Singular linear statistics of the Laguerre Unitary Ensemble and Painlevé III ($P_{III}$): Double scaling analysis.

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Abstract
We continue with the study of the Hankel determinant,

$$D_n(t, \alpha) := \det \left( \int_0^\infty x^{i+k} w(x; t, \alpha) dx \right)_{j,k=0}^{n-1},$$

generated by singularly perturbed Laguerre weight,

$$w(x; t, \alpha) := x^\alpha e^{-x} e^{-t/x}, \quad 0 \leq x < \infty, \quad \alpha > 0, \quad t > 0,$$

obtained through a deformation of the Laguerre weight function,

$$w(x; 0, \alpha) := x^\alpha e^{-x}, \quad 0 \leq x < \infty, \quad \alpha > 0,$$

via the multiplicative factor $e^{-t/x}$.

An earlier investigation was made on the finite $n$ aspect of the problem, this has appeared in [18]. There, it was found that the logarithm of the Hankel determinant has an integral representation in terms of a particular $P_{III}$, and its derivative with $t$. In this paper we show that, under a double scaling, where $n$, the order of the Hankel matrix tends to $\infty$, and $t$, tends to $0$, the scaled—and therefore, in some sense, infinite dimensional—Hankel determinant, has an integral representation in terms of the $C$ potential, and its derivatives. The second order non-linear differential equation which the $C$ potential satisfies, after a minor change of variables, is another $P_{III}$, albeit with fewer number of parameters.

Expansions of the double scaled determinant for small and large parameter are obtained.

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1 Introduction.

The studies of the Hankel determinant play an important role in the random matrix theory \[27\]. The second author and his collaborators made use of a theorem in linear statistics to investigate Hankel determinants and the associated orthogonal polynomials, see \[3\], \[5\], \[6\], \[19\], \[21\] based on Dyson’s Coulomb Fluid \[25\]. See also \[15\], \[20\]. Bonan and others used orthogonal polynomials theory and their corresponding ladder operators, found the expressions for the Hankel determinants \[2\], \[8\], \[9\], \[10\], \[11\], \[12\], see also \[4\], \[13\], \[16\], \[17\], \[23\], \[24\]. Tracy and Widom, in their fundamental contributions characterized the logarithmic derivatives of operator determinants and obtained Painlevé equations associated with the Airy and Bessel kernels, after suitable double scaling, in the study of the level spacing distributions \[31\], \[32\], \[33\]. Adler and Van Moerbeke obtained differential equations using a multi-time approach, \[1\].

The Hankel determinant, generated by the singularly perturbed Laguerre weight, defined by,

\[
D_n(t, \alpha) = \det \left( \int_0^\infty x^{j+k} w(x; t, \alpha) dx \right)_{j,k=0}^{n-1},
\]

where the weight reads,

\[
w(x; t, \alpha) = x^\alpha e^{-x-t/x}, \quad t \geq 0, \quad \alpha > 0,
\]

is motivated in part by a finite temperature integrable quantum field theory, \[7\]. Recently this has appeared as the generating function for the Wigner delay time distribution in chaotic cavities \[30\], modelled as a Hermitian random matrix ensembles. Mathematically, such a singular perturbation, introduces an infinitely strong zero on the weight at 0, and has the effect of pushing the left edge of the equilibrium density (or the charge density, if we view the eigenvalues of the our random matrix as charges) further away from 0, in contrast with the situation of Laguerre weight, \(x^\alpha e^{-x}\).

To set the stage for later development, we note that Hankel can also be expressed as

\[
D_n(t, \alpha) = \prod_{j=0}^{n-1} h_j(t)
\]

where \(\{ h_j(t) : j = 0, ..., n - 1 \} \) are the squares of the weighted norms of monic polynomials \(P_n(x)\) orthogonal with respect to \(w(x; t, \alpha)\), namely,

\[
\int_0^\infty P_j(x) P_k(x) w(x; t, \alpha) dx = h_j(t) \delta_{jk}.
\]

Although the polynomials also depend on \(t\), through its coefficients, we do not always display this.

From the Heine (multiple-)integral, the orthogonal polynomials, maybe represented as follows:

\[
P_n(z; t, \alpha) = \frac{1}{n! D_n(t, \alpha)} \int_{(0, +\infty)^n} \prod_{m=1}^{n} (z - x_m) \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{\ell=1}^{n} x^{x_{\ell}} e^{-x_{\ell} - t/x_{\ell}} dx_{\ell}
\]

2
where the Hankel determinant can also be represented as a multiple integral,

\[
D_n(t, \alpha) = \frac{1}{n!} \int_{(0, +\infty)^n} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{\ell=1}^n x_\ell^\alpha e^{-x_\ell - \frac{t}{x_\ell}} \, dx_\ell.
\]

We see that,

\[
(-1)^n P_n(0; t, \alpha) = \frac{D_n(t, \alpha + 1)}{D_n(t, \alpha)},
\]

and for the monic Laguerre polynomials, there is a closed form evaluation, see \((5.1.7), [29]\),

\[
(-1)^n P_n(0; 0, \alpha) = \frac{\Gamma(n + 1 + \alpha)}{\Gamma(1 + \alpha)}.
\]

