Topological and Frame Properties of Certain Pathological $C^*$-Algebras

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Abstract. We introduce a classification of locally compact Hausdorff topological spaces with respect to the behavior of $\sigma$-compact subsets and, relying on this classification, we study properties of the corresponding $C^*$-algebras in terms of frame theory and the theory of $\mathcal{A}$-compact operators in Hilbert $C^*$-modules; some pathological examples are constructed.

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INTRODUCTION

Originally, the concept of frames was introduced for Hilbert space theory and then it was generalized by Frank and Larson in [1] and [2] to the case of Hilbert $C^*$-modules. During the last decade, this topic has been developing intensively (see, e.g., [3, 4]). It turns out that there is a connection between frame theory and the theory of $\mathcal{A}$-compact operators and uniform structures.

As is well-known, a bounded linear operator in Hilbert space is compact (i.e., can be norm approximated by operators of finite rank) if and only if it maps the unit ball to a totally bounded set with respect to the norm. In the case of Hilbert $C^*$-modules, i.e., if we consider a $C^*$-algebra instead of the scalar field $\mathbb{C}$ (in this case, the operators are called $\mathcal{A}$-compact), this is not true: indeed, even in the case of any infinite-dimensional unital $C^*$-algebra $\mathcal{A}$, the identity operator has finite rank (which is equal to one), but the unit ball is not totally bounded. Thus, a question how to describe the $\mathcal{A}$-compactness in geometric terms arose.

First steps were made in [5] and [6]. In the paper [7] by E. V. Troitsky, a significant advancement was obtained, namely, a specific uniform structure was constructed and it was proved that, if $\mathcal{N}$ is a countably generated Hilbert $C^*$-module, then an adjointable operator $F: \mathcal{M} \to \mathcal{N}$ is $\mathcal{A}$-compact if and only if the image of the unit ball of $\mathcal{M}$ is totally bounded in $\mathcal{N}$ with respect to this uniform structure.

In [8], it was proved that the $\mathcal{A}$-compactness of an operator implies total boundedness of the image of the unit ball with respect to this uniform structure for any Hilbert $C^*$-module $\mathcal{N}$. The inverse statement was proved for modules $\mathcal{N}$ which could be represented as an orthogonal direct summand in the standard module over the unitalized algebra $\mathcal{A}$ (which is equal to $\mathcal{A}$ itself in the unital case) for some cardinality; that is, in the module of the form $\bigoplus_{\lambda \in \Lambda} \mathcal{A}$. In particular, this holds for modules which could be represented as orthogonal direct summands in $\bigoplus_{\alpha \in \Lambda} \mathcal{A}$ for the case in which the $C^*$-algebra $\mathcal{A}$ is countably generated as a module over itself (this is in fact equivalent to the condition that $\mathcal{A}$ is $\sigma$-unital, i.e., has a countable approximate unit, see. [9, Prop. 2.3]).

However, in the case when $\mathcal{A}$ is not $\sigma$-unital, a counterexample was constructed in [10], namely, an algebra $\mathcal{A}$ regarded as a module over itself and such that the identity operator from $\mathcal{A}$ into itself is not $\mathcal{A}$-compact but the unit ball is totally bounded with respect to the introduced uniform structure (and even with respect to a stronger one). The constructed algebra is commutative; this result makes the study of the underlying topological space interesting.

It turns out also that the existence of a representation of a module $\mathcal{N}$ as an orthogonal direct summand in $\bigoplus_{\lambda \in \Lambda} \mathcal{A}$ is equivalent to the existence of a standard frame in $\mathcal{N}$, and thus there is a connection between the posed problem on $\mathcal{A}$-compact operators and the frame theory in Hilbert $C^*$-modules; we study the constructed $C^*$-algebras from this point of view, too.

More precisely, the existence of a standard frame is sufficient to satisfy the $\mathcal{A}$-compactness criterion. Hence, by proving that the criterion is not satisfied for some module, we show, as a consequence, that there is no standard frame for this module (which, by the way, does not mean that there is no outer standard frame in the sense of [3], see [10, Remark 3.3]). In [9, 11, 12], and [13], there is also a search for modules without frames; more precisely, for algebras $\mathcal{A}$ such that it is possible to say surely whether there is an $\mathcal{A}$-module without frames or not.
We introduce a scale of classes of locally compact Hausdorff topological spaces that decreases with respect to some property.

\( \mathcal{K}_1 \) is the class of \( \sigma \)-compact spaces (i.e., spaces which could be covered by a countable family of compact subsets);

\( \mathcal{K}_{II} \) is the class of non-\( \sigma \)-compact spaces which have a dense \( \sigma \)-compact subset;

\( \mathcal{K}_{III} \) is the class of spaces in which no \( \sigma \)-compact subset is dense, i.e., the complement to any \( \sigma \)-compact subset has an inner point, but the point at infinity (in the one-point compactification) may not be inner for the complement; equivalently, there exists a \( \sigma \)-compact subset which is not precompact;

\( \mathcal{K}_{IV} \) is the class of spaces in which the complement to any \( \sigma \)-compact subset has an inner point, and (in the one-point compactification) the point at infinity is always inner for the complement; equivalently, every \( \sigma \)-compact subset is precompact.

In Section 1, some preliminaries on Hilbert \( C^* \)-modules, frames and topological spaces are given and some properties of \( \mathcal{K}_1 \) and \( \sigma \)-unital algebras are established.

In Section 2, we obtain some properties of algebras \( C_0(K) \) for \( K \in \mathcal{K}_{II} \); we prove that such algebras never have standard frames (Theorem 3). Nevertheless, this does not imply any conclusion about the existence of a nonstandard frame (which we define below): it is proved that, in the case of a separable space, such a frame does not exist (Theorem 4), but we also give an example of a space for which such a frame exists (Example 5).

In Section 3, we study the algebras \( C_0(K) \) for \( K \in \mathcal{K}_{II} \). This case seems to be worse (from the point of view of the behavior of \( \sigma \)-compact subsets), but the situation is better than in cases \( \mathcal{K}_{II} \) and \( \mathcal{K}_{IV} \); we can find an example of an algebra for which there exists a standard frame (Example 6), and hence, the \( \mathcal{A} \)-compactness criterion is satisfied (one can say that, in this context, algebraic properties “are not monotonic with respect to topological properties”). On the other hand, there is an example for which the criterion does not hold (and so there is no standard frame), and there is also no nonstandard frame (Example 7 and Theorem 10). However, there is also an intermediate example, for which there is no standard frames, but a nonstandard frame exists (Example 8).

In Section 4, it is established that, for the class \( \mathcal{K}_{IV} \), there are no nonstandard frames in the algebra \( C_0(K) \). Earlier (in [10]), it was proved that, for this class, Troitsky’s criterion never holds, which also means that there are no standard frames.

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1. PRELIMINARIES AND PROPERTIES OF \( \mathcal{K}_1 \)

Let us recall basic notions and facts about Hilbert \( C^* \)-modules and operators in them, which one can find in [14–16].

**Definition 1.** A (right) pre-Hilbert \( C^* \)-module over a \( C^* \)-algebra \( \mathcal{A} \) is an \( \mathcal{A} \)-module equipped with an \( \mathcal{A} \)-inner product \( \langle \cdot, \cdot \rangle : \mathcal{M} \times \mathcal{M} \to \mathcal{A} \) being a sesquilinear form on the underlying linear space such that, for any \( x, y \in \mathcal{M}, a \in \mathcal{A} \):

1. \( \langle x, x \rangle \geq 0; \)
2. \( \langle x, x \rangle = 0 \) if and only if \( x = 0; \)
3. \( \langle y, x \rangle = \langle x, y \rangle^*; \)
4. \( \langle x, y \cdot a \rangle = \langle x, y \rangle a. \)

A Hilbert \( C^* \)-module is a pre-Hilbert \( C^* \)-module over \( \mathcal{A} \), which is complete w.r.t. its norm \( \|x\| = \|\langle x, x \rangle\|^{1/2} \).

A pre-Hilbert \( C^* \)-module \( \mathcal{M} \) is called *countably generated* if there is a countable collection of its elements such that their \( \mathcal{A} \)-linear combinations are dense in \( \mathcal{M} \).

