On Spectral Properties of some Class of Non-selfadjoint Operators

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Blessed memory of Isai I. Mikaelian is devoted

Abstract

In this paper we explore a certain class of non-selfadjoint operators acting in a complex separable Hilbert space. We consider a perturbation of a non-selfadjoint operator by an operator that is also non-selfadjoint. Our consideration is based on known spectral properties of the real component of a non-selfadjoint compact operator. Using a technic of the sesquilinear form theory we establish the compactness property of the resolvent, obtain the asymptotic equivalence between the real component of the resolvent and the resolvent of the real component for some class of non-selfadjoint operators. We obtain a classification of non-selfadjoint operators in accordance with belonging their resolvent to the Schatten-von Neumann class and formulate a sufficient condition of completeness of the root vectors system. Finally we obtain an asymptotic formula for eigenvalues of the considered class of non-selfadjoint operators.

1 Introduction

It is remarkable that initially the perturbation theory of selfadjoint operators was born in the works of M. Keldysh [14]-[16] and had been motivated by the works of such famous scientists as T. Carleman [8] and Ya. Tamarkin [38]. Over time many papers were published within the framework of this theory, for instance F. Browder [7], M. Livshits [26], B. Mukminov [33], I. Glazman [9], M. Krein [25], B. Lidsky [27], A. Marcus [28],[29], V. Matsaev [30]-[31], S. Agmon [2], V. Katzenelson [13]. Nowadays there exists a huge amount of theoretical results formulated in the work of A. Shkalikov [37]. However for applying these results for a concrete operator we must have a representation of one by the sum of the main part (in the other words a so-called non-perturbing operator) and the operator-perturbation. It is essential that the main part must be an operator of a special type either a selfadjoint or a normal operator. If we consider a case where in the representation the main part is neither selfadjoint nor normal and we cannot approach the required representation in an obvious way, then it is possible to use another technique based on properties of the real component of the initial operator. This is a subject to consider in the second section. In the third section we demonstrate the significance of the obtained abstract results and consider concrete operators. Note that the relevance of such consideration is based on
the following. The eigenvalue problem is still relevant for the second order fractional differential operators. Many papers were devoted to this question, for instance the papers [34], [3]-[6]. The singular number problem for the resolvent of the second order differential operator with the Riemann-Liouville fractional derivative in the final term is considered in the paper [4]. It is proved that the resolvent belongs to the Hilbert-Schmidt class. The problem of root functions system completeness is researched in the paper [5], also a similar problem is considered in the paper [6]. We would like to study spectral properties of some class of non-selfadjoint operators in the abstract case. Via obtained results we research a multidimensional case of the second order fractional differential operator which can be reduced to the cases considered in the papers listed above. For this purpose we deal with the extension of the Kipriyanov fractional differential operator considered in detail in the papers [17]-[19].

2 Preliminaries

Let \( C, C_1, \ i \in \mathbb{N}_0\) be positive real constants. We assume that the values of \( C \) can be different in various formulas but the values of \( C_1, \ i \in \mathbb{N}_0\) are certain. Everywhere further we consider linear densely defined operators acting in a separable complex Hilbert space \( \mathcal{H} \). Denote by \( \mathcal{B}(\mathcal{H}) \) the set of linear bounded operators acting in \( \mathcal{H} \). Denote by \( D(L), \ R(L), \ N(L) \) the domain of definition, the range, and the inverse image of zero of the operator \( L \) accordingly. Let \( P(L) \) be a resolvent set of the operator \( L \). Denote by \( R_L(\zeta), \ \zeta \in P(L), [R_L := R_L(0)] \) the resolvent of the operator \( L \). Let \( \lambda_i(L), \ i \in \mathbb{N} \) denote the eigenvalues of the operator \( L \). Suppose \( L \) is a compact operator and \( |L| := (\langle L^*L \rangle)^{1/2}, \ r(|L|) := \dim R(|L|) \); then the eigenvalues of the operator \( |L| \) are called the singular numbers (s-numbers) of the operator \( L \) and are denoted by \( s_i(L), \ i = 1, 2, \ldots, r(|L|) \). If \( r(|L|) < \infty \), then we put by definition \( s_i = 0, \ i = r(|L|) + 1, 2, \ldots \). According to the terminology of the monograph [10] the dimension of the root vectors subspace corresponding to a certain eigenvalue \( \lambda_k \) is called the algebraic multiplicity of the eigenvalue \( \lambda_k \). Let \( \nu(L) \) denote the sum of all algebraic multiplicities of the operator \( L \). Denote by \( \mathcal{S}_p(\mathcal{H}), \ 0 < p < \infty \) the Schatten-von Neumann class and let \( \mathcal{S}_\infty(\mathcal{H}) \) denote the set of compact operators. By definition, put

\[
\mathcal{S}_p(\mathcal{H}) := \left\{ L : \mathcal{H} \rightarrow \mathcal{H}, \sum_{i=1}^{\infty} s_i^p(L) < \infty, \ 0 < p < \infty \right\}.
\]

Suppose \( L \) is an operator that has a compact resolvent and \( s_n(R_L) \leq C n^{-\mu}, \ n \in \mathbb{N}, \ 0 \leq \mu < \infty \); then we denote by \( \mu(L) \) order of the operator \( L \) in accordance with the definition given in the paper [37]. Denote by \( L_\Re := (L + L^*)/2, L_\Im := (L - L^*)/2i \) the real and the imaginary component of the operator \( L \) accordingly and let \( \tilde{L} \) denote the closure of the operator \( L \). In accordance with the terminology of the monograph [12] the set \( \Theta(L) := \{ z \in \mathbb{C} : z = (Lf, f)_\mathcal{H}, \ f \in D(L), \ |f|_\mathcal{H} = 1 \} \) is called the numerical range of the operator \( L \). We use the definition of the sectorial property given in [12] p.280. An operator \( L \) is called a sectorial operator if its numerical range belongs to a closed sector \( [\gamma, \theta) := \{ \zeta : |\arg(\zeta - \gamma)| \leq \theta < \pi/2 \} \), where \( \gamma \) is the vertex and \( \theta \) is the semi-angle of the sector \( [\gamma, \theta) \). We shall say that the operator \( L \) has a positive sector if \( \text{Im} \gamma = 0, \ \gamma > 0 \). According to the terminology of the monograph [12] an operator \( L \) is called strictly accretive if the following relation holds \( \text{Re}(Lf, f)_\mathcal{H} \geq C\|f\|_\mathcal{H}^2, \ f \in D(L) \). In accordance with the definition [12] p.279 an operator \( L \) is called \( m \)-accretive if the next relation holds \( (A + \zeta)^{-1} \in \mathcal{B}(\mathcal{H}), \ \|(A + \zeta)^{-1}\| \leq (\text{Re}\zeta)^{-1}, \ \text{Re}\zeta > 0 \). An operator \( L \) is called \( m \)-sectorial if \( L \) is sectorial and \( L + \beta \) is \( m \)-accretive for some constant \( \beta \). An operator \( L \) is called symmetric if one is densely defined and the next
equality holds \((Lf, g)_\theta = (f, Lg)_\theta, f, g \in D(L)\). A symmetric operator is called positive if the values of its quadratic form are nonnegative. Denote by \(\mathcal{H}_L, \| \cdot \|_L\) the energetic space generated by the operator \(L\) and the norm on this space respectively (see [39, 32]). In accordance with the denotation of the paper [12] we consider a sesquilinear form \(t[\cdot, \cdot]\) defined on a linear manifold of the Hilbert space \(\mathcal{H}\) (further we use the term form). Denote by \(\Theta(\cdot)\) the quadratic form corresponding to the sesquilinear form \(t[\cdot, \cdot]\). Let \(\mathcal{R} t = (t + t^*)/2, \mathcal{I} m t = (t - t^*)/2i\) be the real and imaginary component of the form \(t\) respectively, where \(t^*[u, v] = t[v, u], D(t^*) = D(t)\). According to these definitions, we have \(\mathcal{R} t t[\cdot] = \mathcal{R} t[\cdot], \mathcal{I} m t[\cdot] = \mathcal{I} m t[\cdot]\). Denote by \(t\) the closure of the form \(t\). The range of a quadratic form \(t[f, f], f \in D(t), \|f\|_\theta = 1\) is called the range of the sesquilinear form \(t\) and is denoted by \(\Theta(t)\). A form \(t\) is called sectorial if its range belongs to a sector having the vertex \(\gamma\) situated at the real axis and the semi-angle \(0 \leq \theta < \pi/2\). Suppose \(I\) is a closed sectorial form; then a linear manifold \(D' \subset D(I)\) is called the core of \(I\) if the restriction of \(I\) to \(D'\) has the closure \(I\). Due to Theorem 2.7 [12, p.323] there exist unique \(m\)-sectorial operators \(L_I, L_{\Theta(I)}\) associated with the closed sectorial forms \(I, \mathcal{R} t I\) respectively. The operator \(L_{\Theta(I)}\) is called the real part of the operator \(L_I\) and is denoted by \(Re L_I\). Suppose \(L\) is a sectorial densely defined operator and \(t[u, v] := (Lu, v)_\theta, D(t) = D(L)\); then due to Theorem 1.27 [12, p.318] the form \(t\) is closable, due to Theorem 2.7 [12, p.323] there exists a unique \(m\)-sectorial operator \(T_t\) associated with the form \(t\). In accordance with the definition [12, p.325] the operator \(T_t\) is called the Friedrichs extension of the operator \(L\).

