EXTENSIONS OF CONTINUOUS FUNCTION IN \( LG \)-TOPOLOGY

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Abstract. In this article we introduce three maps, \( OLG \), \( CLG \) and \( LG \) in \( LG \)-space literature and show that these maps are extension of the continuous function in \( LG \)-spaces and have the almost properties of the continuous functions. Also, it has been introduced and studied the natural notions, quotient, decomposition, weak \( LG \)-topology and isomorphism, related to the continuous function.

1. Introduction

A lattice is said to be complete if every subset of \( L \) has the supremum, so a complete lattice has the greatest element \( 1 \) and the smallest element \( 0 \). A frame \( F \) is a complete lattice such that for each \( a \in F \) and \( \{ b_i \}_{i \in I} \subseteq F \), \( a \land (\bigvee_{i \in I} b_i) = \bigvee_{i \in I} (a \land b_i) \), if \( a \lor (\bigwedge_{i \in I} b_i) = \bigwedge_{i \in I} (a \lor b_i) \), also holds, then \( F \) is called symmetric frame. For each element \( a \in F \), \( a^\ast \) is defined \( \bigvee a^\ast \), where \( a^\perp = \{ x \in F : x \land a = 0 \} \). An element \( b \) of a frame \( F \) is called the complement of an element \( a \) of \( F \), if \( a \land b = 0 \) and \( a \lor b = 1 \). Clearly, the complement of an element \( a \), if exists, is unique. We denote the complement of \( a \) by \( a^\ast \). It is easy to see that if the complement of an element \( a \) exists, then \( a^\ast = a^\ast \). If each element of a frame \( F \) has a complement, we say that \( F \) is a complemented frame. A subset \( G \) of a frame \( F \) is said to be a subframe of \( F \), if \( G \) is closed under finite meets and arbitrary joins.

Suppose that \( F \) is a frame, we say \( \tau \subseteq F \) is a lattice generalized topology (briefly \( LG \)-topology) on \( F \), if \( \tau \) is a subframe of \( F \), then \( (F, \tau) \) (briefly \( F \)) is said to be an \( LG \)-space, every element of \( \tau \) is called open element and every element of \( \tau^* = \{ t^* : t \in \tau \} \) is called closed element. Clearly, for each family \( F \subseteq \tau^* \) we have \( \bigwedge F \in \tau^* \). Furthermore, if \( \tau^* \) is sublattice of \( F \), then we call \( \tau \) an LT-space. An \( LG \)-topology \( \tau \) on a frame \( F \) is said to be discrete (trivial), if \( \tau = F \) (\( \tau = \{ 0, 1 \} \)). If \( (F, \tau) \) is an \( LG \)-space, then for each element \( a \in F \), the interior and closure of \( a \), denoted by \( a^\circ \) and \( \bar{a} \), are defined by \( \bigvee \{ t \in \tau : t \leq a \} \) and \( \bigwedge \{ f \in \tau^* : a \leq f \} \), respectively. It is clear to see that, \( \pi \in \tau^* \), for each \( a \in F \) and \( f^* = f \), for each \( f \in \tau^* \). If for some \( a \in F \), \( \pi = 1 \), then we say \( a \) is dense in \( F \). Clearly, if \( (F, \tau) \) is an \( LG \)-space and \( a \in F \), then \( (F_a, \tau_a) \) is also an \( LG \)-space, in which \( F_a = \downarrow a = \{ x \in F : x \leq a \} \) and \( \tau_a = \{ s \land a : s \in \tau \} \). We call \( (F_a, \tau_a) \) a subspace of \( (F, \tau) \). A \( LG \)-space \( (F, \tau) \) is called compact if for each family of open elements \( \{ t_\alpha \}_{\alpha \in A} \) such that \( 1 = \bigvee_{\alpha \in A} a_\alpha \), there is a subfamily \( \{ t_\alpha \}_{\alpha \in A} \) of \( \{ t_\alpha \}_{\alpha \in A} \) such that \( 1 = \bigvee_{\alpha \in A} a_\alpha \). Similarly, countably compact and Lindelöf spaces are defined. Suppose that \( \{ (F_\alpha, \tau_\alpha) \}_{\alpha \in A} \) is a family of \( LG \)-spaces, then \( \tau_F = \{ t \in \prod_{\alpha \in A} F_\alpha : \forall \alpha \in A, \ t_\alpha \in \tau_\alpha \ and \ just \ for \ finitely \ many \ \alpha \in A, \ t_\alpha \neq 1 \} \) is an \( LG \)-topology on
\[ \prod_{\alpha \in A} F_{\alpha}. \] (\[ \prod_{\alpha \in A} F_{\alpha}, \tau_p \]) is called product LGT-space. Suppose that \((F, \tau)\) is an LGT-space and \(B \subseteq \tau\), we say \(B\) is a base for \(\tau\), if for each \(t \in \tau\), there is some \(B' \subseteq B\) such that \(t = \bigvee B'\).

Suppose that \(F_1\) and \(F_2\) are two frames, we say a map \(\phi : F_1 \to F_2\) is an arbitrary join preserve map if for each \(\emptyset \neq E \subseteq F_1\), we have \(\phi(\bigvee E) = \bigvee \phi(E)\), also we say a map \(\psi : F_2 \to F_1\) is a right adjoint of \(\phi\), if for each \(a \in F_1\) and \(b \in F_2\)

\[
\phi(a) \leq b \iff a \leq \psi(b)
\]

Similarly arbitrary meet preserve and left adjoint are defined. It is easy to show that, if \(\phi\) is an arbitrary join (meet) preserve map, then \(\phi\) has a unique right (left) adjoint map and this right adjoint map, denoted by \(\phi_* (\phi^*)\), is an arbitrary meet (join) preserve.

The reader is referred to \[11, 8\], for undefined terms and notations.

Two kind of pointfree version of topological spaces have been introduced. In the first model, since the set of open sets of a topological space is a frame, the researchers focus on a topology as a frame. Introducing of the first version has been started in \[10, 5, 6, 3, 7, 2, 4, 9\] and then studied in many articles. In the new model, researchers pursue this viewpoint and introduce the new structure. The second structure, introduced and studied just in \[1\].

In the rest of this section we give some elementary properties of the right adjoint of a map which we need them in the main parts of this research. In Section 2, we introduce three kind maps \(\text{OLG, CLG and LG maps}\). We show that these maps have some properties similar to the continuous function and also we prove that they are extensions of the continuous function in the lattice generalization topology literature and finally, we introduce a functor form topological spaces and continuous functions to \(\text{LGT-space and LG-maps}\). Section 3 is devoted to properties of \(\text{OLG, CLG and LG maps}\) on subspaces of an \(\text{LG-space}\). In Section 4, the relation of the \(\text{OLG and product LGT-space}\) studied and then the new notions quotient \(\text{LG-space}\) and decomposition \(\text{LG-space}\) are introduced, by the used of \(\text{OLG, CLG and LG maps}\). Finally, in Section 5, the concepts open map, \(\text{LG isomorphism}\) and related topics introduced and studied.

