Min-Max Q-Learning for Multi-Player Pursuit-Evasion Games

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Abstract—In this paper, we address a pursuit-evasion game involving multiple players by utilizing tools and techniques from reinforcement learning and matrix game theory. In particular, we consider the problem of steering an evader to a goal destination while avoiding capture by multiple pursuers, which is a high-dimensional and computationally intractable problem in general. In our proposed approach, we first formulate the multi-agent pursuit-evasion game as a sequence of discrete matrix games. Next, in order to simplify the solution process, we transform the high-dimensional state space into a low-dimensional manifold and the continuous action space into a feature-based space, which is a discrete abstraction of the original space. Based on these transformed state and action spaces, we subsequently employ min-max Q-learning, to generate the entries of the payoff matrix of the game, and subsequently obtain the optimal action for the evader at each stage. Finally, we present extensive numerical simulations to evaluate the performance of the proposed learning-based evading strategy in terms of the evader’s ability to reach the desired target location without being captured, as well as computational efficiency.

Index Terms—Min-max Q-learning, pursuit-evasion games, matrix games.

I. INTRODUCTION

Interaction between multiple agents is present in nature (e.g. predation) as well as human-built constructs (such as markets, traffic situations, and defense). In order to better understand and predict the potential outcomes of such interactions among multiple agents, each with their own aims, preferences and resources, we have to understand their underlying decision-making mechanisms. Uncertain interactions among agents can be studied within the framework of dynamic non-zero-sum multi-player games. A special class of such problems are pursuit-evasion games (PEGs) with multiple players which seek to capture or evade each other. In this paper, we consider the problem of steering an evader to a target location while avoiding capture by multiple pursuers. Finding the exact solution to such a pursuit-evasion game can be a complex task due to the high dimensionality of the problem. In particular, the state space of the PEG consists of high-dimensional vectors that correspond to the concatenation of the position and velocity vectors of all the players involved. Similarly, the combined action space of all the players can be also high-dimensional.

In this paper, the state space and the action space of the PEG are both continuous. Obtaining exact solutions to this class of problems can be a very complex task and thus, the characterization of approximate solutions is preferred in practice. We use min-max Q-learning to obtain an approximation of a Q-function that can characterize the evader's payoff (reward) for actions taken by the different players from any state. It is desirable to have the results of the learning process be independent of specific parameters of the training instances, particularly in terms of the number of players and their speeds. To this aim, the learning is performed in a low-dimensional manifold, which is obtained by applying a nonlinear transformation to the continuous state space, and a discrete abstraction of the continuous action space (feature-based action space). In the learning space, the interaction of the players in the pursuit-evasion game is described purely in terms of times to capture under different conditions, and the discrete actions are designed to correspond to the intent of the different players. The approximate Q-function is used to construct a matrix that describes a two-player game per stage. The solution to the matrix game yields the evader’s action strategy at the current stage. The pursuers may either employ the strategy induced by the latter game or a predetermined policy such as relay pursuit (i.e., only the nearest pursuer to the evader tries to capture the latter).

Literature survey: Multi-player pursuit evasion games are extensively used to model interactions in economics, biology and defense, to name a few [1]–[3]. Multi-player games are typically non-zero sum, and may be played in continuous or discrete time. Non-cooperative multi-player games, including matrix games, are discussed in detail by Basar and Olsder [4]. Matrix games, also discussed by Zaccour et. al in [5], could be single act (static games) or multi-act (repeated or dynamic games) [6]. Well-known results pertaining to existence of Nash equilibria for single-stage matrix games can be found in [4]. The question of existence of Nash equilibria in repeated games is addressed by the so-called folk theorem [7]. Multi-player pursuit-evasion games with the emphasis placed either on capture or evasion have been addressed in [8]–[10]. The latter references use approaches which are based on geometric arguments, such as dynamic Voronoi partitions, switching strategies, Voronoi-based roadmaps as well as greedy pursuit policies.

A Markov game-formulation of the discrete multi-agent PEG in an uncertain environment is treated using a matrix game approach in [17]. In our previous work [18], the multi-agent dynamic non-zero-sum game was re-formulated as a sequence of discrete zero-sum matrix games. The evader’s myopic strategy corresponded to the solution to a linear program that was solved at each stage. In this paper, we retain this basic solution structure, however, we improve our previous formulation of the matrix game using Q-learning to learn the elements of the payoff/cost matrix. Q-learning techniques are popular tools for the characterization of optimal policies in decision making problems in uncertain environments involving...
either a single player or multiple players [19]–[24]. For two-player zero-sum games that are cast as Markov games, min-max Q-learning is applied where the definition of the Q-function is modified to suit the presence of a second, independent decision-maker. The use of min-max Q-learning has been demonstrated for simple problems with discrete state and action spaces in [20] and an integral Q-learning algorithm for continuous differential games is presented in [23]. An exposition of different Q-learning algorithms for Markov games is found in [22], where the authors also present an improved learning algorithm to compute mixed policies. Multi-agent reinforcement learning, using different value functions for the agents, is presented in [25]. In [24], concurrent Q-learning is used by both pursuing and evading agents to learn their optimal policies. A critical overview of reinforcement learning for multi-agent problems can be found in [26], [27]. Finally, examples of the use of deep learning to solve multi-agent problems in dynamic environments are presented in [28], [29].

Contributions: In this paper, we first propose a systematic way to reduce a multi-agent dynamic non-zero-sum game to a sequence of two-player static zero-sum games each of which can be formulated as a linear program. In this set-up, the payoffs of each static game will determine the action of the evader at that stage, and this, in turn, will determine the performance of the evader’s strategy. We construct the payoffs taking into account (i) a risk metric for the evader with regard to its capture, and (ii) the time required by the evader to reach the target location. In order to refine the performance of the evader’s strategy, we employ a min-max Q-learning algorithm to determine the entries of the payoff matrix at each stage of the game.

In our approach, learning takes place in a low-dimensional nonlinear manifold (learning space) embedded in the original high-dimensional state space. The states in this reduced space capture the salient features of the pursuit-evasion game using time-of-capture parameters rather than position and velocity. Consequently, the new state space is invariant with respect to the number of players or the dynamic model of the players which is a key property of our proposed approach. Furthermore, we associate the continuous action space of the players to a feature-based action space which is a discrete space comprised of a small number of actions. The set of discrete actions is informed by the members of the continuous action space. These actions affect the time-to-capture, and/or the time-to-target, of one or more players. Our choice of discrete actions does not restrict the movement of the evader to pre-specified spatial directions, but rather, allows the evader to move in different spatial directions as required by the configuration of each game.

