Hyper-Dilaton Weyl Multiplets of 5D and 6D Minimal Conformal Supergravity

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Abstract

By extending the recent analysis of [arXiv:2203.12203] for $\mathcal{N} = 2$ conformal supergravity in four dimensions, we define new hyper-dilaton Weyl multiplets for five-dimensional $\mathcal{N} = 1$, and six-dimensional $\mathcal{N} = (1,0)$ conformal supergravities. These are constructed by coupling the five- and six-dimensional standard Weyl multiplets to on-shell hypermultiplets and reinterpreting the systems as new multiplets of conformal supergravity. In the five-dimensional case, we also construct a new hyper-dilaton Poincaré supergravity by coupling to an off-shell vector multiplet compensator. As in four dimensions, a $BF$-coupling induces a non-trivial scalar potential for the five-dimensional dilaton that admits AdS$_5$ vacua.
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1 Introduction

A key ingredient to efficiently engineer off-shell supergravity-matter couplings is the fact that Poincaré gravity can be formulated as conformal gravity coupled to a compensating scalar field [1,2]. This approach plays an equally important role both in the superconformal tensor calculus and in superspace supergravity formalisms — see [3,4] and [5,6] for reviews and references. In the supersymmetric case, conformal gravity is turned into an off-shell representation of the local superconformal algebra containing the vielbein as one of its independent fields. Such multiplet is referred to as the Weyl multiplet of conformal supergravity. The scalar compensator is also lifted to an off-shell locally superconformal multiplet. Depending on the amount of supersymmetry, due to the existence of several possible choices of compensating multiplets, it is possible to obtain several different off-shell Poincaré supergravity theories. Moreover, the fact that the Weyl multiplets themselves are in general not unique adds to the richness of the off-shell representations.

The first instance where variant representations of the Weyl multiplets were presented is six-dimensional (6D) minimal $\mathcal{N} = (1,0)$ supergravity [7] — see also [8–15] for further references on 6D conformal supergravity. In this case, it was noted that the so-called standard Weyl multiplet could be turned into a dilaton Weyl multiplet by reinterpreting the system described by an on-shell tensor multiplet in a standard Weyl multiplet background as a new conformal supergravity multiplet. Such a tensor-dilaton Weyl multiplet plays an important role since, once coupled to a second off-shell conformal compensating multiplet, is the one used to construct the simplest versions of two-derivatives Poincaré supergravity theories. Extending the idea of [7], dilaton Weyl multiplets have been discovered also for five-dimensional (5D) $\mathcal{N} = 1$ [16] and, more recently, for four-dimensional (4D) $\mathcal{N} = 2$ conformal supergravities in [17] and [18].

In 5D $\mathcal{N} = 1$ supergravity the known dilaton Weyl multiplet is constructed by coupling an on-shell vector multiplet to the standard Weyl multiplet [16]. See [19–25] for more discussions of 5D conformal supergravity and its matter couplings. The 4D $\mathcal{N} = 2$ analogue of this type of vector-dilaton Weyl multiplet was constructed in [17]. It is natural to expect that different on-shell multiplets containing a scalar field playing the role of a
dilaton could be used to engineer other multiplets of conformal supergravity. In fact, for
the 4D \( \mathcal{N} = 2 \) case, in [18] a so-called hyper-dilaton Weyl multiplet was constructed by
using an on-shell hypermultiplet. The scope of this paper is to present the extension of
the analysis of [18] to 5D \( \mathcal{N} = 1 \) and 6D \( \mathcal{N} = (1,0) \) supergravities.

Besides a mathematically oriented interest in classifying variant Weyl multiplets, it
is worth to explore new options to define off-shell Poincaré supergravities with an eye
on their broad range of applications. For example, in our opinion, for theories with
eight supercharges it remains an open problem to have a simple, though general, off-
shell approach for gauged supergravities with no physical matter hypermultiplets. In the
presence of physical charged hypermultiplets (with no central charge) one will need to
use representations containing an infinite number of auxiliary/matter fields [14, 15, 19,
20, 26–40], but with only physical vector multiplets it might be beneficial to use simpler
approaches, if possible. The exploration of options to fill this gap was one of the main
migrations for the recent construction in [18].

Interesting applications of off-shell approaches to (gauged) supergravity includes the
construction of locally supersymmetric higher-derivative invariants [8–13, 20, 41–55]. This
topic has recently attracted a renewed attention due to advances in the study of black-
hole entropy and next to leading order AdS/CFT calculations — see the very recent
works [56–61] and references therein. Vector-dilaton Weyl multiplets have been a main
ingredient to construct off-shell higher-derivative supergravities in five and six dimensions,
see [8–10, 13, 20, 41, 47, 48, 51, 54]. We hope new hyper-dilaton Weyl multiplets might play an
interesting role to extend some of these results to gauged supergravity. The construction
of alternative Weyl multiplets could also play an interesting role to develop alternative
approaches to localisation of quantum field theories on curved space-times — see [62] for
a recent extensive review. In this context, off-shell supersymmetry has been a central
ingredient for localisation and new Weyl multiplets could offer alternative starting points.

This paper is organised as follows. In section 2 we use the conformal superspace
approach to 5D \( \mathcal{N} = 1 \) supergravity [20] to review the locally superconformal multiplets used
in our analysis. Specifically, we introduce the 5D standard Weyl multiplet and its geometric
superspace construction, the on-shell hypermultiplet, the linear (or \( \mathcal{O}(2) \)) multiplet,
and the Abelian vector multiplet. Section 3 is devoted to a description of the standard
Weyl multiplet in terms of component fields in the notations of our paper. Section 4
describes the construction of the new 32+32 5D \( \mathcal{N} = 1 \) hyper-dilaton Weyl multiplet. In
section 5 we couple the hyper-dilaton Weyl multiplet to a vector multiplet compensator
to recover a new 40+40 Poincaré supergravity. This can be thought of as a 5D analogue
of the 4D off-shell $\mathcal{N} = 2$ supergravity introduced by Müller in 1986 [63] and redefined by using superconformal techniques in [18]. A distinctive feature of the dilaton Poincaré supergravity is the fact that the off-shell multiplet is irreducible, while the two-derivative supergravity action leads to the minimal on-shell Poincaré supergravity multiplet coupled to an extra physical matter multiplet containing the dilaton. Interestingly, as in the 4D analysis of [18], in subsection 5.2 we show how it is possible to engineer a non-trivial scalar potential for the dilaton and obtain an AdS$_5$ vacua in a framework different than standard gauged supergravity. In sections 6, 7, and 8 we closely repeat the 5D analysis of sections 2, 3, and 4 in the case of 6D $\mathcal{N} = (1,0)$ conformal supergravity. The paper also contains two technical appendices, A and B, where we collect conformal superspace identities from [20] and [11,12] used in our paper.

2 Superconformal multiplets in 5D $\mathcal{N} = 1$ superspace

This section reviews the salient details of several superconformal multiplets pertinent to this work. We first describe the standard Weyl multiplet of conformal supergravity in 5D $\mathcal{N} = 1$ conformal superspace before moving on to the discussion of various matter multiplets: the on-shell hypermultiplet, together with the off-shell linear and Abelian vector multiplets. Here we make use of the formulation and results of [20]. We also refer the reader to the following list of papers for other work on flat and curved superspace and off-shell multiplets in five dimensions [19,33,35,64,67].

2.1 The standard Weyl multiplet

In five dimensions, the standard Weyl multiplet of $\mathcal{N} = 1$ conformal supergravity [16] contains $32 + 32$ physical components and is associated with the gauging of the superconformal algebra $F^2(4)$. Associated respectively with local translations, $Q$-supersymmetry, $\text{SU}(2)_R$ symmetry, and dilatations are the vielbein $e_m^a$, the gravitino $\psi_m^i$, the $\text{SU}(2)_R$ gauge field $\phi_m^{ij}$, and a dilatation gauge field $b_m$. There are three composite connections which are associated with the remaining gauge symmetries: these are the spin connection $\omega_m^{ab}$, the $S$-supersymmetry connection $\phi_m^i$, and the special conformal connection $f_m^a$ that are algebraically determined in terms of the other fields by imposing constraints on some of the curvature tensors. To achieve an off-shell representation of the 5D $\mathcal{N} = 1$ local superconformal algebra, three covariant auxiliary fields are introduced: a real antisymmetric tensor $w_{ab}$, a fermion $\chi^i$, and a real auxiliary scalar $D$. In this subsection,
we show how to embed this in conformal superspace. The component structure of the multiplet is given in section 3.

The 5D $\mathcal{N} = 1$ conformal superspace is parametrised by local bosonic $(x^m)$ and fermionic $(\theta_i)$ coordinates $z^M = (x^m, \theta^\mu_i)$, where $m = 0, 1, 2, 3, 4$, $\mu = 1, \cdots, 4$ and $i = 1, 2$. By gauging the full 5D $\mathcal{N} = 1$ superconformal algebra, we introduce covariant derivatives $\nabla_A = (\nabla_a, \nabla^i_\alpha)$ which take the form

$$\nabla_A = E^a_A - \frac{1}{2} \Omega^a_{\alpha b} M_{ab} - \Phi^{ij}_A J_{ij} - B_A \mathbb{D} - \mathfrak{F}^{ab} K_B \ , \quad \nabla^i_\alpha = E^a_A - \frac{1}{2} \Omega^a_{\alpha b} M_{ab} - \Phi^{ij}_A J_{ij} - B_A \mathbb{D} - \mathfrak{F}^{\alpha i}_b S_{\alpha i} - \mathfrak{F}^{a}_A K_a \ .$$

(2.1a)

(2.1b)

Here $E^a_A = E^a_M \partial_M$ is the inverse super-vielbein, $M_{ab}$ are the Lorentz generators, $J_{ij}$ are generators of the SU(2)$_R$ $R$-symmetry group, $\mathbb{D}$ is the dilatation generator, and $K_A = (K_a, S_{\alpha i})$ are the special superconformal generators. The super-vielbein one-form is $E^a_A = dz^M E^a_M$ with $E^a_M E^b_N = \delta^b_N$, $E^a_M E^b_B = \delta^b_B$. Associated with each structure group generator $X_a = (M_{ab}, J_{ij}, D, S_{\alpha i}, K_a)$ is the following connection super one-form $\omega^a = (\Omega^a_{\alpha b}, \Phi^{ij} B, \mathfrak{F}^{\alpha i}$, $\mathfrak{F}^a) = d z^M \omega^a_M = E^a \omega^a_M$.

To describe the standard Weyl multiplet in conformal superspace, one constrains the algebra of covariant derivatives

$$\{\nabla_A, \nabla_B\} = -\mathcal{T}_{AB}^C \nabla_C - \frac{1}{2} \mathcal{R}(M)_{AB}^{cd} M_{cd} - \mathcal{R}(J)_{AB}^{kl} J_{kl}$$

$$- \mathcal{R}(\mathbb{D})_{AB} \mathbb{D} - \mathcal{R}(S)_{AB}^{\gamma k} S_{\gamma k} - \mathcal{R}(K)_{AB}^c K_c \ ,$$

(2.2)

to be completely determined in terms of the symmetric super-Weyl tensor superfield $W_{\alpha \beta}$, which is a superconformal primary with conformal dimension 1

$$W_{\alpha \beta} = W_{\beta \alpha} \ , \quad K_A W_{\alpha \beta} = 0 \ , \quad \mathbb{D} W_{\alpha \beta} = W_{\alpha \beta} \ ,$$

(2.3)

and obeys the Bianchi identity

$$\nabla^k W_{\alpha \beta} = \nabla^k (\alpha W_{\beta \gamma}) + \frac{2}{5} \varepsilon_{\gamma(\alpha} \nabla^{\delta k} W_{\beta)\delta} \ .$$

(2.4)

The relation $W_{\alpha \beta} = 1/2 (\Sigma^{ab})_{\alpha \beta} W_{ab}$ means that the super-Weyl tensor is equivalent to an antisymmetric rank-2 tensor superfield $W_{ab} = -W_{ba}$. In (2.2) $\mathcal{T}_{AB}^C$ is the torsion, and $\mathcal{R}(M)_{AB}^{cd}$, $\mathcal{R}(J)_{AB}^{kl}$, $\mathcal{R}(\mathbb{D})_{AB}$, $\mathcal{R}(S)_{AB}^{\gamma k}$, and $\mathcal{R}(K)_{AB}^c$ are the curvatures associated with Lorentz, SU(2)$_R$, dilatation, $S$-supersymmetry, and special conformal boosts, respectively. Their expressions in terms of $W_{\alpha \beta}$ and its descendant superfields of dimension 3/2

$$W_{\alpha \beta \gamma} := \nabla^k (\alpha W_{\beta \gamma}) \ , \quad X^i_\alpha := \frac{2}{5} \nabla^{\beta i} W_{\beta \alpha} \ ,$$

(2.5a)
and of dimension 2

\begin{align}
W_{\alpha\beta\gamma\delta} & := \nabla^k_{(\alpha} W_{\beta\gamma\delta)k} , \quad X_{\alpha\beta}^{ij} := \nabla^{(i}_{(\alpha} X^{j)}_{\beta)} , \quad Y := i\nabla^k X_{\gamma k} . \tag{2.5b}
\end{align}

are collected in appendix A. Note that in this paper we will make use of conformal superspace with a redefined vector covariant derivative. This corresponds to choosing the “traceless” frame conventional constraints employed for the first time in subsection 4.4 and appendix C of [20]. The component and superspace structure corresponding to this choice are summarised in section 3 and appendix A.

The superfields \( W_{\alpha\beta\gamma}^k, X^i_{\alpha}, W_{\alpha\beta\gamma\delta}, X_{\alpha\beta}^{ij}, \) and \( Y \) satisfy nontrivial Bianchi identities given in [20]. Thus, only these five superfields and their vector derivatives appear upon taking successive spinor derivatives on \( W_{\alpha\beta} \). Eq. (A.5) gives the action of the \( S \)-generators on these independent descendants that prove to be all annihilated by \( K_a \).

The conformal supergravity gauge group \( \mathcal{G} \) is generated by covariant general coordinate transformations, \( \delta_{\text{cgct}} \), associated with a local superdiffeomorphism parameter \( \xi^A \) and standard superconformal transformations, \( \delta_{\mathcal{H}} \), associated with the following local superfield parameters: the dilatation \( \sigma \), Lorentz \( \Lambda^{ab} = -\Lambda^{ba} \), SU(2) \( \Lambda^{ij} = \Lambda^{ji} \), and special conformal transformations \( \Lambda^A = (\eta^{ai}, \Lambda^a_K) \). The covariant derivatives transform as

\begin{align}
\delta_{\mathcal{G}} \nabla_A = [\mathcal{K}, \nabla_A] , \tag{2.6}
\end{align}

where \( \mathcal{K} \) denotes the first-order differential operator

\begin{align}
\mathcal{K} = \xi^C \nabla_C + \frac{1}{2} \Lambda^{ab} M_{ab} + \Lambda^{ij} J_{ij} + \sigma \mathbb{D} + \Lambda^A K_A . \tag{2.7}
\end{align}

A covariant (or tensor) superfield \( U \) transforms as

\begin{align}
\delta_{\mathcal{G}} U = (\delta_{\text{cgct}} + \delta_{\mathcal{H}}) U = \mathcal{K} U . \tag{2.8}
\end{align}

The superfield \( U \) is said to be superconformal primary of dimension \( \Delta \) if \( K_A U = 0 \) (it suffices to require that \( S_{\alpha i} U = 0 \)) and \( \mathbb{D} U = \Delta U \).

\section{2.2 The on-shell hypermultiplet}

The on-shell realisation for the hypermultiplet contains 4 + 4 degrees of freedom, exactly as in the four-dimensional case [68,69]. In conformal superspace, it is described by a Lorentz scalar superfield \( q^{\dot{a}} \) subject to the constraint

\begin{align}
\nabla^{(i}_{\alpha} q^{j)}_{\beta} = 0 , \tag{2.9}
\end{align}
which is equivalent to
\[ \nabla_{\alpha} q^{k} = -\frac{1}{2} \varepsilon^{ij} \rho^{k}_{\beta} , \quad \rho^{k}_{\beta} := \nabla_{\alpha} q^{k} . \] (2.10)

Here, the index \( i = 1, 2 \) denotes an SU(2) flavour index. The superfield \( q^{k} \) is a Lorentz scalar and superconformal primary,
\[ M_{ab} q^{k} = 0 , \quad K_{\alpha} q^{k} = 0 , \quad J^{jk} q^{k} = \varepsilon^{ij} q^{k} . \] (2.11)

Eqs. (2.9), (2.11), and the relation (A.1k) tell us that \( \square q^{k} = \frac{3}{2} q^{k} \).

The independent descendants of \( q^{k} \) are obtained by acting on it with spinor derivatives. We obtain several implications of the (anti-)commutation relations (A.2), (A.3), along with the constraints (2.10), and (2.11):
\[ \nabla_{\alpha} \nabla_{\beta} q^{k} = -4i \nabla_{\alpha} \rho_{\beta} , \quad \nabla_{\alpha} \rho_{\beta} = 0 , \] (2.12a)
\[ \nabla_{\alpha} \rho_{\beta} = 0 . \] (2.12b)

with \( \nabla_{\alpha \beta} := (\Gamma^{a})_{\alpha \beta} \nabla_{a} \). Next, we shall consider
\[ \{ \nabla_{\alpha} , \nabla_{\beta} \} \rho^{k} = 4i \nabla_{\alpha} \rho_{\beta} q^{k} = 4i \nabla_{\alpha} \rho_{\beta} q^{k} , \] (2.13)
where we have made use of (A.2) and the S-supersymmetry transformation
\[ S_{\alpha \beta} := 12 \varepsilon_{\alpha \beta} q^{k} \implies K_{\alpha} \rho_{\beta} = 0 . \] (2.14)

On the other hand, by virtue of (2.12), we also have that
\[ \{ \nabla_{\alpha} , \nabla_{\beta} \} \rho^{k} = 4i \nabla_{\alpha} \rho_{\beta} q^{k} = 4i \nabla_{\alpha} \rho_{\beta} q^{k} . \] (2.15)

Applying the commutation relation (A.3), we can then equate (2.13) and (2.15) to obtain
\[ (\nabla_{\alpha} \rho^{k} \Gamma^{a})_{\beta} = -\frac{3}{4} (\rho^{k} \Sigma^{bc})_{\beta} W^{bc} = \frac{3}{2} X^{\alpha k} q_{\beta} . \] (2.16)

We can then hit both sides of (2.16) with \( \nabla_{\alpha} \) and make use of (2.12), (A.3), and the identity (A.6a). This results in the equation
\[ \square q^{k} = \frac{3}{16} X^{\alpha k} \rho^{k} + \frac{3}{64} (Y - W^{ab} W_{ab}) q^{k} , \quad \square := \nabla^{a} \nabla_{a} . \] (2.17)

Both (2.16) and (2.17) describe on-shell conditions for the hypermultiplet’s fields when \( W^{ab} = 0 \). However, as we will discuss in more detail later, in a non-trivial curved background these equations can be reinterpreted as conditions linking \( q^{k} \) and \( \rho^{k} \) with fields of the standard Weyl multiplet.
By restricting to $\xi^a \equiv 0$, the local superconformal $\delta = \delta_Q + \delta_H$ transformations of the covariant superfields $q^{i\dot{a}}$ and $\rho^{i\dot{a}}$ can be derived using the relations (2.10), (2.12), and (2.14). This leads to

$$\delta q^{i\dot{a}} = \frac{1}{2} \xi^i \rho^{i\dot{a}} + \Lambda^i_q \sigma q^{i\dot{a}} + \frac{3}{2} \sigma q^{i\dot{a}} ,$$

$$\delta \rho^{i\dot{a}} = -4i(\Gamma^a \xi_i)_a \nabla_a q^{i\dot{a}} + \frac{1}{2} \Lambda_{ab}(\Sigma^{ab} \rho^{i\dot{a}})_a + 2\sigma \rho_{i\dot{a}} - 12\eta_{i\dot{a}} q^{j\dot{a}} .$$  

(2.18a)  (2.18b)

As we will describe later, these will lead to the analogue transformations of the component fields in the hypermultiplet.

### 2.3 The $O(2)$ multiplet

The linear multiplet [69–79], or $O(2)$ multiplet, can be described in 5D $\mathcal{N} = 1$ conformal superspace [20] in terms of the superfield $G^{ij} = G^{ji}$, with $(G^{ij})^* = \varepsilon_{ik} \varepsilon_{jl} G^{kl}$ and satisfies the defining constraint

$$\nabla^i G^{jk} = 0 .$$

(2.19)

Here $G^{ij}$ is a superconformal primary dimension-3 superfield,

$$K_A G^{ij} = 0 , \quad \mathbb{D} G^{ij} = 3 G^{ij} .$$

(2.20)

To elaborate on the component structure of the superfield $G^{ij}$, we list the following useful identities:

$$\nabla^i G^{jk} = 2 \varepsilon^{i(j} \varphi^{k)} ,$$

$$\nabla^i \varphi^{j\beta} = \frac{1}{2} \varepsilon^{ij} \varepsilon_{\alpha\beta} F + \frac{1}{2} \varepsilon^{ij} \mathcal{H}_{\alpha\beta} + i \nabla_{\alpha\beta} G^{ij} ,$$

$$\nabla^i F = -2 \nabla^i \varphi^{j\beta} - 3 W_{\alpha\beta} \varphi^{j\beta} - \frac{3}{2} X_{ij} G^{ij} ,$$

$$\nabla^i \mathcal{H}_a = 4(\Sigma_{ab})_{\alpha\beta} \nabla^i \varphi^{j\beta} - \frac{3}{2} (\Gamma_{a})_{\beta} W_{\gamma\gamma} \varphi^{\gamma} - \frac{1}{2} (\Gamma_{a})_{\gamma} W_{\gamma\beta} \varphi^{\gamma} ,$$

(2.21a)  (2.21b)  (2.21c)  (2.21d)

where we have defined the independent descendant superfields

$$\varphi^{i\dot{a}} := \frac{1}{3} \nabla_{i\dot{a}} G^{\dot{a}j} ,$$

$$F := \frac{i}{12} \nabla^{\gamma i} \nabla^j G_{ij} = -\frac{i}{12} \nabla^{\gamma k} \varphi_{\gamma k} ,$$

$$\mathcal{H}_{abcd} := \frac{i}{12} \varepsilon_{abcde} (\Gamma^c)^{\alpha\beta} \nabla^i \nabla^\beta G_{ij} \equiv \varepsilon_{abcde} \mathcal{H}^e .$$

(2.22a)  (2.22b)  (2.22c)
It can be checked that $\mathcal{H}^a$ obeys the differential condition
\[
\nabla_a \mathcal{H}^a = 0 , \quad \mathcal{H}^a := -\frac{1}{4!} \varepsilon^{abced} \mathcal{H}_{bede} .
\] (2.23)

The descendants (2.22) prove to be annihilated by $K_a$ and to satisfy
\[
S^i_{\alpha} \varphi^j_{\beta} = -6\varepsilon_{\alpha\beta} G^{ij} ,
\] (2.24a)
\[
S^i_{\alpha} F = 6i \varphi^i_{\alpha} ,
\] (2.24b)
\[
S^i_{\alpha} \mathcal{H}_b = -8i (\Gamma_b)^{\alpha}_{\beta} \varphi^i_{\beta} .
\] (2.24c)

We refer the reader to [20] for a superform description of the $\mathcal{O}(2)$ multiplet.

