TORSION GROUP SCHEMES AS ITERATIVE DIFFERENTIAL GALOIS GROUPS

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Abstract. We are considering iterative derivations on the function field $L$ of abelian schemes in positive characteristic $p > 0$, and give conditions when the torsion group schemes of this abelian scheme occur as ID-automorphism groups, i.e. are the ID-Galois groups of $L$ over certain ID-subfields. For an explicit example, we even give a construction of (a family of) such iterative derivations.

1. Introduction

For transcendent field extensions $L/F$ the group of automorphisms $\text{Aut}(L/F)$ is huge and one is far from obtaining a Galois correspondence. By considering derivations on the fields (resp. iterative derivations in positive characteristic), one obtains a natural subgroup of all automorphisms, namely those automorphisms which commute with the (iterative) derivation. These automorphisms are called (iterative) differential automorphisms. In special cases, the group of (iterative) differential automorphisms form a linear algebraic group and one has a Galois correspondence between Zariski-closed subgroups and intermediate differential fields. In Picard-Vessiot theory one considers such cases. Here the extension field $L$ is obtained as the solution field of a linear (iterative) differential equation over the differential field $F$, quite analogous to the classical Galois theory where the extension fields are obtained as solution fields of algebraic equations. By considering the automorphism group not as a group, but as a group scheme, one can deal with nonnormal and even inseparable iterative differential extensions (see [1] and [2], Sect.10). Moreover, this also applies to finite ID-extensions, and one can even obtain an infinitesimal group scheme as ID-Galois group scheme (cf. [3]).

In this article, we will consider special finite group schemes, namely the torsion group schemes of an abelian variety. More precisely, we give iterative differential field extensions having as ID-Galois group scheme the torsion group scheme of an abelian variety. Throughout the article we will stick to positive characteristic.

The rough idea for getting the $n$-torsion scheme $A[n]$ of an abelian variety $A$ over a perfect field $C$ as ID-Galois group scheme is the following. Starting with the abelian variety $A$ we consider the function field $L$ of $A_{C(t)}$ (i.e.

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of $A$ after base change to $C(t)$ as an extension of the rational function field $C(t)$. The field $C(t)$ comes with a standard iterative derivation with respect to $t$ (the characteristic $p$-analog of the derivation $\frac{\partial}{\partial t}$), and this iterative derivation is then extended to an iterative derivation on $L$. By taking care that this extension fulfills the appropriate conditions, one guarantees that the torsion group scheme $A[n]$ indeed acts on $L$ by ID-automorphisms. Hence by Picard-Vessiot theory, one obtains $A[n]$ as the iterative differential Galois group of $L$ over $L^A[n]$, the fixed field under $A[n]$. To be more precise, one should say that the group scheme acts by functorial automorphisms, i.e. $D$-rational points act as ID-automorphisms on the total quotient ring $\text{Quot}(L \otimes_C D)$.

The article is structured as follows. In Section 2, we give the basic notation and some basic properties which will be used in the calculations later on. Furthermore, we give a short summary of the Picard-Vessiot theory used in this article. The theoretical considerations for obtaining the torsion group scheme of an abelian scheme as ID-Galois group are given in Sections 3 and 4. The main theorems are Theorem 3.2 giving a necessary and sufficient condition for the iterative derivation on the function field of an abelian variety to “commute” with the addition map, as well as Theorem 4.1 stating that the torsion group schemes are the ID-Galois group schemes over an appropriate subfield when the iterative derivation satisfies the previous conditions.

In the last sections we do explicit calculations. While Section 5 deals with the extension of an iterative derivation to an overfield in general, Section 6 is dedicated to the example of an explicit elliptic curve in characteristic 2. In this case, we give recursive formulas for constructing an iterative derivation on the function field which satisfies the previously stated conditions (cf. Theorem 6.3).

2. Basic notation

All rings are assumed to be commutative with unit.

We will use the following notation (see also [3]). A higher derivation (HD for short) on a ring $R$ is a homomorphism of rings $\theta : R \to R[[T]]$, such that $\theta(r)|_{T=0} = r$ for all $r \in R$. If there is need to emphasize the extra variable $T$ or if we use another name for the variable, we add a subscript to $\theta$, i.e. denote the higher derivation by $\theta_T$ (resp. $\theta_U$ if the variable is named $U$).

A higher derivation is called an iterative derivation (ID for short) if for all $i, j \geq 0$, $\theta^{(i)} \circ \theta^{(j)} = (i+j)\theta^{(i+j)}$, where the maps $\theta^{(i)} : R \to R$ are defined by $\theta^{(i)}(r) := \sum_{i=0}^{\infty} \theta^{(i)}(r)T^i$. The pair $(R, \theta)$ is then called an HD-ring (resp. ID-ring) and $C_R := \{ r \in R \mid \theta(r) = r \}$ is called the ring of constants of $(R, \theta)$. An HD/ID-ring which is a field is called an HD/ID-field. Higher derivations and iterative derivations are extended to localizations by $\theta\left(\frac{r}{s}\right) := \frac{\theta(r)}{\theta(s)}$. 


θ(r)θ(s)^{-1} and to tensor products by

θ^{(k)}(r \otimes s) = \sum_{i+j=k} \theta^{(i)}(r) \otimes \theta^{(j)}(s)

for all k ≥ 0.

Given a homomorphism of rings f : R → S, we often consider the T-linear extension of f to a homomorphism R[[T]] → S[[T]] of the power series rings. This map will be denoted by f[[T]]. Given two HD-rings \((R, \theta)\) and \((S, \tilde{\theta})\).

A homomorphism of rings f : R → S is called an HD-homomorphism (resp. ID-homomorphism if R and S are ID-rings) if \(\tilde{\theta} \circ f = f[[T]] \circ \theta\).

As a special case of a homomorphism f[[T]], we have the homomorphism \(\theta_U[[T]] : R[[T]] \rightarrow R[[U]]\) induced by the higher derivation \(\theta_U : R \rightarrow R[[U]]\) on R. A short calculation shows (cf. [4]) that a higher derivation \(\theta\) on R is an iterative derivation if and only if the following diagram commutes

\[
\begin{array}{ccc}
R & \xrightarrow{\theta_U} & R[[U]] \\
\downarrow{\theta_T} & & \downarrow{U \mapsto U + T} \\
R[[T]] & \xrightarrow{\theta_U[[T]]} & R[[U,T]],
\end{array}
\]
or in other terms \(\theta_U[[T]] \circ \theta_T = \theta_{T+U}\).

