Decay of weakly charged solutions for the spherically symmetric Maxwell-Charged-Scalar-Field equations on a Reissner–Nordström exterior space-time

Maxime Van de Moortel ∗a

aUniversity of Cambridge, Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge CB3 0WA, United Kingdom

April 27, 2018

Abstract

We consider the Cauchy problem for the (non-linear) Maxwell-Charged-Scalar-Field equations with spherically symmetric initial data, on a fixed sub-extremal Reissner–Nordström exterior space-time. We prove that the solutions are bounded and decay at an inverse polynomial rate towards time-like infinity and along the black hole event horizon, provided the charge of the Maxwell equation is sufficiently small. This condition is in particular satisfied for small data in energy space that enjoy a sufficient decay towards the asymptotically flat end.

Some of the decay estimates we prove are arbitrarily close to the conjectured optimal rate in the limit where the charge tends to 0, following the heuristics of [24].

Our result can also be interpreted as a step towards the stability of Reissner–Nordström black holes for the gravity coupled Einstein–Maxwell-Charged-Scalar-Field model. This problem is closely connected to the understanding of strong cosmic censorship and charged gravitational collapse in this setting.

Contents

1 Introduction 1
2 Equations, definitions, foliations and calculus preliminaries 16
3 Precise statement of the main result 23
4 Reduction of the Cauchy problem to a characteristic problem and global control of the charge 26
5 Energy estimates 29
6 Decay of the energy 41
7 From $L^2$ bounds to point-wise bounds 51
A A variant of Theorem 3.3 for more decaying initial data 57
B Proofs of section 4 59
C Useful computations for the vector field method 69

1 Introduction

The model In this paper, we study the asymptotic behaviour of solutions to the Maxwell-Charged-Scalar-Field equations, sometimes referred to as massless Maxwell–Klein–Gordon, arising from spherically symmetric and asymptotically decaying initial data on a fixed sub-extremal Reissner–Nordström exterior space-time:

\[
\nabla^\mu F_{\mu\nu} = i q_0 \left( \phi D_\nu \phi - \phi D_\nu \phi \right) / 2, \quad F = dA,
\]

(Eq. 1.1)
where \( q_0 \geq 0 \) is a constant called the charge of the scalar field \( \phi \), \( M, \rho \) are respectively the mass and the charge of the Reissner–Nordström black hole with \( 0 \leq |\rho| < M \), \( \nabla_\mu \) is the Levi-Civita connection and \( D_\mu = \nabla_\mu + i q_0 A_\mu \) is the gauge derivative. Note that — due to the interaction between the Maxwell field \( F \) and the charged scalar field \( \phi — this system of equations is non-linear when \( q_0 \neq 0 \), the case of interest for this paper. This is in contrast to the uncharged case \( q_0 = 0 \), where (1.2) is then the linear wave equation.

**Main results** Since global regularity is known for this system, we focus on the asymptotic behaviour of solutions.

The case of a charged scalar field on a black hole space-time that we hereby consider is considerably different from the analogous problem on Minkowski space-time. While on the flat space-time, the charge of the Maxwell equation tends to 0 towards time-like infinity, this is not expected to be the case on black hole space-times. This fact constitutes a major difference and renders the proof of decay harder, already in the spherically symmetric case and when the charge is small.

We show that if the charge in the Maxwell equation and the scalar field energies are initially smaller than a constant depending on the black hole parameters \( M \) and \( \rho \), then

1. the charge in the Maxwell equation is bounded and stays small on the whole Reissner–Nordström exterior space-time.
2. Boundedness of the scalar field energy holds.
3. A local integrated energy decay estimate holds for the scalar field.

If now we relax the smallness hypothesis on the charge, requiring only that the initial charge is smaller than a numerical constant \( I_0 \) and assume \( \Omega \) and \( 3 \) then

4. the energy \( I_0 \) of the scalar field decays at an inverse polynomial rate, depending on the charge.
5. The scalar field enjoys point-wise decay estimates at an inverse polynomial rate consistent with \( I_0 \).

These results are stated in a simplified version in Theorem 1.3 and later in a more precise way in Theorem 3.1, Theorem 3.2 and Theorem 3.3.

The decay rate of the energy — and of some point-wise estimates that we derive — has been conjectured to be optimal in [24], in the limit when the asymptotic charge tends to zero, c.f. section 1.2.1. Other point-wise estimates, notably along the black hole event horizon, are however not sharp in that sense.

In the case of an uncharged scalar field \( q_0 = 0 \) (namely the wave equation on sub-extremal black holes), it is well-known that the long term asymptotics are governed by the so-called Price’s law, first put forth heuristically by Price in [44] and later proved in [1], [2], [12], [18], [40], [48]. Generic solutions then decay in time at an universal inverse polynomial decay rate, in the sense that this rate does not depend on any physical parameter or on the initial data. This is in contrast to our charged case \( q_0 \neq 0 \) where the optimal decay rate is conjectured to be slower and moreover depending on the charge in the Maxwell equation, itself determined by the data.

Roughly, this can be explained by the fact that in the uncharged case \( q_0 = 0 \), the equation effectively looks like the wave equation on Minkowski space in the presence of a potential decaying like \( r^{-3} \).

This decay of the potential-like term is somehow more “forgiving” than in the charged case \( q_0 \neq 0 \). In the latter case, the equation becomes similar to the wave equation in the presence of a potential decaying like \( r^{-2} \). As explained beautifully in pages 7 and 8 of [33], while the system is sub-critical with respect to the conserved energy in dimensions \((3+1)\), the new term coming from the charge induces a form of criticality with respect to decay at space-like infinity, i.e. a criticality with respect to \( r \) weights.

One important consequence — in the black hole case — is that the sharp decay rate is expected to depend on the charge, c.f. section 1.2.1. Interestingly in our paper, in order to deal with this criticality, we need to use the full non-linear structure of the system. Also, the criticality with respect to decay implies the absence of any “extra convergence factor” that facilitates the proof of bounds on long time intervals, in the language of [33]. This is in contrast to the uncharged case and requires a certain sharpness in the estimates, as explained in [33].

---

\[ q^{\mu \nu} D_\mu D_\nu \phi = 0, \]  
\[ g = -\Omega^2 dt^2 + \Omega^{-2} dr^2 + r^2 [d\theta^2 + \sin(\theta)^2 d\varphi^2], \]  
\[ \Omega^2 = 1 - \frac{2M}{r} + \frac{\rho^2}{r^2}, \]

\[ q = \mu \rho^2 + \sin^2(\theta)^2, \]

\[ q_0 \geq 0 \text{ is a constant called the charge of the scalar field } \phi, \ M, \ \rho \text{ are respectively the mass and the charge of the Reissner–Nordström black hole with } 0 \leq |\rho| < M, \ \nabla_\mu \text{ is the Levi-Civita connection and } D_\mu = \nabla_\mu + i q_0 A_\mu \text{ is the gauge derivative. Note that — due to the interaction between the Maxwell field } F \text{ and the charged scalar field } \phi — this system of equations is non-linear when } q_0 \neq 0, \text{ the case of interest for this paper. This is in contrast to the uncharged case } q_0 = 0, \text{ where (1.2) is then the linear wave equation.} \]

**Main results** Since global regularity is known for this system, we focus on the asymptotic behaviour of solutions.

The case of a charged scalar field on a black hole space-time that we hereby consider is considerably different from the analogous problem on Minkowski space-time. While on the flat space-time, the charge of the Maxwell equation tends to 0 towards time-like infinity, this is not expected to be the case on black hole space-times. This fact constitutes a major difference and renders the proof of decay harder, already in the spherically symmetric case and when the charge is small.

We show that if the charge in the Maxwell equation and the scalar field energies are initially smaller than a constant depending on the black hole parameters \( M \) and \( \rho \), then

1. the charge in the Maxwell equation is bounded and stays small on the whole Reissner–Nordström exterior space-time.
2. Boundedness of the scalar field energy holds.
3. A local integrated energy decay estimate holds for the scalar field.

If now we relax the smallness hypothesis on the charge, requiring only that the initial charge is smaller than a numerical constant \( I_0 \) and assume \( \Omega \) and \( 3 \) then

4. the energy \( I_0 \) of the scalar field decays at an inverse polynomial rate, depending on the charge.
5. The scalar field enjoys point-wise decay estimates at an inverse polynomial rate consistent with \( I_0 \).

These results are stated in a simplified version in Theorem 1.3 and later in a more precise way in Theorem 3.1, Theorem 3.2 and Theorem 3.3.

The decay rate of the energy — and of some point-wise estimates that we derive — has been conjectured to be optimal in [24], in the limit when the asymptotic charge tends to zero, c.f. section 1.2.1. Other point-wise estimates, notably along the black hole event horizon, are however not sharp in that sense.

In the case of an uncharged scalar field \( q_0 = 0 \) (namely the wave equation on sub-extremal black holes), it is well-known that the long term asymptotics are governed by the so-called Price’s law, first put forth heuristically by Price in [44] and later proved in [1], [2], [12], [18], [40], [48]. Generic solutions then decay in time at an universal inverse polynomial decay rate, in the sense that this rate does not depend on any physical parameter or on the initial data. This is in contrast to our charged case \( q_0 \neq 0 \) where the optimal decay rate is conjectured to be slower and moreover depending on the charge in the Maxwell equation, itself determined by the data.

Roughly, this can be explained by the fact that in the uncharged case \( q_0 = 0 \), the equation effectively looks like the wave equation on Minkowski space in the presence of a potential decaying like \( r^{-3} \).

This decay of the potential-like term is somehow more “forgiving” than in the charged case \( q_0 \neq 0 \). In the latter case, the equation becomes similar to the wave equation in the presence of a potential decaying like \( r^{-2} \). As explained beautifully in pages 7 and 8 of [33], while the system is sub-critical with respect to the conserved energy in dimensions \((3+1)\), the new term coming from the charge induces a form of criticality with respect to decay at space-like infinity, i.e. a criticality with respect to \( r \) weights.

One important consequence — in the black hole case — is that the sharp decay rate is expected to depend on the charge, c.f. section 1.2.1. Interestingly in our paper, in order to deal with this criticality, we need to use the full non-linear structure of the system. Also, the criticality with respect to decay implies the absence of any “extra convergence factor” that facilitates the proof of bounds on long time intervals, in the language of [33]. This is in contrast to the uncharged case and requires a certain sharpness in the estimates, as explained in [33].

---

\[ \frac{q^{\mu \nu} D_\mu D_\nu \phi}{\Omega^2} = 0, \]  
\[ g = -\Omega^2 dt^2 + \Omega^{-2} dr^2 + r^2 [d\theta^2 + \sin(\theta)^2 d\varphi^2], \]  
\[ \Omega^2 = 1 - \frac{2M}{r} + \frac{\rho^2}{r^2}, \]

\[ q = \mu \rho^2 + \sin^2(\theta)^2, \]

\[ q_0 \geq 0 \text{ is a constant called the charge of the scalar field } \phi, \ M, \ \rho \text{ are respectively the mass and the charge of the Reissner–Nordström black hole with } 0 \leq |\rho| < M, \ \nabla_\mu \text{ is the Levi-Civita connection and } D_\mu = \nabla_\mu + i q_0 A_\mu \text{ is the gauge derivative. Note that — due to the interaction between the Maxwell field } F \text{ and the charged scalar field } \phi — this system of equations is non-linear when } q_0 \neq 0, \text{ the case of interest for this paper. This is in contrast to the uncharged case } q_0 = 0, \text{ where (1.2) is then the linear wave equation.} \]
**Motivation**

Our result can be viewed as a first step towards the understanding of the analogous Einstein–Maxwell-Charged-Scalar-Field model, where the Maxwell and Scalar Field equations are now coupled with gravity, c.f. equations (1.16), (1.17), (1.18), (1.19), (1.20) when $m^2 = 0$. In this setting, the asymptotic behaviour of the scalar field is important, in particular because it determines the black hole interior structure, c.f. section 1.2.2. This is also closely related to the so-called Strong Cosmic Censorship Conjecture, c.f. section 1.2.3.

Before discussing the relevance of our result for the interior of black holes, let us mention that the Einstein–Maxwell–Charged–Scalar-Field system possesses a number of new features compared to its uncharged analogue $q_0 = 0$. One of them is the existence of one-ended charged black holes solutions, which are of great interest to study the formation of a Cauchy horizon during gravitational collapse. Indeed one-ended initial data are required to study gravitational collapse. Also, unlike their uncharged analogue in spherical symmetry, charged black holes admit a Cauchy horizon, a feature which is also expected when no symmetry assumption is made, c.f. [11]. While previously studied models in spherical symmetry admit either uncharged one-ended black holes (c.f. [4], [5], [6]) or charged two-ended black holes (c.f. [5], [9], [12], [36], [37]), the Einstein–Maxwell–Charged–Scalar-Field model admits charged one-ended black holes. This is permitted by the coupling between the Maxwell-Field and the charged scalar field, which allows for a non constant charge.

We now return to the interior structure of black holes. Various previous works for different models have highlighted that the interior possesses both stability and instability$^2$ features that depend strongly on the asymptotic behaviour of the scalar field on the event horizon c.f. [8], [9], [11], [35], [36], [37]. For the Einstein–Maxwell–Charged–Scalar-Field model, stability and instability results in the interior have been established in [31]. Concerning stability, it is proven in [31] that a scalar field decaying point-wise at a strictly integrable inverse-polynomial rate on the event horizon gives rise to a $C^0$ stable Cauchy horizon over which the metric is continuously extendible. In the present paper, we establish such strictly integrable point-wise decay for the scalar field on a fixed black hole. If those bounds can be extrapolated to the Einstein–Maxwell–Charged–Scalar-Field case, then the hypotheses of [31] are verified and the stability result applies. In particular, the continuous extendibility result of [31] would then disprove the continuous formulation of the so-called Strong Cosmic Censorship Conjecture as discussed in more details in section 1.2.4.

Another important aspect is the singularity structure of the black hole interior, which is related to the instability feature we stated earlier. This is relevant to a (weaker) $C^2$ formulation of the Strong Cosmic Censorship Conjecture we mentioned before, c.f. section 1.2.2. More specifically, it is proven in [31] that lower bounds for the energy on the event horizon imply the formation of a singular Cauchy horizon. It was proven in [36] for the uncharged analogue that the positive resolution of the $C^2$ strong cosmic censorship conjecture actually follows from this singular nature of the Cauchy horizon. For this, lower bounds on the event horizon must be proven for solutions of a Cauchy problem, c.f. [35], [37] for the uncharged model. Even though in the present paper, we only focus on proving upper bounds for the Cauchy problem, the energy estimates that we carry out are conjectured to be sharp in the limit when the charge tends to zero, c.f. section 1.2.4. Therefore, our work can also be seen as a first step towards the understanding of energy lower bounds, and consequently towards the resolution of strong cosmic censorship in spherical symmetry, for the Einstein–Maxwell–Charged–Scalar-Field model. While this philosophy has been successfully applied to uncharged fields on two-ended black holes [30], [37], the analogous question in the charged case, both for one-ended and two-ended black holes, remains open.

**Outline**  

The introduction is outlined as followed: first in section 1.1 we describe a rough version of the main results, namely asymptotic decay estimates for small decaying data. Then in section 1.2.3, we review previous works on related topics. More specifically in section 1.2.2, we state the conjectured asymptotic behaviour of charged scalar fields on black holes. Then in section 1.2.3, we sum up the known results in the black hole interior for this model, as a motivation for the present study. Then in section 1.2.4, we review previous results for the Maxwell-Charged-Scalar-Field equations on Minkowski space-time and for small energy data. Then section 1.2.5 deals with previous works on the wave equations on black holes space-times and strategies to reach Price’s law optimal decay. After in section 1.2.6 we explain the main ideas of the proof. This includes mostly the strategy to prove energy boundedness, integrated energy estimates and energy decay. Finally in section 1.4 we outline the rest of the paper, section by section.

### 1.1 Simplified version of the main results

We now give a first version of the main results. The formulation of this section is extremely simplified and a more precise version is available in section 3. See also remark [1] for important precisions.

**Theorem 1.1.** Consider spherically symmetric regular data $(\phi_0, Q_0)$, for the Maxwell-Charged-Scalar-Field equations on a sub-extremal Reissner-Nordström exterior space-time of mass $M$ and charge $\rho$. Assume that $\phi_0$ and its derivatives decay sufficiently towards spatial infinity.

---

$^2$In spherical symmetry, the initial data is one-ended if it is diffeomorphic to $\mathbb{R}^3$ and two-ended if it is diffeomorphic to $\mathbb{R} \times S^2$.

$^3$See the recent [21] that investigates a very different kind of scalar field instability on Schwarzschild black hole, which is somehow stronger but specific to uncharged and non-rotating black holes.
If $Q_0$ and $\phi_0$ are small enough in appropriate norms then energy boundedness \[3.1\] is true and an integrated local energy estimate \[3.2\] holds.

Also we can define the future asymptotic charge $e$ such that on any curve constant $r$ curve $\gamma_{r_0} := \{r = R_0\}$ for $r_+ \leq R_0 \leq +\infty$, we have $Q_{\gamma_{r_0}}(t) \to e$ as $t \to +\infty$.

As a consequence of the smallness hypothesis, one can assume that $q_0|e| < \frac{1}{4}$.

Then we have energy decay: for all $R_0 > r_+$, for all $u > 0$ and for all $\epsilon > 0$:

$$E(u) + \int_{\gamma_{R_0} \cap [u, +\infty]} |\phi|^2 + |D_{\nu} \phi|^2 + \int_{\partial [u, +\infty]} |\phi|^2 + |D_{\nu} \phi|^2 \lesssim u^{-2 - \sqrt{1 - 4q_0|e|^4}|+| + \epsilon},$$

where $\gamma_{R_0} := \{r = R_0\}$, $(u, v)$ are defined in section 2.1, $v_R(u) = u + R^2$ is defined in section 2.4 and $E$ is defined in section 2.5.

We also have the following \[19\] point-wise decay, for any $R_0 > r_+$, for $u > 0$, $v > 0$ and for all $\epsilon > 0$:

$$r^\frac{1}{2} |\phi(u, v) + r^\frac{1}{2} D_{\nu} \phi(u, v) \lesssim (\min\{u, v\})^{-1 - \sqrt{1 - 4q_0|e|^4}|+| + \epsilon},$$

$$|\phi|_{L^+} \lesssim u^{-1 - \sqrt{1 - 4q_0|e|^4}|+| + \epsilon},$$

$$|D_{\nu} \phi| \lesssim D^2 \cdot u^{-1 - \sqrt{1 - 4q_0|e|^4}|+| + \epsilon},$$

$$|D_{\nu} \phi|_{\{v \geq 2u + R^2\}(u, v) \lesssim u^{-1 - \sqrt{1 - 4q_0|e|^4}|+| + \epsilon},$$

$$|Q - e|(u, v) \lesssim u^{-1 - \sqrt{1 - 4q_0|e|^4}|+| + \epsilon} 1_{\{r \geq R_0\}}(r \leq R_0).$$

where $\psi := r^\delta$ denotes the radiation field and $Q$ is the Maxwell charge \[11\] defined by $F_{uv} = 2q_0\delta^2$.

Remark 1. In reality, this theorem —which is a broad version of Theorem \[6.1\]— contains a different two intermediate results.

The first one, Theorem 3.2, proves simultaneously energy boundedness and the integrated local energy estimate, on condition that the charge $Q$ is everywhere smaller than a constant depending on the black hole parameters. In particular, this is the case if the $r$ weighted energies of the scalar field data and the initial charge are sufficiently small.

The second one, Theorem 3.3, proves the decay of the energy at a polynomial rate and subsequent point-wise estimates. The decay rate depends only on the dimensionless quantity $q_0\delta$ and is conjectured to be almost optimal when $q_0\delta \to 0$. This theorem only requires that the boundedness of the energy and the integrated local energy estimate are verified, together with the bound $|q_0\delta| < \frac{1}{4}$. In particular, this is the case if the limit of the initial charge $\epsilon_0$ verifies $|q_0\delta| < \frac{1}{4}$ and if the $r$ weighted energies of the scalar field data are sufficiently small. In a sense, the charge smallness—which determines the decay rate—is more precise in these theorems and independent of the black hole mass. This relates to the physical expectation that the decay rate depends only on the asymptotic charge, c.f. section 1.2.1.

Remark 2. Estimates \[1.3\], \[1.7\], \[1.10\] are expected to be sharp \[12\] when $q_0\delta$ tends to 0, c.f. section 1.2.1.

Estimate \[1.6\] is also expected to be sharp in a region of the form $\{v \geq 2u + R^2\}$.

Remark 3. It is also possible to prove an alternative to \[1.9\], c.f. Theorem A.1 in Appendix A. Essentially, if we require more point-wise decay of the initial data, we can prove that $r^\delta D_{\nu} \psi, r^2 D_{\nu} (r^2 D_{\nu} \psi)$ etc. admit a finite limit on $\mathcal{I}^+$, say in the gauge $A_\nu = 0$.

Broadly speaking, the present paper contains three different new ingredients: a non-degenerate energy boundedness statement, an integrated local energy decay Morawetz estimate and a hierarchy of $r^\delta$ weighted estimates.

The presence of an interaction between the Maxwell charge and the scalar field renders the problem non-linear, which makes the estimates very coupled. In particular, we need to prove energy boundedness and the Morawetz estimate together, and the $r^\delta$ estimates hierarchy also depends on them.

This coupling represents a major difficulty that we overcome proving that the charge is small. Even then however, the absorption of the interaction term requires great care. This is essentially due to the presence of a non-decaying quantity, the charge $Q$, that is not present for the uncharged problem or on Minkowski, c.f. sections 1.2.3 and 1.2.4 and to the criticality of the equations, c.f. section 1.2.3 and the introduction.

Remark 4. Following the conjecture of [23], one expects that outside spherical symmetry, the spherically symmetric mode dominates the late time behaviour like in the uncharged case, c.f. Remark 1. One can hope that some of the main ideas of the present paper can be adapted to the case where no symmetry assumption is made. In particular — as it can be seen in the statement of Theorem 3.3 — the decay of the energy is totally independent of point-wise bounds which do not propagate easily without any symmetry assumption.

\footnote{Note that $\epsilon \neq 0$ in general.}

\footnote{Note that estimate \[1.6\] implies a limiting point-wise decay rate of \[\frac{3}{2}\] as $q_0\delta|e| \to 0$ for $\phi$ and $D_{\nu} \phi$ on the event horizon.}

\footnote{The charge $Q$ from the Maxwell equation, should not be confused with the charge of the black hole $\rho$. Because the problem is not coupled with gravity, these are different objects, in contrast to the model of [23]. For definitions, c.f. also section 2.2.}

\footnote{At least for the 0th order term of the Taylor expansion in $q_{uv}$.}
Remark 5. One of the novelty of the present work is the expression of the decay rate in terms of $q_0e$. This was not present in previous work \cite{3, 33, 19} precisely because on Minkowski space-time $e = 0$ so the long time effect of the charge is not as determinant.

Remark 6. It should be noted that everything said in the present paper also works for the case of a spherically symmetric charged scalar field on a Schwarzschild black hole, i.e. when $\rho = 0$.

1.2 Review of previous work and motivation

The motivations to study the Maxwell-Charged-Scalar-Field model are multiple.

First charged scalar fields trigger a lot of interest in the Physics community, sometimes in connection with problems of Mathematical General Relativity, as an example see the recent \cite{25}, which discusses strong cosmic censorship for cosmological space-times in the presence of a charged scalar field.

Second, the difficulty of this problem, due to its criticality with respect to decay at space-like infinity, c.f. section \ref{sec:1.2.3} demands a certain robustness in the estimates. Therefore, the methods of proof may be used in different, potentially more complicated situations where traditional strategies are insufficient.

Finally, if the estimates of the paper can be transposed to the problem where the Maxwell and scalar fields are coupled with gravity, this proves the stability of Reissner-Nordström black holes against charged perturbations. Additionally studying this model on black holes space-time can be considered as a first step towards understanding strong cosmic censorship and gravitational collapse for charged scalar fields in spherical symmetry, which may have different geometric characteristics from its uncharged analogue, see the introduction and section \ref{sec:1.2.2}.

The goal of this section is to review previous works related to the problem of the present manuscript. This allows us to motivate the problem and to compare our results to what already exists in the literature.

The latter is the sole object of section \ref{sec:1.2.1} where we express the conjectured asymptotic behaviour of charged scalar fields, obtained by heuristic considerations in \cite{24}. The former is the object of section \ref{sec:1.2.2} which summarizes the conditions to apply the results of \cite{51}, as one of the main motivation for the study of the exterior we perform.

The previous known results for this model are essentially all proved on Minkowski space-time, although outside spherical symmetry \cite{5}. They either count crucially on conformal symmetries, a method that cannot be generalized easily to black hole space-times, or on the fact that the charge in the Maxwell equations has to tend to 0 towards time-like infinity. This fact greatly simplifies the analysis on Minkowski but is not true on black hole space-times because the charge asymptotes a finite generically non-zero value $e$. These works are discussed in section \ref{sec:1.2.3}.

Finally in section \ref{sec:1.2.4}, we review some of the numerous works on wave equations on black holes space-time. This is the uncharged analogue $q_0 = 0$ of the equations we study. This is the occasion to review the $r^p$ method which is central to our argument and its application to proving exact Price’s law tail in \cite{2}.

1.2.1 Conjectured asymptotic behaviour of weakly charged scalar fields on black hole space-times

We now state the expected asymptotics for a charged scalar field on the event horizon of asymptotic flat black hole space-times. The Physics literature is surprisingly scarce. We base this section on \cite{24}, which is a heuristics-based work and the subsequent papers of the same authors. They state inverse polynomial decay estimates for the charged scalar field on Reissner-Nordström space-time when the asymptotic charge of the Maxwell equation is arbitrarily close to 0.

It is argued that the limiting decay rate 2 is a consequence of multiple scattering already present in flat space-time. Therefore, they suggest that the limit decay rate on black holes space-time —when the charge tends to zero— is the same as on Minkowski, which is consistent with the best decay rate on a constant $r$ curves found in \cite{33}, c.f. section \ref{sec:1.2.3}.

This slow decay stands in contrast to the faster rate prescribed by Price’s law for uncharged perturbations, c.f. Theorem \ref{thm:1.3}. This is because for charged perturbations, the curvature term coming directly from the black hole metric decays faster than the term proportional to the charge of the scalar field. The work \cite{24} was one of the first to notice, in the language of Physics, that charged scalar hairs decay slower than neutral ones.

From an heuristic argument, one can understand from the wave equation \ref{eq:2.0} why the limit decay rate is 2 : we state the charged scalar field equations in spherically symmetry on a $(M, \rho)$ Reissner-Nordström background:

$$D_u(D_v \psi) = \frac{\Omega^2}{r^2} \psi \left( i q_0 Q - \frac{2M}{r^2} + \frac{2p^2}{r^2} \right),$$

while its uncharged analogue when $q_0 = 0$ is

\footnote{Outside spherical symmetry, dynamics of the Maxwell equation notably are much richer and the Maxwell field cannot be reduced to the charge, unlike in the present paper.}
\[ \partial_v (\partial_v \psi) = -\frac{\Omega^2}{r^2} \psi \left( -2M + \frac{2\nu^2}{r} \right). \]

If we expect that the radiation field \( \psi \) and the charge \( Q \) are bounded, we can infer that the charged equations may give a \( t^{-2} \) decay of \( \phi \) on a constant \( r \) curve for compactly supported data, while the uncharged analogue gives \( t^{-3} \) decay, because of the different \( r \) power. This is because for asymptotically flat hyperbolic problems, we expect the \( r \) decay at spatial infinity to be translated into \( t \) decay towards time-like infinity.

It should also be noted that, since higher angular modes are expected to decay faster than the spherically symmetric average, the decay of scalar fields without any symmetry assumption is expected to be the same.

Another interesting discovery made in [24] is the oscillatory behaviour of the scalar field on the event horizon. This is connected to the fact that the scalar field is complex, otherwise the dynamics of the Maxwell equation is trivial as it can be seen in equation (1.1). This makes the proof of decay much more delicate than in the uncharged case, c.f. section 1.2.4 for a discussion.

The result of [24] can be summed up as follows:

**Conjecture 1.2 (Asymptotic behaviour of weakly charged scalar fields, [24]).** Let \((\phi, F)\) be a solution of the Maxwell-Charged-Scalar field system with no particular symmetry assumption on a Reissner-Nordström space-time, and define the Maxwell charge \( Q \) with the relationship \( F_{uv} = \frac{2Q^2}{r^2} \).

Suppose that the data for \( |\phi| \) is sufficiently decaying towards spatial infinity.

Denote the asymptotic charge \( e = \lim_{r \to +\infty} Q_{M^+}(v) \).

Let \( \epsilon > 0 \). Then there exists \( \delta > 0 \), such if \( q_0|\epsilon| < \delta \) we have \[ \phi \] on the event horizon of the black hole, parametrized by an advance time coordinate \( v \), as defined in section 2.7,

\[ \phi|_{M^+}(v) \sim \Gamma_{0} \cdot e^{-q_0e \cdot \frac{v}{r}} \cdot v^{-2+\eta(q_0e)}, \quad (1.11) \]

\[ \psi|_{I^+}(u) \sim \Gamma_{0} \cdot u^{-1+\eta(q_0e)}, \quad (1.12) \]

\[ \phi|_{\gamma}(t) \sim \Gamma_{0} \cdot t^{-\eta(q_0e)} \cdot v^{-2+\eta(q_0e)}, \quad (1.13) \]

as \( v \to +\infty \) and where \( \Gamma_{0}, \Gamma_{0}', \Gamma_{0}'' \) are constants, \( 0 < \eta(q_0e) < \epsilon \) and \( \gamma \) is a far-away curve on which \( |t - u| \sim v \) \( \varepsilon \), defined in section 2.4.

This also implies that the energy on say the \( V \) foliation of section 2.4 behaves like

\[ E(u) \sim E_0 \cdot u^{-3+2\eta(q_0e)}, \quad (1.14) \]

where \( E_0 \) is a constant.

**Remark 7.** Note that the decay of the charged scalar field depends on the dimension-less quantity \( q_0|\epsilon| \) only.

**Remark 8.** Notice also that the conjecture of [24] does not imply any spherical symmetry assumption. It is actually argued that — due to a better decay of the higher angular modes — generic solutions decay as the same as spherically symmetric ones.

The present paper partially solves this conjecture : in particular we are able to prove the upper bounds corresponding to (1.12), (1.13) and (1.14), up an \( u^\epsilon \) loss for any \( \epsilon > 0 \). Moreover, we give an explicit expression \( \eta(q_0e) = 1 - \sqrt{1-4q_0|\epsilon|} \).

The upper bound corresponding to (1.11) is more difficult to prove due to the degeneration of \( r \) weights in the bounded \( r \) region. We indeed prove

\[ r^{\frac{1}{2}}|\phi|(u, v) \lesssim (E(u))^\frac{1}{2} \lesssim u^{-\frac{7}{2}+\eta(q_0e)}, \]

which gives the optimal \( t^{-2+\eta(q_0e)} \) decay on \( \gamma \) but only \( v^{-\frac{7}{2}+\eta(q_0e)} \) on \( H^+ \). While this issue can be circumvented for the uncharged scalar field using the better decay of the derivative, such a strategy is hopeless here, c.f. section 1.2.4. Nonetheless, the strictly integral point-wise estimates on the event horizon are sufficient for future work on the interior of black holes, c.f. section 1.2.3.

The decay of charged scalar fields should be compared to its uncharged analogue prescribed by Price’s law, which is faster :

**Theorem 1.3 (Price’s law, [2], [14], [37], [10], [41], [18]).** Let \( \phi \) be a finite energy solution of the wave equation on a Reissner-Nordström space-time.

Then \( \phi \) is bounded everywhere on the space-time and generically decays in time on the event horizon parametrized by the coordinate \( v \) of section 2.7,

\[ \phi|_{H^+}(v) \sim_{\infty} \Gamma_{0} \cdot v^{-3}, \quad (1.15) \]

where \( \sim_{\infty} \) denotes the numerical equivalence relation as \( v \to +\infty \) for two functions and their derivatives at any order and \( \Gamma_{0} \neq 0 \) is a constant.

\[ ^{11} (1.11) \text{ has to be understood as } \lim_{v \to +\infty} \phi(u, v) e^{-\eta(q_0e) v^2} \sim_{v \to +\infty} \Gamma_{0} \cdot v^{-2+\eta(q_0e)}. \]

\[ ^{12} \text{Although this is not explicitly stated in [24].} \]

\[ ^{13} \text{That may not be sharp.} \]
1.2.2 Towards the gravity coupled model in spherical symmetry

In this section, we mention some of the consequences of a future adaptation of this work to the case of the Einstein–Maxwell-Charged-Scalar-Field equations in spherical symmetry.

If all carries through, this proves in particular the non-linear stability of Reissner-Nordström space-time against small charged perturbations.

Additionally, one can understand the interior structure of black holes for this model. To that effect, we briefly summarize the results of [51] on the black hole interior for the Einstein–Maxwell-Charged-Scalar-Field equations in spherical symmetry. In particular, if the results of the present paper can be extrapolated to the gravity coupled case, it would give interesting information on the structure of one-ended black holes, not modelled by the uncharged analogous equations.

The Einstein–Maxwell-Klein-Gordon equations, whose Einstein–Maxwell-Charged-Scalar-Field is a particular case where the constant $m^2 = 0$, can be formulated as

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T^{EM}_{\mu\nu} + T^{KG}_{\mu\nu}, \]  
\[ T^{EM}_{\mu\nu} = g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu}, \]  
\[ T^{KG}_{\mu\nu} = \mathcal{R}(D_{\mu}\phi D_{\nu}\phi) - \frac{1}{2} (g^{\alpha\beta} D_{\alpha}\phi D_{\beta}\phi + m^2 |\phi|^2) g_{\mu\nu}, \]

\[ \nabla^\mu F_{\mu\nu} = i_{0} \left( \frac{D_{\mu}\phi - 5 D_{\nu}\phi}{2} \right), \quad F = dA, \]

\[ g^{\mu\nu} D_{\mu} D_{\nu} \phi = m^2 \phi. \]

We then have the following result:

**Theorem 1.4 ([51]).** Let $(M, g, F, \phi)$ a spherically symmetric solution of the Einstein–Maxwell-Klein-Gordon system. Suppose that for some $s > \frac{1}{2}$ the following bounds hold, for some advanced time $v$ coordinate $v$ on the event horizon:

\[ |\phi(0, v)|_{H^+} + |D_{v}\phi(0, v)|_{H^+} \lesssim N^{-s}. \]

Then near time-like infinity, the solution remains regular up to its Cauchy horizon. If in addition $s > 1$ then the metric extends continuously across that Cauchy horizon. If moreover the following lower bound on the energy holds, for a $p$ such that $2s - 1 < p < \min\{2s, 6s - 3\}$

\[ v^{-p} \lesssim \int_{0}^{\infty} |D_{v}\phi|_{H^+}^2 (0, v') dv', \]

then the Cauchy horizon is $C^2$ singular. Hence the metric is (locally) $C^2$ inextendible across the Cauchy horizon.

Therefore, if the results of the present manuscript can be extended to the gravity coupled problem, it would imply the continuous extendibility of the metric, at least for small enough data (in particular small initial charge).

Moreover, if we assume that an energy boundedness statement and an integrated local energy estimate hold, then we can prove the continuous extendibility result for a larger class of initial data, namely for an initial asymptotic charge in the range $\Theta_{0} [\epsilon] \in [0, \frac{1}{2}]$.

This is relevant to the so-called Strong Cosmic Censorship Conjecture. Its $C^k$ formulation states that for generic admissible initial data, the maximal globally hyperbolic development is inextendible as a $C^k$ Lorentzian manifold. While its continuous formulation — for $k = 0$ — is often conjectured, it has been disproved in the context of the Einstein–Maxwell-Uncharged-Scalar-Field black holes in spherical symmetry, c.f. [8], [9], [12] and more recently for the Vacuum Einstein equation with no symmetry assumption in the seminal work [11]. Roughly, the continuous formulation of Strong Cosmic Censorship is false in these contexts because one expects dynamic rotating or charged black hole interiors to admit a null boundary — the Cauchy horizon — over which tidal deformations are finite. Therefore, provided that the decay rate assumed in [11] can be proved for Cauchy data, the continuous extendibility result of Theorem 1.4 disapproves the continuous version of Strong Cosmic Censorship for the Einstein–Maxwell-Charged-Scalar-Field black holes in spherical symmetry.

---

15 This $v$ is the $v$ coordinate defined in section 2.1, although in the uncoupled case a gauge choice is necessary, c.f. [31].
16 More precisely, the Penrose diagram -locally near timelike infinity- of the resulting black hole solution is the same as Reissner–Nordström’s.
17 A $C^2$ invariant quantity blows up, namely $Ric(V, V)$ where $V$ is a radial null geodesic vector field that is transverse to the Cauchy horizon.
18 We remark that a definition of the initial asymptotic charge is present in section 2.3.
19 For a precise formulation of Strong Cosmic Censorship, where the notion of admissible data is explained, we refer to [50], [57].
20 The authors of [11] assume the widely-believed stability of Kerr black holes and prove that it implies the continuous extendibility of the metric.
While the (strongest) continuous version of the conjecture is false, the (weaker) $C^2$ formulation of Strong Cosmic Censorship has been proven for Einstein–Maxwell-Uncharged-Scalar-Field spherically symmetric black holes in the seminal works [36, 37]. For the analogous Einstein–Maxwell-Charged-Scalar-Field model, provided one can prove that $1.23$ holds for generic data, the result of Theorem 1.4 proves the $C^2$ inextendibility of the metric along a part of the Cauchy horizon, near time-like infinity. This should be thought of as a first step towards the proof of the $C^2$ formulation of Strong Cosmic Censorship in the charged case, c.f. [51] for a more extended discussion.

Remark 9. The case $k = 2$, i.e. the $C^2$ inextendibility property of the metric is of particular physical interest. Indeed, classical solutions of the Einstein equations are considered in the $C^2$ class: therefore, a $C^2$ inextendibility property implies the impossibility to extend the metric as a classical solution of the Einstein equation. On the other hand, in principle the solution can still be extended as a weak solution of the Einstein equation, for which we only require the Christoffel symbols to be in $L^2$. An interesting but unexplored direction would be to prove the equivalent of Strong Cosmic Censorship in the Sobolev $H^1$ regularity class, that would also exclude extensions as weak solutions of the Einstein equation, in that sense.

For a discussion on other motivations to study the Einstein–Maxwell-Charged-Scalar-Field model in spherical symmetry, we refer to section 1.2.1 of [51] and the pages 23-24, section 1.42 of [11].

1.2.3 Previous works on the Maxwell-Charged-Scalar-Field

We now turn to the study of the Maxwell-Charged-Scalar-Field model. It should be noted that, even in spherical symmetry, the equations are non-linear.

Although this problem on Minkowski space-time has received a lot of attention in the last decade, it should be noted that the only quantitative and rigorous result for that model was derived in [31], for black hole interior (cf. section 1.2.2). We also mention that [30] contains many interesting preliminary results and geometric arguments for the Einstein–Maxwell–Klein–Gordon model in spherical symmetry, although no quantitative study is carried out, either in the interior or the exterior of the black hole.

In the rest of this section, we review previous works on flat space-time.

The first step is to study the global existence problem for variously regular data. This question has been extensively studied, starting with the global existence for smooth data first established by Eardley and Moncrief [19, 20]. Since then, various works have made substantial progress in different directions [7, 27, 28, 31, 32, 39, 43, 44].

After global existence, the next natural question is to study the asymptotic behaviour of solutions. While this problem was pioneered by Shu [47], one of the first modern results is due to Lindblad and Sterbenz [33] who establish point-wise inverse polynomial bounds for the scalar and the Maxwell Field, provided the data is small enough. More precisely they prove the following:

**Theorem 1.5** (Lindblad-Sterbenz, [33]). Consider asymptotically flat energy Cauchy data, namely a scalar field/Maxwell form couple $(\phi_0, F_0)$ such that for some $\alpha > 0$ and $s > 0$,

$$\mathcal{E}^{EM+SF} := \|r^\alpha \phi_0\|_{H^s(\mathbb{R}^3)} + \|r^\alpha D_1 \phi_0\|_{H^s(\mathbb{R}^3)} + \|r^\alpha F_0\|_{H^s(\mathbb{R}^3)} < \infty.$$  

We also denote the asymptotic initial charge $c_0 := \lim_{r \to \infty} \frac{r^2 F_0(\partial_0^\alpha, \partial_r^\alpha)}{2}$.

For every $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\mathcal{E}^{EM+SF} < \delta$$

then we have the following estimates, for $u$ and $v$ large enough

$$|\phi|(u, v) \lesssim u^{-1} v^{-\frac{s}{2} - 1 + \epsilon}, \quad (1.23)$$

$$|\psi|(u, v) \lesssim u^{-1 + \epsilon}. \quad (1.24)$$

$$|D_0 \psi|(v \geq 2u + R^\ast)(u, v) \lesssim u^{-2 + \epsilon}. \quad (1.25)$$

$$|Q|(u, v) \lesssim |c_0| \cdot \frac{1}{v \leq \omega_0(R)} + u^{-1 + \epsilon}. \quad (1.26)$$

As explained in pages 9 and 10 of [33], the Maxwell-Charged-Scalar-Field equations are critical with respect to decay at $r \to +\infty$. This makes the estimates very tight, as opposed for instance to the Einstein equations, which allow for more room. This is due to the presence of a non-linear term that scales exactly like the dominant terms in the energy while its sign cannot be controlled. This very fact relates to the dependence of the decay rate on the asymptotic charge $\epsilon$, as we explain in section 1.1.

To overcome this difficult, the authors of [33] make use of a conformal energy, a fractional Morawetz estimate and some $L^2/L^\infty$ Strichartz-type estimates. These arguments rely on the specific form of the Minkowski metric and are difficult to transpose to any black hole space-time. We also mention the work [44] where similar results are derived but with a simpler proof and the very recent [29] that treats the more general case of Minkowski perturbations, still for the gravity uncoupled case.

\[24\] More exactly, for the Einstein–Maxwell-Charged-Scalar-Field and in the context of spherical symmetry.
Recently, this problem has been revisited by Yang [49] using the modern \( r^p \) method invented in [15] to establish decay estimates. While the proven decay was weaker than that of [33], it made the decay of energy more explicit. More importantly, the proof requires weaker hypothesis. In particular, while the scalar field initial data need to be small, the Maxwell field is allowed to be large. This is summed up by:

**Theorem 1.6 (Yang, [49]).** Consider asymptotically flat energy Cauchy data, namely a scalar field/Maxwell form couple \((\phi_0, F_0)\) such that for some \( \alpha > 0 \) and \( s > 0 \),

\[
E^{SF} := \| r^\alpha \phi_0 \|_{H^s(\mathbb{R}^3)} + \| r^\alpha D^\alpha \phi_0 \|_{H^s(\mathbb{R}^3)},
\]

\[
E^{EM} := \| r^\alpha F_0 \|_{H^s(\mathbb{R}^3)} < \infty.
\]

We also denote the asymptotic initial charge \( e_0 := \lim_{r \to +\infty} r^2 F_0(\partial_u \phi_0) \).

For every \( \epsilon > 0 \), there exists \( \delta > 0 \) such that if

\[
E^{SF} < \delta
\]

then we have the following estimates, for \( u \) and \( v \) large enough

\[
r^\frac{p}{2} |\phi|(u, v) \lesssim u^{-1+\epsilon},
\]

\[
|\psi|(u) \lesssim u^{-\frac{1}{2}+\epsilon},
\]

\[
|D_u \psi|(u \geq 2v+R^+) (u, v) \lesssim v^{-1+\epsilon},
\]

\[
|Q - e_0 \cdot 1_{\{u \leq u_0(\mathbb{R})\}}|(u, v) \lesssim u^{-1+\epsilon},
\]

\[
E(u) \lesssim u^{-2+2\alpha}.
\]

where \( E(u) \) is the energy of the scalar field, similar to the one defined in section 2.2, and \( u_0(\mathbb{R}) \) is defined in section 2.4.

While the \( r^p \) method utilized by Yang is very robust, his work cannot be generalized easily to the case of black hole space-times. This is because on Minkowski space-time, the long range effect of the charge manifests itself only in the exterior of a fixed forward light cone, as it can be seen in estimates (1.28), (1.30). In contrast to the charge on black hole space-times that always admits a non-zero asymptotic charge, this fails on any black hole space-time and instead we need to rely on the smallness of the charge in the present paper.

It should be noted that these works [3, 33, 17, 19] are treating the Maxwell-Charged-Scalar-Field outside spherical symmetry, which makes the dynamics of the Maxwell field much richer than for the symmetric case considered in the present paper where the Maxwell form is reduced to the charge.

We also mention the extremely recent [50] extending the results of [49], with better decay rates and remarkably for large Maxwell field and large scalar field.

### 1.2.4 Previous works on wave equations on black hole space-times and \( r^p \) method

The wave equation on black hole space-times has been an extremely active field of research over the last fifteen years, c.f. [11, 12, 14, 15, 19, 17, 55, 57, 12, 18] to cite a few. This is the uncharged analogue — when \( q_0 = 0 \) — of the problem we study in the present paper.

It is related to one of the main open problems of General Relativity, the question of black holes stability for the Einstein equations without symmetries, c.f. [10, 23, 29] for some recent remarkable advances in various directions.

Subsequently, the black holes interior structure could be inferred from the resolution of this problem, c.f. [3, 33, 11]. In addition to the analyst curiosity to understand the wave equation in different contexts, these works also aim at exploring toy models. This may give valuable insight on the mentioned problems coming from Physics. This is also one of the goals of the present paper.

In this section, we review some results that are related to the decay of scalar fields on spherically symmetric space-times, which is the uncharged version of the model considered in the present manuscript. We are going to mention in particular the different uses of the new \( r^p \) method, pioneered in [15].

After the broader discussion of section 1.2.2, we would like to emphasize how the quantitative late time behaviour of scalar fields impacts the geometry of black holes.

This was first understood in [8, 19] in the context of Einstein–Maxwell-Uncharged-Scalar-Field in spherical symmetry. It is proved that Price’s law of Theorem 1.3 implies that generic black holes for this model

---

25The future asymptotic charge \( e \), defined as the limit of \( Q \) towards infinity on \( \mathcal{J}^+ \) or any constant \( r \) curve.

