On the Notion of a Generalized Mapping on Multiset Spaces

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Abstract. A sufficiently generalized concept of mappings on multisets has been introduced, thus resolving a long standing obstacle in structural study of multiset processing. It has been shown that the mapping defined herein can model a vast array of already defined mappings within the domain of Theoretical Computer Science as special cases and also it handles diverse situations in multiset rewriting transformations. Specifically, this paper unifies and generalizes the works of Parikh [18] (1966), Hickman [8] (1980), Khomenko [12] (2003) and Nazmul [17] (2013).

Keywords: Multisets mapping; Multiset process modeling; Parikh mapping; Multiset rewriting, Hickman mapping

1. Introduction

Index object in Pandas, the Data Science library of Python language, is a multiset. Multisets are also important for analysis of various computer algorithms [3, 4, 13, 16]. Although in mathematics world, one can trace the origins of modern interest in multisets as far as the great mathematician Richard Dedekind (d. 1916), but it is only with the recent advent of Computer Science and allied fields that a renewed interest is being witnessed [5]. Today in search of ever more generalization, multisets have become the dominant data structures of contemporary Computer Science [22], Cybernetics [2], Information Science, Combinatorics, and unconventional computational paradigms like membrane [19] and DNA computing, and Petri Nets [12]. A set is a well defined collection of distinct objects. As weakening the condition of well definedness gave birth to Fuzzy Sets, weakening of the condition of ‘distinctness’ produces the notion of multisets. Thus multisets are sets in which repetition of elements is significant. Both fuzzy sets and multisets are generalizations of classical sets. Examples of multisets abound: water molecule $H_2O$ is a multiset as $H_2O = \{H, H, O\} = \{2/H, 1/O\}$, the prime factorization of integers $n > 0$ is a multiset whose elements are primes. Every monic polynomial $f(x)$ over the complex numbers corresponds in a natural way to the multiset of its roots. Zeros and poles of meromorphic functions, invariants of matrices in a canonical form are multisets. Again words in a language are multisets on the set of alphabets $\Sigma$; symbols in a membrane are multisets over the set of alphabets and markings of a Petri net are multisets over the set of places. Thus naturally multisets find numerous applications in diverse fields: Database Management Systems, Cryptography, Membrane Computing, Rewriting Systems, Abstract Chemical Machines and Neural Networks etc [6, 20, 23].

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Despite a historically prolonged presence of multisets in Mathematics and Computer Science, research on the multiset theory has not yet gained ground and is still in its infant stages. The research shows a strong analogy in the behavior of multisets and sets. Singh [21] rightly pointed out that so far all mappings defined for multisets turn into mappings between the root sets and thus become ordinary mappings of insignificant consequences. Since the case in consideration has a wider generality (compared to that of sets), the results obtained for multisets are technically more complicated and should be more general ones. Main obstacle, in a full fledged research, has been the non-availability of a sufficiently generalized notion of mapping between arbitrary collections of multisets.

This paper addresses the problem of extending a mapping defined between the root sets of two multiset spaces to the multiset spaces themselves. Organization of the paper is as follows: Section 2 surveys the related literature and locates the present work in its context. Section 3 collects necessary definitions and presents some new requisite results. Section 4 is first of the two main sections of this work. It formally introduces the new notion of mapping on multiset spaces. This section also presents many important results, properties and insights concerning the new mapping. Section 5 is the other main section which demonstrates the relevance and interconnections of the notion presented herein with other established notions of the field e.g. Parikh [18], Hickman [8], Khomenko [12] and Nazmul [17] mappings also with cardinality, similarity and distance measures on multisets. It shows how the mapping presented herein unifies and generalizes various notions. Section 6 concludes the paper.

2. Related Work

Parikh [18] introduced a mapping which interlinked words in a language with arithmetical vectors. Despite the fact that Parikh mappings find many applications [1], this notion has been found deficient in many aspects by the subsequent researchers [10, 14].

Hickman also introduced a notion of mapping on multisets in [8]. His definition (see Definition 5.3 later in this paper) is somewhat restrictive definition of a multiset mapping, for multisets are supposed to be richer objects in some sense than ordinary sets, and one might expect that this extra richness would be reflected in the definition of a multiset mapping, but the definition assumes identical multiplicity limits.

Manjunath and John [15] have done some preliminary work on multiset relations. They defined multiset as a sub multiset of the generalized Cartesian product. Continuing with same line of thought Girish and John introduced a notion of functions in multiset context in [7]. Girish and John defined functions as a subcollection of Cartesian product of two multisets. In this specific context i.e. function as subset of multiset relation, they have obtained results involving different types of functions. It must be noted that their approach does not consider a function defined on the root sets and its extension to the multiset spaces induced by the root sets.

Singh [21] considers ordinary mappings between the roots sets and multiplicities are left unaltered. This thus makes the results for multisets as exact copies of the classical results of set theory. Whereas, considering multisets a generalization of sets, one naturally expects to see divergences.

