A NOTE ON THE KÜNNETH THEOREM FOR NONNUCLEAR $C^*$-ALGEBRAS

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Abstract. In this mostly expository note, we revisit the Küneth theorem in $K$-theory of nonnuclear $C^*$-algebras. We show that, using examples considered by Skandalis, there are algebras satisfying the Küneth theorem for the minimal tensor product but not for the maximal tensor product and vice versa.

1. Introduction

Let $A$ be a $C^*$-algebra. Suppose that $A$ is nuclear, that is, for any $C^*$-algebra $B$, the algebraic tensor product $A \otimes B$ admits a unique $C^*$-norm. Let $A \otimes B$ denote the completion. Let $K_*$ denote the $\mathbb{Z}/2\mathbb{Z}$-graded topological $K$-theory.

The Küneth theorem, first studied by Atiyah in the abelian case [Ati62] and Schochet in the general (nuclear) case [Sch82], concerns the question of to what extent the natural $\mathbb{Z}/2\mathbb{Z}$-graded product map

$$\alpha: K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$$

is an isomorphism. The following is the original statement of Schochet. See also [Bla98, CEOO04].

Theorem 1.1 (Sch82). Let $A$ and $B$ be $C^*$-algebras with $A$ in the smallest subcategory of the category of separable nuclear $C^*$-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by $\mathbb{Z}$ and by $\mathbb{R}$. Then there is a natural $\mathbb{Z}/2\mathbb{Z}$-graded Küneth exact sequence

$$0 \to K_*(A) \otimes K_*(B) \xrightarrow{\alpha} K_*(A \otimes B) \to \text{Tor}(K_*(A), K_*(B)) \to 0.$$  (1.2)

Remark 1.2. (1) It was shown in [RS87] that the Küneth exact sequence (1.2) always splits.

(2) It is an open problem whether all separable nuclear $C^*$-algebras satisfy the Küneth exact sequence (1.2).

For general $C^*$-algebras $A$ and $B$, the algebraic tensor product $A \otimes B$ can be completed to a $C^*$-algebra in various ways. In this note, we consider the
maximal tensor product $A \otimes_{\text{max}} B$ and the minimal tensor product $A \otimes_{\text{min}} B$ (see [Tak02, BO08]). We let $\pi = \pi_{A,B}$ denote the natural map

$$\pi : A \otimes_{\text{max}} B \longrightarrow A \otimes_{\text{min}} B.$$  \hfill (1.3)

In [Ska88], Skandalis constructed examples of algebras $A$ and $B$ such that the map $\pi_{A,B}$ is not isomorphic on $K$-theory (see Example 5.3). Hence, for the Künneth theorem for general $C^*$-algebras, we need to distinguish the tensor products $\otimes_{\text{max}}$ and $\otimes_{\text{min}}$.

We consider the Künneth theorem for $\otimes_{\text{min}}$ in Section 3 and $\otimes_{\text{max}}$ in Section 4. Counterexamples are discussed in Section 5. We note that these counterexamples are not new and were considered in [Ska88, Ska91, CEO04, HIG04].

For the convenience of the reader, we start by recalling the mapping cone construction and the Puppe exact sequence in Section 2. We remark that we do not assume that our $C^*$-algebras are separable, since it is an unnatural and unnecessary restriction from our point of view. However, we do restrict, for simplicity, to separable algebras when we deal with $KK$ or $E$-theory.

In Appendix A we sketch Skandalis’ examples.

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## 2. Mapping Cones

We recall the Puppe exact sequence in $K$-theory. All the material in this section are well-known. See [Ros82, Sch84, Bla98, CMR07].

Let

$$C_0[0,1] := \{ f : [0,1] \rightarrow \mathbb{C} \mid f(1) = 0 \} $$  \hfill (2.1)

and let

$$ev_0 : C_0[0,1] \rightarrow \mathbb{C}, \quad f \mapsto f(0) $$  \hfill (2.2)

denote the evaluation map at $0 \in [0,1)$.

