Correlation functions in the CFT$_d$/AdS$_{d+1}$ correspondence

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Abstract

Conformal techniques are applied to the calculation of integrals on AdS$_{d+1}$ space which define correlators of composite operators in the superconformal field theory on the $d$–dimensional boundary. The 3–point amplitudes for scalar fields of arbitrary mass and gauge fields in the AdS supergravity are calculated explicitly. For 3 gauge fields we compare in detail with the known conformal structure of the SU(4) flavor current correlator $\langle J^a_i J^b_j J^c_k \rangle$ of the $\mathcal{N} = 4$, $d = 4$ SU(N) SYM theory. Results agree with the free field approximation as would be expected from superconformal non–renormalization theorems. In studying the Ward identity relating $\langle J^a_i O^I J^j_j \rangle$ to $\langle O^I O^J \rangle$ for (non–marginal) scalar composite operators $O^I$, we find that there is a subtlety in obtaining the normalization of $\langle O^I O^J \rangle$ from the supergravity action integral.

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1 Introduction

The fact that the near horizon geometry of typical brane configurations in string/M theory is the product space $AdS_{d+1} \times S_p$ with $d + 1 + p = 10/11$ has suggested an intriguing conjecture relating gauged supergravity theory on $AdS_{d+1}$ with a superconformal theory on its $d$–dimensional boundary. See also [13]–[17] for earlier appearance of this correspondence in the context of black hole physics and [32]–[75] for recent relevant work on the subject.

Precise forms of the conjecture have been stated and investigated in [2, 3] (see also [4]) for the $AdS_5 \times S_5$ geometry of $N$ 3–branes in Type–IIB string theory. The superconformal theory on the world–volume of the $N$ branes is $\mathcal{N} = 4$ SUSY Yang–Mills with gauge group $SU(N)$. The conjecture holds in the limit of a large number $N$ of branes with $g_{st}N \sim g_{YM}^2 N$ fixed but large. As $N \to \infty$ the string theory becomes weakly coupled and one can neglect string loop corrections; $N g_{st}$ large ensures that the $AdS$ curvature is small so one can trust the supergravity approximation to string theory. In this limit one finds the maximally supersymmetric 5–dimensional supergravity with gauged $SU(4)$ symmetry together with the Kaluza–Klein modes for the “internal” $S_5$. There is a map between elementary fields in the supergravity theory and gauge invariant composite operators of the boundary $\mathcal{N} = 4$ $SU(N)$ SYM theory. This theory has an $SU(4)$ flavor symmetry which is part of its $\mathcal{N} = 4$ superconformal algebra. Correlation functions of the composite operators in the large $N$ limit with $g_{YM}^2 N$ fixed but large are given by certain classical amplitudes in supergravity.

To describe the conjecture for correlators in more detail, we note that correlators of the $\mathcal{N} = 4$ $SU(N)$ SYM theory are conformally related to those on the 4–sphere which is the boundary of (Euclidean) $AdS_5$. Consider an operator $\mathcal{O}(\vec{x})$ of the boundary theory, coupled to a source $\phi_0(\vec{x})$ ($\vec{x}$ is a point on the boundary $S_4$), and let $e^{-W[\phi_0]}$ denote the generating functional for correlators of $\mathcal{O}(\vec{x})$. Suppose $\phi(z)$ is the field of the interior supergravity theory which corresponds to $\mathcal{O}(\vec{x})$ in the operator map. Propagators $K(z, \vec{x})$ between the bulk point $z$ and the boundary point $\vec{x}$ can be defined and used to construct a perturbative solution of the classical supergravity field equation for $\phi(z)$ which is determined by the boundary data $\phi_0(\vec{x})$. Let $S_{cl}[\phi]$ denote the value of the supergravity action for the field configuration $\phi(z)$. Then the conjecture is precisely that $W[\phi_0] = S_{cl}[\phi]$. This leads to a graphical algorithm, see Fig.1, involving $AdS_5$ propagators and interaction vertices determined by the classical supergravity Lagrangian. Each vertex entails a 5–dimensional
integral over $AdS_5$.

Actually, the prescriptions of [2] and [3] are somewhat different. In the first [2], solutions $\phi(z)$ of the supergravity theory satisfy a Dirichlet condition with boundary data $\phi_0(\vec{x})$ on a sphere of radius $R$ equal to the $AdS$ length scale. In the second method [3], it is the infinite boundary of (Euclidean) $AdS$ space which is relevant. Massless scalar and gauge fields satisfy Dirichlet boundary conditions at infinity, but fields with $AdS$ mass different from zero scale near the boundary like $\phi(z) \to z_0^{d-\Delta} \phi_0(\vec{x})$ where $z_0$ is a coordinate in the direction perpendicular to the boundary and $\Delta$ is the dimension of the corresponding operator $O(\vec{x})$. This is explained in detail below. Our methods apply readily only to the prescription of [3], although for 2–point functions we will be led to consider a prescription similar to [2].

To our knowledge, results for the correlators presented so far include only 2–point functions [2, 3, 45], and the purpose of the present paper is to propose a method to calculate multi–point correlators and present specific applications to 3–point correlators of various scalar composite operators and the flavor currents $J_i^a$ of the boundary gauge theory. Our calculations provide explicit formulas for $AdS_{d+1}$ integrals needed to evaluate generic supergravity 3–point amplitudes involving gauge fields and scalar fields of arbitrary mass. Integrals are evaluated for $AdS_{d+1}$, for general dimension, to facilitate future applications of our results. The method uses conformal symmetry to simplify the integrand, so that the internal $(d+1)$–dimensional integral can be simply done. This technique, which uses a simultaneous inversion of external coordinates and external points, has been applied to many two–loop Feynman integrals of flat four–dimensional theories [21, 22, 26]. The method works well in four flat dimensions, although there are difficulties for gauge fields, which arise because the invariant action $F_{\mu\nu}^2$ is inversion symmetric but the gauge–fixing term is not [21]. It is a nice surprise that it works even better in $AdS$ because the inversion is an isometry, and not merely a conformal isometry as in flat space. Thus the method works perfectly for massive fields and for gauge interactions in $AdS_{d+1}$ for any dimension $d$.

It is well–known that conformal symmetry severely restricts the tensor form of 2– and 3–point correlation functions and frequently determines these tensors uniquely up to a constant multiple. (For a recent discussion, see [27]). This simplifies the study of the 3–point functions.

One of the issues we are concerned with are Ward identities that relate 3–point correlators with one or more currents to 2–point functions. It was a surprise to us this requires

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1Very recently, a paper has appeared [76] which computes special cases of 3– and 4–point functions of scalar operators.
a minor modification of the prescription of [3] for the computation of $\langle O^I O^J \rangle$ for gauge–invariant composite scalar operators.

It is also the case that some of the correlators we study obey superconformal non–renormalization theorems, so that the coefficients of the conformal tensors are determined by the free–field content of the $\mathcal{N} = 4$ theory and are not corrected by interactions. The evaluation of $n$–point correlators, for $n \geq 4$, contains more information about large $N$ dynamics, and they are given by more difficult integrals in the supergravity construction. We hope, but cannot promise, that our conformal techniques will be helpful here. The integrals encountered also appear well–suited to Feynman parameter techniques, so traditional methods may also apply. In practice, the inversion method reduces the number of denominators in an amplitude, and we do apply standard Feynman parameter techniques to the “reduced amplitude” which appears after inversion of coordinates.

2 Scalar amplitudes

It is simplest to work [3] in the Euclidean continuation of $AdS_{d+1}$ which is the $Y_{-1} > 0$ sheet of the hyperboloid:

$$-(Y_{-1})^2 + (Y_0)^2 + \sum_{i=1}^{d}(Y_i)^2 = -\frac{1}{a^2} \tag{1}$$

which has curvature $R = -d(d+1)a^2$. The change of coordinates:

$$z_i = \frac{Y_i}{a(Y_0 + Y_{-1})} \tag{2}$$

$$z_0 = \frac{1}{a^2(Y_0 + Y_{-1})}$$

brings the induced metric to the form of the Lobachevsky upper half–space:

$$ds^2 = \frac{1}{a^2z_0^2} \left( \sum_{\mu=0}^{d} (dz_\mu)^2 \right) = \frac{1}{a^2z_0^2} \left( dz_0^2 + \sum_{i=1}^{d} (dz_i)^2 \right) = \frac{1}{a^2z_0^2} \left( dz_0^2 + dz^2 \right) \tag{3}$$
We henceforth set \( a \equiv 1 \). One can verify that the inversion:

\[
z'_\mu = \frac{z_\mu}{z^2}
\]

is an isometry of (3). Its Jacobian:

\[
\frac{\partial z'_\mu}{\partial z_\nu} = (z')^2 \left( \delta_{\mu\nu} - 2 \frac{z'_\mu z'_\nu}{(z')^2} \right)
\]

\[
\equiv (z')^2 J_{\mu\nu}(z') = (z')^2 J_{\mu\nu}(z)
\]

has negative determinant showing that it is a discrete isometry which is not a proper element of the \( SO(d + 1, 1) \) group of (3) and (4). Note that we define contractions such as \((z')^2\) using the Euclidean metric \( \delta_{\mu\nu} \), and we are usually indifferent to the question of raised or lowered coordinate indices, i.e. \( z^\mu = z_\mu \). When we need to contract indices using the \( AdS \) metric we do so explicitly, e.g., \( g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \), with \( g^{\mu\nu} = z_0^2 \delta_{\mu\nu} \).

