Entropy–Stable No–Slip Wall Boundary Conditions for the Eulerian Model for Viscous and Heat Conducting Compressible Flows

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Nonlinear (entropy) stability analysis is used to derive entropy–stable no–slip wall boundary conditions at the continuous and semi–discrete levels for the Eulerian model proposed by Svärd in 2018 (Physica A: Statistical Mechanics and its Applications, 2018). The spatial discretization is based on discontinuous Galerkin summation-by-parts operators of any order for unstructured grids. We provide a set of two–dimensional numerical results for laminar and turbulent flows simulated with both the Eulerian and classical Navier–Stokes models. These results are computed with a high-performance $hp$–entropy–stable solver, that also features explicit and implicit entropy–stable time integration schemes.

I. Nomenclature

| Symbol | Definition |
|--------|------------|
| $q$ | vector of conserved quantities |
| $\mathbf{q}$ | vector of conserved quantities on collocated points |
| $w$ | vector of entropy quantities |
| $\Theta_{x_j}$ | vector of the gradient of entropy quantities in the $x_j$ direction |
| $w$ | vector of entropy quantities on collocated points |
| $S$ | entropy function |
| $\mathcal{T}$ | entropy flux |
| $\Omega$ | generic domain |
| $\Gamma$ | generic boundary of $\Omega$ |
| $f(I)$ | inviscid flux |
| $f(V)$ | viscous flux |
| $\nu$ | kinematic viscosity |
| $c$ | speed of sound |
| $R$ | gas constant |
| $\gamma$ | adiabatic constant |
| $c_v, c_p$ | heat capacity at constant volume and pressure, respectively |

II. Introduction

The classical Navier–Stokes (CNS) equations are a Lagrangian formulation for compressible fluid flows. In the Lagrangian sense, diffusion between air pockets is non-existent and thus, the continuity equation is hyperbolic. On the other hand, in the Eulerian model [1], air molecules diffuse into other parts of the domain and thus, the continuity equation is modeled as a parabolic equation.

Entropy-stability is used to preserve the second-law of thermodynamics in the mathematical sense, i.e., the mathematical entropy function, $S$, decreases monotonically outside the equilibrium. This yields entropy estimates and bounds on $S$, which can be translated into bounds for the conservative variables, $\mathbf{u}$, of the underlying model [2,3]. It can be shown that the condition for entropy stability boils down to Lax’s entropy condition for the augmented system of
equations over a domain $\Omega$, 
\[
\int_{\Omega} \frac{\partial S}{\partial t} + \frac{\partial F_i}{\partial x_i} \, dx \leq 0,
\]
where $F$ is the entropy flux \[4\][5]. The Eulerian model is fully parabolic and thus, it eases the entropy analysis of the viscous flux. Furthermore, we show that the viscous flux is symmetric positive definite in both the continuous and the discrete cases.

Nonlinear entropy–stability and a summation–by–parts (SBP) framework are used to derive entropy–stable wall boundary conditions for the Eulerian model for the viscous and heat conducting compressible flows proposed by \[11\]. As done in \[6\] for the standard compressible Navier–Stokes equations, a semi–discrete entropy estimate for the entire domain is achieved when the new boundary conditions are coupled with an entropy–stable discrete interior operator. The data at the boundary are weakly imposed using a penalty flux approach and a simultaneous-approximation-term (SAT) penalty technique. The work in \[6\] was generalized for internal interfaces in \[7\]. In our discrete analysis, we follow this generalization discretizations constructed using SBP operators at the Legendre–Gauss–Lobatto (LGL) points.

The paper is organized as follows. In section \[11\] we present the Eulerian model in a general form and we show its entropy stability analysis. Then, in section \[14\] we derive the wall boundary condition explicitly. Later, in section \[15\] we discretize the system using SBP-SAT operators and present a discrete entropy analysis of the Eulerian model. Lastly, in section \[20\] we present numerical results that demonstrate the effect of the new wall boundary condition on the density in the boundary layer near a wall and validate the result with analytic results.

