Intrinsic Dynamical Fluctuation Assisted Symmetry Breaking in Adiabatic Following

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(Dated: February 6, 2022)

PACS numbers: 03.65.Vf, 05.40.-a, 05.45.-a, 37.10.Gh, 45.20.Jj

Classical adiabatic invariants in actual adiabatic processes possess intrinsic dynamical fluctuations. The magnitude of such intrinsic fluctuations is often thought to be negligible. This widely believed physical picture is contested here. For adiabatic following of a moving stable fixed-point solution facing a pitchfork bifurcation, we show that intrinsic dynamical fluctuations in an adiabatic process can assist in a deterministic and robust selection between two symmetry-connected fixed-point solutions, irrespective of the rate of change of adiabatic parameters. Using a classical model Hamiltonian also relevant to a two-mode quantum system, we further demonstrate the formation of an adiabatic hysteresis loop in purely Hamiltonian mechanics and the generation of a Berry phase via changing one single-valued parameter only.

Introduction – Adiabatic theorem is about the dynamical behavior of a Hamiltonian system whose parameters are changing slowly with time. It constitutes a fundamental topic in Hamiltonian mechanics. For example, Einstein was among the first to recognize the importance of classical adiabatic invariants in understanding quantization. In recent years, there are still considerable interests in several aspects of adiabatic theorem in both quantum mechanics and classical mechanics.

Classical adiabatic following is the subject of this study, but our findings are also relevant to certain quantum systems. We start from the fact that classical adiabatic theorem is not an exact theorem: an actual adiabatically evolving trajectory fluctuates around an idealized solution predicted by the adiabatic theorem. Adiabatic invariants hence possess intrinsic dynamical fluctuations (IDF’s). The magnitude of such fluctuations, typically proportional to the rate of change of adiabatic parameters, becomes extremely small in truly slow adiabatic processes. So except for special quantities that can accumulate IDF during an adiabatic process (e.g., in calculations of dynamical angles), IDF does not seem to be interesting or physically relevant. As shown in this Letter via both theory and computational examples, this perception is about to change.

Bifurcation phenomenon is ubiquitous in nonlinear systems and it is of fundamental interest to many topics in physics, among which we mention localization-delocalization phase transitions and symmetry breaking. Here we consider the adiabatic following of a stable fixed-point solution, which, as a result of a varying adiabatic parameter, moves towards a supercritical pitchfork bifurcation. As schematically shown in Fig. 1(a), two new stable fixed points and one unstable fixed point emerge after the adiabatic parameter (denoted $R$) slowly passes the bifurcation point at $R = R_1$. It is then curious to know among the three fixed points, which fixed-point solution the system will land on and whether the selection is predictable. It is found, both theoretically and

![FIG. 1: A supercritical pitchfork bifurcation in a classical Hamiltonian system parameterized by $R$. The solid (dashed) lines represent the phase space locations of stable (unstable) fixed points as a function of $R$. (a) When $R$ exceeds $R_1$, a stable fixed point splits into two stable and one unstable fixed points. One then wonders which branch the system will land on as $R$ adiabatically increases beyond $R_1$. (b) The bifurcated fixed points merge back when $R$ exceeds $R_2$. If $R$ first increases from $R < R_1$ to $R > R_2$ and later returns to $R < R_1$, then an adiabatic hysteresis loop may be formed.](image-url)
numerically, that IDF is crucial for a deterministic and robust selection between the symmetry-connected pair of stable fixed-point solutions, regardless of how slow the adiabatic process is. As such, through crossing the bifurcation point, a tiny IDF is amplified to a macroscopic level after the bifurcation: it determines the fate of the trajectory afterwards by “forcing” the system to make a selection between symmetry-breaking solutions. We term this as deterministic symmetry breaking, because there is no need for external noise to initiate the symmetry-breaking. The selection process is robust because, unlike symmetry-breaking selection processes studied previously [14], it is independent of the dynamical details. Figure 1(b) depicts an interesting situation that involves previously 

\[ \Gamma = \left( \begin{array}{cc} \frac{\partial^2 H}{\partial p \partial q} & \frac{\partial^2 H}{\partial q \partial p} \\ \frac{\partial^2 H}{\partial p \partial q} & \frac{\partial^2 H}{\partial q \partial p} \end{array} \right)_{p=\bar{p},q=\bar{q}}, \]

\[ \langle \delta p, \delta q \rangle \]

and \( \langle \cdot \rangle \) denotes an average over all possible initial conditions of \( (\delta q, \delta p) \). As seen from Eq. (2), so long as \( \Gamma^{-1} \) exists, then the scale of IDF is proportional to \( \frac{d R}{dt} \equiv V \), the speed of adiabatic manipulation. The magnitude of \( (\delta q, \delta p) \) is then deceptively small (for a sufficiently small \( V \)), but nonzero in general.