The Jacobian tensor \( J_{\mu\nu} \) obeys a number of identities that will be very useful below. These include the pretty inversion property

\[
J_{\mu\nu}(x - y) = J_{\mu\rho}(x') J_{\rho\sigma}(x' - y') J_{\sigma\nu}(y')
\]

and the orthogonality relation

\[
J_{\mu\nu}(x) J_{\nu\rho}(x) = \delta_{\mu\rho}
\]

The (Euclidean) action of any massive scalar field

\[
S[\phi] = \frac{1}{2} \int d^dz dz_0 \sqrt{g} \left[ g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + m^2 \phi^2 \right]
\]

is inversion invariant if \( \phi(z) \) transforms as a scalar, i.e. \( \phi(z) \rightarrow \phi'(z) = \phi(z') \). The wave equation is:

\[
\frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi) - m^2 \phi = 0
\]

\[
z_0^{d+1} \frac{\partial}{\partial z_0} \left[ z_0^{-d+1} \frac{\partial}{\partial z_0} \phi(z_0, z) \right] + z_0^2 \frac{\partial^2}{\partial z^2} \phi(z_0, z) - m^2 \phi(z_0, z) = 0
\]

A generic solution which vanishes as \( z_0 \rightarrow \infty \) behaves like \( \phi(z_0, z) \rightarrow z_0^{d-\Delta} \phi_0(z) \) as \( z_0 \rightarrow 0 \), where \( \Delta = \Delta_+ \) is the largest root of the indicial equation of (10), namely \( \Delta_\pm = \frac{1}{2}(d \pm \sqrt{d^2 + 4m^2}) \). Witten [3] has constructed a Green’s function solution which explicitly realizes the relation between the field \( \phi(z_0, z) \) in the bulk and the boundary configuration \( \phi_0(\bar{x}) \).
The normalized bulk–to–boundary Green’s function, for $\Delta > \frac{d}{2}$:

$$K_{\Delta}(z_0, \vec{z}, \vec{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta$$ \hspace{1cm} (11)

is a solution of (10) with the necessary singular behavior as $z_0 \to 0$, namely:

$$z_0^{\Delta - d} K_{\Delta}(z_0, \vec{z}, \vec{x}) \to 1 \cdot \delta(\vec{z} - \vec{x}) \hspace{1cm} (12)$$

The solution of (10) is then related to the boundary data by:

$$\phi(z_0, \vec{z}) = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \int d^d x \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta \phi_0(\vec{x})$$ \hspace{1cm} (13)

Note that the choice of $K_{\Delta}$ that we have taken is invariant under translations in $\vec{x}$. This choice corresponds to working with a metric on the boundary of the $\text{AdS}$ space that is flat $R^d$ with all curvature at infinity. Thus our correlation functions will be for CFT$_d$ on $R^d$. Correlation function for other boundary metrics can be obtained by multiplying by the corresponding conformal factors.

It is vital to the CFT$_d$/AdS$_{d+1}$ correspondence, and to our method, that isometries in AdS$_{d+1}$ correspond to conformal isometries in CFT$_d$. In particular the inversion isometry of AdS$_{d+1}$ is realized by the well-known conformal inversion in CFT$_d$. A scalar field (a scalar source from the point of view of the boundary theory) $\phi_0(\vec{x})$ of scale dimension $\alpha$ transforms under the inversion as $x_i \to x'_i/|\vec{x}'|^2$ as $\phi_0(\vec{x}) \to \phi'_0(\vec{x}) = |\vec{x}'|^{2\alpha} \phi_0(\vec{x}')$. The construction (13) can be used to show that a bulk scalar of mass $m^2$ is related to boundary data $\phi_0(\vec{x})$ with scale dimension $d - \Delta$. To see this one uses the equalities:

$$d^d x = \frac{d^d x'}{|\vec{x}'|^{2d}}$$

$$\left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta = \left( \frac{z'_0}{(z'_0)^2 + (\vec{z}' - \vec{x}')^2} \right)^\Delta |\vec{x}'|^{2\Delta}$$ \hspace{1cm} (14)

and $\phi'_0(\vec{x}) = |\vec{x}'|^{2(d-\Delta)} \phi_0(\vec{x}')$. We then find directly that:

$$\frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}} \Gamma(\Delta - \frac{d}{2})} \int d^d x \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\Delta \phi'_0(\vec{x}) = \phi(\vec{z}')$$ \hspace{1cm} (15)

2The special case $\Delta = \frac{d}{2}$ corresponds to the lowest AdS mass allowed by unitarity, i.e. $m^2 = -\frac{d^2}{4}$. In this case $\phi(z_0, \vec{z}) \to -\frac{4}{z_0^2} \ln z_0 \phi_0(\vec{z})$ as $z_0 \to 0$ and the Green’s function which gives this asymptotic behavior is $K_{\frac{d}{2}}(z_0, \vec{z}, \vec{x}) = \frac{\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}} \Gamma(\frac{d}{2} - \frac{d}{2})} \left( \frac{z_0}{z_0^2 + (\vec{z} - \vec{x})^2} \right)^\frac{d}{2}$. All the formulas in the text assume the generic normalization (11) valid for $\Delta > \frac{d}{2}$, obvious modifications are needed for $\Delta = \frac{d}{2}$. 

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Thus conformal inversion of boundary data with scale dimension \( d - \Delta \) produces the inversion isometry in \( AdS_{d+1} \). In the \( CFT_d/AdS_{d+1} \) correspondence, \( \phi_0(\vec{x}) \) is viewed as the source for a scalar operator \( \mathcal{O}(\vec{x}) \) of the CFT\(_d\). From \( \int d^d x \mathcal{O}(\vec{x}) \phi_0(\vec{x}) \) one sees that \( \mathcal{O}(\vec{x}) \to \mathcal{O}'(\vec{x}) = |\vec{x}'|^{2\Delta} \mathcal{O}(\vec{x}') \) so that \( \mathcal{O}(\vec{x}) \) has scale dimension \( \Delta \).

Let us first review the computation of the 2–point correlator \( \mathcal{O}(\vec{x})\mathcal{O}(\vec{y}) \) for a CFT\(_d\) scalar operator of dimension \( \Delta \) [3]. We assume that the kinetic term (8) of the corresponding field \( \phi \) of \( AdS_{d+1} \) supergravity is multiplied by a constant \( \eta \) determined from the parent 10–dimensional theory. We have, accounting for the 2 Wick contractions:

\[
\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y}) \rangle = -2 \frac{\eta}{\pi} \int_0^{d+1} \frac{dz_0}{z_0} \left( \partial_\mu K_{\Delta}(z, \vec{x}) z_0^2 \partial_\mu K_{\Delta}(z, \vec{y}) + m^2 K_{\Delta}(z, \vec{x}) K_{\Delta}(z, \vec{y}) \right) \tag{16}
\]

We integrate by parts; the bulk term vanishes by the free equation of motion for \( K \), and we get:

\[
\langle \mathcal{O}(\vec{x})\mathcal{O}(\vec{y}) \rangle = +\eta \lim_{\epsilon \to 0} \int d^d x \epsilon^{1-d} K_{\Delta}(\epsilon, \vec{z}, \vec{x}) \left[ \frac{\partial}{\partial z_0} K_{\Delta}(z_0, \vec{z}, \vec{y}) \right]_{z_0 = \epsilon} \tag{17}
\]

where (12) has been used. We warn readers that considerations of Ward identities will suggest a modification of this result for \( \Delta \neq d \). One indication that the procedure above is delicate is that the \( \partial_\mu K \partial_\mu K \) and \( m^2 KK \) integrals in (16) are separately divergent as \( \epsilon \to 0 \).

