Probability Distribution Function of the
Coarse-grained Scalar Field at Finite Temperature

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PACS number : 98.80.Cq, 11.10.Wx

Abstract

We present a formalism to calculate the probability distribution function of a scalar field coarse-grained over some spatial scales with a Gaussian filter at finite temperature. As an application, we investigate the role of subcritical fluctuations in the electroweak phase transition in the minimal standard model. It is concluded that the universe was in a mixed state of true and false vacua already at the critical temperature.
1 Introduction

Quantum field theories with spontaneous symmetry breaking usually exhibit symmetry restoration at high temperature \([1]\), and hence our universe is supposed to have experienced several phase transitions in the early stage of its evolution as the cosmic temperature decreased with expansion. Much work has been done on these phase transitions and their generic properties such as the transition order and the critical temperature, \(T_c\), have been extracted from the effective potential. The dynamics of phase transitions, on the other hand, is less understood, partly because it is difficult to clarify the role of thermal fluctuations.

Thermal fluctuations often play a conspicuous role on the dynamics of phase transitions. For example, at formation of topological defects associated with some symmetry breaking \([2]\), magnitude of thermal fluctuations fixes the temperature when the phase distributed randomly on each coherent volume becomes stabilized. Also, their statistical property is important to set the initial distribution of topological defects. Furthermore, thermal fluctuations called subcritical bubbles have been considered important in the context of the electroweak phase transition, where the one-loop improved finite-temperature effective potential shows that it is of first order \([3] - [5]\). Gleiser, Kolb, and Watkins \([6] - [7]\), however, suggested that the barrier between the two minima of the effective potential at \(T_c\) was so shallow that thermal fluctuations called subcritical bubbles might be dominant and that the standard bubble nucleation picture of the first order phase transition might not work.

Thus, it is indispensable to understand the property of thermal fluctuations in order to elucidate the real dynamics of the phase transition. Among the property of thermal fluctuations, their typical scale and amplitude are particularly important. The former gives us the scale we should focus on, and the latter determines whether thermal fluctuations dominate the dynamics. First, we should notice that in evaluating the magnitude of thermal fluctuations, we should pay attention to fluctuations coarse-grained over some characteristic coherent scale of thermal fluctuations rather than raw fluctuations. If we estimate the typical amplitude out of raw fluctuations, random infinitesimal ones make a dominant contribution to them and we cannot obtain a meaningful value to understand the role of thermal fluctuations.

In this paper we present a formalism to calculate the probability distribution function (PDF) of a coarse-grained scalar field at finite temperature. The PDF of coarse-grained fluctuation has also been studied by Hindmarsh and Rivers \([8]\) and by Bettencourt \([9]\), but they gave only the upper limit of probability.

As an application, we apply our formalism to the electroweak phase transition which may be cosmologically important for the possibility of baryogenesis during it \([10]\). For successful baryogenesis, it should be of first order to realize an out-of-equilibrium condition \([11]\). However, if subcritical fluctuations dominate the dynamics of the phase transition it proceeds in a similar manner to a second-order transition and baryogenesis becomes impossible. Here we evaluate PDF of the Higgs field on the relevant scales in the minimal standard model and investigate whether the universe is homogeneously in the symmetric phase at the critical temperature. We have also studied this issue in the previous paper \([20]\), solving a phenomenological Langevin equation for the classical expectation value of the Higgs field numerically. But the PDF we formulate here is based on finite-temperature
field theory and quantum effects are taken into account more appropriately.

In the next section, we present a formalism of evaluating the PDF for the coarse-grained fluctuations. In Sec. 3, the method established in Sec. 2 is applied to the electroweak phase transition.

2 Formulation of the probability distribution function

We start with the partition function $Z$ of a scalar field $\varphi(x)$ at the finite temperature $T = \beta^{-1}$, defined by

$$Z = \text{tr} \rho = \int D\varphi(x) \rho_\beta[\varphi(x), \varphi(x)],$$

where $\rho_\beta[\varphi(x), \varphi(x)]$ is the diagonal component of the density matrix, $\rho_\beta$, which gives the relative probability distribution of the configuration $\varphi(x)$. Then the PDF of the field, $\tilde{P}[\varphi(x)]$, is expressed by path integral as

$$\tilde{P}[\varphi(x)] = \frac{1}{Z} \rho_\beta[\varphi, \varphi] = \frac{1}{Z} \int_{B_1} D\phi(x) \exp \left( - \int d^4x L_E[\phi(x)] \right),$$

where $x = (x, \tau)$, $L_E[\phi]$ is the Euclidean Lagrangian density given by

$$L_E[\phi(x)] = \frac{1}{2} (\partial\phi)^2 + V[\phi],$$

and the boundary condition $B_1$ implies $\phi(x, \tau = 0) = \phi(x, \tau = \beta) = \varphi(x)$.

