STABILITY AND BIFURCATION ANALYSIS OF FILIPPOV FOOD CHAIN SYSTEM WITH FOOD CHAIN CONTROL STRATEGY

SOLIMAN A. A. HAMDALLAH*
School of Mathematics and Information Science, Shaanxi Normal University
Xi’an 710062, China
Department of Mathematics, Faculty of Science, Al-Azhar University
Assiut 71524, Egypt

SANYI TANG
School of Mathematics and Information Science, Shaanxi Normal University
Xi’an 710062, China

ABSTRACT. In the present work, we introduce a control model to describe three species food chain interaction model composed of prey, middle predator, and top predator. The middle predator preys on prey and the top predator preys on middle predator. The control techniques of the exploited natural resources are used to modulate the harvesting effort to avoid high risks of extinction of the middle predator and keep stability of the food chain, by prohibiting fishing when the population density drops below a prescribed threshold. The behavior of the system stability of the regular, virtual, pseudo-equilibrium and tangent points are discussed. The complicated non-smooth dynamic behaviors (sliding and crossing segment and their domains) are analyzed. The bifurcation set of pseudo-equilibrium and the sliding crossing bifurcation have been investigated. Our analytical findings are verified through numerical investigations.

1. Introduction. Non-smooth or switched dynamical systems exhibit a wide variety of complex phenomena which cannot be explained by the classical theory. It is naturally used to model a great variety of engineering devices, as well as used in applied sciences to model a wide variety of physical systems and technological devices, for more details see [1, 4, 6, 7, 8, 9, 10, 23, 26, 31, 33]. The interactions among organisms and their environment, and the evolution of species are the most important biological processes in ecology and population. Therefore, most of the ecological systems have the elements to produce bifurcations and dynamics behavior. The food chain are ecosystems with extremely simple structure. The smooth food chain model was studied in detail by [2, 5, 19, 20, 22, 25, 27, 28, 30, 29, 35]. The theory of Filippov systems used to describe the mathematical modeling of many real-world phenomena. When biological systems are presented as non-smooth model, we can derive useful results from them by extracting dynamic behavior. The smooth food

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* Corresponding author: Soliman A. A. Hamdallah.
chain model is given as:

\[
\begin{align*}
\dot{x}(t) &= (a_1 - b_1 y(t)) x(t), \\
\dot{y}(t) &= (b_2 x(t) - a_2 - \frac{c_1 z(t) y(t)}{d^2 + y(t)^2}) y(t), \\
\dot{z}(t) &= \left(\frac{c_2 y(t)^2}{d^2 + y(t)^2} - a_3\right) z(t),
\end{align*}
\]

where \( x(t) \) is the prey population, \( y(t) \) is the middle predator population, and \( z(t) \) is the top predator population. Furthermore, \( a_1, a_2, a_3, b_1, b_2, c_1, c_2, \) and \( d \) are the intrinsic growth rate of the prey, the death rate of the middle predator, the death rate of the top predator, predation rate of the middle predator, the conversion rate, the maximal growth rate of the middle predator, conversion factor, and the half saturation constant, respectively. The middle predator \( y(t) \) preys on prey \( x(t) \), and the top predator \( z(t) \) preys on middle predator \( y(t) \). Therefore, the main purpose of this paper is to extend the smooth food chain model to Filippov system by using a piecewise saturating function to replace the linear consumption rate for considering the observed experimental results theoretically. This function is defined as \( h(X) = z(t) - ET \), where \( X(t) = (x(t), y(t), z(t))^T \in \mathbb{R}^3_+ \), and \( ET \) describes the critical top predator population threshold. According to the above definition, the extended smooth food chain model with a piecewise saturating function can be defined as the following.

If \( h(X) < 0 \), then the model is given as:

\[
\begin{align*}
\dot{x}(t) &= (a_1 - b_1 y(t)) x(t), \\
\dot{y}(t) &= (b_2 x(t) - a_2 - \frac{c_1 z(t) y(t)}{d^2 + y(t)^2}) y(t), \\
\dot{z}(t) &= \left(\frac{c_2 y(t)^2}{d^2 + y(t)^2} - a_3\right) z(t),
\end{align*}
\]

If \( h(X) > 0 \), then the model becomes

\[
\begin{align*}
\dot{x}(t) &= (a_1 - b_1 y(t)) x(t), \\
\dot{y}(t) &= (b_2 x(t) - a_2 - \frac{c_1 z(t) y(t)}{d^2 + y(t)^2}) y(t), \\
\dot{z}(t) &= \left(\frac{c_2 y(t)^2}{d^2 + y(t)^2} - (a_3 + E)\right) z(t),
\end{align*}
\]

where, \( E \) is the harvesting effort. Therefore, if the density of the top predator population falls below the threshold \( ET \), i.e. \( h(X) < 0 \), then forbids the exploitation. If the density of the top predator population increases and exceeds the threshold \( ET \), i.e. \( h(X) > 0 \), then permit the exploitation. The rest of the paper is organized as follows: In Section 2, we set up the mathematical model of the Filippov food chain system, and give some basic definitions, and preliminaries regarding dynamical behavior. In Section 3 we investigate the dynamics of the subsystems, including the existence of all the possible equilibria, and their stabilities. In Section 4, the sliding and crossing domains are provided. In Sections 5 and 6, respectively the bifurcation set of pseudo-equilibrium and the conditions for the existence of sliding crossing bifurcation are given. The relations between the existence of stable equilibria, unstable equilibria and the a pseudo-equilibrium when the bifurcation parameter varies are also discussed. Finally, the biological conclusion on the results of this work is discussed in Section 7.

