THE CARDINALITY OF ORTHOGONAL EXPONENTIALS OF PLANAR SELF-AFFINE MEASURES WITH THREE-ELEMENT DIGIT SETS

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ABSTRACT. In this paper, we consider the planar self-affine measures $\mu_{M,D}$ generated by an expanding matrix $M \in M_2(\mathbb{Z})$ and an integer digit set $D = \{(0, 0), (\alpha_1, \alpha_2), (\beta_1, \beta_2)\}$ with $\alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$. We show that if $\det(M) \notin 3\mathbb{Z}$, then the mutually orthogonal exponential functions in $L^2(\mu_{M,D})$ is finite, and the exact maximal cardinality is given.

1. Introduction

Let $M \in M_n(\mathbb{R})$ be an expanding matrix (that is, all the eigenvalues of $M$ have moduli > 1), and let $D \subset \mathbb{R}^n$ be a finite subset with cardinality $|D|$. Let $\{\phi_d\}_{d \in D}$ be an iterated function system (IFS) on $\mathbb{R}^n$ defined by

$$\phi_d(x) = M^{-1}(x + d) \quad (x \in \mathbb{R}^n, \; d \in D).$$

Then the IFS arises a natural self-affine measure $\mu := \mu_{M,D}$ satisfying

$$\mu = \frac{1}{|D|} \sum_{d \in D} \mu \circ \phi_d^{-1}. \tag{1.1}$$

The measure $\mu_{M,D}$ is supported on the attractor of the IFS $\{\phi_d\}_{d \in D}$ [13].

For a countable subset $\Lambda \subset \mathbb{R}^n$, let $E_\Lambda = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$. We call $\mu$ a spectral measure, and $\Lambda$ a spectrum of $\mu$ if $E_\Lambda$ is an orthogonal basis for $L^2(\mu)$. We also say that $(\mu, \Lambda)$ is a spectral pair. The existence of a spectrum for $\mu$ is a basic problem in harmonic analysis, it was initiated by Fuglede in his seminal paper [11]. After the original work of Fuglede, the spectral problem has been investigated in a variety of different mathematical fields. The first example of a singular, non-atomic, spectral measure which is supported on $\frac{1}{4}$ Cantor set was given by Jorgensen and Pedersen in [15]. This surprising discovery received a lot of attention, and the spectrality of self-affine measures has become a hot topic. Many new spectral measures were found in [3]-[10], [14], [17]-[18], [21] and references therein. For more general cases such as Moran measures, the reader can refer to [11]-[2].

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On the other hand, the non-spectral problem of self-affine measures is also very interesting. In [8], Dutkay and Jorgensen showed that if $M = \begin{bmatrix} p & 0 \\ 0 & p \end{bmatrix}$ with $p \in \mathbb{Z} \setminus 3\mathbb{Z}$, $p \geq 2$ and $\mathcal{D} = \{(0,0)^t, (1,0)^t, (0,1)^t\}$, then there are no 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. Moreover, they proved that if $M = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then there exist at most 7 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$. In [16], Li proved that if the expanding integer matrix $M = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix}$ with $\det(M) \notin 3\mathbb{Z}$, then there exist at most 3 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 3 is the best. More recently, Liu, Dong and Li [20] extended the above upper triangular matrix to $M = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$ with $\det(M) \notin 3\mathbb{Z}$, and proved that there exist at most 9 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 9 is the best.

Let $D = \{(0,0)^t, (\alpha_1, \alpha_2)^t, (\beta_1, \beta_2)^t\}$, if we assume that $\alpha_1\beta_2 - \alpha_2\beta_1 = 1$, it can be easily seen that there exist at most 9 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$ by Theorem 1.1 of [20]. A natural question is whether the number 9 is suitable for any three-element integer digit set? Motivated by the previous research, we will give a complete answer in this paper. Without loss of generality, we may assume that $\gcd(\alpha_1, \alpha_2, \beta_1, \beta_2) = 1$ by Lemma 2.2.

**Theorem 1.1.** For an expanding matrix $M \in M_2(\mathbb{Z})$ with $\det(M) \notin 3\mathbb{Z}$ and an integer digit set $D = \{(0,0)^t, (\alpha_1, \alpha_2)^t, (\beta_1, \beta_2)^t\}$, let $\mu_{M,D}$ be defined by (1.1). The following hold.

(i) If $2\alpha_1 - \beta_1 \notin 3\mathbb{Z}$ or $2\alpha_2 - \beta_2 \notin 3\mathbb{Z}$, then there exist at most 9 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 9 is the best.

(ii) If $2\alpha_1 - \beta_1, 2\alpha_2 - \beta_2 \in 3\mathbb{Z}$, then there exist at most $3^{2\eta}$ mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number $3^{2\eta}$ is the best, where $\eta = \max\{\eta : 3^{2\eta} \geq |\{\alpha_1\beta_2 - \alpha_2\beta_1\}|\}$.

The case (ii) of Theorem 1.1 actually follows from a more general result. Before stating the result, we need some definitions and notations.

For positive integers $p, q \geq 2$ and $s \geq 1$, let

$$E^n_q := \frac{1}{q} \{ (l_1, l_2, \cdots, l_n)^t : 0 \leq l_1, \cdots, l_n \leq q - 1, l_i \in \mathbb{Z} \}, \quad \hat{E}^n_q := E^n_q \setminus \{0\} \quad (1.2)$$

and

$$\mathcal{A}_p(s) := \frac{1}{p^s} \{ (p^{s-1}, l)^t : 0 \leq l \leq p^s - 1, l \in \mathbb{Z} \}. \quad (1.3)$$

For a finite digit set $D \subset \mathbb{R}^n$, let

$$m_D(x) = \frac{1}{|D|} \sum_{d \in D} e^{2\pi i d(x)}, \quad x \in \mathbb{R}^n, \quad Z(m_D) := \{ x \in \mathbb{R}^n : m_D(x) = 0 \}. \quad (1.4)$$
where \( m_D(x) \) is called the mask polynomial of \( D \) as usual. Define
\[
\mathcal{Z}_D^n := \mathcal{Z}(m_D) \cap [0, 1)^n.
\] (1.5)

It is easy to see that \( m_D \) is a \( \mathbb{Z}^n \)-periodic function if \( D \subset \mathbb{Z}^n \). In this case, \( \mathcal{Z}(m_D) = \mathcal{Z}_D^n + \mathbb{Z}^n \).

**Theorem 1.2.** Assume integers \( p \geq 2, \tilde{\eta} \geq 1 \) and a finite digit set \( D \subset \mathbb{R}^2 \). Let \( M \in M_2(\mathbb{Z}) \) be an expanding matrix with \( \gcd(\det(M), p) = 1 \), and let \( \mu_{M,D}, \hat{\mathcal{E}}_{\hat{\eta}}^2, \mathcal{A}(\tilde{\eta}) \) of \( (m_D) \) be defined by (1.1), (1.2), (1.3) and (1.4), respectively. If \( \mathcal{Z}(m_D) \subset \hat{\mathcal{E}}_{\hat{\eta}}^2 + \mathbb{Z}^2 \), then there exist at most \( p^{2\eta} \) mutually orthogonal exponential functions in \( L^2(\mu_{M,D}) \). Moreover, if \( p \geq 3 \) is a prime and there exists \( \mathcal{N} \in \mathbb{Z} \setminus p\mathbb{Z} \) such that \( \mathcal{N}(\mathcal{A}(\tilde{\eta}) + \mathbb{Z}^2) \subset \mathcal{Z}(m_D) \), then the number \( p^{2\eta} \) is the best.

We arrange the paper as follows. In Section 2, we recall a few basic concepts and notations, establish several lemmas that will be needed in the proof of our main results. In Section 3, we give the detailed proofs of Theorems 1.1 and 1.2.

### 2. Preliminaries

In this section, we give some preliminary definitions and lemmas. We will start with an introduction to the Fourier transform. For a \( n \times n \) expanding real matrix \( M \) and a finite digit set \( D \subset \mathbb{R}^n \), let \( \mu_{M,D} \) be defined by (1.1). The Fourier transform \( \hat{\mu}_{M,D}(\xi) = \int e^{2\pi i (x, \xi)} d\mu_{M,D}(x) (\xi \in \mathbb{R}^n) \) of \( \mu_{M,D} \) plays an important role in the study of the spectrality of \( \mu_{M,D} \). It follows from [8] that
\[
\hat{\mu}_{M,D}(\xi) = \prod_{j=1}^{\infty} m_D(M^{-j} \xi), \quad \xi \in \mathbb{R}^n,
\] (2.1)

where \( M^* \) denotes the transposed conjugate of \( M \), and
\[
m_D(x) = \frac{1}{|D|} \sum_{d \in D} e^{2\pi i (d, x)}, \quad x \in \mathbb{R}^n.
\]

For any \( \lambda_1, \lambda_2 \in \mathbb{R}^n, \lambda_1 \neq \lambda_2 \), the orthogonality condition
\[
0 = \langle e^{2\pi i (\lambda_1, x)}, e^{2\pi i (\lambda_2, x)} \rangle_{L^2(\mu_{M,D})} = \int e^{2\pi i (\lambda_1 - \lambda_2, x)} d\mu_{M,D}(x) = \hat{\mu}_{M,D}(\lambda_1 - \lambda_2)
\]
relates to the zero set \( \mathcal{Z}(\hat{\mu}_{M,D}) \) directly. It is easy to see that for a countable subset \( \Lambda \subset \mathbb{R}^n \), \( \mathcal{E}_\Lambda = \{ e^{2\pi i (\lambda, x)} : \lambda \in \Lambda \} \) is an orthonormal family of \( L^2(\mu_{M,D}) \) if and only if
\[
(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\hat{\mu}_{M,D}).
\] (2.2)

From (2.1), we have \( \mathcal{Z}(\hat{\mu}_{M,D}) = \{ \xi \in \mathbb{R}^n : \exists j \in \mathbb{N} \text{ such that } m_D(M^{-j} \xi) = 0 \} \). Hence
\[
\mathcal{Z}(\hat{\mu}_{M,D}) = \bigcup_{j=1}^{\infty} M^{-j}(\mathcal{Z}(m_D)),
\] (2.3)
where $\mathcal{Z}(m_D) = \{x \in \mathbb{R}^n : m_D(x) = 0\}$.