Now we consider the PDF of the field coarse-grained over an arbitrary scale $R$ as,

$$\varphi_R \equiv \int d^3x \varphi(x) I(x; R),$$

where $I(x; R)$ is a Gaussian window function with the width $R$ and given by

$$I(x; R) = \frac{1}{(2\pi R^2)^{3/2}} \exp \left( -\frac{x^2}{2R^2} \right).$$

Then the probability that $\varphi_R$ is equal to $\bar{\varphi}$, $P[\varphi_R = \bar{\varphi}]$, is formally given by

$$P[\varphi_R = \bar{\varphi}] = \int D\varphi(x) \tilde{P}[\varphi(x)] \delta(\varphi_R - \bar{\varphi}).$$

Inserting (2) into this equation, we get the following expression,

$$P[\varphi_R = \bar{\varphi}] = \frac{1}{Z} \int D\varphi(x) \int_{B_1} D\phi(x) \exp \left( - \int d^4x L_E[\phi(x)] \right) \delta(\varphi_R - \bar{\varphi})$$

$$= \frac{1}{Z} \int_{B_2} D\phi(x) \int_{-\infty}^{+\infty} \frac{d\alpha}{2\pi} \exp(-i\alpha \bar{\varphi}) \exp \left( - \int d^4x L_E[\phi(x)] + \beta \int d^3x J(x) \phi(x, 0) \right),$$

where $J(x)$ is a Lagrange density.
where $J(x) \equiv i\alpha I(x; R)/\beta$ and the boundary condition $B_2$ implies $\phi(x, \tau = 0) = \phi(x, \tau = \beta)$.

Now, we represent the field, $\phi(x, \tau)$, in terms of the Matsubara frequency,

$$\phi(x, \tau) \equiv \varphi_0(x) + \sum_{n \neq 0} \varphi_n(x)e^{2\pi in\tau/\beta}$$

$$\equiv \varphi_0(x) + \varphi_h(x, \tau).$$

(8)

Here, $\varphi_n$ represents a heavy mode with an effective mass-squared $m_n^2 = V''[\varphi_0] + (2\pi n)^2$ and $\varphi_h$ is the collection of heavy modes. In the high temperature regime, the heavy modes have large effective masses so that fluctuations of the heavy modes are much smaller than that of the zero mode, $\varphi_0(x)$, and can safely be integrated out. For this purpose, the Lagrangian is expanded around the zero-mode,

$$\mathcal{L}_E[\varphi(x)] = \mathcal{L}_E[\varphi_0(x) + \varphi_h(x, \tau)]$$

$$= \mathcal{L}_E[\varphi_0(x)] + \frac{\delta \mathcal{L}_E}{\delta \varphi_0}\varphi_h(x) + \frac{1}{2}\varphi_h \frac{\delta^2 \mathcal{L}_E}{\delta \varphi_0^2}\varphi_h + \mathcal{L}_{E,\text{int}}(\varphi_h; \varphi_0).$$

(9)

where $\mathcal{L}_{E,\text{int}}(\varphi_h; \varphi_0)$ represents higher order terms in $\varphi_h$ which generate multi-loop effects and will be omitted hereafter. Here, $\frac{\delta^2 \mathcal{L}_E}{\delta \varphi_0^2}$ is given by

$$\frac{\delta^2 \mathcal{L}_E}{\delta \varphi_0^2} = -\frac{\partial^2}{\partial \tau^2} - \nabla^2 + V''[\varphi_0],$$

(10)

and for each $n$ mode it acts as

$$\frac{\delta^2 \mathcal{L}_E}{\delta \varphi_0^2} \bigg|_n = \left(\frac{2\pi n}{\beta}\right)^2 - \nabla^2 + V''[\varphi_0].$$

(11)