26In [33] and [35], it has been argued that it is not sufficient to study compactly supported data to understand how decaying data behave. This is because the main charge term cancels for the former and not for the latter. This fact is not any longer true on a black hole space-time.
This apparent paradox is resolved once one realizes that stability estimates are proven for a very weak norm, whereas instability implies the formation of a \(C^0\) regular Cauchy horizon, with no symmetry assumption.

Once the first step — namely understanding (almost) sharp upper bounds — has been carried out, the next step is to understand lower bounds. This is the object of the work \[37\] for the Einstein–Maxwell-Uncharged-Scalar-Field in spherical symmetry, in which \(L^2\) lower bounds are proved on the event horizon. Then, in \[29\], it is shown that these lower bounds propagate to the interior of the black hole. The result implies that generic black holes possess a \(C^2\) singular \(\tau\) Cauchy horizon. This means that the \(C^2\) formulation of Strong Cosmic Censorship conjecture is true for this model. Note that the upper bounds of \[12\] are extremely useful in the instability proof of \[37\].

The main intiate of this short review is the idea that the fine geometry of the black hole is determined by the decaying quantities on the event horizon, which makes the study of black hole exteriors all the more important. The problem gathers both \(\Phi\) stability and instability features (c.f. also \[51\]) : stability of scalar fields implies the formation of a \(C^0\) regular Cauchy horizon while instability ensures its \(C^2\) singular nature.

Now we start a short review of the \(r^p\) method from \[15\]. This should be thought of as a new vector field method, which makes use of \(r\) weights as opposed to conformal vector fields that were used more traditionally, like in \[53\]. The objective is to prove that energy decays in time. Nevertheless, it is a well-known fact that the energy on constant time slice is constant and does not decay. The idea is to consider instead the energy on a \(J\)-shaped foliation \(\Sigma_t = \{ r \leq R, t' = t \} \cup \{ r \geq R, u = u_R(t) \}\) as depicted in \[13\] or \[49\]. We can then establish a hierarchy of \(r^p\) weighted energy from which time decay can be obtained, using the pigeon-hole principle.

In this article we are going to consider a \(V\)-shaped foliation instead \(V_u = \{ r \leq R, v = v_R(u) \} \cup \{ r \geq R, u' = u \}\), c.f. Figure 1 and section 2.4. This is purely for the sake of simplicity and does not change anything.

**Theorem 1.7** (\(r^p\) method for 0 \(<\) p \(<\) 2, Dafermos-Rodnianski, \[15\]). Let \(\phi\) be a finite energy solution of
\[
\Box_g \phi = 0,
\]
where \(g\) is a Schwarzschild exterior metric. The following hierarchy of \(r^p\) weighted estimates is true : for all \(u_0(R) \leq u_1 < u_2\):
\[
\int_{u_1}^{u_2} E_1[\psi](u)\, du + E_2[\psi](u_2) \lesssim E_2[\psi](u_1),
\]
\[
\int_{u_1}^{u_2} E_0[\psi](u)\, du + E_1[\psi](u_2) \lesssim E_1[\psi](u_1),
\]
where \(E_p[\psi]\) is defined in section 2.5. Therefore\(^{27}\) the following estimates are true:
\[
r^{\frac{1}{2}} |\phi|_{\{ r \geq R \}}(u, v) \lesssim u^{-1},
\]
\[
|\psi|(u, v) \lesssim u^{-\frac{3}{2}},
\]
\[
E(u)[\phi] \lesssim u^{-2},
\]
where \(E(u) = E(u)[\phi]\) is defined in section 2.5.

The method was then subsequently extended to the case of \(n\)-dimensional Schwarzschild black holes for \(n \geq 3\) by Schlué in \[40\]. The main novelty is the existence of a better energy decay estimate for \(\partial_t \phi\) which proves a better point-wise for \(\phi\) as well : for all \(\epsilon > 0\)
\[
r^{\frac{1}{2}} |\phi|_{\{ r \geq R \}}(u, v) \lesssim u^{-\frac{3}{2} + \epsilon},
\]
\[
|\psi|(u, v) \lesssim u^{-1 + \epsilon},
\]
\[
E(u)[\partial_t \phi] \lesssim u^{-1 + 2\epsilon}.
\]

The work of Moschidis \[42\], which generalizes the \(r^p\) method and point-wise decay estimates to a very general class of space-times, should also be mentioned.

The \(r^p\) hierarchy has been subsequently extended to \(p < 5\) in \[11\]. This has led to the proof of the (almost) optimal decay for the scalar field and its derivatives on Reissner–Nordström space-time.

\(^{27}\) Precisely, there exists a geodesic vector field \(\partial_\nu\) that is transverse to the Cauchy horizon and regular, so that \(\text{Ric}(\partial_\nu, \partial_\nu) = \infty\).

\(^{28}\) This apparent paradox is resolved once one realizes that stability estimates are proven for a very weak norm, whereas instability estimates originate from a blow-up of stronger norms. c.f. \[36\] and \[51\].

\(^{29}\) We must then crucially make use of a Morawetz estimate and of the energy boundedness on Schwarzschild space-time.
Theorem 1.8 (r p method for 0 ≤ p < 5, Angelopoulos-Aretakis-Gajic, [1]). Let φ be a compactly supported solution of
\[ \Box_p \phi = 0, \]
where g is a sub-extremal Reissner–Nordström exterior metric.
For all 0 ≤ p < 5, the following hierarchy of r p weighted estimates is true : for all u_0(R) ≤ u_1 < u_2 :
\[ p \int_{u_1}^{u_2} E_{p-1} [\phi](u) du + E_p [\phi](u_2) \lesssim E_p [\phi](u_1). \]
For all 2 ≤ q < 6, the following hierarchy of r q weighted estimates is true : for all u_0(R) ≤ u_1 < u_2 :
\[ \int_{u_1}^{u_2} E_{q-1} [\partial_t \psi](u) du \lesssim E_q [\partial_t \psi](u) + E_{q-2} [\psi](u) + E(q \phi) + E(u)[\partial_t \phi]. \]
For all 6 ≤ s < 7, the following hierarchy of r s weighted estimates is true : for all u_0(R) ≤ u_1 < u_2 :
\[ \int_{u_1}^{u_2} E_{s-1} [\partial_t \psi](u) du + E_s [\partial_t \psi](u_2) \lesssim E_s [\partial_t \psi](u) + E_{s-2} [\psi](u) + E(u)[\phi] + E(u)[\partial_t \phi]. \]
Therefore for all \( \epsilon > 0 \), we have the following energy decay
\[ E(u)[\phi] \lesssim v^{-5+\epsilon}. \quad (1.38) \]
\[ E(u)[\partial_t \phi] \lesssim v^{-7+\epsilon}. \quad (1.39) \]
Remark 10. This gives an alternative proof of estimate 1.15 for the linear problem, i.e. the wave equation on a Reissner–Nordström background. Actually, after commuting twice with \( \partial_t \), one can also obtain the estimate \( |\partial_t \phi|_{H^s} \lesssim v^{-4+\epsilon} \). This is therefore a better estimate than in [12], although obtained only for the linear problem.
This strategy to prove the almost optimal energy decay is the first step towards understanding lower bounds. The second step, carried out in [2], is to identify a conservation law that allows for precise estimates.
Corollary 1.9 (Angelopoulos-Aretakis-Gajic, [2]). With the same hypothesis as for Theorem 1.8, for every \( r_0 > r_+ \), where \( r_+ \) is the radius of the black hole, there exists a constant \( C > 0 \) and \( \epsilon > 0 \) such that on a \( \{ r = r_0 \} \) curve :
\[ \phi(r_0, t) = \frac{C}{t^3} + O(t^{-3-\epsilon}). \]
In the present paper and although we do not explicitly use any of the techniques of [1] and [2], we intend to pursue the same program for the non-linear Maxwell-charged scalar field equations. In our case the decay mechanism is more complicated, in particular the decay rate is not universal and depends on the asymptotic charge \( c \).
We find that the maximum \( p \) for which we can derive a hierarchy of \( r^p \) weighted energies is
\[ p = 2 + \sqrt{1 - 4q[3]} \]. This is arbitrarily close to the optimal energy decay rate predicted by [24] as \( q \rightarrow 0 \) tends to 0. However, we cannot retrieve the optimal point-wise bound, close to 2, on the event horizon from this : our method only proves \( \frac{2}{3} \) as \( q \rightarrow 0 \) tends to 0. This is because it is not clear whether \( E(u)[\partial_t \phi] \) enjoys a better \( u \) decay than \( E(u)[\phi] \), in contrast to the uncharged problem as it can be seen in equations 1.37, 1.39.
The physical explanation behind this phenomenon is the presence of an oscillatory term, because the scalar field is complex, as explained in [24]. This makes the analysis more difficult. Therefore, while in the uncharged problem \( \phi \) and \( \partial_t \phi \) decay like \( v^{-3} \) and \( v^{-4} \) respectively, it is not certain that an analogous property is true in the charged case.
Notice however that the bounds we prove on the charge —depending only on the energy— and the bounds towards null infinity are expected to be almost sharp. Moreover, the point-wise bounds on the event horizon are always strictly integrable, which is crucial to study the interior of the black hole, see section 12.2.

1.3 Methods of proof
We now briefly discuss some of the main ideas involved in the proofs.

1.3.1 Smallness of the charge
During the whole paper, we require \( q_0 Q \) to be smaller than some constant. This smallness originates from that of the initial data. More precisely, we prove that provided the asymptotic initial charge \( c_0 \) and of the \( r \) weighted initial scalar field energies are small, then so is the charge, everywhere.
We explain heuristically why this is the case. Schematically, the Maxwell equation looks like
\[ \Box_p \phi = 0, \]
where g is a sub-extremal Reissner–Nordström exterior metric.
For all 0 ≤ p < 5, the following hierarchy of r p weighted estimates is true : for all u_0(R) ≤ u_1 < u_2 :
\[ p \int_{u_1}^{u_2} E_{p-1} [\phi](u) du + E_p [\phi](u_2) \lesssim E_p [\phi](u_1). \]
For all 2 ≤ q < 6, the following hierarchy of r q weighted estimates is true : for all u_0(R) ≤ u_1 < u_2 :
\[ \int_{u_1}^{u_2} E_{q-1} [\partial_t \psi](u) du \lesssim E_q [\partial_t \psi](u) + E_{q-2} [\psi](u) + E(u)[\phi] + E(u)[\partial_t \phi]. \]
For all 6 ≤ s < 7, the following hierarchy of r s weighted estimates is true : for all u_0(R) ≤ u_1 < u_2 :
\[ \int_{u_1}^{u_2} E_{s-1} [\partial_t \psi](u) du + E_s [\partial_t \psi](u_2) \lesssim E_s [\partial_t \psi](u) + E_{s-2} [\psi](u) + E(u)[\phi] + E(u)[\partial_t \phi]. \]
Therefore for all \( \epsilon > 0 \), we have the following energy decay
\[ E(u)[\phi] \lesssim v^{-5+\epsilon}. \quad (1.38) \]
\[ E(u)[\partial_t \phi] \lesssim v^{-7+\epsilon}. \quad (1.39) \]
This term is the source of the criticality with respect to $r$ decay that we describe in the introduction and in section 1.2.3.

This is why this charge smallness step is carried out first, as a preliminary estimate, before the much more precise versions later required to prove decay. This first step carried out in section 4 should be thought of as the analogue of a boundedness proof.

Doing so, we reduce the Cauchy problem to a characteristic initial value problem on the double null surface $\{ u = u_0(R) \} \cup \{ v = v_0(R) \}$. Therefore, section 4 is also the only part of the paper where estimates cover the whole space-time, including the region $\{ u \leq u_0(R) \} \cup \{ v \leq v_0(R) \}$. In later sections dealing with decay, we use the results of this first part and only consider a characteristic initial value problem on the domain $D(u_0(R), +\infty)$, the complement of $\{ u < u_0(R) \} \cup \{ v < v_0(R) \}$.

### 1.3.2 Overview of decay estimates

Now we turn to the core of the present paper: decay estimates. As explained earlier, they rely on

1. Degenerate energy boundedness, c.f. section 1.3.3
2. An integrated local energy decay, also called a Morawetz estimate, see section 1.3.4.
3. Non-degenerate energy boundedness, using the red-shift effect c.f. section 1.3.5.
4. A $r^p$ weighted hierarchy of estimates, see section 1.3.6.
5. A pigeon-hole principle like argument from which time decay can be retrieved for the un-weighted energy, using the last three estimates.
6. Point-wise decay estimates, using crucially the energy decay, c.f. section 1.3.7

Steps 1, 2 and 3 are inter-connected and must be carried out all simultaneously, in contrast to the uncharged case (the wave equation) c.f. 13. Step 4 and 5 are also connected and moreover rely crucially on the results of steps 2 and 3. The last step 6 requires the results of 4 and 5 together with some additional point-wise decay hypothesis of the scalar field Cauchy data.

The distinction between the degenerate and non-degenerate energy is due to the causal character of $\partial_t$, the Killing vector field which allows for energy conservation. While on Minkowski space-time $\partial_t$ is everywhere time-like, this is not true on black hole space-times since $\partial_t$ then becomes null on the event horizon. For this reason, the energy conserved by $\partial_t$ is called degenerate. To obtain the so-called non-degenerate energy, which is the most natural to consider, the use of red-shift estimates is required, c.f. section 1.3.5 for a more precise description.

To prove time decay of the energy — one of the main objectives of this paper — we use the $r^p$ method: from the boundedness of the $r^p$ weighted energies, one can roughly retrieve time decay $t^{-p}$ of the un-weighted energy.

For steps 1 to 4 we mainly use the vector field method, which is a robust technique to establish $L^2$ estimates with the use of geometry-inspired vector fields and the divergence theorem, c.f. Appendix C. The principal difficulty, when we apply the energy identity to a vector field $X$, is to absorb an interaction term between the scalar field and the electromagnetic part of the form

$$\frac{q_0 Q}{r^2} \Delta (\phi(X^\mu D_\nu \phi - X^\nu D_\mu \phi)).$$

It comes from the second term of the identity $\nabla^\mu (\nabla^\mu F_{\mu \nu}) = \nabla^\mu F_{\mu \nu} X^\nu + F_{\mu \nu} X^\mu J^\nu(\phi)$, c.f. (C.11) and Appendix C.

This term must be absorbed by a controlled quantity to close the energy identities of steps 1 to 4. However it has the same $r$ weight as the positive main term controlled by the energy. The strategy is then to apply Cauchy-Schwarz inequality to turn this interaction term into a product of $L^2$ norms. Thereafter we use the smallness of $q_0 Q$ to absorb it. For a more precise description, c.f. for instance section 1.3.6.

Because the main term controlled by the energy is proportional to $|D\phi|^2$, we also need to use Hardy-type inequalities throughout the paper, to absorb any term proportional to $|\phi|^2$. These estimates are proven in section 4.

More details on each step are provided in the subsequent sub-sections.

### 1.3.3 Degenerate energy boundedness

We define the degenerate energy of the scalar field on our $\mathcal{V}_u$ foliation by

$$E_{\deg}(u) = \int_{u_0}^{+\infty} r^2 |D_u \phi|^2(u', v_R(u))du' + \int_{v_R(u)}^{+\infty} r^2 |D_v \phi|^2(u, v)dv,$$

This term is the source of the criticality with respect to $r$ decay that we describe in the introduction and in section 1.2.3.
c.f. section 2.3 and section 2.5.

We want to prove an estimate of the form $E_{\text{deg}}(u_2) \lesssim E_{\text{deg}}(u_1)$ for any $u_1 < u_2$. For this, we make use of the Killing vector field $T = \partial_t$ and notice that $\nabla u(T_{\mu}T^{\nu}) = 0$ where $T = T^{\mu\nu} + 2 T^{EM}$, c.f. section C.1 and section C.2.

While the analogous boundedness estimate in the uncharged case of the wave equation is trivial, even on Reissner-Nordström space-time, this is in the charged case one of the technical hearts of the paper.

Indeed, the method we described above now gives rise to an equation of the form

$$E_{\text{deg}}(u_2) + \int_{\partial R(u_1)} r^2 |D_v\phi|^2 (v) dv + \int_{\partial R(u_1)} \int_{\mathbb{R}^3} r^2 |D_o\phi|^2 (u) du + \int_{\partial R(u_1)} \int_{\mathbb{R}^{3+}} 2 \Omega^2 Q^2 (u, vR(u_2)) du$$

$$+ \int_{vR(u_2)}^{+\infty} \frac{2 \Omega^2 Q^2}{r^2} (u_2, v) dv = E_{\text{deg}}(u_1) + \int_{\partial R(u_1)} \int_{\mathbb{R}^3} r^2 |D_o\phi|^2 (u, vR(u_1)) du + \int_{vR(u_2)}^{+\infty} \frac{2 \Omega^2 Q^2}{r^2} (u_1, v) dv.$$  \hspace{1cm} (1.40)

This is problematic, since the quadratic terms involving the charge do not decay, $Q$ tending to a finite limit at infinity in the black hole case. This is of course in contrast to the Minkowski case where $Q$ tends to 0 towards time-like infinity. However, there is a hope that the difference of such terms, e.g. a term like $\int_{vR(u_2)}^{+\infty} \frac{2 \Omega^2 Q^2}{r^2} (u_2, v) dv - \int_{vR(u_1)}^{+\infty} \frac{2 \Omega^2 Q^2}{r^2} (u_1, v) dv$ can be absorbed into the energy of the scalar field.

To prove this, we require the charge to be small and we have to estimate the difference carefully. More precisely, we need to transport some charge differences towards constant $r$ curves and then to use the Morawetz estimate of step 3.

### 1.3.4 Integrated local energy decay

It has been known since 14 that an integrated local energy decay estimate — now called Morawetz estimate — is useful to prove time decay of the energy. This is classically an estimate roughly of the form $\int_{\text{space-time}} r^{-1-\delta} |\phi|^2 + r^2 |D\phi|^2 \lesssim E(u)$ where $E(u)$ is the energy coming from $\partial_t$ and $\delta > 0$ can be taken arbitrarily small. Note that this is a global estimate with sub-optimal $r$ weights but involving all derivatives. We prove such an estimate in the step 3 but for $\delta > 0$ that can be actually large. This does not make the later proof of decay harder, since our argument — carried out in the $r$ bounded region $\{ r \leq R \}$ — is unaffected by the value of $\delta$.

The Morawetz estimate is probably — together with the degenerate energy boundedness of step 3 — probably one of the most delicate point of the present paper. This is because in the charged case, the customary use of the vector field method with $f(r) \partial_r$ now involves a supplementary term of the form $\text{error} = \int_{\text{space-time}} \phi Q \cdot f(r) \cdot \delta (\phi \partial_t \phi)$ that was not present in the uncharged case. This creates additional difficulties:

1. The zero order term $A_0 = \int_{\text{space-time}} r^{-1-\delta} |\phi|^2$ cannot be controlled independently. This is in contrast to the uncharged case where one can first control $\int_{\text{space-time}} r^{-1-\delta'} \cdot r^2 |D\phi|^2$ for some $\delta' > \delta$ and then use this preliminary bound to finally control $\int_{\text{space-time}} r^{-1-\delta} |\phi|^2$, c.f. 37.

2. The integrand of $\text{error}$ decays in $r$ at the same rate as the main controlled term, for any reasonable choice of $f(r)$. To absorb $\text{error}$ in the large $r$ region, we require $r \cdot f'(r) \gtrsim |f(r)|$ as $r \to +\infty$. This is because, unlike on Minkowski space-time where $Q$ tends to 0 towards time-like infinity, we can only rely on $|Q| \lesssim e$, where $e > 0$ is a (small) constant. This roughly gives an estimate of the form

$$\int_{\text{space-time}} f'(r) \cdot r^2 |D\phi|^2 \lesssim A_0 + q e \int_{\text{space-time}} f(r) \cdot |\phi| \cdot |D\phi| \lesssim A_0 + q e \int_{\text{space-time}} r \cdot f(r) \cdot |D\phi|^2,$$

where for the last inequality, we used Cauchy-Schwarz and the Hardy inequality roughly under the form $\int_{\text{space-time}} r^{-1} |f(r)| \cdot |\phi|^2 \lesssim \int_{\text{space-time}} r |f(r)| \cdot |D\phi|^2$. The condition $r \cdot f'(r) \gtrsim |f(r)|$ together with the smallness of $q e$ then allows us to close the estimate, up to the zero order term $A_0$. This line of thought suggests that $f(r) \approx -r^{-\delta}$, $\delta > 0$ is an appropriate choice.

3. On a black hole space-time, the zero order term $A_0$ is harder to control than on Minkowski space. This is because (1.41) can actually be written in a more precise manner as

$$\int_{\text{space-time}} f'(r) \cdot r^2 |D\phi|^2 + r^2 \Box_\phi (\phi^2 \cdot r^{-1} f(r)) \cdot |\phi|^2 \lesssim q e \int_{\text{space-time}} r |f(r)| \cdot |D\phi|^2 + E(u).$$

For $f(r) = r^{-\delta}$, we compute $r^2 \Box_\phi (\phi^2 \cdot r^{-1} f(r)) = r^{-\delta-4} P_{M,\rho}(r)$ where $P_{M,\rho}(r)$ is a second order polynomial in $r^{-1}$ that is positive on $[r_+, \rho(\delta)] \cup [\rho(\delta), +\infty)$ and negative on $(\rho(\delta), R(\delta))$ for some $r_+ < r(\delta) < R(\delta)$. An analogous computation on Minkowski gives a strictly positive constant polynomial $R_{0,\rho}(r) = (\delta + 1)(\delta + 4)$.

To deal with these difficulties, we first need to prove an estimate for $A_0$ in a region $\{ r_+ \leq r \leq R_0 \}$ for $R_0$ close enough from $r_+$, using the vector field $-\partial_r$ and the smallness of $q e$. We then rely on the crucial but elementary fact that $R(\delta) \to r_+$ as $\delta \to +\infty$. Therefore, for $\delta$ large enough, we get a positive control of $A_0$ on $[R_0, +\infty]$ using (1.42). We conclude combining this with the estimate on $[r_+, R_0]$. 

\[37\] whose coefficients involve $M$ and $\rho$, the parameters of the Reissner–Nordström black hole.
1.3.5 Non-degenerate energy boundedness and red-shift

We now define the non-degenerate energy of the scalar field on our \( \mathcal{V}_u \) foliation by

\[
E(u) = \int_{\mathcal{V}_u} r^2 \frac{|D_v \phi|^2}{\Omega^2} (u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v \phi|^2 (u, v) dv,
\]

c.f. section 2.3 and section 2.5.

This is called non-degenerate precisely because on a fixed \( v \) line, \( \Omega^{-2} \partial_v \) is a non-degenerate vector field across the event horizon \( \{ u = +\infty \} \). Therefore, we expect (and prove in step 6) a bound of the form \( |D_v \phi| \lesssim \Omega^p \), consistent with the boundedness of the quantity \( E(u) \). This also means that \( D_v \phi = 0 \) on the event horizon, which explains why \( \partial_v \) is degenerate and needs to be renormalized to obtain a finite limit.

Our goal is then to prove an estimate of the form \( E(u_2) \lesssim E(u_1) \) for any \( u_1 < u_2 \).

For this, we use a so-called red-shift estimate, pioneered in [12], [13], [14]. In our context, it boils down to using the vector field method with \( X = \Omega^{-2} \partial_v \). While this is not the hardest part of the paper, we still need to use the Morawetz estimate of step 3 to conclude, unlike in the uncharged case. This is because in our case, we must absorb a bulk term coming from the charge (c.f. section 1.3.2) into a controlled scalar field bulk term, while for the wave equation, no control of the bulk term is needed at this stage, c.f. [19].

1.3.6 Energy decay and \( r^p \) method

To prove time decay of the energy, we use the \( r^p \) method, pioneered in [15].

The idea is to prove the boundedness of \( r^p \) weighted energies

\[
E_p(u) := \int_{\mathcal{V}_u} r^p |D_v \psi|^2 (u, v) dv,
\]

for a certain range of \( p \), ultimately responsible for time decay \( t^{-p} \). Actually, we prove a hierarchical estimate of the form

\[
\int_{u_1}^{u_2} E_{p-1}(u) du + E_p(u_2) \lesssim E_p(u_1) \lesssim 1.
\]

This kind of estimate is obtained by applying the vector field method in a region \( \{ r \geq R \} \) with the vector field \( r^2 \partial_r \).

Thereafter applying the mean-value theorem or a pigeon-hole like argument, we can retrieve an estimate of the form \( E_{p-1}(u) \lesssim u^{-1} \) and eventually \( E(u) \lesssim u^{-p} \).

Now in the case of the Maxwell-Charged-Scalar-Field model, we also have to control an interaction term coming from the charge c.f. section 1.3.2. We now have an estimate of the form

\[
\int_{u_1}^{u_2} E_{p-1}(u) du + E_p(u_2) \lesssim E_p(u_1) + \text{error},
\]

where \( \text{error} \approx q_0 e \int_{\{ r \geq R \}} r^{p-2} \Delta (\bar{\psi} D_v \psi) \) is the interaction term we mentioned earlier.

Due to the shape of this term, as we explained already, we need a Hardy inequality to absorb to \( |\phi| \) into the energy term.

More explicitly we roughly estimate \( \text{error} \) using the following type of bounds:

\[
| \int_{\{ r \geq R \}} r^{p-2} \Delta (\bar{\psi} D_v \psi) | \lesssim \left( \int_{\{ r \geq R \}} r^{p_1} |\psi|^2 \right)^{\frac{1}{2}} \left( \int_{\{ r \geq R \}} r^{p_2} |D_v \psi|^2 \right)^{\frac{1}{2}},
\]

where \( p_1 + p_2 = 2p - 4 \), simply using Cauchy-Schwarz. We then apply a Hardy inequality to roughly find if \( p_1 < -1 \):

\[
| \int_{\{ r \geq R \}} r^{p-2} \Delta (\bar{\psi} D_v \psi) | \lesssim [1 + p_1]^{-1} \left( \int_{\{ r \geq R \}} r^{p_1+2} |D_v \psi|^2 \right)^{\frac{1}{2}} \left( \int_{\{ r \geq R \}} r^{p_2} |D_v \psi|^2 \right)^{\frac{1}{2}}.
\]

Now because \( \int_{\{ r \geq R \}} r^{p_1+2} |D_v \psi|^2 \) is already controlled for \( p_2 = p - 1 \), the choice \( (p_1, p_2) = (p-3, p-1) \) seems natural, c.f. section 6.2.

We then roughly need to absorb a term \( q_0 |e| \cdot [2 - p]^{-1} \int_{u_1}^{u_2} E_{p-1}(u) du \) into \( \int_{u_1}^{u_2} E_{p-1}(u) du \) which is essentially doable if \( q_0 |e| \) is small but requires \( p \in [0, 2 - \epsilon(q_0e)] \) in particular \( p < 2 \). Calling \( p_0 \) the maximal \( p \) for which we can do this, we then essentially prove for \( u > 0 \) large:

\[
E_{p_0-1}(u) \lesssim \frac{E_{p_0}(u)}{u}.
\]

We employ this strategy in section 6.2. While the estimates we get are necessary to “start” the argument, they are insufficient to reach the best possible decay advertised in the theorems.

This is why in section 6.3 we adopt a completely different strategy. This time we chose \( (p_1, p_2) = (p-4, p) \) for \( p = p_0 + 1 \). Then the error term we need to absorb is roughly of the form
\[ q_0|e| \cdot |2 - p_0|^{-1} \left( \int_{u_1}^{u_2} E_{p-2}(u) \, du \right) ^{\frac{1}{2}} \left( \int_{u_1}^{u_2} E_p(u) \, du \right) ^{\frac{1}{2}} \lesssim q_0|e| \cdot |3 - p|^{-1} \left( \frac{E_{p-1}(u)}{u} \right) ^{\frac{1}{2}} \left( \int_{u_1}^{u_2} E_p(u) \, du \right) ^{\frac{1}{2}}, \]

where we used the equation \((1.43)\) and the fact that \(p_0 = p - 1\).

Now we need to absorb the right-hand side into \( \int_{u_1}^{u_2} E_{p-1}(u) \, du + E_p(u_2) \). This requires a Grönwall-like method and a careful bootstrap argument on \( E_{p-1}(u) \), in which we experience a controlled \( u \) growth to eventually get \( E_p(u) \lesssim u^\epsilon \) and \( E(u) \lesssim u^{-p} \) for all \( \epsilon > 0 \).

With this last argument, a \( r^p \) weighted hierarchy is proven for \( p \in [0, 3 - \epsilon(q_0, \epsilon)] \) which is a significant improvement with respect to the first method and allows us to claim a stronger time decay of the energy.

### 1.3.7 Point-wise bounds

To prove point-wise bounds on the scalar field and the charge, we need two essential ingredients: the weighted energy decay of step \( 5 \) and a point-wise estimate on a fixed light cone of the form \( |D_u \psi([u_0](R), v)| \lesssim v^{-\omega} \) for some \( \omega > 0 \).

The latter comes from the point-wise decay of the scalar field Cauchy data, that implies consistent point-wise bounds in the past of a fixed forward light cone, c.f. section \( 4 \). We then use the former to “initiate” some decay estimate for \( \phi \). For this we essentially use Cauchy-Schwarz under the form \( r^{\frac{1}{2}}|\phi| \leq E(u) \lesssim u^{-p} \), for the maximum \( p \) in the \( r^p \) hierarchy.

Point-wise bounds are then established in the rest of the space-time integrating \((2.6)\) and \((2.7)\) along constant \( u \) and constant \( v \) lines, after carefully splitting the space-time into regions where the scalar field behaves differently.

These regions are roughly:

1. A far away region — including \( I^+ = \{ r = +\infty \} \) — where \( r \sim v \), which is somehow the easiest.
2. An intermediate region \( \{ R \leq r \leq v \} \) where \( R > r_+ \) is a large constant.
3. The bounded \( r \) region \( \{ r_+ \leq r \leq R \} \), which includes \( H^+ = \{ r = r_+ \} \).

The far away region is the one where \( r \) weights are strong so the “conversion” between the \( L^2 \) and the \( L^\infty \) occurs “with no loss”.

The bounded \( r \) region is also not so difficult due to a point-wise version of the red-shift effect: there is again no loss between the estimates on the curve \( r = R \) and the event horizon \( r = r_+ \).

However, in the intermediate region, the \( r \) weights, strong on \( \{ r \sim v \} \) degenerate to a mere constant near \( \{ r = R \} \). For this reason, the estimates imply a loss of \( v^{\frac{1}{2}} \), which explains why the point-wise decay rate obtained on the event horizon is not expected to be optimal, while the rates on the energy and in the far away region are, at least in the limit \( q_0|e| \to 0 \).

In every section and subsection, the precise strategy of the proof is discussed. We refer the reader to these paragraphs for more details.

### 1.4 Outline of the paper

The paper is outlined as follows: after introducing the equations, some notations and our foliation in section \( 2 \) we announce in section \( 3 \) a more precise version of our results.

Because our techniques (with the \( V \) shaped foliation of section \( 2 \)) only deal with characteristic initial value data, we explain in section \( 4 \) how the Cauchy problem can be reduced to a characteristic initial value problem, with the correct assumptions. This is also the occasion to derive a priori smallness estimates on the charge, useful to close the harder estimates of the following sections. To avoid repetition and because the estimates are easier than the ones in the next sections, we postpone the proof to Appendix \( B \).

Then in section \( 4 \) we prove Theorem \( 4.2 \). This section is divided as follows: first the subsection \( 4.1 \) where the integrated local energy decay is established, modulo boundary terms. Then the subsection \( 4.2 \) where the red-shift effect is used to establish a non-degenerate energy boundedness statement, modulo degenerate energy boundary terms. Finally we close together the energy boundedness and the Morawetz estimate in subsection \( 5.3 \), using crucially the results of the former two subsections.

Then, we turn to the proof of the second main result of the present paper, Theorem \( 5.3 \). Establishing energy decay is the object of section \( 6 \). It is divided as follows: first in subsection \( 6.1 \) we carry out preliminary computations. We also re-prove explicitly the version of the \( r^p \) method needed for our purpose, with the notations of the paper. Then a first energy decay estimate is proven, for \( p < 2 \), in section \( 6.2 \) using the smallness of the future asymptotic charge under the form \( q_0|e| \lesssim \frac{1}{n} \).

Finally, in section \( 6.3 \) we prove a hierarchy of \( r^p \) estimates for \( p < 3 \). The employed strategy differs radically from the one of section \( 6.2 \) but we use crucially the \( r^p \) hierarchy proven in the previous section. This proves Theorem \( 5.3 \).

Section \( 6.3 \) is one of the technical hearts of the paper, and the part that allows eventually for strictly integrable bounds on the event horizon.
The intake of sections 6.2 and 6.3 is that energy decay at a rate \( p = 2 + \sqrt{1 - 4q_0|e|} \) holds, provided energy boundedness is true. This rate tends to 3 as \( q \) tends to 0, which is the expected optimal limit. It also tends to 2 as \( q_0|e| \) tends to \( \frac{1}{4} \).

We can then retrieve point-wise estimates from the energy decay in section 7. We use methods that only work in spherical symmetry and which are optimal in a region near null infinity. However, the presence of \( r \) weights that degenerate in the bounded \( r \) region creates a \( \frac{1}{r^2} \) loss in the decay that cannot be compensated otherwise, like it was \( \frac{1}{r^2} \) in the uncharged case, c.f. Section 1.2.4. This is why the optimal energy decay does not give rise to optimal point-wise bounds on the event horizon. However, since the decay rate of the energy is always superior to 2, we retrieve **strictly integrable** point-wise decay rate for \( \phi \) on the event horizon, for the full range of \( q_0|e| \) we consider.

Finally, we prove a slight variant \(^{34}\) of Theorem 3.3 in Appendix A, the proofs of Section 4 are carried out in Appendix B and a few computations are made explicit in Appendix C.

### 1.5 Acknowledgements

I express my profound gratitude to my PhD advisor Jonathan Luk for proposing this problem, for his guidance and for numerous extremely stimulating discussions.

I am grateful to Stefanos Aretakis, Dejan Gajic and Georgios Moschidis for insightful conversations related to this project.

My special thanks go to Haydée Pacheco for drawing the Penrose diagrams.

I gratefully acknowledge the financial support of the EPSRC, grant reference number EP/L016516/1.

This work was completed while I was a visiting student in Stanford University and I gratefully acknowledge their financial support and their hospitality.

## 2 Equations, definitions, foliations and calculus preliminaries

### 2.1 Coordinates and vector fields on Reissner–Nordström’s space-time

The sub-extremal Reissner–Nordström exterior metric can be written in coordinates \((t, r, \theta, \phi)\)

\[
g = -\Omega^2 dt^2 + \Omega^{-2} dr^2 + r^2 [d\theta^2 + \sin(\theta)^2 d\psi^2],
\]

\[
\Omega^2 = 1 - \frac{2M}{r} + \frac{\rho^2}{r^2},
\]

for some \(0 \leq |\rho| < M\) and where \((r, t, \theta, \phi) \in (r_+, +\infty) \times \mathbb{R}^+ \times [0, \pi] \times [0, 2\pi]\).

\[r_+(M, \rho) := M + \sqrt{M^2 - \rho^2}\]

is one of the two positives roots of \(\Omega^2(r)\) and represents the radius of the event horizon, defined to be

\[\mathcal{H}^+ := \{r = r_+, \ (t, \theta, \phi) \in \mathbb{R}^+ \times [0, \pi] \times [0, 2\pi]\}.\]

Symmetrically we define future null infinity to be

\[\mathcal{I}^+ := \{r = +\infty, \ (t, \theta, \phi) \in \mathbb{R}^+ \times [0, \pi] \times [0, 2\pi]\}.\]

\(\mathcal{H}^+\) and \(\mathcal{I}^+\) are then null and complete hyper-surfaces for the Reissner–Nordström metric.

The other root \(r_-(M, \rho) := M - \sqrt{M^2 - \rho^2}\) corresponds to the locus of the Cauchy horizon, inside the black hole, c.f. [51]. Therefore \(r_-\) does not play any role in the exterior.

We want to built a double null coordinate system \((u, v)\) on spheres that can replace \((t, r)\).

One possibility is to define a function \(r^*(r)\), sometimes called tortoise radial coordinate, such that

\[
\frac{dr^*}{dr} = \Omega^{-2}(r),
\]

\[
\lim_{r \to +\infty} r^* = 1.
\]

Notice that the new coordinate \(r^*(r)\) is an increasing function of \(r\) that takes its values in \((-\infty, +\infty)\).

We denote by \(\partial_t\) and \(\partial_r\) the corresponding vector fields in the \((r^*, t, \theta, \phi)\) coordinate system. Notice that \(\partial_t\) is a time-like **Killing vector field** for this metric.

We can then define the functions \(u(r^*, t)\) and \(v(r^*, t)\) as

\[
v = \frac{t + r^*}{2}, \quad u = \frac{t - r^*}{2}.
\]

Notice that \(u\) takes its values in \((-\infty, +\infty)\) and \(\{u = +\infty\} = \mathcal{H}^+\), \(v\) takes its values in \((-\infty, +\infty)\) and \(\{v = +\infty\} = \mathcal{I}^+\).

---

\(^{33}\)The reason is that in the uncharged case, \(\partial_t \phi\) decays better than \(\phi\) but in this charged case, \(D\phi\) decays like \(\phi\), due to oscillations.

\(^{34}\)The main intake is that, for slightly more decaying data than required in Theorem 3.3 one can prove that \(v^2 D_v \psi\) admits a bounded limit when \(v \to +\infty\), for fixed \(u\).
Spatial infinity \( \rho^0 \) is \( \{ u = -\infty, v = +\infty \} \) and the bifurcation sphere is \( \{ u = +\infty, v = -\infty \} \).

Defining \( \partial_u \) and \( \partial_v \), the corresponding vector fields in the \((u,v,\theta,\varphi)\) coordinate system, we can check that \((\partial_u, \partial_v, \partial_\theta, \partial_\varphi)\) is a null frame for the Reissner–Nordström metric.

Notice that we have

\[
\partial_r = \partial_t + \partial_r \cdot, \partial_u = \partial_t - \partial_r \cdot.
\]

The Reissner–Nordström’s metric can then be re-written as

\[
g = -2\Omega^2(du \otimes dv + dv \otimes du) + r^2 [d\theta^2 + \sin(\theta)^2 d\varphi^2].
\]

Note that in this coordinate system we also have

\[
\Omega^2(r) = \partial_r r = -\partial_u r.
\]

Then we define the quantity \(2K\), the log derivative of \(\Omega^2\) that plays a role in the present paper:

\[
2K(r) := \frac{2}{r^2} (M - \rho^2 \rho) = \partial_r \log(\Omega^2) = -\partial_u \log(\Omega^2).
\]

We also define the surface gravity of the event horizon \(2K_+ := 2K(r_+)\) and that of the Cauchy horizon \(2K_- := 2K(r_-)\). These definitions allow for an explicit and simple expression of \(r^*\) as

\[
r^* = r + \frac{\log(r - r_+) + \log(r - r_-)}{2K^+}. \tag{2.1}
\]

This implies that when \(r \rightarrow r_+\):

\[
\Omega^2(r) \sim C^2 e^{2K^+ (v-u)}, \tag{2.2}
\]

where \(C^+ = C^+(M, \rho) > 0\) is defined by \(C^2 = e^{-2K^+ + \frac{1}{2} - \frac{K_-}{r_-}}\).

### 2.2 The spherically symmetric Maxwell-Charged-Scalar-Field equations in null coordinates

We now want to express Maxwell-Charged-Scalar-Field system

\[
\nabla^\mu F_{\mu\nu} = iq_0 \left( \phi D_\nu \phi - D_\nu \phi \right) / 2, \quad F = dA, \tag{2.3}
\]

\[
g^{\mu\nu} D_\mu \phi = 0 \tag{2.4}
\]

in the \((u,v)\) coordinates of section [2.1] and for spherically symmetric solutions.

Here \(\phi\) represents a complex valued function while \(F\) is a real-valued 2-form.

In what follows, for a spherically symmetric solution \((\phi, F)\), we are going to consider the projection of \((\phi, F)\) —that we still denote \((\phi, F)\)— on the 2-dimensional manifold indexed by the null coordinate system \((u,v)\) and on which every point represents a sphere of radius \(r(u,v)\) on Reissner–Nordström space-time. Much more details can be found on the procedure in [51], section 2.2.

It can be shown (c.f. [30]) that in spherical symmetry, we can express the two-form \(F\) as

\[
F = \frac{2Q \Omega^2}{\rho^2} du \wedge dv. \tag{2.5}
\]

where \(Q\) is a scalar function that we call the charge of the Maxwell equation.

**Remark 11.** Notice that this formula, that defines \(Q\), differs from a multiplicative factor 4 from the formula of [23, 30] and [51]. This is because, in the present paper, \(\Omega^2\) is defined to be \(\Omega^2 = 1 - \frac{2M}{\rho} + \frac{\rho^2}{\rho^2}\). In the previous papers, \(\Omega^2\) was actually defined as \(\Omega^2 = 4\left(1 - \frac{2M}{\rho} + \frac{\rho^2}{\rho^2}\right)\), hence the difference.

\(F = dA\) also allows us to chose a spherically symmetric potential \(A\) 1-form written as :

\[
A = A_0 du + A_v dv.
\]

We then define the gauge derivative \(D\) as \(D_\mu = \partial_\mu + iq_0 A_\mu\).

The equations (2.3), (2.4) are invariant under the following gauge transformation :

\[
\phi \rightarrow \tilde{\phi} = e^{-iq_0 f} \phi \\
A \rightarrow \tilde{A} = A + df,
\]

where \(f\) is a smooth real-valued function.

There is therefore a gauge freedom. However, all the estimates derived in this paper are essentially gauge invariant, so we **do not need to choose a gauge**.

Notice that for the gauge derivative \(\tilde{D} := \nabla + \tilde{A}\), we have
\[ \tilde{D} \phi = e^{\phi} D \phi. \]

Therefore, as explained in \[22\], the quantities \(|\phi|\) and \(|D\phi|\) are gauge invariant and so are the energies defined in section \[2.4\].

It should be noted that gauge derivatives do not compute; indeed one can show that

\[ [D_u, D_v] = i q_0 F_{uv} Id = \frac{2 i q_0 Q \Omega^2}{r^2} Id. \]

We now express equation \[(2.4)\] in \((u, v)\) coordinates. For this, it is convenient to define the radiation field \(\psi := \phi \rho\). We then find:

\[
D_u(D_v \psi) = \frac{\Omega^2}{r} \phi \left( i q_0 Q - \frac{2 M}{r} + \frac{2 \rho^2}{r^2} \right)
\]

\[
D_v(D_u \psi) = \frac{\Omega^2}{r} \phi \left( -i q_0 Q - \frac{2 M}{r} + \frac{2 \rho^2}{r^2} \right)
\]

Maxwell’s equation \[(2.3)\] can also be written as \((u, v)\) coordinates as

\[
\partial_u Q = -q_0 r^2 \Im(\tilde{\phi} D_u \phi)
\]

\[
\partial_v Q = q_0 r^2 \Im(\tilde{\phi} D_v \phi).
\]

Notice that in the spherically symmetric case, the Maxwell form is reduced to the charge \(Q\).

Then finally the existence of an electro-magnetic potential \(A\) can be written as:

\[
\partial_u A_v - \partial_v A_u = F_{uv} = \frac{2 Q \Omega^2}{r^2}. \tag{2.10}
\]

Finally we would like to finish this section by a Lemma stated in \[22\] (Lemma 2.1) that says that “gauge derivatives can be integrated normally”. More precisely:

**Lemma 2.1.** For every \(u_1 \in (-\infty, +\infty)\) and \(v_1 \in (-\infty, +\infty)\) and every function \(f\),

\[
|f(u, v)| \leq |f(u_1, v)| + \int_{u_1}^{u} |D_u f|(u', v) du',
\]

\[
|f(u, v)| \leq |f(u, v_1)| + \int_{v_1}^{v} |D_v f|(u, v') dv'.
\]

We refer to \[22\] for a proof, which is identical in the exterior case. This lemma will be used implicitly throughout the paper.

### 2.3 Notations for different charges

In this paper, we make use of several quantities that we call “charge” although they may not be related. This section is present to clarify the notations and the relationships between these different quantities.

First we work on a Reissner–Nordström space-time of parameters \((M, \rho)\).

The Reissner–Nordström charge \(\rho\) is defined by the expression of the metric \[(1.3), (1.4)\].

We assume the sub-extremality condition \(0 \leq |\rho| < M\), where \(M\) is the Reissner–Nordström mass.

Note that \(\rho\) is here **constant**, as a parameter of the black hole that does not vary dynamically because we study the gravity uncoupled problem.

Note also that the value of \(\rho\) plays no role in the paper, provided \(|\rho| < M\).

In this paper we consider the Maxwell-Charged-Scalar-Field model. This means that the charged scalar field interacts with the Maxwell field.

The interaction constant \(q_0 \geq 0\) is called the charge of the scalar field, as appearing explicitly in the system \[(2.3), (2.4)\].

We require the condition \(q_0 \geq 0\) uniquely for aesthetic reasons and this is purely conventional. Everything said in this paper also works if \(q_0 < 0\), after replacing \(q_0 |e|\) by \(|q_0 |e|\) and \(q_0 |e|\) by \(|q_0 |e|\).

Note that we consider \(q_0\) as a universal constant, not as a parameter. Note also that \(q_0\) has the dimension of the inverse of a charge: \(q_0 Q\) is a dimensionless quantity.

Note also that the uncharged case \(q_0 = 0\) corresponds to the well-known linear wave equation.

Now we define the charge of the Maxwell equation \(Q\), defined explicitly from the Maxwell Field form \(F\) by \[(2.5)\], itself appearing in \[(2.3)\].

\(Q\) is a scalar dynamic function on the space-time —in contrast to the uncharged case \(q_0 = 0\) where \(Q\) is forced to be a constant — that determines completely the Maxwell Field in spherical symmetry.
Note also that the Maxwell equations can be completely written in terms of $Q$, c.f. equations (2.8), (2.9).

Because we consider a Cauchy initial value problem, we also consider the initial charge of the Maxwell equation $Q_0$ defined as $Q_0 = Q|_{\Sigma_0}$ where $\Sigma_0$ is the initial space-like Cauchy surface. $Q_0$ is then one part of the initial data $(\phi_0, Q_0)$.

Then we define the initial asymptotic charge $e_0$ as

$e_0 = \lim_{r \to +\infty} Q_0(r)$,

when it exists. This is the limit value of the Maxwell charge at spatial infinity.

