Causal inference with observational studies trimmed by the estimated propensity scores

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Abstract

Causal inference with observational studies often relies on the assumptions of unconfoundedness and overlap of covariate distributions in different treatment groups. The overlap assumption is violated when some units have propensity scores close to zero or one, and therefore both theoretical and practical researchers suggest dropping units with extreme estimated propensity scores. We advance the literature in three directions. First, we clarify a conceptual issue of sample trimming by defining causal parameters based on a target population without extreme propensity score. Second, we propose a procedure of smooth weighting, which approximates the existing sample trimming but has better asymptotic properties. The new weighting estimator is asymptotically linear and the bootstrap can be used to construct confidence intervals. Third, we extend the theory to the average treatment effect on the treated, suggesting trimming samples with estimated propensity scores close to one.

Some key words: Bootstrap; Lack of overlap; Non-smoothness; Potential outcome; Unconfoundedness.

1 Introduction

Under the potential outcomes framework (Rubin, 1974), causal effects are comparisons of the potential outcomes corresponding to different treatments. There is an extensive literature on estimating average treatment effects based on the assumption of unconfoundedness and sufficient overlap in the covariate distributions (Rosenbaum and Rubin, 1983; Imbens and Rubin, 2015). Unfortunately, in many applications it is common to have limited overlap in covariates between the treatment and control groups, i.e., there are regions of the covariate space with low probability of receiving treatment or control. Lack of overlap affects the credibility of all methods attempting to estimate causal effects for the common population. A consequence in weighting (Rosenbaum and Rubin, 1983; Imbens and Rubin, 2015) is that extreme propensity scores induce substantively large weights, which can result in a large variance and poor finite sample properties (Kang and Schafer, 2007; Khan and Tamer, 2010). In this case, it is desirable to modify the estimand to averaging only over the part of the covariate space with all treatment probabilities away from zero. For example, Crump et al. (2009) suggested dropping subjects from the analysis with estimated propensity score close to zero and one, which generally alters the estimand by changing the reference population (Li et al., 2016). In the current practice, researchers often first trim the samples based on the estimated propensity scores, and then characterize the target population and estimand based on the sample estimates. This ad hoc definition of the treatment effect is problematic, because using different samples may change the target estimand.

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The objective of this article is to clarify a conceptual issue of sample trimming, which arises frequently in practice, leading to unambiguous definitions of causal parameters based on a well-defined target population. The non-smooth nature of trimming makes inference complicated. Therefore, instead of making binary decisions to include or exclude subjects from analysis, we propose to use a smooth weighting function so that all subjects are weighted continuously. This smooth weighting approximates the existing sample trimming, but allows us to derive the asymptotic properties of the corresponding causal estimators using conventional linearization methods for two-step statistics. We formally show that the new weighting estimator is asymptotically linear and the bootstrap can be used to construct confidence intervals. Moreover, by smoothing the indicator function, the resulting estimators gain precision, as demonstrated in the asymptotic analysis and simulation study. In addition to the average treatment effect, we extend Crump et al. (2009) to develop an optimal rule to select subpopulation for which the average treatment effect on the treated can be estimated most precisely, and establish asymptotic inference when support reduction is based estimated propensity scores.

2 Notation

For each subject \( i \), the treatment is \( A_i \in \{0, 1\} \), where 0 and 1 are labels for control and treatment. There are two potential outcomes, one for treatment and the other for control, denoted by \( Y_i(1) \) and \( Y_i(0) \), respectively. The observed outcome is \( Y_i = Y_i(A_i) \). Let \( X_i \) be the observed pre-treatment confounders. We assume that \( \{A_i, X_i, (Y_i(1), Y_i(0))\}_{i=1}^{N} \) are independent draws from the distribution of \( \{A, X, Y(1), Y(0)\} \). Given the observed confounders \( X \), the conditional average causal effect is \( \tau(X) = E\{Y(1) - Y(0) \mid X\} \). The average treatment effect is \( \tau = E\{\tau(X)\} \), where the expectation is taken with respect to the whole population. The common assumptions to identify \( \tau \) are as follows [Rosenbaum and Rubin 1983].

Assumption 1 (Unconfoundedness) \( Y(a) \perp A \mid X \) for \( a = 0, 1 \).

Assumption 2 (Sufficient overlap) There exist constants \( c_1 \) and \( c_2 \) such that with probability 1, \( 0 < c_1 \leq e(X) \leq c_2 < 1 \), where \( e(X) = \Pr(A = 1 \mid X) \) is the propensity score.

Assumption 3 \( E\{Y(a)^2\} < \infty \), for \( a = 0, 1 \).

In observational studies the propensity score is not known and therefore has to be estimated from data. Following Rosenbaum and Rubin (1983) and most of the empirical literature, we assume that the propensity score is correctly specified by a generalized linear model \( e(X) = e(X^\theta^*) \). Let \( \hat{\theta} \) be the maximum likelihood estimator of \( \theta^* \). Our method is also applicable to other asymptotically linear estimators of \( \theta \). Then, a simple weighting estimator of \( \tau \) is \( N^{-1} \sum_{i=1}^{N} \hat{\tau}(X_i) \), where

\[
\hat{\tau}(X_i) = \frac{A_iY_i}{e(X_i^\hat{\theta})} - \frac{(1 - A_i)Y_i}{1 - e(X_i^\hat{\theta})}.
\]

The augmented weighting estimator [Lunceford and Davidian 2004; Bang and Robins 2005] augments the simple weighting estimator by further estimating \( \mu(a, X) = E(Y \mid A = a, X) \) by \( \hat{\mu}(a, X) \), and using \( N^{-1} \sum_{i=1}^{N} \hat{\tau}^{\text{aug}}(X_i) \), where

\[
\hat{\tau}^{\text{aug}}(X_i) = \left[ \frac{A_iY_i}{e(X_i^\hat{\theta})} + \left\{ 1 - \frac{A_i}{e(X_i^\hat{\theta})} \right\} \hat{\mu}(1, X_i) \right] - \left[ \frac{(1 - A_i)Y_i}{1 - e(X_i^\hat{\theta})} + \left\{ 1 - \frac{1 - A_i}{1 - e(X_i^\hat{\theta})} \right\} \hat{\mu}(0, X_i) \right].
\]
The augmented weighting estimator features a double robustness property in the sense that under Assumptions 1–3, if either $e(X)$ or $\mu(a, X)$ is correctly specified, $\hat{\tau}^{\text{aug}}$ is consistent for $\tau$.

The weighting estimators suffer from large variability especially when Assumption 2 is violated or close to be violated. In the presence of lack of overlap, define the set with sufficient overlap to be $O = \{x \mid \alpha \leq e(x) \leq 1 - \alpha\}$, where $\alpha$ is a fixed cut-off value, e.g., a rule of thumb is $\alpha = 0.1$, as suggested by Crump et al. (2009). The target population is then represented by $\{x \mid x \in O\}$, and the estimand of interest becomes $\tau(O) = E(\tau(X) \mid X \in O)$. This estimand does not depend on the sample, which is more straightforward to interpret.

In existing sampling trimming, the inclusion weight is

$$\omega(x_i) = \mathbb{1}_{\{x_i \in O\}} = \mathbb{1}_{\{\alpha \leq e(x_i) \leq 1 - \alpha\}},$$

(3)

where $\mathbb{1}_{\{\cdot\}}$ being the indicator function, and $O = \{x \mid \alpha \leq e(x_i) \leq 1 - \alpha\}$ is the trimmed sample based on the estimated propensity scores. The weighting estimators of $\tau(O)$ become

$$\hat{\tau}(\hat{\theta}) = \left\{\sum_{i=1}^{N} \omega(x_i') \hat{\theta}\right\}^{-1} \sum_{i=1}^{N} \omega(x_i') \tau(x_i),$$

(4)

$$\hat{\tau}^{\text{aug}}(\hat{\theta}) = \left\{\sum_{i=1}^{N} \omega(x_i') \hat{\theta}\right\}^{-1} \sum_{i=1}^{N} \omega(x_i') \hat{\tau}^{\text{aug}}(x_i),$$

(5)

where $\tau(x_i)$ and $\hat{\tau}^{\text{aug}}(x_i)$ are defined in (1) and (2), respectively. We write $\hat{\tau} = \hat{\tau}(\hat{\theta})$ and $\hat{\tau}^{\text{aug}} = \hat{\tau}^{\text{aug}}(\hat{\theta})$ in shorthand.

