On the continuity of Weil-Petersson volumes of the moduli space weighted points on the projective line

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Abstract
In this work we show that the Weil-Petersson volume (which coincides with the CM degree) in the case of weighted points in the projective line is continuous when approaching the Calabi-Yau geometry from the Fano geometry. More specifically, the CM volume computed via localization converges to the geometric volume, computed by McMullen with different techniques, when the sum of the weights approaches the Calabi-Yau geometry.

1 Introduction
When considering a moduli space of weighted pairs it becomes interesting to study how such moduli space changes towards the geography of the weights. In this direction, we study the behaviour of the volume function of the moduli space of points in $\mathbb{P}^1$. We denote the latter by $M_d$, where $d$ denotes the weight vector $(d_1, ..., d_n)$ of a configuration of $n$-points in $\mathbb{P}^1$. We assume, henceforth, that the components of such vector are rational numbers in the open interval $(0, 1)$. A point of $M_d$ is a pair $(\mathbb{P}^1, \sum_{i=1}^n d_ip_i)$, where $p_i$ are points in general position in $\mathbb{P}^1$, $\forall \{1, 2, ..., n\}$. The sum of the weights determine the nature of the pair. Namely, given the pair $(\mathbb{P}^1, \sum_{i=1}^n d_ip_i)$, a choice of the rational weights correspond to three distinguished geometries

\[
\begin{align*}
\sum_{i=1}^n d_i &< 2 \quad \log \text{Fano}, \\
\sum_{i=1}^n d_i &\geq 2 \quad \log \text{Calabi-Yau}, \\
\sum_{i=1}^n d_i &> 2 \quad \log \text{General type}.
\end{align*}
\]

We want to study how the Weil-Petersson volume (induced by canonical constant curvature metrics) of the moduli space of $n$ points in the projective line varies in the above geometries, henceforth denoted by $M_d$. This study is motivated by the relations among the first Chern class of the CM line bundle and the Weil-Petersson metric. The aim of this work is to show that the CM volume of $M_d$

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converges to the geometric volume computed by McMullen in [33] (Theorem 8.1), when the sum of the weights approaches the log Calabi-Yau geometry. To do so, we take the below explicit description of $M_d$ as GIT quotient

$$M_d = (\mathbb{P}^1)^n//\mathbb{C}^*\text{SL}(2),$$

where $d = (d_1,...,d_n)$ is a weight vector. This compact space parametrizes configurations of points in $\mathbb{P}^1$ which carry canonical metrics. Indeed, when $\sum_{i=1}^{n} d_i = 2$, it parametrizes flat metrics (with cone angle) thanks to the work of [32], [29]. When $\sum_{i=1}^{n} d_i < 2$, by works of Li in [72] and Fujita in [80], the latter moduli space is actually a Fano K-moduli space, parametrizing K-polystable Fano pairs (positive constant curvature metric). Finally, when $\sum_{i=1}^{n} d_i = 2 + \epsilon$, and generic then it is a Hassett/KSBA type moduli space ([11] Theorem 5.5.2), ([25] Theorem 1.5]) of constant negative curvature metric.

Volumes of $M_d$ have been computed in many works, via localisation by [38], [35], [37], [34], using the isomorphism with the moduli space of polygons. The correspondence between the Weil-Petersson volume and the degree of the CM line bundle it is shown in [77] in the absolute case, and the latter result is extended in the log case in [94].

In this work we will prove the following

**Theorem 1.1.** The volume of $M_d$ with $d_i \in (0,1) \cap \mathbb{Q}$, for all $i \in \{1, 2, ..., n\}$, converges to the volume computed by McMullen in [33] (Theorem 8.1) when the sum of the weights approaches 2.

The above proves the continuity of the volume function when passing from the Fano geometry to the Calabi-Yau geometry. The continuity result holds also when the sum of the weights is $2 + \epsilon$, due to [11] Theorem 5.5.2], [25] Theorem 1.5]. When the sum of the weights exceed significantly two, then we do not have the above explicit description of the moduli space of weighted points in $\mathbb{P}^1$. Namely, the natural K-moduli compactification is not simply given by the above GIT quotient. However, in this work, as an example, we fully examine the simple case of four points in $\mathbb{P}^1$ giving an explanation of the continuity of the volume function across the above mentioned geometries through the picture of $K$-stability.

We organised this work as follows

- In section 2 we recall the notion of CM line bundle for pairs and we relate it to the moduli of log Fano hyperplane arrangements.
- In section 3 we fully examine the case of four points on the projective line, by calculating the volume of the moduli space of four points in $\mathbb{P}^1$. We prove Theorem 1.1 in this case. Later we discuss the continuity of the volume function across the above mentioned three geometries.
- In section 4 we give a proof of Theorem 1.1 by using a continuity argument arising from the continuity of the construction of the Weil-Petersson metrics.
In the final remark, as a miscellanea we show that the CM volume can be expressed as the sum of Donaldson-Futaki invariant of some special test configurations.

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2 The log CM line bundle

2.1 The CM line bundle for pairs

Fix \( f : X \to B \) to be a proper flat morphism of scheme of finite type over \( \mathbb{C} \). Let \( L \) be a relatively ample on \( X \) and assume that \( f \) has relative dimension \( n \geq 1 \), namely \( \forall b \in B, \ (X_b, L_b) \) has constant dimension \( n \).

Definition 2.1. [91] Let \( D_i, \forall i = 1, 2, \ldots, k \) be a closed subscheme of \( X \) such that \( f|_{D_i} : D_i \to B \) has relative dimension \( n - 1 \), and \( f|_{D_i} \) is proper and flat. Let \( d_i \in [0, 1] \cap \mathbb{Q} \) we define the log CM \( \mathbb{Q} \)-line bundle of the data \( (f, D := \sum_{i=1}^{k} d_i D_i), L) \) to be

\[
\lambda_{CM,D} := \lambda_{CM} - \frac{n L_b^{n-1} \cdot D_b}{(L_b^n)} \lambda_{CH} + (n + 1) \lambda_{CH,D},
\]

Where \( \lambda_{CM} \) is the CM line bundle defined in Chapter two and \( \lambda_{CH} \) is the Chow line bundle, defined as the leading order term of the Hilbert Mumford expansion, namely \( \lambda_{CH} = \lambda_{n+1} \), and \( \lambda_{CH,D} = \bigotimes_{i=1}^{k} \lambda_{D_i}^{d_i} \).

