EQUIVALENCIES BETWEEN BETA-SHIFTS AND S-GAP SHIFTS

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Abstract. Let $X_\beta$ be a sofic $\beta$-shift for $\beta \in (1, 2]$. We show that there is an $S$-gap shift $X(S)$ such that $X_\beta$ and $X(S)$ are right-resolving almost conjugate. Conversely, a condition on $S \subseteq \mathbb{N} \cup \{0\}$ is given such that for this $S$, there is a $\beta$ such that $X(S)$ and $X_\beta$ have the same equivalency. We show that if $X_\beta$ is SFT, then there is an $S$-gap shift conjugate to this $X_\beta$; however, if $X_\beta$ is not SFT, then no $S$-gap shift is conjugate to $X_\beta$. Also we will investigate the existence of these sort of equivalencies for non-sofics.

Introduction

Two important classes of symbolic dynamics are $\beta$-shifts and $S$-gap shifts. Both are rich families and highly chaotic with application in coding theory, number theory and a source of examples for symbolic dynamics. There has been some independent studies of these classes. See [6, 14, 17, 19] for $\beta$-shifts and [1, 6, 12] for $S$-gap shifts. There are some common properties among these two classes. For instance, both of them are coded systems having positive entropies for $|S| \geq 2$ and every subshift factor of them is intrinsically ergodic [6]. There are disparities as well: $\beta$-shifts are all mixings, though this is not true for $S$-gap shifts [12]; and $S$-gap shifts are synchronized, a property which does not hold for all $\beta$-shifts. Even among sofic $\beta$-shifts and $S$-gap shifts, which are of our most interest here, there are some major differences. An important class of sofic $S$-gap shifts are almost-finite-type (AFT) [1], but no $\beta$-shift is AFT [19].

Here we let $\beta \in (1, 2]$ and search for an $S$-gap shift $X(S)$ which has some sort of equivalencies with our $\beta$-shift denoted by $X_\beta$. A well known equivalency between dynamical systems is conjugacy and in transitive systems, almost conjugacy which is virtually a conjugacy between transitive points has been much considered. However, in coding theory the equivalencies which are right-resolving - called deterministic in computer science - are more natural and applicable. Our systems can have application in coding theory and so we investigate the equivalency of this sort among them.

Here, we summarize our results. Let $\beta \in (1, 2]$ and let $X_\beta$ be the corresponding $\beta$-shift. We will associate to $X_\beta$ a unique $S$-gap shift denoted by ASS($X_\beta$) and will show that when $X_\beta$ is sofic, then $X_\beta$ and $X(S) = ASS(X_\beta)$ are right-resolving almost conjugate and when $X_\beta$ is SFT, they are conjugate as well (Theorem 3.7). On
the other side, for a given \( S \)-gap shift, it does not necessarily exist a \( \beta \)-shift holding the same equivalencies with \( X(S) \). However, we give a necessary and sufficient condition on \( S \) to have this true (Theorem 3.10).

In section 4, we extend the results obtained for sofic to non-sofic. For instance, \( X_\beta \) and \( \text{ASS}(X_\beta) \) have a common extension which is an almost Markov synchronized system whose maps are entropy-preserving (Corollary 4.2). Additionally, when \( X_\beta \) is synchronized, \( X_\beta \) and \( \text{ASS}(X_\beta) \) have a common synchronized 1-1 a.e. extension with right-resolving legs (Theorem 4.5).

Theorem 4.3 gives the the zeta function of \( X_\beta \) in terms of the zeta function of \( \text{ASS}(X_\beta) \) and Theorem 4.4 states that \( \{ \text{ASS}(X_\beta) : \beta \in (1, 2) \} \) is a Cantor set in the set of all \( S \)-gap shifts.

1. Background and Notations

The notations has been taken from [13] and the proofs of the claims in this section can be found there. Let \( A \) be an alphabet, that is a nonempty finite set. The full \( A \)-shift denoted by \( A^\mathbb{Z} \), is the collection of all bi-infinite sequences of symbols from \( A \). A block (or word) over \( A \) is a finite sequence of symbols from \( A \). The shift function \( \sigma \) on the full shift \( A^\mathbb{Z} \) maps a point \( x \) to the point \( y = \sigma(x) \) whose \( i \)th coordinate is \( y_i = x_{i+1} \).

Let \( B_n(X) \) denote the set of all admissible \( n \)-blocks. The Language of \( X \) is the collection \( B(X) = \bigcup_{n=0}^{\infty} B_n(X) \). A word \( v \in B(X) \) is synchronizing if whenever \( uv \) and \( uvw \) are in \( B(X) \), we have \( uvw \in B(X) \).

Let \( A \) and \( D \) be alphabets and \( X \) a shift space over \( A \). Fix integers \( m \) and \( n \) with \(-m \leq n \). Define the \((m + n + 1)\)-block map \( \Phi : B_{m+n+1}(X) \to D \) by

\[
y_{i} = \Phi(x_{i-m}x_{i-m+1}...x_{i+n}) = \Phi(x_{[i-m,i+n]})
\]

where \( y_i \) is a symbol in \( D \). The map \( \phi : X \to D^\mathbb{Z} \) defined by \( y = \phi(x) \) with \( y_i \) given by (1.1) is called the sliding block code with memory \( m \) and anticipation \( n \) induced by \( \Phi \). An onto sliding block code \( \phi : X \to Y \) is called a factor code. In this case, we say that \( Y \) is a factor of \( X \). The map \( \phi \) is a conjugacy, if it is invertible.

An edge shift, denoted by \( X_G \), is a shift space which consists of all bi-infinite walks in a directed graph \( G \). A labeled graph \( G \) is a pair \((G, L)\) where \( G \) is a graph with edge set \( \mathcal{E} \), vertex set \( \mathcal{V} \) and the labeling \( L : \mathcal{E} \to A \). Each \( e \in \mathcal{E} \) starts at a vertex denoted by \( i(e) \in \mathcal{V} \) and terminates at a vertex \( t(e) \in \mathcal{V} \).

When the set of forbidden words is finite, the space is called subshift of finite type (SFT). A sofic shift \( X_G \) is the set of sequences obtained by reading the labels of walks on \( G \),

\[X_G = \{ L_\infty(\xi) : \xi \in X_G \} = L_\infty(X_G).\]

We say \( G \) is a presentation of \( X_G \).

A labeled graph \( G = (G, L) \) is right-resolving if for each vertex \( I \) of \( G \) the edges starting at \( I \) carry different labels. A minimal right-resolving presentation of a sofic shift \( X \) is a right-resolving presentation of \( X \) having the fewer vertices among all right-resolving presentations of \( X \). Any two minimal right-resolving presentations of an irreducible sofic shift must be isomorphic as labeled graphs [13, Theorem 3.3.18]. So we can speak of “the” minimal right-resolving presentation of an irreducible sofic shift \( X \) which we call it the Fischer cover of \( X \).

Let \( w \in B(X) \). The follower set \( F(w) \) of \( w \) is defined by \( F(w) = \{ v \in B(X) : uvw \in B(X) \} \). A shift space \( X \) is sofic if and only if it has a finite number of follower
sets [13, Theorem 3.2.10]. In this case, we have a labeled graph $G = (G, \mathcal{L})$ called the \textit{follower set graph} of $X$. The vertices of $G$ are the follower sets and if $wa \in B(X)$, then draw an edge labeled $a$ from $F(w)$ to $F(wa)$. If $wa \notin B(X)$ then do nothing.

