AN INTRODUCTION TO EQUIVARIANT COHOMOLOGY AND
HOMOLOGY, FOLLOWING GORESKY, KOTTWITZ, AND
MACPHERSON

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Abstract. This paper provides an introduction to equivariant cohomology and homology using the approach of Goresky, Kottwitz, and MacPherson. When a group $G$ acts suitably on a variety $X$, the equivariant cohomology of $X$ can be computed using the combinatorial data of a skeleton of $G$-orbits on $X$. We give both a geometric definition and the traditional definition of equivariant cohomology. We include a discussion of the moment map and an algorithm for finding a set of generators for the equivariant cohomology of $X$. Many examples and explicit calculations are provided.

1. Introduction

Homology can be thought of as a way of using certain subspaces of a variety $X$ to describe topological invariants of $X$. Equivariant homology arises in the setting of a variety $X$ with the action of a group $G$. Since group actions typically “fatten” a space (see Figure 1), information about both the group and an underlying skeleton in $X$ can be used to recover topological information about $X$.

\begin{center}
\includegraphics[width=0.3\textwidth]{figure1.png}
\end{center}

\textbf{Figure 1.} An $S^1$-action

In this paper, we follow the approach of Goresky, Kottwitz, and MacPherson in [GKM], who identify a particular skeleton inside $X$ which they use to turn the problem of computing equivariant cohomology into a combinatorial puzzle. The simplest possible skeleton is a point, but it would be a terribly dull world if the topology of every variety with a group action could be reduced to a point. The next simplest is a graph, to which the GKM method reduces all suitable varieties.

Their main result is to show that if the $G$-action on $X$ satisfies certain conditions, including having a finite number of fixed points and a finite number of one-dimensional orbits, then the equivariant cohomology of $X$ is a submodule of the equivariant cohomology of the set of fixed points in $X$, which is naturally a module over a polynomial ring. Moreover, this submodule can be identified explicitly using relations imposed by the one-dimensional orbits. All of this data can
be encoded in a graph whose vertices correspond to fixed points and whose edges correspond to one-dimensional orbits.

Group actions on varieties had been used to obtain topological data long before the paper of Goresky, Kottwitz, and MacPherson. Bialynicki-Birula used group actions to decompose varieties into a collection of attracting cells that could be used to describe cohomology, in something like an algebraic analogue of Morse theory [B-B]. Working in symplectic geometry, Kirwan and many others had studied symplectic reductions, namely quotients of a variety by its group action, and had extracted cohomological information using fixed points and their attracting sets. [K1] Theorem 5.4] in fact proves in the symplectic setting that the equivariant cohomology of $X$ is a submodule of the equivariant cohomology of the fixed points. An early paper of Chang and Skjelbred identifies the image of the equivariant cohomology of $X$ inside the equivariant cohomology of the fixed points, for more general topological spaces $X$ [CS] Lemma 2.3. Even earlier contributions came in different forms from Borel, Atiyah, Hsiang, and Quillen, among others, as described in [GKM Section 1.7].

The combinatorial data associated to group actions on a variety had also been studied, especially via the moment map. The moment map, discussed at length in Section 5, can be thought of here as a combinatorial way to collect data about the group orbits in $X$. Kostant showed that if a torus $T$ acts on a coadjoint orbit in the cotangent space $t^*$ to $T$, then the image of the moment map is a convex polytope [Ko]. This theorem inspired a series of papers by various authors, including [A], [H], and [GuSt] which extended the result to more general groups acting on more general varieties, collectively proving that the image of the moment map is some union of convex polytopes.

What Goresky, Kottwitz, and MacPherson do in their approach to equivariant cohomology is to put these two schools together, using the combinatorial data of the orbits of the group action from moment map theory to encode and interpret topological data of the attracting sets for fixed points from the study of algebraic group actions.

Many authors have since built on their work. We mention a sample of examples in this introduction and give others at relevant moments in the exposition. GKM theory has been extended to other cohomology theories: to equivariant intersection homology in [BM] and to equivariant K-theory in [R(Kn)]. The class of varieties for which the theory applies has been expanded, for instance in [GuHo] to torus actions without isolated fixed points. GKM theory is used to calculate the equivariant cohomology rings of Grassmannians [KnTao] and regular varieties (which have a unique fixed point with respect to a solvable group action) in [BC]. The graphs used in GKM theory are studied in their own right in [GuZ].

In order to make this exposition as elementary as possible, we make several assumptions throughout this paper. The first is that we work over the complex numbers, so the variety $X$ is a complex algebraic variety, the group $G$ is a complex linear algebraic group, and all cohomology has complex coefficients. The second is that the group $G$ is in fact an algebraic torus $T$, so $T$ is a product of $k$ copies of $\mathbb{C}^*$. When a result is easier to present using the compact torus $(S^1)^k$, we will use that instead, since the equivariant cohomology is the same regardless of whether $(\mathbb{C}^*)^k$ or $(S^1)^k$ is used. The third assumption is that $X$ is nonsingular and complete. All of these assumptions can be relaxed, but we make no further mention of this. We
also note that there are many things in [GKM] which we do not discuss at all in this paper.

The rest of this paper provides an introduction to equivariant cohomology following GKM theory. Section 2 gives several definitions of equivariant homology and cohomology. Section 3 then presents the main theorem due to Goresky, Kottwitz, and MacPherson. In Section 4 we give three different examples calculating equivariant cohomology. We include in Section 5 a discussion of the moment map in general, which is an important context for the combinatorial graphs used in GKM theory. Finally, Section 6 describes how to find a minimal set of generators for the equivariant cohomology, simplifying the explicit calculations that went before.

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2. Two definitions of equivariant cohomology and homology

2.1. Equivariant homology, topologically defined. Let $X$ be an algebraic variety with an algebraic action of $T = (\mathbb{C}^*)^k$. To define the ordinary homology of $X$, we would use the chain group consisting of formal linear combinations of maps $f : C \rightarrow X$, where $C$ is a not-too-singular space whose precise characteristics we do not detail here. To incorporate the action of $T$, we modify the definition of the chain group as follows:

$$C^T_i(X) = \left\{ \text{formal linear combinations of } T\text{-equivariant maps } f : C \rightarrow X, \right. $$

where $C$ is an $i + k$-dimensional space from a collection $\mathcal{C}$ of oriented spaces with a free $T$-action and with a notion of oriented boundaries.

Details about one possible collection $\mathcal{C}$ may be found in [GKM] Sections 3 and 4]. The notion of oriented boundaries means that the map $f|_{\partial C} : \partial C \rightarrow X$ is in $C^T_{i-1}(X)$. Orientations cancel in the usual way, so $\partial \partial C = 0$. Consequently, we may take the homology of the chain complex

$$\cdots C^T_{i+1}(X) \xrightarrow{\partial_{i+1}} C^T_i(X) \xrightarrow{\partial_{i}} C^T_{i-1}(X) \cdots$$

to get the equivariant homology $H^T(X)$.

For instance, consider the topological torus with the “hula-hoop” action of $S^1$ indicated in Figure 2.

