ON THE CONWAY POTENTIAL FUNCTION INTRODUCED BY KAUFFMAN

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Abstract. We show two results about the Conway potential function which is known as the normalized multivariable Alexander polynomial. We first show that the Conway potential function introduced by Kauffman in “Formal Knot Theory” is indeed a link invariant. Next we show that Kauffman’s potential function equals Hartley’s potential function. We will prove it by using Murakami’s axioms for the multivariable Alexander polynomial.

1. Introduction

In [1] J.H. Conway introduced the Conway potential function by some axioms. But his axioms are not sufficient to determine his potential function. L.H. Kauffman [3] showed how to define the one variable reduced potential function in terms of a Seifert matrix. R. Hartley [2] gave a precise definition of the multivariable potential function. At the same time Kauffman [4] introduced another definition of the multivariable potential function without using Seifert matrices. However he did not show that his definition gave a link invariant. In this paper we show it in Section 3.

On the other hand J. Murakami [5, 6] gave axioms of the potential function which are sufficient for the definition. In Section 4 we show that Kauffman’s potential function equals Hartley’s potential function by J. Murakami’s axioms. In Section 2 we confirm some necessary definitions and theorems in [4].

2. Preparation

In this section we introduce some definitions and theorems in [4] to define Kauffman’s potential function.

In this paper we regard a link diagram as a regular projection of a link with over and under informations at the vertices. On the other hand a link projection is without such information.

Definition 2.1 (Kauffman state [4]). Let $U$ be a link projection on $\mathbb{R}^2$ or $S^2$. A pair of a vertex of $U$ and a local region around the vertex is represented as a marker as in Figure 1. A local region means intersection of a region and an interior of the dotted circle.

If $U$ is connected, the number of regions of $U$ is two more than the number of the vertices since the Euler characteristic of $S^2$ is two. Then we call a set of markers a Kauffman state or a state if it satisfies the following conditions.
(i) Two adjacent regions have no markers. We put a star (\(\ast\)) in each of these regions.
(ii) Every vertex has only one marker.
(iii) Every region without a star (\(\ast\)) contains only one marker.

**Example 2.2.** Figure 2 shows an example of a knot projection and a state.

![Kauffman state](image)

**Figure 2.** Kauffman state

**Definition 2.3** (transposition [4]). The operation as shown in Figure 3 is called a **transposition**. In particular the operation from left to right is called a **clockwise transposition**, and that from right to left is called a **counter-clockwise transposition**. The dotted circle contains a part of a projection and (possibly empty) markers. Moreover a state is said to be a **clocked state** if it admits only clockwise transpositions and a **counter-clocked state** if it admits only counter-clockwise transpositions.

![transposition](image)

**Figure 3.** transposition

**Theorem 2.4** (Clock Theorem [4]). Let \(U\) be a connected link projection and \(\mathcal{S}\) be the set of states of \(U\) for a given choice of adjacent stars.

Then the set \(\mathcal{S}\) has the following properties.

(i) It has a unique clocked state and a unique counter-clocked state.
(ii) Any state in $\mathcal{S}$ can be reached from the clocked (counter-clocked respectively) state by a series of clockwise (counter-clockwise respectively) transpositions.

**Definition 2.5** (state polynomial [4]). Let $U$ be an oriented link projection with $n$ vertices. Let $\mathcal{S}$ be the set of states of $U$ with fixed adjacent stars. We label the vertices as $v_1, v_2, \ldots, v_n$ and put variables $I_1, O_1, U_1, D_1, \ldots, I_n, O_n, U_n, D_n$ in local regions around the vertices as indicated in Figure 4. Then we define the $n$ vertices. We label the vertices and the regions as

$$\text{Figure 4.}$$

We define $\sigma(S)$ to be $\langle U|\mathcal{S}\rangle$ replacing all the $I_k$ with $-1$ and the $O_k$, $U_k$ and $D_k$ with $1$. We regard $\langle U|\mathcal{S}\rangle$ as an element of $\mathbb{C}[I_1, O_1, U_1, D_1, \ldots, I_n, O_n, U_n, D_n]$. If $U$ is connected, then the state polynomial for $U$ is defined by the formula:

$$\langle U|\mathcal{S}\rangle = \sum_{S \in \mathcal{S}} \sigma(S)\langle U|S\rangle \in \mathbb{C}[I_1, O_1, U_1, D_1, \ldots, I_n, O_n, U_n, D_n].$$

If $U$ is non-connected, we define $\langle U|\mathcal{S}\rangle = 0$.

Moreover let $\langle U|\mathcal{S}\rangle' \in \mathbb{C}[B, W]$ be obtained from $\langle U|\mathcal{S}\rangle$ replacing all the $I_k$, $O_k$, $U_k$ and $D_k$ with $1$, $B$ or $W$ as indicated in Figure 5.

$$\text{Figure 5.}$$

**Definition 2.6** (Alexander matrix [4]). Let $U$ be a connected link projection with $n$ vertices. We label the vertices and the regions as $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_{n+2}$ respectively. We put variables as indicated in Figure 4. Then we define the Alexander matrix $A(U)$ as an $n \times (n+2)$ matrix with $(i,j)$ entry $A(U)_{ij}$, where $A(U)_{ij}$ is the sum of the variables around the vertex $v_i$ in the region $r_j$. Let $A(U)(i_1, i_2)$ be the $n \times n$ matrix obtained from $A(U)$ by deleting the $i_1$th and $i_2$th columns. It is called the reduced Alexander matrix. Moreover let $A'(U)$ be obtained from $A(U)$ by replacing all the $I_k$, $O_k$, $U_k$ and $D_k$ with $1$, $B$ or $W$ as indicated in Figure 5.

