COUNTING PROPER MERGINGS OF CHAINS AND ANTICHAINS

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Abstract. A proper merging of two disjoint quasi-ordered sets $P$ and $Q$ is a quasi-order on the union of $P$ and $Q$ such that the restriction to $P$ and $Q$ yields the original quasi-order again and such that no elements of $P$ and $Q$ are identified. In this article, we consider the cases where $P$ and $Q$ are chains, where $P$ and $Q$ are antichains, and where $P$ is an antichain and $Q$ is a chain. We give formulas that determine the number of proper mergings in all three cases, and introduce two new bijections from proper mergings of two chains to plane partitions and from proper mergings of an antichain and a chain to monotone colorings of complete bipartite digraphs. Additionally, we use these bijections to count the Galois connections between two chains, and between a chain and a Boolean lattice respectively.

1. Introduction

Given two quasi-ordered sets $(P, \preceq_P)$ and $(Q, \preceq_Q)$, a merging of $P$ and $Q$ is a quasi-order $\preceq$ on the union of $P$ and $Q$ such that the restriction of $\preceq$ to $P$ or $Q$ yields $\preceq_P$ respectively $\preceq_Q$ again. In other words, a merging of $P$ and $Q$ is a quasi-order on the union of $P$ and $Q$, which does not change the quasi-orders on $P$ and $Q$.

In [3] a characterization of the set of mergings of two arbitrary quasi-ordered sets $P$ and $Q$ is given. In particular, it turns out that every merging $\preceq$ of $P$ and $Q$ can be uniquely described by two binary relations $R \subseteq P \times Q$ and $S \subseteq Q \times P$. The relation $R$ can be interpreted as a description, which part of $P$ is weakly below $Q$, and analogously the relation $S$ can be interpreted as a description, which part of $Q$ is weakly below $P$. A merging is called proper if $R \cap S^{-1} = \emptyset$, and hence if no element of $P$ is identified with an element of $Q$.

The characterization in [3] uses techniques of Formal Concept Analysis (FCA, see [4]), a branch of mathematics, which investigates binary relations, so-called formal contexts, between two sets. The starting point of FCA is the construction of a closure system from such a formal context. Then, this closure system induces a complete lattice, when ordering the closures by inclusion. (A complete lattice is a possibly infinite lattice which has a unique top and a unique bottom element.) The basic theorem of FCA states that every complete lattice can be

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derived from a formal context. In [3], it was shown that the mergings of two quasi-ordered sets $P$ and $Q$ form a distributive lattice, and can thus be described by a formal context. Notably, this formal context can be constructed easily from the quasi-orders $\leftarrow_P$ and $\leftarrow_Q$. The proper mergings of $P$ and $Q$ form a distributive sublattice of the previous lattice.

Unfortunately, the formal context provides only very little information about the cardinality of its associated lattice. Hence, although the set of mergings of two quasi-ordered sets $P$ and $Q$ can be described completely, not much is known about its cardinality. This article provides a first enumerative analysis of the set of proper mergings of two special classes of quasi-ordered sets, namely chains and antichains. The actual genesis of this article was the observation that the number of proper mergings of two $n$-chains is given by

$$F_c(n) = \frac{(2n)!(2n + 1)!}{(n!(n + 1)!)^2}.$$  

It is stated in [2] that $F_c(n)$ also determines the number of plane partitions with $n$ rows, $n$ columns and largest part at most 2. (See [9, Sequence A000891] for some other objects counted by this number.) It is not hard to define a bijection between these plane partitions, and the proper mergings of two $n$-chains, as will be described in Section 3.2. It is then straightforward to extend this bijection to the set of plane partitions with $m$ rows, $n$ columns and largest part at most 2, and the set of proper mergings of an $m$-chain and an $n$-chain. Since the number of such plane partitions can be derived from MacMahon’s formula, see (10), this bijection easily allows for counting the proper mergings of two chains. Interestingly, we can use this bijection for counting the Galois connections between two chains. The key theorem for this correspondence is [4, Theorem 53], which states that the Galois connections between two concept lattices correspond to dual bonds between the corresponding formal contexts.

After succeeding in enumerating proper mergings of chains, we became curious whether we can count proper mergings of two antichains in a similar way. Unfortunately, we cannot give a bijection between the set of proper mergings of two antichains and any other known mathematical object. However, we are able to enumerate the proper mergings of two antichains with the help of a generating function, which was found by Christian Krattenthaler. See Section 4 for the details.

The third part of this article is devoted to the enumeration of proper mergings of an $m$-antichain and an $n$-chain. When computing the number of these proper mergings with the help of Daniel Borchmann’s FCA-tool CONEXP-CLJ [1], we recovered the sequence [9, A085465]. The formula generating this sequence is a special case of the following
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formula.

\[ F_{a,c}(m,n) = \sum_{i=1}^{n+1} \left( (n+2-i)^m - (n+1-i)^m \right) \cdot i^m. \]

It is stated in [5] that \( F_{a,c}(m,n) \) also determines the number of monotone \( (n+1) \)-colorings of the complete bipartite digraph \( \vec{K}_{m,m} \). In Section 5.1, we construct a bijection between the set of proper mergings of an \( m \)-antichain and an \( n \)-chain, and the set of monotone \( (n+1) \)-colorings of \( \vec{K}_{m,m} \). We can also use this bijection, in order to count the number of Galois connections between a chain and a Boolean lattice.

The precise statements of the results described in the previous paragraphs are the following.

**Theorem 1.1.** Let \( \mathcal{M}^*_{P,Q} \) denote the set of proper mergings of two quasi-ordered sets \( P \) and \( Q \).

(i) Let \( P \) and \( Q \) be chains. If \( |P| = m, |Q| = n \), then

\[ |\mathcal{M}^*_{P,Q}| = \frac{1}{n+m+1} \binom{n+m+1}{m+1} \binom{m+1}{m}. \]

(ii) Let \( P \) and \( Q \) be antichains. If \( |P| = m, |Q| = n \), then

\[ |\mathcal{M}^*_{P,Q}| = \sum_{n_1+m_1+k_1=m} \binom{m}{n_1, m_1, k_1} (-1)^{k_1} \left( 2^{n_1} + 2^{m_1} - 1 \right)^n. \]

(iii) Let \( P \) be an antichain, and let \( Q \) be a chain. If \( |P| = m, |Q| = n \), then

\[ |\mathcal{M}^*_{P,Q}| = \sum_{i=1}^{n+1} \left( (n+2-i)^m - (n+1-i)^m \right) i^m. \]

In Theorem 1.1 (iii), we need to be careful with the case \( m = 0 \). In this case, there appears a term of the form \( "0^0" \) in the sum. Since there is exactly one proper merging of an empty antichain and some chain, we need to interpret this term as being equal to zero.

This article is organized as follows: in Section 2, we give a short introduction to Formal Concept Analysis in order to make the reader familiar with notions such as cross-table, intent, extent, bond, and other terminology from FCA. Moreover, we formally define mergings of two quasi-ordered sets. In Section 3, we define the bijection between proper mergings of two chains, and plane partitions with largest part at most 2. We conclude Theorem 1.1 (i) in Section 3.3, and exploit this bijection in order to count the Galois connections between two chains in Section 3.4. In Section 4, we compute the generating function for the proper mergings of two antichains and conclude Theorem 1.1 (ii). In Section 5, we construct the bijection between proper mergings of an antichain and a chain, and monotone colorings of a complete bipartite digraph. We conclude Theorem 1.1 (iii) in Section 5.1, and exploit this
bijection in order to count the Galois connections between chains and Boolean lattices in Section 5.2.

2. Preliminaries

In this section we recall the basic notations and definitions needed in this article. For a detailed introduction to Formal Concept Analysis, we refer to [4].

2.1. Formal Concept Analysis. The theory of Formal Concept Analysis (FCA) was introduced in the 1980s by Rudolf Wille (see [12]) as an approach to restructure lattice theory. The initial goal was to interpret lattices as hierarchies of concepts and thus to give meaning to the lattice elements in a fixed context. Such a formal context is a triple \((G, M, I)\), where \(G\) is a set of so-called objects, \(M\) is a set of so-called attributes and \(I \subseteq G \times M\) is a binary relation that describes whether an object has an attribute. Given a formal context \(K = (G, M, I)\), we define two derivation operators

\[
\begin{align*}
(\cdot)^\mathcal{I} : \wp(G) &\rightarrow \wp(M), \quad A \mapsto A^\mathcal{I} = \{ m \in M \mid g I m \text{ for all } g \in A \}, \\
(\cdot)^\mathcal{I} : \wp(M) &\rightarrow \wp(G), \quad B \mapsto B^\mathcal{I} = \{ g \in G \mid g I m \text{ for all } m \in B \},
\end{align*}
\]

where \(\wp\) denotes the power set. The notation \(g I m\) is to be understood as \((g, m) \in I\). It shall be mentioned that these derivation operators form a Galois connection between \(\wp(G)\) and \(\wp(M)\), and hence, the composition \((\cdot)^\mathcal{I}\mathcal{I}\) is a closure operator on \(\wp(G)\) respectively on \(\wp(M)\). (See Section 3.4 for an explicit definition of Galois connections.) We notice the natural duality between these operators, which justifies the use of the same symbol for both of them.

