On the chiral anomaly in non-Riemannian spacetimes

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Abstract

The translational Chern-Simons type three-form coframe $\wedge$ torsion on a Riemann-Cartan spacetime is related (by differentiation) to the Nieh-Yan four-form. Following Chandia and Zanelli, two spaces with non-trivial translational Chern-Simons forms are discussed. We then demonstrate, firstly within the classical Einstein-Cartan-Dirac theory and secondly in the quantum heat kernel approach to the Dirac operator, how the Nieh-Yan form surfaces in both contexts, in contrast to what has been assumed previously.

1. INTRODUCTION

We will address a question within metric-affine gravity, the gauge theory of the affine group $R^4 \cong GL(4, R)$. In this framework, the linear groups $SL(3, R)$ [1] and $SL(4, R)$ [2]...
play decisive roles for the representation of matter fields. Very early, L. Biedenharn and his collaborators [3] or students [4] investigated the half-integer representations (of the covering group) of the $SL(3, R)$. These representations turned out to be important for metric-affine gravity, see [5]. We dedicate our article to the memory of L. Biedenharn.

In [6], following the earlier paper [7], it has been proposed that, in a Riemann-Cartan spacetime, the translational Chern-Simons term [8,5] may play a role in the chiral anomaly. However, in the context of this derivation, a massless Dirac field is used which is, in addition to the Poincaré invariance, also invariant under scale transformations. Therefore the appropriate spacetime arena is a Weyl-Cartan space. This space has an additional Weyl covector $Q$ which is expected to occur in the corresponding anomaly term.

In this paper we analyze geometrical and physical models which provide some further evidence in support of possible “topological” manifestations of torsion and nonmetricity on the classical and the quantum level.

2. TOPOLOGICAL INVARIANTS IN FOUR DIMENSIONS

In metric-affine gravity, the $GL(4, R)$ Chern-Simons form reads (see, e.g., ref. 8)

$$C_{RR} := -\frac{1}{2} \left( \Gamma_\alpha^\beta \wedge d\Gamma_\beta^\alpha - \frac{2}{3} \Gamma_\alpha^\beta \wedge \Gamma_\gamma^\gamma \wedge \Gamma_\gamma^\alpha \right),$$  \hspace{1cm} (2.1)

where $\Gamma_\alpha^\beta$ is the linear connection. The usual Pontrjagin (Chern) topological invariant is the exterior derivative of it,

$$-\frac{1}{2} R_\alpha^\beta \wedge R_\beta^\alpha = dC_{RR}.$$  \hspace{1cm} (2.2)

It is straightforward to prove that the cohomology class of the Pontrjagin four-form (2.2) is independent of the connection. In the metric-affine space, we can perform a decomposition of all geometrical objects in Riemannian and post-Riemannian contributions. Denote the distortion one-form by $N_\alpha^\beta$. It describes the difference between the general linear connection and the Levi-Civita (Riemannian) connection $\tilde{\Gamma}_\alpha^\beta$:
\[ \Gamma_\alpha^\beta = \tilde{\Gamma}_\alpha^\beta + N_\alpha^\beta. \] (2.3)

According to (2.3), the curvature decomposes as follows:

\[ R_\alpha^\beta = \tilde{R}_\alpha^\beta + \tilde{D}N_\alpha^\beta - N_\alpha^\lambda \wedge N_\lambda^\beta. \] (2.4)

Accordingly we find

\[ R_\alpha^\beta \wedge R_\beta^\alpha = \tilde{R}_\alpha^\beta \wedge \tilde{R}_\beta^\alpha + d \left[ N_\alpha^\beta \wedge \left( 2\tilde{R}_\beta^\alpha + \tilde{D}N_\beta^\alpha - \frac{2}{3}N_\beta^\lambda \wedge N_\lambda^\alpha \right) \right]. \] (2.5)

Thus the difference of the total Pontrjagin form and the purely Riemannian one is an exact form which proves their topological equivalence, see [9]. Incidentally, deformed Euler and Pontrjagin forms are discussed in [10] in the context of establishing the dynamical scheme for Poincaré gauge gravity.

Direct calculations shows that the axial anomaly is expressed in terms of the Riemannian Pontrjagin form. However, it is well known that there exists an additional topological invariant in four dimensions: The ‘Nieh–Yan four-form’ [11] which is defined as the exterior derivative of the translational Chern-Simons term

\[ C_{\text{TT}} := \frac{1}{2\ell^2} \vartheta^\alpha \wedge T_\alpha, \] (2.6)

see [8]. In a generic metric–affine spacetime, the Nieh–Yan form reads:

\[ dC_{\text{TT}} = \frac{1}{2\ell^2} \left( T_\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta - Q_{\alpha\beta} \wedge \vartheta^\alpha \wedge T^\beta \right), \] (2.7)

where \( \ell \) is the Planck length. In a Hamiltonian formulation à la Ashtekar, this serves as a generating function for self-dual or chiral variables in gravity [12].

