EQUIVARIANT ALGEBRAIC COBORDISM AND EQUIVARIANT
FORMAL GROUP LAW

CHUN LUNG LIU

Abstract. We introduce an equivariant algebraic cobordism theory \( \Omega^G(-) \) for algebraic varieties with \( G \)-action, where \( G \) is a split diagonalizable group scheme over a field \( k \). It is done by combining the construction of the algebraic cobordism theory \( \Omega(-) \) by F. Morel and M. Levine, with the notion of \( "(G, F)\)-formal group law" with respect to a complete \( G \)-universe and complete \( G \)-flag \( F \) as introduced by M. Cole, J. P. C. Greenlees and I. Kriz. In particular, we use their corresponding representing ring \( L_G(F) \) in place of the Lazard ring \( L \). We show that localization property and homotopy invariance property hold in \( \Omega^G(-) \). We also prove the surjectivity of the canonical map \( L_G(F) \to \Omega^G(\text{Spec } k) \). Moreover, we give some comparison results with \( \Omega(-) \), the equivariant algebraic cobordism theory introduced by J. Heller and J. Malagón-López, the equivariant K-theory and Tom Dieck’s equivariant cobordism theory (when \( k = \mathbb{C} \)). Finally, we show that our definition of \( \Omega^G(-) \) is independent of the choice of \( F \).

Contents

§1. Introduction
§2. Notations and assumptions
§3. Preliminaries
§4. Definition of the equivariant algebraic cobordism theory
§5. Basic properties
§6. The equivariant algebraic cobordism ring of a point
§7. Fundamental properties
§8. Comparison with other algebraic cobordism theories
§9. More on \( \Omega^G_{\text{Tot}}(-) \)
§10. Comparison with the equivariant K-theory
§11. Realization functor
§12. Flag dependency
References

Date: May 10, 2013.
1. Introduction

In their paper \[LMo\], M. Levine and F. Morel define an algebraic cobordism theory \( \Omega(-) \), which is an analogue of the complex cobordism theory, in spite of the absence of notion of boundary in algebraic geometry. Roughly speaking, if \( X \) is a separated scheme of finite type over the ground field \( k \), then we consider elements of the form \((f : Y \to X, L_1, \ldots, L_r)\) where \( f \) is projective, \( Y \) is an irreducible smooth variety over \( k \) and \( L_i \) are invertible sheaves over \( Y \) (the order of \( L_i \) does not matter and \( r \) can be zero). There is a natural notion of isomorphism of elements of this form. Denote the free abelian group generated by isomorphism classes of such elements by \( Z(X) \). Let \( \Omega(X) \) be the quotient of \( Z(X) \) by the subgroup corresponding to imposing the \((\text{Dim})\) and \((\text{Sect})\) axioms (following the notations in \[LMo\]). The algebraic cobordism group \( \Omega(X) \) is then defined to be the quotient of \( \mathbb{L} \otimes \mathbb{Z} \Omega(X) \), where \( \mathbb{L} \) is the Lazard ring, by the \( \mathbb{L} \)-submodule corresponding to imposing the formal group law \((\text{FGL})\).

This cobordism theory satisfies a number of basic properties and some more advanced properties like the localization property and the homotopy invariance property. Moreover, the cobordism ring \( \Omega(\text{Spec} \ k) \) is isomorphic to the Lazard ring \( \mathbb{L} \) when the characteristic of \( k \) is 0, which is what we expect from the complex cobordism theory (see Corollary 1.2.11 and Theorem 4.3.7 in \[LMo\]).

It is also possible to construct an algebraic cobordism theory via a more geometric approach. Suppose \( X \) is a smooth variety over \( k \). One may consider the abelian group \( M(X)^+ \) generated by isomorphism classes of projective morphisms \( f : Y \to X \), where \( Y \) is a smooth variety over \( k \). A relation called “double point relation” is introduced in \[LP\] and it is shown that the theory \( \omega(-) \) obtained by imposing this relation on \( M(-)^+ \) is canonically isomorphic to the theory \( \Omega(-) \) under the assumption that the characteristic of \( k \) is 0 (see Theorem 1 of \[LP\]).

The current paper contributes to the development of equivariant algebraic cobordism theory for varieties with group action. Following the pattern in topology, we can expect to have several different approaches to defining equivariant algebraic cobordism theory. For the analogue of one of the homotopy theoretic cobordism theories in the algebraic geometry setup, one can employ Totaro’s approximation of \( EG \), which leads to a definition given by taking inverse limit of a system of “good pairs” (see \[HeMa\] for details). Another, possibly equivalent, approach is pursued by Krishna in \[Kri\].

We are more interested in a geometric approach, i.e., by considering varieties with \( G \)-action. One idea is to impose the \( G \)-action on the double point relation. This approach is pursued in \[Li\]. Due to the lack of transversality in the equivariant setting, a generalized version of the double point relation is introduced in \[Li\] and an equivariant algebraic cobordism theory \( \mathcal{U}^G(-) \) is defined accordingly. It is also shown that this generalized double point relation holds in the non-equivariant theory \( \omega(-) \) (see Corollary 3.7 in \[Li\]).
The theory \(U_G(-)\) has a very strong geometric flavor, but it is insufficient to prove the localization property. This is mainly due to the absence of the Chern class operator for arbitrary \(G\)-linearized invertible sheaves in this theory. For this reason, it seems appropriate to define an equivariant algebraic cobordism theory following the original ideas in [LMo], and that is our approach in this paper.

The crucial part in trying to define an equivariant version of \(\Omega(-)\) is on finding the proper notion of "\(G\)-equivariant formal group law" and its representing ring. Fortunately, this issue is addressed by M. Cole, J. P. C. Greenlees and I. Kriz in their paper [CGKr]. Suppose \(G\) is a compact abelian Lie group. For a complete \(G\)-universe \(U\) and complete \(G\)-flag \(F\), they give a definition of \((G,F)\)-equivariant formal group law (definition 12.2 in [CGKr]) and prove that (Corollary 14.3 in [CGKr]) there is a corresponding representing ring \(L_G(F)\) (For a flag-independent definition, see definition 11.1 in [CGKr]).

Similarly to the theory \(\Omega(-)\), one may define an equivariant algebraic cobordism theory by imposing the (Sect), (Dim) axioms and the equivariant formal group law (EFGL) on \(L_G(F)\otimes\mathbb{Z}Z_G(X)\), where \(Z_G(X)\) is the free abelian group generated by isomorphism classes of elements of the form \((f : Y \to X, L_1, \ldots, L_r)\). Unfortunately, the canonical map from \(L_G(F)\) to the equivariant algebraic cobordism ring over \(\text{Spec } k\) will not be an isomorphism as we want (see Remark 13). Therefore, it seems that the only reasonable approach is to drop the (Dim) axiom.

Roughly speaking, our equivariant algebraic cobordism theory \(\Omega^G(-)\) is defined as follow. For a \(G\)-variety \(X\), we define \(LZ^{G,F}(X)\) to be the \(L_G(F)\)-module generated by infinite sums of the form

\[\sum_{I \geq 0} a_I[f : Y \to X, V_{i_1}^{j_1}(L_1), V_{i_2}^{j_2}(L_2), \ldots, V_{i_r}^{j_r}(L_r)]\]

where \(I\) is the multi-index \((i_1, \ldots, i_r)\), \(a_I\) are elements in \(L_G(F)\) and \(V_{i_j}^{j_j}(L_j)\) is a “twisted sequence” of \(L_j\) (see equation (1) for details). Then \(\Omega^G(X)\) is defined as the quotient of \(LZ^{G,F}(X)\) by imposing the (Sect) and (EFGL) axioms. It is worth mentioning that, if \(F'\) is another complete \(G\)-flag and \(\Omega^{G,F}(-), \Omega^{G,F'}(-)\) are the theories defined upon \(F, F'\) respectively, then \(\Omega^{G,F}(-)\) and \(\Omega^{G,F'}(-)\) are canonically isomorphic (see Proposition 12.4). Hence, our definition is indeed flag-independent.

With the aid of the canonically defined Chern class operator, we are able to show many interesting results in this theory (when \(\text{char } k = 0\)). We show that the canonical map \(L_G(F) \to \Omega^G(\text{Spec } k)\) is surjective (Theorem 6.12). Moreover, if the completion map \(\hat{L}_G(F) \to \hat{L}_G(F)\), with respect to a canonically defined ideal, is injective, then \(L_G(F) \to \Omega^G(\text{Spec } k)\) is an isomorphism (Corollary 9.3). We also prove that the localization property and the homotopy invariance property hold in our theory.

We also established some interesting comparison results between our equivariant algebraic cobordism theory and other theories. In particular, we show that the “forgetful map” \(\Omega^G(-) \to \Omega(-)\) is well-defined and it is an isomorphism when \(G\) is the trivial group (see Proposition 8.3 and Corollary 8.4). In addition, when \(\text{char } k = 0\), we show that there is an
abelian group homomorphism
\[ \Omega^G(-) \xrightarrow{\Psi_{\text{Tot}}} \Omega^G_{\text{Tot}}(-) \]
where \( \Omega^G_{\text{Tot}}(-) \) is the equivariant algebraic cobordism theory defined by J. Heller and J. Malagón-López using the Totaro’s approximation of \( EG \) as in [HeMa] (see Proposition 8.5), which can be thought as an analogue of the well-known map in Topology
\[ MU_G(-) \to MU(- \times^G BG) \]
where \( MU_G(-) \) is Tom Dieck’s equivariant cobordism theory. As an analogue to Corollary 4.2.12 in [LMo], we also prove the following Conner-Floyd type result, i.e., there is a canonical, surjective map from \( \Omega^G(-) \) to the equivariant K-theory \( K_0(G;-) \) when \( \text{char } k = 0 \) (see Theorem 10.8).

As mentioned in section 13 of [G], there is a canonical ring homomorphism
\[ \mathbb{L}_G(F) \to MU_G \]
and it is conjectured to be an isomorphism. Therefore, we believe our theory \( \Omega^G(-) \) can be considered as an algebraic analogue of Tom Dieck’s equivariant cobordism \( MU_G(-) \). This is justified by the realization functor
\[ \Omega^*_G(-) \to MU^*_G(-) \]
when \( k = \mathbb{C} \) (Theorem 11.3).

Here is the outline of this paper. In section 2, we introduce some notations and fix some basic assumptions that we use throughout the paper. In section 3, we state and prove a number of basic, relatively general facts. Then we give a formal definition of our equivariant algebraic cobordism theory \( \Omega^G(-) \) in section 4. In section 5, we prove some basic properties and show that the equivariant versions of the double point relation, the blow up relation and the extended double point relation hold in \( \Omega^G(-) \).

In section 6, we investigate the equivariant algebraic cobordism ring \( \Omega^G(\text{Spec } k) \). In particular, we show the following Theorem (Theorem 6.12 in the text):

**Theorem 1.** Suppose \( \text{char } k = 0 \). Then the canonical \( \mathbb{L}_G(F) \)-algebra homomorphism
\[ \mathbb{L}_G(F) \to \Omega^G(\text{Spec } k), \]
which sends \( a \) to \( [I_{\text{Spec } k}] \), is surjective.

Under the assumption that \( \text{char } k = 0 \), we prove some more advanced properties, namely, the localization property and the homotopy invariance property in section 7.

In section 8, we compare our theory \( \Omega^G(-) \) to M. Levine and F. Morel’s non-equivariant theory \( \Omega(-) \) and J. Heller and J. Malagón-López’s equivariant algebraic cobordism theory \( \Omega^G_{\text{Tot}}(-) \) (when \( G \) is a split torus). In section 9, we extend the definition of \( \Omega^G_{\text{Tot}}(-) \) to allow \( G \) to be split diagonalizable, compute the ring structure of \( \Omega^G_{\text{Tot}}(\text{Spec } k) \) and generalize our results in section 8. As a consequence, we prove the following Theorem (Corollary 9.3 in the text):

...
Theorem 2. Suppose \( \text{char } k = 0 \). If the completion map \( \mathbb{L}_G(F) \to \hat{\mathbb{L}}_G(F) \) is injective, then the canonical ring homomorphism

\[
\mathbb{L}_G(F) \to \Omega^G(\text{Spec } k)
\]

is an isomorphism.

In section 10, we compare our theory to the equivariant K-theory. To be more precise, we prove the following Theorem (Theorem 10.8 in the text):

Theorem 3. Suppose \( \text{char } k = 0 \). Then there is a canonical ring homomorphism \( \mathbb{L}_G(F) \to R(G)[v, v^{-1}] \), where \( R(G) \) is the character ring of \( G \). Moreover, there is a canonical, surjective, \( R(G)[v, v^{-1}] \)-module homomorphism

\[
\Psi_K : R(G)[v, v^{-1}] \otimes \mathbb{L}_G(F) \Omega^G(X) \to K_0(G; X)[v, v^{-1}],
\]

for any smooth \( G \)-variety \( X \), and it commutes with projective push-forward, smooth pull-back, Chern class operators and external product.

In section 11, we recall the definition of Tom Dieck’s equivariant cobordism theory \( MU_G(\cdot) \) and the Gysin homomorphism (projective push-forward) and show that there is a canonical realization functor, when \( k = \mathbb{C} \) (Theorem 11.3 in the text):

Theorem 4. There is a canonical \( \mathbb{L}_G(F) \)-homomorphism

\[
\Psi_{\text{Top}} : \Omega_G(X) \to MU_G(X),
\]

for any smooth, projective \( G \)-variety \( X \), and it commutes with projective push-forward, smooth pull-back, Chern class operators and external product. When \( X \) is equidimensional, there is a canonical grading on \( \Omega_G(X) \) and \( \Psi_{\text{Top}} : \Omega^*_G(X) \to MU^*_G(X) \).

Finally, we devote the last section to showing that our definition of \( \Omega^G(\cdot) \) is actually independent of the choice of the complete \( G \)-flag \( F \).

2. Notations and assumptions

In this paper, all schemes are over a ground field \( k \). \( G \) is a split diagonalizable group scheme, i.e., the product of a finite abelian group scheme, denoted by \( G_f \), and a split torus, denoted by \( G_t \). We will assume \( \text{char } k \) is either zero or relatively prime to the order of \( G_f \). We will also assume that \( k \) contains a primitive \( e \)-th root of unity, where \( e \) is the exponent of \( G_f \). Hence, any \( G \)-representation can be written as direct sum of 1-dimensional \( G \)-representations. We call such a pair \((G, k)\) split.

We denote the category of smooth, quasi-projective schemes over \( k \) with \( G \)-action by \( G\text{-Sm} \) and the category of reduced, quasi-projective schemes over \( k \) with \( G \)-action by \( G\text{-Var} \). The identity morphism will be denoted by \( \mathbb{1}_X : X \to X \). We will often use the symbol \( \pi_i \) to denote the projection of \( X_1 \times \cdots \times X_n \) onto its \( i \)-th component \( X_i \) and \( \pi_X \) to denote the structure morphism \( X \to \text{Spec } k \). If \( X, Y \) are two objects in \( G\text{-Var} \), then \( X \times Y \) is
considered to be in $G$-$\text{Var}$ with $G$ acting diagonally. An object $Y \in G$-$\text{Var}$ is called $G$-irreducible if there exists an irreducible component $Y'$ of $Y$ such that $G \cdot Y' = Y$. The set of isomorphism classes of $G$-linearized invertible sheaves over $X$ will be denoted by $\text{Pic}^G(X)$.

We will fix a complete $G$-universe $U$ and a complete $G$-flag $F$ given by

$$0 = V^0 \subseteq V^1 \subseteq V^2 \subseteq \cdots.$$  

We denote the 1-dimensional $G$-characters $V^i / V^{i-1}$ by $\alpha_i$. For technical reasons, we will assume $\alpha_1 = \epsilon$, the trivial character. All $G$-characters are 1-dimensional unless stated otherwise. Each character $\alpha$ defines a 1-dimensional $G$-representation, and hence a $G$-linearized invertible sheaf over $\text{Spec} k$, which will still be denoted by $\alpha$. Moreover, for an object $X \in G$-$\text{Var}$, we will simply denote the sheaf $\pi_X^* \alpha$ by $\alpha$, if there is no confusion.

For a morphism $f : X \to Y$ between schemes and a point $y \in Y$, we denote the fiber product $\text{Spec} k(y) \times_Y X$ by $f^{-1}(y)$ where $k(y)$ is the residue field of $y$ and $\text{Spec} k(y) \to Y$ is the morphism corresponding to $y$. Similarly, if $Z$ is a subscheme of $Y$, then we denote $Z \times_Y X$ by $f^{-1}(Z)$. If $A, B$ are both subschemes of $X$, then we denote $A \times_X B$ by $A \cap B$.

In this paper, for a $G$-irreducible object $X \in G$-$\text{Var}$, a $G$-prime divisor $D$ on $X$ is a $G$-invariant, $G$-irreducible, reduced, codimension 1, closed subscheme of $X$. A $G$-invariant (Weil) divisor $D$ on $X$ is a linear combination $\sum_i m_i D_i$ where $D_i$ are distinct, $G$-prime divisors on $X$. We call such a divisor smooth if all the multiplicities $m_i$ are 1 and $D_i$ are smooth and disjoint. We call a $G$-invariant divisor $A_1 + \cdots + A_n$ reduced strict normal crossing divisor if each $A_i$ is a smooth $G$-invariant divisor and, for each $I \subseteq \{1, \ldots, n\}$, the closed subscheme $\cap_{i \in I} A_i$ is smooth with codimension $|I|$ in $X$. We say two $G$-invariant divisors $A, B$ on $X$ are $G$-equivariantly linearly equivalent if $A - B = \text{div} f$ for some $f \in H^0(X, K^\times)^G$ where $K$ is the sheaf of total quotient rings on $X$ (assuming $X$ is regular in codimension 1).

For a locally free sheaf $E$ of rank $r$ over a $k$-scheme $X$, the corresponding vector bundle $E$ over $X$ will be given by

$$E \overset{\text{def}}{=} \text{Spec} \text{Sym} E^\vee.$$

The same applies to the case when $X$ is a $G$-scheme over $k$ and $E$ is $G$-linearized.

### 3. Preliminaries

Let us begin by stating some basic facts about objects in $G$-$\text{Var}$ and $G$-linearized invertible sheaves over such objects.

**Proposition 3.1.** Suppose $f : X \to Y$ is a smooth morphism between schemes of finite type over $k$. If $Y$ is reduced, then so is $X$.

**Proof.** By standard arguments. \qed
Proposition 3.2. For any morphism \( f : X \to X' \) in \( G\text{-Var} \), there exist a \( G \)-representation \( V \) and a \( G \)-equivariant immersion \( i : X \hookrightarrow \mathbb{P}(V) \times X' \) such that \( f = \pi_2 \circ i \). If we further assume \( f \) to be projective, then \( i \) will be a closed immersion.

Proof. Since \( X \) is quasi-projective, there exists an (not necessarily equivariant) immersion \( i_0 : X \hookrightarrow \mathbb{P}^n \). Define \( \mathcal{L} \overset{\text{def}}{=} i_0^*\mathcal{O}(1) \) as an (not necessarily \( G \)-linearized) invertible sheaf over \( X \). By Theorem 1.6 in [S], there exists an integer \( m \) such that \( \mathcal{L}^\otimes m \) is \( G \)-linearizable. Fix a \( G \)-linearization of \( \mathcal{L}^\otimes m \). Then, \( \mathcal{L}' \overset{\text{def}}{=} \otimes_{g \in G} g^*(\mathcal{L}^\otimes m) \) will be a \( G \)-linearized very ample invertible sheaf over \( X \). By Proposition 1.7 in [MuFK], there exists an \( G \)-equivariant immersion \( i_1 : X \hookrightarrow \mathbb{P}(V) \) for some \( G \)-representation \( V \) such that \( i_1^*\mathcal{O}(1) \cong \mathcal{L}' \). Then, the map \( (i_1, f) : X \to \mathbb{P}(V) \times X' \) will be the equivariant immersion we want. If \( f \) is projective, then \( (i_1, f) \) will be a closed immersion. \( \square \)

Proposition 3.3. Suppose \( \text{char } k = 0 \).

1. Suppose \( Y \) is in \( G\text{-Sm} \), \( X \) is in \( G\text{-Var} \) and \( U \subseteq X \) is a \( G \)-invariant open subscheme. If \( f : Y \to U \) is a projective morphism in \( G\text{-Var} \), then there exist a \( G \)-representation \( V \) and a \( G \)-equivariant closed immersion \( i : Y \hookrightarrow \mathbb{P}(V) \times U \) such that its closure in \( \mathbb{P}(V) \times X \) is smooth and \( f = \pi_2 \circ i \).

2. For any \( Y \in G\text{-Sm} \), there exist a \( G \)-representation \( V \) and a \( G \)-equivariant immersion \( i : Y \hookrightarrow \mathbb{P}(V) \) such that its closure is smooth.

Proof. For part (1), see the proof of Proposition 4.14 in [Li]. For part (2), by Proposition 3.2 there exists a \( G \)-representation \( V' \) and a \( G \)-equivariant immersion \( Y \hookrightarrow \mathbb{P}(V') \). Denote its closure by \( \overline{Y} \). By applying part (1) with \( U = Y, X = \overline{Y} \) and \( f = \mathbb{I}_Y \), we have a \( G \)-equivariant immersion \( Y \hookrightarrow \mathbb{P}(V'') \times \overline{Y} \) with smooth closure. Then the \( G \)-equivariant immersion we want is

\[
Y \hookrightarrow \mathbb{P}(V'') \times \overline{Y} \hookrightarrow \mathbb{P}(V'') \times \mathbb{P}(V') \hookrightarrow \mathbb{P}(V),
\]

where the last morphism is the Segre embedding and \( V \) is the corresponding \( G \)-representation. \( \square \)

Lemma 3.4. Suppose \( X \) is a Noetherian \( G \)-scheme and \( S = \oplus_{d \geq 0} S_d \) is a \( G \)-linearized, graded, \( \mathcal{O}_X \)-algebra such that \( S_0 = \mathcal{O}_X, S_1 \) is a \( G \)-linearized coherent sheaf over \( X \) and \( S \) is locally generated by \( S_1 \). If \( \mathcal{L} \) is a sheaf in \( \text{Pic}^G(X) \), \( p : P \overset{\text{def}}{=} \text{Proj } S \to X \) and \( p' : P' \overset{\text{def}}{=} \text{Proj } \oplus_{d \geq 0} S_d \otimes \mathcal{L}^\otimes d \to X \) are the projections, then there is a natural isomorphism \( \phi : P' \to P \), commuting with \( p \) and \( p' \), such that

\[
\mathcal{O}_{P'}(1) \cong \phi^*\mathcal{O}_P(1) \otimes p'^*\mathcal{L}.
\]

Proof. See Lemma 7.9 in Chapter II in [H] or Proposition 3.3 in [Li]. \( \square \)

Proposition 3.5. Suppose \( X \) and \( Y \) are \( G \)-irreducible objects in \( G\text{-Var} \).
(1) If $f : X \to Y$ is a $G$-equivariant, projective, birational morphism, then there is a $G$-invariant closed subscheme $Z \subseteq Y$ such that $X$ is isomorphic to $\text{Blow}_Z \! Y$ (blow up of $Y$ along $Z$) and $f$ corresponds to $\pi : \text{Blow}_Z \! Y \to Y$.

(2) If $Y$ is projective and $f : X \to Y$ is a $G$-equivariant, rational morphism, then there is a $G$-invariant closed subscheme $Z \subseteq X$ such that $f$ can be extended to a $G$-equivariant morphism $\tilde{f} : \text{Blow}_Z \! X \to Y$.

Proof. (1). This is basically an equivariant version of Theorem 7.17 in Chapter II in \cite{H}. By Proposition 3.2, there exist a $G$-representation $V$ and a $G$-equivariant immersion $i' : X \to \mathbb{P}(V)$. Then, $i \overset{\text{def}}{=} (i', f)$ defines a $G$-equivariant closed immersion $X \hookrightarrow \mathbb{P}(V) \times Y$ such that $f = \pi_2 \circ i$. Therefore, $X \cong \text{Proj} S$ for some $G$-linearized graded $O_Y$-algebra $S$. Let $L \overset{\text{def}}{=} i^* O(1)$ and $S' \overset{\text{def}}{=} \oplus_{d \geq 0} f_* (L^{{\otimes d}})$. By composing $i'$ with some $m$-uple embedding of $\mathbb{P}(V)$, for some large $m$, we may assume $S \cong S'$ as $G$-linearized graded $O_Y$-algebras.

Without loss of generality, we may assume there is a $G$-invariant hyperplane $H$ in $\mathbb{P}(V)$ which does not contain any irreducible component of $X$. So, if we consider $H \times Y$ as a $G$-invariant Cartier divisor on $\mathbb{P}(V) \times Y$, its restriction will also define a $G$-invariant Cartier divisor on $X$. Since $O(1) \cong O(H \times Y) \otimes \beta$ for some character $\beta$, we have $L \cong O(i^*(H \times Y)) \otimes \beta$. Notice that the sheaf associated to a $G$-invariant Cartier divisor can always be embedded into the sheaf of total quotient rings. Therefore, we have a $G$-equivariant embedding $L \hookrightarrow K_X \otimes \beta$. By Lemma 3.2, $\text{Proj} \oplus S_d' \cong \text{Proj} \oplus S_d' \otimes (\beta^d)^{{\otimes d}}$. So, by replacing $i'$ by $X \hookrightarrow \mathbb{P}(V) \to \mathbb{P}(V \otimes \beta^d)$, we may assume $L \subseteq K_X$. Hence, $f_* L \subseteq f_* K_X \cong K_Y$, where $K_X, K_Y$ are the sheaves of total quotient rings on $X, Y$ respectively.

Since $Y$ is quasi-projective, by Proposition 3.2, there exists a very ample sheaf $M \in \text{Pic}^G (Y)$. By a similar argument, we may assume $M \subseteq K_Y$. Also, for a large enough $n$, $M^n \cdot f_* L = \mathcal{I}$ for some $G$-invariant ideal sheaf of $Y$ because $f_* L$ is a $G$-invariant, coherent, subsheaf of $K_Y$. Again, by Lemma 3.4, $\text{Proj} \oplus S_d' \cong \text{Proj} \oplus S_d' \otimes M^{{\otimes nd}}$. Hence, it is enough to show $\oplus f_* L^d \otimes M^{{\otimes nd}} \cong \oplus \mathcal{I}^{{\otimes d}}$ as $G$-linearized graded $O_Y$-algebras. But this is true because all sheaves involved are considered as subsheaves of $K_Y$ and tensor product becomes product.

(2). Let $U$ be a $G$-invariant open subscheme of $X$ such that $f|_U : U \to Y$ is a $G$-equivariant morphism. Let $W$ be closure of the graph of $f|_U$ inside $X \times Y$. Then $W$ is a $G$-irreducible object in $G$-Var. Moreover, we have a $G$-equivariant, projective, birational morphism $p : W \to X$ and a $G$-equivariant morphism $\tilde{f} : W \to Y$, which can be considered as an extension of $f$. The result then follows by applying part (1) on $p$. \hfill \square

Suppose $X \in G$-Var is $G$-irreducible and regular in codimension 1. Denote the set of $G$-invariant Weil divisors and $G$-invariant Cartier divisors, up to $G$-equivariantly linear equivalence, by $\text{Cl}^G (X)$ and $\text{CaCl}^G (X)$ respectively.

Proposition 3.6. Suppose $X \in G$-Var is $G$-irreducible and regular in codimension 1.
(1) The natural map \(\text{CaCl}^G(X) \to \text{Pic}^G(X)\) is injective. If we further assume \(X\) to be locally factorial, \(\text{Cl}^G(X) \cong \text{CaCl}^G(X)\).

(2) The kernel of the forgetful map \(\text{Pic}^G(X) \to \text{Pic}(X)\) is given by \(\pi_X^* \text{Pic}^G(\text{Spec} \ k)\).

(3) Any sheaf \(\mathcal{L} \in \text{Pic}^G(X)\) can be written as \(\mathcal{L} \cong \mathcal{O}_X(D) \otimes \beta\) for some divisor \(D \in \text{CaCl}^G(X)\) and character \(\beta\).

Proof. (1). Standard arguments.

(2). It is enough to consider the set of \(G\)-linearizations on \(\mathcal{O}_X\). Without loss of generality, the \(G\)-action on \(X\) is faithful. Then there exists a subgroup \(H \subseteq G_f\) such that \(G_f/H\) permutes the irreducible components of \(X\) and \(H \times G_t\) acts on each on them. Therefore, we may further assume that \(X\) is irreducible.

Each \(G\)-linearization is given by collection of isomorphisms \(g^* : H^0(X, \mathcal{O}_X) \to H^0(X, \mathcal{O}_X)\) for each \(g \in G\), which is uniquely determined by the element \(a_g \defeq g^*(1)\). For simplicity, denote \(H^0(X, \mathcal{O}_X)\) by \(A\). Since \(X\) is integral, \(A\) is a domain.

Now, consider \(A\) as a \(G\)-representation with \(G\)-action given by multiplication by \(a_g\). Since \((G, k)\) is split, there are 1-dimensional \(G\)-characters \(\beta_1, \ldots, \beta_r\) and non-zero elements \(e_1, \ldots, e_r \in A\) such that 
\[1 = \sum_{i=1}^r e_i \text{ and } a_g = g \cdot 1 = \sum_{i=1}^r \beta_i(g)e_i.\]
Notice that for any \(i \neq j\),
\[\beta_i(g)e_ie_j = (g \cdot e_i)e_j = a_ge_ie_j = (g \cdot e_j)e_i = \beta_j(g)e_ie_j\]
for all \(g \in G\). Therefore, either \(\beta_i = \beta_j\) or \(e_ie_j = 0\). But \(A\) is a domain, so \(\beta_i\) are all isomorphic. Hence, \(k - \text{span}\{1\}\) is \(G\)-invariant and \(a_g = \beta_1(g)\). That proves part (2).

(3). By Proposition \ref{prop_32} there is a \(G\)-equivariant immersion \(i : X \hookrightarrow \mathbb{P}(V)\) for some \(G\)-representation \(V\). So \(\mathcal{L}\) can be expressed as the difference of two \(G\)-linearized very ample sheaves over \(X\) and it is enough to prove the statement on such sheaf. By Proposition 1.7 in \[\text{MuFKi}\], without loss of generality, we may assume \(\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}(V)}(1)\). We may further assume there is a \(G\)-invariant hyperplane \(H \subseteq \mathbb{P}(V)\) which does not contain any irreducible component of \(X\), and hence its restriction on \(X\) defines a \(G\)-invariant Cartier divisor \(D\). Since \(V\) can be expressed as the direct sum of 1-dimensional \(G\)-representations, \(\mathcal{O}_{\mathbb{P}(V)}(1) \cong \mathcal{O}_{\mathbb{P}(V)}(H) \otimes \beta\) for some \(G\)-character \(\beta\). Hence,
\[\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}(V)}(1) \cong \mathcal{O}_{\mathbb{P}(V)}(H)|_X \otimes \beta \cong \mathcal{O}_X(D) \otimes \beta.\]

\[\square\]

**Theorem 3.7.** Suppose \(\text{char} \ k = 0\). For any \(G\)-irreducible \(X \in G\text{-Sm}\) and \(G\)-linearized locally free sheaf \(\mathcal{E}\) over \(X\) of rank \(r\), there exist a \(G\)-equivariant morphism \(f : \tilde{X} \to X\), which is the composition of a series of blow ups along \(G\)-invariant smooth centers, and a \(G\)-linearized invertible subsheaf \(\mathcal{L} \hookrightarrow f^*\mathcal{E}\) over \(\tilde{X}\) such that the sequence
\[0 \to \mathcal{L} \to f^*\mathcal{E} \to (f^*\mathcal{E})/\mathcal{L} \to 0\]
is exact and \((f^*\mathcal{E})/\mathcal{L}\) is locally free of rank \(r - 1\).
Proof. It is a generalization of Theorem 6.3 in [Li], see section 6.1 in [Li] for the details. \[\square\]

4. Definition of the equivariant algebraic cobordism theory

Recall the following notion from [CGKr] (definition 12.2). A \((G,F)\)-formal group law over a commutative ring \(R\) is a topological \(R\)-module
\[
\mathcal{R}\{\{F\}\}^\text{def} = \mathcal{R}\{\{1, y(V^1), y(V^2), \ldots\}\}
\]
with product, coproduct and a \(G^s\)-action satisfying
\[
y(V^i)y(V^j) = \sum_{s \geq 0} b_{s}^{ij} y(V^s) \\
l_\alpha y(V^i) = \sum_{s \geq 0} d(\alpha)^i_s y(V^s) \\
\Delta y(V^i) = \sum_{s,t \geq 0} f_{s,t}^i y(V^s) \otimes y(V^t),
\]
for some elements \(b_{s}^{ij}\), \(d(\alpha)^i_s\), \(f_{s,t}^i\) in \(R\), and some other natural properties. The elements \(b_{s}^{ij}\), \(d(\alpha)^i_s\), \(f_{s,t}^i\) are called structure constants. According to Corollary 14.3 in [CGKr], there is a representing ring \(\mathbb{L}_G(F)\) for \((G,F)\)-formal group laws and it is generated, as a \(\mathbb{Z}\)-algebra, by the structure constants.

