Generalized Backward Doubly Stochastic Differential Equations
Driven by Lévy Processes with Continuous Coefficients

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Abstract  A new class of generalized backward doubly stochastic differential equations (GBDSDEs in short) driven by Teugels martingales associated with Lévy process are investigated. We establish a comparison theorem which allows us to derive an existence result of solutions under continuous and linear growth conditions.

Keywords  Backward doubly stochastic differential equations, Lévy processes, Teugels martingales, comparison theorem, continuous and linear growth conditions

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1 Introduction
Backward Stochastic Differential Equations (BSDEs) have been introduced (in the non-linear case) by Pardoux and Peng [1]. Originally, the study of the BSDEs has been motivated by its connection with partial differential equations (PDEs, in short). Indeed, BSDEs provide the probabilistic interpretation for solutions of both parabolic and elliptic semi-linear partial differential equations generalizing the well-known Feynman–Kac formula (see Pardoux and Peng [2] and Peng [3]). Very quickly this kind of equations has gained importance because of their many applications in the theory of mathematical finance (El Karoui et al. [4]), in stochastic control (El Karoui and Hamadène [5]) and stochastic games (Hamadène and Lepeltier [6]). Roughly speaking, BSDEs are in the form:

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \]

where \( f \) is the generator, \( \xi \) is the terminal value and \( W \) is the brownian motion. All of them are the given data. Denote by \( (\mathcal{F}_t)_{0 \leq t \leq T} \) the natural filtration generated by \( W \), the solution of BSDE \( (\xi, f) \) is the \( \mathcal{F}_t \)-adapted process \((Y, Z)\) satisfying (1.1) in a appropriate space. In [1], Pardoux and Peng derived existence and uniqueness result to BSDE \((\xi, f)\) under uniformly Lipschitz generator. They used the martingale representation theorem which is the main tool in the theory of BSDEs. A few years later, further researches weak the Lipschitz condition. Lepeltier and San Martin [7] studied BSDEs with continuous coefficients, Kobylianski [8] introduced BSDEs with the quadratic coefficients in \( z \), Briand and Carmona [9] considered BSDEs with polynomial growth generators.

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On the other hand, applying the idea used in [1], Pardoux and Peng introduced in [10] the so-called backward doubly stochastic differential equations (BDSDEs, in short). This kind of BDSDEs gives a probabilistic representation for a class of quasilinear stochastic partial differential equations (SPDEs, in short). Next, Bally and Matoussi [11] used also BDSDEs to give the probabilistic representation of the weak solutions of parabolic semi linear SPDEs in Sobolev spaces; Matoussi and Scheutzow [12] introduced another kind of BDSDEs to derive a probabilistic representation for the solution of SPDEs with nonlinear noise term given by the Itô–Kunita stochastic integral; Boufoussi et al. [13] recommended a class of generalized BDSDEs (GBDSDEs, in short) which involved an integral with respect to an adapted continuous increasing process and gave the probabilistic representation for stochastic viscosity solutions of semi-linear SPDEs with a Neumann boundary condition.

In [14], Nualart and Schoutens proved a martingale representation theorem associated to a Lévy process. This progress allows them to establish in [15] the existence and uniqueness result for BSDEs associated with a Lévy process. In continuation of all this works, Hu and Ren [16] showed existence and uniqueness result to GBDSDEs driven by Lévy process (GBDSDEL, in short) under Lipschitz on the generator. Moreover, the probabilistic interpretation for solutions of a class of stochastic partial differential integral equations (SPDIEs, in short) with a nonlinear Neumann boundary condition has been established.

In this note, we consider GBDSDEL

\[ Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T h(s, Y_s) dA_s + \int_t^T g(s, Y_s) d\bar{B}_s - \sum_{i=1}^\infty \int_t^T Z_s^{(i)} dH_s^{(i)}, \quad 0 \leq t \leq T. \]  

(1.2)

More precisely, we establish the existence result to BDSDEs (2.1) under continuous condition on the generators. The proof is strongly linked to the comparison theorem which does not hold in the general case (see [17] for BDSDE and the counter-example in [18] for BSDEs driven by Lévy processes). To overcome this difficulty, we assume relation (2.3) between the generator \( f \) and Lévy process \( L \) which have only \( m \) different jump size with no continuous part.

The rest of the paper is organized as follows. In Section 2, we introduce some preliminaries and deal with a comparison theorem for GBSDEL under Lipschtiz generators. Section 3 proves the existence result to GBDSDEs driven by Lévy processes under continuous generators.

2 Preliminaries

2.1 Notations and Definition

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space on which is defined all the processes stated in this paper and \( T \) be a fixed final time.