In this paper we are concerned with the evaluation, as \(n \to \infty\) of the Hankel determinant \(D_n(t, \alpha)\) and the evaluation of the orthogonal polynomials \(P_n(z; t, \alpha)\), \(z \in \mathbb{C}\) but \(z \notin [a, b]\). Here \(a\) and \(b\), depending on \(n\) and \(t\) are the left and right edges of the equilibrium density,

\[
\sigma(x) = \frac{\sqrt{(b - x)(x - a)}}{2\pi} \left[ \left( \frac{\alpha}{\sqrt{ab}} + \frac{t(a + b)}{2(ab)^{3/2}} \right) \frac{1}{x} + \frac{t}{x^2\sqrt{ab}} \right], \quad a \leq x \leq b,
\]

obtained by solving the singular integral equations,

\[
v'(x) - 2P \int_a^b \frac{\sigma(y)}{x - y} \, dy = 0, \quad x \in [a, b],
\]

with the condition that the equilibrium density, \(\sigma\), vanishes at the end points of the support. A description of such formalism, can be found for example, in \([15]\) and \([19]\). Here \(v(x) = -\ln w(x, t)\), and consequently

\[
v'(x) = -\frac{x}{x} + 1 - \frac{t}{x^2}.
\]

The end points of the single interval \([a, b]\) are determined by the normalization condition

\[
\int_a^b \sigma(x) \, dx = n,
\]

and a further supplementary conditions, which can also be found for example, in \([15]\) and \([19]\). These are

\[
\int_a^b \frac{xv'(x)}{\sqrt{(b - x)(x - a)}} \, dx = 2\pi n,
\]

and

\[
\int_a^b \frac{v'(x)}{\sqrt{(b - x)(x - a)}} \, dx = 0.
\]
With the $v'(x)$ displayed, and with the aid of the integrals in the Appendix $A$, one finds, $a$ and $b$, satisfy the algebraic equations,

$$
2n + \alpha + \frac{t}{\sqrt{ab}} = \frac{a + b}{2},
$$

(1.7)

and

$$
\frac{(a + b)t}{2(ab)^{3/2}} + \frac{\alpha}{\sqrt{ab}} = 1.
$$

(1.8)

We state a lemma here, which gain us insight into a particular $P_{III}$ in the finite $n$ setting.

**Lemma 1.** The geometric mean of boundary points $a$ and $b$, namely, $\tilde{X} := \sqrt{ab}$ satisfies the quartic

$$
\tilde{X}^4 - \alpha \tilde{X}^3 - (2n + \alpha)t \tilde{X} - t^2 = 0.
$$

(1.9)

**Proof.** Substituting $\tilde{X} = \sqrt{ab}$ into (1.7), (1.8) and making use of

$$
\frac{a + b}{2} = 2n + \alpha + \frac{t}{\tilde{X}},
$$

to eliminate $a + b$ which gives (1.9). □

## 2 Double Scaling.

In this section we study the effect of double scaling on the finite $n$ Hankel determinant which originates from our singular weight

$$
w(x; t) = x^\alpha e^{-x - t/x}, \quad x \in [0, \infty), \quad t > 0, \text{ and } \alpha > 0.
$$

In order to characterize the asymptotic behaviour of the Hankel determinant, we recall some results in [18].

For convenience, we use $t$ and $y_n(t)$ instead of $s$ and $a_n(s)$, respectively. See (Theorem 1, [18]).

**Theorem 1.** The diagonal recurrence coefficients, $\alpha_n(t)$, maybe expressed as,

$$
\alpha_n(t) = 2n + 1 + \alpha + y_n(t),
$$

where the auxiliary quantity $y_n(t)$, $n = 0, 1, 2, \ldots$ satisfies

$$
y_n'' = \frac{(y_n')^2}{y_n} - \frac{y_n'}{t} + (2n + 1 + \alpha)\frac{y_n^2}{t^2} + \frac{y_n^3}{t^3} + \frac{\alpha}{t} - \frac{1}{y_n},
$$

(2.10)
with the initial conditions
\[ y_n(0) = 0, \quad y_n'(0) = \frac{1}{\alpha}, \quad \alpha > 0. \] (2.11)

If \( y_n(t) := -q(t) \), then \( q(t) \) is a solution of \( P_{III}(-4(2n + 1 + \alpha), -4\alpha, 4, -4) \), following the convention of [28].

Substituting \( y_n(t) := \frac{t}{X_n(t)} \), then \( X_n(t) \) satisfies,
\[ X_n'' = \frac{(X_n')^2}{X_n} - \frac{X_n'}{t} - \frac{\alpha(X_n')^2}{t^2} - \frac{2n + 1 + \alpha}{t} + \frac{X_n^3}{t^2} - \frac{1}{X_n}, \] (2.12)
which is recognized to another particular \( P_{III}(-4\alpha, -4(2n + 1 + \alpha), 4, -4) \).

If we disregard the derivatives in (2.12), then \( X_n \) satisfies a quartic equation,
\[ X_n^4 - \alpha X_n^3 - (2n + 1 + \alpha) t X_n - t^2 = 0, \]
and note that that the quartic obtained above becomes the quartic obtained previously via
the equilibrium problem, (1.9), if we replace \( 2n + 1 \) by \( 2n \).

Theorem 2 in [18] reveals an important relation between \( X_n \) and the logarithmic derivative
of the Hankel determinant which we recall as follows:

Theorem 2.
\[ \ln \frac{D_n(t, \alpha)}{D_n(0, \alpha)} = \int_0^t \left( \frac{\xi}{2} - \frac{1}{4} (X_n - \alpha)^2 - \frac{\alpha}{2} \frac{\xi}{X_n} - \frac{\xi^2}{4X_n^2} + \frac{\xi^2(X_n')^2}{4X_n^2} \right) d\xi. \] (2.13)

The next theorem recalls ((3.23) and (3.24), in [18]),

Theorem 3. If
\[ H_n(t) := t \frac{d}{dt} \ln D_n(t, \alpha), \] (2.14)
then
\[ (tH_n'')^2 = [n - (2n + \alpha) H_n']^2 - 4 [n(\alpha + \alpha) + tH_n - H_n] H_n' (H_n' - 1). \] (2.15)