Consider the first Bianchi identity, \( DT^\alpha = R_\beta^\alpha \wedge \vartheta^\beta \). Its third irreducible piece supplies only one equation [3],

\[ ^{(3)}R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta = d \left( g_{\alpha\beta} \vartheta^\alpha \wedge T^\beta \right) + \left( Q_{\alpha\beta} \wedge \vartheta^\beta - T_\alpha \right) \wedge T^\alpha \]
\[ = d \left( \vartheta_\alpha \wedge \left( ^{(3)}T^\alpha \right) \right) + \left( ^{(2)}Q_{\alpha\beta} \wedge \vartheta^\beta - (1)^{(1)}T_\alpha \right) \wedge (1)^{(1)}T^\alpha \]
\[ + \left( ^{(3)}Q_{\alpha\beta} \wedge \vartheta^\beta + (4)^{(4)}Q_{\alpha\beta} \wedge \vartheta^\beta - 2^{(2)}T_\alpha \right) \wedge (3)^{(3)}T^\alpha. \] (2.8)
Here the irreducible pieces are denoted by superscripts enclosed by parentheses. Specifically, the axial torsion is described by the third irreducible part, \((3)T^\alpha = \frac{1}{3}*(\vartheta^\alpha \wedge A)\), where the axial one-form is defined by \(A := -*(\vartheta^\alpha \wedge T^\alpha)\). Note that \((3)R_{\alpha\beta} = \frac{1}{2}R_{[\alpha\beta\gamma\delta]}\vartheta^\gamma \wedge \vartheta^\delta\), i.e., its components correspond to the totally antisymmetric part of the curvature. The expression in the last line of (2.8) is typical for the post-Riemannian decompositions,

\[
(3)Q_{\alpha\beta} \wedge \vartheta^\beta + (4)Q_{\alpha\beta} \wedge \vartheta^\beta - 2\ (2)T_\alpha = \frac{1}{3}\vartheta_\alpha \wedge (\Lambda - 3Q - 2T),
\]

where

\[
\Lambda := (e^\beta |Q_{\alpha\beta})\vartheta^\alpha, \quad Q_{\alpha\beta} := Q_{\alpha\beta} - Qg_{\alpha\beta}, \quad Q := \frac{1}{4}Q_\alpha^\alpha, \quad T := e_\alpha |T^\alpha.
\]

If the torsion is totally antisymmetric, \(T^\alpha = (3)T^\alpha\) (pure “axial” torsion), and the non-metricity is purely Weyl, i.e., \(Q_{\alpha\beta} = 0\), then, in this Weyl-Cartan spacetime, we have

\[
(3)R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta = (d + Q)\left(\vartheta_\alpha \wedge (3)T^\alpha\right)
\]

or

\[
\frac{1}{2}R_{[\alpha\beta\gamma\delta]}\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma \wedge \vartheta^\delta = (d + Q)*A.
\]

The proof of (2.8) goes along the lines mentioned in the context of [3] Eqs.(B.2.19) and (B.5.14).

3. SPACETIME WITH NONTRIVIAL NIEH-YAN FOUR-FORM

Consider the \(SO(4)\)-invariant Riemannian metric (with Euclidean signature),

\[
g = h^2 dr^2 + f^2 \left[ d\psi^2 + \sin^2 \psi \left( d\theta^2 + \sin^2 \theta d\phi^2 \right) \right],
\]

where \(h = h(r), f = f(r)\), and \((r, \psi, \theta, \phi)\) are the standard hyperspherical coordinates which parameterize the unit three-sphere \(S^3\),

\[
0 \leq \phi < 2\pi, \quad 0 \leq \theta < \pi, \quad 0 \leq \psi < \pi.
\]
In order to describe the metric-affine spacetime, we need to specify the gravitational potentials \( (g_{\alpha\beta}, \vartheta^{\alpha}, \Gamma_{\alpha}^{\beta}) \). We choose these fields as follows: the metric as

\[
g_{\alpha\beta} = \delta_{\alpha\beta},
\] (3.3)

the coframe as the “square root” of (3.1),

\[
\vartheta^3 = h \, dr, \quad \vartheta^1 = f \, d\psi, \quad \vartheta^2 = f \sin \psi \, d\theta, \quad \vartheta^3 = f \sin \psi \sin \theta \, d\phi,
\] (3.4)

and the linear connection eventually as

\[
\Gamma_1^{\ 2} = - \Gamma_2^{\ 1} = \cos \psi \, d\theta - \sin \psi \sin \theta \, d\phi, \quad (3.5)
\]
\[
\Gamma_3^{\ 1} = - \Gamma_1^{\ 3} = - \sin \psi \, d\theta - \cos \psi \sin \theta \, d\phi, \quad (3.6)
\]
\[
\Gamma_2^{\ 3} = - \Gamma_3^{\ 2} = - d\psi + \cos \theta \, d\phi. \quad (3.7)
\]

