A novel semi-analytical solution to Jeffery-Hamel equation

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Abstract
A new approach based on the Adomian decomposition and the Fourier transform is introduced. The method suggests a solution for the well-known magneto-hydrodynamic (MHD) Jeffery-Hamel equation. Results of Adomian decomposition method combined with Fourier transform are compared with exact and numerical methods. The FTADM as an exclusive and new method satisfies all boundary and initial conditions over the entire spatial and temporal domains. Moreover, using the FTADM leads to rapid approach of approximate results toward the exact solutions is demonstrated. The second derivative of Jeffery–Hamel solution related to the similar number of items of recursive terms under a vast spatial domain shows the maximum error in the order of $10^{-5}$ comparing to exact and numerical solutions. The results also imply that the FTADM can be considered as a precise approximation for solving the third-order nonlinear Jeffery–Hamel equations.

1. Introduction
The Jeffery-Hamel mathematical foundation is the most suitable way of modeling the incompressible viscous fluid flow between two inclined plates under a magnetic field. Generally, engineering applications such as fluid mechanics and environmental science utilize Jeffery-Hamel flows. Because of its significant applications in liquid metals-based cooling systems’ designation, magnetohydrodynamic generators, accelerators, pumps, and flow meters are of major interest of researchers [1–4]. For MHD Jeffery–Hamel flow, Jeffery [5] and Hamel [6] formulated a mathematical foundation for the problem solution in detail. To be or not to be an external magnetic field applied on the fluid flow leads to quite different behaviors of a conducting fluid flowing [7, 8]. Controlling of the fluid flow by variations of the external magnetic field intensity is the main reason of the significance of the Jeffery–Hamel problems’ study. Moreover, the approach of the problem solution must include the non-dimensional factors (the magnetic Reynolds and the Hartmann numbers) as well as the angle of the plates. Therefore, a wide range of the solutions exists comparing to classical problem. Heretofore, to obtain an approximate solution for classical Jeffery-Hamel flow equation, a few of approaches are suggested [9, 10]. A variety of approaches such as the homotopy perturbation [11, 12], the differential transformation [13] and homotopy analysis [14–38] are introduced to obtain analytical solution for the nonlinear problems, because of its strong nonlinearity so far.

The aim of the present work is to introduce a precise analytical solution for the third-order nonlinear Jeffery-Hamel equation by using FTADM [39]. The exclusivity of the FTADM, as a new method, is the satisfaction of all boundary and initial conditions over the entire spatial and temporal domains. Moreover, the well-known strongly nonlinear third-order Jeffery–Hamel type equations are solved using the FTADM where the trend of rapid approach of the approximate results toward the exact solutions is demonstrated. We solve the problem with the Reynolds number equal to 10 and the plates with the angles of $\alpha = \pm 5^\circ$ and a set of the Hartmann numbers includes: 0, 200, 400, 800, 1000, 2000. Furthermore, we investigate the Jeffery-Hamel problem’s solutions by changing the values of the effective physical factors.
2. The Adomian decomposition method

The MHD Jeffery-Hamel problem, includes a confined fluid flow between two inclined sheets. The configuration of the flow depicts in figure 1.

ADM is introduced by Adomian [40–42]. The Solution of a differential equations (ordinary or partial) with the ADM is given as follows. Suppose a differential equation as below:

\[ G(u) = g(t), \]

where \( G \) is general operator. \( G \) could be separated into two individual linear and the nonlinear operators:

\[ G(u) = L(u) + N(u) = g(t) \]

(2)

The unknown function \( u(x, t) \) of the linear operator may be replaced by a series solution [43]:

\[ u = \sum_{n=0}^{\infty} u_n \]

(3)

The recursive equations give the solution components \( (u_n, n \geq 0) \). By equation (3), we can rewrite the linear part in the below form:

\[ L(u) = L \left( \sum_{n=0}^{\infty} u_n \right) \]

(4)

Adomian polynomials can convert the nonlinear part into the infinite series as below [44–50]:

\[ N(u) = \sum_{n=0}^{\infty} A_n = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dx^n} \left[ N \left( \sum_{i=0}^{\infty} X_i \right) \right]_{x=0}, \quad n = 0, 1, 2, \ldots \]

(5)

With some algebra, equation (5) may be rewritten as:

\[ N(u) = N(u_0) + (u - u_0)N'(u_0) + \frac{1}{2!}(u - u_0)^2 N''(u_0) + \frac{1}{3!}(u - u_0)^3 N'''(u_0) + \ldots \]

(6)

It is noticeable: equation (6) is similar to Taylor series expansion of the function \( u \) in the vicinity of a function \( u_0 \) (and not around a point as usual). By replacing equations (6) and (4) into equation (2), we rewrite the solution of equation (1) as:

\[ L \left( \sum_{n=0}^{\infty} u_n \right) + \sum_{n=0}^{\infty} A_n = g(t). \]

(7)

Deriving of The \( u \) components is done by solution of the recursive equations given in equation (7).