There is only one free $0 + 1$-cycle up to homology, shown in Figure 3. Once any given point $p$ on the $0 + 1$-chain is mapped into the torus, the $S^1$-equivariance of $f$ determines the rest of the image. Any two such cycles $f_1$ and $f_2$ differ by the boundary of a $1 + 1$-chain, and so represent the same homology class. The class of this cycle thus generates $H^T_0(X)$.

Similarly, there is only one free $1 + 1$-cycle up to homology. The image of this cycle under $f$ must surject onto $X$, since any vertical strip in the image is wrapped
around the torus by $S^1$-equivariance of the map $f$. Hence, the class of the cycle given by $X$ itself generates $H_1^T(X)$.

The reader can observe that the equivariant homology of $X$ with this $S^1$-action is exactly the same as the ordinary homology of $S^1$. This is no coincidence. It is true because the torus is $S^1$ (vertically oriented), fattened by a free hula-hoop $S^1$-action. The more general statement also holds.

**Proposition 2.1.** If a torus $T$ acts freely on the algebraic variety $X$ and $X/T$ denotes the quotient space of $X$ under this $T$-action, then $H^*_T(X) = H_*(X/T)$.

The proof follows from these definitions. In fact, $X$ need only be a principal $T$-bundle for this proposition to hold.

The equivariant cohomology of $X$ may be defined as the cohomology of the chain complex $\text{Hom}(C_T, C)$. It is an exercise for the reader that Proposition 2.1 holds for equivariant cohomology, too.

**2.2. Equivariant cohomology, traditionally defined.** The following definition is more common in the literature. Fix a $T$-action on $X$. Proposition 2.1 says that equivariant cohomology would be easy to define if this $T$-action were free. Consequently, our strategy is to modify $X$ slightly to obtain a space which does carry a free $T$-action.

To do this, take $ET$ be a contractible space with a free $T$-action. For instance, the group $T = S^1$ acts naturally on each sphere $S^{2n+1}$ by rotation. Viewing $S^{2n+1}$ as the unit sphere in $\mathbb{C}^{n+1}$, the element $e^{i\theta}$ in $S^1$ sends $(z_0, z_1, \ldots, z_n)$ to $(e^{i\theta}z_0, e^{i\theta}z_1, \ldots, e^{i\theta}z_n)$. This action is free, but unfortunately the space $S^{2n+1}$ is not contractible. To correct for this, observe that $S^{2n+1}$ sits inside $S^{2n+3}$ as the equator, and that all cycles of dimension at most $2n - 2$ are contractible in $S^{2n+3}$. By taking the union of $S^{2n+1}$ for all $n$, we obtain the infinite sphere $S^\infty$, which is contractible. This space $S^\infty$ is $ES^1$.

Define $BT$ to be the quotient $ET/T$; it is called the classifying space of $T$. For example, we may construct the classifying space $BS^1$ using the example of $ES^1$. 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{torus.png}
\caption{The torus with the hula action of $S^1$}
\end{figure}

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{generator.png}
\caption{The generator for $H_0^{S^1}(X)$}
\end{figure}
given before. For each \( n \), the quotient of \( S^{2n+1} \) by the rotation action of \( S^1 \) is \( \mathbb{C}P^n \). Taking the union as before gives a space \( \mathbb{C}P^{\infty} \). Intuitively, the space \( \mathbb{C}P^{\infty} \) is constructed by adjoining cells as if to form \( \mathbb{C}P^n \), but without ever stopping.

The diagonal action of \( T \) on \( X \times ET \) is free, since the action on \( ET \) is free. Define \( X \times_T ET \) to be the quotient \( (X \times ET)/T \). This is the total space of a fiber bundle with base \( BT \) and fiber \( X \):

\[
\begin{array}{c}
X \rightarrow X \times_T ET \\
\downarrow \\
BT.
\end{array}
\]

We define the equivariant cohomology of \( X \) to be

\[ H^*_T(X) = H^*(X \times_T ET). \]

To see what this definition means, we work through the example when \( X \) is a point and \( T = S^1 \). In this case \( H^*_T(X) = H^*(pt \times_T ET) = H^*(ET/T) \). This is \( H^*(BT) \) by definition, and equals \( H^*(\mathbb{C}P^{\infty}) \). Recall that \( H^*(\mathbb{C}P^k) = \mathbb{C}[t]/(t^{k+1}) \) is generated by the class \([t]\) of an embedded \( \mathbb{C}P^{k-1} \) together with a normal vector. The argument does not generalize precisely, because it uses Poincaré duality. However, the restriction \( H^i(\mathbb{C}P^{k+i}) \rightarrow H^i(\mathbb{C}P^k) \) is an isomorphism as long as \( i \) is at most \( 2k \).

It follows that \( H^*(\mathbb{C}P^{\infty}) = \mathbb{C}[t] \) is the polynomial ring generated by a single class \([t]\) of complex dimension one.

Equivariant cohomology satisfies the same canonical properties as ordinary cohomology. In particular, the following hold:

1. functoriality;
2. a ring structure;
3. excision;
4. the Mayer-Vietoris sequence;
5. the Künneth formula;
6. the Leray spectral sequence;
7. for smooth orientable \( X \), Poincaré duality; and
8. existence of Chern classes,

where all subsets and maps are assumed to be equivariant.

For instance, we can use these properties to show that \( H^*_T(pt) \) is a module over \( H^*_T(pt) \). Applying functoriality to the map \( X \rightarrow pt \) allows us to pull back each class in \( H^*_T(pt) \) to \( H^*_T(X) \). The ring structure then permits us to multiply classes coming from \( H^*_T(pt) \) with classes in \( H^*_T(X) \).

Of course, the same procedure can be followed in ordinary cohomology, which brings us to a major difference between equivariant and ordinary cohomology: \( H^*_T(pt) \) is a very interesting ring, rather than a field. (One result of this is that the Künneth formula and Poincaré duality in equivariant cohomology may look slightly unfamiliar.)

To identify the ring \( H^*_T(pt) \), suppose the point is acted upon by \( T = (S^1)^k \). Write \( t \) for the Lie algebra of \( T \), namely the tangent to \( T \) at the identity, and write \( t^* \) for its dual, the cotangent space. Then by the same argument as before \( H^*_T(pt) = H^*(\prod_{i=1}^k \mathbb{C}P^{\infty}) \), which by the Künneth formula is \( \bigotimes_{i=1}^k \mathbb{C}[t_i] \). We prefer to write this as the symmetric algebra over the cotangent to \( T \), namely \( S(t^*) \). (The isomorphism \( H^*_T(pt) \cong S(t^*) \) is canonical, though the splitting of \( BT = \prod_{i=1}^k \mathbb{C}P^{\infty} \)
This shows that for each finite-dimensional torus the equivariant cohomology of a point $H^*_T(pt)$ is $S(t^*)$.

One way to compute the ordinary cohomology of a fiber bundle $X \times_T ET$ is to use a Leray spectral sequence. This is in general a complicated problem. Consequently, we restrict to a special class of varieties $X$ for which this spectral sequence degenerates in the sense of the next definition.

**Definition 2.2.** Fix a variety $X$ with an action of $T$. We say that $X$ is equivariantly formal with respect to this $T$-action if $E_2^2 = E_\infty^2$ in the spectral sequence associated to the fibration $X \longrightarrow X \times_T ET \longrightarrow BT$.