Thus the exponent of integrant of the path integral in (7) becomes

$$-\int d^4x \mathcal{L}_E[\phi(x)] + \beta \int d^3x J(x)\phi(x, 0) =$$

$$-\int d^4x \left\{ \mathcal{L}_E[\varphi_0(x)] + \frac{\delta \mathcal{L}_E}{\delta \varphi_0}\varphi_h(x) + \frac{1}{2}\varphi_h(x) \frac{\delta^2 \mathcal{L}_E}{\delta \varphi_0^2}\varphi_h(x) - J(x) \left[ \varphi_0(x) + \sum_{n \neq 0} \varphi_n(x) \right] \right\}$$

$$= \int d^4x \left\{ -\mathcal{L}_E[\varphi_0(x)] - \frac{1}{2} \sum_{n \neq 0} \varphi_n(x) \frac{\delta^2 \mathcal{L}_E}{\delta \varphi_0^2} \bigg|_n \varphi_n(x) + \left[ \varphi_0(x) + \sum_{n \neq 0} \varphi_n(x) \right] J(x) \right\},$$

where

$$\int d^4x \equiv \int_0^\beta d\tau \int d^3x.$$

Note that heavy modes also attach to the external field $J(x)$. If we first evaluate only the probability distribution functional $\tilde{P} [\varphi(x)]$ for the raw field $\varphi(x)$ as was done by Hindmarsh and Rivers [8], the above term is omitted in integrating out heavy modes. Therefore our formalism is more rigorous than theirs to this end. Also, as a result, the propagator in evaluating the connected generating functional $W[J]$, defined below, is the
finite-temperature propagator in Hindmarsh and Rivers [8], while in our formalism it is the zero-temperature propagator.

After integrating over heavy modes, the PDF is given up to one-loop order by

\[ P[\varphi_R = \bar{\varphi}] = \frac{1}{Z} \int D\varphi_0(x) \int \frac{d\alpha}{2\pi} \exp(-i\alpha \bar{\varphi}) \times \exp \left[ -\beta H[\varphi_0] + \frac{\beta}{2} \int dx \sum_{n \neq 0} J(x) \left( \frac{\delta^2 L_E}{\delta \varphi^2_0} \right)_n \right]^{-1} J(x) + \beta \int dx J(x) \varphi_0(x) \right] . \tag{13} \]

where \( H[\varphi_0] \) is the one-loop finite-temperature three-dimensional effective action obtained by integrating over heavy modes only. In the case \( \varphi \) is coupled to other quantum fields, integration over these fields in all modes should also be performed at this stage to obtain the effective action of \( \varphi_0 \).

Here we comment on the difference between the effective action integrated over only heavy modes and that over all modes. We concentrate on the homogeneous part, namely the effective potential for simplicity. The effective potential integrated over all modes is written by the zeta-function as [12]

\[ F_{\text{all}}[\varphi_0] = \frac{1}{2\beta U} \frac{d\zeta_O(s)}{ds} \bigg|_{s=0} , \tag{14} \]

where \( U \) is the box size of the system, or, \( \int d^3x \), and \( \zeta_O(s) \) is defined by

\[ \zeta_O(s) = \sum_n \frac{1}{a_n^s} , \tag{15} \]

with \( a_n \) a series of positive real discrete eigenvalues of an operator \( O \). In our case, the operator \( O \) is identified with \( \frac{\delta^2 L_E}{\delta \varphi^2_0} \).

Then the corresponding zeta-function becomes

\[ \zeta_O(s) = \frac{U}{\Gamma(s)} \left( \frac{\mu\beta}{(4\pi\beta^2)^{3/2}} \right)^{2s} \int_0^\infty ds \sigma^{s-5/2} \sum_{n=-\infty}^{\infty} e^{-(4\pi^2 n^2 + V''[\varphi_0])\sigma} , \tag{16} \]

where \( \mu \) is the renormalization scale and \( n \) represents each mode. Since the zero-mode contribution to \( \zeta_O(s) \) is given by

\[ \zeta_O(s)_{n=0} = \frac{UV''[\varphi_0]^{3/2}}{(8\pi)^{3/2}} \left( \frac{\mu}{V''[\varphi_0]^{1/2}} \right)^{2s} \frac{\Gamma(s - 3/2)}{\Gamma(s)} , \tag{17} \]

that to \( \frac{d\zeta_O(0)}{ds} \) becomes

\[ \frac{d\zeta_O(0)}{ds} \bigg|_{n=0} = \frac{UV''[\varphi_0]^{3/2}}{6\pi} . \tag{18} \]

After all, the difference of the two quantities, namely the zero-mode contribution to the effective potential, is given by
We now return to the evaluation of eq. (13). Below we omit the suffix 0. We first note the convergency of \( \alpha \)-integration in eq. (13). Using the high temperature limit, we find

\[
\sum_{n \neq 0} \left( \frac{\delta^2 L_E}{\delta \phi_0^2} \right)_n^{-1} \approx \frac{\beta^2}{12}.
\]  

(20)