The Hilbert sum of Hilbert \( C^* \)-modules in the evident sense will be denoted by \( \oplus \).

**Definition 2.** An operator is a bounded \( \mathcal{A} \)-homomorphism. An operator having an adjoint (in an evident sense) is adjoinable (see [15, Sec. 2.1]). We will denote the Banach space of all operators \( F : \mathcal{M} \to \mathcal{N} \) by \( \mathbf{L}(\mathcal{M}, \mathcal{N}) \) and the Banach space of adjoinable operators by \( \mathbf{L}^*(\mathcal{M}, \mathcal{N}) \).

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Definition 3. Denote by \( \theta_{x,y} : \mathcal{M} \to \mathcal{N} \), where \( x \in \mathcal{N} \) and \( y \in \mathcal{M} \), an elementary \( \mathcal{A} \)-compact operator, which is defined by formula \( \theta_{x,y}(z) := x(y, z) \). Then the Banach space \( \mathbf{K}(\mathcal{M}, \mathcal{N}) \) of \( \mathcal{A} \)-compact operators is the closure of the subspace generated by all elementary \( \mathcal{A} \)-compact operators in \( \mathbf{L}(\mathcal{M}, \mathcal{N}) \).

To define a uniform structure, that is, s system of pseudometrics, we recall the notions of the multiplier algebra and multiplier module (see [17], and [18] for details).

Recall that \( M(\mathcal{A}) \) is a \( C^* \)-algebra of multipliers of \( C^* \)-algebra \( \mathcal{A} \) (see, e.g., [14, Chap. 2]); \( M(\mathcal{A}) = \mathcal{A} \) if \( \mathcal{A} \) is unital. If \( K \) is a locally compact Hausdorff topological space and \( \mathcal{A} = C_0(K) \), then \( M(\mathcal{A}) = C_b(K) \) is the algebra of all continuous bounded functions on \( K \).

For every Hilbert \( \mathcal{A} \)-module \( \mathcal{N} \), there exists a Hilbert \( M(\mathcal{A}) \)-module \( M(\mathcal{N}) \) (which is called the multiplier module of the module \( \mathcal{N} \)) containing \( \mathcal{N} \) as an ideal submodule associated with \( \mathcal{A} \), i.e., \( \mathcal{N} = M(\mathcal{N})\mathcal{A} \). Moreover, \( (x, y) \in \mathcal{A} \) holds for any \( x \in \mathcal{N} \), \( y \in M(\mathcal{N}) \). \( M(\mathcal{N}) = \mathcal{N} \) if the algebra \( \mathcal{A} \) is unital. Since every element of \( \mathcal{N} \) can be represented in the form \( y \cdot a \) for some \( y \in \mathcal{N} \), \( a \in \mathcal{A} \) (see [15, Subsec. 1.3.10]), it follows that the module \( \mathcal{N} \) and any its submodule can be treated as \( M(\mathcal{A}) \)-modules.

If we regard \( \mathcal{N} = C_0(K) \) as a module over itself, then, as in the case of the multiplier algebra, \( M(C_0(K)) = C_b(K) \).

The uniform structures on submodules of \( \mathcal{N} \) are defined as follows (see [10] and [7] for details).

Definition 4. Consider a Hilbert \( C^* \)-module \( \mathcal{N} \) over \( \mathcal{A} \). A countable system \( X = \{x_i\} \) of elements of the multiplier module \( M(\mathcal{N}) \) is said to be (outer) admissible for a (possibly not closed) submodule \( \mathcal{N}^0 \subseteq \mathcal{N} \) (or outer \( \mathcal{N}^0 \)-admissible) if

1. for every \( x \in \mathcal{N}^0 \), the series \( \sum_i (x, x_i) x_i \) converges in \( \mathcal{A} \);
2. its sum is bounded by \( (x, x) \), that is, \( \sum_i (x, x_i) x_i \leq (x, x) \);
3. \( \|x_i\| \leq 1 \) for any \( i \).

Definition 5. Denote by \( \Phi \) a countable collection \( \{\varphi_1, \varphi_2, \ldots\} \) of states on \( \mathcal{A} \) (i.e., positive linear functionals of norm 1). For each pair \( (X, \Phi) \) with an outer \( \mathcal{N}^0 \)-admissible \( X \), consider a nonnegative function defined by the equality

\[
\nu_{X, \Phi}(x)^2 := \sup_{k} \sum_{i=k}^{\infty} |\varphi_k((x, x_i))|^2, \quad x \in \mathcal{N}^0.
\]

It can be proved that this is a seminorm on the module \( \mathcal{N}^0 \). Denote the system of all these functions by \( \mathcal{OSN}(\mathcal{N}, \mathcal{N}^0) \). We write \( (X, \Phi) \in \mathcal{OA}(\mathcal{N}, \mathcal{N}^0) \) for pairs with outer admissible \( X \).

Definition 6. For \( (X, \Phi) \in \mathcal{OA}(\mathcal{N}, \mathcal{N}^0) \), consider the following function \( d_{X, \Phi} : \mathcal{N}^0 \times \mathcal{N}^0 \to [0, +\infty) \):

\[
d_{X, \Phi}(x, y)^2 := \nu_{X, \Phi}(x - y)^2 = \sup_{k} \sum_{i=k}^{\infty} |\varphi_k((x - y, x_i))|^2, \quad x, y \in \mathcal{N}^0.
\]

We write \( d_{X, \Phi} \in \mathcal{OPM}(\mathcal{N}, \mathcal{N}^0) \).

Evidently, \( d_{X, \Phi} \) are pseudometrics in the sense of [7, Definition 2.10] (and [19, Chap. IX, Sec. 1]), and thus they form a uniform structure.

If \( X \) contains only elements of the module \( \mathcal{N} \), then the word “outer” is not used and, in this case, one may write \( \mathcal{SN}, \mathcal{A}, \) and \( \mathcal{PM} \) instead of \( \mathcal{OSN}, \mathcal{OA}, \) and \( \mathcal{OPM}, \) respectively.

The definition of totally bounded sets for the uniform structure under consideration (or for the system \( \mathcal{OPM}(\mathcal{N}, \mathcal{N}^0) \)) has the following form.

Definition 7. A set \( Y \subseteq \mathcal{N}^0 \subseteq \mathcal{N} \subseteq M(\mathcal{N}) \) is totally bounded with respect to the above uniform structure if, for any \( (X, \Phi) \), where \( X \subseteq M(\mathcal{N}) \) is outer \( \mathcal{N}^0 \)-admissible, and for any \( \varepsilon > 0 \), there exists a finite collection \( y_1, \ldots, y_n \) of elements of \( Y \) such that the sets

\[
\{y \in Y \mid d_{X, \Phi}(y_i, y) < \varepsilon\}
\]
form a cover of \( Y \). This finite collection is called an \( \varepsilon \)-net in \( Y \) for \( d_{X, \Phi} \).

If this holds, then we say that \( Y \) is externally \( (\mathcal{N}, \mathcal{N}^0) \)-totally bounded (or \( (M(\mathcal{N}), \mathcal{N}^0) \)-totally bounded).
In these terms, the $\mathcal{A}$-compactness of operators for some class of modules can be described as follows.

**Theorem 1.** ( [8, Th. 3.5]) Suppose that $\mathcal{M}$, $\mathcal{N}$, and $\mathcal{K}$ are Hilbert $\mathcal{A}$-modules, $\mathcal{N} \oplus \mathcal{K} \cong \bigoplus_{\lambda \in \Lambda} \mathcal{A}$ for some $\Lambda$, $F: \mathcal{M} \to \mathcal{N}$ is an adjointable operator, and $F(B)$ is $(\mathcal{N}, \mathcal{N})$-totally bounded, where $B$ is the unit ball of $\mathcal{M}$. Then $F$ is $\mathcal{A}$-compact as an operator from $\mathcal{M}$ to $\mathcal{N}$.

The inverse statement holds for arbitrary modules ( [8, Th. 2.4]). For the case in which $\mathcal{N}$ is a countably generated module, a similar result was stated and proved as a criterion for $\mathcal{A}$-compactness by Troitsky ( [7, Th. 2.13]).