**Definition 2.1.** Let $\mu$ be a Borel probability measure with compact support on $\mathbb{R}^n$. Let $\Lambda$ be a finite or a countable subset of $\mathbb{R}^n$, and let $\mathcal{E}_\Lambda = \{e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda\}$. We denote $\mathcal{E}_\Lambda$ by $E'_\Lambda$ if $E'_\Lambda$ is a maximal orthogonal set of exponential functions in $L^2(\mu)$. Let

$$n''(\mu) := \sup \{ |\Lambda| : E'_\Lambda \text{ is a maximal orthogonal set} \},$$

and call $n''(\mu)$ the maximal cardinality of the orthogonal exponential functions in $L^2(\mu, M, D)$.  

The following lemma indicates that the spectral properties of $\mu_{M, D}$ are invariant under a linear transform. The proof is the same as that of Lemma 4.1 of [8].

**Lemma 2.2.** Let $D, \tilde{D} \subset \mathbb{R}^n$ be two finite digit sets with the same cardinality, and $M, \tilde{M} \in M_n(\mathbb{R})$ be two expanding matrices. If there exists a matrix $Q \in M_n(\mathbb{R})$ such that $\tilde{D} = QD$ and $\tilde{M} = QMQ^{-1}$, then

(i) a set $\Lambda \subset \mathbb{R}^n$ is an orthogonal set for $\mu_{M, D}$ if and only if $\tilde{\Lambda} := Q^{-1}\Lambda$ is an orthogonal set for $\mu_{\tilde{M}, \tilde{D}}$. In particular, $n'(\mu_{M, D}) = n'(\mu_{\tilde{M}, \tilde{D}})$;

(ii) the $\mu_{M, D}$ is a spectral measure with spectrum $\Lambda$ if and only if the $\mu_{\tilde{M}, \tilde{D}}$ is a spectral measure with spectrum $Q^{-1}\Lambda$.

For any three-element digit set $D = \{(0, 0)', (\alpha_1, \alpha_2)', (\beta_1, \beta_2)'\}$ and an expanding matrix $M$ with $\gcd(\det(M), 3) = 1$, if there exists an invertible matrix $Q$ such that $\tilde{D} = QD = \{(0, 0)', (1, 0)', (0, 1)\}$ and $\tilde{M} = QMQ^{-1}$ is an expanding integer matrix, then the following lemma can be used to judge the maximum number of the orthogonal exponential functions in $L^2(\mu_{M, D})$ by lemma 2.2.

**Lemma 2.3.** [20 Corollary 4.1] For an expanding matrix $\tilde{M} \in M_2(\mathbb{Z})$ and a digit set $D = \{(0, 0)', (1, 0)', (0, 1)\}$, let $\mu_{\tilde{M}, D}$ be defined by (1.1). If $\det(\tilde{M}) \not\in 3\mathbb{Z}$, then there exist at most 9 mutually orthogonal exponential functions in $L^2(\mu_{\tilde{M}, D})$, and the number 9 is the best.

In [8, Theorem 3.1], Dutkay and Jorgensen established a criterion for the non-spectrality of self-affine measures $\mu_{M, D}$, which require that the elements of matrix $M$ and digit set $D$ are all integers. The following lemma is a little generalization and can be proved similarly.

**Lemma 2.4.** For a $n \times n$ expanding integer matrix $M$ and a finite digit set $D \subset \mathbb{R}^n$, let $\mu_{M, D}$, $\mathcal{Z}(m_D)$ be defined by (1.1) and (1.4), respectively. If there exists a set $0 \neq Z' \subset \{0, 1\}^n$ with finite cardinality $|Z'|$, which does not contain 0, such that $\mathcal{Z}(m_D) \subset Z' + \mathbb{Z}^n$ and

$$M''(Z' + \mathbb{Z}^n) \subset Z' + \mathbb{Z}^n,$$

then there exist at most $|Z'| + 1$ mutually orthogonal exponential functions in $L^2(\mu_{M, D})$. In particular, $\mu_{M, D}$ is not a spectral measure.
Lemma 2.5. We have $M(20, \text{Proposition } 2.2)$.

Lemma 2.5. implies that $M$.

Euler's theorem information about the Euler's phi function, the reader can refer to [23]. The following Remark 2.6.

Remark 2.6. Since $\lambda \in \mathbb{Z}$, the principle. But this will contradict (2.5), because $0 \neq \text{element}$.

By (2.3), there exists an integer $k \geq 1$ such that $M^{\ast k}(\lambda_1 - \lambda_2) \in \mathbb{Z}(m_D) \subset \mathbb{Z} + \mathbb{Z}^n$. By the hypothesis, we get $\lambda_1 - \lambda_2 \in M^{\ast k}(\mathbb{Z}(m_D)) \subset M^{\ast k}(\mathbb{Z}' + \mathbb{Z}^n) \subset \mathbb{Z}' + \mathbb{Z}^n$, and hence

$$\Lambda \setminus \{0\} \subset (\Lambda - \Lambda) \setminus \{0\} \subset \mathbb{Z}' + \mathbb{Z}^n.$$  

(2.5)

Since $|\Lambda| > |\mathbb{Z}'| + 1$, there exist $\lambda_1 \neq \lambda_2 \in \Lambda$ such that $\lambda_1 - \lambda_2 \in \mathbb{Z}^n$ by using pigeonhole principle. But this will contradict (2.5), because $0 \notin \mathbb{Z}'$. □

Lemma 2.5. [20] Proposition 2.2] For an integer $q \geq 2$, let $\hat{E}^2_q$ be defined by (1.2) and $M \in M_2(\mathbb{Z})$ with $\gcd(\det(M), q) = 1$. Then $M(\hat{E}^2_q) = q\hat{E}^2_q \pmod{q\mathbb{Z}^2}$, equivalently, $M(\hat{E}^2_q) = \hat{E}^2_q \pmod{\mathbb{Z}^2}$.

Remark 2.6. Let $p, s \geq 2$ be integers and $M \in M_2(\mathbb{Z})$ with $\gcd(\det(M), p) = 1$. By Lemma 2.5, we have $M(\hat{E}^2_{p^s}) = \hat{E}^2_{p^s} \pmod{\mathbb{Z}^2}$ and $M(\hat{E}^2_{p^{s-1}}) = \hat{E}^2_{p^{s-1}} \pmod{\mathbb{Z}^2}$. This implies that $M(\hat{E}^2_{p^s} \setminus \hat{E}^2_{p^{s-1}}) = \hat{E}^2_{p^s} \setminus \hat{E}^2_{p^{s-1}} \pmod{\mathbb{Z}^2}$. Therefore, for any $\xi = \frac{1}{p}(l_1, l_2) \in \hat{E}^2_p$, with $p \nmid \gcd(l_1, l_2)$, $M\xi = \frac{1}{p}(l'_1, l'_2) \hat{E}^2_p$ must satisfy $p \nmid \gcd(l'_1, l'_2)$.

For a positive number $m$, let $\varphi(m)$ denote the Euler's phi function which equal to the number of integers in the set $\{1, 2, \cdots, m-1\}$ that are relatively prime to $m$. For more information about the Euler’s phi function, the reader can refer to [23]. The following lemma is the famous Euler’s theorem.

Lemma 2.7. [23] Theorem 2.12] Let $m$ be a positive integer, and let $N$ be an integer relatively prime to $m$. Then $N^{\varphi(m)} = 1 \pmod{m}$.

For a prime $p$, let $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ denote the residue class fields and $\mathbb{F}_p^n$ denote the vector space of dimension $n$ over $\mathbb{F}_p$. All nonsingular $n \times n$ matrices over $\mathbb{F}_p$ form a finite group under matrix multiplication, called the general linear group $GL(n, \mathbb{F}_p)$.

Definition 2.8. Let $f(x) \in \mathbb{F}_p[x]$ be a nonzero polynomial. If $f(0) \neq 0$, then the least positive integer $n$ for which $f(x)$ divides $x^n - 1$ is called the order of $f$ and denoted by $\text{ord}_p(f)$.

The order of the polynomial $f$ is sometimes also called the period of $f$ or the exponent of $f$. There are many conclusions about the order of polynomial in the third chapter of [19].

Definition 2.9. Let $M \in GL(n, \mathbb{F}_p)$, then the least positive integer $e$ for which $M^e = I$ is called the order of $M$ and denoted by $O_p(M)$, where $I$ is the identity matrix in $GL(n, \mathbb{F}_p)$.

The following lemma reflects the relationship between the order of the matrix $M$ and the order of the characteristic polynomial of $M$. 

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**Lemma 2.10.** [12, Proposition 3.1] Let $M \in GL(n, \mathbb{F}_p)$ and let $f(x) \in \mathbb{F}_p[x]$ be the minimal polynomial of $M$ and $\chi(x) \in \mathbb{F}_p[x]$ be the characteristic polynomial of $M$. Then $f(0) \neq 0$ and $O_p(M) = \text{ord}_p(f)$. In particular, $O_p(M) \leq p^n - 1$, and moreover, if the equality holds, then $f(x) = \chi(x)$. Also, we have: $O_p(M) = p^n - 1 \iff f(x)$ is primitive of degree $n \iff \chi(x)$ is primitive.

It is well known that there exist $\varphi(p^n - 1)/n$ primitive polynomials with degree $n$ over $\mathbb{F}_p$ (see $P_87$ Theorem 4.1.3 of [22]), where $\varphi$ is the Euler’s phi function. Consequently, Lemma 2.10 implies that the matrix $M \in GL(n, \mathbb{F}_p)$ with $O_p(M) = p^n - 1$ always exists.

**Definition 2.11.** We call $M \in GL(n, \mathbb{F}_p)$ an ergodic matrix if 

$\{Mv, M^2v, \cdots, M^{p^n-1}v\} = p\hat{E}_p^n \pmod{p\mathbb{Z}^n}$ for any $v \in \mathbb{F}_p^n \setminus \{0\}$. 

The ergodic matrices have been widely used in Cryptography. The following lemma shows that the ergodic matrices attain to the maximum order of matrices in $GL(n, \mathbb{F}_p)$.

