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Wee Teck Gan and Gordan Savin

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An exceptional Siegel–Weil formula and poles of the Spin $L$-function of $\text{PGSp}_6$

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Abstract

We show a Siegel–Weil formula in the setting of exceptional theta correspondence. Using this, together with a new Rankin–Selberg integral for the Spin $L$-function of $\text{PGSp}_6$ discovered by Pollack, we prove that a cuspidal representation of $\text{PGSp}_6$ is a (weak) functorial lift from the exceptional group $G_2$ if its (partial) Spin $L$-function has a pole at $s = 1$.

1. Introduction

Let $F$ be a totally real number field, and $\mathbb{A}$ its ring of adèles. Let $\pi \simeq \bigotimes_v \pi_v$ be an irreducible cuspidal automorphic representation of the group $\text{PGSp}_6(\mathbb{A})$, which is unramified outside a finite set $S$ of places (including all real places). Since the Langlands dual group of $\text{PGSp}_6$ is $\text{Spin}_7(\mathbb{C})$, there is an associated semisimple conjugacy class $s_v$ in $\text{Spin}_7(\mathbb{C})$ for $v \notin S$; this is the Satake parameter of the local component $\pi_v$. If $r$ denotes the eight-dimensional spin representation of $\text{Spin}_7(\mathbb{C})$, the partial Spin $L$-function corresponding to $\pi$ is defined to be the product

$$L^S(s, \pi, \text{Spin}) = \prod_{v \notin S} \frac{1}{\det(1 - r(s_v)q_v^{-s})},$$

where $q_v$ is the order of the residual field of the local field $F_v$.

It is well known that the stabilizer in $\text{Spin}_7(\mathbb{C})$ of a generic vector in the spin representation is the exceptional group $G_2(\mathbb{C})$, giving a well-defined conjugacy class of embeddings

$$\iota : G_2(\mathbb{C}) \rightarrow \text{Spin}_7(\mathbb{C}).$$

Therefore, as a special case of the Langlands functoriality principle, if $L^S(s, \pi, \text{Spin})$ has a simple pole at $s = 1$, then one expects $\pi$ to be a functorial lift from an exceptional group of absolute type $G_2$ defined over $F$. We note that every such group is given as the automorphism group of an octonion algebra $\mathbb{O}$ over $F$, and by the Hasse principle, the number of isomorphism classes of such groups is $2^n$, where $n$ is the number of real places of $F$.

As explained in a recent paper of Chenevier [Che19, §6.12], if $\pi$ is a tempered cuspidal representation of $\text{PGSp}_6$ such that for almost all places $v$, the Satake parameter $s_v$ of $\pi_v$ belongs to $\iota(G_2(\mathbb{C}))$ (or more accurately, the conjugacy class $s_v$ meets $\iota(G_2(\mathbb{C}))$), then $L^S(s, \pi, \text{Spin})$ will

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have a pole at \( s = 1 \) and so one expects such a tempered \( \pi \) to be a functorial lift from \( G_2 \). In this paper we also prove a slightly weaker version of this expectation.

**Theorem 1.1.** In the above setting, suppose that \( \pi \) is a cuspidal automorphic representation of \( \text{PGSp}_6 \) such that \( L^S(s, \pi, \text{Spin}) \) has a pole at \( s = 1 \). Then there exist an octonion algebra \( \mathbb{O} \) over \( F \) and a cuspidal automorphic representation \( \pi' \) of \( \text{Aut}(\mathbb{O}) \) such that the Satake parameters of \( \pi' \) are mapped by \( \iota \) to those of \( \pi \) (i.e. \( \pi \) is a weak functorial lift of \( \pi' \)).

If the cuspidal representation \( \pi \) of \( \text{PGSp}_6 \) is tempered, then the following statements are equivalent:

(a) for almost all places \( v \), the Satake parameter \( s_v \) of \( \pi_v \) is contained in \( \iota(G_2(\mathbb{C})) \);
(b) there exist an octonion algebra \( \mathbb{O} \) over \( F \) and a cuspidal automorphic representation \( \pi' \) of \( \text{Aut}(\mathbb{O}) \) such that \( \pi \) is a weak functorial lift of \( \pi' \).

Since the local Langlands classification is not known for \( G_2 \) or for \( \text{PGSp}_6 \), this is essentially the best possible result one can expect at the moment. However, if \( \pi \) is unramified everywhere or if it corresponds to a classical Siegel modular form of level one, then \( \pi \) is a functorial lift. Special cases of this result were previously obtained by Ginzburg and Jiang [GJ01], Gan and Gurevich [GG09] and Pollack and Shah [PS18].

Our proof of Theorem 1.1 is based on the following three ingredients:

(1) an exceptional theta correspondence for the dual pair \( \text{Aut}(\mathbb{O}) \times \text{PGSp}_6 \) arising from the minimal representation \( \Pi \) of a group of absolute type \( E_7 \) [GS05, HKM14, Sah92];
(2) a Siegel–Weil formula proved in this paper (see Theorem 1.2 below);
(3) an integral representation of the Spin \( L \)-function of \( \pi \) recently discovered by Pollack [Pol17].

In greater detail, let \( J \) be the exceptional Jordan algebra of \( 3 \times 3 \) hermitian symmetric matrices with coefficients in an octonion algebra \( \mathbb{O} \). By the Koecher–Tits construction, the algebra \( J \) gives rise to an adjoint group \( G \) of absolute type \( E_7 \), with a maximal parabolic subgroup \( P = MN \), such that the unipotent radical \( N \) is commutative and isomorphic to \( J \). Since \( G \) is adjoint, the conjugation action of \( M \) on \( N \) is faithful, and \( M \) is isomorphic to the similitude group of the natural cubic norm form on \( J \). Thus the natural action of \( \text{Aut}(\mathbb{O}) \) on \( J \) gives an embedding of \( \text{Aut}(\mathbb{O}) \) into \( M \). The centralizer of \( \text{Aut}(\mathbb{O}) \) is \( \text{PGSp}_6 \). To see this, observe that the centralizer of \( \text{Aut}(\mathbb{O}) \) in \( J \) is the Jordan subalgebra \( J_F \) of \( 3 \times 3 \) symmetric matrices with coefficients in \( F \). The group \( \text{PGSp}_6 \) arises from \( J_F \) by the Koecher–Tits construction. This gives the dual pair

\[
\text{Aut}(\mathbb{O}) \times \text{PGSp}_6 \subset G
\]

alluded to in item (1) above.

We can now describe another dual pair in \( G \). Let \( D \) be a quaternion algebra over \( F \), and assume that we have an embedding \( i : D \to \mathbb{O} \). The centralizer of \( D \) in \( \text{Aut}(\mathbb{O}) \) is isomorphic to \( D^1 \), the group of norm-1 elements in \( D \). Conversely, the centralizer (i.e. the pointwise stabilizer) of \( D^1 \) in \( \mathbb{O} \) is \( i(D) \subset \mathbb{O} \). Thus the centralizer of \( D^1 \) in \( J \) is the Jordan subalgebra \( J_D \) of \( 3 \times 3 \) hermitian symmetric matrices with coefficients in \( D \), and the centralizer of \( D^1 \) in \( G \) is a group \( G_D \) of absolute type \( D_6 \) arising from \( J_D \) by the Koecher–Tits construction. Thus we have a dual pair

\[
D^1 \times G_D \to G.
\]

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Indeed, the two dual pairs we have described fit into the following seesaw diagram, where the vertical lines represent inclusions of groups:

\[
\begin{array}{ccc}
\text{Aut(}\mathcal{O}\text{)} & \rightarrow & G_D \\
\downarrow & & \downarrow \\
D^1 & \rightarrow & \text{PGSp}_6
\end{array}
\]

The Siegel–Weil formula mentioned in item (2) above concerns the global theta lift \(\Theta(1)\) of the trivial representation of \(D^1\) to \(G_D\), obtained by restricting the minimal representation \(\Pi\) of \(G_D\) to the dual pair \(D^1 \times G_D\). Roughly speaking, \(\Theta(1)\) is the space of automorphic functions on \(G_D\) obtained by averaging the functions in \(\Pi\) over \(D^1(F) \backslash D^1(\mathbb{A})\). We prove that \(\Theta(1)\) is an irreducible automorphic representation of \(G_D\) and determine its local components (as abstract representations) by computing the corresponding local theta lifts. We have not computed the local theta lift for complex groups, and this is the source of the restriction in the paper to totally real fields \(F\). The Siegel–Weil formula identifies the functions in \(\Theta(1)\) as residues of certain Siegel–Eisenstein series.

More precisely, since \(G_D\) arises from \(J_D\) by the Koecher–Tits construction, it contains a maximal parabolic subgroup with abelian unipotent radical isomorphic to \(J_D\). Let \(E_D(s, f)\) be the degenerate Eisenstein series attached to this maximal parabolic subgroup, where \(s \in \mathbb{R}\) and \(f\) varies over all standard sections of the corresponding degenerate principal series representation \(I_D(s)\). In [HS20], it was proved that \(E_D(s, f)\) has at most a simple pole at \(s = 1\), and the residual representation

\[
\mathcal{E}_D := \{\text{Res}_{s=1} E_D(s, f) : f \in I_D(s)\}
\]

was completely determined. Our main result is the following Siegel–Weil identity in the space of automorphic forms of \(G_D\).

**Theorem 1.2.** For fixed quaternion \(F\)-algebra \(D\), we have:

\[
\mathcal{E}_D = \bigoplus_{i:D\rightarrow \mathcal{O}} \Theta(1).
\]

Here the sum is taken over all isomorphism classes of embeddings \(i:D \rightarrow \mathcal{O}\) into octonion algebras over \(F\).

We emphasize that \(D\) is fixed here but \(\mathcal{O}\) varies. If \(D\) is split (i.e. a matrix algebra) then \(\mathcal{O}\) is also split, and there is only one term on the right. In general, the number of summands on the right is equal to \(2^m\), where \(m\) is the number of real places \(v\) of \(F\) such that \(D_v\) is a division algebra.

At this point, we need the result of Pollack [Pol17]: there exists a quaternion algebra \(D\) such that the partial spin-\(L\)-function \(L^S(\pi, s, \text{Spin})\) is given as an integral, over \(\text{PGSp}_6\), of a function \(h \in \pi\) against the Eisenstein series \(E_D(s, f)\). Thus, if the \(L\)-function has a pole at \(s = 1\), then the integral of \(h\) against the elements of \(\mathcal{E}_D\) is non-zero. The Siegel–Weil identity (i.e. Theorem 1.2) then implies that \(\pi\) appears in the exceptional theta correspondence for the dual pair \(\text{Aut}(\mathcal{O}) \times \text{PGSp}_6\), for some \(\mathcal{O}\) containing \(D\). Since this exceptional theta correspondence is known to be functorial for spherical representations (see [LS19] and [SW15]), this completes the proof that \(\pi\) is a weak lift from a group of absolute type \(G_2\).
2. Groups

2.1 Octonion algebra
Let $F$ be a field of characteristic 0, and $D$ be a quaternion algebra over $F$. It is a four-dimensional associative and non-commutative algebra over $F$ which comes equipped with a conjugation map $x \mapsto \overline{x}$ with associated norm $N(x) = xx = x\overline{x}$ and trace $\text{tr}(x) = x + \overline{x}$. Moreover, $N : D \to F$ is a non-degenerate quadratic form.

An octonion algebra $\mathcal{O}$ over $F$ is obtained by doubling the quaternion algebra $D$. More precisely, fix a non-zero element $\lambda$ in $F$. As a vector space over $F$, $\mathcal{O}$ is a set of pairs $(a, b)$ of elements in $D$. The multiplication is defined by the formula

$$(a, b) \cdot (c, d) = (ac + \lambda bd, ad + cb).$$

If $x = (a, b)$, then the conjugation map is $\overline{x} = (a, -b)$, so that $N(x) = x \cdot \overline{x} = N(a) - \lambda N(b)$ is the norm and $\text{tr}(x) = x + \overline{x} = \text{tr}(a)$ the trace on $\mathcal{O}$. In particular, $\mathcal{O}$ is split if $\lambda$ is a norm of an element in $D$. Every element $x$ of $\mathcal{O}$ satisfies its characteristic polynomial $t^2 - \text{tr}(x)t + N(x)$.

The automorphism group $\text{Aut}(\mathcal{O})$ of the $\mathcal{F}$-algebra $\mathcal{O}$ is an exceptional group of the Lie type $G_2$. It is a simple linear algebraic group of rank 2 which is both simply connected and adjoint. The algebra $D$ is naturally a subalgebra of $\mathcal{O}$, consisting of all $x = (a, 0)$. Let $D^1$ be the group of norm-1 elements in $D$. Then any $g \in D^1$ acts as an automorphism of $\mathcal{O}$ by $g \cdot (a, b) = (a, bg)$ for all $(a, b) \in \mathcal{O}$. The subgroup $D^1 \subset \text{Aut}(\mathcal{O})$ is precisely the pointwise stabilizer of the subalgebra $D \subset \mathcal{O}$.

2.2 Albert algebra
An Albert algebra is an exceptional 27-dimensional Jordan algebra $J$ over $F$. It can be realized as the set of matrices

$$A = \begin{pmatrix} \alpha & x & \bar{z} \\ \bar{x} & \beta & y \\ z & \bar{y} & \gamma \end{pmatrix},$$

where $\alpha, \beta, \gamma \in F$ and $x, y, z \in \mathcal{O}$. The determinant $A \mapsto \text{det} A$ defines a natural cubic form on $J$. Let $M$ be the similitude group of this cubic form. It is a reductive group of semisimple type $E_6$. The $M$-orbits in $J$ are classified by the rank of the matrix $A$. Without going into a general definition of the rank, we say that $A \neq 0$ has rank 1 if $A^2 = \text{tr}(A) \cdot A$. Explicitly, this means that the entries of $A$ satisfy the equalities

$$N(x) = \alpha \beta, \quad N(y) = \beta \gamma, \quad N(z) = \gamma \alpha, \quad \gamma \bar{x} = yz, \quad \alpha \bar{y} = zx, \quad \beta \bar{z} = xy.$$

2.3 Dual pairs
Assume that $G$ is a reductive group over $F$, adjoint and of absolute type $E_7$, arising from the Albert algebra $J$ via the Koecher–Tits construction. For our purposes it will be more convenient to realize $G$ as a quotient, modulo one-dimensional center $C \cong F^\times$, of a reductive group $\tilde{G}$ acting on the 56-dimensional representation $W = F + J + J + F$. In particular, $G$ acts on the projective space $\mathbb{P}(W)$. Let $P$ be a maximal parabolic and $\overline{P}$ its opposite, defined as fixing the points $(1, 0, 0, 0)$ and $(0, 0, 0, 1)$ in $\mathbb{P}(W)$. Then $P = MN$, where $N$ is the unipotent radical and $M = P \cap \overline{P}$ a Levi subgroup. Then $M$ is isomorphic to the similitude group of the cubic form det on $J$, and $N \cong J$, as $M$-modules.

