Determinant and Weyl anomaly of the Dirac operator: a holographic derivation

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Abstract
We present a holographic formula relating functional determinants: the fermion determinant in the one-loop effective action of bulk spinors in an asymptotically locally AdS background and the determinant of the two-point function of the dual operator at the conformal boundary. The formula originates from AdS/CFT heuristics that map a quantum contribution in the bulk partition function to a subleading large-$N$ contribution in the boundary partition function. We use this holographic picture to address questions in spectral theory and conformal geometry. As an instance, we compute the type-A Weyl anomaly and the determinant of the iterated Dirac operator on round spheres, express the latter in terms of Barnes’ multiple gamma function and gain insight into a conjecture by Bär and Schopka.

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1. Introduction

Ever since its appearance almost 15 years ago in the form of Maldacena’s conjecture, the AdS/CFT correspondence [1–3] has been a successful tool to address questions concerning strongly coupled systems. Many developments depart from the original canonical formulation in pure anti-de Sitter (AdS) spacetime (mostly still restricted to classical (super)gravity in the bulk); for instance, finite-temperature effects on the boundary theory led to consideration of AdS black holes as bulk background geometries. This novel holographic approach, phenomenological in nature, covers an increasing number of physical situations ranging from strongly coupled quark–gluon plasma and condensed matter systems to cosmological singularities and black hole physics. At present, this ambitious but conjectural program seems to succeed at a qualitative level (cf [4]).

In contrast, quantitative and exact results in AdS/CFT correspondence are to be found in the interplay with mathematics, in for instance conformal geometry and spectral theory. Geometric roots of AdS/CFT date back to the seminal work of Fefferman and Graham [5, 6]
that addresses conformal geometry on a compact manifold as the geometry at the conformal infinity of space-filling Poincaré metrics. Conversely, AdS/CFT revealed interesting conformal invariants, e.g. $Q$-curvature, that arise in the volume renormalization of these Poincaré metrics and triggered new developments in conformal geometry (cf [7, 8]).

The present contribution will focus precisely on these latter aspects of the duality, where the foreseeable progress seems modest but solid. We deal with ‘holographic formulas’ as special entries in the AdS/CFT dictionary, relating one-loop determinants for bulk fields in asymptotically AdS backgrounds and determinants of correlation functions of the dual operators at the boundary. They originate in quantum refinements of the duality where one-loop corrections on the gravity side are mapped to sub-leading terms in the large-$N$ expansion of the boundary theory. Interesting effects in for instance thermodynamics and transport phenomena on the boundary are captured by the holographic correspondence only after inclusion of quantum one-loop effects in the bulk (cf [9, 10]). A systematic study of bulk scalars has led to a holographic formula which has been verified in certain cases amenable to analytic evaluation; these bulk geometries include pure and thermal AdS, the BTZ black hole and other quotients or orbifolds of AdS [11–14].

Our aim now is to show that also for bulk spinors an analogous holographic formula can be established. Explicit computations are performed for the ball model of hyperbolic space which embrace several scattered results in the literature. In this case, the bulk side and the role of Barnes’ gamma function, already explored in [15, 16], can be further exploited to get a closed formula for the determinant of the Dirac operator on round spheres, an interesting result that seems to have escaped notice. A universal formula for the type-A trace anomaly [17] of the Dirac operator is also obtained in this holographic way. These explicit results contain previous ones found in relation to proposals for a $c$-theorem in dimensions other than 2; they include Cardy’s $a$-theorem [18, 19], universal terms in entanglement entropy [20, 21] and F-theorem [22].

We start in section 2 with a review of bulk spinors in AdS/CFT and its double quantization to predict an $O(1)$ quantum contribution to the partition functions. Next we analyze the dual picture at the boundary in section 3 in order to detect this $O(1)$ contribution to the partition function on the CFT side. In section 4, we write down the spinor holographic formula. In section 5, the pure AdS bulk geometry is considered as an instance where both sides of the formula can be worked out in detail. Section 6 is concerned with the Dirac operator at the boundary and the application of the holographic formula to read off the universal part in the associated Polyakov formulas, the type-A trace anomaly as well as the functional determinant on round spheres. In section 7, we examine several scattered results closely related to our calculations. Concluding remarks are given in section 8, and conventions and useful identities for Barnes’ gammas are collected in an appendix.

2. Bulk spinors and double quantization

The role of bulk spinors in AdS/CFT has, of course, been extensively studied since the early days of the correspondence; a non-exhaustive list includes [23–26]. To begin with, we choose the bulk side and review the features relevant to our present concern, namely, the spinor version of the holographic formula.

Consider a bulk metric that approaches asymptotically the Poincaré half-space model for the Euclidean section of $\text{AdS}_{n+1}$, that is, hyperbolic space $H^{n+1}$:

$$ds^2 = \frac{dz^2 + d\vec{x}^2}{z^2}.$$ (1)
The solutions of the Dirac equation \((\nabla + m)\psi = 0\), with positive mass \(m > 0\) for definiteness, behave near the conformal boundary \(z = 0\) as

\[
\psi \sim z^\lambda \psi_o(\vec{x}) + z^{-\lambda} \chi_o(\vec{x}),
\]

with \(\lambda_{\pm} = \frac{\lambda}{2} \pm \frac{m}{2}\), and the boundary data \(\psi_o\) and \(\chi_o\) belong to the eigenspace of the flat Dirac gamma associated with the \(z\)-direction, \(\Gamma^n\), with eigenvalues \(-1\) and \(+1\), respectively.

The requirement of regularity at the deep interior, \(z \to \infty\), imposes a linear relation between \(\psi_o\) and \(\chi_o\) given by convolution with the scattering operator \(\chi_o = S(\lambda) * \psi_0\), or equivalently, with the kernel associated with the two-point function of the dual operator at the boundary. It is this on-shell relation which ultimately leads, upon functional differentiation of the action with respect to the boundary source, to the corresponding two-point correlator

\[
\langle O_+ O_+ \rangle \sim \frac{\Gamma^n \cdot (\vec{x} - \vec{y})}{|\vec{x} - \vec{y}|^{1+n+2m}}.
\]

2.1. \(O(1)\) contribution to the partition function

That is very much the picture at the classical level in the bulk. Considering now quantum fluctuations, we go off-shell and there are two possible AdS-invariant quantizations of the bulk spinor: the conventional one with \(\psi_o\) set to zero and another, ‘alternate’ one, with \(\chi_o\) set to zero. Whenever the mass of the bulk spinor lies in the window \(0 < m < \frac{1}{2}\), both kinds of modes are normalizable (finite energy configurations [27]). Their contribution to the partition function can be computed in the standard way via the Green function for the conventional modes \((\lambda_+)\) and analytic continuation to account for the alternate modes \((\lambda_-)\). With this choice at hand, we can emulate the scalar case [28], since now double quantization for bulk spinors is established. The relative change in the partition function, upon functional integration of the quantum fluctuations at quadratic order, is then given by the ratio of the associated functional determinants

\[
\frac{Z_{grav}^+}{Z_{grav}^-} = \frac{\det_{\{\nabla_x + m\}}}{\det_{\{-\nabla_x + m\}}}.
\]

3. Boundary double-trace deformation

These two choices of asymptotic behavior correspond in AdS/CFT to two CFTs [27, 29] that share the same field content but differ in the dimension of the fermionic operator \(O\), dual to the bulk spinor. Despite its appearance, the situation is by no means symmetric; the UV CFT with \(O_-,\) perturbed by the relevant double-trace deformation \(O_2^-\), flows into the IR CFT with \(O_+\) (cf [19, 22, 27]).

\(^1\) Strictly speaking, valid for \(0 < m < \frac{1}{2}\).
3.1. $O(1)$ contribution to the partition function: a shortcut

To get a handle on the relative change in the CFT partition functions at the endpoints of the RG flow, instead of considering the auxiliary field trick as in [19, 22], we simply adapt the heuristic argument given in [30] to relate $Z_{UV}$ to $Z_{IR}$. Namely, in the path integral of the UV CFT, we promote the sources $\eta$ and $\bar{\eta}$ to dynamical fields and integrate over them

$$\frac{Z_{IR}}{Z_{UV}} = \int \mathcal{D}\bar{\eta} \mathcal{D}\eta \exp \left( \bar{\eta} \mathcal{O}_- + \mathcal{O}_- \eta \right).$$

(6)

The expectation value can be approximated, at leading large $N$ due to the factorization of the correlation functions, by

$$\exp \int \bar{\eta} \langle \mathcal{O}_- \mathcal{O}_- \rangle \eta,$$

(7)

and the Gaussian integral results in the functional determinant of the two-point function

$$\frac{Z_{IR}}{Z_{UV}} = \det \langle \mathcal{O}_- \mathcal{O}_- \rangle,$$

(8)

or, alternatively,

$$\frac{Z_{UV}}{Z_{IR}} = \det \langle \mathcal{O}_+ \mathcal{O}_+ \rangle.$$

(9)

4. The holographic formula

We have now all necessary ingredients to write down the ‘spinor holographic formula’ that stems from the postulated equality of the partition functions in AdS/CFT correspondence, at subleading order $O(1)$:

$$\frac{\det_- \{ \nabla X + m \}}{\det_+ \{ \nabla X + m \}} = \det \langle \mathcal{O}_- \mathcal{O}_- \rangle_M.$$

(10)

The ‘+’ means that we compute with the standard $\lambda$ in the asymptotic behavior of the bulk spinor, whereas ‘−’ means the analytic continuation $\lambda \to n - \lambda$. As usual, to make sense out of this formula, one needs to tame the divergencies that arise due to the IR divergent volume of AdS and the UV divergent short-distance singularities. Both divergencies turn out to be tied by the IR/UV connection in AdS/CFT correspondence [31].

This formula conjecturally applies to bulk geometries $X$ which are Euclidean sections of asymptotically locally AdS (ALAdS). In particular, when the conformal infinity $M$ belongs to the conformal class of the standard round spheres, and therefore conformally flat, the bulk is locally AdS and the IR-divergent volume of AdS factorizes. One can then read off an $O(1)$ contribution to the holographic trace anomaly in even $n$, just as in the case of a bulk scalar [3, 7, 32]. Alternatively, from the difference of one-loop effective actions, one can compute the holographic type-A trace anomaly coefficient $a$ following the general recipe of [33]. The behavior of this coefficient for even $n$ and of a related quantity for odd $n$, in the cases we will explore, gives support to a conjectured c-theorem valid in all dimensions [21]$^2$ and to an F-theorem proposal [22] as well.

$^2$ This promises to settle the disparity pointed out in [34] that although the computation on the AdS side contemplates even and odd dimensions on equal footing, it is not clear how to translate the ‘holographic’ central charge into field theory language in the case of odd-dimensional CFTs.
5. The canonical case

As might be expected, the ball model for hyperbolic space (Euclidean AdS) turns out to be the simplest bulk background where calculations can be spelled out in detail and related to the CFT on the conformal boundary (the conformally flat class of the standard round sphere).

5.1. Bulk

The effective action for a Dirac spinor in hyperbolic space has been recently revisited [15, 16] in connection with a curious gauge–gravity duality where Barnes’ multiple gamma function plays a central role. We briefly survey the relevant steps in the computation of the effective action

\[ \mathcal{S}^+_{\text{grav}} = -\log \det (\nabla \mp m). \]

(11)

In terms of the Green function, one has \((\nabla \mp m)D = -\mathbb{I}:

\[ \mathcal{S}^+_{\text{grav}} = \int m \text{tr} D^{(n+1)_\uparrow}, \]

(12)

where also the spinor indices are traced out. There is a subtlety regarding the dimensionality of the representations of the gamma matrices: for \(n\) odd, bulk and boundary representations share the same dimensionality, whereas for \(n\) even, in order to have a Dirac fermion on the boundary, the dimensionality of the bulk representation must be doubled [27].

We refer to [15, 16] for details of the implementation of dimensional regularization and the nontrivial role of the bulk volume. In all, one gets a remarkable result, valid for both even and odd dimensions, in terms of Barnes’ multiple gamma

\[ \log \frac{\det_{+} (\nabla \mp m)}{\det_{-} (\nabla \mp m)} = -2^{1+|\frac{n}{2}|} \cdot \log \frac{\Gamma_{n+1} \left( \frac{n+1}{2} + m \right)}{\Gamma_{n+1} \left( \frac{n+1}{2} - m \right)}. \]

(13)

In addition, whenever \(n\) is even, one can read off the trace anomaly as in the scalar case [11, 12]. In the present case, one essentially gets the integral of the spinor Plancherel measure (cf [35]) times the volume anomaly \(L_{n+1} = 2^{(-\pi)^2/\Gamma(1 + \frac{n}{2})}:

\[ \left[ \frac{2}{(2\pi)^{\frac{n}{2}}} \int_0^m d\mu \left( \frac{1}{\mu} + \frac{1}{\frac{n}{2} + \mu} \right) \cdot \left( \frac{1}{\frac{n}{2} - \mu} \right) \right] \cdot L_{n+1}. \]

(14)

5.2. Boundary

For the round \(n\)-sphere as the conformal boundary, the knowledge of the eigenvalues of the two-point function \(\langle \mathcal{O}, \overline{\mathcal{O}} \rangle_n\) and their degeneracies allows a brute-force computation of the corresponding functional determinant. The appropriate basis is that of spinor spherical harmonics, and the eigenvalues\(^3\) and degeneracies have been recently computed in connection with fermionic double-trace deformations [19]

\[ \text{eigenvalues: } \pm \frac{\Gamma(l + n/2 + \nu + 1/2)}{\Gamma(l + n/2 - \nu + 1/2)}, \]

(15)

\[ \text{degeneracies: } 2^{1+|\frac{n}{2}|} \frac{(l + n - 1)!}{l! (n-1)!}. \]

(16)

\(^3\) Difference of the effective Lagrangians. The shorthand notation involves Pochhammer’s symbol \((x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}\).

\(^4\) The eigenvalues can also be read off from the scattering problem in \(H^{n+1}\) [36].
The formal trace is then assembled as follows:

\[
\log \det \langle O_\lambda O_\lambda \rangle_{S^n} = 2^{1 + \frac{1}{2} n} \sum_{l=0}^{\infty} \frac{(n)_l l!}{l!} \log \frac{\Gamma(l + \frac{n+1}{2} + \nu)}{\Gamma(l + \frac{n+1}{2} - \nu)},
\]

and it can be worked out within dimensional regularization as in [11, 22]:

\[
\log \det \langle O_\lambda O_\lambda \rangle_{S^n} = -2^{1 + \frac{1}{2} n} \Gamma(-n) \int_0^\nu d\mu \left\{ \frac{\Gamma\left(\frac{n+1}{2} + \mu\right)}{\Gamma\left(\frac{n+1}{2} - \mu\right)} + (\mu \to -\mu) \right\}.
\]

This very same regularized answer has already been found in [16] (equation (5)), where Barnes’s multiple gamma turned up. The renormalized value can be written, modulo polynomial and logarithmic terms (we refer again to [16] for further details and equation (6)), as the following quotient of Barnes’ multiple gamma functions:

\[
\log \det \langle O_\lambda O_\lambda \rangle_{S^n} = 2^{1 + \frac{1}{2} n} \log \frac{\Gamma\left(n + \frac{n+1}{2} + \nu\right)}{\Gamma\left(n + \frac{n+1}{2} - \nu\right)}.
\]

As explained in [16], the gamma factor in front of (18) is deceiving: only for \( n \) even is there a pole and from the residue one can read off the (integrated) conformal anomaly under conformal rescaling of the metric

\[
\frac{2^{2+\frac{3}{2} n} (-1)^n/2}{n!} \int_0^\nu d\mu \left( \frac{1}{2} + \mu \right)^{\frac{1}{2} n} \cdot \left( \frac{1}{2} - \mu \right)^{\frac{1}{2} n}.
\]

At this point, we have a perfect match between bulk and boundary computations. To illustrate the usefulness of this result, besides being an explicit corroboration of the holographic formula, we study a particular value of the spinor mass that unveils the Dirac operator on the boundary and connects with a vast mathematical literature on determinants of differential operators on spheres (cf [37–43]).

6. Holographic life of the Dirac operator

There are two direct ways to identify the Dirac operator at the boundary.

- First, consider the action of the Dirac operator on the flat space two-point function (equation (3))

\[
\mathcal{V} \langle O_+ (\vec{x}) O_+ (\vec{0}) \rangle_{S^n} \sim \frac{1}{|\vec{x}|^{n+1+2m}}.
\]

Here one can recognize the Laplacian \( \mathcal{V}^2 \) in the limit \( m \to \frac{1}{2} \), in a distributional sense.

- Second, by simple inspection of the eigenvalues of the kernel \( \langle O_+ O_+ \rangle_{S^n} \) on the \( n \)-sphere (equation (15)) in the same limit \( m \to \frac{1}{2} \)

\[
\pm \left( \frac{n}{2} + l \right).
\]
6.1. Type-A trace anomaly and Polyakov formulas

We are interested in the universal type-A component of the trace anomaly, the coefficient of the Euler density or Pfaffian according to the classification in [17]. For conformally related metrics \( \hat{g} = e^{2w} g \), in the conformally flat class, Branson (see, e.g., [37]) conjectured the following Polyakov-like formula:

\[
- \log \det \frac{\hat{\nabla}^2}{\nabla^2} = c(n) \int_M w(\hat{Q}_n \, dv_{\hat{g}} + Q_n \, dv_g) + \cdots ,
\]

where the Pfaffian is traded by Branson’s \( Q \)-curvature \( Q_n \), a central object in conformal geometry with a much simpler (in fact, linear in \( w \)) transformation rule under conformal rescalings.

We read this very same structure from the bulk computation, the \( Q \)-curvature terms come from the finite conformal variation of the renormalized volume [44–46] and the overall coefficient \( c(n) / \nabla^2 \) is obtained from the effective Lagrangian at \( m = \frac{1}{2} \):

\[
\frac{c(n)}{\nabla^2} = 4 \cdot \frac{c_k}{(2\pi)^2} \int_0^{\frac{1}{2}} d\nu \left( \frac{1}{2} + \nu \right)^{\frac{1}{2}} \cdot \left( \frac{1}{2} - \nu \right)^{\frac{1}{2}} \frac{\Gamma_{\nu/2} \Gamma_{1/2}}{\Gamma_{\nu+1/2} \Gamma_{\nu-1/2}} ,
\]

where \( c_k = \frac{(-1)^k}{2\pi^2 k!} \). All values reported in [37] for \( c(n) / \nabla^2 \) are correctly reproduced by this formula. Interesting spectral invariants, such as the zeta function at zero argument and conformal index (cf [37, 12]), are easily obtained from this coefficient.

6.2. Determinant of iterated Dirac on spheres

The particular value of the determinant of the scattering operator at the mass value \( \frac{1}{2} \) results in the following remarkable expression in terms of Barnes’ multiple gamma function, after use of recurrence (A.3),

\[
- \log \det \frac{\hat{\nabla}^2}{\nabla^2} = 4 \cdot 2^{\frac{1}{2}} \cdot \log \Gamma_{\frac{n}{2}} \left( \frac{n}{2} \right) .
\]

Barnes’ multiple gamma function is known to occur in functional determinants of Laplacians on spheres (see, e.g., references in [40]). And yet this compact expression, which correctly reproduces all zeta-regularized values reported in the literature (cf [39, 43]), does not seem to have been noticed until now.

A small digression on a conjecture is put forward by Bär and Shopka [39]: they observed that the numerical values of these determinants tend to 1 as the dimension \( n \) grows; this was proved in [41] not only for the Dirac operator, but also for the Yamabe or conformal Laplacian on the \( n \)-sphere. Our result indicates that the above findings amount to establishing the limiting value of Barnes’ gamma \( \Gamma_{\nu}(n/2) \) as \( n \to \infty \). Furthermore, we can interpret this limiting value quite naturally by looking at the bulk side of the holographic formula: the two boundary conditions coincide as \( n \) grows, or alternatively, \( \lambda_{\pm} \to \frac{2}{n} \) so that both determinants in the quotient approach one another. This very same argument would predict the same limiting value for all other operators on the right side of similar holographic formulas; this is the case for the determinants of GJMS operators on round spheres [12]. As a consequence, we predict the same limiting value of unity, a prediction that can be probed by the asymptotic analysis of the explicit results obtained in [42] (see equation (19) therein). In fact, the quotient of Barnes’s gammas in the limit of large dimension \( d \) and fixed order \( k \) tends to 1, as was the case for the Dirac operator; at the same time, the reported numerical values for the multiplicative anomaly \( M(d, k) \) hint at a vanishing limit value, so that the exponential of both contributions in equation (19) of [42] should again render 1 in the limit.
7. C-theorem proposals and entanglement entropy

Our computations contain and, at times, generalize several scattered results that had been independently derived, most of them in pursuit of extensions of the C-theorem to higher dimensions and in connection with certain universal terms in entanglement entropy. We briefly list a few of them.

*Holographic C-charge at $\mathcal{O}(1)$*

- $n = \text{even}$, relative change at order $\mathcal{O}(1)$ in Cardy’s central charge computed in [19]. $C_{\text{UV}} - C_{\text{IR}} > 0$ in the mass window $0 < m \leq \frac{1}{2}$. Agreement with the universal log-term in entanglement entropy [21].
  We find agreement between bulk and boundary outcomes—tables 1 and 2 in [19]—and our equations (14) and (20), respectively. Our calculation accounts for the overall coefficient as well.
- $n = \text{odd}$, relative change at order $\mathcal{O}(1)$ in a one-loop effective action, candidate for central charge, computed in [19]. $C_{\text{UV}} - C_{\text{IR}} > 0$ in the mass window $0 < m \leq \frac{1}{2}$. Agreement with the universal constant term in entanglement entropy [21].
  We find agreement between bulk outcome—integral of equation (3.37) in [19]—and our equation (13). We are also able to fill the ‘hole’ left in [19] by identifying this finite contribution on the CFT side, including the overall coefficient as well (equation (19)).

*F-coefficient of odd-dimensional CFTs*

- $F$-coefficients for free massless Dirac fermions on odd-spheres, table 2 in [22].
  This agrees with our result in equation (25) and, of course, with the values reported in [39, 41]. The decrease of the numerical values in the above table is precisely the hint for the conjecture by Bär and Schopka.
- Under RG flow triggered by a fermionic double-trace deformation, at leading large $N$, the change in free energy in three dimensions is given by equation (82) in [22]. $F_{\text{UV}} - F_{\text{IR}} > 0$ in the mass window $0 < m \leq \frac{1}{2}$.
  This coincides again with our bulk (equation (13)) and boundary (equation (19)) results.

*Entanglement entropy*

- Universal terms in entanglement entropy [21]: logarithmic and constant in even and odd dimensions, respectively. In the case of a free boson, they coincide with the holographic anomaly and determinant of the Yamabe operator or conformal Laplacian, for even [42] and odd [47] dimensions, respectively.
  This matching should hold as well for the entanglement entropy of a free Dirac spinor and the trace anomaly and determinant of the Dirac operator on $n$-spheres. Now under the guise of holographic C-charge at $\mathcal{O}(1)$, in the notation of [21], they verify $(a'^{2}\bar{a})_{\text{UV}} > (a'^{2}\bar{a})_{\text{IR}}$.

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6 The evaluation of the corresponding Barnes’ gammas can be performed with the relations collected in the appendix, and one can rewrite in terms of Riemann zeta by the use of the relation $\zeta'(-2n) = \frac{-1}{2} \frac{\Gamma(2n)}{\pi^{2n}} \zeta(1 + 2n)$. 8
8. Conclusion

Our main contribution has been the spinor version of the holographic formula that connects the functional determinant of a bulk spinor with that of the two-point function of the dual operator at the boundary. The case of pure AdS allows for explicit computations which, in turn, encompass several results independently derived in other contexts. In particular, contact is made in the case of the Dirac operator on the boundary; here we have obtained a compact expression for the determinant on spheres in terms of Barnes’ gamma function as well as the generic type-A trace anomaly in any even dimension.

We have also gained insight into the conjecture by Bär and Schopka and unveiled its possible holographic roots. It seems plausible that the conjecture should also apply to the functional determinant of any conformally covariant differential operator, provided a corresponding holographic formula exits. Natural candidates for the bulk side of such holographic formula are the functional determinants in the one-loop effective action of higher-spin fields (see, e.g., the recent progress in [48]).

Further study of the holographic formula, for instance in black-hole backgrounds, remains a challenge. It seems worth exploring the connection of the functional determinants involved in the formula with regularized products of quasinormal frequencies [9] or scattering resonances. Another possible avenue concerns an intriguing feature of Barnes’ double gamma at integer values, it equals the result of ordinary determinants of Hankel matrices; this might well be a sign for localization of the functional integrals that we have been computing in AdS (see, in this respect, [49]).

We conclude by noting that the explicit expressions, as obtained via the holographic approach, come out in a gracious, simple form. This was the case for the trace anomaly of GJMS operators [12] and now for the determinant and trace anomaly of the Dirac operator.

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Appendix. Barnes’ multiple gamma: disambiguation

There are several choices for the normalization of Barnes’ multiple gamma function \( \Gamma_n(z) \). For definiteness, we stick to the convention in [43, 50, 51] and list a few properties that are relevant to our calculations.

- Its logarithm can be written in terms of derivatives of Hurwitz zeta function

  \[
  \log \Gamma_n(z) = \sum_{k=0}^{n-1} b_{n,k}(z) \cdot \zeta'(-k, z),
  \]

  \( (A.1) \)

  where \( b_{n,k}(z) \) is a polynomial in \( z \) with Stirling numbers of the first kind \( s(n, j) \) in its coefficients

  \[
  b_{n,k}(z) = \frac{(-1)^{n-1-k}}{(n-1)!} \sum_{j=k}^{n-1} \binom{j}{k} \cdot s(n, j+1) \cdot z^{j-k}.
  \]

  \( (A.2) \)
• Recurrence or ladder relation:

\[ \Gamma_{n+1}(1 + z) = \frac{\Gamma_n(z)}{\Gamma_{n+1}(1 + z)} \]  

(A.3)

• Pascal triangle by successive applications of the ladder relation:

\[ \log \Gamma_n(m + z) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} \cdot \log \Gamma_{n-i}(z), \quad 0 \leq m \leq n - 1. \]  

(A.4)

• Particular values in terms of derivatives of the Riemann zeta function:

\[ \log \Gamma_n(1) = \sum_{k=0}^{n-1} b_{n,k}(1) \cdot \zeta'(-k), \]  

(A.5)

\[ \log \Gamma_n \left( \frac{1}{2} \right) = \sum_{k=0}^{n-1} b_{n,k} \left( \frac{1}{2} \right) \cdot (2^{-k} - 1) \cdot \zeta'(-k) - \log 2 \sum_{k=0}^{n-1} b_{n,k} \left( \frac{1}{2} \right) \frac{2^{-k} B_{k+1}}{k+1}, \]  

(A.6)

where \( B_n \) are Bernoulli numbers.

References

[1] Maldacena J 1998 The large-N limit of superconformal field theories and supergravity Adv. Theor. Math. Phys. 2 231–52 (arXiv:hep-th/9711200)

[2] Gabser S S, Klebanov I R and Polyakov A M 1998 Gauge theory correlators from non-critical string theory Phys. Lett. B 428 105–14 (arXiv:hep-th/9802109)

[3] Witten E 1998 Anti-de Sitter space and holography Adv. Theor. Math. Phys. 2 253–91 (arXiv:hep-th/9802150)

[4] Karch A 2011 Recent progress in applying gauge/ gravity duality to quark–gluon plasma physics arXiv:1108.4014 [hep-ph]

[5] Fefferman C and Graham C R 2002 Q-curvature and Poincaré metrics Rend. Circ. Mat. Palermo (2) 74 31–42 (arXiv:math.DG/0010042)

[6] Alm´on 2005 Renormalizing curvature integrals on Poincaré–Einstein manifolds arXiv:math.DG/0504161

[7] Denef F, Hartnoll S A and Sachdev S 2010 Black hole determinants and quasinormal modes Class. Quantum Grav. 27 125001 (arXiv:0908.2657 [hep-th])

[8] Bolognesi S and Tong D 2011 Magnetic catalysis in AdS3 arXiv:1110.5902v1 [hep-th]

[9] Diaz D E and Dorn H 2007 Partition functions and double-trace deformations in AdS/CFT J. High Energy Phys. JHEP07(2007)046 (arXiv:hep-th/0702163)

[10] Diaz D E and Dorn H 2009 Holographic formula for the determinant of the scattering operator in thermal AdS J. Phys. A: Math. Theor. 42 365404 (arXiv:0903.0571 [hep-th])

[11] Diaz D E 2009 Holographic formula for the determinant of the scattering operator in thermal AdS J. Phys. A: Math. Theor. 42 365404 (arXiv:0903.0571 [hep-th])

[12] Diaz D E 2009 Holographic formula for the determinant of the scattering operator in thermal AdS J. Phys. A: Math. Theor. 42 365404 (arXiv:0903.0571 [hep-th])

[13] Aros R and Diaz D E 2010 Functional determinants, generalized BTZ geometries and Selberg zeta function J. Phys. A: Math. Theor. 43 205402 (arXiv:1009.0029 [gr-qc])

[14] Basar G and Dunne G V 2010 A gauge-gravity relation in the one-loop effective action J. Phys. A: Math. Theor. 43 072002 (arXiv:0912.1260 [hep-th])

[15] Aros R, Diaz D E and Montecinos A 2010 A note on a gauge-gravity relation and functional determinants J. Phys. A: Math. Theor. 43 295401 (arXiv:1004.1394 [hep-th])

[16] Deser S and Schwimmer A 1993 Geometric classification of conformal anomalies in arbitrary dimensions Phys. Lett. B 309 279–84 (arXiv:hep-th/9302047)

[17] Cardy J L 1988 Is there a C theorem in four-dimensions? Phys. Lett. B 215 749–52

[18] Allais A 2010 Double-trace deformations holography and the c-conjecture J. High Energy Phys. JHEP11(2010)040 (arXiv:1007.2047 [hep-th])

[19] Dowker J S 2010 Entanglement entropy for even spheres arXiv:1009.3854 [hep-th]
[21] Myers R C and Sinha A 2011 Holographic c-theorems in arbitrary dimensions J. High Energy Phys. JHEP01(2011)125 (arXiv:1011.5819 [hep-th])
[22] Klebanov I R, Pufu S S and Saffai B R 2011 F-theorem without supersymmetry arXiv:1105.4598 [hep-th]
[23] Henningson M and Sfetsos K 1998 Spinors and the AdS/CFT correspondence Phys. Lett. B 431 63 (arXiv:hep-th/9803251)
[24] Mueck W and Viswanathan K S 1998 Conformal field theory correlators from classical field theory on anti-de Sitter space: 2. Vector and spinor fields Phys. Rev. D 58 106006 (arXiv:hep-th/9805145)
[25] Henneaux M and Sfetsos K 1998 Spinors and the AdS/CFT correspondence Phys. Lett. B 431 63 (arXiv:hep-th/9805144)
[26] Arutyunov G E and Frolov S A 1999 The origin of supergravity boundary terms in the AdS/CFT correspondence Nucl. Phys. B 544 576 (arXiv:hep-th/9806216)
[27] Laia J N and Tong D 2011 Flowing between fermionic fixed points arXiv: 1108.2216 [hep-th]
[28] Breitenlohner P and Freedman D Z 1982 Stability in gauged extended supergravity Ann. Phys. 144 249
[29] Klebanov I R and Witten E 1999 AdS/CFT correspondence and symmetry breaking Nucl. Phys. B 556 89–114 (arXiv:hep-th/9903204)
[30] Witten E 2003 SL(2, Z) action on three-dimensional conformal field theories with Abelian symmetry From Fields to Strings ed M Shifman et al (Singapore: World Scientific) pp 1173–200 (arXiv:hep-th/0307041)
[31] Henningson M and Skenderis K 1998 The holographic Weyl anomaly J. High Energy Phys. JHEP01(1998)023 (arXiv:hep-th/9806087)
[32] Henningson M and Skenderis K 2000 Holography and the Weyl anomaly Fortschr. Phys. 48 125
[33] Imbimbo C, Schwimmer A, Theisen S and Yankielowicz S 2000 Diffeomorphisms and holographic anomalies Class. Quantum Grav. 17 1129–38 (arXiv:hep-th/9910267)
[34] Gubser S S and Mitra I 2003 Double-trace operators and one-loop vacuum energy in AdS/CFT Phys. Rev. D 67 064018 (arXiv:hep-th/0210093)
[35] Camporesi R 1992 The spinor heat kernel in maximally symmetric spaces Commun. Math. Phys. 148 283
[36] Camporesi R and Higuchi A 1996 On the eigenfunctions of the Dirac operator on spheres and real hyperbolic spaces J. Geom. Phys. 20 1–18 (arXiv:hep-th/9509009)
[37] Branson T 1993 The Functional Determinant, Global Analysis Research Center (Lecture Note Series no 4) (Seoul: Seoul National University)
[38] Quine J R and Choi J 1996 Zeta regularized products and functional determinants on spheres Rocky Mt J. Math. 26 719–29
[39] Bär C and Schopka S 2003 The Dirac Determinant of Spherical Space Forms (Geometric Analysis and Nonlinear PDEs) (Berlin: Springer) pp 39–67
[40] Dowker J S and Kirsten K 2005 The Barnes zeta function, sphere determinants and Gluisher–Kinkel–Bendersky constants Anal. Appl. 3 45–68 (arXiv:hep-th/0301143)
[41] Møller N M 2005 Dimensional asymptotics of determinants on Sn and proof of Bär–Schopka conjecture Math. Ann. 334 35–51
[42] Dowker J S 2011 Determinants and conformal anomalies of GJMS operators on spheres J. Phys. A: Math. Theor. 44 115402 (arXiv:1010.0566 [hep-th])
[43] Yamasaki Y 2010 Factorization formulas for higher depth determinants of the Laplacian on the n-sphere arXiv:1011.3095 [math.NT]
[44] Chang A, Qing J and Yang P 2005 On the renormalized volumes for conformally compact Einstein manifolds arXiv:math.DG/05051276
[45] Carlip S 2005 Dynamics of asymptotic diffeomorphisms in (2+1)-dimensional gravity Class. Quantum Grav. 22 3055–60 (arXiv:gr-qc/0501033)
[46] Aros R, Romo M and Zamorano N 2007 Conformal gravity from AdS/CFT mechanism Phys. Rev. D 75 067501 (arXiv:hep-th/0612028)
[47] Dowker J S 2010 Entanglement entropy for odd spheres arXiv:1012.1548 [hep-th]
[48] Gaberdiel M R, Gopakumar R and Saha A 2011 Quantum W-symmetry in AdSs J. High Energy Phys. JHEP02(2011)004 (arXiv:1009.6807 [hep-th])
[49] Dabholkar A, Gomes J and Murthy S 2011 Localization and exact holography arXiv:1111.1161 [hep-th]
[50] Ruijsenaars S N M 2000 On Barnes’ multiple zeta and gamma functions Adv. Math. 156 107–32
[51] Friedman E and Ruijsenaars S N M 2004 Shintani–Barnes zeta and gamma functions Adv. Math. 187 362–95