3. Fundamentals of FTADM

Fourier transform of both sides of equation (7) is:

\[ \hat{L} \left( \sum_{n=0}^{\infty} u_n \right) + \sum_{n=0}^{\infty} \hat{A}_n = \hat{g}(\omega). \]

(8)
Adomian polynomials, \( A_n \), have the below forms:

\[
\begin{align*}
A_0 &= N(u_0), \\
A_1 &= u_1 N'(u_0), \\
A_2 &= u_2 N'(u_0) + \frac{1}{2!} u_1^2 N''(u_0), \\
A_3 &= u_3 N'(u_0) + u_1 u_2 N''(u_0) + \frac{1}{3!} u_1^3 N'''(u_0), \\
A_4 &= u_4 N'(u_0) + \left( \frac{1}{2!} u_2^2 + u_1 u_3 \right) N''(u_0) + \frac{1}{2!} u_1^2 u_2 N'''(u_0) + \frac{1}{4!} u_1^4 N^{(iv)}(u_0).
\end{align*}
\] (9)

The recursive relations, in light of equation (8), are:

\[
L(\tilde{u}_0) = \tilde{g}, \\
\sum_{i=1}^{\infty} L(\tilde{u}_i) + \sum_{i=0}^{\infty} A_i = 0.
\] (10)

The last relations may be expand as:

\[
L(\tilde{u}_0) = \tilde{g}, \\
L(\tilde{u}_i) + A_0 = 0, \\
L(\tilde{u}_i) + A_1 = 0, \\
L(\tilde{u}_i) + A_2 = 0, \\
\vdots \\
L(\tilde{u}_i) + A_{k-1} = 0.
\] (11)

Here, we describe the solution step by step. Maple software package gives the value of \( \tilde{u}_0 \) for the first part of equation (11). Then, we obtain the value of \( u_0 \) by inverse Fourier transform applying. In light of the first part of equation (9), it defines the Adomian polynomial, \( A_0 \). The second part of equation (11) and Adomian polynomial \( A_0 \) give the value of \( u_1 \). The value of \( u_2 \), arises by the inverse Fourier transform to \( u_0 \). It leads to define the Adomian polynomial \( A_1 \) based on the second part of equation (9) and so on. This in turn results the complete evaluation of the solution components \( u_k \) upon using different corresponding parts of equations (5) and (11).

4. FTADM application on Jeffery-Hamel problem

Demonstration of the merits and validation of FTADM method are examined by solving the one-dimension, third-order and nonlinear Jeffery-Hamel problem in throughout of the domain. An accurate agreement exists between the solution of the Navier–Stokes equations, in the example of two-dimensional flow inside a channel with tilted sheets having a vertex and existence a source or sink, and the Jeffery Hamel problem. The form of the one-dimensional Jeffery-Hamel equation is [4, 11]:

\[
f''(\eta) + 2\alpha \Re f(\eta)f'(\eta) + (4 - H) \alpha^2 f'(\eta) = 0,
\] (12)

where \( \alpha \) is the angle between the two sheets, \( \Re \) and \( H \) are the Reynolds and the Hartmann numbers, respectively. Solving the equation (12) is done with considering the Dirichlet and Neumann boundary conditions as below:

\[
f(0) = 1, \quad f'(0) = 0, \quad f(1) = 0.
\] (13)

4.1. A precise analytical approximation approach (FTADM)

The Fourier transform of equation (12) gives:

\[
F(f''(\eta)) + 2\alpha \Re F(f(\eta)f'(\eta)) + (4 - H)\alpha^2 F(f'(\eta)) = 0,
\] (14)

where \( F \) denotes the FT. The integration by parts in the equation (14) leads to:

\[
F(f''(\eta)) = -f''(0) - i\omega f'(0) + \omega^2 f(0) - i\omega^3 F(f(\omega)), \\
F(f(\eta)f'(\eta)) = -\frac{1}{2} f^2(\eta)|_{\eta=0} + \frac{1}{2} i\omega F(f^2), \\
F(f'(\eta)) = -f'(0) + i\omega F(f(\omega)).
\] (15)
Replacing the equation (15) into the equation (14) gives:

\[-f''(0) - i\omega f'(0) + \omega^2 f(0) - i\omega^3 F(f(\omega)) + 2\alpha \text{Re}\left(-\frac{1}{2}f^2(\eta)\bigg|_{\eta=0} + \frac{1}{2}i\omega F(f^2)\right)\]

\[+ (4 - H)\alpha^2\left(-f(0) + i\omega F(f(\omega))\right) = 0.\]  