We outline some easy and well known properties of the log CM line bundle, for a full description of the below facts see [91].

Proposition 2.1. Let \( f : (X, D) \to B \) be a \( \mathbb{Q} \)-Gorenstein flat family of \( n \) dimensional pairs over a normal proper variety \( B \). Then for any \( L \) relatively \( f \)-ample line bundle we have

\[
c_1(\lambda_{CM,D}) = n \frac{(-K_X - D_b) \cdot L_b^{n-1}}{(L_b^n)} f_* c_1(L)^{n+1} - (n+1) f_* ((-K_{X/B} - D) \cdot c_1(L)^n).
\]

Remark 2.1. In [90], there is another definition of the log CM line bundle for the case of one divisor with weight \( 1 - \beta \). And in the same work (Theorem 2.7
first assertion) there is a calculation of the first Chern class of the defined log CM line bundle. We point out that the definition we gave in this section it is exactly the same for the case of one divisor. Indeed, for the first assertion of Theorem 2.7 we just require to let

\[
\mu(L) := -\frac{K_X \cdot L^{n-1}}{(L)^n}, \quad \mu(L, D) := \frac{D \cdot L^{n-1}}{(L)^n}
\]

and of course \(D = (1 - \beta)D\).

The definition of the log CM line bundle given in the same work \[90\] definition 2.2 is also a functorial definition, namely

\[
\Lambda_{CM, D} := \lambda_n^{n(n+1) + \frac{2(1 - \beta)a_0}{n^2}} \otimes \lambda_{n+1}^{-2(n+1)} \otimes \tilde{\lambda}_{(1 - \beta)(n+1)}
\]

The element \(\tilde{\lambda}_n\) refers to the leading order term of the Hilbert-Mumford expansion for \(f_{!}|D|\), and \(\tilde{a}_0\) to the leading order term coefficient of the corresponding Hilbert polynomial, that is by the Hirzebruch-Riemann-Roch expansion

\[
\tilde{a}_0 = \frac{1}{n!} L_b \cdot D_b.
\]

Unravelling the other terms via the Hirzebruch-Riemann-Roch expansion, we obtain definition \[2.7\]

When computing the intersection number of the log CM line bundle on a \(\mathbb{Q}-\)Gorenstein flat family on the moduli of pairs, it is important to choose a ample line bundle on such family. In the state of the art of these possible choices, mainly we can make two of such. These choices are presented in the following

**Corollary 2.1.** Let \(f : (X, D) \to B\) be a \(\mathbb{Q}-\)Gorenstein flat family over a normal proper variety \(B\). We have

1. \[89\] Given \(L = -K_{X/B} - D\), then

\[
c_1(\lambda_{CM, D}) = -f_*(-K_{X/B})^{n+1}.
\]

2. \[90\] Given \(L = -K_{X/B}\), and \(D_{X_b} \in |-K_{X_b}|, \forall b \in B\), then

\[
c_1(\lambda_{CM, D}) = f_*(c_1(-K_{X/B})^n \cdot (c_1(-K_{X/B}) + \sum_{i=1}^k d_i((n+1)D_i - nC_1(-K_{X/B}))))
\]

Depending on the choice of \(L\) we will get different results in terms of intersection number on a generic curve into the moduli of pairs we want to study. In the below subsection, as an example, we will point out this difference when calculating the intersection number of the log CM line bundle with a generic \(\mathbb{P}^1\)-curve in the moduli space of log Fano hyperplane arrangements. Finally, we wish to make a remark on the log CM line bundle in the case of families of general type varieties. As a direct consequence of Proposition \[2.1\] we get the following
Corollary 2.2. Let \( f : (X,D) \to B \) be a \( \mathbb{Q} \)-Gorenstein family of log general type varieties of relative dimension \( n \), and with relatively ample line bundle \( L = K_{X/B} + D \). Then,

\[
c_1(\lambda_{CM}) = f_* c_1(K_{X/B} + D)^{n+1}
\]

Proof By applying directly Proposition 2.1 we get

\[
n\frac{(-K_X - D_b) \cdot (K_X + D)^{n-1}}{(K_{X_b} + D_b)^n} = -n.
\]

Thus,

\[
c_1(\lambda_{CM,D}) = -n f_* c_1(K_{X/B} + D)^{n+1} - (n + 1) f_* (- (K_{X/B} + D) c_1(K_{X/B} + D)^{n})
\]

\[
= -n f_* c_1(K_{X/B} + D)^{n+1} + (n + 1) f_* c_1(K_{X/B} + D)^{n+1}
\]

\[= f_* c_1(K_{X/B} + D)^{n+1}.
\]

\[
\square
\]

2.2 The intersection number of the log CM line bundle

In this section we compute the intersection number of the log CM line bundle with a family of log Fano hyperplane arrangements with base \( P^1 \). Consider the product \( X := P^n \times P^1 \), We define a divisor \( D \subset X \), as

\[
D := \sum_{i=1}^{m-1} d_i pr_1^* h + d_m (pr_1^* l + pr_2^* h)
\]

Where the \( pr_i \)'s, \( i = 1, 2 \) denotes the projections onto the first and second factor of \( X \). We can think about \( D \) as \( m - 1 \) fixed hyperplanes of \( P^n \) together with a line \( l \) with weight \( d_m \) free to move along the diagonal of \( X \). We assume that \( d_i \in (0,1) \cap \mathbb{Q}, \forall i \in \{1,2,\ldots,m\} \), \( \sum_{i=1}^{m} d_i < n + 1 \), and the chosen line and hyperplanes are in general position.