**Example 1.** (cf. [2])

1. For any field C and F := C(t), the homomorphism of C-algebras \(\theta : F \rightarrow F[[T]]\) given by \(\theta(t) := t + T\) is an iterative derivation on F with field of constants C. This iterative derivation will be called the iterative derivation with respect to t.

2. For any ring R, there is the trivial iterative derivation on R given by \(\theta_0 : R \rightarrow R[[T]], r \mapsto r \cdot T^0\). Obviously, the ring of constants of \((R, \theta_0)\) is R itself.

3. If \((F, \theta)\) is an HD-field and L ≥ F is a finite separable field extension, then \(\theta\) can be uniquely extended to a higher derivation on L. If the higher derivation \(\theta\) is an iterative derivation, then the extension to L is also an iterative derivation.

4. Let \((F, \theta)\) be an HD-field, L/F a finitely generated separable field extension and \(x_1, \ldots, x_k\) a separating transcendence basis of L over \(F\) (i.e. \(F(x_1, \ldots, x_k)/F\) is purely transcendental and L/F(\(x_1, \ldots, x_k\)) is finite separable). Using the previous example, it is easy to see that any choice of elements \(\xi_{i,n} \in L (i = 1, \ldots, k\) and \(n \geq 1\) defines a unique higher derivation \(\theta_L\) on L extending \(\theta\) and satisfying \(\theta_L(x_i) = x_i + \sum_{n=1}^{\infty} \xi_{i,n} T^n\) for all \(i = 1, \ldots, k\).

We now summarize some well known formulas for higher derivations in characteristic \(p > 0\) which will be used later on:

**Lemma 2.1.**

1. \(\theta^{(j)}(x^p) = 0\) if \(p\) does not divide \(j\) and \(\theta^{(j)}(x^p) = (\theta^{(j/p)}(x))^p\) if \(p\) divides \(j\).
We now recall some definitions from Picard-Vessiot theory. Let\( \mathbb{F}, \theta \) be a matrix with \( \mathbb{F} \), \( \theta \) an ID-simple ring, i.e., has no nontrivial \( \theta \)-stable ideals.

\[ \text{Lemma 2.2.} \quad (1) \text{Given an ID-field } (L, \theta), \text{ and } \ell \in \mathbb{N} \cup \{\infty\}. \text{ The set of elements } x \text{ for which the equation } \theta^{(i)} \circ \theta^{(j)}(x) = \left( \begin{array}{c} i+j \\ i \end{array} \right) \theta^{(i+j)}(x) \text{ holds for all } j, i \text{ satisfying } i + j < p^\ell, \text{ is a subfield of } L. \]

\[ (2) \text{Assume that for fixed } \ell \geq 0 \text{ the iteration rule on } L \text{ holds for all } j, i \text{ satisfying } i + j < p^\ell \text{ (i.e., } \theta^{(i)} \circ \theta^{(j)} = \left( \begin{array}{c} i+j \\ i \end{array} \right) \theta^{(i+j)} \text{ for all } i + j < p^\ell \text{)} \text{ and that } L \text{ contains an element } t \text{ satisfying } \theta(t) = t + T \text{. Then for all } x \in L \text{ and all } j < \ell \text{ one has:} \]

\[ \theta^{(p^\ell)} \left( \sum_{m=0}^{p^\ell-1} \theta^{(m)}(x)(-t)^m \right) = 0 \]

Proof. (1) The given condition on the elements \( x \) is equivalent to \( \theta_U [[T]] \circ \theta_T(x) \equiv \theta_{T+U}(x) \mod (U^{p^\ell - j} T^j \mid j \leq p^\ell) \). From this one easily checks that the set of those elements is indeed a subfield.

(2) This is a more complicated, but straightforward calculation using identities of binomial coefficients.

\[ \text{Picard-Vessiot theory.} \quad \text{We now recall some definitions from Picard-Vessiot theory. } (F, \theta) \text{ denotes some ID-field with constants } C. \]

\[ \text{Definition 2.3.} \quad \text{Let } A = \sum_{k=0}^{\infty} A_k T^k \in \text{GL}_n(F[[T]]) \text{ be a matrix with } A_0 = I_n \text{ and for all } k, l \in \mathbb{N}, \left( \begin{array}{c} k+l \\ k \end{array} \right) A_{k+l} = \sum_{i+j=l} \theta^{(i)}(A_k) \cdot A_j. \text{ An equation } \theta(y) = Ay, \]

where \( y \) is a vector of indeterminants, is called an \textit{iterative differential equation} (IDE).

\[ \text{Remark 2.4.} \quad \text{The condition on the } A_k \text{ is equivalent to the condition that } \theta^{(k)}(\theta^{(l)}(Y_{ij})) = \left( \begin{array}{c} k+l \\ k \end{array} \right) \theta^{(k+l)}(Y_{ij}) \text{ holds for a matrix } Y = (Y_{ij})_{1 \leq i, j \leq n} \in \text{GL}_n(E) \text{ satisfying } \theta(Y) = Ay, \text{ where } E \text{ is some ID-extension of } F. \text{ (Such a } Y \text{ is called a \textit{fundamental solution matrix}). The condition } A_0 = I_n \text{ is equivalent to } \theta^{(0)}(Y_{ij}) = Y_{ij}, \text{ and already implies that the matrix } A \text{ is invertible.} \]

\[ \text{Definition 2.5.} \quad \text{An ID-ring } (R, \theta_R) \geq (F, \theta) \text{ is called a \textbf{Picard-Vessiot ring} (PV-ring) for the IDE } \theta(y) = Ay, \text{ if the following holds:} \]

\[ (1) \text{ } R \text{ is an ID-simple ring, i.e. has no nontrivial } \theta_R \text{-stable ideals.} \]

\[ (2) \text{ There is a fundamental solution matrix } Y \in \text{GL}_n(R), \text{ i.e., an invertible matrix satisfying } \theta(Y) = Ay. \]

\[ (3) \text{ As an } F \text{-algebra, } R \text{ is generated by the coefficients of } Y \text{ and by } \det(Y)^{-1}. \]
The quotient field $E = \text{Quot}(R)$ (which exists, since such a PV-ring is always an integral domain) is called a Picard-Vessiot field (PV-field) for the IDE $\theta(y) = Ay$.  