The specifications (3.3)-(3.7) describe a manifold \( M \) with the line element (3.1), with vanishing nonmetricity and curvature,

\[
Q_{\alpha\beta} = 0, \quad R_{\alpha}^{\beta} = 0, \quad (3.8)
\]

but with nontrivial torsion,

\[
T^1 = \frac{1}{f} \left( \frac{df}{dr} \, dr \wedge \vartheta^1 - 2 \vartheta^2 \wedge \vartheta^3 \right),
\] (3.9)
\[
T^2 = \frac{1}{f} \left( \frac{df}{dr} \, dr \wedge \vartheta^2 - 2 \vartheta^3 \wedge \vartheta^1 \right),
\] (3.10)
\[
T^3 = \frac{1}{f} \left( \frac{df}{dr} \, dr \wedge \vartheta^3 - 2 \vartheta^1 \wedge \vartheta^2 \right).
\] (3.11)

For \( 2h = \pm df/dr \), the torsion is self- or anti-self-dual.

Substituting this into (2.7), we find for the Nieh-Yan four-form:

\[
\frac{1}{2\ell^2} T_\alpha \wedge T^\alpha = - \frac{6}{\ell^2} \frac{df}{dr} f \sin^2 \psi \sin \theta \, dr \wedge d\psi \wedge d\theta \wedge d\phi.
\] (3.12)

If integrated, this generically yields a nontrivial value for the invariant

\[
\frac{1}{2\ell^2} \int_M T_\alpha \wedge T^\alpha.
\] (3.13)
For the function $f(r)$ one could expect the instanton type of behavior

$$f = \frac{ar^2}{r^2 + c^2},$$

for example, where $a$ and $c$ are constants. However, since we do not discuss any gravitational field equations, it should be clear that (3.14) is but one example.

**A. Zero connection gauge**

In a metric-affine spacetime, the connection can be “gauged away” at any one point by a suitable local linear transformation of the frame. However, in the teleparallel case under consideration, when the curvature is trivial everywhere, see (3.8), one can always choose a gauge in which the connection is vanishing \textit{globally}, $\Gamma^\alpha_{\alpha\beta} = 0$. A convenient way to demonstrate this is to recall, following [6], that the hyperspherical coordinates are related to the Cartesian coordinates $(x^1, x^2, x^3, x^4)$ via

$$x^1 = r \sin \psi \sin \theta \sin \phi,$$

$$x^2 = r \sin \psi \sin \theta \cos \phi,$$

$$x^3 = r \sin \psi \cos \theta,$$

$$x^4 = r \cos \psi,$$

such that $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$. Now define a new coframe:

$$\vartheta^4 = h \, dr,$$

$$\vartheta^1 = \frac{f}{r^2} \left( -x^2 dx^1 + x^1 dx^2 + x^4 dx^3 - x^3 dx^4 \right),$$

$$\vartheta^2 = \frac{f}{r^2} \left( x^3 dx^1 + x^4 dx^2 - x^1 dx^3 - x^2 dx^4 \right),$$

$$\vartheta^3 = \frac{f}{r^2} \left( x^4 dx^1 - x^3 dx^2 + x^2 dx^3 - x^1 dx^4 \right).$$

It is straightforward to check that (3.19)-(3.22) and

$$g_{\alpha\beta} = \delta_{\alpha\beta}, \quad \Gamma^\alpha_{\alpha\beta} = 0,$$

(3.23)
define the same metric-affine spacetime with (3.1) as line element, trivial nonmetricity and curvature (3.8), and a non-zero torsion which now equals the anholonomy 2-form:

\[ T^\alpha = d\vartheta^\alpha. \] 

(3.24)

In components, we have again (3.9)-(3.11), only the coframe (3.4) is replaced by (3.19)-(3.22), and thus the Nieh-Yan four-form (3.12) and the integral torsion invariant (3.13) are exactly the same. The coframe (3.19)-(3.22) was discussed by Chandia and Zanelli [6] who took, however, \( f = 1 \) which, as is clear from (3.12), makes the Nieh-Yan four-form vanish identically.