Let $\phi = \Phi_\infty : X \to Y$ be a 1-block code. Then $\phi$ is \textit{right-resolving} if whenever $ab$ and $ac$ are 2-blocks in $X$ with $\Phi(b) = \Phi(c)$, then $b = c$.

Let $G$ and $H$ be graphs. A \textit{graph homomorphism} from $G$ to $H$ consists of a pair of maps $\partial \Phi : V(G) \to V(H)$ and $\Phi : E(G) \to E(H)$ such that $\partial \Phi(i(e)) = i(\Phi(e))$ and $\Phi(t(e)) = t(\Phi(e))$ for all $e \in E(G)$. A graph homomorphism is a \textit{graph isomorphism} if both $\partial \Phi$ and $\Phi$ are one-to-one and onto. Two graphs $G$ and $H$ are graph isomorphic (written $G \cong H$) if there is a graph isomorphism between them. Let $E_I(G)$ be the set of all the edges in $E(G)$ starting from $I \in V(G)$. A graph homomorphism $\Phi : G \to H$ maps $E_I(G)$ into $E_{\Phi(I)}(H)$ for each vertex $I$ of $G$.

Thus $\phi = \Phi_\infty$ is right-resolving if and only if for every vertex $I$ of $G$ the restriction $\Phi_I$ of $\Phi$ to $E_I(G)$ is one-to-one. If $G$ and $H$ are irreducible and $\phi$ is a right-resolving code from $X_G$ onto $X_H$, then each $\Phi_I$ must be a bijection. Thus for each vertex $I$ of $G$ and every edge $e \in E_{\Phi(I)}(H)$, there exists a unique “lifted” edge $e' \in E_I(G)$ such that $\Phi(e) = e'$. This lifting property inductively extends to paths: for every vertex $I$ of $G$ and every path $w$ in $H$ starting at $\partial \Phi(I)$, there is a unique path $\pi$ in $G$ starting at $I$ such that $\Phi(\pi) = w$.

Points $x$ and $x'$ in $X$ are left-asymptotic if there is an integer $N$ for which $x_{(-\infty, N]} = x'_{(-\infty, N]}$. A sliding block code $\phi : X \to Y$ is \textit{right-closing} if whenever $x, x'$ are left-asymptotic and $\phi(x) = \phi(x')$, then $x = x'$.

The entropy of a shift space $X$ is defined by $h(X) = \lim_{n \to \infty} (1/n) \log |B_n(X)|$.

2. General Properties of $S$-gap Shifts and $\beta$-shifts

2.1. $S$-gap shifts. To define an $S$-gap shift $X(S)$, fix $S = \{s_i \in \mathbb{N} \cup \{0\} : 0 \leq s_i < s_{i+1}, i \in \mathbb{N} \cup \{0\}\}$. Define $X(S)$ to be the set of all binary sequences for which 1’s occur infinitely often in each direction and such that the number of 0’s between successive occurrences of a 1 is in $S$. When $S$ is infinite, we need to allow points that begin or end with an infinite string of 0’s. Note that $X(S)$ and $X(S')$ are conjugate if and only if one of the $S$ and $S'$ is $\{0, n\}$ and the other $\{n, n+1, n+2, \ldots\}$ for some $n \in \mathbb{N}$ [1, Theorem 4.1]. So we consider $X(S)$ up to conjugacy and by convention $\{0, n\}$ is chosen. Now let $d_0 = s_0$ and $\Delta(S) = \{d_i\}$, where $d_n = s_n - s_{n-1}$. Then an $S$-gap shift is subshift of finite type (SFT) if and only if $S$ is finite or cofinite, is almost-finite-type (AFT) if and only if $\Delta(S)$ is eventually constant and is sofic if and only if $\Delta(S)$ is eventually periodic [1]. Therefore, for sofic $S$-gap shifts we set

$$
\Delta(S) = \{d_0, d_1, \ldots, d_{k-1}, g_0, g_1, \ldots, g_{l-1}\}, \quad g = \sum_{i=0}^{l-1} g_i
$$

where $g_j = s_{k+j} - s_{k+j-1}, 0 \leq j \leq l - 1$. Also $k$ and $l$ are the least integers such that (2.1) holds.

The Fischer cover of any irreducible sofic shift as well as $S$-gap shifts is the labeled subgraph of the follower set graph consists of the finite set of follower sets of synchronizing words as its vertices. For an $S$-gap shift this set is

$$
\{F(1), F(10), \ldots, F(10^{n(S)})\},
$$

where $n(S) = \max S$ for $|S| < \infty$. If $|S| = \infty$, then $n(S)$ will be defined as follow.
(1) For $k = 1$ and $g_{l-1} > s_0$,
   (a) if $g_{l-1} = s_0 + 1$, then $F(10^{s_1+1}) = F(1)$ and $n(S) = s_{l-1}$.
   (b) if $g_{l-1} > s_0 + 1$, then $F(10^g) = F(1)$ and $n(S) = g - 1$.
(2) For $k \neq 1$, if $g_{l-1} > d_{k-1}$, then $F(10^{g_{k-1}+1}) = F(10^{g_{k-1}+1})$ and $n(S) = g + s_{k-2}$.
(3) For $k \in \mathbb{N}$, if $g_{l-1} \leq d_{k-1}$, then $F(10^{g_{k-1}+1}) = F(10^{g_{k-1}+1})$ and $n(S) = s_{k+1}$.

To have a view about the Fischer cover of an $S$-gap shift, we line up vertices in (2.2) horizontally starting from $F(1)$ on the left followed by $F(10)$ then by $F(10^2)$ and at last ending at $F(10^{n(S)})$ as the far right vertex. In all cases, label 0 the edge starting from $F(10^i)$ and terminating at $F(10^{i+1})$, $0 \leq i < n(S) - 1$; also, label 1 all edges from $F(10^i)$ to $F(1)$ for $s \in S$ and $s < n(S)$.

So the only remaining edges to be taken care of are those starting at $F(10^{n(S)})$. In (1a), there are two edges from $F(10^{s(N)})$ to $F(1)$; label one 0 and the other 1. In (1b), there is only one edge from $F(10^{s(N)})$ to $F(1)$ which is labeled 0. In case (2) (resp. (3)), label 0 the edge from $F(10^{s(N)})$ to $F(10^{s(N)+1})$ (resp. $F(10^{s(N)-1})$) and label 1 the edge from $F(10^{s(N)})$ to $F(1)$. For a more detailed treatment see [2].

2.2. $\beta$-shifts. Rényi [15] was the first who considered the $\beta$-shifts. These shifts are symbolic spaces with rich structures and application in theory and practice. We present here a brief introduction to $\beta$-shifts from [19]. For a more detailed treatment, see [5].

