New small gaps between squarefree numbers

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Abstract

In this paper, we show that, for some constant $C > 0$, the interval $(x, x + Cx^{5/26}]$ always contains a squarefree number when $x$ is sufficiently large (in terms of $C$). Our improvement comes from establishing asymptotic relations between the shifts $a$ and $b$ when $mn^2 \approx (m-a)(n+b)^2$. We apply them to study quadruples $(m+a_1)(n-b_1)^2 \approx mn^2 \approx (m-a_2)(n+b_2)^2 \approx (m-a_2-a_3)(n+b_2+b_3)^2$ and generalize Roth differencing and Filaseta-Trifonov differencing by allowing $b_1$ to be different from $b_3$. We also introduce a new differencing and exploit the interplay among these three differencings.

1 Introduction and Main Result

A positive integer is squarefree if it is not divisible by the square of any prime numbers. For example, 6 is squarefree but 12 is not. It is well-known that

$$\sum_{n \leq x, n \text{ squarefree}} 1 = \frac{6}{\pi^2} x + O(x^{1/2}).$$

Thus, the set of squarefree numbers has positive density and it is believed that gaps between successive squarefree numbers is small.

**Conjecture 1** For any $\epsilon > 0$ and $x$ sufficiently large (in terms of $\epsilon$), the short interval $(x, x + x^\epsilon]$ always contains a squarefree number.

Various authors have studied this question. Firstly, Fogel [6] showed that the conjecture is true for $\epsilon > 2/5$ by studying average of arithmetic functions over short intervals. This was greatly improved by Roth [10] to $\epsilon > 1/4$ through an elementary argument and $\epsilon > 3/13$ by incorporating van der Corput’s method. Later, Richert [9], Rankin [8], Schmidt [11], Graham and Kolesnik [7], Filaseta [1, 2], Trifonov [12, 13] and Filaseta and Trifonov [3] improved the result by exponential sum and elementary techniques. The current best result was by Filaseta and Trifonov [4]:

**Theorem 1** There exists a constant $C_0 > 0$ such that, for $x$ sufficiently large, the interval $(x, x + C_0 x^{1/5} \log x]$ contains a squarefree number.

In this paper, we are going to improve Theorem 1 to

**Theorem 2** There exists a constant $C_0 > 0$ such that, for $x$ sufficiently large, the interval $(x, x + C_0 x^{5/26}]$ contains a squarefree number.

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2 Some Preparations

Throughout this paper,

c’s are sufficiently small positive constants.
C’s are sufficiently large positive constants.
L > 0 is a large parameter.
λ = 1 + 1/L.

x is assumed to be sufficiently large in terms of c’s and C’s.
p denotes a prime number.
|S| denotes the cardinality of a set S.
||θ|| denotes the distance between θ and its nearest integer.

θ = 5/26.
δ = 0.0751.
h = C₀x⁰.
f(x) ≪ g(x) or g(x) ≫ f(x) means that there is a constant C independent of C₀ and L.

f(x) ≥ g(x) means that f(x) ≤ f(x) and g(x) ≤ f(x).
f(x) = o(g(x)) means that \( \lim_{x \to \infty} f(x)/g(x) = 0 \).

mn² is some number in the interval \((x, x + h]\).
n is typically in the interval \((N, λN]\).
m typically has size \(M = x/N²\).
R = \(c₀N^{4/3}/x^{1/3}\).
Q = \(\max(R, x^{1/5−δ})\).

We want to show that the interval \((x, x + h]\) contains a squarefree number. We may assume that \(h ≤ 0.1x\) for otherwise the result follows from \([1]\). Our initial treatment follows that of Filaseta and Trifonov \([4]\). Let S denote the set of integers in \((x, x + h]\) which are not squarefree. Then any member of S must be of the form \(mn²\) for some integers \(m ≥ 1\) and \(n > 1\). Let \(M_n\) denote the number of multiples of \(n²\) in the interval \((x, x + h]\). Then, for \(x\) sufficiently large, we have \(M_n = 0\) if \(n > 2\sqrt{x}\). Since every number greater than one has a prime factor,

\[
|S| ≤ \sum_{p ≤ 2\sqrt{x}} M_p = \sum_{p ≤ h} M_p + \sum_{h < p ≤ c h^{3/2}/x^{1/4}} M_p + \sum_{c h^{5/2}/x^{1/4} < p ≤ c h^{2/3} \log² x / h^{5/3}} M_p + \sum_{x^{2/3} \log² x / h^{5/3} < p ≤ 2\sqrt{x}} M_p
=: S₁ + S₂ + S₃ + S₄
\]

(2)

where \(c > 0\) is some small constant and \(C' ≥ 0\) is some large constant. Clearly, \(M_p ≤ h/p² + 1\) and

\[
S₁ ≤ \sum_{p ≤ h} \left( \frac{h}{p²} + 1 \right) < h \sum_{n=2}^{∞} \frac{1}{n²} + \pi(h) = h\left( \frac{π²}{6} - 1 \right) + \frac{2}{3} h
\]

by a Chebyshev estimate. Define

\[S(t₁, t₂) := \{ n ∈ (t₁, t₂) : \text{ there exists an integer m such that } mn² ∈ (x, x + h]\}\]

Suppose \(n ∈ (h, 2\sqrt{x}]\). Then there can be at most one multiple of \(n²\) in the interval \((x, x + h]\) as \(n² > h\). Hence, for \(n ∈ (h, 2\sqrt{x}]\), \(M_n = 1\) if and only if \(n ∈ S(h, 2\sqrt{x})\) and \(M_n = 0\) if and only if \(n ∉ S(h, 2\sqrt{x})\).
Therefore,
\[ S_2 \leq |S(h, ch^{5/2}/x^{1/4})|, \quad S_4 \leq |S(ch^{5/2}/x^{1/4}, x^{2/3}\log^2 x/h^{5/3})|, \quad S_1 \leq |S(x^{2/3}\log^2 x/h^{5/3}, 2\sqrt{x})|, \]
and
\[ |S| \leq \frac{2}{3}h + |S(h, ch^{5/2}/x^{1/4})| + |S(ch^{5/2}/x^{1/4}, x^{2/3}\log^2 x/h^{5/3})| + |S(x^{2/3}\log^2 x/h^{5/3}, 2\sqrt{x})|. \]
Thus, Theorem 2 is established by the following three propositions as they would imply \(|S| \leq \frac{11h}{12}\) and, hence, there must be some squarefree number in \((x, x + h)\).

**Proposition 1** For \(x^{1/6} < h < x^{1/5}\) and some sufficiently small \(c > 0\), we have
\[ |S(h, ch^{5/2}/x^{1/4})| \leq \frac{h}{12}. \]

**Proposition 2** Let \(c > 0\) be a fixed small constant. Then, for \(h = C_0 x^{26}\) with some sufficiently large \(C_0 > 0\), we have
\[ |S(ch^{5/2}/x^{1/4}, x^{2/3}\log^2 x/h^{5/3})| \leq \frac{h}{12}. \]

**Proposition 3** For \(h < x^{1/5}\), we have
\[ |S(x^{2/3}\log^2 x/h^{5/3}, 2\sqrt{x})| \leq \frac{h}{12}. \]

### 3 Strategies in establishing Propositions 1, 2 and 3

Let \(\lambda = 1 + 1/L\) for some big positive parameter \(L\). The general strategy in proving Propositions 1, 2 and 3 is to divide the intervals of consideration for \(n\) into \(\lambda\)-adic intervals \((N, \lambda N)\). Following Filaseta and Trifonov [4], we consider consecutive elements in \(S(N, \lambda N)\) and define
\[ T_{N,\lambda}(b) := \{n : n \text{ and } n + b \text{ are consecutive elements in } S(N, \lambda N)\}, \quad \text{and } t_{N,\lambda}(b) := |T_{N,\lambda}(b)|. \quad (3) \]

First, note that
\[ |S(N, \lambda N)| \leq 1 + \sum_{b_1=1}^{\infty} t_{N,\lambda}(b_1). \]