(16)

For solving the equation (16), we need to have \(f''(0)\). The left side of equation (12) undergoes the triple definite integration with the boundary condition \(f(0) = 1\), we have [11, 12]:

\[\int_0^1 \int_0^\eta \int_0^\eta f''(\eta)d\eta d\eta d\eta + 2\alpha \text{Re}\left(\frac{1}{2} \int_0^1 \int_0^\eta f(\eta)f'(\eta)d\eta d\eta d\eta\right)\]

\[+ (4 - H)\alpha^2\left(\int_0^1 \int_0^\eta f'(\eta)d\eta d\eta - \frac{1}{2}f'(0)\right) = 0.\]  

(17)

By taking the integration by parts, we rewrite equation (17) as:

\[f(1) - f(0) - f'(0) - \frac{1}{2}f''(0) + 2\alpha \text{Re}\left(\frac{1}{2} \int_0^1 \int_0^\eta f(\eta)d\eta d\eta - \frac{1}{4}f^2\bigg|_{\eta=0}\right)\]

\[+ (4 - H)\alpha^2\left(\int_0^1 \int_0^\eta f(\eta)d\eta d\eta - \frac{1}{2}f(0)\right) = 0.\]  

(18)

By applying the boundary conditions (equation (13)) and reordering the terms, the formulation for \(f''(0)\) is:

\[f''(0) = -2 + 2\alpha \text{Re}\left(\int_0^1 \int_0^\eta f(\eta)d\eta d\eta - \frac{1}{2}f^2\bigg|_{\eta=0}\right) + (4 - H)\alpha^2\left(\int_0^1 \int_0^\eta f(\eta)d\eta d\eta - 1\right).\]  

(19)

Replacing the equation (19) into the equation (16) and apply the boundary conditions defined by equation (15), we have:

\[-2 + 2\alpha \text{Re}\left(\int_0^1 \int_0^\eta f(\eta)d\eta d\eta - \frac{1}{2}f^2\bigg|_{\eta=0}\right) + (4 - H)\alpha^2\left(\int_0^1 \int_0^\eta f(\eta)d\eta d\eta - 1\right)\]

\[+ \omega^2 - i\omega^3 F(f(\omega)) + 2\alpha \text{Re}\left(-\frac{1}{2}f^2(\eta)\bigg|_{\eta=0} + \frac{1}{2}i\omega F(f^2(\eta))\right)\]

\[+ (4 - H)\alpha^2(-1 + i\omega F(f(\omega))) = 0.\]  

(20)

Utilizing of equations (12) and (18), (20) may be rewritten as follows:

\[-2 + 2\alpha \text{Re}\left(\int_0^1 \int_0^\eta \sum_{n=0}^{\infty} A_n(\eta)d\eta d\eta - \frac{1}{2} \sum_{n=0}^{\infty} A_n\bigg|_{\eta=0}\right)\]

\[+ (4 - H)\alpha^2\left(2 \int_0^1 \int_0^\eta \sum_{n=0}^{\infty} f_n(\eta)d\eta d\eta - 1\right)\]

\[+ \omega^2 - i\omega^3 \sum_{n=0}^{\infty} f_n(\omega) + 2\alpha \text{Re}\left(-\frac{1}{2} \sum_{n=0}^{\infty} A_n(\eta)\bigg|_{\eta=0}\right)\]

\[+ \frac{1}{2}i\omega^2 \sum_{n=0}^{\infty} A_n(\omega) + (4 - H)\alpha^2\left(-1 + i\omega \sum_{n=0}^{\infty} f_n(\omega)\right) = 0,\]  

(21)
Using equation (21), we establish the recursive equations as below:

\[-i\omega f_0^0(\omega) + \omega^2 = 0,\]
\[-i\omega f_1(\omega) - 2 = 0,\]
\[2\alpha \text{Re} \left( \int_0^1 \int_0^\eta A_0(\eta)d\eta d\eta - \frac{1}{2}A_0 \right) + \]
\[+ (4 - \mathcal{H})\alpha^2 \left( 2 \int_0^1 \int_0^\eta f_0(\eta)d\eta d\eta - 1 \right) + \]
\[-i\omega^2 f_2(\omega) + 2\alpha \text{Re} \left( -\frac{1}{2}A_0(\eta) \right) + \]
\[+ \frac{1}{2}i\omega A_0(\omega) \right) + (4 - \mathcal{H})\alpha^2 \left( -1 + i\omega f_0(\omega) \right) = 0,\]
\[2\alpha \text{Re} \left( \int_0^1 \int_0^\eta \sum_{n=0}^\infty A_0(\eta)d\eta d\eta - \frac{1}{2} \sum_{n=0}^\infty A_1 \right) + \]
\[+ (4 - \mathcal{H})\alpha^2 \left( 2 \int_0^1 \int_0^\eta \sum_{n=0}^\infty f_1(\eta)d\eta d\eta - 1 \right) + \]
\[-i\omega^2 \sum_{n=0}^\infty f_2(\omega) + 2\alpha \text{Re} \left( -\frac{1}{2} \sum_{n=0}^\infty A_1(\eta) \right) + \]
\[+ \frac{1}{2}i\omega \sum_{n=0}^\infty A_1(\omega) \right) + (4 - \mathcal{H})\alpha^2 \left( -1 + i\omega \sum_{n=0}^\infty f_1(\omega) \right) = 0,\]
\[\vdots\]
\[2\alpha \text{Re} \left( \int_0^1 \int_0^\eta A_k(\eta)d\eta d\eta - \frac{1}{2}A_k \right) + \]
\[+ (4 - \mathcal{H})\alpha^2 \left( 2 \int_0^1 \int_0^\eta f_k(\eta)d\eta d\eta - 1 \right) + \]
\[-i\omega^2 \sum_{n=0}^\infty f_{k+1}(\omega) + 2\alpha \text{Re} \left( -\frac{1}{2}A_k(\eta) \right) + \]
\[+ \frac{1}{2}i\omega A_k(\omega) \right) + (4 - \mathcal{H})\alpha^2 \left( -1 + i\omega f_k(\omega) \right) = 0,\]

where \(A_0, A_1, A_2, A_3, \ldots, A_k\) are the FT of the respective Adomian polynomials. The procedures are as follows: the first and second part of equation (22) gives the values of \(f_0^0\) and \(f_1\), respectively. Then applying the inverse Fourier transform, using the Maple software package, gives the values of \(f_0^0\) and \(f_1\). Then applying the Fourier transform to first part of equation (9), \(A_0\) gives the value of \(A_0\). Having the values of \(f_0^0\), \(A_0\) and \(A_0\) will define the third part of equation (22). This provides the value of \(f_2\). Then applying the inverse FT, gives the value of \(f_2\). Then applying the FT to second part of equation (9), \(A_1\) gives the value of \(A_1\). Having the values of \(f_1\), \(A_1\) and \(A_1\) will define the fourth part of equation (22). Having the values of \(f_1\), \(A_1\), \(A_1\), the fourth part of equation (22) will provide the value of \(f_2\). Then, applying the inverse FT provides the value of \(f_2\) and so on. This in turn leads to the
**Table 2.** Comparison between the present data of $f^0(0)$ with with results of [9] (various Hartmann number, $\alpha = 5^5$, $Re = 10$).

| H(Hartmann number) | Numerical results [8] | Exact [9] | Present results |
|--------------------|------------------------|-----------|-----------------|
| 0                  | −2.251 948 58          | −2.251 948 602 981 818 | −2.251 948 51 |
| 200                | −1.984 606 16          | −1.984 606 164 603 458 | −1.984 606 18 |
| 400                | −1.754 093 06          | −1.754 093 033 347 798 | −1.754 093 01 |
| 600                | −1.554 606 00          | −1.554 605 992 057 426 | −1.554 606 20 |
| 800                | −1.381 370 05          | −1.381 369 953 213 575 | −1.381 370 40 |
| 1000               | −1.230 437 21          | −1.230 437 181 792 459 | −1.230 437 29 |
| 2000               | −0.712 584 86          | −0.712 584 949 074 417 | −0.712 646 04 |

**Table 3.** The numerical Runge–Kutta data versus the FTADM results for velocity ($\alpha = -5^5$, $H = 0$, $Re = 10$, solutions components of 12).

| $\eta$ | Numerical | FTADM |
|--------|-----------|-------|
| 0      | 1         | 1     |
| 0.1    | 0.991 064 53 | 0.991 064 53 |
| 0.2    | 0.964 106 07 | 0.964 106 07 |
| 0.3    | 0.918 674 39 | 0.918 674 39 |
| 0.4    | 0.854 039 62 | 0.854 039 62 |
| 0.5    | 0.769 225 88 | 0.769 225 89 |
| 0.6    | 0.663 063 92 | 0.663 063 93 |
| 0.7    | 0.534 268 04 | 0.534 268 05 |
| 0.8    | 0.381 543 72 | 0.381 543 73 |
| 0.9    | 0.203 732 55 | 0.203 732 55 |
| 1.0    | 0         | −2.015 696 372 10$^{-7}$ |

