Convection and Overshoot in Models of $\gamma$ Doradus and $\delta$ Scuti Stars

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Abstract

We investigate the pulsation properties of stellar models that are representative of $\delta$ Scuti and $\gamma$ Doradus variables. We have calculated a grid of stellar models from 1.2 to 2.2 $M_\odot$, including the effects of both rotation and convective overshoot using MESA, and we investigate the pulsation properties of these models using GYRE. We discuss the observable patterns in the frequency spacing for $p$ modes and the period spacings for $g$ modes. Using the observable patterns in the $g$ mode period spacings, it may be possible to observationally constrain the convective overshoot and rotation of a model. We also calculate the pulsation constant ($Q$) for all models in our grid and investigate the variation with convective overshoot and rotation. The variation in the $Q$ values of the radial modes can be used to place constraints on the convective overshoot and rotation of stars in this region. As a test case, we apply this method to a sample of 22 High-Amplitude $\delta$ Scuti stars (HADS) and provide estimates for the convective overshoot of the sample.

Key words: stars: rotation – stars: variables: delta Scuti – stars: variables: general

1. Introduction

$\delta$ Scuti stars are main-sequence stars with masses that are 1.4–2.5 times the mass of the Sun and are located where the classical instability strip intersects the main sequence in the HR diagram. The pulsations are driven by the $\kappa$ mechanism in the He II ionization zone and are either low-order pressure modes ($p$ modes; Xiong et al. 2016) or are mixed modes with both $p$- and $g$-mode properties (Guzik et al. 2000; Dupret et al. 2004, 2005). The observed pulsation periods are usually short, ranging from 30 minutes to 5 hours. Observations using the Kepler spacecraft (Basri et al. 2005) have shown that $\delta$ Scuti stars have very rich pulsation spectra with many identifiable frequencies.

$\delta$ Scuti stars that pulsate with a peak-to-peak amplitude greater than about 0.3 mag are known as High-Amplitude $\delta$ Scuti stars (HADS). This subgroup is generally thought to be slowly rotating ($v \lesssim 30 \text{ km s}^{-1}$), and is thought to pulsate mostly in radial modes. Given their location in the HR diagram, these stars have been viewed as intermediate objects and are located between the normal $\delta$ Scuti stars and Cepheids. However, Balona (2016) found that there is no difference in the distribution between HADS and low-amplitude $\delta$ Scuti stars, suggesting that these two groups of stars are drawn from the same population. However, HADS account for a small fraction of the total population (Lee et al. 2008), i.e., around 0.24%.

One of the theoretical challenges presented by $\delta$ Scuti stars is determining which stars should be pulsating. Although all of the stars in the $\delta$ Scuti instability strip are predicted to show pulsations, observations show that less than half of the stars in this region actually pulsate (Balona et al. 2011; Balona et al. 2015). This difference between the theory and the observation is not presently understood. It has been suggested that nonlinear mode coupling may stabilize the pulsations (Dziembowski et al. 1988) or that the opacity-driving mechanism could be saturated (Nowakowski 2005). Unfortunately, testing these predictions requires nonlinear models of the nonadiabatic, nonradial pulsations, which do not currently exist.

The $\gamma$ Doradus stars are slightly less massive than the $\delta$ Scuti stars, with an instability region that overlaps the cool part of the $\delta$ Scuti instability strip. The $\gamma$ Dor stars pulsate in $g$ modes, which are thought to be triggered by convective blocking (Guzik et al. 2000; Dupret et al. 2004, 2005). The periods are longer than those in $\delta$ Scuti stars, ranging from 0.3 to 3 days. The low amplitudes and pulsation periods of the order of 1 day make the $\gamma$ Doradus stars difficult to observe from the ground. Space-based observations, such as those with COROT (Harét et al. 2010) and Kepler (e.g., Grigahcène et al. 2010; Bradley et al. 2015), have found that a number of $\delta$ Scuti stars are observed to pulsate with $\gamma$ Dor-type frequencies as well, and these are usually described as hybrid stars. Hybrid stars are particularly interesting asteroseismic targets, as the $g$ modes sample deeper regions of the star than the $p$ modes. Hybrid stars, $\gamma$ Dor stars, and $\delta$ Scuti stars collectively span the mass range of 1.4–2.5 $M_\odot$, where the interior structure transitions from convective envelopes and radiative cores in the lower mass stars to convective cores and radiative envelopes in the higher mass stars. Surveys of stars in this mass range that used Kepler have found more hybrid star candidates than is expected by the current stellar pulsation theory (Uytterhoeven et al. 2011). The leading explanation for this discrepancy is that the interaction between the convection and the pulsation in both the deeper layers ($\gamma$ Dor, $g$ modes) and the upper layers ($\delta$ Scuti, $p$ modes) is not fully understood.

