COARSE BAUM-CONNES CONJECTURE AND RIGIDITY FOR ROE ALGEBRAS

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Abstract. In this paper, we connect the rigidity problem and the coarse Baum-Connes conjecture for Roe algebras. In particular, we show that if $X$ and $Y$ are two uniformly locally finite metric spaces such that their Roe algebras are $\ast$-isomorphic, then $X$ and $Y$ are coarsely equivalent provided either $X$ or $Y$ satisfies the coarse Baum-Connes conjecture with coefficients. It is well-known that coarse embeddability into a Hilbert space implies the coarse Baum-Connes conjecture with coefficients. On the other hand, we provide a new example of a finitely generated group satisfying the coarse Baum-Connes conjecture with coefficients but which does not coarsely embed into a Hilbert space.

1. Introduction

Given a metric space $(X, d)$, one defines the uniform Roe algebra of $X$, denoted by $C^*_u(X)$, as the closure of all bounded operators on $\ell_2(X)$ with finite propagation (we refer the reader to Section 2 for all the definitions in this introduction). Similarly, the Roe algebra of $X$, denoted by $C^*(X)$, is defined as the closure of all bounded operators on $\ell_2(X, H_X)$ with finite propagation, where $H_X$ is an infinite dimensional separable Hilbert space. Recently, the study of rigidity properties for those $C^*$-algebras has gained a lot of attention ([ˇSW13, WW19, BF18, BFV18, BFV19, BV19]). Precisely, the following is open.

Problem 1.1 (Rigidity Problem). Let $X$ and $Y$ be uniformly locally finite metric spaces.

1. If $C^*_u(X)$ and $C^*_u(Y)$ are $\ast$-isomorphic, are $X$ and $Y$ coarsely equivalent?

2. If $C^*(X)$ and $C^*(Y)$ are $\ast$-isomorphic, are $X$ and $Y$ coarsely equivalent?

The first positive partial result for Problem 1.1 was proved by J. Špakula and R. Willett, who showed that both items above have a positive answer if the spaces have property A ([ˇSW13, Theorem 4.1]). Later, Problem 1.1 was answered positively for the larger class of spaces which coarsely embed into a Hilbert space ([BF18, Corollary 1.2]), and even more recently it was
shown that one only needs to assume that one of the spaces coarsely embeds into a Hilbert space ([BFV19, Corollary 1.5]). This was done by looking at a technical condition on the metric spaces (Definition 1.2(2)), and noticing that coarse embeddability into a Hilbert space implies that condition. However, after the work of J. Špakula and R. Willett, the Roe algebra has been neglected in the papers mentioned above, i.e., those subsequent articles only dealt with the rigidity question for uniform Roe algebras.

The goal of this paper is two-fold. Firstly, we generalize the main results of [BF18, BFV19] in order to obtain partial answers to Problem 1.1(2) outside of the realm of property A. Secondly, we further study the technical geometric condition introduced in [BF18, BFV18] – i.e, the property that all sparse subspaces of a given metric space $X$ yield only compact ghost projections in their (uniform) Roe algebras (see Definition 1.2) – in order to extend the class of uniformly locally finite metric spaces satisfying this property.

We now describe our main results. Firstly, let us define the main technical geometric condition considered in these notes. For the definition of a ghost operator in $C^*(X)$, we refer to Definition 2.2 below.

Definition 1.2. Let $(X, d)$ be a metric space.

1. $X$ is called sparse if there exists a partition $X = \bigsqcup_n X_n$ such that
   • $|X_n| < \infty$ for all $n \in \mathbb{N}$, and
   • $d(X_n, X_m) \to \infty$ as $n + m \to \infty$.
2. We say that all sparse subspaces of $X$ yield only compact ghost projections in their Roe algebras if for all sparse subspaces $X' \subset X$ all ghost projections in $C^*(X')$ are compact.

Notice that the geometric condition in (2) was already known to be formally weaker than coarse embeddability into a Hilbert space (see [BF18, Lemma 7.3]). However, as we prove below, this is actually a strictly weaker property – see Theorem 1.5 and Theorem 1.6.

Following the terminology introduced in [BF18] for the uniform Roe algebra, some of our results below depend on rigid $\ast$-isomorphisms between different kinds of Roe algebras (see Definition 2.4 for the precise definition). The next theorem summarizes the state of the art of Problem 1.1(2).

Theorem 1.3. Let $X$ and $Y$ be uniformly locally finite metric spaces. Then the following are equivalent:

1. $X$ is coarsely equivalent to $Y$.
2. $C^s_a(X)$ is rigidly $\ast$-isomorphic to $C^s_a(Y)$.
3. $UC^*(X)$ is rigidly $\ast$-isomorphic to $UC^*(X)$.
4. $C^*(X)$ is rigidly $\ast$-isomorphic to $C^*(Y)$.

Definition 2.1 below gives the precise definitions of the Roe algebra, uniform Roe algebra, stable Roe algebra, and uniform algebra – $C^*(X)$, $C^s_a(X)$, $C^s(X)$, and $UC^*(X)$, respectively.
If all sparse subspaces of $Y$ yield only compact ghost projections in their Roe algebras, then the items above are also equivalent to the following.

5. $C^*_u(X)$ is Morita equivalent to $C^*_u(Y)$.
6. $C^*_s(X)$ is $*$-isomorphic to $C^*_s(Y)$.
7. UC$^*(X)$ is $*$-isomorphic to UC$^*(Y)$.
8. $C^*(X)$ is $*$-isomorphic to $C^*(Y)$.

The main contribution of these notes to Theorem 1.3 are the implications $(5) \Rightarrow (1)$, $(6) \Rightarrow (1)$, $(7) \Rightarrow (1)$, and $(8) \Rightarrow (1)$, under the hypothesis that all sparse subspaces of $Y$ yield only compact ghost projections in their Roe algebras.

As for our second goal, we start by making explicit a definition which was already implicit in all the rigidity papers mentioned above.

**Definition 1.4.** Let $X$ be a uniformly locally finite metric space.

1. We say that $X$ is **Roe rigid** if $X$ is coarsely equivalent to any uniformly locally finite metric space $Y$ so that $C^*(X)$ and $C^*(Y)$ are $*$-isomorphic.
2. We say that $X$ is **uniform Roe rigid** if $X$ is coarsely equivalent to any uniformly locally finite metric space $Y$ so that $C^*_u(X)$ and $C^*_u(Y)$ are $*$-isomorphic.

With this terminology, Theorem 1.3 states that if all sparse subspaces of a uniformly locally finite metric space yield only compact ghost projections in their Roe algebras, then $X$ is Roe rigid. Under the same conditions, we can also conclude that $X$ is uniform Roe rigid (Proposition 5.4). We use this in order to prove the following:

**Theorem 1.5.** Let $X$ be a uniformly locally finite metric space and assume that $X$ satisfies the coarse Baum-Connes conjecture with coefficients. Then $X$ is both Roe rigid and uniform Roe rigid.

Since there are uniformly locally finite metric spaces which satisfy the coarse Baum-Connes conjecture with coefficients but which do not coarsely embed into a Hilbert space (see Remark 2.10 and Proposition 2.11), Theorem 1.5 provides new examples of metric spaces for which Problems 1.1(1) and 1.1(2) have a positive answer.

Moreover, we obtain some other technical conditions on a uniformly locally finite metric space so that it is Roe rigid. We refer the reader to Section 5 for the definitions of those technical conditions.

**Theorem 1.6.** Let $X$ be a uniformly locally finite metric space. Then $X$ is both Roe rigid and uniform Roe rigid if any one of the following conditions holds:

1. $X$ is sparse, admits a fibred coarse embedding into a Hilbert space, and satisfies the coarse Baum-Connes conjecture.
2. $X = \square \Gamma$ is the box space of a residually finite, finitely generated discrete group $\Gamma$ that admits a coarse embedding into a Banach space with property $(H)$, and $X$ satisfies the coarse Baum-Connes conjecture.