Clearly,
\[ \frac{N}{L} \geq \sum_{b_1=1}^{\infty} b_1 t_{N,\lambda}(b_1) \geq \sum_{b_1>B} b_1 t_{N,\lambda}(b_1) \geq B \sum_{b_1>B} t_{N,\lambda}(b_1) \]
which implies
\[ \sum_{b_1>B} t_{N,\lambda}(b_1) \leq \frac{N}{LB} \leq \frac{h}{L \log^{1.2} x} \]
for \(B = N \log^{1.2} x/h\). By Lemma 3 below, we know that of every three consecutive elements in \(S(N, \lambda N)\), there are two consecutive elements with distance > \(R = cN^{4/3}/x^{1/3}\) from one another. Hence,
\[ \sum_{b_1 \leq R} t_{N,\lambda}(b_1) \leq 1 + \sum_{b_1>R} t_{N,\lambda}(b_1), \]
and
\[ |S(N, \lambda N)| \leq 2 + 2 \sum_{b_1>R} t_{N,\lambda}(b_1) \ll 1 + \frac{h}{L \log^{1.2} x} + \sum_{R < b_1 \leq N \log^{1.2} x/h} t_{N,\lambda}(b_1). \quad (4) \]
So, from now on, we restrict our attention to $R < b_1 \leq N \log^{1.2} x/h$. In [4], Filaseta and Trifonov established the bound
\[
t_{N,2}(b) \ll \frac{b^{1/3}x^{(3\beta+1)/3} \log x}{N^5} + \frac{b^3 x}{N^4}
\]
by considering certain differencings on the function $f(n) = x/n^2$. In a subsequent paper [5], Filaseta and Trifonov distillated their differencing method and came up with the following general theorem on small fractional parts:

**Theorem 3** Let $N > 1$. Let $r \geq 3$ be an integer. Let $T$ be a positive real number. Suppose that $f$ is a function with at least $r$ derivatives with $f^{(j)}(u) \propto TN^{-\beta}$ for $j \in \{r-2, r-1, r\}$ and $u \in (N, 2N]$. Let $\delta$ be a positive real number with
\[
\delta < k \min\{TN^{-r+2}, TN^{r-2}(r-2)N^{-r+3} + TN^{-r+1}\},
\]
for some sufficiently small constant $k$ depending on $r$ and the implied constants in the asymptotic formulas $f^{(j)}(u) \propto TN^{-\beta}$ above. Let
\[
S = \{u \in (N, 2N] \cap \mathbb{Z} : \|f(u)\| < \delta\}.
\]
Then
\[
|S| \ll TN^{\frac{2}{r(r-1)}N^{-r+1} + N\delta + TN^{1-\frac{1}{r-3}}}.
\]
We will use this to establish Proposition 1. To prove Proposition 2 we will apply the following proposition which allows one to extend the range of validity for $S(N, \lambda N) = o(h)$ to be true.

**Proposition 4** Suppose $N \leq C_\alpha x^\alpha$ for some $\theta < \alpha < 2/5$ and some constant $C_\alpha > 0$. Then
\[
\sum_{Q < b_1 \leq N \log^{1.2} x/h} t_{N, \lambda}(b_1) \ll \frac{N}{h} + \frac{x^{1/4}}{LN^{1/4}} + \frac{h x^{1/3}}{N^{7/5 + 4\alpha/5}} + \frac{x}{N^{10/5 + 4\alpha}} + \frac{h N^{4-2\beta}}{L x^{1-\theta}}
\]
for
\[
C'_\alpha \min(1, x^{1/3}, x^{2/5 + \theta}, x^{27/5 - 2\beta}) < N \leq C_\alpha x^\alpha, \quad 2 - \frac{1 - \theta}{2\alpha} < \beta < 1
\]
and some sufficiently large constant $C'_\alpha > 0$. The lower bound for $\beta$ in (7) is to guarantee that the last error term in (6) is much less than $h$.

To prove Proposition 3 we are going to establish asymptotic relations between shifts $a$ and $b$ when both $mn^2$ and $(m-a)(n+b)^2$ are members of $S(N, \lambda N)$. Then we apply them to study quadruples $(m+a_1)(n-b_1)^2 \approx mn^2 \approx (m-a_2)(n+b_2)^2 \approx (m-a_3)(n+b_3)^2$ and generalize Roth differencing and Filaseta-Trifonov differencing by allowing $b_1$ to be different from $b_3$. Another new ingredient is the introduction of a new differencing and using of the interplay among these three differencings.

Here, let us give some comments on the propositions. Proposition 1 provides a good estimate for small values of $N$ by extending it slightly beyond $N = h$ (Filaseta and Trifonov stopped at $S_1$ in (2) for the initial segment). On the other hand, Proposition 3 provides an $o(h)$ bound for $S(N, \lambda N)$ for $N$ well below $x^{2/5}$. This is a substantial improvement, as previously, this is only known to be true for $N > x^{2/5}$ when $h = x^{1/5}$. Filaseta and Trifonov hinted in [4] that future improvement might depend on extending this range. Finally, Proposition 2 provides a bridge for the middle range through the use of the new differencing and a recursive argument via Proposition 4.

This paper is organized as follows. First, we will prove Proposition 1 via Theorem 3. Then, we will study the asymptotic relations between shifts $a$ and $b$ when $mn^2$ and $(m-a)(n+b)^2$ are members of $S(N, \lambda N)$. Afterwards, we will revisit Roth differencing and Filaseta-Trifonov differencing as introduced in 4 through these new asymptotic relations. We will generalize these two differencings and introduce a third one. Then, we will apply these to prove Proposition 3. Finally, we will prove Proposition 4 by considering consecutive elements $n, n + b_2 \in T_{N, \lambda}(b_1)$ depending on the size of the product $b_1 b_2$. This will then be used to establish Proposition 2.
4 Proof of Proposition [1]

Recall $M_n$ denotes the number of multiples of $n^2$ in the interval $(x, x + h]$. In this section, we will focus on

$$|S(h, ch^{5/2}/x^{1/4})| = \sum_{h < n \leq ch^{5/2}/x^{1/4}} M_n = \sum_{h < n \leq ch^{5/2}/x^{1/4}} \left[ \frac{x + h}{n^2} \right] - \left[ \frac{x}{n^2} \right] \quad (8)$$

We divide $n$ into dyadic intervals, say $n \in (N, 2N]$ with $h \leq N \leq ch^{5/2}/x^{1/4}$. We will focus on bounding

$$|S(N, 2N)| = \sum_{N < n \leq 2N} M_n = \sum_{N < n \leq 2N} \left[ \frac{x + h}{n^2} \right] - \left[ \frac{x}{n^2} \right] \quad (9)$$

By the discussion in section 2, the summand in (9) is 1 if and only if there is some integer $m$ such that

$$\frac{x}{n^2} < m \leq \frac{x + h}{n^2}, \quad (10)$$

and 0 otherwise. One can see that if condition (10) holds for some $n \in (N, 2N]$, then $\| \frac{x}{n^2} \| < \frac{h}{N^2}$. Thus,

$$|S(N, 2N)| \leq \left| \left\{ n \in (N, 2N] : \| \frac{x}{n^2} \| < \frac{h}{N^2} \right\} \right| =: MN. \quad (11)$$

Let $$f(u) = \frac{x}{u^2}, \quad \delta = \frac{h}{N^2}, \quad \text{and} \quad T = \frac{x}{N^2}.$$ Now, we apply Theorem 3 with $r = 4$. One can check that the conditions in Theorem 3 are satisfied for $h \leq N \leq x^{1/4}$ and $h < x^{1/5}$. Therefore,

$$MN \ll T^{1/10} N^{3/5} + N\delta^{1/3} + N \left( \frac{\delta T}{N^3} \right)^{1/8} \ll x^{1/10} N^{2/5} \quad (12)$$

as the first term dominates due to $x^{1/6} < h < x^{1/5}$. Summing (12) over the dyadic intervals, we obtain

$$|S(h, ch^{5/2}/x^{1/4})| < h/12.$$

for sufficiently small $c > 0$. This establishes Proposition [1].