Then \( \alpha \)-integration becomes

\[
\int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \exp\left[ -i\alpha \bar{\phi} + \frac{\beta}{2} \int d^3x \sum_{n \neq 0} J(x) \left( \frac{\delta^2 L_E}{\delta \phi_0^2} \right)_n^{-1} J(x) + \beta \int d^3x J(x) \phi_0(x) \right]
\]

\[ = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \exp\left[ -i\alpha (\bar{\phi} - \phi_R) - \frac{\beta}{24} \alpha^2 \int d^3x I(x; R)^2 \right] \]

\[ = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \exp\left[ -i\alpha (\bar{\phi} - \phi_R) - \frac{\beta}{192\pi^{3/2} R^3} \alpha^2 \right]. \]

(21)

From the above expression, it is evident that, while \( \alpha \) runs from \(-\infty\) to \(+\infty\), the significant contribution comes from a finite region

\[- \sqrt{\frac{96\pi^{3/2} R^3}{\beta}} \lesssim \alpha \lesssim \sqrt{\frac{96\pi^{3/2} R^3}{\beta}} \equiv \alpha_m(R).\]

(22)

But we do not perform \( \alpha \)-integration for the moment. Instead we first define the generating functional for the connected Green functional \( W[J] \) as

\[
W[J] = \beta^{-1} \ln \left\{ \frac{1}{Z} \int \mathcal{D}\phi \exp \left( -\beta H[\phi] + \beta \int d^3x J(x) \phi(x) \right) \right\}.
\]  

(23)

Then the probability distribution function is expressed as

\[
P[\phi_R = \bar{\phi}] = \int \frac{d\alpha}{2\pi} \exp \left( \beta W[J] - i\alpha \phi + \frac{\beta}{2} \int d^3x \sum_{n \neq 0} J(x) \left( \frac{\delta^2 L_E}{\delta \phi^2} \right)_n^{-1} J(x) \right).
\]  

(24)

Thus we can obtain the probability distribution \( P[\phi_R = \bar{\phi}] \) by evaluating \( W[J] \) in principle. The generating functional (23) is often expressed perturbatively since the higher-point connected Green function typically involves higher-powers of (small) coupling constants. In our case with \( J(x) = i\alpha I(x; R) / \beta \) we can formally write

\[
\beta W[J] = \sum_n \frac{\beta^n}{n!} \int d^3x_1 \ldots d^3x_n G^{(n)}_c(x_1 \ldots x_n) J(x_1) \ldots J(x_n)
\]

\[ = \sum_n \frac{(i\alpha)^n}{n!} \langle (\phi_R)^n \rangle,
\]  

(25)

where \( \langle (\phi_R)^n \rangle \) is defined by
\[
\langle (\varphi_R)^n \rangle \equiv \int d^3x_1 \ldots d^3x_n G_c^{(n)}(x_1, \ldots, x_n) I(x_1; R) \ldots I(x_n; R).
\] (26)

As mentioned above, since the dominant contribution comes from a finite range of |\alpha|, in many cases it is suitable to use only lower-order terms of (24) in (26).

### 3 Application to the Electroweak Phase Transition

Having established a generic formalism, we now apply our formalism to the electroweak phase transition. The one-loop finite-temperature effective potential of the Higgs field in the minimal standard model is well approximated by [3]-[5]

\[
V_{EW}[\phi] = D(T^2 - T^2_c)\phi^2 - ET\phi^3 + \frac{1}{4}\lambda_T \phi^4,
\] (27)

where \(D = 0.169\) and \(E = 0.00965\). For \(M_H = 60\)GeV, which is the lower bound allowed experimentally [23], we find \(\lambda_T = 0.035\), \(T_2 = 92.65\)GeV, and the critical temperature is given by \(T_c = 93.39\)GeV. The temperature dependence of the potential is depicted in Fig. 1. At the critical temperature, the effective potential has two minima, at \(\phi = 0\) and \(\phi \equiv \phi_+(T_c) = \frac{2RT_c}{\lambda_T} = 51.5\)GeV, and the inflection point is equal to \(\phi_{inf} = 10.9\)GeV.

Although the shape of the effective potential exhibits a typical feature of a first-order phase transition, the potential barrier, which gets smaller for larger \(M_H\), is so shallow that it has been suggested that the universe was in a mixed state of true and false vacua already at \(T = T_c\) due to subcritical fluctuations [3] [7] [13]-[16] as mentioned in §1. Furthermore, analytical cross-over is observed for \(M_H > \sim 80\)GeV in recent results of lattice Monte Carlo simulations [17] [18]. On the other hand, by estimating the amplitude of fluctuations on the so-called correlation length, namely the curvature scale at the origin of the potential, or on even larger scales, it has also been claimed subcritical fluctuations is small enough and the conventional picture of the phase transition works [3] [9] [19].