Recently Nazmul et al. have introduced yet another notion of mapping on multisets in [17]. This definition assumes that multiplicities of domain and range multiset spaces are same, in fact identical.
Our work encompasses the works of Parikh, Hickman and Nazmul as special cases, while at the same time, being distinct from the approach adopted by Girish and John in [7]. We consider mappings between root sets extended to arbitrary multiset spaces.

3. Preliminaries

In this section requisite definitions, notations and some other results have been collected. Interested reader may find further material in [7][9] and in the references therein.

A well defined collection of elements containing duplicates is called a multiset. Formally, if $X$ is a set of elements, a multiset $M$ drawn from the set $X$ is represented by a count function $C_M$ defined as $C_M : X \to N$ where $N$ represents the set of non-negative integers. For each $x \in X, C_M(x)$ is the characteristic value of $x$ in $M$ and indicates the number of occurrences of the element $x$ in $M$. A multiset $M$ is a set if $C_M(x) \leq 0$ or $1 \forall x \in X$. The multiset space $X^m$ is the set of all multisets whose elements are in $X$ such that no element in the multiset occurs more than $m$ times; more formally, if $X = \{x_1, x_2, ..., x_k\}$ then $X^m = \{(m_1/x_1, m_2/x_2, ..., m_k/x_k) \mid i = 1, 2, ..., k; m_i \in \{0, 1, 2, ..., m\}\}$. The set $X^\infty$ is the set of all multisets over a domain $X$ such that there is no limit to the number of occurrences of an element in a multiset.

**Notation 1.** (1) A multiset may safely be identified as the same as its count function. In the context of this paper, a multiset $A \in X^m$ and its count function will be denoted by the same letter $A$. Intended use would be clear by context i.e. by $A$ we shall mean multiset and by $A(x)$ we shall mean the count of $x$ in multiset $A$.

(2) In the sequel $m^*$ will denote the set $\{0, 1, 2, ..., m\}$, for any arbitrary $m \in \mathbb{N}$.

Let $M$ and $N$ be two multisets drawn from a multiset space $X^m$. $M$ is a sub multiset of $N (M \subseteq N)$ if $M(x) \leq N(x)$ for all $x \in X$. $M$ is a proper sub multiset of $N (M \subset N)$ if $M(x) \leq N(x) \forall x \in X$ and there exists at least one $x \in X$ such that $M(x) < N(x)$. The union (respectively, intersection, difference) of two multisets $M$ and $N$ drawn from a set $X$ is a multiset $P$ denoted by $M \cup N$ (resp., $M \cap N$, $M - N$) such that $\forall x \in X, (M \cup N)(x) = \max\{M(x), N(x)\}$, $(M \cap N)(x) = \min\{M(x), N(x)\}$, $(M - N)(x) = \max\{0, M(x) - N(x)\}$. The complement $M^c$ of $M$ in $X^m$ is given as $M^c(x) = m - M(x) \forall x \in X$.

We now define the notion of constant multiset and observe that the notion of empty multiset, as defined by Jena et al. (Definition 0.1(v) [9]), is a special kind of constant multiset.

**Definition 3.1.** A constant multiset $\tilde{t}$ in $X^m$ is defined as $\tilde{t}(x) = t \forall x \in X$ where $t$ is some integer such that $0 \leq t \leq m$. Following special constant multisets seem to be interesting:

(i) Multiset $\tilde{0}$, defined as $\tilde{0}(x) = 0 \forall x \in X$. This is empty multiset [9].

(ii) Multiset $\tilde{1}$, defined as $\tilde{1}(x) = 1 \forall x \in X$. Clearly $\tilde{1}$ is the ordinary set $X$.

(iii) Multiset $\tilde{m}$, defined as $\tilde{m}(x) = m \forall x \in X$. This is called absolute multiset in $X^m$.

For a few basic properties of multiset union and intersection, we refer to Theorem 1.1 [9]. Moreover, we have some more properties:
Theorem 3.1. Let $M, N \in X^m$ then, we have:

(1) $M \cup \emptyset = M.$
(2) $M \cup \tilde{m} = \tilde{m}.$
(3) $M \cup M = M.$
(4) $M \cup (N \cup P) = (M \cup N) \cup P.$
(5) $M \subseteq N \iff M \cup N = N.$
(6) $\tilde{0} = \tilde{m}.$
(7) $M \subseteq N \iff N^c \subseteq M^c.$
(8) $M - N = (M \cup N) - N.$
(9) $(M - N) - P \subseteq (M - N) \cup (M \cap P).$
(10) $M - (N \cup P) \subseteq (M - N) \cap (M - P).$
(11) $M \cap \emptyset = \emptyset.$
(12) $M \cap \tilde{m} = M.$
(13) $M \cap M = M.$
(14) $M \cap (N \cap P) = (M \cap N) \cap P.$
(15) $M \subseteq N \iff M \cap N = M.$
(16) $\tilde{m}^c = \emptyset.$
(17) $M - N = M - (M \cap N).$
(18) $(M - N) - P \subseteq M - (N \cup P).$
(19) $(M - P) \cup (N - P) \subseteq (M \cup N) - P.$

Proof. We only prove (10), (17) and (18), remaining proofs are similar.