5 Asymptotic relations between shifts

Suppose $mn^2$ and $(m - a)(n + b)^2$ are members in $S$ for some integers $1 \leq a < m/2$, $1 \leq b < n/2$ and $n \geq h$. Consider

$$d := (m - a)(n + b)^2 - mn^2. \quad (13)$$

Then $d = (2bm - an)n + mb^2 - 2abn - ab^2$ which gives

$$\Delta := 2bm - an = 2ab - \frac{mb^2}{n} + \frac{ab^2}{n} + \frac{d}{n} = \frac{3}{2} ab + \left( \frac{an - 2mb}{2n} \right) + \frac{ab^2}{n} + \frac{d}{n}.$$ 

Hence,

$$\left(1 + \frac{b}{2n}\right) \Delta = \frac{3}{2} ab + \frac{ab^2}{n} + \frac{d}{n}$$

which gives

$$\Delta = \frac{3ab}{2(1 + \frac{b}{2n})} + \frac{ab^2}{n(1 + \frac{b}{2n})} + \frac{d}{n(1 + \frac{b}{2n})} \quad (14)$$

5
In particular, \( ab \leq \Delta \leq 3ab \) for \( x \) (and hence \( n \)) sufficiently large. From \([13]\), we also have \( \frac{d}{(n+b)} = m - a - m\left(\frac{n}{n+b}\right)^2 \) which gives

\[
a = m\left[1 - \frac{1}{(1 + \frac{b}{n})^2}\right] - \frac{d}{(n+b)} = \frac{2mb}{n} - \frac{1 + \frac{b}{n}}{(1 + \frac{b}{n})^2} - \frac{d}{(n+b)^2}. \tag{15}
\]

Using the geometric series formula \( \frac{1-x^k}{1-x} = 1 + x + x^2 + \ldots + x^{k-1} \), one obtains

\[
a = \frac{2mb}{n} - \frac{3mb^2}{n^2} + \frac{4mb^3}{n^3} - \frac{5mb^4}{n^4} + \frac{6mb^5}{n^5} + O\left(\frac{mb^6}{n^6}\right) - \frac{d}{(n+b)^2}. \tag{16}
\]

Summarizing all these, we have the following lemma.

**Lemma 1** Suppose \( mn^2 \) and \((m-a)(n+b)^2\) are members in \( S \) for some integers \( n \geq h \) and \( 1 \leq b < n/2 \). Then \( a \) and \( b \) satisfy the following relations:

\[
a = \frac{2mb}{n} - \frac{3mb^2}{n^2} + \frac{4mb^3}{n^3} - \frac{5mb^4}{n^4} + \frac{6mb^5}{n^5} + O\left(\frac{mb^6}{n^6}\right) - \frac{d}{(n+b)^2},
\]

and

\[\Delta := 2bm - an = \frac{3ab}{2(1 + \frac{b}{2n})} + \frac{ab^2}{n(1 + \frac{b}{2n})} + \frac{d}{n(1 + \frac{b}{2n})}\]

where \( d = (m-a)(n+b)^2 - mn^2 \).

A similar calculation yields

**Lemma 2** Suppose \( mn^2 \) and \((m+a)(n-b)^2\) are members in \( S \) for some integers \( n \geq h \) and \( 1 \leq b < n/2 \). Then \( a \) and \( b \) satisfy the following relations:

\[
a = \frac{2mb}{n} + \frac{3mb^2}{n^2} + \frac{4mb^3}{n^3} + \frac{5mb^4}{n^4} + O\left(\frac{mb^5}{n^5}\right) + \frac{d'}{(n-b)^2},
\]

and

\[\Delta := 2bm - an = -\frac{3ab}{2(1 - \frac{b}{2n})} + \frac{ab^2}{n(1 - \frac{b}{2n})} - \frac{d'}{n(1 - \frac{b}{2n})}\]

where \( d' = (m+a)(n-b)^2 - mn^2 \).

The idea of using differencing was introduced by Roth \([10]\) and is the building block for subsequent improvements via elementary method. We are going to illustrate how one can combine differencing and our asymptotic relations in Lemmas \([14] \) and \([15] \) to recover the following result used in previous works (cf. \([33]\)).

**Lemma 3** If \( I \) is a subinterval of \((N, \lambda N)\) with \( N \geq h \) and \( |I| \leq \frac{c_0 N^{1/3}}{e^{1/3}} \) for some small \( 0 < c_0 < 1 \), then \( |S(N, \lambda N) \cap I| \leq 2 \).

Proof: Suppose \((m+a_1)(n-b_1)^2\), \( mn^2 \) and \((m-a_2)(n+b_2)^2\) are members in \( S(N, \lambda N) \) for some integers \( 1 \leq b_1, b_2 < N/L \) and \( h \leq n \). We consider the difference

\[D := b_1(m-a_2)(n+b_2)^2 - (b_1 + b_2)mn^2 + b_2(m+a_1)(n-b_1)^2.
\]

Also, denote

\[
\Delta_1 = 2b_1m - a_1n, \quad \text{and} \quad \Delta_2 = 2b_2m - a_2n.
\]

By some simple algebra, one gets

\[D = b_1b_2(b_1+b_2)m + (a_1b_2-a_2b_1)n^2 - 2b_1b_2(a_1+a_2)n + b_1b_2(a_1b_1-a_2b_2).
\]
Substituting $a_1 = \frac{2a_1m}{m} - \frac{a_1}{n}$ and $a_2 = \frac{2a_2m}{m} - \frac{a_2}{n}$ into the third term above, we have
\[
D = (a_1b_2 - a_2b_1)n^2 - 3b_1b_2(b_1 + b_2)m + 2b_1b_2(\Delta_1 + \Delta_2) + b_1b_2(a_1b_1 - a_2b_2).
\] (17)

By the formulas of $\Delta$ in Lemmas 1 and 2,
\[
2b_1b_2(\Delta_1 + \Delta_2) + b_1b_2(a_1b_1 - a_2b_2) = -2b_1b_2(a_1b_1 - a_2b_2) + \frac{b_1b_2(a_1b_1^2 + a_2b_2^2)}{2n} + O\left(\frac{b_1b_2(a_1b_1^2 + a_2b_2^2)}{n^2} + \frac{b_1b_2h}{n}\right)
\] (18)

which has size much smaller than the term $3b_1b_2(b_1 + b_2)m$ by the formulas of $a$ in Lemmas 1 and 2 as $b_1, b_2 < N/L$. Putting (18) into (17), one has
\[
D = (a_1b_2 - a_2b_1)n^2 - 3b_1b_2(b_1 + b_2)m - 2b_1b_2(a_1b_1 - a_2b_2) + \frac{b_1b_2(a_1b_1^2 + a_2b_2^2)}{2n} + O\left(\frac{b_1b_2(a_1b_1^2 + a_2b_2^2)}{n^2} + \frac{b_1b_2h}{n}\right).
\] (19)

As $|D| \leq (b_1 + b_2)h \leq 2n^2/L$,
\[
a_1b_2 - a_2b_1 = \frac{3mb_1b_2(b_1 + b_2)}{n^2} + \frac{2b_1b_2(a_1b_1 - a_2b_2)}{n^2} - \frac{b_1b_2(a_1b_1^2 + a_2b_2^2)}{2n} + O\left(\frac{b_1b_2(a_1b_1^2 + a_2b_2^2)}{n^2} + \frac{b_1 + b_2}{n^2}\right)
\] (20)