3. $X = \Gamma$ is a countable discrete group which satisfies the Baum-Connes conjecture with coefficients.

This paper is organized as follows. In Section 2, we deal with the definitions and background necessary for these notes. In particular, in Subsection 2.3, we talk about the Baum-Connes conjectures. In Section 3, we present the main tool in order to generalize the results for uniform Roe algebras obtained in [BF18, BFV19] to the context of Roe algebras (Lemma 3.1). Section 4 starts by dealing with embeddings between Roe algebras, which is the essential step so that we can have the asymmetry of Theorem 1.3, i.e., the fact that only $Y$ has to satisfy a geometric condition. The proof of Theorem 1.3 is also presented in Section 4. Finally, Section 5 deals with the coarse Baum-Connes conjecture, the proof of Theorems 1.5 and 1.6, and provides examples of new spaces for which the rigidity problem has a positive answer.

2. Preliminaries

If $H$ is a Hilbert space, we denote the closed unit ball of $H$ by $B_H$. Moreover, $B(H)$ and $K(H)$ denote the spaces of bounded and compact operators on the Hilbert space $H$, respectively.

Let $X$ be a set and $H_X$ be a Hilbert space. Then the Hilbert space $\ell_2(X) \otimes H_X$ is canonically isometric to $\ell_2(X, H_X)$. Given $x, y \in X$, the operator $e_{xy} \in B(\ell_2(X, H_X))$ is defined by

$$e_{xy} \delta_z \otimes w = \langle \delta_z, \delta_y \rangle \delta_y \otimes w$$

for all $z \in X$ and all $w \in H_X$. Given $a \in B(\ell_2(X, H_X))$, we let $a_{xy} = e_{yy}ae_{xx}$ for all $x, y \in X$, and define

$$\text{supp}(a) = \{(x, y) \in X \times X \mid a_{xy} \neq 0\}.$$  

Clearly, for each $x, y \in X$, $a_{xy}$ can be canonically identified with an element of $B(H_X)$, and we do so without further mention throughout this paper. Given $v, u \in H_X$, the operator $e_{(x,v),(y,u)} \in B(\ell_2(X, H_X))$ is defined by

$$e_{(x,v),(y,u)} \delta_z \otimes w = \langle \delta_z \otimes w, \delta_x \otimes v \rangle \delta_y \otimes u$$

for all $z \in X$ and all $w \in H_X$. Given $A \subset X$, define $\chi_A = \sum_{x \in A} e_{xx}$. If $A = X$, we simply write $1 = \chi_X$.

2.1. Roe algebras. If $(X, d)$ is a metric space and $a \in B(\ell_2(X, H_X))$, the propagation of $a$ is defined by

$$\text{prop}(a) = \sup \{d(x, y) \mid e_{yy}ae_{xx} \neq 0\}.$$
We say the metric space $X$ is \textit{uniformly locally finite}, which we abbreviate by \textit{u.l.f.}, if
\[
\sup_{x \in X} |\{y \in X \mid d(x, y) \leq r\}| < \infty
\]
for all $r > 0$.

We now define the central object of this work, i.e., the Roe algebra of a u.l.f. metric space. We also define variants of this algebra whose rigidity properties are closely related to the ones of the Roe algebra (Theorem \ref{THM:Roe}).

\textbf{Definition 2.1.} Let $X$ be a u.l.f. metric space and $H_X$ be an infinite dimensional separable Hilbert space.

1. The \textit{Roe algebra of $X$ over $H_X$}, denoted by $C^*_X$, is defined as the closure of all $a \in B(\ell_2(X, H_X))$ so that $\text{prop}(a) < \infty$ and $a_{xy}$ is compact for all $x, y \in X$.

2. The \textit{uniform algebra of $X$ over $H_X$}, denoted by $UC^*_X$, is defined as the closure of all $a \in B(\ell_2(X, H_X))$ so that $\text{prop}(a) < \infty$ and so that there exists $N \in \mathbb{N}$ such that $\text{rank}(a_{xy}) \leq N$ for all $x, y \in X$.

3. The \textit{stable Roe algebra of $X$ over $H_X$}, denoted by $C^*_s(X)$, is defined as the closure of all $a \in B(\ell_2(X, H_X))$ so that $\text{prop}(a) < \infty$ and so that there exists a finite dimensional subspace $H \subset H_X$ such that $a_{xy} \in B(H)$ for all $x, y \in X$.

4. If $H_X = \mathbb{C}$, all the algebras above coincide and it is called the \textit{uniform Roe algebra of $X$}, denoted by $C^*_u(X)$.

Notice that, in order to simplify notation, the space $H_X$ is omitted in the notations of the algebras above.

Clearly, $C^*_u(X) \subset UC^*_u(X) \subset C^*(X)$. Moreover, fixing a rank 1 projection on $B(H_X)$, there is a canonical embedding of $C^*_u(X)$ into $C^*_s(X)$ for all u.l.f. metric spaces $X$. Also, notice that $C^*_s(X)$ is canonically isomorphic to $C^*_u(X) \otimes K(H_X)$ for all metric spaces $X$.

The concept of ghost operators plays an essential role in these notes.

\textbf{Definition 2.2.} Let $X$ be a u.l.f. metric space. An operator $a \in C^*(X)$ is a \textit{ghost} if for all $\epsilon > 0$ there exists a finite $A \subset X$ so that $\|a_{xy}\| < \epsilon$ for all $x, y \not \in A$.

Notice that, given any u.l.f. metric space $X$, all compact operators on $\ell_2(X)$ are ghosts. Also, under the canonical embeddings $C^*_u(X) \hookrightarrow C^*_s(X)$ described above, the concept of a ghost operator in $C^*_s(X)$ is also well-defined for all u.l.f. metric space $X$.

Since we are interested in several types of “Roe algebras” – $C^*_u(X)$, $C^*_s(X)$, $UC^*_u(X)$, and $C^*(X)$ – the following definition has the purpose of simplifying the statements of our technical lemmas below.

\textbf{Definition 2.3.} Let $X$ be a u.l.f. metric space. A $C^*$-subalgebra $A \subset C^*(X)$ is called \textit{Roe-like} if $C^*_s(X) \subset A$.

Following \cite{BFTS}, we introduce the notion of rigid $\ast$-homomorphisms and rigid $\ast$-isomorphisms between Roe-like algebras.
Definition 2.4. Let $X$ and $Y$ be metric spaces, and $A \subset C^*(X)$ and $B \subset C^*(Y)$ be Roe-like $C^*$-subalgebras. A $*$-homomorphism $\Phi : A \to B$ is said to be a rigid $*$-homomorphism if

$$\sup_{u \in B_{HY}} \inf_{x \in X} \sup_{y \in Y} \sup_{v \in B_H} \|\Phi(e(x,u),(x,u))\delta_y \otimes v\| > 0.$$ 

A $*$-isomorphism $\Phi : A \to B$ is called a rigid $*$-isomorphism if both $\Phi$ and $\Phi^{-1}$ are rigid $*$-homomorphisms. In this case, the algebras $A$ and $B$ are called rigidly $*$-isomorphic.

In the definition above, an assignment $x \in X \mapsto (y_x, v_x) \in Y \times B_H$ witnesses that $\Phi$ is rigid if there exist $u \in B_{HX}$ and $\delta > 0$ such that

$$\|\Phi(e(x,u),(x,u))\delta_y \otimes v_x\| > \delta,$$

for all $x \in X$.