Recent observations by Balona (2014) and Balona et al. (2015) have suggested that the low frequencies that are typical of $\gamma$ Dor variables are found in all $\delta$ Scuti stars. They argue that, from the ground, only the higher-amplitude $\delta$ Scuti modes are visible, but in fact, all pulsating stars in this region are “hybrid.” If this is true, this would only deepen the discrepancy between theoretical predictions and the observed pulsation. The internal convection, both in the core and the subsurface, is likely to play an important role in determining the pulsation properties of these stars. For the models to correctly predict the pulsations, the interaction between the convection and the pulsation must be included.
Rotation is also an important factor in δ Scuti stars, as these stars can have rotation rates as high as 200–250 km s⁻¹ (Breger 2007). The rotation can affect the small-scale features of convection; in particular, it can affect the dynamics of the shear layers at the boundaries of the convective zones. The details of these small-scale features are likely to be a key part in correctly predicting pulsation frequencies in these rapidly rotating stars. On larger scales, the rotation can increase the size of the convective core, as parameterized in our stellar models by the convective core overshoot.

Many types of variable stars are found to obey a period-mean density relation (see, for example, Cox 1980). This relationship is usually written as

\[ Q \propto \left( \frac{P}{P^*} \right)^{1/2} = Q. \]  

\( Q \), known as the pulsation constant, is approximately constant for stars that are similar in size and structure. The pulsation constant is expected to apply to both radial and nonradial δ Scuti. The relationships between the pulsation constant and the observable properties of the star have been determined by Breger (1990), and these can be used to identify an observed mode in these stars. However, this relationship is calibrated to the Sun and neglects the effects of the rotation and the convective overshoot on the pulsation.

The behavior of \( Q \) has previously been studied in δ Scuti stars by Fitch (1981). In that work, Fitch studied models of 1.5, 2.0, and 2.5 \( M_\odot \), and the pulsation periods were calculated for \( \ell \leq 3 \). Based on these models, Fitch calculated \( Q \) for the first seven overtones and then determined the coefficients for interpolation formulae. These formulae can be used to predict the expected value of \( Q \) for a given mode based on the observed period and the effective temperature of a star.

In this work, we investigate the pulsation characteristics of stellar models in the range of 1.2–2.2 \( M_\odot \). Our model grid, which was calculated using MESA (Paxton et al. 2011) and GYRE (Townsend & Teitler 2013), is described in Section 2. In Section 3, we discuss the pulsation constant (\( Q \)) of the models and the potential for using \( Q \) to constrain the core convective overshoot and the rotation of the populations of stars. We present the patterns in the frequency and the period spacings in Section 4. Finally, we summarize our conclusions in Section 5.

2. Models

We calculate a grid of stellar evolution models using MESA (Paxton et al. 2011) for masses from 1.2 to 2.2 \( M_\odot \). These models were calculated using the metallicity of \( Z = 0.02, Y = 0.28 \) (from Grevesse & Sauval 1998). MESA uses the OPAL equation of state (Rogers & Nayfonov 2002), which is supplemented at low temperatures and densities with tables from Saumon et al. (1995). The convection is treated using the mixing length theory formalism (Böhm-Vitense 1958), with \( \alpha = 2.0 \).