**Theorem 2.7** ([4]). Let $U$ be a connected link projection with labeled vertices and regions as $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_{n+2}$ respectively. Let $\mathcal{S}(i_1, i_2)$ be the set of states of $U$ with fixed adjacent stars in the regions $r_{i_1}$ and $r_{i_2}$. Then state polynomial and the Alexander matrix satisfy the following formula:

$$\langle U|\mathcal{S}(i_1, i_2)\rangle \doteq \det A(U)(i_1, i_2).$$

The symbol $\doteq$ means equality up to sign.
Definition 2.8 (multiple Alexander index \[4\]). Let $L = L_1 \cup L_2 \cup \cdots \cup L_N$ be an oriented link diagram on $\mathbb{R}^2$. Let $U$ be the projection of $L$. Then each region of $U$ is assigned an element $p = (p_1, p_2, \ldots, p_N)$ of $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ as follows.

The unbounded region is assigned index $(0, 0, \ldots, 0)$. The $K$th component, $p_K$, increases if one crosses the $K$th strand from left to right and decrease if one crosses the $K$th strand from right to left. Then each element $p = (p_1, p_2, \ldots, p_N)$ is called the multiple Alexander index of $U$. Moreover the sum of the components of the multiple Alexander index in a region is called the (non-multiple) Alexander index.

\[ L_K \]

\[ (p_1, \ldots, p_K, \ldots, p_N) \quad (p_1, \ldots, p_K+1, \ldots, p_N) \]

**Figure 6.** multiple Alexander index

The following lemma is proved in \[4\]. Since we use the technique in the proof later, we give a proof following Kauffman.

**Lemma 2.9 (\[4\]).** Let $U$ be an oriented link projection. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two sets of states with adjacent stars. Then the state polynomials satisfy the formula $\langle U|\mathcal{S}_1 \rangle' = \langle U|\mathcal{S}_2 \rangle'$.

**Proof.** If $U$ is not connected, $\langle U|\mathcal{S}_1 \rangle' = 0 = \langle U|\mathcal{S}_2 \rangle'$. So we assume that $U$ is connected. Now we give the non-multiple Alexander index to $U$, and label the vertices and regions as $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_{n+2}$ respectively. Let $m(j)$ be the Alexander index in the region $r_j$.

First let $\alpha$ and $\alpha^{-1}$ be the roots of $X^2 + (B + W)X + 1 = 0$, where $X$ is a variable and $B$ and $W$ are regarded as constants. Let $\beta(x, y) = \alpha^{m(x)-m(y)} - \alpha^{-m(x)+m(y)}$ and $\hat{a}_j$ be the $j$th column of the Alexander matrix $A'(U)$. Then we have

\[
\sum_{j=1}^{n+2} \beta(j, k)\hat{a}_j = 0
\]

for a fixed integer $k$ from the relation between the Alexander index and the Alexander matrix (See p.58 of \[4\]). The symbol 0 means a zero-vector. Therefore we have

\[
\beta(l, j) \det A'(U)(j, k) \\
= \det (\hat{a}_1, \ldots, \hat{a}_j, \ldots, \hat{a}_k, \ldots, \beta(l, j)\hat{a}_l, \ldots, \hat{a}_{n+2}) \\
= \det (\hat{a}_1, \ldots, \hat{a}_j, \ldots, \hat{a}_k, \ldots, -\sum_{j' \neq j} \beta(j', j)\hat{a}_{j'}, \ldots, \hat{a}_{n+2}) \\
= \det (\hat{a}_1, \ldots, \hat{a}_j, \ldots, \hat{a}_k, \ldots, -\beta(j, j)\hat{a}_j - \beta(k, j)\hat{a}_k, \ldots, \hat{a}_{n+2}) \\
= -\beta(k, j) \det (\hat{a}_1, \ldots, \hat{a}_j, \ldots, \hat{a}_k, \ldots, \hat{a}_n, \ldots, \hat{a}_{n+2}) \\
\overset{\text{def}}{=} \beta(k, j) \det A'(U)(j, l)
\]
for any fixed $j$, $k$ and $l$ so that $j \neq k$ and $j \neq l$. Similarly we have

$$\beta(h, l) \det A'(U)(l, j) = \beta(j, l) \det A'(U)(l, h)$$

for any fixed $h$ so that $l \neq h$. Therefore we have the following formula:

$$\frac{\beta(l, j)}{\beta(k, j)} \det A'(U)(j, k) = \frac{\beta(j, l)}{\beta(h, l)} \det A'(U)(l, h).$$

Now we have $\beta(l, j) = -\beta(j, l)$ from the definition of $\beta$. So we have

$$\frac{\det A'(U)(j, k)}{\beta(k, j)} = \frac{\det A'(U)(l, h)}{\beta(h, l)}.$$

Now we assume that the regions $r_j$ and $r_k$ are adjacent and that $r_l$ and $r_h$ are adjacent. Then we have the formulas $\det A'(U)(j, k) = (U|\mathcal{S}(j, k))'$ and $\det A'(U)(l, h) = (U|\mathcal{S}(l, h))'$. Moreover we have $\beta(k, j) = \alpha - \alpha^{-1} = \beta(h, l)$. Therefore we have $\langle U|\mathcal{S}(j, k)\rangle' = (U|\mathcal{S}(l, h))'$. On the other hand we have $\langle U|\mathcal{S}(j, k)\rangle' = (U|\mathcal{S}(l, h))'$. Thus we have $\langle U|\mathcal{S}(j, k)\rangle' = (U|\mathcal{S}(l, h))'$. □

Now we prepare the following propositions to prove the invariance of Kauffman’s potential function later.

