Prime bound of a graph

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Abstract

Given a graph $G$, a subset $M$ of $V(G)$ is a module of $G$ if for each $v \in V(G) \setminus M$, $v$ is adjacent to all the elements of $M$ or to none of them. For instance, $V(G)$, $\emptyset$ and $\{v\}$ ($v \in V(G)$) are modules of $G$ called trivial. Given a graph $G$, $m(G)$ denotes the largest integer $m$ such that there is a module $M$ of $G$ which is a clique or a stable set in $G$ with $|M| = m$.

A graph $G$ is prime if $|V(G)| \geq 4$ and if all its modules are trivial. The prime bound of $G$ is the smallest integer $p(G)$ such that there is a prime graph $H$ with $V(H) \supseteq V(G)$, $H[V(G)] = G$ and $|V(H) \setminus V(G)| = p(G)$. We establish the following. For every graph $G$ such that $m(G) \geq 2$ and $\log_2(m(G))$ is not an integer, $p(G) = \lceil \log_2(m(G)) \rceil$. Then, we prove that for every graph $G$ such that $m(G) = 2^k$ where $k \geq 1$, $p(G) = k$ or $k + 1$. Moreover $p(G) = k + 1$ if and only if $G$ or its complement admits $2^k$ isolated vertices. Lastly, we show that $p(G) = 1$ for every non-prime graph $G$ such that $|V(G)| \geq 4$ and $m(G) = 1$.

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1 Introduction

A graph $G = (V(G), E(G))$ is constituted by a vertex set $V(G)$ and an edge set $E(G) \subseteq \binom{V(G)}{2}$. Given a set $S$, $K_S = (S, \binom{S}{2})$ is the complete graph on $S$ whereas $(S, \emptyset)$ is the empty graph. Let $G$ be a graph. With each $W \subseteq V(G)$ associate the subgraph $G[W] = (W, \binom{W}{2} \cap E(G))$ of $G$ induced by $W$. A graph $H$ is an extension of $G$ if $V(H) \supseteq V(G)$ and $H[V(G)] = G$. Given $p \geq 0$, a $p$-extension of $G$ is an extension $H$ of $G$ such that $|V(H) \setminus V(G)| = p$. The

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complement of $G$ is the graph $\overline{G} = (V(G), \left\{ \binom{V(G)}{2} \setminus E(G) \right\})$. A subset $W$ of $V(G)$ is a clique (respectively a stable set) in $G$ if $G[W]$ is complete (respectively empty). The largest cardinality of a clique (respectively a stable set) in $G$ is the clique number (respectively the stability number) of $G$, denoted by $\omega(G)$ (respectively $\alpha(G)$). Given $v \in V(G)$, the neighbourhood $N_G(v)$ of $v$ in $G$ is the family $\{ w \in V(G) : \{ v, w \} \in E(G) \}$. Its degree is $d_G(v) = |N_G(v)|$. We will consider $N_G$ as a function from $V(G)$ in $2^{V(G)}$. A vertex $v$ of $G$ is isolated if $N_G(v) = \emptyset$. The family of isolated vertices of $G$ is denoted by $\text{Iso}(G)$.

We use the following notation. Let $G$ be a graph. For $v \neq w \in V(G)$,

$$(v,w)_{G} = \begin{cases} 0 & \text{if } \{v,w\} \notin E(G), \\ 1 & \text{if } \{v,w\} \in E(G). \end{cases}$$

Given $W \subseteq V(G)$, $v \in V(G) \setminus W$ and $i \in \{0,1\}$, $(v,W)_{G} = i$ means $(v,w)_{G} = i$ for every $w \in W$. Given $W, W' \subseteq V(G)$, with $W \cap W' = \emptyset$, and $i \in \{0,1\}$, $(W,W')_{G} = i$ means $(w,W')_{G} = i$ for every $w \in W$. Given $W \subseteq V(G)$ and $v \in V(G) \setminus W$, $v \sim_G W$ means that there is $i \in \{0,1\}$ such that $(v,W)_{G} = i$. The negation is denoted by $v \not\sim_G W$.

Given a graph $G$, a subset $M$ of $V(G)$ is a module of $G$ if for each $v \in V(G) \setminus M$, we have $v \sim_G M$. For instance, $V(G)$, $\emptyset$ and $\{v\} \ (v \in V(G))$ are modules of $G$ called trivial. Clearly, if $|V(G)| = 2$, then all the modules of $G$ are trivial. On the other hand, if $|V(G)| = 3$, then $G$ admits a non-trivial module. A graph $G$ is then said to be prime if $|V(G)| \geq 4$ and if all its modules are trivial. For instance, given $n \geq 4$, the path $\{(1, \ldots, n), \{p,q\} : |p-q| = 1\}$ is prime. Given a graph $G$, $G$ and $\overline{G}$ share the same modules. Thus $G$ is prime if and only if $\overline{G}$ is.

Let $S$ be a set with $|S| \geq 2$. Given $p \geq 1$, consider a $p$-extension $G$ of $K_S$. If $|S| \geq 2^p$, then $G$ is not prime. Indeed, for each $s \in S$, we have $N_G(s) \subseteq V(G) \setminus S$ because $S$ is a stable set in $G$. So if $|S| \geq 2^p$, then $(N_G)_{|S|} : S \rightarrow 2^{V(G) \setminus S}$ is not injective. Thus $\{s,t\}$ is a non-trivial module of $G$ for $s \neq t \in S$ such that $N_G(s) = N_G(t)$. Furthermore, if $(N_G)_{|S|} : S \rightarrow 2^{V(G) \setminus S}$ is injective and if $|S| = 2^p$, then there is $s \in S$ such that $N_G(s) = \emptyset$. Therefore $s \in \text{Iso}(G)$ and $V(G) \setminus \{s\}$ is a non-trivial module of $G$. On the other hand, the following is well known and is easily verified (see Sumner [17] Theorem 2.45 and also Corollary 4 below). Given a set $S$ with $|S| \geq 2$, $K_S$ admits a prime $\left\lfloor \log_2(|S|+1) \right\rfloor$-extension. This is extended to any graph by Brignall [2] Theorem 3.7 as follows.

**Theorem 1.** A graph $G$, with $|V(G)| \geq 2$, admits a prime extension $H$ such that $|V(H) \setminus V(G)| \leq \left\lfloor \log_2(|V(G)|+1) \right\rfloor$.

Following Theorem 1, we introduce the notion of prime bound. Let $G$ be a graph. The prime bound of $G$ is the smallest integer $p(G)$ such that $G$ admits a prime $p(G)$-extension. Obviously $p(G) = 0$ when $G$ is prime.

A prime extension $H$ of $G$ is minimal [14] [9] [18] [1] [10] if for every $W \not\subseteq V(H)$ such that $H[W]$ is prime, $H[W]$ does not admit an induced subgraph isomorphic to $G$. Given a graph $G$, a prime $p(G)$-extension of $G$ is clearly
minimal. By Theorem 1 \( p(G) \leq \lceil \log_2(|V(G)| + 1) \rceil \). Observe also that \( p(G) = p(G) \) for every graph \( G \). By considering the clique number and the stability number, Brignall [2, Conjecture 3.8] conjectured the following.

**Conjecture 1.** For each graph \( G \) with \( |V(G)| \geq 2 \),

\[
p(G) \leq \lceil \log_2(\max(\omega(G), \alpha(G)) + 1) \rceil.
\]

We answer the conjecture positively by refining the notions of clique number and of stability number as follows. Given a graph \( G \), the **modular clique number** of \( G \) is the largest integer \( \omega_M(G) \) such that there is a module \( M \) of \( G \) which is a clique in \( G \) with \( |M| = \omega_M(G) \). The **modular stability number** of \( G \) is \( \alpha_M(G) = \omega_M(G) \). Obviously \( \omega_M(G) \leq \omega(G) \) and \( \alpha_M(G) \leq \alpha(G) \). For convenience, set

\[
m(G) = \max(\alpha_M(G), \omega_M(G)).
\]

We establish

**Theorem 2.** For every graph \( G \) such that \( m(G) \geq 2 \),

\[
[\log_2(m(G))] \leq p(G) \leq \lceil \log_2(m(G) + 1) \rceil.
\]

On the one hand, it follows that \( p(G) = \lceil \log_2(m(G)) \rceil \) for a graph \( G \) such that \( m(G) \geq 2 \) and \( \log_2(m(G)) \) is not an integer. On the other, if \( \log_2(m(G)) \) is a positive integer, that is, \( m(G) = 2^k \) where \( k \geq 1 \), then \( p(G) = k \) or \( k + 1 \). We prove

**Theorem 3.** For every graph \( G \) such that \( m(G) = 2^k \) where \( k \geq 1 \),

\[
p(G) = k + 1 \text{ if and only if } |Iso(G)| = 2^k \text{ or } |Iso(G)| = 2^k.
\]

Lastly, we show that \( p(G) = 1 \) for every non-prime graph \( G \) such that \( |V(G)| \geq 4 \) and \( m(G) = 1 \) (see Proposition 7).

The case of directed graphs is quite different. Recall that a **tournament** \( T = (V(T), A(T)) \) is a directed graph such that \( \{(v, w), (w, v)\} \cap A(T) = \emptyset \) for any \( v \neq w \in V(T) \). For instance, the 3-cycle \( C_3 = \{(1, 2), (2, 3), (3, 1)\} \) is a tournament. A tournament \( T \) is **transitive** provided that for any \( u, v, w \in V(T) \), if \( (u, v), (v, w) \in A(T) \), then \( (u, w) \in A(T) \). Given a tournament \( T \), a subset \( I \) of \( V(T) \) is an **interval** \( [1, 11] \) (or a clan \( [5] \)) of \( T \) if for any \( x, y \in I \) and \( v \in V(T) \setminus I \), we have: \( (x, v) \in A(T) \) if and only if \( (y, v) \in A(T) \). Once again, \( V(T) \), \( \emptyset \) and \( \{v\} \) (\( v \in V(T) \)) are intervals of \( T \) called trivial. For instance, all the intervals of \( C_3 \) are trivial. On the other hand, a transitive tournament with at least 3 vertices admits a non-trivial interval. A tournament with at least 3 vertices is **indecomposable** \( [6, 11] \) (or simple \( [6, 7, 13] \)) if all its intervals are trivial. The **indecomposable bound** of a tournament \( T \) is defined as the prime bound of a graph. It is still denoted by \( p(T) \).

