ON EXCEPTIONAL ENRIQUES SURFACES

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Abstract

We give a complete description of all classical ("Z/2") Enriques surfaces with non-zero global vector fields. In particular we show that such surfaces exist. The result that we obtain also applies to supersingular ("α2") surfaces that fulfil a rather special condition. In the course of the classification we study some properties of genus 1 fibrations special to characteristic 2 as well as make a close study of the genus 1 fibrations on the surfaces in question.

1 Introduction

Whether a classical Enriques surface has a non-zero global vector field is of interest in, for example, the deformation theory of Enriques surfaces. In [SB96] it was claimed that this cannot occur. However, there is an error in the proof and in fact the truth is the opposite: such surfaces do exist. It turns out that a certain condition, which, in the case of a classical Enriques surface, is equivalent to the existence of a vector field, is also of interest in the non-classical case; we call surfaces that satisfy this condition exceptional. (See 2.6 for the precise definition.) Our main results are summarized in Theorems 1.1 to 1.3, where the conductrix is a certain effective divisor supported on the image of the singular locus of the canonical double cover. In particular, we shall see that the exceptional surfaces are just those that possess a special genus 1 fibration with a double fibre of type $\tilde{E}_6,7,8$. Such surfaces have been classified by Salomonsson [Sa03]. We shall say that an exceptional surface is of type $T$ if the support of the conductrix is a $T$-configuration. We also say that a genus 1 fibration on an Enriques surface is special if it has a 2-section isomorphic to $\mathbb{P}^1$ (it always has a 2-section of arithmetic genus at most 1). Such a 2-section is called special. (Classically surfaces with such a pencil are referred to as special, and in [CD89] the terminology degenerate U-pair is used.)

If $\Delta$ is a simply laced affine Dynkin diagram then $F_\Delta$ will denote the corresponding Kodaira–Néron fibre.

If $A$ is a divisor whose support is a diagram of type $T_{p,q,r}$ then we write

$$A = (a; b_1, \ldots, b_{p-1}; c_1, \ldots, c_{q-1}; d_1, \ldots, d_{r-1})$$

in an obvious way. For example, $F_{\tilde{E}_r}$ is $(3; 2, 1; 2, 1; 2, 1), (4; 2, 3, 2, 1; 3, 2, 1)$ and $(6; 3; 4, 2; 5, 4, 3, 2, 1)$ according as $r = 6, 7$ or 8.
Theorem 1.1 Suppose that $X$ is an Enriques surface in characteristic 2.

1. $X$ is exceptional if and only if its conductrix $A$ is of type $T_{p,q,r}$, $(p,q,r)$ is one of $(2,3,7), (2,4,5), (3,3,3)$ and $A$ is, accordingly,

\[(5; 2; 3, 2; 4, 4, 3, 3, 2, 1), (3; 1; 2, 2, 1; 2, 2, 1, 1), (2; 1, 1; 1, 1, 1, 1).\]

2. $X$ is exceptional if it has a quasi–elliptic fibration with a simple $\widetilde{E}_{7,8}$ fibre. $X$ is of type $T_{3,3,3}$ or $T_{2,4,5}$ accordingly.

3. $X$ is exceptional if and only if it has a special genus 1 fibration with a double fibre of type $\widetilde{E}_{6,7,8}$. It is then of type $T_{3,3,3}, T_{2,4,5}$ or $T_{2,3,7}$, respectively.

1.1 1 is Proposition 3.11 and Theorem 3.14, 1.1 3 is Proposition 3.16 and 1.1 2 is Theorem 4.2.

The definition of an exceptional Enriques surface is given by a simple condition on the conductrix but we also give the following elaboration of that condition.

Theorem 1.2 An exceptional Enriques surface $X$ is either a $\mathbb{Z}/2$-surface or an $\alpha_2$-surface. A $\mathbb{Z}/2$-surface $X$ is exceptional precisely when it has global vector fields, and then $\dim H^0(X, T_X) = 1$. An $\alpha_2$-surface is exceptional exactly when the cup product on $H^1(X, \mathcal{O}_X) \times H^0(X, \Omega^1_X)$ is zero.

This is proved in Propositions 2.7 and 2.8.

Remark: The presence of vector fields on a $\mathbb{Z}/2$-surface clearly makes its deformation theory “pathological”. We shall show elsewhere that an Enriques surface is exceptional exactly when a versal deformation of it as a unipotent Enriques surface is singular.

We go on to discuss the classification of exceptional surfaces and show that all three types exist, both for $\mathbb{Z}/2$-surfaces and for $\alpha_2$-surfaces. We also describe all genus 1 fibrations on them. This description is complicated for surfaces $X$ of type $T_{3,3,3}$, where we need to distinguish between surfaces of different $MW$-rank $MW(X)$. By definition,

\[MW(X) = 8 - \sum_s n(s) - 1,\]

where $s$ runs over the fibres of the unique elliptic pencil on $X$ and $n(s)$ is the number of irreducible components of $s$. ($MW(X)$ equals the Mordell–Weil rank of the Jacobian surface associated to $X$.)

Theorem 1.3

1. An exceptional Enriques surface of type $T_{2,3,7}$ has a unique genus 1 fibration. This fibration is quasi–elliptic.

2. An exceptional Enriques surface of type $T_{2,4,5}$ has 2 or 3 genus 1 fibrations. These fibrations are all quasi–elliptic.

3. An exceptional Enriques surface $X$ of type $T_{3,3,3}$ has a unique elliptic 1 fibration. There exist quasi–elliptic fibrations on it; these fibrations are arranged in triples and the set of triples is a torsor under a discrete group $G$. The group
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G is trivial if $\text{MW}(X) = 0$, while $G = \mathbb{Z}$ if $\text{MW}(X) = 1$ and $G$ is the Weyl group of type $\tilde{A}_2$ if $\text{MW}(X) = 2$, the maximum possible value. Each quasi-elliptic fibration appears in 2 of these triples if $\text{MW}(X) \neq 0$.

1.3 1 is Theorem 3.15, 1.3 2 is Theorem 4.4 and 1.3 3 is a combination of Theorems 5.11, 5.13 and 5.15.

We also give, in Theorem 4.4, a description of the $(-2)$ curves on an exceptional surface of type $(2, 4, 5)$ or $(2, 3, 7)$. For surfaces of type $(3, 3, 3)$ they are described by Theorems 5.11, 5.13 and 5.15.

The base field of all varieties appearing here will be, unless explicitly noted otherwise, algebraically closed and of characteristic 2.

We shall name the types of Enriques surfaces after the $\tau$ in their $\text{Pic}^\tau$.

In other terminology $\mu_2$-surfaces are singular, $\mathbb{Z}/2$-surfaces are classical and $\alpha_2$-surfaces are supersingular. When $\tau = \mathbb{Z}/2$ or $\alpha_2$ we refer to the surface as unipotent. After Proposition 2.7 all surfaces will, unless stated otherwise, be unipotent.

We shall use the extended Dynkin diagram notation for the normal crossing singular fibres of a relatively minimal genus 1 fibration. The $E$-series of (extended) Dynkin diagrams are also graphs of type $T_{*,*,*}$ (cf. [CD89], p. 105) and we shall pass freely between the two kinds of notation.

2 Preliminaries

Lemma 2.1 Suppose that $S$ is an affine Noetherian scheme and that $\pi : X \to S$ is a proper morphism of relative dimension $\leq n$. Assume that $H^n(X, \mathcal{O}_X) \neq 0$ and that $H^n(Z, \mathcal{O}_Z) = 0$ for all closed subschemes $Z$ of $X$ such that $Z \neq X$.

Then

(1) $H^0(X, \mathcal{O}_X)$ is a field and

(2) $\mathcal{O}_X$ has no non-zero subsheaves whose support is of relative dimension less than $n$.