B. Parallelizability and higher-dimensional example

Since the Nieh-Yan form is constructed from the translational Chern-Simons 3-form,

\[ C_{TT} = \frac{1}{2\ell^2} \vartheta^\alpha \wedge T_\alpha = -\frac{1}{2\ell^2} * A, \] 

(3.25)

the new invariant (3.13) can be calculated as the integral

\[ \int_{S^3} C_{TT} \] 

(3.26)

over the three-sphere of infinite radius. In our example,

\[ C_{TT} = -\frac{\ell^2(\infty)}{2\ell^2} C_{TT}, \quad C_{TT} = \vartheta^a \wedge D\vartheta_a, \] 

(3.27)

where \( \vartheta^a \) is a three-coframe which lives on the three-sphere \( S^3 \) of unit radius. Here, the underline denotes the geometrical objects on this \( S^3 \), and the Latin indices, from the beginning of the alphabet, run over \( a, b, \ldots = 1, 2, 3 \). In our construction, we are thus naturally using the parallelizability of the three-sphere. Everything looks particularly simple in the zero-connection gauge, when

\[ C_{TT} = \vartheta^a \wedge d\vartheta_a = -6 \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3. \] 

(3.28)

Thus integration yields
\[ \int_{S^3} C_{TT} = -12 \pi^2, \]  

(3.29)

since the volume of a sphere is

\[ \text{vol}(S^{n-1}) = \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2} + 1\right)}, \]  

(3.30)

that is, \( \text{vol}(S^3) = 2\pi^2 \) and \( \text{vol}(S^7) = \frac{4}{3}\pi^4 \).

These observations may be useful in discovering a further higher-dimensional example. As was noticed in [6], the next higher dimension, in which an analog exists of the Nieh-Yan form, is 8. As is well known, among the spheres only \( S^1, S^3, \) and \( S^7 \) exhibit teleparallelism, i.e. are “parallelizable”. Trautman [13] suggested to us to investigate the Dirac operator on the parallelizable non-trivial spheres \( S^3 \) and \( S^7 \). One of us [14] did this and those techniques are also useful here in our context.

The anticipated generalization of the translational Chern-Simons term is a seven-form

\[ C_{TTTT} = \vartheta^a \wedge d\vartheta_a \wedge d\vartheta^b \wedge d\vartheta_b, \]  

(3.31)

and we need to integrate it over a seven-sphere. The indices evidently run now from \( a, b = 1, \ldots, 7 \).

A convenient unified framework for treating all cases for \( S^n, n = 1, 3, 7, \) is provided by the Clifford algebra approach. For \( n = 1, 3, 7, \) the Clifford algebra over flat Euclidean space \( \mathbb{R}^n \) permits exactly an \( n + 1 \)-dimensional real representation. The corresponding generators are real \((n+1) \times (n+1)\) matrices \( \gamma_1, \ldots, \gamma_n \) which satisfy the relations:

\[ \gamma_a \gamma_b + \gamma_b \gamma_a = -2 \delta_{ab}, \]  

(3.32)

\[ \langle \gamma_a v, \gamma_a w \rangle = \langle v, w \rangle \quad \text{for all } v, w \in \mathbb{R}^{n+1}, \]  

(3.33)

\[ \gamma_1 \cdots \gamma_n = \pm 1 \quad \text{(for } n = 3, 7). \]  

(3.34)

Here \( \langle, \rangle \) denotes the Euclidean metric on \( \mathbb{R}^{n+1} \). Note that \( \gamma_a \) are different from the Dirac \( \gamma \)-matrices which will appear in the next sections.

One can use \( \gamma_a \) to construct a natural \( n \)-bein for \( S^n \) which we define as a submanifold \( S^n := \{ p \in \mathbb{R}^{n+1} | \langle p, p \rangle = 1 \} \). First, using (3.32), we rewrite (3.33) as
\[ \langle \gamma, v, w \rangle = -\langle v, \gamma, w \rangle. \] (3.35)

Now, with \( p \in S^n \), we define the vector fields
\[ e_a|_p := \gamma_a p, \quad a = 1, \ldots, n. \] (3.36)

In view of the equations (3.32) and (3.35), the \( n+1 \) vectors \( (p, e_1|_p, \ldots, e_n|_p) \) form an orthonormal frame on \( R^{n+1} \). In particular, \( e_1, \ldots, e_n \) is an orthonormal \( n \)-bein on \( S^n \). The dual coframe is defined, for any \( v \), by
\[ \vartheta_a(v) = \langle e_a, v \rangle. \] (3.37)

Now it is straightforward to find its exterior derivative:
\[ d\vartheta_a(e_b, e_c) = -\vartheta_a([e_b, e_c]) = -2 \langle \gamma_a p, \gamma_b \gamma_c p \rangle = 2 \phi_{abc}, \] (3.38)

where we introduced the functions \( \phi_{abc}(p) := \langle \gamma_a \gamma_b \gamma_c p, p \rangle \) on \( R^{n+1} \), which comprise a completely antisymmetric tensor. Thus, for \( n = 1, 3, 7 \), we find
\[ \vartheta^a \wedge d\vartheta_a = \phi_{abc} \vartheta^b \wedge \vartheta^c. \] (3.39)

For \( n = 3 \) the functions \( \phi_{abc} \) are constant over \( S^3 \), namely, \( \phi_{abc} = \pm \varepsilon_{abc} \); then, with the minus sign, (3.39) reproduces (3.28).

