Methods for the Numerical Analysis of Boundary Value Problem of Partial Differential Equations Based on Kolmogorov Superposition Theorem

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Abstract

This research introduces a new method for the transition from partial to ordinary differential equations that is based on the Kolmogorov superposition theorem. In this paper, we discuss the numerical implementation of the Kolmogorov theorem and propose an approach that allows us to apply the theorem to represent partial derivatives of multivariate function as a combination of ordinary derivatives of univariate functions. We tested the method by running a numerical experiment with the Poisson equation. As a result, we managed to get a system of ordinary differential equations whose solution coincides with a solution of the initial partial differential equation.

Keywords: Kolmogorov superposition theorem, partial differential equations, boundary value problem of ordinary differential equations.
1 Introduction

Partial differential equations are a powerful instrument of mathematics that allows to model various processes of different complexities. PDEs are applied to many real-life problems. For example, this family of equations may be used for modeling blood flow in the human venous system or for price evaluation of some financial instrument. However, very often partial differential equations are very hard to solve. Some of them we even cannot solve at all, so we use modern methods of machine learning to approximate the solution. Obviously, bifurcation analysis of partial differential equations is even more complex task to solve and mathematicians have not managed to develop a general procedure for how to do it.

Ordinary differential equation is a much simpler class of equations. The number of processes that can be described by them is limited, but we can solve any type of them using methods of numerical integration and analyse behaviour of a dynamic system generated by them using bifurcation analysis.

Thus in this research, we worked on the derivation of the method for the transition from partial differential equations to equivalent ordinary differential equations, as this will allow us to apply all the existing methods of solution and analysis of ODEs to PDEs.

The main research aim of our work is to develop a method, based on the Kolmogorov superposition theorem, for the transition from partial to ordinary differential equations, such that we will be able not only to solve PDE’s using method of simple numerical integration, but we will be able to apply existing methods of bifurcation analysis to partial differential equations.

2 Problem Statement

The problem can be formulated as follows. Consider a boundary problem of partial differential equations that is given by (2.1) and corresponding boundary conditions (2.2)

\[ F\left(x_1, \ldots, x_n, f(x_1, \ldots, x_n), \frac{\partial f(x_1, \ldots, x_n)}{\partial x_{i_1}}, \frac{\partial^2 f(x_1, \ldots, x_n)}{\partial x_{i_1} \partial x_{i_2}}, \ldots, \frac{\partial^N f(x_1, \ldots, x_n)}{\partial x_{i_1} \ldots \partial x_{i_N}} \right) = 0 \] (2.1)

\[ \varphi_{k_1}\left(x_1, \ldots, x_n, f(x_1, \ldots, x_n), \frac{\partial f(x_1, \ldots, x_n)}{\partial x_{i_1}}, \frac{\partial^2 f(x_1, \ldots, x_n)}{\partial x_{i_1} \partial x_{i_2}}, \ldots, \frac{\partial^{N-1} f(x_1, \ldots, x_n)}{\partial x_{i_1} \ldots \partial x_{i_N}} \right) \bigg|_{x_{i_1}=x_{i_1}^{\min}} = 0 \] (2.2)

\[ \varphi_{k_2}\left(x_1, \ldots, x_n, f(x_1, \ldots, x_n), \frac{\partial f(x_1, \ldots, x_n)}{\partial x_{i_1}}, \frac{\partial^2 f(x_1, \ldots, x_n)}{\partial x_{i_1} \partial x_{i_2}}, \ldots, \frac{\partial^{N-1} f(x_1, \ldots, x_n)}{\partial x_{i_1} \ldots \partial x_{i_N}} \right) \bigg|_{x_{i_2}=x_{i_2}^{\max}} = 0 \]

where \(k_1, k_2 \in \mathbb{N}, t_1, t_2 \in \{1, \ldots, n\}, i_j \in \{1, \ldots, n\}, j = 1, \ldots, N\). We seek to find ordinary differential equation

\[ \tilde{F}\left(x, y(x), \frac{dy(x)}{dx}, \frac{d^2 y(x)}{dx^2}, \ldots, \frac{d^N y(x)}{dx^N} \right) = 0, \] (2.3)

such that solution of the equation coincides with the solution of initial problem (2.1, 2.2). To solve this kind of problems we developed a method based on KST\(^3\) that we present in the sections 4-7.

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1Partial differential equations
2Ordinary differential equation
3Kolmogorov superposition theorem
3 Related Works

3.1 Method for transition from partial to ordinary differential equations

As partial differential equations have been used for mathematical modeling in various fields for a long time, the idea of reducing them to ordinary differential equations have been studied by many mathematicians. In a review paper [2] group of researchers give a detailed review of existing methods for reduction of PDEs to ODEs. They carry various experiments in order to compare and outline pros and cons of each method. Two other mathematicians in their work [6] write about superiority of the Kontorovich method on other existing methods supporting their opinions with several numerical experiments. However, existing methods have their own disadvantages. Some of them use approximations that lead to uncertainties. Other methods that use orthogonal basis, such as Fourier based methods, break bifurcation picture of the equations making conclusions of bifurcation theory irrelevant. Our method is based on a strict equality that is known as Kolmorogov superposition theorem.

