Exact Flow Equations and the $U(1)$-Problem

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Abstract

The effective action of a $SU(N)$-gauge theory coupled to fermions is evaluated at a large infrared cut-off scale $k$ within the path integral approach. The gauge field measure includes topologically non-trivial configurations (instantons). Due to the explicit infrared regularisation there are no gauge field zero modes. The Dirac operator of instanton configurations shows a zero mode even after the infrared regularisation, which leads to $U_A(1)$-violating terms in the effective action. These terms are calculated in the limit of large scales $k$.

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1 Introduction

One of the outstanding problems in QCD is the $U(1)$-problem. This problem can be reformulated as the question about the missing ninth light pseudo-scalar meson, the $\eta'$. There have been several attempts to solve this problem at least qualitatively (see reviews [1], [2] and references therein). However, a quantitative description is still missing and discrepancies between the different approaches have not been clarified yet. This is not surprising as a quantitative description must include non-perturbative QCD-effects.

One approach to tackle this intricate problem is to use flow equations (or exact renormalisation group equations [3]) in continuum QFT. They were exploited by Polchinski and subsequently by many other authors to simplify proofs of perturbative renormalisability [4]. More recently they have been investigated also as a powerful tool to study non-perturbative physics. The flow equation describes the scale dependence of the effective action $\Gamma_k$, where $k$ is an infrared cut-off scale. The starting point is an effective action $\Gamma_{k_0}$ which depends only on modes with momenta larger than $k_0$. Then the flow equation is used to integrate out successively the momenta smaller than the cut-off scale $k_0$.

This method has been applied also to gauge theories [5]-[10]. Here one must employ a gauge invariant effective action. An appropriate way to deal with gauge invariance is to use the background field formalism as presented in [11]. Within this formalism it is possible to define an infrared regularised version of the effective action, which is gauge invariant [3]. However, the background field dependence of the cut-off term leads to additional terms in the Slavnov-Taylor identities (STI). A second possibility was proposed in [7], [9], where the infrared cut-off term explicitly breaks gauge invariance. In both approaches one derives so-called modified STIs. It has been shown, that the original STI are restored in the limit, where the cut-off is removed ($k \to 0$). Moreover, the modified STI are compatible with the flow equations. Thus, given an effective action fulfilling the modified STI at the starting scale, the integrated effective action fulfills the modified STI at arbitrary scales [9, 10].

In using flow equations for explicit calculations one has to deal with the following problems: In principle the flow equation consists of an infinite system of coupled differential equations. In most interesting cases it is not possible to solve these equations analytically and one has to truncate the system. Thus the problem is how to find truncations valid in the regime of interest and to formulate general validity checks for these truncations. Furthermore one needs to know the effective action at the starting point $k_0$. Even for large scales $k_0$ the effective action consists of an infinite number of terms. In principle it should be possible to start at a large initial scale $k_0$ with the (finite) number of terms which depend on relevant couplings. The corrections due to the terms neglected at $k_0$ remain of the order $1/k_0$ for all scales $k$ because of universality. But this argument is only valid if one is dealing with the non-truncated flow equation. Since one has to truncate the system of differential equations it is important to calculate the effective action at the starting scale $k_0$ as accurately as possible.

To be more explicit, we study the behaviour of the flow equation under global $U_A(1)$-variations. For that purpose let us briefly outline the derivation of the flow equation: In the path integral formulation we achieve a smooth infrared cut-off by adding a scale-dependent
term to the action (e.g. \[\int \Phi \mathrm{d}^4 \Phi \]

$$\Delta_k S[\Phi] = \frac{1}{2} \int \Phi R_k^\Phi [P_\Phi] \Phi, \quad (1.1)$$

where \(P_\Phi^{-1}\) is proportional to the bare propagator of \(\Phi\) and \(\Phi\) is a shorthand notation for all fields. The regulator \(R_k^\Phi\) has the following properties:

$$R_k^\Phi [x] \xrightarrow{k^2 \to 0} k^{4-2d_\Phi} \frac{x}{|x|}, \quad R_k^\Phi [x] \xrightarrow{k^2 \to \infty} 0, \quad (1.2)$$

where \(d_\Phi\) are the dimensions of the fields \(\Phi\). Hence the cut-off term \((1.1)\) effectively suppresses modes with momenta \(p^2 \ll k^2\) in the generating functional. For modes with large momenta \(p^2 \gg k\) the cut-off term vanishes and in this regime the theory remains unchanged. In the limit \(k \to 0\) we approach the full generating functional since the cut-off term is removed. An infinitesimal variation of the generating functional with respect to \(k\) is described by the flow equation. For the generating functional of 1PI Green functions, the effective action \(\Gamma_k\), the flow equation can be written in the form (see for example \([3, 9]\))

$$\partial_t \Gamma_k[\Phi] = \frac{1}{2} \mathrm{Tr} \left\{ \partial_t R_k^\Phi [P_\Phi] (\Gamma_k^{(2)}[\Phi] + R_k^\Phi [P_\Phi])^{-1} \right\}, \quad (1.3)$$

where \(t = \ln k\) and the trace \(\mathrm{Tr}\) denotes a sum over momenta, indices and the different fields \(\Phi\). \(\Gamma_k^{(2)}\) is the second derivative of \(\Gamma_k\) with respect to the fields \(\Phi\). Note that \(\partial_t R_k^\Phi\) serves as a smeared-out \(\delta\)-function in momentum space peaked about \(p^2 \approx k^2\). Thus by varying the scale \(k\) towards smaller \(k\) according to \((1.3)\) one successively integrates out momentum degrees of freedom related to \(p^2 \approx k^2\). This leads us to the conclusion that the flow equation \((1.3)\) and an initial effective action \(\Gamma_{k_0}\) given at a momentum scale \(k_0\) may be used as a definition of the quantum theory.

Now the question arises how to incorporate in this approach symmetries which are preserved on the classical level but are violated on quantum level. If the bare propagator \(P_\Phi^{-1}\), the full propagator \((\Gamma_k^{(2)}[\Phi] + R_k^\Phi [P_\Phi])^{-1}\) and the regulator \(R_k^\Phi\) are invariant or transform covariantly under a set of (global) symmetry transformations, the flow equation will preserve this symmetry. It is of course simple to introduce symmetry breaking via the regulator. However this makes it hard or even impossible to distinguish between effects due to this explicit symmetry breaking and quantum effects at intermediate steps and for finite \(k\). Additionally symmetry breaking due entirely to the regulator should disappear in the limit where the cut-off is removed. Hence the symmetry breaking has to be introduced on the level of the initial effective action. Therefore a suitable approximation of \(\Gamma_{k_0}\) should contain not only all terms with relevant couplings but as well the leading order in \(1/k_0\) of terms which break the symmetry of the classical action due to quantum effects. Further terms are neglectable in \(\Gamma_{k_0}\) since corrections due to them should remain of the order \(1/k_0\) by means of universality even for lower scales \(k\). Note that this is only valid for the initial value of the related couplings.
In the present case the regulator $R_k^Φ$ and the bare propagator $P^{−1}_Φ$ are invariant under global $U_A(1)$-variations if the effective action $Γ_{k_0}$ at the starting point $k_0$ is $U_A(1)$-invariant. Thus no $U_A(1)$-violating terms would arise during the flow. However we know from the $U_A(1)$-anomaly equation that $U_A(1)$ is broken on the quantum level. As explained above we have to introduce $U_A(1)$-violating terms to the effective action at the initial scale $k_0$ even though the couplings of these terms seem to be irrelevant at large scales.

$U_A(1)$-violating terms due to quantum effects have been derived in instanton calculations [12]. It is well known that the semi-classical approximation used in instanton calculations breaks down in the region where these effects become important. In addition one has infrared problems due to the integration over the zero modes (integration over the width of the instanton) (see [13, 14] and references therein). We can deal with both problems in the framework of flow equations. The infrared cut-off ensures the infrared finiteness and it is possible to go beyond the semi-classical approximation by integrating the flow equation to lower scales. The $U_A(1)$-violating terms calculated for the initial effective action $Γ_{k_0}$ act as the source for generating $U_A(1)$-violation in the flow equation via $\left(Γ_{k_0}^{(2)} + R_{k_0}\right)^{−1}$. After integration of (1.3) they should be responsible for the anomalous large mass of the $η'$-meson after integration.

In the present paper we concentrate on the derivation of the $U_A(1)$-violating terms within the framework of the infrared regularised effective action. We show that the fermionic sector contains zero modes generating $U_A(1)$-violating terms. The effective action is derived in an expansion in orders $1/k_0$, valid at large cut-off scales $k_0$.

