A NEW FAMILY OF TRANSPORTATION COSTS WITH APPLICATIONS TO REACTION-DIFFUSION AND PARABOLIC EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract. This paper introduces a family of transportation costs between non-negative measures. This family is used to obtain parabolic and reaction-diffusion equations with drift, subject to Dirichlet boundary condition, as the gradient flow of the entropy functional \[ \int_{\Omega} \rho \log \rho + V \rho + 1 \, dx. \] In [5], Figalli and Gigli study a transportation cost that can be used to obtain parabolic equations with drift subject to Dirichlet boundary condition. However, the drift and the boundary condition are coupled in that work. The costs in this paper allow the drift and the boundary condition to be detached.

Keywords: transportation distances, gradient flows, reaction-diffusion equations, boundary conditions.

1. Introduction

The use of optimal transport for the study of evolutionary equations has proven to be a powerful method in recent years. More precisely, one of the most surprising achievements of [3, 10, 11] has been that many evolution equations of the form

\[ \frac{d}{dt} \rho(t) = \text{div} \left( \nabla \rho(t) + \rho(t) \nabla V + \rho(t) (\nabla W * \rho(t)) \right), \]

can be seen as gradient flows of some entropy functional on the space of probability measures with respect to the Wasserstein distance:

\[ W_2(\mu, \nu) = \inf \left\{ \int |x - y|^2 \, d\gamma(x, y) : \pi_1 \# \gamma = \mu, \pi_2 \# \gamma = \nu \right\}. \]

In addition to the fact that this interpretation allows one to prove entropy estimates and functional inequalities (see [13, 16] for more details on this area, which is still very active and in constant evolution), this point of view provides a powerful variational method to prove the existence of solutions to the above equations: given a time step \( \tau > 0 \), and an initial measure \( \rho_0 \), construct an approximate solution by iteratively minimizing

\[ \rho \to \frac{W_2(\rho, \rho_n)^2}{2\tau} + \int \left[ \rho \log \rho + \rho V + \frac{1}{2} \rho (W * \rho) \right] \, dx = L[\rho|\rho_n], \]

where \( \rho_n \) is a minimum for \( L[\rho|\rho_{n-1}] \).

This approach will always produce solutions to parabolic equations with Neumann boundary conditions. More recently, Figalli and Gigli [5] introduced a distance among positive
measures in an open domain \( \Omega \). Such a distance allows one to use this approach to build solutions to the problem:

\[
\begin{align*}
\frac{\partial}{\partial t} \rho(t) &= \text{div} \left( \nabla \rho(t) + \rho(t) \nabla V \right) \quad \text{in } \Omega, \\
\rho &= e^{-V} \quad \text{on } \partial\Omega,
\end{align*}
\]

in bounded domains. Note, however, that the boundary condition for \( \rho \) is decided by the drift term appearing in the equation. Our goal here is to decouple the equation and the boundary condition. Also, we want to allow for the presence of a reaction term. Hence, inspired by [5], we introduce a new family of transportation costs in a bounded open domain \( \Omega \). This family allows us to build weak solutions to

\[
\begin{align*}
\partial_t \rho &= \text{div} \left( \nabla \rho(t) + \rho(t) \nabla V \right) - F'(\rho) \quad \text{in } \Omega, \\
\rho &= \rho_0 \quad \text{on } \partial\Omega.
\end{align*}
\]

Here, \( F \) is a function on \([0, \infty) \times \overline{\Omega}\). We will use the notation \( F_x := F(\cdot, x) \). Also, we denote the first and second partial derivatives with respect to the first variable by \( F'_x \) and \( F''_x \). Our method works for a wide class of reaction terms \( F'_x \). Some examples include

\[
\begin{align*}
F'_x(\rho) &= W(x)\rho^{1+\beta} - Q(x), \\
F'_x(\rho) &= W(x) \log \rho - Q(x), \\
F'_x(\rho) &= W(x)(\rho - 1)|1 - \rho|^{\alpha - 1} - Q(x),
\end{align*}
\]

with \( \alpha \in (0, 1), \beta \geq 0, W \) Lipschitz and strictly positive, and \( Q \) Lipschitz and non-negative.

(Note that when \( V = W = 1 \) and \( Q = 0 \), the last example is equivalent to the equation \( \partial_t u = \Delta u - u^\alpha \) via the change of variable \( u = \rho - 1 \), for non negative initial data.)

Now, we list sufficient conditions on \( F \):

\begin{enumerate}
\item[(F1)] \( F_x \) is strictly convex for every \( x \) in \( \overline{\Omega} \).
\item[(F2)] For every \( x \) in \( \overline{\Omega} \), \( F'_x \) is a homeomorphism from \((0, \infty)\) to \((\inf_{r>0} F'_x(r), \infty)\).
\item[(F3)] For every \( r \) in \((0, \infty)\) the map \( F'_x \) is a continuous function of \( x \).
\item[(F4)] \( \lim_{r \to \infty} [F'_x](r) = \infty \) uniformly in \( x \).
\item[(F5)] There exist positive constants \( s, s_1, B_0, \) and \( C_0 \) such that,

\[
F'_x(\rho) \leq C_0 r,
\]

for every \((r, x)\) in \((0, s) \times \overline{\Omega}\) and

\[
||\nabla_x [F'_x](e^{-V}(x))||_{L^\infty(\Omega)} \leq B_0,
\]

for every \((p, x)\) in \((-\infty, -s_1) \times \overline{\Omega}\).
\item[(F6)] The map

\[
(h, x) \to \int_{[F'_x]^{-1}(0)}^{[F'_x]^{-1}(h)} (\log r + V) F''_{x}(r)dr,
\]

is Lipschitz on any compact subset of \( \{(h, x) \in \mathbb{R} \times \overline{\Omega} : [F'_x]^{-1}(h(x)) > 0\} \).
(F7) For every \( x \) in \( \Omega \), \( F'_x \) satisfies that either
\[
\lim_{r \to 0} F'_x(r) = -\infty,
\]
or
\[
\lim_{r \downarrow 0} F'_x(r) = F'_x(0).
\]
We will assume that the drift, the domain, and the boundary data satisfy:

(B1) \( V \) is Lipschitz.

(B2) \( \Omega \) is Lipschitz, open, bounded, and satisfies the interior ball condition.

(B3) \( \rho_D \) is Lipschitz and uniformly positive.

These transportation costs, that we shall define later, were found through a set of heuristic arguments (see Section 2). These arguments explore costs that are related to a larger class of problems. Examples of these problems include:

\[
\begin{align*}
\partial_t \rho &= \text{div} \left( \nabla \rho(t) + \rho(t) \nabla V \right) - F'_x(\rho)m(\rho) \quad \text{in} \quad \Omega, \\
\rho &= \rho_D \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
and

\[
\begin{align*}
\partial_t \rho &= \text{div} \left( \nabla \rho(t) + \rho(t) \nabla V \right) - F'_x(\rho)m(\rho) \quad \text{in} \quad \Omega, \\
-\langle \nabla \rho - \nabla V \rho, \nu \rangle &= g_R(\rho - \rho_R) \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
Here, the functions \( g_R \) and \( \rho_R \) are assumed to be uniformly positive. Also, \( m : [0, \infty) \to [0, \infty) \) is concave.

The author found this heuristic by combining several previous works. First, the work of Felix Otto on the formal Riemannian structure in the space of probability measures [12, Section 3]. Second, the work of John Milnor [9, Part III] on the formal Riemannian structure in the space of paths of a Riemannian manifold. Third, the work of Francesco Rossi and Benedetto Piccoli [13] on the generalization of the Benamou Brenier formula [2] for positive measures. The last ingredient is a paper by Figalli, Gangbo, and Yolcu [4], in which they successfully follow the minimizing movement scheme for Lagrangian cost. The addition of nonlinear mobilities and the corresponding notion of generalized geodesics has been studied in a different context by J.A. Carrillo, S. Lisini, G. Savare, and D. Slepcev [3].

The heuristic arguments are developed in the second section of this paper. These are made rigorous only for the costs induced by Problem 1.1. These costs produce solutions to (1.1) and (1.7) via the minimizing movement scheme. This is the main result of the paper: Theorem 4.1.

Our family of costs depend on a positive number \( \tau \) and two functions
\[
e : \mathbb{R} \times \overline{\Omega} \to \mathbb{R} \cup \{\infty\},
\]
and
\[
\Psi : \overline{\Omega} \to \mathbb{R}.
\]
We will use the notation \( e_x := e(\cdot, x) \). We will denote the derivative of \( e \) with respect to its first entry by \( e'_x \). Additionally, for each fixed \( x \), \( [e'_x]^{-1} \) denotes the inverse of such a derivative as a function of its own first entry. Analogous notation will be used for \( F \). We will denote the interior of the set of points such that \( e \) is finite by \( D(e) \) and the interior of
the set of points \( z \) such that \( e(z, x) \) is finite by \( D(e_x) \). We require that the functions \( \Psi \) and \( e \) satisfy the following properties:

(C1) \( \Psi \) is Lipschitz.
(C2) For each \( x \) in \( \Omega \), \( e_x := e(\cdot, x) \) is strictly convex and lower semicontinuous.
(C3) For each \( L \in \mathbb{R} \), there exists \( C(L) \) such that
\[
e(z, x) \geq L |z| + C(L) \quad \forall (z, x) \in \mathbb{R} \times \overline{\Omega}.
\]
(C4) The map \( e \) is Lipschitz in any compact subset of \( D(e) \). (We regard \( \Omega \) as a topological space: Hence, the interior of any set of the form \( A \times \Omega \), where \( A \) is an open subset of \( \mathbb{R} \), is given by \( A \times \Omega \)).
(C5) For each \( x \) in \( \Omega \), the sets \( D(e_x) \) are of the form \((a(x), \infty)\), with \( a(x) \) being either a constant or negative infinity.
(C6) For each \( x \), the map \( e_x' \) is a homeomorphism between \( D(e_x) \) and \( \mathbb{R} \).
(C7) For each \( r \) in \( \mathbb{R} \), the map \( [e_x']^{-1}(r) \) is a continuous function of \( x \) and
\[
\lim_{r \to \infty} [e_x']^{-1}(r) = \infty,
\]
uniformly in \( x \).
(C8) There exist positive constants \( s, s_1, B_0, \) and \( C_0 \) such that
\[
[e_x']^{-1}(\log r + V(x)) \leq C_0 r,
\]
for every \((r, x)\) in \((0, s) \times \overline{\Omega}\) and
\[
||\nabla_x e_x'\|^{-1}(p)||_{L^\infty(\Omega)} \leq B_0,
\]
for every \((p, x)\) in \((-\infty, s_1) \times \overline{\Omega}\).
(C9) The function \( e \) satisfies that
\[
\int_\Omega e(0, x) \, dx = 0.
\]

Item (C9) can be easily be relaxed by adding a constant to \( e \); we have just assumed it for convenience. The notations \( e(h(x), x) \), \( e(h) \), \( e \circ h \), and \( e_x(h) \) will be used interchangeably. Similarly, we will freely interchange \( e'(h(x), x) \), \( e'(h) \), \( e_x'(h) \), and \( e' \circ h \).

We will use \( \Psi \) to obtain the desired boundary condition and \( e \) to control the reaction term. We define the cost \( W_{\gamma}^{e, \Psi, r} \) on the set of positive measures with finite mass \( \mathcal{M}(\Omega) \), as a result of Problem 1.1, below.

**Problem 1.1 (A variant of the transportation problem).** Given \( \mu, \rho \, dx \in \mathcal{M}(\Omega) \), we consider the problem of minimizing

\[
(1.4) \quad C_\tau(\gamma, h) := \int_{\Omega \times \Omega \setminus \Omega \times \partial \Omega} \left( \frac{1}{2} \frac{|x - y|^2}{\tau} + \Psi(y)1_{\Omega \times \partial \Omega} - \Psi(x)1_{\partial \Omega \times \Omega} \right) \, d\gamma + \tau \int_\Omega e(h) \, dx,
\]
in the space \( \text{ADM}(\mu, \rho) \) of admissible pairs \((\gamma, h)\). An admissible pair consists of a positive measure \( \gamma \) in \( \Omega \times \Omega \) and a function \( h \) in \( L^1(\Omega) \). We require the pair to satisfy

\[
(1.5) \quad \pi_2#\gamma_{\Omega}^\mu = \rho \, dx + \tau h \, dx \quad \text{and} \quad \pi_1#\gamma_{\Omega}^\mu = \mu.
\]

Here, the measure \( \gamma_{\Omega}^\mu \) denotes the restriction of \( \gamma \) to \( A \times B \subset \overline{\Omega} \times \overline{\Omega} \). Also, the functions \( \pi_1 \) and \( \pi_2 \) are the canonical projections of \( \Omega \times \Omega \) into the first and second factor.
Hence, (1.4) provides a transportation cost between $\mu$ and $\rho$ given by

$$Wb_2^{e,\Psi,\tau}(\mu, \rho) := \inf_{(\gamma, h) \in \text{ADM}} C_\tau(\gamma, h).$$

Additionally, we will denote by $\text{Opt}(\mu, \nu)$ the set of minimizers of Problem 1.1 with $\mu$ and $\nu$ given.

The main objective is the following: given an initial measure $\rho_0$, we build a family of curves $t \to \rho^\tau(t)$, indexed by $\tau > 0$. We will do this by iteratively minimizing

$$\rho \to \int_\Omega \left[ \rho \log \rho - \rho + V(x)\rho + 1 \right] dx + Wb_2^{e,\Psi,\tau}(\rho^\tau_n, \rho) = E^\tau[\rho|\rho^\tau_n],$$

where $\rho^\tau_n$ is a minimum of $E^\tau[\rho|\rho^\tau_{n-1}]$ in $\mathcal{M}(\Omega)$. We define the discrete solutions by

$$\rho^\tau(t) := \rho^\tau_{\lfloor t/\tau \rfloor}.$$ 

We then show that as $\tau \downarrow 0$, we can extract a subsequence converging to a weak solution to the problem:

$$\begin{align*}
\partial_t \rho &= \text{div} \left( \nabla \rho + \rho \nabla V \right) - [e'_x]^{-1}(\log \rho + V), &\text{in } \Omega, \\
\rho &= e^{\Psi - V}, &\text{in } \partial \Omega, \\
\rho(0) &= \rho_0.
\end{align*}$$

In particular when we set

$$e(z, x) = \begin{cases} [F'_x]_{-1}(0) \left( \log (r + V(x)) \right) F''_x(r) dr, &\text{if } z > \inf_{r>0} F'_x(r), \\
\lim_{z \downarrow F'_x(0)} [F'_x]_{-1}(0) \left( \log (r + V(x)) \right) F''_x(r) dr, &\text{if } z = \inf_{r>0} F'_x(r), \\
+\infty &\text{otherwise,}
\end{cases}$$

and

$$\Psi = \log \rho' + V \quad \text{on } \partial \Omega,$$

we obtain a weak solution to (1.1).

Whenever the reaction term satisfies (F1)-(F7), the drift, the boundary, and the boundary data satisfy (B1)-(B3), and $\Psi$ and $e$ are as above, then properties (C1)-(C9) are satisfied as well.

We will require $\rho_0$ to be bounded and uniformly bounded away from zero. Using Proposition 5.3, we will show the existence of positive constants $\lambda$ and $\Lambda$ such that the weak solution satisfies

$$\frac{\lambda}{\sup x \in \overline{\Omega} e^{-C_0 t} e^{-(C_0 t + V)}} \leq \rho(x, t) \leq \frac{\Lambda}{\inf x \in \overline{\Omega} e^{-V}} e^{-V},$$

for almost every $x$.

The paper is organized as follows: Section 2 introduces the heuristics used to find the transportation costs. There, we explain the process used to relate the cost with the boundary conditions and reaction term. Section 3 is devoted to the study of Problem 1.1 and characterization of its solutions in terms of convex functions. Section 4 is devoted to the proof of the main result, Theorem 4.1, which states the convergence of the minimizing movement scheme to the weak solution. Section 5 is devoted to the study of properties of the minimizers of $E^\tau[\rho|\rho_0]$ that we use to prove the main Theorem. Finally, Appendix A is used to prove some technical properties of solutions to Problem 1.1 that are necessary in Section 3.
Acknowledgments: All the works used to find the new heuristics, with the exception of the work of Milnor, were introduced to me by my PhD advisor Alessio Figalli. I want to express my gratitude to him for his invaluable support and orientation. The work of Milnor was introduced to me by my undergraduate advisor Lazaro Recht. I also wish to express my gratitude to him for his invaluable support and orientation.

2. Heuristics

We define the cost $W_{b^2}^{e,m,\tau}$, as a result of Problem 2.1, below.

Problem 2.1 (A variant of the transportation problem). Given $\mu, \nu \in \mathcal{M}(\Omega)$ we consider the problem of minimizing

$$\check{C}_\tau(V_t, h_t, \bar{h}_t) = \int_0^\tau \left[ \frac{1}{2} \int_\Omega |V_t|^2 \rho_t \, dx + \int_\Omega e(h_t)m(\rho_t) \, dx + \int_{\partial\Omega} \nu(\bar{h}_t) \, d\mathcal{H}^{d-1} \right] dt,$$

among all positive measured valued maps from $[0, \tau]$ to $\mathcal{M}(\Omega)$, satisfying $\rho_0 \, dx = \mu$ and $\rho_\tau \, dx = \nu$. Here, the measures $\rho_t$ and the triplets $(V_t, h_t, \bar{h}_t)$ are indexed by $t$ in $[0, \tau]$. We require them to satisfy the constraint

$$\frac{d}{dt} \int_\Omega \zeta \rho_t \, dx = \int_\Omega (\nabla \zeta, V_t) \rho_t \, dx - \int_\Omega \zeta h_t m(\rho) \, dx - \int_{\partial\Omega} \zeta \bar{h}_t \, d\mathcal{H}^{d-1}, \quad \forall t \in [0, \tau] \text{ and } \forall \zeta \in C^\infty(\Omega).$$

This provides a transportation cost between $\mu$ and $\nu$ given by

$$W_{b^2}^{e,m,\tau}(\mu, \nu) := \inf \check{C}_\tau(V_t, h_t, \bar{h}_t).$$

Henceforth, a path is defined as a measured valued map from $[0, \tau]$ to $\mathcal{M}(\Omega)$. We apply the minimizing movement scheme to this cost: given an initial measure $\rho_0$, we build a family of curves $t \to \rho^\tau(t)$, indexed by $\tau > 0$, iterating the minimization of the map

$$\rho \to \int_\Omega [\rho \log \rho - \rho + V(x)\rho + 1] \, dx + W_{b^2}^{e,m,\tau}(\rho^\tau_n, \rho) = E^\tau[\rho|\rho^\tau_n],$$

where $\rho^\tau_n$ is a minimum of $E^\tau[\rho|\rho^\tau_{n-1}]$, in $\mathcal{M}(\Omega)$. We define the discrete solutions by

$$\rho^\tau(t) := \rho^\tau_{[t/\tau]}.$$

Then, as $\tau \downarrow 0$, we extract a subsequence converging to a weak solution of the problem:

$$\begin{cases}
\partial_t \rho = \text{div} \left( \nabla \rho + \rho \nabla V \right) - [e']^{-1}(\log \rho + V)m(\rho), & \text{in } \Omega, \\
-\langle \nabla \rho - \nabla V \rho, \nu \rangle = [\tau']^{-1}(\log \rho + V) & \text{in } \partial\Omega, \\
\rho(0) = \rho_0.
\end{cases}$$

In particular, when we set

$$e(h, x) = \begin{cases}
\int_{[F_x']^{-1}(0)} [F_x']^{-1}(h) (\log r + V) F_x''(r) \, dr, & \text{if } [F_x']^{-1}(h) \geq 0, \\
+\infty, & \text{otherwise},
\end{cases}$$

and

$$\pi(\bar{h}, x) = \begin{cases}
g_R \left( \frac{l(\bar{h}) \log l(\bar{h}) + (V - 1)l(\bar{h}) + 1}{l(\bar{h})} \right), & \text{if } l(\bar{h}) \geq 0, \\
+\infty, & \text{otherwise},
\end{cases}$$

we obtain the following:

$$\begin{cases}
\partial_t \rho = \text{div} \left( \nabla \rho + \rho \nabla V \right) - [e']^{-1}(\log \rho + V)m(\rho), & \text{in } \Omega, \\
-\langle \nabla \rho - \nabla V \rho, \nu \rangle = [\tau']^{-1}(\log \rho + V) & \text{in } \partial\Omega, \\
\rho(0) = \rho_0.
\end{cases}$$
we obtain a weak solution for the problem (1.3). Here,

\[ l(r) = \frac{r}{gR} + \rho R. \]

Also, we will show that when we set

\[
e(h, x) = \begin{cases} \int_{[F_x']^{-1}(h)} (\log r + V) F_x''(r) \, dr, & \text{if } [F_x']^{-1}(h) \geq 0, \\
+\infty, & \text{otherwise,} \end{cases}
\]

and

\[ \overline{e}(h, x) = (\log \rho_D + V) \overline{h}, \]

we obtain a weak solution to (1.2).

The heuristic is presented as follows. Section 2.1 characterizes optimal triplets in terms of potentials. Section 2.2 describes a characterization of minimal paths in terms of an equation for the potentials. Section 2.3 describes how the equation for minimal paths can be used to perform the minimizing movement scheme. Section 2.4 describes how to match the cost with the boundary conditions. Finally, section 2.5 describes how to match the cost with the reaction term.

2.1. Optimal triplets. In this section, we show an heuristic argument that characterizes minimizing triplets for Problem 2.1. For such triplets, there exist functions \( \varphi_t \) indexed in \([0, \tau]\), such that:

(a) \( \nabla \varphi_t = V_t \).

(b) \( \varphi_t = -\overline{e}(\overline{h}_t) \) on \( \partial \Omega \) and \( \overline{\rho}_t \overline{\varphi}_t = \langle \rho \nabla \varphi_t, \nu \rangle \).