We are now ready to apply conformal methods to simplify the integrals in \( AdS_{d+1} \) which give 3–point scalar correlators in CFT\(_d\). We consider 3 scalar fields \( \phi_I(z) \), \( I = 1, 2, 3 \), in the supergravity theory with masses \( m_I \) and interaction vertices of the form \( \mathcal{L}_1 = \phi_1 \phi_2 \phi_3 \) and \( \mathcal{L}_2 = \phi_1 g^{\mu \nu} \partial_\mu \phi_2 \partial_\nu \phi_3 \). The corresponding 3–point amplitudes are:

\[
A_1(\vec{x}, \vec{y}, \vec{z}) = -\int \frac{d^d w dw_0}{w_0^{d+1}} K_{\Delta_1}(w, \vec{x}) K_{\Delta_2}(w, \vec{y}) K_{\Delta_3}(w, \vec{z}) \tag{18}
\]

\[
A_2(\vec{x}, \vec{y}, \vec{z}) = -\int \frac{d^d w dw_0}{w_0^{d+1}} K_{\Delta_1}(w, \vec{x}) \partial_\mu K_{\Delta_2}(w, \vec{y}) w_0^2 \partial_\mu K_{\Delta_3}(w, \vec{z}) \tag{19}
\]

where \( K_{\Delta_i}(w, \vec{x}) \) is the Green function (11). These correlators are conformally covariant and are of the form required by conformal symmetry:

\[
A_i(\vec{x}, \vec{y}, \vec{z}) = \frac{a_i}{|\vec{x} - \vec{y}|^{\Delta_1+\Delta_2-\Delta_3}|\vec{y} - \vec{z}|^{\Delta_2+\Delta_3-\Delta_1}|\vec{z} - \vec{x}|^{\Delta_3+\Delta_1-\Delta_2}} \tag{20}
\]

so the only issue is how to obtain the coefficients \( a_1, a_2 \).

The basic idea of our method is to use the inversion \( w_\mu = \frac{w'_\mu}{w_0'} \) as a change of variables. In order to use the simple inversion property (14) of the propagator, we must also refer
boundary points to their inverses, e.g. \( x_i = x_i' \). If this is done for a generic configuration of \( \bar{x}, \bar{y}, \bar{z} \), there is nothing to be gained because the same integral is obtained in the \( w' \) variable. However, if we use translation symmetry to place one boundary point at 0, say \( \bar{z} = 0 \), it turns out that the denominator of the propagator attached to this point drops out of the integral, essentially because the inverted point is at \( \infty \), and the integral simplifies.

Applied to \( A_1(\bar{x}, \bar{y}, 0) \), using (14), these steps immediately give:

\[
A_1(\bar{x}, \bar{y}, 0) = -\frac{1}{|\bar{x}|^{2\Delta_1}} \frac{1}{|\bar{y}|^{2\Delta_2}} \frac{\Gamma(\Delta_3)}{\pi^2} \left( \int \frac{dz'}{(w'_0)^{d+1}} K_{\Delta_1}(w', \bar{x}') K_{\Delta_2}(w', \bar{y}') (w'_0)^{\Delta_3} \right)
\]

The remaining integral has two denominators, and it is easily done by conventional Feynman parameter methods. We will encounter similar integrals below so we record the general form:

\[
\int_0^\infty dz_0 \int d^d\bar{z} \frac{z_0^{\Delta_0}}{(z_0^2 + (\bar{z} - \bar{x})^2)(z_0^2 + (\bar{z} - \bar{y})^2)^c} \equiv I[a, b, c, d]|\bar{x} - \bar{y}|^{1+a+d-2b-2c}
\]

We thus find that \( A_1(\bar{x}, \bar{y}, 0) \) has the spatial dependence:

\[
\frac{1}{|\bar{x}|^{2\Delta_1} |\bar{y}|^{2\Delta_2} |\bar{x} - \bar{y}|^{(\Delta_1 + \Delta_2 - \Delta_3)}} = \frac{1}{|\bar{x}|^{\Delta_1 + \Delta_3 - \Delta_2} |\bar{y}|^{\Delta_2 + \Delta_3 - \Delta_1} |\bar{x} - \bar{y}|^{(\Delta_1 + \Delta_2 - \Delta_3)}}
\]

which agrees with (20) after the translation \( \bar{x} \to (\bar{x} - \bar{z}), \bar{y} \to (\bar{y} - \bar{z}) \). The coefficient \( a_1 \) is then:

\[
a_1 = -\frac{\Gamma\left(\frac{1}{2}(\Delta_1 + \Delta_2 - \Delta_3)\right) \Gamma\left(\frac{1}{2}(\Delta_2 + \Delta_3 - \Delta_1)\right) \Gamma\left(\frac{1}{2}(\Delta_3 + \Delta_1 - \Delta_2)\right)}{2\pi^d \Gamma\left(\Delta_1 - \frac{d}{2}\right) \Gamma\left(\Delta_2 - \frac{d}{2}\right) \Gamma\left(\Delta_3 - \frac{d}{2}\right)} \Gamma\left(\frac{1}{2}(\Delta_1 + \Delta_2 + \Delta_3 - d)\right)
\]

We now turn to the integral \( A_2(\bar{x}, \bar{y}, \bar{z}) \) in (19). It is convenient to set \( \bar{z} = 0 \). Since the structure \( \partial_\mu K_2 w'_0 \partial_\mu K_3 \) is an invariant contraction and the inversion of the bulk point \( a \) is diffeomorphism, we have, using (14):

\[
\partial_\mu K_2(w, \bar{y}) w'_0 \partial_\mu K_3(w, 0) = |\bar{y}|^{2\Delta_2} \partial_{w'_0} K_{\Delta_2}(w', \bar{y}') (w'_0)^2 \partial_{w'_0} K_{\Delta_3}(w', 0)
\]

\[
\sim |\bar{y}|^{2\Delta_2} \frac{\partial w'_0}{\partial w_0} \left( \frac{w'_0}{(w'_0)^2 + (w' - \bar{y})^2} \right)^{\Delta_2} (w'_0)^2 \frac{\partial w'_0}{\partial w_0} (w'_0)^{\Delta_3}
\]

\[
= \Delta_2 \Delta_3 |\bar{y}|^{2\Delta_2} (w'_0)^{(\Delta_2 + \Delta_3)} \frac{1}{((w'_0)^2 + (w' - \bar{y})^2)^{\Delta_2}} - \frac{2(w'_0)^2}{((w'_0)^2 + (w' - \bar{y})^2)^{\Delta_2 + 1}}
\]
where the normalization constants are temporarily omitted. We then find two integrals of the form

\[ I(a, b, c, d) \]

with different parameters. The result is:

\[ a_2 = a_1 \left[ \Delta_2 \Delta_3 + \frac{1}{2} (d - \Delta_1 - \Delta_2 - \Delta_3) (\Delta_2 + \Delta_3 - \Delta_1) \right] \tag{29} \]

As described by Witten \[3\], massive AdS$_5$ scalars are sources of various composite gauge–invariant scalar operators of the $\mathcal{N} = 4$ SYM theory. The values of the 3–point correlators of these operators can be obtained by combining our amplitudes $A_1(\vec{x}, \vec{y}, \vec{z})$ and $A_2(\vec{x}, \vec{y}, \vec{z})$ weighted by appropriate couplings from the gauged supergravity Lagrangian.

## 3 Flavor current correlators

### 3.1 Review of field theory results

We first review the conformal structure of the correlators $\langle J^a_i(x)J^b_j(y) \rangle$ and $\langle J^a_i(x)J^b_j(y)J^c_k(z) \rangle$ and their non-renormalization theorems.\(^3\) The situation is best understood in 4-dimensions, so we mostly limit our discussion to this physically relevant case. The needed information probably appears in many places, but we shall use the reference best known to us \[29\].

Conserved currents $J^a_i(x)$ have dimension $d - 1$, and transform under the inversion as

\[ J^a_i(x) \rightarrow (\Delta_2 \Delta_3 + \frac{1}{2} (d - \Delta_1 - \Delta_2 - \Delta_3) (\Delta_2 + \Delta_3 - \Delta_1)) \]

The two-point function must take the inversion covariant, gauge–invariant form

\[ \langle J^a_i(x)J^b_j(y) \rangle = B \delta^{ab} \frac{2(d-1)(d-2)}{(2\pi)^d} \frac{J_{ij}(x-y)}{(x-y)^{2(d-1)}} \tag{30} \]

where $B$ is a positive constant, the central charge of the $J(x)J(y)$ OPE.

In 4 dimensions the 3–point function has normal and abnormal parity parts which we denote by $\langle J^a_i(x)J^b_j(y)J^c_k(z) \rangle_\pm$. It is an old result \[28\] that the normal parity part is a superposition of two possible conformal tensors (extensively studied in \[29\]), namely

\[ \langle J^a_i(x)J^b_j(y)J^c_k(z) \rangle_+ = f^{abc}(k_1 D_{ijk}^{\text{sym}}(x,y,z) + k_2 C_{ijk}^{\text{sym}}(x,y,z)), \tag{31} \]

where $D_{ijk}^{\text{sym}}(x,y,z)$ and $C_{ijk}^{\text{sym}}(x,y,z)$ are permutation–odd tensor functions, obtained from the specific tensors

\[ D_{ijk}(x,y,z) = \frac{1}{(x-y)^2(z-y)^2} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_j} \log(x-y)^2 \frac{\partial}{\partial z_k} \log \left( \frac{(x-z)^2}{(y-z)^2} \right) \tag{32} \]

\(^3\)In this subsection, $x, y, z$ always indicate $d$–dimensional vectors in flat $d$–dimensional Euclidean space–time.
\[ C_{ijk}(x, y, z) = \frac{1}{(x - y)^4} \frac{\partial}{\partial x_i} \frac{\partial}{\partial z_l} \log(x - z)^2 \frac{\partial}{\partial y_j} \frac{\partial}{\partial z_l} \log(y - z)^2 \frac{\partial}{\partial z_k} \log \left( \frac{(x - z)^2}{(y - z)^2} \right) \]

by adding cyclic permutations

\[ D_{ijk}^{\text{sym}}(x, y, z) = D_{ijk}(x, y, z) + D_{jki}(y, z, x) + D_{kij}(z, x, y) \quad (33) \]

\[ C_{ijk}^{\text{sym}}(x, y, z) = C_{ijk}(x, y, z) + C_{jki}(y, z, x) + C_{kij}(z, x, y). \]

Both symmetrized tensors are conserved for separated points (but the individual permutations are not); \( \frac{\partial}{\partial z_k} D_{ijk}^{\text{sym}}(x, y, z) \) has the local \( \delta^4(x - z) \) and \( \delta^4(y - z) \) terms expected from the standard Ward identity relating 2- and 3-point correlators, while \( \frac{\partial}{\partial z_k} C_{ijk}^{\text{sym}}(x, y, z) = 0 \) even locally. Thus the Ward identity implies \( k_1 = \frac{B}{16\pi^2} \), while \( k_2 \) is an independent constant.