In the second section we formulate the infrared regularised effective action within an approach to flow equations developed in [6] for pure Yang-Mills theory. However we would like to emphasise that the results are independent of the approach chosen. We derive the effective action in the zero instanton sector. For later purpose the topological $θ$-term is included, where we choose the $θ$-angle to be space-time dependent (for a first discussion of the $θ$-parameter within the framework of flow equations see [15]). This is a natural extension, as the strong $CP$-problem is deeply connected to the $U_A(1)$-problem. We show that the regularised effective action in the absence of zero modes (topological effects) interpolates between the classical action ($k → ∞$) and the full effective action ($k → 0$). In the third section we calculate the $U_A(1)$-violating terms in the 1-instanton sector. In the fourth section we derive the effective action at a large scale $k_0$ to leading order in $1/k_0$ by means of universality. Multi-instanton contributions are of sub-leading order in $1/k_0$. In the last section we give a summary of the results and an outlook. In appendix A the existence of a fermionic zero mode for the infrared regularised Dirac operator is proven. The appendices B,C contain some technical details.

2 Derivation of the effective action

Throughout the paper we work in Euclidean space-time. In this section we only deal with topologically trivial configurations, since we want to derive the general properties of the effective action in this theory. Hence the full propagators have only trivial zero modes.
The classical action of a $SU(N)$-gauge theory coupled to fermions is given by

$$S[A, \psi, \bar{\psi}] = S_A[A] + S_\theta[A] + S_\psi[A, \psi, \bar{\psi}]$$

(2.1)

where $S_A$ is the pure Yang-Mills action and $S_\psi$ is the fermionic action:

$$S_A[A] = \frac{1}{2g^2} \int d^4x \text{ tr} F^2(A), \quad S_\psi[A, \psi, \bar{\psi}] = \int d^4x \bar{\psi} D \psi(A) \psi.$$

(2.2)

g is the coupling, the trace $\text{tr}$ denotes a sum over Lorentz indices and the trace in the fundamental representation of the Lie algebra, $\text{tr} t^a t^b = -\frac{1}{2} \delta^{ab} \delta_{\mu\mu}$. The field strength is given by

$$F^a_{\mu\nu}(A) = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu, \quad F = F^a t^a, \quad [t^a, t^b] = f^{abc} t^c.$$

(2.3)

The fermions $\psi = (\psi^A_s, \xi)$ are in the fundamental representation, where $A$ denotes the gauge group indices, $s = 1, \ldots, N_f$ the flavors and $\xi$ the spinor indices. The Dirac operator is given by

$$D^{AB}_{\xi\xi'}(A) = (\gamma_\mu \xi' D^A_\mu, \quad D^A_\mu(A) = \partial_\mu \delta^{AB} + A^c_\mu(t^c)^{AB}, \quad \{\gamma_\mu, \gamma_\nu\} = -2 \delta_{\mu\nu}.$$  

(2.4)

We have also introduced the topological $\theta$-term in (2.1)

$$S_\theta[A] = \frac{i}{16\pi^2} \int d^4x \theta(x) \text{tr} F \tilde{F}(A),$$

(2.5)

where $\tilde{F}$ denotes the dual field strength. $S_\theta$ depends only on global properties of the gauge field for constant $\theta$. In this case we may lose differentiability of $S_\theta$ with respect to the gauge field. The effective action $\Gamma_k$ is defined by the Legendre transformation of the infrared regularised Schwinger functional $W_k$. For non-differentiable $W_k$ one has to use the general definition of the Legendre transformation

$$\Gamma_k[\Phi] = \inf_{J_\Phi} \left\{ \int d^4x J_\Phi \Phi - W_k[J_\Phi] \right\}.$$  

(2.6)

However, it is difficult to use this definition for practical purpose. We circumvent this problem by allowing for an $x$-dependent $\theta$. $\theta$ can be seen as a source of the index $[15]$. We derive the effective action $\Gamma_k$ for this theory within the path integral formalism. The effective actions for gauge theories and for fermionic systems in the present approach have been derived elsewhere ([1, 17, 18, 19, 17]), so it is sufficient to outline this computation. We deal with the gauge degrees of freedom by performing the Fadeev-Popov procedure. Given a gauge fixing condition $\mathcal{F}^a[A, \bar{A}] = 0$ we introduce

$$S_{g.f.}[A, \bar{A}] = \frac{1}{2\alpha} \int d^4x \text{ tr} \mathcal{F}^2[A, \bar{A}], \quad \mathcal{M}^{ab}[A, \bar{A}](x, y) = D^a_{\mu}(y) \frac{\delta}{\delta A^c_\mu(y)} \mathcal{F}^b[A, \bar{A}](x).$$

(2.7)
where $D_{\mu}^{ac}$ is the covariant derivative in the adjoint representation. In (2.7) we have introduced the background field $\bar{A}$ (e.g. [11]). Within the background field approach to gauge theories the generating functional of connected Green functions is given by

$$
\exp W[J, \bar{A}, \eta, \bar{\eta}, \xi, \bar{\xi}] = \frac{1}{\mathcal{N}} \int \mathcal{D}a(c, \bar{c}) \ d\bar{\chi} \ d\chi \ \exp \left\{ -S[a + \bar{A}, \chi, \bar{\chi}] - S_{g.f.}[a, \bar{A}] + \int_x \left( \bar{\eta}^A_{s,\xi} \chi_{s,\xi}^A + \chi_{s,\xi}^A \bar{\eta}_{s,\xi}^A + J^a_\mu a^a_\mu + (\bar{\xi}^a c^a + c^a \xi^a) \right) \right\},
$$

(2.8)

where $\mathcal{N}$ is an appropriate normalisation and the integral in the source terms is over space-time ($\int_x = \int d^4x$). Note also that the source $J$ only couples to the fluctuation field $a$. The gauge field measure $\mathcal{D}a(c, \bar{c})$ in (2.8) depends on the chosen gauge and includes the ghost terms:

$$
\mathcal{D}a(c, \bar{c}) = \det d\bar{\chi} d\chi \ \exp \left\{ -\int_{x,y} \bar{c}^a(x) \mathcal{M}^{ab}[a, \bar{A}](x, y) c^b(y) \right\}.
$$

(2.9)

The interesting quantity is the generating functional of 1PI Green functions $\hat{\Gamma}$ which is the Legendre transformation of $W$. For the sake of convenience we introduce a shorthand notation for contractions, denoting them with a dot whenever the meaning is clear. (e.g. $J \cdot a = J^\mu a_\mu, \bar{\eta} \cdot \psi = \bar{\eta}_{s,\xi}^A \cdot \psi_{s,\xi}^A$). With this abbreviation we get

$$
\hat{\Gamma}[A, \bar{A}, \psi, \bar{\psi}, \rho, \bar{\rho}] = -W[J, \bar{A}, \eta, \bar{\eta}] + \int_x \left( \bar{\eta} \cdot \psi + \bar{\psi} \cdot \eta + J \cdot A + (\bar{\rho} \cdot \xi + \bar{\xi} \cdot \rho) \right)
$$

(2.10)

with

$$
\psi_{s,\xi}^A = \frac{\delta}{\delta \bar{\eta}^A_{s,\xi}} \hat{\Gamma}, \quad \bar{\psi}_{s,\xi}^A = -\frac{\delta}{\delta \eta^A_{s,\xi}} \hat{\Gamma}, \quad A^a_\mu = \frac{\delta}{\delta J^a_\mu} \hat{\Gamma}, \quad \rho^a = \frac{\delta}{\delta \xi^a} \hat{\Gamma}, \quad \bar{\rho}^a = -\frac{\delta}{\delta \bar{\xi}^a} \hat{\Gamma}.
$$

In the following we concentrate on properties of the fermionic integration with the ghosts only being spectators. Thus for the sake of simplicity we will perform the calculation with vanishing ghost fields in the effective action, $\rho, \bar{\rho} = 0$ ($\xi, \bar{\xi} = 0$). The further use of the Schwinger functional $W$ and the effective action $\hat{\Gamma}$ has to be understood in this sense. Using (2.8), $\hat{\Gamma}$ is defined implicitly by the integro-differential equation

$$
\exp -\hat{\Gamma}[A, \bar{A}, \psi, \bar{\psi}] = \frac{1}{\mathcal{N}} \int \mathcal{D}a d\bar{\chi} d\chi \ \exp \left\{ -S[a + \bar{A}, \chi, \bar{\chi}] - S_{g.f.}[a, \bar{A}] + \int_x \left( \frac{\delta \hat{\Gamma}}{\delta A} \cdot (a - A) - \frac{\delta \hat{\Gamma}}{\delta \psi} \cdot (\chi - \psi) + (\bar{\chi} - \bar{\psi}) \cdot \frac{\delta \hat{\Gamma}}{\delta \bar{\psi}} \right) \right\},
$$

(2.11)

where we have used

$$
\eta^A_{s,\xi} = \frac{\delta}{\delta \psi_{s,\xi}^A} \hat{\Gamma}, \quad \bar{\eta}^A_{s,\xi} = -\frac{\delta}{\delta \bar{\psi}_{s,\xi}^A} \hat{\Gamma}, \quad J^a_\mu = \frac{\delta}{\delta A^a_\mu} \hat{\Gamma}.
$$
However, these expressions are only formal notations and do not make sense without a suitable regularisation. The regularisation is introduced in the following way: We use an explicit infrared regularisation dependent on the cut-off scale $k$ (see (1.1), (1.2)). The flow equation (1.3) describes the $k$-dependence of $\Gamma_k$. The ultraviolet regularisation, dependent on the cut-off parameter $\Lambda$, is only used implicitly. We allow for the class of ultraviolet regularisations preserving the gauge symmetry. These regularisations violate the axial symmetry due to a general no-go theorem (see e.g. [18]). Within the framework of flow equations this is equivalent to a fixing of the (axial) anomaly by choosing appropriate ultraviolet boundary conditions [8].