For an object \(X \in G-\text{Var}\), we consider elements of the form \([f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_s]\), where \(f\) is a projective morphism in \(G-\text{Var}\), \(Y\) is smooth and \(G\)-irreducible, \(\mathcal{L}_j\) are \(G\)-linearized invertible sheaves over \(Y\) (\(s\) can be zero and the order of \(\mathcal{L}_j\) does not matter). We call such an element a cycle, or more specifically, a cycle with \(s\) line bundles. We define its geometric dimension to be \(\dim Y\), denoted by \(\text{geodim}\). If \(s = 0\), we call it a geometric cycle. There is a natural notion of isomorphism on such elements. Define \(\mathbb{L}Z_G^F(X)\) to be the free \(\mathbb{L}_G(F)\)-module generated by isomorphism classes of cycles with \(s\) line bundles and \(\mathbb{L}Z_G^F(X)\) defined as
\[
\mathbb{L}Z_G^F(X) = \prod_{s \geq 0} \mathbb{L}Z_s^G(F)(X).
\]

Then we define our basic \(\mathbb{L}_G(F)\)-module \(\mathbb{L}Z_G^F(X)\) to be the submodule of \(\mathbb{L}Z_G^F(X)\) generated by elements of the form
\[
\sum_{I \geq 0} a_I[f : Y \to X, V_{S_1}^i(\mathcal{L}_1), V_{S_2}^{i_2}(\mathcal{L}_2), \ldots, V_{S_r}^{i_r}(\mathcal{L}_r)]
\]
where \(r \geq 0\), \(I\) is the multi-index \((i_1, \ldots, i_r)\), \(a_I\) are elements in \(\mathbb{L}_G(F)\) and \(S_j\) are finite subsets of the set of positive integers (The sets \(S_j\) are independent of \(I\)). Here we consider \(V_{S_j}^{i_j}(\mathcal{L}_j)\) as the abbreviation for \(\mathcal{L}_j \otimes \alpha_1, \mathcal{L}_j \otimes \alpha_2, \ldots, \mathcal{L}_j \otimes \alpha_{i_j}\) omitting \(\mathcal{L}_j \otimes \alpha_k\) whenever \(k \in S_j\). We adopt the convention that
\[
[f : Y \to X, V_{S_1}^i(\mathcal{L}_1), V_{S_2}^{i_2}(\mathcal{L}_2), \ldots, V_{S_r}^{i_r}(\mathcal{L}_r)] = 0
\]
if \( \text{max} S_j > i_j \) for some \( j \). When \( S_j \) is empty, we denote \( V_{S_j}^{i_j}(\mathcal{L}_j) \) as \( V^{i_j}(\mathcal{L}_j) \). We call an element of the form \( \prod \) an infinite cycle. Notice that cycles are infinite cycles (by taking suitable \( r, a_I \) and \( S_j \)).

We will adopt the convention that

\[
[f : Y \sqcup Y' \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r] = [f|_Y : Y \to X, \mathcal{L}_1|_Y, \ldots, \mathcal{L}_r|_Y] + [f|_{Y'} : Y' \to X, \mathcal{L}_1|_{Y'}, \ldots, \mathcal{L}_r|_{Y'}].
\]

Next, we will define four basic operations in \( \mathbb{L}Z^{G,F}(-) \). For a projective morphism \( g : X \to X' \) in \( G\text{-Var} \), we define a push-forward

\[
g_* : \mathbb{L}Z^{G,F}(X) \to \mathbb{L}Z^{G,F}(X')
\]
as the restriction of the push-forward \( g_* : \overline{\mathbb{L}Z^{G,F}(X)} \to \overline{\mathbb{L}Z^{G,F}(X')} \) which sends \([f : Y \to X, \ldots] \) to \([g \circ f : Y \to X', \ldots] \).

For a smooth morphism \( g : X' \to X \) in \( G\text{-Var} \), we define a pull-back

\[
g^* : \mathbb{L}Z^{G,F}(X) \to \mathbb{L}Z^{G,F}(X')
\]
as the restriction of the pull-back \( g^* : \overline{\mathbb{L}Z^{G,F}(X)} \to \overline{\mathbb{L}Z^{G,F}(X')} \) which sends \([f : Y \to X, \mathcal{L}_1, \ldots] \) to \([f' : Y' \to X', g^* \mathcal{L}_1, \ldots] \), where \( f', g' \) are given by the following Cartesian square:

\[
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
\downarrow f' & & \downarrow f \\
X' & \xrightarrow{g} & X
\end{array}
\]

The third operation is called infinite Chern class operator. For an object \( X \in G\text{-Var} \), sheaves \( \mathcal{L}_1, \ldots, \mathcal{L}_r \in \text{Pic}^G(X) \), elements \( a_I \in \mathbb{L}_G(F) \) and \( S_j \) as in \( \mathbb{L} \), we define an operator

\[
\sigma = \sum_I a_I V_{S_1}^{i_1}(\mathcal{L}_1) \cdots V_{S_r}^{i_r}(\mathcal{L}_r) : \mathbb{L}Z^{G,F}(X) \to \mathbb{L}Z^{G,F}(X)
\]

by sending \( \sum_I b_I [f : Y \to X, V_{T_1}^{j_1}(\mathcal{M}_1), \ldots, V_{T_s}^{j_s}(\mathcal{M}_s)] \) to

\[
\sum_I b_I a_I [f : Y \to X, V_{T_1}^{j_1}(\mathcal{M}_1), \ldots, V_{T_s}^{j_s}(\mathcal{M}_s), V_{S_1}^{i_1}(f^* \mathcal{L}_1), \ldots, V_{S_r}^{i_r}(f^* \mathcal{L}_r)]
\]

where \( IJ \) is the multi-index \((i_1, \ldots, i_r, j_1, \ldots, j_s)\). Here we adopt a similar convention that \( \sigma = 0 \) if \( i_k < \text{max} S_k \) for some \( k \). Notice that if \( r = 1, S_1 = \emptyset, a_I = \delta_1^I \) (Kronecker delta), then \( \sigma = V^1(\mathcal{L}) = c(\mathcal{L}) \), which is the usual (first) Chern class operator as in \( \mathbb{L}M_0 \).

**Remark 4.1.** If we denote the subring of \( \text{End}(\mathbb{L}Z^{G,F}(X)) \) generated by all infinite Chern class operators by \( \text{End}(\mathbb{L}Z^{G,F}(X))_{\text{inf}} \), then any infinite cycle can be written as \( f_* \circ \sigma[\mathbb{L}Y] \) for some projective morphism \( f : Y \to X \) in \( G\text{-Var} \), where \( Y \) is smooth and \( G \)-irreducible, and \( \sigma \in \text{End}(\mathbb{L}Z^{G,F}(Y))_{\text{inf}} \).
Finally, we define the external product
\[ \times : \mathbb{L}Z^{G,F}(X) \times \mathbb{L}Z^{G,F}(X') \to \mathbb{L}Z^{G,F}(X \times X') \]
by sending the pair
\[ \left( \sum_I a_I[f : Y \to X, V_{S_1}^{i_1}(\mathcal{L}_1), \ldots, V_{S_r}^{i_r}(\mathcal{L}_r)], \sum_J b_J[f' : Y' \to X', V_{T_1}^{j_1}(\mathcal{M}_1), \ldots, V_{T_s}^{j_s}(\mathcal{M}_s)] \right) \]
to
\[ \sum_{IJ} a_Ib_J[f \times f' : Y \times Y' \to X \times X', V_{S_1}^{i_1}(\pi_1^*\mathcal{L}_1), \ldots, V_{S_r}^{i_r}(\pi_r^*\mathcal{L}_r), V_{T_1}^{j_1}(\pi_1^*\mathcal{M}_1), \ldots, V_{T_s}^{j_s}(\pi_2^*\mathcal{M}_s)]. \]

**Remark 4.2.** With this external product, \( \mathbb{L}Z^{G,F}(\text{Spec } k) \) becomes a unitary, associative, commutative \( \mathbb{L}G(F) \)-algebra and \( \mathbb{L}Z^{G,F}(X) \) becomes a \( \mathbb{L}Z^{G,F}(\text{Spec } k) \)-module.

Then, our equivariant algebraic cobordism group \( \Omega^G(-) \) is defined to be the quotient (as \( \mathbb{L}G(F) \)-modules) of \( \mathbb{L}Z^{G,F}(-) \) corresponding to imposing the following two axioms:

**(Sect)** For all \( G \)-irreducible \( Y \in G\text{-Sm} \) and \( \mathcal{L} \in \text{Pic}^G(Y) \) such that there exists an invariant section \( s \in H^0(Y, \mathcal{L})^G \) that cuts out an invariant smooth divisor \( Z \) on \( Y \),
\[ [I_Y, \mathcal{L}] = [Z \hookrightarrow Y]. \]

**(EFGL)** For all \( G \)-irreducible \( Y \in G\text{-Sm} \), \( i, j \geq 0 \), character \( \alpha \in G^* \) and sheaves \( \mathcal{L}, \mathcal{M} \in \text{Pic}^G(Y) \),
\[
V^i(\mathcal{L})V^j(\mathcal{L})[I_Y] = \sum_{s \geq 0} b_s^{i,j} V^s(\mathcal{L})[I_Y],
\]
\[
V^i(\mathcal{L} \otimes \alpha)[I_Y] = \sum_{s \geq 0} d(\alpha)^s_i V^s(\mathcal{L})[I_Y],
\]
\[
V^i(\mathcal{L} \otimes \mathcal{M})[I_Y] = \sum_{s,t \geq 0} f^{i}_{s,t} V^s(\mathcal{L})V^t(\mathcal{M})[I_Y].
\]

**Remark 4.3.** Suppose one defines \( \Omega^G(-) \) by imposing the **(Dim)**, **(Sect)** and **(EFGL)** axioms on \( \mathbb{L}G(F) \otimes \mathbb{Z} Z^G(-) \), where \( Z^G(X) \) is the free abelian group generated by isomorphism classes of elements of the form \( (f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r) \). Then the canonical map \( \mathbb{L}G(F) \to \Omega^G(\text{Spec } k) \), which sends \( a \) to \( a[I_{\text{Spec } k}] \), will send the Euler class \( e(\alpha) \overset{\text{def}}{=} d(\alpha)_0^1 \) to
\[ e(\alpha)[I_{\text{Spec } k}] = V^1(\alpha)[I_{\text{Spec } k}] = [I_{\text{Spec } k}, \alpha] \]
(by Proposition 3.1), which is equal to zero by the **(Dim)** axiom. Therefore, the canonical map \( \mathbb{L}G(F) \to \Omega^G(\text{Spec } k) \) will not be injective as we want.

**Remark 4.4.** If \( X_j \) are the \( G \)-irreducible components of \( X \), then there is a canonical surjective map
\[ \oplus_j i_{j,*} : \oplus_j \mathbb{L}Z^{G,F}(X_j) \to \mathbb{L}Z^{G,F}(X) \]
where \( i_j : X_j \hookrightarrow X \) are the immersions. Moreover, it respects (Sect) and (EFGL). Therefore, it defines a canonical surjective map

\[
\oplus_j i_{j*} : \oplus_j \Omega^G(X_j) \to \Omega^G(X).
\]

Also, if \( X_j \) are disjoint, this map becomes an isomorphism.

Although it may seem that our definition of \( \Omega^G(\cdot) \) depends on the choice of the complete \( G \)-flag \( F \), we will see in section 12 that our theory is actually independent of this choice.

5. Basic properties

In our equivariant algebraic cobordism theory \( \Omega^G(\cdot) \), following the notation in [LMo], we also have basic properties (A1) - (A8). For properties involving Chern class operators, we will consider infinite Chern class operators instead. In the following list of properties, all objects and morphisms are assumed to be in \( G\text{-Var} \).

For a morphism \( f : X' \to X \) in \( G\text{-Var} \) and an element \( \sigma = \sum_I a_I V^{i_1}_{S_I}(L_1) \cdots V^{i_r}_{S_r}(L_r) \) in \( \text{End}(\Omega^G(X))_{\inf} \), we define the “pull-back” of \( \sigma \) via \( f \) as

\[
\sigma^f \overset{\text{def}}{=} \sum_I a_I V^{i_1}_{S_I}(f^*L_1) \cdots V^{i_r}_{S_r}(f^*L_r),
\]

as an element in \( \text{End}(\Omega^G(X'))_{\inf} \).

(A1) If \( f : X \to X' \) and \( g : X' \to X'' \) are both smooth, then

\[
(g \circ f)^* = f^* \circ g^*
\]

and \( (\mathbb{I}_X)^* \) is the identity map.

(A2) Suppose \( f : X \to Z \) is projective, \( g : Y \to Z \) is smooth and \( f' \), \( g' \) are given by the following Cartesian square :

\[
\begin{array}{ccc}
X \times_Z Y & \overset{g'}{\longrightarrow} & X \\
\downarrow f' & & \downarrow f \\
Y & \overset{g}{\longrightarrow} & Z
\end{array}
\]

Then the object \( X \times_Z Y \) is in \( G\text{-Var} \) and we have \( g^* \circ f_* = f'_* \circ g'^* \).

(A3) If \( f : X \to X' \) is projective and \( \sigma \) is an element in \( \text{End}(\Omega^G(X'))_{\inf} \), then

\[
f_* \circ \sigma^f = \sigma \circ f_*.
\]

(A4) If \( f : X \to X' \) is smooth and \( \sigma \) is an element in \( \text{End}(\Omega^G(X'))_{\inf} \), then

\[
f^* \circ \sigma = \sigma^f \circ f^*.
\]
(A5) If \(\sigma, \sigma'\) are both in \(\text{End}(\Omega^G(X))_{\inf}\), then 
\[\sigma \circ \sigma' = \sigma' \circ \sigma.\]

(A6) If \(f, g\) are both projective, then 
\[\times \circ (f_\ast \times g_\ast) = (f \times g)_\ast \circ \times.\]

(A7) If \(f, g\) are both smooth, then 
\[\times \circ (f^\ast \times g^\ast) = (f \times g)^\ast \circ \times.\]

(A8) Suppose \(x, y\) are elements in \(\Omega^G(X), \Omega^G(X')\) respectively and \(\sigma\) is an element in \(\text{End}(\Omega^G(X))_{\inf}\). Then we have 
\[\sigma(x) \times y = \sigma^{\pi_1}(x \times y)\]
where \(\pi_1 : X \times X' \to X\) is the projection.

Proof. The fact that \(X \times_Z Y\) is reduced in (A2) follows from Proposition 3.1. Everything else can be easily derived from the definitions, similar to [L Mo]. \(\square\)

In addition to the list above, we also have a number of basic facts for computational purpose. In particular, the double point relation (see definition 0.1 in [LP]), the blow up relation (see lemma 5.1 in [LP]) and the extended double point relation (see lemma 5.2 in [LP]) hold in our theory \(\Omega^G(-)\).

For a character \(\alpha\), we will call the element \(e(\alpha) \overset{\text{def}}{=} d(\alpha)_{0} \in L_G(F)\) the Euler class of \(\alpha\). These Euler classes are some very special elements in \(L_G(F)\) and we will see in section 10 and 11 that they correspond exactly to the Euler classes in the equivariant K-theory and Tom Dieck’s equivariant cobordism theory.

**Proposition 5.1.** As an operator on \(\Omega^G(X)\),

1. \(c(O_X) = 0\).
2. For any character \(\alpha\), we have \(c(\alpha) = e(\alpha)\).

Proof. Part (1) follows from the (Sect) axiom. For part (2), by definition,
\[c(\alpha) = V^1(O_X \otimes \alpha) = \sum_{s \geq 0} d(\alpha)^1_s V^s(O_X)\]
by the (EFGL) axiom, which is equal to \(d(\alpha)^0_0 = e(\alpha)\) by part (1). \(\square\)

**Remark 5.2.** By Lemma 16.7 in [CGKr], \(f^1_{i,0} = f^1_{0,i} = \delta^i_1\). That means 
\[c(\mathcal{L} \otimes \mathcal{M}) = c(\mathcal{L}) + c(\mathcal{M}) + \sum_{s, t \geq 1} f^1_{s,t} V^s(\mathcal{L}) V^t(\mathcal{M}).\]
Moreover, the representing ring \(L_G(F)\), as a \(\mathbb{Z}\)-algebra, is generated by \(f^1_{s,t}\) and the Euler classes \(e(\alpha)\) by Theorem 16.1 in [CGKr].
Proposition 5.3. For all $X \in G$-Var and $\mathcal{L} \in \text{Pic}^G(X)$, there exists $\sigma \in \text{End}(\Omega^G(X))_{\text{inf}}$ such that

$$c(\mathcal{L}^\vee) = \sigma \circ c(\mathcal{L}).$$

Proof. By Proposition 5.1 and Remark 5.2, we have

$$0 = c(\mathcal{O}_X) = c(\mathcal{L} \otimes \mathcal{L}^\vee) = c(\mathcal{L}) + c(\mathcal{L}^\vee) + \sum_{s,t \geq 1} f_{s,t}^1 V^s(\mathcal{L}) V^t(\mathcal{L}^\vee).$$

Hence,

$$c(\mathcal{L}^\vee) = -c(\mathcal{L}) - \sum_{s,t \geq 1} f_{s,t}^1 V^s(\mathcal{L}) V^t(\mathcal{L}^\vee) = (-1 - \sum_{s,t \geq 1} f_{s,t}^1 V^s(\mathcal{L}) V^t(\mathcal{L}^\vee)) \circ c(\mathcal{L}).$$

Lemma 5.4. Suppose $Y$ is an object in $G$-Sm and $E_1$, $E_2$ are two invariant divisors on $Y$ such that $E_1 + E_2$ is a reduced strict normal crossing divisor. Denote the intersection $E_1 \cap E_2$ by $D$. Then, as elements in $\Omega^G(D)$, we have

$$\sum_{s,t \geq 1} f_{s,t}^1 V^s_{E_1}(\mathcal{O}_D(E_1)) V^t_{E_1}(\mathcal{O}_D(E_2)) [\mathbb{I}_D] = -[\mathbb{P}_D \to D],$$

where $\mathbb{P}_D \overset{\text{def}}{=} \mathbb{P}(\mathcal{O}_D \oplus \mathcal{O}_D(E_1))$.

Proof. By a similar argument as in the proof of Lemma 3.3 in [LP].

Proposition 5.5. Suppose $A$, $B$, $C$ are invariant divisors on $Y \in G$-Sm such that $A + B \sim C$, $C$ is disjoint from $A \cup B$ and $A + B + C$ is a reduced strict normal crossing divisor. Then, as elements in $\Omega^G(Y)$, we have

$$[C \to Y] = [A \to Y] + [B \to Y] - [\mathbb{P}(\mathcal{O}_D \oplus \mathcal{O}_D(A)) \to D \to Y],$$

where $D \overset{\text{def}}{=} A \cap B$.

Proof. Without loss of generality, we may assume $Y$ to be $G$-irreducible. Then we have

$$[C \to Y] = c(\mathcal{O}(C)) [\mathbb{I}_Y]$$

$$= c(\mathcal{O}(A + B)) [\mathbb{I}_Y]$$

$$= c(\mathcal{O}(A)) [\mathbb{I}_Y] + c(\mathcal{O}(B)) [\mathbb{I}_Y] + \sum_{s,t \geq 1} f_{s,t}^1 V^s(\mathcal{O}(A)) V^t(\mathcal{O}(B)) [\mathbb{I}_Y]$$

by Remarks 5.2. Therefore,

$$[C \to Y] = [A \to Y] + [B \to Y] + \sum_{s,t \geq 1} f_{s,t}^1 V^s_{E_1}(\mathcal{O}(A)) V^t_{E_1}(\mathcal{O}(B)) [D \to Y]$$

$$= [A \to Y] + [B \to Y] + i_{*}(\sum_{s,t \geq 1} f_{s,t}^1 V^s_{E_1}(\mathcal{O}_D(A)) V^t_{E_1}(\mathcal{O}_D(B)) [\mathbb{I}_D])$$

where $i : D \to Y$. The result then follows from Lemma 5.4.
Proposition 5.6. Suppose $Z$ is an invariant closed subscheme of $Y$ such that $Z, Y$ are both in $G$-Sm. Then, as elements in $\Omega^G(Y)$, we have
\[
\text{[Blow}_Z Y \to Y] - [I_Y] = -[P_1 \to Z \hookrightarrow Y] + [P_2 \to Z \hookrightarrow Y],
\]
where $P_1 \overset{\text{def}}{=} P(O_Z \oplus N_{Z \to Y}^\vee) \to Z$ and $P_2 \overset{\text{def}}{=} P(O \oplus O(1)) \to P(N_{Z \to Y}^\vee) \to Z$ with $O(1)$ being the tautological line bundle over $P(N_{Z \to Y}^\vee)$.

Proof. By the same argument as in the proof of Lemma 5.1 in [LP].

Proposition 5.7. Suppose $A, B, C$ are invariant divisors on $Y \in G$-Sm such that $A + B \sim C$ and $A + B + C$ is a reduced strict normal crossing divisor. Then, as elements in $\Omega^G(Y)$, we have
\[
[C \hookrightarrow Y] = [A \hookrightarrow Y] + [B \hookrightarrow Y] - [P_1 \to Y] + [P_2 \to Y] - [P_3 \to Y],
\]
where $D \overset{\text{def}}{=} A \cap B$, $E \overset{\text{def}}{=} A \cap B \cap C$ and
\[
\begin{align*}
P_1 &\overset{\text{def}}{=} P(O_D \oplus O_D(A)) \to D, \\
P_2 &\overset{\text{def}}{=} P(O \oplus O(1)) \to P(O_E(-B) \oplus O_E(-C)) \to E, \\
P_3 &\overset{\text{def}}{=} P(O_E \oplus O_E(-B) \oplus O_E(-C)) \to E
\end{align*}
\]
with $O(1)$ being the tautological line bundle over $P(O_E(-B) \oplus O_E(-C))$.

Proof. By the same argument as in the proof of Lemma 5.2 in [LP].

Remark 5.8. From now on, we will refer to Proposition 5.5, 5.6 and 5.7 as the double point relation, the blow up relation and the extended double point relation respectively.

6. The equivariant algebraic cobordism ring of a point

In this section, we will show that the equivariant algebraic cobordism group $\Omega^G(X)$, as a $\mathbb{L}_G(F)$-module, is generated by geometric cycles. Moreover, we will show that the canonical $\mathbb{L}_G(F)$-algebra homomorphism
\[
\mathbb{L}_G(F) \to \Omega^G(\operatorname{Spec} k)
\]
is surjective. Since we need to employ the embedded desingularization theorem (see [BM]) and the weak factorization theorem (Theorem 0.3.1 in [AKMW]), we will assume $\operatorname{char} k = 0$ throughout this section.

First of all, we have the following result, which is an analogue of the (Nilp) axiom in [LM] (see Remark 2.2.3 in [LMO]).
**Proposition 6.1.** Suppose char k = 0. For any G-irreducible \( Y \in G\text{-Sm}, \mathcal{L} \in \text{Pic}^G(Y) \) and finite set \( S \) as in (7),

\[
V^S_Y(\mathcal{L})[\mathbb{I}_Y] = 0
\]
as elements in \( \Omega^G(Y) \), for sufficiently large \( n \).

**Proof.** We will proceed by induction on \( \dim Y \). If \( \dim Y = 0 \), then, by Proposition 3.6, \( \mathcal{L} \cong \beta \) for some character \( \beta \). Take \( N \) to be an integer such that \( N > \max S \) and \( \alpha_N \cong \beta^N \).

By definition,

\[
V^N_S(\mathcal{L})[\mathbb{I}_Y] = V^{N-1}_S(\mathcal{L}) \circ c(\mathcal{L} \otimes \alpha_N) \circ \cdots \circ c(\mathcal{L} \otimes \alpha_1)[\mathbb{I}_Y] = 0
\]
because \( \mathcal{L} \otimes \alpha_N \cong 0_Y \).

Suppose \( \dim Y > 0 \). By Proposition 3.6, \( \mathcal{L} \cong \mathcal{O}(\sum_i \pm D_i) \otimes \beta \) for some invariant \( G \)-prime divisors \( D_i \) and character \( \beta \). Apply the embedded desingularization theorem on \( \cup_i D_i \hookrightarrow Y \), we got a map \( \pi: \tilde{Y} \to Y \) which is the composition of a series of blow ups along invariant smooth centers such that the strict transforms of \( D_i \), denoted by \( \langle D_i \rangle \), are smooth. For simplicity, assume \( \pi \) is given by a single blow up along \( Z \subseteq Y \). By the blow up relation,

\[
[\tilde{Y} \to Y] - [\mathbb{I}_Y] = -[\mathbb{P}_1 \to Z \to Y] + [\mathbb{P}_2 \to Z \to Y].
\]

Apply \( V^N_S(\mathcal{L}) \) on both sides, we have

\[
\begin{align*}
\pi_* \circ V^N_S(\pi^* \mathcal{L})[\mathbb{I}_{\tilde{Y}}] & - V^N_S(\mathcal{L})[\mathbb{I}_Y] \\
& = -V^N_S(\mathcal{L})[\mathbb{P}_1 \to Z \to Y] + V^N_S(\mathcal{L})[\mathbb{P}_2 \to Z \to Y] \\
& = -i_* \circ p_{1*} \circ p_1^* \circ V^N_S(\mathcal{L}|_Z)[\mathbb{I}_Z] + i_* \circ p_{2*} \circ p_2^* \circ V^N_S(\mathcal{L}|_Z)[\mathbb{I}_Z]
\end{align*}
\]

where \( i: Z \hookrightarrow Y \) is the immersion and \( p_i: \mathbb{P}_i \to Z \) are the projections. By the induction assumption, it is enough to consider \( V^N_S(\pi^* \mathcal{L})[\mathbb{I}_{\tilde{Y}}] \). Since \( \pi^* \mathcal{L} \cong \mathcal{O}(\sum_i \pm \langle D_i \rangle + \sum_j \pm E_j) \otimes \beta \) where \( E_j \) are the strict transforms of the exceptional divisors, which are also invariant and smooth, without loss of generality, we may assume \( D_i \) are smooth.

By the induction assumption, there exist integers \( N_i \) such that \( V^N_S(\mathcal{L}|_{D_i})[\mathbb{I}_{D_i}] = 0 \) for all \( n \geq N_i \). Now take \( N \) to be an integer which is greater than \( \max S \) and \( \max N_i \), and also \( \alpha_N \cong \beta^N \). Then, for all \( n \geq N \),

\[
V^N_S(\mathcal{L})[\mathbb{I}_Y] = V^N_S(\mathcal{O}(\sum_i \pm D_i) \otimes \beta)[\mathbb{I}_Y]
\]

for some \( \sigma \in \text{End } (\Omega^G(Y))_{\text{inf}} \), hence,

\[
V^N_S(\mathcal{L})[\mathbb{I}_Y] = V^{N-1}_S(\mathcal{L}) \circ \sum_i \sigma_i[D_i \hookrightarrow Y] = 0
\]
because \( N - 1 \geq N_i \).

**Remark 6.2.** Suppose, for each \( G \)-irreducible \( Y \in G\text{-}Sm \) and \( \mathcal{L} \in \text{Pic}^G(Y) \), we fix a choice of \( G \)-prime divisors \( D_i \) and character \( \beta \) such that \( \mathcal{L} \cong \mathcal{O}((\sum_{i=1}^k m_i D_i)) \otimes \beta \), and a choice of \( Z \) while applying the embedded desingularization theorem on \( \cup_i D_i \to Y \). If \( \mathcal{L} \) is admissible (see Definition 6.4), we pick \( D_i \) to be smooth. Then, for any \( S \), there is a positive integer \( \text{Nilp}(Y, \mathcal{L}, S) \), suggested by the proof, such that \( V_{S, \text{Nilp}(Y, \mathcal{L}, S)}(\mathcal{L})[\mathbb{I}_Y] = 0 \) as elements in \( \Omega^G(Y) \).

**Corollary 6.3.** Suppose \( \text{char } k = 0 \). For all \( X \in G\text{-}Var \), the \( \mathbb{L}_G(F) \)-module \( \Omega^G(X) \) is generated by cycles.

**Proof.** The module \( \Omega^G(X) \) is generated by infinite cycles, which can be written as \( f_\ast \circ \sigma[\mathbb{I}_Y] \) for some \( \sigma \in \text{End}(\Omega^G(Y))_{\text{inf}} \). By Proposition 6.1, it is indeed a finite sum. \( \square \)

Next, we will employ a technique called “reduction of tower”, which is similar to the technique discussed in section 6.3 in [Li]. In spite of the assumption that \( G \) is a finite abelian group scheme in section 6.3 in [Li], most arguments work for our more general setup too. Therefore, we will not give separate proofs here. Recall the following definitions from [Li].

**Definition 6.4.** Suppose \( Y \) is an object in \( G\text{-}Sm \). A morphism \( \mathbb{P} \to Y \) in \( G\text{-}Sm \) is called a quasi-admissible tower over \( Y \) with length \( n \) if it can be factored into
\[
\mathbb{P} = \mathbb{P}_n \to \mathbb{P}_{n-1} \to \cdots \to \mathbb{P}_1 \to \mathbb{P}_0 = Y
\]
such that, for all \( 0 \leq i \leq n-1 \), \( \mathbb{P}_{i+1} = \mathbb{P}(\mathcal{E}_i) \) where \( \mathcal{E}_i \) is the direct sum of sheaves which is either the pull-back of a \( G \)-linearized locally free sheaves over \( Y \), or the pull back of \( \mathcal{O}_{\mathbb{P}_i}(m) \) for some integer \( m \) and \( 1 \leq j \leq i \).

A sheaf \( \mathcal{L} \in \text{Pic}^G(Y) \) is called admissible if there exist smooth, \( G \)-prime divisors \( D_1, \ldots, D_k \) on \( Y \) and character \( \beta \) such that
\[
\mathcal{L} \cong \mathcal{O}_Y(\sum_{i=1}^k m_i D_i) \otimes \beta
\]
for some integers \( m_i \). Denote the subgroup of \( \text{Pic}^G(Y) \) generated by admissible invertible sheaves by \( \text{APic}^G(Y) \). Also, define the group of admissible invertible sheaves over \( \mathbb{P}_i \) by
\[
\text{APic}^G(\mathbb{P}_i) \overset{\text{def}}{=} \text{APic}^G(Y) + \mathbb{Z}\mathcal{O}_{\mathbb{P}_i}(1) + \cdots + \mathbb{Z}\mathcal{O}_{\mathbb{P}_i}(1).
\]
We then call a quasi-admissible tower \( \mathbb{P} \to Y \) admissible if all sheaves involved in the construction are admissible invertible sheaves.

