Long-Distance Entanglement of Spin-Qubits via Ferromagnet

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We propose a mechanism of coherent coupling between distant spin qubits interacting dipolarly with a ferromagnet. We derive an effective two-spin interaction Hamiltonian and estimate the coupling strength. We discuss the mechanisms of decoherence induced solely by the coupling to the ferromagnet and show that there is a regime where it is negligible. Finally, we present a sequence for the implementation of the entangling CNOT gate and estimate the corresponding operation time to be a few tens of nanoseconds. A particularly promising application of our proposal is to atomistic spin-qubits such as silicon-based qubits and NV-centers in diamond to which existing coupling schemes do not apply.

I. INTRODUCTION

Quantum coherence and entanglement lie at the heart of quantum information processing. One of the basic requirements for implementing quantum computing is to generate, control, and measure entanglement in a given quantum system. This is a rather challenging task, as it requires to overcome several obstacles, the most important one being decoherence processes. These negative effects have their origin in the unavoidable coupling of the quantum systems to the environment they are residing in.

A guiding principle in the search for a good system to encode qubits is the smaller the system the more coherence, or, more precisely, the fewer degrees of freedom the weaker the coupling to the environment. Simultaneously, one needs to be able to coherently manipulate the individual quantum objects, which is more efficient for larger systems. This immediately forces us to compromise between manipulation and decoherence requirements.

Following this principle, among the most promising candidates for encoding a qubit we find atomistic two-level systems, such as NV-centers and silicon-based spin qubits. The latter are composed of nuclear (electron) spins of phosphorus atoms in a silicon nanostructure. They have very long $T_2$ times of 60 ms for nuclei and of 200 µs for electrons. Recently, high fidelity single qubit gates and readout have been demonstrated experimentally. Nitrogen-vacancy centers in diamond have also been demonstrated experimentally to be very stable with long decoherence times of $T_2^* \approx 20 \mu s$ and $T_2 \approx 1.8 \text{ ms}$.

Both types of spin qubits have the additional advantage that noise due to surrounding nuclear spins can be avoided by isotopically purifying the material.

Unfortunately, it is hardly possible to make these spin qubits interact with each other in a controlled and scalable fashion. They are very localized and their position in the host material is given and cannot be adjusted easily. Therefore, if during their production two qubits turn out to lie close to each other they will always be coupled, while if they are well-isolated from each other they will never interact. It is thus of high interest to propose a scheme to couple such atomistic qubits in a way that allows a high degree of control.

We fill this gap in the present work by proposing a setup to couple two spin qubits separated by a relatively large distance on the order of micrometers, see Fig. 1. The coupling is mediated via a ferromagnet with gapped excitations to which the spin qubits are coupled by magnetic dipole-dipole interaction. Since the ferromagnet is gapped only virtual magnons are excited but in order to obtain the sizable coupling one needs to tune the splitting of the qubit close to resonance with the gap of the ferromagnet. The on and off switching of the qubit-qubit interaction is therefore achieved by tuning qubits off resonance (see below). The resulting system is thus realizable with present state-of-the-art technologies. We point out that our analysis is not restricted to a precise type of spin qubit but is in principle applicable to any system that dipolarly interact with the spins of a ferromagnet. In particular, our proposal is also applicable to an electron spin localized in a semiconductor quantum dot, gate-defined or self-assembled. While other schemes exist to couple such qubits over large distances, none of them is applicable to atomistic qubits. The main novelty of our proposal is thus the possibility to also couple atomistic qubits that are of high technological relevance.

Before we proceed with the quantitative analysis, let us first give an intuitive picture of the qubit-qubit coupling. The coupling between two distant qubits is mediated via a coupler system. The relevant quantity of this coupler is its spin-spin susceptibility—in order to have a long-range coupling, a slowly spatially decaying susceptibility is required. The dimensionality of the coupler plays an important role since, in general, it strongly influences the spatial decay of the susceptibility, which can be anticipated from purely geometric considerations. Furthermore, since the coupler interacts with the qubits via magnetic dipolar forces, we require that a large part of the coupler lies close to the qubits. To this end we immediately see that a dog-bone shape depicted in Fig. 1 satisfies these two requirements—strong dipolar coupling to the qubits and slow spatial, practically 1D, susceptibility decay between the qubits.
FIG. 1. The schematics of the ferromagnetic coupler setup. The orange dog-bone shape denotes the ferromagnet that is coupled via magnetic dipole interaction to spins of nearby quantum dots (red sphere with green arrow). The ferromagnet is assumed to be a monodomain and its magnetization is denoted by blue arrows (\( \vec{M} \)) that can take arbitrary orientation. \( L \) is the length of the quasi-1D ferromagnetic channel that is approximately equal to the distance between the qubits. The shape of the ferromagnetic coupler is chosen such that it enables strong coupling to the spin-qubits while maintaining the spatially slowly decaying 1D susceptibility between the two discs.

II. MODEL

The system we consider consists of two spin-\( \frac{1}{2} \) qubits coupled dipolarly to the ferromagnet

\[
H = H_\sigma + H_F + H_1,
\]

where \( H_F \) is for the moment unspecified Hamiltonian of the dog-bone shaped ferromagnet that is assumed to be polarized along the \( x \)-axis. We first assume that the qubits are also polarized along the \( x \)-axis, \( H_\sigma = \sum_{i=1,2} \frac{\Delta_i}{2} \sigma_i^z \), while the ferromagnet disc axes are along \( z \), see Fig. 1. The magnetic dipole coupling between the ferromagnet and the spin-qubits can be written as

\[
H_1 = \frac{\mu_0 \mu_B I}{4\pi a^3} \sum_{i=1,2} \int dx S_\sigma \left[ \left( \frac{3A_i'x_i}{2} + \frac{3C_i''x_i}{4} \right) \sigma_i^z + h.c. \right]
+ \frac{1}{2} \left( B_i - 3C_i'x_i \right) \sigma_i^z
+S_i^+ \left[ \left( \frac{3}{8} C_{i,x} - \frac{3}{2} A_{i,y} x_i + \frac{3}{8} B_i \right) \sigma_i^+ \right]
- \frac{1}{8} \left( B_i - 3C_i' \right) \sigma_i^- + \left( \frac{3C_i''}{4} + \frac{3iA_i'}{2} \right) \sigma_i^- \right]
+ h.c.,
\]

where \( A_r, B_r, C_r \) are given by

\[
A_r = \frac{1}{a^3} \frac{r + r^+}{r^{-}}, \quad B_r = \frac{1}{a^3} \frac{1}{r^+} \left( 2 - \frac{3r + r^+}{r^-} \right),
\]

\[
C_r = \frac{1}{a^3} \left( r^+ \right)^2,
\]

with \( S_i^\pm = S_i^x \pm iS_i^z \) and lattice constant \( a \). Here we denote the real part of a complex number with prime and the imaginary part with double prime. The operator \( S_\sigma \) describes the spin of the ferromagnet at the position \( r \).

Next, we release the assumptions about the mutual orientation of the disc axes, the axes of polarization of the ferromagnet, and the direction of the qubits splitting and assume that these can take arbitrary directions. Now the interaction Hamiltonian reads

\[
H_1 = \frac{\mu_0 \mu_B I}{4\pi a^3} \sum_{i=1,2} \int dx S^\sigma_\sigma \left[ a_{i,r} \sigma_i^z + b_{i,r} \sigma_i^+ + h.c. \right] +
S_i^+ \left[ c_{i,r} \sigma_i^z + d_{i,r} \sigma_i^+ + e_{i,r} \sigma_i^- \right] + h.c.,
\]

where \( S_\sigma \) and \( \sigma_\sigma \) have, in general, different quantization axes. The expressions of the coefficients in Eq. (6) are now more complicated, nevertheless it is important to note that the integrals of these coefficients are experimentally accessible. The qubits can be used to measure the stray field of the ferromagnet which is given by \( B_s = \{ b_i, b_i', a_i \} \), where \( \{ a_1, \ldots, e_i \} = \frac{\mu_0 \mu_B I}{4\pi a^3} \int dr \{ a_i, \ldots, e_i \} \).

In order to measure the remaining coefficients, one needs to apply the magnetic field externally in order to polarize sequentially the ferromagnet along the two perpendicular directions to the ferromagnet easy axis. The coefficients are obtained then by measuring again the stray fields (with the aid of the qubits) which now are given by \( (d' + e', d'' - e', c'') \) and \( (d'' + e', d' - e', c') \). Furthermore, all the results that we are going to obtain for the qubit-qubit coupling as well as the estimates of the decoherence will depend only on the integrals of the coefficients, i.e., on \( \{ a_i, \ldots, e_i \} \) rather than \( \{ a_i, \ldots, e_i \} \).