The cut-off term for the gauge-field is introduced as follows:

$$\Delta_k S_A[a, \bar{A}] = -\frac{1}{2} \int_x a \cdot R_A^k[D_T(\bar{A})] \cdot a, \quad R_{k\mu\nu}^A[D_T] = \left( D_T \frac{1}{e^{D_T/k^2} - 1} \right)^{\mu\nu}. \quad (2.12)$$

The operator $D_T$ is defined by

$$D_T^{ab}_{\rho\mu\nu} = -D^{ac}_{\rho} D^{cb}_{\rho} \delta_{\mu\nu} - 2g f^{ab}_c F_{\mu\nu} \quad (2.13)$$

where $D^{ab}_\rho$ is the covariant derivative in the adjoint representation. For the details of the background field approach to flow equations we refer the reader to [6]. The infrared cut-off for the fermions is defined in a similar way. As discussed in the introduction we use a fermionic cut-off term having the same Dirac structure as the inverse of the bare fermionic propagator. Hence it is proportional to $\varphi(\bar{A})$. We would like to emphasise that the results obtained here easily extend to more general cut-off terms. However in the general case the technical details are more involved. Thus a convenient choice is

$$\Delta_k S_\psi[\psi, \bar{\psi}, \bar{A}] = \int_x \bar{\psi} \cdot R^\psi_k[\varphi(\bar{A})] \cdot \psi, \quad R_{k,\xi\xi'}^\psi[D_T(\bar{A})] = \left( \varphi(\bar{A}) \left( e^{\varphi(\bar{A})/k^2} - 1 \right)^{1/2} \right)^{\xi\xi'} \quad (2.14)$$

The cut-off terms (2.12,2.14) have the limits presented in (1.2). We only refer implicitly to the cut-off term $R^c_k$ of the ghosts by keeping the gauge field measure $k$-dependent ($D a_k$). Note that although the ghosts are spectators in the present derivation $R^c_k$ leads to ghost contributions in the flow equation (1.3) (e.g. [9]).

The $\Lambda$-dependence of the effective action is not specified, because we are working in the limit $\Lambda \to \infty$. In appendix [4] we discuss the properties of the ultraviolet cut-off in more detail. As a result the functional measures of the gauge field and the fermions are $\Lambda$-dependent. For the sake of brevity we will drop any reference to the ultraviolet cut-off. We get for the regularised Schwinger functional $W_k$

$$\exp W_k[J, \bar{A}, \eta, \bar{\eta}] = \frac{1}{\mathcal{N}_k} \int D a_k d\bar{\chi} d\chi \exp \left\{ -S_k[a, \bar{A}, \chi, \bar{\chi}] + \int_x (\bar{\eta} \cdot \chi + \bar{\chi} \cdot \eta + J \cdot a) \right\} \quad (2.15)$$

where the action $S_k$ and the normalisation $\mathcal{N}_k$ are given by

$$S_k = S + S_\text{g.f.} + \Delta_k S, \quad \Delta_k S = \Delta_k S_A + \Delta_k S_\psi, \quad \mathcal{N}_k = \exp \left\{ W_k[0, \bar{A}, 0, 0] \right\}. \quad (2.16)$$
The limit \( k \to \infty \) of the Legendre transformation of \( W_k \) is divergent since it is proportional to \( \Delta_k S \). However it is possible to define the effective action \( \Gamma_k \) in a way that guarantees a finite \( k \to \infty \) limit by

\[
\Gamma_k[A, \bar{A}, \psi, \bar{\psi}] = -W_k[J, \bar{A}, \bar{\eta}, \eta] + \int_x \left( \bar{\eta} \cdot \psi + \bar{\psi} \cdot \eta + J \cdot A \right) - \Delta_k S[A, \bar{A}, \psi, \bar{\psi}]. \tag{2.16}
\]

Now the regularised analogue of \( (2.11) \) follows immediately from \( (2.15) \) and \( (2.16) \) as

\[
\exp \left\{ -\Gamma_k[A, \bar{A}, \psi, \bar{\psi}] \right\} = \frac{1}{N_k} \int \mathcal{D}a_k d\bar{\chi} d\chi \exp \left\{ -S_k[a, \bar{A}, \chi, \bar{\chi}] + \Delta_k S[A, \bar{A}, \psi, \bar{\psi}] \right. \\
\left. + \int_x \left( \bar{\eta} \cdot (\chi - \psi) + (\bar{\chi} - \bar{\psi}) \cdot \eta + J \cdot (a - A) \right) \right\}. \tag{2.17}
\]

The sources \((\eta, \bar{\eta}, J)\) in \( (2.17) \) have to be understood as functions of \( \Gamma_k \) and follow from \( (2.16) \) as

\[
\eta^A_{s,\xi} = \frac{\delta}{\delta \psi^A_{s,\xi}}(\Gamma_k + \Delta_k S_\psi), \quad \bar{\eta}^A_{s,\xi} = -\frac{\delta}{\delta \bar{\psi}^A_{s,\xi}}(\Gamma_k + \Delta_k S_\psi), \quad J^a_\mu = \frac{\delta}{\delta A^a_\mu}(\Gamma_k + \Delta_k S_A). \tag{2.18}
\]

Hence \( \Gamma_k \) is implicitly defined by \( (2.17) \). The effective action can be written in a more convenient form by using the following shift of variables:

\[
a \to a - A, \quad \bar{\chi} \to \bar{\chi} - \bar{\psi}, \quad \chi \to \chi - \psi. \tag{2.19}
\]

Applying \( (2.19) \) to \( (2.17) \) we get

\[
\exp \left\{ -\Gamma_k[A, \bar{A}, \psi, \bar{\psi}] \right\} = \frac{1}{N_k} \int \mathcal{D}a_k d\bar{\chi} d\chi \exp \left\{ -S[a + \bar{A} + A, \bar{\chi} + \bar{\psi}, \chi + \psi] \right. \\
\left. - S_{g.f.}[a + A, \bar{A}] - \Delta_k S[a, \bar{A}, \chi, \bar{\chi}] + \int_x (\bar{\eta} \cdot \chi + \bar{\chi} \cdot \eta + J \cdot a) \right\}. \tag{2.20}
\]

In the limit \( k \to \infty \) the path integral is dominated by the \( \Delta_k S \) terms and to leading order we can neglect the dependence of the other terms on the fields \( a, \chi, \bar{\chi} \) (e.g. \( S[a + A + \bar{A}, \chi + \bar{\psi}, \chi + \psi] \to S[A + \bar{A}, \psi, \bar{\psi}] \)). What is left is a Gaussian integral which is canceled by the normalisation \( N_k \). Thus \( \Gamma_k \) approaches the classical action for \( k \to \infty \) (including the gauge fixing). In the limit \( k \to 0 \) the \( k \)-dependence of \( \Gamma_k \) vanishes and \( \Gamma_k \) approaches the full effective action:

\[
S + S_{g.f.} \xleftarrow{k \to \infty} \Gamma_k \xrightarrow{k \to 0} \Gamma. \tag{2.21}
\]

Hence we have derived the regularised effective action for a gauge theory coupled to fermions in the gauge field sector with trivial topology. \( \Gamma_k \) interpolates between the classical action for \( k \to \infty \) and the full effective action for \( k = 0 \).
3 Instanton-induced terms

The result (2.21) of the last section remains valid in gauge field sectors with non-trivial topology. However we have to take into account also $U_A(1)$-violating terms which may be suppressed with powers of $1/k$. It can be shown that the Dirac operator has a non-trivial zero mode in the 1 instanton sector even after the infrared regularisation (see appendix $A$). This serves as the source for $U_A(1)$-violating terms. For the calculation of these terms we consider gauge field configurations with instanton number $\pm 1$ and carefully examine the zero mode dependence of the path integral.