The amplitude of thermal fluctuations depends largely on the coarse-graining scale. In our recent work [20], by solving the simple Langevin equation, we have shown that although the curvature scale of the potential or the Compton wavelength is a good measure of correlation length for a non-selfinteracting massive scalar field in thermal equilibrium, it is not sensible to take it as the coarse-graining scale for a field with a more complicated potential such as the electroweak Higgs field. We need to determine it from a fundamental point of view.

In [20] we have reconsidered derivation of the Langevin equation for the expectation value of a scalar field at finite temperature originally developed in [21] and [22] using non-equilibrium quantum field theory. We have discussed that properties of stochastic noise force in the Langevin equation have distinct features depending on whether it arises from interactions with bose fields or from fermi fields, and shown that the latter is more effective to disturb the system from a homogeneous configuration. Hence we expect the correlation properties of these fermionic noises play an important role in determining the correlation length of the scalar field.

As shown in [20], the spatial correlation function of the stochastic thermal noise force, \(\xi(\mathbf{x}, t)\), generated by a massive fermion \(\psi\) through Yukawa coupling, \(f\phi \bar{\psi} \psi\), is given by

\[
\xi(\mathbf{x}, t) \approx \frac{f^2}{4\lambda_T} \left(1 - \frac{t}{\tau}\right) \exp\left(-\frac{t}{\tau}\right) \left(1 - \frac{\mathbf{x}^2}{\Lambda^2}\right)
\]
\[
\langle \xi(x,t)\xi(y,t) \rangle = \frac{m_\psi^4 f^2}{4\pi^4 r^2} \left\{ \left[ K_2(m_\psi r) + 2 \sum_{n=1}^{\infty} (-1)^n \frac{r^2}{\sqrt{r^2 + n^2 \beta^2}} K_2(m_\psi \sqrt{r^2 + n^2 \beta^2}) \right] \right\}^2 
- \left[ K_1(m_\psi r) + 2 \sum_{n=1}^{\infty} (-1)^n \frac{r}{r^2 + n^2 \beta^2} K_1(m_\psi \sqrt{r^2 + n^2 \beta^2}) \right]^2 \right\} , \quad r \equiv |x - y| , \quad (28)
\]

where \( f \) is a coupling constant and \( m_\psi \) is the mass of the coupled fermion \( \psi \). It damps exponentially above the inverse mass scale for \( m_\psi \beta \gg 1 \), and above the scale \( \beta \) for \( m_\psi \beta \lesssim 1 \). In the case with \( m_\psi = 0 \), we find

\[
\langle \xi(x,t)\xi(y,t) \rangle = \frac{f^2}{4\pi^4 r^2} \left\{ \left[ \frac{1}{\beta r \sinh \left( \frac{r}{\beta} \pi \right)} + \frac{\pi^2}{\beta^2 \sinh^2 \left( \frac{r}{\beta} \pi \right)} \right]^2 \right\} , \quad (29)
\]

\[
\simeq \frac{f^2}{4\beta^4} \frac{e^{-2r/\beta}}{r^2} \quad \text{for} \quad r \gtrsim \frac{\beta}{\pi} . \quad (30)
\]

So the typical damping scale is given by \( \beta/(2\pi) \). In both cases the temporal correlation damps exponentially if the temporal difference becomes larger than \( \beta/(2\pi) \).

Therefore we should primarily calculate the PDF taking the coarse-graining scale equal to this correlation length of the stochastic noise field acting on \( \phi \) because it is the only source of inhomogeneous evolution of the field. In practice, however, we calculate how the PDF changes as we increase the coarse-graining scale from \( \beta/(2\pi) \) to the curvature scale at the origin of the potential to compare with other work. Note that it is a good approximation to treat the noise field as arising from a massless fermion because, even if the two phases \( \phi = 0 \) and \( \phi_+ \) are completely mixing the average mass of the top quark, which is the most strongly-coupled fermion to the Higgs field and provides the dominant source of thermal noise, is only about 18 GeV then, much smaller than the critical temperature.

In order to obtain the proper PDF, we need evaluate the generating functional \( W[J] \). For this purpose, we must calculate the finite-temperature three dimensional effective action, which is a formidable task, and so we approximate it by the standard kinetic term and the effective potential. That is, the one-loop three-dimensional effective action is replaced by

\[
H[\varphi] = \int d^3 x \left[ \frac{1}{2} \nabla \varphi(\mathbf{x}) \cdot \nabla \varphi(\mathbf{x}) + V_{\text{eff}}[\varphi] \right] . \quad (31)
\]

Strictly speaking, this replacement is justified only if deviation from homogeneous configuration is sufficiently small. When the field is in a mixed state between the two minima, the effective potential should be modified. It is well-known, however, that this modification reduces the potential barrier \[23\], which induces more phase-mixing. Therefore, even if we evaluate the probability distribution function by using the above approximation and reach the conclusion that two phases are mixed, the conclusion is consistent.

