Modular functions and Ramanujan sums for the analysis of $1/f$ noise in electronic circuits

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Abstract

A number theoretical model of $1/f$ noise found in phase locked loops is developed. The dynamics of phases and frequencies involved in the nonlinear mixing of oscillators and the low-pass filtering is formulated thanks to the rules of the hyperbolic geometry of the half plane. A cornerstone of the analysis is the Ramanujan sums expansion of arithmetical functions found in prime number theory, and their link to Riemann hypothesis.

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1. INTRODUCTION

$1/f$ noise, already discovered by Nyquist in resistors at the dawn of electronic age, still fascinates experimentalists and theoreticians. This is because none physical principle or mathematical function allows to predict easily the observed hyperbolic $1/f$ power spectral density. Nevertheless the occurrence of $1/f$ fluctuations in areas as diverse as electronics, chemistry, biology, cognition or geology claims for an unifying mathematical principle [1].

A newly discovered clue for $1/f$ noise lies in the concept of a phase locked loop (or PLL) [2]. In essence two interacting oscillators, whatever their origin, attempt to cooperate by locking their frequency and their phase. They can do it by exchanging continuously tiny amounts of energy, so that both the coupling coefficient and the beat frequency should fluctuate in time around their average value. Correlations between amplitude and frequency noise were observed [3].

Fortunately one can gain a good level of understanding of phase locking from quartz crystal oscillators used in high frequency synthesizers, ultrastable clocks and communication engineering (mobile phones or so). The PLL used in a FM radio receiver is a genuine generator of $1/f$ noise. Close to phase locking the level of $1/f$ noise scales approximately as $\tilde{\sigma}^2$, where $\tilde{\sigma} = \sigma K / \tilde{\omega}_B$ is the ratio between the open loop gain $K$ and the beat frequency $\tilde{\omega}_B$ times a constant coefficient $\sigma$. The relation above is explained from a simple non linear model of the PLL known as Adler’s equation

$$\dot{\theta}(t) + KH(P) \sin \theta(t) = \omega_B,$$  

(1)

where at this stage $H(P) = 1$, $\omega_B = \omega(t) - \omega_0$ is the angular frequency shift between the two quartz oscillators at the input of the non linear device (a Schottky diodes double balanced mixer), and $\theta(t)$ is the phase shift of the two oscillators versus time $t$. Solving (1) and differentiating one gets the observed noise level $\tilde{\sigma}$ versus the one $\sigma = \delta \omega_B / \tilde{\omega}_B$ for the open loop case. Thus the model doesn’t explain the existence of $1/f$ noise but correctly predicts its dependance on the physical parameters of the loop [2].

2. LOW PASS FILTERING AND $1/f$ NOISE

Besides one can get detailed knowledge of harmonic conversions in the PLL by accounting for the transfer function $H(P)$, where $P = \frac{d}{dt}$ is the Laplace operator. If $H(P)$ is a low pass
filtering function with cut-off frequency \( f_c \), the frequency at the output of the mixer + filter stage is such that

\[
\mu = f_B(t)/f_0 = q_i |\nu - p_i/q_i| \leq f_c/f_0, \quad p_i \text{ and } q_i \text{ integers.} \tag{2}
\]

The beat frequency \( f_B(t) \) results from the continued fraction expansion of the input frequency ratio

\[
\nu = f(t)/f_0 = [a_0; a_1, a_2, \ldots a_i, a, \ldots], \tag{3}
\]

where the brackets mean expansions \( a_0 + 1/(a_1 + 1/(a_2 + 1/\ldots + 1/(a_i + 1/(a + \ldots)))) \). The truncation at the integer part \( a = [f_0/f_c] \) defines the edges of the basin; they are located at \( \nu_1 = [a_0; a_1, a_2, \ldots a_i, a] \) and \( \nu_2 = [a_0; a_1, a_2, \ldots a_i - 1, 1, a] \). The two expansions in \( \nu_1 \) and \( \nu_2 \), prior to the last filtering partial quotient \( a \), are the two allowed ones for a rational number. The convergents \( p_i/q_i \) at level \( i \) are obtained using the matrix products

\[
\begin{bmatrix}
  a_0 & 1 \\
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  a_1 & 1 \\
  1 & 0
\end{bmatrix}
\ldots
\begin{bmatrix}
  a_i & 1 \\
  1 & 0
\end{bmatrix}
= 
\begin{bmatrix}
p_i & p_{i-1} \\
q_i & q_{i-1}
\end{bmatrix}.
\tag{4}
\]