Recall that we have constructed $\mathcal{O}$ by doubling a quaternion subalgebra $D$. Let $J_F$ and $J_D$ be the subalgebras consisting of all elements in $J$ with off-diagonal entries in $F$ and $D$, respectively.
Let $J_0 = F$ be the scalar subalgebra of $J$. Consider a sequence of simple, simply connected groups

$$D^1 \subset \text{Aut}(\mathfrak{O}) \subset \text{Aut}(J),$$

where an element in $\text{Aut}(\mathfrak{O})$ acts on the off-diagonal entries of elements in $J$. The pointwise stabilizers in $J$ of these three groups are, respectively,

$$J_D \supset J_F \supset J_0 = F.$$

Observe that $\text{Aut}(J)$ naturally acts on $W$, giving an embedding $\text{Aut}(J) \subset \tilde{G}$. The centralizers in $\tilde{G}$ of the three groups in the sequence are, respectively,

$$\tilde{G}_D \supset \text{GSp}_6(F) \supset \text{GL}_2(F).$$

These three groups act on the 32-, 14- and four dimensional subspaces of $W$ obtained by replacing $J$ by $J_D$, $J_F$ and $J_0$, respectively. It is worth mentioning that the four-dimensional representation of $\text{GL}_2(F)$ is the symmetric cube of the standard two-dimensional representation, twisted by $\det^{-1}$. The group $\tilde{G}_D$ acts on

$$W_D = F + J_D + J_D + F.$$

It is worth noting that the action of $\tilde{G}_D$ on $W_D$ is not faithful (it has $\mu_2 \subset D^1$ as its kernel).

A detailed description of $\tilde{G}_D/\mu_2$ and its action on $W_D$ can be found in Pollack’s paper [Pol17]. Let $G_D$ be the quotient of $\tilde{G}_D$ by the center $C \cong F^\times$ of $\tilde{G}$. Then $D^1 \times G_D$ is a dual pair in $G$, as mentioned in the introduction.

Let $P_D = M_D N_D = G_D \cap P$. With the identification $N \cong J$ fixed, we have $N_D \cong J_D$. The group $P_D$ is a maximal parabolic subgroup of type $A_5$.

3. Minimal representation

Let $F$ be a real or $p$-adic field. Let $I(s)$ be the degenerate principal series representation of $G$ attached to $P$, where $s \in \mathbb{R}$. We normalize $s$ as in [Wei03] so that the trivial representation is a quotient and a submodule at $s = 9$ and $s = -9$ respectively, whereas the minimal representation $\Pi$ is a quotient and a submodule at $s = 5$ and $s = -5$, respectively. Note, however, that the group $G$ is simply connected in [Wei03], whereas our $G$ is adjoint here.

3.1 Unitary model

Fix $\psi : F \rightarrow \mathbb{C}^\times$, a non-trivial additive character, unitary if $F = \mathbb{R}$. After identifying $N \cong J$ and $\tilde{N} \cong J$ (note that the resulting actions of $M$ on $J$ are dual to each other), any $A \in J \cong \tilde{N}$ defines a character of $N$ given by

$$\psi_A(B) = \psi(\text{tr}(A \circ B)) = \psi_B(A)$$

for $B \in J \cong N$, where $A \circ B$ denotes the Jordan multiplication. Every unitary character of $N$ is equal to $\psi_A$ for some $A$. Let $\Omega \subseteq J \cong \tilde{N}$ be the set of rank-1 elements in $J$. A unitary model of the minimal representation is $\mathcal{H} = L^2(\Omega)$ [Sah92]. Here only the action of the maximal parabolic $P = MN$ is obvious: the group $M$ acts geometrically,

$$\pi(m)(f)(A) = \chi(m)f(m^{-1}A),$$

for $f \in \Pi$ and for some character $\chi : M \rightarrow \mathbb{R}^\times$, while $B \in J \cong N$ acts on $f$ by multiplying it by $\psi_B$. 

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3.2 Smooth model
We have the following theorem [KS15].

**Theorem 3.1.** Let $\Pi$ be the subspace of $G$-smooth vectors in the unitary minimal representation $\mathcal{H}$. Then

$$C^\infty_c(\Omega) \subset \Pi \subset C^\infty(\Omega).$$

If $F$ is $p$-adic, then

$$\Pi_N \cong \Pi/C^\infty_c(\Omega) \text{ as } M\text{-modules}.$$  

If $A \in J$ is non-zero, then any continuous functional $\ell$ on $\Pi$ such that $\ell(B \cdot f) = \psi_A(B) \cdot \ell(f)$ for all $B \in N$ and $f \in \Pi$ is equal to a multiple of the evaluation map $\delta_A(f) = f(A)$. In particular, $\ell = 0$ if $A$ is not of rank 1.

3.3 Spherical vector
It is not so easy to characterize the subspace $\Pi \subset C^\infty(\Omega)$. However, we can describe a spherical vector in $\Pi$ in the split case. The algebra $\mathcal{O}$ is obtained by doubling the matrix algebra $D = M_2(F)$ with $\lambda = 1$. Assume firstly that $F$ is a $p$-adic field. Let $\mathcal{O}$ be the ring of integers in $F$ and $\varpi$ a uniformizing element. We have an obvious integral structure on $D$ (the lattice of integral matrices), and hence on $\mathcal{O}$, the integral lattice being the set of pairs $(a, b) \in M_2(\mathcal{O})$. This lattice is a maximal order in $\mathcal{O}$. Now we have an integral structure on $J$ so that $J(\mathcal{O})$ is the set of elements $A \in J$ such that the diagonal entries are integral, and the off-diagonal entries are contained in the maximal order in $\mathcal{O}$. The greatest common divisor (GCD) of entries of $A \in J(\mathcal{O})$ is simply the largest power $\varpi^n$ dividing $A$, that is, such that $A/\varpi^n$ is in $J(\mathcal{O})$. We have the following theorem [SW07].

**Theorem 3.2.** Assume that $G$ is split and $F$ a $p$-adic field. Assume the conductor of $\psi$ is $O$. Then the spherical vector in $\Pi$ is a function $f^o \in C^\infty(\Omega)$ supported in $J(O)$. Its value at $A \in \Omega$ depends on the GCD of entries of $A$. More precisely, if the GCD of the entries of $A$ is $\varpi^n$, and $q$ is the order of the residual field, then

$$f^o(A) = 1 + q^3 + \cdots + q^{3n}.$$  

Since $\Pi$ is generated by $f^o$ as a $P$-module, and the action of $P$ on $\Pi$ is easy to describe, this theorem gives us a good handle on $\Pi$.

Assume now that $F = \mathbb{R}$; in this case, one has a similar result due to Dvorsky and Sahi [DS99]. For every $a \in M_2(\mathbb{R})$, let $\|a\|^2$ be the sum of squares of its entries. For $x = (a, b) \in \mathcal{O}$, let $\|x\|^2 = \|a\|^2 + \|b\|^2$. Extend this to $A \in J$ by

$$\|A\|^2 = \alpha^2 + \beta^2 + \gamma^2 + \|x\|^2 + \|y\|^2 + \|z\|^2.$$  

Let $K_{3/2}(u)$ denote the modified Bessel function of the second kind. Recall that $K_{3/2}(u)$ is greater than 0, for $u > 0$, and rapidly decreasing as $u \to +\infty$. Then we have the following result [DS99, Theorem 0.1].

**Theorem 3.3.** Assume that $G$ is split and $F = \mathbb{R}$. Then the spherical vector in $\Pi$ is a function $f^o \in C^\infty(\Omega)$ given by

$$f^o(A) = \|A\|^{-3/2}K_{3/2}(\|A\|).$$
In this section let $F$ be a $p$-adic field, so that the octonion algebra $\mathbb{O}$ is split. We are interested in understanding the theta lift of the trivial representation of $D^1$ to the group $G_D$.

4.1 $N_D$-spectrum

A crucial step is to understand the $N_D$-spectrum of the minimal representation $\Pi$. In this case we have an exact sequence of $P$-modules

$$0 \to C_c^\infty(\Omega) \to \Pi \to \Pi_{N_D} \to 0.$$ 

The characters of $N_D \cong J_D$ are identified with the elements in $J_D$ using the trace pairing, as we did for $J$. We shall only need three characters, denoted by $\psi_1$, $\psi_2$ and $\psi_3$, corresponding to the elements

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

of rank 1, 2 and 3, respectively. We need to allow signs to capture all possible orbits of rank 1, 2 and 3 in the real case. The following lemma is one of the key ingredients in this paper, and we emphasize that we do not assume that $D$ is split here.

**Lemma 4.1.** Let $\Pi$ be the minimal representation of $G$. Then:

(i) $\Pi_{N_D,\psi_3} = 0$;

(ii) $\Pi_{N_D,\psi_2} \cong C_c^\infty(D^1)$, as $D^1$-modules;

(iii) if $D$ is a division algebra, then $\Pi_{N_D,\psi_1} \cong \mathbb{C}$, as $D^1$-modules.

**Proof.** Let $\omega_i \subseteq \Omega$ be the set of all $A \in \Omega$ such that the restriction of $\psi_A$ to $N_D$ is equal to $\psi_i$. Because $\psi_i$ is not the trivial character, the set $\omega_i$ is (Zariski) closed in $\Omega$. Hence,

$$\Pi_{N_D,\psi_i} \cong C_c^\infty(\omega_i).$$

It remains to determine each $\omega_i$. Let us start with $i = 3$. Then $\omega_3$ consists of all $A \in \Omega$ such that

$$A = \begin{pmatrix} \pm 1 & x & -z \\ -x & \pm 1 & y \\ z & -y & \pm 1 \end{pmatrix},$$

where $x = (0,a), y = (0,b)$ and $z = (0,c)$ for some $a,b,c \in D$. Since $A \in \Omega$, we further have $A^2 = \text{tr}(A)A$. Looking at the off-diagonal terms, we get the equations

$$yx = \pm z, \quad zy = \pm x \quad \text{and} \quad xz = \pm y.$$ 

But the products $yx$, $zy$ and $xz$ have the second coordinate equal to 0. Hence $z = x = y = 0$. But then $A$ cannot be a rank-1 matrix. Hence $\omega_3$ is empty, and this proves (i).

For (ii) we see analogously that $y = z = 0$. Now $A$ has rank 1 if and only if the first $2 \times 2$ minor is 0. This gives $x^2 = \pm 1$. Writing this out, with $x = (0,a)$ we see that $\lambda a \bar{a} = \pm 1$. Hence $\omega_2$ is identified with the set of all elements in $D$ with a fixed non-zero norm. This is a principal homogeneous space for $D^1$. This establishes (ii). In the last case it is easy to see that $x = y = z = 0$. \qed
We now derive a consequence. Let \( \Theta(1) \) be the maximal quotient of \( \Pi \) on which \( D^1 \) acts trivially; it is naturally a \( G_D \)-module. Lemma 4.1 implies that
\[
\Theta(1)|_{N_D, \psi_3} = 0 \quad \text{and} \quad \Theta(1)|_{N_D, \psi_2} = \mathbb{C}.
\]
Let \( I_D(s) \) be the degenerate principal series representation of \( G_D \) attached to \( P_D \) normalized as in [Wei03]. In particular, the trivial representation is a quotient for \( s = 5 \) and a submodule for \( s = -5 \). The inclusion \( \Pi \to I(-5) \) composed with the restriction of functions from \( G \) to \( G_D \) gives a non-zero \( D^1 \)-invariant map \( \Pi \to I_D(-1) \), which clearly factors through \( \Theta(1) \). By [Wei03] and [HS20], \( I_D(-1) \) has a composition series of length 2. The unique irreducible submodule \( \Sigma \) has \( N_D \)-rank 2. We have the following corollary.

**Corollary 4.2.** The above construction gives a surjective \( G_D \)-equivariant map
\[
\Theta(1) \to \Sigma \subseteq I_D(-1)
\]
whose kernel has \( N_D \)-rank no greater than 1. If \( D \) is a division algebra, then \( \Theta(1) \cong \Sigma \).

**Proof.** It remains to prove the last statement. The spherical, rank-2 representation \( \Sigma \) is the classical theta lift of the trivial representation of the quaternionic form of \( \text{Sp}(4) \) [Yam11]. Using the theta correspondence, it is easy to check that \( \Sigma|_{N_D, \psi_1} \cong \mathbb{C} \). Thus, from Lemma 4.1(iii) it follows that the kernel of the map \( \Theta(1) \to \Sigma \) has \( N_D \)-rank 0, that is, \( N_D \) acts trivially. Since \( D^1 \) is compact, \( \Theta(1) \) is a summand of the minimal representation. By the classical result of Howe and Moore the minimal representation cannot contain non-zero vectors fixed by \( N_D \). Thus the kernel is trivial. \( \square \)

### 4.2 Local lifts for split \( D \)

We shall strengthen here the result of Corollary 4.2 by showing that \( \Theta(1) \cong \Sigma \) even when \( D \) is split, in which case \( G \) is also split.

Let \( T \subseteq G \) be a maximal split torus, so we have the associated root groups. Furthermore, \( D^1 \cong \text{SL}_2 \) and it is conjugated to a root \( \text{SL}_2 \). Without loss of generality, we can assume that \( \text{SL}_2 \) corresponds to the highest root for some choice of positive roots. Let \( T_1 = \text{SL}_2 \cap T \). Then the centralizer of \( T_1 \) in \( G \) is a Levi subgroup \( L \) of semisimple type \( D_6 \). The Levi subgroup \( L \) is contained in two maximal parabolic subgroups: \( Q = LU \) and its opposite \( \bar{Q} = LU \). The unipotent radical \( U \) is a two-step unipotent group with the center \( U_1 \) given by the root group corresponding to the highest root. Similarly, the center of \( \bar{U} \) is the root subgroup \( \bar{U}_1 \) corresponding to the lowest root. These two root groups \( U_1 \) and \( \bar{U}_1 \) generate \( \text{SL}_2 \). We identify \( T_1 \cong \text{GL}_1 \) so that \( x \in \text{GL}_1 \) acts on \( U/U_1 \) as multiplication by \( x \).