A. Coherent coupling

We proceed to derive the effective qubit-qubit coupling by performing a Schrieffer-Wolff (SW) transformation. We assume that the excitations in the ferromagnet are gapped due to some magnetic anisotropy (e.g. shape-anisotropy), with the gap being denoted by \( \Delta_F \). This is important because when the qubit splitting \( \Delta \) is smaller than \( \Delta_F \), flipping the qubit spin cannot excite magnons in the ferromagnet, thus there are only virtual magnons excited via coupling to the qubits—otherwise such a coupling would lead to strong decoherence in the qubits. Due to the presence of the gap in the ferromagnet, its transversal susceptibility \( \chi_{\perp}(\omega, r) \) decays exponentially for \( \omega < \Delta_F \) with the characteristic length \( l_F \propto 1/\sqrt{\Delta_F - \omega} \), thus we take into account only terms with \( \omega \sim \Delta_F \), see Appendix. Straightforward application of lowest order SW transformation accompanied by tracing out the degrees of freedom of the ferromagnet yields the effective qubit-qubit coupling Hamiltonian

\[
H_{\text{eff}} = H_\sigma + \chi_{\perp}^{\text{1D}}(\Delta_1, L) e_1 \sigma_i^z \left( c_2 \sigma_2^z + d_2 \sigma_2^+ + e_2 \sigma_2^- \right)^{\dagger} + h.c.,
\]

where \( \chi_{\perp}^{\text{1D}} \) is the transverse susceptibility (i.e. transverse to the \( \vec{z} \) direction) of a quasi-1D ferromagnet, since we assumed a dog-bone shaped ferromagnet. We have neglected the longitudinal susceptibility \( \chi_{||} \) since it is
smaller by factor of $1/S$ compared to the transverse one and it is suppressed by temperature. It is readily seen from the above expression that in order to obtain a sizable coupling between the qubits we have to tune at least one of the qubits close to resonance, $\Delta_1 \sim \Delta_F$. This can be achieved by conveniently positioning the qubit such that the Zeeman splitting produced by the stray field of the ferromagnet is close to the excitation gap of the ferromagnet. The fine tuning can be then achieved by applying locally a small external magnetic field from a coil. The on resonance requirement offers an elegant way to switch on/off the coupling between the qubits. The idea is to tune the qubit splitting close to resonance to switch on the mediated interaction and to tune it off resonance to switch off the mediated interaction.[23]

For the sake of completeness, in the Appendix we present a detailed discussion of the effective coupling mediated by the dog-bone when the qubits are exchange coupled to the ferromagnet which requires a tunnel coupling between spin qubit and ferromagnet.

### B. Implementation of two-qubit gates

Two qubits interacting via the ferromagnet evolve according to the Hamiltonian $H_{\text{eff}}$, see Eq. (7). The Hamiltonian is therefore the sum of Zeeman terms and qubit-qubit interaction. These terms, by and large, do not commute, making it difficult to use the evolution to implement standard entangling gates. Nevertheless, if we assume that $\Delta_1 = \Delta_2$, $H_\sigma$ acts only in the subspace spanned by $\{|\uparrow, \uparrow\rangle, |\downarrow, \downarrow\rangle\}$ and the Zeeman splitting of the qubits is much larger than the effective qubit-qubit coupling, we can neglect the effect of $H_{\text{eff}}$ in this part of the subspace and approximate it by its projection in the space spanned by the vectors $\{|\uparrow, \downarrow\rangle, |\downarrow, \uparrow\rangle\}$

$$H'_{\text{eff}} = H_\sigma + \alpha(\sigma^x_1 \sigma^x_2 - \sigma^y_1 \sigma^y_2) + \beta(\sigma^x_1 \sigma^y_2 + \sigma^y_1 \sigma^x_2),$$  

(8)

where $\alpha = -8\text{Re}(e_i e^*_2)$ and $\beta = -4\text{Re}(d_1 e_2^* + d_2 e_1^*)$. Within this approximation, the coupling in $H'_{\text{eff}}$ and the Zeeman terms now commute. From here we readily see that the stray field components, $a_i, b_i$, as well as the coefficient $e_i$ do not determine the operation time of the two qubit gates—the operation time depends only on $d_1$ and $e_1$. To proceed we perform a rotation on the second qubit around the $z$-axis by an angle $\tan \theta = \beta/\alpha$ and arrive at the Hamiltonian

$$H''_{\text{eff}} = H_\sigma + \sqrt{\alpha^2 + \beta^2}(\sigma^x_1 \sigma^x_2 - \sigma^y_1 \sigma^y_2).$$  

(9)

We consider the implementation of the iSWAP gate

$$U_{\text{iSWAP}} = e^{i(\sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y)\pi/4},$$

which can be used to implement the CNOT gate.[23] The Hamiltonian $H'$ can be transformed to the desired form by changing the sign of $\sigma^x_1 \sigma^x_2$ term. This is achieved with the following sequence[27]

$$U_{\text{iSWAP}} = \sigma^y_1 e^{iH_{\text{eff}}t} e^{-iH''_{\text{eff}}t} \sigma^y_1,$$

(10)

where $t = \pi/(4\sqrt{\alpha^2 + \beta^2})$. When iSWAP is available, the CNOT gate can be constructed in the standard way[23]

$$U_{\text{CNOT}} = e^{-i\tilde{\sigma}_1^z \sigma_2^z} e^{i\tilde{\sigma}_1^x \sigma_2^x} U_{\text{iSWAP}} e^{-i\tilde{\sigma}_1^x \sigma_2^x} U_{\text{iSWAP}} e^{i\tilde{\sigma}_1^z \sigma_2^z}.$$  

(11)

Since $H''_{\text{eff}}$ is an approximation of $H_{\text{eff}}$, the above sequence will yield approximate CNOT, $U_{\text{CNOT}}'$, when used with the full the Hamiltonian. The success of the sequences therefore depends on the fidelity of the gates, $F(U'_{\text{CNOT}})$. Ideally this would be defined using a minimization over all possible states of two qubits. However, to characterize the fidelity of an imperfect CNOT it is sufficient to consider the following four logical states of two qubits,[23] $|+, \uparrow\rangle, |+, \downarrow\rangle, |-, \uparrow\rangle$, and $|-, \downarrow\rangle$. These are product states which, when acted upon by a perfect CNOT, become the four maximally entangled Bell states $|\Phi^+\rangle, |\Psi^+\rangle, |\Phi^-\rangle$, and $|\Psi^-\rangle$, respectively. As such, the fidelity of an imperfect CNOT may be defined,

$$F(U'_{\text{CNOT}}) = \min_{i\in\{+, -\}, j\in\{0, 1\}} |\langle i, j | U'_{\text{CNOT}} U_{\text{CNOT}} | i, j \rangle|^2.$$  

(12)

The choice of basis used here ensures that $F(U'_{\text{CNOT}})$ gives a good characterization of the properties of $U_{\text{CNOT}}'$ in comparison to a perfect CNOT, especially for the required task of generating entanglement. For realistic parameters, with the Zeeman terms two order of magnitude stronger than the qubit-qubit coupling, the above sequence yields fidelity for the CNOT gate of 99.976%.

To compare these values to the thresholds found in schemes for quantum computation, we must first note that imperfect CNOTs in these cases are usually modeled by the perfect implementation of the gate followed by depolarizing noise at a certain probability. It is known that such noisy CNOTs can be used for quantum computation in the surface code if the depolarizing probability is less than 1.1%.[23] This corresponds to a fidelity, according to the definition above, of 99.17%. The fidelities that may be achieved in the schemes proposed here are well above this value and hence, though they do not correspond to the same noise model, we can expect these gates to be equally suitable for fault-tolerant quantum computation.

### III. DECOHERENCE

In this section we study the dynamics of a single qubit coupled to the ferromagnet. In particular we want to answer the question whether the effective coupling derived in the previous section is coherent, i.e., whether the decoherence time solely due to the dipolar coupling to the ferromagnet is larger than the qubit operation time.

