INTEGER SETS OF LARGE HARMONIC SUM WHICH AVOID LONG ARITHMETIC PROGRESSIONS

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Abstract. We give conditions under which certain digit-restricted integer sets avoid \( k \)-term arithmetic progressions. These sets are reasonably efficient to compute and therefore enable large-scale search. We identify a set with no arithmetic progression of four terms and with harmonic sum 4.43975, which improves an earlier “greedy” construction.

The Erdős–Turán conjecture on arithmetic progressions proposes that integer sets with divergent harmonic sums (so-called \textit{large} sets) must contain arithmetic progressions of arbitrary (finite) length. This conjecture is known to hold for integer sets of positive density (Szemerédi’s theorem, [Szem75]) and for the set of primes (the Green–Tao theorem, [GT08]).

Let \( S_k \) denote the collection of sets of positive integers which avoid arithmetic progressions of length \( k \). Such sets will be called \( k \)-free hereafter. The Erdős–Turán conjecture implies that the harmonic sum of any member of \( S_k \) is bounded. In fact, under Erdős–Turán, these harmonic sums would be \textit{uniformly} bounded as a function of \( k \), due to a result of Gerver [Ger77]. Let

\[
M_k := \sup_{T \in S_k} \sum_{t \in T} 1/t,
\]

which is finite for each \( k \) if and only if the Erdős–Turán conjecture holds. Recent work of Bloom–Sisask [BS19] shows that \( M_3 \) is finite, but the finiteness of \( M_k \) is otherwise unknown.

If finite, the growth rate of \( M_k \) represents a refinement to the Erdős–Turán conjecture. For this reason, lower bounds for \( M_k \) appear in several works. Berlekamp gave a construction in [Ber68] which proved \( M_k \geq \frac{1}{2} k \log 2 \). This was improved in [Ger77], which showed that for all \( \epsilon > 0 \), there exists cofinitely many \( k \) for which \( M_k > (1 - \epsilon) k \log k \).

Numerical lower bounds for \( M_k \) for small \( k \) have also received attention. The current record for \( M_3 \) is held by Wróblewski, who proved \( M_3 \geq 3.00849 \) by interlacing sets constructed through greedy algorithms and a denser 3-free packing due to Behrend [Beh46, Wró84].

Let \( G_k \) denote the lexicographically earliest \( k \)-free set. The sets \( G_k \) have reasonable large harmonic sums, especially when \( k \) is prime and \( G_k \) exhibits fractal self-similarity. When \( k \) is composite, the harmonic sums are less impressive. Heuristics from [GR79] suggest that \( G_4 \) and \( G_6 \) have harmonic...
sums of \(\approx 4.3\) and \(\approx 6.9\), respectively. Notably, the harmonic sum of \(G_6\) is predicted to be less than that of \(G_5\), which is 7.866.

This article provides a new construction for infinite \(k\)-free sets. Fix an integer \(b \geq 2\) and a proper subset of integers \(S \subset \{0, b - 1\}\). We define the Kempner set \(K(S, b)\) as the set of non-negative integers whose base-\(b\) digits are contained in \(S\). Kempner sets first appeared in [Kem14], and their arithmetic properties have been studied in [EMS98], [EMS99], and [May19]. The connection between Kempner sets and arithmetic progressions was first developed in [WW20]. In particular, [WW20] proved that every Kempner set is \(k\)-free for some \(k\).

Kempner sets are useful in the experimental study of \(M_k\) because the lengths of their longest arithmetic progressions are easy to compute and their harmonic sums are computable to arbitrary precision (due to an algorithm of Baillie–Schmelzer in [SB08]). Most importantly, they are also capable of producing large harmonic sums. For example, we show that the set
\[K\left(\{0, 1, 2, 4, 5, 7\}, 11\right) + 1 = \{1, 2, 3, 5, 6, 8, 12, 13, 14, 16, 17, 19, 23, 24, \ldots\}\]
is 4-free and has harmonic sum 4.421746. This simple set exceeds the estimated harmonic sum of \(G_4\) by a considerable margin and already sets a new lower bound for the supremum \(M_4\).

We describe and implement algorithms which use Kempner sets to search for lower bounds for \(M_k\). Even with pruning, this search is time-consuming: the number of Kempner sets \(K(S, b)\) grows exponentially in \(b\) and \(b\) must be taken large before interesting results are found. Our search is most successful in the case \(k = 4\), where our best result is the set
\[K\left(\{0, 1, 2, 4, 5, 9, 10, 11, 14, 16, 17, 18, 21, 24, 30, 37, 39, 41, 42, 45, 47\}, 55\right) + 1.\]
This set has harmonic sum 4.43975 and sets a new record among 4-free sets.