Proposition 2.2. With the above data we have

\[
c_1(\lambda_{CM,D}) = (n + 1) d_j \left(n + 1 - \sum_{i=1}^{m} d_i \right)^n, \forall j \in \{1,2,\ldots,m\}.
\]

Proof We see that the projection \( pr_2 : X \to P^1 \) gives a proper and flat family of relative dimension \( n \). Choose \( \mathcal{L} = -K_{X/B} - D \) and use assertion 1. of 2.1 we find that

\[
\mathcal{L} = \left(n + 1 - \sum_{i=1}^{m} d_i \right) pr_1^* h - d_m pr_2^* l
\]
Using the binomial expansion, the only term that survives is
\[ c_1(\lambda_{CM,D}) = -\text{pr}_2^*c_1(\mathcal{L})^{n+1} = (n + 1)d_m \left( n + 1 - \sum_{i=1}^{m} d_i \right)^n \text{pr}_2^* (\text{pr}_1^*h \cdot \text{pr}_1^*l). \]

By using the projection formula and the arbitrariness of the choice of the weight on the line, the claim follows. \( \square \)

**Remark 2.2.** If we would have chosen \( \mathcal{L} = -K_{X/B} \), we can see that the hypothesis \( D_{|X_b} \in | -K_{X_b}| \) would not be satisfied. Indeed, it is true if and only if \( \sum_{i=1}^{m} d_i = (n + 1) \), i.e. a Calabi-Yau log hyperplane arrangement, but this is in contradiction with the log Fano assumption. However, \( -K_{X/B} \) it is still a relatively ample line bundle on \( \text{pr}_2 : X \to \mathbb{P}^1 \). We can compute the CM line bundle with this choice, and relaxing the hypothesis for which \( D_{|X_b} \in | -K_{X_b}| \) as the next result will show.

**Proposition 2.3.** With the above data and \( \mathcal{L} = -K_{X/B} \), we have
\[ c_1(\lambda_{CM,D}) = (n + 1)^2 d_j. \]

**Proof** The first summand is zero, since \( X/\mathbb{P}^1 \) is a trivial fibration. From the second summand we get
\[ c_1(\lambda_{CM,D}) = - (n + 1)\text{pr}_2^* \left[ \left( n + 1 - \sum_{i=1}^{m} d_i \right) \text{pr}_2^*h - d_m \text{pr}_1^*l \right] \cdot (n + 1)\text{pr}_2^*h = (n + 1)^2 d_m. \]

Because of the arbitrariness of choices the claim follows \( \forall j \in \{1, 2, ..., m\} \). \( \square \)

From the choice of the two ample line bundles on the given flat and proper family we see that they mainly differ by a factor which involves the volume of the fiber, that is
\[ \text{Vol}(\text{pr}_{2,*}) := \left( n + 1 - \sum_{i=1}^{m} d_i \right)^n. \]

This difference might become important when calculating the volume of the moduli space of weighted hyperplane arrangements. To see that, we shall calculate the log CM line on to the mentioned moduli space. Recall that the moduli space of weighted log Fano hyperplane arrangement is the GIT quotient.
\[ M_d := (\mathbb{P}^n)^m / \mathcal{L}_d \mathcal{S} \mathcal{L}_{n+1} \]

where \( \mathcal{L}_d = \mathcal{O}(d_1, ..., d_m) \), is the linearization and \( d_i \in (0, 1) \cap \mathbb{Q}, \forall i \in \{1, 2, ..., m\} \).

It is known that, see for example [8, Chapter 11, Lemma 11.1]. Therefore

\[ c_1(\lambda_{CM,D}) \in \text{Pic}(M_d) \Rightarrow c_1(\lambda_{CM,D}) = \mathcal{O}(r_1, ..., r_m), \text{ for some } (r_1, ..., r_m) \in \mathbb{Z}^m. \]

In order to compute the \( r_j \)'s we can use the results of 2.2 and 2.3. Namely, in both cases we compute the following intersection number

\[ c_1(\lambda_{CM,D}) \cdot (\mathcal{L} \to \mathbb{P}^1) = k, \]

which means, \( \forall j \in \{1, 2, ..., m\} \)

\[ \int_{\mathbb{P}^1} \mathcal{O}(r_1, ..., r_m) = r_j = k. \]

Therefore, from [2.2] we get \( r_j = (n+1)d_j \text{Vol}(\text{pr}_{2,i}) \), and from [2.3] we get \( r_j = (n+1)^2d_j \). Hence, we just proved the following result.

**Proposition 2.4.** The log CM line bundle on the moduli space of log Fano hyperplane arrangement is given by

- When \( \mathcal{L} = -K_{X/B} - D \), we have that
  \[ c_1(\lambda_{CM,D}) = \text{Vol}(\text{pr}_{2,i})\mathcal{O}((n+1)d_1, ..., (n+1)d_m) \]

- When \( \mathcal{L} = -K_{X/B} \), we have
  \[ c_1(\lambda_{CM,D}) = (n+1)\mathcal{O}((n+1)d_1, ..., (n+1)d_m). \]

**Remark 2.3.** In both cases of 2.4 we can see that the log CM line bundle is a multiple of the weighted prequantum line bundle on \((\mathbb{P}^n)^m\) hence of its Kähler form, we recall that

\[ \left[ \frac{\omega}{2\pi} \right] = c_1(-K_{(\mathbb{P}^n)^m}) = c_1(\mathcal{O}((n+1)d_1, ..., (n+1)d_m))) \]

Where the Kähler form \( \omega \) is given by the weighted sum of the Fubini Study metrics on each \( \mathbb{P}^n \), namely \( \omega = \sum_{i=1}^{m} d_i \omega_{FS_i} \).

