ALTERNATING SUPER-POLYNOMIALS AND SUPER-COINVARIANTS OF FINITE REFLECTION GROUPS

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Abstract. Motivated by a recent conjecture of Zabrocki [19], Wallach [18] described the alternants in the super-coinvariant algebra of the symmetric group in one set of commuting and one set of anti-commuting variables under the diagonal action. We give a type-independent generalization of Wallach’s result to all real reflection groups $G$. As an intermediate step, we explicitly describe the alternating super-polynomials in $\mathbb{C}Z[V] \otimes \Lambda(V)$ for all complex reflection groups, providing an analogue of a classic result of Solomon [12] which describes the invariant super-polynomials in $\mathbb{C}Z[V] \otimes \Lambda(V^*)$. Using our construction, we explicitly describe the alternating harmonics and coinvariants for all real reflection groups.

1. Introduction

The classical coinvariant algebra of a complex reflection group $G \leq \text{GL}(V)$ is the quotient $\mathbb{C}Z[V]/\mathcal{I}_G$ where $\mathbb{C}Z[V] = \text{Sym}(V^*)$ is the $\mathbb{C}$-algebra of polynomial functions on $V$, $\mathbb{C}$ is a subfield of $\mathbb{C}$, and $\mathcal{I}_G$ is the ideal generated by all homogeneous non-constant $G$-invariants. Chevalley [2] showed that $\mathbb{C}Z[V]/\mathcal{I}_G$ as an ungraded module carries the regular representation of $G$. The full graded representation theory of coinvariant algebras and their generalizations is extremely rich and has resulted in a vast body of work (see e.g. [4, 6, 7, 13, 14, 17]).

Chevalley [2] and Shephard–Todd [11] further showed that $\mathbb{C}Z[V]^G = \mathbb{C}[f_1, \ldots, f_n]$ where $f_1, \ldots, f_n$ are $n = \dim(V)$ homogeneous algebraically independent $G$-invariants, and $\mathcal{I}_G^G = \langle f_1, \ldots, f_n \rangle$. Solomon generalized this result to the following explicit description of the $G$-invariants of the Cartan algebra of differential forms on $V$, $\mathbb{C}Z[V] \otimes \Lambda(V^*)$. Here $d$ denotes the exterior derivative.

Theorem 1.1 (Solomon [12]). For a complex reflection group $G$, the $G$-invariants of $\mathbb{C}Z[V] \otimes \Lambda(V^*)$ have $\mathbb{C}$-basis

$$\{ f_1^{i_1} \cdots f_n^{i_n} df_{i_1} \cdots df_{i_r} : \{ i_1 < \cdots < i_r \} \subset [n], \alpha \in \mathbb{Z}_{\geq 0}^n \}. \leqno{(1)}$$

Corollary 1.2. For a complex reflection group $G$ with degrees $d_1, \ldots, d_n$,

$$\text{Hilb}((\mathbb{C}Z[V] \otimes \Lambda(V^*))^G; q, t) = \prod_{i=1}^n \frac{1 + q^{d_i-1}t}{1 - q^{d_i}}. \leqno{(2)}$$

When $G = S_n$, the Cartan algebra $\mathbb{C}Z[V] \otimes \Lambda(V^*)$ may be interpreted as the ring of “super-polynomials” $\mathbb{C}[x_n, \theta_n]$ in commuting variables $x_1, \ldots, x_n$ and anti-commuting variables
\( \theta_1, \ldots, \theta_n \) where \( \sigma(x_i) = x_{\sigma(i)}, \sigma(\theta_i) = \theta_{\sigma(i)} \). The \( t = 0 \) specialization of a recent conjecture of Zabrocki [19] concerns the bigraded \( S_n \)-module structure of the “super-coinvariant algebra” \( \mathbb{k}[x_n, \theta_n]/J_+ \), where \( J_+ \) is the ideal generated by all homogeneous \( S_n \)-invariant super-polynomials. Zabrocki’s conjecture provides an explicit module for the Delta conjecture of Haglund–Remmel–Wilson [5], generalizing the relationship between the diagonal coinvariants and the \( n! \) theorem [3]. As a special case, Zabrocki’s conjecture predicts that the bigraded Hilbert series of the alternating component of \( \mathbb{k}[x_n, \theta_n]/J_+ \) is

\[
Hilb((\mathbb{k}[x_n, \theta_n]/J_+)^{\text{det}}, q, t) = \prod_{i=1}^{n-1} (t + q^i).
\]

Wallach [18, Thm. 13] has recently proven (3). Our primary objective is to give a type-independent generalization of (3) valid for all real reflection groups.

Our first main result is an analogue of Theorem 1.1 for unitary reflection groups \( G \) and the alternating component of \( \mathbb{k}[V] \otimes \Lambda(V) \). Let \( \Delta \in \mathbb{k}[V] \) be the generalized Vandermonde of \( G \) (see §2.3) and let \( \odot \) denote partial differentiation and multiplication operators (see §4).

**Theorem 1.3.** Let \( G \leq U(n, \mathbb{k}) \) be a unitary reflection group. Then the det-isotypic component of \( \mathbb{k}[V] \otimes \Lambda(V) \) has \( \mathbb{k} \)-basis

\[
\{ df_{i_1} \cdots df_{i_r} \odot (f_1^{a_1} \cdots f_n^{a_n} \Delta) : \{ i_1 < \cdots < i_r \} \subseteq \{ n \}, \alpha \in \mathbb{Z}_{\geq 0}^n \}.
\]

**Corollary 1.4.** For a complex reflection group \( G \) with degrees \( d_1, \ldots, d_n \),

\[
Hilb((\mathbb{k}[V] \otimes \Lambda(V))^{\text{det}}, q, t) = \prod_{i=1}^{n} \frac{q^{d_i-1} + t}{1 - q^{d_i}}.
\]

Solomon’s description of \( (\mathbb{k}[V] \otimes \Lambda(V^*))^G \) motivates the following natural generalization of Zabrocki’s super-coinvariant algebra.

**Definition 1.5.** The super-coinvariant ideal \( J_+^G \) is the ideal in \( \mathbb{k}[V] \otimes \Lambda(V^*) \) generated by bi-homogeneous non-constant \( G \)-invariant super-polynomials. The super-coinvariant algebra of \( G \) is \( (\mathbb{k}[V] \otimes \Lambda(V^*))/J_+^G \). By Solomon’s result, \( J_+^G = \langle f_1, \ldots, f_n, df_1, \ldots, df_n \rangle \).

The super-coinvariant algebra is a bigraded \( G \)-module. It may be represented as a set of polynomials \( \mathcal{H}_G \subset \mathbb{k}[V] \otimes \Lambda(V^*) \), namely the harmonics of \( J_+^G \) relative to a certain non-degenerate Hermitian form (see §5). Precisely, the inclusion of \( \mathcal{H}_G \) into \( (\mathbb{k}[V] \otimes \Lambda(V^*))/J_+^G \) is an isomorphism of bigraded \( G \)-modules. When \( \mathbb{k} \subset \mathbb{R} \) and \( G \leq O(n, \mathbb{k}) \) is an orthogonal reflection group, the representations \( \Lambda(V) \) and \( \Lambda(V^*) \) coincide, and in this case we may explicitly describe the alternating component of \( \mathcal{H}_G \) as follows.

**Theorem 1.6.** For a real reflection group \( G \leq O(n, \mathbb{k}) \) with fundamental invariants \( f_1, \ldots, f_n \) of degrees \( d_1 \leq \cdots \leq d_n \), with \( f_1 = x_1^2 + \cdots + x_n^2 \), the det-isotypic component of \( \mathcal{H}_G \) has \( \mathbb{k} \)-basis

\[
\{ df_{i_1} \cdots df_{i_r} \odot \Delta : \{ i_1 < \cdots < i_r \} \subseteq \{ n \} \}.
\]

Consequently, the projection of these elements yields a \( \mathbb{k} \)-basis for \( (\mathbb{k}[V] \otimes \Lambda(V^*))^{\text{det}}/J_+^G \).
Corollary 1.7. For a real reflection group $G$ as in Theorem 1.6,

$$(7) \quad \text{Hilb}((k[V] \otimes \Lambda(V^*)/\mathcal{F}_{+}^{G})_{\text{det}}; q, t) = \prod_{i=1}^{n}(q^{d_i-1} + t).$$

Wallach’s result (3) is the specialization of Corollary 1.7 where $G = S_n$ acts irreducibly on $\mathbb{R}^n/(1, \ldots, 1)$ via the standard representation with degrees $2, \ldots, n$.