Structure of the paper: In Section II, we define the target-seeking evasion problem, and in Section III, we formulate the corresponding multi-act two-person zero-sum game. The setup for min-max Q-learning in the context of our problem, and the learning process are described in Section IV and Section V respectively. Numerical simulations and observations are given in Section VI. In Section VII, we present concluding remarks.

II. FORMULATION OF TARGET-SEEKING EVASION PROBLEM

Consider a pursuit-evasion game with N pursuers and one evader that takes place in a domain $D \subseteq \mathbb{R}^2$. We will assume that there is an upper bound on the duration of the game which is known a priori and is denoted by $T_f > 0$. For the sake of simplicity, we demonstrate the proposed solution technique in a reach-avoid game where all the players have single integrator dynamics. At any given time $t \in [0, T_f]$, the state (position) of the $i^{th}$ pursuer $p_i$, where $i \in I := \{1, \ldots, N\}$, is denoted by $x_{ic}(t) \in \mathbb{R}^2$, whereas the state (position) of the single evader $E$ at time $t$ is denoted by $x_{ec}(t) \in \mathbb{R}^2$. The dynamic equations are given by:

$$\dot{x}_{ic}(t) = v_p u_{ic}(t), \quad x_{ic}(0) = \bar{x}_i,$$

$$\dot{x}_{ec}(t) = v_e u_{ec}(t), \quad x_{ec}(0) = \bar{x}_e,$$

where $v_p > 0$ and $v_e > 0$ denote respectively the maximum speeds of the $i^{th}$ pursuer and the evader. Further, $u_{ic}(t) \in \mathbb{R}^2$ and $u_{ec}(t) \in \mathbb{R}^2$ denote the inputs of the $i^{th}$ pursuer and the evader at time $t$, respectively, and are assumed to take values in the set $U := \{u \in \mathbb{R}^2 : ||u|| = 1 \text{ or } u = 0\}$. The zero control input will be used by the players only when the game terminates (either by capture of the evader by at least one pursuer, or by the evader reaching its target location).

The framework of matrix games applied to a continuous-time dynamic game requires the choice of actions, repeatedly, at each instant of time. While the state space of the game and the action space are both continuous, we will consider a discrete-time state-space model for the players in Eq. (1), wherein the control input is piece-wise constant. That is, the control input is a constant vector in each time interval $[k\Delta t, (k+1)\Delta t)$, where $k$ is a positive integer and $\Delta t \geq 0$ is the sampling period. The game consists of $K + 1$ finite stages at most, with a constant time step $\Delta t > 0$, such that $T_f := K\Delta t$. Note that at time $t = k\Delta t$, the discrete-time state of the $i^{th}$ pursuer, is denoted by $x_{id}(k)$, and is defined as $x_{id}(k) := x_{ic}(k\Delta t)$. Similarly, the discrete-time states are defined for the other players. If $k \in \{0, ..., K\}$ denotes the current stage of the game, and $x_{id}(k)$ is the state vector of the $i^{th}$ pursuer at stage $k$ and $x_{id}(k)$ is the state vector of the evader, the discrete-time state-space model for the players is described by the following equations:

$$x_{id}(k+1) = x_{id}(k) + (v_p \Delta t) u_{id}(k), \quad x_{id}(0) = \bar{x}_i,$$

$$x_{ed}(k+1) = x_{ed}(k) + (v_e \Delta t) u_{ed}(k), \quad x_{ed}(0) = \bar{x}_e,$$

where $u_{id}(k), u_{ed}(k) \in U$ denote the inputs of the $i^{th}$ pursuer and the evader at stage $k$, respectively. The discrete-time control input corresponds to a piece-wise constant continuous-time signal, that is, $u_{ic}(\tau) := u_{id}(k)$ for all $\tau \in [k\Delta t, (k+1)\Delta t)$ for all $k \in \{0, ..., K-1\}$.

In this paper, we assume that the pursuers have a preferred strategy. In particular, they engage in what is known as “relay pursuit” with corresponding relay metric the minimum time-to-capture [11]. At each instant of time, the active pursuer is the one corresponding to the smallest time-to-capture (among the pursuers). In relay pursuit, only one pursuer is active, that is, engages in pursuit, whereas the other pursuers are stationary. The assignment of the active pursuer can change with time based on which pursuer has the least time-to-capture.
If the active pursuer is designated by the index $i^*$, then, for all $t \in [0, t_f]$, the feedback strategy of the $i^{th}$ pursuer is

$$u_{id}(\bar{x}_{ed}, \bar{x}_{id}) = \begin{cases} r_{id}/\|r_{id}\|, & \text{if } i = i^*, \\ 0, & \text{otherwise}, \end{cases}$$

where $r_{id} := \bar{x}_{ed} - \bar{x}_{id}$ is the relative position vector of the evader with respect to the $i^{th}$ pursuer. Relay pursuit is a suitable choice of strategy for a group of pursuers in which each agent wishes to remain spatially localized, or to conserve resources.

Capture occurs when the evader’s distance from at least one pursuer is less than the radius of capture $\ell > 0$. That is, the game will terminate in capture at stage $K \in \{0, \ldots, \bar{K}\}$ (for a given positive integer $\bar{K}$), if there exists $i \in I : \|\bar{x}_{id}(K) - \bar{x}_{ed}(K)\| \leq \ell$. The target is denoted by $\bar{x}_T \in \mathbb{R}^2$. The evader is considered to be successful in reaching the target if $\|\bar{x}_{ed}(K) - \bar{x}_T\| \leq \epsilon$, where $\epsilon > 0$. The multi-player reach-avoid problem in discrete-time is stated as follows:

Problem 1: Let $\bar{K}$ be a positive integer and let us assume that the initial positions of the pursuers $\bar{x}_i$, for $i \in I$, and the initial position of the evader, $\bar{x}_e$, be given. Find a sequence of control inputs $(u_{id}(k))_{k=0}^{K-1}$, where $K$ is a positive integer with $K \leq \bar{K}$ (K corresponds to the free terminal stage), which will guide the evader to the target $\bar{x}_T \in \mathbb{D}$ within the desired tolerance, while avoiding capture, that is, $\|\bar{x}_{ed}(K) - \bar{x}_T\| \leq \epsilon$, and $\|\bar{x}_{ed}(k) - \bar{x}_T\| > \epsilon$, $\forall k \in \{0, 1, \ldots, K\}$.

We assume that the players have perfect information about the states of all the players of the game at all times. In addition, the target $\bar{x}_T$ is known only to the evader. Note that the evader has two goals: (a) reaching the target location, and (b) avoiding capture, whereas the group of pursuers has only one: to achieve capture of the evader as soon as possible.