The conjugation action of \( L \) on \( U_1 \) and \( \bar{U}_1 \) is given by a character and its inverse; this character is given by \( x \mapsto x^2 \) when restricted to \( T_1 \subseteq L \). Hence \( G_D \) is the kernel of this character, which is the derived group of \( L \). Since \( G \) is of adjoint type, \( G_D \) acts faithfully on \( U/U_1 \) (a 32-dimensional spin representation). Note that the representation \( U/U_1 \) is not \( W_D \), the 32-dimensional representation of \( \bar{G}_D \), from §2.3.

More precisely, recall that the center of \( \text{Spin}_{12} \) can be identified with \( \mu_2 \times \mu_2 \) in such a way that the outer automorphism exchanges the two \( \mu_2 \), and fixes the diagonal \( \mu_2^D \). The quotient of \( \text{Spin}_{12} \) by \( \mu_2^D \) is the special orthogonal group \( \text{SO}_{12} \). On the other hand, the quotient of \( \text{Spin}_{12} \) by \( \mu_2 = \mu_2 \times \{1\} \) and that by \( \mu_2' = \{1\} \times \mu_2 \) are isomorphic (being isomorphic via the outer automorphism). Then one has
\[
G_D \cong \text{Spin}_{12}/\mu_2 \quad \text{and} \quad L \cong T_1 \times \mu_2 \ G_D \cong \text{GL}_1 \times \mu_2' (\text{Spin}_{12}/\mu_2),
\]
so that $L$ has connected center. On the other hand, the group $\tilde{G}_D$ from §2.3 is given by

$$\tilde{G}_D \cong \text{GL}_1 \times_{\mu_2} \text{Spin}_{12}.$$ 

As we mentioned in §2.3, the action of $\tilde{G}_D$ on $W_D$ is not faithful.

We now need a result on the restriction of $\Pi$ to the maximal parabolic subgroup $Q = LU$. By [MS97, Theorem 6.1], the space of $U_L$-coinvariants of $\Pi$, an $L$-module, sits in an exact sequence

$$0 \to \mathcal{C}_c^\infty(\omega) \to \Pi_{U_1} \to \Pi_U \to 0,$$

where $\omega$ is the $L$-orbit of highest weight vectors in $\tilde{U}$. The action of $L$ on $\mathcal{C}_c^\infty(\omega)$ arises from the natural action of $L$ on $\omega$ twisted by an unramified character.

Let $Q_D = L_D U_D$ be a maximal parabolic subgroup in $G_D$ stabilizing the line through a point $v \in \omega$. Note that the Levi factor $L_D$ of $Q_D$ is also of type $A_5$ (like that of $P_D$). The action of $Q_D$ on the line gives a homomorphism $\chi : Q_D \to \text{GL}_1$. Thus the stabilizer in $G_D \times \text{GL}_1$ of $v$ consists of all pairs $(g, x)$ such that $g \in Q_D$ and $\chi(g) = x$. Since $G_D \times \text{GL}_1$ acts transitively on $\omega$, it is easy to see that the following theorem holds.

**Theorem 4.3.** The normalized Jacquet functor $\Pi_{U_1}$, as a $G_D \times \text{GL}_1$-module, has a two-step filtration with the following quotient and submodule, respectively:

- $\Pi_U = \Pi(G_D) \otimes |\cdot|^2 \oplus |\cdot|^3$, where $\Pi(G_D)$ is the minimal representation of $G_D$, and $|\cdot|$ is the absolute value character of $\text{GL}_1$;
- $\text{Ind}_{Q_D}^{G_D} \mathcal{C}_c^\infty(\text{GL}_1)$, where $\mathcal{C}_c^\infty(\text{GL}_1)$ is the regular representation of $\text{GL}_1$ (and the induction is normalized).

Now we can prove the following result which strengthens Corollary 4.2 and which is needed later.

**Proposition 4.4.** Assume that we are in the $p$-adic case with $D$ split. Then $\Theta(1)$ is irreducible and isomorphic to $\Sigma$, the representation of $G_D$ of $N_D$-rank 2 that appears as the unique irreducible quotient of $I_D(1)$.

**Proof.** Let $\pi$ be an irreducible representation of $\text{SL}_2$ and $\Theta(\pi)$ the corresponding big theta lift. We first note that $\Theta(\pi)$ is always non-trivial, as a simple consequence of Lemma 4.1. Moreover, $\Theta(\pi)_{N_D, \psi_2}$ is isomorphic to $\pi^{\vee}$, so that it is infinite-dimensional if and only if $\pi$ is.

Let $J(s)$ be the principal series for $\text{SL}_2$ normalized so that the trivial representation is a quotient for $s = 1$ and a submodule for $s = -1$. Likewise, let $J_D(s)$ denote the degenerate principal series associated to $Q_D$, normalized so that the trivial representation occurs at $J_D(\pm 5)$.

If $-s \neq 2, 3$, then Theorem 4.3 implies by way of the Frobenius reciprocity that

$$\text{Hom}(\Theta(J(-s)), \mathbb{C}) \cong \text{Hom}_{\text{SL}_2}(\Pi, J(-s)) \cong \text{Hom}(J_D(s), \mathbb{C})$$

as $G_D$-modules. For generic $s$, both $J(-s)$ and $J_D(s)$ are irreducible and the above identity implies that

$$\Theta(J(-s)) \cong J_D(s)$$

for such $s$. It follows from Lemma 4.1 that $J_D(s)_{N_D, \psi_2}$ is infinite-dimensional for such $s$. However, since the restriction of $J_D(s)$ to $N_D$ is independent of $s$, it follows that $J_D(s)_{N_D, \psi_2}$ is in fact infinite-dimensional for all $s$.
Now if $\pi$ is a submodule of $J(-s)$ with $-s \neq 2, 3$, then it follows that
\[
\text{Hom}(\Theta(\pi), \mathbb{C}) \cong \text{Hom}_{\text{SL}_2}(\Pi, \pi) \subseteq \text{Hom}_{\text{SL}_2}(\Pi, J(-s)) \cong \text{Hom}(J_D(s), \mathbb{C}),
\]
so that $\Theta(\pi)$ is a quotient of $J_D(s)$. In particular, for $\pi = 1$ the trivial representation, we may take $s = 1$ to deduce that $\Theta(1)$ is a quotient of $J_D(1)$. Since we know that $\Theta(1)_{N_D, \psi_2}$ is one-dimensional whereas $J_D(1)_{N_D, \psi_2}$ is infinite-dimensional, we conclude that $\Theta(1)$ is isomorphic to the unique irreducible quotient of $J_D(1)$ which has $N_D$-rank 2. In particular, $\Theta(1)$ is isomorphic and isomorphic to $\Sigma$, the unique quotient of $I_D$.

As a side remark, the representations $J_D(s)$ have $U_D$-rank 3. However, since $\Pi$ has $N_D$-rank 2, it follows that the two parabolic subgroups $P_D$ and $Q_D$ are not conjugate in $G_D$. But the two principal series $I_D(s)$ and $J_D(s)$ share all small-rank subquotients – the trivial representation, the minimal representation and the rank-2 representation $\Sigma$ – as the above argument shows.

### 5. Global lifting

Assume now that $F$ is a global field, with its local completions denoted by $F_v$, and let $\mathbb{A}$ be the ring of adèles over $F$.

#### 5.1 Global theta lifting

Let $\Pi = \otimes \Pi_v$ be the restricted tensor product of minimal representations over all local places $v$ of $F$, where $\Pi_v \subset C^\infty(\Omega_v)$, as in Theorem 3.1. Every element in $\Pi$ is a finite linear combination of pure tensors $f = \otimes f_v$, where $f_v = f_v^\#$ for almost all places $v$. There is a unique (up to a non-zero scalar) embedding $\theta : \Pi \to \mathcal{A}(G(F)\backslash G(\mathbb{A}))$ of $\Pi$ into the space of automorphic functions of uniform moderate growth.

We restrict $\theta(f)$ to the dual pair $D^1 \times G_D$ and, for every $h \in \mathcal{A}(D^1(F)\backslash D^1(\mathbb{A}))$, consider the function $\Theta(f, h)$ on $G_D$ defined by
\[
\Theta(f, h)(g_D) = \int_{D^1(F)\backslash D^1(\mathbb{A})} \theta(f)(g_D g) \cdot \bar{h}(g) \, dg.
\]

If this is to be of any use, we require the function $\theta(f)(g_D g) \cdot \bar{h}(g)$ to be of rapid decay on $D^1(F)\backslash D^1(\mathbb{A})$ and of moderate growth on $G_D(F)\backslash G_D(\mathbb{A})$. This condition is clearly satisfied if $D^1$ is anisotropic or if $h$ is a cusp form. It is also satisfied for a regularized theta lift, to be constructed in the next section. Namely, for any finite place $v$, we will construct an element $z$ in the Bernstein center of $\text{SL}_2(F_v)$, such that for any $f \in \Pi$, the function $\theta(z \cdot f)(g_D g)$ is of rapid decay on $D^1(F)\backslash D^1(\mathbb{A})$ and of moderate growth on $G_D(F)\backslash G_D(\mathbb{A})$. (See Proposition 6.1, and the discussion of this particular dual pair thereafter.) In particular, in all these cases, the following integral is convergent:
\[
\int_{N_D(F)\backslash N_D(\mathbb{A})} \int_{D^1(F)\backslash D^1(\mathbb{A})} |\theta(z \cdot f)(ng) \cdot \bar{h}(g)| \, dg \, dn.
\]

#### 5.2 Fourier expansion

Let $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ be a non-trivial character. Then any $A \in J(F)$ defines a character $\psi_A$ of $N(F)\backslash N(\mathbb{A})$ by $\psi_A(B) = \psi(\text{tr}(A \circ B))$ for all $B \in N(\mathbb{A}) \cong J(\mathbb{A})$. For every $\varphi \in \mathcal{A}(G(F)\backslash G(\mathbb{A}))$, let
\[
\varphi_A(g) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ng) \cdot \bar{\psi_A(n)} \, dn.
\]
be the Fourier coefficient corresponding to $A$. We have a Fourier expansion

$$
\theta(f)(g) = \theta(f)_0(g) + \sum_{A \in \Omega(F)} \theta(f)_A(g).
$$

By uniqueness of local functionals (Theorem 3.1), for every $A \in \Omega(F)$ there exists a non-zero scalar $c_A$ such that

$$
\theta(f)_A(g) = c_A \prod_v (g_v \cdot f_v)(A).
$$

This formula is particularly useful if $g_v \in M(F_v)$, for then $(g_v \cdot f_v)(A) = \chi_v(g_v) \cdot f_v(g_v^{-1} \cdot A)$ for the character $\chi_v$ of $M(F_v)$.

Let $\psi_2$ and $\psi_3$ be the rank-2 and rank-3 characters of $N_D(A)$, as in the local case. Recall that $x \in \mathcal{O}$ is a pair $x = (y, z)$ of elements in $D$, and $N(x) = N(y) - \lambda N(z)$ for some $\lambda \in F^\times$. Let $\varphi_{N_D, \psi_l}$ denote the global Fourier coefficient with respect to these two characters. Let $\omega_2(F)$ be the set of all rank-1 matrices

$$
\begin{pmatrix}
\pm 1 & x & 0 \\
-x & \pm 1 & 0 \\
0 & 0 & 0
\end{pmatrix} \in J(F)
$$

such that $x = (0, a)$ and $\lambda N(a) = \pm 1$ (for only one choice of sign, depending on $\psi_2$), that is, the $2 \times 2$ minor is 0. Then we have a global version of Lemma 4.1.

**Lemma 5.1.** For every $f \in \Pi$, $\theta(f)_{N_D, \psi_3} = 0$ and

$$
\theta(f)_{N_D, \psi_2}(g) = \sum_{B \in \omega_2(F)} \theta(f)_{B}(g).
$$

**5.3 Non-vanishing of the theta lift**

We shall prove non-vanishing of the (regularized) theta lift by computing the Fourier coefficient

$$
\Theta(f, h)_{N_D, \psi_2}(1) = \int_{N_D(F) \setminus N_D(A)} \int_{D^1(F) \setminus D^1(A)} \theta(f)(ng) \cdot \bar{h}(g) \cdot \bar{\psi}_2(n) \, dn.
$$

Since this integral is absolutely convergent, we can reverse the order of integration. Then, using Lemma 5.1, we obtain

$$
\Theta(f, h)_{N_D, \psi_2}(1) = \int_{D^1(F) \setminus D^1(A)} \sum_{B \in \omega_2(F)} \theta(f)_{B}(g) \cdot \bar{h}(g) \, dg.
$$

**Lemma 5.2.** Fix

$$
A = \begin{pmatrix}
\pm 1 & x & 0 \\
-x & \pm 1 & 0 \\
0 & 0 & 0
\end{pmatrix} \in \omega_2(F),
$$

where $x = (0, a)$, $a \in D$ satisfies $\lambda N(a) = \pm 1$.

For every automorphic form $h$ and every $f \in \Pi$ we have

$$
\int_{D^1(F) \setminus D^1(A)} \sum_{B \in \omega_2(F)} \theta(f)_{B}(g) \bar{h}(g) \, dg = c_A \int_{D^1(A)} f(g^{-1} \cdot A) \bar{h}(g) \, dg,
$$

where the second integral is absolutely convergent.
Proof. Since \( \omega_2(F) \) is a principal homogeneous \( D^1(F) \)-space, the identity formally follows by unfolding the left-hand side and using the formula for \( \theta(f)_A(g) \) as a product of local functionals given above. Hence it remains to discuss the issue of absolute convergence.

We may assume that \( f = \bigotimes_v f_v \) is a pure tensor. For each place \( v \), observe that if \( g \in SL_2(F_v) \), then \( g^{-1}A \) is obtained from \( A \) by replacing \( x \) by \( xg \). Hence, \( g \mapsto g^{-1}A \) gives a closed embedding of \( SL_2(F_v) \) into \( J(F_v) \), with image contained in \( \Omega_v \). In particular, this image is bounded away from the vertex 0 of the cone \( \Omega_v \). Since \( h \) is of moderate growth on \( SL_2(F_v) \), to show that the integral in question is absolutely convergent, we need to show that \( g \mapsto f_v(g^{-1}A) \) is a Schwartz function on \( SL_2(F_v) \). For this, it suffices to show that as a function on the cone \( \Omega_v \), \( f_v \) is rapidly decreasing toward infinity, as we shall explain below.