Remark 1.8. We may essentially always assume $G$ is unitary or orthogonal (see §2.1). By choosing a suitable basis, we may further arrange for $x_1^2 + \cdots + x_n^2$ to be a fundamental invariant in the real case. The requirement $d_i \neq 1$ simply means $\{v \in V : \forall \sigma \in G, \sigma(v) = v\} = 0$, which is automatically satisfied for irreducible $G$. The assumptions in Theorem 1.3 and Theorem 1.6 are thus quite mild.

The rest of the paper is organized as follows. In Section 2 we give background on complex reflection groups. In Section 3 we use Molien series to contrast different notions of super-alternants. Section 4 describes two actions $\cdot$ and $\odot$ on super-polynomials involving differential operators. Section 5 generalizes standard terminology to super-coinvariants. Finally, Section 6 proves Theorem 1.3 and Section 7 proves Theorem 1.6.

2. Preliminaries

Let $V$ be a finite-dimensional vector space over a field $\mathbb{K} \subset \mathbb{C}$.

2.1. Invariant Hermitian forms. We first summarize some very well-known results concerning $G$-invariant positive-definite Hermitian forms. We describe them in an unusual amount of detail in order to make the choice of basis underlying Theorem 1.6 explicit.

Definition 2.1. A Hermitian form on $V$ is a $\mathbb{Z}$-bilinear map $\langle -, - \rangle : V \times V \to \mathbb{K}$ such that

$$\langle \alpha u, \beta v \rangle = \bar{\alpha} \beta \langle u, v \rangle \quad \text{and} \quad \langle u, v \rangle = \langle v, u \rangle$$

for all $\alpha, \beta \in \mathbb{K}$ and $u, v \in V$, where $\bar{\alpha}$ is the complex conjugate of $\alpha$. Such a form is positive-definite if additionally

$$\langle v, v \rangle > 0, \quad \forall v \in V - \{0\}.$$

If $v_i$ denotes the $i$th coordinate of $v \in V$ with respect to some basis $B$, then $\sum_i \overline{v_i} v_i$ is the standard positive-definite Hermitian form on $V$ associated with $B$, so such forms exist. If $G \leq \text{GL}(V)$ is any finite subgroup and $\langle -, -, \rangle'$ is any positive-definite Hermitian form, it is easy to see that

$$\langle u, v \rangle := \sum_{\sigma \in G} \langle \sigma(u), \sigma(v) \rangle'$$

is a $G$-invariant positive-definite Hermitian form, i.e.

$$\langle \sigma(u), \sigma(v) \rangle = \langle u, v \rangle$$

for all $\sigma \in G$. That is, $G$ consists of unitary transformations with respect to $\langle -, - \rangle$.

Given a Hermitian form $\langle -, - \rangle$ on $V$ and an ordered basis $B$, there is a unique matrix $A$ such that

$$\langle u, v \rangle = u^T A v$$
where \( \nu \) is the column vector of \( \nu \) with respect to \( B \) and \( \dagger \) denotes the conjugate-transpose. The matrix \( A \) is Hermitian, i.e. \( A^\dagger = A \). From the spectral theorem, \( A \) is unitarily diagonalizable, so \( A = P^\dagger DP \) for \( P^\dagger = P^{-1} \) and \( D \) diagonal. Since \( \langle \cdot, \cdot \rangle \) is positive-definite, the eigenvalues of \( A \) are positive reals. By extending scalars if necessary, we may assume the square roots of the (positive, real) eigenvalues of \( A \) are in \( \mathbb{K} \), so we may write \( D = M^\dagger M \).

Setting \( Q := MP \), we have \( A = Q^\dagger Q \) where \( Q \) is non-singular, so \( \langle u, v \rangle = (Qu)^\dagger (Qv) \). Consequently, we may replace \( B \) with a new basis \( B' \) for which \( \langle u, v \rangle = u^\dagger v \) is the standard Hermitian form. That is, we may assume \( G \subset U(n, \mathbb{K}) \) consists of unitary matrices, so that \( \sigma^{-1} = \sigma^\dagger \) for all \( \sigma \in G \). Furthermore, for the natural \( G \)-action on the dual space \( V^* \) with respect to the dual basis of \( B' \), \( \sigma \in G \subset U(n, \mathbb{K}) \) is represented by \( \overline{\sigma} \), the conjugate of \( \sigma \), which remains unitary.

To summarize, we have the following. In practice, \( G \) is often defined by generalized permutation matrices, which are automatically unitary.

Lemma 2.2. For any finite subgroup \( G \) of \( GL(V) \), possibly after extending scalars by square roots of positive reals, there exists a basis \( B \) for which the actions of \( G \) on \( V \) and \( V^* \) with respect to \( B \) and the dual basis of \( B \) are unitary.

2.2. Complex reflection groups and super-polynomials.

Definition 2.3. A pseudoreflection is an element \( \sigma \in GL(V) \) such that
\[
\exists m \in \mathbb{Z}_{\geq 1} \text{ s.t. } \sigma^m = 1 \quad \text{and} \quad \text{codim}\{v \in V : \sigma \cdot v = v\} = 1.
\]
A complex reflection group is a finite subgroup \( G \leq GL(V) \) generated by pseudoreflections.

Definition 2.4. Let \( \mathbb{K}[V] := \text{Sym}(V^*) \) be the ring of polynomial functions on \( V \), namely the symmetric algebra on \( V^* \) over \( \mathbb{K} \). If \( V \) has basis \( e_1, \ldots, e_n \) and \( V^* \) has dual basis \( x_1, \ldots, x_n \), we have
\[
\mathbb{K}[V] = \mathbb{K}[x_1, \ldots, x_n] =: \mathbb{K}[x_n].
\]
The group \( G \leq GL(V) \) acts naturally on the dual \( V^* \) via
\[
(\sigma \cdot x)(v) := x(\sigma^{-1}(v))
\]
for all \( \sigma \in G, x \in V^*, v \in V \). Similarly, \( \mathbb{K}[V] \) is naturally a graded \( G \)-module where \( \sigma(fg) = \sigma(f)\sigma(g) \) for all \( f, g \in \mathbb{K}[V] \).

Definition 2.5. Chevalley [2] showed that the ring of polynomial \( G \)-invariants \( \mathbb{K}[V]^G \) is itself a polynomial ring generated by \( n = \text{dim}(V) \) homogeneous, algebraically independent elements \( f_1, \ldots, f_n \) called fundamental invariants, which are not unique. The multiset \( d_1, \ldots, d_n \) of degrees of the fundamental invariants are the degrees of \( G \), which is unique.

Definition 2.6. Let \( \Lambda(V^*) \) be the algebra of alternating multilinear functions on \( V \) with values in \( \mathbb{K} \) under the wedge product, which can be realized as the exterior algebra of \( V^* \) over \( \mathbb{K} \). \( \Lambda(V^*) \) is naturally a graded \( G \)-module where \( \sigma(f \wedge g) = \sigma(f) \wedge \sigma(g) \) for all \( f, g \in \Lambda(V^*) \). To avoid confusion, we write \( \theta_i \) instead of \( x_i \) for the generators of \( \Lambda(V^*) \), and we omit \( \wedge \). Consequently, \( \Lambda(V^*) \) has \( \mathbb{K} \)-basis
\[
\theta_1 := \theta_{i_1} \cdots \theta_{i_v}\]
for $I = \{i_1 < \cdots < i_r\} \subset [n]$. By an abuse of notation, we write
\[ \Lambda(V^*) = \mathbb{k}[\theta_1, \ldots, \theta_n] =: \mathbb{k}[\theta_n]. \]

**Definition 2.7.** Let $\Lambda(V)$ be the exterior algebra on $V$ over $\mathbb{k}$, which is again a graded $G$-module. We write $\psi_i$ instead $e_i$ for the generators of $\Lambda(V)$. As before, $\Lambda(V)$ has $\mathbb{k}$-basis $\{\psi_I\}$ and we write
\[ \Lambda(V) = \mathbb{k}[\psi_1, \ldots, \psi_n] =: \mathbb{k}[\psi_n]. \]

**Definition 2.8.** The super-polynomial rings are $\mathbb{k}[V] \otimes \Lambda(V^*)$ and $\mathbb{k}[V] \otimes \Lambda(V)$. We may write $\mathbb{k}[V] \otimes \Lambda(V^*)$ as the associative $\mathbb{k}$-algebra
\[ \mathbb{k}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_n] =: \mathbb{k}[x_n, \theta_n] \]
generated by indeterminates $x_1, \ldots, x_n$ and $\theta_1, \ldots, \theta_n$ where
\[ x_i x_j = x_j x_i \quad \text{and} \quad \theta_i \theta_j = \theta_j \theta_i. \]
Similarly we may realize $\mathbb{k}[V] \otimes \Lambda(V)$ as $\mathbb{k}[x_n, \psi_n]$. The super-polynomial rings are bigraded $G$-modules. We typically let $q$ track $x$-degree and $t$ track $\theta$ or $\psi$-degree.

