Generalized Stirling Numbers I

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Abstract
We consider generalized Stirling numbers of the second kind $S_{a,b,r}^{s_2,\beta_s,p_s}(p,k)$, $k = 0,1,\ldots,rp + \sum_{s=2}^{L} r_s p_s$, where $a,b,\alpha_s,\beta_s$ are complex numbers, and $r,p,r_s,p_s$ are non-negative integers given, $s = 2,\ldots,L$. (The case $a = 1, b = 0, r = 1, r_s p_s = 0$, corresponds to the standard Stirling numbers $S(p,k)$.) The numbers $S_{a,b,r}^{s_2,\beta_s,p_s}(p,k)$ are connected with a generalization of Eulerian numbers and polynomials we studied in previous works. This link allows us to propose (first, and then to prove, specially in the case $r = r_s = 1$) several results involving our generalized Stirling numbers, including several families of new recurrences for Stirling numbers of the second kind. In a future work we consider the recurrence and the differential operator associated to the numbers $S_{a,b,r}^{s_2,\beta_s,p_s}(p,k)$.

1 Introduction
Throughout the work, $L$ will denote an arbitrary positive integer $\geq 2$ given, $r,r_s$ and $p,p_s$ will denote non-negative integers given, and we will write $\sigma$ for the sum $\sum_{s=2}^{L} r_s p_s$.

Stirling numbers are nice mathematical objects studied along the years: they contain important combinatorial information, and they have shown to be connected with many other important mathematical and physical objects. (For a comprehensive study of Stirling numbers, including connections with physics, see [16] and the hundreds of references therein.) Many generalizations of Stirling numbers are known nowadays. In 1998 Hsu and Shiue [14] presented a unified approach containing several important generalizations of Stirling numbers studied before by different authors (Carlitz [3,4,5], Howard [13], Gould-Hopper [10], Riordan [21], Charalambides [6,7,8], Koutras [15], among others). We also mention the work of P. Blasiak [1]. Both, the Hsu and Shiue, and Blasiak works have some (natural) intersections with the generalization we consider in this work. However, all these works run in different directions.

We will be dealing with the Z-Transform (see [11],[24]), which is a function $Z$ that maps complex sequences $a_n = (a_0,a_1,\ldots)$ into complex functions $Z(a_n)(z)$ (or simply $Z(a_n)$), given by the Laurent series $Z(a_n)(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ (called Z-Transform of the sequence $a_n$, defined in the exterior of the circle of convergence of the Taylor series $\sum_{n=0}^{\infty} a_n z^n$ —the generating function of the sequence $a_n$—). If $Z(a_n) = A(z)$, we can also write $a_n = Z^{-1}(A(z))$, and we say that the complex sequence $a_n$ is the Inverse Z-Transform of the complex function $A(z)$. For example, the sequence $\lambda^n$ (where $\lambda$ is a given non-zero complex number), has Z-transform

$$Z(\lambda^n) = \sum_{n=0}^{\infty} \frac{\lambda^n}{z^n} = \frac{1}{1-\frac{\lambda}{z}} = \frac{z}{z-\lambda}, \quad (1)$$
defined for \(|z| > |\lambda|\). In particular, the Z-transform of the constant sequence 1 is

\[
Z(1) = \frac{z}{z - 1}. \tag{2}
\]

Besides the natural properties of linearity and injectivity, the Z-transform has the following two important properties to be used in this work (which formal proofs are easy exercises left to the reader):

1. (Advance-shifting property) If \(Z(a_n) = A(z)\), and \(k\) is a non-negative integer given, we have

\[
Z(a_{n+k}) = z^k \left( A(z) - \sum_{j=0}^{k-1} \frac{a_j}{z^j} \right). \tag{3}
\]

2. (Multiplication by the sequence \(n\)) If \(Z(a_n) = A(z)\), then

\[
Z(na_n) = -z \frac{d}{dz} A(z). \tag{4}
\]