### 2.4 The Abelian vector multiplet

In conformal superspace [20], an Abelian vector multiplet is described by a superfield $W$, which is superconformal primary of dimension 1, $K_A W = 0$ and $\mathbb{D} W = W$. Moreover, it is real, $(W)^* = W$, and obeys the Bianchi identity
\[
\nabla^{(i} \nabla^{j)} W = \frac{1}{4} \varepsilon_{\alpha\beta} \nabla^{(i} \nabla^{j)} W .
\] (2.25)

It is useful to introduce the following descendant superfields constructed from spinor derivatives of $W$:
\[
\lambda^i_{\alpha} := -i \nabla^i_{\alpha} W , \quad X^{ij} := \frac{i}{4} \nabla^{(i} \nabla^{j)} W = -\frac{1}{4} \nabla^{(i} \lambda^{j)}_{\alpha} .
\] (2.26a)

These superfields, along with
\[
\mathcal{F}_{\alpha\beta} := -\frac{i}{4} \nabla^{(k} \nabla^{\beta)}_{(\alpha} W - W_{\alpha\beta} W = \frac{1}{4} \nabla^{k} (\lambda_{\beta)}_{\alpha} - W_{\alpha\beta} W ,
\] (2.26b)

satisfy the following identities:
\[
\nabla^i_{\alpha} \lambda^j_{\beta} = -2\varepsilon^{ij} (\mathcal{F}_{\alpha\beta} + W_{\alpha\beta} W) - \varepsilon_{\alpha\beta} X^{ij} - \varepsilon^{ij} \nabla_{\alpha\beta} W ,
\] (2.27a)
\[
\nabla^i_{\alpha} \mathcal{F}_{\beta\gamma} = -i \nabla_{\alpha(\beta} \lambda^i_{\gamma)} - i \varepsilon_{\alpha(\beta} \nabla_{\gamma)} \delta \lambda^i_{\delta} - \frac{3i}{2} W_{\beta\gamma} \lambda^i_{\alpha} - W_{\alpha\beta\gamma} X^i_{\alpha} W
\]
\[
+ \frac{i}{2} W_{\alpha(\beta} \lambda^i_{\gamma)} - \frac{3i}{2} \varepsilon_{\alpha(\beta} W_{\gamma)} \delta \lambda^i_{\delta} ,
\] (2.27b)
\[
\nabla^i_{\alpha} X^{jk} = 2i\varepsilon^{(ij} \left( \nabla^{k)}_{\beta} \lambda^i_{\beta} - \frac{1}{2} W_{\alpha\beta} \lambda^{jk} + \frac{3i}{4} X^k_{\alpha} W \right) .
\] (2.27c)
We also note the relation $F_{\alpha \beta} = \frac{1}{2} (\Sigma_{ab})_{\alpha \beta} F_{ab}$. The $S$-supersymmetry generator acts on these descendants as

$$S^i_{\alpha \lambda} = -2i \varepsilon_{\alpha \beta} \varepsilon^{ij} W,$$

$$S^i_{\alpha} F_{\beta} = 4 \varepsilon_{\alpha (\beta \lambda)^i},$$

$$S^i_{\alpha} X^j = -2 \varepsilon^{ij \lambda^k},$$

(2.28)

while all the fields are annihilated by the $K_a$ generators.

For the construction of Poincaré supergravity models later in subsection 5.2, it is important to note that given a system of $n$ Abelian vector multiplets $W^I$, with $I = 1, 2, \ldots, n$, we can construct the following composite $O(2)$ multiplet and its descendant superfields [20]:

$$G^{ij}_{IJK} = C^{IJK} \left( 2 W^J X^{ij K} - i \lambda^{\alpha J (i} \lambda^{j K)} \right),$$

(2.29a)

$$\varphi^j_{\alpha I} = C^{IJK} \left( i X^{ij K}_{\alpha J} - 2i F_{\alpha \beta}^{J} \lambda^{\beta i K} - \frac{3}{2} \lambda^{i K}_{\alpha} W^J W^K - 2i W^J \nabla_{\alpha \beta} \lambda^{\beta i K} - i \nabla_{\alpha \beta} W^J \lambda^{\beta i K} - 3i W_{\alpha \beta} W^J \lambda^{\beta i K} \right),$$

(2.29b)

$$F_I = C^{IJK} \left( X^{ij K}_{\alpha J} - F^{ab J} F^K_{ab} + 4 W^J \nabla^a W^a W^K + 2 (\nabla^a W^J) \nabla_a W^K ight. + 2i (\nabla^a \lambda^{\beta i}) \lambda^{a K} - 6 W^{ab} F^{J}_{ab} W^K - \frac{39}{8} W^{ab} W^{a} W^{J} W^{K} + \frac{3}{8} Y W^{J} W^{K} 

+ 6 X^{\alpha i} \lambda^{J}_{\alpha i} W^K - 3i W_{\alpha \beta} \lambda^{\alpha i J} \lambda^{\beta K} \right),$$

(2.29c)

$$H_{aI} = C^{IJK} \left( - \frac{1}{2} \varepsilon_{abde} F^{bc J}_{ab} F^{de K} + 4 \nabla^b (W^J F^K_{ba} + \frac{3}{2} W_{ba} W^J W^K) 

+ 2i (\Sigma_{ba})^{\alpha \beta} \nabla^b (\lambda^{ij K}_{\alpha} \lambda^{\beta i}) \right),$$

(2.29d)

where $C^{IJK} = C^{(IJK)}$ is a completely symmetric constant. These are the superspace analogue of the composite linear multiplets constructed in [16].

### 3 The standard Weyl multiplet in components

We begin by identifying the various component fields of the 5D $\mathcal{N} = 1$ standard Weyl multiplet [16] within the superspace geometry described in subsection 2.1. Let us start with the vielbein ($e_m^a$) and gravitino ($\psi_m^i \alpha$). These appear as the coefficients of $dx^m$ of the super-vielbein $E_A = (E^a, E^\alpha_i) = dz^M E_M^A$,

$$e_m^a (x) := E_m^a (z), \quad \psi_m^i \alpha (x) := 2 E_m^i \alpha (z),$$

(3.1)
where a single vertical line next to a superfield denotes the usual component projection to \( \theta = 0 \), i.e. \( V(z) | : = V(z) |_{\theta = 0} \). This operation can be written in a coordinate-independent way using the so-called double-bar projection

\[
e^a = dx^m e_m^a = E^a | , \quad \psi^i_a = dx^m \psi_{m}^i = 2 E^i_a | ,
\]

where the double-bar denotes setting \( \theta = d\theta = 0 \). Analogously, the remaining fundamental and composite one-forms are obtained by taking the projections of the corresponding superspace one-forms,

\[
\phi^{ij} := \Phi^{ij} | , \quad b := B | , \quad \omega^{ab} := \Omega^{ab} | , \quad \phi^{\alpha i} := 2 \delta^{\alpha i} | , \quad f^a := \delta^a | .
\]

The covariant auxiliary matter fields are contained within the super-Weyl tensor \( W_{\alpha \beta} \) and its independent descendants,

\[
w_{\alpha \beta} := W_{\alpha \beta} | , \quad \chi^i_\alpha := \frac{3i}{32} X^i_\alpha | = \frac{3i}{80} \nabla^B W_{\beta \alpha} | ,
\]

\[
D := -\frac{3}{128} Y | = -\frac{3}{320} \nabla^k W_{\beta \alpha} |.
\]

The other components of \( W_{\alpha \beta} = \frac{1}{2} \left( \Sigma^{ab} \right)_{\alpha \beta} W_{ab} \) are given by \( W_{ab}^i = \left( \Sigma^{ab} \right)^{\beta \gamma} W_{\beta \alpha}^i \) and by \( X_{ab}^{ij} = \left( \Sigma^{ab} \right)^{\alpha \beta} X_{\alpha \beta}^{ij} \). These will turn out to be composite and expressed in terms of the component curvatures.

Taking the double-bar projection of \( \nabla = E^A \nabla_A \), we define the component vector covariant derivative \( \nabla_a \) to coincide with the projection of the superspace derivative \( \nabla_a \),

\[
e_m^a \nabla_a = \partial_m - \frac{1}{2} \psi^i_m \nabla^i_a | - \frac{1}{2} \omega^{ab} M_{ab} - b_m \bar{D} - \phi^{ij}_m J_{ij} - \frac{1}{2} \phi^{\alpha i}_m S_{\alpha i} - f^a_m K_a .
\]

Here, the projected spinor covariant derivative \( \nabla^i_a | \) corresponds to the generator of \( Q \)-supersymmetry. It is defined such that if \( U = U | \), then \( Q^i_a U := \nabla^i_a | U := (\nabla^i_a U) | \). For the other generators, e.g. \( M_{ab} U = (M_{ab} U) | \), there is no ambiguity in identifying the bar projection; hence, local diffeomorphisms, \( Q \)-supersymmetry transformations, and so forth descend naturally from their corresponding rule in superspace.

The component supercovariant curvature tensors are given by

\[
[\nabla_a , \nabla_b] = -R(P)_{ab}^c \nabla_c - R(Q)_{ab}^a \nabla^i_a | - \frac{1}{2} R(M)_{ab}^{cd} M_{cd} - R(J)_{ab}^{ij} J_{ij}
\]

\[ - R(\bar{D})_{ab} \bar{D} - R(S)_{ab}^{\gamma k} S_{\gamma k} - R(K)_{ab}^c K_c . \]

\[ 11 \]
We have introduced $R(P)_{ab}{}^c = \mathcal{T}_{ab}{}^c$, $R(Q)_{ab}{}^i = \mathcal{T}_{ab}{}^i$, and $R(M)_{ab}{}^{cd}$, $R(J)_{ab}{}^{ij}$, $R(\mathbb{D})_{ab}$, $R(S)_{ab}{}^k$, and $R(K)_{ab}{}^c$ coinciding with the lowest components of the corresponding superspace curvature tensors given in appendix A.

The constraints on the superspace curvatures determine how to supercovariantise a given component curvature by taking the double-bar projection of the superspace torsion and curvature two forms. This leads to

\begin{align}
R(P)_{ab}{}^c &= 2 \epsilon_a{}^m \epsilon_b{}^n \mathcal{D}_{[m} \epsilon_{n]}{}^c - \frac{i}{2} \psi_{aj} \Gamma^c \psi_b{}^j, \\
R(Q)_{ab}{}^i &= \epsilon_a{}^m \epsilon_b{}^n \mathcal{D}_{[m} \psi_{n]}{}^i + i(\Gamma_{[a} \phi_{b]}{}^i)_{\alpha} \\
&\quad + \frac{1}{8} \psi_{[a} \epsilon_{b]} \left[ \frac{3}{2} \left( \Sigma^{cd} \Gamma_{[a}{}^\beta - \Gamma_{[a} \Sigma^{cd} \right)_{\alpha}{}^\beta \right) \psi_{b]}{}^i, \\
R(M)_{ab}{}^{cd} &= \mathcal{R}(\omega)_{ab}{}^{cd} + 8 \delta_{[a} \epsilon_{b]} \phi^i \chi_i - 2 \psi_{[a} \Sigma^{cd} \psi_{b]}{}^i \\
&\quad + \frac{16i}{3} \delta_{[a} \epsilon_{b]} \Gamma^{[a} \chi_i - i \psi_{[a} \left( \Gamma_{b]} R(Q)^{d]}{}^i + 2 \epsilon_{[e} R(Q)_{b]}{}^{d]} \right) \\
&\quad + \frac{i}{2} \psi_{[a} \epsilon_{b]} \Sigma^{cd} \psi_{b]}{}^i - \frac{i}{4} \left( \epsilon_{[a} \phi_{b]} \psi_{b]}{}^i \right) \Sigma^{cd} \psi_{b]}{}^i, \\
R(J)_{ab}{}^{ij} &= \mathcal{R}(\phi)_{ab}{}^{ij} - 3 \psi_{[a} \phi_{b]} \chi_j - 8i \psi_{[a} \Gamma_{b]} \chi_j, \\
R(\mathbb{D})_{ab} &= 2 \epsilon_a{}^m \epsilon_b{}^n \partial_{[m} \mathbb{D}_{n]} + 4 \tilde{f}_{[a} + \psi_{[a} \phi_{b]} + \frac{8i}{3} \psi_{[a} \Gamma_{b]} \chi_j. 
\end{align}

In the above we have introduced the spin, dilatation, and SU(2)$_R$ covariant derivative

\begin{equation}
\mathcal{D}_m = \partial_m - \frac{1}{2} \omega_m{}^{bc} M_{bc} - b_m \mathbb{D} - \phi_m{}^{ij} J_{ij}, \quad \mathcal{D}_a = \epsilon_a{}^m \mathcal{D}_m,
\end{equation}

along with the curvatures

\begin{align}
\mathcal{R}(\omega)_{ab}{}^{cd} &= 2 \epsilon_a{}^m \epsilon_b{}^n \left( \partial_{[m} \omega_{n]}{}^{cd} \right), \\
\mathcal{R}(\phi)_{ab}{}^{ij} &= 2 \epsilon_a{}^m \epsilon_b{}^n \left( \partial_{[m} \phi_{n]}{}^{ij} \right). \quad (3.9b)
\end{align}

The component curvatures turn out to obey “traceless” conventional constraints

\begin{equation}
R(P)_{ab}{}^c = 0, \quad (\Gamma^a)_{\alpha}{}^\beta R(Q)_{ab}{}^i = 0, \quad R(M)_{ab}{}^{cd} = 0, \quad (3.10)
\end{equation}

which allow us to solve for the composite connections as follows:

\begin{align}
\omega_{abc} &= \omega(e)_{abc} + \frac{i}{4} \left( \psi_{ak} \Gamma_c \psi_b{}^k + \psi_{ck} \Gamma_b \psi_a{}^k - \psi_{bk} \Gamma_a \psi_c{}^k \right) + 2b_{[p} \eta_{c]} a, \\
i \phi_m{}^i &= \frac{2}{3} (\Gamma_{[p} \delta_{m]}{}^q + \frac{1}{4} \Gamma_m \Sigma^{pq}) \left( \mathcal{D}_{[p} \psi_{q]}{}^i + \frac{1}{8} \psi_{cd} \left( 3 \Sigma^{cd} \Gamma_{[p} \psi_{q]}{}^i - \Gamma_{[p} \Sigma^{cd} \psi_{q]}{}^i \right) ight), \\
\tilde{f}_a{}^b &= -\frac{1}{6} \mathcal{R}(\omega)_{ac}{}^{bc} + \frac{1}{48} \delta_a{}^b \mathcal{R}(\omega)_{cd} - \frac{i}{12} \psi_{cj} \Gamma_{[b} R(Q)_{a]}{}^{c]} - \frac{i}{12} \psi_{cj} \Gamma_{a} R(Q)^{bcj}.
\end{align}
\[
\begin{align*}
+ \frac{1}{3} \psi_{[aj} \Sigma^{bd} \phi_{d]}^j - & \frac{1}{24} \delta_a b (\psi_{cj} \Sigma^{cd} \phi_d^j) - \frac{2i}{3} (\psi_{aj} \Gamma^b \chi^j) \\
- & i \frac{1}{12} \psi_{aj} \psi_c^j w_{bc} + \frac{1}{24} (\psi_{aj} \Gamma_c \psi_d^j) \tilde{w}^{bcde} \\
+ & \frac{i}{192} \delta_a b \left(2 (\psi_{cj} \psi_d^j) w^{cd} - (\psi_{aj} \Gamma_c \psi_d^j) \tilde{w}^{cde}\right),
\end{align*}
\]

where \( \omega(e)_{abc} = -\frac{1}{2} (\Sigma_{abc} + \Sigma_{cab} - \Sigma_{bca}) \) is the usual torsion-free spin connection in terms of the anholonomy coefficient \( \Sigma_{mn} a := 2 \partial_m [\epsilon_n]^a. \) We have also defined

\[
\tilde{w}_{abc} = \frac{1}{2} \Sigma_{abc} w^{de}.
\]

The curvature \( R(\mathbb{D})_{ab} \) now vanishes due to eqs. (3.11). We stress that in the traceless frame \( \phi_m^i \) has no dependence upon the matter field \( \chi^i \) and \( \tilde{f}_a b \) has no dependence upon \( D. \) This choice minimises the dependence of the covariant derivatives upon some matter "auxiliary" fields and will simplify part of the analysis in the coming sections.

The supersymmetry transformations of the fundamental gauge connections of the Weyl multiplet can be derived from the transformations of their corresponding superspace one-forms. We restrict to all local superconformal transformations except local translations (covariant general coordinate transformations). We denote such transformations by \( \delta = \delta_Q + \delta_H \) and define the operator

\[
\delta = \xi_i Q_i^a + \frac{1}{2} \Lambda^{ab} M_{ab} + \lambda^i J_{ij} + \lambda_D \mathbb{D} + \lambda^a K_a + \eta^{ai} S_{ai}.
\]

Here, the local component parameters are respectively defined as the \( \theta = 0 \) components of the corresponding superfield parameters, \( \xi_i := \xi_i^a \), \( \lambda^{ab} := \Lambda^{ab} \), \( \lambda^i := \Lambda^i \), \( \lambda_D := \sigma \), \( \lambda^a := \Lambda_K^a \), and \( \eta^{ai} := \eta^{ai} \). The local superconformal transformations of the independent connection fields of the standard Weyl multiplet are given by [20]

\[
\begin{align*}
\delta e_m^a &= i (\xi_i \Gamma^a \psi_m^i) - \lambda_D e_m^a + \lambda^a b e_m^b, \\
\delta \psi_m^i &= 2 D_m \xi_i^a - \frac{1}{4} w_{cd} \left( (\Gamma_m \Sigma^{cd})_{\alpha}^{\beta} - 3(\Sigma^{cd} \Gamma_m)_{\alpha}^{\beta} \right) \xi_i^\beta + 2i (\Gamma_m \eta^i)_{\alpha} \\
&+ \frac{1}{2} \lambda^{ab} (\Sigma_{ab} \psi_m^i)_{\alpha} + \lambda^i_j \psi_m^j_{\alpha} - \frac{1}{2} \lambda_D \psi_{m\alpha}, \\
\delta \phi_m^{ij} &= \partial_m \lambda^{ij} - 2 \phi_m^i (k \lambda^j)_{\phantom{k}k} + 3 \xi^{(i} \phi_m^{j)} - 3 \eta^{(i} \psi_m^{j)} + 8i \xi^{(i} \Gamma_m \chi^j), \\
\delta b_m &= \partial_m \lambda_D - \frac{8i}{3} \xi_i \Gamma^i \chi - \xi_i \phi_{m}^i - \psi_m^i \eta - 2 \lambda_m.
\end{align*}
\]

In like fashion, one can derive the transformations \( \delta \omega_m^{ab} \), \( \delta \phi_{ma}^i \), and \( \delta f_{ma} \), which we omit since these fields are composite. For the covariant matter fields, the transformations are given by [20]

\[
\begin{align*}
\delta w_{ab} &= 2i \xi_i R(Q)_{ab}^i - \frac{32i}{3} \xi_i \Sigma_{ab} \chi^i - 2 \lambda_{[a} \epsilon_{b]c} w_{bc} + \lambda_D w_{ab},
\end{align*}
\]
\[
\delta \chi^{ai} = \frac{1}{2} \xi^{ai} D - \frac{1}{16} (\xi_i \Sigma^{ab}) \alpha R(J)^{ij} - \frac{3}{128} (\nabla_a w_{bc}) \left( 3(\xi^i \Gamma^a \Sigma^{bc}) \alpha + (\xi^i \Sigma^{bc} \Gamma^a) \alpha \right) \\
+ \frac{3}{256} w_{ab} w_{cd} \varepsilon^{abcd} (\xi^i \Gamma^a \alpha) + \frac{3i}{16} (\eta^i \Sigma^{ab}) w_{ab} \\
- \frac{1}{2} \chi^{ab} (\chi^i \Sigma_{ab}) \alpha + \lambda^j \chi^{aj} + \frac{3}{2} \lambda_D \chi^{ai}, \\
\delta D = 2i \xi^i \nabla_a \chi^i + iw_{ab} (\xi_i \Sigma^{ab} \chi^i) + 2\eta_i \chi^i + 2\lambda_D D,
\]

(3.14f)

\[
\delta D = 2i \xi^i \nabla_a \chi^i + iw_{ab} (\xi_i \Sigma^{ab} \chi^i) + 2\eta_i \chi^i + 2\lambda_D D,
\]

(3.14g)

where

\[
\nabla_a w_{bc} = D_a w_{bc} - i\psi_{ai} R(Q)^{ij} + \frac{16i}{3} \psi_{ai} \Sigma_{bc} \chi^i,
\]

(3.15a)

\[
\nabla_a \chi^{ai} = D_a \chi^{ai} - \frac{1}{4} \psi_{ai} D - \frac{3i}{32} (\phi^i \Sigma^{bc}) \alpha w_{bc} + \frac{1}{32} (\psi_{ij} \Sigma^{bc}) \alpha R(J)^{ij} \\
+ \frac{3}{256} (\nabla_b w_{cd}) \left( 3(\psi^i \Gamma^a \Sigma^{cd}) \alpha + (\psi^i \Sigma^{cd} \Gamma^a) \alpha \right) - \frac{3}{512} w_{bc} w_{de} \varepsilon^{bedf} (\psi_{i} \Gamma^f) \alpha.
\]

(3.15b)

## 4 The hyper-dilaton Weyl multiplet in 5D

The aim of this section is to construct a new 32 + 32 hyper-dilaton Weyl multiplet of off-shell \( \mathcal{N} = 1 \) conformal supergravity in five dimensions. The analysis closely follows the 4D \( \mathcal{N} = 2 \) case of [18].