When $t$ is a real number we denote by $|t|$ the largest integer which is smaller than $t$. Let $\beta$ a real number greater than 1. Set
\[ 1_\beta = a_1 a_2 a_3 \cdots \in \{0, 1, \ldots, \lfloor \beta \rfloor \}^N, \]
where $a_1 = \lfloor \beta \rfloor$ and
\[ a_i = \lfloor \beta^i (1 - a_1 \beta^{-1} - a_2 \beta^{-2} - \cdots - a_{i-1} \beta^{-i+1}) \rfloor \]
for $i \geq 2$. The sequence $1_\beta$ is the expansion of 1 in the base $\beta$, that is, $1 = \sum_{i=1}^{\infty} a_i \beta^{-i}$. Let $\leq$ be the lexicographic ordering of $(\mathbb{N} \cup \{0\})^N$. The sequence $1_\beta$ has the property that
\[ \sigma^k 1_\beta \leq 1_\beta, \quad k \in \mathbb{N}, \]
where $\sigma$ denotes the shift on $(\mathbb{N} \cup \{0\})^N$. It is a result of Parry [14], that this property characterizes the elements of $(\mathbb{N} \cup \{0\})^N$ which are the $\beta$-expansion of 1 for some $\beta > 1$. Furthermore, it follows from (2.3) that
\[ X_\beta = \{ x \in \{0, 1, \ldots, \lfloor \beta \rfloor \}^Z : x_{[i, \infty)} \leq 1_\beta, i \in \mathbb{Z} \} \]
is a shift space of $\{0, 1, \ldots, \lfloor \beta \rfloor \}^Z$, called the $\beta$-shift. The $\beta$-shift is SFT if and only if the $\beta$-expansion of 1 is finite and it is sofic if and only if the $\beta$-expansion of 1 is eventually periodic [3]. Since we are dealing with the case where $\beta \in (1, 2]$, $a_1 = 1$ throughout this paper.

3. Equivalencies between a Beta-shift and its Associate, Sofic Case
A sliding block code $\phi : X \to Y$ is finite-to-one if there is an integer $M$ such that $\phi^{-1}(y)$ contains at most $M$ points for every $y \in Y$. Shift spaces $X$ and $Y$ are finitely equivalent if there is an SFT say $W$ together with finite-to-one factor codes $\phi_X : W \to X$ and $\phi_Y : W \to Y$. We call $W$ a common extension and
Let $P$ nested: each atom in $M$ prevent the confusion between the total number of edges from $J$ for which $H \preceq G$. This ordering naturally determines an ordering which we still call $\preceq$ on $R_G$. Let $M_G$ be the smallest element in the partial ordering $(R_G, \preceq)$.

Now we recall from [13] how $M_G$ can be constructed. Let $V = V(G)$ be the set of vertices of $G$ and define a nested sequence of equivalence relations $\sim_n$ on $V$ for $n \geq 0$. The partition of $V$ into $\sim_n$ equivalence classes is denoted by $P_n$. To define $\sim_{n-1}$, first let $I \sim_0 J$ for all $I, J \in V$. For $n \geq 1$, let $I \sim_{n-1} J$ if and only if for each class (or atom) $P \in P_{n-1}$ the total number of edges from $I$ to vertices in $P$ equals the total number of edges from $J$ to vertices in $P$. Note that the partitions $P_n$ are nested: each atom in $P_n$ is a union of atoms in $P_{n+1}$.

We have $V$ finite and $P_n$ nested; so $P_n$’s will be equal for all sufficiently large $n$. Let $P$ denote this limiting partition. Then $P$ will be the set of states of $M_G$. To prevent the confusion between $M_G$ and $G$, we call a vertex in $M_G$ a state and of $G$ just vertex.

Since for all large enough $n$, $P = P_n = P_{n+1}$, for each pair $P, Q \in P$ there is $k$ such that for each $I \in P$ there are exactly $k$ edges in $G$ from $I$ to vertices in $Q$. We then assign $k$ edges in $M_G$ from $P$ to $Q$.

Therefore to have $M_G$, for each $n$, we refine the atoms of $P_n$ and when $P_n = P$, then for each $P, Q \in P$ and $I, J \in P$, the total number of paths from $I$ and $J$ to vertices in $Q$ and also the length of these paths (with respect to $G$) for both $I$ and $J$ are equal. Another way to obtain $M_G$ arises from this as follows.

We have $P_0 = V(G)$. Then $\sim_1$ partitions vertices by their out-degrees where for $X_\beta$ and $X = X(S)$, $\sim_1$ partitions vertices into two atoms, one atom containing the vertices with out-degree one and the other with out-degree two. If $P \neq P_1$, for the next step if $P \in P_1$ is refined, then it is turn for $Q$ to be refined where $Q \in P_1$ is any atom having edges terminating to vertices in $P$.

**Theorem 3.1.** [13, Theorem 8.4.7] Suppose that $X$ and $Y$ are irreducible sofic shifts. Let $G_X$ and $G_Y$ denote the underlying graphs of their Fischer covers respectively. Then $X$ and $Y$ are right-resolving finitely equivalent if and only if $M_G \cong M_{G_Y}$. Moreover, the common extension can be chosen to be irreducible.

A point in $X$ is doubly transitive if every word in $B(X)$ occurs infinitely often to the left and to the right of its representation. Shift spaces $X$ and $Y$ are almost conjugate if there is a shift of finite type $W$ and 1-1 a.e. factor codes $\phi_X : W \to X$ and $\phi_Y : W \to Y$ (1-1 a.e. means that any doubly transitive point has exactly one pre-image). Call an almost conjugacy between sofic shifts in which both legs are right-resolving (resp. right-closing) a right-resolving almost conjugacy (resp. right-closing almost conjugacy).
Let r-r and r-c stand for right resolving and right closing respectively. We summarize the relations amongst mentioned properties in the following diagram.

\[
\begin{array}{c}
\text{conjugacy} \\
\Rightarrow \\
\text{r-r almost conjugacy} \\
\Rightarrow \\
\text{r-r finite equivalence}
\end{array}
\]

\[
\begin{array}{c}
\Rightarrow \\
\text{r-c almost conjugacy} \\
\Rightarrow \\
\text{r-c finite equivalence}
\end{array}
\]

There are examples to show that in general the above properties are different [13].

**Definition 3.2.** Let \( w = w_0w_1 \ldots w_{p-1} \) be a block of length \( p \). The least period of \( w \) is the smallest integer \( q \) such that \( w = (w_0w_1 \ldots w_{q-1})^m \) where \( m = \frac{p}{q} \) must be an integer. The block \( w \) is primitive if its least period equals its length \( p \).

Now we set up to picture out the graph \( M_G \) of \( X(S) \). First suppose \( |S| < \infty \). Let \( S = \{s_0, s_1, \ldots, s_{k-1}\} \subseteq \mathbb{N}_0, k > 1 \) and

\[
(3.2) \quad D(S) = d_1d_2 \cdots d_{k-2}(d_{k-1} + s_0 + 1)
\]

where \( d_i = s_i - s_{i-1}, 1 \leq i \leq k - 1 \). Note that if \( I, J \in \mathcal{V}(G) \) are in the same state of \( M(G) \), then both of \( I \) and \( J \) have the same out-degree which is one or two. Also the out-degree of any vertex \( F(10^n), 0 \leq i \leq k - 1 \) is two except that the last one. Hence \( d_i, 1 \leq i \leq k - 2 \) measures the distance between any two vertices with out-degree two.

To pick the next vertex after \( F(10^{s_k-2}) \) with out-degree two we continue to right to \( F(10^{s_k-1}) \) and then along the graph to \( F(1) \) and then again to right to \( F(10^{s_0}) \) that is after \( d_{k-1} + s_0 + 1 \) steps.