Further, if it is not stated otherwise we use the notations of the monographs [10], [12], [36]. Consider a pair of complex separable Hilbert spaces \(\mathcal{H}_+, \mathcal{H}_+\) such that

\[ \mathcal{H}_+ \hookrightarrow \mathcal{H}_+ \]  

This denotation implies that \(\mathcal{H}_+\) is dense in \(\mathcal{H}\) and we have a bounded embedding provided by the inequality

\[ \|f\|_\theta \leq \|f\|_{\mathcal{H}_+}, f \in \mathcal{H}_+, \]  

moreover any bounded set in the space \(\mathcal{H}_+\) is a compact set in the space \(\mathcal{H}\). We consider non-selfadjoint operators which can be represented by a sum \(W = T + A\). The operators \(T\) and \(A\) are called a main part and an operator-perturbation respectively, both these operators act in \(\mathcal{H}\). We assume that: there exists a linear manifold \(\mathcal{M} \subset \mathcal{H}_+\) that is dense in \(\mathcal{H}_+\), the operators \(T, A\) and their adjoint operators are defined on \(\mathcal{M}\). Further, we may assume that \(D(W) = \mathcal{M}\). This gives us the opportunity to prove that \(D(W) \subset D(W^*)\). Suppose the operator \(W^+\) is the restriction of \(W^*\) to \(D(W)\); then the operator \(W^+\) is called a formal adjoint operator with respect to \(W\). Denote by \(W^+\) the closure of the operator \(W^+\). Further, we assume that the following conditions are fulfilled

\[ \begin{align*}
\text{i) } & \Re(Tf, f)_{\theta} \geq C_0 \|f\|_{\mathcal{H}_+}^2, \\
\text{ii) } & |(Tf, g)_{\theta}| \leq C_1 \|f\|_{\mathcal{H}_+} \|g\|_{\mathcal{H}_+}, \\
\text{iii) } & \Re(Af, f)_{\theta} \geq C_2 \|f\|_{\mathcal{H}_+}^2, \\
\text{iv) } & |(Af, g)_{\theta}| \leq C_3 \|f\|_{\mathcal{H}_+} \|g\|_{\mathcal{H}_+}, f, g \in \mathcal{M}. 
\end{align*} \]

Due to these conditions it is easy to prove that the operators \(W, W_\mathcal{M}\) are closeable (see Theorem 3.4 [12, p.268]). Denote by \(W_\mathcal{M}\) the closure of the operator \(W_\mathcal{M}\). To make some formulas readable we also use the following form of notation \(V := (R_W)_\mathcal{M}, H := \tilde{W}_\mathcal{M}\).

3 Main results

In this section we formulate abstract theorems that are generalizations of some particular results obtained by the author. First we generalize Theorem 4.2 [22] establishing the sectorial property
of the second order fractional differential operator.

**Lemma 3.1.** The operators $\tilde{W}$, $\tilde{W}^+$ have a positive sector.

**Proof.** Due to inequalities (2),(2) we conclude that the operator $W$ is strictly accretive, i.e.

$$\text{Re}(Wf,f)_\mathcal{H} \geq C_0 \|f\|_{\mathcal{H}}^2, \ f \in D(W).$$

Let us prove that the operator $\tilde{W}$ is canonical sectorial. Combining (2) (ii) and (2) (iii), we get

$$\text{Re}(Wf,f)_\mathcal{H} = \text{Re}(Tf,f)_\mathcal{H} + \text{Re}(Af,f)_\mathcal{H} \geq C_0 \|f\|_{\mathcal{H}_+} + C_2 \|f\|_{\mathcal{H}}, \ f \in D(W).$$

Obvioulsy we can extend the previous inequality to

$$\text{Re}(\tilde{W}f,f)_\mathcal{H} \geq C_0 \|f\|_{\mathcal{H}_+} + C_2 \|f\|_{\mathcal{H}}, \ f \in D(\tilde{W}).$$

By virtue of (6), we obtain $D(\tilde{W}) \subset \mathcal{H}_+^+$. Note that we have the estimate

$$|\text{Im}(Wf,f)_\mathcal{H}| \leq |\text{Im}(Tf,f)_\mathcal{H}| + |\text{Im}(Af,f)_\mathcal{H}| = I_1 + I_2, \ f \in D(W).$$

Using inequality (2) (ii), the Jung inequality, we get

$$I_1 = |(Tv,u)_\mathcal{H} - (Tu,v)_\mathcal{H}| \leq |(Tv,u)_\mathcal{H}| + |(Tu,v)_\mathcal{H}| \leq 2C_1 \|u\|_{\mathcal{H}_+} \|v\|_{\mathcal{H}_+} \leq C_1 \|f\|_{\mathcal{H}_+}^2,$$

where $f = u + iv$. Consider $I_2$. Applying the Cauchy Schwartz inequality and inequality (2) (iv), we obtain for arbitrary positive $\varepsilon$

$$|(Av,u)_\mathcal{H}| \leq C_3 \|v\|_{\mathcal{H}_+} \|u\|_{\mathcal{H}} \leq C_3 \left\{ \frac{1}{\varepsilon} \|u\|_{\mathcal{H}}^2 + \varepsilon \|v\|_{\mathcal{H}_+}^2 \right\};$$

$$|(Au,v)_\mathcal{H}| \leq C_3 \left\{ \frac{1}{\varepsilon} \|v\|_{\mathcal{H}_+}^2 + \varepsilon \|u\|_{\mathcal{H}_+}^2 \right\}.$$}

Hence

$$I_2 = |(Av,u)_\mathcal{H} - (Au,v)_\mathcal{H}| \leq |(Av,u)_\mathcal{H}| + |(Au,v)_\mathcal{H}| \leq C_3 \left\{ \frac{1}{\varepsilon} \|f\|_{\mathcal{H}}^2 + \varepsilon \|f\|_{\mathcal{H}_+}^2 \right\}.$$}

Finally, we have the following estimate

$$|\text{Im}(Wf,f)_\mathcal{H}| \leq C_3 \varepsilon^{-1} \|f\|_{\mathcal{H}}^2 + \left( \frac{C_3}{2} \varepsilon + C_1 \right) \|f\|_{\mathcal{H}_+}^2, \ f \in D(W).$$

Thus, we conclude that the next inequality holds for arbitrary $k > 0$

$$\text{Re}(Wf,f)_\mathcal{H} - k |\text{Im}(Wf,f)_\mathcal{H}| \geq$$

$$\geq \left[ C_0 - k \left( \frac{C_3}{2} \varepsilon + C_1 \right) \right] \|f\|_{\mathcal{H}_+}^2 + \left( C_2 - k \frac{C_3}{2} \varepsilon^{-1} \right) \|f\|_{\mathcal{H}}^2, \ f \in D(W).$$

Using the continuity property of the inner product, we can extend the previous inequality to the set $D(\tilde{W})$. It follows easily that

$$|\text{Im}( [\tilde{W} - \gamma(\varepsilon)]f,f)_\mathcal{H}| \leq \frac{1}{k(\varepsilon)} \text{Re}( [\tilde{W} - \gamma(\varepsilon)]f,f)_\mathcal{H}, \ f \in D(\tilde{W}),$$
\[ k(\varepsilon) = C_0 \left( \frac{C_3}{2} \varepsilon + C_1 \right)^{-1}, \quad \gamma(\varepsilon) = C_2 - k(\varepsilon) \frac{C_4}{2} \varepsilon^{-1}. \tag{7} \]

The previous inequality implies that the numerical range of the operator \( \tilde{W} \) belongs to the sector \( \mathcal{L}_\gamma(\theta) \) with the vertex situated at the point \( \gamma \) and the semi-angle \( \theta = \arctan(1/k) \). Solving system of equations (3) relative to \( \varepsilon \) we obtain the positive root \( \xi \) corresponding to the value \( \gamma = 0 \) and the following description for the coordinates of the sector vertex \( \gamma \)

\[
\gamma := \begin{cases} 
\gamma < 0, & \varepsilon \in (0, \xi), \\
\gamma \geq 0, & \varepsilon \in [\xi, \infty), \\
\end{cases} \quad \xi = \sqrt{\left( \frac{C_1}{C_3} \right)^2 + \frac{C_0}{C_2} - \frac{C_1}{C_3}}.
\]

It follows that the operator \( \tilde{W} \) has a positive sector. The proof corresponding to the operator \( \tilde{W}^+ \) follows from the reasoning given above if we note that \( \tilde{W}^+ \) is formal adjoint with respect to \( W \).

**Lemma 3.2.** The operators \( \tilde{W}, \tilde{W}^+ \) are m-accretive, their resolvent sets contain the half-plane \( \{ \zeta : \zeta \in \mathbb{C}, \text{Re} \zeta < C_0 \} \).