In constructing such a hyper-dilaton Weyl multiplet, our starting point is the component structure of the on-shell hypermultiplet. This can be readily extracted from the previous superspace realisation (see subsection 2.2) via the bar projection. As was shown before, taking successive spinor derivatives of \( q^{i\dot{a}} \) leads to \( \rho^i_{\dot{a}} \) or the vector derivatives of \( q^{i\dot{a}} \) and \( \rho^i_{\dot{a}} \). Hence, the independent components of the on-shell hypermultiplet are simply the Lorentz scalar field \( q^{i\dot{a}} \), which is superconformal primary, and the spinor field \( \rho^i_{\dot{a}} \).

In what follows, we will associate the same symbol for the covariant component fields and the corresponding superfields, when the interpretation is clear from the context. The superfields \( q^{i\dot{a}} \) and \( \rho^i_{\dot{a}} \) are all annihilated by \( K^a \); hence, all their bar projections are \( K \)-primary fields. The local superconformal transformations of the component fields follow directly from the projections of (2.18), which give

\[
\delta q^{i\dot{a}} = \frac{1}{2} \xi^i \rho^{i\dot{a}} + \chi^i_k q^{k\dot{a}} + \frac{3}{2} \lambda_D q^{i\dot{a}},
\]

(4.1a)

\[
\delta \rho^i_{\dot{a}} = -4i (\Gamma^a \xi_i)_\alpha \nabla_a q^{i\dot{a}} + \frac{1}{2} \lambda_{ab} (\Sigma^{i\dot{a}} \rho^j_{\dot{b}})_{\alpha} + 2\lambda_D \rho^i_{\dot{a}} - 12\eta_i \rho^{i\dot{a}},
\]

(4.1b)

where

\[
\nabla_a q^{i\dot{a}} = D_a q^{i\dot{a}} - \frac{1}{4} \psi_{ai} \rho^{i\dot{a}}.
\]

(4.2)
The algebra of the local transformations (4.1) closes only when the following equations of motion for the fields \( q^{ij} \) and \( \rho^i_\alpha \) are imposed:

\[
(\nabla_a \rho^i_\alpha \Gamma^a) = -\frac{3}{4} (\rho^i_\alpha \Sigma^i) w_{bc} + 16i \chi^{ak} q_k^i, \tag{4.3a}
\]

\[
\Box q^{ij} = 2\chi^i \rho^j - 2D q^{ij} - \frac{3}{64} w_{ab} w_{ab} q^{ij}, \tag{4.3b}
\]

The above equations are obtained by bar-projecting (2.16), (2.17), and using the definitions (3.4). The expressions for \( \nabla_a \rho^i_\alpha \) and \( \Box q^{ij} \) in terms of the derivatives \( D_a \) are given by

\[
\nabla_a \rho^i_\alpha = D_a \rho^i_\alpha + 2i (\psi_a \Gamma^b \rho^b)^a - \frac{1}{4} \psi^b \rho^a \phi^a + 6 \phi^a \rho^a \chi^a \]

\[
\Box q^{ij} = D_a D_a q^{ij} - 3f^{a}_{ab} q^{ij} - \frac{1}{2} \rho^i_\alpha D_b D_a \psi^a_\alpha - \frac{1}{2} \psi^a_\alpha D_a \rho^j_\alpha
\]

\[
+ \frac{i}{4} \phi^a \Gamma^a \rho^j_\alpha + \frac{i}{2} (\psi^a_\alpha \Gamma^b \psi^b_\alpha) D_b q^{ik} + 4i (\psi^a_\alpha \Gamma^b \psi^b_\alpha) (\psi^a_\alpha \Gamma^b \psi^b_\alpha)
\]

Equations (4.3) can then be interpreted as algebraic equations for the standard Weyl multiplet that determine the auxiliary fields \( \chi^a_\alpha \) and \( D \) in terms of \( q^{ij} \) and \( \rho^i_\alpha \), together with the other independent fields of the standard Weyl multiplet. Assuming that \( q^{ij} \) is an invertible matrix,

\[
q^2 := q^{ik} q^j_\alpha = \varepsilon_{ij} \varepsilon_{\alpha\beta} q^i_\alpha q^j_\beta = 2 \det q^{ij} \neq 0, \tag{4.5}
\]

then the following relations hold

\[
\chi^a_\alpha = \frac{1}{8} q^{-2} q^{ij} \left[ - (\nabla_a \rho^i_\alpha \Gamma^a) + \frac{3}{4} w_{cd} (\rho^i_\alpha \Sigma^{cd}) \right] \tag{4.6a}
\]

\[
D = -\frac{1}{2} q^{-2} q^{ij} \Box q^{ij} + \frac{i}{16} q^{-2} \left[ - (\nabla_a \rho^i_\alpha \Gamma^a) + \frac{3}{4} w_{cd} (\rho^i_\alpha \Sigma^{cd}) \right] \rho^i_\alpha - \frac{3}{128} w_{ab} w_{ab}. \tag{4.6b}
\]

Note that so far, we have only used one of the four equations that are equivalent to (4.3b) to solve for \( D \) in eq. (4.6b). It is simple to show that the remaining independent three equations are equivalent to the following

\[
\nabla_a (q^{ij} \nabla_a q_{ij}) = 0. \tag{4.7}
\]

As in the 4D \( N = 2 \) case [18], this equation is solved by turning the SU(2)\( R \) connection \( \phi^m_{kl} \) into a composite field. Let us describe how.
Following the analysis of the \( \mathcal{N} = 2 \) hyper-dilaton Weyl multiplet in 4D \([18]\), here we also note that, associated to an on-shell hypermultiplet, there is always a triplet of composite linear multiplets \([69, 74, 75, 77]\). The covariant component fields of the 5D \( \mathcal{N} = 1 \) off-shell linear (or \( O(2) \)) multiplet are defined in terms of the bar projections of (2.22): an SU(2) \( \mathbf{R} \) triplet of Lorentz scalar fields \( G_{ij} = G_{ij} \); a spinor field \( \varphi_{a i} = \varphi_{a i} \); a scalar field \( F = F \); and a covariant closed anti-symmetric four-form field strength \( H_{abcd} : = H_{abcd} \). The latter is equivalent to a conserved dual vector \( H^a : = -1/4! \varepsilon^{abde} H_{bcde} \). At the component level it holds that

\[
H^a = h^a + 2\psi_{bi} \Sigma^{ab} \varphi^i + \frac{i}{2} \varepsilon^{abde} \psi_{bi} \Sigma_{cd} \psi_{ej} G^{ij} .
\]

The covariant conservation equation for \( H^a \) is

\[
\nabla^a H_a = 0 .
\]

The constraint implies the existence of a gauge three-form potential, \( b_{mnp} \), and its exterior derivative \( h_{mnpq} := 4\partial_{[m} b_{npq]} \), with \( b_{mnp} := B_{mnp} \).

In the standard Weyl multiplet background, the local superconformal transformations of the covariant fields can be derived using the relations (2.21) and (2.24), which lead to

\[
\delta G_{ij} = -2 \xi (i) \varphi_j - 2\lambda (i \phi_{ji}) + 3 \lambda D G_{ij} ,
\]

\[
\delta \varphi_{ai} = -\frac{i}{2} \xi_{ai} F - \frac{i}{2} H_a (\Gamma^a \xi)_a - i (\Gamma^a \xi)_a \nabla_a G_{ij} + 6 \eta_{a j} G_{ij} + \frac{1}{2} \lambda^{ab} (\Sigma_{ab} \varphi_i)_a - \lambda^i_j \varphi_{ja} + \frac{7}{2} \lambda D \varphi_{ai} ,
\]

\[
\delta F = 2 \xi \Gamma^a \nabla_a \varphi_i - \frac{3}{2} (\xi \Sigma^{ab} \varphi_i) w_{ab} + 16 i (\xi^i \chi^j) G_{ij} + 6 i \eta^i \varphi_i + 4 \lambda D F ,
\]

\[
\delta H_a = -4 \xi \Sigma_{ab} \nabla_b \varphi_i + \frac{3}{2} (\xi \Gamma^b \varphi_i) w_{ab} + \frac{1}{2} \varepsilon_{abcde} w^{bc} (\xi \Sigma^{de} \varphi_i) + \lambda^b_a H_b + 4 \lambda D H_a - 8 \eta^i \Gamma_a \varphi_i ,
\]

where

\[
\nabla_a G_{ij} = D_a G_{ij} + \frac{3}{4} \psi_{ai} (i \phi_j) ,
\]

\[
\nabla_a \varphi_{ai} = D_a \varphi_{ai} - \frac{i}{4} \psi_{a ai} F - \frac{i}{4} (\Gamma^b \psi_{ai})_a H_b + \frac{i}{2} (\Gamma^b \psi_{ai})_a \nabla_b G_{ij} - 3 \phi_{ai} G_{ij} .
\]

\[\varepsilon_{mnpqr} := \varepsilon^{abde} e^m_a e^n_b e^p_c e^q_d e^r_e ,\] such that \( \varepsilon_{abde} \) and \( \varepsilon^{abde} \) are normalised as \( \varepsilon_{01234} = -\varepsilon^{01234} = 1.\]
The locally superconformal transformations of $b_{mn}$ are

$$\delta b_{mn} = 2 \varepsilon_{abcde} e^a_m e^b_n e^c_p (\xi^d \Sigma^{de} \varphi^i) - 12i(\psi_{[m} i \Sigma_{np]} \xi^j) G_{ij} + 3 \partial_{[m} l_{np]} , \quad (4.12)$$

where we have also included the gauge transformation $\delta b_{mn} = 3 \partial_{[m} l_{np]}$ leaving $h_{mn}$ and $H^a$ invariant. For convenience, we have summarised the dilatation weights of the fields of the $O(2)$ multiplet in Table 1.

|   | $G_{ij}$ | $\varphi_{ai}$ | $F$ | $H_a$ | $b_{mn}$ |
|---|--------|---------|-----|------|--------|
| $\mathbb{D}$ | 3    | 7/2   | 4   | 4    | 0      |

Table 1: Dilatation weights of the off-shell $O(2)$ multiplet.

Given that $q^i$ and $\rho^a_i$ describe an on-shell hypermultiplet in a standard Weyl multiplet background with transformation rules (4.11), it can be verified that the following composite fields define a triplet of $O(2)$ multiplets

$$G_{ij}^a = q_i^a q_j^a = q_i^a (q_j^a) , \quad (4.13a)$$

$$\varphi_{ai}^a = - \frac{1}{2} q_i^a (q_j^a) , \quad (4.13b)$$

$$F^a_{ij} = \frac{i}{8} \rho^a_{(i} \rho^a_{j)} , \quad (4.13c)$$

$$H^a_{ij} = 2 q^a_{(i} \nabla^a_{j)} - \frac{i}{8} \rho^a_{(i} \Gamma^a_{j)} . \quad (4.13d)$$

These fields all transform according to (4.10) and each of the previous fields is symmetric in $i$ and $j$. The field $H^a_{ij}$ can be used to express the $SU(2)_R$ connection $\phi_{m}^{ij}$ as a composite field. To see this, we introduce a new covariant derivative

$$D_a = e^a_m \left( \partial_m - \frac{1}{2} \omega^c_{m} M_{cd} - b_m \mathbb{D} \right) = D_a + e^a_m \phi_{m}^{ij} J_{ij} , \quad (4.14)$$

which then allows us to rearrange eq. (4.13d) for the $SU(2)_R$ gauge connection:

$$\phi_{a}^{ij} = 4 q^{-4} q_{i (l} q_{j)} q_{k d} q_{d}^{d} \left[ q^{k d} D_{a} q_{k d} - \frac{1}{4} q^{k d} (\psi_{a k l} \rho^l) - \frac{i}{16} \rho^d \Gamma^a_{d l} - \frac{1}{2} H_{a}^{i l} \right] . \quad (4.15)$$

Our analysis demonstrates that the hyper-dilaton Weyl multiplet defines a new representation of the off-shell local 5D $\mathcal{N} = 1$ superconformal algebra. The multiplet comprises of the following independent fields: $e^a_m$, $b_m$, $w_{ab}$, $q^i$, $b_{m n p}$, $\psi_{m i}$, and $\rho^i$. It also possesses the same number of off-shell degrees of freedom as the standard Weyl multiplet, $32 + 32$. 17
Table 2: Degrees of freedom and symmetries of the hyper-dilaton Weyl multiplet. Row one gives all the component fields. Row two gives the number of independent components of these fields — composite connections are counted with zero degrees of freedom. Row three gives the gauge symmetries. Note that $\lambda_{mn}^{ij} = -\lambda_{mn}^{ji}$ corresponds to the symmetry associated with the gauge three-forms $b_{mnpi}^{ij}$ with field strength four-forms $h_{mnqp}^{ij}$ and $H^{ij}$. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom.

Table 2 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields.

Note that with the ingredients provided so far, it is a straightforward exercise to obtain the locally superconformal transformations of the fields of the hyper-dilaton Weyl multiplet written only in terms of the independent fields and they are given as follows:

$$\delta e_m^a = i (\xi_i \Gamma^a \psi_m^i) - \lambda_{\Box} e_m^a + \lambda^a_b e_m^b,$$

$$\delta \psi_{m\alpha} = 2D_m \xi_\alpha + 8q^{-4}q^{ij}q^{ij} \left[ q^{k\ell} D_m q_{k\ell} - \frac{1}{4} q^{k\ell}(\psi_{mk\ell})^2 - \frac{i}{16} \beta^2 H_{m\ell} - \frac{1}{2} H_{m\ell} \right] \xi_{\alpha j},$$

$$\delta b_m = \partial_m \lambda_{\Box} + \frac{1}{3} q^{-2} q^{ij} (\xi_i \Gamma \psi_j)^a [\left( \nabla_a \rho \right)^a \epsilon + \frac{3}{4} w_{cd}(\rho \Sigma_{cd})],$$

$$\delta w_{ab} = 2i \xi_i R(Q)_{ab} + \frac{4}{3} q^{-2} q^{ij} (\xi_i \Sigma_{ab})^a [\left( \nabla_a \rho \right)^a \epsilon + \frac{3}{4} w_{cd}(\rho \Sigma_{cd})],$$

$$\delta q^{ij} = \frac{1}{2} \xi_{ij} \epsilon^{ij} + \lambda_{\Box} q^{ij} + \frac{3}{2} \lambda_{\Box} q^{ij},$$

$$\delta \rho_{\alpha}^i = -4i (\Gamma^a \xi_i) \nabla_a \rho_{\alpha}^i + \frac{1}{2} \lambda_{ab} (\Sigma_{ab} \rho_{\alpha}^i) + 2 \lambda_{\Box} \rho_{\alpha}^i - 12 \eta_{\alpha}^i \eta_{\alpha}^j,$$

$$\delta b_{mnp} = 2 \varepsilon_{abcd} e_m^a e_n^b e_p^c (\Sigma_{de} \psi^i) - 12i (\psi_{m i \Sigma_{np} \xi}^j) G_{ij} + 3 \partial_{m i l_{np}},$$

It would be useful to have (4.14) expressions in terms of the derivative $D_a$ instead of $D_a$. 

| $e_m^a$ | $\omega_{mn}^{ab}$ | $b_m$ | $f_m^a$ | $\phi_{mij}$ | $\psi_{mi}^i$ | $\phi_{m}^i$ | $w_{ab}$ | $\rho^i_j$ | $q^{ij}$ | $b_{mnpq}^{ij}$ |
|--------|------------------|------|--------|-------------|-------------|-------|--------|-----------|--------|----------------|
| $25B$  | 0                | $5B$ | 0      | 0           | 40F         | 0     | $10B$  | 8F        | 4B     | 30B           |
| $P_a$  | $M_{ab}$         | $\mathbb{D}$ | $K_a$  | $J_{ij}$   | $Q$          | $S$   | $\lambda_{mn}^{ij}$-sym |
| $-5B$  | $-10B$           | $-1B$| $-5B$  | $-3B$      | $-8F$       | $-8F$ | $-18B$ |           |        |                |
which has an implicit dependence on this new composite field \( \phi_a^i \). It holds that

\[
\nabla_a q^{ij} = \frac{1}{2} D_a q^{ij} - \frac{1}{8} \psi^j_a \rho^i - q^{-2} q^j_i q^{kl} D_a q_{kl} + \frac{1}{4} q^{-2} q^j_i q^{kl} (\psi_{ak} \rho^l)
\]

\[
+ \frac{i}{8} q^{-2} q^j_i (\rho^4 \Gamma_a \rho^j) + q^{-2} q^j_i H_a q^j
\]

\[
\nabla_a \rho^a q^j = D_a \rho^a q^j + 2i (\psi_{ak} \Gamma^b)^a \nabla_b q^j + 6 \phi_a^a q_{kj}
\]

\[
\Box q^{ij} = D^a \nabla_a q^{ij} - 3 f^i_a q^{ij} + (\nabla_a q^j k) \left[ 2 q^{-2} q^j i (D^a q^{ik}) q^j_k - 2 q^{-2} q^j_i q^{kl} H^{a} q^j_k - \frac{1}{2} q^{-2} q^j_i (\psi^a \Gamma^b) q^j b \right]
\]

\[
- \frac{i}{4} q^{-2} q^j_i q^{kJ} \rho^4 \Gamma^a \rho^j
\]

\[
- \frac{1}{8} w^a_c (\psi_c^i \Gamma_d \rho^j) + 4i (\psi^i \Gamma^a \chi_k) q^{ij} - \frac{1}{4} \psi^a \nabla_a \rho^i + \frac{i}{4} \phi_a^a \Gamma^a \rho^i
\]

(4.17a)

where the composite connection \( \phi_m^i \) and \( f_a^b \) are now given in terms of \( D_a \) by:

\[
\phi_m^i = \frac{2}{3} (\Gamma^{[p} \delta_m^{q]} + \frac{1}{4} \Gamma_m \Sigma^{pq}) \left[ D_q \psi^i_q \right] + \frac{1}{12} w_{cd} (3 \Sigma^{cd} \Gamma_{[p} \psi_q^{i]} - \Gamma_{[p} \Sigma^{cd} \psi_q^{i]})
\]

\[
+ 4 q^{-4} q^j_i q^j_l \left\{ q^k_i D_m q^l_k - \frac{1}{4} q^k_i (\psi_{pk} \rho^l) - \frac{i}{16} \rho^l \Gamma_m \rho^l - \frac{1}{2} H^{i} q^j \right\} \psi_{[ij]}
\]

(4.18a)

\[
f_a^b = - \frac{1}{6} R(\omega)_{ac}^{bc} + \frac{1}{48} \delta_a^b R(\omega)_{cd}^{ad} - \frac{i}{6} \psi_{cj} \Gamma^{[b} \Gamma^{c]} q^j_q - \frac{i}{12} \psi_{cj} \Gamma_a R(Q)^{b c j}
\]

\[
+ \frac{1}{12} q^{-2} q^j_i q^j_l \left[ \nabla_a \rho^j q_{a j} \Gamma^a \epsilon \right] + \frac{3}{4} w_{cd} (\rho^2 \Sigma^{cd} \epsilon)_{a j}
\]

\[
+ \frac{1}{12} \psi_{[a j} \Sigma^{b d} \phi_{d]} + \frac{1}{12} \delta_a^b (\psi_{cj} \Sigma^{cd} \phi_{d j}) - \frac{i}{2} \psi_{a j} \psi_{d j} w_{b c} + \frac{i}{12} \psi_{a j} \Gamma_e \psi_{d j} w_{b c}
\]

\[
+ \frac{i}{192} \delta_a^b \left[ 2 (\psi_{c j} \psi_{d j}) w_{a c d e} - (\psi_{c j} \Gamma_e \psi_{d j}) w_{a c d e} \right]
\]

(4.18b)

Note that the expression of \( f_a^b \) has explicit as well as implicit dependence on the composite connection \( \phi_a^i \) via \( \nabla_a \rho^j \), which can now be substituted from (4.18a). For later use, it is convenient to have the bosonic expression of \( \Box q^{ij} \), which, by also using that \( D^a H_a q^j = 0 \) up to fermions, is given by:

\[
\Box q^{ij} = \frac{1}{2} D^a D_a q^{ij} + \frac{3}{4} q^{-2} q^j_i (D^a q^{kl}) D_a q_{kl} - q^{-2} q^j_i q^{kl} D_a q_{kl} + \frac{1}{4} q^{-2} q^j_i (D^a q^j) D_a q^j
\]

\[
+ q^{-4} q^j_i q^{kl} (D^a q^{ik}) D_a q^j_k - \frac{1}{2} q^{-2} q^j_i H^{a i k} H_{a j k} + \frac{3}{16} \mathcal{R} q^{ij} + \text{fermions}
\]

(4.19)

## 5 Recovering Poincaré supergravity in 5D

The goal of this section is to explicitly show that our hyper-dilaton Weyl multiplet constructed in the previous section can be used to derive a 40 + 40 off-shell multiplet.
of 5D $\mathcal{N} = 1$ Poincaré supergravity, by making use of superconformal approaches. We first elaborate on the structure of the multiplet and then explain how to construct a Poincaré supergravity action, pointing out some peculiarities which do not hold in the 4D $\mathcal{N} = 2$ supergravity case [18]. As an extension of the results of [18], we describe a new type of $BF$-coupling which induces a scalar potential for the dilaton without a standard $R$-symmetry gauging that admits AdS$_5$ vacua.

