Nonconvex Regularized Robust Regression with Oracle Properties in Polynomial Time

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Abstract

This paper investigates tradeoffs among optimization errors, statistical rates of convergence and the effect of heavy-tailed random errors for high-dimensional adaptive Huber regression with nonconvex regularization. When the additive errors in linear models have only bounded second moment, our results suggest that adaptive Huber regression with nonconvex regularization yields statistically optimal estimators that satisfy oracle properties as if the true underlying support set were known beforehand. Computationally, we need as many as $O(\log s + \log \log d)$ convex relaxations to reach such oracle estimators, where $s$ and $d$ denote the sparsity and ambient dimension, respectively. Numerical studies lend strong support to our methodology and theory.

Keywords: Adaptive Huber regression, heavy-tailed noise, nonconvex regularization, optimization error, oracle property, statistical rate, tradeoff.

1 Introduction

Suppose we have collected independent and identically distributed (i.i.d.) copies $\{(y_i, x_i) : 1 \leq i \leq n\}$ of $(y, x)$ that follows the linear model

$$y = \langle x, \beta^* \rangle + \varepsilon = \beta^*_0 + \langle x_-, \beta_-^* \rangle + \varepsilon, \quad (1)$$

where $\beta^*_0$ is the intercept, $\beta_-^* \in \mathbb{R}^d$ denotes the coefficient vector, $x = (1, x')' \in \mathbb{R}^{\bar{d}}$ ($\bar{d} := d + 1$) is the covariate vector, and $\varepsilon$ is an error term satisfying $E(\varepsilon|x) = 0$. This general setting includes the location-scale model in which $\varepsilon = \sigma(x)e$, and $\sigma(\cdot) : \mathbb{R}^d \to \mathbb{R}_+$ is a positive function and the random variable $e$ is independent of $x$ and satisfies $E(e) = 0$. For simplicity, we use $\beta^* = (\beta^*_0, \beta_1^*, \ldots, \beta_p^*)' = (\beta^*_0, \beta_-^*)' \in \mathbb{R}^d$ to denote the vector of unknown parameters. Of particular interest is the case where the error variable is heavy-tailed with only bounded second moment.
We are interested in the high-dimensional regime, where the number of features $d$ exceeds the sample size $n$ and $\beta^*$ is sparse. Since the invention of Lasso two decades ago (Tibshirani, 1996; Santosa and Symes, 1986), a variety of variable selection methods have been developed for finding a small group of covariates that are associated with the response from a large pool. The Lasso estimator $\hat{\beta}_{\text{Lasso}}$ solves the convex optimization problem

$$\argmin_{\beta=(\beta_0, \beta^\top)^\top \in \mathbb{R}^d} \left\{ \frac{1}{2n} \| y - X\beta \|^2_2 + \lambda \| \beta_\parallel \|_1 \right\},$$

where $\lambda > 0$ is a tuning parameter and $X = (x_1, \ldots, x_n)^\top$ is the $n \times d$ design matrix. Essentially, the Lasso is an $\ell_1$-regularized least squares method: the quadratic loss is used as a goodness of fit measure and the $\ell_1$-norm induces sparsity. To achieve better performance under different circumstances, several Lasso variants have been proposed and studied; see, Fan and Li (2001), Zou and Hastie (2005), Zou (2006), Yuan and Lin (2006), Belloni, Chernozhukov and Wang (2011), Sun and Zhang (2012) and Bogdan et al. (2015), to name a few. We refer to Bühlmann and van de Geer (2011) and Hastie, Tibshirani and Wainwright (2015) for comprehensive reviews of high-dimensional statistical methods and theory.

As a general regression analysis method, the Lasso, along with many of its variants, has two potential downsides. First, the regularized least squares methods are sensitive to the tails of error distributions, even though various alternative penalties have been proposed to achieve better model selection performance. Consider a Lasso-type estimator that solves the penalized empirical risk minimization

$$\min_{\beta=(\beta_0, \beta^\top)^\top} \left\{ \frac{1}{n} \sum_{i=1}^n \ell(y_i - \langle x_i, \beta \rangle) + \lambda \| \beta_\parallel \|_1 \right\},$$

where $\ell(x) : \mathbb{R} \mapsto [0, \infty)$ is a general loss function. The effects of the loss and noise on estimation error are coded in the vector $\{\ell'(\varepsilon_i)^n_{i=1}\}$. If $\ell$ is the quadratic loss, this vector is likely to have relatively many large coordinates when $\varepsilon$ is heavy-tailed (Mendelson, 2018). As a result, the combination of the rapid growth of $\ell$ with heavy-tailed sampling distribution inevitably leads to outliers, which will eventually be translated into spurious discoveries. Second, it has been recognized (Fan and Li, 2001; Zou, 2006) that convex penalties, typified by the $\ell_1$-norm, introduce nonnegligible estimation bias. Due to the bias of the $\ell_1$ penalty, the Lasso typically selects far larger model size since the visible bias in Lasso forces the cross-validation procedure to choose a smaller value of $\lambda$.

In the presence of heavy-tailed noise, outliers occur more frequently and may have a significant impact on (regularized) empirical risk minimization when the loss grows quickly. To reduce the ill-effects of outliers, a widely recognized strategy is to use a robust loss function that is globally Lipschitz continuous and locally quadratic. A prototypical example is the Huber loss (Huber, 1964):

$$\ell_\tau(x) = \begin{cases} 
\frac{x^2}{2} & \text{if } |x| \leq \tau, \\
\tau|x| - \frac{\tau^2}{2} & \text{if } |x| > \tau.
\end{cases}$$

The Huber loss $\ell_\tau(\cdot)$ is parametrized by $\tau > 0$, which is referred to as the robustification parameter that controls the tradeoff between the robustness and bias. The second important issue is the choice of sparsity-inducing penalty. In order to eliminate the nonnegligible estimation bias introduced by convex regularization, Fan and Li (2001) introduced a family of folded-concave penalties, including
the smoothly clipped absolute deviation (SCAD) (Fan and Li, 2001), minimax concave penalty (MCP) (Zhang, 2010a), and the capped \( \ell_1 \)-penalty (Zhang, 2010b). These ideas motivate us to propose the nonconvex (concave) regularized robust \( M \)-estimator of the form

\[
\hat{\beta} = \arg\min_{\beta_0, \beta_1, \ldots, \beta_d \in \mathbb{R}^d} \left\{ \mathcal{L}_\tau(\beta) + \sum_{j=1}^{d} p_{\lambda}(\beta_j) \right\},
\]

where \( \mathcal{L}_\tau(\beta) := (1/n) \sum_{i=1}^{n} \ell_\tau(y_i - \langle x_i, \beta \rangle) = (1/n) \sum_{i=1}^{n} \ell_\tau(y_i - \beta_0 - \langle x_i, \beta \rangle) \) is the empirical loss function, \( \tau > 0 \) is a robustification parameter, and \( p_{\lambda} : \mathbb{R} \mapsto [0, \infty) \) is a concave penalty function with a regularization parameter \( \lambda > 0 \). We refer to Zhang and Zhang (2012) for a comprehensive survey of concave regularized methods.

In practice, it is inherently difficult to solve the nonconvex optimization problem (4) directly. Theoretically, statistical properties, such as the rate of convergence under various norms and oracle properties, are established for either the hypothetical global optimum that is unobtainable by any practical algorithm in polynomial time, or a local optimum that exists somewhere like a needle in a haystack. To close the gap between statistical theory and computational complexity, we propose a two-stage (contraction and tightening) regularized robust regression procedure, which yields a solution with desired oracle properties and is computationally efficient as it only involves solving a sequence of adaptive convex programs. Our work builds upon Fan et al. (2018), who studied the tradeoff between algorithmic complexity and statistical error when fitting high-dimensional models with a general but non-robust loss function. The aim of this paper is to explore robust loss functions, not merely for the purpose of generality but owing to a real downside of the squared loss. Typified by the Huber loss, our general principle applies to a class of robust loss functions as will be discussed in Section 4. Software implementing the proposed procedure and reproducing our computational results is available at https://github.com/XiaoouPan/ILAMM.

1.1 Related literature

For classical linear models with heavy-tailed errors, Mendelson (2018) and Sun, Zhou and Fan (2019) studied the performance of empirical risk minimization (ERM) with a convex robust loss function; the former considered a class of sufficiently smooth convex loss functions and the latter focused on the Huber loss. From a non-asymptotic viewpoint, a well-chosen loss, calibrated to fit the noise level, sample size and dimensionality of the problem, alleviates the ill-effects of outliers. For nonconvex loss functions, typified by Tukey’s bisquare loss, Mei, Bai and Montanari (2018) investigated the statistical consistency of \( \ell_2 \)-constrained \( M \)-estimators to stationary points.

In the high-dimensional regime that \( d \gg n \), Minsker (2015) and Fan, Li and Wang (2017), respectively, proposed a median-of-means estimator based on Lasso and \( \ell_1 \)-penalized Huber’s \( M \)-estimator. Both estimators achieve sub-Gaussian deviation bounds when the regression error only has finite variance. Loh (2017) studied statistical consistency and asymptotic normality of nonconvex regularized robust \( M \)-estimators. Assuming a symmetric error distribution, The author proved that there exists a stationary point with \( \ell_2 \) and \( \ell_1 \) error bounds in the order of \( \sqrt{s \log(d)/n} \) and \( s \sqrt{\log(d)/n} \), respectively; and then established uniqueness of the stationary point. Loh (2017) applied a projected gradient algorithm to solve an \( \ell_1 \)-constrained optimization problem. In this paper, we simultaneously analyze the statistical property and algorithmic complexity of the solutions produced by our algorithm. Specifically, we show that as many as \( O(\log s + \log \log d) \) iterations are
needed to deliver a statistically optimal estimator with $\ell_2$ and $\ell_1$ errors in the order of $\sqrt{s/n}$ and $s/\sqrt{n}$, respectively.

### 1.2 Notation

Let us summarize our notation. For every integer $k \geq 1$, we use $\mathbb{R}^k$ to denote the $k$-dimensional Euclidean space. The inner and Hadamard products of any two vectors $u = (u_1, \ldots, u_k)^T$, $v = (v_1, \ldots, v_k)^T \in \mathbb{R}^k$ are defined by $u^T v = \langle u, v \rangle = \sum_{i=1}^k u_i v_i$ and $u \circ v = (u_1 v_1, \ldots, u_k v_k)^T$, respectively. We use $\| \cdot \|_p$ ($1 \leq p \leq \infty$) to denote the $\ell_p$-norm in $\mathbb{R}^k$: $\| u \|_p = (\sum_{i=1}^k |u_i|^p)^{1/p}$ and $\| u \|_\infty = \max_{1 \leq i \leq k} |u_i|$. Moreover, we write $\| u \|_{\min} = \min_{1 \leq i \leq k} |u_i|$. For $k \geq 2$, $S^{k-1} = \{ u \in \mathbb{R}^k : \| u \|_2 = 1 \}$ denotes the unit sphere in $\mathbb{R}^k$. For any function $f : \mathbb{R} \mapsto \mathbb{R}$ and vector $u = (u_1, \ldots, u_k)^T \in \mathbb{R}^k$, we write $f(u) = (f(u_1), \ldots, f(u_k))^T \in \mathbb{R}^k$.

Throughout this paper, we use bold capital letters to represent matrices. For $k \geq 2$, $I_k$ represents the identity/unit matrix of size $k$. For any $k \times k$ symmetric matrix $A \in \mathbb{R}^{k \times k}$, $\| A \|_2$ denotes the operator norm of $A$, and we use $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ to denote the minimal and maximal eigenvalues of $A$, respectively. For a positive semidefinite matrix $A \in \mathbb{R}^{k \times k}$, $\| A \|$ denotes the norm linked to $A$ given by $\| A \| = \| A^{1/2} u \|_2$, $u \in \mathbb{R}^k$. For any two real numbers $u$ and $v$, we write $u \vee v = \max(u, v)$ and $u \wedge v = \min(u, v)$. For two sequences of non-negative numbers $\{a_n\}_{n \geq 1}$ and $\{b_n\}_{n \geq 1}$, $a_n \leq b_n$ indicates that there exists a constant $C > 0$ independent of $n$ such that $a_n \leq C b_n$; $a_n \geq b_n$ is equivalent to $b_n \leq a_n$; $a_n \approx b_n$ is equivalent to $a_n \leq b_n$ and $b_n \leq a_n$. For two numbers $C_1$ and $C_2$, we write $C_2 = C_2(C_1)$ if $C_2$ depends only on $C_1$. For any integer $d \geq 1$, we write $[d] = \{1, \ldots, d\}$. For any set $S$, we use $|S|$ to denote its cardinality, i.e. the number of elements in $S$.

### 2 Regularized Huber $M$-estimation

We first revisit the $\ell_1$-penalized Huber estimator in Section 2.1, and point out two interesting regimes for the robustification parameter $\tau$. In Section 2.2, we propose a two-stage procedure, which is closely related to nonconvex regularized Huber regression, for fitting high-dimensional sparse models with heavy-tailed noise. This two-stage robust regression method not only is computationally efficient, but also achieves optimal rate of convergence and oracle properties, as will be studied in Sections 2.3 and 2.4. Throughout, $S = \text{supp}(\beta^*) = \{1 \leq j \leq d : \beta_j^* \neq 0\} \subseteq [d]$ denotes the active set and $s = |S|$ is the sparsity.

#### 2.1 $\ell_1$-regularized Huber regression

Given i.i.d. observations $\{(y_i, x_i)\}_{i=1}^n$ from the linear model (1), consider the $\ell_1$-regularized Huber $M$-estimator, which we refer to as the *Huber-Lasso*,

$$\widehat{\beta}_{\text{H-Lasso}} = \arg\min_{\beta=(\beta_0, \beta_1)^T \in \mathbb{R}^d} \{L(x, \beta) + \lambda \| \beta \|_1\},$$

where $L(x, \beta)$ is the empirical loss function defined in (4) and $d = d + 1$. Statistical properties of the penalized Huber $M$-estimator have been studied by Lambert-Lacroix and Zwald (2011), Fan, Li and Wang (2017) and Loh (2017) from different perspectives. A less-noticed problem, however, is the connection between the robustification parameter and the error distribution, which in turn quantifies the tradeoff between robustness and unbiasedness. In fact, recent studies by Sun, Zhou and Fan
Assume that Condition 1 holds for model (1). There exist Massart, 2013), while we allow the regression errors to be heavy-tailed and diverging \( \tau \). To begin with, we impose the following assumption on the data generating process. The co-variate (random) vectors are assumed to be sub-exponential or asymmetric.

For estimation purpose, \( \hat{\beta}^{H-\text{Lasso}} \) is a regularized \( M \)-estimator of

\[
\hat{\beta}_\tau^* = (\hat{\beta}_{0,\tau}^*, \hat{\beta}_{-\tau}^*)^T = \arg\min_{\beta=(\beta_0, \beta^\tau)^T} \mathbb{E} L_\tau(\beta).
\]

(6)

The true parameter \( \beta^* = (\beta_0^*, \beta^\tau)^T \), however, is identified as \( \arg\min_{(\beta_0, \beta^\tau)^T} \sum_{i=1}^n \mathbb{E}( Y_i - \beta_0 - \langle x_i, \beta^\tau \rangle)^2 \). It is easy to see that, if the distribution of \( \varepsilon \) is symmetric around zero, \( \beta_\tau^* = \beta^* \) for any \( \tau > 0 \). This is not always true with general asymmetric noise variables. To have a better understanding of the role of \( \tau \) in the Huber loss, we present the next result (Proposition 5 of Wang et al. (2018)) to establish the connection between \( \beta^* \) and \( \beta_\tau^* \) in a fairly general context.

**Proposition 1.** Assume that the function \( \alpha \mapsto \mathbb{E}(\ell_\tau(\varepsilon - \alpha|x)) \) has a unique constant minimizer \( \alpha_\tau \) almost surely, which satisfies \( \mathbb{P}(|\varepsilon - \alpha_\tau| \leq \tau) > 0 \). Assume further that \( \mathbb{E}(xx^\top) \) is positive definite. Then the pseudo parameter \( \beta_{\tau}^* = (\beta_{0,\tau}^*, \beta_{-\tau}^*)^T \) defined in (6) satisfies

\[
\beta_{0,\tau}^* = \beta_0^* + \alpha_\tau \quad \text{and} \quad \beta_{-\tau}^* = \beta^*.
\]

(7)

For Huber regression in low dimensions (e.g. \( d \) is fixed or \( d \) slowly grows as a function of \( n \), the calibration of \( \tau \) is typically determined by the asymptotic 95% efficiency rule. While in the absence of symmetry assumption, Proposition 1 demonstrates that \( \alpha_\tau \) also quantifies the approximation bias induced by the Huber loss, and its magnitude decays as \( \tau \) increases. Therefore, throughout this paper we study the statistical properties of regularized Huber \( M \)-estimators under two scenarios: diverging \( \tau \) and fixed \( \tau \).

To begin with, we impose the following assumption on the data generating process. The co-variate (random) vectors are assumed to be sub-exponential/sub-gamma (Boucheron, Lugosi and Massart, 2013), while we allow the regression errors to be heavy-tailed and/or asymmetric.

**Condition 1.** There exist \( c_0, \upsilon_0 \geq 1 \) such that \( \mathbb{P}(\|\mathbf{u}, x\| \geq \upsilon_0 \|\Sigma \cdot t\|) \leq c_0 e^{-t} \) for all \( \mathbf{u} \in \mathbb{R}^d \) (\( d = d + 1 \)), where \( \Sigma = (\sigma_{jk})_{0 \leq j, k \leq d} = \mathbb{E}(xx^\top) \) is positive definite with \( \lambda_{\min}(\Sigma) \geq \lambda_i > 0 \). For simplicity, we assume \( c_0 = 1 \) and let \( \sigma_x^2 = \max_{0 \leq j \leq d} \sigma_{jj} \). The regression error \( \varepsilon \) satisfies \( \mathbb{E}(\varepsilon|x) = 0 \) and \( \mathbb{E}(\varepsilon^2|x) \leq \sigma_x^2 \) almost surely.

**Theorem 1.** Assume that Condition 1 holds for model (1).

(I) (Diverging \( \tau \)) Any optimal solution \( \hat{\beta}^{H-\text{Lasso}} \) to the convex program (5) with \( \tau = \sigma_x \sqrt{n \log(d)} \) and \( \lambda \) scaling as \( \upsilon_0 \sigma_x \sigma_x \sqrt{\log(d)/n} \) satisfies

\[
||\hat{\beta}^{H-\text{Lasso}} - \beta^*||_2 \leq \lambda_i^{-1} s^{-1/2} \lambda \quad \text{and} \quad ||\hat{\beta}^{H-\text{Lasso}} - \beta^*||_1 \leq \lambda_i^{-1} s \lambda
\]

(8)

with probability at least \( 1 - 4d \) as long as \( n \geq c_1 s \log(d) \), where \( c_1 > 0 \) is a constant depending only on \( (\upsilon_0, \sigma_x, \lambda_i) \).

(II) (Fixed \( \tau \)) Assume further that the function \( \alpha \mapsto \mathbb{E}(\ell_\tau(\varepsilon - \alpha|x)) \) has a unique constant minimizer \( \alpha_\tau \) almost surely, and that there exist constants \( \bar{\rho}_\tau \geq \rho_\tau > 0 \) such that

\[
\rho_\tau \leq \mathbb{P}(|\varepsilon - \alpha_\tau| \leq \tau/2|x) \leq \bar{\rho}_\tau \quad \text{almost surely.}
\]
Then any optimal solution $\tilde{\beta}_{\text{H-Lasso}}$ with $\tau = \sigma_x$ and $\lambda = \nu_0 \sigma_x \sqrt{\log(d)/n}$ satisfies the bounds
\[ \|\tilde{\beta}_{\text{H-Lasso}} - \beta^*\|_2 \leq L_t^{-1} \lambda^{1 - s} t \lambda^{1/2} \quad \text{and} \quad \|\tilde{\beta}_{\text{H-Lasso}} - \beta^*\|_1 \leq L_t^{-1} \lambda^{-1} s \lambda \] (9)
with probability at least $1 - 4d^{-1}$ as long as $n \geq c_2 s \log(d)$, where $\beta^*_t = (\beta^*_0 + \alpha_t, \beta^*_{-t})^\top$ and $c_2 > 0$ is a constant depending only on $(\nu_0, \sigma_x, \lambda, \tilde{\rho}/\rho)$.

2.2 Two-stage procedure: tightening after contraction

For sparse linear regression, it is known that regularized $M$-estimators with convex penalties typically exhibit a suboptimal statistical rate of convergence, as compared to the oracle rate achieved by nonconvex regularization. However, as noted previously, directly solving the nonconvex optimization problem (4) is computationally challenging. Moreover, statistical properties are only established for the hypothetical global optimum (or some stationary point), which is typically unobtainable by any polynomial time algorithm.

Inspired by the local linear approximation to the folded concave penalty (Zou and Li, 2008), here we propose a two-stage procedure, contraction and tightening, that solves a sequence of convex programs up to a prespecified optimization precision. Let $p_1(\cdot)$ be a differentiable penalty function as in (4) and recall that $L_t(\cdot)$ is the empirical loss function. Starting with an initial estimate $\tilde{\beta}^{(0)} = (\tilde{\beta}^{(0)}_0, \tilde{\beta}^{(0)}_1, \ldots, \tilde{\beta}^{(0)}_d)^\top$, consider a sequence of convex optimization programs $(P_{\ell})_{\ell \geq 1}$:
\[
\min_{\beta = (\beta_0, \beta_1, \ldots, \beta_d)^\top} \left\{ L_t(\beta) + \sum_{j=1}^d p_j'(|\tilde{\beta}_j|)|\beta_j| \right\} \quad (P_\ell) \tag{10}
\]
for $\ell = 1, 2, \ldots$, where $\tilde{\beta}^{(\ell)} = (\tilde{\beta}^{(\ell)}_0, \tilde{\beta}^{(\ell)}_1, \ldots, \tilde{\beta}^{(\ell)}_d)^\top$ is the optimal solution to program $(P_\ell)$. Following Zhang and Zhang (2012), we assume the following conditions on the penalty function $p_j$.

**Condition 2.** The penalty function $p_j$ is of the form $p_j(t) = \lambda^j p(t/\lambda)$ for $t \in \mathbb{R}$, where $p : \mathbb{R} \mapsto [0, \infty)$ satisfies: (i) $p(t) = p(-t)$ for all $t$ and $p(0) = 0$; (ii) $p$ is nondecreasing on $[0, \infty)$; (iii) $p$ is differentiable almost everywhere on $(0, \infty)$ and $\lim_{t \to 0^+} p'(t) = 1$; (iv) $p'(t_1) \leq p'(t_2)$ for all $t_1 \geq t_2 \geq 0$.

For each $\ell \geq 1$, program $(P_\ell)$ corresponds to a weighted $\ell_1$-regularized empirical Huber loss minimization of the form
\[
\min_{\beta = (\beta_0, \beta_1, \ldots, \beta_d)^\top} \{ L_t(\beta) + \|\lambda \circ \beta_-\|_1 \}, \tag{11}
\]
where $\lambda = (\lambda_1, \ldots, \lambda_d)^\top$ be a $d$-vector of regularization parameters with $\lambda_j \geq 0$. By convex optimization theory, any optimal solution $\tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_1^\top)$ to the convex program (11) satisfies the first-order optimality condition
\[ \partial_{\beta_0} L_t(\tilde{\beta}) = 0, \quad \nabla_{\beta_-} L_t(\tilde{\beta}) + \lambda \circ \xi = 0_d \quad \text{for some} \quad \xi \in \partial \|\tilde{\beta}_-\|_1 \subseteq [-1, 1]^d, \]
where we use the notation $\nabla L_t(\beta) = (\partial_{\beta_0} L_t(\beta), \nabla_{\beta_-} L_t(\beta)^\top)^\top \in \mathbb{R}^d$ with
\[
\partial_{\beta_0} L_t(\beta) = -\frac{1}{n} \sum_{i=1}^n \ell_i'(y_i - \langle x_i, \beta \rangle) \quad \text{and} \quad \nabla_{\beta_-} L_t(\beta) = -\frac{1}{n} \sum_{i=1}^n \ell_i'(y_i - \langle x_i, \beta \rangle)x_{-i} \in \mathbb{R}^d.
\]
and tightening ($\ell$ we will describe an iterative local adaptive majorize-minimization (I-LAMM) algorithm which produces reasonably good estimator whose statistical error is of the order $\sqrt{\log(d) \cdot \sigma^2 / n}$. Moreover, assume that the regression error $y$ in model (1) satisfies $E(y|x) = 0$ and $E(y^2|x) \leq \sigma^2_x$ almost surely. Moreover, assume that $\Sigma = (\sigma_{jk})_{0 \leq j,k \leq d} = E(x|x^\tau)$ is positive definite. Then

$$|b^*| \leq \frac{\sigma^2_x}{\tau}. \quad (13)$$

Moreover, $\tau|b^*| \to 0$ as $\tau \to \infty$. 

**Definition 1.** Following the terminology in Fan et al. (2018), for a prespecified tolerance level $\epsilon > 0$, we say $\hat{\beta}$ is an $\epsilon$-optimal solution to (11) if $\omega(\hat{\beta}) \leq \epsilon$, where

$$\omega(\beta) := \max \left\{ \left| \frac{\partial \beta}{\partial \beta} L_\tau(\beta) \right|, \min_{\xi \in \partial \beta} \| \nabla_{\beta} L_\tau(\beta) + \lambda \circ \| \xi \|_\infty \right\}, \quad \beta \in \mathbb{R}^d. \quad (12)$$

In view of Definition 1, for a prespecified sequence of tolerance levels $\{\epsilon_t\}_{t \geq 1}$, we use $\hat{\beta}^{(t)} = (\hat{\beta}_0^{(t)}, \hat{\beta}_1^{(t)}, \ldots, \hat{\beta}_d^{(t)})^T$ to denote an $\epsilon_t$-optimal solution to program $P_t$, that is,

$$\min_{\beta=(\beta_0, \beta_1)^T} \left\{ \mathcal{L}_\tau(\beta) + \| \lambda^{(t-1)} \circ \beta_2 \|_1, \right\}$$

where $\lambda^{(t-1)} := p'(\hat{\beta}^{(t-1)}_2)$. For simplicity, we consider a trivial initial estimator $\hat{\beta}^{(0)} = \mathbf{0}$. Since $p'(\hat{\beta}^{(0)}_2) = p'(0) = \lambda$ for $j = 1, \ldots, d$, the program $P_1$ coincides with that in (5). In Section 3, we will describe an iterative local adaptive majorize-minimization (I-LAMM) algorithm which produces an $\epsilon$-optimal solution to (11) after a few iterations.

The above procedure is sequential, and can be categorized into two stages: contraction ($\ell = 1$) and tightening ($\ell \geq 2$). As we will see in the next subsection, even starting with a trivial initial estimator that can be far from the true parameter, the contraction stage will produce a reasonably good estimator whose statistical error is of the order $\sqrt{\log(d) \cdot \sigma^2 / n}$. Essentially, the contraction stage is equivalent to the $\ell_1$-regularized Huber regression in (5). A tightening stage further refines this coarse contraction estimator consecutively, and eventually gives rise to an estimator that achieves the oracle rate $\sqrt{\sigma^2 / n}$.

### 2.3 Deterministic analysis

To analyze the statistical properties of $\{\hat{\beta}^{(t)}\}_{t \geq 1}$, we first introduce a local curvature parameter, which is closely related to the restricted strong convexity property of the empirical Huber loss over a local $\ell_1$-cone.

**Definition 2.** For $r, L > 0$ and $\vartheta \in \mathbb{R}^d$ ($d = d + 1$), define

$$\kappa(r, L, \vartheta) = \inf \left\{ \frac{\nabla \mathcal{L}_\tau(\beta) - \nabla \mathcal{L}_\tau(\vartheta), \beta - \vartheta}{\| \beta - \vartheta \|^2_\Sigma} : \beta \in \vartheta + \mathcal{B}(r) \cap \mathcal{C}(L) \right\},$$

where $\mathcal{B}(r) := \{ \delta \in \mathbb{R}^d : \| \delta \|_\Sigma \leq r \}$ and $\mathcal{C}(L) := \{ \delta \in \mathbb{R}^d : \| \delta \|_1 \leq L \| \delta \|_\Sigma \}$.