2. Filippov food chain model and preliminaries. Let

\[
F_{S_1}(X) = \begin{pmatrix}
(a_1 - b_1 y(t)) x(t) \\
(b_2 x(t) - a_2 - \frac{c_1 z(t) y(t)}{d^2 + y(t)^2}) y(t) \\
\left(\frac{c_2 y(t)^2}{d^2 + y(t)^2} - a_3\right) z(t)
\end{pmatrix},
\]
Then, combinatory the food chain systems (1), (2) yields the following Filippov system [17, 18].

\[
F_{S_2}(X) = \begin{pmatrix}
(a_1 - b_1 y(t)) x(t) \\
(b_2 x(t) - a_2 - \frac{c_1 z(t) y(t)}{x^2 + y(t)^2}) y(t) \\
\frac{c_2 y(t)}{x^2 + y(t)^2} - (a_3 + E) z(t)
\end{pmatrix}.
\]

The dynamics on the separating manifold Σ (crossing or sliding mode dynamics) for Filippov system (3) is determined by the Filippov’s extension [18]:

\[
\dot{X}(t) = \begin{cases}
F_{S_1}(X), & X \in S_1; \\
F_{S_2}(X), & X \in S_2,
\end{cases}
\]

where, \(F_{S_1}, F_{S_2} : \mathbb{R}^3_+ \times \mathbb{R} \to \mathbb{R}^3_+\) are sufficiently smooth functions and \(\mathbb{R}^3_+ = \{X = (x, y, z)^T | x \geq 0, y \geq 0, z \geq 0,\}\). Furthermore, \(\mathbb{R}^3_+\) is split into two regions \(S_1\) and \(S_2\) by the separation manifold \(\Sigma\) such that \(\mathbb{R}^3_+ = S_1 \cup \Sigma \cup S_2\). The regions \(S_1\), \(S_2\) and \(\Sigma\) are defined as:

\[
S_1 = \{X \in \mathbb{R}^3_+ | h(X) < 0\}, \quad S_2 = \{X \in \mathbb{R}^3_+ | h(X) > 0\},
\]

and \(\Sigma = \{X \in \mathbb{R}^3_+ | h(X) = 0\}\), with \(h(X) = z(t) - ET\).

The dynamics on the separating manifold \(\Sigma\) (crossing or sliding mode dynamics) for Filippov system (3) is determined by the Filippov’s convex of vector fields \(F_{S_1}(X)\) and \(F_{S_2}(X)\).

Let

\[
\rho(X) = \langle n^T(X), F_{S_1}(X) \rangle \langle n^T(X), F_{S_2}(X) \rangle,
\]

where, the normal vector \(n(X)\) perpendicular to the manifold \(\Sigma\) is given by \(n(X) = \frac{\nabla h(X)}{\|\nabla h(X)\|}\), and \(\langle \cdot \rangle\) denotes the standard scalar product. Then, the direct crossing set is defined as:

\[
\Sigma^C = \{X \in \Sigma | \rho(X) > 0\},
\]

and the sliding mode set is defined as:

\[
\Sigma^S = \{X \in \Sigma | \rho(X) \leq 0\}.
\]

The sliding mode domain \(\Sigma^S\) can be distinguished by the following sets:

- Attracting sliding mode set if \(\Sigma^S = \{X \in \Sigma^S | \langle n^T (X), F_{S_1}(X) \rangle > 0\}\).
- Repulsive sliding mode set if \(\Sigma^S = \{X \in \Sigma^S | \langle n^T (X), F_{S_1}(X) \rangle < 0\}\).

The flow in \(\Sigma^S\) itself is governed by Filippov’s extension [18]:

\[
F_{\Sigma^S}(X) = \alpha(X) F_{S_1}(X) + (1 - \alpha(X)) F_{S_2}(X),
\]

where

\[
\alpha(X) = \frac{\langle n^T(X), F_{S_1}(X) \rangle}{\langle n^T(X), (F_{S_2}(X) - F_{S_1}(X)) \rangle}.
\]

In the following we show definitions of all types of equilibria of Filippov system [15, 13, 14].

**Definition 2.1.** A point \(X^*\) is called a regular equilibrium of system (3) if \(F_{S_1}(X^*) = 0, h(X^*) < 0\) or \(F_{S_2}(X^*) = 0, h(X^*) > 0\). A point \(X^*\) is called a virtual equilibrium of system (3) if \(F_{S_1}(X^*) = 0, h(X^*) > 0\) or \(F_{S_2}(X^*) = 0, h(X^*) < 0\).