### III. Entropy analysis of the Eulerian model

In our journey to analyze the Eulerian model derived in \[11\], we begin by presenting its general form, and then we present its entropy analysis. The analysis highlights the boundary conditions required at a wall.

#### A. General form of the Eulerian model

In \[11\], Svärd arrives at the form
\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \frac{\partial \rho u_j}{\partial x_j} &= \frac{\partial}{\partial x_j} \left( \nu \frac{\partial \rho}{\partial x_j} \right), \\
\frac{\partial \rho u_i}{\partial t} + \frac{\partial \rho u_i u_j}{\partial x_j} + \frac{\partial p}{\partial x_i} &= \frac{\partial}{\partial x_j} \left( \nu \frac{\partial \rho u_i}{\partial x_j} \right), \\
\frac{\partial \rho E}{\partial t} + \frac{\partial \rho E u_j}{\partial x_j} + \frac{\partial p u_j}{\partial x_j} &= \frac{\partial}{\partial x_j} \left( \nu \frac{\partial \rho E}{\partial x_j} \right),
\end{align*}
\]
where $i, j = 1 : \text{DIM}$ (in MATLAB notation), and
\[
E = c_v T + \frac{1}{2} u_i u_j
\]
is the equation of state with $c_v$ being the heat capacity at constant volume. $c_v$ is related to $c_p$ by the gas constant $R = c_p - c_v$ and $\gamma = \frac{c_p}{c_v}$, where $c_p$ is the heat capacity at constant pressure.

\[
p = \rho R T,
\]
is the pressure for perfect gases, and
\[
\nu = \frac{\alpha \mu}{\rho(x, t)} + \beta(p, T),
\]
is a generalized form of the kinematic viscosity, where $\alpha \in [1, \frac{4}{3}]$ and $\beta$ is an additional diffusion coefficient. We also define
\[
c^2 = \gamma R T
\]
as the square of speed of sound.

**Remark 1** Equation \[2\] is presented in its general form to show the general form of the Eulerian equation. However, in practice, Svärd chooses $\beta = 0$ and thus equation \[2\] reduces to a scaled kinematic viscosity coefficient \[11\].
B. Entropy analysis

Herein, we cover the entropy analysis of the Eulerian model mentioned above. The analysis paves the road to the derivation of the wall boundary conditions. We start by showing the entropy condition by which the model is entropy–stable

$$\int_\Omega \frac{\partial S}{\partial t} + \frac{\partial F}{\partial x_i} \, dx \leq 0.$$  \hspace{1cm} (3)

over a generic domain $\Omega$, where $(S, F)$ is the entropy–entropy flux pair [5]. The Eulerian form can be written as

$$\frac{\partial q}{\partial t} + \frac{\partial f_i^{(l)}}{\partial x_i} = \frac{\partial f_i^{(v)}}{\partial x_i},$$  \hspace{1cm} (4)

where $f_i^{(l)}$, and $f_i^{(v)}$ are the inviscid and viscous fluxes respectively. Using the change of variables $q := q(w)$ and $g(w) := f(q(w))$, where $w^\top = \frac{dS}{dq}$, we get the symmetric form

$$\frac{dq}{dt} + \frac{\partial g_i^{(l)}}{\partial x_i} = \frac{\partial g_i^{(v)}}{\partial x_i},$$

where the inviscid flux is such that

$$w^\top \frac{\partial g_i^{(l)}}{\partial x_i} = \frac{\partial F_i}{\partial x_i},$$

and the viscous flux is given as

$$g_i^{(v)} = \nu \frac{dq}{dw} \frac{\partial w}{\partial x_j}.$$

Consequently,

$$w^\top \frac{dq}{dt} = \frac{dS}{dq} \frac{dq}{dw} \frac{\partial w}{\partial t} = \frac{\partial S}{\partial t}$$

by the chain rule. Multiplying the symmetric form by $w^\top$ and integrating over the domain $\Omega$ gives

$$\int_\Omega \frac{\partial S}{\partial t} + \frac{\partial F}{\partial x_i} \, dx = \int_\Omega w^\top \frac{\partial g_i^{(v)}}{\partial x_i} \, dx.$$