We now examine how nonzero \( (\delta q, \delta p) \) impacts the adiabatic following of \( [\bar{q}(R), \bar{p}(R)] \) that eventually undergoes a supercritical pitchfork bifurcation at \( R = R_1 \). First, right before the adiabatic parameter \( R \) reaches the bifurcation point \( R_1 \), the system’s actual state \( (q, p) \) must deviate from the instantaneous solutions \( [\bar{q}(R), \bar{p}(R)] \) by \( (\delta q, \delta p) \). Without loss of generality and based on Eq. (2), the actual time-evolving state \( (q, p) \) is assumed to be slightly shifted to the left side of the instantaneous fixed point \( [\bar{q}(R), \bar{p}(R)] \). This is illustrated in Fig. 2(a), where the periodic orbits (associated with a fixed \( R \)) around \( [\bar{q}(R), \bar{p}(R)] \) are also shown. The second stage is illustrated by Fig. 2(b), where \( R \) only slightly exceeds \( R_1 \). There the bifurcation already occurs but the actual state does not “feel” the bifurcation yet: it continues to stay on the left side of all the three new fixed points. So the actual left-shifted state is located on an orbit (if \( R \) were fixed) surrounding all the fixed points (this is confirmed in our numerical studies). The line shown in Fig. 2(b) passing through the unstable fixed point represents the separatrix. In the last stage, \( R \) further increases, the two stable fixed points split further, with the stable fixed point moving to the left capturing the actual state [see Fig. 2(c)]. During the ensuing adiabatic following, the actual state then adiabatically follows the instantaneous fixed point on the left. Clearly then, if, as \( R \) approaches a bifurcation point, IDF can induce a definite shift (to the left or the right) of the actual state with respect to \( [\bar{q}(R), \bar{p}(R)] \), then the system will be trapped, deterministically, by one of the two stable fixed-point solutions.

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**FIG. 2:** (color online) Impact of intrinsic dynamical fluctuations on the crossing of a pitchfork bifurcation. (a) The system’s actual state is on the left of the instantaneous fixed point. (b) Immediately after the bifurcation, the actual state is still shifted to the left of all the fixed points. (c) As two stable fixed points split further, the actual state gets trapped by, and starts to adiabatically follow, the stable fixed-point solution on the left.
after the bifurcation. Note that, exactly because of IDF, the actual state always stays away from the vicinity of the unstable fixed point (the extremely slow part of a separatrix). As a result the diverging time scale associated with the whole separatrix does not affect adiabatic following here. This understanding will be confirmed in our following numerical experiments.

Model Hamiltonian – Here we turn to a concrete Hamiltonian system with supercritical pitchfork bifurcations. Specifically, we choose

$$H = -\frac{c}{2}q^2 - R\sqrt{1-q^2}\cos(p) + \Delta\sqrt{1-q^2}\sin(p)$$

in dimensionless units, with $c > \Delta > 0$. Interestingly, this system has two bifurcation points at $R_1 = -\sqrt{c^2 - \Delta^2}$ and $R_2 = \sqrt{c^2 - \Delta^2}$. Define $\eta = \sqrt{1 - (R^2 + \Delta^2)/c^2}$ and $\mu = \arctan(-\Delta/R)$. In the negative regime of $R$, the stable fixed point is $(\bar{q}, \bar{p}) = (0, \mu - \pi)$ for $R < R_1$, which then bifurcates into two stable fixed points at $(\bar{\eta}, \mu - \pi)$. In the positive regime of $R$, the two stable fixed points are at $(\bar{\eta}, \mu)$ for $R < R_2$ and then merge back to one stable fixed point at $(0, \mu)$. Note that this model Hamiltonian is invariant under the joint operation of space reflection $(P)$ $q \rightarrow -q$ and time reversal $(T)$ $t \rightarrow -t$. Therefore, for cases with only one stable fixed point, the solution itself has the same $PT$ symmetry, and for cases with two stable fixed points, the pair are transformed to each other under the $PT$ operations, with each individual fixed point being a symmetry-breaking solution. The phase space locations of these stable fixed-point solutions for various fixed values of $R$ are shown by the solid lines in Fig. 3.

Another motivation to choose $H$ in Eq. (1) is that it describes a two-mode many-body quantum system on the mean-field level. Consider the following mean-field Hamiltonian in dimensionless units ($\hbar = 1$),

$$\hat{H}_m = \begin{pmatrix} c(|b|^2 - |a|^2) & -R - i\Delta \\ -R + i\Delta & -c(|b|^2 - |a|^2) \end{pmatrix},$$

where $a$ and $b$ are quantum amplitudes on two modes, $c$ represents the self-interaction strength, and $R \pm i\Delta$ represents inter-mode coupling. By rewriting $a = |a|e^{i\delta_a}$, $b = |b|e^{i\delta_b}$, $p = \delta_b - \delta_a$, and $q = |b|^2 - |a|^2$, the time evolution of $q$ and $p$, as obtained from the Schrödinger equation for this quantum model, becomes precisely that under the classical model Hamiltonian $\hat{H}$ in Eq. (1). The fixed points of $H$ become eigenstates of $\hat{H}_m$: the adiabatic following when crossing a bifurcation for $H$ is mapped to the issue of adiabatic following as degenerate eigenstates of $\hat{H}_m$ emerge. One possible realization of $\hat{H}_m$ is a Bose-Einstein condensate (BEC) in a double-well potential, with the imaginary coupling constant $\Delta$ implemented via phase imprinting on one well. $\hat{H}_m$ may also be realized in nonlinear optics by using two nonlinear optical waveguides with biharmonic longitudinal modulation of the refractive index. So our detailed results below are relevant to both classical and quantum physics.

Theory and numerical experiments – Applying the theory of IDF to the Hamiltonian in Eq. (1), one obtains $\langle \delta p \rangle = 0$ and $\langle \delta q \rangle = -\frac{\Delta}{\sqrt{R^2 + \Delta^2}}$ for $R < R_1$. Therefore, as $R$ increases from $R < R_1$, the system should adiabatically follow the left symmetry-breaking solution $(\bar{q}, \bar{p}) = (-\eta, \mu - \pi)$ for $R_1 < R < 0$ and $(\bar{q}, \bar{p}) = (-\eta, \mu)$ for $0 < R < R_2$. Similar theoretical results show that, if $R$ decreases from $R > R_2$, the actual state must be slightly right-shifted from the fixed point at $(0, \mu - \pi)$. So after passing the bifurcation at $R = R_1$, the system is expected to adiabatically follow the left symmetry-breaking solution $(\bar{q}, \bar{p}) = (-\eta, \mu - \pi)$ for $R_1 < R < 0$ and $(\bar{q}, \bar{p}) = (-\eta, \mu)$ for $0 < R < R_2$. As shown in Fig. 3 (solid or empty dots), our numerical results based only on Hamilton’s equation of motion confirm this prediction. For the results shown we have set $V = 10^{-6}$ to ensure a slow process. Indeed, at all times the difference between the actual states (dots)
and the instantaneous fixed points (solid lines) is invisible to our naked eyes. Yet, the small IDF does assist in a symmetry-breaking choice regarding which of two stable fixed-point solutions is adiabatically followed by the system. In the language of $H_m$, after the system passes the bifurcation at $R = R_1$ owing to an increasing $R$, $q = |b|^2 - |a|^2$ becomes appreciably negative, so one mode develops more population than the other, signaling a clear delocalization-localization transition induced by IDF. The opposite population imbalance occurs when the system passes the bifurcation at $R = R_2$ with a decreasing $R$. Furthermore, joining these two manipulation steps together so that $R$ returns to its very original value in the end (which is already the case in Fig. 3), we clearly observe the formation of an adiabatic “hysteresis” loop in phase space. That is, by increasing and then decreasing $R$, a navigation loop in phase space is formed because the adiabatic following in the forward step and backward step lands on different symmetry-breaking branches [18].

There is one subtle point to be clarified: Our theory of IDF [see Eq. (2)] is about quantities $\langle \delta q \rangle$ averaged over all initial conditions of $(\delta q, \delta p)$, but what determines the adiabatic following is the actual $\delta q$ in a single process. The Supplementary Material contains a detailed analysis on this point. In particular, we show that if initially there is a deviation of $\delta q$ from $\langle \delta q \rangle$, then the difference $\delta q - \langle \delta q \rangle$ oscillates with time and later, as $R$ approaches the bifurcation point, this deviation becomes negligible as compared with $\langle \delta q \rangle$. It is for this reason that the definite sign of $\langle \delta q \rangle$ is equivalent to the definite sign of $\delta q$, which hence justifies our theory based on $\langle \delta q \rangle$.