**Lemma 6.5.** Suppose \( Y \in G\text{-}Sm \) is \( G \)-irreducible and \( D \) is a smooth \( G \)-prime divisor on \( Y \). Furthermore, suppose \( \mathbb{P} \to Y \) is an admissible tower with length \( n \) and \((i+1)\)-th level \( \mathbb{P}_{i+1} = \mathbb{P}(\oplus_{j=1}^{r} \mathcal{L}_j) \), and \( \mathcal{M}_1, \ldots, \mathcal{M}_s \) are sheaves in \( \text{Pic}^G(\mathbb{P}) \). Then there exist an admissible tower \( \mathbb{P}' \to Y \) with length \( n \), quasi-admissible towers \( Q_0, Q_1, Q_2, Q_3 \to D \) and \( G \)-linearized invertible sheaves \( \mathcal{M}_k' \), \( \mathcal{M}'_k \) such that
\[
\mathbb{P}' = \mathbb{P}'_n \to \cdots \to \mathbb{P}'_{i+1} \to \mathbb{P}'_i \to \cdots \to \mathbb{P}_0 = Y
\]
Hence, the result follows by applying Chern class operators $c_Q$ where

$$\dim P = \dim P' = \dim Q_j$$

and we have the following equality in $\Omega^G(Y)$:

$$\left[ P' \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s \right] = \left[ P \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s \right]$$

We then denote $\left[ P \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s \right]$ by elements of the form $\mathcal{G}$.

Suppose $[\hat{M}]$ and we have the following equality in $\Omega^G(Y)$:

$$\left[ P' \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s \right] = \left[ P \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s \right]$$

We then have $\left[ \hat{M} \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s \right]$.

\begin{proof}

Since this result is very similar to Lemma 6.10 in \cite{Li}, we will only give a sketch of proof here. First of all, we construct an admissible tower $\hat{P} \to Y$ with length $n$ by defining $\hat{P}_{i+1} \equiv P((\oplus_{j=1}^{i+1} \mathcal{L}_j) \oplus \mathcal{L}_r(D))$ and all higher levels are constructed in the same manner as $P$.

We then have $\hat{P} \to P$. Since $\text{Pic}(\hat{P}) \to \text{Pic}(P)$ is surjective, $\text{Pic}^G(\hat{P}) \to \text{Pic}^G(P)$ is surjective (by Proposition 3.6). So there are $G$-linearized invertible sheaves $\hat{\mathcal{M}}_1, \ldots, \hat{\mathcal{M}}_s$ over $\hat{P}$ that extends $\mathcal{M}_1, \ldots, \mathcal{M}_s$ respectively. Then, we construct the admissible tower $P' \to Y$ and the quasi-admissible tower $Q_0 \to D \to Y$ by restricting $\hat{P}$ via $P_{i+1} \to P_{i+1}$ and $D \to Y$ respectively.

By Lemma 6.9 in \cite{Li} (still holds for our group $G$), we have $Q_0 + P \sim P'$ as invariant divisors on $\hat{P}$ and the sum of them is a reduced strict normal crossing divisor. By the extended double point relation, we have

$$[P' \to \hat{P}] = [Q_0 \to \hat{P}] + [P \to \hat{P}] - [(Q_0 \cap P) \times_{\hat{P}} P^1 \to \hat{P}]$$

and

$$[(Q_0 \cap P \cap P') \times_{\hat{P}} P^2 \to \hat{P}] - [(Q_0 \cap P \cap P') \times_{\hat{P}} P^3 \to \hat{P}]$$

where

$$P^1 \equiv \text{Pic}(\mathcal{O} \oplus \mathcal{O}(Q_0)) \to \hat{P},$$

$$P^2 \equiv \text{Pic}(\mathcal{O} \oplus \mathcal{O}(1)) \to \text{Pic}(\mathcal{O}(-P) \oplus \mathcal{O}(-P')) \to \hat{P},$$

$$P^3 \equiv \text{Pic}(\mathcal{O} \oplus \mathcal{O}(-P) \oplus \mathcal{O}(-P')) \to \hat{P}.$$
projective morphisms $Y_i \to Y$ in $G$-Sm with $\dim Y_i \leq \dim Y$ and sheaves $\mathcal{M}_j$ such that
\[ [\mathbb{P} \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s] = \sum_i x_i [Y_i \to Y, \mathcal{M}_j] \]
as elements in $\Omega^G(Y)$. Moreover, $\dim \mathbb{P} = \geodim x_i + \dim Y_i$. If there is no invertible sheaf on the left hand side, i.e., $s = 0$, then the same result holds with no invertible sheaves on the right hand side.

**Proof.** Here is a sketch of the proof (See Proposition 6.17 in [Li] for details). We will prove the statement by induction on the dimension of $Y$. We will handle the induction step first. Suppose $\dim Y \geq 1$. Let $\Omega^G(Y)'$ be the subgroup of $\Omega^G(Y)$ generated by elements of the form
\[ [\mathbb{P}' \to Y', \mathcal{M}_1', \ldots, \mathcal{M}_s'] \]
where $Y' \in G$-Sm is $G$-irreducible with dimension less than $\dim Y$, $\dim \mathbb{P}' = \dim \mathbb{P}'$ and $\mathbb{P}' \to Y'$ is a quasi-admissible tower. So, elements in $\Omega^G(Y)'$ will be handled by the induction assumption. Let $\mathbb{P} \to Y$ be a quasi-admissible tower with length $n$. If $n = 0$, then we are done. Suppose $n \geq 1$.

Step 1 : Reduction to a quasi-admissible tower constructed only by $G$-linearized invertible sheaves.

Suppose $\pi : \tilde{Y} \to Y$ is the composition of a series of blow up along smooth invariant centers and let $\tilde{\mathbb{P}} \overset{\text{def}}{=} \mathbb{P} \times_Y \tilde{Y}$. By the blow up relation, it can be shown that the difference
\[ \pi_*[\tilde{\mathbb{P}} \to \tilde{Y}, \pi'_*\mathcal{M}_1, \ldots, \pi'_*\mathcal{M}_s] - [\mathbb{P} \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s], \]
where $\pi' : \tilde{\mathbb{P}} \to \mathbb{P}$, lies inside $\Omega^G(Y)'$. Therefore, we may blow up $Y$ along any invariant smooth center if necessary.

If $\mathbb{P}_i = \mathbb{P}(\mathcal{E}' \oplus \mathcal{E})$ such that $\text{rank } \mathcal{E} > 1$ (in particular, it comes from $Y$), then, by Theorem 3.7, we may assume we have a splitting
\[ 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}/\mathcal{L} \to 0 \]
of $G$-linearized locally free sheaves over $Y$. Define
\[ \hat{\mathbb{P}}_i \overset{\text{def}}{=} \mathbb{P}(\mathcal{E}' \oplus \mathcal{E} \oplus \mathcal{L}) \quad \text{and} \quad \mathbb{P}'_i \overset{\text{def}}{=} \mathbb{P}(\mathcal{E}' \oplus (\mathcal{E}/\mathcal{L}) \oplus \mathcal{L}). \]
and construct towers $\hat{\mathbb{P}}$, $\mathbb{P}' \to Y$ in a similar manner as in the proof of Lemma 6.5. Also define sheaves $\hat{\mathcal{M}}_i \in \text{Pic}^G(\hat{\mathbb{P}})$ and $\mathcal{M}'_i \in \text{Pic}^G(\mathbb{P}')$ similarly. Hence, by Lemma 6.12 in [Li] (still holds for our group $G$), we have
\[ [\mathbb{P} \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s] = [\mathbb{P}' \to Y, \mathcal{M}'_1, \ldots, \mathcal{M}'_s] \]
as elements in $\Omega^G(Y)$ and the result follows by applying this argument repeatedly until all sheaves involved are of rank 1.

Step 2 : Reduction to an admissible tower.
For each \( \mathcal{L} \in \text{Pic}^G(Y) \) used in the construction of \( \mathbb{P} \), by Proposition 3.6, there are \( G \)-prime divisors \( D_{\mathcal{L},i} \) on \( Y \) and character \( \beta \) such that \( \mathcal{L} \cong \mathcal{O}_Y(\sum_i \pm D_{\mathcal{L},i}) \otimes \beta \). By applying the embedded desingularization Theorem on \( \cup_{\mathcal{L},i} D_{\mathcal{L},i} \hookrightarrow Y \), we may assume \( D_{\mathcal{L},i} \) are all smooth.

Step 3: Reduction to an admissible tower with locally free sheaf over \( \text{Spec} k \).

Consider the first level \( \mathbb{P}_1 = \mathbb{P}(\oplus_{j=1}^r \mathcal{L}_j) \). Since the sheaves \( \mathcal{L}_j \) are admissible, they can be expressed by invariant smooth divisors on \( Y \). By lemma 5.5 for any invariant smooth divisor \( D \) on \( Y \), we can twist \([ \mathbb{P} \to Y, \mathcal{M}_1, \ldots, \mathcal{M}_s ] \) to \([ \mathbb{P}' \to Y, \mathcal{M}'_1, \ldots, \mathcal{M}'_s ] \) so that \( \mathbb{P}' = \mathbb{P}(\mathcal{L}_j) + \mathcal{L}_k(D) \) and the difference will be given by elements inside \( \Omega^G(Y) \).

Hence, we may assume there exists a sheaf \( \mathcal{L}' \in \text{APic}^G(Y) \) such that \( \mathcal{L}_j \cong \mathcal{L}' \otimes \beta_j \) for all \( j \).

The result then follows by defining \( \mathcal{E}_1 = \oplus_{j=1}^r \beta_j \) and observing that \( \mathbb{P}(\mathcal{L}' \otimes \pi_Y^* \mathcal{E}_1) \cong \mathbb{P}(\pi_Y^* \mathcal{E}_1) \) (by Lemma 3.4). Notice that the isomorphism \( \mathbb{P}(\mathcal{L}' \otimes \pi_Y^* \mathcal{E}_1) \cong \mathbb{P}(\pi_Y^* \mathcal{E}_1) \) may take admissible invertible sheaf to non-admissible invertible sheaf. But we may then apply step 2 to ensure the sheaves involved are admissible.

Step 4: Finish the induction step.

By applying the argument in step 3 on all levels, we have an admissible tower

\[ Q = Q_n = \mathbb{P}(\mathcal{E}_n) \to \cdots \to \mathbb{P}(\mathcal{E}_1) \to Q_0 = \text{Spec} k \]

such that \( \mathbb{P} \cong Y \times Q \). Hence, for each \( i \), the invertible sheaf \( \mathcal{M}_i \) over \( \mathbb{P} \) is isomorphic to \( \pi_Y^* \mathcal{M}_{Y,i} \otimes \pi_Q^* \mathcal{M}_{Q,i} \) for some \( G \)-linearized invertible sheaves \( \mathcal{M}_{Y,i}, \mathcal{M}_{Q,i} \) over \( Y, Q \) respectively. For simplicity, assume \( s = 1 \),

\[
\begin{align*}
[ \mathbb{P} \to Y, \mathcal{M}] &= \pi_*(\mathcal{M}[\mathbb{P}]) \\
&= \pi_*(\pi_Y^* \mathcal{M}_Y \otimes \pi_Q^* \mathcal{M}_Q)[\mathbb{P}] \\
&= \pi_*(\pi_Y^* \mathcal{M}_Y)[\mathbb{P}] + \pi_*(\pi_Q^* \mathcal{M}_Q)[\mathbb{P}] + \pi_* \sum_{s,t \geq 1} f_{s,t}^1 V^s(\pi_Y^* \mathcal{M}_Y)V^t(\pi_Q^* \mathcal{M}_Q)[\mathbb{P}],
\end{align*}
\]

which is a finite sum by Proposition 6.1. Hence,

\[
\begin{align*}
[ \mathbb{P} \to Y, \mathcal{M}] &= [Q][\mathbb{I}_Y, \mathcal{M}_Y] + [Q, \mathcal{M}_Q][\mathbb{I}_Y] + \sum_{s,t \geq 1} f_{s,t}^1 \pi_*(V^s(\mathcal{M}_Q)[\mathbb{I}_Y] \times V^t(\mathcal{M}_Y)[\mathbb{I}_Y]) \\
&= x_1 [\mathbb{I}_Y, \mathcal{M}_Y] + x_2 [\mathbb{I}_Y] + \sum_{s,t \geq 1} x_{s,t} V^s(\mathcal{M}_Y)[\mathbb{I}_Y]
\end{align*}
\]

by letting \( x_1 \equiv [Q], x_2 \equiv [Q, \mathcal{M}_Q] \) and \( x_{s,t} \equiv f_{s,t}^1 [Q, V^t(\mathcal{M}_Q)] \). That finishes the induction step (when \( \dim Y > 0 \)).

Step 5: \( \dim Y = 0 \) case.
In this case, $P \cong Y \times Q$ for some admissible tower $Q$ over Spec $k$. Apply step 4 and we are done. □

Our first goal in this section is to show that $\Omega^G(X)$, as a $L_G(F)$-module, is generated by geometric cycles. To this end, we need two more technical Lemmas.

**Lemma 6.8.** Suppose $Y \in G$-$\text{Sm}$ is $G$-irreducible and $D$ is an invariant divisor on $Y$ which can be expressed as the sum of smooth, $G$-prime divisors on $Y$. Then $c(O(D))[I_Y]$ is equal to the finite sum of elements of the form $a[Y' \to Y, \ldots]$ where $a$ is in $L_G(F)$ and $\dim Y' < \dim Y$.

**Proof.** Let $D = \sum_i \pm D_i$ where $D_i$ are smooth, $G$-prime divisors. By Remark 5.2 and Proposition 5.3, for some $\sigma_i \in \text{End} (\Omega^G(Y))_{\text{inf}}$,

$$c(O(D))[I_Y] = c(O(\sum_i \pm D_i))[I_Y] = \sum_i \sigma_i \circ c(O(D_i))[I_Y] = \sum_i j_{i*} \circ \sigma_i^j[I_{D_i}]$$

where $j_i : D_i \hookrightarrow Y$ are the immersions. The result then follows from Proposition 6.1. □

**Lemma 6.9.** Suppose $P = P_n = P(\mathcal{E}_n \oplus \mathcal{L}_n) \to \cdots \to P(\mathcal{E}_1 \oplus \mathcal{L}_1) = P_1 \to P_0 = \text{Spec } k$ is an admissible tower over $\text{Spec } k$ with length $n$. Denote the invariant smooth divisors $P(\mathcal{E}_i)$ on $P_i$ by $H_i$. Then, for all $\mathcal{L} \in \text{Pic}^G(P)$, there exist integers $m_1, \ldots, m_n$ and character $\beta$ such that

$$\mathcal{L} \cong O(m_1 H_1 + \cdots + m_n H_n) \otimes \beta.$$

**Proof.** By Proposition 3.6 it is enough to show

$$\mathcal{L} \cong O(m_1 H_1 + \cdots + m_n H_n)$$

as sheaves in $\text{Pic}(P)$. Since $\text{Pic}(P)$ is generated by sheaves of the form $O_{P_i}(1)$, we may assume $\mathcal{L} \cong O_{P_i}(1)$. For $i \geq 1$, let $U$ be the complement $P_i - \bigcup_{j=1}^n P_{ij}$, which is just $\mathbb{A}^N$ for some $N$. So $O_{P_i}(1)$ is trivial over $U$ (by the Quillen–Suslin theorem). Hence, there exist integers $m_1, \ldots, m_i$ such that

$$O_{P_i}(1) \cong O(m_1 H_1 + \cdots + m_i H_i).$$

□

For the rest of this section, for a $G$-irreducible object $Y \in G$-$\text{Sm}$, we will say $\beta$ is a twisting character of a sheaf $\mathcal{L} \in \text{Pic}^G(Y)$ if $\mathcal{L} \cong O_Y(D) \otimes \beta$ for some invariant divisor $D$ on $Y$. Twisting character may not be unique, but it always exists (Proposition 3.6).

**Theorem 6.10.** Suppose $\text{char } k = 0$. For any $X \in G$-$\text{Var}$, the $L_G(F)$-module $\Omega^G(X)$ is generated by geometric cycles.
More precisely, any element \([Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r] \in \Omega^G(X)\), such that \(Y\) is \(G\)-irreducible, can be expressed in the following form:

\[
\left( \prod_{i=1}^r e(\beta_i) \right) \cdot [Y \to X] + \sum_j a_j [\mathbb{P}_j][Y_j \to X] + \sum_k a_k' [Y_k' \to X]
\]

where \(\beta_i\) is a twisting character of \(\mathcal{L}_i\), \(a_j, a_k'\) are elements in \(L_G(F)\), \(\mathbb{P}_j\) are admissible towers over \(\text{Spec} \, k\), \(Y_j, Y_k' \in G\text{-Sm}\) are \(G\)-irreducible, \(\dim Y_j, \dim Y_k' < \dim Y\) and \(\dim Y = \dim \mathbb{P}_j + \dim Y_j\).

**Proof.** By Corollary 6.3 it is enough to consider cycles. Since \([f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r] = f_*[[Y, \mathcal{L}_1, \ldots, \mathcal{L}_r]]\), it is enough to show the statement on elements of the form \([\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_r]\) where \(Y \in G\text{-Sm}\) is \(G\)-irreducible. We will proceed by induction on \(d \overset{\text{def}}{=} \dim Y\). Within this proof, for a cycle \(x \in \Omega^G(-)\), we will say \(x \equiv 0\) if \(\text{geodim} \, x < d\).

If \(\dim Y = 0\), then \([\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_r] = [\mathbb{I}_Y, \beta_1, \ldots, \beta_r] = e(\beta_1) \cdots e(\beta_r)[\mathbb{I}_Y]\)
and we are done.

Suppose \(\dim Y > 0\). Let \(\mathcal{L}_i' \overset{\text{def}}{=} \mathcal{L}_i \otimes \beta_i'\). Then,

\[
[\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_r] = c(\mathcal{L}_r)[[\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_{r-1}]
= c(\mathcal{L}_r' \otimes \beta_r)[[\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_{r-1}]
= e(\beta_r)[[\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_{r-1}] + \sum_{j \geq 1} d(\beta_r)^j [V^j(\mathcal{L}_r')][\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_{r-1}]
= e(\beta_r)[[\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_{r-1}] + \sigma \circ c(\mathcal{L}_r')[\mathbb{I}_Y]
\]

for some \(\sigma \in \text{End} \, (\Omega^G(Y))_{\inf}\). Inductively, we have

\[
[\mathbb{I}_Y, \mathcal{L}_1, \ldots, \mathcal{L}_r] = \left( \prod_{i=1}^r e(\beta_i) \right) \cdot [\mathbb{I}_Y] + \sum_{i=1}^r \sigma_i \circ c(\mathcal{L}_i')[\mathbb{I}_Y]
\]

for some \(\sigma_i\). Therefore, it suffices to prove that if \(\mathcal{L}\) has a trivial twisting character, then

\[
\sigma \circ c(\mathcal{L})[\mathbb{I}_Y] = \sum_j a_j [\mathbb{P}_j][Y_j \to Y] + \sum_k a_k' [Y_k' \to Y]
\]

where \(a_j, a_k', \mathbb{P}_j, Y_j\) and \(Y_k'\) are as described in the statement.

By the blow up relation, for any \(G\)-irreducible, invariant, smooth closed subscheme \(Z \subseteq Y\), we have

\[
[p : \text{Bl}_Z Y \to Y, p^* \mathcal{L}] = [\mathbb{I}_Y, \mathcal{L}] = -[p_1 : \mathbb{P}_1 \to Z \hookrightarrow Y, p_1^* \mathcal{L}] + [p_2 : \mathbb{P}_2 \to Z \hookrightarrow Y, p_2^* \mathcal{L}]
\]

where \(p_1, p_2 \to Z\) are both quasi-admissible towers. By proposition 6.7

\[
[\mathbb{P}_1 \to Z \hookrightarrow Y, p_1^* \mathcal{L}] = \sum_j x_j [Y_j \to Z \hookrightarrow Y, \ldots]
\]

where \(x_j \in \Omega^G(\text{Spec} \, k)', \dim Y_j \leq \dim Z < d\) and \(d = \dim \mathbb{P}_1 = \text{geodim} \, x_j + \dim Y_j\).
Let us consider elements of the form \([\mathbb{P}, \mathcal{M}_1, \ldots, \mathcal{M}_s]\) where \(\mathbb{P}\) is an admissible tower over \(\text{Spec } k\) with length \(n\) and dimension \(d' \leq d\). By Lemma \[6.9\] we have
\[
\mathcal{M}_s \cong \mathcal{O}(m_1H_1 + \cdots + m_nH_n) \otimes \beta
\]
for some integers \(m_i\), character \(\beta\) and invariant smooth divisors \(H_i\) on \(\mathbb{P}_i\) \((i\)-th level of \(\mathbb{P}\)). Therefore,
\[
\begin{align*}
[\mathbb{P}, \mathcal{M}_1, \ldots, \mathcal{M}_s] &= \pi_{\mathbb{P}}_*[\mathbb{P}, \mathcal{M}_1, \ldots, \mathcal{M}_s] \\
&= \pi_{\mathbb{P}}_* c(\mathcal{O}(m_1H_1 + \cdots + m_nH_n) \otimes \beta) [\mathbb{P}, \mathcal{M}_1, \ldots, \mathcal{M}_{s-1}] \\
&= e(\beta)[\mathbb{P}, \mathcal{M}_1, \ldots, \mathcal{M}_{s-1}] + \pi_{\mathbb{P}}_* \sigma c(\mathcal{O}(m_1H_1 + \cdots + m_nH_n)) [\mathbb{P}],
\end{align*}
\]
for some \(\sigma \in \text{End}(\Omega^G(\mathbb{P}))_{\text{inf}}\). By Proposition \[6.4\] and Lemma \[6.8\]
\[
\pi_{\mathbb{P}}_* \sigma c(\mathcal{O}(m_1H_1 + \cdots + m_nH_n)) [\mathbb{P}] \equiv 0
\]
mod elements with geometric dimension \(< d'\) (by abuse of notation). By repeating this process, we have \([\mathbb{P}, \mathcal{M}_1, \ldots, \mathcal{M}_s] \equiv a [\mathbb{P}]\) for some \(a \in \mathbb{L}_G(F)\). Hence, equation (3) becomes
\[
[\mathbb{P}_1 \to Z \leftarrow Y, p^*_\mathbb{L}] \equiv \sum_j a_j [\mathbb{P}_j][Y_j \to Y, \ldots].
\]
By the induction assumption and the fact that the product of two admissible towers over \(\text{Spec } k\) is again an admissible tower over \(\text{Spec } k\), we can replace \([Y_j \to Y, \ldots]\) by \([Y_j \to Y]\). The same equation holds for \(\mathbb{P}_2\). Therefore, we have
\[
[\text{Blow}_Z Y \to Y, \pi^*\mathcal{L}] - [\mathcal{L}_Y, \mathcal{L}] \equiv \sum_j a_j [\mathbb{P}_j][Y_j \to Y]
\]
where \(\text{dim } Y_j < d\) and \(\text{dim } \mathbb{P}_j + \text{dim } Y_j = d\).

By the same argument used in step 2 in the proof of Proposition \[6.7\] there is a map \(\pi : \tilde{Y} \to Y\) given by a series of blow ups along \(G\)-irreducible, invariant, smooth centers such that \(\pi^*\mathcal{L} \cong \mathcal{O}(\sum_i \pm D_i)\) for some smooth, \(G\)-prime divisors \(D_i\) on \(\tilde{Y}\). By Lemma \[6.8\] we have \([\mathcal{L}_Y, \pi^*\mathcal{L}] \equiv 0\). Therefore,
\[
[\mathcal{L}_Y, \mathcal{L}] = \sum_j a_j [\mathbb{P}_j][Y_j \to Y] + \sum_k a'_k [Y'_k \to Y, \ldots]
\]
for some \(a'_k\) and \(Y'_k\) as described in the statement. The result then follows by applying \(\sigma\) on both sides and the induction assumption. \(\square\)

Next, we would like to show that the canonical map \(\mathbb{L}_G(F) \to \Omega^G(\text{Spec } k)\) is surjective. To that end, we need a better understanding of the function field of a \(G\)-irreducible object \(Y \in G\text{-Sm}\) when \(G\) is a split torus.

**Lemma 6.11.** Suppose \(G\) is a split torus of rank \(r\), \(Y \in G\text{-Sm}\) is irreducible and the \(G\)-action on \(Y\) is faithful. Then
\[
k(Y) \cong k(Y)^G(t_1, \ldots, t_r)
\]
where $k(Y)$ is the function field of $Y$, $t_i$ are algebraically independent over $k(Y)^G$ and the $G$-actions on $t_i$ are given by 1-dimensional characters.

**Proof.** By Proposition 3.2, we can embed $Y$ into some $\mathbb{P}(V')$ with minimal $\dim \mathbb{P}(V')$. Without loss of generality, we may assume $Y$ is projective (by Proposition 3.3). Take an invariant affine part $V = \text{Spec } k[x_1, \ldots, x_m, y_1, \ldots, y_n] \subseteq \mathbb{P}(V')$ where $G$ acts on $x_i$ trivially and the $G$-actions on $y_i$ are given by non-trivial 1-dimensional characters. Consider the invariant open subset $U \subseteq V$ with coordinates $y_i$ being non-zero, i.e.,

$$U = \text{Spec } A \overset{\text{def}}{=} \text{Spec } k[x_1, \ldots, x_m, y_1, \ldots, y_n][y_1^{-1}, \ldots, y_n^{-1}].$$

Since $\dim \mathbb{P}(V')$ is minimal, $Y \cap U$ is non-empty. Also, since the action on $Y$ is faithful, so is the action on $U$. Moreover, by Nagata’s theorem, $U/G = \text{Spec } A^G$ exists as a variety over $k$ and so is $(Y \cap U)/G$.

Claim 1: There are monomials $f_i(y_1, \ldots, y_n) \in A$, which are algebraically independent over $k(U/G)$, such that

$$A = A^G[f_1, \ldots, f_r][f_1^{-1}, \ldots, f_r^{-1}].$$

Consider the case when $r = 1$ first. Fix an isomorphism $G \cong \mathbb{G}_m$. Then the actions on each $y_i$ have weight $a_i$. We claim that we can pick such $f_1$ with weight $\text{gcd}(a_1, \ldots, a_n)$.

If $n = 1$, we just take $f_1 \overset{\text{def}}{=} y_1$. Suppose $n > 1$. Apply the induction assumption on

$$B = k[x_1, \ldots, x_m, y_1, \ldots, y_{n-1}][y_1^{-1}, \ldots, y_{n-1}^{-1}],$$

we have $B = B^G[f_1][f_1^{-1}]$ and $f_1$ has weight $b \overset{\text{def}}{=} \text{gcd}(a_1, \ldots, a_{n-1})$. By the division algorithm, there are integers $q, r$ with $0 \leq r < a_n$ such that $b = qa_n + r$. If we let $g \overset{\text{def}}{=} f_1y_n^q$, then the pair $(g, y_n)$ (and their inverse) will generate $(f_1, y_n)$ and hence $A = A^G[g, y_n][g^{-1}, y_n^{-1}]$. Also, the weights for $(g, y_n)$ are $(r, a_n)$, instead of $(b, a_n)$ for $(f_1, y_n)$. By repeated applications of the division algorithm, we get a pair of monomials $(g, g')$ with weights $(c, 0)$ where $c = \text{gcd}(a_1, \ldots, a_n)$ and $A = A^G[g, g'][g^{-1}, g'^{-1}]$. Since $g'$ has weight zero, it is in $A^G$. So $g$ is the generator we want. Since the action on $U$ is faithful, the weight of $g$ can not be zero. Therefore, $g$ has to be algebraically independent over $k(U/G)$. That handles the $r = 1$ case. The general case then follows by considering the quotients

$$U \to U/G_m \to U/G_m^2 \to \cdots \to U/G_m^r = U/G$$

and repeated applications of the $r = 1$ case at each level. △

Let $I \subseteq A^G$ be the ideal defining the closed subvariety $(Y \cap U)/G \subseteq U/G$. Since $Y \cap U \cong (Y \cap U)/G \times_{U/G} U$, by claim 1,

$$Y \cap U \cong \text{Spec } (A^G/I)[f_1, \ldots, f_r][f_1^{-1}, \ldots, f_r^{-1}].$$

The result then follows by defining $t_i \overset{\text{def}}{=} f_i$. □

We are now ready to prove our first main result in this paper.
Theorem 6.12. Suppose $\text{char } k = 0$. Then the canonical $\mathbb{L}_G(F)$-algebra homomorphism
\[ \mathbb{L}_G(F) \to \Omega^G(\text{Spec } k), \]
which sends $a$ to $a[\text{Spec } k]$, is surjective.

Proof. For simplicity, we will say $x \equiv 0$ if $x$ is in the image of the canonical map. By Theorem 6.10 it is enough to consider elements of the form $[Y]$ where $Y$ is $G$-irreducible. We will proceed by induction on its geometric dimension. Suppose dim $Y \geq 1$. By the blow up relation, for any $G$-irreducible, invariant, smooth closed subscheme $Z \subseteq Y$,
\[ [\text{Blow}_Z Y] - [Y] = -[P_1 \to Z \to \text{Spec } k] + [P_2 \to Z \to \text{Spec } k] \]
for some quasi-admissible towers $P_i \to Z$. Suppose the statement is true for elements of the form $[P]$ where $P$ is an admissible tower over $\text{Spec } k$ of dimension $n$. By Proposition 6.17 and the induction assumption, we have $[\text{Blow}_Z Y] \equiv [Y]$. By the equivariant weak factorization theorem (Theorem 0.3.1) in [AKMW], whenever two projective schemes $Y, Y' \in G$-Sm are equivariantly birational, they define the same equivalence class, i.e., $[Y] \equiv [Y']$.

Step 1: Reduction to an admissible tower over $\text{Spec } k$.
Without loss of generality, $G$ acts on $Y$ faithfully. Then, there is a subgroup $H \subseteq G_f$ and an object $X \in (H \times G_f)$-Sm such that $Y \equiv (G_f/H) \times X$. Also, the $(H \times G_f)$-action on $X$ is faithful.

Since $H$ is abelian, we can write
\[ H \cong H_1 \times \cdots \times H_a \]
where $H_i$ is a cyclic group of order $M_i$. Consider the field extensions $k(X)$ over $k(X)^{G_i}$ and $k(X)^{G_t}$ over $k$. By Lemma 6.11
\[ k(X) \cong k(X)^{G_i}(t_1, \ldots, t_r) \]
where $r$ is the rank of $G_t$, $t_i$ are algebraically independent over $k(X)^{G_t}$ with actions given by non-trivial $G_t$-characters $\beta_i$. By claim 1 in the proof of Theorem 6.22 in [Li],
\[ k(X)^{G_i} \cong k(x_1, \ldots, x_{n-r}, x_{n-r+1}, \ldots, v_a) / (f, v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a) \]
for some $f, g_i \in k[x_1, \ldots, x_{n-r}]$ such that the $H$-action on $x_i$ is trivial and $H_j$ acts on $v_i$ non-trivially if and only if $i = j$. By the same arguments in the proof of Theorem 6.22 in [Li], we may assume
\[ X \cong \text{Proj } k[t_0, \ldots, t_r] \times \text{Proj } k[x_0, \ldots, x_{n-r+1}, v_1, \ldots, v_a] / (f, v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a) \]
where the $(H \times G_t)$-action on $t_0$ is trivial and $f, g_i$ are generic homogeneous polynomials with degree $d$ and $M_i$ respectively (as long as $X$ is smooth).