If $a_1b_2 - a_2b_1 = 0$, then (20) implies
\[
\frac{mb_1b_2(b_1 + b_2)}{n^2} \ll \frac{(b_1 + b_2)h}{n^2} \quad \text{or} \quad \frac{mb_1b_2}{n^2} \ll h
\]
as $b_1, b_2 < N/L$ and $a_1, a_2 < 10M/L$. If $m \geq x^{1/3}$, then $x^{1/3} \leq mb_1b_2 \ll h \ll x^{1/5}$ which is a contradiction. If $m \leq x^{1/3}$, then $n \geq x/m \geq x^{1/3}/m \geq m$. However, since $a_1, a_2$ are positive integers, Lemmas 1 and 2 imply that $b_1, b_2 \gg n/m$. Hence $x^{1/3} \ll n \ll n^2/m = m(n/m)^2 \ll mb_1b_2 \ll h \ll x^{1/5}$ which is impossible. Therefore, $a_1b_2 - a_2b_1 \neq 0$ and (20) can be rewritten as
\[
a_1b_2 - a_2b_1 + O\left(\frac{(b_1 + b_2)h}{n^2}\right) = \frac{3mb_1b_2(b_1 + b_2)}{n^2} + O\left(\frac{b_1b_2(a_1b_1 + a_2b_2)}{n^2}\right).
\]

One can easily see that the error term on each side is much smaller than the main term on each side. Hence, $a_1b_2 - a_2b_1 \geq 1$ as the right hand side is clearly positive. Thus,
\[
n^2 \leq (a_1b_2 - a_2b_1)n^2 \ll b_1b_2(b_1 + b_2)m
\] (21)

which implies
\[
\max(b_1, b_2) \gg \left(\frac{n^2}{m}\right)^{1/3} \gg \frac{N^{4/3}}{x^{1/3}}
\] (22)

and, hence, the lemma by picking $c_0$ sufficiently small.

6 Roth, Filaseta-Trifonov and a new differencing

Suppose $mn^2, (m - a)(n + b)^2 \in S(N, \lambda N)$, Roth [10] considered the difference
\[
D_1 := m(2n - b) - (m - a)(2n + 3b) = 2an - 4bm + 3ab.
\]

By Lemma 1
\[
D_1 = 2n\left[\frac{2mb}{n} - \frac{3mb^3}{n^2} + \frac{4mb^4}{n^3} - \frac{5mb^5}{n^4} + \frac{6mb^6}{n^5}\right] - 4bm + 3b\left[\frac{2mb}{n} - \frac{3mb^3}{n^2} + \frac{4mb^4}{n^3} - \frac{5mb^5}{n^4}\right]
\]
\[
+ O\left(\frac{h}{n} + \frac{mb^6}{n^5}\right)
\]
\[
= -\frac{mb^3}{n^2} + \frac{2mb^4}{n^3} - \frac{3mb^5}{n^4} + O\left(\frac{h}{n} + \frac{mb^6}{n^5}\right).
\] (23)
as initial terms cancel out. This and the quantity $\Delta$ in Lemma [1] might indicate how Roth came up with such a differencing. Similarly, suppose $(m + a)(n - b)^2, mn^2 \in S(N, \lambda N)$. Then

$$D'_1 = (m + a)(2n - 3b) - m(2n + b) = 2an - 4bm - 3ab$$

$$D'_1 = -\frac{mb^3}{n^2} - \frac{2mb^4}{n^3} - \frac{3mb^5}{n^4} + O\left(\frac{h + mb^5}{n^5}\right) \quad (24)$$

Now, suppose $(m + a_1)(n - b_1)^2, mn^2, (m - a_2)(n + b_2)^2$ and $(m - a_2 - a_3)(n + b_2 + b_3)^2$ are four members in $S(N, \lambda N)$ for some integers $R < b_1, b_2, b_3$ and $h \leq n$. Clearly, $b_1, b_2, b_3 < N/L$ as $\lambda = 1 + 1/L$. Applying [24] to $(m + a_1)(n - b_1)^2, mn^2$ and [25] to $(m - a_2)(n + b_2)^2, (m - a_2 - a_3)(n + b_2 + b_3)^2$, we obtain

$$D_2 := (m + a_1)(2n - 3b_1) - m(2n + b_1) = 2a_1n - 4b_1m - 3a_1b_1$$

$$D_2 = -\frac{mb_1^3}{n^2} - \frac{2mb_1^4}{n^3} - \frac{3mb_1^5}{n^4} + O\left(\frac{h + mb_1^5}{n^5}\right) \quad (25)$$

and

$$D_3 := (m - a_2)(2(n + b_2) - b_3) - (m - a_2 - a_3)(2(n + b_2) + 3b_3) = 2a_3n - 4b_3m + 2a_3b_2 + 4a_2b_3 + 3a_3b_3$$

$$D_3 = -\frac{(m - a_2)b_3^3}{(n + b_2)^3} + \frac{2(m - a_2)b_3^4}{(n + b_2)^4} - \frac{3(m - a_2)b_3^5}{(n + b_2)^5} + O\left(\frac{h + (m + b_3^5}{n^5}\right) \quad (26)$$

Subtracting [25] from [26], we have

$$D_4 := D_3 - D_2$$

$$= 2(a_3 - a_1)n - 4(b_3 - b_1)m + 2a_3b_2 + 4a_2b_3 + 3a_1b_1 + 3a_3b_3$$

$$= \frac{mb_1^3}{n^2} - \frac{mb_1^3+b_3^3}{n^2} - \frac{mb_3^3}{n^2} - \frac{(m - a_2)b_3^3}{n^2} + \frac{(m - a_2)b_3^5}{n^2} - \frac{(m - a_2)b_3^5}{n^2} + \frac{(m - a_2)b_3^5}{n^2} - \frac{(m - a_2)b_3^5}{n^2} + O\left(\frac{h + (m + b_3^5}{n^5}\right) \quad (27)$$

by Lemma [1]. This differencing, $D_4$, was introduced and used in [3] and we call it Roth differencing.

To supplement Roth differencing, Filaseta and Trifonov [4] section 3] also introduced the following second differencing (with slightly different notations):

$$D' := b_2(m + a_1) - (b_1 + b_2 + b_3)m + (b_1 + b_2 + b_3)(m - a_2) - b_2(m - a_2 - a_3)$$

$$= (b_2a_1 - b_1a_2) - (b_2a_2 - b_2a_3).$$

It is more general here as we do not require $b_1 = b_3$. In any case, we call this Filaseta-Trifonov differencing. By [20] in Lemma [3]

$$D' = \frac{3mb_1b_2(b_1 + b_2)}{n^2} - \frac{3(m - a_2)b_2b_3(b_2 + b_3)}{(n + b_2)^2} + \frac{2b_1b_2(a_1b_1 - a_2b_2)}{n^2} - \frac{2b_2b_3(a_2b_2 - a_3b_3)}{n^2}$$

$$- \frac{b_1b_2(a_1b_1^2 + a_2b_2^2)}{2n^3} + \frac{b_2b_3(a_2b_2^2 + a_3b_3^2)}{2(n + b_2)^3} + O\left(\frac{b_2(b_1 + b_3)(a_1b_1^3 + a_2b_2^3 + a_3b_3^3)}{n^4} + \frac{(b_1 + b_2 + b_3)h}{n^2}\right). \quad (28)$$
By Lemmas 1 and 2, the first two terms in (28) can be rewritten as
\[
\left[\frac{3mb_1 b_2 (b_1 + b_2)}{n^2}\right] + \left[\frac{3mb_2 b_3 (b_1 + b_2)}{n^2}\right] + \left[\frac{3mb_2 b_3 (b_1, f + b_2)}{n^2}\right] = \frac{3m(b_1 + b_3) b_2 (b_1 + b_2)}{n^2} + \frac{12mb_3^2 b_2 (b_1 + b_3)}{n^3} - \frac{30mb_3^2 b_3 (b_1 + b_3)}{n^3} + O\left(\frac{mb_3^2 b_3 (b_1 + b_3)}{n^5} + \frac{(b_2 + b_3) h}{n^2}\right) ,
\]
(29)

the third and fourth terms in (28) can be rewritten as
\[
\frac{2a_1 b_1^2 b_2 - 2a_2 b_2^3 (b_1 + b_3) + 2a_3 b_3^2 b_2}{n^2} + \frac{4b_3^2 b_3 (a_2 b_2 - a_3 b_3)}{n^3} + O\left(\frac{b_3^2 b_3 (a_2 b_2 - a_3 b_3)}{n^4}\right)
= \frac{4m(b_3^3 b_3 - b_2^3 (b_1 + b_3) + b_3^3 b_2)}{n^3} + \frac{m(6b_3^4 b_2 + 6b_3^2 b_1 + 14b_3 b_3 - 28b_2 b_1 - 6b_2 b_3)}{n^4} + O\left(\frac{m(b_1 + b_3) (b_1^3 + b_2^3 + b_3^3)}{n^5} + \frac{(b_1 + b_2 + b_3) h}{n^2}\right) ,
\]
(30)

and the fifth and sixth terms in (28) can be rewritten as
\[
\frac{-a_1 b_1^3 b_2 - a_2 b_2^3 (b_1 + b_3) + a_3 b_3^3 b_2}{2n^3} + \frac{O\left(b_3^3 b_3 (a_1 b_1^3 + a_2 b_2^3 + a_3 b_3^3)\right)}{n^4}
= \frac{-b_1^3 b_2 - b_1 b_2^3 b_3 + b_3^3 b_2}{n^4} + \frac{O\left(mb_2^3 b_3 (b_1^3 + b_2^3 + b_3^3)\right)}{n^5}.
\]
(31)

