Tunnelling geometries II. Reduction methods for functional determinants

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Abstract

The reduction algorithms for functional determinants of differential operators on spacetime manifolds of different topological types are presented, which were recently used for the calculation of the no-boundary wavefunction and the partition function of tunnelling geometries in quantum gravity and cosmology.

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1. Introduction

In this paper we present the calculational technique for functional determinants of differential operators arising in the one-loop approximation of local field theory on curved spacetime manifolds of different topological types. The purpose of this technique consists in the reduction method which allows one to obtain these determinants in terms of quantities of lower functional dimensionality, that is the determinants and traces on the space of functions defined at some spatial slice and not involving the integration over time. This method, well known in the context of the semiclassical WKB theory as a Pauli-Van Vleck-Morette formula [1], has a rather nontrivial generalization to Hartle-Hawking boundary conditions describing the quantum tunnelling geometries in quantum cosmology. It was applied in [2, 3] for the calculation of the no-boundary wavefunction and served as a link between the unitary Lorentzian theory and the gravitational instantons in the Euclidean quantum gravity.

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The organization of the paper is as follows. In Sect.2 we give a short overview of the functional determinants arising in the one-loop approximation for the path integral which gives the transition amplitude for different types of boundary conditions. Sect.3 contains the derivation of the known Pauli-Van Vleck-Morette formula for a one-loop preexponential factor, appropriate for its generalization to the case of the no-boundary wavefunction in quantum cosmology. In Sect.3 we obtain the reduction algorithm for its preexponential factor in terms of the regular basis functions of the Euclidean ”wave” operator on the underlying spacetime of the no-boundary type. In Sect.4 we derive a similar reduction algorithm for the preexponential factor on the closed Euclidean spacetime without boundary, having a topology of a four-dimensional sphere. We also discuss a deep analogy of this result with the known algorithm for a transition amplitude between IN and OUT vacua of the S-matrix theory in the asymptotically flat Lorentzian spacetime, which underlies the unification of the unitary Lorentzian quantum gravity with its Euclidean version considered in \[3\].

2. Functional determinants in the one-loop approximation of quantum theory

The one-loop functional determinants in Euclidean quantum theory, and quantum gravity in particular, arise in context of the heat equation

\[
\hbar \frac{\partial}{\partial \tau_+} K(\tau_+, \phi_+ | \tau_-, \phi_-) = -\hat{H} K(\tau_+, \phi_+ | \tau_-, \phi_-),
\]

\[
K(\tau_-, \phi_+ | \tau_-, \phi_-) = \delta(\phi_+ - \phi_-)
\]

for a kernel of transition between the configurations $\phi_\pm$ at the ”moments” of the Euclidean time $\tau_\pm$, generated by the Hamiltonian of the theory $\hat{H} = \hat{H}(\phi_+, p_\phi)$ which is a differential operator in the coordinate representation of $\phi_+$, $p_\phi = \hbar \partial / i \partial \phi_+$. This kernel can be represented by the path integral over histories $\phi(\tau)$ subject to boundary conditions $\phi(\tau_\pm) = \phi_\pm$

\[
K(\tau_+, \phi_+ | \tau_-, \phi_-) = \int D\mu[\phi] \ e^{-\frac{1}{\hbar}I[\phi]}
\]

with the Euclidean action

\[
I[\phi(\tau)] = \int_{\tau_-}^{\tau_+} d\tau \mathcal{L}_E(\phi, d\phi / d\tau, \tau)
\]
and local integration measure

\[ D\mu[\phi] = \prod_\tau d\phi(\tau) \left[ \det a \right]^{1/2}(\tau) + O(\hbar), \quad d\phi = \prod_i d\phi^i. \] (2.5)

\[ \det a = \det a_{ik}, \quad a_{ik} = \frac{\partial^2 L_E}{\partial \phi^i \partial \phi^k}, \] (2.6)

where the Euclidean Lagrangian is related to the Hamiltonian \( H(\phi_+, p_\phi) \) by the Legendre transform with respect to \( ip_\phi \).  

In theories with the Hamiltonian and Lagrangian quadratic respectively in momenta and velocities, the integration measure (2.3) does not include the higher-order loop corrections \( O(\hbar) \) and reduces to the determinant of the Hessian matrix \( a_{ik} \). The latter is understood with respect to indices \( i \) and \( k \) labelling the physical variables \( \phi = \phi^i \). Throughout the paper we shall use DeWitt notations, in which these indices have a condensed nature and include, depending on the representation of field variables, either continuous labels of spatial coordinates or discrete quantum numbers labelling some complete infinite set of harmonics on a spatial section of spacetime. Correspondingly the contraction of these indices, the trace operation, \( \text{tr} A \equiv A^i_i \), with respect to them, etc. will imply either the spatial integration or infinite summation over such a set. Therefore the above determinant in field systems is functional, but its functional nature is restricted to a spatial slice of constant time \( \tau \). The product over time points of \( \det a(\tau) \) can be regarded as a determinant of higher functional dimensionality associated with the whole spacetime if we redefine \( a_{ik} \) as a time-ultralocal operator \( \mathbf{a} = a_{ik} \delta(\tau - \tau') \). We shall denote such functional determinants for both ultralocal and differential operators in time by \( \text{Det} \). Thus, in view of the ultralocality of \( \mathbf{a} \), the contribution of the one-loop measure equals

\[ \prod_t \left[ \det a \right]^{1/2}(t) = \left[ \text{Det} \mathbf{a} \right]^{1/2} = \exp \left\{ \frac{1}{2} \int_{t_-}^{t_+} dt \delta(0) \ln \det a(t) \right\}. \] (2.7)

With these notations and local measure, the one-loop approximation for the path integral (2.3) takes the form

\[ K_{1\text{-loop}}(\tau_+, \phi_+| \tau_-, \phi_-) = \left( \frac{\text{Det} \mathbf{F}}{\text{Det} \mathbf{a}} \right)^{-1/2} e^{-\frac{1}{\hbar} I[\phi]} \bigg|_{\phi(\tau) = \phi(\tau, \phi_\perp)}, \] (2.8)

\[ \text{1The equations (2.1) - (2.6) are related to the physical theory in Lorentzian time by the Wick rotation of time variable } \tau = it + \text{const accounting, in particular, for the occurrence of imaginary unit in the transform from the Hamiltonian to the Euclidean Lagrangian.} \]
where $\phi (\tau, \phi_\pm)$ is a solution of classical equations of motion passing the points $\phi_{\pm}$ at $\tau_{\pm}$

$$\delta I [\phi] \over \delta \phi (\tau) = 0, \quad \phi (\tau_, \phi_\pm) = \phi_-, \quad \phi (\tau_+, \phi_\pm) = \phi_+$$

(2.9)

and $F$ is an operator of small disturbances on its background – a matrix-valued second-order differential operator $F (d/d\tau) \equiv F_{ik} (d/d\tau)$

$$F \equiv F (d/d\tau) \delta (\tau - \tau’) = \frac{\delta^2 I [\phi]}{\delta \phi (\tau) \delta \phi (\tau’)}.$$ 

(2.10)

Generically it has the form

$$F (d/d\tau) = - \frac{d}{d\tau} a \frac{d}{d\tau} - \frac{d}{d\tau} b + b^T \frac{d}{d\tau} + c,$$

(2.11)

where the coefficients $a = a_{ik}$, $b = b_{ik}$ and $c = c_{ik}$ are the (functional) matrices acting in the space of field variables $\phi (\tau) = \phi^k (\tau)$, and the superscript T denotes their (functional) transposition $(b^T)_{ik} \equiv b_{ki}$. These coefficients can be easily expressed as mixed second-order derivatives of the Euclidean Lagrangian with respect to $\phi^i$ and $\dot{\phi}^i = d\phi^i / d\tau$. In particular, the matrix of the second order derivatives $a_{ik}$ is given by the Hessian matrix (2.6): $a_{ik} = \partial^2 L_E / \partial \dot{\phi}^i \partial \dot{\phi}^k$.

