TEMPERLEY–LIEB, BIRMAN–MURAKAMI–WENZL AND ASKEY–WILSON ALGEBRAS AND OTHER CENTRALIZERS OF $U_q(\mathfrak{sl}_2)$

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Abstract. The centralizer of the image of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of three irreducible representations is examined in a Schur–Weyl duality spirit. The aim is to offer a description in terms of generators and relations. A conjecture in this respect is offered with the centralizers presented as quotients of the Askey–Wilson algebra. Support for the conjecture is provided by an examination of the representations of the quotients. The conjecture is also shown to be true in a number of cases thereby exhibiting in particular the Temperley–Lieb, Birman–Murakami–Wenzl and one-boundary Temperley–Lieb algebras as quotients of the Askey–Wilson algebra.

1. Introduction

The objective of this paper is to establish precisely the connections between the Askey–Wilson algebra and the centralizers of the quantum algebra $U_q(\mathfrak{sl}_2)$ such as the Temperley–Lieb and Birman–Murakami–Wenzl algebras.

In previous works, the connections between the Racah, Temperley–Lieb and Brauer algebras and other centralizers of $\mathfrak{sl}_2$ were studied in the spirit of the Schur–Weyl duality [4]. In a similar fashion, the Bannai-Ito algebra was connected to the centralizers of the superalgebra $\mathfrak{osp}(1|2)$, and in particular to the Brauer algebra [1]. The present paper generalizes the results of [4] by examining their $q$-deformation.

The Askey–Wilson algebra was first introduced in [22] and is defined by three generators satisfying some $q$-commutation relations. This algebra encodes the properties of the Askey–Wilson polynomials [13] and is related to the Racah problem for $U_q(\mathfrak{sl}_2)$ [8]. Due to this connection, a centrally extended Askey–Wilson algebra can be mapped to the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ into $U_q(\mathfrak{sl}_2)^{\otimes 3}$ [21, 9]. In the $q$-deformation of the universal enveloping algebra of $\mathfrak{sl}_2$ to the quantum algebra $U_q(\mathfrak{sl}_2)$, the Askey–Wilson algebra plays a role analogous to that of the Racah algebra.

From the Schur–Weyl duality, the centralizer of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of its fundamental representation is connected to the Hecke algebra. In the case of the threefold tensor product, the centralizer is known [12] to be isomorphic to the Temperley–Lieb algebra [20], which is a quotient of the Hecke algebra. In fact, the algebra $U_q(\mathfrak{sl}_2)$ has infinitely many finite irreducible representations, labeled by a half-integer or integer spin $j$. In the case of the tensor product of three spin-1 representations, it is also known [16] that the centralizer of $U_q(\mathfrak{sl}_2)$ is isomorphic to the Birman–Murakami–Wenzl algebra [11], which is a $q$-deformation of the Brauer algebra. However, in the general case of three irreducible representations of spins $j_1, j_2$ and $j_3$, an algebraic description of the centralizer is not known.
The present paper provides an attempt to describe the centralizer of the image of the diagonal embedding of \(U_q(\mathfrak{sl}_2)\) in the tensor product of any three irreducible representations in terms of generators and relations, by using the connections with the Askey–Wilson algebra. It is first shown that there is a surjective map (between generators) from the Askey–Wilson algebra to the centralizer. This statement corresponds in invariant theory to the first fundamental theorem \([14]\). A conjecture is then proposed in order to obtain an isomorphism between a quotient (given in terms of relations) of the Askey–Wilson algebra and the centralizer of \(U_q(\mathfrak{sl}_2)\) – this relates to the second fundamental theorem in invariant theory \([14]\). The conjecture is proved for three spin-\(\frac{1}{2}\) representations, in which case the Temperley–Lieb algebra is obtained explicitly as a quotient of the Askey–Wilson algebra. Similarly, for three spin-1 representations, it is shown that the conjecture holds and that the Birman–Murakami–Wenzl algebra is isomorphic to a quotient of the Askey–Wilson algebra. The conjecture is also verified for three spin-\(\frac{3}{2}\) representations, and for one spin-\(j\) and two spin-\(\frac{1}{2}\) representations, for \(j\) any spin greater than \(\frac{1}{2}\). In the latter case, it is shown that the centralizer is isomorphic to the one-boundary Temperley–Lieb algebra \([17, 18, 19]\).

The plan of this paper is as follows. Section 2 gives the precise connection between the centralizer of \(U_q(\mathfrak{sl}_2)\) and the Askey–Wilson algebra. Subsection 2.1 presents the quantum algebra \(U_q(\mathfrak{sl}_2)\) and its properties. The centralizer of \(U_q(\mathfrak{sl}_2)\) in \(U_q(\mathfrak{sl}_2)^{\otimes 3}\) and the intermediate Casimirs are defined in Subsection 2.2. A homomorphism between the centrally extended Askey–Wilson algebra \(AW(3)\) and this centralizer is given in Subsection 2.3. Section 3 is concerned with the representations of \(U_q(\mathfrak{sl}_2)\) and their tensor product decomposition rules are recalled in Subsection 3.1. Subsection 3.2 introduces the object of main interest, that is the centralizer of the image of the diagonal embedding of \(U_q(\mathfrak{sl}_2)\) in the tensor product of three irreducible representations. Section 4 aims to describe this centralizer in terms of generators and relations. Subsection 4.1 maps \(AW(3)\) to this centralizer, and this is shown to be a surjection in Subsection 4.2. The kernel of this map is discussed in Subsection 4.3 and a conjecture proposing that a quotient of \(AW(3)\) is isomorphic to the centralizer is formulated. Subsection 4.4 contains the proof that the conjecture does not depend on the ordering of the three spins \(j_1, j_2, j_3\). In order to support the conjecture, Section 5 studies the finite irreducible representations of the quotient of \(AW(3)\). The remaining sections contain the proofs of the conjecture for some particular cases. Section 6 focuses on the case \(j_1 = j_2 = j_3 = \frac{1}{2}\). It is shown in Subsection 6.1 that the conjecture holds in this case, and the precise connection with the Temperley–Lieb algebra is given in Subsection 6.2. Section 7 considers the case \(j_1 = j_2 = j_3 = 1\). The proof of the conjecture is given in Subsection 7.1. An isomorphism between the quotient of \(AW(3)\) and the Birman–Murakami–Wenzl algebra is obtained in Subsection 7.2. The conjecture for the case \(j_1 = j_2 = j_3 = \frac{3}{2}\) is proved in Section 8. Finally, Section 9 studies the case \(j_1 = j\) for \(j = 1, \frac{3}{2}, \ldots\) and \(j_2 = j_3 = \frac{1}{2}\). The conjecture is verified in Subsection 9.1 and the connection with the one-boundary Temperley–Lieb algebra is described in Subsection 9.2.

2. Centralizer of \(U_q(\mathfrak{sl}_2)\) and Askey–Wilson algebra

In this section, we recall well-known properties of the quantum algebra \(U_q(\mathfrak{sl}_2)\) to fix the notations. Then, the definition of the centralizer of the diagonal embedding of \(U_q(\mathfrak{sl}_2)\) in \(U_q(\mathfrak{sl}_2)^{\otimes 3}\) is recalled and its homomorphism with the centrally extended Askey–Wilson algebra \(AW(3)\) is presented.
2.1. \(U_q(sl_2)\) algebra. The associative algebra \(U_q(sl_2)\) is generated by \(E\), \(F\) and \(q^H\) with the defining relations

\[
q^H E = q E q^H, \quad q^H F = q^{-1} F q^H \quad \text{and} \quad [E, F] = [2H]_q,
\]

where \([X]_q = q^X - q^{X-1}\). Throughout this paper, \(q\) is a complex number not root of unity. There is a central element in \(U_q(sl_2)\), called quadratic Casimir element, given by

\[
\Gamma = (q - q^{-1})^2 FE + q q^{2H} + q^{-1} q^{-2H}.
\]

There exists also an algebra homomorphism \(\Delta : U_q(sl_2) \rightarrow U_q(sl_2) \otimes U_q(sl_2)\), called comultiplication, defined on the generators by

\[
\Delta(E) = E \otimes q^{-H} + q^H \otimes E, \quad \Delta(F) = F \otimes q^{-H} + q^H \otimes F \quad \text{and} \quad \Delta(q^H) = q^H \otimes q^H.
\]

This comultiplication is coassociative

\[
(\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta =: \Delta^{(2)}.
\]

We define the opposite comultiplication \(\Delta^{op} = \sigma \circ \Delta\), where \(\sigma(x \otimes y) = y \otimes x\), for \(x, y \in U_q(sl_2)\). It is a homomorphism from \(U_q(sl_2)\) to \(U_q(sl_2) \otimes U_q(sl_2)\) different from \(\Delta\). Both are related by the universal \(R\)-matrix \(R \in U_q(sl_2) \otimes U_q(sl_2)\) satisfying

\[
\Delta(x)R = R\Delta^{op}(x) \quad \text{for} \quad x \in U_q(sl_2).
\]

The universal \(R\)-matrix also satisfies the Yang-Baxter equation

\[
R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}.
\]

We have used the usual notations: if \(R = R^\alpha \otimes R_\alpha\), then \(R_{12} = R^\alpha \otimes R_\alpha \otimes 1\), \(R_{23} = 1 \otimes R^\alpha \otimes R_\alpha\) and \(R_{13} = R^\alpha \otimes 1 \otimes R_\alpha\) (the sum w.r.t. \(\alpha\) is understood).

2.2. Centralizer of \(U_q(sl_2)\) in \(U_q(sl_2)^{\otimes 3}\). The centralizer \(C_3\) of the diagonal embedding of \(U_q(sl_2)\) in \(U_q(sl_2)^{\otimes 3}\) is

\[
C_3 = \{X \in U_q(sl_2)^{\otimes 3} \mid [\Delta^{(2)}(x), X] = 0, \quad \forall x \in U_q(sl_2)\}.
\]

This centralizer is a subalgebra of \(U_q(sl_2)^{\otimes 3}\) and we want to describe this subalgebra with some generators and defining relations. Let us first give some elements of \(C_3\) by using the Casimir element \(\Gamma\) which is central in \(U_q(sl_2)\). We define the following Casimir elements of \(U_q(sl_2)^{\otimes 3}\)

\[
\Gamma_1 = \Gamma \otimes 1 \otimes 1, \quad \Gamma_2 = 1 \otimes \Gamma \otimes 1, \quad \Gamma_3 = 1 \otimes 1 \otimes \Gamma.
\]

These elements are central in \(U_q(sl_2)^{\otimes 3}\) and thus belong to \(C_3\). We also define the total Casimir

\[
\Gamma_{123} = \Delta^{(2)}(\Gamma).
\]

This element belongs to \(C_3\) because \([\Delta^{(2)}(\Gamma), \Delta^{(2)}(x)] = \Delta^{(2)}([\Gamma, x]) = 0\) for all \(x \in U_q(sl_2)\). Let us notice that \(\Gamma_{123}\) is central in \(C_3\) since it is also an element of the diagonal embedding of \(U_q(sl_2)\).

We then define the intermediate Casimirs associated to the recoupling of the two first or the two last factors of \(U_q(sl_2)^{\otimes 3}\)

\[
\Gamma_{12} = \Delta(\Gamma) \otimes 1 \quad \text{and} \quad \Gamma_{23} = 1 \otimes \Delta(\Gamma).
\]
One uses the properties of the comultiplication to show that $\Gamma_{12}$ and $\Gamma_{23}$ are in $\mathfrak{c}_3$; indeed, for all $x \in U_q(\mathfrak{sl}_2)$,

\begin{align}
(2.11) \quad & [\Gamma_{12}, \Delta^{(2)}(x)] = [\Delta(\Gamma) \otimes 1, (\Delta \otimes \text{id})\Delta(x)] = (\Delta \otimes \text{id})[\Gamma \otimes 1, \Delta(x)] = 0, \\
(2.12) \quad & [\Gamma_{23}, \Delta^{(2)}(x)] = [1 \otimes \Delta(\Gamma), (\text{id} \otimes \Delta)\Delta(x)] = (\text{id} \otimes \Delta)[1 \otimes \Gamma, \Delta(x)] = 0. 
\end{align}

In the limit $q \to 1$, it can be shown that the element

\begin{equation}
(2.13) \quad \Gamma_{13} = \sum_\alpha \Gamma_\alpha \otimes 1 \otimes \Gamma_\alpha,
\end{equation}

where $\Delta(\Gamma) = \sum_\alpha \Gamma_\alpha \otimes \Gamma_\alpha$, belongs to the centralizer of $U(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$. However, this is not the case for the quantum algebra $U_q(\mathfrak{sl}_2)$. This difficulty that arises in the $q$-deformation of the algebra $U(\mathfrak{sl}_2)$ was addressed in [2] where a definition of the third intermediate Casimir element of $U_q(\mathfrak{sl}_2)$ is provided with the help of the universal $R$-matrix. It is shown in [2] that the following elements

\begin{align}
(2.14) \quad & \Gamma_{13}^{(0)} = R_{12} \Gamma_{13} R_{12}^{-1} = R_{32}^{-1} \Gamma_{13} R_{32}, \\
(2.15) \quad & \Gamma_{13}^{(1)} = R_{23} \Gamma_{13} R_{23}^{-1} = R_{21}^{-1} \Gamma_{13} R_{21},
\end{align}

are in the centralizer $\mathfrak{c}_3$.

2.3. Connection with the Askey–Wilson algebra. The intermediate Casimir elements $\Gamma_{12}$, $\Gamma_{23}$, $\Gamma_{13}^{(0)}$ and $\Gamma_{13}^{(1)}$ do not commute pairwise but satisfy certain relations which are identified as those of the Askey–Wilson algebra $AW(3)$.