The symmetrized tensors are characterized by relatively simple forms in the limit that one coordinate, say \( y \), tends to infinity:

\[ D_{ijk}^{\text{sym}}(x, y, 0) \xrightarrow{y \to \infty} \frac{-4}{y^6 x^4} J_{ji}(y) \left\{ \delta_{ik} x_l - \delta_{il} x_k - \delta_{kl} x_i - 2 \frac{x_i x_j x_l}{x^2} \right\} \quad (34) \]

\[ C_{ijk}^{\text{sym}}(x, y, 0) \xrightarrow{y \to \infty} \frac{8}{y^6 x^4} J_{ji}(y) \left\{ \delta_{ik} x_l - \delta_{il} x_k - \delta_{kl} x_i + 4 \frac{x_i x_j x_l}{x^2} \right\} \]

In a superconformal–invariant theory with a fixed line parametrized by the gauge coupling, such as \( \mathcal{N} = 4 \) SYM theory, the constant \( B \) is exactly determined by the free field content of the theory, i.e. 1–loop graphs. This is the non-renormalization theorem for flavor central charges proved in \[25\]. The argument is quite simple. The fixed point value of the central charge is equal to the external trace anomaly of the theory with source for the currents \[23, 22\]. Global \( \mathcal{N} = 1 \) supersymmetry relates the trace anomaly to the R-current anomaly, specifically to the \( U(1)_R F^2 \) (\( F \) is for flavor) which is one-loop exact in a conformal theory. Its value depends on the \( r \)-charges and the flavour quantum numbers of the fermions of the theory, and it is independent of the couplings. For an \( \mathcal{N} = 1 \) theory with chiral superfields \( \Phi^i \) with (anomaly–free) \( r \)-charges \( r_i \) in irreducible representations \( R_i \) of the gauge group, the fixed point value of the central charge was given in (2.28) of \[24\] as

\[ B \delta^{ab} = 3 \sum_i (\text{dim} R_i)(1 - r_i) \text{Tr}_i(T^a T^b). \quad (35) \]

For \( \mathcal{N} = 4 \) SYM we can restrict to the \( SU(3) \) subgroup of the full \( SU(4) \) flavour group that is manifest in an \( \mathcal{N} = 1 \) description. There is a triplet of \( SU(N) \) adjoint \( \Phi^i \) with \( r = \frac{2}{3} \). We thus obtain

\[ B = 3(N^2 - 1) \cdot \frac{1}{3} \cdot \frac{1}{2} = \frac{1}{2} (N^2 - 1). \quad (36) \]
We now discuss the 1–loop contributions in the field theory and obtain the values of $k_1$ and $k_2$ for later comparison with $AdS_5$.

Spinor and scalar 1-loop graphs were expressed as linear combinations of $D^{sym}$ and $C^{sym}$ in [29]. For a single $SU(3)$ triplet of left handed fermions and a single triplet of complex bosons one finds

\[
\langle J_i^a(x) J_j^b(y) J_k^c(z) \rangle_{\text{formi}} = \frac{4 f^{abc}}{3 (4\pi^2)^3} (D_{ijk}^{sym} (x, y, z) - \frac{1}{4} C_{ijk}^{sym} (x, y, z))
\]

The sum of these, multiplied by $N^2 - 1$ is the total 1-loop result in the $N = 4$ theory:

\[
\langle J_i^a(x) J_j^b(y) J_k^c(z) \rangle_{\text{N=4}}^{32\pi^6} = \frac{(N^2 - 1) f^{abc}}{32\pi^6} (D_{ijk}^{sym} (x, y, z) - \frac{1}{8} C_{ijk}^{sym} (x, y, z)).
\]

We observe the agreement with the value of $B$ in (36) and the fact that the free field ratio of $C^{sym}$ and $D^{sym}$ tensors is $-\frac{1}{8}$.

Since the $SU(4)$ flavor symmetry is chiral, the 3–point current correlator also has an abnormal parity part $\langle J_i^a J_j^b J_k^c \rangle$. It is well–known that there is a unique conformal tensor–amplitude [28] in this section, which is a constant multiple of the fermion triangle amplitude, namely

\[
\langle J_i^a(x) J_j^b(y) J_k^c(z) \rangle_\ell = -\frac{N^2 - 1}{32\pi^6} \epsilon^{ijklm} \frac{\text{Tr} [\gamma_i \gamma_j (\partial_x - \partial_y) \gamma_k (\partial_y - \partial_z) \gamma_l (\partial_z - \partial_x)]}{(x-y)(y-z)(z-x)}
\]

where the $SU(N)$ $f$ and $d$ symbols are defined by $\text{Tr}(T^a T^b T^c) \equiv \frac{1}{4} (i f^{abc} + d^{abc})$ with $T^a$ hermitian generators normalized as $\text{Tr} T^a T^b = \frac{1}{2} \delta^{ab}$. The coefficient is again “protected” by a non–renormalization theorem, namely the Adler–Bardeen theorem (which is independent of SUSY and conformal symmetry). After bose–symmetric regularization [26] of the short distance singularity, one finds the anomaly

\[
\frac{\partial}{\partial z_k} \langle J_i^a(x) J_j^b(y) J_k^c(z) \rangle_\ell = -\frac{N^2 - 1}{48\pi^2} \delta^{ijklm} \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_m} \delta(x-y) \delta(y-z)
\]

If we minimally couple the currents $J_i^a(x)$ to background sources $A_i^a(x)$ by adding to the action a term $\int d^4 x J_i^a(x) A_i^a(x)$, this information can be presented as the operator equation:

\[
(D_i J_i^a(z)) = \frac{\partial}{\partial z_i} J_i^a(z) + f^{abc} A_i^b(z) J_i^c(z) = \frac{N^2 - 1}{96\pi^2} \delta^{ijklm} \epsilon_{jklm} \partial_j \frac{\partial}{\partial z_i} A_i^m + \frac{1}{4} f^{abc} A_k^a A_l^d A_m^e
\]
where the cubic term in $A_i^a$ is determined by the Wess–Zumino consistency conditions (see e.g. [33]).

The CFT$_4$/AdS$_5$ correspondence can also be used to calculate the large $N$ limit of correlators $\langle J_i^a(x)O^I(y)O^J(z) \rangle$ and $\langle J_i^a(x)J_j^b(y)O^I(z) \rangle$ where $O^I$ is a gauge–invariant composite scalar operator of the $\mathcal{N} = 4$ SYM theory. For example, one can take $O^I$ to be a $k$–th rank traceless symmetric tensor $\text{Tr} X^{\alpha_1 \cdots \alpha_k}$ (the explicit subtraction of traces is not indicated) formed from the real scalars $X^{\alpha}$, $\alpha = 1, ..., 6$, in the 6–dimensional representation of $SU(4) \cong SO(6)$, and there are other possibilities in the operator map discussed by Witten [3]. We will compute the corresponding supergravity amplitudes in the next section, and we record here the tensor form required by conformal symmetry.

For $\langle J_i^a O^I O^J \rangle$ there is a unique conformal tensor for every dimension $d$ given by

$$\langle J_i^a O^I (x)O^J(y) \rangle = \xi S_1^{IJ}(z,x,y)$$

$$\equiv -\xi (d-2) T^{IJ} \frac{1}{(x-y)^{2\Delta-d+2}} \frac{1}{(x-z)^{d-2}(y-z)^{d-2}} \left[ \frac{(x-z)_i}{(x-z)^2} - \frac{(y-z)_i}{(y-z)^2} \right]$$

where $\xi$ is a constant and $T^{IJ}$ are the Lie algebra generators. This correlator satisfies a Ward identity which relates it to the 2–point function $\langle O^I(x)O^J(y) \rangle$. Specifically:

$$\xi \frac{\partial}{\partial z_i} S_1^{IJ}(z,x,y) = \xi \frac{(d-2)2\pi^\frac{d}{2}}{\Gamma(\frac{d}{2})} T^{IJ} \left( \delta^d(x-z) - \delta^d(y-z) \right) \frac{1}{(x-y)^{2\Delta}}$$

$$= \delta^d(x-z) T^{IK} \langle O^K(x)O^J(y) \rangle + \delta^d(y-z) T^{JK} \langle O^I(x)O^K(y) \rangle$$

There is also a unique tensor form for $\langle J_i J_j O \rangle$ (we suppress group theory labels) which is given in [22]:

$$\langle J_i(x)J_j(y)O(z) \rangle = \zeta R_{ij}(x,y,z) \equiv \zeta \frac{(6 - \Delta)J_{ij}(x-y) - \Delta J_{ik}(x-z)J_{kj}(y-z)}{(x-y)^{\alpha - \Delta}(x-z)^{\Delta}(y-z)^{\Delta}}$$

where $\zeta$ is a constant.

