anQCD: a Mathematica package for calculations in general analytic QCD models

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Abstract

We provide a Mathematica package that evaluates the QCD analytic couplings (in the Euclidean domain) $A_\nu(Q^2)$, which are analytic analogs of the powers $a(Q^2)^\nu$ of the underlying perturbative QCD (pQCD) coupling $a(Q^2) \equiv \alpha_s(Q^2)/\pi$, in three analytic QCD models (anQCD): Fractional Analytic Perturbation Theory (FAPT), Two-delta analytic QCD ($2\delta$anQCD), and Massive Perturbation Theory (MPT). The analytic (holomorphic) running couplings $A_\nu(Q^2)$, in contrast to the corresponding pQCD expressions $a(Q^2)^\nu$, reflect correctly the analytic properties of the spacelike observables $D(Q^2)$ in the complex $Q^2$ plane as dictated by the general principles of quantum field theory. They are thus more suited for evaluations of such physical quantities, especially at low momenta $|Q^2| \sim 1 \text{ GeV}^2$.

PACS numbers: 12.38.Bx, 11.15.Bt, 11.10.Hi, 11.55.Fv

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Program Summary

Title of program: anQCD

The main program (anQCD.m) and supplementary modules (Li_nu.m and sOr.m), and the zipped file containing all three files (anQCD.zip), available from the webpage:
gcvetic.usm.cl

Computer for which the program is designed and others on which it is operable: Any work-station or PC where Mathematica is running.

Operating system or monitor under which the program has been tested: Linux and Mac OS X, Mathematica 9.0.1.

No. of bytes in distributed program including test data etc.: 54 kB (main module anQCD.m), 2 kB (supplementary module Li_nu.m), 18 kB (supplementary module sOr.m);

Distribution format: ASCII

Keywords: Analyticity, Fractional Analytic Perturbation Theory, Two-delta analytic QCD model, Massive Perturbation Theory, Perturbative QCD, Renormalization group evolution.

Nature of the physical problem: Evaluation of the values for analytic couplings $A_\nu(Q^2; N_f)$ in analytic QCD [the analytic analog of the power $(\alpha_s(Q^2; N_f)/\pi)^\nu$] based on the dispersion relation; $A_\nu$ represents a physical (holomorphic) function in the plane of complex squared momenta $-q^2 \equiv Q^2$. In anQCD.m we collect the formulas for three different analytic models depending on the energy scale, $Q^2$, number of flavors $N_f$, the QCD scale $\Lambda_{N_f}$, and the (nonpower) index $\nu$. The considered models are: Analytic Perturbation theory (APT), Two-delta analytic QCD ($2\delta$anQCD) and Massive Perturbation Theory (MPT).

Method of solution: anQCD uses Mathematica functions to perform numerical integration of spectral function for each analytic model, in order to obtain the corresponding analytic images $A_\nu(Q^2)$ via dispersion relation.

Restrictions on the complexity of the problem: It could be that for an unphysical choice of the input parameters the results are meaningless.

Typical running time: For all operations the running time does not exceed a few seconds.
1. Introduction

The perturbative approach to QCD (pQCD) works well for evaluations of physical quantities at high momentum transfer \(|q^2| \gtrsim 10^1 \text{ GeV}^2\). However, it is unreliable at low momenta \(|q^2| \sim 1 \text{ GeV}^2\), the principal reason for this being the existence of singularities of the pQCD coupling parameter \(a(Q^2) \equiv \alpha_s(Q^2)/\pi\) at such complex spacelike momenta \(Q^2\), \(|Q^2| \lesssim 1 \text{ GeV}^2\) and \(Q^2 \lesssim 0\). These (Landau) singularities reappear in evaluations of the spacelike observables \(D(Q^2)\) for small \(|Q^2|\). For example, if \(D(Q^2)\) is dominated by the leading-twist term of dimension zero, its evaluated expression is \(f(a(\kappa Q^2))\) where \(f\) is a (truncated) power series in \(a(\kappa Q^2)\) and the positive \(\kappa (\sim 1)\) is the renormalization scale parameter. Hence \(f(a(\kappa Q^2))\) has the same region of singularities as \(a(\kappa Q^2)\). This does not reflect correctly the true analyticity structure of the spacelike observable \(D(Q^2)\). Such an observable must be, by the general principles of the (local) quantum field theory \([1, 2]\), a holomorphic (analytic) function in the complex \(Q^2\) plane except on (parts of) the negative semiaxis where it has a cut; i.e., analyticity for \(Q^2 \in \mathbb{C}\backslash(−\infty, 0]\). Therefore, the coupling parameter \(A_1(Q^2)\), that is to be used instead of \(a(Q^2)\) to evaluate the spacelike observables \(D(Q^2)\), should have qualitatively the same analyticity properties, i.e., \(A_1(Q^2)\) a holomorphic function for \(Q^2 \in \mathbb{C}\backslash(−\infty, 0]\). Such an analytic function \(A_1(Q^2)\) defines what is called analytic QCD (anQCD) model.

The finiteness of the QCD coupling in the infrared regime and, in general, the holomorphic behavior of it in the \(Q^2\) complex plane, are suggested by various independent lines of research in QCD, among them: by the Gribov-Zwanziger approach \([3]\); by analyses of Dyson-Schwinger equations in QCD \([4, 5]\) and by other functional methods \([6, 7]\); by models using the AdS/CFT correspondence modified by a dilaton background \([8]\); in various other approaches such as those in Refs. \([10–15]\).

The first anQCD model, constructed explicitly in the aforementioned sense, is the Analytic Perturbation Theory (APT) of Shirkov, Solovtsov et al. \([16–19]\). The underlying pQCD discontinuity function \(\rho_1^{(\text{pt})}(\sigma) \equiv \text{Im}\alpha(Q^2 = -\sigma - i\epsilon)\) was kept unchanged on the entire negative axis in the \(Q^2\)-plane, i.e., \(\text{Im}A_1^{(\text{APT})}(-\sigma - i\epsilon) = \rho_1^{(\text{pt})}(\sigma)\) for all \(\sigma \geq 0\). On the other hand, the Landau discontinuity region (at \(-\Lambda_L^2 \leq \sigma < 0\)) was eliminated, i.e., \(\text{Im}A_1^{(\text{APT})}(-\sigma - i\epsilon) = 0\) for \(\sigma < 0\). The resulting coupling \(A_1^{(\text{APT})}(Q^2)\) for \(Q^2 \in \mathbb{C}\backslash[−\infty, 0]\) was then obtained by the use of a dispersion relation involving \(\rho_1^{(\text{pt})}(\sigma)\) at \(\sigma \geq 0\). The analogs \(A_n^{(\text{APT})}(Q^2)\) of integer powers \(a(Q^2)^n\) were also constructed in the aforementioned works. An extension to the analogs \(A_n^{(\text{APT})}(Q^2)\) of noninteger powers \(a(Q^2)^\nu\) in this model were obtained and used in the works \([20, 23]\); hence this anQCD model is also called Fractional APT (FAPT).

Later on, other analytic QCD models were constructed, which fulfill certain additional physically motivated restrictions, such as Refs. \([24, 32]\). Analytic QCD models, as well as related dispersive approaches, have been used in evaluations of various low-momentum QCD quantities, cf. Refs. \([33, 39]\). Reviews of the analytic QCD approaches are given in Refs. \([40–45]\).

In addition to FAPT, we will consider here the Two-delta analytic QCD (2\(\delta\)anQCD) \([31]\) and Massive Perturbation Theory (MPT) \([32]\). The 2\(\delta\)anQCD model \([31]\)
is similar to FAPT model in the sense that it is (partially) based on the underlying pQCD coupling $a(Q^2)$: $\text{Im}A_1^{(2\delta)}(-\sigma-i\epsilon) = \rho_1^{(pQCD)}(\sigma)$ for large enough $\sigma \geq M_0^2$ (where $M_0 \sim 1$ GeV is a “pQCD-onset” scale). On the other hand, in the (otherwise unknown) low-$\sigma$ regime, $0 < \sigma < M_0^2$, the behavior of the discontinuity function $\rho_1(\sigma) \equiv \text{Im}A_1^{(2\delta)}(Q^2 = -\sigma - i\epsilon)$ is parametrized by two positive delta functions. The coupling $A_1^{(2\delta)}(Q^2)$ is then obtained by the use of a dispersion relation involving $\rho_1(\sigma)$. The parameters for the delta functions and $M_0$ are determined by requiring that the model effectively merges with the pQCD for large $|Q^2| > \Lambda^2$ (where $\Lambda^2 \sim 0.1$ GeV$^2$). On the other hand, Massive Perturbation Theory (MPT) \cite{32} is defined via the identity $A_1(Q^2) = a(Q^2 + M^2)$, where $M \sim 1$ GeV is an effective dynamical gluon mass.

In general anQCD models, such as $2\delta anQCD$ or MPT, the formalism for construction of analytic analogs $A_\nu(Q^2)$ of the powers $a(Q^2)^\nu$ was formulated in Refs. \cite{27, 28} for the case of integer index $\nu$, and in Ref. \cite{16} for general (noninteger) index $\nu$.

Presently, there exist programs for numerical evaluation of the APT and “massive” APT (MAPT) \cite{47}, and of FAPT couplings \cite{48}. The purpose of this work is to offer an extended program in Mathematica which numerically evaluates the couplings in FAPT, $2\delta anQCD$ and in MPT, in order to correctly evaluate (truncated) perturbation series of physical quantities in these anQCD models. Our program evaluates the FAPT couplings in a similar way as the program of Ref. \cite{48}; but the part of our program which evaluates the $2\delta anQCD$ and MPT couplings is new.

We summarize in Sec. 2 the calculation of the running coupling of the underlying pQCD, the threshold matching, and the corresponding QCD scales $\Lambda_{N_f}$. In Sec. 3 we present a general method for calculation of the analytic analogs $A_\nu(Q^2)$ of powers $a(Q^2)^\nu$ in anQCD models, and a description of the three mentioned anQCD models: FAPT, $2\delta anQCD$ (with new extension for $N_f \geq 4$), and MPT. In addition, curves of some of the resulting couplings as a function of $Q^2$, at positive $Q^2$, are presented. Finally, in Sec. 4 the main procedures of the calculational program and some practical examples are presented, with more detailed definitions of the procedures included in Appendix A.

### 2. Running coupling in the underlying perturbative QCD

The differential equation that defines the beta function and therefore the running coupling in perturbative QCD (pQCD) is given by the renormalization group equation (RGE, at renormalization scale $\mu^2 = Q^2$)

$$
\beta(a(Q^2)) = Q^2 \frac{\partial a(Q^2)}{\partial Q^2} = -\sum_{j=2}^{\infty} \beta_{j-2}(N_f)a^j(Q^2),
$$

(1)

with the notation: $a(Q^2) \equiv \alpha_s(Q^2)/\pi = g_s(Q^2)^2/(4\pi^2)$ and $N_f$ is the number of active quarks flavors. The first two beta coefficients ($\beta_0$ and $\beta_1$, \cite{49, 50}) are scheme independent, i.e., they are universal in the mass independent renormalization schemes

$$
\beta_0(N_f) = \frac{1}{4} \left(11 - \frac{2}{3}N_f\right), \quad \beta_1(N_f) = \frac{1}{16} \left(102 - \frac{38}{3}N_f\right).
$$

(2)
The next coefficients ($\beta_2, \beta_3, \ldots$) are scheme dependent; in fact, they define the renormalization scheme \cite{footnote1}. In the MS scheme, $\beta_2$ and $\beta_3$ are known \cite{footnote2, footnote3}:

\begin{align}
\beta_2(N_f) &= \frac{1}{64} \left( \frac{2857}{2} - \frac{5033}{18} N_f + \frac{325}{54} N_f^2 \right), \\
\beta_3(N_f) &= \frac{1}{256} \left[ \left( \frac{149753}{6} + 3564 \zeta_3 \right) - \left( \frac{1078361}{162} + \frac{6508}{27} \zeta_3 \right) N_f \\
&\quad + \left( \frac{50065}{162} + \frac{6472}{81} \zeta_3 \right) N_f^2 + \frac{1093}{729} N_f^3 \right].
\end{align}

(3a) (3b)

where $\zeta_\nu$ is the Riemann zeta function, in particular $\zeta_3 \simeq 1.202057$.