From (2) and (4), we see that the Z-transform of the sequence \(n\) is

\[
Z(n) = -z \frac{d}{dz} \frac{z}{z - 1} = \frac{z}{(z - 1)^2}. \tag{5}
\]

The Z-transform of the sequence \(\binom{n}{r}\), where \(r\) is a non-negative integer given, is

\[
Z\left(\binom{n}{r}\right) = \frac{z}{(z - 1)^{r+1}}. \tag{6}
\]

(The cases \(r = 0\) and \(r = 1\) correspond to (2) and (5), respectively. The rest is an easy induction on \(r\) left to the reader.) According to the advance-shifting property (3), together with (6), we see that for \(0 \leq k \leq r\), we Z-transform of the sequence \(\binom{n+k}{r}\) is

\[
Z\left(\binom{n+k}{r}\right) = \frac{z^{k+1}}{(z - 1)^{r+1}}. \tag{7}
\]

In Section 2 we introduce the generalized Stirling numbers of the second kind \(S_{\alpha_s,\beta_s, r_s, p_s}^{a,b,r} (p, k)\). The definition we give for \(S_{\alpha_s,\beta_s, r_s, p_s}^{a,b,r} (p, k)\) is related to generalized Eulerian numbers \(A_{\alpha_s,\beta_s, r_s, p_s}^{a,b,r} (p, k)\) (formula (17)), but soon we show that our generalized Stirling numbers have an explicit formula that generalizes the known explicit formula for \(S(p, k)\) (proposition 1). In Section 3 we consider the case \(r = r_s = 1\), and we prove several results involving the GSN \(S_{\alpha_s,\beta_s, 1, p_s}^{a,b,1} (p, k)\). Some of these results are proved by induction: we mention that the way we arrived to them came from a previous work (not included here) with generalized Eulerian numbers and polynomials [17] [18], together with the connection described in Section 2. Finally, in Section 4 we state the results about the recurrence and the differential operator associated to the generalized Stirling numbers \(S_{\alpha_s,\beta_s, r_s, p_s}^{a,b,r} (p, k)\) (these results are the main topics of the second part of this work [19].) Also, in a future work we consider the generalized Bell numbers resulting of the generalization of Stirling numbers of this work [20].

2 The Generalized Stirling Numbers

In a recent work [17], we considered the Generalized Eulerian Numbers (GEN, for short) \(A_{\alpha_s,\beta_s, r_s, p_s}^{a,b,r} (p, i)\), \(i = 0, 1, \ldots, rp + \sigma\) (where \(a, b, \alpha_s, \beta_s\) are complex numbers, and \(r, p, r_s, p_s\) are non-negative integers), defined
as the coefficients in the expansion of the \((rp + \sigma)\)-th degree polynomial \((an + b)^p \prod_{s=2}^{L} \left(\alpha_s n + \beta_s\right)^{p_s}\) in terms of the basis \(B_1 = \left\{ \binom{n+rp+\sigma-i}{rp+\sigma}, i = 0, 1, \ldots, rp + \sigma \right\}\) (of the vector space \(P_{rp+\sigma}\) of \(n\)-polynomials of degree \(\leq rp + \sigma\); see [22], p. 208, 4.3(b)), that is

\[
\binom{an + b}{r}^p \prod_{s=2}^{L} \left(\frac{\alpha_s n + \beta_s}{r_s}\right)^{p_s} = \sum_{i=0}^{rp+\sigma} A_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, i) \left(\frac{n + rp + \sigma - i}{rp + \sigma}\right).
\]

(8)

When \(r_s p_s = 0, s = 2, 3, \ldots, L\), we write the GEN as \(A_{a,b,r} (p, i), i = 0, 1, \ldots, rp\). Clearly, the numbers \(A_{1,0,1} (p, i), i = 0, 1, \ldots, p\), correspond to the standard Eulerian numbers, that we denote as \(A (p, i)\). In this case, expression (5) is just Worpitzky identity [26]: \(n^p = \sum_{i=0}^{p} A (p, i) \left(\binom{p+i}{p}\right)\). We have the explicit formula (see [17])

\[
A_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, i) = \sum_{j=0}^{i} (-1)^j \binom{rp + \sigma + 1}{j} \left(\binom{a(i-j) + b}{r}\right)^p \prod_{s=2}^{L} \left(\frac{\alpha_s (i-j) + \beta_s}{r_s}\right)^{p_s}.
\]