**Proposition 1.1.** Suppose that \(F_1, F_2\) and \(F_3\) are frames, \(\phi : F_1 \to F_2\) and \(\psi : F_2 \to F_3\) are two arbitrary join preserve maps. Then

1. \(\phi_* (b) = \bigvee_{\phi(x) \leq b} x\), for every \(b \in F_2\).
2. \(\phi_*(b) = \max\{x \in F_1 : \phi(x) \leq b\}\), for each \(b \in F_2\).
3. \(\phi(\phi_*(b)) = b\), for every \(b \in F_2\).
4. \(\phi \circ \psi\) is an arbitrary join preserve map and \((\phi \circ \psi)_* = \psi_* \circ \phi_*\).

**Proof.** (a). Since \(\phi\) is an arbitrary join preserve map, \(\phi\) has a unique right adjoint map, so it is sufficient to show that \(v : F_2 \to F_1\), defined by \(v(b) = \bigvee_{\phi(a) \leq b} a\), is a right adjoint of \(\phi\). Now, suppose that \(\phi(a) \leq b\), then \(a \leq \bigvee_{\phi(x) \leq b} x = v(b)\).

Conversely, If \(a \leq v(b) = \bigvee_{\phi(x) \leq b} x\), then

\[
\phi(a) = \phi(\bigvee_{\phi(x) \leq b} x) = \bigvee_{\phi(x) \leq b} \phi(x) \leq b.
\]

Hence \(v\) is a right adjoint of \(\phi\).

(b), (c) and (d). They are evident. \(\square\)
We can give and prove a proposition similar to the above proposition for the arbitrary meet preserve and left adjoint maps

**Lemma 1.2.** Suppose that $F_1$ and $F_2$ are two frames and $\phi : F_1 \to F_2$. If $\phi$ and $\phi_*$ are arbitrary join preserve maps, then $\phi_* = \phi$.

**Proof.** By Proposition 1.1 $\phi_* (\phi_*(x)) = x$, so $\phi (\phi_*(\phi_*(x))) = \phi (x)$. In the other hand, again, by Proposition 1.1 $\phi (\phi_*(\phi_*(x))) = \phi_*(x)$, hence $\phi(x) = \phi_*(x)$. □

**Lemma 1.3.** Suppose that $F_1$ and $F_2$ are two frames. If $\phi$ is a one-to-one map from $F_1$ onto $F_2$, then the following statements are equivalent

(a) $\phi$ is arbitrary join preserve.
(b) $\phi$ is arbitrary meet preserve.
(c) $\phi_*$ is arbitrary join preserve.
(d) $\phi_*$ is arbitrary meet preserve.

**Proof.** It is easy, by this fact that, if a one-to-one onto map $\phi$ has either a right or a left adjoint, then $\phi^{-1}$ is the right and left adjoint. □

2. Extensions

In this section we introduce some maps and, by some examples, show that the categories of these maps are not equal together. Then we prove that they have the most property of the continuous functions. After that, it has been shown that these introduced maps are extensions of the continuous function. Finally, a functor has been introduced between categories of the topological spaces and the $LGT$-spaces.

**Definition 2.1.** Suppose that $(F_1, \tau_1)$ and $(F_2, \tau_2)$ are two $LGT$-spaces. An arbitrary join preserve map $\phi : F_1 \to F_2$ is called an OLG (CLG) map if for each $b \in \tau_2 (b \in \tau_2^*)$, $\phi_*(b) \in \tau_1 (\phi_*(b) \in \tau_1^*)$. We say $\phi$ is an LG map if it is both OLG and CLG.

Suppose that $(F_1, \tau_1)$ and $(F_2, \tau_2)$ are two $LGT$-spaces. It easy to see that, if a map $\phi : F_1 \to F_2$ is OLG, then for each $u \in \tau_2$,

$$\phi_*(u) = \bigvee_{\phi(x) \leq u} x = \bigvee_{\phi(t) \leq u} t = \max \{ t \in \tau_1 : \phi(t) \leq u \}.$$ 

**Proposition 2.2.** Suppose that $(F, \tau_1)$ and $(F, \tau_2)$ are two $LGT$-spaces.

(a) The identity map $I_F : (F, \tau_1) \to (F, \tau_2)$ is OLG, if and only if $\tau_2 \subseteq \tau_1$.
(b) The identity map $I_F : (F, \tau_1) \to (F, \tau_2)$ is CLG, if and only if $\tau_2^* \subseteq \tau_1^*$.

**Proof.** They are same and straightforward. □

In the following example, by the use of the above proposition, we show that a CLG map need not be an OLG map.

**Example 2.3.** Suppose that $F = \{0, a, 1 \}$, $\tau_1 = \{0, 1 \}$ is the trivial $LG$-topology and $\tau_2 = \{0, a, 1 \}$ is the discrete $LG$-topology. Then $\tau_2^* = \{0, 1 \}$ and $\tau_1^* = \{0, 1 \}$.

By the above proposition, $I_F : (F, \tau_1) \to (F, \tau_2)$ is CLG and is not OLG.

**Corollary 2.4.** Let $(F, \tau)$ be an $LGT$-space.

(a) $\tau$ is the discrete $LG$-topology on $F$; if and only if for each $LGT$-space $(F', \tau')$, each arbitrary join preserve map $\phi : F \to F'$ is OLG.
(b) \( \tau^* = F \); if and only if for each \( \text{LGT}-\text{space} \ (F', \tau') \), each arbitrary join preserve map \( \phi : F \rightarrow F' \) is \( \text{CLG} \).

(c) If for each \( \text{LGT}-\text{space} \ (F', \tau') \), each arbitrary join preserve map \( \phi : F' \rightarrow F \) is \( \text{OLG} \), then \( \tau \) is the trivial LG-topology on \( F \).

(d) If for each \( \text{LGT}-\text{space} \ (F', \tau') \), each arbitrary join preserve map \( \phi : F' \rightarrow F \) is \( \text{CLG} \), then \( \tau^* = \{0, 1\} \).

Proof. (a \( \Rightarrow \)). It is clear.

(a \( \Leftarrow \)). Since the identity map \( I_F : (F, \tau) \rightarrow (F, F) \) is \( \text{OLG} \), by Proposition 2.2, \( \tau = F \) is the discrete LG-topology on \( F \).

(b). It is similar to (a).

(c). Suppose that \( \tau' \) is the trivial topology on \( F \). Since \( I_F : (F, \tau') \rightarrow (F, \tau) \) is \( \text{OLG} \), by Proposition 2.2, \( \tau = \tau' \).