Below we proceed the discussion by using the above approximation and adopt \( V_{\text{EW}}[\varphi] \) as the effective potential \( V_{\text{eff}}[\varphi] \). The alert reader may notice that \( V_{\text{eff}}[\varphi] \) should be that
integrated only over heavy modes for the Higgs self-interaction. But, since the effect of the Higgs self-interaction is so small that it has already been omitted in evaluating $V_{EW}[\varphi]$, it makes no change. We fix $T = T_c$ in order to investigate whether the universe is in a homogeneous state of the false vacuum at the onset of the phase transition. We also set the Higgs mass $M_H = 60 \text{GeV}$. Then $V_{EW}[\varphi]$ is given by

$$V_{EW}[\varphi] = \frac{1}{2} M^2 \varphi^2 - \frac{1}{3!} \mu T_c \varphi^3 + \frac{1}{4!} \lambda \varphi^4 , \quad (32)$$

where $M \approx 0.073 T_c$ and the dimensionless coupling constants read $\mu \approx 0.058$ and $\lambda \approx 0.21$. Since it is impossible to evaluate the generating functional $W[J]$ exactly even in the case [31] is adopted, we expand $W[J]$ perturbatively. Also, we neglect the gauge-nonsinglet nature of the Higgs field for simplicity and consider only its real-neutral component to treat $\varphi$ as if it was a real singlet field.

Up to the first order of the coupling constant, $W[J]$ is made up of the graphs as depicted in Fig. 2. That is,

$$\beta W[J] = \frac{1}{2} \left\langle J_x \Delta_{xy} J_y \right\rangle_{xy} + \frac{\mu}{3! \beta^3} \left[ 3 \left\langle \Delta_{xx} \Delta_{xa} J_a \right\rangle_{xa} + \left\langle \Delta_{xa} \Delta_{xb} \Delta_{xc} J_a J_b J_c \right\rangle_{xabc} \right] - \frac{\lambda}{4! \beta^3} \left[ 3 \left\langle \Delta^2 \right\rangle_x + 6 \left\langle \Delta_{xx} \Delta_{xa} J_a J_b \right\rangle_{xab} + \left\langle \Delta_{xa} \Delta_{xb} \Delta_{xc} \Delta_{xd} J_a J_b J_c J_d \right\rangle_{xabcd} \right] + \text{(higher - order terms)}, \quad (33)$$

where $\Delta_{xy}$ is given by

$$\Delta_{xy} = \beta \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + M^2} e^{i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})}. \quad (34)$$

Here $\left\langle \cdots \right\rangle_{xab...}$ implies integration over $d^3x d^3b d^3c \cdots$, and $\beta = 1/T_c$. Each loop graph can be renormalized into the definition of $\tilde{\varphi}$, the normalization of the probability, and the mass, $M$. Therefore, we have only to evaluate three tree graphs.

First we consider the two-point function. Adding the last term of the exponent in eq.(24), it is given by

$$\left\langle (\tilde{\varphi} R)^2 \right\rangle_T = \left\langle (\varphi R)^2 \right\rangle + \frac{1}{\beta} \int d^3x \sum_{n \neq 0} I(\mathbf{x}; R) \left( \frac{\delta^2 \mathcal{L}_E}{\delta \varphi^2} \right)_{(n)}^{-1} I(\mathbf{x}; R)$$

$$= \int d^3x \int d^3y I(\mathbf{x}; R) \frac{1}{\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \frac{1}{k^2 + M^2 + (2\pi n/\beta)^2} e^{i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} I(\mathbf{x}; R)$$

$$= \int d^3x \int d^3y \int \frac{d^3k}{(2\pi)^3} \left[ \frac{1}{2} + \frac{1}{e^{\beta\omega} - 1} \right] \frac{1}{\omega} e^{i \mathbf{k} \cdot (\mathbf{x} - \mathbf{y})} I(\mathbf{x}; R) I(\mathbf{y}; R)$$

$$= \frac{1}{2\pi^2} \int_0^{\infty} dk \left[ \frac{1}{2} + \frac{1}{e^{\beta\omega} - 1} \right] k^2 \omega I(k; R)^2, \quad \omega \equiv \sqrt{k^2 + M^2}, \quad (35)$$

where $I(k; R)$ is the Fourier transform of the window function,
\[ \tilde{I}(k; R) = \int d^3x I(x; R)e^{i k \cdot x} = \exp \left( -\frac{1}{2}R^2 k^2 \right). \] (36)