(10) $(M - (N \cup P)) (x) = \max \{0, M (x) - (N \cup P) (x)\} = \max \{0, M (x) - \max (N (x), P (x))\} \leq \min \{(M (x) - N (x)) \cup (M (x) - P (x))\} = ((M - N) \cap (M - P)) (x).$

(17) $(M - (M \cap N)) (x) = \max \{0, M (x) - (M \cap N) (x)\} = \max \{0, M (x) - \min (M (x), N (x))\} = \max \{0, M (x) - N (x)\} = (M - N) (x).$

(18) $(M - N) - P (x) = \max \{0, (M - N) (x) - P (x)\} = \max \{0, \max \{0, M (x) - N (x)\} - P (x)\} \leq \max \{0, M (x) - \max (N (x), P (x))\} = \max \{0, M (x) - \cup (N \cup P) (x)\} = (M - (N \cup P)) (x).$

Multi Set Theory (MST) is an extension of Classical Set Theory (CST). Thus one naturally expects divergences between the new theory and its classical counterpart. One such case in point, is the Law of Excluded Middle and Law of Non-Contradiction. It is notable that the set theoretic forms of these laws do not hold i.e. for $A \in X^m,$ in general, we have: $A \cap A^c \neq \emptyset$ and $A \cup A^c \neq \tilde{m}.$ Counterexamples supporting above statements may be seen by choosing $A = \{4/a, 2/b, 0/c, 0/d, 3/e\} \in \{a, b, c, d, e\}^4$ and calculating

\[
A \cap A^c = \{0/a, 1/b, 0/c, 0/d, 2/e\} \neq \emptyset,
A \cup A^c = \{4/a, 3/b, 4/c, 4/d, 2/e\} \neq \tilde{m},
\]

In classical sets we have two sets $A$ and $B$ disjoint, symbolically, $A \cap B = \phi$ if and only if $A \subseteq B^c.$ Since disjointedness does not make sense in multisets, naturally one looks for some alternate condition which can guarantee the subsedhhood in complement of the bigger multiset. This thought is formalized below in the notion of $m$-coincidence as:

Definition 3.2. Two multisets $A, B \in X^m$ are said to be $m$-coincident if we have $A (x) + B (x) > m$ for some $x \in X.$ We denote this as $AmB.$ If $A$ is not $m$-coincident then we write $A\not\perp B.$

Some immediate consequences of the above definition are:

1. $A\not\perp B \iff A \subseteq B^c.$
2. $A \subseteq B \Rightarrow A\not\perp B^c.$

An interesting result regarding $m$-coincidence is given as Theorem 4.1 (15). For a detailed study of the same notion we refer to [11].
4. The Mappings

In this section we intend to define a suitable notion of mapping on multisets. By suitable notion we mean that the mapping should be generalized enough to be defined between arbitrary choices of \(X, Y, m, n\) and thus arbitrary multiset spaces \(X^m\) and \(Y^n\). Moreover a multiset carries some resemblance with a fuzzy set, in the sense that both notions assign a numerical value to the elements of some arbitrary set. That is why one naturally expects the mapping on multiset to be somewhat similar to standard mapping on fuzzy sets as defined by Zadeh in 1965. Difference between fuzzy and multisets is that of positive integral values in a multisets. So for a truly generalized notion of multiset mapping, one has to ensure order preservation between the special sets \(m^*\) and \(n^*\) related to multiset spaces \(X^m, Y^n\), respectively. In fact, such a truly generalized mapping is needed to model, for example, multiset rewriting systems. We shall further discuss this point in subsequent discussion.

For this we first introduce the notion of order preserving maps and then use these for defining multiset mappings. Recall that \(m^* = \{0, 1, 2, ..., m\}\). We shall call a mapping \(p : m^* \to n^*\) order preserving (briefly, OP), if it satisfies:

\[(op1)\quad p(0) = 0\]
\[(op2)\quad p(m) = n\]
\[(op3)\quad p(i) \geq p(i - 1)\]

An OP map does not allow a crossing of arrows in a traditional mapping diagram as shown in figure below:

Following are immediate observations from the definition of OP maps:

