A projection-based model checking for heterogeneous treatment effect

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Abstract

In this paper, we investigate the hypothesis testing problem that checks whether part of covariates / confounders significantly affect the heterogeneous treatment effect given all covariates. This model checking is particularly useful in the case where there are many collected covariates such that we can possibly alleviate the typical curse of dimensionality. In the test construction procedure, we use a projection-based idea and a nonparametric estimation-based test procedure to construct an aggregated version over all projection directions. The resulting test statistic is then interestingly with no effect from slow convergence rate the nonparametric estimation usually suffers from. This feature makes the test behave like a global smoothing test to have ability to detect a broad class of local alternatives converging to the null at the fastest possible rate in hypothesis testing. Also, the test can inherit the merit of local smoothing tests to be sensitive to oscillating alternative models. The performance of the test is examined by numerical studies and the analysis for a real data example for illustration.

Keywords: Dimension reduction, Projection-based test, Treatment effect hypothesis

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1. Introduction

In this paper, we consider the testing problem for treatment effect model. Let $D$ be the indicator variable of treatment and $Y$ the outcome. $D_i = 0, 1$ respectively means the $i$th individual does not receive or receives treatment. The corresponding potential outcomes are then defined as $Y_i(0)$ and $Y_i(1)$. The observed outcome can then be written as $Y_i = D_i Y_i(1) + (1 - D_i) Y_i(0)$. An important quantity of interest in the literature is the average treatment effect (ATE): $E(Y(1) - Y(0))$ [see, e.g. Rosenbaum and Rubin (1983), Hahn (1998)]. To check the heterogeneity of ATE over a set $W$ of collected covariates, conditional (or heterogeneous) average treatment effect $E(Y(1) - Y(0) | W)$ (CATE) has been investigated. Note that CATE can capture the heterogeneity of a treatment effect across the subpopulations defined by the index set of $W$. CATE($W$) is also called a contrast function in the precision medicine literature such as Shi et al. (2019), which plays an important role in estimating optimal individualized treatment regime. In order to identify this function, other than the common support assumption, the unconfoundedness assumption is very important:

- Assumption 1 (Unconfoundedness): $(Y(0), Y(1)) \perp D | W$.

Here $W = (X, Z)$ with $X$ and $Z$ being respectively $p$– and $q$–dimensional vectors of covariates, and $\perp$ stands for statistical independence. To make the paper self-contained, we write down the common support assumption as follows:

- Assumption 2 (Common support): For some very small $c > 0$, $c < \pi(W) < 1 - c$

where $\pi(W) = E(D | W)$ is the propensity score function.

Based on these assumptions and others, most of existing estimation methods are for CATE($W$) conditional on all covariates $W$. See e.g. Crump et al. (2008), Abrevaya et al. (2015), Hsu (2017) and Wager and Athey (2018). However, it may be the case that only the subset $X$ of $W$ is significantly useful for the average treatment effect such that $CATE(W) = CATE(X)$ in this case. This can then very much alleviate the curse of dimensionality in estimation and other further
statistical analyses. Such a dimension reduction structure needs an accompany of model checking to prevent the working model possibly too parsimonious to lose some important covariates. More specifically, the hypotheses are:

\[
H_0 : P(E[Y(1) - Y(0)|W] = E[Y(1) - Y(0)|X]) = 1, \\
H_1 : P(E[Y(1) - Y(0)|W] = E[Y(1) - Y(0)|X]) < 1. 
\] (1)

There are no tests available for this issue in the literature although some are relevant. Crump et al. (2008) focused on testing whether CATE($W$) equals zero or a given constant. Chang et al. (2015) and Hsu (2017) respectively proposed tests for the null hypothesis that CATE($W$) or CATE($X$) is non-negative for all values of covariates. However, these tests cannot be used for the above testing problem. To the best of our knowledge, the research described herewith is the first attempt to handle such a problem in the literature. Further, we consider the situation that the dimensions of $X$ and $Z$, that is, $p$ and $q$ are fixed, but $p$ could be much smaller than $q$. Then, under $H_0$, CATE can be estimated only conditional on a much lower dimensional covariates vector, $X$.

The following are two special cases of $H_0$.

- **Example 1 (Treatment effect heterogeneity)** Set $X = \emptyset$. In this very special case, the above null hypothesis becomes

\[
H_{02} : P(E[Y(1) - Y(0)|W] = E[Y(1) - Y(0)]) = 1.
\]

That is, under the null hypothesis, the treatment effect will not change with the value of $W$, thus the treatment effect does not have heterogeneity across the subpopulation defined by the value of $W$. The rejection of $H_{02}$ implies the necessity of estimating conditional treatment effect. Note that this test has been discussed by Crump et al. (2008).

- **Example 2 (Significant conditional treatment effect)** Set $X = \emptyset$ and consider $E[Y(1) - Y(0)] = 0$, we are still interested in testing:

\[
H_{03} : P(E[Y(1) - Y(0)|W] = 0) = 1.
\]
That is, when receiving a treatment has no effect on outcomes for the overall population, we want to check whether the treatment is still significant for some subpopulations.

Since the test construction for \( H_{02} \) and \( H_{03} \) can be relatively easier in our methodology than that for \( H_0 \), we then only deal with \( H_0 \) in the following.

Certainly, we first need to estimate the conditional mean functions \( E(Y(1) - Y(0) \mid X) \) and \( E(Y(1) - Y(0) \mid W) \). As we do not assume any parametric model structure for these functions, nonparametric estimation is applied. As commented above, when the dimension \( p \) of the covariates \( X \) is high, any nonparametric estimation would be inefficient and thus has negative effect for the performance of constructed test. We then review some typical methods for regressions first. There are a number of proposals available in the literature, but we only name a few to comment on their pros and cons. For local smoothing tests for regressions in the literature, one of methods was proposed by Zheng (1996). But it can only detect local alternatives distinct from the null at the rate of order \( n^{-1/2}h^{-(p+q)/4} \) where \( h \) is the bandwidth going to zero at a certain rate in nonparametric estimation. This drawback can be found in other typical local smoothing test literature, see e.g. Fan and Li (1996), Zhang and Dette (2004) and Guo et al. (2016) although the rate could be proved, in some special model structures, to \( n^{-1/2}h^{-1/4} \) when some dimension reduction approaches are applied. Thus, we also wish that it have the nice properties of global smoothing-based tests in the literature for regressions to detect local alternatives distinct from the null at the fastest possible rate of order \( n^{-1/2} \) in hypothesis testing. See Stute et al. (1998), Zhu (2003) and Khmaladze et al. (2009) for such types of tests. To make a test sensitive, to a certain extent, to oscillating/high-frequency alternative models, it would be good to construct a test that is based on a local smoothing test structure by using some nonparametric estimation for the involved functions. On the other hand, to have the test more powerful to detect smooth local alternatives, we also wish it to have the features global smoothing tests share. Therefore, we combine two ideas to achieve these goals. First, to
alleviate this dimensionality difficulty, we suggest a projection-based test that uses projected covariates $\beta^T W$. It is clear that we cannot simply use only one or a few projections to construct a test otherwise, it will be a directional test. To make test omnibus against all alternatives, we then use the projected covariates at all projection directions in an aggregation manner. From Zhu and Li (1998), Escanciano (2006), Stute et al. (2008), and Lavergne and Patilea (2012), we anticipate that the dimensionality issue could be largely alleviated as for regressions, these tests can reach the rate much faster than $n^{-1/2} h^{-(p+q)/4}$. Thus, all these tests can very much improve the performance in high-dimensional scenarios. But these tests are either still typical nonparametric estimation-based local smoothing tests that can detect local alternatives at slower rate than $1/\sqrt{n}$ or typical empirical process-based global smoothing tests that are less sensitive to high-frequency alternative models. Taking this issue into consideration, we consider constructing a test that is based on local smoothing technique and then is transferred to a final pairwise distance-based test. Under certain regularity conditions, the limiting null distribution of this test statistic can then be free of the nonparametric estimation for the conditional moment on the whole $W$ such that the test behaves like a global smoothing test and at the same time, shares the sensitivity to high-frequency models to certain extent. This will be demonstrated in the numerical studies. Another feature of the test is worthwhile to mention: although the function $E[Y(1) - Y(0)|X]$ under the null hypothesis indispensably requires nonparametric estimation, it does not make a slow-down of the resulting rate of convergence and the test can still share all features global smoothing tests have.

The rest of this paper is organized as follows. In Section 2, we describe the test statistic construction. The asymptotic properties of the test statistic under the null, global and local alternative hypothesis are investigated in Section 3. In Section 4, we examine the finite sample performance of our test through simulations and apply it to a real data example for illustration in Section 5. Some conclusions are presented in Section 5, and the proofs of the theoretical results are postponed to Appendix.
2. The test statistic construction

Note that under the unconfoundedness assumption and common support assumption, the conditional treatment effect can be identified as:

\[ E[Y(1) - Y(0) | W] = E[DY/\pi(W) - (1 - D)Y/(1 - \pi(W)) | W], \quad E[Y(1) - Y(0) | X] = E[DY/\pi(W) - (1 - D)Y/(1 - \pi(W)) | X]. \]

Let \( Y^* = DY/\pi(W) - (1 - D)Y/(1 - \pi(W)). \) Then \( H_0 \) can be rewritten as follows:

\[ H_{01} : P(E[Y^* | W] = E[Y^* | X]) = 1. \]  

Note that we consider the case where the propensity score is a function of \( W \), rather than \( X \). This is because the propensity score is a probability for treatment \( D \) when the covariates are given. Thus, the decision on whether giving treatment is based on all covariates / confounders. While the testing problem is for treatment effect after giving the decision on treatment. Thus, this is a reasonable scenario.

Define \( g(X) = E(Y^* | X) \) and \( e = Y^* - g(X) \), \( u = Y^* - E(Y^* | W) \). Thus, under the null hypothesis, \( e = u \) with \( E(e | W) = 0 \), otherwise \( E(e | W) \neq 0 \). Hence, it is reasonable to directly construct a test statistic based on the sample analogue of \( E[eE(e | W)] = E\{[E(e | W)]^2 \} \geq 0 \) with the equality holds if and only if \( E(e | W) = 0 \). This idea is similar to that in Zheng (1996). However, in order to get the sample analogue of \( E[eE(e | W)] \) without a model misspecification risk, a nonparametric estimation of \( E(e | W) \) is required. Note that \( W = (X^T, Z^T)^T \) with \( X \in R^p \) and \( Z \in R^q \). Hence, any nonparametric estimation of \( E(e | W) \) suffers from the curse of dimensionality when the dimension \( q \) of possible insignificant variables \( Z \) is large, even moderate. This motivates us to construct test statistic based on a method with projection directions. To this end, we first give a lemma about the equivalence between function with original covariates and that with projected covariates below.

**Lemma 1.** \( E(Y^* | W) = E(Y^* | X) \) holds if and only if \( E(e | \alpha^TW) = 0 \) holds for all \( \alpha \in R^{p+q} \). Further the equality \( E(e | \alpha^TW) = 0 \) holds if and only
if \( \int \{ |E(e \mid \alpha^TW)|^2 f_\alpha(\alpha^TW) \} \mu(\alpha) d\alpha = 0 \) when the function \( E(e \mid \alpha^TW) \) is continuous about \( \alpha \).

Similar conclusion can be also found in Zhu and Li (1998), Escanciano (2006), Lavergne and Patilea (2012) and Li et al. (2019).