For the non-zero modes the derivation of the last section holds without modification. Due to the infrared cut-off $\Delta k S_A$ there are no gauge field zero modes. The scale invariance of the action is broken and the minimum of the action is at vanishing instanton width. This can be seen as follows: Let $a$ be a configuration with instanton number 1. Additionally we choose $A, \bar{A}$ to be in the trivial sector. It follows that

$$S_A[a + A + \bar{A}] \geq \frac{8\pi^2}{g^2}, \quad \Delta k S_A[a, \bar{A}] \geq 0. \quad (3.1)$$

In addition, $\Delta k S_A$ is vanishing only for gauge fields with vanishing norm (see appendix $C$). Thus the gauge field sector has no infrared problems even if topologically non-trivial gauge field configurations are considered. In the limit $k \to \infty$ the gauge field integration becomes trivial. This remains valid for $A, \bar{A}$ with arbitrary instanton number. To simplify the following calculations we assume $A, \bar{A}$ to be in the trivial sector. First we have a closer look at the fermionic part of the action. We shall argue by using the limit $\Gamma_k \to \infty S + S_{g.f.}$ that only the source terms couple to the fermionic zero mode. Therefore the zero mode integration can be done explicitly. After shifting the fields as in (2.19) the part of the exponent in (2.20) depending on fermionic variables reads for instanton configurations $a_I$

$$-S_\psi[a_I + A + \bar{A}, \chi' + \psi' + \bar{\psi}'] - \Delta k S_\psi[\chi' + \psi' + \bar{\psi}'] - \Delta k S_\psi[\chi', \bar{\chi}, \bar{\psi}'] + \int_x (\bar{\eta} \cdot \chi + \bar{\chi} \cdot \eta) + \Delta k S_\psi[\psi, \bar{\psi}, \bar{A}], \quad (3.2)$$

where the primed fermionic fields are the non-zero modes of the infrared regularised Dirac operator $\bar{D}(a_I + A + \bar{A}) + B_k^\psi[\bar{D}(\bar{A})]$ and the zero modes are denoted by $\chi_0, \psi_0$. In the limit $k \to \infty$ we get for the sources

$$\eta \to \frac{\delta}{\delta \psi}(S_\psi + \Delta k S_\psi), \quad \bar{\eta} \to -\frac{\delta}{\delta \bar{\psi}}(S_\psi + \Delta k S_\psi). \quad (3.3)$$

Using (3.3) we deduce from (3.2)

$$-S_\psi[a_I + A + \bar{A}, \psi', \bar{\psi}'] - S_\psi[a_I + A + \bar{A}, \chi', \bar{\chi}'] - \Delta k S_\psi[\chi', \bar{\chi}', \bar{A}] + \int_x (\bar{\eta} \cdot \chi_0 + \bar{\chi}_0 \cdot \eta) + \Delta k S_\psi[\psi, \bar{\psi}, \bar{A}] - \Delta k S_\psi[\psi', \bar{\psi}', \bar{A}], \quad (3.4)$$

where the zero modes are denoted by $\chi_0, \psi_0$. In the limit $k \to \infty$ we get for the sources

$$\eta \to \frac{\delta}{\delta \psi}(S_\psi + \Delta k S_\psi), \quad \bar{\eta} \to -\frac{\delta}{\delta \bar{\psi}}(S_\psi + \Delta k S_\psi). \quad (3.3)$$

Using (3.3) we deduce from (3.2)
The terms linear in $\bar{\chi}_0$, $\chi_0$ remain. There is no counterterm in the action, since $S_\psi + \Delta_k S_\psi$ does not depend on the zero mode. The cross terms $\Delta_k S_\psi[\bar{\chi}'', \bar{\psi}']$, $\Delta_k S_\psi[\bar{\phi}', \chi', \bar{A}]$ are canceled by similar ones derived from the source terms. We have dropped the cross terms $S_\psi[a_I + A + \bar{A}, \bar{\chi}'', \psi'']$, $S_\psi[a_I + A + \bar{A}, \bar{\psi}'', \chi'']$, since they are suppressed with $1/k$ in the limit $k \to \infty$. We have, with $\mathcal{D} \psi_0 = -R_k^\psi \psi_0$,

$$S_\psi[a_I + A + \bar{A}, \psi', \bar{\psi}'] = \Delta_k S_\psi[\psi, \bar{\psi}, \bar{A}] + \Delta_k S_\psi[\psi', \bar{\psi}', \bar{A}] = S_\psi[a_I + A + \bar{A}, \psi, \bar{\psi}]. \quad (3.5)$$

The final result for the fermionic part of the exponent in (2.20) is

$$- S_\psi[a_I + A + \bar{A}, \psi, \bar{\psi}] - S_\psi[a_I + A + \bar{A}, \chi', \bar{\chi}'] - \Delta_k S_\psi[\chi', \bar{\chi}', \bar{A}] + \int_x (\bar{\eta} \cdot \chi_0 + \bar{\chi}_0 \cdot \eta) \quad (3.6)$$

The last term in (3.6) depends on the instanton $a_I$ via the zero mode. However only instantons $a_I$ with width $\rho \sim 1/k$ contribute. The infrared regularisation of the gauge field supresses instantons with width $\rho \gg 1/k$ (see appendix [C]). We split the gauge field measure into the measure of collective coordinates of the instanton and the measure of fluctuations about the instanton. Let $da_{I,k}$ be the $(k$-dependent) measure of the collective coordinates of the $SU(N)$-instanton [12, 13]. The zero mode contribution factorises in the limit $k \to \infty$. Hence taking into account the trivial sector and the $\pm 1$ instanton sectors the effective action is given by

$$\exp \left\{ -\Gamma_k[A, \bar{A}, \psi, \bar{\psi}] \right\} = \exp \left\{ -S[A + \bar{A}, \psi, \bar{\psi}] + S_{g.f.}[A, \bar{A}] \right\} \left( 1 + \left[ \int d\mu_1(\theta) d\bar{\chi}_0 d\chi_0 \right. \right.$$

$$\left. \times \exp \left( \int x (\bar{\eta} \cdot \chi_0 + \bar{\chi}_0 \cdot \eta) + \text{h.c.} \right) \right) + O(1/k) \quad (3.7)$$

with

$$d\mu_1(\theta) = da_{I,k} \frac{N_k'[a_I]}{N_k}, \quad N_k'[a_I] = \int da_k d\chi d\chi' \exp \left\{ -S_k[a + a_I, 0, \chi', \bar{\chi}'] \right\}. \quad (3.8)$$

We have dropped the dependence of the zero mode contribution on $A$ and $\bar{A}$, since it is only next to leading order in $1/k$ (see appendix [B]). We also have used that the contribution from the sector with instanton number $-1$ is the hermitean conjugate of the sector with instanton number 1. Thus we concentrate on the sector with instanton number $+1$ and compute

$$\int \prod_{s=1}^{N_f} d\bar{b}_0^s d\phi_0^s \exp \int_x (\bar{\eta} \cdot \chi_0 + \bar{\chi}_0 \cdot \eta) = \prod_{s=1}^{N_f} \left( \left( \int_x \bar{\eta}_s \cdot \phi_0 \right) \right) \left( \int_x \phi_0^+ \cdot \eta_s \right) \quad (3.9)$$

with

$$(\chi_0)^A_{s,\xi} = a_0^s \phi_0^A_{0,\xi}, \quad (\bar{\chi}_0)^A_{s,\xi} = (\phi_0^+)^A_{s,\xi} b_0^s, \quad \int_x \phi_0^+ \cdot \phi_0 = 1, \quad d\bar{\chi}_0 d\chi_0 = \prod_{s=1}^{N_f} d\bar{b}_0^s d\phi_0^s. \quad (3.10)$$
Higher powers of \((\int \bar{\eta} \cdot \chi_0)(\int \bar{\chi}_0 \cdot \eta)\) vanish because of the properties of Grassmann variables. These calculations result in an effective action \(\Gamma_k\), which is given in terms of an integro-differential equation even in the limit \(k \to \infty\) as opposed to (2.21).

\[
\Gamma_k[A, \bar{A}, \psi, \bar{\psi}] \xrightarrow{k \to \infty} S[A + \bar{A}, \psi, \bar{\psi}] + S_{g.f.} + P_k[A, \bar{A}, \psi, \bar{\psi}] + O(1/k), \tag{3.11}
\]

where the \(U_A(1)\) violating term \(P_k\) is

\[
P_k[A, \bar{A}, \psi, \bar{\psi}] = \int d\mu_1(\theta) \prod_{s,t} \left( \int_x \bar{\eta}_s \cdot \phi_0 \right) \left( \int_x \bar{\phi}_0^t \cdot \eta_s \right) + \text{h.c.} \tag{3.12}
\]

The sources \(\eta, \bar{\eta}\) are given in (2.18) as functional derivatives of \(\Gamma_k\) with respect to \(\psi, \bar{\psi}\).