(c) \( \varphi_t = -e'(h_t) \) in \( \Omega \).

In order to see this, we fix \( t \in [0, \tau] \) and minimize

\[
\frac{1}{2} \int_{\Omega} |V_t|^2 \rho_t \, dx + \int_{\Omega} e(h_t) m(\rho_t) \, dx + \int_{\partial \Omega} e(\overline{h}_t) \, dH^{d-1},
\]

under the constraint (2.10).

First, we prove (a).

Let us assume that we have a minimizer for Problem 2.1. Let \((V_t, h_t, \overline{h}_t)\) be the corresponding minimal triplet at the given time. We proceed as in the classical case, [1, Proposition 2.30]. Let \( W \) be a compactly supported vector field in the interior of \( \Omega \), with \( \text{div}(\rho_t W) = 0 \). Then, \((V_t + sW, h_t, \overline{h}_t)\) still satisfies the constraint, for every \( s \). Hence, by minimality, we must have

\[
\frac{d}{dt} \left|_{s=0} \right. \frac{1}{2} \int_{\Omega} |V_t + sW|^2 \rho_t \, dx + \int_{\Omega} e(h_t) m(\rho_t) \, dx + \int_{\partial \Omega} e(\overline{h}_t) \, dH^{d-1} = \int_{\Omega} \langle V_t, W \rangle \rho_t \, dx = 0.
\]

Since \( W \) was an arbitrary vector field satisfying \( \text{div}(\rho_t W) = 0 \), by the Helmholtz-Hodge Theorem we obtain \( \nabla \varphi_t = V_t \) for some \( \varphi_t : \Omega \to \mathbb{R} \).

Second, we prove (b).

Let \( \omega : \partial \Omega \to \mathbb{R} \) be a smooth function. Also, let \( \alpha \) solve the elliptic problem

(2.11)

\[
\begin{cases}
\text{div}(\rho_t \nabla \alpha) = 0 & \text{in } \Omega, \\
\langle \rho_t \nabla \alpha, \nu \rangle = \omega & \text{in } \partial \Omega.
\end{cases}
\]
Then, \((V_t + s\nabla \alpha, , h_t, \overline{h}_t + s\omega)\) satisfies the constraint for any \(s\). By minimality, we must have
\[
\frac{d}{ds}\bigg|_{s=0} \int_\Omega |\nabla \varphi_t + s\nabla \alpha|^2 \rho_t\ dx + \int_\Omega e(h_t)m(\rho_t)\ dx + \int_{\partial \Omega} \overline{e}(\overline{h}_t + s\omega)\ d\mathcal{H}^{d-1} = 0.
\]
Hence,
\[
\int_\Omega \langle \nabla \varphi_t, \nabla \alpha \rangle \rho_t\ dx + \int_{\partial \Omega} \omega \overline{e}(\overline{h}_t)\ d\mathcal{H}^{d-1} = 0.
\]
Integrating by parts and using (2.11), we obtain
\[
\int_{\partial \Omega} \omega(\overline{e}'(\overline{h}_t) + \varphi_t)\ d\mathcal{H}^{d-1} = 0.
\]
Since \(\omega\) was arbitrary, we conclude
\[
\overline{e}'(\overline{h}_t) = -\varphi_t\quad \text{on} \quad \partial \Omega.
\]
By (2.10), we must have
\[
\int_\Omega \zeta \partial_t \rho_t\ dx = \int_\Omega \langle \nabla \zeta, \nabla \varphi_t \rangle \rho_t\ dx - \int_\Omega \zeta h_t \rho\ dx - \int_{\partial \Omega} \zeta \overline{h}_t \rho\ d\mathcal{H}^{d-1}
\]
\[
= -\int_\Omega \zeta \div(\nabla \varphi_t) - \int_\Omega \zeta h_t \rho\ dx + \int_{\partial \Omega} \zeta \left(\langle \nabla \varphi_t, \nu \rangle - \overline{h}_t\right)\ d\mathcal{H}^{d-1},
\]
for any \(\zeta : \Omega \to \mathbb{R}\).

Thus, we conclude
\[
\langle \nabla \varphi_t, \nu \rangle = \overline{h}_t\quad \text{on} \quad \partial \Omega.
\]

Third, we show (c).

Let \(\beta, \eta : \Omega \to \mathbb{R}\) be smooth compactly supported functions satisfying
\[
(2.12)\quad -\div(\nabla \beta \rho_t) = m(\rho_t)\eta.
\]

Then, for any \(s\), the triplet \((\nabla \varphi + s\nabla \beta, h + s\eta, \overline{h})\) is admissible. Consequently, we must have
\[
\frac{d}{ds}\bigg|_{s=0} \int_\Omega |\nabla \varphi_t + s\nabla \beta|^2 \rho_t\ dx + \int_\Omega e(h_t + s\eta)m(\rho_t)\ dx + \int_{\partial \Omega} \overline{e}(\overline{h}_t)\ d\mathcal{H}^{d-1} = 0.
\]
Hence,
\[
\int_\Omega \langle \nabla \varphi_t, \nabla \beta \rangle \rho_t\ dx + \int_\Omega \eta \overline{e}'(h_t)m(\rho_t)\ dx = 0.
\]
Integrating by parts and using (2.12), we obtain
\[
\int_\Omega \left[\varphi_t + \overline{e}'(h_t)\right] \eta m(\rho_t)\ dx = 0.
\]
Since \(\eta\) was arbitrary, we conclude
\[
\overline{e}'(h_t) = -\varphi_t\quad \text{in} \quad \Omega.
\]
2.2. Optimal paths. In this section, we will show an heuristic argument that characterizes minimizers of Problem 2.1.

Let $\rho_t$, indexed in $[0, \tau]$, be a minimizer of Problem 2.1. Also, for each $t$ in $[0, \tau]$, let $\varphi_t$ be the potential generating the corresponding optimal triplet: $(\nabla \varphi_t, [e']^{-1}(-\varphi_t), [\tau]^{-1}(-\varphi_t))$. Then,

$$\partial_t \varphi + \frac{1}{2} |\nabla \varphi|^2 - [\varphi_t[e']^{-1}(-\varphi_t) + e([e']^{-1}(-\varphi_t))] m'(\rho_t) = 0,$$

and

$$\frac{d}{dt} \int_{\Omega} \zeta \rho_t \, dx = \int_{\Omega} (\nabla \zeta, \nabla \varphi_t) \rho_t \, dx - \int_{\Omega} [e']^{-1}(-\varphi_t) m'(\rho_t) \, dx - \int_{\partial \Omega} \zeta [\tau]^{-1}(-\varphi_t) \, d\mathcal{H}^{-1},$$

for every $\zeta$ in $C^\infty_c(\overline{\Omega})$.

In order to see this, we proceed by perturbing such minimizers. For each $t \in [0, \tau]$, we consider optimal triplets $(\nabla \omega_t, [e']^{-1}(-\omega_t), [\tau]^{-1}(-\omega_t))$. We require $\omega_t$ to be identically 0 in the complement of a compact subset of $(0, \tau)$.

Then, for each $s$, we let $t \rightarrow \rho_{t,s}$ and $t \rightarrow (\nabla \varphi_{t,s}, [e']^{-1}(-\varphi_{t,s}), [\tau]^{-1}(-\varphi_{t,s}))$ satisfy constraint (2.10). Additionally, for each $t$, we require the map $s \rightarrow \rho_{t,s}$ to satisfy

$$\frac{d}{ds} \int_{s=0}^{\tau} \int_{\Omega} \zeta \rho_{t,s} \, dx = \int_{\Omega} (\nabla \zeta, \nabla \varphi_{t,s}) \rho_{t,s} \, dx - \int_{\Omega} [e']^{-1}(-\omega_t) m(\rho_{t,s}) \, dx - \int_{\partial \Omega} \zeta [\tau]^{-1}(-\omega_t) \, d\mathcal{H}^{-1},$$

and

$$\rho_{t,0} = \rho_t, \quad \varphi_{t,0} = \varphi_t.$$

Since $t \rightarrow \rho_t$ is a minimizer, we must have

$$\frac{d}{ds} \int_{s=0}^{\tau} \frac{1}{2} \left[ \int_{\Omega} |\nabla \varphi_{t,s}|^2 \rho_{t,s} \, dx + \int_{\Omega} e([e']^{-1}(-\varphi_{t,s})) m(\rho_{t,s}) \, dx \right] \, ds = \int_{\partial \Omega} \tau([\tau]^{-1}(-\varphi_{t,s})) \, d\mathcal{H}^{-1} \, dt = 0.$$

Consequently,

$$\int_{0}^{\tau} \left[ \int_{\Omega} (\nabla \varphi_{t,s}, \nabla \partial_s \varphi_{t,s}) \rho_{t,s} \, dx + \frac{1}{2} \int_{\Omega} |\nabla \varphi_{t,s}|^2 \partial_s \rho_{t,s} \, dx - \int_{\Omega} \varphi_{t,s} \partial_s [e']^{-1}(-\varphi_{t,s}) m(\rho_{t,s}) \, dx 

+ \int_{\Omega} e([e']^{-1}(-\varphi_{t,s})) m'(\rho_{t,s}) \partial_s \rho_{t,s} \, dx - \int_{\partial \Omega} \varphi_{t,s} \partial_s [\tau]^{-1}(-\varphi_{t,s}) \, d\mathcal{H}^{-1} \right] \, dt = 0.$$

Then,

$$\int_{0}^{\tau} \left[ \frac{d}{ds} \left( \int_{\Omega} |\nabla \varphi_{t,s}|^2 \rho_{t,s} \, dx - \int_{\Omega} \varphi_{t,s} [e']^{-1}(-\varphi_{t,s}) m(\rho_{t,s}) \, dx - \int_{\partial \Omega} \varphi_{t,s} [\tau]^{-1}(-\varphi_{t,s}) \, d\mathcal{H}^{-1} \right) 

- \left( \int_{\Omega} (\nabla \partial_s \varphi_{t,s}, \nabla \varphi_{t,s}) \rho \, dx - \int_{\partial \Omega} \partial_s \varphi_{t,s} [e']^{-1}(-\varphi_{t,s}) m(\rho_{t,s}) \, dx \right) 

- \int_{\partial \Omega} \partial_s \varphi_{t,s} [\tau]^{-1}(-\varphi_{t,s}) \, d\mathcal{H}^{-1} \right) 

- \frac{1}{2} \int_{\Omega} |\nabla \varphi_{t,s}|^2 \partial_s \rho_{t,s} \, dx 

+ \int_{\Omega} \varphi_{t,s} [e']^{-1}(-\varphi_{t,s}) m'(\rho_{t,s}) \partial_s \rho_{t,s} \, dx 

+ \int_{\Omega} e([e']^{-1}(-\varphi_{t,s})) m'(\rho_{t,s}) \partial_s \rho_{t,s} \, dx \right] \, dt = 0.$$
Recall that \( h_{t,s} = [e']^{-1}(-\varphi_{t,s}) \) and \( \overline{h}_{t,s} = [\overline{e}']^{-1}(-\varphi_{t,s}) \). By (2.10), we get

\[
\int_0^\tau \left[ \frac{d}{ds} \int_\Omega \varphi_{t,s} \partial_t \rho_{t,s} \, dx - \int_\Omega \partial_s \varphi_{t,s} \partial_t \rho_{t,s} \, dx \right. \\
- \left. \int_\Omega \left( \frac{1}{2} |\nabla \varphi_{t,s}|^2 - [\varphi_{t,s}] [e']^{-1}(-\varphi_{t,s}) + [\overline{e}']^{-1}(-\varphi_{t,s}) \right) \partial_s \rho_{t,s} \, dx \right] dt = 0.
\]

By construction \( \partial_s \varphi_{t,s} = \partial_s \rho_{t,s} = \partial_s \varphi_{0,s} = \partial_s \rho_{0,s} = 0 \). Hence, if we integrate by parts in \( t \), we obtain

\[
- \int_0^\tau \left[ \int_\Omega \left( \partial_t \varphi + \frac{1}{2} |\nabla \varphi_{t,s}|^2 - [\varphi_{t,s}] [e']^{-1}(-\varphi_{t,s}) + [\overline{e}']^{-1}(-\varphi_{t,s}) \right) \partial_s \rho_{t,s} \, dx \right] dt = 0.
\]

This gives the desired result.

### 2.3. The minimizing movement scheme.

Given \( \rho_0 \in \mathcal{M}(\Omega) \) and \( \tau > 0 \), we provide heuristic arguments to characterize the minimizers of (2.15)

\[
\{\rho_t\}_{t \in [0,\tau]} \rightarrow \int_0^\tau \left( \frac{1}{2} \int_\Omega |V_t|^2 \rho_t \, dx + \int_\Omega e(h_t) m(\rho_t) \, dx + \int_{\partial \Omega} \tau(h_t) \, d\mathcal{H}^{d-1} \right) dt \\
+ \int_\Omega \left[ \rho_0 \log \rho_0 + (V - 1) \rho_0 + 1 \right] \, dx.
\]

Here, the triplets \((V_t, h_t, \overline{h}_t)\) satisfy (2.10). Also, \( \rho_0 \) is fixed and \( \rho_\tau = \rho \).

In Section 2.1, we saw that for minimizing triplets we have for each \( t \in [0,\tau] \) a function \( \varphi_t \), such that

\[
(V_t, h_t, \overline{h}_t) = (\nabla \varphi_t, [e']^{-1}(-\varphi_t), [\overline{e}']^{-1}(-\varphi_t)).
\]

In Section 2.2, we found that optimal paths satisfy

\[
\partial_t \varphi_t + \frac{1}{2} |\nabla \varphi_t|^2 - [\varphi_t [e']^{-1}(-\varphi_t) + [\overline{e}']^{-1}(-\varphi_t)] m'(\rho_t) = 0.
\]

In this section, we will show that minimizers of (2.15) must satisfy

\[
\varphi_\tau = -\log \rho_\tau - V.
\]

In order to see this, we suppose that we have a minimizer, \( \rho_\tau \) and a path, \( t \rightarrow \rho_t \), with corresponding triplets \( t \rightarrow (\nabla \varphi_t, [e']^{-1}(-\varphi_t), [\overline{e}']^{-1}(-\varphi_t)) \).

We proceed by perturbing the path \( t \rightarrow \rho_t \). For each \( t \) we choose a function \( \omega_t \). We require these functions to be identically 0 in the complement of a compact subset of \( (0,\tau] \).

This generates for each \( s \) a path, \( t \rightarrow \rho_{t,s} \), as in Section 2.2, with the difference that now the end point \( \rho_{\tau,s} \) is free.

For a minimizer, we must have

\[
\frac{d}{ds} \bigg|_{s=0} \left( \int_0^\tau \frac{1}{2} \int_\Omega |\nabla \varphi_{t,s}|^2 \rho_{t,s} \, dx + \int_\Omega e([e']^{-1}(-\varphi_{t,s})) m(\rho_{t,s}) \, dx \\
+ \int_{\partial \Omega} \tau([\overline{e}']^{-1}(-\varphi_{t,s})) \, d\mathcal{H}^{d-1} dt + \int_\Omega \left[ \rho_{\tau,s} \log \rho_{\tau,s} + (V - 1) \rho_{\tau,s} + 1 \right] \, dx \right) = 0.
\]
By (2.14) we have
\[
\int_0^\tau \left[ \frac{d}{ds} \int_\Omega \varphi_{t,s} \partial_t \rho_{t,s} \, dx - \int_\Omega \partial_s \varphi_{t,s} \partial_t \rho_{t,s} \, dx \right. \\
\left. - \int_\Omega \left( \frac{1}{2} |\nabla \varphi_{t,s}|^2 - [\varphi_{t,s}]^{-1}(-\varphi_{t,s}) + e([\varphi_{t,s}]^{-1}(-\varphi_{t,s})) \right) m'(\rho_{t,s}) \partial_s \rho_{t,s} \, dx \right] dt \\
+ \int_\Omega [\log \rho_{\tau} + V] \partial_s \rho_{\tau,s} \, dx = 0.
\]

Then, if we use (2.13) we obtain
\[
\int_0^\tau \left[ \int_\Omega \varphi_{t,s} \partial_t \partial_s \rho_{t,s} \, dx + \int_\Omega \partial_t \varphi_{t,s} \partial_s \rho_{t,s} \, dx \right] dt + \int_\Omega (\log \rho_{\tau} + V) \partial_s \rho_{\tau,s} \, dx = 0.
\]
Recall that by construction \( \partial_s \varphi_{0,s} = \partial_s \rho_{0,s} = 0 \). Integrating by parts we get
\[
\int_\Omega \left( \varphi_{\tau} + \log \rho_{\tau} + V \right) \partial_s \rho_{\tau,s} \, dx = 0.
\]
Thus, we obtain the desired result.

2.4. **The Boundary Conditions.** In Section 2.3, we showed that for minimizers of (2.15), we have that \( \varphi_{\tau} = -\log \rho_{\tau} - V \). In Section 2.1, we showed that for optimal triplets,
\[
-\varphi = \bar{\tau}(\overline{h}) \quad \text{on} \quad \partial \Omega.
\]
Hence, if we set \( \bar{\tau}(\overline{h}) = \Psi \overline{h} \), we obtain the boundary condition
\[
\rho_{\tau} = e^{\Psi - V} \quad \text{on} \quad \partial \Omega.
\]
This concludes the analysis for the boundary condition for (1.2).

In order to derive the boundary condition for (1.3), we proceed as follows: In Section 2.1, we showed that for minimizers of Problem 2.1, we must have
\[ 
\overline{t}_t = \langle \rho \nabla \varphi_t, \nu \rangle. 
\]
Hence, we expect the limit of the minimizing movement scheme to satisfy the relation
\[ 
-\langle \nabla \rho, \nu \rangle - \langle \rho \nabla V, \nu \rangle = [\bar{\tau}']^{-1}(\log \rho + V).
\]
Our goal is to obtain the boundary condition
\[ 
-\langle \nabla \rho, \nu \rangle - \langle \rho \nabla V, \nu \rangle = g_R(\rho - \rho_R).
\]
For this purpose, we would need
\[ 
[\bar{\tau}']^{-1}(\log \rho + V) = g_R(\rho - \rho_R).
\]
Thus,
\[ 
V + \log \rho = [\bar{\tau}']\left(g_R(\rho - \rho_R)\right).
\]
Hence, if we set
\[ 
l(r) = \frac{r}{g_R} + \rho_R,
\]
we obtain
\[ 
[\bar{\tau}'](l^{-1}(\rho)) = \log \rho + V.
\]
Then, it follows that
\[ e'(l^{-1}(\rho))[l^{-1}(\rho)]' = g_R(\log \rho + V). \]
Integrating, we obtain
\[ \bar{\tau}(l^{-1}(\rho)) = \int_0^\rho g_R(\log r + V)dr + C. \]
Here, \( C \) is a constant that will be chosen later. This implies
\[ \bar{\tau}(\rho) = g_R \int_{l(0)}^{l(\rho)} (\log r + V)dr + C. \]
Thus, it suffices to set
\[ \bar{\tau}(\rho) = g_R \left( l(\rho) \log l(\rho) + \left( V - 1 \right) l(\rho) + 1 \right). \]

2.5. **The reaction term.** In Section 2.1, we showed that optimal triplets satisfy
\[ e'(h_\tau) = -\varphi_\tau. \]
In Section 2.3, we showed that minimizers of [2.15] satisfy
\[ \varphi_\tau = -\log \rho_\tau - V. \]
Thus, in order to obtain
\[ h = F'(\rho), \]
we set
\[ e'(F'(\rho)) = \log \rho + V. \]
This implies
\[ e'(F'(\rho))F''(\rho) = (\log \rho + V)F''(\rho). \]
Integrating we obtain
\[ e(F'(\rho)) = \int_0^\rho (\log r + V)F''(r)dr + C, \]
for some constant \( C \). Thus, it suffices to set
\[ e(\rho) = \int_{[F']^{-1}(0)}^{[F']^{-1}(\rho)} (\log r + V)F''(r)dr. \]
This concludes the heuristic arguments. In the following sections we make these arguments rigorous for the case described in the introduction.

3. **Properties of \( W_{\mathbb{L}_2}^{\mathcal{E},\Psi,\tau} \)**

In this section we study minimizers of Problem 1.1. We begin by showing their existence.

**Lemma 3.1.** (Existence of Optimal pairs) Let \( \mu \) and \( \nu \) be absolutely continuous measures in \( \mathcal{M}(\Omega) \). Then there exists a minimizing pair for Problem 1.1.
Proof. We claim the following:

Claim 1: There exists a minimizing sequence of admissible pairs \( \{ (\gamma_n, h_n) \}_{n=1}^{\infty} \) for which the mass of \( \{ \gamma_n \}_{n=1}^{\infty} \) and \( \{ h_n \} \) is equibounded and the plans in the sequence have no mass concentrated on \( \partial \Omega \times \partial \Omega \).

We assume this claim and postpone its proof until the end of the argument. By (1.5) the claim gives us a uniform bound in the total variation of \( \{ (\gamma_n, h_n) \}_{n=1}^{\infty} \). Then, by compactness of \( \Omega \) and \( \Omega \times \Omega \), for a subsequence \( \{ (\gamma_n, h_n) \}_{n=1}^{\infty} \), not relabeled, we have weak convergence to regular Borel measures, with finite total variation, \( \gamma \) and \( h \). This convergence is in duality with continuous bounded functions in \( \Omega \times \Omega \) and \( \Omega \), respectively.

