \hspace{1cm} (27)

which is absolutely convergent on compacts $K \subset B_R$, and where all functions $f_{k,\ell}$ are analytic in the one-dimensional disc $R \cdot \mathbb{D}$, where $\mathbb{D}$ denotes the unit disc in $\mathbb{C}$. In a similar way, we say that the function $f(\zeta, \theta)$ is KDQ-holomorphic in the exterior of the compact ball $B_R$, if $f$ is representable by the series

$$f(\zeta \theta) = \sum_{k,\ell} f_{k,\ell}(\zeta^2) \frac{1}{\zeta^k} Y_{k,\ell}(\theta), \text{ for } \zeta \theta \in D,$$  \hspace{1cm} (28)

which is absolutely convergent on compacts $K \subset \text{KDQ} \setminus \overline{B_R}$. Here $\overline{B_R}$ denotes the topological closure of $B_R$. 

9
Remark 7. The topological boundary $\partial(B_1)$ of $B_1$ is a natural boundary for the space of $\mathcal{KDQ}$-holomorphic functions in $B_1$. In particular, if a function $f$ is $\mathcal{KDQ}$-holomorphic and satisfies
\[
\sum_{k,\ell} \int_0^{2\pi} |f_{k,\ell}(\zeta^2)|^2 \, d\varphi < \infty, \quad \text{for} \quad \zeta = e^{i\varphi},
\]
then formula (26) holds where $P$ is replaced by $f$. Let us remark that $\partial(B_1)$ is also the Shilov boundary of the so-called fourth domain of Cartan (or "Lie ball"), and formula (26) has been proved in this context for holomorphic functions of several variables in the Lie ball in $\mathbb{C}^n$, (see [2], p. 125, Corollary 1.1).

Let $d\mu(x)$ be a signed measure defined on $\mathbb{R}^n$ or on some domain $D \subset \mathbb{R}^n$.

Definition 8. We define the (multidimensional) Stieltjes-Markov transform as the following function on $\mathcal{KDQ}$, whenever the integral is well defined:
\[
\hat{\mu}(\zeta, \theta) := \int \frac{\zeta^{n-1} d\mu(x)}{r(\zeta\theta - x)} \quad \text{for} \quad \zeta\theta \in \mathcal{KDQ}. \tag{29}
\]

Remark 9. In Proposition 13 below we shall provide a condition under which the integral on the right hand side of (29) is absolutely convergent for all $\text{Im} \, \zeta^2 > 0$, for $|\zeta|$ sufficiently large.

Further, let us introduce the following important notion of component measures, see [13] or [12]:

Definition 10. For every index $(k, \ell), k \in \mathbb{N}_0, \ell = 1, 2, \ldots, d_k$, the one-dimensional component measure $\mu_{k,\ell}(r)$ on $\mathbb{R}_+$ is uniquely defined as the measure on $\mathbb{R}_+$, such that for every continuous function $F$ with a compact support in $\mathbb{R}_+$ it satisfies the following identity:
\[
\int_0^\infty F(r) \, d\mu_{k,\ell}(r) = \int_{\mathbb{R}^n} F(r) \, Y_{k,\ell}(\theta) \, d\mu(x),
\]
where $x = r\theta$, $r = |x|$, $\theta \in S^{n-1}$. Formally, we will write
\[
d\mu_{k,\ell}(r) := \int_{S^{n-1}} Y_{k,\ell}(\theta) \, d\mu(r\theta).
\]

The transform $\hat{\mu}(\zeta, \theta)$ will be interesting for us only for large values of $|\zeta|$.

However, let us note that a priori it is not clear for which values of the parameters $\zeta, \theta$ does the function $\hat{\mu}(\zeta, \theta)$ make sense, i.e. the integral in (29) is well defined. If we assume for simplicity that the signed measure $\mu$ has a compact support in the ball $B_R$ in $\mathbb{R}^n$ for some $R > 0$, then by (24) $\hat{\mu}(\zeta, \theta)$ is well defined for all $\zeta$ with $|\zeta| > R$. But for a general measure singularities can arise for all possible values of $\zeta$ as far as $|\zeta| = |x|$ on the support of $\mu$. The following representation is remarkable since it shows that the multidimensional Stieltjes-Markov transform $\hat{\mu}(\zeta, \theta)$ is representable through infinitely many one-dimensional Stieltjes transforms, those of the measures $r^k d\mu_{k,\ell}(r)$. More precisely, we have:

10
Proposition 11 Let the signed measure \( \mu \) have a compact support in the ball \( B_R \subset \mathbb{R}^n \) and have a finite variation there. Then the following expansion

\[
\hat{\mu} (\zeta, \theta) = \sum_{k, \ell} \zeta^{-k+1} Y_{k, \ell} (\theta) \times \int_{B_R} \frac{1}{\zeta^2 - r^2} r^k d\mu_{k, \ell} (r) \quad \text{for all } |\zeta| > R, \theta \in S^{n-1}
\]

is absolutely convergent on compacts in \( KDQ \setminus B_R \).

