Corrigendum: Renormalization group functions of QCD in the minimal MOM scheme (2013 J. Phys. A: Math. Theor. 46 225403)

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There was an error in the derivation of the four loop quark mass anomalous dimension in the minimal momentum scheme. Using the conversion function and the four loop MS quark mass anomalous dimension, the four loop term of the latter was inadvertently subtracted instead of added in applying (2.9). Accordingly several equations need to be replaced by their correct versions. First, the correct version of equation (3.11) is

\[
\gamma_{\text{MS}}(a, a) = -3C_F a + \left[ a^2 C_A - 67C_A - 6C_F + 8N_f T_F \right] \frac{C_F a^2}{4} + \left[ -3a^2 C_A^2 + 24a^2 C_A C_F - 54\zeta(3) a^2 C_A^2 + 411a^2 C_A C_F 
- 48a^2 C_A N_f T_F + 396\zeta(3) a C_A^2 + 15a C_A^2 + 72a C_A C_F + 48a C_A N_f T_F 
+ 5634\zeta(3) C_A^2 - 10095C_A^2 - 4224\zeta(3) C_A C_F + 244C_A C_F 
- 1152\zeta(3) C_A N_f T_F + 3888C_A N_f T_F - 3096C_F^2 + 1536\zeta(3) C_F N_f T_F 
+ 736C_F N_f T_F - 384N_f T_F \right] \frac{C_F a^2}{48} + \left[ -126\zeta(3) a^4 C_A^3 C_F + 315\zeta(5) a^4 C_A^3 C_F - 1125a^4 C_A^3 C_F 
+ 954a^4 C_A^2 C_F^2 + 72a^4 C_A^2 C_F N_f T_F - 288a^4 C_A^2 C_F \right] \frac{C_F a^2}{48} 
\]

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\[-3924 \zeta(3) a^3 C_A^3 C_F - 180 \zeta(5) a^3 C_A^3 C_F + 5058 a^3 C_A^3 C_F \]
\[-1728 \zeta(3) a^3 C_A^3 C_F^2 + 4860 a^3 C_A^3 C_F^2 - 1008 a^3 C_A^3 C_F N_f T_F \]
\[+ 3456 \zeta(3) a^3 C_A C_F^3 - 864 a^3 C_A C_F^3 - 53 \ 928 \zeta(3) a^3 C_A^3 C_F \]
\[-3150 \zeta(5) a^3 C_A^3 C_F + 99 225 a^2 C_A^3 C_F + 18 648 \zeta(3) a^2 C_A^3 C_F^2 \]
\[+ 14 976 a^2 C_A^3 C_F^2 + 9504 \zeta(3) a^2 C_A^3 C_F N_f T_F - 34 128 a^2 C_A^3 C_F N_f T_F \]
\[+ 17 856 \zeta(3) a^2 C_A C_F^3 + 27 432 a^2 C_A C_F^3 - 9216 \zeta(3) a^2 C_A C_F^3 N_f T_F \]
\[-10 080 a^2 C_A C_F^3 N_f T_F + 34 560 a^2 C_A C_F^3 N_f T_F^2 - 4608 \zeta(3) a^2 C_A^3 N_f T_F \]
\[+ 88 884 \zeta(3) aC_A^3 C_F - 31 860 \zeta(5) aC_A^3 C_F + 32 839 aC_A^3 C_F \]
\[-46 816 \zeta(3) aC_A^3 C_F^2 + 24 852 aC_A^3 C_F^2 - 73 152 \zeta(3) aC_A^3 C_F N_f T_F \]
\[+ 12 824 aC_A^3 C_F N_f T_F - 91 584 \zeta(3) aC_A C_F^3 + 48 096 aC_A C_F^3 \]
\[+ 51 456 \zeta(3) aC_A C_F^3 N_f T_F - 27 840 aC_A C_F^3 N_f T_F \]
\[+ 18 432 \zeta(3) aC_A C_F N_f T_F^2 - 5120 aC_A C_F N_f T_F^2 + 27 648 \zeta(3) aC_A^3 N_f T_F \]
\[+ 2304 aC_A^3 N_f T_F - 12 288 \zeta(3) aC_A^3 N_f T_F + 3072 aC_A^3 N_f T_F^2 \]
\[+ 1600 326 \zeta(3) aC_A^3 C_F - 196 965 \zeta(5) aC_A^3 C_F - 2247 471 C_A^3 C_F \]
\[-939 240 \zeta(3) aC_A^3 C_F^2 - 126 720 \zeta(5) aC_A^3 C_F^2 + 846 270 C_A^3 C_F^2 \]
\[-719 136 \zeta(3) aC_A^3 C_F N_f T_F + 1399 224 C_A^3 C_F N_f T_F - 118 656 \zeta(3) aC_A C_F^3 \]
\[+ 760 320 \zeta(5) aC_A C_F^3 - 1992 360 C_A C_F^3 + 364 032 \zeta(3) aC_A C_F^3 N_f T_F \]
\[+ 46 080 \zeta(5) aC_A C_F^3 N_f T_F + 130 272 C_A C_F^3 N_f T_F \]
\[+ 82 944 \zeta(3) aC_A C_F N_f T_F^2 - 255 552 aC_A C_F N_f T_F^2 \]
\[+ 193 536 \zeta(3) aC_A^3 T_F^2 + 90 792 C_A^3 T_F^2 - 88 552 \zeta(3) aC_A^3 N_f T_F \]
\[+ 276 480 \zeta(5) aC_A^3 N_f T_F + 497 472 C_A^3 N_f T_F + 36 864 \zeta(3) aC_A^3 N_f T_F^2 \]
\[+ 80 256 C_A^3 N_f T_F^2 + 9216 C_A C_F N_f T_F^3 - 138 240 \zeta(3) aC_A^3 N_f T_F^3 + 18 432 \zeta(3) aC_A^3 N_f T_F^3 \]
\[+ 276 480 \zeta(3) d_{FF}^{(3)} N_f - 36 864 \zeta(3) d_{FF}^{(3)} N_f \right] \frac{a^4}{576} + O \left(a^5 \right). \]
Subsequently equation (4.4) should be replaced by
\[
\mathcal{r}_m^{\text{MOM}}(a, \alpha) = -4.0 a + \left[ a^2 + 1.333 333 \alpha N_f - 69.666 667 \right] a^2 \\
+ \left[ 1.916 667 \alpha^3 - 2.000 000 \alpha^2 \alpha N_f + 98.522 232 \alpha^2 + 2.000 000 \alpha N_f \right] a^3 \\
+ \left[ 0.750 000 \alpha^4 N_f - 36.419 738 \alpha^4 - 10.500 000 \alpha^3 \alpha N_f \right] a^4 \\
+ \left[ 127.577 470 \alpha^5 + 6.000 000 \alpha^2 \alpha N_f - 345.848 075 \alpha^2 \alpha N_f \right] a^5 \\
+ \left[ 3588.203 465 \alpha^6 + 20.550 022 \alpha N_f^2 - 551.791 827 \alpha N_f \right] a^6 \\
+ \left[ 5040.515 124 \alpha + 2.666 667 \alpha N_f^3 - 298.304 558 \alpha N_f^2 \right] a^7 \\
+ 8709.238 844 \alpha N_f - 59 996.997 838 \alpha^4 + O(a^5). \tag{4.4}
\]

The remaining results are unaffected by this change. Finally, the associated supplementary data file stacks.iop.org/jpa/48/119501/mmedia has also been corrected.
Renormalization group functions of QCD in the minimal MOM scheme

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Abstract

We provide the full set of renormalization group functions for the renormalization of QCD in the minimal MOM scheme to four loops for the colour group $SU(N_c)$.

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Online supplementary data available from stacks.iop.org/JPhysA/46/225403/mmedia

1. Introduction

The main renormalization scheme used in quantum field theory is the modified minimal subtraction, $\overline{\text{MS}}$, scheme introduced in [1, 2]. It has many elegant features which can be exploited to determine the renormalization group functions to a very high loop order. One of these is that of calculability. Briefly, only the divergences with respect to the regulator are removed, together with a specific finite part, $\ln(4\pi e^{-\gamma})$ where $\gamma$ is the Euler–Mascheroni constant [2]. Ordinarily for conventional perturbation theory calculations one uses dimensional regularization in $d = 4 - 2\epsilon$ dimensions and $\epsilon$ plays the role of the regulator. Given this one need only consider the underlying massless quantum field theory safe in the knowledge that in this renormalization scheme the divergences will be mass independent. Thus, since massless Feynman graphs are significantly easier to compute than massive ones, then one can extract the ultraviolet divergences to high loop order. Moreover, using the scheme in gauge theories with massless fields gauge symmetry is preserved, [1]. Thus $\overline{\text{MS}}$ has been established as the favoured scheme for many years. However, for certain problems it is not necessarily the best choice. For instance, in lattice gauge theory computations it is not practical to implement since, for example, it is expensive for calculating Green’s functions involving derivatives. Instead physical schemes such as the modified regularization invariant (RI) scheme have been introduced, [3, 4]. These are asymmetric in that the definition is related to the choice of momentum configuration of three-point functions. Related types of physical schemes, but not motivated by lattice considerations, are the momentum subtraction schemes of [5] denoted by MOM. These differ from RI schemes in that the three-point function momentum configuration is completely symmetric. Hence they do not suffer from infrared issues as the configuration
both RI′ and MOM schemes differ from $\overline{\text{MS}}$ in that finite pieces are absorbed into the renormalization constants which therefore depend on external momentum scales. As a corollary they are more difficult to calculate in analytically to high loop order. Irrespective of which scheme one chooses to use for an analysis, through the structure of the renormalization group equation it is possible to relate results. Thus within perturbation theory one can compute the conversion functions which allow one to map, for example, the coupling constant defined in one scheme to that in another. The other parameters and renormalization group functions can equally be related by the same formalism.