Integrating the right hand side by parts yields

$$\int_\Omega w^\top \frac{\partial g_i^{(v)}}{\partial x_i} \, dx = \int_{\Gamma} w^\top \frac{dq}{dw} \frac{\partial w}{\partial x_i} \, d\Gamma - \int_\Omega \frac{\partial w}{\partial x_i} \nu \frac{dq}{dw} \frac{\partial w}{\partial x_i} \, dx.$$

Let $\frac{\partial F}{\partial x_i} = 0$, $u = 0$ and $\frac{\partial p}{\partial x_i} = 0$ in $\Gamma$, then

$$\int_\Omega w^\top \frac{\partial g_i^{(v)}}{\partial x_i} \, dx = - \int_\Omega \frac{\partial w^\top}{\partial x_i} \nu \frac{dq}{dw} \frac{\partial w}{\partial x_i} \, dx,$$

and $F_i = 0$. The proof of this result is presented in section IV. Now, since $\nu$ is positive definite and $\frac{dq}{dw}$ is symmetric positive definite, then, their multiplication is positive definite. Thus,

$$\int_\Omega \frac{\partial S}{\partial t} \, dx \leq 0.$$

Therefore, the Eulerian form satisfies the entropy condition (3).
IV. Entropy–stable wall boundary condition

In this section, we derive the entropy–stable wall boundary condition for a specific entropy–entropy flux pair \((S, \mathcal{F}_i) = (-\rho s, -\rho u_i s)\) as chosen in [16]. This selection will allow us to compare the difference in the viscous flux and wall boundary condition between the Eulerian and CNS (used in [6, 7]) models.

**Remark 2** The choice of the entropy–entropy flux pair \((S, \mathcal{F}_i) = (-\rho s, -\rho u_i s)\), used in [6, 7], is due to the fact that it is the only pair that admits a diffusive entropy flux for the CNS [1].

**Remark 3** Since the Eulerian model provides a symmetric positive definite viscous flux, the entropy pair is not restricted to the choice by [6, 7] due to Lax’s condition for the entropy flux [3]. Svärd mentions that the Eulerian model admits a diffusive entropy flux for all Harten’s generalized entropies [1].

A. Derivation of the entropy–stable wall boundary condition

The goal of the following theorem is to compute a bound on the viscous terms on a no–slip wall. This leads to the entropy–stable wall boundary condition specified in [1, 8].

**Note 1** In this section and beyond, we will use \(g(t)\) to indicate the boundary terms.

**Theorem 1** For the entropy–entropy flux pair \(S = -\rho s, \mathcal{F}_i = -\rho u_i s\), the term

\[
g(t) = \kappa \frac{1}{T} \frac{\partial T}{\partial x_i} + \mu (s - R) \frac{1}{\rho} \frac{\partial \rho}{\partial x_i}
\]

(5)

bounds the viscous terms in the time derivative, where

\[
s = \frac{R}{\gamma - 1} \log \left( \frac{T}{T_{\infty}} \right) - R \log \left( \frac{\rho}{\rho_{\infty}} \right),
\]

and \(\kappa, \mu, R, T, \rho > 0\) and \(\gamma > 1\).

**Proof.** Define the entropy–entropy flux pair as [6]

\[
S = -\rho s, \quad \mathcal{F}_i = -\rho u_i s.
\]

We can write the Eulerian form as

\[
\frac{\partial q}{\partial t} + \frac{\partial f_i^{(t)}}{\partial x_i} = \frac{\partial}{\partial x_i} \cdot (v \frac{\partial q}{\partial x_i}).
\]

Multiplying this equation by the entropy variables \(w := \frac{\partial S}{\partial x_i}\) gives

\[
\frac{\partial S}{\partial t} + \frac{\partial \mathcal{F}_i}{\partial x_i} = w^\top \frac{\partial}{\partial x_i} \cdot (v \frac{\partial q}{\partial x_i}).
\]