(d). It is similar to (b). \( \square \)

In the following example, we show that the converse of the statements (c) and (d) in the above theorem need not be true.

Example 2.5. Suppose that \( \phi \) and \( \tau_1^* = \{0, a_1, 1\} \) and \( \tau_2^* \) is the trivial LG-topology. Then \( \tau_1^* = \{0, 1\} \) and \( \tau_2^* = \{0, 1\} \). Clearly, \( \phi \) is arbitrary join preserve and it is easy to check that \( \phi \) is not neither \( \text{OLG} \) nor \( \text{CLG} \).

Lemma 2.6. Suppose that \( (F, \tau) \) is an \( \text{LGT}-\text{space} \). If \( \tau^* = F \), then \( \tau \) is the discrete LG-topology.

Proof. Suppose that \( a \in F \), then \( t, u, v \in \tau \) exist such that \( a = t^* \), \( t = u^* \) and \( u = v^* \), then, by [1 Remark 1.1],

\[
\phi
\]

Thus \( \tau = F \) is the discrete LG-topology. \( \square \)

Now the above corollary and the above lemma conclude the following corollary.

Corollary 2.7. Suppose that \( (F, \tau) \) is an \( \text{LGT}-\text{space} \). If for each \( \text{LGT}-\text{space} \ (F', \tau') \), each arbitrary join preserve is \( \text{CLG} \), then \( \tau \) is the discrete LG-topology.

In the following example we give an \( \text{OLG} \) map which is not \( \text{CLG} \) map and we show that the converse of the above corollary need not be true.

Example 2.8. Suppose that
and $\tau_1 = F_1$ and $\tau_2 = F_2$ are the trivial LG-topologies. Then the map $I_F : F_1 \to F_2$, defined by $\phi(0) = 0$, $\phi(a_1) = b_1$, $\phi(a_2) = b_3$ and $\phi(1) = 1$ is an OLG map, by Corollary 2.4. But $b_2 \in \{0, b_1, b_2, 1\} = \tau_2^*$ and $\bigvee_{\phi(x) \leq b_2} x = a_1 \notin \{0, 1\} = \tau_1^*$, so $\phi$ is not CLG map.

**Proposition 2.9.** Suppose that $(F_1, \tau_1)$ and $(F_2, \tau_2)$ are two LGT-spaces. If $\phi : F_1 \to F_2$ is an OLG map and $\phi_*(a^*) = (\phi_*(a))^*$, then $\phi$ is an LG map.

**Proof.** It is clear. □

**Theorem 2.10.** Suppose that $(F_1, \tau_1)$ and $(F_2, \tau_2)$ are two LGT-spaces and $\phi : F_1 \to F_2$ is an arbitrary join preserve map. Then the followings are equivalent.

1. $\phi$ is CLG.
2. $\phi(\bar{a}) \leq \bar{\phi}(a)$, for each $a \in F_1$.
3. $\phi_*(b) \leq \phi_*(\bar{b})$, for each $b \in F_2$.

**Proof.** ($a \Rightarrow b$). Since $\phi(a) \leq \phi(\bar{a})$, $\bar{a} \leq \phi_*(\phi(\bar{a}))$. Since $\phi(\bar{a}) \in \tau_2^*$, $\phi_*(\phi(\bar{a})) \in \tau_2^*$, so $\bar{a} \leq \phi_*(\phi(a))$ and therefore $\phi(\bar{a}) \leq \phi(a)$.

($b \Rightarrow c$). Since $\phi_*(b) \leq \phi_*(\phi_*(b)) \leq b$, then by the assumption $\phi_*(\phi_*(b)) \leq \phi_*(\phi_*(\phi_*(b))) \leq \phi_*(\phi_*(\phi_*(b)))$, so $\phi_*(t^*) = \phi_*(t^*)$, hence $\phi_*(t^*) \in \tau_1^*$. Consequently, $\phi$ is a CLG map. □

**Corollary 2.11.** Suppose that $(F_1, \tau_1)$ and $(F_2, \tau_2)$ are two LGT-spaces and $\phi : F_1 \to F_2$ is an onto CLG map. If $a \in F_1$ is dense in $F_1$, then $\phi(a)$ is dense in $F_2$.

**Proof.** It follows easily from the above theorem. □

**Theorem 2.12.** Suppose that $(F_1, \tau_1)$ and $(F_2, \tau_2)$ are two LGT-spaces and $\phi : F_1 \to F_2$ is an arbitrary join preserve map. Then $\phi$ is OLG; if and only if $\phi_*(b^o) \leq \phi_*(\phi_*(b))$, for each $b \in F_2$.

**Proof.** ($\Rightarrow$). Since $b^o \in \tau_2$, $\phi_*(b^o) \in \tau_1$, so $\phi_*(b^o) = \phi_*(\phi_*(b^o)) \leq \phi_*(\phi_*(b^o))$.

($\Leftarrow$). For each $t \in \tau_2$, $\phi_*(t) = \phi_*(t^o) \leq \phi_*(t^o)$, so $\phi_*(t) = \phi_*(t^o) \in \tau_1^*$, and therefore $\phi$ is OLG. □

**Theorem 2.13.** Suppose that $(F_1, \tau_1)$, $(F_2, \tau_2)$ and $(F_3, \tau_3)$ are LGT-spaces.

1. If $\phi : F_1 \to F_2$ and $\psi : F_2 \to F_3$ are OLG maps, then $\psi \circ \phi : F_1 \to F_3$ is an OLG map.
(b) If $\phi : F_1 \to F_2$ and $\psi : F_2 \to F_3$ are CLG maps, then $\psi \circ \phi : F_1 \to F_3$ is a CLG map.
(c) If $\phi : F_1 \to F_2$ and $\psi : F_2 \to F_3$ are LG maps, then $\psi \circ \phi : F_1 \to F_3$ is an LG map.

Proof. (a). By Proposition 1.11 and Theorem 2.12 for every $b \in F_2$, we have

$$(\phi \circ \psi)_* (a^\circ) = \psi_* (\phi_* (a^\circ)) \leq (\psi_* (\phi_*(a)))^\circ = ((\phi \circ \psi)_*(a))^\circ$$

Hence, by Theorem 2.12, $\psi \circ \phi$ is an OLG map.

(b). By Theorem 2.10 for every $a \in F_1$, we have $\psi(\phi(\pi)) \leq \psi(\phi(a)) \leq \psi(\phi(a))$.

Hence, by Theorem 2.10 $\psi \circ \phi$ is a CLG map.

(c). It follows immediately from (a) and (b).

In the following proposition, we show that the CLG, OLG and LG maps are extensions of the continuous function.

Suppose that $f : X \to Y$; we put $\bar{\mathcal{O}}(f) : \mathcal{P}(X) \to \mathcal{P}(Y)$, defined by $\bar{\mathcal{O}}(f)(S) = f(S)$, for each $S \in \mathcal{P}(X)$.

