A SPECTRAL COLLOCATION METHOD FOR NONLOCAL DIFFUSION EQUATIONS

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Abstract. Nonlocal diffusion model provides an appropriate description of the diffusion process of solute in the complex medium, which cannot be described properly by classical theory of PDE. However, the operators in the nonlocal diffusion models are nonlocal, so the resulting numerical methods generate dense or full stiffness matrices. This imposes significant computational and memory challenge for a nonlocal diffusion model. In this paper, we develop a spectral collocation method for the nonlocal diffusion model and provide a rigorous error analysis which theoretically justifies the spectral rate of convergence provided that the kernel functions and the source functions are sufficiently smooth. Compared to finite difference methods and finite element methods, because of the high order convergence rates, the numerical cost of spectral collocation methods will be greatly decreased. Numerical results confirm the exponential rate of convergence.

1. Introduction

Nonlocal models given in terms of integral equations in spatial variables have received much attention in recent years \[11,12,13,19,20,22,24,27\], from both theoretical and computational point of view \[2,3,6,5,8,26,9,10,14,16,32\]. The modeling of central nonlocal diffusion is based on a radially symmetric kernel function, in the nonlocal operator, which describes the statistical nature of a stochastic process by assuming the probabilities of a particle moving in arbitrary directions are the same so that the processes is determined by the dependence of jump rates on jump sizes. Our goal is to study nonlocal-convection diffusion models (of integral-type) and their effective numerical solutions.

Let $I$ be a finite bar in $\mathbb{R}$. Without loss of generality, we take $I = [-1, 1]$. A nonlocal operator $L_\delta$ is defined as, for any function $u = u(x) : I \rightarrow \mathbb{R}$,

$$L_\delta u = \int_{B_\delta(x)} (u(y) - u(x))\gamma(x,y)dy$$

with $B_\delta(x) = \{ y \in \mathbb{R} : |y - x| < \delta \}$ denoting a neighborhood centered at $x$ of radius $\delta$ which is the horizon parameter, and $\gamma(x,y) : x \times y \rightarrow \mathbb{R}$ being a symmetric nonlocal kernel (influence function), i.e., $\gamma(x,y) = \gamma(y,x)$, and $\gamma(x,y) = 0$ if $y \notin B_\delta(x)$. In this paper, we assume that the kernel function satisfies transition invariance, i.e., $\gamma(x,y) = \gamma(|y - x|)$.

The following nonlocal diffusion model of one-dimension steady case is our main subject of interests here:

$$\begin{cases} L_\delta u = f(x), \ x \in I := [-1, 1], \\ u(x) = g(x), \ x \in I_c := (-1 - \delta, -1) \cup (1, 1 + \delta). \end{cases}$$

We refer to \[7\] for connections between nonlocal diffusion equations and stochastic jump processes. The well-posedness of \[1\] was studied in \[3\]. Moreover, it is known that, under proper assumptions...
of the kernel function $\gamma(x,y)$, the nonlocal problem (1) converges to the local problem, as the horizon $\delta \to 0$.

Recently, there have been a lot of efforts on developing numerical methods for nonlocal diffusion models (1), like finite difference methods, finite element methods and meshless methods. However, spectral method for nonlocal diffusion (ND) models has received remarkably little attention. In fact, spectral methods have been broadly applied to many integral equations like Voltera integral equations [29, 28] and fraction PDE [30, 31]. The purpose of this paper is to give new insights into spectral collocation algorithms for nonlocal diffusion model. The main contributions reside in the following aspects:

- We construct a spectral-collocation scheme for a nonlocal diffusion model.

  Firstly, because of their nonlocality, numerical methods for the nonlocal diffusion models usually generate dense stiffness matrices in which the bandwidths increase to infinity as the mesh size decreases to zero. Direct solvers are widely used in the nonlocal diffusion modeling, which have $O(N^2)$ memory requirement to store the stiffness matrix and $O(N^3)$ computations to find the numerical solutions. If we apply spectral methods to nonlocal diffusion model, the stiff matrices would still be dense, however the scale of the stiff matrix would be greatly decreased, because of the high accuracy of spectral methods. In this way, the computation and storage

  Secondly, the accuracy and convergence of the numerical methods for the nonlocal diffusion models depends heavily on the accurate evaluation of the integrals, which is defined on $B_\delta(x_i) \cap \Omega_j$, where $B_\delta(x_i)$ is the neighborhood of a collocation point $x_i$ and $\Omega_j$ is the supporting area of a basis function $\phi_j(x)$. For numerical methods like finite element methods, finite difference methods, and collocation methods, the basis function corresponds to a local supporting area. When $\Omega_j$ is on the edge of $B_\delta(x_i)$, $B_\delta(x_i) \cap \Omega_j$ could be highly irregular, which will degrade the accuracy of the numerical integration presented above. In contrast to finite element method and other related methods, the support area of spectral basis functions is the whole computational area. Thus, the intersection area is $B_\delta(x_i)$. The influence area is a regular sphere in 2-D or a ball in 3-D. So these numerical integrations in spectral method can achieve high order accuracy. Hence, this spectral-collocation scheme for nonlocal diffusion model can be extended to the high dimensional cases easily.