**Definition 2.1.** Let $\phi : A \rightarrow B$ be a $*$-homomorphism. The mapping cone $C_{\phi}$ of $\phi$ is the pullback

$$
\begin{array}{ccc}
C_{\phi} & \longrightarrow & C_0[0,1) \otimes B \\
\downarrow & & \downarrow ev_0 \otimes \text{id}_B \\
A & \phi \longrightarrow & B \\
\end{array}
$$  \hfill (2.3)

**Theorem 2.2** (Puppe Exact Sequence). Let $\phi : A \rightarrow B$ be a $*$-homomorphism. Then there is a natural 6-term exact sequence

$$
\begin{array}{ccc}
K_0(C_{\phi}) & \longrightarrow & K_0(A) \xrightarrow{\phi_*} K_0(B) \\
\downarrow & & \downarrow \\
K_1(B) & \xleftarrow{\phi_*} & K_1(A) \longrightarrow K_1(C_{\phi})
\end{array}
$$  \hfill (2.4)
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Proof. See [Ros82, Theorem 3.8 & 4.1] or [CMR07, Theorem 2.38]. □

Corollary 2.3. Let $\phi: A \to B$ be a $*$-homomorphism. Then $\phi$ induces an isomorphism $\phi_*: K_*(A) \cong K_*(B)$ if and only if $K_*(C_\phi) = 0$. □

The following properties of the mapping cone are folklores and follow immediately from Proposition 2.6.

Proposition 2.4. Let $\phi: A \to B$ be a $*$-homomorphism and let $D$ be a $C^*$-algebra. Then we have natural isomorphisms

\[
C_\phi \otimes_{\max} D \cong C_\phi \otimes_{\max} \text{id}_D, \quad (2.5)
\]

\[
C_\phi \otimes_{\min} D \cong C_\phi \otimes_{\min} \text{id}_D. \quad (2.6)
\]

Proposition 2.5. Let $G$ be a locally compact topological group and let $\phi: A \to B$ be a morphisms of $G$-$C^*$-algebras. Then $C_\phi$ is a $G$-$C^*$-algebra and

\[
C_\phi \rtimes G \cong C_\phi \rtimes G. \quad (2.7)
\]

Proposition 2.6. Consider a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & I \\
\longrightarrow & & \Downarrow \\
0 & \longrightarrow & I \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & X \\
\longrightarrow & \longrightarrow & A \\
\longrightarrow & \longrightarrow & 0 \\
\end{array}
\begin{array}{ccc}
\longrightarrow & D \\
\longrightarrow & B \\
\longrightarrow & 0 \\
\end{array}
\]

Suppose that the lower row is exact. Then the right-hand square is a pullback diagram if and only if the upper row is exact.

Proof. See [Ped99, Proposition 3.1]. □

3. THE MINIMAL TENSOR PRODUCT

The following is the Küneth theorem for the minimal tensor product. The equivalence (2) $\iff$ (3) is shown in [CEOO04]. The condition (1) is an analogue of the condition (iii) of [Ska88, Proposition 5.3].

Theorem 3.1 (Küneth theorem for $\otimes_{\min}$). Let $A$ be a $C^*$-algebra. Then the following conditions on $A$ are equivalent.

1. For any $C^*$-algebra $B$, if $K_*(B) = 0$ then $K_*(A \otimes_{\min} B) = 0$.

2. For any $C^*$-algebra $B$, if $K_*(B)$ is free then the product map

$\alpha_{\min}: K_*(A) \otimes K_*(B) \to K_*(A \otimes_{\min} B)$

is an isomorphism.

3. For any $C^*$-algebra $B$, there is a (natural) short exact sequence

$0 \to K_*(A) \otimes K_*(B) \xrightarrow{\alpha_{\min}} K_*(A \otimes_{\min} B) \to \text{Tor}(K_*(A), K_*(B)) \to 0$. 

Proof. See [Ped99, Proposition 3.1]. □
Proof. The implications \((3) \Rightarrow (2) \Rightarrow (1)\) are clear. The implication \((2) \Rightarrow (3)\) is due to Schochet and follows from the existence of a geometric resolution (cf. Proof of [Sch82, Theorem 4.1] or [CEOO04, Theorem 3.3]).

For the implication \((1) \Rightarrow (2)\), let \(B\) be a \(C^*\)-algebra with \(K^*_s(B)\) free. Let \(\Sigma^n := C_0(\mathbb{R}^n), n \geq 0\). In the following, we abbreviate \(\otimes\) by \(\otimes\).

