On an explicit Skorokhod embedding for spectrally negative Lévy processes

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Abstract
We present an explicit solution to the Skorokhod embedding problem for spectrally negative Lévy processes. Given a process $X$ and a target measure $\mu$ satisfying an explicit admissibility condition we define functions $\varphi_{\pm}$ such that the stopping time $T = \inf\{t > 0 : X_t \in (-\varphi_-(L_t), \varphi_+(L_t))\}$ induces $X_T \sim \mu$, where $(L_t)$ is the local time in zero of $X$. We also treat versions of $T$ which take into account the sign of the excursion straddling time $t$. We prove that our stopping times are minimal and we describe criteria under which they are integrable. We compare our solution with the one proposed by Bertoin and Le Jan [5]. In particular, we compute explicitly the quantities introduced in [5] in our setup.

Our method relies on some new explicit calculations relating scale functions and the Itô excursion measure of $X$. More precisely, we compute the joint law of the maximum and minimum of an excursion away from 0 in terms of the scale function.

1 Introduction

The Skorokhod embedding problem was first introduced and solved by Skorokhod [19], where it served to realize a random walk as a Brownian motion stopped at a sequence of stopping times. Since then, it remains an active field of study and the original problem has been generalized in a number of ways and has known many different solutions. We refer to Obłój [14] for a comprehensive survey paper.

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The embedding problem can be phrased in the following general manner: given a stochastic process \((X_t)\) and a measure \(\mu\) on its state space, find a stopping time \(T\) which embeds the measure: \(X_T \sim \mu\). To make the problem interesting one requires \(T\) to be small in some sense.

When \((X_t)\) is a continuous martingale and \(\mu\) is centered, one typically asks that \((X_{T \wedge t})\) is a uniformly integrable martingale. When \(\mu\) has finite second moment this is equivalent to the expectation of the quadratic variation being finite, \(E[\langle X \rangle_T] < \infty\), which for Brownian motion reads simply \(E[T] < \infty\). A more general condition requires \(T\) to be minimal, that is if \(S\) is a stopping time with \(S \leq T\) and \(X_S \sim X_T\) then \(S = T\). Monroe [13] showed that the two conditions are equivalent when \(X\) is a continuous local martingale and \(\mu\) is a centered probability measure. In contrast, Cox and Obłoj [8] showed that for discontinuous processes, even in the simplest case of a symmetric random walk, this no longer holds true. In fact, these authors showed that the set of measures which can be embedded in a uniformly integrable way may be a complex fractal subset of the set of measures which can be embedded using minimal stopping times.

In this paper we solve the Skorokhod embedding problem for spectrally negative Lévy processes. Our solution is based on the general framework developed by Obłoj [15], and recently used in Pistorius [17]. Here, we have to extend the setup to account for the presence of jumps. In order to do so we carry out excursion theoretical computations which have an interest in their own. More precisely, given a target measure \(\mu\) from a certain class, we find functions \(\varphi_\pm\) such that the stopping time

\[
T_{\varphi_\pm} = \inf\{t > 0 : X_t \in \{-\varphi_-(L_t), \varphi_+(L_t)\}\},
\]

(1.1)

where \(L_t\) is the local time at zero, satisfies \(X_{T_{\varphi_\pm}} \sim \mu\). We also look at versions of (1.1) which take into account the sign of the excursion straddling time \(t\). We compute the joint law of the maximum and minimum of an excursion away from zero in terms of the scale function, and this yields the functions \(\varphi_\pm\) explicitly in terms of \(\mu\) and the characteristics of \(X\). Finally we prove that our stopping times are minimal – and this in spite of possible waiting for a ‘comeback’ after an undershoot. To the best of our knowledge, so far only one solution to our embedding problem exists, proposed by Bertoin and Le Jan [5]. In [5] a general solution is presented and then the case of symmetric Lévy processes is treated in detail. We complement this by providing explicit computations for the case of spectrally negative Lévy processes.

In [5] stopping times are constructed using local times at all levels simultaneously, while our construction uses only the local time in zero. The difference in complexity is best seen when \(X_t = B_t\) is a Brownian motion, comparing the formulae in [5] with the solution of Vallois [20], to which our solution reduces in this simplest setup. Naturally, there is a price we have to pay for having a simple explicit construction, namely we can only embed measures satisfying a certain admissibility criterion. We discuss this in detail and state the criterion in terms of the scale function.
2 Preliminaries

Let \( X = (X_t, t \geq 0) \) be a spectrally negative Lévy process defined on a filtered probability space \((\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\). Here the filtration \( \mathbb{F} \) is the completion of the standard filtration generated by \( X \). To avoid trivialities, we exclude the case that \( X \) has monotone paths. Since the jumps of \( X \) are all non-positive, the moment generating function \( \mathbb{E}[e^{\theta X_t}] \) exists for all \( \theta \geq 0 \) and is given by

\[
\psi(\theta) = t^{-1} \log \mathbb{E}[e^{\theta X_t}]
\]

for some function \( \psi(\theta) \). The function \( \psi \) is well defined at least on the positive half-axis where it is strictly convex with the property that \( \lim_{\theta \to \infty} \psi(\theta) = +\infty \). Moreover, \( \psi \) is strictly increasing on \([\Phi(0), \infty)\), where \( \Phi(0) \) is the largest root of \( \psi(\theta) = 0 \). We shall denote the right-inverse function of \( \psi \) by \( \Phi : [0, \infty) \to [\Phi(0), \infty) \). Note that \( \Phi(0) > 0 \) if and only if \( X \) drifts to \(-\infty\) (see Bertoin \([3, \text{Cor. VII.2}]\)).

The continuous martingale component of \( X \) is a Brownian motion with variance \( \sigma^2 \), called the Gaussian coefficient of \( X \), which can be recovered from \( \psi \) by

\[
\sigma^2 = \lim_{\theta \to \infty} \frac{2 \psi(\theta)}{\theta^2}.
\]

In what follows we make the following assumption on the process \( X \):

**Assumption 1** \( X \) is a spectrally negative Lévy process that has unbounded variation and does not drift to \(-\infty\).

The hitting time of a set \( \Gamma \subset \mathbb{R} \) is denoted \( H_\Gamma = \inf\{t : X_t \in \Gamma\} \). We write \( H_\eta \) and \( H_{\eta,\delta} \) respectively for \( H_{\{\eta\}} \) and \( H_{\{\eta,\delta\}} \). An important role in the fluctuation theory of spectrally negative Lévy processes is played by the so-called \( q \)-scale functions \( W^{(q)} : \mathbb{R} \to [0, \infty), \ q \geq 0 \), that are zero on the negative half-axis and continuous and increasing on \([0, \infty)\) with Laplace transforms

\[
\int_0^\infty e^{-\theta x} W^{(q)}(x) dx = (\psi(\theta) - q)^{-1}, \quad \theta > \Phi(q),
\]

(2.1)

See e.g. Bingham \([6]\), Bertoin \([3, \text{Thm. VII.8}]\) or Kyprianou \([10, \text{Thm 8.1}]\) for proofs of the existence of \( W^{(q)} \). Bertoin \([4]\) has shown that for every \( x \geq 0 \), the mapping \( q \to W^{(q)}(x) \) can be analytically extended to the complex plane by the identity

\[
W^{(q)}(x) = \sum_{k \geq 0} q^k W^{*(k+1)}(x),
\]

(2.2)

where \( W^{*k} \) denotes the \( k \)-th convolution power of \( W = W^{(0)} \).
The scale function $W$ is closely linked to the law of $X_{H_{[a,b]}}$:

$$P_x[X_{H_{[a,b]}} = b] = \frac{W(x-a)}{W(b-a)}, \quad a < x < b,
$$

(2.3)

where $P_x[\cdot] = P[\cdot | X_0 = x]$. Moreover, from Bertoin [4, Cor. 1] it follows that

$$E_x[H_{[a,b]}] = W(x-a)W(b-a) - W(x-a), \quad a < x < b,
$$

(2.4)

where $W(x) = \int_0^x W(y)dy$. We note that the Gaussian coefficient can be recovered from $W$ as $W'(0) := W'(0+) = 2/\sigma^2$ (see [16, Lemma 1]).