5.1 Hyper-dilaton multiplet of Poincaré supergravity

To recover a multiplet of Poincaré supergravity, compensating multiplets must be coupled to an off-shell conformal supergravity multiplet to gauge fix some of the local superconformal symmetries. As will be discussed below, from the symmetry point of view, it suffices to use the components of the new hyper-dilaton Weyl multiplet alone to appropriately gauge fix and eliminate all symmetries except local supersymmetry, Lorentz, and the gauge symmetry of the gauge three-forms $b_{mnp}$. This peculiar feature is different from the construction of the 4D $\mathcal{N} = 2$ hyper-dilaton Poincaré multiplet [18]. In the latter case, we are required to gauge fix the scalar field of the compensating vector multiplet in order to fix the extra $U(1)_R$ symmetry. In the 5D case, the only purpose to couple to a compensator is simply to obtain the Einstein-Hilbert kinetic term in a Poincaré supergravity action. The simplest choice is to couple our hyper-dilaton Weyl multiplet to a single off-shell, Abelian vector multiplet compensator. The scalar field in the compensator is assumed to be nowhere vanishing.

Let us first define an off-shell 5D $\mathcal{N} = 1$ Abelian vector multiplet in a standard Weyl multiplet background. Its component structure follows directly from the superfield definitions (2.26). The multiplet contains a real scalar field $\phi := W$, gaugini $\lambda^i := \lambda^i|$, a triplet of auxiliary fields $X^{ij} := X^{ij}|$, and a real Abelian gauge connection $v_m := \mathcal{V}_m$ or, equivalently, its real field strength $f_{mn} := \mathcal{F}_{mn} = 2\partial_m v_n$. The field strength $f_{mn}$ may be expressed in terms of the bar-projected, covariant field strength $F_{ab} := \mathcal{F}_{ab}|$ via the relation

$$F_{ab} = f_{ab} + i\Gamma_{[a}^{[\alpha} \psi_{b]k}^{\alpha} \lambda^k \phi + \frac{i}{2} \psi_{[a|k} \psi_{b|\gamma} \phi , \quad f_{ab} := e^m_a e^n_b f_{mn} . \tag{5.1}$$

The dilatation weights of the vector multiplet fields are summarised in Table 3.

The transformation rules of the vector multiplet fields in a standard Weyl multiplet background can be obtained from the corresponding superfields. They read

$$\delta \phi = i \xi^i \lambda^i + \lambda_D \phi , \quad \tag{5.2a}$$
Table 3: Dilatation weights of the Abelian vector multiplet.

|   | $\phi$ | $\lambda_i^a$ | $X_{ij}$ | $F_{ab}$ | $v_m$ |
|---|------|----------|--------|--------|------|
| $\mathbb{D}$ | 1     | 3/2      | 2      | 2      | 0    |

$$
\delta \lambda_i^a = -(\Sigma^{ab} \xi^i)_{\alpha} F_{ab} - (\Sigma^{ab} \xi^i)_{\alpha} w_{ab} \phi + \xi_{\alpha j} X^{ij} + (\Gamma^a \xi^i)_{\alpha} \nabla_a \phi + \frac{1}{2} \lambda^{ab}(\Sigma_{ab} \lambda^i)_{\alpha} + \lambda^i_j \lambda^j_i \alpha + \frac{3}{2} \lambda_D \lambda^i_{\alpha} + 2i \eta_{\alpha} i \phi , \quad (5.2b)
$$

$$
\delta X^{ij} = -2i \xi^{(i} \Gamma^{a} \nabla_a \lambda^{j)} + \frac{3i}{2} \xi^{(i} \Sigma^{ab} \lambda^j w_{ab} - 16i \xi^{(i} \chi^{j)} \phi \\
+ 2\lambda^{(i} X^{j)k} - 2\eta^{(i} \lambda^{j)} + 2\lambda_D X^{ij} , \quad (5.2c)
$$

$$
\delta v_m = i(\xi^i \psi_{mi}) \phi - i(\xi^i \Gamma_m \lambda_i) + \partial_m \lambda_V , \quad (5.2d)
$$

where

$$
\nabla_a \phi = D_a \phi - \frac{i}{2} \psi_{ai} \lambda^i , \quad (5.3a)
$$

$$
\nabla_a \lambda^i_\alpha = D_a \lambda^i_\alpha + \frac{1}{2} (\Sigma^{bc} \psi^{i}_a)_{\alpha} \left( F_{bc} + w_{bc} \phi \right) - \frac{1}{2} \psi_{aa j} X^{ij} - \frac{1}{2} (\Gamma^b \psi^{i}_a)_{\alpha} \nabla_b \phi - i \phi_{aa} i \phi . \quad (5.3b)
$$

Note that we have also included in (5.2d) the gauge field transformation parametrised by the local real parameter $\lambda_V$. The transformations of the vector multiplet in a hyper-dilaton Weyl multiplet background are precisely the same as above. The only subtlety is that one has to interpret several standard Weyl multiplet fields as composites of $q^{\bar{i} \bar{j}}$, $\rho^{\bar{i} \bar{j}}_a$ and $b_{mnp}^{\bar{i} \bar{j}}$. It should be emphasised that, being a compensator, one may require that the lowest component of the vector multiplet is non-zero and positive, $\phi > 0$.

### 5.1.1 Gauge fixing in a string frame

We now describe the structure of the supergravity multiplet by imposing several gauge fixing constraints. In a string frame, we choose the gauge fixing condition

$$
q^{\bar{i} \bar{j}} = -\varepsilon^{\bar{i} \bar{j}} \iff q_{\bar{i} \bar{j}} = \delta_{\bar{i} \bar{j}} \iff q_{\bar{i}} = -\delta_{\bar{i}} \iff q_{\bar{i} \bar{j}} = \varepsilon_{\bar{i} \bar{j}} . \quad (5.4a)
$$

This condition fixes dilatation and SU(2)R symmetries. By imposing

$$
b_m = 0 , \quad (5.4b)
$$
the special conformal $K^a$ symmetry is now fixed. In order to fix $S$-supersymmetry, we impose the constraint

$$\rho^i_a = 0 .$$  \hspace{1cm} (5.4c)

The compensating vector multiplet contains $8 + 8$ off-shell degrees of freedom. Once added to the remaining fundamental fields in the hyper-dilaton Weyl multiplet, we obtain $40 + 40$ off-shell degrees of freedom of a Poincaré supergravity multiplet, as shown in Table 4. The fundamental fields are the vielbein $e^a_m$, the gravitino $\psi^\alpha_m$, a real antisymmetric tensor $w_{ab}$, a real scalar field that plays the role of a dilaton $\phi$, a real triplet of scalar fields $X^{ij}$, a triplet of gauge three forms $b_{mnp}^{ij}$, a gauge field $v_m$ that plays the role of the graviphoton, and a spinor field $\lambda^i_\alpha$. Note that we kept the distinction of SU(2)$_R$ and SU(2) flavour indices. However, the gauge condition (5.4a) implies that the two indices can be identified, after gauge fixing.

| $e^a_m$ | $\omega^a_{mb}$ | $\psi^\alpha_m$ | $w_{ab}$ | $b_{mnp}^{ij}$ | $\phi$ | $\lambda^i_\alpha$ | $X^{ij}$ | $v_m$ |
|--------|----------------|-----------------|---------|----------------|------|----------------|--------|-------|
| 25B    | 0              | 40F             | 10B     | 30B            | 1B   | 8F             | 3B     | 5B    |
| $P_a$  | $M_{ab}$       | $Q$             | $(\lambda_{mn}^{ij})$ | $(\lambda_V)$ |
| $-5B$  | $-10B$         | $-8F$           | $-18B$  | $-1B$          |
|        |                |                 |         |                |
| Result: 40 + 40 degrees of freedom |

Table 4: A Poincaré supergravity multiplet. Row one gives all fields in the multiplet. Row two gives the number of independent components of these fields. Row three gives the surviving gauge symmetries. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom. The parameter $\lambda_{mn}^{ij}$ describes the symmetry associated with the triplet of gauge three-form $b_{mnp}^{ij}$. The gauge parameter $\lambda_V$ describes the scalar symmetry of $v_m$.

The transformation rules of the resulting Poincaré supergravity multiplet are those that preserve the previous set of gauge conditions (5.4). Since we fix $\rho^i_a = 0$, to preserve (5.4a), we require $\lambda_B = \lambda_V = 0$. To preserve (5.4c), it can be shown that any $Q$-supersymmetry transformation must be accompanied by a compensating $S$-supersymmetry transformation with the following parameter

$$\eta^i_\alpha (\xi) = -\frac{i}{3} (\Gamma^a \xi^i)_\alpha \phi^i_\alpha ,$$  \hspace{1cm} (5.5)

A similar analysis shows that to preserve the condition $b_m = 0$ one needs to enforce nontrivial compensating special conformal $K$-transformations with a parameter $\lambda^a(\xi)$. However, since all the other supergravity fields are conformal (not necessarily superconformal) primaries, not transforming under special conformal boosts, in practice we will
never have to worry about inserting the compensating $\lambda^a(\xi)$ parameter (whose expression is quite involved) in any Poincaré supergravity transformations.

5.1.2 Gauge fixing in the Einstein frame

It is possible to choose a different gauge fixing to Poincaré supergravity, where we also impose constraints on some of the fields of the compensating vector multiplet. This gauge fixing choice, which is analogous to that in the 4D $\mathcal{N} = 2$ case [18], corresponds to the Einstein frame and leads to a different Poincaré supergravity multiplet.

We now adopt the gauge where

$$\phi = 1 \quad (5.6a)$$
$$b_m = 0 \quad (5.6b)$$

Condition (5.6a) fixes dilatation symmetry, while (5.6b) fixes special conformal $K^a$ symmetry. In order to fix $S$-supersymmetry, we impose

$$\lambda^i_\alpha = 0 \quad (5.6c)$$

A characterising feature of the hyper-dilaton Weyl multiplet is that it contains an SU(2)$_R$ compensator, the $q^{i\underline{2}}$ fields. By imposing

$$q^{i\underline{2}} = -\varepsilon^{i\underline{2}} e^{-U} \quad \iff \quad q^{i\underline{2}} = \delta^{i\underline{2}} e^{-U} \quad \iff \quad q^{i\underline{2}} = -\delta^{i\underline{2}} e^{-U} \quad \iff \quad q^{i\underline{2}} = \varepsilon^{i\underline{2}} e^{-U}, \quad (5.6d)$$

we break the SU(2)$_R$ symmetry. The resulting Poincaré supergravity multiplet is shown in Table 5. The fundamental fields are the vielbein $e^a_m$, the gravitini $\psi^\alpha_m$, a real anti-

| $e^a_m$ | $\omega^a_{m\underline{b}}$ | $\psi^\alpha_m$ | $w_{ab}$ | $\rho^i_\alpha$ | $U$ | $b_{mnp}^{i\underline{2}}$ | $X^{ij}$ | $v_m$ |
|--------|----------------|----------------|--------|--------------|-----|----------------|-------|-----|
| 25B    | 0              | 40F            | 10B    | 8F           | 1B  | 30B            | 3B    | 5B  |
| $P_a$  | $M_{ab}$       | $Q$            |        | $(\lambda_{mn}^{i\underline{2}})$ | $(\lambda_V)$ |
| -5B    | -10B           | -8F            |        | -18B         | -1B |

Result: 40 + 40 degrees of freedom

Table 5: A variant Poincaré supergravity multiplet. Row one gives all the fields in the multiplet. Row two gives the number of independent components. Row three gives the surviving gauge symmetries. Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom. The parameter $\lambda_{mn}^{i\underline{2}}$ corresponds to the symmetry associated with the triplet of gauge three-form $b_{mnp}^{i\underline{2}}$. The gauge parameter $\lambda_V$ describes the scalar symmetry of $v_m$. 23
symmetric tensor $w_{ab}$, a real scalar field that plays the role of a dilaton $U$, a real triplet of scalar fields $X^{ij}$, a triplet of gauge three forms $b_{mnp}^i$, a gauge field $v_m$ that plays the role of the graviphoton, and a spinor field $\rho^i_\alpha$.

The transformation rules of the resulting Poincaré supergravity multiplet are those preserving (5.6). To preserve (5.6a), we require $\lambda_D \equiv 0$. Since $Q$-supersymmetry does not preserve the gauge fixing conditions, it is necessary to accompany these transformations with appropriate $S$-supersymmetry, special conformal, and SU(2)$_R$ compensating transformations. To preserve (5.6c), it can be shown that any $Q$-supersymmetry transformation must be accompanied by a compensating $S$-supersymmetry transformation with the following parameter

$$r_a^i(\xi) = -\frac{i}{2} (\Sigma^{ab} \xi^i)_a \left( F_{ab} + w_{ab} \right) + \frac{i}{2} \xi_{i\beta} X^{ij} .$$

(5.7)

A similar analysis shows that to preserve the condition $b_m = 0$, one needs to enforce nontrivial compensating special conformal $K$-transformations with a parameter $\lambda^a(\xi)$. However, since all the other supergravity fields are conformal (not necessarily superconformal) primaries, not transforming under special conformal boosts, in practice we will never have to worry about inserting the compensating $\lambda^a(\xi)$ parameter (whose expression is quite involved) in any Poincaré supergravity transformations. Finally, we can easily check that the requirement $\delta q^{ij}_\alpha = 0$ is satisfied by implementing in (4.1a) a compensating SU(2)$_R$ transformation with the parameter

$$\lambda^{ij}(\xi) = -\frac{1}{2} e^U \xi^{(i} \rho^{j)} ,$$

(5.8)

where $\rho^i = \delta^i_\rho^\rho^\phi$.

### 5.2 Hyper-dilaton Poincaré supergravity action and dilaton potential

We turn to deriving a Poincaré supergravity action by considering the two-derivative action of the vector multiplet compensator [20] in a hyper-dilaton Weyl multiplet background and then imposing appropriate gauge fixing conditions leading to the two frames described above. As shown in [20], the component form of such a vector multiplet action may be derived from the bosonic part of the $BF$ Lagrangian

$$e^{-1} L_{BF|\text{bosonic}} = -\frac{1}{4} \left( F\phi + G_{ij} X^{ij} - \frac{1}{12} \epsilon^{abcde} f_{ab} b_{cde} \right)$$
\[= -\frac{1}{4} \left( F\phi + G_{ij}X^{ij} + v^ah_a \right), \quad (5.9)\]

with the fields of the \(O(2)\) multiplet being composite. More precisely, this amounts to taking the bosonic sector of the bar projection of eqs. (2.29):

\[
G^{ij}\big|_{\text{bosonic}} = 2\phi X^{ij}, \quad (5.10a)
\]

\[
F\big|_{\text{bosonic}} = X^{ij}X_{ij} - f^{ab}f_{ab} + 4\phi\nabla^a\nabla_a\phi + 2(\nabla^a\phi)\nabla_a\phi - 6\phi w^{ab}f_{ab} - \frac{39}{8} \phi^2 w^{ab}w_{ab} - 16\phi^2 D, \quad (5.10b)
\]

\[
h_a\big|_{\text{bosonic}} = -\frac{1}{2} \varepsilon_{abcdef} f^{bc} f^{de} + 4\nabla^b(\phi f_{ba} + \frac{3}{2} \phi^2 w_{ba}), \quad (5.10c)
\]

and plugging (5.10) back into (5.9). This procedure results in

\[
e^{-1} L_{BF}\big|_{\text{bosonic}} = \frac{1}{8} \phi^3 \mathcal{R} + \frac{3}{2} \phi(D^a\phi)D_a\phi - \frac{3}{4} \phi X^{ij}X_{ij} + \frac{1}{8} \varepsilon_{abcdef} v^a f^{bc} f^{de} + \frac{3}{4} \phi f^{ab}f_{ab} + \frac{9}{4} \phi^2 w^{ab}f_{ab} + \frac{39}{32} \phi^3 w^{ab}w_{ab} + 4\phi^3 D. \quad (5.11)
\]

The expression (5.11) can be written in terms of the degauged covariant derivative \(D_a\), where we note the following relation

\[
\nabla^a\nabla_a\phi = D^aD_a\phi + \frac{1}{8} \phi \mathcal{R}. \quad (5.12)
\]

After performing integration by parts, one then arrives at the following bosonic Lagrangian

\[
e^{-1} L_{BF}\big|_{\text{bosonic}} = -\frac{1}{8} \phi^3 \mathcal{R} + \frac{3}{2} \phi(D^a\phi)D_a\phi - \frac{3}{4} \phi X^{ij}X_{ij} + \frac{1}{8} \varepsilon_{abcdef} v^a f^{bc} f^{de} + \frac{3}{4} \phi f^{ab}f_{ab} + \frac{9}{4} \phi^2 w^{ab}f_{ab} + \frac{39}{32} \phi^3 w^{ab}w_{ab} + 4\phi^3 D. \quad (5.13)
\]

The action (5.13) is given in the standard Weyl multiplet background. When working with a hyper-dilaton Weyl multiplet, we need to take into account that the auxiliary field \(D\) is composite (and that (4.6b) has to be used). The algebraic expression for \(D\) takes the following form

\[
D = -\frac{3}{32} \mathcal{R} - \frac{3}{128} w^{ab}w_{ab} - \frac{1}{2q^2} q_a D^a D_a q^i q^i + \text{fermionic terms}. \quad (5.14)
\]

Upon substituting this, one obtains

\[
e^{-1} L|_{\text{bosonic}} = -\frac{1}{2} \phi^3 \mathcal{R} - \frac{2}{q^2} \phi^3 q_i q^i D^a D_a q^i q^i + \frac{3}{2} \phi(D^a\phi)D_a\phi - \frac{3}{4} \phi X^{ij}X_{ij} + \frac{1}{8} \varepsilon_{abcdef} v^a f^{bc} f^{de} + \frac{3}{4} \phi f^{ab}f_{ab} + \frac{9}{4} \phi^2 w^{ab}f_{ab} + \frac{9}{8} \phi^3 w^{ab}w_{ab}. \quad (5.15)
\]
In eq. (5.15), there is still a dependence upon the triplet of gauge three-forms $b_{mnpi}^j$, which is hidden in the $SU(2)_R$ connection inside the $D_a$ derivatives. It is straightforward to obtain the analogue expressions in terms of $D_a$.

As a direct generalisation, coupling to $n$ vector multiplets leads to the following action

$$e^{-1} L_{bosonic} = C_{IJK} \left( -\frac{1}{2} \phi^I \phi^J \phi^K R - \frac{2}{q^2} \phi^I \phi^J \phi^K q_{ij} D^a D_a q^{ij} + \frac{3}{2} \phi^I (D^a \phi^J) D_a \phi^K \
- \frac{3}{4} \phi^I X^{ij} J X^{K} + \frac{1}{8} \varepsilon_{abcde} v^a f^{bc} J f^{de} K + \frac{3}{4} \phi^I f^{ab} J f^{K} \
+ \frac{9}{4} \phi^I \phi^J w^{ab} f^{K} + \frac{9}{8} \phi^I \phi^J \phi^K w^{ab} w_{ab} \right). \quad (5.16)$$

The final step to obtain the bosonic sector of the Poincaré supergravity action is to impose the set of gauge fixing conditions on (5.15) or appropriate generalisations when physical matter multiplets are included. Here we give the gauge-fixed supergravity action in both the string and Einstein frames.

### 5.2.1 String frame

Upon implementing the constraints (5.4), the resulting BF action turns out to be

$$e^{-1} L_{bosonic} = -\frac{1}{2} \phi^3 R + \frac{1}{8} \varepsilon_{abcde} v^a f^{bc} f^{de} + \frac{3}{4} \phi f^{ab} f_{ab} + \frac{9}{4} \phi^2 w^{ab} f_{ab} + \frac{9}{8} \phi^3 w^{ab} w_{ab} \
+ \frac{3}{2} \phi (\partial^m \phi) \partial_m \phi + \frac{1}{4} h^a_{ij} h_{a}^{ij} - \frac{3}{4} \phi X^{ij} X_{ij}. \quad (5.17)$$

Here $h^a_{ij} = \delta^a_i \delta^a_j$ since we have stopped distinguishing between underlined and non-underlined $SU(2)$ indices after gauge fixing. We also stress that $\phi > 0$ as the compensator is nowhere vanishing.

We can further analyse the on-shell structure of (5.17). It is clear that $w_{ab}$ and $X^{ij}$ are auxiliary fields that can be algebraically integrated out by using the equations of motion

$$f_{ab} + \phi w_{ab} = 0, \quad X^{ij} = 0. \quad (5.18)$$

The on-shell Lagrangian then reads

$$e^{-1} L_{bosonic} = -\frac{1}{2} \phi^3 R + \frac{1}{8} \varepsilon_{abcde} v^a f^{bc} f^{de} - \frac{3}{8} \phi f^{ab} f_{ab} \
+ \frac{3}{2} \phi (\partial^m \phi) \partial_m \phi + \frac{1}{4} h^a_{ij} h_{a}^{ij}. \quad (5.19)$$

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The first three terms are kinetic terms for minimal on-shell $\mathcal{N} = 1$ Poincaré supergravity with a dynamical graviton and graviphoton, described in a string frame. The last two terms describe a dilaton and a triplet of dynamical gauge three-forms which are not part of the minimal on-shell $\mathcal{N} = 1$ Poincaré supergravity multiplet.