A more recent development has been the introduction of another variant within the RI′ and MOM family of physical renormalization schemes, [6]. It is called the minimal MOM scheme and is motivated by a property of the ghost–gluon vertex of QCD in the Landau gauge. This property is the non-renormalization of the vertex, [7]. However, the scheme is an extension of the concept beyond this specific gauge in a way which preserves a definition of the coupling constant in terms of the ghost and gluon form factors, [6]. This effective running coupling constant has been the subject of intense interest in recent years due to interesting features at medium and low energies which were noted earlier in [8]. For instance, it is believed that there are dimension 2 deviations from the expected running when compared to pure perturbation theory. More recent work, [9], appears to reaffirm this property. With the minimal MOM scheme the effective coupling constant is not only simple to define but from a practical point of view does not require full knowledge of the ghost–gluon vertex function as would be necessary in other schemes [6]. Indeed in spite of being a non-exceptional momentum configuration it is numerically harder to extract a clean signal from the lattice for a fully symmetric vertex such as in the MOM context. In developing the minimal MOM scheme, [6], the four loop QCD $\beta$-function was determined for $SU(N_c)$ not only in the Landau gauge but also for a particular formulation of a linear covariant gauge. The full set of renormalization group functions as far as they could be calculated were not given. Therefore, it is the purpose of this note to provide the wave function renormalization group functions to as far a loop order as is possible. For an arbitrary colour group this is to three loops for the wave functions and four loops for the $\beta$-function and quark mass anomalous dimension. Though we will also provide the former to four loops for $SU(N_c)$. We will do this in two ways. The first is the direct evaluation of all the three loop Green’s functions where the minimal MOM renormalization scheme definition is implemented directly. The second is by construction of the associated conversion functions and use of the renormalization group equations. This will serve as a check on our computations and allow us to deduce four loop information. One of the reasons for the direct renormalization is that it provides a non-trivial independent check on the results of [6]. There the $\beta$-function was adduced from known finite parts of Green’s functions given in [10]. As a separate exercise we choose to work in a minor variation of the original minimal MOM scheme and that is to renormalize the gauge parameter, $\alpha$, in the full ethos of the minimal MOM scheme. In [6] the renormalization of the gauge parameter is completely equivalent to that of the $\overline{\text{MS}}$ scheme. Though our results will equate for the Landau gauge. A similar issue for $\alpha$ arises in the RI′ case [11]. Finally, given the interest in the behaviour of the effective coupling constant in the Landau gauge and its power law deviation, we will provide the minimal MOM anomalous dimension of the operator thought to be associated with the dimension two correction which is $\frac{1}{4}A_\mu^a A^{a\mu}$ where $A_\mu^a$ is the gluon. This is in order to allow one to perform a complete renormalization group running analysis in the minimal MOM scheme for such infrared problems.

The paper is organized as follows. We recall the definition and properties of the minimal MOM scheme in section 2 before recording our results in the subsequent section. These include the renormalization group functions and the conversion functions for an arbitrary colour group. For $\overline{\text{MS}}$ renormalization group functions which are only known at four loops for $SU(N_c)$, we
provide the corresponding minimal MOM scheme results in section 4 together with those for the dimension two operator anomalous dimension. A conclusion is provided in section 5.

2. Formalism

We begin by recalling the definition of the minimal MOM scheme, [6]. First, if we denote bare quantities in the QCD Lagrangian by the subscript \( \alpha \), then in our notation the renormalization constants, \( Z_i \), are given by

\[
\begin{align*}
  A^a_\mu &= \sqrt{Z_A} A^a_\mu, \\
  c^a_i &= \sqrt{Z_c} c^a_i, \\
  \psi^i_I &= \sqrt{Z_\psi} \psi^i_I \\
  \alpha_\alpha &= Z_\alpha^{-1} \alpha, \\
  m_0 &= m Z_m, \\
  g_0 &= \mu^2 Z_g g
\end{align*}
\]

(2.1)

where \( A^a_\mu \) is the gluon, \( c^a_i \) is the Faddeev–Popov ghost and \( \psi^i_I \) is the quark. The indices have the ranges \( 1 \leq a \leq N_A \), \( 1 \leq i \leq N_F \) and \( 1 \leq I \leq N_f \) where \( N_F \) and \( N_A \) are the respective dimensions of the fundamental and adjoint representations of the colour group and \( N_f \) is the number of massive quarks each of the same mass \( m \). The coupling constant is \( g \) and \( \alpha \) is the gauge parameter of the linear covariant gauge. The Landau gauge corresponds to \( \alpha = 0 \). We use the above definition of the renormalization of \( \alpha \) to be consistent with [11]. Throughout we use dimensional regularization in \( d = 4 - 2\epsilon \) spacetime dimensions and the mass scale \( \mu \) is introduced to ensure the coupling constant is dimensionless in \( d \)-dimensions. With these formal definitions of the renormalization constants they are then determined explicitly by specifying a scheme to absorb the infinities in the various two and three-point functions of the theory. For instance, \( \overline{\text{MS}} \) corresponds to removing only the poles in \( \epsilon \) together with a certain finite part at some subtraction point.

For momentum subtraction schemes, denoted generally by MOM, the scheme is defined such that at the subtraction point the poles in \( \epsilon \) together with all the finite part are absorbed into the renormalization constant, [5]. For the QCD Lagrangian this produces several different schemes since there are several vertices which one can use to define the coupling constant renormalization. Choosing one, say, means that the remaining vertex functions are finite and consistent with the Slavnov–Taylor identities. The variation on this approach introduced in [6] is that the two-point functions are renormalized using the \( \overline{\text{MS}} \) criterion of [5] but the three-point vertices are treated differently. Specifically, to ease comparison with lattice analyses the completely symmetric subtraction point of [5] is not used. Instead the asymmetric point is used where the external momentum of an external leg is nullified. Moreover, partly motivated by the non-renormalization of the ghost–gluon vertex in the Landau gauge, [7], the coupling constant renormalization is defined by ensuring that this vertex renormalization constant is the same as the \( \overline{\text{MS}} \) one. One benefit of this, [6], is that to define the scheme one only needs to know the vertex structure in the \( \overline{\text{MS}} \) scheme which reduces work for non-perturbative applications. In our notation, (2.1), this corresponds to, [6],

\[
\begin{align*}
  Z_{g}^{\overline{\text{MS}}} \sqrt{Z_{A}^{\overline{\text{MS}}} Z_{c}^{\overline{\text{MS}}}} &= Z_{g}^{\text{mMOM}} \sqrt{Z_{A}^{\text{mMOM}} Z_{c}^{\text{mMOM}}}
\end{align*}
\]

(2.2)

where \( \text{mMOM} \) denotes the minimal MOM scheme. Though, in this formal definition it is important to appreciate that the variables \( g \) and \( \alpha \) on either side of the equation are in different schemes. We note that throughout our convention is that when a scheme is specified as a label on a quantity then it is a function of the parameters \( g \) and \( \alpha \) in that scheme. With (2.2) then all the renormalization constants of massless QCD are defined for the minimal MOM scheme, [6]. As noted earlier in [6] the gauge parameter renormalization was treated as an \( \overline{\text{MS}} \) one rather than define it as the full MOM renormalization as used in [5]. Therefore, we will follow the approach of [5] here and have a minimal MOM \( \alpha \). From the practical point of view our results in the Landau gauge will be the same and differ only in the \( \alpha \) dependent part.
The procedure we have used is to apply the MINCER algorithm, [12], to the massless QCD Lagrangian and compute all the two-point functions as well as the ghost–gluon vertex at the asymmetric point. The quark mass anomalous dimension will be discussed later. This algorithm evaluates massless three loop two-point functions to the finite part in dimensional regularization. It has been encoded, [13], in the symbolic manipulation language FORM, [14], which is our main computational tool. The Feynman diagrams are generated by QGRAF, [15], and the output converted into FORM input notation. As all graphs are evaluated to the finite parts then we can extract the explicit renormalization constants in the minimal MOM. We do this first by renormalizing the two-point functions before defining the coupling constant renormalization we have to relate the parameters between the schemes.

For the gauge parameter this is given by

\[ Z_{\text{MOM}} \]

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\[ \alpha_{\text{MOM}}(\mu) = \frac{Z_{\text{MOM}}^\mu}{Z_{\alpha}^\mu} \alpha_{\text{MS}}(\mu) \]  

where we used the fact that we are in a linear covariant gauge which implies \( Z_{\alpha} = 1 \). To ensure a finite expression in \( \epsilon \) emerges the parameters within \( Z_{\text{MOM}}^\mu \) have to be converted to their \( \overline{\text{MS}} \) partners. This is achieved order by order in perturbation theory. We have determined these to three loops and, with \( a = g^2/(16\pi^2) \), found

\[
ad_{\text{MOM}} = a + [9\alpha^2C_A + 18\alpha C_A + 169C_A - 80N_f T_F] \frac{a^2}{36} + [405\alpha^3C_A^2 - 486\xi(3)\alpha^2C_A^2 + 2835\alpha^2C_A^2 + 3564\xi(3)\alpha C_A^2 + 2421\alpha C_A^2 - 1440\alpha C_A N_f T_F - 6318\xi(3) C_A^2 + 76063C_A^2 - 10368\xi(3) C_A N_f T_F - 50656C_A N_f T_F + 20736 \xi(3) C_f N_f T_F - 23760C_f N_f T_F + 6400N_f^2 T_F^2] \frac{a^3}{1296} + [10 - 692\xi(3)\alpha^4C_A^3 + 8505\xi(5)\alpha^4C_A^3 + 41067\alpha^4C_A^3 - 59292\xi(3)\alpha^4C_A^3 - 4860\xi(5)\alpha^2C_A^3 + 293301\alpha^2C_A^3 - 138024\xi(3)\alpha^2C_A^3 - 85050\xi(3)\alpha^2C_A^3 + 1315035\alpha^2C_A^3 + 46656\xi(3)\alpha^2C_A^3 N_f T_F - 322056\alpha^2C_A^3 N_f T_F + 3355020\xi(3)\alpha C_A^3 - 860220\xi(5)\alpha C_A^3 + 1277496\alpha C_A^3 - 2115072\xi(3)\alpha C_A^3 N_f T_F - 581760\alpha C_A^3 N_f T_F + 373248\xi(3)\alpha C_A C_f N_f T_F - 427680\alpha C_A C_f N_f T_F + 331776\xi(3)\alpha C_A N_f^2 T_F^2 + 32256\alpha C_A N_f^2 T_F^2 - 6552900\xi(3)\alpha C_A^2 - 1896615\xi(5)\alpha C_A^2 + 42074947C_A^2 - 4499712\xi(3)\alpha C_A^2 N_f T_F + 2488320\xi(3)C_A^2 N_f T_F - 38975242C_A^2 N_f T_F + 15303168\xi(3)C_A C_f N_f T_F + 3732480\xi(5)C_A C_f N_f T_F - 23755968C_A C_f N_f T_F + 3068928\xi(3)C_A N_f^2 T_F^2 + 9209280C_A N_f^2 T_F^2 + 4603392C_A N_f^2 T_F - 7464460\xi(5)C_A N_f T_f T_f F + 1482624C_f N_f^2 T_f F - 6469632\xi(3)C_f N_f^2 T_f F + 8065152C_f N_f^2 T_f F - 512000N_f^3 T_f F] \frac{a^4}{46656} + O(a^5) \]  

and

\[
ad_{\text{MOM}} = a + [-9\alpha^2C_A - 18\alpha C_A - 97C_A + 80N_f T_F] \frac{a^2}{36} + [18\alpha^2C_A^2 - 18\alpha^2C_A^2 + 190\alpha^2C_A^2 - 320\alpha^2C_A^2 - 576\xi(3)\alpha C_A^2 + 463\alpha C_A^2 - 320\alpha C_A N_f T_F + 864\xi(3) C_A^2 - 7143C_A^2 + 2304\xi(3) C_A N_f T_F + 4248C_A N_f T_F - 4608\xi(3) C_f N_f T_F + 5280C_f N_f T_F] \frac{a^2}{288} \]
where $\zeta(z)$ is the Riemann zeta function. The group Casimirs are defined by

$$\text{Tr}(T^aT^b) = T_F\delta^{ab}, \quad T^aT^a = CF I, \quad f^{acd}f^{bcd} = C_A\delta^{ab}$$

where $T^a$ are the generators of the colour group whose structure functions are $f^{abc}$. In (2.4) and (2.5) the variables on the right-hand side are in the $\overline{\text{MS}}$ scheme. For the Landau gauge it is easy to see that then the parameters coincide. We have checked that (2.4) agrees with the alternative definition of the mapping given in [6] based on the actual finite parts of the gluon and ghost two-point functions after their $\overline{\text{MS}}$ renormalization.