In the general case, when the spectrum of a differential operator is unknown, the only definition for its functional determinant can be given via the following variational equation

$$\delta \ln \text{Det} F = \delta \text{Tr} \ln F = \text{Tr} \delta F G,$$

(2.12)

where Tr denotes the functional trace, which includes the integration over time of the coincidence limit of the corresponding operator kernel, and $G = G (\tau, \tau’)$ is an inverse of $F$ or the Green’s function satisfying

$$F (d/d\tau) G (\tau, \tau’) = \delta (\tau, \tau’).$$

(2.13)

This definition is obviously incomplete unless one fixes uniquely the Green’s function by the appropriate boundary conditions and also specifies the functional composition law $\delta F G$ in the functional trace. One should remember that the kernel $G (\tau, \tau’)$ is not a smooth function of its arguments, and its irregularity enhances when it is acted upon by two derivatives contained in $\delta F = \delta F (d/d\tau)$, therefore one has to prescribe the way how these derivatives act on both arguments of $G (\tau, \tau’)$ and how to take the coincidence limit of the resulting singular kernel in the functional trace.
Both the boundary conditions and the specification of trace operation must follow from the way of calculating the gaussian path integral over quantum disturbances $\varphi(\tau)$ in the vicinity of $\phi(\tau) = \phi(\tau, \phi_{\pm})$, which gives rise to the one-loop functional determinants of the above type:

$$\int D\varphi \exp \left\{-\frac{1}{2\hbar} \int_{\tau_-}^{\tau_+} d\tau \varphi^T \overset{\leftrightarrow}{F} \varphi\right\} = \text{Const} \left[ \text{Det} F \right]^{-1/2}. \quad (2.14)$$

Here the notation $\overset{\leftrightarrow}{F}$ implies that the two time derivatives of the operator $F$ are acting symmetrically on the both functions $\varphi$ and $\varphi^T$, so that the exponentiated time integral contains the quadratic part of the Euclidean Lagrangian $L_E^{(2)} = (1/2) \varphi^T F \varphi$, generating the "wave" operator $F$. For arbitrary two test functions $\varphi_1$ and $\varphi_2$ and for the operator of the form (2.11) this notation reads

$$\varphi^T_1 \overset{\leftrightarrow}{F} \varphi_2 = \varphi^T_1 a \varphi_2 + \varphi^T_1 b \varphi_2 + \varphi^T_1 b^T \varphi_2 + \varphi^T_1 c \varphi_2 \quad (2.15)$$

and implies the following integration by parts

$$\varphi^T_1 \overset{\leftrightarrow}{F} \varphi_2 = \varphi^T_1 (F \varphi_2) + \frac{d}{d\tau} \left[ \varphi^T_1 (W \varphi_2) \right], \quad (2.16)$$

$$W \equiv W(d/d\tau) = a \frac{d}{d\tau} + b. \quad (2.17)$$

We shall call $W$ the Wronskian operator which enters the following Wronskian relation for the operator $F$

$$\varphi^T_1 (F \varphi_2) - (F \varphi_1)^T \varphi_2 = -\frac{d}{d\tau} \left[ \varphi^T_1 (W \varphi_2) - (W \varphi_1)^T \varphi_2 \right] \quad (2.18)$$

and also participates in the variational equation for the Euclidean momentum $\partial L_E/\partial \dot{\phi}$ valid for arbitrary variations $\delta \phi(\tau)$ of field histories

$$\delta \frac{\partial L_E}{\partial \dot{\phi}} = W(d/d\tau) \delta \phi(\tau). \quad (2.19)$$

As shown by Feynman [4], the functional determinant of the differential operator generated by the gaussian path integral (2.14) is determined by the variational equation

$$\delta \ln \text{Det} F = \text{Tr} \overset{\leftrightarrow}{\delta F} G \quad (2.20)$$

where the Green’s function $G(\tau, \tau')$ satisfies the same boundary conditions at $\tau = \tau_{\pm}$ as the integration variables $\varphi(\tau)$ in (2.14) and the functional composition law $\overset{\leftrightarrow}{\delta F} G$ implies a
symmetric action of time derivatives on both arguments of \( G(\tau, \tau') \) similar to (2.15):

\[
\text{Tr} \delta \dot{F} G = \int_{\tau_-}^{\tau_+} d\tau \, \text{tr} \left[ \delta \dot{F} G(\tau, \tau') \right]_{\tau' = \tau} \\
\equiv \int_{\tau_-}^{\tau_+} d\tau \, \text{tr} \left[ (\delta a \frac{d^2}{d\tau d\tau'} + \delta b \frac{d}{d\tau'} + \delta b^T \frac{d}{d\tau} + \delta c) G(\tau, \tau') \right]_{\tau' = \tau} .
\] (2.21)

Here \( \text{tr} \) denotes the matrix trace operation with respect to condensed indices of \( \delta a = \delta a_{ik}, \delta b = \delta b_{ik}, \delta b^T = \delta b_{ki}, \delta c = \delta c_{ik} \) and \( G(\tau, \tau') = G_{ki}(\tau, \tau') \). As discussed above, in field theory this trace has also a functional nature because it involves either a spatial integration or infinite summation over quantum numbers of spatial harmonics, but its functional dimensionality is lower than in \( \text{Tr} \) for it does not involve integration over time.

The purpose of this paper will be to integrate the variational equation (2.21) and, thus, obtain the closed algorithm for the functional determinant \( \text{Det} F \) in terms of quantities of lower functional dimensionality (involving the matrix trace (\( \text{tr} \)) and determinant (\( \text{det} \)) operations on condensed indices \( i, k, \ldots \) of the above type). We shall do it for three different types of boundary conditions in the gaussian path integral (2.14). The first type corresponds to the calculation of the kernel (2.3) of transition between two regular hypersurfaces \( \Sigma_{\pm} \) of finite size at \( \tau_{\pm} \) (see Fig.1) with fixed ”end point” configurations \( \phi_{\pm} \), which imply the Dirichlet boundary conditions on quantum integration variables \( \varphi(\tau_{\pm}) = 0 \). The second type incorporates the calculation of the wavefunction of Hartle and Hawking with fixed field at \( \tau_{+} \), \( \varphi(\tau_{+}) = 0 \), and the so-called no-boundary condition at \( \tau_{-} \). In this case the integration goes over all \( \varphi(\tau_{-}) \) satisfying the necessary regularity properties providing that \( \tau_{-} \) is a regular inner point of the Euclidean spacetime which has as the only boundary the surface \( \Sigma_{+} \) at \( \tau_{+} \). Finally, we consider the third type incorporating the ”no-boundary” conditions at the both ends of time segment \( \tau = \tau_{\pm} \) and corresponding to the one-loop preexponential factor of the theory on the closed Euclidean spacetime of spherical topology. The first type represents the path-integral derivation of the well-known Pauli - Van Vleck - Morette formula [1] for the kernel of the heat equation, and we shall begin our considerations in the next section with this case.