**Theorem 3.3.** Let \( |S| < \infty \). Then \( D(S) \) is primitive if and only if \( M_G \cong G \).

**Proof.** Suppose \( D(S) \) is not primitive. Let \( \mathcal{V} = \mathcal{V}(M_G) \) be the set of states of \( M_G \). Then by the Fischer cover of \( X(S) \), each state in \( M_G \) consists of \( m = \frac{|S|-1}{q} \) vertices of graph \( G \) where \( q - 1 \) is the least period \( D(S) \) and \( |V| = \sum_{i=1}^{q-1} d_i = s_{q-1} - s_0 \). In fact, if \( \mathcal{V} = \{P_i : 0 \leq i \leq s_{q-1} - s_0 - 1\} \), then

\[
P_i = \{F(10^{s_0+i}), F(10^{s_0+i+|V|}), \ldots, F(10^{s_0+i+(m-1)|V|} \mod u)\}
\]

where \( u = s_{k-1} + 1 \). Since \( |V| = s_{q-1} - s_0 < s_{k-1} + 1 = |\mathcal{V}(G)|, M_G \not\cong G \).

Now suppose \( M_G \not\cong G \). So there are at least two different vertices of \( G \) say \( I = F(10^p) \) and \( J = F(10^q) \) such that \( I \) and \( J \) are in the same state of \( M_G \). Assume \( p < q \). There exists an edge from \( I \) (resp. \( J \)) to \( F(10^{(p+1)}) \) (resp. \( F(10^{(q+1)} \mod u) \)). Therefore, by the fact that \( I \) and \( J \) are equivalent, we have that the vertices \( F(10^{(p+1)}) \) and \( F(10^{(q+1)} \mod u) \) are equivalent. By the same reasoning, for each \( i \geq 2 \), \( F(10^{(p+i)} \mod u) \) and \( F(10^{(q+i)} \mod u) \) are equivalent. So \( D(S) \) is not primitive. \( \square \)

**Theorem 3.4.** Let \( X(S) \) be a sofic shift with \( |S| = \infty \) and the Fischer cover \( G = (G, \mathcal{L}) \). Then \( M_G \cong G \).

**Proof.** We consider the three cases appearing for \( |S| = \infty \) in subsection 2.1. We claim that the last vertex \( F(10^n(S)) \) is not equivalent with any other vertex. Otherwise, we will show that at least one of \( k \) or \( l \) will not be the least integer in \( (2.1) \).

So the state of \( M_G \) containing this last vertex, contains only this vertex which this in turn implies that other states of \( M_G \) also have one vertex. So \( M_G \cong G \).
We prove our claim for the most involved case, that is case (3). First suppose there is a vertex
\[(3.3) \quad v_0 = F(10^0) \sim F(10^n(S)), \quad s_{k-1} - g_{l-1} + 1 \leq t_0 < n(S).\]
Without loss of generality assume that this \(t_0\) is the largest integer with this property. Recall that there is an edge from \(F(10^n(S))\) to \(F(10^{n-1} - g_{l-1} + 1)\); so it is convenient to set \(t_1 := n(S), t_1 + 1 := s_{k-1} - g_{l-1} + 1\) and \(v_1 := F(10^n(S))\). By (3.3), \(v_2 := F(10^{i_1 + 1}) \sim F(10^n(S))\) and moving horizontally to right \(F(10^{i_1 + i}) \sim F(10^n(S)), 2 \leq i \leq t_1 - t_0\). Moreover, none of \(F(10^n(S))\) will be equivalent to \(v_0\), for this will violate the way we have picked \(t_0\). If \(v_2 \sim v_0\) we are done, for then \(l\) will not be the least integer. Observe that there are only finitely many vertices; therefore, there must be \(v_i \neq v_0, 2 \leq i < p\) and \(v_p \sim v_0\). Applying the same reasoning, we deduce that again \(l\) is not the least integer.

If \(F(10^n(S))\) is not equivalent to any vertex \(F(10^n)\) for \(s_{k-1} - g_{l-1} + 1 \leq t < n(S)\), it will be equivalent to \(F(10^{n+1} - g_{l-1})\). This implies \(k\) is not the least integer. □

Theorems 3.3 and 3.4 imply the following.

**Corollary 3.5.** Let \(X(S)\) be a sofic shift with the Fischer cover \(G = (G, \mathcal{L})\). Then any state of \(M_G\) has the same number of vertices of \(G\).

When \(|S| < \infty\), there may be cases with \(M_G \not\sim G\). The difference with \(|S| = \infty\) is that for \(|S| < \infty\), the last vertex \(F(10^n(S))\) has always out-degree one with label 1 while for \(|S| = \infty\), the label of edge starting from the vertex with out-degree one is 0.

Now let \(X\) be a sofic shift with the Fischer cover \(G = (G, \mathcal{L})\). Then by definition, \(\mathcal{L}_\infty\) is right-resolving and also it is almost invertible [13, Proposition 9.1.6]. So

**Lemma 3.6.** If \(X\) and \(Y\) are sofic with Fischer covers \(G_X = (G_X, \mathcal{L}_X)\) and \(G_Y = (G_Y, \mathcal{L}_Y)\) respectively, such that \(G_X \equiv G_Y\), then \(X\) and \(Y\) will be right-resolving almost conjugate with legs \(\mathcal{L}_\infty\) and \(\mathcal{L}_\infty\).

**Theorem 3.7.** Let \(X_\beta\) be a sofic \(\beta\)-shift for \(\beta \in (1, 2)\). Then there is \(S \subseteq \mathbb{N}_0\), determined in terms of coefficients of \(1_\beta\), such that \(X_\beta\) and \(X(S)\) are right-resolving almost conjugate. Moreover, if \(X_\beta\) is SFT, then \(X(S)\) can be chosen to be conjugate to \(X_\beta\).

**Proof.** For a given sofic \(\beta\)-shift, \(\beta \in (1, 2)\), we claim that there is \(S \subseteq \mathbb{N}_0\) such that the \(S\)-gap shift \(X(S)\) and \(X_\beta\) have the same underlying graph for their Fischer covers. Then by Lemma 3.6, \(X_\beta\) and \(X(S)\) will be right-resolving almost conjugate.

Let \(1_\beta = a_1 a_2 \cdots a_n (a_{n+1} \cdots a_{n+p})^\infty\) and \(\{i_1, i_2, \ldots, i_t\} \subseteq \{1, 2, \ldots, n\}\) where \(a_{i_v} = 1\) for \(1 \leq v \leq t\). Note that \(i_1\) is always 1. Similarly, let \(\{j_1, j_2, \ldots, j_u\} \subseteq \{n+1, \ldots, n+p\}\) where \(a_{j_w} = 1\) for \(1 \leq w \leq u\). We consider two cases:

1. \(X_\beta\) is SFT. In this case \((a_{n+1} \cdots a_{n+p})^\infty = 0^\infty\) and \(a_n = 1\). So \(i_t = n\) and \(X(S)\) with
\[(3.4) \quad S = \{0, i_2 - 1, \ldots, i_{t-1} - 1, i_t - 1\}.
\]
is the required \(S\)-gap shift as has been claimed. Since both \(X_\beta\) and \(X(S)\) are SFT with the same underlying graph \(G\) for their Fischer covers, they are both conjugate to \(X_G\) [13, Theorem 3.4.17], and so conjugate to each other.
(2) $X_\beta$ is strictly sofic. Then $(a_{n+1} \cdots a_{n+p})^\infty \neq 0^\infty$. Relabel any edge on $G_\beta$ ending at the first vertex for 1 and other edges for 0. Shift space corresponding to this labeling is an $S$-gap shift where

$$S = \{0, i_2 - 1, \ldots, i_t - 1, j_1 - 1, \ldots, j_u - 1, j_1 + p - 1, \ldots\}.$$  

(Observe that then

$$\Delta(S) = \{0, i_2 - 1, \ldots, i_t - i_{t-1}, j_1 - i_t, j_2 - j_1, \ldots, j_u - j_{u-1}, j_1 - j_u + p\}$$

Which shows that $X(S)$ is sofic [1, Theorem 3.4].)  