For \( n = 7 \), we have to work out
\[ d\vartheta^a \wedge d\vartheta_a = \phi_{abc} \phi_{a'b'c'} \vartheta^b \wedge \vartheta^c \wedge \vartheta^{b'} \wedge \vartheta^{c'}. \] (3.40)

For \( b \neq c, b' \neq c' \) we get:
\[ (\phi_{abc} \phi_{a'b'c'})(p) = \langle \gamma_b \gamma_c p, \gamma_a p \rangle \langle \gamma_{a'} \gamma_{b'} p, \gamma_{c'} p \rangle = \langle \gamma_{b'} \gamma_{a'} p, \gamma_b \gamma_{c'} p \rangle \gamma_a p \gamma_{c'} p = \langle \gamma_{b'} \gamma_{a'} p, \gamma_{b'} \gamma_{c'} p \rangle = \langle \gamma_{b'} \gamma_{a'} p, \gamma_{b'} \gamma_{c'} p \rangle. \] (3.41)

Let \( b, c, b', c' \) be pairwise different and \( \sigma \) be a permutation of \( 1, \ldots, 7 \), with \( \sigma(4) = b, \sigma(5) = c, \sigma(6) = b', \sigma(7) = c' \). Then (3.41) yields
\[ (\varphi^a_{\,\bc} \varphi_{abc'}) (p) = -\langle \gamma_a \gamma_b \gamma_c p, p \rangle = -\langle \gamma_{(a(4)} \gamma_{(5)} \gamma_{(6)} \gamma_{(7)} p, p \rangle \]
\[ = \mp \text{sign}(\sigma) \langle \gamma_{\sigma(1)} \gamma_{\sigma(2)} \gamma_{(3)} p, p \rangle = \mp \text{sign}(\sigma) \varphi_{\sigma(1) \sigma(2) \sigma(3)} (p), \] (3.42)

where we used (3.34). Transvecting now (3.41) with \( \delta^{bb'} \), we find

\[ \varphi^a_{\,\bc} \varphi_{abc} = 6 \delta_{cc'}, \quad \varphi^{abc} \varphi_{abc} = 42. \] (3.43)

For the seven-form (3.31), equations (3.39)-(3.43) yield

\[ C_{TTTT} = -\sum_{a, b, c, \sigma} \text{sign}(\sigma) \varphi_{abc} \varphi_{\sigma(1) \sigma(2) \sigma(3)} \varphi^{a} \wedge \varphi^{b} \wedge \varphi^{c} \wedge \varphi^{(4)} \wedge \varphi^{(5)} \wedge \varphi^{(6)} \wedge \varphi^{(7)} \]
\[ = \mp 4! \varphi_{abc} \varphi^{abc} \varphi^{1} \wedge \varphi^{2} \wedge \varphi^{3} \wedge \varphi^{4} \wedge \varphi^{5} \wedge \varphi^{6} \wedge \varphi^{7} \]
\[ = \mp 1008 \varphi^{1} \wedge \varphi^{2} \wedge \varphi^{3} \wedge \varphi^{4} \wedge \varphi^{5} \wedge \varphi^{6} \wedge \varphi^{7}. \] (3.44)

Two sign factors arise from rearranging the products of the coframe one-forms: one is equal to the sign of the permutation \((\sigma(1), \sigma(2), \sigma(3))\) of \((a, b, c)\) and, analogously, another is equal to the sign of the permutation \((a, b, c, \sigma(4), \sigma(5), \sigma(6), \sigma(7))\) of \((1, 2, 3, 4, 5, 6, 7)\). Their product yields \(\text{sign}(\sigma)\), and thus all sign factors drop out. Eventually, using (3.30), the integral turns out to be

\[ \int_{S^7} C_{TTTT} = \mp 24 \times 42 \times \text{vol}(S^7) = \mp 336 \pi^4. \] (3.45)

The 8-dimensional generalization of our example thus reads as follows: Add the radial coordinate \(r\) to the seven angular coordinates, which parameterize \(S^7\) and are used implicitly in \(\varphi^a\). Then in the gauge \(\Gamma_{\alpha \beta} = 0\), we take the metric \(g_{\alpha \beta} = \delta_{\alpha \beta}\) and describe the coframe one-form by

\[ \varphi^8 = h dr, \quad \varphi^1 = f \varphi^1, \ldots, \quad \varphi^7 = f \varphi^7, \] (3.46)

where \(h = h(r), f = f(r)\) are two \(SO(8)\)-symmetric functions. A similar construction has been developed in [15] for the \(Ricci-flat\) Riemann–Cartan geometry on a seven-sphere.
4. AXIAL CURRENT IN THE EINSTEIN–CARTAN–DIRAC THEORY