Finally we define the future asymptotic charge $e$ as

$e = \lim_{t \to +\infty} Q(t, r)$,

for all $r \in [r_+, +\infty[$, when it exists. It can be proven\footnote{The proof is made in section 7.4.} that the limit $e$ does not depend on $r$. This is the limit value of the Maxwell charge at time-like and null infinity.

### 2.4 Foliations, domains and curves

In this section we define the foliation over which we control the energy, c.f. section 2.5. This is a $V$-shaped foliation, similar to the one of \cite{37} but different from the $J$-shaped foliation of \cite{15} and subsequent works.

For any $r_+ \leq R_1$ we define the curve $\gamma_{R_1} = \{ r = R_1 \}$.

For any $u$, we denote $v_{R_1}(u)$ the only $v$ such that $(u, v_{R_1}(u)) \in \gamma_{R_1}$. Such a $v$ is given explicitly by the formula $v = u - R_1^2$. Similarly for every $v$, we introduce $u_{R_1}(v)$.

Then for some $R > r_+, \text{ large enough to be chosen in course of the proof, we considered the curve } \gamma_R$.

We also denote $(u_0(R), v_0(R))$, the coordinates of the intersection of $\gamma_R$ and $\{ t = 0 \}$:

$v_0(R) = -u_0(R) = \frac{R^2}{2}$.

We then define the foliation $V$ : for every $u \geq u_0(R)$

$V_u = (\{v_{R}(u)\} \times [u, +\infty)) \cup ([v_{R}(u), +\infty] \times \{u\})$.

This foliation is composed from a constant $v$ segment in the region $\{ r \leq R \}$ joining a constant $u$ segment in the region $\{ r \geq R \}$. It is illustrated in Figure 4.

Notice that the foliation does not cover the regions $\{ u \leq u_0(R) \}$ and $\{ v \leq v_0(R) \}$ as it can be seen in Figure 5. This is why we need section 4 to connect the energy on this foliation, and in particular on $V_{u_0(R)}$ to the energy on the initial space-like hyper-surface $E$.

We now defined the domain $D(u_1, u_2)$, for all $u_0(R) \leq u_1 < u_2$ as

$D(u_1, u_2) := \cup_{u_1 \leq u \leq u_2} V_u$. \hfill (2.11)

Numerous $L^2$ identities of this paper are going to be integrated either on $D(u_1, u_2)$, or $D(u_1, u_2) \cap \{ r \leq R \}$ or $D(u_1, u_2) \cap \{ r \geq R \}$.

We also define the initial space-like hyper-surface $\Sigma_0 = \{ t = 0 \} = \{ v = -u \}$ on which we set the Cauchy data $(\phi_0, Q_0)$ c.f. section 4.

We are going to make use of the following notations $u_0(v) = -v, v_0(u) = -u, r_0(u)$ the radius corresponding to $(r_0(u))^2 = 2u$ and $r_0(v)$ the radius corresponding to $(r_0(v))^2 = 2v$, when there is no ambiguity between $u$ and $v$.

Finally we will also need a curve close enough to null infinity, in particular to retrieve point-wise bounds. We define $\gamma = \{ r^+ = \frac{\pi}{2} + R^* \}$. Notice that on this curve, $r \sim u \sim \frac{\pi}{2}$ as $v \to +\infty$.

For any $u$, we then define $v_r(u)$, the only $v$ such that $(u, v_r(u)) \in \gamma$. Similarly, we introduce $u_v(v)$.

Remark 12. Note that because $\gamma$ and $\gamma_{R_1}$ are time-like curves, talking of their future domain is not very interesting. Instead, notably in section 4, we are going to make use of the domain “to the right” of $\gamma : \{ r^+ \geq \frac{\pi}{2} + R^* \}$ and the domain “to the right” of $\gamma_{R_1} : \{ r \geq R_1 \}$.\footnote{The proof is made in section 7.4.}
Figure 1: Illustration of the foliation $\mathcal{V}_u, u \geq u_0(R)$

$\Sigma_0 = \{t = 0\}$

Figure 2: Curves $\gamma_{R_0}$ for $r_+ < R_0 = R_0(M, \rho) < R$, $\gamma_R$ and $\gamma$
2.5 Energy notations

In this section, we gather the definitions of all the energies used in the paper. Those definitions are however repeated just before being used for the first time.

The main definition concerns the non-degenerate energy, for all $u \geq u_0(R)$:

$$E(u) = E_R(u) = \int_u^{+\infty} r^2 |D_u \phi|^2 \left( u', v_R(u) \right) du' + \int_{v_R(u)}^{+\infty} r^2 |D_e \phi|^2 (u, v) dv.$$

By default, $E(u)$ is the energy related to the $\gamma_R$-based foliation $\mathcal{V}$ we mention in section 2.1 Sometimes however, we may talk of $E_{R_1}(u)$ for a $R_1$ that is different from $R$, in section 3

We also define the degenerate energy:

$$E_{deg}(u) = \int_u^{+\infty} r^2 |D_u \phi|^2 \left( u', v_R(u) \right) du' + \int_{v_R(u)}^{+\infty} r^2 |D_e \phi|^2 (u, v) dv.$$

It will be convenient to use for all $u_0(R) \leq u < u_2$:

$$E^+(u_1, u_2) := E(u_2) - E(u_1) = \int_{u_1}^{u_2} r^2 |D_u \phi|^2 (v) dv + \int_{u_1}^{u_2} r^2 |D_e \phi|^2 (u) du,$$

$$E_{deg}^+(u_1, u_2) := E_{deg}(u_2) - E_{deg}(u_1) = \int_{u_1}^{u_2} r^2 |D_u \phi|^2 (v) dv + \int_{u_1}^{u_2} r^2 |D_e \phi|^2 (u) du.$$

For the $r^p$ hierarchy we are also going to use for all $u_0(R) \leq u_1 < u_2$:

$$E_p[v](u) := \int_{v_R(u)}^{+\infty} r^p |D_u \phi|^2 (u, v) dv,$$

$$\tilde{E}_p(u) := E_p[v](u) + E(u).$$

Finally we define the initial non-degenerate energy on the initial slice $\Sigma_0 := \{ t = 0 \}$

$$\mathcal{E} = \int_{\Sigma_0} r^2 |D_u \phi_0|^2 + r^2 |D_e \phi_0|^2 \frac{dr^*}{\Omega^2}. \quad (2.12)$$

**Remark 13.** Notice that this energy is proportional to the regular $H^1$ norm for $\phi$ on $\Sigma_t$. Indeed the vector field $\Omega^{-1} \partial_t = \frac{\partial_t}{\sqrt{g}}$ — time-like and unitary — is regular across the bifurcation sphere. But from (2.2), we see that towards the future or the past event horizon $\{ r = r_+ \}$, $\Omega^2 = (C_+)^2 \cdot e^{2K+} \cdot e^{-2K+}$. Therefore, for an expression in $(\Omega, u)$ slice transverse to the future event horizon, $(C_+)^{-1} e^{2K+} \partial_\nu$ is regular. Symmetrically on any constant $u$ slice transverse to the past event horizon, $(C_+)^{-1} e^{-2K+} \partial_\nu$ is regular. We then see that the $(2C_+)^{-1} (\epsilon^{2K+} \partial_u + e^{-2K+} \partial_\nu)$ is then also a regular vector field across the bifurcation sphere, actually proportional to $\Omega^{-1} \partial_t$ because on $\Sigma_t, v = -u$.

Notice that consistently, to prove point-wise bounds, we are going assume $|D_u \phi_0|(r) \lesssim e^{-2K+} \sim \Omega(u, v_0(u))$ — c.f. Hypothesis 3 — as opposed to the too strong but somehow naive intuitive hypothesis $|D_u \phi_0|(r) \lesssim \Omega^2$.

Actually, the $L^2$ bound is “morally” slightly stronger than the point-wise one. Indeed $\Omega^{-2} |D_u \phi_0|^2 dr^* = \Omega^{-4} |D_u \phi_0|^2 dr$ is integrable if $\Omega^{-4} |D_u \phi_0|^2 \lesssim (r - r_+)^{-1+}$ for any $\epsilon > 0$, i.e. $|D_u \phi_0|(u, v_0(u)) \lesssim e^{-2K+(1+\epsilon)}$.

We also define the initial $r^p$ weighted energy for $\psi_0 := r \phi_0$:

$$\mathcal{E}_p := \int_{\Sigma_0} (r^p |D_u \psi_0|^2 + r^p |D_e \psi_0|^2 + \Omega^2 r^p |\phi_0|^2) dr^*.$$ 

Note that $\mathcal{E}_p$ includes a 0 order term $r^p |\phi_0|^2$, of the “same homogeneity” as $r^p |D_u \psi_0|^2$. Notice also that

$$\int_{\Sigma_0} r^p |D \psi_0|^2 dr^* \leq \mathcal{E}_p.$$

Finally we regroup the two former definitions into

$$\tilde{\mathcal{E}}_p = \mathcal{E}_p + \mathcal{E}.$$

We also want to recall that Stress-Energy momentum tensors are defined as:

$$T^{SF}_{\mu \nu} = \mathcal{R}(D_\mu \phi D_\nu \phi) - \frac{1}{2} (g^{\alpha \beta} D_\alpha \phi D_\beta \phi) g_{\mu \nu}, \quad (2.14)$$

$$T^{EM}_{\mu \nu} = g^{\alpha \beta} F_{\alpha \mu} F_{\beta \nu} - \frac{1}{4} F_{\alpha \beta} F_{\alpha \beta} g_{\mu \nu}. \quad (2.15)$$

For an expression in $(u, v)$ coordinates and explicit computations, c.f. Appendix C.
2.6 Hardy-type inequalities

In this paper, we use Hardy-type inequalities numerous times. As explained in section 1.4, this is mainly due to the necessity to absorb the 0 order term that appears in the interaction term with the Maxwell field. In this section, we are going to state and prove the different Hardy-type inequalities that we use throughout the paper.

**Lemma 2.2.** For $R > 0$ as in the definition of the foliation and all $u_0(R) \leq u_1 < u_2$, the following estimates are true, for any $q \in [0, 2)$, and any $r_+ < R_1 < R$:

$$
\int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, vr(u_i))du \leq 4 \int_{u_i}^{+\infty} \Omega^2 \frac{r^2 |D_v \phi|^2(u, vr(u_i))}{r^2 - 2K} du + 2\Omega^2 (R) \cdot |\phi|^2(u_i, vr(u_i)). \quad (2.16)
$$

$$R|\phi|^2(u_i, vr(u_i)) \leq \Omega^{-2}(R) \int_{u_i}^{+\infty} r^2 |D_v \phi|^2(u_i, v)dv \quad (2.17)
$$

$$\int_{u_1}^{u_2} \Omega^2 |\phi|^2(u, vr(u_i))du \leq 4 \frac{\Omega^{-2}(1)}{\Omega^{-2}(R_1)} \int_{u_1}^{u_2} r^2 |D_v \phi|^2(u, vr(u_i))du + 2R|\phi|^2(u_i, vr(u_i)). \quad (2.18)
$$

$$\int_{vr(u_i)}^{+\infty} |\phi|^2(u, v)dv \leq \frac{4}{\Omega^2(R_1)} \int_{vr(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v)dv. \quad (2.19)
$$

$$\left(\int_{D(u_1, u_2) \cap (r \geq R)} r^{n-2} |\phi|^2 dv \right) \leq \frac{2}{(2-q)\Omega^2(R)} \left(\int_{D(u_1, u_2) \cap (r \geq R)} r^{-1-2} |D_v \phi|^2 dv \right)^\frac{1}{2} + \left(\frac{R^{q-2}}{2-q} \int_{u_1}^{u_2} |\phi|^2(u_i, vr(u_i))du \right)^\frac{1}{2} \quad (2.20).
$$

**Proof.** We start by (2.16): after noticing that $\Omega^2 \cdot 2K = (-\partial_v \Omega^2)$ we integrate by parts and get:

$$\int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, vr(u_i))du = \int_{u_i}^{+\infty} (-\partial_u \Omega^2)|\phi|^2(u, vr(u_i))du \leq 2 \int_{u_i}^{+\infty} \Omega^2 \Omega(D_v \phi)(u, vr(u_i))du + \Omega^2 (R) \cdot |\phi|^2(u_i, vr(u_i)).$$

Then using Cauchy-Schwarz, we end up having an inequality of the form $A^2 \leq 2AB + C^2$, where

$$A^2 = \int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, vr(u_i))du, \quad B^2 = \int_{u_i}^{+\infty} \Omega^2 \frac{r^2 |D_v \phi|^2(u, vr(u_i))}{2K} du, \quad C^2 = \Omega^2 (R) \cdot |\phi|^2(u_i, vr(u_i)).$$

Then using what we know on roots of second order polynomials we get (2.16) under the form:

$$A^2 \leq (B + \sqrt{B^2 + C^2})^2 \leq 4B^2 + 2C^2.$$

Now we prove (2.17). Since $\lim_{r \to +\infty} \phi(u, v) = 0$, we can write

$$|\phi|(u, v) = -\int_u^{+\infty} e^{\int_v^u (\phi)_{eu} + A_v} D_v \phi(u, u')dv',$$

where we used the fact that $\partial_v(e^{\int_v^u (\phi)_{eu} + A_v} \phi) = e^{\int_v^u (\phi)_{eu} + A_v} D_v \phi$.

Now using Cauchy-Schwarz we have

$$|\phi|(u, v) \leq \left(\int_v^{+\infty} \Omega^2 r^{-2}(u, u')dv'\right)^\frac{1}{2} \left(\int_{u_1}^{+\infty} \Omega^{-2} r^2 |D_v \phi|^2(u, u')dv'\right)^\frac{1}{2}.$$

After squaring, this directly implies (2.17) under the form

$$r|\phi|^2(u, v) \leq \Omega^{-2}(u, v) \int_{v}^{+\infty} r^2 |D_v \phi|^2(u, u')dv'.$$

Then we turn to (2.18): after an integration by parts we get

$$\int_{u_i}^{u_1} \Omega^2 |\phi|^2(u, vr(u_i))du \leq 2 \int_{u_i}^{+\infty} rR(D_v \phi)(u, vr(u_i))du + R|\phi|^2(u_i, vr(u_i)).$$

Then using Cauchy-Schwarz, we find an inequality of the form $A^2 \leq 2AB + C^2$, where

$$A^2 = \int_{u_i}^{u_1} \Omega^2 |\phi|^2(u, vr(u_i))du, \quad B^2 = \int_{u_i}^{u_1} \frac{r^2 |D_v \phi|^2}{\Omega^2}(u, vr(u_i))du, \quad C^2 = R|\phi|^2(u_i, vr(u_i)).$$

Similarly to (2.16), this concludes the proof of (2.18), after we notice that $\Omega^{-2} \leq \Omega^{-2}(R_1)$.

We now turn to (2.19): after notice that $\Omega^2 (R) \leq \partial_u r$ we integrate by parts and get

\[\text{[Further details of the proof are omitted for brevity.]}\]

\[\text{[Refer to the original text for the complete proof.]}\]
By Hypothesis 3, one can in particular assume that
\[ \int_{\mathbb{R}^n(\Omega)} |D_\nu| (u, v) \, dv \leq \Omega^{-1} (R) \int_{\mathbb{R}^n(\Omega)} |D_\nu r| (u, v) \, dv \leq -2 \Omega^{-1} (R) \int_{\mathbb{R}^n(\Omega)} r \mathcal{R} (D_\nu u, \phi)(u, v) \, dv. \]

Notice that the sum of the boundary terms is negative because \[ \lim_{r \to +\infty} r |\phi|^2 = 0. \] Then Cauchy-Schwarz inequality directly gives (2.19). We used the fact that for all \( q \),
\[ \int_{\mathbb{R}^n(\Omega)} |D_\nu r| (u, v) \, dv \leq -2 \Omega^{-1} (R) \int_{\mathbb{R}^n(\Omega)} r \mathcal{R} (D_\nu u, \phi)(u, v) \, dv. \]

Finally we prove (2.20) after noticing that \( r^{q-3} \Omega^2 = -(2-q)^{-1} \partial_q (r^{q-2}) \) and an integration by parts in \( v \) we get, using that \( q < 2 \)
\[ \int_{D(\mathbb{R}^n(\Omega))} r^{q-3} \Omega^2 |u|^2 \, dv \leq \frac{2}{(2-q)} \int_{D(\mathbb{R}^n(\Omega))} r^{q-2} \mathcal{R} (\psi D_\nu \psi) \, dv + \frac{R_0^{q-2}}{2-q} \int_{\Omega} |\psi|^2 (u, v_R(u)) \, du. \]

Then using Cauchy-Schwarz, we end up having an inequality of the form \( A^2 \leq 2AB + C^2 \), where
\[ A^2 = \int_{D(\mathbb{R}^n(\Omega))} r^{q-3} \Omega^2 |u|^2 \, dv, \quad B^2 = \frac{1}{(2-q)} \int_{D(\mathbb{R}^n(\Omega))} r^{q-2} |D_\nu \psi|^2 \, dv, \quad C^2 = \frac{R_0^{q-2}}{2-q} \int_{\Omega} |\psi|^2 (u, v_R(u)) \, du. \]

Then, as seen for the proof of (2.16), we have \( A \leq B + \sqrt{B^2 + C^2} \). This implies that \( A \leq 2B + C \), which is exactly (2.20). We used the fact that for all \( a, b \geq 0 \), \( \sqrt{a} + \sqrt{b} \leq \sqrt{a+b} \). This concludes the proof.

\[ \square \]

3 Precise statement of the main result

3.1 The main result

We now state the results of the present paper, with precise hypothesis. The main result is an almost-optimal energy and point-wise decay statement, for small and point-wise decaying initial data.

It should be noted that all the required hypotheses are only needed to prove all the claims.

For example, to prove only energy decay, point-wise decay rates are not necessary, and one can assume weaker initial energy boundedness, c.f. the theorems of next sub-sections.

Theorem 3.1 (Energy and point-wise decay for small and decaying data). Consider spherically symmetric Cauchy data \((\phi_0, \rho_0)\) on the surface \( \Sigma_0 = \{ t = 0 \} \) — on a Reissner-Nordström exterior space-time of mass \( M \) and charge \( \rho \) with \( 0 \leq |\rho| < M \) — that satisfy the constraint equation (4.1). Assume that the data satisfy the following regularity hypotheses:

1. The initial energy is finite: \( \mathcal{E} < \infty \), where \( \mathcal{E} \) is defined in section 2.3
2. \( \rho_0 \in L^\infty (\Sigma_0) \) and admits a limit \( \rho_0 \in \mathbb{R} \) as \( r \to +\infty \).
3. We have the following data smallness assumption, for some \( \delta > 0 \) and \( \eta > 0 \):
   \[ \| \rho_0 \|_{L^\infty (\Sigma_0)} + \tilde{\mathcal{E}}_{1+\eta} < \delta. \]
4. Defining \( \psi_0 := r \phi_0 \), assume \( \psi_0 \in C^4 (\Sigma_0) \) enjoys \( 3^\circ \) point-wise decay:
   for all \( \epsilon_0 > 0 \), there exist \( C_0 = C_0 (\epsilon_0) > 0 \), such that
   \[ r |D_\nu \psi_0| (r) + |\psi_0| (r) \leq C_0 \cdot r^{-\frac{1}{4} - \frac{\epsilon_0}{\sqrt{\psi_0 (0)}}} + \epsilon_0. \]

Then, there exists \( \delta_0 = \delta_0 (M, \rho) > 0 \), \( C = C (M, \rho) > 0 \) such that if \( \delta < \delta_0 \), the following are true:

1. The charge \( Q \) admits a future asymptotic charge: there exists \( \epsilon \in \mathbb{R} \) such that for every \( R_1 > r_+ \):
   \[ \lim_{t \to +\infty} Q_{\mathcal{R}} (t) = \lim_{t \to +\infty} Q_{\mathcal{P}} (t) = \lim_{t \to +\infty} Q_{\mathcal{F}} (t) = \epsilon. \]

Moreover the charge is small on the whole-space-time:
\[ |Q(u, v) - \epsilon_0| \leq C \cdot (\| \rho_0 \|_{L^\infty (\Sigma_0)} + \tilde{\mathcal{E}}_{1+\eta}) \]

in particular \( |\epsilon - \epsilon_0| \leq C \cdot \delta \) and \( |\epsilon| \leq (C + 1) \cdot \delta \).

\[ ^3 \text{This comes from the finiteness of } \tilde{\mathcal{E}}_{1+\eta}, \text{ c.f. the proof of Proposition 4.1.} \]

\[ ^4 \text{By Hypothesis } 2 \text{ one can in particular assume } \rho_0 |\epsilon_0| < \frac{1}{4}, \text{ without loss of generality.} \]
2. **Non-degenerate energy boundedness** holds: for all $u_0(R) \leq u_1 < u_2$:

$$\int_{v_1(u_1)}^{v_2(u_2)} r^2 |D_v \phi|^2 + r^2 |D_u \phi|^2 + E(u_2) \leq C \cdot E(u_1) \leq C^2 \cdot E,$$  \hspace{1cm} (3.1)

where $E$ and $E$ are defined in section 2.5.

3. **Integrated local decay estimate** holds: there exists $\sigma = \sigma(M, \rho) > 1$ — potentially large — such that for all $u_0(R) \leq u_1 < u_2$:

$$\int_{D(u_1, u_2)} \left( \frac{r^2 |D_v \phi|^2 + r^2 \Omega^{-2} |D_u \phi|^2 + |\phi|^2}{r^\sigma} \right) dv \leq C \cdot E(u_1) \leq C^2 \cdot E.$$ \hspace{1cm} (3.2)

4. **Weighted $r^p$ energies are bounded and decay**: For all $0 \leq p < 2 + \sqrt{1 - 4\rho_0|e|}$, there exists $\delta_p = \delta_p(M, \rho, p, e) > 0$, such that if $\delta < \delta_p$, then for all $\epsilon > 0$ there exists $R_0 = R_0(\epsilon, e, M, \rho, r_+)$ such that if $R > R_0$, there exists $C_p = C_p(p, M, \rho, r_+, e) > 0$ such that for all $u \geq u_0(R)$:

$$\tilde{E}_p(u) \leq C_p \cdot |u|^{-2 - \sqrt{1 - 4\rho_0|e|} + \epsilon},$$

where $\tilde{E}_p$ is defined in section 2.5.

5. **In particular the non-degenerate energy decays in time**: for all $u \geq u_0(R)$:

$$E(u) \leq C_0 \cdot |u|^{-2 - \sqrt{1 - 4\rho_0|e|} + \epsilon}. \hspace{1cm} (3.3)$$

We also have the decay of the scalar field $L^2$ flux along constant $r$ curves and the event horizon: for all $R > r_+$ and all $\epsilon > 0$, there exists $\tilde{C} = \tilde{C}(R_0, \epsilon, M, \rho, R) > 0$ such that for all $v \geq v_0(R)$:

$$\int^{+\infty}_{v} \left[ |\phi|^2(u_R(v), v') + |D_v \phi|^2(u_R(v), v') \right] dv' \leq \tilde{C} \cdot v^{-2 - \sqrt{1 - 4\rho_0|e|} + \epsilon},$$ \hspace{1cm} (3.4)

$$\int^{+\infty}_{v} \left[ |\phi|^2(u_R(v), v) + |D_v \phi|^2(u_R(v), v) \right] dv' \leq \tilde{C} \cdot v^{-2 - \sqrt{1 - 4\rho_0|e|} + \epsilon}. \hspace{1cm} (3.5)$$

6. Finally, we obtain **point-wise decay estimates**: for all $\epsilon > 0$, there exists $R_0 = R_0(\epsilon, e, M, \rho, r_+)$ such that if $R > R_0$ then there exists $C' = C'(\epsilon, e, R, M, \rho, r_+)$ such that for all $u \geq u_0(R)$:

$$r^\frac{1}{2} |\phi|(u, v) + r^\frac{3}{2} |D_v \phi|(u, v) \leq C' \cdot \left( \min(u, v) \right)^{-\frac{1}{2} - \frac{\sqrt{1 - 4\rho_0|e|}}{2} + \epsilon},$$ \hspace{1cm} (3.6)

$$|D_v \psi| \leq C' \cdot \Omega^2 \cdot \left( \min(u, v) \right)^{-1 - \frac{\sqrt{1 - 4\rho_0|e|}}{2} + \epsilon}, \hspace{1cm} (3.7)$$

$$|\psi|_{L^2(u_R)}(u) \leq C' \cdot u^{-\frac{1}{2} - \frac{\sqrt{1 - 4\rho_0|e|}}{2} + \epsilon},$$ \hspace{1cm} (3.8)

$$|D_v \psi|_{|v|^{2} \geq 2 + R^+}(u, v) \leq C' \cdot v^{-1 - \sqrt{1 - 4\rho_0|e|} + \epsilon} \cdot \log(u), \hspace{1cm} (3.9)$$

$$|Q - e|(u, v) \leq C' \cdot \left( u^{-1 - \sqrt{1 - 4\rho_0|e|} + \epsilon} \cdot \log(u) \right)_{|v|^{2} \geq 2 + R^+} + v^{-2 - \sqrt{1 - 4\rho_0|e|} + \epsilon} \cdot |v| \leq 2 + R^+ \}, \hspace{1cm} (3.10)$$

**Remark 14.** As explained in sections 1.1 and 1.2.1, most of the estimates are expected to be (almost) optimal \(^{39}\) in the limit $\epsilon \to 0$, including (3.4), (3.5), (3.8), (3.10). While (3.6) is expected to be optimal in a region near null infinity $\{v \geq 2u + R^+ \}$, it is not optimal on any constant $r$ curve or on the event horizon for reasons explained in sections 1.2.1 and 1.2.4. However, the $L^2$ bound on the event horizon from (3.4) gives point-wise bounds $|\phi|(u, v) + |D_v \phi|(u, v) \leq u^{-\frac{1}{2} - \frac{\sqrt{1 - 4\rho_0|e|}}{2} + \epsilon}$ along a dyadic sequence $(u_n)$ that are expected to be almost optimal in the limit $\epsilon \to 0$, c.f. section 1.2.1.

**Remark 15.** Notice that we stated (3.7) for $v > 0$, in particular far away from $-\infty$. What happens near the bifurcation sphere is much more subtle, as explained in Remark 14 and Remark 25. Indeed we can prove that for any $V_0 \in \mathbb{R}$, $|D_v \psi|(u, v) \leq \Omega^2$ in $\{v \geq V_0 \}$. The constant in the inequality blows up as $V_0 \to -\infty$ however, remarkably one can still prove that $|D_v \psi|(u, v) \leq e^{-2R^+ - \epsilon}$ on the whole space-time, in conformity with our hypothesis \[^{32}\] that the regular derivative of the scalar field $\Omega^{-1} D_v \phi_0 \sim e^{2R^+} D_v \phi_0$ is initially bounded. Notice that at fixed $v = V_0$, $\Omega^{-2} \phi_0$ is a regular vector field transverse to the event horizon and it degenerates when approaching the bifurcation sphere $v = -\infty$.

This unconditional result is issued as a combination of several other statements, that we believe to be of independent interest. These are the object of the following sub-sections.
3.2 Energy boundedness and integrated local energy decay for small charge and scalar field energy data, no point-wise decay assumption

First we want to emphasize that energy boundedness and integrated local energy decay have nothing to do with point-wise property of the data. This is the content of the following short boundedness theorem:

**Theorem 3.2.** [Boundedness of the energy and integrated local energy decay for small data] In the context of Theorem 3.1, assume hypothesis (4) and (5) for some \( \eta > 0 \) and \( \delta < \delta_0(M, \rho) \) sufficiently small. Suppose also that \( \lim_{r \to +\infty} \phi_0(r) = 0 \).

Then statements (1), (2) and (3) are true: the charge stays small, the energy is bounded and local integrated energy decay holds.

**Remark 16.** Notice that the energy boundedness statement (3.1) actually consists of two estimates: the first inequality is much more subtle \(^{41}\) to prove than the second, but necessary to later prove decay. The only difficulty of the second inequality is to prove the boundedness of the non-degenerate energy \(^{42}\) using the red-shift effect, which is more difficult in this context than for the uncharged case, c.f. section 4.

Then there are a few conditions we would like to relax in Theorem 3.1, at the cost of other assumptions. This is the object of the next section.

3.3 Energy time decay without arbitrary charge smallness or point-wise decay assumption, conditional on energy boundedness

In this section, we try to relax some of the assumptions made in Theorem 3.1. The most important is that energy decay should not rely on the point-wise decay of the initial data. This is because we aim at providing estimates with no symmetry assumptions, where point-wise bounds are harder to propagate. We can prove that this is indeed the case, c.f. Theorem 3.3.

Another important physical fact is that the decay rate should only depend on the future asymptotic charge of the Maxwell equation \( e \), and in particular not on the mass of the black hole. \(^{43}\) The first theorem that we state does not do justice to this fact, for the charge is required to be smaller than a constant depending on the parameters of the black holes, in particular on the mass. This is however due to the difficulty to prove energy boundedness estimates in the presence of a large charge. We overcome it using the red-shift effect and the argument requires such a smallness assumption on the initial charge.

In the following result, assuming the boundedness of the energy and an integrated local energy decay estimate, we can retrieve energy decay for \( q_0|e| \) smaller than a universal numeric constant. \(^{44}\)

We now state the theorem, in which we assume energy boundedness (3.1) and an integrated local energy decay estimate (3.2), but no point-wise decay of the data. Smallness of the scalar field initial energy is still required but the initial charge \( e_0 \) is only required to satisfy \( q_0|e_0| < \frac{1}{4} \), which makes in principle the class of admissible initial data larger.

**Theorem 3.3.** [Almost Optimal decay for a small scalar field energy and larger \( q_0|e| \)]. In the same context as for Theorem 3.1, we make the different following assumptions:

1. The initial energy is finite: \( \mathcal{E} < \infty \), where \( \mathcal{E} \) is defined in section 2.3.
2. \( q_0(r) \) admits a limit \( e_0 \in \mathbb{R} \) as \( r \to +\infty \).
3. The initial asymptotic charge is smaller than an universal constant:
   \[ q_0|e_0| < \frac{1}{4} \]
4. We exclude constant solutions by the condition \( \lim_{r \to +\infty} \phi_0(r) = 0 \).
5. We have the finiteness condition: for every \( 0 \leq p < 2 + \sqrt{1 - 4q_0|e_0|} \),
   \[ \mathcal{E}_p < +\infty \]
6. We have the smallness condition: for some \( \eta > 0 \) and some \( \delta > 0 \),
   \[ \mathcal{E}_{1+\eta} < \delta \]
7. Energy boundedness (3.1) and integrated local decay (3.2) hold.

Then there exists \( C = C(M, \rho) > 0 \), \( \delta_0 = \delta_0(e_0, M, \rho) > 0 \) such that if \( \delta < \delta_0 \) we have the following

---

41This is because we want to have a right-hand-side that depends only on the scalar field and not on the charge.

42The boundedness of the degenerate energy is easy, using the vector field \( \partial_t \), if we do not care of initial charge terms.

43Because we study the gravity uncoupled problem, it should not depend on the charge of the black hole \( \rho \) either. However, in the gravity coupled problem \( \rho = e \) so the decay should depend on the charge of the black hole this time.

44That we do not prove to be optimal.
1. The charge is bounded and there exists a future asymptotic charge $e \in \mathbb{R}$ such that for every $R_1 > r_+$

$$\lim_{t \to +\infty} Q_{\gamma+}(t) = \lim_{t \to +\infty} Q_{x+}(t) = \lim_{t \to +\infty} Q_{\gamma R_1}(t) = e.$$  

Moreover the charge is close to its initial asymptotic value: on the whole-space-time

$$|Q(u, v) - e_0| \leq C \cdot \tilde{\mathcal{E}}_{1 + \eta},$$

in particular $|e - e_0| \leq C \cdot \delta$ and therefore if $\delta$ is small enough, $q_0|e| < \frac{1}{4}$.

2. Moreover, boundedness of $r^p$ weighted energies and energy decay hold:

for every $0 \leq p < 2 + \sqrt{1 - 4q_0|e|}$, if $\delta \leq \delta_p$ then statements 1 and 2 of Theorem 3.1 are true.

3. If additionally, one assumes initial point-wise decay under the form:

for all $\epsilon_0 > 0$, there exist $C_0 = C_0(\epsilon_0) > 0$, such that

$$r|D_v \psi_0(r)| + |\psi_0(r)| \leq C_0 \cdot r^{-1 - \sqrt{1 - 4q_0|e|} + \epsilon_0},$$

$$|D_v \phi_0(r)| \leq C_0 \cdot \Omega,$$

then one can retrieve point-wise decay estimates: statement 6 of Theorem 3.1 is true.

Remark 17. Finally, note that the requirement $q_0|e_0| < \frac{1}{4}$ should be understood — together with the initial scalar field energy smallness— as $q_0|Q| < \frac{1}{4}$ everywhere.

Therefore, it is also equivalent to state it as $q_0|e| < \frac{1}{4}$, which is what we do in section 6.

Remark 18. Note that in both theorems, the only smallness initial energy condition is on $\mathcal{E}_{1 + \eta}$, to control the variations of $Q$. In particular, no smallness condition is imposed on the initial $r^p$ energies for $p > 1 + \eta$, even though we require them to be finite. This is related to the fact that the only smallness needed for this problem is that of the charge $Q$ but not of the scalar field.

Remark 19. Another version of Theorem 3.3 is proven in Appendix A. While this different version requires more point-wise decay of the initial data, it also proves more $v$ decay for $D_v \psi$, at the expense of growing $u$ weights. In particular, we can prove that in $r^p$ the gauge $A_v = 0$, denoting $X = r^2 \partial_u$, $X^n \psi(u, v)$ admits a finite limit when $v \to +\infty$, for fixed $u$.

4 Reduction of the Cauchy problem to a characteristic problem and global control of the charge

In this section, we explain how the Cauchy problem can be reduced to a characteristic problem, with suitable hypothesis. This is the step that we described earlier as “boundedness in terms of initial data”, which is simpler than boundedness with respect to be past values that we prove in later sections.

The main object is to show how the smallness of the initial charge propagates to the whole space-time, providing the initial energy of the scalar field is also small.

The results split into four parts: in the first Proposition 4.1, we show how the smallness of the scalar field energy and the smallness of the initial charge imply energy boundedness and integrated local energy decay, because the charge stays small everywhere. This is the boundedness part of Theorem 3.2.

Then in Proposition 4.2, we show that if the two latter hold, then, if the initial energy is small enough, the charge stays close to its limit value at spatial infinity $e_0$ on the whole space-time, provided $q_0|e_0| < \frac{1}{4}$, a bound which is independent of the black holes parameters. This part is related to Theorems 3.3, more precisely to statement 4.

In Proposition 4.3, we show how certain initial data point-wise decay rates can be extended towards a fixed forward light cone. This is useful for point-wise decay rates in Theorem 3.3 or as an alternative the finiteness assumption of higher order $r^p$ weighted initial energy, like in the statement of Theorem 4.1, although this is technically a stronger statement and that initial point-wise decay is not needed for energy decay.

Finally in Proposition 4.4, we prove that — provided the initial $r^p$ weighted energies of large order are finite — they are still finite on a constant $u$ hyper-surface transverse to null infinity.

To do this, we make use of a $r^n u^s$ weighted energy estimate, for $s > 0$.

In the next sections, whose goal is to prove energy $u$ decay, the strategy is in contrast to what we do here: we will aim at estimating the energy in terms of its past values. This is sensibly more difficult than to bound the energy in terms of the data.

This is why in this section, we only state the results, while their proofs are postponed to appendix B. This allows us to focus on the main difficulty of the paper — the energy decay — and to avoid repeating very similar arguments.

We denote $\Sigma_0 = \{ t = 0 \}$ the 3-dimensional Riemannian manifold on which we set the Cauchy data $(\phi_0, D_v \phi_0, Q_0)$ satisfying the following constraint equation

45[The result that we prove is actually gauge invariant but we state it here in the gauge $A_v = 0$ for simplicity.]}
\[ \partial_t Q_0 = q \alpha r^2 \mathcal{A}(\phi_0 D_t \phi_0). \]  

(4.1)

We first show energy boundedness and global smallness of the charge, on condition that the data is small.

**Proposition 4.1.** Suppose that there exists \( p > 1 \) such that \( \hat{E}_p < \infty \) and that \( Q_0 \in L^\infty(\Sigma_0) \).

Assume also that \( \lim_{r \to +\infty} \phi_0(r) = 0 \).

We denote \( Q_0^\infty = \| Q_0 \|_{L^\infty(\Sigma_0)} \).

There exists \( r_0 = R_0(M, \rho) \) large enough, \( \delta = \delta(M, \rho) > 0 \) small enough and \( C = C(M, \rho) > 0 \) so that for all \( R_1 > R_0 \), and if

\[ Q_0^\infty + \hat{E}_p < \delta, \]

then for all \( u \geq u_0(R_1) \):

\[ E_{R_1}(u) \leq C \cdot \mathcal{E}. \]  

(4.2)

Also for all \( v \leq v_{R_1}(u) \):

\[ \int_{\tilde{u}(v)}^{+\infty} r^2 |D_v \phi|^2 \frac{d\nu}{\Omega^2} (u', v) du' \leq C \cdot \mathcal{E}, \]

(4.3)

where \( \tilde{u}(v) = u_0(v) \) if \( v \leq v_0(R_1) \) and \( \tilde{u}(v) = u_{R_1}(v) \) if \( v \geq v_0(R_1) \).

Moreover for all \( (u, v) \) in the space-time :

\[ |Q|(u, v) \leq C \cdot \left( Q_0^\infty + \hat{E}_p \right). \]  

(4.4)

Finally there exists \( C_1 = C_1(R_1, M, \rho) > 0 \) such that for all \( u \geq u_0(R_1) \):

\[ \int_{\{ t \leq v_{R_1}(u) \} \cap \{ r \leq R_1 \} } \left( |D_t \phi|^2 (u', v) + |D_v \phi|^2 (u', v) + |\phi|^2 (u', v) \right) \Omega^2 du dv \leq C_1 \cdot \mathcal{E}, \]

(4.5)

**Remark 20.** As a by product of our analysis, one can show that

1. \( Q_0 \) admits a limit \( e_0 \in \mathbb{R} \) when \( r \to +\infty \). This just comes from \( \hat{E}_p < \infty \) for \( p > 1 \).
2. The future asymptotic charge exists: there exists \( e \in \mathbb{R} \) such that for every \( R_1 > r_+ \)

\[ \lim_{t \to +\infty} Q_{v_{R_1}}(t) = e. \]

3. The asymptotic charges are small: \( \max \{|e|, |e_0|\} \leq C \cdot \left( Q_0^\infty + \hat{E}_p \right) < C \cdot \delta. \)

**Remark 21.** Notice also that no qualitative strong decay is required on the data for the statement \(^{46}\) The condition \( \lim_{r \to +\infty} \phi_0(r) = 0 \) is present simply to exclude constant solutions that do not decay.

**Remark 22.** Notice that \[ \int_{\tilde{u}(v)}^{+\infty} r^2 |D_v \phi|^2 \frac{d\nu}{\Omega^2} (u', v) du' \leq e^{-2K_+v} \int_{\tilde{u}(v)}^{+\infty} e^{-2K_+u} |D_v \phi|^2 du \leq e^{-2K_+v} \int_{\tilde{u}(v)}^{+\infty} e^{-2K_+u} du \leq 1. \]

This subtlety is related to the degenerescence of the vector field \( \Omega^2 \partial_u \) as \( v \) tends to \( -\infty \), c.f. Remark \[13\].

Now, we want to prove the following fact: in the far away region, the only smallness condition required on the charge is \( q_0 |Q| < \frac{1}{4} \). Provided energy boundedness and the integrated low energy decay hold, one can prove that decay of the energy follows, in the spirit of Theorem \[12, 13\].

For this, we want to show that, if \( q_0 |e_0| < \frac{1}{4} \), then for small enough initial energies, \( |Q - e_0| \) is small:

**Proposition 4.2.** Suppose that there exists \( 1 < p < 2 \) such that \( \hat{E}_p < \infty \).

It follows that there exists \( e_0 \in \mathbb{R} \) such that

\[ \lim_{r \to +\infty} Q_0(r) = e_0. \]

Without loss of generality one can assume that \( 1 < p < 1 + \sqrt{1 - 4q_0 |e_0|} \).

Assume also that \( \lim_{r \to +\infty} \phi_0(r) = 0 \).

Now assume \( \int_{\tilde{u}(v)}^{+\infty} r^2 |D_v \phi|^2 \frac{d\nu}{\Omega^2} (u', v) du' \leq C \cdot \mathcal{E}, \)

(4.6)

(4.7)

\(^{46}\) Even though the finiteness of the energy gives already some mild decay towards spatial infinity.
where \( \tilde{u}(v) = u_0(v) \) if \( v \leq v_0(R) \) and \( \tilde{u}(v) = u_R(v) \) if \( v_0(R) \leq v \leq v_R(\cdot) \).

Assume also \( (1.3) \) : there exists \( R_0 \equiv R_0(M, \rho) > r_+ \) such that for all \( R_1 > R_0 \), there exists \( C_1 = C_1(R_1, M, \rho) > 0 \) such that

\[
\int_{\{ \omega \leq u \cap \{ r \leq R_1 \}} \left( |D_r\phi|^2(u', v) + |D_r\phi|^2(u', v) + |\delta|^2(u', v) \right) \Omega^2 dudv \leq C_1 \cdot \mathcal{E}.
\]  

(4.8)

Make also the following smallness hypothesis : for some \( \delta > 0 \):

\[
\tilde{E} < \delta,
\]

(4.9)

\[
q_0|e_0| < \frac{1}{4}.
\]

There exists \( \delta_0 = \delta_0(e_0, M, \rho) > 0 \) and \( C = C(M, \rho) > 0 \) such that if \( \delta < \delta_0 \) then for all \( (u, v) \) in the space-time:

\[
|Q(u, v) - e_0| \leq C \cdot \tilde{E}_p.
\]

(4.10)

Moreover, there \( (4.11) \) exists \( \delta_p = \delta_p(e_0, p, M, \rho) > 0 \) and \( C' = C'(e_0, p, M, \rho) > 0 \) such that if \( \delta < \delta_p \) then for all \( u \geq u_0(R) \):

\[
E_p[\psi](u) \leq C' \cdot \tilde{E}_p.
\]

(4.11)

Remark 23. Note that in this context, we also have

\[
|\epsilon - \epsilon_0| \leq C \cdot \tilde{E}_p.
\]

(4.12)

so the difference between the initial charge and the asymptotic charge is arbitrarily small, when the initial energies are small. This fact will be used extensively, in particular in the statement of the theorems.

In spherical symmetry, some point-wise decay rates propagate, at least in the past of a forward light cone. This is important to derive point-wise decay estimate, because in the proofs of section \( \ref{section:4} \), an initial decay estimate is needed for \( D_r\psi \) on the forward light cone \( \{ u = u_0(R) \} \).

This is also useful to obtain the finiteness \( \ref{4.11} \) of the \( r^p \) weighted energies, for larger \( p \).

We now prove point-wise bounds in the past of a forward light cone in the following proposition :

**Proposition 4.3.** In the conditions of Proposition \( \ref{4.3} \), assume moreover that there exists \( \omega \geq 0 \) and \( C_0 > 0 \) such that

\[
r|D_r\psi| + |\psi| \leq C_0 \cdot r^{-\omega}.
\]

(4.13)

Then in the following cases

1. \( \omega = 1 + \theta \) with \( \theta > \frac{\min(\omega)}{4} \)

2. \( \omega = \frac{1}{2} + \beta \) with \( \beta \in \left( -\frac{\sqrt{1 - 4\rho|e_0|\rho|e_0|}}{2}, \frac{\sqrt{1 - 4\rho|e_0|\rho|e_0|}}{2} \right) \), if \( q_0|e_0| < \frac{1}{4} \),

there exists \( \delta = \delta_0(\epsilon_0, \omega, M, \rho) > 0 \) and \( R_0 = R_0(\omega, e_0, M, \rho) > r_+ \) such that for all \( u \leq u_0(R) \):

\[
|D_r\psi|(u, v) \leq C_0' \cdot r^{-1-\omega'},
\]

(4.14)

where \( \omega' = \min(\omega, 1) \).

In that case, for every \( 0 < p < 2\omega' + 1 \), we have the finiteness of the \( r^p \) weighted energy on \( \mathcal{V}_{u_0(R)} \):

\[
E_p[\psi](u_0(R)) < \infty.
\]

**Remark 24.** Notice that, even for very decaying data, one cannot obtain a better \( r \) decay for \( D_r\psi \) than \( r^{-2} \). This is in contrast to the uncharged case \( q_0 = 0 \) where decay rate was \( r^{-3} \) in spherical symmetry, due to the sub-criticality with respect to \( r \) decay of the uncharged wave equation, c.f. the discussion in section \( \ref{1.2.1} \)

**Remark 25.** Making an hypothesis on point-wise decay can be thought of as an alternative to circumvent the assumption that \( \tilde{E} \to \infty \) to prove that \( E_p[\psi](u_0(R)) \to \infty \), like we do in the next proposition. This is why we do not need to assume any higher order initial \( r^p \) weighted energy boundedness for Theorem \( \ref{3.1} \) : the result we need is already included in the point-wise assumption, that is stronger.

---

\(^{47}\) The dependence of \( \delta_0 \) on \( p \) only exists as \( p \) approaches \( 1 + \sqrt{1 - 4\rho|e_0|\rho|e_0|} \).

\(^{48}\) Notice that we do not prove or require the boundedness of such energy by the initial data. Indeed, for the decay, only the finiteness of these energies is required but not their smallness, unlike for the smaller \( p \) energies which ensure that the charge \( Q \) is small.
Finally, we prove that higher order $r^{p}$ weighted energies boundedness holds on a characteristic constant $u$ surface transverse to null infinity, provided it holds on the initial surface. This proves the boundedness of higher $r^{p}$ weighted energies, for $p$ close to 3, in order to close the argument of section 4.6.

This also allows us to avoid making initial point-wise decay assumptions on the data, starting with “weaker” weighted $L^{2}$ boundedness hypotheses.

The proof mainly makes use of a $r^{p}|u|^{q}$ weighted estimate in the past of a forward light cone.

