Generation and transmission of arbitrary multipartite entangled W states via an optomechanical interface

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We propose a universal and nontrivial scheme to generate and transmit an arbitrary multipartite W state for multiple cavities via an optomechanical interface. In generation and transmission processes, high fidelity can be obtained by optimizing the time-dependent coupling strengths between the cavities and the mechanical resonator. With a group of optimal couplings, an arbitrary entangled W state in the multipartite system can be mapped to the pulse shape of a single photon and transmitted out of the system. In the time reversal process, an arbitrary W state can be generated with an incident single photon with certain pulse shape. The functions of the optimal couplings only depend on the parameters of the system, which does not change with the arbitrary entangled W states and the pulse shape of the single photons.

I. INTRODUCTION

Quantum entanglement involves non-local correlations between subsystems, which has been recognized as one of the core resources in quantum technologies [11][13]. The bipartite entanglement state has been studied extensively, however, multipartite entanglement has more complicated structure and has been experimentally prepared in various quantum systems [5-7]. As is well known, the GHZ states and W states are two representative multipartite entangled states. A remarkable property of the W state is that, it is robust against losses of qubits since tracing out any part from a W state, there exists entanglement in the rest parties [8, 9]. This property makes W state a useful entanglement resource in quantum information [10, 11]. To date, a number of theoretical schemes have been proposed to generate W states in various systems [12-17], and experimentally, there are a few implementations for W states in superconducting qubits [18], photons [19], trapped ions [20] and atomic ensembles [21]. The entangled W state can also be generated via optomechanical interface.

Optomechanical interactions can take place via the radiation pressure force induced by the optical fields. The optomechanical interface, in which the mechanical oscillator couples to the optical cavities or microwave cavities, can mediate a quantum state transfer between light and matter or microwave and optical fields [22-28]. The entanglement generation via optomechanical interface has also been studied intensively between two cavity modes, one cavity mode and one mechanical mode or two cavity modes and one mechanical mode [29-36]. It is still appealing to propose a method to map the multipartite entangled states prepared in multiple microwave or optical cavities to the pulse shape of single photons via the optomechanical interface.

An arbitrary multipartite entangled W state for the cavity modes \( a_i \) \( (i = 1, 2, ..., n) \) can be expressed as

\( W_n = w_1 |100...0\rangle_{a_1 a_2 ... a_n} + w_2 |010...0\rangle_{a_1 a_2 ... a_n} + ... + w_n |000...1\rangle_{a_1 a_2 ... a_n} \) \( (\sum_{i=1}^{n} |w_i|^2 = 1) \). In this work, we propose a method to generate an arbitrary entangled W state in an open multipartite system composed of multiple cavities coupling to an optomechanical interface. The cavities for the optomechanical interface or the W state being prepared in can be optical or microwave, and the quantum states can be converted between single photons and cavity modes with vastly different frequencies. In our scheme, we suppose that, the optomechanical interface is composed of a mechanical resonator and an optical cavity, and the cavities for the W states preparation are microwave cavities. In the preparation process, we can only change the pulse shape of the incident single photons to generate arbitrary W states. These entangled W states can also be transmitted out of the multiple-microwave-cavity system and mapped to the pulse shape of single optical photons leaving the optical cavity. In the generation and transmission processes, the characteristics of both quantum state transfer and entanglement preparation of the optomechanical interface are applied, and the strength of the couplings between the cavities and the mechanical resonator are time-dependent and optimized. With a group of optimal time-dependent couplings, the mapping and transmission of an arbitrary entangled W state can be proceeded, and the time reversal process is just the generation process for the multipartite entangled W state. This method can be used to transfer quantum information between different quantum systems.

