THE DIMER PARTITION FUNCTION

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Abstract

We apply the Ginzburg criterion to the dimer problem and we solve the apparent contradiction of a system with mean field $\alpha = \frac{1}{2}$, the typical value of tricritical systems, and upper critical dimension $D_{cr} = 6$. We find that the system has upper critical dimension $D_{cr} = 6$, while for $D \leq 4$ it should undergo a first order phase transition. We comment on the latter wrong result examining the approximation we used.

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In this letter we would like to show how it is possible to recover the upper critical dimension of the dimer system without using renormalization group arguments but with the use of the Ginzburg criterion. In this way we solve what could appear as a contradiction: the mean field critical exponent $\alpha = \frac{1}{2}$ and the upper critical dimension $D_{cr} = 6$; in fact $\alpha = \frac{1}{2}$ is the typical value of the mean field critical exponent of the tricritical transitions that have upper critical dimension 3. Differently from previous works ([3, 4]) we do not use renormalization group arguments.

We consider the following action defined on a lattice $G$:

$$Z_{\text{dimer}} = \int \prod_{l \in G} d\tilde{\epsilon}_l \ d\epsilon_l \ \exp \left( -\mu \sum_i \tilde{\epsilon}_i \epsilon_i - \frac{K}{2} \sum_{i,j} A_{ij} \tilde{\epsilon}_i \epsilon_i \tilde{\epsilon}_j \epsilon_j \right)$$

(1)

where $A_{ij}$ is the adjacency matrix of the lattice $G$, $\tilde{\epsilon}_i = (\epsilon_i)^*$ are complex grassman variables defined on the vertex $i$. It is easy to show that the partition function (1) is the generating function for the the dimer problem with negative activity, more explicitly:

$$Z_{\text{dimer}} = \mu V \sum_{D} N(D) \left( -\frac{K}{\mu^2} \right)^D$$

(2)

where $D$ is the number of dimers and $N(D)$ is the number of possible configurations with $D$ dimers ($V$ is the number of lattice sites).

To prove this assertion we consider the high temperature expansion (HTE) in the variable $\frac{K}{\mu^2} \sim \frac{1}{T}$. Now if a term of the HTE gives a non vanishing contribution to the partition function $Z$, it must have at most one active link (dimer) per each vertex since we cannot put two dimers on the lattice sharing a vertex because of the relation $(\tilde{\epsilon}_i \epsilon_i)^2 = 0$. As an observation we want to notice that setting $\mu = 0$ we can get the close packing dimer problem that it is known to be equivalent to Ising and hence, in this case, the theory has critical dimension $D_{cr} = 4$ ([2, 3]).

Before we can use Ginzburg criterion, we must rewrite the partition function (1) in a suitable form for the application of the saddle point expansion.

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1 We use $i, j, \ldots$ to indicate sites (vertices) of the lattice.

2 To this purpose we have to introduce the terminal lattice of the expanded version of the original lattice, then we divide the vertices into two sets: the first one $G_1$ containing the vertices of the original lattice and the other $G_2$ with the new vertices. Finally we
To this purpose we introduce the variable \( R_i = \sqrt{K} \bar{\epsilon}_i \epsilon_i \) and we use the identity
\[
\int d\bar{\epsilon} d\epsilon \delta(R - \sqrt{K} \bar{\epsilon} \epsilon) = \sqrt{K} \frac{d\delta(R)}{dr}
\] (3)
in order to rewrite (1) as follows
\[
Z_{\text{dimer}} = (-\sqrt{K})^\nu \left( \prod_{k \in G} \frac{\partial}{\partial R_k} \right) \exp\left( -\frac{\mu}{\sqrt{K}} \sum_i R_i - \frac{1}{2} \sum_{i,j} A_{ij} R_i R_j \right) \bigg|_{R_k = 0}
\]

If we perform the substitutions \( R_i \rightarrow -R_i \) and \( \frac{\mu}{\sqrt{K}} \rightarrow \mu \), we use the fact that the derivatives act on an analytic function, we can finally rewrite the previous equation as
\[
Z_{\text{dimer}} = (\sqrt{K})^\nu \oint_{\Gamma} \prod_{k \in G} \frac{dz_k}{2\pi i} \exp\left( \sum_i \mu z_i - 2 \log(z_i) - \frac{1}{2} \sum_{i,j} A_{ij} z_i z_j \right)
\] (4)
where \( \Gamma \) is an hypersurface surrounding the origin.

We introduce the notation
\[
S = \sum_i \mu z_i - 2 \log(z_i) - \frac{1}{2} \sum_{i,j} A_{ij} z_i z_j
\]
\[
S_i = -2 \frac{z_i}{z_i} + \mu - \sum_j A_{ij} z_j
\]
\[
S_{ij} = 2 \frac{\delta_{ij}}{z_i^2} - A_{ij}
\]

Under the hypothesis of a translationally invariant solution, the saddle point equations \( S_i = 0 \) yield the solutions
\[
z_{\pm} = \frac{\mu \pm \sqrt{\mu^2 - 8q}}{2q}
\] (5)
generalize the kinetic term as follows:
\[
A_{ij} \mapsto \tilde{A}_{ij} = \begin{cases} J & \text{if } i \text{ and } j \text{ are neighbours and both belong to lattice } G_1 \\ 1 & \text{if } i \text{ and } j \text{ are neighbours} \end{cases}
\]
then there exists a critical value of \( J \) at which a second order phase transition occurs.
where for \( \mu > 0 \) \( z_- \) is a minimum and \( z_+ \) is a maximum and the critical value is attained for \( \mu_{cr}^2 = 8q \) where the two solutions coalesce.

