Large scale evolution of the curvature perturbation in Hořava-Lifshitz cosmology

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In the non-relativistic theory of gravity recently proposed by Hořava, the Hamiltonian constraint is not satisfied locally at each point in space. The absence of the local Hamiltonian constraint allows the system to have an extra dark-matter-like component as an integration constant. We discuss consequences of this fact in the context of cosmological perturbations, paying a particular attention to the large scale evolution of the curvature perturbation. The curvature perturbation is defined in a gauge invariant manner with this “dark matter” taken into account. We then clarify the conditions under which the curvature perturbation is conserved on large scales. This is done by using the evolution equations.

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I. INTRODUCTION

A power-counting renormalizable theory of gravity, proposed recently by Hořava [1, 2], has attracted much attention. The essential aspect of the theory is broken Lorentz invariance in the ultraviolet (UV), where it exhibits a Lifshitz-like anisotropic scaling, \( t \rightarrow r_t \), \( \vec{x} \rightarrow r \vec{x} \), with the dynamical critical exponent \( z = 3 \). This will bring an interesting change in the physics of the early universe since the UV effect may play an important role there. The study of the cosmology based on Hořava gravity, which is called Hořava-Lifshitz (HL) cosmology, has been initiated by Refs. [3, 4], and since then various aspects of HL cosmology have been explored, including the generation of chiral gravitational waves [5], a new mechanism to generate a scale invariant primordial spectrum without inflation [6], the bouncing scenario [7], and others [8]. Aside from cosmology, other interesting works can also be found in [9].

Several versions of Hořava gravity have been known, which are classified according to whether or not the detailed balance and the projectability conditions are imposed. Among them the theory with projectability and without detailed balance is argued to evade the problems [10, 11] pointed out in the literature [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28]. Wang and Maartens were the first to study cosmological perturbations in this version of the theory [27]. In general relativity, the curvature perturbation on uniform density hypersurfaces, commonly denoted as \( \zeta \), is conserved on large scales, provided that the non-adiabatic pressure perturbation is negligible. This fact can be proven by utilizing the energy conservation law only [28], and hence the conservation of \( \zeta \) holds true in a wide range of gravity theories such as brane-world gravity [29]. In Hořava gravity, however, an extra degree of freedom mimicking dark matter, or, what is dubbed as “dark matter as an integration constant” in [10, 11], appears as a natural outcome of the lack of the local Hamiltonian constraint. This forces one to consider effectively a multi-fluid system even if the system is composed of a single (real) fluid, which implies that the effect of the entropy perturbation may not be negligible. Furthermore, individual components are in general not conserved separately, while the total energy including “dark matter” thus introduced is shown to be conserved locally (by invoking the evolution equations). Therefore, we start with defining the gauge invariant curvature perturbation, emphasizing the projectability condition and taking into account the presence of “dark matter as an integration constant.” Then, we discuss the conditions under which the curvature perturbation is conserved on large scales by using the evolution equations.

This paper is organized as follows. In the next section, the basic equations in Hořava’s non-relativistic theory of gravity is provided. We review the background cosmology in Hořava gravity in Sec. III, emphasizing the consequence of the absence of the local Hamiltonian constraint. Then, in Sec. IV, we study the large scale evolution of the curvature perturbation in the absence of the local Hamiltonian constraint and clarify the conditions under which
it is conserved on large scales. We draw our conclusions in Sec. V.

II. HOŘAVA GRAVITY

We consider the projectable version of Hořava gravity without detailed balance. The dynamical variables are \( \mathcal{N}, \mathcal{N}_i, \) and \( \gamma_{ij} \), in terms of which the metric can be written as

\[
d^2s = -\mathcal{N}^2dt^2 + \gamma_{ij} \left( dx^i + \mathcal{N}_i dt \right) \left( dx^j + \mathcal{N}_j dt \right), \tag{1}
\]

with \( \mathcal{N}^i = \gamma^{ij} \mathcal{N}_j \). The projectability condition states that the lapse function depends only on the time coordinate, \( \mathcal{N} = \mathcal{N}(t) \), while \( \mathcal{N}_i \) and \( \gamma_{ij} \) may depend on \( t \) and \( \vec{x} \). The theory is invariant under the foliation-preserving diffeomorphism: \( t \to t(t), x^i \to \hat{x}^i(t, \vec{x}) \). Under the infinitesimal transformation,