Putting (29), (30) and (31) into (28), we get
\[
D' = \frac{4mb_2 (b_1^3 + b_3^3 + 3b_2^3 b_3 - 3b_1 b_2^3 - b_1 b_2^3)}{n^3} + \frac{3m(b_1 + b_3) b_2 (b_1 + b_2 + b_3)}{n^2} + \frac{m(5b_1^4 b_2 + 5b_1 b_3 - 15b_3^3 b_1 - 30b_3^2 b_3 - 38b_2^2 b_1 - 5b_2 b_3)}{n^4} + O\left(\frac{m(b_1 + b_3) (b_1^3 + b_2^3 + b_3^3)}{n^5} + \frac{(b_1 + b_2 + b_3) h}{n^2}\right).
\]
(32)

For \(h = C_2 x^{5/26}\) and \(n \gg x^{1/4} \gg h^{6/5}\), it is worth mentioning that when \(b_1 = b_3 \) and \(b_2 \gg b_1\),
\[
D' = \frac{4mb_1 (b_2 + b_1) (b_2 + b_1)}{n^3} + O\left(\frac{mb_1 (b_1^3 + b_2^3)}{n^4} + \frac{(b_1 + b_2) h}{n^2}\right) > 0
\]
(33)
as the main term is much bigger than the error terms by \(b_1 + b_2 < n/L\) and (21).

Now, we introduce a new differencing, namely:
\[a_1 - a_3.\]

Note that \(b_1 + b_2 + b_3 < N/L\). By Lemmas 1 and 2
\[
a_1 = \frac{2mb_1}{n} + \frac{3m b_2^2}{n^2} + O\left(\frac{h}{n^2} + \frac{m b_2^3}{n^3}\right)
\]
(34)
and
\[
a_3 = \frac{2(m - a_2) b_3}{n + b_2} - \frac{3(m - a_2) b_2^3}{(n + b_2)^2} + O\left(\frac{h}{n^2} + \frac{m b_2^3}{n^3}\right).
\]
(35)
As \(b_1, b_2, b_3 < N/L\), Lemma 1 implies \(0 < a_1, a_2, a_3 < 10M/L\). Subtracting (35) from (34), we obtain

\[
\begin{align*}
    a_1 - a_3 &= \left[\frac{2mb_1}{n} - \frac{2(m - a_2)b_1}{n}\right] + \left[\frac{2(m - a_2)b_1}{n + b_2}\right] + \left[\frac{2(m - a_2)b_1}{n + b_2}\right]
    
    + \left[\frac{3mb_1^2}{n^2} + 3(m - a_2)b_1^2\right] - \left[\frac{3(m - a_2)b_1^2}{n + b_2} + 3(m - a_2)b_1^3\right]
    
    + O\left(\frac{h}{n^2} + \frac{m(b_1^3 + b_3^3)}{n^3}\right)
    
    = \frac{6mb_1(b_1 + b_2)}{n^2} + \frac{2mb_1(b_1 - b_3)}{n} + O\left(\frac{h}{n^2} + \frac{m(b_1 + b_3)(b_1 + b_2 + b_3)^2}{n^3}\right) + \frac{m(b_1 - b_3)(b_1 + b_2 + b_3)}{n^2}
    
    (36)
\end{align*}
\]

by Lemma 1 and (20). Equation (34) also implies that \(\frac{mb_1}{n} \gg a_1 \geq 1\) or \(b_1 \gg \frac{n}{m}\).

Case 1: \(N \leq x^{2/5}\). We use the bound \(b_1 > R\) and get

\[
\frac{mb_1(b_1 + b_2)}{n^2} \gg \frac{m}{n^2} \frac{b_1^2}{1} \gg \frac{x}{N^4} \left(\frac{N^{4/3}}{x^{1/3}}\right)^2 = \frac{x^{1/3}}{N^{4/3}} \geq \frac{x^{1/5}}{x}.
\]

Case 2: \(N > x^{2/5}\). We use the bound \(b_1 \gg \frac{n}{m} \gg \frac{N^2}{x}\) and get

\[
\frac{mb_1(b_1 + b_2)}{n^2} \gg \frac{m}{n^2} \frac{b_1^2}{1} \gg \frac{N^2}{x} \gg \frac{x^{1/5}}{x}.
\]

In either cases, the main term \(\frac{6mb_1(b_1 + b_2)}{n^2}\) is much bigger than the error term \(O\left(\frac{m}{n}\right)\) while the term \(\frac{2mb_1(b_1 - b_3)}{n^2}\) is much bigger than the error term \(O\left(\frac{m(b_1 + b_2 + b_3)}{n^2}\right)\) unless \(b_1 = b_3\) (in which case both terms vanish).

### 7 Proof of Proposition 3

Here we consider \(x^{2/3} \log^2 x / h^{5/3} < n \leq 2\sqrt{x}\). We break this interval into intervals of the form \((N, \lambda N)\) with \(x^{2/3} \log^2 x / h^{5/3} \leq N \leq 2\sqrt{x}\). Note that, as \(h < x^{1/3}\), we have \(N \geq x^{1/3} \log^2 x\) and \(M \ll x^{1/3} / \log^4 x\). Suppose \((m + a_1)(n - b_1)^2, mn^2, (m - a_2)(n + b_2)^2\) and \((m - a_4 - a_3)(n + b_2 + b_3)^2\) are four members in \(S(N, \lambda N)\) with \(n - b_1 \in T_N, \lambda(b_1), n + b_2 \in T_N, \lambda(b_2)\) for some integers \(R < b_1, b_3\) and

\[
|b_1 - b_3| \leq \frac{0.0001N}{M}. \tag{37}
\]

Recall (36)

\[
a_1 - a_3 = \frac{6mb_1(b_1 + b_2)b_1}{n^2} + \frac{2mb_1(b_1 - b_3)}{n} + O\left(\frac{h}{n^2} + \frac{m(b_1 + b_3)(b_1 + b_2 + b_3)^2}{n^3}\right) + \frac{m(b_1 - b_3)(b_1 + b_2 + b_3)}{n^2}. \tag{38}
\]

If \(a_1 - a_3 = 0\), then we have

\[
b_1 - b_3 + O\left(\frac{|b_1 - b_3|}{n}(b_1 + b_2 + b_3)\right) = -\frac{3(b_1 + b_2)b_1}{n} + O\left(\frac{h}{mn} + \frac{(b_1 + b_3)(b_1 + b_2 + b_3)^2}{n^2}\right) \tag{39}
\]

