A New Extension to the Intuitionistic Fuzzy Metric-like Spaces

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Abstract: In this manuscript, we introduce the concept of intuitionistic fuzzy controlled metric-like spaces via continuous t-norms and continuous t-conorms. This new metric space is an extension to intuitionistic fuzzy controlled metric-like spaces, controlled metric-like spaces and controlled fuzzy metric spaces, and intuitionistic fuzzy metric spaces. We prove some fixed-point theorems and we present non-trivial examples to illustrate our results. We used different techniques based on the properties of the considered spaces notably the symmetry of the metric. Moreover, we present an application to non-linear fractional differential equations.

Keywords: symmetric metric spaces; fuzzy metric spaces; fixed point theorems; non-linear fractional differential equations

1. Introduction

In today’s multifaceted environment, uncertainty and fuzziness are widespread in many applications. Zadeh [1] pioneered the concept of fuzzy sets (FSs) to capture the ambiguity and fuzziness of information. Since its origin, many extensions of FSs have been proposed to better represent sophisticated information, including intuitionistic fuzzy sets (IFSs), picture FSs, q-rung orthopair FSs, and neutrosophic sets.

Recently, Harandi [2] initiated the concept of metric-like spaces, which generalized the notion of metric spaces in a nice way. Alghamdi et al. [3] used the concept metric-like spaces to introduce the notion of b-metric-like spaces. Mlaiki [4] introduced the concept of controlled metric type spaces and proved various results. Mlaiki et al. [5] proposed the notion of controlled metric-like spaces (CMLSs) and proved several results for contractive mappings.

Kramosil and Michalek [6] proposed the approach of fuzzy metric spaces (FMSs), while George and Veeramani [7] introduced the concept of FMSs. Garbicec [8] gave the fuzzy interpretation of the Banach contraction principle in FMSs. Dey and Saha [9] gave an extension of the Banach fixed point theorem in the context of FMSs. Nadabhan [10] introduced the notion of fuzzy b-metric spaces (FBMSs) and proved several theorems. Schweizer and Sklar [11] carried out some work for statistical metric spaces. Gregory and Sapena [12] proved various fixed-point results in the context of FMSs. Mehmood et al. [13] initiated the notion of fuzzy extended b-metric space (FEBMS). Recently, Sezen [14]...
generalized the concept of controlled type metric spaces and introduced the concept of Controlled fuzzy metric spaces (CFMS). In this sequel, Shukla and Abbas [15] generalized the concept of metric-like spaces and introduced fuzzy metric-like spaces (FMLSs). Javed et al. [16] proposed fuzzy b-metric-like spaces. Shukla et al. [17] proposed an amazing notion of I-M complete FMSs and proved various theorems.

The approach of intuitionistic fuzzy metric spaces (IFMSs) via continuous t-norms and continuous t-conorms was presented by Park in [18]. Rafi and Noorani [19] proved several fixed-point results in the context of IFMSs. Sintunavarat and Kumam [20] proved fixed point theorems for a generalized intuitionistic fuzzy contraction in IFMSs. Later, Konwar [21] presented intuitionistic fuzzy b-metric space (IFBMS). Alaca et al. [22] and Mohamad [23] proved several fixed-point results. Saadati and Park [24] did amazing work on intuitionistic fuzzy topological spaces. In addition, Sezen in [14], introduced the concept of controlled fuzzy metric spaces.

The goal of this manuscript is to introduce intuitionistic fuzzy controlled metric-like spaces (IFCMLSs) by using the approach in [5], also to extend various fixed point (FP) results for contraction mappings, which is an improvement of the present literature's methodology using different techniques based on the properties of contractions and the considered metric such as the triangle inequality and the symmetry. In closing, and inspired by work carried out in [25–29], we present an application of our results to fractional differential equations.

2. Preliminaries

Now, we start this section by listing various helpful definitions for readers and \( I = [0,1] \) be used in this study.

**Definition 1** [1]. A binary operation \(* : I \times I \rightarrow I\) is called a continuous t-norm (CTN) if:

(a1) \( v \ast \omega = \omega \ast v, \forall v, \omega \in I \);

(b1) \( * \) is continuous.

(c1) \( v \ast 1 = v, \forall v \in I \);

(d1) \( (v \ast \omega) \ast \kappa = v \ast (\omega \ast \kappa), \forall v, \omega, \kappa \in I \);

(e1) If \( v \leq \kappa \) and \( \omega \leq d \), with \( v, \kappa, d \in I \), then \( v \ast \omega \leq \kappa \ast d \).

**Definition 2** [1]. A binary operation \( \circ : I \times I \rightarrow I\) is called a continuous t-conorm (CTCN) if:

(i). \( v \circ \omega = \omega \circ v, \text{ for all } v, \omega \in I \);

(ii). \( \circ \) is continuous.

(iii). \( v \circ 0 = 0, \text{ for all } v \in I \);

(iv). \( (v \circ \omega) \circ \kappa = v \circ (\omega \circ \kappa), \text{ for all } v, \omega, \kappa \in I \);

(v). If \( v \leq \kappa \) and \( \omega \leq d \), with \( v, \omega, \kappa, d \in I \), then \( v \circ \omega \leq \kappa \circ d \).

**Definition 3** [23]. Let \( K \neq \emptyset \). A mapping \( L : K \times K \rightarrow [1, \infty) \), fulfilling the following assertions:

a. \( L(p, \bar{a}) = 0 \) implies \( \bar{p} = \bar{a} \);

b. \( L(p, \bar{a}) = L(\bar{a}, \bar{p}) \);

c. \( L(p, \bar{a}) \leq L(p, \bar{d}) + L(\bar{d}, \bar{p}) \) for all \( p, \bar{a}, \bar{d} \in K \). Then \( (K, L) \) is called a metric-like space.

**Definition 4** [24]. Let \( K \neq \emptyset \). A function \( \psi : K \times K \rightarrow [1, \infty) \) and a mapping \( L : K \times K \rightarrow \mathbb{R}^+ \), fulfilling the following assertions:

I. \( L(p, \bar{a}) = 0 \) implies \( \bar{p} = \bar{a} \);

II. \( L(p, \bar{a}) = L(\bar{a}, \bar{p}) \);

III. \( L(p, \bar{a}) \leq \psi(p, \bar{d})L(p, \bar{d}) + \psi(\bar{d}, \bar{p})L(\bar{d}, \bar{p}) \) for all \( p, \bar{a}, \bar{d} \in K \). Then \( (K, L) \) is named a CMSL.

**Definition 5** [21]. Let \( K \neq \emptyset \). Suppose \( \ast \) be a CTN and \( \mathbb{N}_b \) be a FS on \( K \times K \times (0, \infty) \). A three tuple \( (K, \mathbb{N}_b, \ast) \) is called FMLS, if it is fulfilling the following assertions, for all \( p, \bar{a} \in K \) and \( \bar{d}, \sigma \geq 0 \):

(F1). \( \mathbb{N}_b(p, \bar{a}, \sigma) > 0 \);
(F2). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) = 1 \) implies \( p = \bar{a} \);
(F3). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) = \mathbb{K}_\phi(\bar{a}, p, \sigma) \);
(F4). \( \mathbb{K}_\phi(p, \delta, (\sigma + \theta)) \geq \mathbb{K}_\phi(p, \bar{a}, \sigma^\ast) \mathbb{K}_\phi(\bar{a}, \delta, \theta) \);
(F5). \( \mathbb{K}_\phi(p, \bar{a}, \cdot) : (0, \infty) \to [0, 1] \) is continuous.