Since \(K^*_s(B)\) is free, there is an abelian \(C^*\)-algebra of the form \(D = \oplus \Lambda_1 \Sigma^2 \oplus \oplus \Lambda_2 \Sigma^3\) and a \(\ast\)-homomorphism \(\varphi: D \rightarrow \Sigma^2 \otimes B \otimes K\) inducing isomorphism in \(K\)-theory, where \(K\) is the \(C^*\)-algebra of compact operators on a suitable Hilbert space. Let \(C_\varphi\) denote the mapping cone of \(\varphi\). Then \(K^*_s(C_\varphi) = 0\) by Corollary 2.3. Since \(A \otimes C_\varphi \cong C_{id_A \otimes \varphi}\), we see that \(K^*_s(C_{id_A \otimes \varphi}) = K^*_s(A \otimes C_\varphi) = 0\) by (1), hence \(id_A \otimes \varphi\) induces an isomorphism in \(K\)-theory, again by Corollary 2.3. The top map in the following commutative diagram is clearly an isomorphism, thus it follows that \(\alpha_{\min}\) is an isomorphism for \((A, \Sigma^2 \otimes B \otimes K)\).

\[
\begin{array}{ccc}
K^*_s(A) \otimes K^*_s(D) & \overset{\cong}{\longrightarrow} & K^*_s(A \otimes D) \\
\downarrow K^*_s(A) \otimes \varphi^* & & \downarrow (id_A \otimes \varphi)^* \\
K^*_s(A) \otimes K^*_s(\Sigma^2 \otimes B \otimes K) & \overset{\alpha_{\min}}{\longrightarrow} & K^*_s(A \otimes \Sigma^2 \otimes B \otimes K)
\end{array}
\] (3.1)

Now Bott periodicity completes the proof. \(\square\)

Remark 3.2 (Separable algebras). Let \(A\) be a \(C^*\)-algebra. The proof of Theorem 3.1 shows that the following conditions are equivalent\(^1\). See also [CEOO04, Theorem 3.3].

(1') For any separable \(C^*\)-algebra \(B\), if \(K^*_s(B) = 0\) then \(K^*_s(A \otimes_{\min} B) = 0\).

(2') For any separable \(C^*\)-algebra \(B\), if \(K^*_s(B)\) is free then the product map

\[
\alpha_{\min}: K^*_s(A) \otimes K^*_s(B) \rightarrow K^*_s(A \otimes_{\min} B)
\]

is an isomorphism.

(3') For any separable \(C^*\)-algebra \(B\), there is a (natural) short exact sequence

\[
0 \rightarrow K^*_s(A) \otimes K^*_s(B) \overset{\alpha_{\min}}{\longrightarrow} K^*_s(A \otimes_{\min} B) \rightarrow \text{Tor}(K^*_s(A), K^*_s(B)) \rightarrow 0.
\]

Moreover, it is easy to see that the a priori weaker condition (3') is equivalent to (3). Indeed, write \(B\) as the inductive limit of its separable \(C^*\)-subalgebras under inclusions: \(B \cong \varprojlim_{B' \subseteq B} B'\), \(B'\) separable. Then

\[
A \otimes_{\min} B \cong \varprojlim_{B' \subseteq B} A \otimes_{\min} B'
\] (3.2)

and the implication (3') \(\Rightarrow (3)\) follows from the continuity of \(K\)-theory and the fact that tensor products of abelian groups commute with direct limits. It follows that all six conditions are equivalent, hence we may restrict to \(B\) separable in Theorem 3.1.

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\(^1\) If \(A\) is also separable, the proof can be shortened using \(KK\)-theory.
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**Definition 3.3.** Let \( \mathcal{N}_{\text{min}} \) denote the class of C*-algebras \( A \) satisfying the equivalent conditions of Theorem 3.1 and Remark 3.2.

Now we survey some results about \( \mathcal{N}_{\text{min}} \) and list some examples.

**Lemma 3.4** ([Sch82, Section 2], [Bla98, 23.4] or [CEOO04, Lemma 4.4]).

The class \( \mathcal{N}_{\text{min}} \) enjoys the following properties.