The $q$-potential measure $U_q(dx) = \int_0^\infty e^{-qt}P(X_t \in dx)dt$ of $X$ is absolutely continuous with density $u_q$ related to the $q$-scale function $W(q)$ by

$$u_q(x) = \Phi'(q)e^{-\Phi(q)x} - W(q)(-x), \quad q > 0
$$

(2.5)

(see Bingham [6] or Pistorius [16]). Since the measure $U_q$ is absolutely continuous with bounded density $u_q$ and 0 is regular for $\{0\}$ if $X$ has unbounded variation ([3, Cor VII.5]), the limit, a.s. and in $L^2(P)$,

$$L^y_t = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{\{|X_{s-y}|<\epsilon\}} ds
$$

exists for every $y \in \mathbb{R}$ and $t \geq 0$ and $t \mapsto L^y_t$ is continuous a.s. (cf. Bertoin [3] Thm. II.16, Thm V.1, Prop V.2), so that, in particular, $E_x[L^y_t] < \infty$. The process $(L^y_t, t \geq 0)$ is called the local time of $X$ at $y$. The expectation of the Laplace-Stieltjes transform in time of the local time $L^y_t$ is related to the potential density via

$$E\left[\int_0^\infty e^{-qt}dL^y_t\right] = u_q(y).
$$

(2.6)

Denote by $\tau = (\tau_\ell, \ell \geq 0)$ the right-continuous inverse of $L = L^0$,

$$\tau_\ell = \inf\{t > 0 : L_t > \ell\}.
$$

### 3 The Skorokhod embedding problem

Henceforth we consider the Skorokhod embedding problem for a measure $\mu$ on $\mathbb{R} \setminus \{0\}$ under the condition of minimality on a solution $T$ (i.e. $R = T$ for any stopping time $R \leq T$ with $X_R \sim X_T$):

$$(S) \quad \text{Find a minimal almost surely finite $\mathcal{F}$-stopping time $T$ such that $X_T \sim \mu$.}
$$

We will present two solutions to $(S)$: one which works only in the presence of a positive Gaussian component and a general one, which is however less explicit. The two coincide and simplify for measures concentrated on $\mathbb{R}_+$. The explicit formulae in our solutions are a consequence of the excursion theoretical computations presented in Section 4.
3.1 Solution in the presence of a Gaussian component

Assume \( \sigma^2 > 0 \). In this case our solution extends the ideas of Obloj \[15\] to the discontinuous setup and in particular it simplifies to Vallois’s solution \[20\] when \( X \) is a Brownian motion. We follow the approach of Cox, Hobson and Obloj \[7\] to account for measures with atoms. We note also that an atom in zero can easily be treated (see \[20, 7\]).

Let \( a_\mu < 0 < b_\mu \) be the infimum and supremum of the support of \( \mu \) respectively and denote by \( F_\mu(x) = \mu((\infty, x]) \) the cumulative distribution function of \( \mu \), \( F_\mu^{-1} \) its right-continuous inverse and let \( a_* = F_\mu(0) \). We impose the following admissibility criterion on the measure \( \mu \):

\[
\int_{0}^{\infty} W(s) \mu(ds) = W'(0) \int_{-\infty}^{0} \frac{W(-s) \mu(ds)}{W'(-s)}. \tag{3.1}
\]

Define \( \alpha : [a_*, 1] \to [0, a_*] \) via

\[
\int_{a_*}^{a} W(F_\mu^{-1}(s)) ds = W'(0) \int_{\alpha(a)}^{a_*} \frac{W(-F_\mu^{-1}(s))}{W'(-F_\mu^{-1}(s))} ds. \tag{3.2}
\]

Note that \( \alpha \) is a strictly decreasing, absolutely continuous function with \( \alpha(a_*) = a_* \) and \( \alpha(1) = 0 \). Define \( \xi = \xi_\mu \) via

\[
\xi(a) = \int_{a_*}^{a} \frac{W(F_\mu^{-1}(s))}{\alpha(s) + (1 - s)} ds \quad a_* \leq a \leq 1
\]

and

\[
\xi(a) = \int_{a}^{a_*} \frac{-W'(0) W(-F_\mu^{-1}(s)) ds}{W'(-F_\mu^{-1}(s))(s + 1 - \alpha^{-1}(s))} \quad 0 \leq a \leq a_*,
\]

and put \( \tilde{\psi}_\mu(x) = \xi(F_\mu(x)) \).\(^1\) Note that \( \tilde{\psi} \) is an increasing function on \( \mathbb{R} \). We denote by \( \tilde{\varphi} \) the right-continuous inverse of \( \tilde{\psi} \) and write \( \varphi_\pm(l) = \pm \tilde{\varphi}(\pm l) \).

**Theorem 1** Let \( \sigma > 0 \) and suppose \( 3.1 \) holds. Then

\[
\hat{T}_{\pm \varphi} = \inf \{ t > 0 : X_t = \varphi_+(L_t) \text{ or } X_t = -\varphi_-(L_t) \text{ and } X_{(t,L_t-t)} < 0 \} \quad \tag{3.3}
\]

solves \( S \). If \( E[X_1^2] < \infty \), \( X \) drifts to \( +\infty \) and

\[
\int_{0}^{\infty} y W(y) \mu(dy) - \int_{-\infty}^{0} y W(-y) \mu(dy) < \infty \quad \tag{3.4}
\]

then \( E[\hat{T}_{\pm \varphi}] < \infty \).

\(^1\)A simplified expression for \( \tilde{\psi} \) in the case when \( \mu \) has no atoms is given in Section 5.
3.2 Solution for $\sigma \geq 0$ and measures with a density

Let $\sigma^2 \geq 0$ be arbitrary. Note that when there is no Gaussian component ($\sigma = 0$) it is not possible to separate the excursions into positive and negative ones: every excursion starts positive and either stays always positive or becomes negative and then ends. We will restrain ourselves to probability measures $\mu$ with a positive density function $f_\mu$ on $(a_\mu, b_\mu)$, where $a_\mu$ and $b_\mu$ are respectively the lower and the upper bound of the support of $\mu$. Let $g : [0, b_\mu) \to (a_\mu, 0]$ be a continuous decreasing function given via

$$
\frac{dg}{dy}(y) = - \frac{W(y - g(y)) - W(-g(y))}{W(-g(y))} \frac{f_\mu(y)}{f_\mu(g(y))}, \quad y \in (0, b_\mu),
$$

with $g(0) = 0$. We impose the following admissibility assumption on $\mu$:

$$
g \text{ is well defined and finite with } g(x) \to a_\mu \text{ as } x \to b_\mu.
$$

Let $h$ denote the right-continuous inverse of $g$ and set for $y, z \geq 0$, $\psi_+(y) = \int_y^b W(s) \mu(ds)$ and $\psi_-(z) = \int_{-z}^0 W(h(s)) W(-s) \mu(ds)$.

Theorem 2 Suppose $\mu$ is a probability measure on $\mathbb{R}$ with a positive density $f_\mu$ on $(a_\mu, b_\mu)$ and which satisfies (3.6). Then

$$
T_{\varphi_\pm} = \inf \{ t > 0 : X_t \in \{ -\varphi_-(L_t), \varphi_+(L_t) \} \},
$$

solves (S), where $\varphi_+$, $\varphi_-$ are the right-continuous inverses of respectively $\psi_+$ and $\psi_-$ from (3.7).

Note that the functions $\varphi_\pm$ in (3.8) and (3.9) are different. It should be clear which functions we mean depending on which stopping time is discussed.

3.3 Solution for measures concentrated on $(0, \infty)$

We specialize now to the case where the target measure $\mu$ is a probability measure on $(0, \infty)$, in which case the solution further simplifies. Set $T_\mu$ equal to the stopping time

$$
T_\mu = \inf \{ t > 0 : X_t \geq \varphi_\mu(L_t) \}
$$

where $\varphi_\mu$ denotes the right-continuous inverse of the map $\psi_\mu : [0, \infty) \to [0, \infty]$ that is defined by

$$
\psi_\mu(y) = \int_y^0 W(s) d \left( \log \overline{\mu}(s) \right).
$$

Theorem 3 Suppose $\mu(0, \infty) = 1$. Then $T_\mu$ solves (S) and $\sup_{t \leq T_\mu} X_t = X_{T_\mu}$.
3.4 On minimality of stopping times and admissibility of target measures

We want to stress the minimality property of the stopping times in Theorems 1 and 2. It may seem surprising as the following example demonstrates. Consider the first hitting time $H_{\Gamma}$ of a region $\Gamma = (-\infty,a] \cup [b,\infty)$ for some $a < 0 < b$, which embeds a distribution $\mu$ that has an atom in $b$ and the rest of the mass in $(-\infty,a]$. Now if we try to develop an embedding of $\mu$ with Theorems 1 or 2 it seems that we may very well stop later (also in the local time scale) than $H_{\Gamma}$. The answer to this apparent paradox is that the measure $\mu$ can not be treated within our framework. The admissibility assumptions (3.1) and (3.6) are crucial and they determine the set of measures that can be treated with our methodology. The restriction of the set of admissible measures is a natural price to pay for having a simple explicit form of the stopping time that involves only the local time at zero.

Condition (3.1) requires that $\mu$ is centered relative to some density on $\mathbb{R}$ expressed in terms of the scale function. In particular when $X_t = B_t$ is a Brownian motion $W(s) = 2s - s^2$ for $s \geq 0$, and (3.1) simplifies to

$$\int_{-\infty}^{0} |x| \mu(dx) = \int_{0}^{\infty} x \mu(dx),$$

which is a necessary and sufficient condition for $\varphi_{\pm}$ to be well defined and finite on $\mathbb{R}_+$. In general when $X$ is recurrent any measure on $\mathbb{R}$ can be embedded in $X$. An extension of our construction to arbitrary measures would involve explosion of one of the functions $\varphi_{\pm}$ and the resulting stopping times may not be minimal (as example described above illustrates).