**Definition 2.1.** The centrally extended Askey–Wilson algebra $AW(3)$ is generated by $A$, $B$, $D$ and central elements $\alpha_1$, $\alpha_2$, $\alpha_3$ and $K$ subject to the following defining relations

\begin{align}
(2.16) \quad & A + \frac{[B, D]_q}{q^2 - q^{-2}} = \frac{\alpha_1 \alpha_2 + \alpha_3 K}{q + q^{-1}}, \\
(2.17) \quad & B + \frac{[D, A]_q}{q^2 - q^{-2}} = \frac{\alpha_2 \alpha_3 + \alpha_1 K}{q + q^{-1}}, \\
(2.18) \quad & D + \frac{[A, B]_q}{q^2 - q^{-2}} = \frac{\alpha_1 \alpha_3 + \alpha_2 K}{q + q^{-1}},
\end{align}

where $[X, Y]_q = qXY - q^{-1}YX$. We also define the element $D' \in AW(3)$ by the following relation

\begin{equation}
(2.19) \quad D' + \frac{[B, A]_q}{q^2 - q^{-2}} = \frac{\alpha_1 \alpha_3 + \alpha_2 K}{q + q^{-1}}.
\end{equation}

The algebra $AW(3)$ has a Casimir element given by

\begin{equation}
(2.20) \quad \Omega = qA(\alpha_1 \alpha_2 + \alpha_3 K) + q^{-1}B(\alpha_2 \alpha_3 + \alpha_1 K) + qD(\alpha_1 \alpha_3 + \alpha_2 K) - q^2 A^2 - q^{-2}B^2 - q^2 D^2 - qABD.
\end{equation}

The connection between the centralizer $\mathfrak{c}_3$ defined by (2.7) and the Askey–Wilson algebra is given in the following proposition.

**Proposition 2.1.** The map $\varphi : AW(3) \to \mathfrak{c}_3$ defined by

\begin{equation}
(2.21) \quad \varphi(\alpha_i) = \Gamma_i, \quad \varphi(A) = \Gamma_{12}, \quad \varphi(B) = \Gamma_{23}, \quad \varphi(K) = \Gamma_{123},
\end{equation}

is an algebra homomorphism. We deduce that

\begin{equation}
(2.22) \quad \varphi(D) = \Gamma_{13}^{(0)}, \quad \varphi(D') = \Gamma_{13}^{(1)}.
\end{equation}
The homomorphism has been proved in [8]; a direct computation shows that the intermediate Casimir elements satisfy all the relations of \( AW(3) \). Relations (2.22) have been proved more recently in [2] and a simpler proof of the homomorphism using the universal \( R \)-matrix has also been given. Let us remark that a similar proof has also been simplified in the case of the Bannai–Ito algebra and the centralizer for the super Lie algebra \( \mathfrak{osp}(1|2) \) [6].

Using (2.18) to replace \( D \) in (2.16) and (2.17), one shows that the following relations provide an equivalent presentation of \( AW(3) \) which will be useful for later computations

\[
\frac{[B, [A, B]_q]_q}{(q - q^{-1})^2} = (q + q^{-1})^2 A + (\alpha_1 \alpha_3 + \alpha_2 K) B - (q + q^{-1})(\alpha_1 \alpha_2 + \alpha_3 K),
\]

\[
\frac{[A, B]_q; A]_q}{(q - q^{-1})^2} = (q + q^{-1})^2 B + (\alpha_1 \alpha_3 + \alpha_2 K) A - (q + q^{-1})(\alpha_2 \alpha_3 + \alpha_1 K).
\]

Furthermore, noticing that \([X, [Y, X]]_q = [[X, Y]_q, X]_q \) and using the element \( D' \) defined in (2.19), one finds that (2.23) and (2.24) imply

\[
A + \frac{[D', B]_q}{q^2 - q^{-2}} = \frac{\alpha_1 \alpha_2 + \alpha_3 K}{q + q^{-1}},
\]

\[
B + \frac{[A, D']_q}{q^2 - q^{-2}} = \frac{\alpha_2 \alpha_3 + \alpha_1 K}{q + q^{-1}}.
\]

Relations (2.19), (2.25) and (2.26) provide another \( \mathbb{Z}_3 \) symmetric presentation of \( AW(3) \).

**Remark 2.1.** Upon performing the affine transformation \( X = (q - q^{-1})^2 \tilde{X} + q + q^{-1} \) on the elements \( X = A, B, D, D', \alpha_i, K \) of \( AW(3) \), one sees that relations (2.23) – (2.24) can be written as

\[
[\tilde{B}, [\tilde{A}, \tilde{B}]_q]_q = (q + q^{-1}) \left( -\tilde{B}^2 - \{ \tilde{A}, \tilde{B} \} + (\tilde{K} + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) \tilde{B} + (\tilde{\alpha}_1 - \tilde{K})(\tilde{\alpha}_3 - \tilde{\alpha}_2) \right)
\]

\[
+ (q - q^{-1})^2(\tilde{\alpha}_1 \tilde{\alpha}_3 + \tilde{\alpha}_2 K) \tilde{B},
\]

\[
[[\tilde{A}, \tilde{B}]_q, \tilde{A}]_q = (q + q^{-1}) \left( -\tilde{A}^2 - \{ \tilde{A}, \tilde{B} \} + (\tilde{K} + \tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3) \tilde{A} + (\tilde{\alpha}_3 - \tilde{K})(\tilde{\alpha}_1 - \tilde{\alpha}_2) \right)
\]

\[
+ (q - q^{-1})^2(\tilde{\alpha}_1 \tilde{\alpha}_3 + \tilde{\alpha}_2 K) \tilde{A},
\]

where \( \{X, Y\} = XY + YX \). By taking the limit \( q \to 1 \) of (2.27) and (2.28), one recovers the defining relations of the Racah algebra used in [1]. Relations (2.18) and (2.19) are transformed into

\[
\frac{[\tilde{A}, \tilde{B}]_q}{q - q^{-1}} = \tilde{\alpha}_1 \tilde{\alpha}_3 + \tilde{\alpha}_2 \tilde{K} + \frac{q + q^{-1}}{(q - q^{-1})^2}(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{K} - \tilde{A} - \tilde{B} - \tilde{D}),
\]

\[
\frac{[\tilde{B}, \tilde{A}]_q}{q - q^{-1}} = \tilde{\alpha}_1 \tilde{\alpha}_3 + \tilde{\alpha}_2 \tilde{K} + \frac{q + q^{-1}}{(q - q^{-1})^2}(\tilde{\alpha}_1 + \tilde{\alpha}_2 + \tilde{\alpha}_3 + \tilde{K} - \tilde{A} - \tilde{B} - \tilde{D}').
\]

In the limit \( q \to 1 \), the elements \( \tilde{D} \) and \( \tilde{D}' \) are equal, and the images by \( \varphi \) of (2.29) and (2.30) both reduce to the well-known linear relation \( \tilde{\Gamma}_1 + \tilde{\Gamma}_2 + \tilde{\Gamma}_3 + \tilde{\Gamma}_{123} - \tilde{\Gamma}_{12} - \tilde{\Gamma}_{23} - \tilde{\Gamma}_{13} = 0 \) that holds in \( U(\mathfrak{sl}_2)^{\otimes 3} \).

### 3. Decomposition of tensor product of representations and centralizer

In the previous section, we introduced the centralizer \( \mathfrak{C}_3 \) of the diagonal embedding of \( U_q(\mathfrak{sl}_2) \) in \( U_q(\mathfrak{sl}_2)^{\otimes 3} \) and showed the connection of this subalgebra of \( U_q(\mathfrak{sl}_2)^{\otimes 3} \) with the Askey–Wilson algebra.
AW(3). We now focus on the corresponding objects when each $U_q(\mathfrak{sl}_2)$ in $U_q(\mathfrak{sl}_2)^{\otimes 3}$ is taken in a finite irreducible representation.

3.1. Finite irreducible representations of $U_q(\mathfrak{sl}_2)$. The quantum algebra $U_q(\mathfrak{sl}_2)$ has finite irreducible representations of dimension $2j + 1$ that we will denote by $M_j$, with $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. The representation map will be denoted by $\pi_j : U_q(\mathfrak{sl}_2) \to \mathrm{End}(M_j)$. We will use the name spin-$j$ representation to refer to $M_j$. The representation of the Casimir element (2.2) in the space $M_j$ is

$$\pi_j(\Gamma) = \chi_j \mathbb{I}_{2j+1} \quad \text{where} \quad \chi_j = q^{2j+1} + q^{-2j-1},$$

and $\mathbb{I}_{2j+1}$ is the $(2j + 1)$ by $(2j + 1)$ identity matrix. We define the following sets, for three half-integers or integers $j_1$, $j_2$ and $j_3$

$$\mathcal{J}(j_1, j_2) = \{ |j_1 - j_2|, |j_1 - j_2| + 1, \ldots, j_1 + j_2 \},$$

$$\mathcal{J}(j_1, j_2, j_3) = \bigcup_{j \in \mathcal{J}(j_1, j_2)} \mathcal{J}(j, j_3).$$

Notice that there are no repeated numbers in $\mathcal{J}(j_1, j_2, j_3)$, and this set is invariant under any permutation of $j_1$, $j_2$ and $j_3$.

For $q$ not a root of unity, the tensor product of two irreducible representations of $U_q(\mathfrak{sl}_2)$ decomposes into the following direct sum of irreducible representations

$$M_{j_1} \otimes M_{j_2} = \bigoplus_{j \in \mathcal{J}(j_1, j_2)} M_j.$$

Similarly, the threefold tensor product of irreducible representations of $U_q(\mathfrak{sl}_2)$ decomposes into the following direct sum

$$M_{j_1} \otimes M_{j_2} \otimes M_{j_3} = \bigoplus_{j \in \mathcal{J}(j_1, j_2, j_3)} d_j M_j,$$

where $d_j \in \mathbb{Z}_{>0}$ is the degeneracy of $M_j$ and is referred to as the Littlewood–Richardson coefficient.

3.2. Centralizer of $U_q(\mathfrak{sl}_2)$ in $\mathrm{End}(M_{j_1} \otimes M_{j_2} \otimes M_{j_3})$. From now on, we fix three half-integers or integers $j_1$, $j_2$ and $j_3$. The centralizer $C_{j_1, j_2, j_3}$ of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in $\mathrm{End}(M_{j_1} \otimes M_{j_2} \otimes M_{j_3})$ is

$$C_{j_1, j_2, j_3} = \{ m \in \mathrm{End}(M_{j_1} \otimes M_{j_2} \otimes M_{j_3}) \mid [\pi_{j_1, j_2, j_3}(\Delta^{(2)}(x)), m] = 0, \forall x \in U_q(\mathfrak{sl}_2) \},$$

where we have used the shortened notation $\pi_{j_1, j_2, j_3} = \pi_{j_1} \otimes \pi_{j_2} \otimes \pi_{j_3}$. This centralizer as a subalgebra of $\mathrm{End}(M_{j_1} \otimes M_{j_2} \otimes M_{j_3})$ is the object of interest of this paper. In the next section, we conjecture a presentation of this centralizer in terms of generators and relations by using the connections with the Askey–Wilson algebra $AW(3)$. Let us first recall some known properties of this centralizer.

The knowledge of the centralizer permits to write the decomposition rule (3.6) as follows

$$M_{j_1} \otimes M_{j_2} \otimes M_{j_3} = \bigoplus_{j \in \mathcal{J}(j_1, j_2, j_3)} M_j \otimes V_j,$$
where $V_j$ is a finite irreducible representation of dimension $d_j$ of $C_{j_1,j_2,j_3}$. The set $\{V_j \mid j \in \mathcal{J}(j_1, j_2, j_3)\}$ is the complete set of non-equivalent irreducible representations of $C_{j_1,j_2,j_3}$. In particular, one deduces that the dimension of the centralizer is

\begin{equation}
\dim(C_{j_1,j_2,j_3}) = \sum_{j \in \mathcal{J}(j_1,j_2,j_3)} d_j^2 .
\end{equation}

These representations $V_j$ are explicitly given in Subsection 4.2.

We now define the images of the centralizing elements $\{2.8\}–\{2.10\}$ and $\{2.14\}–\{2.15\}$ of $U_q(\mathfrak{sl}_2)^{\otimes 3}$ in the representation $\text{End}(M_{j_1} \otimes M_{j_2} \otimes M_{j_3})$ as follows

\begin{equation}
\pi_{j_1,j_2,j_3} : C_i \rightarrow C_{j_1,j_2,j_3}, \quad \Gamma_{ij,k} \rightarrow C_i, C_{ij}, C_{123} .
\end{equation}

Therefore, $C_{j_1,j_2,j_3}$ contains the elements $C_i$, $C_{ij}$ and $C_{123}$. According to (3.1), the elements $C_i$ are simply constant matrices of value $\chi_j$, for $i = 1, 2, 3$. The intermediate Casimirs $C_{12}, C_{23}, C^{(0)}_{13}$ and $C^{(1)}_{13}$, and the total Casimir $C_{123}$ of $C_{j_1,j_2,j_3}$ can be diagonalized if $q$ is not a root of unity.

Since $C_{12}$ is the Casimir associated to the recoupling of the two first factors of the threefold tensor product of $U_q(\mathfrak{sl}_2)$, one finds (using the decomposition rule (3.1)) that its eigenvalues are $\chi_j$ for $j \in \mathcal{J}(j_1, j_2)$. Similarly, the eigenvalues of $C_{23}$ (resp. $C_{123}$) are $\chi_j$ for $j \in \mathcal{J}(j_2, j_3)$ (resp. $\mathcal{J}(j_1, j_2, j_3)$). The same argument cannot be applied directly to the intermediate Casimirs $C^{(0)}_{13}$ and $C^{(1)}_{13}$ since they are not trivial in the space 2. However, the element $C_{13}$ defined in (2.13) only couples the spaces 1 and 3 such that its eigenvalues are $\chi_j$ for $j \in \mathcal{J}(j_1, j_3)$. From the definitions (2.14) and (2.15), we see that $C^{(0)}_{13}$ and $C^{(1)}_{13}$ are both conjugations of $C_{13}$ by an $R$-matrix. Hence, their eigenvalues are the same as those of $C_{13}$. The previous discussion implies that the minimal polynomials of the intermediate Casimirs and the total Casimir take the following form

\begin{equation}
\prod_{j \in \mathcal{J}(j_1, j_2)} (C_{12} - \chi_j) = 0 , \quad \prod_{j \in \mathcal{J}(j_2, j_3)} (C_{23} - \chi_j) = 0 , \quad \prod_{j \in \mathcal{J}(j_1, j_2, j_3)} (C_{123} - \chi_j) = 0 ,
\end{equation}

\begin{equation}
\prod_{j \in \mathcal{J}(j_1, j_3)} (C^{(0)}_{13} - \chi_j) = 0 , \quad \prod_{j \in \mathcal{J}(j_1, j_2, j_3)} (C^{(1)}_{13} - \chi_j) = 0 .
\end{equation}

Because $C_{123}$ is central in $C_{j_1,j_2,j_3}$, it can be diagonalized simultaneously with $C_{12}$, $C_{23}$, $C^{(0)}_{13}$ or $C^{(1)}_{13}$. Therefore, one gets the following minimal polynomials

\begin{equation}
\prod_{m \in \mathcal{M}(j_1,j_2,j_3)} (C_{123} - C_{12} - m) = 0 , \quad \prod_{m \in \mathcal{M}(j_2,j_3,j_1)} (C_{123} - C_{23} - m) = 0 ,
\end{equation}

\begin{equation}
\prod_{m \in \mathcal{M}(j_1,j_3,j_2)} (C_{123} - C^{(0)}_{13} - m) = 0 , \quad \prod_{m \in \mathcal{M}(j_1,j_3,j_2)} (C_{123} - C^{(1)}_{13} - m) = 0 ,
\end{equation}

where

\begin{equation}
\mathcal{M}(j_a, j_b, j_c) = \bigcup_{j \in \mathcal{J}(j_a,j_b)} \{\chi_\ell - \chi_j \mid \ell \in \mathcal{J}(j, j_c)\} .
\end{equation}

In the previous set $\mathcal{M}(j_a, j_b, j_c)$, there are no repeated numbers.