### 3 A first study: the case of four points on the complex projective line.

As a first study, we consider the case of four points \( \{p_i\}_{i=1}^{4} \) in \( \mathbb{P}^1 \) with rational weights \( d_i \in (0, 1) \cap \mathbb{Q}, i \in \{1, 2, 3, 4\} \). Given the pair \((\mathbb{P}^1, \sum_{i=1}^{4} d_ip_i)\), a choice of the rational weights correspond to three distinguished geometries.
We want to study how the Volume of the moduli space of four points in the projective line varies in the above geometries. We begin with the Calabi-Yau geometry. The moduli space of four points in $\mathbb{P}^1$ can be described as the following GIT quotient

$$M_d = (\mathbb{P}^1)^4 // _d \text{SL}_2 \mathbb{C},$$

(4)

with linearization $L_d = \mathcal{O}(d_1, d_2, d_3, d_4)$. Furthermore, $M_d$ is isomorphic to the moduli space of marked curves of genus 0, denoted by $M_{0,4}$. For clarity, we recall the following result

**Theorem 3.1.** ([33], Theorem 8.1) Let $M_{0,n}$ be the moduli space of $n$ ordered points on the Riemann sphere. Then the complex hyperbolic volume of $M_{0,n}$ is given by

$$\text{Vol}(M_{0,n}) = C_{n-3} \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|+1} (|\mathcal{P}| - 3)! \prod_{B \in \mathcal{P}} \max \left( 0, 1 - \sum_{i \in B} \mu_i \right)^{|B|-1},$$

(5)

where $\sum_{i=1}^n \mu_i = 2$, $0 < \mu_i < 1$, $\forall i \in \{1, 2, ..., n\}$, and $C_n = \left(\frac{-4\pi}{n+1}\right)^n$. Here $\mathcal{P}$ ranges over all partitions of the indices $(1, ..., n)$ into blocks $B$.

From the above theorem, we can see that the only partitions making sense are those whose size is greater or equal than three. Therefore, in this considered case, are those of size 3, and 4. Recall that the size of a partition $\mathcal{P}$ equals the number of blocks $B$ of the same partition. Therefore, if the partition has size 4 then there is only one partition with four blocks each of which has size one. If the size of the partition is 3, then we have six different partitions of two blocks of size one and one block of size two. From 3.1 we have $C_4 = -2\pi$, set $\alpha_B = \max (0, 1 - \sum_{i \in B} d_i)$. If a block has size one then $\alpha^{B-1} = 1$. The contribution to the sum from the partition of size 4 is just $-2\pi$, and the contribution to the sum from the partitions of size 3 is the sum of all $d_i$’s whose indexing set is a block of size two. Namely $-2\pi \sum_{B, |B|=2} \alpha_B$. Then, as a final result we get

$$\text{Vol}(M_{0,4}) = -2\pi \sum_{I \subset \{1,2,3,4\}, |I|=2} (1 - \sum_{i \in I} d_i)$$

(6)

**Remark 3.1.** The same result given in Equation 6 can be achieved from a result of Thurston [32] which shows that the general moduli space of marked curves of genus 0, with the additional condition that the weights sum up to 2, is a complex hyperbolic cone manifold. The latter, in general, have a natural stratification, and the strata of a cone manifold are connected and Riemannian.
The angle of a cone manifold is constant on each strata. We know that for a closed complex hyperbolic manifold $\mathcal{M}$ of dimension $n$, its volume can be expressed by its Euler characteristic

$$\text{Vol}(\mathcal{M}) = \frac{(-4\pi)^n}{(n+1)!} \chi(\mathcal{M})$$

(7)

The dimension of $\mathcal{M}_{0,4}$ is 1, and it is homeomorphic to $\mathbb{P}^1$. Considering the analogy with the 2-sphere with four marked points with weights $d_1, d_2, d_3, d_4$, respectively, we can associate to these latter a cell complex, where its skeleton consist of four distinct points and six triangles. These latter correspond to the four points partitions of size two and three. With this in mind, and by applying Equation 7 we get Equation 6.

In the case of four points in $\mathbb{P}^1$ the result provided by Madini in [38] becomes

$$\text{Vol}(\mathcal{M}_d) = -\pi \sum_{k=0}^{3} (-1)^k \sum_{I \subset \mathcal{I}, |I| = k} (D_I^+ - D_I^-).$$

(8)

Now assume that $\sum_{i=1}^{4} d_i = 2$. Unravelling the definition of $D_I^+$ and $D_I^-$, we have

$$\sum_{i \notin I} d_i - \sum_{i \in I} d_i = 2 - \sum_{i \notin I} d_i - \sum_{i \in I} d_i$$

$$= 2(1 - \sum_{i \in I} d_i).$$

By substituting in Equation 8 we find

$$\text{Vol}(\mathcal{M}_d) = -2\pi \sum_{k=1}^{4} (-1)^k \sum_{I \subset \mathcal{F}, |I| = k} \max \left(0, 1 - \sum_{i \in I} d_i \right)$$

(9)

Proposition 3.1. The volume function of the moduli space of four points in $\mathbb{P}^1$ is continuous when passing from the Fano geometry to the Calabi-Yau geometry.

Proof. All we need to show is that Equation 8 equals Equation 6. When looking at Equation 8 the subsets $I$ of size one will contribute $-2$ to the sum. Indeed, by assumption $d_i \in (0, 1) \cap \mathbb{Q}, \forall i \in \{1, 2, 3, 4\}$, therefore $\alpha_I = 1 - d_i, \forall i \in \{1, 2, 3, 4\}$. By taking the sum of the four subset of size one, and observing that the sum of all the $d_i$ is by assumption two, we have

$$-\pi \sum_{i=1}^{4} (1 - d_i) = -2\pi(4 - 2) = -2\pi.$$  

The subsets of size three have zero contribute to the sum. Indeed, without loss of generality, suppose $I = \{1, 2, 3\}$, then $1 - d_1 - d_2 - d_3 = 1 - 2 + d_4 = d_4 - 1$. Therefore,
\[ \alpha_{\{1,2,3\}} = \max(0, d_4 - 1) = 0. \]