The beta function on the right-hand side of Eq. (1) is usually approximated as a truncated perturbation series of coupling $a$. The resulting differential equation for $a$ is solved, either analytically (if possible) or numerically. For example, the one-loop order equation can be integrated explicitly, giving the well known solution

$$ a(Q^2) = \frac{1}{\beta_0 \ln(Q^2/\Lambda^2)}, \quad \Lambda^2 = \mu^2 e^{-1/(\beta_0 a(\mu^2))}. $$

(4)

One way to solve the RGE to two loops is iterating with respect to the one-loop formula, which gives us an approximate coupling as an expansion in powers of $L^{-1}$, where $L \equiv \ln(Q^2/\Lambda^2)$. If we truncate at $L^{-2}$, we obtain

$$ a^{(2,\mathcal{L}2)}(Q^2) = \frac{1}{\beta_0 L} \left( 1 - \frac{\beta_1 \ln(L)}{\beta_0^2} \right). $$

(5)

The iterative method can be performed to any loops, and is straightforward to find the next levels. For example, when truncating the expansion of $M$-loop coupling at $L^{-\mathcal{N}} \equiv 1/\ln^{\mathcal{N}}(Q^2/\Lambda^2)$, we obtain

$$ a^{(M,\mathcal{L}N)}(Q^2) = \frac{1}{\beta_0 L} \left\{ 1 - \frac{\beta_1 \ln(L)}{\beta_0^2} + \frac{1}{\beta_0^2 L} \left[ \frac{\beta_2^2}{\beta_0^4} (\ln^2(L) - \ln(L) - 1) + \frac{\beta_2}{\beta_0} \right] + \frac{1}{\beta_0^3 L^3} \left[ \frac{\beta_3^3}{\beta_0^6} \left( -\ln^3(L) + \frac{5}{2} \ln^2(L) + 2 \ln(L) - \frac{1}{2} \right) - 3 \frac{\beta_1 \beta_2}{\beta_0^2} \ln(L) + \frac{\beta_3}{2 \beta_0} \right] + \frac{1}{\beta_0^{N-1} L^{N-1}} \left[ \frac{\beta_1^{N-1}}{\beta_0^{N-1}} (-1)^{N-1} \ln^{N-1} L + \ldots \right] \right\}.
$$

(6)

There is a way to find the two-loop coupling as a solution of RGE exactly. The two-loop RGE leads to a transcendental equation. Namely, integrating (1), with $\beta_k = 0$ for

\footnote{The superscript notation $(M,\mathcal{L}N)$ in Eq. (6) means that the expansion is truncated at $1/L^{\mathcal{N}}$, and that $M$-loop $\beta$-function is taken, i.e., $\beta_j = 0$ for $j \geq M$. For consistency reasons, we must have $\mathcal{N} \geq M$. In practice, the expansion gives us expression which, for $Q^2 > \Lambda^2$, tends toward the exact $M$-loop coupling $a^{(M)}(Q^2)$ when $\mathcal{N} \to \infty$ (i.e., $\mathcal{N} \gg M$).}
\(k = 2, 3, \ldots\), we have
\[
\int_{a(\mu^2)}^{a(Q^2)} \frac{da}{a^2 \left(1 + \frac{a}{\beta_0} a\right)} = -\beta_0 \int_{0}^{\frac{1}{2} \ln(Q^2/\mu^2)} d\ln(Q^2/\mu^2).
\]

(7)

So, the transcendental equation gets the form
\[
\ln(Q^2/\mu^2) = C + \frac{1}{\beta_0 a(Q^2)} + \frac{\beta_1}{\beta_0} \ln \left(1 + \frac{\beta_1}{\beta_0} a(Q^2)\right) - \frac{\beta_1}{\beta_0} \ln \left(1 + \frac{\beta_1}{\beta_0} a(Q^2)\right),
\]

(8)

here \(C\) contains the coupling in \(\mu^2\).

Now we want to introduce (in the same way as in the one-loop case) a new invariant mass parameter \(\Lambda\), given by
\[
\ln\left(\frac{Q^2}{\Lambda^2}\right) = \frac{1}{\beta_0 a(Q^2)} - \frac{\beta_1}{\beta_0} \ln \left(\frac{\beta_1}{\beta_0^2} + \frac{1}{\beta_0 a(Q^2)}\right),
\]

(9)

\[\Lambda^2 = \mu^2 \exp \left[ C - \frac{\beta_1}{\beta_0} \ln(\beta_0) \right].\]

At this stage, we need to invert this relation, but this is difficult and serious problems related to the singularity structure appear. The solution was made with the help of the so-called Lambert \(W\) function defined by
\[
W(z) \exp[W(z)] = z
\]

(10)

The singularity structure of the Lambert function is made up of infinite number of branches; it satisfies the following symmetry relation: \(W_n^*(y^*) = W_n(y)\).

With this function the solution to the coupling is \([34, 54]\)
\[
a^{(2)}(Q^2) = -\frac{1}{c_1} \frac{1}{1 + W_{\pm 1}(z_{\pm})},
\]

(11)

where \(c_1 = \beta_1/\beta_0\), \(Q^2 = |Q^2|e^{i\phi}\), and the upper sign refers to the case \(0 \leq \phi \leq +\pi\), the lower sign to \(-\pi \leq \phi \leq 0\), and
\[
z_{\pm} = \frac{1}{c_1} e \left(\frac{|Q^2|}{\Lambda^2}\right)^{-\beta_0/c_1} \exp \left[i \left(\pm\pi - \frac{\beta_0}{c_1} \phi\right)\right].
\]

(12)

This idea can be extended to the higher-loop case, using an approximation via Padé [3/1] for the beta function \(\beta(a)\)
\[
\beta_{[3/1]}(a) = -\beta_0 a^2(Q^2) \frac{1 + (c_1 - c_2/c_1)^2 a(Q^2)}{1 - (c_2/c_1)^2 a(Q^2)}
\]

(13a)

\[= -\beta_0 a^2(Q^2) \left[1 + c_1 a(Q^2) + c_2 a(Q^2)^2 + \frac{c_3}{c_1} a(Q^2)^2 + \frac{c_3}{c_1^2} a(Q^2)^3 + \ldots\right].\]
where the renormalization scheme parameters are: \( \beta_2 = \beta_0 c_2 \) and \( \beta_j = \beta_0 c_2^{j-1}/c_1^{j-2} \) \( (j \geq 3) \). We call this scheme \( c_2 \)-Lambert scheme. When \( c_2 \) in the beta function (13) is chosen to be in \( \overline{\text{MS}} \) scheme, i.e., \( c_2 = \overline{c}_2 = (\beta_2/\beta_0) \), we will refer to this scheme, somewhat loosely, as 3-loop \( \overline{\text{MS}} \) in pQCD, FAPT and MPT. With this, the solution of the coupling to three loops in terms of the Lambert function takes the form \(^2\)

\[
a(Q^2) = -\frac{1}{c_1} \frac{1}{1 - c_2/c_1^2 + W_{\pm 1}(z_\pm)}.
\]

The Lambert function \( W = W(z) \) is defined via the inverse relation (10), cf. Fig. 1(a).

Figure 1: (a) The defining relation \( z = W e^W \) for the Lambert function \( W(z) \), for \( -1/e < z < 0 \); (b) The branch \( W_{-1}(z) \) for the same \( z \)-interval; when \( c_2 < 0 \), the denominator of Eq. (14) becomes zero at a \( z(u_L) \) in this interval.

The two branches \( W_{\pm 1}(z) \) of the Lambert function are related via complex-conjugation \( W_{+1}(z^*) = W_{-1}(z)^* \), and the point \( z = -1/e \) is the branching point of these functions. In the interval \( -1/e < z < 0 \), \( W_{-1}(z) \) is a decreasing function of \( z \), cf. Fig. 1(b). When \( z \rightarrow -1/e \), the scale \( Q^2 \) tends to \( Q^2 \rightarrow +\infty \), and \( W_{-1}(z) \rightarrow -\infty \), this reflecting the asymptotic freedom of \( a(Q^2) \) of Eq. (14).

The coupling (14) with \( c_2 = \overline{c}_2(N_f) \) will be the underlying pQCD coupling in the analytic models (what we call) : 3-loop FAPT\( _{N_f} \), 3-loop global FAPT, and 3-loop MPT\( _{N_f} \). In \( 2\delta\text{anQCD} \), the underlying pQCD coupling will also be that of Eq. (14), but with the scheme parameter \( c_2 \) in the interval \( -5.6 < c_2 < -2 \), cf. Table 2 later (with \( c_2 = -4.9 \) being the preferred illustrative value).

2.1. Thresholds and global coupling

We note that the dependence on the number of effective quark flavors \( (N_f) \) is in the beta coefficients (2)-(3). We use the following notations: the \( N_f \)th quark flavor has the MS

\(^2\)In the expression (12), the “Lambert” scale \( \Lambda \) is different from the scale \( \overline{X} \) appearing in the expansion (6). Therefore, as we use the latter as an input, the program relates these two scales by equating Eq. (6) [with: \( \beta_j/\beta_0 = c_j = c_2^{j-1}/c_1^{j-2} \) \( (j = 2,3,\ldots) \), cf. the expansion (13b)] with Eq. (14) at high \( Q^2 \) \( (\sim 10^{10}\text{GeV}^2) \).
mass \( m_{N_f} \equiv m_q \equiv \overline{m}_q(\overline{m}_q) \), where \( q = c, b, t \) for \( N_f = 4, 5, 6 \), respectively (we consider \( \overline{m}_u, \overline{m}_d, \overline{m}_s \approx 0 \)). QCD\(_N_f\) is applied, in principle, at the scales \( \mu \equiv \sqrt{|Q^2|} \) such that \( m_{N_f} \ll \mu \ll m_{N_f+1} \); in practice, it is applied at \( \mu \)'s such that \( m_{N_f} \lesssim \mu \lesssim m_{N_f+1} \). If the threshold scale is chosen to be \( Q^2_{\text{thr}} = m^2_{N_f} \), the one-loop quark threshold condition is the

continuity of the coupling \( a(Q^2) \) there; i.e., if we denote \( a(Q^2; N_f) \equiv f_{N_f}(Q^2/\Lambda^2_{N_f}) \) and \( a(Q^2; N_f-1) \equiv f_{N_f-1}(Q^2/\Lambda^2_{N_f-1}) \), at \( Q^2_{\text{thr}} = m^2_{N_f} \) we have

\[
f_{N_f-1}(m^2_{N_f}/\Lambda^2_{N_f-1}) = f_{N_f}(m^2_{N_f}/\Lambda^2_{N_f}) .
\]  

(15)

At a higher loop level, a noncontinuous matching has to be performed between the couplings in the effective theories QCD\(_N_f\) and QCD\(_{N_f-1}\). If the coupling runs according to the results of Ref. \[56\], the 3-loop matching condition (for the case of 4-loop \( \overline{\text{MS}} \) beta function, the \((N-1)\)-loop matching condition should be used. According to the results of Ref. \[56\], the 3-loop matching condition (for the case of 4-loop \( \overline{\text{MS}} \) RGE running) has the form

\[
a' = a - a^2 \frac{\ell_h}{6} + a^3 \left( \frac{\ell_h^2}{36} - \frac{19}{24} \ell_h + \widetilde{c}_2 \right) + a^4 \left[ - \frac{\ell_h^3}{216} \right]
\]

\[
- \frac{131}{576} \ell_h^2 + \frac{\ell_h}{1728} ( -6793 + 281(N_f-1) ) + \widetilde{c}_3 ,
\]

(16)

where: \( \ell_h = \ln[\mu^2_{N_f}/\overline{m}_q^2] ; \ a' = a(\mu^2_{N_f}; N_f-1) \) and \( a = a(\mu^2_{N_f}; N_f) \) in \( \overline{\text{MS}} \); and

\[
\widetilde{c}_2 = \frac{11}{72}, \quad \widetilde{c}_3 = - \frac{82043}{27648} \zeta_3 + \frac{564731}{124416} - \frac{2633}{31104}(N_f-1) .
\]

(17)

The threshold scale is \( \mu^{(N_f)} = k \overline{m}_q ( \ell_h = 2 \ln \kappa ) \), where \( q = c, b, t \) for \( N_f = 4, 5, 6 \), respectively; and usually \( 1 \leq \kappa \leq 3 \) is taken.\(^3\)

In Table 2.1, we present the results for various scales \( \Lambda_{N_f} \) in pQCD, for the case of the 4-loop RGE running in \( \overline{\text{MS}} \) scheme and the corresponding 3-loop threshold matching with \( \kappa = 2 \) [thresholds at \( Q = \kappa \overline{m}_q \)], i.e., the 4/3-loop case; and for the 2-loop RGE running and 1-loop threshold matching with \( \kappa = 2 \) and \( \kappa = 1 \), i.e. the 2/1-loop case. For the starting value in the numerical integration of the RGE, we used the present world average value \( a(M_Z^2; N_f = 5) = 0.1184/\pi \) \[57\] in \( \overline{\text{MS}} \). In all cases, the values of \( \Lambda_{N_f} \) were determined by equating the numerically obtained (“exact”) values of \( a(Q^2) \) with those of the expansion \[6\] with \( N = 8 \); the matchings for \( N_f = 5, 4, 3 \) were made at the corresponding positive maximal values of the \( N_f \)-range, i.e., at \( Q^2 = (\kappa \overline{m}_q)^2 \), where \( \overline{m}_q = \overline{m}_t, \overline{m}_b, \overline{m}_c \) for \( N_f = 5, 4, 3 \), respectively. The used values of the \( \overline{\text{MS}} \) masses \( m_q \equiv m_q(\overline{m}_q) \) were: 1.27 GeV, 4.2 GeV, 163 GeV, respectively. The value of the scale \( \Lambda_6 \) was determined by equating the expansion \[6\] with the numerical values \( a(Q^2; N_f = 6) \) at large momenta \( (Q \gtrsim 10^3 \text{ GeV}) \). The 4/3-loop results change insignificantly when the

\(^3\) For the evaluation of \( a(Q^2) \) at a complex \( Q^2 \), the \( N_f \) value assigned is determined by \( (k \overline{m}_{N_f})^2 < |Q^2| < (k \overline{m}_{N_f+1})^2 \), i.e., with such \( N_f \) we have \( a(Q^2) = a(Q^2; N_f) \).
threshold matching parameter changes from $\kappa = 2$ to $\kappa = 1$: $\overline{\Lambda}_3$ value decreases by 1.2 MeV, and $\overline{\Lambda}_4$ value by 0.4 MeV. The 2/1-loop values, however, change significantly when we change $\kappa = 2$ to $\kappa = 1$: $\overline{\Lambda}_3$ decreases from 375.3 to 361.8 MeV; $\overline{\Lambda}_4$ decreases from 312.6 to 308.1 MeV. In all cases (4/3 and 2/1-loop), the value $\overline{\Lambda}_5$ is independent of $\kappa$ (because the initial value is at $Q^2 = M_Z^2$, i.e., where $N_f = 5$); the value of $\overline{\Lambda}_6$ varies insignificantly in the 4/3-loop case, and in the 2/1-loop case it increases by 1 MeV when $\kappa$ changes from 2 to 1.