Evidently, a $(M(\mathcal{N}), \mathcal{N})$-totally bounded set is $(\mathcal{N}, \mathcal{N})$-totally bounded, and thus, all results stating that the $(\mathcal{N}, \mathcal{N})$-total boundedness of the image of the unit ball implies the $\mathcal{A}$-compactness of corresponding adjointable operator still hold if we use the $(M(\mathcal{N}), \mathcal{N})$-total boundedness.

Let us recall the notion of a frame in a Hilbert $C^*$-module (see, e.g., [1] and [2]). Among all frames, standard ones are also considered.

**Definition 8.** Let $\mathcal{N}$ be a Hilbert $C^*$-module over a unital $C^*$-algebra $\mathcal{A}$ and let $J$ be some set. A family $\{x_j\}_{j \in J}$ of elements of $\mathcal{N}$ is said to be a standard frame in $\mathcal{N}$ if there exist positive constants $c_1, c_2$ such that, for any $x \in \mathcal{N}$, the series $\sum_j \langle x, x_j \rangle \langle x_j, x \rangle$ converges in norm in $\mathcal{A}$, and the following inequalities hold:

$$c_1 \langle x, x \rangle \leq \sum_j \langle x, x_j \rangle \langle x_j, x \rangle \leq c_2 \langle x, x \rangle.$$

A frame is said to be tight if $c_1 = c_2$ and normalized if $c_1 = c_2 = 1$. If the series converges in the ultraweak topology only (also known as $\sigma$-weak one) to some element of the universal enveloping von Neumann algebra $\mathcal{A}'''$, then the frame is said to be nonstandard. Unlike the case of a standard frame, in this case, the number of nonzero elements of the series can be uncountable; the convergence in this case is treated as the convergence of a net consisting of all finite partial sums (see remarks before [20, Subsec. 1.2.19] and [20, Subsec. 5.1.5]). We write just “frame” if it is at least nonstandard.

If the algebra $\mathcal{A}$ is not unital, then $\mathcal{N}$ can be regarded as a module over its unitalization $\mathcal{A}$, and a frame can be defined in $\mathcal{N}$ as an $\mathcal{A}$-module. Below we assume that frames are defined in this way.

**Remark 1.** For a system $\{x_j\}_{j \in J}$, there is a connection between the so-called reconstruction formula and the property of being a frame: if $x = \sum_j x_j \langle x_j, x \rangle$ for any $x \in \mathcal{N}$, then $\{x_j\}_{j \in J}$ is a normalized frame, and the convergence takes place in the same sense (see [2, Example 3.1]).

**Remark 2.** If there exists a positive constant $c$ such that the inequality $\sum_j \langle x, x_j \rangle \langle x_j, x \rangle \leq c \langle x, x \rangle$ holds for any $x \in \mathcal{N}$ and for any partial sum of the series in question (cf. the right-hand side of the inequality for the frame), then $\{x_j\}_{j \in J}$ is called a Bessel system. Due to [21, Subsec. 2.4.19], the series with respect to a Bessel system always converges in the ultraweak topology and, as a consequence, also in the ultraweak one.

Recall the following characterization of nonstandard frames.

**Lemma 1.** ( [11, Proposition 3.1]) A system $\{x_j\}_{j \in J}$ is a frame in $\mathcal{N}$ if and only if there exist positive constants $c_1, c_2$ such that the following inequalities hold for any $x \in \mathcal{N}$ and any state $\varphi$ on $\mathcal{A}$:

$$c_1 \varphi(\langle x, x \rangle) \leq \sum_j \varphi(\langle x, x_j \rangle \langle x_j, x \rangle) \leq c_2 \varphi(\langle x, x \rangle).$$

By this property and by the definition of standard frame, it is clear that any frame in the algebra $C_0(K)$ treated as a module over itself must separate the points of the space $K$.

In our context, the main structural result of the frame theory is the following: it follows from the results of Frank and Larson ( [2, Subsec. 3.5, 4.1, and 5.3], see also [11, Th. 1.1]) that the Hilbert $C^*$-module $\mathcal{N}$ over the $C^*$-algebra $\mathcal{A}$ can be represented as an orthogonal direct summand in the standard module of some cardinality over the unitalization algebra $\bigoplus_{\lambda \in \Lambda} \mathcal{A}$ if and only if there exists a standard frame in $\mathcal{N}$.

Kasparov’s stabilization theorem ( [22] or [15, Th. 1.4.2]) implies that every countably generated module has a standard frame. The construction of a module that does not have a standard frame shows that the stabilization property of Kasparov type does not hold for this module.

Note that frames are well-defined even for the case in which the algebra is not unital; however, to use this stabilization property, we need to take its unitalization.
Remark 3. If an algebra $\mathcal{A}$ has a standard frame as a module over itself, then any $\mathcal{A}$-module $\mathcal{M}$ which can be represented as an orthogonal direct summand in the standard module $\bigoplus_{\lambda \in \Lambda} \mathcal{A}$ also has a standard frame (and hence, can be represented as an orthogonal direct summand in the standard module $\bigoplus_{\lambda \in \Lambda'} \mathcal{A}$). For some special case, this was noted in [8, Remark 3.6].

Let us recall now some preliminaries from topology.

For a locally compact Hausdorff topological space $K$, denote by $\alpha K = K \cup \{t_\infty\}$ its one-point compactification (see [23, Subsec. 29.1]). We often use the fact that the $C^*$-algebra $C_0(K)$ of continuous functions vanishing at infinity (see [24, 436]) is isomorphic to an ideal in $C^*$-algebra $C(\alpha K)$ consisting of functions vanishing at the point $t_\infty$ ([25, Lemma 3.44]). Denote by $\beta K$ the Stone-Čech compactification of $K$; as is known, $C_0(K) \cong C(\beta K) \cong M(C_0(K))$ (see [32, Chap. 1]).

Lemma 2. ([10, Corollary 1.3]) Let $K$ be a locally compact Hausdorff space, let $A$ be a closed subset of $K$, and let $f \in C_0(A)$. Then $f$ can be extended to a function in $C_0(K)$. Moreover, $f$ is bounded, and the extended function can be chosen to be bounded by the same constant. In particular, for every compact set $K' \subset K$, there is a function $g \in C_0(K)$ such that $g = 1$ on $K'$ and $|g| \leq 1$ on $K$.

We need the following useful examples of topological spaces: $[0, \omega_1]$ is the space of all ordinals $\alpha$ such that $\alpha \leq \omega_1$ with order topology, where $\omega_1$ is the first uncountable ordinal; this space is uncountable and $[0, \omega_1] \in \aleph_{IV}$. $[0, \omega_0]$ is the space of all ordinals $\alpha$ such that $\alpha \leq \omega_0$ with order topology, where $\omega_0$ is the first infinite ordinal (this space is homeomorphic to $\{1/n \cup \{0\}, \{0\} \cup \{\infty\}\}$). See [10] or [29, Chaps. VI–VII] and [30, Subsec. 3.1.27] for details.

The following statement, well-known for experts, is useful for our purposes.

Lemma 3. The $C^*$-algebra $C_0(K)$ is $\sigma$-unital if and only if $K$ is $\sigma$-compact.

Proof. Recall that the $\sigma$-unitality is equivalent to the existence of a strictly positive element; in our case, of an everywhere positive function.

If the algebra $C_0(K)$ is $\sigma$-unital, then it contains an everywhere positive function $f$. The sets $\{t \in K : |f(t)| \geq \frac{1}{n}\}$ are compact (as closed subsets of such corresponding compact sets outside of which $|f(t)| < \frac{1}{n}$), and therefore, form a countable cover of $K$ by compact set.

Conversely, if there are compact sets $K_n$ such that $K = \bigcup_{n \in \mathbb{N}} K_n$, then there exist functions $f_n \in C_0(K)$ such that $f_n(t) = 1$ on $K_n$ and $|f_n(t)| \leq 1$ everywhere on $K$. The function $\sum_{n \in \mathbb{N}} \frac{1}{2^n} |f_n(t)|$ is everywhere positive on $K$, since for any point, there is a compact set $K_n$ in which it is contained. \hfill \Box

Lemma 4. Let $K$ be a locally compact non-$\sigma$-compact space. Then, for any countable family of functions $\{f_n\} \subset C_0(K)$, the set $F = \{t \in K : f_n(t) = 0 \ \forall n \in \mathbb{N}\}$ is nonempty, and even uncountable.