For compactness of notation, subsequently, the subscript $d$ is dropped from the state and control vector notations at the corresponding stage $k$.

III. MATRIX GAME FORMULATION OF THE MULTI-PLAYER PEG

In this section, we re-formulate the discrete-time non-zero-sum game with $N + 1$ players described in Problem 1 as a multi-act two person zero-sum matrix game. The group of pursuers is considered as a single entity, $P$, with the ability to deploy exactly one of the pursuers at a given instant of time. At each stage $k$ of the latter game, we consider a matrix $M_k \in \mathbb{R}^{N \times (N+1)}_{> 0}$, whose entries are the payoffs to $E$ at that stage. Each row of $M_k$ represents a pure strategy played by $P$ and each column, a pure strategy played by $E$. The decision space available to the players (the space of control inputs for $P$ and $E$) is continuous and contains infinite number of actions. We consider a restricted decision space for $P$, including only the actions that appear “integral” to the pursuers’ goal of capturing the evader. In particular, $P$ has exactly $N$ pure strategies, where the $i^{th}$ pure strategy corresponds to the case where only the $i^{th}$ pursuer goes after the evader. Similarly, $E$’s restricted decision space consists of $N + 1$ actions, where the first $N$ correspond to evasion from each pursuer in turn (the $j^{th}$ action is to avoid only the $j^{th}$ pursuer), and the $(N+1)^{th}$ action is the target-seeking behavior, which means that the evader directly heads towards the target.

Let $i$ be the row index of $M_k$ and $j$ be the column index, where $i \in I$ and $j \in J := \{1, \ldots, N + 1\}$. If we consider the first $N$ columns of $M_k$, each entry $M_k(i, j)$ with $i = j$ is the payoff for the two-player zero-sum game between only the $i^{th}$ pursuer and the evader. Every other entry $M_k(i, j)$, with $i \neq j$, represents a case when $E$ tries to evade from the $j^{th}$ pursuer, when actually the $i^{th}$ pursuer is active. This situation can happen because while the $E$ knows the states of all the pursuers, it does not know the action chosen by $P$ at the same stage. Finally, the last column of the matrix $M_k$ represents cases where the evader is directly headed towards the target $x_T$, and only one pursuer is active per row. A schematic construction of the matrix $M_k$ is shown in Table I.

| \(i\) | \(j\) | \(M_k(i, j)\) |
|------|------|----------------|
| 1    | 1    | \(P_1\) pursuer |
| 2    | 2    | \(E\) evading \(P_1\) |
| N    | N    | \(P_N\) pursuer |

TABLE I: Entries in the payoff matrix $M_k$, for the case of $N = 2$.

The payoff matrix of the matrix game has to be updated at every stage of the game because the players move between every pair of successive stages and the entries of the payoff matrix must change to reflect the underlying dynamic evolution of the game. For this reason, our matrix game formulation differs from the class of games known as repeated static games. Constructing and updating the entries of the payoff matrix is central to developing a successful strategy for the evader that can lead it to the target while avoiding capture. The values in the payoff matrix must represent the two-fold goal of the evader.

1) Time metrics of the reach-avoid game: Next, we introduce a number of time-metrics that will be used in the subsequent analysis. All these metrics are associated with the minimum positive solution to the following equation [6]:

\[
(v_i^2 - v_{p_i}^2)\phi^2 + 2(r_i, v_e u_e) - \ell v_{p_i}\phi + \langle r_i, r_i \rangle - \ell^2 = 0, \tag{4}
\]

where $r_i := \bar{x}_e - \bar{x}_i$ and $\bar{x}_e$ and $\bar{x}_i$ correspond to the position of the evader and the $i^{th}$ pursuer at the beginning of the game (at stage $k$, where $0 \leq k < \bar{K}$), for different values of the vector $u_e$. In particular, the min-max time-to-capture of the evader by the $i^{th}$ pursuer, ignoring all other players, is denoted as $\phi_i(x_e, x_i)$ and is defined as the minimum positive real solution to the equation (4) when $u_e := r_i/\|r_i\|$ (the $i^{th}$ pursuer uses the control input $u_i := r_i/\|r_i\|$). Similarly, let $\phi_a(x_e, x_i, u_e)$ denote the minimum time-to-capture of the evader by the $i^{th}$ pursuer when the evader is moving in a randomly chosen fixed direction denoted by $u_e$. Ideally, this particular maneuver does not avoid any pursuer in particular, but may delay capture by exploiting the relative positioning of the pursuers around the evader. It is possible to improve this particular evasion maneuver by taking into account the dynamic models of the players. In particular, $\phi_a(x_e, x_i, u_e)$
is the minimum real positive solution to Eq. (3), where $u_c$ represents the direction of the evader’s random maneuver.

Finally, let $\phi_i(x_e, x_i, x_T)$ denote the minimum time-to-capture of the evader by the $i^{th}$ pursuer when the evader is directly headed towards the target location, in which case, $\phi_i(x_e, x_i, x_T)$ is the minimum real positive solution to Eq. (1) with $u_c = (x_T - x_e) / \|x_T - x_e\|$. Note that in this case, the evader does not maneuver to avoid the pursuer, and only the pursuer is maneuvering to minimize the capture time. Finally, let $\phi_f(x_e, x_T)$ denote the minimum time for the evader to reach the target.

### A. Elements of the payoff matrix

In the current matrix game formulation, each element of the payoff matrix is a numerical value that reflects the two-fold objective of the evader: (a) to avoid capture and (b) to reach the target location $x_T$. The two components of each entry are the time that $P$ would take to capture $E$, and the extent to which $E$’s heading is towards $x_T$ from its current location. The target-seeking component of $E$’s velocity is equal to $\cos \theta$, where $\theta$ is the angle between the vectors $u_e$ and $x_T - x_e$.

Because at least two players have moved in the time between the current stage and the previous stage, one has to re-construct the payoff matrix at each stage. While a future payoff is as important as the present payoff for consideration, it is difficult to estimate the payoff that the evader will receive at the end of $K$ stages, since the payoffs at each stage are dependent on the players’ states in the current stage. Thus, the history of moves in previous play is reflected in the changing payoff values, although this information is not available directly to the players as a strategy recall.

The inputs are designed to reflect the long-term effects of each action. In this case, since we have an upper bound $\bar{T}_f$ on the duration of the game, the time-of-capture component is bounded. For each pair of pure strategies $(i, j)$, we calculate the minimum time-to-capture of $E$ by the pursuing player $i$. Note that $E$ will play the strategy corresponding to evasion from the pursuer $j$. If capture is not possible, we set the value to $\bar{T}_f$. The target-seeking component is given by $\cos \theta$, as described earlier. Algorithm 1 shows the main steps for the assignment of payoffs to $M_k$.