2.2. Coarse geometry. Let $(X, d)$ and $(Y, \partial)$ be metric spaces, and $f : X \to Y$ be a map. We say that $f$ is coarse if for all $s > 0$ there exists $r > 0$ so that

$$d(x, y) < s \text{ implies } \partial(f(x), f(y)) < r \text{ for all } x, y \in X,$$

and we say that $f$ is expanding if for all $r > 0$ there exists $s > 0$ so that

$$d(x, y) > s \text{ implies } \partial(f(x), f(y)) > r \text{ for all } x, y \in X.$$

If $f$ is both coarse and expanding, $f$ is said to be a coarse embedding. If $f$ is a coarse embedding so that $\sup_{y \in Y} \partial(y, f(X)) < \infty$, then $f$ is called a coarse equivalence. It is easy to check that $f$ is a coarse equivalence if and only if there exists a coarse map $g : Y \to X$ so that

$$\sup_{x \in X} d(x, g \circ f(x)) < \infty \text{ and } \sup_{y \in Y} d(y, f \circ g(y)) < \infty.$$

In this case, we say that $X$ and $Y$ are coarsely equivalent.

2.3. Baum-Connes conjectures. The coarse Baum-Connes conjectures are defined in terms of certain assembly maps. Due to the high technicality of the precise statements of those conjectures and since we do not make explicit use of the description of those assembly maps, we will not present the formal definitions in this paper but only direct the reader to appropriate sources.

Let $G$ be a locally compact, $\sigma$-compact, Hausdorff groupoid with a Haar system\footnote{See [Tu00] Section 1 for the definition and main properties of groupoids and Haar systems.} $A$ a $C^*$-algebra endowed with an action of $G$ is called a $G$-$C^*$-algebra. Loosely speaking, given a $G$-$C^*$-algebra $A$, the Baum-Connes conjecture for $G$ with coefficients in $A$ states that a certain assembly map

$$\mu_{r, A} : K^\text{top}_*(G, A) \to K_* (A \rtimes_r G)$$
is an isomorphism (we refer the reader to [Tu99b, Définition 5.2] for details). Therefore, this provides means of calculating the $K$-theory groups of the reduced crossed product of $A$ by $G$.

**Definition 2.5.** Let $G$ be a locally compact, $\sigma$-compact, Hausdorff groupoid with a Haar system.

1. Let $A$ be a $G$-$C^*$-algebra. We say that $G$ satisfies the Baum-Connes conjecture for $A$ if the assembly map $\mu_{r,A}$ above is an isomorphism.
2. We say that $G$ satisfies the Baum-Connes conjecture with coefficients if the assembly map $\mu_{r,A}$ above is an isomorphism for all $G$-$C^*$-algebras $A$.

**Remark 2.6.** Any countable discrete group $\Gamma$ can be regarded as a groupoid satisfying the properties listed in Definition 2.5. Hence, it makes sense to say that $\Gamma$ satisfies the Baum-Connes conjecture with coefficients. Moreover, this agrees with the original formulation of the Baum-Connes conjecture for groups in [BCH94, Conjecture 9.6].

In these notes, we will need the coarse Baum-Connes conjecture with coefficients for a u.l.f. metric space, which can be defined in terms of the Baum-Connes conjecture with coefficients for a certain groupoid related to the metric space. Precisely, let us first recall the coarse groupoid $G(X)$ associated with a u.l.f. metric space $(X, d)$. For every $R > 0$, we consider the $R$-neighborhood of the diagonal in $X \times X$, i.e., $\Delta_R := \{(x, y) \in X \times X : d(x, y) \leq R\}$. Define

$$G(X) := \bigcup_{R>0} \Delta_R \subseteq \beta(X \times X),$$

where $\beta(X \times X)$ denotes the Stone-Čech compactification of $X \times X$. It turns out that the domain, range, inversion and multiplication maps on the pair groupoid $X \times X$ have unique continuous extensions to $G(X)$. With respect to these extensions, $G(X)$ becomes a principal, étale, locally compact, $\sigma$-compact Hausdorff topological groupoid whose unit space $G(X)_0$ is $\beta X$ (see [STY02, Proposition 3.2] or [Roe03, Theorem 10.20]). Since $G(X)_0 = \beta X$ is totally disconnected, $G(X)$ is also ample (see [Exel10, Proposition 4.1]). Moreover, there is a canonical isomorphism between $C^*_r(X)$ and the reduced groupoid C*-algebra of $G(X)$ (see [Roe03, Proposition 10.29] for a proof).

Recall that a metric space $(X, d)$ is uniformly discrete if $\inf_{x \neq y} d(x, y) > 0$.

**Definition 2.7 ([Tu12, FSW14]).** Let $X$ be a uniformly discrete u.l.f. metric space.

1. $X$ satisfies the coarse Baum-Connes conjecture if the coarse groupoid $G(X)$ of $X$ satisfies the Baum-Connes conjecture for $\ell_\infty(X, K(\ell_2(\mathbb{N})))$.  

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3We refer the reader to [Tu00] for a survey of the Baum-Connes conjecture for groupoids, and to [GJY19] for a survey of the Baum-Connes conjecture for groups. We also refer the reader to [WO93] for facts about $K$-theory of $C^*$-algebras.
2. $X$ satisfies the coarse Baum-Connes conjecture with coefficients if $G(X)$ satisfies the Baum-Connes conjecture with coefficients.

3. $X$ satisfies the boundary coarse Baum-Connes conjecture if the boundary groupoid $G(X)|_{\beta X \backslash X}$ of $X$ satisfies the Baum-Connes conjecture for $\ell_\infty(X,K(\ell_2(N))/c_0(X,K(\ell_2(N))).$

If $X$ is a u.l.f. metric space, we say that $X$ satisfies the coarse Baum-Connes conjecture (resp. coarse Baum-Connes conjecture with coefficients, or boundary coarse Baum-Connes conjecture) if $X$ is bijectively coarsely equivalent to a uniformly discrete metric space which satisfies the coarse Baum-Connes conjecture (resp. coarse Baum-Connes conjecture with coefficients, or boundary coarse Baum-Connes conjecture).

4. Remark 2.8. In part (2) of the definition above, we require the groupoid $G(X)$ to satisfy the Baum-Connes conjecture for all coefficients but in fact, this is equivalent to requiring $G(X)$ to satisfy the Baum-Connes conjecture for all separable coefficients.

Indeed, if $G$ is a second countable, étale, ample groupoid, and $A$ is a $G$-$C^*$-algebra, then $A$ may be written as an inductive limit $\lim_{i \in I} A_i$ of separable $G$-$C^*$-algebras, where the connecting homomorphisms $\phi_i : A_i \to A$ are injective. Hence, $A \rtimes_r G \cong \lim_{i \in I} A_i \rtimes_r G$, and $\lim_i K_\text{top}(G, A_i) \cong K_\text{top}(G, A)$ by [BD] Lemma 5.1 and Theorem 5.2, and we also have $\lim_i K_*(A_i \rtimes_r G) \cong K_*(A \rtimes_r G)$ by [WO93] Proposition 6.2.9 and Proposition 7.1.7. Moreover, the connecting homomorphisms are compatible with the Baum-Connes assembly maps. Thus, if $G$ satisfies the Baum-Connes conjecture with coefficients in $A_i$ for all $i$, then $G$ satisfies the Baum-Connes conjecture with coefficients in $A$.

Although the coarse groupoid $G(X)$ is not second countable in general, [STY02] Lemma 3.3] gives a second countable, ample groupoid $G'$ such that $G(X) = \beta X \rtimes G$. Then any $G(X)$-$C^*$-algebra $A$ is also a $G'$-$C^*$-algebra in a canonical way, and $A \rtimes_r G(X) = A \rtimes_r (\beta X \rtimes G') \cong A \rtimes_r G'$. By [STY02] Lemma 4.1], the assembly map for $G(X)$ with coefficients in $A$ agrees with the assembly map for $G'$ with coefficients in $A$.

In Section 5, we will make use of some well-known results about the various Baum-Connes conjectures, and we summarize them here for ease of reference.

Theorem 2.9.

1. If a uniformly discrete, u.l.f. metric space $X$ satisfies the coarse Baum-Connes conjecture with coefficients, then so does any subspace of $X$ [Tu12 Theorem 4.2].

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4 Notice that if $X$ is bijectively coarsely equivalent to a metric space satisfying one of those properties, then all metric spaces which are bijectively coarsely equivalent to $X$ also satisfy the same property.