II. MODEL

The coupling scheme we investigated is illustrated in Fig. 1. The microwave cavities \( a_i \) \( (i = 1, 2, ..., n) \) with cavity frequency \( \omega_i \) and optical cavity \( a_0 \) with frequency \( \omega_c \) are coupled to a mechanical mode \( b_m \) with mechanical frequency \( \omega_m \). The coupling strength between the microwave cavity mode \( a_i \) and the mechanical mode \( b_m \) is \( g_i \) with the driving pulse on the microwave cavity with the frequency \( \omega_d \), and the coupling strength between the optical mode and the mechanical mode is \( g_0 \) with
the driving pulse on the optical cavity with frequency \( \omega_{d1} \). The coupling strength can be enhanced by the driving pulse and be tuned by varying it. The mechanical resonator and optical cavity compose an optomechanical interface with the optical cavity damping rate \( \kappa_0 \) and mechanical damping rate \( \gamma_m \). The damping rate of the microwave cavity \( \alpha_i \) is \( \kappa_i \), and the conditions \( \kappa_0 \gg \gamma_m, \kappa_i \) and \( \omega_m \gg \kappa_0 \) are satisfied. Assume that, the pump fields are red detuned as \( \omega_j - \omega_{ij} = \omega_m \) (\( j = 0, 1, 2, ..., n \)). And for simplicity, all of the values of the couplings are real, then the linearized interaction Hamiltonian for the closed system is (\( \hbar = 1 \)) \cite{37}

\[
H_I = \sum_{j=0}^{n} g_j \left( \hat{a}_j^\dagger \hat{b}_m + \hat{b}_m^\dagger \hat{a}_j \right),
\]

where \( \hat{a}_j \) (\( j = 0, 1, 2, ..., n \)) and \( \hat{b}_m \) are the annihilation operators of the cavity modes and mechanical mode, respectively. In the basis of \( |100...0\rangle_{a_0a_1...a_{n-1}b} \), \( |010...0\rangle_{a_0a_1...a_{n-1}b} \), ..., \( |000...1\rangle_{a_0a_1...a_{n-1}b} \), this interaction Hamiltonian can be written as a matrix with \( n+2 \) eigenvalues, \( n \) of them are degenerate with the eigenvalues of \( \lambda_i = 0 \) (\( i = 1, 2, ..., n \)), the other two eigenvalues are \( \lambda_{n+1} = s_n \) and \( \lambda_{n+2} = -s_n \), where \( s_n = \sqrt{\sum_{j=0}^{n} g_j^2} \). All of the eigenvalues define \( n+2 \) adiabatic eigenstates. In the adiabatic basis, the Hilbert space is decomposed into two subspaces: the eigenstates \( |\phi_i\rangle \) (\( i = 1, 2, ..., n \)) with the eigenvalues of \( \lambda_i = 0 \), these \( n \) eigenstates are dark states, for which, the probability for the mode \( b_m \) being excited is 0. The dark states can be given by \( |\phi_1\rangle = \frac{1}{\sqrt{s_n}}[g_0, -g_0, 0, 0, 0]^T \), \( |\phi_2\rangle = \frac{1}{\sqrt{s_n}}[g_0g_2, g_1g_2, -s_n^2, 0, 0]^T \), \( |\phi_3\rangle = \frac{1}{\sqrt{s_n}}[g_0g_3, g_1g_3, g_2g_3, -s_n^2, 0, 0]^T \), ..., \( |\phi_n\rangle = \frac{1}{\sqrt{s_n}}[g_0g_n, g_1g_n, ..., -s_n^2, 0, 0]^T \). Another subspace is a two-dimensional bright space, the two bright eigenstates can be expressed as \( |\phi_{n+1}\rangle = \frac{1}{\sqrt{2s_n}}[g_0, g_1, g_2, ..., g_n, g_n]^T \) and \( |\phi_{n+2}\rangle = \frac{1}{\sqrt{2s_n}}[g_0, g_1, g_2, ..., g_n, -g_n]^T \), where \( s_i = \sqrt{\sum_{j=0}^{n} g_j^2} \) (\( i = 0, 1, 2, ..., n \)) \cite{38}.

### III. Entangled W States Generation and Transmission

#### A. Adiabatic evolution

Consider the coupling of the system to the output with the damping rate of \( \kappa_j \) (\( j = 0, 1, 2, ..., n \)) and \( \gamma_m \), the conditional Hamiltonian is

\[
H_c = -\sum_{j=0}^{n} \frac{i\kappa_j}{2} \hat{a}_j^\dagger \hat{a}_j - \frac{i\gamma_m}{2} \hat{b}_m^\dagger \hat{b}_m + H_I.
\]