Note that Lemma 1 implies that, under the null hypothesis,

\[
\int E\{ |E(e \mid \alpha^TW)|^2 f_\alpha(\alpha^TW) \} \mu(\alpha) d\alpha = 0.
\]  

(3)

While under the alternative hypothesis, there exist some \( \alpha^* \in \mathbb{R}^p \) such that \( E(e \mid \alpha^*^TW) \neq 0 \) and by the continuity of this function with respect to \( \alpha \), there is a neighborhood \( \alpha \) whose measure is positive and \( E(e \mid \alpha^TW) \neq 0 \) for all \( \alpha \) in the neighborhood. Thus, it follows that

\[
\int E\{ |E(e \mid \alpha^TW)|^2 f_\alpha(\alpha^TW) \} \mu(\alpha) d\alpha > 0.
\]  

(4)

Also note that \( E\{ |E(e \mid \alpha^TW)|^2 f_\alpha(\alpha^TW) \} = E\{ eE(e \mid \alpha^TW) f_\alpha(\alpha^TW) \} \). The above argument implies that we can use the sample analogue of \( \int E\{ eE(e \mid \alpha^TW) f_\alpha(\alpha^TW) \} \mu(\alpha) d\alpha \) to construct a test statistic.

When an independent and identically distributed (i.i.d.) random sample \( \{ (Y_1, D_1, X_1, Z_1) \}_{i=1}^n \) is available with a parametric propensity score function \( \pi(W, r_0) \) where \( r_0 \) is an unknown parameter vector of dimension \( d \), we first estimate \( Y_i^* \) by \( \hat{Y}_i^* = \left( \frac{D_i}{\pi(W_i, \hat{r})} - \frac{1-D_i}{1-\pi(W_i, \hat{r})} \right) Y_i \), where \( \hat{r} \) in \( \pi(W, \hat{r}) \) is a maximum likelihood estimator of \( r_0 \). The test statistic is defined as

\[
\hat{T}_n = \int \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} \hat{e}_i \hat{e}_j H_{h_1}(\alpha^TW_i - \alpha^TW_j) \mu(\alpha) d\alpha.
\]  

(5)

This is the sample version of \( \hat{T} \) where \( \hat{E}(e_j \mid \alpha^TW_i) = \frac{1}{(n-1)} \sum_{j \neq i} \hat{e}_j H_{h_1}(\alpha^TW_j - \alpha^TW_i) / f_\alpha(\alpha^TW_i) \) is the kernel estimator of \( E(e \mid \alpha^TW) \) with \( H(\cdot) \) being a kernel function, \( H_{h_1}(\cdot) = H(\cdot/h_1)/h_1 \) and \( h_1 \) is the bandwidth. \( \hat{e}_i = \hat{Y}_i^* - \hat{g}(X_i) \) with \( \hat{g}(X_i) = \sum_{j \neq i} w_{ij} \hat{Y}_j^* \), \( w_{ij} = \frac{1}{n(n-1)} K_h(X_j - X_i) / \hat{f}(X_i) \). The density estimator is \( \hat{f}(X_i) = \frac{1}{n-1} \sum_{j \neq i} K_h(X_j - X_i) \), where \( K \) is a multivariate kernel function and \( K_h(\cdot) = \frac{1}{h^p} K \left( \frac{\cdot}{h} \right) \) with a bandwidth \( h \).
Although nonparametric kernel estimation for $E(Y^* | X)$ is inevitable, this test statistic only involves the integral of univariate $\alpha^\top W$ over all $\alpha$ rather than the original high-dimensional $W$. Thus, $\tilde{T}_n$ could greatly mitigate the dimensionality problem due to the nonparametric estimator of $E(e | \alpha^\top W)$. However, it can be expected that the asymptotic properties of $\tilde{T}_n$ will still be related to the bandwidth $h_1$ in a nonparametric estimation nature so that the convergence rate would be slower than $1/\sqrt{n}$. To tackle this problem, we adapt the idea in [Li et al., 2019] to transform the nonparametric estimation based test into the pairwise distance-based one so that the convergence rate can be free of the bandwidth parameter $h_1$. To be specific, let $H(u) = \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{u^2}{2} \right)$ and consider $\alpha \sim N(0, h_1^2 I_p)$, using the direct consequence of Lemma 2 in [Li et al., 2019]. It can be shown that $\tilde{T}_n$ is proportional to $T_n$, i.e. $\tilde{T}_n = \frac{1}{h_1} T_n$ with

$$\tilde{T}_n = \frac{1}{h_1 n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \hat{e}_i \hat{e}_j B_{ij},$$

(6)

where $B_{ij} = \frac{1}{\sqrt{1+d_{ij}}}$ with $d_{ij} = \|W_i - W_j\|^2$. Here $\| \cdot \|$ denotes the Frobenius norm throughout this paper. Note that $h_1$ is just a constant outside the sum in $\tilde{T}_n$, therefore, we can use $T_n = h_1 \tilde{T}_n$ that is then free of the bandwidth $h_1$.

**Remark 1.** It is worth mentioning that the constructed test could still inherit some features of existing local smoothing tests. Recall that $B_{ij} = \frac{1}{\sqrt{1+d_{ij}}}$ with $d_{ij} = \|W_i - W_j\|^2$. Thus, $T_n$ captures more information from closely related observations. This property ensures that no matter the alternatives are either highly frequent or lowly frequent, the test $T_n$ could be workable to detect them. This merit can be confirmed by the numeric studies below.

3. Asymptotic properties

In order to get the asymptotic behaviour of $T_n$, the following assumptions are required:

- Assumption 3(Sampling): The observation data, $\{(Y_i, D_i, X_i, Z_i)\}_{i=1}^{n}$, is an independent and identically distributed random sample of size $n$ from the joint distribution of the vector $(Y, D, X, Z)$. 


- Assumption 4 (Distribution): the density of $X$, $f(x)$, is bounded away from zero and infinity, $s$-times continuously differentiable on its support $\Omega$. $E(Y^* \mid X) = g(X)$ is continuously differentiable.

- Assumption 5 (Moments): $E(u^4) < \infty$, $E(\eta^4) < \infty$ and $E\|W\|^2 < \infty$.

- Assumption 6 (Kernel): For $p$-dimensional $u$, $K(u)$ is a bounded kernel that is symmetric around zero, and $s$ times continuously differentiable and of order $s$: $\int K(u)du = 1$, $\int u_1^{p_1} \cdots u_p^{p_p} K(u)du = 0$ for all nonnegative integers $p_1, \ldots, p_p$ such that $1 \leq \sum p_i < s$, and nonzero when $\sum p_i = s$.

- Assumption 7 (Bandwidth): $h \to 0$, $nh^2 p \to \infty$ and $nh^{2s} \to 0$ as $n \to \infty$.

- Assumption 8 (Propensity score estimator): The propensity score $\pi(W) = E(D \mid W)$ has a parametric form $\pi(W, r_0)$, and the function $\pi(W, r)$ is bounded away from zero and has bounded continuous partial derivatives up to order 2 with respect to $r \in \Theta \subset \mathbb{R}^d$, $d < \infty$.

Here $\eta = \bar{Y} - E(\bar{Y} \mid X)$ with $\bar{Y} = \left( \frac{D}{\pi(W, r_0)^2} - \frac{1-D}{(1-\pi(W, r_0))^2} \right) Y \nabla \pi(W, r_0)$, and $\nabla \pi(W, r_0)$ stands for the partial derivative with respect to $r_0$.

Assumptions 3 ~ 5 are commonly used to guarantee the asymptotic normality of the test statistic. As our test statistic uses nonparametric estimation for $E(Y^* \mid X)$, assumptions 6 ~ 7 are designed to ensure the nonparametric estimator well-behaved, which are also widely used in the nonparametric estimation literature. The condition on $K$ is for convenience of theoretical analysis. The following proof can be extended to kernels with exponential tails. Assumption 8 is standard in the literature to obtain $\sqrt{n}$-consistent estimation for $r_0$ in $\pi(W, r_0)$, see e.g. Yao et al. (2010) and Lin et al. (2018). Based on this assumption, we can get the following lemma.

**Lemma 2.** Under Assumption 8, the maximum likelihood estimator $\hat{r}$ has the following asymptotically linear representation:

$$\hat{r} - r_0 = \frac{1}{n} \sum_{i=1}^{n} R_i + o_p(n^{-1/2}).$$

(7)
and its asymptotic distribution is
\[\sqrt{n}(\hat{r} - r_0) \xrightarrow{D} N(0, \Sigma^{-1}).\] (8)

where \( R_i = \Sigma^{-1} \sum_{j=1}^{r_i} \frac{\varphi_j(W_i, r_0)(D_i - \pi(W_i, r_0))}{\pi(W_i, r_0)(1 - \pi(W_i, r_0))} \), \( \Sigma = E \left[ \sum_{j=1}^{r_i} \varphi_j(W_i, r_0)^T \varphi_j(W_i, r_0) (1 - \pi(W_i, r_0)) \right]^{-1} \).

This lemma can be found in Yao et al. (2010). Based on this lemma, we return to investigating the asymptotic distribution of \( T_n \) under the null and alternative hypothesis.

3.1. Asymptotic behavior under the null hypothesis

We have the following asymptotic results under the null hypothesis.

**Theorem 1.** Under Assumptions 1 \( \sim 8 \) and \( E[Y(1) - Y(0) \mid W] = E[Y(1) - Y(0) \mid X] \) holds with probability 1, the test statistic \( T_n \) in (6) satisfies
\[nT_n \xrightarrow{D} \sum_{i=1}^{\infty} \lambda_i (Z_i^2 - 1) + 2
\nu_1^T \nu_2 + \nu_1^T A \nu_1 + \mu^*\] (9)

Here \( \nu_1 \sim N(0, \Sigma^{-1}) \), \( \nu_2 \sim N(0, \Sigma_1) \) with \( \Sigma_1 = E(e_i^2 \tilde{H}_1 \tilde{H}_1^T) \) and \( \tilde{H}_1 = E(\eta_2 B_{12} \mid W_1) - \frac{1}{2} E(\eta_2 \tilde{w}_{31} + \eta_3 \tilde{w}_{21} B_{32} \mid W_1) \). \( \mu^* = -E[(e_i^2 \tilde{w}_{12} + e_i^2 \tilde{w}_{21})B_{12}] + \frac{1}{3} E(e_i^2 \tilde{w}_{21} \tilde{w}_{31} B_{23} + e_i^2 \tilde{w}_{12} \tilde{w}_{21} B_{23} + e_i^2 \tilde{w}_{12} \tilde{w}_{23} B_{12}) \) with \( \tilde{w}_{ij} = \mathcal{K}_h (X_j - X_i) / f(X_i) \).

And \( A = E(\eta_1^2 B_{12}) \).

Let \( Z_i \)'s be independent standard normal random variables and \( \lambda_i \)'s the eigenvalues of the integral equation
\[\int L(\chi_1, \chi_2) \phi_i(\chi_2) dF(\chi_2) = \lambda_i \phi_i(\chi_1)\] (10)

with \( \chi_i = (e_i, W_i) \) and \( \phi_i(\chi) \) being the associated orthonormal eigenfunctions.

Let
\[\tilde{B}_{ij} = \frac{1}{2} E(\tilde{w}_{it} B_{jt} + \tilde{w}_{it} B_{it} \mid W_i, W_j)\]

and
\[\tilde{B}_{ij} = \frac{1}{2} E(\tilde{w}_{ti} \tilde{w}_{kj} B_{tk} + \tilde{w}_{ki} \tilde{w}_{ij} B_{kt} \mid W_i, W_j).\]

Write \( L(\chi_i, \chi_j) = e_i e_j (B_{ij} - 2 \tilde{B}_{ij} + \tilde{B}_{ij}) \)
Obviously, under the null hypothesis \( H_0 \), \( T_n = O_p \left( \frac{1}{n} \right) \) implies the fact that \( T_n \) converges to zero very quickly when \( H_0 \) is true. This will lead to a sensitive test to detect local alternatives close to the null at a fastest possible rate in hypothesis testing. As \( T_n \) contains the nonparametric estimation of \( g(X) \) and the parametric propensity score estimation, the limiting null distribution of \( T_n \) is intractable. Thus we use the wild bootstrap to approximate the null distribution. See the details in Section 4.