When $X$ drifts to infinity it is not possible to embed all measures on $\mathbb{R}$ in $X$. More precisely, Rost’s balayage condition [18] states that there exists an embedding of $\mu$ in $X$ if and only if

$$E_0 \left[ \int_0^{\infty} f(X_t) dt \right] \geq E_\mu \left[ \int_0^{\infty} f(X_t) dt \right], \text{ for all } f \geq 0. \quad (3.11)$$

Naturally this is equivalent to a restriction on the potential density of $X$, which we can rewrite using (2.5) as

$$\int_0^{\infty} (W(-y) - W(x-y)) \mu(dx) \leq 0 \text{ a.e.}$$

Our embedding works for measures which satisfy (3.1), which is thus a subclass of all measures which satisfy Rost’s condition (note that it may not be easy to show this directly).

As an example, consider Brownian motion with drift $X_t = B_t + \delta t$, $\delta > 0$. Then $W(s) = (1 - \exp(-2\delta s))/\delta$, $s \geq 0$, and (3.1) simplifies to $\int_{\mathbb{R}} (1 - \exp(-2\delta x)) \mu(dx) = m = 0$, while a necessary and sufficient condition (3.11) on $\mu$ for existence of an embedding in $X$ is $m \geq 0$ (see Obloj [14, Sec. 9]). Note that if the last integral is equal to $m > 0$ then we can still embed $\mu$ by using our construction for the shifted process. More precisely, let $c = -\ln(1 - m)/2\delta > 0$ and $\tilde{\mu}(du) = \mu(du + c)$. Then $\tilde{\mu}$ satisfies (3.1) and embedding $\tilde{\mu}$ in the process $\tilde{X}_t := X_{t+H_{\Gamma} - c}$ we embed $\mu$ in $X$.

For a continuous process $X$ with $\sigma > 0$ Theorems 1 and 2 coincide. However, for a discontinuous process $X$ with $\sigma > 0$ the sets of measures that can be embedded...
embedded using Theorem 1 and Theorem 2 are different. We will come back to this discussion when presenting examples in Section 3.6.

The key ingredient for the proof of minimality of our stopping times is the observation that they minimise the expectation of the local time at zero among all stopping times which embed a given law. This is closely related to the work of Bertoin and Le Jan [5] and is discussed in the subsequent section.

3.5 On the solution of Bertoin and Le Jan [5]

Bertoin and Le Jan [5] presented a general solution to the Skorokhod embedding problem. We explore now briefly their solution in the context of spectrally negative Lévy processes.

For a given probability measure \( \mu \), Bertoin and Le Jan [5] defined the function

\[
V_\mu(y) = \int_{\mathbb{R}} v(x, y) \mu(dx), \quad \text{and} \quad \lambda_\mu = \sup_y V_\mu(y),
\]

(3.12)

where \( v(x, y) = \mathbb{E}_x[L_y^0] \). Bertoin and Le Jan [5] proved that when \( \lambda_\mu < \infty \) the stopping time

\[
T^\mu_{BLJ} = \inf \left\{ t : \int_{\mathbb{R}} \frac{\lambda_\mu L_t^0}{\lambda_\mu - V_\mu(x)} \mu(dx) > L^0_t \right\}
\]

(3.13)

solves the Skorokhod embedding problem, i.e. \( X_{T^\mu_{BLJ}} \sim \mu \), when \( X \) is recurrent. They also proved that among all solutions to the Skorokhod embedding problem \( T^\mu_{BLJ} \) minimizes the expected value of additive functionals and that it holds that \( \mathbb{E}[L^0_{T^\mu_{BLJ}}] = \lambda_\mu \). We recover this bound in our setting:

**Proposition 4**  
(i) For any a.s. finite stopping time \( S \) with \( X_S \sim \mu \),

\[
\mathbb{E}[L_S] \geq \lambda_\mu \geq \int_0^\infty W(x) \mu(dx).
\]

(ii) If \( X \) drifts to \( +\infty \), then \( \lambda_\mu < \infty \).

(iii) Assume \( X \) oscillates and that \( \sigma^2 > 0 \) and let \( \mu \) be a probability measure satisfying (3.1). If \( \int_0^\infty x \mu(dx) < \infty \) then \( \lambda_\mu < \infty \). If, in addition \( \mathbb{E}X_1^2 = \psi''(0+) < \infty \), then \( \lambda_\mu < \infty \) if and only if \( \int_0^\infty x \mu(dx) < \infty \) in which case

\[
\int_{\mathbb{R}} |x| \mu(dx) < \infty.
\]

The rest of this section is devoted to the proof of Proposition 4. The proof is based on two auxiliary results which are of independent interest:

**Lemma 5**  
(i) If \( X \) drifts to \( +\infty \) then \( W \) is bounded by \( 1/\psi'(0+) < \infty \).

(ii) If \( X \) oscillates then \( W(a)/a \to 2/\psi''(0+) \) as \( a \to \infty \).

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\(^3\)Note that our \( V_\mu \) corresponds to \( \hat{V}_\mu \) in [5].
(iii) Let $\Lambda$ the Lévy measure of $X$ and $C = 1 + \overline{W}(a) \int_{-\infty}^{-1} |x| \Lambda(dx)$. Then, if $X$ oscillates, for any $a, x > 0$,

$$a \wedge x - C \leq a \left(1 - \frac{W(a - x)}{W(a)}\right) \leq x. \quad (3.14)$$

Proof (i) If $X$ drifts to infinity, then $\psi'(0+) > 0$ and it follows by a Tauberian theorem (e.g. [3, p.10]) and the definition of $W$ that $\lim_{x \to \infty} W(x) = 1/\psi'(0+)$. 

(ii) Again appealing to a Tauberian theorem and noting that $\psi'(0+) = 0$ in this case, it follows that $W(x)/x \to 2/\psi''(0+)$ as $x \to \infty$.

(iii) Since $E[X_1^+] \leq E[\sup_{t \leq 1} X_t] < \infty$ (Bertoin [3, Ch VII.1]) and $E[X_1^+] = E[X_1^-]$ (as $X$ oscillates), it follows that $E[|X_1|]$ is finite.

Write $\rho_{-a,0} = H_{\mathbb{R} \setminus [-a,0]}$ for the first exit time from $[-a,0]$. We next show that the expectation of the undershoot $o_a := X_{\rho_{-a,0}} - X_{H_{-a,0}}$ is bounded by $C$, which is finite since $E[|X_1|] < \infty$. We have

$$E_{-x}[o_a] \leq 1 + E_{-x}[o_a 1_{\{|o_a| \geq 1\}}] \leq 1 + E_{-x} \left[\sum_{s \geq 0} |\Delta X_s| 1_{\{|\Delta X_s| \leq 1, s = \rho_{-a,0}\}}\right] \leq 1 + E_{-x} \left[\sum_{s \geq 0} |\Delta X_s| 1_{\{|\Delta X_s| \leq 1, s \leq \rho_{-a,0}\}}\right] = 1 + c \int_{-\infty}^{-1} |z| \Lambda(dz) \leq C,$$

with $c = E_{-x}[\rho_{-a,0}]$, where we applied the compensation formula to the Poisson point process $(\Delta X_s)$ and used that $c \leq \overline{W}(a)$ (see (2.4)).

Using (2.3) yields that

$$E_{-x}[X_{\rho_{-a,0}} - o_a] = -a P_{-x}[\rho_{-a,0} < H_0] = -a \left(1 - \frac{W(a - x)}{W(a)}\right). \quad (3.15)$$

Since $X$ oscillates and $E[|X_1|] < \infty$, we note that $X$ is a martingale. An application of Doob’s optional stopping theorem in conjunction with the dominated convergence theorem implies that

$$E_{-x}[X_{\rho_{-a,0}}] = \lim_{t \to \infty} E_{-x}[X_{1 \wedge \rho_{-a,0}}] = -x.$$

The assertion in (3.14) follows instantly from (3.15). Note in particular that for $x > a$ it trivializes. \quad \square

We consider next the expected discounted local time up to $H_0$:

$$u^q(x,y) = E_x \left[ \int_0^{H_0} e^{-qt} dL_t^y \right], \quad q > 0.$$
Note that, when \( q \downarrow 0 \), \( v^q(x, y) \) converges to \( v(x, y) = E_x[L_{H_0}^y] \), by monotone convergence. Recall that the Laplace transform of the first hitting time \( H_0 \) is given by (Bertoin [3 Thm II.19])

\[
E_x[e^{-qH_0}1_{\{H_0 < \infty\}}] = \frac{u^q(-x)}{u_q(0)}, \quad q > 0,
\]

which is equal to \( e^{q(x,y)} \) for \( x < 0 \).

**Lemma 6** Let \( q > 0 \) and \( y \in \mathbb{R} \). The following hold true:

(i) The process \( v(X_t, y) + L_t^y - W(y)L_t^0 \), with \( W(y):= \lim_{t \downarrow 0} \frac{u^q(y)}{u_q(0)} \), is a martingale.