Our models include varying rotations and the convective overshoot. The overshoot is included using the exponential model of Herwig (2000), where the diffusion coefficient is given by

\[ D_{ov} = D_0 \exp \left( \frac{-2r}{f_{ov} H_P} \right), \]

where \( D_0 \) is the diffusion coefficient at the convective boundary, \( r \) is the radial distance from the core boundary, and \( H_P \) is the pressure scale height at the core boundary. The amount of overshooting is defined in terms of a fraction of the pressure scale height, \( H_P \). The fraction is parameterized using the overshooting parameter \( f_{ov} \). The overshoot parameter \( f_{ov} \) is allowed to vary between 0 and 0.1 in our models. Overshooting is permitted both above convective cores and below convective envelopes.

The rotation is included by imposing a surface rotation rate and by forcing the zero-age main-sequence model to be uniformly rotating. The surface rotation rate (\( \Omega \)) is defined as a fraction of the critical rotation (\( \Omega_c \)), or the point at which the gravitational force is balanced by the centripetal acceleration at the equator. We parameterize the rotation rate as \( \Omega/\Omega_c \) and allow it to vary between 0 and 0.6 in our models. MESA uses the shellular treatment of rotation (Meynet & Maeder 1997), which does not take into account the centrifugal deformation of the star. Although this 1D treatment breaks down at very high rotation rates, it is sufficient for the rotation rates discussed here. The transport of angular momentum is implemented via a diffusion approximation (Paxton et al. 2013).

The pulsation frequencies for each model were calculated using GYRE (Townsend & Teitler 2013), with frequencies that range from 2 to 400 \( \muHz \) (0.17–34.56 c/d), with \( \ell = 0, 1, 2 \) for all of the allowed values of \( m \). This covers the radial orders of the \( g \) modes from 1 to approximately 100, depending on the rotation rate, and the first few radial orders of the \( p \) modes. We calculated the nonadiabatic frequencies, which gives us estimates for the growth rates of all calculated modes. GYRE includes rotation using the Traditional Approximation. In Prat et al. (2017), the authors compare the Traditional Approximation to more accurate ray-tracing calculations, and find that the resulting period spacings are accurate to high rotation rates, at least for the low-frequency \( g \) modes. For \( p \) modes, the traditional perturbation methods have been shown to be reliable up to about 0.4 \( \Omega/\Omega_c \) (Lignières et al. 2006; Lovekin & Dupree 2008). Above this limit, the frequencies can differ by more than 10%, depending on the radial order and the \( \ell \) of the mode.

3. Pulsation Constants

We have calculated \( Q \) values for all \( p \) modes with an \( n \leq 7 \) for the models in our grid and compared our calculated \( Q \) values to the \( Q \) values predicted by Fitch’s (1981) interpolation formulae. For each evolution sequence, we selected every 20th model along that evolution sequence. Sample tracks are shown in Figure 1 for models with \( f_{ov} = 0 \) (solid) and \( f_{ov} = 0.1 \) (dashed), spanning the range of our parameter space. The models used to calculate the pulsation constants are indicated by the diamonds. For the radial \( n = 1 \) modes shown in Figure 2, the Fitch (1981) predictions agree well for models with \( \log T_{\text{eff}} \gtrsim 3.83 \), and none were cooler than \( \log T_{\text{eff}} = 3.80 \). Below this temperature, the parameterization predicts much lower values, as our models show a significant change in the slope of \( \log Q \) as a function of \( \log T_{\text{eff}} \). This change in the slope was not accounted for by Fitch (1981). In the original
increasing trend in $n$.

A small amount of the scatter is seen at the overshoot well for an $f_{ov} = 0$ scatter at $f$. These two sets of tracks cover the extremes of our parameter space, with other overshoot parameters and rotation rates falling within this range in log $T_{\text{eff}}$ and log $L/L_{\odot}$.

