Constraint Dynamics and RQM Bound States

Mikhail N. Sergeenko

Fr. Skaryna Gomel State University, BY-246019, Gomel, Belarus
mnsergey@tut.by

Abstract

Flavored mesons containing quarks of unequal masses are studied. The appropriate tool is the Bethe-Salpeter formalism, but its inherent complexity leads to series of difficulties mostly related to the central role played in it by the relative time or energy. We consider bound states in the spirit of “Constraint Relativistic Quantum Mechanics (RQM)”. Interaction of quarks is described by the funnel-type potential with the distant dependent strong coupling, $\alpha_s(r)$. Relativistic bound-state problem is formulated with the use of symmetries, energy-momentum conservation laws in Minkowskian space. Relativistic two-body wave equation with position dependent particle masses is derived and used to describe the flavored mesons. Free particle hypothesis for the bound state is developed: quark and antiquark move as free particles in of the bound system. Solution of the equation for the system in the form of a standing wave is given. Interpolating complex-mass formula for two exact asymptotic eigenmass expressions is obtained. Mass spectra for some leading-state flavored mesons are calculated.

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I. INTRODUCTION

Most numerous of hadrons summarized in the Particle Data Group (PDG) tables [1] are mesons. Simplest of them are quarkonia, quark-antiquark ($q\bar{q}$ and $Q\bar{Q}$) bound states containing quarks of equal masses. It is believed that physics of light and heavy mesons is different, but this is true only in asymptotic limits of large and small distances. In case of heavy-light $Q\bar{q}$ mesons situation is much more complicated. Most mesons listed in the PDG being unstable and are resonances, exited quark-antiquark states. There are great amount and variety of experimental data and the different approaches used to extract the properties of the mesons [2–4].

The strict description of mesons as quark-antiquark bound states in a way fully consistent with all requirements imposed by special relativity and Quantum Mechanics (QM) is one of the great challenges in theoretical elementary particle physics. Such description can be done within the framework of Quantum Field Theory (QFT). The appropriate tool to achieve this goal is the Bethe-Salpeter (BS) formalism [5]. However, attempts to apply the BS formalism to relativistic bound-state problems lead to a number of various difficulties. The usual practice consists in eliminating the relative time variable (3D reduction). The 3D reduction of the two-fermion BS equation has been performed in many works [6–11]; all these methods are theoretically equivalent. There were suggested noncovariant instantaneous truncations of the BS equation [12, 13]. The most well-known of the R2B equations is the one proposed by Salpeter [6]. The equivalence of simplified equations with the original BS equation can be proved exactly [14].

In QED, the Coulomb gauge is the most convenient for treating the R2B problem, since it allows the optimal expansion of the BS equation around the NR theory [15–17]. The main disadvantage of the Coulomb gauge is its noncovariant nature. Constraint theory leads to a manifestly covariant 3D description of relativistic two-body (R2B) systems [9, 18–22] and has opened a new perspective.

It was shown [9, 11] that the expansion of the BS equation around the constraint theory wave equations in the Feynman gauge (as well as for scalar interactions) is free of the above mentioned diseases of covariant gauges and allows a systematic study of infrared leading effects of multiphoton exchange diagrams; the latter can then be represented in three-dimensional (3D) x-space as local potentials. Summing the series of these leading terms one obtains a local potential in compact form [11], which is well suited for a continuation to the strong coupling domain (QCD) of the theory or for a generalization to other effective interactions.

In this work, we study $Q\bar{q}$ mesons and their excitations (resonances) as R2B systems from unified point of view in the framework of the RQM [23, 24]. Difficulties encountered here are related to 1) two-particle relativistic equation of motion and 2) absence of a strict definition of the potential in relativistic theory. We begin our consideration of the R2B problem with relativistic classical mechanics. Relativistic bound state problem is formulated with the use of symmetries, energy-momentum conservation laws in Minkowskian space. The potential of
interaction is treated as the Lorentz-scalar function of the spatial variable \( r \), the distance between particles. The concept of position dependent particle mass is developed. Using the correspondence principle, we deduce, from the R2B classic equation, the two-particle wave equation. The free particle hypothesis for the bound state is developed: particles inside the system move as free ones. Complex eigenmasses for the bound system are obtained. The relative motion of quarks in eigen states is described (in the physical region) by the standing wave of the form \( C_n \sin(k_n x + \delta_n) \) for each spatial degree of freedom. To verify the model, the R2B wave equation with position dependent quark masses is used to describe the flavored \( Q\bar{q} \) mesons.