- We provide a rigorous error analysis which theoretically justifies the spectral rate of convergence. We also present more numerical evidences to demonstrate this surprising convergence behavior.

The rest of paper is organized as follows. In Section 2, we review basic properties of Legendre polynomials and the related quadrature rules, cardinal bases. In Section 3, we introduce the spectral approaches for 1-D nonlocal diffusion model. Maximum principle for the convergence analysis will be provided in Section 4. The convergence analysis in $L^\infty$ space will be given in Section 5. Numerical experiments are carried out in Section 6 to verify the theoretical results obtained in Section 5.

2. Mathematical preliminaries

In this section, we introduce some notation and review the relevant properties of the Legendre polynomials, the associate quadrature rules, cardinal basis (cf. [25]).

2.1. Notation.
Let } \omega^{\alpha,\beta}(x) = (1 - x)^{\alpha}(1 + x)^{\beta} \text{ be the Jacobi weight function defined in } I := (-1, 1), \text{ and let } L_{\omega^\alpha,\beta}^2(I) \text{ be the Hilbert space with the inner product and norm }
\langle u, v \rangle_{\omega^\alpha,\beta} = \int_I u(x)v(x)\omega^{\alpha,\beta}(x)dx, \quad \|u\|_{\omega^\alpha,\beta} = \sqrt{\langle u, u \rangle_{\omega^\alpha,\beta}}.

For any integer } r \geq 0, \text{ we define the weighted Sobolev space: }
H_{\omega^\alpha,\beta}^r(I) = \{ u \in L_{\omega^\alpha,\beta}^2(I) : u^{(k)} \in L_{\omega^\alpha,\beta}^2(I), 0 \leq k \leq r \},

equipped with the norm and semi-norm:
\|u\|_{r,\omega^\alpha,\beta} = \left( \sum_{k=0}^r \|u^{(k)}\|_{\omega^\alpha,\beta}^2 \right)^{1/2}, \quad |u|_{r,\omega^\alpha,\beta} = \|u(r)\|_{\omega^\alpha,\beta}.

For any real } r > 0, \text{ the space } H_{\omega^\alpha,\beta}^r(I) \text{ and its norm } \| \cdot \|_{r,\omega^\alpha,\beta} \text{ are defined by space interpolation as in } [1]. \text{ In particular, we have } L_{\omega^\alpha,\beta}^2(I) = H_{\omega^\alpha,\beta}^0(I) \text{ and denote its inner product and norm by } \langle \cdot, \cdot \rangle \text{ and } \| \cdot \|, \text{ respectively.}

We use } \partial^k_x u(x) \text{ to denote the ordinary derivative } \frac{d^k}{dx^k} u(x) = u^{(k)}(x) \text{ for } k \geq 1.

We introduce the non-uniformly (or anisotropic) Jacobi-weighted Sobolev space:
B_{\omega^\alpha,\beta}^m(I) := \{ u : \partial^k_x u \in L_{\omega^{\alpha+k,\beta+k}}^2(I), 0 \leq k \leq m \}, \quad m \in \mathbb{N},

equipped with the inner product, norm and semi-norm
\langle u, v \rangle_{B_{\omega^\alpha,\beta}^m} = \sum_{k=0}^m \langle \partial^k_x u, \partial^k_x v \rangle_{\omega^{\alpha+k,\beta+k}},
\|u\|_{B_{\omega^\alpha,\beta}^m} = (u, u)_{B_{\omega^\alpha,\beta}^m}^{1/2}, \quad |u|_{B_{\omega^\alpha,\beta}^m} = \|\partial^m_x u\|_{\omega^{\alpha+m,\beta+m}}.

We denote by } \mathbb{P}_N \text{ the set of all algebraic polynomials of degree } \leq N.