By (21), the error terms on the right hand side are much less than the corresponding main term. Thus, \(b_1 - b_3\) must be negative and \(b_3 = b_1(1 + O(1/L))\) as \(b_1 + b_2 + b_3 < n/L\). Putting (39) into (32) and (27), we have

\[
D' = -\frac{5mb_1b_2(b_2 + b_1)(b_2 + 2b_1)}{n^3} \left(1 + O\left(\frac{1}{L}\right)\right) + O\left(\frac{(b_1 + b_2)}{n^2}\right) \tag{40}
\]

and

\[
D_4 = -\frac{5mb_1^2(b_1 + b_2)^2}{n^3} \left(1 + O\left(\frac{1}{L}\right)\right) + O\left(\frac{h}{n}\right). \tag{41}
\]
This implies $|D'| \geq 1$ by (21), and $D_4$ has size $mb_1^2(b_1 + b_2)/n^3$ plus an error of $O(h/n)$. If $a_1 - a_3 \neq 0$, then (38) and condition (37) imply

$$(b_1 + b_2)b_1 \geq \frac{n^2}{2m} \quad \text{and} \quad |b_1 - b_3| \leq \frac{0.0001N}{M} \leq \frac{0.001b_1(b_1 + b_2)}{n} \ll \frac{b_1}{L}.$$  

Putting this into (32) and (27), we have

$$D' \geq \frac{3.999mb_1^2(b_2 + b_1)(b_2 + 2b_1)}{n^3} \left(1 + O\left(\frac{1}{L}\right)\right) + O\left(\frac{(b_1 + b_2)h}{n^2}\right)$$  

and

$$D_4 \geq \frac{3.999mb_1^2(b_1 + b_2)}{n^3} \left(1 + O\left(\frac{1}{L}\right)\right) + O\left(\frac{h}{n}\right).$$

Again, this implies $|D'| \geq 1$ by (21), and $D_4$ has size $mb_1^2(b_1 + b_2)/n^3$ plus an error of $O(h/n)$.

Suppose $b_1 \in (B, 2B)$. We break this dyadic interval $(B, 2B]$ into subintervals $I$ of length $0.0001N/M$ according to condition (37). We want to bound $\sum_{b_1 \in I} t_{N,\lambda}(b_1)$. Following [4], we break $(N, \lambda N]$ into subintervals $J$ of length $\frac{N^3}{MB^2}$ and consider the set $U = \left( \bigcup_{b_1 \in I} T_{N,\lambda}(b_1) \right) \cap J$. Define

$$U_{N,\lambda,I,J}(b_2) := \{ n : n + b_2 \text{ are consecutive elements in } U \}, \quad \text{and } u_{N,\lambda,I,J}(b_2) := |U_{N,\lambda,I,J}(b_2)|.$$ 

Clearly, one has

$$\sum_{b_1 \in I} t_{N,\lambda}(b_1) \leq \sum_{J} \left( 1 + \sum_{b_2} u_{N,\lambda,I,J}(b_2) \right).$$

Say, for some $b_1 \in I$, $n - b_1 \in T_{N,\lambda}(b_1) \cap J$ is an element in $U_{N,\lambda,I,J}(b_2)$ for some $b_2$. Then $n + b_2 \in T_{N,\lambda}(b_3) \cap J$ for some $b_3 \in I$. We must have $b_2 > 0$, $b_1 + b_2 \leq \frac{N^3}{MB^2}$ and $D_4 = 0$ if $c > 0$ is sufficiently small. The contribution from those $b_2 \leq C'b_1$ in (44) is small because (10) and (42) imply

$$1 \leq |D'| \ll \frac{Mb_1^4}{N^3} \quad \text{or} \quad b_1 \gg N^{3/4} M^{1/4},$$

and

$$\sum_{b_1} t_{N,\lambda}(b_1) \ll \sum_{b_1 \gg N^{3/4} M^{1/4}} \frac{N/L}{N^{3/4} M^{1/4}} = \frac{x^{1/4}}{LN^{1/4}}.$$  

For $b_2 > C'b_1$ with large enough $C' > 0$, $D_4 = 0$. (11) and (13) give $b_1 + b_2 \ll \frac{hN}{MB}$ while (10) and (42) imply $b_2 \gg b_1 + b_2 \gg \frac{h}{MB^2}$. These give upper and lower bounds for the distance $b_1 + b_2$ on any such quadruples and is a method introduced in [4]. Therefore,

$$\sum_{J} \left( 1 + \sum_{b_2 \gg C'b_1} u_{N,\lambda,I,J}(b_2) \right) \ll \left( \frac{hN^2}{MB^2} \right) + 1 \frac{N/L}{M^{1/3} B^{1/3}} \ll \frac{hM^{1/3} B^{1/3}}{LN} + \frac{MB^3}{LN^2}$$

and, summing over intervals $I$ in $(B, 2B]$ and over dyadic intervals $(B, 2B]$ up to $N \log^{1/2} x/h$ and combining with (45), we have

$$\sum_{b_1 \in (B, 2B]} t_{N,\lambda}(b_1) \ll \sum_{B \ll N \log^{1/2} x/h} \left( \frac{hM^{1/3} B^{1/3}}{LN} + \frac{MB^3}{LN^2} \right) \frac{B}{0.001N} + \frac{x^{1/4}}{LN^{1/4}} \ll \frac{M^{4/3} \log^{1/3} x}{LN^{2/3} h^{1/3}} + \frac{M^2 N \log^{4/3} x}{Lh^4} + \frac{x^{1/4}}{LN^{1/4}} = \frac{x^{4/3} \log^{1/3} x}{LN^{10/3} h^{1/3}} + \frac{x^{2} \log^{4/3} x}{LN^{3} h^{4}} + \frac{x^{1/4}}{LN^{1/4}} \ll \frac{h}{13}$$

as $x^{1/6} < h < x^{1/5}$ and $N \geq x^{2/3} \log^{2} x/h^{5/3}$ by choosing $L$ sufficiently large. Together with (4), we have the proposition.
8 Proof of Proposition 4

From this point onward, we will consider quadruples \((m + a_1)(n - b_1)^2, \quad mn^2, \quad (m - a_2)(n + b_2)^2\) and \((m - a_3)(n + b_3)^2\) with \(n - b_1, n + b_2 \in T_{N, \lambda}(b_1)\) for some integers \(b_1 > R\) and \(b_2 > 0\) (i.e. \(b_1 = b_3\)). If \(b_2 \leq C'b_1\) for some large constant \(C' > 0\), then (42) and (43) imply

\[
1 \leq D' \ll \frac{M_2 b_1^3}{N^3} \quad \text{or} \quad b_1 \gg \frac{N^{3/4}}{M^{1/4}},
\]

and the contribution from such quadruples with \(b_2 \leq C'b_1\) is

\[
\leq \sum_{b_1 \gg N^{3/4}/M^{1/4}} t_{N, \lambda}(b_1) \ll \frac{N/L}{N^{3/4}/M^{1/4}} = \frac{x^{1/4}}{LN^{1/4}}.
\]

(46)

Now, we focus on those quadruples with \(b_2 > C'b_1\). We break \((N, \lambda N)\) into subintervals of length \(\frac{cN^3}{M^{b_1}}\) for some sufficiently small \(c > 0\) so that \(D_4 = 0\) by (27). Then (27) further implies

\[
\frac{Mb_1 b_2}{N^3} \ll \frac{h}{N} \quad \text{or} \quad b_2 \ll \frac{hN^2}{Mb_1^3}
\]

which gives an upper bound of the distance between any two members in \(T_{N, \lambda}(b_1)\) when \(n\) is in such a subinterval. Meanwhile, by (33), \(D' \geq 1\) which implies

\[
\frac{Mb_1 b_2^3}{N^3} \gg 1 \quad \text{or} \quad b_2 \gg \frac{N}{M^{1/3}b_1^{1/3}},
\]

a lower bound for such distance. If \(b_1 \leq N^{1/5-\delta}\), then the contribution

\[
\sum_{b_1 \leq N^{1/5-\delta}} t_{N, \lambda}(b_1) \ll \sum_{b_1 \leq N^{1/5-\delta}} \left( \frac{hN^2}{M^{b_1}b_1} + 1 \right) \frac{N/L}{N^{3/4}/M^{b_1}} \ll \frac{hM^{1/3}}{N^{11/15+4\delta/3}} + \frac{M}{N^{6/5+4\delta}} \ll \frac{hN^{1/3}}{N^{7/5+4\delta/3}} + \frac{x}{N^{16/5+4\delta}}.
\]