The main question addressed in this article is how the estimated support affects the inference. To study the asymptotic behaviors of $\hat{\tau}$ and $\hat{\tau}^{\text{aug}}$, we need to take into account of first the sampling variability in $\hat{\theta}$, which induces variability of the estimated set $O$ and second the sampling variability in $\hat{\tau}$ and $\hat{\tau}^{\text{aug}}$. We can not directly apply conventional asymptotic linearization methods because the weight function $\omega(x_i')$ is non-smooth. To avoid this difficulty, we consider a smoothed version of the weight function

$$\omega_{\epsilon}(x_i') = \Phi_{\epsilon} \left\{e(x_i') - \alpha\right\} \Phi_{\epsilon} \left\{1 - \alpha - e(x_i')\right\},$$

(6)

where $\Phi_{\epsilon}(z)$ is a normal cumulative distribution with mean zero and variance $\epsilon$. The new weight function imposes a soft threshold, instead of a hard threshold which results in weights either zero or one, for the estimated propensity scores close to $\alpha$ and $1 - \alpha$. An important issue regarding this weight function is the choice of $\epsilon$. As $\epsilon \to 0$, the smooth weight function (6) coverages to the indicator weight function (3); see Figure S1 in the Supplementary Material for visualization of the weight functions. Therefore, for small $\epsilon$, the behaviors of the estimators (4) and (5) with the smooth weight function (6) are similar to those with the indicator weight function (3). We derive the asymptotic results for the smoothed estimators.

3 Main Results

Based on data $\{(A_i, X_i)\}_{i=1}^{N}$, let the score function and the Fisher information matrix of $\theta$ be

$$S(\theta) = \frac{1}{N} \sum_{i=1}^{N} X_i \left(\frac{A_i - e(X_i') \hat{\theta}}{e(X_i') \{1 - e(X_i')\}}\right) f(X_i', \theta), \quad I_{\theta} = E \left[\frac{f(X_i', \theta)^2}{e(X_i') \{1 - e(X_i')\}} X X'\right],$$

respectively, where $f(t) = de(t)/dt$. Because $\hat{\theta}$ is the solution to the score equation $S(\theta) = 0$, under certain regularity conditions, $\theta - \theta^* = I_{\theta}^{-1} S(\theta^*) + o_p(N^{-1/2})$. Let $\sigma^2(a, X) = \text{var}(Y \mid A = a, X)$, respectively.
for $a = 0, 1$. Denote $\hat{\tau}_e$ to be the weighting estimator \(4\) with the smooth weight function \(6\), and $\tau_e = E\{\omega_e(X')\tau(X)\}$.

**Theorem 1** Under Assumptions 1 and 3, $\hat{\tau}_e$ is asymptotically linear. Moreover,

$$N^{1/2}(\hat{\tau}_e - \tau_e) \to N\left(0, \sigma^2 + b_1^2 \tau_{aug}^{-1} b_1, c - b_2^2 \tau_{aug}^{-1} b_2 \right),$$

in distribution, as $N \to \infty$, where

$$b_1, c = E\left\{\frac{\partial}{\partial \theta} \left[ \frac{\omega_e(X')}{E\{\omega_e(X')\}} \right] \tau(X) \right\}, \quad (7)$$

$$b_2 = \frac{1}{E\{\omega_e(X')\}^2} E\left\{ \omega_e(X') \left( \frac{X\mu(1, X) e(X) + E\{X\mu(0, X) | e(X)\}}{1 - e(X)} \right)^{1/2} \mu(1, X) + \left( \frac{e(X)}{1 - e(X)} \right)^{1/2} \mu(0, X) \right\}^2$$

$$\sigma^2 = \frac{1}{E\{\omega_e(X')\}^2} \text{var} \left\{ \omega_e(X') \tau(X) \right\}$$

$$+ \frac{1}{E\{\omega_e(X')\}^2} E\left\{ \omega_e(X') \left( \frac{1 - e(X)}{e(X)} \right)^{1/2} \mu(1, X) + \left( \frac{e(X)}{1 - e(X)} \right)^{1/2} \mu(0, X) \right\}^2$$

$$+ \frac{1}{E\{\omega_e(X')\}^2} E\left\{ \omega_e(X') \left( \sigma^2(1, X) e(X) + \sigma^2(0, X) \frac{1}{1 - e(X)} \right) \right\}.$$

**Remark 1** The term $b_1^2 \tau_{aug}^{-1} b_1$ is the increased variability due to estimating the support. We now show that this term is close to zero for small $e$. We note

$$\frac{\partial}{\partial \theta} \left[ \frac{\omega_e(X')}{E\{\omega_e(X')\}} \right] = \frac{\dot{\omega}_e(X') e\{X' \} - E\{\dot{\omega}_e(X') \} \omega_e(X')}{E\{\omega_e(X')\}^2}, \quad (8)$$

where

$$\dot{\omega}_e(X') = \frac{\partial}{\partial \theta} \left[ \Phi_e \left\{ e(X') - \alpha \right\} \Phi_e \left\{ 1 - \alpha - e(X') \right\} \right]$$

$$= \phi_e \left\{ e(X') - \alpha \right\} \Phi_e \left\{ 1 - \alpha - e(X') \right\} f(X') X$$

$$- \Phi_e \left\{ e(X') - \alpha \right\} \phi_e \left\{ 1 - \alpha - e(X') \right\} f(X') X,$$

and $\phi_e(x) = d\Phi_e(x)/dx$. As $e \to 0$, because $\phi_e(x) \to 0$, the right hand side of \(5\), and therefore $b_1$, go to zero. The increased variability due to the estimated support is close to zero with small $e$.

**Remark 2** The term $-b_2^2 \tau_{aug}^{-1} b_2$ implies that the estimated propensity score increases the precision of the simple weighting estimator $\tau$ based on the true propensity score, which has been demonstrated in the missing data and causal literature; see, e.g., \(\text{Rubin and Thomas (1992)}\) and \(\text{Abadie and Imbens (2016)}\).

**Remark 3** Assuming that $\tau(X)$ is integrable, by the Dominated Convergence Theorem, $\tau_e$ converges to $\tau_O$ as $e \to 0$. This implies that our inference based on $\hat{\tau}_e$, by choosing a small $e$, can be drawn for the target population represented by $O$.

Let $\hat{\tau}_{e}\text{aug}$ be the weighting estimator \(5\) with the smooth weight function \(6\).
Theorem 2 Under Assumptions 1 and 3, \( \hat{\tau}^{\text{aug}} \) is asymptotically linear. Moreover,

\[
N^{1/2}(\hat{\tau}^{\text{aug}} - \tau_e) \to N\left\{ 0, \sigma^2_e + b'_1 \mathcal{I}_\theta \cdot b_1 \epsilon + (C_0 + C_1) \mathcal{I}_\theta (C_0 + C_1) + \bar{B}'(C_0 - C_1) \right\},
\]

in distribution, as \( N \to \infty \), where \( b_1 \epsilon \) is defined in \([7]\).

\[
\hat{\sigma}^2_e = \frac{1}{E\{\omega_e(X'\theta)\}^2} \text{var}\{\omega_e(X'\theta)\tau(X)\}
\]

\[
+ \frac{1}{E\{\omega_e(X'\theta)\}^2} E\left[\omega_e(X'\theta)\right]^2 \left\{ \frac{\sigma^2(1, X)}{e(X)} + \frac{\sigma^2(0, X)}{1 - e(X)} \right\},
\]

\[
C_0 = E\left\{ X \omega_e(X'\theta) f(X'\theta) \mu(0, X) - \mu(0, X) \right\},
\]

\[
C_1 = E\left\{ X \omega_e(X'\theta) f(X'\theta) \mu(1, X) - \mu(1, X) \right\},
\]

with \( \mu(a, X) \to \tilde{\mu}(a, X) \) in probability, for \( a = 0, 1 \), and \( \bar{B} = b_1 \epsilon - C_0 - C_1 \).

Remark 4 The term \( b'_1 \mathcal{I}_\theta \cdot b_1 \epsilon \) can be made small by choosing a small \( \epsilon \), as shown in Remark 7.

If the outcome model is correctly specified, \( \hat{\mu}(a, X) = \mu(a, X) \), and consequently, \( C_0 = C_1 = 0 \). The asymptotic variance of \( \hat{\tau}^{\text{aug}} \) reduces to \( \hat{\sigma}^2_e + b'_1 \mathcal{I}_\theta \cdot b_1 \epsilon \), which is more efficient than \( \hat{\tau}_e \). Intuitively, this occurs because by regressing \( Y \) on \( X \) and \( A \), we are essentially using the residual as the new outcome, which in general has smaller variance than \( Y \).

Remark 5 Because \( \hat{\tau}_e \) and \( \hat{\tau}^{\text{aug}} \) are asymptotically linear, the bootstrap can be used to estimate the variances of \( \hat{\tau}_e \) and \( \hat{\tau}^{\text{aug}} \) (Shao and Tu, 2012). Let \( S = \{x \mid e(x'\theta^*) = \alpha \) or \( 1 - \alpha \} \). If \( \text{pr}(X \in S) = 0 \), we conjecture that the bootstrap works for the weighting estimator with the indicator function. This is demonstrated in the simulation study.

4 Average treatment effect on the treated

Another estimator of interest is the average treatment effect for the treated \( \tau_{\text{ATT}} = E\{Y(1) - Y(0) \mid A = 1\} = E\{\tau(X) \mid A = 1\} \) (Rubin, 1977; Hirano and Imbens, 2001). The outcome distribution for the treated is empirically identifiable \( E\{Y(1) \mid A = 1\} = E(Y \mid A = 1) \), and therefore Assumptions 1 and 2 can be weakened (Heckman et al., 1997).