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1. Modular Arithmetic Progressions

A set \(S \subset [0, b - 1]\) is called an arithmetic progression mod \(b\) of length \(k\) if there exists an arithmetic progression \(A\) (in the ordinary sense) of length \(k\) and common difference \(\Delta\) for which \(A \mod b\) lies in \(S\) and \(b \nmid \Delta\). By extension, a set \(S \subset [0, b - 1]\) is called \(k\)-free mod \(b\) if it contains no arithmetic progressions mod \(b\) of length \(k\). Note that we do not require \(\gcd(\Delta, b) = 1\). This has some counter-intuitive implications; for example, it implies that \(\{1, 3, 5\}\) is an arithmetic progression mod 6 of infinite length.

One can test if a set \(S\) is \(k\)-free mod \(b\) by testing if the associated union of translates \(\bigcup_{j=0}^{n-1} (S + jb)\) has no increasing \(k\)-term arithmetic progression.
Theorem 2.1. Let harmonic sum of a (shifted) $k$-ing result shows that any lower bound for $M_k$.

There exist Remark 1.3. a smaller common difference. We conclude by infinite descent. □

Proof. For the sake of contradiction, suppose that $S \subset [0,M]$ is $k$-free but that $S$ admits a $k$-term arithmetic progression mod $b$ for some $b > 2M$.

To be precise, suppose that $B = \{c + \Delta j : j \in [1,k]\}$ is equivalent mod $b$ to a subset of $S$ and that $0 < \Delta < b$. For $j \leq k$, let $q_j$ and $r_j$ be the quotient and remainder of $c + \Delta j$ upon division by $b$. Our assumption on $\Delta$ implies that $q_{j+1}$ equals $q_j$ or $q_j + 1$. If $q_{j+1} = q_j$ for all $j$, then $\{r_j\}$ is an arithmetic progression in $S$ of length $k$, a contradiction. Thus $q_{j+1} = q_j + 1$ for some $j$, hence $\Delta \geq b - M$ since $r_j \in [0,M]$.

On the other hand, if $q_{j+1} = q_j + 1$ for all $j$, then $\{r_j\}$ is a (decreasing) arithmetic progression in $S$ of length $k$, again a contradiction. Thus $q_{j+1} = q_j$ for some $j$, hence $\Delta \leq M$. It follows that $b - M \leq \Delta \leq M$, hence $b \leq 2M$, which contradicts that $b > 2M$. □

Sets which are $k$-free mod $b$ can be used to produce $k$-free Kempner sets. This is made precise in the following.

Theorem 1.2. Fix $b \geq 3$. If $S \subset [0, b - 1]$ is $k$-free mod $b$ and $0 \in S$, then $K(S, b)$ is $k$-free.

Proof. Suppose that $S$ is $k$-free mod $b$ and that $K(S, b)$ contains the arithmetic progression $A = \{c + \Delta j : j \in [0, k-1]\}$. If $b \nmid \Delta$, then $A$ and therefore $S$ contains residues in an arithmetic progression mod $b$. Thus $S$ contains a progression mod $b$ of length $k$ or of infinite length, so $S$ is not $k$-free.

Alternatively, suppose that $b \mid \Delta$ and let $c_0$ denote the base-$b$ units digit of $c$. The arithmetic progression $(A - c_0)/b$ is contained in $K(S, b)$ and has a smaller common difference. We conclude by infinite descent. □

Remark 1.3. There exist $k$-free Kempner sets $K(S, b)$ for which $S$ is not $k$-free mod $b$. One simple example is the 3-free set $K\{(0, 2, 5), 7\}$. Examples like this rely on gaps in the digit set $S$ (to avoid ‘carrying’) and do not seem to produce large harmonic sums.

2. Harmonic Sums of (Shifted) Kempner Sets

We now turn our attention to the harmonic sums of Kempner sets. In general, let $H(S)$ denote the harmonic sum of the integer set $S$. The following result shows that any lower bound for $M_k$ can be approximated by the harmonic sum of a (shifted) $k$-free Kempner set,

Theorem 2.1. Let $S$ be $k$-free, with a convergent harmonic series. Given $\epsilon > 0$, there exists a $k$-free Kempner set $K$ such that $H(K + 1) > H(S) - \epsilon$. 