(ii) For \( q > 0 \) it holds that

\[
v^q(x, y) = u^q(y - x) - \frac{u^q(y)u^q(-x)}{u_q(0)}.
\]

As a consequence, if \( X \) does not drift to \( -\infty \),

\[
v(x, y) = W(x) + W(-y) - W(x - y) - W(x)W(-y)\psi'(0+).
\]

(iii) If \( X \) drifts to \( +\infty \), \( E[\tau_{L(\infty)-}] = \psi''(0+)/[\psi'(0+)]^2 \).

**Proof** (i) In view of the strong Markov property, it follows that

\[
E_x \left[ \int_0^\infty e^{-qt}dL_t^y \right] = v^q(x, y) + E_x[e^{-qH_0}1_{\{H_0 < \infty\}}]E_0 \left[ \int_0^\infty e^{-qt}dL_t^y \right].
\]

Taking note of (3.16), (2.6) and the spatial homogeneity of a Lévy process, we see that (3.17) holds. Applying again the Markov property it follows that, for \( q > 0 \), the process \( M^q = \{M^q_u, u \geq 0\} \) is a UI martingale:

\[
M^q_u = E \left[ \int_0^\infty e^{-qt}dL_t^y - \int_0^\infty e^{-qt}dL_t^0 \frac{u^q(y)}{u_q(0)} \right] F_u
\]

\[
= \int_0^u e^{-qt}dL_t^y - \int_0^u e^{-qt}dL_t^0 \frac{u^q(y)}{u_q(0)} + e^{-qu} \left[ u^q(y - X_u) - u^q(-X_u) \frac{u^q(y)}{u_q(0)} \right]
\]

\[
= \int_0^u e^{-qt}dL_t^y - \frac{u^q(y)}{u_q(0)} \int_0^u e^{-qt}dL_t^0 + e^{-qu} v^q(X_u, y).
\]

Observe that \( M^q_u \to M^0_u \) pathwise a.s. as \( q \downarrow 0 \). Furthermore, \( v^q(x, y) \leq v^q(y, y) \leq v(y, y) \) and the expectation of the other two integrals is bounded by \( E_x[L_t^y + L_t^0] \) which is finite (cf. Section [2]). It now follows by the dominated convergence theorem that \( M_u^0 = L_t^y - \frac{u^q(y)}{u_q(0)} L_t^0 + v(X_t, y) \) is a martingale. Notice
that, if $X$ oscillates, $u(y)/u(0) = 1$ and the resulting martingale coincides with the one used by Bertoin and Le Jan \[5\].

(ii) In view of (2.5) and (3.17) it follows that $e^q(x, y) = W(q)(-y)e^\Phi(x) + W(q)(x)e^{-\Phi(y)} - W(q)(x - y) - \Phi'(q)^{-1}W(q)(-y)W(q)(x)$. Letting then $q \downarrow 0$ equation (3.18) follows by continuity of $W(\cdot)$ and since $\lim_{q \downarrow 0} \Phi'(q) = \psi'(0^+)^{-1}$ in the case when $X$ does not drift to $-\infty$ (see Section 2).

(iii) If $X$ drifts to $+\infty$, it holds that $L(\infty)$ follows an exponential distribution and $(\tau_\ell, \ell < L(\infty))$ has the same law as a subordinator $\tau$ killed at an independent exponential time, in view of \[3\], Thm. IV.8. Further, it follows by a change of variables, (2.6) and (2.3) that

$$
\Phi'(q) = \int_0^\infty E[e^{-q\tau}1_{\{\tau < L(\infty)\}}]d\ell.
$$

From (3.19) we deduce that $L(\infty) \sim \exp(1/\Phi'(0))$ and that Laplace exponent of $\tau$, $\kappa(q) = -\log E[e^{-q\tau}]$, is equal to $\kappa(q) = \Phi'(q)^{-1} - \Phi'(0^+)^{-1}$. In particular,

$$
E[\tau_\ell] = t\kappa'(0^+) = \frac{d}{dq}\bigg|_{q=0^+} \frac{t}{\Phi'(q)} = \frac{\psi''(0^+)}{\psi'(0^+)}t,
$$

using that $\psi'(\Phi(q)) = [\Phi'(q)]^{-1}$ and $\Phi(0) = 0$ if $X$ drifts to $+\infty$. Finally, if $\eta(x)$ denotes an independent exponential random variable with parameter $a = 1/\Phi'(0)$, it follows from (3.20) and the relation between $\tau$ and $\tau$ described above that $E[\tau_{\eta(x)}] = E[\tau_{\eta(x)}] = \int_0^\infty ae^{-at}E[\tau_\ell]d\ell = \psi''(0^+)/\psi'(0)^2$.

Proof of Proposition \[4\]. Using (3.18) for $y > 0$ we have that $V_\mu(y) = \int_0^\infty (W(x) - W(x - y))\mu(dx)$ which increases to $\lambda_\mu^+: = \int_\infty^\infty W(x)\mu(dx)$ as $y \nearrow \infty$. It follows that $\lambda_\mu \geq \lambda_\mu^+$.

(ii) Suppose first that $X$ drifts to infinity. Then, in view of Lemma \[5\], (3.18) and (3.12), we see that $V_\mu$ is bounded. In consequence $\lambda_\mu < \infty$.

(iii) Part (i) which we prove below and the fact that $E[L_{\eta(x)}] = \int_0^\infty W(x)\mu(dx)$, which we shall derive in (5.4) below, immediately yield that $\lambda_\mu = \int_\infty^\infty W(x)\mu(dx)$. Suppose that $X$ oscillates. In view of the asymptotics in Lemma (3.2(i)) and the fact that $W'(0) = 2/\sigma^2 < \infty$, it then follows that the continuous function $s \mapsto W(s)/s$ attains on $[0, \infty]$ a finite positive maximum $c_+$ and finite non-negative minimum $c_-$ (which is positive if $\psi''(0^+) < \infty$). As a consequence,

$$
c_- \int_0^\infty x\mu(dx) \leq \int_0^\infty W(x)\mu(dx) \leq c_+ \int_0^\infty x\mu(dx),
$$

$$
c_- \int_{-\infty}^0 |x|\mu(dx) \leq \int_{-\infty}^0 W(-x)\mu(dx) \leq c_+ \int_{-\infty}^0 |x|\mu(dx).
$$

Next, noting that $W'(a) \leq W'(0)$, which follows by the fact (e.g. \[16\], eq. (13)) and \[3\], Ch. VI] that

$$
Px \left[X_{H_{(-\infty, 0)}} = 0\right] = \frac{W'(x)}{W'(0)} \quad \text{for } x > 0,
$$
we deduce from (3.1) that
\[ \int_{-\infty}^{0} W(-x) \mu(dx) \leq \int_{0}^{\infty} W(x) \mu(dx). \]  
(3.23)

The statements in (iii) follows by combining (3.21), (3.22) and (3.23).

(i) Recall from Lemma 6 that \( N_y^t = v(X_t, y) + L_y^t - \frac{\alpha(y)}{\alpha(0)} L_0^0 \) is a martingale with \( N_y^0 = 0 \). Localizing and using monotone and dominated convergence theorem (note that \( v(x, y) \leq v(y, y) \)) we obtain from Doob’s optional sampling theorem that \( \frac{\alpha(y)}{\alpha(0)} E[L_y^0] = V_\mu(y) + E[L_y^0] \geq V_\mu(y) \). Taking the supremum over \( y \), and noting that \( \frac{\alpha(y)}{\alpha(0)} \leq 1 \), we deduce that \( E[L_0^0] \geq \lambda_\mu \). □

Taking into account (3.18), the solution proposed by Bertoin and Le Jan [5] is explicit. The only practical drawback is that one has to observe simultaneously the local times at all levels which might be hard to implement. In contrast, solutions presented in Theorems 1 and 2 only depend on the local time at zero, the sign of the present excursion and the present position of the process. We stress however that, unlike the solution of Bertoin and Le Jan, these results only apply to restricted sets of target measures. We give now some examples.

3.6 Examples

Consider \( X_t = B_t + t - J_t \) where \( B \) is a standard Brownian motion and \( J \) is a compound Poisson process that jumps at rate 1 with exponentially distributed jump sizes. Then \( X \) is a martingale, it oscillates and its Laplace exponent is given by \( \psi(\theta) = \theta^2/2 + \theta - \theta/(1 + \theta) = \frac{\theta^2(3+\theta)}{2(1+\theta)}. \) The scale function of \( X \) thus reads as
\[ W(x) = \frac{2x}{3} + \frac{4}{9}(1 - e^{-3x}), \quad x \geq 0. \]  
(3.24)

For \( a < 0 < b \) let us determine which measures \( \mu \) on \( \{a, b\} \), \( \mu = (1-p)\delta_a + p\delta_b \) can be embedded in \( X \). Such measures are covered by Theorem 1 and invoking assumption (3.1) we see that it must hold that
\[ pW(b) = 2(1-p) \frac{W(-a)}{W'(-a)} \quad \text{thus} \quad p = \frac{2W(-a)}{W(b)W'(-a) + 2W(-a)}. \]

In particular, for \( a = -b \) we would have \( p = 3/(4 + 2e^{-3b}) \). The measure \( \mu \) is embedded with the stopping time
\[ \tilde{T}_{a,b} = \inf\{t : X_t = b \text{ or } X_t = a \text{ and } X|_{(\tau_L, \tau_L)} < 0\}. \]

It is possible to work out explicitly the stopping time of Bertoin and Le Jan in (3.13) for this special case. Using \( W'(-a) \leq 2 \), we obtain \( \lambda_\mu = pW(b) = V_\mu(b) > V_\mu(a) = (1-p)W(-a) \) and
\[ T_{BLJ}^a = \inf\left\{ t : X_t = b \text{ or } L_t^a > \frac{pW(b) - (1-p)W(-a)}{p(1-p)W(b)} L_0^0 \right\}. \]
The difference with $\hat{T}_{a,b}$ is in the mechanism that determines whether to stop or not when visiting $a$.

