Seiberg-Witten invariants for manifolds with $b_+ = 1$

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Abstract – In this note we describe the Seiberg-Witten invariants, which have been introduced in [W], for manifolds with $b_+ = 1$. In this case the invariants depend on a chamber structure, and there exists a universal wall crossing formula. For every Kähler surface with $p_g = 0$ and $q=0$, these invariants are non-trivial for all Spin$^c$-structures of non-negative index.

Les invariants de Seiberg-Witten pour les variétés avec $b_+ = 1$

Résumé – Dans cette note nous décrivons les invariants de Seiberg-Witten introduits dans [W], pour les variétés telles que $b_+ = 1$. Ces invariants dépendent d’un paramètre auxiliaire variant dans l’ensemble des chambres, et il existe une formule universelle de passage à travers un mur. Pour les surfaces kählériennes telles que $p_g = 0$ et $q = 0$, les invariants associés à toute Spin$^c$-structure d’index non-négatif sont non-triviaux.
qui se projette sur \((g, c)\) et tout \(\beta\) dans un sous-ensemble ouvert dense de \(b\). Une telle forme \(\beta\) sera appelée régulière. La variété \(W^\gamma_{X, \beta}\) peut alors être orientée en choisissant une orientation \(\sigma\) du fibré \(\det(H^1(X, \mathbb{R})) \otimes \det(\mathbb{H}^2_{+g}(X)^\vee)\). Soit \([W^\gamma_{X, \beta}]_\sigma \in H_{w_\sigma}(\mathcal{B}(c)^*, \mathbb{Z})\) la classe fondamentale associée à \(\sigma\).

La forme de Seiberg-Witten associée à la donnée \((\sigma, (g, b), c)\) est l’élément \(SW^{(g, b)}_{X, \sigma}(c) \in \Lambda^*H^1(X, \mathbb{Z})\) défini par \(SW^{(g, b)}_{X, \sigma}(c)(l_1 \wedge \ldots \wedge l_r) = \left< \nu(l_1) \cup \ldots \cup \nu(l_r) \cup u^{\frac{\omega_g}{\omega_0}}[W^\gamma_{X, \beta}]_\sigma \right>\) sur les éléments décomposables \(l_1 \wedge \ldots \wedge l_r\) avec \(r \equiv w_c\) (mod 2). Ici \(\gamma \in \mathcal{C}\) se projette sur la paire \((g, c)\) et \(\beta \in b\) est régulière. \(SW^{(g, b)}_{X, \sigma}(c)\) ne dépend pas du choix de \(\gamma \in \beta\).

Si \(b_+ > 1\), \(SW^{(g, b)}_{X, \sigma}(c)\) ne dépend pas même du choix de la paire \(c\)-bonne \((g, b)\), donc on peut désigner cet invariant simplement par \(SW_{X, \sigma}(c) \in \Lambda^*H^1(X, \mathbb{Z})\). Si \(b_1 = 0\), on obtient des nombres que nous désignons par \(n^c_\sigma\); ces nombres peuvent être considérés comme des raffinements des nombres \(n^c_\sigma\) introduits en [W]. En effet \(n^c_\sigma = \sum n^c_\sigma\), où la somme est faite par rapport à \(c \in \pi_0(\mathcal{C}/\text{Aut}(\mathcal{P}))\).

Supposons maintenant que \(b_+ = 1\). Il y a une application naturelle \(\text{Met}_X \to \mathbb{P}(H^2_{\text{DR}}(X))\) qui associe à \(g \in \text{Met}_X\) la droite \(\mathbb{R}[\omega_+] \subset H^2_{\text{DR}}(X)\), où \(\omega_+\) est une 2-forme \(g\)-autoduale harmonique non-triviale. L’espace hyperbolique \(\mathcal{H} := \{\omega \in H^2_{\text{DR}}(X) | \omega^2 = 1\}\) a deux composantes connexes, et le choix d’une composante \(\mathcal{H}_0\) définit une orientation de la droite \(H^2_{\perp g}(X)\) pour toute métrique \(g\). Soit \(\omega_g\) le générateur de \(\mathbb{H}^2_{\perp g}(X)\) tel que \([\omega_g] \in \mathcal{H}_0\).

