Meson spectroscopy from lattice QCD

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Sínead Ryan and M.P. (TCD)
A short introduction to QCD on the lattice
The lattice vacuum with quark dynamics
Spectroscopy measurements
  - Excited states
  - Quark-field smearing
  - Spin
Isovector mesons
Isoscalar mesons
Scattering, resonances and decay widths
Conclusions
Regularising QCD on a lattice
Lattice regularisation

- Lattice provides a **non-perturbative, gauge-invariant** regulator for QCD
- Quarks live on sites
- Gluons live on links
- $a$ - lattice spacing
- $a \sim 0.1$ fm

The Nielsen-Ninomiya theorem means chirally symmetric quarks are missing, but can discretise quarks by trading-off some symmetry. In a finite volume $V = L^4$, finite number of degrees of freedom and path-integral is an ordinary (but large) integral.
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Minkowski, Wick and Euclid

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
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\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
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\]

Analytic continuation: 
\[t \rightarrow i \gamma, -i/\mathcal{S} \rightarrow 1/\mathcal{S}.\]

Enables Importance sampling Monte Carlo

Lose direct contact with dynamical properties of field theory, such as decay widths.
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- Analytic continuation: \( t \rightarrow i \tau, \quad \frac{-i}{\hbar} S \rightarrow \frac{1}{\hbar} S. \)
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- Lose direct contact with **dynamical** properties of field theory, such as decay widths.
The lattice QCD vacuum
Monte Carlo sampling the QCD lattice vacuum

\[ C_\pi(t_1, t_0) = \frac{\int DUD\bar{\psi}D\psi \; \bar{\psi}_u(t_1)\gamma_5\psi_d(t_1)\bar{\psi}_d(t_0)\gamma_5\psi_u(t_0) \; e^{-S_G - \bar{\psi}_uM\psi_u - \bar{\psi}_dM\psi_d}}{\int DUD\bar{\psi}D\psi \; e^{-S_G - \bar{\psi}_uM\psi_u - \bar{\psi}_dM\psi_d}} \]

- Too hard to deal with grassmann algebra directly
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\[ C_\pi(t_1, t_0) = \frac{\int DU \; \text{Tr} \; \gamma_5 M^{-1}(t_1, t_0) \gamma_5 M^{-1}(t_0, t_1) \; \det M[U]^2 \; e^{-S_G}}{\int DU \; \det M[U]^2 \; e^{-S_G}} \]

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- Integrate out quark fields
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- Too hard to deal with grassmann algebra directly
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- Quenched approximation was: ignore \( \det M^2 \)
- \( N_f = 2 \) importance sampling
- Non-negative, thanks to Euclidean metric
Cost of vacuum sampling

- The “noughties” have seen a lot of progress:
  \[
  C_{\text{flops}} \propto \left( \frac{m_\pi}{m_\rho} \right)^{-6} \times L^5 \times a^{-7}
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- Dramatic change in *scaling with quark mass* from algorithm improvements [Hasenbusch ’01, Lüscher ’03,’04]

- Coupled to continued fall in price of CPU cores, means simulations at the physical quark masses are possible [PACS-CS arXiv:0911.2561]
- $N_f = 2 \oplus 1$ quark dynamics
- Strange quark mass tuned (about) right
- Light quarks - from $SU(3)$ point to $m_\pi \approx 220$ MeV
- Spatial volumes $12^3 \rightarrow 32^3$
- Non-perturbatively tuned anisotropic lattice with $a_s/a_t = 3.5$
- Improved gauge and quark actions (SW, tadpole improved)
- Quark propagation on a stout-link background.
Spectroscopy measurements
Computing the spectrum (1)

- Energies of colourless QCD states extracted from \textbf{two-point functions} in Euclidean time

\[ C(t) = \langle \Phi(t) | \Phi^\dagger(0) \rangle \]