In parallel, extending the approach of Theorem 2 to atomic measures, we are led to consider $H_{a,b} = \inf\{t : X_t \in \{a, b\}\}$. Note that the measure $\nu \sim X_{H_{a,b}}$ places less mass in $b$ than $\mu$. One can verify independently, using (2.3) and (3.24), that indeed

$$\nu(\{b\}) = \frac{W(-a)}{W(b-a)} < \frac{2W(-a)}{W(b)W'(-a) + 2W'(-a)} = \mu(\{b\}).$$

Finally, it is instructive to verify that the measure $\rho \sim X_{H_{R\setminus[a,b]}}$ does not verify assumption (3.1) and can not be treated in our setup. Note that $\rho$ is given by

$$\rho(dx) = p_+\delta_b(dx) + p_-\delta_a(dx) + (1 - p_+ - p_-)e^{x-a}1_{x<a}dx,$$

where $p_+$ and $p_-$ are given by (2.3) and

$$P_x[X_{H_{R\setminus[a,b]}} = a] = \frac{W'(b-a)}{W'(0)} \left[ \frac{W'(x-a)}{W'(b-a)} - \frac{W(x-a)}{W(b-a)} \right],$$

respectively (the latter identity was shown in [16, Prop. 1]). One can also compute explicitly $V_\rho$ to find that it is a constant $V_\rho \equiv p_+W(b)$ and thus the stopping time introduced by Bertoin and Le Jan [5] coincides with $H_{R\setminus[a,b]}$.

### 4 Excursion theoretical calculations

In this section we derive a number of excursion measure identities which are essential to obtain the formulae presented in Sections 3.1-3.3. We first set the notation and briefly recall the main concepts of the excursion theory of a spectrally negative Lévy process away from zero, following Bertoin [3, Ch. IV].

The excursion process $e = \{e_\ell, \ell \geq 0\}$ of $X$ away from 0 is defined as

$$e_\ell = (X_u : \tau_\ell- \leq u < \tau_\ell\} \quad \text{if} \quad \tau_\ell- < \tau_\ell$$

and $e_\ell = \partial$ (a graveyard state) else. The excursion process takes values in the space $\mathcal{E} \cup \mathcal{E}^{(\infty)} \cup \{\partial\}$ where

$$\mathcal{E} = \{\epsilon \in D[0,\infty) : \exists \zeta = \zeta(\epsilon) < \infty : \epsilon(s) \neq 0, s \in (0,\zeta)\}$$

is the space of excursions with finite lifetime and

$$\mathcal{E}^{(\infty)} = \{\epsilon \in D[0,\infty) : \epsilon(s) > 0, s > 0\}.$$

are the excursions with infinite lifetime. Write $\zeta = \zeta(\epsilon)$ for the lifetime of an excursion $\epsilon \in \mathcal{E} \cup \mathcal{E}^{(\infty)}$ and define the sign of an excursion via

$$\text{sgn}(\epsilon) = \lim_{s \to 0^+} \frac{\epsilon(s)}{|\epsilon(s)|}.$$  \hfill (4.1)
According to the fundamental result of Itô [9], if 0 is recurrent, \( e \) is a Poisson point process with characteristic measure denoted by \( n \); if 0 is transient, \( \{e_\ell, \ell \leq L(\infty)\} \) is a Poisson point process stopped at its first entrance into \( \mathcal{C}(\infty) \).

The process \( X \) (with \( X_0 = 0 \)) is said to creep downwards across \( x < 0 \) if \( X_{H(\infty,x)} = x \). According to Millar [12] a spectrally negative Lévy process can creep downwards only if its Gaussian coefficient \( \sigma^2 \) is not zero. Therefore, if \( \sigma = 0 \), the Lévy process always enters \( (-\infty,0) \) by a jump and never enters \( (-\infty,0] \) by hitting 0. In this case an excursion away from 0 is thus either infinite and all the time strictly positive (after the starting point) or it is first positive and then jumps into \( (-\infty,0) \) and finally returns to 0 (recalling that we assume that \( X \) oscillates or drifts to \( +\infty \)). Note that according to our definition (4.1) the latter is also of positive sign. In the case that \( \sigma > 0 \), there are two additional forms of excursions, namely those that stay positive or negative all the time and hit zero in finite time.

### 4.1 Supremum and infimum

In the literature there are several results on the law of the supremum of excursions of spectrally negative Lévy processes: Bertoin [2] and Avram et al. [1] calculated this law for excursion of \( X \) away from its infimum and away from its supremum respectively. Lambert [11] calculated the law, under the excursion measure, of the supremum of the excursions away from a point of a spectrally negative Lévy process confined to a finite interval. In this section we extend these results by deriving the joint law of the supremum \( \tau \) and the infimum \( \varsigma \) of an excursion \( e \) away from zero,

\[
\tau = \sup_{s \leq \zeta} \epsilon_s, \quad \varsigma = \inf_{s \leq \zeta} \epsilon_s,
\]

under the excursion measure \( n \).

**Lemma 7** For \( \delta, \eta > 0 \) it holds that

\[
n(1 - 1_{(\tau < \eta, \varsigma > -\delta)} e^{-q\zeta}) = \frac{W(q)(\eta + \delta)}{W(q)(\eta)W(q)(\delta)}. \tag{4.2}
\]

In particular,

\[
n(\tau \geq \eta) = \frac{1}{W(\eta)}, \tag{4.3}
\]

\[
n(\varsigma \leq -\delta) = \frac{1}{W(\delta)} - \psi'(0+) \tag{4.4}
\]

\[
n(\tau < \eta, \varsigma \leq -\delta) = \frac{1}{W(\eta)} \left[ \frac{W(\eta + \delta)}{W(\delta)} - 1 \right]. \tag{4.5}
\]

**Proof** Let us first derive the identities (4.3) — (4.5) as consequences of (4.2).
Letting \( q \downarrow 0 \) in (4.2) we see that
\[
n(\overline{\tau} \geq \eta \text{ or } \xi \leq -\delta \text{ or } \zeta = \infty) = \frac{W(\eta + \delta)}{W(\eta)W(\delta)}.
\] (4.6)

As \( X \) does not drift to \(-\infty\), either \( X \) oscillates in which case \( \zeta < \infty \) n.a.s., or \( X \) drifts to infinity in which case \( \overline{\tau} \geq \eta \) - thus the previous display is also equal to \( n(\overline{\tau} \geq \eta \text{ or } \xi \leq -\delta) \). The identity (4.3) then follows by letting \( \delta \to +\infty \) in (4.6) and using that \( W(\delta + \eta)/W(\delta) \) converges to 1 as \( \delta \to \infty \) if \( X \) does not drift to \(-\infty\) (to see the latter recall that \( \lim_{x \to \infty} W(x) = 1/\psi'(0+) < \infty \) if \( X \) drifts to infinity and refer to Lemma 5 if \( X \) oscillates). To obtain (4.3) we note that the identity (4.3) follows by letting \( \eta \to \infty \) in (4.5). To show (4.2) the first step is to establish a link between the excursion measure and the expected discounted local time. We show that for \( \eta, \delta > 0 \)
\[
n \left(1 - e^{-q\zeta}1_{[\xi < \eta, \xi > -\delta, \zeta < \infty]} \right) = \int_0^{H_{\eta,-\delta}} e^{-qt} dL_t \]
Indeed, changing from time scale it follows that
\[
E \left[ \int_0^{H_{\eta,-\delta}} e^{-qt} dL_t \right] = E \left[ \int_0^{\infty} e^{-qt}1_{\{\tau_t < H_{\eta,-\delta}\}} d\ell \right] = \int_0^{\infty} d\ell E \left[ \exp \left( -q \sum_{0 \leq u \leq \ell} (\tau_u - \tau_{u-})\chi_{\{\overline{\tau} < \eta, \xi > -\delta\}} \right) \right]
\]
where \( \chi_A(\omega) \) is the indicator of the set \( A \) that is one if \( \omega \in A \) and \(+\infty\) else. By the exponential formula for Poisson point processes it then follows that the last line of the previous display is equal to
\[
\int_0^{\infty} d\ell \exp \left( -\ell n(1 - e^{-q\zeta}1_{[\xi < \eta, \xi > -\delta, \zeta < \infty]}) \right)
= \left[ n(1 - e^{-q\zeta}1_{[\xi < \eta, \xi > -\delta, \zeta < \infty]}) \right]^{-1}.
\]
where we used that \( e^{-\infty} = 0 \). By the Markov property
\[
E \left[ \int_0^{\infty} e^{-qt} dL_t \middle| F_t \right] = \int_0^t e^{-qt} dL_u + e^{-qt} u^q (-X_t).
\]
Applying the optional stopping theorem at \( H_{\eta,-\delta} \) shows that
\[
E \left[ \int_0^{H_{\eta,-\delta}} e^{-qt} dL_t \right] = u^q(0) - u^q(-\eta) E[e^{-qH_{\eta,-\delta}} 1_{\{H_{\eta,-\delta} < \infty\}}] + u^q(+\delta) E[e^{-qH_{\eta,-\delta}} 1_{\{H_{\eta,-\delta} < \infty\}}].
\]
The expressions (4.10) and (4.11) follow by evaluating the derivative with that series representation (2.2) of $W$ with respect to $q$.