1. If \( A \in \mathcal{N}_{\text{min}} \) and \( B \) is Morita dominated by \( A \), then \( B \in \mathcal{N}_{\text{min}} \). In particular, \( \mathcal{N}_{\text{min}} \) is stable under Morita equivalence.
2. In a semi-split short exact sequence of C*-algebras, if two of the algebras are in \( \mathcal{N}_{\text{min}} \), then so is the third.
3. If \( A, B \in \mathcal{N}_{\text{min}} \), then \( A \otimes_{\text{min}} B \in \mathcal{N}_{\text{min}} \).
4. If \( A = \lim A_i \), such that all structure maps are injective and all \( A_i \in \mathcal{N}_{\text{min}} \), then \( A \in \mathcal{N}_{\text{min}} \).

The following result of Chabert-Echterhoff-Oyono-Oyono generalises the \( \mathbb{Z} \) and \( \mathbb{R} \) case considered by Schochet.

**Theorem 3.5** ([CEOO04, Corollary 0.2]). Let \( G \) be a second countable locally compact topological group satisfying the Baum-Connes conjecture with coefficients (cf. [BCH94, Conjecture 9.6]). Let \( A \) be a separable \( G \)-algebra.

**Theorem 3.5.** If \( A \rtimes K \in \mathcal{N}_{\text{min}} \) for all compact subgroups \( K \subseteq G \), then \( A \rtimes_{\text{red}} G \in \mathcal{N}_{\text{min}} \).

The following is essentially a repackaging of their proof.

**Proof.** By [MN06, Theorem 9.3], the Baum-Connes conjecture with coefficients can be restated as follows:

For any \( G \)-C*-algebra \( A \), if \( K_*(A \rtimes K) = 0 \) for all compact subgroups \( K \subseteq G \), then \( K_*(A \rtimes_{\text{red}} G) = 0 \).

Now the proof is easily completed by appealing to Theorem 3.1.

**Example 3.6.** (1) Type I algebras are in \( \mathcal{N}_{\text{min}} \) (Schochet [Sch82, Theorem 2.13]).

(2) Any separable C*-algebra in the bootstrap category of C*-algebras KK-equivalent to an abelian C*-algebra is in \( \mathcal{N}_{\text{min}} \). (Rosenberg-Shochet [RSS72].)

The groupoid C*-algebra of an amenable groupoid (Tu [Tu99, Proposition 10.7]) and the full and reduced group C*-algebras \( C^*(G) \) and \( C^*_\lambda(G) \) of an almost-connected group (Chabert-Echterhoff-Oyono-Oyono [CEOO04, Proposition 5.1]) are in the bootstrap category, hence in \( \mathcal{N}_{\text{min}} \).

(3) Let \( G \) be a separable locally compact group such that the component group \( G/G_0 \) satisfies the Baum-Connes conjecture with coefficients. Then the reduced group algebra \( C^*_\lambda(G) \) is in \( \mathcal{N}_{\text{min}} \). (Chabert-Echterhoff-Oyono-Oyono [CEOO04, Corollary 0.3]).

However, as we see below, there are non-nuclear (in fact non-exact, see Remark 3.10) C*-algebras that are not in \( \mathcal{N}_{\text{min}} \).
Definition 3.7. We say that a $C^*$-algebra $A$ is $K$-exact if the functor $B \mapsto K_0(A \otimes_{\min} B)$ is half-exact.

Clearly, exact $C^*$-algebras are $K$-exact.

The following remark is due to Skandalis (c.f. [CEO04, Remark 4.3]).

Remark 3.8. Associated to an extension
$$0 \rightarrow I \rightarrow B \rightarrow D \rightarrow 0,$$ (3.3)
there is a double-cone algebra $C$ such that the sequence
$$K_*(A \otimes_{\min} I) \rightarrow K_*(A \otimes_{\min} B) \rightarrow K_*(A \otimes_{\min} D)$$ (3.4)
is exact in the middle if and only if $K_*(A \otimes_{\min} C) = 0$ (see [HLS02, p. 335-336]).

It follows that all $C^*$-algebras in $\mathcal{N}_{\min}$ are $K$-exact.