While we will perform a direct evaluation of the renormalization constants in the minimal MOM, there are several checks which will be carried out. One is to exploit properties of the renormalization group equation which allows one to map the anomalous dimensions deduced in each scheme via conversion functions which are denoted by $C_i(a, \alpha)$ where $i$ will be a label corresponding to a field or a parameter. First, we will perform the explicit renormalization in the minimal MOM and deduce the anomalous dimensions directly. Then we will compute the conversion functions and from these construct the anomalous dimensions indirectly. Thus if we define the conversion functions by

$$C_n^{\text{mMOM}}(a, \alpha) = \frac{Z_n^{\text{mMOM}}}{Z_n^{\overline{\text{MS}}}}, \quad C_n^{\phi}(a, \alpha) = \frac{Z_n^{\phi}}{Z_n^{\overline{\text{MS}}}}$$

where $\phi \in \{A, c, \psi\}$, then the minimal MOM renormalization group functions are given by

$$\beta^{\text{mMOM}}(\alpha_{\text{mMOM}}, \alpha_{\text{mMOM}}) = \left[\beta^{\overline{\text{MS}}}(\alpha_{\overline{\text{MS}}}) \frac{\partial \alpha_{\text{mMOM}}}{\partial \alpha_{\overline{\text{MS}}}} + \alpha_{\overline{\text{MS}}} \gamma_{\overline{\text{MS}}}^{\overline{\text{MS}}}(\alpha_{\overline{\text{MS}}}, \alpha_{\text{mMOM}}) \frac{\partial \alpha_{\text{mMOM}}}{\partial \alpha_{\overline{\text{MS}}}}\right]^{\overline{\text{MS}} \rightarrow \text{mMOM}}$$

(2.7)
and
\[
\gamma^\text{mMOM}_\phi (\beta_{\text{mMOM}}, \alpha_{\text{mMOM}}) = \left[ \gamma^\text{MS}_\phi (\beta_{\text{MS}}, \alpha_{\text{MS}}) + \beta^\text{MS}_\phi \frac{\partial}{\partial \beta_{\text{MS}}} \ln c^\text{mMOM}_\phi (\beta_{\text{MS}}, \alpha_{\text{MS}}) + \alpha_{\text{MS}} \beta^\text{MS}_\phi \frac{\partial}{\partial \alpha_{\text{MS}}} \ln c^\text{mMOM}_\phi (\beta_{\text{MS}}, \alpha_{\text{MS}}) \right]_{\text{MS} \to \text{mMOM}}.
\]

Here MS → mMOM means that after computing the right-hand side the expression will be a function of MS variables and these must therefore be converted to minimal MOM ones. The relations are given by inverting (2.4) and (2.5). One benefit of this formalism is that it can be exploited to produce the four loop anomalous dimensions and β-function. The reason for this is that the three loop conversion functions give a four loop contribution to the minimal MOM scheme independent. Moreover, the dependent renormalization scheme and only in mass independent schemes is the two loop term since that term is scheme independent. This includes the MS versions of these are known, [16–26], then the left-hand side can be deduced at four loops.

3. Results

We now formally record our results. The one loop expressions will be the same as the MS ones since that term is scheme independent. This includes the β-function as we are using a mass dependent renormalization scheme and only in mass independent schemes is the two loop term scheme independent. Moreover, the β-function will be gauge dependent for the same reason. Therefore, we have

\[
\beta^\text{mMOM} (a, \alpha) = \frac{1}{2} \left[ 11 C_A - 4 N_f T_F \right] a^2 + \left[ -3 a^2 C_A^3 + 10 a^2 C_A^2 + 8 a^2 C_A^2 N_f T_F + 13 a C_A^2 + 8 a C_A N_f T_F - 136 C_A^2 
+ 80 C_A N_f T_F + 48 C_f N_f T_F \right] a^3 
+ \left[ -165 a^4 C_A^3 + 244 a^4 C_A^2 + 108 \xi (3) a^4 C_A^3 - 189 a^4 C_A^3 
- 144 a^3 C_A^3 N_f T_F - 468 \xi (3) a^3 C_A^3 + 2175 a^3 C_A^3 + 144 \xi (3) a^3 C_A^3 N_f T_F 
- 1656 a^2 C_A^3 N_f T_F - 864 a^2 C_A N_f T_F - 188 \xi (3) a^2 C_A^3 + 3291 a C_A^3 
- 1776 a C_A^3 N_f T_F - 1152 a C_A N_f T_F + 5148 \xi (3) C_A^3 - 38620 C_A^3 
+ 6576 \xi (3) C_A^3 N_f T_F + 32 144 C_A^3 N_f T_F - 16 896 \xi (3) C_A N_f T_F 
+ 20 512 C_A N_f T_F + 3072 \xi (3) C_A N_f T_F - 4416 C_A^2 N_f T_F 
- 576 C_A^2 N_f T_F + 6144 \xi (3) C_A N_f T_F - 5888 C_A^2 N_f T_F \right] a^4 
+ \left[ 864 \xi (3) a^5 C_A^4 - 3780 \xi (3) a^5 C_A^4 - 11 745 a^5 C_A^4 + 1728 a^5 C_A^4 N_f T_F 
+ 32 472 \xi (3) a^5 C_A^4 + 4140 \xi (3) a^5 C_A^4 - 81 549 a^5 C_A^4 
- 1440 \xi (3) a^5 C_A^4 N_f T_F - 5040 \xi (3) a^5 C_A^4 N_f T_F + 7200 a^5 C_A^4 N_f T_F 
+ 7776 a^5 C_A^4 N_f T_F + 47 052 \xi (3) a^5 C_A^4 + 19 800 \xi (3) a^5 C_A^4 
- 81 873 a^4 C_A^4 + 18 432 \xi (3) a^4 C_A^4 N_f T_F + 1440 \xi (3) a^4 C_A^4 N_f T_F 
- 67 752 a^4 C_A^4 N_f T_F - 7776 a^4 C_A^4 N_f T_F - 397 368 \xi (3) a^4 C_A^4 \right].
\]

1 A data file is attached which gives an electronic version of all our expressions (see stacks.iop.org/JPhysA/46/225403/mmedia).
+ 152 280\xi (5)\alpha^2 C_A^4 + 1028 898\alpha^2 C_A^4 - 36 576\xi (3)\alpha^2 C_A^4 N_f T_F
- 1098 936\alpha^2 C_A^4 N_f T_F + 639 360\xi (3)\alpha^2 C_A^4 N_f T_F
- 790 272\alpha^2 C_A^4 N_f T_F + 73 728\xi (3)\alpha^2 C_A^4 N_f^2 T_F^2 + 133 632\alpha^2 C_A^4 N_f^2 T_F^2
+ 20 736\alpha^2 C_A^4 N_f T_F + 221 184\xi (3)\alpha^2 C_A^4 N_f^2 T_F^2
+ 211 968\alpha^2 C_A^4 N_f^2 T_F^2 - 2400 708\xi (3)\alpha C_A^4 + 987 660\xi (5)\alpha C_A^4
+ 1719 423\alpha C_A^4 + 1655 712\xi (3)\alpha C_A^4 N_f T_F - 254 880\xi (5)\alpha C_A^4 N_f T_F
- 1817 880\alpha C_A^4 N_f T_F + 798 336\xi (3)\alpha C_A^4 N_f T_F
- 1030 752\alpha C_A^4 N_f T_F - 617 472\xi (3)\alpha C_A^4 N_f^2 T_F^2 + 391 488\alpha C_A^4 N_f^2 T_F^2
+ 31 104\alpha C_A^4 N_f^2 T_F - 331 776\xi (3)\alpha C_A^4 N_f^2 T_F^2
+ 317 952\alpha C_A^4 N_f^2 T_F^2 + 98 304\xi (3)\alpha C_A^4 N_f^2 T_F^3 - 24 576\alpha C_A^4 T_F^3
+ 5509 416\xi (3)\alpha C_A^4 - 22 106 704\alpha C_A^4
- 1217 376\xi (3)\alpha C_A^4 N_f T_F - 5178 960\xi (5)\alpha C_A^4 N_f T_F + 23 501 280\alpha C_A^4 N_f T_F
- 7050 240\xi (3)\alpha C_A^4 N_f T_F - 6082 560\xi (5)\alpha C_A^4 N_f T_F
+ 17 477 280\alpha C_A^4 N_f T_F - 1654 272\xi (3)\alpha C_A^4 N_f^2 T_F^2
+ 1474 560\xi (5)\alpha C_A^4 N_f^2 T_F^2 - 5719 680\alpha C_A^4 N_f^2 T_F^3
- 7907 328\xi (3)\alpha C_A^4 N_f T_F + 12 165 120\xi (5)\alpha C_A^4 C_F N_f T_F
- 607 104\alpha C_A^4 N_f T_F + 4755 456\xi (3)\alpha C_A^4 N_f T_F^2
+ 2211 840\xi (5)\alpha C_A^4 N_f^2 T_F^2 - 10 861 056\alpha C_A^4 N_f^2 T_F^2
+ 344 064\xi (3)\alpha C_A^4 N_f^2 T_F^2 + 229 376\alpha C_A^4 N_f^2 T_F^2 - 476 928\alpha C_A^4 N_f T_F
+ 3538 944\xi (3)\alpha C_A^4 N_f^2 T_F^2 - 4423 680\alpha C_A^4 N_f^2 T_F^3 + 267 264\alpha C_A^4 N_f^2 T_F^3
- 884 736\xi (3)\alpha C_A^4 N_f^2 T_F^3 + 1327 104\alpha C_A^4 N_f^3 T_F^3 - 2433 024\xi (3)\alpha C_A^4 N_f^3 T_F^3
+ 92 160\alpha C_A^4 N_f^3 T_F^3 + 5750 784\xi (3)\alpha C_A^4 N_f^3 T_F^3 - 589 824\alpha C_A^4 N_f^3 T_F^3
- 1769 472\xi (3)\alpha C_A^4 N_f^3 T_F^3 + 811 008\alpha C_A^4 N_f^3 T_F^3 \frac{d^{(4)}\alpha}{N_A^3} + \frac{d^{(4)}\alpha}{N_A}
+ \frac{\alpha}{6} \text{Tr}(T^a T^b T^c T^d) \right)
(3.1)
As the four loop \(\bar{MS}\) \(\beta\)-function was computed for an arbitrary colour group, [23], the general
colour group Casimirs appear. In our notation they are defined by
\[
d^{(4)}_{FF} = d^{abcd}_{FF} d^{abcd}_{FF}, \quad d^{(4)}_{FA} = d^{abcd}_{FF} d^{abcd}_{FF}, \quad d^{(4)}_{A} = d^{abcd}_{FF} d^{abcd}_{FF}
(3.2)
\]
and the totally symmetric rank 4 colour tensor is defined by, [27],
\[
d^{abcd}_{R} = \frac{1}{6} \text{Tr}(T^a T^b T^c T^d)
(3.3)
\]
where the group generators are in the \(R\) representation.