**Proposition 2.10.** Let $U$ be an oriented and connected link projection with labeled vertices and regions as $v_1, v_2, \ldots, v_n$ and $r_1, r_2, \ldots, r_{n+2}$ respectively so that the regions $r_{n+1}$ and $r_{n+2}$ are adjacent as indicated in Figure 7 and that the Alexander matrix $A(U)$ satisfies $\langle U|\mathcal{S}(n+1, n+2)\rangle = \det A(U)(n+1, n+2)$. Then the Alexander matrix and the state polynomial satisfy the following formulas:

$$\begin{align*}
(\text{4a}) \quad \det A(U)(i, j) &= \begin{cases} 
(1)^{i+j+1} \langle U|\mathcal{S}(i, j)\rangle & \text{if } j > i. 
\end{cases} 
\end{align*}$$

$$\begin{align*}
(\text{4b}) \quad \det A(U)(i, j) &= \begin{cases} 
(-1)^{i+j} \langle U|\mathcal{S}(i, j)\rangle & \text{if } j > i. 
\end{cases} 
\end{align*}$$

where $j > i$.

![Figure 7](image_url)

![Figure 8](image_url)

![Figure 9](image_url)

In order to prove this proposition we prepare the following lemma.

**Lemma 2.11.** We have $\det A(U)(i, j) = (-1)^a \langle U|\mathcal{S}(i, j)\rangle$ if $\det A'(U)(i, j) = (-1)^a \langle U|\mathcal{S}(i, j)\rangle'$ for some integer $a$.

**Proof.** We can prove this since $\langle U|\mathcal{S}\rangle' = \sum_{S \in \mathcal{S}} (-1)^b(S)B^b(S)W^w(S) \neq 0$ for a non-empty set $\mathcal{S}$. □
Proof of Proposition 2.11. We consider $A'(U)$ instead of $A(U)$ in this proof from Lemma 2.11. We replace $j$ with $n + 2$, $k$ with $n + 1$, $l$ with $i$ and $h$ with $j$ in the proof of Theorem 2.7 and we discuss the sign. Then from (2) we have

$$\beta(i, n + 2) \det A'(U)(n + 2, n + 1)$$

$$= \det (\hat{a}_1 \ldots \beta(i, n + 2)\hat{a}_i \ldots \hat{a}_{n+1} \hat{a}_{n+2})$$

$$= -\beta(n + 1, n + 2) \det (\hat{a}_1 \ldots \hat{a}_{n+1} \ldots \hat{a}_{n+2})$$

$$= (-1)^{n-i+1} \beta(n + 1, n + 2) \det A'(U)(i, n + 2),$$

where $i \leq n + 1$. Similarly we have

$$\beta(j, i) \det A'(U)(i, n + 2)$$

$$= \det (\hat{a}_1 \ldots \hat{a}_i \ldots \beta(j, i)\hat{a}_j \ldots \hat{a}_{n+2})$$

$$= -\beta(n + 2, i) \det (\hat{a}_1 \ldots \hat{a}_i \ldots \hat{a}_{n+1} \ldots \hat{a}_{n+2})$$

$$= (-1)^{n-j} \beta(n + 2, i) \det A'(U)(i, j).$$

Therefore we have

$$\frac{\beta(i, n + 2)}{\beta(n + 1, n + 2)} \det A'(U)(n + 2, n + 1) = (-1)^{i+j+1} \frac{\beta(n + 2, i)}{\beta(j, i)} \det A'(U)(i, j).$$

On the other hand we have $\beta(i, n + 2) = -\beta(n + 2, i)$ and $\beta(n + 1, n + 2) = \alpha^{-1} - \alpha$ from the definitions of $\beta$ and the Alexander index. If the regions $r_i$ and $r_j$ are as shown in Figure 8, we have $\beta(j, i) = \alpha - \alpha^{-1}$. So we have $\det A'(U)(n + 2, n + 1) = (-1)^{i+j+1} \det A'(U)(i, j)$ from (7). Then we have

$$\langle U|\mathcal{I}(i, j)\rangle = \langle U|\mathcal{I}(n + 1, n + 2)\rangle'$$

$$= \det A'(U)(n + 2, n + 1)$$

$$= (-1)^{i+j+1} \det A'(U)(i, j).$$

Similarly if the regions $r_i$ and $r_j$ are as shown in Figure 9 we can get (11). This proof is complete. \qed

3. Kauffman’s potential function

In this section we will define the potential function introduced by Kauffman and discuss its invariance.

Definition 3.1 (potential function [4]). Let $L = L_1 \cup L_2 \cup \cdots \cup L_N$ be an oriented link diagram. Let $U$ be the projection of $L$ with labeled vertices as $v_1, v_2, \ldots, v_n$ and the multiple Alexander index. Let $\mathcal{I}$ be the set of states of $U$ whose fixed stars share the $K$th strand with indices $(p_1, \ldots, p_K, \ldots, p_N)$ and $(p_1, \ldots, p_K+1, \ldots, p_N)$. First we put $N$ variables $X_1, X_2, \ldots, X_N$ in local regions around the vertices as shown in Figure 10. Then let $\langle L|\mathcal{I}\rangle$ be the polynomial obtained from $\langle U|\mathcal{I}\rangle$ by replacing all the $I_k$, $O_k$, $D_k$ and $U_k$ with $X_1, X_2, \ldots, X_N$ as indicated in Figure 10. Second let $|\mathcal{I}|$ be $X_1^{-2p_1}X_2^{-2p_2}\cdots X_N^{-2p_N}X_K^{-1}(X_K - X_K^{-1})$. Third let $c(L)$ be the curvature of the sublink $L_J$, which counts how many times the sublink rotates counter-clockwise.

Then we define $\square_L$ to be

$$\square_L = \frac{X_1^{c(L_1)}X_2^{c(L_2)}\cdots X_N^{c(L_N)}}{|\mathcal{I}|}\langle L|\mathcal{I}\rangle.$$
and we call □_L Kauffman’s potential function.