In FAPT, which is an analytic QCD model with exceptionally fast convergence properties, the more simple approach (2/1-loop) gives the results close to (within a few per cent) the approaches using the higher-loop versions for the underlying pQCD. Therefore, in FAPT model, we can use various levels (2/1-, 3/2- and 4/3-loop), while for the other two versions of analytic QCD (2δanQCD and MPT) the preferred versions are 4/3-loop.

In addition, in FAPT, the program allows to choose either the usual version (i.e., at a fixed chosen $N_f$), or a "global" version [19] for which the underlying pQCD coupling $a(Q^2; N_f)$ [and its discontinuity function $\rho_1(N_f; pt)(\sigma) \equiv \text{Ima}(-\sigma - i\epsilon; N_f)$] is replaced by a new, "global", pQCD coupling

$$
a^{(\text{glob})}(Q^2) = a(Q^2; N_f = 3; \overline{\Lambda}_3)\Theta(|Q^2| \leq \mu^{(4)2}) + a(Q^2; N_f = 4; \overline{\Lambda}_4)\Theta(\mu^{(4)2} < |Q^2| \leq \mu^{(5)2}) + a(Q^2; N_f = 5; \overline{\Lambda}_5)\Theta(\mu^{(5)2} < |Q^2| \leq \mu^{(6)2}) + a(Q^2; N_f = 6; \overline{\Lambda}_6)\Theta(\mu^{(6)2} < |Q^2|) \quad (18)
$$

where $\mu^{(3)} = \kappa m_3 = \kappa \overline{m}_c(\overline{m}_c)$, etc., and the scales $\overline{\Lambda}_{N_f}$ and the RGE-running of $a(Q^2; N_f)$ are determined by $N/(N - 1)$-loop approach in $\overline{\text{MS}}$ (in the following referred to simply as $N$-loop approach; $N = 1, 2, 3, 4$). However, in such a global FAPT the values of the scales $\overline{\Lambda}_{N_f}$ differ somewhat from those of the actually valid pQCD [in the latter, the world average value $a(M_Z^2; 5) = 0.1184/\pi$ in $\overline{\text{MS}}$ scheme fixes the scale $\overline{\Lambda}_5$, see the second line of Table 2.1]. The preferred values in the global FAPT are $\overline{\Lambda}_5 \approx 0.260$ GeV [19, 22, 23, 43], corresponding to $\overline{\Lambda}_3 \approx 0.435$ GeV [and $\alpha_s(M_Z^2; 5; \overline{\text{MS}}) \approx 0.1218$] in 2/1-loop approach with $\kappa = 2$, and to $\overline{\Lambda}_3 \approx 0.400$ GeV in 4/3-loop approach with $\kappa = 2$.

We mention some practical aspects of the program, concerning the $\overline{\Lambda}_{N_f}$ scales and the

| Method           | $\overline{\Lambda}_6$ | $\overline{\Lambda}_5$ | $\overline{\Lambda}_4$ | $\overline{\Lambda}_3$ | $\pi(N_f)$ | $\pi(N_f - 1)$ |
|------------------|-------------------------|-------------------------|-------------------------|-------------------------|------------|---------------|
| 4/3-loop, $\kappa = 2$ | 90.6                    | 213.3                   | 297.0                   | 341.8                   | 0.0318(0.03161) | 0.03185(0.03162) |
| 2/1-loop, $\kappa = 2$ | 89.7                    | 216.7                   | 312.6                   | 375.3                   | 0.03185(0.03162) | 0.05934(0.05852) |
| 2/1-loop, $\kappa = 1$ | 90.7                    | 216.7                   | 308.1                   | 361.8                   | 0.03465(0.03465) | 0.07154(0.07154) |
treatment of quark thresholds. The input parameter in the program is $\Lambda_{N_f}^2$ (in GeV$^2$) for fixed-$N_f$ MPT models and $\Lambda_3^2$ for global FAPT\textsuperscript{4} in (fixed-$N_f$) 2$\delta$anQCD models, the scales $\Lambda_{N_f}$ ($\Leftrightarrow \Lambda_{N_f}$, Lambert scales) are fixed by the world average value $a(M_Z^2; \overline{\text{MS}}; N_f = 5) = 0.1184/\pi$ [57]. In addition, the scheme parameter $c_2 ((\beta_2/\beta_0))$ in 2$\delta$anQCD $N_f=3$ can be adjusted by hand and can vary in the interval $-5.6 < c_2 < -2$ (see later). The quark threshold parameter is fixed to $\kappa = 2$ in the program for 2$\delta$anQCD (kappa2d=2), and also in FAPT (kappa=2). On the other hand, in MPT, at a given $N_f$, there is no $\kappa$ appearing, the scale $\Lambda_{N_f}$ is an input parameter. However, the value of $\kappa$ in FAPT can be adjusted by hand in the program\textsuperscript{7} while in 2$\delta$anQCD it should remain unchanged by construction (kappa2d=2). If $N$ is the number of loops in the RGE running ($N = 1, 2, 3$ or 4), the input will be $\Lambda_3^2 = L2N1nf3$, and other scales (for other $N_f \equiv N_f$) are then given by the following functions: $\Lambda_{N_f}^2 = L2N1[N_f L2N1nf3]$ with $N_f = 4, 5, 6$ which is obtained via the $(N - 1)$-loop matching condition, i.e., the relation [16] where, on the right-hand side, the last included term is $\sim a^N$.

Now, we consider an example of our pQCD running coupling and their value of Lambda QCD parameter, where the perturbative $N$-loop running coupling for $N_f$ is given by functions ($a1l, a2l, a3l, a4l$), where

$$aNl[N_f, Q2, L2, \phi] \equiv a(Q^2 = Q2 \times e^{i\phi}; N_f = N_f; L2 = \Lambda_{N_f}^2; N - \text{loop}; \overline{\text{MS}}), \quad (19)$$

where $Q2 = |Q^2|$, and $-\pi < \phi < \pi$. The global running perturbative QCD coupling is

$$aNl\text{glob}[N_f, Q2, L23, \phi] \equiv a^{(\text{glob})}(Q^2 = Q2 \times e^{i\phi}; L23 = \Lambda_3^2; N - \text{loop}; \overline{\text{MS}}). \quad (20)$$

Our Mathematica package is called by the command

\textbf{In}[1] := <<anQCD.m}

Comment: We defined the physical parameters (mc= $\overline{m}_c$, etc.) inside of the NumDefanQCD function:

\textbf{In}[2] := {mc/.NumDefanQCD, mb/.NumDefanQCD, mt/.NumDefanQCD, MZ/.NumDefanQCD}

\textbf{Out}[2] := {1.27, 4.2, 163., 91.1876}

\textsuperscript{4} In global FAPT, the other $\Lambda_{N_f}$ ($N_f > 3$) are fixed from $\Lambda_3$ by using for $a(Q^2)$ only the expansion [6] with $N = 8$ (and not the RGE-numerically obtained "exact" values). But the effect of this additional approximation in comparison to Table 2.1 is small. For example, for $\Lambda_3 = 341.8$ MeV case with 4/3-loop approach and $\kappa = 2$ (the first line in Table 2.1), the resulting $\Lambda_{N_f}$ become 296.5 MeV, 212.8 MeV, 90.3 MeV for $N_f = 4, 5, 6$, respectively, i.e., by about 0.5 MeV lower than in Table 2.1. In 2/1-loop approach with $\kappa = 2$, for $\Lambda_3 = 375.3$ MeV value (i.e., the second line of Table 2.1), the values of $\Lambda_{N_f}$ in this approach are 311.9 MeV, 215.8 MeV and 89.4 MeV for $N_f = 4, 5, 6$, respectively, i.e., lower than in Table 2.1 by less than 1 MeV.

\textsuperscript{7} Physically, $1 \leq \kappa \leq 3$ appears to be a reasonable interval of possible values. The values of various $\Lambda_{N_f}$ change very little when $\kappa$ is varied.
Comment: Lambda squared QCD parameter $\Lambda^2$ can be fixed by the value $a(M_Z^2;\overline{\text{MS}}) = 0.1184/\pi$

```
In[3] := L2nf5=L25/.FindRoot[a4l[5,91.1876^2/L25,0] == 0.1184/Pi,{L25,0.1}]
Out[3] := 0.0455164
```

3. Analytic QCD models

3.1. General formalism

In analytic QCD models, the dispersion relation between the discontinuity function $\rho_1(\sigma) \equiv \text{Im} A_1(-\sigma - i\epsilon)$ and the coupling itself $A_1(Q^2)$ plays usually a fundamental role, where the discontinuity function $\rho_1(\sigma)$ is proportional to the discontinuity of $A_1$ across the cut at $Q^2 = -\sigma$ ($< 0$). In pQCD such dispersion relation also exists. Namely, when the function $a(Q^2)/(Q^2 - Q'^2)$ is integrated in the $Q'^2$ complex plane along an appropriate closed contour which avoids all the cuts and encloses the pole $Q'^2 = Q^2$ (cf. Fig. 2(a)), and the Cauchy theorem is applied, the following dispersion relation is obtained:

$$a(Q^2) = \frac{1}{\pi} \int_{\sigma = -\Lambda^2_{\text{Land}} - \eta}^{\infty} \frac{d\sigma \rho_1^{(\text{pt})}(\sigma)}{\sigma + Q^2}, \quad (\eta \to +0).$$ (21)

Here, $\rho_1^{(\text{pt})}(\sigma) \equiv \text{Im} a(-\sigma - i\epsilon)$ is the discontinuity function of the pQCD coupling $a$ along the entire cut axis, and $Q'^2 = \Lambda^2_{\text{Land}}$ ($> 0$) is the branching point of the Landau cut of the pQCD coupling $a(Q^2)$.

In general analytic QCD models the dispersion relation has the form

$$A_1(Q^2) = \frac{1}{\pi} \int_{\sigma = M_{\text{thr}}^2}^{\infty} \frac{d\sigma \rho_1(\sigma)}{\sigma + Q^2}, \quad \text{where: } \rho_1(\sigma) \equiv \text{Im} A_1(-\sigma - i\epsilon).$$ (22)

The discontinuity function $\rho_1(\sigma)$ is defined for $\sigma \geq 0$; usually, the discontinuity cut is nonzero below a threshold value $-\sigma \leq -M_{\text{thr}}^2$ where $M_{\text{thr}} \sim M_{\pi}$. Therefore $Q^2$ can have any value in the complex plane except on the cut $(-\infty, -M_{\text{thr}}^2)$ (cf. Fig. 2(b)).
We regard either the discontinuity function $\rho_1(\sigma)$, or the coupling function $A_1(Q^2)$, as the quantity which defines the anQCD model. Below we describe how one constructs from them other quantities, such as analytic analogs $A_\nu(Q^2)$ of powers $a(Q^2)^\nu$ (where $\nu$ is a real number) once the function $\rho_1(\sigma)$ or $A_1(Q^2)$ is known.