II. BOUND STATES IN CONSTRAINT THEORY

The constraint theory first successfully yielded a covariant yet canonical formulation of the R2B problem for two interacting spinless classical particles [21]. The manifestly covariant formalism with constraints in the R2B problem leads to a Poincaré invariant description of the dynamics of the system [10, 22]. The potentials that appear in the corresponding wave equations can be calculated in terms of the kernel of the BS equation, therefore allow one to deal with QFT problems.

The R2B state of two scalar particles can be described by two independent wave equations, which are generalizations of the individual Klein-Gordon equation of each particle, including the mutual interaction potential [11]. The compatibility condition of the two equations imposes certain restrictions on the structure of the potential and leads, in a covariant form, to an elimination of the relative energy variable. This results in the manifestly covariant, 3D eigenvalue equation that describes the relative motion of the two particles [11]. This equation is very similar in form to the Schrödinger (or Klein-Gordon) equation: it is a second order differential equation in the three spacelike coordinates and therefore the usual techniques of NR QM are applicable to it.

For two fermions, the system is described by two independent Dirac type equations [22]. In this case, the compatibility condition imposes restrictions on the structure of the potentials and eliminates the relative energy variable; however, because of the Dirac matrices, the reduction to a final eigenvalue equation is not straightforward. The reduction process is rather complicated and depends on the way of eliminating the components of the spinor wave function in terms of one of them. Up to now, no single Pauli-Schrödinger-type equation was obtained from this procedure.

The principal difficulty in the treatment of the BS equation (1) comes from the existence of unphysical relative time and energy variables. In applications to both quantum electrodynamics (QED) and quantum chromodynamics (QCD), as well as NR reduction, some simplified equations are usually used. Such equations are usually obtained using certain restrictions or constraints.
A. The spinless Salpeter equation

The manifestly covariant BS equation obtained directly from QFT governs all the bound states and the scattering. However, attempts to apply the BS formalism to real relativistic R2B problems lead to a number of various difficulties. These are the impossibility to determine the BS interaction kernel beyond the tight limits of perturbation theory, appearance of abnormal solutions that are difficult to interpret in the framework of quantum physics. Two body BS equation \[13, 25, 26\] for spin-zero bound states is

\[
G_0^{-1}\Psi \equiv \left(p_1^2 + m_1^2\right)\left(p_2^2 + m_2^2\right)\Psi = K\Psi,
\]

where \(G_0 = G_{0,1}G_{0,2}\) is free propagator of particles. The irreducible BS kernel \(K\) would in general contain charge renormalization, vacuum polarization graphs and could contain self-energy terms transferred from the inverse propagators. The kernel \(K\) is obtained from the off-mass-shell scattering amplitude which is defined by the equation \(T = K + KG_0T\). Recent work with static models has indicated, that abnormal solutions disappear if one includes all ladder and cross ladder diagrams \[11\]. This supports Wick’s conjecture on defects of ladder approximations.

Numerous 3D quasipotential reductions of the BS equation had been proposed. The most well-known of the resulting bound-state equations is the one proposed by Salpeter \[6\]. The Salpeter equation is historically first 3D reduction of the BS equation. This is a noncovariant instantaneous truncations of the BS equation (1). The general coordinate-space relativistic spinless Salpeter (SS) equation for R2B system in the c.m. rest frame is \((\bar{\hbar} = \bar{c} = 1)\)

\[
\left[ \sqrt{(-i\nabla)^2 + m_1^2} + \sqrt{(-i\nabla)^2 + m_2^2} + V(r) \right] \psi(\vec{r}) = M\psi(\vec{r}),
\]

where \(V(r)\) is the potential of interaction and \(M\) is the bound-system mass (the c.m. rest energy \(E^* = M = w\)). However, as in the case of the BS equation, it is a problem to find the analytic solution of the equation (2). The problem originates from two square root operators which cause a serious difficulties. It can not be reduced to the second-order differential equation of the Shrődinger type \[9\].

There exist many other approaches to bound-state problem. One of the promising among them is the Regge method in hadron physics \[27\]. All hadrons and their resonances in this approach are associated with Regge poles which move in the complex angular momentum \(J\) plane. Moving poles are described by the Regge trajectories, \(\alpha(s)\), which are the functions of the invariant squared mass \(s = w^2 = (E^*)^2\) (Mandelstam’s variable). Hadrons and resonances populate their Regge trajectories which contain all the dynamics of the strong interaction in bound state and scattering regions.