PROOF: By assumption, $H^n(X, -)$ is right exact on quasi-coherent sheaves. Suppose that $0 \neq \lambda \in R := H^0(X, \mathcal{O}_X)$ and let $X_\lambda$ denote the closed subscheme of $X$ defined by $\lambda$. So there is an exact sequence

$$\mathcal{O}_X \xrightarrow{\lambda} \mathcal{O}_X \to \mathcal{O}_{X_\lambda} \to 0$$

whose cohomology gives an exact sequence

$$H^n(X, \mathcal{O}_X) \xrightarrow{\lambda} H^n(X, \mathcal{O}_X) \to H^n(X_\lambda, \mathcal{O}_{X_\lambda}) \to 0.$$

By our assumptions, $H^n(X_\lambda, \mathcal{O}_{X_\lambda}) = 0$, so that multiplication by $\lambda$ is surjective on $H^n(X, \mathcal{O}_X)$. Now $H^n(X, \mathcal{O}_X)$ is a non-zero finitely generated $\Gamma(S, \mathcal{O}_S)$-module, and so a finitely generated $R$-module. Therefore it has a non-zero quotient $M$
killed by some maximal ideal $m$ of $R$. If $0 \neq \lambda \in m$ then $M = \lambda M = 0$, contradiction. So $m = 0$ and $R$ is a field, as claimed.

(2): If $Z$ is the closed subscheme of $X$ defined by $I$ then $H^n(Z, O_Z) = H^n(X, O_X) \neq 0$, while $H^n(Z, O_Z) = 0$ by assumption.

We apply this lemma to a particular divisor on an Enriques surface.

**Lemma 2.2** Suppose that $D$ is an effective divisor on an Enriques surface $X$, that $h^0(X, O(D)) = 1$ and that $h^1(O_D) \neq 0$. Then $D$ contains a half-fibre of a genus 1 fibration.

**Proof:** Note that, by Riemann–Roch and the assumption that $h^0(X, O(D)) = 1$, $D$ contains no effective subdivisor $E$ with $E^2 > 0$.

By Lemma 2.1 and the Noetherian property, there exists $0 \neq E \subset D$ that is minimal for the condition that $h^1(E, O_E) \neq 0$. By Lemma 2.1 again, $h^0(E, O_E) = 1$. So $\chi(E, O_E) \leq 0$ and then, by Riemann–Roch, $E^2 \geq 0$. So $E^2 = 0$ and now the result follows from [CD89], Th. 3.2.1.

Suppose that $X$ is a Gorenstein scheme and that

\[
\begin{array}{ccc}
\tilde{Z} & \xrightarrow{\pi} & Z \\
\downarrow{\beta} & & \downarrow{\alpha} \\
X & & 
\end{array}
\]

is commutative. Suppose also that $\alpha, \beta$ are finite and flat of degree 2 and that $\pi$ is finite and birational. Suppose that $I_C \subset O_Z$ is the conductor of $\pi$; then $I_C$ is also an ideal in $O_{\tilde{Z}}$.

**Lemma 2.3** $I_C$ is an invertible $O_{\tilde{Z}}$-module and there is an effective Cartier divisor $A$ on $X$ such that $I_C = \beta^* O_X(-A)$.

**Proof:** Note first that the $O_X$-modules $L = O_Z/O_X$ and $L' = O_{\tilde{Z}}/O_X$ are invertible and there is a commutative diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & O_X & \longrightarrow & O_Z & \longrightarrow & L & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & O_X & \longrightarrow & O_{\tilde{Z}} & \longrightarrow & L' & \longrightarrow & 0.
\end{array}
\]

So $L' = L \otimes O_X(A)$ for some effective Cartier divisor $A$ on $X$. By adjunction, $\omega_{Z/X} \cong \alpha^* L^{-1}$ and $\omega_{\tilde{Z}/X} \cong \beta^* L'^{-1}$. Since also $\omega_{\tilde{Z}/X} = I_C \pi^* \omega_{Z/X}$ the result follows.

We shall call $A$ the *conductrix* of the $X$-morphism $\pi$ and $B := 2A$ the *biconductrix*.

**Lemma 2.4** Assume that $\alpha$ is inseparable, so that there is a factorization

\[
\tilde{Z} \xrightarrow{\beta} X \xrightarrow{\alpha} \tilde{Z}^{(1)}
\]
of the Frobenius $\tilde{Z} \to \tilde{Z}^{(1)}$ and $\gamma : X \to \tilde{Z}^{(1)}$ is the quotient of $X$ by a 2-closed foliation $\mathcal{F}$ of rank 1. Then $\mathcal{F} \cong \omega_X(B)$, so that $c_1(\mathcal{F})$ is numerically equivalent to $B$.

**PROOF:** There is an exact sequence

$$0 \to \mathcal{O}_X \to \beta_! \mathcal{O}_{\tilde{Z}} \to \omega_X(A) \to 0.$$ 

Recall that the map

$$\beta_! \mathcal{O}_{\tilde{Z}} \to \Omega^1_X : f \mapsto df^2$$

induces an injective map $\mathcal{O}_X(B) \cong \text{Frob}^*_{\tilde{X}}(\omega_X(A)) \to \Omega^1_X$ which is saturated, since $\tilde{Z}$ is normal. That is, there is a short exact sequence

$$0 \to \mathcal{O}_X(B) \to \Omega^1_X \to \mathcal{I}_W \omega_X(-B) \to 0.$$  

(2.5)

The dual of this is

$$0 \to \mathcal{F} \to T_X \to \mathcal{I}_W(-B) \to 0.$$ 

This foliation $\mathcal{F}$ is the *natural foliation* on $X$.

When $X$ is an Enriques surface, $Z$ its canonical double cover and $\tilde{Z} \to Z$ is the normalization we shall also speak of the conductrix and biconductrix of $X$.

If $f : X \to S$ is a genus 1 fibration in characteristic 2 the we get a map $f' : X' \to S$ which is the pullback of $f$ by the Frobenius on $S$, and we also have the normalization $\rho : \tilde{X} \to X'$. The conductrix of $f$ is then, by definition, the conductrix of $\rho$. This leads to slight ambiguity because there are two conductrices, one of the surface and one of the fibration. However, this should cause no confusion.

**Definition 2.6** An Enriques surface whose biconductrix is $B$ is exceptional if $H^1(B, \mathcal{O}_B) \neq 0$.

**Proposition 2.7** An Enriques surface which is not unipotent has empty conductrix and so is not exceptional.

**PROOF:** The canonical double cover of a surface that is not unipotent is étale.

So from now on we shall only consider Enriques surfaces $X$ that are unipotent. There is a diagram

$$
\begin{array}{ccc}
\hat{Z} & \xrightarrow{\delta} & \tilde{Z} \\
& \searrow^{\beta} & \swarrow^\pi \\
& Z & \\
\end{array}
$$

where $\alpha : Z \to X$ is the canonical double cover and is inseparable, $\tilde{Z} \to Z$ is the normalization and $\hat{Z} \to \tilde{Z}$ the minimal resolution. The conductrix is $A$ and the biconductrix is $B$. We shall assume that $A \neq 0$. 


Proposition 2.8

(1) \( B \) is the divisorial part of the zero locus of any global 1-form.
(2) \( h^0(B, \mathcal{O}_B) = 1 \). In particular, \( A \) cannot contain a fibre or half-fibre in any genus 1 fibration on \( X \) and \( \text{Supp} \ A \) is a normally crossing configuration \( \Gamma \) of \((-2)\)-curves.
(3) \( A \) is 1-connected, \( D^2 < 0 \) for all effective \( D \leq A \), \( A^2 = -2 \) and \( \Gamma \) is a tree.
(4) \( \tilde{Z} \) is rational and has either 4 singularities of type \( A_1 \) or 1 of type \( D_4 \).
(5) \( X \) is exceptional if and only if \( B \) contains a half-fibre of some genus 1 fibration.
(6) If \( X \) is a \( \mathbb{Z}/2 \)-surface then \( X \) is exceptional if and only if it has a vector field. In any case \( h^0(X, T_X) \leq 1 \).
(7) If \( X \) is an \( \alpha_2 \)-surface then it is exceptional if and only if the cup product

\[
H^1(X, \mathcal{O}_X) \otimes H^0(X, \Omega^1_X) \to H^1(X, \Omega^1_X)
\]

is identically zero.