3.2 Kolmogorov superposition theorem

The method that we propose in this paper relies heavily on Sprecher’s formulation of the Kolmogorov superposition theory that he introduced in the article [7]. This formulation is more convenient in terms of numerical realisation. In the same article he proposes algorithms for calculating inner and outer functions that are required in order to apply the theorem on practice.

Despite that Sprecher’s formulation of the theorem was correct, the algorithm for calculating internal functions was wrong. For certain values of parameters the monotonicity property of these functions is violated. Another Mathematician Mario Koppen who also dedicated part of his work to study the Kolmogorov theorem, spotted this mistake and corrected. Proper algorithm for constructing inner functions was introduced in the paper [5].

Jonas Actor in his PhD thesis [1] on the study of the properties of the Kolmogorov theorem, give a review of existing formulations of Kolmogorov theorem, and proposes a new type of improved inner functions that are Lipschitz continuous. Initial functions that were proposed by David Sprecher have very steep slope on some segments. This can lead to computational problems. In contrast, Lipschitz continuous functions have more controlled slope.

T. Hedberb also studied properties of the Komogorov superposition theorem. In the work [3] he derived his own formulation of the theorem that does not use strictly defined inner functions. Instead, he have showed his reformulation is correct for quasi-all functions if only they satisfy certain conditions described in [3].

4 Numerical implementation of Kolmogorov superposition theorem

4.1 Theorem formulation review

The method for transition from partial to ordinary differential equations is based on Sprecher’s reformulation of Kolmogorov theorem. We start with providing a quick overview of the theorem. Consider a function of \( n \) variables \( f(x_1, \ldots, x_n) \). As stated in the paper [7], \( f(x_1, \ldots, x_n) \) can be written as a sum of functions of one variable [4.1]

\[
f(x_1, x_2, \ldots, x_n) = \sum_{q=0}^{2^n} \Phi_q \left[ \sum_{p=1}^n \alpha_p \psi(x_p + aq) \right],
\]

(4.1)
where constants $a, \alpha_p, p = 1, \ldots, n$ and inner function $\psi(x)$ are the same for different $f(x_1, \ldots, x_n)$. Moreover, function $\psi(x)$ is a specific function that can be calculated numerically using algorithm of Mario Koppen described in [5]. Initial algorithm proposed by David A. Sprecher contained a mistake for some combination of algorithm parameters. Further, we provide a quick review of a correct algorithm.

### 4.2 Numerical implementation of inner $\psi(x)$ function

Let’s introduce notation $D$ for the set of terminating rational numbers that are defined according to equation \[ (4.2) \]

$$d_k = \sum_{j=1}^{k} \frac{i_j}{\gamma^j}, i_j = 0, \ldots, \gamma - 1,$$

where $k, \gamma$ are parameters of the algorithm such that $\gamma \geq 2n + 2$ and $k \in \mathbb{N}$. Next, we define constants $a, \alpha_p$ according to the formulas \[ (4.3) \]

$$a = \frac{1}{\gamma(\gamma - 1)} \quad \alpha_p = \sum_{j=1}^{\infty} \gamma^{-(p-1)\frac{j}{n-1}}$$

Then, the inner function $\psi(x)$ is recursively defined by the formula \[ (4.4) \]

$$\psi^k(d_k) = \begin{cases} d_k, & \text{for } k = 1 \\ \psi^{k-1}(d_k - \frac{i_k}{\gamma^k}) + \frac{\psi^{k-1}(d_k + \frac{1}{\gamma^k})}{\gamma^{-(n-1)/(n-1)}} & \text{for } k > 1, i_k < \gamma - 1 \\ \frac{1}{2} \left( \psi(d_k - \frac{1}{\gamma}) + \psi(d_k + \frac{1}{\gamma}) \right) & \text{for } k > 1, i_k = \gamma - 1 \end{cases}$$

where $i_k$ are decimal integers of rational number $d_k = 0.i_1i_2\ldots i_k$. In most of the problems the function $f(x_1, \ldots, x_n)$ is defined beyond the unit cube. For $x > 1$, function $\psi(x)$ is calculated using formula \[ (4.5) \]

$$\psi(x) = \psi(x - \lfloor x \rfloor) + \lfloor x \rfloor,$$

where $\lfloor x \rfloor$ is the integer part of $x$.

According to the work [5] of Mario Koppen $\gamma = 10$ is the best value of parameter for $n = 2$. Graphs for function $\psi(x)$ for $k = 2, 3, 4$ and $\gamma = 10$ are depicted in the **Figure 1**. Function $\psi(x)$ is defined only at certain points $d_k$ but for the better representation we use linear interpolation.