Terms which are suppressed with powers of \(1/k\) but do not violate the \(U_A(1)\) are contained in \(O(1/k)\). Although we will see in the next section that \(P_k\) is also suppressed with powers of \(1/k\), it is not possible to neglect it since it introduces \(U_A(1)\)-violation.

### 4 Effective action in the large scale limit

Equation (3.11) is a functional differential equation for \(\Gamma_k\). In the limit \(k \to \infty\) we are able to solve this equation. First we note that for \(k \to \infty\) the \(U_A(1)\)-violating term takes a local form. The explicit calculation is given in appendix [B]. As mentioned before, we do not have gauge field zero modes. For quantitative purposes one should work within the valley method (see [14] and references therein). However for our purpose it is enough to estimate of the value of the coupling of the \(U_a(1)\)-violating term. As we will see later on only instantons with width \(\rho \sim 1/k \to 0\) contribute. The qualitative behaviour of the corresponding fermionic zero modes does not change in the presence of the cut-off term. The zero modes have width \(\rho \to 0\) and are peaked about the centre of the instanton. It follows (see appendix [B])

\[
P_k[\psi, \bar{\psi}] = \int_z \Delta[k, \theta] \det \bar{\eta}_k(z) \frac{1 - \gamma_5}{2} \bar{\eta}_k(z) + \text{h.c.} + O(\Delta[k, \theta]/k) \tag{4.1}
\]

with

\[
\Delta[k, \theta] = (2^5 \pi^2 \rho^4)^{N_f} \int d\mu_1(\theta) a[N, \bar{N}] \sim k^{-5N_f+4}. \tag{4.2}
\]

\(P_k\) does not depend on \(A, \bar{A}\) to leading order (see appendix [B], so the \(U_A(1)\)-violation is purely fermionic to leading order. Now we concentrate on the measure \(d\mu_1(\theta)\). The fluctuation fields \(a', \chi', \bar{\chi}'\) decouple approximately from the instanton for large scales \(k\) (see discussion about the use of the valley method). This can be used to effectively remove the gauge fixing term for \(a_I\). We have in the limit \(k \to \infty\) (see (3.8))

\[
\frac{N_k''[a_I]}{N_k} \sim \frac{1}{N_k} \int D\alpha' d\chi' d\bar{\chi}' e^{-S_k[\alpha', \bar{\alpha}', \chi', \bar{\chi}']} \exp \left\{ -\Delta_k S_A[a_I, 0] - S_A[a_I] - S_\theta[a_I] \right\}, \tag{4.3}
\]
where the measures do not include the zero (or quasi-zero) modes related to \(a_I\). Note for explicit calculations that the gauge field integration also includes the ghost contribution with a suitable regulator \(R_c^k\) for the ghosts. The integrals in (4.3) become Gaussian for \(k \to \infty\). Taking into account the normalisation \(N_k\) they lead to a factor \(e^{-\Delta k S_A[a_I,0]}\) provides an exponential suppression of the zero mode contribution for \(\rho \gg 1/k\) due to the infrared regularisation of the gauge field (C.6). This ensures the infrared finiteness of the \(\rho\) integration. Therefore we can assume \(\rho\) to be of order \(1/k\) or smaller. The term 

\[
\exp \left\{ -S_A[a_I] \right\} = \exp \left\{ -\frac{8\pi^2}{g^2} \right\}
\]

(4.4)
is well known from instanton calculations (see [14] and references therein). The exponent \(S_\theta\) of the remaining factor is related to the instanton number \(16\pi^2 \int \text{tr} F \tilde{F} = 1\). In the limit \(\rho \to 0\) the density \(\text{tr} F \tilde{F}[a_I(x)]\) serves as a \(\delta\)-function which is peaked at the centre \(z\) of the instanton. Hence we get for \(\rho \sim 1/k \to 0\)

\[
\exp \left\{ -i \frac{1}{16\pi^2} \int_x \theta(x) \text{tr} F \tilde{F}[a_I(x)] \right\} \to \exp \left\{ -i \theta(z) \frac{1}{16\pi^2} \int_x \text{tr} F \tilde{F}[a_I] \right\} = \exp \left\{ -i \theta(z) \right\}.
\]

Thus the \(\theta\)-term leads to the following modification of the \(\det_{s,t}\)-term:

\[
\int_z \Delta[k,\theta] \det_{s,t} \tilde{\eta}_s \frac{1 - \gamma_5}{2} \tilde{\eta}_t = \int_z \Delta[k,0]e^{-i\theta(z)} \det_{s,t} \eta_s \frac{1 - \gamma_5}{2} \eta_t \left(1 + O(1/k)\right).
\]

(4.6)

Anti-instantons have instanton number \(-1\), so in this case one picks up a factor \(\exp\{i\theta(z)\}\). Using these results in (3.11) we end up with

\[
\Gamma_k[A, \bar{A}, \psi, \bar{\psi}] = S[A + \bar{A}, \psi, \bar{\psi}] + S_{gf}[A, \bar{A}] + P_k[\psi, \bar{\psi}] + O(1/k)
\]

(4.7)

with

\[
P_k[\psi, \bar{\psi}] = \int_z \Delta[k,\theta] \det_{s,t} \left[ -\frac{\delta}{\delta \psi_s} (\Gamma_k + \Delta k S_\psi) \frac{1 - \gamma_5}{2} \frac{\delta}{\delta \psi_t} (\Gamma_k + \Delta k S_\psi) \right] + \text{h.c.}
\]

(4.8)

In (4.8) we have used the explicit dependence of \(\eta, \bar{\eta}\) on \(\Gamma_k\) as given in (2.18). The term \(O(1/k)\) includes sub-leading orders of \(U_A(1)\)-conserving contributions and \(U_A(1)\)-violating contributions. The factor \(\Delta[k,\theta]\) provides a suppression of \(P_k\) proportional to \(k^{-5N_f+4}\).

The properties of (4.7) lead to an effective action \(\Gamma_k\), which is well-defined in the limit \(k \to \infty\). In addition an explicit expression for \(\Gamma_k\) can be derived. Note that we have used in the derivation that (2.21) is also valid in the instanton sectors. Hence proving the existence of a well-defined limit of \(\Gamma_k\) serves as a self-consistency check. The only source
for a diverging contribution is $P_k$, which is purely fermionic. In the limit $k \to \infty$ we have (see (2.14))

$$\frac{\delta}{\delta \psi} \Delta k S_\psi \to k \frac{\partial}{\partial \psi} \psi, \quad -\frac{\delta}{\delta \bar{\psi}} \Delta k S_\psi \to k \bar{\psi} \frac{\partial}{\partial \bar{\psi}} \psi.$$  \hspace{1cm} (4.9)

Combined with $\Delta[k, \theta]$ these terms are still suppressed with powers of $1/k$ ($N_f > 1$). In addition, (4.7) is inconsistent for $\frac{\delta}{\delta \psi} \Gamma_k, \frac{\delta}{\delta \bar{\psi}} \Gamma_k \sim k n, \ n \neq 0$. \hspace{1cm} (4.10)

This follows by using $\Delta[k, \theta] \sim k^{-5N_f+4}$ and (1.3). The only consistent choice in (4.10) is $n = 0$ which also ensures the finiteness of $\Gamma_k$ in the limit $k \to \infty$. Thus we drop contributions in $P_k$ dependent on $\frac{\delta}{\delta \psi} \Gamma_k, \frac{\delta}{\delta \bar{\psi}} \Gamma_k$ since they are of sub-leading order. With (1.2) and (2.14) we get

$$\bar{\psi}_s R_k \frac{1 \pm \gamma_5}{2} R_k \psi_t = \bar{\psi}_s R_k \frac{1 \pm \gamma_5}{2} R_k \psi_t \to k^2 \bar{\psi}_s \frac{1 \pm \gamma_5}{2} \psi_t \hspace{1cm} (4.11)$$

and the final result for the effective action for large scales $k$ is

$$\Gamma_k[A, \bar{A}, \psi, \bar{\psi}] = S[A + \bar{A}, \psi, \bar{\psi}] + S_{g.f.}[A, \bar{A}] + \int_z \Delta[k, \theta] k^{2N_f} \det \bar{\psi}_s \frac{1 \pm \gamma_5}{2} \psi_t$$

$$+ \int_z \Delta^*[k, \theta] k^{2N_f} \det \bar{\psi}_s \frac{1 \mp \gamma_5}{2} \psi_t + O(1/k, \Delta[k, \theta] k^{2N_f-1}). \hspace{1cm} (4.12)$$

with $\Delta[k, \theta] k^{2N_f} \sim k^{-3N_f+4}$ and $\Delta^*[k, \theta]$ is the hermitean conjugate of $\Delta[k, \theta]$. The term $O(1/k, \Delta[k, \theta] k^{2N_f-1})$ includes sub-leading orders of $U_A(1)$-conserving contributions (first argument) and $U_A(1)$-violating contributions (second argument).