The final remark, which is in order here, concerns the ultraviolet infinities in the one-loop preexponential factors inalienable in any local field theory and necessarily arising in the transition from quantum mechanical to field models with infinite number of degrees of freedom. In this paper we shall not consider this problem implying that any kind of
ultraviolet regularization can be performed in both sides of the reduction algorithms which we are going to obtain. The justification of this strategy was partly explained in the previous paper of this series \cite{3} (see Sect.8 of that paper) and based on the fact that these reduction algorithms, being intrinsically non-covariant, serve as a bridge between the non-covariant manifestly unitary Lorentzian theory and its Euclidean version accumulating in a manifestly covariant form all divergent quantum corrections. It is the latter theory which must be covariantly regulated to give physically reasonable results, in contrast to the ultraviolet regularization of non-covariant quantities \cite{5, 6, 7} which may lead to the discrepancies with their manifestly covariant counterparts \cite{8, 9, 9a}.

3. The Pauli - Van Vleck - Morette formula for the one-loop preexponential factor

Let us consider the operator (2.11) on a finite segment of the Euclidean time \( \tau_- \leq \tau \leq \tau_+ \) such that all its coefficients \( a, b \) and \( c \) are regular functions on this segment. This operator is formally symmetric under the operation of integration by parts (and matrix transposition with respect to indices \( i \)) and, therefore, it is selfadjoint under boundary conditions in the space of fields, which provide the vanishing of the total-derivative terms. The boundary conditions on the integration variables \( \varphi(\tau_{\pm}) = 0 \) fall into this category and correspond to the calculation of the one-loop contribution for the Euclidean transition kernel from \( \tau_- \) to \( \tau_+ \).

For the integration of eq.(2.20) let us introduce two sets of basis functions \( u_- \) and \( u_+ \) of the operator \( F \)

\[
Fu_\pm = 0, \quad u_\pm = u_\pm^i(\tau)
\]  

satisfying the Dirichlet boundary conditions respectively at \( \tau_- \) and \( \tau_+ \):

\[
u_-(\tau_-) = 0, \quad u_+(\tau_+) = 0.
\]  

In the DeWitt notations, which will be used throughout the paper, we regard these basis functions

\(^2\)See the disagreement between the covariant calculation of gravitational \( \zeta \)-functions in \cite{8} and their non-covariant calculation in terms of physical variables \cite{9, 9a, 9b}, which implies that the unification of the covariant Euclidean quantum gravity and its Lorentzian unitary counterpart requires a deeper consideration accounting for subtleties of the ultraviolet regularization in covariant and unitary gauges.

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functions, enumerated by the condensed index $A$ of arbitrary nature, as forming the square matrices with the first (contravariant) index $i$ and the second (covariant) index $A$. According to the discussion of the previous paper [3], containing examples of such functional matrix structure, these matrices can be regarded invertible at general $\tau$ except certain particular values of time including $\tau_-$ for $u_-$ and $\tau_+$ for $u_+$. As is shown in the Appendix, if the operator $F$ does not have zero eigenvalues at the segment $\tau_- \leq \tau \leq \tau_+$, these matrices are granted to be invertible at the ends of this segment opposite to the points of their boundary conditions (i.e. $u_{\pm}(\tau_{\pm})$) and can be used to form other two invertible matrices related by a simple transposition law

$$
\Delta_{+-} = u_+^T(Wu_-) - (Wu_+)^Tu_-, \quad \Delta_{+-} \equiv (\Delta_{+-})_{AB}, \quad (3.3)
$$

$$
\Delta_{-+} = u_-^T(Wu_+) - (Wu_-)^Tu_+, \quad \Delta_{-+} \equiv (\Delta_{-+})_{AB}, \quad (3.4)
$$

$$
\Delta_{+-}^T = -\Delta_{-+}. \quad (3.5)
$$

In view of the Wronskian relation (2.18) for the operator $F$ these matrices are constants of motion

$$
\frac{d}{d\tau}\Delta_{+-} = 0, \quad \frac{d}{d\tau}\Delta_{-+} = 0 \quad (3.6)
$$

and enter the following important relations for equal-time bilinear combinations of basis functions $u_{\pm} = u_{\pm}(\tau)$ (see Appendix)

$$
u_+(\tau) (\Delta_{-+})^{-1}u_-(\tau) + u_-(\tau) (\Delta_{+-})^{-1}u_+^T(\tau) = 0, \quad (3.7)
$$

$$a \left[ \dot{u}_+(\tau) (\Delta_{-+})^{-1}u_+^T(\tau) + \dot{u}_-(\tau) (\Delta_{+-})^{-1}u_+^T(\tau) \right] = I. \quad (3.8)
$$

Here $I = \delta^i_k$ denotes the unity matrix in the space of indices $i$. Just to clarify the use of indices in these equations and in what follows, note that in the transposed matrix $u_+^T = u_{\pm A}^i$, the covariant index $A$ is considered to be the first one (in contrast to $u_\pm$), so that the matrix composition law with $(\Delta_{+-})^{-1} = [(\Delta_{+-})^{-1}]_{AB}$ gives rise in eqs.(3.7) to the matrices with two field indices $i$ and $k$.

The basis functions and their Wronskian matrices (3.3) - (3.4) can be used to construct the following expression for the Green’s function of the operator $F$ subject to Dirichlet boundary conditions

$$G(\tau, \tau') = -\theta(\tau - \tau') u_+(\tau) (\Delta_{+-})^{-1}u_+^T(\tau') + \theta(\tau' - \tau) u_-(\tau) (\Delta_{+-})^{-1}u_+^T(\tau'), \quad (3.9)
$$

$$G(\tau_{\pm}, \tau') = 0, \quad (3.10)
$$
where $\theta(x)$ represents the step function: $\theta(x) = 1$ for $x \geq 0$ and $\theta(x) = 0$ for $x < 0$. This expression, the derivation of which is presented in Appendix, is nothing but the analogue of positive-negative frequency decomposition for the Feynman propagator in asymptotically-flat spacetime \cite{10}: $u_\pm$ play the role of positive and negative frequency basis functions of the Lorentzian Klein-Gordon equation and the boundary conditions (3.2) serve as a Euclidean counterpart to the IN-OUT Feynman boundary conditions propagating positive frequencies to the future and negative ones to the past \cite{1}.