3.2. Power study

To examine the power performance of \( T_n \), we consider the following sequence of local alternative hypotheses as:

\[
H_{1n} : P(E(Y(1) - Y(0) \mid W) = g(X) + a_nH(W)) = 1, \ W = (X, Z). \quad (11)
\]

Recall that \( g(X) = E(Y(1) - Y(0) \mid X) \). Thus fixed \( a_n \) corresponds to the global alternative model and when \( a_n \) goes to zero, the sequence is about the local alternative hypotheses. To smooth the theoretical analysis, the following assumption is added:

- Assumption 9(Alternatives): \( E(H(W)^2) < \infty \) and \( E[H(W) \mid X] = 0 \).

The moment condition of \( H(W) \) is commonly assumed and the condition \( E[H(W) \mid X] = 0 \) can be also found in Lavergne and Patilea (2015). This condition is parallel to the unconditional one in Lavergne and Vuong (2000) that is \( H(W) \equiv 0 \) when \( a_n = 0 \). We have the following theorem.

**Theorem 2.** Suppose Assumptions 1 ~ 9 hold. Then under the local alternative hypotheses in (11), the following results can be obtained:

1. Under the global alternative hypothesis \( H_{1n} \) with a fixed \( a_n > 0 \),

\[
T_n \xrightarrow{p} \mu > 0. \quad (12)
\]

Here \( \mu = a_n^2E(H_1H_2B_{12}) \).
(2) Under the local alternative hypothesis $H_{1n}$ with $\sqrt{n}a_n \to \infty$,
\[
T_n/a_n^2 \xrightarrow{P} \mu_0 > 0. 
\] (13)
Here $\mu_0 = E(H_1H_2B_{12})$.

(3) Under the local alternative hypothesis $H_{1n}$ with $a_n = n^{-1/2}$,
\[
nT_n \xrightarrow{D} \sum_{i=1}^{\infty} \lambda_i(Z_i^2 - 1) + 2\nu_1^T \nu_2 + \nu_1^T A \nu_1 + N(\tilde{\mu}, \sigma^2). 
\] (14)
\[
\tilde{\mu} = \mu_0 + \mu^*, \quad \sigma^2 = E(u_1^2\tilde{H}_1^2) + \alpha^T \Sigma^{-1} \alpha. 
\] Here $\tilde{H}_1 = 2E(H_2B_{12} \mid W_1) + [(H_2\tilde{w}_{31} + H_3\tilde{w}_{23})B_{23} \mid W_1]$ and $\alpha = E(H_1\eta_2B_{12})$.

This theorem implies that when the local alternatives converge to the null hypothesis at a slower rate $a_n = O(n^{-c})$ than $O(n^{-1/2})$ for $0 \leq c < 1/2$, $nT_n \to \infty$ in probability at the rate of $n^{1/2-c}$. Thus the test is consistent as its asymptotic power tends to 1. Further, the test $T_n$ can still detect the local alternatives that are distinct from the null hypothesis at a fastest possible rate $\sqrt{n}$. This is the typical feature existing global smoothing tests in the literature for regressions share.

**Remark 2.** From Theorem 1 we can see that its limiting null distribution is rather complicated in formula. It is partly because of the effect from the estimation of propensity score function. But it is interesting that the nonparametric estimation does not cause a slowdown of the convergence rate of $T_n$ to its weak limit.

4. Numerical studies

In this section we carry out two sets of simulation studies to evaluate the performance of the proposed test under different model settings. As the limiting null distribution of the proposed test is intractable, we use the wild bootstrap approximation to determine critical values. The procedure is given as follows:
(1) For a given random sample \( \{(Y_i, W_i, D_i) : i = 1, \cdots, n\} \), obtain \( \hat{e}_i = \hat{Y}_i^* - \hat{g}(X_i) \) with \( \hat{Y}_i^* = \left( \frac{D_i}{\pi(W_i, r)} - \frac{1 - D_i}{1 - \pi(W_i, r)} \right) Y_i \) and \( \hat{g}(X_i) = \frac{1}{(n-1) \sum_{j \neq i} w_{ij} \hat{Y}_j^*} \). Here \( \hat{r} \) is the maximum likelihood estimator of \( r_0 \) in \( \pi(W, r_0) \).

(2) Generate a bootstrap sample \( \{(\tilde{Y}_i^*, W_i, D_i) : i = 1, \cdots, n\} \) with the outcome variables as \( \tilde{Y}_i^* = \hat{g}(X_i) + \tilde{e}_i \). Here \( \tilde{e}_i = \zeta_i \hat{e}_i \) and \( \zeta_i \) are i.i.d. variables independent of the initial sample with \( E(\zeta_i) = 0 \) and \( E(\zeta_i^2) = E(\zeta_i^3) = 1 \).

(3) Obtain a bootstrapped statistic \( T_n \) based on the sample \( \{(\tilde{Y}_i^*, W_i, D_i) : i = 1, \cdots, n\} \). Repeat this scheme a large number of times, say, \( B \) times, the bootstrap critical value at a given level \( \alpha \) is the empirical \( (1 - \alpha) \)-th quantile of the bootstrapped distribution \( \{T_{n,j} : j = 1, \cdots, B\} \) of the test statistic.

Here we use the two point distribution proposed by Mammen (1993):

\[
P(\zeta_i = \frac{1 - \sqrt{5}}{2}) = \frac{5 + \sqrt{5}}{10}, \quad P(\zeta_i = \frac{1 + \sqrt{5}}{2}) = \frac{5 - \sqrt{5}}{10}
\]

All reported results are based on 1000 simulation runs with 500 bootstrap replications. The sample size \( n \) equals 100 and 200. In the following, we report the simulation results at the \( \alpha = 0.05 \) significance level. Without loss of generality, we only consider the case of \( X \in R \), i.e. \( p = 1 \) and two dimensions of \( Z \): \( q = 3, 7 \). As for nonparametric estimation of \( g(x) \), we use the kernel function \( K(u) = \frac{3}{4}(1 - u^2) \) if \( |u| \leq 1 \), and \( K(u) = 0 \) otherwise. Thus the order kernel function is \( s = 2 \).

**Study 1.** Consider the potential outcomes \((Y(1), Y(0))\) are discrete, taking both low-frequency and high-frequency alternative models into account. The specific data generating processes (DGPs) are as follows:

- **DGP 1:** \( Y(1) = I(Y(1)^* > 0) \) and \( Y(0) = 0 \) with \( Y(1)^* = X + a(\beta^T Z)^3 + \epsilon \).
- **DGP 2:** \( Y(1) = I(Y(1)^* > 0) \) and \( Y(0) = 0 \) with \( Y(1)^* = X + 2a \sin(\beta^T Z) + \epsilon \).

The observed outcome: \( Y = DY(1) + (1 - D)Y(0) \) with \( E(D \mid X) = \frac{\exp(\alpha^T W)}{1 + \exp(\alpha^T W)} \).
Here $\beta^T = (1, \cdots, 1)/\sqrt{q}$, $\alpha^T = (1, 1, \cdots, 1)/\sqrt{1+q}$. The observations $\{X_i\}_{i=1}^n \sim i.i.d.N(0, 1)$ and $\{Z_i\}_{i=1}^n$, which are independent of $\{X_i\}_{i=1}^n$, are independently generated from $N(0, \Sigma)$, $\Sigma = \{\sigma_{jj'}^2\}$, $\sigma_{jj'}^2 = 0.5|j-j'|$ for $1 \leq j, j' \leq q$, $i = 1, \cdots, n$. In the following studies, the errors $\{\epsilon_i\}_{i=1}^n$ are independently drawn from the standard normal distribution. Obviously, the null and alternative hypothesis respectively respond to $a = 0$ and $a \neq 0$. Further let $a \in \{0.2, 0.4, 0.6, 0.8, 1\}$. Based on DGP 1, consider the low-frequency alternative model, which is in favour of global smoothing tests. DGP 2 is a high-frequency model under the alternative, which is in favour of local smoothing tests.

Before carrying out simulation procedure, we first check the sensitivity of bandwidth selection and choose a reasonable bandwidth. The candidate bandwidths are set to equal $h_c \sigma n^{-1/4}$ for $h_c \in \{0.6, 0.8, \cdots, 1.6\}$ and $\sigma = sd(X)$. To save space, we only investigate the bandwidth impact on DGP 1 under $n = 200$ and $q = 3$. As shown in Table 1 the different bandwidths have little effect on empirical power, while the empirical size can be still under control when the bandwidth $h$ is not too large. Thus, we choose the bandwidth $h = \sigma n^{-1/4}$ to conduct the following simulation studies and will see that such a choice can also suitable for other models in the numericla studies.

Table 1: The empirical sizes and powers results vary with the bandwidth $h$ and $n = 200$, $q = 3$ for DGP 1.

| $h_c$ | 0.6 | 0.8 | 1  | 1.2 | 1.4 | 1.6 |
|------|-----|-----|----|-----|-----|-----|
| $a=0$ | 0.044 | 0.06 | 0.055 | 0.046 | 0.047 | 0.037 |
| $a=1$ | 1  | 1  | 1  | 0.999 | 1  | 1  |

Through the simulations results of Study 1 presented in Table 2 we have the following observations. First, the sample sizes reasonably have significant impact on the power performance of $T_n$: large size of sample results in high power, and empirical size of the test is also close to the significance level.
The proposed test is sensitive to both high and low frequency alternatives. It is worth mentioning that even for small $a = 0.2$, the empirical power of $T_n$ is high enough. To investigate the dimensionality effect of $Z$ on the test performance, we can see that when $q$ increases from 3 up to 7, the power performance of $T_n$ is slightly negatively affected, while the empirical size seems to be stable against this dimensionality increasing. Note that for such sample sizes, the total dimension 8 is already large as $8 \approx 200^{2/5} = 8.33$. Finally, we can see that the test $T_n$ is still powerful even when dealing with high frequency data. This would be because of the benefit from inheriting its original local smoothing feature.

| Table 2: Empirical sizes and powers of $T_n$ under study 1. |
|---|---|---|---|---|
| | DGP 1 | | DGP 2 | |
| | $p+q=4$ | $p+q=8$ | $p+q=4$ | $p+q=8$ |
| | $a$ | $n=100$ | $n=200$ | $n=100$ | $n=200$ | $n=100$ | $n=200$ | $n=100$ | $n=200$ |
| 0 | 0.048 | 0.046 | 0.045 | 0.056 | 0.049 | 0.041 | 0.06 | 0.049 |
| 0.2 | 0.589 | 0.906 | 0.631 | 0.921 | 0.243 | 0.336 | 0.209 | 0.267 |
| 0.4 | 0.829 | 0.996 | 0.844 | 0.992 | 0.578 | 0.806 | 0.468 | 0.654 |
| 0.6 | 0.929 | 1 | 0.887 | 1 | 0.849 | 0.938 | 0.725 | 0.827 |
| 0.8 | 0.967 | 1 | 0.947 | 1 | 0.933 | 0.978 | 0.82 | 0.886 |
| 1 | 0.985 | 1 | 0.952 | 1 | 0.946 | 0.979 | 0.906 | 0.89 |

**Study 2.** In this study, consider the potential outcomes $(Y(1), Y(0))$ are continuous. Both low-frequency and high-frequency alternative models are considered via the following data generating processes (DGPs):

*DG* *P* 3 : $Y(1) = 2X^2 - X + a(\beta^T Z)^3 + \epsilon$, and $Y(0) = 0$.

*DG* *P* 4 : $Y(1) = 2X^2 - X + 4a \sin(\beta^T Z) + \epsilon$, and $Y(0) = 0$.

The observed outcome : $Y = DY(1) + (1 - D)Y(0)$ with $E(D \mid X) = \frac{\exp(\alpha^T W)}{1 + \exp(\alpha^T W)}$.

Here $\beta^T = (1, \ldots, 1)/\sqrt{q}, \alpha^T = (1, 1, \ldots, 1)/\sqrt{1 + q}$. The observations $\{X_i\}_{i=1}^n$. 