5 Notice that this was proved in [BD] Lemma 5.1 and Theorem 5.2] for a sequence of $C^*$-algebras but the proof still works for a net of $C^*$-algebras.
2. Let $G$ be a locally compact groupoid isomorphic to $X \rtimes G'$, where $X$ is a compact space and $G'$ is a locally compact, second countable and étale groupoid. Let $H$ be a closed, étale subgroupoid of $G$. If $G$ satisfies the Baum-Connes conjecture with coefficients, then so does $H$ [Tu12, Theorem 3.14].

3. If a sparse uniformly discrete u.l.f. metric space $X$ admits a fibred coarse embedding into Hilbert space, then the boundary groupoid $G(X)|_{\beta X \setminus X}$ is a-T-menable ([FS14, Theorem 22] and Proposition 5.7 below). Hence, $X$ satisfies the boundary coarse Baum-Connes conjecture [Tu99a, Théorème 9.3].

4. A uniformly discrete, u.l.f. metric space $X$ admits a coarse embedding into Hilbert space if and only if $G(X)$ is a-T-menable [STY02, Theorem 5.4]. Hence, if $X$ admits a coarse embedding into Hilbert space, then $X$ satisfies the coarse Baum-Connes conjecture with coefficients [Tu99a, Théorème 9.3]. However, the converse is false (see Remark 2.10 and Proposition 2.11).

5. For a countable discrete group $\Gamma$ with a proper left-invariant metric, write $|\Gamma|$ for the underlying metric space. Then $G(\{\Gamma\}) = \beta|\Gamma| \rtimes \Gamma$ [STY02, Proposition 3.4]. Moreover, if $\Gamma$ satisfies the Baum-Connes conjecture with coefficients, then $|\Gamma|$ satisfies the coarse Baum-Connes conjecture with coefficients [STY02, Lemma 4.1].

**Remark 2.10.** In [AT18, Theorem 1.2], G. Arzhantseva and R. Tessera provide an example of a finitely generated group $\Lambda$ which is a split extension of an (infinite rank) abelian group by a finitely generated group with the Haagerup property such that $\Lambda$ does not coarsely embed into a Hilbert space. However, $\Lambda$ satisfies the Baum-Connes conjecture with coefficients (see [CE01, Corollary 3.14] and [HK01, Theorem 1.1]).

Besides the example in Remark 2.10 above, the authors of [AT18] also provide an additional example of a finitely generated group $\Gamma$ which is a split extension of a finitely generated group with property A by a finitely generated group with the Haagerup property such that $\Gamma$ does not coarsely embed into a Hilbert space (see [AT18, Theorem 1.3]). More precisely, $\Gamma = \mathbb{Z}/2\mathbb{Z} \rtimes_{\Lambda} (H \times F)$ is a restricted wreath product, where

- $G$ is a finitely generated group which contains an expander graph isometrically in its Cayley graph (see [Osa14, Theorem 4]);
- $H$ is a finitely generated group with the Haagerup property but it does not have property A (see [Osa14, Theorem 2]) and there is a surjective homomorphism $H \twoheadrightarrow G$ (see [AT18, Proposition 2.15]);
- $F$ is a finitely generated free group with a surjective homomorphism $F \twoheadrightarrow G$;
- $G$ is the $H \times F$-set, where $H$ acts by the left translation and $F$ acts by the right translation via their surjections onto $G$. 
To the best of our knowledge, it was unknown whether $\Gamma$ satisfies the Baum-Connes conjecture with coefficients. We provide an affirmative answer here even though $\Gamma$ does not coarsely embed into a Hilbert space:

**Proposition 2.11.** The group $\Gamma$ above satisfies the Baum-Connes conjecture with coefficients. In particular, $|\Gamma|$ satisfies the coarse Baum-Connes conjecture with coefficients.

*Proof.* Since the $H$-action and the $F$-action commute, we have that $\Gamma = (\mathbb{Z}/2\mathbb{Z} \wr G) \rtimes F$. Observe that $\mathbb{Z}/2\mathbb{Z} \wr G$ satisfies the Baum-Connes conjecture with coefficients as it is an extension of two groups with the Haagerup property (see [CE01, Corollary 3.14] and [HK01, Theorem 1.1]).

Since the free group $F$ is torsion-free and has the Haagerup property, we conclude that $\Gamma$ satisfies the Baum-Connes conjecture with coefficients as well (see [OO01, Theorem 7.1]). The last statement follows from Theorem 2.9(5). \qed

### 3. Rigidity of homomorphisms

In this section, we show that, under our main geometric condition, strongly continuous compact preserving $*$-homomorphisms between Roe-like algebras are rigid (see Corollary 3.5).

From now on, in order to simplify statements, every time metric spaces $X$ and $Y$ are mentioned, infinite dimensional separable Hilbert spaces $H_X$ and $H_Y$ over which the Roe algebras $C^*(X)$ and $C^*(Y)$ are defined will be automatically implicitly considered.

The next result is a version of [BFV18, Lemma 4.1] for Roe-like algebras, and it is our main tool in order to prove rigidity of such $*$-homomorphisms. Moreover, we point out that the main obstacle in order to obtain this generalization of [BFV18, Lemma 4.1] is Claim 3.2 below.

**Lemma 3.1.** Let $Y$ be a u.l.f. metric space and assume that all sparse subspaces of $Y$ yield only compact ghost projections in their Roe algebras. Let $(p_n)_n$ be an orthogonal sequence of nonzero finite rank projections such that SOT-$\sum_{n \in M} p_n \in C^*(Y)$ for all $M \subset \mathbb{N}$. Then

$$\inf_{n \in \mathbb{N}} \sup_{y \in Y, v \in B_{H_Y}} \| p_n \delta_y \otimes v \| > 0.$$  

*Proof.* If this fails, by going to a subsequence, assume that $\| p_n \delta_y \otimes v \| < 2^{-n}$ for all $n \in \mathbb{N}$, all $y \in Y$ and all unit vectors $v \in H_Y$.

**Claim 3.2.** For each finite $F \subset Y$, $\lim_n \chi_F P_n = 0$.

*Proof.* Suppose the claim fails for a finite $F \subset Y$. Then there exists $y \in F$ such that

$$\lim_{n} \inf \| \chi_y P_n \| > 0.$$
By going to a subsequence, assume that \( \delta = \inf_n \| \chi_{\{y\}} p_n \| > 0 \). There exists a finite \( E \subset Y \) such that \( \inf_n \| \chi_{\{y\}} p_n \chi_E \| > 0 \). Indeed, if this does not hold, by going to a further subsequence, we can pick a sequence of disjoint finite subsets \((E_n)_n\) of \( Y \) such that \( \| \chi_{\{y\}} p_n \chi_{E_n} \| \geq \delta / 2 \) and \( \| \chi_{\{y\}} p_n \chi_{E_m} \| \leq 2^{-n-2} \delta \) for all \( n \neq m \) in \( \mathbb{N} \). Hence, we must have

\[
\inf_n \| \chi_{\{y\}} \left( \sum_n p_n \right) \chi_{E_n} \| \geq \| \chi_{\{y\}} p_n \chi_{E_n} \| - \sum_{n \neq m} \| \chi_{\{y\}} p_n \chi_{E_m} \| > \frac{\delta}{4}
\]

for all \( m \in \mathbb{N} \). Since \((E_n)_n\) is a disjoint sequence and \( Y \) is locally finite, it follows that \( \lim_n d(y, E_n) = \infty \). Hence, as \( \sum_n p_n \in C^*(Y) \), this gives us a contradiction.