**Proposition 2.14.** If $(X, \tau)$ and $(Y, \tau')$ are topological spaces and $f : X \to Y$, then the followings are equivalent:

(a) $\bar{\mathcal{O}}(f)$ is an OLG map.
(b) $\bar{\mathcal{O}}(f)$ is a CLG map.
(c) $\bar{\mathcal{O}}(f)$ is an LG map.
(d) $f$ is a continuous function.

Proof. Since $\bar{\mathcal{O}}(f)_*(A) = \bigvee_{(f(A) \subseteq S)} A = \bigcup_{f(A) \subseteq S} A = f^{-1}(S)$, the above statements are equivalent.

It is easy to check that every constant map is an LG map. Suppose that $(X, \tau)$ and $(Y, \tau')$ are topological spaces and $\emptyset \neq B \subseteq Y$. Then the constant map $\phi : \mathcal{P}(X) \to \mathcal{P}(Y)$; defined by $\phi(S) = B$, for each $S \subseteq X$, is an LG map and there are not any map $f : X \to Y$ such that $\phi = \bar{\mathcal{O}}(f)$.

For each topological space $(X, \tau)$, we put $\bar{\mathcal{O}}(X, \tau) = (\mathcal{P}(X), \tau)$. Now we can conclude the following corollary from Theorem 2.13, Proposition 2.14 and this fact that $\bar{\mathcal{O}}(I_X) = I_{\mathcal{P}(X)} = I_{\mathcal{U}(X)}$.

**Corollary 2.15.** If $\text{Top}$ is the category of topological spaces and continuous maps and $\text{Lgt}$ is the category of LGT-spaces and LG maps, then $\bar{\mathcal{O}} : \text{Top} \to \text{Lgt}$ is a functor.

3. Subspace

This section has been devoted to studying the relations between the OLG, CLG and LG maps and the subspaces. It has been shown some versions of the relations between a continuous function and a subspace satisfy for these extension. Also, it has been gave some counterexample for some general versions which does not satisfy.

**Lemma 3.1.** Suppose that $(F, \tau)$ and $(F', \tau')$ are two LGT-spaces, $a \in F$ and $\phi : (F, \tau) \to (F', \tau')$. Then $(\phi|_{F_a})_*(y) = \phi_*(y) \wedge a$, for each $y \in F'$.
Suppose that Lemma 3.4.

Proof. Suppose that Theorem 3.5.

(a) It is straightforward.

Proof. Suppose that

Hence \( (\phi|_{F_a})_*(y) = \phi_*(y) \land a \)

\( \square \)

Theorem 3.2. Suppose that \((F, \tau)\) and \((F', \tau')\) are two LGT-spaces, \(a \in F\) and \(\phi : (F, \tau) \to (F', \tau')\).

(a) If \(\phi\) is an OLG map, then \(\phi|_{F_a} : (F_a, \tau_a) \to (F', \tau')\) is an OLG map.

(b) If \(a \in F^*\) and \(\phi\) is a CLG map, then \(\phi|_{F_a} : (F_a, \tau_a) \to (F', \tau')\) is a CLG map.

(c) If \(a \in F^*\) and \(\phi\) is an LG map, then \(\phi|_{F_a} : (F_a, \tau_a) \to (F', \tau')\) is an LG map.

Proof. (a). Suppose that \(t \in \tau'\), then \(\phi_*(t) \in \tau\), so by Lemma 3.1, \(\phi|_{F_a}_*(t) = \phi_*(t) \land a\), hence \(\phi|_{F_a}_*(t) \in F_a\). Consequently, \(\phi|_{F_a}\) is OLG.

(b). By Proposition 3.4, the proof is similar to the proof of part (a).

(c). It follows immediately from parts (a) and (b).

\( \square \)

Clearly, since each element of a complemented frame is a complement of some element, the above theorem implies the following corollary.

Corollary 3.3. Suppose that \((F, \tau)\) and \((F', \tau')\) are two LGT-spaces, \(a \in F\) and \(\phi : (F, \tau) \to (F', \tau')\). If \(\phi\) is an LG map and \(F\) is complemented, then \(\phi|_{F_a} : (F_a, \tau_a) \to (F', \tau')\) is an LG map.

Lemma 3.4. Suppose that \((F, \tau)\) is an LGT-space and \(a \in \tau\).

(a) For each \(t \in F_a\), \(t \in \tau_a\), if and only if \(t \in \tau\).

(b) For each \(b \in F_a^*, b \in \tau_a^*, \) if and only if \(b \in \tau^*\).

Proof. (a). It is straightforward.

(b). By Proposition 3.4, the proof is similar to part (a).

\( \square \)

Theorem 3.5. Suppose that \(\phi : F \to F'\) is an arbitrary join preserve map.

(a) If \((F, \tau)\) and \((F', \tau')\) are two LGT-spaces, \(s, t \in \tau,\) \(\phi|_{F_s}\) and \(\phi|_{F_t}\) are OLG maps and \(s \lor t = 1\), then \(\phi\) is an OLG map.

(b) If \((F, \tau)\) and \((F', \tau')\) are two LT-spaces, \(a, b \in \tau^*_a\), \(\phi|_{F_a}\) and \(\phi|_{F_b}\) are CLG maps and \(a \lor b = 1\), then \(\phi\) is a CLG map.

Proof. (a). Suppose that \(u \in \tau'\). Then \((\phi|_{F_s})_*(u), (\phi|_{F_t})_*(u) \in \tau\), thus, by Lemma 3.1,

\[
\phi_*(u) = \phi_*(u) \land (s \lor t)
= (\phi_*(u) \land s) \lor (\phi_*(u) \land t)
= (\phi|_{F_s})_*(u) \lor (\phi|_{F_t})_*(u) \in \tau
\]

Hence \(\phi\) is OLG.

(b). Since \((F, \tau)\) and \((F', \tau')\) are two LT-spaces, the proof is similar to the proof of part (a).

\( \square \)

Proposition 3.6. Suppose that \((F, \tau)\) and \((F', \tau')\) are two LGT-spaces. If \(\phi : F' \to F\) is an OLG map such that \(\phi(F') = F_a\), for some \(a \in F\), then \(\phi : F \to F_a\) is OLG.
Proof. Suppose that \( s \in \tau_a \), then \( t \in \tau \) exists such that \( s = t \land a \). Since \( \phi(1) = a \), \( \phi_*(a) = 1 \), so
\[
\phi_*(s) = \phi_*(t \land a) = \phi_*(t) \land \phi_*(a) = \phi_*(t) \land 1 = \phi_*(t) \in \tau'
\]
Hence \( \phi : F' \to F_a \) is OLG. \( \square \)

In the following example we show that the LG map image of a compact LGT-space need not be compact.