As a consequence, we see that the expression (4.8) holds true.

In view of (3.16), (4.2) and (4.7) it follows that the left-hand side is equal to $e^{-\Phi(q)\theta}$.

In view of (2.5) the identity (4.2) follows after some algebra. $\square$

Inserting now the expression of the Laplace transform of the first hitting time $H_{\eta,-\delta}$ in terms of the potential density $u^q$ from Pistorius [10 Cor. 3], it follows that

$$E\left[ \int_0^{H_{\eta,-\delta}} e^{-q^2 t} dL_t \right] = u^q(0) - \frac{u^q(-\eta)[u^q(\eta)u^q(0) - u^q(-\delta)u^q(\eta - \delta)]}{u^q(0)^2 - u^q(\delta + \eta)u^q(-\delta - \eta)} - \frac{u^q(\delta)[u^q(-\delta)u^q(0) - u^q(\eta)u^q(-\delta - \eta)]}{u^q(0)^2 - u^q(\delta + \eta)u^q(-\delta - \eta)}$$

In view of (2.7) the identity (4.2) follows after some algebra.

Next we turn to the calculation of the laws of $H_{\eta}$ and $H_{-\delta}$ under the excursion measure $n$. When excursion’s time scale is considered $H_{\eta}$ refers to $H_{\eta}(\varepsilon) = \inf\{s : \varepsilon(s) = \eta\}$.

**Lemma 8** For $\eta, \delta > 0$ it holds that

$$n(e^{-qH_{\eta}}1_{\{\tau \geq \eta\}}) = \frac{1}{W(q)(\eta)}$$

$$n(e^{-qH_{-\delta}}1_{\{\tau < \eta, \tau \leq -\delta\}}) = \frac{e^{\Phi(q)\delta}}{W(q)(\eta)} \left[ W(q)(\delta + \eta) - e^{\Phi(q)\eta} \right]$$

In particular, writing $W \ast W$ for the convolution,

$$n(H_{\eta}1_{\{\tau \geq \eta\}}) = \frac{W \ast W(\eta)}{W(\eta)^2}$$

$$n(H_{-\delta}1_{\{\tau < \eta, \tau \leq -\delta\}}) = \frac{1}{W(\eta)} \left[ \frac{W \ast W(\eta + \delta)}{W(\eta + \delta)} - \frac{W(\eta + \delta)W \ast W(\delta)}{W(\delta)^2} - \Phi'(0)\eta \right]$$

**Proof** The expressions (4.10) and (4.11) follow by evaluating the derivative with respect to $q$ at $q = 0$ of the expressions (4.8) and (4.9) respectively, using the series representation (2.7) of $W(q)$.

Appealing to the compensation formula it follows that for $\eta > 0$

$$E[e^{-qH_{\eta}}] = E\left[ \int_0^{H_{\eta}} e^{-q^2 t} dL_t \right] n(e^{-qH_{\eta}(\varepsilon)}1_{\{\tau \geq \eta\}}).$$

In view of (4.10), (4.2) and (4.7) it follows that the left-hand side is equal to $e^{-\Phi(q)\theta}$ and the first factor on the right-hand side is equal to $e^{-\Phi(q)\theta}W(q)(\eta)$.

As a consequence, we see that the expression (4.8) holds true.

Similarly, an application of the compensation formula shows that

$$E[e^{-qH_{\eta,-\delta}}] = E\left[ \int_0^{H_{\eta,-\delta}} e^{-q^2 t} dL_t \right] \times \left[ n(e^{-qH_{\eta}(\varepsilon)}1_{\{\tau \geq \eta\}}) + n(e^{-qH_{-\delta}(\varepsilon)}1_{\{\tau < \eta, \tau \leq -\delta\}}) \right].$$
The first factor on the right-hand side is equal to the reciprocal of \(4.2\), while a short calculation employing [16, Cor. 3] and (2.5) shows that the left-hand side is equal to

\[
\frac{W^{(q)}(\eta + \delta)e^{\Phi(q)\delta} + W^{(q)}(\delta)(1 - e^{\Phi(q)(\delta + \eta)})}{W^{(q)}(\delta + \eta)}
\]

and the proof of (4.9) is complete. \(\square\)

### 4.2 Further computations in the presence of a Gaussian component

Assume now specifically that \(\sigma^2 > 0\): a Gaussian component is present. In this case the process \(X\) can creep both to positive and negative levels and we can split excursions into negative and positive ones. Recall that we defined the sign of an excursion \(\epsilon\) as its sign at \(t = 0^+\), i.e. \(\text{sgn}(\epsilon) = \lim_{\eta \downarrow 0} \epsilon(s)/|\epsilon(s)|\). The signed maximum functional then reads as

\[
M(\epsilon) = \frac{1}{2}(\text{sgn}(\epsilon) + 1) \sup_{s \leq \zeta(\epsilon)} \epsilon(s) + \frac{1}{2}(\text{sgn}(\epsilon) - 1) \inf_{s \leq \zeta(\epsilon)} \epsilon(s). \tag{4.12}
\]

In a positive excursion we thus only look at the maximum (and ignore the infimum that may be attained in the excursion) and find the following results for the law of \(M\) under \(n\):

#### Lemma 9
Assume \(\sigma > 0\). For \(a > 0\), it holds that

\[
n(M > a) = \frac{1}{W(a)} \quad n(M < -a) = \frac{W'(a)}{W(a)} \cdot \frac{1}{W'(0)}. \tag{4.13}
\]

**Proof** The first identity in (4.13) follows from (4.18). Since in a negative excursion \(\tau = 0\), the second identity in (4.13) follows by taking the limit \(\eta \downarrow 0\) in (4.15)

\[
n(\tau = 0, \zeta \leq -\delta) = \lim_{\eta \downarrow 0} n(\tau < \eta, \zeta \leq -\delta) = \frac{1}{W'(0)} \frac{W'(\delta +)}{W(\delta)}. \tag{4.14}
\]

Finally, we record for later use the form of the first moment of \(H_{-\delta}\) under \(n\):

#### Lemma 10
Assume \(\sigma > 0\) and \(\delta, \eta > 0\). It holds that

\[
n(H_{-\delta}1_{(\tau = 0, \zeta \leq -\delta)}) = \frac{e^{\Phi(q)\delta}}{W^{(q)}(\eta)} \left[ \frac{W^{(q)}(\delta + \eta) - W^{(q)}(\delta)}{\eta W^{(q)}(\delta)} + 1 - e^{\Phi(q)\eta} \right].
\]

**Proof** Rewriting (4.9) as

\[
\eta e^{\Phi(q)\delta} \left[ \frac{W^{(q)}(\delta + \eta) - W^{(q)}(\delta)}{\eta W^{(q)}(\delta)} + 1 - e^{\Phi(q)\eta} \right]
\]
and then taking the limit \( \eta \downarrow 0 \) shows that

\[
\eta(q) = \frac{e^{\Phi(q)\delta}}{W'(\delta)} \left[ W'(q)\delta - \Phi(q) \right].
\]

The identity follows by subsequently calculating the right-derivative with respect to \( q \) in \( q = 0 \), using the series representation (2.2) of \( W(q) \) and the facts that \( \Phi(0) = 0 \) if \( X \) does not drift to \( -\infty \) and that \( W'(0) = W'(0) \), again in view of (2.2).

\[\square\]

5 Proofs of Theorems 1, 2 and 3

Proof of Theorem 1

As Gaussian component is present (\( \sigma > 0 \)) the process \( X \) can creep both to positive and negative levels and excursions are either positive and hit zero, negative and hit zero, positive and jump below zero and then hit zero. Recall our definition of the sign of an excursion given in (4.1) and the signed maximum functional in (4.12) and note that we only look at the infimum along negative excursions. The process \( M(e^\ell) \) is a Poisson point process with characteristic measure \( n(M \in [d\alpha]) \). The embedding part of the theorem can be proved similarly as Theorem 1 in Obloj [15] (accounting for atoms as in Cox, Hobson and Obloj [7]) using the excursion measure calculations in (4.13). Before carrying out the proof let us specialize briefly to the case of non-atomic measures. Define for \( x < 0 < y \)

\[
D_\mu(y) = \int_{[0,y]} W(s)\mu(ds) \quad \text{and} \quad G_\mu(x) = W'(0) \int_{[x,0]} \frac{W(-s)\mu(ds)}{W'(-s)}. \]

Then \( \tilde{\psi}(x) \) is given by \( \psi_+(x) \) for \( x \geq 0 \) and by \( -\psi_-(x) \) for \( x < 0 \) with

\begin{align*}
\psi_+(y) &= \int_{[0,y]} \frac{W(s)\mu(ds)}{(1 + \frac{1}{\mu(s)} - \frac{1}{\mu(g(s)))}} \quad (5.1) \\
\psi_-(z) &= \int_{[z,0]} \frac{W'(0)W(-s)\mu(ds)}{W'(-s)(1 + \frac{1}{\mu(f(s))} - \frac{1}{\mu(s)})}
\end{align*}

where \( g(s) = G_\mu^{-1}(D_\mu(s)) \) and \( f(s) = D_\mu^{-1}(G_\mu(s)) \). The assumption (3.1) simplifies to \( D_\mu(\infty) = G_\mu(-\infty) \) which is the admissibility criterion of Obloj [15].

