EXISTENCE AND STABILITY RESULTS TO A CLASS OF FRACTIONAL RANDOM IMPLICIT DIFFERENTIAL EQUATIONS INVOLVING A GENERALIZED HILFER FRACTIONAL DERIVATIVE

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Abstract. In this paper, the existence, uniqueness and stability of random implicit fractional differential equations (RIFDs) with nonlocal condition and impulsive effect involving a generalized Hilfer fractional derivative (HFD) are discussed. The arguments are discussed via Krasnoselskii’s fixed point theorems, Schaefer’s fixed point theorems, Banach contraction principle and Ulam type stability. Some examples are included to ensure the abstract results.

1. Introduction. Dynamic problems appearing in commercial or herbal sciences rely on the precision of the information we have regarding the parameters that explain those problems. If the approximation of a dynamic problem is accurate then a deterministic dynamical problem occurs. Unfortunately, in most of the instances, the accessible statistics for the explanation and valuation of parameters of a dynamic problem are imprecise, inexact or difficult. When our perceptives about the parameters of a dynamic problem are of arithmetical nature, that is, in order are probabilistic; the general approach in mathematical modeling of such problems is utilized of random differential equations (RDEs) or stochastic differential equations. RDEs, as natural extensions of deterministic ones, come up in several applications and have been observed by many mathematicians. We refer the reader to [20, 17, 23] and the mentions therein.

There are actual world phenomena with uncharacteristic dynamics such as signals transmissions through strong magnetic fields, network traffic, the effect of speculations on the profitability of stocks in financial markets, atmospheric diffusion of pollution, and so on where the usual models are not suitably good to depict these

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characters. In this case, the theory of fractional differential equations (FDEs) including derivatives with (classical) singular kernels [15] is an excellent device for modeling such phenomena. Recently, for the sake of a better description of such model, scientists paid attention to fractional operators with nonsingular kernels [2, 3, 4, 5, 6, 7, 10].

The revise of the FDEs with random parameters seems to be a usual one. However, there are few basic results in the hypothesis of fractional calculus and FDEs, see [9, 11, 13, 15, 18]. Permanence is the one of the most common terms used in literature whenever we treat with the dynamical systems and their behaviors. Here Ulam’s theory of steadiness is utilized to the RDEs. The ideas of Ulam’s stability are adopted from [1, 14, 22, 24, 25].

Alternatively, the nonlocal Cauchy problems, in several cases, take much better effect in uses than the usual problems with a local initial datum. For more aspect information about the significance of nonlocal initial conditions in uses, one can refer to [8].

Initially, we consider the existence, uniqueness and stability of solutions of RIDES with nonlocal condition of the structure

\[
\begin{align*}
\mathcal{D}_{g}^{\alpha,\beta} h(t, \vartheta) &= f(t, \vartheta, h(t, \vartheta), \mathcal{D}_{g}^{\alpha,\beta} h(t, \vartheta)), \quad t \in J := [0, b], b > 0, \\
\mathcal{I}_{g}^{\frac{1}{\nu}} h(t, \vartheta) |_{t=0} &= \sum_{i=1}^{m} c_{i} h(\tau_{i}, \vartheta),
\end{align*}
\]

where \( \mathcal{D}_{g}^{\alpha,\beta} \) is the generalized HFD of orders \( \alpha \in (0, 1) \) and type \( \beta \in [0, 1] \). Further, \( h \) is a random function, \( \vartheta \) is the random variable and \( \mathcal{I}_{g}^{\frac{1}{\nu}} \) is the fractional integral of orders \( 1 - \nu \) with respect to the continuous increasing function \( g(t), g'(t) \neq 0 \) on \( [0, b] \). Let \( \Omega \) be the probability space with \( f : J \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) is a given continuous function with \( \vartheta \in \Omega, c_{i}, i = 0, 1, ..., m \) are prefixed points satisfying \( 0 < \tau_{1} \leq ... \leq \tau_{m} < b \) and \( c_{i} \) are real numbers. We establish an equivalent mixed type integral equation given by

\[
h(t, \vartheta) = \left\{ \begin{array}{l}
\frac{T}{\Gamma(\nu)} (g(t))^{\nu-1} \sum_{i=1}^{m} c_{i} \int_{\tau_{i}}^{t} g' (s) (g(t) - g(s))^{\alpha-1} f_{b}(s) ds \\
+ \frac{1}{\Gamma(\nu)} \int_{0}^{t} g' (s) (g(t) - g(s))^{\alpha-1} f_{b}(s) ds,
\end{array} \right.
\]

where

\[
T = \frac{1}{\Gamma(\nu) - \sum_{i=1}^{m} c_{i} (g(\tau_{i})))^{\nu-1}}.
\]

The impulsive differential equations (IDEs) describe the evolution systems whose state changes rapidly in sometimes, which cannot be modeled by usual initial value problems. The IDEs are possible mathematical models of several genuine processes and phenomena studied by researchers of physics, biology, population dynamics, neural networks, industrial robotics and economics. For instance, it is usually accepted that a few kinds of impulsive effects are predictable in population connections. Also, in best control of economic systems, frequency-modulated signal processing systems, and some soaring object motions, many systems are considered by sudden varies in their states at certain on the spots. Practical impulsive mathematical models have become an active research theme in nonlinear science and have attracted more attention in many fields [1, 16, 19].

Next, we discuss the existence, uniqueness and stability of solutions of RIDES with impulsive involving generalized Hilfer fractional derivative of the form
The integral equation of the problem (3) is of the form

\[
\mathfrak{h}(t, \vartheta) = \frac{h_0(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1} + \sum_{0 < t_k < t} h_{t_k}(\vartheta) (g(t))^{\nu-1} + \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} f_\vartheta(s, \vartheta) ds.
\]

We introduce the following hypotheses which will be used later to prove the existence, uniqueness and stability results of problems (1) and (3).

(H1) There exist functions \( l, k : J \times \Omega \to R^+ \) such that

\[
|f(t, \vartheta, \mathfrak{h}(t, \vartheta), \mathfrak{h}(t, \vartheta)) - f(t, \vartheta, \mathfrak{h}(t, \vartheta), \mathfrak{h}(t, \vartheta))| \leq l(t, \vartheta) |\mathfrak{h} - \mathfrak{h}| + k(t, \vartheta) |\vartheta - \vartheta|,
\]

for any \( \mathfrak{h}, \vartheta, \mathfrak{h}, \vartheta \in R \), and \( L(\vartheta) = \sup_{t \in J} l(t, \vartheta) \) and \( K(\vartheta) = \sup_{t \in J} k(t, \vartheta) \), where \( t \in J \) and \( \vartheta \in \Omega \).