### 3.2 Calculations in AdS supergravity

The boundary values $A_i^a(\bar{x})$ of the gauge potentials $A_i^a(x)$ of gauged supergravity are the sources for the conserved flavor currents $J_i^a(\bar{x})$ of the boundary SCFT$_4$. It is sufficient for our purposes to ignore non-renormalizable $\phi^n F_{\mu\nu}^2$ interactions and represent the gauge sector of the supergravity by the Yang–Mills and Chern–Simons terms (the latter for $d + 1 = 5$)

$$S_{d}[A] = \int d^dz d\bar{z} \left[ \sqrt{g} \frac{F_{\mu\nu}^a F^{\mu\nu a}}{4g_{SG}} + \frac{ik}{96\pi^2} \left( d^{abc} \epsilon^\mu_\nu \epsilon^\rho_\sigma A^a_\mu \partial_\nu A^b_\rho \partial_\sigma A^c_\sigma + \cdots \right) \right]$$

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The coefficient $\frac{k}{96\pi^2}$, where $k$ is an integer, is the correct normalization factor for the 5-dimensional Chern–Simons term ensuring that under a large gauge transformation the action changes by an unobservable phase $2\pi in$ (see e.g. [30]). The couplings $g_{SG}$ and $k$ could in principle be determined from dimensional reduction of the parent 10 dimensional theory, but we shall ignore this here. Instead, they will be fixed in terms of current correlators of the boundary theory which are exactly known because they satisfy non-renormalization theorems.

To obtain flavor–current correlators in the boundary CFT from AdS supergravity, we need a Green’s function $G_{\mu i}(z, \bar{x})$ to construct the gauge potential $A^a_\mu(z)$ in the bulk from its boundary values $A^a_\mu(\bar{x})$. We will work in $d$ dimensions. There is the gauge freedom to redefine $G_{\mu i}(z, \bar{x}) \rightarrow G_{\mu i}(z, \bar{x}) + \frac{\partial}{\partial z^\mu} \Lambda_i(z, \bar{x})$ which leaves boundary amplitudes obtained from the action (46) invariant. Our method requires a conformal–covariant propagator, namely

$$G_{\mu i}(z, \bar{x}) = C^d \frac{z_0^{d-2}}{[z_0^2 + (z - \bar{x})^2]^{d-1}} J_{\mu i}(z - \bar{x})$$

(47)

which satisfies the gauge field equations of motion in the bulk variable $z$. The normalization constant $C^d$ is determined by requiring that as $z_0 \to 0$, $G_{ji}(z, \bar{x}) \to 1 \cdot \delta_{ji} \delta(\bar{x})$:

$$C^d = \frac{\Gamma(d)}{2 \pi^\frac{d}{2} \Gamma(\frac{d}{2})}$$

(49)

This Green’s function does not satisfy boundary transversality (i.e. $\frac{\partial}{\partial z^i} G_{\mu i}(z, \bar{x}) = 0$), but the following gauge–related propagator does:[4]

$$\bar{G}_{\mu i}(z, \bar{x}) = G_{\mu i}(z, \bar{x}) + \frac{\partial}{\partial z^\mu} \left\{ \frac{C^d z_0^{d-2-d}}{(d-2)(d-1)(\Gamma(\frac{d}{2})^2 \partial z_i^d \left[ d - 1, \frac{d}{2} - 1, -1, \frac{d}{2} - \frac{1}{2}, \frac{(z - \bar{x})^2}{z_0^2} \right] \right\}$$

(50)

(Both $G_{\mu i}(z, \bar{x})$ and $\bar{G}_{\mu i}(z, \bar{x})$ differ by gauge terms from the Green’s function used by Witten [3]). The gauge equivalence of inversion–covariant and transverse propagators ensures that the method produces boundary current correlators which are conserved.

Notice that in terms of the conformal tensors $J_{\mu i}$ the abelian field strength made from the Green’s function takes a remarkably simple form:

$$\partial_{[\mu} G_{\nu i]}(z, \bar{x}) = (d - 2)C^d \frac{z_0^{d-3}}{[z_0^2 + (z - \bar{x})^2]^{d-1}} J_{0[j}^{[\mu} J_{\nu i]}^{\mu]}(z - \bar{x})$$

(51)

[4] For even $d$, the hypergeometric function in (50) is actually a rational function. For instance for $d = 4$, $\bar{G}_{\mu i}(z, \bar{x}) = G_{\mu i}(z, \bar{x}) + \frac{\partial}{\partial z^\mu} \left\{ \frac{2z_0^2}{[z_0^2 + (z - \bar{x})^2]} \right\}$. 12
as easily checked by using for \( G_{\mu i} \) the representation (18).

We stress again that the inversion \( z_{\mu} = z'_{\mu}/(z')^2 \) is a coordinate transformation which is an isometry of \( AdS_{d+1} \). It acts as a diffeomorphism on the internal indices \( \mu, \nu, \ldots \) of \( G_{\mu i}, G_{i \nu j}, \ldots \). Since these indices are covariantly contracted at an internal point \( z \), much of the algebra required to change integration variables can be avoided. The inversion \( \vec{x} = x'/ (\vec{x}')^2 \) of boundary points is a conformal isometry which acts on the external index \( i \) and also changes the Green’s function by a conformal factor. Thus the change of variables amounts to the replacement:

\[
G_{\mu i}(z, \vec{x}) = (z')^2 J_{\mu o}(z') \cdot (\vec{x}')^2 J_{ki}(\vec{x}') \cdot (\vec{x}')^{2d-2} \frac{C^d}{[(z')^2 + (\vec{x}' - \vec{x})^2]^{d-1}} \tag{52}
\]

\[= \frac{\partial z'_{\mu}}{\partial z_{\mu}} \frac{\partial x'_{\nu}}{\partial x_{\nu}} \cdot (\vec{x}')^{2d-2} G_{\nu k}(z', \vec{x}') \]

\[= \frac{\partial z'_{\mu}}{\partial z_{\mu}} \frac{\partial x'_{\nu}}{\partial x_{\nu}} \cdot G'_{\nu k}(z', \vec{x}')
\]

\( \partial_{[\mu} G_{\nu i]}(z, \vec{x}) \) will also transform conformal–covariantly under inversion (compare equ.(52)):

\[
\partial_{[\mu} G_{\nu i]}(\vec{x}, z) = (z')^2 J_{\mu p}(z') \cdot (z')^2 J_{\nu o}(z') \cdot (\vec{x}')^{2d-2} \frac{\partial_{[\mu} G_{\nu o]}(\vec{x}', z')}{\partial_{[\mu} G_{\nu o]}(\vec{x}', z')}
\]

as one can directly check from (51) using the identity (6).

\[\langle J^a_i \bar{J}^b_j \rangle \): To obtain the current–current correlator we follow the same procedure [3] as for the scalar 2–point function, eq.(16–17):

\[
\langle J^a_i (\vec{x}) \bar{J}^b_j (\vec{y}) \rangle = -\delta^{ab} 2 \cdot \frac{1}{4 g_{SG}^2} \int \frac{d^dz_0}{z_0^d} \partial_{[\mu} G_{|i}|j]}(z, \vec{x}) \frac{A_{\mu} \partial_{[\mu} G_{|i]}(z, \vec{y})}{\partial_{[\mu} G_{|i]}(z, \vec{y})}
\]

\[= +\delta^{ab} \frac{C^d(d-2)}{g_{SG}^2} \int d^d \vec{z} \epsilon^{3-d} 2 G_{\mu \nu}(\epsilon, \vec{z}, \vec{x}) \left[ \partial_{[\mu} G_{|i]}(z_0, \vec{z}, \vec{y}) \right]_{z_0=\epsilon}
\]

\[= \delta^{ab} \frac{C^d(d-2)}{g_{SG}^2} \frac{J_{ij}(\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^{2(d-1)}}
\]

which is of the form (31) with \( B = \frac{1}{g_{SG}^2} \frac{2^{d-2} \pi^d d!}{(d-1)!^2} \). According to the conjecture [1, 2, 3, 5], (34) represents the large–N value of the 2–point function for \( g_{YM}^2 N \) fixed but large. Let us now consider the case \( d = 4 \). By the non–renormalization theorem proven in [23], the coefficient in (34) is protected against quantum corrections. Hence, at leading order in \( N \), the strong–coupling result (54) has to match the 1–loop computation (38). We thus learn:

\[
g_{SG}^{d+1=5} = \frac{4 \pi}{N}
\]
\[ \langle J_i^a(x)J_j^b(y)J_k^c(z) \rangle_+ \] The vertex relevant to the computation of the normal parity part of \( \langle J_i^a(x)J_j^b(y)J_k^c(z) \rangle \) comes from the Yang–Mills term of the action (56), namely
\[
\frac{1}{2g_{SG}^2} \int \frac{d^d w dw_0}{u_0^{d+1}} \frac{f_{abc}}{w_0} \partial_{\mu} A^a_\mu(w) w_0^4 A^b_\mu(w) A^c_\mu(w)
\]
(56)