Let \( \{B_t; 0 \leq t \leq T\} \) be a standard Brownian motion, with values in \( \mathbb{R} \) and \( \{L_t; 0 \leq t \leq T\} \) be a \( \mathbb{R} \)-valued Lévy process independent of \( \{B_t; 0 \leq t \leq T\} \) corresponding to a standard Lévy measure \( \nu \) such that \( \int_\mathbb{R} (1 \wedge |y|) \nu(dy) < \infty \).

Let \( \mathcal{N} \) denote the class of \( \mathbb{P} \)-null sets of \( \mathcal{F} \). For each \( t \in [0, T] \), we define

\[ \mathcal{F}_t \overset{\Delta}{=} \mathcal{F}_t^L \lor \mathcal{F}_t^D, \]

where for any process \( \{\eta_t; F^n_{s,t} = \sigma(\eta_r - \eta_s; s \leq r \leq t) \lor \mathcal{N}, F^0_t = F^0_{0,t} \). Note that
Similarly, \( \mathcal{S}_{1}^{2}, t \in [0, T] \) is an increasing filtration and \( \mathcal{F}_{T}^{0}, t \in [0, T] \) is a decreasing filtration. Thus the collection \( \{ \mathcal{F}_{t}, t \in [0, T] \} \) is neither increasing nor decreasing so it does not constitute a filtration.

In the sequel, \( \{ A_{t}; 0 \leq t \leq T \} \) is an \( \mathcal{F}_{t} \)-measurable, continuous and increasing real valued process such that \( A_{0} = 0 \).

Let us introduce some spaces:

For any \( m \geq 1 \), \( \mathcal{M}^{2}(0, T, \mathbb{R}^{m}) \) denotes the space of \( \mathbb{R}^{m} \)-valued random process satisfying:

(i) \( \| \varphi \|_{\mathcal{M}^{2}(\mathbb{R}^{m})}^{2} = \sum_{i=1}^{m} \mathbb{E}( \int_{0}^{T} \| \varphi_{t}^{(i)} \|^{2} dt ) < \infty \);

(ii) \( \varphi \) is \( \mathcal{F}_{t} \)-measurable, for any \( t \in [0, T] \).

Similarly, \( \mathcal{S}^{2}(0, T) \) stands for the set of real-valued random processes which satisfy:

(i) \( \| \varphi \|_{\mathcal{S}^{2}}^{2} = \mathbb{E}( \sup_{0 \leq t \leq T} | \varphi_{t} |^{2} ) < \infty \);

(ii) \( \varphi_{t} \) is \( \mathcal{F}_{t} \)-measurable, for any \( t \in [0, T] \).

\( \mathcal{A}^{2}(0, T) \) denotes the set of (class of \( d\mathbb{P} \otimes dA_{t} \) a.e. equal) real-valued measurable random processes \( \{ \varphi_{t}; 0 \leq t \leq T \} \) such that

(i) \( \| \varphi \|_{\mathcal{A}^{2}}^{2} = \mathbb{E}( \int_{0}^{T} | \varphi_{t} |^{2} dA_{t} ) < \infty \);

(ii) \( \varphi_{t} \) is \( \mathcal{F}_{t} \)-measurable, for any \( t \in [0, T] \).

The space \( \mathcal{E}_{m}(0, T) = (\mathcal{S}^{2}(0, T) \cap \mathcal{A}^{2}(0, T)) \times \mathcal{M}^{2}(0, T, \mathbb{R}^{m}) \) endowed with norm

\[
\| (Y, Z) \|_{\mathcal{E}_{m}}^{2} = \mathbb{E} \left( \sup_{0 \leq t \leq T} |Y_{t}|^{2} + \int_{0}^{T} |Y_{t}|^{2} dA_{t} + \int_{0}^{T} |Z_{t}|^{2} ds \right)
\]

is a Banach space.

Furthermore, let us consider the Teugels Martingale \( (H^{(i)})_{i \geq 1} \) associated with the Lévy process \( \{ L_{t}; 0 \leq t \leq T \} \) defined by

\[
H_{t}^{(i)} = c_{i,1}T_{t}^{(i)} + c_{i,i-1}T_{t}^{(i-1)} + \cdots + c_{i,1}T_{t}^{(1)},
\]

where \( T_{t}^{(i)} = L_{t}^{(i)} - \mathbb{E}(L_{t}^{(i)}) = L_{t}^{(i)} - t\mathbb{E}(L_{t}^{(i)}) \) for all \( i \geq 1 \). Let us remark that the process \( L_{t}^{(i)} \) has power jump, for all \( i \geq 1 \). More precisely, denoting \( \Delta L_{s} = L_{s} - L_{s-} \), we have

\[
L_{t}^{(1)} = L_{t} \quad \text{and} \quad L_{t}^{(i)} = \sum_{0 \leq s \leq t} (\Delta L_{s})^{i} \quad \text{for} \quad i \geq 2.
\]

In [15], Nualart and Schoutens proved that the coefficients \( c_{i,k} \) correspond to the orthonormalization of the polynomials \( 1, x, x^{2}, \ldots \) with respect to the measure \( \mu(dx) = x^{2} \nu(dx) + \sigma^{2} \delta_{0}(dx) \), i.e.,

\[
g_{i}(x) = c_{i,i}x^{i-1} + c_{i,i-1}x^{i-2} + \cdots + c_{i,1}.
\]

The martingale \( (H^{(i)})_{i \geq 1} \) can be chosen to be pairwise strongly orthonormal martingale. That is, for all \( i, j \), \( (H^{(i)}, H^{(j)})_{t} = \delta_{ij}t \).