**Proof.** Due to Lemma 3.1 we know that the operator \( \tilde{W} \) has a positive sector, i.e. the numerical range of \( \tilde{W} \) belongs to the sector \( \mathcal{L}_\gamma(\theta) \), \( \gamma > 0 \). In consequence of Theorem 3.2 [12, p.268], we have \( \forall \zeta \in \mathbb{C} \setminus \mathcal{L}_\gamma(\theta) \), the set \( R(\tilde{W} - \zeta) \) is a closed space, and the next relation holds

\[ \text{def}(\tilde{W} - \zeta) = \eta, \quad \eta = \text{const}. \]

Due to Theorem 3.2 [12, p.268] the inverse operator \( (\tilde{W} + \zeta)^{-1} \) is defined on the subspace \( R(\tilde{W} + \zeta) \), \( \text{Re} \zeta > 0 \). In accordance with the definition of m-accretive operator given in the monograph [12, p.279] we need to show that

\[ \text{def}(\tilde{W} + \zeta) = 0, \quad \| (\tilde{W} + \zeta)^{-1} \| \leq (\text{Re} \zeta)^{-1}, \quad \text{Re} \zeta > 0. \]

For this purpose assume that \( \zeta_0 \in \mathbb{C} \setminus \mathcal{L}_\gamma(\theta) \), \( \text{Re} \zeta_0 < 0 \). Using (4), we get

\[ \text{Re} \left( f, [\tilde{W} - \zeta_0] f \right)_\mathcal{H} \geq (C_0 - \text{Re} \zeta_0) \| f \|^2_\mathcal{H}, \quad f \in D(\tilde{W}). \tag{8} \]

Since the operator \( \tilde{W} - \zeta_0 \) has the closed range \( R(\tilde{W} - \zeta_0) \), it follows that

\[ \mathcal{H} = R(\tilde{W} - \zeta_0) \oplus R(\tilde{W} - \zeta_0)^\perp. \]

Note that the intersection of the sets \( \mathcal{M} \) and \( R(\tilde{W} - \zeta_0)^\perp \) is zero. If we assume otherwise, then applying inequality (3) for any element \( u \in \mathcal{M} \cap R(\tilde{W} - \zeta_0)^\perp \) we get

\[ (C_0 - \text{Re} \zeta_0) \| u \|^2_\mathcal{H} \leq \text{Re} \left( u, [\tilde{W} - \zeta_0] u \right)_\mathcal{H} = 0, \]

hence \( u = 0 \). Thus the intersection of the sets \( \mathcal{M} \) and \( R(\tilde{W} - \zeta_0)^\perp \) is zero. It implies that

\[ (g, v)_\mathcal{H} = 0, \quad \forall g \in R(\tilde{W} - \zeta_0)^\perp, \quad \forall v \in \mathcal{M}. \]

Since \( \mathcal{M} \) is a dense set in \( \mathcal{H}_+ \), then taking into account (2), we obtain that \( \mathcal{M} \) is a dense set in \( \mathcal{H} \). Hence \( R(\tilde{W} - \zeta_0)^\perp = 0, \text{def}(\tilde{W} - \zeta_0) = 0 \). Combining this fact with Theorem 3.2 [12, p.268], we
get $\text{def}(\tilde{W} - \zeta) = 0$, $\zeta \in \mathbb{C} \setminus \mathcal{P}(\theta)$. It is clear that $\text{def}(\tilde{W} + \zeta) = 0$, $\forall \zeta$, $\text{Re}\zeta > 0$. We must notice that

$$(C_0 + \text{Re}\zeta)\|f\|_H^2 \leq \text{Re}\left(\langle f, [\tilde{W} + \zeta]f \rangle_\mathcal{H} \right) \leq \|f\|_H\|([\tilde{W} + \zeta]f\|_H, \ f \in D(\tilde{W}), \ \text{Re}\zeta > 0.$$ 

By virtue of the fact $\text{def}(\tilde{W} + \zeta) = 0$, $\forall \zeta$, $\text{Re}\zeta > 0$ we know that the resolvent is defined. Therefore

$$\|([\tilde{W} + \zeta]^{-1}f\|_H \leq (C_0 + \text{Re}\zeta)^{-1}\|f\|_H \leq (\text{Re}\zeta)^{-1}\|f\|_H, \ f \in \mathcal{H}.$$ 

It implies that

$$\|([\tilde{W} + \zeta]^{-1}\| \leq (\text{Re}\zeta)^{-1}, \ \forall \zeta, \ \text{Re}\zeta > 0.$$ 

If we combine inequality (6) with Theorem 3.2 [12, p.268], we get $P(\tilde{W}) \ni \{\zeta : \zeta \in \mathbb{C}, \ \text{Re}\zeta < C_0\}$. The proof corresponding to the operator $\tilde{W}^+$ is absolutely analogous. □

**Lemma 3.3.** The operator $\tilde{W}_{2\mathbb{R}}$ is strictly accretive, $m$-accretive, selfadjoint.

**Proof.** It is obvious that $\tilde{W}_{2\mathbb{R}}$ is a symmetric operator. Due to the continuity property of the inner product we can conclude that $\tilde{W}_{2\mathbb{R}}$ is symmetric too. Hence $\Theta(\tilde{W}_{2\mathbb{R}}) \subset \mathbb{R}$. By virtue of (5), we have

$$(\tilde{W}_{2\mathbb{R}}f, f)_{\mathcal{H}} \geq C_0\|f\|^2_{\mathcal{H}^+}, \ f \in D(\tilde{W}).$$ 

Using inequality (2) and the continuity property of the inner product, we obtain

$$(\tilde{W}_{2\mathbb{R}}f, f)_{\mathcal{H}} \geq C_0\|f\|^2_{\mathcal{H}^+} \geq C_0\|f\|^2_{\mathcal{H}^+}, \ f \in D(\tilde{W}_{2\mathbb{R}}). \ (9)$$ 

It implies that $\tilde{W}_{2\mathbb{R}}$ is strictly accretive. In the same way as in the proof of Lemma 3.2 we come to conclusion that $\tilde{W}_{2\mathbb{R}}$ is $m$-accretive. Moreover we obtain the relation $\text{def}(\tilde{W}_{2\mathbb{R}} - \zeta) = 0$, $\text{Im}\zeta \neq 0$. Hence by virtue of Theorem 3.16 [12, p.271] the operator $\tilde{W}_{2\mathbb{R}}$ is selfadjoint. □

**Theorem 3.4.** The operators $\tilde{W}_{2\mathbb{R}}, \tilde{W}, \tilde{W}^+$ have compact resolvents.

**Proof.** First note that due to Lemma 3.3 the operator $\tilde{W}_{2\mathbb{R}}$ is selfadjoint. Using (9), we obtain the estimates

$$\|f\|_H \geq \sqrt{C_0}\|f\|_{\mathcal{H}^+} \geq \sqrt{C_0}\|f\|_{\mathcal{H}}, \ f \in \mathcal{H}.$$ 

where $H := \tilde{W}_{2\mathbb{R}}$. Since $\mathcal{H}^+ \hookrightarrow \hookrightarrow \mathcal{H}$, then we conclude that each set bounded with respect to the energetic norm generated by the operator $\tilde{W}_{2\mathbb{R}}$ is compact with respect to the norm $\|\cdot\|_{\mathcal{H}}$. Hence in accordance with Theorem 3.2 [32, p.216] we conclude that $\tilde{W}_{2\mathbb{R}}$ has a discrete spectrum. Note that in consequence of Theorem 5 [32, p.222] we have that a selfadjoint strictly accretive operator with discrete spectrum has a compact inverse operator. Thus using Lemma 3.3 Theorem 6.29 [12, p.187] we obtain that $\tilde{W}_{2\mathbb{R}}$ has a compact resolvent.

Further, we need the technique of the sesquilinear form theory stated in [12]. Consider the sesquilinear forms

$$t[f, g] = \langle \tilde{W}f, g \rangle_{\mathcal{H}}, \ f, g \in D(\tilde{W}), \ \ h[f, g] = (\tilde{W}_{2\mathbb{R}}f, g)_{\mathcal{H}}, \ f, g \in D(\tilde{W}_{2\mathbb{R}}).$$ 

Recall that due to inequality (6) we came to the conclusion that $D(\tilde{W}) \subset \mathcal{H}^+$. In the same way we can deduce that $D(\tilde{W}_{2\mathbb{R}}) \subset \mathcal{H}^+$. By virtue of Lemma 3.1 Lemma 3.3 it is easy to prove that the sesquilinear forms $t, h$ are sectorial. Applying Theorem 1.27 [12, p.318] we get that these forms
are closable. Now note that \( \Re \tilde{t} \) is a sum of two closed sectorial forms. Hence in consequence of Theorem 1.31 [12, p.319], we have that \( \Re \tilde{t} \) is a closed form. Let us show that \( \Re \tilde{t} = \tilde{h} \). First note that this equality is true on the elements of the linear manifold \( \mathfrak{M} \subset \mathfrak{N}_+ \). This fact can be obtained from the following obvious relations

\[
\tilde{t}[f, g] = (Wf, g)_{\mathfrak{N}}, \quad \tilde{t}[g, f] = (W^*f, g)_{\mathfrak{N}}, \quad f, g \in \mathfrak{M}.
\]

On the other hand

\[
\tilde{h}[f, g] = (\tilde{W}_3f, g)_{\mathfrak{N}} = (W_3f, g)_{\mathfrak{N}}, \quad f, g \in \mathfrak{M}.
\]

Hence

\[
\Re \tilde{t}[f, g] = \tilde{h}[f, g], \quad f, g \in \mathfrak{M}.
\]

Using (10), we get

\[
C_0\|f\|_{\mathfrak{N}^+}^2 \leq \Re \tilde{t}[f] \leq C_4\|f\|_{\mathfrak{N}^+}^2, \quad C_0\|f\|_{\mathfrak{N}^+}^2 \leq \tilde{h}[f] \leq C_4\|f\|_{\mathfrak{N}^+}^2, \quad f \in \mathfrak{M},
\]

where \( C_4 = C_1 + C_3 \). Since \( \Re \tilde{t}[f] = \Re \tilde{h}[f], \quad f \in \mathfrak{M} \), the sesquilinear forms \( \Re \tilde{t}, \tilde{h} \) are closed forms, then using (11) it is easy to prove that \( D(\Re \tilde{t}) = D(\tilde{h}) = \mathfrak{N}^+ \). Using estimates (11), it is not hard to prove that \( \mathfrak{M} \) is a core of the forms \( \Re \tilde{t}, \tilde{h} \). Hence using (10), we obtain \( \Re \tilde{t}[f] = \tilde{h}[f], \quad f \in \mathfrak{N}^+ \).