In (4.12) the contributions of the trivial sector and the sector with instanton number $\pm 1$ are included. However, contributions $P_k(n)$ of sectors with instanton number $n, \ |n| \geq 2$ are only sub-leading terms in $1/k$. Due to the cut-off term for the gauge field (2.12) the contributions exhibit a natural size $\approx 1/k \to 0$. This locality is sufficient to allow qualitatively the same arguments as in the derivation of the $U_A(1)$-violating terms in the one instanton sector and end up with powers of the flavor determinant

$$P_k(n) \sim k^{-3n \cdot N_f+4} \int_z \left( \det \bar{\psi}_s \frac{1 \mp \gamma_5}{2} \psi_t \right)_s^t \mathcal{T}_n^{A_1 B_1 \cdots A_n N_f B_n N_f}, \hspace{1cm} (4.13)$$

where $\mathcal{T}_n$ denotes the color structure. These are sub-leading terms.

\section{Discussion}

We have calculated the fermionic $U_A(1)$-violating terms contributing to the effective action $\Gamma_k$ in the presence of instantons to leading order in $1/k$. Due to the infrared cut-off term
of the gauge field there are no problems with infrared divergences, so there are no gauge field zero modes. Although we have introduced infrared cut-off terms for the fermionic fields, there is still a fermionic zero mode for instanton configurations. For this reason the integration of the fermionic zero mode sector factorises and we end up with the well known ‘t Hooft determinant as the first order correction in $1/k$. The coupling $\Delta[k, \theta]$ of the ‘t Hooft determinant is infrared finite due to the gauge field regularisation. Further corrections are of sub-leading order in $1/k$. In addition they show the same flavor structure as the ‘t Hooft determinant. Inclusion of fermionic mass terms is straightforward. The effective action (4.12) with suitable wave function renormalisations, an explicit ghost sector (see [3, 4]) and additional $U_A(1)$-conserving terms may serve as an appropriate input to the exact flow equation in order to study the $U(1)$-problem.

It is possible to calculate $\mu[k, \theta]$ numerically based on the results of appendix [3]. The result does not allow quantitative statements, but provides a validity bound of the $1/k$-expansion. If the expansion is still valid at about $k \sim 700$ MeV, it would be possible to take the value of $\mu[k, \theta]$ at $k \sim 700$ MeV as an input for a phenomenological quark-meson model (see [10]). We briefly discuss the approximations used in the numerical calculation and comment on the result. The renormalisation scheme proposed by the framework of flow equations (with appropriate boundary conditions) can be related to the $\overline{\text{MS}}$-scheme [20]. Therefore one works in a first approximation with the well known results at one-loop level of instanton calculations and introduces as the only new ingredient the infrared cut-off in the gauge field sector. Clearly these approximations allow only a rough estimate of the value of $\mu[k, \theta]$. Moreover one can determine the validity-bound of the $1/k$-expansion in the case of QCD. For scales $k \sim 1 - 1.3$ GeV the $1/k$ approximation breaks down, and one has to use the flow equation to extrapolate to lower energies. It is known from instanton calculations, that in this region corrections which are proportional to the gluonic condensate $\langle 1/g^2 \cdot F^2 \rangle$ become important (see [11, 12]). Since the flow should be smooth (as opposed to the case of correlation functions connected with phase transitions), one expects fewer problems with the numerical integration of the flow equation for the relevant correlation functions (e.g. $\Delta[k, \theta]$, which is connected to the $\eta'$-mass). On the other hand the value of $\Delta[k, \theta]$ in the physical region should be dominated by the contributions collected during the flow, otherwise the value would depend on the initial scale $k_0$, which has no physical meaning. Therefore the calculation of correlation functions connected with the instanton-induced terms is interesting for two reasons: It is a good check for the computational power of flow equations and it would be a great success to derive the $\eta'$-mass quantitatively from first principles.

We have also discussed the leading order corrections due to the $\theta$-term. It leads to an additional phase factor in the ‘t Hooft determinant breaking $CP$-invariance in the presence of massive fermions. In the case of massless fermions the $\theta$-angle can be absorbed in a redefinition of the fields. For massive fermions one can calculate the flow of $\theta$. To solve the strong $CP$-problem, one has to calculate $\theta$ in the full quantum theory. For pure QCD the flow of $\theta$ has been studied in [13] using a rough approximation. Also fermionic contributions to the flow in a one-flavor model have been discussed. These calculations indicate a non-trivial scale dependence of the $\theta$-parameter. In addition it is shown that the
infrared limit \( k \to 0 \) has to be studied to give an final answer to this problem. For this issue one needs both a truncation of the flow equation leading to a consistent flow in the regime of interest and an initial effective action \( \Gamma_{k0} \) including all important terms. Certainly the \( U_A(1) \)-violating terms derived in the present paper are necessary initial inputs.

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A Zero modes of the regularised Dirac operator

In this appendix we discuss the behaviour of the regularised fermionic functional integral. We are interested in

\[
Z_{\psi,k}[a, \bar{A}, \bar{\eta}, \eta] = \frac{1}{N_k} \int d\bar{\chi} d\chi \exp \int_x \left\{ -\bar{\chi} (\mathcal{D}(a + \bar{A}) + R_k^\psi(\mathcal{D}(\bar{A}))) \chi + \bar{\eta} \cdot \chi + \bar{\chi} \cdot \eta \right\}, \tag{A.1}
\]

with the normalisation given by \( N_k = Z_{\psi,k}[0, 0, 0, 0] \). The Dirac operator \( \mathcal{D}(a + \bar{A}) \) has one zero eigenvalue (non-degenerate) for a configuration \( a + \bar{A} \) with instanton number \( \pm 1 \). First we discuss the ultraviolet regularisation of (A.1). For this purpose we concentrate on

\[
Z_{\psi,k}[a, \bar{A}, 0, 0] = \left( \frac{\det_\Lambda (\mathcal{D}(a + \bar{A}) + R_k^\psi(\mathcal{D}(\bar{A})))}{\det_\Lambda (\mathcal{D}(0) + R_k^\psi(\mathcal{D}(0)))} \right)^{N_f}. \tag{A.2}
\]

The subscript \( \Lambda \) is related to the fact that an ultraviolet regularisation of the determinants is needed. An appropriate regularisation in (A.2) would be the \( \zeta \)-function regularisation. More generally, high momenta should be suppressed in a gauge-invariant way. These conditions are fulfilled by the regularisations \( g_{\Lambda}[\mathcal{D}(a + \bar{A}) + R_k^\psi(\mathcal{D}(\bar{A}))] \) of the Dirac operator with the properties

\[
g_{\Lambda}[0] = 0, \quad g_{\Lambda}[x] \xrightarrow{x^2 \gg \Lambda^2} 0, \quad \{g_{\Lambda}[x], \gamma_5\} = 0 \text{ if } \{x, \gamma_5\} = 0. \tag{A.3}
\]

An explicit \( g_{\Lambda} \) fulfilling (A.3) is

\[
g_{\Lambda}[x] = xe^{-x^2/\Lambda^2}. \tag{A.4}
\]

\( g_{\Lambda} \) does not influence the infrared behaviour of the Dirac operator. In particular it vanishes for zero modes. Hence we use \( \mathcal{D}(a + \bar{A}) + R_k^\psi(\mathcal{D}(\bar{A})) \) for the discussion of the zero mode. \( \mathcal{D}(a + \bar{A}) + R_k^\psi(\mathcal{D}(\bar{A})) \) is not invertible on the one-dimensional subspace of the zero mode \( \chi_0 \) of \( \mathcal{D}(a + \bar{A}) \), i.e. acting with the inverse of \( \mathcal{D}(a + \bar{A}) + R_k^\psi(\mathcal{D}(\bar{A})) \) on \( \chi_0 \) does not lead
to a square-integrable function. This indicates the existence of a zero mode. To prove the existence of a zero mode for the infrared regularised Dirac operator we introduce

$$H_t = \mathcal{D}(a + A) + t R^\psi_k [\mathcal{D}(A)].$$ (A.5)

This operator is the usual Dirac operator for $t = 0$ and the infrared regularised Dirac operator for $t = 1$. Now we concentrate on the evaluation of the zero eigenvalue for $t \in [0, 1]$. We are dealing with the eigenfunctions $\psi_n(t)$ of $H_t$ with

$$H_t \psi_n(t) = E_n(t) \psi_n(t).$$ (A.6)