Note that we normalize the evasion component which is given by the matrix $M_{k1}$ using the maximum entry of the matrix. This ensures that all evasion components have values between zero and unity, with magnitude similar to the target-seeking component. The summation of the two components in this manner is a standard practice in multi-objective optimization where the objectives are combined into one global criterion [30].

### B. Solution to the matrix game

We proceed to solve for the equilibrium strategies for the players. An equivalent non-zero sum formulation for our problem would consider the whole $N + 1$ player game, with cost assignments that are functions of the states of all players. Consequently, verifying the existence of an equilibrium set of pure strategies is a hard problem, in the sense that it would require an exhaustive search among all possibilities. However, we know that a two-player zero-sum multi-action game is finite, admits a saddle point solution in mixed strategies [4].

The min-max solution to the matrix game at stage $k$ consists to two vectors of probabilities (discrete probability distributions), one for the evader $E$ which is denoted by $\pi_{ek}$, and one for the group of pursuers as a whole which is denoted by $\pi_{pk}$. These vectors correspond to the mixed strategies of the two players. Each entry of the vector representing the mixed strategy of a player corresponds to the probability with which each pure strategy is to be chosen, when the game is played with infinite turns. The discrete action chosen at each stage $k$ is a random sample from the probability distribution assigned by the mixed strategy. Note that the matrix formulation enables the conversion of the $(N + 1)$-player dynamic game to the two-player decision-making problem with discrete action choices for each player.

In particular, at stage $k$, the mixed strategy for $P$, $\pi_{pk}$, $\in \mathbb{R}^N$, given by $\pi_{pk} := [\pi^{1}_{pk}, \ldots, \pi^{N}_{pk}]^T$, with $\sum_{i=1}^{N} \pi^{i}_{pk} = 1$, and $\pi_{pk}^j \geq 0$. In other words, the mixed strategy for the $N$ pursuers can be defined, by considering a member of the simplex of dimension $N - 1$, for the first $N - 1$ probabilities (the $N^{th}$ probability is obtained as the difference of their sum from unity). Similarly for $E$, the mixed strategy is denoted by $\pi_{ek} \in \mathbb{R}^{N+1}$, where $\pi_{ek} := [\pi^{1}_{ek}, \ldots, \pi^{N+1}_{ek}]^T$, and $\sum_{j=1}^{N+1} \pi^{j}_{ek} = 1$, and $\pi_{ek}^j \geq 0$. The $i^{th}$ (respectively, $j^{th}$) entry of the vector $\pi_{pk}$ (respectively, $\pi_{ek}$) represents the probability of the $i^{th}$ (respectively, the $j^{th}$) pure strategy being employed by $P$ (respectively, $E$). We only choose one pure strategy for each player per stage of the game. However, in the limiting case (for a game of infinite stages), the frequency at which each pure strategy is chosen converges to the probabilities assigned to the pure strategies by the mixed strategies $\pi_{pk}$ and $\pi_{ek}$. The problems of computing the vectors $\pi_{pk}$ and $\pi_{ek}$ that correspond to the saddle point of the (min-max) game can be formulated as Linear Programming (LP) problems [5], which can be solved using readily available solvers. In particular, $\pi_{pk}^*$ corresponds to the solution to the following LP problem:

$$
\min_{(p, \pi_{pk})} p
$$

subject to $p1 \geq M_k^T \pi_{pk},$

| input : $x_e, x_T, x_i \forall i \in I, k, \bar{T}_f$
| output : $M_k$
| for $i \leftarrow 1$ to $N$
| for $j \leftarrow 1$ to $N + 1$
| $T_e = \min(\phi_i(x_e, x_i, u_j), \bar{T}_f)$
| $G_c = \frac{\|u_e\|}{\|x_T - x_e\|}$
| $M_{k1}(i, j) = T_e$
| $M_{k2}(i, j) = G_c$
| $M_{k1} = \max_{i,j} M_{k1}(i, j)$
| $M_k = M_{k1} + M_{k2}$

**Algorithm 1**: Payoff Assignment to $M_k$
where 1 := [1, ..., 1]T ∈ ℝN. Similarly, for the computation of π ek, one has to solve the following LP problem:

\[
\begin{align*}
\max_{q, \pi ek} & \quad q \\
\text{subject to} & \quad q 1 \leq M_k \pi ek, \\
& \quad \pi ek \geq 0, \\
& \quad 1^T \pi ek = 1.
\end{align*}
\] (5)

Note that the LP problem whose solution is π ek is dual to the LP problem whose solution is π pk. From strong duality, we have that \(\max_{q, \pi ek} q = \min_{\pi pk} 1^T M_k \pi ek\).

The two LPs and are solved at each stage of the game after we recompute \(M_k\) to account for the change in the players’ locations between successive stages. In practice, the actions (pure strategies) for a particular stage of the game are obtained as random samples from the discrete distributions given by \(\pi pk\) and \(\pi ek\) for that stage of the game.

IV. NEW MATRIX FORMULATION OF THE MULTI-AGENT PEG WITH Q-LEARNING

In the formulation of the matrix game described so far, the size of the matrix depends on \(N\), the number of pursuers. In addition, the action space of the evader is comprised of actions that restrict the evader to switch between pure evasion (try to delay as much as possible the capture by the active pursuer) and goal-seeking (reach the goal destination as fast as possible). Next, we propose a new payoff matrix, whose structure renders it independent of the number of pursuers, as well as the number of actions that are available to the evader. Further, we are interested in improving the payoff function that determines the payoff matrix, so that the evader’s strategy performs better in terms of guiding the evader to the target without getting captured. To construct this new payoff function, we propose to use Q-learning techniques, with the intention of recognizing possible patterns of play in the multi-agent game, which lead to a better evasion strategy.

At each stage \(k\) of the game, we consider a new matrix \(M_k \in \mathbb{R}^N_k \times N_e\), whose entries are the payoffs (rewards) to \(E\) at that stage. Alternatively, the entries of \(M_k\) can be viewed as the costs incurred by the group of pursuers as a whole. Again, since the payoffs to the evader and the pursuer group sum to zero, the matrix represents a zero-sum game with the evader as the maximizing agent and the group of pursuers as the minimizers. Each row of \(M_k\) represents a pure strategy, \(p_i, i \in \{1, ..., N_p\}\), played by the group of pursuers and each column represents a pure strategy, \(e_j, j \in \{1, ..., N_e\}\), which is played by \(E\). Each pure strategy qualitatively describes an action taken by the corresponding player or players. Note that the learning space which is used in the learning process is the output of a nonlinear transformation applied to the original (high-dimensional) state space. Further, the action space used for learning corresponds to a feature-based action set, which is an abstraction of the physical action set available to the players.