Example 3.7. Suppose that \( F_2 = [0, 1] \) with ordinary relation and \( F_1 = [0, \frac{1}{2}] \cup \{a, b, 1\} \) with the following illustrated relation

\[
\begin{array}{c}
1 \\
\downarrow \\
\downarrow \\
\downarrow \\
\downarrow \\
F_1
\end{array}
\begin{array}{c}
\text{b} \\
\text{a} \\
\text{1} \\
\text{?} \\
\text{0}
\end{array}
\]

It is easy to check that \( F_1 \) and \( F_2 \) are frames. Consider \( \tau_1 = F_1 \) and \( \tau_2 = F_2 \). Clearly \( \tau_1^* = \{0, 1, a, b\} \), \( \tau_2^* = \{0, 1\} \), \( (F_1, \tau_1) \) is a compact LGT-space, \( (F_2, \tau_2) \) is not a compact LGT-space and the map \( \phi : (F_1, \tau_1) \to (F_2, \tau_2) \), defined by

\[
\phi(x) = \begin{cases} 
1 & x \notin [0, \frac{1}{2}] \\
2x & x \in [0, \frac{1}{2}]
\end{cases}
\]

is an onto LG map.

Proposition 3.8. Suppose that \( (F_1, \tau_1) \) and \( (F_2, \tau_2) \) are two LGT-spaces, \( \phi : F_1 \to F_2 \) is an onto OLG and \( \bigvee_{\alpha \in A} u_\alpha = 1 \) implies that \( \bigvee_{\alpha \in A} \phi_*(u_\alpha) = 1 \), for every subfamily \( \{u_\alpha\}_{\alpha \in A} \) of \( \tau_2 \).

(a) If \( F_1 \) is compact, then \( F_2 \) is compact.

(b) If \( F_1 \) is countably compact, then \( F_2 \) is countably compact.

(c) If \( F_1 \) is Lindelöf, then \( F_2 \) is Lindelöf.

Proof. (a). Suppose that \( \{u_\alpha\}_{\alpha \in A} \) is a family of open elements of \( F_2 \) and \( \bigvee_{\alpha \in A} u_\alpha = 1 \). Since \( \phi \) is an OLG map, \( t_\alpha = \phi_*(u_\alpha) \in \tau_\alpha \). By the assumption, \( \bigvee_{\alpha \in A} t_\alpha = 1 \). Since \( F_1 \) is compact, there are \( \alpha_1, \alpha_2, \ldots, \alpha_n \in A \) such that \( 1 = \bigvee_{i=1}^n t_{\alpha_i} \), so

\[
1 = \phi\left(\bigvee_{i=1}^n t_{\alpha_i}\right) = \bigvee_{i=1}^n \phi(t_{\alpha_i}) = \bigvee_{i=1}^n u_{\alpha_i}.
\]

Consequently, \( F_2 \) is compact.

(b) and (c). They are similar to (a). \( \square \)
Suppose that, \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two frames, \(\phi : F_1 \to F_2\) is an arbitrary join preserve map and \(B\) is a base for \(\tau_2\). In the following example we show that if \(\phi_* (b) \in \tau_1\), for each \(b \in B\), then \(\phi\) need not be OLG.

**Example 3.9.** Suppose that

\[
\begin{array}{c}
\begin{array}{c}
1 \\
\downarrow \\
\hline \\
\uparrow \\
1
\end{array}
\end{array}
\]

\(F_1\) \hspace{1cm} \(F_2\)

\(\tau_2 = F_2, \tau_1 = [0, 1] \cup \{1\}, B = \{0, d, e, 1\}\) and \(\phi : F_1 \to F_2\) is defined by \(\phi|_{[0, \frac{1}{2}]} = 0, \phi(1) = 1\) and \(\phi(a) = \phi(b) = \phi(c) = f\). Then \(\phi_* (1) = c \notin \tau_1\), so \(\phi\) is not OLG, but \(\phi_* (b) \in \tau_1, \) for each \(b \in B\).

We finish this section by a proposition, in which, by adding a condition; we show that if \(\phi_* (b)\) is an open element, for each element \(b\) of a base, then \(\phi\) is OLG.

**Proposition 3.10.** Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two frames, \(\phi : F_1 \to F_2\) an arbitrary join preserve map, \(B\) is a base for \(\tau_2\) and \(\phi_* (\bigvee_{\alpha \in A} b_{\alpha}) = \bigvee_{\alpha \in A} \phi_* (b_{\alpha})\), for each subfamily \(\{b_{\alpha}\}_{\alpha \in A}\) of \(B\). If \(\phi_* (b) \in \tau_1, \) for each \(b \in B\), then \(\phi\) is OLG.

**Proof.** It is straightforward. \(\square\)

### 4. Product and Quotient

In this section, we study some related items to product LGT-spaces. Then by inspired of the well-known concepts in the topology literature, the new concepts, LGT-space generated by some family of maps, quotient LGT-space and decomposition topology have been introduced and studied.

**Theorem 4.1.** Let \(\{(F_\alpha, \tau_\alpha)\}_{\alpha \in A}\) be a family of LGT-spaces and \((\prod_{\alpha \in A} F_\alpha, \tau_p)\) be the product LGT-space. Then for each \(\beta \in A\), the projection map \(\pi_\beta : \prod_{\alpha \in A} F_\alpha \to F_\beta\) is an OLG map.

**Proof.** Clearly \(\pi_\beta\) is an arbitrary join preserve map. Suppose that \(t_\beta \in \tau_\beta\), pick \(v \in \prod_{\alpha \in A} F_\alpha\), in which

\[
v_\alpha = \begin{cases} \\
1 & \alpha \neq \beta \\
t_\beta & \alpha = \beta \\
\end{cases}
\]

For each \(x \in \prod_{\alpha \in A} F_\alpha\),

\[
x \leq (\pi_\beta)_* (t_\beta) \iff \pi_\beta (x) \leq t_\beta \iff x_\beta \leq t_\beta \iff x \leq v
\]

Hence \((\pi_\beta)_* (t_\beta) = v \in \tau_p. \square\)
**Corollary 4.2.** Let \( \{ (F_\alpha, \tau_\alpha) \}_{\alpha \in A} \) be a family of LGT-spaces, \( (\prod_{\alpha \in A} F_\alpha, \tau_p) \) be the product LGT-space and \( (F, \tau) \) is an LGT-space. An arbitrary join preserve map \( \phi : F \to \prod_{\alpha \in A} F_\alpha \) is OLG if and only if \( \pi_\alpha \circ \phi \) is OLG, for every \( \alpha \in A \).