In algebraic terms, equivariant formality implies that $H^*_T(X)$ is a free module over $H^*_T(pt)$. We will use this crucial fact repeatedly in what follows. The next proposition follows directly from the definition and is the reason we consider equivariant formality.

**Proposition 2.3.** If $X$ is equivariantly formal with respect to a given $T$-action then there is a natural $H^*_T(pt)$-module isomorphism $H^*_T(X) = H^*(X) \otimes H^*_T(pt)$.

When $X$ is equivariantly formal with respect to $T$, the ordinary cohomology of $X$ can be reconstructed from its equivariant cohomology. To do this, let $M$ be the augmentation ideal in $S(t^*)$. (Recall that $M$ is the maximal homogeneous ideal in $S(t^*)$, so if $t^*$ is spanned by basis elements $t_1, t_2, \ldots, t_k$ then $M = \langle t_1, \ldots, t_k \rangle$.) The ordinary cohomology can be reconstructed as the quotient

$$H^*(X) = \frac{H^*_T(X)}{M \cdot H^*_T(X)}$$

which in effect simply sets each $t_i = 0$. This is an equality of rings and not just of modules [GKM 1.2.4]. Whenever we compute an example of $H^*_T(X)$, we will also give generators for $H^*_T(X)$ as a module over $H^*_T(pt)$ that coincide with module generators for $H^*(X)$.

In practice, many varieties of interest are equivariantly formal. For instance, [GKM] note that all of the following are equivariantly formal:

1. a smooth complex projective algebraic variety (with respect to any linear algebraic $T$-action);
2. a variety whose ordinary cohomology vanishes in odd degree (with respect to any $T$-action);
3. a variety with a $T$-invariant CW-decomposition; and
4. a compact symplectic manifold with a Hamiltonian $T$-action, where $T$ is a compact torus.

These are listed in [GKM Section 1.2 and Theorem 14.1].

The quintessential non-equivariantly formal space is shown in Figure 4; the points of intersection are poles of the spheres. The relevant action of $\mathbb{C}^*$ fixes the points of intersection and rotates each copy of $\mathbb{C}P^1$ so that when acted on by $t$ in $\mathbb{C}^*$, as $t \to 0$ the points of each $\mathbb{C}P^1$ flow from the north pole to the south pole, and so that the north pole of each sphere is glued to the south pole of the next.

3. The main theorem

We now come to the main construction in this paper. Suppose $X$ has a $T$-action with respect to which three conditions hold:
Figure 4. A complete variety which is not equivariantly formal

(1) $X$ is equivariantly formal;
(2) $X$ has finitely many fixed points; and
(3) $X$ has finitely many one-dimensional orbits.

In this situation, Goresky, Kottwitz, and MacPherson show that the combinatorial data encoded in the graph of fixed points and one-dimensional orbits of $T$ in $X$ implies a particular algebraic characterization of $H^*_T(X)$. For the rest of this paper, we will use the algebraic torus $(\mathbb{C}^*)^k$ instead of its deformation retract $(S^1)^k$.

Given a space $X$ with a $T$-action, the graph of fixed points and one-dimensional orbits can often be computed directly. For instance, let $X = \mathbb{C}P^2$, written with homogeneous coordinates. Define an action of $(\mathbb{C}^*)^2$ on $X$ by $(t_1, t_2) \cdot [x_0, x_1, x_2] = [x_0, t_1 x_1, t_2 x_2]$. The point $[x_0, x_1, x_2]$ lies in a two-dimensional orbit unless it is zero in one or two coordinates. Thus, there are exactly three fixed points under this action, namely $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$, and exactly three one-dimensional orbits, namely $[* , *, 0]$, $[0, *, *]$, and $[* , 0, *]$, where $*$ denotes any nonzero element in $\mathbb{C}$. The closure of each one-dimensional orbit is isomorphic to $\mathbb{C}P^1$ and has exactly two fixed points in its closure, one at the north pole and one at the south pole. Note that the fixed points and one-dimensional orbits form a subvariety isomorphic to that of Figure 4, although the $T$-action is different.

In fact, the basic properties of this subvariety hold for the subvariety of fixed points and one-dimensional orbits of any variety $X$ satisfying the same hypotheses.

**Proposition 3.1.** Let $X$ be a complete variety with a fixed action of $T$, with respect to which $X$ is equivariantly formal, and which has finitely many fixed points and finitely many one-dimensional orbits. The following all hold:

1. The closure of each one-dimensional orbit of $T$ in $X$ is isomorphic to $\mathbb{C}P^1$.
2. There are exactly two fixed points in the closure of each one-dimensional orbit, one at the north pole and the other at the south pole.
3. The torus $T$ acts on each one-dimensional orbit by rotation.
4. For each one-dimensional orbit $O$, there is a subtorus $T' \subseteq T$ of codimension one which fixes $O$ pointwise.

These facts can all be found in [GKM, Section 7.1].

This proposition implies that the fixed points and one-dimensional orbits of such a variety can be represented by a graph, each of whose vertices corresponds to a fixed point and each of whose edges corresponds to a one-dimensional orbit whose poles are the endpoints of the edge. This graph is called the moment graph of $X$, discussed further in Section 5. Note that the proposition further states that each edge is associated to a unique subtorus of codimension one that fixes the corresponding one-dimensional orbit pointwise. We typically label each edge with
the annihilator in the cotangent space of the tangent to this stabilizer. With our assumptions on $X$, the edges adjacent to a given vertex will be distinct.

There is a natural inclusion from the set of $T$-fixed points in $X$ to all of $X$. This map induces a map on equivariant cohomology $H^*_T(X) \longrightarrow \bigoplus_p \text{is } T\text{-fixed} H^*_T(p)$. From what we said earlier, it follows that $H^*_T(X)$ maps to $\bigoplus_p \text{is } T\text{-fixed} S(t^*)$. Given our hypotheses on $X$ and the $T$-action, this map is actually an injection.

**Theorem 3.2.** Suppose $X$ carries a $T$-action with respect to which $X$ is equivariantly formal, and which has finitely many fixed points and finitely many one-dimensional orbits. Then the map

$$H^*_T(X) \longrightarrow \bigoplus_{p \text{ is } T\text{-fixed}} S(t^*)$$

induced by inclusion from the $T$-fixed points into $X$ is an injection.

This is proven in [GKM], but was also proven previously in [K] Theorem 5.4. There are earlier results that anticipate this, including those of Borel, Atiyah, Hsiang, and Quillen, listed along with a more thorough historical bibliography in [GKM] Section 1.7. A particularly pretty proof in the case of symplectic manifolds is given in [10W].

The reader is encouraged to treat the next example as an exercise, though a sketch of the proof is included.