Rewrite $\Delta(S)$ in (3.6) as

$$\Delta(S) = \{0, d_1, \ldots, d_t, g_0, \ldots, g_u - 1\}.$$  

We claim that $G_S = (G_S, L_S)$ is follower-separated. Otherwise, there are two cases.

(a) There is $1 \leq i \leq t$ such that $d_{t+1-j} = g_{u-j}, 1 \leq j \leq i$. Then $G_\beta = (G_\beta, L_\beta)$ is not follower-separated and so it is not the Fischer cover of $X_\beta$ which is absurd.  

(b) $g_0g_2\cdots g_{u-1}$ is not primitive. This implies that $a_{n+1} \cdots a_{n+p}$ is not primitive which is again absurd.  

This establishes the claim and $S$ is completely determined.  

Now the following is immediate.

**Corollary 3.8.** Let $X_\beta$ be a sofic shift whose underlying graph of its Fischer cover is $G$. Then $M_G \cong G$.

**Proof.** Suppose $M_G \not\cong G$. For this $X_\beta$, find the $S$-gap shift satisfying the conclusion of Theorem 3.7. Then by Theorem 3.4, this $X(S)$ (as well as our $X_\beta$) must be SFT and $D(S)$ is not primitive. But this will not allow to have (2.3) which is a necessary condition.  

**Lemma 3.9.** Let $|S| = \infty$ and $X(S)$ be a sofic shift satisfying (1a) in Subsection 2.1. Then there does not exist any $\beta$-shift being right-resolving finite equivalent with $X(S)$.

**Proof.** Suppose there is some $\beta \in (1, 2]$ such that $X(S)$ and $X_\beta$ are right-resolving finite equivalent and $G_S = (G_S, L_S)$ and $G_\beta = (G_\beta, L_\beta)$ are the Fischer covers of $X(S)$ and $X_\beta$ respectively. By Theorem 3.4 and Corollary 3.8, $G_S \cong G_\beta$. Then $G_\beta$ is the underlying graph of $G_S$ and $1_\beta = (a_1a_2\cdots a_n)^\infty$.

Now by hypothesis, $g_{t-1} = 1$, so $1 \notin S$ and this implies that $a_2 = 0$ while $a_1 = a_n = 1$. This means $(a_1a_2\cdots a_n)^\infty$ does not satisfy (2.3) and we are done.  

Let $X(S)$ be an $S$-gap shift where $s_0 = 0$ and $d_i = s_i - s_{i-1}, i \in \mathbb{N}$ and also $D(S)$ as (3.2). Define

$$d_1d_2d_3\cdots = \begin{cases} (d_1d_2\cdots (d_{k-1} + 1))^N = (D(S))^N, & |S| = k; \\ d_1d_2\cdots, & |S| = \infty. \end{cases}$$

**Theorem 3.10.** Suppose $X(S)$ is a sofic shift. Then $X(S)$ is right-resolving almost conjugate to a $\beta$-shift if and only if

$$d_n\cdots d_{n+1} \geq d_1d_2\cdots$$

for all $n \geq 1$. 


Proof. Let $\beta \in (1, 2]$ with $1 = a_1 a_2 \cdots$ be so that $X(S)$ and $X_\beta$ are right-resolving almost conjugate. This means they are right-resolving finite equivalent. First suppose $M_{G_\beta} \cong G_\beta$. By Corollary 3.8, $G_\beta \cong G_\beta$ and so (3.8) follows from the fact that $a_1 a_2 \cdots$ satisfies (2.3).

If $M_{G_\beta} \not\cong G_\beta$, then by Theorems 3.3 and 3.4, $|S| < \infty$. So $X_\beta$ is right-resolving finite equivalent to $X(S')$ with $S' = \{0, s_1, \ldots, (s_q - 1)\}$ and $D(S) = D(S')^m$ where $m = \frac{|S| - 1}{2}$ as in the proof of Theorem 3.3. Moreover, $M_{G_\beta} \cong G_\beta$ which gives again $d_n d_{n+1} \cdots \geq d_1 d_2 \cdots$ for all $n \geq 1$. Now this fact reflects to $D(S)$ and (3.8) holds.

To prove the sufficiency suppose $G_\beta = (G_\beta, L_\beta)$ is the Fischer cover of $X(S)$ and $V = V(G_\beta)$ the set of vertices of $G$. Relabel $G_\beta$ by labeling 0 any edge terminating at vertex $F(1)$ and any edge whose initial vertex has out-degree 1, and assign 1 all other edges.

Recall that we have lined up the vertices horizontally from $F(1)$ in left to $F(10^n(S))$ on right. First let $|S| < \infty$ and $a_1 a_2 \cdots a_n(S)$ be the assigned label of the horizontal path from $F(1)$ to the last vertex with $a_i = 0$ or 1 as determined above. Then (3.8) implies that $a_1 a_2 \cdots a_n(S)1$ is the $\beta$-expansion of 1 for some $\beta \in (1, 2]$ and $G_\beta$ is the Fischer cover of $X_\beta$.

When $|S| = \infty$, assign the label $a_1 a_2 \cdots a_n(S)$ to the horizontal path from $F(1)$ to the last vertex and label $a_n(S) + 1$ to the edge starting from $F(10^n(S))$ and terminating at $F(10^n(S)+1)$. Again (3.8) implies that $a_1 a_2 \cdots a_n(a_n(S)+1) \infty$ is the $\beta$-expansion of 1 for some $\beta \in (1, 2]$ where the index $n$ depends on $S$. Then $G_\beta$ is the Fischer cover of $X_\beta$ (one needs similar arguments as in the proof of Theorem 3.7 to see this fact). So Lemma 3.6 implies that $X(S)$ and $X_\beta$ are right-resolving almost conjugate.

Let $S = \{n, n + 1, \ldots\}$, $n \geq 2$ and $1 = 10^n$. Then $X_S$ and $X_\beta$ are right-resolving almost conjugate; however, the condition (3.8) does not hold. This is not a contradiction, for we are considering $X(S)$ up to conjugacy and in this exceptional case, we consider $X(S') = \text{ASS}(X_\beta)$ where $S' = \{0, n\}$.

Remark 3.11. $X_\beta$ can be explicitly determined in terms of $S$. If $S = \{0, s_1, \ldots, s_k - 1\}$, then it is sufficient to set $1 = a_1 a_2 \cdots a_{k+1}$ such that $a_1 = a_{s+1} = 1$, $1 \leq i \leq k - 1$. When $|S| = \infty$, different cases of Subsection 2.1 must be considered. Case (1a) has been ruled out by Lemma 3.9, so other cases will be considered.

1. If $k = 1$ and $g_{k-1} > 1$, then $F(10^g) = F(1)$. So $\beta = a_1 a_2 \cdots a_s$ such that $a_{s+1} = 1$, $0 \leq i \leq l - 1$.
2. If $k \neq 1$ and $g_{k-1} > d_{k-1}$, then $F(10^{g_{k-1}+2}) = F(10^{g_{k-1}+2})$. So $\beta = a_1 a_2 \cdots a_{k-2}(a_{k-2}+1) \cdots a_{g+g_{k-2}+1} \infty$ which $a_{s+1} = 1$, $0 \leq i \leq k + l - 2$.
3. If $g_{k-1} \leq d_{k-1}$, then $F(10^{g_{k-1}+2}) = F(10^{g_{k-1}+2})$. So $\beta = a_1 a_2 \cdots a_{s+1-2}(a_{s+1-2}+1) \cdots a_{s+1} \infty$ which $a_{s+1} = 1$, $0 \leq i \leq k + l - 2$ and $a_{s+1} = 1$.