**Proposition 4.1.** If \(p : m^* \to n^*\) is an OP map then

1. \(p\) is a constant map iff \(n = 0\).
2. \(p\) may be a surjective map, only if \(n \leq m\).
3. \(p\) may be an injective map only if \(n \geq m\).
4. \(p\) may be an bijective map only if \(m = n\).
5. If \(m = n\) and \(p\) is surjective, then \(p\) is bijective.
6. If \(m = n\) and \(p\) is injective, then \(p\) is bijective.
Example 4.1. The above definition illustrates the necessary calculations involved in the implementation of arbitrary choice of multisets with \( m, n \in \mathbb{N} \) and \( X, Y \), arbitrary sets. Let \( f = (u, p) : X^m \to Y^n \) be a mapping, the component mapping \( u : X \to Y \) is an ordinary map and \( p : m^* \to n^* \) is an OP map. Furthermore, a multiset map \( f \) is said to be \( u \)-injective (resp., \( u \)-surjective, \( u \)-bijective, \( p \)-injective, \( p \)-surjective, \( p \)-bijective) if \( u \) (resp., \( p \)) is an injective (resp., surjective, bijective) map. \( f \) is said to be \( u \)-injective (resp., \( u \)-surjective, \( u \)-bijective) and \( p \)-injective (resp., \( p \)-surjective, \( p \)-bijective).

Definition 4.2. For a multiset mapping \( f = (u, p) : X^m \to Y^n \), the image and preimage of multisets \( A \in X^m \), \( M \in Y^n \) are given, respectively, as

\[
f(A)(y) = \begin{cases} 
  p \left( \bigvee_{x \in u^{-1}(y)} A(x) \right) & \text{if } u^{-1}(y) \neq \phi \\
  0 & \text{otherwise}
\end{cases}
\]

\[
f^{-1}(M)(x) = \begin{cases} 
  \bigvee_{y \in u^{-1}(x)} p^{-1}(M(u(x))) & \text{if } p^{-1}(M(u(x))) \neq \phi \\
  0 & \text{otherwise}
\end{cases}
\]

Our notion is general one and does not restrict the size of \( Y \) and \( n \), thus a totally arbitrary choice of multiset spaces may be made to model a vast number of situations. Following example illustrates the necessary calculations involved in implementation of above definition.

Example 4.1. Let \( X = \{a, b, c, d\} \) and \( Y = \{s, t, x, y, z\} \). Consider \( f = (u, p) : X^4 \to Y^5 \) with

\[
u(a) = y, u(b) = y, u(c) = z, u(d) = s, \quad p(0) = 0, p(1) = 1, p(2) = 5, p(3) = 5, p(4) = 5
\]

Choose

\[
A = \{1/a, 4/b, 2/c, 4/d\} \in X^4, \quad M = \{1/s, 2/t, 1/x, 1/y, 5/z\} \in Y^5.
\]

Then calculations show

\[
f(A)(s) = p \left( \bigvee_{x' \in u^{-1}(s)} A(x') \right) = p \left( \bigvee A(d) \right) = p \left( \bigvee 4 \right) = p(4) = 5
\]

\[
f(A)(t) = 0 \text{ as } u^{-1}(t) = \phi
\]

\[
f(A)(x) = 0 \text{ as } u^{-1}(x) = \phi
\]

\[
f(A)(y) = p \left( \bigvee_{x' \in u^{-1}(y)} A(x') \right) = p \left( \bigvee_{x' \in \{a, b\}} A(x') \right) = p \left( \bigvee (A(a), A(b)) \right) = p \left( \bigvee (1, 4) \right) = p(4) = 5
\]

\[
f(A)(z) = p \left( \bigvee_{x' \in u^{-1}(z)} A(x') \right) = p \left( \bigvee A(c) \right) = p \left( \bigvee 2 \right) = p(2) = 5
\]
Hence \( f(A) = \{5/s, 0/t, 0/x, 5/y, 5/z\} \). Similarly for the preimage of \( M \) we have
\[
\begin{align*}
    f^{-1}(M)(a) &= \bigvee p^{-1}(M(u(a))) = \bigvee p^{-1}(M(y)) = \bigvee p^{-1}(1) = 1 \\
    f^{-1}(M)(b) &= \bigvee p^{-1}(M(u(b))) = \bigvee p^{-1}(M(y)) = \bigvee p^{-1}(1) = 1 \\
    f^{-1}(M)(c) &= \bigvee p^{-1}(M(u(c))) = \bigvee p^{-1}(M(z)) = \bigvee p^{-1}(5) = \bigvee (2, 3, 4) = 4 \\
    f^{-1}(M)(d) &= \bigvee p^{-1}(M(u(d))) = \bigvee p^{-1}(M(s)) = \bigvee p^{-1}(1) = 1 \\
    \Rightarrow f^{-1}(M) &= \{1/a, 1/b, 4/c, 1/d\}.
\end{align*}
\]