![Diagram](image-url)

**Figure 10.**

**Theorem 3.2.** Kauffman’s potential function is a link invariant.

In order to prove this theorem we prove the following two lemmas.

**Lemma 3.3.** The function \( \langle L|\mathcal{S}\rangle / |\mathcal{S}| \) is independent of the choice of fixed stars.

**Proof.** Let \( U \) be the projection of \( L \) with labeled vertices and regions as \( v_1, v_2, \ldots, v_n \) and \( r_1, r_2, \ldots, r_{n+2} \) respectively so that they satisfy the condition of Proposition 2.10. Let \( A(L) = (\hat{a}_1, \ldots, \hat{a}_{n+2}) \) be the Alexander matrix corresponding to Figure 10. Let \( m_K(j) \) be the \( K \)th component of the multiple Alexander index in the region \( r_j \).

First we show the following two formulas:

\[
\sum_{j=1}^{n+2} (-1)^{m_1(j)+m_2(j)+\cdots+m_N(j)} \hat{a}_j = 0, \tag{8a}
\]

\[
\sum_{j=1}^{n+2} (-1)^{m_1(j)+m_2(j)+\cdots+m_N(j)} X_1^{-2m_1(j)} X_2^{-2m_2(j)} \cdots X_N^{-2m_N(j)} \hat{a}_j = 0. \tag{8b}
\]

Let \( a_{ij} \) be the \((i, j)\) entry of \( A(U) \). For any vertex \( v_i \) with a positive crossing we have the following two formulas from Figure 11 where \((\delta_i(1), \ldots, \delta_i(N))\) is the multiple Alexander index in the region touching the vertex \( v_i \) such that the region is above the vertex:

\[
\sum_{j=1}^{n+2} (-1)^{m_1(j)+m_2(j)+\cdots+m_N(j)} a_{ij} = 0,
\]

\[
\sum_{j=1}^{n+2} (-1)^{m_1(j)+m_2(j)+\cdots+m_N(j)} X_1^{-2m_1(j)} X_2^{-2m_2(j)} \cdots X_N^{-2m_N(j)} a_{ij} = 0.
\]
On the other hand for any vertex $v_i$ with a negative crossing we have the following two formulas from Figure 12:

\[
\sum_{j=1}^{n+2} (-1)^{m_1(j)+m_2(j)+\cdots+m_N(j)} a_{ij} = 0,
\]

\[
\sum_{j=1}^{n+2} (-1)^{\sum_{I=1}^{N} m_I(j)} \prod_{I=1}^{N} X_j^{-2m_I(j)} \hat{a}_{ij} = 0.
\]

Therefore we get (8a) and (8b). From (8a) and (8b) we have the following formulas:

\[
\sum_{j=1}^{n+2} (-1)^{\sum_{I=1}^{N} m_I(j)-m_I(k)} \hat{a}_j = 0,
\]

and

\[
\sum_{j=1}^{n+2} (-1)^{\sum_{I=1}^{N} m_I(j)-m_I(k)} \prod_{I=1}^{N} X_j^{-2(m_I(j)-m_I(k))} \hat{a}_j = 0
\]

for any $k$. So letting $\beta'(x, y) = (-1)^{\sum_{I=1}^{N} m_I(x)-m_I(y)} \{1-\prod_{I=1}^{N} X_j^{-2(m_I(x)-m_I(y))}\}$, we have

\[
\sum_{j=1}^{n+2} \beta'(j, k) \hat{a}_j = 0.
\]
Then we can get the following formula since (11) is equal to (1):
\[
\frac{\beta'(i, n+2)}{\beta'(n+1, n+2)} \det A(L)(n+2, n+1) = (-1)^{i+j+1} \frac{\beta'(n+2, i)}{\beta'(j, i)} \det A(L)(i, j).
\]
Moreover we have
\[
\frac{\beta'(i, n+2)}{\beta'(n+2, i)} = \frac{1 - \prod_{j=1}^N X_j^{-2(m_j(i) - m_j(n+2))}}{1 - \prod_{j=1}^N X_j^{-2(m_j(n+2) - m_j(i))}} = - \frac{X_j^{-2m_1(i)} X_2^{-2m_2(i)} \cdots X_N^{-2m_N(i)}}{X_1^{-2m_1(n+2)} X_2^{-2m_2(n+2)} \cdots X_N^{-2m_N(n+2)}} = -X_j^{-2} \frac{X_1^{-2m_1(i)} X_2^{-2m_2(i)} \cdots X_N^{-2m_N(i)}}{X_1^{-2m_1(n+1)} X_2^{-2m_2(n+1)} \cdots X_N^{-2m_N(n+1)}},
\]
where the regions \(r_{n+1}\) and \(r_{n+2}\) are as shown in Figure 7 with the strand labeled as \(J\). So we have
\[
- X_j^{-2} \frac{\det A(L)(n+2, n+1)}{\beta'(n+1, n+2)X_1^{-2m_1(n+1)} X_2^{-2m_2(n+1)} \cdots X_N^{-2m_N(n+1)}} = (-1)^{i+j+1} \frac{\beta'(j, i)X_1^{-2m_1(i)} X_2^{-2m_2(i)} \cdots X_N^{-2m_N(i)}}{\beta'(n+2, i)X_1^{-2m_1(n+2)} X_2^{-2m_2(n+2)} \cdots X_N^{-2m_N(n+2)}}.
\]
Then we have \(\beta'(n+1, n+2) = -(1 - X_j^{-2})\) and if the regions \(r_i\) and \(r_j\) are as shown in Figure 8 with the strand labeled as \(M\), we have \(\beta'(j, i) = -(1 - X_M^{-2})\). Therefore we have
\[
\frac{\det A(L)(n+2, n+1)}{X_1^{-2m_1(n+1)} X_2^{-2m_2(n+1)} \cdots X_N^{-2m_N(n+1)} X_j^{-1}(X_j - X_j^{-1})} = (-1)^{i+j+1} \frac{\det A(L)(i, j)}{X_1^{-2m_1(i)} X_2^{-2m_2(i)} \cdots X_N^{-2m_N(i)} X_M^{-1}(X_M - X_M^{-1})}.
\]
From Proposition 2.10 we get the following formula:
\[
\frac{\langle L|\mathcal{P}(n+1, n+2)\rangle}{\langle \mathcal{P}(n+1, n+2)\rangle} = \frac{\langle L|\mathcal{P}(i, j)\rangle}{\langle \mathcal{P}(i, j)\rangle}.
\]
Similarly we can get the above formula if the regions \(r_i\) and \(r_j\) are as shown in Figure 8.
So the proof is complete. \(\square\)