Figure 1. Sample evolution tracks for stellar models that range from 1.2 to $2.1 M_\odot$. Models that evolved with no overshoot ($f_{ov} = 0$) are shown with solid lines, while the models with $f_{ov} = 0.1$ are shown with dashed lines. The individual models used for the calculation of the pulsation constants are indicated with diamonds. These two sets of tracks cover the extremes of our individual models used for the calculation of the pulsation constants are shown with solid lines. The solid lines show the piecewise linear scaling law predictions, as shown with squares, (+, $\times$, and $\ast$, agree well for $f_{ov} \leq 0.04$ (squares) for temperatures greater than log $T_{\text{eff}}$ $\approx$ 3.83. A small amount of the scatter is seen at the overshoot $f_{ov} = 0.06$ (+), a larger scatter at $f_{ov} = 0.08$ ($\times$), and the largest amount of the scatter corresponds to the largest overshoot ($f_{ov} = 0.1, \ast$). Below log $T_{\text{eff}}$ $\approx$ 3.83, we see an increasing trend in $Q$ that is not seen in Fitch (1981).

Figure 2. Calculated $Q$ values based on the MESA/GYRE models for the radial $n = 1$ mode for all of the nonrotating models in our grid (diamonds). The Fitch (1981) scaling law predictions, as shown with squares, +, $\times$, and $\ast$, agree well for $f_{ov} \leq 0.04$ (squares) for temperatures greater than log $T_{\text{eff}}$ $\approx$ 3.83. A small amount of the scatter is seen at the overshoot $f_{ov} = 0.06$ (+), a larger scatter at $f_{ov} = 0.08$ ($\times$), and the largest amount of the scatter corresponds to the largest overshoot ($f_{ov} = 0.1, \ast$). Below log $T_{\text{eff}}$ $\approx$ 3.83, we see an increasing trend in $Q$ that is not seen in Fitch (1981).

parameterization, the predicted $Q$ depends only on the period and is independent of the temperature. Indeed, only two models in their grid had a temperature below log $T_{\text{eff}}$ $\approx$ 3.83, where we find the change in the slope for the radial $n = 1$ mode. In the temperature range spanned by Fitch (1981), the $Q$ value calculated from our models is also independent of the effective temperature. We also find that given the parameters of our models, the parameterization of Fitch’s produces a much larger scatter in $Q$ than what we observe based on our calculations. This scatter exists only in models with $f_{ov} \geq 0.06$ and increases dramatically as a function of the overshoot. Increasing the overshoot tends to increase the pulsation period in our models. The scaling relations presented by Fitch (1981) for the $\ell = 0$ modes depend only on the period, which means that these relations tend to predict a higher $Q$. Since the overshoot also decreases the mean density, this increase in $Q$ is not seen in our models. The higher the overshoot, the more pronounced this effect becomes, resulting in higher predicted $Q$ parameters at higher values of the convective overshoot.

The slope in $Q$ below log $T_{\text{eff}}$ $\approx$ 3.83, as seen in Figure 2, is related to the depth of the convective envelope. In Figure 3, we show the fractional depth of the convective envelope as a function of the effective temperature. In the models hotter than log $T_{\text{eff}} = 3.83$, the convective envelope remains very shallow. In models that are either initially cooler than log $T_{\text{eff}} = 3.83$ or evolve to lower temperatures, the convective envelope beings to grow, extending deeper into the star. The variations in the temperature and sound speed gradient changes the frequencies, resulting in the observed slope in log $Q$.

Figure 3. Normalized depth of the convection zone as a function of effective temperature for all of the models in our grid. The convective envelope begins to grow as models become cooler than log $T_{\text{eff}} = 3.83$. This boundary corresponds to the change in the slope seen in the log $Q$–log $T_{\text{eff}}$ plots (see, for example, Figure 2).

As the radial overtone of the modes increases, we find that the change in the slope becomes less sharp and actually changes sign at $n \approx 4$, as shown in Figure 4 for the $\ell = 0$ modes. In all cases, the change in the slope is found near log $T_{\text{eff}} = 3.83$. On the cool side of this break are the models
with an $M \leq 1.4M_\odot$ and the later stages of the main-sequence evolution for the more massive models.