**Remark 2.1** Since the Lévy process \( L \) has only \( m \) different jump size with no continuous part, the Teugels martingales \( H^{(i)} = 0, \forall i \geq m + 1 \). In this context, BDSDEs (1.2) can be written rigorously as

\[
Y_{t} = \xi + \int_{t}^{T} f(s, Y_{s-}, Z_{s}) ds + \int_{t}^{T} h(s, Y_{s-}) dA_{s} + \int_{t}^{T} g(s, Y_{s-}) d\bar{B}_{s} - \sum_{i=1}^{m} \int_{t}^{T} Z_{s}^{(i)} dH_{s}^{(i)}, \quad 0 \leq t \leq T.
\]  

(2.1)

**Definition 2.2** A pair of \( \mathbb{R} \times \mathbb{R}^{m} \)-valued process \( (Y, Z) \) is called a solution of GBDSDEL \( (\xi, f, g, h, A) \) driven by Lévy processes if \( (Y, Z) \in \mathcal{E}_{m}(0, T) \) and verifies (2.1).
2.2 GBDSDE with Lipschitz Coefficients

For memory, we recall the existence and uniqueness result for GBDSDEL under Lipschitz condition due to Hu and Ren [16]. Here, the function $g$ depends on $z$ and we have the following assumptions:

(A1) The terminal value $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R})$ such that for all $\lambda > 0$, $\mathbb{E}(e^{\lambda \mathcal{A} T} |\xi|^2) < \infty$;

(A2) The generators $f, g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$ and $h : \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}$ satisfy, for $\beta_1, \beta_2 \in \mathbb{R}$, $K > 0$, $0 < \alpha < 1$ and three $\mathcal{F}_t$-measurable processes $\{f_t, g_t, h_t : 0 \leq t \leq T\}$ with values in $[1, \infty]$ and for all $(t, y, z) \in \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m$, $\lambda > 0$,

(i) $f(\cdot, y, z), g(\cdot, y, z)$ and $h(\cdot, y)$ are jointly measurable,

(ii) $|f(t, y, z)| \leq f_t + K(|y| + ||z||)$,

(iii) $\|g(t, y, z)\| \leq g_t + K(|y| + ||z||)$,

(iv) $|h(t, y)| \leq h_t + K|y|$,

(v) $\beta_2 < 0$,

(A3) $|f(t, y, z) - f(t, y', z')|^2 \leq K(|y - y'|^2 + ||z - z'||^2)$,

$|g(t, y, z) - g(t, y', z')|^2 \leq K|y - y'|^2 + \alpha||z - z'||^2$,

$|h(t, y) - h(t, y')|^2 \leq K|y - y'|^2$.

Theorem 2.3 (Hu and Ren [16]) Under the assumptions (A1)–(A3), the GBDSDEL (2.1) has a unique solution.

Remark 2.4 (i) Whenever $(Y_t, Z_t)$ satisfies (2.1), $(\hat{Y}_t, \hat{Z}_t) = (e^{\lambda A_t} Y_t, e^{\lambda A_t} Z_t)$ satisfies an analogous GBDSDEL, with $f$, $g$ and $h$ replaced by

$\hat{f}(t, y, z) = e^{\lambda A_t} f(t, e^{-\lambda A_t} y, e^{-\lambda A_t} z),$  

$\hat{g}(t, y, z) = e^{\lambda A_t} g(t, e^{-\lambda A_t} y, e^{-\lambda A_t} z),$  

$\hat{h}(t, y) = e^{\lambda A_t} h(t, e^{-\lambda A_t} y) - \lambda y.$

Hence, if $h$ satisfies (iv) with a possibly non-negative $\beta_2$, we can always choose $\lambda$ such that $\hat{h}$ satisfies (iv) with a strictly negative $\beta_2$. Consequently, (v) is not a severe restriction.

(ii) To assure the existence and uniqueness of the solution to the GBDSDEL (2.1), there is no need to have the assumptions (iv) and (v) of (A2). It is just needed to simplify the calculation in the proof of a priori estimate.