Let $W \overset{\text{def}}{=} (G_f/H)\times\text{Proj } k[t_0, \ldots, t_r] \times \text{Proj } k[x_0, \ldots, x_{n-r+1}, v_1, \ldots, v_a] / (v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a)$. By Lemma 6.21 in [Li], $W$ is smooth.
Claim 1: Without loss of generality, we may assume $D' \overset{\text{def}}{=} \{ x_{n-r+1} = 0 \}$ is an invariant smooth divisor on $W$.

Consider the map $W \to (G_f/H) \times \text{Proj} \ k[t_0, \ldots, t_r] \times \text{Proj} \ k[x_0, \ldots, x_{n-r+1}]$ given by projection. Then, $D'$ is just the pull-back of a generic hyperplane. △

Therefore, $D \overset{\text{def}}{=} \{ f = 0 \}$ and $D'$ are both invariant smooth divisors on $W$. Then we have

$$ [Y] = (G_f/H) \times [X] = \pi_{W,*} \circ c(\mathcal{O}(D)) [\mathbb{I}_W] = \pi_{W,*} \circ c(\mathcal{O}(dD')) [\mathbb{I}_W] = \pi_{W,*} \circ \sigma \circ c(\mathcal{O}(D')) [\mathbb{I}_W]$$

for some $\sigma \in \text{End} (\Omega_{G/W})_{\text{trivial}}$. Hence, $[Y] \equiv a [D']$ for some $a \in \mathbb{L}_G(F)$, by Theorem 6.10, the induction assumption and the assumption that the statement holds for admissible towers over Spec $k$. So, without loss of generality, we may assume

$$X \cong \text{Proj} \ k[t_0, \ldots, t_r] \times \text{Proj} \ k[x_0, \ldots, x_{n-r}, v_1, \ldots, v_a] / (v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a).$$

Let $X_k \cong \text{Proj} \ k[t_0, \ldots, t_r] \times \text{Proj} \ k[x_0, \ldots, x_{n-r}, v_1, \ldots, v_k] / (v_1^{M_1} - g_1, \ldots, v_k^{M_k} - g_k)$.

Claim 2: For generic $g_i$, the $(H \times G)$-schemes $X_k$, where $1 \leq k \leq a$, are all smooth. Moreover, the sum of the invariant divisors $\{ g_i = 0 \}$ on $\text{Proj} \ k[t_0, \ldots, t_r] \times \text{Proj} \ k[x_0, \ldots, x_{n-r}]$ is a reduced strict normal crossing divisor.

The first part follows from Lemma 6.21 in [Li] with the fact that the choice of $g_i$ there is actually generic. The second part is just Bertini’s Theorem. △

Now, let $W$ be the $G$-scheme

$$(G_f/H) \times \text{Proj} \ k[t_0, \ldots, t_r] \times \text{Proj} \ k[x_0, \ldots, x_{n-r}, v_1, \ldots, v_a] / (v_1^{M_1} - g_1, \ldots, v_a^{M_a} - g_a),$$

which may not be smooth. By claim 2, we can consider $Y$ as an invariant smooth divisor $\{ v_a^{M_a} = g_a \}$ on $W$.

Claim 3: The singular locus of $W$ is given by $\{ x_0 = \cdots = x_{n-r} = v_1 = \cdots = v_{a-1} = 0 \}$.

Without loss of generality, we may assume that $G_f = H$, $r = 0$ and $k$ is algebraically closed. Consider the projective space $\text{Proj} \ k[x_0, \ldots, x_n] = \mathbb{P}^n$ with trivial action. By claim 2, the invariant divisors $\{ g_i = 0 \}$ are smooth and the sum of them is a reduced strict normal crossing divisor. Suppose $p \overset{\text{def}}{=} (a_0; \cdots; a_n; b_1; \cdots; b_a)$ is a $k$-rational point in $\text{Sing}(W)$. By reordering, suppose the vectors $\nabla(v_1^{M_1} - g_1), \ldots, \nabla(v_k^{M_k} - g_k)$, where $1 \leq k \leq a - 1$, are linearly dependent at $p$ and $k$ is the smallest among such choices. Then the coordinates $b_1, \ldots, b_k$ are necessarily all zero, which implies that $g_1(p) = \cdots = g_k(p) = 0$. Now, assume $a_0, \ldots, a_n$ are not all zero. Then, $\overline{p} \overset{\text{def}}{=} (a_0; \cdots; a_n)$ will be a $k$-rational point in $\mathbb{P}^n$ which lies in the intersection of the divisors $\{ g_1 = 0 \}, \ldots, \{ g_k = 0 \}$. In addition, the vectors $\nabla g_1, \ldots, \nabla g_k$ will be linearly dependent at $\overline{p}$, which contradicts with the choice of
Apply resolution of singularities on $W$ to obtain $\tilde{W}$. By claim 3, $Y$ is disjoint from $\text{Sing}(W)$ and hence can be considered as an invariant smooth divisor on $\tilde{W}$. Now, consider the invariant divisor $D \overset{\text{def}}{=} \{ v_0 = 0 \}$ on $W$, which is smooth by claim 2. For the same reason as $Y$, it can also be considered as a divisor on $\tilde{W}$. Then, as before, we have

$$[Y] = \pi_{\tilde{W}} \circ c(\mathcal{O}(Y))|_{\tilde{W}} = \pi_{\tilde{W}} \circ c(\mathcal{O}(M_a D))|_{\tilde{W}} \equiv a[D]$$

for some $a \in \mathbb{L}_G(F)$. Since $[D] = [(G_f/H) \times X_{a-1}]$, by repeating the same argument, we may assume

$$Y \cong (G_f/H) \times X_0 = (G_f/H) \times \text{Proj} k[t_0, \ldots, t_r] \times \mathbb{P}^{n-r} = (G_f/H) \times \mathbb{P}(V) \times \mathbb{P}^{n-r}$$

for some $G_r$-representation $V$. The result then follows from our assumption that the statement is true for admissible towers. That finishes step 1.

Step 2 : Reduction to an element of the form $[\mathbb{P}(V)]$ where $V$ is a $(n+1)$-dimensional $G$-representation.

We will now consider the admissible towers over $\text{Spec} k$. Suppose

$$\mathbb{P} = \mathbb{P}_m \rightarrow \mathbb{P}_{m-1} \rightarrow \cdots \rightarrow \mathbb{P}_0 = \text{Spec} k$$

is an admissible tower over $\text{Spec} k$ with dimension $n$ and length $m$. Without loss of generality, we may assume $\dim \mathbb{P}_m = n > \dim \mathbb{P}_{m-1}$. We will proceed by induction on $m$ and the dimension of its $(m-1)$-th level. If $m = 1$, then $\mathbb{P} = \mathbb{P}(V)$ for some $G$-representation $V$ and we are done. If $\dim \mathbb{P}_{m-1} = 0$, then $\mathbb{P}_{m-1} \cong \text{Spec} k$ and we are also done.

Suppose $m \geq 2$ and $\dim \mathbb{P}_{m-1} \geq 1$. Following the notation in Lemma 6.9, $\mathbb{P}_i = \mathbb{P}(\mathcal{E}_i \oplus \mathcal{L}_i)$ and $D_i \overset{\text{def}}{=} \mathbb{P}(\mathcal{E}_i)$. Then, by Lemma 6.9, there exist integers $m_1, \ldots, m_{m-1}$ and character $\beta$ such that

$$\mathcal{L}_m \cong \mathcal{O}(m_1 D_1 + \cdots + m_{m-1} D_{m-1}) \otimes \beta.$$ 

Moreover, for $1 \leq i \leq m-1$, if we define $\hat{\mathbb{P}} \overset{\text{def}}{=} \mathbb{P}(\mathcal{E}_m \oplus \mathcal{L}_m(D_i))$ and $\mathbb{P}' \overset{\text{def}}{=} \mathbb{P}(\mathcal{E}_m \oplus \mathcal{L}_m(D_i))$, then, by Lemma 6.9 in [Li] (still holds for our group $G$), the sum of the invariant divisors $\hat{\mathbb{P}}$, $\mathbb{P}'$ and $\hat{\mathbb{P}}|_{D_i}$ on $\hat{\mathbb{P}}$ is a reduced strict normal crossing divisor and $\hat{\mathbb{P}}|_{D_i} + \mathbb{P} \sim \mathbb{P}'$. Hence, by applying the (EFGL) axiom on $\hat{\mathbb{P}}$ and pushing everything down to $\text{Spec} k$, we have

$$[\mathbb{P}'] = [\hat{\mathbb{P}}|_{D_i}] + [\mathbb{P}] + \pi_{\hat{\mathbb{P}}} \circ \sigma \circ c(\mathcal{O}(\hat{\mathbb{P}}|_{D_i})) \circ c(\mathcal{O}(\mathbb{P}))|_{\hat{\mathbb{P}}}$$

for some $\sigma \in \text{End}(\Omega^G(\hat{\mathbb{P}}))_{\text{inf}}$. Therefore,

$$[\mathbb{P}'] = [\hat{\mathbb{P}}|_{D_i}] + [\mathbb{P}] + \pi_{\hat{\mathbb{P}}} \circ \sigma |_{\mathbb{P}} \rightarrow \hat{\mathbb{P}} \equiv [\hat{\mathbb{P}}|_{D_i}] + [\mathbb{P}]$$

by Theorem 6.10 and the induction assumption. Moreover, notice that

$$\hat{\mathbb{P}}|_{D_i} = \mathbb{P}(\mathcal{E}_m \oplus \mathcal{L}_m \oplus \mathcal{L}_m(D_i)) \rightarrow \mathbb{P}_{m-1}|_{D_i}$$
is an admissible tower over \( \text{Spec} \, k \) with dimension \( n \) and length \( m \), but with \( \dim \mathbb{P}_{m-1}|_{D_i} = \dim \mathbb{P}_{m-1} - 1 \). Hence, by the induction assumption, \([\hat{P}|_{D_i}] \equiv 0\).

Hence, we can twist \( \mathbb{P}_m \) until there is a sheaf \( L \in \text{Pic}^G(\mathbb{P}_{m-1}) \) such that

\[
\mathbb{P}_m = \mathbb{P}(L \otimes \beta_1) + \cdots + (L \otimes \beta_p) \equiv \mathbb{P}((\mathcal{O}_{\mathbb{P}_{m-1}} \otimes \beta_1) + \cdots + (\mathcal{O}_{\mathbb{P}_{m-1}} \otimes \beta_p)).
\]

Then, \([P] = [\mathbb{P}(\beta_1 + \cdots + \beta_p)] [\mathbb{P}_{m-1}] \) with \( \dim \mathbb{P}_{m-1} < n \) and we are done.

Step 3: Reduction to \( n = 0 \) case.

It remains to consider elements of the form \([\mathbb{P}(V)]\) where \( V \) is an \((n + 1)\)-dimensional \( G \)-representation. The following proof is an analogue of the proof of Lemma 4.2.3 in \([LMG]\).

Let \( X \) be an \((n+1)\)-dimensional \( G \)-representation. The following proof is an analogue of the proof of Lemma 4.2.3 in \([LMG]\).

Let \( X \) be an \((n+1)\)-dimensional \( G \)-representation. The following proof is an analogue of the proof of Lemma 4.2.3 in \([LMG]\).

We then consider \( D \) as an invariant smooth divisor on \( X \). Let \( \hat{D} = \mathbb{P}(O \otimes O(D)) \to X \). As before, the sum of the invariant divisors \( A \equiv \hat{D}|_{D_i} \), \( B \equiv \mathbb{P}(O_X) \) and \( C \equiv \mathbb{P}(O_X(D)) \) on \( \hat{D} \) is a reduced strict normal crossing divisor and \( A + B \sim C \). Therefore, we have

\[
[\mathbb{P}(O_X(D))] = [\hat{D}|_{D}] + [\mathbb{P}(O_X)] + \pi_{\hat{D}} \circ \sigma \mathbb{P}(O_X)|_{D} \to \hat{D}
\]

for some \( \sigma \). Again, by Theorem \([6,10]\) and the induction assumption on dimension, we have

\[
[X] = [\mathbb{P}(O_X(D))] \equiv [\hat{D}|_{D}] + [\mathbb{P}(O_X)] = [\mathbb{P}(O_D \oplus O_D(D)) \to D \to \text{Spec} \, k] + [X].
\]

Hence,

\[
[\mathbb{P}(O \oplus O(1) \otimes \beta_n^\vee)] \to \mathbb{P}(e \oplus \beta_1 \oplus \cdots \oplus \beta_{n-1}) \to \text{Spec} \, k) \equiv 0
\]

because \( O_X(D) \equiv O(1) \otimes \beta_n^\vee \).

Claim 4: \([\mathbb{P}(O \oplus (\oplus_{i=0}^{p-1} O(1) \otimes \beta_{n-i}^\vee)) \to \mathbb{P}(e \oplus \beta_1 \oplus \cdots \oplus \beta_{n-p}) \to \text{Spec} \, k) \equiv 0 \) for all \( 1 \leq p \leq n \).

We just showed the statement for \( p = 1 \). For \( 1 \leq p \leq n-1 \), let \( X \equiv \mathbb{P}(e \oplus \beta_1 \oplus \cdots \oplus \beta_{n-p}) \) and \( D \equiv \mathbb{P}(e \oplus \beta_1 \oplus \cdots \oplus \beta_{n-p-1}) \). Apply a similar argument on

\[
\hat{D} \equiv \mathbb{P}(O \oplus (\oplus_{i=0}^{p-1} O(1) \otimes \beta_{n-i}^\vee)) \oplus O(D) \to X,
\]

we have

\[
[\mathbb{P}(\oplus_{i=0}^{p-1} O(1) \otimes \beta_{n-i}^\vee) \oplus O(D)) \to X \to \text{Spec} \, k]
\equiv [\hat{D}|_{D} \to D \to \text{Spec} \, k] + [\mathbb{P}(O \oplus (\oplus_{i=0}^{p-1} O(1) \otimes \beta_{n-i}^\vee)) \to X \to \text{Spec} \, k].
\]

On one hand,

\[
[\mathbb{P}((\oplus_{i=0}^{p-1} O(1) \otimes \beta_{n-i}^\vee) \oplus O(D)) \to X \to \text{Spec} \, k]
= [\mathbb{P}((\oplus_{i=0}^{p-1} \beta_{n-i}^\vee) \oplus \beta_{n-p}^\vee) \times X]
= [\mathbb{P}((\oplus_{i=0}^{p-1} \beta_{n-i}^\vee) \oplus \beta_{n-p}^\vee)[X] \equiv 0
\]
by the induction assumption on dimension. On the other hand, by the induction assumption on $p$,

$$\hat{P} \mid D \rightarrow D \rightarrow \text{Spec } k + [\mathcal{O} \oplus (\oplus_{i=0}^{p-1} \mathcal{O}(1) \otimes \beta_{n-i}^\vee)] \rightarrow X \rightarrow \text{Spec } k$$

$$\equiv [\hat{P} \mid D \rightarrow D \rightarrow \text{Spec } k]$$

$$= [\mathcal{O} \oplus (\oplus_{i=0}^{p} \mathcal{O}(1) \otimes \beta_{n-i}^\vee)] \rightarrow \mathbb{P}(\epsilon \oplus \beta_1 \oplus \cdots \oplus \beta_{n-p-1}) \rightarrow \text{Spec } k].$$

$\triangle$

By setting $p = n$ in claim 4, we obtain

$$0 \equiv [\mathcal{O} \oplus (\oplus_{i=0}^{n-1} \mathcal{O}(1) \otimes \beta_{n-i}^\vee)] \rightarrow \mathbb{P}(\epsilon) \rightarrow \text{Spec } k = [\mathbb{P}(\epsilon \oplus (\oplus_{i=0}^{n-1} \beta_{n-i}^\vee))].$$

The result then follows because $\beta_i$ are arbitrary.

Step 4 : $n = 0$ case.

Finally, it remains to show the statement when $\dim Y = 0$. Without loss of generality, $G$ acts on $Y$ faithfully. In that case, $G_t$ is necessarily trivial. By the same arguments as in step 1, we reduce it to the case when $[Y] = [G/H]$. Let $G_i$ be cyclic groups of order $M_i$ such that $G/H \cong G_1 \times \cdots \times G_a$. Then, $[G/H] = [G_1] \cdots [G_a]$ where $[G_i]$ are considered as elements in $G$-Sm equipped with the natural $G$-action. Let $g_i$ be a generator of $G_i$ and $\beta_i : G \rightarrow G/H \rightarrow k$ be the $G$-character such that $\beta_i(g_j) = 1$ for $i \neq j$. Let $W \overset{\text{def}}{=} \mathbb{P}(\epsilon \oplus \beta_i)$. Then, for some $\sigma \in \text{End}(\Omega^G(W)_{\text{inf}})$ and $\sigma' \in \text{End}(\Omega^G(\mathbb{P}(\beta_i))_{\text{inf}})$,

$$[G_i] = \pi_{W*} \circ c(\mathcal{O}(M_i))[\mathbb{I}_W] = \pi_{W*} \circ \sigma \circ c(\mathcal{O}(1))[\mathbb{I}_W] = \pi_{\mathbb{P}(\beta_i)*} \circ \sigma' [\mathbb{I}_{\mathbb{P}(\beta_i)}] \equiv 0.$$

That finishes the proof of Theorem 6.12.

$\square$

### 7. Fundamental properties

As in the algebraic cobordism theory $\Omega(\dashv)$ in [LM0], we also have the localization property and homotopy invariance property, when $\text{char } k = 0$ (see section 3.2 and 3.4 in [LM0]). Since we need to use the embedded desingularization theorem and the weak factorization theorem, we will assume $\text{char } k = 0$ for the rest of this section.

Let us start with the localization property. Since $\mathbb{L}Z^{G,F}(-)$ is generated by infinite cycles, which is relatively difficult to work with, our strategy is to first define another equivariant algebraic cobordism theory $\Omega^G(-)_{\text{fin}}$ (which does not involve infinite sum), prove that $\Omega^G(-)_{\text{fin}}$ and $\Omega^G(-)$ are canonically isomorphic and then show that the localization property holds in $\Omega^G(-)_{\text{fin}}$.

For an object $X \in G$-Var, let

$$\mathbb{L}Z^{G,F}_{\text{fin}}(X) \overset{\text{def}}{=} \oplus_{s \geq 0} \mathbb{L}Z^{G,F}_s(X).$$
We then define the basic operations: projective push-forward, smooth pull-back, external product as in $\mathbb{L}Z^{G,F}(-)$, and the Chern class operator

$$c(\mathcal{L})[f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r] \overset{\text{def}}{=} [f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r, f^*\mathcal{L}],$$

which is also consistent with our definition of infinite Chern class operator in $\mathbb{L}Z^{G,F}(-)$. For simplicity of notation, let $\text{End}(\mathbb{L}Z^{G,F}(X)_{\text{fin}})$ be the $\mathbb{L}G(F)$-subalgebra of $\text{End}(\mathbb{L}Z^{G,F}(X)_{\text{fin}})$ generated by Chern class operators.

Then we define $\mathbb{L}Z^{G,F}(-)_B$ as the quotient of $\mathbb{L}Z^{G,F}(-)_{\text{fin}}$ by imposing the following axiom:

**Blow** For all $G$-irreducible $Y \in G\text{-}Sm$ and invariant, smooth closed subscheme $Z \subseteq Y$,

$$[\text{Blow}_YZ \to Y] - [\mathbb{I}_Y] = -[\mathbb{P}(\mathcal{O} \oplus \mathcal{N}_{Z \to Y}) \to Z \leftarrow Y] + \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{N}_{Z \to Y}) \to Z \leftarrow Y].$$

Next, we define $\mathbb{L}Z^{G,F}(-)_N$ as the quotient of $\mathbb{L}Z^{G,F}(-)_B$ by imposing the following axiom:

**Nilp** For all $G$-irreducible $Y \in G\text{-}Sm$, $\mathcal{L} \in \text{Pic}^G(Y)$ and set $S$ as in (11),

$$V_S^{\text{Nilp}(Y, \mathcal{L}, S)}(\mathcal{L})[\mathbb{I}_Y] = 0$$

where $\text{Nilp}(Y, \mathcal{L}, S)$ is the positive integer given in Remark 6.2.

Finally, we define $\Omega^G(-)_{\text{fin}}$ as the quotient of $\mathbb{L}Z^{G,F}(-)_N$ by imposing the (Sect) and (EFGL) axioms as in the definition of $\Omega^G(-)$. Notice that the equations in the (EFGL) axiom are all finite sums because of the (Nilp) axiom.

**Lemma 7.1.** Suppose $\text{char } k = 0$. The two theories $\Omega^G(-)$ and $\Omega^G(-)_{\text{fin}}$ are canonically isomorphic.

**Proof.** Let $\Phi : \mathbb{L}Z^{G,F}(X)_{\text{fin}} \to \Omega^G(X)$ be the natural map that sends cycles to cycles. It clearly commutes with the four basic operations. This map descends to a map $\Phi : \Omega^G(-)_{\text{fin}} \to \Omega^G(X)$ because the blow up relation holds in $\Omega^G(-)$ and by Remark 6.2.

The inverse map $\Psi : \mathbb{L}Z^{G,F}(X) \to \Omega^G(X)_{\text{fin}}$ is also natural (infinite cycle becomes a finite sum of cycles because of the (Nilp) axiom), which clearly descends to a map $\Psi : \Omega^G(X) \to \Omega^G(X)_{\text{fin}}$. Moreover, $\Psi \circ \Phi$ is the identity and $\Phi$ is surjective by Corollary 6.3.

Now we can proceed to prove the localization property in $\Omega^G(-)_{\text{fin}}$. We first need the following Lemma.

**Lemma 7.2.** Suppose $X \in G\text{-}Var$ is projective and $Z \subseteq G\text{-}Var$ is an invariant closed subscheme of $X$. Let $U$ be the complement $X - Z$. Furthermore, $\mathcal{L}$ is a sheaf in $\text{Pic}^G(X)$ such that $\mathcal{L}|_U \cong \mathcal{O}_U$. Then, for all $x \in \Omega^G(X)_{\text{fin}}$, the element $c(\mathcal{L})(x)$ lies inside $i_*\Omega^G(Z)_{\text{fin}}$ where $i : Z \hookrightarrow X$ is the immersion.
Proof. Without loss of generality, \( x = [f : Y \to X, \ldots] \) where \( Y \in G\text{-}Sm \) is \( G \)-irreducible. Consider the following Cartesian diagram in the category \( G\text{-}Var \):

\[
\begin{array}{ccc}
(Y|_Z)_{\text{red}} & \xrightarrow{i'} & Y \\
\downarrow f' & & \downarrow f \\
Z & \xrightarrow{i} & X
\end{array}
\]

("red" stands for the reduced structure) Then, \( f' \) is projective, \( i' \) is a closed immersion and \( Y|_U = Y - (Y|_Z)_{\text{red}} \) with \( (f^*\mathcal{L})|_{Y|_U} \cong \mathcal{O}_{Y|_U} \). Moreover, \( c(\mathcal{L})(x) = f_* c(f^*\mathcal{L})[\mathbb{I}_Y, \ldots] \).

Therefore, it is enough to show that \( c(f^*\mathcal{L})[\mathbb{I}_Y, \ldots] \) lies inside \( i'_* \Omega^G((Y|_Z)_{\text{red}})_{\text{fin}} \). In other words, we may assume \( X \) is smooth and \( G \)-irreducible, and \( x = [\mathbb{I}_X, \ldots] \).

Let \( L \overset{\text{def}}{=} \text{Spec} \text{Sym} \mathcal{L}^\vee \) be the corresponding equivariant line bundle of \( \mathcal{L} \) over \( X \) and \( \pi : L \to X \) be the projection. Then, \( \mathcal{L}|_U \cong \mathcal{O}_U \) means that there is an equivariant, non-vanishing section \( s : U \to L|_U \). Denote the closure of \( s(U) \) inside \( L \) by \( \overline{s(U)} \), which is projective. Then \( s \) defines an equivariant rational map \( s : X \dashrightarrow \overline{s(U)} \). By Proposition 3.5, there is an equivariant, projective, birational map \( \phi_1 : X_1 \to X \) that extends \( s \) and induces a map \( s_1 : X_1 \to \overline{s(U)} \to L \). Moreover, \( \pi \circ s_1 = \phi_1 \) because it is a closed condition and it holds over \( X_1|_U \). Therefore, \( s_1 \) defines an invariant global section of \( \mathcal{L}_1 \overset{\text{def}}{=} \phi_1^* \mathcal{L} \). Moreover, if we let \( Z_1 \) be the closed subscheme of \( X_1 \) cut out by \( s_1 \), then \( Z_1 \) is disjoint from \( X_1|_U \cong U \).

Now, let \( \phi_2 : X_2 \to X_1 \) be the blow up of \( X_1 \) along \( Z_1 \) and let \( Z_2 \) be the total transform of \( Z_1 \). Then, \( s_2 = \phi_2^* (s_1) \) is an invariant global section of \( \mathcal{L}_2 \overset{\text{def}}{=} \phi_2^* \mathcal{L}_1 \) that cuts out an invariant effective divisor \( Z_2 \) on \( X_2 \), namely, \( \mathcal{L}_2 \cong \mathcal{O}_{X_2}(Z_2) \). By the resolution of singularities and embedded desingularization theorem, there is an equivariant birational map \( \phi_3 : X_3 \to X_2 \) such that \( X_3 \) is smooth and the total transform of \( Z_2 \) is given by smooth, \( G \)-prime divisors (with multiplicities). Hence, we have constructed a projective, equivariantly birational map

\[
\phi \overset{\text{def}}{=} \phi_1 \circ \phi_2 \circ \phi_3 : X_3 \to X
\]

in \( G\text{-}Sm \) such that \( X_3 \) is \( G \)-irreducible, the immersion \( U \hookrightarrow X \) lifts to an immersion \( U \hookrightarrow X_3 \) and \( \phi^* \mathcal{L} \cong \mathcal{O}_{X_3}(\sum_i \pm D_i) \) where \( D_i \) are smooth, \( G \)-prime divisors on \( X_3 \) which are disjoint from \( U \).

By the weak factorization theorem and the (Blow) axiom, the difference

\[
\phi_* c(\phi^* \mathcal{L})[\mathbb{I}_{X_3}, \ldots] - c(\mathcal{L})[\mathbb{I}_X, \ldots]
\]

lies inside \( i_* \Omega^G(Z)_{\text{fin}} \). So, without loss of generality, we may assume \( \phi = \mathbb{I}_X \). Hence,

\[
c(\mathcal{L})[\mathbb{I}_X, \ldots] = c(\mathcal{O}(\sum_i \pm D_i))[\mathbb{I}_X, \ldots] = \sum_i \sigma_i \circ c(\mathcal{O}(D_i))[\mathbb{I}_X, \ldots],
\]

for some \( \sigma_i \in \text{End} (\Omega^G(X)_{\text{fin}})_{\text{inf}} \) (the infinite Chern class operator acts as a finite sum in \( \Omega^G(-)_{\text{fin}} \)), which clearly lies inside \( i_* \Omega^G(Z)_{\text{fin}} \). \( \square \)

We are now ready to prove the localization property.
Theorem 7.3. Suppose \( \text{char} \, k = 0 \). Let \( X \) be an object in \( G\text{-Var} \), \( Z \in G\text{-Var} \) be an invariant closed subscheme of \( X \) and \( U \) be the complement \( X - Z \). Then the following sequence is exact

\[
\Omega^G(Z) \xrightarrow{i_*} \Omega^G(X) \xrightarrow{j^*} \Omega^G(U) \rightarrow 0
\]

where \( i : Z \hookrightarrow X \) and \( j : U \hookrightarrow X \) are the immersions.

Proof. By Lemma 6.1, we can consider \( \Omega^G(-)_{\text{fin}} \) instead. Our proof is similar to the proof of Theorem 3.2.7 in [LMO]. First of all, the composition \( j^* \circ i_* \) is clearly zero. Moreover, for any element \([f : Y \rightarrow U, \mathcal{L}_1, \ldots, \mathcal{L}_r]\) in \( \Omega^G(U)_{\text{fin}} \), by Proposition 3.6, there exists an equivariant projective map \( j : Y \rightarrow X \) such that \( Y \in G\text{-Sm} \) and \( j|_U = f \). Also, the restriction map \( \text{Pic}^G(Y) \rightarrow \text{Pic}^G(X) \) is surjective (by Proposition 3.6). So, the map \( j^* \) is surjective. Hence, it is enough to show kernel \( j^* \subseteq \text{image} \, i_* \). Without loss of generality, we may assume \( X, U \) and \( Z \) are all non-empty. By Proposition 3.2, we can embedded \( X \) into some \( \mathbb{P}(V) \). It will be enough to show the exactness of the sequence

\[
\Omega^G(X - U) \rightarrow \Omega^G(X) \rightarrow \Omega^G(U) \rightarrow 0
\]

where \( X \) is the closure of \( X \) inside \( \mathbb{P}(V) \). So, we may further assume \( X \) to be projective.

Consider the following commutative diagram:

\[
\begin{array}{ccccccccc}
\mathbb{L}Z^G,F(Z)_{\text{fin}} & \xrightarrow{i_*} & \mathbb{L}Z^G,F(X)_{\text{fin}} & \xrightarrow{j^*} & \mathbb{L}Z^G,F(U)_{\text{fin}} & \rightarrow & 0 \\
\phi_{Z,1} & & \phi_{X,1} & & \phi_{U,1} & & \\
\mathbb{L}Z^G,F(Z)_B & \xrightarrow{i_*} & \mathbb{L}Z^G,F(X)_B & \xrightarrow{j^*} & \mathbb{L}Z^G,F(U)_B & \rightarrow & 0 \\
\phi_{Z,2} & & \phi_{X,2} & & \phi_{U,2} & & \\
\mathbb{L}Z^G,F(Z)_N & \xrightarrow{i_*} & \mathbb{L}Z^G,F(X)_N & \xrightarrow{j^*} & \mathbb{L}Z^G,F(U)_N & \rightarrow & 0 \\
\phi_{Z,3} & & \phi_{X,3} & & \phi_{U,3} & & \\
\Omega^G(Z)_{\text{fin}} & \xrightarrow{i_*} & \Omega^G(X)_{\text{fin}} & \xrightarrow{j^*} & \Omega^G(U)_{\text{fin}} & \rightarrow & 0 \\
\end{array}
\]

where \( \phi_{-1}, \phi_{-2}, \phi_{-3} \) are the quotient maps.

Claim 1:

\( j^* \) kernel \( \phi_{X,3} \supseteq \text{kernel} \, \phi_{U,3} \).

The kernel of \( \phi_{U,3} \) is generated by elements corresponding to the (Sect) and (EFGL) axioms. For the (EFGL) axiom, the elements are of the form

\[
(4) \quad f_* \circ \sigma \circ g^*(V^i(\mathcal{L})V^j(\mathcal{L})[\mathbb{I}_T]) \rightarrow \sum_{s \geq 0} b_{i,j}^{s} V^s(\mathcal{L})[\mathbb{I}_T]
\]

\[
(4) \quad f_* \circ \sigma \circ g^*(V^i(\mathcal{L} \otimes \mathcal{A})[\mathbb{I}_T]) \rightarrow \sum_{s \geq 0} d_{i,j}^{s} V^s(\mathcal{L})[\mathbb{I}_T]
\]

\[
(4) \quad f_* \circ \sigma \circ g^*(V^i(\mathcal{L} \otimes \mathcal{M})[\mathbb{I}_T]) \rightarrow \sum_{s,t \geq 0} f_{s,t}^{j} V^s(\mathcal{L})V^t(\mathcal{M})[\mathbb{I}_T]
\]
where \( f : Y \to U \) is projective, \( T \in G\text{-Sm} \) is \( G \)-irreducible, \( g : Y \to T \) is smooth, \( \sigma \in \text{End}(\mathbb{L}Z^{G,F}(Y))_{\text{fin}} \), \( i, j \geq 0 \), \( \alpha \) is a character and \( \mathcal{L}, \mathcal{M} \) are sheaves in \( \text{Pic}^G(T) \) (see subsection 2.1.3 in [LMo] for details). For simplicity, we will only handle (4). The other two will follow from similar arguments.

Since \( g \) is smooth, \( Y \) is in \( G\text{-Sm} \). Without loss of generality, we may assume \( Y \) is also \( G \)-irreducible. Extend \( f: Y \to U \) to \( f: Y \to X \) as before. Consider the following commutative diagram:

\[
\begin{array}{cccccc}
\mathbb{L}Z^{G,F}(\{(Y|Z)\}_{\text{red}})_N & \xrightarrow{i_*} & \mathbb{L}Z^{G,F}(Y)_N & \xrightarrow{j^*} & \mathbb{L}Z^{G,F}(Y)_N & \longrightarrow 0 \\
\mathbb{L}Z^{G,F}(Z)_N & \xrightarrow{i_*} & \mathbb{L}Z^{G,F}(Y)_N & \xrightarrow{j^*} & \mathbb{L}Z^{G,F}(U)_N & \longrightarrow 0
\end{array}
\]

(By abuse of notation, we will denote the immersions \( Y \hookrightarrow \overline{Y} \) and \( (Y|Z)_{\text{red}} \hookrightarrow \overline{Y} \) by \( j \) and \( i \) respectively as well). By some diagram chasing, it is enough to lift the element

\[
\sigma \circ g^* (V^i(\mathcal{L})V^j(\mathcal{L})[I_T] - \sum_{s \geq 0} b_{i,j}^s V^s(\mathcal{L})[I_T])
\]

to an element in \( \mathbb{L}Z^{G,F}(Y)_N \) which is congruent to zero mod image \( i_* + \text{kernel } \phi_{\overline{Y},3} \). By extending \( \sigma \) to \( \overline{\sigma} \) as before, we may further assume \( \sigma = I \).

Since \( g^* \circ V^i(\mathcal{L}) = V^i(g^*\mathcal{L}) \circ g^* \) and \( g^*[I_T] = [I_T] \), we may assume \( g = I_Y \). By extending \( \mathcal{L} \in \text{Pic}^G(Y) \) to \( \overline{\mathcal{L}} \in \text{Pic}^G(\overline{Y}) \), the lifting we want is

\[
V^i(\overline{\mathcal{L}})V^j(\overline{\mathcal{L}})[I_T] - \sum_{s \geq 0} b_{i,j}^s V^s(\overline{\mathcal{L}})[I_T].
\]

For the (Sect) axiom, by similar arguments, it is enough to lift an element of the form \( [I_Y, \mathcal{L}] - [D \hookrightarrow Y] \), where \( \mathcal{L} \) is a sheaf in \( \text{Pic}^G(Y) \) and \( D \) is an invariant smooth divisor on \( Y \) cut out by some invariant section \( s \in H^0(Y, \mathcal{L})^G \), to an element in \( \mathbb{L}Z^{G,F}(\overline{Y})_N \) which is congruent to zero.

Let \( \overline{D} \) be the closure of \( D \) in \( \overline{Y} \). Apply the embedded desingularization Theorem on \( \overline{D} \hookrightarrow \overline{Y} \) to obtain the following commutative diagram:

\[
\begin{array}{ccc}
\langle D \rangle & \longrightarrow & \hat{Y} \\
\pi \downarrow & & \downarrow \pi \\
\overline{D} & \longrightarrow & \overline{Y}
\end{array}
\]

Let \( E_k \) be the strict transforms of the exceptional divisors.
On one hand, we have
\[ \pi_* \circ c(\pi^*\mathcal{O}_{\mathcal{Y}}(\mathcal{D})) \equiv \pi_* \circ c(\mathcal{O}_{\mathcal{Y}}(\langle \mathcal{D} \rangle + \sum_k m_k E_k)) \equiv \pi_* \circ c(\mathcal{O}(\langle \mathcal{D} \rangle)) \]
for some integers \( m_k \)
\[ \equiv \pi_* \circ c(\mathcal{O}(\langle \mathcal{D} \rangle)) + \sum_k \pi_* \circ \sigma_k \circ c(\mathcal{O}(E_k)) \equiv [\langle \mathcal{D} \rangle \to \mathcal{Y}] + \sum_k \pi_* \circ \sigma_k [E_k \to \mathcal{Y}], \]
which is congruent to \([\langle \mathcal{D} \rangle \to \mathcal{Y}]\) because all \( E_k \) lie over \((\mathcal{Y}|_Z)_{\text{red}}\).

On the other hand, by the (Blow) axiom, we have
\[ \tilde{\mathcal{Y}} \to \mathcal{Y}, \pi^*\mathcal{O}_{\mathcal{Y}}(\mathcal{D}) \equiv [\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(\mathcal{D})] \]
because the towers created by each blow up lie over the smooth center, which lies over \((\mathcal{Y}|_Z)_{\text{red}}\). Hence, we have
\[ [\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}(\mathcal{D})] - [\langle \mathcal{D} \rangle \to \mathcal{Y}] \equiv 0, \]
which is the lifting we want. △

Claim 2 : For all \( G \)-irreducible \( Y \in G \cdot Sm, \mathcal{L} \in \text{Pic}^G(Y) \) and \( S \), there exist projective maps \( f_i, \) smooth maps \( g_i, \sigma_i \) and \( \mathcal{L}_i \in \text{Pic}^G(Z_i) \) such that
\[ V_{S}^{\text{Nilp}(Y, \mathcal{L}, S)}(\mathcal{L})[\mathcal{Y}] = \pi_* \circ V_{S}^{\text{Nilp}(Y, \mathcal{L}, S)}(\pi^* \mathcal{L})[\mathcal{Y}] + \sum_i f_{i*} \circ \sigma_i \circ g_i^* \circ V_{S}^{\text{Nilp}(Z_i, \mathcal{L}_i, S)}(\mathcal{L}_i)[\mathcal{Z}_i] \]
as elements in \( \mathbb{L}_{Z}^{G,F}(Y)_B \), where \( \pi : \tilde{\mathcal{Y}} \to \mathcal{Y} \) and \( Z_i \) are given as part of the definition of \( \text{Nilp}(Y, \mathcal{L}, S) \) (see Remark 6.2) and \( \dim Z_i < \dim Y \).

See the proof of Proposition 6.1 △

Claim 3 :
\[ j^* \text{ kernel } \{ \phi_{X,3} \circ \phi_{X,2} \} \supseteq \text{ kernel } \phi_{U,2}. \]

We need to lift an element of the form
\[ f_* \circ \sigma \circ g^* V_{S}^{\mathcal{L}}(\mathcal{T})[\mathbb{T}], \]
where \( f : Y \to U, \sigma, g : Y \to T \) are as in claim 1, \( Y \) and \( T \) are both \( G \)-irreducible and smooth, \( \mathcal{L} \) is a sheaf in \( \text{Pic}^G(T) \), \( S \) is a finite set as in (1) and \( r = \text{Nilp}(T, \mathcal{L}, S) \), to an element in \( \mathbb{L}_{Z}^{G,F}(X)_B \) which is congruent to zero mod image \( i_* + \text{ kernel } \{ \phi_{X,3} \circ \phi_{X,2} \} \). We will prove this by induction on \( \dim T \). By claim 2 and the definition of \( \text{Nilp}(-) \), we may assume \( \mathcal{L} \cong O(\sum_i m_i \mathcal{D}_i) \otimes \beta \) such that \( \mathcal{D}_i \) are distinct invariant smooth divisors, \( \alpha_i \cong \beta^V \) and \( r > \text{Nilp}(\mathcal{D}_i, \mathcal{L}|_{\mathcal{D}_i}, S) \) for all \( i \).

By Proposition 3.3 we can embedded \( T \) into some \( \mathbb{P}(V) \) such that its closure \( \overline{T} \) is smooth. By the embedded desingularization theorem, we may also assume \( \overline{\mathcal{D}_i} \) are smooth. Let \( \mathcal{L} \equiv O(\sum_i m_i \mathcal{D}_i) \otimes \beta \in \text{Pic}^G(\mathcal{T}) \) be an extension of \( \mathcal{L} \). By the same argument as in the
proof of claim 1, we can extend \( f : Y \to U \) to \( \tilde{f} : \tilde{Y} \to X \) and it is enough to lift \( g^* V_S^* (L)[I_T] \) to an element in \( L \mathcal{Z}^{G,F} (\tilde{Y}) \) which is congruent to zero.

Since \( g \circ j^{-1} : Y \to \tilde{T} \) is an equivariant rational map and \( \tilde{T} \) is projective, by Proposition 3.5, we can blow up an invariant center to obtain \( Y' \) so that \( g \circ j^{-1} \) lifts to an equivariant, regular map. By resolution of singularities, we may assume \( Y' \) is smooth. Also, \( j \) lifts to an immersion \( Y \hookrightarrow Y' \). Therefore, we may assume \( Y' = \tilde{Y} \) and \( g : Y \to \tilde{T} \) has an extension \( \tilde{g} : Y \to \tilde{T} \).

If \( \dim T = 0 \), then \( L \cong \beta \) and the element \( V^*_S (\beta)[I_T] \) will be a lifting of \( g^* V_S^* (L)[I_T] \).

Also, \[
V^*_S (\beta)[I_T] = V^*_S (\beta)[I_T] \equiv 0.
\]

Suppose \( \dim T > 0 \). First of all, we lift \( g \circ V^*_S (L)[I_T] \) to \( V^*_S (\tilde{g}^* \tilde{L})[I_T] \). Since \( g \) is smooth, \( g^{-1} D_i \) are also distinct invariant smooth divisors on \( Y \). Let \( W \subseteq Y \) be the reduced scheme associated to the closed subscheme \( \cup_{i \neq j} (g^{-1} D_i \cap g^{-1} D_j) \). Then \( \cup_i g^{-1} D_i \) becomes an invariant smooth divisor when restricted on \( Y - W \). Let \( C_j \) be the invariant smooth centers to blow up if we apply the embedded desingularization Theorem on \((\cup_i g^{-1} D_i) \to (\tilde{Y} - W) \). Then the centers \( C_j \) all lie over \((\tilde{Y} | Z)_\text{red} \). Apply the same blow ups on \( Y \) to obtain \( \tilde{Y} \) and denote the composition \( Y \to \tilde{Y} \to \tilde{T} \) by \( \tilde{g} \). Then it is clear that the immersion \( Y \hookrightarrow \tilde{Y} \) lifts to an immersion \( Y \hookrightarrow \tilde{Y} \), the map \( \tilde{g} \) extends \( g \) and, for all \( i \),

\[
\tilde{g}^* O_{\tilde{Y}} (D_i) = O_Y (g^{-1} D_i + \sum_k m_{i,k} D_{i,k})
\]

for some integers \( m_{i,k} \) and invariant smooth divisors \( D_{i,k} \) (exceptional divisors) on \( \tilde{Y} \) which lie over \((\tilde{Y} | Z)_\text{red} \). Moreover, \( g^{-1} D_i \), the closure of \( g^{-1} D_i \) in \( \tilde{Y} \), is also smooth. Again, without loss of generality, we may assume \( \tilde{Y} = \tilde{Y} \) and \( \tilde{g} = \tilde{g} \).

Let \( r' \) be the max of \( \text{Nilp}(D_i, L|_{D_i}, S) \). Then, for some \( \sigma_1 \), we have

\[
V^*_S (\tilde{g}^* \tilde{L})[I_T] = V^*_S (L)[I_T] \equiv 0.
\]

Moreover, \( g^{-1} D_i \), the closure of \( g^{-1} D_i \) in \( \tilde{Y} \), is also smooth. Again, without loss of generality, we may assume \( \tilde{Y} = \tilde{Y} \) and \( \tilde{g} = \tilde{g} \).

Let \( V^*_S (\tilde{g}^* \tilde{L})[I_T] = \sum_i \sigma_{2,i} \circ V^*_S (\tilde{g}^* \tilde{L}) [A_i \hookrightarrow \tilde{Y}] \)

for some \( \sigma_{2,i} \) and \( \sigma_{3,i,k} \). Denote \( \tilde{g}^{-1} D_i \) by \( A_i \) for simplicity. Then,

\[
V^*_S (\tilde{g}^* \tilde{L})[I_T] = \sum_i \sigma_{2,i} \circ V^*_S (\tilde{g}^* \tilde{L}) [A_i \hookrightarrow \tilde{Y}]
\]

for some \( \sigma_{4,i} \), where \( h_i \) are the immersions \( A_i \hookrightarrow \tilde{Y} \). Notice that \( h_i \circ \sigma_{4,i} \circ V^*_S (\tilde{g}^* \tilde{L}|_{A_i}) [I_{A_i}] \) is a lifting of

\[
(h_i |_{\tilde{Y}}) \circ \sigma_{5,i} \circ g^* V^*_S (L|_{D_i}) [I_{D_i}]
\]
where $\sigma_{r,i}$ is the restriction of $\sigma_{4,i}$ over $g^{-1}D_i$. Since $r' \geq \text{Nilp}(D_i, L|_{D_i}, S)$, by the induction assumption ($\dim D_i = \dim T - 1$), it has a lifting which is congruent to zero.

To summarize, we lift $a \overset{\text{def}}{=} g \ast V^S_\delta(L)[[T]]$ to $b \overset{\text{def}}{=} V^S_\delta(\mathfrak{L})[[T]]$, which is congruent to the sum of $c_i \overset{\text{def}}{=} h_{x,i} \circ \sigma_{4,i} \circ V^{r'}_{S}(\mathfrak{L})[[T]]$. Then, by the induction assumption, $j^*(c_i)$ has a lifting $d_i$ which is congruent to zero. Hence, $b - \sum c_i + \sum d_i$ is the lifting we want. $\triangle$

Claim 4:

\[ j^* \text{ kernel } \{\phi_{X,3} \circ \phi_{X,2} \circ \phi_{X,1}\} \supseteq \text{ kernel } \phi_{U,1}. \]

It is enough to lift an element of the form

\[ g^*([\text{Blow}_{W}T \rightarrow T] - [T] + [Q_1 \rightarrow W \leftarrow T] - [Q_2 \rightarrow W \leftarrow T]), \]

where $Y$, $T$, and $g : Y \rightarrow T$ are as in claim 1, $W$ is an invariant, smooth closed subscheme of $T$, $Q_1 = \mathbb{P}(O \oplus N_{W \leftarrow T}) \rightarrow W$ and $Q_2 = \mathbb{P}(O \oplus O(1)) \rightarrow \mathbb{P}(N_{W \leftarrow T}) \rightarrow W$, to an element in $\mathbb{L}z^{G,F}(Y)_\text{fin}$ which is congruent to zero mod image $i^* + \text{ kernel } \{\phi_{Y,3} \circ \phi_{Y,2} \circ \phi_{Y,1}\}$.

Since $g$ is smooth, we may assume $g = I_Y$. In that case, denote the closure of $W$ in $Y$ by $W$. By applying the embedded desingularization theorem on $W \leftarrow Y$, we may assume $W$ is smooth. Then we can take

\[ [\text{Blow}_{W}Y \rightarrow Y] - [Y] + [Q'_1 \rightarrow W \leftarrow Y] - [Q'_2 \rightarrow W \leftarrow Y], \]

where $Q'_1 = \mathbb{P}(O \oplus N^{\vee}_{W \leftarrow Y}) \rightarrow W$ and $Q'_2 = \mathbb{P}(O \oplus O(1)) \rightarrow \mathbb{P}(N^{\vee}_{W \leftarrow Y}) \rightarrow W$, as our lifting. $\triangle$

By claims 1, 3, 4 and some diagram chasing, if we can show that

\[ \text{ kernel } j^* \subseteq \text{ image } i^* + \text{ kernel } \{\phi_{X,3} \circ \phi_{X,2} \circ \phi_{X,1}\} \]

in the sequence

\[ \mathbb{L}z^{G,F}(Z)_\text{fin} \overset{i^*}{\longrightarrow} \mathbb{L}z^{G,F}(X)_\text{fin} \overset{j^*}{\longrightarrow} \mathbb{L}z^{G,F}(U)_\text{fin} \longrightarrow 0, \]

then we will have kernel $j^* \subseteq \text{ image } i^*$ in the sequence

\[ \Omega^{G}(Z)_\text{fin} \overset{i^*}{\longrightarrow} \Omega^{G}(X)_\text{fin} \overset{j^*}{\longrightarrow} \Omega^{G}(U)_\text{fin} \longrightarrow 0 \]

and we are done.

Claim 5 : It is enough to consider elements of the form $x - x'$ where $x, x'$ are cycles and $j^*(x) = j^*(x')$.

Notice that $\mathbb{L}z^{G,F}(-)_\text{fin}$ is a free $\mathbb{L}_G(F)$-module with a basis given by isomorphism classes of cycles. If an element in $\mathbb{L}z^{G,F}(X)_\text{fin}$ lies inside kernel $j^*$, we may assume it is of the form $\sum_{i=1}^{n} a_n x_n$, where $a_n$ are elements in $\mathbb{L}_G(F)$, $x_i$ are cycles, $j^*(x_i)$ belongs to the same isomorphism class for all $i$ and $\sum_{i=1}^{n} a_n = 0$. We can then rearrange the terms in the following way:

\[ \sum_{i=1}^{n} a_n x_n = a_1(x_1 - x_n) + \cdots + a_{n-1}(x_{n-1} - x_n) + (a_1 + \cdots + a_{n-1} - a_n)x_n. \]
That proves the claim. △

By claim 5, it is enough to consider elements of the form

$$[f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r] - [f' : Y' \to X, \mathcal{L}'_1, \ldots, \mathcal{L}'_r]$$

where $Y, Y' \in G\text{-}Sm$ are both $G$-irreducible and there is an equivariant isomorphism $\psi : Y|_U \to Y'|_U$ such that $f' \circ \psi = f$ and $\psi^*(\mathcal{L}'_i|_{Y'|_U}) \cong \mathcal{L}_i|_{Y|_U}$ for all $i$. Let $Y'' \subseteq Y \times_X Y'$ be the closure of the graph of $\psi$. By resolution of singularities, we may assume $Y''$ to be smooth and we have a commutative diagram

$$\begin{array}{ccc}
Y'' & \xrightarrow{\mu} & Y \\
\downarrow \mu' & & \downarrow f \\
Y' & \xrightarrow{f'} & X
\end{array}$$

with equivariant, projective, birational maps $\mu, \mu'$ which are isomorphisms over $U$. By weak factorization Theorem, $\mu : Y'' \to Y$ can be factored into a series of blow ups or blow downs along invariant smooth centers. By the (Blow) axiom,

$$[Y'' \to X, \mu^*\mathcal{L}_1, \ldots, \mu^*\mathcal{L}_r] - [Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r] \equiv 0$$

mod image $i_* + \text{kernel} \{\phi_{X,3} \circ \phi_{X,2} \circ \phi_{X,1}\}$. Similarly,

$$[Y'' \to X, \mu'^*\mathcal{L}'_1, \ldots, \mu'^*\mathcal{L}'_r] - [Y' \to X, \mathcal{L}'_1, \ldots, \mathcal{L}'_r] \equiv 0.$$

That reduces the case to $Y = Y'$ and $f = f'$, with sheaves $\mathcal{L}_i, \mathcal{L}'_i \in \text{Pic}^G(Y)$ such that $\mathcal{L}_i \cong \mathcal{L}'_i$ over $U$. In this case,

$$[f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r] = f_* \circ c(\mathcal{L}_1) \circ \cdots \circ c(\mathcal{L}_r)[I_Y]$$

$$= f_* \circ c(\mathcal{L}_1) \circ \cdots \circ c(\mathcal{L}_{r-1}) \circ c(\mathcal{L}_r \otimes (\mathcal{L}_r \otimes \mathcal{L}'_r))[I_Y]$$

$$\equiv f_* \circ c(\mathcal{L}_1) \circ \cdots \circ c(\mathcal{L}_{r-1}) \circ c(\mathcal{L}_r)[I_Y] + f_* \circ \sigma \circ c(\mathcal{L}_r \otimes \mathcal{L}'_r)[I_Y]$$

for some $\sigma$

$$\equiv f_* \circ c(\mathcal{L}_1) \circ \cdots \circ c(\mathcal{L}_{r-1}) \circ c(\mathcal{L}'_r)[I_Y]$$

by Lemma 7.2 (X is projective, so is Y). Hence, by repeating the same argument for all $\mathcal{L}_i$, we have

$$f_* \circ c(\mathcal{L}_1) \circ \cdots \circ c(\mathcal{L}_r)[I_Y] \equiv f_* \circ c(\mathcal{L}'_1) \circ \cdots \circ c(\mathcal{L}'_r)[I_Y],$$

as we want. That finishes the proof of Theorem 7.3. □

The next property we would like to show is the homotopy invariance property. We will handle the surjectivity first.

**Proposition 7.4.** Suppose $X$ is an object in $G\text{-}Var$, $V$ is a finite dimensional $G$-representation and $\pi : X \times V \to X$ is the projection. Then the induced map

$$\pi^* : \Omega^G(X) \to \Omega^G(X \times V)$$

is surjective.
Proof. Without loss of generality, we may assume dim \( V = 1 \). By Theorem 6.10, it is enough to consider elements of the form \([f : Y \to X \times V]\) where \( Y \in G\text{-Sm} \) is \( G\)-irreducible. By Proposition 3.2 and the localization property, we may assume \( X \) is projective.

Consider the projective map \( \pi_2 \circ f : Y \to V \). Since \( Y \) is \( G \)-irreducible, image \( \pi_2 \circ f \) is a reduced, \( G \)-irreducible, closed subscheme of \( V \). So, the image is either \( V \) or a finite disjoint union of copies of Spec \( k \).

Case 1 : image \( \pi_2 \circ f \) is a finite disjoint union of copies of Spec \( k \).

Let \( f_1 : Y \to X \) and \( f_2 : Y \to V \) be the maps \( \pi_1 \circ f \) and \( \pi_2 \circ f \) respectively. Since \( f_2 : Y \to \) image \( f_2 \) is projective, \( Y \) is projective. Consider the map \( \phi : Y \times V \to X \times V \) defined by sending \((y, v)\) to \((f_1(y), f_2(y) - v)\). It is clear that \( \phi \) is equivariant. Notice that sending \((y, v)\) to \((y, f_2(y) - v)\) defines an isomorphism \( g : Y \times V \to Y \times V \) and \( \phi = (f_1 \times \mathbb{I}_V) \circ g \).

Since \( Y \) is projective, \( f_1 \) is projective and so is \( \phi \). Hence,

\[
[Y \times 0 \hookrightarrow Y \times V] = c(\mathcal{O}(Y \times 0))[\mathbb{I}_{Y \times V}],
\]

in which we consider \( Y \times 0 \) as an invariant divisor on \( Y \times V \). Since the divisor \( Y \times 0 \) is the pull-back (along \( \pi_2 : Y \times V \to V \)) of the divisor \([0]\) on \( V \) and \( \mathcal{O}_V([0]) \cong \beta \) for some character \( \beta \), we have

\[
[Y \times 0 \hookrightarrow Y \times V] = c(\beta)[\mathbb{I}_{Y \times V}] = e(\beta)[\mathbb{I}_{Y \times V}]
\]

by Proposition 5.1. Apply \( \phi_* \) on both sides, we have

\[
[f : Y \to X \times V] = e(\beta)[\phi : Y \times V \to X \times V] = e(\beta)\pi^*[f_1 : Y \to X]
\]

and we are done.

Case 2 : image \( \pi_2 \circ f = V \).

We will proceed by induction on \( \dim Y \). If \( \dim Y = 0 \), then \( \pi_2 \circ f \) can not be surjective. Suppose \( \dim Y > 0 \). Consider the following Cartesian square:

\[
\begin{array}{ccc}
Z & \xrightarrow{p_1} & V \times \mathbb{A}^1 \\
| & p_2 \downarrow & \downarrow m \\
Y & \xrightarrow{\pi_2 \circ f} & V
\end{array}
\]

where \( \mathbb{A}^1 \) is equipped with trivial \( G \)-action and \( m(x, y) \overset{\text{def}}{=} xy \). Denote the fiber of \( \pi_2 \circ f \) over zero by \( Y_0 \) and the fibers of \( \pi_2 \circ p_1 : Z \to \mathbb{A}^1 \) over \( 0, 1 \) by \( Z_0, Z_1 \) respectively. Notice that \( \pi_2 \circ f, \pi_2 \circ p_1 \) are both flat and \( Z \) is equidimensional.

Claim 1 : There exist invariant closed subschemes \( D_i \subseteq Y_0, D'_i \subseteq Z_0 \) and integers \( m_i \) such that \( D_i, D'_i \) are \( G \)-prime divisors on \( Y, Z \) respectively, \( D'_i \cong D_i \times V \) as \( G \)-schemes,

\[
(\pi_2 \circ f)^*\mathcal{O}_V([0]) \cong \mathcal{O}_Y(\sum_i m_i D_i),
\]

\[
(\pi_2 \circ p_1)^*\mathcal{O}_{\mathbb{A}^1}([0]) \cong \mathcal{O}_Z(\sum_i m_i D'_i)
\]
and \( \text{Sing}(Z) \subseteq Z_0 \) with dimension \( \dim D_i \).

Let \( \text{Spec} R \) be an irreducible, affine open subscheme of \( Y \) and \( m^* : k[t] \to k[x, y] \) be the ring homomorphism corresponding to \( m \). Then, \( \pi_2 \circ f \) corresponds to an injective ring homomorphism \( k[t] \to R \) which sends \( t \) to some non-zero element \( a \in R \). Therefore, \( Z \) is locally given by \( \text{Spec} R[x, y] / (a - xy) \). Let \( P_i \) be the minimal prime ideals of \( R/(a) \).

Observe that \((\pi_2 \circ f)^*[0]\) defines a Cartier divisor, which induces a Weil divisor. If we take \( D_i \) to be the divisor given by \( P_i \subseteq R \) and \( m_i \) to be the length of \( R_{P_i}/(a) \), then \((\pi_2 \circ f)^*[0] = \sum m_i D_i \). Similarly, since the minimal prime ideals of

\[
R[x, y] / (a - xy, y) \cong R[x]/(a)
\]

are \( Q_i \overset{\text{def}}{=} P_i \cdot (R[x]/(a)) \), if we take \( D_i' \) to be the divisor given by \( Q_i \subseteq R[x, y] / (a - xy) \), then \( D_i' \cong D_i \times V \) and \((\pi_2 \circ p_1)^*[0] = \sum m_i D_i' \).

Over \( V \times (\mathbb{A}^1 - 0) \), \( Z \) is locally given by \( R[x, y][y^{-1}] / (a - xy) \cong R[y][y^{-1}] \), which is regular. So, \( p_1^{-1}(V \times (\mathbb{A}^1 - 0)) \) is smooth. For the same reason, \( p_1^{-1}((V - 0) \times \mathbb{A}^1) \) is also smooth. Hence, \( \text{Sing}(Z) \subseteq p_1^{-1}(0 \times 0) \), which is a closed subscheme of \( Z \) with codimension \( 2 \) (\( p_1 \) is flat).

By resolution of singularities, there exists an equivariant, projective, birational map \( Z' \to Z \) such that \( Z' \in G\text{-Sm} \). Then, we apply the embedded desingularization on \( \cup_i \langle D_i' \rangle \to Z' \) to obtain \( Z'' \to Z' \). Denote the composition \( Z'' \to Z' \to Z \) by \( q \). We then have

\[
(\pi_2 \circ p_1 \circ q)^\ast O_{\mathbb{A}^1}(0) = O_{Z''}(\sum m_i \langle \langle D_i' \rangle \rangle + \sum \pm \langle E_j \rangle)
\]

for some exceptional divisors \( E_j \). Resolve the singularities of \( D_i \) to obtain \( \tilde{D}_i \in G\text{-Sm} \).

Now, consider the following commutative diagram:

\[
\begin{array}{cccc}
\langle \langle D_i' \rangle \rangle & \to & Z'' & \to & Z' & \to & Z \\
\phi \downarrow & & \downarrow & & \downarrow & & \\
\tilde{D}_i \times V & \to & D_i \times V & \overset{\text{iso.}}{\to} & D_i' & \to & Z
\end{array}
\]

where \( \phi \) is an equivariant, birational map (may not be regular). Let \( g : Z \to X \times V \times \mathbb{A}^1 \) be the map defined by sending \( z \) to \((\pi_1 \circ f \circ p_2(z), p_1(z))\). It is projective because \( p_1 \) is projective. Precomposing it with \( g \), we got a projective map \( h : Z'' \to X \times V \times \mathbb{A}^1 \). By Proposition 3.3, we can extend it to a projective map \( \overline{h} : \overline{Z''} \to X \times V \times \mathbb{P}^1 \) (trivial action on \( \mathbb{P}^1 \)). Let \( p : \overline{Z''} \to X \times V \) be the projective map given by composing \( \overline{h} \) with projection.
Then we have

\[
[f : Y \to X \times V] = [Y \cong (\pi_3 \circ h)^{-1}(1) \hookrightarrow \overline{Z}\] \to X \times V] \\
= p_*(c((\pi_3 \circ h)^*\mathcal{O}_F(1))[[\overline{Z}]]) \\
= p_*(c((\pi_3 \circ h)^*\mathcal{O}_F(0))[[\overline{Z}]]) \\
= p_*(c(\mathcal{O}_{\overline{Z}}(\sum m_i \langle D_i' \rangle + \sum j \pm (E_j)))[[\overline{Z}]])
\]

by equation (5) \((Z'', \overline{Z}'' \text{ have the same fibers over } \mathbb{A}^1)\). Hence,

\[
[f : Y \to X \times V] = \sum_i p_\ast \circ \sigma_i[\langle\langle D_i' \rangle\rangle] \hookrightarrow \overline{Z}'' + \sum_j p_\ast \circ \sigma_j[\langle E_j \rangle] \hookrightarrow \overline{Z}''
\]

for some \(\sigma_i, \sigma_j \in \operatorname{End}(\Omega^G(\overline{Z}''))\). Notice that \(\dim Y = \dim (\langle D_i' \rangle) = \dim (\langle E_j \rangle)\).

For simplicity, we will say \(x \equiv 0\) if \(x\) is an element in image \(\pi^* \subseteq \Omega^G(X \times V)\). By Proposition 6.10, Theorem 6.10 and the induction assumption,

\[
[f : Y \to X \times V] \equiv \sum_i p_\ast a_i[\langle\langle D_i' \rangle\rangle] \hookrightarrow \overline{Z}'' + \sum_j p_\ast a'_j[\langle E_j \rangle] \hookrightarrow \overline{Z}''
\]

for some \(a_i, a'_j \in \mathbb{L}_G(F)\). In other words, it is enough to show

\[
\langle\langle D_i' \rangle\rangle \to \overline{Z} \to X \times V] \equiv [\langle E_j \rangle] \to \overline{Z} \to X \times V] \equiv 0.
\]

By the weak factorization Theorem, the rational map \(\phi : \langle\langle D_i' \rangle\rangle \dasharrow \overline{D}_i \times V\) can be considered as the composition of a series of blow ups or blow downs along invariant smooth centers. For simplicity, assume \(\phi\) is given by a single blow up along some invariant smooth center \(C\). Then, by the blow up relation, the difference \([\langle\langle D_i' \rangle\rangle \to X \times V] - [\overline{D}_i \times V \to X \times V]\) is given by elements of the form \(\mathbb{P} \to C \to X \times V\) where \(\mathbb{P} \to C\) is a quasi-admissible tower. Since \(\dim Y = \dim (\langle D_i' \rangle) = \dim \mathbb{P} > \dim C\), we have \([\mathbb{P} \to C \to X \times V] \equiv 0\) by Proposition 6.10 and the induction assumption. Hence,

\[
\langle\langle D_i' \rangle\rangle \to X \times V] \equiv [\overline{D}_i \times V \to X \times V] = [\overline{D}_i \to D_i \hookrightarrow Y \to X] \times [\mathbb{I}_V] \equiv 0.
\]

Since \((E_j)\) is the strict transform of the exceptional divisor \(E_j\) corresponding to a certain blow up in the process of blowing up \(Z\) to obtain \(Z'\), or blowing up \(Z'\) to obtain \(Z''\), there is an object \(C \in G-\mathbb{S}m\) such that \(E_j \to C\) is a quasi-admissible tower, \(\dim E_j > \dim C\) and the map \(E_j \to X \times V\) factors through \(E_j \to C\). Hence, by the same argument as before,

\[
[\langle E_j \rangle] \to E_j \to X \times V] \equiv [E_j \to X \times V] = [E_j \to C \to X \times V] \equiv 0.
\]

That finishes the proof of Proposition 7.4. \(\square\)

**Theorem 7.5.** Suppose \(\text{char } k = 0\). For any object \(X \in G-\text{Var}\) and finite dimensional \(G\)-representation \(V\), the map

\[
\pi^* : \Omega^G(X) \to \Omega^G(X \times V)
\]

induced by the projection \(\pi : X \times V \to X\) is an isomorphism.
Proof. By Proposition 7.4, it is enough to prove the injectivity of \( \pi^* \). Without loss of generality, we may assume \( \dim V = 1 \). By the localization property, we have the following exact sequence:

\[
\Omega^G(X) \xrightarrow{i_*} \Omega^G(X \times \mathbb{P}(V \times \mathbb{A}^1)) \xrightarrow{j^*} \Omega^G(X \times V) \to 0
\]

where \( \mathbb{A}^1 \) is equipped with trivial \( G \)-action, \( i : X \cong X \times \mathbb{A}^1 \to X \times \mathbb{P}(V \times \mathbb{A}^1) \) and \( j : X \times V \hookrightarrow X \times \mathbb{P}(V \times \mathbb{A}^1) \) are the immersions.

Let \( p : X \times \mathbb{P}(V \times \mathbb{A}^1) \to X \) be the projection and \( \phi : \Omega^G(X \times \mathbb{P}(V \times \mathbb{A}^1)) \to \Omega^G(X) \) be the map \( p_* \circ c(L) \) where \( L \overset{\text{def}}{=} \pi^* \mathcal{O}_{\mathbb{P}(V \times \mathbb{A}^1)(\{0\})} \). Then, for any element \( x \in \Omega^G(X) \),

\[
\phi \circ i_*(x) = p_* \circ c(L) \circ i_*(x) = p_* \circ i_* \circ c(L|_{X \times \infty})(x) = p_* \circ i_* \circ c(\mathcal{O}_X)(x) = 0.
\]

Therefore, \( \phi \) defines a map \( \Omega^G(X \times V) \to \Omega^G(X) \). Now, for any element \([Y \to X] \in \Omega^G(X)\),

\[
\phi \circ \pi^*[Y \to X] = \phi [Y \times V \to X \times V] = p_* \circ c(L)[Y \times \mathbb{P}(V \times \mathbb{A}^1) \to X \times \mathbb{P}(V \times \mathbb{A}^1)] = p_*([Y \to X] \times c(\mathcal{O}_{\mathbb{P}(V \times \mathbb{A}^1)(\{0\})[\mathbb{P}(V \times \mathbb{A}^1)])) = p_*([Y \to X] \times [0 \hookrightarrow \mathbb{P}(V \times \mathbb{A}^1)]) = [Y \to X].
\]

Hence, \( \pi^* \) is injective by Theorem 6.10. \[\square\]

8. Comparison with Other Algebraic Cobordism Theories

In this section, we will compare our equivariant algebraic cobordism theory \( \Omega^G(-) \) to other algebraic cobordism theories, namely, the non-equivariant algebraic cobordism theory \( \Omega(-) \) as in \[\text{LMo}\] and the equivariant algebraic cobordism theory, which will be denoted by \( \Omega^G_{\text{Tot}}(-) \) to avoid confusion, defined in \[\text{HeMa}\].

Let us consider the theory \( \Omega(-) \) first. We need to understand the relation between the universal representing ring \( \mathbb{L}_G(F) \) and \( \mathbb{L} \). We will use the same assumptions on \( G \) and \( k \) as in section 2. In particular, we will not assume \( \text{char } k = 0 \).

Recall definition 12.2 in \[\text{CGKr}\]. For a commutative ring \( R \), denote the topological \( R \)-module obtained as the inverse limit of the free \( R \)-modules with basis \( 1 = y(V^0), y(V^1), y(V^2), \ldots, y(V^s) \) by \( R\{\{F\}\} \). Then, a \((G, F)\)-formal group law over a commutative ring \( R \) is a topological \( R \)-module \( R\{\{F\}\} \) with product given by

\[
y(V^i) \cdot y(V^j) = \sum_{s \geq 0} b_{i,j}^s y(V^s)
\]
with $b_{i,j}^s \in R$, a $G^*$-action given by
\[ l_\alpha y(V^i) = \sum_{s \geq 0} d(\alpha)_s y(V^u) \]
with $d(\alpha)_s \in R$ and coproduct given by
\[ \Delta y(V^i) = \sum_{s,t \geq 0} f_{s,t}^i y(V^u) \otimes y(V^t) \]
with $f_{s,t}^i \in R$ satisfying the following conditions:

1. For all $i, s$, the coefficients $b_{i,j}^s = 0$ for sufficiently large $j$, and similarly with $i$ and $j$ exchanged.
2. For all $\alpha, s$, the coefficients $d(\alpha)_s = 0$ for sufficiently large $i$.
3. For all $s, t$, the coefficients $f_{s,t}^i = 0$ for sufficiently large $i$.
4. The product is commutative, associative and unital.
5. The action is through ring homomorphisms, associative and unital.
6. The coproduct is through ring homomorphisms, equivariant in the sense that
\[ \Delta \circ l_{\alpha \beta} = (l_\alpha \otimes l_\beta) \circ \Delta, \]
commutative, associative and unital.
7. For all $i$, the ideal $(y(V^i))$ has additive topological basis $y(V^i)$, $y(V^{i+1})$, ...
sheaves in Pic\(^H\)(\(Y\)). Notice that if we consider \(LZ^H_F(X)\) as a \(L_G(F)\)-module via the map \(\Phi_{H \to G}\), then \(\Psi_{H \to G}\) will be a \(L_G(F)\)-module homomorphism.

**Proposition 8.2.** Suppose \(H\) is a closed subgroup of \(G\) such that the pair \((H,k)\) is split. Then, for all \(X \in G\text{-Var}\), \(\Psi_{H \to G}\) defines a canonical \(L_G(F)\)-module homomorphism

\[ \Omega^G(X) \to \Omega^H(X) \]

and it commutes with the four basic operations.

**Proof.** The map \(\Psi_{H \to G}\) clearly commutes with the basic operations. The fact that it respects the \(\text{(Sect)}\) and \(\text{(EFGL)}\) axioms (by identifying \(b_{i,j}^s\) with \(\Phi_{H \to G}(b_{i,j}^s)\) and similarly for other structure constants) follows immediately from the definition. \(\square\)

As pointed out in example 12.3 (ii) in \([CGK]\), if \(G\) is the trivial group \(\{1\}\), then the notion of “\((G,F)\)-formal group law” agrees with the notion of “formal group law” as in \([LMo]\). More precisely, \(R\{\{F\}\} = R[[y]]\) with \(y(V^i) = y^i\) and the coproduct

\[ \Delta : R[[y]] \to R[[u]] \hat{\otimes} R[[v]] \cong R[[u,v]] \]

is given by

\[ \Delta(y) = F(u,v) = \sum_{s,t \geq 0} a_{s,t} u^s v^t. \]

Therefore, there is a canonical ring isomorphism \(\overline{\Phi} : \mathbb{L}^{(1)}(F) \to \mathbb{L}\) which sends \(b_{i,j}^s\) to \(\delta_{i+j}^s\), \(d(\epsilon)^i_s\) to \(\delta^i_s\) and \(f_{s,t}^1\) to \(a_{s,t}\). In addition, since

\[ (\sum_{s,t \geq 0} f_{s,t}^1 y^s \otimes y^t)^i = (\Delta(y))^i = \Delta(y^i) = \sum_{s,t \geq 0} f_{s,t}^i y^s \otimes y^t, \]

the map \(\overline{\Phi}\) will send \(f_{s,t}^i\) to \(a_{s,t}^i\) if we define elements \(a_{s,t}^i \in \mathbb{L}\) by the equation:

\[ (\sum_{s,t \geq 0} a_{s,t}^i u^s v^t)^i = \sum_{s,t \geq 0} a_{s,t}^i u^s v^t. \]

Thus, for any object \(X \in G\text{-Var}\), we have a canonical \(\mathbb{L}\)-module homomorphism

\[ \overline{\Psi} : \mathbb{L}Z^{(1),F}(X) \to \Omega(X) \]

which sends \(\sum_I a_I [Y \to X, V_{S_1}^j(L_1), \ldots, V_{S_r}^j(L_r)]\) to

\[ \sum_I \overline{\Psi}(a_I) [Y \to X, L_1, \ldots, L_1, L_2, \ldots, L_2, \ldots, L_r, \ldots, L_r] \]

(for each \(1 \leq j \leq r\), there are \(i_j - \#S_j\) copies of \(L_j\)). Notice that it is a finite sum because of the \(\text{(Dim)}\) axiom in \(\Omega(-)\).

**Proposition 8.3.** Suppose \(G\) is the trivial group. Then, for all \(X \in G\text{-Var}\), \(\overline{\Psi}\) defines a canonical \(\mathbb{L}\)-module isomorphism \(\Omega^G(X) \to \Omega(X)\) and it commutes with the four basic operations.
Proof. The map \( \Psi \) clearly commutes with the basic operations and respects the \((\text{Sect})\) axiom. For the \((\text{EFGL})\) axiom,

\[
\Psi(V^i(L)V^j(L)[I_Y]) = c(L)^i c(L)^j[I_Y] = \sum_{s \geq 0} \delta^{i+j}_s c(L)^s[I_Y] = \Psi(\sum_{s \geq 0} b^i_j V^s(L)[I_Y])
\]

and the rest is similar. Hence, \( \Psi \) defines a canonical \( L \)-module homomorphism \( \Omega^G(X) \rightarrow \Omega(X) \).

Its inverse is also naturally defined:

\[
\Psi^{-1}: L \otimes \mathbb{Z} Z(X) \rightarrow \Omega^G(X),
\]

which sends \( a[Y \rightarrow X, L_1, \ldots, L_r] \) to \( \Psi^{-1}(a)[Y \rightarrow X, L_1, \ldots, L_r] \).

Recall that \( \Omega(-) \) is defined by imposing the \((\text{Dim})\), \((\text{Sect})\) and \((\text{FGL})\) axioms on \( L \otimes \mathbb{Z} Z(-) \). For the \((\text{Dim})\) axiom, we need to show the following claim.

Claim 1: Suppose \( Y \in Sm \) (the category of smooth, quasi-projective schemes over \( k \)) is irreducible, \( r > \dim Y \) and \( L_1, \ldots, L_r \) are invertible sheaves over \( Y \).

\[
\Psi^{-1}([I_Y, L_1, \ldots, L_r]) = 0.
\]

Let \( M_1, M_2 \) be two very ample invertible sheaves over \( Y \) such that \( L_r \cong M_1 \otimes M_2^\vee \). Then,

\[
\Psi^{-1}([I_Y, L_1, \ldots, L_r]) = c(L_r)[I_Y, L_1, \ldots, L_{r-1}]
= c(M_1 \otimes M_2^\vee)[I_Y, L_1, \ldots, L_{r-1}]
= c(M_1)[I_Y, L_1, \ldots, L_{r-1}] + \sigma \circ c(M_2)[I_Y, L_1, \ldots, L_{r-1}]
\]

for some \( \sigma \). So, without loss of generality, we may assume \( L_1, \ldots, L_r \) are all very ample. Then the result follows from an equivariant version (with trivial group \( G \)) of Lemma 2.3.9(1) in [LMo]. \( \triangle \)

The map \( \Psi^{-1} \) clearly respects the \((\text{Sect})\) axiom. For the \((\text{FGL})\) axiom, \( \Psi^{-1} \) sends \( c(L \otimes M)[I_Y] \) and \( \sum_{s,t \geq 0} a_{s,t} c(L)^s c(M)^t[I_Y] \) to \( c(L \otimes M)[I_Y] \) and \( \sum_{s,t \geq 0} f_{s,t}^{L,M} V^s(L)V^t(M)[I_Y] \) respectively. Hence, it defines a canonical map

\[
\Psi^{-1}: \Omega(X) \rightarrow \Omega^G(X).
\]

Clearly, \( \Psi \circ \Psi^{-1} \) is the identity. By claim 1, for any \( Y \in G\text{-Sm} \), \( L \in \text{Pic}^G(Y) \) and \( S \) as in [1], we have

\[
V^S_{\Delta}(L)[I_Y] = 0,
\]

as elements in \( \Omega^G(Y) \), for sufficiently large \( n \). Therefore, \( \Psi^{-1} \) is surjective and that finishes the proof. \( \square \)
Corollary 8.4. For all $X \in G\text{-Var}$, the “forgetful map” given by $\Psi \circ \Psi_{\{1\}} \to G$ defines a $\mathbb{L}_{G}(F)$-module homomorphism

$$\Omega^{G}(X) \to \Omega(X)$$

and it commutes with the four basic operations.

Our second goal is to compare our theory $\Omega^{G}(-)$ to the equivariant algebraic cobordism theory $\Omega^{G}_{\text{Tot}}(-)$ defined in [HeMa] using the Totaro’s approximation of $EG$. Following the basic assumptions in [HeMa], $G$ will be a split torus of rank $n$ and char $k = 0$ for the rest of this section.

Recall that $\Omega^{G}_{\text{Tot}}(X)$ is defined to be the inverse limit of $\Omega(X \times^{G} U_{i})$ where $\{(W_{i}, U_{i})\}$ is a good system of $G$-representations (in particular, the $G$-actions on $U_{i}$ are free) and

$$X \times^{G} U_{i} \overset{\text{def}}{=} (X \times U_{i})/G$$

(see definition 1 in [HeMa] for details). When $G = (\mathbb{G}_{m})^{n}$, there is a simple choice of $\{(W_{i}, U_{i})\}$. Let $W_{i} \overset{\text{def}}{=} (\mathbb{A}^{i})^{n}$ be a $G$-representation with action given by

$$(g_{1}, \ldots, g_{n}) \cdot (a_{1}, \ldots, a_{n}) \overset{\text{def}}{=} (g_{1}a_{1}, \ldots, g_{n}a_{n})$$

and $U_{i} \overset{\text{def}}{=} (\mathbb{A}^{i} - 0)^{n}$. Then $\{(W_{i}, U_{i})\}$ forms a good system of $G$-representations. Also notice that $U_{i}/G \cong (\mathbb{P}^{i-1})^{n}$. For $1 \leq j \leq n$, let

$$D_{ij} \overset{\text{def}}{=} \mathbb{P}^{i-1} \times \cdots \times \mathbb{P}^{i-1} \times H \times \mathbb{P}^{i-1} \times \cdots \mathbb{P}^{i-1},$$

where $H \subseteq \mathbb{P}^{i-1}$ is a hyperplane in the $j$-th copy of $\mathbb{P}^{i-1}$, and we will consider $D_{ij}$ as a smooth divisor on $U_{i}/G$.

Let $\mathbb{L}_{G}(F)$ be the completion of $\mathbb{L}_{G}(F)$ with respect to the ideal generated by the Euler classes and $\gamma_{1}, \ldots, \gamma_{n}$ be the natural set of generators of $G^{*} \cong \mathbb{Z}^{n}$. By Theorem 6.5 in [G], there is a ring isomorphism $\mathbb{L}_{G}(F) \to \mathbb{L}[[z_{1}, \ldots, z_{n}]]$ which sends $e(\gamma_{j})$ to $z_{j}$. Therefore, we have a ring homomorphism

$$\phi : \mathbb{L}_{G}(F) \to \mathbb{L}[[z_{1}, \ldots, z_{n}]],$$

which sends $e(\gamma_{j})$ to $z_{j}$.

Now, for an object $X \in G\text{-Var}$, we define an abelian group homomorphism

$$\Psi_{\text{Tot}} : \mathbb{L}Z^{G,F}(X) \to \Omega^{G}_{\text{Tot}}(X)$$

by sending $a[f : Y \to X, \mathcal{L}_{1}, \ldots, \mathcal{L}_{r}]$ to

$$\sum_{K} a_{K}^{r} c(\mathcal{O}(\hat{D}_{1}))^{k_{1}} \cdots c(\mathcal{O}(\hat{D}_{m}))^{k_{n}} [\hat{f} : Y \times^{G} U_{i} \to X \times^{G} U_{i}, \hat{\mathcal{L}}_{1}, \ldots, \hat{\mathcal{L}}_{r}],$$

where $a_{K}^{r} \in \mathbb{L}$ is given by the equation $\phi(a) = \sum_{K} a_{K}^{r} z_{1}^{k_{1}} \cdots z_{n}^{k_{n}}$ with $K$ as multi-index, $\hat{D}_{ij}$ is the pull-back of $D_{ij}$ via $X \times^{G} U_{i} \to U_{i}/G$, $\hat{f} = f \times^{G} U_{i}$ and $\hat{\mathcal{L}}_{j}$ are the sheaves naturally induced by $\mathcal{L}_{j}$. Notice that, by the (Dim) axiom in $\Omega(-)$, this gives a well-defined element in $\Omega(X \times^{G} U_{i})$, for all $i$, and the case of infinite cycle is covered. Moreover, by its naturality, $\Psi_{\text{Tot}}$ is also compatible with the inverse system. Hence, the map $\Psi_{\text{Tot}}$ is well-defined.
Proposition 8.5. Suppose $G$ is a split torus and $\operatorname{char} k = 0$. Then $\Psi_{\text{Tot}}$ defines an abelian group homomorphism

$$\Psi_{\text{Tot}} : \Omega^G(X) \to \Omega^G_{\text{Tot}}(X),$$

for any $X \in G\text{-Var}$, and it commutes with the projective push-forward, smooth pull-back and Chern class operator.

Proof. The only non-trivial part is the well-definedness of

$$\Psi_{\text{Tot}} : \Omega^G(X) \to \Omega(X \times^G U_i)$$

for all $i$.

Let $I \subseteq \mathbb{L}_G(F)$ be the ideal generated by Euler classes and $\hat{\mathbb{L}}_G^{G,F}(X)$, $\hat{\Omega}^G(X)$ be the analogues of $LZ^{G,F}(X)$, $\Omega^G(X)$ respectively, defined by using coefficient ring $\hat{\mathbb{L}}_G(F)$ instead of $\mathbb{L}_G(F)$. Notice that there is an analogue $\hat{\Psi}_{\text{Tot}} : \hat{\mathbb{L}}_G^{G,F}(X) \to \Omega^G_{\text{Tot}}(X)$ of $\Psi_{\text{Tot}}$ and the map $\Psi_{\text{Tot}}$ factors through $\hat{\mathbb{L}}_G^{G,F}(X)$. Hence, it will be enough to show the well-definedness of

$$\hat{\Psi}_{\text{Tot}} : \hat{\Omega}^G(X) \to \Omega(X \times^G U_i).$$

Also, it is not hard to see that it respects the (Sect) axiom. So it remains to show that it respects the (EFGL) axiom.

Let $n$ be the rank of $G$. By Theorem 6.5 in [G],

$$\hat{\mathbb{L}}_G(F)\{\{F\}\} \cong \mathbb{L}[\left[z_1, \ldots, z_n\right]]\{\{F\}\} \cong \mathbb{L}[\left[z_1, \ldots, z_n, \overline{y}\right]].$$

Claim 1 : For any character $\beta$, there are unique elements $g(\beta)_j \in \mathbb{L}[\left[z_1, \ldots, z_n\right]]$ such that

$$y(\beta) = \sum_j g(\beta)_j \overline{y}.$$

By Theorem 6.5 in [G],

$$\hat{\mathbb{L}}_G(F)\{\{F\}\} \cong \mathbb{L}[\left[z_1, \ldots, z_n\right]][\overline{y}],$$

where $\overline{y}$ corresponds to $y(\epsilon)$. For any character $\beta$,

$$y(\beta) = e(\beta) + \overline{y} \sum_{i \geq 1} d(\beta_i) y(\alpha_2) \cdots y(\alpha_i).$$

For each $y(\alpha_j)$, we can apply the above equation with $\beta = \alpha_j$. By repeating this argument, we got

$$y(\beta) = \sum_{i \geq 0} g(\beta)_i \overline{y}$$

for some elements $g(\beta)_i \in \hat{\mathbb{L}}_G(F)$ (Each $g(\beta)_i$ is given by a finite sum of elements in $\hat{\mathbb{L}}_G(F)$ because the flag $F$ is complete and $y(\epsilon) = \overline{y}$). These elements are unique by equation (7). $\triangle$
Claim 2: For all \( n \geq 0, \ x \in \Omega^G(X) \), the element \( c(\mathcal{L})^m(x) \) lies inside \( I^n\Omega^G(X) \) for sufficiently large \( m \).

By Theorem 6.10 we may assume \( x = [f : Y \to X] \). If \( \dim Y = 0 \), then \( f^*\mathcal{L} \cong \beta \) for some character \( \beta \) and the statement is clearly true. The result then follows from Theorem 6.10 and the induction assumption on \( \dim Y \). △

Claim 3: For all \( G \)-irreducible \( Y \in G\text{-Sm}, \mathcal{L} \in \text{Pic}^G(Y) \), the operator \( \sum_j g(\beta)_j c(\mathcal{L})^j \) is well-defined (in the theory \( \hat{\Omega}^G(\cdot) \)) and

\[
(8) \quad c(\mathcal{L} \otimes \beta) = \sum_j g(\beta)_j c(\mathcal{L})^j.
\]

For its well-definedness, since

\[
\hat{\Omega}^G(X) = \lim_{\leftarrow n} \Omega^G(X) / I^n\Omega^G(X),
\]

it suffices to show its well-definedness as a map \( \hat{\Omega}^G(X) \to \Omega^G(X) / I^n\Omega^G(X) \) and it commutes with the maps in the inverse system. Then it follows from claim 2.

For the second part, in \( \Omega^G(\cdot) \),

\[
c(\mathcal{L} \otimes \beta) = c(\beta) + c(\mathcal{L}) \sum_{i \geq 1} d(\beta)_i^1 c(\mathcal{L} \otimes \alpha_2) \cdots c(\mathcal{L} \otimes \alpha_i).
\]

Then the result follows from a similar argument as in claim 1. △

Observe that \( y(V^i) = y(\alpha_1) \cdots y(\alpha_i) \), which can be expressed in terms of \( \pi \) by equation (6). Therefore, the equation

\[
y(V^i)y(V^j) = \sum_s b_s^{i,j}y(V^s)
\]

can be expressed in terms of \( \pi \) (the right hand side in terms of \( \pi \) is well-defined because the flag \( F \) is complete). On the other hand, \( V^i(\mathcal{L}) = c(\mathcal{L} \otimes \alpha_1) \cdots c(\mathcal{L} \otimes \alpha_i) \), which can also be expressed in terms of \( c(\mathcal{L}) \) by equation (5). Hence, by equation (8), we have

\[
V^i(\mathcal{L})V^j(\mathcal{L})[I_Y] = \sum_s b_s^{i,j}V^s(\mathcal{L})[I_Y]
\]

(well-definedness while expressed in terms of \( c(\mathcal{L}) \) follows from claim 2). The rest of the (EFGL) axiom follows from similar arguments. □

**Corollary 8.6.** Suppose \( G \) is a split torus and \( \text{char } k = 0 \). If the completion map \( \mathbb{L}_G(F) \to \hat{\mathbb{L}}_G(F) \) is injective, then the canonical ring homomorphism

\[
\mathbb{L}_G(F) \to \Omega^G(\text{Spec } k)
\]

is an isomorphism.
Proof. The surjectivity is given by Theorem 6.12. For the injectivity, let \( n \) be the rank of \( G \) and \( f : \mathbb{L}_G(F) \to \Omega^G(\text{Spec } k) \) be the canonical map. As pointed out in section 3.3 in [HeMa],

\[
\Omega^G_{\text{Tot}}(\text{Spec } k) \cong \mathbb{L}[\![z_1, \ldots, z_n]\!]
\]

with \( z_j \) corresponding to the element \([D_{ij} \hookrightarrow U_i / G] \in \Omega(U_i / G)\). Then, by Proposition 8.5, we have a composition of maps

\[
\mathbb{L}_G(F) \xrightarrow{f} \Omega^G(\text{Spec } k) \xrightarrow{\Psi_{\text{Tot}}} \Omega^G_{\text{Tot}}(\text{Spec } k) \cong \mathbb{L}[\![z_1, \ldots, z_n]\!],
\]

which is nothing but the completion map \( \mathbb{L}_G(F) \to \hat{\mathbb{L}}_G(F) \). Hence \( f \) is injective. \( \square \)

9. More on \( \Omega^G_{\text{Tot}}(\_ \_ \_ \_ \_ \_ \_ \) \)

In Heller and Malagón-López’s paper [HeMa], they define an equivariant algebraic cobordism theory \( \Omega^G_{\text{Tot}}(\_ \_ \_ \_ \_ \_ \_ \) \) for connected linear algebraic group \( G \) over a field of characteristic zero. In order to have a full comparison with our theory, we need to first extend their definition to allow \( G \) to be a split diagonalizable group (not connected). Since Corollary 3 in [HeMa] still holds, \( \Omega^G_{\text{Tot}}(\_ \_ \_ \_ \_ \_ \_ \) \) is well-defined for such \( G \) and its definition is still independent of the choice of good system of representations. In this section, we will compute \( \Omega^G_{\text{Tot}}(\text{Spec } k) \) and generalize Proposition 8.5 and Corollary 8.6. As in [HeMa], we will assume \( \text{char } k = 0 \).

Let us first compute \( \Omega^G_{\text{Tot}}(\text{Spec } k) \) when \( G \) is a cyclic group of order \( n \). Let \( \alpha \) be a generator of \( G^* \). For all \( i > 0 \), let \( W_i \) be \( \mathbb{A}^i \) with \( G \)-action given by \( g \cdot w = \alpha(g)w \) and \( U_i \overset{\text{def}}{=} W_i - 0 \). Then \( \{(W_i, V_i)\} \) forms a good system of representations. Let \( E_i \) be the equivariant line bundle over \( \mathbb{P}(W_i) \) corresponding to the sheaf \( \mathcal{O}_{\mathbb{P}^i}(n) \) and \( s : \mathbb{P}(W_i) \to E_i \) be the zero section.

Now, \( W_i = \text{Spec } k[x_1, \ldots, x_i] \) and \( U_i \supseteq D(x_j) = \text{Spec } k[x_1, \ldots, x_i][x_j^{-1}] \). On the other hand, \( \mathbb{P}(W_i)|_{D(x_j)} = \text{Spec } k[\frac{x_1}{x_j}, \ldots, \frac{x_i}{x_j}] \) and

\[
E_i|_{D(x_j)} = \text{Spec } k[\frac{x_1}{x_j}, \ldots, \frac{x_i}{x_j}][v_j]
\]

where \( v_j \) corresponds to the section \( x_j^{-n} \). Notice that the \( G \)-actions on \( \mathbb{P}(W_i) \) and \( E_i \) are both trivial. Define an equivariant map

\[
q_j : k[\frac{x_1}{x_j}, \ldots, \frac{x_i}{x_j}][v_j][v_j^{-1}] \to k[x_1, \ldots, x_i][x_j^{-1}]
\]

by sending \( \frac{x_k}{x_j} \) to \( \frac{x_k}{x_j} \) and \( v_j \) to \( x_j^{-n} \) and let

\[
p_j : D(x_j) \to (E_i - s(\mathbb{P}(W_i)))|_{D(x_j)}
\]

be the corresponding equivariant map.
Lemma 9.1. The equivariant maps $p_j$ patch together to define an equivariant map

$$p : U_i \to E_i - s(\mathbb{P}(W_i))$$

and it is isomorphic to the quotient map $U_i \to U_i/G$.

Proof. The patching follows from the naturality of the definition of $p_j$. Also, it is not hard to see that $q_j$ is injective and its image is precisely $(k[x_1, \ldots, x_i][x_j^{-1}])^G$. Hence, $p_j$ is isomorphic to the quotient map and the result then follows. \hfill \Box

By the projective bundle formula,

$$\Omega(\mathbb{P}(W_i)) = \bigoplus_{k=0}^{i-1} \mathbb{L} \cdot c(O_{\mathbb{P}(W_i)}(1))^k[I_{\mathbb{P}(W_i)}].$$

By the localization property, the sequence

$$\Omega(\mathbb{P}(W_i)) \xrightarrow{s^*} \Omega(E_i) \xrightarrow{} \Omega(U_i/G) \xrightarrow{} 0$$

is exact. By the extended homotopy property,

$$\Omega(E_i) = \bigoplus_{k=0}^{i-1} \mathbb{L} \cdot c(L_i)^k[I_{E_i}]$$

where $\pi : E_i \to \mathbb{P}(W_i)$ is the projection and $L_i \overset{\text{def}}{=} \pi^*O_{\mathbb{P}(W_i)}(1)$. Notice that

$$s_*[I_{\mathbb{P}(W_i)}] = c(\pi^*O_{\mathbb{P}(W_i)}(n))[I_{E_i}] = F^n(c(L_i))[I_{E_i}],$$

where $F^n(u) \in \mathbb{L}[u]$ is defined inductively by $F^0(u) \overset{\text{def}}{=} 0$ and $F^{n+1}(u) \overset{\text{def}}{=} F(F^n(u), u)$. Therefore,

$$\Omega(U_i/G) \cong (\bigoplus_{k=0}^{i-1} \mathbb{L} \cdot c(L_i)^k[I_{E_i}] \setminus (c(L_i)^l \circ F^n(c(L_i)))[I_{E_i}]),$$

where $0 \leq l \leq i - 1$, as $\mathbb{L}$-modules. Moreover, these isomorphisms commute with the maps in the inverse system. Hence,

$$\Omega^G_{\text{Tot}}(\text{Spec } k) = \lim_{\leftarrow i} \Omega(U_i/G) \cong \mathbb{L}[t]/(F^n(t))$$

where $t^j$ is identified with $c(L_i)^j[I_{E_i}]$.

In general, suppose

$$G \cong G_f \times G_t = (\prod_{j=1}^s G_j) \times G_t$$

where $G_j$ is a cyclic group of order $n_j$ and $G_t$ is a split torus of rank $r$. Let $\beta_j$ be a generator of $G_j^*$, $\gamma_1, \ldots, \gamma_r$ be the standard set of generators of $G_t^*$. Also, let $W(\beta_j)_i$, $W(\gamma_k)_i$ be $A^i$ with $G$-actions given by $\beta_j$, $\gamma_k$ respectively. $U(\beta_j)_i \overset{\text{def}}{=} W(\beta_j)_i - 0$, $U(\gamma_k)_i \overset{\text{def}}{=} W(\gamma_k)_i - 0$, $W_i \overset{\text{def}}{=} \prod_{j=1}^s W(\beta_j)_i \times \prod_{k=1}^r W(\gamma_k)_i$ and $U_i \overset{\text{def}}{=} \prod_{j=1}^s U(\beta_j)_i \times \prod_{k=1}^r U(\gamma_k)_i$. Then $\{U_i, U_i\}$ forms a good system of representations. Similarly, we have

$$U_i/G = (\prod_{j=1}^s U(\beta_j)_i/G_j) \times (\prod_{k=1}^r U(\gamma_k)_i/G_m) \cong (\prod_{j=1}^s (E(\beta_j)_i - s(\mathbb{P}(W_i^j)))) \times (\prod_{k=1}^r \mathbb{P}(W(\gamma_k)_i).$$
where $E(\beta_j)_i$ is the line bundle over $\mathbb{P}(W(\beta_j)_i)$ corresponding to the sheaf $\mathcal{O}_{\mathbb{P}(W(\beta_j)_i)}(n_j)$ and $s : \mathbb{P}(W(\beta_j)_i) \to E(\beta_j)_i$ is the zero section. As before, let

$$\pi : F_i \overset{\text{def}}{=} \left( \prod_{j=1}^r E(\beta_j)_i \right) \times \prod_{k=1}^r \mathbb{P}(W(\gamma_k)_i) \to \prod_{j=1}^r \mathbb{P}(W(\beta_j)_i),$$

$$\pi' : F_i \to \prod_{k=1}^r \mathbb{P}(W(\gamma_k)_i)$$

be the projections, $\mathcal{L}(\beta_j)_i \overset{\text{def}}{=} \pi^* \circ \pi^*_j \mathcal{O}_{\mathbb{P}(W(\beta_j)_i)}(1)$ and $\mathcal{L}(\gamma_k)_i \overset{\text{def}}{=} \pi'^* \circ \pi^*_k \mathcal{O}_{\mathbb{P}(W(\gamma_k)_i)}(1)$. By similar calculations, we have

$$\Omega^G_{\text{Tot}}(\text{Spec } k) \cong \mathbb{L}[t_1, \ldots, t_s, z_1, \ldots, z_r]/(F^{n_1}(t_1), \ldots, F^{n_s}(t_s))$$

where $t^p_j$ is identified with $c(\mathcal{L}(\beta_j)_i)^p [\mathbb{P}_{F_i}]$ and $z^q_k$ is identified with $c(\mathcal{L}(\gamma_k)_i)^q [\mathbb{P}_{F_i}]$.

Now, as in section \[S\] let

$$\phi : \mathbb{L}_G(F) \to \mathbb{L}_{G}(F) \cong \mathbb{L}[t_1, \ldots, t_s, z_1, \ldots, z_r]/(F^{n_1}(t_1), \ldots, F^{n_s}(t_s))$$

which sends $e(\beta_j)$, $e(\gamma_k)$ to $t_j$, $z_k$ respectively. For an object $X \in G\text{-Var}$, we define an abelian group homomorphism

$$\Psi_{\text{Tot}} : \mathbb{L}Z^{G,F}(X) \to \Omega^G_{\text{Tot}}(X)$$

by sending $a[f : Y \to X, M_1, \ldots]$ to

$$\sum_{PQ} a'_{PQ} \prod_{j=1}^s (c(\mathcal{O}(\mathcal{L}(\beta_j)_i)^{P_j}) \circ \prod_{k=1}^r c(\mathcal{O}(\mathcal{L}(\gamma_k)_i)^{Q_k}) [\hat{f} : Y \times^G U_i \to X \times^G U_i, \hat{M}_i, \ldots],$$

where $a'_{PQ} \in \mathbb{L}$ is given by the equation $\phi(a) = \sum_{PQ} a'_{PQ} t^{p_1}_{PQ} \cdots t^{p_s}_Z q_1 \cdots q_r$ with $P, Q$ as multi-indices, $\mathcal{L}(\beta_j)_i$, $\mathcal{L}(\gamma_k)_i$ are the pull-backs of $\mathcal{L}(\beta_j)_i$, $\mathcal{L}(\gamma_k)_i$ respectively, via $X \times^G U_i \to U_i/G$, $\hat{f} = f \times^G \mathbb{L}_{U_i}$ and $\hat{M}_i$ are the sheaves naturally induced by $M_i$. For the same reason as in section \[S\] the map $\Psi_{\text{Tot}}$ is well-defined.

**Proposition 9.2.** Suppose $\text{char } k = 0$. Then $\Psi_{\text{Tot}}$ defines an abelian group homomorphism

$$\Psi_{\text{Tot}} : \Omega^G(X) \to \Omega^G_{\text{Tot}}(X),$$

for any $X \in G\text{-Var}$, and it commutes with the projective push-forward, smooth pull-back and Chern class operator.

**Proof.** See the proof of Proposition \[S.5\].

**Corollary 9.3.** Suppose $\text{char } k = 0$. If the completion map $\mathbb{L}_G(F) \to \mathbb{L}_G(F)$ is injective, then the canonical ring homomorphism

$$\mathbb{L}_G(F) \to \Omega^G(\text{Spec } k)$$

is an isomorphism.

**Proof.** See the proof of Corollary \[S.6\].

\[\square\]
10. Comparison with the equivariant K-theory

In this section, we will compare our equivariant algebraic cobordism theory to the equivariant K-theory. Recall the following definition of equivariant K-theory from [Me]. Suppose $X$ is in $G\text{-Var}$. Denote the abelian category of $G$-equivariant, coherent sheaves over $X$ by $M(G; X)$. Then define

$$K'_n(G; X) \overset{\text{def}}{=} K_n(M(G; X)).$$

Also, denote the abelian category of $G$-equivariant, locally free coherent sheaves over $X$ by $P(G; X)$ and define

$$K_n(G; X) \overset{\text{def}}{=} K_n(P(G; X)).$$

We then have the following list of basic results (see section 2 in [Me]):

1. $K'_n(G; X)$ has flat pull-back and projective push-forward.
2. If $G$ is the trivial group, then $K'_n(G; X) = K'_n(X)$ (the ordinary $K$-theory).
3. There is a natural isomorphism $R(G) \sim \rightarrow K'_0(G; \text{Spec } k)$.
4. If $X$ is smooth and quasi-projective over $k$, then the natural homomorphism $K_n(G; X) \rightarrow K'_n(G; X)$ is an isomorphism (Proposition 2.20 in [Me]).

Remark 10.1. In this paper, we only focus on the $K'_0(G; -)$ and $K_0(G; -)$ theories. In order to have projective push-forward, we need to consider $K'_0(G; -)$. But for external product, we need the ring structure on $K_0(G; -)$. Hence, we will focus on the category $G\text{-Sm}$.

For an object $X \in G\text{-Sm}$ and $\mathcal{L} \in \text{Pic}^G(X)$, define

$$c_K(\mathcal{L}) \overset{\text{def}}{=} ([\mathcal{O}_X] - [\mathcal{L}^\vee])v^{-1}$$

and $V_{K,S}^j(\mathcal{L})$ analogously as elements in $K_0(G; X)[v, v^{-1}]$. Then $K_0(G; -)[v, v^{-1}]$ is a theory on $G\text{-Sm}$ with four basic operations, i.e.,

$$f_! [\mathcal{E}] \overset{\text{def}}{=} v \dim f \sum_i (-1)^i [R^i f_* \mathcal{E}]$$

(when $f$ is projective and equidimensional),

$$f^* [\mathcal{E}] \overset{\text{def}}{=} [f^* \mathcal{E}]$$

(when $f$ is flat),

$$c_K(\mathcal{L})[\mathcal{E}] \overset{\text{def}}{=} c_K(\mathcal{L}) \cdot [\mathcal{E}],$$

$$[\mathcal{E}_1] \times [\mathcal{E}_2] \overset{\text{def}}{=} [\pi'_1^* \mathcal{E}_1] \cdot [\pi'_2^* \mathcal{E}_2]$$

where $\mathcal{E}_1, \mathcal{E}_2$ are $G$-linearized locally free coherent sheaves over $X_1, X_2$ respectively.

Also recall the notion of “multiplicative formal group law” in section 7 of [G] (also see [G2]). A $(G, F)$-equivariant formal group law (over $R$) is multiplicative if its coproduct has the property

$$\Delta y(\epsilon) = y(\epsilon) \otimes 1 + 1 \otimes y(\epsilon) - v y(\epsilon) \otimes y(\epsilon)$$
for some element $v \in R$. In other words, $f_{1,1}^1 = -v$ and $f_{s,t}^1 = 0$ if $s$ or $t > 1$. Denote its representing ring by $L_G^m(F)$. Then we have a natural surjective map $L_G(F) \to L_G^m(F)$ (by sending $f_{1,1}^1$ to $-v$ and $f_{s,t}^1$ to zero if $s$ or $t > 1$). In addition, by Proposition 4.5 in [G2], there is a natural isomorphism

$$L_G^m(F)[v^{-1}] \to R(G)[v, v^{-1}]$$

which sends Euler classes $d(\alpha)_0 = e(\alpha)$ to Euler classes $e_K(\alpha) \overset{\text{def}}{=} (1 - \alpha^\vee)v^{-1}$ (by Remarks [5.2]. $L_G^m(F)$ is generated by $v$ and the Euler classes, so the map is uniquely determined). Hence, there is a natural ring homomorphism

$$\Phi_K : L_G(F) \to L_G^m(F) \to R(G)[v, v^{-1}].$$

Our main objective in this section is to define a canonical map from our equivariant algebraic cobordism theory $\Omega^G(\cdot)$ to the equivariant K-theory $K_0(G; -)[v, v^{-1}]$. To that end, we need some basic results in $K_0(G; -)$. As in $\Omega^G(\cdot)$, let $\text{End}(K_0(G; -)[v, v^{-1}])_{\text{fin}}$ be the $R(G)[v, v^{-1}]$-subalgebra of $\text{End}(K_0(G; -)[v, v^{-1}])$ generated by $c_K(L)$.

**Lemma 10.2.** Axioms (A1)-(A8) hold in the equivariant K-theory.

**Proof.** Follows from the definitions and some basic facts about $G$-equivariant sheaves. □

**Lemma 10.3.** For any $X \in G$-Sm, $\mathcal{L}, \mathcal{M} \in \text{Pic}^G(X)$ and character $\alpha$,

1. $c_K(O_X) = 0$
2. $c_K(\alpha) = e_K(\alpha)$
3. $c_K(\mathcal{L} \otimes \mathcal{M}) = c_K(\mathcal{L}) + c_K(\mathcal{M}) - v c_K(\mathcal{L})c_K(\mathcal{M})$
4. $c_K(\mathcal{L}^\vee) = \sigma_K \cdot c_K(\mathcal{L})$ for some $\sigma_K \in \text{End}(K(G; X)[v, v^{-1}])_{\text{fin}}$.

**Proof.** Part (1), (2), (3) follow directly from the definition. Part (4) is an analogue of Proposition [5.3]. □

**Lemma 10.4.** Suppose $X \in G$-Sm is $G$-irreducible and $\mathcal{L}$ is a sheaf in $\text{Pic}^G(X)$. Moreover, if there exists an invariant section $s \in H^0(X, \mathcal{L})^G$ that cuts out an invariant smooth divisor $Z$ on $X$, then the following equality holds in $K_0(G; X)[v, v^{-1}]$:

$$c_K(\mathcal{L}) = i_*[O_Z]$$

where $i : Z \hookrightarrow X$ is the immersion.

**Proof.** It follows from the facts that $\mathcal{L} \cong O_X(Z)$, the functor $i_*$ is exact and the following sequence is exact:

$$0 \to O_X(-Z) \to O_X \to i_*O_Z \to 0.$$ □

Next, we will show that the double point relation holds in $K_0(G; -)[v, v^{-1}]$ by following the same recipe as in section [5].
Lemma 10.5. Suppose $Y$ is an object in $G$-$Sm$, $E_1$, $E_2$ are two invariant smooth divisors on $Y$ with transverse intersection $D$. Then, as elements in $K_0(G;D)[v,v^{-1}]$, 

$$-v = -p!\left[\mathcal{O}_D\right]$$

where $p : \mathbb{P}_D \overset{\text{def}}{=} \mathbb{P}(\mathcal{O}_D \oplus \mathcal{O}_D(E_1)) \to D$.

Proof. This result is an analogue of Lemma 10.4. First of all, the morphism $\mathcal{O}_D \oplus \mathcal{O}_D(E_1) \to \mathcal{O}_D(E_1)$ defines an invariant section $D \hookrightarrow \mathbb{P}_D$. Let $Y'$ be the blow up of $\mathbb{P}_D \times \mathbb{P}^1$ (trivial action on $\mathbb{P}^1$) along $D \times 0$. Then we have a map $Y' \to \mathbb{P}_D \times \mathbb{P}^1 \to \mathbb{P}^1$. By Lemma 10.4 we have

$$i_\infty![\mathcal{O}_{Y'}] = c_K(\mathcal{O}_{Y'}(Y'_\infty))$$

where $i_\infty : Y'_\infty \hookrightarrow Y'$

$$= c_K(\mathcal{O}_{Y'}(Y'_0))$$

where $E$ is the exceptional divisor

$$= c_K(\mathcal{O}_{Y'}(\mathbb{P}_D)) + c_K(\mathcal{O}_{Y'}(E)) - v c_K(\mathcal{O}_{Y'}(\mathbb{P}_D)) c_K(\mathcal{O}_{Y'}(E)), $$

which is, by Lemma 10.4, equal to

$$i_{\mathbb{P}_D}[\mathcal{O}_{\mathbb{P}_D}] + i_E[\mathcal{O}_E] - v c_K(\mathcal{O}_{Y'}(\mathbb{P}_D)) \cdot i_E[\mathcal{O}_E]$$

where $i_{\mathbb{P}_D}$ and $i_E$ are the corresponding immersions. Moreover,

$$c_K(\mathcal{O}_{Y'}(\mathbb{P}_D)) \cdot i_E[\mathcal{O}_E] = i_E(c_K(\mathcal{O}_E(\mathbb{P}_D))) = i_E \circ i_{\mathbb{P}_D \cap E_1}[\mathcal{O}_{\mathbb{P}_D \cap E}]$$

So, we have

$$i_\infty![\mathcal{O}_{Y'}] = i_{\mathbb{P}_D}[\mathcal{O}_{\mathbb{P}_D}] + i_E[\mathcal{O}_E] - v i_E \circ i_{\mathbb{P}_D \cap E_1}[\mathcal{O}_{\mathbb{P}_D \cap E}]$$

Notice that $Y'_\infty \cong E \cong \mathbb{P}_D$ and $\mathbb{P}_D \cap E \cong D$. By pushing the above equality down to $K_0(G;D)[v,v^{-1}]$, we have

$$p![\mathcal{O}_D] = p![\mathcal{O}_{\mathbb{P}_D}] + p![\mathcal{O}_{\mathbb{P}_D}] - v$$

and we are done. \hfill \Box

Lemma 10.6. Suppose $A$, $B$, $C$ are invariant smooth divisors on $Y \in G$-$Sm$ such that $A + B \sim C$, $C$ is disjoint from $A \cup B$ and $A + B + C$ is a reduced strict normal crossing divisor. Then, as elements in $K_0(G;Y)[v,v^{-1}]$, 

$$i_C![\mathcal{O}_C] = i_A![\mathcal{O}_A] + i_B![\mathcal{O}_B] - i_D \circ p![\mathcal{O}_{\mathbb{P}_D}]$$

where $D \overset{\text{def}}{=} A \cap B$, $i_A$, $i_B$, $i_C$ and $i_D$ are the corresponding immersions and $p : \mathbb{P}_D \overset{\text{def}}{=} \mathbb{P}(\mathcal{O}_D \oplus \mathcal{O}_D(A)) \to D$.  

Proof. By Lemma 10.4, we have
\[
i_C![\mathcal{O}_C] = c_K(\mathcal{O}_Y(C))
\]
\[
= c_K(\mathcal{O}_Y(A + B))
\]
\[
= c_K(\mathcal{O}_Y(A)) + v c_K(\mathcal{O}_Y(B)) - v c_K(\mathcal{O}_Y(A))c_K(\mathcal{O}_Y(B))
\]
\[
i_A![\mathcal{O}_A] + i_B![\mathcal{O}_B] - v i_B! \circ j_d[\mathcal{O}_D]
\]
where \(j : D \hookrightarrow B\). The result then follows from Lemma 10.5. \(\square\)

Hence, the double point relation holds in \(K_0(G; Z)[v, v^{-1}]\), so does the blow up relation. We also need an analogue of Proposition 6.1.

Lemma 10.7. Suppose \(\text{char } k = 0\). For any \(G\)-irreducible \(Y \in G\)-Sm, \(\mathcal{L} \in \text{Pic}^G(Y)\) and \(S\) as in (7),
\[
V^n_{K,S}(\mathcal{L}) = 0,
\]
as elements in \(K_0(G; Y)[v, v^{-1}]\), for sufficiently large \(n\).

Proof. See the proof of Proposition 6.1. \(\square\)

Now, suppose \(\text{char } k = 0\). For any \(X \in G\)-Sm, since \(K_0(G; X)\) can be considered as a \(K_0(G; \text{Spec } k)\)-module and \(K_0(G; \text{Spec } k) \cong R(G)\), we have a natural map
\[
\Psi_K : R(G)[v, v^{-1}] \otimes_{\mathbb{Z}[G]} \mathbb{Z}^G(X) \rightarrow K_0(G; X)[v, v^{-1}]
\]
declared by sending \(b \otimes \sum_I a_I [f : Y \rightarrow X, V^i_{S_1}(\mathcal{L}_1), \ldots, V^i_{S_r}(\mathcal{L}_r)]\) to
\[
\sum_I b \Phi_K(a_I) f_1(V^i_{K,S_1}(\mathcal{L}_1)) \cdots V^i_{K,S_r}(\mathcal{L}_r)),
\]
which is actually a finite sum by Lemma 10.7.

Theorem 10.8. Suppose \(\text{char } k = 0\). Then \(\Psi_K\) defines a canonical, surjective, \(R(G)[v, v^{-1}]\)-module homomorphism
\[
\Psi_K : R(G)[v, v^{-1}] \otimes_{\mathbb{Z}[G]} \Omega^G(X) \rightarrow K_0(G; X)[v, v^{-1}],
\]
for any \(X \in G\)-Sm, and it commutes with the four basic operations.

Proof. First of all, it is not hard to see that \(\Psi_K\) commutes with the four basic operations. By Lemma 10.4, the map \(\Psi_K\) respects the (Sect) axiom. For the (EFGL) axiom, we need to show that, as elements in \(K_0(G; Y)[v, v^{-1}]\) where \(Y \in G\)-Sm is \(G\)-irreducible,
\[
V^i_K(\mathcal{L}) \cdot V^j_K(\mathcal{L}) = \sum_{s \geq 0} b^{i,j}_s V^s_K(\mathcal{L}),
\]
\[
V^i_K(\mathcal{L} \otimes \alpha) = \sum_{s \geq 0} d(\alpha)^i_s V^s_K(\mathcal{L}),
\]
\[
V^i_K(\mathcal{L} \otimes \mathcal{M}) = \sum_{s,t \geq 0} f^{i}_{s,t} V^{s}_K(\mathcal{L}) \cdot V^{t}_K(\mathcal{M}),
\]
where \( b_{s,l}^{i,j}, d(\alpha)^i_s, f_{s,t} \) are considered to be elements in \( R(G)[v, v^{-1}] \) via \( \Phi_K \).

Notice that

\[ V_K^i(\mathcal{L}) \cdot V_K^j(\mathcal{L}) = V_K^{i+1}(\mathcal{L}) \cdot c_K(\mathcal{L} \otimes \alpha) \cdot V_K^j(\mathcal{L}) \]

\( \text{where } \beta = \sum_{j+1} \otimes \alpha_i \)

\[ V_K^{i+1}(\mathcal{L}) \cdot (c_K(\mathcal{L} \otimes \alpha_j) + c_K(\beta) - v c_K(\beta) c_K(\mathcal{L} \otimes \alpha_j)) \cdot V_K^j(\mathcal{L}) \]

\[ = e_K(\beta) \cdot V_K^{i+1}(\mathcal{L}) \cdot V_K^j(\mathcal{L}) + (1 - v e_K(\beta)) \cdot V_K^{i+1}(\mathcal{L}) \cdot V_K^{j+1}(\mathcal{L}) \]

So, inductively, we can write

\[ V_K^i(\mathcal{L}) \cdot V_K^j(\mathcal{L}) = \sum_{s \geq 0} B_s^{i,j} V_K^s(\mathcal{L}) \]

for some elements \( B_s^{i,j} \in R(G)[v, v^{-1}] \) uniquely determined by the above process. Thus, it is enough to show \( B_s^{i,j} = b_{s,l}^{i,j} \).

Consider the multiplicative formal group law

\[ R(G)[v, v^{-1}]\{\{F\}\} \cong L_C^m(F)[v^{-1}]\{\{F\}\}. \]

By Lemma 2.1 in \([G2]\),

\[ l_\alpha y(\epsilon) = y(\alpha) = e(\alpha) + (1 - ve(\alpha)) y(\epsilon). \]

In other words, \( d(\alpha)^1_0 = e(\alpha), d(\alpha)^1_1 = 1 - ve(\alpha) \) and \( d(\alpha)^1_s = 0 \) when \( s > 1 \). Also, if we apply \( l_\beta \) on the above equation, we have

\[ l_\beta y(\alpha) = e(\alpha) + (1 - ve(\alpha)) l_\beta y(\epsilon). \]

Therefore,

\[ l_\alpha y(\beta) = e(\alpha) + (1 - ve(\alpha)) y(\beta). \]

Now, we apply an analogue of the procedure in \([11]\) to the element \( y(V^i) \cdot y(V^j) \in L_C^m(F)[v^{-1}]\{\{F\}\} : \)

\[ y(V^i) \cdot y(V^j) = y(V^{i-1}) \cdot y(\alpha_i) \cdot y(V^j) \]

\[ = y(V^{i-1}) \cdot l_\beta y(\alpha_{j+1}) \cdot y(V^j) \]

\( \text{where } \beta = \sum_{j+1} \otimes \alpha_i \)

\[ = y(V^{i-1}) \cdot (e(\beta) + (1 - ve(\beta)) y(\alpha_{j+1})) \cdot y(V^j) \]

by equation \([12]\)

\[ = e(\beta) \cdot y(V^{i-1}) \cdot y(V^j) + (1 - ve(\beta)) \cdot y(V^{i-1}) \cdot y(V^{j+1}). \]
Again, inductively, we can write

\[ y(V^i) \cdot y(V^j) = \sum_{s \geq 0} B^{i,j}_s y(V^s) \]

for some elements \( B^{i,j}_s \in L_G^m(F)[v^{-1}] \). Since \( e(\beta) \) is identified with \( e_K(\beta) \), we have \( B^{i,j}_s = B^{i,j}_s \). Also,

\[ y(V^i) \cdot y(V^j) = \sum_{s \geq 0} b^{i,j}_s y(V^s) \]

by definition. Since \( \{y(V^s)\} \) is a basis, \( b^{i,j}_s = B^{i,j}_s \).

Similarly, we have

\[ V_K^i(\mathcal{L} \otimes \alpha) = V_K^{i-1}(\mathcal{L} \otimes \alpha) \cdot c_K(\mathcal{L} \otimes \beta) \]

where \( \beta \overset{\text{def}}{=} \alpha \otimes \alpha_i \)

\[ = e_K(\beta) \cdot V_K^{i-1}(\mathcal{L} \otimes \alpha) + (1 - ve_K(\beta)) \cdot V_K^{i-1}(\mathcal{L} \otimes \alpha) \cdot c_K(\mathcal{L}). \]

By induction, \( V_K^{i-1}(\mathcal{L} \otimes \alpha) \) can be expressed by \( \{V_K^s(\mathcal{L})\} \) with uniquely determined coefficients and

\[ V_K^s(\mathcal{L}) \cdot c_K(\mathcal{L}) = V_K^s(\mathcal{L}) \cdot V_K^1(\mathcal{L}) = \sum_t b_t V_K^t(\mathcal{L}). \]

Hence,

\[ V_K^i(\mathcal{L} \otimes \alpha) = \sum_{s \geq 0} D(\alpha)^i_s V_K^s(\mathcal{L}) \]

for some uniquely determined elements \( D(\alpha)^i_s \in R(G)[v, v^{-1}] \). On the other hand, in \( L_G^m(F)[v^{-1}] \{\{F\}\} \), we have

\[ l_\alpha y(V^i) = l_\alpha y(V^{i-1}) \cdot l_\alpha y(\alpha_i) \]

\[ = l_\alpha y(V^{i-1}) \cdot l_\beta y(\epsilon) \]

where \( \beta \overset{\text{def}}{=} \alpha \otimes \alpha_i \)

\[ = e(\beta) \cdot l_\alpha y(V^{i-1}) + (1 - ve(\beta)) \cdot l_\alpha y(V^{i-1}) \cdot y(V^1). \]

Hence, by the same reason, \( D(\alpha)^i_s = d(\alpha)^i_s \).

Again,

\[ V_K^i(\mathcal{L} \otimes \mathcal{M}) = V_K^{i-1}(\mathcal{L} \otimes \mathcal{M}) \cdot c_K(\mathcal{L} \otimes \mathcal{M} \otimes \alpha_i) \]

\[ = e_K(\alpha_i) \cdot V_K^{i-1}(\mathcal{L} \otimes \mathcal{M}) + (1 - ve_K(\alpha_i)) \cdot V_K^{i-1}(\mathcal{L} \otimes \mathcal{M}) \cdot c_K(\mathcal{L} \otimes \mathcal{M}). \]

By induction, we can express \( V_K^{i-1}(\mathcal{L} \otimes \mathcal{M}) \) in terms of \( V_K^s(\mathcal{L})V_K^t(\mathcal{M}) \). Also,

\[ c_K(\mathcal{L} \otimes \mathcal{M}) = V_K^1(\mathcal{L}) + V_K^1(\mathcal{M}) - vV_K^1(\mathcal{L})V_K^1(\mathcal{M}). \]

Express the products by \( b^{i,j}_s \) as before and we get

\[ V_K^i(\mathcal{L} \otimes \mathcal{M}) = \sum_{s,t \geq 0} F^{i}_{s,t} V_K^s(\mathcal{L}) V_K^t(\mathcal{M}) \]
for some uniquely determined elements $f_{s,t}^i$. On the other hand, in $\mathbb{L}_{\mathbb{G}_m(F)}^{\bullet}([F])$, we have

$$\Delta y(V^i) = \Delta y(V^{i-1}) \cdot \Delta y(\alpha_i)$$

$$= \Delta y(V^{i-1}) \cdot (\Delta(e(\alpha_i) + (1 - ve(\alpha_i))y(\epsilon))$$

$$= e(\alpha_i) \cdot \Delta y(V^{i-1}) + (1 - ve(\alpha_i)) \cdot \Delta y(V^{i-1}) \cdot \Delta y(\epsilon).$$

Again, by induction, we can express $\Delta y(V^{i-1})$ in terms of $y(V^s) \otimes y(V^t)$. Also, we have

$$\Delta y(\epsilon) = y(\epsilon) \otimes 1 + 1 \otimes y(\epsilon) - v \cdot y(\epsilon) \otimes y(\epsilon).$$

Hence, $f_{s,t}^i = f_{s,t}^j$ by the same reason. That finishes the proof of the well-definedness of

$$\Psi_K : R(G)[v, v^{-1}] \otimes_{\mathbb{L}_{\mathbb{G}_m(F)}} \Omega^G(X) \to K_0(G; X)[v, v^{-1}].$$

For the surjectivity of $\Psi_K$, we proceed by induction on the dimension of $X$. Without loss of generality, we may assume $X$ to be $G$-irreducible. As a $R(G)[v, v^{-1}]$-module, $K_0(G; X)[v, v^{-1}]$ is generated by elements of the form $[E]$ where $E$ is a $G$-linearized locally free sheaf of finite rank over $X$. If $\dim X = 0$, then $E$ splits. So, we may assume the rank of $E$ is 1. Then, we have

$$(13) \quad \Psi_K([\mathbb{I}_X] - v [\mathbb{I}_X, E^\vee]) = [\mathcal{O}_X] - v e_K(E^\vee) = [E].$$

That handles the $\dim X = 0$ case.

Suppose $\dim X > 0$. By the blow up relation in the equivariant K-theory, if $Z \subseteq X$ is an invariant, smooth closed subscheme, then we have

$$\pi_1[\mathcal{O}_Z] - [\mathcal{O}_X] = -i_{Z!} \circ p_1![\mathcal{O}_{\mathbb{P}_1}] + i_{Z!} \circ p_2![\mathcal{O}_{\mathbb{P}_2}]$$

where $\pi : \tilde{X} \to X$ is the blow up of $X$ along $Z$, $i_Z : Z \hookrightarrow X$ is the immersion, $p_1$ and $p_2$ are the maps $\mathbb{P}_1 = \mathbb{P}(\mathcal{O} \oplus \mathcal{N}_{\mathbb{P}_1}) \to Z$ and $\mathbb{P}_2 = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)) \to \mathbb{P}(\mathcal{N}_{\mathbb{P}_2}) \to Z$ respectively. Since $\dim \pi = \dim(i_Z \circ p_1) = \dim(i_Z \circ p_2) = 0$, if we multiply the above equation by $[E]$, we have

$$\pi_1[\pi^*E] - [E] = -[E] \cdot i_{Z!} \circ p_1![\mathcal{O}_{\mathbb{P}_1}] + [E] \cdot i_{Z!} \circ p_2![\mathcal{O}_{\mathbb{P}_2}]$$

$$= -i_{Z!} \circ p_1![p_1^*(E|_Z)] + i_{Z!} \circ p_2![p_2^*(E|_Z)].$$

Therefore,

$$\pi_1[\pi^*E] - [E] = i_{Z!}(x)$$

for some $x \in K_0(G; Z)[v, v^{-1}]$. By the induction assumption, the right hand side is in the image of $\Psi_K$. So it is enough to show that $[\pi^*E]$ lies inside the image of $\Psi_K$. By Theorem 3.37, we may assume $E$ splits. Hence, it is enough to show the surjectivity when $\text{rk} E = 1$, which follows from equation (13). That finishes the proof of Theorem 10.8. \qed
11. Realization functor

In this section, we will establish a realization functor from our equivariant algebraic
co bordism theory to Tom Dieck’s equivariant complex cobordism theory, when $k = \mathbb{C}$ and
$G$ is a compact abelian Lie group.

Recall from section 1 in [T] that the Tom Dieck’s equivariant complex cobordism abelian
group for a $G$-manifold $X$ is defined to be

$$MU^{2k}_G(X) \overset{\text{def}}{=} \widetilde{MU}^{2k}_G(X^+) \overset{\text{def}}{=} \lim_{\longrightarrow V} \left[S^V \wedge X^+, MU(\dim_G V + k, G)\right]$$

where $V$ runs through all finite dimensional complex $G$-representations, $S^V$ is the one-
point compactification of $V$, $MU(r, G)$ is the Thom space of the universal rank $r$ complex
$G$-vector bundle and $[-,-]$ denotes pointed $G$-homotopy set.

This theory has a naturally defined pull-back for any $G$-map and a cup product :

$$MU^{2r}_G(X) \times MU^{2s}_G(Y) \xrightarrow{\cup} MU^{2r+2s}_G(X \times Y).$$

With this cup product, $MU_G(X)$ becomes a commutative ring, or $MU_G$-algebra, with unity
$[X^+ \to MU(0, G) = S^0]$. Also, pull-back is a ring homomorphism. For any equivariant line
bundle $L \to X$, we have an element

$$c_1(L) \overset{\text{def}}{=} [X^+ \xrightarrow{a} M(L) \xrightarrow{b} MU(1, G)] \in MU^2_G(X)$$

where $M(L)$ is the Thom space of $L$, $a$ is given by the zero section and $b$ is given by the
classifying map. We can then define a (first) Chern class operator

$$c_1(L) : MU^{2k}_G(X) \to MU^{2k+2}_G(X)$$

by sending $x$ to $c_1(L) \cup x$. It also has a naturally defined external product :

$$MU^{2r}_G(X) \times MU^{2s}_G(Y) \xrightarrow{\wedge} MU^{2r+2s}_G(X \times Y).$$

It then forms a multiplicative equivariant cohomology theory.

For any $r$-dimensional complex $G$-representation $W$, we have a suspension isomorphism

$$MU^{2k}_G(X) \xrightarrow{\sim} \widetilde{MU}^{2k+2r}_G(S^W \wedge X^+).$$

More generally, for any rank $r$ equivariant vector bundle $E \to X$, the map $[M(E) \to MU(r, G)]$ given by the classifying map defines an element in $\widetilde{MU}^{2r}_G(M(E))$, which is called
the Thom class of $E$ and will be denoted by $Th(E)$. Then, we have the Thom isomorphism

$$MU^{2k}_G(X) \xrightarrow{\sim} \widetilde{MU}^{2k+2r}_G(M(E))$$

defined by sending $x$ to $d^*(x \wedge Th(E))$ where $d : M(E) \to X^+ \wedge M(E)$ is given by projection
(see section 2 in [FuKa]). This theory also has a canonical $MU_G$-orientation (following the terminology in [Kat]) :

$$Th(O_{\mathcal{P}(U)}(1)) \in \widetilde{MU}^2_G(M(O_{\mathcal{P}(U)}(1))) = \widetilde{MU}^2_G(MU(1, G))$$
where $\mathcal{O}_{\mathbb{P}(\mathcal{U})}(1)$ is the universal complex $G$-line bundle. Indeed, $MU_G(-)$ is the universal complex oriented cohomology theory with orientation in degree 2 (Theorem 1.2 in [CGKr2]).

As pointed out in Example 11.3 in [CGKr], for any complex oriented cohomology theory $E^*_G(-)$, the pair $E_G$, $E_G(\mathbb{P}(\mathcal{U}))$ with $y(\epsilon)$ being the complex orientation, defines a $G$-formal group law. In particular, for a complete $G$-flag $F$, the pair $MU_G$, $MU_G(\mathbb{P}(\mathcal{U})) \cong MU_G\{\{F\}\}$ (as $MU_G$-module, by Theorem 9.6 in [G]) forms a $(G,F)$-formal group law. Hence, there is a canonical ring homomorphism

$$\nu : \mathbb{L}_G(F) \to MU_G.$$ 

By Theorem 13.1 in [G], $\nu$ is surjective when $G$ is finite and it is conjectured to be an isomorphism for all compact abelian Lie group (see section 13 in [G]).

Our goal in this section is to define a realization functor $\Psi_{Top} : \Omega_G(-) \to MU_G(-),$ which commutes with the four basic operations. To this end, we first need to have a projective push-forward (also known as Gysin homomorphism) in $MU_G(-)$.

Recall the following definition of Gysin homomorphism from section 2 in [Kat]. Suppose $f : X \to Y$ is a map in $G$-$Sm$ such that $X$, $Y$ are both equidimensional and projective. (In [Kat], $X$, $Y$ are required to be $MU_G$-oriented, i.e., their tangent bundles are $MU_G$-oriented. But in $MU_G(-)$, all equivariant vector bundles have Thom classes and hence, $MU_G$-oriented). Define the Gysin homomorphism

$$f^! : MU_G^{2k}(X) \to MU_G^{2k-2\dim f}(Y)$$

as the composition

$$MU_G^{2k}(X) \xrightarrow{a} MU_G^{2k+2r}(M(N_f)) \xrightarrow{b^*} MU_G^{2k+2r}(S^{V_X} \wedge Y^+) \xrightarrow{c} MU_G^{2k-2\dim f}(Y)$$

where $V_X$ is a $G$-representation, $X \hookrightarrow V_X$ is an equivariant embedding (in the sense of Topology), $N_f$ is the rank $r$ normal bundle of the embedding $X \hookrightarrow Y \times V_X$, which will be identified with the tubular neighborhood of $X$ inside $Y \times V_X$ and it is assumed to be inside $Y \times \text{Int } D(V_X)$ (interior of the unit disk), $a$ is the Thom isomorphism, $b$ is the collapsing map and $c$ is the suspension isomorphism. By Lemma 2.2 in [Kat], the above definition is independent of all choices made.

Since we would like to consider $MU_G(-)$ as a theory with four basic operations: projective push-forward, smooth pull-back, Chern class operator and external product, we will focus on the full subcategory of $G$-$Sm$ consisting of projective objects, denoted by $G$-$\text{ProjSm}$, for the rest of this section.

**Proposition 11.1.** For the theory $MU_G(-)$ on $G$-$\text{ProjSm}$,
(1) \( c(f^*L) = f^*(c(L)) \)

(2) If \( x \in MU_G(X) \) and \( y \in MU_G(Y) \), then \( x \wedge y = \pi_1^*(x) \cup \pi_2^*(y) \)

(3) If \( x, y \in MU_G(X) \), then \( x \cup y = \Delta^*(x \wedge y) \) where \( \Delta \) is the diagonal map

(4) \( f_!(x \cup f^*(y)) = f_!(x) \cup y \)

(5) (D1)-(D4), (A1)-(A8) hold (see [LMo])

Proof. (1), (2), (3) follow directly from definition. (D2), (D3), (D4) are what we call pull-back, Chern class operator and external product, which we just defined.

(D1) and (4) : From Lemma 2.2 in [Kat].

(A1) : Follows from the fact that \( MU_G(\cdot) \) is a cohomology theory.

(A2) : Consider the following diagram

\[
\begin{array}{ccc}
W & \xrightarrow{g'} & X \\
\downarrow f' & & \downarrow f \\
Y & \xrightarrow{g} & Z
\end{array}
\]

which is Cartesian and \( W, X, Y, Z \) are objects in \( G\text{-ProjSm} \). Choose equivariant embeddings \( X \hookrightarrow V' \) and \( Y \hookrightarrow V_Y \). Then,

\[
W = X \times_Z Y \hookrightarrow X \times Y \hookrightarrow V' \times V_Y \overset{\text{def}}{=} V_W.
\]

By using the embedding \( X \hookrightarrow V' \hookrightarrow V_W \overset{\text{def}}{=} V_X \), we then have a Cartesian diagram

\[
\begin{array}{ccc}
W & \longrightarrow & X \\
\downarrow & & \downarrow \\
Y \times V_W & \longrightarrow & Z \times V_X
\end{array}
\]

Since \( W \) is smooth, the set of regular value of \( Y \times V_W \to Z \times V_W \) contains \( X \). Therefore, \( N_{f'} \cong g'^*N_f \).

For an element \( x = [S^V \wedge X^+ \to MU_G(k, G)] \in MU_G(X) \),

\[
g^* \circ f_!(x) = [S^V \wedge S^{V_X} \wedge Y^+ \to S^V \wedge S^{V_X} \wedge Z^+ \\
\to S^V \wedge M(N_f) \to S^V \wedge X^+ \wedge M(N_f) \to MU(r + k, G)]
\]

where \( r = \text{rk} N_f \). On the other hand,

\[
f'_! \circ g'^*(x) = [S^V \wedge S^{V_W} \wedge Y^+ \to S^V \wedge M(N_{f'}) \to S^V \wedge W^+ \wedge M(N_{f'}) \to MU(r + k, G)]
\]

So they agree.

(A3) : Follows from (1) and (4).

(A4) : Follows from (1) and the fact that pull-back is a ring homomorphism.

(A5) : Follows from the fact that \( MU_G(X) \) is a commutative ring.

(A6) : If \( X \hookrightarrow V_X, Y \hookrightarrow V_Y \), then \( X \times Y \hookrightarrow V_X \times V_Y \) and \( N_{f \times g} \cong (\pi_1^*N_f) \oplus (\pi_2^*N_g) \).

(A7) : Follows from definition.

(A8) : Follows from (2).
Next, we need an analogue of Proposition 6.1 in $MU_G(-)$. For an equivariant line bundle $L$ over $Y \in G\text{-ProjSm}$, let
\[ V^n(L) \overset{\text{def}}{=} c(L \otimes \alpha_1) \circ \cdots \circ c(L \otimes \alpha_n) \]
(Here we are considering $\alpha_i$ as an equivariant line bundle over $Y$).

**Lemma 11.2.** (Sect), (EFGL), double point relation, blow up relation, extended double point relation hold in $MU_G(-)$. Moreover, for all $G$-irreducible $Y \in G\text{-ProjSm}$, $L$ equivariant line bundle over $Y$, finite set $S$ as in [4],
\[ V^n_S(L)(1_Y) = 0 \]
in $MU_G(Y)$, for sufficiently large $n$.

**Proof.** For (Sect), let $D \subseteq Y \in G\text{-ProjSm}$ be an invariant smooth divisor and $L$ is the line bundle corresponding to $O_Y(D)$. We need to show $c(L) = i_!(1_D)$ where $i : D \hookrightarrow Y$ is the closed immersion. Choose an equivariant embedding $D \hookrightarrow V_D$. Let $r$ be the rank of its normal bundle $N_i$. Then,
\[ i_!(1_D) = [S^{V_D} \wedge Y^+ \to M(N_i) \to MU(r,G)] \]
and
\[ c(L) = [Y^+ \to M(L) \to MU(1,G)] = [S^{V_D} \wedge Y^+ \to M(L \times V_D) \to MU(r,G)]. \]
So they agree.

For (EFGL), $Y \in G\text{-ProjSm}$ and $L$ is an equivariant line bundle over $Y$. Let $cl_L : Y \to \mathbb{P}(U)$ be the classifying map. Then, $cl_L^* \langle O(1) \rangle \cong L$. Since $y(\epsilon) = c(O(1)) \in MU_G(\mathbb{P}(U))$, $y(\alpha) = c(O(1) \otimes \alpha)$. We have a canonical Hopf algebra homomorphism
\[ \mathbb{L}_G(F)\{\{F\}\} \to MU_G(\mathbb{P}(U)) \cong MU_G\{\{F\}\}. \]
Since
\[ y(V^i)y(V^j) = \sum b^i_j y(V^*) \]
in $\mathbb{L}_G(F)\{\{F\}\}$, we have
\[ V^i(O(1))V^j(O(1)) = \sum b^i_j V^*(O(1)) \]
in $MU_G(\mathbb{P}(U))$ (under the identification of coefficients by $\nu$). By pulling back via $cl_L$, we have
\[ V^i(L)V^j(L) = \sum b^i_j V^*(L) \]
(and the well-definedness of the right hand side). The rest of (EFGL) follows from similar arguments.

The other properties follow from the same proofs as in section 5 and Proposition 6.1 with the correspondence
\[ [f : Y \to X] = f_!(1_Y). \]
\[ \square \]
Now, for all $X \in G$-ProjSm, we define a $L_G(F)$-module homomorphism
\[
\Psi_{\text{Top}} : L_ZG,F(X) \to MU_G(X)
\]
by sending $a[f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r]$ to $\nu(a)f_i(c(L_1) \cdots c(L_r))$ where $L_i$ is the equivariant line bundle corresponding to $\mathcal{L}_i$. The case of infinite cycles is covered because of Lemma 11.2.

**Theorem 11.3.** $\Psi_{\text{Top}}$ descends to a canonical $L_G(F)$-homomorphism
\[
\Omega_G(X) \to MU_G(X),
\]
for all $X \in G$-ProjSm, and it commutes with the four basic operations. When $X$ is equidimensional, there is a canonical grading on $\Omega_G(X)$ and $\Psi_{\text{Top}} : \Omega^*_G(X) \to MU^*_G(X)$.

**Proof.** It clearly commutes with the operations. By Lemma 11.2 (Sect) and (EFGL) hold in $MU_G(-)$ and hence, $\Psi_{\text{Top}} : \Omega_G(X) \to MU_G(X)$ is well-defined.

For the (cohomological) grading, first of all, $L_G(F)$ has a natural grading, by Remark 8.1. We then define
\[
\deg(a[f : Y \to X, \mathcal{L}_1, \ldots, \mathcal{L}_r]) = \deg a + r - \dim f
\]
and
\[
\deg(\nu(a)f_i(c(L_1) \cdots c(L_r))) = \deg(\nu(a)) + 2r - 2\dim f.
\]
Therefore, it is enough to show $\nu : L_G(F)^* \to MU^*_G$. Notice that $L_G(F)$ is generated by $e(\alpha)$ and $f^1_{s,t}$. Moreover, $\deg e(\alpha) = 1$ and
\[
\nu(e(\alpha)) = \Psi_{\text{Top}}(c(\alpha)[[\text{Spec } k]]) = c(\alpha),
\]
which is of degree 2. It remains to show the statement for $f^1_{s,t}$.

Let $X = \mathbb{P}(\alpha_1^\vee \oplus \alpha_2^\vee)$ and $Y = X \times X$. Then, in $\Omega^G(Y)$, we have
\[
\begin{align*}
V^1(O(1,1))[[\mathcal{Y}]] & = \sum_{s,t} f^1_{s,t} V^s(O(1,0))V^t(O(0,1))[[\mathcal{Y}]] \\
& = c(O(1,0))[[\mathcal{Y}]] + c(O(0,1))[[\mathcal{Y}]] + f^1_{1,1} V^1(O(1,0))V^1(O(0,1))[[\mathcal{Y}]] \\
& + f^2_{2,1} V^2(O(1,0))V^1(O(0,1))[[\mathcal{Y}]] + \cdots \\
& = [\mathbb{P}(\alpha_2^\vee) \times X \leftarrow Y] + [X \times \mathbb{P}(\alpha_2^\vee) \leftarrow Y] + f^1_{1,1} [\mathbb{P}(\alpha_2^\vee) \times \mathbb{P}(\alpha_2^\vee) \leftarrow Y]
\end{align*}
\]
Push this equality down to Spec $k$, we have
\[
[Y, O(1,1)] = 2[X] + f^1_{1,1}.
\]
Hence,
\[ \nu(f_{1,1}^1) = \Psi_{\text{Top}}([Y, \mathcal{O}(1,1)] - 2[X]). \]
It shows that \( \deg \nu(f_{1,1}^1) = -2 = 2 \deg f_{1,1}^1 \). The result for general \( f_{s,t}^1 \) follows from similar arguments, inductively. \( \square \)

**Remarks 11.4.** The proof of Theorem 11.3 also gives a geometric description to the elements \( f_{s,t}^1 \in \Omega^G(\text{Spec } k) \) and \( \nu(f_{s,t}^1) \in MU_G \).

### 12. Flag dependency

In this last section, we will show that our definition of \( \Omega^G(\mathcal{C}) \) is indeed independent of the choice of the complete \( G \)-flag \( F \). We will use the same notations and assumptions on \( G, k \) and \( F \) as in section 2 except that we will not assume \( \alpha_1 = \epsilon \) anymore.

Suppose \( F' \) is another complete \( G \)-flag given by
\[ 0 = W^0 \subseteq W^1 \subseteq W^2 \subseteq \cdots. \]
Denote the 1-dimensional characters \( W^i/W^{i-1} \) by \( \beta_i \). Again, we will not assume \( \beta_1 = \epsilon \).

According to the results in section 13 of [CGKr], there is a canonical map
\[ \phi : \mathbb{L}_G(F) \rightarrow \mathbb{L}_G(F') \]
defined as follow.

By definition, \( \mathbb{L}_G(F')\{\{F'\}\} \) is a \((G,F')\)-equivariant formal group law over \( \mathbb{L}_G(F') \). By Lemma 13.1 in [CGKr], it defines a \( G \)-equivariant formal group law over \( \mathbb{L}_G(F') \). By Lemma 13.2 in [CGKr], it then defines a \((G,F)\)-equivariant formal group law over \( \mathbb{L}_G(F') \). Then, \( \phi : \mathbb{L}_G(F) \rightarrow \mathbb{L}_G(F') \) is the unique map given by the universal property of \( \mathbb{L}_G(F) \). By symmetry, there is also a canonical map \( \phi' : \mathbb{L}_G(F') \rightarrow \mathbb{L}_G(F) \).

Notice that we have the following commutative diagram:

\[
\begin{array}{ccc}
R\{\{F\}\} & \xrightarrow{\overline{\phi}} & R'\{\{F\}\} \\
\uparrow & & \uparrow \\
R & \xrightarrow{\phi} & R'
\end{array}
\quad
\begin{array}{ccc}
R'\{\{F'\}\} & \xrightarrow{\overline{\psi}} & R'\{\{F'\}\} \\
\uparrow & & \uparrow \\
R' & \xrightarrow{\phi'} & R
\end{array}
\]

where \( R \overset{\text{def}}{=} \mathbb{L}_G(F), \ R' \overset{\text{def}}{=} \mathbb{L}_G(F') \), the maps \( \psi, \psi' \) are the isomorphisms given by Lemma 13.1 and 13.2 in [CGKr] and \( \overline{\phi}, \overline{\phi}' \) are the maps given by the universal properties of \( R, R' \) respectively. Observe that \( \overline{\phi}, \overline{\phi}', \psi \) and \( \psi' \) all preserve product, \( G^* \)-action and coproduct.

**Proposition 12.1.** The compositions \( \psi' \circ \overline{\phi} \circ \psi \circ \overline{\phi} \) and \( \phi' \circ \phi \) are both identity maps. In addition, \( \psi \circ \overline{\phi} \) and \( \phi \) are both isomorphisms.
Proof. First of all, let \( f \overset{\text{def}}{=} \psi' \circ \overline{\phi} \circ \psi \circ \overline{\phi} \). Since

\[
f(y(\alpha_1)) = \psi' \circ \overline{\phi}(l_{\alpha_1\beta_1}^{-}\ y(y(\beta_1))) = l_{\alpha_1\beta_1}^{-} \circ l_{\beta_1\alpha_1}^{+} \ y(\alpha_1) = y(\alpha_1)
\]

and

\[
y(V^i) = y(\alpha_1) \cdots y(\alpha_i) = y(\alpha_1) \cdot (l_{\alpha_1\alpha_2}^{-}\ y(\alpha_1)) \cdots (l_{\alpha_1\alpha_i}^{-}\ y(\alpha_1)),
\]

the map \( f \) fixes \( y(V^i) \). In \( R(\{F\}) \), we have

\[
y(V^i)y(V^j) = \sum_s b^i_{s,j}y(V^s).
\]

Then, \( f \) fixes the left hand side and sends the right hand side to \( \sum_s (\phi' \circ \phi(b^i_{s,j})) y(V^s) \). Therefore,

\[
\sum_s b^i_{s,j}y(V^s) = \sum_s (\phi' \circ \phi(b^i_{s,j})) y(V^s)
\]

and hence, \( \phi' \circ \phi(b^i_{s,j}) = b^i_{s,j} \). By similar arguments, \( \phi' \circ \phi \) fixes all structure constants and hence is an identity. By symmetry, \( \phi \circ \phi' \) is also an identity. So \( \phi \) is an isomorphism. Since \( f \) fixes \( y(V^i) \) and elements in \( R \), \( f \) is an identity. Again, by symmetry, \( \psi \circ \overline{\phi} \) is an isomorphism. \( \square \)

Let us denote the equivariant algebraic cobordism theories corresponding to \( F \) and \( F' \) by \( \Omega^{G,F}(-) \) and \( \Omega^{G,F'}(-) \) respectively. Moreover, define the theories \( \overline{\Omega^{G,F}(-)} \) and \( \overline{\Omega^{G,F'}(-)} \) by imposing the \((\text{Sect})\) and \((\text{EFGL})\) axioms on \( \overline{LZ^{G,F}(-)} \) and \( \overline{LZ^{G,F'}(-)} \) respectively. Notice that since the \( \mathbb{L}_G(F)\)-submodule of \( \overline{LZ^{G,F}(-)} \) corresponding to imposing the axioms is actually a submodule of \( \mathbb{L}Z^{G,F}(-) \), we have \( \Omega^{G,F}(-) \subseteq \overline{\Omega^{G,F}(-)} \) and similarly, \( \Omega^{G,F'}(-) \subseteq \overline{\Omega^{G,F'}(-)} \).

For an object \( X \in G \text{-} \text{Var} \), let

\[
\Psi_{F,F'} : \overline{LZ^{G,F}(X)} \to \overline{LZ^{G,F'}(X)}
\]

be the canonical map which sends \( a[f : Y \to X, \ldots] \) to \( \phi(a)[f : Y \to X, \ldots] \), which induces a map

\[
\Psi_{F,F'} : \mathbb{L}Z^{G,F}(X) \leftarrow \overline{LZ^{G,F}(X)} \xrightarrow{\Psi_{F,F'}} \overline{LZ^{G,F'}(X)} \to \Omega^{G,F'}(X).
\]

Our goal is to show that it descends to a map \( \Omega^{G,F}(X) \to \overline{\Omega^{G,F'}(X)} \) with image inside \( \Omega^{G,F'}(X) \).

First of all, for each infinite Chern class operator on \( \overline{LZ^{G,F}(-)} \), we need to define an associated infinite Chern class operator on \( \overline{LZ^{G,F'}(-)} \). For any \( i \geq 0 \) and \( S \) as in \((1)\), let

\[
y(V^i_S) \overset{\text{def}}{=} \prod_{1 \leq j \leq i} y(\alpha_j)
\]

as an element in \( \mathbb{L}_G(F)\{\{F\}\} \), if \( i \geq \max S \). Otherwise, set it to zero. Also, let \( a^i_{j,S} \in \mathbb{L}_G(F^j) \) be the unique coefficients satisfying the following equation :

\[
(14) \quad \psi \circ \overline{\phi}(y(V^i_S)) = \sum_{j \geq 0} a^i_{j,S} y(W^j).
\]
Lemma 12.2. For all \( j \geq 0 \) and \( S \) as in (1),
\[
a_{i,S}^{j} = 0
\]
for sufficiently large \( i \).

Proof. Since \( F' \) is a complete \( G \)-flag, for sufficiently large \( i \),
\[
\psi \circ \phi(y(V^i_S)) = \psi \circ \phi(y(W^{j+1})y(\gamma_1) \cdots y(\gamma_n))
\]
for some \( n \) and characters \( \gamma_k \)
\[
y(W^{j+1}) \sum_{k \geq 0} a'_k y(W^k),
\]
for some \( a'_k \in L_G(F') \). Therefore,
\[
\psi \circ \phi(y(V^i_S)) = \sum_{k,l \geq 0} a'_k b^{j+1,k}_{i,l} y(W^l).
\]
The result then follows from the fact that \( b^{j+1,k}_{i,l} = 0 \) for all \( k \) (by Proposition 14.1 in [CGKr]). \( \square \)

Now, we define an operator \( W^i_S(\mathcal{L}) \) on \( \Omega^{G,F}(X) \) as an analogue of \( V^i_S(\mathcal{L}) \) (use \( \beta_j \) instead of \( \alpha_j \)). Then, for each infinite Chern class operator \( \sigma = \sum_I a_I V^{i_1}_{S_1} (\mathcal{L}_1) \cdots V^{i_r}_{S_r} (\mathcal{L}_r) \) on \( \Omega^{G,F}(X) \), we define an associated infinite Chern class operator on \( \Omega^{G,F}(X) \) :
\[
\Psi_{F',F}(\sigma) \equiv \sum_{j} \sum_{I} \phi(a_I) a_{j_1}^{i_1,S_1} \cdots a_{j_r}^{i_r,S_r} W^{j_1}(\mathcal{L}_1) \cdots W^{j_r}(\mathcal{L}_r)
\]
(it is well-defined by Lemma 12.2).

Lemma 12.3. For all \( i \geq 0 \) and \( S \) as in (1),
\[
V^i_S(\mathcal{L}) = \sum_{j \geq 0} a_{j,S}^{i} W^{j}(\mathcal{L})
\]
as operators on \( \Omega^{G,F}(X) \).

Proof. If \( i < \text{max} S \), then the statement is trivially true. Suppose \( i \geq \text{max} S \). Denote the indices from 1 to \( i \) which is not in \( S \) by \( j_1, \ldots, j_n \). Then, on one hand,
\[
\psi \circ \phi(y(V^i_S)) = \psi \circ \phi(y(\alpha_{j_1}) \cdots y(\alpha_{j_n}))
\]
\[
= l'_{\alpha_{j_1}} \beta_{j_1}^{\gamma_1} y(W^1) \cdots l'_{\alpha_{j_n}} \beta_{j_n}^{\gamma_1} y(W^1)
\]
\[
= \left( \sum_{k_1} d'_{\alpha_{j_1}} \beta_{j_1}^{\gamma_1} k_1 y(W^{k_1}) \right) \cdots \left( \sum_{k_n} d'_{\alpha_{j_n}} \beta_{j_n}^{\gamma_1} k_n y(W^{k_n}) \right),
\]
which can then be expressed in terms of \( \{y(W^j)\} \) by the equation
\[
y(W^k) y(W^l) = \sum_p b^{k,l}_p y(W^p).
\]
On the other hand,  
\[
V^j_S(\mathcal{L}) = c(\mathcal{L} \otimes \alpha_{j_1}) \cdots c(\mathcal{L} \otimes \alpha_{j_n}) \\
= W^1(\mathcal{L} \otimes \alpha_{j_1} \beta_1^Y) \cdots W^1(\mathcal{L} \otimes \alpha_{j_n} \beta_1^Y) \\
= (\sum_{k_1} d'(\alpha_{j_1} \beta_1^Y)_{k_1} W^{k_1}(\mathcal{L})) \cdots (\sum_{k_n} d'(\alpha_{j_n} \beta_1^Y)_{k_n} W^{k_n}(\mathcal{L})),
\]
which can be expressed in terms of \{W^j(\mathcal{L})\} by the equation  
\[
W^k(\mathcal{L})W^l(\mathcal{L}) = \sum_p b^{k,l}_p W^p(\mathcal{L}).
\]
The result then follows from matching the coefficients and equation (14).

By the map \(\phi : \mathbb{L}_G(F) \to \mathbb{L}_G(F')\), we may consider \(\Omega^{G,F'}(\cdot)\) as a \(\mathbb{L}_G(F)\)-module.

**Proposition 12.4.** For any \(X \in G\text{-}Var\), \(\Psi_{F,F'}\) defines a canonical \(\mathbb{L}_G(F)\)-module isomorphism  
\[
\Psi_{F,F'} : \Omega^{G,F}(X) \to \Omega^{G,F'}(X)
\]
and it commutes with the projective push-forward, smooth pull-back, external product and the infinite Chern class operator, i.e., \(\Psi_{F,F'} \circ \sigma = \Psi_{F,F'}(\sigma) \circ \Psi_{F,F'}\).

**Proof.** The map  
\[
\Psi_{F,F'} : \mathbb{L}Z^{G,F}(\cdot) \to \Omega^{G,F'}(\cdot)
\]
clearly commutes with the projective push-forward, smooth pull-back and external product. For the infinite Chern class operator,  
\[
\Psi_{F,F'} \circ \sigma[\mathbb{I}_Y] = \Psi_{F,F'}(\sum_I a_I V^{ij}_{S_1}(\mathcal{L}_1) \cdots V^{ij}_{S_r}(\mathcal{L}_r)[\mathbb{I}_Y])
\]
\[
= \sum_I \phi(a_I) V^{ij}_{S_1}(\mathcal{L}_1) \cdots V^{ij}_{S_r}(\mathcal{L}_r)[\mathbb{I}_Y]
\]
\[
= \sum_I \sum_{j_1} \cdots \sum_{j_r} \phi(a_I a_{i_1,j_1,s_1} \cdots a_{i_r,j_r,s_r}) W^{ij_1}(\mathcal{L}_1) \cdots W^{ij_r}(\mathcal{L}_r)[\mathbb{I}_Y]
\]
by Lemma \[12.3\], which is equal to \(\Psi_{F,F'}(\sigma) \circ \Psi_{F,F'}[\mathbb{I}_Y]\). So, \(\Psi_{F,F'}\) commutes with the basic operations. Clearly, it respects the \((\text{Sect})\) axiom. Moreover, by Lemma \[12.3\] and the dictionary between \(y(V^j_S)\) and \(V^j_S(\mathcal{L})\), the map \(\Psi_{F,F'}\) also respects the \((\text{EFGL})\) axiom. Therefore, \(\Psi_{F,F'}\) descends to a map \(\Omega^{G,F}(\cdot) \to \Omega^{G,F'}(\cdot)\). Since \(\Omega^{G,F}(X)\) is generated by elements of the form \(f_\ast \circ \sigma[\mathbb{I}_Y]\) and \(\Psi_{F,F'}\) commutes with the basic operations, the image of \(\Psi_{F,F'}\) lies inside \(\Omega^{G,F'}(X)\). Hence, we have a canonical \(\mathbb{L}_G(F)\)-module homomorphism  
\[
\Psi_{F,F'} : \Omega^{G,F}(X) \to \Omega^{G,F'}(X).
\]
By symmetry, we also have \(\Psi_{F',F} : \Omega^{G,F'}(X) \to \Omega^{G,F}(X)\). Then, \(\Psi_{F,F'}\) is an isomorphism because \(\Psi_{F',F} \circ \Psi_{F,F'}\) and \(\Psi_{F,F'} \circ \Psi_{F',F}\) are both identity maps (by Proposition \[12.1\]).
References

[AKMW] Abramovich, D.; Karu, K.; Matsuki, K.; Wlodarczyk, J.: Torification and factorization of birational maps. J. Amer. Math. Soc. 15 (2002), no. 3, 531–572.

[BMi] Bierstone, E.; Milman, P. D.: Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. Invent. Math. 128 (1997), no. 2, 207–302.

[CGKr] Cole, Michael; Greenlees, J. P. C.; Kriz, I.: Equivariant formal group laws. Proc. London Math. Soc. (3) 81 (2000), no. 2, 355–386.

[CGKr2] Cole, Michael; Greenlees, J. P. C.; Kriz, I.: The universality of equivariant complex bordism. Math. Z. 239 (2002), no. 3, 455–475.

[FuKa] Fujii, Michikazu; Kamata, Masayoshi: On the completion of the $G$-equivariant unitary cobordism rings of $G$-spaces. Publ. Res. Inst. Math. Sci. 19 (1983), no. 2, 577–600.

[G] Greenlees, J. P. C.: Equivariant formal group laws and complex oriented cohomology theories. Equivariant stable homotopy theory and related areas (Stanford, CA, 2000). Homology Homotopy Appl. 3 (2001), no. 2, 225–263.

[G2] Greenlees, J. P. C.: Multiplicative equivariant formal group laws. J. Pure Appl. Algebra 165 (2001), no. 2, 183–200.

[H] Hartshorne, R.: Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer–Verlag, New York–Heidelberg, 1977. xvi+496 pp.

[HeMa] Heller, J.; Malagón-López, J.: Equivariant algebraic cobordism. Preprint. [arXiv:1006.5509v1 [math.AG]]

[Kat] Kawakubo, Katsuo: Equivariant Riemann–Roch theorems, localization and formal group law. Osaka J. Math. 17 (1980), no. 3, 531–571.

[Kri] Krishna, A.: Equivariant cobordism of schemes. Preprint. [arXiv:1006.3176v2 [math.AG]]

[Li] Liu, C. L.: Equivariant Algebraic Cobordism and Double Point Relations. Preprint. [arXiv:1110.5282 [math.AG]]

[LMo] Levine, M.; Morel, F.: Algebraic cobordism. Springer Monographs in Mathematics. Springer, Berlin, 2007. xii+244 pp.

[LP] Levine, M.; Pandharipande, R.: Algebraic cobordism revisited. Invent. Math. 176 (2009), no. 1, 63–130.

[Me] Merkurjev, A. S.: Comparison of the equivariant and the standard K-theory of algebraic varieties. (Russian) Algebra i Analiz 9 (1997), no. 4, 175–214; translation in St. Petersburg Math. J. 9 (1998), no. 4, 815–850.

[MuFKi] Mumford, D.; Fogarty, J.; Kirwan, F.: Geometric invariant theory. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], 34. Springer-Verlag, Berlin, 1994. xiv+292 pp.

[S] Sumihiro, H.: Equivariant completion. II. J. Math. Kyoto Univ. 15 (1975), no. 3, 573–605.

[T] Tom Dieck, Tammo: Bordism of $G$-manifolds and integrality theorems. Topology 9 1970 345–358.