For the anomalous dimensions only the three loop \(\bar{MS}\) expressions are known for an
arbitrary colour group, [21]. Thus to the same order the minimal MOM expressions are
\[
\gamma^{\text{mom}}_{\alpha} = \{3\alpha C_A - 13\alpha C_A + 8 N_f T_F\} \frac{a}{6}
+ \left[ -6 \alpha^2 C_A^3 + 17 \alpha^2 C_A^3 - 16 \alpha^2 C_A N_f T_F + 17 \alpha C_A^3 - 16 \alpha C_A N_f T_F - 170 C_A^2
+ 136 C_A N_f T_F + 96 C_A N_f T_F \right] \frac{a^2}{24}
+ \left[ -165 \alpha^3 C_A^4 + 24 \alpha^3 C_A^4 N_f T_F + 54 \alpha^3 C_A^4 - 126 \alpha^3 C_A^4 - 144 \alpha^3 C_A^4 N_f T_F
\]
\[ \gamma_{\text{mMOM}}(a, \alpha) = \left[ \alpha - 3 \frac{C_{AA}}{4} + [3\alpha^2 C_A - 3\alpha C_A - 34C_A + 8Nf T_F]\frac{C_{AA}a^2}{16} \right. \\
\left. + \left[ 54\xi(3)\alpha^2 C_A^2 - 45\alpha^3 C_A^2 - 36\xi(3)\alpha^2 C_A + 216\alpha^2 C_A + 84\alpha^2 C_A Nf T_F \right. \right. \\
\left. \left. - 42\xi(3)\alpha C_A^2 + 109\alpha C_A^2 + 96\xi(3)\alpha C_A Nf T_F - 152\alpha C_A Nf T_F \right. \right. \\
\left. \left. + 564\xi(3)\alpha C_A Nf T_F - 5196 C_A^2 + 96\xi(3) C_A Nf T_F + 3608C_A Nf T_F + 288C_A Nf T_F \right. \right. \\
\left. \left. - 640 N_f^2 T_F^2 \right] \frac{C_{AA}a^3}{192} + O(a^4) \right. \\
\gamma_{\text{mMOM}}^\prime(a, \alpha) = \alpha C_F a + C_F [3\alpha^2 C_A + 6\alpha C_A + 25C_A - 6C_A - 8Nf T_F] \frac{C_F a^2}{4} \\
+ \left[ 18\xi(3)\alpha^2 C_A^2 + 27\alpha^3 C_A^2 - 24\alpha^3 C_A^2 + 90\xi(3)\alpha^2 C_A + 123\alpha^2 C_A \right. \\
\left. - 36\alpha C_A C_F + 48\alpha^2 C_A Nf T_F - 618\xi(3)\alpha C_A^2 + 395\alpha C_A^2 + 72\alpha C_A C_F \right. \\
\left. + 192\xi(3)\alpha C_A Nf T_F - 64\alpha C_A Nf T_F - 1470\xi(3) C_A^2 + 3843 C_A \right. \\
\left. + 576\xi(3) C_A Nf T_F - 1260C_A C_F + 384\xi(3) C_A Nf T_F - 1840C_A Nf T_F \right. \\
\left. + 72 C_A^2 - 96 C_A Nf T_F + 128 N_f^2 T_F^2 \right] \frac{C_F a^3}{48} + O(a^4). \] 

We have checked explicitly that the gauge parameter satisfies

\[ \gamma_{a, \text{mMOM}}(a, \alpha) = -\gamma_{a, \text{mMOM}}^\prime(a, \alpha) \] 

which is a check on our calculation.

Having provided the anomalous dimensions we have checked that they are completely reproduced using the conversion function approach. The explicit forms of these functions are

\[ C_A(a, \alpha) = 1 + \left[ -9\alpha^2 C_A - 18\alpha C_A - 169 C_A + 80Nf T_F \right] \frac{a}{72} \]

\[ + \left[ 243\alpha^2 C_A^2 - 648\alpha^2 C_A^2 + 1944\xi(3)\alpha^2 C_A - 1242\alpha^2 C_A^2 - 4320\alpha^2 C_A Nf T_F \right] \frac{a^2}{72} \]

\[ + \left[ -3645\alpha^2 C_A^2 + 21870\alpha^2 C_A^3 + 33048\xi(3)\alpha^4 C_A^3 - 68040\xi(5)\alpha^4 C_A^3 \right. \]

\[ - 183951\alpha^2 C_A^3 + 97200\alpha^2 C_A^4 Nf T_F + 754272\xi(3)\alpha^4 C_A^3 + 38880\xi(5)\alpha^4 C_A^3 \]

\[ - 1501740\alpha^3 C_A^3 + 155520\alpha^3 C_A^4 Nf T_F + 206064\xi(3)\alpha^4 C_A^3 + 680400\xi(5)\alpha^4 C_A^3 \]

\[ - 710235\alpha^3 C_A^3 + 1026432\xi(3)\alpha^4 C_A^4 Nf T_F - 1887840\alpha^4 C_A^4 Nf T_F \]

\[ + 2239488\xi(3)\alpha^4 C_A^4 C_F Nf T_F - 2566080\alpha^4 C_A^4 C_F Nf T_F - 172800\alpha^4 C_A^4 N_f^2 T_F^2 \]

\[ - 20977056\xi(3)\alpha^4 C_A^4 + 6881760\xi(5)\alpha^5 C_A^4 + 3407958\alpha^5 C_A^4 \]

\[ + 11259648\xi(3)\alpha^4 C_A^4 Nf T_F - 4231296\alpha^4 C_A^4 Nf T_F + 1492992\xi(3)\alpha C_A C_F Nf T_F \]

\[ - 1710720\alpha C_A C_F Nf T_F - 2654208\xi(3)\alpha C_A N_f^2 T_F^2 + 778752\alpha C_A N_f^2 T_F^2 \]
\[ C_{A}(\alpha, \alpha) = 1 + 9(\alpha^3 C_A + 18\alpha C_A + 97 C_A - 80 N_f T_F) \frac{a}{36} \]

\[ + 810 \alpha^3 C_A^2 + 2430\alpha^2 C_A^3 + 5184\xi (3) \alpha C_A^4 + 2817\alpha C_A^2 - 2880\alpha C_A N_f T_F \]

\[ - 7776\xi (3) C_A^2 + 83 105\alpha C_A^4 - 20 736\xi (3) C_A N_f T_F - 69 272 C_A N_f T_F \]

\[ + 41 472\xi (3) C_A N_f T_F - 47 520 C_A N_f T_F + 12 800 N_f^2 T_F^2 \frac{a^2}{2592} \]

\[ + [-12636\xi (3) \alpha^4 C_A^2 + 17 010\xi (5) \alpha^5 C_A^3 - 51 516\xi (3) \alpha^3 C_A^2 \]

\[ + 19 440\xi (5) \alpha^2 C_A^4 + 322 947\alpha^3 C_A^3 + 203 148\xi (3) \alpha^2 C_A^3 - 8784\xi (4) \alpha^4 C_A^2 \]

\[ - 374 220\xi (5) \alpha^2 C_A^3 + 1094 553\alpha^3 C_A^2 - 15 552\xi (3) \alpha^2 C_A N_f T_F \]

\[ - 303 912\alpha C_A^2 N_f T_F + 4636 764\xi (3) \alpha C_A^2 - 34 992\xi (4) \alpha C_A^2 \]

\[ - 134 360\xi (5) \alpha C_A^3 + 1457 685\alpha C_A^3 - 3670 727\xi (3) \alpha C_A N_f T_F \]

\[ - 890 064\alpha C_A N_f T_F + 774 496\xi (3) \alpha C_A C_A N_f T_F - 855 360\alpha C_A C_A N_f T_F \]

\[ + 663 552\xi (3) \alpha C_A N_f^2 T_f^2 + 64 512\alpha C_A N_f^2 T_f^2 - 7531 056\xi (3) C_A^2 \]

\[ - 26 244\xi (4) C_A^2 - 3414 150\xi (5) C_A^3 + 44 961 125\xi (3) C_A^2 N_f T_F \]

\[ + 839 808\xi (4) C_A^2 N_f T_F + 4976 640\xi (5) C_A^2 N_f T_F - 49 928 712 C_A N_f T_F \]

\[ + 23 514 624\xi (3) C_A C_A N_f T_F - 1119 744\xi (4) C_A C_A N_f T_F \]

\[ + 746 960\xi (5) C_A C_A N_f T_F - 37 099 872 C_A C_A N_f T_F + 597 196\xi (3) C_A N_f^2 T_f^2 \]

\[ + 13 873 536 C_A N_f^2 T_f^2 + 9206 784\xi (3) C_A N_f T_F - 14 929 920\xi (5) C_A N_f T_F \]

\[ + 2965 248 C_A N_f T_F - 12 939 264\xi (3) C_A N_f^2 T_f^2 + 16 130 304 C_A N_f^2 T_f^2 \]

\[ - 1024 000 N_f^2 T_f^2 \frac{a^3}{93 312} + O(a^4) \]
and

\[ C_F(a, \alpha) = 1 - \alpha C_F a \]
\[ + \left[ -9\alpha^2 C_A + 8\alpha^2 C_F + 24\alpha(3)\alpha C_A - 52\alpha C_A + 24\alpha (3) C_A \right. \]
\[ - 82C_A + 5C_F + 28N_f T_F \] \[ \frac{C_F a^2}{8} \]
\[ + \left[ 1728\alpha^3 C_A^2 - 11808\alpha^3 C_A^2 - 5184\alpha^3 C_A C_F + 12312\alpha^3 C_A C_F \right. \]
\[ + 3456\alpha^3 C_F^2 - 5184\alpha^3 C_F^2 + 25272\alpha^3 C_F^2 - 972\alpha^2 C_A \]
\[ - 6480\alpha^2 C_A - 63747\alpha^2 C_A - 31104\alpha^2 C_A C_F + 59616\alpha^2 C_A C_F \]
\[ + 181440\alpha C^2 C_A - 1944\alpha C^2 C_A - 12960\alpha C^2 C_A - 358 \]
\[ + 57024\alpha C^2 C_A - 103680\alpha C^2 C_A + 85536\alpha C^2 C_A \]
\[ - 41472\alpha C^2 C_A N_f T_F + 124056\alpha C^2 C_A N_f T_F - 110160\alpha C^2 C_A N_f T_F \]
\[ + 678024\alpha C^2 C_A N_f T_F + 22356\alpha C^2 C_A N_f T_F - 213840\alpha C^2 C_A N_f T_F \]
\[ - 228096\alpha C^2 C_A N_f T_F - 31104\alpha C^2 C_A N_f T_F + 103680\alpha C^2 C_A N_f T_F \]
\[ + 89856\alpha C^2 C_A N_f T_F + 760768\alpha C^2 C_A N_f T_F - 31536\alpha C^2 C_A N_f T_F \]
\[ + 68256C_A N_f T_F - 100480N_f^2 T_f^2 \] \[ \frac{5C_F a^3}{5184} + O(\alpha^4). \] (3.9)

We use the convention that the variables on the right-hand side are in the \( \overline{\text{MS}} \) scheme.