When $G$ consists of orthogonal matrices, we have $\Lambda(V) \cong \Lambda(V^*)$ via $\psi_i \mapsto \theta_i$ as $G$-modules, so in the real case there is only one super-polynomial ring. We must be more careful in the complex case.

**Definition 2.9.** The differential (or exterior derivative) on $\mathbb{k}[V] \otimes \Lambda(V^*)$ is the $\mathbb{k}$-linear map
\[ d: \mathbb{k}[x_n, \theta_n] \rightarrow \mathbb{k}[x_n, \theta_n] \]
\[ d(f(x_n)\theta_I) := \sum_{i=1}^{n} \frac{\partial f(x_n)}{\partial x_i} \theta_I. \]
In particular, we have $dx_i = \theta_i$. The following observation is a routine verification.

**Lemma 2.10** ([12, p. 58]). For all $\sigma \in G$ and $f \in \mathbb{k}[V] \otimes \Lambda(V^*)$, $\sigma \cdot d(f) = d(\sigma \cdot f)$.

**Remark 2.11.** If $G$ consists of orthogonal matrices, then $x_1^2 + \cdots + x_n^2$ and $x_1 \theta_1 + \cdots + x_n \theta_n$ are both $G$-invariant elements of $\mathbb{k}[V] \otimes \Lambda(V)$. If $G$ consists of unitary matrices, then $|x_1|^2 + \cdots + |x_n|^2$ is $G$-invariant, which is not in general an element of either super-polynomial ring. However, $x_1 \psi_1 + \cdots + x_n \psi_n \in \mathbb{k}[V] \otimes \Lambda(V)$ is $G$-invariant in general. For this reason, $\mathbb{k}[V] \otimes \Lambda(V)$ may be the more “natural” super-polynomial ring. Note, however, that Solomon’s description of the $G$-invariants, Theorem 1.1, applies to $\mathbb{k}[V] \otimes \Lambda(V^*)$ and not $\mathbb{k}[V] \otimes \Lambda(V)$.

2.3. **Vandermondes and Jacobians.** Let $G$ be a complex reflection group with fixed fundamental invariants $f_1, \ldots, f_n$.

**Definition 2.12.** The Vandermonde of $G$ (relative to $f_1, \ldots, f_n$) is the element $\Delta \in \mathbb{k}[V]$ defined by
\[ df_1 \cdots df_n =: \Delta \theta_{[n]} \in \mathbb{k}[V] \otimes \Lambda(V^*). \]
$\Delta$ may be thought of as the Jacobian determinant of $f_1, \ldots, f_n$ with respect to $x_1, \ldots, x_n$. Since $f_1, \ldots, f_n$ are algebraically independent, it follows that $\Delta \neq 0$. 
Example 2.13. When $G = S_n$ and $f_i = \sum_j x_j^i$, we have $\Delta = c \prod_{i < j} (x_j - x_i)$ for $c = n^{tn}$.

Remark 2.14. In fact, $\Delta$ has a very explicit description. Let $\ell_1, \ldots, \ell_r \in \mathbb{k}[V]$ be linear forms which vanish on the $r$ reflecting hyperplanes of $G$. Let $m_1, \ldots, m_r$ be the orders of the cyclic subgroups of $G$ fixing $\ker \ell_1, \ldots, \ker \ell_r$. Then (see [11, p. 283], [16])

$$\Delta = c \prod_{i=1}^r \ell_i^{m_i-1}$$

for some $c \in \mathbb{k} - \{0\}$.

Lemma 2.15 ([12, p. 59]). We have $\sigma(\Delta) = \det_V(\sigma) \Delta$ for all $\sigma \in G$.

3. MOLIEN SERIES AND ALTERNANTS

A classical theorem of Molien gives a succinct, beautiful, and remarkably powerful description of the Hilbert series of the invariants of the $G$-action on $\mathbb{k}[V] = \mathbb{k}[x_n]$. An analogous result for the $G$-action on $\Lambda(V^*) = \mathbb{k}[\theta_n]$ is less frequently encountered, though the proof is no harder. We require a bigraded generalization of these results for relative invariants over fields other than $\mathbb{C}$ for both types of super-polynomials. Since it is difficult to find all of the relevant pieces for such a generalization in the literature, we sketch a proof. We then use Molien’s theorem, Solomon’s theorem, and a generalization of Solomon’s theorem due to Orlik–Solomon [8] to analyze several possible notions of alternants.

In this subsection only, $\mathbb{k}$ denotes an arbitrary field of characteristic zero, i.e. not necessarily a subfield of $\mathbb{C}$. For a $\mathbb{k}G$-module $M$ and $\sigma \in G$, let $\Tr_M(\sigma)$ denote the character of $M$ at $\sigma$, and similarly let $\det_M(1 - \sigma q) \in \mathbb{k}[q]$ denote the characteristic polynomial of $\sigma$ acting on $M$. Note that $1/\det_M(1 - \sigma q) \in \mathbb{k}[[q]]$ may be regarded as a formal power series and that $\mathbb{k}[[q]] \supset \mathbb{Q}[[q]]$.

Definition 3.1. If $M$ is irreducible, the Molien series of a bigraded $\mathbb{k}G$-module $S$ relative to $M$ is the formal power series

$$F_M(S; q, t) := \sum_{i,j \geq 0} (\text{multiplicity of } M \text{ in the bidegree } (i, j) \text{ piece of } S)q^it^j.$$ 

Theorem 3.2 (Molien). Let $G$ be a finite subgroup of $\text{GL}(V)$ where $V$ is a finite-dimensional vector space over a field $\mathbb{k}$ of characteristic 0. Suppose $M$ is an irreducible $\mathbb{k}G$-module. Then

$$F_M(\mathbb{k}[V] \otimes \Lambda(V^*); q, t) = \frac{1}{|G| \dim_{\mathbb{k}} \mathbb{D}} \sum_{\sigma \in G} \Tr_M(\sigma) \frac{\det_V(1 + \sigma t)}{\det_V(1 - \sigma q)},$$

$$F_M(\mathbb{k}[V] \otimes \Lambda(V); q, t) = \frac{1}{|G| \dim_{\mathbb{k}} \mathbb{D}} \sum_{\sigma \in G} \Tr_M(\sigma) \frac{\det_V(1 + \sigma^{-1} t)}{\det_V(1 - \sigma q)},$$

where $\mathbb{D} := \text{End}_{\mathbb{k}G}(M)$ is the division ring of $\mathbb{k}G$-linear endomorphisms of $M$.

Proof. The following claim is well-known:

$$\frac{\dim_{\mathbb{D}}(M)}{|G|} \sum_{\sigma \in G} \Tr_M(\sigma^{-1}) \sigma$$
is the unique $G$-invariant projection operator onto $M$-isotypic components. Indeed, by the Artin–Wedderburn theorem, $kG \cong \bigoplus_{i=1}^{r} \text{Mat}_{n_i}(D_i)$ where $M_1, \ldots, M_r$ are the inequivalent irreducible $kG$-modules, $D_i := \text{End}_{kG}(M_i)$ is a right $k$-module, and $n_i = \dim_{D_i}(M_i)$. Consequently, $\text{Tr}_{kG}(\sigma) = \sum_{i=1}^{r} \dim_{D_i}(M_i) \text{Tr}_{M_i}(\sigma)$. Let $e_1, \ldots, e_r \in kG$ be the primitive central orthogonal idempotents, so that for any $kG$-module $N$, $e_i$ is the unique $G$-invariant projection from $N$ to the $M_i$-isotypic component of $N$. Suppose $e_i = \sum_{\sigma \in G} a_{\sigma} \sigma$. We have for each $\sigma \in G$,

$$a_{\sigma}|G| = \text{Tr}_{kG}(e_i \sigma^{-1}) = \sum_{j=1}^{r} \dim_{D_j}(M_j) \text{Tr}_{M_j}(e_i \sigma^{-1}) = \dim_{D_i} \text{Tr}_{M_i}(\sigma^{-1}),$$

which proves the claim.