A ferromagnet has two types of fluctuations—longitudinal and transverse ones. The longitudinal noise stems from fluctuations of the longitudinal $S^z$ component (we recall that the ferromagnet is polarized along $\tilde{z}$), while the transverse one is related to fluctuations of $S^x$. In what follows we study these two noise sources
separately. The general noise model that describes both types of noise is then given by

\[ H = H_F + \frac{\Delta}{2} \sigma^z + \sigma^z \otimes X + \sigma^+ \otimes Y + \text{h.c.}, \quad (13) \]

where the ferromagnet operators \( X (Y) \) with zero expectation value couple longitudinally (transversally) to the qubit. The noise model given in Eq. (13) leads to the following relaxation and decoherence times within Born-Markov approximation\[^\text{30}\]

\[ T_1^{-1} = S_Y (\omega = \Delta), \quad (14) \]
\[ T_2^{-1} = \frac{1}{2} T_1^{-1} + S_X (\omega = 0), \quad (15) \]

where we defined the fluctuation power spectrum of an operator \( A \) in the following way, \( S_A (\omega) = \int dt e^{-i \omega t} \{ A^\dagger (t), A (0) \} \). In order to obtain estimates for the decoherence times we need a specific model for the ferromagnet Hamiltonian, herein taken to be a gapped Heisenberg model \( H_F = -J \sum_{r,r'} S_r \cdot S_{r'} + \Delta_F \sum_r S_r^z \), \( J \) being the exchange coupling and \( \Delta_F \) the excitation gap induced by some magnetic anisotropy.

A. Longitudinal noise

The power spectrum of longitudinal fluctuations is given by the following expression (see Appendix)

\[ S_\parallel^{3D} (\omega) = \frac{\alpha \sqrt{3} \omega}{2 \beta^2 D^3} e^{-\beta \Delta_F} \coth (\beta \omega / 2), \quad (16) \]

where \( D = 2JS \). We readily observe that the power spectrum is sub-ohmic, i.e., it diverges at low frequencies \( S_\parallel^{3D} (\omega) \propto 1 / \sqrt{\omega} \) —this is a direct consequence of the fact that longitudinal fluctuations are gapless. Due to this divergence, the perturbation theory (Born approximation) cannot be used when there is longitudinal coupling to the longitudinal noise. In order to deal with this singularity, we study transverse (Y) and longitudinal (X) coupling separately. The transverse coupling can be treated perturbatively, while for the longitudinal coupling we solve the problem exactly.

1. Transverse coupling to longitudinal noise

The part of the Hamiltonian that describes transverse coupling to the longitudinal noise reads

\[ H = H_F + \sigma^x \otimes \int dr b_r S_r^z + \text{h.c.} \quad (17) \]

Using Eq. (15) and the inequality

\[ S_\parallel^{3D} (\omega, r) \leq S_\parallel^{3D} (\omega, r = 0), \]

we obtain the relaxation time

\[ T_1^{-1} = \int \! dr dr' b_r b_{r'} S_\parallel^{3D} (\Delta, r - r') \]
\[ \leq \int \! dr dr' b_r b_{r'} S_\parallel^{3D} (\Delta, r = 0) = b^2 S_\parallel^{3D} (\Delta). \quad (18) \]

The above expression readily shows that relaxation time can be tailored arbitrarily small by choosing the ratio \( T / \Delta_F \) sufficiently small.

2. Longitudinal coupling to longitudinal noise

Here we consider only longitudinal coupling to longitudinal noise thus the Hamiltonian reads

\[ H = H_F + \sigma^z \otimes V, \quad (19) \]

with \( V = \int dr a_r S_r^z \). To simplify the problem further\[^\text{31}\] we substitute \( S_r^z \rightarrow \bar{S}_r^z \) since the latter is linear in magnon operators while the former is quadratic. When the final formula for the decoherence time is obtained we substitute back the power spectrum of \( S_r^z \) instead of \( \bar{S}_r^z \).

In order to study decoherence we have to calculate the following quantity\[^\text{33}\]

\[ \langle \sigma^- (t) \rangle = e^{i (t / \hbar) \langle \sigma^- (0) \rangle} \times \]
\[ \times \left\langle \bar{T} \exp \left( i \int_0^\epsilon Vdt' \right) \exp \left( i \int_0^\epsilon Vdt' \right) \right\rangle, \quad (20) \]

with \( \bar{T} \) the (anti-) time ordering operator. The average in the above expression can be evaluated using a cluster expansion\[^\text{29}\] and since the perturbation \( V \) is linear in the bosonic operators, only the second order cluster contributes. Therefore, the final exact result for the time-evolution of \( \sigma^- (t) \) reads

\[ \langle \sigma^- (t) \rangle = e^{i (t / \hbar) \langle \sigma^- (0) \rangle} e^{-i \frac{1}{2} \int_0^\epsilon \int_0^\epsilon S(t_2-t_1)dt_1dt_2}, \quad (21) \]

where \( S(t) = \langle [V(t), V(0)]_+ \rangle \). After performing the Fourier transformation we obtain

\[ \langle \sigma_- (t) \rangle = e^{i (t / \hbar) \langle \sigma_- (0) \rangle} \times \]
\[ \times \exp \left( -\frac{1}{2} \int \frac{d \omega}{2 \pi} S(\omega) \sin^2 (\omega t / 2) \frac{\sin^2 (\omega t / 2)}{\omega^2} \right). \quad (22) \]

Note that this expression is of exactly the same form as the one for a classical Gaussian noise. Now we substitute back \( S_r^z \rightarrow \bar{S}_r^z \)

\[ \langle \sigma^- (t) \rangle = e^{i (t / \hbar) \langle \sigma^- (0) \rangle} \times \]
\[ \times \exp \left( -\frac{1}{2} \int \frac{d \omega}{2 \pi} \int dr dr' a_r a_{r'} S_\parallel^{3D} (\omega, r - r') \frac{\sin^2 (\omega t / 2)}{\omega^2} \right). \quad (23) \]
For long times $t \gg \hbar/T$ the dynamics is of the form
\[ \langle \sigma^-(t) \rangle \sim e^{-2\sqrt{2\pi a^2} T^{5/2} e^{-\beta \Delta_F t^{3/2}}/(3D^3) + i \Delta t}, \] (24)
where we have used the inequality $S^{3D}_\parallel(\omega, \mathbf{r}) \leq S^{3D}_\parallel(\omega, \mathbf{r} = 0)$. Thus, this type of decoherence can be suppressed by choosing the ratio $T/\Delta_F$ sufficiently small.

**B. Transverse noise**

The power spectrum of transverse fluctuations of the ferromagnet is gapped and thus vanishes for $\omega < \Delta_F$ (see Appendix),
\[ S^{3D}_\perp(\omega) = \frac{S^{3D}(\omega - \Delta_F)}{D^{3/2}} \coth(\beta \omega/2), \quad \omega > \Delta_F. \] (26)

Since the transverse fluctuations are gapped and the precession frequency of the qubits is below the gap, this noise source does not contribute in the second order (Born approximation) because only virtual magnons can be excited. In this section we choose the quantization axes such that qubit splitting is along the $z$-axis, while the ferromagnet is polarized along the $x$-axis (see Fig. 1), this is done solely for simplicity and all the conclusions are also valid for the most general case. The Hamiltonian of the coupled system is of the form Eq. (13) with operators $X$ ($Y$)
\[ X = \frac{i}{2} \int d\mathbf{r} c_\mathbf{r}(S^+_\mathbf{r} - S^-_{\mathbf{r}}), \] (27)
\[ Y^+ = \frac{i}{8} \int d\mathbf{r} (a_\mathbf{r} S^+_\mathbf{r} + b_\mathbf{r} S^-_{\mathbf{r}}), \] (28)
with $S^\pm_{\mathbf{r}} = S^y_{\mathbf{r}} \pm i S^z_{\mathbf{r}}$ and the definitions
\[ a_\mathbf{r} = B_\mathbf{r} + 3C_\mathbf{r} - 6A_\mathbf{r}, \] (29)
\[ b_\mathbf{r} = B_\mathbf{r} + 3C_\mathbf{r} + 6A_\mathbf{r}, \] (30)
\[ c_\mathbf{r} = B_\mathbf{r} - 3A'_\mathbf{r}, \] (31)
where $A_\mathbf{r}, B_\mathbf{r}, C_\mathbf{r}$ are given by Eqs. (12), (13). To proceed further we perform the SW transformation on the Hamiltonian given by Eq. (13). We ignore the Lamb and Stark shifts and obtain the effective Hamiltonian
\[ H = H_F + \frac{\Delta}{2} \sigma^z + \sigma^z \otimes \tilde{X}_2 + \sigma^+ \otimes \tilde{Y}_2^- + \sigma^- \otimes \tilde{Y}_2^+, \] (32)
where
\[ \tilde{X}_2 = X_2 - \langle X_2 \rangle, \] (33)
\[ \tilde{Y}_2^\pm = Y_2^\pm - \langle Y_2^\pm \rangle. \] (34)
with the following notation
\[ X_2 = 4(Y_\Delta^2 Y^2 + Y^+ Y^-), \] (35)
\[ Y_2^+ = 2(Y_\Delta^2 X - X_0 Y^+), \] (36)
\[ X_\omega = \frac{i}{2} \int d\mathbf{r} r' \chi_\omega(\omega, \mathbf{r} - \mathbf{r}') (S^+_\mathbf{r} - S^-_{\mathbf{r}}), \] (37)
\[ Y_\omega^+ = -\frac{i}{8} \int d\mathbf{r} r' \chi_\omega(\omega, \mathbf{r} - \mathbf{r}') (a_\mathbf{r} S^+_\mathbf{r} + b_\mathbf{r} S^-_{\mathbf{r}}). \] (38)