Since \( E \) is finite, by going to a further subsequence, pick \( z \in E \) such that \( \gamma = \inf_n \| \chi_{\{y\}} p_n \chi_{\{z\}} \| > 0 \). For each \( n \in \mathbb{N} \), write \( q_n = \chi_{\{y\}} p_n \chi_{\{z\}} \), so \( \| q_n \| \in [\gamma, 1] \). Under the natural identification, we view each \( q_n \) as an operator in \( B(H_Y) \). Since \((p_n)_n\) is orthogonal, it converges to zero in the strong operator topology, and so does \((q_n)_n\). As \( \dim(H_Y) = \infty \), there exists a normalized weakly null sequence \((u_n)_n\) in \( H_Y \) such that \( \| q_n u_n \| \geq \gamma / 2 \) for all \( n \in \mathbb{N} \).

Since each \( p_n \) has finite rank, so does \( q_n \). Therefore, \( \lim_n q_n u_m = 0 \) for all \( n, m \in \mathbb{N} \). By passing to a subsequence, we can assume that \( \| q_n u_m \| \leq 2^{-n-4} \gamma^2 \) for all \( n \neq m \). Define \( q = \chi_{\{y\}} \left( \sum_n p_n \right) \chi_{\{z\}} \).

Fix \( a \in K(H_Y) \) with \( \| a \| \leq 2 \). Since \( a \) is compact, there exists an increasing sequence of natural numbers \((m_k)_k\) making \((a u_{m_k})_k\) convergent, say \( v = \lim_k a u_{m_k} \). So, \( \| v \| \leq 2 \). Since \((p_n)_n\) converges to zero in the strong operator topology and each \( p_n \) is self-adjoint, \((q_n)_n\) converges to zero in the strong operator topology. This gives us

\[
\limsup_k \left| \langle q u_{m_k}, v \rangle \right| \leq \limsup_k \sum_n \left| \langle q_n u_{m_k}, v \rangle \right| \\
\leq \limsup_k \left( \left| \langle u_{m_k}, q^*_n v \rangle \right| + \sum_{n \neq m_k} \left| \langle q_n u_{m_k}, v \rangle \right| \right) \\
\leq \frac{\gamma^2}{16}.
\]
Hence, we have that

\[ \|q - a\|^2 \geq \liminf_k \| (q - a)(u_{m_k}) \|^2 \]
\[ \geq \liminf_k \left( \|q u_{m_k}\|^2 + \|a u_{m_k}\|^2 \right) - \frac{\gamma^2}{8} \]
\[ \geq \liminf_k \left( \|q u_{m_k}\|^2 - \sum_{n \neq m_k} \|q_n u_{m_k}\| \right)^2 + \|v\|^2 - \frac{\gamma^2}{8} \]
\[ \geq \frac{9}{64} \gamma^2 + \|v\|^2 - \frac{\gamma^2}{8} \]
\[ = \frac{\gamma^2}{64} + \|v\|^2. \]

This shows that

\[ d\left( q, \{ b \in \mathcal{K}(H_Y) \mid \|b\| \leq 2 \} \right) > 0. \]

As \((p_n)_n\) is an orthogonal sequence of projections, \(\|\sum_n p_n\| \leq 1\). Hence, \(\|q\| \leq 1\) and we have that \(d(q, \mathcal{K}(H_Y)) > 0\). So, \(q = \chi_{\{y\}}(\sum_n p_n)\chi_{\{z\}}\) is not compact; contradiction, since \(\sum_n p_n \in C^*(Y)\).

The next claim should be compared with [BFV18, Claim 4.2]. Let \(\partial\) denote the metric of \(Y\).

**Claim 3.3.** By going to a subsequence of \((p_n)_n\), there exists a sequence \((Y_n)_n\) of disjoint finite subsets of \(Y\) and a sequence of finite rank projections \((q_n)_n\) in \(C^*(Y)\) such that

1. \(\partial(Y_k, Y_m) \to \infty\) as \(k + m \to \infty\),
2. \(\|p_n - q_n\| < 2^{-n}\), and
3. \(q_n \in \mathcal{K}(\ell_2(Y_n, H_Y))\), for all \(n \in \mathbb{N}\).

**Proof.** We construct sequences \((q_k)_k\), \((Y_k)_k\) and \((n_k)_k\) by induction as follows. Since \(p_1\) has finite rank, pick a finite rank projection \(q_1 \in C^*(Y)\) with a finite support such that \(\|p_1 - q_1\| < 2^{-1}\) and set \(n_1 = 1\). Pick a finite \(Y_1 \subseteq Y\) so that \(\text{supp}(q_1) \subseteq Y_1 \times Y_1\). Fix \(k > 1\) and assume that \(Y_j, n_j\) and \(q_j\) have been defined for all \(j \leq k - 1\).

Let

\[ Z = \left\{ y \in Y \mid \partial\left( y, \bigcup_{j \leq k-1} Y_j \right) > k \right\}. \]

Since \(Y\) is uniformly locally finite, \(Z\) is finite. By Claim 3.2 for all large enough \(m\) we have \(\|\chi_{Z} p_m\| < 2^{-k-2}\). Fix such \(m\). For a sufficiently large finite \(Y_k \subseteq Y \setminus Z\), the operator \(a = \chi_{Y_k} p_m \chi_{Y_k}\) is a positive contraction in \(C^*(Y)\) and it satisfies \(\|a - p_m\| < 2^{-k-1}\). Hence \(\text{Sp}(a) \subset [0, 1/2) \cup (1/2, 1]\), so the map \(f : \text{Sp}(a) \to \{0, 1\}\) defined by \(f(t) = 0\) if \(t < 1/2\) and \(f(t) = 1\) if \(t > 1/2\) is continuous. By the continuous functional calculus, \(q_k = f(a)\) is a projection and \(\|q_k - a\| < 2^{-k-1}\) (cf. [BFV18, Claim 4.2]). Therefore \(\|q_k - p_m\| < 2^{-k}\) and \(q_k \in C^*(a) \subseteq \mathcal{K}(\ell_2(Y_k, H_Y))\). Hence, \(\text{supp}(q_k) \subseteq Y_k \times Y_k\).
as required. Let \( n_k = m \). This completes the definition of \((q_k)_k\), \((Y_k)_k\) and \((n_k)_k\).

Let \((Y_n)_n\) and \((q_n)_n\) be given by Claim 3.3. Let \( Y' = \bigsqcup q_n Y_n \), so \( Y' \) is a sparse subspace of \( Y \). Clearly, \( \sum_n q_n \) converges in the strong operator topology to a noncompact projection \( q \in \mathcal{B}(\ell_2(Y', H_Y)) \). By our choice of \((q_n)_n\), \( \sum_n (q_n - p_n) \) converges in norm, so \( \sum_n (q_n - p_n) \in C^*(Y) \). Hence, 
\[
q = \sum_n p_n + \sum_n (q_n - p_n) \in C^*(Y),
\]
and as \( q \in \mathcal{B}(\ell_2(Y', H_Y)) \), we have \( q \in C^*(Y') \).

The next claim is [BF18, Claim 2 in proof of Theorem 6.1], so we omit its proof.

**Claim 3.4.** \( \sum_n p_n \) is a ghost.

Since \( \|p_n - q_n\| < 2^{-n} \) for all \( n \), this shows that \( q \) is also a ghost; contradiction.

**Corollary 3.5.** Let \( X \) and \( Y \) be u.l.f. metric spaces and assume that all sparse subspaces of \( Y \) yield only compact ghost projections in their Roe algebras. Let \( A \subset C^*(X) \) be a Roe-like C*-subalgebra. Then every strongly continuous compact preserving \(*\)-homomorphism \( \Phi : A \to C^*(Y) \) is a rigid \(*\)-homomorphism.

**Proof.** Fix a unit vector \( u \in H_X \) and for each \( x \in X \), let \( p_x = \Phi(e_{(x,u), (x,u)}) \). Then each \( p_x \) is a projection and, as \( \Phi \) is compact preserving, each \( p_x \) has finite rank. Moreover, \( (p_x)_{x \in X} \) is an orthogonal sequence. Since \( \Phi \) is strongly continuous, SOT-\( \sum_{n \in M} p_n \in C^*(Y) \) for all \( M \subset \mathbb{N} \). Let \( x \in X \mapsto (y_x, v_x) \in Y \times B_{H_Y} \) be given by Lemma 3.1, i.e., an assignment satisfying

\[
\inf_{x \in X} \|\Phi(e_{(x,u), (x,u)}) \delta_{y_x} \otimes v_x\| > 0.
\]

This assignment witnesses that \( \Phi \) is a rigid \(*\)-homomorphism.

