A Stackelberg Game: Exploration and Development of Entrepreneurial Resources

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Abstract. In this paper, a mathematical model of entrepreneurial resource development was established. Entrepreneurs are mainly faced with competition from other companies in the same industry and mature large-sized developing enterprises. These players not only possess more initial resources, but also are more capable at occupying resources. Then the equilibrium solution to zero-sum differential games of entrepreneurial resources was investigated, and the optimal strategy and its relationship with the economic environment were analyzed. The cons and pros of the equilibrium solutions under different scenarios were compared by considering small-sized enterprises and mature developing large-sized enterprises as the leader or the follower of Stackelberg games.

Introduction

Entrepreneurship is a long-standing social phenomenon, and its success results in the continuous change of social economy and promotes the development of social economy. As for local economy, economic growth should be targeted at private economy, and private economy must encourage entrepreneurship. Along with deepening into entrepreneurship knowledge, problems of entrepreneurship risks and finite resource utilization have gradually emerged. Whether entrepreneurs can grasp and use the integrated resources largely decides if they can triumphantly create opportunities and establish new enterprises or expand new business. One method of resource utilization by entrepreneurs is to exploit and integrate the existing resources. This method is low-risk, but the entrepreneurs have to face resource competition from other entrepreneurs and must conquer the large resource occupation, large productivity and strong marketing ability of large-sized enterprises. Another method is to explore new resources. This method is featured by large investment, high yield and large risks, but avoids competition from other entrepreneurs. Moreover, if it succeeds, the entrepreneurs also have to decide whether to run alone or cooperate with other large-size enterprises.

The Britain management expert Elton Penrose pointed out in 1959 the importance of corporate resources to corporate growth and how to utilize corporate resources in the growing strategies. Wernerfel agreed with this view and stated from the perspective of resources rather than products that corporate growth strategy was a balance between “to use the existing resources” and “to develop new resources” [1]. Currently, some active management experts (e.g. Zhang Yuli, Cai Li) stated that entrepreneurs should pay more attention to the use of limited resources into entrepreneurship; though they cannot compete with mature large-size companies for resources, they possess unique creativity and other strategic resources [2,3].

Entrepreneurship researchers have made profound theoretical achievements in resource management and clarified the principles of entrepreneurial resource integration. However, this qualitative method is generalized and does not obey the change of the dynamic market environment, and thus is unable to respond to emergent resource variation. The commonly-used case-study result analysis is non-specific and dependent on the unique properties and personality of entrepreneurs, and thus could not guide so many future entrepreneurs. To depict the dynamic market environment and
quantitatively study the variation of entrepreneurial resources, we established a mathematical model of resource distribution, and used the differential game theory to investigate entrepreneurial resource management. We presented the equilibrium strategy between entrepreneurs and mature large-sized enterprises, and thereby precisely and efficiently assessed the resource decision-making, providing some guidance suggestions.

**Modeling**

We think that a differential game has two state functions \( x(t), y(t) \), where \( x(t) \) and \( y(t) \) are the resource conditions of a small-sized enterprise under entrepreneurship and a mature large-sized developing enterprise, respectively. The mature large-sized developing enterprise and the small-sized enterprise under entrepreneurship were the players 1 and 2 of a differential game. In other words, the control function \( u(t), (0 < u(t) < 1) \) of player 1 means its resource occupying ability; then \( 1 - u(t) \) is its ability of resource development. \( v(t) \) is the control function of player 2 and means its ability to occupy resources. The dynamic system equation is:

\[
\begin{align*}
\dot{x}(t) &= bx(t) - (u(t)y(t))^a(v(t))^b, x(0) = x_0 > 0, \\
\dot{y}(t) &= \alpha(1-u(t))y(t) - fv(t), y(0) = y_0 > 0.
\end{align*}
\]

where \( 0 < a < 1 < s \), \( x_0, y_0 \) are the initial values of \( x(t), y(t) \) at time \( t = 0 \), respectively. We suppose \( y(t) > 0 \) is true in the whole time interval \([0, +\infty)\). And \( b, \alpha, f \) parameters are constants larger than 0. \((uy)^a v^s\) is a function conforming to Inada Conditions. Essentially, the Cobb Douglas production function in the field of economics was introduced into a reciprocal process.

We considered zero-sum differential games in which the target functions of the two players were:

\[ J(u, v) = \int_0^\infty e^{-\rho t} \left[ ly(t) - cx(t) \right] dt \]

where \( \rho, l, c \) are constants larger than 0. We considered a perfect competition model.