We then have
\[
\langle J_i^a(x)J_j^b(y)J_k^c(z) \rangle_+ = - \frac{f_{abc}}{2g_{SG}^2} 2 \cdot F_{ijk}^\text{YM}(x, y, z)
\]
(57)
\[
\equiv - \frac{1}{2g_{SG}^2} 2 [F_{ijk}(x, y, z) + F_{jkl}(y, z, x) + F_{kij}(z, x, y)]
\]

where
\[
F_{ijk}(x, y, z) = \int \frac{d^d w dw_0}{u_0^{d+1}} \partial_{\mu} G_{ij}^\mu(w, x) w_0^4 G_{kl}^\mu(w, y) G_{ij}^\mu(w, z)
\]
(58)

(The extra factor of 2 in (57) correctly accounts for the 3! Wick contractions.) To apply the method of inversion, it is convenient to set \( \vec{x} = 0 \). Then, changing integration variable \( w_\mu = \frac{w_\mu}{(w')^2} \) and inverting the external points, \( y_i = \frac{\vec{y}_i}{|\vec{y}|^2}, z_i = \frac{\vec{z}_i}{|\vec{z}|^2} \), we achieve the simplification (using (52), (53), (51)):
\[
F_{ijk}(0, \vec{y}, \vec{z}) = \frac{(\vec{y})^{2(d-1)}|\vec{z}|^{2(d-1)}}{|\vec{z}|^{2(d-1)}} |\vec{z}|^{2(d-1)} J_{jl}(\vec{y}) J_{km}(\vec{z})
\]
\[
\cdot \int \frac{d^d w' dw_0'}{(w_0')^{d+1}} \partial_{\mu} G_{ij}^\mu(w', 0) (w_0')^4 G_{kl}^\mu(w, \vec{y}) G_{jm}^\mu(w', \vec{z})
\]
(59)
\[
= (Cd)^3 \frac{J_{jl}(\vec{y}) J_{km}(\vec{z})}{|\vec{y}|^{2(d-1)} |\vec{z}|^{2(d-1)}} \int \frac{d^d w' dw_0'}{(w_0')^{d+1}} \left[ \partial_{\mu}^\prime (w_0')^{d-2} \partial_{\nu}^\prime (w_0')^4 \right]
\]
\[
\cdot \frac{(w_0')^{d-2}}{(w_0' - \vec{y})^{2(d-2)}} \frac{(w_0' - \vec{z})^{2(d-2)}}{(w_0' - \vec{z})^{2(d-2)}} J_{lm}(w, \vec{y}) J_{mn}(w', \vec{z})
\]
(60)

where in the last step we have defined \( \vec{t} = \vec{y} - \vec{z} \). Observe that in going from (58) to (59) we just had to replace the original variables with primed ones and pick conformal Jacobians for the external (Latin) indices: the internal Jacobians nicely collapsed with each other (recall the contraction rule (51) for \( J_{\mu \nu} \) tensors) and with the factors of \( w' \) coming from the inverse metric. The integrals in (59) now have two denominators and through straightforward manipulations can be rewritten as derivatives with respect to the external coordinate \( \vec{t} \) of standard integrals of the form (23). We thus obtain:
\[
F_{ijk}(\vec{x}, \vec{y}, \vec{z}) = - \frac{J_{jl}(\vec{y} - \vec{x})}{|\vec{y} - \vec{x}|^{2(d-1)}} \frac{J_{km}(\vec{z} - \vec{x})}{|\vec{z} - \vec{x}|^{2(d-1)}} (Cd)^3 \frac{\partial_{\mu} G_{ij}^\mu(w, x) w_0^4 G_{kl}^\mu(w, y) G_{ij}^\mu(w, z)}{(u_0')^{d+1}}
\]
\[
\cdot \frac{(d-2)(w_0')^{2d-4} J_{jl}(w - \vec{t}) J_{km}(w')}{(u_0')^2 + (w_0' - \vec{t})^2} d \frac{d-2}{d-1} \frac{\Gamma \left[ \frac{d}{2} \right]}{\Gamma \left[ \frac{d+1}{2} \right]^2}
\]

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where we have restored the $\vec{x}$ dependence, so that now $\vec{t} \equiv (\vec{y} - \vec{x})' - (\vec{z} - \vec{x})'$. We now add permutations to obtain $F_{ijk}^{\text{sym}}(\vec{x}, \vec{y}, \vec{z})$ in (57). The final step is to express $F_{ijk}^{\text{sym}}$ as a linear combination of the conformal tensors $D_{ijk}^{\text{sym}}$ and $C_{ijk}^{\text{sym}}$ of Section 3.1. It is simplest, and by conformal invariance not less general, to work in the special configuration $\vec{z} = 0$ and $|\vec{y}| \to \infty$. After careful algebra we obtain

$$F_{ijk}^{\text{sym}}(\vec{x}, |\vec{y}| \to \infty, 0) = -(C^d)^3 \pi^{\frac{d+2}{2}} 2^{2-2d} (2d-3) \left( \frac{d-2}{d-1} \right) \frac{\Gamma \left[ \frac{d}{4} \right]}{\Gamma \left[ \frac{d+1}{2} \right]} \frac{J_{ij}(\vec{y})}{|\vec{y}|^{2(d-1)}|\vec{x}|^d} \left\{ \delta_{ik}x_l - \delta_{il}x_k - \delta_{kl}x_i - \frac{d}{2d-3} \frac{x_i x_j x_l}{x^2} \right\}$$

Now take $d = 4$; comparison with (34) gives

$$F_{ijk}^{\text{sym}}(\vec{x}, \vec{y}, \vec{z}) = \frac{1}{\pi^4} \left( D_{ijk}^{\text{sym}}(\vec{x}, \vec{y}, \vec{z}) - \frac{1}{8} C_{ijk}^{\text{sym}}(\vec{x}, \vec{y}, \vec{z}) \right)$$

and finally, from (57) and (55):

$$\langle J^a_i(\vec{x}) J^b_j(\vec{y}) J^c_k(\vec{z}) \rangle_+ = \frac{f^{abc}}{2\pi^4 g_5^2} \left( D_{ijk}^{\text{sym}}(\vec{x}, \vec{y}, \vec{z}) - \frac{1}{8} C_{ijk}^{\text{sym}}(\vec{x}, \vec{y}, \vec{z}) \right)$$

which, at leading order in $N$, precisely agrees with the 1–loop result (38).

The correlator (64) calculated from AdS$_5$ supergravity is supposed to reflect the strong–coupling dynamics of the $\mathcal{N} = 4$ SYM theory at large $N$. The exact agreement found with the free–field result therefore requires some comment. As discussed in Section 3.1, the coefficient of the $D$ tensor is fixed by the Ward identity that relates it to the constant $B$ in the 2–point function, and we matched the latter to the 1–loop result by a non–renormalization theorem. So agreement here is just a check that we have done the integral correctly. However, the fact that the ratio of the $C$ and $D$ tensors coefficients also agrees with the free field value was initially a surprise. Upon further thought, we see that our argument that the value of $k_2$ was a free parameter used only $\mathcal{N} = 0$ conformal symmetry, and superconformal symmetry may impose some constraint. Indeed, in an $\mathcal{N} = 1$ description of the $\mathcal{N} = 4$ SYM theory, we have the flavor $SU(3)$ triplet $\Phi^i$ of $(SU(N) \text{ adjoint})$ chiral superfields, together with their adjoints $\bar{\Phi}^i$. The $SU(3)$ flavor currents are the $\bar{\theta} \theta$ components of composite scalar superfields $K^a(\vec{x}, \theta, \bar{\theta}) = \text{Tr} \bar{\Phi} T^a \Phi$, where $T^a$ is a fundamental $SU(3)$ matrix. Just as
\( \mathcal{N} = 0 \) conformal invariance constrains the tensor form of 2– and 3–point correlators, \( \mathcal{N} = 1 \) superconformal symmetry will constrain the superfield correlators \( \langle K^a K^b \rangle \) and \( \langle K^a K^b K^c \rangle \). We are not aware of a specific analysis, but it seems likely [31] that there are only two possible superconformal amplitudes for \( \langle K^a K^b K^c \rangle \), one proportional to \( f^{abc} \) and the other to \( d^{abc} \). The \( f^{abc} \) amplitude contains the normal parity \( \langle J^a_i J^b_j J^c_k \rangle \) in its \( \theta \)-expansion, and this would imply that the ratio \(-\frac{1}{8}\) of the coefficients of the \( C \) and \( D \) tensors must hold in any \( \mathcal{N} = 1 \) superconformal theory.