**Definition 2.2.** A point \(X^*\) is called pseudo-equilibrium of system (3), if \(F_{\Sigma^S}(X^*) = 0,\) and \(X^* \in \Sigma^S\).

**Definition 2.3.** Tangency points of system (3) are given through \(X^* \in \Sigma\) with \(\rho(X^*) = 0\), i.e., one or both of the vectors \(F_{S_i}(X), i = 1, 2\) become tangent to the separation manifold,

\[
\Sigma^0_{S_i} = \{X \in \Sigma^S | \langle n^T(X), F_{S_i}(X) \rangle = 0\}, i = 1, 2.
\]
Definition 2.4. A point $X^*$, is called boundary point of system (3) provided that $F_{S_1}(X^*) = 0, h(X^*) = 0$ or $F_{S_2}(X^*) = 0, h(X^*) = 0$.

From now on, we call Filippov system (3) defined in region $S_1$ as subsystem $S_1$ (i.e., system (1)), and defined in region $S_2$ as subsystem $S_2$ (i.e., system (2)).

3. Equilibria of Filippov system. The equilibria of the subsystem $S_1$ are obtained as non-negative solutions of the following algebraic system

$$
\begin{cases}
(a_1 - b_1 y) x(t) = 0, \\
(b_2 x(t) - a_2 - c_1 z(t)y(t)) y(t) = 0, \\
\frac{cz(t)^2}{(x^2+y(t)^2)} z(t) = 0,
\end{cases}
$$

which mean that subsystem $S_1$ has at most three equilibria which are given by

$E_{S_1}^0 = (0,0,0)$, $E_{S_1}^1 = \left(\frac{a_2}{b_2}, \frac{a_1}{b_1}, 0\right)$, $E_{S_1}^2 = (\bar{x}_{21}, \bar{y}_{21}, \bar{z}_{21})$,

where $\bar{x}_{21} > 0$, $\bar{y}_{21} > 0$, $\bar{z}_{21} > 0$, and satisfy

$$
\bar{y}_{21} = \frac{a_1}{b_1}, \\
\bar{x}_{21} = \frac{a_1 b_1 c_1 \bar{z}_{21} + a_2 (a_1^2 + b_1^2 d^2)}{b_2 (a_1^2 + b_1^2 d^2)}, \\
\bar{z}_{21} = \frac{a_1^2 c_2 - a_3 (a_1^2 + b_1^2 d^2)}{a_1^2 + b_1^2 d^2} = 0.
$$

Theorem 3.1. The equilibria of the subsystem $S_1$ satisfied the following:

1. The equilibrium $E_{S_1}^0$ is always a regular equilibrium and it is a saddle point.
2. The equilibrium $E_{S_1}^1$ is always a regular equilibrium and it is asymptotically stable if

$$
a_3 (a_1^2 + b_1^2 d^2) > c_2 a_1^2,
$$

while $E_{S_1}^1$ is a Hopf bifurcation point if the following condition holds:

$$
a_3 (a_1^2 + b_1^2 d^2) = c_2 a_1^2.
$$

3. The equilibrium $E_{S_1}^2$ is exist if $a_3 = \frac{(a_1^2 + b_1^2 d^2)}{c_2 a_1^2}$, and it is a regular stable equilibrium if

$$
\bar{z}_{21} < ET, \text{ and } b_1 d > a_1.
$$

Proof of Theorem 3.1. The jacobian matrix $J_{S_1}$ of subsystem $S_1$ is given as:

$$
J_{S_1} = \begin{pmatrix}
    a_1 - b_1 y(t) & -b_1 x(t) & 0 \\
    b_2 y(t) & -a_2 + b_2 x(t) - \frac{2c_1 z(t)y(t)}{(x^2+y(t)^2)} + \frac{2c_1 z(t)y(t)^3}{(x^2+y(t)^2)^2} & -c_2 y(t)^2 \\
    0 & \frac{2c_3 d^2 z(t)y(t)}{(x^2+y(t)^2)^2} - a_3 + \frac{c_2 y(t)^2}{(x^2+y(t)^2)^2} & -a_3 - c_2 y(t)^2
\end{pmatrix}.
$$