For a PV-ring $R/F$ one defines the functor 

$$\text{Aut}^{ID}(R/F) : (\text{Algebras}/C) \to (\text{Groups}), D \mapsto \text{Aut}^{ID}(R \otimes_C D/F \otimes_C D)$$

where $D$ is equipped with the trivial iterative derivation. In [2], Sect. 10, it is shown that this functor is representable by a $C$-algebra of finite type, and hence, is an affine group scheme of finite type over $C$. This group scheme is called the (iterative differential) Galois group scheme of the extension $R$ over $F$ – denoted by $\text{Gal}(R/F)$, or also, the Galois group scheme of the extension $E$ over $F$, $\text{Gal}(E/F)$, where $E = \text{Quot}(R)$ is the corresponding PV-field. Furthermore, $\text{Spec}(R)$ is a $(\text{Gal}(R/F) \times_C F)$-torsor and the corresponding isomorphism of rings

$$\gamma : R \otimes_F R \to R \otimes_C C[\text{Gal}(R/F)]$$

is an $R$-linear ID-isomorphism. Again, $C[\text{Gal}(R/F)]$ is equipped with the trivial iterative derivation.

On the other hand, if $(R, \theta_R)$ is an ID-simple ID-ring extending $(F, \theta)$ with the same constants, and if there is an $R$-linear ID-isomorphism $\gamma : R \otimes_FR \to R \otimes_C C[G]$ for some affine group scheme $G \leq \text{GL}_{n,C}$ corresponding to an action of $G$, then $R/F$ is indeed a Picard-Vessiot ring for some IDE (cf. [2], Prop. 10.12).

For later purposes, also keep in mind that for a finite Picard-Vessiot extension $R/F$, the PV-ring $R$ already is a field. Hence, in that case the quotient field $E$ coincides with the PV-ring $R$.

### 3. Iterative Derivations compatible with addition

Let $C$ be a field of positive characteristic $p$, $k = C(t)$ the rational function field with iterative derivation by $t$, and let $A/C$ be a connected abelian scheme over $C$. The addition map on $A$ will be denoted by $\oplus : A \times A \to A$ (and the subtraction by $\ominus$).

Let $K_A$ denote the function field of $A$. Let $(L, \theta)$ be the field $L = K_A(t)$ with some higher derivation $\theta$ extending the one on $k = C(t)$, and let $D$ be the field $K_A$ equipped with the trivial higher derivation. The higher derivations of $L$ and $D$ are extended to a higher derivation (also denoted by $\theta$) on $LD = \text{Quot}(L \otimes_C D)$.  

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1The PV-rings and PV-fields defined here were called pseudo Picard-Vessiot rings (resp. pseudo Picard-Vessiot fields) in [2] and [3]. This definition, however, is the most natural generalisation of the original definition of PV-rings and PV-fields to non algebraically closed fields of constants.
The map \( \oplus \) induces a homomorphism of the function fields \( K_A \to K_{A \times A} = K_A \cdot K_A \) and also a homomorphism \( L \to L \cdot D \) by \( t \)-linear extension. Extending again \( D \)-linearly, we obtain an isomorphism \( \rho : LD \to LD \). This isomorphism fixes exactly the elements in \( D(t) \subseteq LD \), i.e. \( D(t) = \{ x \in LD \mid \rho(x) = x \} \). Actually \( \rho \) is nothing else than the homomorphism on the generic fibers corresponding to \( A_{C(t)} \times A \to A_{C(t)} \times A, (p_1, p_2) \mapsto (p_1 \oplus p_2, p_2) \).

**Lemma 3.1.** With notation as above, let \( \eta_L \in A(L) \) be the generic point, and let \( \theta_* : A(L) \to A(L[[T]]) \) be the map induced by \( \theta \). Then \( \rho \) is an HD-homomorphism if and only if \( \eta_L \oplus \theta_*(\eta_L) \in A(C(t)[[T]]) \).

**Proof.** Since in any case \( \eta_L \oplus \theta_*(\eta_L) \in A(L[[T]]) \), the condition is equivalent to saying that \( \eta_L \oplus \theta_*(\eta_L) \in A(D(t)[[T]]) \subseteq A(LD[[T]]) \).

Let \( \eta_D \) denote the generic point of \( A \) in \( A(D) \), and \( \rho_* : A(LD) \to A(LD) \) the map induced by \( \rho \). Then by construction, one has \( \rho_*(\eta_L) = \eta_L \oplus \eta_D \), and therefore, \( \theta_*(\rho_*(\eta_L)) = \theta_*(\eta_L \oplus \eta_D) = \theta_*(\eta_L) \oplus \eta_D \), since \( \theta \) acts trivially on \( D \).

Hence:

\[
\begin{align*}
\eta_L \oplus \theta_*(\eta_L) \in A(D(t)[[T]]) &\iff \rho_*[[T]](\eta_L \oplus \theta_*(\eta_L)) = \eta_L \oplus \theta_*(\eta_L) \\
&\iff \rho_*(\eta_L) \oplus \rho_*[[T]](\theta_*(\eta_L)) = \eta_L \oplus \theta_*(\eta_L) \\
&\iff (\eta_L \oplus \eta_D) \oplus \rho_*[[T]](\theta_*(\eta_L)) = \eta_L \oplus \theta_*(\eta_L) \\
&\iff \theta_*\eta_L \oplus \eta_D = \rho_*[[T]](\theta_*(\eta_L)) \\
&\iff \theta_*(\rho_*(\eta_L)) = \rho_*[[T]](\theta_*(\eta_L))
\end{align*}
\]

Since \( \eta_L \) is the generic point of \( A \), the last equality is equivalent to \( \theta \circ \rho = \rho[[T]] \circ \theta \), i.e. to the condition that \( \rho \) is an HD-homomorphism. \( \square \)

**Theorem 3.2.** We use notation as above. Let \( C(t)[[T,U]] \) be the power series ring over \( C(t) \) in two variables \( T \) and \( U \) and let \( R \) denote the subring of \( C(t)[[T,U]] \) of those power series \( P(t,T,U) \) such that \( P(t+U,T,0) = P(t,T,U) \).