The model given by Eq. (13) yields the following expressions for the relaxation and decoherence times
\[ T_1^{-1} = S_{2 \gamma}^- (\omega = \Delta), \] (39)
\[ T_2^{-1} = \frac{1}{2} T_1^{-1} + S_{\chi 2}^- (\omega = 0). \] (40)

After a lengthy calculation we obtain the following expressions for $T_1$ and $T_2$ (see Appendix for a detailed derivation)
\[ T_1^{-1} \leq \frac{B^4 S^2 \Delta_F^2}{2 D^3} \left( \frac{1}{\Delta_F} + \frac{1}{\Delta_F - \Delta} \right)^2 f \left( \frac{\Delta}{\Delta_F}, \beta \Delta_F \right), \] (41)
\[ T_2^{-1} \leq \frac{B^4 S^2 \Delta_F^2}{4 D^3} \left( \frac{1}{\Delta_F} + \frac{1}{\Delta_F - \Delta} \right)^2 f \left( \frac{\Delta}{\Delta_F}, \beta \Delta_F \right) + \frac{B^4 S^2 \Delta_F^2}{2 D^3 (\Delta_F - \Delta)^2} f(0, \beta \Delta_F), \] (42)
with the function $f(x, y)$ defined as follows
\[ f(x, y) = \int_1^\infty dz \sqrt{z - 1} \sqrt{z - x - 1} e^{yz - 1} e^{y(z-x) - 1}. \] (43)

It is important to note that $f(x, y) \propto e^{-y}$, i.e., we obtain, as before for the longitudinal noise, that the effect of transverse fluctuations can be suppressed by choosing the temperature much smaller than the excitation gap of the ferromagnet. As anticipated, Eq. (42) shows that the transverse noise becomes more important as the resonance is approached ($\Delta \sim \Delta_F$).

**IV. ESTIMATES**

In this section we give numerical estimates for the coherent coupling mediated by the ferromagnet and the associated decoherence times. These estimates are valid for both silicon-based and NV-center qubits.

Assume that the qubits lie close to the disc axis at a distance $h = 25 nm$ below the disc and that the ferromagnet has in-plane polarization (along $x$-axis). Assume the thickness of the disk to be $20 nm$, its radius to be $50 nm$, and a lattice constant of $4A$. In this case the stray field at the plane $x = 0$ is along $x$ and has a magnitude that can reach values up to $1T$ depending on the precise position of the qubit. Similarly, when the ferromagnet is polarized out-of-plane (along the $z$-axis), then the stray
field at position \( x = y = 0 \) is along \( z \) and can take values up to \( 1 \, T \). For these cases and when the qubit splitting is brought close to resonance, \( \Delta F - \Delta \approx 10^{-2} \mu eV \), we obtain operation times on the order of tens of nanoseconds when the qubits are separated by a distance of about 1 \( \mu m \). The decoherence times \( T_2 \) depend strongly on the ratio \( k_B T/\Delta F \) and the additional decoherence source can be made negligible if this ratio is sufficiently small. For a magnon gap \( \Delta F = 100 \mu eV \) (corresponding to a magnetic field of about 1 \( T \)) and a temperature \( T = 0.1 K \), we obtain decoherence times solely due to the coupling to the ferromagnet that are much bigger than the operation times and the typical decoherence times of the qubits.

V. CONCLUSIONS

We propose a scheme to coherently couple two atomistic qubits separated over distances on the order of a micron. We present a sequence for the implementation of the entangling CNOT gate and obtain operation times on the order of a few tens of nanoseconds. We show that there is a regime where all fluctuations of the ferromagnet are under control and the induced decoherence is non-detrimental: this is achieved when the temperature is smaller than the excitation gap of the ferromagnet. The main novel aspect of our proposal is its applicability to the technologically very important silicon qubits and NV-centers to which previous coupling methods do not apply.

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Appendix A: Holstein-Primakoff transformation

For the sake of completeness we derive in this Appendix explicit expressions for the different spin-spin correlators used in this work

\[
C^{\alpha \beta}(\omega, \mathbf{q}) = \langle S^\alpha_{\mathbf{q}}(\omega) S^\beta_{-\mathbf{q}}(0) \rangle .
\]

For this purpose, we make use of a Holstein-Primakoff transformation

\[
S^z_i = -n_i, \quad S^+ = \sqrt{2S} \sqrt{1 - \frac{n_i}{2S}} a_i, \quad \text{and} \quad S^- = (S^+)\dagger
\]

in the limit \( n_i \ll 2S \), with \( a_i \) satisfying bosonic commutation relations and \( n_i = a_i^{\dagger} a_i \). The creation operators \( a_i^{\dagger} \) and annihilation operators \( a_i \) satisfy bosonic commutation relations and the associated particles are called magnons. The corresponding Fourier transforms are straightforwardly defined as

\[
\epsilon_\mathbf{q} = \omega_\mathbf{q} + \Delta F = 4JS[3 - (\cos(q_x) + \cos(q_y) + \cos(q_z))] + \Delta F \text{ is the spectrum for a cubic lattice with lattice constant } a = 1 \text{ and the gap } \Delta F \text{ is induced by the external magnetic field or anisotropy of the ferromagnet.}
\]

Appendix B: Transverse correlators \( \langle S^+_\mathbf{q}(t) S^-_{-\mathbf{q}}(0) \rangle \)

Let us now define the Fourier transforms in the harmonic approximation

\[
S^\pm_\mathbf{q} = \frac{1}{\sqrt{N}} \sum_i e^{-i\mathbf{q}\mathbf{r}_i} S^\pm_i = \sqrt{\frac{2S}{N}} \sum_i e^{-i\mathbf{q}\mathbf{r}_i} a^\pm_i = \sqrt{2S} a^\pm_\mathbf{q},
\]

\[
S^\pm_{-\mathbf{q}} = \frac{1}{\sqrt{N}} \sum_i e^{i\mathbf{q}\mathbf{r}_i} S^\pm_i = \sqrt{\frac{2S}{N}} \sum_i e^{i\mathbf{q}\mathbf{r}_i} a^\pm_i = \sqrt{2S} a^\pm_{-\mathbf{q}}
\]

From this it directly follows that

\[
C^{+-}(t, \mathbf{q}) = \langle S^+_\mathbf{q}(t) S^-_{-\mathbf{q}}(0) \rangle = 2S \langle a^+_\mathbf{q}(t) a^-_{-\mathbf{q}} \rangle = 2S e^{i\epsilon_\mathbf{q}t} n_\mathbf{q},
\]

with \( \epsilon_\mathbf{q} \approx Dq^2 + \Delta F \) in the harmonic approximation.