**Theorem 4.1.** For a multimap \( f = (u, p) : X^n \to Y^n \) and \( A, B \in X^n, M, N \in Y^n \), we have
\[
\begin{align*}
    &1) f(\emptyset) = \emptyset \\
    &2) f(\tilde{n}) \subseteq \tilde{n}, \text{equality holds if } f \text{ is } u\text{-injective} \\
    &3) f(A \cup B) = f(A) \cup f(B) \\
    &4) f(A \cap B) \subseteq f(A) \cap f(B), \text{ equality holds if } f \text{ is } u\text{-injective} \\
    &5) A \subseteq B \Rightarrow f(A) \subseteq f(B) \\
    &6a) f(A) \cap f(B) \subseteq f(A \cap B) \text{ if } f \text{ is } u\text{-surjective} \\
    &6b) f(A^c) \subseteq (f(A))^c \text{ if } f \text{ is } u\text{-injective} \\
    &7) \emptyset \subseteq f^{-1}(\emptyset) \\
    &8) \tilde{n} = f^{-1}(\tilde{n}) \\
    &9) f^{-1}(M \cup N) \subseteq f^{-1}(M) \cup f^{-1}(N), \text{ equality holds if } f \text{ is } p\text{-surjective} \\
    &10) f^{-1}(M) \cap f^{-1}(N) \subseteq f^{-1}(M \cap N), \text{ equality holds if } f \text{ is } p\text{-surjective} \\
    &11) M \subseteq N \Rightarrow f^{-1}(M) \subseteq f^{-1}(N) \text{ if } f \text{ is } p\text{-surjective} \\
    &12) f^{-1}(M^c) = (f^{-1}(M))^c \text{ if } f \text{ is } p\text{-bijective.} \\
    &13) A \subseteq f^{-1}(f(A)), \text{ equality holds if } f \text{ is } u\text{-injective} \\
    &14) f(f^{-1}(M)) \subseteq M, \text{ equality holds if } f \text{ is surjective} \\
    &15) f(A) \cap f(B) \Rightarrow A \cap B \text{ if } f \text{ is } p\text{-bijective. Bi-implication holds if } f \text{ is } u\text{-injective, } p\text{-bijective.}
\end{align*}
\]

**Proof.** We only prove (4, 5) and (9), other proofs are similar.

(4) We only consider the non-trivial case when \( u^{-1}(y) \neq \emptyset \). So \( f(A \cap B)(y) = \)
\[
\begin{align*}
    p \left( \bigvee_{x \in u^{-1}(y)} \min (A(x), B(x)) \right) &\leq p \left( \min \left( \bigvee_{x \in u^{-1}(y)} (A(x), B(x)) \right) \right) = \min \left( p \left( \bigvee_{x \in u^{-1}(y)} (A(x), B(x)) \right) \right) = \\
    \min \left( p \left( \bigvee_{x \in u^{-1}(y)} A(x) \right), p \left( \bigvee_{x \in u^{-1}(y)} B(x) \right) \right) &= (f(A) \cap f(B))(y). \text{ Hence we have } f(A \cap B) \subseteq f(A) \cap f(B).
\end{align*}
\]

(5) \( f(A) = p \left( \bigvee_{x \in u^{-1}(y)} A(x) \right) \leq p \left( \bigvee_{x \in u^{-1}(y)} B(x) \right) = f(B), \text{ since } A(x) \leq B(x) \forall x \in X. \)

(6a) For each \( y \in Y \), if \( f^{-1}(y) \) is not empty, then
\[
\begin{align*}
    f(A^c)(y) &= p \left( \bigvee_{x \in u^{-1}(y)} A^c(x) \right) = p \left( \bigvee_{x \in u^{-1}(y)} (m - A(x)) \right) = n - p \left( \bigvee_{x \in u^{-1}(y)} A(x) \right)
\end{align*}
\]
and
\[(f(A))^c(y) = \bigvee p^{-1}(n - M(u(x))) = \bigvee (p^{-1}(n) - p^{-1}(M(u(x))))\]
\[= p^{-1}(n) - \bigvee p^{-1}(M(u(x))) = m - \bigvee p^{-1}(M(u(x)))\]
therefore
\[f(A)^c(y) \geq (f(A))^c(y)\]

(9) \(f^{-1}(M \cup N)(x) = \bigvee p^{-1}(\max (M(u(x)), N(u(x))))(x) = \bigvee p^{-1}(\max (M(u(x)), N(u(x)))) \subseteq p^{-1}(\max M(u(x))) \bigvee p^{-1}(\max N(u(x))) = f^{-1}(M)(x) \bigvee f^{-1}(N)(x) = (f^{-1}(M) \cup f^{-1}(N))(x).\)

(11)
\[f^{-1}(M)(x) = \bigvee p^{-1}(M(u(x)))\]

since \(M \subseteq N, \bigvee p^{-1}(M(u(x))) \subseteq \bigvee p^{-1}(N(u(x)))\) for all \(x \in X\). Hence \(f^{-1}(M) \subseteq f^{-1}(N).\)

(13) \(f^{-1}(f(A)) (y) = \bigvee p^{-1}(f(A)(u(x))) = \bigvee p^{-1}\left(p\left(\bigvee_{z \in u^{-1}(u(x))} f^{-1}(M)(z)\right)\right) \geq A(x)\)

(14) If \(u^{-1}(y)\) is not empty,
\[f\left(f^{-1}(M)\right)(y) = p\left(\bigvee_{z \in u^{-1}(y)} f^{-1}(M)(x)\right) = p\left(\bigvee_{z \in u^{-1}(y)} \left(\bigvee p^{-1}(M(u(x)))\right)\right) \leq M(y).\]

Suitable counterexamples may be constructed to show the direction of inclusions.