Then \( \theta \) is an iterative derivation and \( \rho \) is an ID-homomorphism if and only if \( \theta_{U,*}(\eta_L) \oplus \theta_{T+U,*}(\eta_L) \in A(R) \).

As already mentioned earlier, \( \theta_U : LD \to LD[[U]] \) and \( \theta_{T+U} : LD \to LD[[T,U]] \) denote the maps \( \theta \) with \( T \) replaced by \( U \) and \( T+U \), respectively.

**Proof.** First, let \( \theta \) be an iterative derivation such that \( \rho \) is an ID-homomorphism. Since \( \theta \) is an ID, one has \( \theta_{T+T} = \theta_U[[T]] \circ \theta_T \), and therefore \( \theta_{U,*}(\eta_L) \oplus \theta_{T+U,*}(\eta_L) = \theta_{U,*}[[T]](\eta_L \oplus \theta_{T,*}(\eta_L)) \). Since \( \rho \) is an ID-homomorphism, one has \( \eta_L \oplus \theta_{T,*}(\eta_L) \in A(C(t)[[T]]) \) by the previous lemma. Hence, we obtain \( \theta_{U,*}[[T]](\eta_L \oplus \theta_{T,*}(\eta_L)) \in A(C(t)[[T,U]]) \). Furthermore, since the map \( \theta_U[[T]] \) on \( C(t)[[T]] \) is nothing else than replacing \( t \) by \( t+U \), we obtain

\[
\begin{align*}
\theta_{U,*}[[T]](\eta_L \oplus \theta_{T,*}(\eta_L)) &\xrightarrow{U \mapsto 0} \eta_L \oplus \theta_{T,*}(\eta_L) \\
&\xrightarrow{t \mapsto t+U} \theta_{U,*}[[T]](\eta_L \oplus \theta_{T,*}(\eta_L)),
\end{align*}
\]

i.e. the point \( \theta_{U,*}[[T]](\eta_L \oplus \theta_{T,*}(\eta_L)) \) is indeed \( R \)-valued.
Now, let $\theta_{U,*}(\eta_L) \oplus \theta_{T+U,*}(\eta_L) \in A(R)$. Mapping $U$ to 0 leads to $\eta_L \oplus \theta_{T,*}(\eta_L) \in A(C(t)[[T]])$, hence $\rho$ is an HD-homomorphism by the previous lemma. As before, the condition that the expression is in $A(R)$ implies that we obtain the same element when mapping $U \mapsto 0$ and applying $\theta_{U,*}([T])$. Hence

$$\theta_{U,*}(\eta_L) \oplus \theta_{T+U,*}(\eta_L) = \theta_{U,*}([T]) (\theta_{0,*}(\eta_L) \oplus \theta_{T+0,*}(\eta_L))$$

This means $\theta_{T+U,*}(\eta_L) = \theta_{U,*}([T]) (\theta_{T,*}(\eta_L))$. Since $\eta_L$ is the generic point, this implies $\theta_{T+U} = \theta_U[[T]] \circ \theta_T$, and therefore $\theta$ is iterative. \hfill \Box

Remark 3.3. So far, we didn’t use commutativity of $\oplus$. Hence, all the statements made are also valid for non-commutative connected group schemes instead of abelian schemes.

4. Torsion schemes as Galois group schemes

We use the notation of the previous section. In particular, $A/C$ is an abelian scheme and $L$ is the function field of $A_{C(t)}$ equipped with a higher derivation $\theta$ extending the iterative derivation w.r.t. $t$ on $C(t)$.

Theorem 4.1. Let $\theta$ be an iterative derivation on $L$ such that $\rho$ is an ID-homomorphism. Also assume that the constants of $(L, \theta)$ are $C$. For $n \in \mathbb{N}$, let $[n] : A \to A$ denote multiplication by $n$, $A[n] = \text{Ker}([n])$ the $n$-torsion scheme, and $[n]^\# : L \to L$ the corresponding map on the function fields of $A_{C(t)}$. Then

1. the subfield $[n]^\#(L) \subseteq L$ is an ID-subfield of $L$,
2. the extension $L/[n]^\#(L)$ is a PV-extension and the iterative differential Galois group scheme is given as

$$\text{Gal}_\theta(L/[n]^\#(L)) \cong A[n]$$

as affine group schemes over $C$.

Proof. The addition $A \times A[n] \to A$ induces a homomorphism $\hat{\rho} : \mathcal{O}_A(U) \to \mathcal{O}_A(U) \otimes_C A[n]$ for an appropriate (affine) open subset $U \subseteq A$. The subring $[n]^\#(\mathcal{O}_A(U))$ is then the equalizer of $\hat{\rho}$ and $\text{id} \otimes 1$.

Furthermore, $\hat{\rho}$ can be extended to a homomorphism $\hat{\rho} : L \to L \otimes_C A[n]$ by $\hat{\rho}(t) = t \otimes 1$ and by localisation. This map $\hat{\rho}$ is actually a specialisation of the map $\rho : L \to LD$. By assumption $\rho$ is an ID-homomorphism and therefore $\hat{\rho}$ is an ID-homomorphism when $C(A[n])$ is equipped with the trivial iterative derivation.

This shows that the equalizer $[n]^\#(L) \subseteq L$ is an ID-subfield of $L$.

The $L$-linear extension of $\hat{\rho}$ leads to an ID-homomorphism $\tilde{\rho}_L : L \otimes [n]^\#(L) \to L \otimes_C A[n]$ which is a monomorphism, since $[n]^\#(L)$ is the equalizer of $\hat{\rho}$ and $\text{id} \otimes 1$.

As the degree of the extension $L/[n]^\#(L)$ equals the dimension $\dim_C(C(A[n]))$, this monomorphism is indeed an ID-isomorphism.
In this section, we will assume that the constants of \( L \) for constructing an iterative derivation on \( L \) with respect to Condition 4 of the previous theorem, gives a recursive rule with respect to Theorem 5.1. 