We fit each $n$ value separately with a piecewise linear function:

$$
\log Q = m_{\text{low}} \log T_{\text{eff}} + b_1 \quad \log T_{\text{eff}} < T_{\text{break}}
$$

$$
\log Q = m_{\text{high}} \log T_{\text{eff}} + b_2 \quad \log T_{\text{eff}} \geq T_{\text{break}}.
$$

(3)

with the restriction that $b_1 = (m_{\text{low}} - m_{\text{high}})T_{\text{break}} + b_2$. The four free parameters are the break point $T_{\text{break}}$, the slope for $x < x_b$ ($m_{\text{low}}$), the slope for $x \geq x_b$ ($m_{\text{high}}$), and the intercept of the fit ($b_2$). These fits are also shown in Figure 4. The parameter $m_{\text{high}}$ was not found to vary significantly with any of the parameters and is very close to 0 in all cases. For this reason, we focus on the other three parameters. We used the best fitting values of $m_{\text{low}}$, $T_{\text{break}}$, and $b_2$ to investigate the effects of rotation and overshoot on log $Q$.

We found that the parameters $b_2$ and $T_{\text{break}}$ were basically constant as the overshoot was increased. Increasing the overshoot does increase $m_{\text{low}}$, with the exception of the $f_{ov} = 0$ models. However, this particular set of models shows a higher amount of scatter, particularly near the break point. This scatter appears to artificially decrease the slope, which should in fact be steeper if these points were excluded. If we only consider the only parameter $m_{\text{low}}$ over the range $f_{ov} = 0.02$--0.1, then the variation in $m_{\text{low}}$ as a function of the overshoot can be well-fit by a straight line, with

$$
m_Q = 14f_{ov} - 2.8
$$

(4)

for the radial $n = 1$ mode, where $m_Q$ is the predicted value of $m_{\text{low}}$. We found that for a given $n$, there was not a significant variation in the slopes ($m_{\text{low}}$) with the rotation. At $\ell = 0$, the exception is that the $Q$ values for models with $\Omega / \Omega_c = 0.5$ are significantly lower than those with other rotation rates. To fit $m_{\text{low}}$, we averaged the $Q$ values at each overshoot over all of the rotation rates and fit the averages, excluding the $Q$s for the models with $\Omega / \Omega_c = 0.5$. The variation in the resulting $m_{\text{low}}$ as a function of the overshoot can be fit by

$$
m_Q = (12 \pm 2)f_{ov} - (2.8 \pm 0.1)
$$

(5)

for the radial $n = 1$ mode. We fit the variation in this slope for the first seven radial orders of the $\ell = 0$ modes in our models. The mean and standard deviation for $m_{\text{low}}$, the slope of the $m_Q - f_{ov}$ relationship, and $b_{ov}$, the intercept of this relationship are summarized in Table 1.

Similarly, we note that the intercept of the fit ($b_2$) appears to be sensitive to the amount of rotation in the models. The amount of convective overshoot does not strongly affect this part of the plot, so we average over all overshoot values when fitting. As for the variation in $m_{\text{low}}$, the variation in $b_2$ shows a linear variation with rotation rate, although the scatter at high rotation rates ($\Omega / \Omega_c \geq 0.5$) is large. We fit $b_2$ as a function of rotation rate with a straight line, and present the coefficients of the fit (the slope, $m_{b2}$ and the intercept, $b_{b2}$) at each $n$ in Table 2.

We also find some indications that there is a relationship between the break point in our fit to the $Q$ data and the rotation rate of the models. However, the scatter in this relationship is high, and the variation in the break point is small. For this reason, we do not present fits to the trends here.