In order to find the correct analogs $A_n$ of the powers $a^n$, the logarithmic derivatives are needed, and from RGE it is straightforward to obtain

$$
\tilde{A}_{n+1}(Q^2) = \frac{(-1)^n}{\beta_0^n n!} \left( \frac{\partial}{\partial \ln Q^2} \right)^n A_1(Q^2), \quad (n = 0, 1, 2, \ldots) .
$$

(23)

We note that for $n = 0$ we have $\tilde{A}_1 \equiv A_1$. We can write the logarithmic derivatives in the following form [46]:

$$
\tilde{A}_{n+1}(Q^2) = \frac{1}{\pi} \frac{(-1)}{\beta_0^n \Gamma(n+1)} \int_0^\infty \frac{d\sigma}{\sigma} \rho_1(\sigma) \text{Li}_{-n}(\sigma/Q^2) .
$$

(24)

This relation is valid for $n = 0, 1, 2, \ldots$. Analytic continuation in $n \mapsto \nu$ ($\nu \in \mathbb{R}$) gives us the logarithmic noninteger derivatives [46]

$$
\tilde{A}_{\nu+1}(Q^2) = \frac{1}{\pi} \frac{(-1)}{\beta_0^n \Gamma(\nu+1)} \int_0^\infty \frac{d\sigma}{\sigma} \rho_1(\sigma) \text{Li}_{-\nu}\left(\frac{-\sigma}{Q^2}\right) (-1 < \nu) .
$$

(25)

We note that the integral converges for $\nu > -1$. Namely, at high $\sigma$ ($|z| \gg 1$ where $z = \sigma/Q^2$) we have in the integrand of equation (25): $\rho_1(\sigma) \approx \rho_1^{(\text{pt})}(\sigma) \sim \ln^{-2} \sigma \sim \ln^{-2} z$ and $\text{Li}_{-\nu}(z) \sim \ln^{-\nu} z$ (for noninteger $\nu$). Therefore, the integral converges at $\sigma \to \infty$ if $\nu > -1$. The integral obviously converges at low $\sigma$, too.

We can recast the result (25) into an alternative form involving the spacelike coupling $A_1$ instead of the discontinuity function $\rho_1(\sigma)$. This gives us (for $\nu = n+\delta$, with $0 < \delta < 1$ and $n = -1, 0, 1, 2, \ldots$) [46]

$$
\tilde{A}_{n+1}(Q^2) \equiv \tilde{A}_{n+1+\delta}(Q^2)
$$

$$
= \frac{1}{\beta_0^n \Gamma(1 + \nu) \Gamma(1 - \delta)} \left( -\frac{d}{d \ln Q^2} \right)^{n+1} \int_0^1 \frac{d\xi}{\xi} A_1(Q^2/\xi) \ln^{-\delta}\left(\frac{1}{\xi}\right) \quad (26a)
$$

$$
= \frac{1}{\beta_0^n \Gamma(n + 1 + \delta)} \frac{\Gamma(1 + \delta)}{\sin(\pi \delta)} \left( -\frac{d}{d \ln Q^2} \right)^{n+1} \int_0^\infty \frac{dt}{t^\delta} A_1(Q^2 e^t) ,
$$

(26b)

where the last form (26b) was obtained from the previous one by the substitution $t = \ln(1/\xi)$ and using the identity $\Gamma(1 + \delta) \Gamma(1 - \delta) = \pi \delta/ \sin(\pi \delta)$.

---

6 In Mathematica [58], the $\text{Li}_{-\nu}(z)$ function is implemented as PolyLog$[-\nu, z]$. However, at large $|z| > 10^7$, PolyLog$[-\nu, z]$ appears to be unstable. For such $z$ we should use the identities relating $\text{Li}_{-\nu}(z)$ with $\text{Li}_{-\nu}(1/z)$, which can be found, for example, in [59]. Our supplementary module Li_{-nu}.m gives such stable functions $\text{Li}_{-\nu}(z) = \text{polylog}[-\nu, z]$.

7 A related, but somewhat lengthier, formula for $\tilde{A}_{n+1}(Q^2)$ in terms of $\rho_1(\sigma)$ which is valid in an extended interval $\{ -2 < \nu \}$, was also obtained in Ref. [49] [cf. Eq.(22) there]. Our Mathematica package uses that lengthy formula.
The analytic analogs $\mathcal{A}_\nu(Q^2) \equiv (a''(Q^2))_{an}$ of powers $a(Q^2)^\nu$ can be constructed as linear combinations of $\bar{\mathcal{A}}_{\nu+m}$'s:

$$\mathcal{A}_\nu = \bar{\mathcal{A}}_\nu + \sum_{m \geq 1} \bar{\kappa}_m(\nu)\bar{\mathcal{A}}_{\nu+m},$$

(27)

where the coefficients $\bar{\kappa}_m(\nu)$ were obtained in Refs. [46] for general $\nu$.

The approach [25] with [27] with [26] for the case of integer $\nu$ was constructed in Refs. [27, 28], and for general real $\nu$ in Ref. [60].

Specifically, let us consider a general spacelike scale- and scheme-invariant physical quantity $\mathcal{D}(Q^2)$ which has the available truncated perturbation (power) series of the form

$$\mathcal{D}^{[N]}(Q^2; \kappa)_{pt} = a(\kappa Q^2)^{\nu_0} + d_1(\kappa)a(\kappa Q^2)^{\nu_0+1} + \ldots + d_{N-1}(\kappa)a(\kappa Q^2)^{\nu_0+N-1},$$

(28)

where $0 < \kappa \sim 1$ is the renormalization scale parameter. The evaluation of this quantity in a general analytic QCD model is then performed by the substitution $a^{\nu_0+n} \mapsto \mathcal{A}_{\nu_0+n}$

$$\mathcal{D}^{[N]}(Q^2; \kappa)_{an} = \mathcal{A}_{\nu_0}(\kappa Q^2) + d_1(\kappa)\mathcal{A}_{\nu_0+1}(\kappa Q^2) + \ldots + d_{N-1}(\kappa)\mathcal{A}_{\nu_0+N-1}(\kappa Q^2),$$

(29)

with the quantities $\mathcal{A}_{\nu_0+n}$ constructed according to Eq. (27) where the truncations are made, in general, at the highest available order of the series (28), i.e., at $\sim a^{\nu_0+N-1} \sim \bar{\mathcal{A}}_{\nu_0+N-1}$

$$\mathcal{A}_{\nu_0+n} = \bar{\mathcal{A}}_{\nu_0+n} + \sum_{m=1}^{N-1-n} \bar{k}_m(\nu_0+n)\bar{\mathcal{A}}_{\nu_0+n+m}. $$

(30)

We refer for more details to Refs. [46, 60]. It is important to note that $\mathcal{A}_{\nu_0+n} \neq (\mathcal{A}_1)^{\nu_0+n}$, i.e., the series (29) is a nonpower series in any analytic QCD which is not perturbative. If, instead, we used in such analytic QCD the powers $(\mathcal{A}_1)^{\nu_0+n}$, the resulting truncated power series would show increased renormalization scale dependence and (for low $|Q^2|$) strongly divergent behavior when $N$ increases, a consequence of incorrect treatment of the nonperturbative contributions contained in the difference $\mathcal{A}_1(\mu^2) - a(\mu^2)$, as emphasized in Refs. [60].

Further, the result (29)-(30) can be reexpressed in terms of $\bar{\mathcal{A}}_{\nu_0+n}$'s

$$\mathcal{D}^{[N]}(Q^2; \kappa)_{an} = \bar{\mathcal{A}}_{\nu_0}(\kappa Q^2) + \bar{d}_1(\kappa)\bar{\mathcal{A}}_{\nu_0+1}(\kappa Q^2) + \ldots + \bar{d}_{N-1}(\kappa)\bar{\mathcal{A}}_{\nu_0+N-1}(\kappa Q^2),$$

(31)

where

$$\bar{d}_M(\kappa) = d_M(\kappa) + \sum_{q=1}^{M} \bar{k}_q(\nu_0 + M - q)d_{M-q}(\kappa), \quad (M = 1, 2, \ldots, N - 1),$$

(32)

and the convention $d_0(\kappa) = 1$ is taken. Comparing the expressions (29) and (31), it becomes clear that in anQCD the basic quantities in perturbation expansion are the (generalized) logarithmic derivatives $\bar{\mathcal{A}}_\nu$, and not the (nonpower) analogs $\mathcal{A}_\nu$ of pQCD powers $a''$. These aspects have been presented and emphasized in more detail in Refs. [60].
3.2. Fractional Analytic Perturbation Theory (FAPT)

The APT procedure [16] is the elimination of the contributions of the Landau cut $0 < (-\sigma) \leq \Lambda_{\text{Landau}}$. This gives the APT analytic analog $A_1^{(\text{APT})}(Q^2; N_f)$ of $a(Q^2; N_f)$

$$A_1^{(\text{APT})}(Q^2; N_f) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\sigma \rho_1^{(\text{pt})}(\sigma; N_f)}{(\sigma + Q^2)}.$$  \hspace{1cm} (33)

This procedure can be extended to the construction of the APT-analogs $A_n^{(\text{APT})}(Q^2)$ of $n$-integer powers $a(Q^2)^n$ [17, 19] and their combinations (see also [61]). The APT analogs of general powers $a^\nu$ ($\nu$ a real exponent) are known as Fractional APT (FAPT) [20–23]; following the same procedure, they are

$$A_\nu^{(\text{FAPT})}(Q^2; N_f) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} \frac{d\sigma \rho_\nu^{(\text{pt})}(\sigma; N_f)}{(\sigma + Q^2)}, \hspace{1cm} (34)$$

where (that is general, for $\ell$-loop order)

$$\rho_\nu^{(\text{pt})}(\sigma; N_f) = \text{Im} \ a(Q^2 = -\sigma - i\epsilon; N_f)\nu. \hspace{1cm} (35)$$

It turns out that in FAPT, where the approach [(34) can be applied if it is equivalent with the approach of Eqs. (25) and (27)] or, equivalently, Eqs. (26) and (27) that can be applied in general anQCD models, if in the sums on the right-hand side of Eq. (27) we do not make truncations of the type of Eq. (30), but rather include as many terms as possible. We refer to Refs. [27, 28, 46] for more details on these points.

In the global version of FAPT, the coupling $A_\nu^{(\text{FAPT})\text{glob}}(Q^2)$ is obtained by applying the dispersion relation to the discontinuity function of the power $\nu$ of the global coupling [18], for $\sigma \geq 0$

$$\rho_\nu^{(\text{pt})\text{glob}}(\sigma) = \text{Im} \ a^{\text{glob}}(Q^2 = -\sigma - i\epsilon)\nu \equiv \rho_\nu^{(\text{pt})}(\sigma; \bar{\Lambda}_3)\Theta(|Q^2| \leq \mu^{(4)2}) + \rho_\nu^{(\text{pt})}(\sigma; \bar{\Lambda}_4)\Theta(|Q^2| \leq \mu^{(5)2}) + \rho_\nu^{(\text{pt})}(\sigma; \bar{\Lambda}_5)\Theta(|Q^2| \leq \mu^{(6)2}) + \rho_\nu^{(\text{pt})}(\sigma; \bar{\Lambda}_6)\Theta(|Q^2| \leq |Q^2|). \hspace{1cm} (36)$$

If the underlying pQCD running coupling $a(Q^2)$ runs according to the one-loop perturbative RGE, the corresponding explicit expressions for $A_\nu^{(\text{FAPT})}$ exist and were obtained and used in Ref. [20]

$$A_\nu(Q^2)^{(\text{FAPT,1-}\ell)} = \frac{1}{\beta_0} \left( \frac{1}{\ln^\nu(z)} - \frac{\text{Li}_{-\nu+1}(1/z)}{\Gamma(\nu)} \right). \hspace{1cm} (37)$$

\footnote{We note that in anQCD models other than FAPT as defined by Eq. (33), the approach of the type (34) to the calculation of $A_\nu$’s is not applicable. This is so because in such anQCD models $\rho_1(\sigma) \equiv \text{Im} A_1(-\sigma - i\epsilon) \neq \text{Im} a(-\sigma - i\epsilon)$ and, for $\nu \neq 1$ we have: $\rho_\nu(\sigma) \equiv \text{Im} A_\nu(-\sigma - i\epsilon)$. Therefore, $\rho_\nu(\sigma) \neq \text{Im} a(-\sigma - i\epsilon)\nu$ and $\rho_\nu(\sigma) \neq \text{Im} A_\nu(-\sigma - i\epsilon)\nu$. The former inequality holds because the model is not FAPT; the latter inequality holds because $A_\nu \neq A_\nu(\nu \neq 1)$ in general anQCD models which are simultaneously not pQCD. For models which are anQCD and simultaneously pQCD (i.e., anpQCD), we refer to Refs. [62].}
Here, \( z \equiv Q^2/\Lambda^2 \) and \( \text{Li}_{-\nu+1}(x) \) is the polylogarithm function of order \(-\nu+1\). Explicit extensions to approximate higher loops were performed by expanding the one-loop result in a series of derivatives with respect to the index \( \nu \).\[^{20}\,^{22}\,^{23}\] We refer for reviews of FAPT to Refs. \[43\,^{45}\].