**Définition.** - Soit \(X\) une 4-variété avec \(b_+ = 1\), et \(c \in H^2(X, \mathbb{Z})\) un élément caractéristique. Le mur associé à \(c\) est l’hypersurface \(c^\perp := \{(\omega, b) \in \mathcal{H} \times H^2_{\text{DR}}(X) | (c - b) \cdot \omega = 0\}\). Les composantes connexes de \(\mathcal{H} \setminus c^\perp\) seront appelées chambres du type \(c\).

Les murs ne sont pas linéaires! Tout élément caractéristique \(c\) définit précisément quatre chambres du type \(c : \mathcal{C}_{H_0, \pm} := \{(\omega, b) \in H^2_{\text{DR}}(X) | (c - b) \cdot \omega < 0\}\), où \(H_0\) est l’une des deux composantes connexes de \(\mathcal{H}\). Toute chambre contient des paires de la forme \(([\omega_g], b)\). Choisissons maintenant une orientation \(\sigma_1\) de \(H^1(X, \mathbb{R})\).

**Définition.** - L’invariant de Seiberg-Witten associé à la donnée \((\sigma_1, H_0, c)\) est la fonction \(SW_{X, (\sigma_1, H_0)}(c) : \{\pm\} \to \Lambda^*H^1(X, \mathbb{Z})\) définie par \(SW_{X, (\sigma_1, H_0)}(c)(\pm) := SW^{(g, b)}_{X, \sigma}(c)\), où \(\sigma\) est l’orientation induite par \((\sigma_1, H_0)\), et \(([\omega_g], b)\) est une paire appartenant à \(\mathcal{C}_{H_0, \pm}\).

On vérifie facilement les relations \(SW_{X, (\sigma_1, H_0)}(c)(\pm) = -SW_{X, (\sigma_1, -H_0)}(c)(\mp)\) et \(SW_{X, (\sigma_1, H_0)}(c)(\mp) = -SW_{X, (\sigma_1, -H_0)}(c)(\pm)\). Soit \(u_c(a, b) := \frac{1}{2}(c \cup a \cup b, [X])\), pour \(a, b \in H^1(X, \mathbb{Z})\).

**Théorème.** - Supposons \(b^+(X) = 1\), soit \(\sigma_1\), le générateur de \(\Lambda^6(H^1(X, \mathbb{Z}))\) défini par l’orientation \(\sigma_1\), et soit \(r \geq 0, r \equiv w_c\) (mod 2) : Pour tout \(\lambda \in \Lambda^r(H^1(X, \mathbb{Z})/\text{Tors})\) on a

\[SW_{X, (\sigma_1, H_0)}(c)(\pm)(\lambda) - SW_{X, (\sigma_1, H_0)}(c)(\mp)(\lambda) = \left(\frac{\lambda \wedge u_c}{\|u_c\|^2}\right)(\lambda \wedge u_c^{\frac{\lambda \wedge u_c}{\|u_c\|^2}}, \sigma_1)\]

si \(r \leq \min(b_1, c_1)\), et la différence est nulle dans les autres cas.

Soit \((X, g)\) une surface kählérienne munie de sa Spîn\(^{(4)}\)-structure canonique et soit \(\omega_g\) sa forme de Kähler. Il y a une bijection naturelle entre les classes de Spîn\(^{(4)}\)-structures \(c\) de classe de Chern \(c\) et les fibrés en droites \(M\) dont la classe de Chern vérifie \(2c_1(M) - c_1(K_X) = c\). On désigne par \(\epsilon_M\) la classe définie par \(M\). Soit \(D\) l’espace de Douady des diviseurs effectifs \(D\) sur \(X\) tels que \(c_1(\mathcal{O}_X(D)) = m\).

**Théorème.** - Soit \((X, g)\) une surface kählérienne connexe, et soit \(\epsilon_M\) la classe des Spîn\(^{(4)}\)-structures associée au fibré en droites \(M\) de classe de Chern \(c_1(M) = m\). Soit
\( \beta \in A_{1,1}^\mathbb{Z} \) une forme représentant la classe \( b \) telle que \( (2m - c_1(K_X) - b) \cup [\omega_2] < 0 \) \((> 0)\).

i) Si \( c \notin \text{NS}(X) \), on a \( \mathbb{W}_{X,\beta}^M = 0 \). Si \( c \in \text{NS}(X) \), il existe un isomorphisme réel analytique naturel \( \mathbb{W}_{X,\beta}^M \cong \text{Dou}(m) \) \((\text{Dou}(c_1(K_X) - m))\).

ii) \( \mathbb{W}_{X,\beta}^M \) est lisse en un point correspondant à \( D \in \text{Dou}(m) \) si et seulement si \( h^0(\mathcal{O}_D(D)) = \dim_D \text{Dou}(m) \). Cette condition est toujours satisfaite si \( h^1(\mathcal{O}_X) = 0 \).

iii) Si \( \mathbb{W}_{X,\beta}^M \) est lisse en un point correspondant à \( D \in \text{Dou}(m) \), il a la dimension prédéterminée en ce point si et seulement si \( h^1(\mathcal{O}_D(D)) = 0 \).