2.3 Comparison Theorem

The comparison theorem is one of the principal tools in the theory of the BSDEs which does not hold in general for BSDEs with jumps (see the counter-example in Buckdahn et al. [18]). With an additional property of the jumps size (2.3) as in [19], we derive the comparison theorem for GBDSDEs driven by Lévy processes under Lipschitz condition which generalizes the work of Shi et al. [20] for GBDSDEs with non jumps. In this fact, given $\xi^k$ and $f^k, h^k, g$ for $k = 1, 2,$
we consider
\[ Y_t^k = \xi_t^k + \int_t^T f_k(s, Y_s^k, Z_s^k)ds + \int_t^T h_k(s, Y_s^k)dB_s + \int_t^T g(s, Y_s^k)d\tilde{B}_s - \sum_{i=1}^m \int_t^T Z_s^{(i)}dh_s^{(i)}, \quad t \in [0, T]. \] (2.2)

Under assumptions (A1), (A2) and (A3), it follows from Theorem 2.3 that \((Y^k, Z^k)\) is a unique solution of BDSDEL (2.2).

**Theorem 2.5** Assume (A1)–(A3) and let \((Y^1, Z^1)\) and \((Y^2, Z^2)\) be the solutions of equations (2.2) for \(k = 1, 2\). We suppose that

- \(\xi^1 \geq \xi^2, \mathbb{P}\text{-a.s.},\)
- \(f^1(t, y, z) \geq f^2(t, y, z), \) and \(h^1(t, y) \geq h^2(t, y) \mathbb{P}\text{-a.s.}, \) for all \((t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m,\)
- \(\beta^i_t = f^1(t, y^2, \bar{z}^{(i-1)}) - f^1(t, y^2, \bar{z}^{(i)}) \mathbb{1}_{\{\bar{z}^{(i)} \neq \bar{z}^{(i)}\}},\)

where \(\bar{z}^{(i)} = (z^{(2)}(1), z^{(2)}(2), \ldots, z^{(i)}(i+1), \ldots, z^{(m)}(i))\) such that
\[ \sum_{i=1}^m \beta^i_t \Delta H_t^{(i)} > -1, \quad dt \otimes d\mathbb{P}\text{-a.s.} \] (2.3)

Then, we have for all \(t \in [0, T], Y^1_t \geq Y^2_t, \) a.s.

Moreover, for all \((t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m,\) if \(\xi^1 > \xi^2, \) or \(f^1(t, y, z) > f^2(t, y, z), \) or \(h^1(t, y) > h^2(t, y), \) a.s., \(Y^1_t > Y^2_t, \) a.s., \(\forall t \in [0, T].\)

**Proof** Set
\[ a_t = \frac{f^1(t, Y^1_t, Z^1_t) - f^1(t, Y^2_t, Z^1_t)}{(Y^1_t - Y^2_t)\mathbb{1}_{\{Y^1_t \neq Y^2_t\}}}, \]
\[ b_t = \frac{h^1(t, Y^1_t) - h^1(t, Y^2_t)}{(Y^1_t - Y^2_t)\mathbb{1}_{\{Y^1_t \neq Y^2_t\}}}, \]
\[ c_t = \frac{g(s, Y^1_s) - g(s, Y^2_s)}{(Y^1_t - Y^2_t)\mathbb{1}_{\{Y^1_t \neq Y^2_t\}}} \]

Next, it follows from (A3) that the processes \((a_t)_{t \in [0, T]}, (b_t)_{t \in [0, T]} \) and \((c_t)_{t \in [0, T]} \) are measurable and bounded.

Therefore, for \(0 \leq s \leq t \leq T,\) the linear BDSDE
\[ \Gamma_{s, t} = 1 + \int_s^t \Gamma_{s, r} \cdot dX_r \]

with
\[ X_t = \int_0^t a_r dr + \int_0^t b_r dA_r + \int_0^t c_r dB_r + \sum_{i=1}^m \int_0^t \beta^i_r dH_r^{(i)} \]

has (cf. Doléans-Dade exponential formula) a unique \(\mathcal{F}_t\)-measurable solution
\[ \Gamma_{s, t} = \exp \left( \int_s^t a_r dr + \int_s^t b_r dA_r + \int_s^t c_r dB_r - \frac{1}{2} \int_s^t |c_r|^2 dr \right) \]
\[ \times \prod_{s < r \leq t} \left( 1 + \sum_{i=1}^m \beta_r^i \Delta H_r^{(i)} \right) \exp \left( -\sum_{i=1}^m \beta_r^i \Delta H_r^{(i)} \right). \] (2.4)
Furthermore, denoting \( \tilde{\xi} = \xi_1 - \xi_2 \), \( \tilde{Y}_t = Y_t^1 - Y_t^2 \), \( \tilde{Z}_t = Z_t^1 - Z_t^2 \), \( \tilde{f}_t = f^1(t, Y_t^2, Z_t^2) - f^2(t, Y_t^1, Z_t^1) \) and \( \tilde{h}_t = h^1(t, Y_t^2) - h^2(t, Y_t^1) \), we have
\[
\begin{align*}
\tilde{Y}_t &= \tilde{\xi} + \int_t^T a_s \tilde{Y}_s - \sum_{i=1}^m \beta_s Z_s^{(i)} + \tilde{f}_s \, ds + \int_t^T [b_s \tilde{Y}_s + \tilde{h}_s] \, dA_s + \int_t^T c_s \tilde{Y}_s \, d\tilde{B}_s \\
- \sum_{i=1}^m \int_t^T \tilde{Z}_s^{(i)} \, dH_s^{(i)} \quad t \in [0, T].
\end{align*}
\]