**Lemma 2.12.** [24, Lemma 3.3] A matrix $M \in GL(n, \mathbb{F}_p)$ is ergodic if and only if $O_p(M) = p^n - 1$.

### 3. Main Results

In this section, we first prove Theorem 1.2 and then prove Theorem 1.1 by using the result of Theorem 1.2. In the proof of Theorem 1.2, the “at most” is easy to get by Lemma 2.4; the main difficulty is to show that the number $p^{2h}$ is the best. In order to get this, we will prove that there exists an expanding integer matrix $M$ with $\text{gcd}(\text{det}(M), p) = 1$ such that 

$\bigcup_{j=1}^{p^{s-1}p^{h-1}} M^{s-3} = \hat{E}_p^2$, where $\hat{E}_p^2, M^{s-3}$ are defined by (1.2) and (1.3), respectively. For simplicity, in the later of this paper, we let $I \in M_2(\mathbb{Z})$ denote the identity matrix.

**Theorem 3.1.** For a prime $p \geq 3$ and an integer $s \geq 1$, let $B = p\begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} + I$ be an integer matrix with $l_{i_0} \notin p\mathbb{Z}$ for some $1 \leq i_0 \leq 4$. Suppose $e$ is the least integer such that $B^e = p\begin{bmatrix} \tilde{l}_1 & \tilde{l}_2 \\ \tilde{l}_3 & \tilde{l}_4 \end{bmatrix} + I$ satisfies $\tilde{l}_{i_0} \in p^{s-1}\mathbb{Z}$, then $e = p^{s-1}$. Meanwhile, $B^{p^{s-1}} = p^s\begin{bmatrix} \tilde{l}_1 & \tilde{l}_2 \\ \tilde{l}_3 & \tilde{l}_4 \end{bmatrix} + I$ with $\tilde{l}_i = l_i \pmod{p}$ for all $1 \leq i \leq 4$.

**Proof.** For any integer $m \geq 2$, write $B = pA + I$, we have 

$B^m = (pA + I)^m = p^2\tilde{A}_m + mpA + I$, 

where 

$\tilde{A}_m = p^{m-2}A^m + C_1 p^{m-3}A^{m-1} + \cdots + C_{p-3} pA^3 + C_{p-2} A^2$. 

Now we prove the theorem by induction. Without loss of generality, we assume $l_2 \notin p\mathbb{Z}$. 


It is easy to see that the theorem holds for \( s = 1 \). We then consider \( s = 2 \). Since \( p \) is a prime and \( l_2 \not\in p\mathbb{Z} \), we infer from (3.1) that \( m = p \) is the least integer such that 
\[ B^m = p \begin{bmatrix} \tilde{l}_1^{(2)} & \tilde{l}_2^{(2)} \\ \tilde{l}_3^{(2)} & \tilde{l}_4^{(2)} \end{bmatrix} + I \] satisfies \( \tilde{l}_2^{(2)} \in p\mathbb{Z} \). Obviously, 
\[ B^p = p^2 \tilde{A}_p + p^2 A + I = p^2 \begin{bmatrix} \tilde{l}_1^{(2)} & \tilde{l}_2^{(2)} \\ \tilde{l}_3^{(2)} & \tilde{l}_4^{(2)} \end{bmatrix} + I. \]

In the following, we prove \( \tilde{l}_i^{(2)} = l_i \pmod{p} \) (1 \( \leq i \leq 4 \)). Note that \( C_p^{p-2} = C_p^2 = \frac{p(p-1)}{2} \in p\mathbb{Z} \) for any prime \( p \geq 3 \), we conclude from (3.2) that there exists an integer matrix \( \tilde{B}_p \) such that \( \tilde{A}_p = p\tilde{B}_p \). Hence \( B^p = p^2(p\tilde{B}_p + A) + I \), and therefore \( \tilde{l}_i^{(2)} = l_i \pmod{p} \) (1 \( \leq i \leq 4 \)). We then consider 
\[ \tilde{l}_1^{(2)} = l_1 \pmod{p} \] (1 \( \leq i \leq 4 \)).

Inductively, we assume that the theorem holds for \( s = k \). That is, \( e = p^{k-1} \) is the least integer such that 
\[ B^e = p \begin{bmatrix} \tilde{l}_1^{(k)} & \tilde{l}_2^{(k)} \\ \tilde{l}_3^{(k)} & \tilde{l}_4^{(k)} \end{bmatrix} + I \] satisfies \( \tilde{l}_2^{(k)} \in p^{k-1}\mathbb{Z} \), moreover, 
\[ B^{p^{k-1}} = p^k \begin{bmatrix} \tilde{l}_1^{(k)} & \tilde{l}_2^{(k)} \\ \tilde{l}_3^{(k)} & \tilde{l}_4^{(k)} \end{bmatrix} + I := p^k A_k + I \]
with \( \tilde{l}_i^{(k)} = l_i \pmod{p} \) (1 \( \leq i \leq 4 \)).

For \( s = k + 1 \), by inductive hypothesis and the same discussion as \( s = 2 \), we can easily show that 
\[ B^{p^k} = (B^{p^{k-1}})^p = (p^k A_k + I)^p = p^{k+1} \begin{bmatrix} \tilde{l}_1^{(k+1)} & \tilde{l}_2^{(k+1)} \\ \tilde{l}_3^{(k+1)} & \tilde{l}_4^{(k+1)} \end{bmatrix} + I \]
with \( \tilde{l}_i^{(k+1)} = \tilde{l}_i^{(k)} \) (1 \( \leq i \leq 4 \)). We now prove that \( e = p^k \) is the least integer such that 
\[ B^e = p \begin{bmatrix} \tilde{l}_1^{(k+1)} & \tilde{l}_2^{(k+1)} \\ \tilde{l}_3^{(k+1)} & \tilde{l}_4^{(k+1)} \end{bmatrix} + I \] satisfies \( \tilde{l}_2^{(k+1)} \in p^k\mathbb{Z} \). Suppose that \( n < p^k \) is the least integer which satisfies the above. By the assumption \( \tilde{l}_2^{(k)} \not\in p\mathbb{Z} \) for \( s = k \), we have \( n > p^{k-1} \) and rewrite \( n = \tau p^{k-1} + r \), where \( 1 < \tau < p \) and \( 0 \leq r < p^{k-1} \). It is easy to see that there exist integers \( \tilde{l}_1, \tilde{l}_2, \tilde{l}_3 \) and \( \tilde{l}_4 \) such that 
\[ B^{\tau p^{k-1}} = \left(p^\tau A_k + I\right)^\tau = p^{\tau^2} A_k^\tau + \cdots + C^2 \tau^{2k} A_k^2 + C^1 \tau p^{k} A_k + I = p^k \begin{bmatrix} \tilde{l}_1 & \tilde{l}_2 \\ \tilde{l}_3 & \tilde{l}_4 \end{bmatrix} + I \]
with \( \tilde{l}_i = \tau \tilde{l}_i^{(k)} \pmod{p} \) (1 \( \leq i \leq 4 \)). Especially \( \tilde{l}_2 \not\in p\mathbb{Z} \), because \( \tilde{l}_2^{(k)} \not\in p\mathbb{Z} \) and \( 1 < \tau < p \). By using (3.1), we can denote 
\[ B^r = p \begin{bmatrix} \tilde{m}_1 & \tilde{m}_2 \\ \tilde{m}_3 & \tilde{m}_4 \end{bmatrix} + I \] for some integers \( \tilde{m}_1, \tilde{m}_2, \tilde{m}_3 \) and \( \tilde{m}_4 \).
Hence

\[ B^n = B^{p^{k-1} + r} = p^{k+1}(\bar{m}_1\bar{l}_1 + \bar{m}_3\bar{l}_2) + p^k\bar{l}_1 + p\bar{m}_1 + 1 \]

where the above equation implies \( \bar{m}_2 \in p^{k-1}\mathbb{Z} \) since \( \bar{l}_2^{k+1} \in p^k\mathbb{Z} \). So \( r \geq p^{k-1} \) by the inductive hypothesis for \( s = k \), this contradicts with \( r < p^{k-1} \). Therefore, the theorem holds for \( s = k + 1 \). This completes the proof of Theorem 3.1. \( \square \)

**Remark 3.2.** Theorem 3.1 may not hold if \( p = 2 \). For example, let \( B = 2\left[ \begin{array}{ll} 1 & 1 \\ 1 & 0 \end{array} \right] + I = 2\left[ \begin{array}{ll} 1 & 1 \\ 1 & 0 \end{array} \right] + I \), then \( l_2 \notin 2\mathbb{Z} \) and \( B^2 = 2\left[ \begin{array}{ll} 2 \cdot 3 & 2^2 \cdot 1 \\ 2^2 \cdot 1 & 2 \cdot 1 \end{array} \right] + I \). This shows that \( k = 2^{3-1} \) is not the least integer such that \( B^k = 2\left[ \begin{array}{ll} 1 & 1 \\ 1 & 0 \end{array} \right] + I \) with \( \bar{l}_2 \in 2^{3-1}\mathbb{Z} \). However, we can add some restriction on \( l_i \), such as \( l_2 \notin 2\mathbb{Z} \) and \( (l_1 + 1_4) \in 2\mathbb{Z} \), and get a similar result as Theorem 3.1 for \( p = 2 \). We do not prove it, because it will not be used in the proof of Theorem 1.1 and can be similarly proved as we did in the cases \( p \geq 3 \).

**Theorem 3.3.** For a prime \( p \geq 3 \) and an integer \( s \geq 1 \), let \( M \in \text{GL}(2, \mathbb{F}_p) \). If \( O_p(M) = \iota \) and \( M^s = pA + I \), let \( k \) be the least integer such that \( M^k = I (\mod M_2(p^s\mathbb{Z})) \), then \( k \leq \iota p^{s-1} \), i.e., \( O_p(M) \leq \iota p^{s-1} \). Furthermore, if there exist \( l_i \notin p\mathbb{Z} \) for some \( 1 \leq i_0 \leq 4 \), then \( k = \iota p^{s-1} \).