2.2. Legendre polynomials. The Legendre polynomials, denoted by } L_n(x), \text{ are the are mutually orthogonal with respect to } \omega^{0,0} = 1, \text{ and normalized so that
\int_{-1}^1 L_m(x)L_n(x)dx = \gamma_n\delta_{mn}, \quad \gamma_n = \frac{2}{2n + 1},

where } \delta_{mn} \text{ is Kronecker symbol. They satisfy the three-term recurrence relation:
(n + 1)L_{n+1}(x) = (2N + 1)xL_n(x) - nL_{n-1}(x), \quad n \geq 1,
L_0(x) = 1, \quad L_1(x) = x. \quad (2)

2.3. Legendre-Gauss-Type Quadratures and cardinal basis. The Legendre-Gauss-type nodes and weights } \{x_j, \omega_j\}_{j=0}^N \text{ can be derived from the following formulas:

- For the Legendre-Gauss (LG) quadrature,
\{x_j\}_{j=0}^N \text{ are the zeros of } L_{N+1}(x);
\omega_j = \frac{2}{(1 - x_j^2)[L'_{N+1}(x_j)]^2}, \quad 0 \leq i \leq N, \quad (3)

- For the Legendre-Gauss-Radau (LGR) quadrature,
\{x_j\}_{j=0}^N \text{ are the zeros of } L_N(x) + L_{N+1}(x);
\omega_j = \frac{1}{(N + 1)^2} \frac{1 - x_j}{[L_N'(x_j)]^2}, \quad 0 \leq j \leq N, \quad (4)
For the Legendre-Gauss-Lobatto (LGL) quadrature,
\[ \{x_j\}_{j=0}^{N} \text{ are the zeros of } (1-x^2)L_N(x); \]
\[ \omega_j = \frac{2}{N(N+1)} \frac{1}{[L_N(x_j)]^2}, \quad 0 \leq j \leq N, \]
(5)

With the above quadrature nodes and weights, there holds
\[ \int_{-1}^{1} p(x) dx = \sum_{j=0}^{N} p(x_j) \omega_j, \quad \forall p \in P_{2N+\delta}, \]
(6)

where \( \delta = 1, 0, -1 \) for LG, LGR and LGL, respectively. Moreover, the conventional choice of grid points for Legendre spectral-collocation methods, is the Legendre Gauss-Lobatto points.

The spectral-collocation method is usually implemented in the physical space by seeking approximate solution in the form \( u_N \in P_N \) such that
\[ u_N(x) = \sum_{k=0}^{N} u_N(x_k) h_k(x), \]
where \( \{h_k\} \) are the Lagrange basis polynomials (also referred to as nodal basis functions), i.e., \( h_k \in P_N \) and \( h_k(x_j) = \delta_{kj} \). We write
\[ h_k(x) = \sum_{p=0}^{N} \beta_{p,k} L_p(x), \quad 0 \leq p, k \leq N. \]

and determine the coefficients \( \beta_{p,k} \) from \( h_k(x_j) = \delta_{kj}, \quad 0 \leq k, j \leq N \). More precisely,
\[ \beta_{p,k} = \frac{1}{\gamma_p} \sum_{i=0}^{N} h_k(x_i) L_p(x_i) / \gamma_p = L_p(x_k) / \gamma_p, \]
where
\[ \gamma_p = \sum_{i=0}^{N} L^2_p(x_i) / (p + 1/2)^{-1}, \quad \text{for } p < N \]

and \( \gamma = (N + 1/2)^{-1} \) for the Gauss and Gauss-Radau formulas, and \( \gamma_N = 2/N \) for the Gauss-Lobatto formula.

3. Numerical Algorithm

Firstly, we make the change of variable
\[ y = x + s, \quad s \in [-\delta, \delta], \]
(7)

under which (1) is transformed into
\[ \int_{x-\delta}^{x+\delta} u(y) \gamma(|y-x|) dy - u(x) \int_{-\delta}^{\delta} \gamma(|s|) ds = f(x), \quad x \in I := [-1, 1]. \]
(8)
eqs

Let \( \{x_i\}_{i=0}^{N} \) be a set of Legendre-Gauss-Lobatto points, and a approximation to (1) using a Legendre collocation approach is
\[ \left\{ \begin{array}{l}
\text{Find } u_N \in P_N \text{ such that } \\
\int_{-\delta}^{\delta} u_N(y) \gamma(|y-x_i|) dy - u_N(x_i) \int_{-\delta}^{\delta} \gamma(|s|) ds = f(x_i) - \int_{\Lambda_c} g(y) \gamma(|y-x_i|) dy, \quad 0 \leq i \leq N,
\end{array} \right. \]
eqxs

where \( \Lambda := (x_i - \delta, x_i + \delta) \cap (-1, 1) \) and \( \Lambda_c := (x_i - \delta, x_i + \delta)/\Lambda \). To compute the integral term in eqxs accurately, we will transfer the integral interval \( \Lambda \) to a fixed interval \([-1, 1]\) and then make use
of some appropriate quadrature rule. Firstly, define \( \Lambda := (a_i, b_i) = (x_i - \delta, x_i + \delta) \cap (-1, 1) \) and make a simple linear transformation:

\[
y = \frac{b_i - a_i}{2} t + \frac{a_i + b_i}{2}, \quad t \in (-1, 1).
\]