The complete evaluation of the components of $f_k$, $k \geq 0$, upon using different corresponding parts of equations (9) and (22). Solving the recursive equation, equation (22), gives:

$$f_0 = 1,$$
$$f_1 = -x^2,$$
$$f_2 = -\frac{1}{2}(166 666 666 7(4 - H))\alpha^2 + .333 333 333 3\alpha Re)x^2+$$
$$+ \frac{1}{24}((2(4 - H))\alpha^2 + 4\alpha)x^4$$
$$f_3 = -\frac{1}{2}(-2(4 - H))\alpha^2(-0.004 166 666 670(4 - H))\alpha^2+$$
$$+ 0.005 555 555 55\alpha - 0.013 888 888 89\alpha Re)$$
$$+ 2\alpha (-0.008 333 333 333(4 - H))\alpha^2 + 0.011 111 11\alpha+$$
$$+ 0.033 333 333 3 - 0.027 777 777 77 Re\alpha)x^2+$$
$$+(4 - H)\alpha^2(0.166 666 666 7(4 - H))\alpha^2 + 0.333 333 333 3\alpha Re)+$$
$$+ 3.333 333 333 333 10^{-11} Re\alpha(10^{10} Ha^2 - 2 10^{10} Re\alpha - 4 10^{10} Ha^2)0.041 666 666 67 x^4-$$
$$-(4 - H)\alpha^2(2(4 - H))\alpha^2 + 4\alpha + 3.333 333 333 333 10^{-11} Re\alpha(-4.8 10^{13} \alpha^2+$$
$$+ 2.4 10^{13}\alpha - 7.2 10^{11}+ 1.2 10^{11}Ha^2)0.001 388 888 889x^6$$
$$f_4 = -\frac{1}{2}(-2Re\alpha(-0.000 049 603 174 61(4 - H))\alpha^2((2(4 - H))\alpha^2 + 4\alpha)-$$
$$- 1.653 439 154 10^{-12} Re\alpha(-4.8 10^{13} \alpha^2 - 2.4 10^{13} \alpha - 7.2 10^{11}+ 1.2 10^{11}Ha^2)+$$
$$+ 0.002 579 365 081 (4 - H)\alpha^2 - 0.005 952 380 953\alpha + 0.002 777 777 778$$
$$\times (4 - H)\alpha^2(0.166 666 666 7(4 - H))\alpha^2 + 0.333 333 333 3\alpha+$$
$$+ 9.259 259 260 10^{-14} Re\alpha(10^{10} Ha^2 - 2 10^{10} Re\alpha - 4 10^{10} Ha^2)+$$
$$+ 0.011 111 111 11 Re\alpha + 0.166 666 666 6(4 - H)\alpha^2(-0.004 166 666 670 (4 - H))\alpha^2+$$
$$+ 0.005 555 555 55\alpha - 0.013 888 888 89 Re\alpha + 0.166 666 666 6 R a (-0.008 333 333 333 (4 - H)\alpha^2+$$
$$+ 0.011 111 111 11\alpha + 0.033 333 333 3 - 0.027 777 777 77 Re\alpha)-$$
Table 4. The numerical Runge–Kutta data and the FTADM results for velocity ($\alpha = 5^\circ$, $H = 0$, $Re = 10$, solution components of 12).

| $\eta$ | Numerical | FTADM |
|--------|------------|-------|
| 0      | 1          | 1     |
| 0.1    | 0.988 756 87 | 0.988 756 87 |
| 0.2    | 0.955 224 65 | 0.955 224 67 |
| 0.3    | 0.899 978 37 | 0.899 978 42 |
| 0.4    | 0.823 922 90 | 0.823 922 99 |
| 0.5    | 0.728 217 39 | 0.728 217 53 |
| 0.6    | 0.614 177 89 | 0.614 178 07 |
| 0.7    | 0.483 164 93 | 0.483 165 11 |
| 0.8    | 0.336 462 63 | 0.336 462 77 |
| 0.9    | 0.175 154 64 | 0.175 154 66 |
| 1.0    | 0          | $-2.027 421 781 10^{-7}$ |

Table 5. The numerical Runge–Kutta data and the FTADM results for velocity ($\alpha = 5^\circ$, $H = 200$, $Re = 10$ and solution components of 14).