A. Discrete action sets for the players

The set of actions for the evader, which is denoted by \(A_e\), is a finite set comprised of \(N_e\) elements. Similarly, a finite set \(A_p\) comprising \(N_p\) actions is considered for the group of pursuers. To streamline the subsequent discussion and analysis, we choose \(N_e = 4\) and \(N_p = 2\). The choice of actions for each player (either as an individual or as a group) is motivated by the qualitative effect of each action on the game. The actions that are considered for the group of pursuers are denoted by \(p_i \in \{1, ..., N_p\}\), and are described qualitatively as follows:

1) \(p_1\): the pursuers engage in relay pursuit, that is, only the pursuer closest to the evader engages in pursuit while the others maintain their current state of operation (remaining static).
2) \(p_2\): all pursuers actively engage in pursuit of the evader simultaneously and non-cooperatively.

Note that the focus of this work is on the characterization of an evasion strategy rather than a pursuit strategy. For this reason, we have assumed that the pursuers have a preferred strategy, namely relay pursuit. However, in the construction of the matrix game in our approach, the pursuers are not required to adopt the relay pursuit strategy \((p_1)\) at all times. They can adopt instead an alternative pursuit strategy \((p_2)\). Not having the pursuit strategy fixed at all times is important in the learning process, to increase the robustness of the evader’s learned policy. The evader’s actions, denoted by \(e_j \in \{1, ..., N_e\}\), are described as follows:

1) \(e_1\): the evader engages in pure evasion from the nearest pursuer.
2) \(e_2\): the evader performs a “collective evasion” maneuver which considers all the pursuers at once.
3) \(e_3\): the evader heads directly towards the target.
4) \(e_4\): the evader moves normal to the line of sight from the closest pursuer, that is, normal to the direction in action \(e_1\).

The physical realization of the discrete actions can be any vector direction. The physical realization of each of these actions is straightforward except for the action \(e_2\), which is explained below. The schematic physical realizations of the discrete actions of the evader are illustrated in Fig. 1.

Characterization of action \(e_2\): In a frame with the evader at the origin, consider all the angles formed between the lines of sight to two adjacent pursuers in order. The direction of motion of the evader corresponds to the bisector of the largest angle formed at the evader’s position. This is equivalent to the evader moving towards the largest gap between the pursuers. The intent of this evasion action is to make the evader move away from the entire group of pursuers at once (group evasion action), instead of evading only one in particular. This action is useful when the direction of optimal evasion from one pursuer (say the \(i^{th}\) one) puts the evader directly in the line of sight of another pursuer (say the \(j^{th}\) one, \(j \neq i\)). If \(\theta_i\) is the angle between the lines of sight from the evader to the \(i^{th}\) pursuer and to the \((i + 1)^{th}\) pursuer, then we calculate the resultant direction of motion of the evader as follows:

\[
\begin{bmatrix}
\cos(\theta_r + \omega_{im}) \\
\sin(\theta_r + \omega_{im})
\end{bmatrix}
\]
where \( \omega_i := \Delta r_i \) denotes the polar angle of the vector \( r_i := x_e - x_i \), and
\[
\theta_i := \omega_{i+1} - \omega_i, \quad \theta_i \in [0, 2\pi],
\]
\[
i_m := \arg \max_i \theta_i,
\]
\[
\theta_r := \frac{\theta_{i_m}}{2},
\]
for \( i = \{1,...,N-1\} \). If \( i = N \), we set \( \omega_{N+1} := \omega_1 \) to preserve the order of rotation. The evasion action is illustrated in Fig. 2.

The evader (shown in blue) should move along the blue arrow according to the action \( e_2 \). In Fig. 2a, the evader is outside the polygon formed by the pursuers (shown in red) as vertices, and the evader’s direction of motion is between pursuers \( P_1 \) and \( P_2 \). In Fig. 2b, the evader is inside the polygon of the pursuers, and the preferred bisector lies between \( P_2 \) and \( P_3 \). If two angular gaps are equal, the evader favors the angular gap formed by the longer line of sight to a pursuer.

B. Q-learning for the matrix payoffs

The payoff matrix \( M_k \) at each stage \( k \) has a fixed size for any number of pursuers, and the discrete action sets comprehensively describe the qualitative effects of the players’ actions in the game. However, for the multi-agent game \( (N > 1) \), the optimal payoff function, and consequently the entries of the payoff matrix, are not readily available analytically. Hence, we use min-max Q-learning to construct the entries of the payoff matrix \( M_k \) at each stage.

Before we describe the Q-learning procedure, we must introduce a new state space where the Q-learning process can take place. This is because it is not practical to carry out the learning process effectively using position vectors as states and physical directions as corresponding actions (this would be an intractable approach). In the position space, a set of optimal actions learned by the players from a particular initial configuration, does not enable them to maneuver correctly from other initial configurations, because the environment is not static. Further, the speeds of the players and their number largely influence the weights learned, which in turn influence the optimality of each action for the players. Consequently, we are unable to extend the results (learned weights) to other environments that are similar but with a different number of players and with different speeds. The rewards for the reinforcement learning procedure also cannot be explicitly position-dependent, since the reward for the same state-action pair in another game with different initial conditions would be different.

In this context, we introduce a new state space in which the states are transformed representations of the players’ position vectors and also encode other properties of the pursuit-evasion game. In particular, we recognize that the times-to-capture and the time-to-target are sufficient to accurately describe and decide the progress of the game. Moreover, it is the ratio of the time-to-target to the time-to-capture or the time to intercept \( \phi_T/\phi_e \) (or \( \phi_T/\phi_s \) respectively), which reflects the safety (in the sense of staying sufficiently far away from the pursuers) and target-reaching ability of the evader at all times. With these considerations, we next describe the state space in which the learning process is carried out.

C. Learning state space

The learning state space \( \Psi \) is a subset of \( \mathbb{R}^m_{\geq 0} \), where \( \mathbb{R}^m_{\geq 0} := \{ z \in \mathbb{R}^m : z_j \geq 0, \; j \in \{1,...,m\} \} \). The learning state variables are implicitly dependent on the position and velocity of the players at a given stage. For the multi-agent
PEG, \( m = 4 \). At any stage \( k \in \{1, \ldots, K\} \), let us introduce the following variables:

1) The least min-max time-to-capture of the evader over all the pursuers, which is given by
\[
\psi_c(k; x_c) := \min_{i \in i} \phi_c(x_c, x_i). \tag{7}
\]

2) The least minimum time-of-intercept of the evader by a pursuer when the evader is moving in a randomly chosen and fixed direction that is known to pursuers, which is defined as follows
\[
\psi_a(k; x_c, u_c) := \min_{i \in i} \phi_a(x_c, x_i, u_c), \tag{8}
\]
where \( \psi_a(k; x_c, u_c) \) corresponds to the time-of-intercept when the evader executes action \( e_2 \).