- Euclidean time: \[ \Phi(t) = e^{Ht} \Phi e^{-Ht} \] so \[ C(t) = \langle \Phi | e^{-Ht} | \Phi \rangle \]
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- Insert a complete set of states then:

\[ C(t) = \sum_{k=0}^{\infty} |\langle \Phi | k \rangle|^2 \ e^{-E_k t} \]
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\]

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\]

- Then \( \lim_{t \to \infty} C(t) = Ze^{-E_0 t} \)
- If the large-\( t \) exponential fall-off of \( C(t) \) can be observed, the **energy** of a state can be measured
• **Excited-state** energies measured from matrix of correlators:

\[ C_{ij}(t) = \langle \Phi_i(t) | \Phi_j^\dagger(0) \rangle \]
Computing the spectrum (2)

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- Solve generalised eigenvalue problem:
  \[ C(t_1) \mathbf{v} = \lambda C(t_0) \mathbf{v} \]

for different \( t_0 \) and \( t_1 \)

[M. Lüscher & U Wolff, C. Michael]
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- Then \( \lim_{(t_1-t_0) \to \infty} \lambda_n = e^{-E_n(t_1-t_0)} \)
- Method constructs optimal ground-state creation operator, then orthogonal states
Isovector meson correlation functions

- To create a meson, we need to build functions that couple to quarks.
- Meson can be created by a quark bilinear. Appropriate gauge invariant creation operator (for isospin $I = 1$) would be
  \[ \Phi_{\text{meson}}(t) = \sum_x \bar{u}(x, t) \Gamma U_C(x, y; t) d(y, t) \]
  where $\Gamma$ is some appropriate Dirac structure, and $U_C$ a product of (smeared) link variables.
- Operators that transform irreducibly under the lattice rotation group $O_h$ are needed.
- Complication: we do not have direct access to the fermion integration variables in the computer.
- The quark action is bilinear so:
  \[ \langle \psi^\alpha_a(x, t) \bar{\psi}^\beta_b(y, t') \rangle = [M^{-1}]_{ab}^{\alpha, \beta}(x, t; y, t') \]
Now the elementary component in the correlation function is

\[ \langle 0 | \Phi(t) \Phi^\dagger(0) | 0 \rangle = \langle \text{Tr} \ M^{-1}(z, 0; x, t) \Gamma U_C(x, y, t) M^{-1}(y, t; w, 0) \Gamma^\dagger U_C'(w, z, 0) \rangle \]

In general, this is still expensive to compute, since it requires knowing many entries in the inverse of the fermion operator, \( M \).

If the choice of operator at the source is restricted and no momentum projection is made, only the bilinear at (eg) the origin on time-slice 0 is needed.

Quark propagation from a single site to any other site is computed by solving \( M \psi = e_0^{\alpha, \alpha} \) where \( e_0 \) are the 12 vectors that only have non-zero components at the origin.

Can we get away from this restriction?
Isovector meson correlation functions (3)

The most general operator.

A restricted correlation function accessible to one point-to-all computation.
If we are interested in measuring isoscalar meson masses, extra diagrams must be evaluated, since four-quark diagrams become relevant. The Wick contraction yields extra terms, since

\[
\langle \psi_i \bar{\psi}_j \psi_k \bar{\psi}_l \rangle = M_{ij}^{-1} M_{kl}^{-1} - M_{il}^{-1} M_{jk}^{-1}
\]

Now

\[
\langle 0 | \Phi_{l=0}(t) \Phi_{l=0}^\dagger(0) | 0 \rangle = \\
\langle 0 | \Phi_{l=1}(t) \Phi_{l=1}^\dagger(0) | 0 \rangle - \langle 0 | \text{Tr} \ M^{-1} \Gamma U_C(t) \text{Tr} \ M^{-1} \Gamma U_C(0) | 0 \rangle
\]
Smearing - an essential ingredient for precision