In accordance with the polarization principle (see (1.1) [12, p.309]), we have that \( \Re \tilde{t} = \tilde{h} \). Now recall that the forms \( \tilde{t}, \tilde{h} \) are generated by the operators \( \tilde{W}, \tilde{W}_3 \) respectively. Note that in consequence of Lemmas 3.1, 3.2, Theorem 3.2 [12, p.337] there exist the selfadjoint operators \( \Re \tilde{t}, \tilde{h} \) which are m-sectorial. Hence by virtue of Theorem 2.9 [12, p.326], we get \( T_\tilde{t} = \tilde{W}, T_\tilde{h} = \tilde{W}_3 \). Since we have proved that \( \Re \tilde{t} = \tilde{h} \), then \( T_{\Re \tilde{t}} = \tilde{W}_3 \). Therefore by definition we have that the operator \( \tilde{W}_3 \) is the real part of the m-sectorial operator \( \tilde{W} \), by symbol \( \tilde{W}_3 = \Re \tilde{W} \). Since we proved above that \( \tilde{W}_3 \) has a compact resolvent, then using Theorem 3.3 [12, p.337] we conclude that the operator \( \tilde{W} \) has a compact resolvent. The proof corresponding to the operator \( \tilde{W}^+ \) is absolutely analogous.

**Theorem 3.5.** The following two-sided estimate holds

\[
\|S\|^{-2} \lambda_i(R_H) \leq \lambda_i (V) \leq \|S^{-1}\| \lambda_i (R_H), \quad i \in \mathbb{N},
\]

where \( H := \tilde{W}_3, V := (R_{\tilde{W}_3})_+ \), and \( S \) is a bounded selfadjoint operator defined by the operator \( W \).

**Proof.** It was shown in the proof of Theorem 3.4 that \( H = \Re \tilde{W} \). Hence in consequence of Lemma 3.1, Lemma 3.2, Theorem 3.2 [12, p.337] there exist the selfadjoint operators \( B_i := \{B_i \in \mathfrak{B}(\mathfrak{N}), \|B_i\| \leq \tan \theta\} \), \( i = 1, 2 \) (where \( \theta \) is the semi-angle of the sector \( \mathfrak{L}_0(\theta) \subset \Theta(\tilde{W}) \)) such that

\[
\tilde{W} = H^\frac{1}{2}(I + iB_1)H^\frac{1}{2}, \quad \tilde{W}^+ = H^\frac{1}{2}(I + iB_2)H^\frac{1}{2}.
\]

Since the set of linear operators generates ring, it follows that

\[
Hf = \frac{1}{2} \left( H^\frac{1}{2}(I + iB_1) + H^\frac{1}{2}(I + iB_2) \right) H^\frac{1}{2} = \frac{1}{2} \left( H^\frac{1}{2} ((I + iB_1) + (I + iB_2)) \right) H^\frac{1}{2} = Hf + \frac{i}{2} \tilde{W} \left( B_1 + B_2 \right) \tilde{W} f, \quad f \in \mathfrak{M}.
\]
Consequently
\[ H^{\frac{1}{2}} (B_1 + B_2) H^{\frac{1}{2}} f = 0, \ f \in \mathcal{M}. \] (14)

Let us show that \( B_1 = -B_2 \). In accordance with Lemma 3.3 the operator \( H \) is \( m \)-accretive, hence we have \((H + \zeta)^{-1} \in \mathcal{B} (\mathcal{H})\), \( \text{Re} \zeta > 0 \). Using this fact, we get
\[
\text{Re} \left( [H + \zeta]^{-1} H f, f \right)_\mathcal{H} = \text{Re} \left( [H + \zeta]^{-1} H f, f \right)_\mathcal{H} - \text{Re} \left( [H + \zeta]^{-1} f, f \right)_\mathcal{H} \geq \]
\[
\geq \|f\|_\mathcal{H}^2 - |\zeta| \cdot \| (H + \zeta)^{-1} f \|_\mathcal{H}^2 = \|f\|_\mathcal{H}^2 \left( 1 - |\zeta| \cdot \|(H + \zeta)^{-1}\| \right),
\]
\( \text{Re} \zeta > 0, \ f \in \mathcal{D}(H) \).

Applying inequality (9), we obtain
\[
\|f\|_\mathcal{H} \|(H + \zeta)^{-1} f\|_\mathcal{H} \geq |(f, [H + \zeta]^{-1} f)| \geq (\text{Re} \zeta + C_0) \|(H + \zeta)^{-1} f\|_\mathcal{H}^2, \ f \in \mathcal{H}.
\] (15)

It implies that
\[
\|(H + \zeta)^{-1}\| \leq (\text{Re} \zeta + C_0)^{-1}, \ \text{Re} \zeta > 0.
\]

Combining this estimate and (3), we have
\[
\text{Re} \left( [H + \zeta]^{-1} H f, f \right)_\mathcal{H} \geq \|f\|_\mathcal{H}^2 \left( 1 - \frac{|\zeta|}{\text{Re} \zeta + C_0} \right), \ \text{Re} \zeta > 0, \ f \in \mathcal{D}(H).
\] Applying formula (3.45) [12, p.282] and taking into account that \( H^{\frac{1}{2}} \) is selfadjoint, we get
\[
\left( H^{\frac{1}{2}} f, f \right)_\mathcal{H} = \frac{1}{\pi} \int_0^\infty \zeta^{-1/2} \text{Re} \left( [H + \zeta]^{-1} H f, f \right)_\mathcal{H} d\zeta \geq
\]
\[
\geq \|f\|_\mathcal{H}^2 \cdot \frac{C_0}{\pi} \int_0^\infty \frac{\zeta^{-1/2}}{\zeta + C_0} d\zeta = \sqrt{C_0 \|f\|_\mathcal{H}^2}, \ f \in \mathcal{D}(H).
\] (16)

Since in accordance with Theorem 3.35 [12, p.281] the set \( \mathcal{D}(H) \) is a core of the operator \( H^{\frac{1}{2}} \), then we can extend (3) to
\[
\left( H^{\frac{1}{2}} f, f \right)_\mathcal{H} \geq \sqrt{C_0} \|f\|_\mathcal{H}^2, \ f \in \mathcal{D}(H^{\frac{1}{2}}).
\] (17)

Hence \( \text{N}(H^{\frac{1}{2}}) = 0 \). Combining this fact and (14), we obtain
\[
(B_1 + B_2) H^{\frac{1}{2}} f = 0, \ f \in \mathcal{M}.
\] (18)

Let us show that the set \( \mathcal{M} \) is a core of the operator \( H^{\frac{1}{2}} \). Note that due to Theorem 3.35 [12, p.281] the operator \( H^{\frac{1}{2}} \) is selfadjoint and \( \mathcal{D}(H) \) is a core of the operator \( H^{\frac{1}{2}} \). Hence we have the representation
\[
\|H^{\frac{1}{2}} f\|_\mathcal{H}^2 = (H f, f)_\mathcal{H}, \ f \in \mathcal{D}(H).
\] (19)

To achieve our aim, it is sufficient to show the following
\[
\forall f_0 \in \mathcal{D}(H^{\frac{1}{2}}), \ \exists \{f_n\}_1^\infty \subset \mathcal{M} : f_n \overset{\mathcal{H}}{\rightarrow} f_0, \ H^{\frac{1}{2}} f_n \overset{\mathcal{H}}{\rightarrow} H^{\frac{1}{2}} f_0.
\] (20)
Since in accordance with the definition the set $\mathfrak{M}$ is a core of $H$, then we can extend second relation (11) to $\sqrt{C_0}\|f\|_{\mathfrak{B}_+} \leq (Hf,f)_{\mathfrak{B}} \leq \sqrt{C_4}\|f\|_{\mathfrak{B}_+}, f \in D(H)$. Applying (19), we can write
\[ \sqrt{C_0}\|f\|_{\mathfrak{B}_+} \leq \|H^{1/2}f\|_{\mathfrak{B}} \leq \sqrt{C_4}\|f\|_{\mathfrak{B}_+}, f \in D(H). \] (21)

Using lower estimate (21) and the fact that $D(H)$ is a core of $H^{1/2}$, it is not hard to prove that $D(H^{1/2}) \subset \mathfrak{B}_+$. Taking into account this fact and using upper estimate (21), we obtain (20). It implies that $\mathfrak{M}$ is a core of $H^{1/2}$. Note that in accordance with the definition the set $\mathfrak{B}_+$ is $m$-accretive. Hence combining Theorem 3.2 [12, p.268] with (17), we obtain $R(H^{1/2}) = \mathfrak{B}$. Taking into account that $\mathfrak{M}$ is a core of the operator $H^{1/2}$, we conclude that $R(\tilde{R})$ is dense in $\mathfrak{B}$, where $\tilde{R}$ is the restriction of the operator $H^{1/2}$ to $\mathfrak{M}$. Finally, by virtue of (18), we have that the sum $B_1 + B_2$ equal to zero on the dense subset of $\mathfrak{B}$. Since these operators are defined on $\mathfrak{B}$ and bounded, then $B_1 = -B_2$. Further, we use the denotation $B_1 := B$.