Assumption 4 \( Y(0) \perp A \mid X \).

Assumption 5 There exists a constant \( c_2 \) such that with probability 1, \( e(X) \leq c_2 < 1 \).

A simple weighting estimator of \( \tau_{\text{ATT}} \) (Hirano et al., 2003) is

\[
\widehat{\tau}_{\text{ATT}} = \frac{\sum_{i=1}^{N} A_i Y_i}{\sum_{i=1}^{N} e(X'_i \theta)} - \frac{\sum_{i=1}^{N} (1 - A_i) Y_i e(X'_i \bar{\theta}) / \{1 - e(X'_i \bar{\theta})\}}{\sum_{i=1}^{N} e(X'_i \theta)} = \frac{\sum_{i=1}^{N} e(X'_i \bar{\theta}) \tau(X_i)}{\sum_{i=1}^{N} e(X'_i \theta)}. \tag{9}
\]

By the above expression, \( \widehat{\tau}_{\text{ATT}} \) is a special case of the weighting estimator (4) by choosing \( \omega(X'_i \bar{\theta}) = e(X'_i \bar{\theta}) \). Analogously, we propose the augmented weighting estimator of \( \tau_{\text{ATT}}, \)

\[
\hat{\tau}_{\text{ATT}}^{\text{aug}} = \frac{\sum_{i=1}^{N} e(X'_i \bar{\theta}) \hat{\tau}^{\text{aug}}(X_i)}{\sum_{i=1}^{N} e(X'_i \theta)}. \tag{10}
\]
There is a limited literature dealing with the lack of overlap for \( \tau_{\text{ATT}} \) when Assumption 5 may not hold. Similar to Crump et al. (2009), assuming that \( \sigma^2(1,X) = \sigma^2(0,X) \), we can show that the optimal overlap for estimating \( \tau_{\text{ATT}} \) is of the form \( O = \{ x \mid 1 - e(x) \geq \alpha \} \) for some \( \alpha \), for which the estimators have smallest asymptotic variance. Intuitively, for the treated subjects with \( e(X) \) close to one, there are no similar subjects in the control group that can provide adequate information to infer \( Y(0) \) for these treated subjects. Statistically, the control subjects with \( e(X) \) close to one contribute to large weights. Therefore, it is reasonable to drop these subjects with \( e(X) \) close to one. By restricting the focus to the optimal set, the estimand of interest becomes \( \tau_{\text{ATT}}(O) = E\{\tau(X) \mid A = 1, X \in O\} \), for which we propose two estimators with smooth inclusion weights as

\[
\hat{\tau}_{\text{ATT},\epsilon} = \frac{\sum_{i=1}^{N} \Phi_\epsilon\{1 - \alpha - e(X'_i \hat{\theta})\}e(X'_i \hat{\theta}) \hat{\tau}(X_i)}{\sum_{i=1}^{N} \Phi_\epsilon\{1 - \alpha - e(X'_i \hat{\theta})\}e(X'_i \hat{\theta})},
\]

\[
\hat{\tau}_{\text{ATT},\epsilon}^{\text{aug}} = \frac{\sum_{i=1}^{N} \Phi_\epsilon\{1 - \alpha - e(X'_i \hat{\theta})\}e(X'_i \hat{\theta}) \hat{\tau}_{\text{aug}}(X_i)}{\sum_{i=1}^{N} \Phi_\epsilon\{1 - \alpha - e(X'_i \hat{\theta})\}e(X'_i \hat{\theta})}.
\]

The asymptotic properties can be derived similarly as in Theorems 1 and 2 by recognizing \( \omega_\epsilon(X' \hat{\theta}) = \Phi_\epsilon\{1 - \alpha - e(X'_i \hat{\theta})\}e(X'_i \hat{\theta}) \) for \( \hat{\tau}_{\text{ATT},\epsilon} \) and \( \hat{\tau}_{\text{ATT},\epsilon}^{\text{aug}} \). In particular, the asymptotic linearity enables the bootstrap for inference. See the Supplementary Material for details. Similar discussion applies to estimating the average treatment effect on the control.

### 5 Simulation study

We assess the performance of the new weighting estimators of the average treatment effect over a target population. We consider six covariates \( X_j \) (\( j = 1, \ldots, 6 \)), where \( X_1, X_2, \) and \( X_3 \) are multivariate normal with means \((0,0,0)\), variances \((2,1,1)\) and covariances \((1,-1,-0.5)\), \( X_4 \sim \text{Uniform}[-3,3] \), \( X_5 \sim \chi^2_1 \), and \( X_6 \sim \text{Bernoulli}(0.5) \). Let \( X = (X_1, X_2, X_3, X_4, X_5, X_6)' \) be the 6-component vector of covariates. The treatment indicator \( A \) is generated from a Bernoulli distribution with probability \( e(X) \). We consider four propensity score designs:

(P1) \( e(X) = \logit\{0.1 \times (X_1 + X_2 + X_3 + X_4 + X_5 + X_6)\} \).

(P2) \( e(X) = \logit\{0.3 \times (X_1 + X_2 + X_3 + X_4 + X_5 + X_6)\} \).

(P3) \( e(X) = \logit\{0.1 \times (X_1 + X_2^2 + X_3^2 + X_4 + X_5 + X_6)\} \).

(P4) \( e(X) = \logit\{0.3 \times (X_1 + X_2^2 + X_3^2 + X_4 + X_5 + X_6)\} \).

(P1) and (P3) represent weak separations of propensity score distributions between the treatment and control groups; (P2) and (P4) represent strong separations. See Figure S.2 in the Supplementary Material for visualization of propensity score distributions. We consider two outcome designs:

(O1) \( Y(a) = a(X_1 + X_2 + X_3 - X_4 + X_5 + X_6) + \eta, \text{ with } \eta \sim \mathcal{N}(0,1), \text{ for } a = 0,1. \)

(O2) \( Y(a) = a(X_1 + X_2 + X_3)^2 + \eta, \text{ with } \eta \sim \mathcal{N}(0,1), \text{ for } a = 0,1. \)

(O1) is a linear case, and (O2) is a non-linear case. The sample size is set to be \( N = 500 \). The target population is represented by \( \mathcal{O} = \{ x \mid \alpha < e(x) < 1 - \alpha \} \) with \( \alpha = 0.1 \), and the estimand of interest \( \tau(\mathcal{O}) \) is the average treatment effect over the target population.

We consider the weighting estimators with indicator weight function and smooth weight function, and \( \tau(\mathcal{O}) = \left( \sum_{i=1}^{N} 1_{\{X_i \in \mathcal{O}\}} \right)^{-1} \sum_{i=1}^{N} 1_{\{X_i \in \mathcal{O}\}} \{Y_i(1) - Y_i(0)\} \) for benchmark comparison. The
Table 1: Results: mean, variance (var), and variance estimate (ve) by 100 bootstrapping under eight combinations of outcome design and propensity score design: c indicates the corresponding model is correctly specified, and w indicates the corresponding model is incorrectly specified.

| Scenario | i (OD1c, PSD1c) | ii (OD1c, PSD2c) | iii (OD1c, PSD3w) | iv (OD1c, PSD4w) |
|----------|----------------|-----------------|------------------|-----------------|
| $\tau(O)$ | 1.46 | 1.33 | 1.44 | 1.37 |
| $\hat{\tau}(\hat{\theta})$ | 1.45 | 0.0341 | 0.0336 | 1.33 | 0.0471 | 0.0518 | 1.48 | 0.0285 | 0.0282 | 1.45 | 0.0399 | 0.0405 |
| $\hat{\tau}_{\text{aug}}(\hat{\theta})$ | 1.46 | 0.0282 | 0.0267 | 1.32 | 0.0343 | 0.0342 | 1.50 | 0.0263 | 0.0253 | 1.49 | 0.0331 | 0.0315 |
| $\hat{\tau}_{\epsilon}(\hat{\theta})$ | $10^{-4}$ | 1.45 | 0.0333 | 0.0331 | 1.33 | 0.0445 | 0.0474 | 1.48 | 0.0284 | 0.0278 | 1.45 | 0.0386 | 0.0382 |
| $\hat{\tau}_{\epsilon}(\hat{\theta})$ | $10^{-5}$ | 1.45 | 0.0339 | 0.0331 | 1.33 | 0.0464 | 0.0503 | 1.48 | 0.0285 | 0.0281 | 1.45 | 0.0394 | 0.0397 |
| $\hat{\tau}_{\text{aug}}(\hat{\theta})$ | $10^{-5}$ | 1.46 | 0.0282 | 0.0267 | 1.32 | 0.0343 | 0.0342 | 1.50 | 0.0263 | 0.0253 | 1.49 | 0.0331 | 0.0315 |
| $\tau(O)$ | 7.58 | 6.69 | 7.62 | 5.96 |
| $\hat{\tau}(\hat{\theta})$ | 7.58 | 0.9400 | 0.8912 | 6.69 | 0.8983 | 0.9811 | 8.75 | 0.9201 | 0.9122 | 8.93 | 1.4198 | 1.3808 |
| $\hat{\tau}_{\text{aug}}(\hat{\theta})$ | 7.59 | 0.8538 | 0.7652 | 6.67 | 0.7919 | 0.8417 | 8.82 | 0.8493 | 0.7925 | 9.06 | 1.2260 | 1.0958 |
| $\hat{\tau}_{\epsilon}(\hat{\theta})$ | $10^{-4}$ | 7.57 | 0.8861 | 0.8408 | 6.70 | 0.8528 | 0.8967 | 8.75 | 0.9106 | 0.8828 | 8.94 | 1.3418 | 1.2842 |
| $\hat{\tau}_{\epsilon}(\hat{\theta})$ | $10^{-5}$ | 7.58 | 0.8268 | 0.7473 | 6.68 | 0.7663 | 0.7941 | 8.82 | 0.8441 | 0.7839 | 9.07 | 1.1896 | 1.0554 |
| $\hat{\tau}_{\epsilon}(\hat{\theta})$ | $10^{-5}$ | 7.57 | 0.9203 | 0.8732 | 6.69 | 0.8879 | 0.9525 | 8.75 | 0.9192 | 0.9020 | 8.93 | 1.3997 | 1.3479 |
| $\hat{\tau}_{\text{aug}}(\hat{\theta})$ | $10^{-5}$ | 7.59 | 0.8405 | 0.7591 | 6.68 | 0.7868 | 0.8249 | 8.82 | 0.8474 | 0.7896 | 9.06 | 1.2171 | 1.0824 |