Player 1 wanted to use control function \( u(t) \) to increase the target function \( J(u, v) \) as much as possible. On the contrary, player 2 wanted to use control function \( v(t) \) to reduce the target function \( J(u, v) \) as much as possible. \( \rho \) is the discount rate. Let \( \rho > b, \rho > \alpha \). In real life, mature developing large-sized enterprises expect to possess more resources \( y(t) \) and wish the small-sized enterprises have fewer resources \( x(t) \). However, the small-sized enterprises under entrepreneurship expect the opposite way.

**Solutions to Stackelberg Games**

**Small-sized Enterprises under Entrepreneurship as the Leader; Mature Developing Large-sized Enterprises as the Follower**

In this section, player 2 was considered as the upper layer and player 1 as the lower layer of a Stackelberg game.

**Optimization of Player 1.** We first discussed the optimization of the follower.

\[
\text{max}_u \int_0^\infty e^{-\rho t} \left( ly(t) - cx(t) \right) dt
\]

subject to:

\[
\begin{align*}
\dot{x} &= bx - (uy)^a v^s, \\
\dot{y} &= \alpha(1-u)y - fv.
\end{align*}
\]

Then Hamiltonian function was introduced:

\[
H_1(u) = ly(t) - cx(t) + p_x[bx(t) - (uy)^a v^s(t)] + p_y[\alpha(1-u)y(t) - fv(t)]
\]
where $p_1$ and $p_2$ are the state adjoint variables of players 1 and 2, respectively. According to the principle of Pontryagin maximum value, the optimal control function $u$ and corresponding state function $x$ should meet the following first-order condition:

$$\frac{\partial H}{\partial u} = -ap_1y^\alpha v^\alpha u^{\alpha-1} - \alpha p_2y = 0,$$

(4)

The corresponding adjoint system is:

$$\dot{p}_1 = \rho p_1 - \frac{\partial H_1}{\partial x} = (\rho - b)p_1 + c,$$

(5)

$$\dot{p}_2 = \rho p_2 - \frac{\partial H_1}{\partial y} = [\rho - \alpha(1-u)]p_2 + ap_1y^\alpha v^\alpha y^{\alpha-1} - l.$$

(6)

From Eq.(4), we got:

$$u^* = \alpha p_2 \left[\frac{1}{ap_1y^\alpha v^\alpha y^{\alpha-1}}\right]^{\alpha-1}$$

(7)

Substituting Eq. (7) into Eq.(6), then

$$\dot{p}_2 = (\rho - \alpha)p_2 - l.$$

(8)

Optimization of Player 2. Then we discussed the optimization of the leader.

$$\max_v \int_0^\infty e^{-\alpha t}(-ly + cx)dt,$$

$$\begin{cases}
\dot{x} = bx - (u^*v)^\alpha v^\alpha, \\
\dot{y} = \alpha(1-u^*)y - fv, \\
\dot{p}_1 = (\rho - b)p_1 + c, \\
\dot{p}_2 = (\rho - \alpha)p_2 - l.
\end{cases}$$

(9)

Proposition 1. For the Stackelberg game involving the small-sized enterprises under entrepreneurship as the leader and the mature developing large-sized enterprises as the follower, the optimal game is:

$$u^* = \frac{afv^*}{\alpha s y}, v^* = \left(-\frac{\alpha s^\alpha p_2}{a^\alpha f^\alpha p_1}\right)^{1/\alpha^\alpha-1} \text{ and } y^* = e^{\alpha t_y} \left(y_0 - \frac{g}{\alpha} + \frac{g}{\alpha} \right), \text{ where } g = \left(\frac{a}{s} + 1\right)fv^*.$$

According to Proposition 1, we find decision-making with the mature developing large-sized enterprises as the follower and the small-sized enterprises as the leader is not beneficial to large-sized enterprises. The equilibrium solution of the Stackelberg game is the same as the Nash equilibrium solution of the concurrent selection game. The reason is the information asymmetry between the two parties, as the large-sized enterprises are at the bright side and the small-sized enterprises are at the dark side (the equilibrium solution is associated only with state $x$, but not with state $y$). The mature large-sized enterprises as the follower possess information similar to that as concurrent selection, but are not dominant. Thus, no more favorable prediction can be made according to the current situations. Similarly, the small-sized enterprises as the leader, when they know current state information of the large-sized enterprises, would select the optimal game, which is still a constant. Maintaining this resource-occupying ability would maximize the benefits of small-sized enterprises. Moreover, in the Stackelberg game with the small-sized enterprises as the leader and the large-sized enterprises as the follower, the optimal two-party game has the same characteristics as the Nash equilibrium game.