3) The minimum time for the evader to reach the target state in the absence of pursuers which is defined as follows:
\[
\psi_T(k; x_c, x_T) := \phi_T(x_c, x_T). \tag{9}
\]

4) The least minimum time-of-intercept of the evader by a pursuer when the evader heads directly towards the target location which is given by
\[
\psi_s(k; x_c, x_T) := \min_{i \in i} \phi_s(x_c, x_i, x_T). \tag{10}
\]

For ease of notation, we denote the learning state variables as explicit functions of the stage \( k \), while in fact, they are functions of the position vectors at time \( t = k\Delta t \). Let \( S: \mathbb{R} \to [0,1] \) be a sigmoid function. At stage \( k \), the state vector in the learning space \( \Psi \subseteq \mathbb{R}^m_{\geq 0} \) is denoted by \( \psi(k; x_c, x_T) \) and defined as follows:
\[
\psi(k; x_c, x_T) := \left[ S\left( \frac{\psi_a}{\psi_T} \right) S\left( \frac{\psi_s}{\psi_T} \right) S\left( \frac{\psi_c}{\psi_T} \right) \right]^T.
\]

The first three state variables depend on the ratios of the time-metrics introduced in Section III with respect to the time-to-target (this choice is justified by the fact that the evader is the maximizing agent in the two-player static game). We wish to maximize the ratio of the time-to-capture (or intercept) to the time to reach the target. The following are the desirable attributes of the learning state space:

1) The learning state space is a subset of \( \mathbb{R}^m_{\geq 0} \) for all \( N \). This means that for any number of pursuers, \( N \), the learning state space has a constant dimension. This greatly reduces the computational expense associated with solving the matrix game at each stage.

2) The state variables of the learning space furnish a feature-based representation of the original position space. In other words, we capture the essential features of the game using just \( m \) variables, instead of \( 2(N+1) \) position variables. The feature-based representation is important in identifying states in the game which are far apart in the position space, but are similar in terms of imminent capture or proximity to the target.

Note that to propagate the dynamics forward in time and perform state updates, we are still required to operate in the physical space. In the next section, we will describe in detail how we use min-max Q-learning to obtain the entries of the payoff matrix. The training of the players in the transformed state space enables the players in the game to maneuver correctly in environments other than the one used for training. The schematic of the learning process applied to the matrix game is shown in Fig. 3.

![Fig. 3: Learning applied to matrix game framework for pursu evasion, shown for a single stage.](image)

V. APPROXIMATION OF THE PAYOFF MATRIX WITH MIN-MAX Q-LEARNING

Let \( T_q : \mathbb{R}^m_{\geq 0} \times \mathcal{A}_e \times \mathcal{A}_p \to \mathbb{R}^m_{\geq 0} \) be the state transition function which maps the state \( \psi(k) \) at stage \( k \) to the state at the next stage, \( k+1 \), if the actions chosen by \( P \) and \( E \) at stage \( k \) are \( p_i \) and \( e_j \), respectively. We write
\[
\psi(k + 1) = T_q(\psi(k), p_i, e_j).
\]

The transition function is determined by the dynamics of the players. In practice, we choose to perform this state update in the position space rather than directly in the learning space. Let the function \( R_q : \mathbb{R}^m_{\geq 0} \times \mathcal{A}_p \times \mathcal{A}_e \to \mathbb{R} \) be the reward received by \( E \) for each state transition. Further, let \( H_q : \mathbb{R}^m_{\geq 0} \times \mathcal{A}_e \times \mathcal{A}_p \rightarrow [-1,1] \) be the “heading” function. The heading function maps the state \( \psi(k) \) to the component \( \theta \) of the evader’s unit velocity, obtained by projecting \( E’s \) velocity on the line of sight to the target \( x_T \). For instance, if the action chosen by \( E \) at stage \( k \) is \( e_j \), we have
\[
H_q(\psi(k), e_j, x_c; x_T) := -\frac{1}{v_e} \phi_\psi(x_c(t); x_T)|_{t=k\Delta t}.
\]

The above relation follows directly from the definition of the time-to-target for the evader in the absence of all pursuers, that is, \( \phi_T(x_c(t); x_T) := \|x_T - x_c(t)\|/v_e \). The Q-function, denoted by \( Q : \mathbb{R}^m_{\geq 0} \times \mathcal{A}_p \times \mathcal{A}_e \to \mathbb{R} \), maps every tuple (state, pursuers’ action, evader’s action) to a payoff value for \( E \). Given the definition of the Q-function at stage \( k \), for all \( p_i \)
for \(i \in \{1, \ldots, N_p\}\) and for all \(e_j\), for \(j \in \{1, \ldots, N_e\}\), the payoff matrix \(M_k = [M_k(i, j)]\) is set to be:

\[
M_k(i, j) := Q(\psi(k), p_i, e_j; x_e, x_1, \ldots, x_N, x_T).
\]  

(11)

Since the state space is continuous, it is not practical to use a table of Q-values to determine the optimal state-action-action tuples. If such a grid of states is used to generate a look-up table for the Q-function, we must rely on interpolation to calculate the payoff values at intermediate states. A very fine grid of discrete states would have large computational overhead, while we might still miss out on identifying an essential “feature” of a given state. For this reason, we desire generalization, wherein we can operate on the qualitative abstraction of a state rather its exact quantitative value which might not have been experienced during learning [19]. For instance, a state may be encountered during the game, which has times to capture and intercept that were never encountered during the training process. However, this new state with the combination of times to capture and intercept may be relatable to another known state with its own set of capture and intercept times, based on the criterion that in both cases, the evader can safely reach the target. This is one such example of extracting the qualitative information given by a state. Following generalization, we use an approximation of the Q-function as a weighted linear combination of a finite number of feature-based functions of the states and actions. In this case, the weights are learned iteratively using gradient-descent. In our implementation, there is no restriction on the magnitude and sign of the weights, which are all real numbers.