Assumption (C3) and the Dunford-Pettis Theorem allows us to conclude that \( h = h \), for some \( h \) in \( L^1(\Omega) \) and that \( \{ h_n \}_{n=1}^{\infty} \) converges to \( h \) in duality with functions in \( L^\infty(\Omega) \). Since \( \pi_{2#}(\gamma_n) = \rho dx + h_n \tau \), we have that for any \( \zeta \in C_c(\Omega) \),

\[
\int_{\Omega} \zeta \circ \pi_2 d\gamma_n = \lim_{n \to \infty} \int_{\Omega} \zeta \circ \pi_2 d\gamma_n = \lim_{n \to \infty} \int_{\Omega} \zeta \rho dx + \tau \int_{\Omega} \zeta h_n dx = \int_{\Omega} \zeta \rho dx + \tau \int_{\Omega} \zeta h dx.
\]

Hence, \( \pi_{2#}(\gamma_n) = \rho dx + h \tau \). It can also be shown that \( \pi_{1#}(\gamma) = \mu \) in an analogous way. This implies that \( (\gamma, h) \) is in \( ADM(\mu, \nu) \).

Since the sequence \( \{ h_n \}_{n=1}^{\infty} \) converges weakly in \( L^1(\Omega) \) to \( h \), using assumptions (C2)-(C6) and [13, Theorem 1], we get

\[
\liminf_{n \to \infty} \int_{\Omega} e(h_n(x), x)dx \geq \int_{\Omega} e(h(x), x)dx.
\]

We also claim the following:

Claim 2: there exists a further subsequence \( \{ \gamma_n \}_{n=1}^{\infty} \), not relabeled, with the property that \( \{ (\gamma_n)_{\Omega \times \Omega} \}_{n=1}^{\infty} \), \( \{ (\gamma_n)_{\partial \Omega \times \Omega} \}_{n=1}^{\infty} \), and \( \{ (\gamma_n)_{\partial \Omega \times \partial \Omega} \}_{n=1}^{\infty} \) converge weakly to \( \gamma_{\Omega \times \Omega} \), \( \gamma_{\partial \Omega \times \Omega} \), and \( \gamma_{\partial \Omega \times \partial \Omega} \) in duality with continuous and bounded functions in \( C(\Omega \times \Omega), C(\partial \Omega \times \Omega) \), and \( C(\partial \Omega \times \partial \Omega) \), respectively. We will also postpone the proof of this claim until the end of the argument.

Since \( \Psi \) is bounded and continuous, this claim implies that

\[
\lim_{n \to \infty} \left[ \int_{\Omega \times \Omega} \frac{|x-y|^2}{2\tau} d(\gamma_n)_{\Omega} + \int_{\partial \Omega \times \Omega} \left( \frac{|x-y|^2}{2\tau} - \Psi(x) \right) d(\gamma_n)_{\partial \Omega} \right]
\]

\[
+ \int_{\Omega \times \partial \Omega} \left( \frac{|x-y|^2}{2\tau} + \Psi(y) \right) d(\gamma_n)_{\partial \Omega} = \int_{\Omega \times \Omega} \frac{|x-y|^2}{2\tau} d\gamma_{\Omega} + \int_{\partial \Omega \times \Omega} \frac{|x-y|^2}{2\tau} d\gamma_{\partial \Omega} + \int_{\partial \Omega \times \partial \Omega} \left( \frac{|x-y|^2}{2\tau} + \Psi(y) \right) d\gamma_{\partial \Omega}.
\]

Hence, this shows the existence of minimizers, provided we prove the two claims. In order to prove the first one, we note that due to (1.4) and (1.5) we can assume, without loss of generality, that the plans in the minimizing sequence have no mass concentrated on \( \partial \Omega \times \partial \Omega \).
Also, due to (C3) and (1.5),
\[ C_{\tau}(\gamma, h) \geq -||\Psi||_{\infty} \left( |\gamma_{\Omega}| + |\gamma_{\Omega}^0| \right) + K \int_{\Omega} |h| \, dx + C(K)|\Omega| \]
\[ \geq -||\Psi||_{\infty} \left( |\gamma_{\Omega}| + |\gamma_{\Omega}^0| \right) + K|h|\Omega + C(K)|\Omega| \]
\[ \geq -||\Psi||_{\infty} \left( \mu(\Omega) + \nu(\Omega) + \tau|h|\Omega \right) + K|h|\Omega + C(K)|\Omega|, \]
for any \( K \). Taking \( K \) large enough, we obtain a uniform bound on \(|h|\Omega\) and consequently on \(|\gamma|\), for any minimizing sequence. This proves the first claim.

As previously explained, this claim gives us a subsequence, not relabeled \( \{(\gamma_n, h_n)\}_{n=1}^{\infty} \), that converges weakly to \((\gamma, h)\). To prove the second claim, we note that the measures in the sequence \( \{(\gamma_n)_{\Omega}, (\gamma_n)_{\Omega}, (\gamma_n)_{\Omega}^0\}_{n=1}^{\infty} \) have uniformly bounded mass. Then, by compactness of \( \Omega \times \Omega, \Omega \times \Omega, \) and \( \Omega \times \Omega \) we can find a further subsequence \( \{(\gamma_n)_{\Omega}, (\gamma_n)_{\Omega}, (\gamma_n)_{\Omega}^0\}_{n=1}^{\infty}, \) not relabeled, weakly converging to the measures \( \sigma_0, \sigma_1, \) and \( \sigma_2 \). This convergence is in duality with continuous and bounded functions in \( C(\Omega \times \Omega), C(\partial \Omega \times \Omega), \) and \( C(\Omega \times \partial \Omega) \), respectively. Using the definition of weak convergence, it is easy to verify that we must have
\[
(3.16) \quad \gamma = \sigma_0 + \sigma_1 + \sigma_2.
\]
We will prove the second claim by showing that \( \sigma_0 = \gamma_{\Omega}, \sigma_1 = \gamma_{\Omega}, \) and \( \sigma_2 = \gamma_{\Omega}^0 \). By (3.16), this is a consequence of the measures \( \pi_2\#\sigma_0, \pi_1\#\sigma_0, \pi_2\#\sigma_1, \) and \( \pi_1\#\sigma_2 \) having no mass concentrated in \( \partial \Omega \). In order to see that these measures have this property, we let \( A \subset \partial \Omega \) be a compact set and we take a sequence \( \{\eta_k\}_{k=1}^{\infty} \) of uniformly bounded functions in \( C(\Omega) \) that decreases monotonically to \( 1_A \). Additionally, we require that the sets \( \text{supp}(\eta_k) \) decrease monotonically to \( A \). Since \( \Omega \) is bounded, by the dominated convergence Theorem,
\[
\int_A d\pi_2\#\sigma_0 = \int_{\Omega} 1_A \circ \pi_2 \, d\sigma_0 = \lim_{k \to \infty} \int_{\Omega} \eta_k \circ \pi_2 \, d\sigma_0.
\]
Also, by construction we have
\[
\int_{\Omega} \eta_k \circ \pi_2 \, d\sigma_0 = \lim_{n \to \infty} \int_{\Omega} \eta_k \circ \pi_2 \, d(\gamma_n)_{\Omega} \leq \lim_{n \to \infty} \int_{\Omega \times \Omega} \eta_k \circ \pi_2 \, d(\gamma_n)_{\Omega} \]
\[
= \lim_{n \to \infty} \int_{\Omega} \eta_k \rho \, dx + \tau \int_{\Omega} \eta_k \rho h_n \, dx = \int_{\Omega} \eta_k \rho \, dx + \tau \int_{\Omega} \eta_k \rho h \, dx
\]
\[
= \int_{\text{supp}(\eta_k)} \eta_k (\rho + \tau h) \, dx \leq \sup(\eta_k) \int_{\text{supp}(\eta_k)} |\rho + \tau h| \, dx.
\]
Since \( \{\eta_k\}_{k=0}^{\infty} \) is uniformly bounded and \( \text{supp}(\eta_k) \) converges monotonically to the set \( A \subset \partial \Omega \) with zero \( L^d \) measure, we have
\[
\int_A d\pi_2\#\sigma_0 \leq \limsup_{k \to \infty} \sup(\eta_k) \int_{\text{supp}(\eta_k)} |\rho + \tau h| \, dx = 0.
\]
Thus, we conclude that \( \pi_2\#\sigma_0(A) = 0 \), for any measurable subset \( A \subset \partial \Omega \); the proof for the measures \( \pi_1\#\sigma_0, \pi_2\#\sigma_1, \) and \( \pi_1\#\sigma_2 \) is analogous. This establishes the second claim. Consequently, the Lemma is proven. \( \square \)
We will use the following definitions:

Given an admissible pair \((\gamma, h)\), we define
\[
d_{\gamma, h}(x) = \begin{cases} 
\inf_{y \in \partial \Omega} \frac{|x-y|^2}{2\tau} + \Psi(y) & \text{if } x \in \Omega, \\
0 & \text{otherwise}, 
\end{cases}
\]
and
\[
d_{-\gamma, h}(y) = \begin{cases} 
\inf_{x \in \partial \Omega} \frac{|x-y|^2}{2\tau} - \Psi(x) & \text{if } y \in \Omega, \\
0 & \text{otherwise}.
\end{cases}
\]

For any \(x\) and \(y\) in \(\Omega\) we denote by \(P_{\gamma, h}(x)\) and \(P_{-\gamma, h}(y)\) the sets where the infima are respectively attained. Henceforth, \(P_{\gamma, h}\) and \(P_{-\gamma, h}\) will be measurable maps from \(\overline{\Omega}\) to \(\partial \Omega\) such that
\[
d_{\gamma, h}(x) = \frac{|x - P_{\gamma, h}(x)|^2}{2\tau} + 1_\Omega(x)\Psi(P_{\gamma, h}(x)),
\]
and
\[
d_{-\gamma, h}(y) = \frac{|y - P_{-\gamma, h}(y)|^2}{2\tau} - 1_\Omega(y)\Psi(P_{-\gamma, h}(y)).
\]

It is well known that such maps are uniquely defined on \(L^d\)-a.e. in \(\Omega\). (Indeed, \(P_{\gamma, h}(x)\) and \(P_{-\gamma, h}(y)\) are uniquely defined whenever the Lipschitz functions \(d_{\gamma, \tau}\) and \(d_{-\gamma, \tau}\) are differentiable and they are given by \(P_{\gamma, h}(x) = x - \nabla_x d_{\gamma, h}\) and \(P_{-\gamma, h}(y) = y - \nabla_y d_{-\gamma, h}\). Here, we are just defining them on the whole \(\overline{\Omega}\) via a measurable selection argument (we omit the details).

Henceforth, \(P : \overline{\Omega} \to \partial \Omega\) will be a measurable map defined in the whole \(\overline{\Omega}\) with the property that
\[
|x - P(x)| = d(x, \partial \Omega) \quad \forall x \in \overline{\Omega}.
\]

We define the costs
\[
\tilde{c}(x, y) = \frac{|x - y|^2}{2\tau} 1_{(\partial \Omega \times \partial \Omega)\cap} - 1_{\partial \Omega \times \Omega} \Psi(x) + 1_{\Omega \times \partial \Omega} \Psi(y),
\]
\[
c(x, y) = \frac{|x - y|^2}{2\tau},
\]
\[
c_1 = c_{\partial \Omega \times \overline{\Omega}},
\]
and
\[
c_2 = c_{\overline{\Omega} \times \partial \Omega}.
\]

Also, we define the set
\[
\mathcal{A} = \left\{ (x, y) \in \overline{\Omega} \times \overline{\Omega} : d_{\gamma, h}(x) + d_{-\gamma, h}(y) \geq \tilde{c}(x, y) \right\}.
\]

We will work with the topological space \((\overline{\Omega} \times \overline{\Omega}, G)\). The topology of this space built by considering the product topology, in the spaces \(\Omega \times \Omega, \partial \Omega \times \Omega, \Omega \times \partial \Omega, \) and \(\partial \Omega \times \partial \Omega\), and then taking the disjoint union topology. In other words, the space \(\overline{\Omega} \times \overline{\Omega}\) is equipped with the topology
\[
\partial \Omega \times \partial \Omega \coprod \partial \Omega \times \Omega \coprod \Omega \times \partial \Omega \coprod \Omega \times \Omega.
\]

Hence, if we are given continuous functions \(\{f_i\}_{i=1}^4\) from the spaces \(\Omega \times \Omega, \partial \Omega \times \Omega, \Omega \times \partial \Omega, \) and \(\partial \Omega \times \partial \Omega\) to any other topological space \(Y\), then there exists a unique continuous function \(f : \overline{\Omega} \times \overline{\Omega} \to Y\) such that
\[
f_i = f \circ \phi_i.
\]
Here, \(\{\phi_i\}_{i=1}^4\) are the canonical injections of \(\Omega \times \Omega, \partial \Omega \times \Omega, \Omega \times \partial \Omega,\) and \(\partial \Omega \times \partial \Omega\) into \(\overline{\Omega} \times \overline{\Omega}\). The support of the measures \(\gamma\) in \(\overline{\Omega} \times \overline{\Omega}\) will be taken with respect to this topology. Hence, given a positive \(\gamma\) measure in \(\overline{\Omega} \times \overline{\Omega}\), \(\text{supp}(\gamma)\) is defined to be set of points \((x, y)\) in \(\overline{\Omega} \times \overline{\Omega}\) such that for every \(G\) in \(\mathcal{G}\) containing \((x, y)\), we have \(\gamma(G) > 0\).

Additionally, we will use the notions of \(c\)-cyclical monotonicity, \(c\)-transforms, \(c\)-concavity, and \(c\)-superdifferential. We refer the reader to [1, Definitions 1.7 to 1.10].

The following Proposition characterizes solutions of Problem 1.1 satisfying some hypotheses. We remark that Proposition A.4 provides conditions under which these hypotheses are satisfied.

**Proposition 3.2. (Characterization of optimal pairs)** Let \(\mu\) and \(\nu\) be absolutely continuous measures in \(\mathcal{M}(\Omega)\). Also, let \((\gamma, h)\) be in \(\text{ADM}(\mu, \nu)\). Assume that \(\mu\) and \(\nu + \tau h\) are strictly positive. Then, the following are equivalent:

(i) \(C_\tau(\gamma, h)\) is minimal among all pairs in \(\text{ADM}(\mu, \nu)\) with \(h\) fixed.

(ii) \(\gamma\) is concentrated on \(A\) and \(\text{supp}(\gamma) \cup \partial \Omega \times \partial \Omega\) is \(\hat{c}\)-cyclically monotone.

(iii) There exist functions \(\varphi, \varphi^* : \overline{\Omega} \to \mathbb{R}\) having the following properties:

\(\varphi|_{\Omega}\) is \(c_1\) concave, \(\varphi|_{\Omega} = (\varphi^*)^{c_1}\), \(\varphi^*|_{\Omega}\) is \(c_2\) concave, and \(\varphi^*|_{\Omega} = \varphi^{c_2}\).

\(\text{supp}(\gamma|_{\Omega})\) \(\subset \partial \varphi_1\) and \(\text{supp}(\gamma^*|_{\Omega})\) \(\subset \partial \varphi^*\).

\(\varphi_{|\Omega} = \Psi\) and \(\varphi^*_{|\Omega} = -\Psi\).

Moreover, \((\gamma, h)\) is optimal in \(\text{ADM}(\mu, \nu)\) if and only if \(\varphi^*|_{\Omega} = -e' \circ h + \kappa, \mathcal{L}^d\) a.e., for some constant \(\kappa\).

**Proof.** We start by proving that \((i) \implies (ii)\). Define the plan \(\tilde{\gamma}\) by

\[
\tilde{\gamma} := \gamma|_A + (\pi^1, P_{\Psi, \tau} \circ \pi^1)^\# \left( \gamma|_{\overline{\Omega} \times \overline{\Omega} \setminus A} \right) + (P_{-\Psi, \tau} \circ \pi^2, \pi^2)^\# \left( \gamma|_{\overline{\Omega} \times \overline{\Omega} \setminus A} \right).
\]

Observe that \(\tilde{\gamma} \in \text{ADM}(\mu, \nu)\) and

\[
C_\tau(\gamma, h) = \int_{\overline{\Omega} \times \overline{\Omega}} \left( \frac{|x - y|^2}{2\tau} 1_{(\partial \Omega \times \partial \Omega)^c} + \Psi(y) 1_{\Omega \times \partial \Omega} - \Psi(x) 1_{\partial \Omega \times \Omega} \right) d\tilde{\gamma} = \int_A \tilde{c} d\tilde{\gamma} + \int_{\overline{\Omega} \times \overline{\Omega} \setminus A} \left( d_{\varphi, \tau}(x) + d_{-\varphi, \tau}(y) \right) d\gamma
\]

\[
\leq \int_{\overline{\Omega} \times \overline{\Omega}} \left( \frac{|x - y|^2}{2\tau} 1_{(\partial \Omega \times \partial \Omega)^c} + \Psi(y) 1_{\Omega \times \partial \Omega} - \Psi(x) 1_{\partial \Omega \times \Omega} \right) d\gamma
\]

\[
= C_\tau(\gamma, h),
\]

with strict inequality if \(\gamma|_{\overline{\Omega} \times \overline{\Omega} \setminus A} > 0\). Thus, from the optimality of \(\gamma\), we deduce that it is concentrated on \(A\).

Now we have to prove the \(\hat{c}\)-cyclical monotonicity of \(\text{supp}(\gamma) \cup \partial \Omega \times \partial \Omega\). Note that

\[
C_\tau(\gamma, h) = C_\tau(\gamma + \mathcal{H}^{d-1}_{\partial \Omega} \otimes \mathcal{H}^{d-1}_{\partial \Omega}, h).
\]

Hence, we can assume without loss of generality that \(\partial \Omega \times \partial \Omega \subset \text{supp}(\gamma)\). Let \(\{(x_i, y_i)\}_{i=1}^n \in \text{supp}(\gamma)\). Our objective is to show that

\[
\sum_i \tilde{c}(x_i, y_{\sigma(i)}) - \tilde{c}(x_i, y_i) \geq 0, \quad \text{for all permutations } \sigma \text{ of } \{1, ..., n\}.
\]
We proceed by contradiction. For this purpose, we assume that the above inequality fails for some permutation \( \sigma \). Let
\[
X_i = \begin{cases} 
\partial \Omega & \text{if } x_i \in \partial \Omega, \\
\Omega & \text{otherwise}, 
\end{cases}
\]
and
\[
Y_i = \begin{cases} 
\partial \Omega & \text{if } y_i \in \partial \Omega, \\
\Omega & \text{otherwise}. 
\end{cases}
\]
The cost \( \tilde{c} \) is continuous in \( X_i \times Y_i \), for any \( i \in \{0, \ldots, n\} \). Hence, we can find neighborhoods \( U_i \subset X_i \) and \( V_i \subset Y_i \) of \( x_i \) and \( y_i \) such that
\[
\sum_{i=1}^{N} \tilde{c}(u_i, v_{\sigma(i)}) - \tilde{c}(u_i, v_i) < 0 \quad \forall (u_i, v_i) \in U_i \times V_i \quad \text{and} \quad \forall i \in \{0, \ldots, n\}.
\]
We will build a variation of \( \gamma \), \( \tilde{\gamma} = \gamma + \eta \), in such a way that its minimality is violated. To this aim, we need a signed measure \( \eta \) with:

(A) \( \eta^- \leq \gamma \) (so that \( \tilde{\gamma} \) is non-negative);

(B) \( \pi_\# \eta \Omega = \pi_\# \eta_\Omega = 0 \) (so that \( (\tilde{\gamma}, h) \) is admissible);

(C) \( \int_{\Omega \times \Omega} \tilde{c}(x, y) \, d\eta < 0 \) (so that \( \gamma \) is not optimal).

Let \( \mathcal{C} = \prod_{i=1}^{N} U_i \times V_i \) and \( P \in \mathcal{P}(\mathcal{C}) \) be defined as the product of the measures \( \frac{1}{m_i} \gamma_{U_i \times V_i} \). Here, \( m_i := \gamma(U_i \times V_i) \). Denote by \( \pi_i \) and \( \pi_i \) the natural projections of \( \mathcal{C} \) to \( U_i \) and \( V_i \) respectively. Also, define
\[
\eta := \frac{\min_i m_i}{N} \sum_{i=1}^{N} \left( \pi_i \right)_# \left( \pi_i \right)_# P - \left( \pi_i \right)_# P.
\]
Since \( \eta \) satisfies (A), (B), and (C), the \( \tilde{c} \)-cyclical monotonicity is proven.

Next, we prove that \( (ii) \implies (iii) \).

Arguing as Step 2 of [1, Theorem 1.13], we can produce a \( \tilde{c} \)-concave function \( \tilde{\varphi} \) such that \( \text{supp}(\gamma) \cup \partial \Omega \times \partial \Omega \subset \partial^c \tilde{\varphi} \). Then,

\[
(3.17) \quad \tilde{\varphi}(x) + \tilde{\varphi}^c(y) = \frac{|x-y|^2}{2 \tau} 1_{(\partial \Omega \times \partial \Omega)} c - \Psi(x) 1_{\partial \Omega \times \partial \Omega} + \Psi(y) 1_{\partial \Omega \times \partial \Omega} \quad \forall (x, y) \in \text{supp}(\gamma) \cup \partial \Omega \times \partial \Omega
\]

and
\[
\tilde{\varphi}(x) + \tilde{\varphi}^c(y) \leq \frac{|x-y|^2}{2 \tau} 1_{(\partial \Omega \times \partial \Omega)} c - \Psi(x) 1_{\partial \Omega \times \partial \Omega} + \Psi(y) 1_{\partial \Omega \times \partial \Omega} \quad \forall (x, y) \in \Omega \times \Omega.
\]

After adding a constant, we can assume \( \tilde{\varphi}^c(y_0) = 0 \) for some \( y_0 \in \partial \Omega \). Then, using (3.17) it is easy to show that \( \tilde{\varphi} = 0 \) on \( \partial \Omega \). Consequently, \( \tilde{\varphi} = 0 \) on \( \partial \Omega \) as well.

Set \( \varphi = \tilde{\varphi} + 1_{\partial \Omega} \Psi \) and \( \varphi^* = \tilde{\varphi} - 1_{\partial \Omega} \Psi \). Since the measure \( \mu \) is strictly positive, by (3.17) we have
\[
\inf_{y \in \Omega} c(x, y) - \varphi^*(y) = \varphi(x) \quad \forall x \in \Omega.
\]

Similarly, since \( \pi_{2#}\gamma \) is strictly positive, we have
\[
\inf_{x \in \Omega} c(x, y) - \varphi(x) = \varphi^*(y) \quad \forall y \in \Omega.
\]

Then, all the items in (iii) can be verified using (3.17) (see [1, Definitions 1.7 to 1.10]).