**Figure 1**: $\psi(x)$ for $k = 2, 3, 4$ and $\gamma = 10$

As we are going to apply Kolmogorov theorem to the PDE problems, we should be able to define derivatives of inner functions. $\psi(x)$ cannot be analytically differentiated. Thus, we consider difference analogue of the derivative \[ (4.6) \]
\[ \psi'(x) = \frac{\psi(x + \Delta) - \psi(x)}{\Delta}, \]  

(4.6)

where \( \Delta \) is an increment that depends on the choice of the parameters \( \gamma \) and \( k \). As \( \psi(x) \) is defined only at a finite number of points \( d_k \), we take \( \Delta = \gamma^{-k} \) equal to the size of the step between two neighbour points from the set \( D_k \). Graphs of the first two derivate of the function \( \psi(x) \) with parameters \( k = 4, \gamma = 10 \) are depicted in the Figure 2.

![Graphs of ψ(x), ψ′(x), ψ''(x) with parameters k = 4, γ = 10.](image)

5 Representation of a multivariate function as a sum of functions of one variable

Transition from partial to ordinary differential equations consists of two main parts. First, we should represent function of several variable as a combination of functions of one variable. Second, substitute partial derivatives with corresponding ordinary. To complete the first stage, we use Kolmogorov theorem and the fact that \( a = \frac{1}{\gamma(\gamma - 1)} \) is a small parameter. For recommended value of \( \gamma = 10 \) for \( n = 2 \), \( a \) is equal to \( \frac{1}{90} \). As there are no upper bound for a value of \( \gamma \), we can consider taking \( \gamma \to \infty \), as a result \( a \to 0 \). Thus, we can consider taking Taylor series of (4.1) in the vicinity \( a = 0 \) (see equation (5.1)).

\[
\sum_{q=0}^{2n} \Phi_q \left( \sum_{p=0}^{n} \alpha_p \psi(x_p + qa) \right) = \sum_{m=0}^{M} \frac{a^m}{m! \, da^m} \left( \sum_{q=0}^{2n} \Phi_q \left( \sum_{i=p}^{n} \alpha_p \psi(x_p + aq) \right) \right) \bigg|_{a=0} \tag{5.1}
\]

In order to get representation of Taylor series with \( M \) terms of KST theorem, we use Bell’s polynomials and Faà di Bruno’s formula. Definition of both are provided below.

5.1 Bell’s polynomial

By definition, Bell’s polynomials are defined by the formula

\[
B_{n,k}(x_1, \ldots, x_{n-k+1}) = \sum \frac{n!}{j_1! j_2! \cdots j_{n-k+1}!} \left( \prod_{i=1}^{n-k+1} \left( \frac{x_i}{i!} \right)^{j_i} \right).
\]  

(5.2)

where outer sum is taken over all non-negative numbers \( j_1, j_2, \ldots, j_{n-k+1} \) that satisfy two conditions (5.3, 5.4).

\[
\sum_{i=1}^{n-k+1} j_i = k
\]  

(5.3)

\[
\sum_{i=1}^{n-k+1} ij_i = n
\]  

(5.4)
5.2 Fa`a di Bruno’s formula

Fa`a di Bruno’s formula is given by the equation (5.5)

\[
\frac{d^m}{dx^m} f(g(x)) = \sum \left( \frac{m!}{j_1!j_2! \ldots j_m!} \cdot f^{(j_1 + \ldots + j_m)}(g(x)) \prod_{i=1}^{m} \left( \frac{g^{(i)}(x)}{i!} \right)^{j_i} \right),
\]  

(5.5)

where external sum is taken over all nonnegative integers that satisfy second condition of Bell’s polynomial (5.4). It is used to define high-order derivatives of complex functions. Using Bell’s polynomials, we can rewrite formula (5.5) in the following form.

\[
\frac{d^m}{dx^m} f(g(x)) = \sum_{k=0}^{m} f^{(k)}(g(x)) B_{m,k}(g'(x), g''(x), \ldots, g^{(m-k-1)}(x))
\]  

(5.6)

5.3 Taylor series of Kolmogorov superposition theorem

We can substitute formula (5.6) to achieve compact representation of Taylor series (5.1) as shown in the equation (5.7)

\[
\frac{d^m}{d\alpha^m} \left( \sum_{q=0}^{n} \Phi_q \left( \sum_{p=1}^{n} \alpha_p \psi(x_p + aq) \right) \right) = \sum_{k=0}^{m} \sum_{q=0}^{n} \Phi_q \left( \sum_{p=1}^{n} \alpha_p \psi(x_p) \right) B_{m,k}(q^1 \sum_{p=1}^{n} \alpha_p \psi'(x_p), q^2 \sum_{p=1}^{n} \alpha_p \psi''(x_p), \ldots, q^m \sum_{p=1}^{n} \alpha_p \psi^{(m-k-1)}(x_p))
\]  