**Berry phase generation via one single-valued parameter** – As a final interesting concept, we discuss the generation of a Berry phase using one single-valued adiabatic parameter $R$. This is made possible by IDF and bifurcations. In particular, the navigation loop in Fig. 3 shows that the time-evolving states of $H_g$ trace out a nontrivial geometry after increasing $R$ from $R < R_1$ to $R > R_2$ and then returning $R$ to its initial value. Let $|\psi_{\text{left}}\rangle$ and $|\psi_{\text{right}}\rangle$ be the eigenstates of $H_m$ (with $R_1 < R < R_2$) mapped from the left and right fixed-point solutions of $H$. Assuming exact adiabatic following with the instantaneous adiabatic eigenstates, the Berry phase generated along the navigation loop is analytically found to be

$$\beta_{\text{Berry}} = i \int_{R_1}^{R_2} dR \left( \langle \psi_{\text{left}} | \frac{d}{dR} | \psi_{\text{left}} \rangle - \langle \psi_{\text{right}} | \frac{d}{dR} | \psi_{\text{right}} \rangle \right) = \pi (1 - \Delta / c).$$

In our numerical experiments, we choose to integrate the Berry connection using the actual states during the physical process (this simple method will not account for the nonlinear Berry phase correction studied in Ref. [3]). It is found that for the shown regime in Fig. 4, the agreement between theory and simulations is excellent, with tiny but visible differences. Such visible differences remind us that a direct numerical integration of the Berry connection along the actual time evolution path is not necessarily reliable because it could accumulate the effect of IDF [1]. Nevertheless, the fair agreement shown in Fig. 4 confirms our analytical result, demonstrates the feasibility of Berry phase generation using only one single-valued adiabatic parameter, and verifies from another angle that IDF is important for understanding adiabatic following in the presence of bifurcation.

**Conclusion** – Bifurcation greatly amplifies subtle intrinsic fluctuations that are beyond classical adiabatic invariants. This leads to a selection rule regarding which of two symmetry-connected stable fixed-point solutions may be adiabatically followed. In cases of multiple bifurcation points, adiabatic hysteresis loops in phase space and the generation of Berry phase by manipulating one single-valued parameter are also shown to be possible. Our findings are of fundamental interest to both classical systems and quantum many-body systems on the mean-field level. The implications of this work for symmetry breaking in fully quantum many-body systems should be a fascinating topic in our future studies.

**Appendix**

In this Appendix we discuss the difference between $\langle \delta q \rangle$ and $\delta q$, using the same notation as in the main text. It is important to clarify this point because our theory of intrinsic dynamical fluctuations (IDF) is about the quantity $\langle \delta q \rangle$ averaged over all initial conditions of $(\delta q, \delta p)$, but, as indicated in our main text, what determines the symmetry-breaking adiabatic following is the actual $\delta q$ in a single process without the averaging. Specifically, we need to show that, before the system crosses the bifurcation point, the (positive or negative) sign of $\delta q$ in an actual process is always the same as the sign of $\langle \delta q \rangle$ predicted by our theory. Without loss of generality we choose to discuss the case $R < R_1$ as an example.
FIG. 5: (color online) Comparison between numerical results of $\delta q$ and theoretical $\langle \delta q \rangle$ obtained from theory of IDF. System parameters are $c = 0.2$, $\Delta = 0.1$, and $\frac{d c}{dt} = 10^{-6}$. (a) Initially the canonical variables $(q, p)$ are set exactly on the instantaneous fixed point. The initial $\delta q$ is hence zero, which differs from the theoretical nonzero $\langle \delta q \rangle$. (b) Initially $\delta p$ and $\delta q$ are precisely chosen according to the theory of IDF (see the main text). In this case, $\delta q$ is seen to agree with $\langle \delta q \rangle$ at all times before the bifurcation, which is about to occur at $R = -0.173$ for parameters used here.