In greater detail, assume first that \( v \) is a finite place. Due to \( N_v \)-smoothness, \( f_v \in \Pi_v \) is supported on a lattice in \( J_v \) (and thus vanishes toward infinity). It follows that \( g \mapsto f_v(g^{-1}A) \) is a compactly supported function on \( SL_2(F_v) \). Moreover, let \( S \) be a finite set of places containing all archimedean places such that for \( v \notin S \), all data is unramified: \( D(F_v) \) is split, \( \lambda \in O_v^\times \), \( a \in GL_2(O_v) \), \( \psi_v \) has the conductor \( O_v \), \( f_v = f_v^0 \), and \( h \) is right \( SL_2(O_v) \)-invariant. Here \( O_v \) is the maximal order in \( F_v \). It follows from Theorem 3.2 that \( g \mapsto f_v^0(g^{-1}A) \) is the characteristic function of \( SL_2(O_v) \) for all \( v \notin S \). Thus if we normalize the local measures so that \( \text{vol}(SL_2(O_v)) = 1 \) for all \( v \notin S \), then

\[
\int_{D^1(k)} |f(g^{-1}A)\tilde{h}(g)| \, dg = \int_{D^1(k_S)} |f_S(g^{-1}A)\tilde{h}(g)| \, dg,
\]

where the subscript \( S \) denotes the product of the local data over all places \( v \in S \).

Consider now the case where \( v \) is a real place. We need to show that \( C \mapsto f_v(C) \) is of rapid decay in \( \|C\| \), where \( C \in \Omega(\mathbb{R}) \). To that end, let \( m_v \in M(\mathbb{R}) \) such that \( C = m_v^{-1} \cdot A \). Then, up to a non-zero constant \( c \), independent of \( C \),

\[
f_v(C) = c \cdot \chi_v(m_v)^{-1} \cdot \theta(f)_A(m_v)
\]

for the character \( \chi_v \) of \( M(\mathbb{R}) \). Now observe that \( m_v \) can be taken to be a product of an element \( k_v \) in a maximal compact subgroup of \( M(\mathbb{R}) \) and an element \( z_v \) in \( Z_v \), the identity component of the center of \( M(\mathbb{R}) \). We fix an isomorphism \( \nu : Z_v \rightarrow \mathbb{R}^+ \) such that the conjugation action of \( z_v \in Z_v \) on \( N(\mathbb{R}) \) is given by multiplication by \( \nu(z_v) \). Now, in order to prove that \( f_v \) is rapidly decreasing toward infinity, we shall give a global argument exploiting the automorphic form \( \theta(f) \) (though a local proof is also possible). Namely, it suffices to show that \( z_v \mapsto \theta(f)_A(z_v k_v) \) is rapidly decreasing as \( \nu(z_v) \rightarrow \infty \), with bounds independent of \( k_v \). This can be proved using the usual method of integration by parts, as in [MW95, p. 30, Lemma].

More precisely, if \( X \in J \cong \mathfrak{n} \), then the \( X \)-derivative of the character \( \psi_A \) is a multiple of \( \psi_A \). Using the definition of the Fourier coefficient and integration by parts, one obtains that

\[
\theta(f)_A(z_v k_v) \text{ is a multiple of } (R_Y \cdot \theta(f))_A(z_v k_v) \cdot \nu(z_v)^{-1},
\]

where \( Y = k_v^{-1}Xk_v \) and \( R_Y \) denotes the right \( Y \)-derivative of the automorphic form \( \theta(f) \). We can repeat this procedure to get any negative power of \( \nu(z_v) \). The rapid decay follows from the fact that \( \theta(f) \) is of uniform moderate growth, and the fact that \( Y = k_v^{-1}Xk_v \) is a linear combination of vectors in any fixed basis of \( \mathfrak{n} \), with bounded coefficients, as \( k_v \) runs over the maximal compact subgroup in \( M(\mathbb{R}) \).

Finally, suppose that \( g \in SL_2(\mathbb{R}) \) belongs to the double coset of the diagonal matrix \( \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix} \), \( t > 0 \), in the Cartan decomposition of \( SL_2(\mathbb{R}) \). If we assume for simplicity that \( \lambda = 1 \), so that \( a \) in \( x = (0, a) \) can be taken to be the identity matrix, then \( \|xg\|^2 = t^2 + 1/t^2 \) (on the nose) and
\[ \|g^{-1}A\| = t + 1/t. \] In particular, \( t < \|g^{-1}A\| < t + 1 \) for \( t > 1 \). Hence, the rapid decay toward infinity of \( f_v \) (as a function on \( \Omega_v \)) implies that \( g \mapsto f_v(g^{-1}A) \) has rapid decay on \( \text{SL}_2(\mathbb{R}) \), as desired.

We are now ready to prove the non-vanishing of the global theta lift. Assume firstly that \( h \) is a cusp form. Then we have shown that

\[ \Theta(f, h)_{N_D, \psi_2}(1) = \int_{D^1(\mathbb{A})} f_S(g^{-1}A) h(g) \, dg \]

for some large finite set of places. Since for every \( v \in S \) the local \( f_v \) can be an arbitrary compactly supported smooth function on \( \Omega_v \), the integral will not vanish for some choice of data. Now consider the regularized theta lift \( \Theta(z \cdot f, h) \), where \( h \) is in an automorphic form, not necessarily cuspidal, and \( z \) is an element of the Bernstein center of \( \text{SL}_2(F_v) \) for a particular fixed finite place \( v \) (see the next section for the construction of \( z \)). The corresponding Fourier coefficient is

\[ \Theta(z \cdot f, h)_{N_D, \psi_2}(1) = \int_{D^1(\mathbb{A})} (z \cdot f)(g^{-1}A) h(g) \, dg. \]

Let \( K_v \) be a sufficiently small open compact subgroup of \( \text{SL}_2(F_v) \) such that \( f_v \) is \( K_v \)-invariant. Then \( z \cdot f_v = \alpha \cdot f_v \), where \( \alpha \) is a \( K_v \) bi-invariant, compactly supported function on \( \text{SL}_2(F_v) \). Let \( \alpha^\vee(g) = \overline{\alpha(g^{-1})} \) and define \( z^\vee \cdot h = \alpha^\vee \cdot h \). Using the convergence guaranteed by Lemma 5.2,

\[ \int_{D^1(\mathbb{A})} (z \cdot f)(g^{-1}A) h(g) \, dg = \int_{D^1(\mathbb{A})} f(g^{-1}A)(\overline{z^\vee \cdot h})(g) \, dg, \]

and this can again be arranged to be non-zero, provided \( z^\vee \cdot h \neq 0 \). Hence we have proved the following theorem.

**Theorem 5.3.** If \( h \) is a non-zero cusp form on \( D^1(\mathbb{A}) \), then \( \Theta(f, h) \neq 0 \) for some \( f \in \Pi \). If \( h \) is a (not necessarily cuspidal) automorphic form such that \( z^\vee \cdot h \neq 0 \), then \( \Theta(z \cdot f, h) \neq 0 \) for some \( f \in \Pi \).

**Remark.** The main reason for introduction of the regularized theta lift is to be able to handle the lift of \( h = 1 \) in the case where \( D \) is split. In this case we can take all data to be the simplest possible: \( \lambda = 1 \), the matrix \( A \) with \( a = (0, x) \) with \( x \) the identity matrix, etc. Then the non-vanishing of the theta lift is achieved with the spherical vector \( f_\infty^0 \) at any real place. Indeed, if \( g \in \text{SL}_2(\mathbb{R}) \) belongs to the double coset of the diagonal matrix \( \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \), with \( t > 0 \), in the Cartan decomposition of \( \text{SL}_2(\mathbb{R}) \), then \( \|xg\|^2 = t^2 + 1/t^2 \) and \( \|g^{-1}A\| = t + 1/t \). Write \( u = t + 1/t \) so that

\[ du = \left( t - \frac{1}{t} \right) \frac{dt}{t}. \]

Using the formula for the spherical vector given by Theorem 3.3 and the formula for the Haar measure on \( \text{SL}_2(\mathbb{R}) \) with respect to the Cartan decomposition, we have

\[ \int_{\text{SL}_2(\mathbb{R})} f_\infty^0(g^{-1}A) \, dg = \int_2^\infty \frac{1}{2} \cdot u^{-1/2} K_{3/2}(u) \, du > 0. \]

It will be interesting to compute the value of this integral.
6. Regularizing Theta

Following some ideas of Kudla and Rallis [KR94], the first author introduced in [Gan11] a regularized theta integral for a particular exceptional dual pair. We simplify the arguments so that regularization is now available for a wider class of examples. The notations used in this largely self-contained section will differ from those of the other sections of this paper. We first recall some basic facts about the notions of uniform moderate growth and rapid decay.

6.1 Moderate growth and rapid decay

Let $k$ be a number field and let $A$ denote the corresponding ring of ad` eles. Let $G$ be a reductive group over $k$. In order to keep the notation simple, we shall assume that $G$ is split with a finite center. Fix a maximal split torus $T$ and a minimal parabolic subgroup $P$ containing $T$. Let $N$ be the unipotent radical of $P$. We have a root system $\Phi$, obtained by $T$ acting on the Lie algebra $g$ of $G$ and a set of simple roots in $\Phi$ corresponding to the choice of $P$.

If we fix a place $v$ of $k$, then $G_v$ will denote the group of $k_v$-points of $G$. Similarly, we shall use the subscript $v$ to denote various other subgroups of $G_v$. A smooth function $f$ on $G(\mathbb{A})$ is of uniform moderate growth if there exists an integer $m$ such that for every $X$ in the enveloping algebra of $g$ there exists a constant $c_X$ such that

$$|R_X f(g)| \leq c_X \|g\|^m,$$

where $R_X$ denotes the action of the enveloping algebra on smooth functions obtained by the differentiation from the right and $\|g\|$ is a height function on $G$ defined in [MW95, p. 20]. Since there exists a constant $c$ such that $\|gh\| \leq c\|g\| \cdot \|h\|$ for all $g, h \in G(\mathbb{A})$, it is easy to see that the constants $c_X$ for the right translates $R_h f$ of $f$ are of moderate growth in $h$, or more precisely of growth $\|h\|^{m+d}$ where $d$ is the degree of $X$.

Now assume that $v$ is a real or complex place of $k$. Let $P_v = M_vA_vN_v$ be the Langlands decomposition of $P_v$. For $\epsilon > 0$, let $A_{v,\epsilon}$ be a cone in $A_v$ consisting of $a \in A_v$ such that $\alpha(a) > \epsilon$ for all simple roots $\alpha$. Let $A$ be the product of the $A_v$ and let $A_v$ be the product of the $A_{v,\epsilon}$ over all real and complex places $v$. Let $\omega_N$ be a compact set in $N(\mathbb{A})$ containing the identity element. Let $K$ be a product of maximal compact subgroups $K_v$ of $G_v$, where we have taken $K_v$ to be hyperspecial for all $p$-adic places. Then

$$S = \omega_N A_v K$$

is a Siegel domain in $G(\mathbb{A})$. If $\omega_N$ is sufficiently large and $\epsilon$ is sufficiently small, then $G(\mathbb{A}) = \overline{G(k)} S$.

Let $\Pi$ be an automorphic representation of $G$. Then any smooth $f \in \Pi$ is of uniform moderate growth. In terms of the Siegel domain $S$, this means the following. Let $\rho_P : A \to \mathbb{R}^+$ be the modular character. There exists an integer $m$ such that for every $X$ in the enveloping algebra of $g$, there exists a constant $c_X$ such that

$$|R_X f(nak)| \leq c_X \cdot \rho_P(a)^m$$

on $S$, where the constants $m$ and $c_X$ are not necessarily the same, but related to those above.

Now let $Q \supset P$ be a maximal parabolic with a unipotent radical $U \subseteq N$, corresponding to a simple root $\alpha$. We have a standard Levi factor $L$ of $Q$ defined as the centralizer of a fundamental cocharacter $\chi : \mathbb{G}_m \to T$ (or a power of it). In any case, any element in $A_v$ is uniquely written as a product $\prod \chi(t_\chi)$, over all fundamental cocharacters $\chi$, where $t_\chi \in \mathbb{R}^+$. The element $\prod \chi(t_\chi)$
is contained in the cone $A_{v,ε}$ if $t_χ > ε$ for all $χ$. Let $f_U$ be the constant term of $f$ along $U$. Then, if $f$ has uniform moderate growth, by [MW95, p. 30, Lemma] for every positive integer $i$, there is a constant $c_i$ such that

$$|f - f_U|(nak)| ≤ c_i \cdot ρ_P(a)^m \cdot α^{-i}(a)$$

on $S$. In particular, if $f_U = 0$, then $f$ is rapidly decreasing in the variable $t_χ$. If $f_U = 0$ for all maximal parabolic subgroups, then $f$ is rapidly decreasing on $S$, and that is how the rapid decrease of cusp forms is established. The proof of [MW95, p. 30, Lemma] involves integration by parts, so it is easy to see that the constants $c_i$ for the right translates $R_h f$ of $f$ are of moderate growth in $h$, or more precisely of the growth $∥h∥^{m'}$ where $m'$ depends on $i$: a larger $i$ will demand a larger $m'$.

We highlight another important issue here. Assume that $f$ belongs to an automorphic representation $π$. Then a Frechét space topology on $π$ is given by the family of seminorms

$$∥f∥_X = \sup_{nak \in S} |R_X f(nak)| \cdot ρ_P(a)^{-m},$$

where $m$ depends on $π$ and works for all $X$ in the enveloping algebra. Then [MW95, p. 30, Lemma] says that convergence in these seminorms implies convergence in the seminorm

$$\sup_{nak \in S} |(f - f_U)(nak)| \cdot ρ_P(a)^{-m} \cdot α^i(a).$$

This observation will later imply that the regularized theta integral gives a continuous pairing.

6.2 Restricting to a subgroup

Let $(G_1, G_2)$ be a dual pair in $G$. Let $T_1$ be a maximal split torus in $G_1$ and fix a minimal parabolic subgroup $P_1$ containing $T_1$. Without loss of generality, we can assume that $T_1 ⊆ T$ and $P_1 ⊆ P$. Let $Q_1 ⊇ P_1$ be a maximal parabolic subgroup of $G_1$. Let $χ_1 : G_m → T_1$ be the corresponding fundamental cocharacter (or a multiple thereof) so that the centralizer of $χ_1$ in $G_1$ is a Levi factor $L_1$ of $Q_1$. We make the following assumption.

HYPOTHESIS. For every fundamental cocharacter $χ_1$ of $G_1$, there is a fundamental cocharacter $χ$ of $G$ such that $χ_1$ is a multiple of $χ$.