\[
t \to t + \chi^0(t), \quad x^i \to x^i + \chi^i(t, \vec{x}), \tag{2}
\]

these variables transform as

\[
\begin{align*}
\gamma_{ij} &\to \gamma_{ij} - \dot{\gamma}_{ij} \chi^0 - \gamma_{ik} \nabla_j \chi^k - \gamma_{jk} \nabla_i \chi^k, \\
\mathcal{N} &\to \mathcal{N} - \mathcal{N} \dot{\chi}^0 - \mathcal{N} \chi^0, \\
\mathcal{N}_i &\to \mathcal{N}_i - \chi^i \mathcal{N}_j - \chi^j \mathcal{N}_i - \dot{\chi}^i \gamma_{ij} - \dot{\chi}^j \gamma_{ij} - \chi^0 \mathcal{N}_i - \chi^0 \mathcal{N}_i,
\end{align*}
\]

(3)

where a dot is the derivative with respect to the time coordinate \( t \) and \( \nabla_i \) the covariant derivative associated with the spatial metric \( \gamma_{ij} \). One can see that \( \mathcal{N} \) remains \( \vec{x} \)-independent after the transformation, and thus it is natural to impose the projectability condition.

The dynamical variables are subject to the action \[27\]

\[
S = \frac{1}{16\pi G} \int d^3x \sqrt{-\gamma} \left( K_{ij} K^{ij} - \lambda K^2 + R + \mathcal{L}_{V^2} \right) \\
+ \int d^3x \sqrt{-\gamma} \mathcal{L}_m,
\]

(4)

where \( \mathcal{L}_m \) is the Lagrangian for matter fields,

\[
K_{ij} := \frac{1}{2N} \left( \gamma_{ij} - \nabla_i \mathcal{N}_j - \nabla_j \mathcal{N}_i \right)
\]

(5)

is the extrinsic curvature, \( R = \gamma^{ij} R_{ij} \) is the trace of the Ricci scalar (the spatial curvature scalar), and

\[
\mathcal{L}_{V^2} := \alpha_2 R^2 + \alpha_3 R_{ij} R^{ij} + \alpha_4 R^3 + \alpha_5 R R_{ij} R^{ij} \\
+ \alpha_6 R^i_j R^j_k R^k_i + \alpha_7 R \nabla_i \nabla_j \nabla^i R + \alpha_8 \nabla_i R_{jk} \nabla^i R^{jk}.
\]

(6)

The kinetic term coincides with that of general relativity when \( \lambda = 1 \), but we do not specify the value of \( \lambda \) throughout the paper. Note however that it has been argued that an additional longitudinal degree of freedom of gravitons suffers from ghost-like instabilities for \( 1/3 < \lambda < 1 \) \[2,17\]. One may include a cosmological constant in the above action, but we do not write it explicitly since it can also be thought of as a part of the matter Lagrangian. One can also include a parity-violating term associated with the Cotton tensor, as in original Hořava gravity \[2\].

Variation with respect to \( \mathcal{N} \) yields the Hamiltonian constraint. In the projectable version of Hořava gravity, the Hamiltonian constraint is not satisfied locally at each spatial point, but rather a global equation integrated over the whole space because \( \mathcal{N} \) is a function of \( t \) only. The global Hamiltonian constraint reads

\[
\int d^3x \sqrt{-\gamma} \left[ K_{ij} K^{ij} - \lambda K^2 - R - \mathcal{L}_{V^2} + 16\pi G E \right] = 0, \tag{7}
\]

where

\[
E := -\mathcal{L}_m - \dot{\mathcal{N}} \frac{\delta \mathcal{L}_m}{\delta \mathcal{N}_i}, \tag{8}
\]

Variation with respect to \( \mathcal{N}_i \) leads to the momentum constraint,

\[
\nabla_j P^{ij} = 8\pi G J^j, \tag{9}
\]

where

\[
P^{ij} := K^{ij} - \lambda K \gamma^{ij}, \quad J^j = -\mathcal{N} \frac{\delta \mathcal{L}_m}{\delta \mathcal{N}_i}.
\]

(10)