By construction, the supersymmetric $BF$-action (5.9) is also well defined as an invariant in a hyper-dilaton Weyl background. We can readily construct an invariant of this form by considering the off-shell vector multiplet compensator used in this section and an off-shell linear multiplet given by

$$e^{-1}\mathcal{L}_\xi|_{\text{bosonic}} = -\frac{1}{4} \left( \phi F_\xi + G_{\xi ij} X^{ij} + v_a h^a \right),$$  

(5.20a)

where we have defined

$$G_{\xi ij} := \xi_{ij} G_{ij}, \quad \varphi_{\alpha i} := \xi_{ij} \varphi_{\alpha ij}, \quad F_\xi := \xi_{ij} F^{ij}, \quad b_{\alpha mnp} := \xi_{ij} b_{mnp ij}, \quad h^a := \xi_{ij} h^a_{ij}. \quad (5.20b)$$

Here $G_{ij}, \varphi_{\alpha ij}, F^{ij}, b_{\alpha mnp},$ and $h^a_{ij}$ are fields of the composite triplet of linear multiplets (4.13) constructed in terms of fundamental fields of the hyper-dilaton Weyl multiplet, while $\xi_{ij} = \xi_{ji}$ is a real triplet of (structure group invariant) constants. The bosonic part of the resulting Lagrangian is given by

$$e^{-1}\mathcal{L}_\xi|_{\text{bosonic}} = -\frac{1}{4} \xi_{ij} \left( q_{ij}^{ij} X^{ij} - \frac{1}{12} \varepsilon^{mnpr} b_{mnp i j} f_{qr} \right)$$

$$-\frac{1}{4} \xi_{ij} \left( q_{ij}^{ij} X^{ij} + h^{mij} v_m \right).$$

(5.21)

Upon imposing the gauge fixing (5.4) and adding (5.21) into (5.17), we get

$$e^{-1}\mathcal{L}|_{\text{bosonic}} = -\frac{1}{2} \phi^3 R + \frac{1}{8} \varepsilon_{abcd} v^a f^{bc} f^{de} + \frac{3}{4} \phi f^{ab} f_{ab} + \frac{9}{4} \phi^2 w^{ab} f_{ab} + \frac{9}{8} \phi^3 w^{ab} w_{ab}$$

$$+ \frac{3}{2} \phi (\partial^m \phi) \partial_m \phi + \frac{1}{4} h^{a ij} h^a_{ij} - \frac{3}{4} \phi X^{ij} X_{ij}$$

$$+ \frac{1}{48} \xi_{ij} \varepsilon^{mnpr} b_{mnp i j} f_{qr} - \frac{1}{4} \xi_{ij} X^{ij},$$

(5.22)

where, after gauge fixing, we have used $\xi_{ij} = \delta^{ij}_{\xi ij}$ and $b_{mnp i j} = \delta^{ij}_{\xi i j} b_{mnp}. \xi_{ij}.$ We can then integrate $w_{ab}$ and $X^{ij}$ out as they are auxiliary fields. With the $\xi$-deformation turned on, the equations of motion obtained from (5.22) become

$$f_{ab} + \phi w_{ab} = 0, \quad \phi X^{ij} + \frac{1}{6} \xi^{ij} = 0.$$  

(5.23)
This leads to the on-shell Lagrangian
\[ e^{-1} \mathcal{L}_{\text{bosonic}} = -\frac{1}{2} \phi^3 \mathcal{R} + \frac{1}{8} \varepsilon_{mpqr} v^m f^{np} f^{qr} - \frac{3}{8} \phi f^{mn} f_{mn} + \frac{3}{2} \phi (\partial^m \phi) \partial_m \phi + \frac{1}{4} h^m_{\; kl} h_m^{\; kl} + \frac{1}{24} \xi^2 + \frac{1}{48} \varepsilon_{mpqr} b_{mnp}^{\; ij} f_{qr} , \] (5.24)

where
\[ \xi^2 := \frac{1}{2} \varepsilon_{ij} \xi_{ij} \geq 0 . \] (5.25)

As a result, we have obtained a non-trivial, negatively defined, potential for the dilaton. The previous Lagrangian admits a constant dilaton, AdS5 vacua.

### 5.2.2 Einstein frame

If we instead adopt the gauge fixing conditions (5.6) to (5.15), we obtain the following Poincaré supergravity action
\[ e^{-1} \mathcal{L}_{\text{bosonic}} = -\frac{1}{2} \mathcal{R} + \frac{1}{8} \varepsilon_{abcde} v^a f^{bc} f^{de} + \frac{3}{4} f^{ab} f_{ab} + \frac{9}{4} \epsilon_{ab} f_{ab} + \frac{9}{8} \epsilon_{ab} \epsilon_{ab} - 2 (\partial^m U) \partial_m U + \frac{1}{4} e^{4U} h^a_{\; ij} h_a^{\; ij} - \frac{3}{4} X^{ij} X_{ij} . \] (5.26)

We can further analyse the on-shell structure of (5.26). It is clear that \( w_{ab} \) and \( X^{ij} \) are auxiliary fields that can be algebraically integrated out by using the equations of motion
\[ w_{ab} = - f_{ab} , \quad X^{ij} = 0 . \] (5.27)

The on-shell Lagrangian then reads
\[ e^{-1} \mathcal{L}_{\text{bosonic}} = -\frac{1}{2} \mathcal{R} + \frac{1}{8} \varepsilon_{abcde} v^a f^{bc} f^{de} - \frac{3}{8} f^{ab} f_{ab} - 2 (\partial^m U) \partial_m U + \frac{1}{4} e^{4U} h^a_{\; ij} h_a^{\; ij} . \] (5.28)

The first three terms describe the standard kinetic terms for minimal on-shell \( \mathcal{N} = 1 \) Poincaré supergravity with a dynamical graviton and graviphoton. The last two terms describe a dilaton and a triplet of dynamical gauge three-forms which are not part of the minimal on-shell \( \mathcal{N} = 1 \) Poincaré supergravity multiplet. This is a standard feature of dilaton multiplets, where on-shell a physical dilaton multiplet adds to the degrees of freedom of the multiplet. In the case of hyper-dilaton Poincaré supergravity, the extra multiplet is a hypermultiplet where three of the scalars have been dualised to a triplet of gauge three forms in complete analogy to the 4D \( \mathcal{N} = 2 \) case of [18, 63].
Let us now add the second supersymmetric invariant (5.21) to the action (5.15). Upon imposing the gauge fixing conditions (5.6), we arrive at

$$e^{-1} L_{\text{bosonic}} = -\frac{1}{2}\mathcal{R} + \frac{1}{8}\varepsilon_{abcde}v^a f^{bc}f^{de} + \frac{3}{4}f^{ab}f_{ab} + \frac{9}{4}w^{ab}f_{ab} + \frac{9}{8}w^{ab}w_{ab}$$

$$-2(\partial^m U)\partial_m U + \frac{1}{4}e^{4U}h_{ij}h^{ij} - \frac{3}{4}X^{ij}X_{ij}$$

$$+\frac{1}{48}\xi_{ij}\varepsilon^{mnpqr}b_{mnp}ijf_{qr} - \frac{1}{4}\xi_{ij}e^{-2U}X^{ij}. \quad (5.29)$$

We can then integrate $w_{ab}$ and $X^{ij}$ out as they are auxiliary fields. With the $\xi$-deformation turned on, the equations of motion obtained from (5.29) become

$$w_{ab} = -f_{ab}, \quad X^{ij} = -\frac{1}{6}\varepsilon^{ij}e^{-2U}. \quad (5.30)$$

This leads to the on-shell Lagrangian

$$e^{-1} L_{\text{bosonic}} = -\frac{1}{2}\mathcal{R} + \frac{1}{8}\varepsilon_{mnpr}v^f_{mnp}f^{qr} - \frac{3}{8}r_{mn}f_{mn} - 2(\partial^m U)\partial_m U + \frac{1}{4}e^{4U}h_{kl}h^{kl}$$

$$+\frac{1}{24}e^{-4U}\varepsilon^2 + \frac{1}{48}\xi_{ij}\varepsilon^{mnpqr}b_{mnp}ijf_{qr}. \quad (5.31)$$

6 Superconformal multiplets in 6D $\mathcal{N} = (1, 0)$ superspace

The following section reviews the relevant details of various superconformal multiplets required in this work. We first describe the 6D $\mathcal{N} = (1, 0)$ standard Weyl multiplet of conformal supergravity in superspace before moving on to the discussion of matter multiplets: the on-shell hypermultiplet and linear multiplets. Here we make use of the conformal superspace formulation in the traceless frame [12] and results from [13]. We also refer the reader to the following list of papers for other work on flat superspace and multiplets in six dimensions [80–86] while see also [14, 15, 87–93] for alternative curved superspace approaches to describe supergravity multiplets in six dimensions.

6.1 The standard Weyl multiplet

The standard Weyl multiplet of 6D $\mathcal{N} = (1, 0)$ conformal supergravity [7] contains 40 + 40 physical components, and is associated with the gauging of the superconformal algebra OSp(6,2|1). Associated respectively with local translations, $Q$-supersymmetry,
SU(2)\(_R\) and dilatations are the vielbein \(e_m^a\), the gravitino \(\psi_m^i\), the SU(2)\(_R\) gauge field \(\phi_m^{ij}\), and a dilatation gauge field \(b_m\). There are three composite connections which are associated with the remaining gauge symmetries: these are the spin connection \(\omega_m^{ab}\), the \(S\)-supersymmetry connection \(\phi_m^i\), and the special conformal connection \(f_m^a\) — they are algebraically determined in terms of the other fields by imposing constraints on some of the curvature tensors. To achieve an off-shell representation, one needs to introduce three covariant auxiliary fields: an anti-self-dual tensor \(T\) of the curvature tensors. To achieve an off-shell representation, one needs to introduce three covariant auxiliary fields: an anti-self-dual tensor \(T\) of the curvature tensors. To achieve an off-shell representation, one needs to introduce three covariant auxiliary fields: an anti-self-dual tensor \(T\) of the curvature tensors. To achieve an off-shell representation, one needs to introduce three covariant auxiliary fields: an anti-self-dual tensor \(T\) of the curvature tensors.

The 6D \(\mathcal{N} = (1,0)\) conformal superspace is parametrised by local bosonic \((x^m)\) and fermionic \((\theta^\mu)\) coordinates \(z^M = (x^m, \theta^\mu)\), where \(m = 0,1,2,3,4,5\), \(\mu = 1,2,3,4\) and \(i = 1,2\). By gauging the full 6D \(\mathcal{N} = (1,0)\) superconformal algebra in superspace, we introduce covariant derivatives \(\nabla_A = (\nabla_a, \nabla_i^A)\) which take the form

\[
\nabla_A = E_A - \omega_A^b X_b = E_A - \frac{1}{2} \Omega_A{}^{ab} M_{ab} - \Phi_A{}^{ij} J_{ij} - B_A \mathbb{D} - \tilde{\mathcal{F}}_{AB} K^B ,
\]

\[(6.1)\]

\[
= E_A - \frac{1}{2} \Omega_A{}^{ab} M_{ab} - \Phi_A{}^{ij} J_{ij} - B_A \mathbb{D} - \tilde{\mathcal{F}}_A{}^{i}{}^{c^a} - \tilde{\mathcal{F}}_A{}^{a} K_a . \quad (6.2)
\]

Here \(E_A = E_A{}^M \partial_M\) is the inverse super-vielbein (which plays a role of a connection for local super-translations), \(M_{ab}\) are the Lorentz generators, \(J_{ij}\) are generators of the SU(2)\(_R\) \(R\)-symmetry group, \(\mathbb{D}\) is the dilatation generator and \(K^A = (K^a, S_1^a)\) are the special superconformal generators.\(^2\) The super-vielbein one-form is given by \(E^A = dz^M E_M{}^A\) and satisfies \(E_M{}^A E_A{}^N = \delta^N_M\), \(E_M{}^A E_M{}^B = \delta^B_A\). Associated with each structure group generator \(X_a = (M_{ab}, J_{ij}, \mathbb{D}, S_1^a, K_a)\) there is a connection superfield one-form given by \(\omega^{\hat{a}} = (\Omega^{ab}, \Phi^{ij}, B, \tilde{\mathcal{F}}_a, \tilde{\mathcal{F}}^a) = dz^M \omega_M{}^{\hat{a}} = E^A \omega_A{}^{\hat{a}}\).

To describe the standard 6D \(\mathcal{N} = (1,0)\) Weyl multiplet in conformal superspace, the algebra of covariant derivatives

\[
[\nabla_A, \nabla_B] = -\mathcal{J}_{AB}{}^C \nabla_C - \frac{1}{2} \mathcal{R}(M)_{AB}{}^{cd} M_{cd} - \mathcal{R}(J)_{AB}{}^{kl} J_{kl} - \mathcal{R}(B)_{AB}{}^{kl} J_{kl} - \mathcal{R}(S)_{AB}{}^{k}{}_{\gamma} S_{\gamma}^{\gamma} - \mathcal{R}(K)_{AB}{}^{c} K_{c} , \quad (6.3)
\]

is constrained to be completely determined in terms of the symmetric super-Weyl tensor superfield \(W^{\alpha\beta}\), which is a superconformal primary with conformal dimension one

\[
W^{\alpha\beta} = W^{\beta\alpha}, \quad K^A W^{\alpha\beta} = 0 , \quad \mathbb{D} W^{\alpha\beta} = W^{\alpha\beta} , \quad (6.4)
\]

\(^2\)Note here the change in the SU(2)\(_R\) index structure of the 6D \(S\)-supersymmetry generator, \(S_1^a\), relative to the 5D case where it was originally introduced as \(S_{\dot{a}i}\). Though this difference might seem unnatural, and introduce minus signs in similar expressions in 5D and 6D, we decided to keep adhering to the notations used in [13][20].
obeying the Bianchi identities

\[ \nabla^{(i}_\alpha \nabla^{j)}_{\beta} W^{\gamma\delta} = -\delta^{(\gamma}_{(\alpha} \nabla^{i)}_{\beta} W^{\delta)}_{\rho} , \]  
\[ \nabla^k \nabla_{\gamma k} W^{\beta\gamma} - \frac{1}{4} \delta^\beta_{\alpha} \nabla^k \nabla_{\delta k} W^{\gamma\delta} = 8i \nabla_{\alpha \gamma} W^{\gamma\beta} . \]

(6.5a)

(6.5b)

The relation \( W^{\alpha\beta} = \frac{1}{6} (\tilde{\gamma}^{abc})^{\alpha\beta} W_{abc} \) means that the super-Weyl tensor \( W^{\alpha\beta} \) is equivalent to an anti-self-dual rank-3 tensor superfield \( W^{abc} \). In (6.3) \( T^{ABC} \) is the torsion curvature, and \( R(M)_{AB}^{cd} , R(J)_{AB}^{kl} , R(D)_{AB}^{k} , R(S)_{AB\gamma}^{k} \), and \( R(K)_{AB}^{c} \) are the curvatures associated with Lorentz, \( SU(2) \), dilatation, \( S \)-supersymmetry, and special conformal boosts, respectively. Their expressions in terms of the super-Weyl tensor \( W^{\alpha\beta} \) and its descendant superfields of dimension 3/2

\[ X^{\alpha i} := -\frac{i}{10} \nabla_{\beta}^{i} W^{\alpha\beta} , \quad X^{\alpha k}_{\gamma} := -\frac{i}{4} \nabla^{k} W^{\alpha\beta} - \delta_{\gamma}^{(\alpha} X^{\beta)k} , \]

(6.6)

are given in appendix B. Just like the 5D case, we consider the superspace and component structures for 6D corresponding to the “traceless” choice of conventional constraints, which was first considered in [12]. The component and superspace structures are summarised in section 7 and appendix B.

The superfields \( X^{\alpha i} , X^{\alpha k}_{\gamma} , Y^{\beta ij} , Y , \) and \( Y^{\alpha \beta \gamma \delta} \) are the only independent descendants of \( W^{\alpha\beta} \). All the other higher dimension descendants obtained by the action of spinor derivatives on \( W^{\alpha\beta} \) are vector derivatives of these independent fields as a result of the non-trivial Bianchi identities (6.5). Eq. (B.16a) gives the action of the \( S \)-generators on these independent descendants that prove to all be annihilated by \( K^a \).

The conformal supergravity gauge group \( G \) is generated by covariant general coordinate transformations, \( \delta_{\text{cgt}} \), associated with a local superdiffeomorphism parameter \( \xi^A \) and standard superconformal transformations, \( \delta_{\text{H}} \), associated with the following local superfield parameters: the dilatation \( \sigma \), Lorentz \( \Lambda^{ab} = -\Lambda^{ba} \), \( SU(2)_R \Lambda^{ij} = \Lambda^{ji} \), and special conformal transformations \( \Lambda_A = (\eta^i_A, \Lambda_a) \). The covariant derivatives transform as

\[ \delta_{\theta} \nabla_A = [\mathcal{K}, \nabla_A] , \]

(6.8)
where $K$ denotes the first-order differential operator

$$K = \xi \nabla_C + \frac{1}{2} \Lambda^{ab} M_{ab} + \Lambda^{ij} J_{ij} + \sigma \mathbb{D} + \Lambda_A K^A \, .$$  

(6.9)

A covariant (or tensor) superfield $U$ transforms as

$$\delta_g U = (\delta_{\text{gste}} + \delta_H)U = KU \, .$$  

(6.10)

The superfield $U$ is said to be superconformal primary and of dimension $\Delta$ if $K_A U = 0$ and $\mathbb{D} U = \Delta U$.

### 6.2 The on-shell hypermultiplet

Analogously to the 5D case, our starting point is the on-shell realization for the 6D $\mathcal{N} = (1, 0)$ hypermultiplet with 4 + 4 degrees of freedom. In conformal superspace, it is described by a Lorentz scalar superfield $q_{\hat{i}}$ subject to the constraint

$$\nabla_{\hat{i}} q_{\hat{j}} = 0 \, .$$  

(6.11)

Here, the index $\hat{i} = 1, 2$ denotes an SU(2) flavour index. The superfield $q_{\hat{i}}$ is a Lorentz scalar superconformal primary,

$$M_{ab} q_{\hat{i}} = 0 \, , \quad S^a_{\hat{i}} q_{\hat{j}} = K_a q_{\hat{i}} = 0 \, , \quad J^{\hat{k} \hat{i}} q_{\hat{j}} = \epsilon^{\hat{i} \hat{k} \hat{j}} q_{\hat{l}} \, .$$  

(6.12)

Eqs. (6.11), (6.12), and the relation (B.11) tell us that $\mathbb{D} q_{\hat{i}} = 2q_{\hat{i}}$. The only independent descent superfield of $q_{\hat{i}}$ is a dimension 5/2 spinor superfield

$$\rho_{\hat{i}} := \nabla_{\hat{i}} q_{\hat{j}} \, .$$  

(6.13)

Equation (6.11) can now equivalently be written in terms of this spinor superfield

$$\nabla_{\hat{i}} q_{\hat{j}} = -\frac{1}{2} \epsilon^{\hat{i} \hat{k}} \rho_{\hat{j}} \, .$$  

(6.14)

Applying spinor derivatives on these independent fields leads to several implications of the (anti-)commutation relations (B.13), (B.14), along with the constraints (6.12), and (6.14):

$$\nabla_{\hat{i}} \nabla_{\hat{j}} q_{\hat{k}} = \nabla_{\hat{i}} q_{\hat{j}} = -4i \nabla_{\alpha \beta} q_{\hat{\alpha}} \, .$$  

(6.15)

with $\nabla_{\alpha \beta} := (\gamma^a)_{\alpha \beta} \nabla_a$. Next, we shall consider

$$\epsilon^{\gamma \delta \alpha \beta} \{ \nabla_{\alpha}, \nabla_{\beta k} \} \rho_{\delta} = 8i \nabla^{\gamma \delta} \rho_{\delta} - 24i W^{\gamma \delta} \rho_{\delta} + 288 X^{\gamma k} q_{\hat{k}} \, .$$  

(6.16)
here $\tilde{\nabla}^{\alpha\beta} := (\tilde{\gamma}^a)^{\alpha\beta} \nabla_a$ and we have made use of (B.13) and the $S$-supersymmetry transformation

$$S^\gamma_i \rho^j_\beta = 16\delta^\gamma_\beta q^i_\alpha \implies K_a \rho^j_\beta = 0 . \quad (6.17)$$

On the other hand, by virtue of (6.15), we also have that

$$\epsilon^{\gamma\delta\alpha\beta} \{\nabla^k_\alpha, \nabla^l_\beta\} \rho^j_\delta = 16i(\tilde{\gamma}^a)^{\alpha\gamma} \nabla_a q^i_\dot{\alpha}$$

$$= 16i(\tilde{\gamma}^a)^{\alpha\gamma}[\nabla^j_\alpha, \nabla_a] q^i_\dot{\alpha} + 16i(\tilde{\gamma}^a)^{\alpha\gamma} \nabla_a \rho^j_\alpha . \quad (6.18)$$

Applying the commutation relation (B.14), we can then equate (6.16) and (6.18) to obtain

$$(\tilde{\gamma}^a \nabla_a \rho^j_\alpha)^\alpha = -W^{\alpha\beta} \rho^j_\beta - 4i X^{\alpha k} q^i_k . \quad (6.19)$$

We can then hit both sides of (6.19) with $\nabla_i^\alpha$ and make use of (6.15), (B.14), and the identity (B.17). This results in the equation

$$\Box q^i_\dot{\alpha} = \frac{1}{2} X^i \rho^j_\dot{\alpha} - \frac{1}{2} Y q^i_\dot{\alpha} , \quad \Box := \nabla^a \nabla_a . \quad (6.20)$$

The local superconformal $\delta = \delta_Q + \delta_H$ transformations (except translation, i.e. $\xi^a = 0$) of the covariant superfields $q^i_\dot{\alpha}$ and $\rho^j_\alpha$ can be derived using (6.9) and the relations (6.14), (6.15), and (6.17). This leads to

$$\delta q^i_\dot{\alpha} = \frac{1}{2} \xi^l \rho^j_\dot{\alpha} + \Lambda^{i k} q^k_\dot{\alpha} + 2\sigma q^i_\dot{\alpha} , \quad (6.21a)$$

$$\delta \rho^j_\alpha = -4i(\xi_i \tilde{\gamma}^a)^{\alpha} \nabla_a q^i_\dot{\alpha} - \frac{1}{4} \Lambda_{ab} (\gamma^{ab})_\alpha + \frac{5}{2} \sigma \rho^j_\alpha + 16\eta^j_\alpha q^i_\dot{\alpha} . \quad (6.21b)$$

### 6.3 The $O(2)$ multiplet

The 6D linear multiplet, or $O(2)$ multiplet can be described in terms of an SU(2)$_R$ triplet of Lorentz scalar superfields $L^{ij}$, with $(L^{ij})^* = \epsilon_{ikl} L^{kl}$ and satisfies the defining constraint

$$\nabla_a^{(ij} L^{jk)} = 0 . \quad (6.22)$$

Here $L^{ij}$ is a superconformal primary dimension-4 superfield,

$$S^\gamma_k L^{ij} = K^a L^{ij} = 0 , \quad D L^{ij} = 4L^{ij} , \quad J^{ij} L^{kl} = \epsilon^{k(l} L^{j)l} + \epsilon^{l(i} L^{j)k} . \quad (6.23)$$
The tower of component fields of the superfield $L^{ij}$ is given by the following set of useful identities:

\[ \nabla^i L^{jk} = -2\varepsilon^{ij} \varphi^k, \quad \nabla^i \varphi^j = -\frac{i}{2} \varepsilon^{ij} H_{\alpha\beta} - i \nabla_{\alpha\beta} L^{ij}, \]
\[ \nabla^k H_{\alpha\beta} = -8 \nabla_{[\alpha} \varphi^k_{|\beta]} - 2 \nabla_{\alpha\beta} \varphi^k + 2\epsilon_{\alpha\beta\gamma\delta} W^{\delta\rho} \varphi^k_{\rho}, \]

where we have defined the independent descendant superfields

\[ \varphi^i_{\alpha} := -\frac{1}{3} \nabla_{\alpha j} L^{ij}, \quad H_a := -\frac{i}{4} (\tilde{\gamma}_a)^{\alpha\beta} \nabla_{\alpha\beta} \varphi^k_{\beta k}, \quad H_{\alpha\beta} := (\gamma^\alpha)_{\alpha\beta} H_a. \]

We will be using these results later when analysing the component structure of the multiplet. Further, it can be checked that $H^a$ obeys the differential condition

\[ \nabla_a H^a = 0, \quad H^a := \frac{1}{5!} \varepsilon^{abcd} H_{bcedf}. \]

The descendants (6.25) prove to be annihilated by $K_a$ and to satisfy

\[ S^{\alpha}_i \varphi^j_{\beta} = 8\delta^\alpha_\beta L^{ij}, \quad S^k_{\alpha} H_{\alpha\beta} = -40i\delta^k_{\alpha} \varphi^j_{\beta k}. \]

We refer the reader to [13] for a superform description of the $O(2)$ multiplet.