(H2) There exist \( l, m, n : J \times \Omega \to R^+ \) with \( l^* = \sup_{t \in J} l(t, \vartheta) < 1 \) such that

\[
|f(t, \vartheta, \mathfrak{h}, \vartheta)| \leq l(t, \vartheta) + m(t, \vartheta) |\mathfrak{h}| + n(t, \vartheta) |\vartheta|,
\]

for \( t \in J \) and \( \mathfrak{h}, \vartheta \in R \).

(H3) Let the functions \( h_k : J \times \Omega \to R \) are continuous and there exists a constant \( \ell^* > 0 \), such that

\[
|h_{t_k}(\vartheta) - h_{t_k}(\vartheta)| \leq \ell^* |\mathfrak{h} - \mathfrak{h}|, \quad \text{for all } \mathfrak{h}, \vartheta \in R, \ k = 1, 2, ..., m.
\]

(H4) Let the functions \( h_k : J \times \Omega \to R \) are continuous and there exists a constant \( h^* > 0 \), such that

\[
|h_k(\vartheta)| \leq h^*, \quad \text{for all } h^* \in R, \ k = 1, 2, ..., m.
\]

(H5) There exists an increasing function \( \varphi : J \times \Omega \to R^+ \) and there exists \( \lambda_\varphi > 0 \) such that for any \( t \in J \)

\[
\mathfrak{I}_g^\alpha \varphi(t, \vartheta) \leq \lambda_\varphi \varphi(t, \vartheta).
\]

The rest of this manuscript is systematized as follows. In Section 2, some notations and research are given. In Section 3, some adequate conditions are obtained to certify the existence of solution and stability results for nonlocal RIDEs. In Section 4, some existence results and stability analysis of RIDEs with impulsive is discussed.

2. Preliminaries. In this part, we will begin fractional integral and derivative, some details and definitions.

Throughout this article, we set \((\Omega, A, P)\) be a complete probability space. Let \(([0, b], I, \lambda)\) be a Lebesgue measure space and \(h(\cdot, \cdot) : [0, b] \times \omega \to R\) be a product
measurable function. Denote \( L^p(\varnothing), \ 1 \leq p < \infty \) by a Banach space of equivalence classes of random function \( h(\cdot, \varnothing) \) endowed with norm
\[
\|h\|_{L^p} = \left( \int_\varnothing |h(\varnothing)|^p \, d\varnothing \right)^{\frac{1}{p}}.
\]

Let \( X : [0, b] \to L^p(\varnothing) \) be a measurable function. For each \( t \in [0, b] \), \( X(t) = \tilde{h}(t, \cdot) \) be the equivalence classes of random function \( h(t, \cdot) \). For each \( t \in [0, b] \), we can select a particular function \( h(t, \cdot) \in X(t) \), the resulting function \( h(t, \cdot) : [0, b] \times \Omega \to R \) will be called a representation of the function \( X \).

We set up a Banach space of all continuous random functions space, \( C([0, b] \times \Omega, R) := \{ h : J \times \Omega \to R \} \) with the norm
\[
\|h\|_{C(\varnothing)} = \sup \{ |h(t, \varnothing)| : t \in J, \ \varnothing \in \Omega \}.
\]
We denote the weighted spaces of all continuous random functions space, defined by
\[
C_{1-\nu,p}(\varnothing) = \left\{ h : J \times \Omega \to R : (g(t))^{1-\nu} h(t, \varnothing) \in C(\varnothing) \right\}, 0 \leq \nu < 1,
\]
with the norm
\[
\|h\|_{C_{1-\nu,p}(\varnothing)} = \sup_{t \in J} \left| (g(t))^{1-\nu} h(t, \varnothing) \right|.
\]
Next, we introduce the piecewise continuous space
\[
PC(\varnothing) = \left\{ h : J \times \Omega \to R : h \in C(t_k, t_{k+1}], k = 0, ..., m; \text{ there exists} \ h(t_{k+}^-)(\varnothing) \text{ and } h(t_k^-)(\varnothing) \right\}.
\]
Now, we give the weighted piecewise continuous space of the form \( PC_{1-\nu}(\varnothing) \),
\[
PC_{1-\nu}(\varnothing) = \left\{ h : (\psi(t))^{1-\nu} h(t, \varnothing) |_{t \in [t_k, t_{k+1}]} \in C[t_k, t_{k+1}], k = 0, ..., m, \right. \\
\left. \text{where } 0 \leq \nu < 1 \right\}.
\]
Obviously, which is a Banach space with norm
\[
\|h\|_{PC_{1-\nu}(\varnothing)} = \sup_{t \in [t_k, t_{k+1}]} \left| (\psi(t))^{1-\nu} h(t, \varnothing) \right|, k = 0, ..., m.
\]

**Definition 2.1.** [21] The left-sided fractional integral of a function \( h \) with respect to the function \( g \) on \([0, b]\) is defined by
\[
I_2^g h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} h(s) \, ds, \ t > 0,
\]
where \( g \) is a continuous function such that \( g'(t) > 0 \) on \([0, b]\).

It should be mentioned that when \( g(t) = t \), the integral in (5) is the Riemann-Liouville fractional integral. While when \( g(t) = \ln t \), the integral in (5) is the Hadamard fractional integral.

**Definition 2.2.** [21] Let \( 0 < t < b \leq \infty \) and \( \alpha > 0, \ n \in N \). The Riemann-Liouville fractional derivative of a function \( h \) with respect to \( g \) of order \( \alpha \) is defined by
\[
D_2^g h(t) = \frac{1}{\Gamma(n-\alpha)} \left( \frac{1}{g'(t)} \frac{d}{dt} \right)^n \int_0^t g'(s) (g(t) - g(s))^{n-\alpha-1} h(s) \, ds,
\]
where \( n = [\alpha] + 1 \).
It is worth to mention that when $g(t) = t$, the derivative in (6) is the Riemann-Liouville fractional derivative. While when $g(t) = \ln t$, (6) is the Hadamard fractional derivative.

**Definition 2.3.** [21] Let $\alpha \geq 0$, $n \in N$, $I = [0, b]$, $h, g \in C^n([0, b], R)$ two functions such that $g$ is increasing and $g'(t) \neq 0$, for all $t \in I$. The left Caputo fractional derivative of $h$ with respect to $g$ of order $\alpha$ is given by

$$D_g^\alpha h(t) = I_g^{n-\alpha} \left( \frac{1}{g'(t)^n} \right)^n h(t), \quad (7)$$

where $n = [\alpha] + 1$ for $\alpha \notin N$ and $\alpha = n$ for $\alpha \in N$.