**Proposition 4.4.** Suppose that for all $0 \leq p < 2 + \sqrt{1 - 4q_{0}|e_{0}|}$, $\mathcal{E}_{p} < \infty$.

We also assume the other hypotheses of Theorem 2.3

Then for all $0 \leq p < 2 + \sqrt{1 - 4q_{0}|e_{0}|}$, there exists $\delta_{p} = \delta_{p}(e_{0}, p, M, \rho) > 0$, such that if $\delta < \delta_{p}$ then for all $u \leq u_{0}(R)$:

$$E_{p}[^{\psi}](u) < \infty.$$  

5 Energy estimates

In the former section, we explained how the global smallness of the charge can be monitored from assumptions on the initial data, and how various energies on characteristic surfaces or domains are bounded by the data on a space-like initial surface $\Sigma_{0}$.

Taking this for granted, we turn to the proof of energy decay. We are going to assume that the charge is suitably small — according to the needs — and that the energies on the initial characteristic surface $V_{u_{0}(R)}$ are finite.

The goal of this section and the next section is to prove $u$ decay for a characteristic initial value problem on the domain $\mathcal{D}(u_{0}(R), +\infty)$ — c.f. Figure 4 — with data on $V_{u_{0}(R)}$, assuming that the charge is sufficiently small everywhere.

As explained in the introduction, the proof of decay requires, similarly to the wave equation, three different estimates.

The first is a robust energy boundedness statement, which takes the red-shift effect into account. This allows us in principle to prove point-wise boundedness of the scalar field, following the philosophy of [13].

The second is an integrated local energy estimate, also called Morawetz estimate in reference to the seminal work [11], which is a global estimate on all derivatives but with a sub-optimal $r$ weight at infinity.

Proving and closing these two estimates in the goal of the present section.

Finally the last ingredient which is going to be developed in section 6 is a sufficient condition which is going to be developed in section 6 is a $r^{p}$ weighted estimate, which gives inverse-polynomial time decay of the energy over the new foliation.

The main difficulty, compared to the wave equation, is that these estimates are all very coupled. In particular the energy boundedness statement is not established independently and requires the use of the Morawetz estimate because of the charge terms that cannot be absorbed easily otherwise. These terms are moreover “critical” with respect to $r$ decay, more precisely they possess the same $r$ weight as the positive terms controlled by the energy while their sign is not controllable.

Moreover, whenever an estimate is proven, a term of the form $q_{0}Q(\phi D\phi)$ arises and these terms necessitate a control of both the zero order term and the derivative at the same time to be absorbed, even with a very small $q_{0}Q$. This is already very demanding in the large $r$ region where the estimates are very tight while in the bounded $r$ region, we need to use to use the smallness of $q_{0}Q$ crucially to absorb the $r$ weights.

We start by proving a Morawetz estimate, which bounds bulk terms with sub-optimal weights but does not control the boundary terms, proportional to the degenerate energy.

Then we prove a Red-Shift estimate, which gives a good control of the regular derivative of the scalar field $\Omega^{-2}D_{u}\phi$.

Then we control the boundary terms using the Killing vector field $\partial_{t}$.

The difficulty in this last estimate is the presence of the electromagnetic terms which need to be absorbed in the energy of the scalar field. We require the full strength of the Morawetz and the Red-Shift estimate, together with the smallness of $q_{0}Q$ to overcome this issue.

Before starting, we are going to recall a few notations from section 2.5. We have defined the non-degenerate energy $E(u)$ as

$$E(u) = \int_{u}^{+\infty} r^{2} \frac{|D_{u}\phi|^{2}}{\Omega^{2}}(u', \phi_{R}(u))du' + \int_{\phi_{R}(u)}^{+\infty} r^{2} |D_{v}\phi|^{2}(u, v)dv.$$  

It will also be convenient to have a short notation $E_{+}(u_{1}, u_{2})$ for the sum of all the boundary terms having a role in the energy identity: for all $u_{0}(R) \leq u_{1} \leq u_{2}$:

$$E_{+}(u_{1}, u_{2}) := E(u_{2}) + E(u_{1}) + \int_{r_{n}(u_{1})}^{r_{n}(u_{2})} r^{2} |D_{v}\phi|^{2}_{\|_{L^{+}}}(v)dv + \int_{u_{1}}^{u_{2}} r^{2} |D_{u}\phi|^{2}_{\|_{Z^{+}}}(u)du.$$  

\[\text{The dependence of } \delta_{p} \text{ on } p \text{ only exists as } p \text{ approaches } 2 + \sqrt{1 - 4q_{0}|e_{0}|}.\]

\[\text{The converse is always true even for the wave equation: the Morawetz estimate cannot close without the boundedness of the energy. However, for the wave equation the boundedness of the energy is already a closed independent estimate.}\]
It will also be useful for the Red-Shift estimate to have an equivalent notation for the degenerate energy

$$E_{deg}(u) = \int_{\Omega^\infty(r(u), vR(u))} r^2 |D_u \phi|^2 (u', vR(u)) du' + \int_{\Omega^\infty(\infty, vR(u))} r^2 |D_v \phi|^2 (u, v) dv,$$

and for the sum of all scalar field terms appearing when one contracts $\mathbb{T}^{\mu \nu}_{SP}$ with the vector field $\partial_t$:

$$E_{deg}(u_1, u_2) := E_{deg}(u_2) + E_{deg}(u_1) + \int_{\Omega^\infty(u_1)} r^2 |D_v \phi|^2 (u) dv + \int_{\Omega^\infty(u_2)} r^2 |D_u \phi|^2 (u) du.$$  

5.1 An integrated local energy estimate

The goal of this upcoming Morawetz estimate is to control the derivative of $\phi$ but also $|\phi|^2$ the term of order 0. To do so we need to proceed in two times, using the vector field $X_\alpha = -r^{-\alpha} \partial_r$.

First we bound the zero order bulk term on a region $r \leq R_0$ for some $R_0(M, \rho) > r_+$, at the cost of some boundary terms and using and the modified current $\tilde{J}_{\mu}^\alpha(\phi)$, without obtaining any control of the boundary terms or of the electromagnetic bulk term. It turns out that the boundary term on the time-like boundary $\{ r = R_0 \}$ coming from $\mathbb{T}^{\mu \nu}_{SP}$ has the right sign. Other boundary terms appear due to the use of $\chi$ in the modified current $\tilde{J}_{\mu}^\alpha(\phi)$ but they can be absorbed using the red-shift effect, more precisely the smallness of $\Omega^2$ for $R_0$ close enough to $r_+$.

In a second time, we use $X_\alpha$ — for some large $\alpha > 1$ — and the modified current $\tilde{J}_{\mu}^\alpha(\phi)$ with an appropriate $\chi_\alpha$ on the whole domain $\Omega(u_1, u_2)$. While in the bulk term we control the derivatives of $\phi$ everywhere, we only control the zero order term $|\phi|^2$ near infinity, i.e. on a region $[R_\alpha, +\infty[$. The remarkable key feature is that $R_\alpha$ tends to $r_+$ as $\alpha$ tends to infinity.

Therefore it is enough to take $\alpha > 1$ large enough to have $R_\alpha < R_0 < R$ and we can take a linear combination of the two identities to obtain the control of the zero and first order terms.

The other electromagnetic bulk term are then absorbed using the smallness of the charge.

In more details we are going to prove the following proposition:

**Proposition 5.1.** There exists $\alpha = \alpha(M, \rho) > 1$, $\delta = \delta(M, \rho) > 0$ and $C = C(M, \rho) > 0$ so that if $\|Q\|_{L^\infty(\Omega(u_0(R), +\infty))} < \delta$, then for all $u_0(R) \leq u_1 < u_2$ we have:

$$\int_{\Omega(u_1, u_2)} \left( \frac{|D_u \phi|^2}{r^{\alpha-1}} + \frac{|D_v \phi|^2}{r^{\alpha-1}} \right) \Omega^2 dudv \leq C \cdot E_{deg}(u_1, u_2),$$  

(5.1)

**Proof.** We start by a computation, based on the identities of section C.2 and on (C.11). In the identities we use the vector field $X_\alpha = -r^{-\alpha} \partial_r$, and the function $\chi(r) = \frac{r^{-\alpha-1}}{2} \Omega^2$. We get, for all $\alpha \in \mathbb{R}$

$$\nabla^\mu \tilde{J}_{\mu}^\alpha(\phi) = \frac{\alpha}{r^{\alpha+1}} \left( |D_u \phi|^2 + |D_v \phi|^2 \right) + \frac{\Omega^2}{(\alpha-1)} |\phi|^2 + q_0 Q r^{-\alpha-2} \mathbb{I}(D_1 \phi),$$  

where we also used (C.12) for the last term.

We first take care of the region $\{ r_+ \leq r \leq R_0 \}$ where we only aim at controlling the 0 order term $|\phi|^2$.

For this we are going to prove the following lemma:

**Lemma 5.2.** There exists $C = C(M, \rho) > 0$ and $r_+ < R_0 < R, R_0 = R_0(M, \rho)$ such that

$$\int_{\Omega^\infty(u_1, u_2) \cap \{ r \leq R_0 \}} \Omega^2 r^2 |\phi|^2 dudv \leq C \cdot \left( E_{deg}(u_1, u_2) + \int_{\Omega^\infty(u_1, u_2) \cap \{ r \leq R_0 \}} \Omega^2 q_0 |Q| |\phi| |D_v \phi| dudv \right),$$  

(5.3)

**Proof.** Using (C.13) at $r = r_+$, i.e. where $\Omega^2(r) = 0$, we can prove that for all $\beta \in \mathbb{R}$

$$\Box(\Omega^2 r^{-\beta}) = 4K_+^2 r^{-\beta}.$$

We take $\beta = 1$. Now since $K_+ > 0$, the coefficients of $\Box(\Omega^2 r^{-1})$ only depend on $\rho$ and $M$ and by continuity, there exists $R_\alpha = R_\alpha(M, \rho)$ such that for all $r_+ \leq r \leq R_\alpha$:

$$\Box(\Omega^2 r^{-1}) > 2K_+^2 r^{-1}.$$

Then, we take $r_+ < R_0 < R_\alpha$ with $R_\alpha$ to be chosen later and we integrate $\Box(\Omega^2 r^{-1})$ over $\Omega$ from $u_1$ to $u_2$.

$$\int_{\Omega^\infty(u_1, u_2) \cap \{ r \geq R_0 \}} \Omega^2 r^2 |\phi|^2 dudv + \frac{1}{2} \int_{\Omega^\infty(u_1, u_2) \cap \{ r \geq R_0 \}} \left( r^2 |D_u \phi|^2 + \Omega^2 (\Omega^2 - r \cdot 2K_+) |\phi|^2 - \Omega^2 r \partial_r (|\phi|^2) \right) (u', vR(u_2)) du' + E_{R_\alpha}(u_1, u_2) \leq \frac{1}{2} \int_{\Omega^\infty(u_1, u_2) \cap \{ r \geq R_0 \}} \left( r^2 |D_u \phi|^2 + \Omega^2 (\Omega^2 - r \cdot 2K_+) |\phi|^2 - \Omega^2 r \partial_r (|\phi|^2) \right) (u', vR(u_1)) du' + \int_{\Omega^\infty(u_1, u_2) \cap \{ r \geq R_0 \}} 2q_0 |Q| \Omega^2 |\phi| |D_v \phi| dudv + \int_{vR(u_1)} r^2 |D_u \phi|^2 |D_v \phi|^2 dudv.$$

$^{51}$dvol is defined appendix C
where $\dot{E}_{R_0}(u_1, u_2)$, the $L^2$ flux through $\{r = R_0\}$ is defined by

$$\dot{E}_{R_0}(u_1, u_2) = \int_{r=R_0} R_0^2 \left( \frac{|D_u \phi|^2 + |D_v \phi|^2}{4} - \frac{R_0 \Omega^2(R_0)}{8} (\partial_u |\phi|^2 - \partial_v |\phi|^2) + \frac{\Omega^2(R_0) |\phi|^2}{4} (\Omega^2(R_0) - R_0 \cdot 2 K(R_0)) dt. \right)$$

First we want to absorb all the boundary terms except the $\dot{E}_{R_0}$ term into a $C'(\rho, M) \cdot E_{deg}^+(u_1, u_2)$ term. We start to write, using that $\Omega^2 r|\partial_u |\phi|^2| \leq \Omega^4 |\phi|^2 + r^2 |D_u \phi|^2$, we see that

$$r^2 |D_u \phi|^2 \cdot - \Omega^2 r \cdot K \cdot |\phi|^2 \leq r^2 |D_u \phi|^2 + \frac{\Omega^2}{2} (\Omega^2 - r \cdot 2 K)|\phi|^2 - \Omega^2 r \cdot |\partial_u |\phi|^2).$$

Now we want to control the 0 order term on this constant $v$ boundaries.

For this we use a version of Hardy’s inequality $2.16$ in $u$ of the form

$$\int_{u_1}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v(u(R))) du \leq 4 \int_{u_1}^{+\infty} \Omega^2 r^2 |D_u \phi|^2(u, v(u(R))) du + 2\Omega^2(2) \cdot R \cdot |\phi|^2(u, v(R)).$$

The last term of the right-hand side can be bounded, using Hardy’s inequality $2.17$ in $v$:

$$R |\phi|^2(u, v(R)) \leq \Omega^{-2}(R) \int_{R(R)}^{+\infty} r^2 |D_u \phi|^2(u, v(v)) dv.$$ 

Notice also that $r^2 \cdot 2K(r) = 2(M - \frac{\rho^2}{r^2}) \geq 2K + r^2$. Therefore, we have

$$\int_{u_1}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v(R)(u(R))) du \leq \frac{2}{K + r^2} \int_{u_1}^{+\infty} \Omega^2 r^2 |D_u \phi|^2(u, v(R)(u(R))) du + 2 \int_{R(R)}^{+\infty} r^2 |D_u \phi|^2(u, v(v)) dv.$$ 

Then, taking $R$ large enough so that $\frac{\rho}{R} < \frac{2}{K + r^2}$, we get

$$\int_{u_1}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v(R)(u(R))) du \leq 2 \int_{u_1}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v(R)(u(R))) du \leq \frac{2R_0}{K + r^2} E_{deg}^+(u_1, u_2).$$

Therefore, combining with $4.3$, it is clear that there exists a constant $C' = C'(M, \rho, R_0) > 0$ such that

$$\frac{(K + r^2)^2}{4} \int_{D(u_1, u_2) \cap \{r \leq R_0\}} \Omega^2 r^2 |\phi|^2 d\mu + \dot{E}_{R_0}(u_1, u_2) \leq C' \cdot E_{deg}^+(u_1, u_2) + \int_{D(u_1, u_2) \cap \{r \geq R_0\}} 2\rho \Omega^2 |\phi||D_\phi|d\mu.$$ 

(5.7)

Then we take care of the $\dot{E}_{R_0}$ term. First we want to absorb the $\frac{R_0 \Omega^2(R_0)}{8} (\partial_u |\phi|^2 - \partial_v |\phi|^2)$ into the first and the third term of $\dot{E}_{R_0}$. For this, we simply notice that $\Omega^2 |\partial_u |\phi|^2| \leq \Omega^4 |\phi|^2 + R_0 |D_\mu \phi|^2$. We then get

$$\dot{E}_{R_0}(u_1, u_2) \geq \int_{r = R_0} R_0^2 \left( \frac{|D_u \phi|^2 + |D_v \phi|^2}{8} - \Omega^2(R_0) \frac{R_0 \cdot K(R_0)}{2} \right) dt.$$ 

Then for some small $\epsilon > 0$ to be chosen later, we choose $R_0$ sufficiently close from $r_+$ so that

$$\sup_{r_+ \leq r \leq R_0} \Omega^2 r^2 \frac{K(r)}{2} < \epsilon.$$ 

Then, applying the mean-value theorem in $r$ on $[(1 - \epsilon)R_0, R_0]$ for $\epsilon$ sufficiently small so that $r_+ < (1 - \epsilon)R_0$, we see that there exists $(1 - \epsilon)R_0 < R_0 < R_0$ so that

$$\int_{r = R_0} \Omega^2 r^2 |\phi|^2 d\mu = \int_{D(u_1, u_2) \cap \{(1 - \epsilon)R_0 \leq r \leq R_0\}} \Omega^2 |\phi|^2 d\mu.$$ 

The presence of $\Omega^2$ is the integral is due to the integration in $r$ : indeed $d\mu = \Omega^2 d\mu.$

Thus

$$\int_{r = R_0} \Omega^2(R_0) \frac{R_0 \cdot K(R_0)}{2} dt \leq \epsilon \int_{D(u_1, u_2) \cap \{(1 - \epsilon)R_0 \leq r \leq R_0\}} \Omega^2 |\phi|^2 d\mu \leq \epsilon \int_{D(u_1, u_2) \cap \{r_+ \leq r \leq R_0\}} \Omega^2 |\phi|^2 d\mu.$$ 

Hence if $0 < \epsilon < \frac{(R_0 - r_+)}{r_+}$, applying the former identities for $R_0$, there exists a constant $C = C(M, \rho) > 0$ and a $R_0 = R_0(M, \rho) > r_+$ such that

$$\int_{D(u_1, u_2) \cap \{r \leq R_0\}} \Omega^2 r^2 |\phi|^2 d\mu \leq C \cdot \left( E_{deg}^+(u_1, u_2) + \int_{D(u_1, u_2) \cap \{r \leq R_0\}} q_0 \Omega^2 |\phi||D_\phi|d\mu \right),$$ 

which proves the lemma.
Now that the 0 order bulk term is controlled on a region \( \{ r_+ \leq r \leq R_0 \} \) near the event horizon, we would like a global estimate that can also control the derivative of the scalar field everywhere, and the 0 order term near infinity.

We use the vector field \( X_\alpha \) with \( \alpha \) sufficiently large to get a \( r^{-\alpha} \) weighted control of \( |D\phi|^2 \) on the whole region \( D(u_1, u_2) \). However, this identity alone necessarily comes with a loss of control of the 0 order term in a bounded region \( \{ r_+ \leq r \leq R_0 \} \).

The key point is to notice that \( R(\alpha) \to r_+ \) as \( \alpha \to +\infty \). Therefore, at the cost \( |r^{-\alpha} \phi|^2 \) of a worse \( r^{-\alpha} \) weight, we can take \( \alpha \) large enough so that \( R(\alpha) < R_0 \). As a result, the loss of control of the \( X_\alpha \) estimate can be compensated by the 0 order term estimate obtained prior in \( \{ r_+ \leq r \leq R_0 \} \).

The proof of this key fact is the object of the following lemma:

**Lemma 5.3.** For all \( \Delta > 0 \) and for all \( R_1 > r_+ \), there exists \( \tilde{\alpha}(R_1) > 1 \) sufficiently large such that for all \( \alpha \geq \tilde{\alpha}(R_1) \) and for all \( r \geq R_1 \),

\[
\Box(\Omega^2 r^{-\alpha})(r) > \Delta \cdot r^{-\alpha - 2}. \tag{5.8}
\]

**Proof.** For all \( \alpha \in \mathbb{R} \), we use (4.13) to compute:

\[
\Box(r^{-\alpha}) = \alpha r^{-\alpha - 2} \left( (\alpha - 1) \Omega^2 - r \cdot 2K \right) = \alpha r^{-\alpha - 2} \left( (\alpha - 1) - \alpha \frac{2M}{r} + (\alpha + 1) \frac{\rho^2}{r^2} \right). \tag{5.9}
\]

Then, using the same identity for \( \alpha \) and \( \alpha - 1 \) we get

\[
\Box((1-\frac{2M}{r}) r^{-\alpha}) = \alpha(\alpha - 1)r^{-\alpha - 2} - (2\alpha^2 - 3\alpha + 2)2Mr^{-\alpha - 3} + (4(\alpha - 1)^2 M^2 + \alpha(\alpha + 1)\rho^2) r^{-\alpha - 4} - \alpha(\alpha - 1)(2\rho^2) r^{-\alpha - 5}.
\]

Then using again the identity for \( \alpha - 2 \) we get

\[
\Box(\Omega^2 r^{-\alpha}) = \alpha(\alpha - 1)r^{-\alpha - 2} - (2\alpha^2 - 3\alpha + 2)2Mr^{-\alpha - 3} + (4(\alpha - 1)^2 M^2 + 2(\alpha^2 - 2\alpha + 3)\rho^2) r^{-\alpha - 4} - (2\alpha^2 - 5\alpha + 4)(2\rho^4) r^{-\alpha - 5} + (\alpha - 2)(\alpha - 1)\rho^4 r^{-\alpha - 6}. \tag{5.10}
\]

Now we want to take \( \alpha \) very large. Notice first that

\[
\Box(\Omega^2 r^{-\alpha}) = \alpha^2 r^{-\alpha - 2} \left( 1 - \frac{4M}{r} + 4M^2 + 2r^2 \right) - \frac{4M\rho^2}{r^3} + \frac{\rho^4}{r^4} + p_\alpha(r),
\]

where \( p_\alpha(r) \) is a degree four polynomial in \( r^{-1} \) whose coefficients are all \( O(\alpha^{-1}) \) as \( \alpha \) tends to \( +\infty \).

Now, notice that

\[
1 - \frac{4M}{r} + 4M^2 + 2r^2 - \frac{4M\rho^2}{r^3} + \frac{\rho^4}{r^4} = \Omega^4.
\]

Hence

\[
\Box(\Omega^2 r^{-\alpha}) = \alpha^2 r^{-\alpha - 2} (\Omega^4 + p_\alpha(r)).
\]

We now denote \( b_\alpha(r) := \Box(\Omega^2 r^{-\alpha}) r^{\alpha - 2} = \alpha^2 (\Omega^4 + p_\alpha(r)). \)

Then we have \( \lim_{r \to +\infty} b_\alpha(r) = \alpha(\alpha - 1) \).

Then define \( R(\alpha) \) as the maximum \( r \) such that \( b_\alpha(r) > 0 \) on \( (R(\alpha), +\infty) \), i.e.

\[
R(\alpha) = \sup\{ r_+ \leq r / \forall r' \geq r, b_\alpha(r') > 0 \}.
\]

Because of what precedes, it is clear that for all \( \alpha > 1 \), \( R(\alpha) < +\infty \).

Moreover, because of the continuity of \( b_\alpha \), we also have \( b_\alpha(R(\alpha)) = 0 \).

We want to prove that \( R(\alpha) \) is bounded. Since for all \( \alpha > 1 \), \( R(\alpha) < +\infty \), it is actually enough to prove that \( R(\alpha) \) is bounded for \( \alpha \) in any neighbourhood of \( +\infty \).

Suppose not: then there exists a sequence \( \alpha_n \) such that

\[
\lim_{n \to +\infty} \alpha_n = +\infty,
\]

\[
\lim_{n \to +\infty} R(\alpha_n) = +\infty.
\]

Therefore we have

\[
b_\alpha(R(\alpha_n)) = 0 = \alpha_n^2 \Omega^4(R(\alpha_n)) + p_\alpha(R(\alpha_n)) \sim \alpha_n^2 \to +\infty,
\]

which is a contradiction. To obtain the infinite limit, we used the fact that \( p_\alpha(R(\alpha_n)) \to 0 \) and \( \Omega^4(R(\alpha_n)) \to 1 \) when \( n \to +\infty \).

\[\text{[55]}\text{Which does not matter because we actually want to apply the Morawetz estimate in section 6 to a bounded r region } \{ r_+ \leq r \leq R \}.\]
Now that $R(\alpha)$ is bounded, it admits at least one limit value when $\alpha \to +\infty$. We call $R_{r_{\alpha}} \geq r_{\alpha}$ such a limit value and we take a sequence $\alpha_n$ such that

$$
\lim_{n \to +\infty} \alpha_n = +\infty,
$$

$$
\lim_{n \to +\infty} R(\alpha_n) = R_1.
$$

then we see that because $p_{\alpha_n}(R(\alpha_n)) \to 0$

$$
\lim_{n \to +\infty} \alpha_n^2 \cdot \Omega^2(R_1) = 0,
$$

which automatically implies that $\Omega^2(R_1) = 0$ hence $R_1 = r_{\alpha}$. Since $r_{\alpha}$ is the only admissible limit value for $R(\alpha)$ when $\alpha \to +\infty$, we actually proved that

$$
\lim_{\alpha \to +\infty} R(\alpha) = r_{\alpha}.
$$

More precisely, we have $R_1 = r_{\alpha} + o(\alpha^{-2})$ when $\alpha \to +\infty$.

This implies that for all $\Delta > 0$ and for all $R_1 > r_{\alpha}$, there exists $\alpha(R_1) > 1$ sufficiently large so that for all $r \geq R_1$,

$$
b_u(r) > \Delta,
$$

which proves the lemma. \hfill \square

Now we come back to the main proof: the next step is to establish the global estimate using $X_{\alpha}$.

We integrate identity (5.2) × $d\nu$ on $\mathcal{D}(u_1, u_2)$. We get

$$
\int_{\mathcal{D}(u_1, u_2)} \left[ \frac{2\alpha}{\omega_{\alpha-1}^2} \Omega^2 (|D_v\phi|^2 + |D_x\phi|^2) + \frac{\Omega^2}{2\alpha} \Omega^2 r^{-2\alpha} |\phi|^2 - \frac{\Omega^2}{2} r^{-2\alpha} |\phi|^2 \right] dudv \leq \tilde{E} + \int_{\mathcal{D}(u_1, u_2)} \frac{2\phi|Q|}{r^\alpha} \Omega^2 |\phi||D_v\phi|du dv,
$$

where $\tilde{E}$ accounts for the boundary terms in the identity.

We now need to prove that for some $C' = C'(M, \rho) > 0$,

$$
\tilde{E} \leq C' \cdot E_{deg}(u_1, u_2).
$$

The terms appearing on $\mathcal{H}^+$ is one component of $E_{deg}(u_1, u_2)$, since the terms involving $\chi$ are 0.

The term on $\mathcal{T}^+$ is proportional to

$$
\int_{u_1}^{u_2} \left( r^{2-\alpha}|D_v\phi|^2 + \Omega^2 \left( \Omega^2 - r \cdot 2K \right) |\phi|^2 - \Omega^2 r^{-2\alpha} |\phi|^2 \right) (u') du'.
$$

The first term appears in the $X_{\alpha}$ identity with a positive sign on the left-hand-side. The third term can be absorbed into the first and second term, using that $\Omega^2 r^{1-\alpha} |\partial_u(|\phi|^2)| \leq \Omega^2 r^{-\alpha} |\phi|^2 + r^{2-\alpha} |D_v\phi|^2$. Only remains a $\frac{\Omega^2}{2} (r \cdot 2K) |\phi|^2$ term. Now notice that since $\lim_{r \to +\infty} \phi = 0$, this term is actually 0 on $\mathcal{T}^+$, for any $\alpha > 0$.

The terms on $\{ u \leq u \leq +\infty, \ v = \nu_\alpha(u) \}$, $i = 1, 2$ can be written as :

$$
\int_{u_i}^{+\infty} \left( r^{2-\alpha}|D_v\phi|^2 + \frac{\Omega^2}{2} (\Omega^2 - r \cdot 2K) r^{-\alpha} |\phi|^2 - \frac{\Omega^2}{2} r^{-\alpha} |\phi|^2 \right) (u', v_R(u_i)) du'.
$$

Similarly to what was written for $\mathcal{T}^+$, we can absorb the third term so that we only need to control

$$
\int_{u_i}^{+\infty} (r^{2-\alpha}|D_v\phi|^2 - \Omega^2 \cdot 2K \cdot r^{1-\alpha} |\phi|^2) (u', v_R(u_i)) du'.
$$

Since $\alpha > 0$, the first term can obviously be controlled by a term proportional to $E_{deg}(u_i)$. This only leaves the $\Omega^2 \cdot 2K \cdot r^{1-\alpha} |\phi|^2$ term to control.

Now we proceed in two times: let $r_4 \leq R_1 < R$. On $\{ r_4 \leq r \leq R_1 \}$, we use a similar method to the one leading to (5.6). We find that there exists a constant $C_1 = C_1(M, \rho, R_1) > 0$ such that

$$
\int_{u_i}^{+\infty} \Omega^2 r^{1-\alpha} \cdot 2K |\phi|^2 (u, v_R(u_i)) du \leq C_1 \cdot E_{deg}(u_i). \tag{5.11}
$$

Now we can take care of the region $\{ R_1 \leq r \leq R \}$. Using Hardy inequality (2.18) in $u$ we get :

$$
\int_{u_i}^{+\infty} \Omega^2 |\phi|^2 (u, v_R(u_i)) du \leq \frac{4}{\Omega^2(R_1)} \int_{u_i}^{+\infty} r^2 |D_v\phi|^2 (u, v_R(u_i)) du + 2R |\phi|^2 (u_i, v_R(u_i)).
$$

$54$ $d\nu$ is defined in appendix C

$54$ This comes from the finiteness of $\mathcal{E}$, c.f. the proof of Proposition 4.1
The last term of the right-hand side can be bounded, using Hardy’s inequality 2.17 in $v$:

$$R[\phi^2(u, v_{R(u_i)})] \leq \Omega^{-2}(R) \int_{v_{R(u_i)}}^{+\infty} r^2 |D_\phi|^2(u, v) dv.$$  

Hence there exists a constant $C'_1 = C'_1(M, \rho, R_1) > 0$ such that

$$\int_{u_i}^{u_{R_1}(v_{R(u_i)})} \Omega^2 |\phi|^2(u, v_{R(u_i)}) du \leq C'_1 \cdot E_{deg}(u_i).$$

Now notice that $r^{1-\alpha} \cdot 2K \leq r^{1-\alpha} \cdot 2K_+ r_+^2 \leq r^{1-\alpha} \cdot 2K_+$. Therefore there exists a constant $C''_1 = C''_1(M, \rho, R_1) > 0$ such that

$$\int_{u_i}^{u_{R_1}(v_{R(u_i)})} \Omega^2 r^{1-\alpha} \cdot 2K |\phi|^2(u, v_{R(u_i)}) du \leq C''_1 \cdot E_{deg}(u_i).$$ (5.12)

Combining with (5.11) and after choosing $R_1 = R_1(M, \rho)$, for instance $R_1 = 2r_+$ we see that the boundary terms on $\{u_i \leq u \leq +\infty, v = v_{R(u_i)}\}$, $i = 1, 2$ are controlled by $\bar{C}(M, \rho) \cdot E_{deg}^+(u_i, u_2)$.

Now we take care of the terms on $\{v_{R(u_i)} \leq v \leq +\infty, u = u_i\}$ for $i = 1, 2$.

We can see that it is enough to control $\int_{v_{R(u_i)}}^{\pm\infty} (r^{2-\alpha} |D_\phi|^2(u, v') + r^{-\alpha} |\phi|^2(u, v')) dv'$.

We can use Hardy’s inequality (2.19) under the form

$$\int_{v_{R(u_i)}}^{\pm\infty} |\phi|^2(u, v) dv \leq \frac{4}{\Omega^2(R)} \int_{v_{R(u_i)}}^{\pm\infty} r^2 |D_\phi|^2(u, v) dv.$$  

Therefore,

$$\int_{v_{R(u_i)}}^{\pm\infty} (r^{2-\alpha} |D_\phi|^2(u, v') + r^{-\alpha} |\phi|^2(u, v')) dv' \leq R^{-\alpha} [1 + 4\Omega^{-4}(R)] E_{deg}(u_i) \leq E_{deg}^+(u_i, u_2),$$

after taking $R$ large enough.

Therefore we proved that for some $C' = C'(M, \rho) > 0$,

$$\int_{D(u_1, u_2)} \left[ \frac{2\alpha}{\rho^{\alpha-1}} \Omega^2 (|D_\phi|^2 + |D_\phi|^2) + \frac{2}{\rho} (\Omega \cdot r^{-\alpha-1}) \Omega^2 r^2 |\phi|^2 \right] dv dudv \leq C' \cdot E_{deg}^+(u_1, u_2) + \int_{D(u_1, u_2)} \frac{2\rho|Q|}{\rho^\alpha} \Omega^2 |\phi||D_\phi| dudv.$$ (5.13)

Now take $\Delta = 2$ and $R_1 = R_0$ in Lemma 5.3 where $r_+ < R_0(M, \rho)$ is the radius of Lemma 5.2. Then following Lemma 5.3 we can take $\alpha = \alpha(M, \rho) > 1$ large enough so that

$$\int_{D(u_1, u_2) \cap (r \leq R_0)} \frac{2}{\rho^2} (\Omega \cdot r^{-\alpha-1}) \Omega^2 r^2 |\phi|^2 dv dudv + \int_{D(u_1, u_2) \cap (r \geq R_0)} \Omega^2 r^{-\alpha-1} |\phi|^2 dv dudv < \int_{D(u_1, u_2) \cap (r \geq R_0)} \frac{2}{\rho^2} (\Omega \cdot r^{-\alpha-1}) \Omega^2 r^2 |\phi|^2 dv dudv.$$

Then, since on $[r_+, R_0]$, $\Omega^2 (r^{-\alpha-1})$ is bounded by a constant only depending on $M$ and $\rho$, we find using Lemma 5.2 that there exists $\bar{C} = \bar{C}(M, \rho) > 0$ such that

$$\int_{D(u_1, u_2) \cap (r \leq R_0)} \Omega^2 \left( r^{-\alpha-1} + \frac{2}{\rho^2} \Omega \cdot r^{-\alpha-1} \right) |\phi|^2 dv dudv \leq \bar{C} \cdot \left( E_{deg}^+(u_1, u_2) + \int_{D(u_1, u_2) \cap (r \geq R_0)} q_0|Q| \Omega^2 |\phi||D_\phi| dudv \right).$$

Therefore, combining with (5.13) we get that there exists $\bar{C} = \bar{C}(M, \rho) > 0$ such that

$$\int_{D(u_1, u_2) \cap (r \leq R_0)} \left[ \frac{2\alpha}{\rho^{\alpha-1}} \Omega^2 (|D_\phi|^2 + |D_\phi|^2) + \frac{2\rho}{\rho^\alpha} \Omega^2 |\phi|^2 \right] dv dudv \leq \bar{C} \cdot \left( E_{deg}^+(u_1, u_2) + \int_{D(u_1, u_2) \cap (r \geq R_0)} q_0|Q| \Omega^2 |\phi||D_\phi| dudv \right).$$ (5.14)

Now notice that

$$\frac{2\rho|Q|}{\rho^\alpha} \leq \frac{|\phi|^2}{\rho^\alpha+1} + \frac{|D_\phi|^2 + |D_\phi|^2}{2\alpha}.$$

Then, if $|Q|$ is small enough so that

$$\bar{C} \cdot q_0 \sup_{D(u_1, u_2)} |Q| < \min\{1, 2\alpha\},$$

then the interaction term can be absorbed into the left-hand-side of (5.14), which proves Proposition 5.1.
5.2 A Red-shift estimate

In this section, we are going to prove that the non-degenerate energy near the event horizon is bounded by the degenerate energy. This echoes with the red-shift estimates pioneered in [12] and [13]. The main difference here is that we cannot obtain non-degenerate energy boundedness as an independent statement, due to the presence of the charge term. Indeed we need to use the control of the 0 term order near the event horizon, obtained priorly by the Morawetz estimate (5.1).

This red-shift estimate is proved using the vector field $\Omega^{-2}\partial_\nu$, which is regular across the event horizon.

We integrate the resulting vector field identity on $\{r_+ \leq r \leq R\}$. Although a term appears on the time-like boundary $\{r = R_0\}$, we can control it by the degenerate energy, in the same way we did for the Morawetz estimate. We are borrowing a few important arguments from section 5.1. We are going to prove the following proposition:

**Proposition 5.4.** For all $r_+ < R_0 < R$ and for all $0 < \epsilon < 1$ sufficiently small, there exists $C = C(R_0, \epsilon, M, \rho) > 0$, $0 = \delta = \delta(R_0, \epsilon, M, \rho) > 0$ and $(1 - \epsilon)R_0 < R_0 < (1 + \epsilon)R_0$ such that if $\|Q\|_{2, \infty(D(u_0(R), +\infty))} < \delta$ then for all $u_0(R) \leq u_1 < u_2$:

$$
\int_{u_0(R_0)}^{u_2} \int_{d(u_2)}^{\infty} \frac{r^2|D_u \phi|^2}{\Omega^2}(u, v_R(u_2))\,du + \int_{u_0(R_0)}^{\infty} \frac{|D_u \phi|^2}{\Omega^2} (u, v_R(u_1))\,du
$$

$$
\leq C \left[ E_{\text{deg}}^+(u_1, u_2) + \int_{u_0(R_0)}^{\infty} \frac{r^2|D_u \phi|^2}{\Omega^2} (u, v_R(u_1))\,du \right]
$$

(5.15)

**Proof.** We are going to consider the vector field $X_{RS} = \frac{\partial}{\partial t}$. We also choose $\chi = -\frac{1}{2}$. Then we get

$$
\nabla^u \cdot J^X_{RS}(\phi) = 4K(r)|D_u \phi|^2 \frac{2K(r)|\phi|^2}{r^2} + 2q_0Q\Omega(\partial D_u \phi).
$$

Now we integrate this identity, multiplied \(^{55}\) by $\text{deol}$, on $\{r_+ \leq r \leq R_0\}$:

$$
\int_{D(u_1, u_2) \cap \{r \leq R_0\}} \left( \frac{8K(r)r^2|D_u \phi|^2}{\Omega^2} - 2K(r)\Omega^2|\phi|^2 + 2q_0Q\Omega(\partial D_u \phi) \right)\,du
$$

$$
+ \int_{u_0(R_0)}^{\infty} \frac{r^2|D_u \phi|^2}{\Omega^2} (u, v_R(u_2))\,du' + \hat{E}_{R_0}^R (u_1, u_2)
$$

(5.16)

where $\hat{E}_{R_0}^R (u_1, u_2)$, the $L^2$ flux through $\{r = R_0\}$ is defined by

$$
\hat{E}_{R_0}^R (u_1, u_2) = \int_{r = R_0}^{R_0} \frac{r^2|D_u \phi|^2}{\Omega^2} (u, v_R(u_2))\,du
$$

$$
+ \int_{u_0(R_0)}^{\infty} \frac{r^2|D_u \phi|^2}{\Omega^2} (u, v_R(u_1))\,du' + \hat{C}\cdot E_{\text{deg}}(u_1).
$$

For the constant $v$ boundary term, similarly to what was done in section 5.1, we can absorb the second term into the others using the fact that $r|\partial_\nu|\phi|^2| \leq \Omega^2|\phi|^2 + \frac{r^2|D_u \phi|^2}{\Omega^2}$.

This only leaves the $\Omega^2|\phi|^2$ term to control on $[u_0, v_R(u_1)] \times \{v_R(u_1)\}$. Using 5.6, we see that there exists $\hat{C} = \hat{C}(M, \rho, R_0) > 0$ such that

$$
\int_{u_0(R_0)}^{\infty} \Omega^2|\phi|^2 (u', v_R(u_1))\,du' \leq \hat{C}\cdot E_{\text{deg}}(u_1).
$$

Therefore we proved that for some $\hat{C} = \hat{C}(M, \rho, R_0) > 0$

$$
\int_{D(u_1, u_2) \cap \{r \leq R_0\}} \left( \frac{8K(r)r^2|D_u \phi|^2}{\Omega^2} - 2K(r)\Omega^2|\phi|^2 + 2q_0Q\Omega(\partial D_u \phi) \right)\,du
$$

$$
+ \int_{u_0(R_0)}^{\infty} \frac{r^2|D_u \phi|^2}{\Omega^2} (u, v_R(u_2))\,du' + \hat{E}_{R_0}^R (u_1, u_2)
$$

(5.17)

Now we want to control $\hat{E}_{R_0}^R (u_1, u_2)$ term. Notice that its first term has the right sign. For the other two terms, we use the same technique as for the proof of Lemma 5.2. Therefore, for any $0 < \epsilon < 1$ and after an application of the mean-value theorem on $\{(1 - \epsilon)R_0, (1 + \epsilon)R_0\}$ with the Morawetz estimate proved in former section, we see that there exists $\hat{C}' = \hat{C}'(M, \rho, R_0, \epsilon) > 0$ and $(1 - \epsilon)R_0 < R_0 < (1 + \epsilon)R_0$ such that

\(^{55}\)deol is defined in appendix E

35
\[ | \int_{r = R_0} \frac{\dot{R}_0}{4} (\partial_u |\phi|^2 - \partial_v |\phi|^2) + \frac{\partial^2 (\dot{R}_0)}{\dot{R}_0^2} |\phi|^2 dt | \leq C' \cdot E_{\text{deg}}^+(u_1, u_2). \]

Therefore we have

\[
\left( 8K(r)r^2 |D_u \phi|^2 \Omega^2 - 2K(r)\Omega^2 |\phi|^2 + 2q_0 Q \Delta (\phi D_v \phi) \right) du dv + \int_{u_{R_0}(v_R(u_2))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2} (u', v_R(u_2)) du' \leq \int_{u_{R_0}(v_R(u_1))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2} (u', v_R(u_1)) du' + (C' + C') \cdot E_{\text{deg}}^+(u_1, u_2). \tag{5.17}
\]

Then we use the Morawetz estimate \[5.1\] to control the 0 order term: there exists \( C_0 = C_0(M, \rho, R_0) > 0 \) such that

\[ | \int_{\mathcal{D}(u_1, u_2) \cap \{ r \leq R_0 \}} K(r) \Omega^2 |\phi|^2 du dv | \leq C_0 \cdot E_{\text{deg}}^+(u_1, u_2). \tag{5.18} \]

Then, combining with (5.17) and noticing that \( 2K \cdot r^2 \geq 2K_+ r_+^2 \), we see that there exists \( C'_0 = C'_0(M, \rho, R_0, \epsilon) > 0 \) such that

\[ \int_{\mathcal{D}(u_1, u_2) \cap \{ r \leq R_0 \}} \frac{r^2 |D_u \phi|^2}{\Omega^2} du dv + \int_{u_{R_0}(v_R(u_2))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2} (u', v_R(u_2)) du' \leq C'_0 \cdot \left( \int_{\mathcal{D}(u_1, u_2) \cap \{ r \leq R_0 \}} q_0 |Q| \frac{1}{4} |\phi| D_u \phi dv + \int_{u_{R_0}(v_R(u_1))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2} (u', v_R(v_1)) du' + E_{\text{deg}}^+(u_1, u_2) \right). \tag{5.19} \]

Then for \( |Q| \) small enough and using (5.18), we can absorb the interaction term \( \int_{\mathcal{D}(u_1, u_2) \cap \{ r \leq R_0 \}} \frac{r^2 |D_u \phi|^2}{\Omega^2} du dv \) and \( E_{\text{deg}}^+(u_1, u_2) \). The idea is the same as the one that was used to conclude the proof of Proposition 5.1.

This concludes the proof of Proposition 5.4. \[ \square \]

### 5.3 Boundedness of the energy

This estimate is by far the most delicate in this section. Even though for a charged scalar, a positive conserved quantity—arising from the coupling of the scalar field and the electromagnetic tensors—is still available via the vector field \( \partial_t \), it is of little direct use because one of its component, involving the charge, does not decay in time since the charge approaches a constant value at infinity.

This problem does not exist on Minkowski space-time—where the charge is constrained to tend to 0 towards time-infinity—or in the uncharged case \( q_0 = 0 \) where the conservation laws are uncoupled and the use of \( \partial_t \) gives an estimate only in terms of scalar fields quantities.

Our strategy to deal with this charge term is to absorb the charge difference into a term involving the scalar field. More precisely, the key point is to control the fluctuations of the charge by the energy of the scalar field. This idea already appeared in [22], in the context of the black hole interior.

In the exterior, we do this in four different steps.

In the first step, we take care of the domain \( \mathcal{D}(u_1, u_2) \cap \{ r \geq R \} \) where we integrate \( Q^2 \) towards \( \gamma_R \). This estimate leaves a term \( R^{-1} \cdot (Q^2(u_2, v_R(u_2)) - Q^2(u_1, v_R(u_1))) \) on \( \gamma_R \) to be controlled.

In the second step, we transport in \( u \) the \( R^{-1} \cdot (Q^2(u_2, v_R(u_2)) - Q^2(u_1, v_R(u_1))) \) term towards the curve \( \gamma_R \) for some fixed \( r_0 < R_0 < R \) and \( R_0 = R_0(M, \rho) \). This term can be controlled by the \( r^2 |D_u \phi|^2 \) part of the energy on \( \{ R_0 \leq r \leq R \} \) and by the \( r^2 |D_u \phi|^2 \) part of the energy on \( \{ R \leq r \} \), using a Hardy-type inequality in each region. Every-time a small term \( q_0 |Q| \) multiplies the terms controlled by the energy, which is why they can be absorbed.

This estimate leaves this term \( \int_{\mathcal{D}(u_1, u_2) \cap \{ r \leq R_0 \}} R^{-1} \cdot (Q^2(u_{R_0}(v_R(u_1)), v_R(u_1)) - Q^2(u_{R_0}(v_R(u_2)), v_R(u_2))) \) using the Morawetz estimate of section 5.1. The key point is that \( R_0 \) is chosen "in between" \( r_+ \) and \( R \), so that it is not too close to the event horizon and not too close either from infinity. Since the argument loses a lot of \( \Omega^{-2} \) and \( r \) weights, it is fortunate that these loses are bounded (they only depend on \( \rho \) and \( M \)) and can therefore be absorbed using the smallness of \( q_0 |Q| \). The argument does not depend on the smallness of \( R \), which can still be taken arbitrarily large for the following sections, in particular section 5.

The different curves are summed up by the Penrose diagram of Figure 2.