In particular, a countable family of functions cannot separate points of the space $K$.

Proof. Indeed, if the set $F$ is countable, $F = \{z_n\}_{n \in \mathbb{N}}$, then, for every $n \in \mathbb{N}$, there exists a function $g_n \in C_0(K)$ such that $g_n(z_n) = 1$, $|g_n(t)| \leq 1$, and then the function

$$g(t) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} \frac{|f_n(t)|}{1 + |f_n(t)|} + \sum_{n \in \mathbb{N}} \frac{1}{2^n} |g_n(t)|$$

is everywhere positive on $K$, which contradicts the fact that $K$ is not $\sigma$-compact.

The proof is also valid for the cases in which $F$ is finite or empty. \hfill \Box

Lemma 5. Let $K$ be a locally compact non-$\sigma$-compact space, and let $\{K_n\}_{n \in \mathbb{N}}$ be a countable collection of compact subsets of $K$. Then, for any countable family of functions $\{f_n\} \subset C_0(K)$, the set $F = \{t \in K \setminus \bigcup_{n \in \mathbb{N}} K_n : f_n(t) = 0 \ \forall n \in \mathbb{N}\}$ is nonempty and even uncountable.

Proof. Similarly to the previous lemma, consider the function $g(t) = \sum_{n \in \mathbb{N}} \frac{1}{2^n} |f_n(t)| + \sum_{n \in \mathbb{N}} \frac{1}{2^n} |g_n(t)| + \sum_{n \in \mathbb{N}} \frac{1}{2^n} h_n(t)$, where $h_n \in C_0(K)$ is a nonnegative function which is equal to 1 on $K_n$ and is bounded by 1. \hfill \Box
As is known (see [2, Example 3.5]), every \( \sigma \)-unital algebra as a module over itself has a normalized standard frame. Let us construct an example of a non-\( \sigma \)-unital algebra that has a frame with the same properties.

Recall that \( \mathcal{A} = c_0 - \sum_{\lambda \in \Lambda} \mathcal{A}_\lambda \) is a \( c_0 \)-direct sum of the algebras \( \mathcal{A}_\lambda \) (see [31, Sec. 1.4] or [33, Sec. 1]), the elements \( x = (x_\lambda)_{\lambda \in \Lambda} \) of this sum are enumerated by indices in \( \Lambda \) sets such that \( x_\lambda \in \mathcal{A}_\lambda \), there is at most countable set \( \{x_{\lambda_m}\}_{m \in \mathbb{N}} \) of nonzero elements, and \( \lim_{m \to \infty} ||x_{\lambda_m}||_{\mathcal{A}_\lambda} = 0 \). The norm in this algebra is given by the formula \( ||x||_\mathcal{A} = \sup_{\lambda \in \Lambda} ||x_\lambda||_{\mathcal{A}_\lambda} \). In fact, this algebra is obtained by completing an algebra whose elements are nonzero only for finitely many indices.

**Theorem 2.** Let \( \mathcal{A} = c_0 - \sum_{\lambda \in \Lambda} \mathcal{A}_\lambda \), where every \( \mathcal{A}_\lambda \) has a normalized standard frame for which the reconstruction formula is valid (for example, due to [9, Proposition 2.3], when every \( \mathcal{A}_\lambda \) is \( \sigma \)-unital). Then \( \mathcal{A} \), as a \( \mathcal{A} \)-module, has a normalized standard frame.

**Proof.** Denote by \( \{x_j\}_{j \in J_\lambda} \) the frame in \( \mathcal{A}_\lambda \). For every \( \lambda \in \Lambda \), the elements of the frame can be regarded as elements of the entire algebra \( \mathcal{A} \) by extending them by zero outside the corresponding index. We claim that \( \{x_j\}_{j \in J_\lambda} \) is a frame in \( \mathcal{A} \).

Let \( x \in \mathcal{A} \). Choose an arbitrary \( \varepsilon > 0 \). There exists an element \( x^m \in \mathcal{A} \) which is nonzero only for finitely many indices \( \lambda_1, \ldots, \lambda_m \) and such that \( ||x - x^m|| < \varepsilon \). For every \( \lambda_k, k = 1, \ldots, m \), there exists a finite set \( \{x_j\}_{j \in J_{\lambda_k}} \) of elements of the frame such that

\[
\left\| x_{\lambda_k} - \sum_{j \in J_{\lambda_k}} x_j (x_j, x_{\lambda_k}) \right\| < \varepsilon,
\]

and hence,

\[
\left\| x^m - \sum_{j \in J_{\lambda_k}} x_j (x_j, x^m) \right\| < \varepsilon \quad \text{and} \quad \left\| x - \sum_{j \in \bigcup_{k=1}^m J_{\lambda_k}} x_j (x_j, x^m) \right\| < 2\varepsilon.
\]

Thus, \( x = \sum_{j \in J} x_j (x_j, x) \), and the series converges in norm in \( \mathcal{A} \). Hence, \( \langle x, x \rangle = \sum_{j \in J} \langle x, x_j \rangle (x_j, x) \). \( \square \)

**Corollary 1.** Since \( \mathcal{A} \) has a frame, it can be represented as an orthogonal direct summand in the standard module \( \bigoplus_{\lambda \in \Lambda} \mathcal{A} \) of some cardinality, and thus, by [8, Th. 3.5], it satisfies the \( \mathcal{A} \)-compactness criterion.

Moreover, every \( \mathcal{A} \)-module which can be represented as an orthogonal direct summand in the standard module \( \bigoplus_{\lambda \in \Lambda} \mathcal{A} \) also can be represented in the standard module \( \bigoplus_{\lambda \in \Lambda} \mathcal{A} \) of some cardinality, and hence, it also satisfies the \( \mathcal{A} \)-compactness criterion.

**Remark 4.** If \( \mathcal{A} \) is in addition a commutative algebra, then \( \mathcal{A} \cong c_0 - \sum_{\lambda \in \Lambda} C_0(K_\lambda) \cong C_0\left( \bigcup_{\lambda \in \Lambda} K_\lambda \right) \), and \( C_0(K_\lambda) \) is \( \sigma \)-unital if and only if \( K_\lambda \) is \( \sigma \)-compact; if \( \Lambda \) is uncountable, then \( K = \bigcup_{\lambda \in \Lambda} K_\lambda \) is not \( \sigma \)-compact, because a compact set in \( K \) intersects only finitely many sets \( K_\lambda \). If \( \mathcal{A}_\lambda \) is a unital algebra, then, for a frame in \( \mathcal{A}_\lambda \), we can take just one element, the identity of the algebra (it corresponds to a function which identically equals to one in the commutative case).

### 2. Properties of \( \mathcal{K}_\Pi \)

Let us introduce several examples of spaces in the class \( \mathcal{K}_\Pi \).

**Example 1.** \( K \) is the set of real numbers with rational sequence topology ([27, Sec. 65]). Moreover, it is separable.

**Example 2.** \( K = \beta \mathbb{N} \setminus \{t'\} \), where \( t' \in \beta \mathbb{N} \setminus \mathbb{N} \) (the Stone-Čech compactification of positive integers without an arbitrary point from the growth). It is not \( \sigma \)-compact by [28, Subsec. 9.6], and it is obviously separable. More generally, instead of \( \mathbb{N} \), we can take any separable noncompact space (or just \( \sigma \)-compact, but we can loose the separability).
Example 3. $K = \alpha P \times [0, \omega_0] \setminus \{(p_\infty, \omega_0)\}$, where $P$ is a locally compact, non-$\sigma$-compact Hausdorff space, and $\alpha P = P \cup \{p_\infty\}$ is its one-point compactification.

Theorem 3. Let $K \in \mathcal{X}_{II}$. Then there is no standard frame in the $\hat{C}_0(K)$-module $C_0(K)$.

Proof. Assume that there exists a frame $\{x_j\}_{j \in J}$ of elements in $C_0(K)$. It contains uncountably many nonzero elements, since a countable set cannot separate the points of $K$.