- To build an operator that projects effectively onto a low-lying hadronic state need to use smearing.
- Instead of the creation operator being a direct function applied to the fields in the lagrangian first smooth out the UV modes which contribute little to the IR dynamics directly.
- A popular gauge-covariant smearing algorithm; Jacobi/Wuppertal smearing: Apply the linear operator
  \[ \Box_j = \exp(\sigma \Delta^2) \]

  \( \Delta^2 \) is a lattice representation of the 3-dimensional gauge-covariant laplace operator on the source time-slice

  \[ \Delta^2_{x,y} = 6\delta_{x,y} - \sum_{i=1}^{3} U_i(x)\delta_{x+\hat{i},y} + U_i^\dagger(x - \hat{i})\delta_{x-\hat{i},y} \]

- Correlation functions look like \( \text{Tr} \ \Box_j M^{-1} \Box_j M^{-1} \Box_j \ldots \).
Redefine smearing

• After tuning the free parameter $\sigma$ it turns out $\Box_f$ is a very low rank operator.
• The choice of smearing operator is arbitrary, provided
  1. It is a scalar operator
  2. It is gauge covariant
  3. It is a function of only field on time-slice $t$ (or perhaps a few nearest neighbours?)
• Redefine smearing to be a projection operator onto a low-dimensional space of fields:

$$\Box = \sum_{k=1}^{M} \nu^{(k)} \otimes \nu^{(k)*}$$

• This is distillation.
• How to choose $\nu$? One simple choice is to use the lowest $M$ eigenvectors of $\Delta^2$
Distilled correlation functions

- Why is this helpful? Look at correlation functions such as an isovector meson two-point function

\[ C_{AB}(t_1, t_0) = \text{Tr} \, \Box(t_1) \Gamma_1 \Box(t_1) M_u^{-1}(t_1, t_0) \Box(t_0) \Gamma_0 \Box(t_0) M_d^{-1}(t_0, t_1) \]

- \( \Gamma_{1,2} \) are creation operators that make mesons with appropriate quantum numbers

- Inserting the definition of the distillation operator, the correlation function becomes a trace over a product of rank-M matrices.

\[ C_{AB}(t_1, t_0) = \text{tr} \, \Phi_1(t_1) \tau(t_1, t_0) \Phi_0(t_0) \tau(t_0, t_1) \]

with

\[ \Phi_{a}^{(i,j)} = \nu^{(i)} \ast \Gamma_a \nu^{(j)} \quad \text{and} \quad \tau_{a}^{(i,j)} = \nu^{(i)} \ast (t_1) M^{-1}(t_1, t_0) \nu^{(j)}(t_0) \]
The lowest eigenvector of the laplace operator

- Localised mode - size is confinement scale
Distillation operator is rotationally symmetric and gaussian
A tale of two symmetries

- Continuum: states classified by $J^P$ irreducible representations of $O(3)$.

- Lattice regulator breaks $O(3) \rightarrow O_h$
A tale of two symmetries

- Continuum: states classified by $J^P$ irreducible representations of $O(3)$.

- Lattice regulator breaks $O(3) \to O_h$
- Lattice: states classified by $R^p$ “quantum letter” labelling irrep of $O_h$
Spin on the lattice

- $O_h$ has 10 irreps: $\{A_1^{g,u}, A_2^{g,u}, E^{g,u}, T_1^{g,u}, T_2^{g,u}\}$, where $\{g, u\}$ label even/odd parity.

- Search for degeneracy patterns in the spectrum
Spin on the lattice

- $O_h$ has 10 irreps: $\{A_{1}^{g,u}, A_{2}^{g,u}, E^{g,u}, T_{1}^{g,u}, T_{2}^{g,u}, \}$, where $\{g, u\}$ label even/odd parity.
- Link to continuum: subduce representations of $O(3)$ into $O_h$

|     | $A_1$ | $A_2$ | $E$ | $T_1$ | $T_2$ |
|-----|-------|-------|-----|-------|-------|
| $J = 0$ | 1     |       |     |       |       |
| $J = 1$ |       | 1     |     |       |       |
| $J = 2$ |       |       | 1   | 1     |       |
| $J = 3$ |       |       | 1   | 1     | 1     |
| $J = 4$ |       |       | 1   | 1     | 1     | 1     |

...
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| \(J = 4\) |         |         | 1    | 1      | 1      |
| \(\vdots\) |         |         | \(\vdots\) | \(\vdots\) | \(\vdots\) |

- Search for degeneracy patterns in the spectrum
Nature plays a trick...