When in FAPT the underlying pQCD coupling \( a(Q^2) \) is given by Eqs. (4) and (11), the resulting theory is called 1-loop and 2-loop FAPT, respectively. When \( a(Q^2) \) is given by Eq. (14) with \( c_2 = \bar{c}_2(N_f) \) in \( \overline{\text{MS}} \) scheme, the resulting theory is called, somewhat loosely, 3-loop FAPT. When \( a(Q^2) \) is given by the expansion (6), with \( c_2 = \bar{c}_2(N_f) \) and \( c_3 = \bar{c}_3(N_f) \) (\( c_j = 0 \) for \( j \geq 4 \); and the truncation index \( N = 8 \) is used), the resulting FAPT is called 4-loop.

Due to easiness of numerical implementation, in this model we incorporate the FAPT-analytization of logarithmic powers, too

\[
\mathcal{A}^{(\text{FAPT})}_{\nu,k}(Q^2) = \frac{1}{\pi} \int_{\sigma=0}^{\infty} d\sigma \text{Im} \frac{[a(-\sigma - i\epsilon)^\nu \ln^k a(-\sigma - i\epsilon)]}{(\sigma + Q^2)},
\]

where \( \nu \) is a general (noninteger) index and \( k = 0, 1, 2, \ldots \).

The couplings of FAPT\(_{N_f}\) and of global FAPT are calculated also in the Mathematica program of Ref. \[48\]. The values of couplings \( \mathcal{A}_\nu(Q^2) \) of FAPT\(_{N_f}\) models in our program, when \( \kappa = 2 \) is changed there to \( \kappa = 1 \), practically coincide with the corresponding values of \[48\]. In global FAPT\[^{10}\] there are small differences between our values and theirs, which tend to increase somewhat when \( \nu \) increases: for \( \nu < 1 \) the differences are 1% or less, for \( 1 < \nu < 2 \) are 1-2%, for \( 2 < \nu < 3 \) are 2-3%, for \( 3 < \nu < 4 \) are 4-8%. We note, however, that with increasing \( \nu \) the couplings in FAPT decrease very fast. We believe that one of the principal reasons for the small mentioned differences lies in the fact that in our program the quark thresholds (with \( \kappa = 1 \)) are implemented at the masses \( \bar{m}_q \) while in the program of Ref. \[48\] at the quark pole masses.

### 3.3. Two-delta analytic model (2\( \delta \)anQCD)

#### 3.3.1. 2\( \delta \)anQCD in low momentum regime \((N_f = 3)\)

In this anQCD model \[31\], the discontinuity function \( \rho_1(\sigma) \equiv \text{Im} \ A_1(Q^2 = -\sigma - i\epsilon) \) (for \( \sigma > 0 \)) agrees with the perturbative counterpart \( \rho_1^{(\text{pt})}(\sigma) \equiv \text{Im} \ a(Q^2 = -\sigma - i\epsilon) \) at sufficiently high scales \( \sigma \geq M_0^2 \) (\( M_0^2 \sim 1 \text{ GeV}^2 \)); while in the low-scale regime \( 0 < \sigma < M_0^2 \) its otherwise unknown behavior is parametrized as a linear combination of (two) delta functions (a parametrization motivated by the Padé approximation approach for

---

\[^{9}\] For practical purposes, we use in the integral \( \int \) the \( N \)-loop level \( \rho_\nu^{(\text{pt})}(\sigma) \) (where: \( N \leq 4 \)).

\[^{10}\] We note that our \( \mathcal{A}_\nu \) corresponds to their \( \mathcal{A}_\nu/\pi^\nu \); and what we call (approximate) 3-loop \( (3l) \) they call more rigorously 3-loop-Padé \( (3P) \).
the running coupling \[33\]

\[
\rho_1^{(2\delta)}(\sigma; c_2) = \pi \sum_{j=1}^{2} f_j^2 \Lambda^2 \delta(\sigma - M_j^2) + \Theta(\sigma - M_0^2) \times \rho_1^{(pt)}(\sigma; c_2)
\]

\[
= \pi \sum_{j=1}^{2} f_j^2 \delta(s - s_j) + \Theta(s - s_0) \times r_1^{(pt)}(s),
\]

where we define the dimensionless quantities: \(s = \sigma/\Lambda^2\), \(s_j = M_j^2/\Lambda^2\) (\(j = 0, 1, 2\)), and \(r_1^{(pt)}(s; c_2) = \rho_1^{(pt)}(\sigma; c_2) = \text{Im} a(Q^2 = -\sigma - ie; c_2)\). Here, \(\Lambda^2 (\lesssim 10^{-1} \text{ GeV}^2)\) is the Lambert scale appearing in the expression \[14\] for \(a\) [cf. also Eq. \[12\]]. The underlying pQCD coupling is taken in the form \[14\] where the scheme parameter \(c_2 (\equiv \beta_2/\beta_0)\) is nonzero in general [cf. Eqs. \[12\]-\[13a\]].

The aforementioned branching point of nonanalyticity \(z(s_L) = -1/e\) corresponds, according to Eq. \[12\], to the scale \(Q^2_{s_L} = \Lambda^2 s_L\) with \(s_L = c_1^{-\beta_1/\beta_0}\) \((s_L = 0.6347\) when \(N_f = 3\)). The interval \(Q^2 \in (0, \Lambda^2 s_L)\) is the interval of the unphysical (Landau) singularities of \(a(Q^2)\) of Eq. \[14\]. If \(c_2\) is chosen to be negative (this will be our case), then there is an additional pole-type Landau singularity, at a somewhat higher scale \(Q^2_{u_L} = \Lambda^2 u_L\) \([\Leftrightarrow z = z(u_L) = -u_L^{-\beta_0/c_1} / (c_1 e)]\) -- there the denominator in Eq. \[14\] becomes zero, cf. Fig. \[1b\]. When \(N_f = 3\) and \(c_2 = -4.9\) (this will be our preferred choice of the scheme later), we get \(u_L = 1.0311\) \(> s_L\). For this ("canonical") case, the underlying pQCD discontinuity function \(\rho_1^{(pt)}(\sigma)\) is presented in Fig. 3(a) as a function of \(\sigma\), and the corresponding 2\(\delta\)anQCD discontinuity function \(\rho_1^{(2\delta)}(\sigma)\) is in Fig. 3(b); the Lambert \(\Lambda\) scale value of \(\Lambda = 0.255\) GeV was used because this then corresponds to the world average value \(a(M_Z^2; \overline{\text{MS}}) = 0.1184/\pi\) (see later).

![Figure 3](image-url)

Figure 3: (a) The discontinuity function \(\rho_1^{(pt)}(\sigma) = \text{Im} a(-\sigma - ie)\) of the perturbative coupling \(a\) of Eq. \[14\], for \(c_2 = -4.9\) and \(N_f = 3\); (b) the corresponding 2\(\delta\)anQCD discontinuity function \(\rho_1^{(2\delta)}(\sigma)\), Eq. \[39a\]. The MPT discontinuity function is \(\rho_1^{(MPT)}(\sigma) = \rho_1^{(pt)}(\sigma - m_0^2)\), cf. Eq. \[47\]: when \(m_0^2 = 0.7\) GeV\(^2\), this is just the curve of Fig. (a) shifted by 0.7 GeV\(^2\) toward the right.

In Fig. 3(a) we see that \(a(Q^2; N_f)\), for \(c_2 = -4.9\), has a Landau pole at \(\sigma(\equiv -Q^2) = -u_L \Lambda^2\) \((\approx -0.067\) GeV\(^2\)) and the Landau branching point at \(\sigma(\equiv -Q^2) = -s_L \Lambda^2\) \((\approx -0.066\) GeV\(^2\)).

In Fig. 3(b) we see that \(\rho_1^{(pt)}(s)\), for \(c_2 = -4.9\), has a Landau pole at \(s(\equiv -Q^2) = -u_L \Lambda^2\) \((\approx -1.031\) GeV\(^2\)) and the Landau branching point at \(s(\equiv -Q^2) = -s_L \Lambda^2\) \((\approx -0.634\) GeV\(^2\)).
−0.041 GeV^2). Therefore, the dispersive relation (21) for the underlying perturbative coupling \( a(Q^2; N_f = 3) \) obtains a slightly generalized form [in comparison with Eq. (21)]

\[
a(Q^2) = \frac{1}{\pi} \int_{s = -s_L - \eta}^{\infty} ds \frac{r_1^{(pt)}(s; c_2)}{(s + Q^2/\Lambda^2)} + \text{Res}_{(s = u_L)}(a(z\Lambda^2; c_2)) \left( -u_L + Q^2/\Lambda^2 \right),
\]

which is obtained by application of the Cauchy theorem to the function \( a(Q^2)/\Lambda^2 \) along the contour depicted in Fig. 4 [in contrast to the simple contour Fig. 2(a) leading to Eq. (21)].

Figure 4: The integration contour for the integrand \( a(Q^2)/(Q^2 - Q^2) \) leading to the dispersion relation (40) for \( a(Q^2) \) of Eq. (14) with \( c_2 < 0 \). The radius of the large circular section tends to infinity.

The perturbative discontinuity function \( r_1^{(pt)}(s; c_2) = \text{Im} \ a(Q^2 = -s - i\epsilon; c_2) \), which is nonzero for \( -s_L < s < +\infty \) and at \( s = -u_L \), has the specific form

\[
\begin{align}
r_1^{(pt)}(s; c_2) &= \text{Im} \left[ \frac{(-1)}{c_1} \left[ 1 - (c_2/c_1^2) + W_{+1} \left( \frac{-1}{c_1^2} |s|^{-\beta_0/c_1 - i\epsilon} \right) \right] \right] \quad (s < 0), \quad (41a) \\
&= \text{Im} \left[ \frac{(-1)}{c_1} \left[ 1 - (c_2/c_1^2) + W_{+1} \left( \frac{-1}{c_1^2} |s|^{-\beta_0/c_1} \exp(i\beta_0\pi/c_1) \right) \right] \right] \quad (s > 0). \quad (41b)
\end{align}
\]

The analytic (spacelike) coupling \( A_1^{(2\delta)}(Q^2; c_2) \) of the two-delta anQCD model is constructed on the basis of the discontinuity function (39) [cf. Eq. (41b)] using the dispersion relation. This gives

\[
A_1^{(2\delta)}(Q^2; c_2) = \frac{1}{\pi} \int_{s_0}^{\infty} ds \frac{r_1^{(pt)}(s; c_2)}{(s + u)},
\]

where \( u = Q^2/\Lambda^2 \).
In the Two-delta $N_f = 3$ anQCD model with a chosen value of $c_2$ [2δanQCD$_{N_f=3}(c_2)$], and with $c_1 = c_1(N_f = 3) = (β_1/β_0)_{N_f=3}$, the first three quark flavors are approximated as massless. Most importantly, the model is constructed so that at high $|Q^2|$ it basically coincides with the underlying pQCD$_{N_f=3}(c_2)$, and that it simultaneously reproduces the experimental value of the (canonical) decay ratio $r_τ$ of the strangeless and massless $(V + A)$-channel semihadronic decays of the $τ$ lepton: $r_τ = 0.203$. This is achieved in three steps.

1. The first step is to obtain the value of the Lambert scale $Λ$ appearing in the underlying pQCD$_{N_f=3}(c_2)$ coupling $a(Q^2)$ of Eqs. (14) and (12). This is done in the following way: the world average value $π(M_Z^2) = 0.1184/π$ is evolved by 4-loop $\overline{MS}$ RGE from $Q^2 = M_Z^2$ down to $Q^2 = (2m_c)^2$, obtaining $\bar{a}_\text{in} ≡ \bar{a}((2m_c)^2; N_f = 3) = 0.26535/π$. 3-loop threshold matching (16) is used, at $Q^2 = (2m_b)^2$ and $(2m_c)^2$ ($m_b = 4.2$ GeV and $m_c = 1.27$ GeV). From this value $\bar{a}_\text{in}$, in $\overline{MS}$ scheme, the corresponding value $a_\text{in} ≡ a((2m_c)^2;c_2,c_2^2/c_1,…;N_f = 3)$ in the renormalization scheme of the 2δanQCD$_{N_f=3}(c_2)$ model is obtained, i.e., in the scheme determined by the beta function $β(a)$ of Eq. (13). This is performed using the integrated form of RGE (i.e., implicit solution) in its subtracted form, cf. Appendix A of Ref. [31] (cf. also Appendix A of Ref. [64]).

$$\frac{1}{a_\text{in}} + c_1 \ln \left( \frac{c_1 a_\text{in}}{1 + c_1 a_\text{in}} \right) + \int_0^{a_\text{in}} dx \left[ \frac{β(x) + β_0 x^2(1 + c_1 x)}{x^2(1 + c_1 x)β(x)} \right] =$$

$$\frac{1}{a_\text{in}} + c_1 \ln \left( \frac{c_1 a_\text{in}}{1 + c_1 a_\text{in}} \right) + \int_0^{a_\text{in}} dx \left[ \frac{β(x) + β_0 x^2(1 + c_1 x)}{x^2(1 + c_1 x)β(x)} \right]. \quad (43)$$

For $c_2 = -4.9$ this gives $a_\text{in} = 0.24860/π$. Equating this value with the expression (14) (with $c_2 = -4.9$ and $N_f = 3$) gives the Lambert scale $Λ ≡ Λ_3$ of the model: $Λ = 0.2553$ GeV. For other values of $c_2$, other values of $Λ$ are obtained.