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\( \{Z_i\}_{i=1}^{n} \) and \( \{\epsilon_i\}_{i=1}^{n} \) are generated as before. The values of \( a \) is used to control the deviation from the null hypothesis. DGP 3 and DGP 4 respectively reflect the low-frequency and high-frequency alternative model.

The simulation results are summarized in Table 3. Based on the results, we can get similar conclusions as those from Study 1. The main difference between Study 1 and Study 2 is the property of respondse. It seems that \( T_n \) performs slightly better when the responds are discrete.

Table 3: Empirical sizes and powers of \( T_n \) under study 2.

|                  | DGP 3 | DGP 3 | DGP 4 | DGP 4 |
|------------------|-------|-------|-------|-------|
|                  | p+q=4 | p+q=8 | p+q=4 | p+q=8 |
| \( a \)          |       |       |       |       |
| \( n=100 \)      |       |       |       |       |
| 0                | 0.048 | 0.047 | 0.044 | 0.045 |
| 0.2              | 0.492 | 0.651 | 0.571 | 0.802 |
| 0.4              | 0.79  | 0.914 | 0.808 | 0.925 |
| 0.6              | 0.86  | 0.96  | 0.853 | 0.934 |
| 0.8              | 0.887 | 0.968 | 0.88  | 0.93  |
| 1                | 0.914 | 0.971 | 0.87  | 0.93  |
| \( n=200 \)      |       |       |       |       |
| 0                | 0.044 | 0.045 | 0.052 | 0.043 |
| 0.2              | 0.571 | 0.802 | 0.317 | 0.395 |
| 0.4              | 0.808 | 0.925 | 0.637 | 0.767 |
| 0.6              | 0.853 | 0.934 | 0.835 | 0.924 |
| 0.8              | 0.88  | 0.93  | 0.931 | 0.969 |
| 1                | 0.971 | 0.93  | 0.956 | 0.974 |

5. A real data example

In this section, we consider a data set from AIDS Clinical Trials Group Protocol 175 (ACTG175), which can be obtained from the R package spEff2trial, to illustrate the usefulness of the proposed test. This data set was popularly analyzed in the literature related to treatment effects, such as [Hammer et al. (1996)], [Zhang et al. (2008)], and [Lu et al. (2013)]. There are 2139 HIV-infected subjects in ACTG175 that were randomized to four different treatment groups with equal probability: zidovudine (ZDV) monotherapy, ZDV+didanosine (ddI), ZDV+zalcitabine, and ddI monotherapy. To get more elaborated results, we only consider the subjects under ZDV+didanosine (ddI) and ddI monotherapy groups (1083 subjects) in the following analysis.
Hence the treatment indicator $D$ is a dummy variable such that $D_i = 1$ when the $i$-th subject receives ZDV+ddI treatment and $D_i = 0$ stands for the subject in ddI monotherapy group. The continuous response $Y$ is CD4 count at $20 \pm 5$ weeks post-baseline. Besides, we also consider 12 baseline covariates acted as $W$, including age, weight, Karnofsky score, CD4 count at baseline (CD40) and CD8 count at baseline, hemophilia, homosexual activity, history of intravenous drug use, race, gender, antiretroviral history and symptomatic status, which are also considered in Zhang et al. (2008) and Lu et al. (2013) and more details can be found in their analyses.

The goal of our study is to check whether the average treatment effect conditional on whole $W$ equals the one conditional on $X$, a subset of $W$, i.e.

$$H_0 : P(E[Y(1) - Y(0) | W] = E[Y(1) - Y(0) | X]) = 1.$$  \hspace{1cm} (1)

Lu et al. (2013) pointed out that both age and CD40 are important variables in the contract function $E(Y(1) - Y(0) | W)$. We then check whether each individual age or CD40 is sufficient or whether they are jointly important. Thus, there are three candidates of $X$ being considered and the corresponding null hypotheses are as follows:

- $H^1_0 : P(E[Y(1) - Y(0) | W] = E[Y(1) - Y(0) | X]) = 1$ with $X = \text{age}$;
- $H^2_0 : P(E[Y(1) - Y(0) | W] = E[Y(1) - Y(0) | X]) = 1$ with $X = \text{CD40}$;
- $H^3_0 : P(E[Y(1) - Y(0) | W] = E[Y(1) - Y(0) | X]) = 1$ with $X = \text{(age, CD40)}$.

Similarly as in the simulation studies, we use the Epanechnikov kernel and then the kernels of order $s = 2p$ are derived from it. The bandwidth parameter $h = \frac{\text{tr}(\Sigma_x^{1/2})}{p} n^{-s/2}$ with $\Sigma_x = \text{Var}(X)$ and $\text{tr}(A)$ being the trace of matrix $A$. Also note that the propensity score $\pi(W) \equiv 0.5$. Hence we can get the $p$-values based on (6) and aforementioned wild bootstrap approximation procedure with 500 bootstrap replications. The $p$-values of these three tests are respectively 0.06, 0.082 and 0.28 respectively. Although if we only consider the significance
level 0.05, all three null hypotheses cannot be rejected, the test for $H_0$ provides clearer information on the plausibility that a heterogenous average treatment effect would be on both $X = (age, CD40)$. Thus, we may not use only either one to model the heterogenous average treatment effect to avoid the model misspecification risk. This analysis then provides a formal assessment for the model [Lu et al. (2013)] considered.

6. Conclusion

In this study, we consider the testing problem for conditional average treatment effect to explore whether the equivalence relationship between $E(Y(1) - Y(0) \mid W)$ and $E(Y(1) - Y(0) \mid X)$ holds with $X \subset W$. The proposed test has three useful features. That is, it reduces the risk of model misspecification, mitigates the curse of dimensionality through a projection-based construction procedure and achieves the fastest possible rate $n^{-1/2}$ to have the sensitivity to local alternatives. These features make the test well perform in practice. But this test is not suitable in very large dimension scenarios in the sense that the dimension of covariates is regarded as divergent to infinity. This important research is ongoing.

Appendix

A.1. Preliminary for the proofs.

Before we present the proof, we first define some related quantities.

1. $Y_j^* = \frac{D_j}{\pi(X_j, r_0)} Y_j - \frac{(1-D_j)}{\hat{\pi}(X_j, r_0)} = V_j(r_0) Y_j$ with $V_j(r_0) = \frac{D_j}{\pi(X_j, r_0)} - \frac{1-D_j}{1-\pi(X_j, r_0)}$;
2. $e_j = Y_j^* - g(X_j)$ and $u_j = Y_j^* - E(Y_j^* \mid W_j)$ with $g(X_j) = E(Y_j^* \mid X_j)$;
3. $w_{ij} = \frac{1}{(n-1)} K_h (X_j - X_i) / \hat{f}(X_i)$, $\tilde{w}_{ij} = K_h (X_j - X_i) / \hat{f}(X_i)$ and $\bar{w}_{ij} = K_h (X_j - X_i) / f(X_i)$ with $\hat{f}(X_i) = \frac{1}{(n-1)} \sum_{j \neq i} K_h (X_j - X_i)$ and $K_h(\cdot) = \frac{1}{h} K\left(\frac{\cdot}{h}\right)$;
4. $\hat{e}_j = \hat{Y}_j^* - \hat{g}(X_j)$ with $\hat{Y}_j^* = V_j(\hat{r}) Y_j$ and $\hat{g}(X_j) = \sum_{i \neq j} w_{ij} \hat{Y}_i^*$;
(5) $\hat{e}_j = Y_j^* - \hat{g}(X_j)$ with $Y_j^* = V_j(r_0)Y_j$ and $\hat{g}(X_j) = \sum_{i \neq j} w_{ji}Y_i^*$.

(6) $\eta_j = y_j - G_j$ with $y_j = \nabla V_j(r_0)^T Y_j$ and $G_j = G(X_j) = E(\nabla V_j | X_j)$. Here $\nabla V_j(r_0)$ stands for the Partial derivative with respect to $r$.

(7) $C$ stands for a generic bounded constant and $\Omega$ is the support of $X$.

**Proof of Theorem**

Note that under the null hypothesis $e_i = u_i$ such that $E(u_i | W_i) = 0$. Recall that $\hat{e}_j = \hat{Y}_j^* - \hat{g}(X_j)$ and $\hat{e}_j = Y_j^* - \hat{g}(X_j)$. Hence $T_n$ can be decomposed as

$$
T_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \hat{e}_i \hat{e}_j B_{ij} 
$$

$$
= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i u_j B_{ij} + \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i (\hat{e}_j - u_j) B_{ij} + \frac{1}{n(n-1)} \sum_{i=1}^{n} (\hat{e}_i - u_i) (\hat{e}_j - u_j) B_{ij} 
$$

$$
+ \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i (\hat{e}_j - \hat{e}_i) B_{ij} + \frac{2}{n(n-1)} \sum_{i=1}^{n} (\hat{e}_i - \hat{e}_j) (\hat{e}_j - u_j) B_{ij} 
$$

$$
+ \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\hat{e}_i - \hat{e}_j) (\hat{e}_j - \hat{e}_i) B_{ij} 
$$

$$
:= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i u_j B_{ij} + I_{n1} + \cdots + I_{n5}.
$$

We then deal with $I_{ni}$ for $1 \leq i \leq 5$ in Propositions 1 $\sim$ 5 separately. The proofs of these propositions will be given later.

**Proposition 1.** $I_{n1} = -\frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i u_j \tilde{B}_{ij} - \frac{1}{n} E \left[ \left( u_i^2 \tilde{w}_{12} + u_j^2 \tilde{w}_{21} \right) B_{12} \right] + o_p \left( \frac{1}{n} \right)$ with $\tilde{B}_{ij} = \frac{1}{n} E \left[ \tilde{w}_{ij} B_{ij} + \tilde{w}_{ij} B_{ij} | W_i, W_j \right]$.

**Proposition 2.** $I_{n2} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i u_j \tilde{B}_{ij} + \frac{1}{n} E \left[ u_i^2 \tilde{w}_{21} \tilde{w}_{31} B_{23} + u_j^2 \tilde{w}_{12} \tilde{w}_{32} B_{13} + u_i^2 \tilde{w}_{12} \tilde{w}_{23} B_{12} \right] + o_p \left( \frac{1}{n} \right)$ with $\tilde{B}_{ij} = \frac{1}{n} E \left[ \tilde{w}_{ij} \tilde{w}_{kj} B_{ik} + \tilde{w}_{ki} \tilde{w}_{ij} B_{ik} | W_i, W_j \right]$.

**Proposition 3.** $I_{n3} = (\hat{r} - r_0)^T \frac{2}{n} \sum_{i=1}^{n} u_i H_3(W_i) + o_p \left( \frac{1}{n} \right)$ with $H_3(W_i) = E(\eta_j B_{ij} | W_i)$.

**Proposition 4.** $I_{n4} = - (\hat{r} - r_0)^T \frac{2}{n} \sum_{i=1}^{n} u_i H_4(W_i) + o_p \left( \frac{1}{n} \right)$ with $H_4(W_i) = \frac{1}{n} E \left[ (\tilde{w}_{ij} \tilde{w}_{kj} + \eta_i \tilde{w}_{ij}) B_{ij} | W_i \right]$.

**Proposition 5.** $I_{n5} = (\hat{r} - r_0)^T A (\hat{r} - r_0) + o_p \left( \frac{1}{n} \right)$ with $A = E(\eta_j \eta_j \tilde{B}_{12})$.

Based on these propositions, by the results about the standard first-order degenerate U-statistic in Serfling (1980), the proof is then finished.
**Proof of Theorem 2.** Under the alternative hypothesis $H_{n1}$, $T_n$ can also be decomposed as

$$T_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \hat{e}_i \hat{e}_j B_{ij}$$

$$= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} e_i e_j B_{ij} + \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} e_i (\hat{e}_j - e_j) B_{ij} + \frac{1}{n(n-1)} \sum_{i=1}^{n} (\hat{e}_i - e_i) (\hat{e}_j - e_j) B_{ij}$$

$$+ \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} e_i (\hat{e}_j - e_j) B_{ij} + \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\hat{e}_i - e_i) (\hat{e}_j - e_j) B_{ij}$$

$$+ \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\hat{e}_i - e_i)(\hat{e}_j - e_j) B_{ij}$$

$$:= J_{n0} + J_{n1} + \cdots + J_{n5}.$$  

Similarly as those for $J_{ni}$ for $1 \leq i \leq 5$, we will prove the following Propositions 6 ~ 11 under Assumptions 1 ~ 9.