Finally, variation with respect to \( \gamma_{ij} \) gives the evolution equations,

\[
2 \left[ (K_{ik} K^k_j - \lambda K K_{ij}) - \frac{1}{2} (K_{kl} K^{kl} - \lambda K^2) \right] \gamma_{ij} \\
+ \frac{1}{4N} \sqrt{-\gamma} \gamma_{ij} \theta_i \left( \sqrt{-\gamma} P^{kj} \right) - \frac{1}{N} \sqrt{-\gamma} \delta^{ij} (P_{jk} \mathcal{N}_k) \\
+ \frac{1}{N} \nabla^k (P_{ik} \mathcal{N}_j) + \frac{1}{N} \nabla^k (P_{jk} \mathcal{N}_i) + R_{ij} - \frac{1}{2} R \gamma_{ij} \\
+ F_{ij} = 8\pi G T_{ij},
\]

(11)

where \( F_{ij} := \delta \mathcal{L}_{V^2} / \delta \gamma^{ij} - (1/2) \gamma_{ij} \mathcal{L}_{V^2} \) and

\[
T_{ij} := \mathcal{L}_m \gamma_{ij} - 2 \frac{\delta \mathcal{L}_m}{\delta \gamma^{ij}}.
\]

(12)

The matter action is invariant under the infinitesimal transformation \[3\], which results in the energy-momentum conservation equations:

\[
\int d^3x \left[ \frac{\sqrt{-\gamma}}{2} \gamma_{ij} T^{ij} + \partial_i \left( \sqrt{-\gamma} E \right) + \frac{\mathcal{N}_i}{N} \partial_i \left( \sqrt{-\gamma} J^j \right) \right] = 0, \tag{13}
\]

\[
\nabla^j T_{ij} - \frac{1}{N} \sqrt{-\gamma} \partial_i \left( \sqrt{-\gamma} J_i \right) - \frac{\mathcal{N}_i}{N} \nabla_j J^j \\
- \frac{J^j}{N} (\nabla_j \mathcal{N}_i - \nabla_i \mathcal{N}_j) = 0. \tag{14}
\]

The energy conservation law \[13\] is of the form of the integration over the whole space, as is the case for the Hamiltonian constraint.
III. BACKGROUND EVOLUTION

The background evolution of HL cosmology can be derived by setting $\mathcal{N} = 1$, $\mathcal{N}_i = 0$, and $\gamma_{ij} = a^2(t)\delta_{ij}$. The evolution equation at zeroth order reads

$$\dot{\rho} + 3H(\rho + p) = 0,$$  

(15)

where $H := \dot{a}/a$, and $T_{ij} = p\gamma_{ij}$ has been assumed.

Let us define $\mathcal{E}(t)$ by

$$8\pi G[\mathcal{E}(t) + \rho] = -\frac{3}{2}(1 - 3\lambda)H^2,$$  

(16)

where $\rho$ is the background value of the matter energy density $E$. In the case of $\lambda = 1$, the meaning of $\mathcal{E}$ becomes more transparent by noticing that $8\pi G\mathcal{E} = 8\pi G\mathcal{T}_0^0 - G^0_0$, where $G^0_0$ is the $(00)$ component of the usual Einstein tensor: $\mathcal{E}$ arises because the local Hamiltonian constraint is absent in Hořava gravity. This term corresponds to “dark matter as an integration constant” in Refs. [4, 11].

We emphasize that in this paper the homogeneous background is assumed at least in our observable patch of the universe because nobody can tell what happens beyond the present horizon scale. Under this assumption we can conclude from the global Hamiltonian constraint that $\mathcal{E}$ does not necessarily vanish in the local patch. For example, we may have $\mathcal{E} > 0$ in our patch of the universe, but $\mathcal{E}$ may be negative in a different patch. Our assumption is in contrast to Ref. [23], in which the standard assumption of a homogeneous background is made over the whole space, so that the global Hamiltonian constraint enforces $\mathcal{E} = 0$. In this paper, we do not assume the homogeneity over the whole space, and therefore, Eq. (15) does not constrain the value of $\mathcal{E}$ in our observable patch of the universe.

In terms of $\mathcal{E}$, Eq. (15) can be written in the form of a conservation equation:

$$\dot{\mathcal{E}} + \dot{\rho} + 3H(\mathcal{E} + \rho + p) = 0.$$  

(17)

This does not guarantee the local conservation of the matter energy density. If the matter action respects general covariance, we have an additional conservation equation, $\dot{\rho} + 3H(\rho + p) = 0$. In the case of scalar field matter, the equation of motion leads to $\dot{\rho} + 3H(\rho + p) = 0$. Combining the local conservation of matter energy with Eq. (17), we obtain $\dot{\mathcal{E}} + 3H\mathcal{E} = 0$, implying that $\mathcal{E}$ indeed shows a dust-like behavior [10, 14].