Proof. First, we have the representation formula (25), and we use the definition of the component measures \( \mu_{k, \ell} \), cf. Definition 10, to obtain the following:

\[
\hat{\mu} (\zeta, \theta) = \sum_{k, \ell} \zeta^{-k} Y_{k, \ell} (\theta) \int_{\mathbb{R}^n} Y_{k, \ell} (x) \frac{\zeta}{\zeta^2 - |x|^2} d\mu (x)
\]

\[
= \sum_{k, \ell} \zeta^{-k+1} Y_{k, \ell} (\theta) \int_0^R \frac{\zeta}{\zeta^2 - r^2} r^k d\mu_{k, \ell} (r).
\]

Its absolute convergence on compacts \( K \subset KDQ \setminus B_R \) is obtained by a routine estimation using the bound \( k^{n-2} \) for the spherical harmonics \( Y_{k, \ell} (\theta) \), cf. [29].

At this point we define the very important class of pseudo-positive measures:

Definition 12 We will say that the measure \( \mu \) is pseudo-positive in the ball \( B_R \) in \( KDQ \) (with respect to the fixed basis \( \{ Y_{k, \ell} (\theta) : k \in \mathbb{N}_0, \ell = 1, 2, ..., d_k \} \) if it satisfies

\[
d\mu_{k, \ell} (r) \geq 0 \quad \text{for } 0 \leq r \leq R,
\]

for all indices \( (k, \ell) \), (see Definition 11).

We have the following simple condition for convergence of the multidimensional Stieltjes-Markov transform:

Proposition 13 Assume that for the signed measure \( \mu \), and for some constants \( C > 0 \) and \( D > 0 \), its components \( \mu_{k, \ell} \) satisfy the following growth condition

\[
\left| \int_0^\infty r^k d\mu_{k, \ell} (r) \right| \leq CD^k \quad \text{for all } (k, \ell) \text{ with } k \geq 0,
\]

for some constants \( C, D > 0 \), independent of \( k, \ell \). Then it follows that the multidimensional Stieltjes-Markov transform \( \hat{\mu} (\zeta, \theta) \) in Definition 8 is absolutely convergent for all \( \zeta \) with \( \text{Im} \zeta^2 > 0, |\zeta| > D \) and all \( \theta \in S^{n-1} \).

Proof. The proof follows by a direct estimate of the series (30), taking into account the bound \( k^{n-2} \) for the spherical harmonics \( Y_{k, \ell} (\theta) \), cf. [29].

\[\square\]

Proposition 13 is remarkable in some sense, since it shows that for a reasonable class of measures with non-compact support, one has convergence of the Stieltjes-Markov transform \( \hat{\mu} \) for all sufficiently large values of \( \zeta \), thanks to the expansion (30).
4 Nevanlinna class on the Klein-Dirac quadric
and generalized Hamburger-Nevanlinna theorem

In the present section we will discuss the role of the multidimensional Stieltjes-
Markov transform for the multidimensional pseudo-positive Moment Problem.

Let us recall the classical Nevanlinna class of analytical functions in the half-
plane: A function $f$ defined in the upper half plane $\mathbb{C}$ belongs to the Nevanlinna
class $\mathcal{N}$ iff it satisfies

$$\text{Im } z > 0 \text{ implies } \text{Im } f(z) \geq 0.$$  

The elements $f$ of $\mathcal{N}$ have the following Riesz-Herglotz-Nevanlinna repre-
sentation ([1], p. 92):

$$f(z) = az + b + \int_{-\infty}^{\infty} \frac{1 + u z}{u - z} d\nu(u), \quad \text{for } z \in \mathbb{C}, \text{ Im } z \neq 0,$$

for some non-negative measure $\nu$ with bounded variation (equal to $\text{Im } f(i) - a$),
and real constants $a \geq 0$ and $b$.

We recall the classical Hamburger-Nevanlinna theorem which plays an
important role in the one-dimensional Moment Problem (cf. Theorem 3
2.1 in [1]). It shows that the Stieltjes transform of a measure encodes in a very
appropriate way the information about the moments of the measure, and the
moments may be extracted only by means of an adequate "asymptotic expan-
sion" and not by the usual Taylor expansion, which is asymptotic but in general
notoriously divergent.

**Theorem 14** If the non-negative measure $\sigma$ solves the truncated Moment Prob-
lem

$$s_j = \int_{-\infty}^{\infty} u^j d\sigma(u) \quad \text{for } j = 0, 1, \ldots, 2N, \quad (31)$$

then the Stieltjes transform of the measure $\sigma$ given by

$$f(z) = \int_{-\infty}^{\infty} \frac{d\sigma(u)}{u - z} \quad (32)$$

belongs to the Nevanlinna class $\mathcal{N}$ and for every fixed $\delta > 0 \ (\ < \pi/2 \ )$ the
following limit holds uniformly for $\delta \leq \text{arg } z \leq \pi - \delta$:

$$\lim_{z \to \infty} z^{2N+1} \left\{ f(z) + \sum_{j=0}^{2N-1} \frac{s_j}{z^{j+1}} \right\} = -s_{2N} \quad (33)$$

In particular, the formal series $\sum_{j=0}^{\infty} \frac{s_j}{z^{j+1}}$ gives an asymptotic expansion to all
orders for $f(z)$ as $z \to \infty$.
Conversely, if for some function $f \in \mathcal{N}$ equation (33) holds with $s_j \in \mathbb{R}$, for all $N \in \mathbb{N}$, at least for $z = iy$, $y \to \infty$, then $f$ is representable in the form (32) and the non-negative measure $\sigma$ has finite moments of any order, and in particular satisfies (31).

