Quantum Field Theory Solution for a Short-Range Interacting SO(3) Quantum Spin-Glass

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(Dated: May 21, 2021)

We study the quenched disordered magnetic system, which is obtained from the 2D SO(3) quantum Heisenberg model, on a square lattice, with nearest neighbors interaction, by taking a Gaussian random distribution of couplings centered in an antiferromagnetic coupling, $J > 0$ and with a width $\Delta J$. Using coherent spin states we can integrate over the random variables and map the system onto a field theory, which is a generalization of the SO(3) nonlinear sigma model with different flavors corresponding to the replicas, coupling parameter proportional to $J$ and having a quartic spin interaction proportional to the disorder ($\Delta J$). After deriving the CP$^1$ version of the system, we perform a calculation of the free energy density in the limit of zero replicas, which fully includes the quantum fluctuations of the CP$^1$ fields $z_i$. We, thereby obtain the phase diagram of the system in terms of $(T, J, \Delta J)$. This presents an ordered antiferromagnetic (AF) phase, a paramagnetic (PM) phase and a spin-glass (SG) phase. A critical curve separating the PM and SG phases ends at a quantum critical point located between the AF and SG phases, at $T = 0$. The Edwards-Anderson order parameter, as well as the magnetic susceptibilities are explicitly obtained in each of the three phases as a function of the three control parameters. The magnetic susceptibilities show a Curie-type behavior at high temperatures and exhibit a clear cusp, characteristic of the SG transition, at the transition line. The thermodynamic stability of the phases is investigated by a careful analysis of the Hessian matrix of the free energy. We show that all principal minors of the Hessian are positive in the limit of zero replicas, implying in particular that the SG phase is stable.

PACS numbers: 75.50.Lk

I. INTRODUCTION

A spin glass is a peculiar state, which is presented by certain disordered magnetic materials, when the competition between opposite interactions produces what is known as frustration, namely the incapability of the system to attain the lowest possible energy state that would correspond to each single type of interaction. This situation is described by ascribing to the system a random distribution of coupling constants, allowing interactions of opposite signs. As a consequence of the competition between different types of order, some of the spin glass properties are shared with paramagnetic states and some other with ordered, ferromagnetic or Néel states. Like the first ones, SG states possess zero magnetization and similarly to ordered magnetic states, they present breakdown of ergodicity. The time scale of disorder is typically much larger than the one associated to the dynamics and therefore we must perform the quantum and thermal averages before averaging over the disorder, the so-called “quenched” thermodynamical description, which requires the use of the replica method.$^{1,2}$

A SG state exhibits clear theoretical and experimental signatures. The former has been introduced by Edwards and Anderson (EA), who proposed a model for SG and devised an order parameter that captures one of the basic features of the glassy behavior, namely the occurrence of infinite time correlations for each spin.$^{3}$ The same type of correlations exist in an ordered magnetic state such as the Néel state or the ferromagnetic state, for instance. In the SG state, however, this happens without the associated existence of infinite spatial correlations among the spins and the consequent spontaneous nonzero magnetization (staggered in the case of antiferromagnetic (AF) order). Hence the SG state can be characterized as one presenting infinite time correlations (nonzero EA order parameter) but with zero magnetization order parameter. From the experimental point of view, one of the most distinctive signatures of the SG transition is a very sharp cusp in the magnetic susceptibility as a function of the temperature, right at the transition.$^{4,5}$

An important landmark in the study of SG was the derivation of a mean-field solution of a simplified version of the EA model, obtained by Sherrington and Kirkpatrick (SK)$^{4}$. They considered a system with classical Ising spins with long-range interactions, in which each spin would interact with any other spin in the material, no matter how far apart they might be. This assumption greatly facilitates, technically, the obtainment of the mean-field solution. Soon after, however it was realized that the solution of SK was unstable$^{5}$. This fact has been generally ascribed to the so-called replica symmetry, which was presented by the SK solution. Indeed, later on, Parisi has found a stable replica symmetry breaking solution.$^6$

The long-range interactions of the SK model, however, are likely to be unphysical, to a large extent. Real materials should mostly be short-range interacting, quantum SO(3) Heisenberg spin systems. Despite the large amount of knowledge that we have today about long-
range interacting SG, surprisingly, very little is known about the properties of short-range interacting quantum spin-glasses, especially with SO(3) symmetry. Apart from some numerical calculations, very few analytical approaches exist. An appealing short-range interacting disordered system, which has been thoroughly investigated mostly by numerical methods is the transverse field Ising spin glass. Interesting related results can be found in\textsuperscript{10,16} and\textsuperscript{11}, where a Landau-Ginzburg, phenomenological approach has been developed for the short-range SG problem.

We have proposed a model for a disordered SO(3) Heisenberg-like quantum spin system with nearest neighbor interactions for which an expression for the free energy density can be derived.\textsuperscript{12} In this work, we map the system onto a generalized CP\textsuperscript{1} quantum field theory in the continuum limit. This is a very convenient framework, because nearest neighbor interactions become just derivatives. At this point, however, we must be careful. Indeed, when taking the continuum limit of a quantum system Berry phases will be generated, which in general would not cancel when summed over the lattice.\textsuperscript{11,13} We may tackle this problem by introducing disorder as a perturbation of an antiferromagnetic (AF) 2D Heisenberg model, for which the sum of the quantum phases is known to cancel.\textsuperscript{14,15} Hence, we consider a Gaussian random distribution of couplings centered in an AF coupling \( J > 0 \) and with variance \( \Delta J \), such that \( \Delta J \ll J \). The situation is completely different from the original EA model, where \( J = 0 \), and consequently the disorder cannot be taken as a perturbation of a Heisenberg system.

Using the CP\textsuperscript{1} description, we obtain a solution, which presents replica symmetry. Out of this, we extract the \( T \times J \) phase diagram of the system. This exhibits a Néel phase at \( T = 0 \) and \( \rho_s > \rho_0 \), where \( \rho_s = S^2 J \) is the spin stiffness (\( S \) is the spin quantum number) and \( \rho_0 \) is a quantum critical point. This is displaced by disorder to the right of its original value \( \rho_0(0) = \frac{4}{\pi} (\Lambda = \frac{1}{\alpha}; \alpha: \) lattice spacing) in the pure system.\textsuperscript{19,20} It also contains a spin glass (SG) phase for temperatures below a certain critical line \( T < Tc \) and \( \rho_s < \rho_0 \), characterized by a nonzero EA order parameter and zero staggered magnetization. For \( T > 0 \); \( \rho_s > \rho_0 \) and also for \( T > Tc ; \rho_s < \rho_0 \), we find a paramagnetic (PM) phase. The behavior of the magnetic susceptibility, of the spin-gap and of the EA order parameter are analyzed in detail in each phase as well as on the transition line. The former presents a nice cusp at the transition, in agreement with the typical experimental behavior of spin glasses.\textsuperscript{8} We also show how the phase diagram is modified by varying the amount of disorder.

We make a thorough investigation about the thermodynamic stability of our solution. This is done through a careful analysis of the Hessian matrix of the average free-energy density. We show in detail that all principal minors of the Hessian matrix are strictly positive in the physical limit when the number of replicas reduces to zero. This is a necessary and sufficient condition for the mean-field solution to be a local minimum, hence it guarantees the solution is stable. Furthermore, we can follow the phase transition directly in the Hessian, as the corresponding principal minor determinants cease to be positive if we use the wrong solution for a given phase.

There is an appealing physical motivation for our SG model, in connection to the high-T\textsubscript{c} cuprates. Indeed, these materials, when undoped, are 2D Heisenberg antiferromagnets, which upon doping, develop a SG phase before becoming superconductors. Our model, also being a 2D Heisenberg antiferromagnet for \( \Delta = 0 \), describes precisely the AF-SG transition as we increase the disorder and is therefore potentially very useful for studying the magnetic fluctuations of such materials.

\section*{II. THE MODEL AND THE CONTINUUM LIMIT}

\subsection*{A. The Model}

The model consists of an SO(3) quantum Heisenberg-like Hamiltonian, containing only nearest neighbor interactions of the spin operators \( \hat{S}_i \), on the sites of a 2D square lattice of spacing \( a \),

\begin{equation}
\hat{H} = \sum_{\langle ij \rangle} J_{ij} \hat{S}_i \cdot \hat{S}_j, \tag{2.1}
\end{equation}

The couplings \( J_{ij} \) are random and associated with a Gaussian probability distribution with variance \( \Delta J \) and centered in \( \bar{J} = 0 \), such that \( \Delta J \ll \bar{J} \), namely,

\begin{equation}
P[J_{ij}] = \frac{1}{\sqrt{2\pi(\Delta J)^2}} \exp \left[ -\frac{(J_{ij} - \bar{J})^2}{2(\Delta J)^2} \right], \tag{2.2}
\end{equation}

We consider the quenched situation, in which, according to the replica method, the average free-energy is given by

\begin{equation}
\bar{F} = -k_B T \lim_{n \rightarrow \infty} \frac{1}{n} \langle [Z^n]_{\alpha \nu} \rangle - 1, \tag{2.3}
\end{equation}

where \( Z^n \) is the replicated partition function for a given configuration of couplings \( J_{ij} \)

\begin{equation}
Z^n\{J_{ij}\} = \operatorname{Tr}_\{\hat{S}_i^\alpha\} \exp \left[ -\beta \sum_{\alpha=1}^n \sum_{\langle ij \rangle} J_{ij} \hat{S}_i^\alpha \cdot \hat{S}_j^\alpha \right], \tag{2.4}
\end{equation}

and

\begin{equation}
[Z^n]_{\alpha \nu} = \int \left( \prod_{\langle ij \rangle} dJ_{ij} P[J_{ij}] \right) Z^n\{J_{ij}\}. \tag{2.5}
\end{equation}

is the average thereof with the Gaussian distribution.
We now make use of the coherent spin states $|\Omega(\tau)\rangle$, such that

$$|\Omega(\tau)\rangle \equiv \bigotimes_{i}^{n} \bigotimes_{\alpha=1}^{1} |\Omega_{i}^{\alpha}(\tau)\rangle.$$

with

$$\langle \Omega_{i}^{\alpha}(\tau) | \bar{S}_{i}^{\alpha}(\tau) | \Omega_{i}^{\alpha}(\tau) \rangle = S\Omega_{i}^{\alpha}(\tau),$$

where $i$: lattice sites, $\alpha$: replicas, $\tau$: euclidian time, $S$: spin quantum number, and $\Omega_{i}^{\alpha}(\tau)$ is a classic vector of unit magnitude. With the help of these coherent states we may express $Z^{n}$ as a functional integral over $\Omega_{i}^{\alpha}(\tau)$, namely,

$$Z^{n}\{J_{ij}\} = \int \mathcal{D}\Omega \exp \left\{ -\int_{0}^{\beta} d\tau \sum_{\alpha=1}^{n} L^{\alpha}(\tau) \right\},$$

where $L^{\alpha}(\tau)$

$$L^{\alpha}(\tau) = L_{B}^{\alpha} - S^{2} \sum_{(ij)} J_{ij} \Omega_{i}^{\alpha}(\tau) \cdot \Omega_{j}^{\alpha}(\tau).$$

and

$$L_{B}^{\alpha} = \sum_{i} \langle \Omega_{i}^{\alpha}(\tau) | \frac{d}{d\tau} | \Omega_{i}^{\alpha}(\tau) \rangle,$$

is the Berry phase term.