7 The standard Weyl multiplet in components

Similar to the 5D case, we begin by identifying the various component fields of the 6D $\mathcal{N} = (1, 0)$ standard Weyl multiplet [7] within the geometry of conformal superspace. The vielbein $(e_m^a)$ and gravitino $(\psi_m^a)$ are identified with the coefficients of $dx^m$ of the super-vielbein $E^A = (E^a, E_i^a) = dz^M E_M^A$,

\[ e_m^a(x) := E_m^a(z), \quad \psi_m^a(x) := 2E_m^a(z). \]

In a local coordinate independent way they are given by

\[ e^a = dx^m e_m^a = E^a, \quad \psi^a = dx^m \psi_m^a = 2E^a. \]
Similar to the 5D case, look at (3.1) and (3.2), the single bar denotes setting $\theta = 0$ and the double-bar denotes setting $\theta = d\theta = 0$. Analogously, the remaining fundamental and composite one-forms correspond to double-bar projections of superspace one-forms,

$$\phi^{kl} := \Phi^{kl} \|, \quad b := B \|, \quad \omega^{cd} := \Omega^{cd} \|, \quad \phi^{k} := 2 \tilde{\phi}^{k} \|, \quad f_{c} := \tilde{f}_{c} \|.$$  \hfill (7.3)

The covariant matter fields are contained within the super-Weyl tensor $W_{abc}$ and its independent descendants,

$$T^{-}_{abc} := -2W_{abc} \|,$$  \hfill (7.4a)

$$\chi^{ai} := 15 \frac{1}{2} X^{ai} \| = -\frac{3i}{4} \nabla^{i} W^{\alpha\beta}\|,$$  \hfill (7.4b)

$$D := \frac{15}{2} Y \| = -\frac{3i}{16} \nabla^{k} \nabla^{\beta k} W^{\alpha\beta}\|.$$  \hfill (7.4c)

The lowest components of the other nontrivial descendants of $W^{\alpha\beta}$, specifically $X_{\alpha}^{\beta kl}|$, $Y_{\alpha}^{\beta k}|$ and $Y_{\alpha\beta}^{\gamma\delta}|$, prove to be directly related to component curvatures and hence are composite fields.

The component gauge connections can now be used to define the locally superconformal covariant derivative $\nabla_{a}$, which coincide with the bar projection of the conformal superspace covariant derivative $\nabla_{a} |$

$$e_{m}^{a} \nabla_{a} = \partial_{m} - \frac{1}{2} \psi_{m}^{\alpha}\nabla_{\alpha}^{i} - \frac{1}{2} \omega_{m}^{cd} M_{cd} - \phi_{m}^{ij} J_{ij} - b_{m} \| - \frac{1}{2} \phi_{m}^{i} s_{i}^{\alpha} - \tilde{f}_{ma} K^{a}. \hfill (7.5)$$

This satisfies the algebra

$$[\nabla_{a}, \nabla_{b}] = -R(P)_{a}^{c} \nabla_{c} - R(Q)_{a}^{i} Q_{i}^{c} - \frac{1}{2} R(M)_{a}^{cd} M_{cd} - R(J)_{a}^{kl} J_{kl} - R(\mathbb{D})_{a} \| - R(S)_{a}^{i} s_{i}^{\alpha} - R(K)_{a}^{c} K_{c}\|. \hfill (7.6)$$

where we identified $R(P)_{a}^{c} = \mathcal{T}_{a}^{c} |\$, $R(Q)_{a}^{i} = \mathcal{T}_{a}^{i} |\$, while $R(M)_{a}^{cd}$, $R(J)_{a}^{ij}$, $R(\mathbb{D})_{a}$, $R(S)_{a}^{k}$, and $R(K)_{a}^{c}$ are coinciding with the lowest components of the corresponding superspace curvature tensors given in appendix 13.

The constraints on the superspace curvatures determine the supercovariantised component curvature by taking the double-bar projection of the superspace two forms. This leads to (see [12] for details)

$$R(P)_{ab}^{c} = 2 e_{m}^{a} e_{n}^{b} D_{[m} e_{n]}^{c} + \frac{i}{2} \psi_{[aj} \gamma^{c} \psi_{bj]}^{j}, \hfill (7.7a)$$

$$R(Q)_{ab}^{c} = \frac{1}{2} \Psi_{ab}^{c} + i \tilde{\gamma}_{[a} \phi_{bj]k} + \frac{1}{24} T^{-}_{cde} \tilde{\gamma}^{cde} \gamma_{[a} \psi_{bj]k}, \hfill (7.7b)$$

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\[ R(\mathbb{D})_{ab} = 2e_a^m e_b^n \partial_{[m}b_{n]} + 4f_{[ab]} + \psi_{[a}^i \phi_{ij]} + \frac{i}{15} \psi_{[a}^j \gamma_{b]} \chi_j, \quad (7.7c) \]

\[ R(M)_{ab}^{cd} = \mathcal{R}_{ab}^{cd}(\omega) + 8\delta^{[c}_{[a}b_{d]} + i\psi_{[a}\gamma_{b]} R(Q)^{cdj} + 2i\psi_{[a}^j \gamma^c \mathcal{R}(Q)_b^{d]j} - \psi_{[a}^j \gamma^c \phi_{b]}^j - \frac{2i}{15} \delta^{[c}_{[a}b_{d]} \chi_j + \frac{i}{2} \psi_{[a}^j \gamma^c \psi_{b]}^j T^{-cd}, \quad (7.7d) \]

\[ R(J)_{ab}^{kl} = \mathcal{R}_{ab}^{kl}(\phi) + 4\psi_{[a}^i(k \phi_{l}^j) + \frac{4i}{15} \psi_{[a}^i(k^c \chi^l) , \quad (7.7e) \]

where \( D_m \) is the spin, dilatation, and SU(2)\(_R\) covariant derivative

\[ D_m = \partial_m - \frac{1}{2} \omega_m^{bc} M_{bc} - b_m \mathbb{D} - \phi_m \gamma^i J_{ij}, \quad D_a = e_a^m D_m , \quad (7.8) \]

along with the field strengths and curvature

\[ \Psi_{ab}^\gamma := 2e_a^m e_b^n D_m [\psi_{ij}]^k, \quad (7.9a) \]

\[ \mathcal{R}_{ab}^{cd} := \mathcal{R}_{ab}^{cd}(\omega) = e_a^m e_b^n \left( 2\partial_{[m}n_{n]}^{cd} - 2\omega_m \gamma^e \omega_n^{de} \right), \quad (7.9b) \]

\[ \mathcal{R}_{ab}^{kl} := \mathcal{R}_{ab}^{kl}(\phi) = e_a^m e_b^n \left( 2\partial_{[m}n_{n]}^{kl} + 2\phi_m \gamma^{[k} \phi_{n]}^{p]l} \right). \quad (7.9c) \]

The component curvatures turn out to obey “traceless” conventional constraints \[12\]

\[ R(P)_{ab}^c = 0, \quad \gamma^b R(Q)_{ab} = 0, \quad R(M)_{ab}^{cd} = 0. \quad (7.10) \]

The constraints (7.10) can be solved for the composite connections as follows:

\[ \omega_{abc} = \omega(e)_{abc} - 2\eta_{[a} \eta_{bc]} - \frac{i}{4} \psi_{[a}^k \gamma_{bc] k} - \frac{i}{2} \psi_{a}^k \gamma_{[bc] k} \quad (7.11a) \]

\[ \phi_m^k = \frac{i}{16} \left( \gamma^{bc} \gamma_m - \frac{3}{5} \gamma_m \gamma^{bc} \right) \left( \Psi_{bc}^k + \frac{1}{12} T^{e} \gamma_{def} \psi_{[bc]}^k \right), \quad (7.11b) \]

\[ f_a^b = -\frac{1}{8} \mathcal{R}_a^b(\omega) + \frac{1}{80} \delta_a^b \mathcal{R}(\omega) + \frac{1}{8} \psi_{[a}^j \gamma_{bc] j} + \frac{1}{80} \delta_a^b \psi_{[a}^j \gamma_{bc] j} + \frac{1}{16} \psi_{a}^j \gamma_{bcj} T^{-bcj} - \frac{1}{160} \delta_a^b \psi_{a}^j \gamma_{d} \psi_{e}^j T^{-ade}, \quad (7.11c) \]

where \( \omega(e)_{abc} = -\frac{1}{2}(C_{abc} + C_{cab} - C_{bca}) \) is the torsion-free spin connection given in terms of the anholonomy coefficient \( C_{mn}^a := 2\partial_{[m} e_n] a, \mathcal{R}(\omega) = \mathcal{R}_{ab}^{cd}(\omega) \) is the scalar curvature and \( \mathcal{R}_a^b(\omega) = \mathcal{R}_{ac}^{bc}(\omega) \) is the Ricci curvature. It is important to emphasise that in the traceless frame the composite connection \( \phi_m^k \) does neither depend on \( \chi \) nor on \( D \), however, the composite connection \( f_a^b \) has a dependence on \( \chi \).

The supersymmetry transformations of the fundamental gauge connections of the Weyl multiplet can be derived directly from the transformations of their corresponding superspace one-forms, see [12] for details. In components, the local superconformal transformations, except covariant general coordinate transformations, are identified by the following
operator $\delta$

\[
\delta = \xi^i Q^i + \frac{1}{2} \lambda^{ab} M_{ab} + \lambda^{ij} J_{ij} + \lambda_D \Box + \lambda_a K^a + \eta^i \zeta^a_i .
\]  

(7.12)

Here, the local component parameters are respectively defined as the $\theta = 0$ components of the corresponding superfield parameters, $\xi^i := \xi^i \|$, $\lambda^{ab} := \Lambda^{ab} \|$, $\lambda^{ij} := \Lambda^{ij} \|$, $\lambda_D := \sigma \|$, $\lambda^a := \Lambda^a \|$, and $\eta^i := \eta^i \|$. The local superconformal transformations of the independent fields of the standard Weyl multiplet are given by

\[
\begin{align*}
\delta e^a_m &= -i \xi \gamma^a \psi^i_m + \lambda^a_b e^b_m - \lambda_B e^a_m , \\
\delta \psi^a_{mi} &= 2D_m \xi^a - \frac{1}{12} (\xi \gamma \zeta^{abc})^a T_{abc} - \frac{1}{4} \lambda^b (\psi_{m} \gamma^a) - \lambda^i \psi^a_{mj} - \frac{1}{2} \lambda_B \psi^a_{mi} - 2i (\eta \zeta_m)^a , \\
\delta \phi^a_{km} &= -4 \xi (k \phi^a_m - \frac{4i}{15} \xi \gamma^a \chi^j + \partial_m \lambda^i - 2 \phi^a_{m} (k \lambda^i) + 4 \eta (k \phi^a_m) , \\
\delta b_m &= \frac{i}{15} \xi \gamma \chi^i + \xi i \phi^a_m + \partial_m \lambda_D - \psi^a_m \eta_k - 2 \lambda_m , \\
\delta T_{a\bar{c}} &= -\frac{i}{8} \xi \gamma^{c} \gamma_{a\bar{c}} R(Q)_{c\bar{k}} - \frac{2i}{15} \xi \gamma_{a\bar{c}} \chi^i - 3 \lambda^c_{[a} T_{bc]} - \lambda_D T_{a\bar{c}} , \\
\delta \chi^a &= \frac{1}{2} \xi D^{\alpha} + \frac{3}{4} \mathcal{R}(\xi)_{a}^{ij} (\zeta \gamma^a)^{\alpha} - \frac{1}{4} (\xi \gamma^{a} \gamma_{bcd})^a \nabla_\alpha T_{bcd} - \frac{1}{4} \lambda^c (\chi^{a} \gamma_{cd})^a \\
&+ \lambda^i \chi^{a} j^{ij} + \frac{3}{2} \lambda_B \chi^{a} i + i (\eta \zeta_{a}^{abc})^a T_{a\bar{c}} , \\
\delta D &= -2i \xi \gamma^a \nabla_\alpha \chi^i + 2 \lambda_B D - 4 \chi^i \eta_k ,
\end{align*}
\]

(7.13a)

where

\[
\begin{align*}
\nabla_\alpha T_{abc} &= D_a T_{abc} + \frac{i}{15} \psi_{dk} \gamma_{abc} \chi^k + 4i \psi_{dk} X^{k(\bar{a} \bar{c})} , \\
\nabla_\alpha \chi^{a} j^{ij} &= D_a \chi^{a} j^{ij} - \frac{3}{8} (\psi_{ai} \gamma^a)^{\beta} \mathcal{R}(\xi)_{bc}^{ij} - \frac{1}{8} (\psi_{a}^{j} \gamma^{a} \gamma_{bcd})^a \nabla_\alpha T_{bcd} \\
&+ \frac{1}{4} \psi_{a}^{j} \nabla_\alpha \chi^{a} j^{ij} + \frac{i}{2} T_{bcd} (\phi_{a}^{j} \gamma^{a} \gamma_{bcd})^a ,
\end{align*}
\]

(7.14a)

(7.14b)

and we have defined $X^{k(\bar{a} \bar{c})} := \frac{1}{8} (\gamma_{abc})^{a} \chi^a \chi^b \chi^c$.

Note that the transformations (7.13) form an algebra that closes off-shell on a local extension of OSp(6,2|1). To conclude this subsection, for convenience, we include Table 6 which summarises the non-trivial dilatation weights of the fields and local gauge parameters of the standard Weyl multiplet.

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8 The hyper-dilaton Weyl multiplet in 6D

The aim of this section is to construct the 40 + 40 hyper-dilaton Weyl multiplet of off-shell $\mathcal{N} = (1, 0)$ conformal supergravity in six dimensions. The construction mimics the 5D $\mathcal{N} = 1$ case elaborated earlier in our paper and the 4D $\mathcal{N} = 2$ case of [18].

We begin with the component structure of the on-shell hypermultiplet. This can be readily extracted from the previous superspace realisation via the bar projection. The independent components of the on-shell hypermultiplet are simply: the Lorentz scalar field $q^i\xi_i$ which is a superconformal primary, and the spinor field $\rho^i\alpha\bar{\gamma}^a$, which account for 4 + 4 on-shell degrees of freedom. All other descendants are derivatives of these two fields.

In what follows, we will associate the same symbol for the covariant component fields and the corresponding superfields, when the interpretation is clear from the context. The local superconformal transformations of the component fields follow directly from the projections of (6.21), which give

$$\delta q^{i\bar{z}} = \frac{1}{2}\xi^i\rho^{i\bar{z}} + \lambda^i_k q^{k\bar{z}} + 2\lambda_{\bar{z}}q^{i\bar{z}};$$

$$\delta \rho^i_{\alpha} = -4i(\xi_i\gamma^{\alpha})_{\alpha} \nabla_a q^{i\bar{z}} - \frac{1}{4}\lambda_{ab}(\gamma^{ab}\rho^i_{\alpha})_{\alpha} + \frac{5}{2}\lambda_{\bar{z}}\rho^i_{\alpha} + 16\eta^i_{\alpha}q^{i\bar{z}};$$

where

$$\nabla_a q^{i\bar{z}} = D_a q^{i\bar{z}} - \frac{1}{4}\psi^a_{\alpha}\rho^{i\bar{z}}.$$

Unlike the standard Weyl multiplet, the algebra of the local transformations (8.1) closes only when the equations of motion for the fields $q^{i\bar{z}}$ and $\rho^i_{\alpha}$ are imposed. These equations of motion can be obtained by taking the bar projection of (6.19) and (6.20). Specifically, in the traceless frame, the lowest component of the descendants $\nabla_a \nabla_a q^{i\bar{z}}$ and $\Box q^{i\bar{z}}$ leads to the following two constraints:

$$\Box q^{i\bar{z}} = \frac{1}{15}\chi^i\rho^{i\bar{z}} - \frac{1}{15}D q^{i\bar{z}};$$

$$\Box := \nabla^a \nabla_a.$$

Table 6: Summary of the non-trivial dilatation weights in the standard Weyl multiplet.
The expressions for $\nabla_a \rho_i^\alpha$ and $\Box q^{i\underline{i}}$ in terms of the derivatives $D_a$ are given by

$$\nabla_a \rho_i^\alpha = D_a \rho_i^\alpha + 2i(\psi_{ak} \gamma^c)_\alpha \left(D_c q^{i\underline{k}} - \frac{1}{4} \psi_c^k \rho^\underline{k}\right) - 8\phi_a^k q^{i\underline{k}},$$  \hspace{1cm} (8.4a)

$$\Box q^{i\underline{i}} = D^a D_a q^{i\underline{i}} - 4f_a^i a q^{i\underline{i}} - \frac{1}{4} D_a(\psi^{ai} \rho^j) - \frac{i}{4} \phi_a^i \tilde{\gamma}^a \rho^\underline{i} + \frac{1}{4} \psi_{ak} \gamma^c \left(D_c q^{j\underline{i}} - \frac{1}{4} \psi_{c1} \rho^j\right) - \frac{8i}{15}(\chi^k \gamma^a)_{\alpha k}^{\underline{i}} + \frac{1}{2}(\psi_{aj} \gamma^c \rho^j + 8\phi_{aj} q^{i\underline{j}}).$$  \hspace{1cm} (8.4b)

Equations (8.3) can then be interpreted as algebraic equations for the standard Weyl multiplet that determine the auxiliary fields $\chi^{ai}$ and $D$ in terms of $q^{i\underline{i}}$ and $\rho_i^\alpha$, together with the other independent fields of the standard Weyl multiplet. It can be noted that in the traceless frame equations (8.4a) and (8.4b) do not depend on the fields $\chi^{ai}$ and $D$ respectively making it trivial to find these auxiliary fields. If we assume that $q^{i\underline{i}}$ is an invertible matrix, which is equivalent to imposing

$$q^2 := q^{i\underline{i}} q_{i\underline{i}} = \varepsilon_{ij} \varepsilon_{k\underline{k}} q^{i\underline{i}} q^{j\underline{j}} = 2 \det q^{i\underline{i}} \neq 0,$$  \hspace{1cm} (8.5)

then the following relations hold

$$\chi_i^\alpha = \frac{15i}{4} q^{-2} q^{i\underline{i}} \left[\nabla_a \rho^i \tilde{\gamma}^\alpha + \frac{1}{12}(\rho^i \tilde{\gamma}^{abc}) \alpha T_{abc}\right],$$  \hspace{1cm} (8.6a)

$$D = -15q^{-2} q^{i\underline{i}} \Box q_{i\underline{i}} + \frac{15i}{8} q^{-2} \left[\nabla_a \rho^i \tilde{\gamma}^\alpha \rho^\underline{i} + \frac{1}{12}(\rho^i \tilde{\gamma}^{abc} \rho^\underline{i}) T_{abc}\right].$$  \hspace{1cm} (8.6b)

Once more, we stress that the right-hand side of equation (8.6a) does not have any dependence on field $\chi$, thus making it a composite field. Similarly, the right-hand side of equation (8.6b) does not depend on $D$, however it has an implicit dependence on $\chi$ through the special conformal connection $f_{ab}$, eq. (7.11c) and see (8.4b), that is hidden in the expression of $\Box q^{i\underline{i}}$. It is straightforward to pull out the explicit dependence upon $\chi$ and then use (8.6a). As a result, both $\chi$ and $D$ are composite.