For the remainder of the proof, we write \( T = \tilde{T}_{\psi \pm} \). Using Poisson Point Process properties of the excursion process, we have that

\[
P(L_T > k) = \exp \left( -\int_0^k n(M > \varphi_+(\ell)) + n(M < -\varphi_-(\ell))d\ell \right). \quad (5.2)
\]

Recall the definitions of \( \alpha, \xi, \tilde{\psi} \) and the excursion measure calculation given in (4.13). It follows that, for \( a_* < a < 1 \), \( \varphi_+(\xi(a)) = F_\mu^{-1}(a) \) and \( \varphi_-(\xi(\alpha(a))) = \frac{1}{2} \pi \) if \( a_* < a < 1 \), \( \varphi_+(\xi(a)) = F_\mu^{-1}(a) \) and \( \varphi_-(\xi(\alpha(a))) = \frac{1}{2} \pi \) if \( a_* < a < 1 \).
\[ -F_{\mu}^{-1}(\alpha(a)) \]. Finally note that
\[
\alpha'(a) = -\frac{W'(-F_{\mu}^{-1}(\alpha(a)))W(F_{\mu}^{-1}(a))}{W'(0)W(-F_{\mu}^{-1}(\alpha(a)))} = -\frac{n(M < F_{\mu}^{-1}(\alpha(a)))}{n(M > F_{\mu}^{-1}(a))}.
\]

In consequence we get
\[
P(L_T > \xi(a)) = \exp \left( -\int_{a_s}^{a} \left( \frac{n(M < F_{\mu}^{-1}(\alpha(u)))}{n(M > F_{\mu}^{-1}(u))} + 1 \right) \frac{du}{1 - u + \alpha(u)} \right)
\]
\[= \exp \left( \ln[1 - u + \alpha(u)]_{a_s}^{a} \right) = 1 - a + \alpha(a). \quad (5.3)\]

Let \( a_s < a_0 < 1 \) be such that \( n(M < F_{\mu}^{-1}(\alpha(a_0))) + n(M > F_{\mu}^{-1}(a_0)) \leq 1 \). Then we can write
\[
\xi(1) = \frac{2 \int_{a_s}^{1} \frac{du}{n(M > F_{\mu}^{-1}(u))(1 - u + \alpha(u))}}{\int_{a_0}^{1} \frac{n(M < F_{\mu}^{-1}(\alpha(u))) + n(M > F_{\mu}^{-1}(u))}{n(M > F_{\mu}^{-1}(u))(1 - u + \alpha(u))} du} = -\ln(1 - u + \alpha(u))_{a_0}^{a_s} = \infty.
\]

Likewise we show that \( \xi(0) = -\infty \). We conclude via \( 5.3 \) that \( L_T < \infty \) a.s.

This readily implies that \( T < \infty \) a.s. when \( X \) oscillates. If \( X \) drifts to \(+\infty\) then it a.s. hits the level \( \varphi_+(L_\infty) < b_\mu \) and thus \( T < \infty \) a.s.

Let \( x > 0 \). Then we can write
\[
P(X_T > x) = \int_{0}^{\infty} P(L_T > l) n(M > \varphi_+(l)) 1_{\varphi_+(l) > x} dl
\]
\[= \int_{a_s}^{1} P(L_T > \xi(a)) n(M > F_{\mu}^{-1}(a)) 1_{F_{\mu}^{-1}(a) > x} d\xi(a) = \mu((x, \infty)) \]
and likewise for \( x < 0 \)
\[
P(X_T < x) = \int_{0}^{\infty} P(L_T > l) n(M < -\varphi_-(l)) 1_{-\varphi_-(l) < x} dl
\]
\[= \int_{a_s}^{1} P(L_T > \xi(a)) n(M < F_{\mu}^{-1}(\alpha(a))) 1_{F_{\mu}^{-1}(\alpha(a)) < x} d\xi(a)
\]
\[= \int_{a_s}^{1} \frac{n(M < F_{\mu}^{-1}(\alpha(a)))}{n(M > F_{\mu}^{-1}(a))} 1_{F_{\mu}^{-1}(\alpha(a)) < x} da = \mu(-\infty, x), \]

which proves that \( X_T \sim \mu \).

We turn to the proof of the minimality of the stopping time \( T \). Let \( S \leq T \) be a stopping time with \( X_S \sim X_T \). We will show that \( S = T \) a.s. We start by computing \( E[L_T] \):
\[
E[L_T] = \int_{0}^{\infty} P(L_T \geq k) dk = \int_{a_s}^{1} P(L_T > \xi(u)) d\xi(u)
\]
\[= \int_{a_s}^{1} \frac{du}{n(M > F_{\mu}^{-1}(u))} = \int_{0}^{\infty} W(y) \mu(dy). \quad (5.4)\]

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From Proposition 3 we deduce that $E[L_S] \geq E[L_T]$ and since $0 \leq L_S \leq L_T$ we conclude that $L_S = L_T$ a.s. that is $S$ and $T$ happen in the same excursion away from zero. From the definition of $T$ and since $S \leq T$ we see that $\text{sgn}(X_S) = \text{sgn}(X_T)$. Absence of positive jumps implies $X_S 1_{X_S \geq 0} \leq X_T 1_{X_T \geq 0}$ a.s. and in consequence $S = T$ on the set $\{X_S \geq 0\} = \{X_T \geq 0\}$. For the negative values we have to deal with the undershoot. Let $\varrho = \inf\{t : X_t \leq \varphi_-(L_t) \text{ and } \text{sgn}(e_{L_t}) = -1\}$. Note that on $\{X_T < 0\} = \{T_{\tau_-} < \varrho \leq T\}$ we have $X_u < X_T$ for $u \in (\varrho, T)$. If $P(S \leq \varrho \leq T) = 0$ then $\varrho < S \leq T$ on $\{X_T < 0\}$. Thus $X_S \leq X_T$ on $\{X_T < 0\} = \{X_S < 0\}$ and since $X_S \sim X_T$ we deduce that $S = T$. Suppose next that $P(S \leq \varrho \leq T) = \epsilon > 0$. Then, working conditionally on $\{S \leq \varrho \leq T\}$, we apply the Markov property at $S$ to see that, starting from $X_S$, there is a positive probability of hitting zero before hitting $[-\varphi_-(L_S), -\infty)$ which in turn means that $P(L_T > L_S) > 0$ which gives the contradiction. We conclude that $S = T$ a.s. and therefore $T$ is minimal.

Next we show the finiteness of $E[T]$. Recall that we assume that $X$ drifts to $+\infty$. Conditioning on $L_T$ we can write $E[T]$ as

$$E[T] = E[E[T|L_T]]$$

$$= E[T_{\tau_{L_T}}] + E\left[\left(E\left[(T - T_{\tau_{L_T}}) 1_{\{X_T = \varphi_+(L_T)\}}\right]|L_T\right]\right]$$

$$+ E\left[\left(E\left[(T - T_{\tau_{L_T}}) 1_{\{X_T = -\varphi_-(L_T)\}}\right]|L_T\right]\right].$$

Properties of Poisson point processes imply that

$$p_+(k) := P(X_T = \varphi_+(k)|L_T = k) = \frac{n(\tau > \varphi_+(k))}{n(\tau > \varphi_+(k)) + n(\tau \leq -\varphi_-(k))},$$

with an analogous expression for $p_-(k) = 1 - p_+(k)$. Since the law of the first (positive) excursion away from zero with supremum larger than $\varphi_+(k)$ is given by $n(\tau > \varphi_+(k))$ and, conditional on $L_T$, our functional of the first excursion is independent of $L_T$ (and similarly for the first negative excursion with infimum smaller than $-\varphi_-(k)$) it holds that

$$E[T] = E[T_{\tau_{L_T}}] + \int_0^\infty P(L_T \in dk)\left[n(H_{\varphi_+(k)}(\tau) > \varphi_+(k))p_+(k) + n(H_{-\varphi_-(k)}(\tau) = 0, \tau \leq -\varphi_-(k))p_-(k)\right]$$

$$+ n(H_{-\varphi_-(k)}(\tau) 1_{\{\tau = 0, \tau \leq -\varphi_-(k)\}})\right]dk,$$

where in the last line used the form of $P(L_T \in dk)$ that was displayed in (5.2).