Taking traces, it follows that the $k$-dimension of the $M$-isotypic component of $k[x_n, \theta_n]_{i,j}$ is

$$\frac{\dim_D(M)}{|G|} \sum_{\sigma \in G} \text{Tr}_{M}(\sigma^{-1}) \text{Tr}_{k[x_n, \theta_n]_{i,j}}(\sigma). \tag{14}$$

Dividing (14) by $\dim_k M = \dim_D M \cdot \dim_k D$ and using

$$\sum_{i,j \geq 0} \text{Tr}_{k[x_n, \theta_n]_{i,j}}(\sigma) q^{i} t^{j} = \frac{\det_{V^*}(1 + \sigma t)}{\det_{V^*}(1 - \sigma q)} = \frac{\det_{V}(1 + \sigma^{-1} t)}{\det_{V}(1 - \sigma^{-1} q)}.$$

yields the first Molien series, and the second is similar. (15) is straightforward to prove by diagonalizing $\sigma$. \qed

| $k[V] \otimes \Lambda(V)$ | $\Pi_{i=1}^{n} q^{\frac{i}{n}} + t q^{\frac{i}{n}}$ | $\Pi_{i=1}^{n} \frac{1 + q^{i} t}{1 - q^{i}}$ | $\Pi_{i=1}^{n} \frac{1 + q^{i} t}{1 - q^{i}}$ |
|--------------------------|--------------------------|--------------------------|--------------------------|
| $k[V] \otimes \Lambda(V^*)$ | $q^{\delta} \Pi_{i=1}^{n} q^{\frac{i}{n}} + t q^{\frac{i}{n}}$ | $\Pi_{i=1}^{n} \frac{1 + q^{i} t}{1 - q^{i}}$ | $\Pi_{i=1}^{n} \frac{1 + q^{i} t}{1 - q^{i}}$ |
| $k[V^*] \otimes \Lambda(V)$ | $\Pi_{i=1}^{n} q^{\frac{i}{n}} + t q^{\frac{i}{n}}$ | $\Pi_{i=1}^{n} \frac{1 + q^{i} t}{1 - q^{i}}$ | $q^{\delta} \Pi_{i=1}^{n} \frac{1 + q^{i} t}{1 - q^{i}}$ |
| $k[V^*] \otimes \Lambda(V^*)$ | $q^{\delta} \Pi_{i=1}^{n} q^{\frac{i}{n}} + t q^{\frac{i}{n}}$ | $\Pi_{i=1}^{n} \frac{1 + q^{i} t}{1 - q^{i}}$ | $\Pi_{i=1}^{n} \frac{1 + q^{i} t}{1 - q^{i}}$ |

Table 1. Product formulas for $\text{Hilb}((k[V] \otimes \Lambda(V))^{\text{det}}; q, t)$ and related series.

**Remark 3.3.** For general $G$, there are eight reasonable notions of “alternant super-polynomials”: we may use $k[V]$ or $k[V^*]; \Lambda(V)$ or $\Lambda(V^*)$; and $\text{det}$ or $\text{det}^{-1}$. In the real case, $V \cong V^*$ and $\text{det} = \text{det}^{-1}$, so all eight notions coincide. In the complex case, they may be genuinely different. Theorem 3.2 allows us to relate many of the corresponding Hilbert series along with series for the $G$-invariants by simple transformations. For example, $\det_{V}(\sigma) \det_{V}(1 + \sigma t) = t^n \det_{V}(1 + \sigma^{-1} t)$ gives

$$\text{Hilb}((k[V] \otimes \Lambda(V))^{\text{det}}; q, t) = t^n \text{Hilb}((k[V] \otimes \Lambda(V^*))^{G}; q, t^{-1}).$$
Table 1 lists product formulas for six of the eight types of alternants and all four types of $G$-invariants. In Table 1, we may go down and right one spot by applying $F(t) \mapsto t^n F(t^{-1})$ starting from the first or third rows; we may go down two spots and right one spot by applying $G(q) \mapsto (-q)^{-n} G(q^{-1})$; and we may reflect through the middle using $\sigma \mapsto \sigma^{-1}$, which preserves the Hilbert series. These operations result in three orbits.

The orbit containing $(\mathbb{C}V \otimes \Lambda(V^*))^G$ has four elements and yields product formulas arising from Solomon’s result, Corollary 1.2. The orbit containing $(\mathbb{C}V \otimes \Lambda(V))^G$ has six elements and yields product formulas arising from Orlik–Solomon’s generalization of Solomon’s result [8, Thm. 3.1]. The remaining orbit of two elements is not covered by these results. In Table 1, $e_i := d_i - 1$ are the exponents of $G$, $e_1^*, \ldots, e_n^*$ are the coexponents of $G$ defined by

$$\text{Hilb}((\mathbb{C}[V]/\mathbb{C}[V^*])^G; q) =: q^{e_1^*} + \cdots + q^{e_n^*},$$

and $\delta := e_1 + \cdots + e_n - e_1^* - \cdots - e_n^*$.

**Remark 3.4.** [1, Thm. 2.5.3] gives the relative Molien series for $\mathbb{C}[V]$ (though $\dim_{\mathbb{C}}(M)$ should instead be $\dim_{\mathbb{C}}(M)$). As Benson notes, the result can be generalized as-is to non-modular fields using Brauer characters. When $\text{char} \mathbb{k} = 0$ and $\mathbb{k} = \overline{\mathbb{k}}$, the Schur orthogonality relations may be easily deduced from (13).

## 4. Two Differential Operator Actions

It is well-known that the multivariate polynomial ring acts on itself by polynomial differential operators. For the super-polynomial rings $\mathbb{k}[V] \otimes \Lambda(V) = \mathbb{k}[x_1, \ldots, x_n, \theta_1, \ldots, \theta_r]$ and $\mathbb{k}[V] \otimes \Lambda(V) = \mathbb{k}[x_1, \ldots, x_n, \psi_1, \ldots, \psi_s]$, the anti-commuting variables may act either as a form of partial differentiation, as in [18] and [10], or they may act by multiplication. Here we define actions of these super-polynomial rings on each other and summarize their relationship. Many of these facts (or the special case when $G = S_n$) appear in [18] and [10, §5]. Similar actions appear in [9].

**Definition 4.1.** We have three flavors of $\mathbb{k}$-linear endomorphisms

$$\partial^e_i : \mathbb{k}[x_{\vec{i}}] \to \mathbb{k}[x_{\vec{i}}],$$

$$\partial^\theta_i, m^\theta_i : \mathbb{k}[\theta_{\vec{i}}] \to \mathbb{k}[\theta_{\vec{i}}],$$

$$\partial^\psi_i, m^\psi_i : \mathbb{k}[\psi_{\vec{i}}] \to \mathbb{k}[\psi_{\vec{i}}]$$

given by partial differentiation with respect to $x_i$ and partial differentiation or multiplication with respect to $\theta_i$ or $\psi_i$:

$$\partial_i f(x_{\vec{i}}) := \frac{\partial f(x_{\vec{i}})}{\partial x_i},$$

$$\partial^\theta_i (\theta_{i_1} \cdots \theta_{i_r}) := \begin{cases} (-1)^{m-1} \hat{\theta}_{i_1} \cdots \hat{\theta}_{i_m} \cdots \theta_{i_r} & \text{if } i_1 < \cdots < i_r \text{ and } i = i_m \\ 0 & \text{otherwise} \end{cases},$$

$$m^\psi_i (\theta_{\vec{i}}) := \theta_i \theta_{\vec{i}}.$$
and similarly with $\partial_i^\psi$ and $m_i^\psi$. We extend $\partial_i^x$ to yield $\mathbb{k}[\theta_n]$- or $\mathbb{k}[\psi_n]$-linear endomorphisms of the super-polynomial rings $\mathbb{k}[x, \theta_n]$ and $\mathbb{k}[x, \psi_n]$. We similarly extend $\partial_i^\theta, m_i^\theta, \partial_i^\psi, m_i^\psi$ $\mathbb{k}[x_n]$-linearly to the super-polynomial rings.

**Lemma 4.2.** We have the following commutation and anti-commutation relations.

(i) $[\partial_i^x, \partial_j^x] = 0$, $[\partial_i^\theta, \partial_j^\theta] = [\partial_i^x, m_j^\theta] = 0$, and $[\partial_i^\theta, \partial_j^\psi] = [\partial_i^x, m_j^\psi] = 0$.

(ii) $\partial_i^\theta \partial_j^\theta = -\partial_j^\theta \partial_i^\theta$, $m_i^\theta m_j^\theta = -m_j^\theta m_i^\theta$, $\partial_i^\psi \partial_j^\psi = -\partial_j^\psi \partial_i^\psi$, $m_i^\psi m_j^\psi = -m_j^\psi m_i^\psi$.

(iii) $m_i^\theta \partial_j^\psi + \partial_j^\theta m_i^\psi = \delta_{i,j}$, $m_i^\psi \partial_j^\psi + \partial_j^\psi m_i^\psi = \delta_{i,j}$, where $\delta_{i,j}$ is the identity if $i = j$ and 0 otherwise.

**Proof.** In each case the identities involving $\psi$ are equivalent to the identities involving $\theta$, so we focus on the latter.