The equation (2.15) is the Jimbo-Miwa-Okamoto \( \sigma \)-form of \( P_{III} \), see [26]. From (2.13)
and (2.14), we see that \( H_n(t) \) maybe expressed in terms of \( X_n(t) \) as
\[ H_n = \frac{t}{2} - \frac{1}{4} (X_n - \alpha)^2 - \frac{\alpha}{2} \frac{t}{X_n} - \frac{t^2}{4X_n^2} + \frac{t^2(X_n')^2}{4X_n^2}. \] (2.16)
The double scaling process is obtained by sending \( n \to \infty \) and \( t \to 0 \) in such a way that
\( (2n + 1 + \alpha) t \) is fixed.
Theorem 4. If
\[ s := (2n + 1 + \alpha)t, \]
where \( t \to 0 \) and \( 2n + 1 + \alpha \to \infty \), and \( s \) is finite. Let
\[ C(s) := \lim_{n \to \infty} \frac{1}{X_n(s/(2n+1+\alpha))}, \quad \text{and} \quad \Delta(s, \alpha) := \lim_{n \to \infty} D_n\left(\frac{s}{2n + 1 + \alpha}\right), \quad (2.17) \]
then the \( C \) potential, satisfies a "lesser" \( P_{III} \)
\[ C'' = \frac{(C')^2}{C} - \frac{C''}{s} + \frac{C^2}{s^2} - \frac{1}{s^2 C} + \frac{\alpha}{s^2}, \quad (2.18) \]
and if
\[ \mathcal{H}(s) := \lim_{n \to \infty} H_n(s/(2n + 1 + \alpha)) = s \frac{d}{ds} \ln \Delta(s, \alpha), \]
then \( \mathcal{H}(s) \) satisfies
\[ (s\mathcal{H}'')^2 + 4(\mathcal{H}')^2 (s \mathcal{H}' - \mathcal{H}) - (\alpha \mathcal{H}' + \frac{1}{2})^2 = 0. \quad (2.19) \]
Moreover,
\[ \mathcal{H}(s) = \frac{1}{4} \left( \frac{1}{C} - \alpha \right)^2 + \frac{s C}{2} - \frac{1}{4} \left( \frac{s}{C} \frac{dC}{d\xi} \right)^2. \quad (2.20) \]
Proof. By a straight forward, if tedious computations, we see that (2.12) becomes (2.18) and (2.15) becomes (2.19), and (2.16) becomes (2.20), after the double scaling process. \( \Box \)

Remark 1. Form (2.15) and (2.17), we find
\[ \ln \frac{\Delta(s, \alpha)}{\Delta(0, \alpha)} = \int_0^s \left\{ \frac{1}{4} \left( \frac{1}{C} - \alpha \right)^2 + \frac{\xi C}{2} - \frac{1}{4} \left( \frac{\xi}{C} \frac{dC}{d\xi} \right)^2 \right\} d\xi. \quad (2.21) \]

Remark 2. By the change of variables, the equation (2.18) ("lesser" \( P_{III} \)) becomes,
\[ Y'' = \frac{(Y')^2}{Y} - \frac{Y'}{x} + \frac{Y^2}{x} - \frac{1}{Y} + \frac{\alpha}{x}, \quad (2.22) \]
and see Appendix B for a proof.
Note that from [28], (2.22) is \( P_{III}(1, \alpha, 0, -1) \); the \( P_{III} \) with smaller number of parameters mentioned in the abstract.

Remark 3. Substituting
\[ H(2s) := \mathcal{H}(s) - \frac{\alpha^2}{4}, \]
into (2.19), we find,
\[ (s\mathcal{H}'(s))^2 + 4\mathcal{H}^2(s) (s\mathcal{H}'(s) - \mathcal{H}(s)) - 2s\mathcal{H}'(s) - 1 = 0, \]
which is the same as the equation derived by Ohyama-Kawamuko-Sakai-Okamoto, see ((18), in [28]) with \( \alpha_1 \) replaced by \( \alpha \).
2.1 Coulomb Fluid consideration after double scaling.

Recall that the algebraic equation, \( (1.9) \) derived by the Coulomb Fluid Method reads,

\[
\tilde{X}^4 - \alpha \tilde{X}^3 - (2n + \alpha)t \tilde{X} - t^2 = 0.
\]

Substituting

\[
t = \frac{s}{2n + 1 + \alpha},
\]

into the above quartic equation, followed by \( n \to \infty \) one finds,

\[
\tilde{X}^3 - \alpha \tilde{X}^2 - s = 0,
\]

a cubic equation in \( \tilde{X} \).

Let

\[
\tilde{C} = \frac{1}{\tilde{X}},
\]

the cubic becomes,

\[
\frac{\tilde{C}^3}{s} + \frac{\alpha \tilde{C}}{s^2} - \frac{1}{s^2} = 0.
\]

Note that the above is the algebraic part of \((2.18)\), satisfied by the \( C \) potential.

Retaining only the real solution of the cubic equation in \( \tilde{C} \), we find,

\[
\tilde{C}(s) = -2^{\frac{1}{3}} \alpha \left[ 27s^2 + (729s^4 + 108\alpha^3s^3)^{\frac{1}{3}} \right]^{\frac{1}{3}} + 18^{-\frac{1}{3}}s^{-1} \left[ 9s^2 + (81s^4 + 12\alpha^3s^3)^{\frac{1}{3}} \right]^{\frac{1}{3}}.
\]

For \( s \) near 0, a Taylor series expansion gives,

\[
\tilde{C}(s) = \frac{1}{\alpha} - \frac{1}{\alpha^3} s + \frac{3}{\alpha^4} s^2 - \frac{12}{\alpha^{10}} s^3 + \frac{55}{\alpha^{13}} s^4 + o(s^5),
\]

where \( \alpha \neq 0 \).