IV. LARGE SCALE COSMOLOGICAL PERTURBATIONS

Let us study linear perturbations around the cosmological background. The perturbed metric is given by

$$\mathcal{N}^2 = 1 + 2A(t), \quad \mathcal{N}_i = a^2B_i,$$

$$\gamma_{ij} = a^2[(1 - 2\psi)\delta_{ij} + 2D_{ij}].$$  

(18)

Since we are imposing the projectability condition, the perturbation of the lapse function $A$ does not depend on $\bar{x}$. Cosmological perturbation theory in Hořava gravity without the projectability condition has been studied in [21]. Under the scalar gauge transformation, i.e., the infinitesimal transformation [2] with $\chi' = \partial'\chi(t, \bar{x})$, the metric perturbations transform as

$$A \to A - \chi^0, \quad \psi \to \psi + H\chi^0,$$

$$B \to B - \dot{\chi}, \quad D \to D - \chi.$$  

(19)

Since $\chi^0$ depends only on $t$, inhomogeneous $\psi$ cannot be gauged away, while $A$ can be set to zero by the gauge transformation. This point is in contrast to general relativity. It is convenient to define $\sigma := D - B$, which is gauge invariant.

The evolution equations take the form

$$G \delta t^i_j + \left(\partial_i \partial^j - \frac{1}{3} \nabla^2 \delta_i^j \right) \left[\frac{\psi}{a^2} + \dot{\sigma} + 3H\sigma\right] + \delta F^j_i = 8\pi G \delta T^j_i, \quad (20)$$

where $\nabla^2 := \delta^j_i \partial_j \partial_i$,

$$G := -(1 - 3\lambda) \left[\ddot{\psi} + 3H\dot{\psi} + HA + \left(3H^2 + 2\dot{H}\right)A\right]$$

$$+ (1 - \lambda) \nabla^2 [\dot{\sigma} + 3H\sigma] - \frac{2}{3} \nabla^2 \left[\frac{\psi}{a^2} + \dot{\sigma} + 3H\sigma\right].$$  

(21)

and $\delta F^j_i$ is to be derived from $\mathcal{L}_{V2}$. Since $\delta F^j_i = \mathcal{O}(\nabla^2)$, it is not important as long as one concerns the large scale evolution of cosmological perturbations. (We do not consider the case in which higher spatial derivative terms are much larger than $\mathcal{O}(\nabla^2)$ terms, though $\mathcal{O}(\nabla^6)$ terms have an interesting effect on the spectrum of perturbations [14]..)

The perturbed energy-momentum tensor may be written in terms of isotropic and anisotropic pressure perturbations as

$$\delta T^i_j = \delta p \delta_i^j + \left(\partial_i \partial^j - \frac{1}{3} \nabla^2 \delta_i^j \right) \Pi.$$  

(22)

The momentum constraint is given by

$$\partial^j \left[-(1 - 3\lambda)\ddot{\psi} + (1 - \lambda)\nabla^2 \psi\right] = 8\pi G a^2 \delta J^i.$$  

(23)

Analogously to $\mathcal{E}$ defined in the previous section, let us now define $\mathcal{E}(t, \bar{x})$ by

$$-8\pi G [\mathcal{E}(t, \bar{x}) + \delta \rho] = -3(1 - 3\lambda)H \left(\dot{\psi} + HA\right) - 2\nabla^2 \left[\frac{\psi}{a^2} + H\sigma\right],$$  

(24)

---

1 A straightforward calculation shows that $R_{ij} = \partial_i \partial_j \psi + \nabla^2 \psi \delta_{ij}$ and the variable $D$ does not appear here. Therefore, $\delta F^j_i \supset \nabla^4 \psi, \nabla^6 \psi$. 

where $\delta \rho$ is the perturbation of the matter energy density $E$ given in Eq. (3). In the case of $\lambda = 1$ we have $8\pi G \varepsilon = 8\pi G \delta T_{00}^0 - \delta E_0^0$, from which it is clear again that $\varepsilon$ is a consequence of the absence of the local Hamiltonian constraint. Thus, $\varepsilon$ may be regarded as a energy density perturbation of “dark matter as an integration constant.” It is easy to check that $\varepsilon$ transforms as $\varepsilon \to -\varepsilon - \varepsilon \lambda^0$ under the gauge transformation. In terms of $\varepsilon$, the trace part of the evolution equations, $\delta G = 8\pi G \delta \varepsilon$, can be written as

$$
\dot{\varepsilon} + 3H (\varepsilon + \delta \varepsilon + \delta \rho) - 3\psi (E + \rho + p) = 0.
$$