The average over the disordered couplings $J_{ij}$ can then be performed, yielding

$$[Z^{n}]_{av} = \int \mathcal{D}\Omega \exp \left\{ -S_{J,\Delta} \right\};$$

where $S_{J,\Delta}$ is given by

$$S_{J,\Delta} = \int_{0}^{\beta} d\tau \sum_{\alpha} \left[ L_{B}^{\alpha} - S^{2} \sum_{(ij)} J_{ij} \Omega_{i}^{\alpha}(\tau) \cdot \Omega_{j}^{\alpha}(\tau) \right]$$

$$+ \frac{S^{4}(\Delta J)^{2}}{2} \sum_{(ij)} \int_{0}^{\beta} d\tau d\tau' \Omega_{ia}^{\alpha}(\tau)\Omega_{ib}^{\alpha}(\tau')\Omega_{ja}^{\alpha}(\tau)\Omega_{jb}^{\alpha}(\tau'),$$

where summations in $(\alpha\beta)$, as well as over the SO(3) components of $\Omega$, $(ab)$ are understood.

Using the connectivity matrix, defined as: $(K_{ij} = 1$ if $(ij)$ are nearest neighbors and $K_{ij} = 0$ otherwise), we may write the nearest neighbor sums in the last term of (2.11) as overall sums. This allows us to perform a Hubbard-Stratonovich transformation that replaces the quartic term of (2.11) by

$$S_{\Delta} = S^{4}(\Delta J)^{2} \sum_{\alpha,\beta=1}^{n} \int_{0}^{\beta} d\tau d\tau' \sum_{(ij)} \left[ \frac{1}{2} Q_{i,ab}^{\alpha\beta}(\tau,\tau') Q_{j,ab}^{\alpha\beta}(\tau,\tau') \right].$$

This is no longer a disordered system. The disorder, which was originally present manifests now through the interaction term, proportional to $(\Delta J)^{2}$. In the absence of disorder, we would have $\Delta J \to 0$ and $[Z^{n}]_{av}$ would reduce to the usual coherent spin representation of the AF Heisenberg model, with a coupling $J > (16, 19, 20)$.

### B. Continuum Limit

Since we are only considering the weakly disordered case $(\Delta J \ll J)$ our model, described by the effective action in (2.11), is a perturbation of the AF 2D quantum Heisenberg model. This means we can decompose the classical spin $\Omega_{i,a}^{\alpha}(\tau)$ into antiferromagnetic and ferromagnetic fluctuations as in that model. Using this, then it follows that the sum of the quantum Berry phases, $L_{B,a}^{\alpha}$ over all the lattice sites vanishes, as in the pure system. 14-15.

We can therefore take the continuum limit in the usual way as in the pure AF 2D quantum Heisenberg model 15,16,17 obtaining an SO(3) generalized relativistic nonlinear sigma model (NLSM) describing the field $\mathbf{n}^{\alpha} = (\sigma^{\alpha}, \vec{\pi}^{\alpha})$, which is the continuum limit of the (staggered) spin $\Omega^{\alpha}$:

$$\mathcal{L} = \frac{1}{4} |\nabla \mathbf{n}^{\alpha}|^{2} + \frac{1}{2} \frac{\epsilon_{abc}}{2} |\partial A_{a}^{\alpha}|^{2} + i\lambda_{\alpha}(\mathbf{n}^{\alpha} \cdot \mathbf{n}^{\alpha} - \rho_{s})$$

$$+ \frac{D}{2} \left[ Q_{ab}^{\alpha\beta}(\mathbf{r}, \tau, \tau') Q_{ab}^{\alpha\beta}(\mathbf{r}, \tau, \tau') \right],$$

$$- \frac{2}{\rho_{s}} n_{a}^{\alpha}(\mathbf{r}, \tau) Q_{ab}^{\alpha\beta}(\mathbf{r}, \tau, \tau') n_{b}^{\alpha}(\mathbf{r}, \tau') \right].$$

where $D = S^{4}(\Delta J)^{2}/a^{2}$ ($a$: lattice parameter) and $\rho_{s} = S^{2}J$. The constraint $\mathbf{n}^{\alpha} \cdot \mathbf{n}^{\alpha} = \rho_{s}$ (no sum in $\alpha$), as usual is implemented by the Lagrange multiplier field $\lambda_{\alpha}$.

This generalized NLSM contains a $(\Delta J)^{2}$-proportional trilinear interaction of $n_{a}^{\alpha}$ with the Hubbard-Stratonovich field $Q_{ab}^{\alpha\beta}(\mathbf{r}, \tau, \tau')$, which corresponds to the $\Delta J$-proportional part of (2.11), namely $S_{\Delta}$.

Notice that a null value for $\Delta J$, as we have in the EA model, would make the perturbation around a NLSM meaningless. A negative value, on the other hand, would correspond to the ferromagnetic Heisenberg model, which after taking the continuum limit, is associated to the non-relativistic NLSM. Here perturbation would be possible, however, the Berry’s phases would no longer cancel. We emphasize, therefore, the enormous difference that exists, both from the physical and mathematical points of view, in considering $\Delta J$ as positive, negative or null in the Gaussian distribution of the EA model.

The field $\mathbf{n}^{\alpha} = (\sigma^{\alpha}, \vec{\pi}^{\alpha})$ is the continuum limit of the (staggered) spin $\Omega^{\alpha}$ and satisfies the constraint $\mathbf{n}^{\alpha} \cdot \mathbf{n}^{\alpha} = \rho_{s}$, which is implemented by integration on $\lambda^{\alpha}$.
Decomposing $Q^{\alpha\beta}$ into replica diagonal and off-diagonal parts,

$$Q_{\alpha\beta}^{\gamma}(\vec{r}, \tau, \tau') \equiv \delta_{\alpha\beta}[\delta^{\gamma}\chi(\vec{r}; \tau, \tau') + q^{\alpha\beta}(\vec{r}; \tau, \tau')] \quad (2.14)$$

where $q^{\alpha\beta} = 0$ for $\alpha = \beta$, we get

$$\mathcal{L}_{\bar{J}, \bar{\Delta}} = \frac{1}{2}|\nabla n^{\alpha}|^2 + \frac{1}{2}\bar{c}^2|\partial_{\tau} n^{\alpha}|^2 + \partial_{\tau} n^{\alpha}(\bar{\Delta} n^{\alpha} - \rho_s)
+ \frac{3D}{2} \int d\tau' \left[ \chi(\tau, \tau') + q^{\alpha\beta}(\tau, \tau') q^{\alpha\beta}(\tau, \tau') \right]
- \frac{D}{\rho_s} n^{\alpha}(\tau) q^{\alpha\beta}(\tau, \tau') n^{\beta}(\tau') \right]. \quad (2.15)$$

This will be our starting point for the CP$^1$ formulation. In a previous work, we took a different path. From (2.15), we integrated over the $\bar{\vec{r}}$-field and thereby obtained an effective action for the remaining fields.

### III. THE CP$^1$ FORMULATION

#### A. CP$^1$ Lagrangian

We now introduce the CP$^1$ field in the usual way, namely,

$$n^{\alpha}(\tau) = \frac{1}{\sqrt{\rho_s}} \left[ z^{\alpha}_i(\tau) \sigma_{ij} z^{\alpha}_j(\tau) \right] \quad (3.1)$$

where the $z^{\alpha}_i$ field satisfies the constraint

$$|z^{\alpha}_1|^2 + |z^{\alpha}_2|^2 = \rho_s. \quad (3.2)$$

Using the above two equations, we get the correspondence

$$\frac{1}{2}|\nabla n^{\alpha}|^2 + \frac{1}{2}\bar{c}^2|\partial_{\tau} n^{\alpha}|^2 \Leftrightarrow 2 \sum_{i=1}^2 |D_{\mu} z^{\alpha}_i|^2, \quad (3.3)$$

where $D_{\mu} = \partial_{\mu} + iA_{\mu}$. The above correspondence involves the functional integration over the auxiliary vector field $A_{\mu}$.

Using (3.1), (3.2) and (3.3) in (2.15), we may express the average replicated partition function in terms of the CP$^1$ fields as

$$[Z^n]_{av} = \int \mathcal{D} z \mathcal{D} z^{\ast} \mathcal{D} A_{\mu} \mathcal{D} \chi \mathcal{D} q \mathcal{D} \lambda e^{-S}, \quad (3.4)$$

where $S \left[ z^{\alpha}_i, z^{\ast\alpha}_i, A_{\mu}, \chi(\tau, \tau'), q^{\alpha\beta}(\tau, \tau') \right]$ is the action corresponding to the lagrangian density

$$\mathcal{L}_{J, \Delta, CP^1} = 2 |D_{\mu} z^{\alpha}_i|^2 + i\lambda_a (|z^{\alpha}_i|^2 - \rho_s)
+ \frac{3D}{2} \int d\tau' \left[ \chi(\tau, \tau') + q^{\alpha\beta}(\tau, \tau') q^{\alpha\beta}(\tau, \tau') \right]
- \frac{3}{\rho_s} \int d\tau' \left[ \chi(\tau, \tau') [z^{\alpha}_i(z^{\ast\alpha}_i)]^2 + \frac{1}{n} \ln \det M \right], \quad (3.5)$$

where summation in $i, j, \alpha, \beta$ is understood.