2. The second step is to make the model 2δanQCD$_{N_f=3}(c_2)$ practically coincide with the underlying pQCD$_{N_f=3}(c_2)$ at high $|Q^2| > Λ^2$. In general, $A_1(Q^2;c_2)$ differs from $a(Q^2;c_2)$ at $Q^2 > Λ^2$ by $≈ (Λ^2/Q^2)^0$, as is the case, e.g., with FAPT and MPT. In 2δanQCD we impose the condition

$$A_1(Q^2;c_2) - a(Q^2;c_2) \sim (Λ^2/Q^2)^{n_\text{max}} \quad \text{with } n_\text{max} = 5. \quad (44)$$

The condition (44) represents in practice four conditions, which fix four dimensionless parameters $s_j, f_j^2$ ($j = 1, 2$) in terms of the fifth dimensionless parameter $s_0$.

3. The third step is to ensure that the model 2δanQCD$_{N_f=3}(c_2)$ reproduces the correct central value of the $(V + A)$-channel semihadronic $τ$ decay ratio $r_τ(ΔS = 0, m_q = 0)_{\text{exp}} = 0.203 \pm 0.004$.

---

[11] This quantity is normalized canonically, i.e., its perturbation expansion is $(r_τ)_{\text{pt}} = a + O(a^2)$. For details on $r_τ$ and its evaluation in analytic QCD approaches, we refer to Ref. [31] and Appendices B-E of Ref. [62].
Table 2: Values of the parameters of the considered 2\delta anQCD model, for \( N_f = 3 \) and \(-5.6 \leq c_2 \leq -2.0\). We consider \( c_2 = -4.9 \) (\( M_0 \approx 1.23 \) \( \text{GeV} \)) as the preferred representative case. The value \( \pi \times a_{\text{in}} = \alpha_s(2m_c^2;c_2,\ldots;N_f = 3) \) and the Lambert scale value \( \Lambda \) in the corresponding cases are for the QCD coupling parameter value \( \alpha_s^{\overline{\text{MS}}}(M_Z^2) = 0.1184 \).

| \( c_2 \) | \( \pi \times a_{\text{in}} \) | \( \Lambda \) [GeV] | \( s_0 \) | \( s_1 \) | \( f_j^2 \) | \( s_2 \) | \( f_j^2 \) | \( M_0 \) | \( \mathcal{A}_1(0) \) |
|---|---|---|---|---|---|---|---|---|---|
| -5.60 | 0.2477 | 0.2339 | 24.416 | 17.787 | 0.3013 | 0.6906 | 0.6150 | 1.156 | 0.9999 |
| -5.40 | 0.2480 | 0.2398 | 24.054 | 17.533 | 0.2936 | 0.7179 | 0.5960 | 1.176 | 0.9389 |
| -4.90 | 0.2486 | 0.2552 | 23.076 | 16.839 | 0.2746 | 0.7688 | 0.5505 | 1.226 | 0.8231 |
| -4.00 | 0.2498 | 0.2857 | 21.142 | 15.454 | 0.2416 | 0.8094 | 0.4753 | 1.314 | 0.6916 |
| -3.00 | 0.2512 | 0.3237 | 18.903 | 13.836 | 0.2078 | 0.8003 | 0.4020 | 1.407 | 0.6042 |
| -2.00 | 0.2526 | 0.3668 | 16.708 | 12.241 | 0.1775 | 0.7557 | 0.3388 | 1.499 | 0.5481 |

The scheme parameter \( c_2 \) (\( \equiv \beta_2/\beta_0 \)) can still be varied. Physical considerations guide us to restrict the preferred values of the pQCD-onset scale \( M_0 \) and of the coupling \( \mathcal{A}_1(Q^2) \) at \( Q^2 = 0 \), \( M_0 \leq 1.5 \) \( \text{GeV} \) and \( \mathcal{A}_1(0) < 1 \). This gives us the variation of \( c_2 \) in the interval \(-5.6 < c_2 < -2.0\). In Table 2 we present the results for the parameters of the model for various values of \( c_2 \) in this interval.\(^{12}\) Our preferred choice is \( c_2 = -4.9 \) where \( M_0 \approx 1.23 \) \( \text{GeV} \) and \( \mathcal{A}_1(0) \approx 0.82 \).

The (generalized) logarithmic derivatives \( \tilde{\mathcal{A}}_\nu \) are then constructed by the procedure \(^{25}\), and the power analogs \( \mathcal{A}_\nu \) by the linear combinations \(^{30}\) (where \( \nu_0 = \nu \)) with the truncation (“loop”) index there being \( N = 1, 2, 3, 4, 5 \).

3.3.2. 2\δanQCD for \( N_f \geq 4 \)

The 2\δanQCD model can be constructed also for \( N_f = 4, 5, 6 \). In such cases, for a chosen value of \( c_2 = c_2(N_f) \), the value of \( \Lambda_{N_f} \) is determined by pQCD, as in \( N_f = 3 \) case. Further, the condition \(^{44}\) again gives us the values of the four parameters \( s_j \) and \( f_j^2 \) (\( j = 1, 2 \)) in terms of \( s_0 \). However, since in the case of \( N_f \geq 4 \) the couplings \( \mathcal{A}_\nu(Q^2) \) should be applied only for \( |Q^2| > (2m_{N_f})^2 \) (where: \( m_4 = \overline{m}_c, m_5 = \overline{m}_b, m_6 = \overline{m}_t \)), the low-momentum quantity \( r_\tau \) cannot and should not be evaluated in such framework.

Therefore, for \( N_f \geq 4 \) the value of the \( s_0 \) parameter is free. In our program, we kept the value of \( s_0(N_f) \) equal to the corresponding value of \( s_0(N_f = 3) \). In such cases, the \( N_f = 4 \) 2\δanQCD model still remains formally analytic, while for \( N_f = 5, 6 \) it is formally nonanalytic (since \( s_2 < 0 \) is such a case). Nevertheless, we prefer to keep such, relatively low, values of \( s_0 \) for \( N_f \geq 4 \), because then the coefficient on the right-hand side of Eq. \(^{44}\) in front of \( (\Lambda^2/Q^2)^5 \) is not very large; therefore, the model for \( N_f \geq 4 \) practically agrees with the underlying pQCD. The relative difference between 2\δanQCD values \( \mathcal{A}_1(Q^2; N_f) \)

\(^{12}\) In Ref. \([31]\), the obtained parameters of the model were slightly different. The principal reason for that was that the 3-loop quark threshold conditions in the \( \overline{\text{MS}} \) RGE-running downwards in Ref. \([31]\) were implemented by a version of \([16]\) expressing \( a \) as a truncated power series of \( a' \). However, the numerical results for the coupling, at a given \( c_2 \), are almost indistinguishable from those of Ref. \([31]\).
and the corresponding pQCD values \( a(Q^2; N_f) \), \( \text{rd}(Q^2) \equiv |A_1(Q^2; N_f)/a(Q^2; N_f) - 1| \), as a function of positive \( Q^2 \) and for various \( N_f \), is given in Fig. 5. These differences are extremely small, with the exception of low \( Q^2 \): \( 0 < Q^2 < 1 \text{ GeV}^2 \). When \( N_f = 4 \), the difference \( A_1(Q^2; N_f)/a(Q^2; N_f) - 1 \) changes sign from negative to positive at increasing \( Q^2 \) around \( Q^2 \approx 17 \text{ GeV}^2 \); in the case of \( N_f = 5 \) this occurs around \( Q^2 \approx 6 \text{ GeV}^2 \). In the case of \( N_f = 3 \) we have \( A_1(Q^2; 3)/a(Q^2; 3) - 1 < 0 \) for all positive \( Q^2 \). These differences

\[ \text{rd}(Q^2) \] get smaller when \( N_f \) increases. Therefore, the model 2δanQCD for \( N_f \geq 4 \) can be used in practical calculations of the underlying couplings \( a(Q^2) \) [≈ \( A_1(Q^2) \)] and \( \tilde{a}_\nu(Q^2) \) [≈ \( \tilde{A}_\nu(Q^2) \)]. We note that for any real \( \nu \geq 0 \) we have

\[
\tilde{A}_\nu(Q^2; N_f) - \tilde{a}_\nu(Q^2; N_f) \sim \left( \frac{\Lambda_{N_f}^2}{Q^2} \right)^5,
\]

which is a consequence of Eq. (44). Namely, for integer \( \nu = 2, 3, \ldots \) this can be obtained by applying \( K_\nu(Q^2 d/dQ^2)^{\nu-1} \) to both sides of Eq. (44), where \( K_\nu = (-1)^{\nu-1}/[\beta_0^{\nu-1}(\nu - 1)!] \), cf. Eq. (23). And for \( \nu \) noninteger Eq. (45) follows by analytic continuation of the integer case to \( \nu \). We stress that the exact calculation of the pQCD quantities \( \tilde{a}_\nu(Q^2; N_f) \) for

\[ \text{rd}(Q^2) \]

Figure 5: The relative difference between 2δanQCD coupling and the underlying pQCD coupling, \( \text{rd}(Q^2) \equiv |A_1(Q^2; N_f)/a(Q^2; N_f) - 1| \), as a function of positive \( Q^2 \), for \( N_f = 3, 4, 5 \). The parameter \( c_2 \) of the model is set equal to \( c_2(N_f) = -4.9 \).

\[ \text{rd}(Q^2) \]

We note that in such a case the derivative \( (Q^2 d/dQ^2)^{\nu-1} \) applied to \( (\Lambda^2/Q^2)^5 \) gives \((-5)^{\nu-1}(\Lambda^2/Q^2)^5 \).
noninteger $\nu$ is quite complicated, due to the Landau singularities of the original pQCD coupling $a(Q^2; N_f)$. Therefore, in the evaluations of the series of the type

$$ D(Q^2) = a(Q^2)^{\nu_0} + \sum_{m=1}^{\infty} d_m a(Q^2)^{\nu_0+m} $$

(46a)

$$ = \tilde{a}_{\nu_0}(Q^2) + \sum_{m=1}^{\infty} \tilde{d}_m \tilde{a}_{\nu_0+m}(Q^2) $$

(46b)

with $\nu$ noninteger, the (truncated) expansion in the generalized logarithmic derivatives (46b) can be evaluated in practice by applying the model $2\delta a_{\text{anQCD}}$ (at a given $N_f$), as explained in Eqs. (28)–(31). The (truncated) series in powers (46a) is, certainly, much easier to evaluate technically than the (truncated) series (46b); nonetheless, the latter series may behave in some cases better than the former, and then $2\delta a_{\text{anQCD}}$ can be called upon, with the replacements: $\tilde{a}_{\nu_0+m}(Q^2; N_f) \mapsto \tilde{A}_{\nu_0+m}^{(28)}(Q^2; N_f)$ and $a(Q^2; N_f)^{\nu_0+m} \mapsto \tilde{A}_{\nu_0+m}^{(28)}(Q^2; N_f)$. If the quantity $D(Q^2)$ has low $Q^2$ corresponding to $N_f = 3$, the evaluation of the (truncated) series (46b) with the model $2\delta a_{\text{anQCD}}$ [$\tilde{a}_{\nu_0+m}(Q^2; 3) \mapsto \tilde{A}_{\nu_0+m}^{(28)}(Q^2; 3)$] is then the natural and the preferred way of evaluation, because the (truncated) series (46b) in pQCD are usually numerically badly affected by the vicinity of Landau singularities at such low $|Q^2| < (2m_\text{c})^2$.