**Definition 2.4.** [21] The generalized Hilfer fractional derivative of order $\alpha$ or the Hilfer fractional derivative of function $h$ with respect to $g$ of order $\alpha$ is given by,

$$D_g^{\alpha, \beta} h(t) = I_g^{\beta(1-\alpha)} \left( \frac{1}{g'(t)^{\beta(1-\alpha)}} \right) I_g^{(1-\beta)(1-\alpha)} h(t). \quad (8)$$

The generalized Hilfer fractional derivative as above defined, can be written in the following

$$D_g^{\alpha, \beta} h(t) = I_g^{\nu-\alpha}\nu D_g^\nu h(t).$$

Next, we shall give the definitions and the criteria of Ulam-Hyers (U-H) stability and Ulam-Hyers-Rassias (U-H-R) stability for the $\psi$-Hilfer fractional differential equation with random effects is given by

$$D_g^{\alpha, \beta} h(t, \vartheta) = f(t, \vartheta, h(t, \vartheta), D_g^{\alpha, \beta} h(t, \vartheta)). \quad (9)$$

Let $\epsilon > 0$ be a positive real number and $\varphi : J \times \Omega \rightarrow R^+$ be a continuous function. We consider the following inequalities

$$|D_g^{\alpha, \beta} \eta(t, \vartheta) - f(t, \vartheta, \eta(t, \vartheta), D_g^{\alpha, \beta} \eta(t, \vartheta))| \leq \epsilon. \quad (10)$$

$$|D_g^{\alpha, \beta} \eta(t, \vartheta) - f(t, \vartheta, \eta(t, \vartheta), D_g^{\alpha, \beta} \eta(t, \vartheta))| \leq \epsilon \varphi(t, \vartheta). \quad (11)$$

$$|D_g^{\alpha, \beta} \eta(t, \vartheta) - g(t, \vartheta, \eta(t, \vartheta), D_g^{\alpha, \beta} \eta(t, \vartheta))| \leq \varphi(t, \vartheta). \quad (12)$$

**Definition 2.5.** Eq. (9) is U-H stable if there exists a real number $C_f > 0$ such that for each $\epsilon > 0$ and for each solution $h : \Omega \rightarrow C_{1-\nu, g}(\vartheta)$ of the inequality (10) there exists a solution $\eta : \Omega \rightarrow C_{1-\nu, g}(\vartheta)$ of Eq. (9) with

$$|h(t, \vartheta) - \eta(t, \vartheta)| \leq C_f \epsilon, \quad t \in J, \vartheta \in \Omega.$$

**Definition 2.6.** Eq. (9) is Generalized U-H stable if there exist $\varphi_f \in C([0, \infty), [0, \infty]), \varphi_f(0) = 0$ such that for each solution $h : \Omega \rightarrow C_{1-\nu, g}(\vartheta)$ of the inequality (10) there exists a solution $\eta : \Omega \rightarrow C_{1-\nu, g}(\vartheta)$ of Eq. (9) with

$$|h(t, \vartheta) - \eta(t, \vartheta)| \leq \varphi_f \epsilon, \quad t \in J, \vartheta \in \Omega.$$

**Definition 2.7.** Eq. (9) is U-H-R stable with respect to $\varphi$ if there exists a real number $C_{f, \varphi} > 0$ such that for each $\epsilon > 0$ and for each solution $h : \Omega \rightarrow C_{1-\nu, g}(\vartheta)$ of the inequality (11) there exists a solution $\eta : \Omega \rightarrow C_{1-\nu, g}(\vartheta)$ of Eq. (9) with

$$|h(t, \vartheta) - \eta(t, \vartheta)| \leq C_{f, \varphi} \epsilon \varphi(t, \vartheta), \quad t \in J, \vartheta \in \Omega.$$

**Definition 2.8.** Eq. (9) is generalized U-H-R stable with respect to $\varphi$ if there exists a real number $C_{f, \varphi} > 0$ such that for each solution $h : \Omega \rightarrow C_{1-\nu, g}(\vartheta)$ of the inequality (12) there exists a solution $\eta : \Omega \rightarrow C_{1-\nu, g}(\vartheta)$ of Eq. (9) with

$$|h(t, \vartheta) - \eta(t, \vartheta)| \leq C_{f, \varphi} \varphi(t, \vartheta), \quad t \in J, \vartheta \in \Omega.$$
Lemma 2.9. [27] Suppose $\alpha > 0$, $a(t, \vartheta)$ is a nonnegative function locally integrable on $J \times \Omega$ (some $T \leq \infty$), and let $\gamma(t, \vartheta)$ be a nonnegative, nondecreasing continuous function defined on $J \times \Omega$, such that $\gamma(t, \vartheta) \leq K$ for some constant $K$. Further let $h(t, \vartheta)$ be a nonnegative locally integrable on $J \times \Omega$ function with

$$h(t, \vartheta) \leq a(t, \vartheta) + \gamma(t, \vartheta) \int_a^t g'(s) (g(t) - g(s))^{\alpha - 1} h(s, \vartheta)\, ds, \quad (t, \vartheta) \in J \times \Omega,$$

with some $\alpha > 0$. Then

$$h(t, \vartheta) \leq a(t, \vartheta) + \int_a^t \left[ \sum_{n=1}^{\infty} \frac{\Gamma(\alpha)}{\Gamma(n\alpha)} g'(s) (g(t) - g(a))^{n\alpha - 1} \right] a(s, \vartheta)\, ds.$$

Remark 1. Under the hypothesis of Lemma 2.9 let $a(t, \vartheta)$ be a nondecreasing function on $[0, T]$. Then $h(t, \vartheta) \leq a(t, \vartheta) E_{\alpha}(\gamma(t, \vartheta)\Gamma(\alpha)(g(t))^{\alpha})$, where $E_{\alpha}$ is the Mittag-Leffler function defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + 1)}, \quad z \in C, \quad \text{Re}(\alpha) > 0.$$

Lemma 2.10. [26] Let $h \in PC_{1-\nu}(\vartheta)$ satisfies the following inequality

$$|h(t, \vartheta)| \leq a(t, \vartheta) + \gamma(t, \vartheta) \int_0^t g'(s) (g(t) - g(s))^{\alpha - 1} |h(s, \vartheta)|\, ds + \sum_{0 < s_k < t} |h_{s_k}(\vartheta)|,$$

where $c_1$ is a nonnegative, continuous and nondecreasing function and $c_2, \lambda_i$ are constants. Then

$$|h(t, \vartheta)| \leq a(t, \vartheta) (1 + \lambda E_{\alpha}(\gamma(t, \vartheta)\Gamma(\alpha)(g(t))^{\alpha}))^{k} E_{\alpha}(\gamma(t, \vartheta)\Gamma(\alpha)(g(t))^{\alpha})$$

for $t \in (t_k, t_{k+1})$, where $\lambda = \sup \{\lambda_k : k = 1, 2, 3, ..., m\}$.