Integrating the previous expression over a box \(\Omega = (0, 1) \times (0, 1) \times (0, 1)\) and, without loss of generality, for a wall at \(x_i = 0\), it gives [6]

\[
\int_\Omega \frac{\partial S}{\partial t} \, dx = \int_{x_i = 0} \left[ \mathcal{F}_i - w^\top \left( \frac{\partial q}{\partial x_i} \frac{\partial w}{\partial x_i} \right) \right] \, dx_j dx_k - \int_\Omega \frac{\partial w}{\partial x_i} \left( \frac{\partial q}{\partial w} \right) \frac{\partial w}{\partial x_i} \, dx,
\]

where \(j, k \neq i\). Assuming no–slip boundary conditions, \(u_i = 0\), yields \(\mathcal{F}_i = 0\). Then, the term

\[
-\nu w \frac{\partial q}{\partial w} \frac{\partial w}{\partial x_i} = \kappa \frac{1}{T} \frac{\partial T}{\partial x_i} + \mu (s - R) \frac{1}{\rho} \frac{\partial \rho}{\partial x_i} = g(t)
\]

is the term that contributes positively to the entropy, where equation (5) was computed using Mathematica [9]. □

As mentioned in [1, 8], we require \(\frac{\partial T}{\partial x_i} = 0\), \(u_i = 0\) and \(\frac{\partial \rho}{\partial x_i} = 0\) where the the change is normal to the wall. This requirement is equivalent to the boundary condition

\[
\int_{\Gamma} v w^\top \frac{\partial q}{\partial w} \frac{\partial w}{\partial x_i} \, d\Gamma = 0,
\]

since \(\kappa, \mu, R, T, \rho > 0\) and \(\gamma > 1\). Satisfying this condition gives

\[
\int_{\Omega} \frac{\partial S}{\partial t} \, dx \leq 0
\]

at a wall. Thus, satisfying the entropy condition [3].

V. Semi–discrete entropy–stable framework

In this section, we provide an entropy–stable framework for the semi-discretization of the Eulerian model. The implication of using such a framework is preserving the entropy stability property of the system, discussed in sections III and IV, while allowing for high–order schemes.

In order to discretize the system $H_V$, we extend its conservative form $H_T$ to include boundary and internal fluxes

$$\frac{\partial q}{\partial t} = \frac{\partial}{\partial x_i} \left( f_i(V) - f_i(T) \right) + \left( g(b) + g(in) \right). \quad (6)$$

A. SBP operators

SBP operators are a critical component in the derivation of high–order entropy–stable schemes. Define $D$ as the derivative operator with the property

$$D = P^{-1}Q,$$

where $P$ is positive definite and $Q$ is skew-symmetric with the property $Q + Q^\top = B$, where $B = \text{diag}(-1, 0, \cdots, 0, 1)$. In the following, we restrict $P$ to be diagonal [6, 7, 10].

B. Entropy–stable discontinuous collocation (SSDC)

Here, we provide important results that establish SSDC. We divide this section into three components. The first establishes the semi–discrete time derivative of entropy, $S$. The second establishes an arbitrarily high–order entropy–conservative inviscid flux, guaranteeing no positive contribution to the entropy. The third and last component we cover is the semi–discrete viscous flux that maintains the negative definiteness of the entropy for non–equilibrium points.

1. Time derivative

Following [6], we define $W = \text{diag}(w)$ and write

$$W^\top P \frac{\partial q}{\partial t} = 1^\top WP \frac{\partial q}{\partial t} = 1^\top P \frac{\partial q}{\partial t} = 1^\top P \frac{\partial S}{\partial t},$$

where $WP = PW$ since both are diagonal,

$$W \frac{\partial q}{\partial t} = \frac{\partial S}{\partial q} \frac{\partial q}{\partial t} = \frac{\partial S}{\partial t}$$

by the chain rule, and $1 = (1, 1, \cdots, 1)^\top$ is a vector of length $N$, where $N = p + 1$.