**Proof.** Write \( M^r = pA + I \), it follows from Theorem 3.1 that \( M^{\iota p^{s-1}} = (pA + I)^{p^{s-1}} = I (\mod M_2(p^{s}\mathbb{Z})) \). Thus \( k \leq \iota p^{s-1} \). In particular, we claim that there exists an integer \( k' \) such that \( k = \iota k' \). If otherwise, \( k = \iota k + r \) for some integers \( \iota \) and \( 1 \leq r \leq \iota - 1 \). Note that \( O_p(M) = \iota \), we have \( M^{k'} = I (\mod M_2(p^{s}\mathbb{Z})) \). Consequently, \( I = M^k = M^{k+r} = IM^r = M^r (\mod M_2(p^{s}\mathbb{Z})) \), which contradicts with \( O_p(M) = \iota > r \). Hence \( k = \iota k' \).

Next, we prove \( k = \iota p^{s-1} \) if there exist \( l_i \notin p\mathbb{Z} \) for some \( 1 \leq i_0 \leq 4 \). Due to \( k = \iota k' = O_p(M) \), then there exist integers \( l'_1, l'_2, l'_3 \) and \( l'_4 \) such that

\[ M^k = M^{\iota k'} = \left( p\left[ \begin{array}{ll} l_1 & l_2 \\ l_3 & l_4 \end{array} \right] + I \right)^{k'} = p^s\left[ \begin{array}{ll} l'_1 & l'_2 \\ l'_3 & l'_4 \end{array} \right] + I. \] (3.3)

Since \( l_i \notin p\mathbb{Z} \), we deduce from (3.3) and Theorem 3.1 that \( k' \geq p^{s-1} \). Hence \( k \geq \iota p^{s-1} \), together with \( k \leq \iota p^{s-1} \), shows that \( k = \iota p^{s-1} \). \( \square \)
Let $M \in \text{GL}(2, \mathbb{F}_p)$ with $O_p(M) = \iota$ and let $\xi \in \tilde{E}_{p^s}^2$. According to Theorem 3.3, we have $\bigcup_{j=1}^{\infty} M^l \xi = \bigcup_{j=1}^{\nu} M^l \xi \pmod{\mathbb{Z}^2}$. Assume integers $p \geq 2, s \geq 1$, let

$$T_{p,s} = \{l : 0 \leq l \leq p^s - 1, l \in \mathbb{Z} \setminus p\mathbb{Z}\}. \quad (3.4)$$

For any $l \in T_{p,s}$, define

$$B_{p,s}(l) = \bigcup_{j=1}^{\nu} M^l \left( \frac{p^{j-1}}{l} \right) \pmod{\mathbb{Z}^2} \quad (3.5)$$

and

$$Q_{p,s}(l) = \left\{ l' : B_{p,s}(l) = B_{p,s}(l'), l' \in T_{p,s} \right\}. \quad (3.6)$$

It is clear that if $l' \in Q_{p,s}(l)$, then $Q_{p,s}(l') = Q_{p,s}(l)$. Hence there exists a nonnegative integer $m_{p,s}$ such that the set $T_{p,s}$ can be divided into disjoint union $T_{p,s} = \bigcup_{j=0}^{m_{p,s}} Q_{p,s}(l_i)$. Note that $O_{p^s}(M) \leq sp^{s-1}$, we can easily show that for any $l_1, l_2 \in T_{p,s}$, either $B_{p,s}(l_1) = B_{p,s}(l_2)$ or $B_{p,s}(l_1) \cap B_{p,s}(l_2) = \emptyset$.

Let $|E|$ denote the cardinality of set $E$, the following theorem describes the properties of $B_{p,s}(l)$ and $Q_{p,s}(l)$ for $l \in T_{p,s}$.

**Theorem 3.4.** For a prime $p \geq 3$ and an integer $s \geq 1$, let $\tilde{E}_{p^s}^2, A_p(s), T_{p,s}, B_{p,s}, Q_{p,s}$ be defined by (1.2), (1.3), (5.4), (5.5) and (3.6), respectively. If $O_{p^s}(M) = p^2 - 1$ and $M^{p^s-1} = p \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} + I$ with $l_2 \notin p\mathbb{Z}$, then

(i) $B_{p,s}(l)$ has $(p^2 - 1)p^{s-1}$ different elements for any $l \in T_{p,s}$, i.e., $|B_{p,s}(l)| = (p^2 - 1)p^{s-1}$;

(ii) $|Q_{p,s}(l)| = p - 1$ for any $l \in T_{p,s}$;

(iii) For any integer $\tilde{\eta} \geq 1$,

$$\bigcup_{j=1}^{(p^2 - 1)p^{s-1}} M^j A_p(\tilde{\eta}) = \bigcup_{s=1}^{\tilde{\eta}} \bigcup_{l \in T_{p,s}} B_{p,s}(l) = \tilde{E}_{p^s}^2 \pmod{\mathbb{Z}^2}.$$  

**Proof.** We first prove the following claim.

**Claim 1.** For any $l, l' \in T_{p,s}$ with $l \equiv l' \pmod{p}$, if there exists an integer $1 \leq k \leq (p^2 - 1)p^{s-1}$ such that

$$M^k \frac{1}{p^s} \left( \frac{p^{j-1}}{l} \right) = \frac{1}{p^s} \left( \frac{p^{j-1}}{l'} \right) \pmod{\mathbb{Z}^2}, \quad (3.7)$$

then $k = (p^2 - 1)p^{s-1}$ and $l = l'$.

**Proof of Claim 1.** Since $\nu := (p^s - 1, l') : (p^s - 1, l')' := \nu' \pmod{p\mathbb{Z}^2}$ and $O_{p}(M) = p^2 - 1$, it can be easily proved that $k = (p^2 - 1)k'$ for some integer $1 \leq k' \leq p^{s-1}$. Otherwise $k = (p^2 - 1)k'' + r$ for some integers $k''$ and $1 \leq r < p^2 - 1$. It follows from $O_{p}(M) = p^2 - 1$.
and Lemma 2.12 that $M$ is an ergodic matrix and $M^k v = M^{(p^2 - 1)k + r} v = M^r v = v \pmod{p\mathbb{Z}^2}$. Therefore, $M^{k+1} v = M v \pmod{p\mathbb{Z}^2}$ and

$$\| [M v, \ldots, M^r v, M^{r+1} v, \ldots, M^{r-1} v] \| < p^2 - 1,$$

which contradicts the ergodicity of $M$. Hence $k = (p^2 - 1)k'$.

Next we prove $k' = p^{s-1}$. Denote $M^{p-1} = p A + l$, by (3.1) and (3.7), there exist integers $l_1, l_2, l_3$ and $l_4$ such that

$$M^k \begin{pmatrix} p^{s-1} \\ l \end{pmatrix} = (pA + l) \begin{pmatrix} p^{s-1} \\ l \end{pmatrix} = \begin{pmatrix} p\tilde{l}_1 + p\tilde{l}_2 \\ p\tilde{l}_3 + p\tilde{l}_4 + 1 \end{pmatrix} \begin{pmatrix} p^{s-1} \\ l \end{pmatrix} = 0 \pmod{p^2 \mathbb{Z}^2}.$$

This together with $l \notin p\mathbb{Z}$ yields $l_2 = p^{s-1} m$ for some integer $m$. As $l_2 \notin p\mathbb{Z}$, Theorem 3.1 implies $k' \geq p^{s-1}$. Combining $k' \leq p^{s-1}$ shows that $k' = p^{s-1}$, hence $k = (p^2 - 1)p^{s-1}$. According to $l_2 \notin p\mathbb{Z}$ and Theorem 3.3, we have $O_{p^r}(M) = (p^2 - 1)p^{s-1}$, i.e., $M^{(p^2 - 1)p^{s-1}} = I \pmod{M_2(p^2 \mathbb{Z})}$. Hence $l = l'$, which completes the proof of the claim.

We now continue with the proof of Theorem 3.4.

(i) Suppose $|B_{p,s}(l)| < (p^2 - 1)p^{s-1}$, then there exist $1 \leq k_1 < k_2 \leq (p^2 - 1)p^{s-1}$ such that $M^{k_1} \begin{pmatrix} p^{s-1} \\ l \end{pmatrix} = M^{k_2} \begin{pmatrix} p^{s-1} \\ l \end{pmatrix} \pmod{\mathbb{Z}^2}$. Note that $O_{p^r}(M) = (p^2 - 1)p^{s-1}$, multiplying $M(p^2 - 1)p^{s-1} - k_2$ on both sides of the above equation, we get

$$M^{k_1 + (p^2 - 1)p^{s-1} - k_2} \begin{pmatrix} p^{s-1} \\ l \end{pmatrix} = M^{k_2 + (p^2 - 1)p^{s-1} - k_2} \begin{pmatrix} p^{s-1} \\ l \end{pmatrix} \pmod{\mathbb{Z}^2}.$$ (3.8)

However, Claim 1 shows that (3.8) does not hold because $k_1 + (p^2 - 1)p^{s-1} - k_2 < (p^2 - 1)p^{s-1}$. Hence $B_{p,s}(l)$ has $(p^2 - 1)p^{s-1}$ different elements for any $l \notin T_{p,s}$.

(ii) We first prove $|Q_{p,s}(l)| \leq p - 1$. Assume that there exists $l \in T_{p,s}$ such that $|Q_{p,s}(l)| \geq p$, then there exist $l', l'' \in Q_{p,s}(l)$ satisfies $l' \neq l''$ and $l' = l'' \pmod{p}$. Combining $O_{p^r}(M) = (p^2 - 1)p^{s-1}$ and $B_{p,s}(l') = B_{p,s}(l'')$, we deduce that there exists a positive integer $k < (p^2 - 1)p^{s-1}$ such that

$$M^k \begin{pmatrix} p^{s-1} \\ l' \end{pmatrix} = M^{(p^2 - 1)p^{s-1} - l''} \begin{pmatrix} p^{s-1} \\ l'' \end{pmatrix} \pmod{\mathbb{Z}^2},$$

which contradicts with Claim 1. Hence $|Q_{p,s}(l)| \leq p - 1$.