PROOF: (1) This follows at once from Lemma 2.4.
(2) Computing Chern classes of the sheaves appearing in the exact sequence 2.5 shows that \( \deg W - B^2 = 12 \). Since \( h^{10}(X) = 1 \) we get \( h^0(X, \mathcal{O}_X(B)) = 1 \), which is the first part of (2). The rest follows at once.
(3) By [CD89], Prop. 3.1.2 and Th. 3.2.1, we have \( h^0(X, \mathcal{O}_X(2D)) \geq 2 \) if \( D \) is effective and \( D^2 \geq 0 \), so that \( D^2 < 0 \) for all effective \( D \leq A \).

Castelnuovo’s criterion shows that \( \tilde{Z} \) is rational, since \( A > 0 \), and therefore \( \tilde{Z} \) has du Val singularities. Since also

\[
\chi(\mathcal{O}_{\tilde{Z}}) = \chi(\mathcal{O}_X) + \chi(\mathcal{O}_X(A)) = A^2/2 + 2
\]

we see that \( A^2 = -2 \).

Now suppose that \( A = C + D \) with \( C, D > 0 \). Then

\[
-2 = A^2 = C^2 + D^2 + 2C.D \leq -4 + 2C.D,
\]

so that \( C.D \geq 1 \). Now it is enough to observe that \( A \) cannot contain a cycle \( D \) with \( D^2 \geq 0 \) since \( h^0(X, \mathcal{O}_X(B)) = 1 \).
(4) \( B^2 = -8 \) and \( \deg W = 4 \). If \( P \in \text{Sing} \tilde{Z} \) then we can write

\[
\mathcal{O}_{\tilde{Z}, P} = k[[x, y, z]]/(z^2 - f(x, y))
\]

where \( f \in m^2_{(x, y)} \) and \( 4 \geq \deg_P W = \dim_k k[[x, y]]/(f_p^x, f_p^y) \). Calculation shows that then \( (\tilde{Z}, P) \) is of type either \( A_1 \) or \( D_4 \). Since \( \sum_P \deg_P W = 4 \) the proof of (4) is complete.
(5) If \( 0 < D' \leq D \) are divisors on \( X \) such that \( h^1(D, \mathcal{O}_D) = 0 \), then \( h^1(D', \mathcal{O}_{D'}) = 0 \). This gives one direction, and the other follows from Lemma 2.2.
(6) Notice that, by duality, \( H^1(B, \mathcal{O}_B) \neq 0 \) if and only if \( H^0(B, \omega_B) \neq 0 \). Furthermore there is a short exact sequence

\[
0 \to \omega_X \to \omega_X(B) \to \omega_B \to 0.
\]  

(2.9)

In the \( \mathbb{Z}/2 \) case we exploit the dual of the sequence 2.5. From this it follows that, if \( H^0(X, T_X) \neq 0 \), then \( H^0(X, \omega_X(B)) \neq 0 \). However, \( H^0(X, \omega_X) = 0 \) and we conclude by taking the cohomology of 2.9.

(7) Assume that \( X \) is an \( \alpha_2 \)-surface. Note that

\[ h^1(X, \mathcal{O}_X) = h^0(X, \Omega_X^1) = 1. \]

Suppose that \( \beta \in H^1(X, \mathcal{O}_X) \) and \( \eta \in H^0(X, \Omega_X^1) \). By 2.5 and the fact that \( W \neq 0 \), it follows that \( \eta \) is the image of some \( \eta' \in H^0(X, \omega_X(B)) \). Then \( \eta/\beta \) is the image of \( \eta/\beta \). Since \( H^0(X, \mathcal{O}_X(B)) = 1 \) we can suppose that \( \eta' \) comes from \( 1 \in H^0(X, \mathcal{O}_X) \) under the inclusion of 2.9. Thus \( \eta/\beta \) is the image of \( \beta \) under the map \( H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X(B)) \). Since \( \omega_X \) is trivial the inclusion \( \mathcal{O}_X \hookrightarrow \mathcal{O}_X(B) \) is isomorphic to the inclusion \( \omega_X \hookrightarrow \omega_X(B) \). It follows from 2.9 and the fact that \( h^0(\omega_X) = h^0(\omega_X(B)) = 1 \) that the map \( H^1(X, \omega_X) \to H^1(X, \omega_X(B)) \) is zero exactly when \( h^0(\omega_B) \neq 0 \), which, as we have noted, is equivalent to \( X \) being exceptional.

**Lemma 2.10** If \( X \) is exceptional then \( \Gamma \) is not negative definite.

**PROOF:** If \( \Gamma \) is negative definite then there is a contraction \( h : X \to Y \) of \( \Gamma \) to a du Val singularity. Since \( R^1h_*\mathcal{O}_X = 0 \), it follows that \( H^1(B, \mathcal{O}_B) = 0 \) for all divisors \( B \) supported on \( \Delta \). 

**Lemma 2.11** Suppose that \( g : X \to \mathbb{P}^1 \) is a genus 1 fibration.

(1) \( \alpha : Z \to X \) factors through the pullback \( X_F \) of \( g \) by the Frobenius on \( \mathbb{P}^1 \). The map \( Z \to X_F \) is an isomorphism outside the double fibres of \( g \).

(2) The restriction of \( \alpha \) to a half-fibre of \( g \) is non-trivial.

**PROOF:** We claim that the restriction of \( \alpha \) to a simple fibre \( \Phi \) of \( g \) is trivial.

For this, suppose first that \( X \) is an \( \alpha_2 \)-surface. Since \( H^0(\Phi, \mathcal{O}_\Phi) = k \), the base field, it is enough to show that the map \( H^1(X, \mathcal{O}_X) \to H^1(\Phi, \mathcal{O}_\Phi) \) is zero. This follows from the cohomology of the short exact sequence

\[ 0 \to \mathcal{O}_X(-\Phi) \to \mathcal{O}_X \to \mathcal{O}_\Phi \to 0 \]

and the fact that \( h^1(X, \mathcal{O}_X(-\Phi)) = 1 \).

In the \( \mathbb{Z}/2 \) case we need only observe that \( K_X \) is the difference of the two half-fibres so its restriction to \( \Phi \) is obviously trivial.

Consider now the sheaf \( A = g_*\alpha_*\mathcal{O}_Z \) on \( \mathbb{P}^1 \). Its restriction to \( \Phi \) is a trivial vector bundle, so that \( A \) is of rank 2.
Now assume that $X$ is an $\alpha_2$-surface. Then there is a short exact sequence
\[0 \to \mathcal{O}_X \to \alpha_*\mathcal{O}_Z \to \mathcal{O}_X \to 0\]
whose cohomology gives an exact sequence
\[0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{A} \to \mathcal{O}_{\mathbb{P}^1} \to R^1g_*\mathcal{O}_X.\]
Since rank $\mathcal{A} = 2$ the image of the boundary map is torsion. Now $\text{Tors} R^1g_*\mathcal{O}_X$ has length 1 so $\mathcal{A}/\mathcal{O}_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$. Therefore $\text{Spec} \mathcal{A}$ is obtained by taking the square root of a non-zero quadratic form.

The same statement holds for $\mathbb{Z}/2$-surfaces, and is easier to prove as $R^1g_*\mathcal{O}_X$ is torsion-free and $g^*\omega_X = \mathcal{O}_{\mathbb{P}^1}$.

Up to isomorphism there are only two such covers of $\mathbb{P}^1$, the trivial one and the Frobenius map $F: \mathbb{P}^1 \to \mathbb{P}^1$. Now $\mathcal{A}$ is reduced so the cover is non-trivial, so is the Frobenius. So $Z \to \mathbb{P}^1$ factors through $F: \mathbb{P}^1 \to \mathbb{P}^1$. Finally, the map $g^*\mathcal{A} \to \alpha_*\mathcal{O}_Z$ is an isomorphism away from the double fibres.