Toute surface complexe connexe avec \( p_g > 0 \) est difféomorphe à une surface possédant un diviseur canonique 0-connexe. De là on déduit une démonstration simple du

**Corollaire.** - ([\( W \)]) Tous les invariants de Seiberg-Witten non-triviaux d’une surface kählérienne avec \( p_g > 0 \) sont d’indice 0.

Au contraire, si \( p_g = 0 \), on a

**Corollaire.** - Soit \( X \) une surface kählérienne avec \( p_g = 0 \) et \( q = 0 \). Pour toute donnée \((\mathcal{H}_0, c)\) avec \( w_c \geq 0 \), on a \( \text{SW}_{X,\mathcal{H}_0}(c)(\{\pm\}) = \{0, 1\} \) ou \( \text{SW}_{X,\mathcal{H}_0}(c)(\{\pm\}) = \{0, -1\} \).

1. **The twisted Seiberg-Witten equations.** - Let \( X \) be a closed connected oriented 4-manifold, and let \( c \in H^2(X, \mathbb{Z}) \) be a class with \( c \equiv w_2(X) \pmod{2} \). A compatible Spin\(^c\)-bundle \( \mathcal{P} \) with \( c_1(\mathcal{P} \times_{\det} \mathbb{C}) = c \) such that its \( GL_+^+(4, \mathbb{R}) \)-extension \( \mathcal{P} \times_{\mathbb{R}} GL_+^+(4, \mathbb{R}) \) is isomorphic to the bundle of oriented frames in \( \Lambda_X^{+4} \). Let \( \Sigma^\pm := \mathcal{P} \times_{\det} \mathbb{C}^2 \) be the associated spinor bundles with \( \det \Sigma^\pm = \mathcal{P} \times_{\det} \mathbb{C} \pmod{[O1]} \).

**Définition.** - A **Clifford map** of type \( \mathcal{P} \) is a \( GL_+^+(4, \mathbb{R}) \)-isomorphism \( \gamma : \Lambda_X^{+4} \rightarrow \mathcal{P} \times_{\mathbb{R}} \mathbb{R}^4 \).

The \( SO(4) \)-vector bundle \( \mathcal{P} \times_{\mathbb{R}} \mathbb{R}^4 \) can be identified with the bundle \( \mathbb{R}SU(\Sigma^+, \Sigma^-) \) of real multiples of \( \mathbb{C} \)-linear isometries of determinant 1 from \( \Sigma^+ \) to \( \Sigma^- \). A Clifford map \( \gamma \) defines a metric \( g_\gamma \) on \( X \), a lift \( \mathcal{P} \rightarrow P_{g_\gamma} \) of the associated frame bundle, and it induces isomorphisms \( \Gamma : \Lambda_Z^+ \rightarrow su(\Sigma^\pm) \pmod{[O1]} \). Let \( \mathcal{C} = \mathcal{C}(\mathcal{P}) \) be the space of all Clifford maps of type \( \mathcal{P} \). There is a natural isomorphism \( \mathcal{C}/\text{Aut}(\mathcal{P}) \rightarrow \text{Met}_X \times \pi_0(\mathcal{C}/\text{Aut}(\mathcal{P})) \), where the second factor is a \( \text{Tors}^2 H^2(X, \mathbb{Z}) \)-torsor; it parametrizes the set of equivalence classes of exceptional structures on \( X \) with Chern class \( c \) on \((X, g)\), for an arbitrary metric \( g \). We use the symbol \( c \) to denote elements in \( \pi_0(\mathcal{C}/\text{Aut}(\mathcal{P})) \), and we denote by \( \mathfrak{c}_\gamma \) the connected component defined by \( [\gamma] \in \mathcal{C}/\text{Aut}(\mathcal{P}) \). A fixed Clifford map \( \gamma \) defines a bijection between unitary connections in \( \mathcal{P} \times_{\det} \mathbb{C} \) and Spin\(^c\)-connections in \( \mathcal{P} \) which lift (via \( \gamma \)) the Levi-Civita connection in \( P_{g_\gamma} \), and allows to associate a Dirac operator \( \mathcal{D}_A \) to a connection \( A \in \mathcal{A}(\mathcal{P} \times_{\det} \mathbb{C}) \).