Throughout the following, we assume that the penalty function $p_\lambda$ satisfies Condition 2. Let $\omega = \nabla \mathcal{L}_\tau(\beta^*) - \mathbb{E} \nabla \mathcal{L}_\tau(\beta^*) \in \mathbb{R}^d$ be the centered score function evaluated at $\beta^*$, and define the (deterministic) approximation bias $b^* = \| \Sigma^{-1/2} \mathbb{E} \nabla \mathcal{L}_\tau(\beta^*) \|_2$ that is induced by the Huber loss.

First we characterize the bias term $b^*$, as a function of $\tau$.

**Lemma 1.** Assume that the regression error $\epsilon$ in model (1) satisfies $E(\epsilon|x) = 0$ and $E(\epsilon^2|x) \leq \sigma^2_x$ almost surely. Moreover, assume that $\Sigma = (\sigma_{jk})_{0 \leq j,k \leq d} = E(x|x^\tau)$ is positive definite. Then

$$\| b^* \|_2 \leq \frac{\sigma^2_x}{\tau}. \quad (13)$$

Moreover, $\tau |b^*| \to 0$ as $\tau \to \infty$. 

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That under Condition 1, magnitudes depend on the data generating process as well as \( n \) where \( \| r \| \leq \| w \| \), regardless of the sampling distribution. Indeed, both \( \| r \| \leq \| \kappa \| \), as 

\[
\frac{\| \kappa \|}{\| \beta \|} \rightarrow \infty \quad \text{contraction estimator} \quad \tilde{\beta}^{(1)}
\]

of program (P$_1$) 

\[
E \tilde{\beta}^{(1)} - \beta^* \leq \frac{a_0 \gamma_1^{1/2} s^{1/2} \lambda + b^*}{\kappa_r(r, L_0, \beta^*)},
\]

which completes the proof of (13). The last claim follows from the fact that \( E \{ e^2 I(|e| > \tau) | x \} \to 0 \) as \( \tau \to \infty \).

\(\Box\)

**Proposition 2.** Let \( (\lambda, \epsilon_1, r) \) satisfy

\[
\lambda \geq 2(|w^*|_\infty + \epsilon_1) \quad \text{and} \quad r > \frac{a_0 \gamma_1^{1/2} s^{1/2} \lambda + b^*}{\kappa_r(r, L_0, \beta^*)},
\]

where \( L_0 = 4\lambda^{1/2} s^{1/2} + 2\lambda^{1} b^* \) with \( s = s + 1 \) \( a_0 = 1 + \sqrt{2}/2 \). Then any \( \epsilon_1 \)-optimal solution \( \tilde{\beta}^{(1)} \) of program (P$_1$) satisfies

\[
\| \tilde{\beta}^{(1)} - \beta^* \|_2 \leq \frac{a_0 \gamma_1^{1/2} s^{1/2} \lambda + b^*}{\kappa_r(r, L_0, \beta^*)}.
\]

Proposition 2 is deterministic in the sense that the bound in (15) holds under the assumed scaling regardless of the sampling distribution. Indeed, both \( |w^*|_\infty \) and \( \kappa_r(r, L_0, \beta^*) \) are random and their magnitudes depend on the data generating process as well as \( (n, d) \). In Section 2.4, we will show that under Condition 1,

\[
|w^*|_\infty \leq \sqrt{\frac{\log(d)}{n}} \quad \text{and} \quad \kappa_r(r, L_0, \beta^*) \text{ is bounded away from zero}
\]

with high probability as long as \( n \geq L_0^2 \log(d) \). Consequently, with proper choices of \( (\tau, \lambda, \epsilon_1, r) \), the contraction estimator \( \tilde{\beta}^{(1)} \) satisfies the bound

\[
\| \tilde{\beta}^{(1)} - \beta^* \|_2 \leq \sqrt{\frac{s \log(d)}{n}} \quad \text{with high probability as long as} \ n \geq s \log(d).
\]

Next, we investigate the statistical properties of \( \{ \tilde{\beta}^{(l)} \}_{l \geq 2} \) in the tightening stage. To refine the suboptimal rate obtained in Proposition 2, we further require a minimum signal strength condition on \( \| \beta^* \|_{\min} = \min_{j \in S} \| \beta_j^* \| \), where \( S = \sum_{j=1}^d I(\beta_j^* \neq 0) \). For any subset \( E \subseteq [d] \), we write \( \tilde{E} = \{ 0 \} \cup E \).

**Proposition 3.** For a prespecified \( \delta \in (0, 1) \), assume there exists some constant \( \gamma > 0 \) such that \( p' (\gamma) > 0, \| \beta^* \|_{\min} \geq \gamma \lambda \) and \( \lambda \geq \frac{2}{p' (\gamma)} (|w^*|_\infty + \epsilon_1) \) for all \( \ell \geq 1 \). Moreover, assume there exists \( r > 0 \) such that \( \kappa_r(r, L, \beta^*) \) in Definition 2 with \( L = \left( 2 + \frac{2}{p' (\gamma)} \right) \lambda^{1/2} (3 s)^{1/2} + \frac{2}{p' (\gamma)} \lambda^{1} b^* \) satisfies

\[
\kappa_r(r, L, \beta^*) \geq \frac{1}{\gamma} \max \left\{ \frac{1}{\lambda_0}, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_4} \right\} \quad \text{and} \quad r \geq \frac{a_0 \gamma^{1/2} s^{1/2} \lambda + b^*}{\kappa_r(r, L, \beta^*)}.
\]
where \(a_\gamma = 1 + \sqrt{2}p'(\gamma)\). Then, any \(\varepsilon_\ell\)-optimal solution \(\widehat{\beta}^{(\ell)}(\ell \geq 2)\) satisfies

\[
||\widehat{\beta}^{(\ell)} - \beta^*||_{\Sigma} \\
\leq \delta ||\widehat{\beta}^{(\ell-1)} - \beta^*||_{\Sigma} + \frac{||p'_\lambda(|\beta^*|) - \gamma \lambda||_2 + ||w^*_S||_2 + \bar{s}/2 \varepsilon_\ell + \lambda_{1/2}^* b^*}{\lambda_{1/2}^* \kappa_r(r, L, \beta^*)},
\]

where \(\bar{s} = s + 1\). Furthermore, it holds

\[
||\widehat{\beta}^{(\ell)} - \beta^*||_{\Sigma} \\
\leq \frac{\delta^{\ell-1}}{\kappa_r(r, L_0, \beta^*)} (a_r \lambda_{1/2}^* s^{1/2} \lambda + b^*) + \frac{||p'_\lambda(|\beta^*|) - \gamma \lambda||_2 + ||w^*_S||_2 + \bar{s}/2 \varepsilon_\ell + \lambda_{1/2}^* b^*}{(1 - \delta) \lambda_{1/2}^* \kappa_r(r, L, \beta^*)}.
\]

Proposition 3 unveils how the tightening stage improves the statistical rate: every tightening step shrinks the estimation error from the previous step by a \(\delta\)-fraction. The second term on the right-hand side of (17) or (18) dominates the \(\ell_2\)-error, and up to constant factors, consists of three components,

\[
\begin{align*}
\frac{||p'_\lambda(|\beta^*|) - \gamma \lambda||_2}{\text{shrinkage bias}} & \quad \frac{||w^*_S||_2 + b^*}{\text{oracle rate plus approx. bias}} & \quad \frac{\bar{s}^{1/2} \varepsilon_\ell}{\text{optimization error}}.
\end{align*}
\]

We identify \(||p'_\lambda(|\beta^*|) - \gamma \lambda||_2\) as the shrinkage bias induced by the penalty function. This explains the limitation of the \(\ell_1\)-regularized method in which \(p_\lambda(t) = \lambda |t|\) and \(p'_\lambda(t) = \lambda \text{sign}(t)\). Intuitively, choosing a proper penalty function \(p_\lambda(\cdot)\) with a decreasing derivative reduces the bias. The second term, \(||w^*_S||_2 + b^*\), reveals the oracle property. To see this, consider the oracle estimator defined as

\[
\widehat{\beta}^{\text{ora}} = \arg\min_{\beta, \beta_S \geq 0} \sum_{i=1}^n \tau(y_i - \langle x_i, \beta \rangle).
\]

(19)

Since \(s = |S| \ll n\), the finite sample theory for Huber’s \(M\)-estimation in low dimensions (Wang et al., 2018) applies to \(\widehat{\beta}^{\text{ora}}\), indicating that with high probability,

\[
||\widehat{\beta}^{\text{ora}} - \beta^*||_2 \lesssim ||w^*_S||_2 + b^*.
\]

According to Definition 1, the last term \(\bar{s}^{1/2} \varepsilon_\ell\) demonstrates the optimization error, which will be discussed in greater detail in Section 3.

The above results provide conditions under which the sequence of estimators \(\{\widehat{\beta}^{(\ell)}\}_{\ell \geq 1}\) satisfy the contraction property and, meanwhile, fall in a local neighborhood of \(\beta^*\). However, our previous results for the \(\ell_1\)-regularized method show that, in cases where the regression error is asymmetric and \(\tau\) scales as a constant, the statistical consistency of the resultant estimator is with respect to \(\beta^*_r\) defined in (6) rather than \(\beta^*\). With slight modifications, this fixed \(\tau\) scenario is also covered by the previous theory.

**Proposition 4.** The statements of Propositions 2 and 3 remain valid with \(\beta^*\) replaced by \(\beta^*_r = (\beta^*_{0,r}, \beta^*_{S,r})^\top\), \(S = \text{supp}(\beta^*_{S,r})\) and \(s = |S|\). In this case, the approximation bias \(b^* = 0\).

Another important feature of the proposed procedure is that the resulting estimator satisfies the strong oracle property, as demonstrated by the following result. Let \(\{\widehat{\beta}^{(\ell)}\}_{\ell \geq 1}\) be any optimal
solutions to the convex programs \(\{P_r\}_{r\geq 1}\) in (10) with \(\hat{\beta}^{(0)} = 0\). Similarly to Definition 2, we need the following quantity that quantifies the restricted strong convexity. For \(r, L > 0\) and \(\vartheta = (\vartheta_0, \vartheta_1)^\top \in \mathbb{R}^d\) with \(E = \text{supp}(\vartheta) \subseteq [d]\), define

\[
\bar{r}_r(r, L, \vartheta) = \inf \left\{ \frac{(\nabla L_r(\beta) - \nabla L_r(\beta'), \beta - \beta')}{\|\beta - \beta'\|^2_\Sigma} : \beta \in \beta' + \mathbb{B}_\Sigma(r) \cap C(L), \beta' \in \vartheta + \mathbb{B}_\Sigma(r/2), \beta'_{\underline{S}} = 0 \right\},
\]

where \(\mathbb{B}_\Sigma(r)\) and \(C(L)\) are given in Definition 2. Moreover, define the “oracle” score \(w^{\text{ora}} \in \mathbb{R}^d\) as

\[
w^{\text{ora}} = \nabla L_r(\hat{\beta}^{\text{ora}}),
\]

which satisfies \(w^{\text{ora}} = 0\).

**Proposition 5.** Assume that there exist constants \(\gamma_1 > \gamma_0 > 0\) such that \(p'(\gamma_0) \in (0, 1/2], p'(\gamma_1) = 0, \lambda \geq \frac{2}{p'(\gamma_0)}\|w^{\text{ora}}\|_\infty\) and \(\|\beta^{\text{ora}}\|_{\min} \geq (\gamma_0 + \gamma_1)\lambda\). Let \(L = (2 + \frac{2}{p'(\gamma_0)})\lambda^{-1/2}(2s)^{1/2}\). For a prespecified \(\delta \in (0, 1)\), assume that there exists \(\bar{r} > 0\) such that \(\bar{r}_r(r, L, \beta^*) \geq \bar{r} \geq 1.25/(\delta \lambda \gamma_0)\) for some \(r > 1.25\bar{r}^{-1}L^{-1/2}s^{-1/2}\lambda\). Moreover, suppose that the oracle estimator satisfies the bounds

\[
\|\hat{\beta}^{\text{ora}} - \beta^*\|_\Sigma \leq \frac{r}{2} \quad \text{and} \quad \|\hat{\beta}^{\text{ora}} - \beta^*\|_\infty \leq \frac{1}{\delta} \frac{\lambda}{5\bar{r}L},
\]

Then \(\hat{\beta}^{(\ell)} = \hat{\beta}^{\text{ora}}\) for all \(\ell \geq \lceil \log(s^{1/2}/\delta)/\log(1/\delta) \rceil\).

Similarly to Proposition 4, the strong oracle property prevails if the target parameter is \(\beta^*_r\) instead of \(\beta^*\).

**Proposition 6.** The statement of Proposition 5 remains valid if \(\beta^*\) is replaced by \(\beta^*_r = (\beta^*_0, r)_{\beta^*_1}^{\top}\), \(S = \text{supp}(\beta^*_r)\) and \(s = |S|\).

The proofs of Propositions 2, 3 and 5 are provided in the Appendix. Since the proofs of Propositions 4 and 6 are almost identical, we omit the details to avoid repetition.

### 2.4 Random analysis

In this section, we complement the previous deterministic results with probabilistic bounds on three key quantities, the local curvature parameter \(\kappa_r(r, L, \vartheta)\), \(\|w(\beta)\|_\infty\) and \(\|w(\beta)\|_2\), for either \(\beta = \beta^*\) or \(\beta = \beta^*_r\), where \(w(\beta) = \nabla L_r(\beta) - \mathbb{E}\nabla L_r(\beta)\) is the centered score function. As discussed earlier, \(\|w(\beta)\|_\infty\) determines the order of the regularization parameter and \(\|w(\beta)\|_2\) captures the optimal rate of convergence.

Recall that \(x = (1, x^\top)\) is sub-exponential with \(\Sigma = (\sigma_{jk})_{0 \leq j, k \leq d} = \mathbb{E}(xx^\top)\) positive definite. Moreover, given the true active set \(S \subseteq [d]\) of \(\beta^*\), we define the following \(\hat{s} \times \hat{s} (s = s + 1)\) principal submatrix of \(\Sigma\):

\[
S = \mathbb{E}(x_S x_S^\top), \quad \text{where} \quad x_S = (1, x^\top_S) \in \mathbb{R}^{\hat{s}}.
\]

Throughout, “\(\leq\)” and “\(\geq\)” stand for “\(\leq\)” and “\(\geq\)”, respectively, up to constants that are independent of \((n, d, s)\) but might depend on the constants in Condition 1.
Proposition 7. Assume Condition 1 holds for model (1).

(I) Let \( \tau, r, L > 0 \) satisfy
\[
\tau \geq \max\{4\sigma_x, 24\nu_0^2 r\} \quad \text{and} \quad n \geq (\tau/r)^2 L^2 \log(d).
\] (23)

Then with probability at least \( 1 - d^{-1} \),
\[
\frac{\langle \nabla L_{\tau}(\beta) - \nabla L_{\tau}(\beta^*), \beta - \beta^* \rangle}{\|\beta - \beta^*\|^2_{\Sigma}} \geq \frac{1}{2} \quad \text{for all} \quad \beta \in \beta^* + \mathbb{B}_\Sigma(r) \cap \mathcal{C}(L).
\] (24)

(II) Assume further that the assumptions of Theorem 1, (II) are met. Let \( \tau, r, L > 0 \) satisfy
\[
\tau \geq 16\sigma_x^2 r \quad \text{and} \quad n \geq (\tau/r)^2 L^2 \log(d),
\] (25)

where \( \sigma_x = \sqrt{\beta_{\tau}/L_{\tau}} \). Then with probability at least \( 1 - d^{-1} \),
\[
\frac{\langle \nabla L_{\tau}(\beta) - \nabla L_{\tau}(\beta^*), \beta - \beta^* \rangle}{\|\beta - \beta^*\|^2_{\Sigma}} \geq \frac{\sigma_x}{2} \quad \text{for all} \quad \beta \in \beta^* + \mathbb{B}_\Sigma(r) \cap \mathcal{C}(L).
\] (26)

Part (II) of Proposition 7 follows directly from Lemma 4, and Part (I) can be proved via the same argument. A direct implication of Proposition 7 is that, provided inequality (24) or (26) holds for some choices of \( (r, L) \), the curvature parameter \( \kappa_{(r, L, \vartheta)} \) in Definition 2 can be taken as \( \kappa_{(r, L, \beta^*)} = 1/2 \) or \( \kappa_{(r, L, \beta^*)} = \vartheta/2 \). The next proposition provides upper bounds on \( \|w(\beta)\|_\infty \) and \( \|w(\beta)\|_2 \) for \( \beta = \beta^* \) and \( \beta = \beta^+_s \).

Proposition 8. Assume Condition 1 holds, and let \( w(\beta) = \nabla L_{\tau}(\beta) - \mathbb{E} \nabla L_{\tau}(\beta), \beta \in \mathbb{R}^d \).

(I) With probability at least \( 1 - 2(d^{-1} + d^{-2}) \),
\[
\|w(\beta^*)_\infty \leq 2\nu_0 \sigma_x \left\{ \sigma_x \sqrt{\frac{\log(d)}{n} + 2\tau \frac{\log(d)}{n}} \right\}.
\] (27)

For any \( t > 0 \), it holds with probability at least \( 1 - e^{-2t} \) that
\[
\|w(\beta^*)_S\|_2 \leq 3\nu_0 \lambda_{\max}(S) \left( \sigma_x \sqrt{\frac{\tilde{s} + t}{n} + 2\tau \frac{\tilde{s} + t}{n}} \right),
\] (28)

where \( S \in \mathbb{R}^{d \times d} \) is defined in (22).

(II) Assume further that the assumptions of Theorem 1, (II) are met. Then both inequalities (27) and (28) hold with \( \beta^* \) replaced by \( \beta^+_s \).

Together, Propositions 7 and 8 reveal the impact of the robustification parameter on the statistical properties of the resulting estimator. As discussed in Section 2.3 above, the order of \( \|w(\beta^*)_S\|_2 \) determines the oracle statistical rate. In Theorem 2, we show that after only a small number of iterations, the proposed procedure leads to an estimator that achieves the oracle rate of convergence. Recall from Section 2.2 that \( (\beta^{(l)})_{l=1,2,...} \) is a sequence of \( \epsilon_l \)-optimal solutions of the convex programs (10), initialized at \( \beta^{(0)} = 0 \).
Theorem 2. Assume that Conditions 1 and 2 hold, and for a prespecified $\delta \in (0, 1)$, there exist some $\gamma_1 > \gamma_0 > 0$ such that

$$
\gamma_0 \geq \frac{2}{\lambda_t} \max\{\delta^{-1}, 1 + \sqrt{\frac{t}{2}} p'(\gamma_0)\}, \quad p'(\lambda_0) > 0, \quad p'(t) = 0 \text{ for all } t \geq \gamma_1,
$$

and $\|\beta^*_S\|_{\infty} \geq (\gamma_0 + \gamma_1) \lambda$. Further, suppose that $\epsilon_\ell \geq \sqrt{1/n}$ for all $\ell \geq 1$ and the sample size satisfies $n \geq s \log(d)$.

(I) (Diverging $\tau$) Let $\tau = \sigma_e \sqrt{n/[s + \log(d)]}$. Then, with probability at least $1 - 4d^{-1}$, the estimator $\hat{\beta}^{(T)}$ with $\lambda = \sigma_e \sqrt{\log(d)/n}$ and $T \geq \frac{\log\log(d)}{\log(1/\delta)}$ satisfies the bounds

$$
\|\hat{\beta}^{(T)} - \beta^*\|_2 \leq \sigma_e \sqrt{\frac{s + \log(d)}{n}} \quad \text{and} \quad \|\hat{\beta}^{(T)} - \beta^*\|_1 \leq \sigma_e s^{1/2} \sqrt{\frac{s + \log(d)}{n}}.
$$

(II) (Fixed $\tau$) Suppose that the conditions of Theorem 1, (II) hold for $\tau = c_0 \sigma_e$ and some universal constant $c_0$. For any $t > 0$, the estimator $\hat{\beta}^{(T)}$ with $\lambda = \sigma_e \sqrt{\log(d)/n}$ and $T \geq \frac{\log\log(d)}{\log(1/\delta)}$ satisfies the bounds

$$
\|\hat{\beta}^{(T)} - \beta^*_t\|_2 \leq \sigma_e \sqrt{\frac{s + t}{n}} \quad \text{and} \quad \|\hat{\beta}^{(T)} - \beta^*_t\|_1 \leq \sigma_e s^{1/2} \sqrt{\frac{s + t}{n}}
$$

with probability at least $1 - 3d^{-1} - e^{-t}$.

We refer to the conclusion of Theorem 2 as the weak oracle property in the sense that the proposed estimator shares the statistical rate of convergence with the oracle $\hat{\beta}_{\text{ora}}$ that knows a priori the support $S$ of $\beta^*$. An even more intriguing result, as revealed by the following theorem, is that our estimator achieves the strong oracle property, namely, it coincides with the oracle with high probability. Here we need a slightly stronger moment condition than that in Condition 1, that is, the covariate vector $x$ is sub-Gaussian.

Condition 3. There exists $\nu_1 \geq 1$ such that $P(\|(u, x)\| \geq \nu_1 \|u\| \Sigma : t) \leq 2e^{-t/2}$ for all $u \in \mathbb{R}^d$, where $\Sigma = (\sigma_{ij})_{0 \leq i,j \leq d} = \mathbb{E}(x|x^T)$ is positive definite with $0 < \lambda_t \leq \lambda_{\min}(\Sigma) \leq \lambda_{\max}(\Sigma) \leq \lambda_u < \infty$. Let $\sigma_e^2 = \max_{0 \leq j \leq d} \sigma_{jj}$. The regression error $\epsilon$ satisfies $\mathbb{E}(\epsilon|x) = 0$ and $\mathbb{E}(\epsilon^2|x) \leq \sigma_e^2$ almost surely.

Theorem 3. Assume Conditions 2 and 3 hold. Moreover, for a prespecified $\delta \in (0, 1)$, assume there exists some $\gamma_1 > \gamma_0 \geq 2.5/(\lambda_t \delta)$ such that $p'(\gamma_0) > 0, p'(t) = 0$ for all $t \geq \gamma_1$ and $\|\beta^*_S\|_{\infty} \geq (\gamma_0 + \gamma_1) \lambda$.

(I) (Diverging $\tau$) Let $(\tau, \lambda)$ satisfy $\tau = \sigma_e \sqrt{n/[s + \log(d)]}$ and $\lambda = \sigma_e \sqrt{\log(d)/n}$, and suppose the sample size satisfies $n \geq \max\{s^{3/2}, s \log(d)\}$. Then with probability at least $1 - 5d^{-1}$, $\hat{\beta}^{(T)} = \hat{\beta}_{\text{ora}}$ provided $\ell \geq \lceil \log(s^{1/2}/\delta) / \log(1/\delta) \rceil$.

(II) (Fixed $\tau$) Assume that the conditions of Theorem 1, (II) hold for $\tau = c_0 \sigma_e$ and some universal constant $c_0$. In addition, suppose that there exist constants $L_0, L_1 > 0$ such that

$$
P(\epsilon \leq b|x) \leq \frac{L_0}{\sigma_e} (b - a) \quad \text{for all } a \leq b
$$

(32)
almost surely, and
\[
\max_{j \in S_t} \| S_t^{-1} E[\ell'_t (e - \alpha_t) x_S x_j] \|_2 \leq L_1, \tag{33}
\]
where \( S_t = E[\ell'_t (e - \alpha_t) x_S x_S^T] \). Furthermore, suppose that the sample size is lower bounded as \( n \geq \max[s \log(d), s^2 / \log(d)] \) and the regularization parameter satisfies \( \lambda = \sigma_t \sqrt{\log(d)/n} \). Then with probability at least \( 1 - 4d^{-1} - 2n^{-1} \), \( \beta^{(t)} = \beta^{\text{opt}} \) provided \( \ell \geq [\log(s^{1/2}/\delta) / \log(1/\delta)] \).

### 3 Optimization Algorithm

In this section, we use the local adaptive majorize-minimize (LAMM) principal (Fan et al., 2018) to derive an iterative algorithm for solving each subproblem \((P_t)\) in (10):

\[
\min_{\beta = [\beta_0, \beta_1]} \{ L_t(\beta) + \| \lambda^{(t-1)} \circ \beta_- \|_1 \}. \tag{33}
\]

#### 3.1 LAMM algorithm

To minimize a general function \( f(\beta) \), at a given point \( \beta^{(k)} \), the majorize-minimize (MM) algorithm first majorizes it by another function \( g(\beta | \beta^{(k)}) \), which satisfies

\[
g(\beta | \beta^{(k)}) \geq f(\beta) \quad \text{and} \quad g(\beta^{(k)} | \beta^{(k)}) = f(\beta^{(k)}) \quad \text{for all} \quad \beta,
\]

and then compute \( \beta^{(k+1)} = \arg\min_{\beta} g(\beta | \beta^{(k)}) \) (Lange, Hunter and Yang, 2000). The objective value of such an algorithm is non-increasing in each step, since

\[
f(\beta^{(k+1)}) \leq g(\beta^{(k+1)} | \beta^{(k)}) \leq g(\beta^{(k)} | \beta^{(k)}) \equiv f(\beta^{(k)}). \tag{34}
\]

Fan et al. (2018) observed that the global majorization requirement is not necessary. It only requires the local properties

\[
f(\beta^{(k+1)}) \leq g(\beta^{(k+1)} | \beta^{(k)}) \quad \text{and} \quad g(\beta^{(k)} | \beta^{(k)}) = f(\beta^{(k)}) \tag{35}
\]

for the inequalities in (34) to hold.

Using the above principle, it suffices to locally majorize the objective function \( L_t(\beta) \) in the above minimization problem. At the \( k \)-th step with working parameter vector \( \beta^{(t,k-1)} \), we use an isotropic quadratic function, that is,

\[
F(\beta; \phi, \beta^{(t,k-1)}) := L_t(\beta^{(t,k-1)}) + \langle \nabla L_t(\beta^{(t,k-1)}), \beta - \beta^{(t,k-1)} \rangle + \frac{\phi}{2} \| \beta - \beta^{(t,k-1)} \|^2_2, \tag{36}
\]

to locally majorize \( L_t(\beta) \) such that

\[
F(\beta^{(t,k)}; \phi^{(t,k)}, \beta^{(t,k-1)}) \geq L_t(\beta^{(t,k)}), \tag{37}
\]

where \( \phi^{(t,k)} \) is a proper quadratic coefficient at the \( k \)-th update, and \( \beta^{(t,k)} \) is the solution to

\[
\min_{\beta} [F(\beta; \phi^{(t,k)}, \beta^{(t,k-1)}) + \| \lambda^{(t-1)} \circ \beta_- \|_1].
\]

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Algorithm 1 LAMM algorithm at the $k$-th iteration of the $\ell$-th subproblem.

1: Algorithm: $(\beta^{(\ell,k)}, \phi^{(\ell,k)}) \leftarrow \text{LAMM}(\lambda^{(\ell-1)}, \beta^{(\ell,k-1)}, \phi_0, \phi^{(\ell,k-1)})$
2: Input: $\lambda^{(\ell-1)}, \beta^{(\ell,k-1)}, \phi_0, \phi^{(\ell,k-1)}$
3: Initialize: $\phi^{(\ell,k)} \leftarrow \max\{\phi_0, \gamma_u^{-1} \phi^{(\ell,k-1)}\}$
4: Repeat
5: $\beta^{(\ell,k)} \leftarrow T_{\lambda^{(\ell-1)}, \phi^{(\ell,k)}}(\beta^{(\ell,k-1)})$
6: If $F(\beta^{(\ell,k)}, \lambda^{(\ell-1)}) < L_r(\beta^{(\ell,k-1)})$ then $\phi^{(\ell,k)} \leftarrow \gamma_u \phi^{(\ell,k)}$
7: Until $F(\beta^{(\ell,k)}, \lambda^{(\ell-1)}) \geq L_r(\beta^{(\ell,k-1)})$
8: Return $[\beta^{(\ell,k)}, \phi^{(\ell,k)}]$.