Now we can use eq. (3.9) to calculate the variational functional trace (2.21). The action of derivatives on $G(\tau, \tau')$ in this trace gives rise to additional terms containing the coincidence limit of the delta function \( \delta(\tau - \tau') = (d/dt)\theta(\tau - \tau') \) and its first order derivative \( (d/dt)\delta(\tau - \tau') \). It is remarkable that the total coefficient of \( (d/dt)\delta(\tau - \tau') \) vanishes in virtue of the eq. (3.7), while the rest of the terms give the expression

$$
\text{Tr} \frac{\delta F G}{\delta a} = \int_{\tau_-}^{\tau_+} d\tau \delta(0) \text{tr} \left[ \dot{u}_+ (\Delta_{--})^{-1} u_+^T + \dot{u}_- (\Delta_{++})^{-1} u_-^T \right] - \int_{\tau_-}^{\tau_+} d\tau \text{tr} \left\{ \theta(\tau - \tau') (\Delta_{--})^{-1} u_-^T \delta F u_+ - \theta(\tau' - \tau) (\Delta_{++})^{-1} u_+^T \delta F u_- \right\}_{\tau' = \tau}.
$$

\(3.11\)

Notice that in view of (3.8) the coefficient of $\delta a$ equals $a^{-1}$, so that the first integral on the right-hand side of this equation reduces to the variation of the logarithm of local measure (2.7). On the other hand, the symmetry of the operator $\delta F$, defined by the eq. (2.15) with the coefficients $a, b$ and $c$ replaced by their variations, together with the transposition property (3.5) shows that the expression in the second integral reduces to the form free from discontinuous $\theta$-functions and, thus, has a well-defined coincidence limit. Therefore, combining (2.20), (3.11) and (2.7) one arrives at the following expression for the variation of functional determinants in the one-loop preexponential factor of (2.8)

$$
\delta \ln \frac{\text{Det} F}{\text{Det} a} = - \int_{\tau_-}^{\tau_+} d\tau \text{tr} (\Delta_{--})^{-1} u_-^T \delta F u_+.
$$

\(3.12\)

Here we reserve the same notation $\text{tr}$ for the trace operation in the space of indices $A$ enumerating the basis functions of $F$, so that the integrand of (3.12) should read $[ (\Delta_{--})^{-1}]_{AB} u_-^{T,B} \delta F u_+^A$.

\footnote{Note that the Wick rotation of Lorentzian positive and negative frequency basis functions makes them vanish respectively at the "future" and "past" infinity of the Euclidean time which corresponds to the $\tau_\pm \to \pm \infty$ limit in the boundary conditions (3.2), so that they can be regarded as a generalization of the Euclidean asymptotically-flat boundary conditions to the case of a finite time interval.}
The above equation reveals the role of the local measure (2.7) in the one-loop approximation of quantum theory: it exactly cancels a strongest power divergent part in the functional determinant of the operator of linearized quantum disturbancess. This result was obtained for general dynamical systems by DeWitt [10, 11] and studied for covariant field theories by Fradkin and Vilkovisky [12] with a special emphasis on peculiarities of the ultraviolet regularization for functional traces. The right-hand side of (3.12) does not contain the $\delta(0)$-type divergence (and simply finite for quantum mechanical systems with finite-dimensional configuration space). As we shall show now, it can be further simplified to the expression not involving the integration over time and, thus, effectively reducing the dimensionality of the functional determinant and trace.

Let us use in the right-hand side of (3.12) the equation obtained from varying the operators $F$ and $W$ in the identity (2.16) with a subsequent substitution $\varphi_1^T = u_+^T$ and $\varphi_2 = u_+$:

$$u_+^T \delta F u_+ = u_+^T (\delta F u_+) + \frac{d}{d\tau} \left[ u_+^T (\delta W u_+) \right]. \quad (3.13)$$

We also have the varied version of eq.(3.1) which forms the following boundary-value problem for the variations of basis functions

$$F \delta u_\pm = -\delta F u_\pm, \quad (3.14)$$
$$\delta u_+ (\tau_+) = 0, \quad \delta u_- (\tau_-) = 0. \quad (3.15)$$

This allows to rewrite the first term on the right hand side of (3.13) as $u_+^T (\delta F u_+) = -u_+^T (F \delta u_+)$ and then use the Wronskian relation (2.18) with $\varphi_1^T = u_+^T$ and $\varphi_2 = \delta u_+$. Therefore, the integrand in the right hand side of (3.12) reduces to a total derivative and yields the contribution of surface terms at $\tau_\pm$

$$\delta \ln \frac{\text{Det} F}{\text{Det} a} = -\text{tr} \left( \Delta_- \right)^{-1} \left[ u_+^T \delta (W u_+) \bigg|_{\tau_+} + (W u_-)^T \delta u_+ \bigg|_{\tau_-} \right]. \quad (3.16)$$

Here the other two surface terms vanish in virtue of the boundary conditions for basis functions and the expression $\delta (W u_+) = (\delta W) u_+ + W (\delta u_+)$ involves the variations of both the Wronskian operator and the basis function with respect to arbitrary variations of coefficients in the operator $F$.

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4In field theories the cancellation of the leading volume divergencies by the contribution of local measure is intertwined with the ultraviolet regularization, but in the main it has the same origin as above: Green’s function has a nature of a singular generalized function which generates the power divergencies in the functional trace when it is acted upon by the second order differential operator.
On the other hand, in virtue of the same boundary conditions the conserved matrix (3.4),
\[ \Delta_{-+}, \]
when evaluated at \( \tau^+ \) and \( \tau_- \), has two different representations
\[ \Delta_{-+} = u_+^T (Wu_+) |_{\tau^+}, \]
\[ \Delta_{-+} = -(Wu_-)^T u_+ |_{\tau_-}. \]

They allow us to convert the right hand side of (3.16) into the total variation
\[ \delta \ln \frac{\text{Det} F}{\text{Det} a} = - \text{tr} \left[ (Wu_+)^{-1} \delta (Wu_+) |_{\tau^+} - u_+^{-1} \delta u_+ |_{\tau_-} \right] = - \delta \ln \text{det} I_{++} \]
of the logarithm of the determinant of the following matrix \( I_{++} = (I_{+-})_{ik} \)
\[ I_{++} = (Wu_+) (\tau^+) u_+^{-1}(\tau_-) = u_+^{-1}(\tau^+) \Delta_{-+} u_+^{-1}(\tau_-). \]

Now the functional integration of the equation (3.19) presents no difficulty and gives the following answer for the one-loop preexponential factor
\[ \left( \frac{\text{Det} F}{\text{Det} a} \right)^{-1/2} = \text{Const} \left( \text{det} I_{++} \right)^{1/2}, \]
where Const represents a completely field independent normalization constant.

This equation reduces the functional dimensionality of the determinant from Det to det and represents a well-known Pauli-Van Vleck-Morette formula for the one-loop preexponential factor in the heat kernel or, in its Lorentzian version, the kernel of the unitary Schrödinger evolution [1]. Under a certain Hermitian operator realization of the Hamiltonian \( \hat{H} \) in the heat equation (2.1), this formula for \( K(\tau^+, \phi_+ | \tau_-, \phi_-) \) reads
\[ K(\tau^+, \phi_+ | \tau_-, \phi_-) = \left[ \det \left( \frac{1}{2\pi \hbar} \right) \frac{\partial^2 I}{\partial \phi_+ \partial \phi_-} \right]^{1/2} e^{-\frac{1}{\hbar} I} \left[ 1 + O(\hbar) \right], \]
where
\[ I \equiv I (\tau^+, \phi_+ | \tau_-, \phi_-) = I [ \phi ] |_{\phi(\tau^+) = \phi(\tau, \phi_+)} \] (3.23)
is the Euclidean Hamilton-Jacobi function of the theory – the classical action calculated at the extremal of the equations of motion (2.9) and parametrized in terms of its end points on the segment of the Euclidean time [ \( \tau_-, \tau^+ \) ].