| $\eta$ | Numerical | FTADM |
|--------|------------|-------|
| 0      | 1          | 1     |
| 0.1    | 0.992 098 08 | 0.992 098 08 |
| 0.2    | 0.968 136 87 | 0.968 136 87 |
| 0.3    | 0.927 348 50 | 0.927 348 51 |
| 0.4    | 0.868 449 23 | 0.868 449 24 |
| 0.5    | 0.789 637 18 | 0.789 637 20 |
| 0.6    | 0.688 597 21 | 0.688 597 25 |
| 0.7    | 0.562 521 67 | 0.562 521 75 |
| 0.8    | 0.408 159 94 | 0.408 160 06 |
| 0.9    | 0.221 913 70 | 0.221 913 80 |
| 1.0    | 0          | $-2.261 398 79 10^{-7}$ |

$-(2(4 - H))\alpha^2(-(0.000 024 801 587 30(4 - H))\alpha^2((2(4 - H))\alpha^2 + 4\alpha)+$ $+ 8.267 195 766 10^{-14}I Re \alpha(-4.810^{14}I\alpha^2 - 2.4 10^{14}I\alpha - 7.2 10^{14}I + 1.2 10^{14}I\alpha^2) + + 0.001 388 888 889 (4 - H)\alpha^2$ $\times (0.166 666 666 (4 - H)\alpha^2 + 0.333 333 333 Re \alpha)+$ $+ 4.629 629 630 10^{-14}Re \alpha(10^{10}H\alpha^2 - 2 10^{10}Re \alpha - 4 10^{10}\alpha^2)+$ $+ 0.083 333 333 35 (4 - H)\alpha^2(-0.004 166 666 670 (4 - H)\alpha^2+ + 0.005 555 555 555\alpha - 0.013 888 888 89 Re \alpha)+$ $+ 0.083 333 333 35 Re \alpha(-0.008 333 333 334 (4 - H)\alpha^2$ $+ 0.011 111 111 11\alpha + 0.033 333 333 3 - 0.027 777 777 77 Re \alpha))x^2+ + ((6.666 666 666 10^{-3}-4 Re \alpha(-0.266 666 666\alpha^4 + 0.133 333 333 H\alpha^4$ $+ 0.088 888 888 88\alpha^3 - 0.016 666 666 68 H^2\alpha^3+ + 0.044 444 444 44 Re \alpha^2 + 0.088 888 888 90 Re \alpha^3+- 0.111 111 111 Re^2\alpha^2 - 0.355 555 555 556 Re^3$ $- 0.022 222 222 22 H\alpha^3 + 0.133 333 333 Re \alpha)+$ $(4 - H)\alpha^2(-2(4 - H)\alpha^2(-0.004 166 666 670 (4 - H)\alpha^2$ $+ 0.005 555 555 555\alpha - 0.013 888 888 89 Re \alpha) -$ $2 Re \alpha(-0.008 333 333 334 (4 - H)\alpha^2 + 0.011 111 111 11\alpha$ $+ 0.033 333 333 3 - 0.027 777 777 77 Re \alpha))0.041 666 666 67\alpha^4+ + ((6.666 666 666 10^{-3} Re \alpha(-8.000 000 002 10^{5}\alpha^4 + 6.000 000 001 10^{5}H\alpha^2-$ $- 1.200 000 000 10^{0}Re \alpha - 5.000 000 001 10^{2}H^2\alpha^4$ $J. Phys. Commun. 4 (2020) 075009 S Nourazar et al
Table 6. Comparison between the numerical Runge–Kutta data and the FTADM results for velocity $(\alpha = 5^\circ, H = 200, Re = 10$ and solution components of 12).

| $\eta$ | Numerical | FTADM |
|-------|-----------|-------|
| 0     | 0.990 079 03 | 0.990 079 03 |
| 0.1   | 0.960 339 47 | 0.960 339 47 |
| 0.2   | 0.910 841 02 | 0.910 841 03 |
| 0.3   | 0.841 648 87 | 0.841 648 87 |
| 0.4   | 0.732 782 46 | 0.732 782 47 |
| 0.5   | 0.644 144 20 | 0.644 144 21 |
| 0.6   | 0.515 427 36 | 0.515 427 38 |
| 0.7   | 0.366 001 51 | 0.366 001 53 |
| 0.8   | 0.194 771 14 | 0.194 771 14 |
| 0.9   | 0.000 000 00 | $-1.658 167 918 10^{-7}$ |
| 1.0   | 0          | $-1.658 167 918 10^{-7}$ |

Table 7. The numerical Runge–Kutta data and the FTADM results for velocity $(\alpha = -5^\circ, H = 400, Re = 10$ and solution components of 15).