Theorem 2.11. [12] (Krasnoselskii’s fixed point theorem) Let $X$ be a Banach space, let $B$ be a bounded closed convex subset of $X$ and let $\Psi_1, \Psi_2$ be mapping from $B$ into $X$ such that $\Psi_1 h + \Psi_2 h \in B$ for every pair $h, \eta \in B$. If $\Psi_1$ is contraction and $\Psi_2$ is completely continuous, then the equation $\Psi_1 h + \Psi_2 h = h$ has a solution on $B$.

Theorem 2.12. [12] (Schaefer’s Fixed Point Theorem) Let $K$ be a Banach space and let $\Psi : K \rightarrow K$ be completely continuous operator. If the set $\{h \in K : h = \delta \Psi h \text{ for some } \delta \in (0, 1)\}$ is bounded, then $\Psi$ has a fixed point.

Theorem 2.13. [12] (Banach Fixed Point Theorem) Suppose $Q$ be a non-empty closed subset of a Banach space $E$. Then any contraction mapping $\Psi$ from $Q$ into itself has a unique fixed point.

3. Existence of solution to nonlocal random differential equation. In this part, we obtain the required condition related to the existence of solution to the considered problem.

Theorem 3.1. Assume that hypothesis $[H1]$ is satisfied. Then, Eq.(1) has at least one solution.

Proof. Consider the operator $\Psi(\vartheta) : \Omega \times C_{1-\nu, \psi}(\vartheta) \rightarrow C_{1-\nu, \psi}(\vartheta)$.

Hence, $h$ is a solution for the problem (1) if and only if

$$h(t, \vartheta) = (\Psi h)(t, \vartheta),$$
where the equivalent integral equation which can be written in the operator form

\[
(\mathcal{P} h)(t, \vartheta) = \begin{cases} 
\frac{T}{\Gamma(\alpha)} (g(t))^{\nu - 1} \sum_{i=1}^{m} c_i \int_{0}^{\tau_i} g'(s) (g(\tau_i) - g(s))^{\alpha - 1} f_{h}(s, \vartheta) ds \\
+ \frac{1}{\Gamma(\alpha)} \int_{0}^{t} g'(s) (g(t) - g(s))^{\alpha - 1} f_{h}(s, \vartheta) ds, 
\end{cases}
\]

(13)

where \( f_{h}(t, \vartheta) := \mathcal{D}^{\alpha,\beta}_{t,\vartheta} h(t, \vartheta) = f(t, \vartheta, h(t, \vartheta), f_{h}(t, \vartheta)). \)

Consider the ball \( B_{r} = \left\{ h \in C_{1-\nu, \vartheta} : \|h\|_{C_{1-\nu, \vartheta}} \leq r \right\}. \) Now we subdivide the operator \( \mathcal{P} \) into two operator \( \mathcal{P}_{1} \) and \( \mathcal{P}_{2} \) on \( B_{r} \) as follows

\[
\mathcal{P}_{1}(t, \vartheta) = \frac{T}{\Gamma(\alpha)} (g(t))^{\nu - 1} \sum_{i=1}^{m} c_i \int_{0}^{\tau_i} g'(s) (g(\tau_i) - g(s))^{\alpha - 1} f_{h}(s, \vartheta) ds,
\]

and

\[
\mathcal{P}_{2}(t, \vartheta) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} g'(s) (g(t) - g(s))^{\alpha - 1} f_{h}(s, \vartheta) ds.
\]

Set \( \tilde{f} = f(t, \vartheta, 0, 0). \)

The proof is separated into some steps.

**Step 1.** \( \mathcal{P}_{1} h + \mathcal{P}_{2} v \in B_{r} \) for every \( h, v \in B_{r}. \)

\[
|\mathcal{P}_{1} h(t, \vartheta)| = \left| \frac{T}{\Gamma(\alpha)} (g(t))^{\nu - 1} \sum_{i=1}^{m} c_i \int_{0}^{\tau_i} g'(s) (g(\tau_i) - g(s))^{\alpha - 1} f_{h}(s, \vartheta) ds \right|.
\]

\[
\leq \frac{T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_{0}^{\tau_i} g'(s) (g(\tau_i) - g(s))^{\alpha - 1} \left| f_{h}(s, \vartheta) \right| ds
\]

\[
\leq \frac{T}{\Gamma(\alpha)} \sum_{i=1}^{m} c_i \int_{0}^{\tau_i} g'(s) (g(\tau_i) - g(s))^{\alpha - 1} \left| f_{h}(s, \vartheta) \right| ds
\]

\[
\leq \left( \frac{L(\vartheta)}{1 - K(\vartheta)} \right) \left( \left\| f_{h} \right\|_{C_{1-\nu, \vartheta}} + \left\| \tilde{f} \right\|_{C_{1-\nu, \vartheta}} \right).
\]

This gives

\[
\left\| \mathcal{P}_{1} h \right\|_{C_{1-\nu, \vartheta}} \leq \left( \frac{L(\vartheta)}{1 - K(\vartheta)} \right) \left( \left\| f_{h} \right\|_{C_{1-\nu, \vartheta}} + \left\| \tilde{f} \right\|_{C_{1-\nu, \vartheta}} \right),
\]

(14)
For operator $\mathcal{P}_2$

$$
|\mathcal{P}_2(t, \vartheta)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (\psi(t) - g(s))^{\alpha - 1} f_u(s, \vartheta) ds \right|
$$

$$
|\mathcal{P}_2(t, \vartheta) (g(t))^{1-\nu}| \leq \frac{(g(t))^{1-\nu}}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha - 1} |f_u(s, \vartheta)| ds
$$

$$
\leq \frac{(g(t))^{1-\nu}}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha - 1} \times
$$

$$
(|f(s, \vartheta, h(s, \vartheta), f_h(s, \vartheta)) - f(s, \vartheta, 0, 0)| + |f(s, \vartheta, 0, 0)|) ds
$$

$$
\leq \frac{(g(t))^{1-\nu}}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha - 1} \times
$$

$$
\left( \frac{l(t, \vartheta)}{1 - k(t, \vartheta)} |h(s, \vartheta)| + |\tilde{h}(s, \vartheta)| \right) ds
$$

$$
\leq \frac{(g(t))^{1-\nu}}{\Gamma(\alpha)} B(\nu, \alpha) (g(t))^{\alpha + \nu - 1} \times
$$

$$
\left( \frac{L(\alpha)}{1 - K(\alpha)} \|h\|_{C_{1-\nu,\vartheta}(\vartheta)} + \|\tilde{h}\|_{C_{1-\nu,\vartheta}(\vartheta)} \right).
$$