For large and positive \( s \), one finds

\[
\tilde{C}(s) = s^{-\frac{1}{3}} - \frac{\alpha}{3} s^{-\frac{2}{3}} + \frac{\alpha^3}{81} s^{-\frac{4}{3}} + \frac{\alpha^4}{243} s^{-\frac{5}{3}} - \frac{4\alpha^6}{6561} s^{-\frac{7}{3}} + O(s^{-\frac{8}{3}}).
\]

In the next section, we study small and large \( s \) expansion of the \( C \) potential, assuming appropriate form of asymptotic expansions.
3 Small $s$ and large $s$ behaviour of the $C$ potential.

We first study the solution of the $C$ potential for $s$ near 0, by substituting,

$$C(s) = \sum_{j=0}^{\infty} a_j s^j,$$

into (2.18). After some straightforward computations, we obtain,

$$C(s) = \frac{1}{\alpha} - \frac{1}{\alpha^2(\alpha^2 - 1)} s + \frac{3}{\alpha^3(\alpha^2 - 1)(\alpha^2 - 4)} s^2 - \frac{6(2\alpha^2 - 3)}{\alpha^4(\alpha^2 - 1)^2(\alpha^2 - 4)(\alpha^2 - 9)} s^3$$

$$+ \frac{5(-36 + 11\alpha^2)}{\alpha^5(\alpha^2 - 1)^2(\alpha^2 - 4)(\alpha^2 - 9)(\alpha^2 - 16)} s^4$$

$$+ \frac{3(3600 + 4219\alpha^2 - 1115\alpha^4 - 91\alpha^6)}{\alpha^6(\alpha^2 - 1)^3(\alpha^2 - 4)^2(\alpha^2 - 9)(\alpha^2 - 16)(\alpha^2 - 25)} s^5 + O(s^6),$$

(3.25)

where $\alpha \not\in \mathbb{Z}$.

For large and positive $s$, we substitute the asymptotic expansion

$$C(s) = \sum_{k=1}^{\infty} b_k s^{-\frac{k}{3}},$$

into (2.18). The first term of asymptotic expansion is found to be $b_1 s^{-1/3}$, with $b_1 = 1$. Continuing, we obtain,

$$C(s) = s^{-\frac{1}{3}} - \frac{\alpha}{3} s^{-\frac{2}{3}} - \frac{\alpha(\alpha^2 - 1)}{81} s^{-\frac{4}{3}} + \frac{\alpha^2(\alpha^2 - 1)}{243} s^{-\frac{5}{3}} - \frac{2\alpha^2(\alpha^2 - 1)(2\alpha^2 - 11)}{6561} s^{-\frac{7}{3}}$$

$$- \frac{5\alpha(\alpha^2 - 1)(\alpha^4 - \alpha^2 - 15)}{19683} s^{-\frac{8}{3}} + O(s^{-\frac{10}{3}}).$$

(3.26)

Remark 4. Note that for $\alpha = 0$, $\pm 1$, the asymptotic expansions terminate. In fact, there are only three algebraic solution for the $C$ potential;

$$C(s) = s^{-\frac{4}{3}}, \quad \alpha = 0$$

$$C(s) = s^{-1/3} + \frac{1}{3} s^{-\frac{2}{3}}, \quad \alpha = -1$$

$$C(s) = s^{-1/3} - \frac{1}{3} s^{-\frac{2}{3}}, \quad \alpha = 1.$$

Remark 5. The series expansion for the $C$ potential, given in (3.25), for sufficiently large $|\alpha|$ is the same as the corresponding series for $\tilde{C}$ derived by Coulomb Fluid Method. This phenomenon also holds for the large $s$ asymptotics; seen by comparing (3.26) with (2.24).
3.1 \( \mathcal{H}(s) \) for small and large \( s \) via form Painlevé equation in \( \sigma- \) form.

Recall the \( \sigma \)-form of our Painlevé equation is given by,

\[
(s\mathcal{H}''')^2 + 4(\mathcal{H}')^2 (s\mathcal{H}' - \mathcal{H}) - (\alpha \mathcal{H}' + \frac{1}{2})^2 = 0.
\]

For small \( s \), we assume the series expansion

\[
\mathcal{H}(s) = \sum_{j=0}^{\infty} d_j s^j.
\]

From a direct computation, the coefficients \( d_j, j = 0, 1, 2, \ldots \) may be easily found and,

\[
\mathcal{H}(s) = -\frac{1}{2\alpha} s - \frac{1}{4\alpha^2(\alpha^2 - 1)} s^2 - \frac{1}{2\alpha^3(\alpha^2 - 1)(\alpha^2 - 4)} s^3 + \frac{3(2\alpha^2 - 3)}{4\alpha^4(\alpha^2 - 1)^2(\alpha^2 - 4)(\alpha^2 - 9)} s^4
\]

\[
+ \frac{-36 + 11\alpha^2}{2\alpha^5(\alpha^2 - 1)^2(\alpha^2 - 4)(\alpha^2 - 9)(\alpha^2 - 16)} s^5
\]

\[
+ \frac{-3600 + 4219\alpha^2 - 1115\alpha^4 + 91\alpha^6}{4\alpha^6(\alpha^2 - 1)^3(\alpha^2 - 4)^2(\alpha^2 - 9)(\alpha^2 - 16)(\alpha^2 - 25)} s^6 + O(s^7), \tag{3.27}
\]

where \( \alpha \notin \mathbb{Z} \).