Propensity scores are estimated by a logistic regression model with linear predictors $X$. Therefore, the propensity score model is correctly specified under (P1) and (P2), but it is misspecified under (P3) and (P4). For the augmented weighting estimators, $\mu(a, X)$ is estimated by a simple regression of $Y$ on $X$, separately for $A = a$ ($a = 0, 1$). Therefore, the outcome regression model is correctly specified under (O1) but misspecified under (O2).

Table 1 shows the simulation results. Under Scenarios i, ii, v and vi when the propensity score model is correctly specified, the weighting estimators are unbiased of $\tau(O)$, and the augmented weighting estimators therefore improve the precision. However, under Scenarios iii, iv, vii and viii when the propensity score model is misspecified, all estimators are biased even when the outcome regression model is correctly specified for the augmented weighting estimators. The augmented weighting estimators are not doubly robust in this case, because selecting samples corresponding to the target population relies on correct specification of the propensity score model. We further address the misspecification of propensity score model in the discussion section. The weighting estimators with smooth inclusion weights, $\hat{\tau}_{\epsilon}$ and $\hat{\tau}_{\text{aug}}$, show slightly smaller variances than the counterparts with indicator inclusion weights, $\hat{\tau}$ and $\hat{\tau}_{\text{aug}}$. Moreover, as $\epsilon$ becomes smaller, the performances of $\hat{\tau}_{\epsilon}$ and $\hat{\tau}_{\text{aug}}$ become closer to those of $\hat{\tau}$ and $\hat{\tau}_{\text{aug}}$. The bootstrap works well with variance estimates close to the true variances for all estimators including the weighting estimators with indicator inclusion weights.

In addition, we illustrate our method using two real-life data sets, presented in the Supplementary Material.

6 Discussion

The propensity score model is critical for our weighting estimators. The majority of the literature used a parametric logistic regression model to estimate propensity score. When the propensity score
model is misspecified, the weighting estimators are not consistent to the causal effect defined on the target population $O = \{x | \alpha < e(x) < 1 - \alpha\}$. However, our estimators can still be helpful to inform treatment effects for the population defined as $O^* = \{x | \alpha < e(x'\theta^*) < 1 - \alpha\}$, where $e(x'\theta^*)$ is the propensity score projected to the generalized linear model family. In this case, the smooth weighting estimators are still asymptotically linear and the bootstrap can be used for constructing confidence intervals. See the Supplementary Material for details. Alternatively, we can consider robust nonparametric methods for propensity score estimation such as power series (Hirano et al.; 2003), boosting trees, and random forest (Lee et al.; 2010).

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**Supplementary Material**

Supplementary material includes proofs of Theorems 1 and 2, optimal support for the average treatment effect on the treated, asymptotic linearity under model misspecification, two applications, and figures.

**S7  Proof of Theorem 1**

We write

\[
\hat{\tau}_c = \hat{\tau}_c(\hat{\theta})
\]

\[
= \hat{\tau}_c(\theta^*) + E \left\{ \frac{\partial \hat{\tau}_c(\theta^*)}{\partial \theta} \right\} (\hat{\theta} - \theta^*) \tag{S1}
\]

\[
\approx \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E(\omega_i(X' \theta^*))} \left\{ \frac{A_i Y_i}{e(X'_i \theta^*)} - \frac{(1-A_i) Y_i}{1-e(X'_i \theta^*)} \right\}
\]

\[
+ E \left\{ \frac{\partial \hat{\tau}_c(\theta^*)}{\partial \theta} \right\} S(\theta^*) \tag{S2}
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E(\omega_i(X' \theta^*))} \left\{ \frac{A_i Y_i}{e(X'_i \theta^*)} - \frac{(1-A_i) Y_i}{1-e(X'_i \theta^*)} \right\}
\]

\[
+ B' \frac{1}{N} \sum_{i=1}^{N} X_i \frac{A_i - e(X'_i \theta^*)}{e(X'_i \theta^*) (1-e(X'_i \theta^*))} f(X'_i \theta^*),
\]
where $C \cong D$ means $C = D + O_p(N^{-1/2})$. [S1] follows from the Taylor expansion, [S2] follows from the fact that $\hat{\theta} - \theta^* \cong T_{\theta^*}^{-1} S(\theta^*)$, and

$$
B' = E \left\{ \frac{\partial \hat{\tau}(\theta^*)}{\partial \theta^*} \right\} T_{\theta^*}^{-1}.
$$

Therefore, the asymptotic linearity of $\hat{\tau}$ follows. Moreover,

$$
N^{1/2}(\hat{\tau} - \tau)
\cong N^{-1/2} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E\{\omega_i(X'_i \theta^*)\}} \left[ A_i \{Y_i - \mu(A_i, X_i)\} - \frac{(1 - A_i)\{Y_i - \mu(A_i, X_i)\}}{1 - e(X'_i \theta^*)} \right]
$$

$$
+ N^{-1/2} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E\{\omega_i(X'_i \theta^*)\}} \left( \frac{\{A_i - e(X'_i \theta^*)\}\mu(A_i, e(X'_i \theta^*))}{e(X'_i \theta^*)\{1 - e(X'_i \theta^*)\}} - \tau e(X'_i \theta^*) \right)
$$

$$
+ N^{-1/2} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E\{\omega_i(X'_i \theta^*)\}} \tau e(X'_i \theta^*) - \tau_e
$$

$$
+ N^{-1/2} \sum_{i=1}^{N} B' X_i \frac{A_i - e(X'_i \theta^*)}{e(X'_i \theta^*)\{1 - e(X'_i \theta^*)\}} f(X'_i \theta^*),
$$

where $\tau e(X'_i \theta^*) = E\{Y(1) - Y(0) \mid e(X'_i \theta^*)\}$, and by grouping different terms,

$$
T_0 = N^{-1/2} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E\{\omega_i(X'_i \theta^*)\}} \tau e(X'_i \theta^*) - \tau_e
$$

$$
T_1 = N^{-1/2} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E\{\omega_i(X'_i \theta^*)\}} \left( \frac{\{A_i - e(X'_i \theta^*)\}\mu(A_i, e(X'_i \theta^*))}{e(X'_i \theta^*)\{1 - e(X'_i \theta^*)\}} - \tau e(X'_i \theta^*) \right)
$$

$$
+ N^{-1/2} \sum_{i=1}^{N} B' E\{X_i \mid e(X'_i \theta^*)\} \frac{A_i - e(X'_i \theta^*)}{e(X'_i \theta^*)\{1 - e(X'_i \theta^*)\}} f(X'_i \theta^*),
$$

$$
T_2 = N^{-1/2} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E\{\omega_i(X'_i \theta^*)\}} \left( \frac{\{A_i - e(X'_i \theta^*)\}\mu(A_i, X_i) - \mu(A_i, e(X'_i \theta^*))}{e(X'_i \theta^*)\{1 - e(X'_i \theta^*)\}} \right)
$$

$$
+ N^{-1/2} \sum_{i=1}^{N} B'[X_i - E\{X_i \mid e(X'_i \theta^*)\}] \frac{A_i - e(X'_i \theta^*)}{e(X'_i \theta^*)\{1 - e(X'_i \theta^*)\}} f(X'_i \theta^*),
$$

$$
T_3 = N^{-1/2} \sum_{i=1}^{N} \frac{\omega_i(X'_i \theta^*)}{E\{\omega_i(X'_i \theta^*)\}} \left[ A_i \{Y_i - \mu(A_i, X_i)\} \right.
$$

$$
- \frac{(1 - A_i)\{Y_i - \mu(A_i, X_i)\}}{1 - e(X'_i \theta^*)} \right].
$$
Define
\[ \mathcal{F}_0 = \{ X^i_1, \ldots, X^i_N \}, \quad \mathcal{F}_1 = \{ A_1, \ldots, A_N, X^i_1, \ldots, X^i_N \}, \]
\[ \mathcal{F}_2 = \{ A_1, \ldots, A_N, X^i_1, \ldots, X^i_N, X_1, \ldots, X_N \}. \]

