Bifurcation analysis of the threshold of a solid-state laser

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Abstract. In this paper we describe the qualitative analysis of the dynamic model of a solid-state laser. Initially a brief description of the theory of dynamic systems is made, the part concerning the qualitative solutions of non-linear models, the obtaining of fixed points for second-order systems and the determination of the stability of these points. Later, the dynamic model of the solid-state laser is exposed, where all its inherent parameters and the behavior that follows will be described. Then, the qualitative analysis is developed where the model will be converted to dimensionless equivalent that contains two parameters that condense to all the others. The results obtained are verified with numerical analysis, there clarifies the veracity of the qualitative analysis in conjunction with the description of the behavior of the solid-state laser, to finish with the conclusions obtained from the whole development.

1. Introduction

For all areas of knowledge is essential to study their inherent physical processes, in most cases these processes have an evolution over time and are dependent on parameters that make the process remain at a point of operation (equilibrium) or that act in an undesired way. The previous description describes the problem of the theory of dynamic systems, which studies the modelling of physical systems generally through mathematical modeling by ordinary differential equations to see their changes over time and predict their behavior [1].

Since time immemorial humanity has had a great fascination to know the future of events that pertain to life itself, without entering into philosophical or religious discussions is important to note that with science it has been possible to clarify and give meaning to several discussions. Isaac Newton, great forerunner of mathematics and known as the father of calculus, said that one way to predict the future was to know and solve the differential equations that model dynamic systems, such that he could formulate the whole basis of classical mechanics and knowing with great precision the behavior of several celestial bodies through mathematical representations, based on this fact and thanks to the motivation of great scientists of the following 300 years after Newton, is that it was possible to originate a theory that will point to know the future of physical processes; the dynamic systems [2].

The theory of dynamic systems is properly established by the physicist-mathematician Henri Poincaré with the approach and solution of the problem of the three bodies, the formal study deals with solving techniques for a mathematical model that describes some physical process. Among the solution techniques, there is a wide variety among which the qualitative techniques are distinguished, which will be described and applied in the present document for the analysis of the dynamic second order model of a solid-state laser device.
2. Laser model
The physical process to be analyzed is the dynamics of a type of laser known as a solid-state laser, which consists of a collection of special atoms embedded in a solid-state matrix, which is bounded by partially reflecting mirrors in each extreme, to this is added an external energy source that excites the atoms. Each of these atoms is in some way a small antenna that emits or reeects energy, the process is that initially the source provides a relatively weak energy injection and thus the device acts as a common ashlight, ie atoms excited they oscillate separately from each other and emit light waves in random phase. When the force of energy injection is increased, previously nothing different happens, but later when a certain threshold of energy is exceeded the atoms begin to oscillate in phase and the device acts like a laser, here the atoms cooperate working as one and produce a Ray of radiation much more intense than that produced under the laser threshold. The model of the dynamic system described above obeys to (Equation 1) [3,4].

\[
\begin{align*}
\dot{n} &= Gn - kn \\
\dot{N} &= -Gn - fN + p
\end{align*}
\]

where \( N \) is the number of excited atoms, \( n \) the number of photons in the laser field, \( G \) is the stimulated emission gain coefficient, \( k \) is the rate of decrease due to the loss of photons by mirror transmission or scattering, \( f \) it is the rate of decrease by spontaneous emission and \( p \) is the force of energetic injection, all the parameters are positive except \( p \) that it can be of both signs.

2.1. Model dimensionless
In order to start the analysis of the dynamic system (Equation 2), it is necessary to study its dimensionless form, describing the parameters in Figure 1 and Figure 2.

\[
n = \frac{f}{G}x, N = \frac{f}{G}y, \tau = ft
\]

Finally, a biparametric system is obtained (Equation 3, Equation 4 and Equation 5), where \( \lambda \) represents the rate of decrease of photons by the rate of decrease of spontaneous emission and \( \rho \) is the energy injection multiplied by the gain on the rate of spontaneous decrease [5,6].