The formula (3.22), which is much better known in the Lorentzian context with \( I = -iS \), can be obtained by solving in the semiclassical approximation the heat equation (2.1). Such a general derivation was recently presented in [13], where the precise operator realization of \( H \) corresponding to (3.22) (and usually disregarded in physics literature) was formulated.
that the matrix $\partial^2 I/\partial \phi_+ \partial \phi_-\mathbf{\phi}$ should coincide with the matrix (3.20) in the just obtained algorithm (3.21) for the one-loop preexponential factor. To prove this property, note that this matrix can be written as a variation with respect to $\phi_-$ of the Euclidean momentum of this extremal at $\tau_+ = \partial I/\partial \phi_+ = \partial \mathcal{L}_E/\partial \dot{\phi}(\tau_+)$. Therefore, in virtue of the variational relation (2.19) the above matrix takes the form

$$\frac{\partial^2 I}{\partial \phi_+ \partial \phi_-} = W \left( \frac{d}{d\tau} \frac{\partial \phi(\tau, \phi_{\pm})}{\partial \phi_-} \right) \bigg|_{\tau = \tau_+}. \quad (3.24)$$

On the other hand, the Jacobi matrix $\partial \phi(\tau|\phi_{\pm})/\partial \phi_-$ satisfies the linear boundary value problem, derivable by varying the classical equations (2.9) with respect to the boundary data $\phi_-,$

$$F \left( \frac{d}{d\tau} \frac{\partial \phi(\tau, \phi_{\pm})}{\partial \phi_-} \right) = 0,$$

$$\frac{\partial \phi(\tau_+, \phi_{\pm})}{\partial \phi_-} = 0, \quad \frac{\partial \phi(\tau_-, \phi_{\pm})}{\partial \phi_-} = I \quad (3.25)$$

and, therefore, has the following decomposition in terms of the basis functions $u_+(\tau)$ of the operator $F$

$$\frac{\partial \phi(\tau, \phi_{\pm})}{\partial \phi_-} = u_+(\tau) \ u_-^{-1}(\tau_-) \quad (3.26)$$

whence we get the equality of matrices (3.20) and (3.24) and the needed equivalence of the reduction algorithm (3.21) to the Pauli-Van Vleck-Morette formula (3.22).

4. The functional determinant on spacetime of the no-boundary type

The derivations of the previous section can be now applied to the case of primary interest in this paper – the calculation of the functional determinant subject to the no-boundary regularity conditions. All the functions above were assumed to be regular on the time segment $\tau_- \leq \tau \leq \tau_+$ corresponding in field theory to the calculations on Euclidean spacetime sandwiched between two regular spatial slices $\Sigma_{\pm}.$ For spatially closed cosmology with $\Sigma = S^3$ such a spacetime has a topology of a tube $[\tau_-, \tau_+] \times S^3,$ and, as it was discussed in [3], a no-boundary spacetime can be obtained from this tube by shrinking the surface $\Sigma_- = S^3$ to a point (see Fig.2), so that the variable $\tau - \tau_- = \tau$ in the resulting four-dimensional ball plays the role of a radial coordinate (for simplicity in this section we shall
assume that \( \tau_- = 0 \) and the metric in the vicinity of \( \tau_- \) – the regular internal point of the Euclidean manifold – takes the form

\[
\begin{aligned}
ds^2 &= d\tau^2 + \tau^2 c_{ab} dx^a dx^b + O(\tau^3), \quad \tau \to 0,
\end{aligned}
\]

with \( c_{ab} \) – the round metric of a three-dimensional sphere of unit radius, parametrized with some quasi-angular coordinates \( x^a \). The boundary conditions for the one-loop functional determinant coincide on the boundary of this spacetime, \( \tau = \tau_+ \), with those of the previous section, but in the center of this ball they drastically differ, because instead of a Dirichlet type they correspond to the conditions of regularity. The latter are rather nontrivial because the center of the spacetime ball is a singular point of the radial part of the differential operator \( \mathbf{F}(d/d\tau) \), which results in a special behaviour of basis functions \( u_\pm(\tau) \) in the vicinity of \( \tau_- \).

Indeed, the linearized physical modes of all possible spins, \( s = 0, 1/2, 1, 3/2, 2, \ldots \), are just the transverse-traceless components \( \phi^i = (\varphi, \phi(x), \psi(x), A^T_a(x), \psi^T_a(x), h^T_{ab}(x), \ldots) \) of the three-dimensional tensor fields in the \( \tau \)-foliation of the Euclidean spacetime \(^3\). For them, the coefficient \( a = a_{ik} \) of the second-order derivatives in \( \mathbf{F}(d/d\tau) \) can be collectively written as

\[
a_{ik} = (4g)^{1/2} g_{\tau\tau} g^{a_1 a_2} \cdots g^{a_{2s} a_{2s}} \delta (x_i - x_k),
\]

\( i = (a_1, \ldots a_s, x_i), \ k = (a_2, \ldots a_{2s}, x_k) \), and in the regular metric (4.1) has the following asymptotic behaviour

\[
a = a_0 \tau^k + O(\tau^{k+1}), \quad k = 3 - 2s, \quad \tau \to \tau_- = 0,
\]

where \( a_0 \) is defined by eq.(4.2) with respect to the round metric \( c_{ab} \) on a 3-sphere of the unit radius and the unit lapse \( g_{\tau\tau} = N^{-2} = 1 \). Therefore the equations (3.1) for basis functions have the form

\[
\left( \frac{d^2}{d\tau^2} + f \frac{d}{d\tau} + g \right) u_\pm(\tau) = 0,
\]

with the coefficients \( f \) and \( g \) possessing the following asymptotic behaviour

\[
f = \frac{k}{\tau} I + O(\tau^0), \quad g = \frac{g_0}{\tau^2} + O(\tau^{-1}).
\]
Here the leading singularity in the potential term $g$ originates from the spatial Laplacian $g^{ab}\nabla_a \nabla_b$ entering the operator $\mathbf{F}$, which scales in the metric (4.1) as $1/\tau^2$, and the leading term of $f$ is always a multiple of the unity matrix $\mathbf{I}$ with the same parameter $k = 3 - 2s$ as in (4.3). In the representation of spatial harmonics, the eigenfunctions of a spatial Laplacian, the (functional) matrix $g_0$ can be also diagonalized, $g_0 = \text{diag}\{ -\omega_i^2 \}$, so that, without the loss of generality, the both singularities in (4.4) can be characterised by simple numbers $k$ and $\omega = \omega_i^2$ for every component of $u_{\pm} = u_{\pm}^\prime$.