Since $R^\psi_k$ is a compact operator, the normalisability of $\psi_n(t)$ is guaranteed for every $t$. Moreover the Taylor series in $t$ of $E_n, \psi_n$ are convergent. In the one instanton sector there is one eigenvector $\psi_0$ with

$$H_0 \psi_0(0) = 0.$$ (A.7)

$E_0(0) = 0$. We prove by induction that all derivatives of $E_0$

$$E_0^{(m)}[t] = \partial_t^m E_0(t)$$ (A.8)

are vanishing at $t = 0$. This leads to $E_0(t) = 0$ for $t \in [0, 1]$. We start with $E_0^{(0)} = 0$ and assume that $E_0^{(m)} = 0$ for all $m \leq n - 1$. It follows

$$\partial_t^m [H_0 \psi_0(t)]_{t=0} = 0 \quad \forall m \leq n - 1$$ (A.9)

or

$$\mathcal{D}(a + A) \psi_0^{(m)}(0) = -m R^\psi_k [\mathcal{D}(A)] \psi_0^{(m-1)}(0) \quad \forall m \leq n - 1, \quad \psi_0^{(m)}(t) = \partial_t^m \psi_0(t).$$ (A.10)

As an intermediate result we prove that the $\psi_0^{(m)}(0)$ are chirality eigenstates with the same chirality as $\psi_0(0)$ for all $m \leq n - 1$. We deduce from (A.10)

$$\gamma_5 \psi_0^{(m)} = -m \mathcal{P} \frac{1}{\mathcal{D}(a + A)} \mathcal{P} R^\psi_k [\mathcal{D}(A)] \gamma_5 \psi_0^{(m-1)}(0) + \gamma_5 (1 - \mathcal{P}) \psi_0^{(m)},$$ (A.11)

where $\mathcal{P}$ is the projector on the space of non-zero modes of $\mathcal{D}(a + A)$ and we have used (see (2.4),(2.14))

$$\{ R^\psi_k [\mathcal{D}(A)], \gamma_5 \} = \{ \mathcal{D}(a + A), \gamma_5 \} = [\mathcal{P}, \gamma_5] = 0, \quad \mathcal{P} = \left( \frac{1}{\mathcal{D}} \frac{\partial}{\partial \mathcal{D}} \right) (a + \bar{A}).$$ (A.12)
However $\psi_0^{(0)} (0)$ is a chirality eigenstate, $\gamma_5 \psi_0 (0) = \pm \psi_0 (0)$. Furthermore $(1 - \mathcal{P}) \psi_0^{(m)} (0)$ is proportional to $\psi_0 (0)$. Thus it follows from (A.11) that $\psi_0^{(m)} (0)$ has the same chirality as $\psi_0 (0)$, if $\psi_0^{(m-1)} (0)$ has this property. Starting iteratively with $m = 1$, the claimed chirality properties follow for all $m \leq n - 1$.

With this result and (A.9,A.12) we prove

$$E_0^{(n)} (0) = 0.$$ (A.13)

Therefore the $E_0^{(n)} (0)$ vanish for all $n \in \mathbb{N}$ which leads to $E_0 (t) = 0$, $t \in [0,1]$. This proves that $\psi_0 (t)$ is a zero mode for all $t$, in particular for $t = 1$.

With these preliminaries we can easily factorise the fermionic zero mode contribution as in the case without regularisation. It follows for topologically non-trivial configurations $a + \bar{A}$

$$Z_{\psi,k}[a, \bar{A}, \bar{\eta}, \eta] = \frac{1}{N_k} \int d\bar{\chi}^I d\bar{\chi}^J \exp \left\{ \int_x \bar{\chi}^I \left( \mathcal{D}(a + \bar{A}) + R_k^\psi [\mathcal{D}(\bar{A})] \right) \chi^J + \int_x (\bar{\eta}^I \cdot \chi^J + \bar{\chi}^J \cdot \eta^I) \right\} \int d\bar{\chi}_0 d\chi_0 \exp \int_x (\bar{\eta} \cdot \chi_0 + \bar{\chi}_0 \cdot \eta)$$ (A.14)

with

$$\left( \mathcal{D}(a + \bar{A}) + R_k^\psi [\mathcal{D}(\bar{A})] \right) \chi_0 = 0.$$ (A.15)

**B Zero mode contribution to leading order of $1/k$**

We recall the expression for $P_k$ (see (3.11))

$$P_k[A, \bar{A}, \psi, \bar{\psi}] = \int d\mu_1 (\theta) \prod_{s=1}^{N_f} \int_x \bar{\eta}_s \cdot \phi_0 \int_x \phi_0^+ \cdot \eta_s + \text{h.c.}$$ (B.1)

We shall argue that $P_k$ depends only to sub-leading order in $1/k$ on $A, \bar{A}$. For that purpose we concentrate on the zero mode equation (A.13) with a purely topological configuration $a = a_I$. In the limit $k \to \infty$ only instantons $a_I (x, \rho)$ with width $\rho \sim 1/k$ contribute to (B.1) due to the infrared regularisation of the gauge field present in $d\mu_1$ (see appendix C).
Note that $a_t(x, \rho) = a_t(x/\rho, 1)/\rho$ (see e.g. (C.1), (C.4)) and $R^\psi_k[\theta_x] = R^\psi_{k\rho}[\theta_{x/\rho}]/\rho$ (see (2.14)). Hence after multiplying (A.13) with $\rho \sim 1/k \to 0$ and scaling $x \to \rho x$ we conclude that the fermionic zero mode depends on $A, \bar{A}$ only to sub-leading order. Thus $P_k$ is $A, \bar{A}$-independent to leading order and we write

$$P_k[A, \bar{A}, \psi, \bar{\psi}] = P_k[\psi, \bar{\psi}] + O(1/k). \quad (B.2)$$

$P_k$ is non-local. In the limit $k \to \infty$ it is possible to write it as a sum of a local contribution and terms which are suppressed with powers of $k^{-1}$. In this limit we also calculate the normalisation of $P_k$.

The measure $d\mu_1$ contains integrations over collective coordinates. The interesting collective coordinates are the centre of the instanton $z$, the width $\rho$ and the global gauge rotations $g$. The explicit derivation of the $\rho, z$ dependence of $d\mu_1$ is done in appendix C. Moreover the instanton $a_t$ and the fluctuations $a'$ decouple in the limit $k \to \infty$ which can be used to effectively remove the gauge fixing term for $a_t$ (see derivation of (4.3)). Hence $d\mu_1$ also includes a measure $dg_k$ of local gauge degrees of freedom in this limit. Note that the cut-off term for $a_t$ singles out those local gauge degrees of freedom dependent on large momenta. However this will not effect the following arguments.

We will use well-known results from instanton calculations. For details we refer the reader to the literature (12), (14). The normalised fermionic zero mode is given by

$$\phi_0^A(x; z, \rho) = \frac{\sqrt{2}}{\pi} \frac{\rho}{((x-z)^2 + \rho^2)^{1/2}} u^A_\xi, \quad \sum_A u^A \times \bar{u}^A = \frac{1 - \gamma_5}{2}, \quad ||\phi_0|| = 1 \quad (B.3)$$

and gauge transformations $g(x) \phi_0(x; z, \rho)$ of (B.3), where $g(x)$ could be either $g_k(x)$ or a global gauge rotation $g$. With $\rho \sim 1/k \to 0$ we write

$$\int_x \bar{\eta}_s \cdot \phi_0 = \frac{\sqrt{2}}{\pi} \int_x \frac{\rho}{((x-z)^2 + \rho^2)^{1/2}} \bar{\eta}_s(x) \cdot u = \frac{\sqrt{2}}{\pi} \int_x \frac{\rho}{(x^2 + \rho^2)^{1/2}} \bar{\eta}_s(x+z) \cdot u. \quad (B.4)$$

We are interested in the limit $\rho \to 0$. Therefore we calculate (B.4) in an expansion about $\rho = 0$. The coefficient related to the power $\rho^0$ vanishes. The coefficient proportional to $\rho$ is determined by scaling (B.4) with $\rho^{-1}$

$$\lim_{\rho \to 0} \frac{1}{\rho} \frac{\sqrt{2}}{\pi} \int_x \frac{\rho}{(x^2 + \rho^2)^{1/2}} \bar{\eta}_s(x+z) \cdot u = \frac{\sqrt{2}}{\pi} \int_x \left( \frac{1}{x^2} \right)^{1/2} \bar{\eta}_s(x+z) \cdot u. \quad (B.5)$$