Then \((9)\) becomes

\[
\begin{aligned}
\frac{b_i - a_i}{2} \int_{-1}^{1} u_N(y(x_i, t)) \gamma(x, y(x_i, t_j)) dt - u_N(x_i) \int_{-\delta}^{\delta} \gamma(|s|) ds &= f(x_i) - \int_{\Lambda_c} g(y) \gamma(|y - x_i|) dy, \quad 0 \leq i \leq N. \\
\end{aligned}
\]

We then approximate the integral term by a Legendre-Gauss-Lobatto type quadrature formula with the notes and weights denoted by \( \{t_j, \omega_j\}_{j=0}^{M} \), leading to the Legendre collocation scheme (with numerical integration) for \((10)\):

\[
\begin{aligned}
\text{Find } u_N \in \mathbb{P}_N \text{ such that } &
\bigg\{ \frac{b_i - a_i}{2} \sum_{j=0}^{N} u_N(y(x_i, t_j)) \gamma(x, y(x_i, t_j)) \omega_j - u_N(x_i) \int_{-\delta}^{\delta} \gamma(|s|) ds \bigg\} \\
&= f(x_i) - \int_{\Lambda_c} g(y) \gamma(|y - x_i|) dy, \quad 0 \leq i \leq N.
\end{aligned}
\]

We expand the approximate solution \( u_N \) as

\[
u_N(x) = \sum_{k=0}^{N} u_k h_k(x).
\]

Inserting it into \((11)\) leads to

\[
\begin{aligned}
\frac{b_i - a_i}{2} \sum_{k=0}^{N} \sum_{j=0}^{N} u_k \left( \sum_{j=0}^{N} h_k(y(x_i, t_j)) \gamma(x, y(x_i, t_j)) \omega_j \right) - u_i \int_{-\delta}^{\delta} \gamma(|s|) ds &= f(x_i) - \int_{\Lambda_c} g(y) \gamma(|y - x_i|) dy, \quad 0 \leq i \leq N.
\end{aligned}
\]

Remark 3.1. It is worthwhile to point out that the collocation points \( \{x_j\}_{j=0}^{N} \) and quadrature points \( \{t_j\}_{j=0}^{N} \) could be chosen differently in type and number. As a result, we can also use Legendre-Gauss-Radau or Legendre-Gauss-Lobatto for the integral term.

More precisely, we divide the integral range of \((x_1 - \delta, x_1 + \delta)\) into three cases.

- Case I: \(-1 < x_1 - \delta < x_1 + \delta < 1\). For easy of implementation and analysis, we convert the interval \([x_1 - \delta, x_1 + \delta]\) to \([-1, 1]\) by a linear transformation:

\[
y = x_1 + \delta t, \quad t \in [-1, 1].
\]

The scheme becomes

\[
\delta \int_{-1}^{1} u_N(y(x_i, t)) \gamma(x, y(x_i, t)) dt - u_N(x_i) \int_{-\delta}^{\delta} \gamma(|s|) ds = f(x_i), \quad 0 \leq i \leq N.
\]

Next, we approximate the integral term by a Legendre-Gauss-Lobatto type quadrature formula with the notes and weights denoted by \( \{t_j, \omega_j\}_{j=0}^{M} \), leading to

\[
\frac{\delta}{2} \sum_{j=0}^{M} u_N(y(x_i, t_j)) \gamma(x, y(x_i, t_j)) \omega_j - u_N(x_i) \int_{-\delta}^{\delta} \gamma(|s|) ds = f(x_i), \quad 0 \leq i \leq N.
\]
Let \( \{h_k\}_{k=0}^{N} \) be the Lagrange basis polynomials associated with the Legendre-Gauss-Lobatto-type points \( \{x_i\}_{i=0}^{N} \). We expand the approximate solution \( u_N \) as

\[
u_N = \sum_{k=0}^{N} u_k h_k(x).
\]

Inserting it into (16) leads to

\[
\delta \sum_{k=0}^{N} u_k \sum_{j=0}^{M} h_k(y(x_i,t_j))\gamma(x_i,y(x_i,t_j))\omega_j - u_i \int_{-\delta}^{\delta} \gamma(|s|)ds = f(x_i), \quad 0 \leq i \leq N.
\]

\( \text{Maximum principle} \)