The stability of subsystem $S_1$ at $E_{S_1}^0$ is determined by the eigenvalues of the characteristic polynomial

$$
F_{E_{S_1}^0} (\lambda) = \lambda^3 + (a_2 + a_3 - a_1) \lambda^2 + (a_2a_3 - a_1a_2 - a_1a_3) \lambda - a_1a_2a_3,
$$

and it follows from the Routh-Hurwitz stability criterion that $E_{S_1}^0$ is asymptotically stable if

$$
a_1 < a_2 + a_3, \ a_1a_2a_3 < 0, \text{ and } \frac{a_1a_2a_3}{a_1 - a_3 - a_2} > a_2a_3 - a_1a_2 - a_1a_3,
$$
whereas $a_1, a_2$ and $a_3$ are non-negative then, $E_{S_1}^0$ is unstable. The characteristic polynomial of subsystem $S_1$ at $E_{S_1}^0$ is given as:

$$F_{E_{S_1}^0}(\lambda) = \lambda^3 + \frac{a_3(a_1^2 + b_1^2d^2) - c_2a_1^2}{a_1^2 + b_1^2d^2} \lambda^2 + a_1a_2\lambda + \frac{a_1a_2(a_3a_1^2 + a_3b_1^2d^2 - c_2a_1^2)}{a_1^2 + b_1^2d^2}. \quad (6)$$

The eigenvalues of the characteristic polynomial (7) are given as:

$$\lambda_1 = \frac{c_2a_1^2 - a_3(a_1^2 + b_1^2d^2)}{a_1^2 + b_1^2d^2}, \lambda_{2,3} = \pm \sqrt{a_2a_3}.$$

Since the classical Hopf bifurcation occurs when there is a pair of complex eigenvalues of the characteristic polynomial cross the imaginary axis and a periodic orbit is genetically created. Then, the equilibrium $E_{S_1}^d$ is asymptotically stable if

$$a_3(a_1^2 + b_1^2d^2) > c_2a_1^2,$$

and it is a Hopf bifurcation point if $a_3(a_1^2 + b_1^2d^2) = c_2a_1^2$. The characteristic polynomial of subsystem $S_1$ at $E_{S_1}^d$ is given as:

$$F_{E_{S_1}^d}(\lambda) = \lambda^3 + A_1\lambda^2 + A_2\lambda,$$

where

$$A_1 = \frac{a_1b_1c_1\bar{z}_{21}(b_1^4d^4 - a_1^4)}{(a_1^2 + b_1^2d^2)^3},$$

$$A_2 = a_1a_2 + \frac{a_1^2b_1c_1\bar{z}_{21}(a_1^4 + b_1^4d^4) + 2a_1^3b_1^3c_1d^2\bar{z}_{21}(a_1 + c_2)}{(a_1^2 + b_1^2d^2)^3},$$

from the Routh-Hurwitz stability criterion that $E_{S_1}^d$ is stable if

$$b_1d > a_1, A_1A_2 > 0.$$

According to definition of a regular (a virtual) equilibrium in Section 2 since the discontinuity surface is defined as: $\Sigma = \{X \in \mathbb{R}^3 \mid h(X) = z(t) - ET = 0\}$ with $ET > 0$ then, $h(E_{S_1}^0) > 0, h(E_{S_1}^d) > 0$, therefore, $E_{S_1}^0, E_{S_1}^d$ are always regular equilibria and $E_{S_1}^d$ is regular (a virtual) equilibrium if $\bar{z}_{21} < ET, (\bar{z}_{21} > ET).$

Similarly, the equilibria of the subsystem $S_2$ are obtained as non-negative solutions of the following algebraic system

$$\begin{cases}
(a_1 - b_1y(t))x(t) = 0, \\
(b_2x(t) - a_2 - \frac{c_1z(t)y(t)}{x+y(t)})y(t) = 0, \\
(c_2y(t))^2 - (E + a_3)z(t) = 0.
\end{cases}$$

Then, the subsystem $S_2$ has three equilibria $E_{S_2}^0 = (0, 0, 0), E_{S_2}^1 = (\frac{a_1}{b_1}, 0, 0)$, and $E_{S_2}^2 = (\bar{x}_{22}, \bar{y}_{22}, \bar{z}_{22})$, where $\bar{x}_{22} > 0, \bar{y}_{22} > 0, \bar{z}_{22} > 0$, and satisfy

$$\bar{y}_{22} = \frac{a_1}{b_1},$$

$$\bar{x}_{22} = \frac{a_1b_1c_1\bar{z}_{22} + a_2(a_1^2 + b_1^2d^2)}{b_2(a_1^2 + b_1^2d^2)},$$

$$\bar{z}_{22}(a_3^2c_2 - (a_3 + E)(a_1^2 + b_1^2d^2)) = 0.$$

**Theorem 3.2.** The equilibria of the subsystem $S_2$ satisfies the following:

1. The equilibrium $E_{S_2}^0$ is always a virtual equilibrium and it is a saddle point.
2. The equilibrium $E^1_{S_2}$ is always a virtual equilibrium and it is asymptotically stable if

$$(a_3 + E)(a_1^2 + b_1^2d^2) > c_2a_1^2,$$

while $E^1_{S_2}$ is a Hopf bifurcation point if the following condition holds:

$$(a_3 + E)(a_1^2 + b_1^2d^2) = c_2a_1^2.$$

The equilibrium $E^2_{S_2}$ exist if $a_3 = \frac{(a_1^2 + b_1^2d^2)}{c_2a_1^2} - E$, and it is a regular equilibrium and it is asymptotically stable if