Note that due to Lemma 3.2 there exist the operators $R_{W}^*, R_{W+}^*$. Using the properties of the operator $B$, we get $\|(I \pm iB)f\|_{\mathfrak{B}_+} \geq \|f\|_{\mathfrak{B}_+}$, $f \in \mathfrak{B}$. Hence
\[ \|(I \pm iB)f\|_{\mathfrak{B}} \geq \|f\|_{\mathfrak{B}}, f \in \mathfrak{B}. \]

It implies that the operators $I \pm iB$ are invertible. Since it was proved above that $R(H^{1/2}) = \mathfrak{B}$, $N(H^{1/2}) = 0$, then there exists an operator $H^{-1/2}$ defined on $\mathfrak{B}$. Using representation (13) and taking into account the reasonings given above, we obtain
\[ R_{W} = H^{-1/2}(I + iB)^{-1}H^{-1/2}, \quad R_{W+} = H^{-1/2}(I - iB)^{-1}H^{-1/2}. \] (22)

Note that the following equality can be proved easily $R_{W}^* = R_{W+}$. Hence we have
\[ V = \frac{1}{2}(R_{W} + R_{W+}). \] (23)

Combining (22), (23), we get
\[ V = \frac{1}{2}H^{-1/2}[(I + iB)^{-1} + (I - iB)^{-1}]H^{-1/2}. \] (24)

Using the obvious identity $(I + B^2) = (I + iB)(I - iB) = (I - iB)(I + iB)$, by direct calculation we get
\[ (I + iB)^{-1} + (I - iB)^{-1} = (I + B^2)^{-1}. \] (25)

Combining (24), (25), we obtain
\[ V = \frac{1}{2}H^{-1/2}(I + B^2)^{-1}H^{-1/2}. \] (26)

Let us evaluate the form $(Vf,f)_{\mathfrak{B}}$. Note that there exists the operator $R_{H}$ (see Lemma 3.3). Since $H$ is selfadjoint (see Lemma 3.3), then due to Theorem 3 [11, p.136] $R_{H}$ is selfadjoint. It is clear that $R_{H}$ is positive because $H$ is positive. Hence by virtue of the well-known theorem (see [24, p.174]) there exists a unique square root of the operator $R_{H}$, the selfadjoint operator $\hat{R}$ such that $\hat{R}R = R_{H}$. Using the decomposition $H = H^{1/2}H^{1/2}$, we get $H^{1/2}H^{-1/2}H = I$. Hence $R_{H} \subset H^{1/2}H^{-1/2}$, but $D(R_{H}) = \mathfrak{B}$. It implies that $R_{H} = H^{1/2}H^{-1/2}$. Using the uniqueness property
of square root we obtain $H^{-\frac{1}{2}} = \tilde{R}$. Let us use the shorthand notation $S := I + B^2$. Note that due to the obvious inequality $(\|Sf\|_\beta \geq \|f\|_\beta, f \in \mathcal{H})$ the operator $S^{-1}$ is bounded on the set $R(S)$. Taking into account the reasoning given above, we get

$$(V f, f)_\beta = \left( H^{-\frac{1}{2}} S^{-1} H^{-\frac{1}{2}} f, f \right)_\beta = \left( S^{-1} H^{-\frac{1}{2}} f, H^{-\frac{1}{2}} f \right)_\beta \leq$$

$$\leq \|S^{-1} H^{-\frac{1}{2}} f\|_\beta \|H^{-\frac{1}{2}} f\|_\beta \leq \|S^{-1}\| \cdot \|H^{-\frac{1}{2}} f\|_\beta^2 = \|S^{-1}\| \cdot (R_H f, f)_\beta, f \in \mathcal{H}.$$ 

On the other hand, it is easy to see that $(S^{-1} f, f)_\beta \geq \|S^{-1} f\|_\beta^2, f \in R(S)$. At the same time it is obvious that $S$ is bounded and we have $\|S^{-1} f\|_\beta \geq \|S\|^{-1} \|f\|_\beta, f \in R(S)$. Using these estimates, we have

$$(V f, f)_\beta = \left( S^{-1} H^{-\frac{1}{2}} f, H^{-\frac{1}{2}} f \right)_\beta \geq \|S^{-1} H^{-\frac{1}{2}} f\|_\beta^2 \geq$$

$$\geq \|S\|^{-2} \cdot \|H^{-\frac{1}{2}} f\|_\beta^2 = \|S\|^{-2} \cdot (R_H f, f)_\beta, f \in \mathcal{H}.$$ 

Note that due to Theorem 3.4 the operator $R_H$ is compact. Combining (23) with Theorem 3.4 we get that the operator $V$ is compact. Taking into account these facts and using Lemma 1.1 [10, p.45], we obtain (12).

**Remark 3.6.** Since it was proved above that $R_H$ is selfadjoint and positive, then we have $\lambda_i(R_H) = s_i(R_H), i \in \mathbb{N}$. Note that in accordance with the facts established above the operator $H := \tilde{W}\|_1$ has a discrete spectrum and a compact resolvent. Due to results represented in [35], [3], [11], we have an opportunity to obtain order of the operator $H$ in an easy way in most particular cases.

The following theorem is formulated in terms of order $\mu := \mu(H)$ and devoted to the Schatten-von Neumann classification of the operator $R_W$.

**Theorem 3.7.** We have the following classification

$$R_W \in \mathfrak{S}_p, \ p = \left\{ \begin{array}{ll}
1, & l > 2/\mu, \mu \leq 1, \\
1, & \mu > 1,
\end{array} \right.$$ 

Moreover under the assumption $\lambda_n(R_H) \geq C n^{-\mu}, n \in \mathbb{N}$, we have

$$R_W \in \mathfrak{S}_p \Rightarrow \mu p > 1, \ 1 \leq p < \infty,$$

where $\mu := \mu(H)$.

**Proof.** Consider the case ($\mu \leq 1$). Since we already know that $R^*_W = R_{W^*}$, then it can easily be checked that the operator $R^*_W R_W$ is a selfadjoint positive compact operator. Due to the well-known fact [24, p.174] there exists the operator $|R_W|$. By virtue of Theorem 9.2 [24, p.178] the operator $|R_W|$ is compact. Since $\mathbb{N}(|R_W|^2) = 0$, it follows that $\mathbb{N}(|R_W|) = 0$. Hence applying Theorem 1.1 [11, p.189], we get that the operator $|R_W|$ has an infinite set of the eigenvalues. Using condition (2) (iii), we get

$$\text{Re}(R_W f, f)_\beta \geq C_0 \|R_W f\|_\beta^2, f \in \mathcal{H}.$$ 

Hence

$$(|R_W|^2 f, f)_\beta = \|R_W f\|_\beta^2 \leq C_0^{-1} \text{Re}(R_W f, f)_\beta = C_0^{-1} (V f, f)_\beta, \ V := (R_W)_{\mathfrak{R}}.$$ 

Since we already know that the operators $|R_W|^2, V$ are compact, then using Lemma 1.1 [10, p.45], Theorem 3.5, we get

$$\lambda_i(|R_W|^2) \leq C_0^{-1} \lambda_i(V) \leq C i^{-\mu}, \ i \in \mathbb{N}. \quad (27)$$
Recall that by definition we have \( s_i(R_{\tilde{W}}) = \lambda_i(|R_{\tilde{W}}|) \). Note that the operators \(|R_{\tilde{W}}|, |R_{\tilde{W}}|^2\) have the same eigenvectors. This fact can be easily proved if we note the obvious relation \(|R_{\tilde{W}}| f_i = |\lambda_i(|R_{\tilde{W}}|)|^2 f_i, i \in \mathbb{N}\) and the spectral representation for the square root of a selfadjoint positive compact operator

\[
|R_{\tilde{W}}| f = \sum_{i=1}^{\infty} \sqrt{\lambda_i(|R_{\tilde{W}}|^2)} (f, \varphi_i) \varphi_i, \quad f \in \mathcal{H},
\]

where \( f_i, \varphi_i \) are the eigenvectors of the operators \(|R_{\tilde{W}}|, |R_{\tilde{W}}|^2\) respectively (see (10.25) [24, p.201]). Hence \( \lambda_i(|R_{\tilde{W}}|) = \sqrt{\lambda_i(|R_{\tilde{W}}|^2)}, i \in \mathbb{N}\). Combining this fact with (27), we get

\[
\sum_{i=1}^{\infty} s_i^2(R_{\tilde{W}}) = \sum_{i=1}^{\infty} \lambda_i^2(|R_{\tilde{W}}|^2) \leq C \sum_{i=1}^{\infty} i^{-\mu}.
\]

This completes the proof for the case \( (\mu \leq 1) \).

Consider the case \( (\mu > 1) \). It follows from (23) that the operator \( V \) is positive and bounded. Hence by virtue of Lemma 8.1 [10, p.126], we have that for any orthonormal basis \( \{\psi_i\}_1^\infty \subset \mathcal{H} \) the following equalities hold

\[
\sum_{i=1}^{\infty} \Re(R_{\tilde{W}} \psi_i, \psi_i)_{\mathcal{H}} = \sum_{i=1}^{\infty} (V \psi_i, \psi_i)_{\mathcal{H}} = \sum_{i=1}^{\infty} (V \varphi_i, \varphi_i)_{\mathcal{H}},
\]

where \( \{\varphi_i\}_1^\infty \) is the orthonormal basis of the eigenvectors of the operator \( V \). Due to Theorem 3.5, we get

\[
\sum_{i=1}^{\infty} (V \varphi_i, \varphi_i)_{\mathcal{H}} = \sum_{i=1}^{\infty} s_i(V) \leq C \sum_{i=1}^{\infty} i^{-\mu}.
\]

By virtue of Lemma 3.1, we get \( |\Im(R_{\tilde{W}} \psi_i, \psi_i)| \leq k^{-1}(\xi) \Re(R_{\tilde{W}} \psi_i, \psi_i)_{\mathcal{H}} \). Combining this fact with (28), we get that the following series is convergent

\[
\sum_{i=1}^{\infty} (R_{\tilde{W}} \psi_i, \psi_i)_{\mathcal{H}} < \infty.
\]

Hence by definition [10, p.125] the operator \( R_{\tilde{W}} \) has a finite matrix trace. Using Theorem 8.1 [10, p.127], we get \( R_{\tilde{W}} \in \mathcal{G}_1 \). This completes the proof for the case \( (\mu > 1) \).