Let now \(A \subseteq G\), and \(B \subseteq M\). The pair \(b = (A, B)\) is called formal concept of \(K\) if \(A^\mathcal{I} = B\) and \(B^\mathcal{I} = A\). In this case, we call \(A\) the extent and \(B\) the intent of \(b\). It can easily be seen that for every \(A \subseteq G\), and \(B \subseteq M\), the pairs \((A^\mathcal{I}, A^\mathcal{I})\) and \((B^\mathcal{I}, B^\mathcal{I})\) are formal concepts, respectively. Conversely, every formal concept of \(K\) can be written in such a way. Thus, every formal concept of a given formal context can be seen from an extensional (“Which objects does the concept describe?”) as well as an intensional (“Which attributes describe the concept?”) viewpoint. We denote the set of all formal concepts of \(K\) by \(\mathfrak{B}(K)\), and define a partial order on \(\mathfrak{B}(K)\) by

\[
(A_1, B_1) \leq (A_2, B_2) \quad \text{if and only if} \quad A_1 \subseteq A_2 \quad \text{(or equivalently} \quad B_1 \supseteq B_2). 
\]

Let \(\mathfrak{B}(K)\) denote the poset \((\mathfrak{B}(K), \leq)\). The basic theorem of FCA (see [4, Theorem 3]) states that \(\mathfrak{B}(K)\) is a lattice, the so-called concept lattice of \(K\). Moreover, every finite lattice is a concept lattice\(^1\). This

\(^1\)More precisely, the basic theorem of FCA states that every complete lattice is a concept lattice. A complete lattice is a (possibly infinite) lattice which has a
result implies that every element of a finite lattice can be interpreted as a closure of a suitable closure system.

Usually, a formal context is represented by a cross-table, where the rows represent the objects and the columns represent the attributes. The cell in row $g$ and column $m$ contains a cross if and only if $g \trianglerighteq m$. See Figure 1 for two small examples. The reader is encouraged to compute the concept lattices of both formal contexts in order to see that these lattices are indeed isomorphic to a 4-chain, respectively a Boolean lattice with eight elements.

For every context $\mathbb{K} = (G, M, I)$, there are two maps

\begin{align*}
(4) \quad & \gamma : G \to \mathfrak{B}(\mathbb{K}), \quad g \mapsto (\{g\}^I, \{g\}^F), \quad \text{and} \\
(5) \quad & \mu : M \to \mathfrak{B}(\mathbb{K}), \quad m \mapsto (\{m\}^I, \{m\}^F),
\end{align*}

which map each object, respectively attribute, to its corresponding formal concept. It is common sense in FCA to label the Hasse diagram of $\mathfrak{B}(\mathbb{K})$ in the following way: the node representing a formal concept $b \in \mathfrak{B}(\mathbb{K})$ is labeled with the object $g$ (or with the attribute $m$) if and only if $b = \gamma g$ (or $b = \mu m$). Object labels are attached below unique minimal and a unique maximal element. In particular, every finite lattice is a complete lattice.

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**Figure 1.** Two examples for a formal context associated to a lattice.
the nodes in the Hasse diagram, and attribute labels above. In this presentation, the extent (intent) of a formal concept corresponds to the labels weakly below (weakly above) this formal concept in the Hasse diagram of $\mathcal{B}(\mathcal{K})$. (In Figure 1(a), however, we omitted the attribute labels, since they would be attached to the same formal concept as the corresponding object label.)

There is yet another way to interpret formal contexts. Let $(P, \leq P)$ be a poset. Then, $(P, P, \leq P)$ is a formal context and its cross-table corresponds to the incidence matrix of $(P, \leq P)$, which means that we can read the order-relation of $(P, \leq P)$ from the cross-table. Moreover, the concept lattice $\mathcal{B}(P, P, \leq P)$ is isomorphic to the smallest (complete) lattice that contains $(P, \leq P)$ as a subposet, the so-called Dedekind-MacNeille completion of $(P, \leq P)$. However, not every cross-table of a formal context $(P, P, I)$ can be interpreted as the incidence matrix of a partial order on $P$. (For instance, the cross-table shown in Figure 1(b) does not correspond to a partial order on the set \{a, b, c\}.)

In the remainder of this article, we will usually represent posets (and binary relations in general) by the cross-table of the corresponding formal context. Whenever we speak of a row or column in combination with a poset element $p \in P$, we mean the corresponding set \{p\} $\leq P$ in the sense of (1) (respectively (2)).

2.2. Bonds and Mergings. Let $\mathcal{K}_1 = (G_1, M_1, I_1)$, $\mathcal{K}_2 = (G_2, M_2, I_2)$ be formal contexts. A binary relation $R \subseteq G_1 \times M_2$ is called bond from $\mathcal{K}_1$ to $\mathcal{K}_2$ if for every object $g \in G_1$, the row \{g\}$^R$ is an intent in $\mathcal{K}_2$ and for every $m \in M_2$, the column \{m\}$^R$ is an extent in $\mathcal{K}_1$.

Now let $(P, \leftrightarrow P)$ and $(Q, \leftrightarrow Q)$ be disjoint quasi-ordered sets. Let $R \subseteq P \times Q$, and $S \subseteq Q \times P$. Define a relation $\leftrightarrow_{R, S}$ on $P \cup Q$ as

$$(6) \quad p \leftrightarrow_{R, S} q \quad \text{if and only if} \quad p \leftrightarrow P q \quad \text{or} \quad p \leftrightarrow Q q \quad \text{or} \quad p R q \quad \text{or} \quad p S q,$$

for all $p, q \in P \cup Q$. The pair $(R, S)$ is called merging of $P$ and $Q$ if $(P \cup Q, \leftrightarrow_{R, S})$ is a quasi-ordered set. Moreover, a merging is called proper if $R \cap S^{-1} = \emptyset$. Since for fixed quasi-ordered sets $(P, \leftrightarrow P)$ and $(Q, \leftrightarrow Q)$ the relation $\leftrightarrow_{R, S}$ is uniquely determined by $R$ and $S$, we refer to $\leftrightarrow_{R, S}$ as a (proper) merging of $P$ and $Q$ as well. Let $\circ$ denote the relational product$^2$.

**Proposition 2.1** ([3, Proposition 2]). Let $(P, \leftrightarrow P)$ and $(Q, \leftrightarrow Q)$ be disjoint quasi-ordered sets, and let $R \subseteq P \times Q$, and $S \subseteq Q \times P$. The pair $(R, S)$ is a merging of $P$ and $Q$ if and only if all of the following properties are satisfied:

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$^2$Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be relations between sets $A, B, C$. The relational product is the relation $R \circ S \subseteq A \times C$ that is given by $R \circ S = \{(a, c) \mid (a, b) \in R \text{ and } (b, c) \in S \text{ for some } b \in B\}$. 


(1) \( R \) is a bond from \((P, P, \not\rightarrow_P)\) to \((Q, Q, \not\rightarrow_Q)\),
(2) \( S \) is a bond from \((Q, Q, \not\rightarrow_Q)\) to \((P, P, \not\rightarrow_P)\),
(3) \( R \circ S \) is contained in \( \leftarrow_P \), and
(4) \( S \circ R \) is contained in \( \leftarrow_Q \).

Moreover, the relation \( \leftarrow_{R,S} \) as defined in (6) is antisymmetric if and only if \( \leftarrow_P \) and \( \leftarrow_Q \) are both antisymmetric and \( R \cap S^{-1} = \emptyset \).

In the case that \( P \) and \( Q \) are posets, this proposition implies that \((P \cup Q, \leftarrow_{R,S})\) is a poset again if and only if \((R, S)\) is a proper merging of \( P \) and \( Q \).

Denote the set of mergings of \( P \) and \( Q \) by \( M_{P,Q} \), and define a partial order on \( M_{P,Q} \) by
\[
(R_1, S_1) \preceq (R_2, S_2) \quad \text{if and only if} \quad R_1 \subseteq R_2 \quad \text{and} \quad S_1 \supseteq S_2.
\]

It was shown in [3, Theorem 1] that \((M_{P,Q}, \preceq)\) is a distributive lattice, where \((\emptyset, Q \times P)\) is the unique minimal element, and \((P \times Q, \emptyset)\) the unique maximal element. Let \( M_{P,Q}^* \subseteq M_{P,Q} \) denote the set of all proper mergings of \( P \) and \( Q \). It is also stated in [3, Theorem 1] that \((M_{P,Q}^*, \preceq)\) is a (complete) sublattice of \((M_{P,Q}, \preceq)\), which is still distributive.

Figure 2 shows the lattice of proper mergings of two 2-chains, where the nodes are labeled by the corresponding proper mergings.

3. Proper Mergings of two Chains

In the first part of this article, we provide a closed formula for the number of proper mergings of two chains. In particular, we give a bijective proof of the following theorem.

**Theorem 3.1.** Let \( m, n \in \mathbb{N} \) and let \( E_{m,n}^* \) denote the set of proper mergings of an \( n \)-chain and an \( m \)-chain. Then,
\[
|E_{m,n}^*| = \frac{1}{n + m + 1} \binom{n + m + 1}{m + 1} \binom{n + m + 1}{m}.
\]

In addition, we exploit the bijection constructed in this section to count the number of Galois connections between two chains.

We start with some definitions. Let \( C = \{c_1, c_2, \ldots, c_n\} \) be a set. Consider the \( n \)-chain \((C, \leq)\), where the order \( \leq \) is indicated by the indices, namely \( c_i \leq c_j \) if and only if \( i \leq j \). In the remainder of this section, we abbreviate the poset \((C, \leq)\) by \( c \). The corresponding formal context \((C, C, \leq)\) will be denoted by \( K(c) \). The formal context \((C, C, \not\geq)\) – the so-called contraordinal scale of \( c \) – will be denoted by \( C(c) \).