Let $\{K_n\}_{n \in \mathbb{N}}$ be a sequence of compact sets in $K$ such that $\bigcup_{n \in \mathbb{N}} K_n = K$.

For every $n \in \mathbb{N}$, there exists a function $g_n \in C_0(K)$ such that $g_n(t) = 1$ on $K_n$ and $|g_n(t)| \leq 1$ on $K$. There is at most countable nonempty set $\{x_j\}_{j \in J_n}$ of elements of the frame such that $x_j(t) \neq 0$ identically on $K_n$, because the series $\sum_j \langle g_n, x_j \rangle \langle x_j, g_n \rangle(t) = \sum_j |x_j(t)|^2$ converges uniformly on $K_n$. That is, if $j \in J \setminus J_n$, then $x_j(t) = 0$ on $K_n$.

Hence, there is at most countable set $\{x_j\}_{j \in \bigcup_{n \in \mathbb{N}} J_n}$ such that $x_j(t) \neq 0$ identically on $\bigcup_{n \in \mathbb{N}} K_n$. Hence, for every $j \in J \setminus \bigcup_{n \in \mathbb{N}} J_n$, we have $x_j(t) = 0$ for $t \in \bigcup_{n \in \mathbb{N}} K_n$; however, due to the fact that $\bigcup_{n \in \mathbb{N}} K_n = K$, it also holds for $t \in K$. Hence, only a countable set of frame elements is not identically zero. A contradiction.

Corollary 2. This result implies that $C_0(K)$ cannot be represented as an orthogonal direct summand of a standard module $\bigoplus_{\lambda \in \Lambda} \hat{C}_0(K)$.

Let now $K$ be separable in addition, i.e., finite sets can be taken as compacts in the definition of $\mathcal{X}_{II}$. The spaces in Examples 1 or 2 are spaces of this kind. We claim that, in this case, there are no nonstandard frames either.

Theorem 4. Let $K \in \mathcal{X}_{II}$, and let $K$ be separable. Then there are no frames in the $\hat{C}_0(K)$-module $C_0(K)$.

Proof. Assume that there exists a frame $\{x_j\}_{j \in J}$ of elements in $C_0(K)$. It must contain uncountably many nonzero elements, since a countable set cannot separate the points of $K$.

Let $\{t_n\}_{n \in \mathbb{N}}$ be a countable dense subset of $K$.

For every $n \in \mathbb{N}$, there exists a function $g_n \in C_0(K)$ such that $g_n(t_n) = 1$ and $|g_n(t)| \leq 1$ on $K$. There is at most countable nonempty set $\{x_j\}_{j \in J_n}$ of elements of the frame such that $x_j(t_n) \neq 0$, because taking $x = g_n$ and applying the evaluation $\varphi$ at the point $t_n$, due to Lemma 1, we see that the series $\sum_j \langle g_n, x_j \rangle \langle x_j, g_n \rangle (t) = \sum_j |x_j(t)|^2$ converges at $t_n$. That is, if $j \in J \setminus J_n$, then $x_j(t_n) = 0$.

Hence, there is at most countable set $\{x_j\}_{j \in \bigcup_{n \in \mathbb{N}} J_n}$ such that $x_j(t) \neq 0$ identically on $\bigcup_{n \in \mathbb{N}} \{t_n\}$. Therefore, for every $j \in J \setminus \bigcup_{n \in \mathbb{N}} J_n$, we have $x_j(t) = 0$ for $t \in \bigcup_{n \in \mathbb{N}} \{t_n\}$; however, since $\bigcup_{n \in \mathbb{N}} \{t_n\} = K$, this holds also for $t \in K$. Hence, only a countable set of frame elements is identically zero. A contradiction.

Despite the previous two theorems, it is possible to construct an example of a space $K \in \mathcal{X}_{II}$ such that a nonstandard frame exists in $C_0(K)$.

Example 4. Let $P$ be a non-$\sigma$-compact space such that $C_0(P)$ has a normalized frame $\{u_\beta\}_{\beta \in B}$. We know that such a space exists (see Remark 4).

Take $K = \alpha P \times [0, \omega_0] \setminus \{(p_\infty, \omega_0)\}$ and define $y_\beta \in C_0(K)$, $\beta \in B$, by the formula $y_\beta(p, n) = u_\beta(p)$, where $(p, n) = t \in K$, i.e., $\{y_\beta\}_{\beta \in B}$ is a “copy” of functions $\{u_\beta\}_{\beta \in B}$ on each “row” $P \times \{n\}$, $n \in [0, \omega_0]$. Consider $\{w_n\}_{n \in (0, \omega_0)}$ such that $w_n = 1$ on $\alpha P \times \{n\}$ and $w_n$ vanishes outside $\alpha P \times \{n\}$ (obviously, $w_n \in C_0(K)$ for any $n \in [0, \omega_0)$), since every $\alpha P \times \{n\}$ is a closed-and-open set. Define $\{x_j\}_{j \in J} = \{y_\beta\}_{\beta \in B} \cup \{w_n\}_{n \in (0, \omega_0)}$.

Theorem 5. The system $\{x_j\}_{j \in J}$ is a (nonstandard) frame in $C_0(K)$ in Example 4.

Proof. Take an arbitrary $x \in C_0(K)$. For any partial sum of the series $\sum_j \langle x, x_j \rangle \langle x_j, x \rangle(t) = \sum_j |x(t)|^2 |x_j(t)|^2$, we have $\sum_j |x(t)|^2 |x_j(t)|^2 \leq 2 |x(t)|^2 = 2 \langle x, x \rangle(t)$ (actually, if $p = p_\infty$ or $n = \omega_0$, then $2$ can be replaced by $1$), which means that the series converges in the ultrastrong topology.

Hence, we obtain the right-hand side of the inequality in Lemma 1 with $c_2 = 2$ for any state $\varphi$ (in particular, the corresponding series converges). Indeed, if we have $\sum_j \varphi(\langle x, x_j \rangle \langle x_j, x \rangle) > 2 \varphi(\langle x, x \rangle)$ for some state $\varphi$, then there exists a partial sum of the series for which it holds that $\sum_j \varphi(\langle x, x_j \rangle \langle x_j, x \rangle) > 2 \varphi(\langle x, x \rangle)$.
Then $\varphi(2\langle x, x \rangle - \sum_{j \in J} \langle x, x_j \rangle \langle x_j, x \rangle) < 0$, which contradicts the inequality $2\langle x, x \rangle \geq \sum_{j \in J} \langle x, x_j \rangle \langle x_j, x \rangle$ and the positivity of $\varphi$.

Let us show that the left-hand side holds too.

Let $\varphi$ be a state on $C_0(K)$, i.e., there is a Radon measure $\mu$ on $K$ such that $\varphi(x) = \int_K x(t)d\mu(t)$ (see [24, 436K]). Then $\mu$ can be represented as a sum of the Radon measures $\mu_1$ which support is $P \times \{\omega_0\}$ and $\mu_2$ which support is $\alpha P \times [0, \omega_0)$. Then

$$\varphi((x, x)) = \int_{P \times \{\omega_0\}} |x(t)|^2d\mu_1(t) + \int_{\alpha P \times [0, \omega_0)} |x(t)|^2d\mu_2(t)$$

for any $x \in C_0(K)$. The representation of a measure as a sum corresponds to the representation of $\varphi$ as a sum of states $\varphi_1$ and $\varphi_2$.

Identify the restrictions of $y_{\beta}(p, n)$ to $P \times \{\omega_0\}$ with $u_{\beta}(p)$; then $\{u_{\beta}\}_{\beta \in B}$ is a normalized frame in $C_0(P \times \{\omega_0\})$. For the restriction of $x$ to $P \times \{\omega_0\}$ (and hence, for $x$ itself, since $\mu_1$ vanishes outside this subset), we have $\varphi_1((x, x)) = \sum_{\beta \in B} \varphi_1((x, u_{\beta})) = \sum_{j} \varphi_1((x, x_j) \langle x_j, x \rangle)$ ($\varphi_1$ can be treated as a state on $C_0(P \times \{\omega_0\})$).