![1-dimensional discretization using LGL points of order $p = 4$. * and × denote solution and flux points respectively. Adapted from [11]](image)

2. The inviscid flux

Tadmor in [5] has developed the relation

$$(w_{i+1} - w_i)^T f_i = \psi_{i+1} - \psi_i,$$
where \( \psi = w^T f - F \). In \cite{12}, Fisher has shown that conservation laws discretized with an SBP operator can be cast into a consistent telescoping operator with respect to the divergence form. Since the end point fluxes are consistent (see figure \cite{1} and \( w^T \Delta \tilde{f}^{(I)} \) telescopes, we get the following result

\[
 w^T \Delta \tilde{f}^{(I)} = F(q_N) - F(q_1) = \tilde{F}(q_N) - \tilde{F}(q_1) = 1^T \Delta \tilde{F},
\]

(7)

where \( \Delta \) is a telescoping flux matrix \cite{6,7}. We denote (7) as the entropy conservative condition.

The following three results are adapted from \cite{6} and will only be summarized without details (details are available in \cite{13,14} and the references therein). In addition, we use the index \( i \) here to denote the node in an element. The purpose of these results are to construct inviscid entropy–conservative and stable fluxes of any order of accuracy \( p \).

The first result states that the local condition

\[
(w_{i+1} - w_i)^T \tilde{f}_i = \tilde{\psi}_{i+1} - \tilde{\psi}_i, \quad i = 1, 2, \ldots, N - 1, \quad ; \quad \tilde{\psi}_1 = \psi_1, \quad \tilde{\psi}_N = \psi_N
\]

(8)

when summed, telescope across the domain and satisfy the entropy–conservative condition (7). A flux that satisfies (8) is denoted \( \tilde{f}^{(S)} \) and appropriately called an entropy–conservative flux \cite{14}.

The strategy we adopt in this paper for constructing high–order entropy–conservative fluxes is to construct linear combinations of two-point entropy–conservative fluxes using the coefficients of the SBP operator \( Q \). Parsani et. al. \cite{6} argued that such an approach follows from the telescoping property of diagonal norm SBP operators. Parsani et. al. adds that it is valid for any SBP operator \( Q \) such that \( Q + Q^T = B \), since it only requires the existence of such a flux and an SBP operator \( Q \). This is formalized in the next result that establishes the accuracy of the new fluxes.

The second result states that a two-point entropy–conservative flux can be extended to high–order accuracy with formal boundary closures by using the form

\[
\tilde{f}_i^{(S)} = \sum_{k=i+1}^N \sum_{l=1}^i 2Q_{ik} \tilde{f}_S(q_i, q_k), \quad i = 1, 2, \ldots, N - 1,
\]

(9)

where \( \tilde{f}_S(q_i, q_k) \) satisfies (8).

Fisher and co–authors \cite{14,14} state that the previous result ensures that the design order of the diagonal norm SBP operator is retained when using a high–order entropy conservative flux (7).

The last result, on the other hand, shows that using a high–order entropy conservative flux preserves entropy stability of any diagonal norm SBP operator \( Q \). A two-point high–order entropy–conservative flux satisfying (7) with formal boundary closures can be constructed using (8), where, the two-point entropy–conservative flux satisfies

\[
(w_l - w_k)^T \tilde{f}_S(q_i, q_k) = \psi_l - \psi_k.
\]

(10)

In addition, the high–order entropy–conservative flux satisfies a local entropy–conservation property,

\[
w^T P^{-1} \Delta \tilde{f}^{(S)} = P^{-1} \Delta \tilde{F} = \frac{\partial F(q)}{\partial x} + T_{p+1}
\]

(11)

where \( T_{p+1} \) is the truncation error of the approximation of \( \frac{\partial F(q)}{\partial x} \).

Remark 4 The locality mentioned in the third result refers to a local domain, or more formally an element. Thus, equation (11) was written in vector notation.

Given that we used the same entropy–entropy flux pair as \cite{6}, we get the same results for the Ismael–Roe consistent entropy flux \cite{15}.