Now we show that the conclusion of Theorem 3.7 about conjugacy is not true in non-SFT case. Recall that when $X$ is a shift space with non-wandering part $R(X)$, we can consider the shift space

$$\partial X = \{x \in R(X) : x \text{ contains no words that are synchronizing for } R(X)\};$$
which is called the derived shift space of $X$. The derived shift space is a conjugacy invariant.

**Theorem 3.12.** A non-SFT $\beta$-shift is not conjugate to an $S$-gap shift for any $S \subseteq \mathbb{N}_0$.

**Proof.** All the $S$-gap shifts are synchronized; therefore, a possible conjugacy happens between synchronized $\beta$ and $S$-gap shifts and so we assume that our non-SFT $\beta$-shift is synchronized.

Suppose that there is $S \subseteq \mathbb{N}_0$ such that $\varphi : X(S) \to X_\beta$ is a conjugacy map. Then by [18, Proposition 4.5], we must have $\varphi(\partial X(S)) = \partial X_\beta$. Since $1$ is a synchronizing word for any $S$-gap shift, and $X(S)$ is not SFT, $\partial X(S) = \{0^\infty\}$ (for an SFT $S$-gap shift, $\partial X(S) = \emptyset$). To prove the theorem, we show that

$$\varphi(\{0^\infty\}) \neq \partial X_\beta. \tag{3.9}$$

Recall that the $\omega$-limit set of the sequence $1_\beta$ under the shift map is the derived shift space $\partial X_\beta$ [19, Theorem 2.8]. First assume that $X_\beta$ has specification property. Then there exists some $n \geq 0$ such that $0^n$ is not a factor of $1_\beta$ [4]. So $0^n$ is a synchronizing word for $X_\beta$ [4, Proposition 2.5.2] and $0^\infty \notin \partial X_\beta$. Therefore, $\partial X_\beta \cap P_1(X_\beta) = \emptyset$ while $\varphi(0^\infty) \in P_1(X_\beta)$ and $\varphi(0^\infty) \in \varphi(\partial X(S)) = \partial X_\beta$ and (3.9) holds.

If $X_\beta$ does not have specification, then $\{0^\infty, 10^\infty\} \subseteq \omega(1_\beta) = \partial X_\beta$ and again (3.9) holds. \hfill \Box

**Corollary 3.13.** Let $X_\beta$ be SFT and $X(S_0)$ the unique $S$-gap shift conjugate to $X_\beta$ (Theorem 3.7). Then $X_\beta$ is

1. right-resolving almost conjugate to $X(S_0)$,
2. right-resolving finite equivalent to infinitely many $S$-gap shifts $(X(S_n))_{n \in \mathbb{N}}$ with $D(S_n) = (D(S_0))^{n+1}, n \in \mathbb{N}$,
3. right-resolving almost conjugate to a unique strictly sofic $S$-gap shift.

If $X_\beta$ is strictly sofic, then it is right-resolving almost conjugate to a unique $S$-gap shift.

**Proof.** Let $X_\beta$ be SFT and let $1_\beta = a_1a_2 \cdots a_{n-1}a_n$ and

$$\{i_1, i_2, \ldots, i_t\} \subseteq \{1, 2, \ldots, n\}$$

where $a_{i_j} = 1$, $1 \leq j \leq t$. We will relabel the Fischer cover of $X_\beta$ for possible presentation of an $S$-gap shift.

One of such SFT $S$-gap shifts is $X(S_0)$ characterized in the proof of Theorem 3.7. By that theorem, $X_\beta$ and $X(S_0)$ are right-resolving almost conjugate and conjugate which gives (1). For (2) relabel $\Delta(S_0) = \{0, s_2-1, s_3-s_2, \ldots, s_t-s_{t-1}\}$ as $\Delta(S_0) = \{0, d_1, \ldots, d_{i_t-1}\}$ and observe that $D(S_0) = d_1 \cdots d_{i_t-1}(d_{i_t-1} + 1)$. Set $\mathcal{H}(S_0) := \{0, d_1, d_1 + d_2, \ldots, \sum_{i=1}^{i_t-1} d_i\}$ and let

$$S_1 = (S_0 \setminus \{i_t - 1\}) \cup (i_t + \mathcal{H}(S_0)).$$

Then $D(S_1) = (D(S_0))^2$ is not primitive and we have $M_{G_{S_1}} \cong M_{G_{S_0}}$.

Now for $j \in \mathbb{N}$, let $s_{i_j} = \max\{s : s \in S_{j-1}\}$ and use an induction argument to see that for

$$S_j = (S_{j-1} \setminus \{s_{i_j}\}) \cup (s_{i_j} + 1 + \mathcal{H}(S_0)),$$

$D(S_j) = (D(S_0))^{j+1}$ and $M_{G_{S_j}} \cong M_{G_{S_0}}$.

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To prove (3) note that there is an strictly sofic \( S \)-gap shift with \( k = 1 \) and \( g_{k-1} > 1 \) as in Subsection 2.1 where \( S = \{0, i_2 - 1, \ldots, i_{k-1} - 1, i_k, i_k + i_2 - 1, \ldots\} \). The element \( i_t \) appears in \( S \) because the edge starting from the last vertex and terminating at the first vertex is labeled 0. In fact

\[
\Delta(S) = \{0, i_2 - 1, i_3 - i_2, \ldots, i_{t-1} - i_{t-2}, i_t - i_{t-1} + 1\}.
\]

Hence \( X_\beta \) and \( X(S) \) have the same underlying graph for their Fischer covers and by Lemma 3.6, they are right-resolving almost conjugate.

If there is another \( S \)-gap shift such that \( X_\beta \) and \( X(S) \) are right-resolving finite equivalent, then \( M_{G_\beta} \cong M_{G_S} \) and so \( M_{G_{S_0}} \cong M_{G_S} \). Now Theorems 3.3 and 3.4 imply that \( |S| < \infty \) and \( D(S) \) is not primitive which in turn implies that \( D(S) = (D(S_0))^m \) for some \( m \in \mathbb{N} \). Therefore, \( S = S_{m-1} \) as defined in (3.10).

Now suppose \( X_\beta \) is strictly sofic. A typical Fischer cover of \( X_\beta \) has been shown in Figure 3.1. The existence of loop in the first vertex from left implies that it is the vertex \( F(1) \) in the Fischer cover of \( S \)-gap shift. By Fischer cover of \( S \)-gap shifts [2], there is only one \( X(S) \) with the Fischer cover as appears in Figure 3.1. \( \square \)

![Figure 3.1: A typical Fischer cover of an strictly sofic \( \beta \)-shift for \( 1_\beta = a_1 a_2 \cdots a_n (a_{n+1} \cdots a_{n+p})^{-} \), \( \beta \in (1, 2) \). The edges heading to \( \alpha_1 \) exist if \( a_i = 1 \).](image)

It is not hard to see that for a non-sofic \( \beta \) shift, like strictly sofic case, there exists a unique \( S \)-gap satisfying (3.8) and having the same underlying graph for its Fischer cover as our \( X_\beta \). This motivates the following definition.