#### B. The Quantum Average Free Energy

In order to evaluate $[Z^n]_{av}$ in (3.4), we use the stationary phase approximation. For this, we expand $S \left[ z^{\alpha}_i, z^{\ast\alpha}_i, A_{\mu}, \chi(\tau, \tau'), q^{\alpha\beta}(\tau, \tau') \right]$ around the fields in the stationary point, in such a way that the quadratic fluctuations about the $z^{\alpha}_i$ fields are taken into account, namely,

$$S \left[ z^{\alpha}_i, z^{\ast\alpha}_i, A_{\mu}, \chi, q^{\alpha\beta} \right] = S \left[ z^{\alpha}_{i,s}, z^{\ast\alpha}_{i,s}, A_{\mu}, \chi_s, q^{\alpha\beta}_{s} \right]
+ \frac{1}{2} \int d\tau d\tau' \eta^{\alpha}_i(\tau, \tau') \delta z^{\alpha}_{i,s}(\tau, \tau') \eta^{\alpha}_i(\tau, \tau'), \quad (3.6)$$

where $\eta^{\alpha}_i = z^{\alpha}_i - z^{\alpha}_{i,s}$ and $M$ is the matrix

$$M = \begin{pmatrix} \frac{\delta^2 S}{\delta z^{\alpha}_{i,s}(\tau) \delta z^{\alpha}_{j,s}(\tau)} & \frac{\delta^2 S}{\delta z^{\alpha}_{i,s}(\tau) \delta z^{\alpha}_{j,s}(\tau)} \\ \frac{\delta^2 S}{\delta z^{\alpha}_{i,s}(\tau) \delta z^{\alpha}_{j,s}(\tau)} & \frac{\delta^2 S}{\delta z^{\alpha}_{i,s}(\tau) \delta z^{\alpha}_{j,s}(\tau)} \end{pmatrix}, \quad (3.7)$$

with elements taken at the stationary fields.

These stationary fields are such that $\chi^{\alpha}(\tau, \tau') \rightarrow \lambda_s$ and $m^2 = 2i\lambda_s$, which turns out to be the spin gap. Also $\chi(\tau, \tau') \rightarrow \chi_s(\tau - \tau')$ and $q^{\alpha\beta}(\tau, \tau') \rightarrow q^{\alpha\beta}_s(\tau - \tau')$. The staggered magnetization $\sigma_s^{\alpha}$ is given in terms of the CP$^1$ fields as

$$\sigma_s^{\alpha} = \frac{1}{n} \sum_{\alpha=1}^n \left[ |z^{\alpha}_{i,s}|^2 + |z^{\alpha}_{2,s}|^2 \right] = \frac{1}{n} \sum_{\alpha=1}^n \sigma^{\alpha}_s. \quad (3.8)$$

Finally, the stationary value of the gauge field is $A_{\mu}^{\alpha} = 0$.

Inserting (3.6) in (3.4) we obtain, after integrating over the $z$-fields,

$$[Z^n]_{av} = e^{-nS_{eff}[\sigma^{\alpha}_s, m^2, A_{\mu}^{\alpha}]} \left[ \bar{q}^{\alpha\beta}(\tau - \tau'), \chi_s(\tau - \tau') \right], \quad (3.9)$$

where

$$S_{eff} \left[ \sigma^{\alpha}_s, m^2, A_{\mu}^{\alpha} = 0, q^{\alpha\beta}_s, \chi_s \right] = \beta V \left[ \frac{m^2}{2} [\sigma^{\alpha}_s - \rho_s] \right]
+ V \int_0^{\beta} d\tau d\tau' \left[ \frac{3D}{2} \chi_s^2(\tau - \tau') + \frac{1}{n} q^{\alpha\beta}_s(\tau - \tau') q^{\alpha\beta}_s(\tau - \tau') \right]
- \frac{D}{n\rho_s} \left[ \chi_s(\tau - \tau') \delta^{\alpha\beta} + q^{\alpha\beta}_s(\tau - \tau') \right] \sigma^{\alpha}_s \sigma^{\beta}_s \right] - \frac{1}{n} \ln \det M, \quad (3.10)$$
Notice that the third term in (3.5) is proportional to \( n^2 \sigma_s^2 \) and, therefore, does not contribute to (3.10) in the limit \( n \to 0 \).

By taking the limit \( n \to 0 \) in (3.9) we immediately realize that the average free-energy is given by

\[
\bar{F} = \frac{1}{\beta} \mathrm{S_{\text{eff}}} \left[ \sigma^\alpha, m^2, A^\mu = 0, q_s^{\alpha\beta}(\tau - \tau'), \chi_s(\tau - \tau') \right].
\]

(3.11)

In Appendix A we consider the determinant appearing in the last term of (3.10) This contains the quantum corrections coming from the \( z^a \) fields. This determinant runs over the \( i \) components of these fields, over the replicas and over the field configurations. We are able to exactly calculate the first two determinants by replacing the \( q^{\alpha\beta} \) variables in the last term of (3.3) by their average,

\[
\bar{q} = \lim_{n \to 0} \frac{1}{n(n-1)} \sum_{\alpha\beta} q^{\alpha\beta}
\]

(3.12)

The determinant over the field configurations is most conveniently expressed in the space of Matsubara frequencies \( \omega_r = 2\pi n T, r \in \mathbb{Z} \). For this purpose, we perform the Fourier transformation of the \( \chi \)'s and \( q \)'s. Then, using the expression obtained for the determinant of the quantum fluctuations \( (\alpha, 10) \), we get the average free-energy density as the functional (we henceforth neglect the “s” subscript)

\[
\bar{f} [\sigma^\alpha, m^2, q^{\alpha\beta}(\omega), \chi(\omega)] = \frac{m^2}{2} \left[ \sigma^2 - \rho_s \right]
- \frac{D}{n \rho_s} \left[ \chi(\omega_0) \delta^{\alpha\beta} + q^{\alpha\beta}(\omega_0) \right] \sigma^\alpha \sigma^\beta
+ 3DT \sum_{\omega_r} \left[ \chi(-\omega_r)(\omega_r) + \frac{1}{n} q^{\alpha\beta}(-\omega_r) q^{\alpha\beta}(\omega_r) \right]
+ T \sum_{\omega_r} \int \frac{d^2k}{(2\pi)^2} \left[ \ln (k^2 + M_r) - \frac{A \bar{q}(\omega_m)}{k^2 + M_r} \right]
\]

(3.13)

where \( A = \frac{2D}{\rho_s} \) and

\[
M_r \equiv M(\omega_r) = \omega_r^2 + m^2 - A[\chi_r - \bar{q}_r],
\]

(3.14)

where \( \chi(\omega_r) \) and \( q^{\alpha\beta}(\omega_r) \) are, respectively, the Fourier components of \( \chi(\tau - \tau') \) and \( q^{\alpha\beta}(\tau - \tau') \). We use these, the simplified notation \( \chi_r \equiv \chi(\omega_r) \) and \( q_r^{\alpha\beta} \equiv q^{\alpha\beta}(\omega_r) \).

From (2.12), we can show that

\[
Q^{\alpha\beta}_r (\tau, \tau') = \left( \bar{S}^\beta(\tau) \bar{S}^\alpha(\tau') \right)
\]

and therefore, according to the previous decomposition of \( Q \) into \( \chi \)'s and \( q \)'s, we can identify \( \chi_0 \) as the static magnetic susceptibility, whereas the integrated susceptibility is given by

\[
\chi_1 = \sum_r \chi_r.
\]

(3.16)

The EA order parameter, used to detect the SG phase, accordingly, is given by

\[
\bar{q}_E = T \bar{q}_0
\]

(3.17)

where \( \bar{q}_0 \) is defined in (3.12).

C. The Stationary-Phase Equations

By taking the variations of \( \bar{f} \) with respect to the variables \( \sigma^\alpha, m^2, \chi_r, q_r^{\alpha\beta} \), respectively, we obtain the stationary-phase equations (SPE), which are listed below.

\[
\frac{1}{n} \left[ m^2 - \chi_0 \right] \delta^{\alpha\beta} - A \bar{q}_0^{\alpha\beta} \right] \sigma^\beta = 0
\]

(3.18)

\[
\sigma^2 = \rho_s - \frac{T}{2\pi} \sum_{\omega_r} \left[ 1 + \frac{A^2}{M_r} \right] + 2A \sum_{\omega_r} \bar{q}_r G_r
\]

(3.19)

\[
3DT \chi(-\omega_r) = \frac{TA}{4\pi} \ln \left( 1 + \frac{A^2}{M_r} \right) + A^2 \bar{q}_r G_r - A^2 \delta_{0r}
\]

(3.20)

\[
3DT q_r^{\alpha\beta}(-\omega_r) = A^2 \bar{q}(\omega_r) G_r + A \bar{q}_r \bar{q}_s \delta_{0r}
\]

(3.21)

where

\[
G_r = \frac{T}{4\pi} \left[ \frac{1}{M_r} - \frac{1}{A^2 + M_r} \right]
\]

(3.22)

and \( \Lambda = 1/a \) is the high-momentum cutoff.

Observe that in the absence of disorder \( (D, A \to 0) \) (3.20) and (3.21) disappear and (3.18), (3.19) reduce to the well-known corresponding equations for the continuum limit of the pure Heisenberg model\(^{16,19,20}\). We henceforth will only consider the case \( D \neq 0 \).

Equation (3.21) tells us that in our approximation all the \( q^{\alpha\beta} \)'s are equal, whenever the \( \sigma^\alpha \)'s vanish.