The term of order $\rho^2$ is calculated by subtracting (B.5) times $\rho$ from (B.4). It follows

$$\lim_{\rho \to 0} \frac{1}{\rho} \frac{\sqrt{2}}{\pi} \int_x \left[ \frac{1}{(x^2 + \rho^2)^{1/2}} - \left( \frac{1}{x^2} \right)^{1/2} \right] \bar{\eta}(x+z) \cdot u = -2^{5/2} \pi \bar{\eta}_s(z) \cdot u. \quad (B.6)$$

The final result for the contribution of the fermionic zero mode of an instanton with width $\rho \sim 1/k$ and centre $z$ is

$$\int_x \bar{\eta}^{-1} \cdot \phi_0 = \rho \frac{\sqrt{2}}{\pi} \int_x \left( \frac{1}{x^2} \right)^{1/2} \bar{\eta}_s(x+z) \cdot u - \rho^2 2^{5/2} \pi \bar{\eta}_s(z) \cdot u + O(\rho^3). \quad (B.7)$$
Contributions to $P_k$ dependent on the first (non-local) term on the right hand side of (B.7) vanish because of the integration over local gauge degrees of freedom $g_k$ present in $d\mu_1$. For the evaluation of the second term on the right hand side of (B.7) we concentrate on the integration over global gauge rotations. We get from (B.1) by using (B.7)

$$P_k[\psi, \bar{\psi}] = \int d^4z \int d\bar{\mu}_1(\theta)(2^5\pi^2\rho^4)^{N_f} \int dg \prod_{s=1}^{N_f} (\bar{\eta}^s(z)g^{-1} \cdot u)(\bar{u} \cdot g\eta^s(z)) + \text{h.c.} + O(1/k),(B.8)$$

where $dg$ is the measure of global gauge rotations and $d^4z$ is the measure of the centre of the instanton:

$$d\mu_1(\theta) = d\bar{\mu}_1(\theta) \ dg \ dz, \quad \int dg = 1. \quad \text{(B.9)}$$

For the evaluation of the $g$-integration in $SU(N)$ we use [21]

$$\int dg \prod_{i=1}^{N_f} g_{A_i A_i}^{-1} g_{B_i B_i} = a[N, N_f] \left( \sum_{\sigma} \prod_{i=1}^{N_f} \delta_{A_i B_{\sigma(i)}} \delta_{\bar{A}_i \bar{B}_{\sigma(i)}} + \frac{1}{N} O_{A_1 B_1 \ldots \bar{A}_{N_f} \bar{B}_{N_f}} \right), \quad \text{(B.10)}$$

where $\sigma$ are the permutations of $(1, \ldots, N_f)$. The tensor $O$ is suppressed with $1/N$ and only consists of products of Kronecker deltas $\delta_{A_i B_j} \delta_{\bar{A}_i \bar{B}_m}$. Both $a[N, N_f]$ and $O$ are complicated functions of $N, N_f$. With (B.3, B.10) we get

$$\int dg \prod_{s=1}^{N_f} \bar{\eta}_s g \cdot u\bar{u} \cdot g^{-1} \eta_s \overset{k \rightarrow \infty}{\longrightarrow} a[N, N_f] \det_{s,t} \bar{\eta}^s_{A_s} \left[ 1 - \frac{\gamma_5}{2} \right] \eta^t_{A_t} \left( \delta_{A_s B_t} + \frac{1}{N} U_{A_1 B_1 \ldots \bar{A}_{N_f} \bar{B}_{N_f}} \right)(B.11)$$

The tensor $U$ is related to $O$ and involves only products of Kronecker deltas $\delta_{A_i B_j}$. However from now on we drop the term dependent on $U$. This is a suitable approximation within a $1/N$-expansion since it carries the same flavor structure as the leading term but is suppressed with $1/N$. Note however that this is done more for the sake of convenience and the tensor structure can be added without changing the conclusions of the present paper. Moreover even so tedious the calculation of $U$ is straightforward. This leads to

$$P_k[\psi, \bar{\psi}] \overset{k \rightarrow \infty, N \gg 1}{\longrightarrow} \int d^4z \int d\bar{\mu}_1(\theta) (2^5\pi^2\rho^4)^{N_f} a[N, N_f] \det_{s,t} \bar{\eta}^s_{A_s} \left( 1 - \frac{\gamma_5}{2} \right) \eta^t_{A_t} + \text{h.c.}(B.12)$$

Now we are able to give a final expression for $P_k$

$$P_k[\psi, \bar{\psi}] = \int_z \Delta[k, \theta] \det_{s,t} \bar{\eta}_s(z) \left( 1 - \frac{\gamma_5}{2} \right) \eta_t(z) + O(\Delta[k, \theta]/k) \quad \text{(B.13)}$$

with

$$\Delta[k, \theta] = \int d\bar{\mu}_1(\theta) (2^5\pi^2\rho^4)^{N_f} a[N, N_f] \sim k^{-5N_f + 4}. \quad \text{(B.14)}$$

The gauge field cut-off ensures the finiteness of $\Delta[k, \theta]$. Then the $k$-dependence follows by dimensional arguments.
C Properties of the gauge field regularisation

In this section we examine the properties of the cut-off term for the gauge field in the 1 instanton sector. A general instanton is given by

\[ A^a_{I,\mu}(x; z, \rho) = \eta^a_{\mu\nu} \frac{(x - z)\nu}{(x - z)^2 + \rho^2} \]  

(C.1)

and global gauge rotations of \( A^a_{I,\mu}(x; z, \rho) \). Here \( \eta^a_{\mu\nu} \) are the 't Hooft symbols [12]. In order to stay in contact with [9] we first discuss this approach where the background field is missing. In this case the field \( a \) consists on both, the instanton (C.1) and the fluctuations about the instanton. The configurations (C.1) are not square-integrable because of their infrared behaviour. To see this, let us recall the cut-off term for the gauge field (see (2.12) and (1.2))

\[ \Delta_k S_A[a, 0] = \frac{1}{2} \int_x a \cdot R^A_k[D_T(0)] \cdot a, \quad R^{A,ab}_{k,\mu\nu}[D_T(0)] \xrightarrow{k^2 \to 0} k^2 \delta^{ab} \delta_{\mu\nu}. \]  

(C.2)

If a configuration \( a \) is ultraviolet finite but is not square-integrable due to infrared divergences, then \( \Delta_k S_A[a, 0] \) diverges. These configuration have zero measure in the path integral, since

\[ \exp\{-\Delta_k S_A[a, 0]\} = 0. \]  

(C.3)

We conclude that only configurations which decrease faster than \( 1/x^2 \) can contribute to the infrared regularised path integral. This reflects the fact that within this particular approach [9] the cut-off term introduces trivial (infrared) boundary-conditions. Thus the infrared cut-off term (without background field dependence) introduces a constraint on the class of gauge fixings, i.e. allowing only for those compatible with trivial infrared behaviour. Instantons in the singular gauge fulfill this condition (see for example [14]). They are given by

\[ a^a_{I,\mu}(x; z, \rho) = \eta^a_{\mu\nu} \frac{(x - z)\nu}{(x - z)^2 + \rho^2}. \]  

(C.4)

These configurations are square-integrable, so \( \Delta_k S_A[a, \bar{A}] \) is finite. We write explicitly for an instanton \( a_I(x, z, \rho) \) with centre \( z \) and width \( \rho \)

\[ \Delta_k S_A[a_I(x, z, \rho), 0] = \frac{1}{2} \int_{\tilde{x}} a_I(\tilde{x}, 0, 1) \cdot R^A_k[\partial^2_{\tilde{x}}] \cdot a_I(\tilde{x}, 0, 1), \]  

(C.5)

where we have used the translation invariance of the cut-off term for \( D_T(0) \) and have changed the variable \( x \) to \( \tilde{x} = (x + z)/\rho \). Using the limit in (C.2) we get

\[ \exp\{-\Delta_k S_A[a_I(x, z, \rho), 0]\} \xrightarrow{k\rho \gg 1} \exp\left\{-\frac{1}{2} \int_{\tilde{x}} a_I(\tilde{x}, 0, 1) \cdot a_I(\tilde{x}, 0, 1)\right\} \sim e^{-\#(k\rho)^2}. \]  

(C.6)
Hence in the limit $k \to \infty$ only instantons with width $\rho \sim k$ contribute. This result extends easily to the more general case with non-vanishing background field $\bar{A}$. Moreover the constraint on the class of gauge fixings is related entirely to the introduction of trivial infrared boundary conditions by choosing $\bar{A} = 0$. Following the background field approach to instantons [2, 12] one chooses the background field $\bar{A}$ as the configuration (C.1). In this case the field $a$ consists on fluctuations about the instanton which are square-integrable by definition (after a complete gauge fixing). This leads immediately to (C.6).

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