Dark-like states for the multi-qubit and multi-photon Rabi models

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Abstract
There are special solutions, the well-known dark states, to the even-qubit Dicke models, which are the products of two-qubit singlets and a Fock state, where the qubits are decoupled from the photon field. These spin singlets can be used to store quantum correlations since they preserve entanglement even under dissipation, driving and dipole-dipole interactions. One of the features for these ‘dark states’ is that their eigenenergies are independent of the qubit-photon coupling strength. Here we have obtained a novel kind of exceptional solution—‘dark-like states’ for the multi-qubit and multi-photon Rabi models, whose eigenenergies are also constant in the whole coupling regime. But unlike the ‘dark states’, the qubits and photon field are coupled. Furthermore, the photon numbers are bounded from above commonly at 1,
which is different from the one-qubit case. The existence conditions of these exceptional solutions are simpler than the exact isolated solutions, and may be fine tuned in experiments. While solutions to the single-qubit and multi-photon Rabi model exist only if the photon number $M \leq 2$ and the coupling strength is below a certain critical value, the dark-like eigenstates for the multi-qubit and multi-photon Rabi model exist regardless of these constraints. In view of these properties of the dark-like states, they may find similar applications as ‘dark states’ in quantum information.

Keywords: dark-like states, multi-qubit and multi-photon quantum Rabi models, exceptional solution

(Some figures may appear in colour only in the online journal)

1. Introduction

The Rabi model [1] has been born for 80 years [2] with semi-classical [1] and fully quantized versions [3]. The quantum Rabi model describes a two-level system (qubit) interacting with a single mode photon field, which has found wide applications in solid state [4], quantum optics [5], cavity QED [6], circuit QED [7] and quantum information [8]. Although this model takes a very simple form, its analytical solution had not been found until 2011 by Braak [9]. This is partly due to the fact that there is no closed subspace in its Fock space, which is different from that in the Jaynes–Cummings model [3] with the rotating wave approximation [10]. The reason for considering the full Rabi model lies in that the ultrastrong coupling has been reached in recent circuit QED experiment [11], where the rotating wave approximation fails. This invokes many researches on the Rabi model, which include the analytical solution of the Rabi model retrieved by Chen et al using Bogoliubov operators [12], N-state quantum Rabi model [13, 14], two-photon [15–17], two-qubit [18–22], two-mode[23–26] Rabi model, exact real time dynamics [27, 28], deep strong coupling [29], and anisotropic Rabi model [30], among others [31–35].

One direct and important generalization to the Rabi model is to consider the multi-qubit case[36–38] and multi-photon processes [39, 40]. The reason is that quantum information resources are stored in many qubits and multi-photon processes introduce interesting physical effects [41, 42]. In this case, the full spectrum is very hard to acquire [43], but obtaining some special exceptional solution is possible and interesting, which will be discussed in this paper.

For the single-qubit quantum Rabi model, the eigenstates consist of infinite photon number states, because there are no closed subspaces in the Fock space [9]. But this is not the case for the multi-qubit case, because more qubits will produce closed subspaces. For example, Rodriguez-Lara et al have found ‘trapping states’ (‘dark states’) [44] in the even qubit Dicke model, where two identical qubits form a spin singlet and the eigenstates are just products of these singlets and a Fock state. These singlets are decoupled from the photon field, and will survive even under dissipation, driving, as well as dipole-dipole interactions. Therefore, they can be used to store quantum correlations. Since the qubits and photon are decoupled, the eigenenergies of the ‘dark states’ are constants in the whole qubit-photon coupling regime, which correspond to horizontal lines in the spectrum.

In this paper, we study the multi-qubit and multi-photon quantum Rabi model, and show that there commonly exist some special exceptional solutions, partly like the ‘dark state’, which we call the ‘dark-like state’. These dark-like states possess several features. Firstly, they exist in the whole coupling regime with constant eigenenergies, just like the ‘dark states’.
But surprisingly, the qubit and photon are not decoupled and the wavefunctions are coupling dependent. Secondly, the photon numbers in the eigenstates are bounded from above at \( K \). In particular, \( K = 1 \) for the single-photon case. Thirdly, their existence conditions are simpler than exact isolated solutions, because they can be realized in arbitrary coupling regimes with the same qubit energy, which may be fine tuned in experiment. There has been some researches on the ‘dark-like state’ of the two-qubit and single-photon Rabi model \([20, 45]\), while now we find that they commonly exist, surprisingly not just for even-qubit, but also odd-qubit, and multi-photon cases. On the other hand, normalizable eigenstates of the single-qubit and multi-photon Rabi model exist only if the photon number \( M \leq 2 \) and the coupling strength is below a certain critical value \([39]\), but the multi-qubit case will bring about closed subspaces and the dark-like eigenstates exist in the whole coupling regime and even for \( M > 2 \). We have to point out that we have not found all the exceptional eigenstates in the photon number space, but an interesting one. Although all the exceptional eigenstates in the photon number space consist of finite photon number states, only the ‘dark-like state’ exists in the whole coupling regime with constant eigenenergy and qubit energy. Therefore, just like the ‘dark states’, these ‘dark-like states’ may find possible applications in quantum information technologies.

The paper is organized as follows. In section 2, we search for the ‘dark-like’ eigenstates for the multi-qubit Rabi model. In section 3, we generalize our study to the multi-qubit and multi-photon Rabi models. In section 4, we give some experimental considerations for the implementation in quantum controllable platforms. Finally, we give our conclusions in section 5.

2. Dark-like states for the multi-qubit Rabi model

The Hamiltonian of the \( N \)-qubit quantum Rabi model reads \((\hbar = 1)\) \([19, 20]\)

\[
H_{NQ} = \omega a^\dagger a + \sum_{i=1}^{N} g_i \sigma_{iz} (a + a^\dagger) + \sum_{i=1}^{N} \Delta_i \sigma_{iz},
\]

where \( q^\dagger \) and \( a \) are the single mode photon creation and annihilation operators with frequency \( \omega \), respectively. \( \sigma_{iz} \) are the Pauli matrices corresponding to the \( i \)th qubit. \( 2\Delta_i \) is the energy level splitting of the \( i \)th qubit, and \( g_i \) is the qubit-photon coupling constant. \( \omega \) is set to 1 in the following discussion.