(5.7)

Using definition of Bell’s polynomial and applying sequence of simple transformations, we get more convenient representation (5.8).

\[
B_{m,k}(q \sum_{p=1}^{n} \alpha_p \psi'(x_p), q^2 \sum_{p=1}^{n} \alpha_p \psi''(x_p), \ldots, q^m \sum_{p=1}^{n} \alpha_p \psi^{(m-k-1)}(x_p)) = \sum \left( \frac{m!}{j_1!j_2! \ldots j_{m-k+1}!} \prod_{i=1}^{m-k+1} \left( \frac{q^i \sum_{p=1}^{n} \alpha_p \psi^{(i)}(x_p)}{i!} \right)^{j_i} \right) = q^m \tilde{B}_{m,k}(x_1, x_2, \ldots, x_n)
\]  

(5.8)

where

\[
\tilde{B}_{m,k}(x_1, x_2, \ldots, x_n) = \sum \left( \frac{m!}{j_1!j_2! \ldots j_{m-k+1}!} \prod_{i=1}^{m-k+1} \left( \frac{\sum_{p=1}^{n} \alpha_p \psi^{(i)}(x_p)}{i!} \right)^{j_i} \right)
\]  

(5.9)

Then, Taylor series of KST representation of function of \( n \) variables with arbitrary number of terms \( (M) \) can be represented with the equation (5.10)

\[
\sum_{q=0}^{2n} \Phi_q \left( \sum_{p=0}^{n} \alpha_p \psi(x_p + qa) \right) = \sum_{m=0}^{M} \sum_{k=0}^{m} \tilde{B}_{m,k}(x_1, \ldots, x_n) \sum_{q=0}^{2n} \frac{a^m q^n}{m!} \Phi_q^{(k)} \left( \sum_{p=1}^{n} \alpha_p \psi(x_p) \right)
\]  

(5.10)

Finally, by making a substitution \( z = \sum_{p=1}^{n} \alpha_p \psi(x_p) \), we derive representation of multivariate \( f(x_1, x_2, \ldots, x_n) \) as a sum of univariate functions \( \Phi_q(z) \) (see equation (5.11)).

\[
f(x_1, x_2, \ldots, x_n) = \sum_{m=0}^{M} \sum_{k=0}^{m} \tilde{B}_{m,k}(x_1, x_2, \ldots, x_n) \sum_{q=0}^{2n} \frac{a^m q^n}{m!} \Phi_q^{(k)}(z),
\]  

(5.11)
6 Method for transition from partial to ordinary differential equations

Method for transition from partial to ordinary differential equations that we derived can be applied only to variational formulation of PDE problem. Let’s consider functional \( \Xi \)

\[
\Xi = \int \mathcal{L}(x_1, \ldots, x_n, f(x_1, \ldots, x_n), \frac{\partial f(x_1, \ldots, x_n)}{\partial x_1}, \ldots, \frac{\partial^N f(x_1, \ldots, x_n)}{\partial x_1 \ldots \partial x_N}) dx
\]  

(6.1)

such that solution of variational problem (6.2) coincide with the solution of initial PDE problem (2.1, 2.2).

\[
\Xi \longrightarrow \text{extr}
\]  

(6.2)

In order to get rid of partial derivatives in functional, we use obtained formula (6.11). First, we change variable of integrations. We parameterise \( x_2, \ldots, x_n \) and \( x_1 \) express through other parametrised variables as shown in (6.3).

\[
x_2 = \tilde{x}_2
\]

\[
\ldots \ldots 
\]

\[
x_n = \tilde{x}_n
\]

\[
x_1 = \psi^{-1} \left( \frac{z - \sum_{p=2}^{n} \alpha_p \psi(x_p)}{\alpha_1} \right)
\]

(6.3)

Boundaries of integration change according to formulas (6.4)

\[
x_1^{\min} \mapsto z^{\min} = \alpha_1 \psi(x_1^{\min}) + \alpha_2 \psi(\tilde{x}_2) + \cdots + \alpha_n \psi(\tilde{x}_n)
\]

\[
x_1^{\max} \mapsto z^{\max} = \alpha_1 \psi(x_1^{\max}) + \alpha_2 \psi(\tilde{x}_2) + \cdots + \alpha_n \psi(\tilde{x}_n)
\]

(6.4)

and integration factor according to determinant of Jacobian matrix (6.5). Determinant of a diagonal matrix equal to product of diagonal elements.