(47)

So, from now on, we only consider \(b_1 > Q =: \max(N^{1/5-\delta}, R)\). Suppose \(N \leq C_\alpha x^\alpha\) for some \(\theta < \alpha < 2/5\) with some constant \(C_\alpha > 0\). For a fixed \(b_1 > Q\), define

\[
U_{N, \lambda, b_1}(b_2) := \{n : n \text{ and } n + b_2 \text{ are consecutive elements in } T_{N, \lambda}(b_1)\}, \quad \text{and } u_{N, \lambda, b_1}(b_2) := |U_{N, \lambda, b_1}(b_2)|.
\]

Clearly, one has

\[
\sum_{Q < b_2 \leq N \log^{1/2} x/h} t_{N, \lambda}(b_1) \leq \sum_{Q < b_2 \leq N \log^{1/2} x/h} \left( 1 + \sum_{b_2} u_{N, \lambda, b_1}(b_2) \right)
\ll \frac{N}{h} + \frac{x^{1/4}}{LN^{1/4}} + \frac{hx^{1/3}}{N^{7/5+4\delta/3}} + \frac{x}{N^{16/5+4\delta}} + \sum_{Q < b_1 \leq N \log^{1/2} x/h} \sum_{b_2 > C'/b_1} u_{N, \lambda, b_1}(b_2)
\]

(48)

by (46) and (47). We break the double sum in (48) to the following two pieces:

\[
\sum_{Q < b_1 \leq N \log^{1/2} x/h} \sum_{b_2 > C'b_1} u_{N, \lambda, b_1}(b_2) + \sum_{Q < b_1 \leq N \log^{1/2} x/h} \sum_{b_2 > N^3/b_1} u_{N, \lambda, b_1}(b_2) := S_1 + S_2.
\]

(49)

In \(S_1\), \(b_2 > C'b_1\) and \(b_2 > N^3/b_1\). Again, we break \((N, \lambda N)\) into subintervals of length \(\frac{cN^3}{M^{b_1}}\) for some sufficiently small \(c > 0\) so that \(D_4 = 0\) by (27). Then (27) further implies

\[
\frac{Mb_1^2}{N^{3-\delta}} \ll \frac{Mb_1 b_2}{N^3} \ll \frac{h}{N}
\]

(50)
which gives

\[ b_1 \leq \frac{C_1 h^{1/2} N^{1-\beta/2}}{M^{1/2}} \]  

(51)

for some \( C_1 > 0 \). As \( b_2 > C'b_1 \), (32) and (21) give

\[ \frac{M b_1 b_3^2}{N^3} \gg D' \geq 1 \quad \text{and} \quad b_2 \gg \frac{N}{M^{1/3} b_1^{1/3}} \]  

(52)

for sufficiently large \( C' > 0 \). Moreover, (50) implies

\[ b_2 \ll \frac{h^2 N^{2-2\beta}}{Mb_1^2}. \]  

(53)

Therefore, combining (48), (51), (50), (52) and (53), we obtain

\[ S_1 \ll \sum_{Q < b_1 \leq C_1 h^{1/2} N^{1-\beta/2}/M^{1/2}} \left( \frac{h N^2}{M b_1^2} + 1 \right) \frac{N/L}{M N^{1/3} N(2\beta-1)/3} + \frac{h^5/3}{LM^{1/3}} + \frac{h^2 N^{2-2\beta}}{LM} \]  

(54)

as one can check that \( \frac{N^{1-2\beta/3}}{L^{1-3-2\beta/3}} \leq \frac{N^{1-2\beta}}{L^{1-\theta}} \) when \( N \leq C_\alpha x^\alpha \) and \( \alpha < 2/5 \). By (36) with \( b_1 = b_3 \), we have

\[ 1 \leq a_1 - a_3 \ll \frac{mb_1 b_2}{n}. \]

If \( b_1 b_2 \leq N^{\beta} \), this implies

\[ \frac{N^4}{x} \ll \frac{N^2}{M} \ll b_1 b_2 \ll N^{\beta} \quad \text{or} \quad N \ll x^{1/17}. \]

In view of the first lower bound for \( N \) in (7), this means that the sum \( S_2 \) is empty if \( C'_\alpha \) is sufficiently large, and we have the proposition by (48), (49) and (54).

One can modify the above argument to account for the other bounds in (7). By (80) and (82) with \( b_1 = b_3 \), we have

\[ \frac{N}{b_2^2} \leq D' \frac{N}{b_2^2} \ll a_1 - a_3 \ll \frac{M b_1 b_2}{N^2} \ll \frac{M}{N^{2-\beta}} \]  

when \( b_1 b_2 \leq N^{\beta} \). This implies

\[ \frac{N^{(3-\beta)/2}}{M^{1/2}} \ll b_2 \ll \frac{N^{\beta}}{b_1} \quad \text{or} \quad b_1 \ll \frac{M^{1/2}}{N^{3/2-3\beta/2}}. \]

However, we have \( b_1 > Q \geq R \). This gives

\[ \frac{N^{2/3}}{M^{1/3}} \ll \frac{N^{1/2}}{M^{3/2-3\beta/2}} \quad \text{or} \quad N \ll x^{3/5-\beta}. \]  

(55)

In view of the second lower bound for \( N \) in (7), \( S_2 \) is empty if \( C'_\alpha \) is sufficiently large which gives the proposition.

Similar to (55), one may argue with \( b_1 > Q \geq N^{1/5-\delta} \). Then

\[ \frac{N^{1/5-\delta}}{M^{3/2-3\beta/2}} \ll \frac{M^{1/2}}{N^{3/2-3\beta/2}} \quad \text{or} \quad N \ll x^{27/5-3\beta-28}. \]

In view of the third lower bound for \( N \) in (7), \( S_2 \) is empty if \( C'_\alpha \) is sufficiently large which gives the proposition.

13
9 Proof of Proposition 2

Recall $\delta = 0.0751$. Due to Propositions 1 and 3, we may consider $x^{1/13} \leq \epsilon x^{5/2} / x^{1/4} \leq N \leq x^{2/3} \log^2 x / h^{2/3} \leq x^{9/26} \log^2 x$ as $h = Cx^{5/26}$. Note that the first error term in (63) is $\ll x^{0.16}$ as $N \ll x^{9/26} \log^2 x$, the second error term in (65) is $\ll h x^{0.0009}$ as $\delta = 0.0751$, and $x^{3/13} \ll N$. We apply (4) (similarly for below) and Proposition 4 (using first lower bound in (7)) with $\alpha = 0.34616$ and $\beta = 0.8334$, and obtain

$$ S(C_1 x^{0.3158}, x^{2/3} / h^{5/3}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.0009}} + \frac{x^{1/4}}{L(x^{0.3158})^{1/4}} + \frac{h}{L x^{0.00004}} \ll \frac{hL}{\log^{0.2} x} $$

(56)

for some large $C_1 > 0$ after adding contributions from $\lambda$-adic intervals $(N, \lambda N]$. Next, we apply Proposition 4 (using second lower bound in (7)) with $\alpha = 0.3029$ and $\beta = 0.6668$, and obtain

$$ S(C_2 x^{0.3029}, C_1 x^{0.3158}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.0009}} + \frac{x^{1/4}}{L(x^{0.3029})^{1/4}} + \frac{h}{L x^{0.00002}} \ll \frac{hL}{\log^{0.2} x} $$

(57)

for some large $C_2 > 0$. Next, we apply Proposition 4 (using second lower bound in (7)) with $\alpha = 0.2942$ and $\beta = 0.6274$, and obtain

$$ S(C_3 x^{0.2942}, C_2 x^{0.3029}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.0009}} + \frac{x^{1/4}}{L(x^{0.2942})^{1/4}} + \frac{h}{L x^{0.00003}} \ll \frac{hL}{\log^{0.2} x} $$