Moreover, the construction of a double-cone is functorial and commutes with inductive limits of extensions. Thus $A$ is $K$-exact if the functor $B \mapsto K_0(A \otimes_{\min} B)$ is half-exact on extensions of separable $C^*$-algebras.

Example 3.9. (1) Let $\Gamma$ be an infinite countable discrete group with Khazdan property (T), Kirchberg property (F) and Akemann-Ostrand property (AO) (cf. [AD09]). Then the full group $C^*$-algebra $C^*(\Gamma)$ is not $K$-exact, hence not in $\mathcal{N}_{\min}$. (Skandalis [Ska91]).

(2) The product $\prod_{n \geq 1} M_n$ is not $K$-exact, hence not in $\mathcal{N}_{\min}$. (Ozawa [Oza03, Theorem A.1]).

Remark 3.10. We note that if a separable $C^*$-algebra $A$ is not $K$-exact, then it cannot be $KK$-equivalent to an exact $C^*$-algebra.

Definition 3.11. We say that a $C^*$-algebra $A$ is $K$-continuous if the functor $B \mapsto K_0(A \otimes_{\min} B)$ is continuous i.e. commutes with inductive limits.

Clearly, $C^*$-algebras in $\mathcal{N}_{\min}$ are $K$-continuous. The following is less trivial.

Theorem 3.12. All $K$-continuous algebras are $K$-exact.

Proof. Let $A$ be a $K$-continuous $C^*$-algebra and let $F(B) := K_0(A \otimes_{\min} B)$. Then by [Dad94, Theorem 3.11], $F$ factors through the asymptotic homotopy category of Connes-Higson [CH90]. In particular, for any extension of separable $C^*$-algebras, the inclusion of the kernel into the mapping cone of the quotient map induces an isomorphism on $F$. It follows that $F$ is half-exact on separable $C^*$-algebras. The general case follows from Remark 3.8.

Consequently, Example 3.9 give examples of $C^*$-algebras which are not $K$-continuous.

\footnote{It is the mapping cone of the inclusion of $I$ into the mapping cone of the quotient map $B \rightarrow D$.}
4. The Maximal Tensor Product

The maximal tensor product case is analogous, hence we shall be brief.

**Theorem 4.1** (K"unneth theorem for \(\otimes_{\text{max}}\)). Let \(A\) be a \(C^\ast\)-algebra. Then the following conditions on \(A\) are equivalent.

1. For any \(C^\ast\)-algebra \(B\), if \(K^\ast(B) = 0\) then \(K^\ast(A \otimes_{\text{max}} B) = 0\).
2. For any \(C^\ast\)-algebra \(B\), if \(K^\ast(B)\) is free then the product map
   \[\alpha_{\text{max}} : K^\ast(A) \otimes K^\ast(B) \to K^\ast(A \otimes_{\text{max}} B)\]
   is an isomorphism.
3. For any \(C^\ast\)-algebra \(B\), there is a (natural) short exact sequence
   \[0 \to K^\ast(A) \otimes K^\ast(B) \xrightarrow{\alpha_{\text{max}}} K^\ast(A \otimes_{\text{max}} B) \to \text{Tor}(K^\ast(A), K^\ast(B)) \to 0.\]

□

Needless to say, for nuclear algebras, the K"unneth theorems 3.1 and 4.1 are equivalent.

**Remark 4.2** (Separable algebras). Let \(A\) be a \(C^\ast\)-algebra. The following conditions are equivalent to the (equivalent) conditions in Theorem 4.1.

1. For any separable \(C^\ast\)-algebra \(B\), if \(K^\ast(B) = 0\) then \(K^\ast(A \otimes_{\text{max}} B) = 0\).
2. For any separable \(C^\ast\)-algebra \(B\), if \(K^\ast(B)\) is free then the product map
   \[\alpha_{\text{max}} : K^\ast(A) \otimes K^\ast(B) \to K^\ast(A \otimes_{\text{max}} B)\]
   is an isomorphism.
3. For any separable \(C^\ast\)-algebra \(B\), there is a (natural) short exact sequence
   \[0 \to K^\ast(A) \otimes K^\ast(B) \xrightarrow{\alpha_{\text{max}}} K^\ast(A \otimes_{\text{max}} B) \to \text{Tor}(K^\ast(A), K^\ast(B)) \to 0.\]

□

**Definition 4.3.** Let \(\mathcal{N}_{\text{max}}\) denote the class of \(C^\ast\)-algebras \(A\) satisfying the equivalent conditions of Theorem 4.1 and Remark 4.2.