\[
J = \begin{bmatrix}
z'_1 x_1 & z'_2 x_2 & \cdots & z'_n x_n \\
x'_1 x_1 & x'_2 x_2 & \cdots & x'_n x_n \\
\vdots & \vdots & \ddots & \vdots \\
x'_n x_1 & x'_n x_2 & \cdots & x'_n x_n \\
\end{bmatrix} = \begin{bmatrix}
\alpha_1 \psi'(x_1) & \alpha_2 \psi'(x_2) & \cdots & \alpha_n \psi'(x_n) \\
0 & 1 & \cdots & 0 \\
\cdots & \cdots & \ddots & \cdots \\
0 & 0 & \cdots & 1 \\
\end{bmatrix}
\]

(6.5)

\[
\Rightarrow dx_1 = \frac{dz}{\alpha_1 \psi' \circ \psi^{-1} \left( \frac{z - \sum_{p=2}^{n} \alpha_p \psi(x_p)}{\alpha_1} \right)}
\]

Secondly, we redefine partial derivatives of \( f(x_1, \ldots, x_n) \) as shown in (6.6)

\[
\frac{\partial^N f(x_1, \ldots, x_n)}{\partial x_1 \ldots \partial x_N} = \sum_{m=0}^{M} \sum_{k=0}^{m} \left[ \sum_{j=0}^{N} \frac{\partial^N}{\partial x_1 \ldots \partial x_N} \left( \tilde{B}_{m,k}(x) \sum_{q=0}^{2n} \frac{a^m q^m}{m!} \Phi_q(z) \right) \right]
\]

(6.6)

where \( \tilde{B}_{m,k} \) and \( z \) are known from KST theorem. Thus, in terms of unknown functions \( \Phi_q(z) \) we get expression that contains only ordinary derivatives. Applying described procedure to the functional (6.1), we derive equivalent functional (6.7) that does not contain partial derivatives.

\[
\tilde{\Xi} = \int \int_{z^{\min}}^{z^{\max}} L(z, \tilde{x}_2, \ldots, \tilde{x}_n, \Phi_0(z), \ldots, \Phi_{2n}(z), \frac{d\Phi_0(z)}{dz}, \ldots, \frac{d\Phi_{2n}(z)}{dz}, \frac{d^N \Phi_0(z)}{dz^N}, \ldots, \frac{d^N \Phi_{2n}(z)}{dz^N}) dz d\tilde{x}
\]

(6.7)
Then, initial PDE problem is reduced to a much simpler variational problem with ordinary derivatives (6.8). This problem can be solved using standard methods of calculus of variations.

∀{\tilde{x}_2, \ldots, \tilde{x}_n}, \tilde{x}_2 \in [\tilde{x}_2^{\min}, \tilde{x}_2^{\max}], \ldots, \tilde{x}_n \in [\tilde{x}_n^{\min}, \tilde{x}_n^{\max}]:

\tilde{\Xi} \rightarrow \text{extr} (6.8)

7 Numerical experiment with Poisson equation

7.1 Equivalent variational formulation

In order to check our method, we decided to apply it to the partial differential equation with known solution which is the Poisson equation. Let’s consider the following formulation (7.1) with boundary conditions (7.2).

∇^2 u(x_1, x_2) = \sin(\pi x_1) \sin(\pi x_2) (7.1)

u(0, x_2) = 0, u(1, x_2) = 0, u(x_1, 0) = 0, u(x_1, 1) = 0 (7.2)

Solution for problem (7.1, 7.2) is given by equation (7.3).

\begin{align*}
\quad \quad \quad \quad \quad u^*(x_1, x_2) = \frac{\sin(\pi x_1) \sin(\pi x_2)}{-2\pi^2} (7.3)
\end{align*}

One can easily check that (7.3) satisfies both equation and boundary conditions. As shown in Section 6, we should first find equivalent variational formulation. For PDE problem (7.1, 7.2) corresponding problem of calculus of variations is given by (7.4). The whole derivation procedure is shown in the appendix A.

\begin{align*}
\int_0^1 \int_0^1 \left[ -\left( \frac{\partial u(x_1, x_2)}{\partial x_1} \right)^2 - \left( \frac{\partial u(x_1, x_2)}{\partial x_2} \right)^2 - 2 \sin(\pi x_1) \sin(\pi x_2) u(x_1, x_2) - \\
- 2 \frac{\partial^2 u(x_1, x_2)}{\partial x_1^2} u(x_1, x_2) - 2 \frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} u(x_1, x_2) \right] dx_1 dx_2 \rightarrow \text{extr} (7.4)
\end{align*}

7.2 ODE boundary problem of Poisson equation

In order to derive corresponding system of ordinary differential equations, we apply the procedure that we described in section 6. Let’s consider the simplest case when we remove all terms of Taylor series except the first one. Then, according to (5.11) for \( n = 2, M = 0 \) function \( u(x_1, x_2) \) can be written as (7.5)

\begin{align*}
u(x_1, x_2) = \sum_{q=0}^{1} \Phi_q(z), (7.5)
\end{align*}

where \( z = \alpha_1 \psi(x_1) + \alpha_2 \psi(x_2) \). Further, following algorithm described above we apply substitutions (7.6)

\begin{align*}
x_2 = \tilde{x}_2 \\
x_1 = \psi^{-1} \left( \frac{z - \alpha_2 \psi(\tilde{x}_2)}{\alpha_1} \right) (7.6)
\end{align*}

and change factor and boundaries of integration (7.7, 7.8).