Before concluding this section, let us notice that if one performs the transformation given in Remark 2.1 on the Casimir element $\Gamma$ of $U_q(\mathfrak{sl}_2)$, its value in the representation $\text{End}(M_j)$ is $\tilde{\chi}_j = [j]_q[j+1]_q$. By construction, similar results hold for the eigenvalues of the transformed elements.
\( \tilde{C}, \tilde{C}_{ij} \text{ and } \tilde{C}_{123}. \) In the limit \( q \to 1 \), the minimal polynomials of these transformed elements thus reduce to the ones discussed in [4].

4. Algebraic description of the centralizer \( \mathcal{C}_{j_1,j_2,j_3} \)

Take \( j_1, j_2 \) and \( j_3 \) to be three fixed half-integers or integers. This section contains an attempt to give a definition of the centralizer \( \mathcal{C}_{j_1,j_2,j_3} \) in terms of generators and relations. We rely on the connection with the Askey–Wilson algebra \( AW(3) \).

4.1. Homomorphism with \( AW(3) \). The intermediate Casimir elements \( C_i, C_{ij} \) and \( C_{123} \) belonging to \( \mathcal{C}_{j_1,j_2,j_3} \) satisfy the defining relations of the Askey–Wilson algebra as stated precisely in the following proposition.

**Proposition 4.1.** The map \( \phi : AW(3) \to \mathcal{C}_{j_1,j_2,j_3} \) defined by

\[
(4.1) \quad \phi(\alpha_i) = C_i , \quad \phi(A) = C_{12} , \quad \phi(B) = C_{23} , \quad \phi(K) = C_{123} ,
\]

is an algebra homomorphism.

**Proof.** The result follows from the fact that \( \phi \) is the composition of two homomorphisms \( \phi = \pi_{j_1,j_2,j_3} \circ \varphi \), where \( \varphi \) is defined in Proposition 2.1. \( \square \)

We recall that \( C_i = \chi_{j_i} = q^{2j_i+1} + q^{-2j_i-1} \) times the identity matrix. Therefore, it can be identified as the number \( \chi_{j_i} \). Let us also emphasize that \( \phi(D) = C_{13}^{(0)} \) and \( \phi(D') = C_{13}^{(1)} \). Moreover, the image by \( \phi \) of the Casimir element \( \Omega \) of \( AW(3) \) defined in (2.20) is equal to an expression involving only central elements [8]:

\[
(4.2) \quad \phi(\Omega) = C_1^2 + C_2^2 + C_3^2 + C_{123}^2 + C_1 C_2 C_3 C_{123} - (q + q^{-1})^2.
\]

4.2. Surjectivity. We now show that the intermediate Casimir elements \( C_i, C_{12}, C_{23} \) and \( C_{123} \) generate the whole centralizer \( \mathcal{C}_{j_1,j_2,j_3} \).

**Proposition 4.2.** The map \( \phi : AW(3) \to \mathcal{C}_{j_1,j_2,j_3} \) is surjective.

**Proof.** To reach that conclusion, we prove that the dimension of the image of \( \phi \) is at least \( \sum_{\ell \in J(j_1,j_2,j_3)} d_\ell^2 \), the dimension of \( \mathcal{C}_{j_1,j_2,j_3} \). Let \( \ell \in J(j_1,j_2,j_3) \) and

\[
(4.3) \quad S^\ell(j_1,j_2,j_3) = \{ j \in J(j_1,j_2) \mid \ell \in J(j,j_3) \}.
\]

From the definition (3.2), we deduce that \( S^\ell(j_1,j_2,j_3) = \{j_{\text{min}}, j_{\text{min}} + 1, \ldots, j_{\text{max}}\} \) with

\[
(4.4) \quad j_{\text{min}} = \max(|j_1 - j_2|, |j_3 - \ell|) \quad \text{and} \quad j_{\text{max}} = \min(j_1 + j_2, j_3 + \ell).
\]

The cardinality of this set is \( d_\ell = j_{\text{max}} - j_{\text{min}} + 1 \). We denote by \( M_\ell^+ \) the vector space spanned by the highest weight vectors of the representations \( M_\ell \) in the decomposition (3.5). The dimension of \( M_\ell^+ \) is \( d_\ell \) and we can choose \( d_\ell \) independent vectors \( v_j \in M_\ell^+ \) with \( j \in S^\ell(j_1,j_2,j_3) \) such that

\[
(4.5) \quad \pi_{j_1,j_2,j_3}(\Delta^{(2)}(E))v_j = 0 , \quad \pi_{j_1,j_2,j_3}(\Delta^{(2)}(q^H))v_j = q^\ell v_j , \quad C_{123}v_j = \chi_{j}v_j , \quad C_{12}v_j = \chi_{j}v_j , \quad \text{and}
\]

\[
(4.6) \quad C_{23}v_j = \sum_{k \in S^\ell(j_1,j_2,j_3)} \alpha_{j,k} v_k ,
\]

and...
where $\alpha_{j,k}$ are complex numbers. The elements $C_{12}$ and $C_{23}$ are the images by $\phi$ of $A$ and $B$. Therefore, they satisfy the Askey–Wilson algebra. It is enough to determine the constants $\alpha_{j,k}$ as shown previously in [22]. We reproduce this computation in the particular case needed here. We define the constants

\begin{equation}
\alpha = \chi_{j_1}\chi_{j_2} + \chi_{j_3}\chi_{\ell}, \quad \beta = \chi_{j_2}\chi_{j_3} + \chi_{j_1}\chi_{\ell} \quad \text{and} \quad \gamma = \chi_{j_1}\chi_{j_3} + \chi_{j_2}\chi_{\ell}.
\end{equation}

We act with relation (2.24) on the vector $v_j$ (for $j \in S'(j_1, j_2, j_3)$) and project the result on $v_k$ with $k \neq j$ and on $v_\ell$. We get

\begin{equation}
[j + k + 2]_q[j + k][j - j]_q[k - j + 1]_q\alpha_{j,k} = 0,
\end{equation}

\begin{equation}
\alpha_{j,j} = \frac{\chi_j - b\chi_0}{\chi_j^2 - \chi_0^2} \quad \text{for} \quad j \neq 0.
\end{equation}

The projection on $v_j$ is trivial if $j = 0$. From relation (4.8), we deduce that $\alpha_{j,k} = 0$ for $j \in S'(j_1, j_2, j_3)$ and $k \neq j + 1, j - 1, j$.

Then, we act with relation (2.23) on the vector $v_j$ and project the result on $v_{j-2}, v_{j-1}, \ldots, v_{j+2}$. The projections are trivial except the one on $v_j$ which gives the following relation

\begin{equation}
[2j + 3]_q \alpha_{j,j+1} \alpha_{j+1,j} - [2j - 1]_q \alpha_{j-1,j} \alpha_{j,j-1} = \frac{1}{\chi_0}(c - \chi_j\alpha_{j,j})\alpha_{j,j} + \chi_0\chi_j - a,
\end{equation}

with the boundary conditions $\alpha_{j_{\min},j_{\min} - 1} = 0$ and $\alpha_{j_{\max},j_{\max} + 1} = 0$. By using (4.9), one can show that the recurrence relation (4.10) and the boundary condition $\alpha_{j_{\max},j_{\max} + 1} = 0$ imply

\begin{equation}
\alpha_{j-1,j} \alpha_{j,j-1} = \frac{\prod_{i=1}^{j-1}([j - r_i]_q[j + r_i]_q)}{[2j - 1]_q[2j]_q[2j + 1]_q}(q - q^{-1})^4 \quad \text{for} \quad j \neq 0,
\end{equation}

where $r_1 = j_1 - j_2$, $r_2 = j_3 - \ell$, $r_3 = j_1 + j_2 + 1$ and $r_4 = \ell + j_3 + 1$. We see from (4.4) that the second boundary condition $\alpha_{j_{\min},j_{\min} - 1} = 0$ is automatically satisfied if $j_{\min} > 0$. In the case where $j_{\min} = 0$ (which only happens if $j_1 = j_2$ and $\ell = j_3$), the limit $j \to 0$ of (4.11) vanishes. Moreover, we can deduce from (4.10) that $\alpha_{0,0} = \chi_{j_1}\chi_{j_3}/\chi_0$, which is the limit $j \to 0$ of (4.9).

To conclude the proof, we notice that equation (4.4) implies that the R.H.S. of relation (4.11) is never zero for $j_{\min} < j \leq j_{\max}$, and that the eigenvalues of $C_{12}$ are pairwise distinct. Therefore, for a given $\ell \in J(j_1, j_2, j_3)$, $C_{12}$ and $C_{23}$ generate a vector space of dimension $d^2_\ell$.

\begin{equation}
\square
\end{equation}

4.3. Kernel. The map $\phi$ defined in the Proposition 4.1 is not injective since there are non-trivial elements of $AW(3)$ that are mapped to zero, as seen from the results (3.10)–(3.13). We want to provide a description of the kernel of the map $\phi$ in order to find a quotient of $AW(3)$ that is isomorphic to the centralizer $C_{j_1,j_2,j_3}$. Let us first define a quotient of $AW(3)$.
Definition 4.1. The algebra $\overline{AW}(j_1, j_2, j_3)$ is the quotient of the centrally extended Askey–Wilson algebra $AW(3)$ by the following relations

\begin{align}
\alpha_i &= x_{j_i}, \\
(j_1) & \prod_{j \in J(j_1, j_2)} (A - x_j) = 0, \prod_{j \in J(j_2, j_3)} (B - x_j) = 0, \prod_{j \in J(j_1, j_2, j_3)} (K - x_j) = 0, \\
\prod_{j \in J(j_1, j_3)} (D - x_j) = 0, \prod_{j \in J(j_1, j_3)} (D' - x_j) = 0, \\
\prod_{m \in M(j_1, j_2, j_3)} (K - A - m) = 0, \prod_{m \in M(j_1, j_2, j_3)} (K - B - m) = 0, \\
\prod_{m \in M(j_1, j_3, j_2)} (K - D - m) = 0, \prod_{m \in M(j_1, j_3, j_2)} (K - D' - m) = 0, \\
\Omega &= x_{j_1}^2 + x_{j_2}^2 + x_{j_3}^2 + K^2 + x_{j_1}x_{j_2}x_{j_3}K - (q + q^{-1})^2,
\end{align}

where we recall that $D$ and $D'$ are defined through \((2.18)\)–\((2.19)\), and $\Omega$ is defined in \((2.20)\).

Let us emphasize that all the relations \((4.12)\)–\((4.17)\) are in the kernel of the map $\phi$ in view of the results of Subsections 3.2 and 4.1. We are now in position to state a conjecture that proposes an algebraic description of $C_{j_1, j_2, j_3}$.

Conjecture 4.1. The map $\overline{\phi} : \overline{AW}(j_1, j_2, j_3) \to C_{j_1, j_2, j_3}$ given by

\begin{align}
\overline{\phi}(A) &= C_{12}, \overline{\phi}(B) = C_{23}, \overline{\phi}(K) = C_{123},
\end{align}

is an algebra isomorphism.

To support this conjecture, we remark that by taking the limit $q \to 1$ (as described in Remark 2.1 of relations \((4.12)\)–\((4.16)\), we recover the conjecture proposed in \([4]\) and proved in numerous cases. Let us notice that in this limit, the two relations in \((4.14)\) reduce to only one relation, and similarly for the two relations in \((4.16)\). The relation \((4.17)\) involving the Casimir element of $AW(3)$ is new and will be useful when we discuss the representations of $\overline{AW}(j_1, j_2, j_3)$ in Section 5.

From the previous results, we know that $\overline{\phi}$ is a surjective homomorphism. It remains to prove that it is injective, which can be done by demonstrating that

\begin{align}
\dim(\overline{AW}(j_1, j_2, j_3)) \leq \sum_{j \in J(j_1, j_2, j_3)} d_j^2 = \dim(C_{j_1, j_2, j_3}).
\end{align}

To simplify the demonstration of \((4.19)\), we can decompose $\overline{AW}(j_1, j_2, j_3)$ into a direct sum of simpler algebras. Indeed, let us introduce the following central idempotents, for $k \in J(j_1, j_2, j_3)$,

\begin{align}
K_k = \prod_{r \in J(j_1, j_2, j_3)} \frac{K - x_r}{x_k - x_r},
\end{align}

which satisfy $K_k K_l = \delta_{k,l} K_k$, $\sum_{k \in J(j_1, j_2, j_3)} K_k = 1$ and $KK_k = K K_k = x_k K_k$. We deduce that

\begin{align}
\overline{AW}(j_1, j_2, j_3) = \bigoplus_{k \in J(j_1, j_2, j_3)} K_k \overline{AW}(j_1, j_2, j_3) K_k.
\end{align}
Then, confirming inequality (4.19) amounts to proving the following inequalities, for \( k \in \mathcal{J}(j_1, j_2, j_3) \),
\[
\dim(K_k \mathcal{AW}(j_1, j_2, j_3) K_k) \leq d_k^2.
\]
The algebras \( K_k \mathcal{AW}(j_1, j_2, j_3) K_k \) are simpler to study than \( \mathcal{AW}(j_1, j_2, j_3) \). Roughly speaking, they correspond to replacing in the defining relations of \( \mathcal{AW}(j_1, j_2, j_3) \) the central elements \( \alpha_i \) (resp. \( K \)) by \( \chi_j \) (resp. \( \chi_k \)). One thus gets two annihilating polynomials for \( A \) (similarly for \( B, D \) and \( D' \))
\[
\prod_{j \in \mathcal{J}(j_1, j_2)} (A - \chi_j) = 0, \quad \prod_{m \in \mathcal{M}(j_1, j_2, j_3)} (A - \chi_k + m) = 0,
\]
which reduce to only one, i.e.
\[
\prod_{j \in \mathcal{J}^k(j_1, j_2, j_3)} (A - \chi_j) = 0,
\]
with \( \mathcal{J}^k(j_a, j_b, j_c) = \{ j \in \mathcal{J}(j_a, j_b) \mid \chi_j \in \{ \chi_k - m \mid m \in \mathcal{M}(j_a, j_b, j_c) \} \} \).