**Example 4.2.** We show that (4), (6b), (13) and (14), are in general irreversible. For (4), (13) and (14) consider \(f = (u, p) : \{a, b, c, d\}^5 \rightarrow \{x, y, z\}^7\) where
\[u(a) = z, u(b) = z, u(c) = x, u(d) = y, \quad p(0) = 0, p(1) = 0, p(2) = 4, p(3) = 5, p(4) = 5, p(5) = 7\]
and choose
\[A = \{4/a, 0/b, 0/c, 4/d\}, \quad B = \{1/a, 2/b, 4/c, 4/d\}, \quad M = \{1/x, 2/y, 6/z\}\]

Then the calculations show
\[f(A) \cap f(B) = \{0/x, 5/y, 4/z\} \not\subseteq \{0/x, 5/y, 0/z\} = f(A \cap B)\]
\[f^{-1}(f(A)) = \{4/a, 4/b, 1/c, 4/d\} \not\subseteq \{4/a, 0/b, 0/c, 4/d\} = A\]
\[M = \{1/x, 2/y, 6/z\} \not\subseteq \{0/x, 0/y, 0/z\} = f\left(f^{-1}(M)\right)\]
Again for (6b) set $f = (u, p) : \{a, b, c, d\}^7 \to \{s, t, x, y, z\}^7$ to be a $u$-injective $p$-bijective map by choosing $u(a) = s, u(b) = z, u(c) = x, u(d) = y, \text{ and } p : m^* \to n^*$ is a bijective (hence identity) map.

Then for choosing $A = \{3/a, 2/b, 5/c, 1/d\}$ we have $(f(A))^c = \{4/s, 7/t, 2/x, 6/y, 5/z\} \not\subseteq \{4/s, 0/t, 2/x, 6/y, 5/z\} = f(A^c)$.

For any new generalization one naturally expects divergences from the previous classical theory. For if there are no divergences, generally such generalizations do not prove fruitful. There are many statements in Theorem 4.1 which are either diverging or even reversing the earlier classical results. Statements $(6b - 7, 9 - 11, 13 - 14)$ are unusual in this sense. We see these divergences as potential budding sites for new and richer developments of the theory of multiset computation.

5. **Interconnections**

This section interconnects the multiset mapping presented herein with other important notions. For easy referencing different types of mappings shall be referred by the names of their respective authors e.g. Parikh, Nazmul, Khomenko and Kharal mappings.

Kharal mappings bear many seminal links with other notions of multiset processing: (1) Kharal maps possess enough modeling capability to suitably model other important notions of mappings like representing Parikh and Khomenko mappings. (2) Kharal maps are generalized enough to include other mapping notions as special cases. Specifically the works of Hickman [8] and Nazmul et al. [17] are special cases of Kharal maps. (3) Kharal maps nicely interact with some of the naive measures of pattern recognition on multisets e.g. cardinality, distance and similarity.

5.1. **Kharal Representation of Parikh and Khomenko Mappings.** Parikh mappings (vectors) express properties of words of a context free language as numerical properties of vectors yielding some fundamental language-theoretic consequences. Parikh mapping is used in diverse areas of applications for example in text fingerprinting [1]. Certain shortcomings in the notion of Parikh mapping have also been pointed out in literature [10] [14]. For example it is noted that much information is lost in the transition from a word to a vector. That is why, a sharpening of the Parikh mapping, where more information is preserved than in the original Parikh mapping, was introduced in [13]. Different generalizations of the notion of Parikh Mapping have also been attempted [10].

In the following we establish the connection between Parikh mapping and Kharal mapping based upon multiset processing as a common denominator of the two proposals. The approach adopted is, as usual, to show how to define one formalism in terms of other and vice versa.

**Definition 5.1.** [18] Let $X = \{t_1, t_2, ..., t_k\}$ be a set with the order given by subscripts. The Parikh mapping of a multiset $A : X \to \mathbb{N}$ is denoted by $\psi(A)$ and is defined as $\psi(A) = (A(t_1), A(t_2), ..., A(t_k))$.