The Hamiltonian (1) is usually infinite dimensional in the Fock space, which is exactly the case for just one qubit, but with more qubits it will bring about possible closed subspaces. Working on this finite dimensional subspace, we can obtain the solution of the Hamiltonian (1) with finite photon numbers and the dark-like eigenstates. For this purpose, we must first search for the existence condition of this closed subspace. \( H_{NQ} \) possesses a \( \mathbb{Z}_2 \) symmetry with the transformation \( R = \exp(i\pi a^\dagger a) \prod_{i=1}^{N} \sigma_{iz} \), so we have

\[
R|p\rangle = p|p\rangle
\]

with \( p = \pm 1 \). At the same time, we can categorize the \( N \)-qubit states \( \{|\psi\rangle_{Nq}\} \) into two sets with the eigenvaules of \( \prod_{i=1}^{N} \sigma_{iz} \) being 1 and \(-1\) respectively, and they are denoted by \( 2^{N-1} \) dimensional row vectors \( \{|\psi\rangle_{Nq^+}\} \) and \( \{|\psi\rangle_{Nq^-}\} \). It is easy to find the following relations

\[
|\psi\rangle_{Nq^+} = (|\psi\rangle_{N-1} q^- \otimes |\downarrow\rangle_N, |\psi\rangle_{N-1} q^+ \otimes |\uparrow\rangle_N),
\]

\[
|\psi\rangle_{Nq^-} = (|\psi\rangle_{N-1} q^+ \otimes |\downarrow\rangle_N, |\psi\rangle_{N-1} q^- \otimes |\uparrow\rangle_N),
\]

with the initial states \( |\psi\rangle_{1q^+} = |\uparrow\rangle_1 \) and \( |\psi\rangle_{1q^-} = |\downarrow\rangle_1 \). Then we have two unconnected subspaces.
\[ |0, \psi_{Nq^+} \rangle \leftrightarrow |1, \psi_{Nq^-} \rangle \leftrightarrow |2, \psi_{Nq^+} \rangle \leftrightarrow \ldots \ (p = 1) \]  
(5)

\[ |0, \psi_{Nq^-} \rangle \leftrightarrow |1, \psi_{Nq^+} \rangle \leftrightarrow |2, \psi_{Nq^-} \rangle \leftrightarrow \ldots \ (p = -1) \]  
(6)

Only neighboring states within each parity chain are connected, so \( H^\pm_{NQ} \) will take the following form in even \((p = +1)\) or odd \((p = -1)\) subspace

\[
H^\pm_{NQ} = \begin{pmatrix}
D^\pm_{N0} & O^\pm_{N0} & 0 & 0 & 0 & \ldots \\
O^\pm_{N0} & D^\pm_{N1} & O^\pm_{N1} & 0 & 0 & \ldots \\
0 & O^\pm_{N1} & D^\pm_{N2} & O^\pm_{N2} & 0 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots \\
\end{pmatrix},
\]

(7)

where \( D^\pm_{Nj} \) and \( O^\pm_{Nj} \) \((j = 0, 1, 2, 3, \ldots)\) can be written as

\[
D^\pm_{Nj} = (|j, \psi_{Nq, \pm(-1)}\rangle | \otimes N\downarrow, |j, \psi_{Nq, \pm(-1)}\rangle | \otimes N\uparrow)^T H_{NQ} (|j, \psi_{Nq, \pm(-1)}\rangle),
\]

(8)

\[
O^\pm_{Nj} = ((j + 1, \psi_{Nq, \mp(-1)}\rangle | \otimes N\downarrow, |j, \psi_{Nq, \pm(-1)}\rangle | \otimes N\uparrow)^T H_{NQ} (|j, \psi_{Nq, \pm(-1)}\rangle),
\]

(9)

where \(|j, \psi_{Nq, \pm(-1)}\rangle\) is a \(2^{N-1}\) dimensional row vector, and \( D^\pm_{Nj}, O^\pm_{Nj} \) are \(2^{N-1} \times 2^{N-1}\) matrices. Substituting equations (3) and (4) into equations (8) and (9), we get the following expressions for \( D^\pm_{Nj} \) and \( O^\pm_{Nj} \),

\[
D^\pm_{Nj} = (|j, \psi_{N-1, \mp(-1)}\rangle | \otimes N\downarrow, |j, \psi_{N-1, \pm(-1)}\rangle | \otimes N\uparrow)^T H_{NQ} (|j, \psi_{N-1, \mp(-1)}\rangle | \otimes N\downarrow, |j, \psi_{N-1, \pm(-1)}\rangle | \otimes N\uparrow)
\]

(10)

\[
O^\pm_{Nj} = ((j + 1, \psi_{N-1, \mp(-1)}\rangle | \otimes N\downarrow, |j, \psi_{N-1, \pm(-1)}\rangle | \otimes N\uparrow)^T H_{NQ} (|j, \psi_{N-1, \mp(-1)}\rangle | \otimes N\downarrow, |j, \psi_{N-1, \pm(-1)}\rangle | \otimes N\uparrow)
\]

(11)

where

\[ H_{NQ} = H_{N-1, 0} + \Delta_N \sigma_N + g_N \sigma_N (a + a^\dagger). \]

Then we have

\[
D^\pm_{Nj} = \begin{pmatrix}
D^\pm_{N-1, j} - \Delta_N & 0 \\
0 & D^\pm_{N-1, j} + \Delta_N \\
\end{pmatrix},
\]

(13)

\[
O^\pm_{Nj} = O^{-\pm}_{Nj} = O_{Nj} = \begin{pmatrix}
O_{N-1, j} & \sqrt{1 + g_N} I \\
\sqrt{1 + g_N} I & O_{N-1, j} \\
\end{pmatrix},
\]

(14)

with the initial condition

\[ D^\pm_{1j} = j \pm (-1) / \Delta_1, \]

(15)

\[ O^\pm_{1j} = \sqrt{1 + g_1}. \]

(16)

As seen from equation (14), generally there is no closed subspace if \( O_{Nj} \) is nontrivial, which is exactly the case for single qubit with \( g \neq 0 \). But for the multi-qubit case,
$O_{NJ}$ can be equivalently trivial even for non-zero coupling constant $g_i$, if its eigenvalues are 0, which leads to the closed subspaces. Suppose that a subspace is spanned by $\{|J, \psi_{NJ, \pm(-1)^\gamma}\}, |J+1, \psi_{NJ, \pm(-1)^{\gamma+1}}\}, \ldots, |K, \psi_{NJ, \pm(-1)^K}\}$. If
\begin{equation}
O_{NJ-1} c_{NJ}^\pm |J, \psi_{NJ, \pm(-1)^\gamma}\rangle = 0,
\end{equation}
\begin{equation}
O_{NK} c_{NK}^\pm |K, \psi_{NJ, \pm(-1)^\gamma}\rangle = 0,
\end{equation}
where $c_{NJ}^\pm$ and $c_{NK}^\pm$ are coefficients of $|J, \psi_{NJ, \pm(-1)^\gamma}\rangle$ and $|K, \psi_{NJ, \pm(-1)^\gamma}\rangle$ respectively, then this subspace is closed. Each of $c_{NJ}^\pm$ and $c_{NK}^\pm$ contains $2^{N-1}$ components since $|J, \psi_{NJ, \pm(-1)^\gamma}\rangle$ and $|K, \psi_{NJ, \pm(-1)^\gamma}\rangle$ are $2^{N-1}$ dimensional vectors. So combined with the eigenvalue equation of $H_{NQ}$ in this closed subspace, we obtain
\begin{equation}
\begin{pmatrix}
O_{NJ-1} & 0 & 0 & 0 & \cdots \\
D_{NJ}^+ - E^\pm & O_{NJ} & 0 & 0 & \cdots \\
O_{NJ} & D_{NJ}^+ - E^\pm & O_{NJ} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & D_{NK}^- - E^\pm & O_{NK} \\
0 & \cdots & 0 & O_{NK} & 0 & 0
\end{pmatrix}
\begin{pmatrix}
c_{NJ}^+ \\
c_{NK}^-
\end{pmatrix} = 0.
\end{equation}
Clearly, there are more equations (rows) than variables (columns) in this system of linear homogeneous equations, such that only for some special conditions, we may obtain a solution with finite photon numbers. We can use elementary row transformations to reduce the matrix into row echelon form. Therefore, if the number of the non-zero rows is less than that of the columns, there will be non-trivial solutions. At the same time, equations (17) and (18) are decoupled from other equations in equation (19), which is just the existence condition of the closed subspace, and they differ only by a constant. We can eliminate all the constants and define $O_N = O_{NJ}/\sqrt{J+1} = O_{NK}/\sqrt{K+1}$, such that equations (17) and (18) can be equivalent to the statement that the eigenvalues of $O_N$ are zero, and both $c_{NJ}^\pm |J, \psi_{NJ, \pm(-1)^\gamma}\rangle$ and $c_{NK}^\pm |K, \psi_{NJ, \pm(-1)^\gamma}\rangle$ are its null vectors.