**Definition 6** [4]. Let \( K \neq \emptyset \). Suppose \( * \) be a CTN, \( \circ \) be a CTCN, \( b \geq 1 \) and \( \mathbb{K}_\phi, \mathbb{R}_\phi \) be FSs on \( K \times K \times (0, \infty) \). If \( (K, \mathbb{K}_\phi, \mathbb{R}_\phi, *, \circ) \) verifies the following for all \( p, \bar{a} \in K \) and \( \delta, \sigma > 0 \):

(C1). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) + \mathbb{R}_\phi(p, \bar{a}, \sigma) \leq 1 \);
(C2). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) > 0 \);
(C3). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) = 1 \iff p = \bar{a} \);
(C4). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) = \mathbb{K}_\phi(\bar{a}, p, \sigma) \);
(C5). \( \mathbb{K}_\phi(p, \delta, b(\sigma + \theta)) \geq \mathbb{K}_\phi(p, \bar{a}, \sigma) \mathbb{K}_\phi(\bar{a}, \delta, \theta) \);
(C6). \( \mathbb{K}_\phi(p, \bar{a}, \cdot) \) is a non-decreasing (ND) function of \( \mathbb{R}^+ \) and \( \lim_{\sigma \to \infty} \mathbb{K}_\phi(p, \bar{a}, \sigma) = 1 \);
(C7). \( \mathbb{R}_\phi(p, \bar{a}, \sigma) > 0 \);
(C8). \( \mathbb{R}_\phi(p, \bar{a}, \sigma) = 0 \iff p = \bar{a} \);
(C9). \( \mathbb{R}_\phi(p, \bar{a}, \sigma) = \mathbb{R}_\phi(\bar{a}, p, \sigma) \);
(C10). \( \mathbb{R}_\phi(p, \delta, b(\sigma + \theta)) \leq \mathbb{R}_\phi(p, \bar{a}, \sigma) \mathbb{R}_\phi(\bar{a}, \delta, \theta) \);
(C11). \( \mathbb{R}_\phi(p, \bar{a}, \cdot) \) is a non-increasing (NI) function of \( \mathbb{R}^+ \) and \( \lim_{\sigma \to \infty} \mathbb{R}_\phi(p, \bar{a}, \sigma) = 0 \).

Then \( (K, \mathbb{K}_\phi, \mathbb{R}_\phi, *, \circ) \) is an IFBMS.

### 3. Main Results

In this section, we present the concept of an IFCMS and prove several FP results.

**Definition 7.** Suppose \( K \neq \emptyset \), assume a five tuple \( (K, \mathbb{K}_\phi, \mathbb{R}_\phi, *, \circ) \) where \( * \) is a CTN, \( \circ \) is a CTCN, \( \phi : K \times K \to [1, \infty) \) and \( \mathbb{K}_\phi, \mathbb{R}_\phi \) are FSs on \( K \times K \times (0, \infty) \). If \( (K, \mathbb{K}_\phi, \mathbb{R}_\phi, *, \circ) \) meet the below circumstances for all \( p, \bar{a} \in K \) and \( \delta, \sigma > 0 \):

(CL1). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) + \mathbb{R}_\phi(p, \bar{a}, \sigma) \leq 1 \);
(CL2). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) > 0 \);
(CL3). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) = 1 \) implies \( p = \bar{a} \);
(CL4). \( \mathbb{K}_\phi(p, \bar{a}, \sigma) = \mathbb{K}_\phi(\bar{a}, p, \sigma) \);
(CL5). \( \mathbb{K}_\phi(p, \delta, (\sigma + \theta)) \geq \mathbb{K}_\phi(p, \bar{a}, \sigma) \mathbb{K}_\phi(\bar{a}, \delta, \theta) \);
(CL6). \( \mathbb{K}_\phi(p, \bar{a}, \cdot) \) is ND function of \( \mathbb{R}^+ \) and \( \lim_{\sigma \to \infty} \mathbb{K}_\phi(p, \bar{a}, \sigma) = 1 \);
(CL7). \( \mathbb{R}_\phi(p, \bar{a}, \sigma) > 0 \);
(CL8). \( \mathbb{R}_\phi(p, \bar{a}, \sigma) = 0 \) implies \( p = \bar{a} \);
(CL9). \( \mathbb{R}_\phi(p, \bar{a}, \sigma) = \mathbb{R}_\phi(\bar{a}, p, \sigma) \);
(CL10). \( \mathbb{R}_\phi(p, \delta, (\sigma + \theta)) \leq \mathbb{R}_\phi(p, \bar{a}, \sigma) \mathbb{R}_\phi(\bar{a}, \delta, \theta) \);
(CL11). \( \mathbb{R}_\phi(p, \bar{a}, \cdot) \) is NI function of \( \mathbb{R}^+ \) and \( \lim_{\sigma \to \infty} \mathbb{R}_\phi(p, \bar{a}, \sigma) = 0 \).

Then \( (K, \mathbb{K}_\phi, \mathbb{R}_\phi, *, \circ) \) is an IFCMS.

**Example 1.** Let \( K = \{1, 2, 3\} \) and \( \alpha : K \times K \to [1, \infty) \) be a function given by \( \alpha(p, \bar{a}) = p + \bar{a} + 1 \). Define \( \mathbb{K}_\phi, \mathbb{R}_\phi : K \times K \times (0, \infty) \to [0, 1] \) as,

\[
\mathbb{K}_\phi(p, \bar{a}, \sigma) = \frac{\sigma}{\sigma + \max(p, \bar{a})}
\]

In addition,

\[
\mathbb{R}_\phi(p, \bar{a}, \sigma) = \frac{\max(p, \bar{a})}{\sigma + \max(p, \bar{a})}
\]

Then \( (K, \mathbb{K}_\phi, \mathbb{R}_\phi, *, \circ) \) is an IFCMS with CTN \( a \cdot b = ab \) and CTCN \( a \circ b = \max(a, b) \).

**Proof.** (CL1)–(CL4), (CL6)–(CL9) and (CL11) are obvious, here we prove (CL5) and (CL10).
Let \( \phi = 1, \tilde{a} = 2 \) and \( \delta = 3 \). Then,

\[
K_{\phi}(1,3,\sigma + \vartheta) = \frac{\sigma + \vartheta}{\sigma + \vartheta + \max\{1,3\}}
\]

\[
= \frac{\sigma + \vartheta}{\sigma + \vartheta + 3}
\]

On the other hand,

\[
K_{\phi}\left(1,2,\frac{\sigma}{\alpha(1,2)}\right) = \frac{\sigma}{\sigma(1,2) + \max\{1,2\}}
\]

\[
= \frac{\sigma}{\sigma + 2} = \frac{\sigma}{\sigma + 8}
\]

In addition,

\[
K_{\phi}\left(2,3,\frac{\vartheta}{\alpha(2,3)}\right) = \frac{\vartheta}{\vartheta(2,3) + \max\{2,3\}}
\]

\[
= \frac{\vartheta}{\vartheta + 3} = \frac{\vartheta}{\vartheta + 18}
\]

That is,

\[
\frac{\sigma + \vartheta}{\sigma + \vartheta + 3} \geq \frac{\sigma}{\sigma + 8} \cdot \frac{\vartheta}{\vartheta + 18}.
\]

Then it satisfied, for all \( \sigma, \vartheta > 0 \). Hence,

\[
K_{\phi}(\phi, \delta, \sigma + \vartheta) \geq K_{\phi}\left(\phi, \tilde{a}, \frac{\sigma}{\alpha(\tilde{a})}\right) \ast K_{\phi}\left(\tilde{a}, \delta, \frac{\vartheta}{\alpha(\tilde{a}, \delta)}\right).
\]

Now,

\[
R_{\phi}(1,3, \sigma + \vartheta) = \frac{\max\{1,3\}}{\sigma + \vartheta + \max\{1,3\}}
\]

\[
= \frac{3}{\sigma + \vartheta + 3}
\]

On the other hand,

\[
R_{\phi}\left(1,2,\frac{\sigma}{\alpha(1,2)}\right) = \frac{\max\{1,2\}}{\sigma(1,2) + \max\{1,2\}}
\]

\[
= \frac{2}{\sigma + 2} = \frac{8}{\sigma + 8}
\]

In addition,

\[
R_{\phi}\left(2,3,\frac{\vartheta}{\alpha(2,3)}\right) = \frac{\max\{2,3\}}{\vartheta(2,3) + \max\{2,3\}}
\]

\[
= \frac{3}{\vartheta + 3} = \frac{18}{\vartheta + 18}
\]
Proposition 1. Let $K = \{1, 2, 3\}$ and $\alpha: K \times K \to [1, \infty)$ be a function given by $\alpha(p, \tilde{a}) = p + \tilde{a} + 1$. Define $\mathcal{K}_\phi, \mathcal{R}_\phi: K \times K \times [0, \infty) \to [0, 1]$ as,

$$\mathcal{K}_\phi(p, \tilde{a}, \sigma) = \frac{\sigma + \min\{p, \tilde{a}\}}{\sigma + \max\{p, \tilde{a}\}}, \quad \mathcal{R}_\phi(p, \tilde{a}, \sigma) = 1 - e^{-\frac{\max\{p, \tilde{a}\}}{\sigma^n}}$$

for all $n \in \mathbb{N}, p, \tilde{a} \in K, \sigma > 0$.