\begin{align*}
x_1 = 0 \rightarrow z^{\min} = \alpha_2 \psi(\tilde{x}_2) \\
x_1 = 1 \rightarrow z^{\max} = \alpha_1 + \alpha_2 \psi(\tilde{x}_2) (7.7)
\end{align*}
Next step is to define partial derivatives up to second through ordinary derivatives of functions \( \Phi_q(z) \) (see equation (7.9)).

\[
\frac{\partial u(x_1, x_2)}{\partial x_i} = \alpha_i \psi'(x_1) \sum_{q=0}^{4} \Phi_q'(z) \\
\frac{\partial^2 u(x_1, x_2)}{\partial x_i^2} = \alpha_i \psi''(x_1) \sum_{q=0}^{4} \Phi_q'(z) + \alpha_i^2 \psi^2(x_1) \sum_{q=0}^{4} \Phi_q''(z)
\]

As a result, using all above substitutions, we reformulate PDE Poisson problem as a problem of finding extremum of a functional that contains only ordinary derivatives (7.10)

\[
\int_0^1 \int_{z_{\text{min}}}^{z_{\text{max}}} \frac{1}{\alpha_1 \psi'(x_1)} \left[ - \left( \alpha_1 \psi' \psi^{-1} \left( \frac{\alpha_2 \psi'(x_2)}{\alpha_1} \right) \sum_{q=0}^{4} \Phi_q'(z) \right)^2 - \left( \alpha_2 \psi' \sum_{q=0}^{4} \Phi_q'(z) \right)^2 + 2 \left( \sum_{q=0}^{4} \Phi_q(z) \right) \cdot \left[ f \left( \psi^{-1} \left( \frac{\alpha_2 \psi'(x_2)}{\alpha_1} \right) \right) \right] \right] dz dx_2 \rightarrow \text{extr}
\]

In order to solve problem (7.10) we should take variational derivatives with respect to each independent variable and equate to zero. First, we should identify the number of independent variables by varying functional with respect to each of the functions \( \Phi_q(z) \) and identify rank of the matrix of coefficients of the highest derivatives. To make formulas more readable, we use notation \( \psi^{-1} \left( \frac{\alpha_2 \psi'(x_2)}{\alpha_1} \right) = \tilde{x}_1 \). Then, variational derivative with respect to \( \Phi_1(z) \) is equal to (7.11).

\[
\int_0^1 \int_{z_{\text{min}}}^{z_{\text{max}}} \left[ f \left( \tilde{x}_1, \tilde{x}_2 \right) \frac{\alpha_1^2 \tilde{x}_1 \psi'(\tilde{x}_1) + \alpha_1^2 \psi^2(\tilde{x}_2)}{\alpha_1^2 \psi'(\tilde{x}_1)} \sum_{q=0}^{4} \Phi_q''(z) - \frac{\alpha_1^2 \psi^2(\tilde{x}_1) \psi''(\tilde{x}_1)}{\alpha_1^2 \psi'(\tilde{x}_1)} - \frac{\alpha_1 \alpha_2 \psi^2(\tilde{x}_2) \psi''(\tilde{x}_2)}{\alpha_1^2 \psi'(\tilde{x}_1)} \sum_{q=0}^{4} \Phi_q'(z) - \frac{\alpha_2^2 \psi^2(\tilde{x}_2)}{\alpha_1^2 \psi'(\tilde{x}_1)} \left( \psi'(\tilde{x}_1) - \psi'(\tilde{x}_2) \right) \right] \delta \Phi_1(z) dz d\tilde{x}_2
\]

\[
= \int_0^1 \left[ \frac{\alpha_2 \left( \alpha_2 \psi^2(\tilde{x}_2) \psi''(\tilde{x}_1) + \alpha_1 \psi^2(\tilde{x}_1) \psi''(\tilde{x}_2) \right)}{\alpha_1^2 \psi'(\tilde{x}_1)} \sum_{q=0}^{4} \Phi_q(z) \delta \Phi_1(z) \right]_{z_{\text{min}}}^{z_{\text{max}}} + \left[ \alpha_1 \psi'(\tilde{x}_1) + \frac{\alpha_2^2 \psi^2(\tilde{x}_2)}{\alpha_1^2 \psi'(\tilde{x}_1)} \sum_{q=0}^{4} \Phi_q(z) \delta \Phi_1(z) \right]_{z_{\text{min}}}^{z_{\text{max}}} d\tilde{x}_2 = 0
\]