As it follows from the theory of differential equations with singular points [14], in this case the asymptotic behaviour of the solutions $u_\pm(\tau)$ near $\tau_- = 0$ has the form

$$u_-(\tau) = U_- \tau^{\mu_-} + O\left( \tau^{1+\mu_-} \right), \quad (4.6)$$

$$u_+(\tau) = V_+ \tau^{\mu_+} + O\left( \tau^{1+\mu_+} \right), \quad (4.7)$$

where $\mu_\pm$ are the roots of the following quadratic equation involving only the coefficients of leading singularities

$$\mu^2 + (k - 1) \mu - \omega^2 = 0. \quad (4.8)$$

In view of non-negativity of $\omega^2$ (the eigenvalue of $-c^{ab}\nabla_a \nabla_b$) these roots are of opposite signs, $\mu_-\mu_+ = -\omega^2 \leq 0$, and we can choose $\mu_-$ to be non-negative in order to have $u_-(\tau)$ as a set of regular basis functions at $\tau_- = 0$, the remaining part of them $u_+(\tau)$ being singular. By our assumption that the operator $\mathbf{F}$ does not have zero eigenvalues on the Euclidean spacetime of the no-boundary type, one can be sure that there are no basis functions which are simultaneously regular at $\tau_- = 0$ and vanishing at $\tau_+$. Therefore, all the functions $u_+(\tau)$ subject to Dirichlet boundary conditions at $\tau_+$ have for $\tau \to 0$ a singular asymptotic behaviour (4.7) with $\mu_+ < 0$.

Thus we can use the functions (4.6) and (4.7) satisfying respectively the regularity and the Dirichlet boundary conditions as the basis functions $u_\pm$ of the previous section, construct by the same algorithm (3.9) the corresponding Green’s function on spacetime of the no-boundary type and develope along the same lines the variational technique for the functional determinant. Thus we arrive at the same algorithm (3.21), but due to the above properties of the point $\tau = \tau_-$ it can be further simplified. Indeed, in view of (4.3) the Wronskian operator takes the form

$$W(d/d\tau) = a_0 \tau^k \frac{d}{d\tau} + O\left( \tau^{k+1} \right). \quad (4.9)$$
Therefore, the Wronskian matrix $\Delta_{-+}$ calculated at $\tau_- = 0$ equals

$$\Delta_{-+} = (\mu_+ - \mu_-) U^T_- a_0 V_+,$$  \hspace{1cm}  (4.10)$$

because of a simple property $\mu_- + \mu_+ = 1 - k$ of the roots of eq.(4.8). Using this result in the matrix $I_{+-}$ defined by eq.(3.20) together with the asymptotic behaviour (4.6) - (4.7) we get

$$\left( I_{+-}^T \right)^{-1} = u_-(\tau_+) \left( \Delta_{-+}^T \right)^{-1} u_+^T(\tau_-) = \text{Const} \: u_-(\tau_+) U^{-1}_-. \hspace{1cm} (4.11)$$

Here we have absorbed all field-independent matrix multipliers into the overall coefficient Const. This multipliers include the infinite factor $\tau^\mu_+ \cdot \tau \to \tau_-=0$, the roots $\mu_{\pm}$ of the equation (4.8) and the configuration space metric $a_0$ which are completely field independent, because they are defined with respect to a distinguished geometry of a 3-dimensional sphere of unit radius embedded into flat spacetime with unit radial lapse (in particular, the eigenvalues $\omega^2$ entering (4.8) belong to the covariant Laplacian in this distinguished metric $c_{ab}$).  

Note that the basis functions are generally defined only up to their linear recombinations $u^i_{\pm A}(\tau) \to u^i_{\pm B}(\tau) \Omega_B^A$ with certain time-independent coefficients $\Omega_B^A$ which can be arbitrary functionals of fields entering the operator $F$. In particular, the leading coefficients $U_-$ in (4.6) may be such functionals. Thus, the right-hand side of (4.11) can be viewed as a new set of regular basis functions having such a special normalization that their leading term in a small $\tau$ expansion is field-independent and is proportional to the functional matrix unity $I: u_-(\tau) U^{-1}_- = I \tau^\mu_- + O(\tau^{1+\mu_-}), \tau \to 0$. The nature of the condensed index enumerating these new basis functions coincides with that of $i$: it is a covariant index $k$ originating in (4.11) from projecting the index $A$ of $u^i_{- A}$ with the matrix $(U^{-1}_{-})^A_k$.

Thus we can write down the final algorithm for the one-loop prefactor of the no-boundary type together with the corresponding normalization of the regular basis functions in the center of the Euclidean ball

$$\left( \frac{\det F}{\det a} \right)^{-1/2} = \text{Const} \: [ \det u_-(\tau_+) \: ]^{-1/2}, \hspace{1cm} (4.12)$$

---

7The variational technique of the previous section gives the functional determinant of the differential operator only up to the overall coefficient independent of all the background field variables - the coefficients of $F$. This justifies the above procedure of absorbing such field-independent quantities into the overall normalization which should be defined from other principles, just like the normalization coefficient in the Pauli-Van Vleck-Morette formula follows from the composition law for the kernel (3.22).
\[ u_-(\tau) = I \tau^{\mu_-} + O(\tau^{1+\mu_-}), \quad \tau \to 0. \] (4.13)

Apart from special terms responsible for the renormalization of ultraviolet infinities, which we don’t consider here for reasons explained in Sect.2, this algorithm was previously derived in author’s paper [6]. This derivation was based on the technique of \( \zeta \)-functional regularization for operators with the explicitly unknown spectra on manifolds with a boundary (see also [5, 7]).

5. The functional determinant on closed spacetime without boundary

Let us now consider the case of the closed compact Euclidean spacetime without boundary having a topology \( S^4 \). In analogy with the previous section such a spacetime can be obtained from the tube-like manifold \([\tau_-, \tau_+] \times S^3\) by shrinking to the point both of its boundary surfaces \( S^3_\pm \) and imposing at these two points the no-boundary regularity conditions (Fig.3). Obviously, now the arising two poles \( \tau_\pm \) of this spherical manifold turn to be singular points of the radial part of \( \mathbf{F} \) having similar asymptotic behaviours (4.3) of \( a_{ik} \) (with \( \tau \) replaced by \( \tau - \tau_- \) and \( \tau_+ - \tau \) respectively for \( \tau \to \tau_- \) and \( \tau \to \tau_+ \)). Therefore the two sets of basis functions \( u_-(\tau) \) and \( u_+(\tau) \) regular respectively at the south \( \tau = \tau_- \) and north \( \tau = \tau_+ \) poles of this spacetime have the following asymptotic behaviours

\[ u_- (\tau) = U_- (\tau - \tau_-)^{\mu_-} + O [(\tau - \tau_-)^{1+\mu_-}], \] (5.1)

\[ u_+ (\tau) = V_+ (\tau - \tau_+)^{\mu_+} + O [(\tau - \tau_+)^{1+\mu_+}], \] (5.2)

for \( \tau \to \tau_- \) and

\[ u_- (\tau) = V_- (\tau_+ - \tau)^{\nu_-} + O [(\tau_+ - \tau)^{1+\nu_-}], \] (5.3)

\[ u_+ (\tau) = U_+ (\tau_+ - \tau)^{\nu_+} + O [(\tau_+ - \tau)^{1+\nu_+}], \] (5.4)

for \( \tau \to \tau_+ \). Here \( \mu_\pm \) and \( \nu_\pm \) are pairs of roots of the quadratic equations (4.8) associated with these two poles. They are also completely field-independent and are chosen to satisfy the relations \( \mu_- = \nu_+ \geq 0 \) and \( \mu_+ = \nu_- \leq 0 \) in accordance with the regularity of \( u_- (\tau) \) at \( \tau_- \) and of \( u_+ (\tau) \) at \( \tau_+ \) (note that we have reserved the notation \( U \) for the coefficients of regular asymptotic behaviours in (5.1)-(5.4) and the notation \( V \) for those of singular ones).
Now, in addition to the expression (4.10) for $\Delta_{-+}$ we can write down another expression for the same quantity calculated at $\tau_+$

$$\Delta_{-+} = -(\nu_+ - \nu_-) V_+^T a_0 U_+$$  \hspace{1cm} (5.5)$$

and get in virtue of (4.10) the following relation for the inverse of the matrix \( I_{+-} \)

$$\left(I_{+-}\right)^{-1} = \text{Const} \left(U_+^T\right)^{-1} V^T$$, \hspace{1cm} (5.6)$$

where we again absorb all field-independent matrix multipliers into the coefficient Const.