We proceed to prove that (iii) \implies (i).
Let \((\tilde{y}, h)\) be any admissible pair. We set \(\tilde{\varphi} = \varphi - \Psi 1_{\partial \Omega} \) and \(\tilde{\varphi}^* = \varphi^* + \Psi 1_{\partial \Omega}\). By item (b) of (iii), we have that (3.17) holds with \(\tilde{\varphi}^*\) in place of \(\tilde{\varphi}^c\). Moreover, from (c) we get \(\tilde{\varphi}|_{\partial \Omega} = \tilde{\varphi}|_{\partial \Omega}^* = 0\). From (a), (b), and (B2), we obtain that \(\tilde{\varphi}|_{\Omega}\) and \(\tilde{\varphi}|_{\Omega}^*\) are Lipschitz. Thus, they are integrable against any measure with finite mass. As a consequence of these observations, we deduce

\[
C_{r}(\gamma, h) = \int_{\Omega} \tilde{\varphi} d\gamma + \tau \int_{\Omega} e(h) \, dx \\
= \int_{\Omega} \tilde{\varphi}(x) \, d\mu + \int_{\Omega} \tilde{\varphi}^*(y) \, d\nu + \tau \int_{\Omega} \tilde{\varphi}^*(y) \, dh + \int_{\Omega} e(h) \, dx \\
= \int_{\Omega} \tilde{\varphi}(x) \, d\mu + \int_{\Omega} \tilde{\varphi}^*(y) \, d\nu + \tau \int_{\Omega} e(h) \, dx \\
\leq \int_{\Omega} \tilde{\varphi} d\gamma + \tau \int_{\Omega} e(h) \, dx \\
= C_r(\tilde{\gamma}, h).
\]

In the third and fourth line above, we have used (1.5). This gives us the desired implication. To prove the last part of the Proposition, we suppose the pair \((\gamma, h)\) is optimal. Also, we claim that there exists a set \(L \subset \Omega\) of zero Lebesgue measure such that for every \(x\) in \(\Omega \setminus L\) there exists \(y \in \Omega \setminus L\) such that \((x, y) \in supp(\gamma) \cup \partial \Omega \times \partial \Omega\) and

\[
(3.18) \quad e' \circ h(\tilde{y})1_{\Omega}(\tilde{y}) + \frac{|x - \tilde{y}|^2}{2\tau}1_{(\partial \Omega \times \partial \Omega)^c}(x, \tilde{y}) + \Psi(\tilde{y})1_{\Omega \times \partial \Omega}(x, \tilde{y}) - \Psi(x)1_{\partial \Omega \times \Omega}(x, \tilde{y}) \\
\geq e' \circ h(y)1_{\Omega}(y) + \frac{|x - y|^2}{2\tau}1_{(\partial \Omega \times \partial \Omega)^c}(x, y) + \Psi(y)1_{\Omega \times \partial \Omega}(x, y) - \Psi(x)1_{\partial \Omega \times \Omega}(x, y),
\]

holds for every \(\tilde{y}\) in \(\Omega \setminus L\). We also claim that this set \(L\) can be taken such that for every \(y\) in \(\Omega \setminus L\) there exists \(x\) in \(\Omega \setminus L\) so that \((x, y) \in supp(\gamma) \cup \partial \Omega \times \partial \Omega\) and the above inequality holds for almost every \(\tilde{y}\) in \(\Omega \setminus L\). We will show these claims at the end of the proof. Now, we show how the result follows from them. Define the function

\[
(-e' \circ h)\tilde{\varphi}(x) = \inf_{y \in \Omega \setminus L} \frac{|x - y|^2}{2\tau}1_{(\partial \Omega \times \partial \Omega)^c} + \Psi(y)1_{\Omega \times \partial \Omega}(x, y) - \Psi(x)1_{\partial \Omega \times \Omega}(x, y) + e' \circ h(y)1_{\Omega}(y).
\]

for every \(x\) in \(\Omega \setminus L\). By (3.18) for every \(x\) in \(\Omega \setminus L\) there exists \(y\) in \(\Omega \setminus L\) such that \((x, y) \in supp(\gamma) \cup \partial \Omega\),

\[
(-e' \circ h(y)1_{\Omega}(y)) + (-e' \circ h)\tilde{\varphi}(x) = \Psi(y)1_{\Omega \times \partial \Omega}(x, y) - \Psi(x)1_{\partial \Omega \times \Omega}(x, y) + \frac{|x - y|^2}{2\tau}1_{(\partial \Omega \times \partial \Omega)^c},
\]

and

\[
(-e' \circ h(y)1_{\Omega}(y)) + (-e' \circ h)\tilde{\varphi}(x) \leq \Psi(y)1_{\Omega \times \partial \Omega}(x, y) - \Psi(x)1_{\partial \Omega \times \Omega}(x, y) + \frac{|x - y|^2}{2\tau}1_{(\partial \Omega \times \partial \Omega)^c},
\]

for almost every \(\tilde{x}\) in \(\Omega \setminus L\). Then, we have that

\[
(-e' \circ h(y)1_{\Omega}(y)) = \inf_{x \in \Omega \setminus L} \Psi(y)1_{\Omega \times \partial \Omega}(x, y) - \Psi(x)1_{\partial \Omega \times \Omega}(x, y) + \frac{|x - y|^2}{2\tau}1_{(\partial \Omega \times \partial \Omega)^c} - (-e' \circ h)\tilde{\varphi}(x).
\]
Thus, it follows that the functions $-e' \circ h_{\mid \Omega \setminus L}$ admits a Lipschitz extension to $\Omega$, which we will not relabel. Consequently, for every $y$ in $\Omega \setminus L$ there exists $x \in \overline{\Omega} \setminus L$ such that $(x, y) \in \text{supp}(\gamma)$ and (3.18) holds for every $\tilde{y} \in \Omega$. Then by (3.18), for every $y$ in $\Omega \setminus L$ there exists $x \in \overline{\Omega}$ and a constant $A := A(x, y)$ such that $(x, y)$ is in $\text{supp}(\gamma)$ and

$$
\tau e' \circ h(y) + \frac{|y|^2}{2} + \langle x, \tilde{y} - y \rangle + A(x, y) \leq \frac{|\tilde{y}|^2}{2} + \tau e' \circ \tilde{h}(\tilde{y}),
$$

(3.19) for every $\tilde{y} \in \Omega$. Let $\mathcal{P}$ be the set of affine functions that are below $\tau e' \circ h(y) + \frac{|y|^2}{2}$ in $\Omega$. Then, it follows that

$$
\tau e' \circ h + \frac{|y|^2}{2} = \sup_{p \in \mathcal{P}} p(y),
$$

for every $y$ in $\Omega \setminus L$. This together with the Lipschitz continuity of $-e' \circ h_{\mid \Omega}$ implies that the function $\tau e' \circ h(y) + \frac{|y|^2}{2}$ is convex. In a similar way from (3.17) we can deduce that $\varphi^\ast_{\mid \Omega}$ is Lipschitz, $\frac{|y|^2}{2} - \tau \varphi^\ast(y)$ is convex, and for a.e $y$ in $\Omega$ there exists a point $x \in \overline{\Omega}$ and a constant $B := B(x, y)$ such that $(x, y) \in \text{supp}(\gamma)$ and

$$
-\tau \varphi^\ast(y) + \frac{|y|^2}{2} + \langle x, \tilde{y} - y \rangle + B(x, y) \leq \frac{|\tilde{y}|^2}{2} + \tau \varphi^\ast(\tilde{y}),
$$

(3.20) for every $\tilde{y} \in \Omega$. Recall $\nu + \tau h$ is absolutely continuous and uniformly bounded from below. Consequently, by Lemma A.2 $\frac{\partial}{\partial t} \gamma^\Omega = (S, Id) \# \nu + \tau h$, for a map $S$ that is optimal in the classical sense and is uniquely defined a.e. Thus, it follows from (3.19) and (3.20) that $\frac{|y|^2}{2} - \tau \varphi^\ast(y)$ and $\tau e' \circ h(y) + \frac{|y|^2}{2}$ are Lipschitz, and have the same derivative a.e in $\Omega$. Therefore, we deduce that there exists a constant $\kappa$ such that

$$
\varphi^\ast = -e' \circ h + \kappa \text{ a.e in } \Omega.
$$

In order to prove the opposite implication, suppose $\tilde{\varphi}^\ast = -e' \circ h + \kappa$ and let $(\tilde{\gamma}, \tilde{h}) \in \text{ADM}(\mu, \nu)$. Then we argue as in $(iii) \implies (i)$, we obtain

$$
C_{\tau}(\gamma, h) = \int_{\Omega \times \overline{\Omega}} \tilde{c} \, d\gamma + \tau \int_{\Omega} e(h) \, dx
$$

$$
= \int_{\Omega \times \overline{\Omega}} \left( [\tilde{\varphi}(x) + \kappa] + [\tilde{\varphi}^\ast(y) - \kappa] \right) \, d\gamma + \tau \int_{\Omega} e(h) \, dx
$$

$$
= \int_{\Omega} [\tilde{\varphi}(x) + \kappa] \, d\mu + \int_{\Omega} [\tilde{\varphi}^\ast(y) - \kappa] \, d\nu + \tau \int_{\Omega} [\tilde{\varphi}^\ast(y) - \kappa] \mu \, dx + \tau \int_{\Omega} e(h) \, dx
$$

$$
= \int_{\Omega \times \overline{\Omega}} \left( \tilde{\varphi}(x) + \tilde{\varphi}^\ast(y) \right) \, d\tilde{\gamma} + \tau \int_{\Omega} [\tilde{\varphi}^\ast(y) - \kappa](h - \tilde{h}) \, dx + \tau \int_{\Omega} e(h) \, dx
$$

$$
\leq \int_{\Omega \times \overline{\Omega}} \tilde{c} \, d\tilde{\gamma} + \tau \int_{\Omega} e(h) \, dx + \tau \int_{\Omega} e' \circ h(\tilde{h} - h) \, dx
$$

$$
\leq \int_{\Omega \times \overline{\Omega}} \tilde{c} \, d\tilde{\gamma} + \tau \int_{\Omega} e(\tilde{h}) \, dx.
$$

Here, in the last inequality we used (C2). This completes the proof of the Theorem, provided we can prove the claim.

Finally, we show (3.18). The idea is to use Proposition A.3 and the absolute continuity and uniform positivity of $\mu$ and $\nu + \tau h$. We only prove the statement holds for $x \in \overline{\Omega} \setminus L$;
the corresponding statement for \( y \) is analogous. In order to do this we will use the same notation as in Proposition A.3.

Let \( L_1 \) be a set of zero Lebesgue measure such that every point in \( \Omega \setminus L_1 \) is a Lebesgue point for \( S, \nu + \tau h \), and \( h \). Also let \( L_2 \) be a set of zero Lebesgue measure such that every point in \( \Omega \setminus L_2 \) is a Lebesgue point for \( T \) and the density of \( \mu \). Let \( A = \{ y \in \Omega \setminus L_1 : S(y) \in \Omega \} \) and \( B = \{ x \in \Omega \setminus L_2 : T(x) \in \partial \Omega \} \). Since \( \pi_1#(\gamma_{\Omega \setminus \Omega} + \gamma_{\partial \Omega}) = \mu \) and \( \nu + \tau h \) and \( \mu \) are absolutely continuous and uniformly positive, it follows that \( \Omega \setminus (S(A) \cup T(B)) \) has zero Lebesgue measure. Let \( L_3 = \Omega \setminus (S(A) \cup T(B)) \). Then, for every \( x \in \Omega \setminus L \) we have two possibilities: Either there exists \( y \in \Omega \setminus L \) such that \( x = S(y) \), in which cases the claim follows from (A.43) and (A.44), or \( T(x) \in \partial \Omega \), in which case the claim follows from (A.45) and (A.46). It remains to consider the case when \( x \in \partial \Omega \setminus L \). In such case the statement follows from (A.47) and (A.48). This concludes the proof of the Proposition.

The following result is the analogue in our setting of Brenier’s Theorem on existence and uniqueness of optimal transport maps.

**Corollary 3.3. (On uniqueness of optimal pairs)** Let \( \mu, \nu \in M(\Omega) \) and fix \( (\gamma, h) \in \text{Opt}(\mu, \nu) \) satisfying the hypotheses of the previous Proposition. Additionally, let \( \varphi \) and \( \varphi^* \) be the functions given by Proposition 3.2. Then

(i) The function \( h \) is unique \( \mathcal{L}^d \) a.e.

(ii) The plan \( \gamma_{\Omega \setminus \Omega} \) is unique and it is given by \((\text{Id}, T)_# \mu \). Also, \( T : \Omega \rightarrow \overline{\Omega} \) is the gradient of a convex function and

\[-\nabla \varphi = \frac{T - \text{Id}}{\tau} \quad \text{a.e. in } \Omega.\]

(iii) The plan \( \gamma_{\Omega \setminus \Omega} \) is unique and it is given by \((S, \text{Id})_# \nu \). Also, \( S : \Omega \rightarrow \overline{\Omega} \) is the gradient of a convex function and

\[-\nabla \varphi^* = \frac{S - \text{Id}}{\tau} \quad \text{a.e. in } \Omega.\]

(iv) If \( \gamma \) has no mass concentrated on \( \partial \Omega \times \partial \Omega \), then \( \gamma \) is unique.

**Proof.** By linearity of the constraint (1.5) in \( \text{ADM}(\mu, \nu) \), the uniqueness of \( h \) follows by (C2). Due to the equivalence \((i) \iff (iii)\) of the previous Theorem, using (a) and (b) we get that the functions \( \tau \varphi \) and \( \tau \varphi^* \) are \( \frac{\partial}{\partial t} \)-concave. Here, \( d(x, y) = |x - y| \). Thus, the result follows exactly as in the classical transportation problem (see for example [6, Theorem 6.2.4 and Remark 6.2.11]).

Henceforth we will assume, without loss of generality, that the transportation plans \( \gamma \) have no mass concentrated on \( \partial \Omega \times \partial \Omega \).

4. The weak solution

In this section we follow the minimizing movement scheme described in the introduction. This method yields a map, \( t \rightarrow \rho(t) \), that belongs to \( L^2_{\text{loc}}([0, \infty), W^{1,2}(\Omega)) \). Such a map is a weak solution to (1.7). By this, we mean that the map \( t \rightarrow \rho(t) - e^{\Psi - V} \) belongs to \( L^2_{\text{loc}}([0, \infty), W^{1,2}_0(\Omega)) \),

\[ \rho(0) = \rho_0 \quad \text{in } \Omega, \]
and
\[ \int_{\Omega} \zeta \rho(s) \, dx - \int_{\Omega} \zeta \rho(t) \, dx = \int_{t}^{s} \left( \int_{\Omega} \left[ \Delta \zeta - \langle \nabla V, \nabla \zeta \rangle \right] \rho(r) \, dx - \int_{\Omega} \zeta [\epsilon \cdot]^{-1} \left( \log(\rho(r)) + V \right) \, dx \right) \, dr, \]
for all \( 0 \leq t < s \) and \( \zeta \) in \( C_c^\infty(\Omega) \).

Similarly, we will say that a map \( t \to \rho(t) \) in \( L^2_{\text{loc}}(0, \infty), W^{1,2}(\Omega) \) is a weak solution of (1.1), if there exists a Lipschitz function \( \tilde{\rho} \) such that \( t \to \rho(t) - \tilde{\rho} \) belongs to \( L^2_{\text{loc}}([0, \infty), W^{1,2}_0(\Omega)) \),

\[ \tilde{\rho} = \rho_D \quad \text{on} \quad \partial \Omega, \]

and
\[ \int_{\Omega} \zeta \rho(s) \, dx - \int_{\Omega} \zeta \rho(t) \, dx = \int_{t}^{s} \left( \int_{\Omega} \left[ \Delta \zeta - \langle \nabla V, \nabla \zeta \rangle \right] \rho(r) \, dx - \int_{\Omega} \zeta \mathcal{E}_x'(|\rho(r)|) \, dx \right) \, dr, \]
for all \( 0 \leq t < s \) and \( \zeta \) in \( C_c^\infty(\Omega) \).

\[ E(\mu) := \begin{cases} \int_{\Omega} \mathcal{E}(\rho(x), x) \, dx & \text{if} \quad \mu = \rho \mathcal{L}^d |_{[0, \infty)}, \\ +\infty & \text{otherwise}, \end{cases} \]
where \( \mathcal{E} : [0, \infty) \times \Omega \to [0, \infty) \) is given by

\[ \mathcal{E}(z, x) := z \log z - z + V(x)z + 1. \]

We will denote by \( \mathcal{E}' \) the derivative of \( \mathcal{E} \) with respect to its first variable and by \( D(\mathcal{E}) \) the interior of the sets of points where \( \mathcal{E} \) is finite. The notations \( \mathcal{E}(\rho(x), x) \) and \( \mathcal{E}(\rho) \) will be used interchangeably. Also, we will freely interchange \( \mathcal{E}'(\rho(x), x) \) and \( \mathcal{E}'(\rho) \).

The main result is the following:

**Theorem 4.1.** With the notation from the introduction and assumptions (B1) and (B2), for any pair of functions \( e \) and \( \Psi \) satisfying (C1)-(C9), any uniformly positive and bounded initial data \( \rho_0 \), and any sequence \( \tau_k \downarrow 0 \) there exists a subsequence, not relabeled, such that \( \rho^{\tau_k}(t) \) converges to \( \rho(t) \) in \( L^2(0; \mathcal{T}_f, L^2_{\text{loc}}(\Omega)) \), for any \( \mathcal{T}_f > 0 \). The map \( t \to \rho(t) \) belongs to \( L^2_{\text{loc}}([0, \infty), W^{1,2}(\Omega)) \) and is a weak solution of (1.1). Moreover, there exist positive constants \( \lambda \) and \( \Lambda \) such that

\[ \frac{\lambda}{\inf \{ e^{-V} \}} e^{-\frac{(C_\Omega \tau t + V)}{\inf \{ e^{-V} \}}} \leq \rho(x, t) \leq \frac{\Lambda}{\sup \{ e^{-V} \}} e^{-V}, \]
for almost every \( x \).

**Remark 4.2.** When assumptions (B1)-(B3) and (F1)-(F7) hold, and \( e \) and \( \Psi \) are as in (1.8) and (1.9), properties (C1)-(C9) hold as well and the map \( t \to \rho(t) \) given by the previous Theorem is a weak solution of (1.1).

The proof of Theorem 4.1 is involved. We begin with a technical result.

**Proposition 4.3.** *(A step of the minimizing movement)* Let \( \mu \) be a measure in \( \mathcal{M}(\Omega) \) with the property that \( E(\mu) < \infty \). Also, assume that its density is uniformly positive and bounded. Additionally, let \( \tau \) be a positive number. Then, there exists a minimum \( \mu_\tau \in \mathcal{M}(\Omega) \) of

\[ \rho \to E(\rho) + W^{1,2}_2(\mathcal{E}_x, \mathcal{E}_x')|\rho| \]

Moreover, there exists \( \delta > 0 \) such that if \( \tau < \delta \), then the corresponding optimal pair \( (\gamma, h) \in ADM(\mu, \mu_\tau) \) satisfies:
(i) $\mu_{\tau} = \rho_{\tau} \mathcal{L}^d_{\Omega}$.
(ii) $e' \circ h = \log \rho_{\tau} + V$.
(iii) The restriction of $\gamma$ to $\overline{\Omega} \times \Omega$ is given by $(T, \text{Id})_{\#} \mu_{\tau}$. The map $T$ satisfies
\begin{equation}
\frac{T(y) - y}{\tau} = \nabla \log \rho_{\tau}(y) + \nabla V(y), \quad \mathcal{L}^d - \text{a.e.} x.
\end{equation}
(iv) $\rho_{\tau} \in W^{1,2}(\Omega)$ and $||\text{Tr} [\rho_{\tau} - e^\Psi - V]||_{L^\infty(\partial\Omega)} \leq C \sqrt{\tau}$.

Here, $C$ is a positive constant that depends only on $\Psi$. Also, $\text{Tr} : W^{1,2}(\Omega) \rightarrow L^2(\partial \Omega)$ denotes the trace operator.

**Proof.** Consider a minimizing sequence of measures $\{\rho^n\}_{n=1}^{\infty}$, with corresponding optimal pairs $\{\gamma^n, h^n\}_{n=1}^{\infty}$ in $\text{ADM}(\mu, \rho^n)$. We claim such sequences of measures and optimal pairs have the property that the mass the elements of $\{\rho^n, \gamma^n\}_{n=1}^{\infty}$ and the norm in $L^1(\Omega)$ of the members of $\{h^n\}_{n=1}^{\infty}$ are uniformly bounded. Since $\Omega$ is bounded, the claim allows us to obtain compactness and produce subsequences weakly converging to $\gamma$, $h$, and $\rho_{\tau}$. The previous convergence takes place as described in the proof of Lemma 3.1. We will not relabel these subsequences. The absolute continuity of $h$ and $\rho_{\tau}$ is guaranteed by the superlinearity of $e$ and $E$.

The inequality
\[ \liminf_{n \rightarrow \infty} E(\rho^n) \geq E(\rho_{\tau}), \]

is a consequence of the weak convergence, $\rho_n \rightharpoonup \rho$, and the convexity and superlinearity of the maps $\{r \mapsto E(r, x)\}_{x \in \Omega}$ (See [5, Lemma 9.4.5], for example). To show
\[ \liminf_{n \rightarrow \infty} C_{\tau}(\gamma^n, h^n) \geq C_{\tau}(\gamma, h), \]

and $(\gamma, h) \in \text{ADM}(\mu, \rho_{\tau})$, we argue as in Lemma 5.1. This gives us the existence of a minimum as well as item (i), assuming we can prove the claim.