**Theorem 4** ([6, 7, 13]). For a tournament \( T \) with \( |V(T)| \geq 3 \),

\[
p(T) \leq 2.
\]

Moreover \( p(T) = 2 \) if and only if \( T \) is transitive and \( |V(T)| \) is odd.
2 Preliminaries

We begin with the well known properties of the modules of a graph (for example, see [5, Lemma 3.4, Theorem 3.2, Lemma 3.9]).

**Proposition 1.** Let $G$ be a graph.

1. Given $W \subseteq V(G)$, if $M$ is a module of $G$, then $M \cap W$ is a module of $G[W]$.

2. Given a module $M$ of $G$, if $N$ is a module of $G[M]$, then $N$ is a module of $G$.

3. If $M$ and $N$ are modules of $G$, then $M \cap N$ is a module of $G$.

4. If $M$ and $N$ are modules of $G$ such that $M \cap N \neq \emptyset$, then $M \cup N$ is a module of $G$.

5. If $M$ and $N$ are modules of $G$ such that $M \setminus N \neq \emptyset$, then $N \setminus M$ is a module of $G$.

6. If $M$ and $N$ are modules of $G$ such that $M \cap N = \emptyset$, then there is $i \in \{0, 1\}$ such that $(M, N)_G = i$.

Given a graph $G$, a partition $P$ of $V(G)$ is a modular partition of $P$ if each element of $P$ is a module of $G$. Let $P$ be such a partition. Given $M \neq N \in P$, there is $i \in \{0, 1\}$ such that $(M, N)_G = i$ by the last assertion of Proposition 1. This justifies the following definition. The quotient of $G$ by $P$ is the graph $G/P$ defined on $V(G/P) = P$ by $(M, N)_{G/P} = (M, N)_G$ for $M \neq N \in P$. We use the following properties of the quotient (for example, see [5, Theorems 4.1–4.3, Lemma 4.1]).

**Proposition 2.** Given a graph $G$, consider a modular partition $P$ of $G$.

1. Given $W \subseteq V(G)$, if $|W \cap X| = 1$ for each $X \in P$, then $G[W]$ and $G/P$ are isomorphic.

2. For any module $M$ of $G$, $\{X \in P : M \cap X \neq \emptyset\}$ is a module of $G/P$.

3. For any module $Q$ of $G/P$, the union $\cup Q$ of the elements of $Q$ is a module of $G$.

Given a graph $G$, with each non-empty module $M$ of $G$, associate the modular partition $P_M = \{M\} \cup \{\{v\} : v \in V(G) \setminus M\}$. Given $m \in M$, the corresponding quotient $G/P_M$ is isomorphic to $G(V(G) \setminus M) \cup \{m\}$ by the first assertion of Proposition 2. To associate a unique quotient with any graph and to characterize the corresponding quotient, the following strengthening of the notion of module is introduced. Given a graph $G$, a module $M$ of $G$ is said to be strong provided that for every module $N$ of $G$, we have: if $M \cap N \neq \emptyset$, then $M \subseteq N$ or $N \subseteq M$. We recall the following well known properties of the strong modules of a graph (for example, see [5, Theorem 3.3]).
Proposition 3. Let $G$ be a graph.

1. $V(G)$, $\emptyset$ and $\{v\}$, $v \in V$, are strong modules of $G$.

2. For a strong module $M$ of $G$ and for $N \subseteq M$, $N$ is a strong module of $G$ if and only if $N$ is a strong module of $G[M]$.

With each graph $G$, we associate the family $\Pi(G)$ of the maximal strong modules of $G$ under inclusion among the proper and non-empty strong modules of $G$. The modular decomposition theorem is stated as follows.

Theorem 5 (Gallai [8, 12]). For a graph $G$ with $|V(G)| \geq 2$, the family $\Pi(G)$ realizes a modular partition of $G$. Moreover, the corresponding quotient $G/\Pi(G)$ is complete, empty or prime.

Given a graph $G$ with $|V(G)| \geq 2$, we denote by $\mathcal{S}(G)$ the family of the non-empty strong modules of $G$. As a direct consequence of the definition of a strong module, we obtain that the family $\mathcal{S}(G)$ endowed with inclusion, denoted by $(\mathcal{S}(G), \subseteq)$, is a tree called the modular decomposition tree $\mathcal{T}$ of $G$. Given $M \in \mathcal{S}(G)$ with $|M| \geq 2$, it follows from Proposition 3 that $\Pi(G[M]) \subseteq \mathcal{S}(G)$.

Furthermore, given $W \subseteq V(G)$ (respectively $W \subseteq V(G)$), $\{M \in \mathcal{S}(G) : M \ni W\} \subseteq \mathcal{S}(G)$ (respectively $\{M \in \mathcal{S}(G) : M \ni W\} \subseteq \mathcal{S}(G)$) is a total order. Its smallest element is denoted by $W \uparrow$ (respectively $W \uparrow$). By Proposition 3 if $M \in \mathcal{S}(G) \setminus \{V(G)\}$, then $M \in \Pi(G[M])$.

Let $G$ be a graph with $|V(G)| \geq 2$. Using Theorem 5 we label $\mathcal{S}(G) \setminus \{\{v\} : v \in V(G)\}$ by the function $\lambda_G$ defined as follows. For each $M \in \mathcal{S}(G)$ with $|M| \geq 2$,

$$\lambda_G(M) = \begin{cases} 
\blacksquare & \text{if } G[M]/\Pi(G[M]) \text{ is complete}, \\
\blacktriangle & \text{if } G[M]/\Pi(G[M]) \text{ is empty}, \\
\blacklozenge & \text{if } G[M]/\Pi(G[M]) \text{ is prime}.
\end{cases}$$

Figure 1: $\mathcal{S}(G) \setminus \{\{v\} : v \in V(G)\} = \{C, S, V(G)\}$, $\lambda_G(C) = \blacksquare$, $\lambda_G(S) = \blacktriangle$ and $\lambda_G(V(G)) = \blacklozenge$. 
In Figure $\text{I}$ we depict a graph $G$ defined on $V(G) = \{a, b\} \cup \{c_1, c_2, c_3\} \cup \{s_1, \ldots, s_5\}$ by

$\begin{align*}
C &= \{c_1, c_2, c_3\} \text{ is a clique in } G, \\
S &= \{s_1, \ldots, s_5\} \text{ is a stable set in } G, \\
(a,S)_G &= (a,b)_G = (b,C)_G = 1, \\
(a,C)_G &= (C,S)_G = (b,S)_G = 0.
\end{align*}$

For $i \in \{1,2,3\}$ and $j \in \{1,\ldots,5\}$, $G[\{a,b,c_i,s_j\}]$ is a path and hence is prime. Thus the non-trivial modules of $G$ are the subsets $M$ of $C$ or of $S$ with $|M| \geq 2$. It follows that

$\mathbb{S}(G) = \{\{v\} : v \in V(G)\} \cup \{C, S, V(G)\}$

and

$\begin{align*}
\lambda_G(C) &= \blacksquare, \\
\lambda_G(S) &= \square, \\
\lambda_G(V(G)) &= \cup.
\end{align*}$

### 3 Modular clique number and modular stability number

Given a non-prime graph $G$, we are looking for a prime extension $H$ of $G$. We have to break the non-trivial modules of $G$ as modules of $H$. Precisely, given a non-trivial module $M$ of $G$, there must exist $v \in V(H) \setminus V(G)$ such that $v \not\in_H M$. Moreover, we have only to consider the minimal non-trivial modules of $G$ under inclusion.

Given a graph $G$, denote by $\mathcal{M}(G)$ the family of modules $M$ of $G$ such that $|M| \geq 2$, and denote by $\mathcal{M}_{\text{min}}(G)$ the family of the minimal elements of $\mathcal{M}(G)$ under inclusion.

**Lemma 1.** Let $G$ be a graph. For every $M \in \mathcal{M}(G)$, $M \in \mathcal{M}_{\text{min}}(G)$ if and only if either $|M| = 2$ or $G[M]$ is prime.

**Proof.** To begin, consider $M \in \mathcal{M}_{\text{min}}(G)$. It follows from the second assertion of Proposition $\blacksquare$ that all the modules of $G[M]$ are trivial. Thus $G[M]$ is prime if $|M| \geq 4$. Assume that $|M| \leq 3$. Since a graph on 3 vertices admits a non-trivial module, we obtain $|M| = 2$.

Conversely, consider $M \in \mathcal{M}(G)$ such that $|M| = 2$ or $G[M]$ is prime. Clearly $M \in \mathcal{M}_{\text{min}}(G)$ when $|M| = 2$. So assume that $G[M]$ is prime and consider $N \in \mathcal{M}(G)$ such that $N \subseteq M$. By the first assertion of Proposition $\blacksquare$, $N$ is a module of $G[M]$. As $G[M]$ is prime, $N = M$ and hence $M \in \mathcal{M}_{\text{min}}(G)$. $\blacksquare$

Let $G$ be a graph. Following Lemma $\blacksquare$ we denote by $\mathcal{P}(G)$ the family of modules $M$ of $G$ such that $G[M]$ is prime. In Figure $\blacksquare$ $\mathcal{P}(G) = \emptyset$.

