Quadratic Equations in Hyperbolic Groups are NP-complete

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Abstract

We prove that in a torsion-free hyperbolic group $\Gamma$, the length of the value of each variable in a minimal solution of a quadratic equation $Q = 1$ is bounded by $N|Q|^4$ for an orientable equation, and by $N|Q|^{16}$ for a non-orientable equation, where $|Q|$ is the length of the equation, and the constant $N$ can be computed. We show that the problem, whether a quadratic equation in $\Gamma$ has a solution, is NP-complete, and that there is a PSpace algorithm for solving arbitrary equations in $\Gamma$. We also give a slightly larger bound for minimal solutions of quadratic equations in a toral relatively hyperbolic group.

1 Introduction

The study of quadratic equations (an equation is quadratic if each variable appears exactly twice) over free groups began with the work of Malcev [23]. One of the reasons the research in this topic has been so fruitful is a deep connection between quadratic equations and the topology of surfaces (see, for example [13]).

In [7] the problem of deciding if a quadratic equation over a free group is satisfiable was shown to be decidable. In addition it was shown in [27], [13], and [14] that if $n$, the number of variables, is fixed, then deciding if a standard quadratic equation has a solution can be done in time which is polynomial in the sum of the lengths of the coefficients. (In [14] this was shown for hyperbolic groups.) In [17] it was shown that the problem whether a quadratic equation in a free group has a solution, is NP-complete. In the present paper we will show that similar problem is NP-complete in a torsion-free hyperbolic group. There are still very few problems in topology and geometry which are known to be NP-complete, one of them is 3-manifold knot genus [1]. It was proved in [16] that in a free group, the length of the value of each variable in a minimal solution of a standard quadratic equation is bounded by $2s$ for an orientable equation and by $12s^4$ for a non-orientable equation, where $s$ is the sum of the lengths of the coefficients. Similar result was proved in [21] for arbitrary quadratic equations.
If a group $G$ has a generating set $A$, we will consider a solution of a system of equations $S(X,A) = 1$ in $G$ as a homomorphism $\phi : (F(X) \ast G) / \text{ncl}(S) \to G$. We will prove the following results.

**Theorem 1.** Let $\Gamma$ be a torsion-free hyperbolic group with generating set $A$. It is possible to compute a constant $N$ with the property that if a quadratic equation $Q(X,A) = 1$ is solvable in $\Gamma$, then there exists a solution $\phi$ such that for any variable $x$, $|\phi(x)| \leq N|Q|^4$ if $Q$ is orientable, $|\phi(x)| \leq N|Q|^{16}$, if $Q$ is non-orientable. Here $|Q|$ denotes the length of $Q$ as a word in the free group $F(A,X)$.

A group $G$ that is hyperbolic relative to a collection $\{H_1, \ldots, H_k\}$ of subgroups (see Section 4 for a definition) is called toral if $H_1, \ldots, H_k$ are all abelian and $G$ is torsion-free.

**Theorem 2.** Let $\Gamma$ be a toral relatively hyperbolic group with generating set $A$. It is possible to compute a constant $N$ with the property that if a quadratic equation $Q(X,A) = 1$ is solvable in $\Gamma$, then there exists a solution $\phi$ such that for any variable $x$, $|\phi(x)| \leq N|Q|^8$ if $Q$ is orientable, $|\phi(x)| \leq N|Q|^{80}$, if $Q$ is non-orientable. Here $|Q|$ denotes the length of $Q$, as a word in $F(A,X)$.

**Theorem 3.** Let $\Gamma$ be a torsion-free hyperbolic group. The problem whether a quadratic equation has a solution in $\Gamma$, is NP-complete.

### 2 Reduction of equations in hyperbolic groups to equations in free groups

In [29], the problem of deciding whether or not a system of equations $S(Z) = 1$ (we will often just write system $S(Z)$ skipping the equality sign) over a torsion-free hyperbolic group $\Gamma$ has a solution was solved by constructing canonical representatives for certain elements of $\Gamma$. This construction reduced the problem to deciding the existence of solutions in finitely many systems of equations over free groups, which had been previously solved in [24]. The reduction is described below.

Let $\pi$ denote the canonical epimorphism $F(Z,A) \to \Gamma_S$, where $\Gamma_S$ is the quotient of $F(Z) \ast \Gamma$ over the normal closure of the set $S$.

For a homomorphism $\phi : F(Z,A) \to K$ we define $\overline{\phi} : \Gamma_S \to K$ by

$$\overline{(\pi)}^{\overline{\phi}} = \phi,$$

where any preimage $w$ of $\pi$ may be used. We will always ensure that $\overline{\phi}$ is a well-defined homomorphism. For a system $S(Z) = 1$ without coefficients, $\pi$ denotes the canonical epimorphism $F(Z) \to \langle Z \mid S \rangle$ and $\overline{\phi}$ is defined analogously.

**Proposition 1.** Let $\Gamma = \langle A \mid R \rangle$ be a torsion-free $\delta$-hyperbolic group and $\pi : F(A) \to \Gamma$ the canonical epimorphism. There is an algorithm that, given a
system $S(Z, A) = 1$ of equations over $\Gamma$, produces finitely many systems of equations

$$S_1(X_1, A) = 1, \ldots, S_n(X_n, A) = 1$$

(1)

over $F = F(A)$ and homomorphisms $\rho_i : F(Z, A) \rightarrow F_{S_i}$ for $i = 1, \ldots, n$ such that

(i) for every $F$-homomorphism $\phi : F_{S_i} \rightarrow F$, the map $\rho_i \phi \pi : \Gamma_S \rightarrow \Gamma$ is a $\Gamma$-homomorphism, and

(ii) for every $\Gamma$-homomorphism $\psi : \Gamma_S \rightarrow \Gamma$ there is an integer $i$ and an $F$-homomorphism $\phi : F_{S_i} \rightarrow F(A)$ such that $\rho_i \phi \pi = \psi$.

Further, if $S(Z) = 1$ is a system without coefficients, the above holds with $G = (Z \mid S)$ in place of $\Gamma_S$ and ‘homomorphism’ in place of $\Gamma$-homomorphism.

Moreover, $|S_i| = O(|S|^4)$ for each $i = 1, \ldots, n$.

Proof. The result is an easy corollary of Theorem 4.5 of [29], but we will provide a few details.

We may assume that the system $S(Z, A)$, in variables $z_1, \ldots, z_l$, consists of $m$ constant equations and $q - m$ triangular equations, i.e.

$$S(Z, A) = \left\{ z_{\sigma(j,1)} z_{\sigma(j,2)} z_{\sigma(j,3)} = 1 \quad j = 1, \ldots, q - m \\ z_s = a_s \quad s = l - m + 1, \ldots, l \right\}$$

where $\sigma(j, k) \in \{1, \ldots, l\}$ and $a_s \in \Gamma$. An algorithm is described in [29] which, for every $m \in \mathbb{N}$, assigns to each element $g \in \Gamma$ a word $\theta_m(g) \in F$ satisfying

$$\theta_m(g) = g$$

in $\Gamma$

called its canonical representative. The representatives $\theta_m(g)$ are not ‘global canonical representatives’, but do satisfy useful properties for certain $m$ and certain finite subsets of $\Gamma$, as follows.

Let $L = q \cdot \delta_{5050}(\delta + 1)^{q(2|A|)}$. Suppose $\psi : F(Z, A) \rightarrow \Gamma$ is a solution of $S(Z, A)$ and denote

$$\psi(z_{\sigma(j,k)}) = g_{\sigma(j,k)}.$$ 

Then there exist $h^{(j)}_k, c^{(j)}_k \in F(A)$ (for $j = 1, \ldots, q - m$ and $k = 1, 2, 3$) such that

(i) each $c^{(j)}_k$ has length less than $L$ (as a word in $F$),

(ii) $c^{(j)}_1 c^{(j)}_2 c^{(j)}_3 = 1$ in $\Gamma$.

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\[^1\]The constant of hyperbolicity $\delta$ may be computed from a presentation of $\Gamma$ using the results of [12].

\[^2\]The bound of $L$ here, and below, is extremely loose. Somewhat tighter, and more intuitive, bounds are given in [29].

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(iii) there exists \( m \leq L \) such that the canonical representatives satisfy the following equations in \( F \):

\[
\theta_m(g_{\sigma(j,1)}) = h_1^{(j)} c_1^{(j)} \left(h_2^{(j)} \right)^{-1} \tag{2}
\]
\[
\theta_m(g_{\sigma(j,2)}) = h_2^{(j)} c_2^{(j)} \left(h_3^{(j)} \right)^{-1} \tag{3}
\]
\[
\theta_m(g_{\sigma(j,3)}) = h_3^{(j)} c_3^{(j)} \left(h_1^{(j)} \right)^{-1}. \tag{4}
\]

In particular, when \( \sigma(j,k) = \sigma(j',k') \) (which corresponds to two occurrences in \( S \) of the variable \( z_{\sigma(j,k)} \)) we have

\[
h_k^{(j)} c_k^{(j)} \left(h_k^{(j)} \right)^{-1} = h_k^{(j')} c_k^{(j')} \left(h_k^{(j')} \right)^{-1}. \tag{5}
\]

Consequently, we construct the systems \( S_i(X_i, A) \) as follows. For every positive integer \( m \leq L \) and every choice of \( 3(q - m) \) elements \( c_1^{(j)}, c_2^{(j)}, c_3^{(j)} \in F \) \((j = 1, \ldots, q - m)\) satisfying (i) and (ii) we build a system \( S_i(X_i, A) \) consisting of the equations

\[
x_k^{(j)} c_k^{(j)} \left(x_k^{(j)} \right)^{-1} = x_k^{(j')} c_k^{(j')} \left(x_k^{(j')} \right)^{-1} \tag{6}
\]
\[
x_k^{(j)} c_k^{(j)} \left(x_k^{(j)} \right)^{-1} = \theta_m(a_s) \tag{7}
\]

where an equation of type \( \Box \) is included whenever \( \sigma(j,k) = \sigma(j',k') \) and an equation of type \( \blacksquare \) is included whenever \( \sigma(j,k) = s \in \{l - m + 1, \ldots, l\} \). To define \( \rho_i \), set

\[
\rho_i(z_s) = \begin{cases} 
 x_k^{(j)} c_k^{(j)} \left(x_k^{(j)} \right)^{-1}, & 1 \leq s \leq l - m \text{ and } s = \sigma(j,k) \\
 \theta_m(a_s), & l - m + 1 \leq s \leq l
\end{cases}
\]

where for \( 1 \leq s \leq l - m \) any \( j,k \) with \( \sigma(j,k) = s \) may be used.

If \( \psi : F(Z) \to \Gamma \) is any solution to \( S(Z, A) = 1 \), there is a system \( S(X_i, A) \) such that \( \theta_m(g_{\sigma(j,k)}) \) satisfy \( \Box - \Diamond \). Then the required solution \( \phi \) is given by

\[
\phi(x_j^{(k)}) = h_j^{(k)}.
\]

Indeed, (iii) implies that \( \phi \) is a solution to \( S(X_i, A) = 1 \). For \( s = \sigma(j,k) \in \{1, \ldots, l - m\} \),

\[
z_{s}^{\rho_{i}\phi} = h_k^{(j)} c_k^{(j)} \left(h_k^{(j)} \right)^{-1} = \theta_m(g_{\sigma(j,k)})
\]

and similarly for \( s \in \{l - m + 1, \ldots, l\} \), hence \( \psi = \rho_i \phi \pi \).