4. **Embeddings onto hereditary subalgebras and isomorphisms**

Notice that Theorem 1.3 is asymmetric, since it imposes a geometric condition only on one of the metric spaces. In order to obtain this asymmetric result, we start this section by studying embeddings between Roe-like algebras onto hereditary subalgebras. After providing results on embedding of hereditary subalgebras, we provide a proof for Theorem 1.3.

The next lemma follows completely analogously to [BFV19, Lemma 6.1], so we omit its proof.

**Lemma 4.1.** Let \( X \) and \( Y \) be metric spaces, \( A \subset C^*(X) \) and \( B \subset C^*(Y) \) be Roe-like C*-subalgebras, and \( \Phi : A \to B \) be an embedding onto a hereditary C*-subalgebra of \( B \). Then

\[
\Phi(K(\ell_2(X, H_X))) = K(\ell_2(Y, H_Y)) \cap \Phi(A).
\]

Moreover, there exists an isometry \( U : \ell_2(X, H_X) \to \ell_2(Y, H_Y) \) such that \( \Phi(a) = UaU^* \) for all \( a \in A \). In particular, \( \Phi \) is strongly continuous and \( \ell_2 \)-preserving.
The next lemma should be compared with [BFV19, Lemma 6.2].

**Lemma 4.2.** Suppose $X$ and $Y$ are u.l.f. metric spaces, and $A \subset C^*(X)$ and $B \subset C^*(Y)$ are Roe-like $C^*$-subalgebras so that either $B = C^*_s(Y)$ or $UC^*(Y) \subset B$. Let $\Phi : A \to B$ be a rigid embedding onto a hereditary $C^*$-subalgebra of $B$, and $x \in X \mapsto (y_x, v_x) \in Y \times B_{H_Y}$ be an assignment witnessing that $\Phi$ is a rigid $*$-homomorphism. Then the map $x \in X \mapsto y_x \in Y$ is expanding.

**Proof.** Let $d$ and $\partial$ be the metrics of $X$ and $Y$ respectively. First assume $UC^*(Y) \subset B$. Fix a unit vector $u \in H_X$ and $\delta > 0$ such that

$$\|\Phi(e(x,u),(x,u))\delta_{y_x} \otimes v_x\| \geq \delta$$

for all $x \in X$. Suppose $x \in X \mapsto y_x \in Y$ is not expanding. Then there exist $r > 0$, and sequences $(x^1_n)_n$ and $(x^2_n)_n$ in $X$ such that $d(x^1_n, x^2_n) \geq n$ and $\partial(y^1_n, y^2_n) \leq r$ for all $n \in \mathbb{N}$, where $y^1_n = y^1_{x^1_n}$ and $y^2_n = y^1_{x^2_n}$ for all $n \in \mathbb{N}$. To simplify notation, we also let $v^1_n = v_{x^1_n}$ and $v^2_n = v_{x^2_n}$ for all $n \in \mathbb{N}$.

Since $d(x^1_n, x^2_n) \geq n$ for all $n \in \mathbb{N}$, by going to a subsequence, we can assume that either $(x^1_n)_n$ or $(x^2_n)_n$ is a sequence of distinct elements. Without loss of generality, assume that this is the case for $(x^1_n)_n$. Moreover, going to a subsequence, we can assume that both $(y^1_n)_n$ and $(y^2_n)_n$ are sequences of distinct elements (cf. [BFV19, Claim 6.3]).

By Lemma 4.1, $\Phi$ is rank preserving, so $(\Phi(e(x^1_n,u),(x^1_n,u)))_n$ is an orthogonal sequence of rank 1 projections. Hence, by going to a further subsequence, assume that

$$\|e(y^1_n,v^1_n),(y^2_n,v^2_n)\Phi(e(x^1_n,u),(x^1_n,u))\| < 2^{-n-m-1}\delta^2$$

for all $n \neq m$.

Since $(y^1_n)_n$ and $(y^2_n)_n$ are sequences of distinct elements and $\partial(y^1_n, y^2_n) \leq r$ for all $n \in \mathbb{N}$, $\sum_{n \in \mathbb{N}} e(y^1_n,v^1_n),(y^2_n,v^2_n)$ converges in the strong operator topology to an element in $UC^*(Y)$, and hence, in $B$. As $\Phi(A)$ is a hereditary subalgebra of $B$, there exists $a \in A$ such that

$$\Phi(a) = \Phi(1)\left(\sum_{n \in \mathbb{N}} e(y^1_n,v^1_n),(y^2_n,v^2_n)\right)\Phi(1).$$

Analogously as [BFV19] Claim 6.4, we have that

$$\inf_n \|e(x^1_n,v^1_n),(x^2_n,v^2_n)ae(x^1_n,v^1_n),(x^1_n,v^1_n)\| \geq \delta^2/2.$$ 

Since $a \in C^*(X)$ and $\lim_n d(x^1_n, x^2_n) = \infty$, this gives us a contradiction.

Now assume $B = C^*_s(Y)$. The next claim is essentially [SW13, Lemma 6.4], so we omit its proof.

**Claim 4.3.** There exists a finite rank projection $w$ on $B(H_Y)$ such that

$$\inf_{x \in X} \|\Phi(e(x,u),(x,u))\delta_{y_x} \otimes wv_x\| > 0.$$ 

□
The rest of the proof is just a matter of repeating the proof for the previous case but with this new assignment \( x \in X \mapsto (y_x, v_x) \in Y \times H_Y \).

Proceeding with this strategy, since \( w \) has finite rank, we can guarantee that \( \sum_{n \in \mathbb{N}} e(y_{\delta, wv_\delta}^n), y_{\delta, wv_\delta}^n) \) converges in the strong operator topology to an element in \( C^*_s(Y) \), and the proof works verbatim.

\[ \square \]

**Corollary 4.5.** Suppose \( X \) and \( Y \) are u.l.f. metric spaces, and \( A \subset C^*(X) \) and \( B \subset C^*(Y) \) are Roe-like \( C^* \)-subalgebras so that either \( B = C^*_s(Y) \) or \( UC^*(Y) \subset B \). Let \( \Phi : A \to B \) be a rigid embedding onto a hereditary \( C^* \)-subalgebra of \( B \), and \( x \in X \mapsto (y_x, v_x) \in Y \times B_{H_Y} \) be an assignment witnessing that \( \Phi \) is a rigid \( * \)-homomorphism. Then the map \( x \in X \mapsto y_x \in Y \) is coarse.

**Proof.** By Lemma 4.4 there exists an isometry \( U : \ell_2(X, H_X) \to \ell_2(Y, H_Y) \) such that \( \Phi(a) = UaU^* \) for all \( a \in A \). Hence, since \( x \in X \mapsto (y_x, v_x) \in Y \times B_{H_Y} \) witnesses that \( \Phi \) is a rigid \( * \)-homomorphism, there exists \( \delta > 0 \) and a unit vector \( u \in H_X \) such that \( \| \Phi(e(x,u),(x,u)) \delta_{y_x} \otimes v_x \| > \delta \). In other words,

\[ ||(U \delta_x \otimes u, \delta_{y_x} \otimes v_x)|| > \delta \]

for all \( x \in X \).