By the conditioning argument, we have \( E(T_0) = 0 \), for \( k = 1, \ldots, 3 \), \( E(T_k) = E\{E(T_k | \mathcal{F}_{k-1})\} = 0 \), and for \( k = 1, \ldots, 3 \),
\[
\text{cov}(T_0, T_k) = \text{cov}\{E(T_0 | \mathcal{F}_0), E(T_k | \mathcal{F}_0)\} + E\{\text{cov}(T_0, T_k | \mathcal{F}_0)\} = \text{cov}\{E(T_0 | \mathcal{F}_0), 0\} + E\{0\} = 0,
\]
for \( k = 2, 3 \),
\[
\text{cov}(T_1, T_k) = \text{cov}\{E(T_1 | \mathcal{F}_1), E(T_k | \mathcal{F}_1)\} + E\{\text{cov}(T_1, T_k | \mathcal{F}_1)\} = \text{cov}\{E(T_1 | \mathcal{F}_1), 0\} + E\{0\} = 0,
\]
and
\[
\text{cov}(T_2, T_3) = \text{cov}\{E(T_2 | \mathcal{F}_2), E(T_3 | \mathcal{F}_2)\} + E\{\text{cov}(T_2, T_3 | \mathcal{F}_2)\} = \text{cov}\{E(T_2 | \mathcal{F}_2), 0\} + E\{0\} = 0.
\]

Also, we calculate the variances of \( T_i \), for \( i = 0, \ldots, 3 \), as follows. For \( T_0 \),
\[
\text{var}(T_0) = E(T_0^2) = \frac{1}{E\{\omega_e(X^i_1)\}^2} \text{var}\left[ \frac{\omega_e(X^i_1)}{\theta} \right].
\]
For \( T_1 \),
\[
\text{var}(T_1) = E\{\text{var}(T_1 | \mathcal{F}_0)\} = E\{E(T_1^2 | \mathcal{F}_0)\} = \frac{1}{E\{\omega_e(X^i_1)\}^2} E\{\omega_e(X^i_1)^2 \left[ \frac{1 - e(X^i_1)}{e(X^i_1)} \right]^{1/2} \mu(1, e(X^i_1))
\]
\[+ \left\{ \frac{e(X^i_1)}{1 - e(X^i_1)} \right\}^{1/2} \mu(0, e(X^i_1)) \right\}^2 \]
\[+ \frac{1}{E\{\omega_e(X^i_1)\}^2} B'_e \left( \frac{\omega_e(X^i_1) e(X^i_1)}{e(X^i_1)} \right) \frac{\mu(1, e(X^i_1))}{e(X^i_1)} + \frac{\mu(0, e(X^i_1))}{1 - e(X^i_1)} \right] f(X^i_1)
\[+ B'_e \left[ f(X^i_1)^2 \frac{E\left\{X | e(X^i_1)\right\} E\left\{X' | e(X^i_1)\right\}}{e(X^i_1)} \frac{1}{1 - e(X^i_1)} \right] B.
\]
For \( T_2 \),
\[
\text{var}(T_2) = E\{\text{var}(T_2 | \mathcal{F}_1)\} = E\{E(T_2^2 | \mathcal{F}_1)\} = \frac{1}{E\{\omega_e(X^i_1)\}^2} E\left\{ \omega_e(X^i_1)^2 \left[ \frac{\sigma^2(1, e(X^i_1))}{e(X^i_1)} + \frac{\sigma^2(0, e(X^i_1))}{1 - e(X^i_1)} \right] \right\}
\]
\[+ \frac{1}{E\{\omega_e(X^i_1)\}^2} B'_e \left( \frac{\omega_e(X^i_1) f(X^i_1)}{e(X^i_1)} \right) \frac{\text{cov}\{X, \mu(1, X) | e(X^i_1)\}}{e(X^i_1)}
\]
\[+ \frac{\text{cov}\{X, \mu(0, X) | e(X^i_1)\}}{1 - e(X^i_1)} \right] \right) \left( \frac{\text{var}\{X | e(X^i_1)\}}{e(X^i_1)} \right)^2 B.
\]
For $T_3$,

$$\text{var}(T_3) = E\{\text{var}(T_3 \mid \mathcal{F}_2)\} = E\{E(T_3^2 \mid \mathcal{F}_2)\}$$

$$\cong \frac{1}{E\{\omega_i(X'i^*)\}^2} E\left[\omega_i(X'i^*)^2 \left\{ \frac{\sigma^2_i(X)}{e(X'i^*)} + \frac{\sigma^2_i(X)}{1 - e(X'i^*)} \right\} \right].$$

Because

$$\frac{\partial \hat{\tau}_i(\theta^*)}{\partial \theta'} = \frac{1}{N} \sum_{i=1}^N \frac{\partial}{\partial \theta'} \left[ \frac{\omega_i(X'i^*)}{E\{\omega_i(X'i^*)\}} \right] \left\{ A_i Y_i - \frac{(1 - A_i)Y_i}{e(X'i^*)} \right\}$$

$$- \frac{1}{N} \sum_{i=1}^N \frac{\omega_i(X'i^*)}{E\{\omega_i(X'i^*)\}} \left[ \frac{A_i Y_i}{e(X'i^*)^2} + \frac{(1 - A_i)Y_i}{1 - e(X'i^*)^2} \right] f(X'i^*)X_i,$$

we have

$$E \left\{ \frac{\partial \hat{\tau}_i(\theta^*)}{\partial \theta} \right\} = E \left( \frac{\partial}{\partial \theta} \left[ \frac{\omega_i(X'i^*)}{E\{\omega_i(X'i^*)\}} \right] \tau(X) \right) - \frac{1}{E\{\omega_i(X'i^*)\}} E \left\{ \omega_i(X'i^*) f(X'i^*) \left[ \frac{E[X, \mu(1, X) | e(X'i^*)]}{e(X'i^*)} + \frac{E[X, \mu(0, X) | e(X'i^*)]}{1 - e(X'i^*)} \right] \right\}$$

$$= b_{1,\epsilon} - b_{2,\epsilon},$$

where $b_{1,\epsilon}$ and $b_{2,\epsilon}$ are defined in Theorem 1. Therefore, according to (S3), $B = (b_{1,\epsilon} - b_{2,\epsilon})I_{\theta^{-1}}$. As a result,

$$\text{var}(T_0) + \text{var}(T_1) + \text{var}(T_2) + \text{var}(T_3) = \sum \text{var}[\omega_i(X'i^*)\tau\{e(X'i^*)\}]$$

$$\sum \frac{1}{E\{\omega_i(X'i^*)\}^2} E\left[\omega_i(X'i^*)^2 \left\{ \frac{1 - e(X'i^*)}{e(X'i^*)} \right\}^{1/2} \mu\{1, e(X'i^*)\} \right. \left. + \left\{ \frac{e(X'i^*)}{1 - e(X'i^*)} \right\}^{1/2} \mu\{0, e(X'i^*)\} \right] ^2$$

$$+ \frac{1}{E\{\omega_i(X'i^*)\}^2} E\left[\omega_i(X'i^*)^2 \left\{ \frac{\sigma^2(1, e(X'i^*))}{e(X'i^*)} + \frac{\sigma^2(0, e(X'i^*))}{1 - e(X'i^*)} \right\} \right]$$

$$+ \frac{1}{E\{\omega_i(X'i^*)\}^2} E\left[\omega_i(X'i^*)^2 \left\{ \frac{\sigma^2(X, X)}{e(X'i^*)} + \frac{\sigma^2(X, X)}{1 - e(X'i^*)} \right\} \right]$$

$$+ \frac{2}{E\{\omega_i(X'i^*)\}^2} B' E\left[\omega_i(X'i^*) f(X'i^*) \left[ \frac{E[X, \mu(1, X) | e(X'i^*)]}{e(X'i^*)} \right. \left. + \frac{E[X, \mu(0, X) | e(X'i^*)]}{1 - e(X'i^*)} \right] \right] + B'I_{\theta}B$$

$$= \sigma_i^2 + b'_{1,\epsilon}I_{\theta^{-1}}b_{1,\epsilon} - b'_{2,\epsilon}I_{\theta^{-1}}b_{2,\epsilon},$$

where $\sigma_i^2$ is defined as the terms from (S5) to (S6), and the last equality follows by plugging the expression of $B$,

$$2B'b_{2,\epsilon} + B'I_{\theta}B = 2b'_{1,\epsilon}I_{\theta^{-1}}b_{2,\epsilon} - 2b'_{2,\epsilon}I_{\theta^{-1}}b_{2,\epsilon} + (b_{1,\epsilon} + b_{2,\epsilon})I_{\theta^{-1}}(b_{1,\epsilon} + b_{2,\epsilon})$$