**Lemma 3.4.** Put \(F(L) = \langle L|\mathcal{P}\rangle/\langle \mathcal{P}\rangle\). Then \(F(L)\) is an invariant under the Reidemeister moves II and III. Moreover \(F(L)\) satisfies the following formulas for the Reidemeister move I:
\[
X_k^{-1}F(L) = F(L') = F(L'') \quad \text{and} \quad X_kF(L) = F(\tilde{L}) = F(\tilde{L}),
\]
where \(L, L', L'', \tilde{L}, \tilde{L}\) differ only in one place as shown in Figure 13.

![Figure 13](image_url)
Proof. (i) Reidemeister move II:

Let \( L_2 \) and \( L'_2 \) be two link diagrams which differ only in one place as shown in Figure 14, where \( L_{2,K} \) denotes the component of \( L_2 \) labeled as \( K \). Let \( \mathcal{S} \) and \( \mathcal{S}' \) be the sets of states of the projections \( L_2 \) and \( L'_2 \) respectively with fixed stars as indicated in Figure 14. Then \( \mathcal{S} \) has three subsets as shown in Figure 14 and these subsets are denoted by \( S_0 \), \( S_1 \) and \( S_2 \) from left to right. On the other hand we have \( |\mathcal{S}| = |\mathcal{S}'| \) for any orientations since \( L_2 \) and \( L'_2 \) have the same diagrams in the exteriors.

\[
\begin{align*}
\text{Figure 14.} & \\
\quad & \\
& \begin{array}{c}
L_2 \\
L'_2 \\
S_0 \\
S_1 \\
S_2 \\
\end{array}
\end{align*}
\]

(i) Case where the orientations are as shown in Figure 15.

Figure 16 shows the variables and the signs around the vertices of \( L_2 \). Then we have \( \langle L_2|S_0 \rangle = \langle L'_2|S_0 \rangle \) since \( S_1 \) and \( S_2 \) have the same blank regions. Here a blank region means a region which has the marker outside the picture. In other words a blank region is a region which does not have markers or * in the figure. Moreover we have \( \langle L_2|S_0 \rangle = \langle L'_2|S' \rangle \) since \( S_0 \) and \( \mathcal{S}' \) have the same blank regions. Therefore we have \( \langle L_2|\mathcal{S} \rangle = \langle L_2|S_0 \rangle + \langle L_2|S_1 \rangle + \langle L_2|S_2 \rangle = \langle L'_2|\mathcal{S}' \rangle \). So we have \( F(L_2) = F(L'_2) \).

If the strand over \( L_{2,K} \) has the reversed orientation, \( L_2 \) has the same variables except the signs. In similar way we can get \( F(L_2) = F(L'_2) \) noting the signs.

\[
\begin{align*}
\text{Figure 15.} & \\
\quad & \\
& \begin{array}{c}
L_2 \\
L'_2 \\
X_K \\
X_K' \\
Y_K \\
Y_K' \\
\end{array}
\end{align*}
\]

(ii) Case where the orientations are as shown in Figure 17.

Figure 18 shows the variables and the signs around the vertices of \( L_2 \). As in (i) we can prove \( F(L_2) = F(L'_2) \).

(ii) Reidemeister move III:

Let \( L_3 \) and \( L'_3 \) be two link diagrams which differ only in one place as shown in Figure 19. Let \( \mathcal{S} \) and \( \mathcal{S}' \) be the sets of states of the projections \( L_3 \) and \( L'_3 \) respectively with fixed stars as indicated in Figure 20. Then we have \( |\mathcal{S}| = |\mathcal{S}'| \) for any orientations. On the other hand \( \mathcal{S} \) and \( \mathcal{S}' \) are in one-to-one correspondence for the blank regions as shown in Figure 21. Then we can prove \( F(L_3) = F(L'_3) \) by using the above technique for any orientation.
(iii) Reidemeister move I:

Let $L$, $L'$, $L''$, $\tilde{L}$ and $\tilde{\tilde{L}}$ be link diagrams which differ only in one place as shown in Figure 13. Let $\mathcal{S}$, $\mathcal{S}'$ and $\mathcal{S}''$ be the sets of states with fixed stars as indicated in Figure 22 of the projections $L$, $L'$ (or $L''$) and $\tilde{L}$ (or $\tilde{\tilde{L}}$). Then the markers are determined uniquely as indicated in Figure 22. So we have $|\mathcal{S}| = |\mathcal{S}'|$. Moreover the marker of $\mathcal{S}'$ indicates $X_K^{-1}$ for both of $L'$ and $L''$. Therefore we
have $X_K^{-1}F(L) = F(L') = F(L'')$. On the other hand the marker of $\mathcal{S}$ indicates $X_K$ for both of $\hat{L}$ and $\check{L}$. Therefore we have $X_K F(L) = F(L') = F(L'')$.

**Figure 22.**

This completes the proof of Lemma 3.4.