(9)

According to [7], the \(Z\)-transform of the sequence \((an + b)^p \prod_{s=2}^{L} \left(\alpha_s n + \beta_s\right)^{p_s}\) is

\[
Z \left(\sum_{i=0}^{rp+\sigma} A_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, i) \left(\frac{n + rp + \sigma - i}{rp + \sigma}\right)\right) = \sum_{i=0}^{rp+\sigma} A_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, i) \left(\frac{z^{r+\sigma-i}}{(z-1)^{r+\sigma+1}}\right).
\]

(10)

The polynomial

\[
P_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (z) = \sum_{i=0}^{rp+\sigma} A_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, i) \left(\frac{z^{r+\sigma-i}}{(z-1)^{r+\sigma+1}}\right),
\]

(11)

is the Generalized Eulerian Polynomial (GEP, for short). When \(r_s p_s = 0, s = 2, 3, \ldots, L\), we write the GEP \((11)\) as \(P_{a,b,r} (z)\). These polynomials were studied in [18].

Inspired by the well-known case, in which Stirling numbers of the second kind \(S (p, k), k = 0, 1, \ldots, p\), appear in the coefficients of the expansion of \(n^p\) in terms of the basis \(B_2 = \left\{ \binom{n}{k}, k = 0, 1, \ldots, p \right\}\) (of the vector space \(P_p\) — of polynomials of degree \(\leq p\)), namely \(n^p = \sum_{k=0}^{p} k! S (p, k) \binom{n}{k}\)), we consider the basis \(B_2 = \left\{ \binom{n}{k}, k = 0, 1, \ldots, rp + \sigma \right\}\) of the vector space \(P_{rp+\sigma}\) (see [22], p. 209, 4.3(d)), then write the \((rp + \sigma)\)-th degree \(n\)-polynomial \((an + b)^p \prod_{s=2}^{L} \left(\alpha_s n + \beta_s\right)^{p_s}\) in terms of \(B_2\), and define the Generalized Stirling Numbers of the Second Kind \((GSN, for short) S_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, k), by means of the expansion

\[
\binom{an + b}{r}^p \prod_{s=2}^{L} \left(\frac{\alpha_s n + \beta_s}{r_s}\right)^{p_s} = \sum_{k=0}^{rp+\sigma} k! S_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, k) \binom{n}{k}.
\]

(12)

If \(k < 0\) or \(k > rp + \sigma\), we have \(S_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, k) = 0\). For the rest of the work, “Stirling number(s)” will mean “Stirling number(s) of the second kind”.

By using (6) and according to (12), we see that the \(Z\)-Transform of the sequence \((an + b)^p \prod_{s=2}^{L} \left(\alpha_s n + \beta_s\right)^{p_s}\) is

\[
Z \left(\sum_{k=0}^{rp+\sigma} k! S_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, k) \left(\frac{z}{(z-1)^{k+1}}\right)\right) = \sum_{k=0}^{rp+\sigma} k! S_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, k) \left(\frac{z}{(z-1)^{k+1}}\right).
\]

(13)

that we can write as

\[
Z \left(\sum_{k=0}^{rp+\sigma} k! S_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, k) \left(\frac{z}{(z-1)^{k+1}}\right)\right) = \sum_{k=0}^{rp+\sigma} k! S_{a,b,r}^{\alpha_s,\beta_s,r_s,p} (p, k) \left(\frac{z}{(z-1)^{k+1}}\right).
\]