We can see that coefficient of \( \Phi_1''(z) \) does not depend on the variable of variation. This implies that rank of the matrix of coefficients is 1 and that there is only one independent variable. We define this variable according to
Making this substitution and varying functional with respect to $U(z)$, we get

$$
\int_0^1 \int_{z_{\text{min}}}^{z_{\text{max}}} \left[ \frac{f(\tilde{x}_1, \tilde{x}_2)}{\alpha_1 \psi'(\tilde{x}_1)} - \frac{\alpha_2^2 \psi'(\tilde{x}_1) \psi''(\tilde{x}_1) - \alpha_2^2 \psi'(\tilde{x}_2) \psi''(\tilde{x}_2)}{\alpha_1 \psi'(\tilde{x}_1)} U'(z) - \alpha_1 \alpha_2 \psi'(\tilde{x}_1) \psi''(\tilde{x}_2) + \frac{\alpha_2^2 \psi'(\tilde{x}_2) (3 \psi''(\tilde{x}_1) - \psi(\tilde{x}_1) \psi''(\tilde{x}_1))}{\alpha_1 \psi'(\tilde{x}_1)} U(z) \right] \delta \Phi_i(z) d\tilde{z} d\tilde{x}_2 - \int_0^1 \left[ \frac{\alpha_2^2 (\alpha_2 \psi'(\tilde{x}_2) \psi''(\tilde{x}_1) + \alpha_1 \psi'(\tilde{x}_1) \psi''(\tilde{x}_2))}{\alpha_1 \psi'(\tilde{x}_1)} U(z) \right] \delta U(z) \bigg|_{z_{\text{min}}}^{z_{\text{max}}} d\tilde{x}_2 = 0
$$

(7.13)

Variation of functional is equal to zero if and only if integrand and boundary conditions are equal to zero. Then, we get ordinary differential equation of second degree

$$\frac{\alpha_2^2 \psi'(\tilde{x}_2) + \alpha_1 \psi'(\tilde{x}_1) \psi''(\tilde{x}_1)}{\alpha_1 \psi'(\tilde{x}_1)} U''(z) + \frac{\alpha_2^2 \psi'(\tilde{x}_1) \psi''(\tilde{x}_1) - \alpha_2^2 \psi'(\tilde{x}_2) \psi''(\tilde{x}_2)}{\alpha_1 \psi'(\tilde{x}_1)} U'(z) + \frac{\alpha_1 \alpha_2 \psi'(\tilde{x}_1) \psi''(\tilde{x}_2) + \alpha_2^2 \psi'(\tilde{x}_2) (3 \psi''(\tilde{x}_1) - \psi(\tilde{x}_1) \psi''(\tilde{x}_1))}{\alpha_1 \psi'(\tilde{x}_1)} U(z) - \frac{f(\tilde{x}_1, \tilde{x}_2)}{\alpha_1 \psi'(\tilde{x}_1)} = 0
$$

(7.14)

with boundary conditions

$$\begin{bmatrix}
\alpha_2^2 \psi'(\tilde{x}_2) \psi''(0) + \frac{\alpha_2^2 \psi'(\tilde{x}_1) \psi''(\tilde{x}_1)}{\alpha_1 \psi'(\tilde{x}_1)} U'(0) + \frac{\alpha_2^2 \psi'(\tilde{x}_2)}{\alpha_1 \psi'(\tilde{x}_1)} U(0) + \frac{\alpha_2^2 \psi'(\tilde{x}_2)}{\alpha_1 \psi'(\tilde{x}_1)} U(\alpha_2 \psi(\tilde{x}_2)) = 0 \\
\alpha_2^2 \psi'(\tilde{x}_2) \psi''(1) + \frac{\alpha_2^2 \psi'(\tilde{x}_1) \psi''(\tilde{x}_1)}{\alpha_1 \psi'(\tilde{x}_1)} U'(1) + \frac{\alpha_2^2 \psi'(\tilde{x}_2)}{\alpha_1 \psi'(\tilde{x}_1)} U(\alpha_1 + \alpha_2 \psi(\tilde{x}_2)) = 0
\end{bmatrix}
$$

(7.15)

Making substitution $U'(z) = W(z)$, we get boundary problem of ordinary differential equations of the first degree that can be solved using standard methods of numerical integration.
7.3 Results of simulation

Chosen Poisson equation is two dimensional \( (n = 2) \), then as stated in KST \( \gamma \geq 6 \). We choose \( \gamma = 10 \) and other parameters are defined as follows: \( \alpha = \frac{1}{90}, \alpha_1 = 1, \alpha_2 = 0.10100010000000001 \). For sake of simplicity, we choose \( k = 1 \), as for this value of \( k \), inner function \( \psi(x) \) is a simple identity function. Using linear interpolation, we get \( \psi(x) = x \) as shown in Figure 3.

\[
\begin{align*}
\text{Figure 3: } & \psi(x) \text{ for } k = 1, \gamma = 10. \text{ Orange dots are points in which function } \psi(x) \text{ is defined. Black line is the result of linear interpolation.} \\
\end{align*}
\]