Under the condition of \( g_i \gg \kappa_i \) (\( i = 1, 2, ..., n \)), for simplicity, we first neglect \( \kappa_i \) and only consider the damping rates \( \kappa_0 \) and \( \gamma_m \). In the basis of \( |\phi_1(t)\rangle, |\phi_2(t)\rangle, ..., |\phi_{n+2}(t)\rangle \), the conditional Hamiltonian is an \( n+2 \) by \( n+2 \) matrix \( \hat{H}_c^{(n+2)\times(n+2)} \), where \( [\hat{H}_c^{(n+2)\times(n+2)}]_{ij} = \langle \phi_i| \hat{H}_c |\phi_j\rangle \). In the adiabatic process, if initially, the system is in dark state, there will not be excitations with the bright states, so that, the evolution of the system will only depend on the first \( n \) rows and \( n \) columns of \( \hat{H}_c^{(n+2)\times(n+2)} \). We then extract the first \( n \) rows and \( n \) columns as \( \hat{H}_c^{(n\times n)} = -\frac{i\kappa_0}{2} M \), where \( M = (1 - \frac{g_0^2}{s_n^2})|\phi_0\rangle\langle \phi_0| \) and \( |\phi_0\rangle = \frac{1}{\sqrt{1 - \frac{g_0^2}{s_n^2}}} [\phi_1^{(1)}, \phi_2^{(1)}, ..., \phi_n^{(1)}]_T \), \( \phi_i^{(1)} \) (\( i = 1, 2, ..., n \)) is the first value of \( |\phi_i\rangle \).

First, we study the process without the incident pulse into the system. In general case, assume that, the wave function of the system is \( |\psi(t)\rangle = \sum_{i=1}^{n} c_i(t)|\phi_i(t)\rangle \), substitute the wave function into the schrödinger equation, we can get \( i\frac{d}{dt} |\psi(t)\rangle = -\frac{i\kappa_0}{2} M |\psi(t)\rangle \). In the adiabatic approximation, there is \( \frac{1}{g_0(t)} \frac{d}{dt} \phi_0(t) \right| \ll |g_0(t)| \), so that, \( |\phi_0(t)\rangle \) is negligible, then there is

\[
\frac{d}{dt} C(t) = -\frac{i\kappa_0}{2} M(t) C(t),
\]

where \( C(t) = [c_1(t), c_2(t), ..., c_n(t)]^T \).

There are \( n \) eigenstates for the matrix \( M(t) \), one is \( |\varphi_1(t)\rangle = |\phi_0(t)\rangle \), with the eigenvalue \( 1 - \frac{g_0^2}{s_n^2} \). The other \( n-1 \) eigenstates from \( |\varphi_2(t)\rangle \) to \( |\varphi_n(t)\rangle \) are degenerate with the same eigenvalues of 0. Without loss of generality, we can define the matrix \( U(t) = [\varphi_1(t), \varphi_2(t), ..., \varphi_n(t)] \). The matrix \( M(t) \) can be diagonalized by the matrix \( U(t) \) and \( U^\dagger(t)M(t)U(t) = \Lambda(t) \), where \( \Lambda(t) \) is a diagonal matrix, \( \Lambda_{ij} = 1 - \frac{g_0^2(t)}{s_n^2(t)} \) and \( \Lambda_{ij} = 0 \) (\( i, j = 1, 2, ..., n \) and \( i \neq j \)).

#### B. Time-independent transmission

If at any time \( t \), the coupling strength satisfies \( g_j(t) = f(t)g_j(0) \) (\( j = 0, 1, 2, ..., n \) and \( f(t) \neq 0 \)), there is...
$dM(t) = 0$. And equation \[3\] can be transformed into 
\[
\frac{d\alpha(t)}{dt} = -\frac{\kappa_0}{2} \Lambda(t) - V(t)\alpha(t),
\]
where $\alpha(t) = U^\dag C(t)$. Solving this function, we can get the expression of the wave function as $C(t) = U \exp\left[\int_0^t -\frac{\kappa_0}{2} \Lambda(t') \, dt'\right] U^\dag C(0)$. At time $t = +\infty$, there is $\exp(-\frac{\kappa_0}{2} \Lambda_1 t) = 0$, then we can get 
\[
C(+\infty) = UU^\dag C(0) - |\phi_0\rangle\langle \phi_0| C(0).
\]
Assume that, initially, $C(0) = |\beta_1\rangle |\phi_0\rangle + |\beta_2\rangle |\phi_0\rangle$ and $\langle \phi_0|\phi_0\rangle = 1$ and $\langle \phi_0|\phi_0\rangle = 0$, then there is $C(+\infty) = |\beta_2\rangle |\phi_0\rangle$.