This hypothesis holds in the following examples:

- the dual pair $G_1 × G_2 = D_1 × G_D = SL_2 × G_D$ studied in this paper. Here $G_1 = SL_2$ corresponds to the highest root and the highest weight is also a fundamental weight for $E_7$ (the ambient group $G$).
- the split exceptional dual pairs in $G$ of type $E_n$ where one member of the dual pair is the type $G_2$; see [LS19]. In particular, this includes the case $PGL_3 × G_2$ treated in [Gan11].
- It implies that the cone $A_{1,ε}$ sits as a subcone of $A_ε$; in fact, it is a direct factor in the above cases. In particular, we have an inclusion of Siegel domains $S_1 ⊆ S$.
- Given a fundamental cocharacter $χ_1$ of $G_1$, the associated fundamental cocharacter $χ$ of $G$ given by the hypothesis corresponds to a simple root and so determines a maximal parabolic subgroup $Q_χ = L_χ U_χ$ of $G$. In the following, we will sometimes write $U = U_χ$ to simplify notation.
Now let \( v \) be a \( p \)-adic place and \( z \) an element of the Bernstein center of \( G_1(k_v) \). Then \( z \cdot \Pi \) is naturally a \( G_1(\mathbb{A}) \times G_2(\mathbb{A}) \)-submodule of \( \Pi \). For a fixed cocharacter \( \chi_1 \) of \( G_1 \), with associated maximal parabolic \( Q = LU \) of \( G \), assume that

\[
z \cdot \Pi_v \subset \text{Ker}(\Pi_v \to (\Pi_v)_{U(k_v)}).
\]

We claim that this implies that \((z \cdot f)_U = 0\) on \( G_1(\mathbb{A}) \times G_2(\mathbb{A}) \). Indeed, if \( g \in G_1(\mathbb{A}) \times G_2(\mathbb{A}) \), then

\[
(z \cdot f)_U(g) = (R_g(z \cdot f))_U(1) = (z \cdot R_g(f))_U(1) = 0,
\]

where \( R_g \) denotes the right translation by \( g \). Here, the second equality holds since \( z \) and \( R_g \) commute, and the third equality holds since the projection of \( z \cdot \Pi \) on \( \Pi_U \) vanishes. Write \( g = g_1 \times g_2 \in G_1(\mathbb{A}) \times G_2(\mathbb{A}) \) and assume that \( g_1 \in S_1 \). Using the hypothesis that \( S_1 \subseteq S \) and the estimates for \(|R_{g_2}(z \cdot f) - (R_{g_2}(z \cdot f))_U|\) on \( S \) from the last subsection, it follows that

\[
(z \cdot f)(g_1 \times g_2) = R_{g_2}(z \cdot f)(g_1)
\]

is of moderate growth in both variables, and in the variable \( g_1 \in S_1 \) it is rapidly decreasing in the direction of the fundamental cocharacter \( \chi_1 \). More precisely, we summarize the discussion in this subsection in the following proposition.

**Proposition 6.1.** Assume that:

(i) for every fundamental cocharacter \( \chi_1 \) of \( G_1 \), there is a fundamental cocharacter \( \chi \) of \( G \) such that \( \chi_1 \) is a multiple of \( \chi \), which in turn determines a maximal parabolic subgroup \( Q_{\chi_1} = L_{\chi_1}U_{\chi_1} \);

(ii) one can find an element \( z \) in the Bernstein center of \( G_1(k_v) \) such that for every fundamental cocharacter \( \chi_1 \) of \( G_1 \), the natural projection of \( \Pi_v \) to \((\Pi_v)_{U_{\chi_1}(k_v)}\) vanishes on \( z \cdot \Pi_v \).

Then for every integer \( n \), there exist an integer \( m \) and a constant \( c \) such that

\[
|(z \cdot f)(g_1 \times g_2)| \leq c\|g_1\|^{-n}\|g_2\|^m
\]

for all \( g_1 \in S_1 \) and \( g_2 \in G_2(\mathbb{A}) \).

In the context of the above proposition, a small tradeoff here is that increasing \( n \) can be obtained only by increasing \( m \) at the same time. But this is still good enough to define a regularized theta lift which produces functions of moderate growth as output. To exploit the proposition, it remains then to construct an appropriate \( z \). We also need to ensure that \( z \cdot \Pi_v \neq 0 \), and this may not be always possible, as will be discussed in the next subsection.

**6.3 Bernstein’s center**

We work here locally over a \( p \)-adic field. Thus all our groups are local and we drop the subscript \( v \). For simplicity, we shall discuss only the Bernstein center for the Bernstein component containing the trivial representation of \( G_1 \).

To that end, let \( \hat{T}_1 \) be the complex torus dual to \( T_1 \), and let \( W(G_1) \) be the Weyl group of \( G_1 \). The Bernstein center \( Z(G_1) \) of the said component is isomorphic to the algebra of \( W(G_1) \)-invariant regular functions on \( \hat{T}_1 \). Similarly, the Bernstein center \( Z(L_1) \) of the Levi factor \( L_1 \) is isomorphic to the algebra of \( W(L_1) \)-invariant regular functions on \( \hat{T}_1 \). In particular, we have a natural map \( j : Z(G_1) \to Z(L_1) \). Let \( \pi \) be a smooth representation of \( G_1 \), and let \( p : \pi \to \pi_{U_1} \)
be the natural projection onto the normalized Jacquet module $\pi_U$. Then, for every $z \in Z(G_1)$ and $v \in \pi$, we have

$$p(z \cdot v) = j(z) \cdot (p(v)).$$

Now let $\Pi$ be a smooth representation of $G$. Recall that we want to find a non-zero $z \in Z(G_1)$ such that $z \cdot \Pi$ is in the kernel of the projection of $\Pi$ onto $\Pi_U$. Since $\Pi_U$ is an $L_1$-equivariant quotient of $\Pi_{U^1}$, $j(z) \in Z(L_1)$ acts on $\Pi_U$ and hence we need to find $z \in Z(G_1)$ such that $j(z) = 0$ on $\Pi_U$. This is always possible if $\Pi$ is a finite-length $G$-module, in which case $\Pi_U$ is a finite-length $L$-module. In particular, the center of $L$ acts finitely on $\Pi_U$. Hence, the center of $L_1$ (the center of $L$) acts finitely on $\Pi_U$ and the $Z(L_1)$-spectrum of $\Pi_U$ is contained in a proper subvariety of $\hat{T}_1$. In particular, any non-zero $W(G_1)$-invariant function $z$ vanishing on the subvariety will have the desired property that $j(z)$ vanishes on $\Pi_U$. Hence, a non-trivial $z$ with the desired property always exists.

A potential problem is that such a $z$ may kill the whole $\Pi$. However, if $G$ is split, $G_1$ is the smaller member of the dual pair (also split) and $\Pi$ the minimal representation, then the spherical matrix coefficient $\Phi$ of $\Pi$, when restricted to $G_1$, is typically contained in $L^{2-\epsilon}(G_1)$ for some $\epsilon > 0$. (This is easy to check in any given situation; see [LS19]). Thus, in such situations, it makes sense to integrate $\Phi$ against spherical tempered functions of $G_1$, that is, to consider the spherical transform of $\Phi$ on $G_1(k_v)$. This integral will be non-zero for almost all spherical functions, hence almost all spherical representations of $G_1(k_v)$ will appear as a quotient of $\Pi$. (This argument for the non-vanishing of the theta lift of almost all irreducible spherical tempered representations holds over archimedean fields as well, as we shall exploit in Lemma 7.6 below.) Hence, if $z$ kills $\Pi$, then $z$ kills all spherical tempered representations of $G_1(k_v)$ and hence $z$ must be equal to $0$. Therefore the desired regularization can be carried out in this case.

Let us look at our dual pair $G_1 \times G_2 = SL_2 \times G_D$ in $G$ with $\Pi$ the minimal representation. The Bernstein center is

$$Z(G_1) = \mathbb{C}[x^\pm 1]^{S_2},$$

where $S_2$ acts by permuting $x$ and $x^{-1}$. Let

$$z = (x - q^2)(x^{-1} - q^2)(x - q^3)(x^{-1} - q^3),$$

where $q$ is the order of the residual field. This element satisfies our requirement, since $j(z)$ vanishes on $\Pi_U$ by Theorem 4.3, and the spherical matrix coefficient of $\Pi$ is integrable when restricted to $SL_2$.

### 6.4 Global $\Theta(1)$

Let $z$ be the element in the Bernstein center of $G_1 = SL_2$, as in the previous subsection. We define $\Theta(1)$ as the space of automorphic functions

$$\Theta(f)(g_D) = \int_{D_1(F) \backslash D_1(\mathbb{A})} \theta(z \cdot f)(g_Dg) \, dg \quad \text{with } g_D \in G_D(\mathbb{A}),$$

where we assume that $f_\infty$ is $K_\infty$-finite. (We assume this finiteness since in the next section we will determine the local lift at real places in the language of $(g, K)$-modules.) We want to show that $\Theta(1) \neq 0$, using Theorem 5.3. The input in the theta kernel is $h = 1$, so the first thing is to show that $z^\vee \cdot 1 \neq 0$. In the case at hand, $z^\vee$ is obtained from $z$ by replacing $x$ by $x^{-1}$ in the above expression for $z$. In particular, $z = z^\vee$. Moreover, $z$ acts on the trivial
representation by the scalar obtained by substituting \( x = q \), and this is non-zero. It remains to
argue that we can arrange \( f_\infty \) to be \( K_\infty \)-finite. This follows by the continuity of the regularized
theta integral, which ensures that the non-vanishing for smooth \( f \) implies the non-vanishing for
\( K_\infty \)-finite vectors. Alternatively, by the remark following Theorem 5.3, non-vanishing can be
achieved with \( f_\infty = f_\infty^\circ \) the spherical vector.

7. Correspondence for real groups

In this section we work over the field \( \mathbb{R} \) of real numbers. The goal of this section is to determine
\( \Theta(1) \) explicitly. As in the \( p \)-adic case, the inclusion of the minimal representation \( \Pi \mapsto I(-5) \)
composed with the restriction of functions from \( G \) to \( G_D \) gives a natural non-zero \( G_D \)-equivariant
map \( \Theta(1) \rightarrow I_D(-1) \). We shall show that this natural map is injective and identify its image
as an irreducible submodule of \( I_D(-1) \).

To determine \( \Theta(1) \) in the archimedean setting, we need to consider various cases separately.
Indeed, recall that the adjoint group \( G \) arises from an Albert algebra via the Koecher–Tits
construction. There are two real forms of octonion algebra, the classical Graves algebra and its
split form, and these two algebras can be used to define two Albert algebras of \( 3 \times 3 \) hermitian
symmetric matrices with coefficients in the octonion algebra. The group \( G \) is split or of the
relative rank 3, depending on whether the octonion algebra is split or not (in the \( p \)-adic case, \( G \)
is always split since an octonion algebra is necessarily split).

We shall study the theta correspondence for real groups in the category of \((\mathfrak{g}, K)\)-modules,
so that examination of \( K \)-types plays a key role. Because of this, it will be convenient to work
with the simply connected form \( G' \) of \( G \) rather than the adjoint form, for the maximal compact
subgroup \( K' \) of \( G' \) is then connected (whereas that of \( G \) is not) and its irreducible representations
are classified by highest weights. The inclusion \( D^1 \rightarrow G \) lifts naturally to \( D^1 \rightarrow G' \) (since \( D^1 \)
is simply connected as an algebraic group) and its centralizer in \( G' \) is the simply connected
cover \( G_D' \) of \( G_D \). In particular, \( G_D' \) is isomorphic to Spin_{12} over \( \mathbb{C} \) and we have a dual pair
\( D^1 \times G_D' \rightarrow G' \).

Observe that the Lie group \( G_D \) (i.e. the Lie group \( G_D(\mathbb{R}) \)) has two topological connected
components and \( G_D' \) is a 2-fold cover of the identity component of \( G_D \). The maximal compact
subgroup \( K_D \) has two connected components meeting the connected components of \( G_D \). Hence
there is a natural bijection between irreducible spherical representation of \( G_D \) and \( G_D' \), via the
pullback by the natural map \( G_D' \rightarrow G_D \). We shall use this observation to prove, for example, that
when \( G \) is split, \( \Theta(1) \) is an irreducible, spherical \((\mathfrak{g}_D, K_D)\)-module by computing its \( K_D' \)-types.

7.1 Non-split \( \mathcal{O} \)

Assume firstly that \( \mathcal{O} \) and hence \( G \) is not split. Then the minimal representation of the adjoint
group \( G \), when restricted to the simply connected \( G' \), breaks up as \( \Pi = \Pi_{1,0} \oplus \Pi_{0,1} \), a sum of
a holomorphic and an anti-holomorphic irreducible representation. This sum is the socle of the
degenerate principal series \( I(-5) \) (pulled back to \( G' \)). The maximal compact subgroup \( K' \) is of
semisimple type \( E_6 \), and has one-dimensional center \( U(1) \) that acts on the Lie algebra \( \mathfrak{g} \) with
weights \(-2, 0, 2 \). The weight-2 space is a 27-dimensional representation of \( K' \). Let \( \omega \) be its
highest weight. Then

\[
\Pi_{1,0} = \bigoplus_{n \geq 0} V_{n\omega}(12),
\]

where 12 denotes a twist of the irreducible \( K' \)-module \( V_{n\omega} \) such that \( U(1) \) acts with the weight
\( 2n + 12 \) on it.
In this case, $D$ is necessarily non-split. Recalling that $G'_{D}$ is the simply connected cover of $G_{D}$, its maximal compact subgroup is $K'_{D} \cong U_{6}$. The socle of $I_{D}(-1)$, considered a representation of $G'_{D}$, is a direct sum of three representations $\Sigma_{2,0} \oplus \Sigma_{1,1} \oplus \Sigma_{0,2}$, a holomorphic, spherical and anti-holomorphic representation, respectively [Sah93, Theorem C]. The lowest and the highest $K'_{D}$-types of $\Sigma_{2,0}$ and $\Sigma_{0,2}$ are one-dimensional with $U(1)$-weights 12 and $-12$, respectively. We have the following theorem.

**Theorem 7.1.** If $\O$ is non-split (so $G$ is not split), we have

$$\Pi_{1,0}^{D_{1}} \cong \Sigma_{2,0} \quad \text{and} \quad \Pi_{0,1}^{D_{1}} \cong \Sigma_{0,2}$$

as $(g_{D}, K'_{D})$-modules. In particular, $\Theta(1) \cong \Sigma_{2,0} \oplus \Sigma_{0,2}$ as $(g_{D}, K_{D})$-modules.