**Proposition 6.** $J_{n0} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i u_j \hat{B}_{ij} + 2a_n \frac{1}{n} \sum_{i=1}^{n} u_i \hat{H}_{00}(W_i) + a_n^2 \mu_0$ with $H_{00}(W_i) = E(H_j B_{ij} | W_i)$ and $\mu_0 = E(H_1 H_2 B_{12}).$

**Proposition 7.** $J_{n1} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i u_j \hat{B}_{ij} - \frac{1}{n} E[(u_i^2 \hat{w}_{12} + u_j^2 \hat{w}_{21}) B_{12}] + o_p \left( \frac{1}{n} \right) - 2a_n \frac{1}{n} \sum_{i=1}^{n} u_i H_{10}(W_i) + o_p \left( a_n^2 \right)$ with $H_{10}(W_i) = \frac{1}{n} E[(H_j \hat{w}_{i1} + H_i \hat{w}_{j1}) B_{ij} | W_i].$

**Proposition 8.** $J_{n2} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i u_j \hat{B}_{ij} + \frac{1}{3n} E(u_i^2 \hat{w}_{21} \hat{w}_{31} B_{23} + u_j^2 \hat{w}_{12} \hat{w}_{32} B_{13} + u_i^2 \hat{w}_{12} \hat{w}_{23} B_{12}) + o_p \left( \frac{1}{n} \right) + o_p \left( \frac{a_n}{\sqrt{n}} \right) + o_p \left( a_n^2 \right).$

**Proposition 9.** $J_{n3} = (\hat{r} - r_0)^\top \frac{2}{n} \sum_{i=1}^{n} u_i H_3(W_i) + a_n (\hat{r} - r_0)^\top \alpha + o_p \left( \frac{a_n}{\sqrt{n}} \right)$ with $\alpha = E(H_1 \eta_2 B_{12}).$

**Proposition 10.** $J_{n4} = -(\hat{r} - r_0)^\top \frac{2}{n} \sum_{i=1}^{n} u_i H_4(W_i) + o_p \left( \frac{1}{n} \right) + o_p \left( \frac{1}{\sqrt{n}} \right) + o_p \left( \frac{a_n}{\sqrt{n}} \right).$

**Proposition 11.** $J_{n5} = (\hat{r} - r_0)^\top A(\hat{r} - r_0) + o_p \left( \frac{1}{n} \right).$

Based on these propositions, the conclusion in Theorem 1 and the central limit theorem, we can prove Theorem 2.

□
Proof of Proposition 1. Recall that under the null hypothesis, \( Y_j^* = g(X_j) + u_j \) with \( E(u_j \mid W_j) = 0 \), it follows that

\[
\hat{e}_j - u_j = g(X_j) - \sum_{t \neq j} w_{jt}g(X_t) - \sum_{t \neq j} w_{jt}u_t. \tag{A.1}
\]

Thus we can decompose \( I_{n1} \) as

\[
I_{n1} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i(\hat{e}_j - u_j)B_{ij}
= \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i \left( g(X_j) - \sum_{t \neq j} w_{jt}g(X_t) \right)B_{ij} - \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{t \neq j} u_i w_{jt}u_tB_{ij}
= 2I_{n11} - 2I_{n12}.
\]

Consider the term \( I_{n11} \) first. Without loss of generality, here we consider \( s = 2 \) for ease of exposition. Since \( K \) is a kernel of order \( s = 2 \), by standard non-parametric theory in literature, e.g. [Härdle et al. (2012)], the following formula holds uniformly:

\[
g(X_j) - \sum_{t \neq j} w_{jt}g(X_t) = h^2K_2 \sum_{i=1}^{p} 2\frac{\partial^2g(X_i)}{\partial X_{ji}^2} \frac{\partial f(X_i)}{\partial X_{ji}} + f(X_j) \frac{\partial^2 g(X_i)}{\partial X_{ji} \partial X_{ji}} + o_p(h^2) := h^2F_j + o_p(h^2).
\]

Here \( K_2I_p = \int uu'K(u)du \) with \( I_p \) being a \( p \times p \) identity matrix. Further by \( E(u_i \mid W_i) = 0 \), it follows that

\[
E(I_{n11}^2) = \frac{1}{n^2(n-1)^2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{i' \neq i} \sum_{j' \neq j'} E \left[ \sum_{t \neq j} u_i u_{i'} \left( g(X_j) - \sum_{t \neq j} w_{jt}g(X_t) \right) \times \left( g(X_{j'}) - \sum_{t \neq j'} w_{j't}g(X_{t'}) \right) B_{ij}B_{i'j'} \right]
= \sum_{i=1}^{n} \sum_{j \neq i} \sum_{j' \neq j} E \left[ u_i^2 F_j B_{ij}B_{ij'}' \right] + R_n = o_p \left( \frac{h^4}{n} \right).
\]

Thus we have \( I_{n11} = O_p \left( \frac{h^2}{\sqrt{n}} \right) \). By the assumption \( nh^4 \rightarrow 0 \), \( I_{n11} = o_p \left( \frac{1}{n} \right) \).

When \( s > 2 \), we can similarly get \( I_{n11} = O_p \left( \frac{h^s}{\sqrt{n}} \right) = o_p \left( \frac{1}{n} \right) \).

As for \( I_{n12} \), we can rewrite it as

\[
I_{n12} = \frac{1}{n(n-1)^2} \sum_{i=1}^{n} \sum_{j \neq i} u_i^2 \tilde{w}_{ji}B_{ij} + \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{t \neq j,i} u_i w_{jt}u_tB_{ij}
= I_{n121} + I_{n122}.
\]

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Given Assumptions 5 and 6, \( E|u_i^2 w_{ij} B_{ij}| \leq E|u_i^2| < \infty \) and noting that

\[
\frac{1}{f(X_i)} = \frac{1}{f(X_i)} + O_p(\hat{f}(X_i) - f(X_i)) = \frac{1}{f(X_i)} + o_p(1),
\]

(A.2)

employing the standard U-statistic theory and the results about nonparametric estimation, we have \( (n - 1)I_{n121} - \frac{1}{2}E[((u_i^2 \bar{w}_{12} + u_2^2 \bar{w}_{21})B_{12})| = o_p(1) \) with \( \bar{w}_{12} = \frac{\kappa_2(X_1 - X_2)}{f(X_1)} \). Thus \( I_{n121} = \frac{1}{2n} E[(u_i^2 \bar{w}_{12} + u_2^2 \bar{w}_{21})B_{12}] + o_p(\frac{1}{n}). \)

Consider \( I_{n122} \). Write it as

\[
I_{n122} = \frac{n - 2}{n - 1} U_n
\]

where

\[
U_n = \frac{1}{C_n^3} \sum_{1 \leq i < j < t \leq n} I^*(\chi_i, \chi_j, \chi_t).
\]

Here \( U_n \) is an U-statistic of order 3 and \( \chi_i = (W_i, u_i), I^*(\chi_i, \chi_j, \chi_t) = (I_{ijt} + I_{itj} + I_{jti} + I_{tij} + I_{tji})/6 \) is the kernel with \( I_{ijt} = u_i \bar{w}_{jt} u_t B_{ij} \) and \( \bar{w}_{jt} = \frac{\kappa_2(X_j - X_t)}{f(X_j)}. \)

Note that \( E(u_i | W_1, \cdots, W_n) = 0 \). It follows that \( E(I^*(\chi_i, \chi_j, \chi_t) | \chi_i) = 0 \) and \( E(I^*(\chi_i, \chi_j, \chi_t) | \chi_i, \chi_j) = \frac{1}{6} u_i u_j E(\bar{w}_{it} B_{jt} + \bar{w}_{jt} B_{it} | W_i, W_j) \neq 0 \). Thus \( U_n \) is a degenerate U-statistic. Given assumptions 4 and 5, we can also observe that \( E(u_i^2 u_j^2 \bar{w}_{ij}^2 B_{ij}^2) \leq CE(u_i^2 u_j^2) < \infty \). Therefore, by using the results collected in Serfling (1980) and (A.2), we have

\[
U_n = \frac{2}{n(n - 1)} \sum_{1 \leq i < j \leq n} u_i u_j \bar{B}_{ij} + o_p(\frac{1}{n}) = \frac{1}{n(n - 1)} \sum_{i = 1}^n \sum_{j \neq i} u_i u_j \bar{B}_{ij} + o_p(\frac{1}{n})
\]

Here \( \bar{B}_{ij} = \frac{1}{2} E(\bar{w}_{it} B_{jt} + \bar{w}_{jt} B_{it} | W_i, W_j) \). Taking all the asymptotic results about the terms in the decomposition of \( I_{n1} \) into account, proposition 1 is proved.

\[\square\]
Proof of Proposition 2. Based on (A.1), $I_{n2}$ can be decomposed as

$$I_{n2} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\hat{e}_i - u_i)(\hat{e}_j - u_j) B_{ij}$$

$$= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (g(X_i) - \sum_{t \neq i} w_{it} g(X_t)) \{g(X_j) - \sum_{k \neq j} w_{jk} g(X_k)\} B_{ij}$$

$$- 2 \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i \neq k} \sum_{t \neq j} w_{it} g(X_i) \{w_{jk} u_k B_{ij}\}$$

$$+ \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i \neq k} \sum_{t \neq j} w_{it} u_t w_{jk} u_k B_{ij}$$

$$:= I_{n21} - 2 I_{n22} + I_{n23}.$$ 

As proved above, with $g(X_j) - \sum_{t \neq j} w_{jt} g(X_t) = O_p(h^s)$, we can get $I_{n21} = O_p(h^{2s})$. Further, similarly as the proof for $I_{n11}$, we can show $I_{n22} = O_p(\frac{h^4}{\sqrt{n}}) = o_p\left(\frac{1}{n}\right)$.

For the term $I_{n23}$, we also have the decomposition as

$$I_{n23} = \frac{1}{n(n-1)^3} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{t \neq i,j} \hat{w}_{it} \hat{w}_{jt} u_t^2 B_{ij} + \frac{1}{n(n-1)^3} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{t \neq i,j} \sum_{k \neq j,i,j,t} \hat{w}_{it} \hat{w}_{jk} u_t u_k B_{ij} + R_n$$

$$:= I_{n231} + I_{n232} + R_n.$$ 

Since $E(u_i | W_1, \ldots, W_n) = 0$, the leading term of $I_{n23}$ is $I_{n231} + I_{n232}$ and $R_n$ is a higher order term with $E(R_n) = 0$. In the following, we focus on investigating the asymptotic behaviour of $I_{n231} + I_{n232}$.

Rewrite $I_{n231}$ as a U-statistic of order 3: $I_{n231} = \frac{n-3}{(n-1)^3} V_n$ with

$$V_n = \frac{1}{C_n^3} \sum_{1 \leq i < j < t \leq n} \text{V}^*(\chi_i, \chi_j, \chi_t).$$

Here $V_n$ is an U-statistic of order 3 and $\chi_i = (W_i, u_i)$, $\text{V}^*(\chi_i, \chi_j, \chi_t) = (V_{ijt} + V_{itj} + I_{jiti})/3$ is the kernel with $V_{ijt} = \hat{w}_{jt} \hat{w}_{jt} u_t^2 B_{ij}$. 

Notably, $E|\hat{w}_{jt} \hat{w}_{jt} u_t^2 B_{ij}| \leq CE(u_t^2) < \infty$, and thus employing the U-statistic theory in [Serfling (1984)], we can obtain that $V_n \xrightarrow{p} \mu_v$ with $\mu_v = E(V^*(\chi_i, \chi_j, \chi_t))$. That implies $I_{n231} = \frac{1}{n} \mu_v + o_p\left(\frac{1}{n}\right)$.