We define the multidimensional pseudo-positive Nevanlinna class on the Klein-Dirac quadric:

**Definition 15** Let the function $f$ be KDQ-holomorphic in the exterior of the ball $B_R \subset \text{KDQ}$, in the sense of Definition 6; hence $f$ is representable by the series (28). We say that $f$ belongs to the pseudo-positive Nevanlinna class on $\text{KDQ} \setminus B_R$ iff all its components $f_{k,\ell}$ belong to the one-dimensional Nevanlinna class $\mathcal{N}$.

Now we are able to prove the generalization of the Hamburger-Nevanlinna theorem which shows that the multidimensional Stieltjes-Markov transform encodes in an asymptotic expansion form the information about the multidimensional moments of a measure (cf. [12]). However, since the component measures $\mu_{k,\ell}$ are not so strongly correlated we will need to keep them tight by means of some growth condition.

**Theorem 16** Assume that a pseudo-positive measure $\mu$ is given, and solves the following multidimensional Moment Problem, (22),

$$ s_{k,\ell,j} = \int_{\mathbb{R}^n} r^{k+2j} Y_{k,\ell}(\theta) \, d\mu(x) \quad \text{for all } (k, \ell) \text{ and } j = 0, 1, ..., 2N. $$

Assume that, for some constants $C > 0$ and $D > 0$, independent of $k, \ell$, the following growth condition holds:

$$ |s_{k,\ell,0}| \leq CD^k \quad \text{for all indices } (k, \ell). \quad (34) $$

Then the multidimensional Stieltjes-Markov transform $f(\zeta, \theta) = \hat{\mu}(\zeta, \theta)$ defined by (29) is absolutely convergent and belongs to the pseudo-positive Nevanlinna class on $\text{KDQ} \setminus B_D$. For every index $(k, \ell)$ the following asymptotic expansion holds uniformly for $\delta \leq \arg \zeta^2 \leq \pi - \delta$, and $\zeta \to \infty$ (but not uniformly on the indices $k, \ell$ !):

$$ \zeta^{4N+1} \left\{ \zeta^{k-1} \int_{S^{n-1}} f(\zeta, \theta) Y_{k,\ell}(\theta) \, d\theta - \sum_{j=0}^{2N-1} \frac{s_{k,\ell,j}}{\zeta^{k+2j}} \right\} = s_{k,\ell,2N}. \quad (35) $$

In particular, the formal series $\sum_{j=0}^{2N-1} \frac{s_{k,\ell,j}}{\zeta^{k+2j}}$ provides an asymptotic expansion to all orders for the component $\int_{S^{n-1}} f(\zeta, \theta) Y_{k,\ell}(\theta) \, d\theta$ of $f$ along $Y_{k,\ell}(\theta)$. 

13
Proof. By formula (30) we see that
\[ \zeta^{-1} \int_{S_{n-1}} \hat{\mu}(\zeta, \theta) Y_{k,\ell}(\theta) \, d\theta = \int_0^\infty \frac{1}{\zeta^2 - r^2} r^k \, d\mu_{k,\ell}(r), \]
and the right hand side is a one-dimensional Stieltjes transforms. It follows by Theorem 14 that for every \((k, \ell)\) the component measures \(\mu_{k,\ell}\) (which are non-negative measures) have the moments
\[ s_{k,\ell;j} = \int_0^\infty r^{k+2j} \, d\mu_{k,\ell}(r) \quad \text{for } j = 0, 1, \ldots, 2N, \]
and the asymptotic expansion (35) holds as in the classical case covered by Theorem 14.

There is a converse statement of the above Theorem 16 which we do not formulate.

Remark 17 In the one-dimensional case the Moment Problem provides one of the most important methods for summation of divergent series, cf. [8], [11], chapter 9. Hence, Theorem 16 generalizes known classical results about summation of divergent series. Indeed, in the classical case, by Theorem 14 we attach the function \(f(z)\) to the divergent series \(\sum z^j\) using the fact that the sequence \(\{s_j\}\) is positive-definite. Now, in a similar manner, Theorem 16 provides a new summation method for multidimensional divergent series: The multidimensional Stieltjes-Markov transform \(\hat{\mu}(\zeta, \theta)\) is an asymptotic expansion in polyharmonic functions, associated with the formal Laurent series (divergent in general)
\[ f_N(\zeta\theta) = \frac{1}{\zeta} \sum_{k,\ell} \sum_{j=0}^{2N-1} \frac{s_{k,\ell;j}}{\zeta^{k+2j}} Y_{k,\ell}(\theta), \]
which satisfies formally the polyharmonic equation \(\Delta^{2N} f(\zeta\theta) = 0\), i.e. we may write formally
\[ \lim_{\zeta \to \infty} \zeta^{4N+1} \{ \hat{\mu}(\zeta, \theta) - f_N(\zeta\theta) \} - g_N(\zeta, \theta) = 0 \]
where we have put, again formally,
\[ g_N(\zeta, \theta) = \sum_{k,\ell} \frac{Y_{k,\ell}(\theta)}{\zeta^k} c_{k,\ell;2N}; \]
note that the function \(g_N\) satisfies formally the polyharmonic equation \(\Delta^{2N+1} g_N = 0\).


5 Multidimensional Isospectral Deformation and pseudo-positive Toda lattice

We are about to define our multidimensional generalization of Toda lattice. We proceed by "reverse engineering": we first define the Toda lattice in the model variables \( \lambda \) and \( r \) of the multidimensional Moment Problem, and then by reversion we come to a formulation in the "physical" variables \( x \) and \( y \).