Next, we show the claim. Arguing as in Lemma 3.1 and using Jensen inequality we obtain
\begin{align*}
\int_{\Omega} E(\rho) \, dx + C_{\tau}(\gamma, h) &\geq -||\Psi||_{\infty} \left( \mu(\Omega) + \nu(\Omega) + \tau |h|(\Omega) \right) \\
&\quad + K|h|(\Omega) + (C(K) - 1)|\Omega| + \rho(\Omega) \log \left( \frac{\rho(\Omega)}{\Omega} \right) - (1 + ||V||_{\infty})\rho(\Omega).
\end{align*}

Taking $K$ large enough, we obtain a uniform bound on $\rho(\Omega) + \tau |h|(\Omega)$ and consequently on $|\gamma|$, for any minimizing sequence. (Recall we assume that the plans have no mass concentrated on $\partial \Omega \times \partial \Omega$). This proves the claim.

We proceed to the proof of (ii). Let $\eta$ be a function with compact support in $\Omega$. For each $\varepsilon > 0$, let $\rho^\varepsilon_{\tau} = \rho_{\tau} - \tau \varepsilon \eta$. By Lemma 5.1 for sufficiently small $\varepsilon$ we can guarantee that $\rho^\varepsilon_{\tau}$ is non-negative. Since $(\gamma, h + \varepsilon \eta) \in \text{ADM}(\mu, \rho^\varepsilon_{\tau})$, by minimality must have
\[ E(\rho^\varepsilon_{\tau}) - E(\rho_{\tau}) + C_{\tau}(\gamma, h + \varepsilon \eta) - C_{\tau}(\gamma, h) \geq 0. \]

Dividing by $\varepsilon$ and letting $\varepsilon \downarrow 0$, due to [15], Lemma 5.1 Lemma A.2 the dominated convegence Theorem and the fact that $e$ and $E$ are locally Lipschitz in $D(e)$ and $D(E)$, we get
\[ \int_{\Omega} (e' \circ h) \eta \, dy - \int_{\Omega} (\log \rho_{\tau} + V) \eta \, dy \geq 0. \]

Replacing $\eta$ by $-\eta$ gives the desired result.
Now, we show (iii). Let \( \lambda \) and \( \Lambda \) be positive numbers such that Proposition 5.3 holds. Then, for \( \tau \in (0, 1) \) we have that \( \rho, \rho_\tau > \lambda (\inf e^{-V} / \sup e^{-V})/(1 + C_0) \). We let \( \delta \in (0, 1) \) have the property that Corollary 5.7 and Proposition A.4 hold for any \( \tau \in (0, \delta) \). Now, observe that Corollary 3.3, and Proposition A.3 hold for any \( \Omega \) and \( \Psi \). Finally (iv) follows from (ii), Corollary 3.3, Proposition 3.2 (Note that in Corollary 3.3 \( T \) plays the role of \( S \)).

To show (iv) we note that, by minimality of \( \rho_\tau \),

\[
W_b^\rho,\tau (\mu, \rho_\tau) \leq E(\mu) - E(\rho_\tau),
\]

and thus

\[
\frac{1}{2\tau} \int_{\Omega} |\nabla \log \rho_\tau + \nabla V|^2 (\rho_\tau + \tau h) \, dy \leq \frac{1}{2\tau} \int_{\Omega \times \Omega} |x - y|^2 \, d\gamma_\tau
\]

\[
\leq E(\mu) - E(\rho_\tau) - \tau \int_{\Omega} e(h) \, dy + \int_{\partial \Omega \times \Omega} \Psi(x) \, d\gamma - \int_{\Omega \times \Omega} \Psi(y) \, d\gamma.
\]

Consequently, after making \( \delta \) smaller if necessary, we get

\[
\int_{\Omega} |\nabla \rho_\tau|^2 \, dy = C_2 \int_{\Omega} |\nabla \log \rho_\tau|^2 (\rho_\tau + \tau h) \, dy < \infty.
\]

Here, \( C_2 := C_2(\Psi, e, V, \rho_0) \). Also, we have used the fact that \( \rho_\tau \) is bounded from below by \( \lambda/(1 + C_0) \), \( V \) belongs to \( W^{1,2}(\Omega) \), and Corollary 5.7 holds. Combining (5.36), Lemma 5.4 and Lemma 5.10 we can see that

\[
\frac{|y - P(y)|^2}{2\tau} - C_1 |y - P(y)| - C\sqrt{\tau} \leq -\Psi(P(y)) + \log \rho_\tau(y) + V(y)
\]

\[
\leq C\sqrt{\tau} + C_1 |y - P(y)| + \frac{|y - P(y)|^2}{2\tau},
\]

where \( P(y) \) denotes any of the closest points in \( \partial \Omega \) to \( y \). Also, \( C \) and \( C_1 \) depend only on \( \partial \Omega \) and \( \Psi \). Finally (iv) follows from the previous inequality.

\[\Box\]

**Proof of Theorem 4.1.** Let \( \rho_0 \) be bounded and uniformly positive. Let \( \delta \in (0, 1) \) be such that Propositions 4.3 and 5.9, and Corollary 5.7 hold for any \( \tau \in (0, \delta) \). For any \( n \in \mathbb{N} \), let \( (\gamma^\tau_n, h^\tau_n) \) be the minimizing pair from \( \rho_n^\tau \) to \( \rho^\tau_{n+1} \). Also, let \( T^\tau_n \) be the map that induces \( (\gamma^\tau_n, h^\tau_n, \Omega) \) given by Proposition 4.3 (ii).

Let \( t_f \) be a positive number larger than \( \tau \). Iterating Proposition 5.3 we can see that that there exist positive constants \( \lambda \) and \( \Lambda \) such that

\[
(4.24) \quad \left( (1 + C_0 \tau)^\frac{1}{\gamma} \right)^{-n} \frac{\lambda}{\sup e^{-V} e^{-V}} \leq \rho^\tau_n \leq \frac{\Lambda}{\inf e^{-V} e^{-V}} \quad \forall n \in \mathbb{N}.
\]

Note

\[
\lim_{\tau \to 0} (1 + C_0 \tau)^{\frac{1}{\gamma}} = e^{C_0}.
\]

Hence, for sufficiently small \( \tau \) we obtain a uniform lower bound for \( \rho^\tau_n \) whenever \( n \tau \leq t_f + 1 \). Then, Lemmas 5.3, 5.5, 5.6, 5.8, Corollary 5.7, and Proposition 5.9 can be iterated to hold, with uniform constants \( C, \kappa_1 \), and \( \kappa_2 \), for all these measures. Henceforth, we assume the condition \( n \tau \leq t_f + 1 \). Fix \( \zeta \in C^\infty(\Omega) \).

Recall that given \( \gamma \), we denote by \( \gamma_A^B \) its restriction to \( A \times B \). Note that since

\[
\gamma^\tau_n = (\gamma_n^\tau)_{\Omega} + (\gamma_n^\tau)_{\partial \Omega} + (\gamma_n^\tau)_{\partial \Omega},
\]
by \[13.5\] we have

\[
\mu^\tau_n = (\pi_1)_\#(\gamma^\tau_n)_\Omega + (\pi_1)_\#(\gamma^\tau_n)_{\partial\Omega},
\]

and

\[
\mu^\tau_{n+1} = (\pi_2)_\#(\gamma^\tau_n)_\Omega + (\pi_2)_\#(\gamma^\tau_n)_{\partial\Omega} - \tau h^\tau_n \, dy.
\]

Consequently, we obtain

\[
(4.25) \quad \int_\Omega \zeta \, d\mu^\tau_{n+1} - \int_\Omega \zeta \, d\mu^\tau_n = \int_{\Omega \times \Omega} \zeta \circ \pi_2 \, d(\gamma^\tau_n)_\Omega - \int_{\Omega \times \Omega} \zeta \circ \pi_1 \, d(\gamma^\tau_n)_\Omega
\]

\[
- \tau \int_\Omega \zeta h^\tau_n \, dy + \int_{\Omega \times \Omega} \zeta \circ \pi_2 \, d(\gamma^\tau_n)_{\partial\Omega} - \int_{\Omega \times \Omega} \zeta \circ \pi_1 \, d(\gamma^\tau_n)_{\partial\Omega}.
\]

First, using Proposition \[4.3\] and a Taylor expansion,

\[
(4.26) \quad \int_{\Omega \times \Omega} \zeta \circ \pi_2 \, d(\gamma^\tau_n)_\Omega - \int_{\Omega \times \Omega} \zeta \circ \pi_1 \, d(\gamma^\tau_n)_\Omega
\]

\[
= \int_{\Omega \times \Omega} (\zeta(y) - \zeta(x)) \, d(\gamma^\tau_n)_\Omega
\]

\[
= \int_{\Omega \times \Omega} (\zeta(y) - \zeta(T^n_\tau(y)))1_{\{x=T^n_\tau(y)\}} \, d(\gamma^\tau_n)_\Omega
\]

\[
= \int_{\Omega \times \Omega} (\zeta - \zeta \circ T^n_\tau) \circ \pi_2 \, d(\gamma^\tau_n)_\Omega
\]

\[
= \int_{\Omega \times \Omega} (\zeta - \zeta \circ T^n_\tau) \circ \pi_2 \, d(\gamma^\tau_n)_\Omega + \int_{\Omega \times \Omega} (\zeta - \zeta \circ T^n_\tau)1_{\{T^n_\tau \notin \partial\Omega\}} \circ \pi_2 \, d(\gamma^\tau_n)_{\partial\Omega}
\]

\[
= \int_{\Omega} (\zeta - \zeta \circ T^n_\tau)1_{\{T^n_\tau \notin \partial\Omega\}} \, d\mu^\tau_{n+1} + R_1(\tau, n)
\]

\[
= - \int_{\Omega} \langle \nabla \zeta, T^n_\tau - Id \rangle \rho^\tau_{n+1}1_{\{T^n_\tau \notin \partial\Omega\}} \, dy + R_2(\tau, n) + R_1(\tau, n)
\]

\[
= - \tau \int_{\Omega} \langle \nabla \zeta, \nabla \rho^\tau_{n+1} + \rho^\tau_{n+1} \nabla V \rangle 1_{\{T^n_\tau \notin \partial\Omega\}} \, dy + R_2(\tau, n) + R_1(\tau, n).
\]

Second, by item (iii) of Proposition \[4.3\] we have \(h^\tau_n(y) = [e'_y]^{-1}(\log \rho^\tau_{n+1}(y) + V(y))\) and consequently

\[-\tau \int_{\Omega} \zeta h^\tau_n \, dy = -\tau \int_{\Omega} \zeta [e'_y]^{-1}(\log \rho^\tau_{n+1} + V) \, dy.\]

Third, using Corollary \[3.3\]

\[
\int_{\Omega \times \Omega} \zeta \circ \pi_2 \, d(\gamma^\tau_n)_{\partial\Omega} - \int_{\Omega \times \Omega} \zeta \circ \pi_1 \, d(\gamma^\tau_n)_{\partial\Omega}
\]

\[
= \int_{\Omega \times \Omega} \zeta \circ \pi_21_{\{x=T^n_\tau(y)\}} \, d(\gamma^\tau_n)_{\partial\Omega} - \int_{\Omega \times \Omega} \zeta \circ \pi_11_{\{S^n_\tau(x)=y\}} \, d(\gamma^\tau_n)_{\partial\Omega}
\]

\[
= \int_{\Omega \times \Omega} \zeta(x)1_{\{x=T^n_\tau(y)\}} \, d(\gamma^\tau_n)_{\partial\Omega} - \int_{\Omega \times \Omega} \zeta_{n+1}(y)1_{\{S^n_\tau(x)=y\}} \, d(\gamma^\tau_n)_{\partial\Omega} + R_3(\tau, n)
\]

\[
= \int_{\Omega \times \Omega} \zeta(x) \, d(\gamma^\tau_n)_{\partial\Omega} - \int_{\Omega \times \Omega} \zeta(y) \, d(\gamma^\tau_n)_{\partial\Omega} + R_3(\tau, n).
\]
Here, $S^g_n$ is the map which induces $(\gamma^g_n)^{(g)}_{\Omega}$, given by Corollary 5.8. Putting the above together, we obtain

$$\int_{\Omega} \zeta \, d\mu^g_{n+1} - \int_{\Omega} \zeta \, d\mu^g_n = \tau \left( - \int_{\Omega} \langle \nabla \psi_n+1, \nabla \rho^g_{n+1} + \rho^g_{n+1} \nabla V \rangle 1\{T_n^g \not\subset \partial \Omega \} \, dx \right)$$

(4.27)

$$- \int_{\Omega} \zeta_{n+1} [e]^{-1} (\log \rho^g_{n+1} + V) \, dx \right)$$

$$+ \int_{\Omega \times \Omega} \zeta(x) \, d((\gamma^g_n)^{(g)}_{\partial \Omega} - \int_{\Omega \times \Omega} \zeta(y) \, d((\gamma^g_n)^{(g)}_{\partial \Omega}) + R(n, \tau).$$

Here, $R(n, \tau)$ is given by

$$R(n, \tau) = R_1(n, \tau) + R_2(n, \tau) + R_3(n, \tau)$$

$$= \tau \int_{\Omega} (\zeta(y) - \zeta \circ T^g_n(y))h^g_{n+1} 1\{T^g_n \not\subset \partial \Omega \} \, dy$$

$$+ \int_{\Omega} \left( \int_0^1 \left( \langle \nabla \zeta \circ ((1-s)T^g_n + sId), Id - T^g_n \rangle - \langle \nabla \zeta, Id - T^g_n \rangle \right) \rho^g_{n+1} 1\{T^g_n \not\subset \partial \Omega \} \, dy \right.$$

$$+ \int_{\Omega \times \Omega} \left( \zeta \circ \pi_2 - \zeta \circ \pi_1 \right) 1_{\{x=T^g_n(y)\}} \, d((\gamma^g_n)^{(g)}_{\partial \Omega} - \int_{\Omega \times \Omega} \left( \zeta \circ \pi_1 - \zeta \circ \pi_2 \right) 1\{S^g_n(x)=y\} \, d((\gamma^g_n)^{(g)}_{\partial \Omega}) \right).$$

Recall $\zeta$ is compactly supported. Hence, iterating Lemma 5.4 for sufficiently small $\tau$ we have

$$\left| R(n, \tau) \right| \leq \tau \text{Lip}(\zeta) \int |y - T^g_n(y)||h^g_n| \, dy$$

$$+ \text{Lip}(\nabla \zeta) \int \left| T^g_n - Id \right|^2 \rho^g_{n+1} \, dy$$

$$\leq C_1(\zeta, \Psi, e, V, \rho_0, \Omega) \left[ \tau^{3/2} + \int_{\Omega} \left| T^g_n - Id \right|^2 (\rho^g_{n+1} + \tau h) \, dy \right].$$

Here, we have used Corollary 5.7 and the fact that $\Omega$ is bounded. Now, by Proposition 5.9

$$\frac{1}{2\tau} \int_{\Omega \times \Omega} |x - y|^2 \, d\gamma^g_n \leq C_2(\Psi, e, V, \rho_0) \left( E(\rho^g_n) - \int_{\Omega} \Psi \, d\mu^g_n - E(\rho^g_{n+1}) + \int_{\Omega} \Psi \, d\mu^g_{n+1} + \tau \right).$$

Thus, combining the above inequalities with 1.5, Lemma 5.6 and Corollary 5.7 we get

$$\left| R(n, \tau) \right| \leq C_3(\zeta, \Psi, e, V, \rho_0) \left( \tau^{3/2} + \tau \left[ E(\rho^g_n) - \int_{\Omega} \Psi \, d\mu^g_n - E(\rho^g_{n+1}) + \int_{\Omega} \Psi \, d\mu^g_{n+1} \right] \right).$$

This implies

$$\left| \sum_{n=M}^{N-1} R(n, \tau) \right| \leq C_3(\zeta, \Psi, e, V, \rho_0) \left( \sqrt{\tau(M - N)} \right.$$

$$+ \tau \left[ E(\rho^g_M) - \int_{\Omega} \Psi \, d\mu^g_M - E(\rho^g_N) + \int_{\Omega} \Psi \, d\mu^g_N \right],$$

for sufficiently small $\tau$ and all integers $N$ and $M$ such that $\tau M \leq \tau N \leq t_f + 1.$
Let $\tau = \tau_k$. Also, define
\[
\rho^{\tau_k}(t) = \rho^{\tau_k}_{n+1} \quad \text{for} \quad t \in ((n+1)\tau_k, n\tau_k],
\]
and
\[
\theta_h \rho^{\tau_k}(t) = \rho^{\tau_k}(t + h),
\]
for any positive constant $h$. Now, choose $0 \leq r < s < t_f + 1$ and add up (4.27) from $M = \lceil r/\tau_k \rceil$ to $N = \lceil s/\tau_k \rceil - 1$ to get
\[
(4.29) \quad \int_{\Omega} \zeta \rho^{\tau_k}(s) \, dx - \int_{\Omega} \zeta \rho^{\tau_k}(r) \, dx \\
= \int_{\tau_k\lceil s/\tau_k \rceil}^{\tau_k\lceil r/\tau_k \rceil} \left( - \int_{\Omega} \langle \nabla \zeta, \nabla \rho^{\tau_k}(t) + \nabla V \rho^{\tau_k} \rangle 1_{\{T^{\tau_k}_{n+1} \not\in \partial\Omega; [t/\tau_k] = n\}} \, dx \\
- \int_{\Omega} \zeta [e']^{-1}(\log \rho^{\tau_k}(t) + V) \right) \, dt + \left| \sum_{n=M}^{N} R(n, \tau_k) \right| \\
= \int_{\tau_k\lceil r/\tau_k \rceil}^{\tau_k\lceil s/\tau_k \rceil} \left( \int_{\Omega} \Delta \zeta - \langle \nabla \zeta, \nabla V \rangle \right) \rho^{\tau_k}(t) \, dx - \int_{\Omega} \zeta [e']^{-1}(\log \rho^{\tau_k}(t) + V) \right) \, dt \\
+ \sum_{n=M}^{N} R(n, \tau_k).
\]
Here, we have used the fact that by Lemma 5.4 for sufficiently small $\tau$, $\{T^{\tau_k}_{n+1} \in \partial\Omega; [t/\tau_k] = n\}$ and $\text{supp}(\zeta)$ are disjoint.

The strategy to pass to the limit is to use the Aubin-Lions Theorem [14, Theorem 5]. Let $U$ be an open set with Lipschitz boundary whose closure is compactly contained in $\Omega$. Also, set $p > d + 1$. First, note $L^2(U)$ embeds in the dual of $W^{2,p}(U)$. We will denote this space by $W^{2,p}(U)$. Second, observe $W^{1,2}(U)$ embeds compactly in $L^2(U)$ (recall $\Omega$ is bounded). Thus, in order to use the Aubin-Lions Theorem, we will show $\rho^{\tau_k}$ is bounded in $L^2(0, t_f; L^2(U)) \cap L^1_{\text{loc}}(0, t_f; W^{1,2}(U))$ and
\[
||\theta_h \rho^{\tau_k} - \rho^{\tau_k}||_{L^1(t_1, t_2; W^{2,p}(U))} \to 0 \quad \forall 0 \leq t_1 < t_2 < t_f,
\]
as $h \to 0$, uniformly.