$O_N$ takes different forms for qubit number $N$, but we can find a unified form for its eigenvalues by analyzing its determinant
\begin{equation}
|O_N| = \begin{vmatrix}
O_{N-1} & g_N \\
g_N & O_{N-1}
\end{vmatrix} = \begin{vmatrix}
O_{N-1} & g_N \\
g_N & O_{N-1}
\end{vmatrix} = |O_{N-1} + g_N| |O_{N-1} - g_N|.
\end{equation}
Therefore, if the eigenvalues of $O_{N-1}$ are $e_{N-1,i}$ ($i = 1, 2, \ldots, 2^{N-2}$), then the eigenvalues of $O_N$ would be $e_{N-1,i} + g_N$ and $e_{N-1,i} - g_N$ with the initial condition $e_{1,1} = g_1$. Some eigenvalues and eigenvectors of $O_N$ are shown in table 1.

Setting the eigenvalues of $O_N$ to be 0, which just depends on the coupling strength, and $c_{NJ}^\pm |J, \psi_{NJ, \pm(-1)^\gamma}\rangle$, $c_{NK}^\pm |K, \psi_{NJ, \pm(-1)^\gamma}\rangle$ to be its null vectors, we can simplify (19). Now the relations between all the components of each of $c_{NJ}^\pm$ and $c_{NK}^\pm$ are fixed, such that there is only 1 variable. Meanwhile, using $O_{NJ-1} c_{NJ}^\pm = 0$ and $O_{NK} c_{NK}^\pm = 0$ to simplify equation (19) by elementary row transformation and then put them aside, we will find, if $J \neq 0$, then the
corresponding wavefunction does not depend on the coupling strength and it turns into the ‘dark state’. Thus, in order to obtain other exceptional solutions, we should set \( J = 0 \). Then by solving equation (19), we can obtain all the exceptional solutions in the photon number space.

Clearly \( ON_{c} = 0 \) is decoupled from the other part in equation (19), hence we obtain a necessary but not sufficient condition for a solution

\[
\begin{align*}
D_{N,0}^\pm E_{\pm} = & \quad O_{N,0} \quad 0 \quad 0 \quad 0 \quad \ldots \\
0 \quad & \quad D_{N,1}^\pm E_{\pm} \quad O_{N,1} \quad 0 \quad \ldots \\
\ldots \quad & \quad \ldots \quad \ldots \quad \ldots \quad \ldots \\
0 \quad & \quad \ldots \quad O_{N,K-2} \quad D_{N,K-1}^\pm E_{\pm} \quad 0 \\
0 \quad & \quad \ldots \quad 0 \quad O_{N,K-1} \quad D_{N,K}^\pm E_{\pm} \\
\end{align*}
\]

which is generally dependent both on the qubit energy and coupling strength, except for \( K = 1 \). Therefore, in order to search for the dark-like states, which exist in arbitrary coupling regime with the same qubit energy, we just need to consider the case of \( J = 0 \) and \( K = 1 \). The equation which determines the dark-like solution to \( H_{NQ} \) reads

\[
\begin{pmatrix}
D_{N,0}^\pm E_{\pm} & 0 \\
O_{N,0} & D_{N,1}^\pm E_{\pm} \\
0 & O_{N,1}
\end{pmatrix}
\begin{pmatrix}
\frac{\pm}{\sqrt{c_{N,0}}} \\
c_{N,1}
\end{pmatrix} = 0.
\]

Solving this equation is the key point to obtaining the dark-like eigenstates for the multi-qubit and multi-photon Rabi models.

### 2.1. Two-qubit case

Let us start with the simplest case of \( N = 2 \). For this case, we have \( |\psi\rangle_{2q^+} = (|\downarrow, \downarrow\rangle, |\uparrow, \uparrow\rangle) \), \( |\psi\rangle_{2q^-} = (|\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle) \), and

| \( N \) | Eigenvalues | Transpose of the eigenvectors |
|-------|-------------|-----------------------------|
| 2     | \( g_1 + g_2 \) | (1, 1) |
| 2     | \( g_1 - g_2 \) | (1, -1) |
| 3     | \( g_1 - g_2 - g_3 \) | (1, -1, -1, 1) |
| 3     | \( g_1 + g_2 - g_3 \) | (-1, -1, 1, 1) |
| 3     | \( g_1 - g_2 + g_3 \) | (-1, 1, -1, 1) |
| 3     | \( g_1 + g_2 + g_3 \) | (1, 1, 1, 1) |
| 4     | \( g_1 - g_2 - g_3 - g_4 \) | (-1, 1, -1, -1, -1, 1, 1) |
| 4     | \( g_1 + g_2 - g_3 - g_4 \) | (1, -1, -1, -1, -1, 1, 1) |
| 4     | \( g_1 - g_2 + g_3 - g_4 \) | (1, -1, 1, -1, -1, 1, 1) |
| 4     | \( g_1 + g_2 + g_3 - g_4 \) | (-1, -1, 1, -1, -1, 1, 1) |
| 4     | \( g_1 - g_2 - g_3 + g_4 \) | (1, -1, -1, 1, 1, -1, 1) |
| 4     | \( g_1 + g_2 - g_3 + g_4 \) | (-1, -1, 1, 1, 1, -1, 1) |
| 4     | \( g_1 - g_2 + g_3 + g_4 \) | (-1, 1, -1, -1, -1, 1, 1) |
| 4     | \( g_1 + g_2 + g_3 + g_4 \) | (1, 1, 1, 1, 1, 1, 1, 1) |
whose eigensystem is shown in Table 1. Choosing \( g_1 = g_2 = g/2 \) and \( c_{2,1,1} = -c_{2,1,2} \) to simplify (22), we arrive at

\[
\begin{pmatrix}
\pm \Delta_1 - \Delta_2 - E^\pm \\
0 \\
g/2 \\
g/2 \\
g/2 \\
\end{pmatrix}
\begin{pmatrix}
0 \\
\pm \Delta_1 + \Delta_2 - E^\pm \\
1 + \Delta_1 - \Delta_2 - E^\pm \\
0 \\
\pm \Delta_1 + \Delta_2 - E^\pm \\
-1 + \Delta_1 - \Delta_2 + E^\pm \\
\end{pmatrix}
\begin{pmatrix}
c_{2,0,1}^\pm \\
c_{2,0,2}^\pm \\
c_{2,1,1}^\pm \\
\end{pmatrix}
= 0. 
\tag{24}
\]