\textsuperscript{3}The word problem in hyperbolic groups is decidable.
Conversely, for any solution $\phi(x^{(k)}) = h^{(k)}_j$ of $S(X) = 1$ one sees that by (6),

$$z_{\sigma(j,1)} z_{\sigma(j,2)} z_{\sigma(j,3)} \rightarrow \rho_i \phi h^{(j)}_1 c_1^{(j)} c_2^{(j)} c_3^{(j)} (h^{(j)}_1)^{-1}$$

which maps to 1 under $\pi$ by (ii), hence $\rho_i \phi \pi$ induces a homomorphism.

The statement about the length of the systems $S_i = 1$ will follow from the next proposition.

**Proposition 2.** Let $S = S(Z, A) = 1$ be a system of equations over $\Gamma = \langle A \mid R \rangle$. Then, for the systems $S_i = S_i(X, A)$ defined by equations (6) and (7) we have $|S_i| = O(|S|^2)$. If $S(Z, A)$ is a quadratic system of equations, then the systems $S_i = S_i(X, A)$ are quadratic.

In order to prove the above proposition, we first prove the two following lemmas.

**Lemma 1.** Let $S = S(Z, A) = 1$ be a system of equations over $\Gamma = \langle A \mid R \rangle$. We can rewrite $S$ as a triangular system $S'$ such that $|S'| = O(|S|^2)$. If $S$ is quadratic, then $S'$ is also quadratic.

**Proof.** We can assume that $S$ consists of only one equation of the form $y_1 y_2 \cdots y_n = 1$, where either $y_i \in Z$ or $y_i \in \Gamma$. The general case can be proved by a similar argument. At the first step of triangulation we introduce the new variable $x_1$ and we rewrite $S$ as

$$y_1 y_2 x_1 = 1$$
$$x_1^{-1} y_3 y_4 \cdots y_n = 1$$

If we continue the process we get a triangular system $S'$ of the following form:

$$y_1 y_2 x_1 = 1$$
$$x_1^{-1} y_3 x_2 = 1$$
$$x_2^{-1} y_4 x_3 = 1$$
$$\vdots$$
$$x_n^{-1} y_{n-1} y_n = 1$$

The length of each triangular equation is bounded by $3|S|$ and there are $(|S| - 2)$ such equations. Hence, $|S'| \leq (|S| - 2)(3|S|)$.

If $S$ is quadratic, then there are at most two indices $i, j$ such that $y_i = y_j = z$. Hence each variable $z$ appears at most twice in $S'$. Since each new variable $x_i$ also appears twice in $S'$, we conclude that $S'$ is quadratic. $\square$

**Lemma 2.** Let $S(A, Z) = 1$ be a system of equations over $\Gamma$. If we assume that the system $S(A, Z)$ is in a triangular form, then the systems $S_i$'s defined by equations (6) and (7) are of length $|S_i| = O(|S|^2)$.

If we assume that the system $S(A, Z)$ is quadratic and in a triangular form, then the systems $S_i$'s defined by equations (6) and (7) are quadratic.
Proof. Fix a system $S_i$ defined by equations (6) and (7). First we observe that each equation in $S_i$ of the form $x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1} = x_{k'}^{(j')} c_{k'}^{(j')} \left( x_{k'+1}^{(j')} \right)^{-1}$ corresponds to the triples

$$\theta_m(g_{\sigma(j,1)}) = h_1^{(j)} c_1^{(j)} \left( h_2^{(j)} \right)^{-1},$$
$$\theta_m(g_{\sigma(j,2)}) = h_2^{(j)} c_2^{(j)} \left( h_3^{(j)} \right)^{-1},$$
$$\theta_m(g_{\sigma(j,3)}) = h_3^{(j)} c_3^{(j)} \left( h_1^{(j)} \right)^{-1}.$$

and

$$\theta_m(g_{\sigma(j',1)}) = h_1^{(j')} c_1^{(j')} \left( h_2^{(j')} \right)^{-1},$$
$$\theta_m(g_{\sigma(j',2)}) = h_2^{(j')} c_2^{(j')} \left( h_3^{(j')} \right)^{-1},$$
$$\theta_m(g_{\sigma(j',3)}) = h_3^{(j')} c_3^{(j')} \left( h_1^{(j')} \right)^{-1}.$$

where $\sigma(j, k) = \sigma(j', k')$. If $S$ is quadratic, for each pair $(j, k)$, there is at most one pair $(j', k')$ such that $\sigma(j, k) = \sigma(j', k')$. Hence, there is at most one equation $x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1} = x_{k'}^{(j')} c_{k'}^{(j')} \left( x_{k'+1}^{(j')} \right)^{-1}$, involving $x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1}$.

We conclude that $x_k^{(j)}$ appears at most in two equations of the following form:

$$x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1} = x_{k'}^{(j')} c_{k'}^{(j')} \left( x_{k'+1}^{(j')} \right)^{-1},$$
$$x_{k-1}^{(j)} c_{k-1}^{(j)} \left( x_k^{(j)} \right)^{-1} = x_{k'}^{(j')} c_{k'}^{(j')} \left( x_{k'+1}^{(j')} \right)^{-1}.$$

Hence, $S_i$ is quadratic.

In order to find an upper bound on the length of $S_i$ we have to look at two types of equation which appear in $S_i$. The first type corresponds to variables in $S$. For each occurrence of variable $x_{\sigma(j,k)}^{(j)} \in S$ we have an equation $s : x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1} = x_{k'}^{(j')} c_{k'}^{(j')} \left( x_{k'+1}^{(j')} \right)^{-1}$. The length of such equation is bounded by $4 + (|c_k| + |c_{k'}|)$. We have that $|c_k| + |c_{k'}|$ is bounded by $2L$, where $L$ is the constant introduced in the proof of Proposition I. Hence, we get $|s| \leq 4 + 2L$.

The second type of equations corresponds to constants appearing in $S$. For each constant $a_s \in S$, there is an equation $x_k^{(j)} c_k^{(j)} \left( x_{k+1}^{(j)} \right)^{-1} = \theta_m(a_s)$ in $S_i$, where $\theta_m(a_s) \in F_A$ is the label of a $(\lambda, \mu)$-quasi-geodesic path from 1 to $a_s$ in the
Cayley graph of $\Gamma$, for some fixed $\lambda$ and $\mu$. Thus, $|\theta_m(a_s)| \leq \lambda|a| + \mu \leq \lambda|S| + \mu$. Therefore $2 + L + \lambda|S| + \mu$ is an upper bound for the length of such equation.

Since there are at most $|S|$ equations in $S_i$, we get $|S_i| \leq |S|(4 + 2L + \lambda|S| + \mu) = O(|S|^2)$. This finishes the proof of the lemma.

Now we can prove Proposition 2.

**Proof.** (of Proposition 2) We rewrite $S(Z, A)$ as a triangular system $S'(Z', A)$ and we consider the systems $S'_i$'s defined by equations (6) and (7) for $S'$. From Lemma 1 and Lemma 2, we have $|S'_i| = O(|S'|^2) = O(|S|^4)$ which proves Proposition 2.

Proposition 2 and [21], Theorem 1.1 or [16], Corollary 1 implies the statement of Theorem 1.

Proposition 2 and the result of Gutierres [15] about the PSpace algorithm for solving equations in a free group imply the following result.

**Theorem 4.** [20] Let $\Gamma$ be a torsion-free hyperbolic group. There is a PSpace algorithm for solving equations in $\Gamma$.

### 3 Quadratic equations in free products

In this section we will prove results about quadratic equations in free products. We will use them in the proof of Theorem 2 in the next section.

**Theorem 5.** One can compute a constant $N$ with the following properties. Let $Q$ be a quadratic word. If the equation $Q = 1$ is solvable in a free product of a free group and free abelian groups of finite ranks, then there exists a solution $\alpha$ such that for any variable $x$, $|\alpha(x)| < N((n(Q) + c(Q))^{20}$ for orientable equation and $|\alpha(x)| < N((n(Q) + c(Q))^{20}$ for non-orientable equation. Here $n(Q)$ denotes the total number of variables in $Q$ and $c(Q)$ the total length of coefficients occurring in $Q$.

**Definition 1.** For a group $G$ we define the orientable genus of an $m$-tuple \{ $C_i, C$ \} $\in G$, to be the least integer $g \geq 0$ for which the equation

$$\left( \prod_{i=1}^g [x_i, y_i] \right) \left( \prod_{j=1}^{m-1} z_j C_j z_j \right) C = 1 \quad (8)$$

holds for some $\{x_i, y_i, z_j\} \in G$, where $[x, y] = x^{-1} y^{-1} x y$.

**Definition 2.** For a group $G$, we define the non-orientable genus of an $m$-tuple \{ $C_i, C$ \} $\in G$, to be the least integer $g \geq 0$ for which the equation

$$\left( \prod_{i=1}^g x_i^2 \right) \left( \prod_{j=1}^{m-1} z_j^{-1} C_j z_j \right) C = 1 \quad (9)$$

holds for some $\{x_i, z_j\} \in G$. 

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Definition 3. An orientable quadratic set of words is a quadratic set of cyclic words $w_1,w_2,\ldots,w_k$ (a cyclic word is the orbit of a linear word under cyclic permutations) in some alphabet $a_1^{\pm 1},a_2^{\pm 1},\ldots$ of letters $a_1,a_2,\ldots$ and their inverses $a_1^{-1},a_2^{-1},\ldots$ such that if $a_i^\epsilon$ appears in $w_j$ (for $\epsilon \in \{\pm 1\}$) then $a_i^{-\epsilon}$ appears exactly once in some $w_j'$.

Definition 4. A non-orientable quadratic set of words is a quadratic set of cyclic words $w_1,w_2,\ldots,w_k$ (a cyclic word is the orbit of a linear word under cyclic permutations) in some alphabet $a_1^{\pm 1},a_2^{\pm 1},\ldots$ of letters $a_1,a_2,\ldots$ and their inverses $a_1^{-1},a_2^{-1},\ldots$ such that if $a_i^\epsilon$ appears in $w_j$ (for $\epsilon \in \{\pm 1\}$) then $a_i^{\pm \epsilon}$ appears exactly once in $w_j$, and there is at least one letter which appears twice with the same exponent.

Similar to the case of Wicks forms, we associate an orientable (non-orientable) surface to an orientable (non-orientable) quadratic set of words. For this we take a set of discs with the words of the quadratic set on the boundaries, and identify the edges with the same labels respecting orientation. The genus of a quadratic set of words is defined as the sum of genera of the surfaces obtained from $k$ discs with words $w_1,w_2,\ldots,w_k$ on their boundaries.

Consider the orientable (non-orientable) compact surface $S$ associated to an orientable (non-orientable) quadratic set. This surface carries an embedded graph $\Gamma \subset S$ such that $S \setminus \Gamma$ is a set of open polygons. This construction also works in the opposite direction: Given a graph $\Gamma \subset S$ with $e$ edges on an orientable (non-orientable) compact connected surface $S$ of genus $g$ such that $S \setminus \Gamma$ is a collection of discs, we get an orientable (non-orientable) quadratic set of words of genus $g$ by labelling and orienting the edges of $\Gamma$ and by cutting $S$ open along the graph $\Gamma$. The associated orientable (non-orientable) quadratic set of words can be read on the boundary of the resulting polygons. We henceforth identify, orientable (non-orientable) quadratic sets with the associated embedded graphs $\Gamma \subset S$, allowing the language of of vertices and edges of orientable (non-orientable) of quadratic sets. Moreover, the quadratic set can be associated with a set $T$ of closed paths in $\Gamma$, such every edge of $\Gamma$ is traversed exactly twice. We will call $T$ a quadratic set of circuits.