If \( B = C^*_s(Y) \), it follows from \[\text{SW13 Lemma 6.5(2)}\] that \( x \in X \mapsto y_x \in Y \) is coarse, while if \( UC^*(Y) \subset B \), it follows from \[\text{SW13 Lemma 4.5(2)}\] that \( x \in X \mapsto y_x \in Y \) is coarse. \[\square\]

**Theorem 4.5.** Let \( X \) and \( Y \) be u.l.f. metric spaces and assume that all sparse subspaces of \( Y \) yield only compact ghost projections in their Roe algebras. Let \( A \subset C^*(X) \) and \( B \subset C^*(Y) \) be Roe-like \( C^* \)-algebras such that either \( B = C^*_s(Y) \) or \( UC^*(Y) \subset B \). If \( A \) embeds onto a hereditary \( C^* \)-subalgebra of \( B \), then \( X \) coarsely embeds into \( Y \).

**Proof.** Let \( \Phi : A \to B \) be an embedding onto a hereditary subalgebra of \( C^*(Y) \). By Corollary 4.5, \( \Phi \) is strongly continuous and compact preserving. By Corollary 4.5, \( \Phi \) is a rigid \( * \)-homomorphism. Let \( x \in X \mapsto (y_x, v_x) \in Y \times B_{H_Y} \) be an assignment which witnesses that \( \Phi \) is rigid. Define \( f : X \to Y \) by \( f(x) = y_x \) for all \( x \in X \). By Lemma 4.5 \( f \) is expanding and by Lemma 4.4 \( f \) is coarse. So \( f \) is a coarse embedding. \( \square \)

**Corollary 4.6.** Let \( X \) and \( Y \) be u.l.f. metric spaces and assume that all sparse subspaces of \( Y \) yield only compact ghost projections in their Roe algebras. Let \( A \subset C^*(X) \) and \( B \subset C^*(Y) \) be Roe-like \( C^* \)-algebras such that either \( B = C^*_s(Y) \) or \( UC^*(Y) \subset B \). If \( A \) embeds onto a hereditary \( C^* \)-subalgebra of \( B \), then all sparse subspaces of \( Y \) yield only compact ghost projections in their Roe algebras.

**Proof.** By Theorem 4.5 \( X \) coarsely embeds into \( Y \). Hence, since the property “all sparse subspaces yield only compact ghost projections” passes

\[\text{Notice that in \[\text{SW13}\] the map } U \text{ is unitary. However, this is not necessary and the same proof holds for } U \text{ being an isometry.} \]
through coarse embeddings (see [BFV19 Theorem 7.6 and Remark 7.8]), the result follows.

\[ \square \]

Proof of Theorem 1.3. Since two unital C\(^*\)-algebras \( A \) and \( B \) are Morita equivalent if and only if they are stably *-isomorphic [BGR77 Theorem 1.2], (5) and (6) are equivalent. It was shown in [BNW07 Theorem 4] that (1) implies (4) without the assumption on sparse subspaces. Moreover, it is clear from the proof of [BNW07 Theorem 4] that (1) also implies (2), (3), and (4).

Suppose \( \Phi : C\star(X) \to C\star(Y) \) is a rigid *-isomorphism. Let \( x \in X \mapsto (y_x, v_x) \in Y \times B_{H_Y} \) and \( y \in Y \mapsto (x_y, u_y) \in X \times B_{H_X} \) be assignments witnessing the rigidity of \( \Phi \) and \( \Phi^{-1} \), respectively. Define maps \( f : X \to Y \) and \( g : Y \to X \) by letting \( f(x) = y_x \) and \( g(y) = x_y \) for all \( x \in X \) and all \( y \in Y \). By Lemma 4.4, both \( f \) and \( g \) are coarse.

By [ˇSW13 Lemma 3.1], there exists a unitary \( U : \ell_2(X, H_X) \to \ell_2(Y, H_Y) \) such that \( \Phi(a) = UaU^* \) for all \( a \in C\star(X) \) (cf. Lemma 4.1). Proceeding exactly as in the proof of [ˇSW13 Theorem 4.1], we have that \( f \circ g \) and \( g \circ f \) are close to \( \text{Id}_Y \) and \( \text{Id}_X \), respectively. So \( X \) is coarsely equivalent to \( Y \), and (1) implies (4). The implication (3)\( \Rightarrow \)(1) follows completely analogously, and the implication (2)\( \Rightarrow \)(1) is the same but with [ˇSW13 Theorem 6.1] instead of [ˇSW13 Theorem 4.1].

Assume that all sparse subspaces of \( Y \) yield only compact ghost projections in \( C\star(Y) \). By Corollary 4.6, the same holds for \( X \). Hence, it follows from Lemma 4.4 and Corollary 3.5 that (6) implies (2), that (7) implies (3), and that (8) implies (4), completing the proof.

We finish this section with a simple remark:

Remark 4.7. Let \( X \) be a u.l.f. metric space and \( A \subset C\star(X) \) be a Roe-like C\(^*\)-algebra. We say that \( X \) yields only compact ghost projections in \( A \) if all ghost projections in \( A \) are compact. Proceeding as in the results above, one can obtain the following:

1. Items 1-7 of Theorem 1.3 are all equivalent under the weaker assumption that all sparse subspaces \( Y' \subset Y \) yield only compact ghost projections in \( UC\star(Y') \), and
2. Items 1-6 of Theorem 1.3 are all equivalent under the weaker assumption that all sparse subspaces \( Y' \subset Y \) yield only compact ghost projections in \( C\star_s(Y') \).

5. THE COARSE BAUM-CONNES CONJECTURE AND ROE RIGIDITY

In this section, we prove Theorem 1.5 and Theorem 1.6, which are consequences of Theorem 1.3 and Theorem 5.3 below.

In order to be able to evoke some results in the literature, we must recall the definition of a metric space with only finite coarse components. Let \((X, d)\) be a u.l.f. metric space and \( R > 0 \). The Rips complex of \( X \) associated...
to \( R \) is defined as

\[
P_R(X) = \{ A \subset X \mid \text{diam}(A) \leq R \}
\]

and we define an equivalence relation \( \sim_R \) on \( P_R(X) \) by setting \( A \sim_R A' \) if there exists \( n \in \mathbb{N} \) and \( x_1, \ldots, x_n \in X \) so that \( x_1 \in A \), \( x_n \in A' \) and \( d(x_i, x_{i+1}) \leq R \) for all \( i \in \{1, \ldots, n-1\} \). We say that \( X \) has only finite coarse components if for all \( R > 0 \) every \( \sim_R \)-equivalence class of \( P_R(X) \) is finite.

As we see below, a u.l.f. metric space with only finite coarse components is simply a sparse metric space in disguise.

**Proposition 5.1.** A u.l.f. metric space has only finite coarse components if and only if it is sparse.

**Proof.** Clearly, every space metric space has only finite coarse components. Let \( X = \{ x_n \mid n \in \mathbb{N} \} \) be a countable metric space with metric \( d \) and assume that \( X \) has only finite coarse components. We construct the partition \((X_n)_n \) of \( X \) which witnesses that \( X \) is sparse by induction. Let \( X_1 \) be the union of the elements in the \( \sim_1 \)-equivalence class of \( P_1(X) \) containing \( \{ x_1 \} \). By hypothesis, \( X_1 \) is finite and \( d(X_1, X \setminus X_1) > 1 \). Suppose \( X_1, \ldots, X_n \) have been defined, that \( X_i \) is finite for all \( i \in \{ 1, \ldots, n \} \) and that \( d(X_n, X') > n \), where \( X' = X \setminus \bigcup_{i \leq n} X_i \). Let \( X'' \) be the union of the elements in the \( \sim_{n+1} \)-equivalence class of \( P_{n+1}(X) \) containing \( \{ x_m \} \), where \( m = \min \{ i \in \mathbb{N} \mid x_i \notin \bigcup_{i \leq n} X_i \} \). Set \( X_{n+1} = X'' \cap X' \). Then \( X_{n+1} \) is finite and \( d(X_{n+1}, X' \setminus X_{n+1}) > n + 1 \). This procedure clearly shows that \( X \) is sparse. \( \square \)

**Theorem 5.2.** Let \( X \) be a sparse uniformly discrete u.l.f. metric space. Assume that the boundary coarse Baum-Connes assembly map for \( X \) is injective. If \( [p]_0 \in K_0(C^*(X)) \) is the class of a noncompact ghost projection in the Roe algebra \( C^*(X) \), then \( [p]_0 \) is not in the image of the coarse Baum-Connes assembly map for \( X \).