| $\eta$ | Numerical | FTADM |
|-------|-----------|-------|
| 0     | 0.992 990 68 | 0.992 990 68 |
| 0.1   | 0.971 628 55 | 0.971 628 54 |
| 0.2   | 0.934 900 56 | 0.934 900 53 |
| 0.3   | 0.881 083 92 | 0.881 083 87 |
| 0.4   | 0.807 697 66 | 0.807 697 60 |
| 0.5   | 0.711 441 58 | 0.711 441 57 |
| 0.6   | 0.588 132 27 | 0.588 132 33 |
| 0.7   | 0.432 650 90 | 0.432 651 01 |
| 0.8   | 0.238 927 80 | 0.238 927 50 |
| 0.9   | 0.000 000 00 | $-2.791 960 95 10^{-7}$ |
| 1.0   | 0          | $-2.791 960 95 10^{-7}$ |

\[ + 2.000 000 000 10^{36} Re^2 \alpha^2 + 4.000 000 000 10^{36} H \alpha^4 \\
+ 10^{31} Re \alpha^3 - 2.400 000 000 10^{31} \alpha^2 + (4 - H) \alpha^2 (0.166 666 666 7 \\
\times (4 - H) \alpha^2 + 0.333 333 333 333 Re \alpha) - 3.333 333 333 - 10^{-11} Re \alpha \\
\times ((10^6 H \alpha^2 - 2.10^6 Re \alpha - 4.000 000 001 10^{30} \alpha^2))) \\
\times (-0.001 388 888 889) \alpha^6 + (((6.666 666 666 10^{-3}) Re \alpha (4.800 000 000 10^{31} H \alpha^4 \\
+ 1.8 10^{32} H \alpha^2 + 7.200 000 000 10^{31} Re \alpha + 2.4 10^{31} Re \alpha^2 - 3.6 10^{32} \alpha \\
+ 9.600 000 000 10^{31} \alpha^4 - 1.2 10^{31} Re \alpha^2 - 7.2 10^{32} \alpha^2 \\
+ 1.210^6 H \alpha^3 - 4.8 10^{31} \alpha^3 + 4.8 10^{31} Re \alpha^3 \\
- 6 10^{30} H^2 \alpha^2 + (4 - H) \alpha^2 (2(4 - H) \alpha^2 + 4 \alpha) + 3.333 333 333 \\
\times 10^{-11} Re \alpha (-4.8 10^{31} \alpha^2 - 2.4 10^{31} \alpha - 7.2 10^{11} + 1.2 10^{31} H \alpha^2))) \\
\times 0.000 024 801 5873 \alpha^8, \]
and so on.

5. Results and comparisons

By comparing the results of our method and well-known and valuable results of [8], we show the validation and accuracy of our method. Tables 1 and 2 depict our results for $f^0(0)$ and the true value and the numerical data [8]
for six various Hartmann numbers and \( Re = 10 \) while \( \alpha = \pm 5^\circ \), respectively. Obviously, there is a good similarity between the (FTADM) data, the true value and the numerical data [8]. Tables 3–12 show our data for velocity provide through the FTADM with the numerical Runge–Kutta output for \( Re = 10, \alpha = \pm 5^\circ \) and Hartmann numbers 0, 200, 400, 800, 1000, 2000, respectively. Our comparison indicates brilliant agreement with the numerical Runge–Kutta outputs. The order of the error has the maximum of \( 10^{-5} \). Figures 2 and 3 depict the velocity profiles for a constant \( Re \) and \( \alpha = \pm 5^\circ \) at various Hartmann numbers.

### Table 8. Comparison between the numerical Runge–Kutta data and the FTADM results for velocity (\( \alpha = 5^\circ, H = 400, Re = 10 \) and solution components of 12).

| \( \eta \) | Numerical | FTADM |
|-----------|-----------|-------|
| 0         | 1         | 1     |
| 0.1       | 0.991 220 22 | 0.991 220 22 |
| 0.2       | 0.964 767 88 | 0.964 767 87 |
| 0.3       | 0.920 294 34 | 0.920 294 30 |
| 0.4       | 0.857 185 75 | 0.857 185 70 |
| 0.5       | 0.774 311 43 | 0.774 311 31 |
| 0.6       | 0.670 946 35 | 0.670 946 19 |
| 0.7       | 0.544 661 11 | 0.544 660 94 |
| 0.8       | 0.393 169 72 | 0.393 169 60 |
| 0.9       | 0.213 119 59 | 0.213 119 55 |
| 1.0       | 0         | 2.293 984 2928 \( 10^{-7} \) |

### Table 9. Comparison between the numerical Runge–Kutta data and the FTADM results for velocity (\( \alpha = -5^\circ, H = 800, Re = 10 \) and solution components of 18).