3.4. Massive Perturbation Theory (MPT)

In order to obtain a holomorphically coupling finite in the infrared regime, the author of Ref. [32] proposed a simple change in the momentum

$$ A_{i}^{(\text{MPT})}(Q^2; N_f) = a(Q^2 + m_{\text{gl}}^2; N_f) $$

(47)

The mass scale $m_{\text{gl}} \approx 0.5 - 1$ GeV is in this ansatz a constant and is associated with an effective (dynamical) gluon mass which reflects the infrared dynamics of QCD. The same kind of replacement had been suggested, at one- and two-loop level, in Refs. [10]–[11] as a result of the use of nonperturbative QCD background. It was used in Refs. [12]–[13] in analyses of structure functions (with $m_{\text{gl}} \approx 0.8$ GeV). The relation (47), i.e., the replacement $Q^2 \mapsto Q^2 + m_{\text{gl}}^2$, can be kept even at higher-loop levels, as suggested by the multiplicative renormalizability [65] (and $m_{\text{gl}}^2$ can be expected in general to run with $Q^2$). Such behavior is suggested also by Gribov-Zwanziger approach [3], by analyses of Dyson-Schwinger equations in QCD [4]–[5] and by other functional methods [6]–[7].

The coupling (47) is analytic, because $m_{\text{gl}}^2 > \Lambda_{\text{Lan.}}^2$, where $(-q^2 \equiv) Q^2 = -\Lambda_{\text{Lan.}}^2$ is the branching point of the Landau singularity cut of the corresponding pQCD coupling $a(Q^2)$.

\footnote{The coupling $\tilde{a}_{\nu+1}(Q^2)$ for integer $\nu = n$ is a simple $n$’th logarithmic derivative of $a(Q^2)$, $\tilde{a}_{\nu+1}(Q^2) \equiv (-1)^n/((\beta_0^n n!)) (\partial/\partial \ln Q^2)^n a(Q^2)$ [cf. Eq. (23)]. For noninteger $\nu$, $\tilde{a}_{\nu+1}(Q^2)$ could be obtained by a dispersion integral similar to Eq. (25), by including integration over the Landau cuts and poles ($\sigma < 0$). This integration may be complicated, especially if an additional isolated Landau pole is involved as is the case of the coupling [14] with $c_2 < 0$ used here.}
Therefore, $A_i^{(\text{MPT})}(Q^2)$ can be written in the form of a dispersion integral, typical in any anQCD. At large $|Q^2|$ the coupling $A_i^{(\text{MPT})}(Q^2)$ tends to the pQCD coupling $a(Q^2)$, the difference being

$$A_i^{(\text{MPT})}(Q^2; N_f) - a(Q^2; N_f) \sim \frac{m_\text{gl}^2}{Q^2 \ln^2(Q^2/\Lambda^2)}. \tag{48}$$

It is important to stress that, as $A_i^{(\text{MPT})}(Q^2; N_f)$ is a nonperturbative holomorphic coupling, the evaluation of the (truncated) perturbation power series $D^{[N]}(Q^2)$ of the spacelike scale- and scheme-invariant physical quantities, Eq. [28], should not be performed by replacing $a(\mu^2) \nu \rightarrow A_i^{(\text{MPT})}(\mu^2) \nu$, but by the replacement which is obligatory in any anQCD

$$a(\mu^2) \nu \rightarrow A_\nu(\mu^2), \tag{49}$$

cf. Eq. [29], where the nonpower quantities $A_\nu(\mu^2) = A_\nu^{(\text{MPT})}(\mu^2)$ are constructed via Eqs. [27] and [26], and in the integrands of Eqs. [26] we use for $A_i$ the expression [47]. This use of nonpower expressions, based on the (generalized) logarithmic derivatives $\tilde{A}_\nu(\mu^2)$ presented by Eq. [25] or Eq. [26], has been emphasized in Refs. [27, 28, 32] for the case of integer $\nu$, extended to the case of general (noninteger) $\nu$'s in Refs. [46], and applied in various contexts in Refs. [60].

Since for each given $N_f$ we have a specific underlying pQCD running coupling $a(Q^2; N_f)$ in Eq. [47], we have then the corresponding MPT$_{N_f}$ model. In general, $m_\text{gl}$ may depend on $N_f$, as does the scale $\Lambda_{N_f}$.

The generalized logarithmic derivatives $\tilde{A}_\nu$ are evaluated by Eq. [26] for $0 \leq \nu < 5$, i.e., with $\nu = n + 1 + \delta$ where $n + 1 = 0, 1, 2, 3, 4$ and $0 \geq \delta < 1$. We have $N$-loop MPT$_{N_f}$ ($N = 1, 2, 3, 4$). We call the model 1-loop MPT$_{N_f}$ when $a(Q^2; N_f)$ is 1-loop Eq. [4] and in the construction of $A_i$ in Eq. [30] the right-hand side has only one term: $\tilde{A}_\nu = \tilde{A}_\nu$. We call the model 2-loop MPT$_{N_f}$ when $a(Q^2; N_f)$ is 2-loop Eq. [11] and in the construction of $A_i$ in Eq. [30] the right-hand side has two terms: $A_i = A_i + \tilde{k}_1(\nu)\tilde{A}_{\nu+1}$ (except when $4 \leq \nu < 5$, in which case we take $A_i = \tilde{A}_\nu$). The model is called 3-loop MPT$_{N_f}$ when $a(Q^2; N_f)$ is given by Eq. [14] with $c_2 = \bar{c}_2(N_f)$ $\overline{\text{MS}}$ value and in Eq. [30] the right-hand side has in general four terms: $\tilde{A}_\nu = \tilde{A}_\nu + \sum_{m=1}^{3} \tilde{k}_m(\nu)\tilde{A}_{\nu+m}$ (only three terms when $2 \leq \nu < 3$; etc.).

If we take specific (input) values of the dynamical masses $m_\text{gl}(N_f)$ (for $N_f = 3, 4, 5, 6$), and a specific value of $\Lambda_3$, the values of other scales $\Lambda_{N_f}$ (for $N_f = 4, 5, 6$) can be obtained by applying the quark threshold relations [16] written within MPT model

$$A_i = A_1 - A_2 \frac{\ell_h}{6} + A_3 \left( \frac{\ell_h^2}{36} - \frac{19}{24} \ell_h + \bar{c}_2 \right) + A_4 \left[ -\frac{\ell_h^3}{216} \right], \tag{50}$$

$$- \frac{131}{576} \ell_h^2 + \frac{\ell_h}{1728} \left( -6793 + 281(N_f - 1) + \bar{c}_3 \right).$$
where \( \mathcal{A}_1' \equiv A_1^{(\text{MPT})}(\mu_{N_f}^2; N_f - 1) \) and \( \mathcal{A}_n \equiv A_n^{(\text{MPT})}(\mu_{N_f}^2; N_f) \).

3.5. Examples of various couplings as a function of positive \( Q^2 \)

In Figs. 6 we show the running of \( A_1(Q^2) \) for \( Q^2 > 0 \) and \( N_f = 3 \) for three analytic models: FAPT, \( 2\delta \text{anQCD} \), and MPT (with the choice \( m_{gl}^2 = 0.7 \text{ GeV}^2 \)). For comparison, we show also the underlying pQCD coupling \( a(Q^2) \), i.e., \( a(Q^2) \) in the same renormalization scheme and with the same Lambert scale \( \Lambda \). At low \( Q^2 \), the divergent behavior of \( a(Q^2) \) is evident, due to the Landau singularities. We observe that at \( Q^2 \gtrsim 1 \text{ GeV}^2 \) \( 2\delta \text{anQCD} \) coupling is indistinguishable from the underlying pQCD coupling, cf. also Eq. (44). FAPT and MPT anQCD couplings (presented here in 4-loop MS scheme) are more suppressed in the infrared than \( 2\delta \text{anQCD} \). Figs. 7 represent the couplings at \( \nu = 0.3 \) (and \( N_f = 3 \)), i.e.,

![Figure 6](image1.png)  
**Figure 6**: The couplings \( A_1 \equiv A \) in three anQCD models with \( \nu = 1 \) and \( N_f = 3 \) as a function of \( Q^2 \) (for \( Q^2 > 0 \)), where we include the underlying pQCD coupling \( a \) for comparison: (a) \( 2\delta \text{anQCD} \) coupling and pQCD coupling, in the renormalization scheme with \( c_2 = -4.9 \) (and \( c_j = c_{j-1}^2/c_{j-2} \) for \( j \geq 3 \)); (b) FAPT and MPT in 4-loop MS scheme and with \( \Lambda^2 = 0.1 \text{ GeV}^2 \); MPT with \( m_{gl}^2 = 0.7 \text{ GeV}^2 \).

![Figure 7](image2.png)  
**Figure 7**: The same as in Figs. 6 but now with \( \nu = 0.3 \) (\( A_{\nu = 0.3} \)). The coupling \( A_{0.3} \) is calculated from the couplings \( \tilde{A}_{0.3+m} \) using the relation (30) (with \( \nu_0 = 0.3 \) and \( n = 0 \)) with the truncation index \( N = 5 \) for \( 2\delta \text{anQCD} \) and \( N = 4 \) for MPT; and for FAPT using Eq. (34).
$A_{\nu=0.3}(Q^2)$. We note the same behavior as in Figs. 3 but now MPT coupling increases more quickly when $Q^2$ decreases than in the $\nu = 1$ case.

4. Main procedures in analytic QCD models

We present here general rules on how to use the anQCD.m package. For more detailed description we refer to Appendix A. We present the main functions that we provide to the community:

- $\text{trNl}[N_f, \nu, k, \sigma, \bar{\Lambda}_{N_f}^2]$ returns $N$-loop perturbative spectral density $\rho^{(N)}_{\nu,k}(\sigma; N_f) = \text{Im} \left[ a^\nu \ln^k a \right]_{Q^2=-\sigma-i\epsilon} (N = 1, 2, 3, 4)$ of real power $\nu$ and logarithmic power $k$ at $\sigma$ and at fixed number of active quark flavors $N_f$:

$$\text{trNl}[N_f, \nu, k, \sigma, L2] = \rho^{(N)}_{\nu,k}[\sigma; N_f = N_f; \nu = L2 = \bar{\Lambda}_{N_f}^2] \quad (\nu \in \mathcal{R} ; \ k = 0, 1, \ldots ; \ N = 1, 2, 3, 4 ; \ N_f = 3, 4, 5, 6). \quad (51)$$

- $\text{trNl}^{(N)\text{glob}}[\nu, k, \sigma, \bar{\Lambda}_3^2]$ returns $N$-loop global perturbative spectral density $\rho^{(N)\text{glob}}_{\nu,k}(\sigma; N_f) (N = 1, 2, 3, 4)$ of real power $\nu$ and logarithmic power $k$ at $\sigma$, and with $\bar{\Lambda}_3$ being the QCD $N_f = 3$ scale:

$$\text{trNl}^{(N)\text{glob}}[\nu, k, \sigma, L23] = \rho^{(N)\text{glob}}_{\nu,k}[\sigma; L23 = \bar{\Lambda}_3^2], \quad (N = 1, 2, 3, 4). \quad (52)$$

- $\text{AFAPT}Nl[N_f, \nu, k, |Q^2|, \Lambda_3^2, \phi]$ returns $N$-loop $(N = 1, 2, 3, 4)$ analytic FAPT coupling of real power $\nu$ and logarithmic power $k$ at fixed number of active quark flavors $N_f$, $\mathcal{A}_{\nu,k}^{\text{FAPT},N}(Q^2, N_f) = (a^\nu(Q^2) \ln^k a(Q^2))_{\text{an.FAPT}}$ in the Euclidean domain $[Q^2 = |Q^2| \exp(i\phi) \in \mathcal{C}$ and $Q^2 \neq 0]\$

$$\text{AFAPT}Nl[N_f, \nu, k, Q2, L2, \phi] = \mathcal{A}_{\nu,k}^{\text{FAPT},N}[Q2 = |Q2|, \phi = \arg(Q2); N_f = N_f; \nu = L2 = \bar{\Lambda}_{N_f}^2] \quad (N = 1, 2, 3, 4 ; \ N_f = 3, 4, 5, 6). \quad (53)$$

- In the global FAPT case $\text{AFAPT}Nl^{\text{glob}}[\nu, k, |Q^2|, \bar{\Lambda}_3^2, \phi]$ returns $N$-loop analytic FAPT coupling of real power $\nu$ and logarithmic power $k$ $\mathcal{A}^{(\text{FAPT},N)\text{glob}}_{\nu,k}(Q^2)$ in the Euclidean domain,

$$\text{AFAPT}Nl^{\text{glob}}[\nu, k, Q2, L23, \phi] = \mathcal{A}^{(\text{FAPT},N)\text{glob}}_{\nu,k}[Q2 = |Q2|, \phi = \arg(Q2); L23 = \bar{\Lambda}_3^2] \quad (N = 1, 2, 3, 4). \quad (54)$$

- $\text{A2dNl}[N_f, M, \nu, |Q^2|, \phi]$ returns $N$-loop 2δanQCD coupling of fractional-power $\nu + M$ ($\nu > -1$ and real; $M = 0, 1, \ldots, N - 1$) at fixed number of active quark flavors $N_f$
\( A_{\nu+M}^{(2\delta)}(Q^2, N_f) \) in the Euclidean domain, used for the \( N^{N-1}\)LO truncation approach [cf. Eqs. (28)-(32), where in (30) \( \nu \mapsto \nu_0 \) and \( M \mapsto n \)]

\[
A_{2dN1}[N_f, M, \nu, Q^2, \phi] = A_{\nu+M}^{(2\delta)}[Q^2 = |Q^2|, \phi = \arg(Q^2); N_f = N_f], \\
(N = 1, 2, 3, 4, 5; \ N_f = 3, 4, 5, 6; \ M = 0, 1, \ldots, N - 1).
\] (55)

- AMPTN1\([N_f, \nu, Q^2, m^2_{gl}, \Lambda^2_{N_f}]\) returns \( N \)-loop \((N = 1, 2, 3, 4)\) analytic MPT coupling of fractional-power \( \nu \) \((0 < \nu < 5)\) and at fixed number of active quark flavors \( N_f \)

\[
AMPTN1[N_f, \nu, Q^2, M, L] = \\
= A_{\nu}^{(MPT,N)}[Q^2 = Q^2 \in \mathcal{C}; N_f = N_f; M = m^2_{gl}; L = \Lambda^2_{N_f}] \\
(N = 1, 2, 3, 4; \ N_f = 3, 4, 5, 6; \ 0 < \nu < 5).
\] (56)

With the main procedures and definitions given above, we will provide an example of the use of these quantities for Mathematica 9.0.1.