**Example 3.3.** The variety $X$ in Figure 4 is not equivariantly formal.

*Proof.* Denote the $T$-fixed points of $X$ by $X^T$. By excision, the group $H^*_T(X, X^T)$ is three-dimensional. The long exact sequence for the pair $(X, X^T)$ shows that $H^*_T(X)$ does not vanish. If $X$ were equivariantly formal, then $H^*_T(X) \longrightarrow H^*_T(X^T)$ would be an injection, but $H^*_T(X^T)$ does not exist in odd degrees. \hfill \Box

We wish to identify the image of $H^*_T(X)$ inside the ring $\bigoplus_{p \text{ is } T\text{-fixed}} S(t^*)$. Suppose $O$ is a one-dimensional orbit with north pole $N$ and south pole $S$, and let $f_N$ and $f_S$ be elements of $S(t^*)$ associated to the north and south pole respectively. There is a subtorus $T' \subseteq T$ of codimension one which fixes $O$. We wish to see that in order to be in the image of $H^*_T(X)$, the polynomial functions associated to the poles of $O$ must agree on the tangent space $t'$ of $T'$, namely the restrictions $f_N|_{t'} = f_S|_{t'}$. Equivalently, we need to show that the difference $f_N - f_S$ lies in the ideal generated by the annihilator of $t'$ inside the cotangent $t^*$. The following commutative diagram, all of whose maps are equivariant, proves that this claim is true. (The identification of $H^*_T(O)$ is implied by the construction in Section 2.1.)

$$\begin{array}{ccc}
H^*_T(O) & \longrightarrow & H^*_T(O \cup N) \cong H^*_T(N) \\
\downarrow & & \downarrow \\
H^*_T(S) \cong H^*_T(O \cup S) & \longrightarrow & H^*_T(O) \cong S((t')^*)
\end{array}$$

In fact, the condition that $f_N|_{t'} = f_S|_{t'}$ is sufficient to identify the image of the map $H^*_T(X) \longrightarrow \bigoplus_{p \text{ is } T\text{-fixed}} S(t^*)$. This is the main content of the following theorem of Goresky, Kottwitz, and MacPherson.

**Theorem 3.4.** Let $X$ be an algebraic variety with a $T$-action with respect to which $X$ is equivariantly formal, and which has finitely many fixed points and finitely many one-dimensional orbits. Denote the one-dimensional orbits $O_1, \ldots, O_m$. For
each $i$, denote the poles of $O_i$ by $N_i$ and $S_i$ and denote the stabilizer of $O_i$ in $T$ by $T_i$. The equivariant cohomology ring of $X$ is given by

$$H^*_T(X) \cong \left\{ (f_{p_1}, \ldots, f_{p_m}) \in \bigoplus_{\text{fixed pts}} S(t^*) : f_{N_i}|_{t^i} = f_{S_i}|_{t^i} \text{ for each } i = 1, \ldots, m \right\}.$$ 

We stress that this is a ring isomorphism, unlike the isomorphism in Proposition 2.3. That the relations $f_{N_i}|_{t^i} = f_{S_i}|_{t^i}$ suffice to determine the image of $H^*_T(X)$ was proven earlier in [CS] Lemma 2.3, though at a level of generality which prevented the concrete description which follows. We also remark that [GuHo] contains a very nice sketch of the proof of this theorem in the symplectic setting, assuming the injection of Theorem 5.2.

For instance, let $X$ be $\mathbb{C}P^1$ with an action of $T = \mathbb{C}^*$ given by $t \cdot [x_0, x_1] = [x_0, tx_1]$. There are exactly two fixed points of this action, namely $[1, 0]$ and $[0, 1]$, and there is exactly one one-dimensional orbit given by $[*, *]$. The graph associated to these fixed points and one-dimensional orbits is given in Figure 5. Since the subtorus which fixes the one-dimensional orbit is simply the identity, its tangent space is $\{0\}$, with which we have labelled the edge in Figure 5. We use the polynomial description of the equivariant cohomology of each fixed point. The polynomials associated to each fixed point must agree on this tangent space, so $p_N(0) = p_S(0)$.

By Theorem 5.4 (and used in its proof), the equivariant cohomology of $\mathbb{C}P^1$ is

$$H^*_C(\mathbb{C}P^1) = \{(p_N, p_S) \in S(t^*) \oplus S(t^*) : p_N(0) = p_S(0)\}.$$  

\[1, 0] \quad 0 \quad [0, 1]\]

**Figure 5.** The moment map graph for $\mathbb{C}P^1$

To verify this answer, we first rewrite the condition on the polynomials associated to the fixed points. If $p_N$ is chosen freely from $S(t^*)$ then $p_S = p_N + tp$ for a polynomial $p$ freely chosen in $S(t^*)$. In other words, the equivariant cohomology of $\mathbb{C}P^1$ is generated additively by the classes $(p_N, p_N)$ and $(0, tp)$, where $p_N$ and $p$ run over an additive basis of $S(t^*)$. Using Proposition 2.3 we see that $H^*_C(\mathbb{C}P^1) = H^*(\mathbb{C}P^1) \otimes H^*_C(pt)$. In this case, we know that the ordinary cohomology $H^*(\mathbb{C}P^1)$ is generated by a class $[1]$ in degree zero and $\alpha$ in degree 2. Our calculations identified one family $(p_N, p_N)$ corresponding to $[1] \otimes S(t^*)$ and a second family $(0, tp)$ corresponding to $\alpha \otimes S(t^*)$, so our computation of $H^*_C(\mathbb{C}P^1)$ is confirmed.

4. **Examples**

This section contains more examples computing equivariant cohomology. The calculations here are ad hoc and follow directly from the definitions and theorems in the previous sections. Section 6 gives a more methodical way to construct generators for $H^*_T(X)$, systematizing some of these calculations. The reader is encouraged to treat these examples as exercises.

**Example 4.1.** The equivariant cohomology of $\mathbb{C}P^2$ with an action of $(\mathbb{C}^*)^2$.

If $[x_0, x_1, x_2]$ is a point of $\mathbb{C}P^2$ and $(t_1, t_2)$ is an element of the torus $(\mathbb{C}^*)^2$, then the torus action is given by $(t_1, t_2) \cdot [x_0, x_1, x_2] = [x_0, t_1 x_1, t_2 x_2]$. As described
in Section 3 there are three fixed points and three one-dimensional orbits under this action. The one-dimensional orbit whose closure is \([*, 0, 0]\) is stabilized by the subtorus \(\mathbb{C}^* \times 1\) while the one-dimensional orbit whose closure is \([*, *, 0]\) is stabilized by the subtorus \(1 \times \mathbb{C}^*\). Finally, the one-dimensional orbit \([0, *, *]\) is stabilized by the one-dimensional subtorus given by \((t, t)\) for \(t \in \mathbb{C}^*\). Figure 6 shows the graph of fixed points and one-dimensional orbits for this function, labelling each edge according to the annihilator in the cotangent space of the stabilizer of the orbit. (Note that we use \(t_i\) to denote the standard basis of \(t^*\), which is dual to that of \(t\)).

\[
\begin{bmatrix}
1, 0, 0 \\
0, 1, 0 \\
0, 0, 1 \\
\end{bmatrix}
\]

\[
\begin{array}{ccc}
p_1 & p_2 & \bullet \\
0 & 1 & 0 \\
0 & 0 & \bullet \\
\end{array}
\]

\[
\begin{array}{ccc}
\bullet & \bullet & \bullet \\
t_1 & t_2 & t_1 - t_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
p_3 & \bullet & \bullet \\
0 & 0 & 1 \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Figure 6. The fixed points and one-dimensional orbits for } \mathbb{C}P^2
\end{array}
\]