As for the second part, the case of a $\mathbb{Z}/2$-surface is well known. So suppose that $X$ is an $\alpha_2$-surface and that $\Psi$ is the half-fibre. Consider the exact sequence
\[0 \to \mathcal{O}_X(-\Psi) \to \mathcal{O}_X \to \mathcal{O}_\Psi \to 0;\]
this time $h^1(\mathcal{O}_X(-\Psi)) = 0$ and so the map $H^1(X, \mathcal{O}_X) \to H^1(\Psi, \mathcal{O}_\Psi)$ is injective, which is what is needed.

**Corollary 2.12** Suppose that $g: X \to \mathbb{P}^1$ is a genus 1 fibration.

1. The natural foliation $\mathcal{F}$ is the kernel of the derivative $g_*: T_X \to g^*T_{\mathbb{P}^1}$.
2. $g: X \to \mathbb{P}^1$ factors through the quotient map $X \to X/\mathcal{F} = \mathbb{Z}(1)$.
3. If $g$ is quasi–elliptic and $R = R_g$ is its curve of cusps then $R \subset A$ and $R$ is a $(−2)$-curve on $X$.

**Proof:** (1) and (2) are restatements of part (1) of Lemma 2.11 and (3) is an immediate consequence.

For the rest of this paper $R_g$ will denote the curve of cusps in a quasi–elliptic fibration $g: X \to \mathbb{P}^1$.

### 3 Genus 1 fibrations and the conductrix

Suppose that $X$ is a unipotent Enriques surface whose conductrix $A$ is non-zero. We know that $\Gamma := \text{Supp} A$ is a tree of $(−2)$-curves.

Fix a genus 1 fibration $f: X \to S = \mathbb{P}^1$. For $s \in S$ we write $f^{-1}(s) = d_sX_s$, where $d_s = d = 1$ or 2 is the multiplicity of the fibre and $X_s$ is a Kodaira–Néron divisor. We can write $A = R_1 + \sum A_s$ where each $A_s$ is supported on $X_s$, $R_1$ is horizontal (that is, finite over $S$). Moreover, $R_1 = 0$ if $f$ is elliptic and $R_1 = R_f$ if $f$ is quasi–elliptic.
Lemma 3.1 Assume that $\Gamma$ is not negative definite.

(1) $f$ is elliptic if and only if $\Gamma$ is of type $\tilde{D}$ or $\tilde{E}$.

(2) If $f$ is elliptic then $\Gamma$ is the support of a unique $X_s$.

(3) $X$ has at most one elliptic fibration.

PROOF: Since $f$ is generically smooth, $R_1 = 0$. (1) then follows from the 1-connectedness and the non-negativity of $\Gamma$.

(2) and (3) are immediate.

Lemma 3.2 If $E$ is a $(-2)$-curve on $X$ then $A.E \leq 1$. If $E_1, E_2$ are $(-2)$-curves that meet transversely in at least one point then $A.(E_1 + E_2) \leq 0$.

PROOF: Consider the natural foliation $F \cong \omega_X(2A)$. If $E$ is generically tangent to $F$ then $c_1(F).E \leq 2$; conversely, if $c_1(F).E = 2$ then $F$ is everywhere tangent to $E$. If $E$ is not generically tangent to $F$ then $c_1(F).E \leq -2$. If $E_1$ and $E_2$ are transverse at a point then it is impossible to have both $c_1(F).E_1 = 2$ and $c_1(F).E_2 = 2$, and the lemma is proved.

Proposition 3.3 $\Gamma$ is a chain $A_n$ or a tree $T_{p,q,r}$.

PROOF: We need to show that $\Gamma$ has no vertex of valency at least 4 and does not have two vertices of valency 3; that is, that $\Gamma$ contains no configuration of type $\tilde{D}_n \geq 4$.

If $\Gamma \supset \tilde{D}_4$ then, as an immediate consequence of Lemma 3.2, $A \geq F_{\tilde{D}_4}$. But this contradicts Proposition 2.8.

Similarly, if $\Gamma \supset \tilde{D}_{n \geq 5}$ then, by 2.8, there is a curve $C$ in the spine of $\tilde{D}_n$ of multiplicity 1 in $A$. This leads quickly to a contradiction to Lemma 3.2.

Lemma 3.4 $n \leq 11$ and $p + q + r \leq 12$, respectively.

PROOF: $\rho(X) = 10$.

Lemma 3.5 If $\Gamma$ is not negative definite then there is an affine Dynkin diagram $\Delta \subset \Gamma$ and for every such $\Delta$, $\Gamma - \Delta$ is connected.

PROOF: The existence is clear.

For any such $\Delta$, there is a genus 1 fibration $f : X \to \mathbb{P}^1$ such that $\Delta$ is the support of a fibre. Since $R_f$ is irreducible and each connected component of $\Gamma - \Delta$ contains a vertex corresponding to a component of $R_f$, the lemma is proved.

From now on we assume that $\Gamma$ is not negative definite. So $\Gamma$ is one of $T_{3,3,r \geq 3}$, $T_{2,4,s \geq 4}$ and $T_{2,3,t \geq 6}$.

Lemma 3.6 $r \leq 4$, $s \leq 5$ and $t \leq 7$.

PROOF: Suppose $\Gamma = T_{3,3,r \geq 5} = T_{3,3,3} \cup (R, E_1)$ or $T_{3,3,3} \cup (R, E_1, E_2)$. Then there is a quasi–elliptic fibration $f : X \to \mathbb{P}^1$ such that $R = R_f$ and there is a singular fibre $X_s$ such that $X_s \cap \text{Supp} \ A \supset E_1$. Since $\rho(X) = 10$ the fibre $X_s$ has
at most 3 components, so \( X_s = E_1 + E_2 = Ta \) or \( E_1 + E_2 + E_3 = Tr \) or \( I_3 \). Then either \( A.E_2 \geq 2 \) or \( A.(E_2 + E_3) \geq 2 \), both of which are impossible.

If \( \Gamma = T_{2,4,6} \) then \( \Gamma = (R, E_1) \) and we get a quasi-elliptic fibration \( f : X \to \mathbb{P}^1 \) with a singular fibre \( X_s \) that contains \( E_1 \). Then \( X_s = Ta \) and we get a contradiction as before.

Suppose that \( A' = (a(r); a(r-1), \ldots, a(1); b, b'; c) \) is that part of \( A \) supported on a subdiagram \( T_{2,3,r} \) of \( \Gamma \). So

\[
a(i-1) - 2a(i) + a(i+1) \leq 1 \quad \text{and} \quad a(i-1) - a(i) - a(i+1) + a(i+2) \leq 0
\]

for all \( i \). Define \( d(i) = a(i+1) - a(i) \), so that \( d(i) \leq d(i-2) \) and \( d(i) \leq d(i-1) + 1 \).

Take the least \( j \) such that \( d(j) < 0 \), if such exists. Then \( d(0), \ldots, d(j-1) \geq 0 \) and \( d(j), \ldots, d(r-1) < 0 \).

**Lemma 3.7** \( a(i) \) increases (not necessarily strictly) as \( i \) goes from 1 to \( j+1 \) and then decreases (not necessarily strictly) as \( i \) goes from \( j+1 \) to \( r \).

**Proof:** This is immediate, from the properties of \( d \) just observed.

**Lemma 3.8** \( a(r-1) < a(r) \).

**Proof:** There is a diagram \( \Delta \subset \Gamma \) of type \( D_5 = T_{2,2,3} \) such that the part \( A'' \) of \( A \) that is supported on \( \Delta \) is \( A'' = (a(r); a(r-1); b, b'; c) \) where the entries are all \( > 0 \). Then

\[
a(r-1) - a(r) - c + b \leq 0 \quad \text{and} \quad a(r-1) + b' + c - b - a(r) \leq 0.
\]

Adding these gives

\[
1 \leq b' \leq 2(a(r) - a(r-1)).
\]

Therefore \( a(i) \) increases as \( i \) goes from 1 to \( r \).