Now, assume that \( \lambda_n(R_{H}) \geq C n^{-\mu}, n \in \mathbb{N}, 0 \leq \mu < \infty \). Let us show that the operator \( V \) has the complete orthonormal system of the eigenvectors. Using formula (26), we get

\[
V^{-1} = 2H^{1/2}(I + B^2)H^{1/2}, \quad D(V^{-1}) = R(V).
\]

Let us prove that \( D(V^{-1}) \subset D(H) \). Note that the set \( D(V^{-1}) \) consists of the elements \( f + g \), where \( f \in D(W), g \in D(W^+) \). Using representation (13), it is easy to prove that \( D(W) \subset D(H), D(W^+) \subset D(H) \). This gives the desired result. Taking into account the facts proven above, we get

\[
(V^{-1} f, f)_{\mathcal{H}} = 2(\Re(H^{1/2}f, H^{1/2}f))_{\mathcal{H}} \geq 2\|H^{1/2}f\|^2_{\mathcal{H}} = 2(H f, f)_{\mathcal{H}}, \quad f \in D(V^{-1}),
\]

where \( S = I + B^2 \). Since \( V \) is selfadjoint, then due to Theorem 3 [11, p.136] the operator \( V^{-1} \) is selfadjoint. Combining (29) with Lemma 3.3, we get that \( V^{-1} \) is strictly accretive. Using these facts we can write

\[
\|f\|_{V^{-1}} \geq C\|f\|_{H}, \quad f \in \mathcal{H}.
\]
Since the operator $H$ has a discrete spectrum (see Theorem 5.3 [22]), then any set bounded with respect to the norm $\mathfrak{H}_H$ is a compact set with respect to the norm $\mathfrak{H}$ (see Theorem 4 [32, p.220]). Combining this fact with (30), Theorem 3 [32, p.216], we get that the operator $V^{-1}$ has a discrete spectrum, i.e. it has the infinite set of the eigenvalues $\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_i \leq \ldots$, $\lambda_i \to \infty$, $i \to \infty$ and the complete orthonormal system of the eigenvectors. Now note that the operators $V$, $V^{-1}$ have the same eigenvectors. Therefore the operator $V$ has the complete orthonormal system of the eigenvectors. Recall that any complete orthonormal system is a basis in separable Hilbert space. Hence the complete orthonormal system of the eigenvectors of the operator $V$ is a basis in the space $\mathfrak{H}$. Let $\{\varphi_i\}_{i=1}^\infty$ be the complete orthonormal system of the eigenvectors of the operator $V$ and suppose $R_{\tilde{W}} \in \mathcal{S}_p$; then by virtue of inequalities (7.9) [10, p.123], Theorem 3.5, we get

$$\sum_{i=1}^\infty |s_i(R_{\tilde{W}})|^p \geq \sum_{i=1}^\infty |(R_{\tilde{W}} \varphi_i, \varphi_i)_\mathfrak{H}|^p \geq \sum_{i=1}^\infty |\text{Re}(R_{\tilde{W}} \varphi_i, \varphi_i)_\mathfrak{H}|^p =$$

$$= \sum_{i=1}^\infty |(V \varphi_i, \varphi_i)_\mathfrak{H}|^p = \sum_{i=1}^\infty |\lambda_i(V)|^p \geq C \sum_{i=1}^\infty i^{-\mu p}.$$ 

We claim that $\mu p > 1$. Assuming the converse in the previous inequality, we come to contradiction with the condition $R_{\tilde{W}} \in \mathcal{S}_p$. This completes the proof. 

The following theorem establishes the completeness property of the system of root vectors of the operator $R_{\tilde{W}}$.

**Theorem 3.8.** Suppose $\theta < \pi \mu/2$; then the system of root vectors of the operator $R_{\tilde{W}}$ is complete, where $\theta$ is the semi-angle of the sector $\Sigma_0(\theta) \supset \Theta(\tilde{W})$, $\mu := \mu(H)$.

**Proof.** Using Lemma 3.1, we have

$$|\text{Im}(R_{\tilde{W}} f, f)_\mathfrak{H}| \leq k^{-1}(\xi) \text{Re}(R_{\tilde{W}} f, f)_\mathfrak{H}, f \in \mathfrak{H}. \quad (31)$$

Therefore $\overline{\Theta(R_{\tilde{W}})} \subset \Sigma_0(\theta)$. Note that the map $z : \mathbb{C} \to \mathbb{C}$, $z = 1/\zeta$ takes each eigenvalue of the operator $R_{\tilde{W}}$ to the eigenvalue of the operator $\tilde{W}$. It is also clear that $z : \Sigma_0(\theta) \to \Sigma_0(\theta)$. Using the definition [10, p.302] let us consider the following set

$$\Psi := \left\{ z : z = t \xi, \xi \in \overline{\Theta(R_{\tilde{W}})}, 0 \leq t < \infty \right\}.$$ 

It is easy to see that $\Psi$ coincides with a closed sector of the complex plane with the vertex situated at the point zero. Let us denote by $\vartheta(R_{\tilde{W}})$ the angle of this sector. It is obvious that $\Psi \subset \Sigma_0(\theta)$. Therefore $0 \leq \vartheta(R_{\tilde{W}}) \leq 2\theta$. Let us prove that $0 < \vartheta(R_{\tilde{W}})$, i.e. the strict inequality holds. If we assume that $\vartheta(R_{\tilde{W}}) = 0$, then we get $e^{-\text{arg} z} = \zeta, \forall z \in \Psi \setminus 0$, where $\zeta$ is a constant independent on $z$. In consequence of this fact we have $\text{Im} \Theta(\varsigma R_{\tilde{W}}) = 0$. Hence the operator $\varsigma R_{\tilde{W}}$ is symmetric (see Problem 3.9 [12, p.269]) and by virtue of the fact $\text{D}(\varsigma R_{\tilde{W}}) = \mathfrak{H}$ one is selfadjoint. On the other hand, taking into account the equality $R_{\tilde{W}}^* = R_{\tilde{W}}$ (see the proof of Theorem 3.5), we have $(\varsigma R_{\tilde{W}} f, g)_\mathfrak{H} = (f, \varsigma R_{\tilde{W}}^* g)_\mathfrak{H}, f, g \in \mathfrak{H}$. Hence $\varsigma R_{\tilde{W}} = \varsigma R_{\tilde{W}}^*$. In the particular case we have $\forall f \in \mathfrak{H}, \text{Im} f = 0 : \text{Re} \varsigma R_{\tilde{W}} f = \text{Re} \varsigma R_{\tilde{W}}^* f, \text{Im} \varsigma R_{\tilde{W}} f = -\text{Im} \varsigma R_{\tilde{W}}^* f$. It implies that $\text{N}(R_{\tilde{W}}) \neq 0$. This contradiction concludes the proof of the fact $\vartheta(R_{\tilde{W}}) > 0$. Let us use Theorem 6.2 [10, p.305] according to which we have the following. If the following two
Moreover if \( \nu(R_{W}) = \pi/d \), where \( d > 1 \),

a) \( \vartheta(R_{W}) = \pi/d \), where \( d > 1 \),

b) for some \( \beta \), the operator \( B := (e^{i\beta}R_{W})_{3} \) : \( s_{i}(B) = o(i^{-1/d}) \), \( i \to \infty \).

Let us show that conditions (a) and (b) are fulfilled. Note that due to Lemma 3.1 we have \( 0 \leq \theta < \pi/2 \). Hence \( 0 < \vartheta(R_{W}) < \pi \). It implies that there exists \( 1 < d < \infty \) such that \( \vartheta(R_{W}) = \pi/d \). Thus condition (a) is fulfilled. Let us choose the certain value \( \beta = \pi/2 \) in condition (b) and notice that \( (e^{i\pi/2}R_{W})_{3} = (R_{W})_{3} \). Since the operator \( V := (R_{W})_{3} \) is selfadjoint, then we have \( s_{i}(V) = \lambda_{i}(V) \), \( i \in \mathbb{N} \). In consequence of Theorem 3.5, we obtain

\[
\sum_{i=1}^{n} \frac{1}{i^1/d} = \frac{s_{i}(V)}{i^1/d} \leq C \cdot i^{1/d-\mu}, \quad i \in \mathbb{N}.
\]

Hence to achieve condition (b), it is sufficient to show that \( d > \mu^{-1} \). By virtue of the conditions \( \vartheta(R_{W}) \leq 2\theta, \theta < \pi\mu/2 \), we have \( d = \pi/\vartheta(R_{W}) \geq \pi/2\theta > \mu^{-1} \). Hence we obtain \( s_{i}(V) = o(i^{-1/d}) \).

Since both conditions (a),(b) are fulfilled, then using Theorem 6.2 [10, p.305] we complete the proof.

Proven Theorem 3.7 is devoted to the description of \( s \)-numbers behavior but questions related with asymptotic of the eigenvalues \( \lambda_{i}(R_{W}) \), \( i \in \mathbb{N} \) are still relevant in our work. It is a well-known fact that for any bounded operator with the compact imaginary component there is a relationship between \( s \)-numbers of the imaginary component and the eigenvalues (see [10]). Similarly using the information on \( s \)-numbers of the real component, we can obtain an asymptotic formula for the eigenvalues \( \lambda_{i}(R_{W}) \), \( i \in \mathbb{N} \). This idea is realized in the following theorem.