IV. THE PHASE DIAGRAM

A. Paramagnetic and Spin-Glass Phases

1. Preliminaries

We start by searching for PM and SG phases. In both of them we have \( \sigma = 0 \). Considering this fact and summing \( (3.21) \) in \( \alpha, \beta \), yields

\[
\bar{q}(-\omega_r) \Gamma = \bar{q}(\omega_r) G_r,
\]

(4.1)

where \( \Gamma = \frac{T \gamma}{4\pi A^2} \) and

\[
\gamma = \frac{3\pi \rho_s^2 \Delta^2}{D} = 3\pi \left( \frac{\bar{J}}{\Delta} \right)^2.
\]

(4.2)
Inserting (4.1) in (3.20), we get
\[ m^2 + \omega_r^2 = \frac{\Lambda^2}{e^{6\pi \rho_s (\chi_{\sigma} - \bar{q}_r)} - 1} + A(\chi_r - \bar{q}_r), \quad (4.3) \]
or equivalently
\[ M_r = \frac{\Lambda^2}{e^{6\pi \rho_s (\chi_{\sigma} - \bar{q}_r)} - 1} \quad (4.4) \]

We can use (4.3) in order to determine \( \chi_r - \bar{q}_r \). Let us start with \( r = 0 \). In this case, both \( \chi_0 \) and \( q_0 \) are real and equation (4.3) is depicted in Fig. 1.

![Graph](image)

**FIG. 1:** Function appearing in the rhs of (4.3) \((r = 0)\) for different values of the Gaussian width \( \Delta J \) (a: 220K, b: 210K, c: 200K, d: 190K). The horizontal dashed line represents \( m^2 \). A PM phase will only occur when this line intercepts the function \( m^2 \). The physical solution corresponds to the left branch. \( m \) is in \( K \) and \( \chi_0 \) and \( q_0 \) are in \( K^{-1} \).

The function on the r.h.s. of (4.3) has a minimum at
\[ \chi_0 = \bar{\chi}_c = \frac{1}{6\pi \rho_s} \ln \left[ 1 + \gamma + \frac{\gamma}{2} \left( 1 + \frac{4}{\gamma} \right)^{1/2} - 1 \right], \]

at which the function has the value
\[ m_0^2 = \frac{\Lambda^2}{\gamma} \left[ \frac{1}{1 + \frac{1}{2} \left( 1 + \frac{4}{\gamma} \right)^{1/2} - 1} + \ln \left[ 1 + \gamma + \frac{\gamma}{2} \left( 1 + \frac{4}{\gamma} \right)^{1/2} - 1 \right] \right]. \quad (4.5) \]

It follows that for \( r = 0 \) (4.3) will only have solutions for \( m^2 > m_0^2 \). In this case, however, these solutions of (4.3) are clearly not compatible with the existence of nontrivial solutions \( (q_0 \neq 0) \) of (4.1).

We can see this as follows. A solution \( \bar{q}_0 \neq 0 \) of (4.1) would imply \( G_0 = \Gamma \). In the range of values of \( m^2 \) for which (4.3) has solutions \( (m^2 > m_0^2) \), however these will be such that \( G_0 < \Gamma \). This is so because \( G_0 \) is a monotonically decreasing function of \( M_0 \) and according to (4.4), \( M_0 \) is a monotonically increasing function of \( m^2 \), such that precisely \( G_0(M_0(m_0^2)) = \Gamma \), as can be immediately inferred from (3.22) and from
\[ M_0(m_0^2) = \frac{\Lambda^2}{2} \left[ 1 + \frac{1}{\gamma} \right]^{1/2} - 1, \quad (4.7) \]

It immediately follows that, for \( m^2 > m_0^2 \), we will have \( G_0(M_0(m_0^2)) < \Gamma \). Hence, the only possible solution of (4.1), for \( m^2 > m_0^2 \), is \( \bar{q}_0 = 0 \). For \( m^2 < m_0^2 \), however, (4.3) no longer provides a solution for \( \chi_0 \), hence we may now have \( \bar{q}_0 \neq 0 \). The static susceptibility \( \chi_0 \) is now determined by (4.1), namely, \( G_0 = \Gamma \).

Now consider the case \( r \neq 0 \). We show in Appendix A that we always have \( q_r = 0 \) (for \( r \neq 0 \)), because otherwise (4.3) and (4.1) again become incompatible. We will always have, therefore, the \( \chi_{\sigma} \neq 0 \) determined by (4.3).

Since \( q_{EA} = Tq_0 \) it follows that \( \bar{q}_0 \) is also a SG order parameter and we conclude that the former phase \( (m^2 > m_0^2) \) is a paramagnetic phase \( (\sigma = 0, q_{EA} = 0) \), whereas the latter \( (m^2 < m_0^2) \) is a SG phase \( (\sigma = 0, q_{EA} \neq 0) \). The phase transition occurs at \( m^2 = m_0^2 \).

The ratio \( \frac{1}{\gamma} \), which appears in the expression of the critical mass \( m_0 \), is a measure of the amount of frustration in the system, as we can infer from (4.2) and the actual perturbation parameter. Since we are working in the regime of weak disorder, we take \( \gamma \gg 1 \). In the unperturbed limit where the disorder is removed \( (\gamma \to \infty) \), we would have \( m_0^2 = 0 \) and the SG phase would no longer exist. We see again that a disorder perturbation would be impossible in the original EA model, where \( \gamma = 0 \).

### 2. The Paramagnetic Phase

Let us now use the SPE in order to derive expressions for the susceptibilities \( \chi_1 \) and \( \chi_0 \) in the PM phase, where \( m^2 > m_0^2 \). Inserting (3.20) in (3.19), for \( r = 0 \) and \( q_0 = 0 \) (and also \( q_{r \neq 0} = 0 \)), we readily obtain, for the integrated susceptibility
\[ \chi_{1,PM}^0 = \frac{1}{3T}, \quad (4.8) \]

It follows from (3.20) that
\[ \chi_{0,PM}^0 = \frac{1}{3T} - \frac{1}{6\pi \rho_s} \sum_{\omega_r \neq 0} \left[ \ln(A^2 + M_r) - \ln M_r \right]. \quad (4.9) \]

The previous sums are dominated by large values of \( \omega_r \). In this case, we show in Appendix A that (4.3) or equivalently (4.1) or (4.2) yield the solution
\[ \chi(\omega_r) \cong \frac{\Lambda^2}{6\pi \rho_s (m^2 + \omega_r^2)}. \quad (4.10) \]
We may then evaluate the two sums above, obtaining for them an explicit expression
\[
\sum_{\omega_r \neq 0} \left[ \ln(\Lambda^2 + M_r) - \ln M_r \right] = \ln \Upsilon(m^2, T, \gamma),
\] (4.11)
where
\[
\Upsilon = \frac{\sinh^2 \left( \frac{\sqrt{X_+}}{2T} \right) \sinh^2 \left( \frac{\sqrt{X_-}}{2T} \right)}{\sinh^2 \left( \frac{1}{2T} \sqrt{m^2 + \Lambda^2} \right) \sinh^2 \left( \frac{1}{2T} \sqrt{m^2 - \Lambda^2} \right)} \times \frac{\left( \frac{m + \Lambda^2}{2T} \right) \left( \frac{m - \Lambda^2}{2T} \right)}{X_+ X_-},
\]
(4.12)
with
\[
X_{\pm} = \left( \Lambda^2 + 2m^2 \right) \left[ \frac{1}{2} + \frac{1}{2} \sqrt{\frac{\Lambda^4}{(\Lambda^2 + 2m^2)^2} \left[ 1 + \frac{4}{\gamma} \right]} \right].
\]
(4.13)
The static susceptibility in the PM phase, therefore, is given by
\[
\chi^\text{PM}_0 = \frac{1}{3T} - \frac{1}{6\pi \rho_s} \ln \Upsilon(m^2, T, \gamma).
\]
(4.14)
The function \( \Upsilon(m^2, T, \gamma) \) has the properties
\[
\Upsilon(m^2, T, \gamma) \xrightarrow{T \rightarrow \Lambda} 1
\]
and
\[
\ln \Upsilon(m^2, T, \gamma) \xrightarrow{T \rightarrow 0} \frac{2\pi}{T} \rho_0,
\]
(4.15)
where (for \( \gamma >> 1 \))
\[
\rho_0 = \frac{\Lambda}{2\pi} \left[ 1 + \frac{1}{\gamma} \left[ 1 + \frac{1}{2} \ln(1 + \gamma) \right] \right].
\]
(4.16)
We also have
\[
\frac{1}{3T} - \frac{1}{6\pi \rho_s} \ln \Upsilon(m^2, T, \gamma) \xrightarrow{m^2 \rightarrow m_0^2} \chi_{\text{cr}},
\]
(4.17)
where \( \chi_{\text{cr}} \) is given by (4.5), implying that the critical value of \( \chi^\text{PM}_0 \) is \( \chi_{\text{cr}} \).
We see that \( \chi^\text{PM}_0 \) satisfies the Curie law at high-temperatures
\[
\chi^\text{PM}_0 \xrightarrow{T \gg \Lambda} \frac{1}{3T}
\]
(4.18)
and diverges as
\[
\chi^\text{PM}_0 \xrightarrow{T \rightarrow 0} \frac{1}{3T} \left[ 1 - \frac{\rho_0}{\rho_s} \right]
\]
(4.19)
for \( T \rightarrow 0 \). As we will see this is the expected behavior for \( \rho_s < \rho_0 \), where an AF phase appears at \( T = 0 \). For \( \rho_s > \rho_0 \), conversely, we will see that the PM-SG phase transition occurs at a finite \( T_c \) (Fig. 2 and Fig. 3) and the previous expression is no longer valid.