To solve boundary problem (7.16, 7.15), we use Newton-Raphson method as by introducing slight modifications it allows us to carry bifurcation analysis for boundary value problems of ordinary differential equations. The results of simulation are shown in the Figure 4.

\[
\begin{align*}
\text{Figure 4: } & \text{Solutions of Poisson equation. Blue graph is a solution of system (7.16, 7.15). Orange graph is a theoretical solution of Poisson equation (7.1, 7.2).} \\
\end{align*}
\]

We can see that even using only first term in Taylor series and the simplest variant of inner function \( \psi(x) \), we managed to reduce partial differential equation to the system of ordinary differential equation with the same solution.
8 Conclusion

In this paper, we introduced a new method for the transition from partial to ordinary differential equations. In comparison with existing methods, ours does not use approximation. Instead, it is based on the strict equality - Kolmogorov superposition theorem. We tested the proposed method on the Poisson equation and compared the solution of the resulting ODE system with the solution of the initial partial differential equation. In the zero approximation solutions coincided.

Further, we plan to test the proposed method on non-linear partial differential equations with known solution, such as Karman equation. Secondly, we want to apply this method together with bifurcation analysis to make sure that the method does not influence the behavior of the dynamic system generated by the equation.

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Appendix

A. Variational formulation of Poisson equation

In order to find variational formulation of Poisson equation we start with functional (A.1).

\[
\int_0^1 \int_0^1 \left[ \left( \frac{\partial u(x_1, x_2)}{\partial x_1} \right)^2 + \left( \frac{\partial u(x_1, x_2)}{\partial x_2} \right)^2 - 2 \sin(\pi x_1) \sin(\pi x_2) u(x_1, x_2) \right] dx_1 dx_2 \rightarrow extr
\] (A.1)

To check whether it satisfies partial differential equation (7.1) with boundary conditions (7.2), we solve the given problem by taking variational derivative with respect to \( u(x_1, x_2) \) and equate it to zero.

\[
\int_0^1 \int_0^1 \left[ 2 \frac{\partial u(x_1, x_2)}{\partial x_1} \delta u'_{x_1}(x_1, x_2) + 2 \frac{\partial u(x_1, x_2)}{\partial x_2} \delta u'_{x_2}(x_1, x_2) - 2 \sin(\pi x_1) \sin(\pi x_2) \delta u(x_1, x_2) \right] dx_1 dx_2 = 0
\] (A.2)

Next, we get rid of derivatives in variations using integration by parts.

\[
\int_0^1 \int_0^1 \left[ 2 \nabla^2 u(x_1, x_2) \delta u - 2 \sin(\pi x_1) \sin(\pi x_2) \delta u(x_1, x_2) \right] dx_1 dx_2 + 2 \int_0^1 \frac{\partial u(x_1, x_2)}{\partial x_2} \delta u(x_1, x_2) \bigg|_0^1 dx_1 + 2 \int_0^1 \frac{\partial u(x_1, x_2)}{\partial x_1} \delta u(x_1, x_2) \bigg|_0^1 dx_2 = 0
\] (A.3)

Integrand satisfies initial PDE (7.1) but natural boundary conditions do not match with (7.2). Let’s add boundary terms that do not affect integrand part but change the boundary conditions so that they coincide with the PDE initial conditions (7.2). As a result, we get functional (A.4).

\[
\int_0^1 \int_0^1 \left[ \left( \frac{\partial u(x_1, x_2)}{\partial x_1} \right)^2 + \left( \frac{\partial u(x_1, x_2)}{\partial x_2} \right)^2 - 2 \sin(\pi x_1) \sin(\pi x_2) u(x_1, x_2) \right] dx_1 dx_2 - \int_0^1 2 \frac{\partial u(x_1, x_2)}{\partial x_1} u(x_1, x_2) \bigg|_0^1 dx_2 - \int_0^1 2 \frac{\partial u(x_1, x_2)}{\partial x_2} u(x_1, x_2) \bigg|_0^1 dx_1
\] (A.4)

In order to apply our method for transition from partial to ordinary differential equations, we drag boundary terms under the integral. Finally, we get equivalent variational formulation for Poisson equation.

\[
\int_0^1 \int_0^1 \left[ - \left( \frac{\partial u(x_1, x_2)}{\partial x_1} \right)^2 - \left( \frac{\partial u(x_1, x_2)}{\partial x_2} \right)^2 - 2 \sin(\pi x_1) \sin(\pi x_2) u(x_1, x_2) \right. \\
\left. - 2 \frac{\partial^2 u(x_1, x_2)}{\partial x_1^2} u(x_1, x_2) - 2 \frac{\partial^2 u(x_1, x_2)}{\partial x_2^2} u(x_1, x_2) \right] dx_1 dx_2 \rightarrow extr
\] (A.5)