The potential function is an invariant under the Reidemeister moves II and III from Lemmas 3.3 and 3.4 and that the curvature is an invariant under these moves. Moreover the potential function is an invariant under the Reidemeister move I from Lemmas 3.3 and that the curvature counts how many times the sublink rotates counter-clockwise. So the proof of Theorem 3.2 is now complete properties.

4. Axioms for the Conway potential function

In Section 3 we defined the potential function for links with labeled strands. For a finite set $\Lambda$ let $\mu : \{1, 2, \ldots, N\} \to \Lambda$ be a surjection. A colored link $L = L_1 \cup L_2 \cup \cdots \cup L_N$ with colors in $\Lambda$ is a link where each component $L_K$ is given color $\mu(K)$. Putting $\mu_K = \mu(K)$, the potential function obtained from $\square_L$ by replacing $X_K$ with $X_{\mu_K}$ is an invariant for a colored link. In this section we will show some formulas for Kauffman’s potential function for colored links, and show that Kauffman’s potential function equals Hartley’s potential function.

Let $\nabla_L$ be the potential function defined by Hartley in [2]. J. Murakami showed that $\nabla_L$ can be calculated by using the following six axioms, where letters $\lambda$, $\mu$ and $\nu$ denote colors of the components.

(i) Let $L_+, L_-$ and $L_0$ be three links which differ only in one place as shown in Figure 23. Then the potential function $\nabla$ satisfies

$$\nabla_{L_+} - \nabla_{L_-} = (X_\mu - X_{\mu^{-1}})\nabla_{L_0}.$$  

(ii) Let $L_{++}, L_{--}$ and $L_{00}$ be three links which differ only in one place as shown in Figure 23. Then the potential function $\nabla$ satisfies

$$\nabla_{L_{++}} - \nabla_{L_{--}} = (X_{\mu}X_{\nu} + X_{\mu^{-1}}X_{\nu^{-1}})\nabla_{L_{00}}.$$  

(iii) Let $L_{2112}, L_{1221}, L_{1122}, L_{2211}, L_{111}, L_{22}$ and $L_{000}$ be seven links which differ only in one place as shown in Figure 23. Putting $g_+(x) = x + x^{-1}$ and $g_-(x) = x - x^{-1}$, the potential function $\nabla$ satisfies

$$g_+(X_\lambda)g_-(X_\mu)\nabla_{L_{2112}} - g_-(X_\mu)g_+(X_\nu)\nabla_{L_{1221}} = g_-(X_\mu^{-1}X_\nu)\nabla_{L_{1122}} + \nabla_{L_{2211}} + g_-(X_\mu^{-1}X_\muX_\nu)g_+(X_\nu)\nabla_{L_{11}}$$

$$- g_+(X_\lambda)g_-(X_\lambdaX_\muX_\nu^{-1})\nabla_{L_{22}} - g_-(X_\lambda^{-2}X_\nu^2)\nabla_{L_{000}} = 0.$$  

(iv) For a trivial knot $L$ with color $\mu$, $\nabla_L = \frac{1}{X_\mu - X_{\mu^{-1}}}.$

(v) Let $L_1$ and $L_2$ be two links which differ only in one place as shown in Figure 23. Then the potential function $\nabla$ satisfies

$$\nabla_{L_1} = (X_\mu - X_{\mu^{-1}})\nabla_{L_2}.$$
(vi) For a split link \( L \), \( \nabla L = 0 \).

\[
\begin{array}{cccccccc}
L_+ & L_- & L_0 & L_{++} & L_{--} & L_{00} \\
\end{array}
\]

**Figure 23.**

**Theorem 4.1 ([5, 6]).** The above axioms (i)–(vi) determine Hartley’s potential function.

We can show that Kauffman’s potential function equals Hartley’s potential function by using the above axioms.

**Theorem 4.2.** Kauffman’s potential function is equal to Hartley’s potential function.

**Proof.** From Theorem 4.1 Kauffman’s potential function equals Hartley’s potential function if \( \square \) satisfies the above six axioms. In [5] Kauffman showed the axioms except (iii). So we discuss the axiom (iii).

Let \( \mathcal{F}_{2112}, \mathcal{F}_{1221}, \mathcal{F}_{1122}, \mathcal{F}_{2211}, \mathcal{F}_{11}, \mathcal{F}_{22} \) and \( \mathcal{F}_{000} \) be the sets of states with fixed stars as indicated in Figure 24 of the projections \( L_{2112}, L_{1221}, L_{1122}, L_{2211}, L_{11}, L_{22} \) and \( L_{000} \) respectively. Then we have \(|\mathcal{F}_{2112}| = |\mathcal{F}_{1221}| = |\mathcal{F}_{1122}| = |\mathcal{F}_{2211}| = |\mathcal{F}_{11}| = |\mathcal{F}_{22}| = |\mathcal{F}_{000}|\).

\[
\begin{array}{cccccccc}
\ast & \ast & \ast & \ast & \ast & \ast & \ast & \ast \\
\end{array}
\]

**Figure 24.**

Moreover all the links have the same curvatures. So we will show that \( \langle L, \mathcal{F} \rangle \) satisfies the axiom (iii).

Now we can see that the number of the regions without stars is two more than the number of the vertices for each projections in Figure 24. In other words each states has two blank regions. We can classify \( \mathcal{F}_{2112}, \mathcal{F}_{1221}, \mathcal{F}_{1122}, \mathcal{F}_{2211}, \mathcal{F}_{11}, \mathcal{F}_{22} \) and \( \mathcal{F}_{000} \) as shown in Figure 25–30 where \( \circ \) means a blank region and a dotted line expediently divides the region into a region with a marker outside the picture and a region without markers.
Let $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, and $S_6$ be the sets of states corresponding to Figure 25, Figure 26, Figure 27, Figure 28, Figure 29, and Figure 30, respectively.