3.1. **Intents and Extents of** \( C(c) \). If \( c = (C, \leq) \) is an \( n \)-chain, we can convince ourselves that we can write the corresponding cross-table of \( K(c) \) in a triangular shape, as indicated in Figure 1(a). Since the elements in \( c \) are pairwise comparable, we have for all \( c, c' \in C \) that
$c \not\geq c'$ if and only if $c < c'$. Hence, the cross-table of the context $\mathbb{C}(c)$ is that of $\mathbb{K}(c)$ without crosses on the main diagonal. Thus, for every $i \in \{2, 3, \ldots, n\}$ the set $\{c_i, c_{i+1}, \ldots, c_n\}$ is a row (and thus an intent) of $\mathbb{C}(c)$. At the same time, for every $i \in \{1, 2, \ldots, n-1\}$, the set $\{c_1, c_2, \ldots, c_i\}$ is a column (and thus an extent) of $\mathbb{C}(c)$. By definition,
the empty set and $C$ itself are both intents and extents of $C(e)$. (This follows, since the empty set is an extent (intent) of a formal context if and only if there is no full row (column). The set of objects (attributes) is an extent (intent) of every formal context.) This means that for every $i \in \{0, 1, \ldots, n\}$, the set $\{c_1, c_2, \ldots, c_i\}$ is an extent of $C(e)$, and hence $\mathcal{B}(C(e))$ is isomorphic to an $n + 1$-chain. (The case $i = 0$ is to be interpreted as the empty set.) See Figure 3 for an illustration.

3.2. A Bijection between Plane Partitions and Proper Mergings of Two Chains. Let us recall that a plane partition $\pi = (\pi_{i,j})_{i,j \geq 1}$ is an array of nonnegative integers that is weakly decreasing along rows and columns and has only finitely many nonzero entries. An entry $\pi_{i,j}$ is called part of $\pi$. (We refer the reader to [10, Sections 7.20 and 7.21] for more information on plane partitions.) The next definition is central for this section.

**Definition 3.2.** Let $C_1 = \{a_1, a_2, \ldots, a_m\}$, and $C_2 = \{b_1, b_2, \ldots, b_n\}$ be sets. Consider the chains $e_1 = (C_1, \leq_1)$, and $e_2 = (C_2, \leq_2)$, where the order relations are determined by the indices of the corresponding sets. Let $\pi$ be a plane partition with $m$ rows, $n$ columns, and largest part at most 2. Define relations $R_\pi \subseteq C_1 \times C_2$ and $S_\pi \subseteq C_2 \times C_1$ by

\begin{align}
R_\pi & \quad \text{if and only if } \pi_{i,j} = 2, \quad \text{and} \\
S_\pi & \quad \text{if and only if } \pi_{i,j} = 0,
\end{align}

where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Figure 4 shows a plane partition with five rows, six columns, and largest part 2. Figure 5 shows the corresponding relations $R$ and $S$ in the sense of the previous definition.

**Lemma 3.3.** The relations $R_\pi$ and $S_\pi$ from Definition 3.2 form a proper merging of $e_1$ and $e_2$.
Figure 4. A plane partition with five rows, six columns and largest part 2.

|   | b₁ | b₂ | b₃ | b₄ | b₅ | b₆ |
|---|----|----|----|----|----|----|
| a₁ | ×  | ×  | ×  | ×  | ×  |    |
| a₂ | ×  | ×  | ×  | ×  |    |    |
| a₃ | ×  | ×  | ×  | ×  |    |    |
| a₄ |    |    | ×  |    |    |    |
| a₅ |    |    |    |    |    |    |

|   | a₁ | a₂ | a₃ | a₄ | a₅ |
|---|----|----|----|----|----|
| b₁ | ×  | ×  | ×  | ×  | ×  |
| b₂ | ×  | ×  |    |    |    |
| b₃ | ×  |    | ×  |    |    |
| b₄ | ×  |    |    | ×  |    |
| b₅ |    |    |    |    | ×  |
| b₆ |    |    |    |    |    |

Figure 5. The relations $R$ and $S$ induced by the plane partition in Figure 4.

Proof. It is sufficient to prove that $R_\pi$ and $S_\pi$ satisfy the conditions (1)–(4) in Proposition 2.1. First, let $a_i, a_j \in C_1$, with $(a_i, a_j) \in R_\pi \circ S_\pi$. By definition, there must be some $b_k \in C_2$ satisfying $\pi_{i,k} = 2$ and $\pi_{j,k} = 0$. Since $\pi$ is a plane partition (and hence weakly decreasing along the columns), we can conclude that $i < j$, and hence $a_i < a_j$, which proves condition (3). Now let $b_i, b_j \in C_2$ with $(b_i, b_j) \in S_\pi \circ R_\pi$. By definition, there must be some $a_k \in C_1$ satisfying $\pi_{k,n-j+1} = 0$ and $\pi_{k,n-i+1} = 2$. Again we can conclude that $i < j$, and thus $b_i < b_j$, which proves condition (4).

Now we need to show that $R_\pi$ is a bond from $C \langle c_1 \rangle$ to $C \langle c_2 \rangle$ and $S_\pi$ is a bond from $C \langle c_2 \rangle$ to $C \langle c_1 \rangle$. Hence, we need to show that every row in $R_\pi$ is an intent of $C \langle c_2 \rangle$, and every column in $R_\pi$ is an extent of $C \langle c_1 \rangle$. First we notice that for every $i \in \{1, 2, \ldots, n\}$, the set $\{a_i\}^{R_\pi}$ consists of all $b_j \in C_2$ such that $\pi_{i,j} = 2$. Since, $\pi$ is a plane partition, we can conclude that $\{a_i\}^{R_\pi}$ is of the form $\{b_k, b_k+1, \ldots, b_n\}$ for some $k \in \{1, 2, \ldots, n+1\}$. (The case $k = n + 1$ is to be interpreted as the empty set.) The reasoning in the beginning of this section shows that each such set is indeed an intent of $C \langle c_2 \rangle$. Similarly, we see that for every $b \in C_2$, the set $\{b\}^{S_\pi}$ is of the form $\{a_1, a_2, \ldots, a_k\}$ for some $k \in \{0, 1, \ldots, n\}$. (The case $k = 0$ is to be interpreted as the empty set.) By the same argument as before, we see that these indeed are extents of $C \langle c_2 \rangle$, which proves condition (1). To show that $S_\pi$ is a bond from $C \langle c_2 \rangle$ to $C \langle c_1 \rangle$, we notice that the rows in $S_\pi$ must correspond to intents of $C \langle c_1 \rangle$ and the columns of $S_\pi$ must correspond to extents of
Figure 6. The proper merging of a 5-chain and a 6-chain defined by the relations given in Figure 5.

\( \mathbb{C}(c_2) \). Thus, condition (2) can be shown analogously to the previous case.

Finally, since every cell is labeled by a unique value, we can conclude that \( R_\pi \cap S_\pi^{-1} = \emptyset \), which makes \((R_\pi, S_\pi)\) a proper merging of \(c_1\) and \(c_2\).

Figure 6 shows the poset corresponding to the proper merging shown in Figure 5. We can conclude the following theorem.

**Theorem 3.4.** Let \( PP_{m,n}^{(2)} \) denote the set of plane partitions with \( m \) rows, \( n \) columns and largest part at most 2. Let \( \mathcal{C}_{m,n}^{\bullet} \) denote the set of proper mergings of an \( m \)-chain and an \( n \)-chain. Then, the correspondence described in Definition 3.2 is a bijection between \( PP_{m,n}^{(2)} \) and \( \mathcal{C}_{m,n}^{\bullet} \).

**Proof.** Lemma 3.3 makes immediately clear that each such plane partition induces a proper merging of an \( m \)-chain and an \( n \)-chain.

Conversely, let \((R, S)\) be a proper merging of an \( m \)-chain and an \( n \)-chain. Let \( \pi_{(R,S)} \) be the \((m \times n)\)-array, whose parts \( \pi_{i,j} \) are defined by

\[
\pi_{i,j} = \begin{cases} 
0, & \text{if } b_{n-j+1} S a_i, \\
2, & \text{if } a_i R b_{n-j+1}, \\
1, & \text{otherwise}. 
\end{cases}
\]

Since \((R, S)\) is a proper merging, no cell is labeled twice. Condition (1) in Proposition 2.1 implies that if more than one 2 appears in a row or column of \( \pi_{(R,S)} \), these 2’s appear consecutively. Moreover, it follows that a row (or column), which contains a 2, contains a 2 in its first cell. Condition (2) in Proposition 2.1 implies the analogous properties for 0’s, in particular that a row (or column) that contains a 0, contains a 0.
in its last cell. Condition (3) in Proposition 2.1 implies that every 2 in a column of \( \pi(R,S) \) appears above a 0, and condition (4) in Proposition 2.1 implies that every 0 in a row of \( \pi(R,S) \) appears to the right of a 2. Hence, \( \pi(R,S) \) is a plane partition with \( m \) rows, \( n \) columns and largest part at most 2.

An extensive illustration of this bijection can be found in Appendix A.

3.3. The Number of Proper Mergings of Two Chains. Having the bijection from the previous section in mind, it is now straightforward to determine the number of proper mergings of two chains. Let us recall a classical result by MacMahon.

**Theorem 3.5.** Let \( l, m, n \in \mathbb{N} \). The number \( \pi(m,n,l) \) of plane partitions with \( m \) rows, \( n \) columns and largest part at most \( l \) is given by

\[
\pi(m,n,l) = \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{i + j + k - 1}{i + j + k - 2},
\]

This result was first conjectured in [7] and later proven in [8, Sections XI and X]. The presented form can be derived from [6, Example 13(b)].