**Lemma 4.4.** The class \(\mathcal{N}_{\text{max}}\) enjoys the following properties.

1. If \(A \in \mathcal{N}_{\text{max}}\) and \(B\) is Morita dominated by \(A\), then \(B \in \mathcal{N}_{\text{max}}\). In particular, \(\mathcal{N}_{\text{max}}\) is stable under Morita equivalence.
2. In a short exact sequence of \(C^\ast\)-algebras, if two of the algebras are in \(\mathcal{N}_{\text{max}}\), then so is the third.
3. If \(A, B \in \mathcal{N}_{\text{max}}\), then \(A \otimes_{\text{max}} B \in \mathcal{N}_{\text{max}}\).
4. If \(A = \lim A_i\) and all \(A_i \in \mathcal{N}_{\text{max}}\), then \(A \in \mathcal{N}_{\text{max}}\).

□

We remark that for any \(C^\ast\)-algebra \(A\), the functor \(B \mapsto K_0(A \otimes_{\text{max}} B)\) is half-exact and continuous. Hence we cannot use the same techniques as in Section 3 to construct counterexamples to the K"unneth theorem for \(\otimes_{\text{max}}\). However, see Example 5.3.
5. Counterexamples

The counterexamples exploit the difference between Lemma 3.4(2) and Lemma 4.4(2).

Example 5.1 ($\mathcal{N}_{\text{max}} \setminus \mathcal{N}_{\text{min}} \neq \emptyset$; c.f. [Ska91], [HG04, Theorem 5.4]). Let $C$ be the double-cone of an extension

$$0 \to I \to B \to D \to 0 \quad (5.1)$$

of $C^*$-algebras (see Remark 3.8). Then for any $C^*$-algebra $A$, the tensor product $C \otimes_{\text{max}} A$ is the double-cone of the extension

$$0 \to I \otimes_{\text{max}} A \to B \otimes_{\text{max}} A \to D \otimes_{\text{max}} A \to 0. \quad (5.2)$$

It follows that $K_*(C \otimes_{\text{max}} A) = 0$ for all $A$ and $C$ belongs to $\mathcal{N}_{\text{max}}$.

Let $A$ be a non-$K$-exact algebra and let $C$ be the double-cone of an extension for which $K_*(A \otimes_{\text{min}} C) \neq 0$. Then $C$ does not belong to $\mathcal{N}_{\text{min}}$. Hence $\mathcal{N}_{\text{max}} \setminus \mathcal{N}_{\text{min}} \neq \emptyset$.

Here is a concrete example: Let $\Gamma = \text{SL}_3(\mathbb{Z})$ and let

$$0 \to J \to A \to B \to 0 \quad (5.3)$$

denote the extension of separable commutative $\Gamma$-$C^*$-algebras of [Oza03, Theorem A.1]. Let $C$ denote the double-cone of (5.3). Then $C$ is a separable commutative $\Gamma$-$C^*$-algebra and the full crossed product $C \rtimes \Gamma$ is the double cone of the extension

$$0 \to J \rtimes \Gamma \to A \rtimes \Gamma \to B \rtimes \Gamma \to 0 \quad (5.4)$$

(See Proposition 2.5). Hence $C \rtimes \Gamma \in \mathcal{N}_{\text{max}} \setminus \mathcal{N}_{\text{min}}$.

The following observation is due to Skandalis [Ska88].

Lemma 5.2. Let $A$ be a $C^*$-algebra. Suppose that there is a $C^*$-algebra $B$ such that the natural map

$$\pi: A \otimes_{\text{max}} B \to A \otimes_{\text{min}} B \quad (5.5)$$

does not induce isomorphism in $K$-theory. Then the following statements are true.

1. The algebra $A$ fails one of the Künneth theorems (3.3) or (4.3).
2. If $A$ is separable, then $A$ is not $KK$-equivalent to a nuclear algebra.
3. If $A$ is separable and exact, then $A$ is not $E$-equivalent to a nuclear algebra.