Equation (32) however tells nothing about the large scale evolution of $\varepsilon$ unless the evolution of $\varepsilon_m$ and $\varepsilon_{HL}$ is specified (except for the special case $c_2^2 f(1 - f) \approx 0$). If the matter energy is conserved locally, one finds, from Eqs. (17) and (20), that $f + 3H c_2^2 f(1 - f) = 0$ at zeroth order and $\varepsilon_m \approx 0$ on large scales at perturbative order, assuming that $\delta \epsilon_{\text{nad}} = 0$. In this case it can be shown that $\varepsilon_{HL} \approx 0$ on large scales, but $\varepsilon$ is not conserved in general. Indeed, it follows immediately from the definition $\varepsilon(t, \vec{x}) = [1 - f(t)] \varepsilon_{HL}^{(0)}(\vec{x}) + f(t) \varepsilon_m^{(0)}(\vec{x})$, where $\varepsilon_{HL}^{(0)}$ and $\varepsilon_m^{(0)}$ are the initial conditions for the corresponding variables. For dust-like matter with $\rho \propto a^{-3}$, $f$ is constant since $\varepsilon$ also scales as $a^{-3}$, and hence $\varepsilon$ is conserved. This fact was already clear in Fig. 2(b) with $c_2^2 = 0$. Another case in which $\varepsilon$ is conserved on large scales is $\varepsilon_{HL} = 0$, that is, $\varepsilon_m = \varepsilon_{HL}$. Whether this “adiabatic relation” between “dark matter” and usual matter is likely or not depends upon the specific scenario in the early universe. For example, in the case discussed in Ref. [6], ordinary matter is produced by the decay of the curvaton or the modulus while the initial condition of “dark matter” is determined by quantum fluctuations of the combination of scalar gravitons and (real) matter fields. In this case, there is no relation between them in general, leading to $\varepsilon_{HL} \neq 0$.

Interesting cases with $f \geq 0$ and $f \approx 1$ can be studied without knowing the evolution of $\varepsilon_{HL}$ and hence without relying on the local conservation of the ordinary matter energy density. If radiation dominates the energy density of the universe at early times, we have $(1 - f) \approx 0$, which leads to the conservation of $\varepsilon$ during that period. On the other hand, if “dark matter as an integration constant” accounts for a significant portion of real dark matter and dominates the energy density of the universe at late times, we have $f \approx 0$, which again leads to the conservation of $\varepsilon$. However, in the intermediate regime, $f(1 - f) = O(1)$, so that the curvature perturbation grows provided that $\varepsilon_{HL} \neq 0$. This property is in accordance with what is found in a conventional multifluid system [28]. One should also notice that in the case where “dark matter as an integration constant” constitutes a large portion of real dark matter, $\varepsilon_{HL}$ represents the isocurvature fluctuation between dark matter and radiation, which is strongly constrained by the cosmic mi-
crowave background (CMB) anisotropy. Thus, the scenario in which “dark matter as an integration constant” is really a dark matter component and there is no natural reason to explain $S_{\text{HL}} = 0$ gives rise to a large isocurvature fluctuation, which could be incompatible with the present constraint.

Since $A$ does not depend on $\vec{x}$, taking the spatial gradient of Eq. (33) yields
\[ \partial_i \left( \frac{\partial_{\psi}}{H} \right) \simeq \frac{\dot{H}}{H^2} \partial_i \zeta. \] (34)

(One can do essentially the same thing by making the gauge choice $A = 0$ instead.) This equation is useful for reconstructing the curvature perturbation $\partial_i \psi$ from $\zeta$. Note that $\partial_t \psi$ is gauge invariant because $\psi$ is subject only to the temporal gauge transformation $t \rightarrow t + \phi^0(t)$. If $\partial_t \zeta$ is constant in time, one finds $\partial_t \psi = - \partial_t \zeta(\vec{x}) + H(t) C_1(\vec{x})$, where the second term corresponds to the decaying mode.