**Proof of Theorem 3.1.** Theorem 3.4 and Theorem 3.5 imply

\[
|c^\bullet_{m,n}| = \pi(m,n,2)
= \prod_{i=1}^{m} \prod_{j=1}^{n} \frac{i + j + 1}{i + j - 1}
= \prod_{i=1}^{m} \frac{i + m}{i} \cdot \frac{i + m + 1}{i + 1}
= \frac{1}{m + n + 1} \binom{m + n + 1}{m + 1} \binom{m + n + 1}{m}.
\]

\[\square\]

**Remark 3.6.** Consider the Narayana numbers (see [10, Exercise 6.36 a]), defined by

\[
\text{Nar}(\tilde{m}, \tilde{n}) = \frac{1}{\tilde{n}} \binom{\tilde{n}}{\tilde{m}} \binom{\tilde{n}}{\tilde{m} - 1},
\]

for \( \tilde{m}, \tilde{n} \in \mathbb{N} \), with \( \tilde{m} \leq \tilde{n} \). In view of Theorem 3.4, we obtain

\[
|c^\bullet_{m,n}| = \text{Nar}(m + n + 1, m + 1).
\]

**Remark 3.7.** Let \( \pi = (\pi_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n} \) and \( \sigma = (\sigma_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n} \) be plane partitions with \( m \) rows and \( n \) columns, and largest part 2. Define a partial order \( \leq \) on \( \text{PP}^{(2)}_{m,n} \) as

\[
\pi \leq \sigma \quad \text{if and only if} \quad \pi_{i,j} \leq \sigma_{i,j},
\]

for all \( i, j \).
Figure 7. The lattice of proper mergings of three 1-chains.

for all $1 \leq i \leq m$ and $1 \leq j \leq n$. Let $(R_\pi, S_\pi)$, and $(R_\sigma, S_\sigma)$ denote the proper mergings associated to $\pi$ respectively $\sigma$ in the sense of Definition 3.2. Suppose that $(R_\pi, S_\pi) \preceq (R_\sigma, S_\sigma)$, and hence by definition $R_\pi \subseteq R_\sigma$, and $S_\pi \supseteq S_\sigma$. This implies that if $\pi_{i,j} = 2$, then $\sigma_{i,j} = 2$. If $\pi_{i,j} = 1$, then $\sigma_{i,j} \in \{1, 2\}$, and if $\pi_{i,j} = 0$, then $\sigma_{i,j} \in \{0, 1, 2\}$. Hence, $\pi \preceq \sigma$. This means that the bijection described in Theorem 3.4 is indeed an isomorphism between the lattices $(C_{m,n}, \preceq)$ and $(PP^{(2)}_{m,n}, \leq)$.

Remark 3.8. Christian Meschke proposed the following generalization of mergings of quasi-ordered sets: let $T$ be a linearly ordered set, and let $(P_t, \leftarrow_t)_{t \in T}$ be a family of quasi-ordered sets, indexed by $T$. Define $P = \bigcup_{t \in T} P_t$, and let $R \subseteq P \times P$ be a relation on $P$. We abbreviate $R_{s,t} = R \cap (P_s \times P_t)$. Then, $R$ is called merging of the $P_t$’s if it is a quasi-ordered set on $P$ such that $R_{t,t}$ yields $\leftarrow_t$ again. Moreover, $R$ is called proper if for all $s < t$, we have $R_{s,t} \cap R_{t,s}^{-1} = \emptyset$. Let $\mathcal{M}_T$ denote the set of all mergings of the $P_t$’s. We define a partial order $\sqsubseteq$ on $\mathcal{M}_T$.
as

\[ R \subseteq S \text{ if and only if } \begin{cases} R_{s,t} \subseteq S_{s,t} & \text{if } s < t, \\ R_{s,t} \supseteq S_{s,t} & \text{if } s > t, \end{cases} \]

for all \( R, S \in \mathcal{M}_T \). Then, \((\mathcal{M}_T, \subseteq)\) is again a lattice. However, as we notice from Figure 7, this lattice is in general no longer distributive.

Even more, up to now it is not clear, how to construct the formal context which generates \((\mathcal{M}_T, \subseteq)\) from the quasi-orders \( \preceq \).

We can now think of a generalization of the bijection described in Theorem 3.4 to proper mergings of more than two chains in the following way: let \( c_1, c_2, \ldots, c_t \) be chains, where for all \( i \in \{1, 2, \ldots, t\} \), the chain \( c_i \) has \( n_i \) elements. Consider the standard unit vectors \( e_1, e_2, \ldots, e_t \) in \( \mathbb{R}^t \), and label the points \( e_j - \frac{1}{2}, 2e_j - \frac{1}{2}, \ldots, n_j e_j - \frac{1}{2} \) with the elements of the chain \( c_j \) for all \( j \in \{1, 2, \ldots, t\} \) in the obvious way. For each \( i, j \in \{1, 2, \ldots, t\} \) with \( i < j \), we can insert a plane partition with largest part \( \leq 2 \) into the \((n_i \times n_j)\)-array, spanned by the vectors \( n_i e_i \) and \( n_j e_j \), and call this an arrangement of \( t \) plane partitions.

For an illustration of this construction, we refer to Figure 8. On the left of each figure, there is an arrangement of three plane partitions with one row and one column, together with the labeled coordinate axes. In the middle, the three plane partitions are written next to each other and on the right, there is the merging of three 1-chains which is induced by these plane partitions in the spirit of Definition 3.2. We notice that Figure 8(a) shows a proper merging of the three 1-chains, while Figure 8(b) does not. See Appendix B for an extensive illustration of the case of proper mergings of three 1-chains. In this appendix, we also notice that some arrangements of plane partitions yield the same mergings.

If this construction can indeed be used as a generalization of Theorem 3.4 should be investigated in a subsequent article.

3.4. Counting Galois Connections between Chains. In this section, we describe how we can exploit the bijection given in Definition 3.2 to allow for counting Galois connections between two chains. Let us first recall the definitions. A Galois connection between two posets \((P, \leq_P)\) and \((Q, \leq_Q)\) is a pair \((\varphi, \psi)\) of maps

\[ \varphi : P \rightarrow Q \quad \text{and} \quad \psi : Q \rightarrow P, \]

satisfying

\[(12) \quad p_1 \leq_P p_2 \text{ implies } \varphi p_1 \geq_Q \varphi p_2, \]
\[(13) \quad q_1 \leq_Q q_2 \text{ implies } \psi q_1 \geq_P \psi q_2, \]
\[(14) \quad p \leq_P \psi \varphi p, \quad \text{and} \quad q \leq_Q \varphi \psi q, \]

for all \( p, p_1, p_2 \in P \) and \( q, q_1, q_2 \in Q \). Now, let \((P, \leq_P) \cong \mathfrak{B}(\mathbb{K}_1)\) and \((Q, \leq_Q) \cong \mathfrak{B}(\mathbb{K}_2)\) be concept lattices, where \( \mathbb{K}_1 = (G, M, I) \) and \( \mathbb{K}_2 = \)
(a) An arrangement of three plane partitions, which yields a proper merging of three 1-chains.

(b) An arrangement of three plane partitions, which does not yield a proper merging of three 1-chains.

Figure 8. Two examples of posets induced by an arrangement of three plane partitions.

\( (H, N, J) \) are the corresponding formal contexts. In this particular case, Theorem 3.9 below states that each Galois connection from \( \mathfrak{B}(K_1) \) to \( \mathfrak{B}(K_2) \) corresponds to a dual bond from \( K_1 \) to \( K_2 \). A relation \( R \subseteq G \times H \), is called dual bond from \( K_1 \) to \( K_2 \) if for every \( g \in G \), the set \( \{g\}^R \) is an extent of \( K_2 \) and for every \( h \in H \), the set \( \{h\}^R \) is an extent of \( K_1 \). In other words, \( R \) is a dual bond from \( K_1 \) to \( K_2 \) if and only if \( R \) is a bond from \( K_1 \) to the dual context \( K_2^d \).

**Theorem 3.9** ([4, Theorem 53]). Let \( (G, M, I) \) and \( (H, N, J) \) be formal contexts. For every dual bond \( R \subseteq G \times H \), the maps

\[
\varphi_R(X, X') = (X^R, X^{RJ}), \quad \text{and} \quad \psi_R(Y, Y') = (Y^R, Y^{RI}),
\]

where \( X \) and \( Y \) are extents of \( (G, M, I) \) respectively \( (H, N, J) \), form a Galois connection between \( \mathfrak{B}(G, M, I) \) and \( \mathfrak{B}(H, N, J) \). Moreover, every Galois connection \( (\varphi, \psi) \) induces a dual bond from \( (G, M, I) \) to \( (H, N, J) \) by

\[
R_{(\varphi, \psi)} = \{(g, h) \mid \gamma g \leq \psi h\} = \{(g, h) \mid \gamma h \leq \varphi \gamma g\},
\]

where \( \gamma \) is the map defined in (4). We have

\[
\varphi_{R_{(\varphi, \psi)}} = \varphi, \quad \psi_{R_{(\varphi, \psi)}} = \psi, \quad \text{and} \quad R_{(\varphi, \psi)} = R.
\]

Let \( C_1 = \{a_1, a_2, \ldots, a_m\} \) and \( C_2 = \{b_1, b_2, \ldots, b_n\} \) be sets, and consider the corresponding chains \( c_1 = (C_1, \leq_1) \) and \( c_2 = (C_2, \leq_2) \), where the order relations are given by the indices of the corresponding sets. We can easily deduce from the reasoning in Section 3.1 that a

---

3Let \( K = (G, M, I) \) be a formal context. The dual context \( K^d \) of \( K \) is given by \( (M, G, I^{-1}) \) and satisfies \( \mathfrak{B}(K^d) \cong \mathfrak{B}(K)^d \), where \( \mathfrak{B}(K)^d \) is the (order-theoretic) dual of the lattice \( \mathfrak{B}(K) \).
relation $R \subseteq C_1 \times C_2$ is a dual bond from $\mathcal{K}(c_1)$ to $\mathcal{K}(c_2)$ if and only if it satisfies
\[
\{a\}^R = \{b_1, b_2, \ldots, b_i\}, \quad \text{and} \quad \{b\}^R = \{a_1, a_2, \ldots, a_j\},
\]
for every $a \in C_1, b \in C_2$, and some $i \in \{0, 1, \ldots, n\}$, and some $j \in \{0, 1, \ldots, m\}$. (Again, the cases $i = 0$ and $j = 0$ are to be interpreted as the empty set.)