$$b_1d > a_1, \quad \frac{b_1^2d^2(2c_2a_1^2 + Eb_1^2d^2)}{E} > a_1^4, \quad \text{and} \quad B_1B_2 > B_3,$$

where

$$B_1 = E + \frac{a_1b_1c_1\bar{z}_{22}(b_1^2d^4 - a_1^2)}{(a_1^2 + b_1^2d^2)^3},$$
$$B_2 = a_1a_2 + \frac{a_1^2b_1c_1\bar{z}_{22}}{(a_1^2 + b_1^2d^2)} + \frac{a_1b_1c_1\bar{z}_{22}(Eb_1^4d^4 - Ea_1^4 + 2c_2a_1^2b_1^2d^2)}{(a_1^2 + b_1^2d^2)^3},$$
$$B_3 = Ea_1a_2 + \frac{Eb_1c_1\bar{z}_{22}}{(a_1^2 + b_1^2d^2)}.$$

Proof of Theorem 3.2. The proof is similar to Theorem 3.1.

4. Sliding and crossing segments and their domains. According to the definition of sliding and crossing segment $\Sigma^S, \Sigma^C$ and the function $\rho(X)$ where,

$$\Sigma^S = \{X \in \Sigma | \rho(X) \leq 0\}, \quad \Sigma^C = \{X \in \Sigma | \rho(X) > 0\},$$

where

$$\rho(X) = \{(\frac{c_2y(t)^2}{d^2 + y(t)^2} - a_3)ET\} \cdot \{(\frac{c_2y(t)^2}{d^2 + y(t)^2} - (E + a_3))ET\}.$$

Let

$$G_1(y) = (\frac{c_2y(t)^2}{d^2 + y(t)^2} - a_3)ET,$$
$$G_2(y) = (\frac{c_2y(t)^2}{d^2 + y(t)^2} - (E + a_3))ET,$$

to solving the inequality $\rho(X) \leq 0$ with respect to $y(t)$, we first consider the two algebraic equations

$$\left(\frac{c_2y(t)^2}{d^2 + y(t)^2} - a_3\right)ET = 0,$$
$$\left(\frac{c_2y(t)^2}{d^2 + y(t)^2} - (E + a_3)\right)ET = 0.$$

Solving the above equations (8) with respect to $y(t)$ yields two roots, denoted as follows:

$$y^1_{max} = d \sqrt[3]{\frac{a_3}{c_2 - a_3}}, \quad y^1_{min} = -d \sqrt[3]{\frac{a_3}{c_2 - a_3}}.$$

Similarly, solving (9) with respect to $y(t)$ yields two roots, denoted as follows:

$$y^2_{max} = d \sqrt[3]{\frac{a_3 + E}{c_2 - (a_3 + E)}}, \quad y^2_{min} = -d \sqrt[3]{\frac{a_3 + E}{c_2 - (a_3 + E)}}.$$
Based on the above discussions, for the existence of sliding and crossing segments there are three subcases to consider:

\( H_1 \): If \( c_2 < a_3, \ c_2 < a_3 + E \), for this subcase, either of four roots is not real positive, then the Filippov system (3) has no sliding segment. Therefore, the Filippov system (3) has only crossing segment can be defined as:
\[
\Sigma_1^C = \{ x, y, z \in \mathbb{R}_+^3 | z = ET \}. \tag{10}
\]

see Figure (1) A.

\( H_2 \): If \( c_2 > a_3, \ c_2 > a_3 + E \), for this subcase, there are two positive real roots \((y_{\text{max}}^1, y_{\text{max}}^2)\). Then the Filippov system (3) has one sliding segment can be defined as:
\[
\Sigma_2^S = \{ (x, y, z) \in \mathbb{R}_+^3 | z = ET, y_{\text{max}}^1 \leq y \leq y_{\text{max}}^2 \}, \tag{11}
\]
and crossing segments can be defined as:
\[
\Sigma_2^C = \{ (x, y, z) \in \mathbb{R}_+^3 | z = ET, y \in ([0, y_{\text{max}}^1) \cup (y_{\text{max}}^2, \infty)) \}, \tag{12}
\]
see Figure (1) B.

\( H_3 \): If \( c_2 > a_3, \ c_2 < a_3 + E \), for this subcase, there are one positive real root \( y_{\text{max}}^1 \). Then the Filippov system (3) has one sliding segment can be defined as:
\[
\Sigma_3^S = \{ (x, y, z) \in \mathbb{R}_+^3 | z = ET, y_{\text{max}}^1 \leq y < \infty \}, \tag{13}
\]
and crossing segments can be defined as:
\[
\Sigma_3^C = \{ (x, y, z) \in \mathbb{R}_+^3 | z = ET, y \in [0, y_{\text{max}}^1) \}, \tag{14}
\]
see Figure (1) C.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{Illustration of existence of sliding segment of Filippov system (3): where A \( c_2 < a_3, \ c_2 < a_3 + E \), B \( c_2 > a_3, \ c_2 > a_3 + E \), and C \( c_2 > a_3, \ c_2 < a_3 + E \)}
\end{figure}
4.1. Sliding mode dynamics. The Filippov system (3) may exist as one sliding segment which depend on the parameter space. In the following we use Filippov’s equivalent control method to obtain the differential equation for sliding dynamics, where