In principle, this result could be used to estimate the average rotation and the overshoot for populations of stars. We have tested this method on a small sample of HADS taken from the International Variable Star Index maintained by the American Association of Variable Star Observers (AAVSO). We collected a sample of 22 stars that are known to pulsate in radial modes. The basic parameters of these stars are presented in Table 3. We estimated the effective temperatures for these stars using the $B$ and $V$ magnitudes available in the SIMBAD database and the color-effective temperature relationship determined by Sekiguchi & Fukugita (2000). We used the period of the highest-amplitude mode in the AAVSO catalog as the radial $n = 1$ mode and calculated $Q$ by assuming that all

### Table 1

| $n$ | $m_{\text{low}}$ | $\sigma(m_{\text{low}})$ | $b_{\text{low}}$ | $\sigma(b_{\text{low}})$ |
|-----|-----------------|--------------------------|-----------------|--------------------------|
| 1   | 12              | 2                        | -2.8            | 0.1                      |
| 2   | 3.9             | 0.6                      | -0.59           | 0.04                     |
| 3   | -0.43           | 0.5                      | 0.17            | 0.04                     |
| 4   | -1.7            | 0.6                      | 0.48            | 0.04                     |
| 5   | -2.2            | 0.7                      | 0.67            | 0.02                     |
| 6   | -3.1            | 0.8                      | 0.84            | 0.03                     |
| 7   | -5.5            | 0.5                      | 1.02            | 0.02                     |

### Table 2

| $n$ | $m_{b2}$ | $\sigma(m_{b2})$ | $b_{b2}$ | $\sigma(b_{b2})$ |
|-----|----------|-----------------|----------|-----------------|
| 1   | -0.8     | 0.2             | -1.14    | 0.07            |
| 2   | 0.22     | 0.06            | -1.60    | 0.02            |
| 3   | 0.2      | 0.2             | -1.66    | 0.03            |
| 4   | 0.8      | 0.1             | -2.15    | 0.03            |
| 5   | 0.47     | 0.03            | -2.03    | 0.02            |
| 6   | 0.57     | 0.09            | -2.09    | 0.03            |
| 7   | 0.6      | 0.1             | -2.18    | 0.03            |

### Table 3

| Identifier | Period (day) | $(B - V)$ | log $T_{\text{eff}}$ |
|------------|-------------|-----------|-----------------------|
| AI Vel     | 0.11        | 0.28      | 3.84                  |
| BPS BS 16553-0026 | 0.13 | 0.82 | 3.72                  |
| NSV 14800  | 0.16        | 0.35      | 3.82                  |
| USNO-B1.0 0961-0254829 | 0.05 | -0.20 | 4.01                  |
| V7075 Sco  | 0.12        | 0.33      | 3.83                  |
| BP Peg     | 0.11        | 0.15      | 3.88                  |
| 2MASS J18294745+3745005 | 0.12 | -0.29 | 4.05                  |
| VX Hya     | 0.22        | 0.70      | 3.74                  |
| V8079 Her  | 0.06        | 0.37      | 3.82                  |
| NSVS 7293918 | 0.09 | 0.20 | 3.86                  |
| V4043 Gem  | 0.15        | -0.10     | 3.97                  |
| GSC 03949-00811 | 0.17 | 0.31 | 3.83                  |
| GSC 03949-00386 | 0.10 | 0.56 | 3.77                  |
| ASAS J194803+4146.9 | 0.20 | 0.60 | 3.76                  |
| V1719 Cyg  | 0.27        | 0.42      | 3.80                  |
| GSC 02457-00471 | 0.17 | 0.38 | 3.81                  |
| V8023 Cas  | 0.67        | 0.59      | 3.77                  |
| DO CMi     | 0.19        | 0.43      | 3.80                  |
| VZ Cnc     | 0.18        | 0.17      | 3.87                  |
| 2MASS J06451725+4122158 | 0.05 | 0.08 | 3.90                  |
| RV Ari     | 0.09        | 0.17      | 3.87                  |
| V8029 Aql  | 0.29        | 0.68      | 3.75                  |
stars had the same mean density. The resulting effective temperatures and the $Q$ values are unlikely to have very high accuracies, but should be sufficient for a proof-of-concept test of our method.