We also noticed in Section 3.1 that an $n$-chain $c$ is isomorphic to the concept lattice of the formal context $C(c')$, for some $(n - 1)$-chain $c'$. Hence, if $c_1$ and $c_2$ are $m$- respectively $n$-chains, and $c'_1$ and $c'_2$ are $(m - 1)$- respectively $(n - 1)$-chains, we can interpret each dual bond from $\mathcal{K}(c_1)$ to $\mathcal{K}(c_2)$ as a dual bond from $C(c'_1)$ to $C(c'_2)$. This observation is crucial for the proof of the following proposition.

**Proposition 3.10.** Let $m, n \in \mathbb{N}$. The number of Galois connections between an $m$-chain and an $n$-chain is \(\binom{m+n-2}{m-1}\).

**Proof.** Let $c_1$ be an $m$-chain and let $c_2$ be an $n$-chain. Let $c'_1$ be an $(m - 1)$-chain, and let $c'_2$ be an $(n - 1)$-chain. Note that
\[
\mathcal{R}(\mathcal{K}(c_1)) \cong \mathcal{R}(C(c'_1)), \quad \text{and} \quad \mathcal{R}(\mathcal{K}(c_2)) \cong \mathcal{R}(C(c'_2)).
\]

Since chains are self-dual, the set of dual bonds from $\mathcal{K}(c_1)$ to $\mathcal{K}(c_2)$ is in bijection with the set of bonds from $\mathcal{K}(c_1)$ to $\mathcal{K}(c_2)$. Moreover, it follows immediately from Proposition 2.1 and the reasoning above that $(\emptyset, S)$ is a proper merging of $c'_1$ and $c'_2$ if and only if $S$ is a bond from $\mathcal{K}(c_1)$ to $\mathcal{K}(c_2)$. (Note that every binary relation $S$ satisfies $S \circ \emptyset = \emptyset = \emptyset \circ S$. Moreover, $\emptyset$ is a bond between two formal contexts $\mathcal{K}_1$ and $\mathcal{K}_2$ if and only if $\mathcal{K}_2$ does not contain a full row and $\mathcal{K}_1$ does not contain a full column. Since neither $C(c'_1)$ nor $C(c'_2)$ contain full rows or full columns, the conditions in Proposition 2.1 for $(\emptyset, S)$ to be a proper merging of $c'_1$ and $c'_2$ reduce to $S$ being a bond from $C(c'_1)$ to $C(c'_2)$. The latter is equivalent to $S$ being a bond from $\mathcal{K}(c_1)$ to $\mathcal{K}(c_2)$, when identifying $S$ with the corresponding relation derived from the isomorphisms between $C(c'_1)$ and $\mathcal{K}(c_1)$, respectively $C(c'_2)$ and $\mathcal{K}(c_2)$.) Thus, every Galois connection between $c_1$ and $c_2$ corresponds by Theorem 3.9 and the previous reasoning to a proper merging of $c'_1$ and $c'_2$, which is of the form $(\emptyset, \cdot)$.

Let us make this correspondence more explicit. By the bijection given in Definition 3.2, it is clear that a proper merging of $c'_1$ and $c'_2$, which is of the form $(\emptyset, \cdot)$, corresponds to a plane partition with $m - 1$ rows, $n - 1$ columns and largest part at most 1. Let $\pi$ be such a plane partition, and let $C'_1 = \{a_1, a_2, \ldots, a_{m-1}\}$ be the ground set of $c'_1$ and let $C'_2 = \{b_1, b_2, \ldots, b_{n-1}\}$ be the ground set of $c'_2$. Let $S_\pi \subseteq C'_2 \times C'_1$ be the relation given in Definition 3.2. Define a relation $T_\pi \subseteq C'_2 \times C'_1$.
as

\[(15) \quad b_j T_\pi a_i \quad \text{if and only if} \quad b_j S_\pi a_{n-i+1}, \]

for \(1 \leq i \leq m - 1\), and \(1 \leq j \leq n - 1\). Thus, \(T_\pi\) (as a cross-table) corresponds to a horizontal reflection of \(S_\pi\) (as a cross-table). It is now immediate from the construction that the rows of \(T_\pi\) are of the form \(\{a_1, a_2, \ldots, a_j\}\) for some \(j \in \{0, 1, \ldots, m - 1\}\), and the columns of \(T_\pi\) are of the form \(\{b_1, b_2, \ldots, b_i\}\) for some \(i \in \{0, 1, \ldots, n - 1\}\). Since \(S_\pi\) is a bond between \(C(c'_2)\) and \(C(c'_1)\), we can conclude that \(T_\pi\) is a dual bond between \(C(c'_2)\) and \(C(c'_1)\). By symmetry, \(T_\pi\) induces a Galois connection between \(c'_1\) and \(c'_2\).

The number of plane partitions with \(m - 1\) rows, \(n - 1\) columns and largest part at most 1 can be computed from Theorem 3.5, and it turns out to be \((m+n-2)\). □

Figure 9 shows an example of a Galois connection between a 5-chain and a 7-chain arising from a plane partition with 6 rows and 4 columns and largest part 1. An extensive illustration of the bijection described in the proof of Proposition 3.10 can be found in Appendix C.

4. Proper Mergings of two Antichains

In this section, we investigate the number of the proper mergings of two antichains. In particular, we prove the following theorem.

**Theorem 4.1.** Let \(A_{m,n}^*\) denote the set of proper mergings of an \(m\)-antichain and an \(n\)-antichain. Then,

\[
|A_{m,n}^*| = \sum_{n_1+m_1+k_1=m} \binom{m}{n_1, m_1, k_1} (-1)^{k_1} \left( 2^{n_1} + 2^{m_1} - 1 \right)^n.
\]

Let \(a_1\) and \(a_2\) be antichains. It is obvious that the Hasse diagram of a proper merging of \(a_1\) and \(a_2\) can be regarded as a (not necessarily connected) bipartite graph. Figure 10 shows the lattice of proper mergings of two 2-antichains, where the nodes are labeled by the corresponding proper mergings. In order to prove Theorem 4.1, we construct the generating function of proper mergings of two antichains.

Let \(B(x,y)\) denote the bivariate exponential generating function of bipartite graphs. The vertex set of a bipartite graph can be partitioned into two sets \(V_1\) and \(V_2\). Say that the variable \(x\) counts the cardinality of \(V_1\) and the variable \(y\) counts the cardinality of \(V_2\). Let \(b(m,n)\) denote the number of bipartite graphs with vertex set \(V = V_1 \cup V_2\), and \(|V_1| = m\), \(|V_2| = n\). Clearly, then \(b(m,n) = 2^{mn}\), and we find

\[
(16) \quad B(x,y) = \sum_{n \geq 0} \sum_{m \geq 0} b(m,n) \frac{x^n}{n!} \cdot \frac{y^m}{m!}
\]

\[= \sum_{n \geq 0} \sum_{m \geq 0} 2^{mn} \frac{x^n}{n!} \cdot \frac{y^m}{m!}.\]
Figure 9. A plane partition, the induced proper merging of a 6-chain and a 4-chain, the corresponding dual bond, and the induced Galois connection between a 5-chain and a 7-chain.
Figure 10. The lattice of proper mergings of two 2-antichains.
Let \( B_c(x, y) \) denote the bivariate exponential generating function for connected bipartite graphs. Since every bipartite graph can be seen as a collection of connected bipartite graphs, we obtain

\[
B(x, y) = \exp(B_c(x, y)).
\]

(17)

See for instance [11, Chapter 3] for an explanation of this equality. In particular, this correspondence is a bivariate exponential generating function version of [11, Theorem 3.4.1]. Now we are able to prove Theorem 4.1.

**Proof of Theorem 4.1.** Let \((R, S)\) be a proper merging of an \(m\)-antichain \(a_1\) and an \(n\)-antichain \(a_2\). Denote by \(\{\beta_1, \beta_2, \ldots, \beta_k\}\) the set of connected components of the Hasse diagram of \((R, S)\) (considered as a graph). Clearly, each \(\beta_i\) is a connected bipartite graph. Without loss of generality, we can assume that the vertices of \(\beta_i\) which belong to \(a_1\) are below the vertices of \(\beta_i\) which belong to \(a_2\). Then, we can flip the graph in such a way that the vertices of \(\beta_i\) which belong to \(a_1\) are above the vertices of \(\beta_i\) which belong to \(a_2\), and edges are preserved. This procedure yields another connected bipartite graph, say \(\beta_i^d\). For every \(i \in \{1, 2, \ldots, k\}\), it is clear that the set \(\{\beta_1, \beta_2, \ldots, \beta_i-1, \beta_i^d, \beta_{i+1}, \ldots, \beta_k\}\) is the set of connected components of the Hasse diagram of another proper merging, say \((R, S)^{(i)}\), of \(a_1\) and \(a_2\). It is immediate that \((R, S)\) and \((R, S)^{(i)}\) are different proper mergings of \(a_1\) and \(a_2\) if and only if \(\beta_i\) has more than one vertex. (See Figure 11 for an illustration.)

\[
\begin{align*}
\text{(a) A proper merging of a 7- and a 5-antichain.} & \quad \text{(b) Flipping the second component of Figure 11(a).} \\
\end{align*}
\]