As a next step in the construction of the hyper-dilaton Weyl multiplet, we note that associated to an on-shell hypermultiplet one can construct a triplet of linear multiplets, exactly as in the 4D $\mathcal{N} = 2$ and 5D $\mathcal{N} = 1$ cases. The component fields of the $\mathcal{N} = 1$ off-shell linear (or $\mathcal{O}(2)$) multiplet are defined in terms of the bar projections of (6.25): an SU(2)$_R$ triplet of Lorentz scalar fields $L^{ij} = L^{ij}|$; a spinor field $\phi^i = \phi^i|$, and a closed anti-symmetric five-form field strength $h_{mnpqr} := H_{mnpqr}|$, which is equivalent to a conserved dual vector $h^a := \frac{1}{5!} \varepsilon_{amnpqr} h_{mnpqr}$. Defining $H^a = H^a|$, at the component level
it holds that
\[
H_a = h_a - \psi b_i \gamma^{ab} \varphi^i - \frac{i}{2} \psi b_i \gamma^{abc} \psi c_j L^{ij} .
\] (8.7)

The covariant conservation equation of \(H_a\) is
\[
\nabla^a H_a = 0 .
\] (8.8)

The constraint implies the existence of a gauge four-form potential, \(b_{mn pq}\), and its exterior derivative \(h_{mnpqr} := 5 \partial_m [b_{npqr}]\). The local superconformal transformations of the linear multiplet in a standard Weyl multiplet background are given by
\[
\delta L^{ij} = 2 \xi^a_{(i} \varphi^j_{a)} + 2 \lambda^i k L^{jk} + 4 \lambda_4 L^{ij} ,
\]
\[
\delta \varphi^i = \frac{i}{2} \xi^a_{[i} H_{a j]} - i \xi^a_{[j} \nabla^a H_{i \alpha} + \frac{1}{4} \lambda^a_{(i} \varphi^j_{a) \alpha} - \lambda_{ij} \varphi_{\alpha j} + \frac{9}{2} \lambda_4 \varphi^i + 8 \eta^i_{a \alpha} L^i ,
\]
\[
\delta H_a = 2 (\xi^a_{ab} \varphi^b_\alpha) + \frac{1}{12} (\xi^a_{ij} \varphi^b c d e f) T^{\alpha \beta \gamma \delta} + \lambda_{a d} H_d + 5 \lambda_4 H_a
\]
\[
-10 i \eta^i_{a \alpha} \varphi^i ,
\] (8.9a, 8.9b, 8.9c)

where
\[
\nabla^a L^{ij} = D^a L^{ij} - \psi^a_{(i} \varphi^j_{a)} ,
\]
\[
\nabla^a \varphi^i = D^a \varphi^i - \frac{i}{4} \psi^a_{(i} \gamma^b \varphi^j_{a)} - \frac{i}{2} \psi^a_{(i} \psi^b \varphi^j_{a)} + 4 \phi_{ij} L^{ij} .
\] (8.10a, 8.10b)

The locally superconformal transformations of \(b_{mn pq}\) are
\[
\delta b_{mn pq} = - \varepsilon_{mnpqf} (\xi^e_{[f} \varphi^i_{e]) + 8 i (\psi_{[m} \gamma_{npq]} \xi_{j} L^{ij} + 4 \partial_m [l_{npq}] ,
\] (8.11)

where we have also included the gauge transformation \(\delta b_{mn pq} = 4 \partial_m [l_{npq}]\) leaving \(h_{mn pq}\) and \(H^a\) invariant. For convenience, we have summarised the dilatation weights of the fields of the \(O(2)\) multiplet in Table 7

| \(\mathbb{D}\) | \(L_{ij}\) | \(\varphi_{ai}\) | \(H_a\) | \(b_{mn pq}\) |
|---|---|---|---|---|
| 4 | 9/2 | 5 | 0 |

Table 7: Summary of the dilatation weights in the off-shell \(O(2)\) multiplet.

Now that we have reviewed the structure of a locally superconformal \(O(2)\) multiplet, we return to constructing a triplet of linear multiplets from an on-shell hypermultiplet. Given that \(q^{i \alpha}\) and \(\rho_{a i}\) describe an on-shell hypermultiplet in a standard Weyl multiplet,
background with transformation rules (8.1), it can be shown that the following composite fields define a triplet of $O(2)$ multiplets

\[
L_{ij} \equiv q_i (\mathbf{q}_j) = q_i (i q_j), \quad (8.1a)
\]

\[
\varphi_{ij} \equiv \frac{1}{2} q_i (i \rho_j), \quad (8.1b)
\]

\[
H_{ij} = 2q_i (\mathbf{q} q_j) - \frac{i}{8} \rho_j (i \tilde{\gamma}_i \rho_j). \quad (8.1c)
\]

These fields all transform according to (8.9) and each of the previous fields is symmetric in $i$ and $j$. The field $H_{ij}$, in particular, is interesting as it can be used to express the $SU(2)_R$ connection $\phi_{mij}$ as a composite field. To see this, we introduce a new covariant derivative

\[
D_a = e_a^m \left( \partial_m - \frac{1}{2} \omega_m^{\ cd} M_{cd} - b_m \mathcal{D} \right) = D_a + e_a^m \phi_{mij} J_{ij}, \quad (8.13)
\]

which then allows us to rearrange eq. (8.12c) for the $SU(2)_R$ gauge connection:

\[
\phi_{ij} = 4q_i q_j - \frac{1}{4} q_i q_j \left( \frac{1}{2} \omega_m^{\ cd} M_{cd} - b_m \mathcal{D} \right) + \frac{i}{16} \rho_j (i \tilde{\gamma}_i \rho_j) - \frac{1}{2} H_{ij}. \quad (8.14)
\]

This concludes the definition of the hyper-dilaton Weyl multiplet. Our analysis demonstrates that the hyper-dilaton Weyl multiplet defines a new representation of the off-shell local 6D, $\mathcal{N} = (1, 0)$ superconformal algebra. The multiplet comprises the following independent fields: $e^a_m$, $b_m$, $T_{abc}$, $q^i$, $b_{mnpq} \tilde{q}^i$, $\psi_{mi}$, and $\rho_i$. It also possesses the same number of off-shell degrees of freedom as the standard Weyl multiplet, 40 + 40. Table 8 summarises the counting of degrees of freedom, underlining the symmetries acting on the fields. Note that with the ingredients provided so far, it is a straightforward exercise to obtain the locally superconformal transformations of the fields of the hyper-dilaton Weyl multiplet written only in terms of the independent fields and they are given as follows:

\[
\delta e^a_m = -i \xi_i \gamma^a \psi_m i + \lambda^a b e^b_m - \lambda_D e^a_m, \quad (8.15a)
\]

\[
\delta \psi^{\alpha}_{mi} = 2D_m \xi_i - \frac{1}{12} T_{abc} (\xi_i \gamma^a \gamma_{\alpha}^{abc}) + \frac{1}{4} \lambda^a (\psi_{mi} \gamma_{\alpha}) - \lambda^i \psi^a_{mj},
\]

\[
- \frac{1}{2} \lambda_D \psi^{\alpha}_{mi} - 2i (\eta_i \gamma_{\alpha}^m), \quad (8.15b)
\]

\[
\delta b_m = \frac{1}{4} q^2 q_i (\xi^i \gamma_m) \left[ \frac{1}{12} (\rho^i \gamma_{\alpha}^{abc}) T_{abc} - (\tilde{\gamma}^a \nabla_a \tilde{\rho}^i)^{\alpha} \right]
\]

\[
+ \xi_i \phi_m i + \partial_m \lambda_D - \psi^i m \eta_i - 2\lambda_m, \quad (8.15c)
\]

\[
\delta T_{abc} = -\frac{i}{8} \xi^k \gamma_{ef} \gamma_{abc} R(Q)_{efk}
\]

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\[
\begin{array}{cccccccc}
\varepsilon_m^a & \omega_m^{ab} & b_m & f_m^a & \phi_m^{ij} & \psi_m^i & \phi_m^i & T_{abc}^- \\
36 & 0 & 6B & 0 & 0 & 48F & 0 & 10B \\
\end{array}
\]

\[
\begin{array}{cccccccc}
P_a & M_{ab} & D & K_a & J^{ij} & Q & S & \lambda_{mnpq}^{ij}-\text{sym} \\
-6B & -15B & -1B & -6B & -3B & -8F & -8F & -30B \\
\end{array}
\]

Result: 40 + 40 degrees of freedom

Table 8: Degrees of freedom and symmetries of the hyper-dilaton Weyl multiplet. Row one gives all the fields in the multiplet. Row two gives the number of independent components of these fields — composite connections are counted with zero degrees of freedom. Row three gives the gauge symmetries. Note that the parameter \( \lambda_{mnpq}^{ij} \) describes the symmetry associated with the gauge four-forms \( b_{mnpq}^{ij} \) with field strength five-forms \( h_{mnpq}^{ij} \) and \( E_a^{ij} \). Row four gives the number of gauge degrees of freedom to be subtracted when counting the total degrees of freedom.

\[
-\frac{1}{2} q^{-2} q_{ij} (\xi^i \gamma_{abc})_\alpha \left[ \frac{1}{12} (\rho^k \bar{\gamma}^{abc})^\alpha T_{abc}^- - (\bar{\gamma}^a \nabla_a \rho^i)^\alpha \right] - 3 \lambda^e_{[a} T_{bc]j} + \lambda_B T_{abc}^-, \quad (8.15d)
\]

\[
\delta q_{ij}^i = \frac{1}{2} \epsilon^i \rho^i + \lambda^i_k q^{k2} + 2 \lambda_B q_{ij}^i, \quad (8.15e)
\]

\[
\delta \rho_{ai} = -4 i (\xi^i \gamma^c)_\alpha \nabla_c q_{ij}^i + \frac{1}{4} \lambda_{ab} (\rho^i \gamma^{ab})_\alpha + \frac{5}{2} \lambda_B \rho_{ai} + 16 \eta^k_{ai} q_{ij}^i, \quad (8.15f)
\]

\[
\delta b_{mnpq} = - \varepsilon_{mnpqj} \xi_i \gamma^{ef} \phi_i + 8 i (\psi_{[mi} \gamma_{npq]} \xi_j) L^{ij} + 4 \partial_{[m} l_{npq]} . \quad (8.15g)
\]

For completeness, here we present the expressions relevant to the transformation in terms of this new covariant derivative \( D_a \) instead of \( D_a \), which has an implicit dependence on the composite SU(2)\( _R \) connection \( \phi_a^{ij} \).

\[
\nabla_a q_{ij}^i = \frac{1}{2} D_a q_{ij}^i - \frac{1}{8} \psi_{ai} \rho^i - q^{-2} q_{ij}^i D_a q_{ik}^k + \frac{1}{4} q^{-2} q_{ik}^i (\psi_{ai} k^k) + \frac{1}{8} q^{-2} q_{ij}^i (\rho^k k^a \rho^i) + q^{-2} q_{ij}^i H_{a}^{ij} , \quad (8.16a)
\]

\[
\nabla_a \rho_{ai} = D_a \rho_{ai} + 2 i (\psi_{ai} k^k) \nabla_c q_{ij}^i - 8 \phi_{ai} q_{ij}^i, \quad (8.16b)
\]

\[
\square q_{ij}^i = D_a \nabla_a q_{ij}^i - 4 f_a^a q_{ij}^i + 4 q^{-4} q_{[ij}^i (\nabla_a q_{ij}^i) \left[ q^{kk} D_a q_{jk}^j - \frac{1}{2} H_{a}^{kj} - \frac{1}{4} q^{kk} (\psi_{ai} \rho^i) \frac{i}{16} \rho^k \bar{\gamma}^a \rho^i \right] - \frac{1}{96} (\psi_{ai} \gamma_a \gamma^{bcd} \rho^i) T_{bcd} + \frac{2 i}{15} (\psi_{ai} \gamma^a \chi^k) q_{ij}^i - \frac{1}{4} \psi_{ai} \alpha \nabla_a \rho_{ai} - \frac{i}{4} \phi_{ai} \gamma^a \rho_{ai} , \quad (8.16c)
\]

where the composite connection \( \phi_m^i \) and \( f_a^a \) are now given in terms of \( D_a \) by:

\[
\phi_m^k = \frac{i}{16} \left( \gamma^{bc} \gamma_m - \frac{3}{5} \gamma_m \gamma^{bc} \right) \left[ 2 D_{[b} \psi_{c]}^k + \frac{1}{12} T_{def}^{m} \bar{\gamma}^{def} \gamma_{[b} \psi_{c]}^k \right]
\]
\[ + 8q^{-4}q^{(k}q^{j)} \frac{1}{2} \{ q^{ki}D[a]q_{k}^{j} - \frac{1}{2} Hq^{k}q^{j} - \frac{1}{4} q^{ki}(\psi_{[b}k_{l]}q^{j]) - \frac{i}{16} (\rho^{k}_{\lambda}q^{j}_{\lambda}) \} \psi_{c}^{j} \}, \quad (8.17a) \]

\[ f_{a}^{b} = - \frac{1}{8} \mathcal{R}_{a}^{b}(\omega) + \frac{1}{80} \delta_{a}^{b}R(\omega) + \frac{i}{16} \psi_{c}^{j}\gamma_{a}R(Q)^{[c}j + \frac{i}{8} \psi_{c}^{j}\gamma_{a}R(Q)^{c]j} \]

\[ - \frac{1}{16} q^{-2}q^{j}(\psi_{a}\gamma_{c})_{a} \left[ \frac{1}{12} (\rho^{k}_{\lambda}q^{cde})_{a}^{\alpha} \nabla_{c}^{\alpha} - (\gamma^{c} \nabla_{c}q^{j})^{\alpha} \right] + \frac{1}{8} \psi_{[a}\gamma_{c}^{b} \phi_{c}^{j} \]

\[ - \frac{1}{160} \delta_{a}^{b} \psi_{c}^{j} \gamma_{d} \psi_{e}^{j} T^{-bcd} - \frac{i}{160} \delta_{a}^{b} \psi_{c}^{j} \gamma_{d} \psi_{e}^{j} T^{-cde}. \quad (8.17b) \]

Note that the expression of \( f_{a}^{b} \) has an explicit as well as implicit dependence on the composite connection \( \phi_{a}^{i} \) via \( \nabla_{a}\rho^{i} \), which can now be substituted from (8.17a). It is also convenient to provide the bosonic part of \( \square q^{i} \). By using that \( D^{a}H_{a}^{i} = 0 \) up to fermions, it holds:

\[ \square q^{i} = \frac{1}{2} D^{a}D_{a}q^{i} + \frac{3}{4} q^{-2}q^{i}D^{a}q^{k}D_{a}q_{k} - q^{-2}q^{i}D^{a}q_{k}^{j}D_{a}q^{k} - \frac{1}{2} q^{-2}q^{i}D^{a}q_{j}^{k}D_{a}q^{j} \]

\[ + q^{-4}q_{i}^{i}D^{a}q^{k}D_{a}q_{k} - \frac{1}{2} q^{-4}q^{i}H^{a}D_{a}q^{j} + \frac{1}{5} \mathcal{R}_{a}^{i} + \text{fermions}. \quad (8.18) \]

In analogy to the 5D \( \mathcal{N} = 1 \) hyper-dilaton Weyl multiplet, we will end this subsection by underlining the following two remarks about the 6D \( \mathcal{N} = (1, 0) \) hyper-dilaton Weyl multiplet:

1. From a symmetry point of view, the hyper-dilaton Weyl multiplet contains all the fields that are required to gauge fix the extra symmetries of the superconformal group, i.e., it contains a triplet of scalar field \( q^{i} \), which can be used to gauge fix dilatation and SU(2)\(_{R} \) symmetry; the spinor field \( \psi^{a} \) and the dilatation connection \( b_{m} \) can be used to fix S-supersymmetry and special conformal symmetry, respectively.

An example of such a gauge choice is as follows:

\[ q^{i} = - \epsilon^{i}, \quad (8.19a) \]

\[ \rho^{i}_{a} = 0, \quad (8.19b) \]

\[ b_{m} = 0. \quad (8.19c) \]

This indicates that in the gauge fixed version we would obtain an off-shell irreducible multiplet of Poincaré supergravity.

2. The second point would be to obtain a supersymmetric completion of the Einstein-Hilbert term by using an appropriate compensating multiplet. In 4D \( \mathcal{N} = 2 \) and 5D \( \mathcal{N} = 1 \) this was achieved by using an off-shell vector multiplet compensator. In 6D
the $\mathcal{N} = (1,0)$ vector multiplet has no scalar fields [80][81] that can be used for this purpose. The natural choice would be a tensor multiplet. However, known versions of an off-shell tensor multiplet involve infinite number of auxiliary fields [14][15]. One might wonder whether it suffices to use an off-shell linear multiplet. The action for an improved linear multiplet in a standard Weyl multiplet background take the form of the following $BF$ Lagrangian [7][13]

$$e^{-1} \mathcal{L}_{\text{bosonic}} = -\frac{2}{15} (3\mathcal{R} + D) L - \frac{1}{8L} H^a H_a - \frac{1}{2L} H^a \phi^i_a L_{ij} - \mathcal{D}^a \mathcal{D}_a L$$

$$+ \frac{1}{4L} (\mathcal{D}^a L^{ij}) \mathcal{D}_a L_{ij} - \frac{1}{8L^3} \tilde{b}^{mn} L_{ij} (\partial_m L^{ki}) \partial_n L_{jk} \ . \quad (8.20)$$

When working with a hyper-dilaton Weyl multiplet, we need to take into account that the auxiliary field $D$ and the $SU(2)_R$ connection are composite fields (and that (8.6b) and (8.14) have to be used). The combination $(3\mathcal{R} + D)$ turns out to not depend on the scalar curvature $\mathcal{R}$,

$$3\mathcal{R} + D = -15q^{-2} \left\{ q_{ij} \mathcal{D}_a (D^a q^{ij}) - q^{-2} q^{ij} (D_a q_{ij}) \right\} \ . \quad (8.21)$$

Clearly, by plugging this into (8.20), the result is independent of $\mathcal{R}$ and fails to be a good starting point to engineer a supersymmetric extension of the Einstein-Hilbert term to obtain a two-derivative Poincaré supergravity Lagrangian.

It is worth mentioning that coupling the hyper-Dilaton Weyl multiplet to any number of linear multiplet will encounter the same problem. This is not too surprising since the linear multiplet is on-shell equivalent to the on-shell hypermultiplet up to trading a scalar field with a gauge four-form. We expect that the same would be true by using other variant off-shell hypermultiplets, such as the off-shell charged hypermultiplet, coupled to conformal supergravity [14][15]. We will come back in the future to engineer 6D $\mathcal{N} = (1,0)$ off-shell Poincaré supergravity theories, and their matter couplings, by using our new hyper-dilaton Weyl multiplet in a super-conformal setting.

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A 5D $\mathcal{N} = 1$ conformal superspace identities

In this appendix we collect results about conformal superspace in the traceless frame of [20] focusing on the ingredients relevant to our discussion in the paper. We adhere with the notations and conventions of [20].

The Lorentz generators act on the superspace covariant derivatives $\nabla_A = (\nabla_a, \nabla^i_{\alpha})$ in the following way

\begin{align}
[M_{ab}, M_{cd}] &= 2\eta_{c[a}M_{b]d} - 2\eta_{d[a}M_{b]c} , \\
[M_{ab}, \nabla_c] &= 2\eta_{c[a}\nabla_{b]} , \\
[M_{a\beta}, \nabla^i_{\gamma}] &= \varepsilon_{\gamma(\alpha}\nabla^i_{\beta)} , \\
&= (A.1a) \\
&= (A.1b) \\
&= (A.1c)
\end{align}

where $M_{\alpha\beta} = 1/2(\Sigma^{ab})_{\alpha\beta}M_{ab}$. The SU(2)$_R$ and dilatation generators satisfy

\begin{align}
[J^{ij}, J^{kl}] &= \varepsilon^{k(i}{J^{j]l}}, \\
[J^{ij}, \nabla^k_{\alpha}] &= \varepsilon^{k(i}{\nabla^j_{\alpha}}).
\end{align}

(A.1d)

The Lorentz and SU(2)$_R$ generators act on the special conformal generators $K_A = (K_a, S_{\alpha i})$ according to the rules

\begin{align}
[M_{ab}, K_c] &= 2\eta_{c[a}K_{b]} , \\
[M_{a\beta}, S^i_{\gamma}] &= \varepsilon_{\gamma(\alpha}S^i_{\beta)} \\
[J^{ij}, S^k_{\alpha}] &= \varepsilon^{k(i}{S^j_{\alpha}) , \\
&= (A.1f)
\end{align}

while the dilatation generator acts on $K_A$ as

\begin{align}
[D, K_a] &= -K_a , \quad [D, S_{\alpha i}] = -\frac{1}{2}S_{\alpha i} .
\end{align}

(A.1g)
Among themselves, the generators $K_A$ obey the only nontrivial anti-commutation relation
\[ \{ S^i_\alpha, S^j_\beta \} = -2i \varepsilon^{ij}(\Gamma^c)_{\alpha\beta} K_c. \] (A.1h)

The algebra of $K_A$ with $\nabla_A$ is given by
\[ [K_a, \nabla_b] = 2\eta_{ab}D + 2M_{ab}, \] (A.1i)
\[ [K_a, \nabla^i_\alpha] = i(\Gamma_a)_\alpha^\beta S^i_\beta, \] (A.1j)
\[ \{ S_{\alpha i}, \nabla^j_\beta \} = 2\varepsilon_{\alpha\beta\delta} D - 4\delta_i^j M_{\alpha\beta} + 6\varepsilon_{\alpha\beta} J^j_i, \] (A.1k)
\[ [S_{\beta i}, \nabla_a] = i(\Gamma_a)_\beta^\alpha \nabla_{\alpha i} - \frac{1}{4} W^b_a (\Gamma_b)_\beta^\alpha S_{\alpha i} + \frac{i}{8} (\Gamma_a)_\beta^\gamma X_{\gamma i} K_b - \frac{i}{4} W_{ab\delta} K^b. \] (A.1l)

The anticommutator of two spinor derivatives, $\{ \nabla^i_\alpha, \nabla^j_\beta \}$, has the following non-zero torsion and curvatures
\[ \mathcal{T}^{i j c}_{\alpha\beta} = 2i \varepsilon^{ij}(\Gamma^c)_{\alpha\beta}, \] (A.2a)
\[ \mathcal{R}(M)^{i j cd}_{\alpha\beta} = 2i \varepsilon^{ij}(\Gamma^a)_{\alpha\beta} W^{cd} + i \varepsilon^{ij}(\Gamma^b)_{\alpha\beta} \tilde{W}^{bcd}, \] (A.2b)
\[ \mathcal{R}(S)^{i j \gamma k}_{\alpha\beta} = \frac{3i}{4} \varepsilon^{ij}(\Gamma^a)_{\alpha\beta} X^{\gamma k} + i \varepsilon^{ij}(\Gamma^a)_{\alpha\beta} X^{\gamma k}, \] (A.2c)
\[ \mathcal{R}(K)^{i j c}_{\alpha\beta} = -\frac{1}{2} \varepsilon^{ij}(\Gamma^a)_{\alpha\beta} \nabla^b W^{c}_b + \frac{i}{2} \varepsilon^{ij}(\Gamma^a)_{\alpha\beta} \nabla^d \tilde{W}_{da} - \frac{i}{32} \varepsilon^{ij}(\Gamma^a)_{\alpha\beta} Y + \frac{i}{4} \varepsilon^{ij}(\Gamma^a)_{\alpha\beta} W^{c}_c \left( W_{ad} W^{cd} - \frac{3}{16} W^{bd} W_{bd} \delta^c_{\alpha} \right). \] (A.2d)