(14)
Thus, comparing (14) with (10), we obtain the following expression for the GEP $P_{a,b,r,s}(z)$ in terms of the GSN $S_{a,b,r,s}(p,k)$,

$$P_{a,b,r,s}(z) = \frac{1}{(r!)^p \prod_{s=2}^{L} (r_s)^{p_s}} \sum_{k=0}^{r+p+\sigma} k! S_{a,b,r,s}(p,k) (z - 1)^{r+p+\sigma - k}. \quad (15)$$

The GEP $P_{a,b,r,s}(z)$ written in powers of $z - 1$, is

$$P_{a,b,r,s}(z) = \sum_{k=0}^{r+p+\sigma} \sum_{i=0}^{p} \binom{r+p+\sigma}{r+p-i} \binom{r+p-i}{r+p+\sigma - k} A_{a,b,r,s}(p,i) (z - 1)^{r+p+\sigma - k}. \quad (16)$$

Thus, from (15) and (16), we see that for $0 \leq k \leq r+p+\sigma$ we have

$$S_{a,b,r,s}(p,k) = \frac{(r!)^p \prod_{s=2}^{L} (r_s)^{p_s}}{k!} \sum_{i=0}^{p} \binom{r+p+\sigma}{r+p-i} \binom{r+p-i}{r+p+\sigma - k} A_{a,b,r,s}(p,i). \quad (17)$$

Reciprocally, we can write (14) as

$$Z \left( \left( \frac{an + b}{r} \right)^p \prod_{s=2}^{L} \left( \frac{\alpha_s n + \beta_s}{r_s} \right)^{p_s} \right) \quad (18)$$

$$= \frac{z}{(z - 1)^{r+p+\sigma+1}} \sum_{k=0}^{r+p+\sigma} \sum_{i=0}^{p} \binom{r+p+\sigma}{r+p-i} \binom{r+p-i}{r+p+\sigma - k} A_{a,b,r,s}(p,i) (z - 1)^{r+p+\sigma - k}.$$

In the right-hand side of (18), introduce the new summation index $i = k + j$, to obtain

$$Z \left( \left( \frac{an + b}{r} \right)^p \prod_{s=2}^{L} \left( \frac{\alpha_s n + \beta_s}{r_s} \right)^{p_s} \right) \quad (19)$$

$$= \frac{z}{(z - 1)^{r+p+\sigma+1}} \sum_{i=0}^{p} \binom{-1}{i} \binom{-1}{r+p+\sigma - i} \binom{r+p+\sigma - k}{r+p+\sigma - i} S_{a,b,r,s}(p,k) (z - 1)^{r+p+\sigma - i}.$$

Thus, comparing (19) with (10) (and (13)), we see that for $0 \leq i \leq r+p+\sigma$ we have

$$A_{a,b,r,s}(p,i) = \frac{(-1)^i}{(r!)^p \prod_{s=2}^{L} (r_s)^{p_s}} \sum_{k=0}^{r+p+\sigma} \binom{r+p+\sigma}{r+p+\sigma - i} \binom{r+p+\sigma - k}{r+p+\sigma - i} A_{a,b,r,s}(p,k). \quad (20)$$

In the standard case $a = 1$, $b = 0$, $r = 1$, $r_s = 0$, formulas (17) and (20) are well-known results: $S(p,k) = \frac{1}{k} \sum_{i=0}^{p-k} A(p,i)$, and $A(p,i) = (-1)^i \sum_{k=0}^{p} \binom{p-k}{p-i} k! S(p,k)$. (See [12], p. 269, formulas (6.39) and (6.40)).

The GSN $S_{a,b,r,s}(p,k)$ have an explicit formula (generalizing the known explicit formula $S(p,k) = \frac{1}{k} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^p$).