Itô’s formula to \( \Gamma_{s,T} Y_r \) from \( r = t \) to \( r = T \) provides
\[
\begin{align*}
\Gamma_{s,t} \tilde{Y}_t &= \Gamma_{s,T} \tilde{\xi} - \int_t^T \left[ \Gamma_{s,r} \tilde{Y}_r - \int_t^T \tilde{Y}_r \, d\Gamma_{s,r} - \int_t^T d[\Gamma, \tilde{Y}]_r \right] \, dr \\
&= \Gamma_{s,T} \tilde{\xi} + \int_t^T \Gamma_{s,r} \left[ a_r \tilde{Y}_r - \sum_{i=1}^m \beta_r Z_r^{(i)} + \tilde{f}_r \right] \, dr + \int_t^T \Gamma_{s,r} - [b_r \tilde{Y}_r + \tilde{h}_r] \, dA_r \\
&+ \int_t^T \Gamma_{s,r} - c_r \tilde{Y}_r \, d\tilde{B}_r - \sum_{i=1}^m \int_t^T \Gamma_{s,r} - \tilde{Z}_r^{(i)} \, dH_r^{(i)} \\
- \int_t^T \tilde{Y}_r - \Gamma_{s,r} - b_r \, dA_r - \int_t^T \tilde{Y}_r - \Gamma_{s,r} - c_r \, d\tilde{B}_r + \int_t^T \tilde{Y}_r - \Gamma_{s,r} - |c_r|^2 \, dr \\
- \sum_{i=1}^m \int_t^T \tilde{Y}_r - \Gamma_{s,r} - \beta_r Z_r^{(i)} \, dr - \int_t^T \tilde{Y}_r - \Gamma_{s,r} - |c_r|^2 \, dr - \int_t^T \sum_{i=1}^m \Gamma_{s,r} - \beta_r Z_r^{(i)} \, dr \\
&= \Gamma_{s,T} \tilde{\xi} + \int_t^T \Gamma_{s,r} \tilde{f}_r \, dr + \int_t^T \Gamma_{s,r} \tilde{h}_r \, dA_r - \sum_{i=1}^m \int_t^T \Gamma_{s,r} - \tilde{Z}_r^{(i)} \, dH_r^{(i)}.
\end{align*}
\]

Taking a conditional expectation w.r.t. \( \mathcal{F}_s \), it is not hard to see that for \( s = t \),
\[
\tilde{Y}_t = \mathbb{E} \left( \Gamma_{t,T} \tilde{\xi} + \int_t^T \Gamma_{t,r} \tilde{f}_r \, dr + \int_t^T \Gamma_{t,r} \tilde{h}_r \, dA_r \mid \mathcal{F}_t \right).
\]

Since, according to (2.3), the process \( \Gamma_{t,r} \) is strictly positive, we obtain \( \tilde{Y}_t \geq 0 \), a.s., i.e., \( Y_t^1 \geq Y_t^2 \), a.s. Moreover if \( \tilde{\xi} > 0 \), a.s. or \( \tilde{f}_t > 0 \), a.s., or \( \tilde{h}_t > 0 \), a.s., then \( \tilde{Y}_t > 0 \), a.s., i.e., \( Y_t^1 > Y_t^2 \), a.s.

3 **GBDSDEL with Continuous Coefficients**

In this section, we study the GBDSDEL under the continuous and linear growth condition on the coefficients. Roughly speaking, we prove the existence of a minimal or maximal solution by the well-known approximation method of the functions \( f \) and \( h \) (Lemma 3.2) and the comparison theorem (Theorem 2.5).