**Proof.** By Proposition 5.1, \( X \) has only finite coarse components. The proof is now identical to the proof of [FSW14, Theorem 4.6] (see also the proof of [FS14, Proposition 35]). \( \square \)

**Theorem 5.3.** Let \( X \) be a uniformly discrete u.l.f. metric space. Then all sparse subspaces of \( X \) yield only compact ghost projections in their Roe algebras if any of the following conditions holds:

1. \( X \) satisfies the coarse Baum-Connes conjecture with coefficients.
2. \( X \) is sparse, admits a fibred coarse embedding into a Hilbert space\(^7\), and satisfies the coarse Baum-Connes conjecture.

---

\(^7\)See [CWY13, Definition 2.1] for the definition of fibred embedding into Hilbert spaces.
3. \( X = \square \Gamma \) is any box space\(^8\) of a residually finite, finitely generated discrete group \( \Gamma \) that admits a coarse embedding into a Banach space with property (H)\(^9\), and \( X \) satisfies the coarse Baum-Connes conjecture.

4. \( X = \Gamma \) is a countable discrete group which satisfies the Baum-Connes conjecture with coefficients.

**Proof.** (1): Assume that \( X \) satisfies the coarse Baum-Connes conjecture with coefficients. By Theorem 2.9(1), every sparse subspace \( \tilde{X} \subset X \) also satisfies the coarse Baum-Connes conjecture with coefficients, i.e., the coarse groupoid \( G(\tilde{X}) \) of \( \tilde{X} \) satisfies the Baum-Connes conjecture with coefficients.

By [STY02, Lemma 3.3], \( G(X) = \beta\tilde{X} \rtimes G' \) for some locally compact, second countable, étale groupoid \( G' \), so by Theorem 2.9(2), the closed étale subgroupoid \( G(\tilde{X})|_{\beta\tilde{X}\setminus \tilde{X}} \) satisfies the Baum-Connes conjecture with coefficients. By definition, this implies that \( \tilde{X} \) satisfies the boundary coarse Baum-Connes conjecture. By Theorem 5.2, there cannot be a noncompact ghost projection in \( C^*(\tilde{X}) \).

(2): Since \( X \) is sparse and admits a fibred coarse embedding into a Hilbert space, \( X \) satisfies the boundary coarse Baum-Connes conjecture by Theorem 2.9(3). Since \( X \) also satisfies the coarse Baum-Connes conjecture by assumption, there cannot be a noncompact ghost projection in \( C^*(X) \) by Theorem 5.2.

(3): It follows from [FSW14, Proposition 2.5] that \( G(X)|_{\beta X \setminus X} \cong (\beta X \setminus X) \rtimes \Gamma \). Hence, the boundary coarse Baum-Connes assembly map for \( X \)

\[
K^\text{top}_*(G(X)|_{\beta X \setminus X}, \frac{\ell_\infty(X,K)}{c_0(X,K)}) \to K_* \left( \frac{\ell_\infty(X,K)}{c_0(X,K)} \rtimes_r G(X)|_{\beta X \setminus X} \right)
\]

is injective if and only if the Baum-Connes assembly map

\[
K^\text{top}_*(\Gamma, \frac{\ell_\infty(X,K)}{c_0(X,K)}) \to K_* \left( \frac{\ell_\infty(X,K)}{c_0(X,K)} \rtimes_r \Gamma \right)
\]

is injective by [STY02, Lemma 4.1]. Thus, the conclusion follows directly from [KY12, Theorem 1.2].

(4): If \( \Gamma \) satisfies the Baum-Connes conjecture with coefficients, then it satisfies the coarse Baum-Connes conjecture with coefficients by Theorem 2.9(5), reducing to case (1).

\( \square \)

Before we prove Theorem 1.5 and Theorem 1.6, we need to notice that previous results in the literature already imply that if all sparse subspaces

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\(^8\)See [Roe03, Definition 11.24] for the definition of box spaces.

\(^9\)See [KY12, Definition 1.1] for the definition of property (H). Examples of Banach spaces with property (H) include \( \ell_p(N) \) for \( p \geq 1 \), the Banach space of Schatten \( p \)-class operators on a Hilbert space for \( p \geq 1 \), and Banach spaces with nontrivial cotype admitting an unconditional bases (see [KY12] and [CW18] for more details).
of a u.l.f. metric space yield only compact ghost projections in their Roe algebra, then this space is uniform Roe rigid.

**Proposition 5.4.** Let $X$ be a u.l.f. metric space so that all of its sparse subspaces yield only compact ghost projections in their Roe algebras. Then $X$ is uniform Roe rigid.

**Proof.** Let $Y$ be a u.l.f. metric space so that $C^*_u(X)$ and $C^*_u(Y)$ are $*$-isomorphic. Then $C^*_u(X) \otimes K(H_X)$ and $C^*_u(Y) \otimes K(H_Y)$ are $*$-isomorphic, i.e., $C^*_s(X)$ and $C^*_s(Y)$ are $*$-isomorphic. Hence, Theorem 1.3 gives us that $X$ and $Y$ are coarsely equivalent. □

**Proof of Theorem 1.5 and Theorem 1.6.** First notice that we can assume without loss of generality that $X$ is uniformly discrete. Indeed, let $d$ be the metric on $X$ and set $\partial = d + 1$. The metric $\partial$ is clearly a uniformly discrete u.l.f. metric and the identity map $(X, d) \to (X, \partial)$ is a bijective coarse equivalence. Hence, the Roe algebras of $(X, d)$ and $(X, \partial)$ are canonically isomorphic. Moreover, all the properties considered in Theorem 1.5 and Theorem 1.6 are also shared by $(X, \partial)$ given that $(X, d)$ satisfies them.

The results on Roe rigidity now follow straightforwardly from Theorem 1.3 and Theorem 5.3, and the results on uniform Roe rigidity follow from Proposition 5.3 and Theorem 5.3. □

**Question 5.5.** There is a finitely generated group $\Gamma$ which contains an expander $X$ with large girth isometrically in its Cayley graph (see [Osa14, Theorem 4]). Thus $\Gamma$ does not satisfy the coarse Baum-Connes conjecture with coefficients. Moreover, the sparse subspace $X$ of $\Gamma$ yields a noncompact ghost projection in $C^*(X)$. Is $\Gamma$ Roe rigid?

We finish this paper defining uniform Roe bijectively rigid and listing some related results, some of which are already known from earlier results.

**Definition 5.6.** Let $X$ be a u.l.f. metric space. We say that $X$ is **uniform Roe bijectively rigid** if $X$ is bijectively coarsely equivalent to any uniformly locally finite metric space $Y$ so that $C^*_u(X)$ and $C^*_u(Y)$ are $*$-isomorphic.

**Remark 5.7.** Let $X$ be a u.l.f. metric space. Then $X$ is uniform Roe bijectively rigid if any one of the following conditions holds:

1. $X$ has property A. This case follows from [WW19, Corollary 6.13] and [BFV18, Theorem 1.11].
2. $X$ is non-amenable and satisfies the coarse Baum-Connes conjecture with coefficients. This case follows from Theorem 5.3(1), Theorem 1.3, and [WW19, Theorem 5.1].
3. $X = \Gamma$ is a group which satisfies the coarse Baum-Connes conjecture with coefficients. Indeed, if $\Gamma$ is amenable, then $\Gamma$ has property A, so we apply case (1). If $\Gamma$ is non-amenable as a group, then it is non-amenable as a metric space, so we apply case (2).

**Question 5.8.** Let $X$ be a u.l.f. metric space, which admits a coarse embedding into a Hilbert space. Is $X$ necessarily uniform Roe bijectively rigid?
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