In view of (5.5), however, this relation can be rewritten as

$$\left(I_{+-}\right)^{-1} = \text{Const} \left(U_+^T\right)^{-1} \Delta_{-+} \left(U_+\right)^{-1}$$  \hspace{1cm} (5.7)$$

and similarly to the previous section interpreted as a Wronskian matrix $\Delta_{-+}$ calculated according to its definition (3.4) with respect to the specially normalized basis functions $u_{\pm}(\tau) U_{\pm}^{-1}$ which have matrix-unitary behaviours of their regular asymptotics at $\tau_{\pm}$.

Therefore, the final algorithm for the one-loop prefactor on compact spacetime without boundary takes the form

$$\left( \frac{\det F}{\det a} \right)^{-1/2} = \text{Const} \left[ \det \Delta_{-+} \right]^{-1/2},$$ \hspace{1cm} (5.8)$$

where the Wronskian matrix is calculated with respect to the basis functions $u_{\pm}$ satisfying the following field-independent normalization at the points of their regular asymptotics

$$u_{-}(\tau) = I (\tau - \tau_-)^{-\mu_-} + O \left[ (\tau - \tau_-)^{1+\mu_-} \right], \ \tau \to \tau_-,$$ \hspace{1cm} (5.9)$$

$$u_{+}(\tau) = I (\tau_+ - \tau)^{-\nu_+} + O \left[ (\tau_+ - \tau)^{1+\nu_+} \right], \ \tau \to \tau_+.$$ \hspace{1cm} (5.10)$$

This algorithm has a deep analogy in the S-matrix theory in asymptotically flat and empty Lorentzian spacetime which admits the existence of the so-called standard IN and OUT vacua associated with the remote past and future [10, 11]. These vacua are defined relative to the positive-negative frequency decomposition of linearized quantum fields with respect to the basis functions $v_{IN}$ and $v_{OUT}$ of the Lorentzian wave equation

$$F_L(d/dt) v_{IN}(t) = 0, \quad F_L(d/dt) v_{OUT}(t) = 0,$$ \hspace{1cm} (5.11)$$

their complex conjugates (negative energy modes) having the following behaviour respectively at $t \to -\infty$ and $t \to +\infty$

$$v_{IN}^* \to e^{i\omega(k) t + ikx}, \ t \to -\infty; \quad v_{OUT}^* \to e^{i\omega(k) t + ikx}, \ t \to +\infty.$$ \hspace{1cm} (5.12)
As it was shown by DeWitt [10, 11], the one-loop transition amplitude between these vacua in the presence of external sources, reflecting nontrivial field and geometry configuration in the interior of spacetime, can be represented in terms of the (functional) matrix $\alpha$ of Bogolyubov coefficients relating the IN and OUT sets of the basis functions

$$v_{OUT} = \alpha v_{IN} + \beta v_{IN}^*.$$  \hspace{1cm} (5.13)

Here $\alpha$ is calculable as a set of matrix elements between $v_{IN}$ and $v_{OUT}$ with respect to the (indefinite) inner product in the space of solutions of the Lorentzian wave equation (relative to which the positive-energy basis functions are orthonormal)

$$\alpha = <v_{IN}, v_{OUT}> = i \left[ v_{IN}^\dagger (W_L v_{OUT}) - (W_L v_{IN}^\dagger) v_{OUT} \right],$$  \hspace{1cm} (5.14)

where $W_L$ is a corresponding Lorentzian Wronskian operator of $F_L$ and $v_{IN}^\dagger \equiv (v_{IN}^*)^T$.

In terms of $\alpha$ the one-loop contribution to the IN-OUT transition amplitude

$$<OUT | IN > = e^{i\frac{\hbar}{\alpha} S + iW_{one-loop}} + O(\hbar)$$ \hspace{1cm} (5.15)

looks simply as [10, 11]

$$e^{iW_{one-loop}} = \text{Const} \left[ \det \alpha \right]^{-1/2},$$ \hspace{1cm} (5.16)

which is just the algorithm (5.8) with the Wronskian matrix $\Delta_{++}$ replaced by the Lorentzian Wronskian matrix (5.14) of the Bogolyubov coefficients. On the other hand, this one-loop factor also has a representation

$$e^{iW_{one-loop}} = \left( \frac{\text{Det} F_L}{\text{Det} a} \right)^{-1/2}$$ \hspace{1cm} (5.17)

of the functional determinant of the Lorentzian wave operator $F_L$ (and the local measure) corresponding to the classical action $S$ in eq.(5.13), calculated on the space of functions having asymptotic behaviour (5.12). Thus, our algorithm (5.8) can be regarded as a direct Euclidean closed space analogue of the reduction method for the functional determinant of the hyperbolic wave operator in the Lorentzian asymptotically flat spacetime. Note that in this comparison the asymptotic behaviour (5.12) at $t \to \pm \infty$ plays the role of regularity conditions (5.9) - (5.10) at smooth poles $\tau_{\pm}$ of the Euclidean spherical manifold. This analogy becomes even deeper when one considers the analytic continuation of the theory in the
Lorentzian asymptotically flat spacetime to the Euclidean one with zero boundary conditions at the asymptotically flat infinity – the standard calculational method of Wick rotation \[15\]. Under this rotation, \(\tau = it\), the basis functions (5.12) go over into the basis functions of the Euclidean Klein-Gordon equation \(u^\text{af}(\tau) \sim e^{\imath(\mathbf{k})\tau}, \tau \to -\infty\), and \(u^\text{af}_+(\tau) \sim e^{-\imath(\mathbf{k})\tau}, \tau \to +\infty\), satisfying zero boundary conditions respectively at \(\tau \to -\infty\) and \(\tau \to +\infty\), if we assume the following analytic continuation rule

\[
\begin{align*}
    u^\text{af}_-(it) &= v^*_\text{IN}(t), \\
    u^\text{af}_+(it) &= v^*_\text{OUT}(t).
\end{align*}
\]  

(5.18)