Then $(K, \mathcal{K}_\phi, \mathcal{R}_\phi, \ast, \circ)$ is an intuitionistic fuzzy controlled metric-like space with CTN $a \ast b = ab$ and CTCN $a \circ b = \max(a, b)$.

Remark 2. The above proposition also satisfied for CTN $a \ast b = \min(a, b)$ and CTCN $a \circ b = \max(a, b)$.

Proposition 2. Let $K = [0, 1]$ and $\alpha: K \times K \to [0, 1]$ be a function given by $\alpha(p, \tilde{a}) = 2(p + \tilde{a} + 1)$. Define $\mathcal{K}_\phi, \mathcal{R}_\phi$ as,

$$\mathcal{K}_\phi(p, \tilde{a}, \sigma) = e^{-\frac{\max\{p, \tilde{a}\}}{\sigma^n}}, \quad \mathcal{R}_\phi(p, \tilde{a}, \sigma) = 1 - \left[ e^{-\frac{\max\{p, \tilde{a}\}}{\sigma^n}} \right]^{-1}$$

for all $n \in \mathbb{N}, p, \tilde{a} \in K, \sigma > 0$.

Then $(K, \mathcal{K}_\phi, \mathcal{R}_\phi, \ast, \circ)$ is an IFCMLS with CTN $a \ast b = ab$ and CTCN $a \circ b = \max(a, b)$.

Example 3. Let $K = (0, \infty)$, define $\mathcal{K}_\phi, \mathcal{R}_\phi: K \times K \times [0, \infty) \to [0, 1]$ by,

$$\mathcal{K}_\phi(p, \tilde{a}, \sigma) = \frac{\sigma}{\sigma + \max\{p, \tilde{a}\}}, \quad \mathcal{R}_\phi(p, \tilde{a}, \sigma) = \frac{\max\{p, \tilde{a}\}}{\sigma + \max\{p, \tilde{a}\}}$$

for all $p, \tilde{a} \in K$ and $\sigma > 0$, define CTN $\ast^{\ast}$ by $\nu \ast^\ast \omega = \nu \cdot \omega$ and CTCN $\ast^\circ$ by $\nu \circ^\circ \omega = \max\{\nu, \omega\}$ and define "$\phi$" by,

$$\phi(p, \tilde{a}) = \begin{cases} 1 & \text{if } p = \tilde{a}, \\ 1 + \frac{\max\{p, \tilde{a}\}}{\min\{p, \tilde{a}\}} & \text{if } p \neq \tilde{a} \end{cases}$$

Then $(K, \mathcal{K}_\phi, \mathcal{R}_\phi, \ast, \circ)$ be an IFCMLS.
\textbf{Proof.} (CL1)–(CL4), (CL6)–(CL9) and (CL11) are obvious, here we prove (CL5) and (CL10)

\[ \max\{p, \delta\} \leq \phi(p, \bar{a}) \max\{p, \bar{a}\} + \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\} \]

This implies,

\[ \sigma\theta \max\{p, \delta\} \leq \phi(p, \bar{a})(\sigma\theta + \theta^2) \max\{p, \bar{a}\} + \phi(\bar{a}, \delta)(\sigma\theta + \sigma^2) \max\{\bar{a}, \delta\} \]

Then,

\[ \sigma\theta \max\{p, \delta\} \leq \phi(p, \bar{a})(\sigma + \theta) \theta \max\{p, \bar{a}\} + \phi(\bar{a}, \delta)(\sigma + \sigma\theta) \sigma \max\{\bar{a}, \delta\} \]

Therefore,

\[ \sigma\theta(\sigma + \theta) + \sigma\theta \max\{p, \delta\} \leq \sigma\theta(\sigma + \theta) + \phi(p, \bar{a})(\sigma + \theta) \theta \max\{p, \bar{a}\} + \phi(\bar{a}, \delta)(\sigma + \sigma\theta) \sigma \max\{\bar{a}, \delta\} \]

This implies,

\[ \begin{align*}
\sigma\theta [(\sigma + \theta) + \max\{p, \delta\}] & \leq (\sigma + \theta)[\sigma\theta + \phi(p, \bar{a}) \theta \max\{p, \bar{a}\} + \phi(\bar{a}, \delta) \sigma \max\{\bar{a}, \delta\}] \\
\sigma\theta[(\sigma + \theta) + \max\{p, \delta\}] & \leq (\sigma + \theta)[\sigma\theta + \phi(p, \bar{a}) \theta \max\{p, \bar{a}\} + \phi(\bar{a}, \delta) \sigma \max\{\bar{a}, \delta\}]
\end{align*} \]

Then,

\[ (\sigma + \theta)[(\sigma + \theta) + \max\{p, \delta\}] \leq (\sigma + \theta)[\sigma + \phi(p, \bar{a}) \max\{p, \bar{a}\}] [\theta + \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\}] \]

This implies,

\[ \frac{(\sigma + \theta)}{(\sigma + \theta) + \max\{p, \delta\}} \geq \frac{\sigma\theta}{(\sigma + \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\})} \]

This implies,

\[ \frac{(\sigma + \theta)}{(\sigma + \theta) + \max\{p, \delta\}} \geq \frac{\sigma}{\sigma + \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\}} \]

Then,

\[ \frac{(\sigma + \theta)}{(\sigma + \theta) + \max\{p, \delta\}} \geq \frac{\sigma}{\phi(\bar{a}, \delta) \max\{\bar{a}, \delta\} + \max\{\bar{a}, \delta\}} \]

Therefore,

\[ \max\{p, \delta\} = \max\{p, \delta\} \max\{1, 1\} \]

\[ \max\{p, \delta\} = \max\{p, \delta\} \max\{\max\{p, \bar{a}\} \max\{\bar{a}, \delta\}\} \]

\[ \max\{p, \delta\} \leq [(\sigma + \theta) + \max\{p, \bar{a}\}] \max\{\max\{p, \bar{a}\} \max\{\bar{a}, \delta\}\} \]

Then,

\[ \max\{p, \delta\} \leq [(\sigma + \theta) + \max\{p, \delta\}] \max\{\phi(p, \bar{a}) \max\{p, \bar{a}\} \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\}\} \]

Therefore,

\[ \max\{p, \delta\} \leq [(\sigma + \theta) + \max\{p, \delta\}] \max\{\phi(p, \bar{a}) \max\{p, \bar{a}\} \phi(\bar{a}, \delta) \max\{\bar{a}, \delta\}\} \]

Then,
\[
\max\{p, \delta\} \leq \max\left(\frac{\phi(p, \delta)}{\sigma + \delta} + \max\{p, \delta\}\right)
\]

This implies,
\[
\max\{p, \delta\} \leq \max\left(\frac{\max\{p, \check{a}\}}{\sigma} + \frac{\max\{\check{a}, \delta\}}{\phi(\check{a}, \delta)}\right)
\]

Hence,
\[
\mathcal{R}_\phi\left(p, \delta, (\sigma + \delta)\right) \leq \mathcal{R}_\phi\left(p, \check{a}, \frac{\sigma}{\phi(\check{a}, \delta)}\right) \mathcal{R}_\phi\left(\check{a}, \delta, \frac{\delta}{\phi(\check{a}, \delta)}\right)
\]

(CL10) is satisfied. \(\square\)

**Definition 8.** Let \((K, \mathcal{K}, \mathcal{R}_\phi, *, n)\) be an IFCMSL. Then,

(i). A sequence \(\{p_n\}\) in \(K\) is said to be G-Cauchy sequence (GCS) if and only if for all \(\sigma > 0\),
\[
\lim_{n \to \infty} \mathcal{K}_\phi(p_n, p_{n+q}, \sigma) \quad \text{and} \quad \lim_{n \to \infty} \mathcal{R}_\phi(p_n, p_{n+q}, \sigma),
\]
exists and is finite.