Mesons have been studied in a soft-wall holographic approach AdS/CFT \[28\] using the correspondence of string theory in Anti-de Sitter space and conformal field theory in physical space-time. It is analogous to the Schrödinger theory for atomic physics and provides a precise
mapping of the string modes \( \Phi(z) \) in the AdS fifth dimension \( z \) to the hadron light-front wave functions in physical space-time.

Reductions of the BS equation can be obtained from iterating this equation around a 3D Lorentz-invariant hypersurface in relative momentum \( (p) \) space. This leads to invariant 3D wave equations for relative motion. The resultant 3D wave equation is not unique, but depends on the nature of the 3D hypersurface.

### B. Todorov’s quasipotential equation

Valuable are methods which provide either exact or approximate analytic solutions for various forms of differential equations. They may be remedied in 3D reductions of the BS equation. In most cases the analytic solution can be found if original equation is reduced to the Schrödinger-type wave equation. One can choose Todorov’s quasipotential equation \([9]\) which has the Schrödinger-like form

\[
\left[ \hat{p}^2 + \Phi(x_1 - x_2) \right] \psi = \kappa_w^2 \psi, \quad (3)
\]

where the quasipotential \( \Phi \) is related to the scattering amplitude, 3D hyperfine restriction on the relative momentum \( p \) is defined by \( p \cdot P \psi = 0, \hat{p} \psi = (0, \hat{p}) \psi = 0, P = p_1 + p_2 \). The effective eigenvalue in \( (3) \) is

\[
\kappa_w^2 = \frac{1}{4w^2} \left( w^2 - m_-^2 \right) \left( w^2 - m_+^2 \right), \quad (4)
\]

with \( w = \sqrt{P^2} = E^* \) the c.m. invariant energy, \( m_- = m_1 - m_2, m_+ = m_1 + m_2 \). The forces \( \Phi \) to depend on \( x_1 - x_2 \) only through the transverse component, \( x_{\perp}^\mu = (0, \mathbf{r}) \). Thus, in the c.m. rest frame, the hypersurface restriction \( p \cdot P \psi = 0 \) not only eliminates the relative energy but implies that the relative time does not appear.

In Eikonal approximation for ladder, cross ladder, and constraint diagrams to bound states applied through all orders, it gives for scalar \((S)\) and vector \((V)\) exchanges the quasipotentials

\[
\Phi_s = 2m_w S + S^2, \quad \Phi_V = 2\epsilon_w V - V^2. \quad (5)
\]

The kinematical variables (the reduced mass \( m_w \) and energy \( \epsilon_w \) for the fictitious particle of relative motion),

\[
m_w = \frac{m_1 m_2}{w}, \quad \epsilon_w = \frac{w^2 - m_1^2 - m_2^2}{2w}, \quad (6)
\]

satisfy the Einstein’s relation

\[
\kappa_w^2 = \epsilon_w^2 - m_w^2. \quad (7)
\]

The effects of ladder and cross ladder diagrams thus embedded in their c.m. energy dependencies. The resultant 3D wave equation \((3)\) is not unique, but depends on the nature of the 3D hypersurface.
C. The two-body Dirac equations

For two fermions, the system can be described by two independent Dirac type equations \([20–22]\). In this case, the compatibility condition imposes restrictions on the structure of the potentials and eliminates the relative energy variable; however, because of the Dirac matrices, the reduction to a final eigenvalue equation is not straightforward. The reduction process is rather complicate and depends on the way of eliminating the components of the spinor wave function in terms of one of them. Up to now, no single Pauli-Schrödinger type equation was obtained from this procedure.

The R2B Dirac equations of Constraint Dynamics have dual origins. On the one hand they arise as one of the many quasipotential reductions of the BS equation. On the other they arise independently from the development of a consistent covariant approach to the R2B problem in the framework of RCM independent of QFT \([20]\).