Given $t \in (t_1, t_2)$ set $N = \lceil \frac{t+h}{\tau_k} \rceil - 1$ and $M = \lceil \frac{t}{\tau_k} \rceil$. For each $\zeta \in W^{2,p}(U)$, we consider an extension to $\mathbb{R}^n$ (not relabeled) satisfying $\text{supp}(\zeta) \subset \Omega$ and
\[
\|\zeta\|_{W^{2,p}(\mathbb{R}^n)} \leq C_4 \|\zeta\|_{W^{2,p}(U)}.
\]
Here, \( C_4 := C_4(U, \Omega) \). Then,

\[
\int_\Omega \zeta(\theta_h \rho^\tau_k(t) - \rho^\tau_k(t)) \, dx \\
= \sum_{n=M}^N \int_\Omega \zeta \, d\mu_n^\tau_k - \int_\Omega \zeta \, d\mu_n^\tau_k \\
= \sum_{n=M}^N \int_\Omega (\zeta(y) - \zeta(x)) \, d\gamma_n^\tau_k - \int_\Omega \zeta \tau_n \, d\gamma_n^\tau_k \, dx \\
= \sum_{n=M}^N \int_{\Pi \times \Pi} \int_0^1 (\nabla \zeta(x + s(y - x)), y - x) \, ds \, d\gamma_n^\tau_k - \int_\Omega \zeta \tau_n \, d\gamma_n^\tau_k \, dx \\
\leq \sum_{n=M}^N \int_{\Pi \times \Pi} \left( \int_0^1 |\nabla \zeta(x + s(y - x))|^2 \, ds \, d\gamma_n^\tau_k \right)^{\frac{1}{2}} \left( \int_{\Pi \times \Pi} |y - x|^2 \, d\gamma_n^\tau_k \right)^{\frac{1}{2}} \\
+ C_5 \tau_k \|\zeta\|_{W^{2,p}(U)} \\
\leq C_6 \|\zeta\|_{W^{2,p}(U)} \sum_{n=M}^N \left[ \left( \int_{\Pi \times \Pi} |y - x|^2 \, d\gamma_n^\tau_k \right)^{\frac{1}{2}} + \tau_k \right].
\]

Here, we used Lemma 5.6 and the embedding of \( W^{2,p}(U) \) into \( C^1(U) \). Also, \( C_5 := C_5(t_f, \Psi, e, V, \rho_0) \) and \( C_6 := C_6(\Omega, t_f, U, t_f, \Psi, e, V, \rho_0) \). Consequently, it follows:

\[
(4.30) \quad \|\theta_h \rho^\tau_k(t) - \rho^\tau_k(t)\|_{W^{2,p}(U)} \\
= \sup_{\|\zeta\|_{W^{2,p}(U)} = 1} \int_\Omega \zeta(\theta_h \rho^\tau_k(t) - \rho^\tau_k(t)) \, dy \\
\leq C_6 \sum_{n=M}^N \left[ \left( \int_{\Pi \times \Pi} |y - x|^2 \, d\gamma_n^\tau_k \right)^{\frac{1}{2}} + \tau_k \right] \\
\leq C_6 \left( \tau_k (N - M) + (\tau_k (N - M))^{\frac{1}{2}} \left( \sum_{n=M}^N \left[ \left( \int_{\Pi \times \Pi} |y - x|^2 \, d\gamma_n^\tau_k \right) \right]^{\frac{1}{2}} \right) \right) \\
\leq C_7 \left( h + \sqrt{h} \left[ \sum_{n=M}^N E(\rho_n^\tau_k) - \int_\Omega \Psi \rho_n^\tau_k \, dy - E(\rho_n^\tau_k) + \int_\Omega \Psi \rho_n^\tau_k \, dy \right]^{\frac{1}{2}} \right) \\
\leq C_7 \left( h + \sqrt{h} \left[ E(\rho_M^\tau_k) - \int_\Omega \Psi \rho_M^\tau_k \, dy - E(\rho_M^\tau_k) + \int_\Omega \Psi \rho_M^\tau_k + h \, dy \right]^{\frac{1}{2}} \right). 
\]

Here, we used the Jensen inequality and Proposition 5.9. Also, \( C_7 := C_7(t_f, \Omega, U, \Psi, e, V, \rho_0) \). This shows \( \|\theta_h \rho^\tau_k - \rho^\tau_k\|_{L^1(t_1, t_2; W^{2,p}(U))} \to 0 \) as \( h \to 0 \), uniformly in \( k \). In order to show
that \( \rho^* \) is bounded in \( L^1(0, t_f; W^{1,2}(U)) \) we use (1.5), Proposition 4.3, Lemma 5.6, Corollary 5.7 and Proposition 5.9 to obtain
\[
(4.31)
\int_\Omega \left| \nabla \log \rho_{n+1}^* + \nabla V \right|^2 (\rho_{n+1}^* + \tau_k h_n^r) \, dx \, \tau_k \leq \int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau_k} \, d\gamma_{n+k}
\leq C_8 \left( E(\rho_{n+1}^*) - \int_\Omega \Psi \, d\mu_{n+k}^* - E(\rho_{n+1}^*) + \int_\Omega \Psi \, d\mu_{n+k}^* + \tau_k \right),
\]
for every \( n \) in \( [0, t_f/\tau] \). Here, \( C_8 := C_8(\Psi, e, V, \rho_0) \). By Proposition 4.21, we have that \( \tilde{\Lambda} \geq \rho^*_{n+1} \geq \lambda \), for some positive constants \( \lambda := \lambda(t_f) \) and \( \tilde{\Lambda} \) and every \( n \) in \( [0, t_f/\tau] \). Then, using (1.31), the Young inequality, Corollary 5.7 and the fact that \( V \) is in \( W^{1,2}(\Omega) \), we get
\[
(4.32)
\int_0^{t_f} \left( \int_\Omega |\nabla \rho^*(t)|^2 \, dx \right) \, dt < C_9(t_f, \Psi, e, V, \rho_0)(1 + t_f).
\]

Hence, we conclude that \( \{\rho^*(n)\}_{n=1}^\infty \) is equibounded in \( L^2(0, t_f; W^{1,2}(\overline{\Omega})) \). Also, from Proposition 5.3 we have that \( \{\rho^*(n)\}_{n=1}^\infty \) is equibounded in \( L^2(0, t_f; L^2(\overline{\Omega})) \) as well.

This shows that the hypotheses of the Aubin-Lions Theorem are satisfied. Thus, we obtain a map \( \rho \in L^2(0, t_f; L^2(U)) \) and a subsequence (not relabeled). Such a subsequence satisfies that \( \rho^* \to \rho \) in \( L^2(0, t_f; L^2(U)) \) as \( k \to \infty \). By (1.30) and the Arzela-Ascoli Theorem, this subsequence converges to \( \rho \) in \( C^{1/2}(0, t_f; W^{-2,p}(U)) \).

The final step is to use a diagonal argument along a sequence of sets \( U \) increasing to \( \Omega \). By doing this we obtain a further subsequence converging in \( L^2(0, t_f; L^2(U)) \) and in \( C^{1/2}(0, t_f; L^2_{loc}(\overline{\Omega})) \) to a map \( \rho \in L^2(0, t_f; L^2_{loc}(\overline{\Omega})) \), which we have not relabeled.

Consequently, for any \( \zeta \in C_c^\infty(\Omega) \),
\[
\int_\Omega \zeta \rho^*(s) \, dx - \int_\Omega \zeta \rho^*(r) \, dx \to \int_\Omega \zeta \rho(s) \, dx - \int_\Omega \zeta \rho(r) \, dx.
\]
Let \( U \) be an open set such that \( \text{supp}(\zeta) \subset U \) and \( \overline{U} \) is compactly contained in \( \Omega \). By Proposition 5.3 there exists \( C_{10} := C_{10}(\lambda, \Lambda, t_f) \) such that
\[
\int_\Omega |\zeta \rho^*(t) + V| \, dx \leq C_{10} \int_\Omega ||| \zeta |||_{L^\infty(\Omega)} \, dx < \infty,
\]
and
\[
\int_\Omega \left[ \left| \Delta \zeta - \langle \nabla \zeta, \nabla V \rangle \right| \rho^*(t) \right] \, dx \leq C_{10} \int_\Omega \left[ ||| \Delta \zeta \|\|_{L^\infty(\Omega)} + |\nabla \zeta|^2 + |\nabla V|^2 \right] \, dx < \infty.
\]
Recall that \( V \) is in \( W^{1,2}(\Omega) \). Using the fact that \( \rho^*(t) \to \rho(t) \) in \( L^2(U) \) for almost every \( t \) and the dominated convergence Theorem, we get
\[
\int_\Omega \zeta \rho^*(t) \, dx \to \int_\Omega \zeta \rho(t) \, dx,
\]
and
\[
\int_\Omega \left[ \Delta \zeta - \langle \nabla \zeta, \nabla V \rangle \right] \rho^*(t) \, dx \to \int_\Omega \left[ \Delta \zeta - \langle \nabla \zeta, \nabla V \rangle \right] \rho(t) \, dx.
\]
for almost every $t$ in $[0, t_f]$. Then, a second application of the dominated convergence Theorem gives us

$$\int_{t_k}^{t_{f-k}} \left( \int_{\Omega} [\Delta \zeta - \langle \nabla \zeta, \nabla V \rangle] \rho(t) \, dx - \int_{\Omega} \zeta[e']^{-1}(\log \rho(t) + V) \right) \, dt$$

$$\rightarrow \int_{t}^{s} \left( \int_{\Omega} [\Delta \zeta - \langle \nabla \zeta, \nabla V \rangle] \rho(t) \, dx - \int_{\Omega} \zeta[e']^{-1}(\log \rho(t) + V) \right) \, dt$$

Moreover, (4.32) and the Fatou Lemma yield

$$\int_{0}^{t_f} \lim inf_{k \to \infty} \left( \int_{\Omega} |\nabla \rho \tau_k(t)|^2 \, dx \right) dt < \infty.$$ 

This gives

$$\lim inf_{k \to \infty} \left( \int_{\Omega} |\nabla \rho \tau_k(t)|^2 \, dx \right) < \infty \quad \text{for a.e } t \geq 0.$$ 

Now, for any $t$ such that the above limit inf is finite, consider a subsequence $k_n$ (depending on $t$) such that

$$\sup_{n \in \mathbb{N}} \int_{\Omega} |\nabla \rho \tau_{k_n}(t)|^2 \, dx < \infty.$$ 

This implies that $\rho \tau_k(t)$ is uniformly bounded in $W^{1,2}(\Omega)$. Recall that $\rho \tau_k(t) \to \rho(t)$ in $L^2(0, t_f; L^2_{loc}(\Omega))$. Then, by Proposition 4.3 we get that $\rho(t) - e^{\Psi - V}$ is in $W^{1,2}(\Omega)$. Hence, we have shown that the map $t \to \rho(t)$ is a weak solution of (1.7). Finally, to show (4.21), we use the fact that $\rho \tau_k(t) \to \rho(t)$ in $C^{1/2}(0, t_f; W^{-2,p}_{loc}(\Omega))$ and (4.24).

\[\square\]

5. Properties of Minimizers

In this section, we let $\tau > 0$ be a fixed time step. We set $\mu = \rho |_{\Omega}$ and denote by $\mu_\tau = \rho \tau |_{\Omega}$ a minimizing of (4.22). The density $\rho$ is assumed to be strictly positive. Additionally, we let $(\gamma, h)$ be the associated optimal pair for $(\rho, \rho \tau)$). The objective of the section is to show some properties of $\mu_\tau$ that are necessary to prove the main result, Theorem 4.1.

A priori, it is not immediate that one can obtain a $\tau$ independent positive lower bound for $\mu_\tau$; this is studied in Proposition 5.3. Consequently, we cannot use Proposition 5.2. However, since $\mu$ and $\mu_\tau$ are absolutely continuous, Lemma A.2 guarantees the existence of maps $T$ and $S$ with the property that $(Id, T) \# \mu = \gamma \Omega$ and $(S, Id) \# \mu_\tau = \gamma \Omega$. We will use this maps throughout this section.

(We remark that in the proof of Proposition 4.1, we get existence of $\rho \tau$ without using Proposition 5.2 or any result from this section.)

**Lemma 5.1.** *(Boundedness and Uniform positivity)* The minimizer $\rho \tau$, defined above, is bounded and uniformly positive.

**Proof.** Let $r$ and $R$ be positive constants such that

$$\log r + V < -\frac{\text{diam}(\Omega)^2}{2\tau} - ||\Psi||_\infty,$$
\[
\log R + V > \frac{\text{diam}(\Omega)^2}{\tau} + 2\|\Psi\|_\infty,
\]
and
\[
R + 1 > \tau\|h\|_\infty.
\]
(Recall that due to Lemma A.2, \(h\) is bounded). Now, set \(A_R^\kappa = \{\rho_r > R+1 \cap \{\rho_r + \tau h < \kappa\}\) and \(A_r = \{\rho_r + 1 < r\}\). Define \(\tilde{\gamma}\) by
\[
\tilde{\gamma} = \gamma + \varepsilon(P_{-\psi,\tau}Id)\#1_{A_r} - \varepsilon\gamma_{A_R^\kappa} + \varepsilon(Id, P_{\psi,\tau})\#1_{A_R^\kappa}
\]
If we set \(\tilde{\rho} = \pi_2\tilde{\gamma} - \tau h\), then
\[
\tilde{\rho} = \begin{cases}
  \rho_r & \text{in } \Omega \setminus A_R^\kappa \cup A_r, \\
  \rho_r + \varepsilon & \text{in } A_r, \\
  \rho_r - \varepsilon(\rho_r + \tau h) & \text{in } A_R^\kappa.
\end{cases}
\]
Hence if \(\frac{r+1}{\kappa} > \varepsilon\), then \(\tilde{\rho} \in \mathcal{M}(\Omega)\) and \((\tilde{\gamma}, h) \in ADM(\mu, \tilde{\rho})\). By optimality,
\[
0 \leq E(\tilde{\rho}) - E(\rho_r) + C_r(\tilde{\gamma}, h) - C_r(\gamma, h)
\]
\[
\leq \int_{A_r} [E(\rho_r + \varepsilon) - E(\rho_r)] \, dy + \varepsilon \int_{A_r} \left( \frac{|P_{-\psi,\tau}(y) - y|^2}{2\tau} - \Psi(P_{-\psi,\tau}(y)) \right) \, dy
\]
\[
+ \int_{A_R^\kappa} [E(\rho_r - \varepsilon(\rho_r + \tau h)) - E(\rho_r)] \, dy + \varepsilon \int_{A_R^\kappa} \left( \frac{\text{diam}^2(\Omega)}{2\tau} + \|\Psi\|_\infty \right) \left( \varepsilon(\rho_r + \tau h) \right) \, dy.
\]
Then, by convexity of \(E\) with respect to its first variable
\[
0 \leq \varepsilon \int_{A_r} \left[ \varepsilon'(\rho_r + \varepsilon) + \frac{|P_{-\psi,\tau}(y) - y|^2}{2\tau} - \Psi(P_{-\psi,\tau}(y)) \right] \, dy
\]
\[
+ \varepsilon \int_{A_R^\kappa} \left[ - \varepsilon'(\rho_r - \varepsilon(\rho_r + \tau h)) + \frac{\text{diam}^2(\Omega)}{2\tau} + \|\Psi\|_\infty \right] \left( \varepsilon(\rho_r + \tau h) \right) \, dy.
\]
Now if we let \(\varepsilon < \min(1/\kappa, 1)\), then \(\varepsilon'(\rho_r - \varepsilon(\rho_r + \tau h)) \geq \log R + V \in A_R^\kappa\) and \(\varepsilon'(\rho_r + \varepsilon) < \log r + V \in A_r\). Hence, by construction both integrands are strictly negative. Thus, we conclude \(|A_r| = |A_R^\kappa| = 0\). Since \(\kappa\) was arbitrary and \(\rho_r + \tau h \in L^1(\Omega)\), we obtain the desired result. \(\square\)

In the next Proposition, we will say that a point in \(\Omega\) is a density point for \(\rho_r + \tau h\) if it is a point of with density 1 for the set \(\{\rho_r + \tau h > 0\}\) and it is a Lebesgue point for \(\rho_r\) and \(h\). As before, the interior of the set of points where \(E\) is finite will be denoted by \(D(E)\). The Proposition is the analogue in our context of [3, Proposition 3.7].

**Proposition 5.2.** With the notation introduced at the beginning of this section, the following inequalities hold:

- Let \(y_1\) and \(y_2\) be points in \(\Omega\). Assume that \(y_1\) is a density point for \(\rho_r + \tau h\) and Lebesgue point for \(S\) and that \(y_2\) is a Lebesgue point for \(\rho_r\). Then
(5.33) \[ \log(\rho_r(y_1)) + V(y_1) + \frac{|y_1 - S(y_1)|^2}{2\tau} \leq \log(\rho_r(y_2)) + V(y_2) + \frac{|y_2 - S(y_1)|^2}{2\tau}. \]

- Let \( x \in \Omega \) be a Lebesgue point for \( \rho \), and \( T \) and assume that \( T(x) \in \partial \Omega \). Assume further that \( y \in \Omega \) is a Lebesgue point for \( \rho_r \) and \( h \). Then

\[ \frac{|x - T(x)|^2}{2\tau} + \Psi(T(x)) \leq \log(\rho_r(y)) + V(y) + \frac{|x - y|^2}{2\tau}. \]

- Let \( y_1 \) and \( S(y_1) \) be points in \( \Omega \). Assume that \( y_1 \) is a density point for \( \rho + \tau h \) and a Lebesgue point for \( S \). Then for any \( y_2 \in \partial \Omega \), we have

\[ \log(\rho_r(y_1)) + V(y_1) + \frac{|y_1 - S(y_1)|^2}{2\tau} \leq \frac{|y_2 - S(y_1)|^2}{2\tau} + \Psi(y_2). \]

- Let \( y \in \Omega \) be a density point for \( \rho_r + \tau h \) and a Lebesgue point for \( P_{-\Psi,r} \). Then

\[ \log(\rho_r(y)) + V(y) + \frac{|y - P_{-\Psi,r}(y)|^2}{2\tau} - \Psi(P_{-\Psi,r}(y)) = 0. \]

**Proof.** -Heuristic argument. First we start with (5.31). Consider a point \( y \in \Omega \) and suppose that we take some mass from \( x \in \Omega \) and instead of sending it to \( T(x) \in \partial \Omega \), we send it to \( y \). Then, we are paying \( \log(\rho_r(y)) + V(y) \) in terms of the entropy and \( \frac{|x - y|^2}{2\tau} \) in terms of the cost. We are also saving \( \frac{|x - T(x)|^2}{2\tau} + \Psi(T(x)) \) in terms of the cost. Hence, by minimality, we must have

\[ \frac{|x - T(x)|^2}{2\tau} + \Psi(T(x)) \leq \log(\rho_r(y)) + V(y) + \frac{|x - y|^2}{2\tau}. \]

Now we proceed with (5.35). We take some mass from \( S(y_1) \) and instead of sending it to \( y_1 \), we send it to \( y_2 \). Then, we pay \( \frac{|y_2 - S(y_1)|^2}{2\tau} + \Psi(y_2) \) in terms of the cost. We also save \( \frac{|y_1 - S(y_1)|^2}{2\tau} \) in terms of the cost and \( \log(\rho_r(y_1)) + V(y_1) \) in terms of the entropy. Thus, the desired result follows by minimality; (5.33) is analogous.

To show (5.36), we argue as follows. Pick a point \( y \in \Omega \) and perturb \( \rho_r \) by taking some small mass from a point in \( P_{-\Psi,r}(y) \in P_{-\Psi,r}(y) \) and putting it onto \( y \). In the case that \( S(y) \in \partial \Omega \), we choose the point to be \( S(y) = P_{-\Psi,r}(y) \in P_{-\Psi,r}(y) \). It is easy to verify that the minimality of the pair allows us to do this almost everywhere in \( \Omega \). In this way, we pay \( \log(\rho_r(y)) + V(y) \) in terms of the entropy and \( \frac{|y - P_{-\Psi,r}(y)|^2}{2\tau} - \Psi(P_{-\Psi,r}(y)) \) in terms of the cost. Consequently, by minimality we must have

\[ \log(\rho_r(y)) + V(y) - \Psi(P_{-\Psi,r}(y)) \geq -\frac{|y - P_{-\Psi,r}(y)|^2}{2\tau}. \]

Now consider two cases. First, if \( S(y) \in \Omega \), we stop sending some mass from \( S(y) \) to \( y \). Instead, we send it to \( P_{-\Psi,r}(y) \in \partial \Omega \). By doing this, we earn \( \log(\rho_r(y)) + V(y) \)
in terms of the entropy and $\frac{|S(y) - y|^2}{2\tau}$ in terms of the cost. On the other hand, we pay $\frac{|S(y) - P_{-\psi, \tau}(y)|^2}{2\tau} + \Psi(P_{-\psi, \tau}(y))$ in terms of the cost. Thus,

$$-\Psi(P_{-\psi, \tau}(y)) + \log(\rho_{\tau}(y)) + V(y) + \frac{|S(y) - (y)|^2}{2\tau} \leq \frac{|S(y) - P_{-\psi, \tau}(y)|^2}{2\tau} \leq \frac{|(y - S(y) + |y - P_{-\psi, \tau}(y)|)^2}{2\tau}.$$

Consequently, when we combine this with (5.38), we obtain

$$-\frac{|y - P_{-\psi, \tau}(y)|^2}{2\tau} \leq -\Psi(P_{-\psi, \tau}(y)) + \log(\rho_{\tau}(y)) + V(y) \leq \frac{|y - P_{-\psi, \tau}(y)| |y - S(y)|}{\tau} + \frac{|y - P_{-\psi, \tau}(y)|^2}{2\tau}.$$

Second, if $S(y) \in \partial \Omega$ the above inequality is obtained as a consequence of (5.37) and (5.38). The proof of (5.37) is a sort of converse of (5.38). Indeed, as $S(y) = P_{-\psi, \tau}(y) \in \partial \Omega$, we know that the mass at $y$ comes from the boundary. Hence, we can perturb $\rho_{\tau}$ by taking a bit less mass from the boundary, so that there is less mass in $y$. In this way, we save $\log(\rho_{\tau}(y)) + V(y)$ in terms of the entropy and $\frac{|y - P_{-\psi, \tau}(y)|^2}{2\tau} - \Psi(P_{-\psi, \tau}(y))$ in terms of the cost. Hence,

$$-\left(\log(\rho_{\tau}(y)) + V(y) + \frac{|y - P_{-\psi, \tau}(y)|^2}{2\tau} - \Psi(P_{-\psi, \tau}(y))\right) \geq 0.$$

From (5.38), we get the opposite inequality and thus we conclude the argument.