Proof. Enough to note that we may assume that $B$ is separable. See [Ska88, HG04].

Example 5.3 ($\mathcal{N}_{\text{min}} \setminus \mathcal{N}_{\text{max}} \neq \emptyset$; c.f. [CEO04, Introduction]). Let $\Gamma$ be an infinite countable discrete group with Kazhdan property (T) and Akemann-Ostrand property (AO) (cf. [AD09]). Then the natural map

$$C^*_\lambda(\Gamma) \otimes_{\text{max}} C^*_\lambda(\Gamma) \to C^*_\lambda(\Gamma) \otimes_{\text{min}} C^*_\lambda(\Gamma) \quad (5.6)$$

does not induce isomorphism in $K$-theory (Skandalis [Ska88]).
We specialise to the case $\Gamma$ a lattice in $\text{Sp}(n,1)$. Julg proved that $\Gamma$ satisfies the Baum-Connes conjecture with coefficients $[\text{Jul02}]$. Then by $[\text{CEO04}, \text{Corollary 0.2}]$, we see that $C_\chi^*(\Gamma)$ is in $\mathcal{N}_{\min}$. Consequently, $C_\lambda^*(\Gamma)$ is not in $\mathcal{N}_{\max}$ by Lemma 5.2 and $\mathcal{N}_{\min} \setminus \mathcal{N}_{\max} \neq \emptyset$.

**Example 5.4.** Let $A \in \mathcal{N}_{\min} \setminus \mathcal{N}_{\max}$ and let $B \in \mathcal{N}_{\max} \setminus \mathcal{N}_{\min}$. Then it follows from the 2-out-of-3 property that $A \oplus B$ is neither in $\mathcal{N}_{\max}$ nor in $\mathcal{N}_{\min}$.

Examples 5.1, 5.3 and 5.4 answer some of the questions raised by Blackadar in $[\text{Bla98}, 23.13.2]$.

**Appendix A. Skandalis’ Examples**

We briefly sketch Skandalis’ arguments for the convenience of the reader. See $[\text{Ska88}, \text{Ska91}, \text{HG04}]$ for details.

We refer to $[\text{AD09}]$ for group theoretic terminologies in the following.

**Theorem A.1** (Skandalis). Let $\Gamma$ be an infinite countable discrete group with property (T) and property (AO). Then the natural map

$$C_\chi^\Gamma \otimes_{\max} C^*\Gamma \rightarrow C_\lambda^\Gamma \otimes_{\min} C^*\Gamma$$  \hspace{1cm} (A.1)

does not induce isomorphism in $K$-theory. If in addition, $\Gamma$ has property (F), then $C^*\Gamma$ is not $K$-exact.

See Example 3.9(1) and Example 5.3

**Sketch of Proof.** We assume that $\Gamma$ has (T), (AO) and (F). Then one can construct a commutative diagram of the form

where $\bullet \longrightarrow \bullet$ denotes a quotient map and $\bullet \longrightarrow \bullet$ denotes an extension. Here $I$ is the kernel of $C^*\Gamma \rightarrow C_\chi^\Gamma$.
Let \( q \in C^*\Gamma \otimes_{\max} C^*\Gamma \) denote the image of the Kazhdan projection under the diagonal map

\[
\Delta : C^*\Gamma \to C^*\Gamma \otimes_{\max} C^*\Gamma, \quad \gamma \mapsto \gamma \otimes \gamma.
\]  

(A.3)

Then the image of \( q \) in \( C^*_\lambda \otimes_{\min} C^*\Gamma \) is zero, while the image in \( B(l^2\Gamma) \) is non-zero. Hence \( q \) defines non-zero classes in \( K_0(J) \) and \( K_0(L) \). Moreover, the composition

\[
I \otimes_{\min} C^*\Gamma \longrightarrow L \longrightarrow K_0(l^2\Gamma)
\]  

(A.4)

is zero, since the composition \( I \otimes_{\max} C^*\Gamma \longrightarrow B(l^2\Gamma) \) is zero.

It follows that the class in \( K_0(L) \) cannot come from \( K_0(I \otimes_{\min} C^*\Gamma) \). \( \square \)

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