Hence, it remains to sum the contributions coming from the subsets of size 2. But, putting all together we have

\[ \text{Vol}(M_d) = -2\pi \sum_{I \subset \{1,2,3,4\} : |I| = 2} (1 - \sum_{i \in I} d_i). \]

That proves the claim. \(\square\)

This proves, in the four point case, that the volume function is continuous when passing from the Fano geometry to the Calabi-Yau geometry. When \(\sum_{i=1}^4 d_i > 2\) we may lose this behaviour, since we do not have a GIT quotient, therefore the previous techniques for calculating the volume can not be applied. The work of Alexeev [25] describes the moduli space of hyperplane arrangements by generalising the work of Hassett [24] so that the one dimensional case coincides with the Hassett moduli space. The moduli space of weighted hyperplane arrangements of dimension 1, is the moduli space of points in \(\mathbb{P}^1\). Since the dimension of the considered moduli space is 1, for any choice of weights, then we can calculate the volume in the general type geometry, by computing the degree of the log CM line bundle. To do so, we notice that \(M_d\) has a universal family obtained by considering the product \(\mathbb{P}^1 \times \mathbb{P}^1\), fixing three points, 0, 1, \(\infty\), and allowing the last one to move in the diagonal \(\Delta = \{(s,t) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid s = t\}\). We set \(X = \mathbb{P}^1 \times \mathbb{P}^1\), and \(D = d_1 \cdot \mathbb{P}^1 \times \{0\} + d_2 \cdot \mathbb{P}^1 \times \{1\} + d_3 \cdot \mathbb{P}^1 \times \{\infty\} + d_4 \cdot \Delta\).

Since we fixed three points, then as divisors we can assume that they have the same hyperplane class, that will be called \(h_2\). Hence, we can rewrite \(D\) as follows

\[ D = \sum_{i=1}^4 d_i pr_2^* h_2 + d_4 pr_1^* h_1. \]

Because of the construction of the Hassett’s moduli space we must distinguish several situations. Heuristically, we have the following phenomena: when the point on the diagonal with weight \(d_4\) meets one, two or three points, then a wall is crossed and therefore the universal family becomes singular at that point. This phenomena is encoded in the notion of stability given in [24] that we recall in the following

**Definition 3.1.** Let \(\pi : (C, s_1, \ldots, s_n) \to B\) be a proper and flat morphism of nodal curves of arithmetic genus \(g\), and \(s_1, \ldots, s_n\) are the sections of \(\pi\) corresponding to the marked points on \(C\). A collection of data \((g,A) = (g,a_1,\ldots,a_n)\) consist of an integer \(g \geq 0\) and weights \((a_1,\ldots,a_n) \in \mathbb{Q}^n\), such that \(0 < a_j \leq 1, \forall j \in \{1,2,\ldots,n\}\) and \(2g - 2 + a_1 + a_2 + \cdots + a_n > 0\). We say that \(\pi\) is stable if the following conditions are satisfied.
1. The sections \(s_1, \ldots, s_n\) are in the smooth locus of \(\pi\), and for every subset \(\{s_{i_1}, \ldots, s_{i_r}\}\) with nonempty intersection we have \(a_{i_1} + \ldots + a_{i_r} \leq 1\).

2. \(K_{\pi} + \sum_{i=1}^{n} a_i s_i\) is relatively ample.

Clearly, in our case \(g = 0\), and \(M_{0,4} \simeq \mathbb{P}^1\). The above definition, and the heuristic are rather intuitive as the following picture shows.

In this picture, the dashed vertical lines represents the fibers of the map \(\text{pr}_2\). To the red dots correspond singular points in the fibers, namely those for which the diagonal meets one or more divisors. The blue point is a smooth point for the fiber of \(\{t\}\).

When looking at the above picture, we shall assume that the weight \(d_4\) is the smallest among the weights. So, while \(d_4\) is free to move, and meet one (or more) points in the diagonal, then the rest of the points remain fixed.

- The following inequalities holds, when the diagonal does not meet any of the fixed points

\[
\begin{aligned}
&d_i + d_j > 1 & i, j & \in \{1, 2, 3\} \\
&d_k + d_4 < 1 & \text{for all } k & \in \{1, 2, 3, 4\}
\end{aligned}
\] (10)

- If \(d_4\) grows, then it meets one (or more) points along the diagonal, and a desingularization is needed

\[
\begin{aligned}
&d_i + d_4 > 1 & i & \in \{1, 2, 3\} \\
&d_j + d_4 \leq 1 & i & \neq j
\end{aligned}
\] (11)

As the above picture suggest we can take the projection onto the second factor of \(\mathcal{X}\), namely \(\text{pr}_2 : (\mathcal{X}, \mathcal{D}) \to \mathbb{P}^1\), to get a \(\mathbb{Q}\)-Gorenstein family of log general type varieties. The stability conditions of Hassett suggest that we should choose, as a relatively ample line bundle for \(\text{pr}_2\), the log canonical polarization \(K_{\mathcal{X}/\mathbb{P}^1} + \mathcal{D}\).
We begin by studying the \( K_{X/P^1} + D \) as a relative polarization for the family \( \text{pr}_2: (X, \mathcal{D}) \to \mathbb{P}^1 \). By a direct application of Corollary 2.2 we have that

\[
c_1(\lambda_{CM, D}) = 2d_4 \left( \sum_{i=1}^{4} d_i - 2 \right).
\]  

(12)

In \( \mathbb{P}^4 \) it is the situation when the diagonal meets one point. Suppose, without loss of generality, that the diagonal meets the zeroth fiber. The universal family becomes singular at that point, therefore we take the blowup at \( \{0\} \) of the total space of the universal family. Set \( \tilde{X} = \text{Bl}_0(X) \). The divisor \( D \), modifies as follows

\[
D = d_1(\text{pr}_2^* h_2 - E) + (d_2 + d_3)\text{pr}_2^* h_2 + d_4(\text{pr}_2^* h_2 + \text{pr}_1^* h_1 - E)
\]

\[
= \sum_{i=1}^{4} d_i\text{pr}_2^* h_2 + d_4\text{pr}_1^* h_1 - (d_1 + d_4)E.
\]