(58)

for some large $C_3 > 0$. Next, we apply Proposition 4 (using second lower bound in (7)) with $\alpha = 0.2882$ and $\beta = 0.5988$, and obtain

$$ S(C_4 x^{0.2882}, C_3 x^{0.2942}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.0009}} + \frac{x^{1/4}}{L(x^{0.2882})^{1/4}} + \frac{h}{L x^{0.00005}} \ll \frac{hL}{\log^{0.2} x} $$

(59)

for some large $C_4 > 0$. Next, we apply Proposition 4 (using second lower bound in (7)) with $\alpha = 0.2840$ and $\beta = 0.5781$, and obtain

$$ S(C_5 x^{0.2840}, C_4 x^{0.2882}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.0009}} + \frac{x^{1/4}}{L(x^{0.2840})^{1/4}} + \frac{h}{L x^{0.00004}} \ll \frac{hL}{\log^{0.2} x} $$

(60)

for some large $C_5 > 0$. Next, we apply Proposition 4 (using second lower bound in (7)) with $\alpha = 0.2810$ and $\beta = 0.5629$, and obtain

$$ S(C_6 x^{0.2810}, C_5 x^{0.2840}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.0009}} + \frac{x^{1/4}}{L(x^{0.2810})^{1/4}} + \frac{h}{L x^{0.00005}} \ll \frac{hL}{\log^{0.2} x} $$

(61)

for some large $C_6 > 0$. Next, we apply Proposition 4 (using second lower bound in (7)) with $\alpha = 0.2789$ and $\beta = 0.5521$, and obtain

$$ S(C_7 x^{0.2789}, C_6 x^{0.2810}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.0009}} + \frac{x^{1/4}}{L(x^{0.2789})^{1/4}} + \frac{h}{L x^{0.00004}} \ll \frac{hL}{\log^{0.2} x} $$

(62)

for some large $C_7 > 0$. Next, we apply Proposition 4 (using second lower bound in (7)) with $\alpha = 0.2773$ and $\beta = 0.5521$, and obtain

$$ S(C_8 x^{0.2773}, C_7 x^{0.2789}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.0009}} + \frac{x^{1/4}}{L(x^{0.2773})^{1/4}} + \frac{h}{L x^{0.00005}} \ll \frac{hL}{\log^{0.2} x} $$

(63)
for some large $C_8 > 0$. Next, we apply Proposition 4 (using second lower bound in (7)) with $\alpha = 0.2773$ and $\beta = 0.5437$, and obtain

$$S(C_9x^{0.2762}, C_9x^{0.2773}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2762})^{1/4}} + \frac{h}{Lx^{0.00002}} \ll \frac{hL}{\log^{0.2} x}$$

(64)

for some large $C_9 > 0$. Next, we apply Proposition 4 (using third lower bound in (7)) with $\alpha = 0.2762$ and $\beta = 0.5379$, and obtain

$$S(C_{10}x^{0.2751}, C_9x^{0.2762}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2751})^{1/4}} + \frac{h}{Lx^{0.00002}} \ll \frac{hL}{\log^{0.2} x}$$

(65)

for some large $C_{10} > 0$. Next, we apply Proposition 4 (using the third lower bound in (7)) with $\alpha = 0.2751$ and $\beta = 0.5321$, and obtain

$$S(C_{11}x^{0.2738}, C_{10}x^{0.2751}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2738})^{1/4}} + \frac{h}{Lx^{0.00005}} \ll \frac{hL}{\log^{0.2} x}$$

(66)

for some large $C_{11} > 0$. Next, we apply Proposition 4 (using the third lower bound in (7)) with $\alpha = 0.2722$ and $\beta = 0.5164$, and obtain

$$S(C_{12}x^{0.2722}, C_{11}x^{0.2738}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2722})^{1/4}} + \frac{h}{Lx^{0.00003}} \ll \frac{hL}{\log^{0.2} x}$$

(67)

for some large $C_{12} > 0$. Next, we apply Proposition 4 (using the third lower bound in (7)) with $\alpha = 0.2703$ and $\beta = 0.5060$, and obtain

$$S(C_{13}x^{0.2680}, C_{12}x^{0.2722}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2680})^{1/4}} + \frac{h}{Lx^{0.00003}} \ll \frac{hL}{\log^{0.2} x}$$

(68)

for some large $C_{13} > 0$. Next, we apply Proposition 4 (using the third lower bound in (7)) with $\alpha = 0.2680$ and $\beta = 0.4932$, and obtain

$$S(C_{14}x^{0.2680}, C_{13}x^{0.2703}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2680})^{1/4}} + \frac{h}{Lx^{0.00004}} \ll \frac{hL}{\log^{0.2} x}$$

(69)

for some large $C_{14} > 0$. Next, we apply Proposition 4 (using the third lower bound in (7)) with $\alpha = 0.2653$ and $\beta = 0.4778$, and obtain

$$S(C_{15}x^{0.2653}, C_{14}x^{0.2680}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2653})^{1/4}} + \frac{h}{Lx^{0.00004}} \ll \frac{hL}{\log^{0.2} x}$$

(70)

for some large $C_{15} > 0$. Next, we apply Proposition 4 (using the third lower bound in (7)) with $\alpha = 0.2653$ and $\beta = 0.4592$, and obtain

$$S(C_{16}x^{0.2621}, C_{15}x^{0.2653}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2621})^{1/4}} + \frac{h}{Lx^{0.00001}} \ll \frac{hL}{\log^{0.2} x}$$

(71)

for some large $C_{16} > 0$. Next, we apply Proposition 4 (using the third lower bound in (7)) with $\alpha = 0.2621$ and $\beta = 0.4592$, and obtain

$$S(C_{17}x^{0.2583}, C_{16}x^{0.2621}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2583})^{1/4}} + \frac{h}{Lx^{0.00004}} \ll \frac{hL}{\log^{0.2} x}$$

(72)
for some large $C_{17} > 0$. Next, we apply Proposition 4 (using the third lower bound in (71)) with $\alpha = 0.2539$ and $\beta = 0.4366$, and obtain

$$S(C_{18}x^{0.2539}, C_{17}x^{0.2539}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2539})^{1/4}} + \frac{h}{Lx^{0.00003}} \ll \frac{hL}{\log^{0.2} x}$$

(73)

for some large $C_{18} > 0$. Next, we apply Proposition 4 (using the third lower bound in (71)) with $\alpha = 0.2487$ and $\beta = 0.3762$, and obtain

$$S(C_{19}x^{0.2487}, C_{18}x^{0.2539}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2487})^{1/4}} + \frac{h}{Lx^{0.00003}} \ll \frac{hL}{\log^{0.2} x}$$

(74)

for some large $C_{19} > 0$. Next, we apply Proposition 4 (using the third lower bound in (71)) with $\alpha = 0.2427$ and $\beta = 0.3361$, and obtain

$$S(C_{20}x^{0.2427}, C_{19}x^{0.2487}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2427})^{1/4}} + \frac{h}{Lx^{0.00001}} \ll \frac{hL}{\log^{0.2} x}$$

(75)

for some large $C_{20} > 0$. Finally, we apply Proposition 4 (using the third lower bound in (71)) with $\alpha = 0.2358$ and $\beta = 0.2874$, and obtain

$$S(C_{21}x^{0.2358}, C_{20}x^{0.2427}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(x^{0.2358})^{1/4}} + \frac{h}{Lx^{0.00003}} \ll \frac{hL}{\log^{0.2} x}$$

(76)

for some large $C_{21} > 0$. Finally, we apply Proposition 4 (using the third lower bound in (71)) with $\alpha = 0.2358$ and $\beta = 0.2874$, and obtain

$$S(ch^{5/4}/x^{1/4}, C_{21}x^{0.2358}) \ll \frac{hL}{\log^{0.2} x} + \frac{h}{x^{0.00009}} + \frac{x^{1/4}}{L(ch^{5/2}/x^{1/4})^{1/4}} + \frac{h}{Lx^{0.00003}} \ll \frac{hL}{L^{1/4}}.$$  

(77)

Combining (56) - (77), we have the proposition by picking $L$ sufficiently large.

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