$P \times \{\omega_0\}$ is a closed set in $K$; therefore, $\alpha P \times [0, \omega_0)$ is open, and hence it is a Borel set. Thus, $\mu_2$ is a Radon measure on $\alpha P \times [0, \omega_0)$ ( [24, 416R(b)]). By the monotone convergence theorem (see, e.g, [25, Th. 2.25] or [26, Proposition 8.7(b)]), we have

$$\int_{\alpha P \times [0, \omega_0)} |x(t)|^2d\mu_2(t) = \int_{\alpha P \times [0, \omega_0)} \sum_{n=0}^{\infty} |x(t)|^2|w_n(t)|^2d\mu_2(t) = \sum_{n=0}^{\infty} \int_{\alpha P \times [0, \omega_0)} |x(t)|^2|w_n(t)|^2d\mu_2(t),$$

and hence,

$$\varphi_2((x, x)) = \int_{\alpha P \times [0, \omega_0)} |x(t)|^2d\mu_2(t) = \sum_{n=0}^{\infty} \int_{\alpha P \times [0, \omega_0)} |x(t)|^2|w_n(t)|^2d\mu_2(t)$$

$$= \sum_{n=0}^{\infty} \varphi_2((x, w_n) \langle w_n, x \rangle) \leq \sum_{j} \varphi_2((x, x_j) \langle x_j, x \rangle).$$

Summing the above inequalities, we obtain $\varphi((x, x)) \leq \sum_{j} \varphi((x, x_j) \langle x_j, x \rangle)$.

Therefore, for every state $\varphi$ on $C_0(K)$ and every $x \in C_0(K)$, the inequalities $\varphi((x, x)) \leq \sum_{j} \varphi((x, x_j) \langle x_j, x \rangle) \leq 2\varphi((x, x))$ hold, and thus, $\{x_j\}_{j \in J}$ is a frame in $C_0(K)$ with constants 1 and 2.

**Remark 5.** It is clear that the proof of the previous theorem is still valid if, instead of a normalized frame in $C_0(P)$, we take a frame with arbitrary frame constants.

The existence of a standard frame is sufficient but not necessary for the $\mathcal{A}$-compactness criterion to be satisfied. Therefore, we cannot claim that the criterion is not valid for every topological space in the class $\mathcal{K}_I$, but we can construct an example of a space (more precisely, some subclass of spaces) for which the criterion actually fails.

First, let us consider several properties of topological spaces (it is assumed everywhere that $K$ is a locally compact Hausdorff space).

1. $\beta K = \alpha K$ (in other words, every continuous bounded function has a limit at infinity).
2. Any $\sigma$-compact subset of $K$ is precompact in $K$ (that is, $K \in \mathcal{K}_{I(\sigma)}$).
3. Any continuous function which tends to zero at infinity is constant outside some compact $K'$.
4. Any continuous function that has a limit at infinity is constant outside some compact $K'$.
5. Any continuous function on $K$ is constant outside some compact $K'$.

Let us observe the relationships among these properties.

**Lemma 6.** The properties (2), (2.1), (2.2) are equivalent.
Proof. Indeed, (2.1) obviously follows from (2.2).

Let us prove (2.2) if (2.1) is true. Let \( f \) be a continuous function which has a limit at infinity which is equal to \( f_\infty \). Then the continuous function \( f - f_\infty \) tends to zero at infinity and, therefore, it vanishes outside some compact set, and thus, the function \( f \) is constant and is equal to \( f_\infty \) outside the same compact set.

Let us prove (2) if (2.1) is true. Let \( \{K_n\}_{n \in \mathbb{N}} \) be a sequence of compact sets in \( K \). For every \( n \in \mathbb{N} \), there exists a function \( g_n \in C_0(K) \) such that \( g_n(t) = 1 \) on \( K_n \), \( |g_n(t)| \leq 1 \) on \( K \). Define the function \( g \in C_0(K) \) by the formula \( g(t) = \sum_{n \in \mathbb{N}} \frac{1}{n} g_n(t) \). It tends to zero at infinity, and hence, it vanishes outside some compact set \( K' \); moreover, it is nonzero at \( \bigcup_{n \in \mathbb{N}} K_n \), and thus, \( \bigcup_{n \in \mathbb{N}} K_n \subset K' \).

The implication (2) \( \Rightarrow \) (2.1) was proved in [10, Lemma 1.5].

Obviously, (3) implies (2). The converse is false in general; it suffices to consider a disjoint union of sets with property (2), for example, \([0, \omega_1)\). The same example shows that (2) does not imply (1).

Obviously, (3) implies (1). The converse is not true, as follows from an example below, which does not satisfy (2) either. This example represents a class of spaces for which the criterion of \( \mathcal{A} \)-compactness fails to hold.

Example 5. Take \( K = \alpha P \times [0, \omega_1) \setminus \{(p_\infty, \omega_0)\} \), where \( P \in \mathcal{X}_{IV} \) (i.e., \( P \) satisfies property (2)) and \( \alpha P = P \cup \{p_\infty\} \) is the one-point compactification of \( P \).

A special case of this construction is the deleted Tychonoff plank ( [27, Sec. 87]) for \( P = [0, \omega_1) \) and \( \alpha P = [0, \omega_1] \).

This space does not satisfy (2) (and, as a consequence, does not satisfy (3)), since it contains a countable dense family of compact sets \( \{\alpha P \times \{n\}\}_{n \in \mathbb{N}} \). Let us show that it satisfies (1) (this generalizes the properties of the deleted Tychonoff plank, see [28, Subsec. 8.20]).

Theorem 6. Let \( K \) be a space in Example 5. Then \( \alpha K = \beta K \), i.e., every continuous bounded function has a limit at infinity. Moreover, \( K \) is pseudocompact, i.e., every continuous function on \( K \) is bounded.

Proof. Let \( f \in C(K) \). Then, for every \( n \in \mathbb{N} \), the restriction of \( f \) to \( \alpha P \times \{n\} \) is continuous, and the restriction of \( f \) to \( P \times \{n\} \) is a continuous function that has a limit at infinity. Hence, by property (2), for every \( n \in \mathbb{N} \), there is a compact set \( P_n \subset P \) such that \( f \) is constant (and is equal to some \( p_{n,\infty} \)) outside the compact set \( P_n \times \{n\} \subset P \times \{n\} \). \( \{P_n\}_{n \in \mathbb{N}} \) is a countable family of compact sets in \( P \), and hence, there exists a compact set \( P' \subset P \) which contains all of them. Thus, outside \( P' \times [0, \omega_0) \), the function \( f \) depends only on the number \( n \in \mathbb{N} \) in \( P' \) and is equal to \( p_{n,\infty} \) on the "nth row" \( P \times \{n\} \).

Consider \( (p, \omega_0) \in P \times \{\omega_0\} \) with \( p \notin P' \). Since \( f \) is continuous at \( (p, \omega_0) \), we have \( f(p, \omega_0) = \lim_{n \to \infty} f(p, n) \); this limit does not depend on \( p \) outside \( P' \), and hence, the function \( f(p, \omega_0) \) is constant outside \( P' \). Therefore, the function \( f \) can be extended by continuity to the point \( (p_\infty, \omega_0) \) by \( f(p_\infty, \omega_0) = \lim_{n \to \infty} f(p, n) \). Thus, we have proved property (1).

Corollary 3. For \( K \) from Example 5 we have \( C(K) = C_0(K) = C_0(K) = M(C_0(K)) \equiv C(\alpha K) = C(\beta K) \) (except the first equality, this is also true for any space with property (1)).

To show that Troitsky’s theorem does not hold for \( C_0(K) \) as a module over itself with such \( K \), we need another intermediate step.

Theorem 7. A system \( \{x_j\} \subset C_0(K) \) is \( (C_0(K), C_0(K))\)-admissible if and only if it is \( (C_0(K), C_0(K))\)-admissible.

Proof. The implication \( \Leftarrow \) is obvious; let us prove the inverse.

Let \( \{x_j\} \subset C_0(K) \) be \( (C_0(K), C_0(K))\)-admissible, i.e., for every \( x \in C_0(K) \),

1. the series \( \sum_j \langle x, x_j \rangle \langle x, x \rangle \) converges in norm (i.e., uniformly);
2. its sum is bounded by \( \langle x, x \rangle \);
3. \( \|x_i\| \leq 1 \) for any \( i \).
Take an arbitrary function $x \in C_0(K)$ and show that these conditions are also satisfied for it (obviously, it suffices to show (1) and (2)).