For positive and large \( s \), we substitute asymptotic expansion

\[
\mathcal{H}(s) = \sum_{k=-2}^{\infty} \eta_k s^{-\frac{k}{3}},
\]

into the \( \sigma \)-from. Note that for large \( s \), \( \mathcal{H}(s) \sim (C(s))^{-2} \sim s^{2/3} \). The precise leading term is found by assuming \( \mathcal{H}(s) = \eta_{-2} s^{2/3} \); an easy computation gives \( \eta_{-2} = -3/4 \).

After some computations, the expansion for large \( s \) reads,

\[
\mathcal{H}(s) = -\frac{3}{4} s^2 + \alpha \frac{1}{2} s^\frac{1}{3} + \frac{1 - 6\alpha^2}{36} s^\frac{2}{3} + \frac{\alpha(\alpha^2 - 1)}{54} s^{-\frac{1}{3}} + \frac{\alpha^2(\alpha^2 - 1)}{324} s^{-\frac{2}{3}} + \frac{\alpha(\alpha^2 - 1)}{486} s^{-1}
\]

\[
- \frac{2\alpha^6 - 13\alpha^4 + 11\alpha^2}{8784} s^{-\frac{4}{3}} - \frac{\alpha(\alpha^6 - 2\alpha^4 - 14\alpha^2 + 15)}{13122} s^{-\frac{7}{3}}
\]

\[
- \frac{8\alpha^6 - 41\alpha^4 + 33\alpha^2}{26244} s^{-2} + O(s^{-\frac{7}{3}}). \tag{3.28}
\]

3.2 Verifying the relation satisfied by \( C(s) \) and \( \mathcal{H}(s) \).

Recall the relation satisfied by \( C(s) \) and \( \mathcal{H}(s) \),

\[
\mathcal{H}(s) = \frac{1}{4} \left( \frac{1}{C} - \alpha \right)^2 + \frac{s}{2} \left( \frac{dC}{ds} \right) - \frac{1}{4} \left( \frac{s}{C} \frac{dC}{ds} \right)^2.
\]
To check the solution for small $s$, we proceed as follows. Let

$$g(s) = 1 - \alpha C(s)$$

and the Right Side (RS) of $H(s)$ becomes

$$RS = \frac{1}{4} \left( \frac{1}{C} - \alpha \right)^2 + \frac{s C}{2} - \frac{1}{4} \left( \frac{s}{C} \frac{dC}{ds} \right)^2$$

$$= \frac{\alpha^2}{4} \left( \sum \limits_{t=1}^{\infty} g^t(s) \right)^2 + \frac{s(1 - g(s))}{2\alpha} - \frac{(sg'(s))^2}{4} \left( \sum \limits_{t=0}^{\infty} g^t(s) \right)^2.$$ 

From the series expansion of the $C$ potential, (3.25), it follows that,

$$RS = -\frac{1}{2\alpha}s - \frac{1}{4\alpha^2(\alpha^2 - 1)}s^2 - \frac{1}{2\alpha^3(\alpha^2 - 1)(\alpha^2 - 4)}s^3$$

$$+ \frac{3(2\alpha^2 - 3)}{4\alpha^4(\alpha^2 - 1)(\alpha^2 - 4)(\alpha^2 - 9)}s^4 + O(s^5).$$

Note that this series is identical with the series representation of $H(s)$ for small $s$, obtained in (3.27). So the relation between $C(s)$ and $H(s)$ is verified for small $s$.

For large $s$, let now,

$$g(s) := 1 - s^{1/3} C(s),$$

and

$$RS = \frac{1}{4} \left( \frac{1}{C} - \alpha \right)^2 + \frac{s C}{2} - \frac{1}{4} \left( \frac{s}{C} \frac{dC}{ds} \right)^2$$

$$= \frac{1}{4} \left( \sum \limits_{k=0}^{\infty} g^k(s) - \alpha \right)^2 + \frac{s^{1/3}(1 - g(s))}{2} - \frac{1}{36} \left( (g(s) + 3sg'(s) - 1) \sum \limits_{k=0}^{\infty} g^k(s) \right)^2.$$ 

With $C$ potential given by (3.26), for large $s$, then

$$RS = \frac{3}{4} s^{2/3} + \frac{\alpha}{2} s^{1/3} + \frac{1 - 6\alpha^2}{36} s^{1/3} + \frac{\alpha(\alpha^2 - 1)}{54} s^{-1/3} + \frac{\alpha^2(\alpha^2 - 1)}{324} s^{-2/3} + O(s^{-1}). \quad (3.29)$$

This series is also the same as the series representation of $H(s)$ for large $s$, see (3.28). The relation between $C(s)$ and $H(s)$ is verified for large $s$.

At the end of this section, we provide expansions of the Hankel determinant for small and large $s$.

**Theorem 5.** Under the double scaling scheme the asymptotic expressions of the Hankel determinant for small $s$ and large $s$ are as follows:
For small $s$,

$$
\Delta(s, \alpha) = \exp \left[ c_1 - \frac{1}{2\alpha} s + \frac{1}{8\alpha^2(\alpha^2 - 1)} s^2 - \frac{1}{6\alpha^3(\alpha^2 - 1)(\alpha^2 - 4)} s^3 + \frac{3(2\alpha^2 - 3)}{16\alpha^4(\alpha^2 - 1)^2(\alpha^2 - 4)(\alpha^2 - 9)} s^4 + \frac{10\alpha^5(\alpha^2 - 1)^2(\alpha^2 - 4)(\alpha^2 - 9)(\alpha^2 - 16)}{36 - 11\alpha^2} s^5 + \frac{91\alpha^6 - 1115\alpha^4 + 4219\alpha^2 - 3600}{24\alpha^6(\alpha^2 - 1)^2(\alpha^2 - 9)(\alpha^2 - 16)(\alpha^2 - 25)} s^6 + O(s^7) \right],
$$

(3.30)

where $\alpha \notin \mathbb{Z}$, and $c_1 = c_1(\alpha)$ is an $\alpha$ dependent integration constant.