**Figure 11.** Flipping a connected component of a proper merging of a 7-antichain and a 5-antichain.

Let \(G(x, y)\) denote the bivariate exponential generating function of proper mergings of two antichains. The previous reasoning implies that every proper merging of two antichains can be regarded as a collection of connected bipartite graphs \(\beta_1, \beta_2, \ldots, \beta_k\). Moreover, each connected component \(\beta_i\) can appear in two positions, namely \(\beta_i\) and \(\beta_i^d\), unless it has only one vertex. Again, in the spirit of [11, Theorem 3.4.1], we can
write this down as

\[ G(x, y) = \exp(2 \cdot B_c(x, y) - x - y). \]

Putting (16), (17) and (18) together, we obtain

\[
G(x, y) = \exp(2 \cdot \log B(x, y) - x - y)
= B(x, y)^2 \cdot \sum_{k_1 \geq 0} \frac{(-x)^{k_1}}{k_1!} \cdot \sum_{k_2 \geq 0} \frac{(-y)^{k_2}}{k_2!}
= \left( \sum_{n \geq 0} \sum_{m \geq 0} 2^{mn} \frac{x^n}{n!} \cdot \frac{y^m}{m!} \right)^2 \cdot \sum_{k_1 \geq 0} \frac{(-x)^{k_1}}{k_1!} \cdot \sum_{k_2 \geq 0} \frac{(-y)^{k_2}}{k_2!}
= \sum_{n_1 \geq 0} \sum_{n_2 \geq 0} 2^{n_1 n_2} \frac{x^{n_1}}{n_1!} \cdot \frac{y^{n_2}}{n_2!} \cdot \sum_{m_1 \geq 0} \sum_{m_2 \geq 0} 2^{m_1 m_2} \frac{x^{m_1}}{m_1!} \cdot \frac{y^{m_2}}{m_2!}
\times \sum_{k_1 \geq 0} \frac{(-x)^{k_1}}{k_1!} \cdot \sum_{k_2 \geq 0} \frac{(-y)^{k_2}}{k_2!}
= \sum_{n_1 \geq 0, n_2 \geq 0, m_1 \geq 0, m_2 \geq 0, k_1 \geq 0, k_2 \geq 0} 2^{n_1 n_2 + m_1 m_2} (-1)^{k_1} (-1)^{k_2} \cdot \frac{x^{n_1 + m_1 + k_1}}{n_1! m_1! k_1!} \cdot \frac{y^{n_2 + m_2 + k_2}}{n_2! m_2! k_2!}.
\]

The number of proper mergings of an \( m \)- and an \( n \)-antichain is now given by the coefficient of \( \frac{x^m y^n}{m!n!} \) in \( G(x, y) \). Hence,

\[
|\mathcal{Q}_{m,n}^*| = \left\langle \frac{x^m y^n}{m!n!} \right\rangle G(x, y)
= m! n! \sum_{n_1 + m_1 + k_1 = m, n_2 + m_2 + k_2 = n} 2^{n_1 n_2} 2^{m_1 m_2} (-1)^{k_1} (-1)^{k_2}
\times \sum_{n_2 + m_2 + k_2 = n} \left( \begin{array}{c} n \\ n_2, m_2, k_2 \end{array} \right) \left( 2^{n_1} \right)^{n_2} \left( 2^{m_1} \right)^{m_2} (-1)^{k_1}
= \sum_{n_1 + m_1 + k_1 = m} \left( \begin{array}{c} m \\ n_1, m_1, k_1 \end{array} \right) (-1)^{k_1} \left( 2^{n_1} + 2^{m_1} - 1 \right)^n.
\]

\[ \square \]

5. PROPER MERGINGS OF AN ANTICHAIN AND A CHAIN

In this section, we investigate the family of proper mergings of an antichain and a chain. In particular, we give a bijective proof of the following theorem.
Figure 12. The lattice of proper mergings of a 2-antichain and a 2-chain.
Theorem 5.1. Let $m, n \in \mathbb{N}$, and let $\mathcal{W}_{m,n}$ denote the set of proper mergings of an $m$-antichain and an $n$-chain. Then,

$$\left| \mathcal{W}_{m,n} \right| = \sum_{i=1}^{n+1} \left( (n + 2 - i)^m - (n + 1 - i)^m \right) \cdot i^m. \quad (19)$$

Remark 5.2. We notice that in the case $m = 0$, the equation (19) contains a term of the form “$0^0$” which is per se undefined. Since there exists exactly one proper merging of an empty antichain and some chain (namely the chain itself), it is reasonable to define the term “$0^0$” as being equal to 0. This harmonizes well with Theorem 5.6 below, since there is exactly one monotone coloring of an empty graph.

Figure 12 shows the lattice of proper mergings of a 2-antichain and a 2-chain, where the nodes are labeled by the corresponding proper mergings. Computer experiments show that the number of proper mergings of a 3-antichain and an $n$-chain is (up to a shift) given by $[9, A085465]$. This sequence counts the number of monotone $(n + 1)$-colorings of the complete bipartite digraph $\vec{K}_{3,3}$, and was first mentioned in [5], in a more general form. But let us first recall some definitions.

A directed graph (digraph for short) is a tuple $(V, \vec{E})$, where $V$ is a set of vertices, and $\vec{E} \subseteq V \times V$ is a set of directed edges. A directed edge $(v_1, v_2) \in \vec{E}$ is to be understood as being directed from $v_1$ to $v_2$. We call a digraph $(V, \vec{E})$ complete bipartite if we can partition $V$ into two disjoint sets $V_1$ and $V_2$ such that $\vec{E} = V_1 \times V_2$. In the case $|V_1| = m_1$ and $|V_2| = m_2$, we simply write $\vec{K}_{m_1,m_2}^2$ instead of $(V, \vec{E})$.

A $k$-coloring of $(V, \vec{E})$ is a map $\gamma : V \to \{1, 2, \ldots, k\}$. A $k$-coloring $\gamma$ is called monotone if $(v_1, v_2) \in \vec{E}$ implies $\gamma(v_1) \leq \gamma(v_2)$. See Figure 13 for an illustration. As already mentioned in the beginning of this section, there exists a general formula for the number of monotone $k$-colorings of $\vec{K}_{m_1,m_2}^2$.

Proposition 5.3 ([5, Proposition 4.5]). For every $k, m_1, m_2 \in \mathbb{N}$, let $\eta_k(\vec{K}_{m_1,m_2})$ denote the number of monotone $k$-colorings of the complete
bipartite digraph $\mathcal{G}_{m_1, m_2}$. Then,
\[
\eta_k(\mathcal{K}_{m_1, m_2}) = \sum_{i=1}^{k} \left( (k + 1 - i)^{m_1} - (k - i)^{m_1} \right) \cdot i^{m_2}.
\]
Equivalently,
\[
\eta_k(\mathcal{K}_{m_1, m_2}) = \sum_{i=1}^{k} \left( (k + 1 - i)^{m_2} - (k - i)^{m_2} \right) \cdot i^{m_1}.
\]

In the light of this proposition, we notice immediately that \((19)\) corresponds to $\eta_{n+1}(\mathcal{K}_{m,m})$. Let $\Gamma_{n+1}(\mathcal{K}_{m,m})$ denote the set of monotone $(n+1)$-colorings of $\mathcal{K}_{m,m}$.

5.1. A Bijection between Monotone Colorings and Proper Mergings of an Antichain and a Chain. Let $a_{(m)} = (A,)$, with $A = \{a_1, a_2, \ldots, a_m\}$, denote an $m$-antichain, and let $c_{(n)} = (C, \leq)$, with $C = \{c_1, c_2, \ldots, c_n\}$, denote an $n$-chain, where the order is indicated by the indices. Since $\mathcal{K}_{m,m}$ consists of two independent sets of size $m$, it is obvious to relate these independent sets to the antichain $a_{(m)}$. We recall from Section 3.1 that the contraordinal scale of $c_{(n)}$ has precisely $(n+1)$ extents. Since we consider monotone $(n+1)$-colorings of $\mathcal{K}_{m,m}$, it is quite evident to relate the color of a vertex in $\mathcal{K}_{m,m}$ to an extent of $\mathcal{C}(c_{(n)})$.

**Definition 5.4.** Let $\gamma \in \Gamma_{n+1}(\mathcal{K}_{m,m})$ be a monotone $(n+1)$-coloring of $\mathcal{K}_{m,m}$. Let the vertex set $V$ of $\mathcal{K}_{m,m}$ be partitioned into sets $V_1 = \{v_1^{(1)}, v_2^{(1)}, \ldots, v_m^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \ldots, v_m^{(2)}\}$. Let $a_{(m)} = (A,) = \{a_1, a_2, \ldots, a_m\}$ denote an $m$-antichain with ground set $A = \{a_1, a_2, \ldots, a_m\}$, and let $c_{(n)} = (C, \leq)$ denote an $n$-chain with ground set $C = \{c_1, c_2, \ldots, c_n\}$, where the order is indicated by the indices. Define relations $R_\gamma \subseteq A \times C$, and $S_\gamma \subseteq C \times A$ as

\[
\begin{align*}
(20) & \quad a_i R_\gamma c_j \quad \text{if and only if} \quad \gamma(v_i^{(1)}) = k \text{ and } n + 2 - k \leq j \leq n, \\
(21) & \quad c_j S_\gamma a_i \quad \text{if and only if} \quad \gamma(v_i^{(2)}) = k \text{ and } 1 \leq j \leq n + 1 - k,
\end{align*}
\]

for all $1 \leq i \leq m$, and $1 \leq j \leq n$.

This means that the row $\{a_i\}^R$ corresponds to the $(n+2-k)$-th extent of $\mathcal{B}(\mathcal{C}(c_{(n)}))$ read from bottom to top if and only if the vertex $v_i^{(1)}$ has color $k$. Similarly, the column $\{c_j\}^S$ corresponds to the $(n+2-k)$-th extent of $\mathcal{B}(\mathcal{C}(c_{(n)}))$ read from bottom to top if and only if the vertex $v_i^{(2)}$ has color $k$. See Figure 14 for an illustration.