$$\alpha(t) = \frac{(d^2 + y(t)^2)(a_4 + E) - c_2 y(t)^2}{E(d^2 + y(t)^2)}.$$ 

Therefore, the dynamic on the sliding mode $$\Sigma^S$$ can be determined by the following system

$$F_{\Sigma^S}(X) = \begin{pmatrix} (a_1 - b_1 y(t))x(t) \\ b_2 x(t) - a_2 - \frac{c_1 E y(t)}{d^2 + y(t)^2}y(t) \end{pmatrix},$$

where

$$y(t) \in [y_{\text{max}}^1, y_{\text{max}}^2], x(t) \in [0, \infty) \text{ or } y(t) \in [y_{\text{max}}^1, \infty), x(t) \in [0, \infty).$$

**Theorem 4.1.** The sliding vector filed $$F_{\Sigma^S}$$ has two equilibria $$E_{\Sigma^S}^0, E_{\Sigma^S}^1$$ which are given by

$$E_{\Sigma^S}^0 = (0, 0, 0), E_{\Sigma^S}^1 = \left( \frac{a_2(a_1^2 + b_1^2 d^2) + a_1 b_1 c_1 E T}{b_2(a_1^2 + b_1^2 d^2)}, \frac{a_1}{b_1}, E T \right).$$

The equilibrium $$E_{\Sigma^S}^1$$ is a pseudo-equilibrium of system (3) if

$$\frac{a_1^2 c_2}{(a_1^2 + b_1^2 d^2)} - E \leq a_3 \leq \frac{a_1^2 c_2}{(a_1^2 + b_1^2 d^2)}, \text{ or } a_3 \leq \frac{a_1^2 c_2}{(a_1^2 + b_1^2 d^2)}.$$ 

The pseudo-equilibrium $$E_{\Sigma^S}^1$$ is asymptotically stable if $$b_1 > \frac{a_1}{d},$$ and it is a Hopf bifurcation point if

$$b_1 = \frac{a_1}{d}, \ (a_1 a_2 + a_1^2 b_1 c_1 E T) > \frac{(a_1 b_1 c_1 E T(b_1^2 d^2 - a_1^2))^2}{4(b_1^2 d^2 - a_1^2)^2}.$$ 

**Proof of Theorem 4.1.** The Jacobian matrix $$J_{\Sigma^S}$$ of system $$F_{\Sigma^S}$$ is given as:

$$J_{\Sigma^S} = \begin{pmatrix} a_1 - b_1 y(t) & -b_1 x(t) \\ b_2 y(t) & -a_2 + b_2 x(t) - \frac{2 c_1 E y(t)}{d^2 + y(t)^2} + \frac{2 c_1 E y(t)^3}{(d^2 + y(t)^2)^2} \end{pmatrix}.$$ 

The characteristic polynomial of system $$F_{\Sigma^S}$$ at $$E_{\Sigma^S}^1$$ is given as

$$F_{E_{\Sigma^S}^1}( \lambda ) = \lambda^2 + \frac{a_1 b_1 c_1 E T(b_1^2 d^2 - a_1^2)}{(a_1^2 + b_1^2 d^2)} \lambda + a_1 a_2 + a_1^2 b_1 c_1 E T,$$

(16)

The eigenvalues of the characteristic polynomial (16) are given as:

$$\lambda_{1,2} = \frac{1}{2(a_1^2 + b_1^2 d^2)} (-K_1 \pm \sqrt{K_1^2 - 4K_2}),$$

where

$$K_1 = a_1 b_1 c_1 E T(b_1^2 d^2 - a_1^2), \ K_2 = (a_1 a_2 + a_1^2 b_1 c_1 E T)(b_1^2 d^2 - a_1^2)^2.$$ 

Then, $$E_{\Sigma^S}^1$$ is asymptotically stable if $$b_1 > \frac{a_1}{d},$$ and it is a Hopf bifurcation point if

$$b_1 = \frac{a_1}{d}, \ K_1 > \frac{K_2}{4}.$$ 

According to the domain of sliding mode dynamics (11), (13) then, $$E_{\Sigma^S}^1$$ is a pseudo-equilibrium if

$$\frac{a_1^2 c_2}{(a_1^2 + b_1^2 d^2)} - E \leq a_3 \leq \frac{a_1^2 c_2}{(a_1^2 + b_1^2 d^2)}, \text{ or } a_3 \leq \frac{a_1^2 c_2}{(a_1^2 + b_1^2 d^2)},$$

respectively. \(\square\)
The set of tangent points of Filippov system (3) is given as:

\[ \Sigma_{S_i}^0 = \{ X \in \Sigma^S \mid \langle n^T(X), F_{S_i}(X) \rangle = 0 \}, i = 1, 2, \]

where,

\[ \langle n^T(X), F_{S_i}(X) \rangle = \left( \frac{c_2y(t)^2}{d^2 + y(t)^2} - a_3 \right) ET, \]

\[ \langle n^T(X), F_{S_2}(X) \rangle = \left( \frac{c_2y(t)^2}{d^2 + y(t)^2} - (a_3 + E) \right) ET. \]