In fact, in the quotient of \( \mathcal{C}_{j_1, j_2, j_3} \) where \( C_{123} = \chi_k \), the minimal polynomial of \( C_{12} \) is
\[
\prod_{j \in \mathcal{S}^k(j_1, j_2, j_3)} (C_{12} - \chi_j) = 0,
\]
where we recall that (see proof of Proposition 4.2)
\[
\mathcal{S}^k(j_a, j_b, j_c) = \{ j \in \mathcal{J}(j_a, j_b) \mid k \in \mathcal{J}(j, j_c) \}.
\]
Similar results hold for \( C_{23}, C_{13}^{(0)} \) and \( C_{13}^{(1)} \). Let us emphasize that
\[
\mathcal{S}^k(j_a, j_b, j_c) \subseteq \mathcal{J}^k(j_a, j_b, j_c),
\]
and that the cardinality of \( \mathcal{S}^k(j_a, j_b, j_c) \) is equal to \( d_k \) and does not depend on the ordering of \( j_a, j_b, j_c \). This discussion suggests the definition of another quotient of \( \mathcal{AW}(3) \).

**Definition 4.2.** The algebra \( \overline{\mathcal{AW}}^k(j_1, j_2, j_3) \), where \( k \in \mathcal{J}(j_1, j_2, j_3) \), is the quotient of the centrally extended Askey–Wilson algebra \( \mathcal{AW}(3) \) by \( \alpha_i = \chi_j \), and the following relations
\[
K = \chi_k,
\]
\[
\prod_{j \in \mathcal{S}^k(j_1, j_2, j_3)} (A - \chi_j) = 0, \quad \prod_{j \in \mathcal{S}^k(j_2, j_3, j_1)} (B - \chi_j) = 0,
\]
\[
\prod_{j \in \mathcal{S}^k(j_1, j_3, j_2)} (D - \chi_j) = 0, \quad \prod_{j \in \mathcal{S}^k(j_1, j_3, j_2)} (D' - \chi_j) = 0.
\]

Let us remark that the four annihilating polynomials (4.29)–(4.30) are of degree \( d_k \). These quotients lead to another conjecture.

**Conjecture 4.2.** The direct sum
\[
\overline{\mathcal{AW}}(j_1, j_2, j_3) = \bigoplus_{k \in \mathcal{J}(j_1, j_2, j_3)} \overline{\mathcal{AW}}^k(j_1, j_2, j_3)
\]
is isomorphic to \( \mathcal{C}_{j_1, j_2, j_3} \).
As for Conjecture 4.1, the proof of this conjecture reduces to showing that

\[(4.32) \quad \dim (\overline{AW}^k(j_1, j_2, j_3)) \leq d_k^2.\]

In view of (4.27), we see that Conjecture 4.2 is true if Conjecture 4.1 is. Moreover, in this case, \(\overline{AW}^k(j_1, j_2, j_3)\) is isomorphic to \(K_k\overline{AW}(j_1, j_2, j_3)K_k\).

To conclude this section, let us emphasize that both conjectures presented above would provide an algebraic description of the centralizer \(C_{j_1,j_2,j_3}\). A strategy to prove these conjectures would be to establish inequalities (4.32) and then to derive the isomorphism between \(\overline{AW}^k(j_1, j_2, j_3)\) and \(K_k\overline{AW}(j_1, j_2, j_3)K_k\).

4.4. Invariance under permutations of \(\{j_1, j_2, j_3\}\). The algebras involved in Conjecture 4.1 depend on the choice of three spins \(j_1, j_2\) and \(j_3\). We now show that it is sufficient to check the conjecture for only one ordering for the spins \(j_1, j_2\) and \(j_3\).

**Proposition 4.3.** Let \(j_1, j_2\) and \(j_3\) be three positive half-integers or integers. If Conjecture 4.1 is true for the sequence of spins \(\{j_1, j_2, j_3\}\), then it is also true for every permutation of \(j_1, j_2, j_3\).

*Proof.* For any two representation maps \(\pi_{j_1}\) and \(\pi_{j_2}\) of \(U_q(\mathfrak{sl}_2)\), it is known that there exists an invertible matrix \(P\) such that for all \(x \in U_q(\mathfrak{sl}_2)\) we have \((\pi_{j_2} \otimes \pi_{j_1})(\Delta(x)) = P^{-1}(\pi_{j_1} \otimes \pi_{j_2})(\Delta(x))P\). Therefore, from the definition of the centralizer (3.6) and the coassociativity of the comultiplication (2.4), we deduce that for any permutation \(\sigma\) of the symmetric group \(S_3\), \(C_{j_1,j_2,j_3}\) is isomorphic to \(C_{\sigma(1),\sigma(2),\sigma(3)}\).

We must now show that the quotiened Askey–Wilson algebra \(\overline{AW}(j_{\sigma(1)}, j_{\sigma(2)}, j_{\sigma(3)})\) is isomorphic to \(\overline{AW}(j_1, j_2, j_3)\) for any permutation \(\sigma \in S_3\). Since \(S_3\) is generated by the transpositions (1, 2) and (1, 3), it suffices to prove the isomorphism for these two transformations.

The following maps are algebra isomorphisms

\[(4.33) \quad \phi_1 : \overline{AW}(j_3, j_2, j_1) \to \overline{AW}(j_1, j_2, j_3) \quad \phi_1(\alpha_1) = \alpha_3, \phi_1(\alpha_2) = \alpha_2, \phi_1(\alpha_3) = \alpha_1, \phi_1(A) = B, \phi_1(B) = A, \phi_1(K) = K, \]

\[(4.34) \quad \phi_2 : \overline{AW}(j_2, j_1, j_3) \to \overline{AW}(j_1, j_2, j_3) \quad \phi_2(\alpha_1) = \alpha_2, \phi_2(\alpha_2) = \alpha_1, \phi_2(\alpha_3) = \alpha_3, \phi_2(A) = A, \phi_2(B) = D', \phi_2(K) = K. \]

To see the homomorphism for the defining relations of \(AW(3)\), it is easier to work with the symmetric presentations (2.16)–(2.19) and (2.25)–(2.26). By noticing that \(\phi_1(D) = D'\) and \(\phi_2(D) = B\), the homomorphism immediately follows. In order to preserve relation (1.12), the central elements \(\alpha_i\) have to be permuted in the same way as the spins \(j_i\), which is the case for the two maps given above.

For the quotiened relations (4.13)–(4.16), the homomorphism is checked by observing that \(J(j_a, j_b) = J(j_b, j_a), M(j_a, j_b, j_c) = M(j_b, j_a, j_c)\) and that \(J(j_a, j_b, j_c)\) is invariant under any permutation of its entries. The R.H.S. of relation (1.17) is invariant under any permutation of the elements \(\alpha_i\). By using relations (2.16)–(2.19) and (2.25)–(2.26), it is straightforward to show that \(\phi_1(\Omega) = \Omega\) and \(\phi_2(\Omega) = \Omega\), which proves the homomorphism for relation (1.17). Finally, since the maps \(\phi_1\) and \(\phi_2\) are surjective and invertible, they are bijective. \(\square\)
5. Finite irreducible representations of $\mathbb{AW}(j_1, j_2, j_3)$

To support Conjecture 4.1, we want to show that the sum of the squares of the dimensions of all the finite irreducible representations of $\mathbb{AW}(j_1, j_2, j_3)$ is equal to the dimension of the centralizer. This result implies that Conjecture 4.1 is true if and only if $\mathbb{AW}(j_1, j_2, j_3)$ is semisimple. Moreover, if $\mathbb{AW}(j_1, j_2, j_3)$ is not semisimple, the previous result proves that the missing relations (if there are any) in the kernel of $\phi$ are in a nilpotent radical of $\mathbb{AW}(j_1, j_2, j_3)$.

To identify all the finite irreducible representations of $\mathbb{AW}(j_1, j_2, j_3)$, we use the classification of the representations of the universal Askey–Wilson algebra given in [10] and look for the ones where the different relations of the quotient are satisfied. The universal Askey–Wilson algebra $\Delta_q$, introduced in [21], is generated by three elements $A, B, C$ and has three central elements $\alpha, \beta, \gamma$. There is a surjective algebra homomorphism from $\Delta_q$ to the quotient of $\mathbb{AW}(3)$ by $\alpha_i = \chi_{j_i}$, with the following mappings

\begin{align}
(5.1) & \quad A \mapsto A, \quad B \mapsto B, \quad C \mapsto D, \\
(5.2) & \quad \alpha \mapsto \chi_{j_1} \chi_{j_2} + \chi_{j_3} K, \quad \beta \mapsto \chi_{j_2} \chi_{j_3} + \chi_{j_1} K, \quad \gamma \mapsto \chi_{j_1} \chi_{j_3} + \chi_{j_2} K.
\end{align}

We deduce that the quotient of $\mathbb{AW}(3)$ by $\alpha_i = \chi_{j_i}$ is isomorphic to the quotient of $\Delta_q$ by the relations $(\alpha - \chi_{j_1} \chi_{j_2})/\chi_{j_3} = (\beta - \chi_{j_2} \chi_{j_3})/\chi_{j_1} = (\gamma - \chi_{j_1} \chi_{j_3})/\chi_{j_2}$. We can therefore use the representation theory of $\Delta_q$ in order to determine all the finite irreducible representations of $\mathbb{AW}(j_1, j_2, j_3)$.

The finite irreducible modules of the universal Askey–Wilson algebra $\Delta_q$ for $q$ not a root of unity are classified in [10]. They are given by the isomorphism classes of the $n + 1$-dimensional modules $V_n(a, b, c)$ defined in [10], for $n \geq 0$, under certain conditions on $a, b, c$ (see Theorem 4.7 in [10]). In the representation $V_n(a, b, c)$, the central elements $\alpha, \beta$ and $\gamma$ of $\Delta_q$ take the following values

\begin{align}
(5.3) & \quad \alpha = (q^{n+1} + q^{-n-1})(a + a^{-1}) + (b + b^{-1})(c + c^{-1}), \\
(5.4) & \quad \beta = (q^{n+1} + q^{-n-1})(b + b^{-1}) + (c + c^{-1})(a + a^{-1}), \\
(5.5) & \quad \gamma = (q^{n+1} + q^{-n-1})(c + c^{-1}) + (a + a^{-1})(b + b^{-1}).
\end{align}

The characteristic polynomials of $A, B, C \in \Delta_q$ in this representation are (see Lemma 4.3 of [10]) $K_a(X), K_b(X), K_c(X)$, with

\begin{equation}
(5.6) \quad K_x(X) = \prod_{i=0}^{n}(X - (q^{2i-n}x + q^{n-2i}x^{-1})).
\end{equation}

The Casimir element of the algebra $\Delta_q$ is given by

\begin{equation}
(5.7) \quad qA\alpha + q^{-1}B\beta + qC\gamma - q^2A^2 - q^{-2}B^2 - q^2C^2 - qABC.
\end{equation}

It is straightforward to compute the value $\omega$ of this element in the representation $V_n(a, b, c)$ by using the representation matrices given in [10]. One gets

\begin{equation}
(5.8) \quad \omega = (q^{n+1} + q^{-n-1})^2 + (a + a^{-1})^2 + (b + b^{-1})^2 + (c + c^{-1})^2
\end{equation}

$$ + (q^{n+1} + q^{-n-1})(a + a^{-1})(b + b^{-1})(c + c^{-1}) - (q + q^{-1})^2.$$
representations $V_n(a, b, c)$ to pass to the quotient:

\begin{align}
(5.9) \quad 0 \leq n & \leq \min\{j_1 + j_2 - |j_1 - j_2|, j_2 + j_3 - |j_2 - j_3|, j_1 + j_3 - |j_1 - j_3|\} , \\
(5.10) \quad a = q^{2x+n+1} , \quad b = q^{2y+n+1} , \quad c = q^{2z+n+1} , \quad \text{for } x, y, z \text{ integers or half-integers}, \\
(5.11) \quad |j_1 - j_2| & \leq x \leq j_1 + j_2 - n , \quad |j_2 - j_3| \leq y \leq j_2 + j_3 - n , \quad |j_1 - j_3| \leq z \leq j_1 + j_3 - n .
\end{align}

We recall that $K$ is a central element of the annihilating polynomial given in (4.13). Therefore, $K$ has to be a constant equal to $\chi_\ell$ for some $\ell \in J(j_1, j_2, j_3)$ in any irreducible representation of $\text{AW}(j_1, j_2, j_3)$. We also recall that the Casimir element $\Omega$ of $\text{AW}(3)$ satisfies relation (4.17) in the quotient $\text{AW}(j_1, j_2, j_3)$. From this discussion and from the results (5.2)–(5.5), (5.7), (5.8) and (5.10), we deduce that the following equations must hold so that the representation $V_n(a, b, c)$ passes to the quotient $\text{AW}(j_1, j_2, j_3)$:

\begin{align}
(5.12) \quad \chi_n^a \chi_{x+n} + \chi_n^b \chi_{x+n} + \chi_n^c \chi_{x+n} & = \chi_{j_1} \chi_{j_2} + \chi_{j_3} \chi_\ell , \\
(5.13) \quad \chi_n^a \chi_{y+n} + \chi_n^b \chi_{y+n} + \chi_n^c \chi_{y+n} & = \chi_{j_2} \chi_{j_3} + \chi_{j_1} \chi_\ell , \\
(5.14) \quad \chi_n^a \chi_{z+n} + \chi_n^b \chi_{z+n} + \chi_n^c \chi_{z+n} & = \chi_{j_3} \chi_{j_1} + \chi_{j_2} \chi_\ell , \\
(5.15) \quad \chi_{n}^a \chi_{x+n} + \chi_{n}^b \chi_{x+n} + \chi_{n}^c \chi_{x+n} & = \chi_{j_1} \chi_{j_2} + \chi_{j_3} \chi_\ell + \chi_{j_1} \chi_{j_2} \chi_{j_3} \chi_\ell .
\end{align}

In the case of three identical spins $j_1 = j_2 = j_3 = s$, we find by using mathematical software that there are only 192 possible solutions for $x, y, z, n$ to the system of equations (5.12)–(5.15). The only solutions respecting conditions (5.9)–(5.11) and corresponding to inequivalent representations $V_n(a, b, c)$ are

\begin{align}
(5.16) \quad n & = 2\ell , \\
(5.17) \quad n & = 3s - \ell , \\
(5.18) \quad n & = s + \ell , \\
(5.19) \quad n & = s - \ell - 1 ,
\end{align}

and any permutation of $x, y, z$ in the previous equations is also a solution.