(i) The first equality is essentially classical and the second and third are immediate since $\partial_i^x$ operates on $x$-variables and $\partial_i^\theta, m_i^\theta$ operate on $\theta$-variables,

(ii) The first equality is straightforward to verify on $\theta_I$ directly and extends $\mathbb{k}[x_n]$-linearly to all of $\mathbb{k}[x_n, \theta_n]$. The second is immediate from $\theta_i \theta_j = -\theta_j \theta_i$.

(iii) This is a consequence of the Leibniz rule, Lemma 4.5, which we will prove shortly.

The following actions are fundamental to the rest of our arguments.

**Definition 4.3.**

(a) We have an action of $\mathbb{k}[V] \otimes \Lambda(V^*) = \mathbb{k}[x_n, \theta_n]$ on itself given by

$$f(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n) \cdot g := \mathcal{F}(\partial_1^x, \ldots, \partial_n^x, \partial_1^\theta, \ldots, \partial_n^\theta)(g)$$

for all $f, g \in \mathbb{k}[x_n, \theta_n]$, which is well-defined by Lemma 4.2.

(b) We have an action of $\mathbb{k}[V] \otimes \Lambda(V^*) = \mathbb{k}[x_n, \theta_n]$ on $\mathbb{k}[V] \otimes \Lambda(V) = \mathbb{k}[x_n, \psi_n]$:

$$f(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n) \circ g := \mathcal{F}(\partial_1^x, \ldots, \partial_n^x, m_1^\psi, \ldots, m_n^\psi)(g)$$

for all $f \in \mathbb{k}[x_n, \theta_n], g \in \mathbb{k}[x_n, \psi_n]$.

We emphasize the appearance of the coefficient-wise complex conjugate of $f$ in Definition 4.3(a) and Definition 4.3(b). This will be justified by an equivariance property, Theorem 4.7, which is our next goal. We must first review the Leibniz rule for $\partial_i^\theta$, which was stated in [10], though it was not used and the proof was left as an exercise to the reader. Since the proof is somewhat intricate and we require (17) in an essential way, we include a proof here.

**Definition 4.4.** For $I, J \subseteq [n]$, let

$$\text{inv}(I, J) := \# \{ (i, j) \in I \times J : j < i \}.$$

We see that if $I \cap J = \emptyset$,

$$\theta_I \theta_J = (-1)^{\text{inv}(I, J)} \theta_{I \cup J}.$$

**Lemma 4.5** ([10, (5.4)]). For all $f, g \in \mathbb{k}[x_n, \theta_n]$ where $f$ has $\theta$-degree $r$, we have the Leibniz rule

$$\partial_i^\theta(fg) = \partial_i^\theta(f)g + (-1)^r f \partial_i^\theta(g).$$
Proof. We may suppose \( f = \theta_I, g = \theta_J \) where \( I = \{i_1 < \cdots < i_r\}, J = \{j_1 < \cdots < j_s\}. \) If \( I \cap J \neq \emptyset, \) the left-hand side is 0. If \( I \cap J \supseteq \{i\}, \) both terms on the right-hand side are also 0, so suppose \( I \cap J = \{i\}. \) Suppose \( i = i_\ell = j_m. \) The right-hand side is then

\[
(1) = (1) (-1)^{r + m - 1} \theta_{(i)} \theta_J + (1) (-1)^{-1} \theta_I (1) (-1) \theta_{(i)} \theta_J
\]

The powers on \(-1\) differ by \((\ell - 1) - (r + m - 1). \) Now \( \theta_I \theta_J \) and \( \theta_I \theta_{(i)} \) differ in that \( \theta_i \) has been commuted past \( \theta_{i_{\ell + 1}}, \theta_{i_{\ell + 1}} \cdots \theta_{j_{m-1}}, \) a total of \( m - 1 + r - \ell \) terms. It follows that the two terms are negatives, so they cancel. Thus, we may assume \( I \cap J = \emptyset. \) If \( i \not\in I \cup J, \) then each term is 0, so we may suppose \( i \in I \cup J. \) The left-hand side is then

\[
\partial_i^\theta (1) \theta_{I \cup J} = (1) \theta_{I \cup J - \{i\}}
\]

where \# \( I \cup J < i \) is shorthand for the number of elements of \( I \cup J \) smaller than \( i, \) and similarly with \# \( J < I. \)

If \( i \in I, \) then \( i \not\in J, \) and the right-hand side becomes

\[
(1) \# I < i \theta_{I - \{i\}} \theta_J + 0 = (1) \# I < i (1) \theta_{I \cup J - \{i\}} = (1) \# I < i + \# J < i - \{i\}.
\]

Now

\[
\# I \cup J < i + \# J < I \equiv_2 \# I < i + \# J < I - \{i\}
\]

is equivalent to

\[
\# J < i + \# J < i \equiv_2 0,
\]

which is true.

On the other hand, if \( i \in J, \) then \( i \not\in I, \) and the right-hand side becomes

\[
0 + (1) \# I = (1) \# J < i \theta_J \theta_{(i)} = (1) \# I (1) \# J < (1) \# J < i - \{i\} \theta_{I \cup J - \{i\}} = (1) \# I < i + \# J - \{i\} < I \theta_{I \cup J - \{i\}}.
\]

We see directly that

\[
\# I \cup J < i + \# J < I = \# I + \# J < i + \# J - \{i\} < I
\]

is equivalent to

\[
\# I < i + \# J < I = \# I,
\]

which is true since \( i \not\in I. \) This completes the proof of (17). \( \square \)

**Lemma 4.6.** Suppose \( \sigma \in U(n, \mathbb{K}) \) is unitary and \( x_1, \ldots, x_n \) are the coordinate functions on \( \mathbb{K}^n. \) Then for all \( g \in \mathbb{k}[x_n, \theta_n] \) and \( h \in \mathbb{k}[x_n, \psi_n], \)

\[
(\sigma \circ \partial_x^\sigma \circ \sigma^{-1})(g) = \sigma(x_i) \cdot g \quad \text{and} \quad (\sigma \circ \partial_x^\sigma \circ \sigma^{-1})(h) = \sigma(x_i) \circ h
\]

(18)

\[
(\sigma \circ \partial^\sigma \circ \sigma^{-1})(g) = \sigma(x_i) \cdot g
\]

(19)

\[
(\sigma \circ \partial^\sigma \circ \sigma^{-1})(h) = \sigma(x_i) \circ h.
\]

(20)

\[
(\sigma \circ m^\psi \circ \sigma^{-1})(h) = \sigma(x_i) \circ h.
\]

Proof. For (18), by \( \mathbb{k}[\theta_n]- \) or \( \mathbb{k}[\psi_n]-\) linearity, it suffices to consider the case when \( g = x_1^{a_1} \cdots x_n^{a_n}. \) Let \( \sigma(u_i) := x_i, \) so \( u_1, \ldots, u_n \) forms another basis of \( V^*. \) Suppose \( x_i = \sum_j c_{ij} u_j \) and \( u_i = \sum_j d_{ij} x_j, \) so that \( \frac{\partial x_i}{\partial u_j} = c_{ij} \) and \( \frac{\partial u_i}{\partial x_j} = d_{ij}. \) Furthermore, \( \sigma(x_i) = \sigma(\sum_j c_{ij} u_j) = \sum_j c_{ij} x_j, \) so \( [c_{ij}] \) is the matrix of \( \sigma, \) and \( \sigma^{-1}(x_i) = u_i = \sum_j d_{ij} x_j, \) so
\([d_{ij}]\) is the matrix of \(\sigma^{-1}\). Since \(\sigma\) is assumed unitary, we have \([c_{ij}] = [d_{ij}]^\dagger\), so \(c_{ij} = \overline{d_{ji}}\). Using the multivariate chain rule, we now compute

\[
(\sigma \circ \partial^x_i \circ \sigma^{-1})(x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = \sigma \left( \frac{\partial}{\partial x_i} u_1^{\alpha_1} \cdots u_n^{\alpha_n} \right) = \sigma \left( \sum_{j=1}^n \frac{\partial}{\partial u_j} u_1^{\alpha_1} \cdots u_n^{\alpha_n} \frac{\partial u_j}{\partial x_i} \right)
\]

\[
= \sigma \left( \sum_{j=1}^n \alpha_j u_1^{\alpha_1} \cdots u_j^{\alpha_j-1} \cdots u_n^{\alpha_n} d_{ji} \right)
\]

\[
= \sum_{j=1}^n \alpha_j x_1^{\alpha_1} \cdots x_j^{\alpha_j-1} \cdots x_n^{\alpha_n} c_{ij}
\]

\[
= \sum_{j=1}^n \frac{\partial}{\partial x_j} x_1^{\alpha_1} \cdots x_n^{\alpha_n} c_{ij} = \left( \sum_{j=1}^n c_{ij} x_j \right) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}
\]

\[
= \sigma \left( \sum_{j=1}^n c_{ij} u_j \right) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \sigma(x_i) \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}.
\]

This proves (18).