Case II: \( x_i - \delta < -1 < x_i + \delta < 1, \) we have

\[
\int_{-1}^{x_i+\delta} u_N(y)\gamma(|y-x_i|)dy - u_N(x_i) \int_{-\delta}^{\delta} \gamma(|s|)ds = f(x_i) - \int_{x_i-\delta}^{1} g(y)\gamma(|y-x_i|)dy, \quad 0 \leq i \leq N.
\]

Next, we convert the interval \([-1,x_i+\delta]\) to \([-1,1]\) and approximate the integral term by a Legendre-Gauss-Lobatto type quadrature \( \{t_j, \omega_j\}_{j=0}^{M} \), leading to

\[
\frac{x_i+\delta+1}{2} \sum_{j=0}^{M} u_N(y(x_i,t_j))\gamma(x_i,y(x_i,t_j))\omega_j - u_N(x_i) \int_{-\delta}^{\delta} \gamma(|s|)ds = f(x_i) + \int_{x_i-\delta}^{1} g(y)\gamma(|y-x_i|)dy, \quad 0 \leq i \leq N.
\]

We expand the approximate solution \( u_N \) as

\[
u_N = \sum_{k=0}^{N} u_k h_k(x).
\]

Plugging it into (20) leads to

\[
\frac{x_i+\delta+1}{2} \sum_{k=0}^{N} u_k \sum_{j=0}^{M} h_k(y(x_i,t_j))\gamma(x_i,y(x_i,t_j))\omega_j - u_i \int_{-\delta}^{\delta} \gamma(|s|)ds = f(x_i) + \int_{x_i-\delta}^{1} g(y)\gamma(|y-x_i|)dy, \quad 0 \leq i \leq N.
\]

Case III: \( -1 < x_i - \delta < 1 < x_i + \delta, \) we have

\[
\int_{x_i-\delta}^{1} u_N(y)\gamma(|y-x_i|)dy - u_N(x_i) \int_{-\delta}^{\delta} \gamma(|s|)ds = f(x_i) - \int_{-1}^{x_i-\delta} g(y)\gamma(|y-x_i|)dy, \quad 0 \leq i \leq N.
\]

Then, we can treat this case in the same fashion as above.

4. Maximum principle

The nonlocal diffusion operator is analogous to its local counterpart \( \Delta u \). It is known that the local diffusion equation satisfies the maximum principle. The proposed nonlocal diffusion operator satisfies the following property.

**Theorem 4.1.** (Maximum principle) Suppose that \( \mathcal{L}_\delta u \) is well-defined in \( I := [-1,1] \). If \( \mathcal{L}_\delta u \geq 0 \) in \( I \), then a maximum of \( u \) is attained in the interaction domain \( I_c = (-\delta - 1, -1) \cup (1,1+\delta) \).
Proof. Consider an auxiliary function \( v(x) = u(x) + \epsilon e^{rx} \), where \( \epsilon > 0, r > 0 \), so

\[
L_\delta v(x) = L_\delta u(x) + \epsilon e^{rx} \int_{-\delta}^{\delta} (e^{rs} - 1) \gamma(|s|) \, ds
\]

We can easily get \( L_\delta v(x) > 0 \). We claim \( v(x) \) cannot attain a maximum in \( \Omega_s \). If not, assume that \( v \) attains a nonnegative maximum at \( x_0 \in \Omega_s \), i.e., \( v(x_0) = \max_{x \in \Omega} v(x) \geq 0 \). We have

\[
L_\delta v(x_0) = \int_{x_0-\delta}^{x_0+\delta} (v(y) - v(x_0)) \gamma(|y - x_0|) \, dy
\]

Since \( v(y) - v(x_0) \leq 0 \), it is easy to verify that the integral in (26) satisfies

\[
\int_{x_0-\delta}^{x_0+\delta} (v(y) - v(x_0)) \gamma(|y - x_0|) \, dy \leq 0.
\]

Hence, we get \( L_\delta v(x_0) \leq 0 \), which is a contradiction with the assumption of \( L_\delta v(x_0) > 0 \). Let \( \epsilon \) goes to 0, we can get the desired result. \( \square \)

**Lemma 4.1.** If an integrable function \( e(t) \) satisfies

\[
e(t) \int_{-\delta}^{\delta} \gamma(|s|) \, ds - \int_{t-\delta}^{t+\delta} e(y) \gamma(|y - t|) \, dy = G(t), \quad t \in [-1, 1],
\]

where \( G(t) \) is an integrable function, then

\[
\|e(t)\|_{L^\infty(I)} \leq C\|G\|_{L^\infty(I)}.
\]

**Proof.** It is easy to see, from (24), that

\[
|e(t)| \int_{-\delta}^{\delta} \gamma(|s|) \, ds - \int_{t-\delta}^{t+\delta} |e(y)| \gamma(|y - t|) \, dy \leq \|G(t)\|_{L^\infty},
\]