Similarly to the previous proof, there exists a compact set $P' \subset P$ such that the function $x$ and all the functions $x_j$ are constant outside $P' \times [0, \omega_0]$ on each “row.” There exists a $p' \in P \setminus P'$; write $P'' = P' \cup \{p'\}$.

There exists a function $g \in C_0(P)$ such that $g(p) = 1$ on $P''$ and $|g(p)| \leq 1$ on $P$. Define $\bar{x} \in C_0(K)$ by the formula $\bar{x}(t) = x(t)g(p)$, where $t = (p, n) \in K$. $\bar{x}$ satisfies conditions (1) and (2) on $K$, and hence, on the set $P'' \times [0, \omega_0]$ on which $\bar{x} = x$. Outside $P''$, we have $\sum_i \langle x, x_i \rangle \langle x_i, x \rangle (p, n) = \sum_i \langle x, x_i \rangle \langle x_i, x \rangle (p', n)$. That is, outside $P''$, conditions (1) and (2) are satisfied, since they are satisfied at $p = p'$. If the uniform convergence on each of these two sets holds, then it also holds on their union; if the inequality holds on the sets, then it also holds on their union. Hence, conditions (1) and (2) are satisfied on the entire $K$ for any $x \in C_0(K)$. \hfill \Box

**Corollary 4.** A set $Y \subset C_0(K)$ is $(C_b(K), C_0(K))$-totally bounded if and only if it is $(C_b(K), C_b(K))$-totally bounded.

**Remark 6.** Using [32, Subsec. 4.2], one can construct other complex examples of spaces with the above properties by taking an arbitrary infinite compact set instead of $[0, \omega_0]$ and choosing for $P$ a sufficiently large ordinal if necessary instead of $\omega_1$ to use the condition that the function is eventually constant on every “row.”

**Theorem 8.** The unit ball in $C_0(K)$ (and hence, the image of the unit ball with respect to the identity operator $\text{Id} : C_0(K) \to C_0(K)$) is $(C_b(K), C_0(K))$-totally bounded, but the identity operator is not $C_0(K)$-compact.

**Proof.** The unit ball in $C_0(K)$ is a subset of the unit ball in $C_b(K)$, which is $(C_b(K), C_b(K))$-totally bounded since it is the image of the unit ball with respect to the identity operator $\text{Id} : C_b(K) \to C_b(K)$, which is $C_b(K)$-compact because $C_b(K)$ is unital and countably generated as a module over itself. Hence, the unit ball in $C_b(K)$ is also $(C_b(K), C_b(K))$-totally bounded, and it is $(C_b(K), C_b(K))$-totally bounded by the previous corollary.

The identity operator is not $C_0(K)$-compact since the image of $\mathcal{A}$-compact operator must be countably generated ([7, Lemma 1.10]), and $C_0(K)$ is not. \hfill \Box

Let us also prove that there are no nonstandard frames for the above example, and we should begin with the following useful lemma.

**Lemma 7.** Let $K$ be a locally compact Hausdorff space, and let $A$ be a closed subset of $K$. If there is a frame $\{x_j\}_{j \in J}$ (standard or not) in $C_0(K)$, then its restriction to $A$ $\{y_j\}_{j \in J}$ is a frame in $C_0(A)$ in the same sense.

**Proof.** Since the uniform convergence on a set implies the uniform convergence on a subset, it follows that the proposition is obvious for standard frames. Let us prove it for nonstandard ones.

Take $x \in C_0(A)$, and let $\varphi$ be a state on $C_0(A)$, i.e., a Radon measure $\mu$ on $A$. It can be extended to a measure on the entire $K$ by zero outside $A$, i.e., to the state $\varphi'$ on $C_0(K)$. The function $x$ can be extended to the function $x' \in C_0(K)$ due to Lemma 2. Since $\{x_j\}_{j \in J}$ is a frame, we have

$$c_1 \varphi'((x', x')) \leq \sum_j \varphi'(\langle x', x_j \rangle \langle x_j, x' \rangle) \leq c_2 \varphi'((x', x')).$$

Since $\varphi'$ is a measure which is actually calculated on the restrictions of functions on $A$, it follows that, for $x$, we have

$$c_1 \varphi((x, x)) \leq \sum_j \varphi((x, y_j) \langle y_j, x \rangle) \leq c_2 \varphi((x, x)),$$

i.e., $\{y_j\}_{j \in J}$ is a nonstandard frame in $C_0(K)$. \hfill \Box

**Theorem 9.** For $K$ in Example 5, there are no nonstandard frames in $C_0(K)$.

**Proof.** Suppose that there exists a nonstandard frame $\{x_j\}_{j \in J}$ in $C_0(K)$. Then its restriction $\{y_j\}_{j \in J}$ to $P \times \{\omega_0\}$ is a nonstandard frame in $C_0(P \times \{\omega_0\})$, and thus $C_0(P)$ also has a frame, which is impossible, as is shown below (Theorem 12). A contradiction. \hfill \Box
3. PROPERTIES OF $\mathcal{X}_{III}$

**Example 6.** Due to 2 and 4, it suffices to take $K = \bigcup_{\lambda \in \Lambda} K_\lambda$ with uncountable $\Lambda$, where all $K_\lambda$ are $\sigma$-compact, as a “good” example in which there exists a standard frame. Indeed, any compact set in $K = \bigcup_{\lambda \in \Lambda} K_\lambda$ intersects only finitely many sets $K_\lambda$, which means that a $\sigma$-compact set intersects only countably many of them. Hence, the complement to any $\sigma$-compact subset contains some $K_\beta$, which is an open set in $K$.

Let us now introduce an example in which the $\mathcal{A}$-compactness criterion is not satisfied, which implies that there is no standard frame; there are no frames for it at all either.

**Example 7.** Let $K = P_1 \sqcup P_2$, where $P_1 \in \mathcal{X}_{IV}$ and $P_2 \in \mathcal{X}_{I}, \mathcal{X}_{II},$ or $\mathcal{X}_{III}$. A $\sigma$-compact set in $K$ is a union of $\sigma$-compact sets from $P_1$ and $P_2$, respectively. The complement to a $\sigma$-compact set in $P_1$ has interior points, so the same is true for $K$. However, one can reach a point at infinity with a countable set of compact sets from $P_2$, so $K \in \mathcal{X}_{III}$.

**Theorem 10.** Let $K$ be a space in Example 7. Then the operator $F : C_0(K) \to C_0(K)$ of multiplication by the identity function on $P_1$ and by zero function on $P_2$ is not $C_0(K)$-compact, but the image of the unit ball with respect to this operator is $(C_0(K), C_0(K))$-totally bounded. Obviously, this operator is adjointable. $C_0(K)$ has no frames either.

**Proof.** The image of this operator is an uncountably generated module $C_0(P_1)$, and thus the operator cannot be $C_0(K)$-compact.

The image of the unit ball is the unit ball in $C_0(P_1)$ and, since the restriction of the Radon measure to a measurable subset is the Radon measure ( [24, 416R(b)] ), it follows that the seminorm $\nu_{X, \varphi}$ on the image has the following form:

$$
\nu_{X, \varphi}(x)^2 = \sup_k \sum_{i=1}^{\infty} \left| \int_{K} x(t) \cdot x_i(t) d\mu_k(t) \right|^2 = \sup_k \sum_{i=1}^{\infty} \left| \int_{P_1 \sqcup P_2} x(t) \cdot x_i(t) d\mu_k(t) \right|^2 \tag{1}
$$

i.e., the seminorm on the image is calculated as a seminorm on $C_0(P_1)$. Obviously, the restriction to $P_1$ of any $(C_b(K), C_0(K))$-admissible system is $(C_b(P_1), C_0(P_1))$-admissible. Hence, the unit ball in $C_0(P_1)$ is $(C_b(K), C_0(K))$-totally bounded by the reasons similar to [10, Th. 2.5], because the unit ball in $C_0(P_1)$ is $(C_b(P_1), C_0(P_1))$-totally bounded (more specifically, since the elements of the $\varepsilon$-net, which are functions on $P_1$, can be extended to the entire $K$).