We plotted $\log Q$ versus $\log T_{\text{eff}}$ for each of the 22 stars in our sample and fit them with the same broken power-law fit that we used to fit our models. The result is shown in Figure 5. The broken power law used to fit the model data also appears to fit the data in this case, although the small number of observations leads to large uncertainties. We find the observed slope of the cool stars is $-2.7 \pm 1.2$, and we can use this value in Equation (5) to estimate the convective overshoot of the stars in our sample. For these stars, we estimate that the convective overshoot of $f_{\text{ov}} = 0.008^{+0.004}_{-0.008}$. The errors on this estimate are, of course, extremely large, but it should be possible to reduce the errors using a larger sample of stars that have more accurate measurements of their effective temperatures and mean densities.

We are also able to determine the average $\log Q$ for the stars in this sample that are hotter than $\log T_{\text{eff}} = 3.85$ is $-1.2 \pm 8.3$. However, the sample presented here has very few $(\approx 4)$ stars hotter than $\log T_{\text{eff}} = 3.85$, so the estimate has an extremely high uncertainty. Again, a larger sample of better constrained stars should make more accurate estimates of the rotation rate possible. This is a promising area of future work. In particular, for this technique to be practical for the application to observed stars, we will need to investigate the dependence of the scaling relations on the metallicity and the helium abundance. It would also be interesting to see if these same relations hold true for other radial pulsators, such as classical Cepheids or RR Lyrae variables.

We have also calculated the $Q$ values for modes with $\ell = 1$ and $\ell = 2$, and compared them to the predictions from Fitch (1981). In the nonradial case, Fitch’s predictions do not work as well, as illustrated in Figure 6 for the $\ell = 1$ modes. Our model $Q$ values show a bimodal distribution. The lower curve (shorter $Q$) consists of the main-sequence models, while models in the blue hook phase of the evolution are on the upper sequence. Fitch’s predictions are based on subgiant models and lie between the two sequences. For the $\ell = 2$ models, we find that the Fitch (1981) predictions over-predict the expected $\log Q$ by between 0.1 and 0.2 dex. For both the $\ell = 1$ and $\ell = 2$ models, there are no significant trends with either rotation or overshoot, so these modes cannot be easily used to constrain these parameters.

### 4. Regular Patterns in the Observed Pulsation

#### 4.1. Frequency Spacings

The $p$-mode pulsations in Scuti stars are of a low radial order and are not expected to obey the asymptotic frequency spacing relations of Tassoul (1980). However, regular frequency spacings have been observed in at least some Scuti stars (e.g., Handler et al. 1997; Breger et al. 1999), and various techniques have been proposed to identify the patterns (García Hernández et al. 2013; Paparó et al. 2016). The identification of the frequency patterns can be quite challenging though, as Paparó et al. (2016) find that in some cases, as few as 20% of the frequencies can be located on the echelle ridges.

Since the mode identification for our models is known, it is a straightforward matter to construct echelle diagrams and to determine the large frequency spacing ($\delta \nu$). We have found that an increased rotation and a convective overshoot increase the large frequency spacing, although $\delta \nu$ increases more rapidly with the rotation than with the convective core overshoot. Both rotation and convective core overshoot introduce extra mixing, which tends to smooth out composition gradients surrounding the convective core, making the star appear younger. This is consistent with the results found by Otí Floranes et al. (2005), which shows that the large separation increases as the star evolves. Unfortunately, we have no method for disentangling the effects of the rotation and the overshoot, so although a larger-than-expected $\delta \nu$ indicates the effects of convective overshoot and/or rotation are present, we cannot use the echelle diagrams and the large separation to quantify the processes.

#### 4.2. Period Spacings

The $\gamma$ Doradus variables are expected to display regular patterns in the period spacing (Van Reeth et al. 2015, 2016; Ouazzani et al. 2016). As was found by these previous studies, we find that in the nonrotating models, the period spacings are
nearly flat as a function of period, as shown in Figure 7. Adding a rotation lifts the degeneracy of the $m$ values, so for the $m = 1$ modes, the period spacings have a negative slope, while for $m = -1$ modes, the period spacings have a positive slope. As the age of the model increases, the chemical discontinuity at the core causes regular dips in the period spacings.