Denoting \( E(t) := \frac{|e(t)| \int_{-\delta}^{\delta} \gamma(|s|) \, ds}{\|G(t)\|_{L^\infty(I)}} \), we rewrite above inequality as,

\[
\int_{t-\delta}^{t+\delta} E(y) \gamma(|y - x|) \, dy - E(t) \int_{-\delta}^{\delta} \gamma(|s|) \, ds \geq -1.
\]

For \( \varpi(t) := t(1-t)/2 + \delta(1+\delta), \ t \in [-1, 1] \), it holds that

\[
\int_{t-\delta}^{t+\delta} \varpi(y) \gamma(|y - t|) \, dy - \varpi(t) \int_{-\delta}^{\delta} \gamma(|s|) \, ds = -1.
\]

In view of (26) and (27), we derive that

\[
\int_{t-\delta}^{t+\delta} c(y) \gamma(|y - x|) \, dy - c(t) \int_{-\delta}^{\delta} \gamma(|s|) \, ds \geq 0, \ t \in [-1, 1],
\]

where \( c(t) := E(t) - \varpi(t) \). Since \( e(t) = 0, \ t \in (-1 - \delta, -1) \cup (1, 1 + \delta) \), it holds that

\[
c(t) := E(t) - \varpi(t) \leq 0, \ t \in (-1 - \delta, -1) \cup (1, 1 + \delta).
\]

Consequently, according to Theorem 4.1, we have

\[
e(t) \leq 0, \ \forall t \in I, \ \text{i.e.,} \ E(t) \leq \|\varpi(t)\|_{L^\infty(I)}, \ \forall t \in I.
\]

Therefore, let \( C = \|\varpi(t)\|_{L^\infty(I)} \), we have

\[
\|e(t)\|_{L^\infty(I)} \leq C\|G\|_{L^\infty(I)}, \ \forall t \in I.
\]

\( \square \)
5. Convergence Analysis

5.1. Some Useful Lemmas.

Lemma 5.1. [17, Lemma 4.8] Assume that \( u \in B^m_{1,-1}(I) \) with \( 1 \leq m \leq N + 1 \), then for any \( \phi \in \mathbb{P}_N \), Then the following estimates hold:

\[
|u(\phi) - (u, \phi)_N| \leq c \sqrt{(N-m+1)!}\left( N + m \right)^{-(m+1)/2} \| \partial_x^m u \|_{\omega^{-1,-1}} \| \phi \|,
\]

where \( c \) is a positive constant independent of \( m, N, \phi \) and \( u \).

Lemma 5.2. [17, B.33] Let \( (a, b) \) be a finite interval. There holds the Sobolev inequality:

\[
\max_{x \in [a, b]} |u(x)| \leq \left( \frac{1}{b-a} + 2 \right)^{1/2} \| u \|_{L^2(a, b)} \| u \|_{H^1(a, b)}, \quad \forall u \in H^1(a, b),
\]

which is also known as the Gagliardo-Nirenberg interpolation inequality.

Lemma 5.3. [17, B.44] For any \( u \in H^1(a, b) \) with \( u(x_0) = 0 \) for some \( x_0 \in (a, b) \), the following Poincare inequality holds:

\[
\| u \|_{L^2(a, b)} \leq \| u \|_{L^2(a, b)}, \quad \forall u \in H^1(a, b).
\]

Lemma 5.4. [17, Theorem 3.44] For any \( u \in B^m_{1,-1}(I) \), we have that for \( 1 \leq m \leq N + 1 \),

\[
\| \partial_x(I_N u - u) \| + N \| I_N u - u \|_{\omega^{-1,-1}} \leq c \sqrt{(N-m+1)!}\left( N + m \right)^{(m+1)/2} \| \partial_x^m u \|_{\omega^{-1,-1}}.
\]

5.2. Error analysis in \( L_\infty \).

Theorem 5.1. Let \( u \) be the exact solution of (1), and assume that

\[
\begin{align*}
  u_N &= \sum_{k=0}^N u_k h_k(x), \quad \forall x \in I \\
  u_N &= g(x), \quad \forall x \in (-1 - \delta, -1) \cup (1, 1 + \delta),
\end{align*}
\]

where \( u_k \) is given by [13] and \( h_k(x) \) is the \( k \)-th Lagrange basis function associated with the Gauss-points \( \{x_k\}_{k=0}^N \). If

\[
\gamma(x, y) \in L^\infty(D) \cap L^\infty(I; B^k_{-1,-1}(I)), \quad \partial_x \gamma(x, y) \in L^\infty(D), \quad u \in B^m_{-1,-1}(I),
\]