Since $K = \chi_\ell$ for $\ell \in J(j_1, j_2, j_3)$ in some irreducible representation, the annihilating polynomial of $K - A$ given in (4.15) implies that the annihilating polynomial of $A$ reduces to the relation (4.24) in this representation. If the set $J^\ell(j_1, j_2, j_3)$ is equal to the set $S^\ell(j_1, j_2, j_3)$, then this reduced annihilating polynomial for $A$ leads to the constraint

\begin{equation}
(5.20) \quad \max(|j_1 - j_2|, |j_3 - \ell|) \leq x \leq \min(j_1 + j_2, j_3 + \ell) - n .
\end{equation}

Similar results hold for the annihilating polynomials of $B$ (resp. $D$) and the constraints on $y$ (resp. $z$). In the case of identical spins $j_1 = j_2 = j_3 = s$, this implies that $s - \ell \leq x, y, z \leq s + \ell - n$ for $\ell \leq s$, and $\ell - s \leq x, y, z \leq 2s - n$ for $\ell > s$. The only solutions remaining are (5.16) and (5.17). For $s$ half-integer, we do not find any cases where $J^\ell(s, s, s) \neq S^\ell(s, s, s)$. For $s$ integer, as a consequence of the fact that $0 \in M(s, s, s)$, we find $J^\ell(s, s, s) \setminus S^\ell(s, s, s) = \{\ell\}$ if $\ell < s/2$, and otherwise the previous set is empty. We have verified numerically the sets $J^\ell(s, s, s) \setminus S^\ell(s, s, s)$ given above for at least $s = \frac{1}{2}, 1, \ldots, 10$. In any case, the upper bound on the values of $x, y, z$ remains the same, and we still conclude that the only solutions are (5.16) and (5.17). Therefore, the sum of
the squares of the dimensions \( n + 1 \) of all the irreducible modules of \( \overline{AW}(s, s, s) \) is

\[
(5.21) \quad \sum_{\ell \in J(s, s, s), \ell \leq s} (2\ell + 1)^2 + \sum_{\ell \in J(s, s, s), \ell > s} (3s - \ell + 1)^2 = \frac{1}{2}((2s + 1)(2s + 1)^2 + 1),
\]

which is equal to the dimension of the centralizer \( \dim(C_{s, s, s}) = \sum_{\ell \in J(s, s, s)} d_\ell^2 \).

Let us remark that for \( j_1 = j_2 = j_3 = \frac{1}{2}, 1, \ldots, \frac{17}{2} \), we used mathematical software to test all the possible integer values for \( x, y, z, n \) such that the restrictions (5.9) and (5.11) and the three equations (5.12)–(5.14) are respected. The only solutions we found are those given in (5.16)–(5.19). Hence, equation (5.15) is perhaps not necessary if one wants to find all solutions for \( x, y, z, n \) integers.

Let us also notice that in the general case where \( j_1, j_2, j_3 \) are any three fixed integers or half-integers, we find at least 192 solutions to the system of equations (5.12)–(5.14) with \( n \) integer and \( x, y, z \) integers or half-integers. If these are the only such solutions, then it is possible to argue (in a similar manner as for the case of identical spins) that the sum of the squares of the dimensions of the irreducible representations that pass to the quotient \( \overline{AW}(j_1, j_2, j_3) \) is also equal to the dimension of the centralizer \( C_{j_1,j_2,j_3} \).

Finally, we notice that in the representations \( V_n(a, b, c) \), the element \( D' \) of \( AW(3) \) has the same characteristic polynomial \( K_c(X) \) as the element \( D \). Therefore, the second relations in (4.14) and (4.16) do not provide any additional constraint on the values of \( n, a, b, c \).

6. Quotient \( \overline{AW}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) and Temperley–Lieb algebra

In this section, we consider the case \( j_1 = j_2 = j_3 = \frac{1}{2} \) and show that the quotient \( \overline{AW}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) is isomorphic to the centralizer \( C_{\frac{1}{2},\frac{1}{2},\frac{1}{2}} \), which is known to be the Temperley–Lieb algebra. We give an explicit isomorphism between \( \overline{AW}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) and the Temperley–Lieb algebra.

6.1. \( \overline{AW}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) algebra. From the definitions (3.2)–(3.3) and (3.14), we find the sets

\[
(6.1) \quad J \left( \frac{1}{2}, \frac{1}{2} \right) = \{0, 1\}, \quad J \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) = \left\{ \frac{1}{2}, \frac{3}{2} \right\},
\]

\[
(6.2) \quad M \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) = \{\chi_{\frac{1}{2}} - \chi_1, \chi_{\frac{1}{2}} - \chi_0, \chi_{\frac{1}{2}} - \chi_1, \chi_{\frac{1}{2}} - \chi_0 \}.
\]

The degeneracies are \( d_{\frac{1}{2}} = 2 \) and \( d_{\frac{3}{2}} = 1 \). We find from (3.8) that \( \dim\left( C_{\frac{1}{2},\frac{1}{2},\frac{1}{2}} \right) = 5 \). The central elements \( \alpha_i \) of \( \overline{AW}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \) can all be replaced by the constant \( \chi_{\frac{1}{2}} \). For computational convenience, we perform the transformation \( X = (q - q^{-1})^2 \tilde{X} + q + q^{-1} \) on the elements \( X = A, B, D, D', K \), as in Remark 2.1 and we define the shifted central element \( \tilde{G} = \tilde{K} + [1/2]^2q \). By using the sets given
in (6.1) and (6.2), one finds that the defining relations of $AW(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ are

\begin{align}
(6.3) & \quad [\tilde{B}, [\tilde{A}, \tilde{B}]]_q = [2]_q \left( -\tilde{B}^2 - \{\tilde{A}, \tilde{B}\} \right) + (q^2 + q^{-2})\tilde{G}\tilde{B} + [2]_q^2 \tilde{B} , \\
(6.4) & \quad [[\tilde{A}, \tilde{B}], \tilde{A}]_q = [2]_q \left( -\tilde{A}^2 - \{\tilde{A}, \tilde{B}\} \right) + (q^2 + q^{-2})\tilde{G}\tilde{A} + [2]_q^2 \tilde{A} , \\
(6.5) & \quad \tilde{A}(\tilde{A} - [2]_q) = 0 , \quad \tilde{B}(\tilde{B} - [2]_q) = 0 , \quad (\tilde{G} - 1)(\tilde{G} - [2]_q^2) = 0 , \\
(6.6) & \quad \tilde{D}(\tilde{D} - [2]_q) = 0 , \quad \tilde{D}'(\tilde{D}' - [2]_q) = 0 , \\
(6.7) & \quad (\tilde{G} - \tilde{A} + [2]_q - [2]_q^2)(\tilde{G} - \tilde{A} + [2]_q - 1)(\tilde{G} - \tilde{A} - 1) = 0 , \\
(6.8) & \quad (\tilde{G} - \tilde{B} - [2]_q + [2]_q^2)(\tilde{G} - \tilde{B} + [2]_q - 1)(\tilde{G} - \tilde{B} - 1) = 0 , \\
(6.9) & \quad (\tilde{G} - \tilde{D} - [2]_q + [2]_q^2)(\tilde{G} - \tilde{D} + [2]_q - 1)(\tilde{G} - \tilde{D} - 1) = 0 , \\
(6.10) & \quad (\tilde{G} - \tilde{D}' + [2]_q - [2]_q^2)(\tilde{G} - \tilde{D}' + [2]_q - 1)(\tilde{G} - \tilde{D}' - 1) = 0 , \\
(6.11) & \quad (q^2 + q^{-2})(q - q^{-1})(q\tilde{A} + q^{-1}\tilde{B} + q\tilde{D})\tilde{G} - [2]_q((2]_q - q^3)(\tilde{A} + \tilde{D}) + q^{-3}\tilde{B}) \\
& \quad - (q - q^{-1})(q^2\tilde{A}^2 + q^{-2}\tilde{B}^2 + q^{2}\tilde{D}^2) - q[2]_q(q - q^{-1})(\tilde{A}\tilde{B} + \tilde{A}\tilde{D} + \tilde{B}\tilde{D}) - q(q - q^{-1})^3\tilde{A}\tilde{B}\tilde{D} \\
& \quad = (q - q^{-1})\tilde{G}^2 + (q^5 - q^{-5} - q^2[2]_q)\tilde{G} - q^{-1}[2]_q^2 ,
\end{align}

where

\begin{align}
(6.12) & \quad \tilde{D} = [2]_q + (q^2 + q^{-2})\tilde{G} - \tilde{A} - \tilde{B} - \frac{q - q^{-1}}{q + q^{-1}}[\tilde{A}, \tilde{B}]_q , \\
(6.13) & \quad \tilde{D}' = [2]_q + (q^2 + q^{-2})\tilde{G} - \tilde{A} - \tilde{B} - \frac{q - q^{-1}}{q + q^{-1}}[\tilde{B}, \tilde{A}]_q .
\end{align}

We want to show that $AW(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is isomorphic to the centralizer $C_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}$.

**Proposition 6.1.** The relations defining the quotient $\overline{AW}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ can be given as follows

\begin{align}
(6.14) & \quad \tilde{A}^2 = [2]_q\tilde{A} , \quad \tilde{B}^2 = [2]_q\tilde{B} , \\
(6.15) & \quad \tilde{A}\tilde{B}\tilde{A} = [2]_q\{\tilde{A}, \tilde{B}\} - [3]_q\tilde{A} - [2]_q^2\tilde{B} + [2]_q[3]_q , \\
(6.16) & \quad \tilde{B}\tilde{A}\tilde{B} = [2]_q\{\tilde{A}, \tilde{B}\} - [3]_q\tilde{B} - [2]_q^2\tilde{A} + [2]_q[3]_q .
\end{align}

**Proof.** The two first relations in (6.5) directly lead to (6.14). The third relation in (6.5) implies

\begin{align}
(6.17) & \quad \tilde{G}^2 = ([2]_q^2 + 1)\tilde{G} - [2]_q^2 .
\end{align}

Developing (6.3) and (6.4) and using (6.14), one gets

\begin{align}
(6.18) & \quad \tilde{B}\tilde{A}\tilde{B} = \tilde{G}\tilde{B} , \quad \tilde{A}\tilde{B}\tilde{A} = \tilde{G}\tilde{A} .
\end{align}

Expanding (6.7) and (6.8) and simplifying with the help of (6.14) and (6.17), one gets

\begin{align}
(6.19) & \quad \tilde{G}\tilde{A} = [2]_q\tilde{G} + \tilde{A} - [2]_q , \quad \tilde{G}\tilde{B} = [2]_q\tilde{G} + \tilde{B} - [2]_q ,
\end{align}

which implies

\begin{align}
(6.20) & \quad \tilde{G}\tilde{A}\tilde{B} = [2]_q^2\tilde{G} + \tilde{A}\tilde{B} - [2]_q^2 , \quad \tilde{G}\tilde{B}\tilde{A} = [2]_q^2\tilde{G} + \tilde{B}\tilde{A} - [2]_q^2 .
\end{align}
Equations (6.6) and (6.9)–(6.11) can be simplified using the previous relations, and they lead to
\[(6.21) \quad \tilde{G} = -[2]_q(\tilde{A} + \tilde{B}) + \{\tilde{A}, \tilde{B}\} + [2]_q^2.\]

Substituting (6.21) in (6.19) and (6.20), one finds
\[(6.22) \quad \tilde{G}\tilde{A} = [2]_q\{\tilde{A}, \tilde{B}\} - [3]_q\tilde{A} - [2]_q^2\tilde{B} + [2]_q[3]_q,\]
\[(6.23) \quad \tilde{G}\tilde{B} = [2]_q\{\tilde{A}, \tilde{B}\} - [3]_q\tilde{B} - [2]_q^2\tilde{A} + [2]_q[3]_q,\]
\[(6.24) \quad \tilde{G}\tilde{A}\tilde{B} = [2]_q^2\tilde{B}\tilde{A} + ([2]_q^2 + 1)\tilde{A}\tilde{B} - [2]_q^3(\tilde{A} + \tilde{B}) + [2]_q^2[3]_q,\]
\[(6.25) \quad \tilde{G}\tilde{B}\tilde{A} = [2]_q^2\tilde{B}\tilde{A} + ([2]_q^2 + 1)\tilde{B}\tilde{A} - [2]_q^3(\tilde{A} + \tilde{B}) + [2]_q^2[3]_q.\]

Equations (6.18) and (6.22)–(6.23) imply the relations (6.15)–(6.16) of the proposition.