**Lemma 5.5.** The relations $R_\gamma$ and $S_\gamma$ from Definition 5.4 form a proper merging of $a_{(m)}$ and $c_{(n)}$. 
Proof. Let \( a_{(m)} = (A, =) \) be an antichain, and denote by \( \mathbb{K}(a_{(m)}) \) the corresponding formal context \((A, A, =)\). We initiate the proof with the investigation of the intents and extents of the contraordinal scale \( C(a_{(m)}) = (A, A, \neq) \). Since \( a_{(m)} \) is an antichain, we can write the cross-table of \( \mathbb{K}(a_{(m)}) \) in such a way that there are only crosses on the main diagonal. It is immediate that we can write the cross-table of \( C(a_{(m)}) \) in such a way that there are crosses in every cell which is not on the main diagonal. It is well-known that the concept lattice \( \mathfrak{B}(C(a_{(m)})) \) is isomorphic to the Boolean lattice with \( 2^m \) elements. See Figure 1(b) for an illustration. This implies that every subset of \( A \) is an intent and an extent of \( C(a_{(m)}) \).

It is immediate from Definition 5.4 that every row of \( R_\gamma \) corresponds to an intent of \( C(c_{(n)}) \), and that every column of \( S_\gamma \) corresponds to an extent of \( C(c_{(n)}) \). With the previous reasoning, this implies that \( R_\gamma \) is a bond from \( C(a_{(m)}) \) to \( C(c_{(n)}) \) and \( S_\gamma \) is a bond from \( C(c_{(n)}) \) to \( C(a_{(m)}) \). Hence, conditions (1) and (2) of Proposition 2.1 are satisfied.

We need to show conditions (3) and (4) of Proposition 2.1, namely that \( R_\gamma \circ S_\gamma \) is contained in the order relation of \( a_{(m)} \), and that \( S_\gamma \circ R_\gamma \) is contained in the order relation of \( c_{(n)} \). Let \( a_i, a_j \in A \) satisfy \((a_i, a_j) \in R_\gamma \circ S_\gamma \), and let \( \gamma(v^{(1)}_i) = l_1, \gamma(v^{(2)}_j) = l_2 \). This means that there is an element \( c_k \in C \) with \( n + 2 - l_1 \leq k \leq n \) and \( 1 \leq k \leq n + 1 - l_2 \). Since \( \gamma \) is a monotone coloring, we know that \( l_1 \leq l_2 \). We obtain

\[
k \leq n + 1 - l_2 \leq n + 1 - l_1 < n + 2 - l_1 \leq k,
\]

and thus \( k < k \), which is a contradiction. Hence, \( R_\gamma \circ S_\gamma = \emptyset \), which proves condition (3). Let now, in turn, \( c_i, c_j \in C \) satisfy \((c_i, c_j) \in S_\gamma \circ R_\gamma \). This means, there must be some \( a_k \in A \) such that the colors \( \gamma(v^{(1)}_k) = l_1 \), and \( \gamma(v^{(2)}_k) = l_2 \) satisfy \( n + 2 - j \leq l_1 \) and \( l_2 \leq n + 1 - i \). Since \( \gamma \) is a monotone coloring, we know that \( l_1 \leq l_2 \), which implies

\[
n + 2 - j \leq l_1 \leq l_2 \leq n + 1 - i.
\]

Hence, \( i < j \), and \( S_\gamma \circ R_\gamma \) is contained in the order relation of \( c_{(n)} \) as desired for condition (4).

\[\begin{array}{c|ccc}
R & c_1 & c_2 & c_3 \\
\hline
a_1 & & & \\
a_2 & & & \\
a_3 & X & & \\
a_4 & X & & \\
\end{array}\]

\[\begin{array}{c|cccc}
S & a_1 & a_2 & a_3 & a_4 \\
\hline
c_1 & & X & & \\
c_2 & & X & & \\
c_3 & & & & \\
\end{array}\]

Figure 14. The relations \( R \) and \( S \) induced by the monotone coloring of \( \overline{K}_{4,4} \) depicted in Figure 13.
Figure 15. The proper merging of a 4-antichain $a_{(4)}$ and a 3-chain $c_{(3)}$ defined by the relations given in Figure 14. The green nodes represent $a_{(4)}$, and the black nodes represent $c_{(3)}$.

It remains to show that $R_{\gamma} \cap S_{\gamma}^{-1} = \emptyset$. Assume the opposite, and let $(a_i, c_j) \in R_{\gamma} \cap S_{\gamma}^{-1}$. Let $\gamma(v_i^{(1)}) = l_1$, and $\gamma(v_i^{(2)}) = l_2$. Hence,

$$n + 2 - l_1 \leq j \leq n + 1 - l_2,$$

which implies $l_2 < l_1$. This is a contradiction to $\gamma$ being a monotone coloring. \qed

Figure 15 shows the poset corresponding to the proper merging depicted in Figure 14. We can conclude the following theorem.

**Theorem 5.6.** Let $\Gamma_{n+1}(\vec{K}_{m,m})$ denote the set of monotone $(n + 1)$-colorings of $\vec{K}_{m,m}$. Let $\mathcal{A}^*_{m,n}$ denote the set of proper mergings of an $m$-antichain and an $n$-chain. Then, the correspondence described in Definition 5.4 is a bijection between $\Gamma_{n+1}(\vec{K}_{m,m})$ and $\mathcal{A}^*_{m,n}$.

**Proof.** It follows immediately from Lemma 5.5 that each monotone $(n + 1)$-coloring of $\vec{K}_{m,m}$ induces a proper merging of an $m$-antichain and an $n$-chain.

Let $a_{(m)} = (A, \leq)$ be an $m$-antichain, where $A = \{a_1, a_2, \ldots, a_m\}$, and let $c_{(n)} = (C, \leq)$ be an $n$-chain, where $C = \{c_1, c_2, \ldots, c_n\}$ and the ordering is induced by the indices. Let $(R, S)$ be a proper merging of $a_{(m)}$ and $c_{(n)}$. Consider the complete bipartite graph $\vec{K}_{m,m}$ and let its vertex set be partitioned into $V_1$ and $V_2$, with $V_1 = \{v_1^{(1)}, v_2^{(1)}, \ldots, v_m^{(1)}\}$ and $V_2 = \{v_1^{(2)}, v_2^{(2)}, \ldots, v_m^{(2)}\}$. Define a coloring $\gamma_{(R,S)}$ of $\vec{K}_{m,m}$ via

$$\gamma_{(R,S)}(v_i^{(1)}) = k \quad \text{if and only if} \quad a_i R c_j \quad \text{for all} \quad j \in \{n + 2 - k, n + 3 - k, \ldots, n\},$$

and

$$\gamma_{(R,S)}(v_i^{(2)}) = k \quad \text{if and only if} \quad c_j S a_i \quad \text{for all} \quad j \in \{1, 2, \ldots, n + 1 - k\},$$

for all $i \in \{1, 2, \ldots, m\}$. Since $R$ is a bond from $C(a_{(m)})$ to $C(c_{(n)})$, every subset of $V_1$ can be colored with color $k$, for $1 \leq k \leq n + 1$. (Every subset of $A$ is an extent of $C(a_{(m)})$, and the set $\{c_{n+2-k}, c_{n+3-k}, \ldots, c_n\}$
is an intent of \( C(c_{(n)}) \) for every \( k \in \{1, 2, \ldots, n+1\} \). Since \( S \) is a bond from \( C(c_{(n)}) \) to \( C(a_{(m)}) \), the same property holds for \( V_2 \). Hence, \( \gamma(R,S) \) is an \( (n+1) \)-coloring of \( \tilde{K}_{m,m} \).

Let \( v_1^{(1)} \in V_1 \) and \( v_2^{(2)} \in V_2 \), with \( \gamma(R,S)(v_1^{(1)}) = l_1 \) and \( \gamma(R,S)(v_2^{(2)}) = l_2 \). By definition, it follows that \( a_i R c_{k_1} \) for all \( k_1 \in \{n+2-l_1, n+3-l_1, \ldots, n\} \), and \( c_{k_2} S a_j \) for all \( k_2 \in \{1, 2, \ldots, n+1-l_2\} \). Assume that \( l_1 > l_2 \). Hence, there exists a \( k \in \{1, 2, \ldots, n\} \) with \( a_i R c_k \) and \( c_k S a_j \). Since \( R \circ S \) is contained in the order relation of \( a_{(m)} \), it follows that \( a_i = a_j \). This means in particular that \( a_i R c_{n+2-l_1} \) and \( c_{n+1-l_2} S a_i \), and thus \( (c_{n+1-l_2}, c_{n+2-l_1}) \in S \circ R \). If \( l_1 = l_2 + 1 \), then \( c_{n+2-l_1} = c_{n+1-l_2} \), which is a contradiction to \( R \cap S^{-1} = \emptyset \). If \( l_1 > l_2 + 1 \), then \( c_{n+2-l_1} < c_{n+1-l_2} \), which is a contradiction to \( S \circ R \) being contained in the order relation of \( c_{(n)} \). Thus, \( \gamma(R,S) \) is a monotone coloring of \( \tilde{K}_{m,m} \) with at most \( n+1 \) colors.

An extensive illustration of this bijection can be found in Appendix D.