We now prove $|Q_{p,s}(l)| \geq p - 1$. Suppose on the contrary that there exists $l_0 \in T_{p,s}$ such that $|Q_{p,s}(l_0)| \leq p - 2$, we decompose the set $T_{p,s}$ into disjoint union $Q_{p,s}(l_0) \cup \bigcup_{i=1}^{p-1} Q_{p,s}(l_i)$.
Note that \(|Q_{p,s}(l_0)| \leq p - 2\) and \(|Q_{p,s}(l)| \leq p - 1\) for all \(1 \leq i \leq m_{p,s}\), we have

\[
p^s - p^{s-1} = |\mathcal{T}_{p,s}| = \sum_{i=0}^{m_{p,s}} |Q_{p,s}(l_i)| \leq p - 2 + m_{p,s}(p - 1),
\]

this shows that \(m_s \geq \frac{p^s - p^{s-1} - p + 2}{p - 1}\).

From the definition of \(Q_{p,s}\), we see that \(|\bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l)| = |\bigcup_{i=0}^{m_{p,s}} \bigcup_{l \in \mathcal{Q}_{p,s}(l_i)} \mathcal{B}_{p,s}(l)| = |\bigcup_{i=0}^{m_{p,s}} \mathcal{B}_{p,s}(l_i)|\). Note that \(|\mathcal{B}_{p,s}(l)| = (p^2 - 1)p^{s-1}\) for any \(l \in \mathcal{T}_{p,s}\) and \(\mathcal{B}_{p,s}(l_i) \cap \mathcal{B}_{p,s}(l_j) = \emptyset\) for any \(0 \leq i \neq j \leq m_{p,s}\), we obtain

\[
\left| \bigcup_{i=0}^{m_{p,s}} \mathcal{B}_{p,s}(l_i) \right| = (m_{p,s} + 1)(p^2 - 1)p^{s-1} \geq \left( \frac{p^s - p^{s-1} - p + 2}{p - 1} + 1 \right)(p^2 - 1)p^{s-1}
\]

\[
= p^{2s} - p^{2(s-1)} + p^s + p^{s-1} > p^{2s} - p^{2(s-1)}.
\]

However, by Remark 2.6 we have \(\bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l) \subset \hat{E}_{p^s} - \hat{E}_{p^{s-1}}\). Hence

\[
\left| \bigcup_{i=0}^{m_{p,s}} \mathcal{B}_{p,s}(l_i) \right| = \left| \bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l) \right| \leq |\hat{E}_{p^s} - \hat{E}_{p^{s-1}}| = p^{2s} - p^{2(s-1)}.
\]

This contradiction yields \(|Q_{p,s}(l)| \geq p - 1\). Therefore, \(|Q_{p,s}(l)| = p - 1\) for any \(l \in \mathcal{T}_{p,s}\).

(iii) From the definition of \(\mathcal{A}_p(\bar{\eta})\) and \(\mathcal{B}_{p,s}\), the first equation is clearly established. Next we prove \(\bigcup_{s=1}^{\bar{\eta}} \bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l) \equiv \hat{E}_{p^\bar{\eta}}\) (mod \(\mathbb{Z}^2\)). Since \(|Q_{p,s}(l)| = p - 1\), the set \(\mathcal{T}_{p,s}\) can be decomposed into disjoint union \(\bigcup_{i=1}^{p^{s-1}} Q_{p,s}(l_i)\). Note that \(\mathcal{B}_{p,s}(l_i) \cap \mathcal{B}_{p,s}(l_j) = \emptyset\) for any \(1 \leq i \neq j \leq p^{s-1}\) and \(|\mathcal{B}_{p,s}(l)| = (p^2 - 1)p^{s-1}\) for any \(l \in \mathcal{T}_{p,s}\), we get

\[
\left| \bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l) \right| = \left| \bigcup_{l=1}^{p^{s-1}} \mathcal{B}_{p,s}(l_i) \right| = p^{s-1}(p^2 - 1)p^{s-1} = p^{2s} - p^{2(s-2)}.
\]

For any \(s \neq s'\), it is easy to see that \((\bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l)) \cap (\bigcup_{l \in \mathcal{T}_{p,s'}} \mathcal{B}_{p,s'}(l)) = \emptyset\) by Remark 2.6. Hence

\[
\bigcup_{s=1}^{\bar{\eta}} \bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l) = \sum_{s=1}^{\bar{\eta}} (p^{2s} - p^{2(s-1)}) = p^{2\bar{\eta}} - 1 = |\hat{E}_{p^{\bar{\eta}}}|. \tag{3.9}
\]

It follows from Lemma 2.5 that \(\bigcup_{s=1}^{\bar{\eta}} \bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l) \equiv \hat{E}_{p^\bar{\eta}}\) (mod \(\mathbb{Z}^2\)). Combining this with (3.9), we obtain \(\bigcup_{s=1}^{\bar{\eta}} \bigcup_{l \in \mathcal{T}_{p,s}} \mathcal{B}_{p,s}(l) = \hat{E}_{p^\bar{\eta}}\) (mod \(\mathbb{Z}^2\)).

\[\square\]

**Remark 3.5.** For the matrix \(M\) satisfies the conditions of Theorem 3.4 by Claim 1, we have \(\mathcal{B}_{p,s}(l) \cap \mathcal{B}_{p,s}(l') = \emptyset\) for any \(l, l' \in \mathcal{T}_{p,s}\) with \(l \neq l'\) and \(l = l\) (mod \(p\)). Let \(1 \leq i \leq p - 1\), we define \(\mathcal{R}_{p,s}(i) = \{l : l = i \text{ (mod } p), 0 \leq l \leq p^s - 1, l \in \mathbb{Z}\}\), then \(|\mathcal{R}_{p,s}(i)| = p^{s-1}\). Hence Theorem 3.4(i) implies that

\[
\left| \bigcup_{l \in \mathcal{R}_{p,s}(i)} \mathcal{B}_{p,s}(l) \right| = p^{s-1}(p^2 - 1)p^{s-1} = p^{2s} - p^{2(s-2)}.
\]
for any \(1 \leq i \leq p - 1\). By Remark \[2.6\] we have

\[
\bigcup_{l \in \mathcal{R}_{\mu,M,D}} \mathcal{B}_{p,s}(l) = \hat{E}^2_{\mu} \setminus \hat{E}^2_{\mu - 1} \quad \text{(mod } \mathbb{Z}^2\text{).} \tag{3.10}
\]

Let \(\det(M) = L\) and \(\varphi(q)\) be the Euler’s phi function. If \(\gcd(L, q) = 1\), it follows from Lemma \[2.7\] that there exists an integer \(n\) such that

\[
L^{\varphi(q)} = nq + 1. \tag{3.11}
\]

To prove Theorem \[1.2\], we need the following lemma, which was proved in \[20\].

**Lemma 3.6.** Let \(M \in M_2(\mathbb{Z})\) be an expanding matrix with \(\det(M) = L\) and \(\hat{E}_q^2\) be defined by \[(1.2)\]. If \(\gcd(L, q) = 1\), then

\[
M^{\ast j} (\lambda + \mathbb{Z}^2) \supset L^{\varphi(q)} (M^{\ast j} \lambda + \mathbb{Z}^2) \quad \text{for all } \lambda \in \hat{E}_q^2.
\]

**Proof of Theorem \[1.2\].** First, we prove that \(n^\ast (\mu_{M,D}) \leq p^{2\theta}\). Since \(M\) is an integer matrix and \(\gcd(\det(M), p) = 1\), by Lemma \[2.5\], we have \(M^\ast (\hat{E}_{p^\theta}^2 + \mathbb{Z}^2) = M^\ast (\hat{E}_{p^\theta}^2) + M^\ast (\mathbb{Z}^2) \subset M^\ast (\hat{E}_{p^\theta}^2) + \mathbb{Z}^2 = \hat{E}_{p^\theta}^2 + \mathbb{Z}^2\). It follows from \(\mathcal{Z}(m_D) \subset \hat{E}_{p^\theta}^2 + \mathbb{Z}^2\) and Lemma \[2.4\] that

\[
n^\ast (\mu_{M,D}) \leq |\hat{E}_{p^\theta}^2| + 1 = p^{2\theta}. \tag{3.12}
\]

Second, we will show that there exists an expanding integer matrix \(M_0\) such that \(n^\ast (\mu_{M_0,D}) = p^{2\theta}\) if \(p\) is a prime. Assume that there exists an expanding matrix \(M_0 \in GL(2, \mathbb{F}_p)\) such that \(M_0^\ast\) satisfies Theorem \[3.4\] (we will prove its existence at the end of the proof). Let \(\det(M_0) = L\), then \(\gcd(L, p) = 1\). From Theorems \[3.3\] and \[3.4\] we have \(O_p(M_0^\ast) = p^2 - 1\), \(O_{p^\theta}(M_0^\ast) = (p^2 - 1)p^{\theta - 1}\) and

\[
\bigcup_{j=1}^{(p^2-1)p^{\theta-1}} M_0^{\ast j} \mathcal{A}_{p}(\bar{\eta}) = \hat{E}_{p^\theta}^2 \quad \text{(mod } \mathbb{Z}^2\text{).} \tag{3.13}
\]

Since \(\mathcal{N}(\mathcal{A}_p(\bar{\eta}) + \mathbb{Z}^2) \subset \mathcal{Z}(m_D)\) and \(O_{p^\theta}(M_0^\ast) = (p^2 - 1)p^{\theta - 1}\), by \[2.3\] and Lemma \[3.6\] we have

\[
\mathcal{Z}(\mu_{M_0,D}) = \bigcup_{j=1}^\infty M_0^{\ast j} (\mathcal{Z}(m_D)) \supset \mathcal{N} \bigcup_{j=1}^\infty M_0^{\ast j} (\mathcal{A}_p(\bar{\eta}) + \mathbb{Z}^2)
\]

\[
\supset \mathcal{N} \bigcup_{j=1}^\infty L^{\varphi(p^\theta)} j (M_0^{\ast j} \mathcal{A}_p(\bar{\eta}) + \mathbb{Z}^2) \supset \mathcal{N} \bigcup_{j=1}^{(p^2-1)p^{\theta-1}} L^{\varphi(p^\theta)} j (M_0^{\ast j} \mathcal{A}_p(\bar{\eta}) + \mathbb{Z}^2). \tag{3.14}
\]