Then, the set of tangent points for subsystem \( S_1 \) is denoted as:

\[ \Sigma_{S_1}^0 = \{ X \in \Sigma^S \mid y = d \sqrt{\frac{a_3}{c_2 - a_3}}, c_2 \geq a_3 \}, \quad (17) \]

the set of tangent points for subsystem \( S_2 \) is denoted as:

\[ \Sigma_{S_2}^0 = \{ X \in \Sigma^S \mid y = d \sqrt{\frac{a_3 + E}{c_2 - (a_3 + E)}}, c_2 \geq (a_3 + E) \}. \quad (18) \]

The boundary equilibrium of subsystems \( S_i, i = 1, 2 \), satisfies equations

\[
\begin{align*}
(a_1 - b_1y(t))x(t) &= 0, \\
(b_2x(t) - a_2 - \frac{c_1ETy(t)}{d^2 + y(t)^2})y(t) &= 0, \\
\left( \frac{c_2y(t)^2}{d^2 + y(t)^2} - a_3 \right) ET &= 0,
\end{align*}
\]

\[
\begin{align*}
(a_1 - b_1y(t))x(t) &= 0, \\
(b_2x(t) - a_2 - \frac{c_1ETy(t)}{d^2 + y(t)^2})y(t) &= 0, \\
\left( \frac{c_2y(t)^2}{d^2 + y(t)^2} - (E + a_3) \right) ET &= 0,
\end{align*}
\]

5. **Sliding bifurcations.** In this section, we discuss bifurcation set of pseudo-equilibrium and sliding crossing bifurcation of Filippov system (3).

5.1. **Bifurcation set of pseudo-equilibrium.** Based on the above analyses, the Filippov system (3) has rich dynamics. These dynamics mainly depend on the positions between the bifurcation parameter \( a_3 \) and the equilibria of subsystem \( S_1, S_2 \) and a pseudo-equilibrium. Therefore, it is necessary to investigate the bifurcation set of pseudo-equilibrium and sliding bifurcation of the Filippov system (3). Let

\[ D_1 = \frac{a_2^2c_2}{(a_1^2 + b_1^2d^2)}, \quad (19) \]

\[ D_2 = D_1 - E. \quad (20) \]

In the subcase \( H_2 \) whereas \( c_2 > a_3, c_2 > a_3 - E \), the Filippov system (3) has one sliding segment (11) and bifurcation of pseudo-equilibrium can be occurred with the a bifurcation parameter \( a_3 \) and fixed all other parameters as \( (a_1 = 1, a_2 = 0.5, b_1 = 0.09, b_2 = 5, c_1 = 1.1, c_2 = 1.25, d = 10, E = 0.5, ET = 0.9) \). If we choose \( a_3 = D_1 \) the pseudo-equilibrium \( E_{1_{S_1}} \) can collide at the tangent line (17) see Figure(2)(A). When \( D_2 < a_3 < D_1 \) the pseudo-equilibrium \( E_{1_{S_2}} \) exist in \( \Sigma_2^S \) see Figure(2)(B). If we choose \( a_3 = D_2 \) the pseudo-equilibrium \( E_{1_{S_2}} \) can collide at the tangent line (18) see Figure(3)(A). when \( D_2 > a_3 > D_1 \) the pseudo-equilibrium \( E_{1_{S_2}} \) leave the tangent segment \( \Sigma_1^S \) see Figure(3)(B).
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5.2. Sliding crossing bifurcation. The sliding crossing bifurcation can be occurred with the bifurcation parameter $a_3$ and fixed all other parameters as $(a_1 = 1, a_2 = 0.5, b_1 = 0.09, b_2 = 1.117283951, c_1 = 1.1, c_2 = 1.25, d = 10, E = 0.5 \text{ and } \dot{E}t = 0.9)$. If the bifurcation parameter $a_3$ decreases then a sliding crossing bifurcation occurs at $a_3 = 0.5$, as shown in Figure (4). The subsystem $S_1$ has a limit cycle without sliding segment. when $a_3 = 0.43$, then there exists a sliding cycle with a sliding segment $\Sigma^S$ see Figure (5). When $a_3 = 0.1$, the Filippov system (3) has stable crossing sliding cycle see Figure (6). Therefore, the dynamical behavior of the Filippov system (3) undergoes a crossing-sliding bifurcations.