While [6] provided the renormalization group functions for massless QCD it is possible to deduce the quark mass anomalous dimension to four loops for an arbitrary colour group. This requires the conversion function for the quark mass renormalization and the four loop \( \overline{\text{MS}} \) anomalous dimension. The latter has been provided in [24] and [25]. To deduce the former we work in the massless theory but renormalize the associated mass operator by inserting it in a quark two-point function at zero momentum insertion. This was the procedure used in the original three loop \( \overline{\text{MS}} \) renormalization of [28, 29]. We then use the renormalization condition that there is no finite part at the subtraction point. In this computational setup we can still use the Mincer algorithm, [12, 13]. Thus we can deduce the renormalization constant and hence the three loop quark mass conversion function which is

\[ C_m(a, \alpha) = 1 + C_F[-\alpha - 4]a \]
\[ + \left[ -18\alpha^2 C_A + 24\alpha^2 C_F - 84\alpha C_A + 96\alpha C_F + 432\alpha (3) C_A - 1285 C_A \right. \]
\[ - 288\alpha (3) C_F + 57C_F + 332N_f T_F \] \[ \frac{C_F a^2}{24} \]
\[ + \left[ -13122\alpha^2 C_A^2 + 15552\alpha^2 C_A C_F - 7776\alpha^2 C_A^2 + 8748\alpha^2 C_A^2 \right. \]
\[ - 7160\alpha^2 C_A^2 \]
\[ - 23328\alpha^2 C_A C_F + 89424\alpha^2 C_A C_F + 46656\alpha^2 C_A C_F \]
\[ - 31104\alpha^2 C_A C_F \]
\[ + 103032\alpha^2 C_A C_F + 357777\alpha^2 C_A C_F \]
\[ - 334368\alpha^2 C_A C_F + 573804\alpha C_A C_F \]
\[ - 31104\alpha^2 C_A C_F + 113400\alpha C_A C_F \]
\[ - 46656\alpha C_A C_F \]
\[ - 30132\alpha C_A C_F \]
\[ + 62208\alpha (3) C_A N_f T_F - 123120\alpha C_A N_f T_F + 3368844\alpha (3) C_A N_f T_F \]
\[ - 466560\alpha (5) C_A \]
\[ - 6720046\alpha (5) C_A \]
\[ - 2493504\alpha (5) C_A \]
\[ + 155520\alpha (5) C_A \]
\[ + 2028348C_A \]
\[ - 532224\alpha (3) C_A N_f T_F + 186624\alpha (4) C_A N_f T_F + 3052384C_A \]
\[ + 451008\alpha (3) C_A \]
\[ + 933120\alpha (5) C_A \]
\[ + 2091096C_A \]
\[ - 331776\alpha (3) C_A N_f T_F \]
\[ - 1086624\alpha (4) C_A N_f T_F + 958176C_A N_f T_F - 27648\alpha (3) N_f^2 T_f^2 \]
\[ - 240448N_f^2 T_f^2 \] \[ \frac{C_F a^3}{7776} + O(\alpha^4). \] (3.10)
Equipped with this and the result of [24, 25] we find the minimal MOM quark mass anomalous dimension is

\[ \gamma_{m^{\text{MOM}}}^{\text{MOM}}(\alpha, \alpha) = -3C_F a + \left[ \alpha^2 C_A - 67C_A - 6C_F + 8N_f T_F \right] \frac{C_F a^2}{4} \]

\[ + \left[ -3\alpha^3 C_A^2 + 24\alpha^3 C_A C_F - 54\zeta(3)\alpha^2 C_A^3 + 411\alpha^2 C_A + 108\alpha^2 C_A C_F \right. \]

\[ - 48\alpha^2 C_A N_f T_F + 396\zeta(3)\alpha C_A^2 + 15\alpha C_A^2 + 72\alpha C_A C_F + 48\alpha C_A N_f T_F \]

\[ + 5634\zeta(3) C_A^2 - 10095\alpha C_A + 4224\zeta(3) C_A^2 C_F + 244C_A C_F \]

\[ + 1152\zeta(3) C_A N_f T_F + 3888 C_A N_f T_F - 3096 C_F^2 + 1536\zeta(3) C_A N_f T_F \]

\[ + 736 C_F N_f T_F - 384 N_f^2 T_F^2 \left] \frac{C_F a^3}{48} \right. \]

\[ + \left[ -1134\alpha^4 C_A C_F \zeta(3) + 2835\alpha^4 C_A C_F \zeta(5) - 10 125\alpha^4 C_A C_F \right. \]

\[ + 8586\alpha^4 C_A C_F^2 + 648\alpha^4 C_F^2 N_f T_F - 2592\alpha^4 C_A C_F^3 \]

\[ - 35 316\zeta(3) \alpha^3 C_A^2 C_F - 1620\zeta(5) \alpha^3 C_A^2 C_F + 45 522\alpha^3 C_A C_F \]

\[ - 15 552\zeta(3) \alpha^3 C_A^2 C_F + 43 740\alpha^3 C_A C_F^2 - 9072\alpha^2 C_A^3 C_F - 307 152\alpha^2 C_A^3 C_F \]

\[ + 160 704\zeta(3) \alpha^2 C_A^2 C_F^2 + 246 888\alpha^2 C_A C_F^3 - 82 944\alpha^2 C_A C_F^3 \]

\[ - 90 720\alpha C_A^3 C_F^2 + 31 104\alpha C_A^3 C_F^2 - 41 472\zeta(3) \alpha^2 C_A^3 C_F \]

\[ + 799 956\zeta(3) \alpha C_A^2 C_F^3 - 286 740\zeta(5) \alpha C_A^2 C_F^3 + 295 551\zeta(3) \alpha C_A^2 C_F^3 \]

\[ - 417 744\zeta(3) \alpha C_A^2 C_F^3 + 223 668\alpha C_A^2 C_F^3 - 658 368\zeta(3) \alpha C_A^2 C_F^3 \]

\[ + 115 416\alpha C_A^2 C_F^3 N_f T_F - 824 256\zeta(3) \alpha C_A^2 C_F^3 + 432 864\alpha C_A^2 C_F^3 \]

\[ + 463 104\alpha C_A^2 C_F^3 N_f T_F - 250 560\alpha C_A^2 C_F^3 \]

\[ + 165 888\alpha C_A^2 C_F^3 N_f T_F - 46 080\alpha C_A^2 C_F^3 \]

\[ + 248 832\alpha C_A^2 C_F^3 N_f T_F + 20 736\alpha C_A^2 C_F^3 N_f T_F - 110 592\zeta(3) \alpha C_A^2 N_f T_F^2 \]

\[ + 27 648\alpha C_A^2 N_f T_F^2 + 16 036 470\zeta(3) \alpha C_A^2 C_F^3 - 6334 605\zeta(5) \alpha C_A^3 C_F \]

\[ - 10 139 319\zeta(3) C_A - 10 029 996\zeta(3) \alpha C_A^2 C_F^3 + 342 1440\zeta(5) \alpha C_A^2 C_F^3 \]

\[ + 2188 530\zeta(3) \alpha C_A^2 C_F^3 - 15 748 128\zeta(3) \alpha C_A^2 N_f T_F + 2737 152\zeta(4) \alpha C_A^2 N_f T_F \]

\[ + 4147 200\zeta(5) \alpha C_A^2 N_f T_F + 8403 640\zeta(3) \alpha C_A^2 N_f T_F + 2208 384\zeta(3) \alpha C_A^3 C_F \]

\[ + 6842 880\zeta(5) \alpha C_A^3 C_F + 4669 704\alpha C_A^3 C_F + 7091 712\zeta(3) \alpha C_A^3 N_f T_F \]

\[ + 2737 152\zeta(4) C_A^2 N_f T_F + 1244 160\zeta(5) \alpha C_A^2 N_f T_F \]

\[ - 2214 048 C_A^2 N_f T_F + 2405 376\zeta(3) \alpha C_A^2 N_f T_F^2 \]

\[ - 995 328\zeta(4) C_A C_F N_f T_F^2 + 2128 192 C_A C_F N_f T_F^2 - 1741 824\zeta(3) \alpha C_F \]

\[ + 817 128 C_F + 4935 168\zeta(3) C_A^2 N_f T_F - 7464 960\zeta(5) \alpha C_A^2 N_f T_F \]

\[ + 3509 568 C_A^2 N_f T_F - 1327 104\zeta(3) \alpha C_A^2 N_f T_F^2 + 995 328\zeta(4) C_A^2 N_f T_F^2 \]

\[ - 605 568 C_A^2 N_f T_F^2 + 147 456\zeta(3) \alpha C_A^2 N_f T_F^2 - 2048 C_A N_f T_F^3 \]
which involves the same rank 4 Casimirs as the $\beta$-function. We have checked the three loop part by the direct evaluation of the anomalous dimension using the minimal MOM renormalization constant. Given that we are considering a relatively new scheme we have also renormalized the flavour non-singlet vector current. This is important since it is a conserved physical current and its anomalous dimension is zero to all orders in perturbation theory. This is true in all schemes but we have checked this explicitly to three loops by repeating the above quark mass operator renormalization but using the vector current, $\bar{\psi}\gamma^\mu\psi$, instead. With the minimal MOM quark wave function renormalization constants and isolating the Lorentz channel of the Green function with the inserted current corresponding to the transverse part, we have checked that the vector current renormalization constant is unity to three loops in the minimal MOM scheme. Thus the Slavnov–Taylor identity has been checked to this loop order with the above renormalization.