**Proof.** Since $\Pi_{1,0}$ is unitarizable and $U(1)$-admissible, its restriction to $g_{D}$ is a direct sum of irreducible lowest weight representations. The minimal type of $\Pi_{1,0}$ generates an irreducible lowest weight $(g_{D}, K'_{D})$-module, with the minimal $U_{6}$-type $\det^{2}$ (i.e. $U(1)$-weight 12). Thus $\Sigma_{2,0} \subseteq \Pi_{1,0}^{D_{1}}$. The infinitesimal character of $\Sigma_{2,0}$ is $(3, 2, 1, 0, -1, -2)$, in terms of the standard realization of the $D_{6}$ root system. If the inclusion $\Sigma_{2,0} \subseteq \Pi_{1,0}^{D_{1}}$ is strict, then $\Pi_{1,0}^{D_{1}}$ contains another lowest weight representation with the same infinitesimal character. There is precisely one other irreducible lowest weight $(g_{D}, K'_{D})$-module with this infinitesimal character, with the minimal $U_{6}$-type $\det^{3}$ (i.e. $U(1)$-weight 18). Thus the number of irreducible summands in $\Pi_{1,0}^{D_{1}}$ is bounded by the dimension of $SU_{6} \times D^{1}$-invariants in $\Pi_{1,0}$. By the Cartan–Helgason theorem, a finite-dimensional irreducible representation of $E_{6}$ has a line fixed by $A_{5} \times A_{1}$ if and only if it is self-dual. It follows that the space of $SU_{6} \times D^{1}$-invariants in $\Pi_{1,0}$ is one-dimensional, with the only contribution coming from the trivial type. This proves the theorem.

\[ \square \]

### 7.2 Split $\O$ but non-split $D$

We move on to the case where $\O$ and hence $G$ is split. Let $K'$ be a maximal compact subgroup of $G'$ (the simply connected form of $G$), and $g = \mathfrak{k} \oplus \mathfrak{p}$ the corresponding Cartan decomposition of the complexification of the Lie algebra of $G$. Then $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}_{6}$. Fixing this isomorphism, we see that as a $K' \cong SU_{8}/\mu_{2}$-module, $\mathfrak{p}$ is isomorphic to $V_{\omega_{4}}$, where $\omega_{4}$ is the fourth fundamental weight. The minimal representation $\Pi$ remains irreducible when pulled back to $G'$ and is a direct sum of $K'$-types $V_{n\omega_{4}}$, where $n = 0, 1, 2, \ldots$.

We have two cases depending on $D$. Assume in this subsection that $D$ is a division algebra. In this case $D_{1} \cong SU_{2}$ is compact, and embeds into $SU_{8}$ as a $2 \times 2$ block. The centralizer of $SU_{2}$ in $K' = SU_{8}/\mu_{2}$ is $K'_{D} \cong U_{6}$. The minimal representation $\Pi$ decomposes discretely when restricted to this dual pair. A simple application of the Gelfand–Zetlin rule shows that the $K'_{D}$-types of $\Theta(1)$ are multiplicity-free and the highest weights of the $K'_{D}$-types which occur are

$$(x, x, 0, 0, y, y),$$

where $x \geq 0 \geq y$ are any two integers. Here we are using the standard description of highest weights for $U_{6}$ by sextuples of non-increasing integers. But these are precisely the $K'_{D}$-types of the spherical submodule of $I_{D}(-1)$, that is, the $G'_{D}$-constituent $\Sigma_{1,1}$ in [Sah93, Theorem C]. In view of the natural non-zero $G_{D}$-equivariant map $\Theta(1) \to I_{D}(-1)$, this proves the following theorem.

**Theorem 7.2.** When $\O$ is split but $D$ is non-split, one has

$$\Theta(1) = \Pi_{1}^{D_{1}} \cong \Sigma_{1,1}$$

as $(g_{D}, K_{D})$-modules.
This is the most involved case. Let \((e, h, f)\) be an \(\mathfrak{sl}_2\)-triple spanning the complexified Lie algebra of \(D^1 = \text{SL}_2\). After conjugating by \(G'\), if necessary, we can assume that the triple is stable under the Cartan involution. Then \(e \in \mathfrak{p}\) is a highest weight vector for the action of \(K'\), and \(h \in \mathfrak{k}\). Let \(\Theta(1)\) be the maximal quotient of the \((\mathfrak{g}, K')\)-module of the minimal representation such that the \(\mathfrak{sl}_2\) triple acts trivially.

**Theorem 7.3.** As a \((\mathfrak{g}_D, K_D)\)-module, \(\Theta(1)\) is irreducible and isomorphic to the unique irreducible submodule \(\Sigma\) of \(I_D(-1)\), which is a spherical representation.

The proof of this result will take up the rest of this section. After conjugating by \(K'\), if necessary, we can assume that

\[
h = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{pmatrix} \in \mathfrak{sl}_8.
\]

Then \(G'_D\) is the centralizer of the \(\mathfrak{sl}_2\)-triple in \(G'\) and is isomorphic to \(\text{Spin}(6, 6)\) as an algebraic group. Let \(\mathfrak{g}_D = \mathfrak{k}_D \oplus \mathfrak{p}_D\) be the corresponding Cartan decomposition. Then \(\mathfrak{k}_D \cong \mathfrak{sl}_4 \oplus \mathfrak{sl}_4\) sitting block diagonally in \(\mathfrak{sl}_8\). The centralizer of \(h\) in \(\text{SU}_8/\mu_2\) is

\[
K'_D = \text{SU}_4 \times \text{SU}_4/\Delta \mu_2.
\]

Let \(\Pi\) be the \((\mathfrak{g}, K')\)-module corresponding to the minimal representation of \(G\). Then, as a \(K'\)-module,

\[
\Pi = \bigoplus_{n \geq 0} V_{n\omega_4}.
\]

We shall also need the following facts about the action of \(e\) on \(\Pi\). From the formula for the tensor product \(V_{\omega_4} \otimes V_{n\omega_4}\) it follows that

\[
e \cdot V_{n\omega_4} \subseteq V_{(n-1)\omega_4} \oplus V_{(n+1)\omega_4}.
\]

Since \(\Pi\) is not a highest weight module, by [Vog81, Lemma 3.4], \(e\) is injective on \(\Pi\). The same results hold for \(f\).

Let \(\pi\) be an irreducible \(\mathfrak{sl}_2\)-module such that \(h\) acts semisimply and integrally. Let \(\Theta(\pi)\) be the big theta lift of \(\pi\); it is a \((\mathfrak{g}_D, K'_D)\)-module. We shall now partially determine the structure of \(K'_D\)-types of \(\Theta(\pi)\). In order to state the result, we need some additional notation. A highest weight \(\mu\) for \(\text{SU}_4\) is represented by a quadruple \((x, y, z, u)\) of integers, such that \(x \geq y \geq z \geq u\), and it is determined by the triple

\[
\alpha = x - y, \quad \beta = y - z, \quad \gamma = z - u
\]

of non-negative integers.
Proposition 7.4. Let $V \otimes U$ be a $K'_D \cong SU_4 \times SU_4/\Delta \mu_2$-type of $\Theta(\pi)$. Then $U \cong V^*$, the dual representation of $V$, and the multiplicity of $V \otimes V^*$ in $\Theta(\pi)$ is at most 1. If $\pi = 1$, the trivial representation, and $\mu$ is the highest weight of $V$, then $\alpha = \gamma$.

Proof. We need the following lemma which can be easily deduced from the Gelfand–Zetlin branching rule.

Lemma 7.5. The restriction of $V_{\nu\omega_4}$ to $\mathfrak{sl}_4 \oplus \mathfrak{sl}_4 \oplus Ch$ is multiplicity-free and given by

$$V_{\nu\omega_4} = \bigoplus_{n \geq x \geq y \geq z \geq u \geq 0} V_{\mu} \otimes V_{\mu}^* \otimes \mathbb{C}(m),$$

where $\mu$ is represented by the quadruple $(x, y, z, u)$ and $h$ acts on $\mathbb{C}(m)$ by the integer $m = x + y + z + u - 2n$.

It follows from the lemma that the only $K'_D$-types appearing in the restriction of $\Pi$ are isomorphic to $V \otimes V^*$, as claimed. In order to prove multiplicity 1 of $K'_D$-types in $\Theta(\pi)$, we proceed as follows.

Let $m$ be an integer appearing as an $h$-type in $\pi$. Let $\Omega$ be the Casimir element for $\mathfrak{sl}_2$ and let $\chi : \mathbb{C}[\Omega] \to \mathbb{C}$ be the central character of $\pi$. Let $\Pi(\mu, m)$ be the maximal subspace of $\Pi$ such that $h$ acts as the integer $m$ and $\mathfrak{sl}_4 \oplus \mathfrak{sl}_4$ as a multiple of $V_{\mu} \otimes V_{\mu}^*$. Note that $\Pi(\mu, m)$ is naturally a $\mathbb{C}[\Omega]$-module, and it suffices to show that the maximal quotient of $\Pi(\mu, m)$ such that $\mathbb{C}[\Omega]$ acts on it by $\chi$ is isomorphic to $V_{\mu} \otimes V_{\mu}^*$ as an $\mathfrak{sl}_4 \oplus \mathfrak{sl}_4$-module. We have a canonical isomorphism

$$\Pi(\mu, m) \cong (V_{\mu} \otimes V_{\mu}^*) \otimes \operatorname{Hom}_{\mathfrak{sl}_4 \oplus \mathfrak{sl}_4}(V_{\mu} \otimes V_{\mu}^*, \Pi(m)),$$

and $\mathbb{C}[\Omega]$ acts on

$$\operatorname{Hom}_{\mathfrak{sl}_4 \oplus \mathfrak{sl}_4}(V_{\mu} \otimes V_{\mu}^*, \Pi(m)) = \bigoplus_{n \geq 0} \operatorname{Hom}_{\mathfrak{sl}_4 \oplus \mathfrak{sl}_4}(V_{\mu} \otimes V_{\mu}^*, V_{\nu\omega_4}(m)).$$

Now notice that, given $\mu$ and $m$, $\operatorname{Hom}_{\mathfrak{sl}_4 \oplus \mathfrak{sl}_4}(V_{\mu} \otimes V_{\mu}^*, V_{\nu\omega_4}(m)) \neq 0$ for only one parity of $n$. Furthermore, if this space is non-zero for some $n$, then it is non-zero for $n + 2$, as $\mu$ is also represented by $(x + 1, y + 1, z + 1, u + 1)$ and

$$m = x + y + z + u - 2n = x + 1 + y + 1 + z + 1 + u + 1 - 2(n + 2).$$

Let $n_0$ be the first integer such that $\operatorname{Hom}_{\mathfrak{sl}_4 \oplus \mathfrak{sl}_4}(V_{\mu} \otimes V_{\mu}^*, V_{\nu\omega_4}(m)) \neq 0$ and let $T_0$ be a generator of this one-dimensional space. We then have a natural map

$$A : \mathbb{C}[\Omega] \cdot T_0 \to \operatorname{Hom}_{\mathfrak{sl}_4 \oplus \mathfrak{sl}_4}(V_{\mu} \otimes V_{\mu}^*, \Pi(m)).$$

Lemma 7.6. The map $A$ is an isomorphism.

Proof. Let $i$ be a non-negative integer. Let $\mathbb{C}[\Omega]_i$ be the space of polynomials of degree less than or equal to $i$, and let

$$\operatorname{Hom}_{\mathfrak{sl}_4 \oplus \mathfrak{sl}_4}(V_{\mu} \otimes V_{\mu}^*, \Pi(m))_i = \bigoplus_{j=0}^i \operatorname{Hom}_{\mathfrak{sl}_4 \oplus \mathfrak{sl}_4}(V_{\mu} \otimes V_{\mu}^*, V_{(n_0+2j)\omega_4}(m)).$$

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These two spaces have dimension $i+1$ and define filtrations of $\mathbb{C}[\Omega]$ and $\text{Hom}_{\mathfrak{sl}_2}(V_\mu \otimes V_\mu^*, \Pi(m))$ as $i$ increases. Since $\Omega$ has degree 2, as an element of the enveloping algebra of $\mathfrak{g}$, the map $A$ preserves the two filtrations. Thus, in order to prove the claim, it suffices to show that $A$ is injective.

But if it is not, then there would be a polynomial $p(\Omega)$ acting trivially on $V_\mu \otimes V_\mu^* \subseteq V_{n\omega_4}(m)$. Under the action of $\mathfrak{sl}_2$, this subspace would generate a finite-length representation $F_0 \subset \Pi$ of $\mathfrak{sl}_2$. Let $U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$, and let $U_n(\mathfrak{g}) \subset U(\mathfrak{g})$ be the Poincaré–Birkhoff–Witt filtration. We have $\Pi = U(\mathfrak{g}) \cdot F_0$, since $\Pi$ is irreducible $\mathfrak{g}$-module. Hence $\Pi$ is a union of $F_n = U_n(\mathfrak{g}) \cdot F_0$. Now observe that each $F_n$ is a finite-length $\mathfrak{sl}_2$-module. Hence there could be only countably many irreducible $\mathfrak{sl}_2$-modules appearing as quotients of $\Pi$. But this contradicts the fact that almost all spherical tempered representations of $\mathfrak{sl}_2$ are quotients, as discussed in §6.3. The lemma is proved. □

Lemma 7.6 implies that

$$\Pi(\mu, m) \cong (V_\mu \otimes V_\mu^*) \otimes \mathbb{C}[\Omega]$$

as $\mathbb{C}[\Omega]$-modules. Hence, if we fix a character $\chi$ of $\mathbb{C}[\Omega]$, the maximal quotient of $\Pi(\mu, m)$ such that $\mathbb{C}[\Omega]$ acts by $\chi$ is isomorphic to $V_\mu \otimes V_\mu^*$. This proves that $\Theta(\pi)$ has multiplicity-free $K_D'$-types.

Finally, we proceed to narrow down the $K_D'$-types appearing in $\Theta(1)$. For every $\mu$, the action of $e$ on $\Pi$ gives an injective map

$$e : \Pi(\mu, -2) \rightarrow \Pi(\mu, 0).$$

**Lemma 7.7.** If $e : \Pi(\mu, -2) \rightarrow \Pi(\mu, 0)$ is bijective, then $V_\mu \otimes V_\mu^*$ is not a $K_D'$-type of $\Theta(1)$.

**Proof.** The image of $e$ is necessarily contained in the kernel of the natural surjective map $\Pi(\mu, 0) \rightarrow \Theta(1)(\mu)$. Hence the lemma follows. □

Consider the filtration $\Pi(\mu, m)_i = \bigoplus_{n\leq i} V_{n\omega_4}(\mu, m)$ of $\Pi(\mu, m)$. Then we have an injective map

$$e : \Pi(\mu, -2)_i \rightarrow \Pi(\mu, 0)_{i+1}$$

for all $i$. Hence, if the dimensions of the two spaces are equal for all $i$, then $e$ is bijective. This will happen precisely when $V_\mu \otimes V_\mu^*$ occurs in $V_{n\omega_4}(-2)$ but not in $V_{(n-1)\omega_4}(0)$, for some $n$. The occurrence in $V_{n\omega_4}(-2)$ implies that there exists a unique quadruple $(x, y, z, u)$ representing $\mu$ such that

$$n \geq x \geq y \geq z \geq u \geq 0 \quad \text{and} \quad x + y + z + u - 2n = -2.$$

Then $V_\mu \otimes V_\mu^*$ occurs in $V_{(n-1)\omega_4}(0)$ if and only if

$$n - 1 \geq x \geq y \geq z \geq u \geq 0,$$

that is, $n > x$. Thus, if $n = x$, then $V_\mu \otimes V_\mu^*$ does not appear in $\Theta(1)$.