**Theorem 3.9.** The following inequality holds

\[
\sum_{i=1}^{n} |\lambda_{i}(R_{W})|^{p} \leq \sec^{\theta} \theta \| S^{-1} \| \sum_{i=1}^{n} \lambda_{i}^{p}(R_{H}), \quad (n = 1, 2, ..., \nu(R_{W})), \quad 1 \leq p < \infty.
\]

Moreover if \( \nu(R_{W}) = \infty \) and the order \( \mu(H) \neq 0 \), then the following asymptotic formula holds

\[
|\lambda_{i}(R_{W})| = o\left(i^{-\mu+\varepsilon}\right), \quad i \to \infty, \quad \forall \varepsilon > 0.
\]

**Proof.** Let \( L \) be a bounded operator with a compact imaginary component. Note that according to Theorem 6.1 [10, p.81], we have

\[
\sum_{m=1}^{k} |\text{Im} \lambda_{m}(L)|^{p} \leq \sum_{m=1}^{k} |s_{m}(L_{3})|^{p}, \quad (k = 1, 2, ..., \nu_{3}(L)), \quad 1 \leq p < \infty,
\]

where \( \nu_{3}(L) \leq \infty \) is the sum of all algebraic multiplicities corresponding to the not real eigenvalues of the operator \( L \) (see [10, p.79]). It can easily be checked that

\[
(iL)_{3} = L_{3}, \quad \text{Im} \lambda_{m}(iL) = \text{Re} \lambda_{m}(L), \quad m \in \mathbb{N}.
\]
By virtue of (31), we have \( \Re \lambda_m(R_{\tilde{W}}) > 0, m = 1, 2, ..., \nu(R_{\tilde{W}}) \). Combining this fact with (35), we get \( \nu_2(iR_{\tilde{W}}) = \nu(R_{\tilde{W}}) \). Taking into account the previous equality and combining (34), (35), we obtain

\[
\sum_{m=1}^{k} |\Re \lambda_m(R_{\tilde{W}})|^p \leq \sum_{m=1}^{k} |s_m(V)|^p, \ (k = 1, 2, ..., \nu(R_{\tilde{W}})), \ V := (R_{\tilde{W}})_{\mathbb{R}}.
\]  

(36)

Note that by virtue of (31), we have

\[
|\Im \lambda_m(R_{\tilde{W}})| \leq \tan \theta \Re \lambda_m(R_{\tilde{W}}), \ m \in \mathbb{N}.
\]

Hence

\[
|\lambda_m(R_{\tilde{W}})| = \sqrt{|\Im \lambda_m(R_{\tilde{W}})|^2 + |\Re \lambda_m(R_{\tilde{W}})|^2} \leq \sqrt{\tan^2 \theta + 1} |\Re \lambda_m(R_{\tilde{W}})|, \ m \in \mathbb{N}.
\]

(37)

Combining (36), (3), we get

\[
\sum_{m=1}^{k} |\lambda_m(R_{\tilde{W}})|^p \leq \sec^p \theta \sum_{m=1}^{k} |s_m(V)|^p, \ (k = 1, 2, ..., \nu(R_{\tilde{W}})).
\]

Using (12), we complete the proof of inequality (32).

Suppose \( \nu(R_{\tilde{W}}) = \infty, \mu(H) \neq 0 \) and let us prove (33). Note that for \( \mu > 0 \) and for any \( \varepsilon > 0 \), we can choose \( p \) so that \( \mu p > 1, \mu - \varepsilon < 1/p \). Using the condition \( \mu p > 1 \), we obtain convergence of the series on the left side of (32). It implies that

\[
|\lambda_i(R_{\tilde{W}})|^{1/p} \to 0, \ i \to \infty.
\]

(38)

It is obvious that

\[
|\lambda_i(R_{\tilde{W}})|^{\mu - \varepsilon} < |\lambda_i(R_{\tilde{W}})|^{1/p}, \ i \in \mathbb{N}.
\]

Taking into account (38), we obtain (33).

\[\square\]

4 Applications

1. We begin with definitions. Suppose \( \Omega \) is a convex domain of the \( n \)-dimensional Euclidean space with the sufficient smooth boundary, \( L_2(\Omega) \) is a complex Lebesgue space of summable with square functions, \( H^2(\Omega), H^1(\Omega) \) are complex Sobolev spaces, \( D_i f := \partial f / \partial x_i, 1 \leq i \leq n \) is the weak partial derivatives of the function \( f \). Consider a sum of a uniformly elliptic operator and the extension of the Kipriyanov fractional differential operator of order 0 < \( \alpha < 1 \) (see Lemma 2.5 [22])

\[
Lu := -D_j(a^{ij} D_i f) + \mathcal{D}_{0+}^a f,
\]

\[
D(L) = H^2(\Omega) \cap H^1_0(\Omega),
\]

with the following assumptions relative to the real-valued coefficients

\[
a^{ij}(Q) \in C^1(\bar{\Omega}), a^{ij} \xi_i \xi_j \geq a|\xi|^2, \ a > 0.
\]

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It was proved in the paper [22] that the operator $L^+ f := -D_t (a^{ij} D_j f) + D^a_{\alpha \mu} f$, $D(L^+) = D(L)$ is formal adjoint with respect to $L$. Note that in accordance with Theorem 2 [23] we have $R(L) = R(L^+) = L_2(\Omega)$, due to Theorem 4.2 [22], the operators $L, L^+$ are strictly accretive. Taking into account these facts we can conclude that the operators $L, L^+$ are strictly accretive. Taking into account these facts we can conclude that the operators $L, L^+$ are strictly accretive. Hence in consequence we have that $D(\mathcal{T})$ is a core of the operator $\mathcal{T}$. Consider the trivial decomposition of the operator $L$ on the sum of the main part and the operator-perturbation, where the main part must be an operator of a special type either a selfadjoint or a normal operator. Note that a uniformly elliptic operator of second order is neither selfadjoint no normal in general case. To demonstrate the significance of the method obtained in this paper, we would like to note that a search for a convenient decomposition of the initial operator $L$ on a sum of a selfadjoint operator and the operator-perturbation does not seem to be a reasonable way. Now to justify this claim we consider one of possible decompositions of $L$ on a sum. Consider a selfadjoint strictly accretive operator $\mathcal{T} : \mathcal{H} \rightarrow \mathcal{H}$.

**Definition 4.1.** In accordance with the definition of the paper [37], a quadratic form $a := a[f]$ is called a $\mathcal{T}$ - subordinated form if the following condition holds

$$|a[f]| \leq b t[f] + M \|f\|^2_{L^2}, \quad D(a) \supset D(t), \quad b < 1, \quad M > 0, \quad \text{(39)}$$

where $t[f] = \|\mathcal{T}^{\frac{1}{2}} f\|^2_{L^2}, \quad f \in D(\mathcal{T}^{\frac{1}{2}})$. The form $a$ is called a completely $\mathcal{T}$ - subordinated form if besides of (39) we have the following additional condition $\forall \varepsilon > 0 \exists b, M > 0 : b < \varepsilon$.

Let us consider the trivial decomposition of the operator $L$ on the sum $L = 2L_\text{tr} - L^+$ and let us use the notation $\mathcal{T} := 2L_\text{tr}, \quad A := -L^+$. Then we have $L = \mathcal{T} + A$. Due to the sectorial property proven in Theorem 4.2 [22] we have

$$|(Af, f)_{L^2}| = \sec \theta_f |\text{Re}(Af, f)_{L^2}| = \sec \theta_f \frac{1}{2} (\mathcal{T} f, f)_{L^2}, \quad f \in D(\mathcal{T}), \quad \text{(40)}$$

where $0 \leq \theta_f \leq \theta, \quad \theta_f := \arg(\mathcal{T}^{\frac{1}{2}} f, f)_{L^2}, \quad L^2 := L^2(\Omega)$ and $\theta$ is the semi-angle corresponding to the sector $\Sigma_0(\theta)$. Due to Theorem 4.3 [22] the operator $\mathcal{T}$ is $\mathcal{M}$-accretive. Hence in consequence of Theorem 3.35 [22, p.281] we have that $D(\mathcal{T})$ is a core of the operator $\mathcal{T}^{\frac{1}{2}}$. It implies that we can extend relation (10) to

$$\frac{1}{2} t[f] \leq |a[f]| \leq \sec \theta_f \frac{1}{2} t[f], \quad f \in D(t), \quad \text{(41)}$$

where $a$ is a quadratic form generated by $A$ and $t[f] = \|\mathcal{T}^{\frac{1}{2}} f\|^2_{L^2}$. If we consider the case $0 < \theta < \pi/3$, then it is obvious that there exist constants $b < 1$ and $M > 0$ such that the following inequality holds

$$|a[f]| \leq b t[f] + M \|f\|^2_{L^2}, \quad f \in D(t).$$

Hence the form $a$ is a $\mathcal{T}$ - subordinated form. In accordance with the definition given in the paper [37] it means $\mathcal{T}$ - subordination of the operator $A$ in the sense of form. Assume that $\forall \varepsilon > 0 \exists b, M > 0 : b < \varepsilon$. Using inequality (11), we get

$$\frac{1}{2} t[f] \leq \varepsilon t[f] + M(\varepsilon) \|f\|^2_{L^2}, \quad t[f] \leq \frac{2M(\varepsilon)}{(1 - 2\varepsilon)} \|f\|^2_{L^2}, \quad f \in D(t), \quad \varepsilon < 1/2.$$
Using the strictly accretive property of the operator $L$ (see inequality (4.9) [22]), we obtain

$$\|f\|_{H^1_0}^2 \leq t|f|, \quad f \in D(t).$$

On the other hand, using the results of the paper [22], it is easy to prove that $H^1_0(\Omega) \subset D(t)$.

Taking into account the facts considered above, we get

$$\|f\|_{H^1_0} \leq C\|f\|_{L_2}, \quad f \in H^1_0(\Omega).$$

It cannot be! It is a well-known fact. This contradiction shows us that the form $a$ is not a completely $T$-subordinated form. It implies that we cannot use Theorem 8.4 [37] which could give us an opportunity to describe the spectral properties of the operator $L$. Note that the reasonings corresponding to another trivial decomposition of $L$ on a sum is analogous.