We will consider the measures \( d\mu(x) \) for \( x = r\theta \in \mathbb{R}^n \) and \( r = |x| \) which have the following Laplace-Fourier expansion

\[
d\mu(x) = \sum_{k,\ell} d\mu_{k,\ell}(r) Y_{k,\ell}(\theta) \quad \text{for } k \in \mathbb{N}_0, \ \ell = 1, ..., d_k. \tag{36}
\]

We have seen that in the one-dimensional case the finite Toda lattice is generated by an "isospectral deformation" of a non-negative measure \( \mu(t) \) and the identity (15), namely by

\[
f(\lambda, t) = \int \frac{d\mu(\tau; t)}{\lambda - \tau} = \frac{1}{\lambda - b_N(t)} - \frac{a_N(t)}{\lambda - b_{N-1} - ...} = Q_N(\lambda, t) \frac{P_N(\lambda, t)}{\lambda - \Lambda_j},
\]

for \( \lambda \not\in \sigma(L) \), \( t \in \mathbb{R} \). The spectrum of the measure \( d\mu \) coincides with the set of the constants \( \lambda_j \), cf. [23]. "Isospectrality" is understood as time-independence of the spectrum of the measure \( d\mu(t) \).

In a similar way, we propose to generalize the "isospectral property" by considering a time-dependent pseudo-positive measure \( \mu \) and its multidimensional Stieltjes-Markov transform \( \hat{\mu}(\zeta, \theta; t) \) on the Klein-Dirac quadric. We will have respectively the series

\[
f(\zeta, \theta; t) = \hat{\mu}(\zeta, \theta; t) = \int \frac{\zeta^{n-1} d\mu(x; t)}{r(\zeta - \theta)^n} = \sum_{k,\ell} \frac{Y_{k,\ell}(\theta)}{r} \int_0^\infty \frac{\zeta}{\zeta^2 - \lambda^2} \rho^k d\mu_{k,\ell}(r; t) \tag{37}
\]

\[
= \sum_{k,\ell} \frac{Y_{k,\ell}(\theta)}{\zeta^{k-1}} \sum_{j=1}^N \frac{\lambda_{k,\ell;j}^2(t)}{\zeta^2 - \lambda_{k,\ell;j}^2}, \quad \zeta \in \mathbb{C}, \ \theta \in \mathbb{S}^{n-1},
\]

which converges in the absolute sense. We define the spectrum of the measure \( d\mu \) as the union of the spectral of all measures \( \{\mu_{k,\ell}\}_{k,\ell} \) which is the set of all constants \( \{\lambda_{k,\ell;j}\} \). Now, by analogy with the one-dimensional case, under "isospectrality" we will understand the time-independence of all constants \( \lambda_{k,\ell;j} \).

We will consider the more convenient measures \( d\bar{\mu}_{k,\ell}(\rho; t) \) where we have put \( \rho = r^2 \), hence,

\[
d\bar{\mu}_{k,\ell}(\rho; t) = \rho^k d\mu_{k,\ell}(r; t) = \rho^{k/2} d\mu_{k,\ell}(\sqrt{\rho}; t), \tag{38}
\]

i.e.

\[
d\bar{\mu}_{k,\ell}(\rho; t) = \sum_{j=1}^N \rho_{k,\ell;j}^2(t) \frac{\lambda_{k,\ell;j}^2}{\lambda_{k,\ell;j}^2 - \rho}, \quad \rho \not\in \sigma(L).\]
We put

\[ r_{k,\ell}^2 (t) := r_{k,\ell}^2 (t) \lambda_{k,\ell,j}^k \]

\[ \tilde{\lambda}_{k,\ell,j} := \lambda_{k,\ell,j}^2. \]

By the definition of the measure \( \mu_{k,\ell} \) we obtain the equality

\[ f (\zeta, \theta; t) = \sum_{k,\ell} \frac{Y_{k,\ell} (\theta)}{\zeta_{k-1}} \int_0^\infty \frac{1}{\zeta^2 - \rho} d\tilde{\mu}_{k,\ell} (\rho; t) \]

and the Jacobi matrix for \( d\tilde{\mu}_{k,\ell} (\rho; t) \) will satisfy a one-dimensional Toda lattice equation, called below pseudo-positive Toda lattice. We now give the rigorous definition.

**Definition 18** We say that the pseudo-positive measure \( \mu (x; t), x \in \mathbb{R}^n \), given for every \( t \in \mathbb{R} \) by (36), with \( \mu_{k,\ell} (r) \) replaced by \( \mu_{k,\ell} (r; t) \), represents a pseudo-positive Toda lattice if all component measures \( \mu_{k,\ell} \) are given by

\[ d\mu_{k,\ell} (r; t) = \sum_j r_{k,\ell,j}^2 (t) \delta (r - \lambda_{k,\ell,j}) dr, \quad r \in [0, \infty), \]

where for all indices we have

\[ r_{k,\ell,j} \geq 0, \quad \lambda_{k,\ell,j} \geq 0. \]

We assume further that the system is isospectral, i.e. the spectrum \( \{ \lambda_{k,\ell,j} \}_{j=1}^N \) of all measures \( \mu_{k,\ell} (r; t) \) is constant, i.e. independent of \( t \in \mathbb{R} \). We assume that the masses \( r_{k,\ell,j} (t) \) given by (39) vary with the time \( t \) according to the Toda lattice equations (17)-(18), i.e.