It remains to show that the generator \(\tilde{G}\) can be suppressed from the presentation, or in other words that (6.17) and (6.22)–(6.25) are implied from the relations of the proposition. Suppose that relations (6.14)–(6.16) are true and let \(\tilde{G} = -[2]_q(\tilde{A} + \tilde{B}) + \{\tilde{A}, \tilde{B}\} + [2]_q^2\). Multiplying the expression of \(\tilde{G}\) on the left and on the right by \(\tilde{A}\) and \(\tilde{B}\), one finds
\[(6.26) \quad \tilde{G}\tilde{A} = \tilde{A}\tilde{G} = \tilde{A}\tilde{B}\tilde{A}, \quad \tilde{G}\tilde{B} = \tilde{B}\tilde{G} = \tilde{B}\tilde{A}\tilde{B}.\]

Using (6.15) and (6.16), equations (6.22) and (6.23) are recovered. Multiplying (6.15) on the right by \(\tilde{B}\) and (6.16) on the right by \(\tilde{A}\), one finds
\[(6.27) \quad \tilde{G}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}\tilde{A}\tilde{B} = \tilde{A}\tilde{B} - [2]_q\tilde{B} + [2]_q\tilde{A}\tilde{B},\]
\[(6.28) \quad \tilde{G}\tilde{B}\tilde{A} = \tilde{B}\tilde{A}\tilde{B}\tilde{A} = \tilde{B}\tilde{A} - [2]_q\tilde{A} + [2]_q\tilde{A}\tilde{B},\]
from which one easily recovers (6.24) and (6.25). Finally, it is straightforward to arrive at
\[(6.29) \quad \tilde{G}^2 = -[2]_q^3(\tilde{A} + \tilde{B}) + [2]_q^2\{\tilde{A}, \tilde{B}\} - [2]_q(\tilde{A}\tilde{B}\tilde{A} + \tilde{B}\tilde{A}\tilde{B}) + \tilde{A}\tilde{B}\tilde{A}\tilde{B} + \tilde{B}\tilde{A}\tilde{B}\tilde{A} + [2]_q^4\]
and to use the results (6.27) and (6.28) to recover (6.17). \(\square\)

**Theorem 6.1.** Conjecture 4.1 is verified for \(j_1 = j_2 = j_3 = \frac{1}{2}\).

**Proof.** We already know from proposition 4.2 that the map \(\overline{\phi}\) is surjective. From the previous proposition, it is easy to show that \(\{1, \tilde{A}, \tilde{B}, \tilde{A}\tilde{B}, \tilde{B}\tilde{A}\tilde{B}\}\) is a linearly generating set of \(\overline{\text{AW}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\). Since \(\dim(\mathcal{C}_{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}}) = 5\), this shows the injectivity of the map \(\overline{\phi}\). \(\square\)

### 6.2. Connection with the Temperley–Lieb algebra

It is known that the Temperley–Lieb algebra is isomorphic to the centralizer of the diagonal embedding of \(U_q(\mathfrak{sl}_2)\) in the tensor product of three fundamental representations \(\text{AW}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\). Hence, from the results of the previous subsection, the quotiented Askey–Wilson algebra \(\text{AW}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})\) is isomorphic to the Temperley–Lieb algebra.

**Definition 6.1.** \(\text{[20]}\) The Temperley–Lieb algebra \(TL_3(q)\) is generated by \(\sigma_1\) and \(\sigma_2\) with the following defining relations
\[(6.30) \quad \sigma_1^2 = (q + q^{-1})\sigma_1, \quad \sigma_2^2 = (q + q^{-1})\sigma_2,\]
\[(6.31) \quad \sigma_1\sigma_2 = \sigma_2\sigma_1 = \sigma_1, \quad \sigma_2\sigma_1\sigma_2 = \sigma_2.\]
Theorem 6.2. The quotiented Askey–Wilson algebra $\overline{\mathbb{AW}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ is isomorphic to the Temperley–Lieb algebra $TL_3(q)$. This isomorphism is given explicitly by

$$\overline{\mathbb{AW}} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \rightarrow TL_3(q)$$

(6.32) \[\tilde{A} \mapsto (q + q^{-1}) - \sigma_1,\]

(6.33) \[\tilde{B} \mapsto (q + q^{-1}) - \sigma_2.\]

Proof. It is straightforward to show that the defining relations (6.30) and (6.31) of $TL_3(q)$ are equivalent to the relations (6.14)–(6.16) of $\overline{\mathbb{AW}}(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. \qed

7. Quotient $\overline{\mathbb{AW}}(1, 1, 1)$ and Birman–Murakami–Wenzl Algebra

In this section, we choose $j_1 = j_2 = j_3 = 1$ and prove that the quotient $\overline{\mathbb{AW}}(1, 1, 1)$ is isomorphic to the centralizer $C_{1,1,1}$. In this case, $C_{1,1,1}$ is known to be connected to the Birman–Murakami–Wenzl algebra. We give an explicit isomorphism between $\overline{\mathbb{AW}}(1, 1, 1)$ and a specialization of the BMW algebra.

7.1. $\overline{\mathbb{AW}}(1, 1, 1)$ Algebra. We have the following sets

(7.1) \[\mathcal{J}(1, 1) = \{0, 1, 2\}, \quad \mathcal{J}(1, 1, 1) = \{0, 1, 2, 3\},\]

(7.2) \[\mathcal{M}(1, 1, 1) = \{\chi_1 - \chi_2, \chi_0 - \chi_1, 0, \chi_1 - \chi_0, \chi_2 - \chi_1, \chi_3 - \chi_2\}.\]

The degeneracies are $d_0 = d_3 = 1$, $d_1 = 3$ and $d_2 = 2$, and the dimension of the centralizer is $\dim(C_{1,1,1}) = 15$. The central elements $\alpha_i$ of $\overline{\mathbb{AW}}(1, 1, 1)$ can all be replaced by the constant $\chi_1$. For computational convenience again, we perform the transformation $X = (q - q^{-1})^2 \tilde{X} + q + q^{-1}$ on the elements $X = \tilde{A}, \tilde{B}, D, D', K$ (see Remark 2.1). We recall that the eigenvalues $\chi_j$ are transformed to $\tilde{\chi}_j = [j]_q[j + 1]_q$, and we define the constants $m_1 = \tilde{\chi}_1 - \tilde{\chi}_0$, $m_2 = \tilde{\chi}_2 - \tilde{\chi}_1$ and $m_3 = \tilde{\chi}_3 - \tilde{\chi}_2$. The defining relations (4.13)–(4.16) of $\overline{\mathbb{AW}}(1, 1, 1)$ are written as

(7.3) \[(q^2 + q^{-2})\tilde{B}\tilde{A}\tilde{B} = \tilde{A}\tilde{B}^2 + \tilde{B}^2\tilde{A} - [2]_q \tilde{B}^2 - [2]_q \tilde{A} + [2]_q[2]_q - 3)\tilde{K}\tilde{B} + [2]_q[3]_q\tilde{B},\]

(7.4) \[(q^2 + q^{-2})\tilde{A}\tilde{B}\tilde{A} = \tilde{B}\tilde{A}^2 + \tilde{A}^2\tilde{B} - [2]_q \tilde{A}^2 - [2]_q \tilde{B} + [2]_q[2]_q^2 - 3)\tilde{K}\tilde{A} + [2]_q[3]_q\tilde{A},\]

(7.5) \[\tilde{A}(\tilde{A} - \tilde{\chi}_1)(\tilde{A} - \tilde{\chi}_2) = 0, \quad \tilde{B}(\tilde{B} - \tilde{\chi}_1)(\tilde{B} - \tilde{\chi}_2) = 0,\]

(7.6) \[\tilde{K}(\tilde{K} - \tilde{\chi}_1)(\tilde{K} - \tilde{\chi}_2)(\tilde{K} - \tilde{\chi}_3) = 0,\]

(7.7) \[\tilde{D}(\tilde{D} - \tilde{\chi}_1)(\tilde{D} - \tilde{\chi}_2) = 0, \quad \tilde{D}'(\tilde{D}' - \tilde{\chi}_1)(\tilde{D}' - \tilde{\chi}_2) = 0,\]

(7.8) \[(\tilde{K} - \tilde{A} + m_2)(\tilde{K} - \tilde{A} + m_1)(\tilde{K} - \tilde{A})(\tilde{K} - \tilde{A} - m_1)(\tilde{K} - \tilde{A} - m_2)(\tilde{K} - \tilde{A} - m_3) = 0,\]

(7.9) \[(\tilde{K} - \tilde{B} + m_2)(\tilde{K} - \tilde{B} + m_1)(\tilde{K} - \tilde{B})(\tilde{K} - \tilde{B} - m_1)(\tilde{K} - \tilde{B} - m_2)(\tilde{K} - \tilde{B} - m_3) = 0,\]

(7.10) \[(\tilde{K} - \tilde{D} + m_2)(\tilde{K} - \tilde{D} + m_1)(\tilde{K} - \tilde{D})(\tilde{K} - \tilde{D} - m_1)(\tilde{K} - \tilde{D} - m_2)(\tilde{K} - \tilde{D} - m_3) = 0,\]

(7.11) \[(\tilde{K} - \tilde{D}' + m_2)(\tilde{K} - \tilde{D}' + m_1)(\tilde{K} - \tilde{D}')(\tilde{K} - \tilde{D}' - m_1)(\tilde{K} - \tilde{D}' - m_2)(\tilde{K} - \tilde{D}' - m_3) = 0,\]
where

\begin{align}
\tag{7.12}
\tilde{D} &= [2]_q[3]_q + ([2]_q^2 - 3)\tilde{K} - \frac{(q - q^{-1})}{[2]_q} [\tilde{A}, \tilde{B}]_q - \tilde{A} - \tilde{B}, \\
\tag{7.13}
\tilde{D}' &= [2]_q[3]_q + ([2]_q^2 - 3)\tilde{K} - \frac{(q - q^{-1})}{[2]_q} [\tilde{B}, \tilde{A}]_q - \tilde{A} - \tilde{B}.
\end{align}

**Theorem 7.1.** Conjecture 4.1 is verified for \( j_1 = j_2 = j_3 = 1 \).

*Proof.* We already know from proposition [4.2] that the map \( \bar{\phi} \) is surjective. We only need to prove that it is injective in this case.

We define the set

\begin{equation}
\tag{7.14}
S = \{1, \tilde{A}, \tilde{B}, \tilde{A}^2, \tilde{A}\tilde{B}, \tilde{B}\tilde{A}, \tilde{A}^2\tilde{B}, \tilde{A}\tilde{B}\tilde{A}, \tilde{A}\tilde{B}^2, \tilde{B}\tilde{A}\tilde{B}, \tilde{A}^2\tilde{B}^2, \tilde{A}\tilde{B}\tilde{A}\tilde{B}, \tilde{B}\tilde{A}\tilde{B}\tilde{A}\tilde{B}\tilde{A}\}.
\end{equation}

Using relations (7.3)–(7.6), it can be shown that \( S_r = S \cup \tilde{K}S \cup \tilde{K}^2S \cup \tilde{K}^3S \) is a linearly generating set for \( \mathcal{A}\mathcal{W}(1, 1, 1) \). We can construct the 60 by 60 matrices \( \bar{A}_r, \bar{B}_r \) and \( \bar{K}_r \) corresponding to the regular actions of \( \bar{A}, \bar{B} \) and \( \bar{K} \) on the set \( S_r \). Knowing that \( \bar{A}_r, \bar{B}_r \) and \( \bar{K}_r \) have to satisfy (7.3)–(7.6) and the first of (7.7), we find 32 independant relations between the elements of \( S_r \) and we can reduce the generating set to

\begin{equation}
\tag{7.15}
S'_r = S \cup \tilde{K}\{1, \tilde{B}, \tilde{A}^2, \tilde{A}\tilde{B}, \tilde{B}\tilde{A}, \tilde{A}^2\tilde{B}, \tilde{A}\tilde{B}\tilde{A}, \tilde{A}\tilde{B}^2, \tilde{B}\tilde{A}\tilde{B}, \tilde{A}^2\tilde{B}^2, \tilde{A}\tilde{B}\tilde{A}\tilde{B}\} \cup \tilde{K}^2\{\tilde{A}^2, \tilde{A}^2\tilde{B}\}.
\end{equation}

We repeat the procedure and construct 28 by 28 matrices corresponding to the regular actions on \( S'_r \). Only using again (7.3)–(7.6) and the first of (7.7), we can reduce the generating set to

\begin{equation}
\tag{7.16}
S''_r = S \cup \{\tilde{K}, \tilde{K}\tilde{B}, \tilde{K}\tilde{A}^2, \tilde{K}\tilde{A}\tilde{B}, \tilde{K}\tilde{B}^2, \tilde{K}\tilde{A}\tilde{B}\tilde{A}, \tilde{K}\tilde{A}\tilde{B}\tilde{B}, \tilde{K}\tilde{B}\tilde{A}\tilde{B}\}.
\end{equation}

We repeat and construct 24 by 24 matrices. At this point, relations (7.3)–(7.7) are already satisfied. We must use relations (7.8)–(7.11) to reduce the generating set to

\begin{equation}
\tag{7.17}
S'''_r = S \cup \{\tilde{K}, \tilde{K}\tilde{A}^2, \tilde{K}\tilde{B}^2\}.
\end{equation}

We repeat one last time by constructing 18 by 18 matrices and we use (7.3)–(7.11) to find 3 independant relations which allow to reduce the generating set to \( S \). It can also be verified that the matrices of the regular action satisfy the defining relation (4.17) involving the Casimir element \( \Omega \). We made the previous computations by using a formal mathematical software.