\( \langle J^a_i J^b_j J^c_k \rangle \): Witten [3] has sketched an elegant argument that allows to read the value of the abnormal parity part of the 3–current correlator directly from the supergravity action (11), with no integral to compute. Under an infinitesimal gauge transformation of the bulk \( \Lambda \), any \( J \) correlators in the boundary theory. Since by construction \( S_{cl} = \langle D_\mu \Lambda(z) \rangle \) for finite \( \hat{N} \) and comparison with (65) gives

\[
\delta_\Lambda S_{cl} = \int d^d z \Lambda^a(z) \left( -\frac{ik}{96\pi^2} f^{abc} e^{ijkl} \partial_i A^b_j \partial_k A^c_l + \frac{1}{4} f^{cde} A^b_j A^d_k A^c_l \right)
\]

By the conjecture [1, 2, 3], \( S_{cl}[A^a_\mu(z)] = W[A^a_\mu(z)] \), the generating functional for current correlators in the boundary theory. Since by construction \( J^a_i(x) = \frac{\delta W[A^a_\mu]}{\delta A^a_i(x)} \), one has:

\[
\delta_\Lambda S_{cl}[A^a_\mu(z)] = \delta_\Lambda W[A^a_\mu(z)] = \int d^4 z [D_i \Lambda(z)]^a J^a_i(z) = -\int d^4 z \Lambda^a(z) [D_i J^a_i(z)]^a
\]

and comparison with (65) gives

\[
(D_i J^a_i(z))^a = \frac{ik}{96\pi^2} f^{abc} e^{ijkl} \partial_j (A^b_k \partial_l A^c_m) + \frac{1}{4} f^{cde} A^b_j A^d_k A^c_l
\]

which has precisely the structure (11). Thus the CFT/AdS correspondence gives a very concrete physical realization of the well–known mathematical relation between the gauge anomaly in \( d \) dimensions and the gauge variation of a \((d+1)\)-dimensional Chern–Simons form. Witten [3] has argued that (67) is an exact statement even at finite \( N \) (string–loop effects) and for finite \( \hat{N} \) and coupling \( g^2 \) N (string corrections to the classical supergravity action), which is of course what one expects from the Adler–Bardeen theorem. Matching with the 1–loop result (11) we are thus led to identify \( k = N^2 - 1 \).

\( \langle J^a_i J^b_j \mathcal{O} \rangle \): The next 3–point correlator to be discussed is \( \langle J^a_i(x) J^b_j(y) \mathcal{O} (z) \rangle \). For this purpose we suppress group indices and consider a supergravity interaction of the form

\[
\frac{1}{4} \int d^d w d\sigma \phi \partial_\mu A_{ij} \partial_\nu A_{ij}
\]

This leads to the boundary amplitude

\[
\frac{1}{2} \int \frac{d^d w d\sigma}{w_0^{d+1}} K(\sigma, w,\bar{x}) \phi \partial_\mu G_{ij}(w, \bar{x}) w_0^2 \partial_\nu G_{ij}(w, \bar{y})
\]
We set \( \vec{y} = 0 \), apply the method of inversion and obtain the integral

\[
T_{ij}(\vec{x}, 0, \vec{z}) = (C^d)^2 \frac{\Gamma[\Delta]}{\pi^{3d} \Gamma[\Delta - \frac{d}{2}]} \frac{(d - 2)J_k(\vec{x})}{|\vec{x}|^{2\Delta} |x|^{2(d-1)}} \cdot \int d^d w' dw'_0 \frac{w'_0}{(w' - \vec{x})^2} \frac{\partial}{\partial w'_0} \left( \frac{w'_0}{(w' - \vec{x})^2} \right)^{d-2} \frac{\partial}{\partial w'_{ij}} \frac{(w' - x')^k}{(w' - \vec{x}')^2}
\]

(70)

This can be evaluated as a fairly standard Feynman integral with two denominators. The result is

\[
T_{ij}(\vec{x}, \vec{y}, \vec{z}) = -\frac{\Delta}{8\pi^2 \pi^{3d} \Gamma[\Delta - \frac{d}{2}]} R_{ij}(\vec{x}, \vec{y}, \vec{z})
\]

(71)

where \( R_{ij} \) is the conformal tensor (43).

\( \langle J_i^a \mathcal{O}^I \mathcal{O}^J \rangle \): It is useful to study the correlator \( \langle J_i^a(\vec{z}) \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \rangle \) from the AdS viewpoint because the Ward identity (44) which relates it to \( \langle \mathcal{O}(\vec{y}) \mathcal{O}^I(\vec{z}) \rangle \) is a further check on the CFT/AdS conjecture. We assume that \( \mathcal{O}^I(\vec{x}) \) is a scalar composite operator, in a real representation of the \( SO(6) \) flavor group with generators \( T_i^I \) which are imaginary antisymmetric matrices, and that \( \mathcal{O}^I(\vec{x}) \) corresponds to a real scalar field \( \phi^I(\vec{x}) \) in AdS supergravity. Actually we will present an \( AdS_{d+1} \) calculation based on a gauge-invariant extension of (8), namely

\[
S[\phi^I, A^a_\mu] = \frac{1}{2} \int d^d z d\zeta \sqrt{g} \left[ g^{\mu\nu} D_\mu \phi^I D_\nu \phi^I + m^2 \phi^I \phi^I \right]
\]

(72)

\[
D_\mu \phi^I = \partial_\mu \phi^I - iA^a_\mu T^I_{\mu I} \phi^I
\]

The cubic vertex then leads to the AdS integral representation of the gauge theory correlator

\[
\langle J_i^a(\vec{z}) \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \rangle = T_{IJ} \int \frac{d^d w dw_0}{w_0^{d+1}} G_{\mu i}(w, \vec{z}) w_0^2 \mathcal{K}_\Delta(w, \vec{x}) \frac{\partial}{\partial w_\mu} \mathcal{K}_\Delta(w, \vec{y})
\]

(73)

The integral is easily done by setting \( \vec{z} = 0 \) and applying inversion. We have also shown that \( \vec{y} = 0 \) followed by inversion gives the same final result, which is

\[
\langle J_i^a(\vec{z}) \mathcal{O}^I(\vec{x}) \mathcal{O}^J(\vec{y}) \rangle = \frac{2C^d T_{IJ}^a}{|x|^2 \Delta |y|^2 \Delta} \frac{\partial}{\partial x^i} \int \frac{d^d w' dw'_0}{w'_0} \mathcal{K}_\Delta(w', \vec{x}') \mathcal{K}_\Delta(w', \vec{y}')
\]

(74)

\[
\xi = \frac{(\Delta - \frac{d}{2}) \Gamma[\frac{d}{2}] \Gamma[\Delta]}{\Gamma[\Delta - \frac{d}{2}]}
\]

where \( \tilde{S}_{IJ}^a(\vec{z}, \vec{x}, \vec{y}) \) is the conformal amplitude of (43). Comparing with (44) and (17), we see that the expected Ward identity is not satisfied; there is a mismatch by a factor \( \frac{2\Delta - d}{\Delta} \).
Although we have checked the integral thoroughly, this is an important point, so we now give a heuristic argument that the answer is correct. We compute the divergence of the correlator \((73)\) using the following identity inside the integral:

\[
\frac{\partial}{\partial z_i} G_{\mu i}(w, \bar{z}) = - \frac{\partial}{\partial w_\mu} K_d(w, \bar{z})
\]  

(75)

where \(K_d(w, \bar{z})\) is the Green’s function of a massless scalar, \(\text{i.e. } \Delta = d\). If we integrate by parts, the bulk term vanishes and we find

\[
\frac{\partial}{\partial z_i} \langle J^a_i(z) O^I(x) O^J(y) \rangle = \lim_{\epsilon \to 0} \int d^d w \epsilon^{1-d} K_d(\epsilon, \bar{w}, \bar{z}) \left[ K_\Delta(w, \bar{x}) \frac{\partial}{\partial w_0} K_\Delta(w, \bar{y}) \right]_{w_0=\epsilon}
\]

(76)

where we used the property \(\lim_{w_0 \to 0} K_d = \delta(\bar{w} - \bar{z})\) (see \(12\)). It also follows from \((11–12)\) that

\[
\lim_{w_0 \to 0} \frac{u_0^{2\Delta-d+2}}{(w-\bar{y})^{2(\Delta+1)}} = \frac{\pi^{\frac{\Delta-d}{2}} \Gamma[\Delta + 1]}{\Gamma[\Delta - \frac{d}{2}]} \delta^d(w - \bar{y})
\]

(77)

This gives

\[
\frac{\partial}{\partial z_i} \langle J^a_i(z) O^I(x) O^J(y) \rangle = \xi \frac{(d-2)2\pi^\frac{\Delta-d}{2}}{\Gamma[\frac{\Delta}{2}]} \xi^{ij} T^{ij} \left[ \delta^d(\bar{x} - \bar{z}) - \delta^d(\bar{y} - \bar{z}) \right] \frac{1}{|\bar{x} - \bar{y}|^{2\Delta}}
\]

(79)

which is consistent with \((74)\) and confirms the previously found mismatch between \(\langle J^a_i O^I O^J \rangle\) and \(\langle O^I O^J \rangle\).