For our model Hamiltonian, the $\Gamma$ matrix evaluated at the fixed point is found to have zero diagonal elements. Hence we can write it in the following form

$$\Gamma = \begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix}. \quad \text{(7)}$$

Next we expand the Hamilton’s equation of motion to the first order of $\delta q$ and $\delta p$, we have

$$\begin{pmatrix} \frac{d \delta p}{dt} \\ \frac{d \delta q}{dt} \end{pmatrix} = \begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix} \begin{pmatrix} \delta p \\ \delta q \end{pmatrix} = \begin{pmatrix} 0 & -A \\ B & 0 \end{pmatrix} \begin{pmatrix} p - \bar{p} \\ q - \bar{q} \end{pmatrix}, \quad \text{(8)}$$

where the instantaneous fixed point $(\bar{q}, \bar{p})$ is located at $[0, \arctan(-\Delta/R) - \pi]$. Note that in general the off-diagonal elements $A$ and $B$ are $R$-dependent. Nevertheless, to gain physical insights and to develop a simple analytical result from the above equation, we consider a small time segment during which the $\Gamma$ matrix can be regarded as a constant, and $\bar{p}$ moves at a rate $M$ due to an increasing $R$. We then obtain

$$\frac{dp}{dt} = -Az; \quad \frac{dq}{dt} = Bp - BMt, \quad \text{(9)}$$

a first-order differential equation that can be integrated directly. Upon direct integration we find the analytic solution

$$\begin{align*}
q &= D \cos(\sqrt{AB}t) - \frac{M}{A}, \\
p &= -AD \sin(\sqrt{AB}t) + Mt, \quad \text{(10)}
\end{align*}$$

where $D$ is one integration constant. This then indicates

$$\begin{align*}
\delta q &= D \cos(\sqrt{AB}t) - \frac{M}{A}; \\
\delta p &= -AD \sin(\sqrt{AB}t). \quad \text{(11)}
\end{align*}$$

Equation (11) clearly shows that $\delta q$ and $\delta p$ are oscillating solutions. In particular, the oscillation amplitude in $\delta q$ is seen to be the initial difference between $\delta q$ and the averaged quantity $\langle \delta q \rangle = -M/A$ (for the concerned time segment). Even more importantly, as the system approaches the bifurcation point at $R = R_1$, the term $-M/A$ is the dominating term because $|A|$ sharply decreases for $R$ approaching $R_1$. This makes it clear that the sign of $\delta q$ must agree perfectly with the sign of $\langle \delta q \rangle$.

To verify this theoretical understanding, we performed numerical experiments and quantitatively compare in Fig. 5 the numerically found $\delta q$ with $\langle \delta q \rangle$ obtained from the theory of IDF (see the main text), for an increasing $R (R < R_1)$. As seen in Fig. 5(a), the actual numerical $\delta q$ indeed oscillates around theoretical $\langle \delta q \rangle$. As $R$ approaches closer to $R_1 = -0.173$ (beyond the shown regime of $R$ in Fig. 5), the oscillations of $\delta q$ around $\langle \delta q \rangle$ remain small and yet $\langle \delta q \rangle$ becomes more and more negative. Therefore, the sign of $\delta q$ is always identical with the sign of $\langle \delta q \rangle$ when the system is close to bifurcation crossing. As a second confirmation of our insights above, Fig. 5(b) shows that, if initially we set $\delta q = \langle \delta q \rangle$ and $\delta p = \langle \delta p \rangle$, then the ensuing time dependence of $\delta q$ as found from our numerical calculation agrees exactly with that from the theory of IDF.

We finally note that, the ultimate reason why our analytical solution above for $(\delta q, \delta p)$ during a small time segment is already so useful is again linked with adiabatic following. That is, since $R$ changes slowly, the off-diagonal elements of $\Gamma$ and $M$ will all change slowly, and hence the equation of motion for $\delta q$, or for the deviation $\delta q - \langle \delta q \rangle$, is expected to adiabatically follow the solution given in Eq. (11). This is interesting because it suggests a theory of higher-order fluctuations in the intrinsic fluctuations.

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[17] The expression of $\langle \delta q \rangle$ diverges at $R = R_1$, because in our model, $\Gamma^{-1}$ diverges at isolated points $R_1$ or $R_2$. Such unphysical divergence can be removed by considering a second order expansion of $\delta q$ and $\delta p$ (which then predicts that $\delta q$ and $\delta p$ can be of the order of $\sqrt{|dR/dt|}$). This procedure is unnecessary here because we do not need to estimate the precise magnitude of $\langle \delta q \rangle$ to make symmetry-breaking predictions (rather, only its sign is crucial).

[18] A preliminary observation as a side result in a driven two-mode system in Ref. [13] (coauthored by two of us) also suggested the possibility of adiabatic hysteresis loops. However, therein the origin of symmetry breaking was not well understood due to the lack of connection with IDF.