**Lemma 3.9** If \( a(r) \geq r \) then \( a(i) \geq i \) for all \( i \in [1, r] \).

**Proof:** We know that \( d(i) \geq 0 \) for all \( i \). Suppose that \( d(m) = 0 \) for some \( m \); take \( m \) to be minimal. Since \( d(i) \leq d(i-2) \), it follows that

\[
0 = d(m) = d(m+2) = \cdots = d(m+2s)
\]

for all \( s \). Since also \( d(k+1) \leq d(k) + 1 \), it follows that

\[
d(m+1), \ldots, d(m+2s+1) \leq 1.
\]

So \( a(r-2k) \geq a(r) - k \) and \( a(r-2k-1) \geq a(r) - k - 1 \), so that \( a(i) \geq i \) for all \( i \geq m \).

In the range \( i < m \) we have \( d(i) \geq 1 \), so \( a(i) \geq i \) for all \( i < m \).
Corollary 3.10  Suppose that $\Gamma$ is affine. Then $a(r) < r$.

PROOF: Suppose $a(r) \geq r$. Then $A \geq F_\Gamma$, by Lemma 3.9, which is impossible. □

Corollary 3.10 and Lemmas 3.7 and 3.8 make it easy to enumerate the cycles $A$ such that $h^0(X, 2A) = 1$, $\text{Supp} A$ is an affine diagram $\Gamma$ and $A$ satisfies the conclusions of Lemma 3.2 (but maybe $A^2 \neq -2$).

(1) $\Gamma = T_{3,3,3} : A = (2; 1, 1; 1; 1, 1), A^2 = -2$.

(2) $\Gamma = T_{2,4,4} : A = (3; 2, 2, 1; 2, 2, 1; 1), A^2 = -2$.

(3) $\Gamma = T_{2,3,6} :$

(a) $A = (5; 4, 4, 3, 3, 2; 3, 2; 1), A^2 = -4$.

(b) $A = (5; 4, 4, 3, 3, 1; 3, 2; 2), A^2 = -4$.

(c) $A = (5; 4, 4, 3, 2, 1; 3, 2; 2), A^2 = -2$.

If $\Gamma = T_{p,q,r}$ is hyperbolic then $\Gamma = \Gamma_{aff} \cup \{R_g\}$ where $R_g$ is the curve of cusps in a quasi–elliptic fibration $g : X \to \mathbb{P}^1$ and $\text{mult}_A(R_g) = 1$. The list above of possible cycles in the affine case leads to this classification in the hyperbolic case.

$\Gamma = T_{2,4,5} : A = (3; 2, 2, 1; 1; 2, 2, 1; 1)$.

$\Gamma = T_{2,3,7} : A = (5; 4, 4, 3, 3, 2, 1; 3, 2; 2)$.

Everything else is impossible. In particular, $\Gamma$ cannot be $T_{3,3,4}$.

Finally, since $A^2 = -2$, if $\Gamma = T_{2,3,6}$ then $A = (5; 4, 4, 3, 2, 1; 3, 2; 2)$.

Proposition 3.11  Suppose that $X$ is an Enriques surface.

(1) $X$ is exceptional if and only if the support $\Gamma$ of its conductrix $A$ is not negative definite.

(2) $X$ is exceptional if and only if $\Gamma$ is one of the five diagrams just listed and $A$ is as listed.

PROOF: It is enough to note that if $A$ is one of these cycles then $2A$ contains a Kodaira–Néron fibre $F$, and $h^1(F, \mathcal{O}_F) = 1$. □

We refer to this diagram $T_{p,q,r}$, or the triple $(p, q, r)$, as the type of an exceptional Enriques surface.

Theorem 3.12  An exceptional Enriques surface $X$ has a unique elliptic fibration if its type is affine. In this case the support of its conductrix is a half-fibre. $X$ has no elliptic fibration if its type is hyperbolic.

PROOF: Immediate. □

Recall that a genus 1 fibration $f : X \to \mathbb{P}^1$ is special if it has a 2-section that is a $(-2)$-curve and that, according to [CD89], Th. 3.4.1, if $f$ has no special 2-section then there is another genus 1 fibration $g : X \to \mathbb{P}^1$. 
Lemma 3.13 If \(X\) is exceptional of type \((2, 4, 4)\) or \((2, 3, 6)\) then the unique elliptic fibration \(f : X \to \mathbb{P}^1\) is not special.

PROOF: Suppose that \(C\) is a special 2-section of \(f\). Then \(C.A > 0\), so that, since \(C\) is not contained in \(A\), \(C.A = 1\). Moreover, \(C\) meets \(A\) in a point on a component \(D\) of multiplicity 1 in the Kodaira–Néron half-fibre \(F\) supported on \(A\). So \(D\) is at the end of one of the long arms of the diagram. However, \(A.D = 0\) in each case, and we have a contradiction to Lemma 3.2.

Theorem 3.14 If \(X\) is exceptional then it is of type \((3, 3, 3)\), \((2, 4, 5)\) or \((2, 3, 7)\).

PROOF: It remains to exclude the types \((2, 4, 4)\) and \((2, 3, 6)\).

If \(X\) is of type \((2, 4, 4)\) there is a unique elliptic fibration \(f : X \to \mathbb{P}^1\) and a quasi-elliptic 1 fibration \(g : X \to \mathbb{P}^1\). The curve \(R_g\) has multiplicity 1 in \(A\) and \(A - R_g\) is \(g\)-vertical, so there is a fibre \(X_t\) of \(g\) that contains a cycle \(D\) of type either \(A_7\) or \(E_7\) and \(A = A' + R_g\) where \(A'\) is supported on \(D\).

(1) Suppose \(D_{red}\) is of type \(A_7\). Then \(A' = (1, 2, 2, 3, 2, 1, 1)\) and there is a component \(C\) of \(X_t\) that meets \(A'\). Necessarily \(C\) meets \(A'\) in an end component \(E\), so \(A.C \geq 1\) while \(A.E = 0\), contradiction.

(2) Suppose \(D_{red}\) is of type \(E_7\). Then \(A' = (3, 2, 2, 1, 2, 2, 1)\) and we again get a contradiction by considering a component of \(X_t\) that meets \(A'\).

If \(X\) is of type \((2, 3, 6)\) then \(A = A' + R_g\) and \(A'\) has no component of multiplicity 1, while there must be a \((-2)\)-curve \(C\) in the fibre containing \(A'_{red}\) that meets \(A'\), contradiction.

Theorem 3.15 Suppose that \(X\) is exceptional of type \((p, q, r)\) and that its conductrix is \(A\).

(1) If \((p, q, r) = (3, 3, 3)\) then \(A\) is supported on a half-fibre, of type \(\tilde{E}_6\), of the unique elliptic fibration on \(X\).

(2) If \((p, q, r) = (2, 3, 7)\) then \(X\) has a unique genus 1 fibration \(g : X \to \mathbb{P}^1\), \(g\) is quasi-elliptic and \(A - R_g\) is supported on a half-fibre of \(g\) of type \(\tilde{E}_8\).

(3) If \((p, q, r) = (2, 4, 5)\) then \(X\) has no elliptic fibration and there is a unique quasi-elliptic fibration \(g : X \to \mathbb{P}^1\) such that \(A - R_g\) is supported on the whole of a fibre. This fibre is a half-fibre and is of type \(\tilde{E}_7\).

PROOF: This follows at once from the description of \(A\) and the facts that \(\text{mult}_A(R_g) = 1\) and \(A - R_g\) is \(g\)-vertical.

Proposition 3.16 If \(X\) has a quasi-elliptic fibration \(g : X \to \mathbb{P}^1\) with a fibre \(X_s\) (simple or double) of type \(\tilde{E}_{7,8}\) then \(X\) is exceptional. If the fibre is simple then \(X\) is of type \((3, 3, 3)\) or \((2, 4, 5)\) accordingly.