The R2B Dirac equations of constraint dynamics provide a covariant 3D truncation of the BS equation. It was shown \([29, 30]\) that the BS equation can be algebraically transformed into two independent equations, which for spinless particles are

\[
\begin{align*}
(H_{0,1} + H_{0,2} + 2\Phi_w) \Psi(x_1, x_2) &= 0, \\
(H_{0,1} - H_{0,2}) \Psi(x_1, x_2) &= 2(p \cdot P) \Psi(x_1, x_2) = 0, 
\end{align*}
\]

where \(H_{0,i} = p_i^2 + m_i^2\), \(P = p_1 + p_2\) is the total momentum, \(p = \eta_2 p_1 - \eta_1 p_2\) is the relative momentum, \(w\) is the invariant total c.m. energy with \(P^2 = -w^2\), \(\hat{P}^\mu = P^\mu/w\) is a time-like unit vector \((\hat{P}^2 = -1)\) in the direction of the total momentum. The \(\eta_i\) are chosen so that the relative coordinate \(x = x_1 - x_2\) and \(p\) are canonically conjugate, i.e. \(\eta_1 + \eta_2 = 1\). The first equality (8) is a covariant 3D eigenvalue equation. The second equation (9) overcomes the difficulty of treating the relative time in the c.m. system by setting an invariant condition on the relative momentum \((p \cdot P) \Psi(x_1, x_2) = 0\), that implies \(p^\mu \Psi = p_1^\mu \Psi\).

III. THE INTERACTION POTENTIAL

It is well known that the potential as a function in 3D-space is defined by the propagator \(D(q^2)\) (Green function) of the virtual particle as a carrier of interaction, where \(q = p_1 - p_2\) is the transferred momentum. In case of the Coulomb potential the propagator is \(D(q^2) = -1/q^2\); the Fourier transform of \(4\pi\alpha D(q^2)\) gives the Coulomb potential, \(V(r) = -\alpha/r\). The relative momentum \(q\) is conjugate to the relative vector \(r = r_1 - r_2\), therefore, one can accept that \(V(r_1, r_2) = V(r) [31]\). If the potential is spherically symmetric, one can write \(V(r) = V(|r|)\). Thus, the system’s relative time \(\tau = t_1 - t_2 = 0\) (instantaneous interaction).

The NR QM shows very good results in describing bound states; this is partly because the potential is NR concept. In relativistic mechanics one faces with different kind of speculations around the potential, because of absence of a strict definition of the potential in this theory. In
NR formulation, the hydrogen \((H)\) atom is described by the Schrödinger equation and is usually considered as an electron moving in the external field generated by the proton static electric field given by the Coulomb potential. In relativistic case, the binding energy of an electron in a static Coulomb field (the external electric field of a point nucleus of charge \(Ze\) with infinite mass) is determined predominantly by the Dirac eigenvalue [32]. The spectroscopic data are usually analyzed with the use of the Sommerfeld’s fine-structure formula [33].

One should note that, in these calculations the \(S\) states start to be destroyed above \(Z = 137\), and that the \(P\) states being destroyed above \(Z = 274\). Similar situation we observe from the result of the Klein-Gordon wave equation, which predicts \(S\) states being destroyed above \(Z = 68\) and \(P\) states destroyed above \(Z = 82\). Besides, the radial \(S\)-wave function \(R(r)\) diverges as \(r \to 0\). These problems are general for all Lorentz-vector potentials which have been used in these calculations [34, 35]. In general, there are two different relativistic versions: the potential is considered either as the zero component of a four-vector, a Lorentz-scalar or their mixture [36]; its nature is a serious problem of relativistic potential models [37].

This problem is very important in hadron physics where, for the vector-like confining potential, there are no normalizable solutions [37, 38]. There are normalizable solutions for scalar-like potentials, but not for vector-like. This issue was investigated in [34, 39]; it was shown that the effective interaction has to be Lorentz-scalar in order to confine quarks and gluons. The relativistic correction for the case of the Lorentz-vector potential is different from that for the case of the Lorentz-scalar potential [40].

Quarkonia among all mesons are simplest as quark-antiquark bound states. The quarkonium universal mass formula and “saturating” Regge trajectories were derived in [39] and in [41, 42] applied for gluonia (glueballs). The mass formula was obtained by interpolating between NR heavy \(Q\bar{Q}\) quark system and ultra-relativistic limiting case of light \(q\bar{q}\) mesons for the Cornell potential [43, 44],

\[
V(r) = V_S(r) + V_L(r) \equiv -\frac{4\alpha_s}{3r} + \sigma r. \tag{10}
\]

The short-range Coulomb-type term \(V_S(r)\), originating from one-gluon exchange, dominates for heavy mesons and the linear one \(V_L(r)\), which models the string tension, dominates for light mesons. Parameters \(\alpha_s\) and \(\sigma\) are directly related to basic physical quantities of mesons.