The Einstein–Cartan–Dirac (ECD) theory of a coupled gravitational and spin $1/2$ fermion field provide a classical (i.e., not quantized) understanding of the axial anomaly and establishes a link to the Nieh-Yan topological invariant. The ECD-Lagrangian reads:

$$L = \frac{1}{2} \ell^2 R^{\alpha\beta} \wedge \eta_{\alpha\beta} + L_D,$$

(4.1)

where the Dirac Lagrangian for the massless spinor field $\Psi$ is

$$L_D = \frac{i}{2} \left\{ \overline{\Psi} * \gamma \wedge D\Psi + D\overline{\Psi} \wedge *\gamma \Psi \right\}.$$  

(4.2)

As usually, $\eta^{\alpha\beta} := *(\vartheta^\alpha \wedge \vartheta^\beta)$ and $\eta^\alpha := *\vartheta^\alpha$, overbar denotes the Dirac conjugate, and the constant Dirac matrices $\gamma^\alpha$ are entering via the Clifford-algebra valued exterior forms:

$$\gamma := \gamma_\alpha \vartheta^\alpha, \quad *\gamma = \gamma^\alpha \eta_\alpha.$$  

(4.3)

The covariant exterior derivative $D$ for spinors is introduced by

$$D\Psi = d\Psi + \frac{i}{4} \Gamma^{\alpha\beta} \wedge \sigma_{\alpha\beta} \Psi, \quad \overline{D\Psi} = d\overline{\Psi} - \frac{i}{4} \Gamma^{\alpha\beta} \wedge \overline{\Psi} \sigma_{\alpha\beta},$$

(4.4)

where the Lorentz generator is represented by the components of the Clifford-algebra valued two-form

$$\sigma := \frac{i}{2} \gamma \wedge \gamma = \frac{1}{2} \sigma_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta \quad \Rightarrow \quad \sigma_{\alpha\beta} = \frac{i}{2} (\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha).$$

(4.5)

The spin current of the Dirac field is given by the Hermitian three–form

$$\tau_{\alpha\beta} := \frac{\partial L_D}{\partial \Gamma^{\alpha\beta}} = \frac{1}{8} \overline{\Psi} (\gamma \sigma_{\alpha\beta} + \sigma_{\alpha\beta} *\gamma) \Psi.$$  

(4.6)

From the anticommutation relations for the Dirac matrices we can infer that

$$\tau_{\alpha\beta} = \tau_{\alpha\beta\gamma} \eta^\gamma = \frac{i}{4} \overline{\Psi} \gamma_{[\alpha\beta} \gamma_{\gamma]} \Psi \eta^\gamma = -\frac{1}{4} \eta_{\alpha\beta\gamma\delta} \overline{\Psi} \gamma^\gamma \gamma^\delta \Psi \eta^\gamma.$$  

(4.7)

This implies that the components $\tau_{\alpha\beta\gamma} = \tau_{[\alpha\beta\gamma]}$ of the spin current are totally antisymmetric.
The second field equation of EC-theory, i.e. the algebraic relation between torsion and spin,

\[-\frac{1}{2}\eta_{\alpha\beta\gamma} \wedge T^\gamma = \ell^2 r_{\alpha\beta};\]  

(4.8)
can now be rewritten as

\[\vartheta^\delta \wedge T^\gamma = \frac{\ell^2}{2} \overline{\Psi}_r \gamma^\delta \Psi \eta^\gamma.\]  

(4.9)

The contraction of its free indices involves the *axial current* three-form of the Dirac field

\[j_5 := \overline{\Psi} \gamma^5 \gamma^\alpha \Psi_\alpha = \overline{\Psi} \gamma^5 \gamma^\alpha \Psi.\]  

(4.10)

For the axial torsion one-form \(A\), this implies \(A = -(\ell^2/2) \overline{\Psi}_r \gamma^5 \gamma \Psi\). Substituting it into the translational Chern-Simons term (3.25), we find (cf. [16])

\[C_{TT} = \frac{1}{4} j_5.\]  

(4.11)

From (2.7) we thus find in ECD-theory, if a possible coupling to the Weyl covector is allowed for,

\[dj_5 = 4dC_{TT} = \frac{2}{\ell^2} \left(T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta\right) - Q \wedge j_5.\]  

(4.12)

This result, cf. [16],[17], holds on the level of first quantization. Since the Hamiltonian of the semi-classical Dirac field is not bounded from below, one has to go over to second quantization, where the divergence of the axial current picks up anomalous terms. The question is whether in the vacuum expectation value \(\langle dj_5 \rangle\) similar torsion and Weyl covector terms emerge, besides the usual Pontrjagin term.