**Proof.** (⇒). It follows immediately from Theorems 2.13 and 4.1.

(⇐). Suppose that \( t \in \tau_p \). It easy to see that, for some \( n \in \mathbb{N} \), we have \( t = \bigwedge_{i=1}^n (\pi_{\alpha_i})(t_{\alpha_i}) \), in which \( \alpha_i \in A \) and \( t_{\alpha_i} \in \tau_{\alpha_i} \), for every \( 1 \leq i \leq n \). Since \( \pi_{\alpha_i} \circ \phi \) is OLG, \( (\pi_{\alpha_i} \circ \phi)_\ast(t_{\alpha_i}) \in \tau' \), for \( 1 \leq i \leq n \). Then, by Proposition 4.11

\[
\phi_\ast(t) = \phi_\ast \left( \bigwedge_{i=1}^n (\pi_{\alpha_i})_\ast(t_{\alpha_i}) \right) = \bigwedge_{i=1}^n \phi_\ast(\pi_{\alpha_i})_\ast(t_{\alpha_i}) = \bigwedge_{i=1}^n (\pi_{\alpha_i} \circ \phi)_\ast(t_{\alpha_i}) \in \tau'
\]

Hence \( \phi \) is OLG.

**Definition 4.3.** Suppose that \( \{ (F_\alpha, \tau_\alpha) \}_{\alpha \in A} \) is a family of LGT-spaces, \( F \) is a frame and \( \phi_\ast : F \to F_\alpha \) is an arbitrary join preserve map, for each \( \alpha \in A \). The LG-topology generated by the family \( \{ \phi_\ast(t_\alpha) : \alpha \in A \) and \( t_\alpha \in \tau_\alpha \} \) is called the weak LG-topology generated by \( \{ \phi_\ast \}_{\alpha \in A} \) and \( F \) with this topology is called weak LG-space generated by \( \{ \phi_\ast \}_{\alpha \in A} \).

Actually, the product LG-topology is not a good extension of the product topology. But the weak LG-topology generated by the family \( \{ \pi_\ast(t_\alpha) \}_{\alpha \in A} \) and \( F \) coincides with \( \bar{\bigvee}(\prod_{\alpha \in A} X_\alpha) \), in which \( \tau \) is the product topology.

**Theorem 4.4.** Suppose that \( \{ (F_\alpha, \tau_\alpha) \}_{\alpha \in A} \) is a family of LGT-spaces, \( (F, \tau) \) is the weak LG-topology generated by \( \{ \phi_\ast : F \to F_\alpha \}_{\alpha \in A} \). \( (F', \tau') \) is an LGT-space and \( \psi : F' \to F \). If \( \psi \) and \( \psi_\ast \) are arbitrary join preserve maps; then \( \psi \) is OLG, if and only if \( \phi_\ast \circ \psi \) is OLG, for each \( \alpha \in A \).

**Proof.** By Proposition 4.11 it similar to the proof of Corollary 4.2.

**Proposition 4.5.** Suppose that \( (F', \tau') \) is an LGT-space and \( F \) is a frame. If \( \phi : F' \to F \) is onto, \( \phi_\ast(0) = 0 \) and \( \phi \) and \( \phi_\ast \) are arbitrary join preserve maps, then \( \tau_\phi = \{ t \in F : \phi_\ast(t) \in \tau_1 \} \) is the greatest LG-topology on \( F \), where \( \phi \) is OLG.

**Proof.** By the assumption \( 0 \in \tau_\phi \). Now, suppose that \( \phi(a) = 1 \), for some \( a \in F \), then

\[
\phi(1) = \phi(a \lor 1) = \phi(a) \lor \phi(1) = 1
\]

Hence \( \phi_\ast(1) = \bigvee_{x \in \phi(x) \leq 1} x = 1 \), so \( 1 \in \tau_\phi \). Since \( \phi_\ast(\bigwedge_{\alpha \in A} t_\alpha) = \bigwedge_{\alpha \in A} \phi_\ast(t_\alpha) \) and \( \phi_\ast(t_1 \land t_2) = \phi_\ast(t_1) \land \phi_\ast(t_2) \), for every subfamily \( \{ t_\alpha \}_{\alpha \in A} \) of \( \tau' \) and \( t_1 \) and \( t_2 \) in \( \tau' \), \( \bigwedge_{\alpha \in A} t_\alpha \in \tau_\phi \) and \( t_1 \land t_2 \in \tau_\phi \). Consequently, \( \tau_\phi \) is an LG-topology on \( F \). Clearly, \( \tau_\phi \) is the greatest LG-topology on \( F \), where \( \phi \) is OLG.

**Definition 4.6.** Suppose that \( (F', \tau') \) is an LGT-space and \( F \) is a frame. If \( \phi : F' \to F \) is onto, \( \phi_\ast(0) = 0 \) and \( \phi \) and \( \phi_\ast \) are arbitrary join preserve maps. Then by the above proposition, the LG-topology \( \tau_\phi = \{ t \in F : \phi_\ast(t) \in \tau' \} \) is called the quotient LG-topology on \( F \) induced by \( \phi \).

**Theorem 4.7.** Suppose that \( (F', \tau') \) and \( (F'', \tau'') \) are two LGT-spaces and \( (F, \tau_\phi) \) is the quotient LG-topology induced by \( \phi : F' \to F \). An arbitrary join preserve map \( \psi : F \to F'' \) is OLG, if and only if \( \psi \circ \phi : F' \to F'' \) is OLG.

**Proof.** (⇒). It is trivial, by Theorem 2.13

(⇐). If \( t \in \tau_\phi \), then \( (\psi \circ \phi)_\ast(t) \in \tau' \), so \( \phi_\ast(\psi_\ast(t)) \in \tau' \), by Proposition 4.11. Thus \( \psi_\ast(t) \in \tau_\phi \), and therefore \( \psi \) is OLG.
**Definition 4.8.** Suppose that $F$ is a frame. $D \subseteq F$ is called a partition for $F$, if

(a) $0 \notin D$.
(b) $\bigvee D = 1$.
(c) For every $d_1, d_2 \in D$, $d_1 \neq d_2$ implies that $d_1 \land d_2 = 0$.

**Lemma 4.9.** Suppose that $F$ is a frame, $a \in F$, $D \subseteq F$ is a partition for $F$, $T \subseteq D$ and $T_a = \{d \in D : d \land a \neq 0\}$. $a \leq \bigvee T$, if and only if $T_a \subseteq T$.

**Proof.** If $a = 0$, then it is clear. Now suppose that $a \neq 0$.

(⇒). On contrary, suppose that $d' \in T_a \setminus T$ exists, then

$$a = a \land \left( \bigvee_{d \in T} d \right) = \bigvee_{d \in T} (d \land a) \Rightarrow 0 \neq d' \land a = d' \land \left[ \bigvee_{d \in T} (d \land a) \right] = \bigvee_{d \in T} (d' \land d \land a) = 0$$

which is a contradiction.