**Definition 3.14.** We say that \( X(S) \) is the associated \( S \)-gap shift to a \( \beta \)-shift and is denoted by \( \text{ASS}(X_\beta) \), \( \beta \in [1, 2) \) if

- \( X_\beta \) is SFT and conjugate to \( X(S) \), or
- \( X_\beta \) is not SFT and has the same underlying graph for its Fischer cover as \( X(S) \).

Similarly, for \( X(S) \) satisfying (3.8), a unique \( X_\beta \) exists such that \( X(S) = \text{ASS}(X_\beta) \). This \( X_\beta \) is called the associated \( \beta \)-shift to \( X(S) \) and is denoted by \( \text{ASS}(X(S)) \).

Therefore, \( \text{ASS}(\text{ASS}(X_\beta)) = X_\beta \) and \( \text{ASS}(\text{ASS}(X(S))) = X(S) \) for \( \beta \in [1, 2) \) and \( S \) satisfying (3.8).

**Remark 3.15.** \( X_\beta \) and \( \text{ASS}(X_\beta) \) in Theorem 3.7 have all equivalencies given in diagram (3.1) when they are both SFT and all except conjugacy when they are strictly sofic.

### 4. Common Properties between a Beta-shift and its Associate, Non-sofic Case

By [11, Theorem 4.22], for every \( \beta > 1 \) there exists \( 1 < \beta' < 2 \) such that \( X_\beta \) and \( X(\beta') \) are flow equivalent. But any two flow equivalent shift spaces have the
same Bowen-Franks groups. Therefore, by Theorem 3.7 and [2, Theorems 3.1 and 3.2] which gives a complete account of the Bowen-Franks group of S-gap shifts, we have also a complete characterization of such groups for β-gap shifts for β > 1.

Also when $X_β$ is sofic shift with Fischer cover $G_β = (G_β, L_β)$, by Theorems 3.7 and [2, Theorem 2.2] we can determine the characteristic polynomial of $G_β$ and so we have all eigenvalues.

Now we consider non-sofic β and S-gap shifts. By Theorem 3.12, no conjugacy occurs between a β-shift and any S-gap shift and hence we set up to look for other equivalencies, and in particular, similar to those in diagram (3.1).

First recall that any β-shift is half-synchronized [9] whereas any S-gap shift is synchronized. So the Fischer covers $G_β = (G_β, L_β)$ and $G_S = (G_S, L_S)$ exist (see Figure 4.1).

Let $1_β = a_1 a_2 \cdots$. Relabel $G_β$ by labeling 1 any edge terminating at vertex $α_1$ and 0 all other edges to get an S-gap shift with the same underlying graph as $X_β$. Also using Theorem 3.10, one can relabel a non-sofic S-gap shift satisfying (3.8) to obtain a β-shift. In both cases, $X_S = ASS(X_β)$.

![Figure 4.1: A typical Fischer cover of a non-sofic β-shift for $1_β = a_1 a_2 \cdots$, $β \in (1, 2]$. The edges heading to $α_1$ exist if $a_i = 1$. This cover can be relabeled to give the Fischer cover of ASS($X_β$).](image)

**Theorem 4.1.** $h(X_β) = h(ASS(X_β)), β \in (1, 2]$.

**Proof.** Entropy is an invariant for all the properties given in diagram (3.1). So when $X_β$ is sofic, the proof is obvious (Theorem 3.7).

Now let $X_β$ be a non-sofic shift and let $1_β = a_1 a_2 \cdots$. We have $a_i = 1$ and only if $i - 1 \in S$. But for $1_β = \sum_{i=1}^{∞} a_i β^{-i}$, $h(X_β) = \log β$ and $h(X(S)) = \log λ$ where $λ$ is a nonnegative solution of $\sum_{n \in S} x^{-(n+1)} = 1$ [16]. So $h(X_β) = h(ASS(X_β))$. □

Let $X$ and $Y$ be two coded systems. Then there is a coded system $Z$ which factors onto $X$ and $Y$ with entropy-preserving maps if and only if $h(X) = h(Y)$. In particular, $Z$ can be chosen to be an almost Markov synchronized system [8, Theorem 2.1]. So Theorem 4.1 implies the following.

**Corollary 4.2.** $X_β$ and $ASS(X_β)$ have a common extension which is an almost Markov synchronized system whose maps are entropy-preserving.

For a dynamical system $(X, T)$, let $p_n$ be the number of periodic points in $X$ having period $n$. When $p_n < ∞$, the zeta function $ζ_T(t)$ is defined as

$$ζ_T(t) = \exp \left( \sum_{n=1}^{∞} \frac{p_n}{n} t^n \right).$$

The zeta functions of β-shifts have been determined in [10]. Here we will give the zeta function of $ζ_β$ in terms of $ζ_σ$, where $X(S) = ASS(X_β)$.

Let $X_β$ be sofic and $1_β = a_1 a_2 \cdots a_n (a_{n+1} \cdots a_{n+p})^∞$ such that

$\{i_1 = 1, i_2, \ldots, i_l\} \subseteq \{1, 2, \ldots, n\}$, and $\{j_1, j_2, \ldots, j_u\} \subseteq \{n + 1, \ldots, n + p\}$
where $a_{iv} = a_{jw} = 1$ for $1 \leq v \leq t$, $1 \leq w \leq u$. Now we have the following.

**Theorem 4.3.** Let $X(S) = \text{ASS}(X_\beta)$ for some $\beta \in (1, 2]$. If $X_\beta$ is SFT, then

\begin{equation}
\zeta_{\sigma_\beta}(r) = \zeta_{\sigma_S}(r).
\end{equation}

If $X_\beta$ is not SFT, then

\begin{equation}
\zeta_{\sigma_\beta}(r) = (1 - r)\zeta_{\sigma_S}(r).
\end{equation}

Furthermore, in the case of SFT,

\begin{equation}
\zeta_{\sigma_\beta}(r) = \frac{1}{1 - r^{i_1} - r^{i_2} - \ldots - r^{i_t}}.
\end{equation}

and for strictly sofic,

\begin{equation}
\zeta_{\sigma_\beta}(r) = \frac{1}{(1 - r^{i_1} - r^{i_2} - \ldots - r^{i_t})(1 - r^p) - (r^{j_1} + \ldots + r^{j_u})}.
\end{equation}

**Proof.** First let $X_\beta$ be an SFT shift. Also let $S = \{0, i_2 - 1, \ldots, i_{t-1} - 1, i_t - 1\}$; then by Theorem (3.7), $X(S) = \text{ASS}(X_\beta)$. Since $X_\beta$ and $X(S)$ are conjugate, they have the same zeta function, that is

\[ \zeta_{\sigma_\beta}(r) = \frac{1}{f_S(r^{-1})} = \frac{1}{1 - r^{i_1} - r^{i_2} - \ldots - r^{i_t}} \]

where $f_S(x) = 1 - \sum_{s_n \in S} \frac{1}{x^{n+1}}$ [2, Theorem 2.3].