**Définition.** - Let \( \gamma \) be a Clifford map, and let \( \beta \in \mathbb{Z}_{\text{DR}}^2(X) \) be a closed 2-form. The **\( \beta \)-twisted Seiberg-Witten equations** are

\[
\mathcal{D}_A \Psi = 0 , \quad \Gamma ((F_A + 2\pi i \beta)^+) = 2(\Psi \bar{\Psi})_0 .
\]

These twisted Seiberg-Witten equations arise naturally in connection with certain non-abelian monopoles \([O2]\). They should **not** be regarded as perturbation of \( \text{SW}_0^0 \), since later the cohomology class of \( \beta \) will be fixed.
Let $W_{X,\beta}^{\gamma}$ be the moduli space of solutions $(A, \Psi) \in \mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+)$ of $(SW_{X,\beta}^{\gamma})$ modulo the natural action $((A, \Psi), f) \mapsto (A^f, f^{-1}\Psi)$ of the gauge group $G = C^\infty(X, S^1)$.

The moduli space $W_{X,\beta}^{\gamma}$ depends up to canonical isomorphism only on $(g_\gamma, c_\gamma)$ and $\beta$, since two Clifford maps lifting the same pair $(g, c)$ are equivalent modulo $\text{Aut}(\hat{P})$.

Now fix a class $b \in H^2_{DR}(X)$, consider $(SW_{X,\beta}^{\gamma})$ as equation for a triple $(A, \Psi, \beta, \gamma) \in \mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+) \times b/\mathcal{G}$ be the (infinite dimensional) moduli space of solutions. Finally we need the universal moduli space $W_X \subset \mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+) \times Z^2_{DR}(X) \times C/\mathcal{G}$ of solutions of $(SW_{X,\beta}^{\gamma})$ regarded as equations for tuples $(A, \Psi, \beta, \gamma) \in \mathcal{A}(\det \Sigma^+) \times A^0(\Sigma^+) \times Z^2_{DR}(X) \times C$. We complete the spaces $\mathcal{A}(\det \Sigma^+)$, $A^0(\Sigma^+)$ and $A^2$ with respect to the Sobolev norms $L_2^2$, $L_2^n$ and $L_2^{n-1}$, and the gauge group $\mathcal{G}$ with respect to $L_2^{n+1}$, but we suppress the Sobolev subscripts in our notations. As usual we denote by the superscript $^*$ the open subspace of a moduli space where the spinor component is non-zero.

**Definition.** Let $c \in H^2(X, \mathbb{Z})$ be characteristic. A pair $(g, b) \in \text{Met}_X \times H^2_{DR}(X)$ is $c$-good if the $g$-harmonic representant of $(c - b)$ is not $g$-anti-selfdual.

**Proposition.** Let $X$ be a closed oriented 4-manifold, and let $c \in H^2(X, \mathbb{Z})$ be characteristic. Choose a compatible Spin$^c(4)$-bundle $\hat{P}$ and an element $\epsilon \in \pi_0(C/\text{Aut}(\hat{P}))$.

1. The projections $p : W_X \to Z^2_{DR}(X) \times C$ and $p_{\gamma, b} : W_{X,\beta}^{\gamma} \to b$ are proper for all $\gamma, b$.
2. $W_{X,\beta}^{\gamma}$ and $W_{X,\beta}^{\gamma \ast}$ are smooth manifolds for all $\gamma$ and $b$.
3. $W_{X,\beta}^{\gamma \ast} = W_{X,\beta}^{\gamma}$ if $(g_\gamma, b)$ is $c$-good.
4. If $(g_\beta, b)$ is $c$-good, then every pair $(\beta_0, \beta_1)$ of regular values of $p_{\gamma, b}$ can be joined by a smooth path $\beta : [0, 1] \to b$ such that the fiber product $[0, 1] \times (\beta, p_{\gamma, b}) W_{X,\beta}^{\gamma}$ defines a smooth bordism between $W_{X,\beta_0}^{\gamma}$ and $W_{X,\beta_1}^{\gamma}$.