If $\beta_2 = 0$, the initial state of the system is $|\phi_0\rangle$, we have 
\[
C(+\infty) = 0
\]
and only if $\beta_2 \neq 0$, there is $C(+\infty) = |\beta_2\rangle |\phi_0\rangle$, which means that, when only considering the damping rates $\kappa_0$ and $\gamma_m$, in the time-independent case ($\frac{dM(t)}{dt} = 0$) and adiabatic transition process, only when $C(0) = |\phi_0\rangle$, the final state of the system is empty, if not, at time $t = +\infty$, the probability of transmitting the state $C(0)$ out of the system is $|\beta_2|^2$, and the probability for the system to be at the state of $|\phi_0\rangle$ is $|\beta_2|^2$.

C. Time-dependent transmission and generation

In the case of $\frac{dM(t)}{dt} \neq 0$, equation \[3\] can be transformed into 
\[
\frac{d\alpha(t)}{dt} = -\frac{\kappa_0}{2} \Lambda(t) - V(t)\alpha(t),
\]
where $V(t) = \frac{dM(t)}{dt} U(t)$ and $\alpha(t) = U^\dag(t) C(t)$. Although the Hamiltonian of the system changes slowly, there are $\Lambda_i(t) = 0$ $i = 2, 3, ..., n$, so that $V(t)$ is not negligible. There is $\alpha(t) = [\alpha_1(t), \alpha_2(t), ..., \alpha_n(t)]^T$, where 
\[
\alpha_i = \varphi_i(t) C(t) (i = 1, 2, ..., n)
\]
is the projection of $C(t)$ to $\varphi_i(t)$ at time $t$. Then there is
\[
\frac{d\alpha_i(t)}{dt} = -\frac{\kappa_0}{2} \Lambda_i(t)\alpha_i(t) + \sum_{j=1}^n V_{ij}(t)\alpha_j(t).
\]

As shown in the Appendix, we can prove that $V_{ij}(t) + V_{ji}(t) = 0 \ (V(t)$ is real), and when $\kappa_0 = 0$, there is $d\sum_{i=1}^n \alpha_i^2(t)/dt = 0$. So that, in equation \[4\] the effect of the part $-\frac{\kappa_0}{2} \Lambda(t)$ is to output the quantum state via $\varphi_1(t)$ and the effect of $V(t)$ is to redistribute the populations in every state $\varphi_i(t)$. Then we can conclude that, in case of $\frac{dM(t)}{dt} \neq 0$, the initial state $C(0)$ can not determine whether the state of the system can be output totally out of the system from the cavity mode $a_0$. Then we will study the methods to output or prepare an arbitrary entangled W state.

If there are several different initial states $C_1(0)$, $C_2(0), ..., C_n(0)$, we have that, for $\forall i \in \{1, 2, ..., n\}$ there is 
\[
\frac{dC_i(t)}{dt} = -\frac{\kappa_0}{2} M(t) C_i(t).
\]
For the initial state of $C_i(0)$, there will be a final state $C_i(T)$. Then if the initial state is $C_i(0) = \sum_{i=1}^n p_i C_i(0)$, there is $C_i(T) = \sum_{i=1}^n p_i C_i(T)$. When the $n$ initial states of $C_i(0)$ are linearly independent, if we can find a group of optimal time-dependent couplings $g_0(t), g_1(t), ..., g_n(t)$ to totally output any one of $C_i(0)$ within a limited time of $[0, T]$ from the cavity mode $a_0$, which means that for $\forall i \in \{1, 2, ..., n\}$ there is $C_i(T) = 0$, then, for an arbitrary initial state $C_0(0)$, with this group of optimal time-dependent couplings, there is $C_s(T) = 0$.

With the above method, we can output arbitrary entangled W states in the microwave cavities $a_i (i = 1, 2, ..., n)$ from the cavity mode $a_0$ and mapped them into the pulse shapes of the single photons with a certain group of optimal time-dependent couplings. The function of the output single photon can be denoted as $f(t)$. The time reverse of the output process is just the entangled W state generation process. In the generation process, the initial state of the system is empty, a single photon with the pulse function of $f(T - t)$ is inputted into the system from cavity mode $a_0$. The driving pulse for the cavities $a_0, a_1, ..., a_n$ are $g_0(T - t), g_1(T - t), ..., g_n(T - t)$, respectively. And finally, the system will be in an entangled W state, which is the same with the initial state of the output process.