It is easy to see that $\beta^{(\ell,k)} = (\beta_0^{(\ell,k)}, (\beta_-^{(\ell,k)})^T)$ takes a simple explicit form

$$
\begin{align*}
\beta_0^{(\ell,k)} &= \beta_0^{(\ell,k-1)} - \partial_{\beta_0} L_r(\beta^{(\ell,k-1)})/\phi^{(\ell,k)}, \\
\beta_-^{(\ell,k)} &= S_{\text{soft}}(\beta_-^{(\ell,k-1)} - \nabla_{\beta_-} L_r(\beta^{(\ell,k-1)})/\phi^{(\ell,k)}, \lambda^{(\ell-1)}/\phi^{(\ell,k)}),
\end{align*}
$$

where $S_{\text{soft}}(x, \lambda) := (\max(|x_j| - \lambda, 0) / \lambda)$ is the soft-thresholding operator. For simplicity, we summarize and define the above update as $\beta^{(\ell,k)} = T_{\lambda^{(\ell-1)}, \phi^{(\ell,k)}}(\beta^{(\ell,k-1)})$. Using this simple update formula of $\beta$, we iteratively search for the pair $(\phi^{(\ell,k)}, \beta^{(\ell,k)})$ that ensures the local majorization (37).

Starting with an initial quadratic coefficient $\phi = \phi_0$, say $10^{-4}$, we iteratively increase $\phi$ by a factor of $\gamma_u > 1$ and compute

$$
\beta^{(\ell,k)} = T_{\lambda^{(\ell-1)}, \phi^{(\ell,k)}}(\beta^{(\ell,k-1)}) \text{ with } \phi^{(\ell,k)} = \gamma_u^{-1} \phi_0,
$$

until the local property (37) holds. This routine is summarized in Algorithm 1.

3.2 Complexity theory

To investigate the complexity theory of the proposed algorithm, we first impose the following regularity condition.

**Condition 4.** $\nabla L_r(\beta)$ is $\rho_c$-Lipschitz continuous, that is,

$$
\|\nabla L_r(\beta_1) - \nabla L_r(\beta_2)\|_2 \leq \rho_c \|\beta_1 - \beta_2\|_2,
$$

where $\rho_c$ is the Lipschitz constant.

Our next theorem characterizes the computational complexity in the contraction stage. Recall that $\lambda^{(0)} = (\lambda_1, \ldots, \lambda_d)^T \in \mathbb{R}^d$.

**Theorem 4.** Assume that Condition 4 holds and the optimal solution $\hat{\beta}^{(1)}$ satisfies $\|\hat{\beta}^{(1)} - \beta^*\|_2 \leq s^{1/2} \lambda$. Then, to attain an $\epsilon_c$-optimal solution $\hat{\beta}^{(1)}$, i.e. $\omega_{\lambda^{(0)}}(\hat{\beta}^{(1)}) \leq \epsilon_c$, in the contraction stage, we need as many as

$$
C_1 \rho_c^2 (1 + \gamma_u)^2 (\|\beta^*\|_2 + s^{1/2} \lambda)^2 / \epsilon_c^2
$$

LAMM iterations in (38), where $C_1 > 0$ is a constant independent of $(n, d, s)$. 

$$
\frac{\lambda_1}{\gamma_u} \leq 1
$$
The sublinear rate in the contraction stage is due to the lack of strong convexity of the loss function in this stage, because we start with a naive initial value $0$. Once we enter the contracting region where the estimator is sparse and reasonably close to the underlying true parameter vector, the problem becomes strongly convex (at least with high probability). This endows the algorithm a linear rate of convergence. Our next theorem provides a formal statement on the geometric convergence rate for each subproblem in the tightening stage. We need a form of sparse eigenvalue condition.

**Condition 5.** We say that the LSE condition holds if there exist an integer $\tilde{s} \leq s$ and $r, \tau > 0$ such that

$$0 < \rho_s \leq \rho_+(2s + 2 + 2\tilde{s}, r, \tau) < \rho_+(2s + 2 + 2\tilde{s}, r, \tau, \tau) \leq \rho^* < +\infty$$

and

$$\rho_+(\tilde{s}, r, \tau)/\rho_-(2s + 2 + 2\tilde{s}, r, \tau) \leq 1 + C\tilde{s}/s,$$

where $\rho_s, \rho^*$ and $C$ are positive constants.

**Theorem 5.** Suppose that Condition 5 holds. To obtain an $\epsilon_t$-optimal solution $\tilde{\beta}^{(f)}$, i.e., $\omega_{\lambda^{(f)}}(\tilde{\beta}^{(f)}) \leq \epsilon_t$, in the $\ell$-th subproblem for $\ell \geq 2$, we need as many as $C_1 \log(C_2/\sqrt{s}/\epsilon_t)$ LAMM iterations in (38), where $C_1$ and $C_2$ are positive constants.

Together, the above two results yield the following corollary, which characterizes the computational complexity of the entire algorithm.

**Corollary 1.** Suppose that Condition 5 holds. To achieve a sequence of approximate solutions $\{\tilde{\beta}^{(f)}\}_{f=1}^T$ such that $\omega_{\lambda^{(f)}}(\tilde{\beta}^{(f)}) \leq \epsilon_t \leq \lambda$ and $\omega_{\lambda^{(f)}}(\tilde{\beta}^{(f)}) \leq \epsilon_t \leq \sqrt{T/n}$ for $2 \leq \ell \leq T$, the required number of LAMM iterations is at the order of $C_1 \epsilon_t^{-2} + C_2(T - 1) \log(\epsilon_t^{-1})$, where $C_1$ and $C_2$ are positive constants independent of $(n, d, s)$.

## 4 Extension to General Robust Losses

Thus far, we have restricted our attention to the Huber loss. As a representative robust loss function, the Huber loss has the merit of being (i) globally $\tau$-Lipschitz continuous, and (ii) locally quadratic. A natural question arises that whether similar results, both statistical and computational, remain valid for more general loss functions that possess the above two features. In this section, we introduce a class of loss functions which, combined with nonconvex regularization, leads to statistically optimal and robust estimators.
**Condition 6** (Globally Lipschitz and locally quadratic loss functions). Consider a general loss function \( \ell_t(x) \) that is of the form \( \ell_t(x) = \tau^2 \ell(x/\tau) \) for \( x \in \mathbb{R} \), where \( \ell : \mathbb{R} \to [0, \infty) \) satisfies: (i) \( \ell'(0) = 0 \) and \( |\ell'(x)| \leq c_1 \) for all \( x \in \mathbb{R} \); (ii) \( \ell''(0) = 1 \) and \( \ell''(x) \geq c_2 \) for all \( |x| \leq c_3 \); and (iii) \( |\ell'(x) - x| \leq c_4 x^2 \) for all \( x \in \mathbb{R} \), where \( c_1 \ldots c_4 \) are positive constants.

We first discuss the implications of the three properties in Condition 6. First, since \( \ell_t'(x) = \tau \ell'(x/\tau) \), it follows from property (i) that \( \sup_{x \in \mathbb{R}} |\ell_t'(x)| \leq c_1 \). The boundedness of \( |\ell_t'| \) facilitates the use of Bernstein’s inequality on deriving upper bounds for the random quantities

\[
\|w(\beta)\|_\infty \quad \text{and} \quad \|w(\beta)\|_2
\]
as in Proposition 8, where \( w(\beta) = \nabla L_t(\beta) - \mathbb{E}\nabla L_t(\beta) \) with \( L_t(\beta) = (1/n) \sum_{i=1}^n \ell_t(y_i - \langle x_i, \beta \rangle) \) and \( \beta = \beta^* \) or \( \beta_0^* \). Next, note that \( \ell_t' \) is strongly convex on \([-c_3 \tau, c_3 \tau]\), which turns out to be the key factor in establishing the restricted strong convexity condition on \( L_t \). See Proposition 7, and Lemmas 4, 5 and 7. Lastly, property (iii) is particularly useful in the “diverging \( \tau^* \)” scenario. Even though it can be shown under property (i) that \( \nabla L_t(\beta^*) \) is concentrated around its expected value \( \mathbb{E}\nabla L_t(\beta^*) \) with high probability, in general \( \mathbb{E}\nabla L_t(\beta^*) = -\mathbb{E}[\ell_t'(e|x)] \) is nonzero. However, since \( \mathbb{E}[\ell_t'(e|x)] = 0 \), we have \( \mathbb{E}[\ell_t'(e|x)] = \mathbb{E}[\ell_t'(e|x)] = \mathbb{E}[\ell_t'(e|x) - e|x] = \tau \mathbb{E}[\ell_t'(e/\tau) - e/\tau|x] \). Together with property (iii), this implies

\[
\|\mathbb{E}[\ell_t'(e|x)]\| \leq c_4 \tau \mathbb{E}[(e/\tau)^2|x] = c_4 \sigma^2 \tau^{-1}.
\]
The term \( \sigma^2 \tau^{-1} \), which corresponds to the approximation bias, then arises in Lemma 1.

Below we list four examples of \( \ell \) that satisfy Condition 6.

1. (Huber loss): \( \ell(x) = x^2 / 2 \cdot I(|x| \leq 1) + (|x| - 1/2) \cdot I(|x| > 1) \) with \( \ell'(x) = x I(|x| \leq 1) + \text{sign}(x) I(|x| > 1) \) and \( \ell''(x) = I(|x| \leq 1) \). Moreover,

\[
|\ell'(x) - x| = |x - \text{sign}(x) I(|x| > 1)| \leq x^2.
\]

2. (Pseudo-Huber loss I): \( \ell(x) = \sqrt{1 + x^2} - 1 \), whose first and second derivatives are

\[
\ell'(x) = \frac{x}{\sqrt{1 + x^2}} \quad \text{and} \quad \ell''(x) = \frac{1}{(1 + x^2)^{3/2}},
\]
respectively. It is easy to see that \( \sup_{x \in \mathbb{R}} |\ell'(x)| \leq 1 \) and \( \ell''(x) \geq (1 + x^2)^{-3/2} \) for all \( |x| \leq c \) and \( c > 0 \). Moreover, since \( \ell''''(x) = -3x(1+x^2)^{-5/2} \) satisfies \( |\ell''''(x)| < 0.9 \) for all \( x \), it follows from Taylor’s theorem and Lagrange error bound that \( |\ell'(x) - x| = |\ell'(x) - \ell'(0) - \ell''(0)|x| \leq 0.45x^2 \).

3. (Pseudo-Huber loss II): \( \ell(x) = \log((e^x + e^{-x})/2) \), whose first and second derivatives are, respectively,

\[
\ell'(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}} \quad \text{and} \quad \ell''(x) = \frac{4}{(e^x + e^{-x})^2}.
\]

It follows that \( \sup_{x \in \mathbb{R}} |\ell'(x)| \leq 1 \) and \( \ell''(x) \geq 4(e^x + e^{-x})^{-2} \) for all \( |x| \leq c \) and \( c > 0 \). Moreover, we calculate the third derivative \( \ell'''(x) = -8(e^x - e^{-x})(e^x + e^{-x})^{-3} \) that satisfies \( |\ell'''(x)| < 0.4 \). Again, by Taylor’s theorem and Lagrange error bound, \( |\ell'(x) - x| \leq 0.2x^2 \).
4. (Smoothed Huber loss I): The Huber loss is twice differentiable in $\mathbb{R}$, except at $\pm 1$. Modifying the Huber loss gives rise to the following function that is twice differentiable everywhere:

$$\ell(x) = \begin{cases} 
\frac{x^2}{2} - \frac{|x|^3}{6} & \text{if } |x| \leq 1, \\
\frac{|x|}{2} - \frac{1}{6} & \text{if } |x| > 1,
\end{cases}$$

whose first and second derivatives are

$$\ell'(x) = \begin{cases} 
-x \cdot \text{sign}(x) \cdot \frac{x}{2} & \text{if } |x| \leq 1, \\
\frac{\text{sign}(x)}{2} & \text{if } |x| > 1,
\end{cases} \quad \ell''(x) = \begin{cases} 
1 - |x| & \text{if } |x| \leq 1, \\
0 & \text{if } |x| > 1.
\end{cases}$$

Direct calculations show that $\sup_{x \in \mathbb{R}} |\ell'(x)| \leq 1/2$ and $\ell''(x) \geq 1 - c$ for all $|x| \leq c$ and $0 < c < 1$. Since $\ell''$ is $1$-Lipschitz continuous, we have $|\ell'(x) - x| \leq x^2/2$.

5. (Smoothed Huber loss II): Another smoothed version of the Huber loss function is

$$\ell(x) = \begin{cases} 
\frac{x^2}{2} - \frac{x^4}{24} & \text{if } |x| \leq \sqrt{2}, \\
(2 \sqrt{2}/3)|x| - 1/2 & \text{if } |x| > \sqrt{2}.
\end{cases}$$

The derivative of this function, also known as the influence function, is used in Catoni and Giulini (2017) for mean vector estimation. We compute

$$\ell'(x) = \begin{cases} 
x - \frac{x^3}{6} & \text{if } |x| \leq \sqrt{2}, \\
(2 \sqrt{2}/3) \text{sign}(x) & \text{if } |x| > \sqrt{2},
\end{cases} \quad \ell''(x) = \begin{cases} 
1 - \frac{x^2}{2} & \text{if } |x| \leq \sqrt{2}, \\
0 & \text{if } |x| > \sqrt{2}.
\end{cases}$$

It is easy to see that $\sup_{x \in \mathbb{R}} |\ell'(x)| \leq 2 \sqrt{2}/3$ and $\ell''(x) \geq 1 - c^2/2$ for all $|x| \leq c$ and $0 < c < \sqrt{2}$. Noting that $\ell''$ is $\sqrt{2}$-Lipschitz continuous, it holds $|\ell'(x) - x| \leq x^2 / \sqrt{2}$.

The loss functions discussed above, along with their derivatives up to order three, are plotted in Figure 1 except for the Huber loss. Provided that the loss function $\ell_\tau$ satisfies Condition 6, all the theoretical results in Sections 2.3 and 2.4 remain valid only with different constants. It is worth noticing that the four loss functions described in examples 2–5 and plotted in Figure 1 also have Lipschitz continuous second derivatives. In fact, if the function $\ell$ satisfies $\ell'(0) = 0$, $\ell''(0) = 1$ and has $L_2$-Lipschitz second derivative, then property (iii) in Condition 6 holds with $c_2 = L_2/2$. This additional smoothness sometimes facilitates theoretical analysis.
Theorem 6. Assume Conditions 2 and 3 hold, and that $\epsilon$ and $x$ are independent. For a prespecified $\delta \in (0, 1)$, assume there exist some $\gamma_1 > \gamma_0 \geq 2.5/(\lambda_0 \delta)$ such that $p' (\gamma_0) > 0$, $p'(t) = 0$ for all $t \geq \gamma_1$ and $\| \beta_S^* \|_{\text{min}} \geq (\gamma_0 + \gamma_1)\. \lambda$. Let $\ell_r(x) = \tau^2 \ell(x/\tau)$ be a loss function that satisfies Condition 6 and has Lipschitz continuous second derivative. Let $\tau \propto \sigma_\epsilon$ be such that the function $\alpha \mapsto E[\ell((\epsilon - \alpha)/\tau)]$ has a unique minimizer, denoted by $\alpha_\tau$, and $P(|\epsilon - \alpha_\tau| \leq c_3 \tau/2) > 0$ for $c_3 > 0$ given in Condition 6. Furthermore, suppose that the sample size is lower bounded as $n \gtrsim \max\{s \log(d), s^2/\log(d)\}$. Then, for the choice $\lambda = \sigma_\epsilon \sqrt{\log(d)/n}$, we have, with probability at least $1 - C(n^{-1} + d^{-1})$,

$$\tilde{\beta}^{(t)} = \tilde{\beta}_{\text{ora}} \quad \text{and} \quad \| \tilde{\beta}^{(t)} - \beta_{\epsilon}^* \|_2 \lesssim \sigma_\epsilon \sqrt{s + \log n \over n}$$
for all $\ell \geq [\log(s^{1/2}/\delta)/\log(1/\delta)]$.

Comparing with Theorem 3, (II), we see that the additional Lipschitz continuity of $\ell''$ helps remove the anti-concentration condition (32) on the distribution of $\varepsilon$.

5 Empirical Study

In this section, we compare the empirical performance of the proposed two-stage robust regression approach with several benchmark methods, such as the Lasso (Tibshirani, 1996), the SCAD- and MCP-penalized least squares (Fan and Li, 2001; Zhang, 2010a). All the computational results presented below are reproducible using software available at https://github.com/XiaoouPan/ILAMM.

We generate data vectors $\{(y_i, x_i)\}_{i=1}^n$ from two types of linear models:

1. (Homoscedastic model):
   \[ y_i = \langle x_i, \beta \rangle^* + \varepsilon_i \quad \text{with} \quad x_i \sim \mathcal{N}(0, I_d), \quad i = 1, \ldots, n; \tag{41} \]

2. (Heteroscedastic model):
   \[ y_i = \langle x_i, \beta \rangle^* + c^{-1}(\langle x_i, \beta \rangle^*)^2 \varepsilon_i \quad \text{with} \quad x_i \sim \mathcal{N}(0, I_d), \quad i = 1, \ldots, n, \tag{42} \]

where the constant $c$ is chosen as $c = \sqrt{3}||\beta^*||^2$ such that $\mathbb{E}[c^{-1}(\langle x_i, \beta \rangle^*)^2] = 1$, and therefore the variance of the noise is the same as that of $\varepsilon_i$.

In addition, we consider the following four error distributions:

1. Normal distribution $\mathcal{N}(\mu, \sigma^2)$ with mean $\mu = 0$ and standard deviation $\sigma = 1.5$;

2. Skewed generalized $t$ distribution $\mathcal{SGt}(0, 5, 0.75, 2, 2.5)$ (Theodossiou, 1998) with mean $\mu = 0$, variance $\sigma^2 = q/(q - 2) = 5$, $q = 2.5$, skewness parameter $\lambda = 0.75$ and shape parameter $p = 2$;

3. Lognormal distribution $\mathcal{LN}(\mu, \sigma^2)$ with log location parameter $\mu = 0$ and log shape parameter $\sigma = 1.2$;

4. Pareto distribution $\mathcal{Par}(x_m, \alpha)$ with scale parameter $x_m = 2$ and shape parameter $\alpha = 2.2$.

Except for the normal distribution, all the other three are skewed and heavy-tailed. To meet our model assumption, we subtract the mean from the lognormal and Pareto distributions.

In both homoscedastic and heteroscedastic models, the sample size $n = 100$, the ambient dimension $d = 1000$ and the sparsity parameter $s = 6$. The true vector of regression coefficients is $\beta^* = (4, 3, 2, -2, -2, 2, 0, \ldots, 0)^\top$, where the first 6 elements are non-zero and the rest are all equal to 0. We apply the proposed TAC (Tightening After Contraction) algorithm to compute all the estimators with tuning parameters $\lambda$ and $\tau$ chosen via three-fold cross-validation. To be more specific, we first choose a sequence of $\lambda$ values the same way as in the glmnet algorithm (Friedman, Hastie and Tibshirani, 2010). Next, guided by its theoretically “optimal” choice, the candidate set for $\tau$ is taken to be $\{2^{\gamma_{MAD}} \sqrt{n/\log(nd)} : j = -2, -1, 0, 1, 2\}$, where $\gamma_{MAD} := \text{median}||\hat{R} - \text{median}(\hat{R})||/\Phi^{-1}(3/4)$ is the median absolute deviation (MAD) estimator using the residuals $\hat{R} = (\hat{r}_1, \ldots, \hat{r}_p)^\top$ obtained from the Lasso.

To highlight the tail robustness and oracle property of our algorithm, we consider the following four measurements to assess the empirical performance:
1. True positive, TP, which is the number of signal variables that are selected;

2. False positive, FP, which is the number of noise variables that are selected;

3. Relative error, \( \text{RE}_1 \) and \( \text{RE}_2 \), which is the relative error of an estimator \( \hat{\beta} \) with respect to the Lasso under \( \ell_1 \)- and \( \ell_2 \)-norms:

\[
\text{RE}_1 = \frac{||\hat{\beta} - \beta^*||_1}{||\beta_{\text{Lasso}} - \beta^*||_1} \quad \text{and} \quad \text{RE}_2 = \frac{||\hat{\beta} - \beta^*||_2}{||\beta_{\text{Lasso}} - \beta^*||_2}.
\]

| Error dist. | Lasso | SCAD | Huber-SCAD | MCP | Huber-MCP |
|-------------|-------|------|------------|-----|-----------|
| Normal      |       |      |            |     |           |
| TP          | 6.00  | 6.00 | 6.00       | 6.00| 6.00      |
| FP          | 24.44 | 3.11 | 2.19       | 0.84| 0.53      |
| RE\(_1\)    | 1.00  | 0.23 | 0.22       | 0.19| 0.19      |
| RE\(_2\)    | 1.00  | 0.32 | 0.30       | 0.30| 0.30      |
| Skewed \(t\) |       |      |            |     |           |
| TP          | 4.74  | 4.87 | 4.74       | 3.97| 3.97      |
| FP          | 20.78 | 18.49| 11.48      | 4.28| 2.76      |
| RE\(_1\)    | 1.00  | 0.88 | 0.73       | 0.73| 0.65      |
| RE\(_2\)    | 1.00  | 0.91 | 0.86       | 0.94| 0.88      |
| Lognormal   |       |      |            |     |           |
| TP          | 5.68  | 5.71 | 6.00       | 5.49| 5.97      |
| FP          | 29.70 | 16.75| 3.80       | 4.32| 0.91      |
| RE\(_1\)    | 1.00  | 0.54 | 0.15       | 0.42| 0.13      |
| RE\(_2\)    | 1.00  | 0.62 | 0.23       | 0.60| 0.22      |
| Pareto      |       |      |            |     |           |
| TP          | 5.64  | 5.67 | 6.00       | 5.44| 5.98      |
| FP          | 28.30 | 14.69| 2.91       | 3.48| 0.71      |
| RE\(_1\)    | 1.00  | 0.51 | 0.14       | 0.40| 0.13      |
| RE\(_2\)    | 1.00  | 0.58 | 0.21       | 0.57| 0.22      |

Table 1: Simulation results for the Lasso, SCAD, Huber-SCAD, MCP and Huber-MCP estimators under the homoscedastic model (41).

Tables 1 and 2 summarize the averages of each measurement, TP, FP, RE\(_1\) and RE\(_2\), over 200 replications under models (41) and (42). Here, Huber-SCAD and Huber-MCP signify the proposed two-stage algorithm using the SCAD and MCP penalties, respectively. When the noise distributions are heavy-tailed and/or skewed, we see that Huber-SCAD and Huber-MCP outperform SCAD and MCP, respectively, with fewer spurious discoveries (false positives) and smaller estimation errors. Under the homoscedastic normal model, Huber-SCAD and Huber-MCP perform similarly to their least squares counterparts; while under heteroscedasticity, the proposed algorithm exhibits a notable advantage over existing methods on selection consistency even though the error is normally distributed. In summary, these numerical studies validate our expectations that the proposed robust regression algorithm improves the Lasso as a general regression analysis method on two aspects: robustness against heavy-tailed (and even heteroscedastic) noise and selection consistency.
To further visualize the advantage of our robust regression methods over the existing ones (e.g. Lasso, SCAD and MCP), we draw the receiver operating characteristic (ROC) curve, which is the plot of true positive rate (TPR) against false positive rate (FPR) at various regularization parameters. Specifically, TPR and FPR are defined, respectively, as the ratio of true positive to \( s \) and the ratio of false positive to \( d - s \). We generate data vectors \( \{(y_i, x_i)\}_{i=1}^{n} \) from both homoscedastic and heteroscedastic models (41) and (42) with sample size \( n = 100 \), dimension \( d = 1000 \) and sparsity \( s = 10 \). The true vector of regression coefficients is \( \beta^* = (1.5, 1.5, \ldots, 1.5, 0, \ldots, 0) \top \), where the first 10 elements are non-zero with weaker signals than the previous experiment, and the rest are all equal to 0. We apply the proposed TAC algorithm to implement all the five methods, Lasso, SCAD, MCP, Huber-SCAD and Huber-MCP, with a sequence of \( \lambda \) values chosen as before and \( \tau \) as \( \hat{\sigma}_{MAD} \sqrt{n / \log(nd)} \). For each combination of \( \lambda \) and \( \tau \), the empirical FPR and TPR are computed based on 200 simulations.

Figures 2 and 3 indicate evident advantage of Huber-SCAD and Huber-MCP over their least squares counterparts: the robust methods have a greater area under the curve (AUC) when the noise distribution is heavy-tailed and/or skewed in both homoscedastic and heteroscedastic models. Surprisingly, even in a normal model, the proposed methods still outperform the competitors by a visible margin.

Table 2: Simulation results for the Lasso, SCAD, Huber-SCAD, MCP and Huber-MCP estimators under the heteroscedastic model (42).

| Error dist. | Lasso | SCAD | Huber-SCAD | MCP | Huber-MCP |
|-------------|-------|------|------------|-----|-----------|
| Normal      | TP 6.00 | 6.00 | 5.96       | 6.00 | 5.98      |
|             | FP 22.71 | 3.29 | 0.31       | 0.88 | 0.13      |
|             | RE 1.00 | 0.28 | 0.21       | 0.24 | 0.16      |
|             | RE 1.00 | 0.38 | 0.31       | 0.36 | 0.25      |
| Skewed t    | TP 4.93 | 5.04 | 5.83       | 4.58 | 5.52      |
|             | FP 22.99 | 18.21 | 2.71      | 4.99 | 0.92      |
|             | RE 1.00 | 0.83 | 0.26       | 0.69 | 0.27      |
|             | RE 1.00 | 0.87 | 0.34       | 0.87 | 0.35      |
| Lognormal   | TP 5.74 | 5.77 | 6.00       | 5.65 | 6.00      |
|             | FP 26.61 | 11.28 | 1.23     | 2.62 | 0.30      |
|             | RE 1.00 | 0.45 | 0.14       | 0.35 | 0.12      |
|             | RE 1.00 | 0.53 | 0.21       | 0.50 | 0.19      |
| Pareto      | TP 5.67 | 5.67 | 5.97       | 5.59 | 5.95      |
|             | FP 25.56 | 10.13 | 0.61     | 2.80 | 0.23      |
|             | RE 1.00 | 0.46 | 0.14       | 0.39 | 0.15      |
|             | RE 1.00 | 0.55 | 0.22       | 0.54 | 0.23      |
Figure 2: Plots of ROC curves of the five methods under homoscedastic model (41) with errors generated from four distributions: normal, Student’s t, lognormal and Pareto.
6 Discussion

In this paper, we have presented a general computational framework for fitting high-dimensional linear models with heavy-tailed noise. In such a case, a real side-effect of the quadratic loss has been recognized: the combination of its rapid growth with heavy-tailed sampling distributions inevitably leads to outliers (Fan, Li and Wang, 2017; Mendelson, 2018). To achieve robustness against such outliers, we discussed a general class of loss functions, typified by the Huber loss, which are globally Lipschitz continuous and locally strongly convex. The use of nonconvex regularizers eliminates the
bias introduced by convex penalty, such as the popular $\ell_1$-penalty, and therefore produce oracle estimators. The proposed tightening after contraction algorithm, which bypasses directly solving the nonconvex optimization program, iteratively solves a sequence of adaptive convex programs with controlled algorithmic complexity. Statistically, our algorithm produces estimators that enjoy optimal rates of convergence and oracle properties. Moreover, our theoretical analysis provides key insights on calibrating the robustification parameters properly, which leads to the removal of outliers without sacrificing statistical efficiency.

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Let \( \beta = (\beta_0, \beta_1, \ldots, \beta_d)^\top \in \mathbb{R}^{d+1} \) be a \( d \)-vector of regularization parameters with \( \lambda_j \geq 0 \). Consider the optimization problem

\[
\min_{\beta = (\beta_0, \beta_1, \ldots, \beta_d)^\top \in \mathbb{R}^{d+1}} \{ \mathcal{L}_r(\beta) + ||\lambda \circ \beta_-||_1 \},
\]

(43)

where \( \mathcal{L}_r(\beta) = (1/n) \sum_{i=1}^n \ell_r(y_i - \langle x_i, \beta \rangle) \) and \( \lambda \circ \beta_- = (\lambda_1 \beta_1, \ldots, \lambda_d \beta_d)^\top \).