where \( D = \{(x, y) : -1 \leq x, y \leq 1 \} \) and \( 1 \leq k, m \leq N + 1 \). Then we have

\[
\| u - u_N \|_{L^\infty(I)} \leq c \sqrt{(N-k+1)!}\left( N + k \right)^{-k/2} \| \partial_y^k \gamma(x, \cdot) \|_{\omega^{-k-1,-1}} \| u \| + c \sqrt{(N-m+1)!}\left( N + m \right)^{-m/2} \| \partial_x^m u \|_{\omega^{-m-1,-1}}.
\]

provided that \( N \) is sufficiently large, where \( C \) is a constant independent of \( N \).

Proof. Let \( I_N \) be the Legendre-Gauss-Lobatto interpolation operator. We start from (9) and reformulate it as

\[
\int_{x_i - \delta}^{x_i + \delta} u_N(y) \gamma(|y - x_i|) dy - u_i \int_{-\delta}^{\delta} \gamma(|s|) ds = f(x_i) + J_1(x_i), \quad x \in I.
\]
In what follows, we use the asymptotic estimate of the Lebesgue constant (cf.,[15]):

\[ J_1(x) = \delta \left( \int_{-1}^{1} u_N(y(x, t)) \gamma(x, y(x, t)) dt - \sum_{j=0}^{N} u_N(y(x, t_j)) \gamma(x, y(x, t_j)) \omega_j \right). \]  

(38)

Multiply \( h_i(x) \) on both sides, and take the summation from 0 to \( N \), we have,

\[ I_N \left( \int_{x-\delta}^{x+\delta} u_N(y) \gamma(|y-x|) dy \right) - u_N(x) \int_{-\delta}^{\delta} \gamma(|s|) ds = I_N f + I_N J_1, \quad x \in I. \]  

(39)

Clearly, by \([8]\),

\[ I_N f = I_N \left( \int_{x-\delta}^{x+\delta} u(y) \gamma(|y-x|) dy \right) - I_N u \int_{-\delta}^{\delta} \gamma(|s|) ds. \]  

(40)

Denote \( e = u - u_N \). Inserting \([10]\) into \([39]\) leads to the error equation:

\[ e(x) \int_{x-\delta}^{x+\delta} \gamma(|y-x|) dy - \int_{x-\delta}^{x+\delta} e(y) \gamma(|y-x|) dy = I_N J_1 + J_2(x) + J_3(x), \]  

(41)

where \( e(y) = 0 \) for \( y \in (-1-\delta, -1) \cup (1, 1+\delta) \), and

\[ J_2(x) = (u - I_N u) \int_{x-\delta}^{x+\delta} \gamma(|s|) ds, \]

\[ J_3(x) = \int_{x-\delta}^{x+\delta} e(y) \gamma(|y-x|) dy - I_N \left( \int_{x-\delta}^{x+\delta} e(y) \gamma(|y-x|) dy \right). \]  

(42)

According to Lemma \([4.1]\)

\[ \| e(x) \|_{L^\infty(I)} \leq C(\| I_N J_1 \|_{L^\infty(I)} + \| J_2 \|_{L^\infty(I)} + \| J_3 \|_{L^\infty(I)}). \]  

(43)

It remains to estimate the three terms on the right hand side of \([43]\). By Lemma \([5.1]\)

\[ \| J_1(x) \| = |\delta \left( \int_{-1}^{1} u_N(y(x, \cdot)) \gamma(x, y(x, \cdot)) dt - \sum_{j=0}^{N} u_N(y(x, t_j)) \gamma(x, y(x, t_j)) \omega_j \right)| \]

\[ \leq c \sqrt{(N-k+1)! / N!} (N+k)^{-k+1} \times \delta \| \partial_y^k \gamma(x, y(x, \cdot)) \|_{\omega_{k-1,k-1}} \| u_N(y(x, \cdot)) \| \]

(44)

A direct calculation yields

\[ \| \partial_x^k \gamma(\cdot) \|_{\omega_{k-1,k-1}}^2 = \int_{-1}^{1} |\partial_x^k \gamma(x, y(t))|^2 (1-t^2)^{k-1} dt \]

\[ = \delta \int_{x-\delta}^{x+\delta} |\partial_y^k \gamma(x, y)|^2 (x+\delta-y)^{k-1}((\delta-x+y)^{k+1} dy \]

\[ \leq \| \partial_y^k \gamma(x, \cdot) \|_{\omega_{k-1,k-1}}^2, \]

and

\[ \delta \| u_N(y(x, \cdot)) \|^2 = \int_{x-\delta}^{x+\delta} \| u_N(y) \|^2 dy \leq \| u_N \|^2. \]  