For an illustrative purpose let us consider conserved matter with the equation of state $p = w \rho$. Equation (34) is integrated to give
\[ \partial_t \psi(t, \vec{x}) = - \partial_t \zeta^{(0)}_{\text{HL}}(\vec{x}) + \partial_t \left[ \zeta^{(0)}_{\text{HL}}(\vec{x}) - \zeta^{(0)}_{\text{m}}(\vec{x}) \right] \times \frac{3(1 + w)}{2} H(t) \int^t dt' \frac{dt'}{1 + \mathcal{E}/\rho}. \] (35)

For $\mathcal{E} \ll \rho$, we obtain $\partial_t \psi \simeq - \partial_t \zeta^{(0)}_{\text{m}}(\vec{x})$.

Substituting the curvature perturbation $\partial_i \psi$ into the traceless part of the evolution equations (20), we obtain the metric shear $\partial_i \sigma$. Once these metric perturbations are determined, the (large scale) CMB anisotropies can be computed by making use of the perturbed geodesic equations. The detailed calculation of the CMB temperature anisotropies in HL cosmology is left for further study.

**B. $\mathcal{E} = 0$**

In this case it does not make sense to define $\zeta_{\text{HL}}$, but still one may define $\zeta$ directly by Eq. (33). The trace part of the evolution equations implies
\[ \dot{\zeta} \simeq - \frac{H}{\rho + p} \left( \delta p_{\text{nad}} - c_s^2 \varepsilon \right). \] (36)

(Note that $\varepsilon$ is gauge invariant when $\mathcal{E} = 0$.) Thus, in general $\zeta$ is not conserved even if $\delta p_{\text{nad}} = 0$.

Let us consider again the simple case where the matter energy is conserved locally and $\delta p_{\text{nad}} = 0$. In this case it is easy to see that $\varepsilon = \varepsilon_0 / a^3$. Integrating Eq. (36), we obtain
\[ \zeta \simeq \zeta^{(0)} \left. - \frac{\varepsilon}{3(\rho + p)} \right|_{t = t_0} + \frac{\varepsilon}{3(\rho + p)}, \] (37)
where $t_0$ is some initial time.

**V. CONCLUSIONS**

In this paper, we have studied the large scale evolution of the cosmological curvature perturbation in projectable Hořava gravity, emphasizing the effect of “dark matter as an integration constant” [10] [11] that appears as a consequence of the global Hamiltonian constraint. Our view is that we cannot tell the cosmological dynamics far outside the present Hubble horizon, and hence the global Hamiltonian constraint does not provide any information in our observable patch of the universe. This assumption makes the impact of “dark matter as an integration constant” rather non-trivial. The curvature perturbation $\zeta$ has been defined in a gauge invariant manner with this “dark matter” component taken into account. We then clarified the conditions under which $\zeta$ is conserved on large scales by invoking the evolution equations. In particular, we pointed out that $\zeta$ is sourced by the relative entropy perturbation $S_{\text{HL}}$ between “dark matter as an integration constant” and ordinary matter. This source term is effective during the period when $c_s^2 f(1 - f)$ is not negligible. In that period, we need to know the evolution of $S_{\text{HL}}$ in order to know the evolution of $\zeta$. This is made possible by assuming the local conservation of the energy density of ordinary matter.

If the “dark matter” component constitutes a large portion of real dark matter, $S_{\text{HL}}$ corresponds to the isocurvature fluctuation between radiation and dark matter, which is strongly constrained by the cosmic microwave background anisotropy. In this case, one therefore needs a natural reason to explain $S_{\text{HL}} \simeq 0$, which is a challenge in HL cosmology.

In the present paper, we have focused on the superhorizon evolution of the curvature perturbation for a given initial condition. In order to impose an appropriate initial condition, we need to specify the scenario of the early stage of the universe and then to quantize the cosmological perturbations, which would allow us to give observational prediction for curvature and isocurvature perturbations. Although the procedure is familiar and established in conventional cosmology, it is non-trivial in HL cosmology. It would be interesting to study quantization of the coupled system of a scalar field and metric perturbations and solve its evolution from subhorizon (or WKB) to superhorizon regimes in HL cosmology. Moreover, in HL cosmology we have a novel mechanism to generate a scale invariant spectrum of quantum fluctuations [3], which relies on the modified dispersion relation brought by $O(\nabla^6)$ terms and does not require inflation. A detailed analysis of this mechanism taking into account the effect of the metric perturbations is yet to be done. These issues are left for a future study.

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