(ii). A sequence \(\{p_n\}\) in \(K\) is named to be G-convergent (GC) to \(p\) in \(K\), if and only if for all \(\sigma > 0\),
\[
\lim_{n \to \infty} \mathcal{K}_\phi(p_n, p, \sigma) = \mathcal{K}_\phi(p, p, \sigma) \quad \text{and} \quad \lim_{n \to \infty} \mathcal{R}_\phi(p_n, p, \sigma) = \mathcal{R}_\phi(p, p, \sigma).
\]

(iii). A IFCMIS is named to be complete iff each GCS is convergent, i.e.,
\[
\lim_{n \to \infty} \mathcal{K}_\phi(p_n, p_{n+q}, \sigma) = \mathcal{K}_\phi(p, p, \sigma), \quad \lim_{n \to \infty} \mathcal{R}_\phi(p_n, p_{n+q}, \sigma) = \mathcal{R}_\phi(p, p, \sigma).
\]

**Theorem 1.** Suppose \((K, \mathcal{K}, \mathcal{R}_\phi, *, n)\) be a G-complete IFCMIS in the company of \(\phi: K \times K \to [1, \infty)\) and suppose that,
\[
\lim_{n \to \infty} \mathcal{K}_\phi(p, \check{a}, \sigma) = 1 \quad \text{and} \quad \lim_{n \to \infty} \mathcal{R}_\phi(p, \check{a}, \sigma) = 0 \quad \text{(1)}
\]
for all \(p, \check{a} \in K\) and \(\sigma > 0\). Let \(\xi: K \to K\) be a mapping satisfying,
\[
\mathcal{K}_\phi(\xi p, \xi \check{a}, E\sigma) \geq \mathcal{K}_\phi(p, \check{a}, \sigma) \quad \text{and} \quad \mathcal{R}_\phi(\xi p, \xi \check{a}, E\sigma) \leq \mathcal{R}_\phi(p, \check{a}, \sigma) \quad \text{(2)}
\]
for all \(p, \check{a} \in K\), and \(\sigma > 0\), where \(0 < E < 1\). Furthermore, assume that for every \(\check{a} \in K\),
\[
\lim_{n \to \infty} \phi(p_n, \check{a}) \quad \text{and} \quad \lim_{n \to \infty} \phi(\check{a}, p_n)
\]
In addition,
\[
\lim_{n, m \to \infty} \phi(p_n, p_m) \quad \text{and} \quad \lim_{n, m \to \infty} \phi(p_m, p_n) \quad \text{(3)}
\]
exists and are finite, where \(p_n = \xi^n p_0 = \xi p_{n-1}\), for all \(n \in \mathbb{N}\) and \(p_0\) be arbitrary point of \(K\).

Then \(\xi\) has a unique FP.

**Proof.** Let \(p_0\) be an arbitrary point of \(K\) and define a sequence \(p_n\) by \(p_n = \xi^n p_0 = \xi p_{n-1}\), \(n \in \mathbb{N}\). Using (2) for all \(\sigma > 0\), we examine,
\[
\mathcal{K}_\phi(p_n, p_{n+q}, E\sigma) = \mathcal{K}_\phi(p_{n-1}, \xi p_n, E\sigma) \geq \mathcal{K}_\phi(p_{n-1}, p_n, \sigma) \geq \mathcal{K}_\phi(\xi p_{n-2}, p_{n-1}, \frac{\sigma}{E})
\]
\[
\geq \mathcal{K}_\phi(p_{n-3}, \xi p_{n-2}, \frac{\sigma}{E^2}) \geq \cdots \geq \mathcal{K}_\phi(p_0, p_{n-1}, \frac{\sigma}{E^{n-1}})
\]
In addition,
\[
\mathcal{R}_\phi(p_n, p_{n+q}, E\sigma) = \mathcal{R}_\phi(p_{n-1}, \xi p_n, E\sigma) \leq \mathcal{R}_\phi(p_{n-1}, p_n, \sigma) \leq \mathcal{R}_\phi(p_{n-2}, p_{n-1}, \frac{\sigma}{E})
\]
\[
\leq \mathcal{R}_\phi \left( \frac{p_{n-3} \cdot p_{n-2}}{x^2} \right) \leq \cdots \leq \mathcal{R}_\phi \left( \frac{p_0 \cdot p_1 \cdot \cdots}{x^{n-1}} \right)
\]

We obtain,
\[
\mathcal{K}_\phi(P_n, P_{n+1}, x) \geq \mathcal{K}_\phi \left( \frac{p_n \cdot p_{n+1}}{2(\phi(p_n, p_{n+1}))} \right) \cdot \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_{n+1}, p_{n+2}))} \right)
\]
\[
\geq \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_n, p_{n+1}))} \right) \cdot \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_{n+1}, p_{n+2}))} \right)
\]
\[
\geq \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_n, p_{n+1}))} \right) \cdot \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_{n+1}, p_{n+2}))} \right)
\]
\[
\geq \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_n, p_{n+1}))} \right) \cdot \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_{n+1}, p_{n+2}))} \right)
\]
\[
\geq \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_n, p_{n+1}))} \right) \cdot \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_{n+1}, p_{n+2}))} \right)
\]
\[
\geq \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_n, p_{n+1}))} \right) \cdot \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_{n+1}, p_{n+2}))} \right)
\]
\[
\geq \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_n, p_{n+1}))} \right) \cdot \mathcal{K}_\phi \left( \frac{p_{n+1} \cdot p_{n+2}}{2(\phi(p_{n+1}, p_{n+2}))} \right)
\]
\[
\mathcal{K}_\phi \left( \frac{p_{n+q-2} \cdot p_{n+q-1}}{2^{q-1}(\phi(p_{n+1}, p_{n+2}) \cdot \phi(p_{n+2}, p_{n+3}) \cdots \phi(p_{n+q-2}, p_{n+q-1}))} \right) \cdot \mathcal{K}_\phi \left( \frac{p_{n+q-1} \cdot p_{n+q}}{2^{q-1}(\phi(p_{n+1}, p_{n+2}) \cdot \phi(p_{n+2}, p_{n+3}) \cdots \phi(p_{n+q-1}, p_{n+q}))} \right)
\]