Thus the functions \(u^\text{af}_\pm(\tau)\), vanishing at \(\tau \to \pm\infty\) and singular at the opposite asymptotics \(\tau \to \mp\infty\) serve as a direct asymptotically-flat version of the two sets of basis functions \(u_\pm(\tau)\) regular respectively at the north \(\tau_+\) and south \(\tau_-\) poles of our compact manifold of spherical topology. By the analytic continuation rule (5.18) they give rise to two pairs of complex conjugated basis functions \((v^\text{IN}, v^*_\text{IN})\) and \((v^\text{OUT}, v^*_\text{OUT})\) which determine the standard vacua in Lorentzian quantum field theory, thus relating the notion of regularity and Dirichlet boundary conditions in Euclidean spacetime to the choice of preferred quantum states in the Lorentzian one. As was shown in \[3\], this formal analogy takes in the context of quantum cosmology the form of the physical process of nucleating the Lorentzian Universe with a special vacuum state from the Euclidean spacetime of the no-boundary type \[10, 17\]. The closed Euclidean spacetime also naturally arises in the calculation of the quantum distribution of such Universes as a gravitational quasi-DeSitter instanton carrying the effective action of the theory \[3\]. Such a distribution weighted by this Euclidean action can be viewed as an analogue of the IN-OUT matrix element (5.13) weighted by the Lorentzian effective action \(\Gamma_{\text{1-loop}} = S + \hbar W_{\text{1-loop}}\). This emphasizes a deep interrelation between the Euclidean and Lorentzian quantum theories, well-known in the S-matrix context as a Wick rotation \[15\] and extending it to quantum gravity of tunnelling geometries on spatially closed spacetimes \[3\].

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Appendix. The basis function representation of the Green’s function

Consider two complete sets of basis functions $\boldsymbol{u}_\pm$ of the operator $\boldsymbol{F}$ subject to boundary conditions (3.2). In virtue of these boundary conditions the $\tau$-independent matrix of their Wronskian inner products $\varphi_1^T(W\varphi_2) - (W\varphi_1)^T\varphi_2$, $\varphi_{1,2} = \boldsymbol{u}_\pm(\tau)$, can be rewritten in the form

$$\begin{bmatrix}
\boldsymbol{u}_-^T - (W\boldsymbol{u}_-)^T \\
\boldsymbol{u}_+^T - (W\boldsymbol{u}_+)^T
\end{bmatrix}
\begin{bmatrix}
W\boldsymbol{u}_+ & W\boldsymbol{u}_- \\
\boldsymbol{u}_+ & \boldsymbol{u}_-
\end{bmatrix}
= \begin{bmatrix}
\Delta_{+-} & 0 \\
0 & \Delta_{-+}
\end{bmatrix}, \quad (A.1)$$

where $\Delta_{+-}$ and $\Delta_{-+}$ are mutually transposed matrices (3.3) and (3.4). By taking the determinant of this equation one immediately obtains, due to the factorized nature of its left hand side, the following relation

$$\left(\det \begin{bmatrix}
W\boldsymbol{u}_+ & W\boldsymbol{u}_- \\
\boldsymbol{u}_+ & \boldsymbol{u}_-
\end{bmatrix}\right)^2 = \left(\det \Delta_{+-}\right)^2. \quad (A.2)$$

This relation implies that the matrix $\Delta_{+-}$ can be degenerate only when there is a linear dependence between $\boldsymbol{u}_+$ and $\boldsymbol{u}_-$, which is ruled out by our assumption that the operator $\boldsymbol{F}$ does not have zero modes and is uniquely invertible under the Dirichlet boundary conditions at $\tau_{\pm}$. Calculating this constant matrix at $\tau_{\pm}$ shows also that $\boldsymbol{u}_-(\tau_+)$ and $\boldsymbol{u}_+(\tau_-)$ are guaranteed to be invertible.

The equation (A.1) has a simple corollary relying on the proved invertibility of matrices (3.3) and (3.4)

$$\begin{bmatrix}
W\boldsymbol{u}_+ & W\boldsymbol{u}_- \\
\boldsymbol{u}_+ & \boldsymbol{u}_-
\end{bmatrix}^{-1} = \begin{bmatrix}
(\Delta_{+-})^{-1} & 0 \\
0 & (\Delta_{-+})^{-1}
\end{bmatrix}\begin{bmatrix}
\boldsymbol{u}_-^T - (W\boldsymbol{u}_-)^T \\
\boldsymbol{u}_+^T - (W\boldsymbol{u}_+)^T
\end{bmatrix}. \quad (A.3)$$

When multiplied from the left by the matrix entering the left hand side of (A.2), this equation gives, on account of the form of the Wronskian operator (2.17), the relations (3.7) and (3.8) for equal-time bilinear combinations of basis functions.

To derive the needed basis function representation (3.9) of the Green’s function of $\boldsymbol{F}$ subject to Dirichlet boundary conditions, let us consider the boundary value problem

$$\boldsymbol{F}(d/d\tau)\varphi(\tau) = J(\tau), \quad \varphi(\tau_{\pm}) = 0 \quad (A.4)$$
with arbitrary source \( J(\tau) \) for the function \( \varphi(\tau) \) which can be decomposed in the sets of basis functions \( u_\pm(\tau) \) with some unknown time-dependent coefficients \( \varphi_\pm(\tau) \) satisfying the necessary boundary conditions:

\[
\varphi(\tau) = u_+(\tau) \varphi_+(\tau) + u_-(\tau) \varphi_-(\tau), \quad \varphi_\pm(\tau_\pm) = 0. \tag{A.5}
\]

Substituting this representation into the left hand side of the equation (A.4), one finds that it can be satisfied if these coefficients \( \varphi_\pm(\tau) \) solve the following system of equations

\[
\begin{bmatrix}
Wu_+ & Wu_- \\
u_+ & u_-
\end{bmatrix}
\begin{bmatrix}
\dot{\varphi}_+ \\
\dot{\varphi}_-
\end{bmatrix} = -
\begin{bmatrix}
J \\
0
\end{bmatrix}, \quad \varphi \equiv d\varphi/d\tau. \tag{A.6}
\]

In view of (A.3) this linear system can be easily solved with respect to \( \dot{\varphi}_\pm \) and integrated subject to boundary conditions in (A.5), whence this decomposition takes the form of the integral

\[
\varphi(\tau) = \int_{\tau_-}^{\tau_+} d\tau' G(\tau, \tau') J(\tau') \tag{A.7}
\]

with the needed Green’s function \( G(\tau, \tau') \) given by the equation (3.9).

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Figure captions

Fig.1 The Euclidean spatially closed spacetime interpolating between two regular hypersurfaces $\Sigma_{\pm}$ of constant Euclidean time $\tau_{\pm}$, which underlies the heat-equation transition amplitude between the configurations on these hypersurfaces.

Fig.2 Euclidean spacetime of the no-boundary type originating from the tube-like manifold $\Sigma \times [\tau_{-}, \tau_{+}]$ by shrinking one of its boundaries $\Sigma_{-}$ to a point which becomes a regular internal point $\tau_{-}$ of the resulting 4-dimensional ball.

Fig.3 Obtainig the closed compact spacetime of spherical topology as a continuation of the process depicted on Fig.2: shrinking the remaining boundary $\Sigma_{+}$ to a point $\tau_{+}$ and imposing at this point the regularity conditions of the no-boundary type. The resulting manifold inherits the foliation with slices of constant Euclidean time $\tau$ in the form of its quasi-spherical latitudinal sections $\Sigma$. 
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