**Lemma 2.** Let $G$ be a graph. For any $P \in \mathcal{P}(G)$ and $M \in \mathcal{M}(G)$, either $P \cap M = \emptyset$ or $P \subseteq M$.
Proof. Assume that \( P \cap M \neq \emptyset \). By the first assertion of Proposition \( \ref{prop:module} \), \( P \cap M \) is a module of \( G[P] \). Since \( G[P] \) is prime, either \( |P \cap M| = 1 \) or \( P \cap M = P \). Suppose for a contradiction that \( |P \cap M| = 1 \). As \( |M| \geq 2 \), \( M \setminus P \neq \emptyset \). By the second to last assertion of Proposition \( \ref{prop:module} \), \( P \setminus M \) is a module of \( G \). By the first assertion, \( P \setminus M \) would be a non-trivial module of \( G[P] \). It follows that \( P \cap M = P \). \( \diamond \)

Given a graph \( G \), consider \( u \neq v \in V(G) \) and \( v \neq w \in V(G) \) such that there are \( M_{\{u,v\}}, M_{\{v,w\}} \in \mathcal{M}_{\text{min}}(G) \) with \( u, v \in M_{\{u,v\}} \) and \( v, w \in M_{\{v,w\}} \). First, assume that \( M_{\{u,v\}} \in \mathcal{P}(G) \). By Lemma \( \ref{lem:module} \), \( M_{\{u,v\}} \subseteq M_{\{v,w\}} \) and hence \( M_{\{u,v\}} = M_{\{v,w\}} \) because \( M_{\{v,w\}} \in \mathcal{M}_{\text{min}}(G) \). Similarly, \( M_{\{u,v\}} = M_{\{v,w\}} \) when \( M_{\{v,w\}} \in \mathcal{P}(G) \). Second, assume that \( |M_{\{u,v\}}| = |M_{\{v,w\}}| = 2 \), that is, \( M_{\{u,v\}} = \{u, v\} \) and \( M_{\{v,w\}} = \{v, w\} \). By interchanging \( G \) and \( G \), assume that \( \{u, v\} \) is a stable set in \( G \). We obtain \( (u, \{v, w\}) \in E(G) \) and hence \( (w, \{u, v\}) \in E(G) \). Thus \( \{u, v\} \) is a stable set in \( G \). Furthermore \( \{u, v\} \) is a module of \( G \) by the fourth assertion of Proposition \( \ref{prop:module} \). Since \( \{u, v\} \) is a stable set in \( G \), \( \{u, v\} \) is a module of \( G[\{u, v\}] \). By the second assertion of Proposition \( \ref{prop:module} \), \( \{u, v\} \) is a module of \( G \). By Lemma \( \ref{lem:module} \), \( \{u, v\} \in \mathcal{M}_{\text{min}}(G) \). In both cases, there exists \( M_{\{u,w\}} \in \mathcal{M}_{\text{min}}(G) \) such that \( u, w \in M_{\{u,w\}} \). Consequently, the binary relation \( \approx_G \) defined on \( V(G) \) by

\[
 u \approx_G v \quad \text{if} \quad \begin{cases} u = v \\ u \neq v \text{ and there is } M_{\{u,v\}} \in \mathcal{M}_{\text{min}}(G) \text{ with } u, v \in M_{\{u,v\}}, \end{cases}
\]

for any \( u, v \in V(G) \), is an equivalence relation. The family of the equivalence classes of \( \approx_G \) is denoted by \( \mathfrak{M}(G) \). In Figure \( \ref{fig:module} \), \( \mathfrak{M}(G) = \{C, S, \{a\}, \{b\}\} \).

**Lemma 3.** Let \( G \) be a graph. For every \( C \in \mathfrak{M}(G) \) such that \( |C| \geq 2 \), one and only one of the following holds

- \( C \in \mathcal{P}(G) \),
- \( C \) is a maximal module of \( G \) under inclusion among the modules of \( G \) which are cliques in \( G \),
- \( C \) is a maximal module of \( G \) under inclusion among the modules of \( G \) which are stable sets in \( G \).

**Proof.** Consider \( C \in \mathfrak{M}(G) \) such that \( |C| \geq 2 \). Clearly \( C \) satisfies at most one of the three assertions above. For any \( u \neq v \in C \), there is \( M_{\{u,v\}} \in \mathcal{M}_{\text{min}}(G) \) such that \( u, v \in M_{\{u,v\}} \). By Lemma \( \ref{lem:module} \) either \( |M_{\{u,v\}}| = 2 \) or \( M_{\{u,v\}} \in \mathcal{P}(G) \).

First, assume that there are \( u \neq v \in C \) such that \( M_{\{u,v\}} \in \mathcal{P}(G) \). Let \( w \in C \setminus \{u, v\} \). By Lemma \( \ref{lem:module} \), \( M_{\{u,v\}} \subseteq M_{\{u,w\}} \) and hence \( M_{\{u,v\}} = M_{\{u,w\}} \) because \( M_{\{u,w\}} \in \mathcal{M}_{\text{min}}(G) \). Thus \( C = M_{\{u,v\}} \in \mathcal{P}(G) \).

Second, assume that \( M_{\{u,v\}} = \{u, v\} \) for any \( u \neq v \in C \). Given \( u \neq v \in C \), assume that \( \{u, v\} \) is a stable set in \( G \) by interchanging \( G \) and \( G \). As observed
above, \( \{u, v, w\} \) is a module of \( G \) and a stable set in \( G \) for every \( w \in C \setminus \{u, v\} \).
Therefore \( C \) is a stable set in \( G \) and it follows from the fourth assertion of Proposition \( \Box \) that \( C \) is a module of \( G \). Furthermore consider \( u \in C \) and \( x \in V(G) \setminus C \). Since \( u \not\in G x \), \( \{u, x\} \) is not a module of \( G \) and hence there is \( y \in V(G) \setminus \{u, x\} \) such that \( y \not\in G \{u, x\} \). If \( y \not\in G C \cup \{x\} \) and \( C \cup \{x\} \) is no longer a module of \( G \). If \( y \in C \), then \( (y, x)_G = 1 \) because \( (y, u)_G = 0 \). Thus \( C \cup \{x\} \) is no longer a stable set in \( G \). Consequently \( C \) is a maximal module of \( G \) among the modules of \( G \) which are stable sets in \( G \). \( \Diamond \)

Let \( G \) be a graph. Following Lemma \( \Box \) denote by \( C(G) \) the family of the maximal elements of \( M(G) \) under inclusion among the elements of \( M(G) \) which are cliques in \( G \), and denote by \( S(G) \) the family of the maximal elements of \( M(G) \) under inclusion among the elements of \( M(G) \) which are stable sets in \( G \). In Figure \( \Box \) \( C(G) = \{C\} \) and \( S(G) = \{S\} \). The next is a simple consequence of Lemma \( \Box \)

**Corollary 1.** For a graph \( G \), \( \{C \in M(G): |C| \geq 2\} \in C(G) \cup S(G) \cup P(G) \).

**Proof.** By Lemma \( \Box \)
\[
\{C \in M(G): |C| \geq 2\} \in C(G) \cup S(G) \cup P(G).
\]
For the opposite inclusion, consider \( C \in C(G) \cup S(G) \cup P(G) \). First, assume that \( C \in C(G) \cup S(G) \). By interchanging \( G \) and \( \bar{G} \), assume that \( G \) is a clique of \( G \). For any \( c \not\in d \in C \), \( \{c, d\} \) is a module of \( G[C] \). By the second assertion of Proposition \( \Box \) \( \{c, d\} \) is a module of \( G \). Thus \( \{c, d\} \in M_{\text{min}}(G) \) and hence \( c \equiv_G d \). Therefore there is \( D \in M(G) \) such that \( D \supset C \). By Lemma \( \Box \) \( D \in C(G) \cup S(G) \cup P(G) \). If \( D \in P(G) \), then \( D = C \) because \( D \in M_{\text{min}}(G) \) by Lemma \( \Box \) So assume that \( D \in C(G) \cup S(G) \). As \( C \) is a clique in \( G \), \( D \in C(G) \) and \( C = D \) by the maximality of \( C \in C(G) \).

Second, assume that \( C \in P(G) \). By Lemma \( \Box \) \( C \in M_{\text{min}}(G) \) and hence \( c \equiv_G d \) for any \( c \not\in d \in C \). So there is \( D \in M(G) \) such that \( D \supset C \). By Lemma \( \Box \) \( D \in C(G) \cup S(G) \cup P(G) \). As \( G[C] \) is prime, \( C \) is not included in a clique or a stable set in \( G \). Therefore \( D \in P(G) \). By Lemma \( \Box \) \( D \in M_{\text{min}}(G) \) and hence \( C = D \). \( \Diamond \)

Given a graph \( G \), set \( \mathcal{I}(G) = M(G) \setminus (C(G) \cup S(G) \cup P(G)) \) and \( I(G) = \{v \in V(G): \{v\} \in \mathcal{I}(G)\} \). In Figure \( \Box \) \( I(G) = \{a, b\} \).

**Remark 1.** Given a graph \( G \), consider \( M \in M(G) \). There exists \( N \in M_{\text{min}}(G) \) such that \( N \subseteq M \). By considering \( p \neq q \in N \), we obtain that there exist \( p \neq q \in M \) such that \( p \equiv_G q \). By Corollary \( \Box \) there is \( C \in C(G) \cup S(G) \cup P(G) \) such that \( |C \cap N| \geq 2 \).

Let \( G \) be a graph. If \( \omega_M(G) \geq 2 \), that is, \( C(G) \neq \emptyset \), then \( \omega_M(G) = \max(\{|C|: C \in C(G)\}) \). Similarly \( \alpha_M(G) = \max(\{|C|: C \in S(G)\}) \) if \( \alpha_M(G) \geq 2 \). Consequently, if \( m(G) \geq 2 \), that is, \( C(G) \cup S(G) \neq \emptyset \), then \( m(G) = \max(\{|C|: C \in C(G) \cup S(G)\}) \).
Given a graph $G$, Sabidussi [15] introduced the following equivalence relation $S_{ab}G$ on $V(G)$. Given $u, v \in V(G)$, $u S_{ab}G v$ if $N_G(u) = N_G(v)$. If $\alpha_M(G) \geq 2$, then $\mathcal{S}(G)$ is the family of the equivalence classes of $S_{ab}G$ which are not singletons.

We complete the section with another equivalence relation induced by the modular decomposition tree. It is used by Giakoumakis and Olariu [10] to construct a minimal prime extension of a graph. Given a graph $G$, consider the equivalence relation $\leftrightarrow_G$ defined on $V(G)$ by: given $v, w \in V(G)$, $v \leftrightarrow_G w$ if $\{v\} \not\subset \{w\}$. The set of the equivalence classes of $\leftrightarrow_G$ is denoted by $\mathcal{S}(G)$. In Figure 1 $\mathcal{S}(G) = \{C, S, \{a, b\}\}$.

Precisely, to construct a minimal prime extension of a graph $G$, Giakoumakis and Olariu [10] use only the elements of $\mathcal{S}(G) \cap M(G)$. The remainder of the section is mainly devoted to the study of $\mathcal{S}(G) \cap M(G)$ (see Proposition 4).

**Lemma 4.** Let $G$ be a graph.

1. Let $M \in \mathcal{S}(G)$ with $|M| \geq 2$. If $\{N \in \Pi(G[M]) : |N| = 1\} \neq \emptyset$, then $\{m \in M : \{m\} \in \Pi(G[M])\} \in \mathcal{S}(G)$.
2. Let $C \in \mathcal{S}(G)$ with $|C| \geq 2$. For every $c \in C$, $C \vdash \{c\} \uparrow$ and $C = \{c \in C : \{c\} \in \Pi(G[C])\}$.
3. For each $M \in \mathcal{S}(G)$ with $|M| \geq 2$, there is $C \in \mathcal{S}(G) \cap \mathcal{S}(G)$ such that $|C| \geq 2$ and $C \subseteq M$.

**Proof.** The first assertion follows from the definition of $\leftrightarrow_G$. For the second, consider $C \in \mathcal{S}(G)$ with $|C| \geq 2$. For $c, d \in C$, we have $\{c\} \not\subset \{d\}$. Given $c_0 \in C$, we obtain $C \not\subseteq \{c_0\}$ and hence $C \uparrow \subseteq \{c_0\}$. As $|C| \geq 2$, we have also $\{c_0\} \not\subseteq C \uparrow$ and hence $\{c_0\} \not\subset C \uparrow$. Thus $C \uparrow = \{c\}$ for every $c \in C$. It follows from the definition of $\leftrightarrow_G$ that for each $v \in V(G)$, $v \in C$ if and only if $\{v\} \not\subset C \uparrow$. Furthermore it follows from the second assertion of Proposition 4 that for each $d \in C \uparrow$, $\{d\} \not\subset C \uparrow$ if and only if $\{d\} \in \Pi(G[C \uparrow])$. Therefore $C = \{c \in C \uparrow : \{c\} \in \Pi(G[C \uparrow])\}$.

For the third assertion, consider a strong module $M$ of $G$ with $|M| \geq 2$. Let $N$ be a minimal strong module of $G$ under inclusion among the strong modules $M'$ of $G$ such that $|M'| \geq 2$ and $M' \subset M$. By the minimality of $N$, we obtain $\Pi(G[N]) = \{\{n\} : n \in N\}$. By the first assertion, $N \in \mathcal{S}(G)$.

**Proposition 4.** For a graph $G$,

1. $\mathcal{P}(G) = \{C \in \mathcal{S}(G) \cap \mathcal{S}(G) : \lambda_G(C) = \emptyset\}$,
2. $\mathcal{C}(G) = \{C \in \mathcal{S}(G) : |C| \geq 2, \lambda_G(C \uparrow) = \bigcirc\}$,
3. $\mathcal{S}(G) = \{C \in \mathcal{S}(G) : |C| \geq 2, \lambda_G(C \uparrow) = \square\}$.

**Proof.** For the first assertion, consider $P \in \mathcal{P}(G)$. By Lemma 2, $P \in \mathcal{S}(G)$. By the third assertion of Lemma 4, there is $C \in \mathcal{S}(G) \cap \mathcal{S}(G)$ such that $|C| \geq 2$ and $C \subseteq P$. As $P \in \mathcal{M}_{\text{min}}(G)$ by Lemma 1, $C = P$. Since $G[P]$ is prime, $\Pi(G[P]) = \{\{p\} : p \in P\}$. By the first assertion of Lemma 4, $P \in \mathcal{S}(G)$. 

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Moreover it follows from the first assertion of Proposition 2 that \( G[P]/\Pi(G[P]) \) is prime. Thus \( \lambda_G(P) = \emptyset. \)

Conversely, consider \( C \in \mathcal{S}(G) \cap \mathcal{S}(G) \) such that \( \lambda_G(C) = \emptyset. \) Clearly \( C \uparrow M \) because \( C \in \mathcal{S}(G). \) As \( C \in \mathcal{S}(G) \), it follows from the second assertion of Lemma 3 that \( \Pi(G[C]) = \{ \{ c \} : c \in C \}. \) Furthermore we have \( G[C]/\Pi(G[C]) \) is prime because \( \lambda_G(C) = \emptyset. \) Thus \( G[C] \) is prime as well by the first assertion of Proposition 2. Therefore \( C \in \mathcal{P}(G). \)

For the second assertion, consider \( C \in \mathcal{C}(G). \) Denote by \( Q \) the family of \( M \in \Pi(G[C]) \) such that \( M \cap C \neq \emptyset. \) For each \( M \in Q \), \( C \setminus M \) is not empty because \( M \leq C \). As \( M \) is a strong module of \( G \) by Proposition 3 we obtain \( M \subseteq C \). Thus \( |Q| \geq 2 \) and \( C = \cup Q \). Given \( M \in Q \), consider \( N \in Q \setminus \{ M \}. \) For \( m \in M \) and \( n \in N \), \( \{ m, n \} \) is a module of \( G[C] \) because \( G[C] \) is complete. By the second assertion of Proposition 4 \( \{ m, n \} \) is a module of \( G \). Since \( M \) is a strong module of \( G \) such that \( m \in M \cap \{ m, n \} \) and \( n \in \{ m, n \} \setminus M \), we get \( M \subseteq \{ m, n \} \) and hence \( M = \{ m \}. \) Thus \( \{ c \} \in \Pi(G[C]) \) for each \( c \in C \). Set \( D = \{ c \in C : \{ c \} \in \Pi(G[C]) \}. \) We have \( C \subseteq D \) and \( D \in \mathcal{S}(G) \) by the first assertion of Lemma 4. Moreover, by the first assertion of Proposition 4 \( C \) is a module of \( G[C] \). It follows from the second assertion of Proposition 2 that \( \{ \{ c \} : c \in C \} \) is a module of \( G[C]/\Pi(G[C]) \). Since \( \{ \{ c \} : c \in C \} \) is a clique in \( G[C]/\Pi(G[C]) \), we obtain \( \lambda_G(C) = \emptyset. \) As \( \lambda_G(C) = \emptyset. \) and as \( D = \{ c \in C : \{ c \} \in \Pi(G[C]) \}, \{ \{ d \} : d \in D \} \) is a module of \( G[C]/\Pi(G[C]) \) and a clique in \( G[C]/\Pi(G[C]) \). Consequently \( D \) is a clique in \( G \) and \( D \) is a module of \( G[C] \) by the last assertion of Proposition 2. By the second assertion of Proposition 4 \( D \) is a module of \( G \). Since \( C \in \mathcal{C}(G), \) \( C = D. \)

Conversely, consider \( C \in \mathcal{S}(G) \) such that \( |C| \geq 2 \) and \( \lambda_G(C) = \emptyset. \) By the second assertion of Lemma 4 \( C = \{ c \in C : \{ c \} \in \Pi(G[C]) \}. \) Since \( \lambda_G(C) = \emptyset. \) \( C \) is a clique in \( G \) and, as above for \( D \), it follows from Propositions 1 and 2 that \( C \) is a module of \( G. \) Thus there is \( D \in \mathcal{C}(G) \) such that \( C \subseteq D. \) As already proved, \( D \in \mathcal{S}(G) \) and hence \( C = D. \)

The third assertion follows from the second by interchanging \( G \) and \( G. \) Therefore \( C \in \mathcal{P}(G). \)

It follows from Proposition 4 that

**Corollary 2.** For a graph \( G, \)

\[
\begin{align*}
\omega_M(G) &= \max\{|C| : C \in \mathcal{S}(G), |C| \geq 2, \lambda_G(C) = \emptyset\} \text{ if } \omega_M(G) \geq 2, \\
\alpha_M(G) &= \max\{|C| : C \in \mathcal{S}(G), |C| \geq 2, \lambda_G(C) = \emptyset\} \text{ if } \alpha_M(G) \geq 2.
\end{align*}
\]

The next is a simple consequence of Corollary 4 and Proposition 4.

**Corollary 3.** Let \( G \) be a graph.

1. \( \{ C \in \mathcal{M}(G) : |C| \geq 2 \} \subseteq \{ C \in \mathcal{S}(G) : |C| \geq 2 \}. \)

2. Given \( C \in \mathcal{S}(G) \) with \( |C| \geq 2, \)

\[ C \notin \mathcal{M}(G) \text{ if and only if } C \neq C \uparrow \text{ and } \lambda_G(C) = \emptyset. \]
3. Let \( C \in \mathcal{S}(G) \setminus \mathcal{M}(G) \) with \( |C| \geq 2 \). For each \( c \in C \), \( \{c\} \in \mathcal{M}(G) \setminus \mathcal{S}(G) \).

4. \( I(G) = \{v \in V(G) : \{v\} \in \mathcal{S}(G)\} \cup (\bigcup_{C \in \mathcal{S}(G) \setminus \mathcal{M}(G)} \{\{c\} : c \in C\}) \).

Remark 2. The second assertion of Corollary 3 easily provides graphs \( G \) such that \( \mathcal{S}(G) \setminus \mathcal{M}(G) \neq \emptyset \). For instance, \( \{a, b\} \in \mathcal{S}(G) \setminus \mathcal{M}(G) \) in Figure 1.

Consider a prime graph \( G_0 \). Given \( \alpha \notin V(G_0) \) and \( u \in V(G_0) \), consider a 1-extension \( G \) of \( G_0 \) to \( V(G_0) \cup \{\alpha\} \) such that \( \{u, \alpha\} \) is a module of \( G \), that is, \( \alpha \in X_G(u) \) where \( X = V(G_0) \). Clearly \( \{u, \alpha\} \) is the single non-trivial module of \( G \). Consequently

\[
\mathcal{M}(G) = \{\{u, \alpha\}\} \cup \{\{v\} : v \in V(G_0) \setminus \{u\}\},
\]

\[
\mathcal{S}(G) = \{\{v\} : v \in V(G)\} \cup \{\{u, \alpha\}, V(G)\}.
\]

Since \( \mathcal{S}(G) = \{\{v\} : v \in V(G)\} \cup \{\{u, \alpha\}, V(G)\} \), we get

\[
\mathcal{S}(G) = \{\{u, \alpha\}, V(G_0) \setminus \{u\}\}.
\]

Thus \( V(G_0) \setminus \{u\} \in \mathcal{S}(G) \setminus \mathcal{M}(G) \).

The following summarizes our comparison between \( \mathcal{M}(G) \) and \( \mathcal{S}(G) \) for a graph \( G \).

**Proposition 5.** For a graph \( G \),

\[
\mathcal{S}(G) \cap \mathcal{M}(G) = \{C \in \mathcal{M}(G) : |C| \geq 2\} = \mathcal{L}(G) \cup \mathcal{S}(G) \cup \mathcal{P}(G).
\]

**Proof.** By Corollary 1 \( \{C \in \mathcal{M}(G) : |C| \geq 2\} = \mathcal{L}(G) \cup \mathcal{S}(G) \cup \mathcal{P}(G) \). Furthermore, it follows from Proposition 1 that \( \mathcal{L}(G) \cup \mathcal{S}(G) \cup \mathcal{P}(G) \subset \mathcal{S}(G) \cap \mathcal{M}(G) \).

So consider \( C \in \mathcal{S}(G) \cap \mathcal{M}(G) \). If \( \lambda_G(C^\top) = \square \) or \( \boxdot \), then \( C \in \mathcal{L}(G) \cup \mathcal{S}(G) \) by the last two assertions of Proposition 1. Thus assume that \( \lambda_G(C^\top) = \bigcup \), that is, \( G[C^\top] \cap \Pi(G[C^\top]) \) is prime. By the second assertion of Lemma 1 \( C = \{c \in C^\top : \{c\} \in \Pi(G[C^\top])\} \). Since \( C \) is a module of \( G \), \( C \) is a module of \( G[C^\top] \) by the first assertion of Proposition 1. Therefore \( \{\{c\} : c \in C\} \) is a module of \( G[C^\top] \cup \Pi(G[C^\top]) \) by the second assertion of Proposition 2. As \( G[C^\top] \cup \Pi(G[C^\top]) \) is prime, \( \{\{c\} : c \in C\} = \Pi(G[C^\top]) \). Consequently, \( C = C^\top \) and hence \( C \in \mathcal{P}(G) \) by the first assertion of Proposition 3. \( \square \)

Given a non-primitive and connected graph \( G \), Giakoumakis and Olariu [11] construct a minimal prime extension of \( G \) by adding \(|C|-1\) vertices for each \( C \in \mathcal{L}(G) \cup \mathcal{S}(G) \) and one vertex for each element of \( \mathcal{P}(G) \).

### 4 Some prime extensions

We use the next corollary to prove Theorem 2.
Lemma 5. Let $S$ and $S'$ be disjoint sets such that $|S'| = \lceil \log_2(|S| + 1) \rceil \geq 2$, that is, $1 \leq 2^{|S'| - 1} \leq |S| < 2^{|S'|}$. Consider any graph $G$ defined on $V(G) = S \cup S'$ such that $S$ and $S'$ are stable sets in $G$.

1. Assume that $2^{|S'| - 1} < |S| < 2^{|S'|}$. Then, $G$ is prime if and only if
   \[
   \text{the function } (N_G)_{|S} \text{ is an injection from } S \text{ into } 2^{S'} \setminus \{\emptyset\};
   \] (4.1)

2. Assume that $2^{|S'| - 1} = |S|$. Then, $G$ is prime if and only if (4.1) holds and for any $s \in S$ and $s' \in S'$,
   \[
   \text{if } d_G(s) = d_G(s') = 1, \text{ then } (s,s')_G = 0.
   \]

Proof. First, if there is an injection from $S$ into $S'$ such that $N_G(s) = \emptyset$, then $s \in \text{Iso}(G)$ and hence $V(G) \setminus \{s\}$ is a non-trivial module of $G$. Second, if there are $s \neq t \in S$ such that $N_G(s) = N_G(t)$, then $\{s,t\}$ is a non-trivial module of $G$. Third, if there are $s \in S$ and $s' \in S'$ such that $N_G(s) = \{s'\}$ and $N_G(s') = \{s\}$, then $\{s,s'\}$ is a non-trivial module of $G$.

Conversely, assume that (4.1) holds. Consider a module $M$ of $G$ such that $|M| \geq 2$. We have to show that $M = V(G)$. As $(N_G)_{|S}$ is injective, $M \neq S$, that is, $M \cap S' \neq \emptyset$.

For a first contradiction, suppose that $M \subseteq S'$. Recall that for each $s \in S$, either $M \cap N_G(s) = \emptyset$ or $M \subseteq N_G(s)$. Given $m \in M$, consider the function $f : S \rightarrow 2^{|S' \setminus M|} \setminus \emptyset$ defined by
   \[
   f(s) = \begin{cases} 
   N_G(s) & \text{if } M \cap N_G(s) = \emptyset, \\
   N_G(s) \cup \{m\} & \text{if } M \subseteq N_G(s),
   \end{cases}
   \]
for every $s \in S$. Since $(N_G)_{|S}$ is injective, $f$ is also and we would obtain that $|S| < 2^{|S'| - 1}$. It follows that $M \cap S = \emptyset$.

For a second contradiction, suppose that $S' \setminus M \neq \emptyset$. We have $(S \cap M, S' \setminus M)_G = (S' \setminus M, S' \setminus M)_G = 0$. If $S \subseteq M$, then $(S, S' \setminus M)_G = 0$ so that $(N_G)_{|S}$ would be an injection from $S$ into $2^{S' \setminus M} \setminus \emptyset$ which contradicts $|S| \geq 2^{|S'| - 1}$. Thus $S \setminus M \neq \emptyset$. We obtain $(S \setminus M, S' \cap M)_G = (S \setminus M, S \cap M)_G = 0$. As $(S \cap M, S' \setminus M)_G = 0$, $(N_G)_{|(S \cap M)} : S \cap M \rightarrow 2^{S' \setminus M} \setminus \emptyset$ and hence $|S \cap M| \leq 2^{|S'| - |M|} - 1$. Since $(S \setminus M, S' \cap M)_G = 0$, $(N_G)_{|(S \setminus M)} : S \setminus M \rightarrow 2^{S' \setminus M} \setminus \emptyset$ and hence $|S \setminus M| \leq 2^{|S'| - |M|} - 1$. Therefore $|S| \leq 2^{|S'| - |M|} + 2^{|S' \setminus M|} - 2 \leq 2^{|S'| - 1}$. As $|S| \geq 2^{|S'| - 1}$, we obtain $|S| = 2^{|S'| - 1}$ and $2^{|S' \setminus M|} = 2 + 2^{|S'| - 1}$ so that $|S' \setminus M| = 1$. For instance, assume that $|S' \setminus M| = 1$. Since $|S \cap M| \leq 2^{|S'| - |M|} - 1$, $|S \cap M| = 1$. There exist $s \in S$ and $s' \in S'$ such that $M = \{s, s'\}$. By what precedes, $(s, S' \setminus \{s'\})_G = (s', S \setminus \{s\})_G = 0$. As $N_G(s) \neq \emptyset$, we would obtain $N_G(s) = \{s'\}$, $N_G(s') = \{s\}$ and $(s,s')_G = 1$. It follows that $S' \subseteq M$.

Lastly, suppose that $S \setminus M \neq \emptyset$. For each $s \in S \setminus M$, we would have $(s, S \cap M)_G = (s, S \cap M)_G = 0$ and hence $N_G(s) = \emptyset$. It follows that $S \subseteq M$ and $M = S' \cup S'$. \(\diamondsuit\)
Corollary 4. Let $S$ and $S'$ be disjoint sets such that $|S| \geq 3$ and $|S'| = \lceil \log_2(|S| + 1) \rceil$. There exists a prime graph $G$ defined on $V(G) = S \cup S'$ and satisfying

- $S$ and $S'$ are stable sets in $G$;
- $(N_G)_S : S \to 2^{S'} \setminus \{\emptyset\}$ is injective;
- there exists an injection $\varphi_{S'} : S' \to S$ such that $N_G(\varphi_{S'}(s')) = S' \setminus \{s'\}$ for each $s' \in S'$.

![Figure 2: Corollary 4](image)

Proof. (See Figure 2) As $|S'| = \lceil \log_2(|S| + 1) \rceil$, we have $2^{|S'|-1} \leq |S| < 2^{|S'|}$ and hence $|S'| \leq |S|$. Thus there exists a bijection $\psi_{S'}$ from $S'$ onto $S'' \subseteq S$. Consider the injection $f_{S''} : S'' \to 2^{S'} \setminus \{\emptyset\}$ defined by $s'' \mapsto S' \setminus \{(\psi_{S'})^{-1}(s'')\}$. Let $f_S$ be any injection from $S$ into $2^{S'} \setminus \{\emptyset\}$ such that $(f_S)_{S''} = f_{S''}$. Lastly, consider the graph $G$ defined on $V(G) = S \cup S'$ such that $S$ and $S'$ are stable sets in $G$ and $(N_G)_S = f_S$. Before applying Lemma 5, assume that $|S| = 2^{|S'|-1}$. Since $|S| \geq 3$, $|S'| \geq 3$. For each $s' \in S'$, there are $t' \neq u' \in S' \setminus \{s'\}$. We obtain $N_G(\psi_{S'}(t')) = S' \setminus \{t'\}$ and $N_G(\psi_{S'}(u')) = S' \setminus \{u'\}$. Therefore $s' \in N_G(\psi_{S'}(t')) \cap N_G(\psi_{S'}(u'))$. It follows that $d_G(s') \geq 2$ for every $s' \in S'$. By Lemma 5, $G$ is prime.

We use the following two results to prove Theorem 2 when $P(G) \neq \emptyset$. Given a graph $G$, consider $X \subseteq V(G)$ such that $G[X]$ is prime. We utilize the following subsets of $V(G) \setminus X$ (for instance, see Lemma 5.1)

- $\text{Ext}_G(X)$ is the set of $v \in V(G) \setminus X$ such that $G[X \cup \{v\}]$ is prime;
- $\langle X \rangle_G$ is the set of $v \in V(G) \setminus X$ such that $X$ is a module of $G[X \cup \{v\}]$;
- for $u \in X$, $X_G(u)$ is the set of $v \in V(G) \setminus X$ such that $\{u, v\}$ is a module of $G[X \cup \{v\}]$. 

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The family $\{\text{Ext}_G(X), \{X\}_G\} \cup \{X_G(u) : u \in X\}$ is denoted by $p_G[X]$. The next lemma follows from Proposition[4]

**Lemma 6.** Given a graph $G$, consider $X \subseteq V(G)$ such that $G[X]$ is prime. The family $p_G[X]$ is a partition of $V(G) \setminus X$. Moreover, for each module $M$ of $G$, one and only one of the following holds

- $X \subseteq M$ and $V(G) \setminus M \subseteq (X)_G$;
- there is a unique $u \in X$ such that $M \cap X = \{u\}$ and $M \setminus \{u\} \subseteq X_G(u)$;
- $M \cap X = \emptyset$ and $M$ is included in an element of $p_G[X]$. Moreover, for $v, w \in M$, the function $X \cup \{v\} \rightarrow X \cup \{w\}$, defined by $v \mapsto w$ and $u \mapsto u$ for $u \in X$, is an isomorphism from $G[X \cup \{v\}]$ onto $G[X \cup \{w\}]$.

**Lemma 7.** Let $G$ be a prime graph. For every $\alpha \notin V(G)$, there are

$$2^{|V(G)|} - 2|V(G)| - 2$$

distinct prime 1-extensions of $G$ to $V(G) \cup \{\alpha\}$.

**Proof.** Consider any graph $H$ defined on $V(H) = V(G) \cup \{\alpha\}$ such that $H[V(G)] = G$ and

$$N_H(\alpha) \in 2^{|V(G)|} \setminus \{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}.$$ 

We verify that $H$ is prime. Set $X = V(G)$. We have $H[X] = G$ is prime. If $\alpha \in (X)_H$, then $N_H(\alpha) = \emptyset$ or $V(G)$. Thus $\alpha \notin (X)_H$. If there is $v \in V(G)$ such that $\alpha \in X_H(v)$, then $N_H(\alpha) = N_G(v)$ or $N_G(v) \cup \{v\}$. Therefore $\alpha \notin X_H(v)$ for every $v \in V(G)$. It follows from Lemma[4] that $\alpha \in \text{Ext}_H(X)$, that is, $H$ is prime. Consequently the number of prime 1-extensions of $G$ to $V(G) \cup \{\alpha\}$ equals

$$2^{|V(G)|} \setminus \{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\}.$$ 

Clearly $\emptyset \notin \{N_G(v) \cup \{v\} : v \in V(G)\}$, $V(G) \notin \{N_G(v) : v \in V(G)\}$ and $\{N_G(v) : v \in V(G)\} \cap \{N_G(v) \cup \{v\} : v \in V(G)\} = \emptyset$. Moreover, if there is $v \in V(G)$ such that $N_G(v) = \emptyset$ or $V(G) \setminus \{v\}$, then $V(G) \setminus \{v\}$ would be a non-trivial module of $G$. If there are $v \neq w \in V(G)$ such that $N_G(v) = N_G(w)$ or $N_G(v) \cup \{v\} = N_G(w) \cup \{w\}$, then $\{v, w\}$ would be a non-trivial module of $G$. As $G$ is prime, $V(G) \notin \{N_G(v) \cup \{v\} : v \in V(G)\}$, $\emptyset \notin \{N_G(v) : v \in V(G)\}$ and for $v \neq w \in V(G)$, we have $N_G(v) \neq N_G(w)$ and $N_G(v) \cup \{v\} \neq N_G(w) \cup \{w\}$. Therefore

$$2^{|V(G)|} \setminus \{\emptyset, V(G)\} \cup \{N_G(v) : v \in V(G)\} \cup \{N_G(v) \cup \{v\} : v \in V(G)\} = 2^{|V(G)|} - 2|V(G)| - 2.$$ 

$\diamond$

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5 Proofs of Theorems 2 and 3

Proof of Theorem 2 Consider a graph $G$ such that $|m(G)| \geq 2$. To begin, we show that $p(G) \geq \lceil \log_2(m(G)) \rceil$. By interchanging $G$ and $\overline{G}$, assume that there exists $S \in \mathcal{S}(G)$ with $|S| = m(G)$. Given an integer $p < \log_2(m(G))$, consider any $p$-extension $H$ of $G$. We must prove that $H$ is not prime. We have $2^{V(H) \cdot V(G)} < |S|$ so that the function $S \rightarrow 2^{V(H) \cdot V(G)}$, defined by $s \mapsto N_H(s) \cap (V(H) \setminus V(G))$, is not injective. Thus there are $s \neq t \in S$ such that $N_H(s) \cap (V(H) \setminus V(G)) = N_H(t) \cap (V(H) \setminus V(G))$. In other words, $v \sim_H \{s, t\}$ for every $v \in V(H) \setminus V(G)$. As $S$ is a module of $G$, we have $v \sim_G \{s, t\}$, that is, $v \sim_H \{s, t\}$ for every $v \in V(G) \setminus S$. Since $S$ is a stable set in $G$, there is the $\{v, s\}_{G} = (v, t)_{G} = (v, t)_H$. Therefore $\{s, t\}$ is a module of $H$ and $H$ is a prime.

To prove that $p(G) \leq \lceil \log_2(m(G) + 1) \rceil$, we must construct a prime $\lceil \log_2(m(G) + 1) \rceil$-extension $H$ of $G$. Let $S'$ be a set such that $S' \cap V(G) = \emptyset$ and $|S'| = \lceil \log_2(m(G) + 1) \rceil$. Let $S_0 = S'$. Consider $S_0 \in \mathcal{C}(G) \cup \mathcal{S}(G)$ such that $|S_0| = m(G)$, that is, $2^{2^{|S_0|}} \leq |S_0| < 2^{|S_0|}$. By interchanging $G$ and $\overline{G}$, we can assume that $S_0$ is a clique or a stable set in $G$. (In Figure 3 $m(G) = 5$, $S_0 = S = \{s_1, \ldots, s_5\} \in \mathcal{S}(G)$ and $S' = \{s'_1, s'_2, s'_3\}$.)

We consider any graph $H$ defined on $V(G) \cup S'$ and satisfying the following.

1. $S'$ is a stable set in $H$.
2. The subgraph $H[S_0 \cup S']$ of $H$ is defined as follows

   - Assume that $|S_0| = 2$. We require that the subgraph $H[S_0 \cup S']$ of $H$ is a path on 4 vertices and $S_0$ is a clique in $H[S_0 \cup S']$. Thus $d_{H[S_0 \cup S']}(s') = 1$ for each $s' \in S'$.

(5.1)
6. Let $v \notin S_0$. By Corollary 4 we can consider for $H[S_0 \cup S']$ a prime graph defined on $S_0 \cup S'$ such that

\[
\begin{cases}
S_0 \text{ is a stable set in } H[S_0 \cup S']; \\
(N_{H[S_0 \cup S']})_{S_0} : S_0 \to 2^{S'} \setminus \{\emptyset\} \text{ is injective}; \\
\text{there exists an injection } \varphi_{S'} : S' \to S_0 \text{ with } \tilde{N}_{H[S_0 \cup S']}(\varphi_{S'}(s')) = S' \setminus \{s'\} \text{ for each } s' \in S'.
\end{cases}
\] (5.2)

(In Figure 3 the subgraph $H[S_0 \cup S']$ is also depicted in Figure 2)

Set $X = S_0 \cup S'$. In both cases, $H[X]$ is prime.

3. Let $C \in \mathcal{C}(G) \setminus \{S_0\}$. We have $|C| < 2^{|S'|}$. We consider for $H[C \cup S']$ a graph such that $C$ is a clique in $H[C \cup S']$ and

\[
f_C : C \to 2^{S'}
\]

\[
c \mapsto \tilde{N}_{H[C \cup S']}(c) \cap S'
\]

is an injection from $C$ into $2^{S'} \setminus \{S'\}$. (5.3)

(In Figure 3 we have $C = \{c_1, c_2, c_3\} \in \mathcal{C}(G)$ and $f_C$ is defined by $c_1 \mapsto \{s'_1\}, c_2 \mapsto \{s'_2\}, c_3 \mapsto \emptyset$.)

4. Let $S \in \mathcal{S}(G) \setminus \{S_0\}$. We have $|S| < 2^{|S'|}$. We consider for $H[S \cup S']$ a graph such that $S$ is a stable set in $H[S \cup S']$ and

\[
f_S : S \to 2^{S'}
\]

\[
s \mapsto \tilde{N}_{H[S \cup S']}(S)
\]

is an injection from $S$ into $2^{S'} \setminus \{\emptyset\}$. (5.4)

5. Let $P \in \mathcal{P}(G)$. As $G[P]$ is prime, it follows from Lemma 7 that $G[P]$ admits a prime 1-extension. We consider for $H[P \cup S']$ a graph such that

\[
\begin{cases}
H[P] = G[P] \text{ is prime,} \\
\Ext_{H[P \cup S']}(P) = S' \text{ by using Lemma 7}
\end{cases}
\] (5.5)

6. Let $v \in I(G)$. Since $S_0$ is a module of $G$ such that $v \notin S_0$, there is $i \in \{0,1\}$ such that $(v, S_0)_G = i$. We consider for $H[\{v\} \cup S']$ a graph such that

\[
(v, s'_1)_{H[\{v\} \cup S']} \neq i.
\] (5.6)

(In Figure 3 we have $S_0 = S$ and $I(G) = \{a, b\}$ with

\[
\begin{cases}
(a, S)_G = 1 \text{ and } (a, s'_1)_{H[\{a\} \cup S']} = 0,
(b, S)_G = 0 \text{ and } (b, s'_1)_{H[\{b\} \cup S']} = 1.
\end{cases}
\]
In the construction above, we have $H[N] = G[N]$ for each $N \in \mathcal{M}(G)$. Thus we can also assume that $H[V(G)] = G$.

We begin with the following observation. For every module $M$ of $H$ such that $|M| \geq 2$, we have $M \cap S' \neq \varnothing$. Otherwise suppose that $M \subseteq V(G)$. By the first assertion of Proposition 1, $M \in \mathcal{M}(G)$. By Remark 1 there is $C \in \mathcal{C}(G) \cup \mathcal{S}(G) \cup \mathcal{P}(G)$ such that $|C \cap M| \geq 2$. As $M$ is a module of $H$, it follows from the first assertion of Proposition 1 that $C \cap M$ is a module of $H[C \cup S']$. Since $H[S_0 \cup S']$ is prime, $C \neq S_0$. If $C \in (\mathcal{C}(G) \cup \mathcal{S}(G)) \setminus \{S_0\}$, then $(f_C)_H(C \cap M)$ would be constant which contradicts \((5.3)\) and \((5.4)\). Lastly, suppose that $C \in \mathcal{P}(G)$.

As $C \cap M$ is a module of $H[C] = G[C]$ by the first assertion of Proposition 1, it follows from Lemma 2 that $C \cap M = C$. Thus $C$ would be a module of $H[C \cup S']$ so that $(C)_H[C \cup S'] = S'$ which contradicts \((5.3)\) and Lemma 6. Consequently, $M \cap S' \neq \varnothing$ for every module $M$ of $H$ such that $|M| \geq 2$.

Now, we prove that $H$ is prime. Consider a module $M$ of $H$ such that $|M| \geq 2$. We have to show that $M = V(H)$. As observed above, $M \cap S' \neq \varnothing$. By Lemma 6 either there is $s' \in S'$ such that $M \cap X = \{s'\}$ or $X \subseteq M$.

For a contradiction, suppose that there is $s' \in S'$ such that $M \cap X = \{s'\}$. By the first assertion of Proposition 1, $M \setminus \{s'\}$ is a module of $G$. By the last assertion of Proposition 1, there is $i \in \{0, 1\}$ such that $(M \setminus \{s'\}, S_0)_G = i$. Thus $(s', S_0)_H[S_0 \cup S'] = i$. Since $S'$ is a stable set in $H[S_0 \cup S']$, we have $d_{H[S_0 \cup S']}(s') = 0$ or $d_{H[S_0 \cup S']}(s') \geq 2$ so that \((5.1)\) does not hold. Therefore $|S_0| \geq 3$. Let $t \in S' \setminus \{s'\}$. By \((5.2)\), $(\varphi_S(s'), s')_H[S_0 \cup S'] = 0$ and $(\varphi_S(t'), s')_H[S_0 \cup S'] = 1$ which contradicts $(s', S_0)_H[S_0 \cup S'] = i$. It follows that $X \subseteq M$.

First, consider $C \in \mathcal{C}(G) \setminus \{S_0\}$. Suppose for a contradiction that $C \cap M = \varnothing$. Thus $C \cap S_0 = \varnothing$ and it follows from the last assertion of Proposition 1 that there is $i \in \{0, 1\}$ such that $(C, S_0)_G = i$. As $C \cap M = \varnothing$, we obtain $(C, M)_H = i$. In particular $(C, S')_H[C \cup S'] = i$ and $f_C$ would be constant which contradicts \((5.3)\). Therefore $C \cap M \neq \varnothing$. Suppose for a contradiction that $C \cap M \neq \varnothing$ and consider $c \in C \setminus M$. Since $C$ is a clique of $G$, $(c, C \cap M)_G = 1$. Hence $(c, M)_H = 1$ and in particular $(c, S')_H[C \cup S'] = 1$ which contradicts $f_C(c) \neq S'$. It follows that $C \not\subseteq M$. Similarly $S \subseteq M$ for every $S \in \mathcal{S}(G) \setminus \{S_0\}$.

Second, consider $P \in \mathcal{P}(G)$. By the first assertion of Proposition 1, $M \cap V(G)$ is a module of $G$. As $M \cap V(G) \supseteq S_0$, it follows from Lemma 2 that either $(M \cap V(G)) \cap P = \varnothing$ or $P \subseteq M \cap V(G)$. Suppose for a contradiction that $(M \cap V(G)) \cap P \subseteq \varnothing$. By the last assertion of Proposition 1, there is $i \in \{0, 1\}$ such that $(P, S_0)_G = i$. As $S_0 \subseteq M$ and $M \cap P = \varnothing$, we obtain $(P, M)_H = i$. Thus $(P, S')_H = i$ and hence $(P, S')_H[C \cup S'] = S'$ which contradicts \((5.3)\) and Lemma 6. It follows that $P \subseteq M$.

Lastly, it follows from \((5.6)\) that $I(G) \subseteq M$. Consequently, $M = V(H)$.  

**Corollary 5.** For every graph $G$ such that $|m(G)| \geq 2$, if $\log_2(m(G))$ is not an integer, then $p(G) = [\log_2(m(G))]$.

**Proof.** It suffices to apply Theorem 2 after recalling that $[\log_2(m(G))] = [\log_2(m(G) + 1)]$ if and only if $\log_2(m(G))$ is not an integer.

Before showing Theorem 2 we observe
Lemma 8. Given a graph $G$, if $\text{Iso}(G) \neq \emptyset$ or $\text{Iso}(\overline{G}) \neq \emptyset$, then

$$p(G) \geq \lceil \log_2(\max(\|\text{Iso}(G)\|,\|\text{Iso}(\overline{G})\| + 1)) \rceil.$$  

Proof. Let $G$ be a graph such that $\max(\|\text{Iso}(G)\|,\|\text{Iso}(\overline{G})\|) > 0$. By interchanging $G$ and $\overline{G}$, assume that $\text{Iso}(G) \neq \emptyset$. Given $p < \lceil \log_2(\|\text{Iso}(G)\| + 1) \rceil$, consider any $p$-extension $H$ of $G$. We have $2^{\|V(H)\|} \leq \|\text{Iso}(G)\|$ and we verify that $H$ is not prime.

For each $u \in \text{Iso}(G)$, we have $N_H(u) \subseteq V(H) \setminus V(G)$. Thus $(N_H)_{\text{Iso}(G)}$ is a function from $\text{Iso}(G)$ to $2^{V(H) \setminus V(G)}$. As previously observed, if $(N_H)_{\text{Iso}(G)}$ is not injective, then $\{u, v\}$ is a non-trivial module of $H$ for $u \neq v \in \text{Iso}(G)$ such that $N_H(u) = N_H(v)$. So assume that $(N_H)_{\text{Iso}(G)}$ is injective. As $2^{\|V(H)\| \setminus V(G)} \leq \|\text{Iso}(G)\|$, we obtain that $(N_H)_{\text{Iso}(G)}$ is bijective. Thus there is $u \in \text{Iso}(G)$ such that $N_H(u) = \emptyset$, that is, $u \in \text{Iso}(H)$. Therefore $H$ is not prime. It follows that $p(G) \geq \lceil \log_2(\max(\|\text{Iso}(G)\|,\|\text{Iso}(\overline{G})\| + 1)) \rceil$. \hfill \Box

We prove Theorem 3 when $m(G) = 2$.

Proposition 6. For every graph $G$ such that $m(G) = 2$,

$$p(G) = 2 \text{ if and only if } \|\text{Iso}(G)\| = 2 \text{ or } \|\text{Iso}(\overline{G})\| = 2.$$  

Proof. By Theorem 2, $p(G) = 1$ or 2. To begin, assume that $\|\text{Iso}(G)\| = 2$ or $\|\text{Iso}(\overline{G})\| = 2$. By Lemma 8, $p(G) \geq 2$ and hence $p(G) = 2$. Conversely, assume that $p(G) = 2$. Let $\alpha \notin V(G)$. As $m(G) = 2$, $|C| = 2$ for each $C \in \mathcal{C}(G) \cup \mathcal{S}(G)$. Let $C_0 \in \mathcal{C}(G) \cup \mathcal{S}(G)$. We consider any graph $H$ defined on $V(G) \cup \{\alpha\}$ and satisfying the following.

1. For each $C \in \mathcal{C}(G) \cup \mathcal{S}(G)$, $\alpha \notin H(C)$.

2. Let $P \in \mathcal{P}(G)$. We have $G[P]$ is prime. Using Lemma 7, we consider for $H[P \cup \{\alpha\}]$ a prime 1-extension of $G[P]$ to $P \cup \{\alpha\}$.

3. Let $v \in I(G)$. There is $i \in \{0, 1\}$ such that $(v, C_0)_G = i$. We require that $(v, \alpha)_H \neq i$.

4. $H[V(G)] = G$.

Since $p(G) = 2$, $H$ admits a non-trivial module $M$. First, we verify that $\alpha \in M$. Otherwise $M$ is a module of $G$. By Remark 1, there is $C \in \mathcal{C}(G) \cup \mathcal{S}(G) \cup \mathcal{P}(G)$ such that $|C \cap M| \geq 2$. Suppose that $C \in \mathcal{C}(G) \cup \mathcal{S}(G)$. As $|C| = 2$, $C \in M$ which contradicts $\alpha \notin H(C)$. Suppose that $C \in \mathcal{P}(G)$. By Lemma 2, $C \cap M = C$. Thus $C$ would be a module of $H[C \cup \{\alpha\}]$ which contradicts the fact that $H[C \cup \{\alpha\}]$ is prime. It follows that $\alpha \in M$.

Second, we show that $P \subseteq M$ for each $P \in \mathcal{P}(G)$. Since $H[P \cup \{\alpha\}]$ is prime and since $M \cap (P \cup \{\alpha\})$ is a module of $H[P \cup \{\alpha\}]$ with $\alpha \in M \cap (P \cup \{\alpha\})$, we obtain either $(M \setminus \{\alpha\}) \cap P = \emptyset$ or $P \subseteq M \setminus \{\alpha\}$. In the first instance, there is $i \in \{0, 1\}$ such that $(M \setminus \{\alpha\}, P)_G = i$. Therefore $(\alpha, P)_H = i$ which contradicts the fact that $H[P \cup \{\alpha\}]$ is prime. It follows that $P \subseteq M$.  

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Third, we prove that $C \cap M \neq \emptyset$ for each $C \in \mathcal{C}(G) \cup \mathcal{S}(G)$. Otherwise consider $C \in \mathcal{C}(G) \cup \mathcal{S}(G)$ such that $C \cap M = \emptyset$. There is $i \in \{0,1\}$ such that $(M, C)_i = i$. Thus $(\alpha, C)_H = i$ which contradicts $\alpha \not\equiv_H C$.

In particular we have $C_0 \cap M \neq \emptyset$. Let $v \in I(G)$. Since $(v, C_0 \cap M)_G \neq (v, \alpha)_H$, we obtain $v \in M$.

To conclude, consider $v \in V(H) \setminus M$. By what precedes, there is $C \in \mathcal{C}(G) \cup \mathcal{S}(G)$ such that $v \in C$. By interchanging $G$ and $\overline{G}$, assume that $C \in \mathcal{S}(G)$. Since $v \sim_H M$ and $(v, C \cap M)_G = 0$, we obtain $(v, M)_H = 0$. In particular $(v, I(G))_G = 0$ and $(v, P)_G = 0$ for every $P \in P(G)$. Let $D \in (\mathcal{C}(G) \cup \mathcal{S}(G)) \setminus \{C\}$. As $D \cap M \neq \emptyset$, we have $(v, D \cap M)_G = 0$ and hence $(v, D)_G = 0$ because $D$ is a module of $G$. It follows that $v \in \text{Iso}(G)$. Therefore $(C, V(G) \setminus C)_G = 0$ because $C$ is a module of $G$. Since $C$ is a stable set in $G$, we obtain $C \subseteq \text{Iso}(G)$. Clearly $\text{Iso}(G)$ is a module of $G$ and a stable set in $G$. Thus $|\text{Iso}(G)| \leq m(G) = 2$. Consequently $C = \text{Iso}(G)$.

We use the following notation in the proof of Theorem 3. Given a graph $G$ such that $m(G) \geq 3$, set

$$C_{\max}(G) = \{C \in \mathcal{C}(G) : |C| = m(G)\},$$

$$S_{\max}(G) = \{S \in \mathcal{S}(G) : |S| = m(G)\}.$$

Proof of Theorem 3. Consider a graph $G$ such that $m(G) = 2^k$ where $k \geq 1$. By Theorem 2, $p(G) = k$ or $p(G) = k + 1$. To begin, assume that $|\text{Iso}(G)| = 2^k$ or $|\text{Iso}(G)| = 2^{k+1}$. By Lemma 8, $p(G) \geq k + 1$ and hence $p(G) = k + 1$.

Conversely, assume that $p(G) = k + 1$. If $k = 1$, then it suffices to apply Proposition 6. So assume that $k \geq 2$. With each $C \in C_{\max}(G) \cup S_{\max}(G)$ associate $w_C \in C$. Set $W = \{w_C : C \in C_{\max}(G) \cup S_{\max}(G)\}$ and $G' = G[V(G) \setminus W]$.

We prove that $m(G') = 2^k - 1$. Given $C \in C_{\max}(G) \cup S_{\max}(G)$, $C \setminus \{w_C\}$ is a module of $G'$ and $C \setminus \{w_C\}$ is a clique or a stable set in $G'$. Thus $2^k - 1 \leq |C \setminus \{w_C\}| \leq m(G')$. Consider $C' \in C_{\max}(G') \cup S_{\max}(G')$. We show that $C'$ is a module of $G$. We have to verify that for each $C \in C_{\max}(G) \cup S_{\max}(G)$, $w_C \sim_G C$.

First, assume that there is $c \in (C \setminus \{w_C\}) \setminus C'$. We have $c \sim_G C'$. Furthermore $(c, w_C)$ is a module of $G$. Thus $w_C \sim_G C'$. Second, assume that $C \setminus \{w_C\} \subseteq C'$. Clearly $w_C \sim_G C'$ when $C' \subseteq C \setminus \{w_C\}$. Otherwise assume that $C' \setminus (C \setminus \{w_C\}) \neq \emptyset$. By interchanging $G'$ and $\overline{G}$, assume that $C'$ is a clique in $G'$. As $C \setminus \{w_C\} \subseteq C'$ and $|C \setminus \{w_C\}| \geq 2$, we obtain that $C$ is a clique in $G$. Since $(C \setminus \{w_C\}, C' \setminus C)_G = 1$ and since $C'$ is a module of $G$, we have $(w_C, C' \setminus C)_G = 1$. Furthermore $(w_C, C \setminus \{w_C\})_G = 1$ because $C$ is a clique in $G$. Therefore $(w_C, C')_G = 1$. Consequently $C'$ is a module of $G$. As $C'$ is a clique or a stable set in $G$, there is $C \in \mathcal{C}(G) \cup \mathcal{S}(G)$ such that $C \sim_G C'$. If $C' \notin C_{\max}(G) \cup S_{\max}(G)$, then $|C'| < |C| < m(G)$. If $C' \in C_{\max}(G) \cup S_{\max}(G)$, then $C' \subseteq C \setminus \{w_C\}$ and hence $|C'| < |C| = m(G)$. In both cases, we have $|C'| = m(G') < m(G)$. It follows that $m(G') = 2^k - 1$.

By Corollary 7, $p(G') = k$ and hence there exists a prime $k$-extension $H'$ of $G'$. We extend $H'$ to $V(H') \cup W$ as follows. Let $C \in C_{\max}(G) \cup S_{\max}(G)$. Consider the function $f_C : C \setminus \{w_C\} \rightarrow 2^{V(H') \setminus V(G')}$ defined by $c \mapsto N_{H'}(c) \setminus
follows that $X$ as well. We would obtain $2^k$. 

**Remark 1.** Consequently, there exists $M$ such that $X_H(u) = \emptyset$. In the first instance, $M \leq W$ and $M$ is a module of $G$ which contradicts Remark 1. Consequently $X \leq M$. As $M$ is a non-trivial module of $H$, there exists $C \in C_{max}(G) \cup S_{max}(G)$ such that $w_C \notin M$. By interchanging $G$ and $H$, it follows that $C$ is a stable set in $G$. We have $(w_C, C)_{G} = 0$ so that $(w_C, M)_{G} = 0$ and $(w_C, G) = 0$. Given $D \in C_{max}(G) \cup S_{max}(G)$, we obtain $(w_C, D)_{G} = 0$. Since $D$ is a module of $G$, $(w_C, D)_{G} = 0$. It follows that $w_C \in Iso(G)$. As at the end of the proof of Proposition 6, we conclude by $C = Iso(G)$.

Lastly, we examine the graphs $G$ such that $m(G) = 1$. For these, $C(G) = S(G) = \emptyset$. Thus either $|V(G)| \leq 1$ or $|V(G)| \geq 4$ and $G$ is not prime.

**Proposition 7.** For every non-prime graph $G$ such that $|V(G)| \geq 4$ and $m(G) = 1$, we have $p(G) = 1$.

**Proof.** Since $m(G) = 1$, we have $C(G) = S(G) = \emptyset$. By Corollary 11, $M(G) = \mathcal{P}(G) \cup \mathcal{I}(G)$. By considering $V(G) \in \mathcal{M}(G)$, it follows from Remark 1 that there is $P_0 \in \mathcal{P}(G)$.

Let $\alpha \notin V(G)$. We consider any graph $H$ defined on $V(G) \cup \{\alpha\}$ and satisfying the following.

1. Let $P \in \mathcal{P}(G)$. We have $G[P]$ is prime. Using Lemma 7 we consider for $H[P \cup \{\alpha\}]$ a prime graph such that $H[P] = G[P]$.

Set $X = P_0 \cup \{\alpha\}$. We have $H[X]$ is prime.

2. Let $v \in I(G)$. Since $P_0$ is a module of $G$ such that $v \notin P_0$, there is $i \in \{0,1\}$ such that $(v, P_0)_{G} = i$. We consider for $H[\{v, \alpha\}]$ the graph such that $(v, \alpha)_{H[\{v, \alpha\}]} = i$.

In the construction above, we have $H[N] = G[N]$ for each $N \in M(G)$. Thus we can also assume that $H[V(G)] = G$. As in the proof of Theorem 2, we verify the following. For every module $M$ of $H$ such that $|M| \geq 2$, we have $\alpha \in M$. Now, we prove that $H$ is prime. Consider a module $M$ of $H$ such that $|M| \geq 2$. By what precedes, $\alpha \in M$ and it follows from Lemma 6 that either $M \cap X = \{\alpha\}$
or \( X \subseteq M \). In the first instance, there is \( i \in \{0,1\} \) such that \( (P_0,M \setminus \{\alpha\}) \neq 1 \). Thus \( (P_0,\alpha) \neq 1 \) which contradicts the fact that \( H[P_0 \cup \{\alpha\}] \) is prime. It follows that \( X \subseteq M \). We conclude as in the proof of Theorem 2. Since \( (v,P_0) \neq (v,\alpha) \) for every \( v \in I(G) \), we have \( I(G) \subseteq M \). Lastly, consider \( P \in P(G) \setminus \{P_0\} \). As \( H[P \cup \{\alpha\}] \) is prime, it follows from Proposition 1 that either \( (M \setminus \{\alpha\}) \cap P = \emptyset \) or \( P \subseteq M \setminus \{\alpha\} \). In the first instance, there is \( i \in \{0,1\} \) such that \( (P,P_0) = i \). Therefore \( (P,\alpha) \neq 1 \) which contradicts the fact that \( H[P \cup \{\alpha\}] \) is prime. Consequently \( P \subseteq M \). \( \diamond \)

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