For (19), we begin with an analogue of the multivariate chain rule in this context. Let \(\phi_i := du_i\), so \(\phi_1, \ldots, \phi_n\) is a basis for \(\Lambda^1(V^*)\). We claim that

\[
\frac{\partial}{\partial \theta_i} \phi_{k_1} \cdots \phi_{k_r} = \sum_{j=1}^r \frac{\partial}{\partial \phi_{k_j}} \phi_{k_1} \cdots \phi_{k_r} \frac{\partial \phi_{k_j}}{\partial \theta_i}
\]

(21)

for all \(k_1 < \cdots < k_r\). When \(r = 0\), the result is clear. For \(r > 0\), by induction and (17), we have

\[
\frac{\partial}{\partial \theta_i} \phi_{k_1} \cdots \phi_{k_r} = \frac{\partial \phi_{k_1}}{\partial \theta_i} \phi_{k_2} \cdots \phi_{k_r} - \phi_{k_1} \frac{\partial}{\partial \theta_i} \phi_{k_2} \cdots \phi_{k_r}
\]

\[
= \phi_{k_2} \cdots \phi_{k_r} \frac{\partial \phi_{k_1}}{\partial \theta_i} - \phi_{k_1} \sum_{j=2}^r \frac{\partial}{\partial \phi_{k_j}} \phi_{k_2} \cdots \phi_{k_r} \frac{\partial \phi_{k_j}}{\partial \theta_i}
\]

\[
= \hat{\phi}_{k_1} \phi_{k_2} \cdots \phi_{k_r} \frac{\partial \phi_{k_1}}{\partial \theta_i} - \sum_{j=2}^r (-1)^{j-2} \phi_{k_1} \phi_{k_2} \cdots \hat{\phi}_{k_j} \cdots \phi_{k_r} \frac{\partial \phi_{k_j}}{\partial \theta_i}
\]

\[
= \sum_{j=1}^r \frac{\partial}{\partial \phi_{k_j}} \phi_{k_1} \cdots \phi_{k_r} \frac{\partial \phi_{k_j}}{\partial \theta_i},
\]

proving (21). Now (19) follows from virtually the same calculation as (18) using (21); the details are omitted.

As for (20), we have

\[
(\sigma \circ m_1^\psi \circ \sigma^{-1})(g) = \sigma(\psi_1 \sigma^{-1}(g)) = \sigma(\psi_1)g = \sum_j f_{ij} \psi_j g = \left( \sum_j \overline{f_{ij}} \theta_j \right) \odot g
\]
where \([f_{ij}]\) is the matrix of \(\sigma\) with respect to \(\psi_1, \ldots, \psi_n\). Since \(\psi_1, \ldots, \psi_n\) is the dual basis of \(\theta_1, \ldots, \theta_n\), the matrix of \(\sigma\) acting on \(V\) is the inverse-transpose of the matrix of \(\sigma\) acting on \(V^*\), namely \(((f_{ij})^{-1})^T = (f_{ij})^T = [f_{ij}]\). Thus \(\sum_j f_{ij} \theta_j = \sigma(\theta_i)\), completing the result. 

**Theorem 4.7.** If \(\sigma \in U(n, k)\) is unitary, then for all \(f, g \in k[x_n, \theta_n]\) and \(h \in k[x_n, \psi_n]\),

\[
\begin{align*}
\sigma(f \cdot g) &= \sigma(f) \cdot \sigma(g) \\
\sigma(f \circ h) &= \sigma(f) \circ \sigma(h).
\end{align*}
\]

**Proof.** For (22), by (18) and (19),

\[
(\sigma \circ f(\partial^x_1, \ldots, \partial^x_n, \partial^\theta_1, \ldots, \partial^\theta_n) \circ -1)(g)
= f(\sigma(\partial^x_1, \ldots, \partial^x_n, \partial^\theta_1, \ldots, \partial^\theta_n) \circ -1, \ldots, \sigma(\partial^x_1, \partial^\theta_1, \ldots, \partial^\theta_n)(\circ -1))
= f(\sigma(x_1), \ldots, \sigma(x_n), \sigma(\theta_1), \ldots, \sigma(\theta_n)) \cdot g
= \sigma(f) \cdot g.
\]

Replacing \(g\) with \(\sigma(g)\) gives the result. Similarly (23) follows from (18) and (20).

### 4.1. Hodge duality

The two actions \(\cdot\) and \(\circ\) are related by the following operation. We will not directly use the results of this subsection but include it for completeness.

**Definition 4.8.** The Hodge dual is the \(k[x_n]\)-linear endomorphism

\[
\ast : k[x_n, \theta_n] \to k[x_n, \psi_n]
\]

\[\ast \theta_I := (-1)^{\deg(I)} \psi_J\]

where \(I \uplus J = [n]\) and

\[
\deg(I) := \sum_{i \in I} (i - 1).
\]

**Lemma 4.9.** For \(f, g \in k[x_n, \theta_n]\), we have

\[
f \circ \ast g = \ast(f \cdot g)
\]

**Proof.** By \(k[x_n]\)-sesquilinearity, we may suppose \(f = \theta_I\) and \(g = \theta_{[n] - J}\). The left-hand side becomes

\[
\theta_I \circ (-1)^{\deg([n] - J)} \psi_J = (-1)^{\deg([n] - J)} \psi_I \psi_J = (-1)^{\deg([n] - J) + \text{inv}(I, J)} \psi_{I \uplus J}
\]

or 0 if \(I \uplus J \neq \emptyset\). The right-hand side becomes

\[
\ast(\theta_I \cdot \theta_{[n] - J}) = \ast((-1)^{\text{inv}(I, [n] - J)} \theta_{[n] - J - I}) = (-1)^{\text{inv}(I, [n] - J) + \deg([n] - J - I)} \psi_{I \uplus J}
\]

or 0 if \(I \not\subset [n] - J\). Since \(I \cap J = \emptyset\) if and only if \(I \subset [n] - J\), we may suppose \(I \cap J = \emptyset\). The result is hence equivalent to

\[
\deg([n] - J) + \text{inv}(I, J) \equiv_2 \text{inv}(I, [n] - J) + \deg([n] - I - J).
\]

Since \(I \cap J = \emptyset\), we have \([n] - J = ([n] - J - I) \cup I\) and \(\deg([n] - J) = \deg([n] - J - I) + \deg(I)\). Thus (25) becomes

\[
\deg(I) + \text{inv}(I, J) \equiv_2 \text{inv}(I, [n] - J).
\]
Indeed, for all \( I, J \subset [n] \), we have
\[
\text{inv}(I, J) + \text{inv}(I, [n] - J) = \deg(I)
\]
since
\[
#J < I + #([n] - J) < I = #[n] < I = \deg(I).
\]

Corollary 4.10. Let \( f, g \in \mathbb{k}[x_n, \theta_n] \) have the same bi-degree. Then
\[
f \odot \ast g = \langle f, g \rangle \psi_{[n]}^\ast
\]
where \( \langle -, - \rangle \) is the non-degenerate Hermitian form defined in the next section.

5. HERMITIAN FORMS, COINVARIANTS, AND HARMONICS

We next define a non-degenerate Hermitian form on the super-polynomial ring \( \mathbb{k}[V] \otimes \Lambda(V^*) \). We then summarize the connection between the harmonic super-polynomials and super-coinvariants.

Definition 5.1. We have a \( \mathbb{Z} \)-bilinear form on \( \mathbb{k}[V] \otimes \Lambda(V^*) = \mathbb{k}[x_n, \theta_n] \) given by
\[
\langle f, g \rangle := \text{constant coefficient of } f \cdot g.
\]

Lemma 5.2. The form \( \langle - , - \rangle \) on \( \mathbb{k}[x_n, \theta_n] \) is Hermitian, non-degenerate, and \( G \)-invariant. Moreover, for \( \alpha, \beta \in \mathbb{Z}_{\geq 0}^n \) and \( I, J \subset [n] \),
\[
\langle x^{\alpha} \theta_I, x^{\beta} \theta_J \rangle = \begin{cases} 
(-1)^{|I|/2} \alpha! & \text{if } \alpha = \beta, \ I = J \\
0 & \text{otherwise,}
\end{cases}
\]
where \( x^{\alpha} := x_1^{\alpha_1} \cdots x_n^{\alpha_n}, \ \alpha! := \alpha_1! \cdots \alpha_n! \). Consequently, \( \langle -, - \rangle \) is positive-definite or negative-definite when restricted to \( \theta \)-degree \( r \) depending on the sign of \((-1)^{|I|/2}\).