$$= b'_{1,\epsilon}I_{\theta^{-1}}b_{1,\epsilon} - b'_{2,\epsilon}I_{\theta^{-1}}b_{2,\epsilon}.$$
Moreover, $\sigma_\varepsilon^2$ can be further simplified as

\[
\sigma_\varepsilon^2 = \frac{1}{E\{\omega_\varepsilon(X'\theta^*)\}^2} \text{var}\{\omega_\varepsilon(X'\theta^*)\tau(X)\}
\]

\[
= \frac{1}{E\{\omega_\varepsilon(X'\theta^*)\}^2} E\left\{\omega_\varepsilon(X'\theta^*)^2 \left[\frac{1 - e(X'\theta^*)}{e(X'\theta^*)}\right]^{1/2} \mu(1, X)^2 \right\}
\]

\[
+ \begin{cases} \frac{e(X'\theta^*)}{1 - e(X'\theta^*)} \mu(0, X) \end{cases}^{1/2} \cdot (S7)
\]

\[
+ \frac{1}{E\{\omega_\varepsilon(X'\theta^*)\}^2} E\left\{\omega_\varepsilon(X'\theta^*)^2 \left[\frac{\sigma^2(1, X)}{e(X'\theta^*)} + \frac{\sigma^2(0, X)}{1 - e(X'\theta^*)}\right] \right\}. \quad (S8)
\]

In addition, Assumption 3 is the moment condition for the Central Limit Theorem. Therefore,

\[
N^{1/2}(\hat{\tau}_\varepsilon - \tau_\varepsilon) \rightarrow N\left(0, \sigma_\varepsilon^2 + b_1'\mathcal{I}_{\theta^*}^{-1}b_1, - b_2'\mathcal{I}_{\theta^*}^{-1}b_2\right),
\]

in distribution, as $N \rightarrow \infty$.

**S8 Proof of Theorem 2**

Let $\hat{\mu}(A_i, X_i)$ converge to $\tilde{\mu}(A_i, X_i)$ as $N \rightarrow \infty$. If the model for $\mu(A_i, X_i)$ is correctly specified, $\tilde{\mu}(A_i, X_i) = \mu(A_i, X_i)$. Write

\[
\hat{\tau}_\varepsilon^{\text{aug}} = \hat{\tau}_\varepsilon^{\text{aug}}(\hat{\theta}) \cong \hat{\tau}_\varepsilon^{\text{aug}}(\theta^*) + E\left\{\frac{\partial \hat{\tau}_\varepsilon^{\text{aug}}(\theta^*)}{\partial \theta'} \right\} (\hat{\theta} - \theta^*)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_\varepsilon(X_i'\theta^*)}{E\{\omega_\varepsilon(X'\theta^*)\}} \hat{\varepsilon}_i(X_i) + E\left\{\frac{\partial \hat{\tau}_\varepsilon^{\text{aug}}(\theta^*)}{\partial \theta'} \right\} \mathcal{I}_{\theta^*}^{-1} S(\theta^*)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_\varepsilon(X_i'\theta^*)}{E\{\omega_\varepsilon(X'\theta^*)\}} \left[ A_i y_i \right] + E\left\{\frac{\partial \hat{\tau}_\varepsilon^{\text{aug}}(\theta^*)}{\partial \theta'} \right\} \tilde{\mu}(1, X_i)
\]

\[
- \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_\varepsilon(X_i'\theta^*)}{E\{\omega_\varepsilon(X'\theta^*)\}} \left[ (1 - A_i) \frac{y_i}{1 - e(X_i)} + \left\{1 - \frac{1 - A_i}{1 - e(X_i)}\right\} \tilde{\mu}(0, X_i) \right]
\]

\[
+ \tilde{B}' \frac{1}{N} \sum_{i=1}^{N} X_i \frac{A_i - e(X_i'\theta^*)}{e(X_i'\theta^*)} \{1 - e(X_i'\theta^*)\} f(X_i'\theta^*),
\]

where

\[
\tilde{B}' = E\left\{\frac{\partial \hat{\tau}_\varepsilon^{\text{aug}}(\theta^*)}{\partial \theta'} \right\} \mathcal{I}_{\theta^*}^{-1}. \quad (S9)
\]
Therefore, the asymptotic linearity of \( \hat{\tau}_e^{\text{aug}} \) follows. Moreover,

\[
N^{1/2}(\hat{\tau}_e^{\text{aug}} - \tau_e) \\
\leq N^{-1/2} \sum_{i=1}^{N} \frac{\omega_e(X_i' \theta^*)}{E\{\omega_e(X_i' \theta^*)\}} \left[ A_i \{Y_i - \mu(A_i, X_i)\} - \frac{(1 - A_i) \{Y_i - \mu(A_i, X_i)\}}{1 - e(X_i' \theta^*)} \right] \\
+ N^{-1/2} \sum_{i=1}^{N} \left[ \frac{\omega_e(X_i' \theta^*)}{E\{\omega_e(X_i' \theta^*)\}} \tau(X_i) - \tau_e \right] \\
+ N^{-1/2} \sum_{i=1}^{N} \tilde{B}'X_i \frac{A_i - e(X_i' \theta^*)}{e(X_i' \theta^*)\{1 - e(X_i' \theta^*)\}} f(X_i' \theta^*), \\
+ N^{-1/2} \sum_{i=1}^{N} \frac{\omega_e(X_i' \theta^*)}{E\{\omega_e(X_i' \theta^*)\}} \{1 - \frac{A_i}{e(X_i)}\} \{\tilde{\mu}(1, X_i) - \mu(1, X_i)\} \\
+ N^{-1/2} \sum_{i=1}^{N} \frac{\omega_e(X_i' \theta^*)}{E\{\omega_e(X_i' \theta^*)\}} \{1 - \frac{1 - A_i}{1 - e(X_i)}\} \{\tilde{\mu}(0, X_i) - \mu(0, X_i)\},
\]

where \( \tilde{T}_3 = T_3 \) is defined in \( S4 \),

\[
\tilde{T}_0 = N^{-1/2} \sum_{i=1}^{N} \left[ \frac{\omega_e(X_i' \theta^*)}{E\{\omega_e(X_i' \theta^*)\}} \tau(X_i) - \tau_e \right], \\
\tilde{T}_1 = N^{-1/2} \sum_{i=1}^{N} \tilde{B}'X_i \frac{A_i - e(X_i' \theta^*)}{e(X_i' \theta^*)\{1 - e(X_i' \theta^*)\}} f(X_i' \theta^*),
\]

and

\[
\tilde{T}_2 = N^{-1/2} \sum_{i=1}^{N} \frac{\omega_e(X_i' \theta^*)}{E\{\omega_e(X_i' \theta^*)\}} \left[ 1 - \frac{A_i}{e(X_i)} \right] \{\tilde{\mu}(1, X_i) - \mu(1, X_i)\} \\
+ N^{-1/2} \sum_{i=1}^{N} \frac{\omega_e(X_i' \theta^*)}{E\{\omega_e(X_i' \theta^*)\}} \left[ 1 - \frac{1 - A_i}{1 - e(X_i)} \right] \{\tilde{\mu}(0, X_i) - \mu(0, X_i)\}.
\]

By the same argument as in the proof of Theorem 1, \( E(\tilde{T}_j) = 0 \), for \( j = 0, \ldots, 3 \), and \( \text{cov}(\tilde{T}_j, \tilde{T}_k) = 0 \) for all \( j \neq k \) expect \( \text{cov}(\tilde{T}_1, \tilde{T}_2) \). Moreover,

\[
\text{var}(\tilde{T}_3) + \text{var}(\tilde{T}_0) + \text{var}(\tilde{T}_1) + \text{var}(\tilde{T}_2) + 2\text{cov}(\tilde{T}_1, \tilde{T}_2) \\
= \frac{1}{E\{\omega_e(X' \theta^*)\}^2} E \left[ \omega_e(X' \theta^*)^2 \left\{ \frac{\sigma^2(1, X)}{e(X' \theta^*)} + \frac{\sigma^2(0, X)}{1 - e(X' \theta^*)} \right\} \right] \\
+ \frac{1}{E\{\omega_e(X' \theta^*)\}^2} \text{var} \left\{ \omega_e(X' \theta^*)\tau(X) \right\} + \tilde{B}'\tilde{T}_0 \cdot \tilde{B} \\
+ \frac{1}{E\{\omega_e(X' \theta^*)\}^2} E \left[ \omega_e(X' \theta^*)^2 \left\{ \left[ 1 - e(X' \theta^*) \right] \right\}^{1/2} \{\tilde{\mu}(1, X) - \mu(1, X)\} \right]
\]