The use of the vector field \( \partial_t \) on the domain \( \mathcal{D}(u_1, u_2) \) gives rise to an energy identity, summed up by the following lemma:

\[ \text{The first multiplicative term is not a typo: it is indeed } R \text{ and not } R_0. \]
Lemma 5.5. For all \( u_0(R) \leq u_1 < u_2 \) we have the following energy identity:

\[
\begin{align*}
\int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_\phi|_{H^+}^2(v) dv &+ \int_{u_2}^{+\infty} \left( r^2 |D_\phi|^2(u, v_R(u_2)) + \frac{2Q^2}{r^2}(u, v_R(u_2)) \right) du \\
&+ \int_{v_R(u_2)}^{+\infty} \left( r^2 |D_\phi|^2(u_2, v) + \frac{2Q^2}{r^2} (u_2, v) \right) dv + \int_{u_2}^{+\infty} r^2 |D_\phi|^2 |_{\mathcal{I}^+} (u) du \\
= \int_{u_1}^{+\infty} \left( r^2 |D_\phi|^2(u, v_R(u_1)) + \frac{2Q^2}{r^2} (u, v_R(u_1)) \right) du + \int_{v_R(u_1)}^{+\infty} \left( r^2 |D_\phi|^2(u_1, v) + \frac{2Q^2}{r^2} (u_1, v) \right) dv.
\end{align*}
\]

(5.20)

With the notations of this section, it can also be written as

\[
\begin{align*}
\int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_\phi|_{H^+}^2(v) dv &+ \int_{u_2}^{+\infty} \frac{2Q^2}{r^2} (u, v_R(u_2)) du + \int_{v_R(u_2)}^{+\infty} \frac{2Q^2}{r^2} (u_2, v) dv + \int_{u_2}^{+\infty} r^2 |D_\phi|^2 |_{\mathcal{I}^+} (u) du \\
&+ \int_{u_1}^{+\infty} \frac{2Q^2}{r^2} (u_1, v) dv + \int_{v_R(u_1)}^{+\infty} \frac{2Q^2}{r^2} (u, v_R(u_1)) du + E_{\delta\phi}(u_2) \\
= \int_{u_1}^{+\infty} \frac{2Q^2}{r^2} (u, v_R(u_1)) du + \int_{v_R(u_1)}^{+\infty} \frac{2Q^2}{r^2} (u, v_R(u_1)) dv + E_{\delta\phi}(u_1).
\end{align*}
\]

(5.21)

Proof. The proof is an elementary computation based on the fact that \( \nabla^\nu(T_{E^M}^\nu + T_{E^F}^\nu) = 0 \) and that \( \delta \phi \) is a Killing vector field of the Reissner–Nordström metric, c.f. Appendix [4] for more details. We omit the (easy) argument in which one writes the estimate on a truncated domain \( \mathcal{D}(u_1, u_2) \cap \{ v \leq v_0 \} \) and sends \( v_0 \) toward \(+\infty\) to obtain the claimed boundary terms on \( \mathcal{I}^+ \). We are going to repeat this omission in what follows.

The goal of this section is to prove the following:

**Proposition 5.6.** There exists \( C = C(M, \rho) > 0, \delta = \delta(M, \rho) > 0 \) such that

if \( \|Q\|_{L^\infty(\mathcal{D}(u_0(R), +\infty))} < \delta \), we have for all \( u_0(R) \leq u_1 < u_2 \):

\[
\begin{align*}
\int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_\phi|_{H^+}^2(v) dv &+ \int_{u_2}^{+\infty} \frac{2Q^2}{r^2} (u, v_R(u_2)) du + \int_{v_R(u_2)}^{+\infty} \frac{2Q^2}{r^2} (u_2, v) dv + \int_{u_2}^{+\infty} r^2 |D_\phi|^2 |_{\mathcal{I}^+} (u) du \\
&\leq C \cdot \left( \int_{u_1}^{+\infty} r^2 |D_\phi|^2 (u, v_R(u_1)) du + \int_{v_R(u_1)}^{+\infty} r^2 |D_\phi|^2 (u, v_R(u_1)) dv \right).
\end{align*}
\]

(5.22)

With the notations of this section, it can also be written as

\[
\begin{align*}
\int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_\phi|_{H^+}^2(v) dv + \int_{u_1}^{+\infty} r^2 |D_\phi|^2 |_{\mathcal{I}^+} (u) du + E(u_2) \leq C \cdot E(u_1).
\end{align*}
\]

(5.23)

In short, the strategy is to bound all the terms involving \( Q^2 \) by \( C'(M, \rho) \cdot q_0 |Q| \cdot E^+(u_1, u_2) \) for a constant \( C'(M, \rho) > 0 \) that only depends on the black hole parameters. Then we take \( |Q| \) to be small enough so that \( C'(M, \rho) \cdot q_0 |Q| < 1 \), in order to absorb those terms into the others.

5.3.1 Step 1 : the region \( \mathcal{D}(u_1, u_2) \cap \{ r \geq R \} \)

**Lemma 5.7.**

\[
\begin{align*}
\int_{v_R(u_1)}^{+\infty} \frac{\Omega^2 Q^2}{r^2} (u_2, v) dv &- \int_{v_R(u_1)}^{+\infty} \frac{\Omega^2 Q^2}{r^2} (u_1, v) dv \\
&\leq \frac{Q^2(u_2, v_R(u_2)) - Q^2(u_1, v_R(u_1))}{R} \\
&+ 4q_0 \left( \sup_{\mathcal{D}(u_1, u_2)} |Q| \right) \Omega^{-2}(R) \left[ \int_{v_R(u_1)}^{+\infty} r^2 |D_\phi|^2 (u_2, v) dv + \int_{v_R(u_1)}^{+\infty} r^2 |D_\phi|^2 (u_1, v) dv \right]
\end{align*}
\]

(5.24)

Proof. We start to prove the identity:

\[
\begin{align*}
\int_{v_R(u_1)}^{+\infty} \frac{\Omega^2 Q^2}{r^2} (u, v) dv = \frac{Q^2(u, v_R(u_1))}{R} + 2 \int_{v_R(u_1)}^{+\infty} q_0 Qr^3 |D_\phi|_3 (u, v) dv.
\end{align*}
\]

(5.25)

For this we write using (2.9), for all \( v_R(u_1) \leq v \):

\[
Q^2(u, v) = Q^2(u, v_R(u_1)) + 2 \int_{v_R(u_1)}^{v} q_0 Qr^3 |D_\phi|_3 (u, v') dv'.
\]

Then for the first term, we just need to notice that because \( \Omega^2 = \partial_r r \):

37
Lemma 5.8. For all $r$,

$$\int_{\Omega_R(u_i)}^+ \frac{\Omega^2}{r^2}(u, v) dv = \frac{1}{R}.$$

For the second term we integrate by parts as:

$$\int_{\Omega_R(u_i)}^+ \frac{\Omega^2}{r^2}(u, v) \left( \int_{\Omega_R(u_i)} v Q r^2 \Im(\bar{D}_v \phi)(u, v') dv' \right) dv = \int_{\Omega_R(u_i)}^+ Q r^2 \Im(\bar{D}_v \phi)(u, v) dv.$$  

The boundary terms cancelled because $Q^2$ is bounded:

$$\lim_{v \to +\infty} \frac{1}{r(u, v)} \int_{\Omega_R(u_i)}^v 2 Q r^2 \Im(\bar{D}_v \phi)(u, v') dv' = \frac{Q^2(u, v) - Q^2(u, v R(u_i))}{r(u, v)} = 0.$$  

Therefore (5.25) is proven.

5.3.2 Step 2: transport in $u$ of $Q^2_{\Omega_R}$ to $Q^2_{\Omega_{R_0}}$

Lemma 5.8. For all $r_1 < R_0 < R$ we have

$$\frac{Q^2(u_2, v R(u_2)) - Q^2(u_1, v R(u_1))}{R} \leq \frac{Q^2(u R_0(v R_0(u_2)), v R(u_2)) - Q^2(u R_0(v R(u_1)), v R(u_1))}{R}$$

$$+ 5 q_0 \left( \sup_{\Omega(u_1, u_2)} |Q| \right) \Omega^{-2}(R) \int_{\Omega_R(u_i)}^+ r^2 |D_v \phi|^2(u, v) dv. \quad (5.26)$$

Proof. We start by an identity coming directly from (2.8):

$$Q^2(u, v R(u_i)) = Q^2(u R_0(v R(u_i)), v R(u_i)) + 2 \int_{u_i}^{u R_0(v R(u_i))} q_0 Q r^2 \Im(\bar{D}_v \phi)(u, v R(u_i)) du.$$

Using Hardy inequality (2.18) in $u$ we get:

$$\int_{u_i}^{u R_0(v R(u_i))} \Omega^2 |\phi|^2(u, v R(u_i)) du \leq 4 \int_{u_i}^{u R_0(v R(u_i))} r^2 |D_v \phi|^2(u, v R(u_i)) du + 2 R |\phi|^2(u, v R(u_i)).$$

The last term of the right-hand side can be bounded, using Hardy’s inequality (2.17) in $v$:

$$R |\phi|^2(u, v R(u_i)) \leq \Omega^{-2}(R) \int_{\Omega_R(u_i)}^+ r^2 |D_v \phi|^2(u, v) dv.$$

We then use Cauchy-Schwarz, taking advantage of the fact that $r \leq R$:

$$2 \int_{u_i}^{u R_0(v R(u_i))} q_0 Q r^2 \Im(\bar{D}_v \phi)(u, v R(u_i)) du \leq R \cdot q_0 \left( \sup_{\Omega(u_1, u_2)} |Q| \right) \int_{u_i}^{u R_0(v R(u_i))} \left( \frac{r^2 |D_v \phi|^2}{\Omega^2} + \Omega^{-2} |\phi|^2 \right)(u, v R(u_i)) du.$$

Then we combine with the estimates proven formerly to get

$$2 \int_{u_i}^{u R_0(v R(u_i))} q_0 Q r^2 \Im(\bar{D}_v \phi)(u, v R(u_i)) du \leq R \cdot q_0 \left( \sup_{\Omega(u_1, u_2)} |Q| \right) \left( 5 \int_{u_i}^{u R_0(v R(u_i))} \frac{r^2 |D_v \phi|^2}{\Omega^2}(u, v R(u_i)) du + 2 \Omega^{-2}(R) \int_{\Omega_R(u_i)}^+ r^2 |D_v \phi|^2(u, v) dv \right). \quad (5.27)$$

Then if we take $R$ large enough so that $2 \Omega^{-2}(R) < 5$, this last estimate proves Lemma 5.8.
5.3.3 Step 3: the region $\mathcal{D}(u_1, u_2) \cap \{ r \leq R \}$

Lemma 5.9. For all $r_+ < R_0 < R$ we have

$$\int_{u_2}^{+\infty} \frac{\Omega^2 Q^2}{r^2} (u, v_R(u_2)) du - \int_{u_1}^{+\infty} \frac{\Omega^2 Q^2}{r^2} (u, v_R(u_1)) du \leq \frac{5}{2} \varphi_0 \left( \int_{\mathcal{D}(u_1, u_2)} E^+(u_1, u_2). \right)$$  \hfill (5.28)

Proof. We start to prove an identity for all $r_+ < R_0 < R$ :

$$\int_{u_1}^{+\infty} \frac{\Omega^2 Q^2}{r^2} (u, v_R(u_1)) du = \frac{Q^2(u_R_0(v_R(u_1)), v_R(u_1))}{r_+} \left( 1 - \frac{r_+}{R} \right) + \frac{5}{2} \varphi_0 \left( \int_{\mathcal{D}(u_1, u_2)} |Q| \right) E^+(u_1, u_2).$$ \hfill (5.29)

After dividing $[u_1, +\infty]$ into $[u_1, u_R_0(v_R(u_1))]$ and $[u_R_0(v_R(u_1)), +\infty]$, the method of proof is similar to that of the estimate 5.22 except that we integrate $Q^2$ in a towards $\gamma_{R_0}$ this time.

Taking $R_0 < 2r_+$ and $R$ large enough so that $\frac{R_0}{r_+} < 1 < 1 - \frac{r_+}{R}$, we can write that

$$\int_{u_R_0(v_R(u_1))}^{+\infty} \varphi_0 Q r (1 - \frac{r}{r_+}) \delta D_0 \phi(u, v_R(u_1)) du + 2 \int_{u_1}^{u_R_0(v_R(u_1))} \varphi_0 Q r (1 - \frac{r}{R}) \delta D_0 \phi(u, v_R(u_1)) du$$

$$\leq (1 - \frac{r}{r_+}) \varphi_0 \left( \int_{\mathcal{D}(u_1, u_2)} |Q| \right) \int_{u_1}^{\infty} r \delta D_0 \phi(u, v_R(u_1)) du$$ \hfill (5.30)

Using Hardy inequality 2.18 in $u$ we get :

$$\int_{u_1}^{+\infty} \frac{\Omega^2 |\phi|^2 (u, v_R(u_1)) du}{r^2} \leq \frac{4}{\Omega^2} \int_{u_1}^{+\infty} \frac{r^2 D_0 \phi^2 (u, v_R(u_1)) du}{r^2} + 2 \Omega^2 \varphi^2(u_1, v_R(u_1)).$$

The last term of the right-hand side can be bounded, using Hardy’s inequality 2.17 in $v$ :

$$\Omega^2 |\phi|^2 (u, v_R(u_1)) \leq \Omega^{-2}(R) \int_{v_R(u_1)}^{+\infty} r^2 |D_0 \phi|^2 (u, v) dv.$$ \hfill (5.31)

Combining all the former estimates and using Cauchy-Schwarz, we finally get :

$$\int_{u_1}^{+\infty} r |\delta D_0 \phi(u, v_R(u_1)) du \leq \frac{5}{2} \int_{u_1}^{+\infty} \frac{r^2 D_0 \phi^2 (u, v_R(u_1)) du}{r^2} + 2 \Omega^2 \int_{v_R(u_1)}^{+\infty} r^2 |D_0 \phi|^2 (u, v) dv \leq \frac{5}{2} E(u_1),$$

where we took $R$ large enough so that $2 \Omega^{-2}(R) < \frac{5}{4}$.

We then combine with (5.30), which proves Lemma 5.9. \square

5.3.4 Step 4: control of the $Q^2_{\gamma_{R_0}}$ terms by the Morawetz estimate

Lemma 5.10. For all $r_+ < R_0 < R$ there exists $r_+ < R_0 < R_0'$ and $C_4 = C_4(R_0', M, \rho) > 0$ such that

$$Q^2(u_R_0(v_R(u_2)), v_R(u_2)) - Q^2(u_R_0(v_R(u_1)), v_R(u_1)) \leq C_4 \varphi_0 \left( \sup_{\mathcal{D}(u_1, u_2)} |Q| \right) E^+(u_1, u_2).$$ \hfill (5.32)

Proof. For this proof, we work in $(t, r^*)$ coordinates. We define $t_1 = 2v_R(u_1) - R_0$ such that the space-time point $(t_1, R_0')$ corresponds to the space-time point $(u_R_0(v_R(u_1)), v_R(u_1))$.

Using (2.8), (2.9) we see that on $\gamma_{R_0}$ :

$$Q^2(u_R_0(v_R(u_2)), v_R(u_2)) - Q^2(u_R_0(v_R(u_1)), v_R(u_1)) = Q^2(t_2, R_0') - Q^2(t_1, R_0') = 2R_0^2 \int_{t_1}^{t_2} \varphi_0 Q_3 \delta D_0 \phi(t, R_0') dt.$$

This estimate is valid for any $r_+ < R_0 < R$.

Now fix some $r_+ < R_0 < R$. Using the mean-value theorem in $r$, we see that there exists a $r_+ < R_0 < R_0'$ such that :

\[39\]
\[ \int_{r_+}^{R_0} \left[ \int_{r_1}^{r_2} q_0 Q^3(\tilde{\phi} D_r \phi)(t, r^*) dt \right] dr = (R_0 - r_+) \int_{r_1}^{r_2} q_0 Q^3(\tilde{\phi} D_r \phi)(t, (R_0)^*) dt. \]

From now on, we choose \( R_0 \) to be exactly that \( \tilde{R}_0 \) and we take \( R_0' < 2r_+ \).
Therefore, since \( R_0 < R_0' \) and \( R_0' < 2r_+ \) we have
\[
Q^2(\text{u}_{R_0}(\text{v}_{R}(u_2)), \text{v}_{R}(u_2)) - Q^2(\text{u}_{R_0}(\text{v}_{R}(u_1)), \text{v}_{R}(u_1)) \leq 4R_0' \int_{r_+}^{R_0} \left[ \int_{r_1}^{r_2} q_0 Q^3(\tilde{\phi} D_r \phi)(t, r^*) dt \right] dr.
\]

Then Lemma 5.10 follows from a direct application of Cauchy-Schwarz and the Morawetz estimate 5.1.

5.3.5 Completion of the proof of Proposition 5.6

Proof. Now we choose \( R_0' = 2r_+ \) and take the \( R_0 > r_+ \) of Lemma 5.10. We now apply Lemma 5.8 and Lemma 5.9 with that \( R_0 \). Coupled with Lemma 5.7 and identity (5.20) we find that there exists \( C_5 = C_5(M, \rho) > 0 \) such that
\[
\int_{v_{R}(u_1)}^{v_{R}(u_2)} r^2 |D_v \phi|^2 |_{H_+}(v) dv + \int_{u_2}^{+\infty} r^2 |D_v \phi|^2 (u, v_{R}(u_2)) du + \int_{u_1}^{+\infty} r^2 |D_v \phi|^2 (u_2, v) dv + \int_{u_1}^{+\infty} r^2 |D_v \phi|^2 (u_2, v) dv + C_{5q_0} \left( \sup_{\partial(u_1, u_2)} |Q| \right) E^+(u_1, u_2).
\]

(5.33)

Now we apply the Red-shift estimate (5.15) for say \( R_0 = 2r_+ \) and \( \epsilon = \frac{1}{7} \); there exists \( C_6 = C_6(M, \rho) > 0 \) such that
\[
C_6^{-1} \int_{u_{R_0}(v_{R}(u_2))}^{+\infty} \frac{r^2 |D_v \phi|^2}{\Omega^2} (u, v_{R}(u_2)) du \leq \frac{1}{2} E^+_{d_{gep}}(u_1, u_2) + \int_{u_{R_0}(v_{R}(u_1))}^{+\infty} \frac{r^2 |D_v \phi|^2}{\Omega^2} (u, v_{R}(u_1)) du.
\]

(5.34)

Therefore, adding (5.33) and (5.34) and absorbing the \( \frac{1}{2} E^+_{d_{gep}}(u_1, u_2) \), both in the left and right-hand side we get
\[
\int_{v_{R}(u_1)}^{v_{R}(u_2)} r^2 |D_v \phi|^2 |_{H_+}(v) dv + \int_{u_2}^{+\infty} r^2 |D_v \phi|^2 (u, v_{R}(u_2)) du + 2C_6^{-1} \int_{u_2}^{+\infty} \frac{r^2 |D_v \phi|^2}{\Omega^2} (u, v_{R}(u_2)) du + \int_{u_1}^{+\infty} r^2 |D_v \phi|^2 (u_2, v) dv + \int_{u_1}^{+\infty} r^2 |D_v \phi|^2 (u_2, v) dv + C_{5q_0} \left( \sup_{\partial(u_1, u_2)} |Q| \right) E^+(u_1, u_2). \]

(5.35)

Without loss of generality \(^{57}\) we can take \( C_6 > 20r^2(\tilde{R}_0) \) so that
\[
2C_6^{-1} \int_{u_2}^{+\infty} \frac{r^2 |D_v \phi|^2}{\Omega^2} (u, v_{R}(u_2)) du \leq \int_{u_{R_0}(v_{R}(u_2))}^{+\infty} \frac{r^2 |D_v \phi|^2}{\Omega^2} (u, v_{R}(u_2)) du + 2C_6^{-1} \int_{u_{R_0}(v_{R}(u_2))}^{+\infty} \frac{r^2 |D_v \phi|^2}{\Omega^2} (u, v_{R}(u_2)) du.
\]

Therefore, also using the fact that \( 3E_{d_{gep}}(u_1) + 2 \int_{u_{R_0}(v_{R}(u_1))}^{+\infty} \frac{r^2 |D_v \phi|^2}{\Omega^2} (u, v_{R}(u_1)) du \leq 5E(u_1) \) we get
\[
\int_{v_{R}(u_1)}^{v_{R}(u_2)} r^2 |D_v \phi|^2 |_{H_+}(v) dv + 4C_6^{-1} E(u_2) + \int_{u_1}^{+\infty} r^2 |D_v \phi|^2 (u_2, v) dv + 2C_{5q_0} \left( \sup_{\partial(u_1, u_2)} |Q| \right) E^+(u_1, u_2).
\]

(5.36)

Now choosing \( |Q| \) small enough so that \( C_5C_6q_0 \left( \sup_{\partial(u_1, u_2)} |Q| \right) < 1 \), we can absorb the \( E^+(u_1, u_2) \) term on the left-hand-side, which proves Proposition 5.6.

We indeed proved that:

\(^{57}\) This only makes the former estimate worse, and \( C_6 \) can still be taken to depend on \( M \) and \( \rho \) only.
6 Decay of the energy

We now establish the time decay of the energy. We use the $r^p$ method developed in [3]. The main idea is that the boundedness of $r^p$ weighted energies $E_p$ can be converted into time decay for the standard energy $E$ on the $V$-shaped foliation $V_t$ we introduced in section 2.3.

The papers [1], [3], [2] and [6] all treat the linear wave equation on black hole space-times, in different contexts. Compared to these works, the main difference and difficulty for the charged scalar field model is the presence of a non-linear term, coming from the interaction with the Maxwell field.

The whole objective of this section is to find a way absorb this interaction term, for various values of $p$ and to prove the boundedness of the $r^p$ weighted energy.

We are going to assume the energy boundedness and the Morawetz estimates of the former section. As before, to treat the interaction term, we rely on the smallness of the charge $Q$, however it is not as drastic as in the former section : indeed, $q_0/Q$ must be smaller than a universal constant, independent on the black holes parameters. This is one of the key ingredients of the proof of Theorem 2.3.

Notice also that, thanks to the charge a priori estimates from section 2, provided that the initial energy of the scalar field is sufficiently small, it is equivalent to talk about the smallness of $Q$, of $e$ or of $e_0$, c.f. Remark 17. This remark will be used implicitly throughout this section.

The first step is to establish a $r^p$ hierarchy for $p < 2$. For this, we use a Hardy type inequality and we absorb the interaction term into the $\int_{V_t} E_{p-1}[\psi](u) du$ term if the left-hand-side. The smallness of the charges $Q$ and $e$ is essential and limits the maximal $p$ to $p_{max} < 1 + \sqrt{1 - 4q_0/e}$, which tends to $2$ as $e$ tends to $0$.

This method cannot be extended for $p > 2$ because for a scalar field $\phi$, $r\partial\phi$ admits a finite generically non-zero limit $\psi$ the radiation field — towards $\mathcal{T}^+$. This very fact imposes that the $r^{p-3}$ weight in the Hardy estimate is strictly inferior to $r^{-1}$, hence $p < 2$ is necessary to apply the same argument, even when $e \to 0$.

Still this first step already gives some decay of the energy that will be crucial for the second step.

The second step uses a different strategy : this time, while we still use the Hardy inequality, we apply it to a weaker $r$ weight. This is because the Hardy inequality proceeds with a maximal $r^{-1}$ weight. This allows us to take $p$ larger, up to almost $3$ as $e$ tends to $0$. On the other hand, we now have to absorb the interaction term into $E_p[\psi](u)$, in contrast to what was done in the first step.

This is done in two times : first we bootstrap the expected energy decay\(^{59}\) $\hat{E}_{p-1}(u) \lesssim u^{-1+p}$, for some small $\epsilon$.

Finally we inject this bound in the energy inequality to control the interaction term and get decay of $E_{p-1}(u)$. Then the bootstrapped decay is retrieved, after making use of the smallness of $e$.

It is interesting to notice that the procedure does not close the boundedness of the $r^p$ weighted energy for the largest $p$ that we consider but allows for a small $u$ growth, due to the use of a Grönwall-type argument.

This procedure imposes to take $p < 3 - \epsilon(q_0 e)$, with $\epsilon(q_0 e) \to 0$ as $q_0 e \to 0$ and gives the boundedness of $r^p$ weighted energies for such a $p$ range.

This finally proves that the standard energy decays at a rate that tends to $0$ when $e$ tends to $0$.

More precisely, the goal of this section is to prove the following result :\(^{60}\)

**Proposition 6.1.** Suppose that the energy boundedness (5.23) and the Morawetz estimates (5.38) are valid. Assume that $q_0|e| < \frac{1}{4}$. Assume also that for all $0 \leq p' < 2 + \sqrt{1 - 4q_0|e|}$, $E_{p'}(u_0(R)) < \infty$.

For all $0 \leq p < 2 + \sqrt{1 - 4q_0|e|}$, there exist $\delta = \delta(p, e, M, p) > 0$ such that if $||Q - e||_{L^\infty(D(u_0(R), +\infty))} < \delta$ then for all $\epsilon > 0$, there exists $R_0 = R_0(\epsilon, p, M, p, e) > r_+ \ such that if $R > R_0$, there exists $D = D(M, p, R, e, p, e) > 0$ such that for all $u > 0$:

$$E_p(u) \leq D_p \cdot u^{p - (2 + \sqrt{1 - 4q_0|e|})},$$

in particular,

$$E(u) \leq D_0 \cdot u^{-2 - \sqrt{1 - 4q_0|e|}},$$

\(^{59}\)For a definition of $E_p[\psi](u)$, c.f. section 2.3 or section 6.1.

\(^{60}\)For a definition of $E_{p-1}(u)$, c.f. section 2.3 or section 6.1.
6.1 Preliminaries on the decay of the energy

In this section, we are going to establish a few preliminary results on the energy decay. The main goal is to understand how the boundedness of $r^p$ weighted energies implies the time decay of the un-weighted energy.

The estimates are so tight that they can be closed only with the smallness of the charge. Therefore it is very important to monitor the constants as carefully as possible.

As we will see, the lowest order term, which should be controlled together with the $r^p$ weighted energy, is bounded by a large constant and the smallness of the charge cannot make it smaller. The key point is that this term actually enjoys additional $u$ decay so that the large constant can be absorbed for large $u$.

This motivates the following definitions, some of which have already been encountered by the reader:

$$E(u) = \int_{q}^{+\infty} r^2 |D_u \phi|^2 \Omega^2 (u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v \psi|^2 (u, v) dv,$$

$$E^+(u_1, u_2) := E(u_2) + E(u_1) + \int_{r_R(u_1)}^{r_R(u_2)} r^2 |D_u \phi|^2 (v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|^2 (u, v) du,$$

$$E_p[u](u) := \int_{v_R(u)}^{+\infty} r^p |D_u \psi|^2 (u, v) dv,$$

$$\bar{E}_p(u) := E_p[u](u) + E(u).$$

It should be noted that $\bar{E}_p(u)$ and $E(u)$ are comparable.

In what follows, we make a repetitive implicit use of the trivial inequality $E_p[u](u) \leq \bar{E}_p(u)$.

We now establish a preliminary lemma, that reduces the problem to understanding the interaction term.

Lemma 6.2. There exists $C_1 = C_1(M, \rho) > 0$ such that for every $0 < p < 3$ and for all $u_0(R) \leq u_1 < u_2$:

$$p \int_{u_1}^{u_2} E_p - E_0[u](u) + \bar{E}_p[u](u_2) \geq [1 + P_0(R)] \left( E_p[u](u_1) + \int_{D(u_1, u_2) \cap \{ r \geq R \}} 2q_0 Q \Omega^2 r^{p-2} \Omega(|D_u \psi|) dudv \right) + C_1 \cdot E(u_1),$$

(6.3)

where $P_0(r)$ is a polynomial in $r$ that behaves like $O(r^{-1})$ as $r$ tends to $+\infty$ and with coefficients depending only on $M$ and $\rho$.

Proof. We multiply (2.6) by $2r^p D_u \psi$ and take the real part. We get:

$$r^p \partial_u (|D_u \psi|^2) = \Omega^2 r^{p-3} \left( -2M + \frac{2p^2}{r} \right) \partial_u (|\psi|^2) - 2q_0 Q \Omega^2 r^{p-2} \Omega(|D_u \psi|).$$

We integrate this identity on $D(u_1, u_2) \cap \{ r \geq R \}$ and after a few integrations by parts we get:

$$\int_{D(u_1, u_2) \cap \{ r \geq R \}} \left( p r^{p-1} D_u \psi \right)^2 + \left( 2M(3 - p) - (4 - p) \frac{2p^2}{r} \right) r^{p-4} |\psi|^2 \right) [1 + P_1(r)] dudv + \int_{r_R(u_1)}^{+\infty} r^p |D_u \psi|^2 (u_2, v) dv$$

$$+ \int_{u_1}^{u_2} \Omega^2 \left( 2M - \frac{2p^2}{r} \right) r^{p-3} |\psi|^2_{L^1} (u) du = \int_{D(u_1, u_2) \cap \{ r \geq R \}} 2q_0 Q \Omega^2 r^{p-2} \Omega(|D_u \psi|) dudv + \int_{r_R(u_1)}^{+\infty} r^p |D_u \psi|^2 (u_1, v) dv,$$

(6.4)

where $P_1(r)$ is a polynomial in $r$ that behaves like $O(r^{-1})$ as $r$ tends to $+\infty$ and with coefficients depending only on $M$ and $\rho$.

Since $p < 3$ we can take $R$ large enough so that $|P_1(r)| < 1$ and $2M(3 - p) - (4 - p) \frac{2p^2}{r} > 0$.

We then have established:

$$p \int_{u_1}^{u_2} E_p - E_0[u](u) + \bar{E}_p[u](u_2) \geq [1 + P_0(R)] \left( E_p[u](u_1) + \int_{D(u_1, u_2) \cap \{ r \geq R \}} 2q_0 Q \Omega^2 r^{p-2} \Omega(|D_u \psi|) dudv \right),$$

(6.5)

where $P_0(r)$ is a polynomial in $r$ that behaves like $O(r^{-1})$ as $r$ tends to $+\infty$ and with coefficients depending only on $M$ and $\rho$.

Finally, to obtain the claimed estimate (6.3), it is enough to add (5.22) to the previous inequality. This concludes the proof of the lemma.

We are now going to explain how to the boundedness of the $r^p$ weighted energies $\bar{E}_p(u)$ implies the time decay of the standard energy $E(u)$. This is the core of the new method invented in [15]. We write a refinement of the classical argument with the notations of the paper. This will be important in section 6.6.
Lemma 6.3. Suppose that there exists $1 < p < 2$ and $C > 0$ such that for all $u_0(R) \leq u_1 < u_2$

$$\int_{u_1}^{u_2} E_{p-1}[\psi](u)du + \tilde{E}_p(u_2) \leq C \cdot \tilde{E}_p(u_1), \quad (6.6)$$

and

$$\int_{u_1}^{u_2} E_0[\psi](u)du + \tilde{E}_1(u_2) \leq C \cdot \tilde{E}_1(u_1). \quad (6.7)$$

Then for all $0 \leq s \leq p$ and for all $k \in \mathbb{N}$, there exists $C' = C'(M, \rho, C, R, s, E(u_0(R)), k) > 0$ such that for all $u > 0$

$$E_s(u) \leq C' \cdot \left( u^{-k(1-\frac{s}{p})} + \sup_{2^{-2k-1}u \leq u'} \tilde{E}_p(u') \right). \quad (6.8)$$

Remark 26. In the case $s = 0$, this lemma broadly says that, up to an arbitrarily fast decaying polynomial term and a sup, the energy decays like $u^{-p} \cdot \tilde{E}_p$. This formulation that was not present in the original article [15] will be useful to obtain the almost optimal energy decay in section 6.6.

Remark 27. The lemma is stated for $1 < p < 2$ because it is going to be applied in this paper to $p = 1 + \sqrt{1 - 4\eta},$ with $\eta > 0$ small. Of course, a similar result still holds without that restriction, using a similar method. In the present paper, we will need more than this lemma in section 6.6 where a major improvement of the method will become necessary.

Proof. During this proof, we make use of the following notation: $A \lesssim B$ if there exists a constant $C_0 = C_0(M, \rho, C, R, s, \tilde{E}_p(u_0(R)), k)$ such that $A \leq C_0 B$.

We take $(u_n)_{n \in \mathbb{N}}$ to be a dyadic sequence, i.e. $u_{n+1} = 2u_n$.

We first apply the mean-value theorem on $[u_n, u_{n+1}]$ there exists $u_n < u_n < u_{n+1}$ such that

$$E_{p-1}[\psi](\tilde{u}_n) = \frac{\int_{u_n}^{u_{n+1}} E_{p-1}[\psi](u)du}{u_n}.$$ 

where we used the fact that $u_{n+1} - u_n = u_n$

Applying (6.6) gives

$$E_{p-1}[\psi](\tilde{u}_n) \lesssim \tilde{E}_p(u_n). \quad (6.9)$$

Now notice that because $1 < p < 2$, $(p-1) \in (0,1)$ and $(2-p) \in (0, 1)$. We then apply Hölder’s inequality under the form :

$$E_1[\psi](u) \leq (E_{p-1}[\psi](u))^{p-1} (E_p[\psi](u))^{2-p},$$

where we used the fact that $1 = (p-1)(p-1) + (2-p)p$.

Then we apply (6.9) and the precedent inequality for $u = \tilde{u}_n$ :

$$E_1[\psi](\tilde{u}_n) \lesssim \sup_{u_n \leq u \leq u_{n+1}} \frac{\tilde{E}_p(u)}{(u_n)^{p-1}}. \quad (6.10)$$

We now use (6.7) and (6.10) to get :

$$\int_{u_n}^{u_{n+1}} E_0[\psi](u)du \lesssim E_1(u_n) \lesssim E_1(\tilde{u}_n) \lesssim E(u_n) + \sup_{u_{n-1} \leq u \leq u_n} \frac{\tilde{E}_p(u)}{(u_n)^{p-1}}.$$

We used that $u_n \sim u_{n-1} \leq \tilde{u}_n \leq u_{n}$ and for the last inequality, we also used the energy boundedness (5.22) under the form $E(\tilde{u}_{n-1}) \lesssim E(u_{n-1})$.

Then, following the method developed in [15], we add a multiple of the Morawetz estimate (5.38) to cover for the $\{r \leq R\}$ energy bulk terms. After a standard computation, identical to that done in [15] we get

$$\int_{u_n}^{u_{n+1}} E(u)du \lesssim E(u_{n-1}) + \sup_{u_{n-1} \leq u \leq u_n} \frac{\tilde{E}_p(u)}{(u_n)^{p-1}}.$$

Now using the mean-value theorem again on $[u_{n-1}, u_n]$ there exists $\tilde{u}_{n-1} \in (u_{n-1}, u_n)$ such that

$$E(\tilde{u}_{n-1}) = \frac{\int_{u_{n-1}}^{u_n} E(u)du}{u_n - u_{n-1}} \lesssim E(u_{n-2}) + \sup_{u_{n-2} \leq u \leq u_{n-1}} \frac{\tilde{E}_p(u)}{(u_n)^{p-1}}.$$

Making use of the energy boundedness from (5.22) finally gives

$$E(u_n) \lesssim \frac{E(u_{n-2})}{u_n} + \sup_{u_{n-2} \leq u \leq u_{n-1}} \frac{\tilde{E}_p(u)}{(u_n)^{p-1}}. \quad (6.11)$$

Now take an integer $k \geq 1$ and iterate (6.11) $k$ times.
\[ E(u_n) \lesssim \frac{E(u_{n-k})}{(u_n)^k} + \sum_{i=1}^{l} \frac{\sup_{u_{n-2i} \leq u \leq u_{n-2i+1}} \tilde{E}_p(u)}{(u_n)^{p+r-1}}. \]

Now we can use the energy boundedness to bound \( E(u_{n-2k}) \): there exists a constant \( \tilde{C} = \tilde{C}(M, \rho) > 0 \) such that

\[ E(u_{n-2k}) \leq C \cdot E(u_0(R)). \]

We then have

\[ E(u_n) \lesssim (u_n)^{-k} + \frac{\sup_{u_{n-2k} \leq u \leq u_{n-1}} \tilde{E}_p(u)}{(u_n)^p}. \]

For \( u \in (u_n, u_{n+1}) \), we get

\[ E(u) \lesssim u^{-k} + \frac{\sup_{u_{n-2k-1} \leq u' \leq u_n} \tilde{E}_p(u')}{u^p}, \quad (6.12) \]

where we used the fact that \( \frac{n}{2} \leq u_n \leq u \).

This already proves the lemma for \( s = 0 \).

For the general case \( 0 < s \leq p \), we use the Hölder inequality under the form

\[ E_0[\psi](u) \leq \frac{E_0[\psi](u)}{s} \left( E_p[\psi](u) \right)^\frac{s}{p} \lesssim \left( E(u) \right)^{1-\frac{s}{p}} \left( E_p(u) \right)^{\frac{s}{p}}, \]

where we used the fact that \( s = (1 - \frac{p}{2}) \cdot 0 + \frac{p}{2} \cdot p \). For the last inequality, we also used the fact that for \( R \) large enough we have

\[ E_0[\psi] \leq 12E(u). \]

This is because \( |D_\theta \psi|^2 \leq 2(r^2|D_\theta \phi|^2 + \Omega^4|\phi|^2) \leq 2(r^2|D_\theta \phi|^2 + |\phi|^2) \) and the use of the Hardy inequality (2.19) under the form

\[ \int_{\mathbb{R}(u)} |\phi|^2 (u, v) dv \leq \frac{4}{\Omega^2} \int_{\mathbb{R}(u)} r^2 |D_\theta \phi|^2 (u, v) dv. \]

Then using (6.12) we get

\[ \tilde{E}_s(u) \lesssim u^{-k} + \frac{\sup_{u_{n-2k-1} \leq u' \leq u} \tilde{E}_p(u')}{u^p} + u^{-k(1-\frac{s}{p})} + \frac{\sup_{u_{n-2k-1} \leq u' \leq u} \tilde{E}_p(u')}{u^{p-s}}, \]

where we used the boundedness of \( \tilde{E}_p(u) \) for the third term and the inequality \( (a + b)^\theta \leq C(\theta) \cdot (a^\theta + b^\theta) \), for \( \theta = \frac{s}{p} \) or \( \theta = 1 - \frac{s}{p} \).

Putting everything together and taking \( u \) large we get

\[ \tilde{E}_s(u) \lesssim u^{-k(1-\frac{s}{p})} + \frac{\sup_{u_{n-2k-1} \leq u' \leq u} \tilde{E}_p(u')}{u^{p-s}}, \]

which concludes the proof of the lemma.

\[ \square \]

### 6.2 Application for the \( r^p \) method for \( p < 2 \).

In this section, we establish the boundedness of the \( r^p \) weighted energy for \( p \) slightly inferior to 2. The argument relies on the absorption of the interaction term into the \( \int_{u_0}^{u_2} E_{p-1}[\psi](u) du \) term. To do this, we make use of a Hardy-type inequality, coupled with the Morawetz estimate to treat the boundary terms on the curve \( \{ r = R \} \).

As a corollary, we establish the \( u^{-p} \) decay of the un-weighted energy for the range of \( p \) considered. This step is crucial to establish the almost-optimal decay claimed in section 6.3.

The maximal value of \( q_0(\Omega) \) authorized in this section is \( \frac{1}{2} \).

We start by the main result of this section, which is the computation that illustrates how the interaction term can be absorbed into the left-hand-side.

**Proposition 6.4** \( (r^p \) weighted energy boundedness). Assume that \( q_0(\Omega) < \frac{1}{2} \).

For all \( \eta_0 > 0 \), there exists \( R_0 = R_0(M, \rho, \eta_0, e) > r_{\frac{1}{2}}, \delta = \delta(M, \rho, \eta_0, e) > 0 \) such that if \( ||Q - \ell||_{L^\infty(\mathbb{R}(u_0(R), +\infty))} < \delta \), then for all \( 1 - \sqrt{1 - 4q_0(\Omega)} < p < 1 + \sqrt{1 - 4q_0(\Omega)} \) and \( R > R_0 \), there exists \( C_2 = C_2(M, \rho, R, \eta_0, p, e) > 0 \) such that for all \( u_0(R) \leq u \leq u_2 \), we have

\[ \left( p - \frac{4q_0\Omega}{2-p} - \eta_0 \right) \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq (1 + \eta_0) \cdot E_p[\psi](u_1) + C_2 \cdot E(u_1). \quad (6.13) \]
Proof. We apply Cauchy-Schwarz to control the charge term as:

\[ \left| \int_{D(u_1,u_2) \cap \{ r \geq R \}} \Omega^2 r^{p-2} \Theta(\psi Dv) dudv \right| \leq \left( \int_{D(u_1,u_2) \cap \{ r \geq R \}} r^{p-1} |Dv|^2 dudv \right)^{\frac{1}{2}} \left( \int_{D(u_1,u_2) \cap \{ r \geq R \}} r^{p-3} \Omega^4 |\psi|^2 dudv \right)^{\frac{1}{2}}. \]

Using a version of Hardy’s inequality (2.20), we then establish that

\[ \left( \int_{D(u_1,u_2) \cap \{ r \geq R \}} r^{p-3} \Omega^2 |\psi|^2 dudv \right)^{\frac{1}{2}} \leq \frac{2}{2-p} \left( \int_{D(u_1,u_2) \cap \{ r \geq R \}} r^{p-1} |Dv|^2 dudv \right)^{\frac{1}{2}} + \frac{R^{p-2}}{2-p} \int_{u_1}^{u_2} |\psi|^2(u,v_R(u)) du \right)^{\frac{1}{2}}. \]

Then, using an averaging procedure in \( r \), similar to the one in Lemma 5.10, we find that for all \( R > 0 \), there exists \( \frac{R}{2} < \tilde{R} < R \).

\[ \int_{u_1}^{u_2} |\psi|^2(u,v_R(u)) du \leq \frac{2}{R} \int_{D(u_1,u_2) \cap \{ \frac{R}{2} \leq r \leq R \}} |\psi|^2 dudv. \]

We are going to abuse notations and still call \( R \) this \( \tilde{R} \). Using the Morawetz estimate (6.1), we see that there exists \( C = C(M, \rho, R) > 0 \) such that

\[ \frac{R^{p-2}}{2-p} \int_{u_1}^{u_2} |\psi|^2(u,v_R(u)) du \leq C \cdot \frac{C}{2-p} \cdot E(u_1). \]

Then we apply Cauchy-Schwarz to the charge term and we combine the result with the inequality of Lemma 6.2 to get that for all \( \eta_0 > 0 \):

\[ \left( p - \frac{4q_0}{(2-p)\Omega(R)} \left| Q \right| \right) \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq (1 + P(R)) \cdot E_p[\psi](u_1) + \frac{C}{2\eta_0 \cdot (2-p)} \cdot E(u_1). \]

Then, we can taking \( R_0 \) large enough and \( \delta \) small enough so that:

\[ \left| \frac{4q_0}{(2-p)\Omega(R)} \left| Q \right| \right| - \frac{4q_0}{(2-p)} \right| \leq \frac{\eta_0}{2}, \]

\[ 1 + P(R) \leq 1 + \eta_0. \]

More precisely, it suffices to have \( R_0 > \frac{D(M,\rho,\eta_0)}{(1-\sqrt{1-4\eta_0}|e| \cdot \eta_0)} \) and \( \delta < 2(1 - \sqrt{1-4\eta_0}|e|) \cdot \eta_0 \), for some \( D(M,\rho) > 0 \).

This concludes the proof of Proposition 6.4. \( \square \)

To finish this section, we establish the best decay of the energy that we can attain so far. The methods employed are similar to that of [15]. In particular the constants do not need to be monitored as sharply as in section 6.3.

Corollary 6.5. Suppose that \( q_0 |e| \leq \frac{1}{4} \).

Then for all \( 1 < p' < 1 + \sqrt{1-4\eta_0} |e| \), for all \( 0 \leq s \leq p' \) and for all \( k \in \mathbb{N} \) large enough, there exists \( C_0 = C_0(M,\rho,\eta_0,R,p',s,k,E(u_0(R))) > 0 \) and \( C_0' = C_0'(M,\rho,\eta_0,R,p',s,k,E(u_0(R))) > 0 \) such that for all \( u > 0 \), we have the following energy decay

\[ \tilde{E}_s(u) \leq C_0 \cdot \left( u^{-k(1-\frac{1}{p'})} + \sup_{u \leq u' \leq u^{p'-2}} \tilde{E}_{p'}(u') \right) \leq C_0'. \]

Prove. We make use of the result of Proposition 6.4 and take \( \eta_0 \) sufficiently small so that there exists \( D_0 = D_0(M,\rho,p',\eta_0,R) \) so that for all \( u_0(R) \leq u_1 < u_2 \):

\[ \int_{u_1}^{u_2} E_{p'-1}[\psi](u) du + \tilde{E}_{p'}(u_2) \leq D_0 \cdot \tilde{E}_{p'}(u_1), \]

(6.16)

and

\[ \int_{u_1}^{u_2} E_0[\psi](u) du + \tilde{E}_1(u_2) \leq D_0 \cdot \tilde{E}_1(u_1). \]

(6.17)

Thus the hypothesis of Lemma 6.3 are satisfied for the chosen \( p' \) range. This proves the first inequality of (6.15).

The second one solely relies on the boundedness of the \( r^p \) weighted energy.
We also need to take \( k > p' \), so that \( k(1 - \frac{1}{p'}) > p' - s \).

This concludes the proof of Corollary 6.3.

**6.3 Extension of the \( r^p \) method to \( p < 3 \).**

In this section, we establish the almost optimal decay for the energy, namely almost 3 as \( \epsilon \to 0 \). To be more precise, the maximal decay is \( p = 2 + \sqrt{T - 4q_0|\epsilon|} \). Notice that \( p > 2 \) for \( q_0|\epsilon| < \frac{1}{4} \), which implies an integrable point-wise decay on the event horizon, crucial for the interior study, c.f. section 6.2.

The strategy employed to absorb the interaction term differs radically from that of section 6.2, where the maximal decay was limited to 2.

Indeed, we now aim at absorbing this term inside the \( \tilde{E}_p(u_2) \) term. To do this, we have to solve an ordinary differential inequation and, making use of the relevant smallness, prove an arbitrary growth of \( \tilde{E}_p(u) \). This is actually sufficient to prove the claimed decay.

In the core of the proof, we make use of a bootstrap method and we retrieve the bootstrap assumption using a pigeon-hole type argument already developed in the proof of Lemma 6.3.

The most crucial argument is of a non-linear nature, and is best summarised by the alternative of

**Lemma 6.8.** In short, according to how \( \left( \int u \tilde{E}_p[u] \right)^\frac{2}{p} \) compares with some initial energies, we either have \( (6.27) \) or \( (6.25) \).

(6.27) is an energy boundedness statement, that would be suitably applied for \( u_1 = \tilde{u}_0 \) and \( u_2 = u \), i.e. on an interval \([\tilde{u}_0, u]\) where \( \tilde{u}_0 \) is fixed and \( u \) is the large variable.

In contrast, (6.25) is more adapted to prove \( u \) decay of \( \tilde{E}_p-1 \) with a pigeon-hole like argument. It is suitably applied for \( u_1 = \frac{\epsilon}{2} \) and \( u_2 = u \), because then the term \( \frac{R - \epsilon}{\epsilon} \) is bounded.

We then use a dichotomy argument:

Suppose that we can apply alternative 2, then it is easy to use the boundedness result of Proposition 6.6 for \( p - 1 \) and we broadly get \( u^{-1} \) decay of \( \tilde{E}_p-1 \).