For large $s$,

$$
\Delta(s, \alpha) = \exp \left[ c_2 - \frac{9}{8} s^\frac{2}{3} + \frac{3\alpha}{2} s^\frac{1}{3} + \frac{1 - 6\alpha^2}{36} \ln s + \frac{\alpha(1 - \alpha^2)}{18} s^{-\frac{1}{3}} + \frac{\alpha^2(1 - \alpha^2)}{216} s^{-\frac{2}{3}} + \frac{\alpha^2(2\alpha^4 - 13\alpha^2 - 11)}{15664} s^{-\frac{4}{3}} + \frac{\alpha^6 - 2\alpha^4 - 14\alpha^2 + 15}{21870} s^{-\frac{5}{3}} + O(s^{-\frac{3}{3}}) \right],
$$

(3.31)

where $c_2 = c_2(\alpha)$ is another $\alpha$ dependent integration constant.

Proof. The small $s$ expansion of $H(s)$ is given by (3.27). Integrating,

$$
H(s) = s \frac{d}{ds} \ln \Delta(s, \alpha),
$$

with respect to $s$ then the small expansion of $\Delta(s, \alpha),$ (3.30), is obtained.

For large $s$, $H(s)$ is given by (3.28) and through the similar process and some computations, then the $\Delta(s, \alpha)$ expression of (3.31) is also valid.

At the end of this section, we compute the large $n$ behaviour of the evaluation of the polynomials at 0 through the fact that,

$$
(-1)^n P_n(0; t, \alpha) = \frac{D_n(t, \alpha + 1)}{D_n(t, \alpha)}.
$$

Corollary 1. Under the double scaling, $s = (2n + 1 + \alpha)t$, as $n \to \infty$, $t \to 0$ and such that $s$ is finite, we find,

$$
\lim_{n \to \infty} (-1)^n P_n \left( 0; \frac{s}{2n + \alpha + 1}, \alpha \right) = \frac{\Delta(s, \alpha + 1)}{\Delta(s, \alpha)} = \exp \left( c_3 + \frac{3}{2} s^{\frac{1}{3}} - \frac{1 + 2\alpha}{6} \ln s - \frac{\alpha(\alpha + 1)}{6} s^{-\frac{1}{3}} - \frac{\alpha(\alpha + 1)(2\alpha + 1)}{108} s^{-\frac{2}{3}} - \frac{\alpha(\alpha + 1)}{162} s^{-1} + \frac{(2\alpha + 1)(3\alpha^4 + 6\alpha^3 - 6\alpha^2 - 9\alpha - 11)}{7832} s^{-\frac{4}{3}} + O(s^{-\frac{5}{3}}) \right),
$$

(3.32)

where $c_3 = c_3(\alpha)$ is independent of $s$. 

11
Proof. From the fact that
\[
\lim_{n \to \infty} (-1)^n P_n \left( 0; \frac{s}{2n + \alpha + 1}, \alpha \right) = \lim_{n \to \infty} \frac{D_n(s/(2n + \alpha + 1), \alpha + 1)}{D_n(s/(2n + \alpha + 1), \alpha)} = \frac{\Delta(s, \alpha + 1)}{\Delta(s, \alpha)},
\]
and together with the expression of \( \Delta(s, \alpha + 1) \) and \( \Delta(s, \alpha) \) given by (3.31), the equation (3.32) is found.

In the next section a computation produces the constant \( c_3(\alpha) \).

4 The asymptotic of \( P_n(0; t, \alpha) \).

In this section, we evaluate \( P_n(0; t, \alpha) \), and later give a derivation of \( c_3(\alpha) \). First we state a result for the orthogonal polynomials \( P_n(z; t, \alpha) \), \( z \notin [a, b] \).

As \( n \to \infty \), \( P_n(z) \) can be computed as
\[
P_n(z) \sim \exp[-S_1(z) - S_2(z)], \quad \text{where} \quad z \notin [a, b],
\]
and \( S_1(z) \) and \( S_2(z) \) are given by ((4.6) and (4.7), [19]). These are
\[
S_1(z) = \frac{1}{4} \ln \left[ 16(z - a)(z - b) \left( \frac{\sqrt{z - a} - \sqrt{z - b}}{\sqrt{z - a} + \sqrt{z - b}} \right)^2 \right], \quad z \notin [a, b],
\]
furthermore the above is equivalent to
\[
\exp(-S_1(z)) = \frac{1}{2} \left[ \left( \frac{z - b}{z - a} \right)^{\frac{1}{4}} + \left( \frac{z - a}{z - b} \right)^{\frac{1}{4}} \right], \quad z \notin [a, b],
\]
and
\[
S_2(z) = -n \ln \left( \frac{\sqrt{z - a} + \sqrt{z - b}}{2} \right)^2
+ \frac{1}{2\pi} \int_{a}^{b} \frac{v(x)}{\sqrt{(b - x)(x - a)}} \left[ \frac{\sqrt{z - a}(z - b)}{x - z} + 1 \right] dx, \quad z \notin [a, b].
\]