The Fourier transform is then simply given by

\[
C^{+-}(\omega, \mathbf{q}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{-i\omega t} C^{+-}(t, \mathbf{q})
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt e^{i(\epsilon_\mathbf{q} - \omega)t} 2S n_\mathbf{q}
\]

\[
= \sqrt{2\pi} 2S \delta(\epsilon_\mathbf{q} - \omega) \frac{1}{e^{\beta\omega} - 1}.
\]

The corresponding correlator in real space is then simply given by \((q := |\mathbf{q}|)\)

\[
C^{+-}(\omega, \mathbf{r}) = \frac{1}{(2\pi)^3/2} \int d\mathbf{q} e^{i\mathbf{q}\mathbf{r}} C^{+-}(\omega, \mathbf{q})
\]

\[
= \frac{\sqrt{2\pi}}{(2\pi)^3/2} 2S \frac{1}{e^{\beta\omega} - 1} \int d\mathbf{q} \delta(D\mathbf{q}^2 + \Delta F - \omega) e^{i\mathbf{q}\mathbf{r}}
\]

\[
= \frac{2S}{e^{\beta\omega} - 1} \int_0^1 dq dq x^2 \delta(D\mathbf{q}^2 + \Delta F - \omega) e^{iqx} x
\]

\[
= \frac{4S}{r} \frac{1}{e^{\beta\omega} - 1} \int_0^\infty dq q^2 \delta(D\mathbf{q}^2 + \Delta F - \omega) \sin(qr).
\]

Let us now perform the following substitution

\[
y = Dq^2,
\]

\[
B5
\]
which gives for \( \omega > \Delta_F \)

\[
C^+(\omega, r) = \frac{4S}{2D(e^{\beta \omega} - 1)} \int_0^\infty dy \delta(y + \Delta_F - \omega) \times \\
\times \sin \left( \frac{y}{D} \right) \\
= \frac{2S}{D} \frac{1}{e^{\beta \omega} - 1} \sin \left( \frac{(\omega - \Delta_F)/D}{r} \right). \tag{B6}
\]

We remark that

\[ C^+(\omega, r) = 0, \quad \omega < \Delta_F. \tag{B7} \]

We note the diverging behavior of the above correlation function for \( \Delta_F = 0 \) and \( \omega \to 0 \), namely

\[
\frac{1}{e^{\beta \omega} - 1} \frac{\sin \left( \frac{\sqrt{D}r}{r} \right)}{r} \to \frac{1}{\sqrt{D} \beta} \frac{1}{\sqrt{\omega}}. \tag{B8}
\]

Similarly, it is now easy to calculate the corresponding commutators and anticommutators. Let us define

\[
S_\perp(t, q) := \frac{1}{2} \{ S^+_{q}(t), S^-_{q}(0) \}. \tag{B9}
\]

It is then straightforward to show that

\[
S_\perp(t, q) = Se^{i\omega t}(1 + 2n_q), \tag{B10}
\]

and therefore

\[
S_\perp(\omega, q) = \frac{S}{\sqrt{2\pi}} \int_0^\infty e^{i(\epsilon_q - \omega)t} (1 + 2n_q) \\
= \frac{S\sqrt{2\pi}}{\sqrt{D}} \delta(\epsilon_q - \omega) \left( 1 + 2 \frac{1}{e^{\beta \omega} - 1} \right). \tag{B11}
\]

Following essentially the same steps as the one performed above, we obtain the 3D real space anticommutator for

\[
S_{3D}^\perp(\omega, q) = S \coth(\beta\omega/2) \times \\
\times \int_{-1}^1 \int_0^\infty dxdqq^2 e^{iqr} \delta(\epsilon_q - \omega) \\
= \frac{S}{D} \coth(\beta\omega/2) \frac{\sin \left( \frac{(\omega - \Delta_F)/D}{r} \right)}{r}. \tag{B13}
\]

Let us now finally calculate the transverse susceptibility defined as

\[
\chi_\perp(t, q) = -i\theta(t) \{ S^+_{q}(t), S^-_{q}(0) \}. \tag{B14}
\]

As before, in the harmonic approximation, one finds

\[
\chi_\perp(t, q) = i\theta(t) 2Se^{i\omega t}. \tag{B15}
\]

In the frequency domain, we then have

\[
\chi_\perp(\omega, q) = \frac{2iS}{\sqrt{2\pi}} \int_0^\infty dt e^{i(\epsilon_q - \omega)t - \eta t} \\
= -\frac{2S}{\sqrt{2\pi}} \frac{1}{\epsilon_q - \omega + i\eta}, \tag{B16}
\]

and thus in the small \( q \) expansion

\[
\chi_\perp(\omega, q) = -\frac{2S}{\sqrt{2\pi}} \frac{1}{Dq^2 + \Delta_F - \omega + i\eta}. \tag{B17}
\]

In real space, for the three-dimensional case, we obtain

\[
\chi_{3D}^\perp(\omega, r) = \frac{2S}{\sqrt{2\pi}} \frac{2\pi}{(2\pi)^{3/2}} \int_0^\infty dq \frac{1}{Dq^2 + \Delta_F - \omega + i\eta} e^{iqr} \\
= -\frac{2S}{\sqrt{2\pi}} \frac{2\pi}{(2\pi)^{3/2}} \frac{1}{D} \int_0^\infty dq \frac{1}{Dq^2 + \Delta_F - \omega + i\eta} \sin(qr). \tag{B18}
\]

Making use of the Plemelj formula we obtain for \( \omega > \Delta_F \)

\[
\chi_{3D}^\perp(\omega, r) = -\frac{2S}{\sqrt{2\pi}} \frac{2\pi}{(2\pi)^{3/2}} \frac{1}{D} \int_0^\infty dq \frac{1}{Dq^2 + \Delta_F - \omega + i\eta} \sin(qr) \\
= -\frac{2S}{\sqrt{2\pi}} \frac{2\pi}{(2\pi)^{3/2}} \frac{1}{D} \int_0^\infty dq \frac{q}{Dq^2 + \Delta_F - \omega} \sin(qr) + \frac{2S}{\sqrt{2\pi}} \frac{2\pi}{(2\pi)^{3/2}} \frac{1}{D} \int_0^\infty dq \delta(Dq^2 + \Delta_F - \omega) \sin(qr) \\
= -\frac{S}{D} \cos(r\sqrt{(\omega - \Delta_F)/D}) \frac{1}{r} + i \frac{S}{2D} \frac{\sin \left( \sqrt{(\omega - \Delta_F)/Dr} \right)}{r}. \tag{B19}
\]

It is worth pointing out that the imaginary part of the susceptibility vanishes,

\[
\chi_{3D}^\perp(\omega, r)' = 0, \quad \omega < \Delta_F, \tag{B20}
\]
and therefore the susceptibility is purely real and takes the form of a Yukawa potential

$$\chi^{3D}_{\perp}(\omega, r) = -\frac{S}{D} \frac{e^{-r/l_F}}{r}, \quad \omega < \Delta_F, \quad (B21)$$

where $l_F = \frac{D}{\Delta_F - \omega}$.

Note also that the imaginary part of the transverse susceptibility satisfies the well-know fluctuation-dissipation theorem

$$S^{3D}_{\perp}(\omega, r) = \coth(\beta \omega/2) \chi^{3D}_{\perp}(\omega, r)'' . \quad (B22)$$

In three dimensions the susceptibility decay as $1/r$, where $r$ is measured in lattice constants. For distances of order of $1 \mu m$ this leads to four orders of magnitude reduction.

For quasi one-dimensional ferromagnets such a reduction is absent and the transverse susceptibility reads

$$\chi^{1D}_{\perp}(\omega, r) = -\frac{S}{D} l_F e^{-r/l_F}, \quad \omega < \Delta_F, \quad (B23)$$

where $l_F$ is defined as above and the imaginary part vanishes as above, i.e.,

$$\chi^{1D}_{\perp}(\omega, r)'' = 0, \quad \omega < \Delta_F. \quad (B24)$$

Similarly for $\omega > \Delta_F$ we have

$$\chi^{1D}_{\perp}(\omega, r) = \frac{\sin(\sqrt{(\omega - \Delta_F)/D_F})}{\sqrt{D_F(\omega - \Delta_F^2)}}, \quad (B25)$$

and

$$\chi^{1D}_{\perp}(\omega, r)'' = \frac{S}{2D} \frac{\sqrt{D_F}}{\omega - \Delta_F} \cos\left(\frac{\sqrt{(\omega - \Delta_F)/D_F}}{D_F}\right). \quad (B26)$$

**Appendix C: Longitudinal correlators $\langle S^z_q(t)S^z_{-q}(0) \rangle$**

The longitudinal susceptibility reads

$$\chi^z(q, \omega, t) = -i \theta(t) [S^z_q(t), S^z_{-q}(0)]$$

$$= -\theta(t) \frac{1}{N} \sum_{q', q''} e^{i(q'q'')/\omega} \langle [a^\dagger_{q'} a_{q}, a^\dagger_{q''} a_{-q'}] \rangle . \quad (C1)$$

Applying Wick’s theorem and performing a Fourier transform, we obtain the susceptibility in frequency domain

$$\chi^z(\omega, q) = -\frac{1}{N} \sum_k \frac{n_k - n_{k+q}}{\omega - \epsilon_{k+q} + \epsilon_k + i\eta}, \quad (C2)$$

where $n_k$ is the magnon occupation number given by the Bose-Einstein distribution

$$n_k = \frac{1}{e^{\beta \epsilon_k} - 1}, \quad (C3)$$

where $\epsilon_k$ is again the magnon spectrum ($\epsilon_k = \omega_k + \Delta_F \approx Dk^2 + \Delta_F$ for small $k$). Note that the longitudinal susceptibility is proportional to $1/S$, due to the fact that $\epsilon_k - \epsilon_{k+q} = \omega_k - \omega_{k+q} \propto S$.