Separate consideration of two asymptotic components \(V_S(r)\) and \(V_L(r)\) of the potential (10) for quarkonia results in the complex-mass expression for resonances, which in the center-of-momentum (c.m.) frame is \((\hbar = c = 1)\) [45, 46]:

\[
\mathcal{M}_N^2 = 4 \left[ \left( \sqrt{2\sigma\tilde{N}} + \frac{i\hat{\alpha}m}{\tilde{N}} \right)^2 + \left( m - i\sqrt{2\hat{\alpha}\sigma} \right)^2 \right], \tag{11}
\]

where \(\hat{\alpha} = \frac{4}{3}\alpha_s\), \(\tilde{N} = N + (k + \frac{1}{2})\), \(N = k + l + 1\), \(k\) is radial and \(l\) is orbital quantum numbers; it has the form of the squared energy \(\mathcal{M}_N^2 = 4 [(\pi_N)^2 + \mu^2]\) of two free relativistic particles with the quarks’ complex momenta \(\pi_N\) and masses \(\mu\). This formula allows to calculate in a unified
way the centered masses and total widths of heavy and light quarkonia. In our method the energy, momentum and quark masses are complex.

The Cornell potential (10) is a special in hadron physics and results in the complex energy and mass eigenvalues. As known, operators in ordinary QM are Hermitian and the corresponding eigenvalues are real. It is possible to extend the Hamiltonian in QM into the complex domain while still retaining the fundamental properties of a quantum theory. One of such approaches is complex QM [47]. The complex-scaled method is the extension of theorems and principles proved in QM for Hermitian operators to non-Hermitian operators.

A. Modification of the Cornell potential

The Cornell potential (10) is fixed by the two free parameters, $\alpha_s$ and $\sigma$. However, the strong coupling $\alpha_s$ in QCD is a function $\alpha_s(Q^2)$ of virtuality $Q^2$ or $\alpha_s(r)$ in configuration space. The potential can be modified by introducing the $\alpha_s(r)$-dependence, which is unknown. A possible modification of $\alpha_s(r)$ was introduced in [41],

$$V_{\text{QCD}}(r) = -\frac{4}{3} \alpha_s(r) r + \sigma r, \quad \alpha_s(r) = \frac{1}{b_0 \ln[1/(\Lambda_{\text{QCD}}r)^2 + (2\mu_g/\Lambda_{\text{QCD}})^1]}$$

(12)

where $b_0 = (33 - 2n_f)/12\pi$, $n_f$ is number of flavors, $\mu_g = \mu(Q^2)$ — gluon mass at $Q^2 = 0$, $\Lambda_{\text{QCD}}$ is the QCD scale parameter. The running coupling $\alpha_s(r)$ in (12) is frozen at $r \to \infty$, $\alpha_\infty = \frac{1}{2} [b_0 \ln(2\mu_g/\Lambda_{\text{QCD}})]^{-1}$, and is in agreement with the asymptotic freedom properties, i.e., $\alpha_s(r \to 0) \to 0$.

A more complicate case are flavored heavy-light $Q\bar{q}$ mesons. A simplest example of heavy-light two-body system is the $H$ atom, comprising only a proton and an electron which are stable particles. This simplicity means its properties can be calculated theoretically with impressive accuracy [48]. The spherically symmetric Coulomb potential, with interaction strength parametrized by dimensionless coupling (“fine structure”) constant $\alpha$, is of particular importance in many realms of physics. The $H$ atom can be used as a tool for testing any relativistic two-body theory, because latest measurements for transition frequencies have been determined with a highest precision [32].

IV. THE R2B WAVE EQUATION AND ITS SOLUTION

Standard relativistic approaches for R2B systems run into serious difficulties in solving known relativistic wave equations. The formulation of RQM differs from NR QM by the replacement of invariance under Galilean transformations with invariance under Poincarè transformations. The RQM is also known in the literature as relativistic Hamiltonian dynamics or Poincarè-invariant QM with direct interaction [24]. There are three equivalent forms in the RQM called “instant”, “point”, and “light-front” forms.
The dynamics of many-particle system in the RQM is specified by expressing ten genera-
tors of the Poincaré group, \( \hat{M}_{\mu\nu} \) and \( \hat{W}_\mu \), in terms of dynamical variables. In the constructing
generators for interacting systems it is customary to start with the generators of the corre-
sponding non-interacting system; the interaction is added in the way that is consistent with
Poincare algebra. In the relativistic case it is necessary to add an interaction \( V \) to more than
one generator in order to satisfy the commutation relations of the Poincaré algebra.