Then,

\[
K_{\tilde{X}/\mathbb{P}^1} + D = \left( \sum_{i=1}^{4} d_i - 2 \right) \text{pr}_2^* h_2 + d_4\text{pr}_1^* h_1 - (d_1 + d_4 + 1)E.
\]

By applying directly Corollary 2.2 we find Equation 12. Indeed the cohomology ring of the blow up suggests that the intersection products \( \text{pr}_i^* h_i \cdot E = 0 \), \( i = 1, 2 \). Moreover, the pushforward of the constant term \( (d_1 + d_4 + 1) \) coming from \( E^2 = -1 \) term, is zero, therefore the only term that survives is only \( \text{pr}_2^* 2d_4 \left( \sum_{i=1}^{4} d_i - 2 \right) \text{pr}_2^* h_2 \cdot \text{pr}_1^* h_1 \), that yields to 12. The same holds if more than one desingularization is needed. The formula of the volume function for the log general type case is therefore constant with respect to the log canonical polarization. We notice also that when Equation 12 is evaluated in the Calabi-Yau zone then the volume function is zero. With respect to the anticanonical log polarization given in the Fano case we have a change of sign. This shows that the volume function has a discontinuity point in the Calabi-Yau zone. However, according to [11, Theorem 5.5.2], and [25, Theorem 1.5], if we choose weights whose sum is slightly greater than 2 the moduli space do not change, and it is described as the GIT quotient [4]. Then, at least for \( \mathbb{P}^4 \) we can chose as relative polarization \(-K_{X/P^1}\). Therefore, a fast computation proves that

\[
c_1(\lambda_{CM, D}) = 4d_4.
\]

We can easily observe that up to a normalization factor, \( \pi \), it coincides with the result obtained by applying [6] and [9] with the following order of weights

\[
d_i < \sum_{j \neq i} d_j, \quad \forall i \in \{1, 2, 3, 4\},
\]
\[ d_4 + d_2 < d_3 + d_1 \]
\[ d_4 + d_3 < d_2 + d_1 \]
\[ d_4 + d_4 < d_2 + d_1. \]

We resume all these results in the following

**Proposition 3.2.** Let \( M_d \) be the moduli space of weighted hyperplane arrangements of dimension 1. Suppose that the weights \( d_i \in (0, 1) \cap \mathbb{Q}, i \in \{1, 2, 3, 4\} \) satisfy the following conditions

- \( d_j < \sum_{i \neq j} d_i, \forall j \in \{1, 2, 3, 4\}; \)
- \( d_4 + d_2 < d_3 + d_1 \)
- \( d_4 + d_3 < d_2 + d_1 \)
- \( d_4 + d_4 < d_2 + d_1. \)

Then, for small weights the volume of \( M_d \) changes continuously along the Fano, Calabi-Yau and general type geometry and it is given by \( 4\pi d_4. \)

**Remark 3.2.** In the condition of Proposition 3.2, when the sum of the weights largely exceed two, then we must choose another polarization, i.e. the log canonical polarization, as the Hasset compactification suggests. The calculations show that the volume of \( M_d \) in this case is given by \( 2d_4 \left( \sum_{i=1}^{4} d_i - 2 \right) \). Note that when approaching the Calabi-Yau geometry from the far away general type geometry, then the volume of \( M_d \) goes to zero.

### 4 The general case

In order to prove Theorem 1.1, we recall the following definition

**Definition 4.1.** [94] Let \( f : (X, D) \to B \) be a family of log K-polystable Fano varieties of pure dimension \( n \). Assume that \( D := \sum_k (1 - \beta_k)[s_k = 0] \) is simple normal crossing. Then, we define the log Weil-Petersson metric as the following fiber integral

\[ \omega_{\text{WP}}^0 := - \int_{X^0/B^0} \omega_{X^0}^{n+1}. \]  

(13)

In [94 Theorem 2.3] it is proven that Equation 13 can be extended to the whole family \( f : (X, D) \to B \), and it extends as a curvature for a metric on the descended log CM line bundle \( \lambda_{\text{CM}, D} \).

**Proof** (Of Theorem 1.1). Consider \( n \) fixed points \( \{p_i\}_{i=1}^{n} \) in general position in \( \mathbb{P}^1 \). Each of these points has weight \( d_i \in (0, 1) \cap \mathbb{Q} \), where \( d_i = 1 - \beta_i \) is
related to the conic angle $\beta_i$. Fix a conic KE metric in $\mathbb{P}^1$, $\omega_d \in c_1(\mathcal{O}(1))$, where $d = (d_1, \ldots, d_n)$. Note that when $d = (1, \ldots, 1)$ then $\omega_d$ coincides with the Fubini-Study metric on $\mathbb{P}^1$. When $\sum_{i=1}^n d_i = 2$ then $\omega_d$ is a Calabi-Yau metric, that is

$$\omega_d = i\Omega \wedge \overline{\Omega}$$  \hspace{1cm} (14)

Where, by choosing an affine coordinate $z$ for the points $\{p_i\}_{i=1}^n$, such that $z(p_i) = c_i \in \mathbb{C}, \forall i \in \{1, 2, \ldots, n\}$, the holomorphic 1-form $\Omega$ is given by

$$\Omega = \frac{dz}{\prod_{i=1}^n(z - c_i)^{d_i}}.$$  

In [79, section 4.1] it is proven that (14) is indeed the metric used by McMullen in [33], that is the Weil-Peterson metric. Namely,

$$\omega_{WP} = -i\partial \overline{\partial} \log \int_{\mathbb{P}^1} i\Omega \wedge \overline{\Omega}.$$  