Let \(\Lambda = \mathcal{N} L^{\varphi(p^\theta)} (p^2 - 1)p^{\theta - 1} \hat{E}_{p^\theta}^2\), we will show that \(\mathcal{E}_\Lambda = \{ e^{2\pi i \langle \lambda, x \rangle} : \lambda \in \Lambda \}\) is an orthogonal set of \(L^2(\mu_{M,D})\). For any \(\lambda_1 \neq \lambda_2 \in \Lambda\), there exists \(\lambda' \in \hat{E}_{p^\theta}^2 \text{ (mod } \mathbb{Z}^2\text{) such that } \lambda_1 - \lambda_2 = \mathcal{N} L^{\varphi(p^\theta)} (p^2 - 1)p^{\theta - 1} \lambda'\). By \[3.13\], there exist \(\lambda_0 \in \mathcal{A}_p(\bar{\eta})\) and \(1 \leq j_0 \leq (p^2 - 1)p^{\theta - 1}\) such that
\[
\lambda = M_0^{\ast \lambda_0} \pmod{\mathbb{Z}^2}. \text{ Then} \\
\lambda_1 - \lambda_2 \in NL^{\varphi(p^d)(\varphi(p^d-1))p^d}(M_0^{\ast \lambda_0} + \mathbb{Z}^2) \\
= NL^{\varphi(p^d)}(L^{\varphi(p^d)(\varphi(p^d-1))}(M_0^{\ast \lambda_0} + \mathbb{Z}^2)). \tag{3.15}
\]

Since \(\gcd(L, p^d) = 1\), by using (3.11), we obtain \(L^{\varphi(p^d)(\varphi(p^d-1))} = p^d m' + 1\) for some integer \(m'\). It follows from \(p^d m' M_0^{\ast \lambda_0} \in \mathbb{Z}^2\) that \(L^{\varphi(p^d)(\varphi(p^d-1))}(M_0^{\ast \lambda_0} + \mathbb{Z}^2) = (p^d m' + 1)(M_0^{\ast \lambda_0} + \mathbb{Z}^2) \subset M_0^{\ast \lambda_0} + \mathbb{Z}^2\). Hence, by (3.14) and (3.15), we have
\[
\lambda_1 - \lambda_2 \in NL^{\varphi(p^d)}(M_0^{\ast \lambda_0} + \mathbb{Z}^2) \subset NL^{\varphi(p^d)}(M_0^{\ast \lambda_0} + \mathbb{Z}^2) \subset \mathbb{Z}(\hat{\mu}, D).
\]

This shows that \((\Lambda - \Lambda) \setminus \{0\} \subset \mathbb{Z}(\hat{\mu}, D)\), by (2.2), the elements in \(E_\Lambda\) are mutually orthogonal. Hence \(n'(\mu, D) \geq p^{2h}\), and (3.12) gives \(n'(\mu, D) = p^{2h}\).

Finally, we prove that there exists an expanding matrix \(M_0 \in GL(2, \mathbb{F}_p)\) such that \(M_0^*\) satisfies Theorem 3.4. It is well known that the matrix in \(GL(2, \mathbb{F}_p)\) with order equals \(p^2 - 1\) always exists. Let \(M_0 \in GL(2, \mathbb{F}_p)\) with \(O_p(M_0) = p^2 - 1\) and \(M_0 p^2 - 1 = \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} + I\).

(a) If \(l_2 \notin \mathbb{pZ}\) or \(l_3 \notin \mathbb{pZ}\). Let \(\tilde{M}_0 = \tilde{N} M_0\), where \(\tilde{N} = p^2 n + 1\) is a sufficient large integer such that \(\tilde{M}_0\) is an expanding integer matrix. Then \(O_p(\tilde{M}_0) = O_p(M_0) = p^2 - 1\) and it is clear that there exists an integer \(n'\) such that \((p^2 n + 1)p^{2h - 1} = p^2 n' + 1\), hence
\[
\tilde{M}_0^{p^2 - 1} = \tilde{N} M_0^{p^2 - 1} = p \left[ p^2 n l_1 + pn l_2 + l_1 p^2 n l_2 + l_2 \right] + I. \tag{3.16}
\]

Let \(M_0 = M_0^*\) if \(l_2 \notin \mathbb{pZ}\) or \(M_0 = M_0^*\) if \(l_3 \notin \mathbb{pZ}\). Then (3.16) shows that \(M_0^*\) satisfies Theorem 3.4 and therefore \(n'(\mu, D) = p^{2h}\).

(b) If \(l_2, l_3 \in \mathbb{pZ}\). Let \(\tilde{M}_1 = M_0 + pI\). Then \(O_p(\tilde{M}_1) = O_p(M_0)\) and
\[
(\tilde{M}_1)^{p^2 - 1} = \sum_{r=2}^{p^2 - 1} \frac{C^r}{r} M_0^{p^2 - 1} - r^{p^2 - 1} \cdot (p^2 - 1) p M_0^{p^2 - 2} + M_0^{p^2 - 1} = p \left[ \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} \right] + I. \tag{3.17}
\]

We will prove \(l_2 \notin \mathbb{pZ}\) or \(l_3 \notin \mathbb{pZ}\). Since \(M_0^{p^2 - 1} = p \left[ \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} \right] + I\) with \(l_2, l_3 \in \mathbb{pZ}\), by (3.17), we only need to show that \(M_0^{p^2 - 2} = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}\) satisfies \(m_2 \notin \mathbb{pZ}\) or \(m_3 \notin \mathbb{pZ}\). Suppose on the contrary that \(m_2 = p s_2, m_3 = p s_3\). Then \(M_0 \in GL(2, \mathbb{F}_p)\) implies \(m_1, m_4 \notin \mathbb{pZ}\). Let \(M_0 = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix}\), we have
\[
M_0^{p^2 - 1} = M_0^{p^2 - 2} \cdot M_0 = \begin{bmatrix} p s_2 r_3 + m_1 r_1 & p s_2 r_4 + m_1 r_2 \\ p s_3 r_1 + m_3 r_2 & p s_3 r_4 + m_4 r_4 \end{bmatrix} = \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} + I. \tag{3.18}
\]

Since \(l_2, l_3 \in \mathbb{pZ}\), by \(m_1, m_4 \notin \mathbb{pZ}\) and (3.18), we have \(r_2, r_3 \in \mathbb{pZ}\). However, if \(r_2, r_3 \in \mathbb{pZ}\), according to Lemma 2.7, \(M_0^{p^2 - 1} = \begin{bmatrix} r_1 & 0 \\ 0 & r_4 \end{bmatrix}^{p^2 - 1} = \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} = I \text{ (mod } M_2(p\mathbb{Z}))\). This
Proof of Theorem 1.1. (i) Suppose \( \alpha \) implies \( \beta \) and \( \kappa \) respectively. If \( \alpha \) contradicts with \( \beta \) or \( \kappa \), then Lemma 3.7. Let \( M \in M_2(\mathbb{Z}) \) be an expanding matrix and \( D \subset \mathbb{Z}^2 \) be a finite subset, and let \( \mu_{M,D}, E^2_{\mathbb{Z}}, \mathbb{Z}_{D}, n^*(\mu_{M,D}) \) be defined by (1.1), (1.2), (1.5) and (2.4), respectively. If \( \emptyset \neq \mathbb{Z}^2_D \subset \hat{E}_d^2 \) and \( \text{gcd}(\text{det}(M), q) = 1 \), then

\[
\begin{align*}
\alpha \beta \quad \text{and} \quad \kappa \\
\end{align*}
\]

Proof of Theorem 1.1. (i) Suppose \( 2\alpha_1 - \beta_1 \notin 3\mathbb{Z} \) or \( 2\alpha_2 - \beta_2 \notin 3\mathbb{Z} \). First, we prove \( n^*(\mu_{M,D}) \leq 9 \). Let

\[
A_1 = \begin{bmatrix} \alpha_1\beta_2 - \alpha_2\beta_1 & 0 \\ 0 & \alpha_1\beta_2 - \alpha_2\beta_1 \end{bmatrix},
\]

then \( D_1 = A_1^{-1}D = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}D \) and \( A_1^{-1}MA_1 = M \). By Lemma 2.2, we have \( n^*(\mu_{M,D_1}) = n^*(\mu_{M,D}) \). It is well known that \( 1 + e^{2\pi i\theta_3} + e^{2\pi i\theta_2} = 0 \) if and only if

\[
\begin{align*}
\theta_1 &= 1/3 + k_1, \\
\theta_2 &= 2/3 + k_2, \\
\end{align*}
\]

or

\[
\begin{align*}
\theta_1 &= 2/3 + k_3, \\
\theta_2 &= 1/3 + k_4,
\end{align*}
\]

where \( k_1, k_2, k_3, k_4 \in \mathbb{Z} \). By (3.19), we can easily obtain

\[
\mathcal{Z}(m_{D_1}) = \{ x \in \mathbb{R}^2 : m_{D_1}(x) = 0 \} := Z_0 \cup \bar{Z}_0,
\]

where

\[
Z_0 = \left\{ \left( \frac{\beta_2 - 2\alpha_2}{3}, \frac{2\alpha_1 - \beta_1}{3} \right) + \left( \frac{\beta_2 k_1 - \alpha_2 k_2}{\alpha_1 k_2 - \beta_1 k_1}, k_1, k_2 \in \mathbb{Z} \right) \right\},
\]

and

\[
\bar{Z}_0 = \left\{ \left( \frac{\beta_2 - 2\alpha_2}{3}, \frac{2\alpha_1 - \beta_1}{3} \right) + \left( \frac{\beta_2 \bar{k}_1 - \alpha_2 \bar{k}_2}{\alpha_1 \bar{k}_2 - \beta_1 \bar{k}_1}, \bar{k}_1, \bar{k}_2 \in \mathbb{Z} \right) \right\}.
\]

Since \( 2\alpha_1 - \beta \notin 3\mathbb{Z} \) or \( 2\alpha_2 - \beta_2 \notin 3\mathbb{Z} \), (3.20) yields \( \mathcal{Z}(m_{D_1}) \subset \hat{E}_3^2 + \mathbb{Z}^2 \). As \( \text{gcd}(\text{det}(M), 3) = 1 \), we conclude from Lemma 2.5 that \( M^* (\hat{E}_3^2 + \mathbb{Z}^2) \subset \hat{E}_3^2 + \mathbb{Z}^2 \). Therefore, Lemma 2.4 implies \( n^*(\mu_{M,D}) = n^*(\mu_{M,D_1}) \leq |\hat{E}_3^2| + 1 = 9 \).