For a multiset space $X^m$ set $Y^n$ such that $n = 1$ and $Y = \bigcup_{t_i \in X} \{0, 0, \ldots, A(t_i), 0, \ldots, 0\} \mid 1 \leq i \leq k\}$
then for \( f : X^m \rightarrow Y^n \) define
\[
u(t_i) = (0, 0, \ldots, A(t_i), 0, \ldots, 0)\]
such that \( A(t_i) \) is at the \( i \)-th place
and \( p : m^* \rightarrow n^* = \{0, 1\} \) an identity mapping given as
\[
p(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{otherwise} \end{cases}
\]
Then
\[
\psi(A) = \sum f(A)
\]
where sum is the usual vector sum of members of \( Y \). Observe that \( f \) is \( u \)-injective \( p \)-bijective mapping. Also note that \( Y \) already incorporates all possible orders on \( X \) : for different orders on \( X \), only the assignments of \( u \) are to be changed. Following example illustrates the Kharal representation of a Parikh mapping. Note that many other possibilities may also be handled by Kharal mappings amongst which Parikh mapping is just one.

**Example 5.1.** \( X = \{a, b, c, d, e\}, m = 5 \). Choose \( A = [3/e, 4/a, 3/b, 0/d, 1/c] \in X^5 \), where square brackets denote that \( A \) is an ordered multiset. Then its Parikh mapping is given as \( \psi(A) = (3, 4, 3, 0, 1) \). Now we set
\[
Y = \{(0, 0, 0, 0, 0), (3, 0, 0, 0, 0), (0, 0, 3, 0, 0), (0, 0, 0, 3, 0), (0, 0, 0, 0, 3),
(4, 0, 0, 0, 0), (0, 4, 0, 0, 0), (0, 0, 0, 4, 0), (0, 0, 0, 0, 4),
(1, 0, 0, 0, 0), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0), (0, 0, 0, 1, 0), (0, 0, 0, 0, 1)\}
\]
where parenthesis denote ordered pairs. \( f = (u, p) : X^5 \rightarrow Y^1 \) is given as follows: Order of \( A \) forces following assignments for \( u \):
\[
u(a) = (0, 4, 0, 0, 0), \quad u(b) = (0, 0, 3, 0, 0), \quad u(c) = (0, 0, 0, 0, 1), \quad u(d) = (0, 0, 0, 0, 0), \quad u(e) = (3, 0, 0, 0, 0)
\]
\[
p(0) = 0, \quad p(1) = p(2) = p(3) = p(4) = p(5) = 1
\]
Then the calculations give
\[
f(A) = \{(0, 0, 0, 0, 0), 1/(0, 0, 0, 0, 1)/, 1/(0, 0, 3, 0, 0), 1/(0, 4, 0, 0, 0), 1/(3, 0, 0, 0, 0)\}
\]
Clearly we have
\[
\psi(A) = (3, 4, 3, 0, 1) = (0, 0, 0, 0, 0)+(0, 0, 0, 0, 1)+(0, 0, 0, 3, 0, 0)+(0, 4, 0, 0, 0)+(3, 0, 0, 0, 0) = \sum f(A)
\]
Another notion of mappings is defined and used in [12] by Khomenko, is as follows:

**Definition 5.2.** [12] Let \( A \) be a multiset over \( X \) and \( h : X \rightarrow Y \) is a mapping. Then the image \( h(A) \) is defined as
\[
h(A)(y) = \sum_{x \in X \land h(x) = y} A(x)
\]
Replacing \( \bigvee \) with \( \sum \) and choosing \( p : m^* \rightarrow n^* \) to be an identity mapping, one immediately sees above definition as a variation of Kharal mappings.
5.2. Hickman Mappings. Before considering Hickman’s mapping we have:

**Theorem 5.1.** Let \( f = (u, p) : X^m \to Y^m \) be a Kharal multiset mapping, \( M \in X^m \) and \( N = f(M) \). We have

1. If \( f \) is \( p \)-injective then \( M(x) \leq N(u(x)) \).
2. If \( \#X = \#Y \), and \( f \) is surjective then \( M(x) \geq N(u(x)) \), where \( \# \) denotes the cardinality of a set.

**Corollary 5.1.** If \( f \) is injective then \( M(x) \leq N(u(x)) \).

Hickman introduced following notion of mapping for various applications:

**Definition 5.3.** Let \( M, N \) be multisets. Define a multiset map \( s : M \to N \) to be a function \( \text{Dom}(M) \to \text{Dom}(N) \). We say that \( s \) is \( m \)-injective if \( s \) is injective and \( M(k) \leq N(s(k)) \) for each \( k \in M \) and that \( s \) is \( m \)-surjective if \( s \) is surjective and \( M(k) \geq N(s(k)) \) for each \( k \in M \). We say that \( s \) is \( m \)-bijective if \( s \) is \( m \)-injective and \( m \)-surjective.