Similarly, the three- and the four-point functions are given, respectively, by

\[ \langle \varphi_R \rangle^3 = \frac{\mu}{\beta^3} \int \frac{d^3k_1}{(2\pi)^3} \int \frac{d^3k_2}{(2\pi)^3} f(k_1) f(k_2) f(k_1 + k_2), \]
\[ \langle \varphi_R \rangle^4 = -\frac{\lambda}{\beta^3} \int \frac{d^3k}{(2\pi)^3} g(k)^2 < 0, \] (37)

where

\[ f(k) = \frac{1}{k^2 + M^2} \tilde{I}(k; R), \]
\[ g(k) = \int \frac{d^3q}{(2\pi)^3} f(q) f(k + q). \] (38)

In order to confirm that it is sufficient to consider graphs with lowest-order in coupling constants for the case with the fundamental coarse-graining scale \( R = \beta/(2\pi) \), let us estimate the magnitude of graphs with \( n \)-th order of coupling constants on scale \( R \equiv N\beta/(2\pi) \). For even \( n \), the dominant contribution comes from the graph made of \((n/2 - 1)\) four-point vertices, which reads

\[ \frac{1}{n! \beta^{n-1}} \int \frac{d^3k_1}{(2\pi)^3} \cdots \int \frac{d^3k_{n-1}}{(2\pi)^3} f(k_1) \cdots f(k_{n-1}) \times \]
\[ \frac{1}{(k_1 + k_2 + k_3)^2 + M^2} \cdots \frac{1}{(k_1 + \cdots + k_{n-1})^2 + M^2} e^{-(k_1 + \cdots + k_{n-1})^2 R^2} \]
\[ \lesssim \frac{1}{n! \beta^{n-1}} \left( \int \frac{d^3k}{(2\pi)^3} f(k) \right) \frac{\lambda^{n-1}}{2} \left( \int \frac{d^3k}{(2\pi)^3} \frac{f(k)}{k^2 + M^2} \right)^{\frac{n-1}{2}} \]
\[ \lesssim \frac{1}{n!} \left( \frac{\lambda}{0.073} \right)^{\frac{n}{2} - 1} 2^{-\frac{n}{2} + 2} 3^{\frac{n}{2}} \pi^{-\frac{n}{2} + 1} N^n, \] (39)

where \( R \) is assumed to be much smaller than \( M^{-1} \). For odd \( n \), the graph with one three-point vertex and \((n - 3)/2\) four-point vertices makes dominant contribution to \( W[J] \) and its magnitude can be estimated similarly. As a result we find, for the case \( \alpha \) in \( J \) takes the value \( \alpha_m(R) \) defined in (22), the graphs with the lowest order in the coupling constants make dominant contribution to \( W[J] \), provided that the coarse graining scale satisfies \( N \lesssim 10 \). If the dominant contribution of the \( \alpha \)-integral comes from a narrower region than (22), the constraint on \( N \) can be even weaker.

Thus the PDF of the coarse-grained field \( \varphi_R \) up to the first order of the coupling constants is given by
\[ P[\varphi_R = \varphi] = \int_{-\infty}^{\infty} \frac{d\alpha}{2\pi} \exp \left( -i\alpha\varphi - \frac{\alpha^2}{2} \langle (\varphi_R)^2 \rangle_T - i\frac{\alpha^3}{3!} \langle (\varphi_R)^3 \rangle + \frac{\alpha^4}{4!} \langle (\varphi_R)^4 \rangle \right) \]

\[ = \int_{0}^{\infty} \frac{d\alpha}{\pi} \exp \left( -\frac{\alpha^2}{2} \langle (\varphi_R)^2 \rangle_T + \frac{\alpha^4}{4!} \langle (\varphi_R)^4 \rangle \right) \cos \left( \alpha\varphi + \frac{\alpha^3}{3!} \langle (\varphi_R)^3 \rangle \right) \] (40)

The PDF with \( R = \beta/(2\pi) \) is depicted in Fig. 3. The probability \( \varphi_R \geq \varphi_{\text{inf}} = 0.117T_c \) is 43.8% and the root-mean-square (RMS) is 0.76\( T_c \), which is much larger than the inflection point \( \varphi_{\text{inf}} \) and the maximum point \( \varphi_- = 0.277T_c \) of the potential. This result suggests that there is non-negligible amount of soaking into the asymmetric phase. Therefore we can conclude that the electroweak phase transition does not proceed by the standard bubble nucleation picture.