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{intermodulation_spectrum.png}
\caption{The intermodulation spectrum at the output of the mixer+filter set-up}
\end{figure}

Using \( \mu = f_c/f_0 \) one can get the fractions \( \nu_1 \) and \( \nu_2 \) as \( \nu_1 = \frac{p_a}{q_a} \) and \( \nu_2 = \frac{p_i(2a+1)-p_a}{q_i(2a+1)-q_a} \) so that, with the relation relating convergents \( (p_iq_i - p_{i-1}q_i) = (-1)^{i-1} \), the width of the basin of index \( i \) is \( |\nu_1 - \nu_2| = \frac{2a+1}{q_a(q_a+(2a+1)q_i)} \simeq \frac{1}{q_aq_i} \) whenever \( a > 1 \) (see also Fig. 1).

In our previous publications we proposed a phenomenological model for \( 1/f \) noise in the PLL based on an arithmetical function which is a logarithmic coding for prime numbers \( a \). If one accepts a coupling coefficient evolving discontinuously versus the time \( n \) as \( K = K_0 \Gamma(n) \), with \( \Gamma(n) \) the Mangoldt function which is \( \ln(p) \) if \( n \) is the power of a prime
number $p$ and 0 otherwise, then the average coupling coefficient is $K_0$ and there is an arithmetical fluctuation $\epsilon(t)$

$$\psi(t) = \sum_{n=1}^{t} \Lambda(n) = t(1 + \epsilon(t)),$$

$$t\epsilon(t) = -\ln(2\pi) - \frac{1}{2}\ln(1 - t^{-2}) - \sum_{\rho} \frac{\rho}{\rho}.$$  \hspace{1cm} (5)

The three terms at the right hand side of $t\epsilon(t)$ come from the singularities of the Riemann zeta function $\zeta(s)$, that are the pole at $s = 1$, the trivial zeros at $s = -2l$, $l$ integer, and the zeros on the critical line $\Re(s) = \frac{1}{2}$ \hspace{1cm} (4). Also the power spectral density roughly shows a $1/f$ dependance versus the Fourier frequency $f$. This is an unexpected relation between Riemann zeros (the unsolved Riemann hypothesis is that all zeros should lie on the critical line) and $1/f$ noise.

We improved the model by replacing the Mangoldt function by its modified form $b(n) = \Lambda(n)\phi(n)/n$, with $\phi(n)$ the Euler (totient) function \hspace{1cm} (5). This seemingly insignificant change was introduced by Hardy \hspace{1cm} (6) in the context of Ramanujan sums for the Goldbach conjecture and resurrected recently by Gadiyar and Padma for analyzing the distribution of prime pairs \hspace{1cm} (7). Then by defining the error term $\epsilon_B(t)$ from

$$B(t) = \sum_{n=1}^{t} b(n) = t(1 + \epsilon_B(t)),$$  \hspace{1cm} (6)

its power spectral density $S_B(f) \sim \frac{1}{f^{2\alpha}}$ exhibits a slope close to twice the Golden ratio $\alpha \simeq (\sqrt{5} - 1)/2 \simeq 0.618$ (see Fig. 2).

![Power spectral density](image)

FIG. 2: Power spectral density of the error term in modified Mangoldt function $b(n)$ in comparison to the power law $1/f^{2\alpha}$, with the Golden ratio $\alpha = (\sqrt{5} - 1)/2$.

The modified Mangoldt function occurs in a natural way from the logarithmic derivative
of the quotient
\[ Z(s) = \frac{\zeta(s)}{\zeta(s + 1)} = \sum_{n \geq 1} \frac{\phi(n)}{n^{s+1}}, \tag{7} \]
since \(-Z'(s) = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}\). This replaces the relation from the Riemann zeta function where
\[ -\frac{\zeta'(s)}{\zeta(s)} = \sum_{n \geq 1} \frac{\Lambda(n)}{n^s}. \]