\[
\begin{align*}
\dot{x} &= xy - \delta x, \delta = k/f \\
\dot{y} &= -xy + y + \rho, \rho = pG/f^2
\end{align*}
\]

\[
(x,y) = (0,0), x_1^* \rightarrow (x^*, y^*)_1 = (0, \rho), x_2^* \rightarrow (x^*, y^*)_2 = \left(\frac{\rho - \delta}{\delta}, \delta \right)
\]

\[
J = \left[ \begin{array}{ccc}
\rho - \delta & x & -y \\
-x & 0 & -1
\end{array} \right], J_1 = \left( \begin{array}{ccc}
\rho - \delta & 0 & -1 \\
-\rho & 1 & 1
\end{array} \right)_{(x^*, y^*)_1}, J_2 = \left( \begin{array}{ccc}
0 & \rho - \delta & -1 \\
-\delta & -\rho & 1
\end{array} \right)_{(x^*, y^*)_2}
\]

From the eigenvalues obtained above it can be seen that the stability for the fixed point 1 (Equation 6) depends on the fact that the eigenvalue \( \lambda_{12} \) is less than zero, i.e. \( \rho < \delta \), at the moment in which the two parameters are equal, there is a bifurcation point, since from there is a variation in any of the two values the stability of this point changes, Figure 1. On the other hand, the stability at the fixed point 2 (Equation 7) is affected by the \( \rho > \delta \) and \( \rho > 0 \) ratios, so that only one fixed point will be taken at a time (see Figure 2) [7,8].

\[
P_1: \lambda^2 + (\delta - \rho + 1)\lambda + \delta - \rho = 0, \lambda_{11} = -1, \lambda_{12} = \rho - \delta
\]
\[ P_2: \lambda^2 + \frac{\rho}{\delta} \lambda + \rho - \delta = 0, \quad \lambda_{21} = \frac{-\rho - \sqrt{\rho^2 - 4\delta^2\rho + 4\delta^4}}{2\delta}, \quad \lambda_{22} = \frac{-\rho + \sqrt{\rho^2 - 4\delta^2\rho + 4\delta^4}}{2\delta} \]  

(7)

In Figure 1 and Figure 2 it is possible to see according to (Equation (3)) obtained from the fixed points, the relationship between the parameters \( x, \delta, \rho \) (Equation (8)) the blue bars indicate the values for which the fixed point in question is stable and the blue for when it is not.

\[ x = \frac{\rho}{\delta} - 1 \]  

(8)

**Figure 1.** Stability of the fixed point 1, \( x_1^* \).

**Figure 2.** Stability of the fixed point 2, \( x_2^* \).

In Figure 2 we can see an empty region which indicates that we have complex eigenvalues with imaginary part other than zero, this indicates that the fixed point is attractor or repeller according to the stability ratio: \( \rho > \delta \). In order to verify the analysis carried out previously, numerical results are obtained through the “PPlane8” library of Matlab. Figure 3, Figure 4, Figure 5 and Figure 6 show the behavior of the fixed points found and their stability variation according to its parameters [9,10].

**Figure 3.** Phase diagram with \( \rho > \delta \).

**Figure 4.** Phase diagram with \( \rho < \delta, \rho > 0 \).
According to Figure 7, we can see a summary of the zones that describe the stability of the fixed points according to the parameters, in addition, it was previously that the bifurcation of the system was generated when $\rho = \delta$, however, in Figure 8 shows that there the two fixed points and their eigenvalues are equal, $x_1^* = x_2^*$, $\lambda_{11} = \lambda_{21} = -1$, $\lambda_{12} = \lambda_{22} = 0$, these values of the fixed points according to Figure 7 are equivalent to having a center or an attractor and that from any variation of the parameters. This stability is broken [11-13], except for this result all others are met according to the description of fixed points.

3. Conclusion
The solution of dynamic systems by means of qualitative techniques provides a lot of information about the physical process developed, in addition, if it is required to observe the stability limits of the fixed points, the bifurcation analysis provides even more information about the prediction of the behavior of the system, as it was found based on the supplied numeric response from software qualitative analysis of the stability of fixed points was successful, except for the bifurcation point, for this case it was found that the fixed points converging to one and that its stability should be different from that presented in Figure 2, even though the stability study does not correct the incident in this case can be approached in future work from other perspectives. It must be taken into account that, although this model correctly predicts the existence of a threshold, it ignores the dynamics of excited atoms, the existence of spontaneous emission and several other complications which must be addressed from the field of quantum physics [7].
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