In addition,
\[
\mathcal{R}_\phi (p_n, p_{n+q}, \alpha) \leq \mathcal{R}_\phi \left( p_n, p_{n+1}, \frac{\alpha}{2} \left( \phi(p_n, p_{n+1}) \right) \right)
\]
\[
\circ \mathcal{R}_\phi \left( p_{n+1}, p_{n+q}, \frac{\alpha}{2} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+1}, p_{n+q}) \right) \right)
\]
\[
\leq \mathcal{R}_\phi \left( p_n, p_{n+1}, \frac{\alpha}{2} \left( \phi(p_n, p_{n+1}) \right) \right)
\]
\[
\circ \mathcal{R}_\phi \left( p_{n+1}, p_{n+2}, \frac{\alpha}{2} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+1}, p_{n+2}) \right) \right)
\]
\[
\circ \mathcal{R}_\phi \left( p_{n+2}, p_{n+3}, \frac{\alpha}{2} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+2}, p_{n+3}) \right) \right)
\]
\[
\leq \mathcal{R}_\phi \left( p_n, p_{n+1}, \frac{\alpha}{2} \left( \phi(p_n, p_{n+1}) \right) \right)
\]
\[
\circ \mathcal{R}_\phi \left( p_{n+1}, p_{n+2}, \frac{\alpha}{2} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+1}, p_{n+2}) \right) \right)
\]
\[
\circ \mathcal{R}_\phi \left( p_{n+2}, p_{n+3}, \frac{\alpha}{2} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+2}, p_{n+3}) \right) \right)
\]
\[
\circ \mathcal{R}_\phi \left( p_{n+3}, p_{n+4}, \frac{\alpha}{2} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+2}, p_{n+3}) \phi(p_{n+3}, p_{n+4}) \right) \right) \circ \ldots
\]
\[
\mathcal{R}_\phi \left( p_{n+q-2}, p_{n+q-1}, \frac{\alpha}{2^{q-1}} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+2}, p_{n+q}) \phi(p_{n+3}, p_{n+q}) \phi(p_{n+q-2}, p_{n+q-1}) \right) \right)
\]
\[
\circ \mathcal{R}_\phi \left( p_{n+q-1}, p_{n+q}, \frac{\alpha}{2^{q-1}} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+2}, p_{n+q}) \phi(p_{n+3}, p_{n+q}) \phi(p_{n+q-1}, p_{n+q}) \right) \right)
\]
Using (4) in the above inequalities, we deduce,

\[
\phi_n(P_n, P_{n+q}, \omega) \geq \phi_n\left( P_0, P_1, \frac{\omega}{2(E)^n - 1}(\phi(P_n, P_{n+1})) \right)
\]

\[
\times \phi_n\left( P_0, P_1, \frac{\omega}{(2)^2(E)^n}(\phi(P_{n+1}, P_{n+2})) \right)
\]

\[
\times \phi_n\left( P_0, P_1, \frac{\omega}{(2)^3(E)^{n+1}}(\phi(P_{n+1}, P_{n+2}, P_{n+3})) \right)
\]

\[
\times \phi_n\left( P_0, P_1, \frac{\omega}{(2)^4(E)^{n+2}}(\phi(P_{n+1}, P_{n+2}, P_{n+3}, P_{n+4})) \right)
\]

In addition,

\[
\mathcal{R}_n(P_n, P_{n+q}, \omega) \leq \mathcal{R}_n\left( P_0, P_1, \frac{\omega}{2(E)^n - 1}(\phi(P_n, P_{n+1})) \right)
\]

\[
\times \mathcal{R}_n\left( P_0, P_1, \frac{\omega}{(2)^2(E)^n}(\phi(P_{n+1}, P_{n+2})) \right)
\]

\[
\times \mathcal{R}_n\left( P_0, P_1, \frac{\omega}{(2)^3(E)^{n+1}}(\phi(P_{n+1}, P_{n+2}, P_{n+3})) \right)
\]

\[
\times \mathcal{R}_n\left( P_0, P_1, \frac{\omega}{(2)^4(E)^{n+2}}(\phi(P_{n+1}, P_{n+2}, P_{n+3}, P_{n+4})) \right)
\]

Using (1), for \( n \to \infty \), we deduce,

\[
\lim_{n \to \infty} \mathcal{N}_n(P_n, P_{n+q}, \omega) = 1 \times 1 \times \cdots = 1
\]

In addition,
\[
\lim_{n \to \infty} R_\phi(p_n, p_{n+q}, \omega) = 0 \circ 0 \circ \cdots \circ 0 = 0
\]

i.e., \(\{p_n\}\) is a GCS. Therefore, \((K, \mathcal{N}_\phi, R_\phi, \ast, \circ)\) is a G-complete IFCMLS, there exists,
\[
\lim_{n \to \infty} p_n = p
\]

Now examine that \(p\) is a FP of \(\xi\), using (CL5), (CL10) and (1), we deduce,
\[
\begin{align*}
N_\phi(p, \xi p, \omega) &\geq N_\phi \left(p, p_{n+1}, \frac{\alpha}{2(\phi(p, p_{n+1}))} \right) \ast N_\phi \left(p_{n+1}, \xi p, \frac{\alpha}{2(\phi(p_{n+1}, \xi p))} \right) \\
N_\phi(p, \xi p, \omega) &\geq N_\phi \left(p, p_{n+1}, \frac{\alpha}{2(\phi(p, p_{n+1}))} \right) \ast N_\phi \left(p_{n+1}, \xi p, \frac{\alpha}{2(\phi(p_{n+1}, \xi p))} \right) \\
N_\phi(p, \xi p, \omega) &\geq N_\phi \left(p, p_{n+1}, \frac{\alpha}{2(\phi(p, p_{n+1}))} \right) \ast N_\phi \left(p_{n+1}, \xi p, \frac{\alpha}{2(\phi(p_{n+1}, \xi p))} \right) \\
N_\phi(p, \xi p, \omega) &\geq N_\phi \left(p, p_{n+1}, \frac{\alpha}{2(\phi(p, p_{n+1}))} \right) \ast N_\phi \left(p_{n+1}, \xi p, \frac{\alpha}{2(\phi(p_{n+1}, \xi p))} \right) \to 1 \circ 1 = 1
\end{align*}
\]
as \(n \to \infty\), and,
\[
\begin{align*}
R_\phi(p, \xi p, \omega) &\leq R_\phi \left(p, p_{n+1}, \frac{\alpha}{2(\phi(p, p_{n+1}))} \right) \ast R_\phi \left(p_{n+1}, \xi p, \frac{\alpha}{2(\phi(p_{n+1}, \xi p))} \right) \\
R_\phi(p, \xi p, \omega) &\leq R_\phi \left(p, p_{n+1}, \frac{\alpha}{2(\phi(p, p_{n+1}))} \right) \ast R_\phi \left(p_{n+1}, \xi p, \frac{\alpha}{2(\phi(p_{n+1}, \xi p))} \right) \\
R_\phi(p, \xi p, \omega) &\leq R_\phi \left(p, p_{n+1}, \frac{\alpha}{2(\phi(p, p_{n+1}))} \right) \ast R_\phi \left(p_{n+1}, \xi p, \frac{\alpha}{2(\phi(p_{n+1}, \xi p))} \right) \to 0 \circ 0 = 0
\end{align*}
\]
as \(n \to \infty\). Hence, \(\xi p = p\), a FP. To examine uniqueness, assume that \(\xi c = c\) for some \(c \in K\), then,
\[
1 \geq N_\phi(c, p, \omega) = N_\phi(\xi c, \xi p, \omega) \geq N_\phi \left(c, p, \frac{\alpha}{\varepsilon} \right) = N_\phi \left(c, p, \frac{\alpha}{\varepsilon} \right)
\]
\[
\geq N_\phi \left(c, p, \frac{\alpha}{\varepsilon} \right) \geq \cdots \geq N_\phi \left(c, p, \frac{\alpha}{\varepsilon} \right) \to 1 \text{ as } n \to \infty,
\]
In addition,
\[
0 \leq R_\phi(c, p, \omega) = R_\phi(\xi c, \xi p, \omega) \leq R_\phi \left(c, p, \frac{\alpha}{\varepsilon} \right) = R_\phi \left(c, p, \frac{\alpha}{\varepsilon} \right)
\]
\[
\leq R_\phi \left(c, p, \frac{\alpha}{\varepsilon} \right) \leq \cdots \leq R_\phi \left(c, p, \frac{\alpha}{\varepsilon} \right) \to 0 \text{ as } n \to \infty,
\]
by using (CL3) and (CL8), \(p = c\). □

**Definition 9.** Let \((K, \mathcal{N}_\phi, R_\phi, \ast, \circ)\) be a IFCMLS. A map \(\xi: K \to K\) is intuitionistic fuzzy controlled (IFC) contraction if there exists \(0 < \varepsilon < 1\), such that,
\[
1 - \frac{1}{N_\phi(\xi p, p, \omega)} \leq \varepsilon \left[ 1 - \frac{1}{N_\phi(p, p, \omega)} \right]
\]
(5)

In addition,
\[
R_\phi(\xi p, p, \omega) \leq \varepsilon R_\phi(p, p, \omega),
\]
(6)

for all \(p, \tilde{p} \in K\) and \(\omega > 0\).