In fact, there is a trivial method to realize the entangled W state transmission and generation. In the output process, apart from $g_0$ being always not zero, there is only one coupling strength of $g_i (i = 1, 2, ..., n)$ being not zero for a time. The state of the corresponding microwave cavity will be transmitted out of the system and the other $n - 1$ cavities is decoupled from the system in this period of time. In this way, the state of all the microwave cavities will be transmitted out of the system one by one and the system will be empty at the final time. With this method, the output process are not continuous, the pulse shape of the single photon is divided into $n$ time-bins artificially, and the microwave cavities is not equal in time sequence in the evolution process. To overcome these problems, we can apply the optimization method to optimize all the couplings at the same time to realize the sync output for all the microwave cavities.

D. A numerical example

An example for adiabatic entangled W state transfer and initial state preparation is shown in Fig. 2. In this system, the number of the microwave cavities is 3 and the entangled W state is prepared in these cavities. The couplings $g_i (i = 1, 2, 3)$ are time-dependent and optimized using CRAB optimization method [39]. In the optimization process, $g_0$ is constant with a value of $10^7$ and $g_i(t) = \frac{1}{m} \sum_{k=1}^m A_k^{(i)} \sin(2\pi k(1 + r_k)t/T)$, where $m \in N_+$, $r_k \in [0, 1]$ are random numbers and $A_k^{(i)}$ are optimizable parameters. The sine function can keep the initial values (at $t = 0$) of $g_i$ to be zero, and $g_i$ can also be expanded into other functions to meet different needs. The damping rate $\kappa_0$ of the optical cavity is $10^8$, and for the microwave cavities the damping rate $\kappa_i (i = 1, 2, 3)$ are $10^2$ with high internal quality factors [40]. The mechanical resonator occurs at
\( \omega_m = 10^6 \), with a quality factor of \( 10^5 \), the mechanical damping rate is \( \gamma_m = 10^4 \). The resolved-band condition of \( \omega_m \gg \kappa_0 \) is fulfilled, and in the adiabatic transmission and generation processes, the mechanical mode is a dark mode, the damping rate of which has less effect on the whole processes than the cavity dampings. In the output process, there are two different initial states \( \psi_1 = |0\rangle_{a_0} \left( \frac{1}{\sqrt{2}} |100\rangle_{a_1a_2a_3} + \frac{1}{\sqrt{3}} |010\rangle_{a_1a_2a_3} + \frac{1}{\sqrt{6}} |001\rangle_{a_1a_2a_3} \right)_{b_m} \) and \( \psi_2 = |0\rangle_{a_0} \left( \frac{1}{\sqrt{3}} |100\rangle_{a_1a_2a_3} - \frac{1}{\sqrt{4}} |010\rangle_{a_1a_2a_3} - \frac{1}{\sqrt{6}} |001\rangle_{a_1a_2a_3} \right)_{b_m} \). In Fig. 2(a), \( f_1(t) \) and \( f_2(t) \) are the output single photon pulse shapes leaving from the optical cavity \( a_0 \) with the initial state of \( \psi_1 \) and \( \psi_2 \), respectively, which can be expressed as \( f(t) = \sqrt{\kappa_0} \langle a_0(t) \rangle \). The optimized time-dependent couplings for the output process are shown in Fig. 2(b). With this group of optimal couplings, any entangled W states for the cavities \( a_i \) \( (i=1,2,3) \) can be mapped to the pulse shape of single photons leaving from cavity \( a_0 \) and then transmitted to other quantum systems. Fig. 2(c) and (d) are numerical results for the output processes, (e) and (f) are which for the input processes. For the input processes, we define the fidelity as \( F_p = |\langle \psi_p | \psi_\psi \rangle|^2 \) \( (p=1,2) \), where \( |\psi_p \rangle \) is the final states of the system in the input process with the incident pulse of \( f_p(T-t) \) and time-dependent couplings of \( g_i(T-t) \) \( (i=1,2,3) \). The input processes own the fidelity of \( F_1 = 0.9973 \) and \( F_2 = 0.9971 \).