\[ \tilde{\lambda}_{k,\ell,j} (t) = \tilde{\lambda}_{k,\ell,j} (0) \]

\[ r_{k,\ell,j}^2 (t) = \frac{r_{k,\ell,j}^2 (0) e^{-2\tilde{\lambda}_{k,\ell,j} t}}{\sum_{m=1}^N r_{k,\ell,m}^2 (0) e^{-2\tilde{\lambda}_{k,\ell,m} t}}, \]

with the restriction

\[ \sum_{j=1}^N r_{k,\ell,j}^2 (0) = 1 \quad \text{for all } (k, \ell). \]

(see [21], p. 223, [27]).
From (40) we obtain immediately the equalities
\[
\tilde{r}_{k,\ell}^2(t) = \frac{\lambda_k^{k,\ell} r_{k,\ell}^2(0) e^{-2\lambda_k^{k,\ell} t}}{\sum_{m=1}^{N} \lambda_k^{k,\ell} m r_{k,\ell}^2(0) e^{-2\lambda_k^{k,\ell} m t}}
\]
\[
\tilde{v}_{k,\ell}^2(t) = \frac{\lambda_k^{k,\ell} v_{k,\ell}^2(0) e^{-2\lambda_k^{k,\ell} t}}{\sum_{m=1}^{N} \lambda_k^{k,\ell} m v_{k,\ell}^2(0) e^{-2\lambda_k^{k,\ell} m t}}
\]
and from (41)
\[
\sum_{m=1}^{N} \lambda_k^{k,\ell} m r_{k,\ell}^2(0) = 1
\]

Let us define the Jacobi matrix \( L_{k,\ell} \) related to the Stieltjes transform
\[
\int_{1}^{\infty} \frac{1}{z-\rho} d\tilde{\mu}_{k,\ell}(\rho) \quad \text{for Im } z \neq 0,
\]
by putting
\[
L_{k,\ell}(t) := \left( \left( \tilde{a}_{k,\ell}^j(t), \tilde{b}_{k,\ell}^j(t) \right) \right)_{j=0}^{N};
\]
it generalizes the one-dimensional Jacobi matrix \( L \) in (11), cf. [20], p. 473, also [26]. Now we are able to define the Hamiltonians corresponding to the system of equations (13), by putting
\[
\frac{1}{2} H_{k,\ell} = 2 \left\{ \sum_{j=1}^{N-1} \tilde{a}_{k,\ell}^2 + \frac{1}{2} \sum_{j=1}^{N} \tilde{v}_{k,\ell}^2 \right\} = \sum_{j=1}^{N} \tilde{\lambda}_{k,\ell}^2 = \sum_{j=1}^{N} \lambda_{k,\ell}^2
\]

The corresponding total Hamiltonian is now given by
\[
H = \sum_{k,\ell} H_{k,\ell} = 2 \sum_{k,\ell} \sum_{j=1}^{N} \tilde{\lambda}_{k,\ell}^2 = 2 \sum_{k,\ell} \sum_{j=1}^{N} \lambda_{k,\ell}^4
\]

We obtain

**Corollary 19**

1. The variables \( \left( \tilde{a}_{k,\ell}^j, \tilde{b}_{k,\ell}^j \right) \) satisfy equations
\[
a'_{k,\ell} = a_{k,\ell} \left( b_{k,\ell+1} - b_{k,\ell} \right), \quad j = 1, 2, ..., N - 1
\]
\[
b'_{k,\ell} = 2 \left( a_{k,\ell}^2 - a_{k,\ell-1}^2 \right), \quad j = 1, 2, ..., N
\]
\[
a_{k,\ell}^2 = 0, \quad a_{k,\ell}^2 = 0
\]
which coincide with equations (1)-(3).

2. By using the one-to-one map defined by the equations (3)-(5) we find the variables \( \left( x_{k,\ell}, y_{k,\ell} \right) \) which satisfy the system of equations (20). This represents a Hamiltonian system with Hamiltonian given by (44).
From above the following important statement is easily derived, by using (39):

**Proposition 20** Let us assume that

$$C_\lambda := \sum_{k,\ell} \sum_{j=1}^{N} \lambda_{k,\ell,j}^4 < \infty. \quad (48)$$

Then for every $t \geq 0$ the Hamiltonian $H$ of the pseudo-positive Toda lattice is a real number independent of $t \in \mathbb{R}$ which is bounded by $2C_\lambda$.

The following Theorem shows that a very mild restriction on the spectrum $\{\lambda_{k,\ell,j}\}$ suffices to guarantee convergence of the multidimensional Stieltjes-Markov transform for every $t \geq 0$.

**Theorem 21** Let a pseudo-positive Toda lattice be given in the sense of Definition 18. Then the multidimensional Stieltjes-Markov transform $\hat{\mu}(\zeta, \theta; t)$ defined in (37) (of the associated pseudo-positive measure $\mu(x; t)$ defined by (36) with $\mu_{k,\ell}(r)$ replaced by the measures $\mu_{k,\ell}(r; t)$ in Definition 18) is convergent (in the sense, the integral representing it is convergent) for all $t \geq 0$ and for all $|\zeta| > 1$ with $\text{Im} \zeta^2 > 0$.