For $I_{n232}$, we can also write it as $I_{n232} = \frac{(n-2)(n-3)}{(n-1)^2} V_n^*$ with

$$V_n^* = \frac{1}{C_n^3} \sum_{1 \leq i < j < t < k \leq n} \text{V}^{**}(\chi_i, \chi_j, \chi_t, \chi_k).$$
Here $V^*_n$ is an U-statistic of order 4 and $V^{**}(\chi_i, \chi_j, \chi_t, \chi_k) = \frac{1}{n^2} \sum_{(a)} V^*(\chi_{i1}, \chi_{i2}, \chi_{33}, \chi_{14})$ is a symmetric kernel where $\sum_a$ denotes summation over the 4! permutations $(i_1, \cdots, i_4)$ of $(i, j, t, k)$.

Given assumptions 4 and 5, on the analogy of investigating $I_{n122}$, we can obtain $V^*_n$ is a degenerate U-statistic and

$$V^*_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} u_i u_j B_{ij} + o_p \left( \frac{1}{n} \right) = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i}^{n} u_i u_j B_{ij} + o_p \left( \frac{1}{n} \right)$$

Here $B_{ij} = \frac{1}{n} E \left( \bar{w}_{i\bar{w}} \bar{B}_{ik} + \bar{w}_{k\bar{w}} \bar{B}_{kt} | W_i, W_j \right)$. Altogether, we can conclude the proof of Proposition 2.

\[ \square \]

**Proof of Proposition 3.** Note that \( \hat{e}_j - \bar{e}_j = (V_j(\hat{r}) - V_j(r_0))Y_j - \sum_{t \neq j} w_{jt}(V_t(\hat{r}) - V_t(r_0))Y_t \). Let $\overline{\eta}_j = \nabla V_j(r_0)^T Y_j \), $G_j = G(X_j) = E(\overline{\eta}_j | X_j)$, $\eta_j = \overline{\eta}_j - G_j$ with $E(\eta_j \mid W_i) \neq 0$.

By the Taylor expansion around $r_0$ and $\hat{r} - r_0 = O_p(n^{-1/2})$, it follows that

$$I_{n3} = (\hat{r} - r_0)^T \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{i \neq j}^{n} u_i (\overline{\eta}_j - \sum_{t \neq j}^{n} w_{jt} \overline{\eta}_t) B_{ij} + R_n$$

$$= (\hat{r} - r_0)^T \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{i \neq j}^{n} u_i (G_j - \sum_{t \neq j}^{n} w_{jt} G_t) B_{ij} + (\hat{r} - r_0)^T \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i}^{n} u_i \eta_j B_{ij}$$

$$- (\hat{r} - r_0)^T \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i}^{n} \sum_{t \neq j}^{n} u_i w_{jt} \eta_j B_{ij} + R_n$$

$$:= 2(\hat{r} - r_0)^T I_{n31} + 2(\hat{r} - r_0)^T I_{n32} - 2(\hat{r} - r_0)^T I_{n33} + R_n.$$

Here $R_n = o_p(I_{n31} + I_{n32} - I_{n33})$ is a higher order term.

For the term $I_{n31}$, we can show that

$$E(I_{n31}^2) = \frac{1}{n^2(n-1)^2} \sum_{i,j \neq i}^{n} \sum_{i,j' \neq i}^{n} E(u_i u'_i (G_j - \sum_{t \neq j}^{n} w_{jt} G_t) (G_{j'} - \sum_{t' \neq j'}^{n} w_{j't'} G_{t'})^T B_{ij} B_{ij'}).$$

Noting that \{W_i\}_{i=1}^{n} are i.i.d. samples, $E(u_i \mid W_i) = 0$, thus only the terms $i = i'$ have non-zero expectations, therefore,

$$E(I_{n31}^2) = \frac{1}{n^2(n-1)^2} \sum_{i,j \neq i}^{n} \sum_{j' \neq j}^{n} E(u_i^2 (G_j - \sum_{t \neq j}^{n} w_{jt} G_t) (G_{j'} - \sum_{t' \neq j'}^{n} w_{j't'} G_{t'}) B_{ij} B_{ij'}).$$
Further, note that $G_j = \sum_{t \neq j} w_{jt} G_t$ is also a bias term. Hence similarly as the proof for $I_{n11}$, we have $E(I_{n31}^2) = O(\frac{k^2}{n}) = o_p \left( \frac{1}{n} \right)$. That implies $(\hat{r} - r_0)^\top I_{n31} = o_p \left( \frac{1}{n} \right)$.

As for $I_{n33}$, we can rewrite it as

$$I_{n33} = \frac{1}{n(n-1)^2} \sum_{i=1}^{n} \sum_{j \neq i} u_i \eta_j \tilde{w}_{ij} B_{ij} + \frac{1}{n(n-1)^2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{i' \neq i} u_i \tilde{w}_{ij} \eta_i B_{ij}$$

$$:= I_{n331} + I_{n332}.$$

Similarly as the discussion on $I_{n231}$, we can show that $I_{n331} = \frac{1}{n} \mu_{331} + o_p \left( \frac{1}{n} \right)$ with $\mu_{331} = \frac{1}{n} E(u_i \eta_j \tilde{w}_{ij} B_{ij} + u_j \eta_j \tilde{w}_{ij} B_{ij})$ and then $(\hat{r} - r_0)^\top I_{n331} = o_p \left( \frac{1}{n} \right)$. It is easy to see that

$$E(I_{n332}^2) = \frac{1}{n^2(n-1)^4} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{i' \neq i} \sum_{j' \neq i'} E(u_i u_{i'} \tilde{w}_{ij} \tilde{w}_{ij'} \eta_i \eta_i B_{ij} B_{ij'}).$$

Since $E(\tilde{w}_{ij} \eta_i | W_1, \ldots, W_n) = 0$, only the terms $i = i'$, $t = t'$ have, as discussed before, non-zero expectation, thus

$$E(I_{n332}^2) = \frac{1}{n^2(n-1)^4} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{i' \neq i} \sum_{j' \neq i'} E(u_i^2 \tilde{w}_{ij} \tilde{w}_{ij'} \eta_i^2 \eta_i B_{ij} B_{ij'}).$$

Hence we can obtain $E(I_{n332}^2) = O(\frac{1}{n^2}) = O(\frac{1}{n^2})$. Combining that $\hat{r} - r = O_p(\frac{n^{-1/2}}{\sqrt{n}})$, $(\hat{r} - r_0)^\top I_{n332} = o_p \left( \frac{1}{n^2} \right)$.

As for $I_{n32}$, we can also rewrite it in the form of U-statistic:

$$I_{n32} = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} h_3(\chi_i, \chi_j).$$

Here $h_3(\chi_i, \chi_j) = \frac{1}{2}(u_i \eta_j + u_j \eta_i) B_{ij}$ with $\chi_i = (W_i, u_i, D_i)$. Note that $E(h_3(\chi_i, \chi_j) | \chi_i) = \frac{1}{2} u_i H_3(W_i) \neq 0$ with $H_3(W_i) = E(\eta_i B_{ij} | W_i)$. Obviously, $\xi_1 = E(u_i^2 H_3(W_i)^2) > 0$, thus $I_{n32}$ is not degenerate. Further, we have $E(h_3(\chi_i, \chi_j)^2) \leq CE(u_i^2 \eta_i^2)$.

Then by Holder’s inequality and assumption 5, we have $E(u_i^4 \eta_i^2) \leq (E(u_i^4))^{1/2} (E(\eta_i^2))^{1/2} < \infty$. Again U-statistic theory in [Serfling, 1980] leads to an asymptotically linear representation of $I_{n32}$ as

$$I_{n32} = \frac{2}{n} \sum_{i=1}^{n} u_i H_3(W_i) + o_p \left( \frac{1}{\sqrt{n}} \right).$$
That implies the higher order term $R_n = o_p \left( \frac{1}{n} \right)$. Altogether, Proposition 3 is proved.

\[ \square \]

**Proof of Proposition 4.** As for $I_n4$, we can decompose it as

\[
I_n4 = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\hat{c}_i - \hat{c}_j) \{g(X_j) - \sum_{t \neq j} w_{jt} Y^*_t \} B_{ij}
\]

\[
= (\hat{r} - r_0) \top \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \{\overline{Y}_i - \sum_{k \neq i} w_{ik} \overline{Y}_k \} \{g(X_j) - \sum_{t \neq j} w_{jt} Y^*_t \} B_{ij} + R_n.
\]

Here $R_n$ is a higher order term. Further, we can decompose $\overline{Y}_i - \sum_{k \neq i} w_{ik} \overline{Y}_k = \eta_i + G_i - \sum_{k \neq i} w_{ik} \bar{Y}_k$, then $I_n4$ can be rewritten as

\[
I_n4 = (\hat{r} - r_0) \top \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \{G_i - \sum_{k \neq i} w_{ik} \overline{Y}_k \} \{g(X_j) - \sum_{t \neq j} w_{jt} Y^*_t \} B_{ij}
\]

\[
+(\hat{r} - r_0) \top \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \eta_i \{g(X_j) - \sum_{t \neq j} w_{jt} Y^*_t \} B_{ij} + R_n
\]

\[
:= 2(\hat{r} - r_0) \top I_{n41} + 2(\hat{r} - r_0) \top I_{n42} + R_n.
\]

By the standard results in kernel estimation, see e.g. Abrevaya et al. (2015), we have

\[
\sup_{X_i \in \Omega} |G_i - \sum_{k \neq i} w_{ik} \overline{Y}_k| = O_p \left( \frac{\log n}{nh^p} \right), \quad (A.5)
\]

\[
\sup_{X_j \in \Omega} |g(X_j) - \sum_{t \neq j} w_{jt} Y^*_t| = O_p \left( \frac{\log n}{nh^p} \right). \quad (A.6)
\]

Thus we can derive that

\[
|I_{n41}| \leq \sup_{X_i \in \Omega} |G_i - \sum_{k \neq i} w_{ik} \overline{Y}_k| \sup_{X_j \in \Omega} \sum_{t \neq j} \sum_{i=1}^{n} \sum_{j \neq i} |B_{ij}|.
\]

Since $|B_{ij}| \leq 1$, then under assumption 7, it follows that $I_{n41} = o_p \left( \frac{1}{\sqrt{n}} \right)$. So that $(\hat{r} - r_0) \top I_{n41} = o_p \left( \frac{1}{\sqrt{n}} \right)$.

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As for $I_{n42}$, note that $Y_t^* = g(X_t) + u_t$ and then

$$I_{n42} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \eta_i \{g(X_j) - \sum_{t \neq i} w_{jt} g(X_t)\} B_{ij} - \frac{1}{n(n-1)^2} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{t \neq j, i} \eta_i \tilde{w}_{jt} u_t B_{ij}$$

$$- \frac{1}{n(n-1)^2} \sum_{i=1}^{n} \sum_{j \neq i} \eta_i u_t \tilde{w}_{jt} B_{ij}$$

$$:= I_{n421} - I_{n422} - I_{n423}.$$

As the bias term $\{g(X_j) - \sum_{t \neq j} w_{jt} g(X_t)\}$ is $O_p(h^*)$, under assumption 7, we can easily obtain that $I_{n421} = O_p(h^*) = o_p\left(\frac{1}{\sqrt{n}}\right)$. Thus we have $(\hat{r} - r_0)I_{n421} = o_p\left(\frac{1}{n}\right)$. Further, $I_{n423} = O_p\left(\frac{1}{n}\right) = o_p\left(\frac{1}{\sqrt{n}}\right)$ is obviously derived. Thus we have $(\hat{r} - r_0)I_{n423} = o_p\left(\frac{1}{n}\right)$ as well.