Let us see what this means in terms of $\alpha$, $\beta$ and $\gamma$. We have to find $n$ such that $\mu$ is represented by

$$(x, y, z, u) = (n, n - \alpha, n - \alpha - \beta, n - \alpha - \beta - \gamma).$$

Since the last entry must be non-negative, we have $n \geq \alpha + \beta + \gamma$. On the other hand, $h$ has to act as $-2$, hence $x + y + z + u - 2n = -2$, and this is equivalent to $2n = 3\alpha + 2\beta + \gamma - 2$. Combining with the previous inequality, we obtain $\alpha \geq \gamma + 2$. Hence the types with $\mu$ such that $\alpha \geq \gamma + 2$ do not appear in $\Theta(1)$. Replacing the role of $e$ with $f$, a similar argument shows that
the types such that $\gamma \geq \alpha + 2$ do not appear either. Hence $|\alpha - \gamma| \leq 1$ for all types that appear in $\Theta(1)$. Since $\alpha \equiv \gamma \pmod{2}$ for any type in $\Pi(0)$, $\alpha = \gamma$ for all types that appear in $\Theta(1)$. This completes the proof of Proposition 7.4.

The $K'_D$-types of $\Theta(1)$, as described in Proposition 7.4, are the same as the $K'_D$-types of the irreducible spherical rank-2 submodule in $I_D(-1)$ by [Sah95, Theorem 4B]. In view of the natural non-zero $G_D$-equivariant map $\Theta(1) \longrightarrow I_D(-1)$, this proves Theorem 7.3.

8. Siegel–Weil formula

We are now ready to prove the desired Siegel–Weil formula (Theorem 1.2 in the introduction). Assume that $F$ is a totally real global field and $D$ a quaternion algebra over $F$.

8.1 The representation $\Theta(1)$

We have shown that the global (regularized) theta lift $\Theta(1)$ is a non-zero automorphic representation of $G_D(\mathbb{A})$. We have also studied the abstract local theta lift of the trivial representation of $D$ to $G_D$. The following proposition summarizes what we have shown.

Proposition 8.1.

(i) The automorphic representation $\Theta(1)$ is irreducible and occurs with multiplicity 1 in the space of automorphic forms of $G_D$.

(ii) For every $p$-adic place $v$ of $F$, the local component $\Theta(1)_v$ is isomorphic to the unique irreducible quotient of the local degenerate principal series $I_D(1)$.

(iii) For every real place $v$ of $F$, the local component $\Theta(1)_v$ is an irreducible quotient of $I_D(1)$ as described in Theorems 7.1, 7.2 and 7.3.

Proof. Indeed, we have shown that the abstract local theta lift $\Theta(1_v)$ is irreducible. Hence the global $\Theta(1)$ is an irreducible automorphic representation. The fact that $\Theta(1)$ has multiplicity 1 in the space of automorphic forms follows by [KS15, Theorem 1.1]. Note that the required conditions, as spelled out in the introduction of [KS15], are satisfied by the recent work of Möllers and Schwarz [MS17].

8.2 A Siegel–Weil formula

For a flat section $\Phi \in I_D(s)$, let $E_D(s, \Phi)$ be the associated Eisenstein series. Then $E_D(s, \Phi)$ has at most simple poles at $s = 1, 3$ or 5 and the corresponding residual representations are completely described in [HS20, Theorem 6.4]. Set

$$\mathcal{E} = \{\text{Res}_{s=1} E_D(s, \Phi) : \Phi \in I_D(s)\}.$$

We can now prove Theorem 1.2 in the introduction (which we restate here).

Theorem 8.2. Let $F$ be a totally real global field and $D$ a quaternion algebra over $F$. Then we have the identity

$$\mathcal{E} = \bigoplus_{i:D \to \mathbb{O}} \Theta(1)$$

in the space of automorphic representations $G_D(\mathbb{A})$, where the sum is taken over all isomorphism classes of embeddings $i : D \to \mathbb{O}$ into octonion algebras over $F$. 1253
Proof. Comparing Proposition 8.1 with [HS20, Theorem 6.4], one sees that $\Theta(1)$ is isomorphic, as an abstract representation, to a summand of $E$. In view of the multiplicity-1 result in Proposition 8.1(i), it follows that $\Theta(1)$ is equal to that irreducible summand, as a subspace of the space of automorphic forms.

Now recall that the dual pair $D^1 \times G_D$ arises from an embedding of $D$ into an octonion algebra $\mathfrak{O}$. Every such embedding is unique up to conjugacy by $\text{Aut}(\mathfrak{O})$. However, given $D$, there are multiple octonion algebras over $F$ containing $D$. An isomorphism class of octonion algebras $\mathfrak{O}$ over $F$ is specified by the isomorphism class of its local completions $\mathfrak{O}_v$ for real places $v$. At each real place, we have two choices: the classical octonion algebra and its split form. But $D_v$ embeds into both if and only if it is a quaternion division algebra. Hence the number of octonion algebras over $F$ containing $D$ is $2^m$, where $m$ is the number of real places $v$ such that $D_v$ is the quaternion algebra. Now, by an easy check left to the reader, non-isomorphic $\mathfrak{O}$ give non-isomorphic $\Theta(1)$. Moreover, using our description of $\Theta(1)$ in Proposition 8.1 and [HS20, Theorem 6.4], one sees that all those possible $\Theta(1)$ sum to $E$. This proves the theorem. \qed

9. Spin $L$-function

To complete the proof of the main result of this paper (Theorem 1.1 in the introduction), the remaining ingredient we need is a Rankin–Selberg integral for the degree-8 Spin $L$-function for cuspidal representations of $\text{PGSp}_6$ which was discovered by Pollack [Pol17]. However, since the paper [Pol17] works over $\mathbb{Q}$ whereas we are working over a general number field $F$, we recall some details here for the sake of completeness.

9.1 Global zeta integrals

Suppose that $\pi$ is a cuspidal automorphic representation of $\text{PGSp}_6(\mathbb{A})$. Let $U$ be the unipotent radical of the Siegel maximal parabolic subgroup of $\text{PGSp}_6(F)$. Let $J_F$ be the Jordan algebra of $3 \times 3$ symmetric matrices with coefficients in $F$. Then $U \cong J_F$ and any $T \in J_F \cong \bar{U}_F$ defines an additive character $\phi_T : U(\mathbb{A})/U(F) \to \mathbb{C}^\times$. Since $\pi$ is cuspidal, there exists a non-degenerate $T$ (i.e. $\det(T) \neq 0$) such that the global Fourier coefficient $\phi_T$ is a non-zero function on $\text{PGSp}_6(\mathbb{A})$ for any $\phi \in \pi$. We fix such a $T$ (which depends on $\pi$) in what follows.

The non-degenerate orbits on $U(F) \cong J_F$, under the action of the Siegel Levi factor in $\text{PGSp}_6(F)$, are parameterized by quaternion algebras over $F$. So let $D$ be the algebra corresponding to $T$. Let $\tilde{G}_D$ be the reductive group of type $D_6$ acting on $W_D$, as in §2.3. We shall assume that $\tilde{G}_D$ acts from the right on $W_D$. Let $\omega \subset W_D$ be the $\tilde{G}_D$-orbit of $(1,0,0,0)$, that is, the orbit consisting of highest weight vectors. Following Pollack [Pol17], for every Schwartz function $\Phi = \bigotimes_v \Phi_v$ on $W_D(\mathbb{A})$ define an Eisenstein series on $\tilde{G}_D(\mathbb{A})$ by

$$E_\Phi(g) = \sum_{x \in \omega(F)} \Phi(xg).$$

Recall that $\text{GSp}_6 \subseteq \tilde{G}_D$ and let $\nu$ be the similitude homomorphism of $\text{GSp}_6$. Define a global zeta integral

$$Z(\phi, \Phi, s) = \int_{\text{GSp}_6(F) \backslash \text{GSp}_6(\mathbb{A})} \phi(g) \cdot E_\Phi(g) \cdot |\nu(g)|^s \, dg$$

for $\phi \in \pi$ and $\Phi$ as above. This integral is absolutely convergent for $s \in \mathbb{C}$ with sufficiently large real part. After integrating over the center of $\text{GSp}_6(\mathbb{A})$, we see that

$$Z(\phi, \varphi, s) = \int_{\text{PGSp}_6(F) \backslash \text{PGSp}_6(\mathbb{A})} \phi(g) \cdot E_\Phi(s, g) \, dg,$$
where $\Phi_s \in I_D(2s-5)$, and $s \mapsto \Phi_s$ is a holomorphic section for $s > 0$. Observe that the meromorphic continuation of $E(\Phi_s, g)$ gives a meromorphic continuation of $Z(\phi, \Phi, s)$.

9.2 Unfolding
Let $V \subset U$ be the codimension-1 subgroup such that the character $\psi_T$ is trivial on $V(\mathbb{A})$. There is a $w_T \in \omega(F)$, contained in the third summand of $W_D$, such that the stabilizer of $w_T$ in $\mathrm{Sp}_6$ is $V \subset U$; see [Pol17, Proposition 5.5]. The integral unfolds into

$$Z(\phi, \Phi, s) = \int_{V(\mathbb{A}) \setminus \mathrm{GSp}_6(\mathbb{A})} \phi_T(g)\Phi(w_Tg)|\nu(g)|^s \, dg.$$ 

Furthermore, by Theorem 9.4 in the first arXiv version of the paper [Pol17] (i.e. in arXiv:1506.03406v1), for a sufficiently large set of places $S$, including the set $S_\infty$ of all real places,

$$Z(\phi, \varphi, s) = L^S(s-2, \pi, \text{Spin}) \cdot c(s) \cdot \int_{V(\mathbb{A}) \setminus \mathrm{GSp}_6(\mathbb{A})} \phi_T(g)\Phi_S(w_Tg)|\nu(g)|^s \, dg.$$ 

Here $c(s)$ denotes a product of partial Dedekind zeta functions that we have omitted writing down since they do not affect analytic properties of $Z(\phi, \varphi, s)$ at our point of interest $s = 3$. We note that Pollack works over $F = \mathbb{Q}$. However, he has kindly informed us that:

- the unfolding of the integral representation works over any number field;
- the unramified computation is valid for any non-archimedean place $v$ away from 2 such that $D_v$ is split.

The proof of the latter goes through line by line if one makes the following changes of notation: every time $p$ is used as a uniformizer, replace $p$ with $\varpi$; every time $p$ is used as a magnitude, replace $p$ with $q$. In other words, the above identity holds for $S$ containing all real places, places of even residual characteristic and places where $D$ is ramified.

9.3 Non-vanishing
The following technical result was contained in an earlier version of Pollack’s paper. However, as this particular version is no longer publicly available (even on the arXiv), we reproduce the proof for the sake of completeness (see [GG06, Lemma 15.7] for a similar proof).

**Lemma 9.1.** Let $s_0 \in \mathbb{C}$. For some data $\phi$ and $\Phi_S$,

$$Z(\phi, \Phi_S, s) = \int_{V(\mathbb{A}) \setminus \mathrm{GSp}_6(\mathbb{A})} \phi_T(g)\Phi_S(w_Tg)|\nu(g)|^s \, dg$$

extends to a meromorphic function on $\mathbb{C}$ which is non-vanishing at $s_0$.

**Proof.** Observe that the meromorphic continuation is clear, since the global zeta integral has a meromorphic continuation, and so does the partial $L$-function, since it appears in the constant term of an Eisenstein series on the exceptional group $F_4$ (à la Langlands–Shahidi theory). All that remains is to deal with non-vanishing.

Let $v \in S$. Consider the local version of the zeta integral:

$$Z(\phi, \Phi_v, s) = \int_{V_v \setminus \mathrm{GSp}_6(\mathbb{A})} \phi_T(g) \cdot \Phi_v(w_Tg) \cdot |\nu(g)|^s \, dg.$$
The stabilizer of \( w_T \) in \( \text{GSp}_6 \) is \( V \times A \), where \( A \) is a one-dimensional torus that we can identify with \( \text{GL}_1 \) using the similitude character \( \nu \). Thus \( \text{GSp}_6 \) is a semidirect product of \( A \) and \( \text{Sp}_6 \), and we can write
\[
Z(\phi, \Phi_v, s) = \int_{V_v \backslash \text{Sp}_6(F_v)} \Phi_v(w_T g) \int_{A_v} \phi_T(a g) \cdot |\nu(a)|^{s-5} \, da \, dg
\]
for an invariant measure \( da \) on \( A_v \). Let us set
\[
H(\phi, s) = \int_{A_v} \phi_T(a) \cdot |\nu(a)|^{s-5} \, da.
\]
If \( h \) is a Schwartz function on \( U_v \), let
\[
\hat{h}(a) = \int_{U_v} h(u) \psi_T(a u a^{-1}) \, du,
\]
where \( a \in A_v \). Since \( \phi_T(a u a^{-1}) = \psi_T(a u a^{-1}) \cdot \phi_T(a g) \) for \( u \in U_v \), it follows that
\[
H(h \ast \phi, s) = \int_{A_v} \hat{h}(a) \cdot \phi_T(a) \cdot |\nu(a)|^{s-5} \, da.
\]
Since \( \hat{h} \) can be any compactly supported function on \( A_v \), the integral can be arranged to be non-zero for any \( s_0 \). In fact, if \( v \) is a finite place, then \( \hat{h} \) can be picked so that the integral is 1 for all \( s \). Thus, if \( v \) is a finite place, we can assume that \( \phi \) has been chosen so that \( H(\phi, s) = 1 \), for all \( s \).