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\begin{align*}
&- \left\{ \frac{e(X'\theta^*)}{1 - e(X'\theta^*)} \right\}^{1/2} \left\{ \hat{\mu}(0, X) - \mu(0, X) \right\}^2 \\
&+ \frac{1}{E\{\omega_i(X'\theta^*)\}} \tilde{B}' \left[ \omega_i(X'\theta^*) X f(X'\theta^*) \left\{ \frac{\tilde{\mu}(1, X_i) - \mu(1, X_i)}{e(X_i)} \right\} \right] \\
&+ \frac{1}{E\{\omega_i(X'\theta^*)\}} \tilde{B}' \left[ \omega_i(X'\theta^*) X f(X'\theta^*) \left\{ \frac{\tilde{\mu}(0, X) - \mu(0, X)}{1 - e(X_i)} \right\} \right] \\
&= \tilde{\sigma}_e^2 + \tilde{B}' \mathcal{I}_\theta \tilde{B} + \tilde{B}'(C_0 - C_1) \\
&= \tilde{\sigma}_e^2 + b_{1, \epsilon} \mathcal{I}_\theta b_{1, \epsilon} + (C_0 + C_1)' \mathcal{I}_\theta (C_0 + C_1) + \tilde{B}'(C_0 - C_1)
\end{align*}

where \( \tilde{\sigma}_e^2, C_0 \) and \( C_1 \) are defined in Theorem 2. Because

\[
\frac{\partial \tau_{\text{aug}}(\theta^*)}{\partial \theta} = \frac{1}{N} \sum_{i=1}^{N} \frac{\partial}{\partial \theta} \left[ \frac{\omega_i(X'\theta^*)}{E\{\omega_i(X'\theta^*)\}} \right] \tau_{\text{aug}}(X_i)
\]

\[
- \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X'\theta^*)}{E\{\omega_i(X'\theta^*)\}} X_i f(X'\theta^*) \frac{A_i \{Y_i - \tilde{\mu}(A_i, X_i)\}}{e(X_i)^2}
\]

\[
- \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X'\theta^*)}{E\{\omega_i(X'\theta^*)\}} X_i f(X'\theta^*) \frac{(1 - A_i) \{Y_i - \tilde{\mu}(A_i, X_i)\}}{\{1 - e(X_i)^2\}^2}
\]

we have

\[
E \left\{ \frac{\partial \tau_{\text{aug}}(\theta^*)}{\partial \theta} \right\} = E \left( \frac{\partial}{\partial \theta} \left[ \frac{\omega_i(X'\theta^*)}{E\{\omega_i(X'\theta^*)\}} \right] \tau(X) \right)
\]

\[
- \frac{1}{E\{\omega_i(X'\theta^*)\}} E \left\{ \omega_i(X'\theta^*) X f(X'\theta^*) \frac{\mu(1, X) - \tilde{\mu}(1, X)}{e(X'\theta^*)} \right\}
\]

\[
- \frac{1}{E\{\omega_i(X'\theta^*)\}} E \left\{ \omega_i(X'\theta^*) X f(X'\theta^*) \frac{\mu(0, X) - \tilde{\mu}(0, X)}{1 - e(X'\theta^*)} \right\}
\]

\[
= b_{1, \epsilon} - C_0 - C_1.
\]

Therefore,

\[
N^{1/2}(\tau_{\text{aug}} - \tau_e) \rightarrow \mathcal{N} \left\{ 0, \tilde{\sigma}_e^2 + b_{1, \epsilon} \mathcal{I}_\theta b_{1, \epsilon} + (C_0 + C_1)' \mathcal{I}_\theta (C_0 + C_1) + \tilde{B}'(C_0 - C_1) \right\},
\]

in distribution, as \( N \rightarrow \infty \).

**S9 Improving overlap for the treatment effect on the treated**

Define a general weighting average treatment effect,

\[
\tau_\omega(\mathcal{O}) = \frac{\sum_{i : X_i \in \mathcal{O}} \omega(X_i) \tau(X_i)}{\sum_{i : X_i \in \mathcal{O}} \omega(X_i)}.
\]  \hfill (S10)

The efficiency bound for \( \tau_\omega(\mathcal{O}) \) is

\[
V_\omega(\mathcal{O}) = \frac{1}{E\{\omega(X) \mid X \in \mathcal{O}\}^2} E \left\{ \omega(X)^2 \left\{ \frac{\sigma^2(1, X)}{e(X)} + \frac{\sigma^2(0, X)}{1 - e(X)} \right\} \mid X \in \mathcal{O} \right\}.
\]  \hfill (S11)
\textbf{Crump et al. (2009)} showed that the optimal set with which \( \hat{\tau}_\omega(\mathcal{O}) \) achieves the smallest asymptotic variance over all choices of \( \mathcal{O} \) is
\[
\mathcal{O} = \left\{ x \mid \omega(x) \left\{ \frac{\sigma^2(1,x)}{e(x)} + \frac{\sigma^2(0,x)}{1-e(x)} \right\} \leq \gamma \right\},
\] (S12)
where \( \gamma \) is defined through the following equation:
\[
\gamma = 2 \frac{E \left[ \omega^2(X) \left\{ \frac{\sigma^2(1,X)}{e(X)} + \frac{\sigma^2(0,X)}{1-e(X)} \right\} \mid \omega(X) \left\{ \frac{\sigma^2(1,X)}{e(X)} + \frac{\sigma^2(0,X)}{1-e(X)} \right\} < \gamma \right]}{E \left[ \omega(X) \mid \omega(X) \left\{ \frac{\sigma^2(1,X)}{e(X)} + \frac{\sigma^2(0,X)}{1-e(X)} \right\} < \gamma \right]}.
\] (S13)
We identify that the weighted estimator for the average treatment effect on the treated is (S10) with \( \omega(X) = e(X) \). Assuming that \( \sigma^2(1,X) = \sigma^2(0,X) = \sigma^2 \), the optimal set (S12) reduces to \( \mathcal{O} = \{ x \mid 1-e(x) \geq \alpha \} \) with the cut-off value \( \alpha = \sigma^2/\gamma \).

In practice, \( \alpha \) can be determined by the smallest value of \( \alpha \) that satisfy the empirical estimate of equation (S13):
\[
\frac{1}{\alpha} = 2 \frac{\sum_{i=1}^{N} e^2(X_i) \left\{ \frac{1}{e(X_i)} + \frac{1}{1-e(X_i)} \right\} 1\{1-e(X_i) \geq \alpha \}}{\sum_{i=1}^{N} e(X_i) 1\{1-e(X_i) \geq \alpha \}}.
\]
The choice of \( \alpha \) in \( \mathcal{O} = \{ 1-e(X) \geq \alpha \} \) has two opposite effects on the asymptotic variance in (S11). On the one hand, as \( \alpha \) increases, we reduce the denominator of the right hand side of (S11), \( E\{\omega(X) \mid X \in \mathcal{O}\}^2 = E\{e(X) \mid X \in \mathcal{O}\}^2 \), and therefore increase the asymptotic variance; on the other hand, as \( \alpha \) increases, we decrease the numerator of the right hand side of (S11),
\[
E \left[ \omega(X)^2 \left\{ \frac{\sigma^2(1,X)}{e(X)} + \frac{\sigma^2(0,X)}{1-e(X)} \right\} \mid X \in \mathcal{O} \right] = E \left[ e(X) \sigma^2(1,X) + \frac{e(X)^2 \sigma^2(0,X)}{1-e(X)} \mid X \in \mathcal{O} \right],
\]
and therefore decrease the asymptotic variance. The optimal value of \( \alpha \) balances the two effects.

\textbf{S10 Asymptotic linearity when propensity score model is misspecified}
Because \( \hat{\theta} \) is the solution to the score equation \( S(\theta) = 0 \), under certain regularity conditions, \( \hat{\theta} - \theta^* = \mathcal{J}_{\theta^*}^{-1} S(\theta^*) + o_p(N^{-1/2}) \), where \( \mathcal{J}_{\theta^*} = E\{\partial S(\theta^*)/\partial \theta^*\} \). Here, when propensity score model is misspecified, \( \mathcal{J}_{\theta^*} \) is not necessarily equal to \( I_{\theta^*} \).

We write
\[
\hat{\tau}_e = \hat{\tau}_e(\hat{\theta}) \\
\approx \hat{\tau}_e(\theta^*) + E \left\{ \frac{\partial \hat{\tau}_e(\theta^*)}{\partial \theta^*} \right\} (\hat{\theta} - \theta^*) \\
\approx \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X_i^\theta)}{E\{\omega_i(X_i^\theta)\}} \left\{ \frac{A_i Y_i}{e(X_i^\theta)} - \frac{(1-A_i)Y_i}{1-e(X_i^\theta)} \right\} + E \left\{ \frac{\partial \hat{\tau}_e(\theta^*)}{\partial \theta^*} \right\} \mathcal{J}_{\theta^*}^{-1} S(\theta^*) \\
= \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X_i^\theta)}{E\{\omega_i(X_i^\theta)\}} \left\{ \frac{A_i Y_i}{e(X_i^\theta)} - \frac{(1-A_i)Y_i}{1-e(X_i^\theta)} \right\} \\
+ \Gamma \frac{1}{N} \sum_{i=1}^{N} X_i \left( \frac{A_i - e(X_i^\theta)}{e(X_i^\theta)(1-e(X_i^\theta))} \right) f(X_i^\theta),
\]
\[16\]
where
\[ \Gamma' = E \left\{ \frac{\partial \hat{\tau}_i(\theta^*)}{\partial \theta'} \right\} \mathcal{J}_{\theta'}^{-1}. \]

Therefore, the asymptotic linearity of \( \hat{\tau}_i \) follows.