Next, we deal with $I_{n422}$. As the arguments are very similar, we then only give an outline. Rewrite it as an $U$-statistic $I_{n422} = \frac{n-2}{n-1} V_n^*$ with

$$V_n^* = \frac{1}{C_n^3} \sum_{1 \leq i < j < t \leq n} V^{**}(\hat{\chi}_i, \hat{\chi}_j, \hat{\chi}_t)$$  \hspace{1cm} (A.7)

Here $U_n$ is an $U$-statistic of order 3 and $\hat{\chi}_i = (W_i, u_i, \eta_i)$, $V^{**}(\hat{\chi}_i, \hat{\chi}_j, \hat{\chi}_t) = (V_{ijt}^* + V_{ij}^* + V_{it}^* + V_{tij}^* + V_{tji}^*)/6$ is the kernel with $V_{ijt}^* = \eta_i \tilde{w}_{jt} u_t B_{ij}$. Note that $V_{ij}^*(\hat{\chi}_i) = E(V^{**}(\hat{\chi}_i, \hat{\chi}_j, \hat{\chi}_t) \mid \hat{\chi}_i) = \frac{1}{2} u_i H_4(W_i) \neq 0$ with $H_4(W_i) = \frac{1}{2} E[(\eta_j \tilde{w}_{it} + \eta_i \tilde{w}_{jt}) B_{ij} \mid \hat{\chi}_i]$. We can have

$$V_n^* = \frac{1}{n} \sum_{i=1}^{n} u_i H_4(W_i) + o_p\left(\frac{1}{\sqrt{n}}\right).$$  \hspace{1cm} (A.8)

Similarly $I_{423} = O_p\left(\frac{1}{\sqrt{n}}\right) = o_p\left(\frac{1}{\sqrt{n}}\right)$. Altogether, Proposition 4 is proved.

$\square$

Proof of Proposition 5.

Recall that $\hat{e}_j - \tilde{e}_j = (V_j(\hat{r}) - V_j(r_0))Y_j - \sum_{t \neq j} w_{jt} (V_t(\hat{r}) - V_t(r_0)) Y_t$ and
Thus we can rewrite \( \bar{Y}_i - \sum_{k \neq i} w_{ik} \bar{Y}_k = \eta_i + G_i - \sum_{k \neq i} w_{ik} \bar{Y}_k \). Again decompose \( I_{n5} \) as:

\[
I_{n5} = (\hat{r} - r_0)^{T} \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\bar{Y}_i - \sum_{t \neq i} w_{it} \bar{Y}_t) (\bar{Y}_j - \sum_{k \neq j} w_{jk} \bar{Y}_k) \right)^{T} (\hat{r} - r_0) B_{ij} + R_n
\]

\[
= (\hat{r} - r_0)^{T} \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (G_i - \sum_{t \neq i} w_{it} \bar{Y}_t) (G_j - \sum_{k \neq j} w_{jk} \bar{Y}_k) \right)^{T} B_{ij} (\hat{r} - r_0)
\]

\[
+ (\hat{r} - r_0)^{T} \left( \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \eta_i (G_j - \sum_{k \neq j} w_{jk} \bar{Y}_k) \right)^{T} B_{ij} (\hat{r} - r_0)
\]

\[
+ (\hat{r} - r_0)^{T} \left( \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \eta_i \eta_j \right)^{T} B_{ij} (\hat{r} - r_0) + R_n
\]

\[
:= I_{n51} + I_{n52} + I_{n53} + R_n.
\]

Here \( R_n \) is a higher order term. By \( \hat{r} - r_0 = O_p \left( \frac{1}{\sqrt{n}} \right) \), \( \sup_{X_i \in \Omega} |G_i - \sum_{k \neq i} w_{ik} \bar{Y}_k| = O_p \left( h^* + \sqrt{\frac{\log n}{nh^*}} \right) \) and assumption 7, it can be easily obtained that \( I_{n51} + I_{n52} = o_p \left( \frac{1}{n} \right) \), and the higher order term \( R_n = o_p \left( \frac{1}{n} \right) \). Also note that \( \left[ \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \eta_i \eta_j B_{ij} \right] - E(\eta_i \eta_j B_{12}) = o_p(1) \). Thus we can finish the proof of Proposition 5.

\[\square\]

**Proof of Proposition 6.** Note that \( e_i = a_n H(W_i) + u_i \) with \( E(u_i \mid W) = 0 \).

Thus we can rewrite \( J_{n0} \) as

\[
J_{n0} = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} e_i e_j B_{ij}
\]

\[
= \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i u_j B_{ij} + 2a_n \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i H_j B_{ij} + a_n^2 \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} H_i H_j B_{ij}
\]

\[
:= J_{n01} + 2a_n J_{n02} + a_n^2 J_{n03}
\]  

(A.9)

Obviously, \( J_{n01} \) is a degenerate U-statistic and thus \( J_{n01} = O_p \left( \frac{1}{n} \right) \). Given Assumptions 4, 5, 9, U-statistic theory yields that

\[
J_{n02} = \frac{1}{n} \sum_{i=1}^{n} u_i H_{00}(W_i) + o_p \left( \frac{1}{\sqrt{n}} \right),
\]

(A.10)

\[
J_{n03} - \mu_0 = o_p(1),
\]

(A.11)
where \( \mu_0 = E(H_1 H_2 B_{12}) \) and \( H_{00}(W_1) = E(H_2 B_{12} \mid W_1) \). Therefore \( J_{n0} = O_p \left( \frac{1}{n} \right) + O_p \left( \frac{a_n}{\sqrt{n}} \right) + o_n^2 \mu_0 \). Thus, when \( a_n \) is fixed, \( J_{n0} = a_n^2 \mu_0 + o_p(1) \) and when \( \sqrt{n}a_n \to \infty \), \( J_{n0}/a_n^2 = O_p \left( \frac{1}{\sqrt{n}a_n} \right) + O_p \left( \frac{1}{\sqrt{n}a_n} \right) + \mu_0 = \mu_0 + o_p(1) \). Further if \( a_n = \frac{1}{\sqrt{n}} \), \( nJ_{n0} = O_p(1) + O_p(1) + \mu_0 \). Proposition 6 is proved.

\[ \square \]

**Proof of Proposition 7.** Under the alternative hypothesis, \( e_i = a_n H(W_i) + u_i \) and then \( J_{n1} \) can be decomposed as

\[
J_{n1} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} e_i(\tilde{e}_j - e_j) B_{ij}
\]

\[
= \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i(\tilde{e}_j - e_j) B_{ij} + 2a_n \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} H_i(\tilde{e}_j - e_j) B_{ij}.
\]

Note that \( \tilde{e}_j - e_j = (Y_j^\ast - \sum_{i \neq j} w_{ji} Y_i^\ast) - (Y_j^\ast - g(X_j)) = g(X_j) - \sum_{i \neq j} w_{ji} g(X_i) - \sum_{i \neq j} w_{ji} u_j - a_n \sum_{i \neq j} w_{ji} H_i \). Here \( H_j = H(W_j) \). \( J_{n1} \) can be further rewritten as

\[
J_{n1} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i[g(X_j) - \sum_{i \neq j} w_{ji} g(X_i) - \sum_{i \neq j} w_{ji} u_i] B_{ij} - a_n \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{i \neq j} u_i w_{ji} H_i B_{ij}
\]

\[
+ a_n \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} H_i[g(X_j) - \sum_{i \neq j} w_{ji} g(X_i) - \sum_{i \neq j} w_{ji} u_i] B_{ij} - a_n^2 \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} \sum_{i \neq j} H_i w_{ji} H_i B_{ij}
\]

\[
:= 2J_{n10} - 2a_n J_{n11} + 2a_n J_{n12} - 2a_n^2 J_{n13}.
\]

Obviously, \( J_{n10} \) has a same asymptotic behavior as \( I_{n1} \) and then \( J_{n10} = -\frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i w_{ji} \tilde{B}_{ij} - \frac{1}{n} E[(u_1^2 \tilde{w}_{12} + u_2^2 \tilde{w}_{21}) B_{12}] + o_p \left( \frac{1}{n} \right) \). Note that \( E(\tilde{w}_{ji} H_i \mid X_i, W_j, W_i) = 0 \), and then similarly as the discussion on \( I_{n33} \) in Proposition 3, we can derive that

\[
a_n J_{n11} = O_p \left( \frac{a_n}{n} \right) = o_p \left( a_n^2 \right).
\]

Consider \( J_{n12} \). Similarly as the proof for \( I_{n42} \) in Proposition 4, we can derive that

\[
J_{n12} = -\frac{1}{n} \sum_{i=1}^{n} u_i H_{10}(W_i) + o_p \left( \frac{1}{\sqrt{n}} \right),
\]

(A.12)

where \( \tilde{X}_i = (W_i, u_i) \) and \( H_{10}(W_i) = \frac{1}{2} E [(H_j \tilde{w}_{ti} + H_{ij} \tilde{w}_{ij}) B_{ij} \mid \tilde{X}_i] \). Thus we have \( a_n J_{n12} = O_p \left( \frac{a_n}{\sqrt{n}} \right) \).
Further, under the assumption $E(H_i \mid X_i) = 0$ and assumption 3, we have $E(H_i \tilde{w}_{ji} H_i B_{ij}) = E[H_i \tilde{w}_{ji} B_{ij}E(H_t \mid X_t, W_t, W_j)] = 0$. Thus $J_{n13} = o_p(1)$ and $a_n^2J_{n13} = o_p(a_n^2)$.

Thus, $J_{n1} = O_p \left( \frac{1}{n} \right) + O_p \left( \frac{a_n}{\sqrt{n}} \right) + o_p(a_n^2)$. When $a_n$ is fixed, we have $J_{n1} \to 0$; when $\sqrt{n}a_n \to \infty$, $J_{n1}/a_n^2 \to 0$; and when $a_n = \frac{1}{\sqrt{n}}$, $nJ_{n1} = nJ_{n10} - \frac{2}{\sqrt{n}} \sum_{i=1}^n u_i H_{10}(W_i) + o_p(1)$. The proof of Proposition 7 is completed.

□

**Proof of Proposition 8.** Recall that $\hat{e}_j - e_j = (Y_j^* - \sum_{t \neq j} w_{jt} Y_t^* - (Y_j^* - g(X_j)) = g(X_j) - \sum_{t \neq j} w_{jt} g(X_t) - \sum_{t \neq j} w_{jt} u_t - a_n \sum_{t \neq j} w_{jt} H_t$. Thus we have

$$J_{n2} = \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} (\hat{e}_i - e_i)(\hat{e}_j - e_j)B_{ij}$$

$$= \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} [g(X_i) - \sum_{k \neq i} w_{ik} g(X_k) - \sum_{k \neq i} w_{ik} u_k] [g(X_j) - \sum_{t \neq j} w_{jt} g(X_t) - \sum_{t \neq j} w_{jt} u_t] B_{ij} + J_{n22}$$

$$:= J_{n21} + J_{n22} \quad (A.13)$$

where

$$J_{n22} = -2a_n \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} [g(X_i) - \sum_{k \neq i} w_{ik} g(X_k)] [\sum_{t \neq j} w_{jt} H_t] B_{ij}$$

$$+ 2a_n \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} [\sum_{k \neq i} w_{ik} u_k] [\sum_{t \neq j} w_{jt} H_t] B_{ij}$$

$$+ a_n^2 \frac{1}{n(n-1)} \sum_{i=1}^n \sum_{j \neq i} [\sum_{k \neq i} w_{ik} H_k] [\sum_{t \neq j} w_{jt} H_t] B_{ij}.$$ 

$$:= -2a_n J_{n221} + 2a_n J_{n222} + a_n^2 J_{n223}. \quad (A.14)$$

$J_{n21}$ has the same asymptotic behavior as $I_{n2}$ in Proposition 2 to have $J_{n21} = O_p \left( \frac{1}{n} \right)$.