Thus, we obtain

$$
\|\mathcal{P}_2 h\|_{C_{1-\nu,\vartheta}(\vartheta)} \leq \frac{1}{\Gamma(\alpha)} B(\nu, \alpha) (g(t))^{\alpha} \left( \frac{L(\alpha)}{1 - K(\alpha)} \|h\|_{C_{1-\nu,\vartheta}(\vartheta)} + \|\tilde{h}\|_{C_{1-\nu,\vartheta}(\vartheta)} \right).
$$

Linking (14) and (15), for every $h, v \in B_r$,

$$
\|\mathcal{P}_1 h + \mathcal{P}_2 v\|_{C_{1-\nu,\vartheta}(\vartheta)} \leq \|\mathcal{P}_1 h\|_{C_{1-\nu,\vartheta}(\vartheta)} + \|\mathcal{P}_2 v\|_{C_{1-\nu,\vartheta}(\vartheta)} \leq r.
$$

**Step 2.** $\mathcal{P}_1$ is a contraction mapping.

For any $h, v \in B_r$,

$$
\left| (\mathcal{P}_1 h(t, \vartheta) - \mathcal{P}_1 v(t, \vartheta)) \right|
$$

$$
\leq \frac{T}{\Gamma(\alpha)} (g(t))^{\nu - 1} \sum_{i=1}^m c_i \int_0^{\tau_i} g'(s) (g(\tau_i) - g(s))^{\alpha - 1} |f_h(s, \vartheta) - f_v(s, \vartheta)| ds
$$

$$
\leq \frac{T}{\Gamma(\alpha)} (g(t))^{\nu - 1} \sum_{i=1}^m c_i \int_0^{\tau_i} g'(s) (g(\tau_i) - g(s))^{\alpha - 1} \times
$$

$$
\left( \frac{l(t, \vartheta)}{1 - k(t, \vartheta)} \right) |h(s, \vartheta) - v(s, \vartheta)| ds
$$

$$
\leq \frac{T}{\Gamma(\alpha)} (g(t))^{\nu - 1} B(\nu, \alpha) \sum_{i=1}^m c_i (g(\tau_i))^{\alpha + \nu - 1} \left( \frac{l(t, \vartheta)}{1 - k(t, \vartheta)} \right) \|h - v\|_{C_{1-\nu,\vartheta}(\vartheta)}.
$$

This gives

$$
\| (\mathcal{P}_1 h - \mathcal{P}_1 v) \|_{C_{1-\nu,\vartheta}(\vartheta)} \leq \frac{T}{\Gamma(\alpha)} B(\nu, \alpha) \sum_{i=1}^m c_i (g(\tau_i))^{\alpha + \nu - 1} \left( \frac{L(\alpha)}{1 - K(\alpha)} \right) \times
$$

$$
\|h - v\|_{C_{1-\nu,\vartheta}(\vartheta)}.
$$

Thus, $\mathcal{P}_1$ is a contraction mapping.

**Step 3.** The operator $\mathcal{P}_2$ is compact and continuous.
According to Step 1, we know that
\[
\| \mathfrak{P}_2 h \|_{C_{1-\nu,g}(\vartheta)} \leq \frac{1}{\Gamma(\alpha)} B(\nu, \alpha) \left( \frac{L(\vartheta)}{1 - K(\vartheta)} \right) \left( \| h \|_{C_{1-\nu,f}(\vartheta)} + \| h \|_{C_{1-\nu,g}(\vartheta)} \right).
\]

Hence, operator \( \mathfrak{P}_2 \) is uniformly bounded.

Now, we confirm the compactness of operator \( \mathfrak{P}_2 \).

For \( 0 < t_1 < t_m < T \), we have
\[
\left| \left( (g(t_m))^{1-\nu} \mathfrak{P}_2 h(t_m, \vartheta) - (g(t_1))^{1-\nu} \mathfrak{P}_2 h(t_1, \vartheta) \right) \right| = \left| \frac{1}{\Gamma(\alpha)} \int_0^{t_m} \left( g'(s) (g(t_m) - g(s))^\alpha \right)^{1-\nu} f_h(s, \vartheta) ds \right|
\]
\[
- \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left( g'(s) (g(t_1) - g(s))^\alpha \right)^{1-\nu} f_h(s, \vartheta) ds \right| \]
tending to zero as \( t_1 \to t_m \). Thus \( \mathfrak{P}_2 \) is equicontinuous. Hence, the operator \( \mathfrak{P}_2 \) is compact on \( B_r \) by the Arzela-Ascoli Theorem. It follows from Theorem 2.11 that the problem (1) has at least one solution.

**Lemma 3.2.** Assume that the hypothesis \( (H1) \) is satisfied. If
\[
\frac{B(\nu, \alpha) \left( \frac{L(\vartheta)}{1 - K(\vartheta)} \right)}{\Gamma(\alpha)} \left( \| T \sum_{i=1}^{m} c_i (g(\tau_i))^\alpha + (g(b))^\alpha \right) < 1,
\]
then, (1) has a unique solution.

**Theorem 3.3.** Under the hypotheses \( (H1) \) and \( (H2) \), the solution of Eq.(1) is generalized U-H-R stable.

**Proof.** Let \( h \) be solution of inequality 12 and by Lemma 3.2 there exists a unique solution \( \eta \) for the problem (1). Thus we have
\[
h(t, \vartheta) = \eta_h + \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^\alpha f_h(s, \vartheta) ds,
\]
where
\[
\eta_h = \frac{T}{\Gamma(\alpha)} (g(t))^{\nu-1} \sum_{i=1}^{m} c_i \int_0^{\tau_i} g'(s) (g(\tau_i) - g(s))^\alpha f_h(s, \vartheta) ds.
\]

On the other hand, \( h(\tau_i, \vartheta) = \eta(\tau_i, \vartheta) \), then we get \( \eta_h = \eta_y \).

Thus
\[
h(t, \vartheta) = \eta_y + \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^\alpha f_h(s, \vartheta) ds.
\]

By differentiating inequality (12) for each \( t \in J, \ \vartheta \in \Omega \), we have
\[
\left| \eta(t, \vartheta) - \eta_y - \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^\alpha g_0(s, \vartheta) ds \right| \leq \left( T (g(t))^{\nu-1} m \sum_{i=1}^{m} c_i + 1 \right) \lambda_{\varphi}(t, \vartheta).
\]
Hence, it follows that
\[
|\eta(t, \vartheta) - h(t, \vartheta)| \\
\leq |\eta(t, \vartheta) - \xi_y - \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} f_h(s, \vartheta) ds| \\
\leq |h(t, \vartheta) - \xi_y - \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} f_h(s, \vartheta) ds| \\
+ \left| \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} f_h(s, \vartheta) ds \right|
\]

\[
- \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} f_h(s, \vartheta) ds \leq \left( T (g(t))^{\nu-1} m \sum_{i=1}^m c_i + 1 \right) \lambda_{\varphi}(t, \vartheta) \\
+ \left( \frac{L(\vartheta)}{1 - K(\vartheta)} \right) \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} |h(t, \vartheta) - \eta(t, \vartheta)| ds \leq C_{f, \varphi}(t, \vartheta).
\]

Thus, Eq. (1) is generalized U-H-R stable. \(\square\)

3.1. Example. For \(g(t) = t\) we obtain a particular case of the problem (1) involving the Hilfer fractional derivative
\[
\begin{align*}
\left\{ D_t^{\alpha,\beta} h(t, \vartheta) = \frac{1}{\Gamma(\alpha)} \sin |\vartheta| h(t, \vartheta) + D_t^{\alpha,\beta} \eta(t, \vartheta), \right. \\
\left. I_t^{1-\nu} h(t, \vartheta) |_{t=0} = 2h(\frac{1}{3}, \vartheta), \right. \\
\end{align*}
\]

Denote \(\alpha = \frac{2}{3}, \beta = \frac{1}{2}\) and \(\gamma = \frac{5}{6}\). Moreover,
\[
\left| f(t, \vartheta, h(t, \vartheta), D_t^{\alpha,\beta} h(t, \vartheta)) - f(t, \vartheta, \eta(t, \vartheta), D_t^{\alpha,\beta} \eta(t, \vartheta)) \right| \\
\leq \frac{1}{9} \left( |h(t, \vartheta) - \eta(t, \vartheta)| + |D_t^{\alpha,\beta} h(t, \vartheta) - D_t^{\alpha,\beta} \eta(t, \vartheta)| \right)
\]

(16)

Here \(L(\vartheta) = K(\vartheta) = \frac{1}{9}\), we get \(\frac{L(\vartheta)}{1 - K(\vartheta)} = \frac{1}{8}\).

On the other hand
\[
|T| = \left| \frac{1}{\Gamma(\frac{5}{6}) - 2 \left( \frac{1}{3} \right)^{\frac{5}{6}} - 1} \right| \approx 0.5203.
\]

and
\[
\frac{B(\frac{2}{3}, \frac{5}{6})}{\Gamma(\frac{5}{6})} \left( \frac{1}{8} \right) \left| T \right| \times 2 \left( \frac{1}{3} \right)^{\frac{5}{6}} + (1) \left( \frac{5}{6} \right) < 1.
\]

Next, set \(\varphi(t, \vartheta) = e^{t+\vartheta}\). Thus we have,
\[
\mathcal{U}^\varphi(t, \vartheta) \leq \frac{1}{\Gamma(\frac{5}{6})} \varphi(t, \vartheta).
\]
Thus all the assumptions of Theorem 3.1, 3.2 and 3.3 are satisfied, our result can be applied to the proposed problem.

4. Random implicit differential equation with impulsive effect.

**Theorem 4.1.** Assume that \([H1]\) - \([H4]\) are satisfied. Then, Eq. (3) has at least one solution.

**Proof.** Consider the operator \( \mathfrak{B} : \Omega \times \text{PC}_{1-\nu}(\vartheta) \to \text{PC}_{1-\nu}(\vartheta) \). The operator form of integral equation (4) is written as follows

\[
\mathfrak{h}(t, \vartheta) = \mathfrak{B}\mathfrak{h}(t, \vartheta),
\]

where

\[
(\mathfrak{B}\mathfrak{h})(t, \vartheta) = \frac{\mathfrak{h}_0(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1} + \sum_{0 < t_k < t} \frac{\mathfrak{h}_{t_k}(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1} \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g'(s)(g(t) - g(s))^{\alpha-1}}{s^{\nu}} f_{\mathfrak{h}}(s, \vartheta) \, ds
\]  

(17)

First, we prove that the operator \( \mathfrak{B} \) defined by (17) verifies the conditions of Theorem 2.12.

**Step 1.** The operator \( \mathfrak{B} \) is continuous.

Let \( \mathfrak{h}_n \) be a sequence such that \( \mathfrak{h}_n \to \mathfrak{h} \) in \( \text{PC}_{1-\nu}(\vartheta) \). Then for each \( t \in J \),

\[
\left| (\mathfrak{B}\mathfrak{h}_n(t, \vartheta) - \mathfrak{B}\mathfrak{h}(t, \vartheta))(g(t))^{1-\nu} \right| \\
\leq \frac{1}{\Gamma(\nu)} \sum_{0 < t_k < t} \left| g(\mathfrak{h}_{t_k}(t_k)) - g(\mathfrak{h}(t_k)) \right| \\
+ \frac{(g(t))^{1-\nu}}{\Gamma(\alpha)} \int_0^t \frac{g'(s)(g(t) - g(s))^{\alpha-1}}{s^{\nu}} \left| f_{\mathfrak{h}_n}(s, \vartheta) - f_{\mathfrak{h}}(s, \vartheta) \right| ds.
\]

Since \( g \) is continuous, then we have

\[
\|\mathfrak{B}\mathfrak{h}_n - \mathfrak{B}\mathfrak{h}\|_{\text{PC}_{1-\nu}} \to 0 \quad \text{as} \quad n \to \infty.
\]

This proves the continuity of \( \mathfrak{B} \).

**Step 2.** The operator \( \mathfrak{B} \) maps bounded sets into bounded sets in \( \text{PC}_{1-\nu}(\vartheta) \).

Indeed, it is enough to show that for \( r > 0 \), there exists a positive constant \( l \) such that \( B_r = \left\{ x \in \text{PC}_{1-\nu}[J, R] : \|x\|_{\text{PC}_{1-\nu}} \leq r \right\} \), we have \( \|\mathfrak{B}(\mathfrak{h})\|_{\text{PC}_{1-\nu}} \leq l \).

\[
\left| (\mathfrak{B}\mathfrak{h})(t, \vartheta) (g(t))^{1-\nu} \right| \leq \frac{\mathfrak{h}_0(\vartheta)}{\Gamma(\nu)} + \sum_{0 < t_k < t} \frac{|h_{t_k}(\vartheta)|}{\Gamma(\nu)} \\
+ \frac{(g(t))^{1-\nu}}{\Gamma(\alpha)} \int_0^t \frac{g'(s)(g(t) - g(s))^{\alpha-1}}{s^{\nu}} |f_{\mathfrak{h}}(s, \vartheta)| \, ds \\
\leq \frac{\mathfrak{h}_0(\vartheta)}{\Gamma(\nu)} + \frac{(g(t))^{\nu-1}}{\Gamma(\nu)} \sum_{0 < t_k < t} |f_{\mathfrak{h}}(t)|.
\]
Then,

\[
\Phi: PC_{1^{-\nu}}(\vartheta) \to PC_{1^{-\nu}}(\vartheta)
\]

Step 3. The operator \( \Phi \) maps bounded sets into equicontinuous set of \( PC_{1^{-\nu}}(\vartheta) \).

Let \( t_1, t_m \in J, t_1 > t_m, B_r \) be a bounded set of \( PC_{1^{-\nu}}(\vartheta) \) as in Step 2, and \( h \in B_r \).

Then,

\[
\left| (g(t_1))^{1-\nu} (\Phi h)(t_1, \vartheta) - (g(t_m))^{1-\nu} (\Phi h)(t_m, \vartheta) \right| \leq \frac{\|h_0(\vartheta)\|}{\Gamma(\nu)} + m (g(b))^{1-\nu} \frac{\|h^+\|_{PC_{1^{-\nu}}}}{\Gamma(\nu)} + \int_0^{t_1} g'(s) (g(t_1) - g(s))^{\alpha-1} f_h(s, \vartheta) ds
\]

As \( t_1 \to t_m \), the right hand face of the above inequality tends to zero. As a outcome of Step 1 - 3 together with Arzelà-Ascoli theorem, we can conclude that \( \Phi: PC_{1^{-\nu}}(\vartheta) \to PC_{1^{-\nu}}(\vartheta) \) is continuous and completely continuous.

It is continuous and bounded from Step 1 - 3.

Now, it remains to show that the set

\[
\omega = \{ h \in PC_{1^{-\nu}}(\vartheta) : h = \eta \Phi(t, \vartheta), 0 < \tau < 1 \}
\]

is bounded set.

Let \( h \in \omega, h = \eta \Phi(t, \vartheta) \) for some \( 0 < \eta < 1 \). Thus for each \( t \in J \).
Theorem 4.2. Assume that [H1] and [H3] are satisfied. If

\[ \begin{align*}
\eta(\frac{h_0(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1} + \sum_{0 < t_k < t} \frac{h_t(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1} + \\
+ \frac{1}{\Gamma(\alpha)} \int_0^t g(s) (g(t) - g(s))^{\alpha-1} h_t(s, \vartheta) ds \end{align*} \]

This shows that the set \( \omega \) is bounded. As a consequence of Theorem 2.12, we deduce that \( \Psi \) has a fixed point which is a solution of problem (3).

\[ \square \]

**Theorem 4.2.** Assume that [H1] and [H3] are satisfied. If

\[ \left( \frac{m \ell^*}{\Gamma(\nu)} (g(t))^{1-\nu} + \left( \frac{L(\vartheta)}{1-K(\vartheta)} \right) (g(t))^\alpha B(\nu, \alpha) \right) < 1, \quad (18) \]

then, the Eq. (3) has a unique solution.

We consider the following inequalities

Let \( \epsilon > 0 \) be a positive real number and \( \varphi : J \times \Omega \to R^+ \) be a continuous function. We consider the following inequalities

\[ \left\{ \begin{align*}
|D^{\alpha,\beta}_g \eta(t, \vartheta) - f(t, \vartheta, \eta(t, \vartheta), D^{\alpha,\beta}_g \eta(t, \vartheta))| & \leq \epsilon, \\
|\Delta^{1-\nu}_g \eta(t, \vartheta)|_{t=t_k} - \eta_k(\vartheta) & \leq \epsilon.
\end{align*} \]

\[ \quad (20) \]

\[ \left\{ \begin{align*}
|D^{\alpha,\beta}_g \eta(t, \vartheta) - f(t, \vartheta, \eta(t, \vartheta), D^{\alpha,\beta}_g \eta(t, \vartheta))| & \leq \epsilon \varphi(t, \vartheta), \\
|\Delta^{1-\nu}_g \eta(t, \vartheta)|_{t=t_k} - \eta_k(\vartheta) & \leq \epsilon \varphi(t, \vartheta).
\end{align*} \]

\[ \quad (21) \]

\[ \left\{ \begin{align*}
|D^{\alpha,\beta}_g \eta(t, \vartheta) - f(t, \vartheta, \eta(t, \vartheta), D^{\alpha,\beta}_g \eta(t, \vartheta))| & \leq \varphi(t, \vartheta), \\
|\Delta^{1-\nu}_g \eta(t, \vartheta)|_{t=t_k} - \eta_k(\vartheta) & \leq \varphi(t, \vartheta).
\end{align*} \]

\[ \quad (22) \]

**Definition 4.3.** Eq. (19) is U-H stable if there exists a real number \( C_f > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( \eta : \Omega \to PC_{1-\nu,\alpha}(\vartheta) \) of the inequality (20) there exists a solution \( \eta : \Omega \to PC_{1-\nu,\alpha}(\vartheta) \) of Eq. (19) with

\[ |\eta(t, \vartheta) - \eta(t, \vartheta)| \leq C_f \epsilon, \quad t \in J, \vartheta \in \Omega. \]

**Definition 4.4.** Eq. (19) is Generalized U-H stable if there exists \( \varphi_f \in C([0, \infty), [0, \infty]) \), \( \varphi_f(0) = 0 \) such that for each solution \( \eta : \Omega \to PC_{1-\nu,\alpha}(\vartheta) \) of the inequality (20) there exists a solution \( \eta : \Omega \to PC_{1-\nu,\alpha}(\vartheta) \) of Eq. (19) with

\[ |\eta(t, \vartheta) - \eta(t, \vartheta)| \leq \varphi_f \epsilon, \quad t \in J, \vartheta \in \Omega. \]

**Definition 4.5.** Eq. (19) is U-H-R stable with respect to \( \varphi \) if there exists a real number \( C_{f, \varphi} > 0 \) such that for each \( \epsilon > 0 \) and for each solution \( \eta : \Omega \to PC_{1-\nu,\alpha}(\vartheta) \) of the inequality (21) there exists a solution \( \eta : \Omega \to PC_{1-\nu,\alpha}(\vartheta) \) of Eq. (19) with

\[ |\eta(t, \vartheta) - \eta(t, \vartheta)| \leq C_{f, \varphi} \epsilon \varphi(t, \vartheta), \quad t \in J, \vartheta \in \Omega. \]
Definition 4.6. Eq. (19) is generalized U-H-R stable with respect to $\varphi$ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $h : \Omega \to PC_{1-\nu,\psi}(\vartheta)$ of the inequality (22) there exists a solution $\eta : \Omega \to PC_{1-\nu,\psi}(\vartheta)$ of Eq. (19) with

$$|h(t, \vartheta) - \eta(t, \vartheta)| \leq C_{f,\varphi}\varphi(t, \vartheta), \quad t \in J, \vartheta \in \Omega.$$ 

Remark 2. A function $\eta \in PC_{1-\nu}(\vartheta)$ is a solution of the inequality

$$|D^\alpha_{g,\beta} \eta(t, \vartheta) - f(t, \vartheta, \eta(t, \vartheta), D^\alpha_{g,\beta} \eta(t, \vartheta))| \leq \epsilon,$$

if and only if there exists a function $p \in PC_{1-\nu}(\vartheta)$ and a sequence $p_k, k = 1, 2, \ldots, m$ such that

1. $|p(t)| \leq \epsilon, |p_k| < \epsilon$.
2. $D^\alpha_{g,\beta} \eta(t, \vartheta) = f(t, \vartheta, \eta(t, \vartheta), D^\alpha_{g,\beta} \eta(t, \vartheta)) + p(t)$.
3. $\Delta J^\nu_{g^{-1}}(t)\eta(t, \vartheta)|_{t=t_k} = \eta_{t_k}(\vartheta) + p_k$.