3. The viscous flux

The contribution to the viscous terms of the semi–discrete time derivative of the entropy is

\[
w^T \Delta \tilde{f}^{(V)} = w^T B_v \frac{dq}{dw} Dw - w^T D \cdot \frac{dq}{dw} Dw.
\]

(12)
Since \( \mathcal{P} \), \( \nu \) and \( \frac{\partial \Theta}{\partial x_j} \) are symmetric positive definite, their multiplication is positive definite. This implies that the last term is negative semi-definite. In addition, we used a high-order entropy–conservative inviscid flux. Thus, the only contribution to the production of entropy is from the term

\[
\mathbf{w}^T \mathbf{B} \mathbf{v} \frac{\partial \mathbf{q}}{\partial \mathbf{w}} \mathcal{D} \mathbf{w} = w_{N} \mathbf{v}_N \left( \frac{\partial \mathbf{q}}{\partial \mathbf{w}} \right)_N \left( \mathcal{D} \mathbf{w} \right)_N - w_1 \mathbf{v}_1 \left( \frac{\partial \mathbf{q}}{\partial \mathbf{w}} \right)_1 \left( \mathcal{D} \mathbf{w} \right)_1,
\]

where

\[
w_{N} \mathbf{v}_N \left( \frac{\partial \mathbf{q}}{\partial \mathbf{w}} \right)_N \left( \mathcal{D} \mathbf{w} \right)_N = \kappa \frac{1}{T_N} \left( \mathcal{D} \mathbf{T} \right)_N + \mu (s - R) \frac{1}{\rho_N} \left( \mathcal{D} \mathbf{\rho} \right)_N,
\]

which is appropriately bounded by the entropy–stable wall boundary condition \([5]\).

C. Entropy–stable wall boundary condition for the semi–discrete system

We semi-discretize the system using SBP operators constructed from 1-dimensional operators using tensor product elements. We distribute the nodes within an element using the LGL nodal distribution \([6, 7, 11]\). This gives the semi–discrete counterpart to equation \([6]\)

\[
\frac{\partial \mathbf{q}}{\partial t} = \mathcal{D} \mathbf{v}_j^{(\nu)}(\mathbf{q}) - \mathcal{P}^{-1}_{x_j} \Delta \mathbf{v}_j^{(\nu)} + \mathcal{P}^{-1}_{x_j} \left( \mathbf{g}_{x_j}^{(b)} + \mathbf{g}_{x_j}^{(I_n)} \right),
\]

where the subscript \( x_j \) denotes an SBP operator and boundary terms with respect to the \( j \)th coordinate. The vector \( \mathbf{g}_{x_j}^{(b)} \) enforce boundary conditions, while \( \mathbf{g}_{x_j}^{(I_n)} \) patches the elements using the simultaneous-approximation-term (SAT) approach.

The boundary conditions penalty term is defined as

\[
\mathbf{g}_{x_j}^{(b)} = - \left( \mathbf{f}_{j}^{(I)}(\mathbf{q}) - \mathbf{f}_{j}^{(I)}(\mathbf{q}^{(E)}) \right) + \left( \mathbf{f}_{j}^{(V)} - \mathbf{f}_{j}^{(V,B)} \right) - [M](\mathbf{w} - \mathbf{w}^{(V,el)}).
\]

The first component of each term is computed from the numerical solution, and the second component is constructed from a combination of the numerical solution and the physical boundary data \([6]\).

The first term enforces Euler no–penetration wall, where

\[
\mathbf{q}^{(E)} = \left( (\rho)_{(j,wall)}, (\rho u_j)_{(j,wall)}, -(\rho u_j)_{(j,wall)}, (\rho u_k)_{(j,wall)}, (\rho E)_{(j,wall)} \right)^T,
\]

and the subscript \((j, wall)\) refers to the numerical solution at the wall.

The second enforces the Neuman boundary conditions where

\[
\mathbf{f}_{j}^{(V,B)} = \nu \frac{\partial \mathbf{q}}{\partial \mathbf{w}} \Theta_{x_j}.
\]

The gradient of the entropy variables \( \Theta_{x_j} \) is manufactured, where the tilde represent a manufactured component

\[
\tilde{\Theta}_{x_j}^T = \left( \tilde{\Theta}_{x_j}(1), \tilde{\Theta}_{x_j}(2), \tilde{\Theta}_{x_j}(3), \tilde{\Theta}_{x_j}(4), \tilde{\Theta}_{x_j}(5) \right)^T.
\]

where \( \tilde{\Theta}_{x_j} \) is the manufactured gradient of the entropy variables. We can compute \( \tilde{\Theta}_{x_j}(1) \) as

\[
\tilde{\Theta}_{x_j}(1) = - \left( \frac{R}{(y - 1)} \frac{1}{T} \frac{\partial T}{\partial x_j} - \frac{R}{\rho} \frac{\partial \rho}{\partial x_j} \right) + \frac{w(2 : 4)}{w(5)} \left( \Theta_{x_j}(2 : 4) + \frac{1}{2} \frac{\partial T}{\partial x_j} w(2 : 4) \right),
\]

where the notation \((2 : 4)\) defines a sum over the components 2, 3 and 4. We can simplify \( \tilde{\Theta}_{x_j}(1) \) because we are enforcing no–slip boundary conditions, \( u_i = 0 \), at the same wall , where \( w(2 : 4) = \frac{w_{2 : 4}}{T} = 0 \). So, the term becomes

\[
\tilde{\Theta}_{x_j}(1) = - \left( \frac{R}{(y - 1)} \frac{1}{T} \frac{\partial T}{\partial x_j} - \frac{R}{\rho} \frac{\partial \rho}{\partial x_j} \right). \]

Thus, enforcing \( \frac{\partial \rho}{\partial x_j} = 0 \) and \( \frac{\partial T}{\partial x_j} = 0 \) implies \( \tilde{\Theta}_{x_j}(1) = 0 \). Then, we enforce the heat-flux by

\[
\tilde{\Theta}_{x_j}(5) = - \frac{\partial T}{\partial x_j} w(5),
\]
Fig. 2 The primitive quantities of a Re = 60 flow around a cylinder. This simulation was run with α = 1 for the Eulerian model and p = 7 order of accuracy.

and enforcing \( \frac{\partial T}{\partial x_y} = 0 \) implies \( \Theta_{x_y} = 0 \).

The third term enforces the no-slip boundary condition. Recall that the entropy variables are computed as

\[
\begin{align*}
\omega^\top &= \left( \frac{h}{T} - s - \frac{u_i u_i}{2T}, \frac{u_i}{T}, \frac{u_j}{T}, \frac{u_k}{T}, -1 \right)^\top.
\end{align*}
\]

Thus, we enforce the no-slip boundary condition by

\[
\omega^{(\text{Vel})} = (\omega(1), 0, 0, 0, \omega(5))^\top.
\]

The matrix \([M]\) is a block diagonal matrix defined as

\[
M = -\alpha^B \left( \begin{array}{cccc}
\mathcal{P}_{x_j} & & & \\
& \mathcal{P}_{x_j} & & \\
& & \mathcal{P}_{x_j} & \\
& & & \mathcal{P}_{x_j}
\end{array} \right) (H_v \frac{dg}{dw} H).
\]

where \(\alpha^B\) is a positive constant controlling the strength of the penalty term, the factor \(\mathcal{P}_{x_j}(1,1)\) insures the correct asymptotic order of accuracy, since \(\mathcal{P}\) depends on the order, and \(H = \text{diag}(0, 1, 1, 1, 0)\).

**VI. Numerical results**

This section demonstrates the effect of the new boundary condition on the fluid in the boundary layer of walls. The SSDC framework uses a non-dimensional formalism; thus, all quantities are scaled to units. For both simulations, the kinematic viscosity was scaled using \(\alpha = 1\) for the Eulerian model.

**A. Laminar flow around a cylinder**

Herein, the case of a Re = 60 flow around a cylinder is shown. First, we compute the length of the wake region to demonstrate that inertial fields behave similarly between the Eulerian and CNS models. Then, we compute the mean...
The wake region of a Re = 60 flow around a cylinder. The classical Navier–Stokes equations are denoted as CNS. This simulation was run with $\alpha = 1$ for the Eulerian model and order of accuracy $p = 7$ for both models.

Fig. 3 The wake region of a Re = 60 flow around a cylinder. The classical Navier–Stokes equations are denoted as CNS. Headwind density is shown in the left plot, and tailwind density is shown on the right. This simulation was run with $\alpha = 1$ for the Eulerian model and order of accuracy $p = 7$ for both models.