PROOF: \(R_g\) has multiplicity 1 in \(A\) and \(R_g \cdot X_s = 1\) or 2. Moreover, \((R_g \cap X_s)_{red}\) is a single point, so \(R_g\) meets \(X_s\) in a curve of multiplicity 1 or 2 in \(X_s\).
Assume that $X$ is not exceptional, so that $\text{Supp} A$ is negative definite. Then there are components $C_1, C_2$ of $X_s$ that do not lie in $A$ while $C_2$ meets $A$ and $C_1.C_2 = 1$. Then $A.C_2 \geq 1$ and $A.C_1 \geq 0$, which contradicts Lemma 3.2. So $X$ is exceptional.

If $X_s$ is simple then $R_g$ meets $X_s$ transversely in a component $C$ of multiplicity 2 in $X_s$ and the result follows from inspection of the possibilities provided by Proposition 3.11 and Theorem 3.14.

\section{From configuration to conductrix}

In this section we describe exceptional Enriques surfaces $X$ in terms of the configurations of ($-2$)-curves on them.

\textbf{Lemma 4.1} If $f : X \to \mathbb{P}^1$ is an elliptic fibration and $f^{-1}(s)$ is a fibre that contains exactly one irreducible component $E$ of the conductrix $A$, then $f^{-1}(s)$ is of type $\tilde{D}_4$, $E$ is the branch vertex of the configuration and $\text{mult}_A(E) = 1$.

\textsc{Proof:} This is a consequence of Lemma 3.2. \hfill \Box

\textbf{Theorem 4.2} An Enriques surface $X$ is exceptional if and only if it has a special genus 1 fibration $g : X \to \mathbb{P}^1$ with a double fibre of type $\tilde{E}_{6,7,8}$. The type of $X$ is $(3,3,3), (2,4,5)$ or $(2,3,7)$, accordingly.

\textsc{Proof:} Assume that $X$ is exceptional. We consider the various types separately.

$T_{3,3,3}$: there is an elliptic fibration $f : X \to \mathbb{P}^1$ with a double fibre $X_{f,s}$ of type $\tilde{E}_6$ and $A = (2; 1, 1; 1, 1; 1, 1)$. If $f$ is special we are done, and if not then there is, by Th. 3.4.1 of [CD89], another genus 1 fibration $g : X \to \mathbb{P}^1$. This is quasi-elliptic. Write $A = A' + R_g$, so that $A'$ is $g$-vertical. So there is a singular fibre $g^{-1}(t) = d_t X_{g,t}$ such that $\text{Supp} A' \subset \text{Supp} X_{g,t}$.

If $R_g$ is not an end curve of $A$ then $A'$ is disconnected; say $A' = A'_1 \cup A'_2$, where $A'_2$ is an end curve of $A$. The fibre of $g$ that contains $A'_2$ is of type $\tilde{D}_4$, by Lemma 4.1, so has 5 components. The fibre containing $A'_1$ has at least 6 components. But $\rho(X) = 10$, contradiction.

So $R_g$ is an end curve of $A$, so that $A'$ is of type $E_6$ and $X_{g,t} = \tilde{E}_{r \geq 6}$. If $r = 6$ let $E$ denote the curve such that $X_{g,t} = A' \cup E$, set-theoretically, and $C$ the curve in $A'$ that meets $E$. Then $A.E \geq 1$ and $A.C = 1$, contradiction.

So $r = 7, 8$ and $R_g$ is special.

$T_{2,4,5}$ and $T_{2,3,7}$: the result is clear in these cases.

Conversely, suppose that there is a special genus 1 fibration $g : X \to \mathbb{P}^1$ with a special 2-section $D$ and a double fibre $g^{-1}(s) = 2X_s$ of type $\tilde{E}_r$. Suppose that $\text{dim}(A \cap X_s) \leq 0$. Then $c_1(\mathcal{F}).E_i \geq 0$ for each component $E_i$ of $X_s$, so that $c_1(\mathcal{F}).E_i = 0$ for all $E_i$. Then $E_i$ is generically tangent to $\mathcal{F}$. Let $E_1$ be the curve corresponding to the branch vertex of $\tilde{E}_r$, meeting $E_2, E_3, E_4$. Then $E_1 \cap E_j$
is a singular point of $\mathcal{F}$ for each $j = 2, 3, 4$, but $c_1(\mathcal{F}).E_1 = 0$, contradiction. So $A' := A \cap X_s$ is a non-zero divisor.

(1) $r = 6$. Then $X_s + D$ is a $T_{3,3,4}$ configuration. If $g$ is elliptic then from Lemmas 3.7 to 3.9 it follows that $A = (2; 1, 1; 1, 1, 1, 1)$, and $X$ is exceptional. If $g$ is quasi–elliptic then $R_g \leq A$ and $\Gamma = T_{3,3,4}$ which we know to be impossible.

(2) $r = 7$. If $g$ is elliptic then $\text{mult}_A(D) = 0$ and Lemmas 3.7 to 3.9 give a contradiction. If $g$ is quasi–elliptic then $\text{mult}_A(D) = 1$ from Lemmas 3.7 to 3.9 it follows that $A = (3; 2, 2, 1, 1; 2, 2, 1, 1)$ and $X$ is exceptional.

(3) $r = 8$. As when $r = 7$ we see that $g$ is quasi–elliptic,

$$A = (5; 4, 4, 3, 3, 2, 1; 3, 2; 2)$$

and $X$ is exceptional. \hfill \Box

**Corollary 4.3** An Enriques surface $X$ is exceptional if and only if it contains a configuration of $(-2)$-curves of type $T_{p,q,r}$ where $(p, q, r) = (3, 3, 4), (2, 4, 5)$ or $(2, 3, 7)$. The type of the surface is the type of the configuration except that a configuration $T_{3,3,4}$ gives a surface of type $T_{3,3,3}$.

**Proof:** If $X$ is exceptional then examination of the genus 1 fibration provided by Theorem 4.2 gives the result. Conversely, a $T_{3,3,4}$ configuration yields a special genus 1 fibration with a double fibre of type $\tilde{E}_6$, while $T_{2,4,5}$, resp., $T_{2,3,7}$, gives a special genus 1 fibration with a double fibre of type $\tilde{E}_7$, resp., $\tilde{E}_8$. \hfill \Box

**Theorem 4.4** Suppose that $X$ is an exceptional Enriques surface of type $T$ and conductrix $A$.

1. If $T = (2, 3, 7)$ then the only $(-2)$-curves on $X$ are the ones in $A$.
2. If $T = (2, 4, 5)$ then there are exactly two $(-2)$-curves that are not in $A$. They form a fibre of type $Ta$ in the natural quasi–elliptic fibration $g : X \rightarrow \mathbb{P}^1$ given by Theorem 3.15. If this fibre has multiplicity $d$ then $X$ possesses just $3 - d$ further genus 1 fibrations. Each is quasi–elliptic and has a simple fibre of type $\tilde{E}_8$.

**Proof:** (1) $E$ must meet $A$ since $\rho(X) = 10$. If $E$ is not in $A$ then $A.E = 1$, so $E$ meets $A$ in its unique component $R$ of multiplicity 1. Since $R = R_f$ where $f : X \rightarrow \mathbb{P}^1$ is the unique genus 1 fibration on $X$ and $E$ is disjoint from $A - R$, $E$ is $f$-vertical. But this contradicts $\rho(X) = 10$.

(2) There is a further reducible fibre $f^{-1}(s) = dX_s$ of $f$; it has two components $E_1, E_2$ and $d = 1$ or 2. Since $X_s$ meets $R$ it meets $A$ and since $\rho(X) = 10$ it must be of type $Ta$. If $d = 2$ then $R$ meets just one component and if $d = 1$
then $R$ meets both components in their point of intersection. Examination of the diagram defined by $A \cup (E_1, E_2)$ concludes the proof.

Surfaces of type $(3,3,3)$ require more work and are dealt with in the next section.