If there exists a frame in $C_0(K)$, then its restriction to $P_1$ would also be a frame; however, as is shown below (Theorem 12), $C_0(P_1)$ has no frames, and thus, there are no frames in $C_0(K)$ either.

There is also an intermediate example of a space: there is no standard frame, and a nonstandard frame exists. First, let us prove the following useful lemma.

**Lemma 8.** Let $P_1, P_2$ be locally compact Hausdorff spaces, and let both $C_0(P_1)$ and $C_0(P_2)$ have frames $\{x_j\}_{j \in J}$ (with constants $d_1, d_2$) and $\{y_i\}_{i \in I}$ (with constants $c_1, c_2$), respectively. Then in $C_0(K)$, where $K = P_1 \sqcup P_2$, there exists a frame $\{w_y\}_{y \in G} = \{x_j\}_{j \in J} \cup \{y_i\}_{i \in I}$. If each of the original frames is standard, then $\{w_y\}_{y \in G}$ is also standard.

**Proof.** The uniform convergence on finitely many sets implies the uniform convergence on their union. The same is true for the inequalities; thus, the case when both frames are standard is obvious. Let us prove the lemma for nonstandard frames.

Let $\varphi$ be a state on $C_0(K)$, i.e., a measure on $K$. It can be represented as the sum of measures on $P_1$ and $P_2$, respectively, i.e., $\varphi = \varphi_1 + \varphi_2$, where $\varphi_1, \varphi_2$ are states on $P_1, P_2$, respectively. It is also possible to represent the function $w \in C_0(K)$ in the same way, $w = x + y$, and write $\langle w, w \rangle = |w|^2 = \langle x, x \rangle + \langle y, y \rangle$.

Hence,

$$
\sum_y \varphi(\langle w, w_y \rangle(\langle w_y, w \rangle)) = \sum_j \varphi_1(\langle x, x_j \rangle \langle x_j, x \rangle) + \sum_i \varphi_2(\langle y, y_i \rangle \langle y_i, y \rangle)
$$
Similarly, does not intersect any $\sum C$ for \{.$\}

Assume that there exists a frame $P$ such that $q \neq 0$. Then in any neighborhood of the point $q$, there is a point $t$ such that $|x(t) - x(t)| = |q| > 0$; a contradiction to the continuity of $x$. Let now $j \in \bigcup_{l=1}^{\infty} J_l$, i.e., $j \in J_k$ for some $k \in \mathbb{N}$, and suppose that $x_j(t_0) = 0$. Then $t_0 \in K_k$ (because $x_j = 0$ outside $K_k$). Hence, $x_j(t_{k+l}) = 0$ for all $l \in \mathbb{N}$ (because $t_{k+l} \notin K_k$), $t_0$ is still a limit point for the sequence $\{t_n\}_{n=k+1}^{\infty}$, and then $x_j(t_0) = 0$, similarly to the previous case.

Hence, $\{x_j\}_{j \in J}$ is not a frame.

**Example 8.** Let $K = P_1 \cup P_2$, where $P_1$ is the space in Example 6 and $P_2$ is the space in Example 4. Similarly to the previous discussion, we have $K \in \mathcal{X}_{III}$.

**Theorem 11.** There is no standard frame in $C_0(K)$, and there exists a nonstandard one.

**Proof.** If $C_0(K)$ has a standard frame, then its restriction to $P_1$ would also be a standard frame; however, since $P_2 \in \mathcal{X}_{III}$, it follows that $C_0(P_2)$ has no standard frame (Theorem 3), and hence, the same holds for $C_0(K)$.

We claim that there exists a nonstandard frame. We know that there is a normalized standard frame $\{x_j\}_{j \in J}$ in $C_0(P_1)$, and there is a frame $\{y_l\}_{l \in I}$ in $C_0(P_2)$ with constants $c_1 = 1$ and $c_2 = 2$, respectively. By the previous lemma, their union $\{w_g\}_{g \in G} = \{x_j\}_{j \in J} \cup \{y_l\}_{l \in I}$ is a frame in $C_0(K)$. \hfill $\Box$

**4. NONEXISTENCE OF NONSTANDARD FRAMES IN $\mathcal{X}_{IV}$**

**Theorem 12.** Let $K \in \mathcal{X}_{IV}$. Then the $C_0(K)$-module $C_0(K)$ has no frame.

**Proof.** Assume that there exists a frame $\{x_j\}_{j \in J}$ in $C_0(K)$. Take an arbitrary point $t_1 \in K$. There exists a function $g_1 \in C_0(K)$ such that $g_1(t_1) = 1$, $|g_1(t)| \leq 1$ on $K$. There is a nonempty at most countable set $\{x_j\}_{j \in J}$ of elements of the frame such that $x_j(t_1) \neq 0$ because, by taking $x = g_1$ and using the evaluation $\varphi = \delta_{t_1}$ at the point $t_1$, the series $\sum_j g_1(x_j) x_j(t_1) = \sum_j |x_j(t_1)|^2$ converges at the point $t_1$ due to Lemma 1.

That is, if $j \in J \setminus J_1$, then $x_j(t_1) = 0$. For every $j \in J_1$, there is a compact set $K_{1,j} \subset K$ such that $x_j = 0$ outside $K_{1,j}$. $J_1$ is at most a countable set, and thus, there is a compact set $K_1$ such that $\bigcup_{j \in J_1} K_{1,j} \subset K_1$. That is, $x_j = 0$ outside $K_1$ for any $j \in J_1$.

Assume that we have already found points $t_1, \ldots, t_n$, compact sets $K_1, \ldots, K_n$, and index sets $J_1, \ldots, J_n \subset J$ such that $t_i \notin \bigcup_{l=1}^{i-1} K_l$ for $i = 2, \ldots, n$, $x_j(t_i) \neq 0$ only for $j \in J_i$ (as a consequence, different sets $J_i$ do not intersect), and $x_j = 0$ outside $K_l$ for $j \in J_l$.

Take an arbitrary point $t_{n+1} \in K \setminus \bigcup_{l=1}^{n} K_l$. As in the case of $n = 1$, there exists an at most countable nonempty set $\{x_j\}_{j \in J_{n+1}}$ of elements of the frame such that $x_j(t_{n+1}) \neq 0$ for $j \in J_{n+1}$ (and hence, $J_{n+1}$ does not intersect any $J_l$, $l = 1, \ldots, n$, since the functions $x_j$ for $j \in \bigcup_{l=1}^{n} J_l$ vanish outside $\bigcup_{l=1}^{n} K_l$). There also exists a compact set $K_{n+1}$ such that $x_j = 0$ outside $K_{n+1}$ for any $j \in J_{n+1}$. By induction, we can continue this construction for any $n \in \mathbb{N}$.

The sequence $\{t_n\}_{n \in \mathbb{N}}$ is a $\sigma$-compact set, and thus there exists a compact set $K'$ containing this sequence. Hence, the sequence has a limit point $t_0 \in K'$. Let us show that $x_j(t_0) = 0$ for all $j \in J$, which will contradict the fact that $\{x_j\}_{j \in J}$ is a frame.

First, let $j \in J \setminus \bigcup_{l=1}^{\infty} J_l$. Then $x_j(t_0) = 0$ for all $n \in \mathbb{N}$. Hence, $x_j(t_0) = 0$, because otherwise, if $x_j(t_0) = q \neq 0$, then in any neighborhood of the point $t_0$, there is a point $t_n$ such that $|x_j(t_n) - x_j(t_0)| = |q| > 0$; a contradiction to the continuity of $x_j$.

Let now $j \in \bigcup_{l=1}^{\infty} J_l$, i.e., $j \in J_k$ for some $k \in \mathbb{N}$, and suppose that $x_j(t_0) \neq 0$. Then $t_0 \in K_k$ (because $x_j = 0$ outside $K_k$). Hence, $x_j(t_{k+l}) = 0$ for all $l \in \mathbb{N}$ (because $t_{k+l} \notin K_k$), $t_0$ is still a limit point for the sequence $\{t_n\}_{n=k+1}^{\infty}$, and then $x_j(t_0) = 0$, similarly to the previous case.

Hence, $\{x_j\}_{j \in J}$ is not a frame. \hfill $\Box$
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