5. If $(g_0, b_0)$, $(g_1, b_1)$ are $c$-good pairs which can be joined by a smooth path of $c$-good pairs, then there is a smooth path $(\beta, \gamma) : [0, 1] \to Z^2_{DR}(X) \times C$ with the following properties:
   1. $[\beta_1] = b_1$ and $g_{\gamma_i} = g_i$ for $i = 0, 1$.
   2. $\gamma_t$ lifts $(g_{\gamma_t}, c)$ and $(g_{\gamma_1}, [\beta_1])$ is $c$-good for every $t \in [0, 1]$.
   3. $[0, 1] \times (\gamma, \beta, p_{\gamma, b}) W_X$ is a smooth bordism between $W_{X,\beta_0}^{\gamma}$ and $W_{X,\beta_1}^{\gamma}$.

6. If $b_+ > 1$, then any two $c$-good pairs $(g_0, b_0)$, $(g_1, b_1)$ can be joined by a smooth path of $c$-good pairs.

The proof uses techniques from [DK], [KM] and [T].

2. **Seiberg–Witten invariants for 4-manifolds with $b_+ = 1$.** Let $X$ be a closed connected oriented 4-manifold, $c$ a characteristic element, and $\hat{P}$ a compatible Spin$^c(4)$-bundle. We put $B(c) := \mathcal{A}(\det \Sigma^+) \times (A^0(\Sigma^+) \setminus \{0\})/G$.

The space $B(c)$ has the weak homotopy type of $K(\mathbb{Z}, 2) \times K(H^1(X, \mathbb{Z}), 1)$ and there is a natural isomorphism $\nu : \mathbb{Z}[u] \otimes \Lambda^*(H_1(X, \mathbb{Z})/\text{Tors}) \to H^*(B(c)), \mathbb{Z}$.

Suppose that $(g, b)$ is a $c$-good pair and fix $c \in \pi_0(C/\text{Aut}(\hat{P}))$. The moduli space $W_{X,\beta}^{\gamma}$ is a compact manifold of dimension $w_c := \frac{1}{4}(c^2 - 2e(X) - 3\sigma(X))$ for every lift $\gamma$ of $(g, c)$ and every regular value $b$ of $p_{\gamma, b} : W_{X,\beta}^{\gamma} \to b$. It can be oriented by using the canonical complex orientation of the line bundle $\det(\text{index}(\hat{D}))$ over $B(c)$ together with a chosen orientation $\sigma$ of the line $\det(H^1(X, \mathbb{R}) \otimes \det(\mathbb{H}_{2, g}(X))^\gamma)$. Let $|W_{X,\beta}^{\gamma}| \subset H_{w_c}(B(c)), \mathbb{Z}$ be the fundamental class associated with the choice of $\sigma$.

The Seiberg–Witten form $SW^{(g, b)}(\mathcal{O}) \in \Lambda^*H^1(X, \mathbb{Z})$ associated with $(c, (g, b), c)$ is defined by $SW^{(g, b)}_{X,\beta}(c)(l_1 \wedge \ldots \wedge l_r) = \langle \nu(l_1) \cup \ldots \cup \nu(l_r) \cup u^{w_c}, [W_{X,\beta}^{\gamma}] \rangle$ for decompos-
able elements $l_1 \wedge \ldots \wedge l_r$ with $r \equiv w_c (\text{mod} \ 2)$. Here $\gamma$ lifts the pair $(g, c)$ and $\beta \in b$ is a regular value of $p_{\gamma, b}$. The form $SW^{(g, b)}(c)$ is well-defined, since the cohomology classes $u, \nu(l_\gamma)$, as well as the trivialization of the orientation line bundle extend to $A(\det \Sigma^+) \times (A^0(\Sigma^+) \setminus \{0\}) \times \mathbb{R}_+$, and since the homology class defined by $[W_{X, b}]_{\mathcal{O}}$ in this quotient depends only on $(g, \gamma, c)$ and $b$. Now there are two cases:

If $b_+ > 1$, then $SW^{(g, b)}(c)$ does not depend on $(g, b)$, since the cohomology classes $u, \nu(l_\gamma)$ and the trivialization of the orientation line bundle extend to $\text{Aut}(\tilde{P})$-invariant objects on the quotient $A(\det \Sigma^+) \times (A^0(\Sigma^+) \setminus \{0\}) \times \mathbb{R}_+$, and thus we may simply write $SW_{X,\mathcal{O}}(c) \in \Lambda^* H^1(X, \mathcal{O})$. If $b_1 = 0$, then we obtain numbers $n_c^2$ which can be considered as refinements of the numbers $n_c$ defined in [W]. Indeed, $n_c^2 = \sum n_c^2$, the summation being over all $c \in \pi_0(C/\text{Aut}(\tilde{P}))$.