**Proof.** From (40) it follows

$$\int_0^\infty r^k d\mu_{k,\ell}(r) = \sum_{j=1}^{N} \lambda_{k,\ell,j}^2 r_{k,\ell,j}^2(t) = \sum_{j=1}^{N} \lambda_{k,\ell,j}^2 \frac{r_{k,\ell,j}^2(0) e^{-2\lambda_{k,\ell,j} t}}{\sum_{m=1}^{N} \lambda_{k,\ell,m}^2 r_{k,\ell,m}^2(0) e^{-2\lambda_{k,\ell,m} t}} = 1.$$ 

The proof is finished by Proposition 13 since condition (34) is fulfilled with $C = D = 1$.

6 The functional model for the pseudo-positive Jacobi (Schrödinger) operator

We will define an operator in a functional model generated by a given pseudo-positive measure $\mu$. This model is related to the above multidimensional Toda lattice in a way generalizing the one-dimensional considered by J. Moser in [20], [22].

We recall that in the classical spectral theory, for a non-negative measure $\mu$ on $\mathbb{R}$, the functional model is defined by the space

$$L_2(\mu) = \left\{ f : \int_{\mathbb{R}} |f(t)|^2 d\mu(t) < \infty \right\}$$
and the multiplication operator $A_t f = tf(t)$, for $t \in \mathbb{R}$, cf. [15]. We assume that the polynomials are dense in the space $L_2(\mu)$, i.e., we assume that we have a determinate Moment Problem relative to the measure $\mu$ (cf. [1]). The orthonormal basis is given by polynomials $\{P_j(t)\}_{j \geq 0}$ which are orthonormal with respect to the measure $\mu$. In this basis the operator $A_t$ is represented by the Jacobi matrix obtained through the corresponding 3-term recurrence relations. See [1], [26], [30] for the details.

In the present situation the functional model is constructed as follows: We assume that a pseudo-positive measure $\mu$ is given. We define the space $\bigoplus_{k,\ell} L_2(\mu_{k,\ell})$

$$= \left\{ f = (f_{k,\ell}), k \in \mathbb{N}_0, \ell = 1, 2, \ldots, d_k : \sum_{k,\ell} \int_0^\infty |f_{k,\ell}(r)|^2 d\mu_{k,\ell}(r) < \infty \right\} ,$$

where the measures $\mu_{k,\ell}$ are non-negative component measures on $\mathbb{R}_+$ associated with $\mu$ according to Definition 10. The operator $A_{|x|^2}$ is defined as the multiplication operator

$$A_{|x|^2} f(x) = |x|^2 f(x) \quad \text{for } x \in \mathbb{R}^n,$$

where $f \in D\left(|\cdot|^2\right) \subset L^2(\mathbb{R}^n)$; here $D$ is the appropriate domain of definition of the operator $A_{|x|^2}$. Further, we assume that all Moment Problems defined by the measures $\mu_{k,\ell}$ are determined on $\mathbb{R}_+$, cf. [12]. In this case the basis for the space $\bigoplus_{k,\ell} L_2(\mu_{k,\ell})$ is the set of polynomials

$$r^k P_{k,\ell,j} \left(r^2\right)$$

where the polynomials $\{P_{k,\ell,j}\}_{j \geq 0}$ are orthonormal relative to the measure $\tilde{\mu}_{k,\ell}$ on $\mathbb{R}_+$, defined in terms of the $d\mu_{k,\ell}$ by (38). In this basis the operator $A_{|x|^2}$ is represented by a matrix with components given by the Jacobi matrices $L_{k,\ell}$ in the spaces $L_2(\tilde{\mu}_{k,\ell})$. The representation of the Hamiltonian in (43) means that this operator is bounded in the respective $L^2$ space.

6.1 Physical meaning of the pseudo-positive Toda lattice

We consider here the pseudo-positive Toda lattice in a generalization of the Flaschka variables, and we also provide some "physical meaning". Indeed, we may consider the multidimensional counterpart of the Flaschka variables $(a_j, b_j)$ arising from equations (6)-(8), which we call generalized Flaschka variables,

$$A_j(\theta) := \sum_{k,\ell} \bar{a}_{k,\ell,j} Y_{k,\ell}(\theta) \quad \text{for } j = 1, 2, \ldots, N-1 \quad (49a)$$

$$B_j(\theta) := \sum_{k,\ell} \bar{b}_{k,\ell,j} Y_{k,\ell}(\theta) \quad \text{for } j = 1, 2, \ldots, N. \quad (49b)$$
The coefficients $\tilde{a}_{k,\ell}^j$ and $\tilde{b}_{k,\ell}^j$ may be found from the multidimensional Markov-Stieltjes transform, by means of the continued fraction expansion in (15) for every $(k, \ell)$, with $a_j$ and $b_j$ replaced by $\tilde{a}_{k,\ell}^j$ and $\tilde{b}_{k,\ell}^j$. Thus,

$$\tilde{a}_{k,\ell}^j = \int_{S^{n-1}} A_j(\theta) Y_{k,\ell}(\theta) \, d\theta \quad \text{and} \quad \tilde{b}_{k,\ell}^j = \int_{S^{n-1}} B_j(\theta) Y_{k,\ell}(\theta) \, d\theta.$$ 

**Proposition 22** The series representing $A_j(\theta)$ and $B_j(\theta)$ in (49a)-(49b) are absolutely convergent provided condition (48) is satisfied.