One can easily note that Hickman’s \( s \) map is Kharal’s \( u \) map. Then an injective Kharal map \( f \) guarantees both conditions of Hickman’s \( m \)-injective mappings. Specifically, \( p \)-injectivity implies \( n \geq m \) (by Proposition 4.1.3) and \( \text{OP} \) property \( p(i) \geq p(i-1) \) assures \( M(x) \leq N(u(x)) \), by Corollary 5.1. Also a surjective Kharal mapping guarantees both conditions of Hickman’s \( m \)-surjective maps. Surjectivity of \( p \) assures \( m \geq n \) and by Theorem 5.1.2 we have \( M(x) \geq N(u(x)) \). It is clear from Hickman’s definition that \( \#\text{Dom}(M) = \#\text{Dom}(N) \) which is also assured by Theorem 5.1.2 as \( \#X = \#Y \).

Hickman’s notion is general enough in the sense that it does not restrict \( n \). But the definition is restrictive, first in the sense, that it requires \( \#\text{Dom}(M) = \#\text{Dom}(N) \), secondly, note that Kharal map affords many other variations as well e.g. \( f \) being \( u \)-injective \( p \)-surjective and \( f \) being \( u \)-surjective \( p \)-bijective etc.

5.3. Work of Nazmul et al. If we put \( n = m \) and \( p : m^* \to n^* \) is an identity map and setting \( f = u \) i.e. map \( f \) to be the same as \( u \), then the definition of Kharal maps reduces to

\[
    f(A)(y) = \begin{cases} 
    \bigvee_{x \in u^{-1}(y)} A(x) & \text{if } u^{-1}(y) \neq \phi \\
    0 & \text{otherwise}
    \end{cases}
\]

which is exactly the Nazmul mapping with a slight change of notation as they use symbol \( f \) in the role of \( u \) in Kharal mappings. Note that this notion of mapping restricts the codomain multiset space to be \( Y^m \) only, though \( Y \) may be arbitrary.

5.4. Distance and Similarity Measures.

**Definition 5.4.** The cardinality of a multiset \( A \in X^m \) is \( \#A = \sum_{x \in X} A(x) \). In the sequel we shall use the same symbol for cardinality of an ordinary set as well.
Definition 5.5. A mapping $S : X^m \times X^m \rightarrow [0, 1]$ is said to be similarity measure if it satisfies following axioms:

(s1) $0 \leq S(A, B) \leq 1,$
(s2) if $A = B,$ then $S(A, B) = 1,$
(s3) $S(A, B) = S(B, A),$  
(s4) if $A \subseteq B$ and $B \subseteq C,$ then $S(A, C) \leq S(A, B)$ and $S(A, C) \leq S(B, C).$

Definition 5.6. Distance and similarity between two multisets $A, B \in X^m$ are defined, respectively, as:

$$d(A, B) = \sqrt{\sum_{x \in X} (A(x) - B(x))^2}$$ and  
$$S(A, B) = \frac{1}{1 + d(A, B)},$$

where $d$ is a metric on $X^m.$

It is easy to check that $d : X^m \times X^m \rightarrow \mathbb{R}^+ \cup \{0\}$ and $S : X^m \times X^m \rightarrow [0, 1],$ as defined above, are respectively a metric and a similarity measure. It is also clear that diameter of $X^m$ i.e. the maximum distance between any two members of $X^m$ is given as

$$\text{dia}(X^m) = \sqrt{m \times \#X}$$

where $\#X$ is the cardinality of the ordinary set $X.$

Following result shows that Kharal mapping possesses some nice invariance properties with respect to cardinality, distance and similarity:

Theorem 5.2. Let $f : X^m \rightarrow Y^n$ be a Kharal mapping and $A, B \in X^m$

(1) If $d, S$ are a metric and similarity, respectively, on $X^m$ and $f$ is $u$-injective $p$-bijective, then we have

\begin{align*}
(i) & \quad d(A, B) = d(f(A), f(B)) \\
(ii) & \quad S(A, B) = S(f(A), f(B))
\end{align*}

(2) If $m > n$ and $f$ is $p$-surjective then $\#A \geq \#f(A).$

(3) If $f$ is injective then $\#A \leq \#f(A)$

6. Conclusion

This paper has addressed the problem of defining a suitable notion of mappings on multiset spaces. Main contribution of the paper is twofold: It first defines a mapping on multiset spaces and presents several of its properties and counter examples. Secondly, the new mapping has been shown to possess many nice properties in relation to pattern recognition measures of multisets like cardinality, distance and similarity. This mapping is further shown to encompass Parikh and Khomenko mappings through suitable representation schemes and Nazmul and Hickman mappings as its special cases. The mapping rewrites multisets and thus enables one to model paradigms like $P$ systems, Petri Nets, Abstract Rewriting on Multisets (ARMS) and Abstract Chemical Machines. The paper also gives several fundamental results. By defining the notion of constant multisets, it shows that set theoretic forms of Law of Excluded Middle and Law of Non-Contradiction do not hold for multisets. This is the motivation to introduce $m$-coincidence to handle disjoint multisets.
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