Theorem 6. If \( v(x) = -\ln w(x) = -\alpha \ln x + x + \frac{t}{x}, x \geq 0 \), the evaluation at \( z = 0 \) of \( S_1(z; t, \alpha) \), \( S_2(z; t, \alpha) \), and \( P_n(z; t, \alpha) \) are given by
\[
\exp[-S_1(0; t, \alpha)] \sim 2^{-\frac{1}{4}} n^{\frac{1}{4}} t^{-\frac{1}{4}},
\]
\[
(-1)^n \exp[-S_2(0; t, \alpha)] \sim n^n 2^{-\frac{n}{2}} t^{-\frac{n}{2}} e^{-\frac{n}{2}} \exp \left( -n + 3 \cdot 2^{-\frac{n}{2}} n^{\frac{1}{2}} t^{\frac{1}{2}} + 2\alpha \ln n \right),
\]
and
\[ (-1)^nP_n(0; t, \alpha) \sim (-1)^n \exp[-S_1(0; t, \alpha) - S_2(0; t, \alpha)] \]
\[ \sim n^x(2t)^{-\frac{1}{2} - \frac{\alpha}{2}} e^{-\frac{1}{2}} \exp \left( -n + 3 \cdot 2^{-\frac{3}{2}} n^{\frac{1}{3}} t^{\frac{1}{3}} + \frac{1}{3} (1 + 2\alpha) \ln n \right), \quad (4.38) \]
for large \( n \).

**Proof.** Recall the quartic equation satisfied by \( \tilde{X} \),
\[ \tilde{X}^4 - \alpha \tilde{X}^3 - (2n + \alpha)t \tilde{X} - t^2 = 0, \]
where \( \tilde{X} = \sqrt{ab} \).

We denote \( \tilde{n} = 2n + \alpha \) and found that the relevant solution for large \( \tilde{n} \), reads,
\[ \frac{1}{\tilde{X}} \sim (\tilde{n}t)^{-\frac{1}{4}} - \frac{\alpha}{3} (\tilde{n}t)^{-\frac{5}{4}} + \frac{\alpha^3}{81} (\tilde{n}t)^{-\frac{9}{4}} + \frac{\alpha^4}{243} (\tilde{n}t)^{-\frac{13}{4}} + O((\tilde{n}t)^{-2}). \]

From (1.7) and (1.8), we see that
\[ a = \tilde{n} + \frac{s}{\tilde{X}} - \sqrt{(\tilde{n} + \frac{s}{\tilde{X}})^2 - \tilde{X}^2} \sim \frac{t^{\frac{1}{3}}}{2n^{\frac{1}{3}}} \frac{\alpha t^{\frac{1}{3}}}{\tilde{n}^{\frac{1}{3}}} + \frac{5\alpha^3}{81} \frac{t^{\frac{1}{3}}}{\tilde{n}^{\frac{1}{3}}} + O(\frac{1}{n^{\frac{1}{3}}}) \sim \frac{t^{\frac{1}{3}}}{2(2n + \alpha)^{\frac{1}{3}}}, \]
and
\[ b = \tilde{n} + \frac{s}{\tilde{X}} + \sqrt{(\tilde{n} + \frac{s}{\tilde{X}})^2 - \tilde{X}^2} \sim 2\tilde{n} + \frac{3t^{\frac{1}{2}}}{2n^{\frac{1}{2}}} - \frac{\alpha t^{\frac{1}{2}}}{\tilde{n}^{\frac{1}{2}}} - \frac{\alpha^2}{6\tilde{n}} - \frac{\alpha^3}{27t^{\frac{1}{4}}n^{\frac{1}{4}}} + O(\frac{1}{n^{\frac{1}{3}}}) \sim 2(2n + \alpha), \]
for large \( \tilde{n} \).

Hence
\[ ab \sim (2nt)^{\frac{3}{4}} \quad \text{or} \quad \sqrt{ab} \sim (2nt)^{\frac{1}{4}}. \]

From the expression for \( \exp(-S_1(z)) \), see (4.34), we find,
\[ \exp[-S_1(0; t, \alpha)] \sim \frac{1}{2} \left( \frac{b}{a} \right) \frac{1}{4} \sim 2^{-\frac{3}{2}} n^{\frac{1}{3}} t^{\frac{1}{2}}. \quad (4.39) \]

We evaluate \( S_2(0; t, \alpha) \) by setting \( z = 0 \) in (4.35). With the aid of the integral identities in the Appendix A, followed by some computations, we find,
\[ \exp[-S_2(0; t, \alpha)] \sim (-1)^n \left( n + \frac{\alpha}{2} + \frac{t}{2\sqrt{ab}} + \frac{\sqrt{ab}}{2} \right)^n \left( \frac{n}{\sqrt{ab}} + \frac{\alpha}{2\sqrt{ab}} + \frac{t}{2ab} + \frac{1}{2} \right)^\alpha \]
\[ \times \exp \left[ -n - \frac{\alpha}{2} - \frac{s}{\sqrt{ab}} + \frac{\sqrt{ab}}{2} + \left( n + \frac{\alpha}{2} \right) \frac{t}{ab} + \frac{t^2}{2(ab)^{\frac{3}{2}}} \right]. \]