**E. Fidelity for the generation process**

Since the \( W \) states generation process is the time reversal of the transmission process, the fidelities for the two processes are equivalent. We only consider the fidelity for the generation process. Define the fidelity \( F = \frac{1}{n} \sum_{i=1}^{n} |\langle \psi^{(i)} | \psi^{(i)} \rangle|^2 \) \( (i=1,2,...,n) \) for the system with \( n \) microwave cavities, where \( |\psi^{(i)} \rangle \) are \( n \) linearly independent states for the multiple-microwave-cavity system composed of cavity modes \( a_1 \sim a_n \), and the excitation probabilities for the mode \( a_0 \) and \( b_m \) are 0, e.g. \( |\psi^{(1)} \rangle = |0\rangle_{a_0} |10,...,0\rangle_{a_2a_3...a_n} |0\rangle_{b_m} \) and \( |\psi^{(n)} \rangle = |0\rangle_{a_0} |00,...,1\rangle_{a_2a_3...a_n} |0\rangle_{b_m} \). \( |\psi^{(i)} \rangle \) are the final states of the input process for generating the states \( |\psi^{(i)} \rangle \) with a group of optimized time-dependent couplings. In the entangled state transmission and generation processes, when \( \kappa_i \ll \kappa_0, g_j \) \( (i=1,2,...,n, j=0,1,2,...,n) \), the whole process can be proceeded with high fidelity in a short time duration \( T \), where \( T \ll \frac{1}{\kappa} \). In the adiabatic transmission and generation, the mechanical mode is a dark mode, and high fidelity can be archived with the mechanical damping rate \( \gamma_m < 10^{-2} \gamma_0 \). However, when \( \kappa_i \) and \( \gamma_m \) become larger, the fidelity will reduce significantly, the numerical simulation results are shown in Fig. 3(a).
We again consider the fidelity for the systems with different number of microwave cavities. With the same values of $g_0$, $\kappa_j$ ($j = 0, 1, 2, ..., n$) and $\gamma_m$, in the transmission and generation process, using the trivial method, we can estimate that, to reach the same value of fidelity, it takes longer for the system with more microwave cavities as $T_n = nT_1$, where $T_n$ ($T_1$) is the time length for the system with $n$ (one) microwave cavities. Fig. 2(b) shows the fidelity over the time length $T$ with the number of the microwave cavities $n = 1, 2, 3, 4$. The black lines are for the trivial method and the green lines are for the non-trivial method we proposed in this letter. With both of the two methods, for the system with more microwave cavities, the fidelity over the time length $T_n$ takes longer for the system with more microwave cavities. With the same values of fidelity, it shows better performance in fidelity than the trivial method.

IV. CONCLUSIONS

In this letter, we have proposed a universal and non-trivial method for transmission and generation of the arbitrary entangled W states in multiple microwave cavities via an optomechanical interface. We have demonstrated that, the arbitrary entangled states can be mapped to the pulse shape of single photons and transmitted out of the system, which can be applied for the quantum information transfer. And the entangle W states can also be generated with the single incident photons with certain pulse shapes. In our method, all of the microwave cavities are equivalent in time sequence, and it shows better performance in fidelity than the trivial method.

Appendix A: The characteristics of V matrix

$V(t)$ matrix is defined as $V(t) = \frac{dU^\dagger(t)}{dt}U(t)$ and $\alpha(t) = U^\dagger(t)C(t)$, and there is

$$\frac{d\alpha(t)}{dt} = -\frac{\kappa_0}{2} \Lambda(t) - V(t)\alpha(t)$$  \hspace{1cm} (A1)

Because $\frac{d}{dt}[U^\dagger(t)U(t)] = 0$, so we can get $\frac{dU^\dagger(t)}{dt}U(t) + U^\dagger(t)\frac{dU}{dt} = 0$, then there is $V(t) + V^\dagger(t) = 0$. Since that, the matrix $V(t)$ is real, we have $V^\dagger(t) = V^T(t)$, so we can get $V_{ij}(t) + V_{ji}(t) = 0$. If $\kappa_0 = 0$, there is

$$\frac{d}{dt} \sum_{i=1}^{n} \alpha_i^2(t) = 2 \sum_{i=1}^{n} \alpha_i(t) \frac{d\alpha_i(t)}{dt} = 2 \sum_{i=1}^{n} \alpha_i(t) \left[ \sum_{j=1}^{n} V_{ij}(t)\alpha_j(t) \right] = 2 \sum_{i,j=1}^{n} V_{ij}(t)\alpha_i(t)\alpha_j(t) = 0$$  \hspace{1cm} (A2)

So that, the function of $V(t)$ is only to redistribute of the populations in every $\varphi_i(t)$.

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