Next, there exists a compactly supported function \( \varphi \) on \( \text{Sp}_6(F_v) \) such that \( \varphi \ast \phi = \phi \). Observe that
\[
H(\varphi \ast \phi, s) = \int_{V_v \backslash \text{Sp}_6(F_v)} \varphi'(g) \int_{A_v} \phi_T(a g) \cdot |\nu(a)|^{s-5} \, da \, dg,
\]
where \( \varphi' \) is a smooth compactly supported function on \( V(F_v) \backslash \text{Sp}_6(F_v) \) defined by
\[
\varphi'(g) = \int_{V(F_v)} \varphi(u g) \, du.
\]
Using the Iwasawa decomposition of \( \text{Sp}_6(F_v) \), it is not difficult to see that the map \( g \mapsto w_T g \) gives a locally closed embedding (i.e. an immersion) of \( V_v \backslash \text{Sp}_6(F_v) \) into \( W_D(F_v) \), with the closure of the image containing the extra point 0. Hence, any smooth compactly supported function on \( V_v \backslash \text{Sp}_6(F_v) \) is the restriction of a smooth compactly supported function on \( W_D(F_v) \). In particular, we can pick \( \Phi_v \) such that \( \Phi_v(w_T g) = \varphi'(g) \), for all \( g \in \text{Sp}_6(F_v) \). Then, with this choice of \( \Phi_v \), one has
\[
Z(\phi, \Phi_v, s) = H(\phi, s) = 1.
\]
It follows that
\[
Z(\phi, \Phi_S, \varphi, s) = Z(\phi, \Phi_\infty, s)
\]
for some choice of data.

Let \( F_\infty = F \otimes_{Q} \mathbb{R} \). Assume as above that \( \phi \) has been chosen so that \( H(\phi, s_0) \neq 0 \). While we perhaps cannot write \( \phi = \varphi \ast \phi \), for a compactly supported function on \( \text{Sp}_6(F_\infty) \), by the well-known Dixmier–Malliavin theorem, there exist finitely many compactly supported functions \( \varphi_i \) on \( \text{Sp}_6(F_\infty) \) such that \( \phi = \sum_i \varphi_i \ast \phi_i \) for some \( \phi_i \). Then, as in the finite-place case, there exist compactly supported \( \Phi_{i\infty} \) such that
\[
\sum_i Z(\phi, \Phi_{i\infty}^i, s) = H(\phi, s).
\]
Since $H(\phi, s_0) \neq 0$, we see that $Z(\phi, \Phi^i, s_0) \neq 0$ for some $i$. This completes the proof of the lemma. 

10. Applications to Functoriality

Finally, we are ready to assemble the various ingredients and complete the proof of Theorem 1.1 (which we reproduce here).

**Theorem 10.1.** Suppose that $\pi$ is a cuspidal automorphic representation of $\text{PGSp}_6$ such that $L^S(s, \pi, \text{Spin})$ has a pole at $s = 1$. Then there exist an octonion algebra $\mathcal{O}$ over $F$ and a cuspidal automorphic representation $\pi'$ of $\text{Aut}(\mathcal{O})$ such that the Satake parameters of $\pi'$ are mapped by $\iota$ to those of $\pi$ (i.e. $\pi$ is a weak functorial lift of $\pi'$).

If the cuspidal representation $\pi$ of $\text{PGSp}_6$ is tempered, then the following statements are equivalent.

(a) For almost all places $v$, the Satake parameter $s_v$ of $\pi_v$ is contained in $\iota(G_2(\mathbb{C}))$.

(b) There exist an octonion algebra $\mathcal{O}$ over $F$ and a cuspidal automorphic representation $\pi'$ of $\text{Aut}(\mathcal{O})$ such that $\pi$ is a weak functorial lift of $\pi'$.

**Proof.** As explained in the introduction, we shall make use of the following seesaw dual pair in $G$:

$$
\begin{array}{ccc}
\text{Aut}(\mathcal{O}) & & G_D \\
\downarrow & & \downarrow \\
D^1 & \times & \text{PGSp}_6
\end{array}
$$

Let $\pi$ be an irreducible cuspidal automorphic representation of $\text{PGSp}_6$ and consider its global theta lift $\pi'$ on $\text{Aut}(\mathcal{O})$. It can be shown (by a standard computation of the constant term of the global theta lift) that $\pi'$ is contained in the space of cusp forms on $\text{Aut}(\mathcal{O})$. This was explained in [GJ01, Theorem 3.1], noting that the genericity assumption on $\pi$ was not needed there. See also [GG09, Proposition 5.2] (note, though, that there is a typo in the first paragraph of the proof of [GG09, Proposition 5.2]: the word ‘nonzero’ should be ‘zero’).

Now suppose that the partial (degree-8) spin $L$-function $L^S(s, \pi, \text{Spin})$ of $\pi$ has a pole at $s = 1$. Then, by Lemma 9.1, it follows that $\text{Res}_{s=3} Z(\phi, \Phi, s)$ is non-zero, for some $\phi \in \pi$ and some $\Phi$. At this point we note that Pollack has a slightly different choice of the parameter of the Eisenstein series: his parameter $s'$ and our $s$ are related by $s = 2s' - 5$. Hence the integral of $\phi$ against some residue $\text{Res}_{s=1} E_D(s, \Phi)$ is non-zero. Since the space of residues at $s = 1$ is invariant under the complex conjugation, it follows that the integral of $\overline{\phi}$ against some residue $\text{Res}_{s=1} E_D(s, \Phi)$ is non-zero. Note that the Eisenstein series $E_D(s, \Phi)$ used in the global zeta integral are associated to smooth sections of the degenerate principals series (holomorphic and non-zero at $s = 1$), but the analytic behaviour of these smooth Eisenstein series at $s = 1$ is the same as that of $K$-finite Eisenstein series by [Lap08]. Hence, by the Siegel–Weil formula (Theorem 8.2), it follows that

$$
\int_{\text{PGSp}_6(F) \setminus \text{PGSp}_6(\mathbb{A})} \overline{\phi}(g) \cdot \left( \int_{D^1(F) \setminus D^1(\mathbb{A})} \theta(f)(gh) \, dh \right) \, dg \neq 0
$$

for some $\mathcal{O} \supset D$, $f \in \Pi_{\mathcal{O}}$ and $\phi \in \pi$, where $\theta(f)$ is rapidly decreasing on $D^1(F) \setminus D^1(\mathbb{A})$ and of moderate growth on $\text{PGSp}_6(\mathbb{A})$. Exchanging the order of integration, we deduce that the global
theta lift of $\pi$ to $\text{Aut}(\mathbb{O})$ is non-zero, that is,

$$\phi'(h) = \int_{\text{PGSp}_6(F) \backslash \text{PGSp}_6(A)} \theta(f)(gh) \overline{\phi}(g) \, dg$$

is a non-zero function of uniform moderate growth on $\text{Aut}(\mathbb{O}) \backslash \text{Aut}(\mathbb{O} \otimes_F A)$.

It is given that $\phi$ is an eigenfunction for the center of the enveloping algebra of $\text{PGSp}_6(F_v)$ for every real place $v$ of $F$. By [HPS96] and [Li99], for every element $z$ in the center of the enveloping algebra of $\text{PGSp}_6(F_v)$, there exists an element $z'$ in the center of the enveloping algebra of $\text{Aut}(\mathbb{O}_v)$ such that $z = z'$ when acting on the minimal representation. In particular, $z' \cdot f = z \cdot f$. Thus $\phi'$ is an eigenfunction for the center of the enveloping algebra of $\text{Aut}(\mathbb{O}_v)$ for every real place $v$ of $F$. (At this point we use that $\phi$ is rapidly decreasing to justify that differentiation of $f$ can be carried over to differentiation of $\phi$.)

Similarly, it is given that $\phi$ is an eigenfunction for the Hecke algebra for almost all finite places. But so is $\phi'$ by matching of Hecke operators under the exceptional theta correspondences [SW15]. Moreover, by [SW15, Theorem 1.1], if $s'_v$ are the Satake conjugacy classes in $G_2(C)$ corresponding to $\phi'$ and $s_v$ are the Satake conjugacy classes in Spin$_7(C)$ corresponding to $\phi$, then $s_v = \iota(s'_v)$, where $\iota : G_2(C) \to \text{Spin}_7(C)$ is the natural inclusion. Hence, the submodule generated by all such global theta lifts $\phi'$ gives an automorphic representation $\pi'$ which weakly lifts to $\pi$. This proves the first assertion of the theorem.

For the second part of the theorem, it is clear that (b) implies (a). Conversely, as observed by Chenevier [Che19, Theorem 6.18, equation (6.6)], hypothesis (a) in the theorem implies that

$$L^S(s, \pi, \text{Spin}) = \zeta^S(s) \cdot L^S(s, \pi, \text{Std}),$$

where the last L-function on the right is the degree-7 (partial) standard L-function of $\pi$. Since we are assuming that $\pi$ is tempered, it follows that $L^S(1, \pi, \text{Std})$ is finite and non-zero (by the characterization of the image of the standard functorial lifting of tempered cuspidal representations from $\text{Sp}_6$ to $\text{GL}_7$). Hence $L^S(s, \pi, \text{Spin})$ has a pole at $s = 1$ and the results we have shown above imply that (b) holds, with $\pi'$ the global theta lift of $\pi$ to $\text{Aut}(\mathbb{O})$.

This completes the proof of the theorem.

**Remark.** Let us comment on the relation of Theorem 10.1 with [Che19, Theorem 6.18].

- In [Che19, Theorem 6.18], Chenevier showed the second part of Theorem 10.1 for globally generic cuspidal representations, by reducing it to the first part of Theorem 10.1, which is a result of Ginzburg and Jiang for globally generic cuspidal representations. As Chenevier remarked in [Che19, Remark 6.19], if one has an analog of the endoscopic classification of Arthur for $\text{PGSp}_6$, one would know that any tempered cuspidal representation of $\text{PGSp}_6$ is nearly equivalent to a globally generic cuspidal representation, in which case the second part of the theorem will follow for tempered cuspidal representations by reduction to the globally generic case.

- In our proof of Theorem 10.1, our argument reducing the second part of the theorem to the first follows Chenevier’s. Thus, the main innovation of Theorem 10.1 is a direct proof of the first part of the theorem for all cuspidal representations, regardless of whether they are globally generic or tempered. In particular, this gives the second part of the theorem without resort to an Arthur-type classification for $\text{PGSp}_6$.

We can strengthen our results in the case where $F = \mathbb{Q}$ and $\pi$ is a cuspidal representation of $\text{PGSp}_6(A)$ that corresponds to a classical Siegel holomorphic form of positive weight. Recall that
there are two isomorphism classes of octonion algebras over \( \mathbb{Q} \); the classical octonion algebra \( \mathbb{O}^c \) and its split form \( \mathbb{O}^s \). Then \( \text{Aut}(\mathbb{O}^c) \) is an anisotropic group, while \( \text{Aut}(\mathbb{O}^s) \) is split.

**Theorem 10.2.** Let \( F = \mathbb{Q} \), and \( \pi \) a cuspidal representation of \( \text{PGSp}_6(\mathbb{A}) \) that corresponds to a classical Siegel holomorphic form \( \phi_{2r} \) of weight \( 2r > 0 \). If \( L^S(s, \pi, \text{Spin}) \) has a pole at \( s = 1 \), then \( \pi \) is a lift from \( \text{Aut}(\mathbb{O}^c) \). Moreover, if the level of \( \phi_{2r} \) is 1, then \( \pi \) is a strong functorial lift from \( \text{Aut}(\mathbb{O}^c) \).

**Proof.** Let \( U_3(\mathbb{R}) \) be the maximal compact subgroup of \( \text{Sp}_6(\mathbb{R}) \). By our assumption, \( \pi_\infty \) is a lowest weight module, with the minimal \( U_3(\mathbb{R}) \)-type \( \det^{2r} \), \( r > 0 \). We need the following lemma.

**Lemma 10.3.** Let \( \sigma \) is a lowest weight module of \( \text{Sp}_6(\mathbb{R}) \), with the minimal \( U_3(\mathbb{R}) \)-type \( \det^{2r} \), \( r > 0 \). Then \( \sigma \) does not occur in the exceptional theta correspondence with split \( G_2(\mathbb{R}) \).

**Proof.** Adopting the notation from [LS19], let \( G' = G_2(\mathbb{R}) \), \( g' \) the Lie algebra of \( G' \), \( K' \) a maximal compact subgroup of \( G' \), and \( g' = t' \oplus p' \) the corresponding Cartan decomposition. Let

\[
\Pi = \bigoplus_{n=0}^\infty V_n
\]

be the decomposition of the minimal representation of the split real \( E_7 \) into its \( K \)-types. Let \( V_n^{\det^{2r}} \) be the maximal subspace of \( V_n \) on which \( U_3(\mathbb{R}) \) acts by the character \( \det^{2r} \). If \( r = 0 \), by [LS19, Proposition 5.2], the dimension of this space is equal to the dimension of \( S_n(p') \), the space of the \( n \)-th symmetric tensor power of \( p' \). But this result can be easily generalized to any \( r \): \( V_n^{\det^{2r}} \) is non-trivial only for \( n \geq 3r \), and the dimension of \( V_n^{\det^{2r}} \) is equal to the dimension of \( S_n(p') \). In particular, \( V_n^{\det^{2r}} \) is one-dimensional. Let \( v_r \) be a vector spanning this line. The group \( K' \) acts on this line and the vector \( v_r \) is fixed by \( K' \), since \( K' \) is semisimple. By [LS19, Lemma 3.1], the matrix coefficient of \( v_r \), when restricted to \( G' \), is contained in \( L^{3/2+\epsilon}(G') \). This fact, combined with the dimension of \( \det^{2r} \)-invariants in the types of \( \Pi \), implies that

\[\Pi^{\det^{2r}} = U(g') \cdot v_r \cong U(g') \otimes U(k') \subset \mathbb{C}\]

as explained in the introduction of [LS19], where the case \( r = 0 \) is discussed. After taking \( \det^{2r} \)-invariants in \( \Pi \to \Theta(\sigma) \boxtimes \sigma \), it follows that \( \Theta(\sigma) \) is a quotient of \( U(g') \otimes U(k') \subset \mathbb{C} \). Thus any irreducible quotient \( \sigma' \) of \( \Theta(\sigma) \) is spherical. It was also shown in [LS19] that \( \Theta(\sigma') \) has unique irreducible quotient, and it is spherical. This is a contradiction, since \( \sigma \) is not spherical, and hence it cannot appear in this correspondence.

The correspondence for the dual pair \( \text{Aut}(\mathbb{O}^c) \times \text{Sp}_6(\mathbb{R}) \) was completely determined in [GS98] and is functorial. Thus, if \( \phi_{2r} \) is of level 1 (i.e. spherical at all primes), then \( \pi \) is indeed a (strong) functorial lift from \( \text{Aut}(\mathbb{O}^c) \).

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Wee Teck Gan matgwt@nus.edu.sg
Department of Mathematics, National University of Singapore,
10 Lower Kent Ridge Road, 119076, Singapore

Gordan Savin savin@math.utah.edu
Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA