An application of spectral localization to the critical SQG on a ball

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Abstract. We study the Cauchy problem for the quasi-geostrophic equations in a unit ball of the two-dimensional space with the homogeneous Dirichlet boundary condition. We show existence and uniqueness of the strong solution in the framework of Besov spaces. We also establish a spectral localization technique and commutator estimates.

1. Introduction

We consider the surface quasi-geostrophic equation in a unit ball.

\[ \partial_t \theta + (u \cdot \nabla) \theta + \Lambda_D \theta = 0, \quad u = \nabla \perp \Lambda_D^{-1} \theta, \quad t > 0, x \in B, \]

\[ \theta(0, x) = \theta_0(x), \quad x \in B, \]

where \( B := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 1\} \), \( \nabla \perp := (-\partial_{x_2}, \partial_{x_1}) \), and \( \Lambda_D \) is the square root of the Dirichlet Laplacian. The equations are known as an important model in geophysical fluid dynamics, which is derived from general quasi-geostrophic equations in the special case of constant potential vorticity and buoyancy frequency (see [23,25]).

The purpose of this paper is to establish well-posedness.

Let us recall several known results for when the space is the whole space \( \mathbb{R}^2 \). If we consider the fractional Laplacian of order \( \alpha \), \( (-\partial^2_x)^{\alpha/2} \), with \( 0 < \alpha \leq 2 \), instead of \( (-\partial^2_x)^{1/2} \), then the case when \( \alpha < 1 \), \( \alpha = 1 \), \( \alpha > 1 \) are called supercritical case, critical case, and subcritical case, respectively. It is known that global-in-time regularity is obtained for the subcritical case and the critical case. The subcritical case can be treated by \( L^\infty \)-maximum principle, but the critical case is delicate. In the critical case, regularity with small data was proved by Constantin, Cordoba, and Wu [2] (see also Constantin and Wu [9]). We also refer to the approach in the framework of Besov spaces to [13,16,30]. The problem in the large data case was solved by Caffarelli and Vasseur [1] and Kiselev, Nazarov and Volberg [22]. As another approach, Constantin and Vicol [8] proved global regularity by nonlinear maximum principles in the form of a nonlinear lower bound on the fractional Laplacian. On the other hand, in the

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supercritical case, blow-up for smooth solutions is an open problem and the regularity is only known for small data (see, e.g., [10]).

In bounded domains with smooth boundary, the equations were introduced by Constantin and Ignatova ([3,4]). Let us focus on the critical case. Local existence was shown by Constantin and Nguyen [7], and global existence of weak solutions was proved by Constantin and Ignatova [4] for the critical case (see also the paper by Constantin and Nguyen [6] for the inviscid case). An interesting question here is how to understand the behavior of the solutions; a priori bounds of smooth solutions were obtained by Constantin and Ignatova [3], and interior Lipschitz continuity of weak solutions was studied by Ignatova [12]. Recently, Constantin and Ignatova [5] considered the quotient of the solution by the first eigenfunction to investigate near the boundary and gave a condition to obtain global regularity up to the boundary. Stokols and Vasseur [27] constructed global-in-time weak solutions with Hölder regularity up to the boundary. We should note from the viewpoint of smooth solutions that regularity holds for a short time to the best of our knowledge. As for the half space case, the odd reflection reduces the problem to the whole space case completely, and analyticity up to the boundary is obtained (see [17]).

In this paper, we study the existence of solutions for initial data in the critical Besov spaces $\dot{B}_{\infty,q}^0$ associated with the Dirichlet Laplacian, where the criticality of the spaces comes from the scaling invariance property in the case when the domain is the whole space. Namely, the transformation $\theta(\lambda t, \lambda x) (\lambda > 0)$ maintains the equation (1.1) and we have

$$\|\theta(0)\|_X \simeq \|\theta(0)\|_X$$

for $X = L^\infty(\mathbb{R}^2), \dot{H}^2_{p} (\mathbb{R}^2), and \dot{B}^{2}_{p,q} (\mathbb{R}^2)$. It would be natural that these spaces on domains have some critical structure locally in time at least. We prove the existence of local solutions for arbitrary data and global solutions for small data.

We state our main result for initial data in $\dot{B}_{\infty,1}^0$ to explain the essence of this paper simply and mention that the $\dot{B}_{\infty,q}^0, q > 1$ case follows as well as the whole space case. We also see that in the case when $q = 1$ the functions in $\dot{B}_{\infty,1}^0$ are continuous up to the boundary, and the boundary condition is understood by the boundary value of continuous functions.

We introduce the definition of Besov spaces. Let $\phi_0$ be such that $\phi_0 \in C_0^\infty(\mathbb{R})$ and

$$\text{supp} \phi_0 \subset [2^{-1}, 2]. \quad \phi_0(\lambda) = \phi_0 \left( \frac{\lambda}{2^j} \right), \quad \sum_{j \in \mathbb{Z}} \phi_0 \left( \frac{\lambda}{2^j} \right) = 1 \text{ for any } \lambda > 0,$$

and we define

$$\phi_j(\lambda) = \phi_0 \left( \frac{\lambda}{2^j} \right), \quad \lambda \in \mathbb{R}. \quad \text{and} \quad \phi(\lambda) = \sum_{j \in \mathbb{Z}} \phi_j(\lambda).$$
For \( s \in \mathbb{R} \) and \( 1 \leq p, q \leq \infty \), \( \dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\Lambda_D) \) is defined by

\[
\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\Lambda_D) = \left\{ f \in \mathcal{Z}'_D \left| \| f \|_{\dot{B}^s_{p,q}} = \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{sj} \| \phi_j(\Lambda_D) f \|_{L^p} \right)^q \right\}^{\frac{1}{q}} < \infty \right\},
\]

where \( \mathcal{Z}'_D \) is a space of tempered distribution, and we explain the precise definitions in Sect. 2.1.

**Theorem 1.1.** For every \( \theta_0 \in \dot{B}^0_{\infty,1}(\Lambda_D) \), there exists \( T > 0 \) such that equation (1.1) with the initial condition (1.2) has a unique solution \( \theta \) such that

\[
\theta \in C([0, T], \dot{B}^0_{\infty,1}) \cap L^1(0, T; \dot{B}^1_{\infty,1}), \quad \partial_t \theta \in L^1(0, T; \dot{B}^1_{\infty,1}),
\]

and \( \theta \) is continuous with respect to \( t \geq 0, x \in B \), and

\[
\lim_{|x| \to 1} \theta(t, x) = 0 \quad \text{for } t \geq 0.
\]

If \( \theta_0 \) is small in \( \dot{B}^0_{\infty,1} \), then the solution \( \theta \) exists globally in time.

Let us give some remarks on the proof. Our idea is to establish a method based on that in the case of the whole space ( [16,30]), replacing the Fourier transformation with spectral decomposition. The difference is that we have several problems: how to control or understand boundary values of functions satisfying the Dirichlet boundary condition. Our starting point is the spectral multiplier theorem, which proves boundedness of the operator \( \varphi(-\Delta_D) \) in \( L^p, 1 \leq p \leq \infty \) for all \( \varphi \) in the Schwartz class on the real line, and our method is applying spectral localization (see Proposition 2.9) and commutator estimates (see Proposition 3.2). The spectral localization in this paper is a bound from below, more precisely, at a maximum point \( x_0 \) of \( |\psi_j(\Lambda_D) f| \)

\[
\Lambda_D \psi_j(\Lambda_D) f(x_0) \text{ sign } f(x_0) \geq c 2^j \| \psi_j(\Lambda_D) f \|_{L^\infty},
\]

where \( \psi_j(\Lambda) \) is an operator restricting the spectrum around \( 2^j \). We remark that this is possible for other domains. This kind of localization when the domain is the whole space is established in [30], and it can be generalized in domains as in Proposition 2.9.

The commutator estimate in this paper is a bilinear estimate,

\[
\sum_{j \in \mathbb{Z}} \left\| \nabla^\perp \left[ \Lambda_D^{-1} f \cdot \nabla, \psi_j(\Lambda_D) \right] g \right\|_{L^\infty} \leq C \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^0_{\infty,1}}.
\]

However, it seems very difficult to establish by Littlewood–Paley dyadic decomposition as in the whole space. We can avoid this by using a resolution of unity such that

\[
1 = \sum_{j \in \mathbb{Z}} \left( \frac{1}{1 + 2^{-2j-j^2}} - \frac{1}{1 + 2^{-2j}j^2} \right) =: \sum_{j \in \mathbb{Z}} \psi_j(\lambda), \quad \lambda > 0,
\]
since the resolvent has a property that
\[
\left\[ \nabla \Lambda_D^{-1} f \cdot \nabla, \frac{1}{1 - 2^{-2j} \Delta_D} \right\]
\]
\[
= \frac{-2^{-2j}}{1 - 2^{-2j} \Delta_D} \left( (\nabla \Lambda_D^{-1} f \cdot \nabla) + 2(\nabla \nabla \Lambda_D^{-1} f \cdot \nabla) \nabla \right) \frac{1}{1 - 2^{-2j} \Delta_D} g.
\]

and the right-hand side is justified even if \( g \) does not have weak derivatives, where \( -\Delta_D \) denotes the Dirichlet Laplacian. We also see the equivalency of the norm defined by the dyadic decomposition and the resolvent for \( s \) close to zero, and the commutator estimate can be expected. It is also important in the argument above that if \( f, g \) satisfy the Dirichlet boundary condition and are smooth, then \((\nabla \Lambda_D^{-1} f \cdot \nabla)g\) does and
\[
(\nabla \Lambda_D^{-1} f \cdot \nabla)g = (1 - 2^{-2j} \Delta_D)^{-1}(1 - 2^{-2j} \Delta_D) \left( (\nabla \Lambda_D^{-1} f \cdot \nabla)g \right).
\]

Finally, we mention two basic tools: maximum regularity estimates in Lemma 2.5 (see also [14]) and bilinear estimates in Proposition 2.11. In this paper, we give a simple proof for boundedness of the second derivative near the \( L^\infty \) space in Lemma 2.8 by using the explicit formula of \((-\Delta_D)^{-1}\) in a ball, but it is possible to consider other smooth bounded domains. We also refer to [15,18] for the bilinear estimates in the half space case and for how to understand the boundary value of functions.

It is also possible to obtain a similar result with initial data in \( \dot{B}_{\infty, q}^0 \) with \( q > 1 \), by introducing spaces, whose norms are defined by
\[
\|f\|_{\tilde{L}^p(0, T; \dot{B}_{\infty, q}^0)} := \left\{ \sum_{j \in \mathbb{Z}} \left\| 2^j \| \phi_j (\Lambda_D) f \|_{L^1(0, \infty; L^p(\mathbb{R}^2_+))} \right\|_{\ell^q(\mathbb{Z})} \right\}.
\]

**Theorem 1.2.** (i) Let \( 1 \leq q < \infty \). For every \( \theta_0 \in \dot{B}_{\infty, q}^0 (\Lambda_D) \), there exists \( T > 0 \) such that equation (1.1) with the initial condition (1.2) has a unique solution \( \theta \) such that
\[
\theta \in C([0, T], \dot{B}_{\infty, q}^0) \cap \tilde{L}^\infty(0, T; \dot{B}_{\infty, q}^0) \cap \tilde{L}^1(0, T; \dot{B}_{\infty, q}^1), \quad \partial_t \theta \in \tilde{L}^1(0, T; \dot{B}_{\infty, q}^1),
\]

If \( \theta_0 \) is small in \( \dot{B}_{\infty, q}^0 \), then the solution exists globally in time.

(ii) Let \( q = \infty \). For every \( \theta_0 \in \dot{B}_{\infty, \infty}^0 \) such that \( \| \phi_j (\Lambda_D) \theta_0 \|_{L^\infty} \to 0 \) \( (j \to \infty) \), the same existence result holds.

This kind of local existence when the domain is the whole space is established by Wang-Zhang [30] (see also [16]). It is possible to modify the proof of Theorem 1.1 to handle the case when \( q > 1 \) in a similar way to [16,30]. We leave the proof for readers.

This paper is organized as follows: In Sect. 2, we recall the definition of Besov spaces associated with the Dirichlet Laplacian, and several properties for the boundary value of functions, such as spectral localization and commutator estimates. In Sect. 3, we prove Theorem 1.1. In the “Appendix,” we discuss the equivalence of the Besov
norms defined by the dyadic decomposition and the resolvent when the regularity is close to 0.

**Notations.** We denote by $-\Delta D$ the Dirichlet Laplacian on $L^2(\Omega)$ and on distribution space $Z_D'$ defined in Sect. 2. We write $x = (x_1, x_2)$. Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be the dyadic decomposition of the unity such that $\phi_j$ is a nonnegative function in $C_0^\infty(\mathbb{R})$ and

\[
\supp \phi_0 \subset [2^{-1}, 2], \quad \phi_j(\lambda) = \phi_0(\frac{\lambda}{2^j}), \quad \sum_{j \in \mathbb{Z}} \phi_j(\lambda) = 1 \text{ for any } \lambda > 0.
\]

$\{\psi_j\}_{j \in \mathbb{Z}}$ is another resolution of identity such that

\[
1 = \sum_{j \in \mathbb{Z}} \left( \frac{1}{1 + 2^{-2j-2}\lambda^2} - \frac{1}{1 + 2^{-2j}\lambda^2} \right) =: \sum_{j \in \mathbb{Z}} \psi_j(\lambda), \quad \lambda > 0.
\]

We use the following notations for norms of spaces in space and time as follows:

\[
\| f \|_{B^s_{p,q}(\Lambda D)} = \left\| \left\{ 2^{sj} \| \phi_j(\Lambda D) f \|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})},
\]

\[
\| f \|_{L^r(0,\infty; X)} = \| f(t) \|_{X} \|_{L^r(0,\infty)}, \quad X = L^p(\mathbb{R}_+^2), \quad \dot{B}^s_{p,q}(\Lambda D).
\]

We define the Sobolev spaces $H^s$ using the Besov spaces:

\[
H^s = \dot{B}^s_{2,2}, \quad s \in \mathbb{R}.
\]

We write the domain of the functions only when the function space on the whole space is used, for instance $B^s_{p,q}(\mathbb{R}^2)$, where we will use the theory on $\mathbb{R}^2$. When the domain is the ball $B$, then we omit it. The kernel of $(-\Delta D)^{-1}$ (see, e.g., [11]) is defined by

\[
(-\Delta D)^{-1}(x, y) = \frac{1}{2\pi} \log |x - y| - \Phi(x, y),
\]

where

\[
\Phi(x, y) = \frac{1}{2\pi} \log \left( |x| - \frac{x}{|x|^2} \right).
\]

We write

\[
B := \{ x \in \mathbb{R}^2 \mid |x| < 1 \}, \quad B^c := \{ x \in \mathbb{R}^2 \mid |x| \geq 1 \},
\]

and $\chi_B$ and $\chi_{B^c}$ are the characteristic functions on $B$ and $B^c$, respectively. On the whole space $\mathbb{R}^2$, we denote by $-\Delta_{\mathbb{R}^2}$, $\Lambda_{\mathbb{R}^2}$, the Laplacian and the square root of the Laplacian defined by the Fourier transform defined in the space of tempered distribution. We denote by $-\Delta D$, $\Lambda D$, the Dirichlet Laplacian and the square root of the Dirichlet Laplacian.
2. Preliminary

In Sect. 2.1, we recall the definition of Besov spaces in [20]. In Sect. 2.2, we introduce a spectral multiplier theorem together with derivative estimates and smoothing property such as maximum regularity for $e^{-t\Delta_D}$. In Sect. 2.3, inequalities for the spectral localization are established. In Sect. 2.4, the commutator estimates are proved.

2.1. Besov spaces

We recall the definition of Besov spaces (see [20]). We start by defining the Dirichlet Laplacian $-\Delta_D$ and spaces of test functions, $\mathcal{Z}$, of homogeneous type. We here notice that the infimum of the spectrum is strictly positive, since we consider the bounded domain with the Dirichlet condition and the spaces of homogeneous and non-homogeneous types as equivalent. We just adopt the homogeneous type for simplicity of notation in our proof.

**Definition.** (i) Let $-\Delta_D$ be the Dirichlet Laplacian on $L^2(B)$ defined by

\[
D(-\Delta_D) := \{ f \in H^1_0(B) \mid \Delta f \in L^2(B) \},
\]

\[-\Delta_D f := -\Delta f = - \left( \frac{\partial^2}{\partial x_1^2} f + \frac{\partial^2}{\partial x_2^2} f \right), \quad f \in D(-\Delta_D).
\]

(ii) Let $\mathcal{Z}_D$ be a space of test functions such that

\[
\mathcal{Z}_D := \{ f \in L^1 \cap L^2 \mid q_m(f) < \infty \text{ for all } m \in \mathbb{N} \},
\]

where

\[
q_m(f) := \sup_{j \in \mathbb{Z}} 2^{mj} \| \phi_j(\Lambda_D) f \|_{L^1}.
\]

(iii) Let $\mathcal{Z}_D'$ be the topological duals of $\mathcal{Z}_D$.

It was proved in [20] that the space $\mathcal{Z}_D$ is a Fréchet space and can regard their duals $\mathcal{Z}_D'$ as distribution spaces, which are variants of the space of the tempered distributions and the quotient space by the polynomials in the whole space. We define Besov spaces associated with the Dirichlet Laplacian on the unit ball as follows.

**Definition.** Let $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. $\dot{B}^s_{p,q}(\Lambda_D)$ is defined by

\[
\dot{B}^s_{p,q} = \dot{B}^s_{p,q}(\Lambda_D) := \{ f \in \mathcal{Z}_D' \mid \| f \|_{\dot{B}^s_{p,q}(\Lambda_D)} < \infty \},
\]

where

\[
\| f \|_{\dot{B}^s_{p,q}(\Lambda_D)} := \left\| \left\{ 2^{sj} \| \phi_j(\Lambda_D) f \|_{L^p(\mathbb{R}^2_+)} \right\}_j \right\|_{l^q(\mathbb{Z})}.
\]

It is proved that $\dot{B}^s_{p,q}(\Lambda_D)$ is a Banach space and satisfies standard properties such as lift properties, embedding theorems of Sobolev type as well as the whole space
case. We here recall the uniform boundedness of the frequency restriction operator \( \phi_j(\Lambda_D) \) and some fundamental property of the Besov spaces for the purpose of this paper. This is possible, since operators \( \phi_j(\Lambda) \) restricting the spectrum are uniformly bounded in \( L^p \) for all \( 1 \leq p \leq \infty \) (see Lemma 2.4 for more details).

We here write several properties which are needed in the proof.

**Lemma 2.1.** Let \( |s| < 2 \) and \( 1 \leq p, q \leq \infty \). Then,

\[
f = \sum_{j \in \mathbb{Z}} \phi_j(\Lambda_D) f \quad \text{in} \quad \mathcal{Z}'_D, \quad \|f\|_{\dot{B}^s_{p,q}} \simeq \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{sj} \|\psi_j(\Lambda_D) f\|_{L^p} \right)^q \right\}^{\frac{1}{q}} ,
\]

for all \( f \in \dot{B}^s_{p,q} \).

We prove Lemma 2.1 in “Appendix A.”

**Lemma 2.2.** Let \( 1 \leq p, q, r \leq \infty \). Then,

\[
\begin{align*}
\|f\|_{L^p} &\leq \|f\|_{\dot{B}^0_{p,1}}, \\
\|\Lambda_s^s f\|_{\dot{B}^0_{p,q}} &\leq C \|f\|_{\dot{B}^0_{p,q}} \quad \text{for} \ s \in \mathbb{R}, \quad (2.2) \\
\|f\|_{\dot{B}^0_{p,q}} &\leq C \|f\|_{\dot{B}^1_{r,q}} \quad \text{for} \ s \in \mathbb{R}, \quad (2.3) \\
\|f\|_{\dot{B}^0_{\infty,1}} &\leq C \|f\|_{\dot{B}^1_{\infty,q}} \quad \text{for} \ s > 0. \quad (2.4)
\end{align*}
\]

**Proof.** The first inequality of (2.1) is obtained by the resolution of the identity in Lemma 2.1 and the triangle inequality, and the second inequality is proved by the resolution and the gradient estimate (2.11). The lifting property, the embedding theorem, is already known in [20]. The validity of the last inequality (2.4) is due to the infimum of the spectrum being positive and an elementary boundedness in the sequence spaces.

**Lemma 2.3.**

(i) Every \( f \in \dot{B}^0_{\infty,1} \) is regarded as a continuous function up to the boundary and \( f \equiv 0 \) on the boundary.

(ii) Let \( f, g \in L^\infty \) and \( f_k = \phi_k(\Lambda_D) f, \ g_l := \phi_l(\Lambda_D) g \) for \( k, l \in \mathbb{Z} \). Then, \( (\nabla \perp \Lambda_D f_k \cdot \nabla) g_l \) is regarded as a continuous function up to the boundary and is equal to zero on the boundary.

We give a direct proof of Lemma 2.3 by using the formula of \( (-\Delta_D)^{-1} \) in “Appendix B,” since the proof seems elementary. One can also find the orthogonality due to \( \nabla \perp \) and \( \nabla \) for functions in \( H^1_0 \) on smooth bounded domain in [7].

2.2. Spectral multipliers and smoothing property of \( e^{-t\Lambda_D} \)

We recall boundedness of the spectral multipliers and gradient estimates. We mainly refer to [21], but there is a plenty of literature in this field, and one can refer to [19, 24, 29] for the theory.
Lemma 2.4. ([21]) (Boundedness of spectral multipliers) Suppose that $\psi$ and its all derivatives are bounded and that $\varphi$ belongs to the Schwartz class in the real line and $1 \leq p \leq \infty$. Then, $\delta, C > 0$ exist such that

$$
\|\varphi(2^{-2j}(-\Delta_D))\|_{L^p \rightarrow L^p} \leq C \| (1 - \Delta_D)^{d/4 + \delta} \varphi(\cdot) \|_{L^2(\mathbb{R})}.
$$

(2.5)

Furthermore,

$$
\|\psi(-\Delta_D)\varphi(2^{-2j}(-\Delta_D))\|_{L^p \rightarrow L^p} \leq C \| (1 + |\cdot|^2)^{3d/8 + d/4 + \delta} (1 - \Delta_D)^{d/4 + \delta} \varphi(2^{2j} \cdot) \varphi(\cdot) \|_{L^2(\mathbb{R})}.
$$

(2.6)

We recall maximum regularity estimate.

Lemma 2.5. ([14]) For every $\theta_0 \in \dot{B}^{0}_{\infty,1}$

$$
\| e^{-t\Lambda} \theta_0 \|_{L^\infty(0,T; \dot{B}^{0}_{\infty,1} \cap L^1(0,T; \dot{B}^{1}_{\infty,1}))} \leq C \| \theta_0 \|_{\dot{B}^{0}_{\infty,1}}. \tag{2.7}
$$

If $u \in C([0,T], \dot{B}^{0}_{\infty,1}) \cap L^1(0,T; \dot{B}^{1}_{\infty,1})$ and $f \in L^1(0,T; \dot{B}^{0}_{\infty,1})$ satisfy $\partial_t u \in L^1(0,T; \dot{B}^{1}_{\infty,1}), \partial_t u + \Lambda u = f$, then

$$
\| u \|_{L^\infty(0,T; \dot{B}^{0}_{\infty,1}) \cap L^1(0,T; \dot{B}^{1}_{\infty,1})} \leq C \| u(0) \|_{\dot{B}^{0}_{\infty,1}} + C \| f \|_{L^1(0,T; \dot{B}^{0}_{\infty,1})}. \tag{2.8}
$$

We also use the boundedness of the resolvent.

Lemma 2.6. For every $1 \leq p \leq \infty$

$$
\sup_{j \in \mathbb{Z}} \left\| (1 - 2^{-2j} \Delta_D)^{-1} \right\|_{L^p \rightarrow L^p} < \infty. \tag{2.9}
$$

Proof. Let $f \in L^p$ and $\tilde{\phi}_0 \in C^\infty_0(\mathbb{R})$ be such that

$$
supp \tilde{\phi}_0 \subset [2^{-1}, 2], \quad \tilde{\phi}_0(\lambda^2) = \sum_{j \leq 0} \phi_j(\lambda), \quad \lambda > 0.
$$

We use the resolution

$$
1 = \tilde{\phi}_0(2^{-2j} \lambda^2) + \sum_{k > j} \phi_j(\lambda), \quad \text{for } \lambda > 0.
$$

By $(1 + \lambda^2)^{-1} \tilde{\phi}_0 \in C^\infty_0(\mathbb{R})$ and the boundedness of the spectral multipliers (2.5),

$$
\|(1 - 2^{-2j} \Delta_D)^{-1} f\|_{L^p}
\leq \|(1 - 2^{-2j} \Delta_D)^{-1} \tilde{\phi}_0(-2^{-2j} \Delta_D) f\|_{L^p} + \sum_{k > j} \|(1 - 2^{-2j} \Delta_D)^{-1} \phi_k(\Delta_D) f\|_{L^p}
\leq C \| f \|_{L^p} + \sum_{k > j} \frac{C}{1 + 2^{-2j+2k}} \| f \|_{L^p} \leq C \| f \|_{L^p}.
$$

We use the boundedness of the derivatives.
Lemma 2.7. ([21]) Let \( m = 0, 1, 2, \ldots, \) and \( 1 \leq p \leq \infty. \) Then,
\[
\| \Lambda_D^m \phi_j(\Lambda_D^m) f \|_{L^p} \leq C 2^m j \| \phi_j(\Lambda_D) f \|_{L^p}, \tag{2.10}
\]
\[
\| \nabla \phi_j(\Lambda_D) f \|_{L^p} + \| \phi_j(\Lambda_D) \nabla f \|_{L^p} \leq C 2^j \| \phi_j(\Lambda_D) f \|_{L^p}. \tag{2.11}
\]

Lemma 2.8. (i) There exists a constant \( C > 0 \) such that
\[
\| \nabla^2 (\nabla \phi_j(\Lambda_D))^{-1} f \|_{L^\infty} \leq C \| f \|_{\dot{B}^0_{\infty, 1}}, \quad \text{for } f \in \dot{B}^0_{\infty, 1}. \tag{2.12}
\]
When \( 1 < p < \infty, \)
\[
\| \nabla^2 (\nabla \phi_j(\Lambda_D))^{-1} f \|_{L^p} \leq C \| f \|_{L^p}, \quad \text{for } f \in L^p. \tag{2.13}
\]

(ii) Let \( f, g \in L^\infty \) and \( f_k := \phi_k(\Lambda_D) f, \ g_l := \phi_l(\Lambda_D) g \ (k, l \in \mathbb{Z}). \) Then,
\[
\nabla^\perp f_k \cdot \nabla g_l \in H^1_0 \quad \text{and} \quad (\nabla \phi_j(\Lambda_D))^{-1} \big( \nabla^\perp f_k \cdot \nabla g_l \big) \in L^2. \tag{2.14}
\]

**Proof.** By \( \Phi(x, y) = \Phi(y, x) \) (see (1.3) for the definition) and a change of variable \( y \mapsto y/|y|^2, \) we write
\[
(\nabla \phi_j(\Lambda_D))^{-1} f(x) = \frac{1}{2\pi} \int_{|x|<1} \left( \log |x-y| \right) f(y) \, dy - \frac{1}{2\pi} \int_{B^c} \left( \log |y| - \log |x-y| \right) f \left( \frac{y}{|y|^2} \right) \frac{dy}{|y|^4}
\]

\[
= \frac{1}{2\pi} \int_{\mathbb{R}^2} \left( \log |x-y| \right) \left( \nabla \phi_j f \left( \frac{y}{|y|^2} \right) - \frac{\chi_{\mathbb{R}^2}}{|y|^4} f \left( \frac{y}{|y|^2} \right) \right) dy + \frac{1}{2\pi} \int_{B^c} \frac{\log |y|}{|y|^4} f \left( \frac{y}{|y|^2} \right) dy.
\]

and an extension to \( \mathbb{R}^2 \)
\[
F(y) := \nabla \phi_j f \left( \frac{y}{|y|^2} \right), \quad y \in \mathbb{R}^2.
\]

By considering \( \mathbb{R}^2 \) and the boundedness of the Riesz transform (see, e.g., Stein [26]),
\[
\| \nabla^2 (\nabla \phi_j(\Lambda_D))^{-1} F \|_{L^\infty(\mathbb{R}^2)} \leq \sum_{j \in \mathbb{Z}} \| \nabla^2 (\nabla \phi_j(\Lambda_D))^{-1} \phi_j(\Lambda_D) F \|_{L^\infty(\mathbb{R}^2)}
\]
\[
\leq C \| F \|_{\dot{B}^0_{\infty, 1}(\mathbb{R}^2)}. \tag{2.15}
\]

The real interpolation \( \dot{B}^0_{\infty, 1}(\mathbb{R}^2) = (BMO^{-2}(\mathbb{R}^2), BMO^2(\mathbb{R}^2))_{\frac{1}{2}, 1} \) (see, e.g., [28]) implies that
\[
\| F \|_{\dot{B}^0_{\infty, 1}(\mathbb{R}^2)} \leq \int_0^\infty t^{-\frac{1}{2}} \inf_{F_1 + F_2 = F} \left( \frac{\| (-\Delta_{\mathbb{R}^2})^{-1} F_1 \|_{BMO(\mathbb{R}^2)} + t \| (-\Delta_{\mathbb{R}^2})^{-1} F_2 \|_{BMO(\mathbb{R}^2)}}{t} \right) \frac{dt}{t}.
\]

Here, we restrict the decomposition \( F = F_1 + F_2 \) to \( F_1, F_2 \) such that
\[
F_j = \chi_B f_j \left( \frac{y}{|y|^2} \right), \quad j = 1, 2,
\]
where \( f_1, f_2 \) are functions satisfying \( f = f_1 + f_2 \) in the ball \( B \). We then have

\[
\|(-\Delta_{\mathbb{R}^2})^{-1} f_1\|_{BMO(\mathbb{R}^2)} = \|(-\Delta_{\mathbb{R}^2})^{-1} f_1 + \frac{1}{2\pi} \int_{B^c} \frac{\log |y|}{|y|^4} f_1(\frac{y}{|y|^2}) \, dy\|_{BMO(\mathbb{R}^2)} = \|(-\Delta_D)^{-1} f_1\|_{BMO(\mathbb{R}^2)} \leq C \|(-\Delta_D)^{-1} f_1\|_{L^\infty} \leq C \|f_1\|_{\dot{B}^{-2}_\infty,1},
\]

where

\[
(-\Delta_D)^{-1} f_1(x) = \begin{cases} \left((-\Delta_D)^{-1} f_1\right)(x) & \text{for } x \in B, \\ -\left((-\Delta_D)^{-1} f_1\right)(\frac{x}{|x|^2}) & \text{for } x \in B^c,
\end{cases}
\]

and

\[
\|(-\Delta_{\mathbb{R}^2}) f_2\|_{BMO(\mathbb{R}^2)} \leq C \|f_2\|_{L^\infty} + \|\nabla f_2\|_{L^\infty} + \|(-\Delta_D) f_2\|_{L^\infty} \leq C \|f_2\|_{\dot{B}^{-2}_\infty,1}.
\]

By the real interpolation \( \dot{B}^0_{\infty,1} = (\dot{B}^{-2}_{\infty,1}, \dot{B}^2_{\infty,1})_{\frac{1}{2},1} \), we conclude

\[
\|F\|_{\dot{B}^0_{\infty,1}(\mathbb{R}^2)} \leq C \int_0^\infty t^{-\frac{1}{2}} \inf_{f=f_1+f_2} \left( \|f_1\|_{\dot{B}^{-2}_{\infty,1}} + t \|f_2\|_{\dot{B}^2_{\infty,1}} \right) \frac{dt}{t} \leq C \|f\|_{\dot{B}^0_{\infty,1}},
\]

which proves (2.12). The boundedness in \( L^p \), \( 1 < p < \infty \), follows from a similar argument and the boundedness of the Riesz transform (see, e.g., [26]) in the whole space.

Next, we write

\[
f_k = (-\Delta_D)^{-1}\left(-\Delta_D f_k\right), \quad g_l = (-\Delta_D)^{-1}\left(-\Delta_D g_l\right),
\]

and the boundary value of \( \nabla^\perp f_k \cdot \nabla g_l \) can be checked by the explicit formula of \( (-\Delta_D)^{-1}(x, y) \) and its derivatives (or the orthogonality of \( \nabla^\perp \) and \( \nabla \) (see [7])), and it yields that \( \nabla^\perp f_k \cdot \nabla g_l \) is continuous up to the boundary and \( \nabla^\perp f_k \cdot \nabla g_l = 0 \) on the boundary. We can also see from a similar argument to (2.15) and the spectral multiplier theorem (2.10) that

\[
\|\nabla^2 f_k\|_{L^4} \leq \left\| \nabla^2(-\Delta_{\mathbb{R}^2})^{-1}\left(\chi_B(-\Delta_D)f_k - \frac{\chi_B(-\Delta_D)f_k(\frac{\cdot}{|\cdot|^2})}{|\cdot|^4}\right) \right\|_{L^4(\mathbb{R}^2)} \leq C \|(-\Delta_D)f_k\|_{L^4} \leq C 2^{2k}\|f_k\|_{L^4},
\]

and \( g_l \) satisfies the same inequality, with replacing \( 2^{2k} \) by \( 2^{2l} \). By the Hölder’s inequality and the gradient estimate (2.11), we conclude that

\[
\|\nabla(\nabla^\perp f_k \cdot \nabla g_l)\|_{L^2} \leq \|\nabla^2 f_k\|_{L^4}\|\nabla g_l\|_{L^4} + \|\nabla f_k\|_{L^4}\|\nabla^2 g_l\|_{L^4} \leq C (2^{2k+l} + 2^{k+2l})\|f_k\|_{L^4}\|g_l\|_{L^4} < \infty,
\]

and \( \nabla^\perp f_k \cdot \nabla g_l \in H^1_0 \). Also, \( (-\Delta_D)f \in L^2 \) follows from

\[
(-\Delta_D)(\nabla^\perp f_k \cdot \nabla g_l) = \nabla^\perp(-\Delta_D)f_k \cdot \nabla g_l - 2\nabla \nabla^\perp f_k \cdot \nabla \nabla g_l + \nabla^\perp f_k \cdot \nabla (-\Delta_D)g_l,
\]

\[
\|(-\Delta_D)(\nabla^\perp f_k \cdot \nabla g_l)\|_{L^2} \leq C(2^{3k+l} + 2^{2k+2l} + 2^{k+3l})\|f_k\|_{L^4}\|g_l\|_{L^4} < \infty.
\]
2.3. Spectral localization

**Proposition 2.9.** Let \( \psi_j(\lambda) := (1 + 2^{-2j} \lambda^2)^{-1} - (1 + 2^{-2j} \lambda^2) \). There exists \( c > 0 \) such that for every \( f \in L^\infty \) and \( j \in \mathbb{Z} \), we have at a maximum point \( x_0 \) of \( \phi_j(\Lambda_D f) \)

\[
\left| \left( \Lambda_D \psi_j(\lambda, f) \right)(x_0) \right| \geq c 2^j \| \phi_j(\Lambda_D f) \|_{L^\infty}. \tag{2.16}
\]

**Proof.** We write \( f_j = \psi_j(\Lambda_D) f \) for the sake of simplicity, and we may assume positivity \( f_j(x_0) = \| f_j \|_{L^\infty} \geq 0 \) at the maximum point \( x_0 \), unless we may consider the case when \( f_j(x_0) \) is negative and it suffices to replace \( f \) by \(-f\). We choose a constant \( c_1 > 0 \) such that

\[
\lambda = c_1 \int_0^\infty t^{-\frac{3}{2}} \left( 1 - e^{-t \lambda^2} \right) dt, \quad \text{for} \ \lambda > 0,
\]

and write

\[
\Lambda_D f_j(x_0) = c_1 2^j \int_0^\infty t^{-\frac{3}{2}} (f_j(x_0) - e^{\lambda \Delta_D} f_j(x_0)) dt.
\]

We here notice that the integrand above is nonnegative, since the \( L^\infty \) norm of \( e^{\lambda \Delta_D} f_j \) is non-increasing. By a change of variable \( t \mapsto 2^{-2j} t \),

\[
\Lambda_D f_j(x_0) = c_1 2^j \int_0^\infty t^{-\frac{3}{2}} (f_j(x_0) - e^{2^{2j} \lambda \Delta_D} f_j(x_0)) dt.
\]

We write

\[
ed^{2^{2j} \lambda \Delta_D} f_j = \left( \sum_{l < j} + \sum_{l \geq j} \right) \psi_l(\Lambda_D) e^{2^{2j} \lambda \Delta_D} f_j = \frac{\psi_j(\Lambda_D)}{1 + 2^{2j} \lambda \Lambda_D^2} e^{2^{2j} \lambda \Delta_D} f_j + \sum_{l \geq j} \psi_l(\Lambda_D) e^{2^{2j} \lambda \Delta_D} f_j,
\]

and apply the spectral multiplier theorem (2.6) to have that

\[
\left\| \psi_j(\Lambda_D) \frac{e^{2^{2j} \lambda \Delta_D} f_j}{1 + 2^{2j} \lambda \Lambda_D^2} \right\|_{L^\infty} \leq C e^{-c t} \| f_j \|_{L^\infty} = C e^{-c t} f_j(x_0),
\]

and

\[
\left\| \sum_{l \geq j} \psi_l(\Lambda_D) e^{4^{2j} \lambda \Delta_D} f_j \right\|_{L^\infty} \leq C \sum_{l \geq j} \frac{2^j}{2^l} e^{-c t} \| f_j \|_{L^\infty} \leq C e^{-c t} f_j(x_0).
\]

We can then find a half line \( [a, \infty) \) independent of \( j \) such that

\[
\left\| e^{2^{2j} \lambda \Delta_D} f_j \right\|_{L^\infty} \leq \frac{1}{2} f_j(x_0), \quad \text{for} \ t \in [a, \infty).
\]

We then obtain that

\[
\Lambda_D f_j(x_0) \geq c_1 2^j \int_a^\infty t^{-\frac{3}{2}} \left( f_j(x_0) - \frac{1}{2} f_j(x_0) \right) dt = \left( \frac{c_1}{2} \int_a^\infty t^{-\frac{3}{2}} dt \right) 2^j \| f_j \|_{L^\infty},
\]

which completes the proof.
Corollary 2.10. Let \( \theta, f \) be smooth functions satisfying \( \partial_t \theta + \Lambda_D \theta = f \). Then,

\[
\partial_t \| \psi_j(\Lambda_D) \theta \|_{L^\infty} + c2^j \| \psi_j(\Lambda_D) \theta \|_{L^\infty} \leq \| f \|_{L^\infty}.
\] (2.17)

Proof. At a maximum point of \( |\psi_j(\Lambda_D) f| \), we apply Lemma 3.2 in [30] (see also Lemma 2.2 in [16]) to the time derivative and Proposition 3.11 to the fractional Laplacian, and obtain (2.17).

2.4. Bilinear estimate

Proposition 2.11. For every \( f \in \dot{B}^1_{\infty,1} \) and \( g \in \dot{B}^1_{\infty,1} \)

\[
\left\| (\nabla^j \cdot \nabla) g \right\|_{\dot{B}^0_{\infty,1}} \leq C \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^1_{\infty,1}}.
\] (2.18)

Moreover, for every \( f \in \dot{B}^0_{\infty,1} \) and \( g \in \dot{B}^1_{\infty,1} \)

\[
\left\| (\nabla^j \Lambda_D^{-1} f \cdot \nabla) g \right\|_{\dot{B}^0_{\infty,1}} \leq C \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^1_{\infty,1}}.
\] (2.19)

Proof. By the decomposition of the unity, we write

\[
f_j := \phi_j(\Lambda_D) f, \ g_j = \phi_j(\Lambda_D) g,
\]

\[
(\nabla^j \cdot \nabla) g = \sum_{j,k,l} \phi_j(\Lambda_D) \left( \nabla^j f_k \cdot \nabla g_l \right),
\]

and we divide into two cases when \( j \geq \max\{k, l\} \) and \( j < \max\{k, l\} \).

We start by the case when \( j \geq \max\{k, l\} \). It follows from the term in the domain of

the Dirichlet Laplacian (see (2.14)) that

\[
\phi_j(\Lambda_D) \left( \nabla^j f_k \cdot \nabla g_l \right) = \Lambda_D^{-2} \phi_j(\Lambda_D) \left( (-\Lambda_D) \left( \nabla^j f_k \cdot \nabla g_l \right) \right).
\]

We then apply the Leibniz rule to \( -\Lambda_D \) and the inequalities in Lemmas 2.7 and 2.8 to have

\[
\| \phi_j(\Lambda_D) \left( \nabla^j f_k \cdot \nabla g_l \right) \|_{L^\infty} \leq C 2^{-j} \left( 2^{3k+l} + 2^{k+3l} \right) \| f_k \|_{L^\infty} \| g_l \|_{L^\infty}
\]

By the inequality above, we estimate

\[
\sum_{j \geq \max\{k, l\}} \| \phi_j(\Lambda_D) \left( \nabla^j f_k \cdot \nabla g_l \right) \|_{L^\infty} \leq C \sum_{j \geq \max\{k, l\}} 2^{-j} \| f_k \|_{L^\infty} 2^j \| g_l \|_{L^\infty} + C \sum_{j \geq \max\{k, l\}} 2^{-j} \| f_k \|_{L^\infty} 2^j \| g_l \|_{L^\infty}
\]

\[
\leq C \sum_{j \in \mathbb{Z}, k^\prime \geq 0} 2^{-2j} \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^1_{\infty,1}} + C \sum_{j \in \mathbb{Z}, l^\prime \geq 0} 2^{-2j} \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^1_{\infty,1}}.
\]
We next consider the case when \( j < \max\{k, l\} \), by dividing into two cases \( k > l \) and \( k \leq l \). When \( k > l \), it follows from \( \nabla^\perp f_k \cdot \nabla g_l = \nabla^\perp (f_k \nabla g_l) \) and (2.11) that

\[
\sum_{j<k, k>l} \| \phi_j(\Lambda_D) (\nabla^\perp f_k \cdot \nabla g_l) \|_{L^\infty} = \sum_{j<k, k>l} \| \phi_j(\Lambda_D) \nabla^\perp \cdot (f_k \nabla g_l) \|_{L^\infty} \leq C \sum_{j<k, k>l} 2^j \| f_k \|_{L^\infty} 2^j \| g_l \|_{L^\infty} \leq C \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^1_{\infty,1}}.
\]

Analogously,

\[
\sum_{j<l, k\leq l} \| \phi_j(\Lambda_D) (\nabla^\perp f_k \cdot \nabla g_l) \|_{L^\infty} = \sum_{j<l, k\leq l} \| \phi_j(\Lambda_D) \nabla (\nabla^\perp f_k) g_l \|_{L^\infty} \leq C \sum_{j<l, k\leq l} 2^j \| f_k \|_{L^\infty} \| g_l \|_{\dot{B}^1_{\infty,1}} \leq C \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^1_{\infty,1}}.
\]

We obtain the first inequality (2.18). The second inequality follows from the lifting property (2.2).

### 2.5. Commutator estimates

**Proposition 2.12.** Let \( 1 \leq q \leq \infty \) and

\[
\psi_j(\lambda) := (1 + 2^{-2j-2\lambda^2})^{-1} - (1 + 2^{-2j\lambda^2})^{-1} = \frac{3}{4} \cdot \frac{2^{-2j\lambda^2}}{(1 + 2^{-2j-2\lambda^2})(1 + 2^{-2j\lambda^2})}.
\]

Then, there exists \( C > 0 \) such that

\[
\begin{align*}
&\left\{ \sum_{j \in \mathbb{Z}} \left\| (\nabla^\perp \Lambda^{-1}_D f \cdot \nabla) \psi_j(\Lambda_D) g - \psi_j(\Lambda_D) (\nabla^\perp \Lambda^{-1}_D f \cdot \nabla) g \right\|_{L^\infty}^q \right\}^{\frac{1}{q}} \leq C \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^1_{\infty,q}}, \quad (2.20) \\
&\left\{ \sum_{j \in \mathbb{Z}} \left\| (\nabla^\perp \Lambda^{-1}_D f \cdot \nabla) \psi_j(\Lambda_D) g - \psi_j(\Lambda_D) (\nabla^\perp \Lambda^{-1}_D f \cdot \nabla) g \right\|_{L^\infty}^q \right\}^{\frac{1}{q}} \leq C \| f \|_{\dot{B}^1_{\infty,1}} \| g \|_{\dot{B}^1_{\infty,q}}, \quad (2.21)
\end{align*}
\]

**Proof.** We utilize the Littlewood–Paley dyadic decomposition \( \{ \phi_j \}_{j \in \mathbb{Z}} \) and the resolution of the identity

\[
f = \sum_{k \in \mathbb{Z}} \phi_k(\Lambda_D) f = \sum_{k \in \mathbb{Z}} f_k, \quad g = \sum_{l \in \mathbb{Z}} \phi_l(\Lambda_D) g = \sum_{l \in \mathbb{Z}} g_l.
\]

We mainly discuss the proof of the second inequality (2.21), since the first one (2.20) can be handled rather easily.

We write \( f = S_j f + (1 - S_j) f \) and start with the easier part with \( (1 - S_j) f \). The term with \( (1 - S_j) f \) does not require some cancellation due to commutators, and we estimate the two terms in the left-hand side of (2.21) separately. The first term in the
left-hand side of (2.21) is estimated by using (2.10) and (2.11)

\[
\left\{ \sum_{j \in \mathbb{Z}} \left\| (\nabla \Lambda_D^{-1} (1 - S_j) f \cdot \nabla) \psi_j (\Lambda_D) g \right\|_{L^\infty}^q \right\}^{\frac{1}{q}}
\]

\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{k > j} \| \nabla \Lambda_D^{-1} f_k \|_{L^\infty} \sum_{l \in \mathbb{Z}} \| \nabla \psi_j (\Lambda_D) g_l \|_{L^\infty} \right)^q \right\}^{\frac{1}{q}}
\]

\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{k > j} 2^{-\frac{1}{2} j} \left( \sup_{k > j} 2^{\frac{1}{2} k} \| f_k \|_{L^\infty} \right) \sum_{l \in \mathbb{Z}} 2^l \frac{2^{-2 j + 2 l}}{(1 + 2^{-2 j + 2 l} 2^2 \| g_l \|_{L^\infty})^2} \right)^q \right\}^{\frac{1}{q}}
\]

\[
\leq C \| f \|_{\frac{1}{B^2_{\infty, \infty}}} \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{l \in \mathbb{Z}} 2^{\frac{1}{2} (l - j)} 2^{-2 j + 2 l} \| g_l \|_{L^\infty} \right)^q \right\}^{\frac{1}{q}}
\]

\[
\leq C \| f \|_{\frac{1}{B^2_{\infty, \infty}}} \| g \|_{\frac{1}{B^2_{\infty, \infty}}}.
\]

For the second term in the left-hand side of (2.21), we decompose \( g = S_j g + (1 - S_j) g \) and have that for the term with \( S_j g \)

\[
\left\{ \sum_{j \in \mathbb{Z}} \left\| \psi_j (\Lambda_D) \left( (\nabla \Lambda_D^{-1} (1 - S_j) f \cdot \nabla) S_j g \right) \right\|_{L^\infty}^q \right\}^{\frac{1}{q}}
\]

\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{k > j} \| f_k \|_{L^\infty} \sum_{l \leq j} 2^l \| g_l \|_{L^\infty} \right)^q \right\}^{\frac{1}{q}}
\]

\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{-\frac{1}{2} j} \| f \|_{\frac{1}{B^2_{\infty, \infty}}} \right)^q \left( 2^{\frac{1}{2} j} \right)^q \sum_{l \leq j} \left( 2^{\frac{1}{2} l} \| g_l \|_{L^\infty} \right)^q \right\}^{\frac{1}{q}}
\]

\[
\leq C \| f \|_{\frac{1}{B^2_{\infty, \infty}}} \left\{ \sum_{l \in \mathbb{Z}} \sum_{j \geq l} \left( 2^{-\frac{1}{2} j} \right)^q \left( 2^{\frac{1}{2} l} \| g_l \|_{L^\infty} \right)^q \right\}^{\frac{1}{q}} \leq C \| f \|_{\frac{1}{B^2_{\infty, \infty}}} \| g \|_{\frac{1}{B^2_{\infty, \infty}}}.
\]

For the term with \( (1 - S_j) g \), we write the divergence form, \( (\nabla \Lambda_D^{-1} (1 - S_j) f \cdot \nabla) (1 - S_j) g = \nabla \cdot ((\nabla \Lambda_D^{-1} (1 - S_j) f) (1 - S_j) g) \), and by (2.11)

\[
\left\{ \sum_{j \in \mathbb{Z}} \left\| \psi_j (\Lambda_D) \nabla \cdot ((\nabla \Lambda_D^{-1} (1 - S_j) f) (1 - S_j) g) \right\|_{L^\infty}^q \right\}^{\frac{1}{q}}
\]

\[
\leq \left\{ \sum_{j \in \mathbb{Z}} \sum_{j' \in \mathbb{Z}} \left\| \psi_j (\Lambda_D) \phi_j (\Lambda_D) \nabla \cdot ((\nabla \Lambda_D^{-1} (1 - S_j) f) (1 - S_j) g) \right\|_{L^\infty}^q \right\}^{\frac{1}{q}}.
\]
\[
\leq \left\{ \sum_{j \in \mathbb{Z}} \left\{ \sum_{j' \in \mathbb{Z}} 2^j 2^{-2|j-j'|} \left\| (\nabla^\perp A_D^{-1} (1 - S_j)f)(1 - S_j)g \right\|_{L^\infty} \right\}^{\frac{1}{q}} \right\}.
\]

We then have that

\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{j' \in \mathbb{Z}} 2^j 2^{-2|j-j'|} \left\| f \right\|_{B_{\infty, \infty}^2} \sum_{l > j} \left\| g_l \right\|_{L^\infty} \right\} \right\}^{\frac{1}{q}}.
\]

\[
= C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{j' \in \mathbb{Z}} 2^j 2^{-2|j-j'|} \left\| f \right\|_{B_{\infty, \infty}^2} \sum_{l' > 0} 2^{-\frac{1}{2}l'} 2^j 2^{\frac{1}{2}(j+l')} \left\| g_{j+l'} \right\|_{L^\infty} \right\} \right\}^{\frac{1}{q}}.
\]

\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{j' \in \mathbb{Z}} 2^j 2^{-2|j-j'|} \left\| f \right\|_{B_{\infty, \infty}^2} \sum_{l' > 0} 2^{-\frac{1}{2}l'} 2^j 2^{\frac{1}{2}(j+l')} \left\| g_{j+l'} \right\|_{L^\infty} \right\} \right\}^{\frac{1}{q}}.
\]

We next consider the remainder part having \( S_j f \), and in this case we need a cancellation due to the commutator. Since the first term in the left-hand side belongs to the domain of the Dirichlet Laplacian, the Dirichlet Laplacian can act on it. We then note that

\[
(\nabla^\perp A_D^{-1} S_j f \cdot \nabla) \psi_j (\Delta) g - \psi_j (\Delta) (\nabla^\perp A_D^{-1} S_j f \cdot \nabla) g = R_j(f, g),
\]

and we can write

\[
(\nabla^\perp A_D^{-1} S_j f \cdot \nabla) \psi_j (\Delta) g - \psi_j (\Delta) (\nabla^\perp A_D^{-1} S_j f \cdot \nabla) g = R_{j+1}(f, g) - R_j(f, g),
\]

and it suffices to estimate

\[
\left\{ \sum_{j \in \mathbb{Z}} \left\| R_j(f, g) \right\|_{L^\infty} \right\}^{\frac{1}{q}}.
\]
We apply the boundedness of the resolvent (2.9), the boundedness of spectral multipliers with the first and the second derivatives (2.11) and (2.12) to have that

\[ \| R_j(f, g) \|_{L^\infty} \leq C 2^{-j} \sum_{k \leq j} 2^{k} \| f_k \|_{L^\infty} \sum_{l \in \mathbb{Z}} \frac{2^l}{1 + 2^{-2j + 2l}} \| g_l \|_{L^\infty} + \sum_{k \leq j} 2^{k} \| f_k \|_{L^\infty} \sum_{l \in \mathbb{Z}} \frac{2^{2l}}{1 + 2^{-2j + 2l}} \| g_l \|_{L^\infty} \]

By taking the \( \ell^q \) norm of the term above, we obtain (2.21). We argue for the first inequality (2.20) here, since this part is crucial. Similar to the above,

\[ \| R_j(f, g) \|_{L^\infty} \leq C \| f \|_{\dot{B}_{\infty, \infty}^{1/2}} 2^{-2j} \left\{ 2^j \sum_{l \in \mathbb{Z}} \frac{2^l}{1 + 2^{-2j + 2l}} \| g_l \|_{L^\infty} + \sum_{l \in \mathbb{Z}} \frac{2^{2l}}{1 + 2^{-2j + 2l}} \| g_l \|_{L^\infty} \right\} \]

and by taking the \( \ell^1 \) norm and the Young’s inequality, we conclude (2.20).

3. Proof of Theorem 1.1

Let the initial data \( \theta_0 \in \dot{B}_{\infty, 1}^0 \). Let \( \{ \theta_n \}_{n=1}^\infty \) be defined by for \( n = 1 \)

\[ \begin{cases} \partial_t \theta_1 + \Lambda_D \theta_1 = 0, \\ \theta_1(0) = \sum_{j \leq 1} \phi_j(\Lambda_D) \theta_0, \end{cases} \tag{3.1} \]

and for \( n = 2, 3, \ldots \)

\[ \begin{cases} \partial_t \theta_n + \Lambda_D \theta_n + (\nabla^\perp \Lambda_D^{-1} \theta_{n-1} \cdot \nabla) \theta_n = 0, \\ \theta_n(0) = \sum_{j \leq n} \phi_j(\Lambda_D) \theta_0 =: \theta_{0,n}, \end{cases} \tag{3.2} \]

It is easy to see that \( \theta_1 \) is well defined, since it is a linear solution and we need to prove the existence of \( \theta_n \) for \( n \geq 2 \) for given \( \theta_{n-1} \).

**Proposition 3.1.** For every \( \theta_0 \in \dot{B}_{\infty, 1}^0, \quad T > 0 \) exists such that there exists a function \( \theta_n \in C([0, T), \dot{B}_{\infty, 1}^0) \cap L^1(0, T; \dot{B}_{\infty, 1}^1) \) satisfying \( \partial_t \theta_n \in L^1(0, T; \dot{B}_{\infty, 1}^1) \) and the equation (3.1) for \( n = 1 \), equation (3.2) for \( n \geq 2 \).
Proof. The case when \( n = 1 \) follows obviously due to maximum regularity estimate (2.7) in the time interval \([0, \infty)\), and let \( n \geq 2 \) and we assume that a solution \( \theta_{n-1} \) exists such that \( \theta_{n-1} \in C([0, T), \dot{B}_{\infty,1}^0) \cap L^1(0, T; \dot{B}_{\infty,1}^1) \) satisfying \( \partial_t \theta_{n-1} \in L^1(0, T; \dot{B}_{\infty,1}^1) \), where the existence time \( T \) will be discussed later to be independent of \( n \).

We now show the existence of \( \theta_n \). It is possible to obtain a local solution, where the existence time depends on \( n \) by Galerkin approximations (see [4, 7]), but we give a self-contained proof to make the independence for the existence time clear in our framework. To this end, we approximate solutions by \( \theta_\varepsilon \), a solution of the following equation:

\[
\begin{aligned}
\partial_t \theta_\varepsilon + \Lambda_D \theta_\varepsilon - \varepsilon \Delta \theta_\varepsilon + (\nabla \Lambda^{-1} \theta_{n-1} \cdot \nabla) \theta_\varepsilon &= 0, \\
\theta_\varepsilon(0) &= \theta_{0,n},
\end{aligned}
\]  

(3.3)

where \( \varepsilon > 0 \). We construct a solution \( \theta_\varepsilon \in C([0, T], H^2) \cap C^1([0, T], H^1) \) of (3.3), to obtain a solution \( \theta_n \) of (3.2) by passing to the limit as \( \varepsilon \to 0 \). We mainly discuss in the case when \( n = 2 \) with several estimates possible to be applied to the cases when \( n \geq 3 \).

Step 1 (Solution in a short interval \([0, T_\varepsilon]\) when \( n = 2 \)). We consider the integral equation

\[
\theta_\varepsilon(t) = e^{-t(\Lambda_D + \varepsilon \Lambda_D^2)} \theta_{0,n} + \int_0^t e^{-(t-\tau)(\Lambda_D + \varepsilon \Lambda_D^2)} (\nabla \Lambda^{-1} \theta_{n-1} \cdot \nabla) \theta_\varepsilon \, d\tau, \quad \text{with } n = 2.
\]  

(3.4)

It is easy to check that

\[
\|\theta_{0,n}\|_{H^2} \leq C 2^n \|\theta_0\|_{L^2} \leq C 2^n \|\theta_0\|_{L^\infty} < \infty.
\]

Also, it is not difficult to show the following inequality:

\[
\left\| \int_0^t e^{-(t-\tau)(\Lambda_D + \varepsilon \Lambda_D^2)} (\nabla \Lambda^{-1} f \cdot \nabla) g \, d\tau \right\|_{H^2} \leq \frac{CT \varepsilon^2}{\varepsilon} \|f\|_{L^\infty(0,T;H^2)} \|g\|_{L^\infty(0,T;H^2)}.
\]

In fact,

\[
\|e^{-(t-\tau)(\Lambda_D + \varepsilon \Lambda_D^2)}\|_{H^1 \to H^2} \leq C \left( (t - \tau) \varepsilon \right)^{-\frac{1}{2}},
\]

and by the Hölder’s inequality, (2.1), (2.3), (2.4), and (2.13)

\[
\|\nabla \Lambda^{-1} f \cdot \nabla\|_{H^1} \leq C \left( \|\nabla \nabla \Lambda^{-1} f\|_{L^4} \|\nabla g\|_{L^4} + \|\nabla \Lambda^{-1} f\|_{L^2} \|\nabla^2 g\|_{L^2} \right) 
\leq C \left( \|f\|_{\dot{B}_{4,1}^{\infty}} \|g\|_{\dot{B}_{4,1}^{\infty}} + \|f\|_{\dot{B}_{4,1}^{\infty}} \|g\|_{H^2} \right) 
\leq C \|f\|_{\dot{B}_{4,2}^{\infty}} \|g\|_{\dot{B}_{4,2}^{\infty}},
\]

which prove the inequality above. We can then apply the Banach fixed-point theorem to obtain a solution \( \theta_\varepsilon \in C([0, T_\varepsilon], H^2) \) of (3.4), and it satisfies \( \partial_t \theta_\varepsilon \in C^1([0, T_\varepsilon], H^1) \), where \( T_\varepsilon \leq \varepsilon/(C\|\theta_{0,n}\|_{H^2})^2 \), with \( n = 2 \).
Step 2 (A priori estimate when \( n = 2 \)). Suppose that \( \theta_\varepsilon \) is a solution of (3.4) in the time interval \([0, \infty)\). For sufficiently small \( \delta > 0 \), we prove that there exists \( T > 0 \) independent of \( \varepsilon \) such that
\[
\| \theta_\varepsilon \|_{L^1(0,T; \dot{B}^{1}_{\infty,1})} \leq 2\delta.
\]

\( \psi_j(\Lambda_D) \) acting on the equation, we write
\[
\partial_t \psi_j(\Lambda_D)\theta_\varepsilon + \Lambda_D \psi_j(\Lambda_D)\theta_\varepsilon - \varepsilon \Delta_D \psi_j(\Lambda_D)\theta_\varepsilon + (\nabla ^\perp \Lambda_D^{-1} \theta_{n-1} \cdot \nabla) \psi_j(\Lambda_D)\theta_\varepsilon
\]
\[
= (\nabla ^\perp \Lambda_D^{-1} \theta_{n-1} \cdot \nabla) \psi_j(\Lambda_D)\theta_\varepsilon - \psi_j(\Lambda_D) \left( (\nabla ^\perp \Lambda_D^{-1} \theta_{n-1} \cdot \nabla) \theta_\varepsilon \right) 
\]
\[
= \left[ \nabla ^\perp \Lambda_D^{-1} \theta_{n-1} \cdot \nabla, \psi_j(\Lambda_D) \right] \theta_\varepsilon.
\]

For almost every \( t \), at a maximum point of \( |\psi_j(\Lambda_D)\theta_\varepsilon| \), it follows from \( -\Delta_D \psi_j(\Lambda_D)\theta_\varepsilon \) having the same sign as \( \psi_j(\Lambda_D)\theta_\varepsilon \) and Corollary 2.10 that
\[
\partial_t \| \psi_j(\Lambda_D)\theta_\varepsilon \|_{L^\infty} + c2^j \| \psi_j(\Lambda_D)\theta_\varepsilon \|_{L^\infty} \leq \left\| \left[ \nabla ^\perp \Lambda_D^{-1} \theta_{n-1} \cdot \nabla, \psi_j(\Lambda_D) \right] \theta_\varepsilon \right\|_{L^\infty},
\]
which yields that
\[
\| \psi_j(\Lambda_D)\theta_\varepsilon \|_{L^\infty} \leq e^{-ct2^j} \| \psi_j(\Lambda_D)\theta_0 \|_{L^\infty} + \int_0^t e^{-c(t-\tau)2^j} \left\| \left[ \nabla ^\perp \Lambda_D^{-1} \theta_{n-1} \cdot \nabla, \psi_j(\Lambda_D) \right] \theta_\varepsilon \right\|_{L^\infty} d\tau. \tag{3.6}
\]

By summing over \( j \in \mathbb{Z} \) and applying (2.21), we have
\[
\| \theta(t) \|_{\dot{B}^0_{\infty,1}} \leq C \| \theta_0 \|_{\dot{B}^0_{\infty,1}} + C \sum_{j \in \mathbb{Z}} \int_0^T \left\| \nabla ^\perp \Lambda_D^{-1} \theta_{n-1} \cdot \nabla, \psi_j(\Lambda_D) \right\|_{\dot{B}^0_{\infty,1}} d\tau
\]
\[
\leq C \| \theta_0 \|_{\dot{B}^0_{\infty,1}} + C \int_0^T \| \theta_{n-1} \|_{\dot{B}^{\frac{1}{2}}_{\infty,1}} \| \theta_\varepsilon \|_{\dot{B}^{\frac{1}{2}}_{\infty,1}} d\tau \tag{3.7}
\]
\[
\leq C \| \theta_0 \|_{\dot{B}^0_{\infty,1}} + C \| \theta_{n-1} \|_{L^2(0,T; \dot{B}^{\frac{1}{2}}_{\infty,1})} \| \theta_\varepsilon \|_{L^2(0,T; \dot{B}^{\frac{1}{2}}_{\infty,1})}.
\]

Also, by multiplying (3.6) by \( 2^j \) and taking the \( L^1 \) norm for time variable and the \( \ell^1 \) norm
\[
\| \theta_\varepsilon \|_{L^1(0,T; \dot{B}^{1}_{\infty,1})} \leq C \| e^{-ct\Lambda_D} \theta_0 \|_{L^1(0,T; \dot{B}^{1}_{\infty,1})} + C \int_0^T \| \theta_{n-1} \|_{\dot{B}^{\frac{1}{2}}_{\infty,1}} \| \theta_\varepsilon \|_{\dot{B}^{\frac{1}{2}}_{\infty,1}} d\tau
\]
\[
\leq C \| e^{-ct\Lambda_D} \theta_0 \|_{L^1(0,T; \dot{B}^{1}_{\infty,1})} + C \| \theta_{n-1} \|_{L^2(0,T; \dot{B}^{\frac{1}{2}}_{\infty,1})} \| \theta_\varepsilon \|_{L^2(0,T; \dot{B}^{\frac{1}{2}}_{\infty,1})}. \tag{3.8}
\]

We then take \( \delta > 0 \) such that
\[
\delta + C(2\delta)^2 \leq 2\delta,
\]
and \( T > 0 \) such that
\[
C \max \left\{ \| e^{-ct\Lambda_D} \theta_0 \|_{L^1(0,T; \dot{B}^{1}_{\infty,1})}, \left( \| \theta_0 \|_{\dot{B}^0_{\infty,1}} + (2\delta)^2 \right)^{\frac{1}{2}} \right\} \leq \delta. \tag{3.9}
\]
We have from (3.8) that
\[ \| \theta_\varepsilon \|_{L^1([0,T];\hat{B}_{\infty,1}^1)} \leq \delta + C(2\delta)^2 \leq 2\delta, \quad \text{with } n = 2. \tag{3.10} \]

Step 3 (Independent existence time of \( \varepsilon \) when \( n = 2 \)). By Step 1 above, it is sufficient to have a boundedness in \( H^2 \) independent of \( \varepsilon, n \). \( \Lambda_D^2 = -\Delta_D \) acting on equation (3.3), we have
\[ \partial_t \Lambda_D^2 \theta_\varepsilon + \Lambda(\Lambda_D^2 \theta_\varepsilon) + (\nabla \Lambda_D^{-1} \theta_{n-1} \cdot \nabla) \Lambda_D^2 \theta_\varepsilon = 2(\nabla \nabla \Lambda_D^{-1} \theta_{n-1} \cdot \nabla) \nabla \theta_\varepsilon + (\nabla \Lambda_D \theta_{n-1} \cdot \nabla) \theta_\varepsilon. \]

Multiplication by \( \Lambda_D^2 \theta_\varepsilon \), integration over the domain and the Hölder’s inequality give
\[
\frac{1}{2} \theta_\varepsilon \| \theta_\varepsilon \|_{H^2}^2 \leq \frac{1}{2} \| \partial_t \theta_\varepsilon \|_{H^2}^2 + \| \theta_\varepsilon \|_{H^2}^2 \leq \| \nabla \nabla \Lambda_D^{-1} \theta_{n-1} \|_{L^\infty} \| \nabla \theta_\varepsilon \|_{L^2} \| \Lambda_D^2 \theta_\varepsilon \|_{L^2} + \| \nabla \Lambda_D \theta_{n-1} \|_{L^2} \| \nabla \theta_\varepsilon \|_{L^\infty} \| \Lambda_D^2 \theta_\varepsilon \|_{L^2} \leq C\| \theta_{n-1} \|_{\hat{B}_{\infty,1}^1} \| \theta_\varepsilon \|_{H^2}^2 + C\| \theta_{n-1} \|_{H^2} \| \theta_\varepsilon \|_{\hat{B}_{\infty,1}^1} \| \theta_\varepsilon \|_{H^2}^2.
\]

By Young’s inequality and integrating over a time interval, when \( 0 \leq t \leq T \),
\[
\| \theta_\varepsilon \|_{H^2}^2 \leq \| \theta_0 \|_{H^2}^2 + C \int_0^t \left( \| \theta_{n-1} \|_{\hat{B}_{\infty,1}^1} \| \theta_\varepsilon \|_{H^2}^2 + \| \theta_\varepsilon \|_{\hat{B}_{\infty,1}^1} \| \theta_{n-1} \|_{H^2}^2 + \| \theta_\varepsilon \|_{\hat{B}_{\infty,1}^1} \| \theta_\varepsilon \|_{H^2}^2 \right) \, dt \leq \| \theta_0 \|_{H^2}^2 + C\| \theta_\varepsilon \|_{L^1([0,T];\hat{B}_{\infty,1}^1)} \| \theta_{n-1} \|_{L^\infty([0,T];H^2)}^2 + C \int_0^t \left( \| \theta_{n-1} \|_{\hat{B}_{\infty,1}^1} + \| \theta_\varepsilon \|_{\hat{B}_{\infty,1}^1} \right) \| \theta_\varepsilon \|_{H^2}^2 \, dt,
\]
and the Gronwall’s inequality implies that
\[
\| \theta_\varepsilon(t) \|_{H^2}^2 \leq \left( \| \theta_0 \|_{H^2}^2 + C \cdot 2\delta \cdot (2\delta)^2 \right) \exp \left\{ C \int_0^t \left( \| \theta_{n-1} \|_{\hat{B}_{\infty,1}^1} + \| \theta_\varepsilon \|_{\hat{B}_{\infty,1}^1} \right) \| \theta_\varepsilon \|_{H^2}^2 \, dt \right\}.
\]
For \( T > 0 \) defined by (3.9), we have from (3.10) that
\[
\| \theta_\varepsilon(t) \|_{H^2}^2 \leq \left( \| \theta_0 \|_{H^2}^2 + C \cdot 2\delta \cdot (2\delta)^2 \right) \exp \left\{ C(2\delta + 2\delta) \right\}, \quad \text{as long as } 0 < t \leq T,
\]
which yields the existence of \( \theta_\varepsilon \in C([0,T], H^2) \) for all \( \varepsilon > 0 \).

Step 4 (Convergence as \( \varepsilon \to 0 \) and the existence of \( \theta_n \) when \( n = 2 \)). Let \( 0 < \varepsilon' < \varepsilon \) and the difference of \( \theta_\varepsilon - \theta_{\varepsilon'} \) satisfies
\[
\partial_t (\theta_\varepsilon - \theta_{\varepsilon'}) + \Lambda_D (\theta_\varepsilon - \theta_{\varepsilon'}) - \varepsilon \Delta_D (\theta_\varepsilon - \theta_{\varepsilon'}) - (\varepsilon - \varepsilon') \Delta_D \theta_{\varepsilon'} + (\nabla \nabla \Lambda_D^{-1} \theta_{n-1} \cdot \nabla) (\theta_\varepsilon - \theta_{\varepsilon'}) = 0.
\]

The \( L^2 \) inner product with \( -\Delta_D (\theta_\varepsilon - \theta_{\varepsilon'}) \) gives
\[
\frac{1}{2} \| \theta_\varepsilon(t) - \theta_{\varepsilon'}(t) \|_{H^1}^2 \leq \int_0^t \left( (\varepsilon - \varepsilon') \| \theta_\varepsilon \|_{H^2} \| \theta_\varepsilon - \theta_{\varepsilon'} \|_{H^2} + \| \nabla \nabla \Lambda_D^{-1} \theta_{n-1} \|_{L^\infty} \| \theta_\varepsilon - \theta_{\varepsilon'} \|_{H^1}^2 \right) \, dt \leq 2(\varepsilon - \varepsilon') T \sup_{\tau \in [0,T]} \| \theta_\varepsilon \|_{H^2} \| \theta_{\varepsilon'} \|_{H^2} + C \int_0^t \| \theta_{n-1} \|_{\hat{B}_{\infty,1}^1} \| \theta_\varepsilon - \theta_{\varepsilon'} \|_{H^2}^2 \, dt.
The Gronwall’s inequality implies that
\[
\sup_{t \in [0, T]} \| \theta(t) - \theta(t') \|_{L^1} \leq 4(\varepsilon - \varepsilon') T \sup_{t \in [0, T]} \exp \left\{ C \| \theta_{n-1} \|_{L^1(0, T; \dot{H}^1_{\infty, 1})} \right\} \rightarrow 0 \quad \text{as } \varepsilon, \varepsilon' \rightarrow 0,
\]
which implies that \{\theta(t)\}_{\varepsilon} satisfies a Cauchy condition, and we then obtain a limit function in \( C([0, T], H^1) \) and the uniform boundedness yields that it also belongs to \( L^\infty(0, T; \dot{H}^2) \cap L^\infty(0, T; \dot{B}^0_{\infty, 1}) \cap L^1(0, T; \dot{B}^1_{\infty, 1}) \) and that the time derivative is in \( L^1(0, T \dot{B}^0_{\infty, 1}) \). By taking \( \varepsilon \rightarrow 0 \) for the integral equation (3.4) in the topology of \( L^\infty(0, T; L^2) \), we have that the limit function is a solution of (3.2), and hence we obtain \( \theta_n \) with \( n = 2 \).

Step 5 (The case when \( n \geq 3 \)). We use an induction argument. It is possible to argue analogously to Steps 1, 2, 3, 4 and we obtain \( \theta_n \) under an assumption that \( \theta_{n-1} \) exists as a solution of (3.2) with \( n \) replaced by \( n - 1 \) such that it belongs to \( C([0, T], H^2) \cap L^\infty(0, T; \dot{B}^0_{\infty, 1}) \cap L^1(0, T; \dot{B}^1_{\infty, 1}) \), where the existence time \( T \) is same as Step 3.

**Proposition 3.2.** Let \( \theta_n \) be obtained in Proposition 3.1. Then, there exists \( T_0 \leq T \) such that \( \theta_n \) converges in \( L^\infty(0, T_0; \dot{B}^0_{\infty, 1}) \cap L^1(0, T_0; \dot{B}^1_{\infty, 1}) \) as \( n \rightarrow \infty \), and the limit function, \( \theta \), is a unique solution of (1.1) with the initial condition (1.2) in \( L^\infty(0, T_0; \dot{B}^0_{\infty, 1}) \cap L^1(0, T_0; \dot{B}^1_{\infty, 1}) \).

**Proof.** Similar to (3.5), we write
\[
\partial_t \psi_j(\Lambda_D)(\theta_{n+1} - \theta_n) + \Lambda_D \psi_j(\Lambda_D)(\theta_{n+1} - \theta_n) + (\nabla^\perp \Lambda_D^{-1} \theta_n \cdot \nabla) \psi_j(\Lambda_D)(\theta_{n+1} - \theta_n) = \left[ \nabla^\perp \Lambda_D^{-1} \theta_n \cdot \nabla, \psi_j(\Lambda_D) \right](\theta_{n+1} - \theta_n) - \psi_j(\Lambda_D) \left( \left( \nabla^\perp \Lambda_D^{-1} (\theta_n - \theta_{n-1}) \cdot \nabla \right) \theta_n \right).
\]

We argue similarly to (3.6), (3.7), (3.8), but we apply a commutator estimate (2.20) for the first term in the right-hand side of the equality above instead of (2.21) and a bilinear estimate (2.19) for the second term.

\[
\| \theta_{n+1} - \theta_n \|_{L^\infty(0, t; \dot{B}^0_{\infty, 1}) \cap L^1(0, t; \dot{B}^1_{\infty, 1})} \leq C \| \phi_{n+1}(\Lambda_D) \theta_0 \|_{L^\infty} + C \int_0^t \| \theta_n \|_{\dot{B}^1_{\infty, 1}} \| \theta_{n+1} - \theta_n \|_{\dot{B}^0_{\infty, 1}} \, dt + C \int_0^t \| \theta_n - \theta_{n-1} \|_{\dot{B}^0_{\infty, 1}} \| \theta_n \|_{\dot{B}^1_{\infty, 1}} \, dt.
\]

We here write
\[
D_{n+1}(t) := \| \theta_{n+1} - \theta_n \|_{L^\infty(0, t; \dot{B}^0_{\infty, 1}) \cap L^1(0, t; \dot{B}^1_{\infty, 1})}
\]
and then obtain by the inequality above that
\[
D_n(t) \leq C \| \phi_{n+1}(\Lambda_D) \theta_0 \|_{L^\infty} + C \| \theta_n \|_{L^1(0, t; \dot{B}^1_{\infty, 1})} (D_{n+1}(t) + D_n(t)).
\]
We may have
\[ C \| \theta_n \|_{L^1(0,T;\dot{B}^1_{\infty,1})} \leq \frac{1}{2} \]
between the time interval shorter if necessary, and it yields that
\[ D_{n+1}(t) \leq 2C \| \phi_{n+1}(\Lambda_D)\theta_0 \|_L^\infty + \frac{1}{2} D_n(t). \]
We deduce that
\[ \lim_{n \to \infty} \sum_{k=2}^{n} (\theta_k - \theta_{k-1}) + \theta_1 \quad \text{in} \quad L^\infty(0,T;\dot{B}^0_{\infty,1}) \cap L^1(0,T;\dot{B}^1_{\infty,1}) \]
exists, and let us define \( \theta \) by the limit above. We can also prove that \( \theta \) belongs to \( C([0,T];\dot{B}^0_{\infty,1}) \) and satisfies \( \partial_t \theta \in L^1(0,T;\dot{B}^0_{\infty,1}) \) and the equation \( \partial_t \theta + \Lambda_D \theta + (\nabla \cdot \Lambda_D^{-1} \theta \cdot \nabla)\theta = 0 \).

The uniqueness of solutions follows from an argument starting with a similar equality to (3.11) for the difference of two solutions \( \theta, \tilde{\theta} \) with the same data, which implies
\[ \| \theta - \tilde{\theta} \|_{L^\infty(0,T;\dot{B}^0_{\infty,1}) \cap L^1(0,T;\dot{B}^1_{\infty,1})} \leq C \left( \| \theta \|_{L^1(0,T;\dot{B}^1_{\infty,1})} + \| \tilde{\theta} \|_{L^1(0,T;\dot{B}^1_{\infty,1})} \right) \| \theta - \tilde{\theta} \|_{L^\infty(0,T;\dot{B}^0_{\infty,1}) \cap L^1(0,T;\dot{B}^1_{\infty,1})}. \]

Therefore, we conclude the existence and the uniqueness of the local solution.

**Proof of Theorem 1.1.** Local existence of solutions follows from Proposition 3.2. We also see that global existence for small data holds, since the constants, \( C \), appearing in the proofs of Propositions 3.1 and 3.2 are independent of the time interval and we can have the global existence result. \( \Box \)

**Remark.** We here introduce an alternative simple proof for the interpolation index \( q = 1 \) by using the integral equation, seeking a fixed point of the integral equation.

\[ \theta(t) = e^{-t\Lambda_D}\theta_0 - \int_0^t e^{-(t-\tau)\Lambda_D} \left( (\nabla \cdot \Lambda_D^{-1} \theta \cdot \nabla)\theta \right) d\tau. \]

Let
\[ \Psi(\theta) := e^{-t\Lambda_D}\theta_0 - \int_0^t e^{-(t-\tau)\Lambda_D} \left( (\nabla \cdot \Lambda_D^{-1} \theta \cdot \nabla)\theta \right) d\tau, \]

\[ X := L^\infty(0,\infty;\dot{B}^0_{\infty,1}) \cap L^1(0,\infty;\dot{B}^1_{\infty,1}). \]

We then have from maximum regularity estimate (2.7), (2.8) and the bilinear estimate (2.19) that
\[ \| \Psi(\theta) \|_X \leq C \| \theta_0 \|_{\dot{B}^0_{\infty,1}} + C \int_0^\infty \left\| (\nabla \cdot \Lambda_D^{-1} \theta \cdot \nabla)\theta \right\|_{\dot{B}^0_{\infty,1}} d\tau \leq C \| \theta_0 \|_{\dot{B}^0_{\infty,1}} + C \| \theta \|_X^2, \]
\[ \| \Psi(\theta) - \Psi(\tilde{\theta}) \|_X \leq C (\| \theta \|_X + \| \tilde{\theta} \|_X) \| \theta - \tilde{\theta} \|_X, \]
which allows to apply contraction argument. We conclude the global existence for small initial data.
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Appendix A. Equivalency of two resolutions

Proof of Lemma 2.1. The resolution with \( \{ \phi_j \}_{j \in \mathbb{Z}} \) follows from Lemma 4.5 in [20]. We write

\[
\| f \|_{\tilde{B}^s_{p,q}} := \left\{ \sum_{j \in \mathbb{Z}} \left( 2^{sj} \| \psi_j(\Lambda_D) f \|_{L^p} \right)^q \right\}^{1/q}
\]

and show the equivalency of the norm defined by \( \{ \psi_j \}_{j \in \mathbb{Z}} \), where we see

\[
\psi_j(\lambda) = \frac{3}{4} \cdot \frac{2^{-2j} \lambda^2}{(1 + 2^{-2j} \lambda^2)(1 + 2^{-2j} \lambda^2)}.
\]

Let \( \Phi_j := \phi_{j-1} + \phi_j + \phi_{j+1} \), which satisfies \( \phi_j = \Phi_j \phi_j \). It follows from the resolution by \( \{ \phi_{j'} \}_{j' \in \mathbb{Z}} \) and the boundedness of the spectral multiplier that

\[
\| f \|_{\tilde{B}^s_{p,q}} \leq \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{j' \in \mathbb{Z}} 2^{sj} \| \psi_j(\Lambda_D) \Phi_{j'}(\Lambda_D) \phi_{j'}(\Lambda_D) f \|_{L^p} \right)^q \right\}^{1/q}
\]

\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{j' \in \mathbb{Z}} 2^{sj} \frac{2^{-2j+2j'}}{(1 + 2^{-2j+2j'} \lambda^2)} \| \phi_{j'}(\Lambda_D) f \|_{L^p} \right)^q \right\}^{1/q}
\]

\[
\leq C \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{j' \in \mathbb{Z}} 2^{-2(|s|)|j-j'|} \cdot 2^{sj'} \| \phi_{j'}(\Lambda_D) f \|_{L^p} \right)^q \right\}^{1/q} \leq C \| f \|_{\tilde{B}^s_{p,q}}.
\]
Conversely, we have from the resolution by \( \{ \psi_j' \} \) that
\[
\| f \|_{\dot{B}^s_{p,q}} \leq \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{j' \in \mathbb{Z}} 2^{sj} \| \Phi_j(\Lambda_D) \Phi_j(\Lambda_D) \psi_j(\Lambda_D) f \|_{L^p} \right)^q \right\}^{\frac{1}{q}}.
\]

For \( N \in \mathbb{N} \) to be fixed later, we divide the sum over \( j' \in \mathbb{Z} \) into two cases of \( |j - j'| \leq N \) and \( |j - j'| > N \). For the first case, the uniform boundedness of \( \phi_j(\Lambda) \) implies that a constant \( C_N \) depending on \( N \) exists such that
\[
\left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{|j - j'| \leq N} 2^{sj} \| \Phi_j(\Lambda_D) \Phi_j(\Lambda_D) \psi_j(\Lambda_D) f \|_{L^p} \right)^q \right\}^{\frac{1}{q}} \leq C_N \| f \|_{\dot{B}^s_{p,q}}.
\]

We estimate the second case,
\[
\left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{|j - j'| > N} 2^{sj} \| \Phi_j(\Lambda_D) \Phi_j(\Lambda_D) \psi_j(\Lambda_D) f \|_{L^p} \right)^q \right\}^{\frac{1}{q}} \leq \left\{ \sum_{j \in \mathbb{Z}} \left( \sum_{|j - j'| > N} 2^{-2j' + 2j} \frac{1}{1 + 2^{-2j' + 2j}} \cdot 2^{sj} \| \Phi_j(\Lambda_D) f \|_{L^p} \right)^q \right\}^{\frac{1}{q}} \leq C \| f \|_{\dot{B}^s_{p,q}} \sum_{|j' > N} 2^{-2|j'|}.
\]

The three inequalities above yield that
\[
\left( 1 - C \sum_{|j' > N} 2^{-2|j'|} \right) \| f \|_{\dot{B}^s_{p,q}} \leq C_N \| f \|_{\dot{B}^s_{p,q}},
\]
and we obtain the converse inequality by taking \( N \) sufficiently large. □

Appendix B. Boundary value of \((\nabla^\perp \Lambda_D^{-1} f \cdot \nabla)g\)

Proof of Lemma 2.3. (i) Let \( f \in \dot{B}^0_{\infty,1} \). Then, we have from Lemma 2.1 and \( \dot{B}^0_{\infty,1} \hookrightarrow L^\infty \) that
\[
f = \sum_{j \in \mathbb{Z}} \phi_j(\Lambda_D) \quad \text{in } L^\infty.
\]

We write
\[
\phi_j(\Lambda_D) f(x) = (-\Lambda_D)^{-1} \left( \Lambda_D^2 \phi_j(\Lambda_D) f \right)(x)
= \int_{|y| < 1} \left( \frac{1}{2\pi} \log |x - y| + \Phi(x, y) \right) \left( \Lambda_D^2 \phi_j(\Lambda_D) f \right)(y) \, dy.
\]
and see that \( (\Lambda^2_D \phi_j(\Lambda_D)f) \) is in \( L^\infty \) by the boundedness of the spectral multiplier (2.10). Therefore, the integral above can be regarded as a continuous function for \(|x| \leq 1\) and it is easy to check that the integral is zero when \(|x| = 1\). \( \square \)

**Proof of Lemma 2.3.** (ii) Since \( \Phi(x, y) = \Phi(y, x) \), we write

\[
2\pi(-\Delta^\perp_D)^{-1}(x, y) = 2\pi \partial_{x_1} \left( \frac{1}{2\pi} \log |x - y| - \Phi(x, y) \right) = \frac{x_1 - y_1}{|x - y|^2} - \frac{x_1 - \frac{|x|^2}{|y|^2}}{|x - \frac{y}{|y|^2}|^2}.
\]

When \( x = (0, 1) \),

\[
2\pi \partial_{x_1} \left( \frac{1}{2\pi} \log |x - y| - x \Phi(x, y) \right) = \frac{-y_1}{|x - y|^2} - \frac{-y_1}{|x|^2 |x - \frac{y}{|y|^2}|^2} \bigg|_{x=(0,1)} = \frac{-y_1}{|x - y|^2} - \frac{-y_1}{|x|^2 |y - \frac{x}{|x|^2}|^2} \bigg|_{x=(0,1)} = 0.
\]

Therefore, when \( x = (0, 1) \)

\[
(\nabla^\perp \Lambda^{-1}_D f_k \cdot \nabla) g_l = \left( \nabla^\perp (-\Delta^\perp_D)^{-1} (\Lambda^2_D f_k \cdot \nabla) \right) (-\Delta^\perp_D)^{-1} (\Lambda^2_D g_l) = 0.
\]

We also see that the domain and the derivative \( (\nabla^\perp \Lambda^{-1}_D (\cdot \cdot \nabla)) (\cdot) \) are invariant under the rotation and we see that \( (\nabla^\perp \Lambda^{-1}_D f_k \cdot \nabla) g_l \) is zero on the boundary. \( \square \)

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