Mature Large-sized Enterprises under as the Leader; Small-sized Enterprises as the Follower

In this section, player 1 was considered as the upper layer and player 2 as the lower layer of a Stackelberg game.
Optimization of Player 2

\[
\max_v \int_0^\infty e^{-\rho t} (-ly + cx)dt,
\]
\[\text{s.t. } \begin{cases}
\dot{x} = bx - (uy)^\rho v^* , \\
\dot{y} = \alpha(1-u)y - fv.
\end{cases}
\]  

(10)

Hamiltonian function was introduced,

\[
H_2(v) = -ly + cx + p_3[bx - (uy)^\rho v^*] + p_4[\alpha(1-u)y - fv]
\]

where \( p_3 \) and \( p_4 \) are the state adjoint variables of players 1 and 2, respectively. According to the principle of Pontryagin maximum value, the optimal control function \( v \) and corresponding state function \( x \) should meet the following first-order condition:

\[
\frac{\partial H}{\partial v} = -sp_3(uy)^\rho v^{*-1} - fp_4 = 0,
\]  

(11)

The corresponding adjoint system is:

\[
\dot{p}_3 = \rho p_3 - \frac{\partial H}{\partial x} = (\rho - b)p_3 - c,
\]

\[
\dot{p}_4 = \rho p_4 - \frac{\partial H}{\partial y} = (\rho - \alpha(1-u))p_4 + l + ap_3u^\alpha v^\alpha y^{-1}.
\]

According to Eq. (11),

\[
v^* = \left[ -\frac{fp_4}{sp_3(uy)^\rho} \right]^{1-\frac{1}{\alpha\rho}}
\]  

(12)

Optimization of Player 1. Then we discussed the optimization of the leader.

\[
\max_u \int_0^\infty e^{-\rho t} (ly - cx)dt,
\]
\[\text{s.t. } \begin{cases}
\dot{x} = bx - (uy)^\rho v^* , \\
\dot{y} = \alpha(1-u)y - fv^*,
\end{cases}
\]

\[
\dot{p}_3 = \rho p_3 - \frac{\partial H}{\partial x} = (\rho - b)p_3 - c,
\]

\[
\dot{p}_4 = \rho p_4 - \frac{\partial H}{\partial y} = (\rho - \alpha(1-u))p_4 + l + ap_3u^\alpha v^\alpha y^{-1}v^*.
\]

Proposition 2. For the Stackelberg game involving the large-sized mature developing enterprises as the leader and the small-sized enterprises under entrepreneurship as the follower, the optimal game is:

\[
u^* = \left[ \frac{\alpha(1-s)(y\lambda_k - \rho\lambda_k)}{a\phi} \right] \left( -\frac{sp_3y^\alpha}{fp_4} \right)^{1-\frac{1}{\alpha\rho}} u^* ,
\]

\[
v^* = \alpha(1-s)(y\lambda_k - \rho\lambda_k) u^* ,
\]

where \( \phi = -\frac{af\lambda_k p_4}{sy} + \frac{f^2 p_4}{sp_3} - f\lambda_k .
\]

According to Proposition 2, \( v^* \) and \( u^* \) are directly proportional, which is consistent with the conclusion from Nash equilibrium. Namely, as the resource occupying ability of the small-sized enterprises is enhanced, the large-sized enterprises also would grab and occupy more resources. Furthermore, relationship between the increasing rate \( \alpha \) of resource exploitation by the large-sized
enterprise and $u^*$ or $v^*$ was analyzed. The first equation from (13) shows $u^*$ is proportional to $\frac{1}{s^\alpha}$. When $\alpha$ increases, $u^*$ decreases. Namely when the resource exploitation increasing rate of large-sized enterprises is large, the proportion of the occupied resources decreases. The second equation from (13) shows $v^*$ is proportional to $\frac{a}{s^\alpha}$. When $\alpha$ increases, $v^*$ also rises. Namely when the resource exploitation increasing rate of the large-sized enterprise increases, the the resource occupying ability of small-sized enterprises is also strengthened.

Clearly, in the game between the large-sized mature enterprises and the small-sized enterprises, if the large-sized enterprises became more active to occupy the leading position, it would force the small-sized enterprises to take fewer resources.

Conclusions
The stratified structure of Stackelberg game was analyzed. Two situations were considered: (1) the mature developing large-sized enterprises as the leader and small-sized enterprises under entrepreneurship as the follower; (2) the mature developing large-sized enterprises as the follower and small-sized enterprises under entrepreneurship as the leader. The relationship between feedback equilibrium solutions was analyzed, and the optimal strategy of resource exploration between two enterprises was clarified. In the future, we will focus on the large-sized enterprises dominated multi-party differential games, so as to better adapt to the current economy situations. The game between resource exploration and resource development at this moment not only is a layered Stackelberg game, but also a common Nash equilibrium differential game.

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