Notice that the following general relation involving the integrated susceptibility is automatically satisfied by \( \chi^\text{PM}_1 \) and \( q^\text{PM}(\omega_r) = 0 \):
\[
\chi_1 = \frac{1}{3T} - \frac{1}{3} \sum_{\omega_r} q(\omega_r)
\]
(4.21)

3. The Spin-Glass Phase

We now turn to the the SG phase \( (m^2 < m_0^2) \). Now (4.1), or equivalently \( G_0 = \Gamma \) implies
\[
M_0 = \frac{\Lambda^2}{2} \left[ 1 + \frac{1}{\gamma} \right] \left[ 1 + \frac{1}{\gamma} \left( \ln(\Lambda^2) - 1 \right) \right]
\]
(4.22)
which coincides with (4.7).
Then (3.19) yields (for \( \gamma >> 1 \))
\[
q^\text{SG}_0 = \frac{1}{3T} - \frac{1}{6\pi \rho_s} \ln \Upsilon(m^2, T, \gamma) - \frac{1}{6\pi \rho_s} \ln(1 + \gamma).
\]
(4.23)
From this and (4.22) we obtain
\[
\chi^{\text{SG}}_0 = \frac{1}{3T} - \frac{1}{6\pi \rho_s} \ln \Upsilon(m^2, T, \gamma) - \frac{\rho_0}{2D} \left[ m_0^2 - m^2 \right].
\]
(4.24)
Since (4.20) still holds for \( r \neq 0 \), we still have
\[
\sum_{\omega_r \neq 0} \chi(\omega_r) = \frac{1}{6\pi \rho_s} \ln \Upsilon(m^2, T, \gamma)
\]
(4.25)
and therefore we get
\[
\chi^{\text{SG}}_1 = \frac{1}{3T} - \frac{\rho_0}{2D} \left[ m_0^2 - m^2 \right],
\]
(4.26)
Comparing (4.24) and (4.14) and also (4.26) and (4.8) we can identify a clear cusp at the transition appearing in both susceptibilities. This is an important result, since the presence of these cusps is a benchmark of the SG transition and has been experimentally observed in many materials presenting a SG phase.

B. The Néel Phase

Let us now search for an ordered Néel phase, for which \( \sigma \neq 0 \). We see that in this case the quantity between brackets in (3.18) must vanish. By summing it in \( \alpha, \beta \), we conclude that \( M_0 = 0 \) in this phase. According to (3.19), however, this can only happen at \( T = 0 \), otherwise we would have an unphysical infinite imaginary staggered magnetization \( \sigma \). This is in agreement with the Mermin-Wagner theorem\(^{18} \) and is a clear evidence that our approach goes beyond mean-field.
Now, $M_0 = 0$ implies

$$\chi_0^{AF} - \tilde{q}_0^{AF} = \frac{m^2}{A}. \quad (4.27)$$

On the other hand, for $\sigma \neq 0$, (3.14) and (3.20) imply, instead of (1.8)

$$\chi_1^{AF} = \frac{1}{3T} \left(1 - \frac{2\sigma^2}{\rho_s}\right). \quad (4.28)$$

From this, it follows that

$$\chi_0^{AF} = \frac{1}{3T} - \frac{1}{6\pi \rho_s} \ln \Upsilon(m^2, T, \gamma) - \frac{2\sigma^2}{3T \rho_s}, \quad (4.29)$$

which in the limit $T \to 0$ reduces to

$$\chi_0^{AF} \approx 0 \left[1 - \rho_0 \rho_s\right] - \frac{2\sigma^2}{3T \rho_s}, \quad (4.30)$$

for $\rho_s > \rho_0$.

On the other hand, (4.21) leads to

$$\tilde{q}_0^{AF} = \frac{1}{T} \left(\frac{2\rho_s^2}{\rho_s}\right). \quad (4.31)$$

Equations (4.27) (4.30) and (4.31), allow us to solve for $\chi_0^{AF}$, $\tilde{q}_0^{AF}$ and $\sigma$. We get, for $T \to 0$,

$$\sigma^2 = \frac{1}{8} [\rho_s - \rho_0] \quad (4.32)$$

and

$$\chi_0^{AF} = \tilde{q}_0^{AF} = \frac{1}{4T \rho_s} [\rho_s - \rho_0]. \quad (4.33)$$

This implies, according to (4.27) a zero spin gap: $m^2 = 0$.

From (4.28) and (4.32) we obtain

$$\chi_1^{AF} = \frac{1}{12T} \left[3 + \frac{\rho_0}{\rho_s}\right]. \quad (4.34)$$

Notice that the susceptibilities diverge for $T \to 0$ as they should. The EA parameter, however, remains finite:

$$\tilde{q}_{EA}^{AF} = \frac{1}{4\rho_s} [\rho_s - \rho_0]. \quad (4.35)$$

We see that, indeed, there is an AF phase characterized by ($\sigma \neq 0$, $q_{EA} \neq 0$) on the line ($T = 0$, $\rho_s > \rho_0$), with $\rho_0$ given by (4.17). In the absence of disorder ($\Delta J = 0$, $\gamma \to \infty$), $\rho_0 \to \rho_0(0) = \frac{2\rho_s}{3\pi}$, which is the well-known quantum critical coupling determining the boundary of the AF phase in the pure 2D AF Heisenberg model at $T = 0$.[16,19,20] The effect of disorder on the AF phase is to displace the quantum critical point (QCP) to the right. This result should be expected on physical grounds: in the presence of disorder a larger coupling is required, to stabilize an ordered AF phase.

C. Critical Curve

We have seen that the parameter $m^2$ determines the transition between the PM and SG phases. It is important, consequently, to see how it depends on the control parameters of our system, namely, $T$, $\bar{J}$ and $\Delta J$, or, equivalently, $T$, $\rho_s$ and $\gamma$. We may obtain an equation for $m^2$ by using (4.21), (4.23), (4.26) and the fact that $\tilde{q}_{\neq 0} = 0$. These yield

$$m^2 - m_0^2 = \frac{2D}{3\rho_s} \left[\chi_{cr} - \frac{1}{3T} + \frac{1}{6\pi \rho_s} \ln \Upsilon(m^2, T, \gamma)\right]. \quad (4.36)$$

From this we get

$$\tilde{q}_{0}^{SG} = \frac{3\rho_s}{2D} \left[m_0^2 - m^2\right] \quad (4.37)$$

for $m^2 < m_0^2$. We see that $\tilde{q}_{0}^{SG} \to 0$ at the transition as it should. For $m^2 > m_0^2$ we have $\tilde{q}_{0}^{PM} = 0$ as above.

We may determine the critical curve by observing that the critical condition $m^2 = m_0^2$ implies

$$\frac{1}{3T_c} - \frac{1}{6\pi \rho_s} \ln \Upsilon(m_0^2, T_c, \gamma) = \chi_{cr} \quad (4.38)$$

For $T_c \ll \Lambda$, which corresponds the situation found in realistic systems, this becomes, near the quantum critical point ($\rho_s \lesssim \rho_0$),

$$\frac{T_c}{2\pi} \left[\ln \left(\frac{\Lambda}{T_c}\right)^2 - \ln(1 + \gamma)\right] = \rho_0 - \rho_s, \quad (4.39)$$

which is the equation for the critical curve separating the PM and SG phases. Notice that it meets the $T = 0$ axis, precisely at the quantum critical point $\rho_0$, separating the SG from the AF phase. We plot the $T_c \times \rho_s$ phase diagram corresponding to (4.39), for a fixed value of $\gamma$ in Fig. 2.

In Fig. 3 we plot again $T_c \times \rho_0$, for a fixed value of the amount of disorder, namely, the Gaussian width $\Delta J$.

Now we plot in Fig. 4 the associated $T_c \times \Delta J$ phase diagram, for different values of the spin stiffness. The latter shows the AF-SG transition as a function of increasing disorder. This is the type of transition which is observed in the high-Tc cuprates. In order to describe it, we must relate the doping parameter of these materials to our disorder parameter, $\Delta J$. We are presently investigating this point.

Below, we plot the quantum critical point $\rho_0$ as a function of the amount of disorder, for different values of $\rho_s$.

D. Critical Behavior

The critical behavior of relevant quantities may be determined by analyzing the function $\Upsilon(m^2, T, \gamma)$ for
FIG. 2: Phase diagram for a fixed value of $\gamma$. ($\gamma = 10^2$, $\Lambda = 10^3$). The critical curve corresponds to (4.39) and is valid near the QCP $\rho_0$ (solid curve). $\rho_0(0) = \Lambda/2\pi$ is the QCP of the pure AF system. Notice that disorder besides creating the SG phase, displaces the QCP to the right. The value ascribed to $\Lambda$ is a realistic one in $K$ ($\Lambda \rightarrow \hbar v_s k_B \Lambda$; $v_s$: spin-wave velocity). The resulting temperatures naturally appear with the correct order of magnitude, in $K$, found in real SG systems.

FIG. 3: Phase diagram for $\Lambda = 10^3$ and $\Delta J$ (a: 200K, b: 350K, c: 460K). Notice that the AF phase (thick line) is displaced to the right as we increase the disorder.

FIG. 4: Phase diagram $T_c \times \Delta J$, for $\Lambda = 10^3$ and $\rho_s/\rho_0(0)$ (a: 1.01, b: 1.03, c: 1.05). Notice that as we increase the value of the center of the Gaussian ($\rho_s$), a larger amount of disorder will be required for the SG phase.

FIG. 5: The quantum critical point as a function of the amount of disorder for different values of the average coupling, $\rho_s$ (a: 180K, b: 200K, c: 220K). Notice that in the limit of zero disorder $\rho_0 \rightarrow \rho_0(0) = \Lambda/2\pi$.

The SG order parameter and susceptibilities for $T \gg T_c$ and $\rho_s < \rho_0$:

$$\chi_0^{PM} \sim \left(\frac{T_c}{T}\right) \bar{\chi}_c; \quad \chi_1^{PM} = \frac{1}{3T}; \quad \delta_0^{PM} = 0$$ (4.42)

and for $T \ll T_c$ and $\rho_s < \rho_0$:

$$\chi_0^{SG} \sim \left(\frac{T_c}{T}\right) \bar{\chi}_c - \frac{2\pi\Delta \rho_s}{D} \left[\frac{T_c - T}{T_c}\right][\rho_0 - \rho_s]$$ (4.43)

$$\chi_1^{SG} \sim \frac{1}{3T} - \frac{2\pi\Delta \rho_s}{D} \left[\frac{T_c - T}{T_c}\right][\rho_0 - \rho_s].$$ (4.44)
follows.

We plot $\chi_0$ in Fig. 6. We can see the characteristic cusps of the SG transition occurring in these magnetic susceptibilities.

![Figure 6: The static susceptibility for different values of $\rho_s$.](image)

For $\rho_s > \rho_0$ and $T > 0$, we always have $m^2 > m_0^2$ and $\sigma = \bar{q}_0 = 0$ i.e. the system is in the PM phase for any finite temperature.