Now, let $f : (\mathcal{X}, \mathcal{D}) \rightarrow \mathbb{P}^1$ be a familily of $K$-polystable log Fano hyperplane arrangements of dimension one, like in section 2.2. That is, $\mathcal{X} = \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{D} = \sum_{i=1}^n d_i p_i + d_n (p_1 + p_2)$. Then $f$ is the projection map onto the second factor of $\mathcal{X}$. Furthermore, we assume for this family that the sum of all the weights is two. We choose as a relative ample line bundle for $f$ $L = -K_{\mathcal{X}/\mathbb{P}^1}$. Fiberwise we have

$$-i\partial \overline{\partial} \log \int_{\mathcal{X}_t} i\Omega_t \wedge \overline{\Omega_t} = -i\partial \overline{\partial} \log \int_{\mathcal{X}_t} \omega_{n+1}.$$  

Then, because of Definition [4.1] the above is exactly the fiberwise definition of the log Weil Petersson metric, and hence the curvature for a metric on $\lambda_{CM, \mathcal{D}}$. That means,

$$\text{Vol}(\mathcal{M}_{0,n}) = \text{Vol}(\lambda_{CM, \mathcal{D}, \omega_{WP}}).$$  \hspace{1cm} (15)

Because of the second assertion of Proposition [2.4] we see that the log CM line bundle it is a multiple of the anti-canonical bundle of $\mathbb{P}^1$. Therefore, the volume can be achieved with the Jeffrey-Kirwan residue theorem, that leads to equation in [33] (Theorem 8.1) as wanted. \hspace{1cm} $\Box$

Remark 4.1. A combinatorial proof like in the case of four points it is rather complicated. As a check we provide a Python script that shows numerically the equality of the two formulas, that can be found in APPENDIX A.

5 Final remarks

We wish to conclude this section with an observation. Consider the log Fano pair ($\mathbb{P}^1, \sum_1^4 d(p_i)$). Chose some coordinate $z \in \mathbb{P}^1$, such that for a fixed
\(i \in \{1, 2, 3, 4\}\) we have \(z(p_i) = 0\). Consider the vector field \(v = z\partial_z\), and notice that it generates a one parameter subgroup \(\lambda : \mathbb{C}^* \to GL_1(\mathbb{C})\), that acts on \(\mathbb{P}^1\) in the following fashion \((\lambda(t), z) := t \cdot z\). This actions easily translates on the divisor, \(D_t := t \cdot (\sum_{i=1}^4 d_i p_i)\), and on the anticanonical polarization of \(\mathbb{P}^1\), namely \(\mathcal{O}_{\mathbb{P}^1}(2)\).

This data define a test configuration \(\{(\mathbb{P}^1_t, D_t, -K_{\mathbb{P}^1_t})\}_{t \in \mathbb{C}^*} = (\mathcal{X}, \mathcal{D}, K_{\mathcal{X}})\) whose central fiber \(X_0\) is just a point and the divisor, at the central fiber, behave as

\[
D_0 = d_i \{\infty\} + \sum_{i \neq j} d_j \{0\}.
\]

Then, this latter defines an integral test configuration. As we mentioned in previous discussions and in [78] there exist a bijection between integral test configurations and dreamy prime divisors. We observe that since we are dealing with toric varieties, then every prime divisor is also dreamy ([78]). In particular in our central fiber \(16\) we have the sum of two dreamy prime divisor

\[
W_1 = \{0\}, \text{ and } W_2 = \{\infty\}.
\]

We have that

\[
\text{ord}_{W_1}(D_0) = d_i, \text{ and } \text{ord}_{W_2}(D_0) = \sum_{i \neq j} d_j.
\]

The divisor \(W_1\) leads to \(\text{DF}_D(\mathcal{X}, \mathcal{L}) = \sum_{i \neq j} d_j - d_i\). We will now show that the volume of \(M_{0,4}\) can be obtained by summing all the Donaldson-Futaki invariant of the corresponding integral test configuration. In [8] every single term \(D_F - D_F^-\), by the Jeffrey-Kirwan residue theorem [19] is associated to

\[
\frac{i \mu^* \mathbf{c}_1(L_d)(X)}{e_B(X)} = (-1)^{4-k} \mu(B) X^{-3}
\]

where \(B \in \mathcal{F} = \{(z^1, \ldots, z^4) \in (\mathbb{P}^1)^4 \mid z^i \in \{0, \infty\}\}\) is a fixed point for the maximal torus action. In [17] we used the fact that since \(L_d\) is a prequantum line bundle and the maximal torus is the unit circle then by [12], Lemma 9.31, the image of the moment map \(\mu(B)\) is given by the weighted sum of the height function on each \(\mathbb{P}^1\), namely

\[
\mu(B) = \sum_{i=1}^4 d_i \mu^i(z^i),
\]

where \(\mu^i(z^i)\) is the height function, i.e. the moment map for the maximal torus action on each \(\mathbb{P}^1\). Clearly, the fixed points for this action are the standard basis element \(e_1, e_2\) of \(\mathbb{C}^2\) that corresponds to \(0, \infty\) respectively. We have

\[
\mu^i(e_i) = \begin{cases} 
1 & \text{if } i = 1 \\
-1 & \text{if } i = 2.
\end{cases}
\]

It follows in [8] that the index \(F\) is the one that tells how many \(e_2\) we have in the fixed point \(B\).
If $|F| \geq 2$ we can always think of an integral test configuration for which two points came together into one point, where the weight of the latter is the sum of two or more weights. With this in mind, it is easy to convince ourselves that the size of the index set $F$ tells how many points should come together into one point, and hence how many weights should be summed at that point. Therefore, the general term

$$i_F^* c_1(\mathcal{L}) = -\frac{(1)^{1-k}}{x^3} \text{DF}_D(X, \mathcal{L})(B),$$

as wanted. This observation, can be immediately generalised for configurations of $n-$points in $\mathbb{P}^1$. The key point is that in the work of [80] it is shown that in the case of log Fano hyperplane arrangements the notion of K-stability coincides with the notion of GIT-stability. Moreover by changing the weights, the corresponding moduli spaces are well behaved so that the Jeffrey-Kirwan localization formula of [19] can be applied for calculating the volume. The GIT quotient describing the hyperplane arrangements of dimension one is the following