**Rigorous proof.** We only prove (5.35); the proofs of the other inequalities are analogous.

Let $T_{y_1}^{y_2} : B_r(y_1) \to \partial \Omega$ be identically equal to $y_2$ in $B_r(y_1)$ and let $r$ be positive constant such that $B_r(y_1)$ is contained in $\Omega$. Define the plan $\gamma_{r, \varepsilon}$ by

$$\gamma_{r, \varepsilon} = \gamma^B_{\Omega \#}(y_1, y_2) + (1 - \varepsilon)\gamma^B_{\Omega \#}(y_1, y_2) + \varepsilon(\pi^1, T_{y_1}^{y_2}) \# \gamma^B_{\Omega \#}(y_1),$$

and set

$$\mu_{r, \varepsilon} := \pi^2 _\# \gamma_{r, \varepsilon} - \tau h \ dy.$$

Observe that $\pi^1 _\# \gamma_{r, \varepsilon} = \pi^1 _\# \gamma$, $(\gamma_{r, \varepsilon}, h) \in ADM(\rho, \mu_{r, \varepsilon})$, $\gamma_{r, \varepsilon} = \gamma_{\partial \Omega}$, $\gamma_{r, \varepsilon} = \gamma_{\partial \Omega} - \varepsilon \gamma_{\partial \Omega}$, $\gamma_{r, \varepsilon} = \gamma_{\partial \Omega} + \varepsilon(\pi^1, T_{y_1}^{y_2}) \# \gamma_{\partial \Omega}$, and $\mu_{r, \varepsilon} = \mu_{r, \varepsilon} \mathcal{L}^d$. Here, $\rho_{r, \varepsilon}$ is given by

$$\rho_{r, \varepsilon} = \begin{cases} \rho_{\tau}(y) & \text{if } y \in B_r(y_1), \\ (1 - \varepsilon)(\rho_{\tau}(y) + \tau h) - \tau h & \text{if } y \in B_r(y_1). \end{cases}$$

(We remark that by Lemma A.2, $h$ is in $L^\infty(\Omega)$ and by Lemma 5.1 we that $\rho_{\tau}$ is bounded and uniformly positive. Hence, we can guarantee that for sufficiently small $\varepsilon$, $\rho_{r, \varepsilon}$ is strictly positive.)
From the minimality of \( \rho \) and the relationship between \( \gamma, S, \) and \( T \), we get
\[
0 \leq \int_{\Omega} \mathcal{E}(\rho^x) \, dx + C_T(\gamma^x_1, h) - \int_{\Omega} \mathcal{E}(\rho) \, dx - C_T(\gamma, h)
\]
\[
= \int_{B_r(y_1)} \mathcal{E}((1 - \varepsilon)(\rho(y) + \tau h) - \tau h) - \mathcal{E}(\rho) \, dy
\]
\[
+ \varepsilon \int_{B_r(y_1)} \left( \frac{|T_{y_1}^{y_2}(y) - S(y)|^2}{2\tau} 1_{\{S(y) \in \Omega\}} - \frac{|y - S(y)|^2}{2\tau} \right)(\rho(y) + \tau h) \, dy
\]
\[
+ \varepsilon \int_{B_r(y_1)} \left[ \Psi(T_{y_1}^{y_2}(y)) 1_{\{S(y) \in \Omega\}} + \Psi(S(y)) 1_{\{S(y) \in \partial\Omega\}} \right] (\rho(y) + \tau h) \, dy.
\]
Dividing by \( \varepsilon \) and letting \( \varepsilon \downarrow 0 \), using Lemma 5.1, the dominated convergence Theorem, and the fact that \( \mathcal{E} \) in Lipschitz any compact subset of \( D(\mathcal{E}) \), we obtain
\[
\int_{B_r(y_1)} \mathcal{E}'(\rho(y))(\rho(y) + \tau h) \, dy
\]
\[
\leq \int_{B_r(y_1)} \left( \frac{|T_{y_1}^{y_2}(y) - S(y)|^2}{2\tau} 1_{\{S(y) \in \Omega\}} - \frac{|y - S(y)|^2}{2\tau} \right)(\rho(y) + \tau h) \, dy
\]
\[
+ \int_{B_r(y_1)} \left[ \Psi(T_{y_1}^{y_2}(y)) 1_{\{S(y) \in \Omega\}} + \Psi(S(y)) 1_{\{S(y) \in \partial\Omega\}} \right] (\rho(y) + \tau h) \, dy.
\]
Recall that by assumption \( S(y_1) \notin \partial\Omega \). Now, since \( y_1 \) is a density point for \( \rho + \tau h \), and a Lebesgue point for \( S \) when we divide both sides by \( \mathcal{L}^d(B_r(0)) \) and we let \( r \downarrow 0 \), we obtain \( \rho \).

Henceforth, we will omit the proof of these kinds of perturbation arguments. They can be made rigorous using the ideas contained in the previous Proposition. In the next Proposition, the constants \( C_0 \) and \( s \) are the ones described in the introduction.

**Proposition 5.3.** (\( L^\infty \) Barriers) With the notation introduced at the beginning of this section, the following holds: There exists \( \varepsilon \in (0, 1) \), such that if \( \lambda \) and \( \Lambda \) satisfy \( 0 < \lambda < \varepsilon < \frac{1}{\varepsilon} < \Lambda \) and
\[
\frac{\lambda}{\sup \{e^{-V}\}} e^{-V} \leq \rho \leq \frac{\Lambda}{\inf \{e^{-V}\}} e^{-V},
\]
then
\[
\left( \frac{1}{1 + C_0 r^2} \right) \frac{\lambda}{\sup \{e^{-V}\}} e^{-V} \leq \rho_r \leq \frac{\Lambda}{\inf \{e^{-V}\}} e^{-V}.
\]
Here, \( \varepsilon \) depends only on \( e \), \( ||\Psi||_{\infty} \), and \( ||V||_{\infty} \).

**Proof.** We first prove the lower bound. Assumption \( (C8) \) allows us to choose \( \varepsilon \in (0, s) \) so that
\[
(5.40) \quad [\varepsilon_2']^{-1} (\log r + V(x)) \leq C_0 r \quad \text{in} \quad \Omega,
\]
and
\[
\Psi > \log r + V \quad \text{on} \quad \partial\Omega,
\]
for all \( r \in (0, \varepsilon) \).
Let \( \lambda \in (0, \varepsilon) \) such that \( \lambda e^{-V} / \sup \{e^{-V}\} \leq \rho \) and set
\[
A_\lambda = \left\{ \rho_r < \left( \frac{1}{1 + C_0 r^2} \right) \frac{\lambda}{\sup \{e^{-V}\}} e^{-V} \right\}.
\]
For a contradiction suppose \( \rho_T(A_\lambda) > 0 \) (Note that by Lemma 5.1, \( \rho_T \) is uniformly positive).

For each \( x \in A_\lambda \), we perturb \((\gamma, h)\) by decreasing \( h(x) \) and thus increasing the mass created at \( x \). By optimality, we get

\[
(5.41) \quad \log \rho_T(x) + V(x) - \psi'(h(x)) \geq 0.
\]

Since

\[
\left( \frac{1}{1 + C_0 \tau} \right) \sup \{ e^{-V} \} e^{-V} < s,
\]

when we combine (5.40) and (5.41) we conclude

\[
h < \left( \frac{C_0}{1 + C_0 \tau} \right) \sup \{ e^{-V} \} e^{-V} \quad \text{in} \ A_\lambda.
\]

Let \( C_\lambda = \{ x \in A_\lambda : T(x) \notin A_\lambda \} \) and note that \( \rho(C_\lambda) > 0 \). Otherwise by (1.5) and the previous inequality

\[
\int_{A_\lambda} \left( \frac{1}{1 + C_0 \tau} \right) \sup \{ e^{-V} \} e^{-V} \, dx > \rho_T(A_\lambda) \geq \rho_T(T(A_\lambda))
\]

\[
\geq \rho(T^{-1}(T(A_\lambda))) - \tau \int_{T A_\lambda} h \, dx
\]

\[
\geq \int_{A_\lambda} \left( \frac{C_0 \tau}{1 + C_0 \tau} \right) \sup \{ e^{-V} \} e^{-V} \, dx
\]

\[
= \int_{A_\lambda} \left( \frac{1}{1 + C_0 \tau} \right) \sup \{ e^{-V} \} e^{-V} \, dx.
\]

Define the sets

\[
C_\lambda^1 := \{ x \in C_\lambda : T(x) \in \Omega \} \quad \text{and} \quad C_\lambda^2 := \{ x \in C_\lambda : T(x) \notin \partial \Omega \}.
\]

Since \( C_\lambda = C_\lambda^1 \cup C_\lambda^2 \), we have that either \( \rho(C_\lambda^1) > 0 \) or \( \rho(C_\lambda^2) > 0 \). Suppose we are in the first case. Then, we can find a point \( x \) which is a Lebesgue point for \( T \) such that \( T(x) \) is a Lebesgue point for \( \rho_T \). If we stop sending some mass from \( x \) to \( T(x) \), then, by optimality we obtain

\[
\log \rho_T(x) + V(x) - \log(\rho_T(T(x))) - V(T(x)) - \frac{|x - T(x)|^2}{2\tau} \geq 0.
\]

Since

\[
\log \rho_T(T(x)) \geq \log \left[ \left( \frac{1}{1 + C_0 \tau} \right) \sup \{ e^{-V} \} e^{-V(T(x))} \right],
\]

and

\[
\log \left[ \left( \frac{1}{1 + C_0 \tau} \right) \sup \{ e^{-V} \} e^{-V(x)} \right] > \log \rho_T(x),
\]

we get a contradiction.

Now, suppose \( \rho(C_\lambda^2) > 0 \). We perturb \((\gamma, h)\) by not moving some mass from \( x \) to the boundary. By optimality we must have

\[
\log \rho_T(x) + V(x) - \psi(T(x)) - \frac{|x - T(x)|^2}{2\tau} \geq 0.
\]

Since \( \lambda > \rho_T(x) \) and \( \psi > \log \lambda + V(x) \), we get a contradiction.
Second, we prove the upper bound. By assumptions (C1), (C7), (B1), and (B2), after making \(\varepsilon\) smaller, we can guarantee that
\[
[e']^{-1}(\log r + V) > 0 \quad \text{in} \quad \Omega,
\]
and
\[
\Psi < \log r + V \quad \text{on} \quad \partial\Omega,
\]
for all \(r > 1/\varepsilon\).

Let \(\Lambda > 1/\varepsilon\) satisfy \(\frac{\Lambda}{\inf\{e^{-V}\}} e^{-V} \geq \rho\) and set \(A_{\Lambda} = \left\{ \rho_r > \frac{\Lambda}{\inf\{e^{-V}\}} e^{-V} \right\}\).

In order to get a contradiction, suppose \(\rho_r(A_{\Lambda}) > 0\).

For each \(x \in A_{\Lambda}\), we perturb \((\gamma, h)\) by increasing \(h(x)\) and hence decreasing the amount of mass created in \(x\). By optimality we get
\[
e'(h(x), x) - \log \rho_r(x) - V(x) \geq 0.
\]

Since \(\frac{\Lambda}{\inf\{e^{-V}\}} e^{-V} \geq \Lambda\), we deduce that \(h\) is non-negative in \(A_{\Lambda}\). Now, we consider the following cases:

**Case 1:** the mass of \(\rho_r\) in \(A_{\Lambda}\) does not come from \(\partial\Omega\). Let \(B_{\Lambda} = T^{-1}(A_{\Lambda})\) and observe that due to (1.5),
\[
\int_{A_{\Lambda}} \frac{\Lambda}{\inf\{e^{-V}\}} e^{-V} \, dx < \rho_r(A_{\Lambda}) \leq \rho(B_{\Lambda}) - \tau \int_{A_{\Lambda}} h \, dx < \int_{B_{\Lambda}} \frac{\Lambda}{\inf\{e^{-V}\}} e^{-V} \, dx,
\]
which implies
\[
|A_{\Lambda}| < |B_{\Lambda}|.
\]

Hence, we can find a Lebesgue point \(x \in B_{\Lambda} \setminus A_{\Lambda}\). If we stop transporting some mass from \(x\) to \(T(x)\), then by optimality, we obtain
\[
-\frac{|x - T(x)|^2}{2\tau} + \log \rho_r(x) + V(x) - V(T(x)) - \log \rho_r(T(x)) \geq 0.
\]

Now by construction,
\[
\log \frac{\Lambda}{\sup\{e^{-V}\}} e^{-V(x)} > \log \rho_r(x), \quad \text{and} \quad \log \rho_r(T(x)) > \log \frac{\Lambda}{\sup\{e^{-V}\}} e^{-V(T(x))}.
\]

When we combine this with the previous inequality we reach a contradiction.

**Case 2:** the mass of \(\rho_r\) comes partially from \(\partial\Omega\). Let \(D_{\Lambda} \subset A_{\Lambda}\) be the set of points \(y\) such that the mass \(\rho_r(y)\) comes from the boundary; i.e., \(D_{\Lambda} := \{y \in A : S(y) \in \partial\Omega\}\). Also, let \(y \in D_{\Lambda}\) be a Lebesgue point for \(S\). Then, if we stop moving some mass from \(S(y)\) to \(y\), by optimality we obtain
\[
-\frac{|S(y) - y|^2}{2\tau} + \Psi(S(y)) - \log \rho_r(y) - V(y) \geq 0.
\]

Since \(\rho_r(x) > \Lambda\) and \(\Psi < \log \Lambda + V(x)\), we get a contradiction. This concludes the proof. \(\square\)

For the next Lemma we recall that we have assumed that \(\frac{\partial\Omega}{\partial\Omega} = 0\).

**Lemma 5.4. (Transportation bound)** Let \(\varepsilon, \rho, \lambda, \) and \(\Lambda\) be as in Proposition \(\frac{\partial\Omega}{\partial\Omega}\). Then, there exists \(C > 0\) such that
\[
|y - x| \leq C \sqrt{\tau} \quad \forall (x, y) \in \text{supp}(\gamma).
\]
Here, \(C\) depends only on \(\varepsilon, \lambda, \Lambda, \|\Psi\|_\infty, \) and \(\|V\|_\infty\).
\textbf{Proof.} Let \((x, y)\) be a point in \(\text{supp}(\gamma)\). Then, we perturb the plan \(\gamma\) by not moving some mass from \(x\) to \(y\). By optimality,

\[
\Psi(y)1_{\Omega \times \partial \Omega} - \Psi(x)1_{\partial \Omega \times \Omega} + \left( \log \rho_r(x) + V(x) \right)1_{\Omega(x)} - \left( \log \rho_r(y) - V(y) \right)1_{\Omega(y)} - \frac{|x - y|^2}{2\tau} \geq 0.
\]

Thus, the result follows (B1), (B2), (C1), and Proposition 5.3. \(\square\)

\textbf{Lemma 5.5. (Boundary Mass Flux estimate)} Let \(\varepsilon, \rho, \lambda, \) and \(\Lambda\) be as in Proposition 5.3. Then, there exists \(C > 0\) such that

\[
\gamma(\partial \Omega \times \Omega \cup \Omega \times \partial \Omega) \leq C \sqrt{\tau}.
\]

Here, \(C\) depends only on \(\varepsilon, \lambda, \Lambda, ||\Psi||_\infty, \) and \(||V||_\infty.\)

\textbf{Proof.} By Lemma 5.4, no mass either sent or taken from the boundary travels more than \(C \sqrt{\tau}.\) Then, at most a \(C \sqrt{\tau}\) neighborhood of \(\partial \Omega\) can be sent to the boundary. The mass taken from the boundary can fill at most a \(C \sqrt{\tau}\) neighborhood of \(\partial \Omega.\) Hence, the desired result follows from (B2) and Proposition 5.3. \(\square\)

\textbf{Lemma 5.6. (Interior Mass Creation estimate)} Let \(\varepsilon, \rho, \lambda, \) and \(\Lambda\) be as in Proposition 5.3. Then, there exists \(C > 0\) such that

\[
\int_{\Omega} |h| \, dx \leq C \quad \text{and} \quad |h| \leq C.
\]

Here, \(C\) depends only on \(\varepsilon, \Omega, \lambda, \Lambda, \) and \(||V||_\infty.\)

\textbf{Proof.} By item (iii) of Proposition 4.3, we know

\[
e'(h) = \log \rho_r + V.
\]

Consequently, by Proposition 5.3

\[
\log \left[ \left( \frac{1}{1 + C_0 \tau} \right) \lambda \right] - ||V||_\infty \leq e'(h(x)) \leq \log(\Lambda) + ||V||_\infty, \quad \forall x \in \Omega.
\]

Using assumptions (C2), (C7), (B1) and (B2), we get that \(h\) is bounded. Thus, since \(\Omega\) is bounded, the result follows. \(\square\)

\textbf{Corollary 5.7.} Let \(\varepsilon, \rho, \lambda, \) and \(\Lambda\) be as in Proposition 5.3. Then, there exist positive constants \(\kappa_1, \kappa_2, \) and \(\delta\) such that

\[
\kappa_1 < \frac{\rho_r}{\rho_r + \tau h} < \kappa_2.
\]

for every \(\tau \in (0, \delta).\) Here, \(\kappa_1\) and \(\kappa_2\) depend only on \(\varepsilon, \lambda, \Lambda, ||\Psi||_\infty, \) and \(||V||_\infty.\)

\textbf{Proof.} This follows directly from Proposition 5.3 and Lemma 5.6. \(\square\)

\textbf{Lemma 5.8. (Boundary cost bound)} Let \(\varepsilon, \rho, \lambda, \) and \(\Lambda\) be as in Proposition 5.3. Then, for every \(\epsilon > 0,\) there exists \(C > 0\) such that

\[
\int_{\Omega \times \overline{\Omega}} \Psi(y)1_{\Omega \times \partial \Omega} - \Psi(x)1_{\partial \Omega \times \Omega} \, d\gamma \geq \int_{\Omega} \Psi \, d\mu - \int_{\Omega} \Psi \, d\mu_r - \epsilon \int_{\Omega \times \overline{\Omega}} \frac{|x - y|^2}{2\tau} \, d\gamma - C \left( 1 + \frac{1}{\epsilon} \right) \tau.
\]

Here, \(C\) depends only on \(\varepsilon, \text{Lip} \Psi, \lambda, \Lambda, \) and \(||V||_\infty.\)
Proof. Set $\zeta = \Psi$ in (4.25). By doing this and rearranging terms, we get
\[
\int_{\Omega \times \Omega} \Psi(y)1_{\partial \Omega} - \Psi(x)1_{\partial \Omega} \, d\gamma = \int_{\Omega} \Psi \, d\mu - \int_{\Omega} \Psi \, d\mu_x
\]
\[- \left( \int_{\Omega \times \Omega} \Psi(x) - \Psi(y) \, d\gamma + \tau \int_{\Omega} \Psi \, dx \right) + R(\Psi, \tau),
\]
where,
\[
R(\Psi, \tau) = \int_{\Omega \times \Omega} \left( \Psi \circ \pi_2 - \Psi \circ \pi_1 \right) 1_{\{s(y)\}} \, d\gamma
\]
\[- \int_{\Omega \times \Omega} \left( \Psi \circ \pi_1 - \Psi \circ \pi_2 \right) 1_{\{T(x) = y\}} \, d\gamma.
\]
First, by Lemma 5.4 and Lemma 5.5,
\[
|R(\Psi, \tau)| \leq C_1(\text{Lip}\Psi, \lambda, \Lambda, ||V||_\infty) \tau.
\]
Second, by Lemma 5.6 we have
\[
-\tau \int_{\Omega} \Psi h \, dx \geq -C_3(||\Psi||_\infty, V, \lambda, \Lambda) \tau.
\]
Finally, by the Young inequality, Proposition 4.3, Proposition 5.3, and Lemma 5.6,
\[
- \int_{\Omega \times \Omega} \Psi(x) - \Psi(y) \, d\gamma \geq - \int_{\Omega} \text{Lip}\Psi |x - y| \, d\gamma
\]
\[- \frac{\tau}{2e} \int (\text{Lip}\Psi)^2 \, d\gamma - e \int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} \, d\gamma
\]
\[- \geq -C_4(\epsilon, \Psi, e, V, \lambda, \Lambda, \Omega) \frac{\tau}{\epsilon} - e \int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} \, d\gamma.
\]
Thus, the desired result follows.

\[
\square
\]

Proposition 5.9. (Energy Inequality) Let $\epsilon, \rho, \lambda, \text{ and } \Lambda$ be as in Proposition 5.3. Then, there exist positive constants $C$ and $\delta$ such that
\[
\int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} \, d\gamma \leq C \left( E(\rho) - \int_{\Omega} \Psi \, d\mu - \rho + \int_{\Omega} \Psi \, d\mu_x + \tau \right),
\]
for every $\tau \in (0, \delta)$. Here, $C$ depends only on $\epsilon, \lambda, \Lambda, \text{Lip}\Psi$, and $||V||_\infty$.

Proof. By minimality of $\rho_\tau$, we obtain
\[
\int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} + \Psi(y)1_{\partial \Omega} - \Psi(x)1_{\partial \Omega} \, d\gamma + \tau \int_{\Omega} e(h) \, dx + E(\rho_\tau) \leq E(\rho).
\]
Also, by Lemma 5.6 and the above inequality,
\[
\int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} + \Psi(y)1_{\partial \Omega} - \Psi(x)1_{\partial \Omega} \, d\gamma \leq E(\rho) - E(\rho_\tau) + C_1(\Psi, e, V, \mu, \Omega) \tau.
\]
Now, using the above inequality and Lemma 5.8, we obtain
\[
\int_{\Omega^*} \frac{|x - y|^2}{2\tau} d\gamma + \int_{\Omega} \Psi \, d\mu - \int_{\Omega} \Psi \, d\mu_\tau - \epsilon \int_{\Omega^*} \frac{|x - y|^2}{2\tau} d\gamma \tau - C_2 \left(1 + \frac{1}{\epsilon}\right)^2 \tau \\
\leq E(\rho) - E(\rho_\tau) + C_1(\Psi, e, V, \lambda, \Lambda, \Omega) \tau.
\]
Here, \(C_2 := C_2(\Psi, e, V, \lambda, \Lambda, \Omega)\). Then, the result follows by first choosing \(\epsilon\) and then \(\delta\) appropriately in the above inequality.

For the next proposition, we will need the map \(P : \Omega \to \mathbb{R}^d\), which was defined in Section 3. Such a map satisfies
\[
|x - P(x)| = d(x, \partial \Omega) \quad \forall x \in \Omega.
\]

**Lemma 5.10. (Projection estimate)** Assume \(\Omega\) satisfies the interior ball condition with radius \(r > 0\). Then, for all \(x\) with \(d(x, \partial \Omega) < \frac{r}{2}\), we have
\[
|P(x) - P_{\psi, \tau}(x)| \leq 4\tau \text{Lip}_c \Psi \quad \text{and} \quad |P(x) - P_{\rho, \tau}(x)| \leq 4\tau \text{Lip}_c \Psi.
\]

**Proof.** Let \(x \in \Omega\) such that \(d(x, \partial \Omega) < \frac{r}{2}\). By the interior ball condition, \(P(x)\) is unique. For a contradiction, suppose
\[
|P(x) - P_{\psi, \tau}(x)| > 4\tau \text{Lip}_c \Psi.
\]
Denote by \(Q\) the center of the circle of radius \(r\) that is tangent to \(\partial \Omega\) at \(P(x)\) and is contained in \(\Omega\). Using the cosine law and the fact that \(|Q - P_{\psi, \tau}(x)| \geq r\), we can see that
\[
|x - P_{\psi, \tau}(x)|^2 - |x - P(x)|^2 \\
\geq |P(x) - P_{\psi, \tau}(x)|^2 \left(1 - \frac{|x - P(x)|}{r}\right) \geq \frac{|P(x) - P_{\psi, \tau}(x)|^2}{2}.
\]
Hence,
\[
\frac{|x - P_{\psi, \tau}(x)|^2}{2\tau} - \frac{|x - P(x)|^2}{2\tau} + \Psi(P_{\psi, \tau}(x)) - \Psi(P(x)),
\]
is bounded from below by
\[
\frac{|P(x) - P_{\psi, \tau}(x)|^2}{4\tau} - \text{Lip}_c |P(x) - P_{\psi, \tau}(x)|.
\]
Our assumption implies that the above quantity is strictly positive. This contradicts the minimality of \(P_{\psi, \tau}(x)\). Thus, we get the first inequality of the Lemma. The second inequality can be shown using the same argument.