\( \ell_{i,p} \geq 0 \)

\( \xi_{i,p} \)

\( \theta_{\ell} \)

\( \theta_{\ell}^{-1} \)

\( \theta_{\ell}^{m} \)

\( \theta_{\ell}^{m+ap^\ell} \)

\( \xi_{i,m} \)

Therefore, the second claim follows by [2], Prop. 10.12. (Here we use that the constants of \( L \) are indeed \( C \).) 

5. Extension of iterative derivations

In this section, we will assume that \( C \) is a field of characteristic \( p > 0 \), and \( (F, \theta) \) is an ID-field containing \( C(t) \) such that \( \theta|_{C(t)} \) is the iterative derivation with respect to \( t \) (compare Ex. 111).

**Theorem 5.1.** Let \( L \) be a finitely generated separable field extension of \( F \) with a higher derivation on \( L \) extending \( \theta \) on \( F \), which will also be denoted by \( \theta \). Let \( x_1, \ldots, x_k \) be a separating transcendence basis of \( L \) over \( F \), and \( \theta(x_i) =: x_i + \sum_{n=1}^{\infty} \xi_{i,n} T^n \) for all \( i = 1, \ldots, k \).

Assume that \( \xi_{i,n} \in L^p \cdot F \) for all \( i = 1, \ldots, k \) and all \( n \geq 1 \). Then for any \( \ell_0 \geq 0 \) the following are equivalent:

1. For all \( j, m > 0 \) s.t. \( j + m \leq \ell_0 + 1 \), the iteration rule \( \theta^{(j)} \circ \theta^{(m)} = (j+m) \theta^{(j+m)} \) holds.
2. For all \( 0 \leq \ell \leq \ell_0 \), one has:
   a. for all \( 0 \leq m \leq p^\ell \) and \( 0 < a < p \): \( \theta^{(m+ap^\ell)} = \frac{1}{a!} (\theta^{(p^\ell)})^a \circ \theta^{(m)} \),
   b. \( (\theta^{(p^\ell)})^p = 0 \), and
   c. for all \( 0 \leq j < \ell \): \( \theta^{(p^\ell)} \circ \theta^{(p^j)} = \theta^{(p^\ell)} \circ \theta^{(p^j)} \).
3. Condition (1) holds when evaluated at all \( x_i \) (\( i = 1, \ldots, k \)).
4. Condition (2) holds when evaluated at all \( x_i \) (\( i = 1, \ldots, k \)).

For all \( 0 \leq \ell \leq \ell_0 \) and \( i = 1, \ldots, k \), one has:

\[ \xi_{i,p^\ell} + \sum_{m=1}^{p^\ell-1} \theta^{(p^\ell)}(\xi_{i,m})(-t)^m \in \bigcap_{0 \leq j < \ell} \text{Ker} \left( \theta^{(p^j)} \right) \cap \text{Ker} \left( \theta^{(p^\ell(p^j-1))} \right), \]

for all \( 1 < a < p \): \( \xi_{i,ap^\ell} = \frac{1}{a!} (\theta^{(p^\ell)})^{a-1}(\xi_{i,p^\ell}) \), and for all \( 0 < m < p^\ell \) and \( 0 < a < p \):

\[ \xi_{i,m+ap^\ell} = \frac{1}{a!} (\theta^{(p^\ell)})^{a}(\xi_{i,m}). \]

**Remark 5.2.** Condition 4 of the previous theorem, gives a recursive rule for constructing an iterative derivation on \( L \). In more detail:

1. Choose \( \xi_{i,1} \in (L^p \cdot F) \cap \text{Ker} \left( \theta^{(p^1)} \right) = L^p \cdot (F \cap \text{Ker} \left( \theta^{(p^1)} \right)) \) arbitrarily for all \( i = 1, \ldots, k \).
2. Calculate \( \xi_{i,1+a} := \frac{1}{a!} (\theta^{(1)})^a(\xi_{i,1}) \) for \( 0 < a < p - 2 \).
3. Proceed inductively: Assume that for \( \ell > 0 \), the elements \( \xi_{i,m} \) for \( m < p^\ell \) are already given satisfying condition 4 of the theorem. Then choose

\[ \xi_{i,p^\ell} \in - \sum_{m=1}^{p^\ell-1} \theta^{(p^\ell)}(\xi_{i,m})(-t)^m + \bigcap_{0 \leq j < \ell} \text{Ker} \left( \theta^{(p^j)} \right) \cap \text{Ker} \left( \theta^{(p^\ell(p^j-1))} \right) \cap L^p F \]
and calculate $\xi_{i,ap^j}$ for $1 < a < p$ as well as $\xi_{i,m+ap^j}$ for $0 < m < p^\ell$ and $0 < a < p$, by the rules above. Since for an element $x^p \in L^p$, one has $\theta(p^j)(x^p) = \left(\theta(p^{\ell-1})x\right)^p$, the condition $\xi_{i,m} \in L^p F$ implies that $\theta(p^j)(\xi_{i,m})$ is computable using only the values $\xi_{i,m}$ for $m < p^\ell$. By the same reason the set $\bigcap_{0 \leq j < \ell} \ker \theta(p^j) \cap \ker \theta(p^{(p-1)}) \cap L^p F$ is determined by the elements $\xi_{i,m}$ for $m < p^\ell$.

**Proof of Thm. 5.1**  

(1) $\iff$ (3) We only have to show that (3) implies (1). By the same argument as above, condition (3) implies that for all $n, j \geq 0$, one has $\theta(n) \circ \theta(j)(x_i) = \binom{n+j}{n} \theta(n+j)(x_i)$. Since the set for which the iteration rule holds is a subalgebra of $L$ and since $x_1, \ldots, x_k$ generate $F(x_1, \ldots, x_k)$ over $F$ it is immediate that the iteration rule holds for all elements in $F(x_1, \ldots, x_k)$. But an extension of an ID to a finite separable field extension is unique, and again an ID, the HD on $L$ is indeed an ID.

(2) $\iff$ (3') This is shown in the same way.