This rather particular example does not aim to show the inability of using remarkable methods considered in the paper [37] but only creates prerequisite for some value of another method based on using spectral properties of the real component of the initial operator $L$. Now we would like to demonstrate the effectiveness of this method. Suppose $\tilde{\mathcal{S}} := L_2(\Omega), \tilde{\mathcal{S}}^+ := H^1_0(\Omega), T:\世代_{\tilde{\mathcal{S}}} \to \tilde{\mathcal{S}}^+$, $Af := \mathcal{D}^a_{I}f, \quad D(T), D(A) = H^2(\Omega) \cap H^1_0(\Omega)$; then due to the Rellich-Kondrachov theorem we have that condition (1) is fulfilled. Due to the results obtained in the paper [22] we have that condition (2) is fulfilled. Applying the results obtained in the paper [22] we conclude that the operator $L_0$ has non-zero order. Hence we can apply the abstract results of this paper to the operator $L$. In fact, Theorems 3.7-3.9 describe the spectral properties of the operator $L$.

2. We deal with the differential operator acting in the complex Sobolev space and defined by the following expression

$$\mathcal{L}f := (c_kf^{(k)})^{(k)} + (c_{k-1}f^{(k-1)})^{(k-1)} + \ldots + c_0f,$$

$$D(\mathcal{L}) = H^2(I) \cap H^k(I), \quad k \in \mathbb{N},$$

where $I := (a, b) \subset \mathbb{R}$, the complex-valued coefficients $c_j(x) \in C^{(j)}(\overline{I})$ satisfy the condition $\text{sign}(\text{Re}c_j) = (-1)^j, \quad j = 1, 2, \ldots, k$. It is easy to see that

$$\text{Re}(\mathcal{L}f, f)_{L_2(I)} \geq \sum_{j=0}^{k} |\text{Re}c_j| \|f^{(j)}\|_{L_2(I)}^2 \geq C\|f^{(j)}\|_{H^0_0(I)}^2, \quad f \in D(\mathcal{L}).$$

On the other hand

$$|\langle \mathcal{L}f, f \rangle_{L_2(I)}| = \left| \sum_{j=0}^{k} (-1)^j (c_jf^{(j)}, g^{(j)})_{L_2(I)} \right| \leq \sum_{j=0}^{k} |(c_jf^{(j)}, g^{(j)})_{L_2(I)}| \leq$$

$$\leq C \sum_{j=0}^{k} \|f^{(j)}\|_{L_2(I)} \|g^{(j)}\|_{L_2(I)} \leq \|f\|_{H^0_0(I)} \|g\|_{H^0_0(I)}, \quad f \in D(\mathcal{L}).$$

Consider the Riemann-Liouville operators of fractional differentiation of arbitrary non-negative order $\alpha$ (see [36, p.44]) defined by the expressions

$$D_+^{\alpha} f = \left( \frac{d}{dx} \right)^{[\alpha]+1} I_+^{1-\{\alpha\}} f; \quad D_-^{\alpha} f = \left( -\frac{d}{dx} \right)^{[\alpha]+1} I_-^{1-\{\alpha\}} f.$$
where the fractional integrals of arbitrary positive order \( \alpha \) defined by

\[
(I_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad (I_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b \frac{f(t)}{(t-x)^{1-\alpha}} dt, \quad f \in L_1(I).
\]

Suppose \( 0 < \alpha < 1, f \in AC^{l+1}(\bar{I}), f^{(j)}(a) = f^{(j)}(b) = 0, j = 0, 1, \ldots, l; \) then the next formulas follow from Theorem 2.2 [36, p.46]

\[
D_{a+}^{\alpha+1} f = I_{a+}^{1-\alpha} f^{(l+1)}, \quad D_{b-}^{\alpha+1} f = (-1)^{l+1} I_{b-}^{1-\alpha} f^{(l+1)}.
\]  (42)

Further, we need the following inequalities (see [20])

\[
\text{Re}(D_{a+}^\alpha f, f)_{L_2(I)} \geq C \|f\|^2_{L_2(I)}, \quad f \in I_{a+}^\alpha(L_2),
\]

\[
\text{Re}(D_{b-}^\alpha f, f)_{L_2(I)} \geq C \|f\|^2_{L_2(I)}, \quad f \in I_{b-}^\alpha(L_2),
\]  (43)

where \( I_{a+}^\alpha(L_2), I_{b-}^\alpha(L_2) \) are the classes of the functions representable by the fractional integrals (see [36]). Consider the following operator with the constant real-valued coefficients

\[
D f := p_n D_{a+}^{\alpha_n} + q_n D_{b-}^{\beta_n} + p_{n-1} D_{a+}^{\alpha_{n-1}} + q_{n-1} D_{b-}^{\beta_{n-1}} + \ldots + p_0 D_{a+}^{\alpha_0} + q_0 D_{b-}^{\beta_0},
\]

\[
D(D) = H^{2k}(I) \cap H_0^k(I), \quad n \in \mathbb{N},
\]

where \( \alpha_j, \beta_j \geq 0, 0 \leq [\alpha_j], [\beta_j] < k, j = 0, 1, \ldots, n, \)

\[
q_j \geq 0, \quad \text{sign} p_j = \begin{cases} (-1)^{\frac{[\alpha_j]+1}{2}}, & [\alpha_j] = 2m - 1, \quad m \in \mathbb{N}, \\ (-1)^{\frac{[\alpha_j]}{2}}, & [\alpha_j] = 2m, \quad m \in \mathbb{N}_0. \end{cases}
\]

Using (42), (43), we get

\[
(p_j D_{a+}^{\alpha_j} f, f)_{L_2(I)} = p_j \left( \left( \frac{d}{dx} \right)^{m} D_{a+}^{m+\{\alpha_j\}} f, f \right)_{L_2(I)} = (-1)^{m} p_j \left( I_{a+}^{1-\{\alpha_j\}} f^{(m)} f^{(m)} \right)_{L_2(I)} \geq
\]

\[
\geq C \left\| I_{a+}^{1-\{\alpha_j\}} f^{(m)} \right\|_{L_2(I)}^2 = C \left\| D_{a+}^{\{\alpha_j\}} f^{(m-1)} \right\|_{L_2(I)}^2 \geq C \left\| f^{(m-1)} \right\|_{L_2(I)}^2,
\]

where \( f \in D(D) \) is a real-valued function and \( [\alpha_j] = 2m - 1, m \in \mathbb{N}. \) Similarly, we obtain for orders \( [\alpha_j] = 2m, m \in \mathbb{N}_0 \)

\[
(p_j D_{a+}^{\alpha_j} f, f)_{L_2(I)} = p_j \left( D_{a+}^{2m+\{\alpha_j\}} f, f \right)_{L_2(I)} = (-1)^{m} p_j \left( D_{a+}^{m+\{\alpha_j\}} f, f \right)_{L_2(I)} =
\]

\[
= (-1)^{m} p_j \left( D_{a+}^{\{\alpha_j\}} f^{(m)} f^{(m)} \right)_{L_2(I)} \geq C \left\| f^{(m)} \right\|_{L_2(I)}^2.
\]

Thus in both cases we have

\[
(p_j D_{a+}^{\alpha_j} f, f)_{L_2(I)} \geq C \left\| f^{(s)} \right\|_{L_2(I)}^2, \quad s = \left[ \frac{[\alpha_j]}{2} \right].
\]

In the same way, we obtain the inequality

\[
(q_j D_{b+}^{\beta_j} f, f)_{L_2(I)} \geq C \left\| f^{(s)} \right\|_{L_2(I)}^2, \quad s = \left[ \frac{[\alpha_j]}{2} \right].
\]
Hence in the complex case we have
\[ \Re(\mathcal{D}f, f)_{L_2(I)} \geq C \|f\|_{L_2(I)}^2, \quad f \in \mathcal{D}(\mathcal{D}). \]

Combining Theorem 2.6 \[36\] p.53 with (42), we get
\[ \|p_j D_{a^j} \mathcal{L} + f\|_{L_2(I)} = \|I^{1-\{a_j\}} f^{\{a_j\}+1}\|_{L_2(I)} \leq C \|f\|_{L_2(I)} \leq C \|f\|_{H_0^k(I)}; \]
\[ \|q_j D_{b^j} \mathcal{L} + f\|_{L_2(I)} \leq C \|f\|_{H_0^k(I)}, \quad f \in \mathcal{D}(\mathcal{D}). \]

Hence, we obtain
\[ \|\mathcal{D}f\|_{L_2(I)} \leq C \|f\|_{H_0^k(I)}, \quad f \in \mathcal{D}(\mathcal{D}). \]

Now we can formulate the main result. Consider the operator
\[ G = \mathcal{L} + \mathcal{D}, \]
\[ \mathcal{D}(G) = H^{2k}(I) \cap H_0^k(I). \]

Suppose \( \mathcal{D} := L_2(I), \mathcal{D}^+ := H_0^k(I), \mathcal{T} := \mathcal{L}, A := \mathcal{D}; \) then due to the well-known fact of the Sobolev spaces theory condition (1) is fulfilled, due to the reasonings given above condition (2) is fulfilled. Taking into account the equality
\[ \mathcal{L}f = (\Re c_k f^{(k)})^{(k)} + (\Re c_{k-1} f^{(k-1)})^{(k-1)} + ... + \Re c_0 f, \quad f \in \mathcal{D}(\mathcal{D}) \]

and using the method described in the paper [21], we can prove that the operator \( \hat{G}_f \) has non-zero order. Hence we can successfully apply the abstract results of this paper to the operator \( G \). Indeed, Theorems 3.7-3.9 describe the spectral properties of the operator \( G \).

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