Second, we show that the number 9 is the best. We will prove it in two cases: Case 1, \( \alpha_1\beta_2 - \alpha_2\beta_1 \notin 3\mathbb{Z} \); Case 2, \( \alpha_1\beta_2 - \alpha_2\beta_1 \in 3\mathbb{Z} \).

Case 1, if \( \alpha_1\beta_2 - \alpha_2\beta_1 \notin 3\mathbb{Z} \). Let

\[
A_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}, \quad M_2 = (\alpha_1\beta_2 - \alpha_2\beta_1) \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}
\]
and let $D_2 = A_2^{-1}D = \{(0, 0)' , (1, 0)' , (0, 1)' \}$, $A_2^{-1}M_1A_2 = M_2$. Then $M_1$ is an expanding integer matrix with $\gcd(\det(M_1), 3) = \gcd(\det(M_2), 3) = 1$. By (3.19) and a direct calculation, we obtain $\mathcal{Z}_{D_2}^{3} = \{(1/3, 2/3)' , (2/3, 1/3)' \} \subset \tilde{E}_3^2$ and $\bigcup_{j=1}^{3} M_2^{-1} \mathcal{Z}_{D_2}^{j} = \tilde{E}_3^2 \pmod {\mathbb{Z}^2}$. Hence Lemmas 2.2 and 3.7 imply $n'(\mu_{M_2, D_2}) = n'(\mu_{M_1, D}) = 9$, which shows that the number 9 is the best.

Case 2, $\alpha_1\beta_2 - \alpha_2\beta_1 \in 3\mathbb{Z}$. Observe that the case (ii) of Theorem 4.1 also satisfies $\alpha_1\beta_2 - \alpha_2\beta_1 \in 3\mathbb{Z}$, so the following discussions will also be used in the proof of case (ii).

Let $\alpha_1\beta_2 - \alpha_2\beta_1 = 3^\eta \gamma$ for some integers $\eta \geq 1$ and $3 \nmid \gamma$. Without loss of generality, we assume $\gcd(\alpha_1, \alpha_2) = \sigma$ with $3 \nmid \sigma$ (Otherwise, we can choose $\sigma = \gcd(\beta_1, \beta_2)$ with $3 \nmid \sigma$, because $\gcd(\alpha_1, \alpha_2, \beta_1, \beta_2) = 1$). Let $\alpha_1 = \sigma t_1, \alpha_2 = \sigma t_2$, where $\gcd(t_1, t_2) = 1$, then there exist integers $p$ and $q$ such that $pt_1 + qt_2 = 1$. Clearly, $\sigma = p\alpha_1 + q\alpha_2$ and $\sigma|\gamma$. For convenience, we denote $\omega = p\beta_1 + q\beta_2$ and $\vartheta = \gamma/\sigma$. Let $A_3 = \gamma \begin{bmatrix} t_1 & -q \\ t_2 & p \end{bmatrix}$. By noting that $t_2\alpha_1 = t_1\alpha_2$ and $t_1\beta_2 - t_2\beta_1 = 3^\eta \vartheta$, we have

$$D_3 = A_3^{-1}D = \frac{1}{\gamma} \begin{bmatrix} p & q \\ -t_2 & t_1 \end{bmatrix} \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{1}{\sigma} \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{\omega}{3^\eta} \\ \frac{\vartheta}{\sigma} \end{bmatrix} \right\}$$

(3.21)

and

$$M_3 = A_3^{-1}MA_3 = \begin{bmatrix} (pa + qc)t_1 + (pb + qd)t_2 & (pb + qd)p - (pa + qc)q \\ (ct_1 - at_2)t_1 + (dt_1 - bt_2)t_2 & (dt_1 - bt_2)p - (ct_1 - at_2)q \end{bmatrix}.$$ 

(3.22)

It is obvious that $M_3$ is an expanding integer matrix with $\gcd(\det(M_3), 3) = \gcd(\det(M), 3) = 1$. By Lemma 2.2, we get $n'(\mu_{M_3, D_3}) = n'(\mu_{M, D})$.

In view of (3.19), it is easy to calculate that

$$\mathcal{Z}(m_{D_3}) = \{x \in \mathbb{R}^n : m_{D_3}(x) = 0\} := Z_0 \cup \tilde{Z}_0,$$

(3.23)

where

$$Z_0 = \left\{ \begin{bmatrix} \frac{\vartheta}{\sigma} + k_1 \\ \frac{1}{3^\eta}(2\sigma - \omega - 3\omega k_1 + 3\sigma k_2) \end{bmatrix} : k_1, k_2 \in \mathbb{Z} \right\},$$

and

$$\tilde{Z}_0 = \left\{ \begin{bmatrix} \frac{\vartheta}{\sigma} + \tilde{k}_1 \\ \frac{1}{3^\eta}(\sigma - 2\omega - 3\omega k_1 + 3\sigma k_2) \end{bmatrix} : \tilde{k}_1, \tilde{k}_2 \in \mathbb{Z} \right\}.$$

**Proposition 3.8.** Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \sigma$ and $\omega$ be defined as above. Then

(i) $2\sigma - \omega \not\in 3\mathbb{Z}$ if $2\alpha_1 - \beta_1 \not\in 3\mathbb{Z}$ or $2\alpha_2 - \beta_2 \not\in 3\mathbb{Z}$;

(ii) $2\sigma - \omega \in 3\mathbb{Z}$ if $2\alpha_1 - \beta_1, 2\alpha_2 - \beta_2 \not\in 3\mathbb{Z}$.

**Proof.** Note that $\alpha_1 = \sigma t_1, \alpha_2 = \sigma t_2, pt_1 + qt_2 = 1, \sigma = p\alpha_1 + q\alpha_2, t_1\beta_2 - t_2\beta_1 = 3^\eta \vartheta$ and $\omega = p\beta_1 + q\beta_2$, we have

$$2\sigma - \omega = p(2\alpha_1 - \beta_1) + q(2\alpha_2 - \beta_2).$$

(3.24)
The case (ii) holds from (3.24) immediately.

For case (i), without loss of generality, suppose $2\alpha_1 - \beta_1 \not\in 3\mathbb{Z}$. Since $\alpha_1 = \sigma t_1$, $\alpha_2 = \sigma t_2$ and $t_1^2 - t_2^2 = 3^q \theta$, we have $t_2 \alpha_1 = t_1 \alpha_2$ and
\[ q^3 \theta = q(t_1^2 - t_2^2) = qt_1(\beta_1 - 2\alpha_2) + qt_2(2\alpha_1 - \beta_1). \quad (3.25) \]

Multiplying $2\alpha_1 - \beta_1$ on both sides of $pt_1 + qt_2 = 1$ yields $pt_1(2\alpha_1 - \beta_1) + qt_2(2\alpha_1 - \beta_1) = 2\alpha_1 - \beta_1$. Combining this with (3.25), we get $t_1(p(2\alpha_1 - \beta_1) + q(2\alpha_2 - \beta_2)) = -q^3 \theta + 2\alpha_1 - \beta_1$, which implies $2\sigma - \omega = p(2\alpha_1 - \beta_1) + q(2\alpha_2 - \beta_2) \not\in 3\mathbb{Z}$, because $2\alpha_1 - \beta_1 \not\in 3\mathbb{Z}$. \(\square\)

Let $\mathcal{A}_3(s) = \frac{1}{3^l}((3^k, i^l) : 0 \leq l \leq 3^i - 1, l \in \mathbb{Z})$ be defined by (1.3) and define
\[ \mathcal{A}_3(s) := \frac{1}{3^l}((3^k, i^l) : 0 \leq l \leq 3^i - 1, l = 3k + i, k \in \mathbb{Z}) \quad (3.26) \]
for $i = 0, 1, 2$. Obviously, $\bigcup_{i=0}^2 \mathcal{A}_3(s) = \mathcal{A}_3(s)$.

**Proposition 3.9.** Let $Z_0$, $\eta$, $\sigma$, $\omega$ and $\theta$ be given by (3.23), and let $\mathcal{A}_3$, $\mathcal{A}_3'$ be defined by (1.3) and (3.26), respectively. Then there exist $\tau \not\in 3\mathbb{Z}$ such that the following two conclusions hold:

(i) If $2\sigma - \omega \not\in 3\mathbb{Z}$, then $\tau \theta(\mathcal{A}_3(\eta + 1) + \mathbb{Z}^2) \subset Z_0$ for $i = 1$ or 2;

(ii) If $2\sigma - \omega \in 3\mathbb{Z}$, then $\tau \theta(\mathcal{A}_3(\eta) + \mathbb{Z}^2) \subset Z_0$.