Instead, suppose that alternative 1 holds. Then we can iterate the estimate \( (6.27) \) of alternative 1 on intervals \([2^{-k-1}u, 2^{-k+1}u]\) until we get to an interval where alternative 2 applies, or that we reach the first interval \([u_0, 2u_0]\). In any case, the iterated use of \( (6.27) \) proves, after a choice of relevant small constants, that \( \tilde{E}_p \lesssim u' \) for some arbitrary small \( \epsilon \). This is enough to close the bootstrap assumption and obtain some \( u^{-1+\epsilon} \) decay for \( \tilde{E}_p-1 \).

**Proposition 6.6.** Assume that \( q_0|\epsilon| < \frac{1}{4} \).

Assume also that for all \( 0 \leq p' < 2 + \sqrt{1 - 4q_0|\epsilon|} \), \( \tilde{E}_p(u_0(R)) < \infty \).

For all \( 2 - \sqrt{1 - 4q_0|\epsilon|} < p < 2 + \sqrt{1 - 4q_0|\epsilon|} \), there exist \( \delta = \delta(p, e, M, \rho) > 0 \) such that if \( \|Q - \epsilon\|_{L^\infty(D(u_0(R), +\infty))} < \delta \), then for all \( \epsilon > 0 \) there exists \( R_0 = R_0(\epsilon, p, M, \rho, e) > r_+ \) such that if \( R > R_0 \), there exists \( D = D(M, \rho, R, \epsilon, p, e) > 0 \) such that for all \( u > 0 \):

\[
\tilde{E}_p-1(u) \leq \frac{D}{u^{1+\epsilon}}, \quad (6.18)
\]

\[
\tilde{E}_p(u) \leq D \cdot u^\epsilon, \quad (6.19)
\]

**Proof.** Like in section 6.2, we apply Cauchy-Schwarz to control the charge term but we distribute the \( r \) weights differently as:

\[
\left| \int_{\mathcal{D}(u_1, u_2)} \Omega^2 r^{p-2} \Im(\psi^2 D\psi) \, dudv \right| \leq \left( \int_{\mathcal{D}(u_1, u_2)} r^p |D\psi|^2 \, dudv \right)^{\frac{1}{2}} \left( \int_{\mathcal{D}(u_1, u_2)} r^{p-4} |\psi|^2 \, dudv \right)^{\frac{1}{2}}.
\]

Using a version of Hardy’s inequality \eqref{hardy}, we then establish that

\[
\left( \int_{\mathcal{D}(u_1, u_2)} r^{p-4} |\psi|^2 \, dudv \right)^{\frac{1}{2}} \leq \frac{2}{(3-p)\Omega(R)} \left( \int_{\mathcal{D}(u_1, u_2)} r^{p-2} |D\psi|^2 \, dudv \right)^{\frac{1}{2}} + \left( \frac{R^{p-3}}{3-p} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) \, du \right)^{\frac{1}{2}}.
\]

Combining both inequalities we see that

\[
\left| \int_{\mathcal{D}(u_1, u_2)} \Omega^2 r^{p-2} \Im(\psi^2 D\psi) \, dudv \right| \leq \frac{2}{(3-p)\Omega(R)} \left( \int_{u_1}^{u_2} \tilde{E}_{p-2} |\psi(u)| \, du \right)^{\frac{1}{2}} \left( \int_{u_1}^{u_2} \tilde{E}_p |\psi(u)| \, du \right)^{\frac{1}{2}}
\]

\[
+ \left( \int_{u_1}^{u_2} \tilde{E}_p |\psi(u)| \, du \right)^{\frac{1}{2}} \left( \frac{R^{p-3}}{3-p} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) \, du \right)^{\frac{1}{2}}.
\]

(6.20)
The main contribution of the right-hand-side is the first term.
Using the results \( \text{[6.2]} \) of section 6.2, we see that there exists \( C' = C'(M, \rho, R, p, c) > 0 \) such that for all \( u_0(R) \leq u_1 < u_2 \):
\[
\left( \int_{u_1}^{u_2} E_{p-2}(\psi)(u)du \right)^{\frac{1}{2}} \leq C' \cdot \left[ (E_{p-1}[\psi](u_1))^{\frac{1}{2}} + (E(u_1))^{\frac{1}{2}} \right],
\]
where we used the fact that for all \( \alpha, \beta > 0 \), \( \sqrt{\alpha} + \beta \leq \sqrt{\alpha + \beta} \).
Similarly to the method employed in section 6.2, we deal with the second term in the right-hand-side of \( \text{[6.20]} \) using the Morawetz estimate of section 5.1. We find that there exists a constant \( D = D(M, \rho, p, c, R) > 0 \) such that
\[
| \int_{\Omega \cap \{ \eta > 0 \}} \sqrt{2} \Re(\psi \overline{D_v \psi})dudv | \leq D \cdot \left[ (E_{p-1}[\psi](u_1))^{\frac{1}{2}} + (E(u_1))^{\frac{1}{2}} \right] \cdot \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}}.
\]
(6.21)
Again, the main contribution in the right-hand-side is the first term. Actually we will see that the \( E(u_1) \) term enjoys a better decay in \( u \) than \( E_{p-1}[\psi](u_1) \) term because in this section \( p > 1 \). This will later allow us to absorb \( \eta^2 \) the second term in the right-hand-side into the first term, for \( u_1 \) large enough.
We then combine the bound from Lemma \( \text{[6.2]} \) with the bound on the charge term \( \text{[6.21]} \) to get for any \( \eta > 0 \) small enough:
\[
\begin{align*}
p \int_{u_1}^{u_2} E_{p-1}[\psi](u)du + \hat{E}_p(u_2) & \leq (1 + \eta) \cdot E_p[\psi](u_1) \\ + \hat{D} \cdot \left[ (E_{p-1}[\psi](u_1))^{\frac{1}{2}} \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} + (E(u_1))^{\frac{1}{2}} \right] \cdot \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} + E(u_1),
\end{align*}
\]
(6.22)
where we took \( R > R'_0(M, \rho, \eta) \) so \( P_0(r) \) — defined in the statement of Lemma \( \text{[6.2]} \) — verifies \( [1 + P_0(r)] < (1 + \eta) \) and \( \hat{D} = \hat{D}(M, \rho, p, c, R) > 0 \).
In contrast to the strategy adopted in section 6.2, we now aim at absorbing the error term into the \( E_p[\psi](u_2) \) term of the left-hand-side.
Therefore we are going to drop temporarily the first term of the left-hand-side and consider the differential functional inequality, with \( u_2 \) being the variable, \( u_1 \) the constant and \( \int_{u_1}^{u_2} E_p[\psi](u)du \) the unknown function
\[
E_p[\psi](u_2) \leq \hat{D} \cdot \left[ (E_{p-1}[\psi](u_1))^{\frac{1}{2}} + (E(u_1))^{\frac{1}{2}} \right] \cdot \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} + (1 + \eta) \cdot E_p[\psi](u_1) + \hat{D} \cdot E(u_1).
\]
(6.23)
As usual, the main contribution of the right-hand-side is on the first term, the others being treated as errors.
To include also the error terms, we will require a small technical lemma dealing with integration of certain square-root functions:

**Lemma 6.7.**
\[
\int_{u_1}^{u_2} \frac{du}{a\sqrt{u} + b} = \frac{2}{a} \left[ \sqrt{\frac{b}{a}} - \frac{b}{a} \log(\sqrt{\frac{b}{a}} + \sqrt{\frac{b}{a}}) \right]_{u_1}^{u_2},
\]
(6.24)
where for every function \( f \) we define \( |f(u)|_{u_1}^{u_2} := f(u_2) - f(u_1) \).

**Proof.** The proof is elementary and merely uses the change of variable \( x = \sqrt{u} \). \( \square \)

We are now going to use a bootstrap method.
For any \( \epsilon > 0 \) small enough, as claimed in the proposition statement, we make the following bootstrap assumption for some \( \Delta \) large enough, some \( U_0 > 1 \) large enough to be determined later: for all \( u \geq U_0 \):
\[
\tilde{E}_{p-1}(u) \leq \frac{\Delta}{u^{1-2\epsilon}}.
\]
(6.25)
The set of points for which this estimate is verified is non-empty provided that \( \Delta > \tilde{E}_{p-1}(U_0) \cdot U_0^{1-2\epsilon} \).
For now on, assume that \( \tilde{E}_{p-1}(U_0) < +\infty \). We are going to justify this at the end of the proof.
Using \( \text{[6.25]} \) with Corollary \( \text{[6.5]} \) to \( p' = p - 1 \) for \( 2 - \sqrt{1 - 4q_0|e|} < p < 2 + \sqrt{1 - 4q_0|e|} \) we get the following decay of the energy, after taking \( k \) large enough, with respect to \( \Delta \): for all \( u \geq U_0 \)
\[
E(u) \leq \frac{C_0 \cdot \Delta}{u^{p-2\epsilon}},
\]
(6.26)
\footnote{After choosing \( \eta_0 = \eta_0(M, \rho, p, c) \) appropriately small.}
\footnote{Even though the constant \( D \) can be arbitrarily large.}
Lemma 6.8. Under bootstrap assumption (6.25), for all \( U_0 \leq u_1 < u_2 \) and for all \( \beta > 0 \) we either have

\[
p \int_{u_1}^{u_2} E_{p-1}[\psi](u)du + \tilde{E}_p(u_2) \leq (1 + \beta) \cdot \left( (1 + \eta) \cdot E_p[\psi](u_1) + \frac{D_0 \cdot \Delta}{(u_1)^{p-2}} \right),
\]

or we have

\[
p \int_{u_1}^{u_2} E_{p-1}[\psi](u)du + \tilde{E}_p(u_2) \leq \tilde{D} \cdot \Delta \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right) \cdot \left( \frac{u_2 - u_1}{2u_1} \right),
\]

where \( \tilde{D} := (\tilde{D})^2 \cdot (1 + \sqrt{C_0})^2 \) and \( D_0 = C_0 \cdot \tilde{D} \).

Proof. We now combine the bootstrap assumption with (6.23) to put it in the form

\[
E_p[\psi](u_2) \leq a \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} + b,
\]

with \( a = \tilde{D} \cdot \sqrt{\Delta} \cdot \left( (u_1)^{\frac{1}{2} + \epsilon} + \sqrt{C_0} \cdot (u_1)^{\frac{1}{2} - \epsilon} \right) \) and \( b = (1 + \eta) \cdot E_p[\psi](u_1) + \frac{D_0 \cdot \Delta}{(u_1)^{p-2}} \).

Now using Lemma 6.7 we see that for all \( 0 < u_1 < u_2 \):

\[
\left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} \leq \frac{a}{2}(u_2 - u_1) + \frac{b}{a} \log \left( 1 + \frac{a}{b} \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} \right)
\]

(6.30)

Now for all \( \beta > 0 \) we have the following alternative:

Either

\[
\left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} \leq \beta \cdot \frac{b}{a},
\]

in which case, combining with (6.22) we find

\[
p \int_{u_1}^{u_2} E_{p-1}[\psi](u)du + \tilde{E}_p(u_2) \leq (1 + \beta) \cdot b,
\]

which is (6.27).

Or

\[
\left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} \geq \beta \cdot \frac{b}{a}.
\]

In that case (6.29) and (6.22) give

\[
p \int_{u_1}^{u_2} E_{p-1}[\psi](u)du + \tilde{E}_p(u_2) \leq (1 + \frac{1}{\beta}) \cdot a \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}}.
\]

(6.31)

Now because the function \( \frac{\log(1 + x)}{x} \) is decreasing, we also have

\[
\frac{b}{a} \log \left( 1 + \frac{a}{b} \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} \right) \leq \frac{\log(1 + \beta)}{\beta} \left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}}.
\]

Hence from (6.30) we have

\[
\left( \int_{u_1}^{u_2} E_p[\psi](u)du \right)^{\frac{1}{2}} \leq \frac{a}{2(1 - \frac{\log(1 + \beta)}{\beta})}(u_2 - u_1).
\]

The combination with (6.31) gives (6.28), after using the fact that \( u_1 \geq 1 \).

\( \square \)

We are now going to apply the pigeon-hole principle.

We take \( (\tilde{u}_n) \) to be a dyadic sequence, namely \( \tilde{u}_{n+1} = 2\tilde{u}_n \) and \( \tilde{u}_0 = U_0 \).

We take \( u = u_2 \in [\tilde{u}_n, \tilde{u}_{n+1}] \) and \( u_1 = \tilde{u}_n \).

Using the mean-value theorem on \([\tilde{u}_n, u]\), we see that there exists \( \tilde{u}_n < U < u \) so that

\[
E_{p-1}[\psi](U) = \int_{\tilde{u}_n}^{\tilde{u}_n} E_{p-1}[\psi](u')du' / (u - \tilde{u}_n).
\]

Then we use the result of Proposition 6.3 of section 6.2 to obtain that for some \( C_2 = C_2(M, \rho, R, p, \epsilon) > 0 \), some \( C = C(M, \rho) > 0 \) and for every \( \eta > 0 \) small enough:

48
\[ \hat{E}_{p-1}(u) \leq (1 + \eta) \cdot E_{p-1}[\hat{v}](U(u)) + C_2 \cdot E(U(u)) \leq (1 + \eta) \cdot E_{p-1}[\hat{v}](U(u)) + C \cdot C_2 \cdot E(\tilde{u}_n), \]

where we also used the boundedness of the energy \((5.22)\) in the last inequality.

Then, using that \((\tilde{u}_n)^{-1} \leq 2u^{-1}\) and \((6.26)\), we see that for all \(u \geq U_0\)
\[ \hat{E}_{p-1}(u) \leq (1 + \eta) \cdot \frac{\int_{\tilde{u}_n}^u E_{p-1}[\hat{v}](u')du'}{u - \tilde{u}_n} + \frac{C_2' \cdot \Delta}{u^{p-\epsilon}}, \]

where we set \(C_2' := C_2 \cdot C_0 \cdot 2^{n-\epsilon}\).

We are now going to make a case disjunction, using Lemma \(6.8\). The idea is that, if alternative 2 holds, then the claimed decay holds immediately. If this is not the case, then we descend in the dyadic interval until we find an interval where alternative 2 holds. It may not exist, but in any case we can close the estimates provided we are willing to abandon the boundedness of the \(r^p\) weighted energy, allowing for an arbitrary small \(u\) growth, using the smallness of \(\eta\) and \(\beta\) with respect to \(\epsilon\). This finally retrieves bootstrap \((6.25)\), provided \(U_0\) and \(\Delta\) are larger than a constant depending on \(\epsilon\).

We want to prove that for all \(n \in \mathbb{N}\), on the interval \([\tilde{u}_{n-1}, \tilde{u}_n]\) :
\[ \hat{E}_{p-1}(u) \leq \frac{\Delta}{2 \cdot u^{1-\epsilon}}, \]

(6.33)
\[ E_p(u) \leq \frac{\Delta \cdot u^*}{2}. \]

(6.34)

Now fix \(\beta > 0\) — small to be chosen later, depending on \(\epsilon\) only — and some \(\tilde{u}_n < u < \tilde{u}_{n+1}\).

Consider the interval \([\tilde{u}_n, u]\) and this \(\beta\) to apply Lemma \(6.8\).

For the first case, suppose that alternative 2 holds on \([\tilde{u}_n, u]\) i.e for \(u_2 = u, u_1 = \tilde{u}_n\) and for this \(\beta\) in Lemma \(6.8\).

Then combining \((6.28)\) and \((6.32)\) we see that
\[ \hat{E}_{p-1}(u) \leq \frac{\hat{D} \cdot (1 + \eta) \cdot \Delta}{p \cdot u^*} \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right) + \frac{C_2' \cdot \Delta}{u^{p-\epsilon}}, \]

To prove \((6.33)\), since \(u \geq U_0\), it is sufficient to have
\[ \frac{\hat{D} \cdot (1 + \eta) \cdot \Delta}{p \cdot U_0} \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right) \leq \frac{1}{4}, \]
\[ \frac{C_2' \cdot \Delta}{U_0^{p-1-\epsilon}} \leq \frac{1}{4}. \]

This is equivalent to
\[ U_0 > \left( \frac{4\hat{D} \cdot (1 + \eta)}{p} \right)^{-1} \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right)^{1-1}, \]
\[ U_0 > (4C_2')^{(p-1-\epsilon)^{-1}}. \]

Since \(\epsilon\) can be taken small enough — depending on \(\epsilon\) — so that \(p - 1 > \epsilon\) and since \(\beta\) — allowed to depend on \(\epsilon\) is chosen independently of \(U_0\) and \(\Delta\), \((6.33)\) is true for all \(u \in [\tilde{u}_n, \tilde{u}_{n+1}]\) provided \(U_0\) is larger than a constant depending only on \(\epsilon, p, e, R\) and the parameters.

Remember also that \(\eta\) is a small constant so we can take \(\eta < 1\).

Notice that \((6.28)\) also gives the following :
\[ \hat{E}_p(u) \leq \frac{\hat{D} \cdot \Delta}{p} \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right) \cdot \left( \frac{u - \tilde{u}_n}{2\tilde{u}_n} \right) \leq \frac{\hat{D} \cdot \Delta}{2} \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right). \]

Therefore, if the following condition is satisfied, \((6.34)\) is true :
\[ U_0 > \left( \frac{\hat{D}}{2} \right)^{-1} \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right)^{\epsilon^{-1}}. \]

Now we treat the case where alternative 1 of Lemma \(6.8\) holds on \([\tilde{u}_n, u]\). \((6.27)\) can then be written as
\[ p \int_{\tilde{u}_n}^u E_{p-1}[\hat{v}](u')du' + \hat{E}_p(u) \leq (1 + \beta) \cdot \left( (1 + \eta) \cdot E_p[\hat{v}](\tilde{u}_n) + \frac{D_0 \cdot \Delta}{(\tilde{u}_n)^{p-2\epsilon}} \right). \]

(6.35)

Then there are again two cases. In the first case, alternative 1 of Lemma \(6.8\) applies on \([\tilde{u}_{n-1}, \tilde{u}_n]\), i.e for \(u_2 = \tilde{u}_n, u_1 = \tilde{u}_{n-1}\) and the same \(\beta\) as before. In the second case, alternative 2 applies on \([\tilde{u}_{n-1}, \tilde{u}_n]\).

If the latter applies, then as we saw before we have
\[ \hat{E}_p(\tilde{u}_n) \leq \hat{D} \cdot \Delta \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right). \]
Then, combining with (6.32) and (6.35) we get:

\[
E_{p-1}(u) \leq 2\tilde{D} \cdot (1 + \eta)^2 \cdot (1 + \beta) \cdot \Delta \cdot \frac{1 + \beta}{\beta - \log(1 + \beta)} \cdot u \leq 2\tilde{D} \cdot (1 + \eta) \cdot (1 + \beta) \cdot \Delta \cdot \frac{C_2^p \cdot \Delta}{w_{p-\epsilon}}.
\]

Then (6.33) is true provided the following bounds hold:

\[
U_0 > \left( \frac{12\tilde{D} \cdot (1 + \beta) \cdot (1 + \eta)^2}{p} \right)^{\frac{1}{\gamma}} \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right)^{\frac{1}{\gamma}},
\]

\[
U_0 > \left[ 12D_0 \cdot (1 + \eta) \cdot 2^{p-2\epsilon} \cdot (1 + \beta) \right]^{(n-1-2\epsilon)^{-1}},
\]

\[
U_0 > (6C_2^p)^{(p-1-\epsilon)^{-1}}.
\]

Under similar conditions, we also prove (6.34).

Now for the former, we can define the integer \( k(n) \) as the minimum of all the integer \( k \) in \([0, n-1] \) such that for all integer \( k \leq k' \leq n \), alternative 1 holds on \([\tilde{u}_{k'}, \tilde{u}_{k'+1}] \).

Using (6.35) repetitively we see that for all \( u \in [\tilde{u}_n, \tilde{u}_{n+1}] \):

\[
p \int_{\tilde{u}_n}^u E_{p-1}[\psi](u') du' + \tilde{E}_p(u) \leq (1 + \beta)^n \cdot (1 + \eta)^{n-k(n)+1} \cdot E_p(u_{k(n)}) + D_0 \cdot \Delta \cdot \sum_{i=0}^{n-k(n)} \frac{(1 + \beta)^{i+1}}{(\tilde{u}_{n-i})^{p-2\epsilon}}.
\]

We now simplify a little this inequality taking advantage of the fact that \( k \geq 0 \), making more explicit the geometric sum and using the fact that \( n \leq \frac{\log(u_0) - \log(U_0)}{\log(2)} \) where \( \tilde{u}_n = 2^p U_0 \):

\[
p \int_{\tilde{u}_n}^u E_{p-1}[\psi](u') du' + \tilde{E}_p(u) \leq (1 + \beta)(1 + \eta) \left( \frac{u}{U_0} \right)^{\frac{\log((1 + \beta)(1 + \eta))}{\log(2)}} \cdot \tilde{E}_p(U_0) + \frac{D_0 \cdot \Delta \cdot 2^{p-2\epsilon}(1 + \beta)^2}{(U_0)^{p-2\epsilon} \cdot \left( \frac{u}{U_0} \right)^{\frac{\log(1 + \beta)}{\log(2)}}}.
\]

Now there are two cases: either \( k(n) = 0 \) or alternative 2 of Lemma 6.8 holds on \([\tilde{u}_{k(n)-1}, \tilde{u}_{k(n)}] \). We treat the two cases separately again.

Suppose that \( k(n) = 0 \). Then combining the last estimate with (6.32) and using the fact that \( (u - \tilde{u}_n)^{-1} \leq \frac{1}{n} \) we get

\[
\tilde{E}_{p-1}(u) \leq \frac{2(1 + \beta)(1 + \eta)}{p \cdot U_0} \cdot \tilde{E}_p(U_0) + \frac{D_0 \cdot \Delta \cdot 2^{p-2\epsilon}(1 + \beta)^2}{(U_0)^{p-2\epsilon} \cdot \left( \frac{u}{U_0} \right)^{\frac{\log(1 + \beta)}{\log(2)}}} + C_2^p \cdot \Delta \cdot \frac{u^{-\epsilon}}{w_{p-\epsilon}},
\]

where \( D_0^* = D_0 \cdot (1 + \eta) \cdot \frac{2^{p+1-2\epsilon}(1 + \beta)^2}{p \cdot [2^{p-2\epsilon}(1 + \beta)^2 - 1]} \).

Now we can take \( \eta \) and \( \beta \) small enough so that \( \log((1 + \beta)(1 + \eta)) \leq \epsilon \), giving

\[
\tilde{E}_{p-1}(u) \leq \frac{2(1 + \beta)(1 + \eta)}{p \cdot (U_0)^{\epsilon}} \cdot \tilde{E}_p(U_0) + D_0^* \cdot \Delta \cdot \frac{u^{-1+\epsilon}}{(U_0)^{p-\epsilon}} + C_2^p \cdot \Delta \cdot \frac{u^{-\epsilon}}{w_{p-\epsilon}}.
\]

Then (6.33) holds provided that

\[
\Delta > \frac{12(1 + \beta)(1 + \eta) \cdot \tilde{E}_p(U_0)}{p \cdot (U_0)^{\epsilon}},
\]

\[
U_0 > (6D_0^*)^{(p-1-\epsilon)^{-1}},
\]

\[
U_0 > (6C_2^p)^{(p-1-\epsilon)^{-1}}.
\]

(6.34) is proven under similar conditions.

Finally we treat the last case where alternative 2 of Lemma 6.8 holds on \([\tilde{u}_{k(n)-1}, \tilde{u}_{k(n)}] \), namely for \( u_2 = \tilde{u}_{k(n)} \), \( u_1 = \tilde{u}_{k(n)-1} \) and the \( \beta \) mentioned before. We see that

\[
\tilde{E}_p(\tilde{u}_{k(n)}) \leq D \cdot \Delta \cdot \left( \frac{1 + \beta}{\beta - \log(1 + \beta)} \right).
\]

This implies that

\[
\tilde{E}_{p-1}(u) \leq \frac{2(1 + \beta)(1 + \eta)}{p \cdot (U_0)^{\epsilon}} \cdot \tilde{E}_p(U_0) + D_0^* \cdot \Delta \cdot \frac{u^{-1+\epsilon}}{(U_0)^{p-\epsilon}} + C_2^p \cdot \Delta \cdot \frac{u^{-\epsilon}}{w_{p-\epsilon}},
\]

where we remind that we took \( \eta \) and \( \beta \) small enough so that \( \frac{\log((1 + \beta)(1 + \eta))}{\log(\beta)} < \epsilon \).

Then (6.33) is true on condition that

\[\text{We drop the integral term whenever we apply (6.27), except for the first iteration.}\]
In the sense that the energy bounds are almost sharp when \( e \) or null infinity.

\( u \) decay for large are simultaneously derived in the proofs and do not present any supplementary difficulty.

contain a point-wise boundedness statement for \( v \).

D point-wise decay assumptions for \( \psi \). From \( L \)
here varies slightly.

That \( \tilde{u} \) from \( L \)
that the charge is sufficiently small in \( \phi \). Lemma 7.1.

In this section, we indicate how the decay of the energy implies some point-wise decay and the bounds
\( \gamma \) 7.1 Point-wise bounds near null infinity, “to the right” of \( \gamma \)

Proof. Since \( \lim_{t \to +\infty} \phi(u,v) = 0 \) — this is a consequence of the finiteness of \( E \), c.f. the proof of Proposition

We can now complete the proof of Proposition 6.1: it is enough to combine Proposition 6.6 for \( p = 2 + \sqrt{1 - 4q_0|\epsilon|} - \epsilon \) with Corollary 6.5.

7 From \( L^2 \) bounds to point-wise bounds

In this section we indicate how the energy decay can be translated into point-wise decay, provided initial point-wise decay assumptions for \( D_u \psi_0 \) are available.

It should be noted that this decay is to be understood as \( u \to +\infty \) or \( v \to +\infty \), namely near time-like or null infinity.

All the bounds that we prove in the form of a decay estimate e.g. \( |\phi| \leq v^{-\epsilon} \) for \( v > 0 \) actually also contain a point-wise boundedness statement for \( v \) close to 0 or negative. Because we take more interest in decay for large \( v \), we do not state those explicitly but the reader should keep in mind that these estimates are simultaneously derived in the proofs and do not present any supplementary difficulty.

Note also that later in this section, we assume the energy decay result of section 5 under the form
\[
E_p(u) \leq D_p \cdot u^{p - (3 - \alpha(\epsilon, \rho))},
\]
for all \( 0 \leq p < 3 - \alpha(\epsilon, \rho) \), to make the notations slightly lighter. The best \( \alpha \) that we proved was \( \alpha = 1 - \sqrt{1 - 4q_0|\epsilon|} - \epsilon \), for any \( \epsilon > 0 \) small enough so that \( \alpha > 0 \).

7.1 Point-wise bounds near null infinity, “to the right” of \( \gamma \)

In this section, we indicate how the decay of the energy implies some point-wise decay and the bounds proved are sharp near null infinity \( ^{63} \) according to heuristics c.f. section 1.2.1.

This strategy to prove quantitative decay rates has been initiated in \[34\] although the argument we use here varies slightly.

The main idea is to start by the energy decay and to use a Hardy type inequality to get a point-wise bound of \( r|\phi|^2 \). After, we can integrate the equation to establish \( L^- \) estimates, using the decay of the initial data.

Lemma 7.1. Suppose that the energy boundedness \[5.23\] and the Morawetz estimates \[5.38\] are valid and that the charge is sufficiently small in \( D(u_0(R), +\infty) \) so that the result of Proposition 6.1 applies.

Then for all \( 0 \leq \beta < \frac{1}{2} \), there exists \( C = C(\beta, R, M, \rho) > 0 \) such that for all \( (u, v) \in D(u_0(R), +\infty) \cap \{ r \geq R \} \)
\[
r^{1 + \beta}|\phi|(u, v) \leq C \cdot \left( E_{2\beta}(u) \right)^{\frac{1}{2}}.
\]  \( \quad \) (7.1)

Proof. Since \( \lim_{t \to +\infty} \phi(u, v) = 0 \) — this is a consequence of the finiteness of \( E \), c.f. the proof of Proposition 6.1.

we can write
\[
\phi(u, v) = - \int v^{1 + \beta} e^{\xi_{0}^{v'}} \phi_{0}^{\xi_{0}^{v'}} D_{\psi} \phi(u, v') dv',
\]
after noticing that \( \partial_{v} (e^{\xi_{0}^{v'}} \phi_{0}^{\xi_{0}^{v'}}) = e^{\xi_{0}^{v'}} D_{\psi} \phi \). Now using Cauchy-Schwarz we can write that for every \( 0 < \beta < \frac{1}{2} \)

\( \quad \) \( ^{63} \)In the sense that the energy bounds are almost sharp when \( \epsilon \) tends to 0 and the method does not waste any decay.
The precise way to use it, averaging on \( R \) and using the fact that there exists 64, we have for

\[
\psi
\]

\( \Omega \)

\( \alpha \)

Make the same assumptions as for Lemma 7.1.

\[
\begin{align*}
\int_{v R(u)}^+ & \Omega^{-2+2\beta}|D\phi|^2(u,v)dv' \\
& = \int_{v R(u)}^+ \Omega^{-2+2\beta}|D\psi|^2(u,v)dv' + 2\beta \int_{v R(u)}^+ \Omega^{2\beta}|\phi|^2(u,v)dv' + R^{1+2\beta}|\phi|^2(u,v R(u)).
\end{align*}
\]

Now we use Hardy's inequality 65 coupled with the Morawetz 66 estimate (5.37) : there exists \( D = D(M, \rho, R) > 0 \) such that

\[
R^{1+2\beta}|\phi|^2(u,v R(u)) + \int_{v R(u)}^+ \Omega^{2\beta}|\phi|^2(u,v)dv' \leq \left( \frac{9}{1 - 2\beta} \right) E_{2\beta}|\phi|(u) + D\cdot E(u).
\]

Combining with (7.2), we see that there exists \( C = C(R, M, \rho) \) so that

\[
r^{1+2\beta}|\phi|^2(u,v) \leq \frac{C}{1 - 2\beta} \cdot E_{2\beta}(u).
\]

This concludes the proof of Lemma 7.1.

\[
\begin{align*}
|\phi(u,v)| & \leq \left( \int_{v}^{+\infty} \Omega^2 r^{-2-2\beta}(u,v')dv' \right)^\frac{1}{2} \left( \int_{v}^{+\infty} \Omega^{-2+2\beta}|D\phi|^2(u,v')dv' \right)^\frac{1}{2}.
\end{align*}
\]

This gives

\[
r^{\frac{1}{2}+\beta}|\phi|(u,v) \leq (1 + 2\beta)^{-\frac{1}{2}} \left( \int_{v}^{+\infty} \Omega^{-2+2\beta}|D\phi|^2(u,v')dv' \right)^\frac{1}{2}.
\]

Now we square this inequality and write :

\[
r^{1+2\beta}|\phi|^2(u,v) \leq (1 + 2\beta)^{-1} \left( \int_{v}^{+\infty} \Omega^{-2+2\beta}|D\phi|^2(u,v')dv' \right).
\]

Now, based on the fact that \( D\psi = rD\phi + \Omega^2\phi \), we write the identity

\[
\Omega^{-2}r^{2+2\beta}|D\phi|^2 - \Omega^{-2}r^{2\beta}|D\psi|^2 - \Omega^{-2}r^2|\phi|^2 - r^{1+2\beta}\partial_r(|\phi|^2).
\]

We now integrate on \( \{u\times [v R(u),+\infty] \). After one integration by parts we get

\[
\int_{v R(u)}^+ \Omega^{-2}r^{2+2\beta}|D\phi|^2(u,v)dv' = \int_{v R(u)}^+ \Omega^{-2}r^{2\beta}|D\psi|^2(u,v)dv' + 2\beta \int_{v R(u)}^+ \Omega^{2\beta}|\phi|^2(u,v)dv' + R^{1+2\beta}|\phi|^2(u,v R(u)).
\]

This concludes the proof of Lemma 7.1.

From now on , we are going to assume the energy decay result of section 6 for some \( p = 3 - \alpha(e_4) \), to make the notations slightly lighter. Of course, what we proved was valid for \( \alpha = 1 - \sqrt{1-4\rho_4}|e_4| - \epsilon, \) for any \( \epsilon > 0 \) small enough so that \( \alpha > 0 \).

We can now establish the decay of \( D\psi \) in \( \{ r^* \geq \frac{\gamma}{\rho} + R^* \} \) — which is the “right” of \( \gamma \), c.f. Figure \( 3 \) — together with estimates for the radiation field \( \psi \), in particular on \( I^* \) :

**Corollary 7.2.** Make the same assumptions as for Lemma 7.1.

Suppose also that there exists \( D_0 > 0 \) such that for all \( v_0(R) \leq v \) :

\[
|D_0\psi|(u_0(R),v) \leq D_0 \cdot v^{-2+\frac{\beta}{2}}.
\]

Then there exists \( C = C(M, \rho, R, e, D_0) > 0 \) so that for all \( (u,v) \in \mathcal{D}(u_0(R),+\infty) \cap \{ r^* \geq \frac{\gamma}{\rho} + R^* \} \) we have for \( u > 0 \) :

\[
\begin{align*}
|D\phi| & \leq C \cdot v^{-2+\frac{\beta}{2}} \cdot |\log(u)|, \\
|\psi| & \leq C \cdot u^{-1+\frac{\beta}{2}} \cdot |\log(u)|, \\
|Q-e| & \leq C \cdot u^{-2+\alpha} \cdot |\log(u)|,
\end{align*}
\]

where \( \alpha \in [0, 1) \) is defined by \( \alpha = 1 - \sqrt{1-4\rho_4}|e_4| + \epsilon, \) as introduced in the beginning of the section.

**Proof.** We take \( \beta = \frac{1}{2} - \epsilon \) for \( 0 < \epsilon < \frac{1}{2} \).

We are working to the “right” of \( \gamma \) where \( r \sim v \gtrsim u \). Therefore using 2.6 and the result of Lemma 7.1 there exists \( C = C(R, e, M, \rho) > 0 \) such that

\[
|D\phi(D\psi)| \leq C \cdot r^{-2+\alpha} u^{-1+\frac{\beta}{2}}.
\]

Then we choose \( \epsilon = \frac{\beta}{2} \) and we integrate on \( [u_0(R), u] \). For \( R \) large enough we get

\[
|D\psi|(u,v) \leq |D\psi|(u_0(R),v) + 2C \cdot v^{-2+\frac{\beta}{2}} |\log(u)|.
\]

Making use of the decay of the initial data gives 7.4.

Now we notice that from Lemma 7.1 we have that there exists \( C' = C'(R, M, \rho, e) > 0 \) such that

---

64 Using the fact that \( r^{1+2\beta}\phi \) tends to 0 when \( v \) tends to \(+\infty\), guaranteed by the finiteness of \( \mathcal{E}_{1+} \), c.f. the proofs of Section 4.

65 The precise way to use it, averaging on \( R \), has been carefully explained in section 5.10. We do not repeat the argument.
We now propagate the bounds obtained “to the right” of \( \gamma \) that to the right of \( \gamma \).

**7.1** for \( \beta \)

Make the same assumptions as for Corollary 7.2.

**Proposition 7.3.**

An estimate from Lemma (7.1), the argument is very soft.

**Proof.**

With the work that we have already done, this proof is easy.

Therefore, integrating (7.4) in \( v \) to the right of \( \gamma \), we prove (7.5).

Now we turn to the charge: integrating (2.9) towards null infinity and using Cauchy-Schwarz we get

\[
|Q(u,v)-Q_{I^+}(u)| \leq q_0 \left( \int_v^{v_0} r^{1+\epsilon} |D_{\psi} \psi|^2 (u,v) dv \right)^{\frac{1}{2}} \left( \int_v^{v_0} |\psi|^2 (u,v) dv \right)^{\frac{1}{2}} \leq \frac{D}{\epsilon} (\delta_1, \delta_2) \leq D^2 u^{-2+\alpha},
\]

where \( D = D(R, M, \rho, e) > 0 \) and we used (7.3) and the energy decay of Section 6, for \( \epsilon = \frac{1}{2} \).

Now using (2.8) and Cauchy-Schwarz again we get that for all \( u \geq u_0(R) \):

\[
|Q_{I^+}(u) - e_0| \leq q_0 \left( \int_u^{v_0} r^{2} |D_{\psi} \psi|^2 (u,v) dv \right)^{\frac{1}{2}} \left( \int_u^{v_0} |\psi|^2 (u,v) dv \right)^{\frac{1}{2}} \leq \frac{D}{\epsilon} (E(u)) \leq \frac{D}{\epsilon} \left( \int_u^{v_0} (C \cdot \log(u)) du \right)^{\frac{1}{2}},
\]

where we used (7.5) in the last estimate. Combining the two estimates for \( Q \), it proves (7.6).

This concludes the proof of Corollary 7.2.

Notice that these two results prove estimates (3.8) and (3.9) of Theorem 3.1 together with (3.6), (3.10) to the right of \( \gamma \).

### 7.2 Point-wise bounds in the region between \( \gamma \) and \( \gamma_R \)

We now propagate the bounds obtained “to the right” of \( \gamma \) towards the right of \( \gamma_R \). Since we already have an estimate from Lemma 7.1, the argument is very soft.

**Proposition 7.3.**

Make the same assumptions as for Corollary 7.2.

Then there exists \( C = C(M, \rho, R, e) > 0 \) so that for all \( (u,v) \in D(u_0(R), +\infty) \cap \{ R' \leq r^* \leq \frac{R}{2} + R' \} \) we have for \( u > 0 \):

\[
|Q - e| \leq C \cdot u^{-2+\alpha} \cdot |\log(u)|, \tag{7.8}
\]

where \( \alpha \in [0, 1] \) is defined by \( \alpha = 1 - \sqrt{1 - 4q_0|e|} + e \), as introduced in the beginning of the section.

**Proof.**

With the work that we have already done, this proof is easy.

First notice that in this region \( u \sim v \sim 2u \) when \( v \) tends to \( +\infty \).

Then, making use of the boundedness of \( Q \) it is enough to integrate (2.6), using the estimate of Lemma 7.1 for \( \beta = 0 \) and the bound on \( |D_{\psi} \psi|, \) of Corollary 7.2.

We get for all \( (u,v) \in D(u_0(R), +\infty) \cap \{ R' \leq r^* \leq \frac{R}{2} + R' \} \)

\[
|D_{\psi} \psi| (u,v) \leq C' \cdot r^{-\frac{3}{2}} (u,v) \cdot u^{-2+\alpha},
\]

for some \( C' = C'(M, \rho, R, e) > 0 \) since \( E(u) \) decays like \( u^{-3+\alpha} \). Combining with the estimate of Lemma 7.1 for \( \beta = 0 \) gives directly (7.7), noting that \( D_{\psi} \psi = \Omega^2 \phi + r D_{\psi} \phi \).
Lemma 7.5. Under the same assumptions as in Proposition 7.4, then for any \( K \)

\[ C = \frac{\partial u}{\partial r} \]  

Using the red-shift estimate (4.5) from Proposition 4.1 with the help of Cauchy-Schwarz, one finds

Proof. The first step in the proof is to prove a red-shift estimate with no time decay in a region \( \{ r \leq R \} \). We are going to prove the following proposition, that also includes a red-shift estimate.

Proposition 7.4. Suppose that \( Q \) is bounded in \( \{ r_+ \leq r \leq R \} \) and define \( Q^+ = \|Q\|_\infty \).

By section 4, \( Q^+ \) can be taken arbitrarily close to \( |e| \) or \( |v_0| \). We also have assumed naturally that \( E < \infty \).

Suppose that \( D_0 \phi \) and \( \phi_0 \) in \( L^\infty (\Sigma_0) \). We denote \( N_{\infty} := \|D_0 \phi_0\|_{L^\infty (\Sigma_0)} + ||\phi_0||_{L^\infty (\Sigma_0)} \).

Then there exists \( C = C(M, \rho, R, e, E, N_\infty) > 0 \) such that

for all \( (u, v) \in \{ r_+ \leq r \leq R \} \), \( v > 0 \):

\[ |\phi(u, v) + |D_\phi(u, v) + \frac{|D_\phi(u, v)|}{\Omega^2} \leq C \cdot v^{-\frac{3+\alpha}{2}}. \]  

(7.9)

where \( \alpha \in [0, 1] \) is defined by \( \alpha = 1 - \sqrt{1 - 4|\phi_0|^2} \) or \( \epsilon \), as introduced in the beginning of the section.

Proof. The red-shift estimate (4.1) from Proposition 7.4 with the help of Cauchy-Schwarz, one finds

\[ C = C(M, \rho) > 0 \]  

such that in \( \{ v \leq v_0(R) \} \)

\[ |\phi| \leq ||\phi_0||_\infty + C \cdot \sqrt{E} \leq C \cdot (N_\infty + \sqrt{E}). \]  

(7.11)

Now we write (7.9) as

\[ D_\phi \left( \frac{\partial \phi}{\Omega^2} \right) = -\frac{2K}{\Omega^2} D_\phi \psi + \frac{\phi}{r} (i \psi_0 Q - r \cdot 2K), \]

which can also be expressed as

\[ D_\phi \left( \exp(\int_{v_0(u)}^v 2K(u, v') dv') \cdot \frac{D_\phi}{\Omega^2} \right) = \exp(\int_{v_0(u)}^v 2K(u, v') dv') \cdot \frac{\phi}{r} (i \psi_0 Q - r \cdot 2K). \]

We can then integrate such an estimate on \( \{ u \} \times [v_0(u), v] \) and obtain

\[ \left| \frac{D_\phi}{\Omega^2} \right| \leq \exp \left( - \int_{v_0(u)}^v 2K(u, v') dv' \right) \cdot \Omega^{-1} (u, v_0(u)) \cdot N_{\infty} + C' \cdot (N_\infty + \sqrt{E}) \cdot \int_{v_0(u)}^v \exp \left( - \int_{v_0(u)}^v 2K(u, v') dv' \right) dv', \]

for some \( C' = C'(M, \rho, R) > 0 \) where we used the fact that \( \frac{1}{2} (i \psi_0 Q - r \cdot 2K) \) is bounded and (7.11).

Now remember from (7.2) that \( \Omega(u, v_0(u)) \sim C_+ \cdot e^{-2K_+(v-u)} \) as \( u \to +\infty \).

Also it can be proven easily that there exists \( D_+ = D_+(M, \rho) > 0 \) such that in this region, since \( (r - r_+ - e^{-2K}(v-u)) \) is bounded,

\[ |2K(u, v) - 2K_+| \leq D_+ e^{2K_+(v-u)}. \]

If we integrate this as a lower bound, we get that there exists \( D'_+ = D'_+(M, \rho) > 0 \) such that

\[ \exp \left( - \int_{v_0(u)}^v 2K(u, v') dv' \right) \leq D'_+ e^{-2K_+(v-u)}. \]

Then we get that for some \( D^*_+ = D^*_+(M, \rho) > 0 : \)

\[ \left| \frac{D_\phi}{\Omega^2} \right| \leq D^*_+ e^{-2K_+} \cdot N_{\infty} + C' \cdot (N_\infty + \sqrt{E}) \cdot \int_{v_0(u)}^v \exp \left( - \int_{v_0(u)}^v 2K(u, v') dv' \right) dv', \]

Now, because we stand in a region where \( r \leq R \), one can find a constant \( K_0 = K_0(M, \rho, R) > 0 \) such that \( K(r) > K_0 \). Therefore we can write

\[ \int_{v_0(u)}^v \exp \left( - \int_{v_0(u)}^v 2K(u, v') dv' \right) dv' \leq \int_{v_0(u)}^v \exp \left( -2K_0 \cdot (v - v') dv' \right) \leq \frac{1}{2K_0}. \]  

(7.10)
This concludes the proof of the lemma, after controlling $e^{-2K_t}v$ by $e^{-2K_t}v_0$.

Then we use the time decay of the non-degenerate energy under the form:

for all $(u, v) \in \{ v \geq v_0(R) \} \cap \{ r \leq R \}$

$$|\phi(u, v) - \phi(u_R(v), v)| \leq \left( \int_{u_R(v)}^{u} \Omega^2 dv' \right)^{\frac{1}{2}} \cdot \left( \int_{u_R(v)}^{u} \frac{|D_\nu \phi|^2}{\Omega^2} dv' \right)^{\frac{1}{2}} \leq C_v \cdot v^{-\frac{3+\alpha}{2}},$$

where $C = C(M, \rho, R, e) > 0$ and we took advantage of the fact that $|u_R(v)| \sim v$.

Therefore, using the bounds of the former section we see that there exists $C' = C'(M, \rho, R, e) > 0$ such that for all $(u, v) \in \{ v \geq v_0(R) \} \cap \{ r \leq R \}$

$$|\phi(u, v)| \leq C' \cdot v^{-\frac{3+\alpha}{2}}.$$  

(7.12)

Now integrating (2.6) we can, allowing $C'$ to be larger, derive the same estimate for $D_\nu \psi$:

$$|D_\nu \psi(u, v)| \leq C' \cdot v^{-\frac{3+\alpha}{2}}.$$  

(7.13)

Now using a reasoning similar to one of the lemma, we get that for some $C'' = C''(M, \rho, R, e) > 0$:

$$\frac{|D_\nu \psi(u, v)|}{\Omega^2} \leq \exp\left( - \int_{v_0(R)}^{u} 2K(u, v')dv' \right) \cdot D_0 + C'' \int_{v_0(R)}^{u} (v')^{-\frac{3+\alpha}{2}} \exp\left( - \int_{v'}^{u} 2K(u, v'')dv'' \right) dv'.

Now observe that for all $\beta > 0$, $s > 0$ — using an integration by parts — when $v \to +\infty$:

$$e^{-\beta v} \int_{v_0(R)}^{u} e^{\beta v'} (v')^{-\frac{3+\alpha}{2}} dv' \leq \frac{\beta}{\beta} + O(v^{-s-1}).$$

Now, because there exists $K_0 = K_0(M, \rho, R) > 0$ such that $K > K_0$, this concludes the proof of the proposition, after noticing that $\exp\left( - \int_{v_0(R)}^{u} 2K(u, v')dv' \right) = o(v^{-\frac{3+\alpha}{2}})$.

Remark 28. Notice that (7.9) was stated for $v > 0$ only. This is related to a degenerescence of the $\Omega^2$ weight towards the bifurcation sphere c.f. Remark 15. Actually, from Lemma 7.9 it is not hard to infer that $e^{2K+v}|D_\nu \phi|$ is bounded on the whole space-time, in conformity with Hypothesis 4. For a discussion of the $L^2$ analogue, c.f. also Remark 13.