For comparison, we show the probability distribution function for the field coarse-grained over the inverse mass scale \( R = 1/M \) in Fig. 4. In this case, since \( R \) is large enough, from the physical argument, we can infer that the fluctuation becomes small and approaches the Gaussian configuration. Indeed, the probability of \( \varphi_R \geq \varphi_- \) is extremely small and the RMS is 0.028\( T_c \), much smaller than the inflection point, which is consistent with the above observation and [5]. Table 1 shows how the probability \( \varphi_R \geq \varphi_{\text{inf}} \) and the root-mean-square \( \sqrt{\langle \varphi_R^2 \rangle} \) change as a function of the coarse-graining scale \( R \). Roughly speaking, if \( R \geq 40\beta/(2\pi) \), we can neglect the role of these thermal fluctuations. In the previous work [20], the critical scale is about \( R_L \approx 80\beta/(2\pi) \). The relation between the coarse-graining scale in this paper and that in the previous paper is not so clear because the former is defined in the spherical Gaussian window function and the latter corresponds to that defined in the cubic top-hat window function. But, if the correspondence is decided by the equality of the volume, that is,

\[ R_L^3 = \int d^3x \frac{4\pi r^3}{3} I(0; R) , \] (41)

we find \( R_L \approx 3.0R \). Therefore the classical lattice simulation of [20] is in reasonable agreement with the present analysis.

4 Concluding remark

In this paper, we formulate the probability distribution function for the coarse-grained fluctuations, which can be evaluated by calculating perturbatively the connected generating functional \( W[J] \). In application, we apply our formalism to the electroweak phase transition. We evaluate the probability distribution function of the field coarse-grained over \( 2\pi/\beta_c \) so that we find that the thermal fluctuations called subcritical bubbles play a significant role and that the universe is not homogeneously in the symmetric phase at the onset of the electroweak phase transition. We also calculate the probability \( P[\varphi_R \geq \varphi_{\text{inf}}] \) and the root-mean-square at several coarse-graining scale. The result is depicted in Table. 1. We can observe that as the coarse-graining scale is larger, the probability \( P[\varphi_R \geq \varphi_{\text{inf}}] \) becomes smaller and also the RMS smaller. Thus, the coarse-graining scale is very important in investigating the property of thermal fluctuations. Unless the coarse-graining scale is specified properly, the discussion becomes meaningless and brings out confusion.
Table 1: Property of thermal fluctuations at several scales

| $\beta/(2\pi)$ | $P[\varphi_R \geq \varphi_{\text{inf}}]$ | $\sqrt{\langle \varphi_R^2 \rangle}/T_c$ |
|----------------|---------------------------------|---------------------------------|
| $0.15\beta$    | 44%                             | $0.76T_c$                       |
| $1.59\beta$    | 22%                             | $0.15T_c$                       |
| $3.18\beta$    | 12%                             | $9.8 \times 10^{-2}T_c$         |
| $4.77\beta$    | 5.9%                            | $7.3 \times 10^{-2}T_c$         |
| $6.37\beta$    | 2.6%                            | $5.8 \times 10^{-2}T_c$         |
| $7.96\beta$    | 0.99%                           | $4.8 \times 10^{-2}T_c$         |
| $9.55\beta$    | 0.32%                           | $4.1 \times 10^{-2}T_c$         |
| $11.1\beta$    | $8.4 \times 10^{-2}$%           | $3.5 \times 10^{-2}T_c$         |
| $12.7\beta$    | $1.8 \times 10^{-2}$%           | $3.0 \times 10^{-2}T_c$         |
| $13.7\beta$    | $1.6 \times 10^{-3}$%           | $2.8 \times 10^{-2}T_c$         |
| $1/M$          | $13.7\beta$                     | $2.8 \times 10^{-2}T_c$         |

Acknowledgments

MY would like to thank Dr. T. Shiromizu for discussion. MY is grateful to Professor K. Sato for his continuous encouragement and Professor M. Morikawa for his useful comments. This work was partially supported by the Japanese Grant in Aid for Scientific Research Fund of the Ministry of Education, Science, Sports and Culture Nos. 07304033(JY) and 08740202(JY).

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Figure 1: One-loop improved finite temperature effective potential of the Higgs field, $V_{EW}[\phi]$. 
Figure 2: Contribution to $W[J]$ up to $\mathcal{O}(\mu, \lambda)$.

Figure 3: Probability distribution function for the field coarse-grained over $\beta/(2\pi)$. The inflection point, $\varphi_{\inf}$ in the potential corresponds to $0.117\beta$ and the local maximum, $\varphi_-$ $0.277\beta$. 
Figure 4: That over the inverse mass scale.