| \( \eta \) | Numerical | FTADM |
|-----------|-----------|-------|
| 0         | 1         | 1     |
| 0.1       | 0.994 438 72 | 0.994 438 75 |
| 0.2       | 0.977 318 44 | 0.977 318 55 |
| 0.3       | 0.947 298 99 | 0.947 299 23 |
| 0.4       | 0.902 043 30 | 0.902 043 64 |
| 0.5       | 0.838 058 74 | 0.838 059 14 |
| 0.6       | 0.750 470 28 | 0.750 470 62 |
| 0.7       | 0.632 723 97 | 0.632 724 12 |
| 0.8       | 0.476 226 79 | 0.476 226 64 |
| 0.9       | 0.269 943 47 | 0.269 942 96 |
| 1.0       | 0         | -8.734 435 43 \( 10^{-7} \) |

### Table 10. Comparison between the numerical Runge–Kutta data and the FTADM results for velocity (\( \alpha = 5^\circ, H = 800, Re = 10 \) and solution components of 12).

| \( \eta \) | Numerical | FTADM |
|-----------|-----------|-------|
| 0         | 1         | 1     |
| 0.1       | 0.993 068 26 | 0.993 068 26 |
| 0.2       | 0.971 971 90 | 0.971 971 91 |
| 0.3       | 0.935 789 12 | 0.935 789 22 |
| 0.4       | 0.882 920 65 | 0.882 920 83 |
| 0.5       | 0.810 988 04 | 0.810 988 05 |
| 0.6       | 0.716 674 09 | 0.716 674 52 |
| 0.7       | 0.595 492 04 | 0.595 492 51 |
| 0.8       | 0.441 443 08 | 0.441 443 14 |
| 0.9       | 0.246 523 51 | 0.246 521 67 |
| 1.0       | 0         | -1.111 951 079 \( 10^{-5} \) |
6. Conclusions

In this paper, a new modification of the ADM, the Fourier transform Adomian decomposition method (FTADM), is proposed to solve the nonlinear MHD Jeffery-Hamel equation. The comparison of our results for $f''(0)$ at various Hartmann numbers obtained via the FTADM with the exact and numerical data [8] show...
excellent agreement. The comparison of velocity obtained using the FTADM for $\text{Re} = 10$ and $\alpha = \pm 5^\circ$ at different Hartmann numbers with those obtained by the numerical Runge–Kutta method show excellent agreements and the maximum error is on the order of $10^{-5}$. We conclude that the new FTADM is a capable and precise approximate semi-analytical approach for solving the nonlinear MHD Jeffery–Hamel equation.

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**Table 11.** Comparison between the numerical Runge–Kutta data and the FTADM results for velocity ($\alpha = -5^\circ$, $H = 1000$, $\text{Re} = 10$ and solution components of 16).

| $\eta$ | Numerical | FTADM |
|---|---|---|
| 0.1 | 0.995 027 33 | 0.995 028 21 |
| 0.2 | 0.979 643 09 | 0.979 645 51 |
| 0.3 | 0.952 404 23 | 0.952 408 89 |
| 0.4 | 0.910 767 63 | 0.910 773 49 |
| 0.5 | 0.850 871 69 | 0.850 877 34 |
| 0.6 | 0.767 216 39 | 0.767 217 36 |
| 0.7 | 0.652 228 82 | 0.652 219 28 |
| 0.8 | 0.495 706 50 | 0.495 679 50 |
| 0.9 | 0.284 146 16 | 0.284 094 65 |
| 1.0 | 0 | $-8.224 323 99 10^{-5}$ |

**Table 12.** Comparison between the numerical Runge–Kutta data and the FTADM results for velocity ($\alpha = -5^\circ$, $H = 2000$, $\text{Re} = 10$ and solution components of 14).

| $\eta$ | Numerical | FTADM |
|---|---|---|
| 0.01 | 0.997 065 75 | 0.997 064 53 |
| 0.1 | 0.987 758 89 | 0.987 754 30 |
| 0.3 | 0.970 481 45 | 0.970 472 27 |
| 0.5 | 0.942 271 27 | 0.942 257 63 |
| 0.7 | 0.898 303 82 | 0.898 287 37 |
| 0.9 | 0.831 089 37 | 0.831 072 87 |
| 1.0 | 0.729 248 08 | 0.729 234 40 |
| 0.8 | 0.575 698 39 | 0.575 689 49 |
| 0.9 | 0.345 052 07 | 0.345 048 26 |
| 1.0 | 0 | $3.441 655 708 10^{-5}$ |
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