We now analyze the equivariant cohomology ring using Theorem 3.3. Suppose \(p_1, p_2,\) and \(p_3\) are elements of \(S(t^*)\) associated to the vertices as labelled in Figure 6. Then the theorem tells us that

\[
\begin{align*}
p_2 - p_1 & \in \langle t_1 \rangle, \\
p_3 - p_1 & \in \langle t_2 \rangle, \quad \text{and} \\
p_3 - p_2 & \in \langle t_2 - t_1 \rangle.
\end{align*}
\]

These relations show that we may choose arbitrary elements \(p_1\) and \(q_1\) of \(S(t^*)\) and set \(p_2 = p_1 + t_1 q_1\). Once we have done that, we choose an element \(q_2\) of \(S(t^*)\) and determine the conditions imposed on \(q_2\) by setting \(p_3 = p_1 + t_2 q_2\). Subtracting shows that \(p_3 - p_2 = t_2 q_2 - t_1 q_1\). Since this is in the ideal \(\langle t_2 - t_1 \rangle\), we may rewrite \(q_2 = q_1 + (t_2 - t_1) q\). This equation shows that the most general choice of \(q_2\) in \(S(t^*)\) corresponds to a free choice of \(q\) in \(S(t^*)\). In other words, the equivariant cohomology ring of \(\mathbb{C}P^2\) consists of the triples

\[
H^*_{(\mathbb{C}^*)^2} (\mathbb{C}P^2) = \{ (p_1, p_1 + t_1 q_1, p_1 + t_2 q_1 + t_2 (t_2 - t_1) q) : p_1, q_1, q \in S(t^*) \}.
\]

Note that the equivariant cohomology \(H^*_{(\mathbb{C}^*)^2} (\mathbb{C}P^2)\) is generated as a module over \(S(t^*)\) by the elements \((1, 1, 1)\), \((0, t_1, t_2)\), and \((0, 0, t_2 (t_2 - t_1))\) of degrees 0, 1, and 2 respectively. This confirms our calculation, since \(H^*_{(\mathbb{C}^*)^2} (\mathbb{C}P^2) = H^* (\mathbb{C}P^2) \otimes H^* (\mathbb{C}P^2)\) by Proposition 2.8, which simplifies to \(H^* (\mathbb{C}P^2) \otimes S(t^*)\).

**Example 4.2.** The equivariant cohomology of the full flag variety in \(\mathbb{C}^3\) under the diagonal action of \((\mathbb{C}^*)^2\).

In this example we consider the full flag variety in \(\mathbb{C}^3\), namely the collection of nested vector spaces \(V_1 \subseteq V_2 \subseteq V_3 = \mathbb{C}^3\) with each \(V_i\) an \(i\)-dimensional subspace. To represent the flag \(V_1 \subseteq V_2 \subseteq V_3\), we choose vectors \(g_1, g_2,\) and \(g_3\) so that \(V_i\) is the linear span of \(g_1\) through \(g_i\). We then represent the flag by the matrix whose \(i^{th}\) column vector is \(g_i\). (This representation is not unique.)

Let \(g\) be a matrix whose entries are \(g_{i,j}\) and let \((t_1, t_2)\) be an element of \(\mathbb{C}^* \times \mathbb{C}^*\). The element \((t_1, t_2)\) acts on \(g\) by conjugation, so that \((t_1, t_2) \cdot g\) is given by

\[
\begin{pmatrix}
t_1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & t_2 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\frac{1}{t_1} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & \frac{1}{t_2} \\
\end{pmatrix}
\]
though multiplication on the right by a diagonal matrix fixes each flag.

Suppose \( g \) is a matrix representing a flag. If the flag given by \( g \) is fixed by this torus action, then the first column of \( g \) can only have one nonzero entry \( g_{w,1} \). We may assume that the entries in the columns to the right of \( g_{w,1} \) are zero by choosing \( g_2 \) and \( g_3 \) appropriately. By the same argument, the second column of \( g \) can only have one nonzero entry \( g_{w,2} \), and we may assume that the entry \( g_{w,3} \) is zero without altering the flag. This leaves a unique entry \( g_{w,3} \) nonzero. In sum, the flags fixed by this torus action are exactly the flags generated by permutations of the basis vectors.

A similar argument shows that the one-dimensional orbits are parametrized by the matrices which are a permutation matrix plus a single nonzero entry \( a \). Moreover, the entry \( a \) must lie to the left and above the nonzero entries of the permutation matrix. A set of matrices parametrizing each one-dimensional orbit is given in Figure 7.

\[
\begin{pmatrix}
0 & 0 & 1 \\
a & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
a & 0 & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a & 1 & 0 \\
0 & a & 1 \\
0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
0 & a & 1 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
a & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & a & 1 \\
0 & 1 & 0
\end{pmatrix}
\]

**Figure 7.** Matrices parametrizing one-dimensional orbits of the flag variety in \( \mathbb{C}^3 \), for \( a \) in \( \mathbb{C} \)

Finally, we identify the closures of these one-dimensional orbits. When \( a \) is zero, the permutation is clear. When \( a \) goes to \( \infty \), the one-dimensional orbit approaches a transposition of this permutation. We give an example; all the rest follow the same pattern.

\[
\begin{pmatrix}
a & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
= \text{Flag } \langle ae_1 + e_3 \rangle \subseteq \langle ae_1 + e_3, e_2 \rangle \subseteq \langle e_1, e_2, e_3 \rangle
\]

\[
= \text{Flag } \langle e_1 + a^{-1}e_3 \rangle \subseteq \langle e_1 + a^{-1}e_3, e_2 \rangle \subseteq \langle e_1, e_2, e_3 \rangle
\]

\[
a \rightarrow \infty \quad \text{Flag } \langle e_1 \rangle \subseteq \langle e_1, e_2 \rangle \subseteq \langle e_1, e_2, e_3 \rangle = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

(In fact, the two fixed points in the closure of each one-dimensional orbit are always related by multiplication by a transposition. We do not give a systematic description of this in order to maintain an elementary exposition while avoiding cumbersome notation.)

The stabilizer of each one-dimensional orbit can be determined from Figure 7 using the multiplication rule. This is all of the information needed to build the
The graph of fixed points and one-dimensional orbits for this torus action on the flag variety, shown in Figure 8. The edges have been arranged so that they correspond to the matrices in Figure 7, ordered from left to right and top to bottom according to the endpoint which is highest on the graph. Each edge is labelled by the linear functional in $t^*$ that annihilates that edge’s stabilizer in $T$. Note that if two edges in this graph are parallel then they share the same label.