(i) For $S_1$, $S_2$, $S_3$, $S_4$, $S_5$, and $S_6$:

Figure 31 shows the variables and the signs around the vertices of $L_{2112}$, $L_{1221}$, $L_{1122}$, $L_{2211}$, $L_{11}$, $L_{22}$, and $L_{000}$.
Now the states of $\mathcal{P}_{\square}$ have the same blank regions. So we have the following formula:

$$
\frac{\langle L_{2112}|\mathcal{P}^1_{2112} \rangle}{X_\lambda X_\mu X_\nu} = \frac{\langle L_{1221}|\mathcal{P}^1_{1221} \rangle}{X_\lambda X_\mu X_\nu} = \frac{\langle L_{1122}|\mathcal{P}^1_{1122} \rangle}{X_\lambda X_\mu X_\nu} = \frac{\langle L_{2211}|\mathcal{P}^1_{2211} \rangle}{X_\lambda X_\mu X_\nu} = \frac{\langle L_{11}|\mathcal{P}^1_{11} \rangle}{X_\lambda X_\mu} = \frac{\langle L_{22}|\mathcal{P}^1_{22} \rangle}{X_\lambda X_\mu} = \langle L_{000}|\mathcal{P}^1_{000} \rangle.
$$

Therefore when we replace $\nabla_{L_{\square}}$ with $\langle L_{000}|\mathcal{P}^1_{000} \rangle$ in the left-hand side of (10), we can calculate the value as follows:

$$
g_+(X_\lambda)g_-(X_\mu)\langle L_{2112}|\mathcal{P}^1_{2112} \rangle - g_-(X_\mu)g_+(X_\nu)\langle L_{1221}|\mathcal{P}^1_{1221} \rangle
$$

$$= g_+(X_\lambda)g_-(X_\mu)(\langle L_{1122}|\mathcal{P}^1_{1122} \rangle + \langle L_{2211}|\mathcal{P}^1_{2211} \rangle)
+ g_-(X^{-1}\lambda X_\nu)g_+(X_\nu)\langle L_{11}|\mathcal{P}^1_{11} \rangle - g_+(X_\lambda)g_-(X_\lambda X_\mu X^{-1}_\mu)(\langle L_{22}|\mathcal{P}^1_{22} \rangle
- g_-(X^{-2}_\lambda X^{-2}_\mu)(\langle L_{000}|\mathcal{P}^1_{000} \rangle)
$$

$$=[g_+(X_\mu)g_-(X_\lambda)X_\lambda X_\mu X_\nu^2 - g_-(X_\mu)g_+(X_\nu)X_\lambda^2 X_\mu X_\nu]
- g_-(X^{-1}_\lambda X_\nu)(X_\lambda X^2_\mu X_\nu + X_\lambda X^2_\mu X_\nu) + g_-(X^{-1}_\lambda X_\mu X_\nu)g_+(X_\nu)X_\lambda X_\mu
- g_+(X_\lambda)g_-(X_\lambda X^{-1}_\mu X_\nu - g_-(X^{-2}_\lambda X^2_\mu)\langle L_{000}|\mathcal{P}^1_{000} \rangle
$$

$$=[X_\lambda X_\mu X_\nu(g_-(X_\mu)(g_+(X_\lambda)X_\nu - g_+(X_\nu)X_\lambda) - 2g_-(X^{-1}_\lambda X_\nu)X_\mu)
+ X_\mu(g_-(X^{-1}_\lambda X_\mu X_\nu)g_+(X_\nu)X_\lambda - g_+(X_\lambda)g_-(X_\lambda X^{-1}_\mu X_\nu)X_\nu)
- g_-(X^{-2}_\lambda X^2_\mu)\langle L_{000}|\mathcal{P}^1_{000} \rangle].
$$

Since the first term in the square bracket is

$$X_\lambda X_\mu X_\nu(g_-(X_\mu)(X^{-1}_\lambda X_\nu - X_\lambda X^{-1}_\nu) - 2g_-(X^{-1}_\lambda X_\nu)X_\mu)
$$

$$=X_\lambda X_\mu X_\nu((X_\mu - X^{-1}_\mu)g_-(X^{-1}_\lambda X_\nu) - 2g_-(X^{-1}_\lambda X_\nu)X_\mu)
$$

$$=g_-(X^{-1}_\lambda X_\nu)X_\lambda X_\mu X_\nu(-X^{-1}_\mu - X_\mu)
$$

$$= -g_-(X^{-1}_\lambda X_\nu)g_+(X_\mu)X_\lambda X_\mu X_\nu,
$$

Figure 31.
and the second term is
\[ X_\mu (X_\lambda^{-1} X_\mu X_\nu - X_\lambda X_\mu^{-1} X_\nu^{-1}) (X_\lambda X_\nu + X_\lambda X_\nu^{-1}) \]
\[ - (X_\lambda X_\mu X_\nu^{-1} - X_\lambda^{-1} X_\mu^{-1} X_\nu) (X_\lambda X_\nu + X_\lambda X_\nu^{-1}) \]
\[ = X_\mu (X_\lambda X_\nu X_\mu - X_\lambda^2 X_\mu^{-1} + X_\mu - X_\lambda X_\mu^{-1} X_\nu^{-2} \]
\[ - X_\lambda^2 X_\mu + X_\mu^{-1} X_\nu - X_\mu^2 + X_\lambda^{-2} X_\mu^{-1} X_\nu^{-2} \]
\[ = X_\mu X_\nu (X_\lambda X_\mu X_\nu - X_\lambda X_\mu^{-1} X_\nu^{-1} - X_\lambda X_\mu X_\nu^{-1} + X_\lambda X_\mu^{-1} X_\nu) \]
\[ - X_\lambda^2 X_\mu + X_\mu^{-1} X_\nu + X_\lambda^{-2} X_\mu^{-1} X_\nu^2 \]
\[ = X_\lambda X_\mu X_\nu (g_-(X_\lambda^{-1} X_\nu) X_\mu + g_-(X_\lambda X_\mu^{-1} X_\nu^{-1}) + g_-(X_\lambda^2 X_\mu^2) \]
\[ = g_-(X_\lambda^{-1} X_\nu) g_+(X_\mu) X_\lambda X_\mu X_\nu + g_-(X_\lambda^{-2} X_\mu^2) \]
\[ (\mathbf{10}) \]
vanishes.