6. Numerical simulation. In this section, we present numerical simulation to study and address the richness of the possible equilibria and sliding modes that Filippov system (3) can exhibit. As the rate parameter $a_3$ and $E$ are two key parameters for system (3), so we choose $a_3$ and $E$ as the bifurcation parameters, and all other parameters are fixed at values $(a_1 = 1, a_2 = 0.5, b_1 = 1, b_2 = 5, c_1 = 1.1, c_2 = 1.25, d = 10, \dot{E}t = 0.9)$. Then, the Filippov system (3) has the following cases.

Case(1): If we choose $a_3 = 0.0135$ and $E = 0.02$, then the subsystem $S_1$ has stable equilibrium $E_{S_1}^1$ and the equilibria $E_{S_1}^0, E_{S_1}^2$ are un stable. In this case the top predator $z(t)$ growing up to collide at the sliding segment $\Sigma^S$ and then
decreases stably periodic over time. On the other hand, the prey $x(t)$ and the middle predator $y(t)$ can survive, growing asymptotically stable see Figure (7). If we choose $a_3 = 0.0124$ and $E = 0.02$, then, the Filippov system (3)
has stable pseudo-equilibrium $E_{S_1}^1$. In this case the prey $x(t)$ and the middle predator $y(t)$ and the top predator $z(t)$, can coexist on the sliding surface see Figure (8).

**Figure 7.** The phase portrait of Filippov system (3). We choose $a_3 = 0.0135$, $E = 0.02$.

**Case(2):** If we choose $a_3 = 0.0122$ and $E = 0.0002$, then, the subsystem $S_2$ has stable equilibrium $E_{S_2}^2$ and the equilibria $E_{S_2}^1$, $E_{S_2}^0$ are unstable. In this case the prey $x(t)$ and the middle predator $y(t)$ and the top predator $z(t)$, can growth asymptotically stable and coexist on subsystem $S_2$ see Figure (9). If we choose $a_3 = 0.0123$ and $E = 0.0002$, then, the Filippov system (3) has stable pseudo-equilibrium $E_{S_2}^1$. In this case the top predator $z(t)$ growing up and then stay in the sliding surface. The prey $x(t)$ and the middle predator $y(t)$ can growth asymptotically stable and coexist on the sliding surface see Figure (10).

**Case(3):** If we choose $a_3 = 0.01$ and $E = 0.0002$, then, the subsystem $S_2$ has unstable equilibrium $E_{S_2}^2$. In this case the prey $x(t)$ and the top predator $z(t)$ can survive growing periodic unstable. On the other hand, the middle predator $y(t)$ persist and has populations that vary asymptotically stable see Figure (11). If we choose $a_3 = 0.01$, $E = 0.5$, $a_1 = 1$, $a_2 = 0.5$, $b_1 = 0.09$, $b_2 = 1.117283951$, $c_1 = 1.1$, $c_2 = 1.25$, $d = 10$, and $ET = 0.9$, then, the Filippov system (3) has stable sliding periodic Solution. In this case the top predator $z(t)$, the middle predator $y(t)$ and the prey $x(t)$ growth stable periodic see Figure (12).
Figure 8. The phase portrait of Filippov system (3). We choose $a_3 = 0.0124$, $E = 0.02$.

Figure 9. The phase portrait of Filippov system (3). We choose $a_3 = 0.0122$, $E = 0.0002$. 
Figure 10. The phase portrait of Filippov system (3). We choose $a_3 = 0.0123$, $E = 0.0002$.

Figure 11. The phase portrait of Filippov system (3). We choose $a_3 = 0.01$, $E = 0.02$. 
7. **Conclusion.** The modeling and control of the ecological systems and the evolution of species are known to undergo a variety of bifurcation phenomena ranging from multiplicity and stability of steady states to sustained oscillations. Recently, Filippov systems have gained considerable attention in life science and engineering [3, 10, 11, 12, 16, 20, 21, 24, 31, 32, 34] and they provide a natural and convenient unified framework for mathematical modeling of several real-world problems. In this paper, we proposed a Filippov food chain model with food chain control strategy to describe three species food chain interaction model composed of $x(t), y(t),$ and $z(t)$. Where $x(t), y(t),$ and $z(t)$ denote the non dimensional population density of the prey, predator, and top predator respectively. The control techniques of the exploited natural resources are used to modulate the harvesting effort ($E$) to avoid high risks of extinction of the middle predator $y(t)$ and keep stability of the food chain, by prohibiting fishing when the population density drops below a prescribed threshold. The dynamics of the proposed Filippov food chain (3) with control strategies have been studied in detail by using qualitative analysis techniques of non-smooth Filippov dynamic systems and numerical techniques. The sliding mode dynamics and existence of the several types of equilibria are discussed. The results obtained in Section 5 indicate that the Filippov system (3) can give rise to bifurcation of pseudo-equilibrium and sliding crossing bifurcation. Furthermore, the results presented in Section 6 introduced the relations between the existence of equilibria and a pseudo-equilibrium when the bifurcation parameters value varies.

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E-mail address: s.handallah@azhar.edu.eg
E-mail address: sytang@snnu.edu.cn