(⇐). Since for each $d \in D \setminus T$, $d \land a = 0$,

$$a = a \land 1 = a \land \left( \bigvee_{d \in D} d \right) = \left[ \bigvee_{d \in T} (a \land d) \right] \bigvee_{d \in D \setminus T} (a \land d) = \bigvee_{d \in T} (a \land d) = a \land \left( \bigvee_{d \in T} d \right).$$

Consequently, $a \leq \bigvee_{d \in T} d$. \hfill \Box

**Proposition 4.10.** Suppose that $(F, \tau)$ is an LGT-space. If $D$ is a partition for $F$, then $\tau_D = \{T \subseteq D : \bigvee T \in \tau\}$ is a topology on $D$.

**Proof.** Clearly $\bigvee \emptyset = \emptyset$ and, by the assumption, $\bigvee D = 1$, so $\emptyset, D \in \tau_D$.

Suppose that $T_1, T_2 \in \tau_D$. By Lemma 4.9

$$a \leq \bigvee_{d \in T_1 \cap T_2} d \Leftrightarrow T_a \subseteq T_1 \cap T_2 \Leftrightarrow T_a \subseteq T_1 \text{ and } T_a \subseteq T_2$$

$$\Leftrightarrow a \leq \bigvee_{d \in T_1} d \text{ and } a \leq \bigvee_{d \in T_2} d$$

$$\Leftrightarrow a \leq \left( \bigvee_{d \in T_1} d \right) \land \left( \bigvee_{d \in T_2} d \right).$$

Thus $\bigvee_{d \in T_1 \cap T_2} d = \left( \bigvee_{d \in T_1} d \right) \land \left( \bigvee_{d \in T_2} d \right) \in \tau$, and therefore $T_1 \cap T_2 \in \tau_D$.

Now suppose that $\{T_a\}_{a \in A} \subseteq \tau_D$. Since $\bigvee_{d \in \bigcup_{a \in A} T_a} d = \bigvee_{a \in A} \bigvee_{d \in T_a} d \in \tau$, $\bigcup_{a \in A} T_a \in \tau_D$. Consequently, $(D, \tau_D)$ is a topological space. \hfill \Box

**Definition 4.11.** Suppose that $(F, \tau)$ is an LGT-space and $D$ is a partition for $F$.

By the above proposition, $\tau_D = \{T \subseteq D : \bigvee T \in \tau\}$ is a topology on $D$, $\tau_D$ is called the decomposition topology and $(\mathcal{P}(D), \tau_D)$ is called decomposition LGT-space.

**Theorem 4.12.** Suppose that $(F, \tau)$ is an LGT-space and $D$ is a partition for $F$.

The decomposition topology $\tau_D$ is a quotient LGT-topology on $\mathcal{P}(D)$.

**Proof.** Set $P : F \rightarrow \mathcal{P}(D)$, defined by $P(a) = \{d \in D : d \land a \neq 0\} = T_a$. By Lemma 4.9 $a \leq \bigvee T$, if and only if $T_a \subseteq T$; if and only if $P(a) \subseteq T$ and this is equivalent to say that $a \leq P_*(T)$. Consequently, $P_*(T) = \bigvee T \quad (\ast)$. 

Clearly, \( P(0) = \emptyset \). Suppose that \( \{T_\alpha\}_{\alpha \in A} \subseteq P(D) \), then by (+),
\[
P_\ast \left( \bigcup_{\alpha \in A} T_\alpha \right) = \bigvee_{\alpha \in A} T_\alpha = \bigvee_{\alpha \in A} P_\ast(T_\alpha).
\]
Thus \( P_\ast \) is arbitrary join preserve.

Now suppose that \( \{a_\alpha\}_{\alpha \in A} \). Then for each \( T \subseteq D \),
\[
P(\bigvee_{\alpha \in A} a_\alpha) \subseteq T \iff \bigvee_{\alpha \in A} a_\alpha \leq P_\ast(T)
\]
\[
\iff \bigvee_{\alpha \in A} a_\alpha \leq \bigvee T
\]
\[
\iff \forall \alpha \in A \ a_\alpha \leq \bigvee T
\]
\[
\iff \forall \alpha \in A \ T_{a_\alpha} \subseteq T
\]
\[
\iff \bigcup_{\alpha \in A} T_{a_\alpha} \subseteq T
\]
Thus \( P(\bigvee_{\alpha \in A} a_\alpha) = \bigcup_{\alpha \in A} T_{a_\alpha} = \bigcup_{\alpha \in A} P(a_\alpha) \), hence \( P \) is arbitrary join preserve.

Finally, by (+),
\[
T \in \tau_p \iff P_\ast(T) = \bigvee T \in \tau \iff T \in \tau_p
\]
Hence \( \tau_p = \tau_p \).

Clearly, a non-topological \( \text{LGT} \)-space \((F, \tau)\) exists. Then \( \tau_{IF} = \tau \), in which \( I_F \) is the identity map, hence \( \tau_{IF} \) is a quotient \( \text{LG} \)-topology which is not decomposition topology. Therefore the converse of the above theorem is not true in generally.

5. ISOMORPHISM

In the last section, first we introduce and study open and closed map and then, by the use of the \( \text{LG} \) map, we introduce an isomorphism, called \( \text{LG} \) map, between \( \text{LGT} \)-spaces. Finally some \( \text{LG} \)-properties have been studied.

**Definition 5.1.** Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two \( \text{LGT} \)-spaces. A map \( \phi : F_1 \rightarrow F_2 \) is called open (closed) map if \( \phi(t) \in \tau_2 \ (\phi(t^*) \in \tau_2^*) \), for each \( t \in \tau_1 \).

**Proposition 5.2.** Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two \( \text{LGT} \)-spaces and \( \phi : F_1 \rightarrow F_2 \). If \( \phi \) and \( \phi^* \) are arbitrary join preserve maps, then
(a) \( \phi \) is an open map, if and only if \( \phi^* \) is an \( \text{OLG} \) map.
(b) \( \phi \) is a closed map, if and only if \( \phi^* \) is a \( \text{CLG} \) map.

**Proof.** It is clear, by Lemma 1.2. \( \square \)

**Proposition 5.3.** Let \( \{(F_\alpha, \tau_\alpha)\}_{\alpha \in A} \) be a family of \( \text{LGT} \)-spaces and \( (\prod_{\alpha \in A} F_\alpha, \tau_p) \) be the product \( \text{LGT} \)-space. Then for each \( \beta \in A \), the projection map \( \pi_\beta : \prod_{\alpha \in A} F_\alpha \rightarrow F_\beta \) is an open map.

**Proof.** It is straightforward. \( \square \)

**Proposition 5.4.** Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two \( \text{LGT} \)-spaces, \( \phi : F_1 \rightarrow F_2 \), \( \phi \) and \( \phi^* \) are two arbitrary join preserve and \( \phi(0) = 0 \). If \( \phi \) is an onto open \( \text{OLG} \), then \( \tau_2 = \tau_\phi \).