3. THE HYPERBOLIC GEOMETRY OF PHASE NOISE AND 1/f FREQUENCY NOISE

The whole theory can be justified by studying the noise in the half plane \( H = \{ z = \nu + Iy, \ I^2 = -1, \ y > 0 \} \) of coordinates \( \nu = \frac{f_f}{f_0} \) and \( y = \frac{f_B}{f_c} > 0 \) and by introducing the modular transformations
\[ z \rightarrow \gamma(z) = \frac{p_i z + p'_i}{q_i z + q'_i}, \quad p_i q'_i - p'_i q_i = 1. \tag{8} \]
The set of images of the filtering line \( z = \nu + I \) under all modular transformations can be written as
\[ |z - \left( \frac{p_i}{q_i} + \frac{I}{2q_i^2} \right)| = 1 - \frac{1}{2q_i^2}. \tag{9} \]
Equation (9) defines Ford circles \[1\]-\[10\] (see Fig. 3) centered at points \( z = \frac{p_i}{q_i} + \frac{I}{2q_i^2} \) with radius \( \frac{1}{2q_i^2} \). To each \( \frac{p_i}{q_i} \) belongs a Ford circle in the upper half plane, which is tangent to the real axis at \( \nu = \frac{p_i}{q_i} \). Ford circles never intersect: they are tangent to each other if and only if they belong to fractions which are adjacent in the Farey sequence \( \frac{0}{1} < \ldots < \frac{p_1}{q_1} < \frac{p_1 + p_2}{q_1 + q_2} < \frac{q_2}{q_2} < \ldots < \frac{1}{1} \).

For modular transformations one easily calculates \( \Im(\gamma(z)) = \frac{y}{|q_z + q'_z|}, \quad \frac{dy(z)}{dz} = \frac{1}{(q_z + q'_z)^2} \), where the symbol \( \Im \) means the imaginary part. This implies \( |\frac{dy(z)}{dz}|(\Im(\gamma(z)))^{-1} = (\Im z)^{-1} \) and the invariance of the non-Euclidean metric \( |dz| = \frac{(dy^2 + dz^2)^{1/2}}{y} \).

A basic fact about the modular transformations \[5\] is that they form a discontinuous group \( \Gamma \simeq SL(2, \mathbb{Z})/\{\pm I\} \), which is called the modular group. The action of \( \Gamma \) on the half-plane \( H \) looks like the one generated by two independent linear translations on the Euclidean plane, which is equivalent to tesselate the complex plane \( C \) with congruent parallelograms. Here one introduces the fundamental domain of \( \Gamma \) (or modular surface) \( F = \{ z \in H : |z| \geq 1, \ |\nu| \leq \frac{1}{2} \} \), and the family of domains \( \{ \gamma(F), \gamma \in \Gamma \} \) induces a tesselation of \( H \).
FIG. 3: Ford circles: the mapping of the filtering line under modular transformations \( \mathbb{F} \). The arrows indicate that Ford circles were used as an integration path by Rademacher to compute the partition function \( p(n) \).

FIG. 4: The phase angle \( \theta(k) \) for the scattering of noise waves on the modular surface.

\[
y^2\left(\frac{\partial^2}{\partial \nu^2} + \frac{\partial^2}{\partial y^2}\right), \quad \text{and one can write a non-Euclidean wave equation} \quad \square
\]

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m}(\Delta + \frac{1}{4})\Psi. \quad (10)
\]

Stationary solutions \( \Psi = \phi(\nu, y) \exp(-i\omega t) \) satisfy

\[
(\Delta + \frac{1}{4} + k^2)\phi = 0, \quad \text{with} \quad k^2 = \frac{2mE}{\hbar^2}, \quad E = \hbar\omega. \quad (11)
\]

They are special \( \nu \)-independent solutions \( \phi = y^s \) with \( s = \frac{1}{2} + Ik \), and any other wave can be transformed by \( \mathbb{F} \) to form the new solution

\[
\phi_s(z) = (\Im\gamma(z))^s \frac{y^s}{|q_iz + q_i'|^{2s}}, \quad s = \frac{1}{2} + Ik. \quad (12)
\]
In Gutzwiller’s paper [12] there is a detailed discussion on the geometry of solutions (12). The $\nu$-independent solution represents a wave propagating in the $y$-direction $\Psi(y) = y^{1/2} \exp(Ik \ln y - I\omega t)$. The factor $y^{1/2}$ comes from restricting the total flux in a vertical strip of constant width such as $0 < \nu < 1$, so that with the hyperbolic metric one has $\int_0^1 \! d\nu/y = 1/y$. The phase factor is explained by looking at wavefronts $y = C^\text{at}$ and the distance proportional to them is $\int_1^y \! dy/y = \ln y$.