The following result provides conditions under which an \( \epsilon \)-optimal solution to the convex program (43) falls in an \( \ell_1 \)-cone. Recall that \( \mathcal{S} = \text{supp}(\beta^*_-) \) and \( \mathcal{S}^c = [d] \setminus \mathcal{S} \). Moreover, define

\[
w(\beta) = \nabla \mathcal{L}_r(\beta) - \mathbb{E} \nabla \mathcal{L}_r(\beta) \quad \text{and} \quad b(\beta) = ||\Sigma^{-1/2} \nabla \mathcal{L}_r(\beta)||_2,
\]

which are, respectively, the centered score function and the approximation bias. The covariance matrix \( \Sigma \) is positive definite with \( \lambda_{\min}(\Sigma) \geq \lambda_i > 0 \).

**Lemma 2.** Let \( \mathcal{E} \) be a subset of \( [d] \) satisfying \( \mathcal{S} \subseteq \mathcal{E} \). For any \( \beta = (\beta_0, \beta_1^\top)^\top \in \mathbb{R}^{d+1} \) satisfying \( \beta_{\mathcal{E}^c} = 0 \) and \( \epsilon > 0 \), provided \( \lambda = (\lambda_1, \ldots, \lambda_d)^\top \) satisfies \( ||\lambda_{\mathcal{E}^c}||_{\min} > ||w(\beta)||_\infty + \epsilon \), any \( \epsilon \)-optimal solution \( \tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_1^\top)^\top \) to (43) satisfies

\[
||\tilde{\beta} - \beta||_1 \leq \frac{||\lambda||_{\infty} + ||w(\beta)||_\infty + \epsilon||||\tilde{\beta} - \beta||_\infty|| ||\tilde{\beta} - \beta||_\Sigma}{||\lambda_{\mathcal{E}^c}||_{\min} - ||w(\beta)||_\infty - \epsilon}.
\]

**Proof of Lemma 2.** For any \( \xi \in \partial||\tilde{\beta} - \beta||_1 \), define \( u = u_\xi = \nabla \mathcal{L}_r(\tilde{\beta}) + (0, \lambda \circ \xi)^\top \in \mathbb{R}^{d+1} \). Note that

\[
||u||_\infty ||\tilde{\beta} - \beta||_1 \geq \langle u, \tilde{\beta} - \beta \rangle = \langle \nabla \mathcal{L}_r(\tilde{\beta}) - \nabla \mathcal{L}_r(\beta), \tilde{\beta} - \beta \rangle + \langle \nabla \mathcal{L}_r(\beta) - \mathbb{E} \nabla \mathcal{L}_r(\beta), \tilde{\beta} - \beta \rangle \\
\geq 0 + \langle \mathbb{E} \nabla \mathcal{L}_r(\beta), \tilde{\beta} - \beta \rangle + \langle \lambda \circ \xi, \tilde{\beta}_- - \beta_- \rangle \geq -||w(\beta)||_\infty ||\tilde{\beta} - \beta||_1 - b(\beta)||\tilde{\beta} - \beta||_\Sigma + \langle \lambda \circ \xi, \tilde{\beta}_- - \beta_- \rangle.
\]
Moreover, we have
\[
\langle \lambda \circ \xi, \bar{\beta} - \beta \rangle = \langle (\lambda \circ \xi)_{\bar{E}}, (\bar{\beta} - \beta)_{\bar{E}} \rangle + \langle (\lambda \circ \xi)_{E^c}, (\bar{\beta} - \beta)_{E^c} \rangle \\
\geq \|\lambda_{E^c}\|_{\min} \|\bar{\beta} - \beta\|_{E^c} - \|\lambda_{E^c}\| \|\bar{\beta} - \beta\|_{E^c}.
\]
Together, the last two displays imply
\[
\|u\|_{\infty} \|\bar{\beta} - \beta\|_1 \\
\geq -\|u(\beta)\|_{\infty} \|\bar{\beta} - \beta\|_1 - b(\beta)\|\bar{\beta} - \beta\|_{\Sigma} + \|\lambda_{E^c}\|_{\min} \|\bar{\beta} - \beta\|_{E^c} - \|\lambda_{E^c}\| \|\bar{\beta} - \beta\|_{E^c}.
\]
Since the right-hand side of this inequality does not depend on \(\xi\), taking the infimum with respect to \(\xi \in \partial \|\bar{\beta} - \beta\|_1\) on both sides to reach
\[
\omega_{\lambda}(\bar{\beta}) \|\bar{\beta} - \beta\|_1 \\
\geq -\|u(\beta)\|_{\infty} \|\bar{\beta} - \beta\|_1 - b(\beta)\|\bar{\beta} - \beta\|_{\Sigma} + \|\lambda_{E^c}\|_{\min} \|\bar{\beta} - \beta\|_{E^c} - \|\lambda_{E^c}\| \|\bar{\beta} - \beta\|_{E^c}.
\]
By definition, \(\bar{\beta}\) is an \(\epsilon\)-optimal solution so that \(\omega_{\lambda}(\bar{\beta}) \leq \epsilon\). Putting together the pieces, we obtain
\[
\{\epsilon + \|u(\beta)\|_{\infty} \|\bar{\beta} - \beta\|_1 + b(\beta)\|\bar{\beta} - \beta\|_{\Sigma} \geq \|\lambda_{E^c}\|_{\min} \|\bar{\beta} - \beta\|_{E^c} - \|\lambda_{E^c}\| \|\bar{\beta} - \beta\|_{E^c}.
\]
Decompose \(\|\bar{\beta} - \beta\|_1\) as \(\|\langle \bar{\beta} - \beta \rangle_{\bar{E}}\|_1 + \|\langle \bar{\beta} - \beta \rangle_{E^c}\|_1 + \|\bar{\beta} - \beta_0\|_1\), the stated result follows immediately. \(\square\)

**Lemma 3.** Let \(E \subseteq [d]\) be such that \(S \subseteq E\) and \(\bar{k} = k + 1 = |E| + 1\). For \(\beta = (\beta_0, \beta_{E^c})^T \in \mathbb{R}^d\) satisfying \(\beta_{S^c} = 0\), assume that \(\lambda = (\lambda_1, \ldots, \lambda_d)^T\) satisfies \(\|\lambda\|_{\infty} \leq \lambda\) and \(\|\lambda_{E^c}\|_{\min} \geq \alpha \lambda \geq 2\|u(\beta)\|_{\infty} + \epsilon\) for some \(\alpha \in (0, 1]\). Then any \(\epsilon\)-optimal solution \(\bar{\beta}\) to (43) satisfies \(\bar{\beta} \in \beta + \mathcal{C}(L)\) where \(L = (2 + \frac{3}{2})(\bar{k}/d)^{1/2} + \frac{2}{\alpha} b(\beta)\). Furthermore, provided that \(\kappa_r = \kappa_r(r, L, \beta)\) in Definition 2 satisfies
\[
\kappa_r > \lambda^{-1/2}(s^{1/2} + \bar{k}^{1/2}a/2) \lambda + b(\beta),
\]
we have
\[
\|\bar{\beta} - \beta\|_{\Sigma} \leq \kappa^{-1}_r \lambda^{-1/2} (\|\lambda_S\|_2 + \|u(\beta)\|_{\Sigma} + \bar{k}^{1/2} \lambda) + \kappa^{-1} b(\beta) \\
\leq \kappa^{-1}_r \lambda^{-1/2} (s^{1/2} + \alpha \bar{k}^{1/2} / 2) \lambda + \kappa^{-1} b(\beta).
\]

**Proof of Lemma 3.** For some \(r > 0\) to be specified, let \(\tilde{\beta}_\eta = \eta \bar{\beta} + (1 - \eta)\beta\) \((0 < \eta \leq 1)\) be an intermediate quantity such that (i) \(\tilde{\beta}_\eta \in \beta + \mathcal{B}_\Sigma(r)\), (ii) \(\tilde{\beta}_\eta\) lies on the boundary of \(\beta + \mathcal{B}_\Sigma(r)\) with \(0 < \eta < 1\) if \(\tilde{\beta} \notin \beta + \mathcal{B}_\Sigma(r)\), and (iii) \(\tilde{\beta}_\eta = \bar{\beta}\) with \(\eta = 1\) if \(\bar{\beta} \in \beta + \mathcal{B}_\Sigma(r)\).

Since the Huber loss is convex, from Lemma F.2 in Fan et al. (2018) we obtain that
\[
(\nabla L_r(\tilde{\beta}_\eta) - \nabla L_r(\beta) - \tilde{\beta}_\eta - \beta) \leq \eta(\nabla L_r(\bar{\beta}) - \nabla L_r(\beta), \bar{\beta} - \beta).
\]

First we bound the left-hand side of (46) from below. Under the assumed scaling, Lemma 2 indicates
\[
\|\langle \bar{\beta} - \beta \rangle_{\bar{E}}\|_1 \leq (1 + 2/a) \|\langle \bar{\beta} - \beta \rangle_{\bar{E}}\|_1 + \|\tilde{\beta}_0 - \beta_0\| + (2/a) \lambda^{-1} b(\beta) \|\bar{\beta} - \beta\|_{\Sigma}.
\]
Recall that together, (48)–(51) yield
\[
\langle \lambda, \beta \rangle \leq \lambda \beta_0 + \frac{1}{\alpha} b(\beta) \| \beta - \beta \| \Sigma.
\] In conjunction with Definition 2 and bound \(|u|_\Sigma \geq \lambda_1^{1/2} \| u \|_2\), this implies that \(\tilde{\beta} \in \beta + C(L)\) with \(L = (2 + \frac{\alpha}{\alpha}) (k/\lambda_0)^{1/2} + \frac{1}{\alpha} b(\beta)\). Since \(\tilde{\beta}_\eta - \beta = \eta(\tilde{\beta} - \beta)\), we have \(\tilde{\beta}_\eta \in \beta + \mathbb{B}_\Sigma(r) \cap C(L)\) and
\[
\langle \nabla L_r(\tilde{\beta}_\eta) - \nabla L_r(\beta), \tilde{\beta}_\eta - \beta \rangle \geq \kappa_r(\lambda L, \beta) \| \tilde{\beta}_\eta - \beta \| \Sigma^2.
\] (47)

Next we upper bound the right-hand side of (46). For any \(\xi \in \partial \| \beta \|_1\), write
\[
\langle \nabla L_r(\beta), \beta \rangle - \langle \nabla L_r(\beta), \beta \rangle - \langle \nabla L_r(\beta), \beta \rangle
:= \Pi_1 - \Pi_2 - \Pi_3,
\] (48)
where \(u = \nabla L_r(\beta) + (0, (\lambda_0 \circ \xi)^\top)^\top\). For \(\Pi_3 = \langle \nabla L_r(\beta), \beta \rangle - \beta \), by the decomposition \(\nabla L_r(\beta) = w(\beta) + \nabla E_r(\beta)\) we have
\[
\| \Pi_3 \| \leq \| w(\beta) \|_2 \| \beta - \beta \|_2 + \| u \|_\infty \| \beta - \beta \|_\Sigma + b(\beta) \| \beta - \beta \| \Sigma.
\] (49)
Tuning to \(\Pi_2\), decompose \(\lambda \circ \xi \) and \(\beta \) according to \(S \cup (E \setminus S) \cup \Sigma^e\) to reach
\[
\Pi_2 = \langle \lambda \circ \xi \circ \beta, (\beta - \beta) \rangle + \langle \lambda \circ \xi \circ (\beta - \beta), (\beta - \beta) \rangle + \langle \lambda \circ \xi \circ \beta, (\beta - \beta) \rangle.
\] Since \(\beta \|_e = 0\) and \(\xi \in \partial \| \beta \|_1\), we have \(\langle \lambda \circ \xi \circ \beta, (\beta - \beta) \rangle = \langle \lambda \circ \xi \circ \beta, (\beta - \beta) \rangle \) and \(\beta \|_e \leq b(\beta) \| \beta - \beta \| \Sigma\). Also, \(\langle \lambda \circ \xi \circ \beta, (\beta - \beta), (\beta - \beta) \rangle = \langle \lambda \circ \xi \circ \beta, (\beta - \beta) \rangle \) and \(\beta \|_e \leq b(\beta) \| \beta - \beta \| \Sigma\). Therefore,
\[
\Pi_2 \geq -\| \lambda \|_2 \| \beta - \beta \|_2 + \| \lambda \|_\infty \| \beta - \beta \|_1\|\beta - \beta \|_\Sigma.
\] (50)
Similarly, \(\Pi_1\) satisfies the bound
\[
\| \Pi_1 \| \leq \| u \|_2 \| \beta - \beta \|_2 + \| u \|_\infty \| \beta - \beta \|_1\|\beta - \beta \|_\Sigma.
\] (51)
Together, (48)–(51) yield
\[
\langle \nabla L_r(\beta) - \nabla L_r(\beta), \beta \rangle - \beta \rangle
\leq \| \lambda \|_e \| \beta - \beta \|_2 + \| u \|_\infty \| \beta - \beta \|_1\|\beta - \beta \|_\Sigma + b(\beta) \| \beta - \beta \| \Sigma.
\] (52)
Taking the infimum over \(\xi \in \partial \| \beta \|_1\) on both sides, it follows that
\[
\langle \nabla L_r(\beta) - \nabla L_r(\beta), \beta \rangle - \beta \rangle
\leq -\| \lambda \|_e \| \beta - \beta \|_2 + \| u \|_\infty \| \beta - \beta \|_1\|\beta - \beta \|_\Sigma + b(\beta) \| \beta - \beta \| \Sigma.
\] (53)
Recall that \(\| \lambda \|_e \| \beta - \beta \|_2 \geq 2 \| \lambda \|_e \| \beta - \beta \|_2 \), it follows from (46), (47), (52) and the bound \(|u|_2 \leq \lambda_1^{-1/2} \| u \|_\Sigma\) that
\[
\kappa_r(\lambda L, \beta) \| \beta - \beta \|_\Sigma \leq \| \lambda \|_2 + \| u \|_2 + \| \beta \|_2 \| \beta - \beta \|_2 + b(\beta) \| \beta - \beta \|_\Sigma.
\] (54)
Under the assumed scaling, we have \(\| u \|_2 + \| \beta \|_2 \leq \frac{1}{\alpha} \| a \|_2 \) and \(\| \lambda \|_2 \leq s/\lambda^2 \). Plugging these bounds into (53) leads to
\[
\| \beta - \beta \|_\Sigma \leq \frac{s^{1/2}}{\kappa_r(\lambda L, \beta)} \lambda_1^{-1/2} \frac{a}{\alpha} \| \beta \|_2 + \| \beta \|_2 \frac{\lambda_1^{-1/2}}{\kappa_r(\lambda L, \beta)} < \epsilon.
\]
Hence, \(\tilde{\beta}_\eta \) falls in the interior of \(\beta + \mathbb{B}_\Sigma(r)\), which enforces \(\eta = 1 \) and \(\tilde{\beta}_\eta = \tilde{\beta}\). Consequently, (44) and (45) follow, respectively, from (53) and the last display. □
B Proofs of Propositions

B.1 Proof of Proposition 2

With the initial estimate $\tilde{\beta}^{(0)} = 0_{d+1}$, we have $\lambda^{(0)} = p'_j(0_d) = (\lambda, \ldots, \lambda)^\top \in \mathbb{R}^d$. Then (15) follows immediately from Lemma 3 with $\beta = \beta^*$, $\mathcal{E} = \mathcal{S}$ and $a = 1$. 

B.2 Proof of Proposition 3

In order to improve the statistical rate at step $\ell \geq 1$, we need to control the magnitude of the spurious discoveries from the last step, that is, $\max_{j \in \mathcal{S}} |\tilde{\beta}_j^{(\ell-1)}|$. Recall that $\lambda^{(\ell-1)} = (\lambda_1^{(\ell-1)}, \ldots, \lambda_d^{(\ell-1)})^\top = (p'_j(\tilde{\beta}_j^{(\ell-1)}), \ldots, p'_j(\tilde{\beta}_d^{(\ell-1)}))^\top$ and $p_i(t) = \lambda^2 p(t/\lambda)$ for $t \in \mathbb{R}$. Intuitively, the larger $|\tilde{\beta}_j^{(\ell-1)}|$ is, the smaller $\lambda_j^{(\ell-1)}$ is. Motivated by this observation, we construct an augmented set $\mathcal{E}_\ell$ of $\mathcal{S}$ in each step and control the magnitude of $|\lambda_{\mathcal{E}_\ell^{(\ell-1)}}|_{\min}$.

With $\tilde{\beta}^{(0)} = 0$, we have $\lambda^{(0)} = (\lambda, \ldots, \lambda)^\top \in \mathbb{R}^d$. Under the scaling $\lambda \geq \frac{2}{p'(\gamma)}(||w^*||_{\infty} + \epsilon_1) \geq 2(||w^*||_{\infty} + \epsilon_1)$, it follows from Lemma 3 with $\mathcal{E} = \mathcal{S}$ and $L_0 = 4L_1^{-1/2} \sqrt{3}/2 + 2L_1^{-1}b^*$ that

$$ ||\tilde{\beta}^{(1)} - \beta^*||_{\Sigma} \leq \frac{||\lambda_{\mathcal{S}}^{(0)}||_2 + ||w^*||_2 + \sqrt{3}/2 \epsilon_1}{\lambda_1^{1/2} \kappa_\tau(r, L_0, \beta^*)} + \frac{b^*}{\kappa_\tau(r, L_0, \beta^*)} \leq \frac{s^{1/2} + p'(\gamma)\sqrt{3}/2}{\kappa_\tau(r, L_0, \beta^*)} \lambda_1^{-1/2} \lambda + \frac{b^*}{\kappa_\tau(r, L_0, \beta^*)} \leq a_\gamma \frac{s^{1/2} \lambda + b^*}{\kappa_\tau(r, L_0, \beta^*)} $$ (54)

where $a_\gamma = 1 + \frac{\sqrt{3}}{2} p'(\gamma) \in (1, 1 + \sqrt{3}/2)$. For $\ell \geq 1$, define the augmented set

$$ \mathcal{E}_\ell = \mathcal{S} \cup \{1 \leq j \leq d : \lambda_j^{(\ell-1)} < p'(\gamma)\lambda\} $$ (55)

which depends on the solution $\tilde{\beta}^{(\ell-1)}$ from the previous step. We claim that the above constructed set satisfies

$$ |\mathcal{E}_\ell| \leq 3s - 1 \quad \text{and} \quad ||\lambda_{\mathcal{E}_\ell^{(\ell-1)}}||_{\min} \geq p'(\gamma)\lambda. $$ (56)

Set $\tilde{\mathcal{E}}_\ell = \{0\} \cup \mathcal{E}_\ell$ with $|\tilde{\mathcal{E}}_\ell| \leq 3s$. Under the scaling $\lambda \geq \frac{2}{p'(\gamma)}(||w^*||_{\infty} + \epsilon_1)$, it follows from Lemma 3 with $a = p'(\gamma)$, $\tilde{k} = 3s$ and $L = (2 + \frac{2}{p'(\gamma)})L_1^{-1/2}(3s)^{1/2} + \frac{2}{p'(\gamma)}b^*$ that

$$ ||\tilde{\beta}^{(\ell)} - \beta^*||_{\Sigma} \leq \frac{||\lambda_{\mathcal{S}}^{(\ell)}||_2 + ||w^*||_2 + ||\tilde{\mathcal{E}}_\ell||_2}{\lambda_1^{1/2} \kappa_\tau(r, L, \beta^*)} + \frac{b^*}{\kappa_\tau(r, L, \beta^*)} \leq \frac{s^{1/2} + (3s)^{1/2} p'(\gamma)/2}{\kappa_\tau(r, L, \beta^*)} \lambda_1^{-1/2} \lambda + \frac{b^*}{\kappa_\tau(r, L, \beta^*)} \leq a_\gamma \frac{s^{1/2} \lambda + b^*}{\kappa_\tau(r, L, \beta^*)} < r. $$ (57)

where the last step follows from the condition on $r$ in (16).
We prove the earlier claim (56) by the method of induction. For \( \ell = 1 \), we have \( \lambda^{(0)} = (\lambda, \ldots, \lambda)^T \in \mathbb{R}^d \). Thus, (56) holds with \( \mathcal{E}_1 = S \). Next, assume (56) holds for some \( \ell \geq 1 \), from which (58) follows. To bound the cardinality of \( \mathcal{E}_{\ell+1} \), note that for any \( j \in \mathcal{E}_{\ell+1} \setminus S \), \( p'_j(\tilde{\beta}_j^{(\ell)}) = \lambda_j^{(\ell)} < p'(\gamma)\lambda = p'_j(\gamma \lambda) \). This, together with the monotonicity of \( p'_j \) on \( \mathbb{R}^s \), implies \( |\tilde{\beta}_j^{(\ell)}| > \gamma \lambda \). Recalling that \( \beta'_j = 0 \) for \( j \in \mathcal{E}_{\ell+1} \setminus S \), we obtain

\[
|\mathcal{E}_{\ell+1} \setminus S|^{1/2} < \frac{1}{\gamma \lambda} \|\tilde{\beta}^{(\ell)}_{\mathcal{E}_{\ell+1} \setminus S}\|_2 = \frac{1}{\gamma \lambda} \|\tilde{\beta}^{(\ell)} - \beta^*\|_{\mathcal{E}_{\ell+1} \setminus S}\|_2
\]

\[
\leq \begin{aligned}
&\frac{1}{\gamma \cdot \kappa_s(r, L, \beta^*)} (a \gamma \lambda^2 s^{1/2} + \lambda_j^{-1/2} \lambda^{-1} b^*) \\
&\leq \sqrt{2} s^{1/2},
\end{aligned}
\]

where step (i) applies the bound (58), and step (ii) follows from the condition on \( \kappa_s(r, L, \beta^*) \) in (16). Hence, \( |\mathcal{E}_{\ell+1}| \leq |S| + |\mathcal{E}_{\ell+1} \setminus S| < 3s \), implying \( |\mathcal{E}_{\ell+1}| \leq 3s - 1 \). By (55) and the property \( p'_j(t) = \lambda p'(t/\lambda) \), we are guaranteed that

\[
\lambda_j^{(\ell)} \geq p'(\gamma)\lambda \geq 2(\|w^*\|_\infty + \epsilon_{\ell+1}) \quad \text{for} \quad j \in \mathcal{E}_{\ell+1}^c.
\]

The two hypotheses in (56) then hold for \( \ell + 1 \), which completes the induction step. Consequently, the bounds (57) and (58) hold for any \( \ell \geq 1 \).

We have shown that under proper conditions, all the estimates \( \tilde{\beta}^{(\ell)} \) fall in a local neighborhood of \( \beta^* \), i.e. \( \|\tilde{\beta}^{(\ell)} - \beta^*\|_\Sigma \leq s^{1/2} \lambda + b^* \). To further refine this bound, particularly the statistical error term \( s^{1/2} \lambda \), in view of (57), we need to establish sharper bounds on

\[
\|\lambda_S^{(\ell-1)}\|_2 = \sqrt{\sum_{j \in S} \lambda_j^{(\ell-1)^2}} \quad \text{and} \quad \|w^*_{E_\ell}\|_2 + |\mathcal{E}_{\ell}|^{1/2} \epsilon_{\ell}.
\]

For each \( j \in [d] \), \( \lambda_j^{(\ell-1)} = \max \{ \lambda_j^{(\ell-1)} \} \). If \( |\tilde{\beta}_j^{(\ell-1)} - \beta_j^*| \geq \gamma \lambda \), then \( \lambda_j^{(\ell-1)} \leq \lambda \leq \gamma^{-1} |\tilde{\beta}_j^{(\ell-1)} - \beta_j^*| \); otherwise if \( |\tilde{\beta}_j^{(\ell-1)} - \beta_j^*| \leq \gamma \lambda \), \( \lambda_j^{(\ell-1)} \leq p'_j(\beta_j^*) - \gamma \lambda \) due to monotonicity of \( p'_j \). Putting together the pieces, we conclude that

\[
\|\lambda_S^{(\ell-1)}\|_2 \leq \|p'_j(\beta_j^*) - \gamma \lambda\|_2 + \gamma^{-1} \|\tilde{\beta}^{(\ell-1)} - \beta^*\|_{\mathcal{E}_{\ell} \setminus S}\|_2.
\]

For the remaining terms that involve \( \mathcal{E}_{\ell} \), by the triangle inequality and (59) we obtain that

\[
\|w^*_{E_\ell}\|_2 + |\mathcal{E}_{\ell}|^{1/2} \epsilon_{\ell}
\]

\[
\leq \|w^*_{\mathcal{E}_{\ell}}\|_2 + \|w^*_{\mathcal{E}_{\ell} \setminus S}\|^{1/2} \|w^*\|_\infty + |\mathcal{E}_{\ell} \setminus S|^{1/2} \epsilon_{\ell}
\]

\[
\leq \|w^*_{\mathcal{E}_{\ell}}\|_2 + \|w^*\|_\infty + \epsilon_{\ell} \|\tilde{\beta}^{(\ell-1)} - \beta^*\|_{\mathcal{E}_{\ell} \setminus S}\|_2
\]

\[
\leq \|w^*_{\mathcal{E}_{\ell}}\|_2 + \|w^*\|_\infty + p'(\gamma) \|\tilde{\beta}^{(\ell-1)} - \beta^*\|_{\mathcal{E}_{\ell} \setminus S}\|_2.
\]

Plugging the above refined bounds into (57) yields

\[
\|\tilde{\beta}^{(\ell)} - \beta^*\|_\Sigma \leq \frac{\|p'_j(\beta_j^*) - \gamma \lambda\|_2 + \|w^*_{\mathcal{E}_{\ell}}\|_2 + \gamma^{1/2} \epsilon_{\ell} + \gamma \lambda^2 b^*}{\lambda_j^{1/2} \kappa_s(r, L, \beta^*)} + \frac{\|\tilde{\beta}^{(\ell-1)} - \beta^*\|_{\mathcal{E}_{\ell}}\|_\Sigma}{\lambda_j}.
\]

30
For the last term on the right-hand side, the constraint (16) ensures that \( \gamma \kappa \tau(r, L, \beta^*) \geq (\lambda \rho \delta)^{-1}. \) The contraction inequality (17) then follows immediately. Finally, (18) is a direct consequence of (17) and (54).

\[ \Box \]

### B.3 Proof of Proposition 5

By construction, the oracle estimator \( \hat{\beta} \) is such that \( \hat{\beta}^{\text{ora}} = 0 \) and \( \nabla L_{\ell}(\hat{\beta}^{\text{ora}}) = 0. \) With \( w^{\text{ora}} = \nabla L_{\ell}(\hat{\beta}^{\text{ora}}) \), the proof strategy is similar to that in the proof of Proposition 3 with \( \epsilon = 0 \), because \( \hat{\beta}^{(l)} \) are optimal solutions of \( (P_l) \).

Under the constraints \( \lambda \geq \frac{2}{p'(\gamma_0)} \| w^{\text{ora}} \|_\infty \) and \( \| \hat{\beta}^{\text{ora}} - \beta^* \|_\Sigma \leq r/2 \), and following the proof of Proposition 3 with \( \kappa \) replaced by \( \kappa_\tau \), it can be shown that

\[
\| \hat{\beta}^{(l)} - \beta^{\text{ora}} \|_\Sigma \leq \frac{\| \lambda \|^{(l-1)}_{\Sigma} + \| w^{\text{ora}} \|_2}{\lambda^{1/2} \kappa_\tau(r, L, \beta^*)} \leq \frac{1 + p'(\gamma_0)\gamma}{\kappa_\tau(r, L, \beta^*)} \lambda^{1/2} s^{1/2} \lambda < r, \tag{60}
\]

where \( L = \{ 2 + \frac{2}{p'(\gamma_0)} \lambda \}^{-1/2} (2s)^{1/2}, \) and similarly to (55) and (56), \( E_{\ell} = S \cup \{ 1 \leq j \leq d : \lambda_j^{(l-1)} < p'(\gamma_0)\lambda \} \) is such that \( |E_{\ell} \setminus S| \leq s - 1 \) and \( |E_{\ell}| \leq 2s - 1 \). The constants are slightly different because we don’t have the approximation bias term here. Moreover, define a sequence of subsets

\[
S_\ell = \{ 1 \leq j \leq d : \| \hat{\beta}_{j}(\ell) - \beta^*_j \| \geq \gamma_0 \lambda \}, \ \ell = 0, 1, 2, \ldots.
\]

Starting with \( \hat{\beta}^{(0)} = 0 \), it holds under the minimum signal strength condition that \( S_0 = S \).