A. Linear approximation of the Q-function

Motivated by the previous discussion, the basis functions for the Q-function approximation are chosen as the coordinates of the new state obtained from executing one action of the evader and one action by the group of pursuers simultaneously. In particular, for all \(i \in \{1, \ldots, N_p\}\) and \(j \in \{1, \ldots, N_e\}\),

\[
Q(\psi(k), p_i, e_j) = w_q^T \zeta(\psi(k), p_i, e_j; x_T),
\]  

(12)

where both \(Q(\psi(k), p_i, e_j)\) and \(\zeta(\psi(k), p_i, e_j; x_T)\) are dependent on the states of all the players and the target location \((x_e, x_1, \ldots, x_N, x_T)\), and where

\[
\zeta(\psi(k), p_i, e_j; x_e; x_T) := \begin{bmatrix}
H_q(\psi(k), e_j, x_e; x_T) \\
T_q(\psi(k), p_i, e_j) \end{bmatrix}.
\]

Note that \(w_q\) is independent of \(k\) and \(\psi(k)\), as well as the choice of actions by the players. For a zero-sum game, the min-max Q algorithm aims to iteratively approximate [26] the Q-function \(Q()\) that satisfies the following equation:

\[
Q(\psi(k), p_i, e_j; x_e, x_T) = R_q(\psi(k), p_i, e_j; x_e, x_T) + \max_{i'} \left(\min_i \text{row}_{i'}(M_{k+1})\bar{\pi}_{e_q}\right),
\]  

(13)

where row\(_{i'}\)(\(M_{k+1}\)) denotes the row vector that corresponds to the \(i'\)-th row of the matrix \(M_{k+1} = [M_{k+1}(i', j')]\) with \(M_{k+1}(i', j') := [T_q(\psi(k + 1)), p_{i'}, e_j; x_e, x_T]\). The min-max Q-learning process is outlined in Algorithm 2.

Note that the assumption that the relay pursuit strategy is the preferred strategy for the group of pursuers does restrict their choice of action to \(p_1\) exclusively. If we assume during the learning process that the group of pursuers is required to use relay pursuit at all states, then the problem becomes a decision making problem for a single-agent in a dynamic environment. In that case, the static game at every stage corresponds to a Markov Decision Process (MDP) because there is only one available action for the agent opposing the evader. In our proposed approach, we train the evader to be able to handle both relay pursuit and a completely non-cooperative pursuit strategy. This is choice is intended to increase the robustness of the proposed learning-based solution approach. Hence, we have a Markov Game [22] with the opposing agent having two actions \((p_1, p_2)\) to choose from.

Algorithm 2: Linear approximation of the Q-function using min-max Q-learning.

\[
\text{Input: } \mathcal{A}_e, \mathcal{A}_p, \mathcal{R}_q \text{ and } T_q \text{ (dynamics), } N_{\text{tr}} \text{ (maximum training episodes), } \alpha \text{ (learning rate), } \gamma \text{ (discount factor), } \text{tol} \text{ (convergence tolerance), } \delta\alpha \text{ (decay in learning rate), } \beta \text{ (exploration probability), } \delta\beta \text{ (decay in exploration).}
\]

1. Initialize each element of \(w_q\) to a random sample from a uniform distribution in \((0, 1)\).
2. Number of training game episodes \(n = 0\)
3. While \(n \leq N_{\text{tr}}\) or \(\|\delta w_q\| > \text{tol}\) do
   a. Pick random initial conditions \(x_e, x_T, x_i, \forall i \in \mathcal{I}\)
   b. Initialize \(k = 0\)
   c. While game episode is not over do
      i. Calculate \(\psi(k)\)
      ii. For \(i = 1\) to \(N_e\) do
         For \(j = 1\) to \(N_e\) do
            \(M_k(i, j) = Q(\psi(k), p_i, e_j)\)
      end
      end
      iii. Find mixed strategies \(\pi_{e_j}\) and \(\pi_{p_k}\)
      iv. \(V(\psi(k)) = \min_i \text{row}_i(M_k)\pi_{e_j}\)
      v. With probability \(\beta\) choose an action \(\bar{e}\) for \(E\) at random from set \(\mathcal{A}_e\) otherwise,
         Choose an action \(\bar{e}\) for \(E\) based on \(\pi_{e_j}\)
      vi. Choose an action \(\bar{p}\) for \(P\) based on \(\pi_{p_k}\)
      v. \(\mathcal{D} = R_q(\psi(k), \bar{p}, \bar{e}) + \gamma V(\psi(k + 1)) - Q(\psi(k), \bar{p}, \bar{e})\)
      vii. \(\delta w_q = \alpha \cdot \mathcal{D} \cdot \zeta(\psi(k), \bar{p}, \bar{e})\)
     ii. \(w_q := w_q + \delta w_q\)
   end
   end
   iii. Perform one-step update of the state \(x_i := x_i + \Delta x_i\)
   iv. \(k := k + 1\)
end
\]

Output: \(w_q\)

B. Reward function

The choice of the reward function \(R_q\) directly influences the learning process in terms of learning desired behavior as well as the time taken to converge to a policy. In particular, for
problems with a desired terminal state (such as the one considered in this paper), the shaping of the reward is essential to drive the evader towards the target location while avoiding the pursuers. There is positive as well as negative reinforcement in the reward function. Since the time-to-capture relative to the pursuers. There is positive as well as negative reinforcement to solve the linear programming problem.

At the termination of the learning process, after the weights have converged to a specified tolerance, the performance of the evader using the strategy determined by the solution to the matrix game with Q-learning were compared with its performance when using the strategy determined by the solution to the matrix game with payoff matrix of size $N_p \times N_e$, where the payoff function was taken to be the sum of the heading reward (as described in Section V-B) and the minimum value of the learning state variables, was also compared with the others. The package cvx [31] as well as the built-in solver in MATLAB were used to solve the linear programming problem.

We test the policy based on the learned payoff function with a different number of pursuers than that used in training. It is important to note that the size of the payoff matrix as formulated for Q-learning is independent of the number of agents in the game. In Table III we see the evader’s performance for about $10^3$ episodes of the multi-player game, where $N = 5$, with initial conditions sampled from a uniform distribution within a grid of size 10 units. Note that the grid size is also different from the training grid size. The column headers, which correspond to different solution methods, are to be interpreted as follows:

1) M-1: Matrix game with Q-Learning with a $N_p \times N_e$ payoff matrix.
2) M-2: Matrix game without Learning with a $N \times (N+1)$ payoff matrix.
3) M-3: Matrix game without Learning with a $N_p \times N_e$ payoff matrix.