For equation (12) the wavefronts are obtained by assigning to the quantity $y_{|q_i|z+q_i'|2}$ some constant value that one can choose to be the unity. These wavefronts are circles of equation $(\nu + \frac{q_i'}{q_i})^2 + (y - \frac{1}{4q_i'^2})^2 = \frac{1}{4q_i^2}$. They are tangent to the real axis at $\nu_0 = -\frac{q_i'}{q_i}$ and of radius $\frac{1}{2q_i}$. Up to a constant shift along the real axis $\nu$, they are the same as the Ford circles shown on Fig. 3.

Thus an outgoing plane wave can be said to start at $y = 1$, and move up into $H$ with increasing $y$, its wavefronts parallel to the real axis. All other plane waves start at some circle which touches the real axis at some arbitrary point $\nu_0$ of radius $\frac{1}{2q_i}$. Then they contract by shrinking their radius while maintaining their point of contact $\nu_0$. These wave fronts are horocycles in the hyperbolic geometry of the half-plane. They are perpendicular to the geodesics, which are Euclidean half-circles, hitting the real axis at $\nu_0$, at a right angle.

In addition to (12) they are general solutions [13] of the form

$$\phi_s(z) = \sum_{\gamma \in \Gamma_\infty/\Gamma} (3\gamma(z))^s, \quad s = \frac{1}{2} + Ik,$$

where $\Gamma_\infty$ is the stabilizer in the modular group $\Gamma$ of the cusp at infinity, i.e. the subgroup of integer translations $z \rightarrow z + n$. The solution corresponds to waves scattered again and again from the modular surface $F$. For arbitrary $s$ the series (13) can be rewritten as

$$\phi_s(z) = \frac{1}{2} y^s \sum_{(p_i, q_i) = 1} \frac{1}{|p_i z + q_i|^s},$$

with the summation performed over all coprime numbers $(p_i, q_i) = 1$. The convergence is ensured for $\Re(s) > 1$.

The series (13) satisfies a functional equation

$$\xi(2s)\phi_s(z) = \xi(2 - 2s)\phi_{1-s}(z),$$

where $\xi(s) = \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completed Riemann zeta function. It follows that (13)
is expanded as

\[ \phi_s(z) = y^s + S(s)y^{1-s} + T_s(y), \]

with \( S(s) = A(s)Z(s) \),

\[ Z(s) = \frac{\zeta(s-1)}{\zeta(s)}, \quad A(s) = \frac{\Gamma(1/2)\Gamma(s-1/2)}{\Gamma(s)}, \]

and the remaining term vanishes exponentially when \( y \to \infty \). For \( s = \frac{1}{2} + Ik \), that is \( s \) on the critical line, the remainder is \( T_s(y) = 0 \). In such a case the scattering coefficient equals

\[ S\left(\frac{1}{2} + Ik\right) = \frac{\xi(2Ik)}{\xi(1 + 2Ik)}, \quad \text{with } |S\left(\frac{1}{2} + Ik\right)| = 1, \]

that is the flux of the reflected wave is equal and opposite sign the incoming flux. The phase angle represented on Fig. 4 is a very complicated function of \( k \) and is even considered as a prototype for quantum chaos [12].

It is observed that the term \( A(\frac{1}{2} + Ik) \) in (15) increases while \( |Z(\frac{1}{2} + Ik)| \) decreases monotonously to preserve the modulus 1 of \( S(\frac{1}{2} + Ik) \). All the stochastic dynamics is encoded in the function \( Z(\frac{1}{2} + Ik) \). Fig. 4 represents the phase angle \( \theta(k) \) in the scattering coefficient \( S\left(\frac{1}{2} + Ik\right) = \exp(I\theta(k)) \) in comparison to the one \( \theta_0(k) \) of \( Z\left(\frac{1}{2} + Ik\right) \). Apart for weak changes the two curves looks similar; this is confirmed from the power spectral density which is about the same for the two cases.