Theorem 4.7. The assumptions [H1], [H3], [H5] and (18) hold. Then, Eq. (3) is generalized U-H-R stable.

Proof. Let $\eta$ be solution of inequality (22) and by let $h$ be the unique solution of the problem

$$D^\alpha_{g,\beta} h(t, \vartheta) = f(t, \vartheta, h(t, \vartheta), D^\alpha_{g,\beta} h(t, \vartheta)), \quad t \in J := [0, b], b > 0,$$

$$\Delta J^\nu_{g^{-1}} h(t, \vartheta)|_{t=t_k} = h_{t_k}(\vartheta),$$

$$J^\nu_{g^{-1}} h(t, \vartheta)|_{t=0} = h_0(\vartheta).$$

Then, we have

$$h(t, \vartheta) = \frac{h_0(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1} + \frac{\sum_{0 \leq t < t_k} h_{t_k}(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} f_\nu(s, \vartheta) ds.$$

By differentiating inequality (22), for each $t \in (t_k, t_{k+1}]$, we have

$$\left|\eta(t, \vartheta) - \frac{h_0(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1} + \frac{\sum_{0 < t \leq t_k} h_{t_k}(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1}$$

$$+ \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} f_\nu(s, \vartheta) ds\right|$$

$$\leq \left| \sum_{0 < t \leq t_k} p_k \right| (g(t))^{\nu-1} + \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} \varphi(s, \vartheta) ds\right|$$

$$\leq \frac{m}{\Gamma(\nu)} (g(t))^{\nu-1} \varphi(t, \vartheta) + \lambda \varphi(t, \vartheta)$$

$$\leq \left( \frac{m}{\Gamma(\nu)} (g(t))^{\nu-1} + \lambda \right) \varphi(t, \vartheta).$$
Hence for each \( t \in (t_k, t_{k+1}] \), it follows
\[
|\eta(t, \vartheta) - b(t, \vartheta)| \\
\leq \left| \eta(t, \vartheta) - \frac{b_0(\vartheta)}{\Gamma(\nu)} (g(t))^{\nu-1} \right| \\
- \frac{1}{\Gamma(\alpha)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} f_b(s, \vartheta)ds \\
+ \frac{1}{\Gamma(\gamma)} (g(t))^{\gamma-1} \\
+ \frac{1}{\Gamma(\alpha)} \int_a^t g'(s) (g(t) - g(s))^{\alpha-1} |f_b(s, \vartheta) - b(s, \vartheta)| ds \\
\leq \left( m \frac{(g(t))^{\nu-1}}{\Gamma(\nu)} + \lambda_\varphi \right) \varphi(t, \vartheta) + m \frac{t^\nu}{\Gamma(\nu)} (g(t))^{\nu-1} |\eta(t, \vartheta) - b(t, \vartheta)| \\
+ \frac{L(\vartheta)}{1 - K(\vartheta)} \int_0^t g'(s) (g(t) - g(s))^{\alpha-1} |\eta(s, \vartheta) - b(s, \vartheta)| ds
\]
By Lemma 2.10, there exists a constant \( K > 0 \) independent of \( \lambda_\varphi \varphi(t, \vartheta) \) such that
\[
|\eta(t, \vartheta) - b(t, \vartheta)| \leq K \left( m \frac{(g(t))^{\nu-1}}{\Gamma(\nu)} + \lambda_\varphi \right) \varphi(t, \vartheta) := C_\varphi \varphi(t, \vartheta).
\]
Thus, Eq.(3) is generalized U-H-R stable. \( \square \)

Here we present an illustrative example to demonstrate the afore established results.

4.1. Example \( \bullet \). For \( g(t) = t \) we obtain a particular case of the problem (3) involving generalized Hilfer fractional derivative as
\[
\begin{cases}
D_t^{\alpha, \beta} b(t, \vartheta) = \frac{1 + |b(t, \vartheta)| + D_t^{\alpha, \beta} |b(t, \vartheta)| |b(t, \vartheta)|}{9(1 + |b(t, \vartheta)| + |D_t^{\alpha, \beta} b(t, \vartheta)|)}, & t \in J := [0, 1], \quad t = t_k, \\
\Delta I_t^{1-\nu, \beta} b(t, \vartheta) |_{t=t_k} = b \frac{t}{2} (\vartheta), \\
I_t^{1-\nu} b(t, \vartheta) |_{t=0} = 0.
\end{cases}
\]
Denote \( \alpha = \frac{1}{2} \), \( \beta = \frac{1}{4} \) and \( \nu = \frac{5}{8} \). Moreover,
\[
\left| f(t, \vartheta, b(t, \vartheta), D_t^{\alpha, \beta} b(t, \vartheta)) - f(t, \vartheta, \eta(t, \vartheta), D_t^{\alpha, \beta} \eta(t, \vartheta)) \right| \\
\leq \frac{1}{9} \left( |b(t, \vartheta) - \eta(t, \vartheta)| + |D_t^{\alpha, \beta} b(t, \vartheta) - D_t^{\alpha, \beta} \eta(t, \vartheta)| \right)
\] (23)
Here, \( m = 1 \), \( L(\vartheta) = K(\vartheta) = \frac{1}{9} \), we get \( \frac{L(\vartheta)}{1 - K(\vartheta)} = \frac{1}{8} \).
\[
\left( \frac{m t^\nu}{\Gamma\left(\frac{5}{8}\right)} (t)^{1-\frac{5}{8}} + \frac{1}{8} (t)^{\frac{5}{8}} B\left(\frac{5}{8}, \frac{1}{2}\right) \right) < 1.
\]
Next, set \( \varphi(t, \vartheta) = t^2 \) condition [H5] is satisfied.
Thus all the assumptions of Theorem 4.1, 4.2 and 4.7 are satisfied, our result can be applied to the proposed problem.

5. Conclusion. In this paper, the existence, uniqueness and Ulam stability of RIFDs with nonlocal condition and impulsive effect involving $\psi$-Hilfer fractional derivative (HFD) were discussed. Some classical fixed point methods were used to obtain the conditions for the existence of solution. The established results have been demonstrated by providing suitable examples.

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