**Proposition 1** We have the following explicit formula for the GSN $S_{a,b,r,s}(p,k)$

$$S_{a,b,r,s}(p,k) = \frac{(r!)^p \prod_{s=2}^{L} (r_s)^{p_s}}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} \binom{(k-j)(a+b)}{r} \prod_{s=2}^{L} \binom{(k-j) \alpha_s + \beta_s}{r_s}. \quad (21)$$
Proof. From (17) and (21), we have to show that
\[
\sum_{i=0}^{k} \binom{rp + \sigma - i}{k - i} A_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,i) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} \left( \binom{k-j}{r} a + b \right)^p L \prod_{s=2}^L \left( \binom{k-j}{r_s} \alpha_s + \beta_s \right)^{p_s}. \tag{22}
\]

Beginning with the left-hand side of (22), we use the explicit formula (9) for \( A_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} \), to write
\[
\sum_{i=0}^{k} \binom{rp + \sigma - i}{k - i} A_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,i)
\]
\[
= \sum_{i=0}^{k} \binom{rp + \sigma - i}{k - i} \sum_{j=0}^{k} (-1)^{i+j} \binom{rp + \sigma + 1}{i - j} \binom{a_j + b_j}{r_j} \prod_{s=2}^L \left( \binom{\alpha_s + \beta_s}{r_s} \right)^{p_s}.
\]
\[
= \sum_{j=0}^{k} (-1)^j \sum_{i=0}^{k} (-1)^{k+i} \binom{rp + \sigma - i}{k - i} \binom{rp + \sigma + 1}{i - k+j} \binom{a_j (k-j) + b_j}{r_j} \prod_{s=2}^L \left( \binom{\alpha_s (k-j) + \beta_s}{r_s} \right)^{p_s}.	ag{23}
\]

Now, observe that for \( 0 \leq j \leq k \), we have
\[
\sum_{i=0}^{k} (-1)^{k+i} \binom{rp + \sigma - i}{k - i} \binom{rp + \sigma + 1}{i - k+j} = \sum_{i=0}^{k} (-rp - \sigma + i + k - i - 1) \binom{rp + \sigma + 1}{i - k+j}
\]
\[
= \sum_{i=0}^{j} \binom{k - rp - \sigma - 1}{i} \binom{rp + \sigma + 1}{j - i}
\]
\[
= \binom{k}{j},
\]
where the last step is the Vandermonde Convolution. Thus, expression (23) (the left-hand side of (22)) is precisely the right-hand side of (22), as desired. □

From (21) one can easily see that, for \( p \) given, the first two values \( S_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,0) \) and \( S_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,1) \) are
\[
S_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,0) = \left( r! \binom{b}{r} \right)^p \prod_{s=2}^L \binom{\beta_s}{r_s}^{p_s},
\]
\[
S_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,1) = \left( r! \binom{a+b}{r} \right)^p \prod_{s=2}^L \binom{\alpha_s + \beta_s}{r_s}^{p_s} - S_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,0),
\]
and the last value \( S_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,rp+\sigma) \) is
\[
S_{a,b,r}^{\alpha_s,\beta_s,r_s,p_s} (p,rp+\sigma) = a^rp \prod_{s=2}^L \alpha_s r_s^{p_s}.
\]

3 Case \( r = r_s = 1 \): Main Results

In this section we consider the GSN \( S_{a,b,1}^{\alpha_s,\beta_s,1,p_s} (p,k) \), \( k = 0,1,\ldots,p + \sigma, \sigma = p_2 + \cdots + p_L \), involved in the expansion \( (an + b)^p \prod_{s=2}^L (\alpha_s n + \beta_s)^{p_s} = \sum_{k=0}^{p+\sigma} k! S_{a,b,1}^{\alpha_s,\beta_s,1,p_s} (p,k)(n)_k(p) \). For the sake of a simpler notation, we
will work with the GSN $S_{a_1,b_1}^{a_2,b_2,p_2} (p,k)$, that we write as $S_{a_1,b_1}^{a_2,b_2,p_2} (p,k)$. According to (21) we have the explicit formula

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (a_1 (k-j) + b_1)^{p_1} (a_2 (k-j) + b_2)^{p_2}. \tag{27}$$