Looking at the logarithmic derivative of the scattering coefficient \( S(s) = \exp(i\theta(s)) \), one can get the counting function

\[ \theta'(k) = \frac{d\ln S(s)}{ds} \quad \text{at } s = \frac{1}{2} + Ik, \]

which is related to the stochastic factor \( Z(s) \) as

\[ -\frac{Z'(s)}{Z(s)} = \sum_{n=1}^\infty \frac{b(n)}{n^s} = s \int_1^\infty t^{-s-1}B(t)dt, \]

with \( B(t) = \sum_{n=1}^\infty b(n), b(n) = \Lambda(n)\phi(n)/n \). By inverting the Mellin transform in (15) one gets

\[ B(t) = \frac{1}{2\pi i} \int_{\Re(s)-i\infty}^{\Re(s)+i\infty} -\frac{Z'(s)}{Z(s)} \frac{t^s}{s} ds. \]

This extends the calculation performed in [5] for getting the error term in the summatory Mangoldt function \( \psi(t) \). There the Riemann zeta function \( \zeta(s) \) replaces the quotient \( Z(s) = \zeta(s)/\zeta(s+1) \) given in (7). As reminded in Fig. 2, the error term in the summatory modified Mangoldt function is very close to a \( 1/f^{2\alpha} \) noise, with \( \alpha \) the Golden ratio.
4. CONCLUDING REMARKS

From its definition $1/f$ noise is attached to the use of the fast Fourier transform (FFT). But the FFT refers to the fast calculation of the discrete Fourier transform (DFT) with a finite period $q = 2^l$, $l$ a positive integer. In the DFT one starts with all $q^{th}$ roots of the unity $\exp(2i\pi p/q)$, $p = 1 \ldots q$ and the signal analysis of the arithmetical sequence $x(n)$ is performed by projecting onto the $n^{th}$ powers (or characters of $\mathbb{Z}/q\mathbb{Z}$) with well known formulas.

The signal analysis based on the DFT is not well suited to aperiodic sequences with many resonances (by nature a resonance is a primitive root of the unity: $(p, q) = 1$), and the FFT may fail to discover the underlying structure in the spectrum. We recently introduced a new method based on Ramanujan sums \[5,6\],

$$c_q(n) = \sum_{p=1}^{q} \exp\left(\frac{2i\pi p}{q} n\right) \text{ with } (p, q) = 1,$$

which are $n^{th}$ powers of the $q^{th}$ primitive roots of the unity. The sums are evaluated from the use of Möbius transforms as $c_q(n) = \mu\left(\frac{q}{(q,n)}\right) \frac{\phi(q)}{\phi\left(\frac{q}{(q,n)}\right)}$. Here $(q, n)$ is the greatest common divisor $(q, n)$ of $q$ and $n$ and $\mu(n)$ is the Möbius function. It is 0 if $n$ contains a square, 1 if $n = 1$ and $(-1)^k$ if $n$ is the product of $k$ distinct primes.

The sums are quasiperiodic versus the time $n$ ($c_1 = 1$; $c_2 = -1, 1$; $c_3 = -1, -1, 2$, where the bar indicates the period; for example $c_3(4) = -1$) and aperiodic versus the order $q$ of the resonance. In particular $c_q(n) = \mu(q)$ whenever $(n, q) = 1$.

Möbius function can be considered as a coding sequence for prime numbers, as it is the case of Mangoldt function. Mangoldt function is related to Möbius function thanks to the Ramanujan sums expansion found by Hardy \[7\]

$$b(n) = \frac{\phi(n)}{n} \Lambda(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi(q)} c_q(n).$$

(21)

We call such a type of Fourier expansion a Ramanujan-Fourier transform (RFT). General formulas are given in our recent publication \[5\] and in the paper by Gadiyar \[7\]. This author also reports on a stimulating conjecture relating the autocorrelation function of $b(n)$ and the problem of prime pairs. In the special case (21), it is clear that $\mu(q)/\phi(q)$ is the RFT of the modified Mangoldt sequence $b(n)$.
FIG. 5: Ramanujan-Fourier transform (RFT) of the error term (upper curve) of modified Mangoldt function $b(n)$ in comparison to the function $\mu(q)/\phi(q)$ (lower curve).

Using Ramanujan-Fourier analysis the $1/f^{2\alpha}$ power spectrum gets replaced by a new signature shown on Fig. 5, very close to $\mu(q)/\phi(q)$ (up to a scaling factor). There is thus a deep relationship between $1/f$ noise in phase locking, the Golden ratio, the M"obius function, the modified Mangoldt function, the frequency of windings around the modular surface, and the Riemann hypothesis. All these ingredients arise from the hyperbolic geometry of the half-plane.

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