E. Dependence on Disorder

Let us now examine the $\Delta J$ dependence of the phase diagram. For this we make $\Delta J \rightarrow \Delta J(1 + \epsilon)$, with $|\epsilon| \ll 1$, for fixed $\rho_s$, and study how the relevant quantities change. We find,

$$\rho_0 - \rho_0 = \epsilon \frac{\Lambda}{2\pi\gamma} \ln \gamma. \quad (4.46)$$

For $\rho_s < \rho_0$ and a fixed $\rho_s$, we also obtain

$$T_c - T_c = \frac{[m_0^2] - m_0^2}{4\pi\Lambda\rho_0} = \frac{\rho_0 - \rho_0}{\Delta \rho} = \epsilon \frac{\Lambda}{2\pi\gamma \Delta \rho} \ln \gamma., \quad (4.47)$$

where $\Delta \rho = \rho_0 - \rho_s$.

The SG order parameter changes as

$$[\bar{q}_0^{\text{SG}}]^2 - \bar{q}_0^{\text{SG}} = \frac{\epsilon}{\pi\rho_s} \ln \gamma. \quad (4.48)$$

We see that increasing the amount of disorder, through an increment of the Gaussian width, will increase $\rho_0$ and, for a fixed $\rho_s$, also $T_c$, $m_0^2$ and $\bar{q}_0^{\text{SG}}$. Conversely, decreasing the amount of disorder by narrowing the Gaussian width will produce the opposite effects.

F. Effect of Quantum Fluctuations

As mentioned before, the effect of quantum fluctuations, introduced by integration over the quadratic fluctuations of the CP$^1$ fields and embodied in the last term of (4.13), is essential for the obtainment of a sensible solution for this system. Should we do a pure mean-field approach, by disregarding these fluctuations, we would obtain a Néel state for any temperature, in obvious disagreement with the Mermin-Wagner theorem. Conversely, taking these fluctuations into account washes out the Néel phase to $T = 0$, but leaving a spin glass state below the critical curve. The fact that the AF phase is removed to $T=0$ shows that our approach transcends the mean-field approximation. The SG phase is robust to such fluctuations. In the spirit of the loop expansion that is being done, we assume that higher quantum fluctuations will not change this picture qualitatively. Anyway, considering only the quadratic fluctuations is always a valid first approach to a difficult problem.

V. THE THERMODYNAMIC STABILITY

We finally consider the important question of the thermodynamic stability of the phases. As is well known the replica symmetric mean-field solution of the SK model turned out to be unstable. It is, therefore, absolutely necessary to analyze the stability of any solution to a SG. We will focus on the SG and PM phases. The stability of the AF phase should not be a problem.

For studying the stability of the solution, we must consider the Hessian matrix of the free-energy density

$$f = f[\sigma^\alpha q^{\alpha \beta}(\omega), ..., q^{\alpha \beta}(\omega), \lambda; \chi(\omega), ..., \chi(\omega), ...], \quad (5.1)$$

where the variables $q^{\alpha \beta}(\omega_r)$ and $\chi(\omega_r)$ are complex, such that $\chi_{-r} = \chi^*_r$ and $q^{\alpha \beta}_{-r} = q^{\alpha \beta}_r$. The Hessian is given by

$$H_{ij} = \frac{\partial^2 f}{\partial \phi_i \partial \phi_j}. \quad (5.2)$$

where $\phi_i = \sigma^\alpha q^{\alpha \beta}(\omega_0), \Re q^{\alpha \beta}(\omega_r), \Im q^{\alpha \beta}(\omega_r), \lambda \chi(\omega_0), \Re \chi(\omega_r), \Im \chi(\omega_r)$, where the index $r$ runs from 1 to $\infty$. This is a matrix with entries of dimensions $\lim_{n \rightarrow 0} [n; n(n - 1), ..., 2(n - 1), ..., 1; 1, 2, ...]$, corresponding, respectively, to derivatives with respect to each of the above variables. Since we are interested here in the SG and PM phases, we will take (5.2) at $\sigma = 0$.

The elements of the Hessian matrix are as follows: there are four overall diagonal and six crossed elements, namely $\sigma \sigma^\alpha q, \lambda \lambda, \chi \chi, \sigma q, \sigma \lambda, q \lambda, q \chi$ and $\lambda \chi$.

The $\sigma \sigma$ term is,

$$\frac{\partial^2 f}{\partial \sigma^\alpha \partial \sigma^\beta} = \frac{1}{n} [M_0 \delta^{\alpha \beta} - A\bar{q}_0 C^{\alpha \beta}] \quad (5.3)$$
The \( q, q_s \) terms are proportional to \( \delta_{rs} \). For \( r = 0 \)
\[
\frac{\partial^2 \bar{f}}{\partial q_0^a \partial q_0^a} = [a_0 \delta^{\alpha \gamma} \delta^{\beta \eta} + b_0 C^{\alpha \gamma} C^{\beta \eta}] \equiv K_0, \tag{5.4}
\]
where
\[
a_r = A^2 \frac{\Gamma}{n}; \quad b_r = A^2 \frac{G_r - H_r}{n(n - 1)^2} \tag{5.5}
\]
and
\[
H_r = A \frac{T}{4\pi} \bar{q}_r \left[ \frac{1}{M_r^2} - \frac{1}{(\Lambda^2 + M_r)^2} \right]. \tag{5.6}
\]
For the \( r \neq 0 \) blocks, we have four terms, namely
\[
\frac{\partial^2 \bar{f}}{\partial \text{Re} q^{\alpha \beta}(\omega_r) \partial \text{Re} q^{\alpha \beta}(\omega_s)} =
\]
\[
[a_r \delta^{\alpha \gamma} \delta^{\beta \eta} + \text{Re} \ b_r C^{\alpha \gamma} C^{\beta \eta}] \equiv K_r(11) \delta_{rs}, \tag{5.7}
\]
\[
\frac{\partial^2 \bar{f}}{\partial \text{Im} q^{\alpha \beta}(\omega_r) \partial \text{Im} q^{\alpha \beta}(\omega_s)} =
\]
\[
[a_r \delta^{\alpha \gamma} \delta^{\beta \eta} - \text{Re} \ b_r C^{\alpha \gamma} C^{\beta \eta}] \equiv K_r(22) \delta_{rs}, \tag{5.8}
\]
\[
\frac{\partial^2 \bar{f}}{\partial \text{Re} q^{\alpha \beta}(\omega_r) \partial \text{Im} q^{\alpha \beta}(\omega_s)} =
\]
\[
[-\text{Im} \ b_r C^{\alpha \gamma} C^{\beta \eta}] \equiv K_r(12) \delta_{rs} = K_r(21) \delta_{rs}. \tag{5.9}
\]
The \( \lambda \lambda \) term is,
\[
\frac{\partial^2 \bar{f}}{\partial \lambda \partial \lambda} = - \frac{\partial^2 \bar{f}}{\partial m^2 \partial m^2} = \sum_r \{G_r + H_r\} \equiv \varphi \tag{5.10}
\]
The \( \chi_r \chi_s \) terms are also proportional to \( \delta_{rs} \). For \( r = 0 \), we have
\[
\frac{\partial^2 \bar{f}}{\partial \chi_0 \partial \chi_0} = A^2 [\Gamma - F_0] \equiv L_0 \tag{5.11}
\]
where \( F_r \equiv G_r + H_r \).
For \( r \neq 0 \), we have
\[
\frac{\partial^2 \bar{f}}{\partial \text{Re} \chi_r \partial \text{Re} \chi_s} = A^2 [\Gamma + \text{Re} F_r] \equiv L_r(11) \delta_{rs}, \tag{5.12}
\]
\[
\frac{\partial^2 \bar{f}}{\partial \text{Im} \chi_r \partial \text{Im} \chi_s} = A^2 [\Gamma - \text{Re} F_r] \equiv L_r(22) \delta_{rs}, \tag{5.13}
\]
\[
\frac{\partial^2 \bar{f}}{\partial \text{Re} \chi_r \partial \text{Im} \chi_s} = A^2 [\text{Im} F_r] \equiv L_r(12) \delta_{rs} = L_r(21) \delta_{rs}. \tag{5.14}
\]
The \( \sigma q, \sigma \lambda \) and \( \sigma \chi \) crossed terms vanish for \( \sigma = 0 \):
\[
\frac{\partial^2 \bar{f}}{\partial \sigma^\alpha \partial q_0^\beta} = \frac{\partial^2 \bar{f}}{\partial \sigma^\alpha \partial \lambda} = \frac{\partial^2 \bar{f}}{\partial \sigma^\alpha \partial \chi_r} = 0 \tag{5.15}
\]
The only non-vanishing \( q \lambda \) and \( q \chi \) terms are the ones for which \( r = 0 \). These are respectively,
\[
\frac{\partial^2 \bar{f}}{\partial \lambda \partial q_0^\alpha} = i \frac{\partial^2 \bar{f}}{\partial m^2 \partial q_0^\alpha} = - i A C_0 \tag{5.16}
\]
\[
\frac{\partial^2 \bar{f}}{\partial \chi_0 \partial q_0^\alpha} = - A^2 C_0 \tag{5.17}
\]
where
\[
C_0 = \frac{H_0}{n(n - 1)} \tag{5.18}
\]
The \( r \neq 0 \) terms vanish because \( \bar{q}_r = 0 \) (and consequently \( H_r = 0 \)) for \( r \neq 0 \).
Finally, the \( \lambda \chi \) terms are
\[
\frac{\partial^2 \bar{f}}{\partial \lambda \partial \text{Re} \chi_r} = i \frac{\partial^2 \bar{f}}{\partial m^2 \partial \text{Re} \chi_r} = - i A [\text{Re} F_r] \tag{5.19}
\]
and
\[
\frac{\partial^2 \bar{f}}{\partial \lambda \partial \text{Im} \chi_r} = i \frac{\partial^2 \bar{f}}{\partial m^2 \partial \text{Im} \chi_r} = - i A [\text{Im} F_r]. \tag{5.20}
\]
The complete Hessian is as follows:
Here $\tilde{K}_r$ is the matrix block with elements given by $\sigma_{\alpha\beta}$.