5 (3, 3, 3) exceptional surfaces

Suppose that $X$ is an exceptional Enriques surface of type $(3,3,3)$. We know that $X$ has a unique elliptic fibration $f : X \to \mathbb{P}^1$, that $f$ has a double fibre of type $\tilde{E}_6$ that supports the conductrix $A$ and that $A = (2; 1, 1; 1, 1)$. 

Lemma 5.1 There exists at least one quasi--elliptic fibration on $X$.

PROOF: If $f$ were the only one genus 1 fibration on $X$ then, by Th. 3.4.1 of [CD89], it would have a fibre of type $\tilde{E}_8$. But $\rho(X) = 10$ so this is impossible.

Lemma 5.2 (1) For any quasi--elliptic fibration $g : X \to \mathbb{P}^1$ the curve $R_g$ is an end curve of $A$ and the fibre $X_s$ of $g$ that contains $A - R_g$ is a simple fibre of type $\tilde{E}_7$.

PROOF: (1) See the proof of Theorem 4.2 for the fact that $R_g$ is an end curve of $A$.

Say $g^{-1}(s) = d_sX_s$. Since $X_s$ contains the $E_6$ configuration $A - R_g$ it is of type $\tilde{E}_r$. Lemma 3.2 excludes the cases $r = 6, 8$ and shows that $R_g$ meets the short arm of a $\tilde{E}_7$ fibre transversely. So $d_s = 1$.

(2) There is a genus one fibration $g : X \to \mathbb{P}^1$ such that $e\Psi$ is a fibre and $e = 1$ or 2. $T_0$ is not $h$-vertical, since $\rho(X) = 10$, so that $g \neq f$ and therefore $g$ is quasi--elliptic. $R_g$ is an end curve of $A$ and $A - R_g$ is $g$-vertical; then $A - R_g \subset \Psi$ since again $\rho(X) = 10$ and (2) is proved.

Write $\text{Supp} A = T_0$.

Lemma 5.3 $T_0$ is the unique $T_{3,3,3}$-configuration on $X$.

PROOF: Suppose $S$ is another such configuration. Then there is a quasi--elliptic fibration $g : X \to \mathbb{P}^1$ with a fibre supported on $S$. The curve $R_g$ is an end component of $A$ and $A - R_g$ is $g$-vertical. Since $S$ and $A - R_g$ are connected and $\rho(X) = 10$, it follows that $T_0 - R_g \subset S$. Then there is an end curve $C$ of $S$ that is not contained in $A$ and which meets $A$ in a curve $D$ of multiplicity at least 2. Therefore $C.A \geq 2$, which is impossible.

Let $S$ denote the set of $T_{4,4,4}$ configurations on $X$.

Lemma 5.4 $T_0$ extends to some $S \in S$ and every element of $S$ contains $T_0$.

PROOF: Take the $\tilde{E}_7$ configuration that is the fibre provided by the fibre $X_s$ of Lemma 5.2. There is another reducible fibre of $g$, since $\rho(X) = 10$; it is of type
Ta. A suitable component of this fibre extends $X_s + R$ to a $T_{4,4,4}$ configuration. The rest follows from Lemma 5.3.

Let $\rho : X \to Y_1 = X/F$ be the quotient, so that $Y_1 = \tilde{Z}^{(1)}$. We know that $Y_1$ is a rational surface and that $\text{Sing} Y_1 = 4 \times A_1$ or $1 \times D_4$.

**Lemma 5.5** If $C$ is a curve in $X$ such that $F|_C$ maps isomorphically to either $T_C$ or $N_{C/X}$ then $Y_1$ is smooth along $D = \rho(C)$. If $F|_C = T_C$ then $\rho_* C = 2D$, $\rho^* D = C$ and $D^2 = C^2/2$. If $F|_C = N_{C/X}$ then $\rho_* C = D$, $\rho^* D = 2C$ and $D^2 = 2C^2$.

**PROOF:** This is standard.

**Proposition 5.6** (1) $T_0$ maps to a normally crossing configuration $U_1$ of $\mathbb{P}^1$’s on $Y_1$, disjoint from $\text{Sing} Y_1$ and described by $U_1 = (-4; -1, -4; -1, -4; -1, -4)$.

(2) There is a birational contraction $\pi : Y_1 \to Y$ of the central $D_4$ configuration in $U_1$ to a smooth point. The image of $U_1$ is a Kodaira–Néron cycle $U_0 = C_1 + C_2 + C_3$ of type $Tr$. There is a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y_1 \\
\downarrow{h} & & \downarrow{h_1} \\
\mathbb{P}^1 & & Y \\
\end{array}
\]

(3) $h : Y \to \mathbb{P}^1$ is a relatively minimal elliptic fibration and $U_0 \in | - K_Y |$.

(4) The quasi–elliptic fibrations on $X$ correspond to the rulings on $Y$.

(5) Give a ruling $q : Y \to \mathbb{P}^1$, two of the curves $C_i$ are $q$-vertical and the third is a purely inseparable $2$-section of $q$.

(6) Every $S \in \mathcal{S}$ maps to a configuration $H = U_0 + \sum D_j$ of $\mathbb{P}^1$’s on $Y - \text{Sing}(Y)$ such that

$C_i D_j = -D_i, D_j = \delta_{ij}$.

(7) Write $\text{Pic} Y = L$. Then $L$ is of signature $(1, 5)$ and $L^1 / L \cong (\mathbb{Z}/2)^2$.

(8) $\text{MW}(X)$ is the Mordell–Weil rank of $h : Y \to \mathbb{P}^1$.

**PROOF:** Where this does not follow from what we already know it is easy.

A type $H$ configuration of curves on $Y$ is a configuration $U_0 + \sum D_j$ of irreducible curves as in 5.6 6. A type $H$ configuration of classes on $Y$ is the same thing, except that each $D_j$ is only required to be a class in $\text{Pic}(Y)$. So $\mathcal{S}$ is identified with the set of type $H$ configurations of curves on $Y$.

**Lemma 5.7** Suppose that $E$ is a section of $h : Y \to \mathbb{P}^1$. If $E^2 < 0$ then $Y$ is smooth along $E$ and $E^2 = -1$. Conversely, if $Y$ is smooth along $E$ then $E^2 = -1$.

**PROOF:** $E$ is disjoint from the exceptional locus of $Y_1 \to Y$, so that we can write $E = \rho(C)$ for some curve $C$ on $X$ with $C^2 < 0$. Therefore $C$ is a $(-2)$-curve. Since $C$ meets $A$, $F|_C$ maps isomorphically to $T_C$, and we can apply Lemma 5.5. The converse is well known.
Lemma 5.8  Suppose that $V$ is an RDP surface, that $q : V \to \mathbb{P}^1$ is an elliptic fibration with fibre $\Phi$ and that $D$ is a Cartier divisor class on $V$ such that $D.\Phi = 1$ and that $D^2 = -\chi(\mathcal{O}_V)$. Assume also that $D$ and $K_V$ are nef relative to $q$.

Then $D$ is the class of a section of $q$ that is disjoint from $\text{Sing} V$.

**Proof:** Since $D$ is Cartier, $D.\Phi = 1$ and $D$ is $q$-nef it follows that there is a birational contraction $\tau : V \to V'$ that contracts exactly the $q$-vertical curves $\psi$ with $D.\psi = 0$, $V'$ also has RDPs and $D = \tau^*D'$ for some Cartier divisor class $D'$ on $V'$. Then $q$ factors as $q = q' \circ \tau$ with $q' : V' \to \mathbb{P}^1$ and all fibres of $q' : V' \to \mathbb{P}^1$ are reduced and irreducible. Let $\Phi'$ be a fibre of $q'$. Then there is a divisor class $a$ on $C$ such that $D' + q^*a$ is effective; if $a$ is taken to have minimal degree then $D' + q^*a \sim D_1$ where $D_1$ is a section. Then

$$-\chi(\mathcal{O}_{V'}) = D_1^2 = (D')^2 + 2 \deg a = -\chi(\mathcal{O}_V) + 2 \deg a,$$

so that $a = 0$ and $D'$ is the class of a section. So $D \sim \tau^*D'_1$ and is therefore the class of a section. Since this section is a Cartier divisor on $V$ and is smooth it is disjoint from $\text{Sing} V$.