The convective overshoot has two effects on the predicted period spacings. First, as noted by Van Reeth et al. (2015), the increased mixing caused by convective overshooting tends to wash out the chemical discontinuity, so the dips in the period spacing pattern become smaller, as shown in Figure 7. Second, we find that an increased convective overshoot increases the absolute value of the period spacing. Increased convective overshooting effectively increases the size of the hydrogen burning core, so the star behaves as though it were more massive. This trend for an increasing period spacing is identical to that seen as the mass increases. At the highest overshoots in our grid, $f_{ov} = 0.1$, the increase in the period spacing is the same as that produced by increasing the mass by approximately $0.8 M_\odot$.

We find that the rotation does not affect the overall level of the period spacing, but does affect the slope. For the $m = 0$ modes, the rotation causes the period difference to decrease with an increasing period, which is similar to the results presented by Van Reeth et al. (2016) and Ouazzani et al. (2016). This makes the $m = 0$ modes appear more like the $m = 1$ modes. A similar trend is seen for the $m = 1$ modes, as the increased rotation makes the negative slope steeper. The $m = -1$ modes may provide a useful diagnostic of rotation rates in these stars, as shown in Figure 8. These modes show a clear upward trend at the high periods in the nonrotating models. As the rotation rate increases, the point at which the period difference starts to increase moves to lower periods, and the positive slope becomes steeper.

5. Conclusion

We have calculated a grid of stellar models from 1.2 to $2.2 M_\odot$ using MESA and GYRE. Using these models, we have calculated the pulsation constant ($Q$) for the predicted $p$ modes.

We find that the previous fitting functions for $Q$, as published by Fitch (1981), work well above $\log T \approx 3.83$. Below this temperature, we find a sharp increase in the $Q$ value for the radial modes and a large scatter toward a smaller $Q$ for nonradial modes. This temperature range was outside of the grid of models used by Fitch (1981), and so the trend is not accounted for by these fitting relations. We find that the slope of the $\log Q$–$\log T_{\text{eff}}$ plot is strongly correlated with the amount of convective overshoot, and we provide fitting relations that can be used to determine the convective overshoot of a population of stars in a statistical sense. We apply this method to a small sample of HADS that are known to pulsate radially and provide estimates for the convective overshoot. This estimate has very large uncertainties, which is in part because of the assumptions used to estimate $Q$ for these stars. More accurate measurements as part of a dedicated study should be able to reduce these uncertainties and should place tighter constraints on the convective overshoot.

We also find that the $\log Q$–$\log T_{\text{eff}}$ plot is relatively flat above $\log T_{\text{eff}} \approx 3.83$, and that the level of this region is sensitive to the rotation rate. As for the convective overshoot, we provide a fitting function that can, in principle, be used to determine the rotation rate. Unfortunately, our test sample had too few stars in this region to give a reliable determination of the average rotation rate of the sample.

A comparison of our model $Q$ values with the predictions for the nonradial modes taken from Fitch (1981) do not agree as well as for the radial modes. We find a bimodal distribution of the $Q$ values for the $\ell = 1$ modes, with the lower $Q$ values corresponding to the main-sequence models, and the larger $Q$ values corresponding to the models that evolve through the blue hook. The predicted $Q$ values lie in between these two sequences, but do not agree with either case.

We calculate the frequency and period spacing patterns for these stars, confirming the previous research on the expected patterns. By plotting echelle diagrams for the theoretical $p$ modes, we find that the large separation increases with both the rotation rate and the convective overshoot, but it is difficult to disentangle the effects. Although it seems that the effects of the rotation and the overshoot can be seen in the echelle diagram of $\delta$ Scuti stars, the effects are degenerate, and it is not possible to uniquely determine either parameter individually.
We also plotted the period spacings for the theoretical $g$ modes. We confirm the previous trends detected by Ouazzani et al. (2016) and Van Reeth et al. (2015, 2016). We also find that the $m = -1$ modes may provide a useful diagnostic of the rotation rate, as the upward slope of the period spacings becomes steeper as the rotation rate increases. We also find that the change in the slope occurs at a lower period as the rotation rate increases.

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