Suppose now that $b_+ = 1$. There is a natural map $\mathcal{M}et_X \rightarrow \mathbb{P}(H^2_{\mathcal{DR}}(X))$ which sends a metric $g$ to the line $\mathbb{R} \omega_+ \subset H^2_{\mathcal{DR}}(X)$, where $\omega_+$ is any non-trivial $g$-selfdual harmonic form. The hyperbolic space $\mathcal{H} := \{\omega \in H^2_{\mathcal{DR}}(X) \mid \omega^2 = 1\}$ has two connected components, and the choice of one of them directs the line $\mathbb{R} \omega_+(X)$ for all metrics $g$. Having fixed a component $\mathcal{H}_0$ of $\mathcal{H}$, every metric defines a unique $g$-self-dual form $\omega_+ \subset \omega_+(X)$ with $[\omega_+] \in \mathcal{H}_0$.

**Definition.** - Let $X$ be a manifold with $b_+ = 1$, and let $c \in H^2(X, \mathcal{O})$ be characteristic. The wall associated with $c$ is the hypersurface $c^\perp := \{\omega, b) \in \mathcal{H} \times H^2_{\mathcal{DR}}(X) \mid (c - b) \cdot \omega = 0\}$. The connected components of $\mathcal{H} \setminus c^\perp$ are called chambers of type $c$.

Notice that the walls are non-linear! Every characteristic element $c$ defines precisely four chambers of type $c$, namely $C_{\mathcal{H}_0, \pm} := \{(\omega, b) \in \mathcal{H}_0 \times H^2_{\mathcal{DR}}(X) \mid \pm (c - b) \cdot \omega < 0\}$, where $\mathcal{H}_0$ is one of the components of $\mathcal{H}$. Each of these four chambers contains pairs of the form $([\omega_+], b)$. Let $\sigma_1$ be an orientation of $H^1(X, \mathbb{R})$.

**Definition.** - The Seiberg-Witten invariant associated with $(\sigma_1, \mathcal{H}_0, c)$ is the function $SW_{X, (\sigma_1, \mathcal{H}_0)}(c) : \{\pm\} \rightarrow \Lambda^* H^1(X, \mathbb{Z})$ given by $SW_{X, (\sigma_1, \mathcal{H}_0)}(c)(\pm) := SW^{(g, b)}(c)$, where $\sigma$ is the orientation defined by $(\sigma_1, \mathcal{H}_0)$, and $(g, b)$ is a pair such that $([\omega_+], b) \in C_{\mathcal{H}_0, \pm}$.

Note that, changing the orientation $\sigma_1$ changes the invariant by a factor $-1$, and that $SW_{X, (\sigma_1, -\mathcal{H}_0)}(c)(\pm) = -SW_{X, (\sigma_1, \mathcal{H}_0)}(c)(\mp)$.

**Remark.** - A different approach - adapting ideas from intersection theory to construct "Seiberg-Witten multiplicities" - has been proposed by R. Brussee.

Define $u_c \in \Lambda^2 (H^1(X, \mathcal{O})/\text{Tors})$ by the formula $u_c(a, b) := \frac{1}{2} (c \cup a \cup b, [X])$, for elements $a, b \in H^1(X, \mathcal{O})$. The following wall crossing formula generalizes results of [W], [KM], [LL].

**Theorem.** - Let $b^+(X) = 1$, let $l_{\sigma_1}$ be the generator of $\Lambda^b_1(H^1(X, \mathcal{O}))$ defined by the orientation $\sigma_1$, and let $r \geq 0, r \equiv w_c (\text{mod} \ 2)$. For every $\lambda \in \Lambda^r (H^1(X, \mathcal{O})/\text{Tors})$ we have

$$SW_{X, (\sigma_1, \mathcal{H}_0)}(c)(\pm)(\lambda) - SW_{X, (\sigma_1, \mathcal{H}_0)}(c)(\mp)(\lambda) = (-1)^{\left\lfloor \frac{b^+ - r}{2} \right\rfloor} \lambda \wedge u_c^{\left\lfloor \frac{b^+ - r}{2} \right\rfloor} l_{\sigma_1},$$

if $r \leq \min(b_1, w_c)$, and the difference is 0 otherwise.