Now we use Theorem 3.4 to compute the equivariant cohomology of the flag variety. We represent this calculation in Figure 9 using the previous graph, labelling each vertex this time with an element of $S(t^*)$. If $p_1$ is the function in $S(t^*)$

\[
p_1 + (t_1 - t_2)p_2 - (t_1 - t_2)p_3 + (t_1 - t_2)t_1p_4 + (t_1 - t_2)t_2p_5 + t_1t_2(t_1 - t_2)p_6
\]

associated to the lowest vertex, then the two vertices immediately above the lowest vertex can be labelled as indicated in Figure 9 for any $p_2$ and $p_3$ in $S(t^*)$. The two vertices above those have conditions imposed by the cycles in the graph. For instance, the vertex on the left can be written as $p_1 + t_1p_2 + (t_1 - t_2)q$ by moving up the edge labelled $t_1 - t_2$ from level one. Since this function can also be obtained by moving over the edge labelled $t_1$ from level one, we conclude that

\[
(p_1 + t_1p_2 + (t_1 - t_2)q) - (p_1 + t_2p_3) \in \langle t_1 \rangle.
\]

In other words, the difference $t_2(q + p_3)$ is in $\langle t_1 \rangle$ and so $q = t_1p_4 - p_3$ for $p_4$ chosen freely in $S(t^*)$. We have labelled this vertex accordingly in Figure 9.
calculation for the vertex on the right follows a symmetric argument. The topmost
vertex must be labelled with a function specified by three edge relations, given in
Figure 9.

In other words, the equivariant cohomology of the flag variety is generated as
an $S(t^*)$-module by six elements. We list these generators in Figure 10 associating
each generator to the vertex where the function first appears. Figure 11 displays
$(0, 0, 0, 0, 0, t_1 t_2(t_1 - t_2))$
$(0, 0, t_1(t_1 - t_2), 0, t_1(t_1 - t_2))$
$(0, t_1, 0, t_1, t_1 - t_2, t_1 - t_2)$
$(0, 0, t_2, t_2 - t_1, t_2 - t_1, -t_1 + t_2)$
$(1, 1, 1, 1, 1, 1)$

Figure 10. Generators for the equivariant cohomology of the flag variety

several of these generators in a style used frequently by those who work in this
area. Note that as a module over $S(t^*)$ the equivariant cohomology has one gen-
erator of dimension zero, two of dimension one, two of dimension two, and one of
dimension three. These are the even dimensional Betti numbers of the flag variety,
as Proposition 2.3 assures us they should be.

Example 4.3. The hypersurface $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$ in $\mathbb{C}P^5$.

We remark that this hypersurface is in fact the image of the Grassmannian
$Gr(2, 4)$ under the Plucker embedding.

The torus $(\mathbb{C}^*)^3$ acts coordinatewise on this hypersurface as follows. Given an
element $(x_1, x_2, x_3, y_1, y_2, y_3)$ of the hypersurface, then

$(t_1, 1, 1) \cdot (x_1, x_2, x_3, y_1, y_2, y_3) = (t_1 x_1, t_1 x_2, t_1 x_3, y_1, y_2, y_3)$
$(1, t_2, 1) \cdot (x_1, x_2, x_3, y_1, y_2, y_3) = (t_2 x_1, t_2 x_2, x_3, y_1, y_2, t_2 y_3)$
$(1, 1, t_3) \cdot (x_1, x_2, x_3, y_1, y_2, y_3) = (t_3 x_1, t_2 x_2, t_3 x_3, y_1, t_3 y_2, y_3)$

We now verify that each fixed point under this action has exactly one nonzero
coordinate. In fact, at most one of the $x_i$ is nonzero, since otherwise the action of
the subtorus $1 \times \mathbb{C}^* \times \mathbb{C}^*$ is at least one-dimensional. Likewise, at most one of the
$y_i$ is nonzero, using the action of the same subtorus. Finally, either one of the $x_i$
or one of the $y_i$ is nonzero, but not both, since otherwise the action of $\mathbb{C}^* \times 1 \times 1$
would form a one-dimensional orbit. This means that the six coordinate entries are
the fixed points of this torus action on the hypersurface.

The one-dimensional orbits are identified using a similar argument. If the point
$(x_1, x_2, x_3, y_1, y_2, y_3)$ lies on a one-dimensional orbit then at most two entries among
the $x_i$ and $y_j$ are nonzero. The nonzero coordinates cannot be $x_i$ and $y_i$ for the
same index $i$ because that point is not on the hypersurface. Any other pair of
nonzero coordinates defines a one-dimensional orbit of the torus action. The fixed
points in the closure of each one-dimensional orbit are the two coordinate points
given by the two possible nonzero entries.

We compute the stabilizer of each one-dimensional orbit by examining the torus
action. For instance, the stabilizer of the orbit for which either $x_1$ and $x_3$ are
nonzero or $y_1$ and $y_3$ are nonzero is $\mathbb{C}^* \times 1 \times \mathbb{C}^*$. The other stabilizers can be found
by inspection in just the same way.

The graph of fixed points and one-dimensional orbits is given in Figure 11. Each
vertex is labelled by the coordinate which is nonzero, and each edge is labelled by
the element of \( t^* \) that generates the annihilator of the stabilizer of the orbit. Note again that parallel edges have the same label.

![Figure 11](image1.png)

**Figure 11.** The graph of fixed points and one-dimensional orbits for a quadric hypersurface in \( \mathbb{CP}^5 \)

We now use this graph to compute the equivariant cohomology of the hypersurface \( X \). Each vertex in Figure 11 is labelled by the generic element of \( S(t^*) \) associated to that fixed point by \( H^*_T(X) \). It requires no argument to label the first two vertices \( p_1 \) and \( p_1 + t_3 p_2 \), since no cycles have yet been formed. The vertex for

\[
p_1 + (t_3 - t_1)p_2 + t_1(t_1 - t_3)p_4
\]

the fixed point \( x_3 \) can be reached either by moving over the edge labelled \( t_2 - t_3 \) from the function \( p_1 + t_3 p_2 \) or up the edge labelled \( t_2 \) from \( p_1 \). Consequently, this vertex is labelled \( p_1 + t_3 p_2 + (t_2 - t_3)q \) where \( q \) is an element of \( S(t^*) \) that satisfies \( t_3 p_2 + (t_2 - t_3)q \in \langle t_2 \rangle \). Since this means that \( t_3(p_2 - q) \in \langle t_2 \rangle \), we conclude that \( q = p_2 + t_2 p_3 \) for some \( p_3 \) freely chosen in \( S(t^*) \). The general function associated to the leftmost vertex in the back is identified by a symmetric argument. The rightmost vertex in the back can be reached by going up the edge labelled \( t_1 - t_2 \) from \( p_1 \), going over the edge labelled \( t_2 - t_3 \) from \( p_1 + (t_3 - t_1)p_2 + t_1(t_1 - t_3)p_4 \), or going back the edge labelled \( t_1 \) from \( p_1 + t_2 p_2 + t_2(t_2 - t_3)p_3 \). This means that the generic function labelling this vertex is \( p_1 + t_2 p_2 + t_2(t_2 - t_3)p_3 + t_4 q \) for some
q in $S(t^*)$ satisfying both
\[ t_2p_2 + t_2(t_2 - t_3)p_3 + t_1q \in (t_1 - t_2) \quad \text{and} \quad (t_2p_2 + t_2(t_2 - t_3)p_3 + t_1q) - ((t_3 - t_1)p_2 + t_1(t_1 - t_3)p_4) \in (t_2 - t_3). \]
To satisfy the first relation, we need $q'$ in $(t_1 - t_2)$ so that $q = -p_2 - (t_2 - t_3)p_3 + q'$.
To satisfy the second relation, the function $q' = (t_1 - t_2)p_4 + (t_1 - t_2)(t_2 - t_3)p_5$ where $p_5$ is chosen freely in $S(t^*)$. The generic function labeling the topmost vertex is shown in the graph. It is left to the reader to supply the argument, which requires satisfying the four conditions imposed by the edges leading to the vertex. Figure 14 displays one of these generators in a very commonly used graphical form.