From these results, we have that \( S \) is a linearly generating set for \( \mathcal{A}\mathcal{W}(1, 1, 1) \) with 15 elements. Since \( \text{dim}(C_{1,1,1}) = 15 \), we conclude that \( \bar{\phi} \) is injective. \qed

### 7.2. Connection with the Birman–Murakami–Wenzl algebra.

It is known [16] that the Birman–Murakami–Wenzl algebra is isomorphic to the centralizer of the diagonal embedding of \( U_q(\mathfrak{sl}_2) \) in the tensor product of three spin-1 representations. Hence, from the previous theorem, the quotiented Askey–Wilson algebra \( \mathcal{A}\mathcal{W}(1, 1, 1) \) is isomorphic to the BMW algebra.
Proof. The degeneracies are isomorphic to each other. It can be verified that the image of $\tilde{A}$ is isomorphic to $C_{\frac{2}{3}, \frac{2}{3}, \frac{2}{3}}$. Hence they are isomorphic to each other. It can be verified that the image of $\tilde{A}$ (resp. $\tilde{B}$) is isomorphic to the Birman–Murakami–Wenzl algebra BMW. This isomorphism is given explicitly by

$$\tilde{A} \mapsto (q + q^{-1})(s_1 - q^{-2}e_1) + (q + q^{-1})^2 q^{-1},$$

(7.22)

$$\tilde{B} \mapsto (q + q^{-1})(s_2 - q^{-2}e_2) + (q + q^{-1})^2 q^{-1}.$$  

(7.23)

Proof. The algebras $\AW(1, 1, 1)$ and $BMW_3(q^2, q^4)$ are both isomorphic to $C_{1,1,1}$, hence they are isomorphic to each other. It can be verified that the image of $\tilde{A}$ (resp. $\tilde{B}$) is equal to the image of the R.H.S. of (7.22) (resp. (7.23)), which justifies the explicit mapping. The inverse map is given by

$$s_1 \mapsto q^{-2}(q + q^{-1})^{-2} \tilde{A}^2 - q^{-2}(q + q^{-1})^{-1}(2 + q^{-2}) \tilde{A} + q^{-4},$$

(7.24)

$$s_2 \mapsto q^{-2}(q + q^{-1})^{-2} \tilde{B}^2 - q^{-2}(q + q^{-1})^{-1}(2 + q^{-2}) \tilde{B} + q^{-4}.$$  

(7.25)

8. Quotient $\AW(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$

In this section, we take $j_1 = j_2 = j_3 = \frac{2}{3}$ and show that the $\AW(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ algebra is isomorphic to the centralizer $C_{\frac{2}{3}, \frac{3}{2}, \frac{3}{2}}$.

From the decomposition rules of the tensor product, we find the sets

$$\mathcal{J} \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) = \{0, 1, 2, 3\}, \quad \mathcal{J} \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) = \left\{ \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2} \right\},$$

(8.1)

$$\mathcal{M} \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) = \{\chi_3/2, \chi_3/2 - \chi_3, \chi_3/2 - \chi_2, \chi_3/2 - \chi_3, \chi_3/2 - \chi_2, \chi_3/2 - \chi_1, \chi_3/2 - \chi_3, \chi_3/2 - \chi_2, \chi_3/2 - \chi_2, \chi_3/2 - \chi_1\}.$$  

(8.2)

The degeneracies are $d_{\frac{2}{3}} = 1$, $d_{\frac{3}{2}} = d_{\frac{5}{2}} = 2$, $d_{\frac{5}{2}} = 3$ and $d_{\frac{3}{2}} = 4$, and the dimension of the centralizer is $\dim \left( \mathbb{C}_{\frac{2}{3}, \frac{3}{2}, \frac{3}{2}} \right) = 34$. The central elements $\alpha_i$ of $\AW(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})$ are equal to the constant $\chi_{\frac{3}{2}}$.

In order to prove the injectivity of the map $\tilde{\delta}$ in this case, we will use the strategy described in Subsection 4.3 and show that

$$\dim \left( \mathcal{K}_k \AW \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \mathcal{K}_k \right) \leq d_k^2, \quad \forall k \in \mathcal{J} \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right).$$  

(8.3)
We recall that for each \( k \in J \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \), the central element \( K \) is replaced by the constant \( \chi_k \) in the algebra \( \mathcal{K}_k \overline{\text{AW}} \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \mathcal{K}_k \), and the annihilating polynomials for \( A \) (similarly for \( B, D \) and \( D' \)) reduce to

\[
(8.4) \prod_{j \in J^k \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right)} (A - \chi_j) = 0 ,
\]

where \( J^k \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) = \{ j \in J \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \mid \chi_j \in \{ \chi_k - m \mid m \in \mathcal{M} \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \} \} \). Once again, we perform the transformation \( X = (q - q^{-1})^2 \tilde{X} + q + q^{-1} \) on the elements \( X = A, B, D, D', K \) (see Remark 2.1). Therefore, one finds that the following relations hold in the algebras \( \mathcal{K}_k \overline{\text{AW}} \left( \frac{3}{2}, \frac{3}{2}, \frac{3}{2} \right) \mathcal{K}_k : \)

- \( k = \frac{9}{2} \)

\[
(8.5) \tilde{A} = \tilde{B} = \chi_3 .
\]

- \( k = \frac{7}{2} \)

\[
(8.6) \tilde{B} \tilde{A} \tilde{B} = [2]_q^3 \{ \tilde{A}, \tilde{B} \} + [3]_q^2 (-2[2]_q^2 \tilde{A} + (q^4 + q^{-4} - 1) \tilde{B} + [2]_q^3) ,
\]

\[
(8.7) \tilde{A} \tilde{B} \tilde{A} = [2]_q^3 \{ \tilde{A}, \tilde{B} \} + [3]_q^2 (-2[2]_q^2 \tilde{B} + (q^4 + q^{-4} - 1) \tilde{A} + [2]_q^3) ,
\]

\[
(8.8) (\tilde{A} - \tilde{\chi}_2)(\tilde{A} - \tilde{\chi}_3) = 0 , \quad (\tilde{B} - \tilde{\chi}_2)(\tilde{B} - \tilde{\chi}_3) = 0 ,
\]

\[
(8.9) (\tilde{D} - \tilde{\chi}_2)(\tilde{D} - \tilde{\chi}_3) = 0 , \quad (\tilde{D}' - \tilde{\chi}_2)(\tilde{D}' - \tilde{\chi}_3) = 0 .
\]

- \( k = \frac{5}{2} \)

\[
(8.10) (q^2 + q^{-2}) \tilde{B} \tilde{A} \tilde{B} = \tilde{A} \tilde{B}^2 + \tilde{B}^2 \tilde{A} - [2]_q \tilde{B}^2 - [2]_q \{ \tilde{A}, \tilde{B} \} + (2[2]_q \tilde{\chi}_2 + (q^4 + q^{-4}(\tilde{\chi}_2 + \tilde{\chi}_2)) \tilde{B} ,
\]

\[
(8.11) (q^2 + q^{-2}) \tilde{A} \tilde{B} \tilde{A} = \tilde{B} \tilde{A}^2 + \tilde{A}^2 \tilde{B} - [2]_q \tilde{A}^2 - [2]_q \{ \tilde{A}, \tilde{B} \} + (2[2]_q \tilde{\chi}_2 + (q^4 + q^{-4}(\tilde{\chi}_2 + \tilde{\chi}_2)) \tilde{A} ,
\]

\[
(8.12) (\tilde{A} - \tilde{\chi}_1)(\tilde{A} - \tilde{\chi}_2)(\tilde{A} - \tilde{\chi}_3) = 0 , \quad (\tilde{B} - \tilde{\chi}_1)(\tilde{B} - \tilde{\chi}_2)(\tilde{B} - \tilde{\chi}_3) = 0 ,
\]

\[
(8.13) (\tilde{D} - \tilde{\chi}_1)(\tilde{D} - \tilde{\chi}_2)(\tilde{D} - \tilde{\chi}_3) = 0 , \quad (\tilde{D}' - \tilde{\chi}_1)(\tilde{D}' - \tilde{\chi}_2)(\tilde{D}' - \tilde{\chi}_3) = 0 .
\]

- \( k = \frac{3}{2} \)

\[
(8.14) (q^2 + q^{-2}) \tilde{B} \tilde{A} \tilde{B} = \tilde{A} \tilde{B}^2 + \tilde{B}^2 \tilde{A} - [2]_q \tilde{B}^2 - [2]_q \{ \tilde{A}, \tilde{B} \} + 2([2]_q + q^4 + q^{-4}) \tilde{\chi}_2 \tilde{B} ,
\]

\[
(8.15) (q^2 + q^{-2}) \tilde{A} \tilde{B} \tilde{A} = \tilde{B} \tilde{A}^2 + \tilde{A}^2 \tilde{B} - [2]_q \tilde{A}^2 - [2]_q \{ \tilde{A}, \tilde{B} \} + 2([2]_q + q^4 + q^{-4}) \tilde{\chi}_2 \tilde{A} ,
\]

\[
(8.16) \tilde{A}(\tilde{A} - \tilde{\chi}_1)(\tilde{A} - \tilde{\chi}_2)(\tilde{A} - \tilde{\chi}_3) = 0 , \quad \tilde{B}(\tilde{B} - \tilde{\chi}_1)(\tilde{B} - \tilde{\chi}_2)(\tilde{B} - \tilde{\chi}_3) = 0 ,
\]

\[
(8.17) \tilde{D}(\tilde{D} - \tilde{\chi}_1)(\tilde{D} - \tilde{\chi}_2)(\tilde{D} - \tilde{\chi}_3) = 0 , \quad \tilde{D}'(\tilde{D}' - \tilde{\chi}_1)(\tilde{D}' - \tilde{\chi}_2)(\tilde{D}' - \tilde{\chi}_3) = 0 .
\]

- \( k = \frac{1}{2} \)

\[
(8.18) (q^2 + q^{-2}) \tilde{B} \tilde{A} \tilde{B} = [3]_q([2]_q \{ \tilde{A}, \tilde{B} \} - 2[2]_q^2 \tilde{A} + [2]_q^3) + [2]_q^2(q^4 + q^{-4}) \tilde{B} ,
\]

\[
(8.19) (q^2 + q^{-2}) \tilde{A} \tilde{B} \tilde{A} = [3]_q([2]_q \{ \tilde{A}, \tilde{B} \} - 2[2]_q^2 \tilde{B} + [2]_q^3) + [2]_q^2(q^4 + q^{-4}) \tilde{A} ,
\]

\[
(8.20) (\tilde{A} - \tilde{\chi}_1)(\tilde{A} - \tilde{\chi}_2) = 0 , \quad (\tilde{B} - \tilde{\chi}_1)(\tilde{B} - \tilde{\chi}_2) = 0 ,
\]

\[
(8.21) (\tilde{D} - \tilde{\chi}_1)(\tilde{D} - \tilde{\chi}_2) = 0 , \quad (\tilde{D}' - \tilde{\chi}_1)(\tilde{D}' - \tilde{\chi}_2) = 0 .
\]
For each value of $k$, the elements $\tilde{D}$ and $\tilde{D}'$ are given by

\begin{align}
\tilde{D} &= \frac{(q^1 + q^{-4})}{\binom{2}{q}} (\bar{x}_2^{\pm} + \bar{x}_k) + 2\bar{x}_2\bar{x}_k - \frac{(q - q^{-1})}{\binom{2}{q}} [\tilde{A}, \tilde{B}]_q - \tilde{A} - \tilde{B}, \\
\tilde{D}' &= \frac{(q^1 + q^{-4})}{\binom{2}{q}} (\bar{x}_2^{\pm} + \bar{x}_k) + 2\bar{x}_2\bar{x}_k - \frac{(q - q^{-1})}{\binom{2}{q}} [\tilde{B}, \tilde{A}]_q - \tilde{A} - \tilde{B}.
\end{align}

**Theorem 8.1.** Conjecture 4.1 is verified for $j_1 = j_2 = j_3 = \frac{3}{2}$.

**Proof.** Since we already know that the map $\phi$ is surjective, we only need to prove (8.3). For the case $k = \frac{3}{2}$, all the elements are constants and $\dim(\mathcal{K}_k \tilde{AW}(\frac{3}{2}, \frac{3}{2}, \frac{3}{2})) = 1$. For the case $k = \frac{5}{2}$ (resp. $k = \frac{7}{2}$), one uses (8.22) in the first relation of (8.9) (resp. (8.21)) to find (after some simplifications using the defining relations) $\tilde{B}\tilde{A} = -\tilde{A}\tilde{B} + x_1(\tilde{A} + \tilde{B}) + x_2$, for some constants $x_1$ and $x_2$ that can be computed. Therefore, in both cases we see that a linearly generating set is given by $\{1, \tilde{A}, \tilde{B}, \tilde{A}\tilde{B}\}$. For the case $k = \frac{5}{2}$, we used formal mathematical software to show that $\{1, \tilde{A}, \tilde{B}, \tilde{A}^2, \tilde{A}\tilde{A}, \tilde{B}\tilde{A}, \tilde{B}^2, \tilde{A}\tilde{A}^2, \tilde{B}\tilde{A}^2, \tilde{B}\tilde{B}, \tilde{A}^2\tilde{B}, \tilde{A}\tilde{B}^2, \tilde{B}^2\tilde{B}^2, \tilde{A}\tilde{B}\}$ is a generating set. Similarly, for the case $k = \frac{7}{2}$, a generating set is given by $\{1, \tilde{A}, \tilde{B}, \tilde{A}^2, \tilde{A}\tilde{B}, \tilde{B}\tilde{A}, \tilde{B}^2, \tilde{A}\tilde{B}\tilde{A}, \tilde{A}\tilde{B}\tilde{A}^2, \tilde{B}\tilde{A}\tilde{B}, \tilde{B}^2\tilde{A}, \tilde{A}^2\tilde{B}, \tilde{A}\tilde{B}^2, \tilde{B}^2\tilde{B}^2, \tilde{A}\tilde{B}\tilde{A}\tilde{B}\}$.

From these results and the degeneracies $d_k$ given at the beginning of the section, we see that (8.3) holds, which concludes the proof.

Let us notice that the defining relation (4.17) of $\tilde{AW}(j_1, j_2, j_3)$ which involves the Casimir element $\Omega$ of $\tilde{AW}(3)$ has not been called upon in the previous proof. It is straightforward to verify that this relation is satisfied in each of the algebras $\mathcal{K}_k \tilde{AW}(\frac{3}{2}, \frac{3}{2}, \frac{3}{2}) \mathcal{K}_k$ by using the relations given above.

9. **Quotient $\tilde{AW}(j, \frac{1}{2}, \frac{1}{2})$ and One-Boundary Temperley–Lieb Algebra**

In this section, we consider the case $j_1 = j$, for $j = 1, \frac{3}{2}, 2, \ldots$, and $j_2 = j_3 = \frac{1}{2}$. We show that the algebra $\tilde{AW}(j, \frac{1}{2}, \frac{1}{2})$ is isomorphic to the centralizer $C_{j, \frac{1}{2}, \frac{1}{2}}$. We also find an explicit isomorphism between this quotient of the Askey–Wilson algebra and a specialization of the one-boundary Temperley–Lieb algebra.