Now, we prove the theorem for IFCMLS.
Theorem 2. Let \((K, \mathcal{K}, \mathcal{R}, \mathcal{F}, \sigma, v)\) be a G-complete IFCMLS with \(\phi: K \times K \to [1, \infty)\) and suppose that,
\[
\lim_{\sigma \to \infty} \mathcal{K}_\phi(p, \tilde{a}, \sigma) = 1 \quad \text{and} \quad \lim_{\sigma \to \infty} \mathcal{R}_\phi(p, \tilde{a}, \sigma) = 0
\]  
for all \(p, \tilde{a} \in K\) and \(\sigma > 0\). Let \(\Phi: K \to K\) be an IFC contraction. Further, suppose that for every \(\tilde{a} \in K\),
\[
\lim_{n \to \infty} \phi(p_n, \tilde{a}) \quad \text{and} \quad \lim_{n \to \infty} \phi(\tilde{a}, p_n)
\]
In addition,
\[
\lim_{n,m \to \infty} \phi(p_n, p_m) \quad \text{and} \quad \lim_{n,m \to \infty} \phi(p_m, p_n)
\]
each and are finite, where \(p_n = \xi^n p_0 = \xi p_{n-1}\), \(n \in \mathbb{N}\). Using Equations (5) and (6) for all \(\sigma > 0\), \(n > q\), we acquire,
\[
\frac{1}{\mathcal{K}_\phi(p_n, p_{n+1}, \sigma)} - 1 = \frac{1}{\mathcal{K}_\phi(p_n, p_{n+1}, \sigma)} = \frac{1}{\mathcal{K}_\phi(p_n, p_{n+1}, \sigma)} - \frac{1}{\mathcal{K}_\phi(p_n, p_{n+1}, \sigma)}
\]
Continuing in this way, we acquire,
\[
\frac{1}{\mathcal{K}_\phi(p_n, p_{n+1}, \sigma)} \leq \frac{\epsilon^n}{\mathcal{K}_\phi(p_0, p_1, \sigma)} + \epsilon^{n-1}(1 - \epsilon) + \epsilon^{n-2}(1 - \epsilon) + \cdots + (1 - \epsilon) + (1 - \epsilon)
\]
We obtain,
\[
\frac{\epsilon^n}{\mathcal{K}_\phi(p_0, p_1, \sigma)} + (1 - \epsilon^n) \leq \mathcal{K}_\phi(p_n, p_{n+1}, \sigma)
\]
In addition,
\[
\mathcal{R}_\phi(p_n, p_{n+1}, \sigma) = \mathcal{R}_\phi(\xi p_{n-1}, p_n, \sigma) \leq \epsilon \mathcal{R}_\phi(p_{n-1}, p_n, \sigma) = \mathcal{R}_\phi(\xi p_{n-2}, p_{n-1}, \sigma)
\]
For any \(q \in \mathbb{N}\), using (CL5) and (CL10), we deduce,
\[
\mathcal{K}_\phi(p_n, p_{n+q}, \sigma) \geq \mathcal{K}_\phi\left(\frac{\sigma}{2(\phi(p_n, p_{n+1}))}\right) \mathcal{K}_\phi\left(\frac{p_{n+1}, p_{n+q}}{\phi(p_{n+1}, p_{n+q})}\right)
\]
\[
\geq \mathcal{K}_\phi\left(\frac{\sigma}{2(\phi(p_n, p_{n+1}))}\right) \mathcal{K}_\phi\left(\frac{p_{n+1}, p_{n+2}}{(\phi(p_{n+1}, p_{n+q}))^2}\right)
\]
In addition,

\[ \mathfrak{R}_\phi(p_n, p_{n+q}, \alpha) \leq \mathfrak{R}_\phi \left( p_n, p_{n+1}, \frac{\alpha}{2(\phi(P_n, P_{n+1}))} \right) \]

\[ \circ \mathfrak{R}_\phi \left( p_{n+1}, p_{n+q}, \frac{\alpha}{2(\phi(P_{n+1}, P_{n+q}))} \right) \]

\[ \leq \mathfrak{R}_\phi \left( p_n, p_{n+1}, \frac{\alpha}{2(\phi(P_{n+1}, P_{n+1}))} \right) \]

\[ \circ \mathfrak{R}_\phi \left( p_{n+1}, p_{n+2}, \frac{\alpha}{2(\phi(P_{n+1}, P_{n+2}))} \right) \]

\[ \circ \mathfrak{R}_\phi \left( p_{n+2}, p_{n+q}, \frac{\alpha}{2(\phi(P_{n+2}, P_{n+q}))} \right) \]
\[ \leq R_\phi \left( \frac{\alpha}{2} \left( \phi(p_n, p_{n+1}) \right) \right) \]

\[ \circ R_\phi \left( \frac{\alpha}{(2)^2} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+1}, p_{n+2}) \right) \right) \]

\[ \circ R_\phi \left( \frac{\alpha}{(2)^3} \left( \phi(p_{n+2}, p_{n+3}) \phi(p_{n+2}, p_{n+q}) \phi(p_{n+3}, p_{n+q}) \right) \right) \]

\[ \circ R_\phi \left( \frac{\alpha}{(2)^4} \left( \phi(p_{n+2}, p_{n+3}) \phi(p_{n+2}, p_{n+4}) \phi(p_{n+3}, p_{n+4}) \phi(p_{n+3}, p_{n+4}) \right) \right) \]... 

\[ R_\phi \left( \frac{\alpha}{(2)^q-1} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+2}, p_{n+q}) ... \phi(p_{n+q-2}, p_{n+q-1}) \right) \right) \]

\[ \circ R_\phi \left( \frac{\alpha}{(2)^q-1} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+2}, p_{n+q}) ... \phi(p_{n+q-1}, p_{n+q}) \right) \right) \]

\[ \kappa_\phi(p_n, p_{n+q}, \omega) \geq \frac{1}{\kappa_\phi\left( \frac{\alpha}{2} \left( \phi(p_n, p_{n+1}) \right) \right)} + (1 - \varepsilon^n) \]

\[ \kappa_\phi\left( \frac{\alpha}{2} \left( \phi(p_n, p_{n+1}) \right) \right) \]

\[ \kappa_\phi\left( \frac{\alpha}{(2)^2} \left( \phi(p_{n+1}, p_{n+q}) \phi(p_{n+1}, p_{n+2}) \right) \right) \]

\[ \kappa_\phi\left( \frac{\alpha}{(2)^3} \left( \phi(p_{n+2}, p_{n+3}) \phi(p_{n+2}, p_{n+q}) \phi(p_{n+3}, p_{n+q}) \right) \right) \]

\[ \kappa_\phi\left( \frac{\alpha}{(2)^4} \left( \phi(p_{n+2}, p_{n+3}) \phi(p_{n+2}, p_{n+4}) \phi(p_{n+3}, p_{n+4}) \phi(p_{n+3}, p_{n+4}) \right) \right) \]...
In addition,
\[
\lim_{n \to \infty} \mathcal{N}_\phi(p_n, p_{n+q}, \omega) = 1 * 1 * \cdots * 1 = 1
\]

In addition,
\[
\lim_{n \to \infty} \mathcal{R}_\phi(p_n, p_{n+q}, \omega) = 0 * 0 * \cdots * 0 = 0
\]
i.e., \(\{p_n\}\) is a GCS. Since \(\{K, \mathcal{N}_\phi, \mathcal{R}_\phi, \ast, \circ\}\) be a \(G\)-complete IFCMLS, there exists,
\[
\lim_{n \to \infty} p_n = p.
\]