By the similar discussions as before, we can have that $J_{n221} = O_p(h^*) = o_p \left( \frac{1}{\sqrt{n}} \right)$ and $J_{n223} = o_p(1)$. As for $J_{n222}$, noting that $E(u_k \mid W_1, \cdots, W_n) = 0$ and $E(w_{jt} H_t \mid W_j, W_t) = 0$ when $t \neq j, i$. Thus the similar argument for dealing with $I_{n23}$ in Proposition 2 yields that $J_{n22} = O_p \left( \frac{1}{n} \right)$. Hence we obtain that $J_{n2} = O_p \left( \frac{1}{n} \right) - o_p \left( \frac{a_n}{\sqrt{n}} \right) + O_p \left( \frac{a_n}{n} \right) + o_p(a_n^2)$. When $a_n > 0$ is fixed,
\[ J_{n2} = o_p(1); \text{ when } \sqrt{n}a_n \to \infty, J_{n2}/a_n^2 = o_p(1); \text{ and when } a_n = \frac{1}{\sqrt{n}}, nJ_{n2} = O_p(1) = nJ_{n21} + o_p(1). \text{ Thus, Proposition 8 is proved.} \]

\[ \square \]

**Proof of Proposition 9.** Decompose \( J_{n3} \) as

\[ J_{n3} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} u_i(\hat{\epsilon}_j - \hat{\epsilon}_j)B_{ij} + \frac{2a_n}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} H_i(\hat{\epsilon}_j - \hat{\epsilon}_j)B_{ij} \]

\[ := J_{n30} + J_{n31}. \quad (A.15) \]

Since \((\hat{\epsilon}_j - \hat{\epsilon}_j)\) is not related to the alternative model, the asymptotic behavior of \( J_{n30} \) is the same as \( I_{n3} \) in Proposition 3, that is, \( J_{n30} = O_p \left( \frac{1}{\sqrt{n}} \right) = (\hat{r} - r_0)^T \frac{2}{n} \sum_{i=1}^{n} u_i H_3(W_i) + o_p \left( \frac{1}{n} \right). \)

For \( J_{n31} \), recall that \( \hat{\epsilon}_j - \hat{\epsilon}_j = (\hat{r} - r_0)^T (\eta_j + G_j - \sum_{t \neq j} w_{jt} \mathbf{Y}_t) + o_p(\hat{r} - r_0). \) Then we have

\[ J_{n31} = 2a_n(\hat{r} - r_0)^T \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} H_i \eta_j B_{ij} \]

\[ + 2a_n(\hat{r} - r_0)^T \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} H_i (G_j - \sum_{t \neq j} w_{jt} \mathbf{Y}_t)B_{ij} + R_n \]

\[ := 2a_n(\hat{r} - r_0)^T J_{n311} + J_{n312} + R_n. \]

Here \( R_n \) is a higher order term. By \( \hat{r} - r_0 = O_p \left( \frac{1}{\sqrt{n}} \right) \text{ and sup}_{X_j \in \Omega} |G_j - \sum_{t \neq j} w_{jt} \mathbf{Y}_t| = O_p \left( h^s + \sqrt{\log n \frac{1}{n}} \right), J_{n312} = o_p \left( \frac{a_n}{\sqrt{n}} \right). \) Also, it is easy to derive that \( J_{n311} - E(H_1 \epsilon_2 B_{12}) = o_p(1). \) Thus the higher order term \( R_n = o_p \left( \frac{a_n}{\sqrt{n}} \right). \)

In summary, we have \( J_{n3} = O_p \left( \frac{1}{n} \right) + O_p \left( \frac{a_n}{\sqrt{n}} \right) + o_p \left( \frac{a_n}{\sqrt{n}} \right). \) Thus, when \( a_n > 0 \) is fixed, \( J_{n3} = o_p(1); \text{ when } \sqrt{n}a_n \to \infty, J_{n3}/a_n^2 = o_p(1); \text{ and when } a_n = \frac{1}{\sqrt{n}}, J_{n3} = (\hat{r} - r_0)^T \frac{2}{n} \sum_{i=1}^{n} u_i H_3(W_i) + \frac{2}{n}(\hat{r} - r_0)^T a + o_p \left( \frac{1}{n} \right) \) with \( a = E(H_1 \epsilon_2 B_{12}). \)

The proof of Proposition 9 is completed.

\[ \square \]
Proof of Proposition 10. Note that

\[ J_{n4} = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\hat{e}_i - \tilde{e}_i)(\hat{e}_j - \tilde{e}_j)B_{ij} \]

\[ = \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i} (\hat{e}_i - \tilde{e}_i)[g(X_j) - \sum_{t \neq j} w_{jt}g(X_t) - \sum_{t \neq j} w_{jt}u_i]B_{ij} \]

\[ -a_n \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i \neq j} (\hat{e}_i - \tilde{e}_i)w_{jt}H_tB_{ij} \]

\[ := J_{n41} - a_n J_{n42}. \]

The asymptotic behavior of \( J_{n41} \) is the same as the one of \( I_{n4} \) in Proposition 4 to have \( J_{n41} = O_p \left( \frac{1}{n} \right) = (\hat{r} - r_0)^\top \frac{1}{n} \sum_{i=1}^{n} u_iH_i(W_i) + o_p \left( \frac{1}{n} \right) \). As for \( J_{n42} \), recall that \( \hat{e}_j - \tilde{e}_j = (\hat{r} - r_0)^\top (\eta_j + G_j - \sum_{t \neq j} w_{jt}Y_t) + o_p(\hat{r} - r_0) \), and then we can rewrite \( J_{n42} \) as

\[ J_{n42} = -a_n(\hat{r} - r_0)^\top \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i \neq j} (G_j - \sum_{t \neq j} w_{jt}Y_t)w_{jt}H_tB_{ij} \]

\[ -a_n(\hat{r} - r_0)^\top \frac{2}{n(n-1)} \sum_{i=1}^{n} \sum_{j \neq i \neq j} \eta_j w_{jt}H_tB_{ij} + R_n \]

\[ := J_{n421} - 2a_n(\hat{r} - r_0)^\top J_{n422} + R_n. \]

Here \( R_n \) is a higher order term. Very similarly as the proof for Proposition 9, we can also obtain that \( J_{n4} = (\hat{r} - r_0)^\top \frac{1}{n} \sum_{i=1}^{n} u_iH_i(W_i) + o_p \left( \frac{1}{n} \right) + o_p \left( \frac{\sqrt{\lambda}}{n} \right) \).

Therefore, Proposition 10 is proved.

\[ \square \]

Proof of Proposition 11.

Since \( \hat{e}_i - \tilde{e}_i \) is not related to the local alternative model, the asymptotic behavior of \( J_{n5} \) is the same as that of \( I_{n5} \) in Proposition 5 to have \( J_{n5} = (\hat{r} - r_0)^\top A(\hat{r} - r_0) + o_p \left( \frac{\lambda}{n} \right) \). Thus the proof of proposition 11 is completed.

\[ \square \]

References

Abrevaya, J., Hsu, Y.-C., Lieli, R. P., 2015. Estimating conditional average treatment effects. Journal of Business & Economic Statistics 33 (4), 485–505.
Chang, M., Lee, S., Whang, Y.-J., 2015. Nonparametric tests of conditional
treatment effects with an application to single-sex schooling on academic
achievements. The Econometrics Journal 18 (3), 307–346.

Crump, R. K., Hotz, V. J., Imbens, G. W., Mitnik, O. A., 2008. Nonparamet-
ric tests for treatment effect heterogeneity. The Review of Economics and
Statistics 90 (3), 389–405.

Escanciano, J. C., 2006. A consistent diagnostic test for regression models using
projections. Econometric Theory 22 (6), 1030–1051.

Fan, Y., Li, Q., 1996. Consistent model specification tests: omitted variables and
semiparametric functional forms. Econometrica: Journal of the econometric
society, 865–890.

Guo, X., Wang, T., Zhu, L., 2016. Model checking for parametric single-index
models: a dimension reduction model-adaptive approach. Journal of the Royal
Statistical Society: Series B (Statistical Methodology) 78 (5), 1013–1035.

Hahn, J., 1998. On the role of the propensity score in efficient semiparametric
estimation of average treatment effects. Econometrica 66 (2), 315–331.

Hammer, S. M., Katzenstein, D. A., Hughes, M. D., Schooley, R. T., Haubrich, R. H., Henry, W. K., Lederman, M. M., Phair, J. P., Niu, M., et al., 1996. A trial comparing nucleoside monotherapy with combination
therapy in hiv-infected adults with cd4 cell counts from 200 to 500 per cubic
millimeter. New England Journal of Medicine 335 (15), 1081–1090.

Härdle, W. K., Müller, M., Sperlich, S., Werwatz, A., 2012. Nonparametric and
semiparametric models. Springer Science & Business Media.

Hsu, Y.-C., 2017. Consistent tests for conditional treatment effects. The econo-
metrics journal 20 (1), 1–22.

Khmaladze, E. V., Koul, H. L., et al., 2009. Goodness-of-fit problem for er-
rors in nonparametric regression: Distribution free approach. The Annals of
Statistics 37 (6A), 3165–3185.
Lavergne, Pascal, M. S., Patilea, V., 2015. A significance test for covariates in nonparametric regression. Electronic Journal of Statistics 9 (1), 643–678.

Lavergne, P., Patilea, V., 2012. One for all and all for one: regression checks with many regressors. Journal of business & economic statistics 30 (1), 41–52.

Lavergne, P., Vuong, Q., 2000. Nonparametric significance testing. Econometric Theory 16 (4), 576–601.

Li, L., Chiu, S. N., Zhu, L., 2019. Model checking for regressions: An approach bridging between local smoothing and global smoothing methods. Computational Statistics & Data Analysis 138, 64–82.

Lin, H., Zhou, F., Wang, Q., Zhou, L., Qin, J., 2018. Robust and efficient estimation for the treatment effect in causal inference and missing data problems. Journal of econometrics 205 (2), 363–380.

Lu, W., Zhang, H. H., Zeng, D., 2013. Variable selection for optimal treatment decision. Statistical methods in medical research 22 (5), 493–504.

Mammen, E., 1993. Bootstrap and wild bootstrap for high dimensional linear models. The annals of statistics 21 (1), 255–285.

Rosenbaum, P. R., Rubin, D. B., 1983. The central role of the propensity score in observational studies for causal effects. Biometrika 70 (1), 41–55.

Serfling, R. J., 1980. Approximation theorems of mathematical statistics. Vol. 162. John Wiley & Sons.

Shi, C., Lu, W., Song, R., 2019. A sparse random projection-based test for overall qualitative treatment effects. Journal of the American Statistical Association (just-accepted), 1–41.

Stute, W., Manteiga, W. G., Quindimil, M. P., 1998. Bootstrap approximations in model checks for regression. Journal of the American Statistical Association 93 (441), 141–149.
Stute, W., Xu, W., Zhu, L., 2008. Model diagnosis for parametric regression in high-dimensional spaces. Biometrika 95 (2), 451–467.

Wager, S., Athey, S., 2018. Estimation and inference of heterogeneous treatment effects using random forests. Journal of the American Statistical Association 113 (523), 1228–1242.

Yao, L., Sun, Z., Wang, Q., 2010. Estimation of average treatment effects based on parametric propensity score model. Journal of Statistical Planning and Inference 140 (3), 806–816.

Zhang, C., Dette, H., 2004. A power comparison between nonparametric regression tests. Statistics & probability letters 66 (3), 289–301.

Zhang, M., Tsiatis, A. A., Davidian, M., 2008. Improving efficiency of inferences in randomized clinical trials using auxiliary covariates. Biometrics 64 (3), 707–715.

Zheng, J. X., 1996. A consistent test of functional form via nonparametric estimation techniques. Journal of Econometrics 75 (2), 263–289.

Zhu, L., Li, R., 1998. Dimension-reduction type test for linearity of a stochastic regression model. Acta Mathematicae Applicatae Sinica 14 (2), 165–175.

Zhu, L.-X., 2003. Model checking of dimension-reduction type for regression. Statistica Sinica 13, 283–296.