9.1. **$\tilde{AW}(j, \frac{1}{2}, \frac{1}{2})$ algebra.** From the tensor decomposition rules, we find the sets

\begin{align}
\mathcal{J} \left( j, \frac{1}{2} \right) &= \left\{ j - \frac{1}{2}, j + \frac{1}{2} \right\}, \quad \mathcal{J} \left( \frac{1}{2}, \frac{1}{2} \right) = \left\{ 0, 1 \right\}, \quad \mathcal{J} \left( j, \frac{1}{2}, \frac{1}{2} \right) = \left\{ j - 1, j, j + 1 \right\}, \\
\mathcal{M} \left( j, \frac{1}{2}, \frac{1}{2} \right) &= \{ \chi_{j+1} - \chi_{j+\frac{1}{2}}, \chi_{j} - \chi_{j+\frac{1}{2}}, \chi_{j+1} - \chi_{j-\frac{1}{2}}, \chi_{j-1} - \chi_{j-\frac{1}{2}} \} = \{ m_1, m_2, m_3, m_4 \}, \\
\mathcal{M} \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) &= \{ \chi_{j+1} - \chi_1, \chi_j - \chi_1, \chi_j - \chi_0, \chi_{j-1} - \chi_1 \}.
\end{align}

The degeneracies are $d_{j-1} = d_{j+1} = 1$ and $d_j = 2$, and the dimension of the centralizer is $\dim(C_{j, \frac{1}{2}, \frac{1}{2}}) = 6$. The central elements $\alpha_i$ take the values $\alpha_1 = \chi_j$ and $\alpha_2 = \alpha_3 = \alpha_4 = \chi_{\frac{1}{2}}$ in the quotient $\tilde{AW}(j, \frac{1}{2}, \frac{1}{2})$. As in the previous sections, we perform the transformation $X = (q - q^{-1})^2 X + q + q^{-1}$ on the generators $X = A, B, D, D', K$ (see Remark 2.1). The defining relations of $\tilde{AW}(j, \frac{1}{2}, \frac{1}{2})$ can
be written as follows

\( \tilde{B} \tilde{A} \tilde{B} = (\tilde{x}_j - [2]_q [1/2]_q^2) \tilde{B} + \tilde{K} \tilde{B} , \)

\( \tilde{A} \tilde{A} \tilde{A} = a_1 \{ \tilde{A}, \tilde{B} \} - a_2 \tilde{B} + a_3 \tilde{A} + \tilde{K} (\tilde{A} - a_1) + a_4 , \)

\( \tilde{A} - \tilde{x}_j - \frac{1}{2} (\tilde{A} - \tilde{x}_j + \frac{1}{2}) = 0 , \quad \tilde{B} (\tilde{B} - [2]_q) = 0 , \quad (\tilde{K} - \tilde{x}_{j-1})(\tilde{K} - \tilde{x}_j)(\tilde{K} - \tilde{x}_{j+1}) = 0 , \)

\( \tilde{D} - \tilde{x}_j - \frac{1}{2} (\tilde{D} - \tilde{x}_j + \frac{1}{2}) = 0 , \quad (\tilde{D}' - \tilde{x}_j - \frac{1}{2})(\tilde{D}' - \tilde{x}_j + \frac{1}{2}) = 0 , \)

\( \tilde{K} - \tilde{B} - \tilde{x}_{j+1} + \tilde{x}_1)(\tilde{K} - \tilde{B} - \tilde{x}_j + \tilde{x}_1)(\tilde{K} - \tilde{B} - \tilde{x}_j - \tilde{x}_{j-1} + \tilde{x}_1) = 0 \)

\( \prod_{i=1}^{4} (\tilde{K} - \tilde{A} - m_i) = 0 \quad \prod_{i=1}^{4} (\tilde{K} - \tilde{D} - m_i) = 0 \quad \prod_{i=1}^{4} (\tilde{K} - \tilde{D}' - m_i) = 0 \)

\( \Omega = \chi_j^2 + 2\chi_j - K^2 + \chi_j \chi_j^2 K - \chi_0^2 , \)

where

\( \tilde{D} = \frac{q^2 + q^{-2}}{[2]_q} (\tilde{K} + \tilde{x}_j) + 2\tilde{x}_j^2 - \frac{(q - q^{-1})}{[2]_q} [\tilde{A}, \tilde{B}]_q - \tilde{A} - \tilde{B} , \)

\( \tilde{D}' = \frac{q^2 + q^{-2}}{[2]_q} (\tilde{K} + \tilde{x}_j) + 2\tilde{x}_j^2 - \frac{(q - q^{-1})}{[2]_q} [\tilde{B}, \tilde{A}]_q - \tilde{A} - \tilde{B} , \)

\( \Omega = q(A + D) \chi_j^2 (\chi_j + K) + q^{-1} B (\chi_j^2 + \chi_j K) - q^2 A^2 - q^{-2} B^2 - q^2 D^2 - qABD , \)

and where we have used the following constants

\( a_1 = \frac{[2]_q [j - \frac{1}{2}]_q [j + \frac{1}{2}]_q}{q^2 + q^{-2}} , \quad a_2 = 2 \frac{\tilde{x}_j - \frac{1}{2} \tilde{x}_j + \frac{1}{2}}{q^2 + q^{-2}} , \)

\( a_3 = \frac{[2]_q}{q^2 + q^{-2}} (2\tilde{x}_j^2 - [2]_q [j + \frac{1}{2}]_q^2) + \tilde{x}_j , \quad a_4 = a_1 ([j + \frac{1}{2}]_q^2 + \tilde{x}_j^2) , \)

\( m_1 = \tilde{x}_j - \frac{1}{2} \tilde{x}_j + \frac{1}{2} , \quad m_2 = \tilde{x}_j - \tilde{x}_j - \frac{1}{2} , \quad m_3 = \tilde{x}_j - \tilde{x}_j - \frac{1}{2} , \quad m_4 = \tilde{x}_j - \tilde{x}_j - \frac{1}{2} . \)

**Proposition 9.1.** The quotient \( \overline{AW} (j, \frac{1}{2}, \frac{1}{2}) \) can be presented with the following relations

\( \tilde{A}^2 = (\tilde{x}_j - \frac{1}{2} + \tilde{x}_j + \frac{1}{2}) \tilde{A} - \tilde{x}_j - \frac{1}{2} \tilde{x}_j + \frac{1}{2} , \quad \tilde{B}^2 = [2]_q \tilde{B} , \)

\( \tilde{B} \tilde{A} \tilde{B} = [2]_q \{ \tilde{A}, \tilde{B} \} - [2]_q \tilde{A} - ([j + \frac{3}{2}]_q^2 + [j - \frac{1}{2}]_q^2 - 1)(\tilde{B} - [2]_q) . \)

**Proof.** We first show that the relations (9.1)–(6) imply the relations of the proposition. The two equations in (9.1) follow directly from (9.3). We also deduce from the first relation of (9.3) that

\( (\tilde{A} - a_1) (\tilde{A} - \tilde{x}_j - \frac{1}{2} - \tilde{x}_j + \frac{1}{2} + a_1) = \frac{[2j - 1]_q [2j + 3]_q}{q^2 + q^{-2}} . \)

Since the R.H.S. of the previous relation does not vanish for \( j > \frac{1}{2} \), it can be used in (9.2) to find

\( \tilde{K} = \{ \tilde{A}, \tilde{B} \} - (\tilde{x}_j - \frac{1}{2} + \tilde{x}_j + \frac{1}{2}) \tilde{B} - [2]_q \tilde{A} + (1 + [2]_q)([2j + \frac{3}{2}]_q^2 + [\tilde{x}_j - \frac{1}{2}]) . \)

Using (9.14) in (9.8) and (9.9), and then substituting in (9.4), one obtains expressions for \( \tilde{A} \tilde{B} \tilde{A} \tilde{B} \) and \( \tilde{B} \tilde{A} \tilde{B} \tilde{A} \) in terms of the elements \( 1, \tilde{A}, \tilde{B}, \tilde{A} \tilde{B}, \tilde{B} \tilde{A}, \tilde{A} \tilde{B} \tilde{A} \) and \( \tilde{B} \tilde{A} \tilde{B} \). By using (9.14) and the expressions for \( \tilde{A} \tilde{B} \tilde{A} \tilde{B} \) and \( \tilde{B} \tilde{A} \tilde{B} \tilde{A} \) in the third relation of (9.3), one gets (9.12).
Finally, we want to show that (9.11) and (9.12) imply the defining relations of $\overline{AW}(j, \frac{1}{2}, \frac{1}{2})$. To that end, we suppose that the relations of the proposition are true and we define the element $\tilde{K}$ as in (9.14). It is then straightforward to verify that $\tilde{K}$ is central and that (9.1)–(9.7) hold. □

Theorem 9.1. Conjecture 4.1 is verified for $j_1 = j$ and $j_2 = j_3 = \frac{3}{2}$, where $j = 1, \frac{3}{2}, 2, ...$

Proof. We already know that the map $\bar{\phi}$ is surjective. From the previous proposition, we conclude that \{1, $\bar{A}, \bar{B}, \bar{A}\bar{B}, \bar{B}\bar{A}$\} is a generating set for $\overline{AW}(j, \frac{1}{2}, \frac{1}{2})$. Therefore, $\dim(\overline{AW}(j, \frac{1}{2}, \frac{1}{2})) \leq \dim(C_{j, \frac{1}{2}, \frac{1}{2}}) = 6$, which shows the injectivity of $\bar{\phi}$. □

9.2. Connection with the one-boundary Temperley–Lieb algebra. On the basis of the findings for the limit $q \to 1$ \cite{4}, one might expect that the centralizer in the case of one spin-$j$ and two spin-$\frac{1}{2}$ will be isomorphic to the one-boundary Temperley–Lieb algebra. We can indeed confirm that this algebra is recovered as a quotient of $AW(3)$.

Definition 9.1. \cite{17, 18, 19} The one-boundary Temperley–Lieb algebra $1bTL_2(q, \omega)$ is generated by $\sigma_0$ and $\sigma_1$ with the following defining relations

\begin{equation}
\sigma_0^2 = \frac{[\omega]_q}{[\omega - 1]_q} \sigma_0, \quad \sigma_1^2 = (q + q^{-1}) \sigma_1, \quad \sigma_1 \sigma_0 \sigma_1 = \sigma_1.
\end{equation}

Theorem 9.2. The quotiented Askey–Wilson algebra $\overline{AW}(j, \frac{1}{2}, \frac{1}{2})$, for $j = 1, \frac{3}{2}, 2, ...$, is isomorphic to the one-boundary Temperley–Lieb algebra $1bTL_2(q, 2j + 1)$. This isomorphism is given explicitly by

\begin{equation}
\overline{AW} \left(j, \frac{1}{2}, \frac{1}{2} \right) \to 1bTL_2(q, 2j + 1)
\end{equation}

\begin{align}
\bar{A} &\mapsto \tilde{\chi}_{j+\frac{1}{2}} - [2j]_q \sigma_0, \\
\bar{B} &\mapsto [2]_q - \sigma_1.
\end{align}

Proof. It is easy to see that the map $\varphi$ is bijective. The homomorphism can be directly verified from the relations of the proposition \cite{9.1} □

10. Conclusion and perspectives

Summing up, we have offered a conjecture according to which a quotient of the Askey–Wilson algebra is isomorphic to the centralizer of the image of the diagonal embedding of $U_q(\mathfrak{sl}_2)$ in the tensor product of any three irreducible representations. It has been proved in several cases, and we thus obtained the Temperley–Lieb, Birman–Murakami–Wenzl and one-boundary Temperley–Lieb algebras as quotients of the Askey–Wilson algebra. In the limit $q \to 1$, the results of the paper \cite{4} are recovered. We have provided further evidence in support of the conjecture by studying the finite irreducible representations of the quotient of the Askey–Wilson algebra, more particularly in the case of three identical spins.

Proving the conjecture in the case of three arbitrary spins $j_1, j_2, j_3$ would be an obvious continuation of the work presented here. If true, this conjecture would provide a presentation of the centralizer of $U_q(\mathfrak{sl}_2)$ in terms of generators and relations for any three irreducible representations.
We could first consider, more simply, the case of three identical spins $j_1 = j_2 = j_3 = s$. As for the Temperley–Lieb ($s = \frac{1}{2}$) and the Birman–Murakami–Wenzl ($s = 1$) algebras, we expect that the centralizer for any spin $s$ will be linked to a quotient of the braid group algebra.

In [3], a diagrammatic description of the centralizers of $U_q(gl_n)$ has been proposed. It is based on the notion of fused Hecke algebras. Developing a connection between this diagrammatic approach and the Askey–Wilson algebra could prove fruitful.

Throughout the present paper, we assume $q$ to be not a root of unity. This choice allows to decompose the tensor product of irreducible representations of $U_q(sl_2)$ into a direct sum of irreducible representations (see Subsection 3.1). As a consequence, the matrices $C_i, C_{ij}, C_{123}$ are diagonalizable and their minimal polynomials are those discussed in Subsection 3.2. It could be interesting to study the centralizer when $q$ is a root of unity and to examine how the quotient of $AW(3)$ is affected.

Another generalization of the results presented here would be to consider the $n$-fold tensor product of irreducible representations of $U_q(sl_2)$ and to connect the centralizer to a higher rank Askey–Wilson algebra $AW(n)$. The approach using the $R$-matrix proposed in [2] should be helpful for this purpose. In fact, a simpler starting point could be to generalize either the conjecture given in [4] by studying the connection between the centralizers of $sl_2$ and a higher rank Racah algebra, or the one given in [1] by examining how centralizers of $osp(1|2)$ relate to the higher rank Bannai-Ito algebra $BI(n)$ (see [6]).

Yet another direction to generalize the results of this paper would be to study the centralizer of the diagonal embedding of $g$ or $U_q(g)$ with $g$ a higher rank Lie algebra. A first step in this direction was made recently in [5] where the centralizer $Z_2(sl_3)$ of the diagonal embedding of $sl_3$ in the twofold tensor product of $sl_3$ has been identified. A proposition similar to Proposition 4.2 has also been proved in that case. A quotient of the algebra $Z_2(sl_3)$ that describes the centralizer for any representations of $sl_3$ has still to be investigated. We hope to report on some of this issues in the future.

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