7.4 Remaining estimates for $D_\nu \psi$ near $I^+$ and $|Q - \epsilon|$ near $H^+$

For this last sub-section of section 7 we make a little summary of what we did.

First, we derived an estimate for $r^+\phi$ on the whole region $\{ r \geq R \}$ using energy decay.

Then, with the help of the point-wise decay of $D_\nu \psi$ on $u = u_0(R)$ — itself coming from the point-wise decay of $\psi_0$ and $D_\nu \psi_0$ on $\Sigma_0$, c.f. Proposition 4.3 — we derived $D_\nu \psi$ on $\{ r \geq R \}$.

This gives directly almost optimal point-wise estimate for $\psi$ and $|Q - \epsilon|$ on null infinity and nearby.

Then the $v$ decay of $\phi$ can be propagated from $\gamma_R$ to the event horizon using again the decay of the energy. As a consequence $v$ decay can also be retrieved for $D_\nu \phi$ on $\{ r \leq R \}$. After we use a red-shift estimate to prove the same $v$ decay for the regular derivative $\Omega^{-2}D_\nu \phi$.

Now compared to the statement of our theorems, we are missing four estimates.

The first is an estimate for $D_\nu \psi$ in the large $r$ region $\{ r \geq R \}$. It is the object of Proposition 7.6.

The second is the almost optimal $L^2$ flux bounds on $\phi$ and $D_\nu \phi$ on any constant $r$ curve. We prove them in Proposition 7.7.

The third is the existence of the future asymptotic charge $e$. We proved that $Q$ admits a future limit when $t \to +\infty$ on constant $r$ curves, the event horizon and null infinity but we never proved that they were all the same.

The fourth is related to the third : it is the $v$ decay of $|Q - \epsilon|$ in the bounded $r$ region. These two are the object of Proposition 7.8.

Proposition 7.6. We make the same assumptions as for the former propositions. Then there exists a constant $C = C(M, \rho, R, e, E, N_\infty) > 0$ such that in $\{ r \geq R \} \cap \{ u \geq u_0(R) \}$, for $u > 0$:

$$|D_\nu \psi| \leq C \cdot u^{-\frac{3+\alpha}{2}},$$

where $\alpha \in [0, 1)$ is defined by $\alpha = 1 - \sqrt{1 - 4q0|e| + \epsilon}$, as introduced in the beginning of the section.
Proof. Using the estimates for $\phi$ in $\{r \geq R\} \cap \{u \geq u_0(R)\}$ and (2.7) we get that for some $D = D(M, \rho, R) > 0$

$$|D_\nu D_\lambda \psi| \leq D \cdot r^{-\alpha} \cdot u^{-\frac{2\alpha}{2\alpha - 1}}.$$ 

It is enough to integrate this bound — given that $|\partial_\nu r| \geq \Omega^2(r)$ and make use of the estimate in the past of Proposition 7.4.

\[ \square \]

**Proposition 7.7.** For every $r_+ \leq R_0 \leq R$, there exists a constant $C_0 = C_0(R_0, M, \rho, R, e) > 0$ such that

$$\int_0^{+\infty} \left[ |\phi|^2(u_{R_0}(v'), v') + |D_\nu \phi|^2(u_{R_0}(v'), v') \right] dv' \leq C_0 \cdot v^{-3+\alpha}. \quad (7.14)$$

Proof. Take any $r_+ \leq R_0 < R_1 \leq R$.

First, using Cauchy-Schwarz, we find that there exists $D = D(R_0, R_1, M, \rho) > 0$ such that

$$|\phi|^2(u_{R_1}(v), v) \leq 2|\phi|^2(u_{R_0}(v), v) + D \cdot \int_{u_{R_0}(v)}^{u_{R_1}(v)} \frac{|D_\nu \phi|^2(u, v)}{\Omega^2} du,$$

where we squared the Cauchy-Schwarz estimate and bounded $2 \int_{u_{R_0}(v)}^{u_{R_1}(v)} \Omega^2 du \leq D$. This implies that

$$\int_0^{+\infty} |\phi|^2(u_{R_1}(v'), v') dv' \leq 2 \int_0^{+\infty} |\phi|^2(u_{R_0}(v'), v') dv' + D \cdot \int_{u_{R_0}(v')}^{u_{R_1}(v')} \frac{|D_\nu \phi|^2(u, v)}{\Omega^2} du dv'. \quad (7.15)$$

Now since $\{R_0 \leq r \leq R_1\}$ is included inside $\{r \leq R\}$, one can use the Morawetz estimate (5.38) : there exists $C = C(M, \rho, e, R) > 0$ such that

$$\int_0^{+\infty} |\phi|^2(u_{R_1}(v'), v') dv' \leq 2 \int_0^{+\infty} |\phi|^2(u_{R_0}(v'), v') dv' + C \cdot E(u_{R_0}(v)). \quad (7.16)$$

Similarly, using Cauchy-Schwarz and (2.6), we find that there exists $D' = D'(R_0, R_1, M, \rho) > 0$ such that

$$|D_\nu \phi|^2(u_{R_1}(v), v) \leq 2|D_\nu \phi|^2(u_{R_0}(v), v) + D' \cdot \int_{u_{R_0}(v)}^{u_{R_1}(v)} \Omega^2 |\phi|^2(u, v) du.$$

Therefore one can use again the non-degenerate Morawetz estimate (5.35) : there exists $C' = C'(M, \rho, e, R) > 0$ such that

$$\int_0^{+\infty} |D_\nu \phi|^2(u_{R_1}(v'), v') dv' \leq 2 \int_0^{+\infty} |D_\nu \phi|^2(u_{R_0}(v'), v') dv' + C' \cdot E(u_{R_0}(v)). \quad (7.17)$$

Now we use the mean-value theorem with the Morawetz estimate (5.38), like we did several times in section 5; there exists $r_+ < \tilde{R} < R$ such that

$$\int_0^{+\infty} \left[ |\phi|^2(u_{R}(v'), v') + |D_\nu \phi|^2(u_{R}(v'), v') \right] dv' \leq \int_{u_{R_0}(v')}^{u_{R_1}(v')} \left( r^2 |D_\nu \phi|^2 + |\phi|^2 \right) \Omega^2 du dv \leq (C' \cdot E(u_{R_0}(v))),$$

where $C'' = C''(M, \rho, e, R) > 0$.

Therefore, taking $R_0 = \tilde{R}$, it means that for all $r_+ \leq R_1 \leq R$,

$$\int_0^{+\infty} \left[ |\phi|^2(u_{R_1}(v'), v') + |D_\nu \phi|^2(u_{R_1}(v'), v') \right] dv' \leq C_1 \cdot E(u_{R_0}(v)) \leq C_1^2 \cdot v^{-3+\alpha}, \quad (7.18)$$

for some $C_1 = C_1(M, \rho, e, R) > 0$.

Notice that since $D_\nu \psi = rD_\nu \phi + \Omega^2 \phi$, this estimate is equivalent to the claimed (7.14). This concludes the proof of the proposition.

\[ \square \]

**Proposition 7.8.** The future asymptotic charge $e$ exists — in the sense explained above — and the following estimate is true : for some $C' = C'(M, \rho, R, e) > 0$

$$|Q - e| = C' \cdot \left( u^{-2+\alpha} |\log(u)|1_{(r \geq R)} + v^{-3+\alpha}1_{(r \leq R)} \right). \quad (7.19)$$

Proof. For now on, $e$ is defined to be the asymptotic limit of $Q_{x^+}(t)$ when $t \to +\infty$.

Temporarily we also define $e_{x^+}$ as the asymptotic limit of $Q_{(x^+)}(t)$ when $t \to +\infty$, and $e(R_0)$ the asymptotic limit of $Q_{(x^+)}(t)$ when $t \to -\infty$. One of the goals of the proposition is to prove $e = e_{(x^+)} = e(R_0)$.

We apply the estimate (7.9) of former section to get some $\tilde{C} = \tilde{C}(M, \rho, R, e, \mathcal{E}, N_\infty) > 0$ such that

$$|\partial_\nu Q| \leq \tilde{C} \cdot \Omega^2 \cdot v^{-3+\alpha}.$$

Integrating this gives that for some $\tilde{C} = \tilde{C}(M, \rho, R, e, \mathcal{E}, N_\infty) > 0$ and for all $r_+ \leq R_0 \leq R_1 \leq R$
|Q(u_{R_0}(v), v) - Q(u_{R_0}(v), v)| \leq \tilde{C} \cdot v^{-3+\alpha}. \tag{7.20}

In particular, taking \( v \) to +\( \infty \), it proves that for all \( r_+ \leq R_0 \leq R \leq R \), \( \epsilon(R_0) = \epsilon(R) = \epsilon_{R+} \). But \( \epsilon(R_0) = \epsilon(R_0) = \epsilon_{R+} \) as requested.

Now apply \( \ref{7.14} \) on the event horizon, for \( R_0 = r_+ \) and integrate \( \ref{2.9} \) : we get that for some \( C_0^\prime = C_0^\prime(R_0, M, \rho, R, \epsilon) > 0 \)

\[ |Q_{\tilde{\epsilon}}(v) - \tilde{\epsilon}| \leq C_0^\prime \cdot v^{-3+\alpha}. \]

Combining with estimate with \( \ref{7.20} \) taken in \( R_0 = r_+ \) we get that for all \( v \geq v_0(R) \) and \( u \geq u_R(v) \)

\[ |Q(u, v) - \epsilon| \leq (C_0 + \tilde{C}) \cdot v^{-3+\alpha}, \tag{7.21} \]

which is the claimed estimate on \( \{ r \leq R \} \). The second part in \( \{ r \geq R \} \) was already proven in section \( \ref{7.3} \).

This concludes the proof of the proposition. \( \square \)

### A Variant of Theorem 3.3 for More Decaying Initial Data

While the point-wise results stated in Theorem \( \ref{3.1} \) and Theorem \( \ref{3.3} \) are established assuming the weakest decay of the initial data that made the proofs work, they do not give the same \( r \) decay rate for \( D_\alpha \psi \) towards null infinity as we would expect for say compactly supported data, which is \( |D_\alpha \psi| \lesssim r^{-2} \).

Notice that by what was done in the proof of Theorem 3.3, we already know that \( |D_\alpha \psi| \lesssim v^{-1-\epsilon} \) for some \( \epsilon > 0 \). Therefore \( D_\alpha e^{iq_0 \int_0^r A_\alpha(u, v') dv'} \psi(u, v) \) is integrable in \( v \) so \( e^{iq_0 \int_0^r A_\alpha(u, v') dv'} \psi(u, v) \) admits a finite limit \( \tilde{\varphi}_0(u) \) when \( v \to +\infty \).

In this section, we want to prove that for data decaying slightly more than in the assumption of Theorem 3.3, \( e^{iq_0 \int_0^r A_\alpha(u, v') dv'} \psi(u, v) \) admits a bounded limit towards null infinity, like for the wave equation without any symmetry assumption.

It is equally interesting to notice that — similarly to the wave equation c.f. \cite{2} — \( e^{iq_0 \int_0^r A_\alpha(u, v') dv'} \psi(u, v) \) also admits a bounded limit towards null infinity, providing the data decay even more. In fact, one can apply \( r^2 D_\alpha \) to the radiation field infinitely many times and still find a finite limit towards null infinity\(^{66} \) providing the data decay rapidly.

This feature is reminiscent of what happens for the unchanged wave equation, although the analogous of the Newman-Penrose quantity is no longer conserved in the charged case.

**Theorem A.1.** We make the same assumptions as for Theorem \( \ref{3.3} \) and we also make the following additional assumption : There exists \( \epsilon_0 > 0 \) and \( C_0 > 0 \), such that

\[ |D_\alpha(r^2 D_\alpha \psi_0)| + |D_\alpha \psi_0| r + |\psi_0| r \leq C_0 \cdot r^{-1-\frac{\gamma}{2}} \tag{A.1} \]

then — in addition to all the conclusions of Theorem \( \ref{3.3} \) being true — we can also conclude that for all \( u \in \mathbb{R} \), \( e^{iq_0 \int_0^r A_\alpha(u, v') dv'} v^\alpha D_\alpha \psi(u, v) \) admits a finite limit \( \tilde{\varphi}_1(u) \) as \( v \to +\infty \). Moreover in the whole region \( D(u_0(R), +\infty) \cap \{ r^+ \geq \frac{r}{2} + R^2 \} \), for all \( \epsilon > 0 \), there exists \( C > 0 \) such that for all \( u > 0 \) :

\[ r^2 |D_\alpha \psi| \lesssim C \cdot u^{1-\frac{\gamma}{2}} \tag{A.2} \]

In particular \( |\tilde{\varphi}_1(u)| \leq C \cdot u^{1-\frac{\gamma}{2}} \).

If now we assume that we have rapidly decaying data, i.e. that for all \( \omega > 0 \), there exists \( C_0 = C_0(\omega) > 0 \) such that on \( \Sigma_0 \) :

\[ r |D_\alpha \psi_0| + |\psi_0| \lesssim C_0 \cdot r^{-\omega}, \tag{A.3} \]

and for all \( n \in \mathbb{N} \), on \( \Sigma_0 \) :

\[ |\varphi_n(r)| \leq C_0, \tag{A.4} \]

where we defined \( \varphi_0 := \psi \) and \( \varphi_{n+1} = r^2 D_\alpha \varphi_n \).

Then one can prove that for all \( n \in \mathbb{N} \), for all \( u \in \mathbb{R} \), \( e^{iq_0 \int_0^r A_\alpha(u, v') dv'} \psi_n(u, v) \) admits a finite limit \( \tilde{\varphi}_n(u) \) as \( v \to +\infty \).

**Proof.** First we only assume (A.1). We start by the proof of (A.2), the easiest claim. The hypothesis of Proposition 1.3 are satisfied so (A.13) and (A.14) are true : in particular there exists \( D_0 > 0 \) such that for all \( v \geq v_0(R) \),

\[ |D_\alpha \psi(u_0(R), v)| \leq D_0 \cdot v^{-2}. \tag{A.5} \]

We write (2.6) as :

\[ D_\alpha(r^2 D_\alpha \psi) = -2r \Omega^2 D_\alpha \psi + \Omega^2 \psi \cdot (iq_0 Q - \frac{2M}{r} + \frac{2\rho^2}{r^2}). \]

\(^{66}\)After multiplying the last term by \( e^{iq_0 \int_0^r A_\alpha(u, v') dv'} \).
We are going to apply Corollary 7.2: we recall that for all $\epsilon > 0$ small enough we define $\alpha = 1 - \sqrt{1 - 4q\|\epsilon\|}$ and given that in this region $v \geq |u|$, there exists $D > 0$ such that for $u > 0$:

$$|D_u(r^2D_v\psi)| \leq D \cdot u^{-1 + \frac{\alpha}{2}}.$$ Integrating this in $u$ and using (3.3) gives (2.2) in this region.

Now, to prove that $e^{iq_0r^2A_v(u,v')dv}r^2D_v\psi(u,v)$ admits a limit when $v \to +\infty$, we prove that $\partial_u(e^{iq_0r^2A_v(u,v')dv}r^2D_v\psi(u,v)) = e^{iq_0r^2A_v(u,v')dv}r^2D_v(r^2D_v\psi) = (2K \cdot r^2) + i\Theta(\varphi_i,\varphi_i) + (2M - \frac{4\rho^2}{r^2}) \cdot \Omega^2.$

First, in the region $\{u \leq u_0(R)\}$, we come back to the proof of Proposition 4.3. If we examine more closely the estimate (13.30), we see that we actually proved that for some $D' > 0$ and for all $u \leq u_0(R)$:

$$|\varphi_i| = |r^2D_v\psi| \leq D' \cdot |u|^{1-\omega},$$

where $\omega = 1 + \frac{2|\alpha|}{4} + \epsilon$. From Proposition 4.3 we also have the estimate

$$|\psi| \leq D' \cdot |u|^{-\omega}.$$ Using (4.6) in the region $\{u \leq u_0(R)\}$ and the boundedness of relevant quantities, we see that there exists $D'' > 0$ such that

$$|D_u\varphi_i| \leq D'' \cdot r^{-\omega}.$$ Integrating in $u$ towards $\Sigma_0$, since $1 < \omega < 2$, we get that there exists $C'' > 0$ such that

$$|\varphi_i| = |D_u\varphi_i| \leq D'' \cdot r^{-\omega}.$$

Now we can use the bound on $\varphi_i$ (2.2) with (3.3) and integrate (4.6) in $u$ towards $\{u = u_0(R)\}$ to get that there exists $C'' > 0$ such that for all $(u,v) \in D(u_0(R), +\infty) \cap \{r \geq \frac{a}{2} + R\}$

$$|D_u\varphi_i| \leq D'' \cdot r^{-\omega},$$

where we used the fact that $0 < \alpha < 1$ and (4.8). Now because $\omega > 1$, it is clear that $\partial_u(e^{iq_0r^2A_v(u,v')dv}r^2D_v\psi) = e^{iq_0r^2A_v(u,v')dv}r^2D_v(r^2D_v\psi)$ is integrable. Hence $e^{iq_0r^2A_v(u,v')dv}r^2D_v\psi(u,v)$ admits a limit when $v \to +\infty$, as claimed.

We now assume (3.3), (4.4) and we want to prove that for all $n \geq 2, e^{iq_0r^2A_v(u,v')dv}r^2\varphi_i(u,v)$ admits a limit when $v \to +\infty$.

For this, we need to commute the operator with $r^2D_v$ $n$ times. While the precise formula is very complicated one can write the following:

$$D_uD_v\varphi_n = \frac{2n\Omega^2}{r}D_v\varphi_n + \sum_{k=0}^{n-1} \left( B_k + \sum_{q=k}^{n} iC_{qy}X^{n-q} \cdot \frac{\varphi_k}{r^2} \right),$$

where $B_k(r)$, $C_{qy}$ are real valued entire functions of $r^{-1}$ (therefore bounded on the space-time) whose coefficients depend only on $M$, $p$ and $q_0$.

We check this by a quick induction; by what we derived earlier, the formula is obviously true for $n = 0$ and $n = 1$, since $X(Q) = q_0\Theta(\varphi_i)$. Now if (10.10) is true, we can rewrite it using $\varphi_{n+1} = r^2D_v\varphi_n$ as

$$D_u(\varphi_{n+1}) = \frac{2(n+1)\Omega^2}{r}\varphi_{n+1} + \sum_{k=0}^{n} \left( B_k + \sum_{q=k}^{n} C_{qy}X^{n-q} \cdot \frac{\varphi_k}{r^2} \right).$$

Then we apply $D_v$ to get

$$D_vD_u(\varphi_{n+1}) = -\frac{2(n+1)\Omega^2}{r}D_v\varphi_{n+1} - 2(n+1)\partial_v(\frac{\Omega^2}{r})\varphi_{n+1} + \sum_{k=0}^{n} \left( B_k + \sum_{q=k}^{n} C_{qy}X^{n-q} \cdot \frac{\varphi_k}{r^2} \right) + \sum_{k=0}^{n} \sum_{q=k}^{n} C_{qy}\partial_v(X^{n-q} \cdot \varphi_k).$$
Now, because \( r^2 \partial_r \left( \frac{r^2}{2} \right) \) is an entire function in \( r^{-1}, \partial_r (X^{n-q}Q) = r^{-2}X^{n+1-q}Q \) and 
\[ D_\nu D_\nu (\varphi_{n+1}) = D_\nu D_\nu (\varphi_{n+1}) - \frac{2n^2}{r^2} \varphi_{n+1}, \] it is clear that \( (A.10) \) is true at the level \( (n+1) \), therefore for all \( n \).

Now that the equation is written, let us assume first we have in the region \( \{ u \leq u_0(R) \} \) that for all \( n \), there exists \( D_n \) to show that for all \( v \geq v_0(R) \)

\[ |D_\nu \psi_n (u_0(R), v)| \leq D_n \cdot r^{-2}. \quad (A.11) \]

Then we can prove by induction that \( |\varphi_n| (u, v) \leq u^{n-1+\frac{2}{2}} \) for \( u > 0 \). This is indeed the case for \( n = 0 \) and \( n = 1 \). Now if we assume the result for all integer \( 0 \leq k \leq n \) we want to prove it at the level \( (n+1) \). For this we first write \( (A.10) \) as

\[ D_\nu (r^{-2n} D_\nu \varphi_n) = r^{-2n-2} \sum_{k=0}^{n} \left( \sum_{q=k}^{n} C_{q,k} X^{n-q} \varphi_{n-k} \right). \quad (A.12) \]

Now notice that \( X^n Q \) can be written as a linear combination of the form \( X^n Q = \sum_{i=0}^{q} a_i \mathcal{J}(\varphi_{q-i} \tilde{\varphi}_i) \), with \( a_i \in \mathbb{R} \). Using the induction hypothesis we get that there exits \( C_n > 0 \) such that for \( u > 0 \)

\[ |D_\nu (r^{-2n} D_\nu \varphi_n) (u, v)| \leq C_n \cdot r^{-2n-2} u^{n-1+\frac{2}{2}}. \]

Then integrating in \( u \) towards \( \{ u = u_0(R) \} \) and using \( (A.11) \), we get that for some \( C'_n > 0 \)

\[ |\varphi_{n+1}| = r^2 D_\nu |\varphi_n| \leq C'_n \cdot u^{n+\frac{2}{2}}, \]

which finishes the induction. Now, this proves that for all \( n \), \( |D_\nu \varphi_n| \leq r^{-2} u^{n+\frac{2}{2}} \) hence \( \partial_r (e^{u\phi} \mathcal{J}_\nu \mathcal{J}_\nu \psi(u, v)) \) is integrable in \( v \) so \( e^{u\phi} \mathcal{J}_\nu \mathcal{J}_\nu \psi(u, v) \) admits a finite limit \( \psi_0 (u) \) when \( v \to +\infty \), as claimed.

Now, the last part of the proof is to establish \( (A.11) \). As noticed in \( (A.7) \), we can prove, from Proposition 4.3 that for all \( \omega > 1 \), there exists \( \tilde{D} > 0 \) such that in \( \{ u \leq u_0(R) \} \):

\[ |\psi| = |\varphi| \leq \tilde{D} \cdot |u|^{-\omega}, \]

\[ |\varphi| \leq \tilde{D} \cdot |u|^{-\omega}. \]

Then this time we want to prove by induction that \( |\varphi_n| \leq |u|^{n-\omega} \) for all \( 0 < \omega < n \). It is enough to use \( (A.12) \) with an argument similar to the one developed in the region \( \{ u \geq u_0(R) \} \). We also need the hypothesis \( (A.4) \) to deal with the term on \( \Sigma_0 \) when integrating in \( u \).

This finally gives \( (A.11) \) and concludes the proof.

\[ \square \]

## B Proofs of section 4

In this section, we carry out the proofs of the results claimed in section 4. We are going to state once more the propositions, for the convenience of the reader.

We start by the proof of Proposition 4.11.

**Proposition B.1.** Suppose that there exists \( p > 1 \) such that \( \mathcal{E}_p < \infty \) and that \( Q_0 \in L^\infty (\Sigma_0) \).

Assume also that \( \lim_{r \to +\infty} \mathcal{E}_p (\varphi) = 0 \).

We denote \( Q_0^n = \| Q_0 \|_{L^\infty (\Sigma_0)} \).

There exists \( r_+ < R_0 = R_0(M, \rho) \) large enough, \( \delta = \delta(M, \rho) > 0 \) small enough and \( C = C(M, \rho) > 0 \) so that for all \( R_1 > R_0 \), and if

\[ Q_0^n + \mathcal{E}_p < \delta, \]

then for all \( u \geq u_0 (R_1) \):

\[ E_{R_1}(u) \leq C \cdot \mathcal{E}. \quad (B.1) \]

Also for all \( v \leq v_{R_1}(u) \):

\[ \int_{u(v)}^{+\infty} r^2 |D_\nu \phi|^2 \frac{du'}{u'^2} (u', v) dv' \leq C \cdot \mathcal{E}, \quad (B.2) \]

where \( \tilde{u}(v) = u_0(v) \) if \( v \leq u_0 (R_1) \) and \( \tilde{u}(v) = u_{R_1} (v) \) if \( v \geq v_0 (R_1) \).

Moreover for all \( (u, v) \) in the space-time:

\[ |Q|(u, v) \leq C \cdot (Q_0^n + \mathcal{E}_p). \quad (B.3) \]

Finally there exists \( C_1 = C_1(R_1, M, \rho) \) such that for all \( u \geq u_0 (R_1) \):

\[ \int_{v \leq v_{R_1}(u)} \int_{r \leq R_1} \left( |D_\nu \phi|^2 (u', v) + |D_\nu \phi|^2 (u', v) + |\phi|^2 (u', v) \right) \Omega^2 du dv \leq C_1 \cdot \mathcal{E}, \quad (B.4) \]
Proof. Let $R_0 > r_+$ large to be chosen later and $R_1 \geq R_0$.

We are going to apply various identities to the domain

$\Theta_u(R_1) := \{ v \leq v_{R_1}(u) \, | \, r \leq R_1 \} \cup \{ u' \leq u, \, r \geq R_1 \}$, defined for all $u \geq u_0(R_1)$, having boundaries $\mathcal{H}^+ \cap \{ v \leq v_{R_1}(u) \}$, $[u, +\infty) \times \{ v_{R_1}(u) \}$, $\{ u \} \times [v_{R_1}(u), +\infty)$, $\mathcal{I}^+ \cap \{ u' \leq u \}$ and $\Sigma_0$, c.f. Figure 4.

Notice that for all $u \geq u_0(R_1)$, $\Theta_u(R_1)$ is the complement of the interior of $\mathcal{D}_{R_1}(u)$.

We apply the divergence identity in $\Theta_u(R_1)$ for the Killing vector field $\partial_t$, c.f. Appendix C for more details. We get that for all $u \geq u_0(R_1)$:

$$
\int_{R_0}^{+\infty} \int_{\mathcal{H}^+} r^2 |D_v \phi|^2(v) dv + \int_{-\infty}^{R_0} \int_{\mathcal{H}^+} r^2 |D_v \phi|^2(u) du + \int_{R_0}^{+\infty} \int_{\mathcal{I}^+} r^2 |D_v \phi|^2(u') dv + \int_{R_0}^{+\infty} \int_{\mathcal{I}^-} r^2 |D_v \phi|^2(u) dv
$$

\begin{align}
&+ \int_{u}^{+\infty} 2\Omega^2 Q^2(u', v_{R_1}(u)) \frac{du'}{r^2} + \int_{u}^{+\infty} 2\Omega^2 Q^2(u, v) dv = \int_{\Sigma_0} \frac{r^2}{2} (|D_v \phi|^2 + |D_v \phi|^2) \frac{dr^*}{r^4} + \frac{2\Omega^2 Q^2}{r^4} dr^*.
\end{align}

(B.5)

Therefore it implies that

$$
E_{deg,R_1}^+(u) \leq \frac{\mathcal{E}}{2} + C \cdot (Q_0^\infty)^2,
$$

where $C = C(M, \rho) > 0$ is defined as $C = \int_{\Sigma_0} \frac{2\Omega^2}{r^4} dr^*$ and $E_{deg,R_1}^+(u)$ is defined as

$$
E_{deg,R_1}^+(u) := \int_{R_0}^{+\infty} \int_{\mathcal{H}^+} r^2 |D_v \phi|^2(v) dv + \int_{-\infty}^{R_0} \int_{\mathcal{H}^+} r^2 |D_v \phi|^2(u) du + \int_{R_0}^{+\infty} \int_{\mathcal{I}^+} r^2 |D_v \phi|^2(u') dv + \int_{R_0}^{+\infty} \int_{\mathcal{I}^-} r^2 |D_v \phi|^2(u) dv.
$$

(B.7)

Now we want to prove a similar estimate for the non-degenerate energy $E_{R_1}(u)$. We are going to use a slight variation of the red-shift effect as demonstrated in section 5.2.

We start to prove a Morawetz type estimate in the region $\Theta_u$.

The method of proof is the same as in section 5.1:

**Lemma B.2.** There exists $\tilde{\varepsilon} = \tilde{\varepsilon}(M, \rho) > 0$, $\bar{C} = \bar{C}(M, \rho) > 0$, $\bar{\sigma} = \bar{\sigma}(M, \rho) > 1$ such that

for all $u \geq u_0(R_1)$, if $\sup_{\Theta_u(R_1)} |Q| < \tilde{\varepsilon}$ then

$$
\int_{\Theta_u(R_1)} (|D_t \phi|^2(u', v) + |D_t \phi|^2(u', v) + |\phi|^2(u', v)) \Omega^2 du' dv \leq \bar{C} \cdot E_{deg,R_1}^+(u) + \mathcal{E} \leq (\bar{C})^2 \cdot (\mathcal{E} + (Q_0^\infty)^2),
$$

where $E_{deg,R_1}^+(u)$ is defined in (B.7).

**Proof.** The proof is exactly the same as the one in the interior $\mathcal{D}$. The bulk term is identical, the only difference is the boundary term on $\{ t = 0 \}$, which has already been controlled: we first get that for some $C' = C'(M, \rho) > 0$

$$
\int_{\Theta_u(R_1)} (|D_t \phi|^2(u, v) + |D_t \phi|^2(u', v) + |\phi|^2(u', v)) \Omega^2 du' dv 
\leq C' \cdot \left( \int_{R_0}^{+\infty} \int_{\mathcal{H}^+} r^2 |D_v \phi|^2(v) dv + \int_{-\infty}^{R_0} \int_{\mathcal{H}^+} r^2 |D_v \phi|^2(u) du + \int_{R_0}^{+\infty} \int_{\mathcal{I}^+} r^2 |D_v \phi|^2(u', v_{R_1}(u)) du' + \int_{R_0}^{+\infty} \int_{\mathcal{I}^-} r^2 |D_v \phi|^2(u) dv \right),
$$

and then we apply (B.6) to conclude.

Now we are going to prove a Red-shift type estimate in the region $\{ v \leq v_{R_1}(u) \}$. The method of proof is the same as in section 5.2:

**Lemma B.3.** There exists $R_0 = R_0(M, \rho)$, sufficiently close to $r_+$ such that all $r_+ < \tilde{R}_0 < R_0$, there exists $\varepsilon = \varepsilon(M, \rho, R_0) > 0$, $C = C(M, \rho, R_0) > 0$ such that for all $u \geq u_0(R_1)$, if $\sup_{\Theta_u(R_1)} |Q| < \varepsilon$ then for all $v \leq v_{R_1}(u)$

$$
\int_{u_{R_0}(v)}^{+\infty} \frac{r^2 |D_v \phi|^2(u', v)}{\Omega^2} du' + \int_{v_{R_1}(u)}^{+\infty} \frac{r^2 |D_v \phi|^2(u, v)}{\Omega^2} dudv \leq C \cdot E_{deg,R_1}^+(u) + \mathcal{E} \leq (\bar{C})^2 \cdot (\mathcal{E} + (Q_0^\infty)^2),
$$

(B.9)

where $E_{deg,R_1}^+(u)$ is defined in (B.7).
such that $r_+ \leq R_0(\rho, \kappa) \leq R_1$. We also define $e^+ = e^+(\rho, \kappa) < \max\{\varepsilon, \bar{\varepsilon}\}$, with the notations of the former lemmata. For fixed $u \geq u_0(R_1)$, we bootstrap in $\Theta_u$:

$$|Q| \leq e^+. \quad (B.10)$$

If we assume $Q_0^\infty < e^+$, then the set of points which verify the bootstrap is non-empty. We now want to establish preliminary “weak” estimates. First we write, on the constant $v \equiv V$ surface.

$$|e^{\int_{-V}^{+V} A_u} \phi(u, V) - \phi_0(-V, V)| \leq \left( \int_{-V}^{+V} e^{\int_{-V}^{+V} A_u \phi(u', V) du'} \right)^{\frac{1}{2}} \left( \int_{-V}^{+V} \Omega^2 r^2 |D_u \phi|^2 (u', V) du' \right)^{\frac{1}{2}},$$

where we used the fact that $\partial_u (e^{\int_{-V}^{+V} A_u} \phi) = e^{\int_{-V}^{+V} A_u} D_u \phi$.

Then we take the limit $V \to +\infty$, we use the fact from the hypothesis that $\phi_0$ tends to 0 towards spatial infinity, together with (B.6) to get, for some $C' = C'(\rho, \kappa) > 0$:

$$r^2 |\phi|_{L^2}^2 \leq \left( \int_{-V}^{+V} \Omega^2 r^2 |D_u \phi|^2 (u', V) du' \right)^{\frac{1}{2}} \leq \sqrt{\frac{E}{2} + C \cdot (Q_0^\infty)^2} \leq C' \cdot \sqrt{\frac{E}{2} + Q_0^\infty}.$$}

Hence we get that $\lim_{r \to +\infty} \phi(u, v) = 0$.

Now because we know that $\lim_{r \to +\infty} \phi(u, v) = 0$, one can use Hardy’s inequality under the form (2.17) (the proof is the same although the statement differs slightly) and (B.6) to get that for all $(u, v) \in \{ r \geq R_0 \}$

$$r |\phi|^2(u, v) \leq \int_{v}^{r^\infty} \frac{r^2 |D_u \phi|^2 (u', v') dv'}{\Omega^2} \leq \frac{E}{\Omega^2}. \quad (B.11)$$

We were able to claim the estimate in the whole region $\{ r \geq R_0 \}$ because it is true in $\{ r \geq R_1 \}$ and $R_1$ is allowed to vary in the range $[R_0, +\infty]$

Now using a method that has been made explicit already several times in section 5 we use the mean-value theorem with the Morawetz estimate (B.8): we find that there exists $R_0' \in (0.8 R_0, 0.9 R_0)$ and $C_0' = C_0(M, \rho)$ such that

$$\int_{0}^{t_{R_0 \setminus R_0'}} (|\phi|^2 (R_0', t) + |D_u \phi|^2 (R_0', t)) dt \leq C_0 \cdot (E + (Q_0^\infty)^2),$$

this provided $r_+ < 0.8 R_0$ and where $t_{R_1 \setminus R_0'}(u) := 2v_{R_1}(u) - (R_0')^* = 2u + 2R_1^* - (R_0')^*$ is defined such that $(t_{R_1 \setminus R_0'}(u), (R_0')^*) = \gamma_{R_0'} \cap \{ v = v_{R_1}(u) \}$.

Therefore, using (2.8), (2.9) as $|\partial_t Q| \leq Q_0 r^2 |\phi| |D_u \phi|$ we integrate and find that there exists $C_0' = C_0'(M, \rho) > 0$ so that for all $0 \leq t \leq t_{R_1 \setminus R_0'}(u)$

$$|Q(t, R_0)| \leq Q_0^\infty + C_0' \cdot (E + (Q_0^\infty)^2).$$

Then we integrate $\partial_u Q$ in $u$ towards $\gamma_{R_0'}$, using the red-shift estimate (B.9). Tedious details of the integration are left to the reader.
We find that in \( \{ v_0(R_0) < v \leq v_{R_1}(u) \} \cap \{ r \leq R_0 \} \), there exists \( C_0'' = C_0''(M, \rho) > 0 \) such that

\[
|Q(u, v)| \leq Q_0^{\infty} + C_0'' \cdot (E + (Q_0^{\infty})^2).
\]

Using the same technique with \([B.9]\) in \( \{ v \leq v_0(R_0) \} \) and then in the bounded region \( \{ R_0' \leq r \leq R_0 \} \) we then get that there exists \( C = C(M, \rho) > 0 \) such that in the whole region \( \{ v \leq v_{R_1}(u) \} \cap \{ r \leq R_0 \} \)

\[
|Q(u, v)| \leq Q_0^{\infty} + C \cdot (E + (Q_0^{\infty})^2),
\]

where we used the fact that \( R_0 \) depends only on \( M \) and \( \rho \).

This should be thought of as a local in space smallness propagation of the charge, for potentially large times.

We now want to “globalise” this result in space. For this, we need to control higher \( r^p \) weighted energies, for any \( p > 1 \). We will need to use the \( r^p \) method, as developed in section \([B.3]\).

We establish the \( r^p \) estimate, in the very same way as in section \([B.4]\), but this time on the domain \( \Theta_u \cap \{ r \geq R_0 \} \).

We are going to write two different \( r^p \) estimates, according to \( u < u_0(R_0) \) or \( u > u_0(R_0) \). The proof is however the same.

For this we define \( \Theta_u'(R_0) = \Theta_u(R_0) \cap \{ r \geq R_0 \} \) if \( u \geq u_0(R_0) \) and \( \Theta_u'(R_0) = \{ u' \leq u \} \) if \( u \leq u_0(R_0) \), c.f. Figure \( 5 \).

We also define \( \tilde{v}(u) = v_{R_0}(u) \) if \( u \geq u_0(R_0) \) and \( \tilde{v}(u) = v_0(u) \) if \( u \leq u_0(R_0) \). We can then write

\[
\int_{\Theta_u'(R_0)} \left( p r^{p-2} |D_v \psi|^2 + \left( 2 M (3 - p) - (4 - p) \frac{2p^2}{r} \right) r^{p-4} |\psi|^2 \right) dv + \int_{\tilde{v}(u)}^{+\infty} r^p |D_v \psi|^2 (u, v) dv + \int_{-\infty}^{\tilde{v}(u)} r^p |D_v \psi|^2 (r^*) dr^*,
\]

where \( P_1(r) \) is a polynomial in \( r \) that behaves like \( O(r^{-1}) \) as \( r \) tends to \( +\infty \) and with coefficients depending only on \( M \) and \( \rho \).

Since \( p < 3 \) we can take \( R_0 \) large enough (depending on \( M \) and \( \rho \)) so that

\[
|P_1(r)| < 1 \text{ and } 2 M (3 - p) - (4 - p) \frac{2p^2}{r} > 0.
\]

We get

\[
\int_{\Theta_u'(R_0)} p r^{p-2} |D_v \psi|^2 dv + \int_{\tilde{v}(u)}^{+\infty} r^p |D_v \psi|^2 (u, v) dv \leq \int_{\Theta_u'(R_0)} 2 q_0 Q^2 r^{p-2} \Omega |\psi D_v \psi| dv + \mathcal{E}_p. \quad (B.13)
\]

Now we use a variant of Hardy’s inequality \([2.20]\) for any \( 0 \leq q < 2 \) under the form

\[
\left( \int_{\Theta_u'(R_0)} r^{q-2} |\psi|^q dv \right)^\frac{1}{q} \leq \left( \frac{2}{(2-q)M(R_0)} \int_{\Theta_u'(R_0)} r^{q-2} |D_v \psi|^2 dv \right)^\frac{1}{q} + \left( \frac{1}{2-q} \int_{-\infty}^{\tilde{v}(u)} r^{q-2} |\psi|^2 (u, \tilde{v}(u)) du \right)^\frac{1}{q}.
\]

Now we are free to choose \( 1 < p < 2 \) without loss of generality, and with \( p \) as close to 1 as needed. Taking \( q = p \), this implies, by the hypothesis and using the Morawetz estimate \([B.8]\) that there exists a constant \( C_0 = C_0(M, \rho) > 0 \) such that
\[
\frac{1}{2 - p} \int_{-\infty}^{+\infty} r^{p-2} |\psi|^2(u, \tilde{v}(u)) du \leq C_0 \cdot \tilde{\xi}_p.
\]

Combining this with the Cauchy-Schwarz inequality and the bootstrap assumption \([B.10]\) we see — like in section [6.2] — that for all \(\eta > 0\) small enough we have

\[
\left( p - \frac{4q_0 e^+}{(2 - p)|\Omega(R_0)|} - \frac{\eta}{2} \right) \int_{\partial\Omega(R_0)} r^{p-1} \Omega^2 |D_v \psi|^2 du dv + \int_{\Sigma}^+ r^p |D_v \psi|^2(u, v) dv \leq \left[ 1 + \frac{C_0}{2\eta} \right] \cdot \tilde{\xi}_p, \tag{B.14}
\]

so if say \(4q_0 e^+ < \frac{(2 - p)|\Omega(R_0)|}{2 - p}\) — which can be assumed, taking \(\delta\) small enough — we proved that for some \(C'_0 = C'_0(M, \rho) > 0\) and for all \(u \in \mathbb{R}\):

\[
\int_{\Sigma}^+ r^p |D_v \psi|^2(u, v) dv \leq C'_0 \cdot \tilde{\xi}_p. \tag{B.15}
\]

Now we re-write \([2.20]\) as \(|\partial_v Q| \leq q_0 |\psi||D_v \psi|\) and we integrate towards \(\gamma R_0\) if \(u \geq u_0(R_0)\) and towards \(\Sigma_0\) if \(u \leq u_0(R_0)\), using Cauchy-Schwarz:

\[
|Q(u, v) - Q(u, \tilde{v}(u))| \leq q_0 \left( \int_{\tilde{v}(u)}^0 r^p |D_v \psi|^2(u, v) dv \right)^{\frac{1}{2}} \left( \int_{\tilde{v}(u)}^0 r^{-p} |\psi|^2(u, v) dv \right)^{\frac{1}{2}} . \tag{B.16}
\]

We now use the “boundary term” version of Hardy’s inequality \([2.20]\) and we get that there exists \(C_1 = C_1(M, \rho) > 0\) such that

\[
\left( \int_{\tilde{v}(u)}^0 r^{-p} |\psi|^2(u, v) dv \right)^{\frac{1}{2}} \leq C_1 \cdot \left[ \left( \int_{\tilde{v}(u)}^0 r^p |D_v \psi|^2(u, v) dv \right)^{\frac{1}{2}} + |r(u, \tilde{v}(u))|^{1-p} \cdot |\psi(u, \tilde{v}(u))| \right], \tag{B.17}
\]

where we used the fact that \(2 - p < p\) because \(p > 1\).

Now, to estimate \(\psi_0\) we can use the Cauchy-Schwarz under the following form, taking advantage of the fact that \(p > 1\) : for all \(r \geq R_0\)

\[
|\psi_0(r) - \psi_0(R_0)| \leq \int_{(R_0)^*} |D_r \psi_0| |(r')^*| d(r')^* \leq \sqrt{2} \left( \int_{(R_0)^*} (r')^{-p} d(r')^* \right)^{\frac{1}{2}} (\tilde{\xi}_p)^{\frac{3}{2}},
\]

where we used the fact that \(\int |r^p |D_r \psi_0|^2 dr^* \leq 2\tilde{\xi}_p\).

Using also \([B.11]\) and remembering that \(R_0 = R_0(M, \rho)\) depends only on \(M\) and \(\rho\) we can then write, for some \(\tilde{C} = \tilde{C}(M, \rho) > 0\) :

\[
|\psi_0(r)| \leq \tilde{C} \cdot (\tilde{\xi}_p)^{\frac{3}{2}}.
\]

Coupled with \([B.11]\) applied on \(\gamma R_0\), there exists also \(\tilde{C} = \tilde{C}(M, \rho) > 0\) such that for all \(u \in \mathbb{R}\):

\[
|\psi(u, \tilde{v}(u))| \leq \tilde{C} \cdot (\tilde{\xi}_p)^{\frac{3}{2}}. \tag{B.18}
\]

Now combining \([B.15]\), \([B.16]\), \([B.17]\) and \([B.18]\) we get that there exists \(C_2 = C_2(M, \rho) > 0\) such that

\[
|Q(u, v) - Q(u, \tilde{v}(u))| \leq C_2 \cdot \tilde{\xi}_p.
\]

Now combining with \([B.12]\), we see that for all \(u \in \mathbb{R}\):

\[
|Q(u, v)| \leq Q_0^\infty + C \cdot (Q_0^\infty)^2 + C_2 \cdot \tilde{\xi}_p < \delta \cdot (1 + C_2 + C \cdot \delta), \tag{B.19}
\]

where we defined \(C_2' = C_2 + C\).

Therefore the bootstrap \([B.10]\) is retrieved provided if \(\delta = \delta(M, \rho)\) is small enough so that

\[
\delta \cdot (1 + C_2' + C \cdot \delta) < e^+.
\]

This proves the charge bound claimed in the statement of the Proposition.

The last step we need to carry out is to prove the claimed boundedness of the energy. Indeed we only proved in \([B.6]\)

\[
E_{R_1}(u) \leq \frac{\tilde{\xi}}{2} + C \cdot (Q_0^\infty)^2,
\]

and we would like a right-hand-side that only depends on \(\tilde{\xi}\).

For this, we have to revisit the proof of section [5.3] to absorb the charge terms properly in the energy identity \([B.5]\).

The term on the future boundary of \(\Theta_u\) are treated in the very same way so we do not repeat the argument, c.f. sections [5.3.2, 5.3.3, 5.3.4]. We only take care of the charge term on \(\Sigma_0\).
First, using the same strategy as in section 5.3.3 one can prove that
\[
\left| \int_{(R_1)^+}^{\infty} \frac{\Omega^2 Q_0^2(r)}{r^2} dr^* - \frac{Q_0^2(R_1)}{R_1} \right| \leq \frac{2q_0Q_0^\infty E}{\Omega^2(R_1)}. \tag{B.20}
\]

Then, like in section 5.3.2 one can prove
\[
\left| \frac{Q_0^2(R_1)}{R_1} - \frac{Q_0^2(R_0)}{R_0} \right| \leq \frac{2q_0Q_0^\infty E}{\Omega^2(R_0)}. \tag{B.21}
\]

Now for the analogue of the proof in section 5.3.3, we need to prove a few preliminary estimates.

First, using an argument similar to the one used in the proof of Hardy inequality (2.14) and the fact that \(\phi_0(r) \to 0\) when \(r \to +\infty\) one can prove that there exists \(C_+ = C_+(M, \rho) > 0\) such that
\[
\int_{-\infty}^{\infty} \Omega^2 |\phi_0|^2 dr^* \leq \bar{C} \cdot E.
\]

The rough idea is to apply an Hardy argument to the integral on \([-\infty, (R_0)^+]\) then we pick a term on \(\gamma R_0\) that can be controlled \((B.11)\). The integral on \([(R_0)^+, +\infty] \cap \gamma R_0 \subset \Omega\). Every-time, we lose a weight on \(\gamma R_0\) but that weight depends only on \(M\) and \(\rho\).

Consequently with this estimate, like in section 5.3.3 one can prove
\[
\left| \int_{-\infty}^{(R_1)^+} \frac{\Omega^2 Q_0^2(r)}{r^2} dr^* - \left(1 - \frac{r_+}{R_1} \right) \frac{Q_0^2(R_0)}{R_0} \right| \leq \left(\sqrt{2} R_0 \frac{R_0}{r_+} - 1 \right) + \frac{2(1 - \frac{R_0}{R})}{\Omega^2(R_0)} \cdot q_0Q_0^\infty E, \tag{B.22}
\]

We can then use the Morawetz estimate \((B.8)\) to deal with the charge difference on \(\gamma R_0\), recalling the dependence \(R_0 = R_0(M, \rho)\), everything like in section 5.10. We conclude that — provided \(\delta\) is small enough
\[
E_{R_1}(u) \leq C' \cdot E.
\]

This concludes the proof of the proposition.