In addition, we give the following assumptions:

(H1) The terminal value \( \xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{R}) \) such that for all \( \lambda > 0 \), \( \mathbb{E}(e^{\lambda A_T} | \xi|^2) < \infty \);

(H2) The coefficients \( f: \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R} \) and \( g, h: \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R} \), satisfy, for some constants \( \beta_1 \in \mathbb{R}, \beta_2 < 0, K > 0 \) and three \( \mathcal{F}_t \)-measurable processes \{\( f_t, g_t, h_t : 0 \leq t \leq T \)\},

(i) \( f(\cdot, y, z), g(\cdot, y) \) and \( h(\cdot, y) \) are jointly measurable,

(ii) \( |f(t, y, z)| \leq f_t + K|y| + ||z||, |f(t, y, z) - f(t, y, z')| \leq K ||z - z'||, \forall y \in \mathbb{R}, \)

(iii) \( |h(t, y)| \leq h_t + K|y|, \) for some \( K > 0, \)

(iv) \( \mathbb{E}(\int_0^T e^{\mu t + \lambda A_t} f_t^2 \, dt + \int_0^T e^{\mu t + \lambda A_t} g_t^2 \, dt + \int_0^T e^{\mu t + \lambda A_t} h_t^2 \, dA_t) < \infty, \) for all \( \mu, \lambda > 0, \)
(v) \(|g(t,y) - g(t,y')|\leq K|y - y'|^2;
(vi) \(y \mapsto f(t,y,z)\) and \(y \mapsto h(t,y)\) are continuous for all \(z,\omega,t\).

The main result of this paper is the following theorem.

**Theorem 3.1** Under the assumptions (H1) and (H2), the GBDSDEL (2.1) has a solution \((Y,Z) \in E_m(0,T)\) which is a minimal one, in the sense that, if \((Y^*,Z^*)\) is any other solution, we have \(Y^* \leq Y\), a.s..

To prove this theorem, we need an important result which gives an approximation of continuous functions by Lipschitz functions (see Lepeltier and San Martin [7] to appear for the proof).

**Lemma 3.2** Let \(\phi : [0,T] \times \mathbb{R}^p \to \mathbb{R}\) be a continuous function with linear growth, that is, there exists a constant \(K > 0\) such that \(\forall x \in \mathbb{R}^p, |\phi(t,x)| \leq \phi_t + K \|x\|\). Then the sequence of functions

\[
\phi_n(t,x) = \inf_{y \in \mathbb{R}^p} \{\phi(t,y) + n|x-y|\}
\]

is well defined for \(n \geq K\) and satisfies

(a) Linear growth: \(\forall (t,x) \in \times \mathbb{R}^p, |\phi_n(t,x)| \leq \phi_t + K \|x\|\),
(b) Monotonicity: \(\forall (t,x) \in \times \mathbb{R}^p, \phi_n(t,x) \nearrow\),
(c) Lipschitz condition: \(\forall t \in [0,T], x, y \in \mathbb{R}^p, |\phi_n(t,x) - \phi_n(t,y)| \leq n \|x-y\|\),
(d) Strong convergence: if \(x_n \to x\) as \(n \to \infty\), then \(\phi_n(t,x_n) \to \phi(t,x)\) as \(n \to \infty\) for all \(t\).

**Proof of Theorem 3.1** For fixed \((t,\omega)\), it follows from (H2) that \(f(t,\omega)\) and \(h(t,\omega)\) are continuous and are with linear growth. Hence, by Lemma 3.2, there exist sequences of functions \(f_n(t,\omega)\) and \(h_n(t,\omega)\) associated with \(f\) and \(h\), respectively. Then \(f_n, h_n\) are measurable functions as well as Lipschitz functions. Moreover, since \(\xi\) satisfies (H1), we get from Hu and Ren [16] that there is a unique pair \(\{(Y_n^n, Z_n^n)\}, 0 \leq t \leq T\) of \(\mathcal{F}_t\)-measurable processes taking values in \(\mathbb{R} \times \mathbb{R}^m\) and satisfying

\[
Y_n^t = \xi + \int_t^T f_n(s,Y_{s^-}^n, Z_{s^-}^n)ds + \int_t^T h_n(s,Y_{s^-}^n)dA_s + \int_t^T g(s,Y_{s^-}^n)d\overline{B}_s
- \sum_{i=1}^m \int_t^T Z_{s}^{n(i)}dH_{s}^{(i)}, \quad t \in [0,T],
\]

and

\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_0^T \|Z_s^n\|^2ds\right) < \infty.
\]

Since for fixed \((t,\omega)\), \(f_{n+1}(t,\omega) \geq f_n(t,\omega), h_{n+1}(t,\omega) \geq h_n(t,\omega)\) and inequality (2.3) still holds, for all \(n \geq K\), it follows from the comparison theorem (Theorem 2.5) that for every \(n \geq K\),

\[
Y^n \leq Y^{n+1}, \quad dt \otimes d\mathbb{P}\text{-a.s.}
\]

(3.2)

The idea of the proof of Theorem 3.1 is to establish that the limit of the sequence \((Y^n, Z^n)\) is a solution of the BDSDE (2.1). It follows by the same steps and technics as in [21] (see Theorem 3.1).