For more practical purposes it is useful to provide the explicit numerical expressions for $SU(3)$. Thus we have

\[
C_\gamma(a, \alpha) = 1 + [-0.375000\alpha^2 - 0.750000\alpha + 0.555556N_f - 7.041667]a + \left[0.210937\alpha^4 - 0.562500\alpha^3 - 0.625000\alpha^2 N_f + 0.950.36\alpha^2 - 0.41667\alpha N_f - 7.43795\alpha - 0.154321\alpha N_f^2 + 24.491186N_f - 163.359911\right]a^2 + \left[ -0.131836\alpha^6 + 0.791016\alpha^5 + 0.585937\alpha^4 N_f - 7.768301\alpha^4 - 0.937500\alpha^3 N_f - 20.064612\alpha^3 - 0.173611\alpha^2 N_f^2 - 18.480594\alpha^2 N_f + 8.788748\alpha^2 - 2.423078\alpha N_f^2 + 56.307562\alpha N_f - 530.662942\alpha + 0.085734\right]a^3 + O(a^4)
\]

\[
C_A(a, \alpha) = 1 + [0.750000\alpha^2 + 1.500000\alpha - 1.111111N_f + 8.083333]a + \left[2.812500\alpha^3 + 8.437500\alpha^2 - 1.666667\alpha N_f + 31.418274\alpha + 1.234568\right]a^2 + 53.912928N_f + 256.1034919a^2 + \left[19.411733\alpha^4 + 81.359870\alpha^3 - 13.754707\alpha^2 N_f + 272.349881\alpha^2 - 6.929489\alpha N_f^2 - 254.788806\alpha N_f + 1621.114903\alpha - 1.371742\right]a^3 + O(a^4)
\]

\[
C_\psi(a, \alpha) = 1 + 3.000000\alpha + (1.346529\alpha^2 + 3.072567\alpha - 5.937500N_f + 80.935770)\alpha^2 + 9.344565\alpha^3 + 61.885373\alpha^2 - 5.501305\alpha N_f + 211.462123\alpha + 8.765877\right]a^3 - 431.804136N_f + 2945.691833\alpha^3 + O(a^4)
\]

\[
C_m(a, \alpha) = 1 + [-1.333333\alpha a + -2.722222\alpha^2 - 11.575317\alpha + 2.333333N_f - 25.464206]a^2 + \left[-16.906900\alpha^3 - 72.363802\alpha^2 + 23.739312\alpha N_f - 317.382214\alpha - 6.460905\right]a^3 + O(a^4)
\]

\[
C_n(a, \alpha) = 1 + [-1.333333\alpha a + -2.222222\alpha^2 - 6.888889\alpha + 9.222222N_f - 149.040228]a^2 + \left[-11.953704\alpha^3 - 44.682370\alpha^2 + 14.024098\alpha N_f - 269.395219\alpha - 11.731930\right]a^3 + O(a^4).
\]
Clearly it would appear that the series have large corrections at three loops. Though that for the quark wave function is best.

4. \( SU(N_c) \)

Although we have given the minimal MOM scheme results to as high a loop order as is possible for an arbitrary colour group, it is possible to provide the complete set at four loops for the case of \( SU(N_c) \). This is because the four loop MS anomalous dimensions of the gluon, ghost and quark are known for this colour group for an arbitrary linear covariant gauge fixing, [24, 26]. Using the electronically available data files associated with the latter article we have extended the various three loop minimal MOM results using the same method. This is also possible since we have the mapping for the gauge parameter between the two schemes at three loops. Thus for the gluon we have

\[
\gamma_{\lambda}^{\text{nMOM}}(a, \alpha) = \left[ 3\alpha N_c - 13 N_c + 4 N_f \right] \frac{a^2}{6}
\]

\[
+ \left[-6\alpha^3 N_c^3 + 17\alpha N_c^3 - 8\alpha^2 N_c^2 N_f + 17\alpha N_c^3 - 8\alpha N_c^2 N_f - 170 N_c^3
\right.
\]

\[
+ 92 N_c^2 N_f - 24 N_f \right] \frac{a^2}{24 N_c}
\]

\[
+ \left[-165\alpha^4 N_c^5 + 12\alpha^3 N_c^4 N_f + 54\alpha^4 (3)\alpha^3 N_c^4 - 126\alpha^3 N_c^4 - 72\alpha^3 N_c^4 N_f
\right.
\]

\[
- 576\alpha (3)\alpha^2 N_c^5 + 1761\alpha^2 N_c^5 + 72\alpha (3)\alpha N_c^4 N_f - 972\alpha^2 N_c^4 N_f
\]

\[
+ 216\alpha^2 N_c^2 N_f - 774\alpha (3) N_c^5 + 102\alpha N_c^5 - 144\alpha (3)\alpha N_c^4 N_f - 588\alpha N_c^4 N_f
\]

\[
+ 288\alpha N_c^2 N_f N_f + 3456\alpha (3) N_c^4 - 23.032 N_c^4 - 1080\alpha (3) N_c^4 N_f + 15.500 N_c^4 N_f
\]

\[
- 1360 N_c^3 N_f^2 + 4224\alpha (3) N_c^5 N_f - 4768 N_c^3 N_f^2 - 768\alpha (3) N_c^5 N_f^2 + 736 N_c N_f^2
\]

\[
- 72 N_f \right] \frac{a^3}{288 N_c^2}
\]

\[
+ \left[-1728\alpha (3)\alpha^2 N_c^5 N_f - 7560\alpha (5)\alpha^5 N_f^2 - 23.490 \alpha^5 N_f^2 + 1728\alpha^5 N_c^5 N_f
\right.
\]

\[
+ 52389\alpha (3)\alpha^4 N_c^5 N_f - 2250\alpha (5)\alpha^4 N_f^2 - 130.590 \alpha^4 N_f^2 - 1440\alpha (3)\alpha^4 N_c^5 N_f
\]

\[
- 5040\alpha (5)\alpha^3 N_c^6 N_f + 10.440 \alpha^4 N_c^6 N_f + 3888\alpha (3)\alpha^3 N_c^6 N_f + 2430\alpha (5)\alpha^3 N_c^6 N_f
\]

\[
- 3888\alpha N_c^4 N_f + 38.214\alpha (3)\alpha^3 N_c^5 N_f + 40.410 \alpha (5)\alpha^3 N_c^5 N_f - 20.727 \alpha N_c^4 N_f
\]

\[
\right. + 23616\alpha (3)\alpha^3 N_c^6 N_f + 1440\alpha (5)\alpha^3 N_c^6 N_f - 86.328 \alpha^3 N_c^6 N_f
\]

\[
+ 1944\alpha (3)\alpha^2 N_c^5 N_f^2 + 53460\alpha (5)\alpha^2 N_c^5 N_f + 3888\alpha^2 N_c^5 N_f^2 + 937.188 \alpha^2 N_c^5 N_f^2
\]

\[
+ 656370\alpha (3)\alpha^2 N_c^5 N_f^2 + 1387.395 \alpha^2 N_c^5 N_f^2 + 324.144 \alpha^2 N_c^5 N_f^2
\]

\[
+ 30.240 \alpha^3 N_c^6 N_f^2 - 1222.596 \alpha^3 N_c^6 N_f^2 - 18.432 \alpha^2 \alpha^3 N_c^6 N_f^2
\]

\[
+ 93888\alpha^2 N_c^5 N_f^2 + 93.312 \alpha^2 N_c^5 N_f^2 + 145800 \alpha (5)\alpha^5 N_f^2 - 5832 \alpha^2 N_c^5 N_f^2
\]

\[
- 319.680\alpha (3)\alpha^2 N_c^5 N_f^2 + 373.104 \alpha^2 N_c^5 N_f^2 + 55.296 \alpha (3)\alpha^2 N_c^5 N_f^2
\]

\[
- 52992\alpha^2 N_c^5 N_f^2 + 5184 \alpha^2 N_c^5 N_f^2 - 3727.014 \alpha (3)\alpha N_f^2
\]

\[
+ 1557630\alpha (5)\alpha N_c^5 N_f + 1030995 \alpha N_c^5 N_f + 1860768 \alpha (3)\alpha N_c^5 N_f
\]

\[
- 142560\alpha (5)\alpha N_c^5 N_f - 1254696 \alpha N_c^5 N_f - 400896 \alpha (3)\alpha N_c^5 N_f^2
\]

\[
+ 192288\alpha N_c^5 N_f^2 + 872856 \alpha (3)\alpha N_c^5 N_f^2 - 461700 \alpha (5)\alpha N_c^5 N_f^2 - 23328 \alpha N_c^5 N_f^2
\]

\[
+ 24576\alpha (3)\alpha N_c^5 N_f^2 - 6144 \alpha N_c^5 N_f^2 + 383616 \alpha (3)\alpha N_c^5 N_f
\]

\[
+ 428544 \alpha N_c^5 N_f^2 + 82944 \alpha (3)\alpha N_c^5 N_f^2 - 79488 \alpha N_c^5 N_f^2 + 7776 \alpha N_c^5 N_f
\]

\[
+ 8025711\alpha (3) N_f^2 + 4451400 \alpha (5) N_f^2 - 27205691 N_f^2
\]
where we have substituted the $SU(N_c)$ values for $C_F$ and $C_A$. Similarly, the ghost anomalous dimension is

$$\gamma_{c}^{\text{MOM}}(a, \alpha) = N_c[a - \frac{3a^2}{4}]
+ N_{f}[3a^2 N_{f} - 3a N_{f} - 34 N_{c} + 4N_{f}] \frac{a^2}{16}
+ \left[54 \zeta(3) a^2 N_{c}^3 - 45a^3 N_{c}^3 - 36 \zeta(3) a^3 N_{c}^3 + 216a^2 N_{c}^3 - 24a^2 N_{c}^3 N_{f}
+ 42 \zeta(3) a^4 N_{c}^3 + 109a^4 N_{c}^3 + 48 \zeta(3) a^3 N_{c}^3 N_{f} - 76a^3 N_{c}^3 N_{f} + 564 \zeta(3) N_{c}^3
- 5196 N_{c}^3 + 48 \zeta(3) N_{c}^3 N_{f} + 1876 N_{c}^3 N_{f} - 160 N_{c}^3 N_{f} - 72 N_f \right] \frac{a^3}{192}
+ 3843 \zeta(3) a^4 N_{c}^5 + 2070 \zeta(5) a^4 N_{c}^5 - 8052 a^4 N_{c}^5 + 72 a^4 N_{c}^5 N_{f}
- 3456 \zeta(3) a^4 N_{c}^5 + 1890 \zeta(5) a^4 N_{c}^5 + 8298 \zeta(3) a^4 N_{c}^5 - 6390 \zeta(5) a^3 N_{c}^5
- 9411 a^3 N_{c}^5 - 576 \zeta(3) a^3 N_{c}^5 N_{f} + 192 a^3 N_{c}^5 N_{f} + 10152 \zeta(3) a^3 N_{c}^5 N_{f}
- 18900 \zeta(5) a^3 N_{c}^5 - 108 \zeta(3) a^2 N_{c}^5 - 32 790 \zeta(5) a^2 N_{c}^5 + 73 071 a^2 N_{c}^5
- 2448 \zeta(3) a^2 N_{c}^5 N_{f} + 3360 \zeta(5) a^2 N_{c}^5 N_{f} - 32 388 a^2 N_{c}^5 N_{f} + 2880 a^2 N_{c}^5 N_{f}
+ 3888 \zeta(3) a^2 N_{c}^5 N_{f} - 22 680 \zeta(5) a^2 N_{c}^5 N_{f} + 648 a^2 N_{c}^5 N_{f} + 1296 a^2 N_{c}^5 N_{f}
- 40 458 \zeta(3) a^2 N_{c}^5 + 26 790 \zeta(5) a^2 N_{c}^5 + 88 801 a^2 N_{c}^5 + 1152 \zeta(3) a N_{c}^5 N_{f}
- 12 480 \zeta(5) a^3 N_{c}^5 - 42 248 a^3 N_{c}^5 + 3072 \zeta(3) a^4 N_{c}^5 + 3136 a^3 N_{c}^5 N_{f}
- 1080 \zeta(3) a^4 N_{c}^5 - 52 380 \zeta(5) a^4 N_{c}^5 + 2592 a^4 N_{c}^5 - 1728 \zeta(3) a N_{c}^5 N_{f}
+ 3600 a^2 N_{f}^3 + 310041 \zeta(3) a^2 N_{c}^5 + 192 240 \zeta(5) a^2 N_{c}^5 - 1888 893 N_{c}^5
- 91 728 \zeta(3) a^4 N_{c}^5 - 59 040 \zeta(5) a^4 N_{c}^5 + 997 068 a^4 N_{c}^5 + 13 824 \zeta(3) a^3 N_{c}^5 N_{f}
- 141 984 a^3 N_{c}^5 N_{f} - 526 176 \zeta(3) a^3 N_{c}^5 + 549 990 \zeta(5) a^3 N_{c}^5 + 14 904 N_{c}^5
+ 3840 a^2 N_{c}^5 N_{f} + 54 720 \zeta(3) a^2 N_{c}^5 N_{f} - 34 560 \zeta(5) a^2 N_{c}^5 N_{f} - 104 928 N_{c}^5 N_{f}
- 4608 \zeta(3) a^2 N_{c}^5 N_{f} + 18 816 N_{c}^5 N_{f} - 432 N_f] \frac{a^4}{4608 N_c} + O(a^5).$$