(1), (2) $\implies$ (4) By the iteration rule resp. condition (2)(a), one has

$$\xi_{i,m+ap^j} = \theta(m+ap^j)(x_i) = \frac{1}{a!} \left(\theta(p^j)\right)^a \circ \theta(m)(x_i) = \frac{1}{a!} \left(\theta(p^j)\right)^a (\xi_{i,m}),$$

for all $0 < m \leq p^\ell$ and $0 < a < p$ s.t. $m + ap^j < p^{\ell+1}$. Furthermore by condition (2)(c) and by Lemma 2.2 we have

$$\theta(p^j) \left( \xi_{i,p^j} + \sum_{m=1}^{p^\ell-1} \theta(p^j)(\xi_{i,m})(-t)^m \right) = \theta(p^j)\theta(p^j) \left( x_i + \sum_{m=1}^{p^\ell-1} \theta(m)(x_i)(-t)^m \right) = 0$$

for all $0 \leq j < \ell$ and by condition (2)(b)

$$\theta(p^{(p-1)}) \left( \xi_{i,p^j} + \sum_{m=1}^{p^\ell-1} \theta(p^{(p-1)})(\xi_{i,m})(-t)^m \right) = \theta(p^{(p-1)})\theta(p^j) \left( x_i + \sum_{m=1}^{p^\ell-1} \theta(m)(x_i)(-t)^m \right) = 0.$$

(4) $\implies$ (3') The formulae for $\xi_{i,ap^j}$ and $\xi_{i,m+ap^j}$ imply the conditions (2)(a) evaluated at $x_i$. Furthermore, by induction $\theta(p^{j-1})\theta(p^{j-1}) = \theta(p^{j-1})\theta(p^{j-1})$ for all $j < \ell$ and hence $\theta(p)(\theta(p)(x) = \theta(p)(x)$ for all $x \in L^p F$. 


This implies

$$0 = \theta^{(p^\ell)} \left( \xi_{i,p^\ell} + \sum_{m=1}^{p^\ell-1} \theta^{(p^\ell)}(\xi_{i,m})(-t)^m \right)$$

$$= \theta^{(p^\ell)}(\xi_{i,p^\ell}) - \theta^{(p^\ell)}(\theta^{(p^\ell)}(x_i) + \theta^{(p^\ell)}(\theta^{(p^\ell)}(x_i)) + \theta^{(p^\ell)} \left( \sum_{m=1}^{p^\ell-1} \theta^{(p^\ell)}(\theta^{(m)}(x_i))(t)^m \right)$$

$$= \theta^{(p^\ell)}(x_i) - \theta^{(p^\ell)}(\theta^{(p^\ell)}(x_i))$$

The last step is obtained by a similar calculation as in Lemma 2.2 using the fact that $\theta^{(k)}(\theta^{(p^\ell)}(\xi_{i,m})) = \theta^{(p^\ell)}(\theta^{(k)}(\xi_{i,m}))$ and $\theta^{(k)}(\theta^{(m)}) = (k+m)\theta^{(k+m)}$ for all $k, m < p^\ell$. Similarly, one obtains

$$0 = \theta^{(p^\ell)(p-1)} \left( \xi_{i,p^\ell} + \sum_{m=1}^{p^\ell-1} \theta^{(p^\ell)}(\xi_{i,m})(-t)^m \right)$$

$$= \theta^{(p^\ell)(p-1)}(x_i) + \sum_{m=1}^{p^\ell-1} \theta^{(p^\ell)}(\theta^{(p^\ell)(p-1)}(\xi_{i,m})(-t)^m$$

$$= \frac{1}{(p-1)!}(\theta^{(p^\ell)}(p^{p-1}\theta^{(p^\ell)}(x_i)).$$

\[\square\]

6. Example

In this section we give an example to illustrate the previous sections. In this example it is even possible to give a recursive formula for constructing the iterative derivation $\theta$ (see Theorem 6.3). Indeed, it will be a sharpening of the formula in Thm. 5.1 Item 4.

The example we consider is the elliptic curve $E/C$ in characteristic $p = 2$ given by the equation $x^3 = z^2 + z$, the neutral element of addition being given by the point $(0, 0)$.

As before, $K_{E/C}$ denotes the function field of $E/C$ and $L = K_{E}(t) = C(x, z, t)$ is the HD-field with a higher derivation $\theta$ extending the iterative derivation with respect to $t$ on $C(t)$. The iterative derivatives of $x$ are denoted by $\xi_m$, i.e. $\theta(x) = x + \sum_{m=1}^{\infty} \xi_m T^m$. Furthermore, $D = K_{E}$ denotes the ID-field with trivial iterative derivation.

Lemma 6.1. For two points $(x_1, z_1)$ and $(x_2, z_2)$ the difference $(x_d, z_d) := (x_1, z_1) \oplus (x_2, z_2)$ is given by:

$$x_d = x_2 + \frac{x_1}{1 + z_1} + \left( \frac{z_2 - z_1}{x_2 - \frac{z_1}{1 + z_1}} \right)^2$$
and
\[ z_d = \frac{z_2 - \frac{1}{1+z_1}}{x_2 - \frac{1}{1+z_1}} \cdot (x_d - x_2) + z_2 \]

**Proof.** One only has to check, that the point \((x_d, z_d)\) is the third intersection of the elliptic curve with the line passing through \((x_2, z_2)\) and \(\ominus(x_1, z_1) = (\frac{1}{1+z_1}, \frac{z_1}{1+z_1})\).

Let
\[ f(T) := \sum_{k=0}^{\infty} f_m T^m := \theta(x) + \frac{x}{1+z} + \left( \frac{\theta(z) - \frac{x}{1+z}}{\theta(x) - \frac{x}{1+z}} \right)^2 \in L[[T]]. \]

Then by the previous lemma, \(f(T)\) is the \(x\)-coordinate of \(\eta_L \ominus \theta_*(\eta_L)\). For the coefficients \(f_m\) we have: \(f_0 = 0, f_m = \xi_m\) for odd \(m\) and \(f_m = \xi_m + (\tilde{f}_m)^2\) for even \(m > 0\) and an appropriate element \(\tilde{f}_m \in L\), depending only on \(x, z\) and the elements \(\xi_k\) for \(k \leq m/2\). For convenience, we introduce \(\tilde{f}_m := 0\) for odd \(m\) in order to have \(\tilde{f}_m = \xi_m + (\tilde{f}_m)^2\) for all \(m > 0\).