Proof. It is clear that \( \langle f, g \rangle \) is conjugate-linear in the first argument and linear in the second argument. \( G \)-invariance follows from the first half of Theorem 4.7. Non-degeneracy and the symmetry \( \langle f, g \rangle = \langle g, f \rangle \) both follow from (30). As for (30), we may assume that \( x^{\alpha} \theta_I \) and \( x^{\beta} \theta_J \) have the same bi-degree, in which case we see that
\[
(x^{\alpha} \theta_I) \cdot (x^{\beta} \theta_J) = (x^{\alpha} \cdot x^{\beta})(\theta_I \cdot \theta_J)
= \alpha! \delta_{\alpha = \beta} (\theta_I \cdot \theta_J) \delta_{I = J}.
\]

It is straightforward to check that \( \theta_I \cdot \theta_J = (-1)^{(|I| - 1) + (|J| - 2) + \cdots + 0} = (-1)^{|I|/2} \).

Definition 5.3. Given a subspace \( W \) in \( \mathbb{k}[x_n, \theta_n] \), the orthogonal complement of \( W \) is
\[
W^\perp := \{ g \in \mathbb{k}[x_n, \theta_n] : \langle f, g \rangle = 0 \text{ for all } f \in W \}.
\]

Lemma 5.4. We have
\[
(i) \quad W^\perp = \{ f \in \mathbb{k}[x_n, \theta_n] : \langle f, g \rangle = 0 \text{ for all } g \in W \}
(ii) \quad (W^\perp)^\perp = W
(iii) \quad W \cap W^\perp = 0 \text{ and } W \oplus W^\perp = \mathbb{k}[x_n, \theta_n]
\]

Proof. These are all standard consequences of Lemma 5.2.
**Definition 5.5.** Let \( G \leq U(n, \mathbb{k}) \). The **coinvariant ideal** of \( G \) is the ideal

\[ \mathcal{J}_+^G := (\text{homogeneous non-constant } G\text{-invariants}) \subset \mathbb{k}[V] \otimes \Lambda(V^*) . \]

The **coinvariant algebra** of \( G \) is the quotient

\[ \mathbb{k}[V] \otimes \Lambda(V^*) / \mathcal{J}_+^G , \]

which is a bigraded \( G \)-module. The **\( G \)-harmonics** are

\[ \mathcal{H}_G := (\mathcal{J}_+^G)^\perp \subset \mathbb{k}[V] \otimes \Lambda(V^*) . \]

The classical harmonic polynomials are those for which \( \sum_i \frac{\partial^2 f}{\partial x_i^2} = 0 \). The \( G \)-harmonics are so named because of the following well-known result.

**Lemma 5.6.** If \( \mathcal{J} \subset \mathbb{k}[x_n, \theta_n] \) is an ideal generated by \( j_1, \ldots, j_r \), then

\[ \mathcal{J}^\perp = \{ g \in \mathbb{k}[x_n, \theta_n] : j \cdot g = 0 \text{ for all } j \in \mathcal{J} \} \]

\[ = \{ g \in \mathbb{k}[x_n, \theta_n] : j \cdot g = \cdots = j_r \cdot g = 0 \} . \]

In particular, when \( G \) is a complex reflection group with fundamental invariants \( f_1, \ldots, f_n \),

\[ \mathcal{H}_G = \{ g \in \mathbb{k}[x_n, \theta_n] : f_1 \cdot g = \cdots = f_n \cdot g = df_1 \cdot g = \cdots = df_n \cdot g = 0 \} . \]

**Proof.** If \( j \cdot g = 0 \) for all \( j \in \mathcal{J} \), then trivially \( \langle j, g \rangle = 0 \). The converse also holds since \( \mathcal{J} \) is closed under multiplication by \( x_i \) and \( \theta_i \). Similarly, \( \mathcal{J} \) annihilates \( g \) if and only if the generators annihilate \( g \). The explicit description of \( \mathcal{H}_G \) follows from Solomon’s Theorem 1.1. \( \square \)

**Proposition 5.7.** The projection of \( \mathcal{H}_G \) to \( \mathbb{k}[V] \otimes \Lambda(V^*) / \mathcal{J}_+^G \) is an isomorphism of bigraded \( G \)-modules.

**Proof.** \( \mathcal{H}_G \) is closed under the \( G \)-action since \( \langle -, - \rangle \) and \( \mathcal{J}_+^G \) are \( G \)-invariant. The projection is trivially \( G \)-equivariant and bidegree-preserving. It is a bijection since \( \mathbb{k}[x_n, \theta_n] / \mathcal{J}_+^G = (\mathcal{J}_+^G)^\perp \subset \mathcal{J}_+^G = \mathcal{J}_+^G \). \( \square \)

6. ALTERNANTS IN \( \mathbb{k}[V] \otimes \Lambda(V) \)

We may now prove the claimed classification of the alternants in \( \mathbb{k}[V] \otimes \Lambda(V) \) for \( G \leq U(n, \mathbb{k}) \) from the introduction.

**Proof of Theorem 1.3.** Recall that the claimed \( \mathbb{k} \)-basis for \( (\mathbb{k}[V] \otimes \Lambda(V))^\text{det} \) is

\[ \{ df_I \otimes f^\alpha \Delta : I \subset [n], \alpha \in \mathbb{Z}_{\geq 0}^n \} , \]

where \( df_I := df_{i_1} \cdots df_{i_r} \) for \( I = \{ i_1 < \cdots < i_r \} \) and \( f^\alpha := f_{i_1}^{\alpha_1} \cdots f_{i_r}^{\alpha_r} \). We first show that \( \{ df_I \otimes f^\alpha \Delta \} \) carries the det-representation and is linearly independent.

As is well-known [12], \( \mathbb{k}[V]^\text{det} = \mathbb{k}[V]^G \Delta \), so \( \{ f^\alpha \Delta : \alpha \in \mathbb{Z}_{\geq 0}^n \} \) is a \( \mathbb{k} \)-basis for \( \mathbb{k}[V]^\text{det} \). Since \( f_i \in \mathbb{k}[V]^G \), we have \( df_I \in (\mathbb{k}[V] \otimes \Lambda(V^*))^G \) by Lemma 2.10 and \( df_I \otimes f^\alpha \Delta \in (\mathbb{k}[V] \otimes \Lambda(V))^\text{det} \) by the second part of Theorem 4.7.

Now let \( \{ \tilde{g}_\alpha \} \) be an orthogonal basis for \( \mathbb{k}[V]^\text{det} \) with respect to the non-degenerate Hermitian form \( \langle -, - \rangle \). Since \( \mathbb{k}[V]^\text{det} = \mathbb{k}[V]^G \Delta \), we have \( \tilde{g}_\alpha = g_\alpha \Delta \) for some \( g_\alpha \in \mathbb{k}[V]^G \).
As far as linear independence is concerned, we may replace \( f^\alpha \Delta \) with \( g_\alpha \Delta \). Consequently, suppose
\[
0 = \sum_{\beta} c_{I,\beta} df_I \odot g_\beta \Delta
\]
for some \( c_{I,\beta} \in \mathbb{k} \). By homogeneity in the \( \theta \) variables, we may suppose \(|J|\) is constant. Fixing a particular \( I, \alpha \), apply \( g_\alpha df_{[n]} - I \odot - \) to (32). If \( J \neq I \), then \( df_{[n]} - I df_{I} = 0 \) since some \( df_i \) appears twice. If \( J = I \), we have \( df_{[n]} - I df_{I} = \pm df_{[n]} = \pm \Delta \theta_{[n]} \). Consequently, we’re left with (up to an overall sign)
\[
0 = \sum_{\beta} c_{I,\beta} g_\alpha \Delta \theta_{[n]} \odot g_\beta \Delta = \sum_{\beta} c_{I,\beta} (g_\alpha \Delta \cdot g_\beta \Delta) \psi_{[n]}.
\]
By homogeneity in the \( x \)-variables, we may suppose \( \deg g_\alpha = \deg g_\beta \), so that \( g_\alpha \Delta \cdot g_\beta \Delta = \langle \tilde{g}_\alpha, \tilde{g}_\beta \rangle \). By orthogonality of \( \{ \tilde{g}_\alpha \} \), it follows that \( c_{I,\alpha} = 0 \).