Now examine that \(p\) is a FP of \(\xi\), using (CL5) and (CL10), we deduce,
\[
\frac{1}{\mathcal{N}_\phi(\xi p_n, \xi p, \omega)} - 1 \leq \epsilon \left[ \frac{1}{\mathcal{N}_\phi(p_n, p, \omega)} - 1 \right] = \frac{\epsilon}{\mathcal{N}_\phi(p_n, p, \omega)} - \epsilon
\]
\[
\Rightarrow \frac{1}{\mathcal{N}_\phi(p_n, p, \omega)} + (1 - \epsilon) \leq \mathcal{N}_\phi(\xi p_n, \xi p, \omega)
\]

Using above inequality, we obtain,
\[
\mathcal{N}_\phi(p, \xi p, \omega) \geq \mathcal{N}_\phi(p, p_{n+1}, \omega) \cdot \mathcal{N}_\phi(p_{n+1}, \xi p, \omega)
\]
\[
\geq \kappa_\phi \left( p, p_{n+1}, \frac{\sigma}{2\phi(p, p_{n+1})} \right) * \kappa_\phi \left( \xi p, \xi p, \frac{\sigma}{2\phi(p_{n+1}, \xi p)} \right)
\]

\[
\geq \kappa_\phi \left( p, p_{n+1}, \frac{\sigma}{2\phi(p, p_{n+1})} \right) * \frac{1}{\epsilon} \geq \frac{1}{\kappa_\phi \left( p, p, \frac{\sigma}{2\phi(p, p)} \right) + (1 - \epsilon)} \to 1 \times 1 = 1
\]

as \( n \to \infty \), and,

\[
\Re_\phi(p, \xi p, \omega) \leq \Re_\phi \left( p, p_{n+1}, \frac{\sigma}{2\phi(p, p_{n+1})} \right) * \Re_\phi \left( p_{n+1}, \xi p, \frac{\sigma}{2\phi(p_{n+1}, \xi p)} \right)
\]

\[
\leq \Re_\phi \left( p, p_{n+1}, \frac{\sigma}{2\phi(p, p_{n+1})} \right) * \Re_\phi \left( \xi p, \xi p, \frac{\sigma}{2\phi(p_{n+1}, \xi p)} \right)
\]

\[
\leq \Re_\phi \left( p, p_{n+1}, \frac{\sigma}{2\phi(p, p_{n+1})} \right) * \epsilon \Re_\phi \left( p, p_{n}, \frac{\sigma}{2\phi(p_{n+1}, \xi p)} \right) \to 0 \times 0 = 0 \text{ as } n \to \infty
\]

This shows that \( \xi p = p \), a FP. To examine the uniqueness, assume that \( \xi c = c \) for some \( c \in K \), then,

\[
\frac{1}{\kappa_\phi(p, c, \omega)} - 1 = \frac{1}{\kappa_\phi(\xi p, \xi c, \omega)} - 1
\]

\[
\leq \epsilon \left[ \frac{1}{\kappa_\phi(p, c, \omega)} - 1 \right] < \frac{1}{\kappa_\phi(p, c, \omega)} - 1
\]

a contradiction, and,

\[
\Re_\phi(p, c, \omega) = \Re_\phi(\xi p, \xi c, \omega) \leq \epsilon \Re_\phi(p, c, \omega) < \Re_\phi(p, c, \omega)
\]

a contradiction. Therefore, we must have \( \kappa_\phi(p, c, \omega) = 1 \) and \( \Re_\phi(p, c, \omega) = 0 \), hence \( p = c \). \( \Box \)

**Example 4.** Let \( K = [0,1] \). Define \( \phi \) by,

\[
\phi(p, \tilde{a}) = \begin{cases} 
1 & \text{if } p = \tilde{a}, \\
\frac{1 + \max\{p, \tilde{a}\}}{\min\{p, \tilde{a}\}} & \text{if } p \neq \tilde{a} \neq 0
\end{cases}
\]

Furthermore, take,

\[
\kappa_\phi(p, \tilde{a}, \omega) = e^{-\frac{\max\{p, \tilde{a}\}}{\omega}} \text{ and } \Re_\phi(p, \tilde{a}, \omega) = 1 - e^{-\frac{\max\{p, \tilde{a}\}}{\omega}}
\]

with \( v * \omega = v \cdot \omega \) and \( v \cdot \omega = \max\{v, \omega\} \). Then \( \left( K, \kappa_\phi, \Re_\phi, \ast, \cdot \right) \) is a G-complete IFMLS. Observe that \( \lim_{\omega \to 0} \kappa_\phi(p, \tilde{a}, \omega) \) and \( \lim_{\omega \to 0} \Re_\phi(p, \tilde{a}, \omega) \) exist and finite. Define \( \xi : K \to K \) by,

\[
\xi(p) = \begin{cases} 
0 & \text{if } p \in [0, \frac{1}{2}], \\
\frac{p}{4} & \text{if } p \in \left( \frac{1}{2}, 1 \right]
\end{cases}
\]

Then we have for cases:

I. If \( p, \tilde{a} \in [0, \frac{1}{2}] \), then \( \xi p = \xi \tilde{a} = 0 \);

II. If \( p \in \left[ 0, \frac{1}{4} \right] \) and \( \tilde{a} \in \left[ \frac{1}{2}, 1 \right] \), then \( \xi p = 0 \) and \( \xi \tilde{a} = \frac{3}{4} \);

III. If \( \tilde{a} \in \left[ 0, \frac{1}{2} \right] \) and \( p \in \left[ \frac{1}{2}, 1 \right] \), then \( \xi \tilde{a} = 0 \) and \( \xi p = \frac{3}{4} \);

IV. If \( p, \tilde{a} \in \left[ \frac{1}{2}, 1 \right] \), then \( \xi p = \frac{3}{4} \) and \( \xi \tilde{a} = \frac{3}{4} \);

In all (I)–(IV) cases,

\[
\kappa_\phi(\xi p, \xi \tilde{a}, \omega) \geq \kappa_\phi(p, \tilde{a}, \omega) \text{ and } \Re_\phi(\xi p, \xi \tilde{a}, \omega) \leq \Re_\phi(p, \tilde{a}, \omega)
\]
are satisfied for \( E \in \left[ \frac{1}{2}, 1 \right) \), and also,
\[
\frac{1}{\kappa_F(p, \xi, \omega)} - 1 \leq E \left[ \frac{1}{\kappa_F(p, \zeta, \omega)} - 1 \right] \quad \text{and} \quad \kappa_F(\xi, \zeta, \omega) \leq E \kappa_F(p, \zeta, \omega)
\]
satisfied for \( E \in \left[ \frac{1}{2}, 1 \right) \).

Observe that \( \lim_{n \to \infty} \phi(p_n, \tilde{a}) \) and \( \lim_{n \to \infty} \phi(\tilde{a}, p_n) \) exists and finite. Furthermore, observe that all circumstances of Theorems 1 and 2 are fulfilled, and 0 is a unique FP of \( \xi \).

**Open Problem 1.** It is related to dealing with the Kannan contraction, Chatterjee contraction and Suzuki contraction in the sense of IFMLS.