**Step 1** A priori estimates.

There exists a constant \(C > 0\) independent of \(n\) such that

\[
\sup_{n \geq K} \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_t^n|^2 + \int_0^T \|Z_t^n\|^2dt\right) \leq C.
\]

(3.3)
Indeed, for any \( \mu, \lambda > 0 \), Itô’s formula applied to \( e^{\mu t+\lambda A_t} |Y^n_t|^2 \) provides
\[
e^{\mu t+\lambda A_t} |Y^n_t|^2 + \lambda \int_t^T e^{\mu s+\lambda A_s} |Y^n_s|^2 dA_s + \mu \int_t^T e^{\mu s+\lambda A_s} |Y^n_s|^2 ds = e^{\mu T+\lambda A_T} |\xi|^2 + 2 \int_t^T e^{\mu s+\lambda A_s} Y^n_s f_n(s, Y^n_s, Z^n_s) ds + 2 \int_t^T e^{\mu s+\lambda A_s} Y^n_s g(s, Y^n_s) dB_s + 2 \int_t^T e^{\mu s+\lambda A_s} Y^n_s h_n(s, Y^n_s) dA_s - 2 \sum_{i=1}^m \int_t^T e^{\mu s+\lambda A_s} Y^n_s Z^n_s(i) dH^{(i)}(s) + \int_t^T e^{\mu s+\lambda A_s} |g(s, Y^n_s)|^2 ds - \sum_{i,j=1}^m \int_t^T e^{\mu s+\lambda A_s} Z^n_s(i) Z^n_s(j) d[H^{(i)}(s), H^{(j)}(s)]. \tag{3.4}
\]

Assumption (H2) together with Young’s inequality implies, for any \( \sigma > 0 \) and \( \gamma > 0 \),
\[
2Y^n_s f_n(s, Y^n_s, Z^n_s) \leq \left( 1 + 2K + \frac{1}{\sigma} K^2 \right) |Y^n_s|^2 + \sigma \|Z^n_s\|^2 + f^2_s,
\]
\[
2Y^n_s h_n(s, Y^n_s) \leq (2K + 1) |Y^n_s|^2 + h^2_s,
\]
\[
|g(s, Y^n_s)|^2 \leq (1 + \gamma) C |Y^n_s|^2 + \left( 1 + \frac{1}{\gamma} \right) g^2_s.
\]

Therefore, taking expectation in both sides of (3.4) with the suitable \( \lambda \) and \( \sigma \), we have
\[
\mathbb{E}\left( e^{\mu t+\lambda A_t} |Y^n_t|^2 + \int_t^T e^{\mu s+\lambda A_s} \|Z^n_s\|^2 ds \right) \leq C \mathbb{E}\left( e^{\mu T+\lambda A_T} |\xi|^2 + \int_t^T e^{\mu s+\lambda A_s} h^2_s dA_s + \int_t^T e^{\mu s+\lambda A_s} (f^2_s + g^2_s) ds \right) < \infty,
\]
which, by Burkholder–Davis–Gundy’s inequality, provides
\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} e^{\mu t+\lambda A_t} |Y^n_t|^2 + \int_t^T e^{\mu s+\lambda A_s} \|Z^n_s\|^2 ds \right) \leq C \mathbb{E}\left( e^{\mu T+\lambda A_T} |\xi|^2 + \int_0^T e^{\mu s+\lambda A_s} h^2_s dA_s + \int_t^T e^{\mu s+\lambda A_s} (f^2_s + g^2_s) ds \right) < \infty.
\]

Step 2 Convergence result.

We have the existence of process \( Y \) from (3.2) and (3.3) such that \( Y^n \to Y \) a.s. for all \( t \in [0, T] \). Hence, it follows from Fatou’s lemma together with the dominated convergence theorem that
\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |Y_t| \right) \leq C \quad \text{and} \quad \mathbb{E}\left( \int_0^T |Y^n_s - Y_s|^2 (ds + dA_s) \right) \to 0 \tag{3.5}
\]
as \( n \) goes to infinity. Next, for all \( n \geq n_0 \geq K \), it follows from Itô’s formula, taking \( t = 0 \),
\[
\mathbb{E}|Y^n_0 - Y^{n+1}_0|^2 + \mathbb{E} \int_0^T \|Z^n_s - Z^{n+1}_s\|^2 ds
\]
\[
= 2 \mathbb{E} \int_0^T (Y^n_s - Y^{n+1}_s) (f_n(s, Y^n_s, Z^n_s) - f_{n+1}(s, Y^{n+1}_s, Z^{n+1}_s)) ds
\]
\[
+ 2 \mathbb{E} \int_0^T (Y^n_s - Y^{n+1}_s) (h_n(s, Y^n_s) - h_{n+1}(s, Y^{n+1}_s)) dA_s
\]
\[
+ \mathbb{E} \int_0^T |g(s, Y^n_s) - g(s, Y^{n+1}_s)|^2 ds
\]
\[
\begin{align*}
    &\leq 2\left(\mathbb{E}\int_0^T |Y_n^s - Y_n^{s+1}|^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T |f_n(s, Y_n^s, Z_n^s) - f_{n+1}(s, Y_n^{s+1}, Z_n^{s+1})|^2 ds\right)^{\frac{1}{2}} \\
    &+ 2\left(\mathbb{E}\int_0^T |Y_n^s - Y_n^{s+1}|^2 dA_s\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T |h_n(s, Y_n^s) - h_{n+1}(s, Y_n^{s+1})|^2 dA_s\right)^{\frac{1}{2}} \\
    &+ C\mathbb{E}\int_0^T |Y_n^s - Y_n^{s+1}|^2 ds.
\end{align*}
\]