The discount factor $\gamma \in [0, 1]$ determines if the learning player prioritizes immediate rewards or long-term rewards. Typically, the discount factor is chosen to be closer to 1, indicating that long-term rewards are preferred. The choice of learning rate $\alpha \in [0, 1]$ is critical, because a consistently high learning rate will make the learning process sensitive to every input, thereby making it difficult to obtain convergence of weights. On the other hand, if $\alpha$ is too small, the learning process will take a long time. In general, we start with a nominal value of $\alpha = 0.1$ and decay the learning rate as more learning episodes are covered. Finally, in the beginning of the learning process, some randomness is introduced in the choice of actions of the learning player to ensure exploration of the state-action space by the player.

Next, we train the evader for the case when $v_e = 1$. Then, we test the evader’s performance for a different number of pursuers and a large number of random initial conditions.
TABLE II: Outcome of games - $v_e = 0.9, N = 5$

| Solution Method | M-1 | M-2 | M-3 |
|-----------------|-----|-----|-----|
| Evader Captured | 89.51% | 89.84% | 89.24% |
| Target Reached  | 10.49% | 10.16% | 10.70% |

TABLE III: Outcome of games - $v_e = 1, N = 4$

| Solution Method | M-1 | M-2 | M-3 |
|-----------------|-----|-----|-----|
| Evader Captured | 2.92% | 9.50% | 7.36% |
| Target Reached  | 87.51% | 90.45% | 92.64% |

In Table III, we present the results for $N = 4$. We find that all the solution approaches perform better than the previous case, in terms of reaching the target. In terms of both avoiding capture and reaching the target within the end time, the method using Q-learning is better than the other two methods. Note that the number of games that were inconclusive within our simulation time limit is highest in the case of the evader using the learned strategy. This means that even though the evader was unable to reach the target, it managed to avoid capture or at least delay it until the end of the simulation time. This is a desirable behavior for the evader, which is not observed when the other two strategies, M-2 and M-3, are employed. In the subsequent figures, the pursuers are represented in red and the evader in green. The capture disks around the pursuers and the evader (using large square markers) are shown at certain time instances during the game. The black dot represents the target state. The evolution of the game for two sets of initial conditions for using the solution methods M-1 and M-2 is illustrated in Fig. 4 and Fig. 5.

In simulations, where the evader’s decisions are not always intuitive, but lead to better performance in terms of reaching the target. The evolution of the pursuit-evasion game in Fig. 7 elucidates the evader’s performance when initially the evader is within the convex hull of the pursuers. In particular, the evader’s initial position is at $[4, 4]^T$ and the pursuers are initially at $[5, 4]^T, [3, 4]^T, [4, 4.3]^T$ and $[4, 3.7]^T$. Clearly, the learning-based strategy performs a swerve-like maneuver, which is not mimicked in the other matrix payoff formulations.

Next, we examine the computational efficiency of the proposed approach in terms of the time taken to generate the payoff matrix and solve the matrix game for one stage. The simulations were performed for methods M-1 and M-2 in which the payoff matrices have different sizes. The resulting values shown in Fig. 8 were derived for a grid of size $10 \times 10$, and it is seen that the M-1 method (where the payoff matrix is of constant dimensions $N_p \times N_e$) is clearly advantageous in terms of computational speed, while resulting in better performance than the M-2 method (where the payoff matrix has size $N \times (N + 1)$).

In this case, we expect a much higher rate of success for the evader, since it is as fast as the pursuers.

In Fig. 6, the evader uses the action $e_4$ in the normal direction, to move towards the target, when the pursuer is initially on the evader’s line of sight to the target. The pursuer turns for a tail chase, but the evader is able to reach the target within the required radius of tolerance. The black arrows indicate the direction of motion of the evader.

We see that even in the cases where the other methods give a good policy for the evader, the policy determined using learned payoffs sometimes results in the evader reaching the target quicker. This is another advantage of learning as seen in simulations, where the evader’s decisions are not always intuitive, but lead to better performance in terms of reaching the target. The evolution of the pursuit-evasion game in Fig. 7 elucidates the evader’s performance when initially the evader is within the convex hull of the pursuers. In particular, the evader’s initial position is at $[4, 4]^T$ and the pursuers are initially at $[5, 4]^T, [3, 4]^T, [4, 4.3]^T$ and $[4, 3.7]^T$. Clearly, the learning-based strategy performs a swerve-like maneuver, which is not mimicked in the other matrix payoff formulations.

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Finally, we also observe that the learning method is sensitive to the value of the sampling period \( \Delta t \) relative to the capture radius \( \ell \). While the ratio \( \ell / (v_e \Delta t) \) used during training is maintained constant, the performance of the learning-based method is consistent. When the ratio is changed, the learning-based strategy executes a different sequence of moves for the same initial conditions of the players.

VII. CONCLUDING REMARKS

In this paper, we have proposed a systematic way to compute the evasion strategy for an evader whose goal is to reach its target destination while avoiding capture by multiple pursuers at all times. The proposed solution is based on a novel matrix game formulation of the multi-agent pursuit-evasion game. The first step of the proposed solution is to generate a dynamic payoff matrix, whose elements are computed using a linear combination of the time-to-target and the time-to-capture. We then refine the matrix formulation by representing the game in a different state space. The new state space corresponds to a feature-based representation of the game, and the discrete actions of the players are also feature-based. The actions chosen for each player represent the goals of that player in the game. In the new state space and action space, we use min-max Q-learning to learn the payoff Q-function for the zero-sum-game of the evader against the pursuers.

One of the key results of the proposed learning-based matrix formulation is that it is, in principle, independent of the number of players, and their specific parameters (e.g., speed of the pursuers). In terms of performance, the strategy that uses learning is equally successful as the methods where the payoffs are constructed using weighted sums. Moreover, using learning, the evader can perform a combination of evasion maneuvers that is difficult to explicitly generate otherwise. However, there are dependencies of the Q-learning solution on the variables \( \ell \) and \( \epsilon \) that are used during training.

We propose that the next step is to use a deep neural network to develop an improved non-linear approximation of the Q-function. The structure of the learning algorithm will remain the same, however, the weight update which is specific to the linear approximation of the Q-function will be replaced by the corresponding weight update for a neural network which is appropriately defined. There are some challenges pertaining to the use of deep neural nets for Q-learning for this multi-agent reach-avoid problem, of which two are prominent: (a) stability concerns of a single net Q-function approximator that uses bootstrapping, (b) convergence concerns for multi-agent Q-learning. The former can be addressed by using a double neural network for Q-learning, and for the latter, we can use measures such as reward clipping and input scaling. We can also simplify the problem by embedding the pursuers’ play in the environment of the evader. Finally, the performance of the unsupervised learning algorithm (Q-learning in this context) heavily depends on the reward function associated with each agent. Reward shaping, particularly for the multi-agent reach-avoid problem under consideration, is challenging, as it aims to determine a reward function that guarantees successful evasion, if such evasion is feasible.