A necessary and sufficient condition for the mean field solution to be a local minimum is to have all the principal minors of the Hessian positive. This would be equivalent to having all the eigenvalues of the Hessian positive. The principal minors are the determinants of the matrices obtained from the original matrix by successively stripping the last line and the last column, starting from the matrix itself and ultimately reaching the $(11)$ element. In the $n \to 0$ limit we have the following set of principal minors of $\mathbb{H}$: $D_\sigma, D_{q_0}, ..., D_{q_r}, ..., D_\lambda, D_{\chi_0}, ..., D_{\chi'}, ..., D_{\chi''}$. In the previous expressions, the prime and double prime refer to the two principal minor determinants generated by the sub-matrix $L_r(1, j)$.

We have carefully evaluated each of these determinants (see Appendix C) in the limit $n \to 0$, for $\sigma = 0$ (PM and SG phases).

Defining

$$P_r \equiv \prod_{s=1}^{r} \left[ \frac{(\Gamma^2 - |G_s|^2)}{\Gamma^2} \right]$$ (5.22)

with

$$P = \lim_{r \to \infty} P_r$$ (5.23)

and

$$G \equiv \sum_r G_r = G_0 + 2 \sum_{r=1}^{\infty} \text{Re} G_r$$ (5.24)

(In Appendix D, we demonstrate that both $P$ and $G$ are finite and positive).

we obtain

$$D_\sigma = 1;$$ (5.25)

$$D_{q_0} = \frac{1}{\Gamma} \left[ (\Gamma - G_0) + H_0 \right];$$ (5.26)

$$D_{q_r} = D_{q_0} P_r; \quad D_\lambda = D_{q_0} PG;$$ (5.27)

$$D_{\chi_0} = A^2 \left[ G_0^2 D_{q_0} + G \left( \frac{(\Gamma - G_0)^2}{\Gamma} \right) \right] P;$$ (5.28)

$$D'_{\chi_r} = A^2 [\Gamma + \text{Re} G_r] D''_{\chi_{r-1}} +$$

$$\frac{A^4}{\Gamma} (\Gamma - G_0)^2 \left[ \text{Re} G_r \right]^2 \left[ \prod_{s=1}^{r-1} A^2 \left[ \Gamma^2 - |G_s|^2 \right] \right] P$$ (5.29)

and

$$D''_{\chi_r} = [A^4 (\Gamma^2 - |G_r|^2)] D''_{\chi_{r-1}} +$$
\[
\frac{A^2}{\Gamma} (\Gamma - G_0)^2 A^2 |G_r|^2 \left[ \Gamma - \text{Re} \, G_r \right] \prod_{s=1}^{r-1} A^2 \left[ \Gamma^2 - |G_s|^2 \right] P
\]

or equivalently,

\[
D'_{\chi r} = \prod_{s=1}^{r} \left[ A^4 |\Gamma^2 - |G_s|^2| \right] D_{\chi 0} +
\]

\[
\frac{A^2}{\Gamma} (\Gamma - G_0)^2 \sum_{s=1}^{r} A^4 |G_s|^2 |\Gamma - \text{Re} \, G_s| \prod_{t \neq s}^{r} A^4 \left[ \Gamma^2 - |G_t|^2 \right] P
\]

We see that all the principal minors, except \( D_\sigma = 1 \), can be written in the form

\[
\xi D_{\sigma 0} + \eta |\Gamma - G_0|^2,
\]

where \( \xi \) is a positive factor and \( \eta \) is either positive or zero. These properties follow from the fact that \( \Gamma > |G_r| \), for \( r \neq 0 \), as we demonstrate in Appendix B.

From (5.32), we see that \( D_{\sigma 0} \) is a key piece in the evaluation of the principal minors. In the PM phase we have \( q_0 = 0, H_0 = 0 \) and \( G_0 < \Gamma \), therefore it follows that \( D_{\sigma 0} > 0 \), hence all the principal minors are positive. In the SG phase, conversely, \( q_0 > 0, H_0 > 0 \) and \( G_0 = \Gamma \), implying that again \( D_{\sigma 0} > 0 \). It follows that all the principal minors are positive also in the SG phase. At the transition, all principal minors vanish, except for \( D_\sigma = 1 \).

The above result establishes the thermodynamic stability of the SG and PM phases obtained from our solution. Furthermore, we can see the phase transition occurring by direct inspection of the principal minors, since they vanish at the transition point. This result rules out the usual instabilities found in the long-range interacting replica-symmetric solutions. We can still have, of course, meta-stability, which seems to be a characteristics of spin-glasses. This would deserve further investigation. For this purpose, a promising procedure would be to use the method of quantum annealing, in order to find the absolute minimum of the free energy.

VI. CONCLUDING REMARKS

We have proposed a model for describing a short-range interacting, disordered quantum magnetic systems with SO(3) symmetry on a square lattice. The random distribution of couplings is a Gaussian biased to an AF coupling. A replica symmetric stable solution was obtained, which clearly shows the existence of a stable genuine SG thermodynamical phase, at a finite \( T \). This can be seen directly from the solution, but also by examining the principal minors of the Hessian matrix of the free energy, which are all positive, both in the PM and in the SG phases, but vanish at the transition.

The use of \( \Delta J \ll \bar{J} > 0 \), allowed us to assume the cancelation of the Berry phases. Relaxing this condition, we would obtain a Chern-Simons term for the \( A_\mu \)-field in the final CP1 version of the model. This system deserves a deeper investigation but, presumably, the presence of this term will not modify the phase structure found here.

Our solution takes into account the quantum fluctuations of the CP1 fields and therefore transcends the mean-field approximation. This fact becomes evident, when we note that our solution does predict any ordered AF phase at \( T \neq 0 \), in agreement with the Mermin-Wagner theorem, but contrary to what a mean field approximation would yield.

The stable SG phase derives from a replica-symmetric solution. This indicates that, in the case of short-ranged interactions, there is no basic clash between the replica-symmetry of the SG solution and its stability. We are naturally led to inquire, therefore, whether the instability, which has been found in the SK solution, is actually produced by the long-range interaction itself, rather than by the replica symmetry it possesses.

The plots of the magnetic susceptibilities versus the temperature, exhibit the characteristic cusps, experimentally found at the PM-SG transition, in materials exhibiting the SG phase. By choosing a realistic value for the momentum cutoff \( \Lambda \), we see that the cusps occur precisely at the temperatures values, which are observed experimentally. This provides clear evidence that our model is really capable of describing realistic SG systems and our results are not an artifact of the mean-field approximation.

Our model nicely describes the AF-SG transition, which occurs as we increase the amount of disorder, hence it will be probably useful in the description of the corresponding transition in the high-Tc cuprates. We are currently investigating this point.

Acknowledgments

ECM would like to thank Curt Callan and the Physics Department of Princeton University, where part of this work was done, for the kind hospitality. This work was supported in part by CNPq and FAPERJ. CMSC was supported by FAPERJ. We are grateful to P.R. Wells for the help with the graphics.

APPENDIX A: DETERMINANT OF THE QUANTUM FLUCTUATIONS

Let us evaluate here the determinant of the matrix of quantum fluctuations \( M \) appearing in (3.10). From (3.7) and (3.5) we obtain

\[
M = \begin{pmatrix}
K^{\alpha \beta} + A^{\alpha \beta}_{11} & C^{\alpha \beta}_{12} \\
C^{\alpha \beta}_{21} & K^{\alpha \beta} + A^{\alpha \beta}_{22}
\end{pmatrix},
\]
where, already in momentum-frequency space,

\[ K^{\alpha \beta} = \left[ |\vec{k}|^2 + \omega_r^2 + m^2 \right] \delta^{\alpha \beta}, \quad (A.2) \]

\[ A_{ij}^{\alpha \beta} = A^{\alpha \beta} z_i^* z_j, \quad (A.3) \]

and

\[ C_{ij}^{\alpha \beta} = A^{\alpha \beta} z_i z_j. \quad (A.4) \]

In these expressions,

\[ A^{\alpha \beta} = \frac{2D}{\rho_s^2} \left[ [\bar{q}(\omega_r) - \chi(\omega_r)] \delta^{\alpha \beta} - \bar{q}(\omega_r) C^{\alpha \beta} \right], \quad (A.5) \]

where \( C^{\alpha \beta} \) is the \( n \times n \) matrix with all elements equal to one and \( \chi(\omega_r) \) and \( \bar{q}(\omega_r) \) are, respectively, the Fourier transforms of \( \chi(\tau - \tau') \) and \( \bar{q}(\tau - \tau') \).

We want to calculate

\[ \ln \det M = \ln \det \det \det M, \quad (A.6) \]

where the three determinants run, respectively over the momentum-frequency arguments of the fields, the replicas and the \( z \)-field components. The first determinant can be easily evaluated by diagonalizing \( (A.1) \). We get

\[ \ln \det M = \ln \det \det \left[ K^{\alpha \beta} + \rho_s A^{\alpha \beta} \right] \quad (A.7) \]

The two remaining determinants were evaluated in\(^{12} \). The one over the replicas yields exactly, for \( n \to 0 \),

\[ \det_{\alpha \beta} \left[ K^{\alpha \beta} + \rho_s A^{\alpha \beta} \right] = N_0^n \left[ 1 - n \frac{N_1}{N_0} \right], \quad (A.8) \]

where

\[ N_0 = |\vec{k}|^2 + \omega_r^2 + m^2 - \frac{2D}{\rho_s} (\chi_r - \bar{q}_r) \]

and

\[ N_1 = \bar{q}_r \left( \frac{2D}{\rho_s} \right). \]

Inserting \( (A.8) \) in \( (A.7) \), finally, we can write the last determinant as trace, which for \( n \to 0 \) reads

\[ \ln \det M = n \sum_{\omega_r} \text{Tr} \left[ \ln N_0 - \frac{N_1}{N_0} \right]. \quad (A.9) \]

This is given by

\[ \ln \det M = nV \sum_{\omega_r} \int \frac{d^2k}{(2\pi)^2} \left[ \ln \left( k^2 + M_r \right) - \frac{A\bar{q}(\omega_m)}{k^2 + M_r} \right], \quad (A.10) \]

where \( A = \frac{2D}{\rho_s} \) and \( M_r \) is given by \( (3.14) \).