**Proof.** (i) Since $3 \not\mid \sigma$, let $\sigma = 3a + \kappa$ for some integers $a$ and $\kappa = 1$ or 2. Let $\tau = \kappa \sigma$, we will prove that there exists $i = 1$ or 2 such that $\kappa \sigma \theta(\mathcal{A}_3'(\eta + 1) + \mathbb{Z}^2) \subset Z_0$. In fact, this only need to show that for any $k_1', k_2' \in \mathbb{Z}$, there exist $k_1, k_2 \in \mathbb{Z}$ such that
\[
\begin{cases}
\kappa \sigma \theta(k_1') + k_1 = \theta(k_1 + k_1), \\
\kappa \sigma \theta(i + 3k_2')/3^q = (2\sigma - \omega - 3\omega k_1 + 3\sigma k_2)/3^q,
\end{cases} \quad (3.27)
\]
where $i = 1$ or 2 and it is independent of $k_1', k_2'$. Obviously, (3.27) holds if and only if
\[
\begin{cases}
k_1 = (\kappa \sigma - 1)/3 + \kappa \sigma k_1', \\
k_2 = (\kappa \sigma i + \kappa \omega - 2)/3 + \kappa \omega k_1' + \kappa \sigma k_2'.
\end{cases}
\]

Since $\kappa = 1$ or 2 and $\kappa \sigma = \kappa(3a + \kappa) = 3ak + \kappa^2 = \kappa^2 = 1$ (mod 3), then $k_1 \in \mathbb{Z}$. We then prove that $k_2$ is also an integer. It follows from $\sigma = 3a + \kappa$ and $\kappa^2 = 1$ (mod 3) that $\kappa(2\sigma - \omega) = 6ak + 2\kappa^2 - \kappa\omega = 2 - \kappa \omega$ (mod 3). This together with $2\sigma - \omega, \kappa \not\in 3\mathbb{Z}$ implies $2 - \kappa \omega \not\in 3\mathbb{Z}$. Therefore, by noting that $\kappa, \theta \not\in 3\mathbb{Z}$, we can always choose $i = 1$ or 2 such that $\kappa \sigma i = 2 - \kappa \omega$ (mod 3), which shows that $k_2 \in \mathbb{Z}$. Hence (3.27) holds and $\tau \theta(\mathcal{A}_3'(\eta + 1) + \mathbb{Z}^2) \subset Z_0$ for $i = 1$ or 2.

(ii) Let $\tau = \kappa \sigma$ as case (i), we will show that $\kappa \sigma \theta(\mathcal{A}_3(\eta) + \mathbb{Z}^2) \subset Z_0$. Indeed, we only need to show that for any $k_1', k_2' \in \mathbb{Z}$, there exist $k_1, k_2 \in \mathbb{Z}$ such that (3.27) holds for $i = 0,$
i.e.,
\[
\begin{cases}
    \kappa \sigma \vartheta \left( \frac{1}{3} + k_1' \right) = \vartheta \left( \frac{1}{3} + k_1 \right), \\
    \kappa \sigma \vartheta \cdot k_2'/3^\eta = (2\sigma - \omega - 3\omega k_1 + 3\sigma k_2)/3^{\eta+1}.
\end{cases}
\] (3.28)

(3.28) holds if and only if
\[
\begin{cases}
    k_1 = (\kappa \sigma - 1)/3 + \kappa \sigma k_1', \\
    k_2 = (\kappa \omega - 2)/3 + \kappa \omega k_1' + \kappa \vartheta k_2'.
\end{cases}
\]

Using the similar proof as case (i), we obtain \(k_1, k_2 \in \mathbb{Z}\) by \(\kappa \sigma = 1 \text{ (mod 3)}\), \(2\sigma - \omega \in 3\mathbb{Z}\), and \(0 = \kappa(2\sigma - \omega) = 2 - \kappa \omega \text{ (mod 3)}\). Hence (3.28) holds and \(\tau \vartheta(\mathcal{A}_3(\eta) + \mathbb{Z}^2) \subset \mathbb{Z} \)

We now continue to prove Case 2 of Theorem\([1.1](i)\). By Propositions\([3.8](i)\) and\([3.9](i)\), there exists \(\tau \notin 3\mathbb{Z}\) such that \(\tau \vartheta(\mathcal{A}_3(\eta + 1) + \mathbb{Z}^2) \subset \mathbb{Z}_0\) for \(i = 1\) or \(2\).

Now considering an integer matrix \(M_3\) in (3.22) with its transposed conjugate
\[
M_3' = \tilde{N} \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \tilde{N}M \in GL(2, \mathbb{F}_3),
\] (3.29)

where \(\tilde{N} = 3^2n + 1\) is a sufficient large integer such that \(M_3\) is an expanding integer matrix. We will show that \(M_3\) satisfies \(n^*(\mu_{M_3,D}) = 9\). It is easy to check that \(O_3(M_3') = O_3(M) = 8\) and \(M_3^8 = (3\tilde{n} + 1)^8 \begin{bmatrix} 3 & 4 & 7 \\ 7 & 11 & 1 \end{bmatrix} + I\) := \(3 \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix} + I\). Similar to (3.16), it is easy to check that \(l_i \notin 3\mathbb{Z}\) for all \(1 \leq i \leq 4\), hence Theorem\([3.3]\) implies that \(O_{3^{\tilde{n}+1}}(M_3') = 8 \cdot 3^9\) and \(\bigcup_{j=1}^{\infty} M_3^j \xi = \bigcup_{j=1}^{8 \cdot 3^9} M_3^j \xi \text{ (mod } \mathbb{Z}^2\text{)}\) for any \(\xi \in E_{3^{\tilde{n}+1}}^2\). Let \(\det(M_3') = L\) and \(\varphi\) be the Euler’s phi function. Note that \(\gcd(L, 3) = 1\), \(\varphi(3^{\tilde{n}+1}) = 2 \cdot 3^8\) and \(\tau \vartheta(\mathcal{A}_3(\eta + 1) + \mathbb{Z}^2) \subset \mathbb{Z}_0 \subset \mathbb{Z}(m_{D_3})\) for \(i = 1\) or \(2\), by using the similar proof as (3.14), we obtain
\[
\mathbb{Z}(\tilde{M}_3, D_3) \supset \bigcup_{j=1}^{8 \cdot 3^9} L^{2 \cdot 3^8} \tau \vartheta(M_3^j \mathcal{A}_3(\eta + 1) + \mathbb{Z}^2).
\] (3.30)

Note that \(M_3^*\) satisfies Theorem\([3.4]\), according to (3.5), (3.10) and (3.26), for any \(i = 1\) or \(2\), we have
\[
\bigcup_{j=1}^{8 \cdot 3^9} M_3^j \mathcal{A}_3(\eta + 1) = \bigcup_{\lambda \in \mathcal{R}_{3^\lambda + 1}(l)} \mathcal{B}_{3^\lambda + 1}(l) = \mathring{E}_{3^{\lambda + 1}}^2 \setminus \mathring{E}_{3^\lambda}^2 \text{ (mod } \mathbb{Z}^2\text{)}.
\] (3.31)

Let \(\Lambda = L^{2 \cdot 3^8} \cdot 8 \cdot 3^9 \tau \vartheta E\), where
\[
E = \frac{1}{3^{\eta+1}} \left\{ \left( \begin{array}{c} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 0 \\ 1 \\ 2 \end{array} \right) \right\}.
\]

We will show that \(E_{\Lambda} = \{ e^{2\pi i (\lambda \cdot \lambda')} : \lambda \in \Lambda \} \) is an orthogonal set of \(L^2(\mu_{M_3,D_3})\). For any \(\lambda_1 \neq \lambda_2 \in \Lambda\), there exists \(\lambda' \in \mathring{E}_{3^{\lambda_1}}^2 \setminus \mathring{E}_{3^{\lambda_2}}^2 \text{ (mod } \mathbb{Z}^2\text{)}\) such that \(\lambda_1 - \lambda_2 = L^{2 \cdot 3^8 \cdot 8 \cdot 3^9 \tau \vartheta \lambda'}\).
By (3.31), there exist $\lambda_0 \in \mathcal{A}_3(\eta + 1)$ and $1 \leq j_0 \leq 8 \cdot 3^9$ such that $\lambda' = M_3^{r,j_0} \lambda_0 \pmod{Z^2}$. Using the similar proof as in the last part of Theorem 1.2 and (3.30), we have
\[
\lambda_1 - \lambda_2 \in L^{2^{3\eta}:8 \cdot 3^9} \tau \theta(M_3^{r,j_0} \lambda_0 + Z^2) \subset L^{2^{3\eta}:8 \cdot 3^9} \tau \theta(M_3^{r,j_0} \mathcal{A}_3(\eta + 1) + Z^2)
\subset L^{2^{3\eta}:8 \cdot 3^9} \tau \theta(M_3^{r,j_0} \mathcal{A}_3(\eta + 1) + Z^2) \subset \mathcal{Z}(\mu_{M_3, D_3}).
\]
This shows that $(\Lambda - \Lambda) \setminus \{0\} \subset \mathcal{Z}(\mu_{M_3, D_3})$. It follows from (2.22) that the elements in $\mathcal{E}_\Lambda$ are mutually orthogonal, and hence $n^*(\mu_{M_3, D_3}) \geq 9$. Combining with $n^*(\mu_{M_1, D_1}) = n^*(\mu_{M, D}) \leq 9$, we obtain $n^*(\mu_{M, D}) = 9$ and complete the proof of Theorem 1.1(i).

(ii) Suppose $2\alpha_1 - \beta_1, 2\alpha_2 - \beta_2 \in 3\mathbb{Z}$, then $\alpha_1\beta_2 - \alpha_2\beta_1 \in 3\mathbb{Z}$. We mainly use Theorem 1.2 to complete the proof. By the same transformation as Case 2 of (i), we get the same matrix $M_3$ and digit set $D_3$ in (3.21) and (3.22). Thus $n^*(\mu_{M_1, D_1}) = n^*(\mu_{M, D})$, and the zero set of $m_{D_3}(x)$ is also (3.23). By Proposition 3.8(ii), $\theta \notin 3\mathbb{Z}$ and (3.23), we have
\[
\mathcal{Z}(m_{D_3}) \subset \hat{E}_{3^2} + Z^2.
\]
At the same time, it follows from Propositions 3.8(ii) and 3.9(ii) that there exists $\tau \notin 3\mathbb{Z}$ such that $\tau \theta(\mathcal{A}_3(\eta) + Z^2) \subset Z_0$. Then (3.23) and (3.32) imply that $\tau \theta(\mathcal{A}_3(\eta) + Z^2) \subset \mathcal{Z}(m_{D_3}) \subset \hat{E}_{3^2} + Z^2$, where $\tau \theta \notin 3\mathbb{Z}$. By (3.22) and the definition of matrix $A_3$, it is easy to see that $M_3$ can be any expanding integer matrix with $\gcd(\det(M_3), 3) = 1$. Therefore, by Theorem 1.2 $n^*(\mu_{M, D}) = n^*(\mu_{M_1, D_1}) \leq 3^{2\eta}$ and the number $3^{2\eta}$ is the best.

This completes the proof of Theorem 1.1.

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