Since we are interested in the decoherence processes caused by the longitudinal fluctuations, we calculate the imaginary part of $\chi^z(\omega, q)$ which is related to the fluctuations via the fluctuation-dissipation theorem. Performing a small $q$ expansion and assuming without loss of generality $\omega > 0$, we obtain for the imaginary part

$$\chi^z(\omega, q)'' = \frac{1}{(2\pi)^3} \int dk (n_k - n_{k+q}) \delta(\omega_k - \omega_{k+q} + \omega)$$

$$= \frac{1}{4\pi} \int_0^\infty dk k^2 \int_{-1}^1 dx \left( \frac{1}{e^{\beta(\omega + Dq^2) - 1}} - \frac{1}{e^{\beta(\omega + \Delta_F + Dk^2) - 1}} \right) \delta(\omega - Dq^2 - 2Dkq)$$

$$= \frac{1}{4\pi} \int_0^\infty dk k^2 \int_{-1}^1 dx \left( \frac{1}{e^{\beta(\omega + Dq^2) - 1}} - \frac{1}{e^{\beta(\omega + \Delta_F + Dk^2) - 1}} \right) \left( \frac{k^2}{2Dq^2} \right)$$

$$= \frac{1}{4\pi} \int_{-1}^1 dx \left( \frac{\omega - Dq^2}{2Dq^2} \right)^2 \left( \frac{1}{e^{\beta(\omega + \Delta_F + Dk^2) - 1}} - \frac{1}{e^{\beta(\omega + \Delta_F + Dk^2)/2Dq^2) - 1}} \right)$$

$$= \frac{1}{4\pi} \int_0^1 dx \left( \frac{\omega - Dq^2}{2Dq^2} \right)^2 \left( \frac{1}{e^{\beta(\omega + \Delta_F + Dk^2)/2Dq^2) - 1}} - \frac{1}{e^{\beta(\omega + \Delta_F + Dk^2)/2Dq^2) - 1}} \right). \quad (C4)$$

Next, since we are interested in the regime where $\omega \gg T$ (and thus $\beta \omega \gg 1$), we have $n_k \gg n_{k+q}$. Further-
more, we approximate the distribution function \( n_k = e^{-\beta(\Delta_F + \omega_k)} \) (this is valid when \( \beta \omega_k \ll 1 \)) and arrive at the following expression

\[
\chi^{3D}(\omega, q)'' = \frac{1}{4\pi} \int_0^1 dx \frac{1}{2Dq} \left( \omega - Dq^2 \right)^2 \left( e^{-\beta(\Delta_F + D(\omega/Dq)^2)} \right) \frac{e^{-\beta(\Delta_F + D(\omega/Dq)^2)}}{1 - e^{-\beta \Delta_F + \beta D(\omega/Dq)^2}} \]

\[
= -e^{-\beta \Delta_F - \beta \Delta_F} \frac{4}{3D^2q} Ei \left( e^{-\beta \Delta_F} + \frac{1}{4} \left( -4 - \beta Dq^2 + 2\beta \omega - \frac{\beta \omega^2}{Dq^2} \right) \right), \quad (C5)
\]

where \( Ei(z) \) is the exponential integral function. We also need the real space representation obtained after inverse Fourier transformation,

\[
\chi^{3D}(\omega, r)'' = \sqrt{\frac{2}{\pi D}} \int_0^\infty dq \chi^{3D}(\omega, q)'' \sin(qr). \quad (C6)
\]

In order to perform the above integral we note that the imaginary part of the longitudinal susceptibility, given by Eq. (C5), is peaked around \( q = \sqrt{\omega/D} \) with the width of the peak \((1/\sqrt{3D})\) much smaller than its position in the regime we are working in \((\omega \gg T)\). For \( r = 0 \), the integration over \( q \) can be then performed approximately and yields the following expression

\[
\chi^{3D}(\omega, r = 0)'' = \frac{\sqrt{\pi} e^{-\beta \Delta_F - \beta \Delta_F/2}}{2\beta^2 D^3} \left( e^{-\beta \Delta_F + \beta \Delta_F/2} \right. \]

\[
- e \sqrt{\pi} \sqrt{e^{-\beta \Delta_F} - 1} \left. \times \text{Erfc}(e^{-\beta \Delta_F/2} \sqrt{e^{-\beta \Delta_F} - 1}) \right) \sqrt{\beta \omega}, \quad (C7)
\]

where \( \text{Erfc}(z) \) denotes the complementary error function. It is readily observed from the above expression that the longitudinal fluctuations are exponentially suppressed by the gap. Assuming that \( \Delta_F \gg T \), we obtain the following simplified expression

\[
\chi^{3D}(\omega, r = 0)'' = \frac{\sqrt{\pi} - e^{-\beta \Delta_F} \text{Erfc}(1)}{2\beta^2 D^3} e^{-\beta \Delta_F} \sqrt{\beta \omega}. \quad (C8)
\]

We observe that, since \( J(\omega) = \chi^{3D}(\omega, r)'' \), the longitudinal noise of the ferromagnet is—as the transverse one—sub-ohmic.

It is interesting to obtain the behavior of the longitudinal susceptibility in the opposite limit, when \( \beta \omega \ll 1 \). In this limit, the difference of the two Boltzmann factors in Eq. (C4) can be expanded to the lowest order in the small quantity \( \beta \omega \),

\[
\chi^{3D}(\omega, q)'' = \int_0^1 dx \frac{1}{8\pi Dq} \left( \omega - Dq^2 \right)^2 \frac{e^{-\beta(\Delta_F + D(\omega/Dq)^2)}}{\beta \omega} \frac{1}{\text{ch} \left( \beta \Delta_F + \beta D(\omega/Dq)^2 - 1 \right)} \]

\[
= \frac{\omega}{16\pi D^2 q^2} \left( e^{\beta \Delta_F + \frac{\beta \omega Dq^2}{4Dq^2} - 1} \right). \quad (C9)
\]

In order to calculate the Fourier transform to real space, we note that for \( \beta \omega \ll 1 \) the denominator of the above expression depends only weakly on \( \omega \), thus we ignore this dependence and obtain the Fourier transform for \( r = 0 \)

\[
\chi^{3D}(\omega)'' = \frac{\ln(1 + n_k = 0)}{16\pi \beta D^3} \omega. \quad (C10)
\]

The above formula shows that the longitudinal noise of a ferromagnet at high temperatures \((\beta \omega \ll 1)\) behaves as ohmic rather than sub-ohmic bath.

Next we calculate the longitudinal fluctuations for the case of a quasi-one-dimensional ferromagnet \((\Delta_F \gg T)\) and obtain

\[
\chi^{1D}(\omega, r = 0)'' = \frac{1}{4\pi} \int_{-\infty}^\infty dk \int_{-\infty}^\infty dq \left( \frac{1}{e^{\beta(\Delta_F + Dk^2)} - 1} - \frac{1}{e^{\beta(\omega + \Delta_F + Dk^2)} - 1} \right) \delta(\omega - Dq^2 - 2Dkq) \]

\[
= \int_{-\infty}^\infty dk \left( \frac{1}{1 - e^{-\beta Dk^2}} \frac{1}{D\sqrt{\beta \omega}} \right) e^{-\beta \Delta_F} \frac{1}{D\sqrt{\beta \omega}} \]

\[
= \frac{\gamma}{D\sqrt{\beta \omega}} e^{-\beta \Delta_F}, \quad (C11)
\]

where \( \gamma \) is a numerical factor of order unity.