**Proof.** It is easy, by Propositions 1.1 and 4.5 \( \square \)
Lemma 5.5. Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two LGT-spaces. If \(\phi\) is an arbitrary join preserve one-to-one map from \(F_1\) onto \(F_2\), then

(a) \(\phi(0) = 0\) and \(\phi(1) = 1\).
(b) \(\phi_*(0) = 0\) and \(\phi_*(1) = 1\).
(c) \(\phi_*(b^*) = (\phi_*(b))^*\), for every \(b \in F_2\).
(d) \(\phi(a^*) = (\phi(a))^*\), for every \(a \in F_1\).

Proof. (a) There is some \(a \in F_1\) such that \(\phi(a) = 0\), then
\[
0 = \phi(a) = \phi(a \lor 0) = \phi(a) \lor \phi(0) = 0 \lor \phi(0) = \phi(0)
\]
Similarly, by Lemma 1.3, one can show that \(\phi(1) = 1\).

(b) By Lemma 1.3, it is similar to (a).

(c) By Lemma 1.3,
\[
\phi(a^*) \land \phi(a) = \phi(a^* \land a) = \phi(0) = 0
\]
If \(b \land \phi(a) = 0\), for some \(b \in F_2\), then \(a' \in F_1\) exists such that \(\phi(a') = b\), so, by Lemma 1.3
\[
0 = \phi(a') \land \phi(a) = \phi(a' \land a) \Rightarrow a' \land a = 0 \Rightarrow a' \leq a^* \Rightarrow b = \phi(a') \leq \phi(a^*)
\]
Hence \(\phi(a^*) = (\phi(a))^*\).

(d) By Lemma 1.3, it is similar that (c). \(\square\)

Now we can conclude the following corollary form Propositions 2.9 and 5.2 and the above lemma.

Corollary 5.6. Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two LGT-spaces and \(\phi\) is a join preserve one-to-one map from \(F_1\) onto \(F_2\). Then the following statements are equivalent

(a) \(\phi\) is an OLG map.
(b) \(\phi\) is an LG map
(c) \(\phi_*\) is an open map.

Example 2.3 shows that an arbitrary join preserve one-to-one onto CLG map need not be LG map.

Definition 5.7. Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two LGT-spaces. A one-to-one map \(\phi\) from \(F_1\) onto \(F_2\) is called LGT isomorphism (briefly, isomorphism), if \(\phi\) and \(\phi^{-1}\) are LG maps, then we say \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are isomorphic.

Theorem 5.8. Suppose that \((F_1, \tau_1)\) and \((F_2, \tau_2)\) are two LGT-spaces and \(\phi\) is a join preserve one-to-one map from \(F_1\) onto \(F_2\). Then the following are equivalent

(a) \(\phi\) is an isomorphism.
(b) \(\phi\) is an open LG map.
(c) \(\phi^{-1}\) is an open LG map.

Proof. It follows immediately from Lemma 1.3 Corollary 5.6 and these facts that \(\phi_* = \phi^{-1}\) and \((\phi^{-1})_* = \phi\). \(\square\)

Theorem 5.9. Let \(\{(F_\alpha, \tau_\alpha)\}_{\alpha \in A}\) be a family of LGT-spaces and \((\prod_{\alpha \in A} F_\alpha, \tau_p)\) be the product LGT-space. Then for each \(\beta \in A\), there are some subspaces of \(\prod_{\alpha \in A} F_\alpha\) which are isomorphic to \(F_\beta\).
Proof. Set $F = \prod_{\alpha \in A} F_{\alpha}$ and $a \in F$ such that

$$a_\alpha = \begin{cases} 0 & \alpha \neq \beta \\ 1 & \alpha = \beta \end{cases}$$

Then $\pi_\beta|_{F_a} : F_a \to F_\beta$ is OLG, by Theorems 4.1 and 3.2. Clearly, $\pi_\beta|_{F_a}$ is an one-to-one onto map. Now suppose that $t \land a \in \tau_a$, in which $t \in \tau_\beta$. Then $\pi_\beta|_{F_a}(t \land 1) = t_\beta \in \tau_\beta$, so $\pi_\beta|_{F_a}$ is open, and therefore $\pi_\beta|_{F_a}$ is an isomorphism map, by Proposition 5.6 and Theorem 5.8. Hence $F_\beta$ is isomorphism to the subspace $F_a$ of $\prod_{\alpha \in A} F_{\alpha}$.

\[\square\]

**Definition 5.10.** A property is called LG-property if it preserves by isomorphism.

**Theorem 5.11.** Compactness, countably compactness and Lindelöf, ps-property, $T_0$, $T_1$, $T_2$, regular and $T_3$ properties are LG-properties.

**Proof.** By Lemma 1.3 and Proposition 3.8, the compactness, countably compactness and Lindelöf property are LG-properties. It is easy to proof that the ps-property, $T_0$, $T_1$ and $T_2$ properties are LG-properties. Finally, by the use of Lemma 5.5, one can show that the regular property and therefore $T_3$ property are LG-properties.

In [1, Proposition 4.14], it has been shown that if the product LGT-space of the family of LGT-spaces $(F_\alpha, \tau_\alpha)_{\alpha \in A}$ is $T_0$ $(T_1)$, then $F_\alpha$ is $T_0$ $(T_1)$, for every $\alpha \in A$. Now, by Theorems 5.9 and 5.11 and [1, Proposition 4.12], we can say that they are evident.

**References**

[1] A.R. Aliabad and A Sheykhmiri. LG-topology. Bull. Iranian Math. Soc., 41(1):239-258, 2015.
[2] C.H. Dowker and D. Papert. Quotient frames and subspaces. Proc. Lond. Math. Soc., 3(1):275-296, 1966.
[3] C. Ehresmann. Gattungen von lokalen strukturen, jahresber. d. Dtsch. Math., pages 602, 1957.
[4] J. R. Isbell. Atomless parts of spaces. Math. Scand., 31(1):5-32, 1973.
[5] J. C. C. McKinsey and A. Tarski. The algebra of topology. Ann. of Math., pages 141-191, 1944.
[6] G. Nöbeling. Grundlagen der analytischen Topologie. Springer-Verlag.
[7] S. Papert. An abstract theory of topological subspaces. In Math. Proc. Cambridge Philos. Soc., volume 60, pages 197-203. Cambridge University Press, 1964.
[8] J. Picado and A. Pultr. Frames and Locales: topology without points. Springer Science & Business Media, 2011.
[9] H. Simmons. A framework for topology. In Stud. Logic Found. Math., volume 96, pages 239-251. Elsevier, 1978.
[10] H. Wallman. Lattices and topological spaces. Ann. of Math., pages 112-126, 1938.
[11] S. Willard. General Topology. Addison Wesley Publishing Company, New York, 1970.

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