Furthermore, we let \(g(T)\) denote the \(z\)-coordinate of \(\eta_L \ominus \theta_*(\eta_L\), i.e. \(\eta_L \ominus \theta_*(\eta_L\) = \((f(T), g(T))\) in these local coordinates. Since this point is on \(E\), one has the relation \(f(T)^2 = g(T)^2 + g(T)\), and hence the coefficients \(g_i\) of \(g(T) := \sum_{k=0}^{\infty} g_m T^m\) can be expressed in terms of the \(f_m\). In more detail, \(g_0 = g_1 = g_2 = 0\) and \(g_m\) can be written as a polynomial in \(f_1, \ldots, f_{m-2}\).

**Lemma 6.2.** Assume that \(\theta\) is an ID on \(L\). Then for even \(m, j \in \mathbb{N}_{>0}\) the difference \(\theta^{(m)}(f_j) - (\binom{m+j}{m}) f_{m+j}\) is a polynomial in \((\binom{m+j}{m/2}) f_{(m+j)/2}, f_{(m+j)/2-1}, \ldots, f_1\), whereas for all other choices of \(m, j \in \mathbb{N}\) this difference is 0.

**Proof.** By definition, \(f(T)\) is the \(x\)-coordinate of \(\eta_L \ominus \theta_*(\eta_L\), hence \(\theta_\Omega([T])(f(T))\) is the \(x\)-coordinate of \(\theta_{U*,[T]}(\eta_L \ominus \theta_*(\eta_L))\). But
\[
\theta_\Omega([T])(\eta_L \ominus \theta_*(\eta_L)) = \theta_{U*,[T]}(\eta_L) \ominus \theta_{U*,[T]}(\theta_*(\eta_L)) = \theta_{U*,[T]}(\eta_L) \ominus \eta_L \ominus \theta_{U+T,*}(\eta_L) = (\eta_L \ominus \theta_{U+T,*}(\eta_L)) \ominus (\eta_L \ominus \theta_{U,*}(\eta_L))
\]
Hence, \((\theta_\Omega([T])(f(T)), \theta_\Omega([T])(g(T))) = (f(U + T), g(U + T) \ominus (f(U), g(U))\).

Using the formula for the difference, we obtain
\[
\theta_\Omega([T])(f(T)) = f(U) + \frac{f(T + U)}{1 + g(T + U)} + \left( \frac{g(U) - \frac{g(T + U)}{1 + g(T + U)}}{f(U) - \frac{f(T + U)}{1 + g(T + U)}} \right)^2 \in L[[T, U]].
\]

The coefficient of \(U^m T^j\) on the left hand side is \(\theta^{(m)}(f_j)\). For the right hand side, we first remark that
\[
\frac{f(T + U)}{1 + g(T + U)} = f(T + U) + (g(T + U)/(T + U))^2 \cdot (f(T + U)/(T + U))^{-2},
\]

as power series in \((T+U)\). So the right hand side is \(f(U) + f(T+U)\) modulo squares. This already shows that the coefficient of \(U^m T^j\) on the right hand side is \(\binom{m+j}{m} f_{m+j}\), if \(m\) or \(j\) are odd.

For the other coefficients one has to consider the equation more carefully. We consider the remaining terms as power series in \((T+U)\) with coefficients in \(L((U))\). The coefficient of \(U^m T^j\) in \((g(T+U)/(T+U))^2 \cdot (f(T+U)/(T+U))^{-2}\) is \(\binom{m+j}{m}\) times the coefficient of \((T+U)^{m+j}\) in this expression. Since \((g(T+U)/(T+U))\) is a multiple of \((T+U)^2\), this coefficient depends only on \(f_{(m+j)/2-1}, f_{(m+j)/2-3}, \ldots, f_1\). The last term is the square of

\[
\frac{g(U) - \frac{g(T+U)}{1+g(T+U)}}{f(U) - \frac{f(T+U)}{1+g(T+U)}} = \frac{1}{1+g(U)} \cdot \frac{g(U) + g(U)^2 + (g(U)^2 - 1)g(T+U)}{f(U) + f(U)g(T+U) - f(T+U)}
\]

\[
= \frac{1}{1+g(U)} \cdot \frac{f(U)^3 + (g(U)^2 - 1)g(T+U)}{f(U) + f(U)g(T+U) - f(T+U)}
\]

\[
= \frac{1}{1+g(U)} f(U)^2 \cdot \frac{1 + \sum_{k=1}^{\infty} g_k \frac{(g(U)^2 - 1)(U)^k}{f(U)^2}}{1 + \sum_{k=1}^{\infty} (g_k - \frac{1}{f(U)^2}) (T+U)^k}
\]

\[
= (1 + g(U))^{-1} f(U)^2 \cdot \left( \sum_{n=0}^{\infty} \tau_n (T+U)^n \right)
\]

where \(\tau_n\) is some polynomial in \(f_1, \ldots, f_n\) (and \(g_1, \ldots, g_n\)), \(g(U)\) and \(\frac{1}{g(U)}\). Since the whole expression is a power series, \(f(U)^2 \cdot \tau_n\) is already in \(L[[U]]\). Hence, the coefficient of \(U^m T^j\) in \((\frac{1}{1+g(U)} f(U)^2 \cdot (\sum_{n=0}^{\infty} \tau_n (T+U)^n))^{\frac{1}{2}}\) depends only on \(f_{(m+j)/2}, f_{(m+j)/2-1}, \ldots, f_1\), and \(f_{(m+j)/2}\) only occurs with the factor \(\binom{m+j}{m/2}\).

**Theorem 6.3.** \(\theta\) is an iterative derivation on \(L\) commuting with \(\rho\) if and only if for all \(\ell \geq 0\) and all \(0 < m < 2^\ell\) one has \(\xi_{m+2^\ell} = \theta^{(2^\ell)}(\xi_m)\) and

\[
\xi_{2^\ell} \in \sum_{m=1}^{2^\ell-1} \theta^{(2^\ell)}(\xi_m) t^m + \left( \sum_{m=0}^{2^\ell-1} \theta^{(m)}(\xi_{2^\ell}) t^m \right)^2 + C(t^{2^\ell+1}).
\]

In particular, it is possible to choose/calculate elements \(\xi_m\) recursively for \(m = 1, 2, \ldots\) in order to obtain an iterative derivation on \(L\) commuting with \(\rho\).

**Proof.** First let \(\theta\) be an ID which commutes with \(\rho\). Then \(\xi_{m+2^\ell} = \theta^{(m)}(\xi_\ell)\) for all \(0 < m < 2^\ell\) by **Theorem 5.1**.

Further using the rules in **Theorem 5.1**, we obtain:

\[
\xi_{2^\ell} + \sum_{m=1}^{2^\ell-1} \theta^{(2^\ell)}(\xi_m) t^m = \sum_{m=1}^{2^\ell-1} \theta^{(m)}(\xi_{2^\ell}) t^m = \sum_{m=1}^{2^\ell+1-1} \theta^{(m)}(\xi_{2^\ell}) t^m,
\]
We will first show that \( \theta \bigcap \ell \bigcirc \), for \( 2^\ell \leq m < 2^{\ell+1} \), as well as

\[
\left( \sum_{m=0}^{2^\ell-1} \theta^{(m)}(\tilde{f}_2) t^m \right)^2 = \sum_{m=0}^{2^\ell-1} \left( \theta^{(m)}(\tilde{f}_2) \right)^2 t^{2m} = \sum_{m=0}^{2^\ell-1} \theta^{(2m)}((\tilde{f}_2)^2) t^{2m} = \sum_{m=0}^{2^\ell+1-1} \theta^{(m)}((\tilde{f}_2)^2) t^m,
\]

since \( \theta^{(m)}((\tilde{f}_2)^2) = 0 \) for \( m \) odd. Combining these we get:

\[
\xi_{2^\ell} + \sum_{m=1}^{2^\ell-1} \theta^{(2^\ell)}(\xi_m) t^m + \left( \sum_{m=0}^{2^\ell-1} \theta^{(m)}(\tilde{f}_2) t^m \right)^2 = \sum_{m=0}^{2^\ell+1-1} \theta^{(m)}(\xi_{2^\ell}) t^m + \sum_{m=0}^{2^\ell+1-1} \theta^{(m)}((\tilde{f}_2)^2) t^m = \sum_{m=0}^{2^\ell+1-1} \theta^{(m)}(f_{2^\ell}) t^m
\]

This expression is in \( C(t) \), since \( f_{2^\ell} \in C(t) \) by Lemma 8.1, and it is in \( \bigcap_{0 \leq j < \ell+1} \text{Ker}(\theta^{(j)}) \) by Lemma 2.2, hence in \( C(t^{2^{\ell+1}}) \) as desired.

On the other hand, assume that the conditions on \( \xi_{m+2^\ell} \) and on \( \xi_{2^\ell} \) hold.

We will first show that \( \theta \) is an ID by showing inductively that \( \theta^{(j)} \circ \theta^{(m)} = (j+m) \theta^{(j+m)} \) for all \( j + m \leq 2^{\ell_0} \).

For \( \ell_0 = 0 \), condition \((*_{\ell_0})\) is just \( \xi_1 \in C(t^2) \), which implies \( \theta^{(1)}(\xi_1) = 0 \). Hence by Theorem 5.1, the iteration rule holds for all \( j + m \leq 2 = 2^{\ell_0} + 1 \).

Now, assume by induction that the iteration rule holds for all \( j + m \leq 2^{\ell_0} \). Then it even holds for all \( j + m < 2^{\ell_0+1} \), since \( \xi_{m+2^\ell} = \theta^{(2^\ell)}(\xi_m) \), and we obtain by Lemma 2.2 that \( \theta^{(2^\ell)} \left( \sum_{m=0}^{2^{\ell_0}-1} \theta^{(m)}(a) t^m \right) = 0 \) for all \( a \in L \) and \( 0 \leq j < \ell_0 \), in particular \( \theta^{(2^\ell)} \left( \sum_{m=0}^{2^{\ell_0}-1} \theta^{(m)}(\tilde{f}_{2^\ell_0}) t^m \right) = 0 \) for \( 0 \leq j < \ell_0 \).

Therefore using \((*_{\ell_0})\), \( \xi_{2^\ell_0} + \sum_{m=0}^{2^{\ell_0}-1} \theta^{(2^\ell_0)}(\xi_m) t^m \in \bigcap_{0 \leq j \leq \ell_0} \text{Ker}(\theta^{(2^\ell)}) \). By Theorem 5.1, this shows that the iteration rule holds for \( j + m \leq 2^{\ell_0+1} \).

It remains to show that \( \rho \) is an ID-homomorphism. By Lemma 3.1, this is equivalent to \( f_k \in C(t) \) for all \( k \geq 1 \). Again we will use induction: The case \( k = 1 \) is given by condition \((*)\), since \( f_1 = \xi_1 \). If \( k \) is not a power of 2, i.e. \( k = 2^\ell + m \) for some \( 0 < m < 2^\ell \), then by Lemma 6.2, \( f_{2^\ell+m} \) differs from \( \theta^{(2^\ell)}(f_m) \) by a polynomial in \( f_j \) for \( 1 \leq j \leq 2^\ell - 1 \), and hence is an element of \( C(t) \) by induction. If \( k = 2^\ell \), condition \((*_{\ell})\) and the calculations above imply that

\[
(\dagger) \quad \sum_{m=0}^{2^{\ell+1}-1} \theta^{(m)}(f_{2^\ell}) t^m \in C(t).
\]
By the same argument as in the case \( k = 2^\ell + m \), we obtain that \( f_{2^\ell + m} \in C(t) \) for all \( 0 < m < 2^\ell \). Again using Lemma \[6.2\] we see that \( \theta^{(m)}(f_{2^\ell}) \) is an element of \( C(t) \) for \( 0 < m < 2^\ell \), and even for \( m = 2^\ell \), since \( \binom{2^\ell+1}{2^\ell} \) and \( \binom{2^\ell}{2^\ell-1} \) are both even. For \( 2^\ell < m < 2^\ell+1 \), we have \( \theta^{(m)}(f_{2^\ell}) = \theta^{(m-2^\ell)}(\theta^{(2^\ell)}(f_{2^\ell})) \in C(t) \). Therefore all the terms in (\dag) different from \( f_{2^\ell} \) are in \( C(t) \) and hence \( f_{2^\ell} \in C(t) \). □

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