For \( \| \lambda \|^{(l-1)}_{S} \), note that if \( j \in S \cap S_{\ell-1}, \lambda_{j}^{(l-1)} = p'_j((\hat{\beta}_{j}^{(l-1)})) \leq p'_j(|\beta^*_j| - \gamma_0 \lambda) \) due to monotonicity; otherwise if \( j \in S \cap S_{\ell-1}, \lambda_{j}^{(l-1)} \leq \lambda \). Therefore,

\[
\| \lambda \|^{(l-1)}_{S} \leq \| p'_j(|\beta^*_j| - \gamma_0 \lambda) \|_2 + \lambda |S \cap S_{\ell-1}|^{1/2}.
\]

Since \( \| \beta^*_j \|_{\text{min}} \geq (\gamma_0 + \gamma_1) \lambda \) and \( p'_j(t) = 0 \) for all \( t \geq \gamma_1 \lambda \), \( \| p'_j(|\beta^*_j| - \gamma_0 \lambda) \|_2 \) vanishes. Turning to \( \| w^{\text{ora}} \|_2 \), recalling that \( w^{\text{ora}} = 0 \), we have

\[
\| w^{\text{ora}} \|_2 = \| w^{\text{ora}} \|_2 |S_{\ell} \setminus S|^1/2.
\]

For each \( j \in E_\ell \setminus S, \beta^*_j = 0 \) and \( \lambda_{j}^{(l-1)} = p'_j((\hat{\beta}_{j}^{(l-1)})) < p'(\gamma_0)\lambda = p'_j(\gamma_0 \lambda). \) Hence, \( |\hat{\beta}_{j}^{(l-1)} - \beta^*_j| > \gamma_0 \lambda \) so that \( j \in S_{\ell-1} \setminus S \). Therefore, \( E_\ell \setminus S \subseteq S_{\ell-1} \setminus S \). Combined with the earlier bound, we arrive at

\[
\| w^{\text{ora}} \|_2 \leq \| w^{\text{ora}} \|_2 |S_{\ell-1} \setminus S|^1/2.
\]

Since \( p'(\gamma_0) \leq 1/2 \), substituting the above estimates into (60) yields

\[
\| \hat{\beta}^{(l)} - \beta^{\text{ora}} \|_\Sigma \leq \frac{|S \cap S_{\ell-1}|^{1/2} + |S_{\ell-1} \setminus S|^1/2 / 4}{\lambda^{1/2} \kappa} \leq \frac{\sqrt{17}}{4} \frac{\lambda}{\lambda^{1/2} \kappa} |S_{\ell-1}|^{1/2}, \tag{61}
\]
Next we bound $|S_ℓ|$ ($ℓ ≥ 1$), the cardinality of $S_ℓ$. By (21), it holds for any $j ∈ S_ℓ$ that

$$|\hat{β}_j(ℓ) - \hat{β}_j| ≥ γ_0 λ - \|\hat{β}_{ora} - β^*\|_∞$$

$$≥ \frac{1.25λ}{δλ_k^3} - \frac{λ}{5δλ_k^3}$$

$$> \frac{\sqrt{T}}{4} \frac{λ}{δλ_k^3}.$$  

In conjunction with (61), this implies

$$|S_ℓ|^{1/2} ≤ \frac{λ\|\hat{β}(ℓ) - \hat{β}_{ora}\|_2}{\sqrt{T} 4 λ/(δκ)} ≤ \frac{\sqrt{T}}{4} (λ/κ)|S_{ℓ-1}|^{1/2} = δ|S_{ℓ-1}|^{1/2}, \; ℓ ≥ 1.$$  

(62)

Recall that $S_0 = S$, we have $|S_ℓ|^{1/2} < δ^{1/2}$ for any $ℓ ≥ 1$. As long as $ℓ ≥ T := \log(s^{1/2})/\log(1/δ)$, we are guaranteed that $|S_ℓ| < 1$, i.e. $S_ℓ = \emptyset$. Consequently, it follows from (61) that $\hat{β}(ℓ) = \hat{β}_{ora}$ for all $ℓ ≥ T + 1$. This completes the proof.

B.4 Proof of Proposition 8

Proof of Part (I). Write $S_j = (1/n) \sum_{i=1}^{n} (ξ_i x_i - \mathbb{E}ξ_i x_i)$ with $ξ_i = η_i(ε_i)$. Then $\|w(β^*)\|_∞$ can be written as $\max_{0≤j≤d} |S|$. Note that $\mathbb{E}(ξ_i^2 | x_i) ≤ σ^2 | i ≤ τ$. For each $0 ≤ j ≤ d$, we have $\mathbb{E}(ξ_i x_i)^2 ≤ σ^2 | x_i j|^2$ and

$$\mathbb{E}|ξ_i x_i|^k ≤ τ^{k-2} σ^2 | x_i|^k ≤ τ^{k-2} σ^2 | x_i 0|^k | x_i j|^2 ≤ \frac{k!}{2} σ^2 | x_i 0|^k $$(28), $k = 3, 4, \ldots.$

Bernstein’s inequality, in conjunction with the union bound, implies that for any $x ≥ 0$,

$$\max_{0≤j≤d} |S| ≤ u_0 σ | x \left(σ x \left(\frac{2x}{n} + \frac{2x}{n}\right)\right)$$

with probability at least $1 - 2δe^{-x}$. Taking $x = 2 \log(d)$ proves (27). Next we use a standard covering argument to prove (28). For any $ε ∈ (0, 1)$, there exists an $ε$-net $N_ε$ of the unit sphere in $\mathbb{R}^2$ with cardinality $|N_ε| ≤ (1 + 2/ε)^2$ such that

$$\|w(β)\|_2 ≤ \frac{1}{1 - ε} \max_{u ∈ N_ε} \frac{1}{n} \sum_{i=1}^{n} (ξ_i (x_i S, u) - \mathbb{E}ξ_i (x_i S, u)).$$  

(63)

For every $u ∈ N_ε$, Bernstein’s condition holds: $\mathbb{E}(ξ_i (x_i S, u))^2 ≤ σ^2 | u |_S^2$ and for $k = 3, 4, \ldots,$

$$\mathbb{E}|ξ_i (x_i S, u)|^k ≤ \frac{k!}{2} σ^2 | u |_S^2.$$  

Applying Bernstein’s inequality, yields for any $x > 0$,

$$\frac{1}{n} \sum_{i=1}^{n} (ξ_i (x_i S, u) - \mathbb{E}ξ_i (x_i S, u)) ≤ u_0 | u |_S \left(σ x \left(\frac{2x}{n} + \frac{2x}{n}\right)\right)$$

with probability at least $1 - e^{-x}$. Consequently, from the union bound (63), we have

$$\|w(β)\|_2 ≤ \frac{u_0 λ_{max}(S)}{1 - ε} \left(σ x \left(\frac{2x}{n} + \frac{2x}{n}\right)\right)$$

with probability at least $1 - e^{log(1+2/ε)δ - x}$. Taking $ε = 1/3$ and $x = 2(δ + t)$ proves (28).

Proof of Part (II). Using he bound (70) in the proof of Theorem 1, the proof of Part (II) is essentially identical to that of Part (I), and therefore is omitted.
C Proofs of Theorems

C.1 Proof of Theorem 1

Part (I) is a direct consequence of Theorem B.2 in Appendix B of Sun, Zhou and Fan (2019). In what follows, we provide a self-contained proof of (9). Without loss of generality, we assume $d \geq 4$ and write $\widehat{\beta} = \widehat{\beta}^{init,iso}$ for simplicity. For $t > 0$, define the $\| \cdot \|_\Sigma$-ball $\mathcal{B}_\Sigma(t) = \{ \delta \in \mathbb{R}^d : \| \delta \|_\Sigma \leq t \}$. For some $r > 0$ to be specified, if $\widehat{\beta} \not\in \beta^* + \mathcal{B}_\Sigma(r)$, there exists $\eta \in (0, 1)$ such that $\widehat{\beta} := \beta^* + \eta (\widehat{\beta} - \beta^*) \in \beta^* + \partial \mathcal{B}_\Sigma(r)$; otherwise if $\beta = \beta^* + \mathcal{B}_\Sigma(r)$, we simply take $\eta = 1$ so that $\widehat{\beta} = \beta^*$. Applying Lemma C.1 in Appendix C of Sun, Zhou and Fan (2019) to $L_r(\beta) = (1/n) \sum_{i=1}^n \ell_r(y_i - \langle x_i, \beta \rangle)$ gives

$$\langle \nabla L_r(\widehat{\beta}) - \nabla L_r(\beta^*), \widehat{\beta} - \beta^* \rangle \leq \eta \langle \nabla L_r(\widehat{\beta}) - \nabla L_r(\beta^*), \widehat{\beta} - \beta^* \rangle. \tag{64}$$

We deal with the left-hand and right-hand sides of (64) separately, starting with latter.

Write $\tilde{v} = (\tilde{v}_0, \tilde{v}_1)^T = \tilde{\beta} - \beta^*$. From Proposition 1 we see that $\beta^*_r = \beta^*_r$ and $\text{supp}(\beta^*_r) = \mathcal{S}$. By the convexity of Huber loss, $\beta^*$ satisfies the first-order condition that $\nabla L_r(\beta^*) + \lambda \tilde{v} = 0$, where $\tilde{v} = (0, \tilde{v}_1)^T$ for some $\tilde{v}_1 \in \partial \| \beta \|_1$. Then, under the constraint $\lambda \geq 2 \| \nabla L_r(\beta^*) \|_\infty$, we have

$$0 \leq \langle \nabla L_r(\beta^*) - \nabla L_r(\beta^*_r), \beta^* - \beta^*_r \rangle \leq \lambda \| \beta^* - \beta^*_r \|_1 + \lambda \| \beta^*_r \|_1$$

$$\leq \lambda \| \beta^* \|_1 + \| \tilde{v} \|_1 - \| \beta^*_r \|_1 + \frac{\lambda}{2} \| \tilde{v} \|_1$$

$$\leq \lambda \| \beta^* \|_1 + \| \tilde{v} \|_1 - \| \beta^*_r \|_1 + \frac{\lambda}{2} \| \tilde{v} \|_1$$

$$= \frac{\lambda}{2} \| \tilde{v} \|_1 - \| \beta^*_r \|_1 \leq \frac{3}{2} \| \beta^* \|_1 - \| \beta^*_r \|_1 \tag{65}$$

where $\tilde{s} = s + 1$ and $\mathcal{S} = [\mathcal{S}] \setminus \mathcal{S}$. This in turn implies that provided $\lambda \geq 2 \| \nabla L_r(\beta^*) \|_\infty$, $\beta^* - \beta^*_r$ falls in the $\ell_1$-cone $C := \{ \delta = (\delta_0, \delta_1)^T \in \mathbb{R}^d : \| \delta \|_1 \leq 3 \| \delta \|_1 + \| \delta_0 \| \}$.

Next we bound the left-hand side of (64) from below. Since $\beta^* - \beta^*_r = \eta (\beta^* - \beta^*_r)$, we also have $\beta^* - \beta^*_r \in \mathcal{C}$ as long as $\lambda \geq 2 \| \nabla L_r(\beta^*_r) \|_\infty$. The following proposition indicates that under certain constraints, the empirical Huber loss satisfies the restricted strong convexity condition over $\beta^*_r + \mathcal{B}_\Sigma(r) \cap \mathcal{C}$ with high probability.

**Lemma 4.** Let $r > 0$ be such that

$$c_1 = \tau / (16 \varrho_r \nu_1^2 r) \geq 1, \tag{66}$$

where $\varrho_r = \sqrt{\rho_r / E_r} \geq 1$. Then with probability at least $1 - d^{-1}$,

$$\frac{\langle \nabla L_r(\beta) - \nabla L_r(\beta^*_r), \beta - \beta^*_r \rangle}{\| \beta - \beta^*_r \|_\Sigma^2} \geq \rho_r (1 - c_2) - \delta(n, d) \tag{67}$$

holds uniformly over $\beta \in \beta^*_r + \mathcal{B}_\Sigma(r) \cap \mathcal{C}$, where $c_2 = (16c_1^2 + 8c_1 + 2)e^{\delta c_1}$ and $\delta(n, d) \geq 0$ satisfies

$$\delta(n, d) \leq 5 \sqrt{2} \lambda \nu_1^{1/2} \nu_1 \sigma S^{1/2} \tau^2 \left( \frac{\log(2d)}{n} + \frac{\log(2d)}{n} \right) + m_4^{1/2} \sqrt{\frac{2 \log(d)}{n}} + \frac{\tau^2 \log(d)}{3n^2} \tag{68}$$

with $m_4 = \sup_{u \in \mathcal{S}^d} \mathbb{E}(\Sigma^{-1/2} x, u)^4$.  

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In view of (65), (67) and (68), we take \( r = \tau/(16\rho_2 u_0^2) \) such that with probability at least \( 1 - d^{-1}, \)

\[
\frac{\rho_2}{2} \frac{1}{d} \beta_1 \| \beta - \beta_*^i \|_2 \leq \frac{\rho_2}{2} \| \beta - \beta_*^i \|_2 \leq \frac{3}{2} \beta_1 \| \beta - \beta_*^i \|_2
\]
as long as \( n \geq s \log(d) \). In conjunction with the cone property that \( \beta \in \beta_*^i + \mathbb{C} \), this implies

\[
\| \beta - \beta_*^i \|_2 \leq 3 \lambda_1 \| \beta - \beta_*^i \|_2 \lambda_1 \quad \text{and} \quad \| \beta - \beta_*^i \|_1 \leq 12 \lambda_1 \| \beta - \beta_*^i \|_2 \lambda_1
\]

(69) provided \( \lambda \geq 2 \| \nabla L_r(\beta_*^i) \|_\infty \) and (66) is met. Under the constraint \( \tau > 48 u_0^2 \lambda_1 \| \beta - \beta_*^i \|_2 \lambda_1 \), we are guaranteed that with probability at least \( 1 - d^{-1}, \beta \) falls in the interior of \( \beta_*^i + \mathbb{B}_\Sigma(r) \). Consequently, \( \hat{\beta} \) and \( \tilde{\beta} \) must coincide and the error bounds in (69) hold for \( \hat{\beta} \).

It remains to show that the constraint \( \lambda \geq 2 \| \nabla L_r(\beta_*^i) \|_\infty \) is fulfilled with high probability. Recalling \( \beta_*^i = \arg\min_\beta \mathbb{E} L_r(\beta) \), then by the convexity of \( \beta \mapsto \mathbb{E} L_r(\beta) \) and Proposition 1, we have

\[
0 = \nabla \mathbb{E} L_r(\beta_*^i) = \nabla \mathbb{E} L_r(\beta_*^i) = -\frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \{ \ell'_i(x_i - \alpha_r) x_i \}.
\]

This means that \( \nabla L_r(\beta_*^i) = -(1/n) \sum_{i=1}^{n} \ell'_i(x_i - \alpha_r) x_i \) is a sum of independent mean zero random vectors. Further, by the definition of \( \alpha_r, \mathbb{E} \ell'_i(x - \alpha_r) = 0 \) and

\[
\mathbb{E} \{ \ell'_i(x - \alpha_r)^2 \} \leq 2 \mathbb{E} \ell'_i(x - \alpha_r) x_i \leq 2 \mathbb{E} \ell'_i(x) x_i \leq \sigma_x^2.
\]

(70)

Let \( \xi_i = \ell'_i(x_i - \alpha_r), i = 1, \ldots, n, \) so that \( \| \nabla L_r(\beta_*^i) \|_\infty = \max_{0 \leq j \leq d} \| (1/n) \sum_{i=1}^{n} \xi_i x_i j \| \). For each \( 1 \leq j \leq d, \) as in the proof of Lemma 4, it can be shown that \( \mathbb{E} \xi_i x_i j^k \leq \sigma_x^2 \mathbb{E} x_i j^k \leq \sigma_x^2 \sigma_y^2 j^k 1 \leq k! \mathbb{E} \xi_i x_i j^k 1 \leq \frac{k!}{2} \sigma_x^2 \sigma_y^2 j^k 1 \cdot (2 \sigma_y^2 1^2 \sigma_x^2)^k, \)

\[ k = 3, 4, \ldots \]

Similarly, \( \mathbb{E} \xi_i^2 \leq \sigma_x^2 \) and \( \mathbb{E} \xi_i^k \leq \sigma_x^2 \mathbb{E} x_i j^k 1 \leq \frac{k!}{2} \sigma_x^2 \mathbb{E} x_i j^k 1 \) for \( k = 3, 4, \ldots \). Consequently, it follows from Bernstein’s inequality and the union bound that for any \( x > 0, \)

\[
\| \nabla L_r(\beta_*^i) \|_\infty \leq v_0 \sigma_x \left\{ \sigma_x \sqrt{\frac{2x}{n} + \frac{2\tau x}{n}} \right\}
\]

(71)

with probability at least \( 1 - 2d e^{-x}. \) Taking \( x = 2 \log(d) \) in (71) we see that for any choice of regularization parameter such that

\[
\lambda \geq 4 v_0 \sigma_x \left\{ \sigma_x \sqrt{\frac{\log(d)}{n} + \frac{\log(d)}{n}} \right\},
\]

the event \( \{ \lambda \geq 2 \| \nabla L_r(\beta_*^i) \|_\infty \} \) occurs with probability at least \( 1 - 2(d^{-1} + d^{-2}). \)

Putting together the pieces leads to the stated result (9). \qed

C.1.1 Proof of Lemma 4

Recalling that the Huber loss is convex and continuously differentiable, we have

\[
\langle \nabla L_r(\beta) - \nabla L_r(\beta_*^i), \beta - \beta_*^i \rangle
\]

\[
\geq \frac{1}{n} \sum_{i=1}^{n} \{ \ell'_i(y_i - \langle x_i, \beta_*^i \rangle) - \ell'_i(y_i - \langle x_i, \beta \rangle) \} \langle x_i, \beta - \beta_*^i \rangle I_{\ell_2^r},
\]

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where $E_{i,r} = \{ |e_i - \alpha_r| \leq \tau/2 \cap \langle \langle x, \beta \rangle - \langle x_i, \beta_i \rangle \rangle \} / ||\beta - \beta_i||_\Sigma \leq \tau/(2r)$. On the event $E_{i,r}, |y_i - \langle x_i, \beta_i \rangle| = |e_i + \beta_0 - \beta_{0,i}| = \alpha_r \leq \tau$ and $|y_i - \langle x_i, \beta_i \rangle| \leq \tau$ for all $\beta \in \beta_i + B_\Sigma(r)$. Following the proof of Lemma C.3 in Appendix C of Sun, Zhou and Fan (2019), it can be shown that

$$\langle \nabla L_r((\beta) - \nabla L_r((\beta_i)), \beta - \beta_i \rangle \geq D(\delta) := \frac{1}{n} \sum_{i=1}^n \varphi_{\|\delta\|_\Sigma}(\langle x, \delta \rangle) \psi(x_i)$$ (72)

for all $\delta = \beta - \beta_i \in B_\Sigma(r)$, where $\varphi_R(u) = u^2 I(|u| \leq R/2) + (|u| - R)^2 I(|u| \leq R) and \psi_R(u) = I(|u| \leq R)$ for any $R > 0$. In the following, we bound the expectation $\mathbb{E}D(\delta)$ and the stochastic term $D(\delta) - \mathbb{E}D(\delta)$, separately.

Using the inequalities that $u^2 I(|u| \leq R/2) \leq \varphi_R(u) \leq u^2 I(|u| \leq R)$, we have

$$\mathbb{E} \varphi_{\|\delta\|_\Sigma}(\langle x, \delta \rangle) \psi(x_i) \geq \mathbb{E} \langle x, \delta \rangle^2 \psi(x_i) - \mathbb{E} \langle x, \delta \rangle^2 \psi(x_i) I(|x, \delta| > \|\delta\|_\Sigma) \geq \rho_r ||\delta||_\Sigma^2 - \rho_r ||\delta||_\Sigma^2 \mathbb{E} \xi^2 \mathbb{I}_{|\xi| > \tau/(4r)}$$,

where $\xi_\delta = \langle x, \delta \rangle / ||\delta||_\Sigma$. Consequently,

$$\inf_{\xi \in \mathbb{R}^d} \mathbb{E}D(\delta) \geq \rho_r - \bar{\rho}_r \sup_{||u||_\Sigma = 1} \mathbb{E} \langle x, u \rangle^2 I(|x, u| > \tau/(4r)).$$ (73)

For any $u > 0$,

$$\mathbb{E}[\xi^2 I(|\xi| > u)] = 2 \int_0^u t \cdot \mathbb{P}(|\xi| > t) \cdot \mathbb{P}(|\xi| > u) \, dt + 2 \int_u^{\infty} t \cdot \mathbb{P}(|\xi| > t) \, dt$$

$$= u^2 \mathbb{P}(|\xi| > u) + 2 \mathbb{V}_0 \int_{u/\mathbb{V}_0}^{\infty} t \cdot \mathbb{P}(|\xi| > u) \, dt$$

$$\leq u^2 e^{-u/\mathbb{V}_0} + 2 \mathbb{V}_0 \int_{u/\mathbb{V}_0}^{\infty} t e^{-t} \, dt$$

$$= (u^2 + 2 \mathbb{V}_0 u + 2 \mathbb{V}_0^2) e^{-u/\mathbb{V}_0}.$$

Given our choice of $r$ specified in (66), it follows from the above bound that

$$\rho_r^2 \sup_{||u||_\Sigma = 1} \mathbb{E} \langle x, u \rangle^2 I(|x, u| > \tau/(4r)) \leq \rho_r^2 (\tau^2/(16r^2) + \mathbb{V}_0 \tau/(2r) + 2 \mathbb{V}_0^2) e^{-\tau/(4r\mathbb{V}_0)}$$

$$= (16c_1^2 + 8c_1 + 2\mathbb{V}_0^2) e^{-\tau/(4r\mathbb{V}_0)}$$

$$\leq (16c_1^2 + 8c_1 + 2) e^{-\tau/(4r\mathbb{V}_0)} < 1/2.$$

Plugging this bound into (73) and by our choice of $r$, we obtain that

$$\inf_{\delta \in \mathbb{R}^d} \mathbb{E}D(\delta) \geq \rho_r (1 - c_2).$$ (74)

Next we evaluate the stochastic term

$$\Delta(r) = \sup_{\delta \in \mathbb{R}^d} \{ D_0(\delta) - \mathbb{E}D_0(\delta) \},$$ (75)
where $D^*_0(\delta) = -D(\delta)/\|\delta\|_\Sigma$. Note that $0 \leq \varphi_R(u) \leq \min[(R/2)^2, u^2]$ and $0 \leq \psi_R(u) \leq 1$ for all $u \in \mathbb{R}$. Therefore,

$$0 \leq \chi_i := \frac{1}{\|\delta\|_\Sigma} \varphi_{\|\delta\|_\Sigma}((x_i, \delta)) \psi_{\|\delta\|_\Sigma}(e_i) \leq \frac{\tau^2}{(4r)^2} \sqrt{\langle x_i, \delta/\|\delta\|_\Sigma \rangle^2},$$

from which it follows that $\mathbb{E}\chi_i^2 \leq m_4$. Putting together the pieces, and applying Bousquet’s version of Talagrand’s inequality (see, e.g. Theorem 7.3 in Bousquet (2003)), we conclude that for any $t > 0$,

$$\Delta(r) \leq \mathbb{E}\Delta(r) + (\mathbb{E}\Delta(r))^{1/2} \frac{\tau}{2r} \sqrt{\frac{t}{n}} + \sqrt{\frac{2m_4t}{n}} + \frac{\tau^2}{48r^2n} \leq 1.25 \mathbb{E}\Delta(r) + \frac{\sqrt{2m_4}}{n} + \frac{\tau^2}{3r^2n}$$

(76)

with probability at least $1 - e^{-t}$, where the second step follows from the inequality that $ab \leq a^2/4 + b^2$ for all $a, b \in \mathbb{R}$. It remains to bound $\mathbb{E}\Delta(r)$. Noting that $\varphi_{R}(cu) = c^2 \varphi_{R}(u)$ for any $c > 0$, we define

$$\mathcal{E}(\delta; z_i) = \frac{1}{\|\delta\|_\Sigma} \varphi_{\|\delta\|_\Sigma}((x_i, \delta)) \psi_{\|\delta\|_\Sigma}(e_i) = \varphi_{\|\delta\|_\Sigma}((x_i, \delta/\|\delta\|_\Sigma)) \psi_{\|\delta\|_\Sigma}(e_i), \quad \delta \in \mathbb{R}^d,$$

where $z_i = (x_i, e_i)$. By Rademacher symmetrization,

$$\mathbb{E}\Delta(r) \leq 2\mathbb{E}\left\{ \sup_{\delta \in \Sigma} \frac{1}{n} \sum_{i=1}^{n} e_i \mathcal{E}(\delta; z_i) \right\},$$

where $e_1, \ldots, e_n$ are independent Rademacher random variables. Recalling that $\varphi_R(\cdot)$ is $R$-Lipschitz, we find that $\mathcal{E}(\delta; z_i)$ is a $(\tau/2r)$-Lipschitz function in $\langle x_i, \delta/\|\delta\|_\Sigma \rangle$, i.e. for any sample $z_i = (x_i, e_i)$ and parameters $\delta, \delta' \in \mathbb{R}^d$,

$$|\mathcal{E}(\delta; z_i) - \mathcal{E}(\delta'; z_i)| \leq \frac{\tau}{2r} |\langle x_i, \delta/\|\delta\|_\Sigma \rangle - \langle x_i, \delta'/\|\delta'\|_\Sigma \rangle|. \quad (77)$$

Moreover, observe that $\mathcal{E}(\delta; z_i) = 0$ for any $\delta$ such that $\langle x_i, \delta/\|\delta\|_\Sigma \rangle = 0$, and $\psi_{\|\delta\|_\Sigma}(e_i) \in \{0, 1\}$. Then, applying Talagrand’s contraction principle (see, e.g. Theorem 4.4, Theorem 4.12 and (4.20) in Ledoux and Talagrand (1991)) yields

$$\mathbb{E}\Delta(r) \leq 2\mathbb{E}\left\{ \sup_{\delta \in \Sigma} \frac{1}{n} \sum_{i=1}^{n} e_i \mathcal{E}(\delta; z_i) \right\} \leq \frac{\tau}{r} \mathbb{E}\left\{ \sup_{\delta \in \Sigma} \frac{1}{n} \sum_{i=1}^{n} e_i \langle x_i, \delta/\|\delta\|_\Sigma \rangle \right\} \leq 4\lambda_i^{-1/2} \bar{z}_i^{-1/2} \frac{\tau}{r} \mathbb{E}\left\{ \frac{1}{n} \sum_{i=1}^{n} e_i |x_i| \right\}_\infty,$$

(78)

where the last inequality follows from the cone constraint that $\|\delta\|_\Sigma \leq 4\bar{z}_i^{-1/2} \|\delta\|_2 \leq 4\lambda_i^{-1/2} \bar{z}_i^{-1/2} \|\delta\|_\Sigma$. We will use a maximal inequality for sub-exponential random variables to bound the last term on the right-hand side of (78). For each $0 \leq j \leq d$, define the partial sum $S_j = \sum_{i=1}^{n} e_i x_{ij}$, of which each summand satisfies $\mathbb{E}(e_{i,j}) = 0$ and $\mathbb{E}(e_{i,j})^2 = \sigma_{ij}^2$. In addition, for $k = 3, 4, \ldots$, $\mathbb{E}|e|^k = 1$ and

$$\mathbb{E}|e_{i,j}|^k \leq v_0 \sigma_{jj}^{k/2} \cdot k \int_{0}^{\infty} t^{k-1} e^{-t} dt \leq v_0 \sigma_{jj}^{k/2} \cdot k \int_{0}^{\infty} t^{k-1} e^{-t} dt = k! v_0 \sigma_{jj}^{k/2}.$$
From these moment bounds and by the symmetry of Rademacher random variables, we have

\[
\mathbb{E} e^{\lambda \epsilon_i x_{ij}} = 1 + \frac{1}{2} \sigma_{jj} \lambda^2 + \sum_{k=2}^{\infty} \frac{\lambda^k}{k!} \mathbb{E}(\epsilon_i x_{ij})^k \\
\leq 1 + \frac{1}{2} \sigma_{jj} \lambda^2 + \sum_{\ell=2}^{\infty} \frac{\lambda^{2\ell}}{(2\ell)!} (2\ell) ! \nu_0^2 \sigma_{jj}^\ell \\
= 1 + \frac{\nu_0^2 \sigma_{jj}}{2} \sum_{k=2}^{\infty} \lambda^k (\sqrt{2} \nu_0 \sigma_{jj}^{1/2})^{k-2} \\
\leq 1 + \frac{1}{2} \frac{\nu_0^2 \sigma_{jj}}{1 - \sqrt{2} \nu_0 \sigma_{jj}^{1/2}} \lambda^2, \quad \text{for all } 0 \leq \lambda < \frac{1}{\sqrt{2} \nu_0 \sigma_{jj}^{1/2}}.
\]

Let \( \nu = \nu_0^2 \sigma_{jj}^{1/2} \) and \( c = \sqrt{2} \nu_0 \sigma_{jj}^{1/2} \). Following the proof of Theorems 2.10 and 2.5 in Boucheron, Lugosi and Massart (2013), it can be shown that for all \( 0 \leq j \leq d \) and \( \lambda \in (0, 1/c) \), \( \log \mathbb{E} e^{\lambda S_j} \leq \psi(\lambda) := \frac{\nu_0^2 \sigma_{jj}}{2(1 - c \nu_0 \sigma_{jj}^{1/2})} \lambda^2 \) and

\[
\mathbb{E} \max_{0 \leq j \leq d} |S_j| \leq \inf_{\lambda \in (0, 1/c)} \left\{ \frac{\log (2d) + \psi(\lambda)}{\lambda} \right\} = \sqrt{2} \nu_0 \log (2d) + c \log (2d).
\]

Re-arranging terms and using (78), we find that

\[
\mathbb{E} \Delta(r) \leq 4 \sqrt{2} \lambda_r^{-1/2} \nu_0 \sigma_x \sqrt{1/2} \frac{r}{n} \left\{ \sqrt{\frac{\log (2d)}{n} + \frac{\log (2d)}{n}} \right\}
\]

which, in conjunction with (75) and (76) with \( r = \log (d) \), leads to

\[
\Delta(r) \leq 5 \sqrt{2} \lambda_r^{-1/2} \nu_0 \sigma_x \sqrt{1/2} \frac{r}{n} \left\{ \sqrt{\frac{\log (2d)}{n} + \frac{\log (2d)}{n}} + \sqrt{\frac{2m_4 \log (d)}{n} + \frac{\tau^2 \log (d)}{3r^2 n}} \right\}
\]

with probability at least \( 1 - d^{-1} \). Combining this with (72), (73) and (75) proves (67).

\[\square\]

**C.2 Proof of Theorem 2**

We only prove (30) since (31) can be shown via the same argument with slight modifications. In view of Propositions 3 and 7, we take \( L = \left\{ 2 + \frac{r}{p'(\gamma_0)} \right\} \lambda_r^{-1/2} \sqrt{(3s)^{1/2}} + \frac{r}{p'(\gamma_0)} \lambda_r^{-1} b^* \) and \( r = \tau/(24 \nu_0^2) \). By Lemma 1, the bias term \( b^* \) satisfies \( b^* \leq \sigma_x^2 r^{-1} \). Here \( \gamma_0, \nu_0, \lambda_r \) and \( \sigma_x \) are constants that are independent of \( (n, d, s) \). Applying Part (I) of Proposition 7, we see that with probability at least \( 1 - d^{-1} \), the curvature parameter \( \kappa_\epsilon(r, L, \beta^*) \) given in Definition 2 satisfies \( \kappa_\epsilon(r, L, \beta^*) \geq 1/2 \) as long as \( \tau \geq 4 \sigma_x \) and \( n \geq [s + (\lambda \tau)^{-2}] \log (d) \). Under the assumed bound on \( \gamma_0 \) (see (29)), the constraints in (16) are satisfied with probability at least \( 1 - d^{-1} \) as long as

\[
\lambda \geq 2(1 + \sqrt{2}) \gamma_0^{-1} \lambda_r^{-1/2} \frac{\sigma_x^2}{s^{1/2}} \text{ and } r \geq 2(a_{\gamma_0} \lambda_r^{-1/2} s^{1/2} \lambda + \sigma_x^2 \tau^{-1}), \quad (79)
\]

where \( a_{\gamma_0} = 1 + \frac{\sqrt{2}}{2} p'(\gamma_0) \).

Moreover, the high-level result in Proposition 3 requires the regularization parameter \( \lambda \) to satisfy \( \lambda \geq \frac{r}{p'(\gamma_0)} (||w^*||_\infty + \epsilon_\ell) \) for all \( \ell \geq 1 \), where \( w^* = \nabla L_r(\beta^*) - \mathbb{E} \nabla L_r(\beta^*) \). For the choice
\[ \tau = \sigma_e \sqrt{n/(s + \log(d))}, \] 

it follows from the tail bounds (27) and (28) with \( t = \log(d) \), that with probability at least \( 1 - 3d^{-1} \),

\[ \|w^*\|_\infty \leq 6\nu_0\sigma e \sigma e \sqrt{\frac{\log(d)}{n}} \quad \text{and} \quad \|w^*_S\|_2 \leq 9\nu_0\lambda_{\max}(S)\sigma e \sqrt{\frac{s + \log(d)}{n}}. \]  

(80)

Finally, based on (79), (80) and the assumed upper bound on the optimization error \( \epsilon_\ell \), if we choose the regularization parameter \( \lambda = \sigma_e \sqrt{\log(d)/n} \), then the event \( \{ \lambda \geq \frac{2}{p(u_0)}(\|w^*\|_\infty + \epsilon_\ell) \} \) for all \( \ell \geq 1 \) holds with probability at least \( 1 - 3d^{-1} \). We have thus verified that all the conditions need to apply Proposition 3 are satisfied with high probability. In conjunction with the upper bound on \( \|w^*_S\|_2 \) in (80) and the minimal signal strength condition, the result of Proposition 3 implies that with probability at least \( 1 - 4d^{-1} \),

\[ \|\hat{\beta}^{(\ell)} - \beta^*\|_\Sigma \leq \delta^{\ell-1}\sigma e \sqrt{\frac{s + \log(d)}{n}} + \sigma e \frac{s + \log(d)}{1 - \delta} \] 

for all \( \ell \geq 1 \).

This proves (30) by letting \( \ell \geq \log(d)/\log(1/\delta) \).

C.3 Proof of Theorem 3

Proof of Part (I). The proof is based primarily on Proposition 5, combined with complementary probabilistic analysis. We start with establishing the required statistical properties of the oracle estimator \( \hat{\beta}^{ora} \) defined in (19). Since the oracle \( \hat{\beta}^{ora} \) has direct access to the true active set \( S \), it is essentially an unpenalized Huber estimator computed from the oracle observations \( \{(y_i, x_iS)\}_{i=1}^n \), which satisfy \( y_i = \langle x_iS, \beta^*_S \rangle + \epsilon_i \). Applying Theorem B.1 in Sun, Zhou and Fan (2019) to the oracle \( \hat{\beta}^{ora} \) with \( \tau = \sigma e \sqrt{n/(s + \log(d))} \) guarantees that provided \( n \geq s + \log(d) \),

\[ \|\hat{\beta}^{ora} - \beta^*\|_\Sigma = \|(\hat{\beta}^{ora} - \beta^*)_S\|_S \leq \sigma e \sqrt{\frac{s + \log(d)}{n}} \]  

(81)

holds with probability at least \( 1 - d^{-1} \), where \( S = \mathbb{E}(x_Sx_S^\top) \in \mathbb{R}^{d \times d} \). For the error under \( \ell_\infty \)-norm, it is obvious that

\[ \|\hat{\beta}^{ora} - \beta^*\|_\infty = \|(\hat{\beta}^{ora} - \beta^*)_S\|_S \leq \|(\hat{\beta}^{ora} - \beta^*)_S\|_2 \leq \frac{1}{\sqrt{\lambda_{\min}(S)}} \|(\hat{\beta}^{ora} - \beta^*)_S\|_S. \]  

(82)

In order to apply the high-level result Proposition 5, we need the following two technical lemmas, which provide lower and upper bounds for the curvature parameter \( k_r \) and \( \|\nabla L_r(\hat{\beta}^{ora})\|_\infty \), respectively.

Lemma 5. Write \( m_4 = \sup_{u \in \mathbb{S}^d} \mathbb{E}(u, \Sigma^{-1/2}u)^4 \) and let \( r, L > 0 \) satisfy

\[ \tau \geq 8 \max\{ \sqrt{2}e, 2m_4^{1/2}r \} \quad \text{and} \quad n \geq C \max(\tau/r)^2, (v_1 \sigma \tau/r)^2L^2 \log(d), \]  

(83)

where \( C > 0 \) is a universal constant. Then with probability at least \( 1 - d^{-1} \),

\[ \langle \nabla L_r(\beta_1) - \nabla L_r(\beta_2), \beta_1 - \beta_2 \rangle \geq \frac{1}{2} \|\beta_1 - \beta_2\|_\Sigma^2 \]  

(84)

uniformly over \((\beta_1, \beta_2) \in \mathbb{C}(r, L) := \{ (\beta, \beta') : \beta \in \beta' + \mathbb{B}(\Sigma) \cap \mathbb{C}(L), \beta' \in \beta' + \mathbb{B}(\Sigma(\frac{1}{2}), \beta'_{\Sigma} = 0) \} \).
Lemma 6. Let \( \tau = \sigma_\varepsilon \sqrt{n/[s + \log(d)]} \). Then, with probability at least 1 - 4d^{-1},
\[
\|\hat{\beta}^{ora} - \beta^*\|_\Sigma \leq \sigma_\varepsilon \sqrt{s + \log(d)}\]
\[
\text{and } \|\nabla L_{\tau}(\hat{\beta}^{ora})\|_{\infty} \leq \sigma_\varepsilon \sqrt{s + \log(d)}
\]
hold as long as \( n \geq s + \log(d) \).

According to Lemma 5 and Proposition 5, we take \( r = \tau/(16m_4^{1/2}) \) and \( L = (2 + \frac{2}{p'(y_0)}) \lambda_i^{-1/2}(2s)^{1/2} \) so that under the sample size requirement \( n \geq s \log(d) \), \( \tilde{\kappa}_v(r, L, \beta^*) \geq 1/2 \) with probability at least 1 - d^{-1}. With the above choice of \( r \), it follows from Lemma 6 that \( \|\hat{\beta}^{ora} - \beta^*\|_\Sigma \leq r/2 \) holds with high probability as long as \( n \geq s + \log(d) \). Taking \( \tilde{\kappa} = 1/2 \) in Proposition 5, we see that if \( \lambda \) satisfies
\[
\max\left\{ \frac{2}{p'(y_0)} \|\nabla L_{\tau}(\hat{\beta}^{ora})\|_{\infty}, 2.5 \delta \lambda \|\hat{\beta}^{ora} - \beta^*\|_{\infty} \right\} \leq \lambda \leq \frac{\tau}{2.5s^{1/2}} = \frac{\tau}{40(m_4s)^{1/2}},
\]
then \( \hat{\beta}^{(\ell)} = \hat{\beta}^{ora} \) for all \( \ell \geq [\log(s^{1/2}/\delta)/\log(1/\delta)] \).

Together, (81), (82) and Lemma 6 imply that with \( \lambda = \sigma_\varepsilon \sqrt{s + \log(d)}/n \), (86) holds with probability at least 1 - 4d^{-1} as long as \( n \geq \max(s^{3/2}, s \log(d)) \). Putting together the pieces completes the proof of Part (I).

Proof of Part (II). The proof strategy is similar in spirit to the previous one. A key step is to bound the curvature parameter \( \tilde{\kappa}_v \) and \( \|\nabla L_{\tau}(\hat{\beta}^{ora})\|_{\infty} \) from below and above, respectively. The following two lemmas are in parallel with Lemmas 5 and 6.

Lemma 7. Write \( m_4 = \sup_{u \in \mathbb{R}^d} \mathbb{E}(u, \Sigma^{-1/2}u)^4 \) and let \( r, L > 0 \) satisfy
\[
\tau \geq 8 \sqrt{s} (m_4/L)^{1/2}r \quad \text{and} \quad n \geq \max((\tau/r)^2s, (\nu_1\sigma_\varepsilon^2\tau/r)^2L^2 \log(d)).
\]
Then with probability at least 1 - d^{-1},
\[
\langle \nabla L_{\tau}(\beta_1) - \nabla L_{\tau}(\beta_2), \beta_1 - \beta_2 \rangle \geq \frac{1}{2} L_{\tau}^2 \|\beta_1 - \beta_2\|_{\Sigma}^2
\]
holds uniformly over \( (\beta_1, \beta_2) \in C(r, L) \).

Lemma 8. Let \( \tau \geq \sigma_\varepsilon^2 \). Then, with probability at least 1 - 3d^{-1} - 2n^{-1},
\[
\|\hat{\beta}^{ora} - \beta^*\|_\Sigma \leq \sigma_\varepsilon \sqrt{s + \log(n)}
\]
\[
\|\hat{\beta}^{ora} - \beta^*\|_{\infty} \leq \sigma_\varepsilon \left\{ \frac{s}{n} + \sqrt{\frac{\log(n)}{n}} \right\}
\]
\[
\text{and } \|\nabla L_{\tau}(\hat{\beta}^{ora})\|_{\infty} \leq \sigma_\varepsilon \left\{ \frac{s}{n} + \sqrt{\frac{\log(d)}{n}} \right\}
\]
hold as long as \( n \geq s + \log(d) \).

Based on these lemmas, the remainder of the proof is straightforward. Take \( r = (L_{\tau}/m_4)^{1/2}\gamma \sqrt{s} \) and \( L = (2 + \frac{2}{p'(y_0)}) \lambda_i^{-1/2}(2s)^{1/2} \) so that under the scaling \( n \geq s \log(d) \), \( \tilde{\kappa}_v(r, L, \beta^*) \geq \tau/2 \) with probability at least 1 - d^{-1}. From Proposition 5 with \( \tilde{\kappa} = \tau/2 \), we see that if \( \lambda \) satisfies
\[
\max\left\{ \frac{2}{p'(y_0)} \|\nabla L_{\tau}(\hat{\beta}^{ora})\|_{\infty}, 2.5 \delta \lambda \|\hat{\beta}^{ora} - \beta^*\|_{\infty} \right\} \leq \lambda \leq \frac{\tau}{2.5s^{1/2}} = \frac{\tau}{20(m_4s)^{1/2}},
\]

(87)
then $\hat{\beta}^{(t)} = \hat{\beta}^{na}$ for all $t \geq \lceil \log(s/\delta)/\log(1/\delta) \rceil$. The result of Lemma 8 ensures that under the scaling $\lambda = \sigma_x \sqrt{\log(d)/n}$, (87) holds with probability at least $1 - 3d^{-1} - 2n^{-1}$ provided $n \geq \max\{s^2/\log(d), s \log(d)\}$. Putting together the pieces yields the claim. \hfill \Box

### C.3.1 Proof of Lemma 5

To begin with, note that

$$C.3.1 \text{ Proof of Lemma 5}$$

To begin with, note that

$$D(\beta_1, \beta_2) := \langle \nabla L_x(\beta_1) - \nabla L_x(\beta_2), \beta_1 - \beta_2 \rangle$$

$$= \frac{1}{n} \sum_{i=1}^{n} [\ell'_i(y_i - \langle x_i, \beta_2 \rangle) - \ell'_i(y_i - \langle x_i, \beta_1 \rangle)](x_i, \beta_1 - \beta_2)$$

$$\geq \frac{1}{n} \sum_{i=1}^{n} [\ell'_i(y_i - \langle x_i, \beta_2 \rangle) - \ell'_i(y_i - \langle x_i, \beta_1 \rangle)](x_i, \beta_1 - \beta_2)I_{\epsilon_i}, \quad (88)$$

where $I_{\epsilon_i}$ is the indicator function of the event

$$E_i := [|\epsilon_i| \leq \tau/4] \cap \{|(x_i, \beta_2 - \beta_1) | \leq \tau/4\} \cap \{|(x_i, \beta_1 - \beta_2)| \leq \tau, \sigma_x \|eta_1 - \beta_2\| \leq \frac{\tau}{2R}\}. \quad (89)$$

on which $|y_i - \langle x_i, \beta_1 \rangle| = |\epsilon_i| + |\langle x_i, \beta_2 - \beta_1 \rangle| \leq \tau/2$ and $|y_i - \langle x_i, \beta_1 \rangle| \leq |\langle x_i, \beta_1 - \beta_2 \rangle| + |\langle x_i, \beta_2 - \beta_1 \rangle| + |\epsilon_i| \leq \tau$ for all $\beta_1 \in \beta_2 + \mathbb{R}(r)$. As in the proof of Lemma 4, for any $R > 0$, define the functions

$$\phi_R(u) = u^2I(|u| \leq R/2) + (|u| - R)^2I(R/2 \leq |u| \leq R),$$

and $\psi_R(u) = I(|u| \leq R/2) + [2 - (2u/R) \operatorname{sign}(u)]I(R/2 < |u| \leq R),$ which are smoothed versions of $u^2I(|u| \leq R)$ and $\psi_R(u) := I(|u| \leq R)$, respectively. Recalling that $\ell''(u) = 1$ for $|u| \leq \tau$, by (88) we have

$$D(\beta_1, \beta_2) \geq D_0(\beta_1, \beta_2) := \frac{1}{n} \sum_{i=1}^{n} \varphi_{\frac{\tau}{2[R(\beta_1 - \beta_2)]}}((x_i, \beta_1 - \beta_2))\psi_{\frac{\tau}{2}}((x_i, \beta_2 - \beta_1))\psi_{\frac{\tau}{2}}(\epsilon_i)$$

$$\quad = \mathbb{E}D_0(\beta_1, \beta_2) + D_0(\beta_1, \beta_2) - \mathbb{E}D_0(\beta_1, \beta_2). \quad (90)$$

In what follows, we deal with $\mathbb{E}D_0(\beta_1, \beta_2)$ and $D_0(\beta_1, \beta_2) - \mathbb{E}D_0(\beta_1, \beta_2)$, separately.

Using the properties that

$$u^2I(|u| \leq R/2) \leq \varphi_R(u) \leq u^2I(|u| \leq R) \quad \text{and} \quad \phi_R(u) \geq \psi_{\frac{\tau}{2}}(u),$$

we have

$$\mathbb{E}D_0(\beta_1, \beta_2) \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(x_i, \beta_1 - \beta_2)^2I_{(x_i, \beta_1 - \beta_2) \leq \frac{\tau}{2|R(\beta_1 - \beta_2)|}}(x_i, \beta_1 - \beta_2)$$

$$\geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(x_i, \beta_1 - \beta_2)^2I_{(|x_i, \beta_1 - \beta_2| \leq \frac{\tau}{2R})} - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(x_i, \beta_1 - \beta_2)^2I_{|\epsilon_i| \leq \frac{\tau}{2R}}.$$  \hfill (91)
Write $\delta = \beta_1 - \beta_2$ for $\beta_1 \in \beta_2 + \mathbb{B}_\Sigma(r)$ and $\beta_2 \in \beta^* + \mathbb{B}_\Sigma(\frac{r}{4})$. By Markov’s inequality,
\[
E(x, \delta)^2 I_{(x, \delta) > \frac{\tau}{\Sigma}} \leq (4r/\tau)^2 \mathbb{E}(x, \delta)^4 / \|\delta\|^2_{\Sigma},
\]
\[
E(x, \delta)^2 I_{(x, \delta, \beta - \beta^*) > \frac{\tau}{4}} \leq (8/\tau)^2 \mathbb{E}(x, \delta)^2 (\|x_2 - \beta^*\|^2_{\Sigma}) \leq (4\sigma_x/\tau)^2 \|\delta\|^2_{\Sigma}
\]
and $E(x, \delta)^2 I(|e_i| > \tau/4) \leq (4\sigma_x/\tau)^2 \|\delta\|^2_{\Sigma}$. Substituting these into (91) yields
\[
\mathbb{E}D_0(\beta_1, \beta_2) \geq (1 - 2m_d(4r/\tau)^2 - (4\sigma_x/\tau)^2) \|\delta\|^2_{\Sigma}.
\]
Provided $\tau \geq \max\{8\sqrt{2}\sigma_x, 16m_4^{1/2}/r\}$, this further implies
\[
\mathbb{E}D_0(\beta_1, \beta_2) \geq \frac{3}{4} \|\beta_1 - \beta_2\|^2_{\Sigma}
\]
uniformly over $\beta_1 \in \beta_2 + \mathbb{B}_\Sigma(r)$ and $\beta_2 \in \beta^* + \mathbb{B}_\Sigma(\frac{r}{4})$.

To bound $D_0(\beta_1, \beta_2) - \mathbb{E}D_0(\beta_1, \beta_2)$ from below uniformly over $(\beta_1, \beta_2) \in C(r, L)$, we define
\[
\Delta(r, L) = \sup_{(\beta_1, \beta_2) \in C(r, L)} \frac{-D_0(\beta_1, \beta_2) + \mathbb{E}D_0(\beta_1, \beta_2)}{\|\beta_1 - \beta_2\|^2_{\Sigma}}.
\]
For each pair $(\beta_1, \beta_2)$, write $(-D_0(\beta_1, \beta_2) + \mathbb{E}D_0(\beta_1, \beta_2))/\|\beta_1 - \beta_2\|^2_{\Sigma} = (1/n) \sum_{i=1}^n (x_i - E(x_i))$. Since
\[
0 \leq \varphi_R(u) \leq \min((R/2)^2, u^2)
\]
and $0 \leq \varphi_R(u) \leq 1$ for all $u \in \mathbb{R}$, we have
\[
\varphi_R(u) \leq \min((r/4)^2, (x_i, (\beta_1 - \beta_2))/\|\beta_1 - \beta_2\|^2_{\Sigma}).
\]
By Bousquet’s version of Talagrand’s inequality (Bousquet, 2003), for any $x > 0,$
\[
\Delta(r, L) \leq \mathbb{E}\Delta(r, L) + \sqrt{\frac{2x}{n} \sqrt{m_d + 2(\frac{r}{4})^2 \mathbb{E}\Delta(r, L) + \left(\frac{r}{4}\right)^2 \frac{x}{3n}}} \leq 1.25 \mathbb{E}\Delta(r, L) + \frac{2m_d x}{n} + \frac{r^2 x}{3n}
\]
with probability at least $1 - e^{-x}$.

It suffices to bound the expected value $\mathbb{E}\Delta(r, L)$. Applying the symmetrization inequality for empirical processes and the connection between Gaussian complexity and Rademacher complexity, we obtain
\[
\mathbb{E}\Delta(r, L) \leq 2 \cdot \sqrt{\frac{\pi}{2}} \cdot \mathbb{E}\left(\sup_{(\beta_1, \beta_2) \in C(r, L)} R_{\beta_1, \beta_2}\right),
\]
where
\[
R_{\beta_1, \beta_2} := \frac{1}{n} \sum_{i=1}^n \varphi_{\beta_1 - \beta_2 / \Sigma}(x_i, \beta_1 - \beta_2) \rho_{\beta_1, \beta_2}((x_i, (\beta_1 - \beta_2))/\rho_{\beta_1, \beta_2}(x_i, (\beta_1 - \beta_2))
\]
with $\rho_{\beta_1, \beta_2} = 0$ and $\gamma_i$ are i.i.d. standard normal random variables that are independent of the observations. Let $E^*$ be the conditional expectation given $\{(y_i, x_i)\}_{i=1}^n$. Next, we apply the Gaussian comparison theorem to bound $E^* \left(\sup_{(\beta_1, \beta_2) \in C(r, L)} R_{\beta_1, \beta_2}\right)$, from which an upper bound for $\mathbb{E}\left(\sup_{(\beta_1, \beta_2) \in C(r, L)} R_{\beta_1, \beta_2}\right)$ follows immediately. For $(\beta_1, \beta_2), (\beta_1', \beta_2') \in C(r, L)$, write $\delta = \beta_1 - \beta_2$ and $\delta' = \beta_1' - \beta_2'$. Since $\varphi_R(cu) = c^2 \varphi_R(u)$ for any $c > 0$, $\rho_{\beta_1, \beta_2}$ can be re-written in the simple form
\[
\rho_{\beta_1, \beta_2} = \frac{1}{n} \sum_{i=1}^n g_i \varphi_{\beta_1 - \beta_2 / \Sigma}(x_i, \delta / \rho_{\beta_1, \beta_2}(x_i, (\beta_1 - \beta_2))
\]
with $g_i$ are i.i.d. standard normal random variables that are independent of the observations.
Motivated by (95), (96) and the inequality that

\[ \phi \]

By the Lipschitz properties of \( \phi \), applying Sudakov-Fernique’s Gaussian comparison inequality yields

\[ Z \]

Furthermore, we have

\[ E \]

and

\[ E \]

Motivated by (95), (96) and the inequality that

\[ E'(G_{\beta_1, \beta_2} - G_{\beta'_1, \beta'_2})^2 \leq 2E'(G_{\beta_1, \beta_2} - G_{\beta_2 + \delta, \beta_2}^2 + 2E'(G_{\beta_2 + \delta, \beta_2} - G_{\beta_1, \beta_2})^2, \]

we define the Gaussian process \( Z_{\beta_1, \beta_2} \) as

\[ Z_{\beta_1, \beta_2} = \frac{\sqrt{2\tau}}{2r^2} \cdot \frac{n}{n} \sum_{i=1}^{n} g_i \langle x_i, \beta_2 - \beta' \rangle + \frac{\sqrt{2\tau}}{2r} \cdot \frac{n}{n} \sum_{i=1}^{n} g_i' \langle x_i, \beta_1 - \beta_2 \rangle \]

where \( g_1', \ldots, g_n' \) are i.i.d. standard normal random variables that are independent of all the other variables. We have established that \( E'(G_{\beta_1, \beta_2} - G_{\beta_1', \beta_2'})^2 \leq E'(Z_{\beta_1, \beta_2} - Z_{\beta_1', \beta_2'})^2. \) Then, applying Sudakov-Fernique’s Gaussian comparison inequality yields

\[ E' \left\{ \sup_{(\beta_1, \beta_2) \in (C, L)} G_{\beta_1, \beta_2} \right\} \leq E' \left\{ \sup_{(\beta_1, \beta_2) \in (C, L)} Z_{\beta_1, \beta_2} \right\}, \]

which remains valid if \( E' \) is replaced by \( E \). To bound the supremum of \( Z_{\beta_1, \beta_2} \), using the cone-like constraint \( \| \beta_2 - \beta_1 \| \leq L \| \beta_2 - \beta \| \), we obtain

\[ E \left\{ \sup_{(\beta_1, \beta_2) \in (C, L)} Z_{\beta_1, \beta_2} \right\} \leq \frac{\sqrt{2\tau}}{4r} E \left\| \frac{1}{n} \sum_{i=1}^{n} g_i' S^{-1/2} x_i \right\|_2 + \frac{\sqrt{2\tau} L r}{2r} E \left\| \frac{1}{n} \sum_{i=1}^{n} g_i' x_i \right\|_\infty \]

\[ \leq \frac{\sqrt{2\tau}}{4r} \sqrt{\frac{\tau}{n}} + \frac{\sqrt{2\tau} L r}{2r} E \left\| \frac{1}{n} \sum_{i=1}^{n} g_i' x_i \right\|_\infty. \]
Together, (94), (97) and (98) deliver the bound

$$\mathbb{E}\Delta(r, L) \leq \sqrt{\pi r} \left\{ \frac{1}{2} \sqrt{\frac{s}{n}} + L \mathbb{E} \left( \max_{0 \leq j \leq d} \left| \sum_{i=1}^{n} g_i x_{ij} \right| \right) \right\}.$$  (99)

Finally we bound the right-hand side of (99). Write $S_j = \sum_{i=1}^{n} g_i x_{ij}$ for $j = 0, \ldots, d$. Under the sub-Gaussian condition on $x$, for each $1 \leq j \leq d$ and $m \geq 3$ we have

$$\mathbb{E}|x_j|^m = v_1^m \sigma_j^m |m| \int_0^\infty r^{m-1} \mathbb{P}(|x_j|/\sigma_j^{1/2} \geq v_1 r) \, dr$$

$$\leq 2v_1^m \sigma_j^m |m| \int_0^\infty r^{m-1} e^{-r^2/2} \, dr = 2^m/2v_1^m \sigma_j^m |m| \Gamma(m/2).$$

Let $g \sim \mathcal{N}(0, 1)$ be independent of $x$. Using the Legendre duplication formula, i.e. $\Gamma(s)\Gamma(s+1/2) = 2^{s-1}\sqrt{\pi}\Gamma(2s)$, some algebra then yields

$$\mathbb{E}|g x_j|^m \leq 2^{m/2} \Gamma\left( \frac{m+1}{2} \right) \cdot 2^m/2v_1^m \sigma_j^m |m| \Gamma(m/2) = 2^m/2v_1^m \sigma_j^m |m|!.$$

Hence, for any $\lambda \in (0,(2v_1 \sigma_j^{1/2})^{-1})$,

$$\mathbb{E}e^{\lambda S_j} = 1 + \frac{1}{2} \sigma_j \lambda^2 + \sum_{k=1}^{\infty} \frac{\lambda^k}{k!} \mathbb{E}(g x_j)^k$$

$$\leq 1 + \frac{1}{2} \sigma_j \lambda^2 + 2 \sum_{\ell=2}^{\infty} \frac{\lambda^{2\ell}}{(2\ell)!} v_1^{2\ell} \sigma_j^{2\ell} (2\ell)!$$

$$= 1 + \frac{1}{2} \sigma_j \lambda^2 + 2 \sum_{\ell=2}^{\infty} v_1^{2\ell} \sigma_j^{2\ell} \lambda^{2\ell}$$

$$\leq 1 + \frac{1}{2} \sigma_j \lambda^2 \sum_{k=2}^{\infty} \lambda^k (2v_1 \sigma_j^{1/2})^{k-2}$$

$$\leq 1 + \frac{1}{2} \sigma_j \lambda^2 \frac{\lambda}{2 - 2v_1 \sigma_j^{1/2} \lambda}.$$

It then follows that $\log \mathbb{E}e^{\lambda S_j} \leq \frac{1}{2} \frac{\lambda^2 v_1^2 \sigma_j \mu}{1 - 2v_1 \sigma_j^{1/2} \lambda}$ for any $\lambda \in (0,(2v_1 \sigma_j^{1/2})^{-1})$. By symmetry, the same bound applies to $-S_j$. Consequently, it follows from Corollary 2.6 in Boucheron, Lugosi and Massart (2013) that

$$\mathbb{E} \left( \max_{0 \leq j \leq d} \left| \sum_{i=1}^{n} g_i x_{ij} \right| \right) = \mathbb{E} \max_{0 \leq j \leq d} |S_j|/n \leq v_1 \sigma_x \left( \sqrt{\frac{2 \log(2d)}{n}} + \frac{2 \log(2d)}{n} \right).$$  (100)

Combining (99), (100) with the concentration inequality (93), we determine that with probability at least $1 - d^{-1}$, $\Delta(r, L) \leq 1/4$ as long as $n \geq \max \{(\tau/\tau^2)^2 s, (v_1 \sigma_x \tau/\tau)^2 L^2 \log(d)\}$. This, together with (90) and (92), proves the claim (84). \hfill $\Box$

### C.3.2 Proof of Lemma 6

To begin with, consider the decomposition

$$\nabla L_{\tau}(\bar{\beta}^{ora}) = w(\bar{\beta}^{ora}) - w(\beta^*) + \mathbb{E}\nabla L_{\tau}(\bar{\beta}^{ora}) + w(\beta^*),$$

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In order to bound the local fluctuation sup
that the oracle
have
Next we bound the right-hand side of (101). For
so that
Consequently, we have
In order to bound the local fluctuation sup, we need to control the moment generation function of \( \Delta^0(\delta) \) for each \( \delta \). Using the Lipschitz continuity of \( \ell'_i \), it follows that
The above moment inequalities, combined with the elementary inequality \( |e^n - 1 - u| \leq (u^2/2)e^{u|\sigma_j|^2} \), imply that for any \( \lambda \in \mathbb{R} \) and \( \lambda^* = \lambda/((\sigma_j)^2||\delta||_S) \),
Applying Hölder’s inequality to the exponential-type moments on the right-hand side of (103), we have
and
Substituting these bounds into the earlier inequality (103), we find that for any \( |\lambda| \leq \sqrt{n}/C_1 \),

from which we see that the main difficulty is in controlling the difference \( w(\hat{\beta}^{ora}) - w(\beta^*) \). Recalling that the oracle \( \hat{\beta}^{ora} \) is sparse and shares the active set with \( \beta^* \), define the oracle local neighborhood \( \Theta_S(r) = \{ \beta \in \beta^* + \mathbb{B}_r(S) : \beta_{S^c} = 0 \} \). Conditioned on the event \( ||\hat{\beta}^{ora} - \beta^*||_S \leq r \),

\[
||w(\hat{\beta}^{ora}) - w(\beta^*)||_\infty \leq \sup_{\beta \in \Theta_S(r)} ||w(\beta) - w(\beta^*)||_\infty. \tag{101}
\]
where $C_1, C_2 > 0$ depend only on $\nu_1$ in Condition 3. A similar argument can be used to establish the same bound for each pair $(\delta, \delta')$, that is,

$$
\mathbb{E}e^{\lambda \sqrt{n} (|\Delta^0(\delta)| - |\Delta^0(\delta')|)/(\sigma^1 J_{\beta, \|\delta - \delta'\|_k})} \leq e^{C_2^2 \lambda^2 / 2} \text{ for all } |\lambda| \leq \sqrt{n}/C_1.
$$

The above inequality certifies condition (Ed) in Spokoiny (2012) (see Section 2 in the supplement), so that Corollary 2.2 therein applies to the process \{\Delta^0(\delta)\}_{\delta \in \mathbb{R}^1; \|\delta\|_0 \leq r}$, with probability at least $1 - e^{-t}$,

$$
\sup_{\beta \in \Theta \Sigma(r)} \langle w(\beta) - w(\beta'), e_j \rangle = \sup_{\delta \in \mathbb{R}^1; \|\delta\|_0 \leq r} \Delta^0(\delta) \leq 3 \sqrt{2C_1} \sigma^1 J_{\beta, \|\delta\|_0} \sqrt{\frac{2s + t}{n}}
$$

as long as $n \geq (2C_1/C_2)(2s + t)$. Combined with (102) and the union bound, we find that

$$
\sup_{\beta \in \Theta \Sigma(r)} \|w(\beta) - w(\beta')\|_{\infty} \leq 6C_2 \sigma \sqrt{\frac{s + \log(2d)}{n}} \tag{104}
$$

with probability at least $1 - (2d)^{-1}$ provided $n \geq (4C_1/C_2)(s + \log(2d))$. Recall from (101) that the above bound applies conditioned on the event $\{||\beta^{\text{ora}} - \beta||_{\Sigma} \leq r\}$.

Tuning to $||\mathbb{E} \nabla L_r(\beta^{\text{ora}})||_{\infty}$, again, we control this term conditioned on the same event above. For any $\beta \in \Theta \Sigma(r)$, write $\delta = (\beta - \beta)^{\Sigma} \in \mathbb{R}^d$. From the proof of Lemma 1 and by the Lipschitz continuity of $\ell^r$, we have

$$
||\mathbb{E} \nabla L_r(\beta)||_{\infty} \leq ||\mathbb{E} \nabla L_r(\beta^*)||_{\infty} + ||\mathbb{E}(\ell^r(e - \langle x, \delta \rangle)) - \ell^r(e)||_{\infty} \\
\leq \sigma \sigma^1 J_{\beta, \|\delta\|_0} + \max_{0 \leq j \leq d} \mathbb{E}|x_j| ||x, \delta|| \\
\leq \sigma \sigma^1 J_{\beta, \|\delta\|_0} + \sigma \sigma^1 J_{\beta, \|\delta\|_0}. \tag{105}
$$

Conditioned on the event $||\beta^{\text{ora}} - \beta^*||_{\Sigma} \leq r$ with $r = \sigma \sqrt{s + \log(d)/n}$, the bound (81) ensures that this event holds with probability at least $1 - d^{-1}$. We therefore conclude from (101), (104) and (105) that with probability at least $1 - (2d)^{-1} - d^{-1}$, the following bounds

$$
||w(\beta^{\text{ora}}) - w(\beta^*)||_{\infty} \leq \sigma \sqrt{\frac{s + \log(d)}{n}} \text{ and } ||\mathbb{E} \nabla L_r(\beta^{\text{ora}})||_{\infty} \leq \sigma \sqrt{\frac{s + \log(d)}{n}}
$$

hold as long as $n \geq s + \log(d)$. Combined with (27), the claim (85) follows. \hfill \Box

### C.3.3 Proof of Lemma 7

This proof is almost identical to that of Lemma 5, except that the event $E_i$ in (89) and $D_0(\beta_1, \beta_2)$ in (90) are now replaced by

$$
E_i = \{|\epsilon_i - \alpha_i| \leq \tau / 2\} \cap \{|(x_i, \beta_2 - \beta^*)| \leq \tau / 4\} \cap \left\{ \frac{|(x_i, \beta_1 - \beta_2)|}{||\beta_1 - \beta_2||_{\Sigma}} \leq \frac{\tau}{4r} \right\}
$$

and

$$
D_0(\beta_1, \beta_2) = \frac{1}{n} \sum_{i=1}^n \varphi_{\tau r ||\beta_1 - \beta_2||_{\Sigma}}((x_i, \beta_1 - \beta_2)) \phi_z((x_i, \beta_2 - \beta^*)) \psi_z(\epsilon_i - \alpha_i),
$$

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shown that similar to that used to prove Theorem B.1 in Appendix B of Sun, Zhou and Fan (2019), it can be shown that

$$
\mathbb{E}D_0(\beta_1, \beta_2) \geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}\varphi_{\frac{\tau}{2}}(\|\beta_i - \beta_0\|_{\Sigma})(\langle x_i, \beta_i - \beta_2 \rangle)\psi_{\frac{\tau}{2}}(\langle x_i, \beta_2 - \beta^* \rangle)\psi_{\frac{\tau}{2}}(\epsilon_i - \alpha_r)
$$

$$
\geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(x_i, \beta_1 - \beta_2)^2 I_{[\epsilon_i - \alpha_r] \leq \frac{\tau}{2}} - \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(x_i, \beta_1 - \beta_2)^2 I_{(x_i, \beta_1 - \beta_2) > \frac{\tau}{2}}
$$

$$
\geq \frac{1}{n} \sum_{i=1}^{n} \mathbb{E}(x_i, \beta_1 - \beta_2)^2 I_{(x_i, \beta_1 - \beta_2) > \frac{\tau}{2}}
$$

where the last inequality holds if $\tau \geq 8 \sqrt{5} (m_4/L_r)_{1/2} r$. Keep all other statements the same, we then get the desired result. $\square$

C.3.4 Proof of Lemma 8

From the proof of Lemma 6, inequalities (101) and (104) remain valid with $\beta^*$ replaced by $\bar{\beta}^*$. Our analysis of $\|\nabla L_r(\bar{\beta}^{ora})\|_{ao}$, however, will be more refined. With slight abuse of notation, set $\Theta_S(r) = \{\beta \in \beta^* + \Sigma : \beta^S = 0\}$ and for any $\beta \in \Theta_S(r)$, write $\delta = (\beta - \beta^*)^S$. Recalling that $\nabla L_r(\beta^*) = 0$, we have

$$
\mathbb{E} \nabla L_r(\beta) = \mathbb{E}[\ell''(e - \alpha_r - \langle x, \beta^* \rangle)] x
$$

$$
= \mathbb{E} [\int_{e \in \mathbb{R}} [\ell''(t)] x \cdot t] dt
$$

$$
= \mathbb{E}[\ell''(e - \alpha_r)] x (\bar{\beta}^*, \delta) + \mathbb{E} \int_{0 - (x, \delta)} [\ell''(e - \alpha_r + u)] x - (e - \alpha_r)] d\mu (u) \cdot x.
$$

Let $\mathbb{E}^\tau$ and $\mathbb{P}^\tau$ be the conditional expectation and probability given $x$, respectively. By the anticoncentration property (32) of the distribution of $e$ given $x$, we find that for any $u \in \mathbb{R}$,

$$
\left| \mathbb{E}^\tau [\ell''(e - \alpha_r + u)] - \ell''(e - \alpha_r) \right| = \left| \mathbb{P}^\tau (|e - \alpha_r + u| \leq \tau) - \mathbb{P}^\tau (|e - \alpha_r| \leq \tau) \right| \leq \frac{L_0}{\sigma_e} |u|.
$$

Together, the last two displays imply

$$
\|\nabla L_r(\beta) + \Sigma_{r,S}(\beta - \beta^*)\|_{ao} \leq \frac{1}{2} L_0 \max_{0 \leq \delta \leq d} \mathbb{E}(x, \delta)^2 \leq \frac{L_0}{2\sigma_e} m_4^{1/2} \sigma_e ||\delta||_{\Sigma}^2, \quad (106)
$$

where $\Sigma_{r,S} := \mathbb{E} \{\ell''(e - \alpha_r)x^T x^T \} \in \mathbb{R}^{d \times d}$ is the submatrix of $\Sigma_r = \mathbb{E} \{\ell''(e - \alpha_r)x^T x^T \}$.

Let us now turn to the oracle estimator. Recall that $w(\beta^*) = \nabla L_r(\beta^*)$. Following an argument similar to that used to to prove Theorem B.1 in Appendix B of Sun, Zhou and Fan (2019), it can be shown that $\bar{\beta}^{ora}$ with $\tau \sim \sigma_e$ satisfies

$$
\|\bar{\beta}^{ora} - \beta^*\|_{\Sigma} \leq \sigma_e \sqrt{\frac{s + \log(n)}{n}} \quad \text{and} \quad \|S_r^{-1/2}(\bar{\beta}^{ora} - \beta^*)\|_{\Sigma} \leq \sigma_e \sqrt{\frac{s + \log(n)}{n}} \quad (107)
$$
with probability at least $1 - n^{-1}$, where $S_r = \mathbb{E}[\ell'_{\tau}(x - \alpha_r) x_S x_S^\top]$. Moreover, note that

$$
\|\tilde{\beta}^\text{ora} - \beta_r^\star\|_{\infty} = \|\tilde{\beta}^\text{ora} - \beta_r^\star\|_{S}\leq \|\tilde{\beta}^\text{ora} - \beta_r^\star\|_{S} - S_r^{-1} w(\beta_r^\star)_{S}\|_{\infty} + \|S_r^{-1} w(\beta_r^\star)_{S}\|_{\infty} \\
\|\tilde{\beta}^\text{ora} - \beta_r^\star\|_{S} - S_r^{-1} w(\beta_r^\star)_{S}\|_{\infty} + \|S_r^{-1} w(\beta_r^\star)_{S}\|_{\infty} \\
\leq \frac{1}{\sqrt{\lambda_{\min}(S_r)}} \|S_r^{1/2}(\tilde{\beta}^\text{ora} - \beta_r^\star)_{S} - S_r^{-1/2} w(\beta_r^\star)_{S}\|_{2} + \|S_r^{-1} w(\beta_r^\star)_{S}\|_{\infty}.
$$

Based on techniques similar to the proof of Proposition 8, it can be shown that with probability at least $1 - n^{-1}$,

$$
\|S_r^{-1} w(\beta_r^\star)_{S}\|_{\infty} \lesssim \frac{\sigma_x}{\sqrt{\lambda_{\min}(S)}} \sqrt{\frac{\log(n)}{n}}. \quad (108)
$$

Together with the above two displays, this proves the desired bound on $\|\tilde{\beta}^\text{ora} - \beta_r^\star\|_{\infty}$.

Finally, we use the general bound (106) to control $\|\mathbb{E}[\nabla L_r(\tilde{\beta}^\text{ora})]\|_{\infty}$. The bound (107) reveals that $\tilde{\beta}^\text{ora} - \beta_r^\star$ can be approximated by $S_r^{-1} w(\beta_r^\star)_{S}$ with a higher-order remainder. To see how this approximation plays an role in (106), note that

$$
\Sigma_{r, S} S_r^{-1} = \left[ \mathbb{E}[\ell''_{\tau}(x - \alpha_r) x_S x_S^\top] S_r^{-1} \right],
$$

and for any $\delta \in \mathbb{R}^d$ with $\delta_{S^c} = 0$,

$$
\|\Sigma_{r, S} \delta_{S}\|_{\infty} = \|\Sigma_{r} \delta\|_{\infty} \leq \|\Sigma_{r} \delta\|_{2} \leq \lambda_{u}^{1/2} \|\Sigma_{r}^{1/2} \delta\|_{2} = \lambda_{u}^{1/2} \|S_r^{-1/2} \delta_{S}\|_{2}.
$$

Consequently, combining inequality (106) with the probabilistic bounds (107) and (108), we conclude that with probability at least $1 - 2n^{-1}$,

$$
\|\mathbb{E}[\nabla L_r(\tilde{\beta}^\text{ora})]\|_{\infty} \lesssim \max_{j \in S} \left[ \mathbb{E}[\ell''_{\tau}(x - \alpha_r) x_S x_j] \right] \|\sigma_x \sqrt{\frac{\log(n)}{n}} + \sigma_e \frac{s + \log(n)}{n}.
$$

In conjunction with inequalities (101), (104) and Proposition 8, (II), the claim follows. \hfill \Box

### C.4 Proof of Theorem 4

For simplicity, we write $\beta^{(k)} = \beta^{(1, k)}$, $\phi^{(k)} = \phi^{(1, k)}$ and $\lambda = \lambda^{(0)}$ throughout this section.

#### C.4.1 Technical lemmas

We first present three technical lemmas, which are key ingredients of the proof. The first lemma provides an alternative to the stopping rule.

**Lemma 9.** $\omega_\lambda(\beta^{(k)}) \leq \rho_e(1 + \gamma_{\mu}) \|\beta^{(k)} - \beta^{(k-1)}\|_2$. 

Proof of Lemma 9. For simplicity, we omit the subscript $\tau$ in $L_\tau(\beta)$. Since $\beta^{(k)}$ is the exact solution at the $k$-th iteration when $\ell = 1$, the first-order optimality condition holds: there exists some $\xi^{(k)} \in \partial \|\beta^{(k)}\|_1$ such that
\[
\nabla L(\beta^{(k-1)}) + \phi^{(k)}(\beta^{(k)} - \beta^{(k-1)}) + \bar{\lambda} \circ \xi^{(k)} = 0_{d+1},
\]
where $\xi^{(k)} = (0, \xi^{(k)\top})$ and $\bar{\lambda} = (0, \lambda^\top)$. For any $u$ such that $\|u\|_1 = 1$, we have
\[
\langle \nabla L(\beta^{(k)}), u \rangle + \bar{\lambda} \circ \xi^{(k)} + \langle \nabla L(\beta^{(k-1)}), u \rangle + \phi^{(k)}(\beta^{(k)} - \beta^{(k-1)}), u \rangle
= \langle \nabla L(\beta^{(k)}) - \nabla L(\beta^{(k-1)}), u \rangle - \phi^{(k)}(\beta^{(k)} - \beta^{(k-1)}), u \rangle
\leq \|\nabla L(\beta^{(k)}) - \nabla L(\beta^{(k-1)})\|_\infty + \|\phi^{(k)}\| \|\beta^{(k)} - \beta^{(k-1)}\|_\infty
\leq (\phi^{(k)} + \rho_c) \|\beta^{(k)} - \beta^{(k-1)}\|_2,
\]
where the last inequality is due to the Lipschitz continuity of $\nabla L(\cdot)$. Taking the supremum over all $u$ satisfying $\|u\|_1 \leq 1$, we obtain
\[
\omega(\beta^{(k)}) \leq (\phi^{(k)} + \rho_c) \|\beta^{(k)} - \beta^{(k-1)}\|_2.
\]
It remains to show that $\phi^{(k)} \leq \gamma_d \rho_c$ for any $k$. This is guaranteed by the iterative LAMM algorithm. Otherwise, if $\phi^{(k)} > \gamma_d \rho_c$, then $\phi' \equiv \phi^{(k)}/\gamma_d > \rho_c$ is the quadratic parameter in the previous iteration for searching $\phi$ such that
\[
F(\tilde{\beta}^{(k)}; \phi', \beta^{(k-1)}) < L(\tilde{\beta}^{(k)}),
\]
where $\tilde{\beta}^{(k)}$ is the new updated parameter vector under the quadratic coefficient $\phi'$. On the other hand, it follows from the definition of $F$ and the Lipschitz continuity of $\nabla L$ that
\[
\begin{align*}
F(\tilde{\beta}^{(k)}; \phi', \beta^{(k-1)}) + \lambda \|\beta\|_1 &= L(\beta^{(k-1)}) + \langle \nabla L(\beta^{(k-1)}), \tilde{\beta}^{(k)} - \beta^{(k-1)} \rangle + \phi' \|\tilde{\beta}^{(k)} - \beta^{(k-1)}\|_2^2 \\
&> L(\beta^{(k-1)}) + \langle \nabla L(\beta^{(k-1)}), \tilde{\beta}^{(k)} - \beta^{(k-1)} \rangle + \rho_c \|\tilde{\beta}^{(k)} - \beta^{(k-1)}\|_2^2 \\
&\geq L(\tilde{\beta}^{(k)}),
\end{align*}
\]
This leads to a contradiction, indicating that $\phi^{(k)} \leq \gamma_d \rho_c$. \hfill \Box

The second lemma is a modified version of Lemma E.4 in Fan et al. (2018). We reproduce its proof for completeness. Let $\Psi(\beta, \lambda) = L(\beta) + \|\lambda \circ \beta\|_1$ with $\lambda = \lambda^{(0)}$.

Lemma 10. For any $\beta \in \mathbb{R}^{d+1}$, we have
\[
\Psi(\beta, \lambda) - \Psi(\beta^{(k)}, \lambda) \geq \frac{\phi^{(k)}}{2}\|\beta - \beta^{(k)}\|_2^2 - \|\beta - \beta^{(k-1)}\|_2^2.
\]

Proof of Lemma 10. Because $F(\beta; \phi^{(k)}, \beta^{(k-1)})$ majorizes $L(\beta)$ at $\beta^{(k)}$, we have
\[
\Psi(\beta, \lambda) - \Psi(\beta^{(k)}, \lambda) \geq \Psi(\beta, \lambda) - [F(\beta^{(k)}; \phi^{(k)}, \beta^{(k-1)}) + \|\lambda \circ \beta^{(k)}\|_1]. \tag{109}
\]
By the convexity of $L(\beta)$ and $\|\lambda \circ \beta\|_1$,
\[
\begin{align*}
L(\beta) &\geq L(\beta^{(k-1)}) + \langle \nabla L(\beta^{(k-1)}), \beta - \beta^{(k-1)} \rangle \\
\text{and } \|\lambda \circ \beta\|_1 &\geq \|\lambda \circ \beta^{(k)}\|_1 + \langle \lambda \circ \xi^{(k)}, \beta - \beta^{(k)} \rangle
\end{align*}
\]
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for any $\xi^{(k)} \in \partial||\beta^{(k)}||_1$. Together, (110) and (111) imply
\[
\Psi(\beta, \lambda) \geq \mathcal{L}(\beta^{(k-1)}) + \langle \nabla \mathcal{L}(\beta^{(k-1)}), \beta - \beta^{(k-1)} \rangle + \|\lambda \circ \beta^{(k)}\|_1 + \langle \lambda \circ \xi^{(k)}, \beta - \beta^{(k)} \rangle.
\] (112)

Plugging the expression of $F(\beta^{(k)}; \phi^{(k)}, \beta^{(k-1)})$ in (36) and (112) into (109), we obtain
\[
\Psi(\beta, \lambda) - \Psi(\beta^{(k)}, \lambda) \geq -\frac{\phi_1}{2}\|\beta^{(0)} - \beta^{(1)}\|^2 + \langle \nabla \mathcal{L}(\beta^{(k-1)}), \beta - \beta^{(k)} \rangle + \langle \lambda \circ \xi^{(k)}, \beta - \beta^{(k)} \rangle.
\] (113)

By the first-order optimality condition, there exists some $\xi_\ast \in \partial||\beta^{(k)}||_1$ such that
\[
\nabla \mathcal{L}(\beta^{(k-1)}) + \phi^{(k)}(\beta^{(k)} - \beta^{(k-1)}) + \lambda \circ \xi^{(k)} = 0,
\]
where $\xi^{(k)} = (0, \xi^\top)$. Substituting this into (113), and by direct calculations, we arrive at the conclusion.

Recall that $\Psi(\beta, \lambda) = \mathcal{L}(\beta) + \|\lambda \circ \beta\|_1$ and $\bar{\beta}^{(1)} \in \min_\beta \Psi(\beta, \lambda)$ denotes the optimal solution at the contraction stage.