Write
\[ \hat{\tau}_i^{\text{aug}} = \hat{\tau}_i(\hat{\theta}) \approx \hat{\tau}_i(\theta^*) + E \left\{ \frac{\partial \hat{\tau}_i^{\text{aug}}(\theta^*)}{\partial \theta'} \right\} (\hat{\theta} - \theta^*) \]
\[ \approx \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X_i'\theta^*)}{\text{E}\{\omega(X_i'\theta^*)}\} \hat{\tau}_i \left( X_i \right) + E \left\{ \frac{\partial \hat{\tau}_i^{\text{aug}}(\theta^*)}{\partial \theta'} \right\} \mathcal{J}_{\theta'}^{-1} S(\theta^*) \]
\[ \approx \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X_i'\theta^*)}{\text{E}\{\omega(X_i'\theta^*)}\} \left[ A_i Y_i \text{I} e(X_i) + \left\{ 1 - \frac{A_i}{e(X_i)} \right\} \hat{\mu}(1, X_i) \right] \]
\[ - \frac{1}{N} \sum_{i=1}^{N} \frac{\omega_i(X_i'\theta^*)}{\text{E}\{\omega(X_i'\theta^*)}\} \left[ (1 - A_i) Y_i \text{I} 1 - e(X_i) + \left\{ 1 - \frac{1 - A_i}{1 - e(X_i)} \right\} \hat{\mu}(0, X_i) \right] \]
\[ + \hat{\Gamma}' \frac{1}{N} \sum_{i=1}^{N} X_i \frac{A_i - e(X_i'\theta^*)}{e(X_i'\theta^*)\{1 - e(X_i'\theta^*)\}} f(X_i'\theta^*), \]

where
\[ \hat{\Gamma}' = E \left\{ \frac{\partial \hat{\tau}_i^{\text{aug}}(\theta^*)}{\partial \theta'} \right\} \mathcal{J}_{\theta'}^{-1}. \]

Therefore, the asymptotic linearity of \( \tau_i^{\text{aug}} \) follows.

The asymptotic linearity of the weighting estimators allows for the bootstrap to construct confidence intervals.

**S11 The National Health and Nutrition Examination Survey Data**

We examine a data set from the 2007–2008 U.S. National Health and Nutrition Examination Survey to estimate the causal effect of smoking on blood lead levels. The data set includes 3340 subjects consisting of 679 smokers, denoted as \( A = 1 \), and 2661 nonsmokers, denoted as \( A = 0 \). The outcome variable \( Y \) is the measured lead level in blood, with observed range from 0.18 ug/dl to 33.10 ug/dl. The covariates \( X \) include age, income-to-poverty level, gender, education and race. For details of the data set, see [Hsu and Small (2013)](#).

The propensity score is estimated by a logistic regression model with linear predictors including all covariates. See Figure S3 for illustration of the estimated propensity score distribution by smokers and non-smokers. To help address lack of overlap, for the average smoking effect, because there is little overlap for the propensity score less than 0.05 and greater than 0.6, we restrict our estimand to the target population \( \mathcal{O} = \{ x \mid 0.05 < \epsilon(x) < 0.6 \} \). This results in removal of 794 subjects (23.8% of the sample), with 111 smokers and 683 non-smokers. Thus, the analysis sample includes 2546 subjects, with 568 smokers and 1978 non-smokers. For the average smoking effect on the smokers, subjects are trimmed if their estimated propensity score is greater than 0.7. This results in removal of 36 subjects (1.1% of the sample), with 29 smokers and 7 non-smokers. Thus, the analysis sample includes 3304 subjects, with 650 smokers and 2654 non-smokers. We consider the weighting estimators using both indicator and smoothed inclusion weights with \( \epsilon = 10^{-4} \). For the augmented weighting estimator, we consider the outcome model to be a linear regression model adjusting for all covariates, separately for \( a = 0, 1 \).
Table S2 shows the results from the four estimators for the average smoking effect and the average smoking effect on the smokers, based on the trimmed samples. The weighting estimators with indicator weight function are close to the counterparts with smoothed weight function, which have slightly smaller standard error. The augmented weighting estimators have smaller standard error than the non-augmented ones. From the results, on average, smoking increases the lead level in blood at least by 0.65 over the target population O. Moreover, smoking increases the lead level in blood at least by 0.79 for smokers in the target population with $e(x) < 0.7$.

S12 The Lalonde Data

We examine the Lalonde (1986) data to investigate the treatment effect of the National Support Work Demonstration, a labor training program, on postintervention earnings. The Lalonde data combines the treated units from a randomized evaluation of the National Support Work Demonstration with nonexperimental comparison units drawn from survey datasets. The data includes 185 treated units and 15,992 control units. The outcome variable $Y$ is the postintervention earnings at year 1978. The covariates $X$ include earnings and employment status at two preintervention years 1974 and 1975, education, age, indicators for black and Hispanic, and single versus married. This dataset has been analyzed by many researchers; see, e.g., Dehejia and Wahba (1999); Hainmueller (2012); Imbens and Rubin (2015).

Because the number of control units is much larger than the number of treated units, we first create a matched dataset where we match each treated unit with $M = 5$ control units using based on Mahalanobis distance matching on all covariates. We then apply our methods to the matched dataset. Following Hainmueller (2012), the propensity score is estimated by a logistic regression model with linear predictors including all covariates and their pairwise one-way interactions, and squared terms for age and years of education.

Even after matching, the overlap between the treated and the control is not satisfactory. One implication is that for the region in the right tail of the propensity score distribution, there is a limited number of control units. See Figure S4 for illustration of the estimated propensity score distribution by the treated and the control. To help address lack of overlap, units are trimmed if their estimated propensity score is greater than 0.78, obtained by the methods in the above section. This results in removal of 26 subjects, with 10 treated units and 16 control units. We consider the weighting estimators using both indicator and smoothed inclusion weights with $\epsilon = 10^{-4}$. For the augmented weighting estimator, we consider the outcome model to be a linear regression model adjusting for all covariates, and their pairwise one-way interactions, and squared terms for age and years of education, separately for the treated and the control.

Table S3 shows the results from the four estimators for the average treatment effect on the
Table S3: Results: estimate, standard error by 500 bootstrapping, and 95% confidence interval

|                | estimate | s.e. | 95% c.i.     |
|----------------|----------|------|--------------|
| Hain et al.    | 1571     | –    | (97, 3044)   |
| \( \hat{\tau}_{\text{ATT}}(\hat{\theta}) \) | 1506     | 404  | (733, 2321)  |
| \( \hat{\tau}_{\text{ATT}}^{\text{aug}}(\hat{\theta}) \) | 1312     | 404  | (503, 2121)  |
| \( \hat{\tau}_{\text{ATT},\epsilon}(\hat{\theta}) \) | 1527     | 397  | (697, 2314)  |
| \( \hat{\tau}_{\text{ATT},\epsilon}^{\text{aug}}(\hat{\theta}) \) | 1305     | 400  | (505, 2014)  |

Hain et al. is the results from Hainmueller (2012).

treated, based on the trimmed samples, along with Hainmueller (2012)'s results. The weighting estimators with indicator weight function are close to the counterparts with smoothed weight function, which have slightly smaller standard error. The augmented weighting estimators do not improve the precision of the non-augmented ones, likely because of the difficulty in specifying a correct model for the outcome. Our point estimates are close to the ones obtained by Hainmueller (2012), but the confidence intervals are much narrower, which is consistent with the theoretical result. From the results, on average, the National Support Work Demonstration increases the earning at least by 1305 over the target population \( O \) with \( e(X) < 0.78 \).

S13 Figures

This section presents figures mentioned in the manuscript.
Figure S1: Illustration of weight functions: the solid black line is the indicator weight function, the dash red line is the smoothed weight function with $\epsilon = 0.001$, and the dot blue line is the smoothed weight function with $\epsilon = 0.0001$. As $\epsilon \to 0$, the smoothed weight function converges to the indicator function.
Figure S2: Propensity score distribution by treatment under Propensity Score Design 1–4: Design 1, weak separation and linear predictor; Design 2, strong separation and linear predictor; Design 3, weak separation and non-linear predictor; Design 4, strong separation and non-linear predictor.
Figure S3: Propensity score distribution by $A$: the vertical solid lines mark the cut-off values 0.05 and 0.6 for estimating the average treatment effect, and the vertical dashed line marks the cut-off value 0.7 for estimating the average treatment effect on the treated.
Figure S4: Propensity score distribution by $A$: the vertical dashed line marks the cut-off value 0.75 for estimating the average treatment effect on the treated.