Now suppose $X_\beta$ is an strictly sofic shift and let $1_\beta = a_1 a_2 \cdots a_n (a_{n+1} \cdots a_{n+p})^\infty$. An arbitrary periodic point $x \in X_\beta$ has one presentation in $G_\beta$ unless

\[ x = (a_{n+1} \cdots a_{n+p})^\infty \]

where then it has exactly two presentations. This fact can be deduced from the proof of [11, Proposition 4.7]. So if $m = pk$ ($k \in \mathbb{N}$), then every point in $X_G$ of period $m$ is the image of exactly one point in $X_G$ of the same period, except $p$ points in the cycle of $(a_{n+1} \cdots a_{n+p})^\infty$ which are the image of two points of period $m$. As a result, $p_m(\sigma_{G_\beta}) = p_m(\sigma_{G_\beta}) - p$ where $p_m = |P_m|$ and $P_m$ is the set of periodic points of period $m$. When $p$ does not divide $m$, $p_m(\sigma_{G_\beta}) = p_m(\sigma_{G_\beta})$. Therefore,

\[ \zeta_{\sigma_\beta}(r) = \exp \left( \sum_{m=1}^{\infty} \frac{p_m(\sigma_{G_\beta})}{m} r^m + \sum_{m=1}^{\infty} \frac{p_m(\sigma_{G_\beta}) - p}{m} r^m \right) \]

\[ = \exp \left( \sum_{m=1}^{\infty} \frac{p_m(\sigma_{G_\beta})}{m} r^m - \sum_{m=1}^{\infty} \frac{pr^m}{m} \right) \]

\[ = \zeta_{\sigma_{G_\beta}}(r) \times (1 - r^p). \]

But $G_\beta \cong G_S$ for $S$ as in (3.5). Therefore by [2, Theorem 2.3],

\[ \zeta_{\sigma_{G_\beta}}(r) = \frac{1}{(1 - r^p)f_S(r^{-1})} \]

\[ = \frac{1}{(1 - r^{i_1} - r^{i_2} - \ldots - r^{i_t})(1 - r^p) - (r^{j_1} + \ldots + r^{j_u})}. \]

It remains to consider the case when $X_\beta$ is not sofic. We claim that $P_n(X_S) = P_n(X_\beta) + 1$ for all $n \in \mathbb{N}$. 

**EQUIVALENCIES BETWEEN BETA-SHIFTS AND S-GAP SHIFTS**
Observe that one may assume that the initial vertex of $\pi$, a cycle in the graph of $G_\beta$, is $\alpha_1$ as in Figure 4.1. Now let $x = v^\infty \in P_n(X_\beta)$ with $v = v_1 \cdots v_n \in B_n(X_\beta)$. Pick $\pi_\beta$ a cycle in $G_\beta$ such that $v = L_\beta(\pi_\beta)$ and set $\pi_S$ to be the associated cycle to $\pi_\beta$ in $G_S$ and also let $w = L_\beta(\pi_S)$. Then $w^\infty \in P_n(X(S))$. Now define $\phi_n : P_n(X_\beta) \setminus P_1(X(S)) \to P_n(X(S)) \setminus P_1(X(S))$ for all $n \geq 2$ such that $\phi_n(v^\infty) = w^\infty$.

Clearly, $\phi_n$ is well-defined. Also it is one-to-one; otherwise, for $w^\infty \in P_n(X(S))$, there are two different cycles $\pi_S$ and $\gamma_S$ such that $w = L_\beta(\pi_S) = L_\beta(\gamma_S)$. Let $a_i$ and $a_j$ be the rightmost occurrence of the edges that represent the coefficients of $\alpha_\beta$ in $\pi_S$ and $\gamma_S$ respectively. Then adopting notation in Figure 4.1, the terminal vertex of the edge representing $a_i$ (resp. $a_j$) is $\alpha_{i+1}$ (resp. $\alpha_{j+1}$).

By Fischer cover of $X(S)$, $\alpha_{i+1} \neq \alpha_{j+1}$. Suppose $i < j$ and $L_\infty = L_\beta \infty$. Then $L_\infty(\pi_\beta^\infty) = L_\infty(\gamma_\beta^\infty)$ implies that $a_{i+1} = 1$. On the other hand, there is another edge starting at $\alpha_{i+1}$ and terminating at $\alpha_1$ whose label is 1. Since the Fischer cover is right-resolving, it is a contradiction and so $\phi_n$ is one-to-one. But $P_1(X_\beta) = \{0^\infty\}$ while $P_1(X(S)) = \{0^\infty, 1^\infty\}$. So the claim is proved and we have

$$\zeta_{\pi_\beta}(r) = \exp \left( \sum_{m=1}^{\infty} \frac{p_m(\sigma_{\pi_\beta})}{m} r^m \right) = \exp \left( \sum_{m=1}^{\infty} \frac{p_m(\sigma_{\pi_\beta}) + 1}{m} r^m \right) = \zeta_{\pi_\beta}(r) \times \frac{1}{1 - r}.$$

\[\square\]

Now we investigate the frequency of associated $S$-gap shifts in the space of all $S$-gap shifts by using topology of $S$-gap shifts given in [1]. This topology is obtained by assigning a real number $x_S = [d_0; d_1, d_2, ...]$, where $[d_0; d_1, d_2, ...]$ is the continued fraction expansion of $x_S$, to any $X(S)$ with $d_0 = s_0$ and $d_n = s_n - s_{n-1}$. By that, a one-to-one correspondence between the $S$-gap shifts up to conjugacy and $R = \mathbb{R}^{\geq 0} \setminus \{ \frac{1}{n} : n \in \mathbb{N} \}$, up to homeomorphism, will be established and the subspace topology of $R$ together with its measure structure will be induced on the space of all $S$-gap shifts.

**Theorem 4.4.** Let $S$ be the set of all $S$-gap shifts associated to some $X_\beta$. Then $S$ is a Cantor set on the space of all $S$-gap shifts (a nowhere dense perfect set). Entropy is a complete invariant for the conjugacy classes of $S$.

**Proof.** First suppose $X(S)$ does not satisfy (3.8) and $x_S = [d_0; d_1, ...]$ corresponds to $X(S)$ [1]. Let $N$ be the least integer such that $d_N d_{N+1} \cdots < d_1 d_2 \cdots$ and set $\gamma_i := [d_0; d_1, ..., d_i]$, $i \in \mathbb{N}_0$. If $N$ is even, set $U = (\gamma_N, \gamma_{N+1})$ and otherwise, $U = (\gamma_{N+1}, \gamma_N)$. Then no points of $U$ satisfies (3.8) and so none is an associated $S$-gap shift. This shows $S$ is closed.

Now let $X(S) \in S$ and $V$ be a neighborhood of $x_S$. Note that two real numbers are close if sufficiently large numbers of their first partial quotients in their continued fraction expansion are equal. So we can select two points $x_S', x_S'' \in V$ such that $X(S')$ satisfies (3.8) and $X(S'')$ does not satisfy (3.8). This implies that all points of $S$ are limit points of themselves and $S$ is nowhere dense.

The second part follows from the fact that the entropy is a complete invariant for the conjugacy classes of $\beta$-shifts. \[\square\]

The interesting equivalencies happening in case of sofic is when the two systems under investigation have a common SFT extension. This clearly cannot be done for
non-sofic. The most natural extension of this idea is when two non-sofic systems have a common synchronized extension and in particular, when the legs are right-resolving. This has been studied in [7] and [8]. Recall that when $X$ is a synchronized system with Fischer cover $\mathcal{G} = (G, \mathcal{L})$, then $\mathcal{L}_\infty$ is a.e. 1-1 [9]. So for our case we have the following.

**Theorem 4.5.** $X_\beta$ and $\text{ASS}(X_\beta)$ have a common synchronized 1-1 a.e. extension with right-resolving legs.

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