Fig. 4 The mean density of a Re = 60 flow around a cylinder. The classical Navier–Stokes equations are denoted as CNS. Headwind density is shown in the left plot, and tailwind density is shown on the right. This simulation was run with $\alpha = 1$ for the Eulerian model and order of accuracy $p = 7$ for both models.
Fig. 5  Computational domain, $hp$–nonconforming grid, and solution polynomial degree distribution, $p$, for the simulation of the supersonic flow past a circular cylinder. The underlying figure was adapted from [16].

Fig. 6  The primitive quantities of a $Ma = 3.5$ and $Re = 10000$ flow around a cylinder. This simulation was run with $\alpha = 1$ for the Eulerian model and non-conforming orders of accuracy.
B. Sonic flow around a cylinder

This section shows a $Ma = 3.5$ and $Re = 10000$ flow around a cylinder. First, we show the order distribution for the non-conforing domain. Next, we show the primitive quantities for a developed solution of the Eulerian model. Then, we present the results of shockwave theory and demonstrate that the Eulerian model matches the theoretical behavior of sonic flow around a cylinder. Lastly, we present the normalized distance between the shock and the leading edge of the cylinder and show that it matches the analytic distance. Figure 5 shows the order of accuracy distribution of the domain. The bow shock region is usually appropriately modeled with first-order $p = 1$ accuracy as done in [16]. However, in our case, the solution experiences a numerical instability known as the carbunkle. Thus, we modeled the shock region with second-order $p = 2$ accuracy. Figure 6 shows the primitive quantities of the Eulerian model for the developed solution.

The results of the Eulerian model simulation for high inertial forces are expected to match the CNS result. Thus, we include both results for the shockwave theory and the normalized distance between the shock and the leading edge of the cylinder.

Figure 7 shows the results of the shockwave theory analysis of the sonic flow around a cylinder. Subfigure 7a shows the streamlines of velocity vectors changing angles as they cross the bow shock. These streamlines are used to compute the angles of deflection, $\theta$, and shock, $\beta$, defined in subfigure 7b. Subfigure 7c shows the calculated angles for the Eulerian model in comparison with the analytic shockwave theory curve [18]

$$\tan(\theta) = 2 \cot(\beta) \left[ \frac{Ma^2 \sin^2(\beta) - 1}{Ma^2(\gamma + \cos(2\beta)) + 2} \right].$$

The $\theta - \beta$ results for the Eulerian model follow the theoretical curve and the CNS results. Part of the calculation of the angles were done automatically using Python. However, since they usually cluster near some points, we have manually calculated the angles for sparser regions of the curve.

Figure 8 shows the normalized distance between the shock and the leading edge of the cylinder. In addition, the Eulerian model matches the CNS result of 0.3 obtained in [16]. The figure show that our result follows closely the analytic distance $\Delta = 0.293$ [19].
Fig. 8  The velocity magnitude of Ma = 3.5 and Re = 10000 flow at the leading edge of a cylinder. This figure reveals the normalized distance between the shock and the cylinder leading edge for the numerical simulation of the Eulerian model $\Delta = 0.3$ in comparison with the analytic distance $\Delta = 0.293$ [19].

VII. Conclusion

In the analysis section of the Eulerian model, we confirmed that the choice of boundary conditions in [1] ensures that the system is entropy–stable at a no–slip wall. Then, in the wall boundary condition section, we derived the explicit bound on the entropy producing boundary terms, which justify the choice of boundary conditions mentioned in the entropy analysis of the Eulerian model. After that, in the discrete analysis section, we showed an entropy–stable discretization using SBP-SAT operators that mimics the properties of the continuous model and allows for high–order discretizations without sacrificing accuracy or entropy stability. Also, this discretization follows the same boundary condition derived in the wall boundary condition section. Next, we presented the method of enforcement of the boundary condition using the penalty term introduced in the extended model. Lastly, in the Numerical results section, we demonstrated the validity of the Eulerian model for inertial fields and inertially dominant flow, and where it differs from the CNS for viscous flows.

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