Since \( \sqrt{ab} \sim (2nt)^{\frac{1}{2}} \), the above equation is (4.37). With the expressions for \( \exp(-S_1(0; t, \alpha)) \) and \( \exp(-S_2(0; t, \alpha)) \), the asymptotic expansion for \( P_n(0; t, \alpha) \) follows immediately, which is (4.38). \( \square \)
Remark 6. For convenience, we rewrite the asymptotic expression of \((-1)^nP_n(0; t, \alpha)\), as

\[
(-1)^nP_n(0; t, \alpha) \sim (-1)^n \exp[-S_1(0; t, \alpha) - S_2(0; t, \alpha)]
\]

\[
\sim n^n(2t)^{-\frac{1}{3} + \frac{\alpha}{2}} e^{-\frac{\alpha}{2}} \exp\left(-n + 3 \cdot 2^{-\frac{2}{3}} n^{-\frac{1}{3}} t^\frac{1}{2} + \frac{1}{3}(1 + 2\alpha) \ln n\right)
\]

\[
\sim n^{n+\alpha + \frac{1}{4}} e^{-n} \exp\left(-\frac{\alpha}{2} + \frac{3}{2} (2nt)^{\frac{1}{3}} - \frac{1 + 2\alpha}{6} \ln(2nt)\right)
\]

\[
\sim \Gamma(n + 1 + \alpha) \exp\left(\ln(\sqrt{2\pi} \Gamma(1 + \alpha)) - \frac{\alpha}{2} + \frac{3}{2} (2nt)^{\frac{1}{3}} - \frac{1 + 2\alpha}{6} \ln(2nt)\right)
\]

\[
= \frac{\Gamma(n + 1 + \alpha)}{\Gamma(1 + \alpha)} \exp\left(\ln(\sqrt{2\pi} \Gamma(1 + \alpha)) - \frac{\alpha}{2} + \frac{3}{2} (2nt)^{\frac{1}{3}} - \frac{1 + 2\alpha}{6} \ln(2nt)\right),
\]

(4.40)

and hence

\[
\frac{P_n(0; t, \alpha)}{P_n(0; 0, \alpha)} = \exp\left(c_3(\alpha) + \frac{3}{2} (2nt)^{1/3} - \frac{1 + 2\alpha}{6} \ln(2nt)\right),
\]

from which we may identify \(c_3(\alpha)\) as

\[
\ln\left(\sqrt{2\pi} \Gamma(1 + \alpha)\right) - \frac{\alpha}{2}.
\]

In the pan-ultimate step, we have replaced,

\[
\sqrt{2\pi} \exp^{(n+\alpha+1/2) \ln n - n}
\]

by

\[
\Gamma(n + 1 + \alpha).
\]

We see that the singular perturbation, through the multiplication of \(e^{-t/x}\) on classical Laguerre weight causes a modification,

\[
\frac{P_n(0; t, \alpha)}{P_n(0; 0, \alpha)} \sim \exp\left(\frac{3}{2} (2nt)^{1/3} - \frac{1 + 2\alpha}{6} \ln(2nt) + c_3(\alpha)\right).
\]

If 2nt is replaced by \(s\), then (4.40) is the same with (3.32).

5 Conclusion

The (finite \(n\)) Hankel determinant generated by a singular deformation of the Laguerre weight, intimately related to a \(P_{III}\). This reduces in a double scaling limit described by the \(C\) potential, which in fact satisfies a \(P_{III}\) with fewer parameters.

Asymptotic expansions of the scaled Hankel determinant are found, through the Coulomb
Fluid approach and the relevant Painlevé equations.

Appendix A: Some Integration Identities.
We consult the references [14] and [22], integration identities used in our Coulomb Fluid derivations, valid for $0 < a < b$, are listed below,

$$
\int_a^b \frac{1}{\sqrt{(b-x)(x-a)}} \, dx = \pi. \quad (A1)
$$

$$
\int_a^b \frac{x}{\sqrt{(b-x)(x-a)}} \, dx = \frac{(a+b)\pi}{2}. \quad (A2)
$$

$$
\int_a^b \frac{1}{x\sqrt{(b-x)(x-a)}} \, dx = \frac{\pi}{\sqrt{ab}}. \quad (A3)
$$

$$
\int_a^b \frac{1}{x^2\sqrt{(b-x)(x-a)}} \, dx = \frac{(a+b)\pi}{2(ab)^{3/2}}. \quad (A4)
$$

$$
\int_a^b \frac{\ln x}{\sqrt{(b-x)(x-a)}} \, dx = 2\pi \ln \left( \frac{\sqrt{a} + \sqrt{b}}{2} \right). \quad (A5)
$$

$$
\int_a^b \frac{\ln x}{x\sqrt{(b-x)(x-a)}} \, dx = \frac{2\pi}{\sqrt{ab}} \ln \frac{2\sqrt{ab}}{\sqrt{a} + \sqrt{b}}. \quad (A6)
$$

Appendix B: From the $C(s)$ potential to $P_{III}(1, \alpha, 0, -1)$.
Recall the $C(s)$ potential satisfies,

$$
C'' = \frac{(C')^2}{C} - C' \frac{C}{s} + C^2 \frac{1}{s^2 C} - \frac{1}{s^2} + \frac{\alpha}{s^2}. \quad
$$

If

$$
C(s) = \frac{1}{y(s)},
$$

then we find

$$
y'' = \left( \frac{y'}{y} \right)^2 - \frac{y'}{s} + \frac{y^3}{s^2} - \frac{\alpha y^2}{s^2} - \frac{1}{s}.
$$

Let

$$
s = \hat{x}^2, \quad y = \hat{x} f,
$$

then the above ODE becomes

$$
f'' = \frac{(f')^2}{f} - \frac{f'}{\hat{x}} - \frac{4\alpha f^2}{\hat{x}} - \frac{4}{\hat{x}} + 4f^3,
$$
where \( f' \) denotes derivative with \( \hat{x} \).

At last, if
\[
\hat{x} = \frac{x}{4}, \quad f\left(\frac{x}{4}\right) = \frac{1}{Y(x)},
\]
then we arrive
\[
Y'' = \frac{(Y')^2}{Y} - \frac{Y'}{x} + \frac{Y^2}{x} - \frac{1}{Y} + \frac{\alpha}{x},
\]
where \( Y' \) denotes derivative with respect to \( x \), which is a \( \text{P}_{III}(1, \alpha, 0, -1) \).

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