The proof follows from Proposition 20.

**Remark 23** Thus, the physical interpretation is the following: We have $N$ surfaces $A_j$, considered as functions on the sphere $S^{n-1}$, for $j = 1, 2, ..., N$, which interact in the sense of classical mechanics according to equations (45)-(47); we see that only the neighboring surfaces interact.

We have the following interesting representation for the Hamiltonian in (44):

$$H = \sum_{k,\ell} H_{k,\ell} = \int_{S^{n-1}} \sum_{j=1}^N \left( 4 \times |A_j(\theta)|^2 + 2 \times |B_j(\theta)|^2 \right) \, d\theta.$$ 

Alternative physical meaning may be found if we use the variables $(x_{k,\ell}^j, y_{k,\ell}^j)_{k,\ell,j}$.

### 6.2 Physical meaning in the variables $x$ and $y$

We have reformulated above our pseudo-positive Toda lattice in the generalized Flaschka "variables" $A_j(\theta)$ and $B_j(\theta)$. In view of the complicated non-linear relation between the variables $(x_j, y_j)$ and the Flaschka variables $(a_j, b_j)$ it is natural to ask, if a reasonable physical interpretation exists in the "original" variables, $x$ and $y$. A major criterion for success of the interpretation will be the convergence of the series.

Alternatively to the $(a, b)$ variables, we *introduce* the pseudo-positive Toda lattice by means of the generalized variables given by the series

$$X_j(x; \theta) := \sum_{k,\ell} e^{-x_{k,\ell}^j} Y_{k,\ell}(\theta), \quad \text{for } j = 1, 2, ..., N, \quad (50)$$

and

$$Y_j(y; \theta) := \sum_{k,\ell} y_{k,\ell}^j Y_{k,\ell}(\theta) \quad \text{for } j = 1, 2, ..., N. \quad (51)$$

We assume that the values for $x = (x_{k,\ell}^j)$ and $y = (y_{k,\ell}^j)$ are obtained from $\tilde{a}_{k,\ell}^j$ and $\tilde{b}_{k,\ell}^j$ via the Flaschka change of variables in (49). Let us recall however that the Flaschka map is *not bijective* and one has to identify equivalence classes called configurations in the variables $(x,y)$, in §4, cf. [20], p. 471.

The main justification for representation of the pseudo-positive Toda lattice in the variables $X_j$ and $Y_j$ is the following statement.
Proposition 24 The series representing $X_j(\theta)$ and $Y_j(\theta)$ in (50) and (51) respectively, are absolutely convergent in every configuration class for $(x,y)$ defined by (9), provided condition (48) is satisfied.

Proof. For proving the convergence we will need explicitly the variables $x$ and $y$. From (3)-(5) we have

$$a_j = \frac{1}{2}e^{(x_j - x_{j+1})/2},$$

hence,

$$x_1 - x_2 = 2 \ln 2a_1$$
$$x_2 - x_3 = 2 \ln 2a_2$$
$$\vdots$$
$$x_{n-1} - x_n = 2 \ln 2a_{n-1}.$$ 

Consequently, we get the following equivalent system of equations

$$x_1 - x_2 = 2 \ln 2a_1$$
$$x_1 - x_3 = 2 \ln 2a_1 + 2 \ln 2a_2$$
$$\vdots$$
$$x_1 - x_n = 2 \ln 2a_1 + 2 \ln 2a_2 + \ldots + 2 \ln 2a_{n-1},$$

which implies for $j = 2, 3, \ldots, n$ the expression

$$x_j = x_1 - 2 \ln 2^{j-1} - 2 \ln \prod_{m=1}^{j-1} a_m.$$ 

Hence, we see that $X_j(\theta)$ defined by (50) satisfies

$$X(j\theta) := \sum_{k,\ell} e^{-x_{k,\ell;1}(t)} Y_{k,\ell}(\theta)$$
$$= \sum_{k,\ell} \exp \left(-x_{k,\ell;1} + 2 \ln 2^{j-1} + 2 \ln \prod_{m=1}^{j-1} \tilde{a}_{k,\ell; m} \right) Y_{k,\ell}(\theta)$$
$$= 2^{2(j-1)} \sum_{k,\ell} \left( \prod_{m=1}^{j-1} \tilde{a}_{k,\ell; m}^2 \right) \times \exp (-x_{k,\ell;1}) Y_{k,\ell}(\theta)$$

From the inequality between geometric and arithmetic means we obtain for every integer $M \geq 1$ the following inequality:

$$\left| \prod_{m=1}^{M} \tilde{a}_{k,\ell; m}^2 \right| \leq \left( \frac{\sum_{m=1}^{M} \tilde{a}_{k,\ell; m}^2}{M} \right)^M.$$
From condition (48) we obtain
\[
\sum_{k,\ell} a_{k,\ell;1}^2 < \infty.
\]

On the other hand, recall that the Flaschka map (3)–(5) is not on-to-one and all \(x_{k,\ell;j}\) are only determined up to a constant \(C_{k,\ell}\), i.e. \((x_{k,\ell;j} + C_{k,\ell})_{j=1}^N\) belong to the same class; hence, we may choose \(x_{k,\ell;1}\) in such a way that
\[
|e^{-x_{k,\ell;1}} Y_{k,\ell}(\theta)| \leq C \quad \text{for all } (k, \ell);
\]
in particular we may choose \(e^{-x_{k,\ell;1}} = k^{-(n-2)}\), or \(x_{k,\ell;1} = \ln (k^{n-2})\). Thus for every class of equivalence we find a convergent series for \(X_j(\theta)\).