The interaction of a relativistic particle with the four-momentum \( p_\mu \) moving in the external
field \( A_\mu(x) \) is introduced in QED according to the gauge invariance principle, \( p_\mu \rightarrow P_\mu = p_\mu - eA_\mu \). The description in the “point” form of RQM implies that the mass operators \( \hat{M}_{\mu\nu} \) are the same as for non-interacting particles, i.e., \( \hat{M}_{\mu\nu} = M_{\mu\nu} \), and these interaction terms can
be presented only in the form of the four-momentum operators \( \hat{W}_\mu \) [49].

Consider the R2B problem in Relativistic Classic Theory (RCM). Two particles with four-
momenta \( p_\mu^1, p_\mu^2 \) and the interaction field \( W^\mu(q_1, q_2) \) together compose a closed conservative
system, which can be characterized by the 4-vector \( P_\mu \),
\[
\begin{align*}
P^\mu &= p_\mu^1 + p_\mu^2 + W^\mu(q_1, q_2) = \text{const}, \tag{13} \\
E &= \sqrt{p_{\mu}^2 + m_{\mu}^2} + \sqrt{p_{\mu}^2 + m_{\mu}^2} + W_0(q_1, q_2) = \text{const}, \tag{14} \\
P &= p_1 + p_2 + W(r_1, r_2) = 0, \tag{15}
\end{align*}
\]
describing the energy and momentum conservation laws. The energy (14) and total momentum
(15) of the system are the constants of motion. By definition, for conservative systems, the
integrals (14) and (15) can not depend on time explicitly. This means the interaction \( W(q_1, q_2) \) should not depend on time, i.e., \( W(q_1, q_2) \Rightarrow V(r_1, r_2) \).

Equations (14) and (15) in the c.m. frame are
\[
\begin{align*}
M &= \sqrt{p_{\mu}^2 + m_{\mu}^2} + \sqrt{p_{\mu}^2 + m_{\mu}^2} + V(r), \tag{16} \\
P &= p_1 + p_2 + W(r_1, r_2) = 0, \tag{17}
\end{align*}
\]
where \( p = p_1 = -p_2 \) that follows from the equality \( p_1 + p_2 = 0 \); this means that \( W(r_1, r_2) = 0 \).

The system’s mass (16) in the c.m. frame is Lorentz-scalar. In case of free particles (\( V = 0 \))
the invariant mass \( M = \sqrt{p_{\mu}^2 + m_{\mu}^2} + \sqrt{p_{\mu}^2 + m_{\mu}^2} \) can be transformed for \( p^2 \) as
\[
p^2 = \frac{1}{4s}(s - m_-^2)(s - m_+^2) \equiv k_s^2, \quad s = M^2. \tag{18}
\]
Equation (14) is the zeroth component of the four-vector (13). But, in the c.m. frame the mass (16) is Lorentz-scalar; and what about the potential $V$? Is it still Lorentz-vector? To show that the potential is Lorentz-scalar, let us reconsider (16) as follows. The relativistic total energy $\epsilon_i(p)$ ($i = 1, 2$) of particles in (16) given by $\epsilon_i^2(p) = p^2 + m_i^2$ can be represented as sum of the kinetic energy $\tau_i(p)$ and the particle rest mass $m_i$, i.e., $\epsilon_i(p) = \tau_i(p) + m_i$. Then the system's total energy (invariant mass) (16) can be written in the form $M = \sqrt{p^2 + m_1^2(r)} + \sqrt{p^2 + m_2^2(r)}$, where $m_{1,2}(r) = m_{1,2}^0 + \frac{V(r)}{2}$ are the distance-dependent particle masses [50] and (18) with the use of $m_1(r)$ and $m_2(r)$ takes the form,

$$p^2 = K_s \left[ s - (m_+ + V) \right]^2 \equiv k_s^2 - U(s, r),$$

where $K_s = (s - m_+^2)/4s$, $k_s^2$ is squared invariant momentum given by (18) and $U(s, r) = K_s [2m_+V + V^2]$ is the potential function. The equation (19) is the relativistic analogy of the NR expression $p^2 = 2\mu[E - V(r)] \equiv k_E^2 - U(E, r)$.

The equality (19) with the help of the fundamental correspondence principle gives the two-particle spinless wave equation,

$$\left[ (-i \vec{\nabla})^2 + U(s, r) \right] \psi(r) = k_s^2 \psi(r).$$

The equation (20) can not be solved by known methods for the potential (12). Here we use the quasiclassical (QC) method and solve another wave equation [40, 51].