**APPENDIX B: \( \chi(\omega_r) \) AND \( \bar{q}(\omega_r) \) FOR \( r \neq 0 \)**

1. The Stationary Phase Equations

Let us define \( 6\pi \rho_s [\chi(\omega_r) - \bar{q}(\omega_r)] \equiv \alpha + i\theta \), for an arbitrary \( r \neq 0 \). Then, the mean field equation \( (3.3) \) yields the two equations

\[ 2\theta \left[ \cosh \alpha - \cos \theta \right] = \gamma \sin \theta. \quad (B.1) \]

and

\[ m^2 + \omega_r^2 = \frac{\Lambda^2 \theta - e^{-\alpha}}{\theta} + \frac{\Lambda^2}{\gamma} \alpha. \quad (B.2) \]

These equations imply \( |G_r| < \Gamma \), as we demonstrate below.

2. \( |G_r| < \Gamma \)

Indeed, from \( (B.3) \), we have

\[ \frac{\Lambda^2}{M_r} = e^{\alpha + i\theta} - 1. \quad (B.3) \]

This yields, from \( (622) \)

\[ G_r = \frac{2\Gamma}{\gamma} \left[ \cosh(\alpha + i\theta) - 1 \right], \quad (B.4) \]

which implies

\[ |G_r|^2 = \left( \frac{\Gamma}{\gamma} \right)^2 \left[ 2 \left( \cosh \alpha - \cos \theta \right) \right]^2. \quad (B.5) \]

From \( (B.1) \), we see that, for \( \theta \neq 0 \)

\[ |G_r| = \Gamma \left| \frac{\sin \theta}{\theta} \right| < \Gamma \quad (B.6) \]

For \( \theta = 0 \), conversely, \( G_r \) is real and we see from Fig.1 that \( \alpha_r < \alpha_0 \) (for \( r \neq 0 \)). Since \( G_r \) is a monotonically increasing function of \( \alpha_r \), it follows that \( G_r < G_0 \). Now, as we have seen, \( G_0 \leq \Gamma \), hence \( G_r < \Gamma \) for \( \theta = 0 \). We conclude therefore that, for \( r \neq 0 \), we always have \( |G_r| < \Gamma \).

For a non-vanishing \( \bar{q}_r \), however, Eq. \( (B.1) \) implies \( |G_r| = \Gamma \). We conclude, therefore, that \( \bar{q}_r \neq 0 \), or equivalently \( (B.1) \) and \( (B.2) \) will only admit solutions for \( r \neq 0 \) when \( \bar{q}_r = 0 \). In this work, therefore we always have \( \bar{q}_r \neq 0 \). We also choose the \( \theta = 0 \) solutions of \( (B.1) \) and \( (B.2) \) for all \( r \neq 0 \), which imply real \( \chi_r \)'s.
APPENDIX C: THE PRINCIPAL MINORS OF THE HESSIAN

1. The Determinant $D_\sigma$

According to (5.3), the determinant of the $(\sigma \sigma)$ $n \times n$ block of the Hessian, for finite $n$ is given by

$$D_\sigma(n) = \left( \frac{M_0}{n} \right)^{n-1} \left[ \frac{M_0}{n} - nA \frac{\bar{q}_0}{n} \right]. \quad \text{(C.1)}$$

In the limit $n \to 0$, this gives

$$D_\sigma = \lim_{n \to 0} = 1 - nA \frac{\bar{q}_0}{M_0} = 1, \quad \text{(C.2)}$$

where we used the fact that

$$\lim_{n \to 0} \left( \frac{M_0}{n} \right)^n = 1.$$

2. The Determinant $D_{q_0}$

From (5.27), we have the determinant of the $(q_0q_0)$ $n(n-1) \times n(n-1)$ block of the Hessian, for finite $n$, given by

$$D_{q_0}(n) = a_0^n [a_0 - n(n-1)b_0], \quad \text{(C.3)}$$

where $a_0$ and $b_0$ are given by (5.5), for $r = 0$.

In the $n \to 0$ limit, this gives

$$D_{q_0} = \lim_{n \to 0} \left[ \frac{\Gamma}{n} \left( \frac{G_0 - H_0}{n(n-1)} \right) \right] = \frac{1}{\Gamma} \left[ (\Gamma - G_0) + H_0 \right], \quad \text{(C.4)}$$

which is (5.28).

3. The Determinant $D_{q_r}, r \neq 1$

From (5.27)-(5.9), we can show that the determinant of the $2n(n-1) \times 2n(n-1)$-dimensional, $(q_r q_r)$ block of the Hessian, for finite $n$, given by

$$D_{q_r}^{(r)}(n) = \left[ a_r^{2n(n-1)} + 2n(n-1)b_0 a_r^{2n(n-1)-1}|b_r| \right] \times \left[ a_r^{2n(n-1)} - 2n(n-1)b_0 a_r^{2n(n-1)-1}|b_r| \right], \quad \text{(C.5)}$$

where $a_r$ and $b_r$ are given by (5.5).

The limit $n \to 0$ can be taken in the same way as we did in the previous subsection. Using the fact that $H_r = 0$ for $r \neq 0$, we obtain,

$$D_{q_r}^{(r)} = \frac{\Gamma^2 - |G_r|^2}{\Gamma^2}, \quad \text{(C.6)}$$

Since the $q$-part of the Hessian is block-diagonal, we immediately establish (5.27) for $D_{q_r}$. The limit $r \to \infty$ exists, as we show in Appendix C.

4. The Determinant $D_\lambda$

From (5.2), we have that

$$D_\lambda(n) = \varphi D_{q_0} + A^2 n(n-1) C_0^2 a_0^{n(n-1)-1} \prod_{r \neq 0} \left[ \left( \frac{\Gamma^2 - |G_r|^2}{\Gamma^2} \right) \right], \quad \text{(C.7)}$$

where we used the fact that $c_r = 0$ for $r \neq 0$.

Now, using (5.5) and (5.18), we get

$$\lim_{n \to 0} A^2 n(n-1) C_0^2 a_0^{n(n-1)-1} = -\frac{H_0^2}{\Gamma}. \quad \text{(C.8)}$$

Inserting in (C.7) and using (5.10), we immediately obtain $D_\lambda$, considering that always $H_0|\Gamma - G_0| = 0$.

5. The Determinant $D_{\chi_0}$

From (5.2), we get

$$D_{\chi_0}(n) = A^2 [\Gamma - (G_0 + H_0)] D_\lambda + A^2 F_0^2 D_{q_0}$$

$$+ A^4 (2 F_0 - \varphi) n(n-1) c_0^2 a_0^{n(n-1)-1} P. \quad \text{(C.9)}$$

Taking the limit $n \to 0$, using (C.8) and (5.10), we obtain (5.28), after a little algebra.

6. The Determinants $D_{\chi_1}$ and $D_{\chi_1}''$

From (5.2), using the fact that $H_r = 0$ for $r \neq 0$, we get

$$D_{\chi_1}'(n) = A^2 [\Gamma + \text{Re} G_1] D_{\chi_0}$$

$$+ A^2 (\text{Re} G_1)^2 [A^2 [\Gamma - G_0 - H_0] D_{q_0}$$

$$- A^4 2n(n-1) C_0^2 a_0^{2n(n-1)-1}] P. \quad \text{(C.10)}$$

Using (C.8), after some algebra we obtain, in the limit $n \to 0$,

$$D_{\chi_1}' = A^2 [\Gamma + \text{Re} G_1] D_{\chi_0} + A^4 (\text{Re} G_1)^2 \frac{\Gamma - G_0}{\Gamma} P. \quad \text{(C.11)}$$

Following an analogous procedure, we obtain

$$D_{\chi_1}'' = A^4 \left[ \Gamma^2 - |G_1|^2 \right] D_{\chi_0} +$$

$$A^4 |G_1|^2 \left[ \Gamma - \text{Re} G_1 \right] \frac{\Gamma - G_0}{\Gamma} P. \quad \text{(C.12)}$$

Considering the cases of $D_{\chi_2}$ and $D_{\chi_3}$, it is not difficult to obtain, by induction, the general expression for $D_{\chi_r}$ and $D_{\chi_r}'$, eqs. (5.29)-(5.31).
APPENDIX D: FINITENESS OF G AND P

1. Theorem: $0 < G < \infty$

From (4.4) and (3.22), or directly from (B.5), after choosing the solution $\theta = 0$, we can write

$$G_r = \frac{T}{2\pi A^2} \left[ \cosh(6\pi \rho_r \chi_r) - 1 \right] \geq 0$$  \hspace{1cm} (D.1)

As argued before, according to (4.3) or (4.4), for large values of $r$, we have $\chi_r$ given by (4.10). In this case,$\,$

$$G_r \approx \frac{T}{4\pi A^2} \left[ \frac{A^2}{m^2 + \omega_r^2} \right]^2$$  \hspace{1cm} (D.2)

and, for a sufficiently large but finite $N$, $G$ can be written as

$$G = G_0 + 2 \sum_{r=1}^{N} G_r + \frac{T}{2\pi A^2} \sum_{r>N} \left[ \frac{A^2}{m^2 + \omega_r^2} \right]^2$$  \hspace{1cm} (D.3)

The first two terms are obviously finite and positive. The third term, is clearly smaller than

$$\frac{T}{4\pi A^2} \sum_{r=0}^{\infty} \left[ \frac{A^2}{m^2 + \omega_r^2} \right]^2$$  \hspace{1cm} (D.4)

which is finite. It follows that $0 < G < \infty$.

2. Theorem: $0 < P < 1$

From (5.22) and (5.23), we have

$$\ln P = \sum_{r=1}^{\infty} \ln \left[ 1 - \left( \frac{G_r}{T} \right)^2 \right]$$  \hspace{1cm} (D.5)

Using the same idea of the previous subsection, we can write, for a sufficiently large but finite $N$

$$\ln P = \sum_{r=1}^{N} \ln \left[ 1 - \left( \frac{G_r}{T} \right)^2 \right] - \frac{1}{\gamma^2} \sum_{r>N} \left[ \frac{A^2}{m^2 + \omega_r^2} \right]^4$$  \hspace{1cm} (D.6)

The first term is obviously finite and negative. The modulus of the second term is clearly smaller than

$$\frac{1}{\gamma^2} \sum_{r=0}^{\infty} \left[ \frac{A^2}{m^2 + \omega_r^2} \right]^4$$  \hspace{1cm} (D.7)

which is finite. It follows that $0 < |\ln P| < \infty$, with $\ln P < 0$. We conclude, therefore, that $0 < P < 1$.

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