To show that $E[T]$ is finite we continue now by estimating the three terms in the above display. For the first term note that $E[T_{\tau_{L_T}}] \leq E[T_{\tau_{\infty}}]$ which is finite if $E[X_T^2] < \infty$, in view of Lemma (i). Changing variables in a similar
Proof of Theorem 2

There are several ways in which we can stop. Firstly, an excursion can start positive and have a maximum larger than \( \varphi_+(L_T) \). Secondly, an excursion can start positive, have a maximum smaller than \( \varphi_+(L_T) \) then jump negative and have an infimum smaller than \( -\varphi_-(L_T) \). Finally, an excursion can start negative to achieve an infimum smaller than \( -\varphi_-(L_T) \). The last scenario is possible iff \( \sigma > 0 \).

From standard considerations we obtain the law of \( L_T \):

\[
P(L_T > k) = \exp \left( - \int_0^k n(\varpi \geq \varphi_+(s) \text{ or } \xi \leq -\varphi_-(s)) ds \right).
\]

Given \( L_T = k \) either \( X_T = \varphi_+(k) \) or \( X_T = -\varphi_-(k) \). By the property of Poisson point processes it thus follows that

\[
p_+(k) = \frac{n(\varpi \geq \varphi_+(k))}{n(\varpi \geq \varphi_+(k) \text{ or } \xi \leq -\varphi_-(k))}
\]

and \( p_-(k) = P(X_T = -\varphi_-(L_T)|L_T = k) = 1 - p_+(k) \). Also, for \( h : \mathbb{R} \to \mathbb{R}_+ \),

\[
E[h(X_T)] = \int_0^\infty P(L_T = dk) [h(-\varphi_-(k))p_-(k) + h(\varphi_+(k))p_+(k)].
\]

By choosing \( h(z) = 1_{z \geq y} \) for \( y > 0 \) and writing \( \psi_\pm \) for the inverses of \( \varphi_\pm \) we get

\[
\overline{\mu}(y) = \int_{\psi_+(y)}^\infty dk \ n(\varpi \geq \varphi_+(k))P(L_T \geq k)
\]

and (with \( h(z) = 1_{z > x} \) \((x < 0)\))

\[
\overline{\mu}(x) = P(L_T \leq \psi_-(-x)) + \int_{\psi_-(-x)}^\infty dk \ n(\varpi \geq \varphi_+(k))P(L_T \geq k).
\]

Reasoning as in the proof of Thm. 1 in Óbój [15] we find that

\[
d\psi_+(y) = \frac{-\partial \overline{\mu}(y)}{n(\varpi \geq y)(1 + \overline{\mu}(y) - \overline{\mu}(g(y)))}.
\]
where \( g(y) = -\varphi_-(\psi_+(y)) \) and

\[
d\psi_+(y) = \frac{d\overline{\mu}(g(y)) - d\mu(y)}{n(\tau \geq y \text{ or } \xi \leq g(y))(1 + \overline{\mu}(y) - \overline{\mu}(g(y)))}.
\]

Comparing these two expressions shows that

\[
d\mu(y) = \frac{d\overline{\mu}(g(y))}{n(\tau < y, \xi \leq g(y))} - \frac{d\mu(y)}{n(\tau \geq y)}.
\]

This leads to the following equation for \( g \) that must be satisfied:

\[
\overline{\mu}(g(x)) - \mu(0) = \int_0^x \frac{n(\tau < y, \xi \leq g(y))}{n(\tau \geq y)} \mu(dy).
\]

Assuming that \( \mu \) is absolutely continuous w.r.t. the Lebesgue measure and writing \( f_\mu(x) \) for its density at \( x \) it follows from (4.3) and (4.5) that \( g : \mathbb{R}_+ \to \mathbb{R}_- \) must satisfy (3.5) with \( g(0) = -\varphi_-(\psi_+(0)) = 0 \). We see that if such \( g \) exists then it is plainly a decreasing function, as required. Furthermore, for \( g \) and its inverse to be well defined we have to have \( g(x) \to a_\mu \) as \( x \to b_\mu \), which is the analogue of criterion (3.1) in Theorem 1. The formulae in Theorem 2 then follow.

It remains to see that \( T \) is minimal. Let \( S \leq T \) with \( X_S \sim X_T \). Reasoning presented in the proof of Theorem 1 applies if we can show that \( L_S = L_T \) and \( \text{sgn}(X_S) = \text{sgn}(X_T) \). To this end, note that \( P(L_T > \psi_+(y)) = 1 + \overline{\mu}(y) - \overline{\mu}(g(y)) \), so that

\[
E[L_T] = \int_0^\infty P(L_T > k) = \int_0^\infty P(L_T > \psi_+(k))d\psi_+(k) = \int_0^\infty W(y)\mu(dy)
\]

which by Proposition 4 is the lower bound on \( EL_S \). We thus have \( 0 \leq L_S \leq L_T \) with \( E[L_S] = E[L_T] \) and thus \( L_S = L_T \) a.s. From the definition of \( T \) we see promptly that \( \{X_S \geq 0\} = \{X_T \geq 0\} \), and \( X_S \sim X_T \) implies \( \text{sgn}(X_S) = \text{sgn}(X_T) \) a.s.

\( \Box \)

Proof of Theorem 3

As the running supremum of an excursion is continuous, the embedding part of the theorem follows directly from Obloj [13]. So does the minimality of \( T_{\psi_+} \) and the statement \( \sup_{t \leq T_{\mu}} X_t = X_{T_{\mu}} \) is immediate.

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References

[1] F. Avram, A. E. Kyprianou, and M. R. Pistorius. Exit problems for spectrally negative Lévy processes and applications to (Canadized) Russian options. *Ann. Appl. Probab.*, 14(1):215–238, 2004.
[2] J. Bertoin. An extension of Pitman’s theorem for spectrally positive Lévy processes. *Ann. Probab.*, 20(3):1464–1483, 1992.

[3] J. Bertoin. *Lévy processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.

[4] J. Bertoin. Exponential decay and ergodicity of completely asymmetric Lévy processes in a finite interval. *Ann. Appl. Probab.*, 7(1):156–169, 1997.

[5] J. Bertoin and Y. Le Jan. Representation of measures by balayage from a regular recurrent point. *Ann. Probab.*, 20(1):538–548, 1992.

[6] N. H. Bingham. Fluctuation theory in continuous time. *Advances in Appl. Probability*, 7(4):705–766, 1975.

[7] A. Cox, D. Hobson, and J. Obłoj. Pathwise inequalities of the local time: applications to skorokhod embeddings and optimal stopping. ArXiv: math.PR/0702173, 2007.

[8] A. Cox and J. Obłoj. Classes of Skorokhod embeddings for simple symmetric random walk. ArXiv: math.PR/0609330, 2006.

[9] K. Itô. Poisson point processes attached to Markov processes. In *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. III: Probability theory*, pages 225–239, Berkeley, Calif., 1970. Univ. California Press.

[10] A. E. Kyprianou. *Introductory lectures on fluctuations of Lévy processes with applications*. Universitext. Springer-Verlag, Berlin, 2006.

[11] A. Lambert. Completely asymmetric Lévy processes confined in a finite interval. *Ann. Inst. H. Poincaré Probab. Statist.*, 36(2):251–274, 2000.

[12] P. W. Millar. Exit properties of stochastic processes with stationary independent increments. *Trans. Amer. Math. Soc.*, 178:459–479, 1973.

[13] I. Monroe. On embedding right continuous martingales in Brownian motion. *Ann. Math. Statist.*, 43:1293–1311, 1972.

[14] J. Obłoj. The Skorokhod embedding problem and its offspring. *Probability Surveys*, 1:321–392, 2004.

[15] J. Obłoj. An explicit Skorokhod embedding for functionals of excursions of Markov processes. *Stochastic Process. Appl.*, 117:409–431, 2007.

[16] M. R. Pistorius. A potential-theoretical review of some exit problems of spectrally negative Lévy processes. In *Séminaire de Probabilités XXXVIII*, volume 1857 of *Lecture Notes in Math.*, pages 30–41. Springer, Berlin, 2005.
[17] M. R. Pistorius. An excursion theoretical approach to some boundary crossing problems and the skorokhod embedding for reflected Lévy processes. In Séminaire de Probabilités XL, volume 1899 of Lecture Notes in Math., pages 287 – 308. Springer, Berlin, 2007.

[18] H. Rost. The stopping distributions of a Markov Process. Invent. Math., 14:1–16, 1971.

[19] A. V. Skorokhod. Studies in the theory of random processes. Translated from the Russian by Scripta Technica, Inc. Addison-Wesley Publishing Co., Inc., Reading, Mass., 1965.

[20] P. Vallois. Le problème de Skorokhod sur $\mathbb{R}$: une approche avec le temps local. In Séminaire de Probabilités, XVII, volume 986 of Lecture Notes in Math., pages 227–239. Springer, Berlin, 1983.