### 4. Application to Nonlinear Fractional Differential Equations

In present section, we aim to apply Theorem 3 to obtain the existence and uniqueness of a solution to a nonlinear fractional differential equation (NFDE),

\[
D_p^\alpha p(t) = g(t, p(t)), \quad 0 < t < 1
\]

with the boundary conditions,

\[
p(0) + p'(0) = 0, \quad p(1) + p'(1) = 0,
\]

where \( 1 < p \leq 2 \) is a number, \( D_p^\alpha \) is the Caputo fractional derivative and \( g : [0,1] \times [0,\infty) \to [0,\infty) \) is a continuous function. Let \( K = C([0,1], \mathbb{R}) \) denote the space of all continuous functions defined on \( [0,1] \) equipped with the CTN \( c \circ d = c \circ d \) and CTN \( c \circ d = \max\{c, d\} \) for all \( c, d \in [0,1] \) and define an IFCMLS on \( K \) as follows:

\[
\kappa_F(p, \delta, \omega) = \frac{\omega}{\alpha \omega + \gamma} \max\{\sup_{t \in [0,1]} \phi(t), \sup_{t \in [0,1]} \phi(t)\} \]

\[
\kappa_F(p, \delta, \omega) = \frac{\omega}{\alpha \omega + \gamma} \max\{\sup_{t \in [0,1]} \phi(t), \sup_{t \in [0,1]} \phi(t)\} \]

for all \( \alpha > 0 \) and \( \delta, \omega \in K \) with the control function,

\[
\phi(p, \delta) = 1 + \max\{\sup_{t \in [0,1]} \phi(t), \sup_{t \in [0,1]} \phi(t)\}.
\]

Observe that \( p \in K \) solves (4A) whenever \( q \in K \) solves the below integral equation:

\[
p(t) = \frac{1}{\Gamma(p)} \int_0^1 (t-s)^{p-1}(1-t)g(s, p(s))ds + \frac{1}{\Gamma(p-1)} \int_0^1 (1-s)^{p-2}(1-t)g(s, p(s))ds
\]

\[
+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1}g(s, p(s))ds.
\]

**Theorem 3.** The integral operator \( \xi : K \to K \) is given by,

\[
\xi p(t) = \frac{1}{\Gamma(p)} \int_0^1 (t-s)^{p-1}(1-t)g(s, g(s))ds + \frac{1}{\Gamma(p-1)} \int_0^1 (1-s)^{p-2}(1-t)g(s, g(s))ds
\]

\[
+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1}g(s, g(s))ds,
\]

where \( g : [0,1] \times [0,\infty) \to [0,\infty) \) fulfilling the following criteria:

\[
\max\{\sup_{t \in [0,1]} g(s, p(s)), \sup_{t \in [0,1]} g(s, \delta(s))\} \leq \frac{1}{4} \max\{\sup_{t \in [0,1]} g(s, p(s)), \sup_{t \in [0,1]} g(s, \delta(s))\}, \text{for all } p, \delta \in K;
\]

\[
\sup_{t \in (0,1)} \frac{1}{4096} \left[ \frac{1-t}{\Gamma(p+1)} + \frac{1-t}{\Gamma(p+1)} + \frac{1-t}{\Gamma(p+1)} \right] \leq a < 1.
\]

Then NFDE has a unique solution in \( X \).

**Proof.**
\[ \max[\xi \mathcal{B}(t), \xi \delta(t)]^6 = \left( 1 - t \right) \Gamma(p) \int_0^1 (1 - s)^{p-1} \max\{\sup_{s \in [0,1]} \mathcal{B}(s), \sup_{s \in [0,1]} \delta(s)\} ds \\
+ \frac{1}{\Gamma(p-1)} \int_0^1 (1 - s)^{p-2} \max\{\sup_{s \in [0,1]} \mathcal{B}(s), \sup_{s \in [0,1]} \delta(s)\} ds \\
+ \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} \max\{\sup_{s \in [0,1]} \mathcal{B}(s), \sup_{s \in [0,1]} \delta(s)\} ds \] \\
\leq \left( 1 - t \right) \Gamma(p) \int_0^1 (1 - s)^{p-1} \max\{\sup_{s \in [0,1]} \mathcal{B}(s), \sup_{s \in [0,1]} \delta(s)\} ds \\
+ \frac{1}{\Gamma(p-1)} \int_0^1 (1 - s)^{p-2} \max\{\sup_{s \in [0,1]} \mathcal{B}(s), \sup_{s \in [0,1]} \delta(s)\} ds \\
+ \frac{1}{\Gamma(p)} \int_0^t (t - s)^{p-1} \max\{\sup_{s \in [0,1]} \mathcal{B}(s), \sup_{s \in [0,1]} \delta(s)\} ds \] \\
\leq \frac{1}{4^6} \max\{\sup_{s \in [0,1]} \mathcal{B}(s), \sup_{s \in [0,1]} \delta(s)\}^6 \sup_{t \in [0,1]} \left[ \frac{1 - t}{\Gamma(p + 1)} + \frac{1 - t}{\Gamma(p)} + \frac{t^p}{\Gamma(p + 1)} \right] ^6 \\
= \tilde{\alpha} \max\{\sup_{s \in [0,1]} \mathcal{B}(s), \sup_{s \in [0,1]} \delta(s)\}^6 ,
\]

where,

\[ \tilde{\alpha} = \sup_{t \in (0,1)} \frac{1}{4096} \left[ \frac{1 - t}{\Gamma(p + 1)} + \frac{1 - t}{\Gamma(p)} + \frac{t^p}{\Gamma(p + 1)} \right] ^6 . \]

Therefore, the above inequality,

\[ \max\{\sup_{t \in [0,1]} \xi \mathcal{B}(t), \sup_{t \in [0,1]} \xi \delta(t)\}^6 \leq \tilde{\alpha} \max\{\sup_{t \in [0,1]} \mathcal{B}(t), \sup_{t \in [0,1]} \delta(t)\}^6 \]

\[ \Rightarrow a \sigma + \frac{\gamma}{\tilde{\alpha}} \max\{\sup_{t \in [0,1]} \xi \mathcal{B}(t), \sup_{t \in [0,1]} \xi \delta(t)\}^6 \leq a \sigma + \gamma \max\{\sup_{t \in [0,1]} \mathcal{B}(t), \sup_{t \in [0,1]} \delta(t)\} \]

\[ \Rightarrow \frac{a (\tilde{\alpha} \sigma)}{\alpha (\tilde{\alpha} \sigma) + \gamma \max\{\sup_{t \in [0,1]} \xi \mathcal{B}(t), \sup_{t \in [0,1]} \xi \delta(t)\}^6} \]

\[ \geq \frac{\sigma a + \gamma \max\{\sup_{t \in [0,1]} \mathcal{B}(t), \sup_{t \in [0,1]} \delta(t)\}^6}{\alpha (\tilde{\alpha} \sigma) + \gamma \max\{\sup_{t \in [0,1]} \xi \mathcal{B}(t), \sup_{t \in [0,1]} \xi \delta(t)\}^6} \]

\[ \Rightarrow \mathcal{R}_\phi(\xi \mathcal{B}, \sigma) \geq \mathcal{R}_\phi(\mathcal{B}, \delta, \sigma), \]

In addition,

\[ \Rightarrow \frac{\gamma \sup_{t \in [0,1]} \max\{\sup_{t \in [0,1]} \xi \mathcal{B}(t), \sup_{t \in [0,1]} \xi \delta(t)\}^6}{a (\tilde{\alpha} \sigma) + \gamma \sup_{t \in [0,1]} \max\{\sup_{t \in [0,1]} \xi \mathcal{B}(t), \sup_{t \in [0,1]} \xi \delta(t)\}^6} \]

\[ \leq \frac{\gamma \sup_{t \in [0,1]} \max\{\sup_{t \in [0,1]} \mathcal{B}(t), \sup_{t \in [0,1]} \delta(t)\}^6}{\alpha (\tilde{\alpha} \sigma) + \gamma \sup_{t \in [0,1]} \max\{\sup_{t \in [0,1]} \mathcal{B}(t), \sup_{t \in [0,1]} \delta(t)\}^6} \]

\[ \Rightarrow \mathcal{R}_\phi(\xi \mathcal{B}, \xi \delta, \tilde{\alpha} \sigma) \leq \mathcal{R}_\phi(\mathcal{B}, \delta, \sigma), \]
for some $\alpha, \gamma > 0$. Observe that the conditions of the Theorem 1 are fulfilled. Resultantly, $\xi$ has a unique fixed point; accordingly, the specified NFDE has a unique solution. ☐

5. Conclusions

We present intuitionistic fuzzy controlled metric-like spaces in this paper and established several new types of fixed-point theorems in this new context. Moreover, we provided non-trivial examples and an application to non-linear fractional differential equations is given to demonstrate the viability of the proposed method. Our findings and concepts expand and generalize the existing literature. The structures of intuitionistic fuzzy double controlled metric-like spaces, intuitionistic pentagonal fuzzy controlled metric-like spaces, and neutrosophic controlled metric-like spaces etc. can all be extended using this study.

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