**Remark.** - For manifolds admitting a metric of positive scalar curvature, the invariants are determined by Witten's vanishing result [W] and the wall crossing formula.

3. SEIBERG-WITTEN INVARIANTS OF KÄHLER SURFACES. - Let $(X, g)$ be a Kähler surface with Kähler form $\omega_g$, and let $c_0$ be the class of the canonical Spin$^c(4)$-structure.
of determinant $K^X_\gamma$ on $(X, g)$. The corresponding spinor bundles are $\Sigma^+ = \Lambda^{00} \oplus \Lambda^{02}$, $\Sigma^- = \Lambda^{01}$ [OT1]. There is a natural bijection between classes of Spin$^c(4)$-structures $\gamma$ of Chern class $c$ and isomorphism classes of line bundles with $2c_1(M) - c_1(K_X) = c$. We denote by $\gamma_M$ the class defined by a line bundle $M$. The spinor bundles of $\gamma_M$ are the tensor products $\Sigma^\pm \otimes M$, and the map $\gamma_M : \Lambda^1_X \longrightarrow \mathbb{R}SU(\Sigma^+ \otimes M, \Sigma^- \otimes M)$ given by $\gamma_M(\cdot) = \gamma_0(\cdot) \otimes 1$ is a Clifford map representing $\gamma_M$.

Let $\text{Dou}(m)$ be the Douady space of all effective divisors $D$ on $X$ with $c_1(O_X(D)) = m$.

**Theorem.** - Let $(X, g)$ be a connected Kähler surface, and let $\gamma_M$ be the class of the Spin$^c(4)$-structure associated to a line bundle $M$ with $c_1(M) = m$. Let $\beta \in A_X^{1,1}$ be a closed form representing the class $b$ such that $(2m - c_1(K_X) - b) \cup [\omega_g] < 0$ ($> 0$).

i) If $c \notin NS(X)$, then $W_{X,\beta}^M = 0$. If $c \in NS(X)$, then there is a natural real analytic isomorphism $W_{X,\beta}^M \simeq \text{Dou}(m) : (\text{Dou}(c_1(K_X) - m))$.

This condition is always satisfied when $h^1(O_X) = 0$.

ii) If $W_{X,\beta}^M$ is smooth at a point corresponding to $D \in \text{Dou}(m)$ iff $h^0(D) = \dim D \text{Dou}(m)$.

iii) If $W_{X,\beta}^M$ is smooth at a point corresponding to $D$, then it has the expected dimension in this point iff $h^1(D) = 0$.

It is easy to see that every connected complex surface with $p_g > 0$ is oriented diffeomorphic to a surface which possesses a 0-connected canonical divisor. This yields an easy proof of

**Corollary.** - ([W]) All non-trivial Seiberg-Witten invariants of Kähler surfaces with $p_g > 0$ have index 0.

On the other hand, for $p_g = 0$, we have

**Corollary.** - Let $X$ be a surface with $p_g = 0$ and $q = 0$. For all data $(H_0, c)$ with $w_c \geq 0$, we have $SW_{X, H_0}(c)(\{±\}) = \{0, 1\}$ or $SW_{X, H_0}(c)(\{±\}) = \{0, -1\}$.

**Remark.** - In this situation, the invariants are already determined by their reduction modulo 2. There exist examples of surfaces with $p_g = 0$ and $q = 0$, having infinitely many non-trivial invariants of any given non-negative index.

**Example.** - Let $X = \mathbb{P}^2$, let $h \in H^2(\mathbb{P}^2, \mathbb{Z})$ be the class of the ample generator, and let $H_0$ be the component of $H = \{± h\}$ which contains $h$. The classes of Spin$^c(4)$-structures are labelled by odd integers $c$ and the corresponding index is $w_c = \frac{1}{4}(c^2 - 2)$. We find $SW_{\mathbb{P}^2, H_0}(\pm 1) = 0$, $SW_{\mathbb{P}^2, H_0}(c)(+) = 1$ if $c \geq 3$, and $SW_{\mathbb{P}^2, H_0}(c)(-) = -1$ if $c \leq 3$.

**References**

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