**Example 4.4.** The ordinary cohomology ring of Examples 4.1, 4.2, and 4.3.

Multiplication in the equivariant cohomology ring can be done coordinatewise. Since each coordinate of an equivariant cohomology class is an element of $S(t^*)$, this calculation is straightforward. As an example, we square the class $(0, t_1, t_2)$ in Example 4.1 and find $(0, t_1, t_2)^2 = (0, t_1^2, t_2^2)$, which we rewrite in terms of the original classes as $t_1(0, t_1, t_2) + (0, 0, t_2(t_2 - t_1))$.

Now we use the equality $H^*_T(X) = H^*_T(\mathbb{CP}^2)$ from Section 2. If $u$ represents the equivariant cohomology class $(0, t_1, t_2)$ and $v$ represents the class $(0, 0, t_2(t_2 - t_1))$, then our calculation shows that $u^2 = t_1 u + v$ in equivariant cohomology. Taking the quotient by $M \cdot H^*_T(\mathbb{CP}^2)$ in effect just sets all $t_i = 0$, so in ordinary cohomology $u^2 = v$. This confirms what we already know to be true.

We leave to the reader the exercise of computing full multiplication tables for the ordinary cohomology rings in Examples 4.1, 4.2, and 4.3.

5. The moment map and its image

The graph of fixed points and one-dimensional orbits of the $T$-action on $X$ is actually a small part of a larger collection of combinatorial and geometric information associated to $X$. We describe this for a $T$-action on a variety $X$, though it can be defined for general $G$-actions by replacing appropriate letters in what follows. Each complex projective variety $X$ with an algebraic $T$-action is equipped with a moment map $\mu : X \rightarrow t^*$, where $t^*$ denotes the dual to the Lie algebra of $T$, or equivalently the cotangent space to $T$ at the identity. In this section, we describe the moment map in three different ways: locally, globally, and then combinatorially. The combinatorial description naturally extends the graphs used to compute the equivariant cohomology of $X$.

We first describe the moment map locally at a point $x$ in $X$. The $T$-action on $X$ induces a linear $T$-action on the tangent space $T_x X$, which we recall is diffeomorphic to a neighborhood of $x$ in $X$. In fact this diffeomorphism can be taken to intertwine the local and global moment maps. Locally near $x$, the moment map for $T$ acting on $T_x X$ differs from that for $X$ by a translation [GuSt, Equation 4.5 and preceding]. Since $T_x X$ is a vector space, its moment map is easy to describe explicitly. Indeed, $T$ can be viewed as a subgroup of the general linear group on $T_x X$. This permits us to write $t$ as a subspace of the tangent space to $GL_n$, and in particular as some subset of $n \times n$ matrices. In this language, the moment map is simply

\[ \mu : T_x X \rightarrow t^* \]
\[ y \mapsto \{ M \mapsto y^T M y \}. \]

(This presentation follows that of [GuSt] Equation 4.5 and preceding.)
To describe the moment map in general, we make reference to the fact that a complex projective variety has a naturally associated symplectic form $\omega$ (see [C, pages 96–97] for a construction of $\omega$, the Fubini-Study form). Given this form, the moment map is the $T$-equivariant map $\mu : X \to t^*$ whose differential satisfies

$$d\mu(\xi)(\alpha) = \omega(\xi, \alpha_x),$$

where $\xi$ is a tangent vector in $T_X$, $\alpha$ is in the tangent space to $X$, and $\alpha_x$ is the vector field on $X$ induced by the infinitesimal action along $\alpha$. (See [C, Chapter 22] for more.)

For many purposes, all relevant information about the image of the moment map can be computed without the map itself. First, the $T$-fixed points in $X$ will be mapped to points in $t^*$. Let $T'$ be the codimension-one stabilizer of a one-dimensional orbit $e$ and $t'$ its tangent. Applying Equation (5.1) to each $\alpha$ in $t'$ shows that the moment map sends the orbit $e$ into a line segment between the image of two fixed points in $t^*$. Moreover, the direction of this line segment is determined by the annihilator of $t'$. By construction, two line segments are parallel if their corresponding orbits have the same stabilizer. Up to rescaling along each edge, then, the closures of the orbits determine the image of the moment map. Lest the reader be too cavalier about this rescaling, it does correspond to the choice of a line bundle on the variety.

If $X$ is an irreducible projective variety, the image of the moment map is in fact the convex hull of these fixed points (see [A] and [GuSt, Theorem 4]). In other words, this image is a polytope. Various classes of varieties have been studied using the combinatorics of these moment map images [Gu], most notably toric varieties [F]. From the other direction, combinatorial properties of polytopes have been studied by constructing projective varieties whose moment map images are the desired polytopes [St]. Note that the moment map image of the fixed points and one-dimensional orbits of $X$ are exactly the 0- and 1-dimensional faces of this polytope.

Not only can the image of the moment map be reconstructed from the fixed points and one-dimensional orbits of $X$ under $T$, but the algebraic constructions used in Section 3 can be extended to any graph satisfying certain minimal conditions. We give the description in [BM] here; [GuZ] has a similar construction. A moment graph is defined to be a finite graph for which each edge $e$ is associated to a one-dimensional subspace $V_e$ of the vector space $t^*$, and with a partial order on the vertices so that if $u$ and $v$ are joined by an edge then either $u < v$ or $v < u$. The subspace $V_e$ is called the direction of the edge $e$.

In our case, the direction $V_e$ corresponds to the annihilator of the codimension-one subtorus that stabilizes the one-dimensional orbit $e$. Intuitively, since the closure of $e$ is isomorphic to $\mathbb{CP}^1$, it has both a north pole and a south pole. We would like to define a partial order so that the north pole is greater than the south pole for each one-dimensional orbit. To do this, we need to choose a generic map $\rho : \mathbb{C}^* \to T$, where generic means that the resulting $\mathbb{C}^*$-action on $X$ has the same fixed points as the original $T$-action. Given a one-dimensional orbit $e$ and a point $p$ on $e$, the south pole of $e$ is the fixed point $v = \lim_{t \to 0} \rho(t) \cdot p$ while the north pole is the fixed point $u = \lim_{t \to \infty} \rho(t) \cdot p$. The map $\rho$ in effect chooses a “global north” and is discussed further in the next paragraph. When $X$ and its $T$-action satisfy certain conditions, for instance if the $T$-action induces a Whitney stratification [BM] or if the $T$-action is Hamiltonian [GuZ, Theorem 1.4.2], then the convention that if $u$ is
the north pole then \( u > v \) gives a partial order on the graph. (Note that there is an “opposite” map that is the composition of \( \rho \) with the map \( z \mapsto \frac{1}{z} \) in \( \mathbb{C}^* \). In terms of the moment graph, the opposite map reverses the original partial order.)