Proof. Fix $\epsilon > 0$ and choose $M$ such that $\mathcal{H}(S \cap [1, M]) > \mathcal{H}(S) - \epsilon$. Fix an integer $b > \max(k; 2M)$, so that $S \cap [1, M]$ and hence $(S \cap [1, M]) - \min(S)$ are $k$-free mod $b$ by Proposition 1.1. Then Proposition 1.2 implies that both $K = K(S \cap [1, M] - \min(S), b)$ and the shifted set $K + 1$ are $k$-free.

Yet $K + 1$ contains a copy of $S \cap [1, M]$, shifted by $1 - \min(S) \leq 0$ (i.e. held constant or decreased), hence $\mathcal{H}(K + 1) \geq \mathcal{H}(S \cap [1, M]) > \mathcal{H}(S) - \epsilon$. \hfill \Box

One of the reasons to study Kempner sets is that machinery exists due to [SB08] to compute harmonic sums of Kempner sets with great precision. There is one small difficulty, in that Kempner sets include 0. Rather than exclude 0, we opt to increase our sets termwise by 1. This shift affects harmonic sum in a way that can be addressed with the following lemma.

**Lemma 2.2.** Let $S$ be a set of positive integers and let $H_n$ denote the $n$th harmonic number. Then

$$\sum_{s \in S} \frac{1}{s + n} = \sum_{s \in S} \frac{1}{s} + \sum_{s \notin S} \frac{n}{s(s + n)} - H_n.$$  

Proof. Since $\sum_{s \in S} 1/s - \sum_{s \notin S} 1/(s + n) = \sum_{s \in S} n/(s^2 + ns)$, it suffices after rearranging to show that $H_n = \sum_{m=1}^{\infty} n/(m^2 + nm)$. To prove this last fact, we write the series on $m$ as a telescoping sum. \hfill \Box

3. Implementation

The Baillie–Schmelzer Algorithm described in [SB08] has been fully implemented in the Mathematica language and is freely available from the Wolfram Library Archive [BS08]. This is useful for fine-tuning pruned results but inefficient for larger searches because the Baillie–Schmelzer Algorithm is somewhat time-intensive.

As a complement to the Mathematica implementation of the Baillie–Schmelzer algorithm, the author wrote a family of search algorithms using the C++ language. The core algorithm is a branch-and-bound depth-first search through the subsets of $[0, b - 2]$ which are $k$-free mod $b$. More specifically, states are stored as pairs $(S, T)$, in which $S$ is a $k$-free set mod $b$ and $T$ is the set of possible extensions to $S$:

$$T = \{t \in [0, b - 2] : t > \max(S) \text{ and } S \cup \{t\} \text{ is } k\text{-free mod } b\}.$$  

An upper bound for the branch rooted at $(S, T)$ is then $\mathcal{H}(K(S \cup T, b) + 1)$.

Efficient estimates for the harmonic sums of the associated Kempner sets are obtained using a first-order approximation to the Baillie–Schmelzer algorithm. To be precise, one uses the approximation

$$\mathcal{H}(K(S, b) + 1) \approx \left(\frac{1}{1 - \#S/b}\right) \sum_{s \in S} \frac{1}{s + 1}$$

(3.1)

to rapidly estimate the fitness of a candidate set $S$. Branches whose approximate upper bounds lie below a threshold are pruned, and surviving sets are
recorded for further processing using the full Baillie–Schmelzer algorithm (and Lemma 2.2) in Mathematica.

4. 3-Free Kempner Sets of Large Harmonic Sum

A branch-and-bound search over 3-free sets mod $b \leq 120$ produces only a handfull of sets whose associated shifted Kempner sets have harmonic sums near that of $G_3$. We find 5095 candidate sets with estimated harmonic sum (via (3.1)) of at least 3.0. Of these, the ten of largest (actual) harmonic sum are reproduced in Table 1 below.

Table 1. Notable 3-free Kempner Sets for $b \leq 120$

| $\mathcal{H}(K + 1)$ | $b$ | $S$ |
|----------------------|-----|-----|
| 3.00794              | 3   | $\{0,1\}$ |
| 3.00118              | 82  | $\{0,1,3,4,9,10,12,13,27,28,30,31,36,37,39,40\}$ |
| 2.99461              | 83  | $\{0,1,3,4,9,10,12,13,27,28,30,31,36,37,39,40\}$ |
| 2.99312              | 81  | $\{0,1,3,4,9,10,12,13,27,28,30,31,36,37,39,67\}$ |
| 2.99260              | 81  | $\{0,1,3,4,9,10,12,13,27,28,30,31,36,37,40,66\}$ |
| 2.99146              | 81  | $\{0,1,3,4,9,10,12,13,27,28,30,31,36,39,40,64\}$ |
| 2.99083              | 81  | $\{0,1,3,4,9,10,12,13,27,28,30,31,37,39,40,58\}$ |
| 2.98823              | 81  | $\{0,1,3,4,9,10,12,13,27,28,30,36,37,39,40,57\}$ |

The non-$G_3$ Kempner sets listed in Table 1 above differ from $G_3$ in minor, deleterious ways. This suggests that $G_3$ is an influential local maximum and that far larger search spaces may be needed to improve lower bounds on $M_3$.