Finally, we comment on a physical interpretation of the above representation.

\textbf{Remark 25} In analogy to the classical Toda lattice, we may assign some \textit{physical meaning} (although not so directly understood) to the set of surfaces \(X_j(\theta)\) and \(Y_j(\theta)\) over the sphere, by considering them as mass-surfaces which interact by means of the Hamiltonian system for the infinite set of variables \((x_{k,\ell;j}, y_{k,\ell;j})_{k,\ell;j}\) provided in (2a), with Hamiltonian (44). Let us remark that these equations may be considered as a non-linear system of pseudodifferential equations on the sphere with respect to the functions \(X(j\theta) = X_j(\theta)\) and \(Y(j\theta) = Y_j(\theta)\).

\section{A class of isospectral measures}

There is an important class of measures which has been identified in [12]:

\textbf{Definition 26} We say that the pseudo-positive measure \(\mu\) satisfies the integrability condition iff the following inequality holds
\[
\sum_{k,\ell} \int_0^\infty \frac{d\mu_{k,\ell}(r)}{r^k} < \infty \quad \text{for } k \in \mathbb{N}_0, \ell = 1, ..., d_k. \quad (53)
\]

In [12] it has been proved that for such measures we may construct a Gauss-Jacobi measure on \(\mathbb{R}^n\) which provides a generalization of the Gauss quadrature formula on \(\mathbb{R}\). The new formula has been called polyharmonic Gauss-Jacobi cubature formula.

Further we will consider measures \(\mu(x;t)\) which depend on both the space variable \(x\) and the time variable \(t\) and which have the isospectral property in every component \(\mu_{k,\ell}(r;t)\), namely for every index \((k, \ell)\) we consider the measure on \(\mathbb{R}_+\)
\[
d\mu_{k,\ell}(\rho;t) = \sum_{j=1}^N r_{k,\ell,j}(t) \delta(\rho - \lambda_{k,\ell,j}(t)), \quad \rho \in \mathbb{R}_+, t \in \mathbb{R},
\]
where the functions \( r_{k,\ell,j}(t) \geq 0 \) and \( \lambda_{k,\ell,j}(t) \geq 0 \) satisfy the equations of the Toda flow:

\[
\frac{d}{dt} \lambda_{k,\ell,j}(t) = 0, \quad \frac{d}{dt} r_{k,\ell,j}(t) = -\lambda_{k,\ell,j}(t) r_{k,\ell,j}^2(t).
\]

We see from the first equation that \( \lambda_{k,\ell,j} \) are constants (with respect to the variable \( t \)), hence the second equation becomes

\[
\frac{d}{dt} r_{k,\ell,j}(t) = -\lambda_{k,\ell,j} r_{k,\ell,j}^2(t) \quad (54)
\]

with \( \lambda_{k,\ell,j} = \lambda_{k,\ell,j}(0) \) for \( t \in \mathbb{R} \). We consider the measure defined by the formal series

\[
d\mu(x; t) = \sum_{k,\ell} d\mu_{k,\ell}(\rho; t) Y_{k,\ell}(\theta)
\]

for which it is not \( a \) priori clear that the series is convergent in any sense. We prove its convergence in the following theorem.

**Theorem 27** Let the measure \( d\mu(x; 0) \) satisfy the integrability condition \((53)\). Then for every \( t > 0 \) the measure \( d\mu(x; t) \) satisfies the integrability condition \( (53) \).

**Proof.** Let us consider

\[
S_{k,\ell}(t) := \int_{0}^{\infty} \frac{d\mu_{k,\ell}(\rho; t)}{\rho^k} = \sum_{j=1}^{N} \frac{r_{k,\ell,j}^2(t)}{\lambda_{k,\ell,j}^k}, \quad t \in \mathbb{R}.
\]

We obtain for its derivative, using the equation of motion \((54)\),

\[
\frac{d}{dt} S_{k,\ell}(t) = \frac{d}{dt} \sum_{j=1}^{N} \frac{r_{k,\ell,j}^2(t)}{\lambda_{k,\ell,j}^k} = 2 \sum_{j=1}^{N} r_{k,\ell,j}(t) \frac{d}{dt} r_{k,\ell,j}(t) = -2 \sum_{j=1}^{N} \frac{r_{k,\ell,j}(t)}{\lambda_{k,\ell,j}} \lambda_{k,\ell,j} r_{k,\ell,j}^2(t)
\]

\[
= -2 \sum_{j=1}^{N} \frac{r_{k,\ell,j}^3(t)}{\lambda_{k,\ell,j}^k}, \quad \text{for } k \in \mathbb{N}_0, \ \ell = 1, \ldots, d_k.
\]

By definition the functions \( r_{k,\ell,j}(t) \) and \( \lambda_{k,\ell,j} \) are non-negative, hence it follows that \( S_{k,\ell}(t) \) is a decreasing function of \( t \). Hence,

\[
\sum_{k,\ell} \int_{0}^{\infty} \frac{d\mu_{k,\ell}(\rho; t)}{\rho^k}
\]

is a decreasing function of \( t \) which proves our statement, according to Definition \(20\).

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