Suppose we computed this spectrum:

\[ \begin{array}{cccccc}
A_1 & T_1 & E & T_2 \\
3350 & 3400 & 3450 & 3500 & 3550 & 3600
\end{array} \]

Within errors, it could be the nine degrees of freedom of a spin-4 state. But degeneracy also consistent with 0 ⊕ 1 ⊕ 2. This is the near-degenerate quark-model P-wave triplet!
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Example: $J^{PC} = 2^{++}$ meson creation operator

- Need more information to discriminate spins. Consider continuum operator that creates a $2^{++}$ meson:

$$\Phi_{ij} = \bar{\psi} \left( \gamma_i D_j + \gamma_j D_i - \frac{2}{3} \delta_{ij} \gamma \cdot D \right) \psi$$
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- A reducible representation:

$$\Phi^{T2} = \{ \Phi_{12}, \Phi_{23}, \Phi_{31} \}$$

$$\Phi^{E} = \left\{ \frac{1}{\sqrt{2}} (\Phi_{11} - \Phi_{22}), \frac{1}{\sqrt{6}} (\Phi_{11} + \Phi_{22} - 2\Phi_{33}) \right\}$$
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- Look for signature of continuum symmetry:
  \[ \langle 0 | \Phi^{(T_2)} | 2^{++(T_2)} \rangle = \langle 0 | \Phi^{(E)} | 2^{++(E)} \rangle \]
Spin-3 identification: J. Dudek et. al., Hadron Spectrum Collab.

- $m_{A_2}/m_\Omega = 1.210(5)$
- $m_{A_2}/m_\Omega = 1.626(16)$
- $m_{T_1}/m_\Omega = 1.207(5)$
- $m_{T_1}/m_\Omega = 1.648(23)$
- $m_{T_2}/m_\Omega = 1.204(4)$
- $m_{T_2}/m_\Omega = 1.626(8)$
The continuum-based operator construction

Cross-correlation between operator sets in \( T_{1}^{-1} \)

- “Cross-talk” between sets of operators that should be distinct in the continuum is small
Light quarks still heavy - more analysis underway
The continuum spectrum - strange mesons

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Exotic states - comparison vs other analyses

Comparison is with data from Graz group
Multi-hadron states make things very much more complicated!
Isoscalar $A_{1}^{--}$ correlation function

- Small volume ($12^3$)
- Fit requires a constant: $C(t) = A_0 + A_1 e^{-A_2 t}$ - volume artefact?
- Precise result - $a_t E$ determined to 1% instead of 10%
• No-go: Maiani-Testa theorem tells us matrix elements measured in Euclidean field theory do not contain information about strong-decay widths.
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• Changes to **Energy spectrum** in a finite-box as size of box changes gives information about widths [DeWitt ’56, Lüscher ’86]

• New proposed analysis [Bernard et.al. JHEP0808:024,2008] uses binning algorithm to measure width

• Will need very high-precision - widths are inferred from shifts in energy levels as box-size changes
$\pi\pi_{l=0} - \pi\pi_{l=0}$ correlator (disconnected part)

Initial tests on small lattices

With Andrew Nolan
Glueball - $\pi \pi$ correlation function

Initial test on small lattices

With Andrew Nolan
Conclusions

- HadSpec collaboration have generated $2 \otimes 1$ dynamical anisotropic lattices
- New technology under testing and now in production (for isovector mesons)
- Precision is crucial to go further.
- New method show promise for more precision needed in the isoscalar meson sector.