Let us now compute the critical exponent \( \alpha \). It turns out that in the mean field approximation this critical exponent is independent on the choice of \( z_+ \) or \( z_- \), nevertheless we must choose \( z_- \) (the minimum); an explanation of that is delayed to the computation of the first loop corrections. It is easy to obtain

\[
F^{(0)} = \frac{\log Z^{(0)}}{y} = -\mu z_- - 2 \log(z_) - \frac{q}{2} \frac{z_-^2}{2}. \tag{6}
\]

and if we identify \( T = \mu^2 \), we also get

\[
E^{(0)} = \frac{1}{2} \mu^3 z_-, \\
C^{(0)} = \frac{3}{4} \mu z_- + \frac{1}{4} \mu^2 \frac{dz_-}{d\mu}. \tag{7}
\]

It is now immediate to find the behaviour at the critical point (\( \delta T = \mu^2 - \mu_{cr}^2 = \mu^2 - 8q \)):

\[
E^{(0)} = 2\mu_{cr}^2 - 2\mu_{cr} \sqrt{\delta T} + O(\delta T), \\
C^{(0)} = -\frac{\mu_{cr}}{\sqrt{\delta T}} + O(1). \tag{8}
\]

Form these equations we can read immediately that the mean field value \( \alpha = \frac{1}{2} \) as it should be.

Let us now turn to the exam of the first loop correction. In order to decide which of the two stationary points we must choose, we use the observation that if we cannot find a proper path \( \gamma \) for the function \( S(z) = S|_{z_i = z_-} \), we surely cannot find a proper hypersurface \( \Gamma \). Now since the path \( \gamma \) has to surround the origin and \( z_+ \) is a maximum, we cannot find a proper path \( \gamma \) for \( S(z) \) that crosses its saddle but only paths that raise and descend the saddle on the same side, it follows that there is no proper path \( \gamma \) and hence no proper \( \Gamma \). It follows that we must choose \( z_- \); it is not difficult to check that the path \( \gamma(t) = z_- e^{it} \) with \( 0 \leq t < 2\pi \) has all the characteristics necessary for the application of the saddle point method.

If we compute the first loop correction on an hypercubic lattice, we find

\[
F^{(1)} = \frac{1}{2} \log \pi - \frac{1}{2} \int_{-\pi}^{\pi} \frac{d^D p}{(2\pi)^D} \log \left( \frac{2}{z_-^2} - 2 \sum_\nu \cos(p_\nu) \right). \tag{9}
\]
from which we can compute both the correction to the energy and to the specific heat due to the first loop correction, explicitly we have

\[ E^{(1)} = - \left( \frac{\mu}{z} \right)^3 \frac{dz}{d\mu} \frac{dF^{(1)}}{d(1/z^2)} = \]

\[ = - \left( \frac{\mu}{z} \right)^3 \frac{1}{q} \left( 1 - \frac{2\mu}{\sqrt{\mu^2 - \mu_{cr}^2}} \right) \int_{-\pi}^{\pi} \frac{d^D p}{(2\pi)^D} \frac{1}{2z - 2 \sum \cos(p_\nu)} \]

and

\[ C^{(1)} = \frac{3\mu}{2z^2} \frac{dz}{d\mu} \frac{dF^{(1)}}{dz^2} + \frac{2\mu^2}{z^4} \left( \frac{dz}{d\mu} \right)^2 \frac{dF^{(1)}}{dz^2} - \frac{\mu^2}{2z^2} \frac{d^2z}{d\mu^2} \frac{dF^{(1)}}{dz^2} - \frac{\mu^2}{2z^2} \frac{dz}{d\mu} \frac{d^2F^{(1)}}{dz^2} = \]

\[ = - \frac{\mu^4}{qz^2} \frac{1}{\sqrt{\mu^2 - \mu_{cr}^2}} \int_{-\pi}^{\pi} \frac{d^D p}{(2\pi)^D} \frac{1}{2z - 2 \sum \cos(p_\nu)} \]

\[ - \frac{2\mu^2}{qz^2} \frac{1}{\sqrt{\mu^2 - \mu_{cr}^2}} \int_{-\pi}^{\pi} \frac{d^D p}{(2\pi)^D} \frac{1}{(2z - 2 \sum \cos(p_\nu))^2} + \ldots \]

(11)

Using the fact that \( \frac{1}{z} - \frac{1}{z_{cr}} \sim \frac{2q}{\mu_{cr}^2} \sqrt{\delta T} \) we easily deduce the critical behaviour of these quantities:

\[ F^{(1)} \sim (\sqrt{\delta T})^{\frac{D}{2}} \log(\delta T) \]

\[ E^{(1)} \sim (\sqrt{\delta T})^{\frac{D}{2} - 2} \]

\[ C^{(1)} \sim (\sqrt{\delta T})^{\frac{D}{2} - 4} \]

(12)

From the first equation we deduce immediately that we should have a first order transition for \( D \leq 4 \), while from the comparison between the second one and the second of (8) we get the condition \( \frac{D}{2} - 4 > -1 \), i.e. \( D > 6 \).