Finally, the quark anomalous dimension is

$$\gamma_{q}^{\text{MOM}}(a, \alpha) = a[N_{c}^2 - 1] \frac{a}{2N_c}
+ \left[3 a^2 N_{c}^4 - 3a^2 N_{c}^4 + 6a N_{c}^4 - 6a N_{c}^4 + 22 N_{c}^4 - 4 N_{c}^4 N_{f} - 19 N_{c}^2
+ 4N_{f} N_{c} - 3 \right] \frac{a^2}{8 N_{c}^2}.$$
\[
\begin{align*}
&+ \left[ 18 \zeta(3) a^2 \chi^6 + 15 a^4 \chi^6 - 15 a^3 \chi^6 - 3 a^3 \chi^4 - 12 a^2 \chi^2 \right] \alpha^2 \frac{a^3}{96 \chi^3} \\
&- 90 \zeta(3) a^2 \chi^6 + 105 a^4 \chi^6 + 24 a^2 \chi^6 \chi_f + 90 \zeta(3) a^3 \chi^6 \\
&- 87 a^2 \chi^6 - 24 a^2 \chi^6 \chi_f - 18 a^2 \chi^2 - 618 \zeta(3) a \chi^6 + 431 a \chi^6 \\
&+ 96 \zeta(3) a \chi^6 \chi_f - 32 a \chi^6 \chi_f + 618 \zeta(3) a \chi^6 - 467 a \chi^4 - 96 \zeta(3) a \chi^6 \chi_f \\
&+ 32 a \chi^6 \chi_f + 36 a \chi^2 - 1182 \zeta(3) \chi^6 + 3231 \chi^6 + 192 \zeta(3) \chi^6 \chi_f \\
&+ 944 \chi^6 \chi_f + 32 \chi^2 \chi_f^2 + 894 \zeta(3) \chi^6 - 2637 \chi^6 - 192 \zeta(3) \chi^6 \chi_f \\
&+ 968 \chi^6 \chi_f - 32 \chi^2 \chi_f^2 + 288 \zeta(3) \chi^6 - 576 \chi^6 - 24 \chi^2 \chi_f - 18 \right] \alpha^3 \\
&+ \left[ 330 \zeta(3) a^2 \chi^8 + 75 \zeta(5) a^2 \chi^8 - 151 a^4 \chi^8 - 12 a^4 \chi^6 \chi_f - 420 \zeta(3) a^3 \chi^8 \\
&- 45 \zeta(5) a^2 \chi^8 + 280 a^4 \chi^6 + 12 a^4 \chi^6 \chi_f + 42 \zeta(3) a^3 \chi^6 \\
&- 105 a^4 \chi^4 + 48 \zeta(3) a^4 \chi^4 - 24 a^4 \chi^6 + 1116 \zeta(3) a^3 \chi^6 - 180 \zeta(5) a^3 \chi^8 \\
&- 33 a^3 \chi^8 - 192 \zeta(3) a^3 \chi^6 \chi_f + 184 a^3 \chi^6 \chi_f - 1620 \zeta(3) a^2 \chi^8 \\
&+ 180 \zeta(5) a^2 \chi^6 + 663 a^2 \chi^6 + 192 \zeta(3) a^2 \chi^6 \chi_f - 184 a^2 \chi^2 \chi_f \\
&+ 360 \zeta(3) a^2 \chi^8 + 558 a^4 \chi^6 + 144 \zeta(3) a^3 \chi^6 - 72 a^3 \chi^6 - 1662 \zeta(3) a^2 \chi^8 \\
&- 192 \zeta(3) \chi^8 \chi_f + 3130 \zeta(5) a^2 \chi^8 - 5767 a^2 \chi^6 - 384 \zeta(3) a^3 \chi^6 \\
&- 160 \zeta(5) a^2 \chi^6 \chi_f + 2580 a^4 \chi^6 \chi_f - 96 a^4 \chi^6 \chi_f + 1518 \zeta(3) a^2 \chi^8 \\
&- 4510 \zeta(5) a^2 \chi^6 + 6307 a^4 \chi^6 + 576 \zeta(3) a^3 \chi^6 \chi_f + 160 \zeta(5) a^3 \chi^8 \\
&- 2940 a^4 \chi^6 \chi_f + 96 a^2 \chi^2 \chi_f^2 + 144 a^3 \zeta(3) a^2 \chi^8 + 1380 \zeta(5) a^2 \chi^8 \\
&- 558 a^2 \chi^2 \chi_f - 192 \zeta(3) a^2 \chi^6 \chi_f + 360 a^2 \chi^2 \chi_f \chi_f - 18 a^2 \chi^2 \chi_f \\
&- 33 680 \zeta(3) a^2 \chi^8 + 17 420 \zeta(5) a^2 \chi^8 + 14 292 a^2 \chi^6 + 7040 \zeta(3) a^2 \chi^6 \\
&- 3200 \zeta(5) a^2 \chi^6 \chi_f - 3752 a^4 \chi^6 \chi_f + 320 a^4 \chi^2 \chi_f + 49 248 \zeta(3) a^2 \chi^8 \\
&- 37 500 \zeta(5) a^2 \chi^6 - 13 642 a^2 \chi^6 - 9408 \zeta(3) a^2 \chi^6 \chi_f + 5760 \zeta(5) a^2 \chi^6 \chi_f \\
&+ 2632 a^4 \chi^2 \chi_f - 192 a^4 \chi^2 \chi_f + 15 568 \zeta(3) a^4 \chi^4 + 20 080 \zeta(5) a^4 \chi^4 \\
&- 122 a^4 \chi^4 + 2368 \zeta(3) a^4 \chi^4 - 2560 \zeta(5) a^4 \chi^4 + 1024 a^4 \chi^4 \\
&- 128 a^2 \chi^6 \chi_f^2 - 528 a^2 \chi^2 \chi_f^2 + 96 a^2 \chi^2 \chi_f^2 - 172 560 \zeta(3) a^2 \chi^8 + 61 875 \zeta(5) a^2 \chi^8 \\
&+ 252 104 a^2 \chi^8 + 43 392 \zeta(3) a^2 \chi^8 - 12 000 \zeta(5) a^2 \chi^8 - 118 572 \chi^2 \chi_f \\
&- 1536 \zeta(3) a^2 \chi^8 + 15 392 \chi^2 \chi_f + 118 002 \zeta(3) a^2 \chi^8 - 29 085 \zeta(3) a^2 \chi^8 \\
&- 225 318 \chi^2 \chi_f - 640 \chi^2 \chi_f + 36 684 \zeta(3) a^2 \chi^8 + 480 \zeta(5) a^2 \chi^8 \\
&+ 120 868 \chi^2 \chi_f + 2304 \zeta(3) a^2 \chi^8 - 16 592 \chi^2 \chi_f + 66 462 \zeta(3) a^2 \chi^8 \\
&- 55 830 \zeta(5) a^2 \chi^8 - 19 731 a^4 + 640 a^4 \chi^2 \chi_f - 6912 \zeta(3) a^2 \chi^8 \\
&+ 11 520 \zeta(5) a^2 \chi^8 + 7152 \chi^2 \chi_f - 768 \zeta(3) a^2 \chi^8 + 1200 \chi^2 \chi_f \\
&- 2304 \zeta(3) a^2 \chi^8 + 7680 \zeta(5) a^2 \chi^8 - 3974 \chi^2 \chi_f + 384 \zeta(3) a^2 \chi^8 - 9448 \chi^2 \chi_f \\
&- 9600 \zeta(3) + 15 360 \zeta(5) - 3081 \right] \frac{a^4}{384 \chi^6} + O(a^5). \quad (4.3)
\end{align*}
\]

For practical purposes it is perhaps more appropriate to provide the explicit numerical values for all renormalization group functions at four loops for the SU(3) colour group. Thus

\[ \beta_{\text{MOM}}(a, \alpha) = 0.666667 N_f - 11.00000 \alpha a^2 + [-2.250000 \alpha^3 - \alpha^2 N_f + 7.500000 \alpha^2 - \alpha N_f + 9.750000 \alpha a^2 + 12.66667 N_f - 102.00000 \alpha a^3] \]