A $(-2)$-curve $E$ on $X$ is **extraneous** if it is disjoint from $T_0$. Equivalently, $E$ is extraneous if it is an irreducible component of a reducible fibre of $f$ besides $T_0$. $E$ is **horizontal** if it is not $f$-vertical, or, equivalently, if it is not in $T_0$ and is not extraneous. A curve on $Y$ is extraneous if it is the image of an extraneous curve on $X$. A fibre of $f$ or $h$ is extraneous if it consists of extraneous curves. Extraneous curves (or fibres) exist if and only if $\text{MW}(X) \leq 1$.

**Proposition 5.9** Assume that $\text{MW}(X) = 2$, that $D \in L$, $D^2 = -1$, $D.C_i \geq 0$ for all $i$ and $D.U_0 = 1$. Then $D$ is the class of a section of $h : Y \to \mathbb{P}^1$.

**Proof:** $h$ has no extraneous fibres and the result follows from Lemma 5.8.

Put $M = \sum \mathbb{Z}.C_i \subset L$ and $\Delta = \{ \gamma \in O(L) | (\gamma(C_i) = C_i \forall i) \}$. Fix an $H$-configuration $U_0 \cup \{ D_j \}$ on $Y$ and put $\phi_i = D_j + C_j + C_k + D_k$ when $\{i,j,k\} = \{1,2,3\}$. Put $\alpha_i = D_i - \phi_i/2 \in M^\perp \subset L^\perp$, so that $\alpha_i.\alpha_j = (1 - 3\delta_{ij})/2$. Let $s_i$ be the reflexion in $\alpha_i$; then $s_i \in \Delta$. Define $W = \langle s_1, s_2, s_3 \rangle \subset \Delta$.

**Lemma 5.10**

- (1) $W$ acts on $M^\perp$ as the Weyl group $W(\tilde{A}_2)$.
- (2) $\Delta = W$.

**Proof:** (1) $s_i s_j$ has order $3 - \delta_{ij}$. So there is a surjection $\pi : W(\tilde{A}_2) \to W$. But the reflexion group action of $W(\tilde{A}_2)$ visibly factors through $\pi$.

(2) This follows from the facts that $W(\tilde{A}_2) \times (\pm 1)$ is the full group $O(M^\perp)$ and that $\Delta$ acts effectively on $M^\perp$.

**Theorem 5.11**  Suppose that $\text{MW}(X) = 2$.

- (1) $S$ is a torsor under $W(\tilde{A}_2)$.
- (2) Each horizontal $(-2)$-curve on $X$ lies in exactly six elements of $S$. 


(3) Each \(S \in \mathcal{S}\) defines three quasi–elliptic fibrations on \(X\) and each quasi–elliptic fibration on \(X\) arises from two such diagrams.

PROOF: (1) There are no extraneous curves and therefore any \((-2)\)-curve on \(X\) corresponds to a \((-1)\)-curve on \(Y\) that meets \(U_0\). We know that every type \(H\) configuration of classes on \(Y\) is given by a unique type \(H\) configuration of curves on \(Y\), so that \(\mathcal{S}\) is identified with the set of type \(H\) configurations of classes on \(Y\). Therefore \(\mathcal{S}\) is a torsor under \(\Delta\). (1) now follows from this and Lemma 5.10.

For (2) and (3) regard \(W\) as a symmetry group of a tessellation of the Euclidean plane \(\Pi = (M^1/M \cap M^1) \otimes \mathbb{R}\) by equilateral triangles each of which is a fundamental domain for the action of \(W\) on \(\Pi\). (2) follows from noting that each vertex of the tessellation lies in six triangles while (3) follows from the facts that each triangle has three edges and each edge lies in two triangles. \(\square\)

Lemma 5.12 Suppose that \(\text{MW}(X) = 1\).

(1) \(h\) has one extraneous fibre \(\Phi_0\), say, with components \(\psi_1, \psi_2\).

(2) Suppose that \(D \in L\) with \(D^2 = -1\), \(D.U_0 = 1\) and that \(D.\psi_i \geq 0\). Then \(D\) is the class of a section of \(h: Y \to \mathbb{P}^1\).

PROOF: (1) is clear. For (2), apply Lemma 5.8. \(\square\)

Theorem 5.13 Suppose that \(\text{MW}(X) = 1\).

(1) \(\mathcal{S}\) is a torsor under the infinite cyclic subgroup of \(W\) generated by some glide-reflexion.

(2) Each horizontal \((-2)\)-curve on \(X\) lies in exactly three elements of \(\mathcal{S}\).

(3) Each \(S \in \mathcal{S}\) defines three quasi–elliptic fibrations on \(X\) and each quasi–elliptic fibration on \(X\) arises from two such diagrams.

PROOF: We have to classify type \(H\) diagrams of classes on \(Y\) such that each \(D_i\) is \(h\)-nef. That is, \(D_i.\psi_j \geq 0\) for all \(i, j\). Since each type \(H\) diagram spans \(L\), we require, after renumbering if necessary, that \(D_1.\psi_1 = D_2.\psi_2 = D_3.\psi_2 = 1\) and \(D_i.\psi_j = 0\) otherwise. Again, regard \(W\) as a group of symmetries of \(\Pi\) that preserves a tessellation into equilateral triangles; one sees that the theorem follows and that \(\gamma\) takes one triangle to another with a common edge. So \(\langle \gamma \rangle\) acts on a strip \(\Sigma\) whose width is one triangle. A single triangle forms a fundamental domain for the action of \(\langle \gamma \rangle\) on \(\Sigma\). Since each vertex of the tessellation of \(\Sigma\) lies in three triangles in \(\Sigma\) (1) is proved.

(2) and (3) are proved as in Theorem 5.11. \(\square\)

Lemma 5.14 If \(\text{MW}(X) = 0\) then \(h: Y \to \mathbb{P}^1\) has just one extraneous fibre \(\Phi_0\).

It is simple and has three components.

PROOF: It is enough to prove the analogous result for \(f: X \to \mathbb{P}^1\). Recall that \(T_0\) is a double fibre of \(f\). Suppose that \(v_1, v_2, v_3\) are the end curves in a \(T_{4,4,4}\) diagram. Then \(v_i\) is a special bisection of \(f\). Suppose that \(\Phi_0, \Phi_1\) are reducible fibres each of which has just two components, say \(\psi_1, \psi_2\) and \(\psi_3, \psi_4\), respectively.
Then $A.\psi_j = 0$ and $A.v_i = 1$, so that $v_i$ cannot meet $\psi_j$ transversely. So $\Phi_k$ is a simple fibre, $v_i.\psi_j = 0$ or 2.

After renumbering if necessary, $\psi_1.v_i = 2\delta_{1i}$ and $\psi_3.v_i = 2\delta_{3i}$. Then 

$$(v_1 + \psi_1)^2 = 0 = (v_3 + \psi_3)^2 = (v_1 + \psi_1).(v_3 + \psi_3),$$

so that $v_1 + \psi_1$ and $v_3 + \psi_3$ are proportional. However, if $u_3$ is the end curve in $T_0$ that meets $v_3$ then $u_3.(v_3 + \psi_3) = 1$ while $u_3.(v_1 + \psi_1) = 0$. This contradiction proves the lemma.

**Theorem 5.15** If $MW(X) = 0$ then $S$ has one element and there are only three horizontal $(-2)$-curves on $X$.

**Proof:** As before, we must classify type $H$ diagrams of classes on $Y$ where each $D_i$ is $h$-nef. By Lemma 5.14 $h$ has just one reducible fibre besides $U_0$, with three components $\psi_i$. We can take $D_i.\psi_j = \delta_{ij}$ and now the result is immediate.

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