We now turn to the proof of Proposition 5.2: this time we assume already energy boundedness and the Morawetz estimate but not arbitrary charge smallness. The method of proof is very similar to that of Proposition 4.7.

**Proposition B.4.** Suppose that there exists \(1 < p < 2\) such that \(\tilde{E}_p < \infty\).

It follows that there exists \(e_0 \in \mathbb{R}\) such that
\[
\lim_{r \to +\infty} Q_0(r) = e_0.
\]

Without loss of generality one can assume that \(1 < p < 1 + \sqrt{1 - 4q_0|e_0|}\).

Assume also that \(\lim_{r \to +\infty} \phi_0(r) = 0\).

Now assume \((4.2), (4.7)\) for \(R_1 = R: \) there exists \(C = C(M, \rho) > 0\) such that for all \(u \geq u_0(R)\) and for all \(v \leq v_R(u)\):
\[
E(u) = E_R(u) \leq C \cdot E, \tag{B.23}
\]
\[
\int_{\hat{u}(v)}^{+\infty} \frac{r^2 |D_r \phi|^2}{\Omega^2} (u', v) du' \leq C \cdot E, \tag{B.24}
\]

where \(\hat{u}(v) = u_0(v)\) if \(v \leq v_0(R)\) and \(\hat{u}(v) = u_R(v)\) if \(v_0(R) \leq v \leq v_R(u)\).

Assume also \((4.5)\): there exists \(R_0 = R_0(M, \rho) > r_+\) such that for all \(R_1 > R_0\), there exists \(C_1 = C_1(R_1, M, \rho) > 0\) such that
\[
\int_{\{u' \leq u\} \cap \{r' \leq R_1\}} (|D_r \phi|^2(u', v) + |D_r \cdot \phi|^2(u', v) + |\phi|^2(u', v)) \Omega^2 du dv \leq C_1 \cdot E. \tag{B.25}
\]

Make also the following smallness hypothesis: for some \(\delta > 0\):
\[
\tilde{E}_p < \delta,
\]
\[
q_0 |e_0| < \frac{1}{4}
\]

There exists \(\delta_0 = \delta_0(e_0, M, \rho) > 0\) and \(C = C(M, \rho) > 0\) such that if \(\delta < \delta_0\) then for all \((u, v)\) in the space-time:
\[
|Q(u, v) - e_0| \leq C \cdot \tilde{E}_p, \tag{B.26}
\]
\[
q_0|Q|(u, v) < \frac{1}{4} \tag{B.27}
\]
Moreover, there exists $\delta_p = \delta_p(\mathbf{e}_0, \mathbf{p}, M, \rho) > 0$ and $C' = C'(\mathbf{e}_0, p, M, \rho) > 0$ such that if $\delta < \delta_p$ then for all $u \geq u_0(R)$:

$$E_p[\psi](u) \leq C' \cdot \tilde{E}_p. \quad (B.28)$$

**Proof.** First take $R_0 = R_0(M, \rho)$. Using Cauchy-Schwarz and the Maxwell equation under the form $|\partial_t \cdot Q| \leq q_0 |\psi| |D_t \psi|$, we can show that there exists $\tilde{C}_0 = \tilde{C}_0(M, \rho) > 0$ such that for all $r \geq R_0$,

$$|Q_0(r) - e_0| \leq \tilde{C}_0 \cdot \tilde{E}_p.$$

Similarly on $\{r_+ \leq r \leq R_0\}$ one can prove that there exists $\tilde{C}_0 = \tilde{C}_0(M, \rho) > 0$ such that for all $r \leq R_0,$

$$|Q_0(r) - Q_0(R_0)| \leq \tilde{C}_0 \cdot \tilde{E},$$

where we used Cauchy-Schwarz and the Maxwell equation under the form $|\partial_t \cdot Q| \leq \frac{q_0 R_0^2}{2} (|\phi||D_\omega \phi| + |\phi||D_c \phi|)$. This gives, on the whole $\Sigma_0$:

$$|Q_0(r) - e_0| \leq (\tilde{C}_0 + \tilde{C}_0) \cdot \tilde{E}_p. \quad (B.29)$$

Now assume that $R_0 > \tilde{R}_0(M, \rho)$ so that estimate (B.25) is valid. Now, using the Morawetz estimate (B.25), in a way that was explained numerous times in this paper, one can find $r_+ < \tilde{R}_0 = \tilde{R}_0(M, \rho) < R_0$ and $\tilde{D}_0 = \tilde{D}_0(M, \rho) > 0$ such that for all $t \geq 0$

$$|Q(t, \tilde{R}_0) - Q_0(\tilde{R}_0)| \leq \tilde{D}_0 \cdot \tilde{E}, \quad (B.30)$$

where we used Cauchy-Schwarz and the Maxwell equation under the form $|\partial_t \cdot Q| \leq \frac{q_0 R_0^2}{2} |\phi||D_\omega \phi|$. Then using (B.29) and (B.30), one can prove that there exists $\tilde{D}_0' = \tilde{D}_0'(M, \rho) > 0$ such that for all $(u, v) \in \{r \leq R\}$,

$$|Q_0(u, v) - Q(\tilde{u}(v), v)| \leq \tilde{D}_0' \cdot \tilde{E}_p,$$

where $\tilde{u}(v) = u_0(v)$ if $v \leq u_0(R)$ and $\tilde{u}(v) = u_\tilde{R}_0(v)$ if $v \geq u_0(R)$. This combined with (B.29) and (B.30) proves that there exists $C_0 = C_0(M, \rho) > 0$ such that for all $(u, v) \in \{r \leq R\}$,

$$|Q_0(u, v) - e_0| \leq C_0 \cdot \tilde{E}_p. \quad (B.31)$$

In the same way as in Proposition L.1, we can derive the estimate for $R_0 = R_0(M, \rho)$ large enough, for all $R_1 \geq R_0$ and for all $u \in \mathbb{R}$:

$$\int_{\psi_0(R_1)} \rho^{r-p-1} \Omega^2 |D_\nu \psi|^2 du dv + \int_{\psi_0(R_1)}^{+\infty} r^p |D_\nu \psi|^2 (u, v) dv \leq \int_{\psi_0(R_1)} \frac{2q_0 R_0^2r}{2\rho} (\tilde{\psi} D_\nu \psi) du dv + \tilde{E}_p. \quad (B.32)$$

where all the notations are introduced in the proof of Proposition L.1. Note that this time we consider $\Theta'_\omega(R_1)$ for any $R_1 \geq R_0$ to be chosen later, in contrast to the proof of Proposition L.1 where only $\Theta'_\omega(R_0)$ was considered.

Then we bootstrap for some $\tilde{\epsilon} = |e_0| + 2\epsilon$, $\epsilon > 0$ and in $\Theta'_\omega(R_1)$:

$$|Q| < \tilde{\epsilon}. \quad (B.33)$$

Because $Q_0 \to e_0$ towards spatial infinity, it is clear that the set of points for which this bootstrap is verified is non-empty.

For $\epsilon$ small enough, one can assume that $q_0 \cdot (|e_0| + 2\epsilon) < \frac{1}{4}$. By assumption one can assume that there exists $1 < p < \sqrt{1 - 4q_0\epsilon}$ such that $\tilde{\epsilon}_p < \infty$.

Using this, we find constants $C_{R_1} = C_{R_1}(R_1, M, \rho) > 0$ and $D = D(M, \rho)$ such that for all $\eta > 0$ small enough and for all $u \in \mathbb{R}$:

$$\int_{\psi_0(R_1)}^{+\infty} r^p |D_\nu \psi|^2 (u, v) dv \leq \frac{D \cdot \tilde{E}_p}{\eta \cdot \left( p - \frac{4q_0 \tilde{\epsilon}}{2 - p} \right)}, \quad (B.34)$$

$$\int_{\psi_0(R_1)}^{+\infty} r^{p-2} |\psi|^2 (u, v) \leq \frac{C_{R_1} \cdot \tilde{E}_p}{\eta \cdot \left( p - \frac{4q_0 \tilde{\epsilon}}{2 - p} \right)}. \quad (B.35)$$

where for the second estimate, we used a Hardy inequality coupled with the Morawetz estimate (B.25).

Now we take $R_1 = R_1(e_0, M, \rho)$ large enough so that $4q_0 \tilde{\epsilon} < \Omega(R_1)$. Then we can take temporarily $p > 1$ sufficiently close to 1 and $\eta$ small enough so that $p - \frac{4q_0 \tilde{\epsilon}}{2 - p} > 0$.

We then find using (2.9) that there exists $C = C(e_0, M, \rho) > 0$ such that on $\{r \geq R_1\}$

\footnote{The dependence of $\delta_p$ on $p$ only exists as $p$ approaches $1 + \sqrt{1 - 4q_0|e_0|}$.}
\[ |Q(u, v) - Q(u, \tilde{v}(u))| \leq C \cdot \tilde{\mathcal{E}}_p, \quad (B.36) \]

where \( \tilde{v}(u) = \nu \mathcal{R}_1(u) \) if \( u \geq u_0(R_1) \) and \( \tilde{v}(u) = v_0(u) \) if \( u < u_0(R_1) \). Then with \( (B.31) \) and provided that \( R_1 < R \) — we actually proved that for some \( C' = C'(\epsilon_0, M, \rho) > 0 \) and on the whole \( \mathcal{F}_\psi(R_1) \):

\[ |Q(u, v) - v_0| \leq C' \cdot \mathcal{E}_p < C' \cdot \delta. \quad (B.37) \]

Then it suffices to take \( C' \cdot \delta < \epsilon \) to retrieve bootstrap \( (B.32) \). This evidently gives the first two claims on the whole space-time.

Now come back to general \( 1 < p < \sqrt{1 - 4|q_0|e_0} \) and notice that \( (B.34) \) can be written as, for all \( \eta > 0 \):

\[ \int_{r_0}^{+\infty} r^p |D_{\nu} \psi|^2(u, v) dv \leq \frac{D \cdot \tilde{\mathcal{E}}_p}{\eta} \left( p - \frac{4|q_0|e_0 + \eta}{(2-p)} \right). \quad (B.38) \]

for \( \delta \) small enough and for the choice \( R_1 = R \), for \( R \) large enough. This gives directly \( (B.28) \) and concludes the proof of the proposition.

Now we turn to the proof of Proposition 4.3.

**Proposition B.5.** In the conditions of Proposition 4.3 assume moreover that there exists \( \omega > 0 \) and \( C_0 > 0 \) such that

\[ r|D_{\nu} \psi_0| + |\psi_0| \leq C_0 \cdot r^{-\omega}. \]

Then in the following cases

1. \( \omega = 1 + \theta \) with \( \theta > \frac{\nu_0|e_0|}{\frac{1}{2} + \beta \epsilon} \).
2. \( \omega = \frac{1}{2} + \beta \) with \( \beta \in (-\sqrt{\frac{1}{2} - \frac{\nu_0|e_0|}{\frac{1}{2}}}, \sqrt{\frac{1}{2} - \frac{\nu_0|e_0|}{\frac{1}{2}}}) \), if \( q_0|e_0| < \frac{1}{2} \),

there exists \( \delta = \delta(\epsilon_0, \omega, M, \rho) > 0 \) and \( R_0 = R_0(\omega, \epsilon_0, M, \rho) > r_+ \) such that if \( \mathcal{E}_p < \delta \) and \( R > R_0 \) then the decay is propagated: there exists \( C_0 = C_0(\omega, R, M, \rho, \epsilon_0) > 0 \) such that for all \( u \leq u_0(R) \):

\[ |D_{\nu} \psi|(u, v) \leq C_0 \cdot r^{-1-\omega}, \]

\[ |\psi|(u, v) \leq C_0 \cdot |u|^{-\omega}, \]

where \( \omega' = \min\{\omega, 1\} \).

In that case, for every \( 0 < p < 2\omega' + 1 \), we have the finiteness of the \( r^p \) weighted energy on \( \mathcal{V}_{\nu_0}(R) \):

\[ E_p[\psi](u_0(R)) < \infty. \]

**Proof.** We start the proof in a region \( \{ u \leq \mathcal{U}_0 \} \) for \( |\mathcal{U}_0| \) large enough to be chosen later.

We are going to use the notations \( u_0(\epsilon), u_0(\nu), r_0(\nu), r_0(\epsilon) \), c.f. section 2.4 for a definition.

For some \( B > 0 \) large enough to be chosen appropriately later, we bootstrap the following in \( \{ u \leq \mathcal{U}_0 \} \):

\[ |\psi| \leq B|u|^{-\omega}. \]

Notice that with the assumptions and the fact that \( r_0(\nu) \sim 2|u| \) when \( u \to -\infty \), the set of points for which the bootstrap is verified is non-empty, for \( B \) large enough.

We also denote \( Q^\omega = \sup\{ u \leq \mathcal{U}_0 \} |Q| \). By Proposition 4.3, \( Q^\omega \) can be taken arbitrarily close to \( |e_0| \) or \( |\epsilon| \) for \( \delta \) appropriately small.

Then we use \( (2.3) \) under the form

\[ |D_{\nu} D_{\nu} \psi| \leq \frac{B|u|^{-\omega}}{r^2} (q_0 Q^\omega + |2r \cdot K(r)|). \]

Now take \( \epsilon > 0 \). \( \{ u \leq \mathcal{U}_0 \} \subset \{ r \geq r_0(\mathcal{U}_0) \} \) so by taking \( |\mathcal{U}_0| \) large enough, one can assume that \( |2r \cdot K(r)| < \epsilon \) since this quantity tends to 0 as \( r \) tends to \( +\infty \).

There are two cases: either \( \omega' > 1 \) or \( \omega' \leq 1 \). We start by \( \omega' > 1 \) : we can integrate in \( u \) the inequality above and get

\[ |D_{\nu} \psi|(u, v) \leq C_0 \cdot (r_0(\nu))^{-1-\omega} + \frac{B|u|^{-\omega}}{(\omega - 1)r^2} (q_0 Q^\omega + \epsilon). \quad (B.39) \]

Now we want to integrate this in \( v \) : first notice that \( \frac{d(r_0(\nu))}{dv} = 2\nu^2 (-v, v) \geq 2\nu^2 (r_0(\mathcal{U}_0)) \geq \frac{2}{(1+\nu)} \) if \( |\mathcal{U}_0| \) is large enough. Therefore

\[ \int_{\mathcal{U}_0(u)}^{v} (r_0(\nu'))^{-1-\omega} dv' \leq \frac{(1 + \epsilon)}{2(1-\nu)} \int_{\mathcal{U}_0(u)}^{v} \frac{d[(r_0(\nu'))^{-\omega}]}{dv'} dv' \leq \frac{(1 + \epsilon)}{2\omega} (r_0(\nu))^{-\omega}, \]

\[ \int_{\mathcal{U}_0(u)}^{v} (r_0(\nu'))^{-1} dv' \leq \frac{(1 + \epsilon)}{2} (r_0(\nu))^{-1}, \]

66
after noticing that \( r_0(v_0(u)) = r_0(u) \). Hence we have

\[
|\psi(u, v)| \leq C_0 \cdot (1 + \frac{(1 + \epsilon)}{2\omega}) \cdot (r_0(u))^{-\omega} + \frac{B(q_0Q^+ + \epsilon)(1 + \epsilon)}{2(\omega - 1)} (r_0(u))^{-1}|u|^{1-\omega}.
\]

Since \( r_0(u) \sim 2|u| \) when \( u \to -\infty \), it is clear that for \( |U_0| \) large enough, \( \epsilon \) small enough and \( B \) large enough, the bootstrap is retrieved if

\[
\omega > 1 + \frac{q_0Q^+}{4},
\]

or equivalently if \( \delta \) is small enough,

\[
\omega > 1 + \frac{q_0|\epsilon_0|}{4}.
\]

Now we turn to \( \omega \leq 1 \). We ignore the case \( \omega = 1 \) and consider \( \omega < 1 \). Integrating in \( u \) we get this time

\[
|D_u\psi|(u, v) \leq C_0 \cdot (r_0(v))^{-1-\omega} + \frac{B \cdot v^{1-\omega}}{(1 - \omega)\omega(1 - \epsilon)} |u|^{1-\omega},
\]

where we used that \( |u_0(v)| = v \).

Now, taking \( |U_0| \) large enough, one can assume that everywhere on \( \{ u \leq U_0 \} \) : \( r^{-2} \leq \frac{(r^*)^{-2}}{1-\omega} \leq \frac{v^{-2}}{1-\epsilon} \), since \( u \leq 0 \). Using this, we get

\[
\int_{r_0(u)}^{v} \frac{(v')^{-1-\omega}}{r_2(u,v')} dv' \leq \frac{|u|^{-\omega}}{(1 - \epsilon)\omega},
\]

hence we have

\[
|\psi_0(u, v)| \leq C_0 \cdot (1 + \frac{(1 + \epsilon)}{2\omega}) \cdot (r_0(u))^{-\omega} + \frac{B(q_0Q^+ + \epsilon)}{(1 - \omega)\omega(1 - \epsilon)} |u|^{-\omega}.
\]

We now see that the bootstrap is retrieved on the condition :

\[
q_0Q^+ < (1 - \omega),
\]

Otherwise said

\[
\frac{1 - \sqrt{1 - 4q_0Q^+}}{2} < \omega < \frac{1 + \sqrt{1 - 4q_0Q^+}}{2},
\]

or equivalently if \( \delta \) is small enough :

\[
\frac{1 - \sqrt{1 - 4q_0|\epsilon_0|}}{2} < \omega < \frac{1 + \sqrt{1 - 4q_0|\epsilon_0|}}{2}.
\]

This proves the proposition in the region \( \{ u \leq U_0 \} \), for \( |U_0| \) large enough with respect to \( \omega \) and \( \epsilon_0 \) and independently of \( R \).

Now it is enough to take \( R \) large enough so that \( |u_0(R)| > |U_0| \) and the result is proven, if we accept that the constants now depend on \( R \).

Then we turn to our last step, the proof of Proposition 4.4.

**Proposition B.6.** Suppose that for all \( 0 \leq p < 2 + \sqrt{1 - 4q_0|\epsilon_0|}, \xi_p < \infty \).

We also assume the other hypotheses of Theorem 3.3.

Then for all \( 0 \leq p < 2 + \sqrt{1 - 4q_0|\epsilon_0|} \), there exists \( \delta_p = \delta_p(\epsilon_0, p, M, \rho) > 0 \), such that if \( \delta < \delta_p \) then for all \( u \leq u_0(R) \):

\[
E_p[\psi](u) < \infty.
\]

**Proof.** The proof relies on the generalization of the \( r^p \) weighted estimate, namely a \( r^p u^s \) weighted estimate, for \( s > 0 \).

We are going to state this identity on the neighbourhood of spatial infinity \( \{ u \leq u_0(R) \} \) where \( R \) is large enough so that \( u_0(R) < 0 \). As a consequence, \( |u| = -u \) in this region.

Using (2.6) that we multiply by \(-u)^r D_u\psi\), we take the real part, integrate by parts and get, for all \( u \leq u_0(R) \)

\[
\int_{\{u' \leq u\}} \int_{\{r \geq r_0(v)\}} \left[ \frac{4p-1}{2\omega} |u'|^{s}\right] |D_u\psi|^2 du' dv + |u|^r \int_{r_0(v)}^{+\infty} r^p |D_u\psi|^2(u,v) dv
\]

\[
\leq \left[ 1 + P_0(r) \right] \left[ \int_{\{r \geq r_0(v)\}} \left| u_0(v) \right|^s (r_0(v))^{p} |D_u\psi|^2(u_0(v), v) dv + \int_{\{ u' \leq u\}} 2q_0 q^{-2} |u'|^{s} 3(\psi D_u\psi) du' dv \right],
\]

68 The dependence of \( \delta_p \) on \( p \) only exists as \( p \) approaches \( 2 + \sqrt{1 - 4q_0|\epsilon_0|} \).
where $P_0(r)$ is a polynomial in $r$ that behaves like $O(r^{-1})$ as $r$ tends to $+\infty$ and with coefficients depending only on $M$ and $\rho$.

Now because $|u_0(v)| \sim \frac{\omega(v)}{r}$ as $v \rightarrow +\infty$, for $R$ large enough we can say that

$$\int_{\Sigma_{R}\cap (v\geq v_0(v))} |u_0(v)|^2 (r_0(v))^p |Dv\psi|^2(u,v)dv \leq \mathcal{E}_{p+\varepsilon}.$$  

Denoting $Q^+ = \sup_{u' \leq u_0(R)} |Q|$ : note that $|Q^+-|\varepsilon_0| \lesssim \delta$ by the last proposition.

With this, we find that for all $\eta > 0$, taking $R$ large enough so that $[1 + P_0(r)] < (1 + \eta)$:

$$\int_{\{u' \leq u\}} \left[ p^{p-1} |u|^{s} + sr^{p}|u'|^{s'-1} \right] |Dv\psi|^2 du'dv + |u|^s \int_{v_0(u)}^{+\infty} r^p |Dv\psi|^2(u,v)dv \leq \mathcal{E}_{p+\varepsilon + 2\delta} Q^+ \int_{\{u' \leq u\}} r^{p-2} |u'|^{s} |Dv\psi| du'dv. \tag{B.40}$$

As it was seen in section 6.2 if $1 - \sqrt{1 - 4\delta_0 Q^+} < p' < 1 + \sqrt{1 - 4\delta_0 Q^+}$ the interaction term can be absorbed inside the bulk term using Hardy’s inequality because the presence of the $u$ weight does not change anything. Therefore for all $s' \geq 0$, there exists a constant $D > 0$ such that

$$\int_{\{u' \leq u\}} \left[ r^{p-1} |u|^{s'} + r^{p}|u'|^{s'-1} \right] |Dv\psi|^2 du'dv + |u|^{s'} \int_{v_0(u)}^{+\infty} r^p |Dv\psi|^2(u,v)dv \leq D \cdot \mathcal{E}_{p'+s'}. \tag{B.41}$$

Now we take $2 - \sqrt{1 - 4\delta_0 |\varepsilon_0|} < p < 2 + \sqrt{1 - 4\delta_0 |\varepsilon_0|}$ and come back to (B.40). If $\delta$ is small enough—depending on $p$—one can assume that actually $2 - \sqrt{1 - 4\delta_0 Q^+} < p < 2 + \sqrt{1 - 4\delta_0 Q^+}$.

Now applying Cauchy-Schwarz we find that

$$\int_{\{u' \leq u\}} r^{p-2} |u'|^{s} |\psi||Dv\psi| du'dv \leq \left( \int_{\{u' \leq u\}} r^{p-4} |u|^{s+1} |\psi|^2 du'dv \right)^\frac{1}{2} \left( \int_{\{u' \leq u\}} r^{p} |u'|^{s-1} |Dv\psi|^2 du'dv \right)^\frac{1}{2}.$$ \hfill \Box

Then we prove a Hardy inequality, very similar to (2.20) under the form : for all $u' \leq u$:

$$\int_{v_0(u')}^{+\infty} r^{p-4} |u'|^{s+1} |\psi|^2 du'dv \leq \frac{2}{(3-p)^2 \Omega^4(R)} (r_0(u'))^{p-3} |\psi_0|^2(u',v_0(u')) + \frac{4}{(3-p)^2 \Omega^4(R)} \int_{v_0(u')}^{+\infty} r^{p-2} |Dv\psi|^2(u',v)dv.$$  

Using Fubini’s theorem, one can now prove that for some $D' > 0$:

$$\int_{\{u' \leq u\}} r^{p-4} |u'|^{s+1} |\psi|^2 du'dv \leq D' \cdot \left[ \mathcal{E}_{p+\varepsilon} + \int_{\{u' \leq u\}} r^{p-2} |u'|^{s+1} |Dv\psi|^2 du'dv \right],$$

where we controlled the integral of $|u'|^{s+1}(r_0(u'))^{p-3} |\psi_0|^2(u',v_0(u'))$ by the zero order term of $\mathcal{E}_{p+\varepsilon}$.

Combining with (B.41) for $p' = p - 1$ and $s' = s + 1$ we finally get

$$\int_{\{u' \leq u\}} r^{p-4} |u'|^{s+1} |\psi|^2 du'dv \leq D' \cdot (1 + D) \cdot \mathcal{E}_{p+\varepsilon}. \tag{B.42}$$

Now denoting $D'' = 2Q^+ \cdot (1 + \eta) \cdot \sqrt{D'' \cdot (1 + D)}$ and coming back to (B.40) we get :

$$\int_{\{u' \leq u\}} \left[ p^{p-1} |u|^{s} + sr^{p}|u'|^{s'-1} \right] |Dv\psi|^2 du'dv + |u|^s \int_{v_0(u)}^{+\infty} r^p |Dv\psi|^2(u,v)dv \leq D'' \cdot \mathcal{E}_{p+\varepsilon} \cdot \left( \int_{\{u' \leq u\}} r^{p} |u'|^{s-1} |Dv\psi|^2 du'dv \right)^\frac{1}{2}. \tag{B.43}$$

Once we reach this step, the proof is over if $s > 0$ : it suffices to absorb the last term on the right-hand-side into the first term of the left-hand-side, using the inequality $|ab| \leq \varepsilon a + \frac{1}{\varepsilon} b$ for small enough $\varepsilon > 0$.

This proves that $E_p[v](u) = \int_{v_0(u)}^{+\infty} r^p |Dv\psi|^2(u,v)dv < +\infty$ if $\mathcal{E}_{p+\varepsilon} < +\infty$ which concludes the proof of the proposition.

---

69Since we just need a finiteness statement, the parameters on which $D$ depends do not matter so we do not specify them.

70Since we just need a finiteness statement, the parameters on which $D'$ depends do not matter so we do not specify them.
C Useful computations for the vector field method

This appendix carries out explicitly a few computations to apply the vector field method to the case of Maxwell-Charged-Scalar-Field.

For the linear wave equation, the traditional vector field method proceeds as follows: we can construct a quadratic quantity $T_{\mu \nu}^{\text{Wave}}(\phi) = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (g^{\alpha \beta} \partial_\alpha \phi \partial_\beta \phi) g_{\mu \nu}$. We then see that the wave equation corresponds to a conservation law:

$$\nabla^\mu T_{\mu \nu}^{\text{Wave}} = 0 \iff \Box \phi = 0.$$  

Then for any solution of the wave equation $\phi$ and for a well chosen vector field $X$ we construct the current $J_\nu^X = T_{\mu \nu}^{\text{Wave}}(\phi) X^\nu$ and we integrate $\nabla^\mu J^X_\nu = T_{\mu \nu}^{\text{Wave}}(\nu(X)X^\nu)$ on a space-time domain.

Making use of the divergence theorem, we see that a bulk term, namely the integral on a space-time domain of $T_{\mu \nu}^{\text{Wave}}(\nu(X)X^\nu)$, equals some boundary terms involving the current $J^X_\nu$.

Notice that if $X$ is a Killing vector field, then $\nabla^\nu (X^\nu) = 0$ and the identity only includes boundary terms.

Compared to the classical vector field method, a major difference — in the case we consider in this paper — is the presence of a Maxwell stress-energy tensor that is coupled to the scalar field’s. This still gives rise of a conservation law that couples the scalar field and the charge. While compactly supported scalar fields on asymptotically flat space-time decay, this is not the case for charges on black hole space-times.

Therefore, in many cases we do not apply this conservation law directly, except in subsection 5.3. Instead, we treat the Maxwell term as an error term, that can be controlled by the energy of the scalar field. To put it otherwise, instead of looking at charges —that do not decay— we look at the fluctuation of these charges, that do enjoy time decay estimates. To see the main related computations, c.f. subsection C.3.

C.1 Stress-Energy momentum tensor

For a spherically symmetric scalar field $\phi$ and 2-form $F$ we define the stress energy momentum tensor of the scalar field $T^{SF}_{\mu \nu}(\phi)$ and the one of the Maxwell field $T^{EM}_{\mu \nu}(F)$. Notice that the Maxwell-Charged-Scalar-Field equations (2.3), (2.4) imply the following conservation law:

$$\nabla^\mu (T^{SF}_{\mu \nu} + T^{EM}_{\mu \nu}) = 0,$$

where $T^{SF}_{\mu \nu}(\phi)$ and $T^{EM}_{\mu \nu}(F)$ are defined as:

$$T^{SF}_{\mu \nu} = \Re(D_\mu \phi D_\nu \phi) - \frac{1}{2} (g^{\alpha \beta} D_\alpha \phi D_\beta \phi) g_{\mu \nu},$$

$$T^{EM}_{\mu \nu} = g^{\alpha \beta} F_{\alpha \mu} F_{\beta \nu} - \frac{1}{4} F^{\alpha \beta} F_{\mu \nu} g_{\alpha \beta}.$$

In the $(u, v, \theta, \varphi)$ coordinate system of section 2.1 this gives:

$$T^{SF}_{uv} = |D_u \phi|^2,$$  (C.1)

$$T^{SF}_{uu} = |D_v \phi|^2,$$  (C.2)

$$T^{SF}_{\theta \theta} = \frac{T^{SF}_{\varphi \varphi} \sin^{-2}(\theta)}{r^2} = \frac{r^2 \Re(D_u \phi D_v \phi)}{2 \Omega^2},$$  (C.3)

$$T^{EM}_{uu} = \frac{2 \Omega^2 Q^2}{r^4},$$  (C.4)

$$T^{EM}_{\varphi \varphi} = \frac{Q^2}{r^2},$$  (C.5)

$$T^{SF}_{uv} = T^{SF}_{vu} = T^{EM}_{uv} = T^{EM}_{vu} = 0.$$  (C.6)

While (C.1), (C.2), (C.3) are used everywhere in the paper, particularly in subsections 5.1 and 5.2, (C.4) is only useful in subsection 5.3 while (C.5) is not used.

Notice that in section 6 we do not make use of the divergence theorem but directly of equations (2.6), (2.7). Hence this appendix is mainly useful for section 6.

C.2 Deformation tensors

As we saw in the beginning of this section, to use the divergence theorem we need to compute the derivative of vector fields.

More precisely for any vector field $X$ we define the deformation tensor as

$$\Pi^\nu_X := \nabla(\nu^\nu X^\nu).$$

In the $(u, v, \theta, \varphi)$ coordinate system of section 2.1 we compute:

$$\Pi^\nu_X := \nabla(\nu^\nu X^\nu).$$
As seen in the former section, the use of a modified current involves a

\[
\Pi_X^v = -\frac{2}{\Omega^2} \partial_u X^v, \quad (C.7)
\]

\[
\Pi_X^u = -\frac{2}{\Omega^2} \partial_v X^u, \quad (C.8)
\]

\[
\Pi_X^\theta = -\frac{1}{\Omega^2} \left( \partial_u X^v + \partial_v X^u + \partial_u \log(\Omega^2)X^v + \partial_v \log(\Omega^2)X^u \right), \quad (C.9)
\]

\[
\Pi_X^{\phi} = \Pi_X^\phi \sin^2(\theta) = \frac{1}{r^2} \left( \frac{\partial_u r}{r} X^v + \frac{\partial_v r}{r} X^u \right). \quad (C.10)
\]

Notice that for \( \partial_t = \frac{\partial_u + \partial_v}{2} \),

\[
\Pi_\theta^\phi = 0.
\]

This is because \( \partial_t \) is a Killing vector field, corresponding to the \( t \) invariance of the Reissner–Nordström metric.

### C.3 Computations of bulk terms in the divergence formula

Let \( (\phi, F) \) be a solution to the Maxwell-Charged-Scalar-Field equations \([2.7], [2.14]\), and we consider a vector field \( X \).

Even though the traditional method considers the current \( J_\mu^X = T_{\mu\nu}^S F(\phi)X^\nu \), it is sometimes useful to create a modified current, c.f. \([13], [14]\).

For this we introduce a real-valued scalar function \( \chi \) and define the modified energy current:

\[
j_\mu^X(\phi, \chi) := T_{\mu\nu}^S X^\nu + \chi \cdot \partial_\mu |\phi|^2 - \partial_\mu \chi \cdot |\phi|^2.
\]

Thus we can compute its divergence :

\[
\nabla^\mu j_\mu^X(\phi, \chi) := T_{\mu\nu}^S X^\nu + F_{\mu\nu} X^\mu j^\nu(\phi) + \chi D^\mu \phi D_\mu \phi - \frac{\Box \chi}{2} |\phi|^2,
\]

where the particular current \( j_\mu(\phi) \) is defined by

\[
j_\mu(\phi) = \delta_{\mu}(\phi, D_\mu \phi).
\]

For this computation, we used the fact that \( \nabla^\mu T_{\mu\nu}^S = -\nabla^\mu T_{\mu\nu}^{EM} = F_{\mu\nu} j^\nu(\phi) \), where the last identity makes use of the Maxwell equation \([2.3]\).

In order to compute this term in \((u, v)\) coordinates, recall that \( F_{uv} = \frac{\alpha^2 q}{r^2} \). Then we can establish the following expression for the interaction term :

\[
F_{\mu\nu} X^\mu j^\nu(\phi) = \frac{Q}{r^2} \left[ X^v j_u(\phi) - X^u j_v(\phi) \right]. \quad (C.12)
\]

### C.4 D’Alembertians

As seen in the former section, the use of a modified current involves a \( \Box \chi \) term, multiplying the 0 order term \( |\phi|^2 \).

Notice that the control of this 0 order term is one of the difficulties of this paper.

We compute the expression of the \( \Box \) operator in \((u, v)\) coordinates, for a spherically symmetric \( \chi \) :

\[
\Box(\chi) = -\frac{1}{\Omega^2} \left( \partial_u \partial_v \chi + \frac{\partial_u r}{r} \partial_v \chi + \frac{\partial_v r}{r} \partial_u \chi \right) = -\frac{\partial_u \partial_v \chi}{\Omega^2} + \frac{\partial_u \chi}{r} - \frac{\partial_v \chi}{r}. \quad (C.13)
\]

### C.5 Volume forms, normals and current fluxes

To conclude this appendix, we include a few computations of space-time volume forms and volume forms induced on the curves we use in this paper, together with exterior unit normals.

Note that as far as null surfaces are concerned, there no canonical notion of exterior unit normals or induced volume forms. However, the contraction of an induced volume form with its unit normal does not depend on the normalization choice.

The volume form corresponding to Reissner–Nordström metric is defined by

\[
dvol = 2\Omega^2 r^2 dudvd\sigma_{g^2},
\]

where \( d\sigma_{g^2} \) is the standard volume form on the unit sphere.

We now give consecutively the normals, the induced volumes forms and the current flux for each of the following : constant \( u \) hyper-surfaces, constant \( v \) hyper-surfaces, constant \( r \) hyper-surfaces \( \gamma_H \) and \( \Sigma_\phi = \{t = 0\} \).
For these constant $u$ and constant $v$ hyper-surfaces, we expressed the future-directed normal. Notice that if such a null hyper-surface appears as the future boundary of a space-time domain, than the future-directed is also the exterior normal.

Symmetrically, if such a null hyper-surface appears as the past boundary of a space-time domain, than the exterior normal is the past-directed normal, i.e. the opposite of the future-directed one that we wrote.

Then we carry out the same computations for $\gamma_{R_1} = \{ r = R_1 \}$, for any $r_+ < R_1$:

$$n^u_{\gamma_{R_1}} = \frac{1}{2\Omega(R_1)}(\partial_u - \partial_\sigma),$$

$$dvol_{\gamma_{R_1}} = \Omega(R_1)(dv + du)r^2d\sigma_{S^2},$$

$$J_\mu n^\mu_{\gamma_{R_1}}dvol_{\gamma_{R_1}} = \frac{J_u - J_v}{2}(dv + du)r^2d\sigma_{S^2}.$$  

Notice that if $\gamma_{R_1}$ appears as the right-most boundary of a space-time domain, e.g. for $\{ r \leq R_1 \}$, than the exterior normal is the one we wrote.

Symmetrically, if $\gamma_{R_1}$ appears as the left-most boundary of a space-time domain, e.g. for $\{ r \geq R_1 \}$, than the exterior normal is the opposite of the one we wrote.

Finally we write the same computations for $\Sigma_0 = \{ t = 0 \}$:

$$n^u_{\Sigma_0} = \frac{1}{2\Omega}(\partial_u + \partial_\sigma),$$

$$dvol_{\Sigma_0} = \Omega \cdot (dv - du)r^2d\sigma_{S^2},$$

$$J_\mu n^\mu_{\Sigma_0}dvol_{\Sigma_0} = \frac{J_u + J_v}{2}(dv - du)r^2d\sigma_{S^2}.$$  

We wrote the future directed normal, which is in fact the opposite of the exterior unit normal, because the space-time is included inside $\{ t \geq 0 \}$.

References

[1] Yannis Angelopoulos, Stefanos Aretakis, Dejan Gajic A vector field approach to almost-sharp decay for the wave equation on spherically symmetric, stationary spacetimes. Preprint, 2016. [arXiv:1612.01565]

[2] Yannis Angelopoulos, Stefanos Aretakis, Dejan Gajic Late-time asymptotics for the wave equation on spherically symmetric, stationary spacetimes. Advances in Mathematics, Vol. 323, 529–621, 2018.

[3] Lydia Bieri, Shuang Miao, and Sohrab Shahshahani Asymptotic properties of solutions of the Maxwell Klein Gordon equation with small data. Communications in Analysis and Geometry 25(1), 2014.

[4] Demetrios Christodoulou, The Formation of Black Holes and Singularities in Spherically Symmetric Gravitational Collapse. Comm. Pure Appl. Math. 44 (1991), no. 3, 339–373.

[5] Demetrios Christodoulou, Bounded Variation Solutions of the Spherically Symmetric Einstein-Scalar Field Equations. Comm. Pure Appl. Math. 46 (1993), 1131–1220.

[6] Demetrios Christodoulou, The instability of naked singularities in the gravitational collapse of a scalar field. Ann. of Math. (2) 149 (1999), no. 1, 183–217.

[7] Piotr T. Chruściel, Jalal Shatah Global existence of solutions of the Yang-Mills equations on globally hyperbolic four dimensional Lorentzian manifolds. Asian journal of Mathematics, Vol.1, No.3,pp. 530-548, 1997.

[8] Mihalis Dafermos, Stability and instability of the Cauchy horizon for the spherically symmetric Einstein-Maxwell-scalar field equations. Ann. of Math. 158, 875–928, 2003.

[9] Mihalis Dafermos, The Interior of Charged Black Holes and the Problem of Uniqueness in General Relativity. Comm. Pure Appl. Math. 58, 0445–0504, 2005.

[10] Mihalis Dafermos, Gustav Holzegel, Igor Rodnianski The linear stability of the Schwarzschild solution to gravitational perturbations. Preprint, 2017. [arXiv:1601.06467]
[11] Mihalis Dafermos, Jonathan Luk, The interior of dynamical vacuum black holes I: The $C^0$-stability of the Kerr Cauchy horizon. Preprint, 2017. [arXiv:1710.01772]

[12] Mihalis Dafermos, Igor Rodnianski A proof of Price’s law for the collapse of a self-gravitating scalar field. Invent. math. 162, 381–457, 2005.

[13] Mihalis Dafermos, Igor Rodnianski , Lectures on black holes and linear waves. 2008.

[14] Mihalis Dafermos, Igor Rodnianski The red-shift effect and radiation decay on black hole spacetimes. Comm. Pure Appl. Math., 62(7):539–591, 2009.

[15] Mihalis Dafermos, Igor Rodnianski , A new physical-space approach to decay for the wave equation with applications to black hole spacetimes. pp. 421–432 in XVIth International Congress on Mathematical Physics, edited by P. Exner, World Sci. Publ., Hackensack, NJ, 2010.

[16] Mihalis Dafermos, Igor Rodnianski A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds. Invent. Math., 185, 2011.

[17] Mihalis Dafermos, Igor Rodnianski, Yakov Shlapentokh-Rothman, Decay for solutions to the wave equation on Kerr exterior spacetimes III: the full sub-extremal case $|a| < M$. Ann. of Math., 183, 787–913, 2016.

[18] Roland Donninger, Wilhelm Schlag, Avy Soffer A proof of Price’s Law on Schwarzschild black hole manifolds for all angular momenta. Advances in Mathematics Volume 226, Issue 1, Pages 484–540, 2011.

[19] Douglas M. Eardley, Vincent Moncrief, The global existence of Yang–Mills–Higgs fields in 4- dimensional Minkowski space, I: Local existence and smoothness properties. Comm. Math. Phys., 83:2, 171–191, 1982.

[20] Douglas M. Eardley, Vincent Moncrief, The global existence of Yang–Mills–Higgs fields in 4- dimensional Minkowski space, II: Completion of proof. Comm. Math. Phys., 83:2, 193–212, 1982.

[21] Grigorios Fournodavlos, Jan Sbierski, Generic blow-up results for the wave equation in the interior of a Schwarzschild black hole. Preprint, 2018.

[22] Dejan Gajic, Jonathan Luk The interior of dynamical extremal black holes in spherical symmetry. Preprint, 2017. [arXiv:1709.09137]

[23] Peter Hintz, András Vasy The global non-linear stability of the Kerr-de Sitter family of black holes. Preprint, 2016. [arXiv:1606.04014]

[24] Shahar Hod, Tsvi Piran Late-Time Evolution of Charged Gravitational Collapse and Decay of Charged Scalar Hair- I. Phys.Rev D58 024017, 1998.

[25] Shahar Hod, Strong cosmic censorship in charged black-hole spacetimes: As strong as ever. Preprint, 2018. arXiv 1801.07261

[26] Christopher Kauffman Global Stability for Charged Scalar Fields in an Asymptotically Flat Metric in Harmonic Gauge. Preprint, 2018. [arXiv:1801.09648]

[27] Markus Keel, Tristan Roy, Terence Tao Global well-posedness of the Maxwell–Klein–Gordon equation below the energy norm. Discrete Contin. Dyn. Syst. 30:3, 573–621, 2011.

[28] Sergiu Klainerman, Matei Machedon On the Maxwell–Klein–Gordon equation with finite energy. Duke Math. J. 74:1 , 19–44, 1994.

[29] Sergiu Klainerman, Jérémie Szeftel Global Nonlinear Stability of Schwarzschild Spacetime under Polarized Perturbations. Preprint, 2017. [arXiv:1711.07597]

[30] Jonathan Kommemi, The Global Structure of Spherically Symmetric Charged Scalar Field Spacetimes. Comm. Math. Phys., 323: 35. doi:10.1007/s00220-013-1759-1, 2013.

[31] Joachim Krieger, Jonas Lührmann, Concentration compactness for the critical Maxwell–Klein–Gordon equation. Annals of PDE 1:1, Art. 5, 208, 2015.

[32] Joachim Krieger, Jacob Sterbenz, Daniel Tataru, Global well-posedness for the Maxwell–Klein–Gordon equation in $(+1)$ dimensions: small energy. Duke Math. J. 164:6, 973–1040, 2015.

[33] Hans Lindblad, Jacob Sterbenz Global stability for charged-scalar fields on Minkowski space. IMRP Int. Math. Res. Pap., Art. ID 52976, 109, 2006.

[34] Jonathan Luk, Sung-Jin Oh, Quantitative decay rates for dispersive solutions to the Einstein-scalar field system in spherical symmetry. Analysis and PDE, 8(7):1603:1674, 2015.

[35] Jonathan Luk, Sung-Jin Oh Proof of Linear Instability of the Reissner–Nordström Cauchy Horizon under Scalar Perturbations. Duke Math. J. Volume 166, Number 3, 437–493, 2017.

[36] Jonathan Luk, Sung-Jin Oh, Strong Cosmic Censorship in Spherical Symmetry for two-ended Asymptotically Flat Initial Data I. The Interior of the Black Hole Region. Preprint, 2017.

[37] Jonathan Luk, Sung-Jin Oh, Strong Cosmic Censorship in Spherical Symmetry for two-ended Asymptotically Flat Initial Data II. The Exterior of the Black Hole Region. Preprint, 2017.
[38] Jonathan Luk, Jan Sbierski, Instability results for the wave equation in the interior of Kerr black holes. J. Funct. Anal., 271(7):1948-1995, 2016.

[39] Matei Machedon, Jacob Sterbenz Almost optimal local well-posedness for the (3+1)-dimensional Maxwell–Klein–Gordon equations. Journal of the American Mathematical Society 17:2, 297–359, 2004.

[40] Jason Metcalfe, Daniel Tataru, Mihai Tohaneanu Price's Law on Nonstationary Spacetimes. Adv. Math. 230, no. 3, 995-1028, 2012.

[41] Cathleen Synge Morawetz Time decay for the nonlinear Klein-Gordon equations. Proc. Roy. Soc. Ser. A, 306:291:296, 1968.

[42] Georgios Moschidis The $r^n$-weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications. Annals of PDE 2:6 , 1-194, 2016.

[43] Sung-Jin Oh, Daniel Tataru Global well-posedness and scattering of the (4+1)-dimensional Maxwell–Klein–Gordon equation. Invent. Math. 205:3 , 781–877, 2016.

[44] Richard Price Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations Phys. Rev. D (3) 5, 2419-2438, 1972.

[45] Igor Rodnianski, Terence Tao Global regularity for the Maxwell–Klein–Gordon equation with small critical Sobolev norm in high dimensions . Comm. Math. Phys., 251:2, 377–426, 2004.

[46] Volker Schlue Decay of linear waves on higher dimensional Schwarzschild black holes . Analysis and PDE 6:3, 515–600, 2013.

[47] Wei-Tong Shu Asymptotic properties of the solutions of linear and nonlinear spin field equations in Minkowski space . Comm. Math. Phys. 140:3, 449–480, 1991.

[48] Daniel Tataru Local decay of waves on asymptotically flat stationary space-times . Amer. J. Math. 130, no. 3, 571–634, 2008.

[49] Shiwu Yang Decay of solutions of Maxwell-Klein-Gordon equations with arbitrary Maxwell fields. Analysis and PDE, Vol. 9, No. 8, 2016.

[50] Shiwu Yang, Pin Yu On global dynamics of the Maxwell-Klein-Gordon equations. Preprint, 2018. arXiv 1804.00078.

[51] Maxime Van de Moortel Stability and instability of the sub-extremal Reissner–Nordström black hole interior for the Einstein–Maxwell-Klein-Gordon equations in spherical symmetry. Comm. Math. Phys. , to appear, 2018.