The non-vanishing torsion and curvatures in the spinor-vector commutator $[\nabla_b, \nabla^i_\alpha]$ are:
\[ \mathcal{T}^{i j \gamma}_{b\alpha k} = \frac{1}{4} \delta^i_k \left( 3(\Gamma_b)_{\alpha}^{\beta} W^\beta_{\gamma} - W^\alpha_{\beta} (\Gamma_b)_{\beta}^{\gamma} \right), \] (A.3a)
\[ \mathcal{R}(D)^i_{\alpha} = -\frac{1}{4} \delta^i_k \left( (\Gamma_b)_{\alpha}^{\gamma} X^i_{\gamma} \right), \] (A.3b)
\[ \mathcal{R}(J)^i_{\alpha} = -\frac{3}{4} \delta^i_k \left( (\Gamma_b)_{\alpha}^{\gamma} X^k_{\gamma} \right), \] (A.3c)
\[ \mathcal{R}(M)^{i c \gamma}_{b\alpha} = -\frac{1}{4} \varepsilon^{cde} W^{i}_{e f} \delta^i_{\alpha} + \frac{1}{2} \delta^i_{\alpha} (\Gamma^d)_{\alpha}^{\gamma} X^i_{\gamma}, \] (A.3d)
\[ \mathcal{R}(S)^i_{b\alpha} = \frac{1}{16} \left( X_{c d}^{i j} (\Sigma^{c d} \Gamma_b - 2\Gamma_b \Sigma^{c d})_{\alpha}^{\gamma} - \frac{3i}{8} \varepsilon^{ij} \nabla^b W^{cd} (\Sigma^{cd})_{\alpha}^{\gamma} - \frac{i}{8} \varepsilon^{ij} \nabla^d W^{cd} (\Sigma^{cd})_{\alpha}^{\gamma} \right) \]  
\[ + \frac{3i}{16} \varepsilon^{ij} \nabla^d W^{cd} \delta^\gamma_{\alpha} - \frac{i}{8} \varepsilon^{ij} \nabla^c \tilde{W}^{d}_{cb} (\Gamma_d)_{\alpha}^{\gamma} \]  
\[ + \frac{i}{16} \varepsilon^{ij} \tilde{W}^{c de} (\Sigma^{de} \Gamma_b)_{\alpha}^{\gamma} - \frac{3i}{32} \varepsilon^{ij} \tilde{W}^{bde} W^{cd} \delta^\gamma_{\alpha} \]  
\[ + \frac{i}{4} \varepsilon^{ij} W^{cd} (\Gamma_b)_{\alpha}^{\gamma} - \frac{3i}{64} \varepsilon^{ij} W^{cd} W^{cd} (\Gamma_b)_{\alpha}^{\gamma}, \] (A.3e)
\[\mathcal{R}(K)_{bc}^{i} = \frac{1}{6} (\Gamma^{c})_{\alpha}^{\beta} \nabla^{d} W_{db\beta}^{i} + \frac{1}{12} (\Gamma_{b})_{\alpha}^{\beta} \nabla^{d} W_{dci}^{i\beta} + \frac{1}{6} \nabla_{\alpha}^{\beta} W_{b\beta}^{ci} - \frac{1}{24} \varepsilon_{\beta e f}^{c} \nabla_{d} W_{e f\alpha}^{i} + \frac{1}{8} (\Gamma^{c})_{\alpha}^{\beta} \nabla_{b} X_{\beta}^{i} + \frac{1}{64} W^{de} (3 \Gamma_{b} \Sigma_{de} \Gamma^{c} - \Sigma_{de} \Gamma_{b} \Gamma^{c})_{\alpha}^{\beta} X_{\beta}^{i}\]

\[\mathcal{R}(J)_{ab}^{ij} = -\frac{3i}{4} X_{ab}^{ij}, \quad \mathcal{R}(M)_{ab}^{cd} = -\frac{1}{4} (\Sigma_{ab})^{\alpha\beta} (\Sigma_{cd})^{\gamma\delta} \left(i W_{\alpha\beta\gamma\delta} + 3 W_{(\alpha\beta} W_{\gamma\delta)}\right), \quad \mathcal{R}(S)_{ab}^{i} = -\frac{1}{2} \nabla_{\alpha}^{\beta} W_{ab\beta}^{i} - \frac{1}{2} (\Gamma_{a})_{\alpha}^{\beta} \nabla^{c} W_{b\alpha c}^{i} - \frac{1}{8} W_{\alpha}^{\beta} W_{ab\beta}^{i} + \frac{1}{16} (\Sigma_{ab})_{\alpha}^{\beta} W^{cd} W_{cd\beta}^{i} + \frac{3}{8} W_{\alpha c}^{i} W_{ab\alpha c}^{i} - \frac{1}{8} (\Gamma^{c})_{\alpha}^{\beta} X_{\beta}^{i} - \frac{1}{8} W_{d\alpha j}^{i} (\Gamma_{a})_{\alpha}^{\beta} W_{b\beta d}^{i} + \frac{1}{8} W_{ab}^{i} (\Gamma^{c})_{\alpha}^{\beta} W_{b\beta d}^{i} \]

The commutator of two vector derivatives \([\nabla_{a}, \nabla_{b}]\) has the following non-zero torsion and curvatures:

\[\mathcal{T}_{ab}^{i} = -\frac{i}{2} W_{ab}^{i}, \quad \mathcal{R}(J)_{ab}^{ij} = -\frac{3i}{4} X_{ab}^{ij}, \quad \mathcal{R}(M)_{ab}^{cd} = -\frac{1}{4} (\Sigma_{ab})^{\alpha\beta} (\Sigma^{cd})^{\gamma\delta} \left(i W_{\alpha\beta\gamma\delta} + 3 W_{(\alpha\beta} W_{\gamma\delta)}\right), \quad \mathcal{R}(S)_{ab}^{i} = -\frac{1}{2} \nabla_{\alpha}^{\beta} W_{ab\beta}^{i} - \frac{1}{2} (\Gamma_{a})_{\alpha}^{\beta} \nabla^{c} W_{b\alpha c}^{i} - \frac{1}{8} W_{\alpha}^{\beta} W_{ab\beta}^{i} + \frac{1}{16} (\Sigma_{ab})_{\alpha}^{\beta} W^{cd} W_{cd\beta}^{i} + \frac{3}{8} W_{\alpha c}^{i} W_{ab\alpha c}^{i} - \frac{1}{8} (\Gamma^{c})_{\alpha}^{\beta} X_{\beta}^{i} - \frac{1}{8} W_{d\alpha j}^{i} (\Gamma_{a})_{\alpha}^{\beta} W_{b\beta d}^{i} + \frac{1}{8} W_{ab}^{i} (\Gamma^{c})_{\alpha}^{\beta} W_{b\beta d}^{i} \]

Recall that the descendant superfields \(X_{a}^{i}, W_{\alpha\beta\gamma}^{i}, W_{\alpha\beta\gamma}, X_{\alpha\beta}^{i}, W_{\alpha\beta\gamma}^{i}, Y_{\alpha\beta\gamma}^{i}, \) and \(Y_{\alpha\beta\gamma}^{i}, \) were defined in \([2.5]\). They transform under S-supersymmetry as

\[S_{\alpha i} W_{\beta\gamma\delta}^{i} = 6 \delta_{i}^{j} \varepsilon_{\alpha}(\beta W_{\gamma\delta}), \quad S_{\alpha i} X_{\beta}^{i} = 4 \delta_{i}^{j} W_{\alpha\beta}, \quad S_{\alpha i} W_{\beta\gamma\delta}^{i} = 24 \varepsilon_{\alpha}(\beta W_{\gamma\delta}i), \quad S_{\alpha i} Y_{\alpha\beta\gamma}^{i} = 8i X_{\alpha i}, \quad S_{\alpha i} X_{\beta\gamma}^{k} = -4 \delta_{i}^{j}(\beta W_{\alpha\beta\gamma}^{k}) + 4 \delta_{i}^{j} \varepsilon_{\alpha}(\beta X_{\gamma}^{k}). \]

A complete list of the identities for spinor covariant derivative acting on these descendants can be found in subsection 2.3 of \([20]\). Here we only provide the relations which are useful for our analysis. The following relations are expressed in the traceless frame, which is our convention in this paper:

\[\nabla_{\gamma}^{k} W_{\alpha\beta} = W_{\alpha\beta\gamma}^{k} + \varepsilon_{\gamma(\alpha} X_{\beta)}^{k}, \quad (A.6a)\]

It should be noted that the identities given in \([20]\) were described in a non-traceless frame.
\[ \nabla^i \nabla_i X^j_{\alpha} = X_{\alpha \beta}^{ij} + \frac{1}{8} \varepsilon^{ij} \left( \varepsilon_{\alpha \beta} Y + 4 \varepsilon^{abcde} (\Sigma_{ab})_{\alpha \beta} \nabla_c W_{de} - 4 (\Gamma^b)_{\alpha \beta} \nabla^a W_{ab} + \varepsilon^{abcde} (\Gamma_a)_{\alpha \beta} W_{bc} W_{de} \right). \] (A.6b)

B 6D \( \mathcal{N} = (1, 0) \) conformal superspace identities

In this appendix we collect results about 6D \( \mathcal{N} = (1, 0) \) conformal superspace \[11\] in the traceless frame of \[12,13\] focusing on the ingredients relevant to our discussion in the paper.

The Lorentz generators act on the superspace covariant derivatives \( \nabla_A = (\nabla_a, \nabla^i_\alpha) \) as

\[
[M_{ab}, M_{cd}] = 2 \eta_{cla} M_{bd} - 2 \eta_{d[a} M_{b]c}, \tag{B.1}
\]
\[
[M_{ab}, \nabla_c] = 2 \eta_{cla} \nabla_b,
\]
\[
[M_\alpha^\beta, \nabla_\gamma^k] = -\delta_\beta^\gamma \nabla_\alpha^k + \frac{1}{4} \delta_\alpha^\beta \nabla_\gamma^k, \tag{B.3}
\]

where \( M_\alpha^\beta = -\frac{1}{4} (\gamma^{ab})_{\alpha \beta} M_{ab} \). The SU(2)\(_R\) and dilatation generators satisfy

\[
[J^{ij}, J^{kl}] = \varepsilon^{k(i} J^{j)l} + \varepsilon^{l(i} J^{j)k}, \quad [J^{ij}, \nabla^k_\alpha] = \varepsilon^{k(i} \nabla^j_\alpha), \tag{B.4}
\]
\[
[D, \nabla_a] = \nabla_a, \quad [D, \nabla^i_\alpha] = \frac{1}{2} \nabla^i_\alpha. \tag{B.5}
\]

The Lorentz and SU(2)\(_R\) generators act on the special conformal generators \( K^A = (K^a, S^i_\alpha) \) as

\[
[M_{ab}, K^c] = 2 \delta^c_{[a} K_{b]} , \quad [M_\alpha^\beta, S^i_k] = \delta^i_\alpha S^k_\beta - \frac{1}{4} \delta^i_\beta S^k_\alpha , \quad [J^{ij}, S^i_k] = \delta^i_k S^{ij}, \tag{B.6}
\]

while the dilatation generator acts on \( K^A \) as

\[
[D, K^a] = -K^a , \quad [D, S^i_\alpha] = -\frac{1}{2} S^i_\alpha. \tag{B.7}
\]

Among themselves, the generators \( K_A \) obey the only nontrivial anti-commutation relation

\[
\{S^\alpha_i, S^\beta_j\} = -2i \varepsilon_{ij} (\tilde{\gamma}_c)^{\alpha \beta} K^c. \tag{B.8}
\]

The algebra of \( K^A \) with \( \nabla_A \) is given by

\[
[K_a, \nabla_b] = 2 \eta_{ab} D + 2 M_{ab} , \tag{B.9}
\]
\[
[K^a, \nabla^i_\alpha] = -i (\gamma^a)_{\alpha \beta} S^\beta_i , \tag{B.10}
\]

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\{S_i^\alpha, \nabla^i_j\} = 2\delta^\alpha_\beta \delta^i_j \mathbb{D} - 4\delta^i_j M^{\alpha}_\beta + 8\delta^\alpha_\beta J^i_j , \quad (B.11)

\left[ S_i^\alpha, \nabla_b \right] = -i(\tilde{\gamma}_b)^{\alpha\beta} \nabla_{\beta i} + \frac{1}{10} W_{bcd}(\tilde{\gamma}^{cd})^\alpha_\gamma S_i^\gamma - \frac{1}{4} X_i^\alpha K_b \\
+ \left[ \frac{1}{4}(\tilde{\gamma}_{bc})^{\alpha_\beta} X_i^{\beta} + \frac{1}{2}(\gamma_{bc})_\beta^\gamma X_{\gamma i}^{\beta\alpha} \right] K^c . \quad (B.12)

The anticommutator of two spinor derivatives, \{\nabla^i_\alpha, \nabla^j_\beta\}, has the following non-zero torsion and curvatures

\mathcal{T}^{i,j}_{\alpha\beta} = 2i\varepsilon^{ij}(\gamma^c)_{\alpha\beta} , \quad (B.13a)
\mathcal{R}(M)^{i,j}_{\alpha\beta} = 4i\varepsilon^{ij}(\gamma^c)_{\alpha\beta} W^{acd} , \quad (B.13b)
\mathcal{R}(S)^{i,j}_{\alpha\beta\gamma} = -\frac{3}{2} \varepsilon^{ij} \varepsilon_{\alpha\beta\gamma} X^{\delta k} , \quad (B.13c)
\mathcal{R}(K)^{i,j}_{\alpha\beta\gamma} = i\varepsilon^{ij}(\gamma^c)_{\alpha\beta} \left( \frac{1}{4} \eta_{ac} Y - \nabla^b W_{abc} + W^a_{ef} W_{cef} \right) . \quad (B.13d)

The non-zero torsion and curvatures in the commutator \left[ \nabla_\alpha, \nabla_\beta \right] are:

\mathcal{T}^{j\gamma}_{\alpha\beta k} = -\frac{1}{2}(\gamma_\alpha)_{\beta\delta} W^{\delta\gamma} \delta^j_k , \quad (B.14a)
\mathcal{R}(\mathbb{D})^{j}_{\alpha\beta} = -\frac{i}{2}(\gamma_\alpha)_{\beta\gamma} X^{\gamma j} , \quad (B.14b)
\mathcal{R}(M)^{j}_{\alpha\beta} = i\delta^c_\alpha(\gamma^d)_{\beta\gamma} X^{\gamma j} - i(\gamma^a_{cd})_{\gamma\delta} X^{\gamma j} \gamma^\delta + 2i(\gamma_\alpha)_{\beta\gamma}(\gamma^a_{cd})_{\gamma} X^{\gamma j} \gamma^\delta , \quad (B.14c)
\mathcal{R}(J)^{j}_{\alpha\beta k l} = 2i(\gamma_\alpha)_{\beta\gamma} X^{(k\varepsilon\ell) j} , \quad (B.14d)
\mathcal{R}(S)^{j}_{\alpha\beta\gamma} = -\frac{i}{4}(\gamma_\alpha)_{\beta\delta} Y_{\gamma} \delta_{jk} + \frac{3i}{20}(\gamma_\alpha)_{\gamma\delta} Y_{\beta} \delta_{jk} - \frac{i}{8}(\gamma_\alpha)_{\beta\delta} \nabla_{\gamma\rho} W^{\delta\rho} \varepsilon_{jk} \\
+ \frac{1}{40}(\gamma_\alpha)_{\gamma\delta} \nabla_{\beta\rho} W^{\delta\rho} \varepsilon_{jk} - \frac{i}{8}(\gamma_\alpha)_{\beta\gamma} W^{\delta\rho} W^{\varepsilon\rho} \varepsilon_{jk} , \quad (B.14e)
\mathcal{R}(K)^{j}_{\alpha\beta c} = \frac{i}{4}(\gamma_\gamma)_{\beta\gamma} \nabla_a X^{\gamma j} - \frac{i}{4}(\gamma^a_{\gamma c})_{\delta\gamma} \nabla^d X^{\gamma j} \gamma^\delta + \frac{i}{3}(\gamma_\alpha)_{\beta\delta}(\gamma^a_{cd})_{\rho} \gamma^a \gamma^\delta X_{\gamma j}^{\rho} \\
- \frac{i}{8}(\gamma_\alpha)_{\beta\rho}(\gamma_\gamma)_{\delta\rho} W^{\gamma\delta} X^{\rho j} + \frac{5i}{12}(\gamma_\alpha)_{\beta\gamma}(\gamma_\gamma)_{\delta} W^{\gamma\delta} X^{\rho j}_{\delta} \\
+ \frac{i}{4}(\gamma_\alpha)_{\gamma\rho}(\gamma_\gamma)_{\beta\epsilon} W^{\gamma\delta} X^{\rho j}_{\delta} - \frac{i}{2}(\gamma_\alpha)_{\gamma\rho}(\gamma_\gamma)_{\delta\epsilon} W^{\gamma\delta} X^{\rho j}_{\beta} . \quad (B.14f)

The commutator of two vector derivatives, \left[ \nabla_\alpha, \nabla_\beta \right], has the following non-vanishing torsion and curvatures:

\mathcal{T}^{\alpha\beta}_{abk} = (\gamma_{ab})_{\beta}^\alpha X_{ak}^{\beta\gamma} , \quad (B.15a)
\mathcal{R}(M)^{cd}_{ab} = Y_{ab}^{cd} = \frac{1}{4}(\gamma_{ab})_{\gamma}^\alpha(\gamma^a_{cd})_{\delta} Y_{\alpha\beta}^{\gamma\delta} , \quad (B.15b)
\mathcal{R}(J)^{kl}_{ab} = \frac{1}{2}(\gamma_{ab})_{\delta}^{\gamma} Y_{\gamma}^{\delta kl} = Y_{ab}^{\gamma} kl , \quad (B.15c)
\[
\mathcal{R}(S)_{ab}^k = -\frac{1}{3}(\gamma_{ab})^\delta_{\alpha}^\beta \nabla_\gamma X_{\alpha}^{k\beta \delta} - \frac{1}{6}(\gamma_{abc})_{\alpha\beta}^\delta \nabla_\gamma X_{\alpha}^{k\alpha\beta} - \frac{i}{6} \varepsilon_{\gamma\beta\rho}(\gamma_{ab})_{\delta}^\rho W_{\alpha}^{\alpha\beta} X_{\alpha}^{k\delta \epsilon},
\]

(B.15d)

\[
\mathcal{R}(K)_{abc} = \frac{1}{4} \nabla^d Y_{abcd} + \frac{i}{3} X_{\alpha}^{k\gamma \beta} X_{\beta k}^{\epsilon \alpha \delta} (\gamma_{abc})_{\gamma}^{\beta \delta} + i(\gamma_{ab})_{\epsilon}^\alpha (\gamma_{c})_{\gamma}^\delta X_{\alpha}^{k\beta \gamma} X_{\beta k}^{\delta \epsilon} + \frac{i}{4} X_{\alpha}^{\epsilon \gamma \beta} X_{\beta k}^{\gamma \delta} (\gamma_{ab})_{\gamma}^{\beta} (\gamma_{c})_{\gamma}^\alpha \delta.
\]

(B.15e)

Recall the descendant superfields \(X^{\alpha i}\), \(X^{\alpha i\beta\gamma}\), \(Y\), \(Y^{\alpha\beta kl}\), \(Y^{\alpha\beta\gamma\delta}\) (and equivalently \(Y_{ab\cd}^{cd}\)), were defined in (6.6) and (6.7). They transform under \(S\)-supersymmetry as \([11]\)

\[
S_{\alpha}^{\epsilon} X_{\beta j} = \frac{8i}{5} \delta_{\beta}^{\epsilon} W_{\alpha}^{\alpha\beta}, \quad S_{\alpha}^{\epsilon} X_{\beta j}^{i\delta} = -i\delta_{\beta}^{j} \delta_{\alpha}^{\epsilon} W_{\gamma}^{\gamma\delta} + \frac{2i}{5} \delta_{\gamma}^{j} \delta_{\beta}^{\epsilon} W_{\rho}^{\rho\delta} \alpha,
\]

(B.16a)

\[
S_{\gamma}^{\alpha} Y_{\alpha}^{\beta ij} = -\delta_{k}^{i} (16 X_{\alpha}^{\gamma \beta} - 2\delta_{\beta}^{\gamma} X_{\alpha}^{\gamma j} + 8\delta_{\alpha}^{\gamma} X_{\beta}^{\gamma j}),
\]

(B.16b)

\[
S_{\rho}^{\alpha} Y_{\alpha}^{\beta \gamma\delta} = 24 \left( \delta_{\gamma}^{\beta} \delta_{\alpha}^{\epsilon} W_{\delta}^{\gamma\delta} - \frac{1}{3} \delta_{\gamma}^{\beta} X_{\delta}^{\gamma j} \delta_{\rho}^{\epsilon} \right), \quad S_{\epsilon}^{\alpha} Y = -4X_{\epsilon}^{\alpha}.
\]

(B.16c)

By using (6.5) and the previous definitions, one can derive spinor covariant derivative acting on these descendants of the super-Weyl tensor and can be found in [12, 13] in the traceless frame. Here we only provide the relations which are useful for our analysis, which are

\[
\nabla_{\alpha}^{\gamma \beta} X_{\beta}^{\gamma j} = \frac{2}{5} Y_{\alpha}^{\gamma \beta ij} - \frac{2}{5} \varepsilon_{\gamma \beta}^{ij} \nabla_{\alpha \gamma} W_{\gamma}^{\beta} - \frac{1}{2} \varepsilon_{\gamma \beta}^{ij} \delta_{\alpha}^{\epsilon} Y_{\epsilon},
\]

(B.17a)

\[
\nabla_{\alpha}^{\gamma \beta} X_{\beta}^{\gamma j} = \frac{1}{2} \delta_{\alpha}^{\gamma} (\gamma_{\beta}^{\gamma \beta})^{ij} - \frac{1}{10} \delta_{\beta}^{j} (\gamma_{\alpha}^{\gamma \beta})^{ij} - \frac{1}{2} \varepsilon_{\gamma \beta}^{ij} Y_{\alpha \beta}^{\gamma \delta} - \frac{1}{4} \varepsilon_{\gamma \beta}^{ij} \nabla_{\alpha \beta} W_{\gamma}^{\delta} + \frac{3}{20} \varepsilon_{\gamma \beta}^{ij} \delta_{\alpha}^{\epsilon} \nabla_{\alpha \delta} W_{\epsilon}^{\delta \rho} + \frac{1}{4} \varepsilon_{\gamma \beta}^{ij} \delta_{\epsilon}^{\delta} \nabla_{\epsilon \beta} W_{\delta}^{\delta \rho}.
\]

(B.17b)

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