### A. Solution of the R2B wave equation

Solution of the Shrödinger-type’s wave equation (20) can be found by the QC method developed in [51]. In our method one solves the QC wave equation derivation of which is reduced to replacement of the operator $\vec{\nabla}^2$ in (20) by the canonical operator $\Delta^c$ without the first derivatives, acting onto the state function $\Psi(\vec{r}) = \sqrt{\text{det} g_{ij}} \psi(\vec{r})$, where $g_{ij}$ is the metric tensor. Thus, instead of (20) one solves the QC equation, for the potential (12),

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} + K_s \left[ s - \left( m_+ - \frac{4}{3} \frac{\alpha_s(r)}{r} + \sigma r \right) \right]^2 \right\} \psi(r) = 0.$$  

This equation is separated. Solution of the angular equation was obtained in [51] by the QC method in the complex plane, that gives $M_l = (l + \frac{1}{2})\hbar$, for the angular momentum eigenvalues. These angular eigenmomenta are universal for all spherically symmetric potentials in relativistic and NR cases.

The radial problem has four turning points and cannot be solved by standard methods. We consider the problem separately by the QC method for the short-range Coulomb term (heavy mesons) and the long-range linear term (light mesons). The QC method reproduces the exact energy eigenvalues for all known solvable problems in QM [40, 51]. The radial QC wave
equation of (21) for the Coulomb term has two turning points and the phase-space integral is found in the complex plane with the use of the residue theory and method of stereographic projection [45, 51] that gives

\[ \mathcal{M}_N^2|_C = \left( \sqrt{\varepsilon_N^2} \pm \sqrt{(\varepsilon_N^2)^*} \right)^2 \equiv \Re\{\varepsilon_N^2\} \pm i\Im\{\varepsilon_N^2\}, \]

where \( \varepsilon_N^2 = m_+^2 (1 - v_N^2) + 2im_+m_-v_N, \) \( v_N = \frac{2}{\alpha_\infty}/N, \) \( N = k + l + 1. \)

Large distances in hadron physics are related to the problem of confinement. The radial problem of (21) for the linear term has four turning points, i.e., two cuts between these points. The phase-space integral in this case is found by the same method of stereographic projection as above that results in the cubic equation [50]: \( s^3 + a_1s^2 + a_2s + a_3 = 0, \) where \( a_1 = 16\tilde{\alpha}_\infty\sigma - m_-^2, \) \( a_2 = 64\sigma^2 (\tilde{\alpha}_\infty - \tilde{N}^2 - \tilde{\alpha}_\infty m_-^2/4\sigma), \) \( a_3 = -(8\tilde{\alpha}_\infty\sigma m_-^2), \) \( \tilde{N} = N + k + \frac{1}{2}, \) \( \tilde{\alpha}_\infty = \frac{4}{3}\alpha_\infty, \) \( \alpha_\infty = \alpha_s(r \rightarrow \infty). \) The first root \( s_1(N) \) of this equation gives the physical solution (complex eigenmasses), \( M_1^2|_L = s_1(N) \), for the squared invariant mass.

Two exact asymptotic solutions obtained such a way are used to derive the interpolating mass formula. The interpolation procedure for these two solutions [39] is used to derive the mesons’ complex-mass formula,

\[ \mathcal{M}_N^2 = (m_1 + m_2)^2 \left( 1 - v_N^2 \right) \pm 2im_+m_-v_N + M_1^2|_L. \]  

The real part of the square root of (23) defines the centered masses and its imaginary part defines the total widths, \( \Gamma_N^{\text{TOT}} = -2\Im\{\mathcal{M}_N\}, \) of mesons and resonances [45, 46].

In the QC method not only the total energy, but also momentum of a particle-wave in bound state is the constant of motion. Solution of the QC wave equation in the whole region is written in elementary functions as [51]

\[ R(r) = C_n \left\{ \begin{array}{ll} \frac{1}{\sqrt{2}} e^{|k_n|r - \phi_1}, & r < r_1, \\ \cos(|k_n|r - \phi_1 - \frac{\pi}{4}), & r_1 \leq r \leq r_2, \\ \frac{(-1)^n}{\sqrt{2}} e^{-|k_n|r + \phi_2}, & r > r_2, \end{array} \right. \]

where \( C_n = \sqrt{2|k_n|/\pi(n + \frac{1}{2}) + 1} \) is the normalization coefficient, \( k_n \) is the corresponding eigenmomentum found from solution of (20), \( \phi_1 = -\pi(n + \frac{1}{2})/2 \) and \( \phi_2 = \pi(n + \frac{1}{2})/2 \) are the values of the phase-space integral at the turning points \( r_1 \) and \( r_2, \) respectively.