After elementary row transformation, the coefficient matrix in equation (24) is simplified to

\[
\begin{pmatrix}
g/2 \\
g/2 \\
1 + \Delta_1 - \Delta_2 - E^\pm \\
0 \\
\pm \Delta_1 + \Delta_2 - E^\pm \\
E^\pm - 1 \\
\end{pmatrix}
\begin{pmatrix}
\pm \Delta_1 - \Delta_2 - E^\pm \\
0 \\
1 + \Delta_1 - \Delta_2 - E^\pm \\
0 \\
\pm \Delta_1 + \Delta_2 - E^\pm \\
E^\pm - 1 \\
\end{pmatrix}
\begin{pmatrix}
c_{2,0,1}^\pm \\
c_{2,0,2}^\pm \\
c_{2,1,1}^\pm \\
\end{pmatrix}
= 0. 
\tag{25}
\]

There are three columns, such that only two non-zero rows can exist in its row echelon form, from which we obtain

\[
\Delta_1 + \Delta_2 = E^+ = 1, 
\tag{26}
\]

with eigenstate

\[
|\psi\rangle = \frac{1}{N} \left( \begin{array}{c}
2(\Delta_1 - \Delta_2) \\
0, \uparrow, \uparrow \\
|0, \uparrow, \downarrow\rangle - |1, \downarrow, \downarrow\rangle + |1, \uparrow, \uparrow\rangle
\end{array} \right), 
\tag{27}
\]

for even parity and

\[
\Delta_1 - \Delta_2 = E^- = 1, \text{ or } \Delta_2 - \Delta_1 = E^- = 1 
\tag{28}
\]

with eigenstates

\[
|\psi\rangle_{g_1} = \frac{1}{N'} \left( \begin{array}{c}
2(\Delta_1 + \Delta_2) \\
0, \uparrow, \downarrow \\
|0, \downarrow, \downarrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle
\end{array} \right), 
\tag{29}
\]

\[
|\psi\rangle_{g_2} = \frac{1}{N'} \left( \begin{array}{c}
2(\Delta_1 + \Delta_2) \\
0, \downarrow, \uparrow \\
|0, \downarrow, \downarrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle
\end{array} \right), 
\tag{30}
\]

respectively, for odd parity. Eigenstates (27), (29) and (30) exist for any coupling strength \( g_1 = g_2 = g/2 \) with constant eigenenergy \( E^\pm = 1 \), corresponding to a horizontal line in the spectra, which has been shown in [20]. These properties are just like those for the ‘dark state’ formed by the qubit singlet. However, for these eigenstates, the qubit and photon are not decoupled, and the photon number is bounded from above at 1.

### 2.2. Three-qubit case

Then we consider the case of 3 qubit, where \((|\psi\rangle_{3g^-} = (| \downarrow, \downarrow, \downarrow\rangle, | \uparrow, \uparrow, \downarrow\rangle, | \uparrow, \downarrow, \downarrow\rangle, | \uparrow, \downarrow, \uparrow\rangle, | \downarrow, \downarrow, \uparrow\rangle))\), \((|\psi\rangle_{3g^+} = (| \uparrow, \downarrow, \downarrow\rangle, | \downarrow, \uparrow, \downarrow\rangle, | \downarrow, \downarrow, \downarrow\rangle, | \uparrow, \downarrow, \uparrow\rangle, | \downarrow, \uparrow, \uparrow\rangle))\), and
\[
O_3 = \begin{pmatrix}
g_1 & g_2 & g_3 & 0 \\
g_2 & g_1 & 0 & g_3 \\
g_3 & 0 & g_1 & g_2 \\
0 & g_3 & g_2 & g_1
\end{pmatrix},
\]

whose eigensystem is shown in Table 1. For \( g_1 = g_2 + g_3, g_2 = g_1 + g_3, \) or \( g_3 = g_1 + g_2, \) the eigenvalues are zero, and corresponding eigenvectors are nullvectors. \((D_3^±)^0 E^±) c_{3,0}^± = 0\) in equation (22) is decoupled from the other parts. The coefficient matrix \(D_3^± - E^±\) takes the diagonal form

\[
\begin{pmatrix}
±\Delta_1 - \Delta_2 - \Delta_3 - E^± & 0 & 0 & 0 \\
0 & ±\Delta_1 + \Delta_2 - \Delta_3 - E^± & 0 & 0 \\
0 & 0 & ±\Delta_1 - \Delta_2 + \Delta_3 - E^± & 0 \\
0 & 0 & 0 & ±\Delta_1 + \Delta_2 + \Delta_3 - E^±
\end{pmatrix}.
\]

Choosing \( g_1 = g_2 + g_3\) and \( c_{3,1,1} = -c_{3,1,2} = -c_{3,1,3} = c_{3,1,4}\) to simplify the other part

\[
\begin{pmatrix}
O_3 & D_3^± - E^± \\
0 & O_3
\end{pmatrix}
\begin{pmatrix}
c_{3,0}^± \\
c_{3,1}^±
\end{pmatrix} = 0,
\]

we arrive at

\[
\begin{pmatrix}
g_2 + g_3 & g_2 & g_3 & 0 & 1 ±\Delta_1 - \Delta_2 - \Delta_3 - E^± \\
g_2 & g_2 + g_3 & 0 & g_3 & -(1 ±\Delta_1 + \Delta_2 - \Delta_3 - E^±) \\
g_3 & 0 & g_2 + g_3 & 0 & -(1 ±\Delta_1 - \Delta_2 + \Delta_3 - E^±) \\
0 & g_3 & 0 & g_2 + g_3 & 1 ±\Delta_1 + \Delta_1 + \Delta_3 - E^±
\end{pmatrix}
\begin{pmatrix}
c_{3,0,1}^± \\
c_{3,0,2}^± \\
c_{3,0,3}^± \\
c_{3,0,4}^± \\
c_{3,1,1}^±
\end{pmatrix} = 0.
\]

After elementary row transformations, the coefficient matrix in equation (34) becomes

\[
\begin{pmatrix}
1 & 0 & 0 & -1 & \frac{1 ±\Delta_1 - E^±}{g_3^2} + \frac{1 ±\Delta_2 - E^±}{g_2^2} \\
0 & 1 & 0 & 1 & -\frac{1 ±\Delta_1 + E^±}{g_3^2} + \frac{1 ±\Delta_2 + E^±}{g_2^2 + g_3^2} \\
0 & 0 & 1 & 1 & -\frac{1 ±\Delta_2 + E^±}{g_3^2} + \frac{1 ±\Delta_3 + E^±}{g_2^2 + g_3^2} \\
0 & 0 & 0 & 0 & 4 - 4E^±
\end{pmatrix}
\]

There are totally 5 variables, such that the total nonzero rows in equations (32) and (35) should be less than 5. By choosing \( E^± = 1\), the nonzero rows in equation (35) reduce to 3. Thus, if only one nonzero row exists in equation (32), we obtain a nontrivial solution. Luckily, there is one such case for odd parity with the following parameters

\[
\Delta_1 = \Delta_2 = \Delta_3 = \Delta = E^− = 1.
\]

Substituting equation (36) into equations (32) and (35), we obtain a dark-like state

\[
|\psi\rangle = \frac{g_3}{g_2 g_3 + g_3} |0, ↑, ↑, ↓\rangle + \frac{g_2}{g_3 (g_2 + g_3)} |0, ↑, ↓, ↓\rangle - \frac{g_2 + g_3}{g_2 g_3} |0, ↓, ↑, ↑\rangle + |1, ↑, ↓, ↓\rangle - |1, ↓, ↑, ↓\rangle - |1, ↓, ↓, ↑\rangle + |1, ↑, ↑, ↑\rangle
\]

(37)
If we choose $g_2 = g_1 + g_3$ and $-c_{3,1,1} = c_{3,1,2} = -c_{3,1,3} = c_{3,1,4}$ to satisfy the condition $O_{3} \neq 0$, the corresponding solution can be obtained just by interchanging the states of the first and second qubits, including the coupling strength, in equation (37). For $g_3 = g_1 + g_2$, we can obtain a solution by interchanging the states of the first and third qubit in equation (37).

Choosing $g_1 = g_2 + g_3$ and $\Delta_1 = \Delta_2 = \Delta_3 = \Delta = 1$, the dark-like state (37) corresponds to the horizontal line $E^+ = 1$ in figure 1.

2.3. Four-qubit case

Now we turn to the case of 4 qubits, with $\langle \psi \rangle_{4q^+} = (|\psi\rangle_{3q^+} \otimes |\downarrow\rangle_4, |\psi\rangle_{3q^+} \otimes |\uparrow\rangle_4)$, $\langle \psi \rangle_{4q^-} = (|\psi\rangle_{3q^-} \otimes |\downarrow\rangle_4, |\psi\rangle_{3q^-} \otimes |\uparrow\rangle_4)$, and

$$O_4 = \begin{pmatrix}
g_1 & g_2 & g_3 & 0 & g_4 & 0 & 0 & 0 
g_2 & g_1 & 0 & g_3 & 0 & g_4 & 0 & 0 
g_3 & 0 & g_1 & g_2 & 0 & 0 & g_4 & 0 
g_4 & 0 & 0 & g_1 & g_2 & g_3 & 0 & 0 
g_4 & 0 & 0 & g_2 & g_1 & 0 & g_3 & 0 
g_4 & 0 & 0 & g_3 & 0 & g_1 & g_2 & 0 
g_4 & 0 & 0 & g_3 & 0 & g_1 & g_2 & 0 
g_4 & 0 & 0 & g_4 & 0 & g_3 & 0 & g_1 
g_4 & 0 & 0 & g_4 & 0 & g_3 & 0 & g_1 
\end{pmatrix}.$$ (38)

As seen from the system of linear homogeneous equations (22), there are 24 rows and just 16 columns in its coefficient matrix. A solution exists if the nonzero rows are less than the columns in its row echelon form. First we consider the $8 \times 8$ diagonal matrix in (22)

$$D_4^+ - E^+ = \begin{pmatrix} D_{3,0}^+ - \Delta_4 - E^+ & 0 
0 & D_3^+ + \Delta_4 - E^+ \end{pmatrix},$$ (39)

Figure 1. The numerical spectrum of three-qubit quantum Rabi model with $\Delta_1 = \Delta_2 = \Delta_3 = 1$, $\omega = 1$, $g_3 = 0.5g_2$, $0 \leq g_1 = g_2 + g_3 \leq 1.5$. $E^+$ and $E^-$ are solutions with even and odd parity respectively.
and then take into account the other part

$$
\begin{pmatrix}
O_4 & D_{41}^\pm - E^\pm \\
0 & O_{41}
\end{pmatrix}
\begin{pmatrix}
\epsilon_{4,0}^\pm \\
\epsilon_{4,1}^\pm
\end{pmatrix} = 0. 
$$

(40)

We can simplify the condition $O_4 + D_{41}^\pm = 0$ by setting one of the eigenvalues of $O_4$ to be 0 and $(c_{2,1}^\pm)$ to be its null vector (shown in table 1). This will eliminate 8 rows and 7 columns in the coefficient matrix in equation (40). If all the coupling strengths $g_i$ ($i = 1, 2, 3, 4$) are nonzero, then there are at least 7 nonzero rows in the row echelon form of this coefficient matrix, because the number of the zero rows in the echelon form of $O_{40}$ is just the same as its null vectors. Together with the diagonal matrix $D_{40}^\pm - E^\pm$, there are 15 rows but only 9 columns totally. It is impossible to obtain a non-trivial solution by analyzing equations (39) and (40).

It seems that there are no dark-like solutions for the 4-qubit Rabi model up to now. However, there are other possibilities by setting more than 1 eigenvalue in table 1 to be 0 simultaneously. Then, $(c_{2,1}^\pm)$ can be the linear superposition of the corresponding two null vectors (shown in table 1), and there will be two free variables. After elementary row transformation, the coefficient matrix in equation (40) reduces to row echelon form

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 + \frac{\pm \Delta_1 - \Delta_2 - E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 + \Delta_2 - E^\pm}{g_1} & 1 + \frac{\pm \Delta_1 - \Delta_2 + E^\pm - 1}{g_1} \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 + \frac{\pm \Delta_1 - \Delta_2 + E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 + \Delta_2 - E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 + \Delta_2 + E^\pm - 1}{g_1} \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & -1 + \frac{\pm \Delta_1 - \Delta_2 + E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 + \Delta_2 - E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 + \Delta_2 + E^\pm - 1}{g_1} \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & -1 + \frac{\pm \Delta_1 + \Delta_2 + E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 - \Delta_2 - E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 - \Delta_2 + E^\pm - 1}{g_1} \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 + \frac{\pm \Delta_1 + \Delta_2 + E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 - \Delta_2 - E^\pm}{g_1} & 1 - \frac{\pm \Delta_1 - \Delta_2 + E^\pm - 1}{g_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{4E^\pm - 4}{g_1} & \frac{4 - 4E^\pm}{g_1} & \frac{4 - 4E^\pm}{g_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4 - 4E^\pm}{g_1} & \frac{4 - 4E^\pm}{g_1} & \frac{4 - 4E^\pm}{g_1}
\end{pmatrix}
$$

(41)

If $E^\pm = 1$, equation (41) reduces to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & \frac{\pm \Delta_1 - \Delta_2}{g_1} & \frac{\pm \Delta_1 + \Delta_2}{g_1} & \frac{\pm \Delta_1 - \Delta_2}{g_1} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\Delta_1 - \Delta_2}{g_1} & \frac{\Delta_1 + \Delta_2}{g_1} & \frac{-\Delta_1 + \Delta_2}{g_1} \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & \frac{-\Delta_1 + \Delta_2}{g_1} & \frac{\Delta_1 + \Delta_2}{g_1} & \frac{-\Delta_1 + \Delta_2}{g_1} \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & \frac{\Delta_1 + \Delta_2}{g_1} & \frac{-\Delta_1 - \Delta_2}{g_1} & \frac{\pm \Delta_1 + \Delta_2}{g_1} \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & \frac{-\Delta_1 + \Delta_2}{g_1} & \frac{\Delta_1 + \Delta_2}{g_1} & \frac{-\Delta_1 + \Delta_2}{g_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{\pm \Delta_1 + \Delta_2}{g_1} & \frac{\pm \Delta_1 + \Delta_2}{g_1} & \frac{\pm \Delta_1 + \Delta_2}{g_1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

(42)

then together with $D_{40}^\pm - E^\pm$, there are 14 rows but just 10 columns in total, such that there should be 5 zero rows in $D_{40}^\pm - E^\pm$. This seems impossible by only analyzing equation (39), but this is indeed not the case, because elementary row transformation will greatly simplify
equations (39) and (42). For even parity, if $\Delta_1 - \Delta_2 = \pm 1 = \pm E$ and $\Delta_3 = \Delta_4$, there are dark-like state solutions by analyzing equations (39) and (42),

$$|\psi\rangle_{g1} = \frac{1}{\mathcal{N}_g} \left( \frac{2(\Delta_1 + \Delta_2)}{g} |0, \uparrow, \downarrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right) \otimes (|\uparrow\downarrow - |\downarrow\uparrow\rangle),$$

(43)

$$|\psi\rangle_{g2} = \frac{1}{\mathcal{N}_g} \left( \frac{2(\Delta_1 + \Delta_2)}{g} |0, \downarrow, \uparrow\rangle + |1, \downarrow, \downarrow\rangle - |1, \uparrow, \uparrow\rangle \right) \otimes (|\uparrow\downarrow - |\downarrow\uparrow\rangle),$$

(44)

where the first two qubits form a two-qubit dark-like state (29) and (30) respectively, and another two qubits form a spin singlet dark state. For odd parity, if $\Delta_1 + \Delta_2 = 1 = E$, there are similar dark-like states formed by

$$|\psi\rangle_e \otimes (|\uparrow\downarrow - |\downarrow\uparrow\rangle),$$

(45)

where $|\psi\rangle_e$ is given by equation (27). We can also interchange the first two qubits and the other two qubits in equations (43), (72) and (45) and their existence conditions, to obtain the equivalent ‘dark-like state’ shown above. At the same time, we believe if we choose $g_1 = g_3$ and $g_2 = g_4$, or $g_1 = g_4$ and $g_2 = g_3$, similar ‘dark-like state’ can also be obtained follow the same procedure, because we just change the order of the qubits.

If $E^\pm \neq 1$, then equation (41) reduces to

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

(46)

For even parity, if $\Delta_1 = \Delta_2$ and $\Delta_3 = \Delta_4$, there is a ‘dark state’ solution

$$|\psi\rangle_d = (|0, \uparrow, \downarrow\rangle - |0, \downarrow, \uparrow\rangle) \otimes (|\uparrow\downarrow - |\downarrow\uparrow\rangle),$$

(47)

which is just the product of the two-qubit singlet.

Finally, we come to the case $g_1 = g_2 = g_3 = g_4$. Now three eigenvalues $g_1 - g_2 + g_3 - g_4$, $g_1 - g_2 - g_3 + g_4$ and $g_1 + g_2 - g_3 - g_4$ are set to be 0, and there are three null vectors shown in table 1, which can be simplified to $(1, 0, 0, -1, 0, 0, 1)^T, (0, 1, 0, -1, 0, 1, 0)^T, (0, 0, 1, -1, 0, 1, 0)^T$. Supposing that $(c_{4,1}^\pm |\psi_{4q}\rangle)$ is the linear superposition of these null vectors, after elementary row transformation, the coefficient matrix in equation (40) reduces to row echelon form.

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This can be easily understood due to the fact that for $\Delta_2 = \Delta_3 = \Delta_4 = \Delta_1 - 1$, there are three independent solutions, each formed by the product of a two-qubit dark-like state and a two-qubit singlet. For $\Delta_2 = \Delta_3 = \Delta_4 = \Delta_1 + 1$, the solution takes the same form as (52) with
the dark-like state substituted by (30). For odd parity, we choose $\Delta_2 = \Delta_3 = \Delta_4 = -\Delta_1 + 1$, and the dark-like state takes the same form as (52) with the dark-like state substituted by (27). Choosing $\Delta_1 = \Delta_2 + 1$, $\Delta_3 = \Delta_4$, $g_1 = g_2$, $g_3 = g_4$, a dark-like state (43) corresponding to the horizontal line $E^+ = 1$ is shown in figure 2.

2.4. N-qubit case

We can follow a similar procedure to find dark-like states for higher qubit cases. The key point is just to solve (22) with different format. It should be pointed out that we have not found a universal existence condition and all the dark-like states for arbitrary qubit number $N$, because it still needs detailed analysis for more qubits. But we have found that one kind of dark-like states commonly exist for arbitrary qubit number $N > 1$

$$|\psi\rangle_{n\text{dark-like}}^{\text{dark}} = |\psi\rangle_{2\text{dark-like}}^{\text{singlet}} \times (|\psi\rangle_{\text{singlet}})^{(N-2)/2} \quad N = 2, 4, 6, 8, \ldots \quad (53)$$

$$|\psi\rangle_{3\text{dark-like}}^{\text{singlet}} \times (|\psi\rangle_{\text{singlet}})^{(N-3)/2} \quad N = 3, 5, 7, 9, \ldots \quad (54)$$

$N = 4$ is an example of equation (53), and all its dark-like states have the form of equation (53).

3. Dark-like states for the multi-qubit and multi-photon Rabi model

The $N$-qubit and $M$-photon Rabi model reads

$$H_{PQ} = \omega a^\dagger a + \sum_{i=1}^{N} g_i \sigma_{iz} (a^M + a^M) + \sum_{i=1}^{N} \Delta_i \sigma_{iz}, \quad (55)$$

where $M$ is a positive integer. This model is of considerable interest because of its relevance to the study of the coupling between multi-qubit and photon field with the qubit making M-photon transitions. Besides, it is known that under rotating wave approximation, the dynamics of the $M$-photon J-C model is qualitatively different from that of the usual single-photon case [39]. As discussed in [39, 46, 47], for the single-qubit case, this model is solvable only if $M \leq 2$ and the coupling parameter is below a certain critical value. But in the following discussion, we will show that the case for more qubits is different: Dark-like eigenstates for $H_{PQ}$ (55) with $N > 1$ still exist, regardless of these constraints, although in usual cases this model is indeed not well-defined.

There are $2M$ invariant subspaces for $H_{PQ}$ (55)

$$\begin{align*}
\{0, \psi_{Nq+}\}, & |M, \psi_{Nq-}\rangle, |2M, \psi_{Nq+}\rangle, \ldots \\
\{0, \psi_{Nq-}\}, & |M, \psi_{Nq+}\rangle, |2M, \psi_{Nq-}\rangle, \ldots \\
\{1, \psi_{Nq+}\}, & |M + 1, \psi_{Nq-}\rangle, |2M + 1, \psi_{Nq+}\rangle, \ldots \\
\{1, \psi_{Nq-}\}, & |M + 1, \psi_{Nq+}\rangle, |2M + 1, \psi_{Nq-}\rangle, \ldots \\
& \ldots \\
\{|M - 1, \psi_{Nq+}\rangle, & |2M - 1, \psi_{Nq-}\rangle, |3M - 1, \psi_{Nq+}\rangle, \ldots \\
\{|M - 1, \psi_{Nq-}\rangle, & |2M - 1, \psi_{Nq+}\rangle, |3M - 1, \psi_{Nq-}\rangle, \ldots, \ldots, \end{align*} \quad (56)$$

each of which can be labeled by $\{i, \pm\}$, where the initial photon number takes the values $i = 0, 1, 2, \ldots, M - 1$, and $\pm$ is the eigenvalue of $\prod_{k=1}^{N} \sigma_{iz}$ for the initial qubit state. $H_{PQ}$ in each subspace has the same form as $H_{NQ}$ (1) except for some constants.
\[ H_{PQ}^\pm = \begin{pmatrix} i + D_{N0}^\pm & \sqrt{\frac{(i+M)!}{i!}} O_{N0}^\pm & 0 & 0 & 0 & \cdots \\ \sqrt{\frac{(i+M)!}{i!}} O_{N0}^\pm & i + M - 1 + D_{N1}^\pm & \sqrt{\frac{(i+2M)!}{(i+M)!}} O_{N1}^\pm & 0 & 0 & \cdots \\ 0 & \sqrt{\frac{(i+2M)!}{(i+M)!}} O_{N1}^\pm & i + 2M - 2 + D_{N2}^\pm & \sqrt{\frac{(i+3M)!}{3(i+2M)!}} O_{N2}^\pm & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \]  \tag{57}

Figure 2. The numerical spectrum of the four-qubit quantum Rabi model with \( \Delta_1 = \Delta_2 = 1.2, \Delta_3 = \Delta_4 = 0.3, \omega = 1, g_1 = g_2 = g_3 = g_4, 0 \leq g_1 = g_2 = g_3 = g_4 = g \leq 1. E_+ \text{ and } E_- \text{ are solutions with even and odd parity respectively.}

where \( D_{Nj}^\pm \) and \( O_{Nj}^\pm \) (\( j = 0, 1, 2, 3, \ldots \)) are just the same as defined in the \( N \)-qubit Rabi model in (8) and (9), respectively.

Now, to find out the dark-like solution in the subspace \( \{i, \pm\} \), we follow the steps for the \( N \)-qubit case to get

\[ \begin{pmatrix} i + D_{N0}^\pm - E_i^\pm & 0 & 0 & 0 & \cdots \\ \sqrt{\frac{(i+M)!}{i!}} O_{N0}^\pm & i + M - 1 + D_{N1}^\pm & 0 & 0 & \cdots \\ 0 & \sqrt{\frac{(i+2M)!}{(i+M)!}} O_{N1}^\pm & i + 2M - 2 + D_{N2}^\pm & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_{N,0}^\pm \\ c_{N,1}^\pm \end{pmatrix} = 0. \tag{58} \]

If we define \( E^\pm = E_i^\pm - (i + M - 1) \) and \( g_k^\pm = \sqrt{\frac{n}{(i+M)!}} g_{ik} \) (\( k = 1, 2, \ldots, N \)), where \( g_{ik} \) is the coupling strength in the subspace \( \{i, \pm\} \), so that \( \sqrt{\frac{(i+M)!}{i!}} O_{N0,1}^\pm \rightarrow O_{N0,1}^\pm \), then we obtain

\[ \begin{pmatrix} D_{N0}^\pm - (E^\pm + M - 1) & 0 & 0 & 0 & \cdots \\ 0 & D_{N1}^\pm - E_i^\pm & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} c_{N,0}^\pm \\ c_{N,1}^\pm \end{pmatrix} = 0. \tag{59} \]

Equation (59) has exactly the same form as equation (22), except for \( [D_{N0}^\pm - (E^\pm + M - 1)]c_{N,0}^\pm = 0 \), which will just determine the relation between \( \Delta_k(k = 1, 2, \ldots, N) \).
and $E^\pm$, such that we can obtain the dark-like solution for equation (59) from the solution to equation (22) for the N-qubit Rabi model just by making the replacement $f(\Delta_k, E^\pm) = 0 \rightarrow f(\Delta_k, (E^\pm + M - 1)) = 0$. To conclude, for a dark-like state of the N-qubit Rabi model, we can get a corresponding dark-like state of the N-qubit and M-photon Rabi model in the subspace labeled by $\{i, \pm\}$, upon using the following relations

$$E_i^\pm = E^\pm + i + M - 1$$

$$g_{i,k} = \sqrt{\frac{(i + M)!}{i!}} g'_k (k = 1, 2, \ldots, N)$$

$$f(\Delta_i^\pm, E_i^\pm - i) = f(\Delta_k, E^\pm) = 0,$$

where $E^\pm$, $g'_k$, $\Delta_k$ are the eigenenergy, coupling strength and qubit energy of the dark-like states in the multi-qubit and single photon Rabi model respectively.

As discussed above, the dark-like eigenstates of $H_{PQ}$ (55) exist for arbitrary photon number $M$ in the whole qubit-photon coupling regime with constant energy, due to the existence of the closed subspace, just like the multi-photon Jaynes–Cummings model [39]. But generally the multi-photon Rabi model is only solvable under some constraints on the coupling strength and photon number $M$.

Let us take the simplest case: two-qubit and two-photon Rabi model for example. We first try to find out the critical value of the coupling strength. We assume that $\Delta_k (k = 1, 2, \ldots, N) = 0$, which does not affect the result [46]. In the basis formed by the eigenstates of $\prod\sigma_jx_j (j = 1, 2, \ldots, N)$, the Hamiltonian (55) with $M = 2$ is turned into the form [46]

$$H_{PQ} = a^\dagger a + \lambda (a^2 + a^\dagger^2),$$

(63)

where $\lambda = \pm g_1 \pm g_2 \ldots \pm g_N$. Defining operators

$$x = \frac{1}{\sqrt{2}}(a + a^\dagger), \quad p = i\sqrt{\frac{1}{2}}(a^\dagger - a),$$

(64)

then $H_{PQ}$ can be rewritten as

$$H_{PQ} = \frac{1 - 2\lambda}{2} p^2 + \frac{1 + 2\lambda}{2} x^2 - \frac{1}{2}.$$  

(65)

Clearly, if $\frac{1 + 2\lambda}{1 - 2\lambda} \sim m^2 \omega^2 > 0$, then $H_{PQ}$ corresponds to a quantum harmonic oscillator and can be diagonalized. On the contrary, if $\frac{1 + 2\lambda}{1 - 2\lambda} = -\omega^2 < 0$, $H_{PQ}$ represents an inverted quantum harmonic oscillator, which cannot be diagonalized using the basis states $|n\rangle$ of the number operator because its eigenstates are not normalizable. Thus $H_{PQ}$ (63) is diagonalizable if $\lambda < \frac{1}{2}$ [46], and correspondingly we have $\max\{\pm g_1 \pm g_2 \ldots \pm g_N\} < \frac{1}{2}$, that is

$$\sum_{k=1}^{N} g_k < \frac{1}{2}.$$  

(66)

At the same time, we can search for the dark-like states following just the same step as the two-qubit and single-photon Rabi model shown in section 2, from which we find six dark-like states

$$|\psi\rangle_{0, +} = \frac{2(\Delta_1 - \Delta_2)}{\sqrt{2}g} |0, \uparrow, \uparrow\rangle - |2, \uparrow, \downarrow\rangle + |2, \downarrow, \uparrow\rangle,$$

(67)
with the conditions $g_1 = g_2 = g/2$, $\Delta_1 + \Delta_2 = 2$ and $E^+ = 2$, 3 respectively, and
\[
|\psi\rangle_{a,-b} = \left( \frac{2(\Delta_1 + \Delta_2)}{\sqrt{2g}} \right) (0, \uparrow, \downarrow) + (|2, \downarrow, \downarrow\rangle - |2, \uparrow, \uparrow\rangle),
\]
(70)
with the conditions $g_1 = g_2 = g/2$, $\Delta_1 - \Delta_2 = 2$ and $E^- = 2$, 3 respectively, and
\[
|\psi\rangle_{a,-b} = \left( \frac{2(\Delta_1 + \Delta_2)}{\sqrt{2g}} \right) (0, \downarrow, \uparrow) + (|2, \downarrow, \downarrow\rangle - |2, \uparrow, \uparrow\rangle),
\]
(71)
\[
|\psi\rangle_{1,-b} = \left( \frac{2(\Delta_1 + \Delta_2)}{\sqrt{6g}} \right) (1, \downarrow, \uparrow) + (|3, \downarrow, \downarrow\rangle - |3, \uparrow, \uparrow\rangle),
\]
(72)
with the conditions $g_1 = g_2 = g/2$, $\Delta_2 - \Delta_1 = 2$ and $E^- = 2$, 3 respectively. We can clearly see all these dark-like states satisfy equations (60)--(62).

Choosing $g_1 = g_2 = g/2$, $\Delta_1 + \Delta_2 = 2$, the numerical spectrum of the two-qubit and two-photon Rabi model is shown in figure 3, where the dark-like states (67) and (68) correspond to the horizontal lines $E^+ = 2$ and $E^+ = 3$, respectively. These special states exist in the whole coupling regime, while other eigenstates exist only for $g < 0.5$. Numerical results show that two eigenenergies converge at $E = 2$ and $E = 2$ respectively even for $g > 0.5$, and the wave-function corresponds to the dark-like states we have found. Besides, they commonly exist even for multi-qubit and M-photon ($M > 2$) Rabi model. Numerical diagonalization shows although the eigenvalues usually will not converge for $N = 2$ and $M = 3$, with regard to dark-like states, the eigenvalue always converges at $E = 3$ with $i = 0$.

4. Experimental considerations

In the past few years there have appeared a series of proposals for the implementation of the quantum Rabi model in all its parameter regimes, via analog or digital-analog quantum simulations, in a variety of quantum platforms including trapped ions [48, 49] and superconducting circuits [50]. Moreover, the multiqubit, single-photon Rabi model may be straightforwardly implemented in superconducting circuits via a digital-analog quantum simulator [51, 52]. Indeed, a set of superconducting qubits capacitively coupled with a coplanar microwave resonator naturally implement a Tavis–Cummings Hamiltonian. Via digital-analog techniques, one can combine this naturally-appearing interaction with local rotations, in order to reproduce the multiqubit Rabi model in all parameter regimes, and with arbitrary inhomogeneous couplings and qubit energies, with polynomial resources [51, 52]. Therefore, a quantum dynamics provided by the Hamiltonian in equation (1) can be carried out in the lab with current technology. In order to probe the dark-like states of the multiqubit, single-photon Rabi model, one may proceed initializing the system in an eigenstate of an easy to initialize Hamiltonian, e.g. the purely qubit and bosonic mode free terms without mutual interaction, and adiabatically turn on the multiqubit Rabi coupling term, via a digitization of the adiabatic evolution, as in [53]. In order to measure the energy, to check its constant character under parameter change, one
may either apply the phase estimation algorithm, or measure term by term of the Hamiltonian, with standard superconducting circuit technology [51, 52].

5. Conclusions

We have found a special kind of exceptional solution—‘dark-like states’ for the multi-qubit and multi-photon quantum Rabi models, which exist in the whole coupling regime with constant eigenenergy, with qubit and photon field being entangled. Besides, their photon numbers are bounded from above, distinctly different from the single-qubit case, because there are closed subspaces in the Fock space due to the interaction between the qubits and the photon field. Their existence conditions are simple, which does not depend on the relation between the qubit energy and coupling strength. And they correspond to horizontal lines in the spectra, which means that for arbitrary coupling $g_i$, we always find one such state by tuning other conditions. This may be used to prepare entangled states. These dark-like states can also serve as benchmarks for numerical techniques and as foundations for perturbative treatments. We have not obtained all the exceptional solutions in the photon number space, but the ‘dark-like state’ is a special one among them and shares similar properties with the well known ‘dark state’. We also believe the other exceptional solutions can be found following the same procedure shown in section 2.

For the single-qubit and multi-photon Rabi model, the solution exists only if the photon number $M \leq 2$ and the coupling strength is below a certain critical value. But multi-qubits make it different. There exist dark-like eigenstates in the whole coupling regime for arbitrary $M$ under certain conditions. This is due to the closed subspace in the photon number representation brought about by the multi-qubit, such that just like the multi-photon J-C model, the multi-photon Rabi model is diagonalizable in this special case.

Dark states can preserve entanglement under dissipation, driving and dipole-dipole interactions, and therefore they can be used to store correlations. Dark-like states have similar properties as dark states in the spectra, but their properties under the influence of environment

Figure 3. The numerical spectrum of two-qubit and two-photon quantum Rabi model with $\Delta_1 = 1.6, \Delta_2 = 0.4, \omega = 1, g_1 = g_2 = g/2, 0 \leq g \leq 0.5$. $E_{0+}, E_{0-}, E_{1+}, E_{1-}$ are eigenvalues of four invariant subspace labeled by $(i, \pm)$ respectively.
(dissipation, dephasing, or the like) need to be explored. Whether this kind of dark-like states has similar applications as dark-like states or has other peculiarities is an interesting problem to study.

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