Thus the observed phenomenon is that the Ward identity relating the correlators \(\langle J^a_i O^I O^J \rangle\) and \(\langle O^I O^J \rangle\), as calculated from \(AdS_{d+1}\) supergravity, is satisfied for operators \(O^I\) of scale dimension \(\Delta = d\), for which the corresponding \(AdS_{d+1}\) scalar is massless, but fails for \(\Delta \neq d\).

We suggest the following interpretation of the problem, namely that the prescription of \([3]\) is correct for \(n\)-point correlators in the boundary \(CFT_d\) for \(n \geq 3\), but 2-point correlators are more singular, so a more careful procedure is required. The fact that the kinetic and mass term integrals in \((16)\) are each divergent has already been noted. In the Appendix we outline an alternate calculation of 2-point functions, very similar to that of \([2]\), in which we Fourier transform in \(\bar{x}\) and write a solution \(\phi(z_0, \tilde{k})\) of the massive scalar field equation which satisfies a Dirichlet boundary–value problem at a small finite value \(z_0 = \epsilon\), compute the 2-point correlator at this value and then scale to \(\epsilon = 0\). This procedure gives a value of \(\langle O^I O^J \rangle\) which is exactly a factor \(\frac{2\Delta-d}{2\Delta}\) times that of \((17)\) and thus agrees with the Ward identity.
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Appendix

For scalars with dimension $\Delta = d$ the correlation functions achieve constant limiting values as we approach the boundary of $AdS$ space. If $\Delta \neq d$ then the correlation function goes to zero or infinity as we go towards the boundary, and must be defined with an appropriate scaling. In this case an interesting subtlety is seen to arise in the order in which we take the limits to define various quantities, and we discuss this issue below.

Let us discuss the 2–point function for scalars. We take the metric (3) on the $AdS$ space, and put the boundary at $z_0 = \epsilon$ with $\epsilon << 1$; at the end of the calculation we take $\epsilon$ to zero. We also Fourier transform the variables $\vec{x}$, and follow the discussion of [2].

The wave equation in Fourier space for scalars with mass $m$ is

$$z_0^{d+1} \frac{\partial}{\partial z_0}[z_0^{-d+1} \frac{\partial}{\partial z_0} \phi(z_0, \vec{k})] - (k^2 z_0^2 + m^2) \phi(z_0, \vec{k}) = 0$$

(80)

where we have written

$$\phi(z_0, \vec{x}) = \frac{1}{(2\pi)^{d/2}} \int d\vec{k} \ e^{i\vec{k} \cdot \vec{x}} \phi(z_0, \vec{k})$$

(81)

The solution to this equation is

$$\phi(z_0, \vec{k}) = z_0^{\frac{d}{2}} F_\nu[ikz_0]$$

(82)

where $F_\nu$ is a solution of the Bessel equation with index

$$\nu = \Delta - \frac{d}{2} = \left[ \frac{d^2}{4} + m^2 \right]^{1/2}$$

(83)

The action in terms of Fourier components is

$$S = \frac{1}{2} \int dz_0 d\vec{k} d\vec{k}' \delta(\vec{k} + \vec{k}') z_0^{-d+1}$$

$$\left[ \frac{\partial}{\partial z_0} \phi(z_0, \vec{k}) \frac{\partial}{\partial z_0} \phi(z_0, \vec{k}') + (k^2 + \frac{m^2}{z_0^2}) \phi(z_0, \vec{k}) \phi(z_0, \vec{k}') \right]$$

(84)
We have to evaluate this action on a solution of the equation of motion with \( \phi(\epsilon, \vec{k}) \equiv \phi_b(\vec{k}) \) given. An integration by parts gives

\[
S = \frac{1}{2} \int d\vec{k}d\vec{k}' \delta(\vec{k} + \vec{k}') \lim_{z_0 \to \epsilon} z_0^{-d+1} [\phi(z_0, \vec{k}) \partial_{z_0} \phi(z_0, \vec{k}')] \quad (85)
\]

If we have a solution to the wave equation \( K^\epsilon(z_0, \vec{k}) \) such that

\[
\lim_{z_0 \to \epsilon} K^\epsilon(z_0, \vec{k}) = 1, \quad \lim_{z_0 \to \infty} K(z_0, \vec{k}) = 0 \quad (86)
\]

then we can write the desired solution to the wave equation as

\[
\phi(z_0, \vec{k}) = K(z_0, \vec{k}) \phi_b(\vec{k}) \quad (87)
\]

Then the 2-point function in Fourier space will be given by

\[
\langle O(\vec{k}) O(\vec{k}') \rangle = -\epsilon^{-d+1} \delta(\vec{k} + \vec{k}') \lim_{z_0 \to \epsilon} \partial_{z_0} K^\epsilon(z_0, \vec{k}) \quad (88)
\]

We have

\[
K^\epsilon(z_0, \vec{k}) = \left( \frac{z_0}{\epsilon} \right)^{d/2} \frac{\mathcal{K}_\nu(kz_0)}{\mathcal{K}_\nu(k\epsilon)} 
\]

where \( \mathcal{K} \) is the modified Bessel function which vanishes as \( z_0 \to \infty \). For small argument \( \mathcal{K}_\nu \) has the expansion

\[
\mathcal{K}_\nu(kz_0) = 2^{\nu-1} \Gamma(\nu)(kz_0)^{-\nu}[1 + \ldots] - 2^{-\nu-1} \frac{\Gamma(1-\nu)}{\nu}(kz_0)^{\nu}[1 + \ldots] \quad (90)
\]

where the terms represented by ‘\( \ldots \)’ are positive integer powers of \( (kz_0)^2 \). Then (88) gives

\[
\langle O(\vec{k}) O(\vec{k}') \rangle =
-\epsilon^{-d+1} \delta(\vec{k} + \vec{k}') \lim_{z_0 \to \epsilon} \partial_{z_0} K^\epsilon(z_0, \vec{k})

= -\epsilon^{2(\Delta-d)} \delta(\vec{k} + \vec{k}') k^{2\nu} 2^{-2\nu} \frac{\Gamma(1-\nu/2)}{\Gamma(1+\nu/2)}(2\nu) + \ldots
\]

Here in the last line we have written only those terms that correspond to the power law behavior of the correlator in position space, and further only the largest such terms in the limit \( \epsilon \to 0 \) have been kept. In particular we have dropped terms that are integer powers in \( k^2 \), even though some of these terms are multiplied by a smaller power of \( \epsilon \) than the term that we have kept. The reason for dropping these terms is that they give delta–function contact terms in the correlator after transforming to position space, and we are interested here in the correlation function for separated points.
The result (91) is the Fourier transform of the function
\[ \frac{1}{\pi^{d/2}} \frac{2^{(\Delta-d)}(2\Delta - d)}{\Delta} \frac{\Gamma(\Delta + 1)}{\Gamma(\Delta - \frac{d}{2})} |\vec{x} - \vec{y}|^{-2\Delta} \]
which should therefore be the correctly normalized 2-point function on the boundary \( z_0 = \epsilon \).

It also agrees with the correctly normalized 2-point function required by the Ward identity (44). The power of \( \epsilon \) indicates the rate of growth of this correlation function as the boundary of AdS space is moved to infinity, and we can define for convenience a scaled correlator that
is the same as above but without this power of \( \epsilon \). The correlation functions given in the
rest of this paper are in fact written after such a rescaling.

We would however have obtained a different result had we taken the limits in the fol-
lowing way. We first take \( \epsilon \to 0 \) in the propagator (89), obtaining
\[ K^\epsilon(z_0) = \left( \frac{z_0}{\epsilon} \right)^{d/2} \frac{1}{2^{2\nu - 1} \Gamma(\nu)(k\epsilon) - \nu} K_{\nu}(kz_0) \]
Using (93) in (88) we get
\[ \langle O(\vec{k})O(\vec{k}') \rangle = -\epsilon^{-2(\Delta - d)} \delta(\vec{k} + \vec{k}') k^{2\nu} 2^{2\nu - 2\nu - \nu + \frac{d}{2}} \Gamma(\nu + \frac{d}{2}) + \ldots \]
which differs from (91) by a factor
\[ \frac{\Delta}{2\Delta - d} \]
The difference between (91) and (94) can be traced to the fact that the terms in \( K^\epsilon(z_0) \)
which are subleading in \( \epsilon \) when \( z_0 \) is order unity, give a contribution that is not subleading
when \( z_0 \to \epsilon \), which is the limit that we actually require when computing the 2-point function.
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