Note that \( S^{1D}(\omega, r) \) is defined through the fluctuation dissipation theorem as

\[
S^{1D}(\omega, r) = \text{coth}(\beta \omega/2) \chi^{1D}(\omega, r)'' . \quad (C12)
\]
Appendix D: Exchange coupling to the ferromagnet

a. Exchange coupling

The Hamiltonian we consider is of the following form

\[ H = H_F + H_\sigma + A \sum_i \sigma_i \cdot S_i , \]

where \( A \) is the exchange coupling constant between the qubit spins and the ferromagnet. The ferromagnet is assumed to be below the Curie temperature with the magnetization pointing along the out-of-plane \( z \)-direction. The qubit Hamiltonian is assumed to be without splitting initially, that is \( H_\sigma^{(0)} = 0 \). Nevertheless, since the ferromagnet is in the ordered phase, there exists a first order effect due to coupling to the ferromagnet which gives rise to the term of the form \( A \sum_i \sigma_i^z \langle S_i^z \rangle \). Such a splitting is undesirable if one is interested in coherent interaction—we remedy this by coupling the spins to another ferromagnet, albeit with anti-parallel magnetization\(^\texttt{19}\). Since we allow for some misalignment between orientation of the magnetization of the two ferromagnets, the final Hamiltonian for the qubits in the spin space after taking into account the first order corrections due to coupling to the ferromagnet reads

\[ H_\sigma = \frac{1}{2} \Delta \sum_i \sigma_i^z . \]  

The splitting in the \( x \)-direction of the qubit (or equivalently along the \( y \)-direction) is beneficial since it reduces decoherence due to longitudinal noise of the ferromagnet: the effect of such noise spectrum can significantly influence decoherence times for the case of no splitting of the qubit because the longitudinal noise is gapless.

b. Coherent coupling

We proceed with the derivation of an effective two-spin interaction Hamiltonian for \( A \ll J \) by employing a perturbative Schrieffer-Wolff transformation\(^\texttt{23}\) up to the second order

\[ H_{\text{eff}} = H_\sigma + A^2 \frac{1}{8} \chi_\perp(\Delta)(2\sigma_y^i \sigma_y^j + \sigma_z^i \sigma_z^j + \sigma_z^i \sigma_z^j) , \]

where we introduced the notation \( \chi_\perp(\omega) = \chi_\perp(\omega, L) \) \((L = |r_2 - r_1|)\) and \( \chi_\perp(\omega, r) \) is the transverse real space spin susceptibility of the ferromagnet. Note that we have neglected \( \chi_\perp^{3D}(-\Delta) \) and \( \chi_\perp^{3D}(0) \) in comparison to \( \chi_\perp^{1D}(\Delta) \), as well as the longitudinal susceptibility \( \chi_\parallel \) since it is smaller by factor of \( 1/S \) compared to the transverse one and it is suppressed by temperature. The real space transverse susceptibility of the 3D ferromagnet is given by

\[ \chi_\perp^{3D}(\omega, r) = -\frac{S}{D} e^{-r/l_F} , \quad \omega < \Delta_F , \]

where \( \Delta_F \) is the gap induced via applied external magnetic field or due to internal anisotropy of the ferromagnet, \( l_F = \sqrt{\frac{D}{\Delta_F}} \) and \( D = 2JS \). In what follows, we assume that the external gap is always larger than the qubit splitting, \( \Delta < \Delta_F \), as this ensures that the transverse noise is not contributing to decoherence in second order since transverse noise is related to the vanishing imaginary part of the transverse susceptibility, \( \chi_\perp(\omega'') = 0 \) \((\omega < \Delta_F)\). The spatial dependence of the effective two spin coupling given by Eq. (D4) is of Yukawa type due to presence of the external gap.\(^\texttt{20}\) We assume a realistic tunnel coupling to the ferromagnet of 100\(\mu\text{eV}\),\(^\texttt{25,27,28}\) the Curie temperature of 550K [as for example for yttrium iron garnet (YIG)] and a gap of \( \Delta_F = 100\mu\text{eV} \), and the qubit splitting close to the resonance \( \Delta_F - \Delta = 3 \times 10^{-3}\mu\text{eV} \) (corresponding to a magnetic field of about \( B = 60\mu\text{T} \)) we obtain for the qubit-qubit coupling strength a value on the order of \( 4 \times 10^{-11}\text{eV} \) for a lattice constant of about 4\(\AA\). This coupling strength gives rise to the operation times of 5\(\mu\text{s}\)—significantly below the relaxation and decoherence times of the spin qubit, \( T_1 = 1.\text{ns} \) and \( T_2 > 200\mu\text{sec} \) respectively. Furthermore, the error threshold—defined as the ratio between the two-qubit gate operation time to the decoherence time—we obtain with such an operation time is about \( 10^{-2} \), which is good enough for implementing the surface code error correction.\(^\texttt{20}\) Here we used \( T_2 \) instead of \( T_2 \) since spin-echo can be performed together with two-qubit gates.\(^\texttt{11}\) Alternatively, the decoherence time of GaAs qubits can be increased without spin-echo by narrowing the state of the nuclear spins\(^\texttt{12,13}\).

The dimensionality of the ferromagnet plays an important role—if we assume 10\(\mu\text{m}\) width of the trench where the ferromagnet is placed, then, for energies below 0.1meV, the ferromagnet behaves as quasi one-dimensional (1D). In this case we obtain

\[ \chi_\perp^{1D}(\omega, r) = -\frac{S}{D} l_F e^{-r/l_F} , \quad \omega < \Delta_F , \]

wherefrom it is evident that at distances \( r \lesssim l_F \) the susceptibility of a quasi-1D ferromagnet is practically constant in contrast to the 3D case, where a 1/r decay is obtained, see Eq. (D1). Additionally, we require \( l_F \lesssim D/(AS) = 2J/A \) for the perturbation theory to be valid. Thus, for the same parameters as above, but without the need to tune very close to the resonance (we set herein \( \Delta_F - \Delta = 0.5\mu\text{eV} \), corresponding to about \( B = 10\text{mT} \)) a coupling strength of \( 10^{-8}\text{eV} \) is obtained.

For 1D case there is yet another rather promising possibility—to use magnetic semiconductors.\(^\texttt{32}\) These materials are characterized by a particularly low Curie temperature of 30K or below\(^\texttt{33}\) and the distance between the ions that are magnetically ordered via RKKY interaction is about 10–100\(\mu\text{m}\). Such a large lattice constant is very beneficial for the long range coupling—if we take the lattice constant to be 10\(\mu\text{m} \), the coupling to the ferromagnet \( A = 15\mu\text{eV} \) and the qubit splitting close to resonance \( \Delta_F - \Delta = 0.5\mu\text{eV} \), corresponding to about
$B = 10 \text{mT}$, the qubit-qubit coupling becomes of the order of $1 \mu \text{eV}$. Such a coupling strength in turn leads to an error threshold on the order of $10^{-8}$. Therefore, even the standard error correction protocol can be used in this case.

c. Derivation of the effective Hamiltonian (exchange coupling)

Here we give a detailed derivation of the qubit-qubit effective Hamiltonian. As stated above, the total Hamiltonian of the system reads

$$H = H_F + H_\sigma + A \sum_i \left( \frac{1}{2}(\sigma_i^+ S^-_r + \sigma_i^- S^+_r) + \sigma_i^z S^z_r \right),$$

where we identify the main part as $H_0 = H_F + H_\sigma$ and the small perturbation as the exchange coupling $V = A \sum_i \sigma_i \cdot S^z_r$. The Hamiltonian of the ferromagnet reads $H_F = -J \sum_{(r,r')} S_r \cdot S_{r'}$, while the Hamiltonian for the two distant qubits is $H_\sigma = \frac{3}{2} \sum_{i=1,2} \sigma_i^z$.

The second order effective Hamiltonian is given by

$$H_{\text{eff}}(2) = H_0 + U,$$

where

$$U = -\frac{i}{2} \lim_{\eta \to 0^+} \int_0^\infty dt e^{-\eta t} [V(t), V],$$

and

$$V(t) = e^{i H_0 t} V e^{-i H_0 t}.$$  

We have

$$\sigma_i^z(t) = \frac{1 + \cos(\Delta t)}{2} \sigma_i^z + \frac{1 - \cos(\Delta t)}{2} \sigma_i^z - i \sin(\Delta t) \sigma_i^z,$$

and

$$\sigma_i^z(t) = \sigma_i^z(t)^\dagger.$$  

Recalling that the $zz$ susceptibility can be neglected and that only the transverse susceptibility contributes, we obtain the following result from Eq. (D7), $U = \lim_{\eta \to 0^+} \int_0^\infty dt e^{-\eta t} \sum_{ij} U_{ij}$

$$U_{ij} = -\frac{i A^2}{8} \left( [\sigma_i^-(t) S^+_r(t), \sigma_j^+ S_r^-(t)] + \text{h.c.} \right)$$

Finally, by rewriting $\cos(\Delta t) = \frac{e^{i \Delta t} + e^{-i \Delta t}}{2}$, $\sin(\Delta t) = \frac{e^{i \Delta t} - e^{-i \Delta t}}{2i}$, and using the definition of the real space transverse spin susceptibility

$$\chi(\omega, r_i - r_j) = -i \lim_{\eta \to 0^+} \int_0^\infty dt e^{-(\omega + \eta) t} \left( [S^+_r(t), S^-_r(t)] \right),$$

we obtain by inserting Eq. (D8) into Eq. (D9)

$$U = \frac{A^2}{8} \sum_{ij} \left( \frac{\chi_+(0)}{2} + \frac{\chi_-(\Delta)}{4} + \frac{\chi_-(\Delta)}{4} \right) \sigma_i^- \sigma_j^+$$

$$+ \frac{A^2}{8} \sum_{ij} \left( \frac{\chi_+(0)}{2} - \frac{\chi_-(\Delta)}{4} + \frac{\chi_-(\Delta)}{4} \right) \sigma_i^+ \sigma_j^-$$

$$- \frac{A^2}{8} \sum_{ij} \chi_+(\Delta) - \frac{\chi_-(\Delta)}{2} \sigma_i^z \sigma_j^z + \text{h.c.} \quad (D11)$$

Since the decay length of the susceptibility $\chi_\omega, r_i - r_j$ is large only close to the resonance, $\Delta_F \sim \Delta$, we can simplify the above equation by neglecting $\chi_-(\Delta, r)$ and $\chi_+(0, r)$ in comparison to $\chi(\Delta, r)$ which is assumed to be close to the resonance. Within this approximation we arrive at Eq. (D3) of the main text.