$$M_d = (\mathbb{P}^1)^n // \mathcal{L}_d \text{SL}_2 \mathbb{C}.$$

The dimension of $M_d$ is $n - 3$. We apply directly the Jeffrey-Kirwan theorem for the case of SU(2), on the top class namely

$$k(c_1(\mathcal{L}_d)^{n-3})e^{\omega_0}[M_d] = \frac{n_0}{2} \text{Res}_{X=0} \left( 4X^2 \sum_{F \in F^+} \int_{\mathcal{L}_d} i_F^* c_1(\mathcal{L}_d)^{n-3}(X) dX \right)$$

since $\mathcal{L}_d$ is a prequantum line bundle and the maximal torus is the unit circle then by Lemma 9.31 of [12] $i_F^* c_1(\mathcal{L}_d)^{n-3}(X) = (-1)^{n-3} \mu(F)^{n-3}(X)$. Hence,

$$\frac{i_F^* c_1(\mathcal{L}_d)^{n-3}}{e_F(X)} = (-1)^{n-k} (\mu(F)(X))^{n-3} X^{-n-3}.$$

Because of the above observation, $\mu(F) = \text{DF}_D(X_F, \mathcal{L}_F) = D^+_F - D^-_F$. We have

$$4X^2 \frac{i_F^* c_1(\mathcal{L}_d)^{n-3}}{e_F(X)} = (-1)^{n-k} \text{DF}_D(X_F, \mathcal{L}_F)^{n-3} X^{-1}. \quad (18)$$

The residue of $X^{-1}$ in (18) is one. By taking the sum, we proved the following result

**Theorem 5.1.** The volume of the moduli space of log Fano hyperplane arrangements of dimension one is the sum of the Donaldson-Futaki invariants associated to test configurations $(X_F, \mathcal{L}_F)$. Where $F \in F_+$ is like in [33], and $(X_F, \mathcal{L}_F)$ is the test configuration for which $|F|$ points come together into one point.

$$\text{Vol}(M_d) = -\frac{(2\pi)^{n-3}}{2(n-3)!} \sum_{i=0}^{n-3} (-1)^k \sum_{F \in F^+, |F|=k} \text{DF}_D(X_F, \mathcal{L}_F)^{n-3}.$$
Remark 5.1. It is natural to ask if more generally one can express the volumes as sums of CM degrees of special unstable families, using the $\Theta$-stratification \cite{28} to generic Kirwan’s conditions \cite[Theorem 5.6]{28}.

APPENDIX A

The Python script for calculating the equality of the two mentioned volume formulas is given below.

Listing 1: Test of Equivalence

```
import numpy as np
import math
from fractions import Fraction
import sympy
import itertools
from itertools import combinations
import random

def subsets_k(collection, k):
    yield from partition_k(collection, k, k)

def partition_k(collection, min, k):
    if len(collection) == 1:
        yield [collection]
    return

first = collection[0]
for smaller in partition_k(collection[1:], min - 1, k):
    if len(smaller) > k: continue
    # insert ‘first’ in each of the sub partition’s subsets
    if len(smaller) >= min:
        for n, subset in enumerate(smaller):
            yield smaller[:n] + [[first] + subset] + smaller[n + 1:]
    # put ‘first’ in its own subset
    if len(smaller) < k: yield [[first]] + smaller

def MandiniVolume(V):
    N = len(V)

    num = Fraction(np.power(2, 2*N-7))
    Prefactor = Fraction(num, math.factorial(N-3)*np.power(sympy.pi, N-3))

    Result = 0

    for I in range(0,N):
        R = list(combinations(V, I))
        factor = 0
        for j in range(0, np.array(R, dtype = 'object').shape[0]):
            Q = 1
            S = np.power(np.max([[0, 1- np.sum(R[j])]], N-3)
            Q *= S
```
```python
factor += (-1)**(I+1)*Q
Result += Fraction(factor)
return (Result*Prefactor)

def McMullenVolume(V):
    N = len(V)
    Prefactor = Fraction((-4)**(N-3),math.factorial(N-2)) * sympy.pi**(N-3)
    Result = 0
    for P in range(3,N+1):
        R = list(subsets_k(V, P))
        factor = 0
        for j in range(0,np.array(R, dtype="object").shape[0]):
            Q = 1
            for i in range(0,P):
                B = len(R[j][1])
                S = np.max([0,1-np.sum(R[j][1])])**I**(B-1)
                Q *=S
                factor += (-1)**(P+1) *math.factorial(P-3)*Q
        Result += factor
    return (Result*Prefactor)

class AnomalyTest:
    def __init__(self,length, total_tests):
        if not isinstance(length, int) and length >= 3:
            raise ValueError('length must be an integer greater or equal than 3')
        if not isinstance(total_tests, int) and total_tests >0:
            raise ValueError(f'total_tests must be a positive integer, found {total_tests}')
        self.length = length
        self.total_tests = total_tests
    
def RandomVector(self):
        V = []
        for i in range(0,self.length):
            n = random.randint(1,30)
            V.append(n)
        V = list(np.asarray(V)*Fraction(2)/np.sum(V))
        return V
    
def run(self):
        test_iterations, anomalies = np.zeros((1,), dtype=int), np.zeros((1,), dtype=int)
while test_iterations < self.total_tests:
    V = self.RandomVector()
    check = any(entry > 1 for entry in V)
    if check:
        continue
    test_iterations += 1
    Vol_1 = MandiniVolume(V)
    Vol_2 = McMullenVolume(V)
    if Vol_1 == 0 and Vol_2 == 0:
        Ratio = 1.0
    else:
        Ratio = float(Vol_1 / Vol_2)
    if Ratio != 1:
        anomalies += 1
        print(V)
        print(f'Vol_1, Vol_2: {Vol_1}, {Vol_2}
        print(f'The ratio of the two volumes of Vol(M_d): {r}
        print(f'Total anomalies', anomalies)

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