(iii) For \( \mathcal{P}_{212}, \mathcal{P}_{121}, \mathcal{P}_{221}, \mathcal{P}_{112}, \mathcal{P}_{22} \) and \( \mathcal{P}_{000} \): From Figure 26 and 31 we have the following formulas:
\[ \langle L_{2212} | \mathcal{P}_{212} \rangle \]
\[ = \langle L_{2211} | \mathcal{P}_{221} \rangle \]
\[ = \langle L_{11} | \mathcal{P}_{11} \rangle \]
\[ = \langle L_{22} | \mathcal{P}_{22} \rangle \]
\[ = \langle L_{11} | \mathcal{P}_{11} \rangle \]
\[ = \langle L_{22} | \mathcal{P}_{22} \rangle \]
\[ = \langle L_{00} | \mathcal{P}_{00} \rangle \]
Therefore when we replace \( \nabla_{L_C} \) with \( \langle L_C | \mathcal{F}_2 \rangle \) in the left-hand side of \( \mathbf{10} \), we can show that the value equals zero.

(iv) For \( \mathcal{P}_{212}, \mathcal{P}_{121}, \mathcal{P}_{221}, \mathcal{P}_{112}, \mathcal{P}_{22} \) and \( \mathcal{P}_{000} \): From Figure 26 and 31 we have the following formulas:
\[ \langle L_{2212} | \mathcal{P}_{312} \rangle \]
\[ = \langle L_{2211} | \mathcal{P}_{321} \rangle \]
\[ = \langle L_{22} | \mathcal{P}_{22} \rangle \]
\[ = \langle L_{00} | \mathcal{P}_{00} \rangle \]
Therefore when we replace \( \nabla_{L_C} \) with \( \langle L_C | \mathcal{F}_2 \rangle \) in the left-hand side of \( \mathbf{10} \), we can show that the value equals zero.
(v) For $\mathcal{F}_{2112}^5$, $\mathcal{F}_{1221}^5$, $\mathcal{F}_{1122}^5$, $\mathcal{F}_{2211}^5$, $\mathcal{F}_{12}^5$, and $\mathcal{F}_{00}^5$. From Figure 29 and 31 we have the following formulas:

$$\langle L_{2112} | \mathcal{F}_{2112}^5 \rangle = \frac{-X^{-1}_\lambda X_{\mu}^{-1} + X_\lambda X_{\mu}^{-1}}{-X_{\mu}^{-1} X_{\nu}^{-1} + X_\lambda X_{\mu}^{-1} X_{\nu}^{-1}}$$

$$= \frac{\langle L_{2221} | \mathcal{F}_{2211}^5 \rangle}{X_{\lambda}^{-1} X_{\mu}^{-2} X_{\nu}^{-1} - X_{\lambda}^{-1} X_{\mu}^{-1} - X_{\lambda} X_{\mu}^{-2} X_{\nu}^{-1} + X_{\lambda} X_{\nu}^{-1}},$$

and

$$\langle L_{1122} | \mathcal{F}_{1122}^5 \rangle = \langle L_{11} | \mathcal{F}_{11}^5 \rangle = \langle L_{22} | \mathcal{F}_{22}^5 \rangle = \langle L_{00} | \mathcal{F}_{00}^5 \rangle = 0.$$

Therefore when we replace $\nabla_{\cal L}$ with $\langle L_{\cal L} | \mathcal{F}_{\cal L}^5 \rangle$ in the left-hand side of (10), we can show that the value equals zero.

(vi) For $\mathcal{F}_{2112}^6$, $\mathcal{F}_{1221}^6$, $\mathcal{F}_{1122}^6$, $\mathcal{F}_{2211}^6$, $\mathcal{F}_{12}^6$, and $\mathcal{F}_{00}^6$. From Figure 29 and 31 we have the following formulas:

$$\langle L_{2112} | \mathcal{F}_{2112}^6 \rangle = \frac{\langle L_{2221} | \mathcal{F}_{2211}^6 \rangle}{X_{\lambda}^{-1} X_{\mu}^{-1} X_{\nu}^{-2} - X_{\lambda} X_{\mu}^{-1} X_{\nu}^{-2}}$$

$$= \frac{\langle L_{1122} | \mathcal{F}_{1122}^6 \rangle}{X_{\lambda}^{-1} X_{\mu}^{-2} X_{\nu}^{-1} - X_{\lambda}^{-1} X_{\nu}^{-1} - X_{\lambda} X_{\mu}^{-2} X_{\nu}^{-1} + X_{\lambda} X_{\nu}^{-1}},$$

and

$$\langle L_{2211} | \mathcal{F}_{2211}^6 \rangle = \langle L_{11} | \mathcal{F}_{11}^6 \rangle = \langle L_{22} | \mathcal{F}_{22}^6 \rangle = \langle L_{00} | \mathcal{F}_{00}^6 \rangle = 0.$$

Therefore when we replace $\nabla_{\cal L}$ with $\langle L_{\cal L} | \mathcal{F}_{\cal L}^6 \rangle$ in the left-hand side of (10), we can show that the value equals zero.

Therefore we showed that $\langle L_{\cal L} | \mathcal{F} \rangle$ satisfies axiom (iii).

Hence the proof of Theorem 4.2 is now complete. □

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