**Appendix A. Minimizers of problem 1.1**

In this section, we study properties of the minimizers of Problem 1.1 that are needed for Section 3. For this purpose, we let \(\mu\) and \(\rho \, dx\) be absolutely continuous measures in \(M(\Omega)\) and let \(\tau\) be a fixed positive number. Additionally, we define \(m_\tau : \Omega \to \mathbb{R}\) by
\[
m_\tau(x) := [\epsilon_x']^{-1}(r),
\]
for any \(r\) in \(\mathbb{R}\).

Henceforth, we will say that a plan is optimal in the classical sense if it is an optimal plan for the cost \(d(x, y) = |x - y|^2\). Whenever \(\gamma\) is an optimal plan in the classical sense and \(\mu = \pi_1 \# \gamma\) is absolutely continuous, we can guarantee the existence of a map \(T\) such that \((Id, T)_{\#} \mu = \gamma\) (see, for example, [6, Theorem 6.2.4 and Remark 6.2.11]). Any map satisfying the previous property will be called optimal in the classical sense.
Lemma A.1. (Refinement of pairs) Let $\mu$ and $\rho$ be absolutely continuous measures in $M(\Omega)$ and let $\tau$ be a positive constant. Then, for any $(\gamma, h)$ in ADM($\mu, \rho$) there exists $(\gamma', h)$ and $(\gamma'', h')$ in ADM($\mu, \rho$) with the following properties:

(i) The plans $(\gamma')_{\Omega}^{\pi}$ and $(\gamma')_{\Omega}^{\rho}$ are optimal in the classical sense, $(\gamma')_{\Omega}^{\rho} = 0$ and

$$C_\tau(\gamma', h) - C_\tau(\gamma, h) = \int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} d\gamma' - \int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} d\gamma.$$  

(ii) We have

$$h'(x) > [e_x']^{-1}\left(-\frac{\text{diam}(\Omega)^2}{\tau} - ||\Psi||_{\infty}\right),$$

for almost every $x$ in $\Omega$ and

(A.42)

$$C_\tau(\gamma'', h') - C_\tau(\gamma', h) \leq \tau \int_{A'_{r}} \left[ e'(m_r) + \frac{|P_{-\Psi,\tau}(y) - y|^2}{2\tau} - \Psi(P_{-\Psi,\tau}(y)) \right] (m_r - h) dy$$

$$+ \tau \delta \int_{A'_{R}} \left[ - e'(h - \delta(\rho + \tau h)) + \frac{\text{diam}^2(\Omega)}{2\tau} - \frac{|S(y) - y|^2}{2\tau} + 2||\Psi||_{\infty} \right] (\rho + \tau h) dy \leq 0.$$  

Here, $S$ is an optimal map, in the classical sense, such that $(S, \text{Id})_{\#}(\rho + \tau h) = (\gamma')_{\Omega}^{\rho}$ (this exists by the absolute continuity of $\rho + \tau h$), $A_r = \{h < m_r\}$, $A'_{R} = \{h > m_R + 1\} \cap \{\rho + \tau h < \kappa\}$, $\kappa$ is a positive constant, and $\delta < \min(1/\kappa, 1/\tau)$. Also, $r$ and $R$ are constants satisfying

$$r < -\frac{\text{diam}(\Omega)^2}{2\tau} - ||\Psi||_{\infty},$$

$$R > \frac{\text{diam}(\Omega)^2}{\tau} + 2||\Psi||_{\infty},$$

and

$$m_R > 0 \quad \text{in} \quad \Omega.$$  

Proof. It is easy to verify that if $\tilde{\gamma}$ satisfies $\pi_i \gamma_{\Omega}^{\pi} = \pi_i (\tilde{\gamma})_{\Omega}^{\pi}$ and $\pi_i \gamma_{\Omega}^{\rho} = \pi_i (\tilde{\gamma})_{\Omega}^{\rho}$ for $i = 1$ and $i = 2$, then $(\tilde{\gamma}, h) \in$ ADM($\mu, \rho$), and

$$C_\tau(\tilde{\gamma}, h) - C_\tau(\gamma, h) = \int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} d\tilde{\gamma} - \int_{\Omega \times \Omega} \frac{|x - y|^2}{2\tau} d\gamma.$$

Consequently, if $(\tilde{\gamma})_{\Omega}^{\pi}$ and $(\tilde{\gamma})_{\Omega}^{\rho}$ are optimal plans in the classical sense, then

$$C_\tau(\tilde{\gamma}, h) \leq C_\tau(\gamma, h).$$  

Now, (i) follows from the observation that $(\tilde{\gamma} - \gamma'_{\Omega}^{\rho}, h) \in$ ADM($\mu, \rho$) and

$$C_\tau(\tilde{\gamma}, h) = C_\tau(\tilde{\gamma} - \gamma'_{\Omega}^{\rho}, h).$$  

We proceed to the proof of (ii). Let $\gamma'$ be the plan given by item (i). Define $h'$ and $\gamma''$ by

$$h' = h + (m_r - h) 1_{A_r} - \delta \pi_2(\gamma')_{\Omega}^{A'_{R}},$$

$$\gamma'' = \gamma' + \tau (P_{-\Psi,\tau,\text{Id}})(m_r - h) 1_{A_r} - \delta(\rho + \tau h) + \tau \delta(\text{Id}, P_{\Psi,\tau,\text{Id}})(\gamma')_{\Omega}^{A'_{R}}.$$
Here, we are using same notation as in the statement of the Lemma. Observe that by (1.5), \((\gamma'', h') \in ADM(\mu, \rho),\)

\[
h' = \begin{cases} h & \text{in } \Omega \setminus A^\rho_R \cup A_r, \\ m_r & \text{in } A_r, \\ h - \varepsilon(\rho + \tau h) & \text{in } A^\rho_R, \end{cases}
\]

and

\[
\pi_2\#\gamma''|_{\Omega} = \begin{cases} \rho + \tau h & \text{in } \Omega \setminus A^\rho_R \cup A_r, \\ \rho + \tau m_r & \text{in } A_r, \\ (1 - \tau \varepsilon)(\rho + \tau h) & \text{in } A^\rho_R. \end{cases}
\]

Hence,

\[
C_\tau(\gamma'', h') - C_\tau(\gamma', h) 
\leq \tau \int_{A_r} [e(m_r) - e(h)] \, dy + \tau \int_{A_r} \left( \frac{|P - \Psi, \tau(y) - y|^2}{2\tau} - \Psi(P - \Psi, \tau(y)) \right) (m_r - h) \, dy 
+ \tau \int_{A^\rho_R} e(h - \varepsilon(\rho + \tau h)) - e(h) \, dy + \varepsilon \tau \int_{A^\rho_R} \left( \frac{\text{diam}^2(\Omega)}{2\tau} + ||\Psi||_\infty \right) (\rho + \tau h) \, dy.
\]

Then, the desired result follows by the convexity of \(e\) with respect to its first variable and the definition of \(r, R, \) and \(\varepsilon.\)

For the next lemma, we will need the set \(D(e)\), which was previously defined to be the interior of the set of points such that \(e\) is finite.

**Lemma A.2. (Optimal maps and Bounds on the created mass)** Let \(\mu\) and \(\rho \, dx\) be absolutely continuous measures in \(M(\Omega)\) and let \(\tau\) be a positive constant. Additionally, let \((\gamma, h)\) be a pair in \(\text{Opt}(\mu, \rho)\). Then

(i) The plans \(\gamma_{\Omega}\) and \(\gamma_{\Omega}^{\Omega}\) are optimal in the classical sense.

(ii) There exist maps \(T\) and \(S\) from \(\Omega\) to \(\Omega\) such that \(\text{(Id, T)}#\mu = \gamma_{\Omega}\),

and

\(\text{(S, Id)}#(\rho + \tau h) = \gamma_{\Omega}^{\Omega}\).

(iii) There exists a compact set \(K \subset \mathbb{R} \times \overline{\Omega}\) contained in \(D(e)\) such that \((x, h(x)) \in K,\)

for a.e \(x\) in \(\Omega.\)

**Proof.** By Lemma A.1, (i) follows by optimality. Since \(\mu\) and \(\rho + \tau h\) are absolutely continuous, (ii) follows from the classical optimal transportation theory (see for example [6, Theorem 6.2.4 and Remark 6.2.11]).

Now, we proceed to the proof of (iii). Let \(r, R, \) and \((\gamma'', h')\) be defined as in the previous Lemma and set \(K = \{(q, x) \in \mathbb{R} \times \overline{\Omega} : m_r(x) \leq q \leq m_R(x) + 1\}.\) By (C7) and (C8), \(K\) is compact. By construction, both integrands in (A.42) are strictly negative. Thus, from
the minimality of \((\gamma, h)\), we conclude that \(|A_r| = |A_r^c| = 0\). Since \(\kappa\) was arbitrary and \(h + \tau \rho \in L^1(\Omega)\), we obtain the desired result. \(\square\)

In the next proposition, we will say that a point in \(\Omega\) is a density point for \(\rho + \tau h\) if it is a point of with density 1 for the set \(\{\rho + \tau h > 0\}\) and it is a Lebesgue point for \(\rho\) and \(h\).

**Proposition A.3.** Let \(\mu\) and \(\rho\) \(dx\) be absolutely continuous measures in \(M(\Omega)\) and let \(\tau\) be a positive constant. Additionally, let \((\gamma, h)\) be a pair in \(\text{Opt}(\mu, \rho)\). If \(T\) and \(S\) are the maps given by Lemma A.2, then the following inequalities hold:

- Let \(y_1 \) and \(y_2\) be points in \(\Omega\). Assume that \(y_1\) is a density point for \(\rho + \tau h\) and a Lebesgue point for \(S\) and that \(y_2\) is a Lebesgue point for \(h\). Then
  \[
  e'(h(y_1)) + \frac{|y_1 - S(y_1)|^2}{2\tau} \leq e'(h(y_2)) + \frac{|y_2 - S(y_1)|^2}{2\tau}.
  \]

- Let \(y_1\) and \(S(y_1)\) be points in \(\Omega\). Assume that \(y_1\) is a density point for \(\rho + \tau h\) and a Lebesgue point for \(S\). Then for any \(y_2 \in \partial \Omega\), we have
  \[
  e' \circ h(y_1) + \frac{|y_1 - S(y_1)|^2}{2\tau} \leq \frac{|y_2 - S(y_1)|^2}{2\tau} + \Psi(y_2).
  \]

- Let \(x_1 \in \Omega\) be a Lebesgue point for the density of \(\mu\), and \(T\) and assume that \(T(x_1) \in \partial \Omega\). Assume further that \(y_1 \in \Omega\) is a Lebesgue point for \(h\). Then
  \[
  \frac{|x_1 - T(x_1)|^2}{2\tau} + \Psi(T(x_1)) \leq e'(h(y_1)) + \frac{|x_1 - y_1|^2}{2\tau}.
  \]

- Let \(x_1 \in \Omega\) be a Lebesgue point for the density of \(\mu\), and \(T\) and assume that \(T(x_1) \in \partial \Omega\). Then for any \(y_1 \in \partial \Omega\),
  \[
  \frac{|x_1 - T(x_1)|^2}{2\tau} + \Psi(T(x_1)) \leq \frac{|x_1 - y_1|^2}{2\tau} + \Psi(y_1).
  \]

- Let \(y_1 \in \Omega\) be a Lebesgue point for \(h\). Then for any \(x_1 \in \partial \Omega\),
  \[
  0 \leq e' \circ h(y_1) + \frac{|y_1 - x_1|^2}{2\tau} - \Psi(x_1).
  \]

- Let \(y_1 \in \Omega\) be a density point for \(\rho + \tau h\) such that \(S(y_1) \in \partial \Omega\). Then, for any \(x_1 \in \partial \Omega\),
  \[
  \frac{|y_1 - S(y_1)|^2}{2\tau} - \Psi(S(y_1)) \leq \frac{|y_1 - x_1|^2}{2\tau} - \Psi(x_1).
  \]

- Let \(y_1 \in \Omega\) be a density point for \(\rho + \tau h\) such that \(S(y_1) \in \partial \Omega\). Then
  \[
  \frac{|y_1 - S(y_1)|^2}{2\tau} - \Psi(S(y_1)) \leq -e'(h(y_1)).
  \]

**Proof.** We only prove (A.43); the proofs of the other inequalities are analogous. Also, Proposition B.2 provides heuristic arguments that illustrate the method used to prove those inequalities. This Proposition is the analogue of [5, Proposition 3.7] in our context. We have decided to include this proof since this is the first times we explain how to make these kinds of arguments rigorous with perturbations that involve mass creation.

**-Heurisitic argument** We provide the idea to show (A.43). First suppose \(S(y_1) \in \Omega\). Then we can take some mass from \(S(y_1)\) and instead of sending it to \(y_1\), we send it to \(y_2\). In order to end up with an admisible pair, we then have to create the missing mass at \(y_1\) and...
remove the extra mass at \( y_2 \). In order to do this, we have to decrease \( h(y_1) \) and increase \( h(y_2) \). By doing this we save \( \frac{|y_1 - S(y_1)|^2}{2\tau} \) and we pay

\[
\frac{|y_2 - S(y_1)|^2}{2\tau} - e'(h_1) + e'(h_2).
\]

Hence, (A.43) follows by minimality. If \( S(y_1) \in \partial \Omega \), when we do the previous perturbation we save \( \frac{|y_1 - S(y_1)|^2}{2\tau} + \Psi(S(y_1)) \) and we pay

\[
\frac{|y_2 - S(y_1)|^2}{2\tau} + \Psi(S(y_1)) - e'(h_1) + e'(h_2),
\]

Thus, we get the same conclusion.

**-Rigorous proof** We define \( \gamma^{r,\varepsilon} \) and \( h^{r,\varepsilon} \) by

\[
\gamma^r = \gamma^{B_r(y_1)} + (1 - \varepsilon)\gamma^{\overline{B_r(y_1)}} + \varepsilon(\pi_1, T^{y_2}_{y_1})\gamma^{B_r(y_1)}
\]

and

\[
h^r = h - \varepsilon\frac{\pi_2\gamma^{B_r(y_1)}}{\tau} + \frac{\varepsilon}{\tau}(T^{y_2}_{y_1} \circ \pi_2\gamma^{B_r(y_1)}).
\]

Here, \( T^{y_2}_{y_1}(y) = y - y_1 + y_2 \), and \( r \) is small enough so that \( B_r(y_1) \) and \( B_r(y_2) \) are disjoint and contained in \( \Omega \).

Note that \( \pi_2\gamma^{(r,\varepsilon)} = \pi_2\gamma \) and \( \pi_2\gamma - \tau h = \pi_2\gamma^{r,\varepsilon} - \tau h^{r,\varepsilon} \). Hence, \( (\gamma^{r,\varepsilon}, h^{r,\varepsilon}) \in ADM(\mu, \nu) \).

By optimality, we must have

\[
0 \leq C(h^{r,\varepsilon}, \gamma^{r,\varepsilon}) - C_\tau(h, \gamma).
\]

Thus,

\[
0 \leq \varepsilon \int_{B_r(y_1)} \left[ \frac{|T^{y_2}_{y_1} - S|^2}{2\tau} - \frac{|Id - S|^2}{2\tau} \right] (\rho + \tau h) \, dy \\
+ \tau \int_{B_r(y_1)} \left[ e\left( h - \frac{\varepsilon}{\tau}(\rho_\tau + \tau h) \right) - e(h) \right] \, dy \\
+ \tau \int_{B_r(y_1)} \left[ e\left( h \circ T^{y_2}_{y_1} + \frac{\varepsilon}{\tau}(\rho + \tau h) \right) - e(h \circ T^{y_2}_{y_1}) \right] \, dx.
\]

If we divide by \( \varepsilon \) and let \( \varepsilon \downarrow 0 \), using Lemma [A.2], the fact that \( e \) is locally Lipschitz in \( D(e) \), and the dominated convergence Theorem, we obtain

\[
0 \leq \int_{B_r(y_1)} \left[ \frac{|T^{y_2}_{y_1} - S|^2}{2\tau} - \frac{|y_1 - S|^2}{2\tau} \right] (\rho + \tau h) \, dy - \int_{B_r(y_1)} e'(h)(\rho + \tau h) \, dy \\
+ \int_{B_r(y_1)} e'(h \circ T^{y_2}_{y_1})(\rho + \tau h) \, dy.
\]

Recall \( y_1 \) is a density point for \( \rho + \tau h \) and a Lebesgue point for \( S \) and \( y_1 \) and \( y_2 \) are Lebesgue points for \( h \). Hence, when we divide by \( L^d(B_r(0)) \) and we let \( r \downarrow 0 \), we obtain (A.43). \( \square \)

Henceforth, as we did in Section 5, we will omit the proof of these kinds of perturbation arguments. They can be made rigorous using the ideas contained in the previous Proposition. For the next proposition, we will need the sets \( D(e_x) \), which were previously defined to be the interior of the set of points \( z \) such that \( e(z, x) \) is finite.
Proposition A.4. (Bounds on the transported mass) With the notations and assumptions from Proposition A.3, the following implication holds: If there there exists a positive constant $\lambda_0$ such that
\[
\lambda_0 \, dx \leq \mu,
\]
and
\[
\lambda_0 \leq \rho,
\]
then there exists a positive number $\delta$ such that
\[
\frac{\lambda_0}{4} \leq \rho + \tau h,
\]
for all $\tau$ in $(0, \delta)$. Here, $\delta$ depends only on $\lambda_0$, $\Psi$, and $e$.

Proof. Let $\tilde{\rho} = \rho + \tau h$. If the sets $D(e_x)$ are of the form $(a, \infty)$ with $a$ finite, then since $h(x) \in D(e_x)$ and (C2), (C5), and (C8) hold, the lower bound follows easily by choosing $\delta$ sufficiently small. Hence, we assume that the sets $D(e_x)$ are of the form $(-\infty, \infty)$. (We remark that due to (C8) the two conditions are mutually exclusive).

By (C6) and (C8), there exit $\delta$ such that for every $\tau < \delta$ there exists $r$ such that
\[
\tau < -||\Psi||_{\infty}
\]
and
\[
\frac{1}{\tau}\left(\frac{\lambda_0}{4} - \rho\right) \leq m_r \leq \frac{1}{\tau}\left(\frac{\lambda_0}{2} - \rho\right)
\]
in $\Omega$.

Set $A_r = \{\tilde{\rho} < \rho + \tau m_r(x)\}$ and $C_r = \{x \in A_r : T(x) \notin A_e\}$. For a contradiction, suppose $|A_r| > 0$. Note that $|C_r| \geq 0$. Otherwise, by (1.3)
\[
\frac{\lambda_0}{2}|A_r| > \tilde{\rho}(A_r) \geq \tilde{\rho}(T(A_r)) = \mu(T^{-1}T(A_r)) \geq \lambda_0 |A_r|.
\]

Define the sets
\[
C^1_r := \left\{ x \in C_r : T(x) \in \Omega \right\} \quad \text{and} \quad C^2_r := \left\{ x \in C_r : T(x) \notin \partial \Omega \right\}.
\]

Since $C_r = C^1_r \cup C^2_r$, we have that either $|C^1_r| > 0$ or $|C^2_r| > 0$. Suppose we are in the first case. Then, we can find a point $x$ which is a Lebesgue point for $T$ such that $T(x)$ is a Lebesgue point for $\tilde{\rho}$. If we stop sending some mass from $x$ to $T(x)$ then we can create the missing mass at $T(x)$ and remove the extra mass at $x$. To do this we have to increase $h(x)$ and decrease $h(T(x))$. By doing this, we produce a pair in $ADM(\mu, \rho)$. Thus, by optimality, we must have
\[
e'(h(x)) - e'(h(T(x))) - \frac{|x - T(x)|^2}{2\tau} \geq 0.
\]

By construction, if we use (1.5), we obtain
\[
e'(h(x)) = e'\left(\frac{\tilde{\rho}(x) - \rho(x)}{\tau}\right) < e'(m_r(x)) = r,
\]
and
\[
e'(h(T(x))) = e'\left(\frac{\tilde{\rho}(T(x)) - \rho(T(x))}{\tau}\right) \geq e'(m_r(x)) = r.
\]
This gives us a contradiction.

Now, suppose $|C^2_r| > 0$. Then, we can find a point $x \in C^2_r$ such that $x$ is a Lebesgue point for $T$ and $h$. If we stop moving some mass from $x$ to the boundary, then we can remove the
extra mass at \( x \). To do this we have to increase \( h(x) \). By doing this, we produce a pair in \( \text{ADM}(\mu, \rho) \). By optimality, we must have
\[
e'\left(h(x)\right) - \Psi(T(x)) - \frac{|x - T(x)|^2}{2\tau} \geq 0.
\]
As before \( e'(h(x)) < r \) and by construction \( r - \Psi < 0 \). This gives us a contradiction. Hence, we conclude that
\[
\rho + \tau h = \bar{\rho} \geq \rho + \tau m_r \geq \frac{\lambda_0}{4} \quad \text{a.e. in } \Omega.
\]
\[ \square \]

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