Definition 5. Let $v$ be a vertex of $\Gamma$ with edges $a_1,\ldots,a_l$ emanating from $v$. If $\Gamma$ corresponds to an orientable quadratic set $Q$, then this set contains subwords $a_1^{-1}a_2,\ldots,a_{l-1}^{-1}a_l,a_l^{-1}a_1$ (of course, these subwords can be contained in different words of the set). If $Q$ is non-orientable, then some of these subwords are reversed. This set of subwords will be called girth of the vertex $v$. We will say, that the vertex $v$ is extended by a cyclic word $W$ of some group $H$, $\psi_1\ldots\psi_l = W$ in $H,\psi_1,\psi_l \in H$, if subwords $a_1^{-1}a_2,\ldots,a_{l-1}^{-1}a_l,a_l^{-1}a_1$ are replaced by $a_1^{-1}\psi_1a_2,\ldots,a_{l-1}^{-1}\psi_{l-1}a_l,a_l^{-1}\psi_1a_1$. If a subword $a_j^{-1}a_{j+1}$ is reversed, i.e. occurs as $a_{j+1}^{-1}a_j$, then the corresponding $\psi$ appears in the product with the negative exponent.

Example 1. $AB^{-1}AC^{-1}BC^{-1}$

$$AB^{-1}\psi_3AC^{-1}\psi_2BC^{-1}\psi_1,\psi_1^{-1}\psi_2\psi_3 = W \in H.$$
**Definition 6.** Let $Q$ be an orientable (non-orientable) quadratic set and $\Gamma$ be the associated graph. Let vertices $v_1, \ldots, v_t$ of $\Gamma$ be extended by words $W_1, \ldots, W_t \in H$. We will say, that an orientable (non-orientable) genus $g$ joint extension $\Delta$ of $Q$ is constructed on these $t$ vertices, if

- the $t$-tuple $(W_1, \ldots, W_t)$ is orientable and the the genus of $(W_1, \ldots, W_t)$ is $l = g - t + 1$ in $H$
- or
- the $t$-tuple $(W_1, \ldots, W_t)$ is non-orientable and the the genus of $(W_1, \ldots, W_t)$ is $l = g - 2t + 2$ in $H$.

The sum of the lengths of $W_i$ will be called the length of the extension.

**Definition 7.** Let $G$ be a free product of groups $G_1, G_2, \ldots$. Let $Q$ be an orientable (non-orientable) quadratic set of genus $k$ and $\Gamma$ be the associated graph. We will separate all vertices of $\Gamma$ into $p + 1$ disjoint sets $B_0, B_1, \ldots, B_r, \ldots, B_p$. We leave the vertices of $B_0$ without changes. The genus $g_i$ orientable (non-orientable) joint extension by some free factor is constructed on the vertices of the set $B_i, i = 1, \ldots, p$ (Different sets can be extended by the same free factor or by different free factors). We consider the following cases:

1. $Q$ is orientable and all $g_i$-extensions are orientable. Then we define $n$ to be
   $$ n = k + \sum_{i=1}^{p} g_i $$

2. $Q$ is non-orientable and at least one of $g_i$-extensions is non-orientable. Then we define $n$ to be
   $$ n = k + \sum_{i=1}^{p} g_i $$

3. $Q$ is orientable and at least one of $g_i$-extensions is non-orientable. Then we define $n$ to be
   $$ n = 2k + \sum_{i=1}^{p} g_i $$

4. $Q$ is non-orientable and all $g_i$-extensions are orientable. Then we define $n$ to be
   $$ n = k + 2 \sum_{i=1}^{p} g_i $$

If the quadratic set and all the extensions are orientable, we will call the resulting set of words $A$ orientable multi-form of genus $n$. If the quadratic set is non-orientable and/or at least one of the extensions if non-orientable, then we get a non-orientable multi-form, and the genus is computed according to the corresponding Euler characteristic.
Definition 8. The set $Q$ from Definition 7 will be called a framing set of words.

Definition 9. Let a quadratic set $Q$ be a collection of words in a group alphabet $B$, and $W$ be a multi-form such that $Q$ is the framing set of words for this multi-form. Let $V_1,\ldots,V_l$ be elements of a free product $G$ which is obtained from a multi-form $W$ over the free product by substitution of letters of the alphabet $B$ by elements of $G$. We will say, that $V$ is obtained from $W$ by a permissible substitution, if elements of the same free factor don't occur in $V$ side by side.

Example 2. We give an example of a non-orientable multi-form of non-orientable genus 15:

$$W_1 = A\xi_1 ED\psi_1 C^{-1} B\xi_3 A^{-1} FB\xi_2 EG_1 G_2 H_1 O_2 O_1^{-1} I_1,$$
$$W_2 = C\psi_2 H_2 O_2 ZG_2 I_2 I_1 F,$$
$$W_3 = D\psi_3 H_2 H_1^{-1} I_2 O_1 ZG_1,$$

where $\xi_1 \xi_2^{-1} \xi_3 = U_1, \psi_1 \psi_2 \psi_3^{-1} = U_2, U_1^e U_2^{-e_2} = U$ in some free factor $G_i$, $U$ has an orientable genus 3 in $G_i$ and $G_i$ is a free group.

Lemma 3. Let $\Delta$ be a connected genus $k$ surface with $t$ holes. Suppose the words $u_1,\ldots,u_t$ are written around the boundaries of these holes. We assume that the set $\{u_1,\ldots,u_t\}$ is quadratic. If we identify the boundaries of the holes in $\Delta$ according to their labels we get another surface $\Delta'$. Let the genus of $\Delta'$ be $n$. We also assume that when we write the words $u_1,\ldots,u_t$ around the boundaries of $t$ disks and identify the 1-cells according to their labels, we get one compact closed surface $\Delta_1$. Then we have the following cases.

(i) If $\Delta$ and the set $\{u_1,\ldots,u_t\}$ are both orientable, then we have
$$g\{u_1,\ldots,u_t\} = g\Delta_1 = n - k - t + 1$$

(ii) If $\Delta$ and the set $\{u_1,\ldots,u_t\}$ are both non-orientable, then we have
$$g\{u_1,\ldots,u_t\} = g\Delta_1 = n - k - 2t + 2$$

(iii) If $\Delta$ is orientable and the set $\{u_1,\ldots,u_t\}$ is non-orientable, then we have
$$g\{u_1,\ldots,u_t\} = g\Delta_1 = n - 2k - 2t + 2$$

(iv) If $\Delta$ is non-orientable and the set $\{u_1,\ldots,u_t\}$ is orientable, then we have
$$g\{u_1,\ldots,u_t\} = g\Delta_1 = \frac{n - k - 2t + 2}{2}$$

Proof. We give a proof for the case (ii). Other cases can be proved in a similar way. Let, as in the statement of the proposition, $\Delta_1$ be the surface that we get by writing the words $u_1\cdots u_t$ around the boundaries of $t$ disks and identifying
the 1-cells according to their labels. Then the surface $\Delta'$ is a connected sum of $\Delta$ and $\Delta_1$ and $t - 1$ tori:

$$\Delta' = \Delta \# \Delta_1 \# \underbrace{T \# \cdots \# T}_{(t - 1)\text{-times}}.$$  

Since $\Delta$ and $\Delta_1$ are both non-orientable, this connected sum is the same as a connected sum of $k + g_{\Delta_1}$ projective planes and $t - 1$ tori. Hence,

$$\Delta' = \underbrace{P \# \cdots \# P}_{(k + g_{\Delta_1})\text{-times}} \# \underbrace{T \# \cdots \# T}_{(t - 1)\text{-times}}.$$  

Since the connected sum of a torus and a projective plane is homeomorphic to the connected sum of three projective planes, we have

$$\Delta' = \underbrace{P \# \cdots \# P}_{(k + g_{\Delta_1} + 2)\text{-times}} \# \underbrace{T \# \cdots \# T}_{(t - 2)\text{-times}}.$$  

If we continue rewriting this connected sum in terms of the connected sum of projective planes, we get that

$$\Delta' = \underbrace{P \# \cdots \# P}_{(k + g_{\Delta_1} + 2(t - 1))\text{-times}}.$$  

Hence, the genus of $\Delta'$ is equal to $k + g_{\Delta_1} + 2(t - 1)$:

$$n = k + g_{\Delta_1} + 2(t - 1).$$

Solving this for $g_{\Delta_1}$, we get

$$g_{\Delta_1} = g_{\{u_1, \ldots, u_t\}} = n - k - 2(t - 1),$$

which proves (ii). \hfill \Box

**Proposition 3.** Any solution of a quadratic equation in a free product of groups can be obtained from a multi-form of genus $g$ over the free product.

**Proof.** Let $V_1, \ldots, V_k$ be a solution of an orientable quadratic equation. Applying Olshanskii’s results \[27\] for quadratic equations in free groups, we may assume, that an orientable quadratic set of words $U_1, \ldots, U_m$ can be written on boundaries of $m$ discs labeled by elements of the free product in normal form, such that these discs define a surface of genus $g$. The boundaries of discs become a graph $\Gamma$ on the surfaces with elements of the free product written on the edges, each element is in normal form of the free product. We divide the edges of $\Gamma$ with vertices of degree two according to the normal forms of the words written on the edges. We can say, that there is a labelling function $\phi$ on the edges of $\Gamma$ by elements of the free product and $\Gamma$ may be equipped by quadratic set of circuits $T$ such that the words $U_1, \ldots, U_m$ can be read along $T$.  

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Let's $p$ be a path in $\Gamma$ such that $\phi(p) = 1$, and $v$ and $w$ be its end points. Vertices $v$ and $w$ don't coincide, since it would decrease the genus of the equation. Now, there is an edge $e_j$, which appears in $p$ exactly once (otherwise the genus of the equation would be smaller). We identify the vertices $v$ and $w$ of $\Gamma$ and delete $e_j$. We say that the new graph $\Gamma'$ is obtained from $\Gamma$ by $\tau_k$-transformation. Since the number of edges in $\Gamma'$ is one less than the number of edges in $\Gamma$, if we continue this process, we get a graph $\Gamma'$ and quadratic set of circuits $T'$. The words $U_1, ..., U_m$ can be read along the circuits $T'$ and $\Gamma'$ does not have any subpath $p$ such that $\phi(p) = 1$.

In the next step, we show that removing all connected components of $\Gamma'$ which are labeled by elements of the same free factor, corresponds to constructing joint extensions of a framing word.

Let $K$ be a connected subgraph of $\Gamma'$ whose edges are labeled by elements of a free factor $G_i$. We take $K$ to be maximal. We call a vertex a boundary vertex if it is incident to both $K$ and $\Gamma' \setminus K$. We refer to the edges labeled by $K$ as $K$-edges and to the edges labeled by $\Gamma' \setminus K$ as $K^c$-edges.

Let $w$ be a boundary vertex. Without loss of generality we assume that all edges incident to $w$ are leaving it. Let

$$a_{1,1}^{-1}a_{1,2}, \ldots, a_{1,t_1}^{-1}a_{1,1}, a_{1,t_1}^{-1}b_{1,1}, b_{1,1}^{-1}b_{1,2}, \ldots, b_{1,s_1}^{-1}a_{2,1},$$

$$\ldots, a_{2,t_2}^{-1}b_{2,1}, \ldots, a_{r,t_r}^{-1}b_{r,1}, \ldots, b_{r,s_r}^{-1}a_{1,1}$$

be a girth of $w$, where $b_{i,j}$ edges are $K$-edges and $a_{i,j}$ edges are $K^c$-edges. We refer to the set of $K$-edges ($K^c$-edges) not separated by $K^c$-edges ($K$-edges) as $K$-bundles ($K^c$-bundles)( see Figure 1).

![Figure 1](image-url)
We replace the vertex \( w \) by a sequence of vertices \( w_1, w_2, \ldots, w', w_2 \) and we insert an edge \( e_i \) between \( w_1 \) and \( w_2 \) pointing towards \( w_1 \) and an edge \( e'_i \) between \( w_1 \) and \( w_{i+1} \) pointing towards \( w_2 \). Each \( K^c \)-bundle \( a_{i,1}, \ldots, a_{i,t_i} \) is incident to \( w_1 \) and each \( K \)-bundle \( b_{i,1}, \ldots, b_{i,s_i} \) to \( w_2 \). We let \( \phi(e_i) = \phi(e'_i) = 1 \) and we consider them as \( K^c \)-edges. We call \( w_1 \) a boundary \( K^c \)-vertex (see Figure 2).

If we do this for all boundary vertices, we get a new graph \( \Gamma'' \) with quadratic set of circuits \( T'' \) such that the words \( U_1, \ldots, U_m \) can be read along \( T'' \). A subpath of \( T'' \) consisting of only \( K(K^c) \)-edges is called \( K(K^c) \)-subpath.

![Figure 2](image)

Let \( v_1 = w_1 \) be a boundary \( K^c \)-vertex with \( a_{1,1}, \ldots, a_{1,t_1} \) and \( e_1 \) leaving \( v_1 \). Then the girth of \( v_1 \) is

\[
e_1^{-1}a_{1,1}, a_{1,1}^{-1}a_{1,2}, \ldots, a_{1,t_1-1}^{-1}a_{1,t_1}, a_{1,t_1}^{-1}a_{1,1}e_1.
\]

Let \( B_1 \) be the \( K \)-subpath which is traversed after \( e_1 \) in \( T'' \). Let \( e_2^{-1} \) be the first edge which is taken after this \( K \)-path by \( T'' \) and \( v_2 \) be the initial vertex of \( e_2 \). Hence, we have the following sequence of subpaths in \( T'' \):

\[
e_1^{-1}a_{1,1}, a_{1,1}^{-1}a_{1,2}, \ldots, a_{1,t_1-1}^{-1}a_{1,t_1}, a_{1,t_1}^{-1}a_1B_1e_2^{-1}a_{2,1}, a_{2,1}^{-1}a_{2,2}, \ldots, a_{2,t_2-1}^{-1}a_{2,t_2}, a_{2,t_2}^{-1}e_2.
\]

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A subpath $B_2$ follows $e_2$ in $T''$ and $e_3$ follows $B_2$. Since $T''$ is finite, at some point it comes back to $v_1$. So there exists a sequence of subpaths of $T''$ of the following order:

\[ a_1^{-1}a_{1,2}, \ldots, a_i^{-1}e_1B_1e_2^{-1}a_{2,3}, \ldots, a_2^{-1}e_2B_2e_3^{-1}a_{3,1}, \ldots, e_m^{-1}a_{m,1}, \ldots, a_m B_m e_1^{-1}a_{1,1}. \quad (10) \]

We will call such a sequence a chain. Each vertex is included in some chain and boundary $K^s$-vertices can be partitioned into disjoint classes according to the chain in which they appear. Let $t$ be the number of these classes.

If we write $T''$ along the boundary of several disks and identify $K^s$-edges and $e$-edges according to their labels, we get a surface $\Delta$ with $t$ holes $\Delta_1, \ldots, \Delta_t$. The labels of these holes are $K$-paths. We denote these labels by $u_i$. For example if we consider the chain (10) mentioned above, the cyclic word $u := B_1 \cdots B_m$ will be written around one of these holes. Let the genus of $\Delta$ be $k$.

Let us assume that the set $\{u_1, \ldots, u_t\}$ is an orientable quadratic set of words. We glue a disk $D_i$ to each hole $\Delta_i$ by identifying the boundary of $D_i$ with the boundary of $\Delta_i$. Then we shrink each $\Delta_i$ to a vertex $v_i$. Let $\Delta'''$ be the surface that we get after this operation. Then $\Delta'''$ has a graph $\Gamma'''$ and a circuit $T'''$ such that the words $U''_1, \ldots, U''_m$ can be read along $T'''$. It is clear that if we cancel all $K$-paths in $T''$, we get $T'''$. The words $U'''_1, \ldots, U'''_m$ can be obtained from $U_1, \ldots, U_m$ by constructing a joint extension on $v_1, \ldots, v_t$ by $u_1, \ldots, u_t \in K$.

If we identify all $K$-edges in $\Delta$ according to their labels, we get a closed orientable surface $\Delta'$ and an associated graph with circuits along which we can read $U_1, \ldots, U_m$. Hence, the genus of $\Delta'$ is $n$. By lemma 3 we get that

\[ g_{G_1}(u_1, \ldots, u_t) = n - t - k + 1. \]

Hence a joint extension of genus $g_{G_1}(u_1, \ldots, u_t) + t + 1$ is constructed on $v_1, \ldots, v_t$ by $u_1, \ldots, u_t \in K$. The resulting multi-form has genus $n$ (see definition 7). All other cases where either the equation or the set $\{u_1, \ldots, u_t\}$ (or both) are non-orientable can be proved similarly.

Now, we go back to $T'$. Let $K_1, \ldots, K_m$ be connected subgraphs of $\Gamma'$ such that each $K_i$ is labeled by elements of some free factors and has no less than two edges. We assume that if $K_i$ and $K_j$ are labeled by elements of the same free factor, they are not sharing any edges nor any vertices. By applying the above process to each $K_i$ we get the statement of the theorem.

\[ \square \]

**Proof of Theorem 5.**

By Proposition 3 we have that any solution of a quadratic equation in a free product of groups can be obtained from a multi-form $A$ of genus $g$ over the free product. Proving the estimates we will refer to the geometric structure of the quadratic set of genus $g$ over the free product.

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Let $G$ be a free product of groups $G_1, G_2, \ldots$. If $U_1, \ldots, U_m$ are left sides of a multi-form, then by Proposition 3 $U_1, U_2, \ldots, U_m$ can be obtained by a permissible substitution. Let $A$ be the corresponding orientable (non-orientable) multi-form, $\Gamma$ be the associated graph. By the definition of the multi-form $A$, it is obtained by separation of all vertices of $\Gamma$ into $p+1$ disjoint sets $B_0, B_1, \ldots, B_p$ and genus $g_i$ extension on the vertices of $B_i$, $i = 1, \ldots, p$. Let a set $B_i$ has $l_i$ vertices.

It means that every of these $l_i$ vertices is extended by some word $W_{i,s}$, where $s = 1, \ldots, l_i$. To proceed we need the following definition:

**Definition 10.** Let the vertex $v$ be extended by a cyclic word $W$ of some group $H$, $\psi_1 \ldots \psi_l = W$ in $H$, $\psi_i \in H$, $a_1^{-1} \psi_1 a_2, \ldots, a_{l-1}^{-1} \psi_{l-1} a_l, a_l^{-1} \psi_1 a_1$. We will say, that the extended vertex is augmented if the words $a_1^{-1} \psi_1 a_2, \ldots, a_{l-1}^{-1} \psi_{l-1} a_l, a_l^{-1} \psi_1 a_1$ are replaced by $A_1^{-1} a_2, \ldots, A_{l-1}^{-1} A_l, A_l^{-1} W A_1$, where $A_1 = a_1$ and $A_i = \psi_1 \psi_2 \ldots \psi_{i-1} a_i$, where $i = 2, \ldots, n - 1$.

**Example 3.** Let $v$ be the vertex extended by a cyclic word $W$ of a group $H$, $\psi_1 \psi_2 \psi_3 \psi_4 = W$ in $H$, $\psi_1 \in H$, $a_1^{-1} \psi_1 a_2, a_2^{-1} \psi_2 a_3, a_3^{-1} \psi_3 a_4, a_4^{-1} \psi_4 a_1$.

Augmentation of $v$: $a_1^{-1} \psi_1 a_2$, $a_2^{-1} \psi_1^{-1} \psi_2 a_3$, $a_3^{-1} \psi_2^{-1} \psi_3 a_4$, $a_4^{-1} \psi_3^{-1} \psi_2^{-1} \psi_1^{-1} \psi_1 \psi_2 \psi_3 \psi_4 a_1$.

Now we can replace $a_1$ by $A_1$, $\psi_1 a_2$ by $A_2$, $\psi_1 \psi_2 a_3$ by $A_3$, $\psi_1 \psi_2 \psi_3 a_4$ by $A_4$ and the product $\psi_1 \psi_2 \psi_3 \psi_4$ by $W$.

Denote the length of the original equation $n(Q) + c(Q)$ by $M$. If we perform augmentation of every extended vertex of $A$, the length of every new letter is estimated from above by $M$ and the result of augmentation of every extended vertex will look like the framing set of words plus $W(i, s)$ which were used for the joint extensions. Words $W(i, s)$ for a fixed $i$ are coefficients for a standard quadratic equation in a free factor.

Consider the case, when all free factors are either free groups, or abelian groups. (Example 4 illustrates the proof.) We denote by $M_i$ the length of the $i$th extension. Then the length of each component of the quadratic equation associated with the extension is bounded linearly by $M_i$ in the orientable case and by a polynomial of degree 4 on $M_i$ in the non-orientable case, see [10, 21]. Let $U_1', \ldots, U_m'$ be the left sides of the multi-form after augmentation. The tuple $U_1', \ldots, U_m'$ is presented by a quadratic set of words in a free group now. The total length of $U_1', \ldots, U_m'$ is bounded by $6M^2$ (in orientable case) and $60M^5$ (in non-orientable case), and, therefore, values of variables in the quadratic set of words are bounded by the same numbers. For the quadratic equation $Q = 1$ in the formulation of the theorem, the tuple $U_1', \ldots, U_m'$ represents the coefficients. Now we may use the results for the quadratic equations in free groups from [10, 21] to obtain the estimates in Theorem 5. Theorem 5 is proved.

**Example 4.** Let a solution of a quadratic equation is obtained from a non-orientable multi-form of Example 2 by a permissible substitution:

$$W_1 = A\xi_1 ED\psi_1 C^{-1} B\xi_3 A^{-1} FBC_2 E\xi_1 G_2 H_1 O_2 O_1^{-1} I_1.$$
\[ W_2 = C\psi_2 H_2 O_2 ZG_2 I_2 I_1 F, \]
\[ W_3 = D\psi_3 H_2 H_1^{-1} I_2 O_1 ZG_1, \]

where \( \xi_1 \xi_2^{-1} \xi_3 = U_1, \psi_1 \psi_2 \psi_3^{-1} = U_2, \) \( U_1^{-1} U_2^{-c_2} = U \) in some free factor \( G_i \) and \( U \) has an orientable genus 3 in \( G_i, |W_1| + |W_2| + |W_3| = s. \)

Now we have to bring the multi-form to a quadratic set in a free group. First of all, we perform augmentations and our multi-form looks the following way:

\[ W_1' = AE_1 DC_1^{-1} B_1 U_1 A^{-1} F B_1 E_1 G_1 G_2 H_1 O_2 O_1^{-1} I_1, \]
\[ W_2' = C_1 H_2 O_2 ZG_2 I_2 I_1 F, \]
\[ W_3' = DU_2^{-1} H_3 H_1^{-1} I_2 O_1 ZG_1, \]

where \( E_1 = \xi_1 E, B_1 = B\xi_2 \xi_1^{-1}, C_1 = C\psi_1^{-1}, H_3 = \psi_1 \psi_2 H_2. \) Without loss of generality we may assume \( c_2 \) to be trivial, so \( U_2 = UU_1^{-1}, U = [x_1, y_1][x_2, y_2][x_3, y_3]. \)

Substituting \( U_2 \) by \( UU_1^{-1} \), we get a quadratic set of non-orientable genus 15 for the free group. The length of every letter in \( W_1', W_2', W_3' \) is bounded by \( s \) and every letter in \( U \) is bounded by \( 2|U_1| + 2|U_2| \leq 2s. \) Using the results for the free group, we get, that the length of the solution is bounded by a polynomial of degree eight in \( s. \)

## 4 Toral relatively hyperbolic groups

We will use the following definition of relative hyperbolicity. A finitely generated group \( G \) with generating set \( A \) is relatively hyperbolic relative to a collection of finitely generated subgroups \( \mathcal{H} = \{H_1, \ldots, H_k\} \) if the Cayley graph \( C(G, A \cup \Pi) \) (where \( \Pi \) is the set of all non-trivial elements of subgroups in \( \mathcal{H} \)) is a hyperbolic metric space, and the pair \( \{G, \mathcal{H}\} \) has **Bounded Coset Penetration** property (BCP property for short). The pair \( \{G, \{H_1, H_2, \ldots, H_k\}\} \) satisfies the BCP property, if for any \( \lambda \geq 1 \), there exists constant \( a = a(\lambda) \) such that the following conditions hold. Let \( p, q \) be \((\lambda, 0)\)-quasi-geodesics without backtrackings in \( C(G, A \cup \Pi) \) (do not have a subpath that joins a vertex in a left coset of some \( H_k \) to a vertex in the same coset (and is not in \( H_k \)) such that their initial points coincide \( (p_- = q_-) \), and for the terminal points \( p_+, q_+ \) we have

\[ d_A(p_+, q_+) \leq 1. \]

1) Suppose that for some \( i, s \) is a \( H_i \)-component of \( p \) such that \( d_A(s-, s+) \geq a; \) then there exists a \( H_i \)-component \( t \) of \( q \) such that \( t \) is connected to \( s \) (there exists a path \( c \) in \( C(G, A \cup \Pi) \) that connects some vertex of \( p \) to some vertex of \( q \) and the label of this path is a word consisting of letters from \( H_i \)).

2) Suppose that for some \( i, s \) and \( t \) are connected \( H_i \)-components of \( p \) and \( q \) respectively. Then \( d_A(s-, t_-) \leq a \) and \( d_A(s+, t_+) \leq a \).

Recall that a group \( G \) that is hyperbolic relative to a collection \( \{H_1, \ldots, H_k\} \) of subgroups is called toral if \( H_1, \ldots, H_k \) are all abelian and \( G \) is torsion-free.

In this section we will prove Theorem 3. Notice that we will use the facts that the word problem and the conjugacy problem in (toral) relatively hyperbolic groups are decidable.
Proposition 4. Let $\Gamma = \langle A | R \rangle$ be a total relatively hyperbolic group and with parabolic subgroups $H_1, \ldots, H_k$ and $\pi : F(A) * H_1 * \ldots * H_k \to \Gamma$ the canonical epimorphism. There is an algorithm that, given a system $S(Z, A) = 1$ of equations over $\Gamma$, produces finitely many systems of equations $S_1(X_1, A) = 1, \ldots, S_n(X_n, A) = 1$ (11) over a free product $P = F * H_1 * \ldots * H_k$ and homomorphisms $\rho_i : F(Z) * P \to P_{S_i} = (F(Z) * P)/\text{ncl}S_i$ for $i = 1, \ldots, n$ such that

(i) for every $P$-homomorphism $\phi : P_{S_i} \to P$, the map $\rho_i \phi \pi : \Gamma \to \Gamma$ is a $\Gamma$-homomorphism, and

(ii) for every $\Gamma$-homomorphism $\psi : \Gamma \to \Gamma$ there is an integer $i$ and an $F$-homomorphism $\phi : P_{S_i} \to P$ such that $\rho_i \phi \pi = \psi$.

Further, if $S(Z) = 1$ is a system without coefficients, the above holds with $G = (Z | S)$ in place of $\Gamma_S$ and ‘homomorphism’ in place of $\Gamma$-homomorphism’. Moreover, $|S_i| = O(|S|^4)$ for each $i = 1, \ldots, n$.

The proof is the same as the proof of Proposition 1 but instead of [29] one has to use Theorem 3.3 in [10] about representatives in a free product of free abelian groups of finite rank.

Now the proof of Theorem 2 is almost identical to the proof of Theorem 1 but instead of the results about length estimates of a minimal solution of a quadratic system of equations in a free group one should use Theorem 5.

5 An equation for which the problem of finding solution is NP-hard

In this section we will complete the proof of Theorem 3 by showing that, for any bin packing problem, there is a corresponding quadratic equation $S$ over a torsion-free hyperbolic group $\Gamma$, such that finding a solution to $S$ gives a solution to the bin packing problem.

Let $\Gamma = < A | R >$ be a torsion-free $\delta$-hyperbolic group. By 2, $\Gamma$ contains a convex free subgroup $F(b, b_1)$ of rank two, which contains a convex free subgroup $F(b, c, d)$ of rank three. We can assume that $b, c, d$ are cyclically reduced. Given $b, c, d$, by [28] there are constants $\lambda, C$ such that any path labeled by $b^t, c^t$ or $d^t$ is $(\lambda, C)$-quasigeodesic for all $t$. By [9] there is a constant $R(\delta, \lambda, C)$ such that any $(\lambda, C)$-quasigeodesic path is in the $R$-neighborhood of a geodesic path with the same endpoints. By [2] there exists an integer $D$, such that the normal closure $N = <<< b^{s_1D}, c^{s_2D}, d^{s_3D} >>>$ in $G$ is free for any $s_i > 0$ (also, $\Gamma/N$ is non-elementary hyperbolic). We can choose numbers $t_i \in \mathbb{Z}^+$ such that $t_i+1$ is much larger than $t_i$ and all $t_i$ are much larger than $R$. Consider words $a$ in $\Gamma$ of the form $a = a^{D}e^{t_1D}d^{D} \ldots a^{D}e^{t_nD}d^{D}$, for each $n > 0$. 

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Let \( i \in \mathbb{Z}, i \geq 3 \) (since \( i \) must be large enough for what will be defined to be stable occurrences). For each \( i \) consider the equation

\[
\prod_{j=1}^{s} z_j^{-1} [a(i), b^{ir_j}] z_j = [a(i)^N, b^{iB}]
\]  

(12)

where \( a(i) = D^D c_{it_1} D^D c_{it_2} D^D \) and the \( r_j, N, B \) are positive integers from the bin packing problem. Call this equation \( S(i) \). By Proposition \([\text{1}]\) from each \( S(i) \) we may obtain (by first transforming to a system of triangular and constant equations, then taking canonical representatives) a system of equations over the free group \( F = F(A) \) of the form:

\[
x_{j,k}^{(i)} c_{j,k}^{(i)} (x_{j,k+1}^{(i)})^{-1} = x_{j',k'}^{(i)} c_{j',k'}^{(i)} (x_{j',k'+1}^{(i)})^{-1}
\]

and also

\[
(x_{j,k}^{(i)} c_{j,k}^{(i)} (x_{j,k+1}^{(i)})^{-1} = \theta_m([a(i), b^{ir_k}]) \text{ or } \theta_m([a(i)^N, b^{iB}])
\]

where \( k + 1 \) is taken with respect to the cyclic group \( \{1, 2, 3\} \). The notation \( x_{j,k}^{(i)}, c_{j,k}^{(i)} \) refers to the canonical representatives and derived variables \( x_j^{(i)}, c_k^{(i)} \) (as in equation 6) from the equation \( S(i) \), not the \( j^{th} \) power of the representatives and variables. \( \theta_m(g) \) is the canonical representative of \( g \), for any \( g \in \Gamma \), as given in Proposition \([\text{1}]\).

Call this system \( S_{\text{can}}(i) \). By Proposition \([\text{2}]\) \( |S_{\text{can}}(i)| = O(|S(i)|^4) \) and \( S_{\text{can}}(i) \) is quadratic. Notice that, as \( i \) varies, only the constants of \( S(i) \) change, so all \( S(i) \) have the same form of decomposition into triangular equations (and so the correspondence of sides of triangles coming from the double occurrences of each variable is always the same), therefore the \( j, k \) corresponding to the \( j', k' \) do not depend on \( i \).

Now since the length of \( c_{j,k}^{(i)} \) is uniformly bounded (by the \( L \) in the proof of Proposition \([\text{1}]\)) for all \( i \), we can take a sequence \( \{d_i\} \) where \( c_{j,k}^{(d_i)} = c_{j,k}^{(3)} \) for all \( d_i \) in the sequence. Let \( c_{j,k} = c_{j,k}^{(3)} \). So we have an infinite sequence of systems \( S_{\text{can}}(d_i) \) over \( F(A) \), each in variables \( x_{j,k}^{(d_i)} \) and coefficients \( c_{j,k}, [a^{(d_i)}, b^{d_{ir_k}}], [a^{(d_i)}]^N, b^{d_iB} \) where each equation in \( S_{\text{can}}(d_i) \) is of one of the following forms:

\[
x_{j,k}^{(d_i)} c_{j,k}^{(d_i)} (x_{j,k+1}^{(d_i)})^{-1} = x_{j',k'}^{(d_i)} c_{j',k'}^{(d_i)} (x_{j',k'+1}^{(d_i)})^{-1}
\]

and also

\[
(x_{j,k}^{(d_i)} c_{j,k}^{(d_i)} (x_{j,k+1}^{(d_i)})^{-1} = \theta_m([a^{(d_i)}, b^{d_{ir_k}}]) \text{ or } \theta_m([a^{(d_i)}]^N, b^{d_iB}])
\]

5.1 Entire Transformations
We now give a rewriting process to transform each \( S_{\text{can}}(d_i) \) to a system where each equation includes (the canonical representative of) exactly one original constant from \( S(d_i) \).
Proposition 5. For each $d_i$, the system $S_{can}(d_i)$ is equivalent to a system of equations over $F$, made up of equations of the form

$$w(x_j, c_{j,k}) = \theta_m([a^{(d_i)}, b^{d_i(r_i)}]) \text{ or } w(x_j, c_{j,k}) = \theta_m([a^{(d_i)} N, b^{d_i} B])$$

where $w(x_j, c_{j,k})$ is some freely reduced word in the variables $x_j, k$ and the constants $c_{j,k}$.

Proof. To prove the result, we will use the process of entire transformations (Makanin rewriting process) [19]. We introduce the language of generalized equations and elementary transformations to establish this rewriting process [19].

Definition 11. Let $F$ be the free group on a finite generating set $A = \{a_1, ..., a_m\}$. A combinatorial generalized equation $\omega_c$ consists of the following:

1) A finite set $BS = BS(\omega_c) = \{\mu_1, ..., \mu_{2n}, \nu_1, ..., \nu_m\}$. Elements of this set are called bases, and the set is equipped with a function $\epsilon : BS \to \{-1, 1\}$ and an involution $\delta : \{\mu_1, ..., \mu_{2n}\} \to \{\nu_1, ..., \nu_m\}$. Denote $\delta(\mu) = \mu$ and call $\mu, \nu$ dual bases. Elements $\nu_i$ with no dual base are called constant bases.

2) A finite set of positive integers $BD = BD(\omega_c) = \{1, 2, ..., s\}$ where $s \geq \rho + m + 1$ for some integers $\rho, m$ with $\rho > 0, m \geq 0$. Elements of this set are called boundaries.

3) Two functions $\alpha : BS \to BD$ and $\beta : BS \to BD$ where for every $\mu \in BS$, $\alpha(\mu) < \beta(\mu)$ if $\epsilon(\mu) = 1$ and $\alpha(\mu) > \beta(\mu)$ if $\epsilon(\mu) = -1$. For each constant base $\nu_i$ we have $\alpha(\nu_i) \geq \rho$ and if $\alpha(\nu_i) = \alpha(\nu_j)$ and $\beta(\nu_i) = \beta(\nu_j)$ then $i = j$.

Now to a combinatorial generalized equation $\omega_c$, one can canonically associate a system of equations $\omega$ over $F$, with $\rho + m + 1$ variables $h_1, ..., h_{\rho+m+1}$ and $m$ constants $a_1, ..., a_m$. The variables may be referred to as items, and the system is called a generalized equation. $\omega$ is generated from $\omega_c$ in the following way:

1. For each pair of dual bases $\mu, \bar{\mu}$ form the basic equation:

$$[h_{\alpha(\mu)} h_{\alpha(\mu)+1} \ldots h_{\beta(\mu)-1}] = [h_{\alpha(\bar{\mu})} h_{\alpha(\bar{\mu})+1} \ldots h_{\beta(\bar{\mu})-1}]$$

if $\epsilon(\mu) = \epsilon(\bar{\mu})$ and

$$[h_{\alpha(\mu)} h_{\alpha(\mu)+1} \ldots h_{\beta(\mu)-1}] = [h_{\alpha(\bar{\mu})} h_{\alpha(\bar{\mu})+1} \ldots h_{\beta(\bar{\mu})-1}]^{-1}$$

if $\epsilon(\mu) = -\epsilon(\bar{\mu})$

2. Form the constant equations $h_{\rho+1} = a_1, ..., h_{\rho+m+1} = a_m$

Notice that the association can also be made in the reverse direction. Given a system of equations $\omega$ over $F$, an associated $\omega_c$ can be formed by taking boundaries such that each interval between boundaries corresponds to a letter
(either variable or constant) in one of the words over \( F(X,A) \) making up \( \omega \) (taking boundaries in the order in which letters appear). Add an additional boundaries for each constant at the end (i.e. greater than \( \rho \)). Then form dual bases covering the words making up the left and right side of each equation, dual bases for each multiple appearance of the same letter and dual bases covering each appearance of a constant in a word and the corresponding interval at the end of the boundaries. Add constant bases (which can be thought of as being labelled by the covered constant) covering each interval past \( \rho \).

Figure 3 shows how to form the combinatorial generalized equation corresponding to \( S_{\text{can}}(d_i) \). In the figure (the upper row is the boundaries, and the lower two rows are the bases), \( \lambda_1 \) and \( \overline{\lambda_1} \) corresponds to an equation of the form \( x_{j,k}y_{j,k}(x_{j,k+1})^{-1} = x_{j',k'}y_{j',k'}(x_{j',k'}+1)^{-1} \). Then, since \( x_{j,k} \) appears as part of another equation, we have \( \lambda_2 \) and \( \overline{\lambda_2} \) corresponding to these two appearances (in the case in the figure, we have \( x_{j,k}^{-1} \) in the second appearance). We see the two constant bases labelled by \( c_{j,k} \) and \( \theta_m([a(i), b^{\text{rs}}]) \). Then \( \mu_1 \) and \( \overline{\mu_1} \) correspond to the constant equation \( y_{j,k} = c_{j,k} \) and \( \mu_2, \overline{\mu_2} \) correspond to an equation of the form \( x_{j,k}c_{j,k}x_{j,k+1} = \theta_m([a(i), b^{\text{rs}}]) \).

Figure 3

**Definition 12.** A triple \((p, \mu, q)\), where \( \mu \) is a base, \( p \) is a boundary on \( \mu \) and \( q \) is a boundary on \( \overline{\mu} \), is called a boundary connection. We say that boundary \( i, \alpha(\lambda) \leq i \leq \beta(\lambda), \) is \( \lambda \)-tied if there exists some boundary \( j = b(i) \) such that \((i, \lambda, j)\) is a boundary connection. We say that all boundaries in \( \lambda \) are \( \lambda \)-tied if \( i \) is \( \lambda \)-tied for all \( \alpha(\lambda) \leq i \leq \beta(\lambda) \).

We now describe the elementary transformations that will be used to form the rewriting process.

An elementary transformation (ET) associates to a generalized equation \( \omega \) a...
family of generalized equations \( ET(\omega) = \{ \omega_1, \ldots, \omega_k \} \) and surjective homomorphisms \( \pi_i : F_R(\omega) \rightarrow F_R(\omega_i) \) such that for any solution \( \delta \) of \( \omega \) and corresponding epimorphism \( \pi_\delta : F_R(\omega) \rightarrow F \) there exists \( i \in \{ 1, \ldots, k \} \) and a solution \( \delta_i \) of \( \omega_i \) such that \( \pi_\delta = \pi_\delta_i \circ \pi_i \). Each elementary transformation is described by a transformation on the combinatorial generalized equation \( \omega_c \).

\[ (ET1) \text{ (Cutting a base) Let } \lambda \text{ be a base in } \omega_c \text{ and } p \text{ an internal boundary of } \lambda, q \text{ an internal boundary of } \lambda. \text{ Then cut } \lambda \text{ at } p \text{ into bases } \lambda_1, \lambda_2 \text{ and cut } \lambda \text{ at } q \text{ into bases } \lambda_1, \lambda_2. \]

\[ (ET2) \text{ (Transfering a base) If for bases } \lambda, \mu \text{ of } \omega_c, \alpha(\lambda) \leq \alpha(\mu) < \beta(\mu) \leq \beta(\lambda) \text{ (i.e. } \lambda \text{ contains } \mu), \text{ and there are boundary connections } (\alpha(\mu), \lambda, p) \text{ and } (\beta(\mu), \lambda, q), \text{ then transfer } \mu \text{ from } \lambda \text{ to } \lambda, \text{ i.e. let } \alpha(\mu) = p \text{ and } \beta(\mu) = q. \]

\[ (ET3) \text{ (Removal of a pair of matched bases) If for some } \lambda \text{ of } \omega_c, \alpha(\lambda) = \alpha(\lambda), \beta(\lambda) = \beta(\lambda) \text{ then remove } \lambda, \lambda \text{ from } \omega_c. \]

\[ (ET4) \text{ (Removal of a lone base) If a base } \lambda \text{ does not intersect any other base, i.e. for all } i, \alpha(\lambda) + 1 \leq i \leq i(\lambda) - 1, \text{ there is no } \mu \text{ such that } \alpha(\mu) \leq i \leq \beta(\mu), \text{ then we can remove the pair of bases } \lambda, \lambda \text{ and the boundaries } \alpha(\lambda) + 1, \ldots, \beta(\lambda) - 1 \text{ and rename the boundaries } \beta(\lambda), \ldots \text{ starting at } \alpha(\lambda) + 1. \]

\[ (ET5) \text{ (Introduction of a boundary) Suppose that a boundary } p \text{ in a base } \lambda \text{ is not } \lambda \text{-tied, } p \text{ may become } \lambda \text{-tied as follows: either choose a boundary } q \text{ on } \lambda \text{ and make a boundary connection } (p, \lambda, q) \text{ or introduce a new boundary } r \text{ between some boundaries } q \text{ and } q + 1 \text{ on } \lambda \text{ and make a boundary connection } (p, \lambda, q). \]

Now by [19] each of these elementary transformations changes a combinatorial generalized equation \( \omega_c \) to a combinatorial generalized equation \( ET_i \omega_c \) for \( i = 1, \ldots, 5 \), where if \( \omega \) has a solution then \( ET_i \omega \) has a solution.

**Definition 13.** Rewriting process (Entire transformation): Given a combinatorial generalized equation \( \omega_c \). Start at \( 1 \in BD \), fix a leading base (the longest base beginning at 1) \( \mu \) and \( \mu \)-tie all boundaries in \( \mu \). Then transfer all bases covered by \( \mu \) to \( \mu \). There exists \( j \), where \( j \) is the minimal boundary with \( \gamma(j) \geq 2 \) (define \( \gamma \)), cut \( \mu \) at \( j \) to \( \mu_1, \mu_2 \) applying \( ET1 \) and delete boundaries \( 1, \ldots, j - 1 \) and bases \( \mu_1 \) and \( \mu_2 \). Renumber BD and all boundary connections accordingly and then repeat \( (j \text{ now equal to } 1) \). Terminate the process if, after renumbering boundaries, \( h_1 \) corresponds to a constant equation in omega, \( h_1 = a \). If this process terminates, call the resulting combinatorial generalized equation \( ET_c \omega_c \) with associated generalized equation \( E(\omega) \).

**Lemma 4.** [19] If \( \omega \) corresponds to a quadratic system of equations, then for a solution of \( \omega \) of minimal length the entire transformation process corresponding to this solution terminates.

**Proof.** This rewriting process always moves forward since the system is quadratic.
Suppose the rewriting process continues infinitely (since more boundaries may be introduced, perhaps the active section does not reach the coefficients), now the number of bases may only decrease and the number of boundaries is bounded from above, so there is only a finite number of distinct generalized equations that may be formed, so in an infinite rewriting process, the same generalized equation must be repeated, but if we assume the solution to the initial generalized equation is minimal, the same generalized equation cannot appear twice.

Remark 1. Notice that the entire transformation does not add more variables, and may only remove a pair of occurrences of some variable (and its inverse), so the resulting system of equations is quadratic.

After the entire transformation process terminates, we obtain a system $E(S_{can}(d_i))$ of equations of the form

$$w(x_{j,k}, c_{j,k}) = \theta_m([a^{(i)}, b^{d_i}])$$

or

$$w(x_{j,k}, c_{j,k}) = \theta_m([a^{(i)}N, b^{d_i}B])$$
in $F$. The proposition is proved.

Figure 4 shows what $E_c(S_{can}(d_i))$ will look like after the entire transformation terminates. Notice that all boundaries are covered exactly twice, once by a constant base, and once by a base with a dual. Also the initial and terminal boundaries of each constant base are also the initial/terminal boundaries of bases with duals (i.e. no dual base overlaps multiple constant bases).

![Figure 4](image)

5.2 Solution Diagram

Fix $i$ in the sequence $\{d_i\}$ and suppose $S(i)$ has a solution in $\Gamma$. So by Propositions 1 and 5, $S_{can}(i)$ has a solution in $F$ and $E(S_{can}(i))$ has a solution $\delta$
in \( F \). Let \( \mu_{j,k} = x_{j,k}^d \). Notice that paths corresponding to each \( w(\mu_{j,k}, c_{j,k}) \) are geodesics in the Cayley graph of \( \Gamma \), and all of the \([a^{(i)}(t), b^{(r)}], [a^{(i)}N, b^{iB}]\) are \((\pi, \epsilon) - \text{quasigeodesics}\) for some \( \pi \) and \( \epsilon \). So by [3] there is a constant \( R(\delta, \pi, \epsilon) \), (where \( \delta \) is the constant of hyperbolicity for \( G \)), such that \( d_H(\gamma, \eta) \leq R \) (where \( d_H \) is the Hausdorff distance) for any geodesic \( \gamma \) and \((\pi, \epsilon) - \text{quasigeodesic} \eta \) with the same endpoints. Therefore the paths in \( \text{Cay}(\Gamma) \) corresponding to \( w(\mu_{j,k}, c_{j,k}) \) and \( [a^{(i)}(t), b^{(r)}] \) \( ([a^{(i)}N, b^{iB}] \) are (pairwise, for each equation in \( E(S_{can}(i)) \)) in the \( R \)-neighborhood of each other.

**Proposition 6.** If \( S(i) \) has a solution in \( \Gamma \) then, for each \( x_{j,k} \) in \( E(S_{can}(i)) \), there are subwords \( g, g' \) of words from \([a^{(i)}(t), b^{(r)}], [a^{(i)}N, b^{iB}] \), with \( g =_F g' \) up to an error of bounded length, meaning

\[
g = c_1 g_1 c_2 g_2 \cdots g_n c_{n+1}
\]

\[
g' = d_1 g_1 d_2 g_2 \cdots g_n d_{n+1}
\]

where \( |c_1| + \ldots + |c_{n+1}|, |d_1| + \ldots + |d_{n+1}| \leq E \) for a constant \( E \) that does not depend on \( i, j \) or \( k \).

**Proof.** Since \( E(S_{can}(i)) \) is quadratic, for each \( j \) and \( k \), \( \mu = \mu_{j,k} \) is the label of two distinct subpaths of two (not necessarily distinct) geodesics \( \gamma, \gamma' \) with labels of the form \( w(\mu_{j,k}, c_{j,k}) \). Each of those geodesics share endpoints with (not necessarily distinct) \((\pi, \epsilon) - \text{quasigeodesics} \eta, \eta' \), which have labels \( w, w' \in \{[a^{(i)}(t), b^{(r)}], [a^{(i)}N, b^{iB}] \}. \) (Later, in the proof of Lemma 5 it is shown that \( w \) and \( w' \) will always be distinct, for large enough \( i \).) So there are minimal words (which may be the empty word) \( s_1, s_1, t_1, t_1 \) and maximal subwords \( g, g' \) of \( w, w' \), respectively, with

\[
s_1 \mu(t_1)^{-1} g^{-1} = 1 \text{ and } s_2 g'(t_2)^{-1} \mu^{-1} = 1
\]

Since they are minimal, we have \( |s_1|, |s_2|, |t_1|, |t_2| \leq R \). Let \( s \) be the label of a geodesic from the initial vertex of \( s_1 \) to the terminal vertex of \( s_2 \) and \( t \) will be the label of a geodesic from the initial vertex of \( t_1 \) to the terminal vertex of \( t_2 \) (henceforth we will often not distinguish between labels and actual geodesics). So we have \( sf^{e_1} g^{-1} = 1 \) and \( |s|, |t| \leq 2R \). Since \( sf^{e_1} = 1 \) and \( s(t_1)^{-1} \) is a \((\pi, \epsilon + 2R) - \text{quasigeodesic} \) and \( g, g' \) are \((\pi, \epsilon) - \text{quasigeodesic} \), \( |g|, |g'| > |\mu| + 2R \) and \( d_H(sf^{e_1} g, g) < R' \) for some constant \( R' = R(\delta, \pi, \epsilon + 2R, R) \).

Now on \( g \) and \( g' \), consider each vertex between any of the words \( b^D, c^D, d^D \). Call such a vertex a \( D \)-vertex. Now from each \( D \)-vertex \( v \) on \( g \) find the shortest geodesic \( \gamma \) to a vertex \( v' \) on \( sg^{e_1} t^{-1} \). Notice that for each \( D \)-vertex \( v \) further than \( 2\pi R_\epsilon + \epsilon \) from either an initial or terminal vertex of \( g, v' \) is actually on \( g' \). Now \( v' \) is within \( D/2(\max\{|b|, |c|, |d|\}) \) of a \( D \)-vertex \( u \) of \( g' \). So for each \( D \)-vertex \( v \) on \( g \) there is a geodesic, of length less than or equal to \( L = R' + D/2(\max\{|b|, |c|, |d|\}) \) to a \( D \)-vertex \( u \) of \( g' \). Call a geodesic from a \( D \)-vertex of \( g \) to a \( D \)-vertex of \( g' \) a \( D \)-geodesic.

There is some constant, call it \( C_1 \), depending on \( 2\pi R_\epsilon + \epsilon \) and \( D \), such that there is at most \( C_1 \) many \( D \)-vertices that are within \( 2\pi R_\epsilon + \epsilon \) of an initial or
terminal vertex of $g_j$. Let $N = N(\Gamma, L)$ be the number of elements in $\Gamma$ of length at most $L$. If $|\mu| \geq D(N + 1)(\max \{|b|, |c|, |d|\}) + C_1 + 2R$ then there are at least $N + 1$ $D$-geodesics from $D$-vertices of $g$ to $D$-vertices of $g'$, each of length less than or equal to $L$. Of these geodesics, at most $N$ of them can have unique labels, so there must multiple $D$—geodesic with the same label in $\Gamma$. Let $T$ be the number of these geodesics that have unique labels.

Now if $h \in G$ is the label of more than one of these geodesics, then we have $hw_2(b^D,c^D,d^D)h^{-1}(w_1(b^D,c^D,d^D))^{-1} = 1$ where $w_1(b^D,c^D,d^D)$ is a subword of $g_j$ and $w_2(b^D,c^D,d^D)$ is a subword of $g_j'$ (and both are words in $b^D,c^D,d^D$). Since the normal closure $N(b^D,c^D,d^D)$ in $G$ is free by $[CH12]$, then $hw_2(b^D,c^D,d^D)h^{-1} = F w_1(b^D,c^D,d^D)$. So for each repeated label we have subwords of $g$ and $g'$ that are equal in the free group up to an error of at most $2L$. For each of the $T$ unique labels of $D$—geodesics we may have an error from graphical equality between $g$ and $g'$ of size at most $2D(\max \{|b|, |c|, |d|\})$. So for the $x_{j,k}$ with $|\mu_{j,k}| \geq D(N + 1)(\max \{|b|, |c|, |d|\}) + C_1 + 2R$, we have $g = F g'$ except for subwords of total length at most $E = 2(L(N - T) + D(\max \{|b|, |c|, |d|\})).$ For $x_{j,k}$ with $|\mu_{j,k}| < D(N + 1)(\max \{|b|, |c|, |d|\}) + C_1 + 2R$, the corresponding $g,g'$ each have $|g|,|g'| \leq \pi |\mu_{j,k}| + \epsilon < \pi(D(N + 1)(\max \{|b|, |c|, |d|\}) + C_1 + 2R) + \epsilon = E$. So for all $x_{j,k}$, $E = \max \{E, E'\}$ is an upper bound on the error of $g_j = F g_j'$.

\[\square\]

**Corollary 1.** $E(S_{can}(i))$ has a solution diagram in the free group with errors of total size at most $O(|S(i)|^4)E$.

**Lemma 5.** There exists a subsequence of indices $\{d_j\}$ of the sequence $\{d_i\}$ such that for each equation $S(d_j)$ we have the same generalized equation $E(S_{can}(d_j))$ and the same (finite) process of entire transformations for a minimal solution $\psi_j$. For this subsequence of indices, the solution diagrams corresponding to solutions $\psi_j$ of the equation $[12]$ have the same combinatorial structure and differ from a solution diagram in a free group by errors of total size at most $M$, where $M$ is a constant that does not depend on the particular subscript $i_j$.

**Proof.** There is only a finite number of possible systems $E(S_{can}(d_j))$ up to a difference in the constants, and a finite number of possible sequences of entire transformations for a minimal solution of the equation $[12]$, therefore for an infinite subset of indices their combination will be the same. \[\square\]

For ease of notation, rename the subsequence $\{d_j\}$ from the lemma as the sequence $\{d_i\}$.

**Lemma 6.** There is a number $N$, depending on $s$, and a subsequence $\{d_k\}$ of the sequence $\{d_i\}$ such that if there is a solution for $E(N)$, then there is a solution for all $S_{can}(d_k)$.

We use a similar idea to that of Bulitko’s Lemma. Given a particular system $S(d_i)$ with a solution we get a solution $\delta$ for $S_{can}(d_i)$, and for any cyclically
reduced word \( w \) appearing in a particular way (stable-occurrence) in \( \delta \), it may be possible to replace each occurence of (a power of) \( w \) with some larger power of \( w \) and still have a solution of the resulting system of equations. What determines this possibility is only the occurences of \( w \) that occur in the center of a triangular equation \( xyz = 1 \), i.e. a subword of \( w \) is a subword of each of \( x^\beta, y^\beta, z^\beta \). Since there are only finitely many centers (finitely many triangular equations), if it can be shown that there are infinitely many solutions to an associated system of linear equations (coming from the requirement that powers of \( w \) through the center of the equation must "match up"), then those solutions give an increasing sequence of powers of \( w \) that may be used. Notice that, after replacing with these larger powers, the system of equations may only possible change in it’s coefficients, which only happens if a (stable-occurrence) of \( w \) occurs in the coefficients, since the formation of the solution diagram does not change. We see then, that in fact the resulting equations with solution are the \( Sc_{an}(d_i) \).

First we need some definitions. For a simple cyclically reduced word \( U \) we say a \( U \)-occurrence in a word \( w \) is an occurence in \( w \) of a word \( U^t \), \( t \geq 1 \). A \( U \)-occurrence is said to be stable if it occurs as part of a subword \( v_1U^tv_2 \) of \( w \) where \( v_1 \) ends with \( U^t \) and \( v_2 \) begins with \( U^t \). It is a maximal stable \( U \)-occurrence if there is no strictly larger stable \( U \)-occurrence containing it, so every stable \( U \)-occurrence is contained in a maximal stable \( U \)-occurrence and two distinct maximal stable \( U \)-occurrences do not intersect. So the following notion is well defined. A \( U \)-decomposition \( D_U(w) \) of a word \( w \) is the unique representation \( w = v_0U^{t_1}v_1...v_{m-1}U^{t_m}v_m \) where the \( U^{t_i} \) are all maximal stable \( U \)-occurrences in \( w \). If there are no stable \( U \)-occurrences then \( D_U(w) = w \).