A similar story unfolds whenever we search for $p$-free sets with $p$ a prime: the greedy set $G_p = K([0, p - 2], p) + 1$ dominates early results and we find little else of interest. However, when $k$ is composite, $G_k$ has a much smaller harmonic sum and the best lower bound on $M_k$ typically comes from the set $[1, k - p] \cup (G_p + (k - p))$, in which $p$ is the largest prime less than $k$. This general construction is least impressive when $k$ is one less than a prime.

5. 4-Free Kempner Sets of Large Harmonic Sum

We next consider $k$-free sets in the case $k = 4$. Heuristics from [GR79] suggest the lower bound $M_4 > 4.3$, as derived from the greedy set $G_4$. (There is no known algorithm to compute $\mathcal{H}(G_4)$ to arbitrary precision; a lower bound from the first 10000 terms of $G_4$ gives $\mathcal{H}(G_4) > 4.19111$.) The lack of obvious structure in $G_4$ suggests that further improvements to $M_4$ may be within reach.

A branch-and-bound search over all 4-free sets mod $b \leq 60$ yields 109 sets with approximate harmonic sums (via (3.1)) of at least 4.5. The ten
sets of largest (actual) harmonic sum are compiled in Table 2. Our best result employs a 21-term 4-free set mod 55 and shows $M_4 \geq 4.43975$, which improves the heuristic record set by [GR79].

### Table 2. Notable 4-free Kempner Sets for $b \leq 60$

| $\mathcal{H}(\mathcal{K} + 1)$ | $b$ | $S$ |
|---------------------------------|-----|-----|
| 4.43975                         | 55  | \{0,1,2,4,5,9,10,11,14,16,17,18,21,24,30,37,39,41,42,45,47\} |
| 4.42175                         | 11  | \{0,1,2,4,5,7\} |
| 4.41989                         | 22  | \{0,1,2,4,5,7,8,9,14,17\} |
| 4.36437                         | 55  | \{0,1,2,4,7,8,9,13,14,16,17,18,26,28,31,32,34,36,43,49,52\} |
| 4.32651                         | 55  | \{0,1,2,4,5,7,8,13,14,16,17,18,26,28,32,34,36,43,49,52\} |
| 4.30467                         | 55  | \{0,1,2,4,5,9,10,11,14,16,17,18,21,24,30,37,39,41,42,47\} |
| 4.30139                         | 55  | \{0,1,2,4,5,9,10,11,14,16,17,18,21,24,30,37,39,41,45,47\} |
| 4.30021                         | 55  | \{0,1,2,4,5,9,10,11,14,16,17,18,21,24,30,37,39,41,45,47\} |
| 4.29770                         | 55  | \{0,1,2,4,5,9,10,11,14,16,17,18,21,24,30,37,39,41,45,47\} |
| 4.29497                         | 55  | \{0,1,2,4,5,9,10,11,14,16,17,18,21,24,30,37,39,41,45,47\} |

The author experimented with several types of constrained search spaces in an attempt to extend these results to larger moduli. These include:

a. Restriction to a single branch, like the branch rooted at \{0, 1, 2\}.

b. Restriction to those sets which deviate from a greedy construction for 4-free sets mod $b$ some bounded number of times.

c. Restrictions based on term-wise upper bounds for the elements of $S$.

Experimentation suggests that (c) offers the best balance between runtime efficiency and result quality as $b$ grows large. A series of constrained searches among bases $b \leq 200$ was unable to improve the $M_4$ bound 4.43975. The ten Kempner sets of largest harmonic sum found during these sparser searches are given in Table 3 below. (We omit several results with base $b = 121$, as these resemble but fail to improve the listed $b = 11$ case.)