However, the Dirac equation cannot be extended to metric-affine gravity with its gauged \(GL(4, R)\) group. Only an additional coupling to a Weyl covector \(Q\) is admitted, which can surface in the Nieh-Yan term. Then the scale invariant covariant derivative \(D := d + Q\) should differentiate the axial current.
5. HEAT KERNEL TECHNIQUE AND AXIAL ANOMALY

As a matter of fact, quantum anomalies both in the Riemannian and in the Riemann-Cartan spacetimes (see [18]) were calculated previously in a large number of papers using different methods (which prove to be all mutually consistent), see e.g. [19] for a list of references and the more recent papers [20,21]. However, recently Chandia and Zanelli [6] have questioned the completeness of the earlier calculations which all seem to demonstrate that the Nieh-Yan four-form is irrelevant to the axial anomaly. In this section we carefully reconsider the derivation of the axial anomaly for the massless Dirac field within the framework of the heat kernel technique. In particular, we will use the results of [18] (though the reader should be careful because of the different notation).

From (4.2) we read off the Dirac operator $\Delta$ in Riemann-Cartan spacetime,

$$
\eta \Delta \Psi := i^* \gamma \wedge \left( D - \frac{1}{2} T \right) \Psi,
$$

(5.1)

where, as usual, $T = e_\alpha \lrcorner T^\alpha$ is the torsion trace one-form. The square of $\Delta$ is a self-adjoint second order differential operator $D^2 := \Delta \Delta^\dagger$,

$$
\eta D^2 = -(D^* D + 2 S \wedge * D + X),
$$

(5.2)

where, as follows from (5.1), the one-form $S$ is defined by

$$
S := \frac{i}{2} \partial_\alpha (* \sigma \wedge T^\alpha),
$$

(5.3)

and the four-form $X$ reads:

$$
X := \frac{1}{4} * \sigma \wedge R_{\alpha \beta} \sigma^{\alpha \beta} + \frac{1}{2} \left( -d^* T + i^* \sigma \wedge d T - \frac{1}{2} T \wedge * T + i (e_\alpha \lrcorner * \sigma) \wedge T^\alpha \wedge T \right).
$$

(5.4)

Among the operator functions which can be constructed from $D^2$, the expression $\exp(-tD^2)$, $t > 0$, is of particular importance. It is formally defined by its kernel four-form $K(t,x,y,D^2)$:

$$
\exp(-tD^2) \psi(x) := \int K(t,x,y,D^2) \psi(y).
$$

(5.5)
Here $x, y$ are points of the spacetime manifold $M$ and the integral is over $M$ parameterized by the local coordinates $y$. By construction, $K$ satisfies the heat equation

$$
\frac{\partial}{\partial t} K(t, x, y, D^2) + D^2 K(t, x, y, D^2) = 0, \quad K(0, x, y, D^2) = \delta(x, y). \tag{5.6}
$$

For small $t \to +0$, there exists an asymptotic expansion,

$$
K(t, x, x, D^2) = \sum_{n=0}^{\infty} t^{n-2}(4\pi)^{-2} K_n(x, D^2), \tag{5.7}
$$

where the coefficients $K_n(x, D^2), n = 0, 1, \ldots$ are the four-forms on spacetime which are completely determined by the form of the positive second-order differential operator $D^2$, namely by the exterior forms $S$ and $X$. In [18] the general expression for these coefficients was derived for a four-dimensional Riemann-Cartan manifold $M$ and an arbitrary operator of the form (5.2) acting on the sections of a bundle with internal curvature two-form $F$. For odd $n = 1, 3, \ldots$ the coefficients are zero, while the first nontrivial terms read

$$
K_0 = 1\eta, \tag{5.8}
$$

$$
K_2 = Z + \frac{1}{6} \tilde{R}\eta, \tag{5.9}
$$

$$
K_4 = \frac{1}{3} (\text{com}) + \frac{1}{6} \tilde{D}^{\alpha\beta} \left( *Z + \frac{1}{5} \tilde{R} \right) + \frac{1}{2} Z^* Z + \frac{1}{6} \tilde{R} Z + \frac{1}{6} Y \wedge Y
$$

$$
+ \frac{1}{180} \left( 2^{*1} \tilde{R}^{\alpha\beta} \wedge (1) \tilde{R}_{\alpha\beta} + 2^{*4} \tilde{R}^{\alpha\beta} \wedge (4) \tilde{R}_{\alpha\beta} + 29^{*6} \tilde{R}^{\alpha\beta} \wedge (6) \tilde{R}_{\alpha\beta} \right). \tag{5.10}
$$