Proof. First we give an example of how to form the linear equations associated to a system of equations over the free group. For any \( Sc_{an}(d_i) \) with a solution \( \phi_i \), let

\[
\begin{align*}
x_i^\phi &= v_1,0c^{t_1,1s_1,1}v_1,1...v_1,l-1c^{t_1,1s_1,1}v_1,l \\
y_i^\phi &= v_2,0c^{t_2,1s_2,1}v_2,1...v_2,m-1c^{t_2,1s_2,1}v_2,m \\
z_i^\phi &= v_3,0c^{t_3,1s_3,1}v_3,1...v_3,n-1c^{t_3,1s_3,1}v_3,n
\end{align*}
\]

be the c-decompositions, for each triangular equation \( xyz = 1 \), and each constant equation \( x = c \) in \( Sc_{an}(d_i) \). Similarly we have the d-decompositions

\[
\begin{align*}
x_i^\phi &= v_1,0b^{t_1,1s_1,1}v_1,1...v_1,l-1b^{t_1,1s_1,1}v_1,l \\
y_i^\phi &= v_2,0b^{t_2,1s_2,1}v_2,1...v_2,m-1b^{t_2,1s_2,1}v_2,m \\
z_i^\phi &= v_3,0b^{t_3,1s_3,1}v_3,1...v_3,n-1b^{t_3,1s_3,1}v_3,n
\end{align*}
\]

Now for each \( U = c, b, \) since each maximal stable \( U \)-occurrence must cancel another maximal stable \( U \)-occurrence, unless it crosses the middle of the tripod in the Cayley graph of \( F \) corresponding to \( x_i^\phi y_i^\phi z_i^\phi = 1 \), in which case it must be cancelled by two maximal stable \( U \)-occurrences, one from each of the other sides of the tripod. Notice that since \( x_i^\phi, y_i^\phi, z_i^\phi \) are each reduced words, at most one
maximal stable U-occurrence may cross the middle of a tripod.

From this system of cancellation over all of the triangular equations of $S_{can}(d_i)$ we obtain a system of linear equations on the natural numbers

$$\{ s_{i,k}, t_{i,k} \}, i = 1, 2, 3, \ k = 1, \ldots, \max\{ l', m', m, n' \}$$

(note: maximum is taken over all $l, l', m, m', n, n'$ generated by all of the equations of $S_{can}(d_i)$), where all coefficients are equal to $\pm 1$. The variables $s_{i,k}$ correspond to the powers of the maximal stable $c - \text{occurrences}$ and the variables $t_{i,k}$ correspond to the maximal stable $b - \text{occurrences}$. We want to show that we have a family of solutions to this system in which certain U-occurrences can be lengthened by a uniform multiple. Since variables corresponding only to edges between middle of tripods must occur in equations of the form $s_{i,k} = s_{i',k'}$ (similarly for $t_{i,k}$), and such equations can solved by any multiple of a solution, we actually only care about to equations corresponding to the middle of each tripod, which must have the form $s_{1,k} + s_{2,k'} + 2 = s_{3,k''}$ or $s_{1,k} + s_{2,k'} + 3 = s_{3,k''}$ (or of the same form with the $t_{i,k,j}$). Also, we know the vaules of the variables corresponding to the $c, b - \text{occurrences}$ coming from the coefficients, since $s_{ik} = d_i - 2$, $t_{ik} = d_i r_j - 2$ for each variable coming from the $c, b - \text{occurrences}$, respectively, of the coefficient $[a^{(d_i)}, b^{(r_j)}]$ and $s_{ik} = d_i - 2$, $t_{ik} = d_i B - 2$ for the variables coming from the $c, b - \text{occurrences}$, respectively, in the coefficient $[a^{(d_i)}N, b^{(B)}]$. Let $M(d_i, S, T)$ be the system of equations corresponding to the middle of tripods, with the variables corresponding to $c, b - \text{occurrences}$ in the coefficients written as above, only with $d_i$ replaced with a variable $d$. So we have a system of equations in possibly fewer variables $s_{ik}, t_{ik}$ with the addition of (possibly) a single variable $d$.

Notice that this system of equations depends on $r_j, N, B, s$, but the total number of possible system of equations obtained depends only on $s$. This is because the system of equations are obtained from at most $s$ (number of coefficients) changes on variables coming from $O(|S(d_i)|^2)$ triangular equations. So there is a finite number of different possible systems $M(d_i, S, T)$ obtained. Some of these systems can have a finite number of solutions, and some have infinite number of solutions (since there is some bin packing problem for the given $s$ that can be solved, giving a system with infinite number of solutions).

We take $N$ larger than the solutions of any of the systems of equations with finite number of solutions. Suppose $E(N)$ has a solution. Then $M_N(d, S, T)$ has an infinite number of solutions. Let $\{ d_k \}$ be an increasing sequence of solutions of the variable $d$ in $M_N(d, S, T)$. Its not hard to see that these solutions give solutions to the $S_{can}(d_k)$ by taking the solution to $S_{can}(d_j)$ and adding powers of $c$ and $b$ accordingly.

\[ \square \]

**Corollary 2.** There is a subsequence $\{ i \}$ of the sequence of natural numbers such that equation

$$\prod_{j=1}^{s} z_j^{-1} [a^{(i)}, b^{(r_j)}] z_j \prod_{j=s+1}^{s+m} z_j^{-1} R_j - z_j = [a^{(i)}N, b^{(B)}], \quad (13)$$

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where $R_1, \ldots, R_m$ are fixed words, that maybe are absent, has a solution in a free group $F(A)$ for each $i$ in the subsequence, if equation (12) has solution in $\Gamma$ for $i = N$.

**Lemma 7.** There is an element $a$ in $\Gamma$ such that equation (13) has a solution in $F(A)$ for all $i$ in the subsequence $\{i\}$, if and only if there is a collection of discs $D_j$ with boundary labels $[a^{(i)}, b^{(i)}]$ for $j = 1, \ldots, s$ and $R_j$ for $j = 1, \ldots, m$ and a disc $D$ with boundary label $[a^{(i)}N, b^B]$ such that, glued together in a way that respects labels and orientation of edges, form a union of spheres, and furthermore, words $a$ are only glued to words $a$.

**Proof.** Using Olshanski's Theorem 2.1 in [17], we conclude that equation (13) has a solution in $F(A)$ if and only if there is a collection of discs $D_j$ with boundary labels $[a^{(i)}, b^{(i)}]$ for $j = 1, \ldots, s$ and $R_j$ for $j = 1, \ldots, m$ and a disc $D$ with boundary label $[a^{(i)}N, b^B]$ such that, glued together in a way that respects labels and orientation of edges, form a union of spheres. Moreover, for all $i$ from the subsequence $\{i\}$, these discs situated relative to each other the same way (as seen in the proof of lemma 6, where there are infinite solutions to an equation generated by changing exponents corresponding to a fixed solution diagram). We now fix $i$ much larger that the total length of $R_1, \ldots, R_m$.

After the entire transformation process for generalized equations corresponding to (13) terminates, we obtain equations of the form

$$\phi([a^{(i)}, b^{(i)}]) = w(x_{j,k, c_{j,k}}), R_j = w(x_{j,k, c_{j,k}})$$

in $F$. The number $m$ in the representation of the word $a = d^D c^{1D} d^D \ldots d^D c^{mD} d^D$ should now be taken large enough so that there is a subword $d^D c^{1D} d^D$ of $a$ that in all occurrences of $a$ are not cut by any boundary (notice that for any $m$ there is still fixed/bounded number of boundary cuts so by increasing the length of $a$ there must be some such $t_1$). Also take the $t_i$ in the representation of $a$ such that $t_1$ is much larger than the total length of $R_1, \ldots, R_m$. This means that in the diagram on the union of spheres these sub words are glued to each other (since having no boundary cuts on the subword forces an exact gluing up to a possible error disk with boundary $R_j$ by Proposition 5, but the $R_1, \ldots, R_m$ are too short to be glued to the subword, and if $R_j$ were to be glued to both occurrences of the subword, then $R_j$ would be an unreduced word). It is possible to change the diagram in such a way that the occurrences of words $a$ are all glued only to words $a$. In particular, for each base $\lambda$ covering $d^D c^{1D} d^D$, assume the covering is exact, i.e. $\alpha(\lambda)$ and $\beta(\lambda)$ correspond to the left and right endpoints of the subword $d^D c^{1D} d^D$ (introduce a boundary connection and cut at those boundaries if necessary) (and so $\lambda$ also exactly covers $d^D c^{1D} d^D$). Then if we have $\lambda \mu_1$ and $\lambda \mu_2$ and the length of $\mu_1$ is greater than the length of $\mu_2$, then we introduce a boundary connection to $\mu_1$ corresponding to the (right) endpoint of $\mu_2$. Cut $\mu_1$ and $\mu_2$ at that boundary connection, creating $\mu_1 = \kappa_1 \nu_1$ and $\mu_2 = \kappa_2 \nu_2$. Now rename $\lambda_1 = \lambda \kappa_1$, $\lambda_2 = \lambda \mu_2$, $\pi_1 = \mu_2$, $\pi_1 = \mu_2$ (this can be done since $\mu_2$ and $\mu_2$ cover the same subword neighboring $d^D c^{1D} d^D$ since the $d^D c^{1D} d^D$ are always located in exactly same place of each label $a$). We now have $\lambda_1$
and \(\tilde{\lambda}_1\) exactly covering strictly larger subwords of \(a\), corresponding to another diagram in which those larger subwords are glued together. By repeating this process (in both the left and right directions) for every base (exactly) covering a subword \(d^D c^L d^D\), we obtain a combinatorial equation with every occurrence of \(a\) completely covered by a base. Since \(i\) is much larger than the length of any \(R_j\), no base covering \(a\) can correspond to a base covering some \(R_j\), otherwise we would have an unreduced label of the disk for \(R_j\). So if \(\lambda\) covers \(a\), then \(\tilde{\lambda}\) must also cover only \(a\). Therefore, by Proposition 5, in the corresponding diagram \(a\) is only glued to \(a\).

\[\text{Lemma 8. If equation (13) has solution in the free group } F(A) \text{ for some subsequence } \{i\} \text{ of the natural numbers, then equation} \]

\[
\prod_{j=1}^{s} z_j^{-1}[a_1, b_1^j] z_j = [a_1^N, b_1^R] \tag{14}
\]

\[\text{has solution in } F(a_1, b_1) \text{ for } a_1 = a^{(i)} \text{ and } b_1 = b^i \text{ for some } i.\]

\[\text{Proof. Since words } a \text{ are only glued to words } a \text{ we have } a\text{-bands in the diagram, as in [17]. Now fix } i \text{ such that } i \text{ is much larger than the total length of } R_1, ..., R_m \text{ (this makes it so that in the diagram there are no annuli of } a\text{-bands with some } R_j \text{ disks filling in the center). So the disks with boundaries } R_j \text{ that glue to } a\text{-bands must by glued between the sides of } a\text{-bands labelled by powers of } b. \]

\[\text{However, since } b \text{ is not a proper power it can not be shifted relative to itself (i.e. no path labelled by } b \text{ may end in the middle of another path labelled by } b, \text{ so any } R_j \text{ disk between two } a\text{-bands partially glued together, must be labelled by powers of } b \text{ and exactly the same subwords of } b \text{ (starting from where the two } a\text{-bands separate), i.e. } R_j = b_1(b_1^{-1}b_2^{-1}b_2^{-1}b^{-n} = b^n-b^{-n} \text{ for subwords } b_1, b_2 \text{ of } b \text{ or } b^{-1}. \text{ Since the labels } R_j \text{ must be reduced words and, again, such disks do not have large enough powers of } b \text{ to glue to any annuli of } a\text{-bands, so they must only glue to other disks labelled } R_{j'}. \text{ Therefore the } R_j\text{-disks appear in the diagram tiling spheres that have no } a\text{-bands, so those spheres may be removed and leave a diagram with spheres tiled only by } a\text{-bands. This new diagram is labelled only with powers of } a^{(i)} \text{ and } b^i \text{ so by [17] equation (14) has a solution in } F(a^{(i)}, b^i) \text{ for this fixed } i. \]

\[\text{So by [17] there is a solution to the bin packing problem. Therefore the satisfiability problem for quadratic equations in } G \text{ is NP-hard. On the other hand, Theorem 1 implies that this problem is in NP. The proof of Theorem 3 is now complete.}\]

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