We have just shown
\[
\text{Hi}lb((\mathbb{k}[V] \otimes \Lambda(V))^{\det}; q, t) \geq \prod_{i=1}^{n} \frac{q^{d_i-1} + t}{1 - q^{d_i}}
\]
where \( \geq \) indicates coefficient-wise inequality as bivariate formal power series. From Table 1, equality holds, so \( \{ df_{I} \odot f^\alpha \Delta \} \) spans \( (\mathbb{k}[V] \otimes \Lambda(V))^{\det} \), completing the proof. □

**Remark 6.1.** When \( G = S_n \), in the preceding proof we may in fact use \( \{ g_\alpha \} = \{ s_\lambda \} \) where \( s_\lambda \) denotes a Schur polynomial in \( n \) variables. More precisely,
\[
\langle s_\lambda(x_1, \ldots, x_n) \Delta_n, s_\mu(x_1, \ldots, x_n) \Delta_n \rangle = (\lambda + \delta_n)! n! \delta_{\lambda, \mu}
\]
where \( \Delta_n := \prod_{1 \leq i < j \leq n} (x_i - x_j) \) and \( \delta_n := (n-1, n-2, \ldots, 0) \). Indeed, the classical bialternant expression [15, §7.15] gives \( s_\lambda \Delta_n = \det(x_i^{\lambda_j + n - j}) = \sum_{\sigma \in S_n} (-1)^{t(\sigma)} \prod_j x_j^{\lambda_i + n - j} \).

Since \( \lambda_j + n - j \) is strictly decreasing, it follows that when \( \lambda \neq \mu \), \( s_\lambda \Delta_n \) and \( s_\mu \Delta_n \) have no monomials in common. (33) now follows from Lemma 5.2.

7. HARMONIC AND COINVARIANT ALTERNANTS IN THE REAL CASE

Throughout this section, we assume \( \mathbb{k} \subset \mathbb{R} \), so \( G \leq O(n, \mathbb{k}) \) consists of orthogonal matrices. Consequently, we may identify the two super-polynomial rings \( \mathbb{k}[V] \otimes \Lambda(V^*) = \mathbb{k}[x_n, \theta_n] \) and \( \mathbb{k}[V] \otimes \Lambda(V) = \mathbb{k}[x_n, \psi_n] \) since \( \theta_i \mapsto \psi_i \) is an isomorphism of \( G \)-modules. In particular, the two differential operator actions \( \cdot \) and \( \odot \) now both act on the same space \( \mathbb{k}[x_n, \theta_n] \). Since \( G \) consists of orthogonal matrices, \( x_1^2 + \cdots + x_n^2 \) is \( G \)-invariant.

**Definition 7.1.** The Laplacian on \( \mathbb{k}[V] \otimes \Lambda(V^*) \) is
\[
\nabla^2 := \sum_{i=1}^{n} (\partial_i^2)^2.
\]
Thus, \( \nabla^2 f = (x_1^2 + \cdots + x_n^2) \cdot f \). By Theorem 4.7, we then have \( \sigma(\nabla^2 f) = \nabla^2(\sigma(f)) \). In particular, if \( f \) is \( G \)-invariant, then so is \( \nabla^2 f \).

The following is an elementary “polarization identity” for the Laplacian.
Lemma 7.2. For all \( f, g \in \mathbb{k}[V] \), we have
\[
\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} = \frac{1}{2} (\nabla^2 (fg) - (\nabla^2 f)g - f(\nabla^2 g)).
\]

Proof. By the classical Leibniz rule,
\[
\frac{\partial^2 (fg)}{\partial x_i^2} = \frac{\partial^2 f}{\partial x_i^2} g + 2 \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} + f \frac{\partial^2 g}{\partial x_i^2}.
\]
Summing over \( i = 1, \ldots, n \) gives
\[
\nabla^2 (fg) = (\nabla^2 f)g + 2 \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_i} + f(\nabla^2 g).
\]

Lemma 7.3. For \( f, g \in \mathbb{k}[V] \) and \( h \in \mathbb{k}[V] \otimes \Lambda(V^*) \), we have
\[
(34) \quad df \cdot (dg \circ h) = -dg \circ (df \cdot h) + \frac{1}{2} (\nabla^2 (fg) - (\nabla^2 f)g - f(\nabla^2 g)) \cdot h.
\]

Proof. We calculate
\[
df(\partial^x, \partial^\theta)dg(\partial^x, m^\theta) = \sum_{a=1}^{n} \frac{\partial f}{\partial x_a} (\partial^x) \partial^\theta_a \sum_{b=1}^{n} \frac{\partial g}{\partial x_b} (\partial^x) m^\theta_b
\]
\[
= \sum_{a,b} \frac{\partial f}{\partial x_a}(\partial^x) \frac{\partial g}{\partial x_b} (\partial^x) \partial^\theta_a m^\theta_b
\]
\[
= \sum_{b,a} \frac{\partial g}{\partial x_b}(\partial^x) \frac{\partial f}{\partial x_a} (\partial^x) (-m^\theta_b \partial^\theta_a + \delta_{a,b})
\]
\[
= - \sum_{b} \frac{\partial g}{\partial x_b}(\partial^x) m^\theta_b \sum_{a} \frac{\partial f}{\partial x_a} (\partial^x) \partial^\theta_a + \sum_{a} \frac{\partial g}{\partial x_a}(\partial^x) \frac{\partial f}{\partial x_a} (\partial^x)
\]
\[
= -dg(\partial^x, m^\theta)df(\partial^x, \partial^\theta) + \sum_{a} \frac{\partial g}{\partial x_a}(\partial^x) \frac{\partial f}{\partial x_a} (\partial^x),
\]
where the third equality follows from Lemma 4.2. The result now follows from Lemma 7.2.

Lemma 7.4. Suppose \( f \in \mathbb{k}[V]^G \) is homogeneous with \( \deg f \geq 2 \). Then \( df \circ \mathcal{H}_G \subset \mathcal{H}_G \).

Proof. Suppose \( g \in \mathcal{H}_G \) is harmonic. By Lemma 5.6, we have \( f_i \cdot g = df_i \cdot g = 0 \) for all \( i \), and we must show \( f_i \cdot (df \circ g) = df_i \cdot (df \circ g) = 0 \). Since \( u \cdot v = u \circ v \) for all \( u \in \mathbb{k}[V] \), we have
\[
f \cdot (df_i \circ g) = df_i \circ (f \cdot g) = df_i \circ 0 = 0.
\]
As for \( df_i \cdot (df \circ g) \), by (34) we have
\[
df_i \cdot (df \circ g) = -df \circ (df_i \cdot g) + \frac{1}{2} (\nabla^2 (ff_i) - \nabla^2 (f)f_i - f(\nabla^2 f_i)) \cdot g.
\]
Since $df_i, f_i \in \mathcal{J}_G^C$, we have $df_i \cdot g = f_i \cdot g = f \cdot g = 0$, so each term except possibly $\nabla^2(ff_i) \cdot g$ vanishes. Since $f, f_i \in \mathbb{K}[V]$ are $G$-invariant, $\nabla^2(ff_i)$ is also $G$-invariant. The result follows trivially if $\nabla^2(ff_i) = 0$, so suppose $\nabla^2(ff_i) \neq 0$. Since $\deg f \geq 2$ and $\deg f_i \geq 1$, we have $\deg \nabla^2(ff_i) \geq 1$, so $\nabla^2(ff_i) \in \mathcal{J}_G^C$. Thus indeed $\nabla^2(ff_i) \cdot g = 0$, completing the proof. □

We are now in a position to prove Theorem 1.6 from the introduction.

**Proof of Theorem 1.6.** From Theorem 1.3, $\{df_I \odot \Delta : I \subset [n]\}$ is $\mathbb{K}$-linearly independent and carries the det-representation. It is well-known that $\Delta \in \mathcal{H}_G$. Indeed, $df_i \cdot \Delta = 0$ since $\Delta$ contains no $\theta$’s, and $f_i \cdot \Delta$ would be an alternant in $\mathbb{K}[V]$ of degree lower than $\Delta$, so $f_i \cdot \Delta = 0$. By Lemma 7.4, $\{df_I \odot \Delta\} \subset \mathcal{H}_G$. We must only show this is a spanning set. By Theorem 1.3, an arbitrary homogeneous element of $\mathcal{H}_G^{\det} \subset \mathbb{K}[x_n, \theta_n]^{\det}$ is of the form

$$\sum_J df_I \odot g_I \Delta \in \mathcal{H}_G$$

for $g_I \in \mathbb{K}[V]^G$ homogeneous and $|J|$ constant. By Lemma 7.4, applying $df_{[n]-I}$ for fixed $I$ gives

$$\Delta \theta_{[n]} \odot g_I \Delta = (\Delta \cdot g_I \Delta) \theta_{[n]} \in \mathcal{H}_G.$$  

If $\deg g_I > 0$, then $g_I \in \mathcal{J}_G^C$, so applying $g_I \cdot -$ gives

$$(g_I \Delta \cdot g_I \Delta) \theta_{[n]} = 0,$$

forcing $g_I \Delta = 0$, so $g_I = 0$, a contradiction. Hence $g_I$ is constant for all $I$, which completes the proof. □

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