In[1]:= <<anQCD.m;
First of all, we use as an input the \( \Lambda^3_{3} \) QCD scale fixed by \( \Lambda^2_{3} = 0.1 \) GeV\(^2\). With this input scale \( \Lambda^2_{3} = 0.1 \) GeV\(^2\) (in FAPT\(_{N_f=3}\), global FAPT, and MPT\(_{N_f=3}\)), we illustrate now how to obtain the values of the analytic couplings at what we call the three-loop level \((N = 3)\), i.e., the underlying pQCD coupling is given by Eq. (14) with \( c_2 = c_2(N_f; \overline{\text{MS}}) \) in FAPT and MPT, and \( c_2 = -4.9 \) in \( 2\delta \)anQCD. Thus, we evaluate \( A_{\nu}^{(FAPT,N)}[Q^2, N_f; L2], \ A_{\nu,k}^{(FAPT,N),glob}[Q^2; L23], \ A_{\nu+M}^{(2\delta)}[Q^2, N_f], \) and \( A_{\nu}^{(MPT,N)}[Q^2, N_f; M, L] \), taking the common parameters: \( L2 = L23 = 0.1 \) GeV\(^2\) (in FAPT and MPT); \( M = 0 \) for \( 2\delta \)anQCD model; and \( M2 = 0.7 \) GeV\(^2\) in MPT; for the low-momentum scale \( Q^2 = 10^{-3} \) GeV\(^2\) \((Nf = 3)\) and for the high scale \( Q^2 = 10^2 \) GeV\(^2\) \((Nf = 5)\). This values will be given below (as the second entry), with the corresponding typical time that they need (as the first entry, varies with various computers):

In[2]:= AFAPT3l[3, 1, 0, 10^-3, 0.1, 0] // Timing
Out[2]= {0.366944, 0.28312}

In[3]:= AFAPT3lglob[1, 0, 10^-3, 0.1, 0] // Timing
Out[3]= {0.729890, 0.287775}

In[4]:= AMPT3l[3, 1, 10^-3, 0.7, 0.1] // Timing
Out[4]= {0.136979, 0.171356}

In[5]:= A2d3l[3, 0, 1, 10^-3, 0] // Timing
Out[5]= {0.798879, 0.809039}
In[6]:= AFAPT3l[5, 1, 0, 10^2, 0.1, 0] // Timing
Out[6]= {0.365945, 0.0624843}

In[7]:= AFAPT3lglob[1, 0, 10^-2, 0.1, 0] // Timing
Out[7]= {0.723890, 0.0559854}

In[8]:= AMPT3l[5, 1, 10^2, 0.7, 0.1] // Timing
Out[8]= {0.098985, 0.0627726}

In[9]:= A2d3l[5, 0, 1, 10^-2, 0] // Timing
Out[9]= {0.728889, 0.0559182}

In[10]:= AFAPT3l[3, 0.3, 0, 10^-3, 0.1, 0] // Timing
Out[10]= {0.446932, 0.990337}

In[11]:= AFAPT3lglob[0.3, 0, 10^-3, 0.1, 0] // Timing
Out[11]= {0.705892, 0.998707}

In[12]:= AMPT3l[3, 0.3, 10^-3, 0.7, 0.1] // Timing
Out[12]= {0.120982, 0.849992}

In[13]:= A2d3l[3, 0, 0.3, 10^-3, 0] // Re // Timing
Out[13]= {3.812421, 1.98302}

In order to make plots of the analytic running couplings as in Fig. 6 and 7, users could construct an interpolation in order to reduce the time of calculation.

Acknowledgments This work was supported by FONDECYT (Chile) Grant No. 1130599 and DGIP (UTFSM) internal project USM No. 11.13.12 (C.A and G.C).

Appendix A. Description of the main procedures

The main functions found in our package are presented and described in the following.

- **trNl[Nf,Nu,k,sig,L2]**:

  **general:** it computes the $N$-loop spectral density including possibly powers of the logarithmic coupling, $\rho^{(N)}_{\nu,k}(\sigma; N_f) = \text{Im}[a(Q^2)^{\nu}\ln^k(a(Q^2))]_{Q^2=-\sigma-i\epsilon}$;

  **input:** the squared momentum argument $\text{sig} = \sigma$, the squared MS Lambda QCD parameter $L_2 = \Lambda_{N_f}^2$, the number of active flavors $N_f$, the power index $Nu = \nu$, and the logarithmic power index $k = k$;
output: $\rho_{\nu,k}^{(N)}$.

example: In order to compute the value of the three-loop spectral density, at 
$\sigma = 1.5$ GeV$^2$ and $N_f = 3$, and with $\Lambda_{N_f}^2 = 0.1$ GeV$^2$, i.e., the quantity 
$\rho_{0.5,0}^{(3)}(1.5, 3) = 0.104393$, one has to use the command 
$\text{tr3l}[3, 0.5, 0, 1.5, 0.1]$.

• $\text{trN1glob}[\nu, k, \sigma, L2nf3]$:

general: it computes the $N$-loop global spectral density incorporating the powers of 
the logarithmic coupling 
$\rho_{\nu,k}^{(N)\text{glob}}(\sigma, N_f) = \text{Im}[a^{(\text{glob})}(Q^2)^\nu \ln^k(a^{(\text{glob})}(Q^2))]_{Q^2=-\sigma-i\epsilon}$; 

input: the squared momentum argument $\sigma = \sigma$, the squared MS Lambda QCD 
parameter at $N_f = 3$ (at the corresponding $N$-loop) $L2nf3 = \Lambda_N^2$, the power 
index $\nu = \nu$, and the logarithmic power index $k = k$;

output: $\rho_{\nu,k}^{(N)\text{glob}}$;

example: In order to compute the value of the three-loop global spectral density at 
$\sigma = 1.5$ GeV$^2$ and with $\Lambda_{N_f}^2 = 0.1$ GeV$^2$, i.e., the quantity 
$\rho_{0.5,0}^{(3)\text{glob}}(1.5, 3) = 0.104393$, one has to use the command 
$\text{tr3l}[0.5, 0, 1.5, 0.1]$.

• $\text{AFAPT}N1[Nf, Nu, k, Q2, L2, Fi]$:

general: it computes the $N$-loop coupling in FAPT$_{N_f}$ incorporating the analytization 
of powers of the logarithmic coupling 
$A^{(\text{FAPT}, N)}_{\nu,k}(Q^2, N_f) = (a^{(\nu)}(Q^2) \ln^k a(Q^2))_{\text{an}, \text{FAPT}}$ in the Euclidean domain;

input: the squared momentum argument $Q2 = |Q^2|$, the squared MS Lambda QCD 
parameter $L2 = \Lambda_{N_f}^2$, the number of active flavors $Nf = N_f$, the phase of the 
complex $Q^2 = |Q^2| e^{i\phi}$, i.e., $Fi = \phi$, the power index $Nu = \nu$, and the logarithmic 
power index $k = k$;

output: $A^{(\text{FAPT}, N)}_{\nu,k}$;

example: In order to compute the value of the three-loop FAPT coupling $A_{\nu}$ at 
$Q^2 = 1.5$ GeV$^2$, with $\nu = 0.5$, $N_f = 3$ and $\Lambda_N^2 = 0.1$ GeV$^2$, i.e., the quantity 
$A_{0.5,0}^{(\text{FAPT}, 3)}(1.5, 3) = 0.324597$, one has to use the command 
$\text{AFAPT3l}[3, 0.5, 0, 1.5, 0.1, 0]$.

• $\text{AFAPT}N1glob[Nu, k, Q2, L2nf3, Fi]$:

general: it computes the $N$-loop global FAPT coupling incorporating the analytization 
of powers of the logarithmic $A^{(\text{FAPT}, N)\text{glob}}_{\nu,k}(Q^2)$ in the Euclidean domain;

input: the squared momentum argument $Q2 = |Q^2|$, the squared MS Lambda QCD 
parameter at $N_f = 3$ $L2nf3 = \Lambda_N^2$, the phase of the complex $Q^2 = |Q^2| e^{i\phi}$, i.e., $Fi = \phi$, the power index $Nu = \nu$, and the logarithmic power index $k = k$;
output: $A_{\nu,k}^{(\text{FAPT},N),\text{glob}}$.

dexample: In order to compute the value of the three-loop FAPT coupling $A_{\nu}$ at $Q^2 = 1.5 \text{ GeV}^2$, with $\nu = 0.5$ and $\Lambda_3^2 = 0.1 \text{ GeV}^2$, i.e., the quantity $A_{0,5,0}^{(\text{FAPT},3),\text{glob}}(1.5) = 0.333458$, one has to use the command $A\text{FAPT3lglob}[0.5, 0, 1.5, 0.1, 0]$.

• $A2dNl[Nf,M,Nu,Q2,Fi]$:

dgeneral: it computes $N$-loop coupling $A_{\nu}^{(2\delta)}(Q^2,N_f)$ in $2\delta\text{anQCD}_N$ in the Euclidean domain;

dinput: the squared momentum argument $Q^2=Q^2=|Q^2|e^{i\phi}$, the number of active flavors $Nf=N_f$, the phase of the complex $Q^2 = |Q^2|e^{i\phi}$, i.e., $Fi=\phi$, the power index $\nu + M$ ($Nu=\nu$ and $M=M$) with $\nu > -1$ and real, and $M$ nonnegative integer.

doutput: $A_{\nu,M}^{(2\delta,N)}$.

dexample: In order to compute the value of the “three-loop” $2\text{danQCD}$ coupling $A_{\nu}$ at $Q^2 = 1.5 \text{ GeV}^2$, with $\nu = 0.5$, $M = 0$ and $N_f = 3$, i.e., the coupling $A_{0.5}^{(2\delta,3)}(1.5) = 0.332313$ one has to use the command $A\text{2d3l}[3, 0.5, 1.5, 0]$.

• $AMPTNl[Nf,Nu,Q2,M2,L2MPT]$:

dgeneral: it computes the $N$-loop coupling $A_{\nu}^{(\text{MPT},N)}(Q^2,m_{gl}^2,N_f)$ in $\text{MPT}_{N_f}$ in the Euclidean domain;

dinput: the squared momentum argument $Q^2=Q^2=|Q^2|e^{i\phi}$, the squared $\overline{\text{MS}}$ Lambda QCD parameter $L2\text{MPT} = \Lambda_{N_f}$, the number of active flavors $Nf=N_f$, the effective mass parameter $M2=m_{gl}^2$ and the power index $Nu=\nu$ ($0 < \nu < 5$);

doutput: $A_{\nu,N_f}^{(\text{MPT},N)}$;

dexample: In order to compute the value of the three-loop MPT coupling $A_{\nu}$ at $Q^2 = 1.5 \text{ GeV}^2$, with $\nu = 0.5$, $m_{gl}^2 = 0.7 \text{ GeV}^2$, $N_f = 3$ and $\Lambda_3^2 = 0.1 \text{ GeV}^2$, i.e., the quantity $A_{0.5}^{(\text{MPT},3)}(1.5,0.7,3) = 0.334118$ one has to use the command $A\text{MPT3l}[3, 0.5, 1.5, 0.7, 0.1]$. All $\Lambda_{N_f=3}$ are in GeV, all squared momenta $Q^2$ (Euclidean), and spectral-integration variables $\sigma$ are in GeV$^2$. The number of loops $N$ is everywhere specified in the names of the procedures.

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