The uniform linear growth condition on the sequence \((f_n, h_n)\) together with inequality (3.3) provide the existence of a constant \(C\) such that
\[
\mathbb{E}\int_0^T \|Z_n^s - Z_n^{s+1}\|^2 ds \leq C\left(\mathbb{E}\int_0^T |Y_n^s - Y_n^{s+1}|^2 (ds + dA_s)\right)^{\frac{1}{2}}.
\]

Thus from (3.5), \(\{Z^n\}\) is a Cauchy sequence in a Banach space \(\mathcal{M}^2(0, T, \mathbb{R}^m)\), and there exists an \(\mathcal{F}_t\)-jointly measurable process \(Z\) such that \(\{Z^n\}\) converges to \(Z\) as \(n \to \infty\).

Similarly, by Itô’s formula together with Burkholder–Davis–Gundy inequality, it follows that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |Y_n^t - Y_t^{n+1}|^2\right) \to 0 \text{ as } n \to \infty,
\]
from which we deduce that \(\mathbb{P}\)-almost surely, \(Y^n\) converges uniformly to \(Y\) which is continuous.

**Step 3** \((Y, Z)\) verifies GBDSDEL (2.1).

Since \(Z^n \to Z\) in \(\mathcal{M}^2(0, T, \mathbb{R}^m)\), along a subsequence which we still denote \(Z^n, Z^n \to Z, dt \otimes d\mathbb{P}\) a.e. and there exists \(\Pi \in \mathcal{M}^2(0, T, \mathbb{R}^m)\) such that \(\forall n, |Z^n| < \Pi, dt \otimes d\mathbb{P}\) a.e. Therefore, by Lemma 3.2, we have
\[
\begin{align*}
    f_n(t, Y^n_t, Z^n_t) &\to f(t, Y_t, Z_t) dt \otimes d\mathbb{P} \text{ a.e.,} \\
    h_n(t, Y^n_t) &\to h(t, Y_t) dA_t \otimes d\mathbb{P} \text{ a.e.}
\end{align*}
\]

Moreover, from (H2) and (3.3), the dominated convergence theorem provides
\[
\begin{align*}
    \mathbb{E}\left(\int_t^T f_n(t, Y^n_t, Z^n_t) ds\right) &\to \mathbb{E}\left(\int_t^T f(t, Y_t, Z_t) ds\right), \\
    \mathbb{E}\left(\int_t^T h_n(t, Y^n_t) dA_s\right) &\to \mathbb{E}\left(\int_t^T h(t, Y_t) dA_s\right)
\end{align*}
\]
as \(n \to \infty\). Furthermore, in virtue of Burkholder–Davis–Gundy inequality, (H2) and (3.5), we obtain
\[
\begin{align*}
    \mathbb{E}\left(\sup_{0 \leq t \leq T} \left|\int_t^T g(s, Y^n_s) dB_s - \int_t^T g(s, Y_s) dB_s\right|\right) &\to 0, \\
    \mathbb{E}\left(\sup_{0 \leq t \leq T} \left|\sum_{i=1}^m \left(\int_t^T Z^{(i)}_s dH^{(i)}_s - \int_t^T Z^{(i)}_s dH^{(i)}_s\right)\right|\right) &\to 0
\end{align*}
\]
as \(n\) goes to infinity. Finally, passing to the limit in (3.1), we conclude that \((Y, Z)\) is a solution of GBDSDEL (2.1).

**Step 4** Minimal solution.

Let \((Y', Z') \in \mathcal{E}_n^\pi(0, T)\) be any solution of GBDSDEL (2.1). By virtue of the comparison theorem (Theorem 2.5), we have \(Y^n \leq Y', \forall n \in \mathbb{N}\). Therefore, \(Y \leq Y'\). That proves that \(Y\) is the minimal solution. \(
\)
Remark 3.3 Using the same arguments and the following approximating sequence
\( \phi_n(t, x) = \sup_{y \in Q} \{ \phi(t, y) - n|x - y| \} \), one can prove that the GBDSDEL (2.1) has a maximal solution

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