The free fit to the data [1] shows a good agreement for the light and heavy \( Q\bar{q} \) meson and their resonances. To demonstrate efficiency of the model we calculate the leading-state masses \( (S = 1) \) of the \( \rho \) and \( D^* \) meson resonances (see tables, where masses are in MeV). Parameters of calculations are also in the tables. The strong coupling constant, \( \alpha_s, \) is given by (12) and expressed via the gluon mass \( m_g \) and the QCD scale parameter \( \Lambda_{\text{QCD}}. \) The gluon mass, \( m_g = 416 \text{ MeV}, \) is the same for all types of mesons: \( \rho^\pm, K, D, B \) mesons and also for glueballs [41]. The QCD scale parameter \( \Lambda_{\text{QCD}}^0 = 638 \text{ MeV}, \) for the \( \rho \) mesons. The relative error of the data description is \( e^\rho = 0.67\%. \) The QCD scale parameter \( \Lambda_{\text{QCD}}^D = 616 \text{ MeV}, \) for the \( D \) mesons. The relative error of the data description is \( e^D = 0.54\%. \)
TABLE I. The masses of the $\rho^\pm$-mesons and resonances

| Meson | $J^{PC}$ | $E_n^{ex}$ | $E_n^{th}$ | Parameters in (23) |
|-------|---------|-----------|-----------|--------------------|
| $\rho$ (1S) | 1-- | 775.5 | 775.3 | $\alpha_s = 1.478$ |
| $a_2$ (1P) | 2++ | 1318.3 | 1317.9 | $\sigma = 0.142 \text{GeV}^2$ |
| $\rho_3$ (1D) | 3-- | 1688.8 | 1695.0 | $m_d = 4.70 \text{MeV}$ |
| $a_4$ (1F) | 4++ | 1996.6 | 2002.2 | $m_u = 2.15 \text{MeV}$ |
| $\rho$ (1G) | 5-- | 2330.0 | 2268.3 | |
| $\rho$ (2S) | 1-- | 1720.0 | 1695.5 | |
| $\rho$ (2P) | 2++ | 2002.2 | |
| $\rho$ (2D) | 3-- | 2268.3 | |

TABLE II. The masses of the $D^\pm*$-mesons and resonances

| Meson | $J^{PC}$ | $E_n^{ex}$ | $E_n^{th}$ | Parameters in (23) |
|-------|---------|-----------|-----------|--------------------|
| $D^*$ (1S) | 1-- | 2010.3 | 2010.3 | $\alpha_s = 1.308$ |
| $D^*_2$ (1P) | 2++ | 2460.1 | 2432.0 | $\sigma = 0.275 \text{GeV}^2$ |
| $D^*_3$ (1D) | 3-- | 2823.2 | | $m_c = 1026.9 \text{MeV}$ |
| $D^*_4$ (1F) | 4++ | 3176.9 | | $m_d = 4.7 \text{MeV}$ |
| $D^*_5$ (1G) | 5-- | 3499.3 | | |
| $D^*$ (2S) | 1-- | 2822.9 | | |
| $D^*$ (2P) | 2++ | 3176.8 | | |
| $D^*$ (2D) | 3-- | 3499.2 | | |

V. CONCLUSION

We have considered bound states as relativistic bound systems in the potential approach without using the QCD BS equation or its reductions. We have modeled mesons containing light and heavy quarks and their resonances in the framework of RQM. We have begun our investigation within relativistic classical mechanics using the basic principles of symmetries, i.e., the energy and momentum conservations’ laws in Minkowskiiy space. The potential of interaction the Lorentz-scalar function of the spatial variable $r$. The concept of position dependent particle mass was used. Using the correspondence principle, we have deduced, from the R2B classic equation, the two-particle wave equation.

We have calculated masses of light-heavy $S = 1$ mesons containing $d$ quark and their resonances, i.e., $\rho^\pm$ and $D^{\ast\pm}$. Quark masses are close to current masses. We have shown that quarks inside the system move as free particles. Using the complex-mass analysis, we have derived the
meson interpolating masses formula, in which the real and imaginary parts are exact expressions. This approach allows to simultaneously describe in the unified way the centered masses of resonances. We have shown here the calculation results for unflavored $\rho$ and $D$ mesons and their resonances, however, other mesons, containing $s$ and $b$ quarks can be described also well.

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