(45)

Hence, we obtain the estimate of \(| J_1 | : \)

\[ |J_1(x)| \leq c \sqrt{(N-k+1)! / N!} (N+k)^{-k+1/2} \| \partial_y^k \gamma(x, \cdot) \|_{\omega_{k-1,k-1}} \| u_N \|. \]  

(47)

In what follows, we use the asymptotic estimate of the Lebesgue constant (cf.,[15]):

\[ \max_{|x| \leq 1} \sum_{j=0}^{N} |h_j(x)| \simeq \sqrt{N}, \quad N \gg 1. \]  

(48)
This implies
\[ \| I_N J_1 \|_{L^\infty} \leq c \| J_1 \|_{L^\infty} \max_{|x| \leq 1} \sum_{j=0}^{N} |h_j(x)| \]
\[ \leq c \sqrt{\frac{(N-k+1)!}{N!}} (N+k)^{-k/2} \| \partial_y^k \gamma(x, \cdot) \|_{L^{\infty, k-1, k-1}} \| u_N \| \]
\[ \leq c \sqrt{\frac{(N-k+1)!}{N!}} (N+k)^{-k/2} \| \partial_y^k \gamma(x, \cdot) \|_{L^{\infty, k-1, k-1}} (\| e \| + \| u \|). \quad (49) \]

Using the inequalities (31) and (32), we obtain from Theorem 5.4 that
\[ \| J_2 \|_{L^\infty} \leq c \| u - I_N u \|^{1/2} \| \partial_x (u - I_N u) \|^{1/2} \]
\[ \leq c \sqrt{\frac{(N-m+1)!}{N!}} (N+m)^{-m/2} \| \partial_x^m u \|_{L^{\infty, m-1, m-1}}. \quad (50) \]
Moreover, using Theorem 5.4 with \( m = 1 \) yields
\[ \| J_3 \| \leq c N^{-1} \| \gamma(x, x + \delta) e(x + \delta) - \gamma(x, x - \delta) e(x - \delta) + \int_{x-\delta}^{x+\delta} \partial_x \gamma(x, y) e(y) dy \| \]
\[ \leq c N^{-1} \left( \max_{|x| \leq 1} |\gamma(x, x + \delta)| + \max_D \| \partial_x \gamma(x, y) \|_{L^\infty} \right) \| e \|. \quad (51) \]

Then, we have
\[ \| J_3 \|_{L^\infty} \leq c \| J_3 \|^{1/2} \| \partial_x J_3 \|^{1/2} \leq c N^{-1/2} \| e \|^{1/2} \| \partial_x \int_{x-\delta}^{x+\delta} \gamma(x, y) e(y) dy \| \]
\[ \leq c N^{-1/2} \| e \| \leq c N^{-1/2} \| e \|_{L^\infty}. \quad (52) \]

Finally, a combination of (49), (50) and (52) leads to the estimate . □

6. Numerical Experiments

Without lose of generality, we will only use the Legendre-Gauss-Lobatto points as the collocation points. Our numerical evidences show that the other two kinds of Legendre-Gauss points produce results with similar accuracy.

Example 1. We first consider the equation (2.1) with
\[ \gamma(x) = \frac{3}{\delta^3}, \quad f_0(x) = -\frac{6e^{4x}}{\delta^3} - \frac{3e^{4x}(e^{-4\delta} - e^{4\delta})}{4\delta^3} \quad (53) \]
The corresponding exact solution is given by \( u(x) = e^{4x} \).

In Figure 6.1 (left), we plot \( \log_{10} \) (Maximum error) against \( N \in [4, 32] \), and observe that the desired spectral accuracy is obtained. We plot in Figure 6.1 (right), \( \log_{10} \) (Maximum error) against \( \log_{10}(\delta) \). The slopes are nearly equal to 2. These results indicate that when we use a relatively high number of collocation points, \( N = 64 \), the error from spectral collocation discretization is relatively negligible. The error left is only the error between nonlocal diffusion models and local diffusion models, which is order of 2 for \( \delta \).

7. Conclusions

In this paper, we presented a spectral method for a nonlocal diffusion model and provide a rigorous error analysis which theoretically justifies the spectral rate of convergence provided that the kernel function and the source function are sufficiently smooth.

We mainly focus on one-dimension case in this paper, and there is no difficulty to extent this algorithm to a higher dimension, especially it will be obvious in spectral collocation methods, the numerical integration to assemble stiff matrices is more accurate. As is seen in the paper, to apply
In the future we will focus on how to develop spectral collocation methods to deal with nonlocal diffusion models with singular kernels.

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