Appendix E: Fourth order contributions to decoherence

In this section we determine the effect of the transverse noise in the lowest non-vanishing order due to coupling dipolarly to the ferromagnet. Here we choose quantizations axes such that the qubit splitting is along the z-axis, while the ferromagnet is polarized along $x$-axis. The Hamiltonian of the coupled system reads

$$H = H_F + \frac{\Delta}{2} \sigma^z + \sigma^z \otimes X + \sigma^+ \otimes Y^- + \sigma^- \otimes Y^+,$$

where the operator $X$ ($Y$) that couples longitudinally (transversally) to the qubit is linear in the transverse operators of the ferromagnet

$$X = \frac{i}{2} \int drc_r (S^+_r - S^-_r),$$

$$Y^+ = -\frac{i}{8} \int drc_r (a_r S^+_r + b_r S^-_r),$$

with $S^z_r = S^z_r \pm i S^x_r$ and the definitions of the coefficients

$$a_r = B_r + 3C_r - 6A_r,$$

$$b_r = B_r + 3C_r + 6A_r,$$

$$c_r = B_r - 3A_r$$

$$A_r = \frac{1}{a^2} r^+ r^-,$$

$$C_r = \frac{1}{a^2} (r^+)^2,$$

$$B_r = \frac{1}{a^2} \frac{1}{r^{5/2}} \left( 2 - \frac{3r^+ r^-}{r^2} \right).$$

To proceed further we perform the SW transformation on the Hamiltonian given by Eq. (E1). We ignore the Lamb and Stark shifts and obtain the effective Hamiltonian

$$H = H_F + \frac{\Delta}{2} \sigma^z + \sigma^z \otimes \tilde{X}_2 + \sigma^+ \otimes \tilde{Y}_2^- + \sigma^- \otimes \tilde{Y}_2^+,$$
where

\[
\dot{X}_2 = X_2 - \langle X_2 \rangle, \quad (E11)
\]

\[
\dot{Y}_2^\pm = Y_2^\pm - \langle Y_2^\pm \rangle, \quad (E12)
\]

with the following notation

\[
X_2 = 4(Y_2^+Y^- + Y^+Y^-), \quad (E13)
\]

\[
Y_\omega^+ = 2(Y_2^+X - X_0Y^+), \quad (E14)
\]

\[
X_\omega = \frac{i}{2} \int drr'\chi_{\perp}(\omega, r - r')(c_r(S^+_{r'} - S^-_{r'})), \quad (E15)
\]

\[
Y_\omega^- = -\frac{i}{8} \int drr'\chi_{\perp}(\omega, r - r')(a_rS^+_r + b_rS^-_r), \quad (E16)
\]

The model given by Eq. (E10) yields the following expressions for the relaxation and decoherence times

\[
T_1^{-1} = S_{\dot{X}_2}(\omega = \Delta), \quad (E17)
\]

\[
T_2^{-1} = \frac{1}{2} T_1^{-1} + S_{\dot{Y}_2}(\omega = 0), \quad (E18)
\]

where, again, \(S_A(\omega) = \int dt e^{-i\omega t}\{A^+(t), A(0)\}\).

After a lengthy calculation we obtain the expressions for \(S_{\dot{X}_2}(\omega = 0)\) and \(S_{\dot{Y}_2}(\omega = \Delta)\),

\[
S_{\dot{X}_2}(0) = \frac{1}{128} \int d\nu d\tau r_3r_4r_5r_6C^+(-\nu, r_3 - r_4)C^+(-\nu, r_1 - r_2) \times
\]

\[
((ar_5, br_5^*) + (ar_4, br_4^*) + (ar_3, br_3^*)) \chi_{\perp}(\Delta, r_1 - r_3) \chi_{\perp}(\Delta, r_2 - r_6) \chi_{\perp}(\Delta, r_4 - r_5) \chi_{\perp}(\Delta, r_5 - r_6)
\]

\[
S_{\dot{Y}_2}(\Delta) = \frac{1}{64} \int d\nu d\tau r_3r_4r_5r_6C^+(-\nu, r_3 - r_4)C^+(-\nu, r_1 - r_2) \times
\]

\[
((cr_1c_1^* + cr_2c_2^*) + (cr_3c_3^*) \chi_{\perp}(0, r_2 - r_3) \chi_{\perp}(0, r_4 - r_6) \chi_{\perp}(0, r_5 - r_6)
\]

In order to obtain the estimates for relaxation and decoherence time, we consider the ferromagnet to be in shape of infinite plane. Furthermore, we are not aiming at performing an exact evaluation of the integrals in Eqs. (E20)-(E21), but rather at finding the lower bound for the relaxation and decoherence times. To this end we note that \(|C^+(-\omega, r = r')| \leq |C^+(-\omega, r = 0)|\) and arrive at the following inequalities

\[
S_{\dot{X}_2}(0) \leq \frac{B^4}{8(\Delta_F - \Delta)^2} \int_{\Delta_F} \nu C^+(-\nu)^2, \quad (E21)
\]

\[
S_{\dot{Y}_2}(\Delta) \leq \frac{B^4}{8} \left( \frac{1}{\Delta_F} + \frac{1}{\Delta_F - \Delta} \right)^2 \times
\]

\[
\int_{\Delta_F + \Delta}^{\infty} d\nu C^+(-\nu)C^+(-\nu - \Delta), \quad (E22)
\]

where we used notation \(B = \int d\tau B_\tau\). Finally we arrive at the estimates for the relaxation and decoherence times

\[
T_1^{-1} \leq \frac{B^4S^2\Delta_F^2}{2D^3} \left( \frac{1}{\Delta_F} + \frac{1}{\Delta_F - \Delta} \right)^2 f \left( \frac{\Delta}{\Delta_F}, \beta \Delta_F \right), \quad (E23)
\]

\[
T_2^{-1} \leq \frac{B^4S^2\Delta_F^2}{4D^3} \left( \frac{1}{\Delta_F} + \frac{1}{\Delta_F - \Delta} \right)^2 f \left( \frac{\Delta}{\Delta_F}, \beta \Delta_F \right) + \frac{B^4S^2\Delta_F^2}{2D^3\left(\Delta_F - \Delta\right)^2} f \left( 0, \beta \Delta_F \right), \quad (E24)
\]

with the function \(f(x, y)\) defined as follows

\[
f(x, y) = \int_{1+x}^{\infty} dz \frac{\sqrt{z - x - 1}}{e^{yz - 1} - e^{yz - x} - 1}. \quad (E25)
\]

Assuming the same parameters as in the main text, we obtain decoherence times of about 0.5 hours, while
the relaxation time is on the order of 1000 hours. It is worth noting that this result depends sensitively on the relaxation time being on the order of 1000 hours. It is not difficult to see that the relaxation time is on the order of 1000 hours. It is not difficult to see that another possibility is to keep one of the qubits off resonance and then tuning the other one on and off.

Another possibility is to keep one of the qubits off resonance and then tuning the other one on and off. The ferromagnets does not need to cancel exactly. We only need to split the qubits around the zero to smaller that $\Delta F$.

Note that the z-component of the magnetization of both the ferromagnets does not need to cancel exactly. We only require the splitting along z to be smaller than $\Delta F$. For instance, if we assume a temperature of 4K, we obtain $T_1 \geq 200\mu s$ and $T_2 \geq 30\mu s$. Note that the z-component of the magnetization of both the ferromagnets does not need to cancel exactly. We only require the splitting along z to be smaller than $\Delta F$. For instance, if we assume a temperature of 4K, we obtain $T_1 \geq 200\mu s$ and $T_2 \geq 30\mu s$.

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