Here $1$ is the unity matrix,

$$
Z := X - d^* S - S \wedge * S, \quad Y := F + dS + S \wedge S, \tag{5.11}
$$

the tildes denote purely Riemannian objects and operators, and the superscripts label the irreducible pieces of the Riemannian curvature (1st represents the Weyl piece, 4th – the traceless Ricci, and 6th – the curvature scalar $\tilde{R} := e_\alpha [e_\beta \tilde{R}^{\alpha\beta}]$). Finally, (com) is an inessential term which satisfies $\text{Tr}(\text{com}) = \text{Tr}(\gamma_5(\text{com})) = 0$.

It is well known that the axial anomaly is closely related to the Atiyah-Singer index theorem [22]. Technically, this leads to the calculation of the trace

$$
\text{Tr} \left( \gamma_5 e^{-tD^2} \right), \quad t \to +0, \tag{5.12}
$$

14
for example, using the Fujikawa method. In view of the heat kernel expansion (5.7), this problem eventually reduces to the computation of $\text{Tr}(\gamma_5 K_n)$.

For the Dirac operator, evidently

$$\text{Tr}(\gamma_5 K_0) = 0,$$

while the result for $n = 4$ was given explicitly in [18] in terms of the Riemannian Pontrjagin four-form,

$$\text{Tr}(\gamma_5 K_4) = \frac{1}{12} \left( \tilde{R}^{\alpha \beta} \wedge \tilde{R}_{\alpha \beta} + \frac{1}{2} dA \wedge dA + d\mathcal{K} \right),$$

where, to recall, $A$ is the axial torsion one-form, and in the boundary term the three-form $\mathcal{K}$ is constructed from $A$.

Finally, substituting (5.3) and (5.4) into (5.11) and (5.9), we find

$$\text{Tr}(\gamma_5 K_2) = \text{Tr}(\gamma_5 Z) = \text{Tr}(\gamma_5 [X - S \wedge *S])$$

$$= \frac{1}{4} \text{Tr}(\gamma_5 *\sigma \sigma^{\alpha \beta}) \wedge \left( R_{\alpha \beta} + \frac{1}{2} T^\gamma * (\eta_{\alpha \beta} \wedge T_\gamma) \right)$$

$$= - \left( R_{\alpha \beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta + T^\alpha \wedge T_\alpha \right),$$

(5.15)

where we used the identity $\eta_{\alpha \beta} \wedge \Phi = - \eta [e_\alpha] [e_\beta] \Phi$ valid for any two-form $\Phi$, and the trace

$$\frac{1}{4} \text{Tr}(\gamma_5 *\sigma \sigma^{\alpha \beta}) = - \vartheta^\alpha \wedge \vartheta^\beta.$$

(5.16)

[Note that a sign factor (and/or $i$) may appear which depends on the signature and the representation of the Dirac matrices; in this section we work with the Euclidean version.]

It is not necessary to compute further terms $\text{Tr}(\gamma_5 K_n)$ for $n > 4$ because they are multiplied by the positive powers of $t$ and drop out when $t \to 0$.

Thus, using the heat kernel technique and the earlier results on the spectral geometry of the Riemann-Cartan spacetime (5.8)-(5.10), we are able to confirm the proposition of Chandia and Zanelli that the Nieh-Yan four-form can, indeed, show up in the study of the chiral anomaly. This is demonstrated explicitly by (5.15).

There is a problem, though: the Nieh-Yan four-form enters the heat kernel expansion with the necessarily divergent coefficient $\frac{1}{t}$, with $t \to 0$. Formal regularization of the axial
anomaly in the zeta-function or dimensional schemes leads to the complete subtraction of this term which explains why it was never reported in the literature. However, in the Fujikawa method, used in [3], this term survives and is proportional to the regulator mass square $M^2 \rightarrow \infty$.

Chandia and Zanelli proposed to absorb this divergence by a proper rescaling of the coframe. We can comment here that such a renormalization should be only possible in a conformally invariant gravity theory with massless fermions. Then, the usual Einstein-Cartan theory is not applicable.

**ACKNOWLEDGMENTS**

We thank Jorge Zanelli for useful discussions and for providing a preliminary draft of Chandia’s and his paper. We also acknowledge interesting discussions with Alfredo Macías. This work was partially supported by KFA–Conacyt Grant No. E130–2924. One of us (EWM) acknowledges the support by the short–term fellowship 961 616 015 6 of the German Academic Exchange Service (DAAD), Bonn. For (YNO) this work was supported by the Deutsche Forschungsgemeinschaft (Bonn) under contract He 528/17-2.
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