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\[ \psi (A_{mMOM}, \alpha) = \sum_{k=0}^{500000} \left[ 500000 - 1000000 \right] a^k + O(a^5) \]

\[ \gamma_A^{mMOM}(a, \alpha) = [1.500000a + 0.666667N_f - 6.500000]a \]
\[ + [-2.250000a^3 - 0.750000a^2 - 0.750000a - 19.125000]a^2 \]
\[ + [0.375000a^4 - 15.468750a^3 - 2.250000a^2 - 5.457924a]a^3 \]
\[ + [0.750000a^4 - 15.468750a^3 - 2.250000a^2 - 5.727087a]a^4 \]
\[ + [2.250000a^5 - 114.265701a^4 + 2.816305a^3 - 298.61620a^4]a^5 \]
\[ + [9.492589a^5 - 2.000000a^4 - 0.296280a^3 - 6.949789a]a^6 \]
\[ + [0.750000a^5 - 61.6500284a^4 - 2.924683a^3 - 210.615973a^4]a^7 \]
\[ + [1.916667a^3 - 2.000000a^2 - 98.522232a^2 + 2.000000a^2]a^8 \]
\[ + 500000 - 1000000]a^9 + O(a^{10}) \]

\[ \gamma_c^{mMOM}(a, \alpha) = [0.750000a - 2.250000]a \]
\[ + [1.687500a^2 - 1.687500a + 0.750000N_f - 19.125000]a^2 \]
\[ + [2.799995a^3 - 1.125000a^2N_f + 24.289587a - 0.857872a]N_f \]
\[ + [2.2427774a - 2.500000N_f + 90.267128N_f - 6.35349362]a^3 \]
\[ + [0.421875a^4N_f - 26.892596a^4 - 2.931942a^3 - 121.006552a^3]a^4 \]
\[ + [5.625000a^5N_f - 185.757176a^5N_f + 648.958414a^5 + 3.337341aN_f]a^5 \]
\[ + [314.266897aN_f - 1090.833574aN_f - 2.500000N_f]a^6 \]
\[ + [4788.563749N_f - 23.240.416706]a^7 + O(a^{10}) \]

\[ \gamma_0^{mMOM}(a, \alpha) = 1.333333a + 1.333333N_f + 2.600000a - 1.333333N_f + 2.333333]a^2 \]
\[ + [9.492589a^3 + 2.000000a^2N_f - 0.296280a^2 + 6.949789a]N_f \]
\[ - 78.967792a + 0.888889N_f^2 - 59.211534N_f + 459.481285]a^3 \]
\[ + [-0.750000a^4N_f + 61.6500284a^4 - 2.924683a^3N_f + 210.615973a^4]a^4 \]
\[ - 2.000000a^5N_f + 121.134099a^5N_f - 869.566016a^5 + 6.962963aN_f^2 \]
\[ + [77.834748aN_f - 1553.524949aN_f - 4.444444N_f^2 + 281.560058N_f^2 \]
\[ - 4934.050666N_f + 20.300.851595]a^4 + O(a^{10}) \]

\[ \gamma_0^{mMOM}(a, \alpha) = -4.0a + [a^2 + 1.333333N_f - 69.666667]a^2 \]
\[ + [1.916667a^3 - 2.000000a^2N_f + 98.522232a^2 + 2.000000a^2]a^3 \]
\[ + 130.753633a - 2.666667N_f^2 + 152.122739N_f - 1520.596003]a^3 \]
\[
+ \left[0.750 \, 000 000 \alpha^4 N_f - 36.419 \, 738 \alpha^4 - 10.500 \, 000 \alpha^3 N_f + 127.577 \, 470 \alpha^3
\right]
+ \left[6.000 \, 000 \alpha^2 N_f^2 - 345.848 \, 075 \alpha^2 N_f + 3588.203 \, 465 \alpha^2 + 20.550 \, 022 \alpha N_f^2
\right]
- \left[551.791 \, 827 \alpha N_f + 5040.515 \, 124 \alpha + 5.632 \, 797 \alpha N_f^2 - 156.909 \, 331 N_f^2
\right]
- \left[1073.781 \, 658 N_f - 9337.969 \, 739 \right] \alpha^4 + O(\alpha^5)
\]

We have checked that the Landau gauge expression for the \(\beta\)-function agrees with that of [6].

One interesting consequence of these expressions is that we can provide the anomalous dimension of a particular dimension two operator which is

\[
O = \frac{1}{2} A^\mu A^{\mu} - \alpha c^a c^a.
\]

It is known, [30–32], that \(O\) has a novel renormalization property. In the Landau gauge the anomalous dimension of \(O\) is the sum of the gluon and ghost anomalous dimensions. Moreover, in an arbitrary linear and nonlinear covariant gauge there is a simple generalization of this Slavnov–Taylor identity which was established in [33]. This was based on the observation given in [34] which were explicit three loop \(\overline{\text{MS}}\) computations. The operator is of interest as it was an attempt to have a gluon mass term in the Lagrangian which while not gauge invariant was constant in the low energy limit, [8, 9]. In [26] the four loop \(\overline{\text{MS}}\) Landau gauge result was given. However, the arbitrary \(\alpha\) \(\overline{\text{MS}}\) expression for a linear covariant gauge fixing was not recorded. As the gluon and ghost propagators are examined in the minimal MOM scheme, [36], it is worth providing the renormalization for the operators explicitly. Though for reasons of space we provide the \(SU(3)\) expression\(^2\)

\[
\gamma_{\overline{\text{MS}}}^{SU(3)}(a, \alpha) = \left[27 \alpha + 8 N_f - 105\right] \frac{a}{12}
+ \left[108 \alpha^2 + 567 \alpha + 548 N_f - 4041\right] \frac{a^2}{48}
+ \left[12 \, 393 \alpha^3 + 4374 \alpha (3) \alpha^2 + 52 \, 488 \alpha^2 - 22 \, 356 \alpha N_f + 17 \, 496 \alpha (3) \alpha
\right]
+ 268 \, 272 \alpha - 12 \, 080 N_f^2 - 28 \, 512 \alpha (3) N_f + 437 \, 304 N_f + 13 \, 122 \alpha (3)
\right]
- 2041 \, 389 \right] \alpha^3 \frac{a^3}{1728}
+ \left[-3011 \, 499 \alpha (3) \alpha^4 + 4133 \, 430 \alpha (5) \alpha^4 + 8030 \, 664 \alpha^4
\right]
+ 20 \, 824 \, 614 \alpha (3) \alpha^3 + 708 \, 588 \alpha (4) \alpha^3 - 10 \, 431 \, 990 \alpha (5) \alpha^3
\right]
+ 39 \, 936 \, 807 \alpha^3 - 2939 \, 328 \alpha (3) \alpha^2 N_f + 314 \, 928 \alpha (4) \alpha^2 N_f
\right]
- 8669 \, 268 \alpha^2 N_f + 93 \, 612 \, 348 \alpha (3) \alpha^2 - 3779 \, 136 \alpha (4) \alpha^2
\right]
- 18 \, 305 \, 190 \alpha (5) \alpha^2 + 159 \, 478 \, 227 \alpha^2 + 3359 \, 232 \alpha (3) \alpha N_f^2
\right]
\left[-2796 \, 768 \alpha \alpha N_f^2 - 60 \, 046 \, 272 \alpha (3) \alpha N_f - 11 \, 337 \, 408 \alpha (4) \alpha N_f
\right]
- 127 \, 188 \, 144 \alpha N_f + 612 \, 180 \, 666 \alpha (3) \alpha - 4 \, 369 \, 660 \alpha (4) \alpha
\right]
- 513 \, 923 \, 130 \alpha (5) \alpha + 11 \, 464 \, 159 \, 210 \alpha + 497 \, 664 \alpha (3) \alpha N_f^2 - 84 \, 646 \, 464 N_f^2
\right]
- 19 \, 558 \, 656 \alpha (3) \alpha N_f^2 - 6158 \, 592 \alpha (4) N_f^2 - 107 \, 248 \, 896 N_f^2
\right]
- 289 \, 945 \, 440 \alpha (3) N_f + 104 \, 451 \, 120 \alpha (4) N_f + 313 \, 061 \, 760 \alpha (5) N_f
\right]
+ 20 \, 828 \, 935 \, 820 N_f + 72 \, 636 \, 831 \alpha (3) - 46 \, 766 \, 808 \alpha (4)
\right]
- 19 \, 084 \, 636 \, 810 \alpha (5) - 7232 \, 776 \, 173 \right] \frac{a^4}{373 \, 248} + O(a^5)
\]

\(^2\) The full expression for \(SU(N_f)\) is given in the attached data file (see stacks.iop.org/JPhysA/46/225403/mmedia).
for non-zero $\alpha$. Equipped with this then the equivalent minimal MOM scheme expression is

$$\gamma_{\text{O}}^{\text{MOM}}(a, \alpha)_{SU(3)} = \left[ 27\alpha + 8N_f - 105 \right] \frac{a^2}{12}$$

$$+ [-108\alpha^2 - 48\alpha^2 N_f + 387\alpha^3 - 48\alpha N_f + 225\alpha + 572N_f - 3978] \frac{a^2}{48}$$

$$+ [648\alpha^4 N_f - 26730a^4 - 3888\alpha^3 N_f + 21870\xi (3)\alpha^3 - 31347\alpha^3$$

$$+ 3888\xi (3)\alpha^2 N_f - 53136\alpha^2 N_f - 102060\xi (3)\alpha^2 + 337770\alpha^2$$

$$- 3888\xi (3)\alpha N_f - 36180\alpha N_f - 115182\xi (3)\alpha + 43011\alpha$$

$$- 1536\xi (3)N_f^2 - 27328N_f^2 - 29088\xi (3)N_f + 959652N_f$$

$$+ 696924\xi (3) - 4993812] \frac{a^3}{1728}$$

$$+ [93312\alpha^6 N_f + 279936\xi (3)\alpha^5 - 1224720\xi (5)\alpha^5 - 3805380\alpha^5$$

$$- 77760\xi (3)\alpha^4 N_f - 272160\xi (5)\alpha^4 N_f + 557928\alpha^4 N_f$$

$$+ 11078613\xi (3)\alpha^4 + 1341360\xi (5)\alpha^4 - 27025488\alpha^4$$

$$+ 1135296\xi (3)\alpha^3 N_f + 77760\xi (5)\alpha^3 N_f - 4591728\alpha^3 N_f$$

$$+ 13097214\xi (3)\alpha^3 + 1319490\xi (5)\alpha^3 - 10218393\alpha^3$$

$$- 221184\xi (3)\alpha^2 N_f^2 + 1817280\alpha^2 N_f^2 + 14990832\xi (3)\alpha^2 N_f$$

$$- 816480\xi (5)\alpha^2 N_f - 71613396\alpha^2 N_f - 149908644\xi (3)\alpha^2$$

$$+ 83215350\xi (5)\alpha^2 + 277974261\alpha^2 + 147456\xi (3)\alpha N_f^3$$

$$- 36864\alpha N_f^3 - 6801408\xi (3)\alpha N_f^3 + 3556224\alpha N_f^3$$

$$+ 98413056\xi (3)\alpha N_f - 10730880\xi (5)\alpha N_f - 75346200\alpha N_f$$

$$- 617646222\xi (3)\alpha + 259312590\xi (5)\alpha - 231547167\alpha$$

$$- 73728\xi (3)N_f^3 + 1125120N_f^3 + 8977920\xi (3)N_f^3$$

$$+ 14254080\xi (5)N_f^3 - 82945888N_f^3 - 276910092\xi (3)N_f$$

$$- 280604160\xi (5)N_f + 1438122060N_f + 1437422031\xi (3)$$

$$+ 816720570\xi (5) - 5780555523] \frac{a^4}{41472} + O(\alpha^5) \quad (4.7)$$

for $SU(3)$.

5. Discussion

We have provided all the renormalization group functions in QCD in the minimal momentum subtraction scheme introduced in [6]. To do this we have explicitly renormalized the theory and applied the renormalization prescription given in [6] to define the scheme. While [6] concentrated on the $\beta$-function the other renormalization group functions are required for other problems such as the infrared structure of propagators and therefore we have provided that information. Currently the results are known at four loops for the $SU(N_f)$ colour group and at three loops for a general group. One feature which differs from [6] rests in the renormalization of the gauge parameter. In [6] $\alpha$ was renormalized in the $\overline{\text{MS}}$ way whereas here we have chosen to follow a fuller approach and renormalize the gauge parameter according to the same criterion as all the two-point functions. While this differs from [6] both sets of results agree in the Landau gauge which is the main gauge of interest for practical studies of the infrared dynamics of the gluon and ghost.

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