Now we would comment about the rigour of these results. The first observation is that we applied the saddle point expansion without any "large" parameter and hence we should keep in account also the higher order, as we will show the higher order give milder divergences on \( \alpha \) provided \( D \geq 6 \). To show this we notice that the coefficient of a generic \( n \)-vertex is \( \sim \frac{1}{z^n} \) and that the generic contribution to the \( L \) loop correction to \( F^{(L)} \) for \( \mu \sim \mu_{cr} \) is

\[ F^{(L)} \sim \prod_{n \geq 3} \left( \frac{1}{z^n} \right)^{V_n} \int d^D p_1 \ldots d^D p_L \frac{1}{\Delta(\frac{1}{z}) + q_1^2} \ldots \frac{1}{\Delta(\frac{1}{z}) + q_L^2} \]

(13)
where $V_n$ is the number of $n$ vertices and $I$ is the number of internal lines. Using $\Delta(\frac{1}{z}) \sim \sqrt{\delta T}$, we get immediately

$$
F^{(L)} \sim \left( \frac{\sqrt{\delta T}}{z} \right)^{D_L - I - 1}
$$

$$
E^{(L)} \sim \left( \frac{\sqrt{\delta T}}{z} \right)^{D_L - I - 2}
$$

$$
C^{(L)} \sim \left( \frac{\sqrt{\delta T}}{z} \right)^{D_L - I - 2}
$$

(14)

With the help of $2I = \sum_n nV_n$ (the graphs have no external legs) and of $L = I - \sum_n V_n + 1$, we find $DL - 2I = D + \frac{1}{2} \sum_{n \geq 3} (D(n - 2) - 2n)V_n$ and hence for $D \geq 6$ it follows that $DL - 2I \geq D$. Finally we get $-\alpha^{(L)} \geq \frac{D}{2} - 2 \geq 1$ and similarly we find that the energy cannot diverge. All this means that the method is selfconsistent and that for $D < 6$ the fluctuation are important and the renormalization group is needed. As far as the dimension where the first order phase transition takes place, we cannot be sure that the found value 4 is the correct one because from the previous discussion we know that this value is deep inside the region where the analysis is not more reliable. The proof of the fact that in $D = 4$ no first order phase transition takes place was done long time ago in (4) in connection to the Yang-Lee edge singularity. In this work it was performed an analysis of the scaling dimension of the irrelevant operators in $D = 6$, like $\phi^4$, near the fixed point in lower dimension: the result is that the $\phi^4$ term is still irrelevant near four dimension. Obviously the dimer problem and the Yang-Lee edge singularity problem are equivalent, i.e. are in the same universality class, until the dimers have a second order phase transition but for $D < 3$ this is no longer true and they can be different as they actually are: for instance in $D = 1$ we have $\sigma_{YL} = -\frac{1}{2}$ and $\sigma_{dimer} - 1 = \sigma_{dimer} = 1$.

There is also another possible approach in order to justify this result: we could imagine of replacing $S$ by $NS$ in (4) and then let $N \to \infty$, this is the point of view advocated in (5), for instance. Performing backward the steps that lead from (4) to (4), we discover that this substitution is equivalent to perform the substitution $\Pi_{l \in G} d\epsilon_l d\epsilon_l \to \Pi_{A=1}^{A=2N-1} d\epsilon^A_i d\epsilon^A_i$ and $\epsilon \to \epsilon^A$.

\footnote{This is easily understandable looking to the number of derivatives implied by $-2N \log(z)$ in (4) and then looking for the number of grassman variables needed to have such a number of derivatives form a formula similar to (4).}
\[ \sum_{A=1}^{A=2N-1} \bar{\bar{\epsilon}}^A \epsilon^A \text{ in (1)}, \]
where \( A \) labels different replica of the original grassman variables \( \bar{\bar{\epsilon}} \) and \( \epsilon \). This in turn would imply a different HTE expansion: we would get a gas of branched polymers with loops made of \( N \) different types of monomers, each type being selfavoiding and having a negative activity. Consequently we would describe very different objects when \( N \gg 1 \) and hence this point of view is not completely satisfactory in this case.

In conclusion we show that there is no contradiction between \( \alpha = \frac{1}{2} \) and \( D_{cr} = 6 \) and that the apparent paradox is due to the fact that the dependence on \( \sqrt{\delta T} \) does not cancel in the free energy differently from what happens in the usual Landau-Ginzburg.

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