**Proof of Theorem 5.1.** This follows immediately from Theorem 5.6 and Proposition 5.3. □

**Remark 5.7.** Let \( \gamma, \delta \in \Gamma_{n+1}(\tilde{K}_{m,m}) \), and let \( V \) denote the vertex set of \( \tilde{K}_{m,m} \). Define a partial order \( \leq \) as

\[
\gamma \leq \delta \quad \text{if and only if} \quad \gamma(v) \leq \delta(v),
\]

for all vertices \( v \in V \). Consider the partition \( V = V_1 \cup V_2 \). Let \( (R_\gamma, S_\gamma) \), and \( (R_\delta, S_\delta) \) denote the proper mergings associated to \( \gamma \) respectively to \( \delta \) in the sense of Definition 5.4. Suppose that \( (R_\gamma, S_\gamma) \preceq (R_\delta, S_\delta) \), and hence by definition \( R_\gamma \subseteq R_\delta \), and \( S_\gamma \supseteq S_\delta \). This implies \( \gamma(v) \leq \delta(v) \) if \( v \in V_1 \), and \( \gamma(v) \leq \delta(v) \) if \( v \in V_2 \), and hence \( \gamma \leq \delta \). This means that the bijection described in Theorem 5.6 is indeed an isomorphism between the lattices \( (\mathcal{B}^*_{m,n}, \preceq) \) and \( (\Gamma_{n+1}(\tilde{K}_{m,m}), \leq) \).

### 5.2. Counting Galois Connections between Boolean Lattices and Chains

Similarly to Section 3.4, we can exploit the bijection described in Theorem 5.6 in order to count the Galois connections between chains with \( n+1 \) elements and Boolean lattices with \( 2^n \) elements. Theorem 3.9 states that every such Galois connection can be described as a dual bond from \( (C, C, <) \) to \( (A, A, \neq) \), where \( A = \{a_1, a_2, \ldots, a_m\} \) and \( C = \{c_1, c_2, \ldots, c_n\} \). Since every extent of \( (A, A, \neq) \) is also an intent of \( (A, A, \neq) \) every dual bond from \( (C, C, <) \) to \( (A, A, \neq) \) corresponds to a bond from \( (C, C, <) \) to \( (A, A, \neq) \). By definition, each such bond corresponds to a proper merging of \( (A, =) \) and \( (C, \leq) \), which is of the form \( (\emptyset, \cdot) \). It follows immediately from Definition 5.4 that each such proper merging corresponds to a monotone coloring of \( \tilde{K}_{m,m} \), where the vertices in \( V_1 \) all have color 1. Hence, each vertex in \( V_2 \) can
take every color \( k \in \{1, 2, \ldots, n + 1\} \). Thus, we can conclude the following proposition.

**Proposition 5.8.** Let \( B_m \) denote the Boolean lattice with \( 2^m \) elements, and let \( c_{(n+1)} \) denote a chain with \( n + 1 \) elements. The number of Galois connections between \( B_m \) and \( c_{(n+1)} \) is \( (n + 1)^m \).

An extensive illustration of this proposition can be found in Appendix E.

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**References**

[1] Daniel Borchmann, conexp-clj – An Extensive Tool for Computations in Formal Concept Analysis. http://daniel.kxpq.de/math/conexp-clj/.

[2] William Y. C. Chen, Sabrina X. M. Pang, Ellen X. Y. Qu, and Richard Stanley, *Pairs of Noncrossing Free Dyck Paths and Noncrossing Partitions*, Discrete Mathematics 309 (2007), 2834–2838.

[3] Bernhard Ganter, Christian Meschke, and Henri Mühle, *Merging Ordered Sets*, Proceedings of the 9th International Conference on Formal Concept Analysis, 2011, pp. 183–203.

[4] Bernhard Ganter and Rudolf Wille, *Formal Concept Analysis: Mathematical Foundations*, Springer, Heidelberg, 1999.

[5] Vladeta Jovović and Goran Kilibarda, *Antichains of Multisets*, Journal of Integer Sequences 7 (2004).

[6] Ian G. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford University Press, Oxford, 1995.

[7] Percy A. MacMahon, *Memoir on the Theory of the Partitions of Numbers – Part 1*, Philosophical Transactions of the Royal Society of London (A) 187 (1897), 619–673.

[8] ______, *Combinatory Analysis, Vol. 2*, Cambridge University Press, Cambridge, 1916.

[9] Neil J. A. Sloane, *The Online Encyclopedia of Integer Sequences*. http://www.research.att.com/~njas/sequences/.

[10] Richard P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 2001.

[11] Herbert S. Wilf, *generatingfunctionology*, A. K. Peters, Ltd., Natick, 2006.

[12] Rudolf Wille, *Restructuring Lattice Theory: An Approach Based on Hierarchies of Concepts*, Ordered Sets (1982), 314–339.
### Appendix A. Illustration of Theorem 3.4, $m = n = 2$

| $\pi \in \mathbb{PP}_{2,2}^{(2)}$ | $R_\pi$ | $S_\pi$ | $(C_1 \cup C_2, \leq_{R_\pi, S_\pi})$ |
|-----------------|------|------|-----------------|
| 0 0             | $\bar{c}_1 \bar{c}_2$ | $c_1 \times \times$ | |
| 0 0             | $\bar{c}_1 \bar{c}_2$ | $c_2 \times \times$ | |
| 1 0             | $\bar{c}_1 \bar{c}_2$ | $c_1 \times \times$ | |
| 0 0             | $\bar{c}_1 \bar{c}_2$ | $c_2 \times \times$ | |
| 1 0             | $\bar{c}_1 \bar{c}_2$ | $c_1 \times \times$ | |
| 1 0             | $\bar{c}_1 \bar{c}_2$ | $c_2 \times \times$ | |
| 1 1             | $\bar{c}_1 \bar{c}_2$ | $c_1 \times \times$ | |
| 0 0             | $\bar{c}_1 \bar{c}_2$ | $c_2 \times \times$ | |
| 1 1             | $\bar{c}_1 \bar{c}_2$ | $c_1 \times \times$ | |
| 1 0             | $\bar{c}_1 \bar{c}_2$ | $c_2 \times \times$ | |
| 1 1             | $\bar{c}_1 \bar{c}_2$ | $c_1 \times \times$ | |
| 1 0             | $\bar{c}_1 \bar{c}_2$ | $c_2 \times \times$ | |
Appendix B. Illustration of Remark 3.8, $n_1 = n_2 = n_3 = 1$

| Collection | $\pi_1$ | $\pi_2$ | $\pi_3$ | Proper Merging |
|------------|---------|---------|---------|----------------|
| 2 1        |         |         |         | ![Diagram](image1) |
| 2 1        |         |         |         | ![Diagram](image2) |
| 2 1        |         |         |         | ![Diagram](image3) |
| 2 1        |         |         |         | ![Diagram](image4) |
| 2 1        |         |         |         | ![Diagram](image5) |
| 2 1        |         |         |         | ![Diagram](image6) |
| 2 1        |         |         |         | ![Diagram](image7) |
| 2 1        |         |         |         | ![Diagram](image8) |
| 2 1        |         |         |         | ![Diagram](image9) |

*Image descriptions for diagrams not provided in the text.*
| 0 1          | 0 1 0          | 0 0 2          |
| 0 2          | 0 2 0          | 0 0 1          |
| 1 0          | 0 1 1          | 0 1 0          |
| 1 1          | 1 0 1          | 1 1 0          |
| 1 1          | 0 1 1          | 1 1 0          |
| 2 1          | 0 1 2          | 0 2 1          |

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Appendix C. Illustration of Proposition 3.10, $m = n = 2$

| $\pi \in \mathbb{P}^{(1)}_{2,2}$ | $T_\pi$ | $\psi_{T_\pi}$ | $\varphi_{T_\pi}$ |
|-------------------------------|--------|----------------|-----------------|
| 0 0                           |        |                |                 |
| 0 0                           |        |                |                 |
| 1 0                           |        |                |                 |
| 0 0                           |        |                |                 |
| 1 0                           |        |                |                 |
| 1 1                           |        |                |                 |
| 0 0                           |        |                |                 |
| 1 1                           |        |                |                 |
| 1 0                           |        |                |                 |
| 1 1                           |        |                |                 |
| 1 1                           |        |                |                 |

Appendix D. Illustration of Theorem 5.6, $m = n = 2$

| $\gamma \in \Gamma_3(\tilde{K}_{2,2})$ | $R_\gamma$ | $S_\gamma$ | $(A \cup C, \leq_{R_\gamma, S_\gamma})$ |
|-----------------------------------------|------------|------------|-----------------------------------|
| 1 1                                     |            |            |                                   |
| 1 1                                     |            |            |                                   |
| 1 1                                     |            |            |                                   |
### Appendix E. Illustration of Proposition 5.8, $m = n = 2$

| $\gamma \in \Gamma_3(K_{2,2}), \gamma(V_1) \equiv 1$ | $S_\gamma$ | $\psi_{S_\gamma}$ | $\varphi_{S_\gamma}$ |
|---|---|---|---|
| 1 1 1 | $\begin{array}{c} a_1 \times \times \\ a_2 \times \times \\ c_1 \\ c_2 \end{array}$ | ![Diagram 1] | ![Diagram 2] |
| 1 1 2 | $\begin{array}{c} a_1 \times \times \\ a_2 \times \times \\ c_1 \\ c_2 \end{array}$ | ![Diagram 3] | ![Diagram 4] |
| 1 1 3 | $\begin{array}{c} a_1 \times \times \\ a_2 \times \times \\ c_1 \\ c_2 \end{array}$ | ![Diagram 5] | ![Diagram 6] |
| 1 2 1 | $\begin{array}{c} a_1 \times \times \\ a_2 \times \times \\ c_1 \\ c_2 \end{array}$ | ![Diagram 7] | ![Diagram 8] |
| 1 2 2 | $\begin{array}{c} a_1 \times \times \\ a_2 \times \times \\ c_1 \\ c_2 \end{array}$ | ![Diagram 9] | ![Diagram 10] |
| 1 2 3 | $\begin{array}{c} a_1 \times \times \\ a_2 \times \times \\ c_1 \\ c_2 \end{array}$ | ![Diagram 11] | ![Diagram 12] |

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