### Table 3. Notable 4-free Kempner Sets of Larger Modulus

| $\mathcal{H}(\mathcal{K} + 1)$ | $b$ | $S$ |
|---------------------------------|-----|-----|
| 4.43975                         | 55  | \{0,1,2,4,5,9,10,11,14,16,17,18,21,24,30,37,39,41,42,45,47\} |
| 4.42175                         | 11  | \{0,1,2,4,5,7\} |
| 4.41989                         | 22  | \{0,1,2,4,5,7,8,9,14,17\} |
| 4.37406                         | 105 | \{0,1,2,4,5,7,8,9,15,16,17,18,19,20,25,26,28,29,31,32,33,36,45,50,51,59,61,63,68,70,72,79\} |
| 4.36953                         | 177 | \{0,1,2,4,5,7,8,9,15,16,17,19,20,26,27,29,30,32,33,34,50,52,55,56,57,59,62,63,64,66,72,75,76,79,87,90,93,101,103,107,109,113,126,133,137,146\} |
| 4.36437                         | 55  | \{0,1,2,4,7,8,9,13,14,16,17,18,26,28,31,32,34,36,43,49,52\} |

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Table 3. Continued from previous page

| $\mathcal{H}(K+1)$ | $b$ | $S$ |
|-----------------|-----|-----|
| 4.36280         | 153 | \{0,1,2,4,5,7,8,9,15,16,17,19,20,26,27,28,30,31,33,34,50,54,55,56,58,59,63,65,68,69,71,72,76,78,91,93,96,98,99,101,103\} |
| 4.36238         | 141 | \{0,1,2,4,5,7,8,9,14,16,17,18,26,28,29,31,32,33,36,37,39,51,52,53,56,57,58,60,61,68,69,70,72,86,94,95,96,129,130\} |
| 4.36233         | 153 | \{0,1,2,4,5,7,8,9,15,16,17,19,20,26,27,28,30,31,33,34,50,54,55,57,58,59,63,65,68,69,71,72,76,78,91,93,96,98,99,101,103\} |
| 4.36022         | 195 | \{0,1,2,4,5,7,8,9,15,16,18,19,20,25,26,28,29,31,32,33,45,49,51,52,53,54,59,63,67,68,72,79,80,82,84,87,90,98,102,104,108,110,112,118,120,122,130\} |

6. Other Notable Results

Limited searches with $k \in \{6, 8, 9, 10\}$ were unable to identify $k$-free sets with harmonic sums exceeding the trivial bounds $\mathcal{H}(\{1\} \cup (G_5 + 1)) = 7.94433$ ($k = 6$), $\mathcal{H}(\{1\} \cup (G_7 + 1)) = 13.5332$ ($k = 8$), $\mathcal{H}(\{1, 2\} \cup (G_7 + 1)) = 13.5638$ ($k = 9$), and $\mathcal{H}(\{1, 2, 3\} \cup (G_7 + 1)) = 13.5905$ ($k = 10$). These attempts came closest in the case $k = 10$, where the author produced the 10-free set

$\mathcal{K}(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 15, 16, 17, 18, 19, 20, 21, 22, 24, 26, 27, 28, 29, 30, 31, 33, 34, 35, 37, 38, 39, 42, 43, 46, 47, 49, 51, 52, 53, 54, 59\}, 61) + 1$

of harmonic sum $13.5865$.

Heuristically, we expect Kempner sets to have large harmonic sum when their digit set includes many digits (cf. (3.1)), i.e. when they have large logarithmic density. By combining a family of particularly dense 3-free packings of Behrend [Beh46] with Proposition 1.1, one may produce Kempner sets with logarithmic density arbitrarily near 1.

Kempner sets $\mathcal{K}(S, b)$ of this construction would require extremely large $b$, but we may nevertheless search for sporadic Kempner sets with small $b$ and unusually large logarithmic density $\delta(\mathcal{K}(S, b))$. By adapting our branch-and-bound algorithm, we can address this search as well. Some notable examples from our limited search are presented in Table 4 below.

Table 4. Some $k$-free Kempner Sets with Large $\delta(\mathcal{K})$

| $\delta(\mathcal{K})$ | $k$ | $b$ | $S$ |
|----------------------|----|----|-----|
| 0.63093              | 3  | 3  | \{0,1\} |
| 0.63767              | 3  | 37 | \{0,1,3,7,17,24,25,28,29,35\} |
| 0.63773              | 3  | 85 | \{0,1,3,4,9,10,13,24,28,29,31,36,40,42,50,66,73\} |

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In particular, the set $\mathcal{K}(\{0, 1, 3, 7, 17, 24, 25, 28, 35\}, 37)$ provides a simple and explicit example of a 3-free set surpassing the classical density $\log_3 2$.

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