**Lemma 11.** For any $k \geq 1$, we have
\[
\Psi(\beta^{(k)}, \lambda) - \Psi(\bar{\beta}^{(1)}, \lambda) \leq \frac{\max_{1 \leq j \leq k} \phi^{(j)}}{2k} \|\beta^{(0)} - \bar{\beta}^{(1)}\|^2.
\]

**Proof of Lemma 11.** For simplicity, we write $\bar{\beta} = \bar{\beta}^{(1)}$, and define $\phi_{\max} = \max_{1 \leq j \leq k} \phi^{(j)}$ and $\phi_{\min} = \min_{1 \leq j \leq k} \phi^{(j)} > 0$. Taking $\beta = \bar{\beta}$ in Lemma 10 gives
\[
0 \geq \Psi(\bar{\beta}, \lambda) - \Psi(\beta^{(j)}, \lambda) \geq \frac{\phi^{(j)}}{2} \|\|\bar{\beta} - \beta^{(j)}\|^2 - \||\beta^{(j-1)} - \bar{\beta}\|^2\|
\]
for all $j \geq 1$. Summing over $j$ from 1 to $k$ yields
\[
\sum_{j=1}^k \frac{2}{\phi^{(j)}} \left(\Psi(\bar{\beta}, \lambda) - \Psi(\beta^{(j)}, \lambda)\right) \geq \sum_{j=1}^k \left(\|\beta^{(j)} - \bar{\beta}\|^2 - \||\beta^{(j-1)} - \bar{\beta}\|^2\right),
\]
which further implies
\[
\frac{2}{\phi_{\max}} \left(\sum_{j=1}^k \Psi(\beta^{(j)}, \lambda) - \Psi(\bar{\beta}, \lambda)\right) \geq \|\beta^{(k)} - \bar{\beta}\|^2 - \|\beta^{(0)} - \bar{\beta}\|^2.
\] (114)

Again, by Lemma 10 with $\beta = \beta^{(j-1)}$ and $k = j$,
\[
\Psi(\beta^{(j-1)}, \lambda) - \Psi(\beta^{(j)}, \lambda) \geq \frac{\phi^{(j)}}{2} \|\beta^{(j)} - \beta^{(j-1)}\|^2 \geq \frac{\phi_{\min}}{2} \|\beta^{(j)} - \beta^{(j-1)}\|^2.
\]

Multiplying both sides of the above inequality by $j - 1$ and summing over $j$, we obtain
\[
\frac{2}{\phi_{\min}} \sum_{j=1}^k (j - 1) \Psi(\beta^{(j-1)}, \lambda) - \Psi(\beta^{(j)}, \lambda) + \Psi(\beta^{(j)}, \lambda) \geq \sum_{j=1}^k (j - 1) \|\beta^{(j)} - \beta^{(j-1)}\|^2.
\]
or equivalently,
\[
\frac{2}{\phi_{\min}} \left\{ -k\Psi(\beta^{(k)}, \lambda) + \sum_{j=1}^{k} \Psi(\beta^{(j)}, \lambda) \right\} \geq \sum_{j=1}^{k} (j-1)\|\beta^{(j)} - \beta^{(j-1)}\|_2^2.
\] (115)

Putting (114) and (115) together to reach
\[
\frac{2k}{\phi_{\min}} \{\Psi(\lambda) - \Psi(\beta^{(k)}, \lambda)\} \geq \frac{\phi_{\max}}{\phi_{\min}} \|\beta^{(k)} - \lambda\|_2^2 + \sum_{j=1}^{k} (j-1)\|\beta^{(j)} - \beta^{(j-1)}\|_2^2 - \frac{\phi_{\max}}{\phi_{\min}} \|\beta^{(0)} - \lambda\|_2^2,
\]
from which it follows immediately that
\[
\frac{2k}{\phi_{\max}} \{\Psi(\beta^{(k)}, \lambda) - \Psi(\lambda)\} \leq \|\beta^{(0)} - \lambda\|_2^2.
\]
This completes the proof. \(\square\)

### C.4.2 Proof of the theorem

Recall that \(\beta^{(k)} = \beta^{(1,k)}\) and \(\phi^{(k)} = \phi^{(1,k)}\). By Lemma 9 and its proof,
\[
\omega_\chi(\beta^{(k)}) \leq (\phi^{(k)} + \rho_\varepsilon)\|\beta^{(k)} - \beta^{(k-1)}\|_2 \leq \rho_\varepsilon (1 + \gamma_u)\|\beta^{(k)} - \beta^{(k-1)}\|_2.
\]
Next, taking \(\beta = \beta^{(k-1)}\) in Lemma 10 yields
\[
\Psi(\beta^{(k-1)}, \lambda) - \Psi(\beta^{(k)}, \lambda) \geq \frac{\phi^{(k)}}{2} \|\beta^{(k-1)} - \beta^{(k)}\|_2^2.
\]
Together, the last two displays lead to a bound for the suboptimality measure
\[
\omega_\chi(\beta^{(k)}) \leq \rho_\varepsilon (1 + \gamma_u) \left[ \frac{2}{\phi^{(k)}} \{\Psi(\beta^{(k-1)}, \lambda) - \Psi(\lambda)\} \right]^{1/2}.
\] (116)

Recall that \(\{\Psi(\beta^{(k)}, \lambda)\}_{k=0}^\infty\) is a non-increasing sequence, i.e.
\[
\Psi(\beta^{(1)}, \lambda) \leq \cdots \leq \Psi(\beta^{(k)}, \lambda) \leq \cdots \leq \Psi(\beta^{(0)}, \lambda).
\]
Then it follows from (116) and Lemma 11 that
\[
\omega_\chi(\beta^{(k)}) \leq \rho_\varepsilon (1 + \gamma_u) \left[ \frac{2}{\phi^{(k)}} \{\Psi(\beta^{(k-1)}, \lambda) - \Psi(\lambda)\} \right]^{1/2}
\leq \frac{\rho_\varepsilon (1 + \gamma_u)}{\sqrt{k}} \varepsilon \sqrt{\max_{1 \leq j \leq k-1} \phi^{(j)}} \|\beta\|_2,
\]
where we used the fact that \(\beta^{(0)} = 0\). By the triangle inequality,
\[
\omega_\chi(\beta^{(k)}) \leq \frac{\rho_\varepsilon (1 + \gamma_u)}{\sqrt{k}} (\|\beta^*\|_2 + \|\beta - \beta^*\|_2).
\]
Therefore, in the contraction stage, we need
\[
k \geq (\rho_\varepsilon (1 + \gamma_u)(\|\beta^*\|_2 + \|\beta - \beta^*\|_2)/\varepsilon_c)^2
\]
to ensure
\[
\omega_\chi(\beta^{(k)}) \leq \varepsilon_c.
\]
This proves the stated result. \(\square\)
C.5 Proof of Theorem 5

For convenience, we omit the index \( \ell \), and use \( \hat{\beta}, \beta^{(k)}, \lambda \) and \( E \) to denote \( \hat{\beta}^{(\ell)}, \beta^{(k,\ell)}, \lambda^{(\ell-1)} \) and \( E_{\ell} \), respectively. Moreover, write \( L(\beta) = L_{E}(\beta) \), and define \( \Psi(\beta, \lambda) = L(\beta) + \|\lambda \circ \beta_{-}\|_1 = L(\beta) + \|\lambda^{(\ell-1)} \circ \beta_{-}\|_1 \) so that \( \hat{\beta} \in \min_{\beta} \Psi(\beta, \lambda) \).

C.5.1 Technical lemmas

We first provide several technical lemmas along with their proofs. Recall the sparse cone \( \mathbb{C}(m, r, \tau) \) given in (40).

Lemma 12. For any \( \beta_{1}, \beta_{2} \in \mathbb{C}(m/2, r, \tau) \cap \beta^{*} + \mathbb{B}_{2}(r) \), we have
\[
\frac{1}{2} \rho_{-}(m, r, \tau) \|\beta_{1} - \beta_{2}\|_{2}^{2} \leq D_{L}(\beta_{1}, \beta_{2}) \leq \frac{1}{2} \rho_{+}(m, r, \tau) \|\beta_{1} - \beta_{2}\|_{2}^{2},
\]
where \( D_{L}(\beta_{1}, \beta_{2}) := L(\beta_{1}) - L(\beta_{2}) - \langle \nabla L(\beta_{2}), \beta_{1} - \beta_{2} \rangle \).

Proof of Lemma 12. By a second-order Taylor series expansion, there exists some \( \gamma \in [0, 1] \) such that \( \hat{\beta} = \gamma \beta_{1} + (1 - \gamma) \beta_{2} \in \beta^{*} + \mathbb{B}_{2}(r) \) and \( D_{L}(\beta_{1}, \beta_{2}) = (1/2)(\beta_{1} - \beta_{2})^{\top} \nabla^{2} L(\hat{\beta})(\beta_{1} - \beta_{2}) \). The stated bounds then follow directly from Definition 3. \( \square \)

The next lemma converts the bound on \( \Psi(\beta, \lambda) - \Psi(\beta^{*}, \lambda) \) to that on \( \|\beta - \beta^{*}\|_{2} \). Recall that for any subset \( E \subseteq [d] \), we write \( \beta_{E} = (\beta_{0}, \beta_{E, r}^{\tau}) \).

Lemma 13. Assume Condition 5 holds. Let \( E \subseteq [d] \) be a subset satisfying \( S \subseteq E \) and \( |E| \leq 2s \). Assume further that \( \lambda \geq \max \{4\|\nabla L(\beta^{*})\|_{\infty}, \|\lambda\|_{\infty} \} \) and \( \|\lambda_{E^{c}}\|_{\min} \geq \lambda/2 \). Then, for any \( \beta \in \beta^{*} + \mathbb{B}_{2}(r) \) satisfying \( \|\beta_{S}\|_{0} \leq \bar{s} \) and \( \Psi(\beta, \lambda) - \Psi(\beta^{*}, \lambda) \leq C \lambda^{2} s \), we have
\[
\|\beta - \beta^{*}\|_{2} \leq C_{1} \lambda \sqrt{s} \quad \text{and} \quad \|\beta - \beta^{*}\|_{1} \leq C_{2} \lambda s,
\]
where \( C_{1}, C_{2} > 0 \) depend only on \( C \) and localized sparse eigenvalues.

Proof of Lemma 13. We omit the arguments in \( \rho_{-}(m, r, \tau) \) and \( \rho_{+}(m, r, \tau) \) whenever there is no ambiguity. Using Lemma 12, we immediately obtain
\[
L(\beta^{*}) + \langle \nabla L(\beta^{*}), \beta - \beta^{*} \rangle + \frac{\rho_{-}}{2} \|\beta - \beta^{*}\|_{2}^{2} \leq L(\beta).
\]

Because \( \Psi(\beta) - \Psi(\beta^{*}) \leq C \lambda^{2} s \), or equivalently,
\[
L(\beta) - L(\beta^{*}) + (\|\lambda \circ \beta_{-}\|_{1} - \|\lambda \circ \beta_{-}^{*}\|_{1}) \leq C \lambda^{2} s,
\]
it holds
\[
\frac{\rho_{-}}{2} \|\beta - \beta^{*}\|_{2}^{2} \leq C \lambda^{2} s - (\nabla L(\beta^{*}), \beta - \beta^{*}) + (\|\lambda \circ \beta_{-}^{*}\|_{1} - \|\lambda \circ \beta_{-}\|_{1}).
\]

After some simple algebra, it can be derived that
\[
\begin{align*}
\|I\| & \leq \max \{4\|\nabla L(\beta^{*})\|_{\infty}, \|\lambda\|_{\infty} \} \|\nabla L(\beta^{*})\|_{\infty} + \|\beta - \beta^{*}\|_{E} \|\nabla L(\beta^{*})\|_{\infty}, \\
\|II\| & \leq \lambda \|\nabla L(\beta^{*})\|_{1} - (\lambda/2) \|\beta - \beta^{*}\|_{E} \|\nabla L(\beta^{*})\|_{1}.
\end{align*}
\]
Combining the above bounds
\[
\frac{\rho_-}{2} \|\beta - \beta^*\|_2^2 + (\lambda/2 - \|\nabla L(\beta^*)\|_\infty)\|(\beta - \beta^*)_E\|_1 \\ \leq \lambda + \|\nabla L(\beta^*)\|_\infty\|(\beta - \beta^*)_E\|_1 + C\lambda^2 s.
\]
which further implies
\[
\frac{\rho_-}{2} \|\beta - \beta^*\|_2^2 \leq \frac{5\lambda}{4}\|(\beta - \beta^*)_E\|_1 + C\lambda^2 s.
\]
To further bound the right-hand side of the above inequality, we discuss two cases regarding the magnitude of \(\|(\beta - \beta^*)_E\|_1\), comparing with \(\lambda s\):

- If \(5\lambda\|(\beta - \beta^*)_E\|_1/4 \leq C\lambda^2 s\), we have
  \[
  \frac{\rho_-}{2} \|\beta - \beta^*\|_2^2 \leq 2C\lambda^2 s, \quad \text{and thus } \|\beta - \beta^*\|_2 \leq 2\sqrt{\frac{C}{\rho_-} \lambda \sqrt{s}}. \quad (118)
  \]

- If \(5\lambda\|(\beta - \beta^*)_E\|_1/4 > C\lambda^2 s\), we have
  \[
  \frac{\rho_-}{2} \|\beta - \beta^*\|_2^2 \leq \frac{5}{2}\lambda\|(\beta - \beta^*)_E\|_1 \leq \frac{5}{2}\lambda(2s + 1)^{1/2}\|\beta - \beta^*\|_2,
  \]
  which further yields
  \[
  \|\beta - \beta^*\|_2 \leq \frac{5 \sqrt{3}}{\rho_-} \lambda \sqrt{s}. \quad (119)
  \]
Combining (118) and (119), we obtain
\[
\|\beta - \beta^*\|_2 \leq \max \left\{ 2\sqrt{\frac{C}{\rho_-} \frac{5\sqrt{3}}{\rho_-}} \lambda \sqrt{s} \right\} \lambda \sqrt{s} \leq \lambda \sqrt{s}.
\]
Since \(\beta - \beta^*\) is at most \((s + 1 + \bar{s})\)-sparse, \(\|\beta - \beta^*\|_1 \leq (s + 1 + \bar{s})^{1/2}\|\beta - \beta^*\|_2\). The stated results then follow immediately.

**Lemma 14.** Assume Condition 5 holds and \(4\|\nabla L(\beta^*)\|_\infty + \epsilon c \vee \epsilon I\|_\epsilon \leq \lambda \leq r/\sqrt{3}\). For any \(\ell \geq 2\), the solution sequence \([\beta^{(\ell,k)}]_{k \geq 0}\) satisfies
\[
\|(\beta^{(\ell,k)})_E\|_0 \leq \bar{s}, \quad \|\beta^{(\ell,k)} - \beta^*\|_2 \leq C_1\lambda \sqrt{s} \quad \text{and} \quad \|\beta^{(\ell,k)} - \beta^*\|_1 \leq C_2\lambda s, \quad (120)
\]
where \(C_1, C_2 > 0\) are constants depending only on the localized sparse eigenvalues.

**Proof of Lemma 14.** We prove the theorem by the method of induction on \((\ell, k)\). Throughout, \(C\) denotes a constant independent of \((n, d, s)\) and may take different values at each appearance. For the 1st subproblem, directly applying Proposition 4.1 and Lemma 5.4 in Fan et al. (2018) we obtain that \(\|\tilde{\beta}^{(1)} - \beta\|_2 \leq C\rho_r^{-1}\lambda \sqrt{s} < r\), \(\|\tilde{\beta}^{(1)} - \beta\|_1 \leq C\rho_r^{-1}\lambda s\) and \(\tilde{\beta}^{(1)}\) is \((s + 1 + \bar{s})\)-sparse, where \(\bar{s} \leq C s\). It follows that \(\beta^{(2,0)} = \tilde{\beta}^{(1)}\) falls in a localized sparse set.

To apply the method of induction, first we assume that for any \(k\), \(\beta^{(2,k)}\) falls in a localized sparse set such that (120) holds. We then use Lemma E.13 in Fan et al. (2018) to show that \(\beta^{(2,k+1)}\)
also falls in a localized sparse set. To this end, we need to verify two conditions. The first one, $\|\mathbf{X}_0\|_{\infty} \geq \lambda / 2$ is guaranteed by Claim (56) in the proof of Proposition 3, when taking $a = 1 / 2$ therein. For the second condition, it suffices to show

$$
\Psi(\beta^{(2,k)}, \lambda^{(1)}) - \Psi(\beta^*, \lambda^{(1)}) \leq (1 + \xi)\rho_s^{-1}\lambda^2 s,
$$

where $\xi = \rho^*/\rho_s$. Using the mean value theorem, there exists some convex combination of $\beta^{(2,k)}$ and $\beta^*$, say $\beta$, such that

$$
\Psi(\beta^{(2,k)}, \lambda^{(1)}) - \Psi(\beta^*, \lambda^{(1)}) = L(\beta^{(2,k)}) - L(\beta^*) + ||\lambda^{(1)} \circ \beta^{(2,k)}||_1 - ||\lambda^{(1)} \circ \beta^*||_1
$$

$$
\leq \langle \nabla L(\beta^*), \beta^{(2,k)} - \beta^* \rangle + \frac{1}{2}(\beta^{(2,k)} - \beta^*)^\top \nabla^2 L(\beta)(\beta^{(2,k)} - \beta^*) + ||\lambda^{(1)} \circ (\beta^{(2,k)} - \beta^*)||_1
$$

$$
\leq ||\nabla L(\beta^*)||_{\infty}||\beta^{(2,k)} - \beta^*||_1 + \frac{1}{2}\rho^* ||\beta^{(2,k)} - \beta^*||_2^2 + \lambda ||\beta^{(2,k)} - \beta^*||_1
$$

$$
\leq C_4 \rho_s^{-1}\lambda^2 s + C_2^2 \rho^* \rho_s^{-1} s + C_2 \rho_s^{-1}\lambda^2 s \leq (1 + \xi)\rho_s^{-1}\lambda^2 s.
$$

With above preparations, it follows from Lemma E.13 in Fan et al. (2018) with slight modification that $||\beta^{(2,k+1)}||_0 \leq s + 1 + \tilde{s}$. Next, we show that $||\beta^{(2,k+1)} - \beta^*||_2 \leq \rho_s^{-1}\lambda \sqrt{s}$. Again, by Lemma 10,

$$
\Psi(\beta^{(2,k+1)}, \lambda^{(1)}) - \Psi(\beta^{(2,k)}, \lambda^{(1)}) \leq -\frac{\rho^{(2,k+1)}}{2} ||\beta^{(2,k+1)} - \beta^{(2,k)}||_2.
$$

This implies that $||\Psi(\beta^{(2,k)}, \lambda^{(1)}) - \Psi(\beta^*, \lambda^{(1)})||_{\infty} \geq 1$ is a non-increasing sequence. By induction, it follows that

$$
\Psi(\beta^{(2,k+1)}, \lambda^{(1)}) - \Psi(\beta^*, \lambda^{(1)}) \leq \Psi(\beta^{(2,k)}, \lambda^{(1)}) - \Psi(\beta^*, \lambda^{(1)}) \leq (1 + \xi)\rho_s^{-1}\lambda^2 s.
$$

Combining this with Lemma 13 gives the desired bounds on $||\beta^{(2,k+1)} - \beta^*||_2$ and $||\beta^{(2,k+1)} - \beta^*||_1$.

Finally, by an argument similar to that in the proof of Lemma 5.4 in Fan et al. (2018), we can derive the stated results for all $\ell \geq 3$.

For $\epsilon > 0$, let $\tilde{\beta} = (\tilde{\beta}_0, \tilde{\beta}_1)^\top$ be an $\epsilon$-optimal solution to the program $\min_{\beta} L(\beta) + ||\lambda \circ \beta||_1$. The following lemma provides conditions under which $\tilde{\beta}$ falls in an $\ell_1$-cone.

**Lemma 15.** Let $\mathcal{E} \subseteq [d]$ be a subset satisfying $\mathcal{S} \subseteq \mathcal{E}$, and assume $\lambda \geq ||\lambda||_{\infty} \vee 4(||\nabla L(\beta^*)||_{\infty} + \epsilon)$ and $||\lambda\phi||_{\min} \geq \lambda / 2$. Then, any $\epsilon$-optimal solution $\tilde{\beta}$ satisfies the cone constraint

$$
||\tilde{\beta} - \beta^*||_{\mathcal{E}} \leq \frac{||\lambda||_{\infty} + ||\nabla L(\beta^*)||_{\infty} + \epsilon}{||\lambda||_{\min} - ||\nabla L(\beta^*)||_{\infty} - \epsilon} ||(\tilde{\beta} - \beta^*)||_{\mathcal{E}} \leq 5(||\tilde{\beta} - \beta^*||_{\mathcal{E}}||_1.
$$

**Proof of Lemma 15.** For any $\xi = (0, \xi^\top)$ with $\xi \in \partial ||\tilde{\beta}||_1$, let $u = \nabla L(\tilde{\beta}) + \lambda \circ \xi$ where $\lambda = (0, \lambda^\top)$. By the convexity of $L$, $\langle \nabla L(\tilde{\beta}) - \nabla L(\beta^*), \tilde{\beta} - \beta^* \rangle \geq 0$. This, together with the inequality $\langle \nabla L(\tilde{\beta}) + \lambda \circ \xi, \tilde{\beta} - \beta^* \rangle \leq ||u||_{\infty} ||\tilde{\beta} - \beta^*||_1$, implies

$$
0 \leq ||u||_{\infty} ||\tilde{\beta} - \beta^*||_1 - \langle \nabla L(\beta^*), \tilde{\beta} - \beta^* \rangle - \langle \lambda \circ \xi, \tilde{\beta} - \beta^* \rangle.
$$

(121)
For I and II, note that 1 \geq -||\nabla L(\beta')||_\infty ||\bar{\beta} - \beta||_1, and

\[ II = \langle \lambda \circ \xi, \bar{\beta}_- - \beta^* \rangle = \langle \lambda \circ \xi, (\bar{\beta} - \beta^*)_\xi \rangle + \langle (\lambda \circ \xi, (\bar{\beta} - \beta^*)_\xi \rangle \geq ||\lambda E||_{\text{min}} ||(\bar{\beta} - \beta^*)_\xi||_1 - ||\lambda E||_{\text{inf}} ||(\bar{\beta} - \beta^*)_\xi||_1. \]

Substituting the above bounds into (121) and taking the infimum over \( \xi \in \partial ||\bar{\beta}||_1 \) yields

\[ 0 \leq -[||\lambda E||_{\text{min}} - ||\|\nabla L(\beta')\|_\infty + \omega \lambda(\bar{\beta})||\|\bar{\beta} - \beta^*\|_\xi]_1 + [||\lambda E||_{\text{inf}} + ||\nabla L(\beta')\|_\infty + \omega \lambda(\bar{\beta})||\|\bar{\beta} - \beta^*\|_\xi]_1, \]

or equivalently,

\[ ||(\bar{\beta} - \beta^*)_\xi||_1 \leq \frac{||\lambda||_{\text{inf}} + ||\|\nabla L(\beta')\|_\infty + \omega \lambda(\bar{\beta})||\|\bar{\beta} - \beta^*\|_\xi}{||\lambda E||_{\text{min}} - (||\nabla L(\beta')\|_\infty + \omega \lambda(\bar{\beta}))} ||(\bar{\beta} - \beta^*)_\xi||_1. \]

This leads to the stated result. \( \Box \)

**C.5.2 Proof of the theorem**

Restricting our attention to the \( \ell \)-th subproblem, we write \( \phi^{(k)} = \phi^{(\ell,k)} \) for simplicity. Define the subset \( S = \{\alpha \bar{\beta} + (1 - \alpha)\bar{\beta}^{(k-1)} : 0 \leq \alpha \leq 1\} \). Due to local majorization, we have

\[ \Psi(\beta^{(k)}, \lambda) \leq \min_{\beta \in S} \left\{ \mathcal{L}(\beta^{(k-1)}) + \langle \nabla \mathcal{L}(\beta^{(k-1)}), \beta - \beta^{(k-1)} \rangle + \frac{\phi^{(k)}}{2} ||\beta - \beta^{(k-1)}||_2^2 + ||\lambda \circ \beta_-||_1 \right\} \]

where we used the convexity of \( \mathcal{L}(\beta) \) in the second inequality. Since \( \Psi(\beta, \lambda) = \mathcal{L}(\beta) + ||\lambda \circ \beta_-||_1 \) is minimized at \( \bar{\beta} \), by convexity we have

\[ \Psi(\beta^{(k)}, \lambda) \leq \min_{\beta \in S} \left\{ \Psi(\beta, \lambda) + \frac{\phi^{(k)}}{2} ||\beta - \beta^{(k-1)}||_2^2 \right\} \]

\[ \leq \min_{0 \leq \alpha \leq 1} \left\{ \alpha \Psi(\bar{\beta}, \lambda) + (1 - \alpha)\Psi(\beta^{(k-1)}, \lambda) + \frac{\alpha^2 \phi^{(k)}}{2} ||\beta^{(k-1)} - \bar{\beta}||_2^2 \right\} \]

\[ = \min_{0 \leq \alpha \leq 1} \left\{ \Psi(\beta^{(k-1)}, \lambda) - \alpha(\Psi(\beta^{(k-1)}, \lambda) - \Psi(\bar{\beta}, \lambda)) + \frac{\alpha^2 \phi^{(k)}}{2} ||\beta^{(k-1)} - \bar{\beta}||_2^2 \right\}. \tag{122} \]

Next, we bound the right-hand side of (122). By Lemma 14,

\[ ||(\beta^{(k-1)} - s)||_0 \leq \bar{s}, \quad ||(\beta^{(k-1)} - \beta^*)||_2 \leq \alpha \sqrt{s} \leq r \quad \text{and} \quad ||(\beta^{(k-1)} - \beta^*)||_2 \leq \alpha s. \]

Similarly, it can be shown the the optimum \( \bar{\beta} \) satisfies the same properties. Hence,

\[ \beta^{(k)}, \bar{\beta} \in C(s + \bar{s} + 1, r, r, \tau) \cap \beta^* + E_2(r). \]

By the first-order optimality condition, there exists some \( \bar{\xi}_- \in \partial ||\bar{\beta}||_1 \) such that \( \nabla \mathcal{L}(\bar{\beta}) + \bar{\lambda} \circ \bar{\xi} = 0 \), where \( \bar{\lambda} = (0, \lambda^T)^T \) and \( \bar{\xi} = (0, \bar{\xi}^T)^T \). Moreover, define \( D_L(\beta_1, \beta_2) = \mathcal{L}(\beta_1) - \mathcal{L}(\beta_2) - \langle \nabla \mathcal{L}(\beta_2), \beta_1 - \beta_2 \rangle \)
\( \beta_2 \). Using Definition 3, Lemma 12, and the convexity of \( L \) and \( \ell_1 \)-norm, \( \Psi(\beta^{(k-1)}, \lambda) - \Psi(\tilde{\beta}, \lambda) \) can be bounded as
\[
\Psi(\beta^{(k-1)}, \lambda) - \Psi(\tilde{\beta}, \lambda) 
\geq (\nabla L(\tilde{\beta}) + \lambda \circ \xi, \beta^{(k-1)} - \tilde{\beta}) + D_L(\beta^{(k-1)}, \tilde{\beta}) \geq \frac{\rho_-}{2}\|\beta^{(k-1)} - \tilde{\beta}\|_2^2,
\]
where \( \rho_- = \rho_-(2s + 2\delta + 2, r, \tau) \). Plugging this bound into (122) yields
\[
\Psi(\beta^{(k)}, \lambda) 
\leq \min_{0 \leq \alpha \leq 1} \left[ \Psi(\beta^{(k-1)}, \lambda) - \alpha[\Psi(\beta^{(k-1)}, \lambda) - \Psi(\tilde{\beta}, \lambda)] + \frac{\alpha^2 \phi^{(k)}}{\rho_-}\{\Psi(\beta^{(k-1)}, \lambda) - \Psi(\tilde{\beta}, \lambda)\} \right]
\leq \Psi(\beta^{(k-1)}, \lambda) - \frac{\rho_-}{4\phi^{(k)}}\{\Psi(\beta^{(k-1)}, \lambda) - \Psi(\tilde{\beta}, \lambda)\}.
\]
Following the proof of Lemma 9, it can be similarly shown that \( \phi^{(k)} \leq \gamma_u \rho^* \) under Condition 5. Consequently,
\[
\Psi(\beta^{(k)}, \lambda) - \Psi(\tilde{\beta}, \lambda) \leq \left( 1 - \frac{1}{4\gamma_u \xi} \right)^k \|\Psi(\beta^{(0)}, \lambda) - \Psi(\tilde{\beta}, \lambda)\|.
\]
where \( \xi = \rho^*/\rho_+ \).

By an argument similar to that in the proof of Lemma 9, we can show that, for \( \ell \geq 2 \),
\[
\omega_{\lambda^{(\ell-1)}}(\beta^{(\ell,k)}) \leq \rho^*(1 + \gamma_u)\|\beta^{(\ell,k)} - \beta^{(\ell,k-1)}\|_2.
\]
Further, using Lemma 10 to bound \( \|\beta^{(\ell,k)} - \beta^{(\ell,k-1)}\|_2 \) from above and noting that \( \phi^{(k)} \geq \rho_+ \), we obtain
\[
\omega_{\lambda^{(\ell-1)}}(\beta^{(\ell,k)}) 
\leq (1 + \gamma_u)\rho^* \sqrt{2/[\rho_+]} \{\Psi(\beta^{(\ell,k-1)}, \lambda^{(\ell-1)}) - \Psi(\beta^{(\ell,k)}, \lambda^{(\ell-1)})\}
\leq (1 + \gamma_u) \sqrt{2\xi \rho^*} \{\Psi(\beta^{(\ell,k-1)}, \lambda^{(\ell-1)}) - \Psi(\tilde{\beta}^{(\ell)}, \lambda^{(\ell-1)})\}
\leq (1 + \gamma_u) \sqrt{2\xi \rho^*} \left( 1 - \frac{1}{4\gamma_u \xi} \right)^{k-1} \{\Psi(\beta^{(\ell,0)}, \lambda^{(\ell-1)}) - \Psi(\tilde{\beta}^{(\ell)}, \lambda^{(\ell-1)})\}
\leq C(1 + \gamma_u) \sqrt{\xi \rho^*} \left( 1 - \frac{1}{4\gamma_u \xi} \right)^{k-1} \lambda^2 s \leq C(1 + \gamma_u) \xi \sqrt{\left( 1 - \frac{1}{4\gamma_u \xi} \right)^{k-1}} \lambda^2 s,
\]
where we used Lemmas 10 and 14 in the last step.

To make the right-hand side of the above inequality smaller than \( \epsilon_t \), we need \( k \) to be sufficiently large that \( k \geq C_1 \log(C_2 \lambda \sqrt{s}/\epsilon_t) \), where \( C_1, C_2 > 0 \) are constants depending only on localized sparse eigenvalues and \( \gamma_u \). This completes the proof.

\( \square \)

### C.6 Proof of Theorem 6

The proof is almost identical to that of Theorem 3, (II), except that to obtain (106), now we use the Lipschitz continuity of \( \ell'' \). Keep all other statements the same, we then get the desired result. \( \square \)