We can also describe this partial order given just the data of the skeleton of fixed points and one-dimensional orbits in the moment map image of \( X \). Each edge \( e \) in \( t^* \) defines a hyperplane in \( t \), namely the tangent \( t_e \) of the stabilizer of the corresponding one-dimensional orbit. Since there are a finite number of edges, the complement of the union \( \bigcup_e t_e \) is dense in \( t \). In particular, we can pick an element \( \xi \) in \( t \) so that \( e(\xi) \) is nonzero for each \( e \). If the endpoints of \( e \) are \( u \) and \( v \), then their difference in \( t^* \) satisfies either \( (u - v)(\xi) > 0 \) or \( (v - u)(\xi) > 0 \). Define a partial order on the graph by \( u > v \) if \( (u - v)(\xi) > 0 \). (Transitivity holds since if \( u > v \) and \( v > w \) then by linearity \( (u - w)(\xi) = (u - v + v - w)(\xi) > 0 \).) This element \( \xi \) can also be constructed as the derivative of the map \( \rho \) mentioned in the previous paragraph. Since the choice of \( \xi \) was only loosely constrained, the reader can see that many partial orders can be imposed on the vertices.

In the language of moment graphs, Theorem 3.1.1 associates a copy of \( S(t^*) \) to each vertex in the moment graph and identifies \( H^*_T(X) \) with the submodule for which whenever \( e \) is an edge with endpoints \( u \) and \( v \) then \( f_u - f_v \in S(t^*)/V_e S(t^*) \). This perspective is further studied in [BM], which describes more general ways of associating \( S(t^*) \)-modules to the vertices of moment graphs so that the edges impose compatibility restrictions. In [BM], Section 1.4, they provide a purely algebraic algorithm to check these compatibility restrictions and show how this can be used to construct the equivariant intersection cohomology of \( X \). We note that [BM] assume that \( X \) has a \( T \)-equivariant stratification by cells, which in turn implies equivariant formality of the variety. However, their construction depends only on the moment graph, regardless of whether it is associated to an algebraic variety or not. Indeed, some moment graphs are the graphs of more than one algebraic variety, not all of which need be equivariantly formal. (Compare, for instance, the moment graph of the variety shown in Figure 4 with that of \( \mathbb{CP}^2 \).) Given a fixed moment graph, it is possible to construct a (noncompact) complex manifold with that moment graph so long as some added conditions are satisfied. [GuZ, Theorem 3.1.1] gives an explicit construction and a characterization of the extra conditions needed.

6. Generators for each equivariant cohomology group

In this section we discuss an algorithm to construct a set of generators for \( H^*_T(X) \) as a module over \( S(t^*) = H^*_T(pt) \). This method is implicit in the calculations from the examples of Section 4. We include here both its limitations and some extensions.

Our approach to describing \( H^*_T(X) \) in Section 4 was to find the most general element of \( H^*_T(X) \) that could be associated to each fixed point. For instance, we began with the lowest vertex and associated a generic element in \( S(t^*) \) to it. We then chose a neighbor \( v \) and associated the most general element of \( S(t^*) \) that would satisfy the edge relation to \( v \) from the lowest vertex. We continued until all vertices were labelled, each time selecting a lowest neighbor of labelled vertices and determining the most general element of \( S(t^*) \) that could be associated to that neighbor while satisfying the edge relations to already labelled vertices.

We now give a computationally simpler and more systematic method for constructing elements of \( H^*_T(X) \) given a moment graph for \( X \). Denote the edge between
vertices \( w \) and \( u \) by \( wu \) and denote the direction of this edge \( V_{wu} \). Fix a vertex \( v \) and label all vertices \( u \) such that \( u < v \) in the partial order with the function \( 0 \). Now, if \( w \) is a vertex for which each neighbor \( u < w \) has been labelled with \( f_u \in S(t^*) \), associate to \( w \) a minimal degree element \( f_w \in S(t^*) \) satisfying
\[
f_w - f_u \in V_{wu} \quad \text{for all neighbors } u < w.
\]
Continue this process until all vertices have been labelled. This gives a cohomology class associated to \( v \), which we will examine in more detail later.

It requires some proof even to see that an element of \( H^*_\mathcal{T}(X) \) can be produced by the process described. However, this algorithm, which is described in [BM, Section 1.4], not only works but in fact gives a minimal set of generators for \( H^*_\mathcal{T}(X) \) as a module over \( S(t^*) \) in many cases. [GuZ2] gives conditions for the moment graph which ensure that these cohomology classes actually generate the cohomology ring.

To see how the algorithm works, first consider the function associated to \( v \) itself by this construction. We indicate the partial order by directing the edges of the moment graph, so the edges pointing down from \( v \) are labelled \( f_1, f_2, \ldots, f_k \) as indicated in Figure 13. The function associated to \( v \) must be in the ideal generated by the \( f_i \). Since each edge is labelled by an element of degree one in \( S(t^*) \) and since the lines that they span meet at a point, the \( f_i \) are distinct. Hence, a natural choice for the minimum degree symmetric function associated to \( v \) is \( f_1 f_2 \cdots f_k \). If the moment graph satisfies the conditions of [GuZ2, Theorem 2.2], then for each fixed point \( v \), the generator associated to \( v \) has degree equal to the number of edges pointing downward from \( v \).

Figure 14 displays more examples of generators found using this algorithm for fixed points from examples computed in Section 4. The generators are the same as those found previously, but the process of computing them is less laborious. The way the generators are displayed in this figure is very often used by practitioners. Note that in these examples, the generator shown is unique up to rescaling by \( \mathbb{C}^* \).

In fact, this algorithm produces a unique set of generators as long as the moment map for \( X \) has a Palais-Smale component. For our purposes, this condition means
that the moment graph can be drawn in the plane so that the vertex \( v \) lies above its neighbor \( u \) if and only if \( v \) has more downward-pointing edges than \( u \). A formal definition can be found in [Kn]; see also [GuZ2, Theorem 2.3]. We observe that all of the varieties \( X \) discussed in this paper satisfy this condition. However, it is easy to find varieties which do not. For instance, recall the flag variety of Example 4.2 and let \( S \) be the diagonal matrix with diagonal entries 1, 2, and 3. The subvariety of the complex flag variety that consists of flags \( V_1 \subseteq V_2 \subseteq \mathbb{C}^3 \) such that \( SV_1 \subseteq V_2 \) has the moment graph shown in Figure 15, by a similar argument to that in Example 4.2.

By inspection, this moment graph cannot satisfy the Palais-Smale condition. We have shown two different minimal-degree generators associated to the same vertex. (This variety is one of a class of subvarieties of the flag variety called Hessenberg varieties; see [T]. The cells of the Bialynicki-Birula decomposition, namely the attracting sets for the torus action, are not in general a stratification of Hessenberg varieties.)

A similar approach to that presented here can be used to find generators for cohomology rings related to those of \( H^*_T(X) \). For instance, in the context of symplectic manifolds, a natural restriction map called the Kirwan map sends the equivariant cohomology \( H^*_T(X) \) onto the ordinary cohomology of the symplectic reduction of \( X \) (which is essentially the quotient of \( X \) by the action of \( T \)). [LoW] use this to identify a subset of generators for \( H^*_T(X) \) that generate the entire cohomology \( H^*(X/T) \) of the symplectic reduction, and [Go] constructs these generators combinatorially.

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