(The case $p_2 = 0$ was studied before by L. Verde-Star [23], and it is particular case of the Hsu and Shiu generalization [14].) We present some Generalized Stirling Number Triangles (GSNT, for short), where the GSN $S_{a_1,b_1}^{a_2,b_2,p_2} (p,k)$ appear in a triangular array ($p \in \mathbb{N}$ stands for lines and $k$ stands for columns, $0 \leq k \leq p + p_2$).

| GSNT1: $S_{a_1,b_1}^{a_2,b_2,p_2} (p,k)$ | GSNT2: $S_{a_1,b_1}^{a_2,b_2,p_2} (p,k)$ |
|---|---|
| $p \setminus k$ | $p \setminus k$ |
| 0 | 1 | 2 | 3 | ... |
| 0 | 0 | 1 |
| 1 | 0 | 2 | 1 | ... |
| 2 | 0 | 4 | 5 | 1 |
| ... | ... | ... | ... | ... |

| GSNT3: $S_{1,1}^{a_1,b_1} (p,k)$ | GSNT4: $S_{1,2}^{a_1,b_1} (p,k)$ |
|---|---|
| $p \setminus k$ | $p \setminus k$ |
| 0 | 1 | 2 | 3 | ... |
| 0 | 1 |
| 1 | 2 | 4 | 1 | ... |
| 2 | 4 | 14 | 8 | 1 |
| ... | ... | ... | ... | ... |

The following facts are obvious from (27):

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,k) = S_{a_1,b_1} (p_1 + p_2,k), \quad S_{a_1,b_1}^{a_2,b_2,p_2} (0,0) = b_2^{p_2},$$

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,k) = S_{a_1,b_1}^{a_2,b_2,p_2} (p_2,k), \quad S_{a_1,b_1}^{a_2,b_2,p_2} (0,k) = S_{a_2,b_2} (p_2,k). \tag{28}$$

Also, one can see easily from (27) that

$$S_{1,1} (p,k) = S (p + 1, k + 1), \quad S_{1,2} (p,k) = S (p + 2, k + 2) - S (p + 1, k + 2). \tag{29}$$

(We will be using (28) and (29) without further comments.)

Some values of the GSN $S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,k)$ are

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,0) = b_1^{p_1} b_2^{p_2}, \tag{30}$$

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,1) = (a_1 + b_1)^{p_1} (a_2 + b_2)^{p_2} - b_1^{p_1} b_2^{p_2},$$

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,2) = \frac{1}{2} (2a_1 + b_1)^{p_1} (2a_2 + b_2)^{p_2} - (a_1 + b_1)^{p_1} (a_2 + b_2)^{p_2} + \frac{1}{2} b_1^{p_1} b_2^{p_2},$$

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p_1, p_1 + p_2) = a_1^{p_1} \cdot a_2^{p_2}.$$  

The GSN $S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,k)$ satisfy the recurrence (to be proved in [19])

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,k) = a_1 S_{a_1,b_1}^{a_2,b_2,p_2} (p_1 - 1,k - 1) + (a_1 k + b_1) S_{a_1,b_1}^{a_2,b_2,p_2} (p_1 - 1, k). \tag{31}$$
Lemma 2 The GSN $S_{a_1,b_1}^{c_2,d_2,2} (p_1,k)$ are related to the GSN $S_{c_1,d_1}^{c_2,d_2,j_2} (p_1,k)$, by the following formula

$$S_{a_1,b_1}^{c_2,d_2,2} (p_1,k) = c_1^{-p_1} c_2^{-p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{j_1} a_2^{j_2} (b_1 c_1 - a_1 d_1)^{p_1-j_1} (b_2 c_2 - a_2 d_2)^{p_2-j_2} S_{c_1,d_1}^{c_2,d_2,j_2} (j_1,k). \tag{32}$$

Proof. It is a straightforward calculation:

$$c_1^{-p_1} c_2^{-p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{j_1} a_2^{j_2} (b_1 c_1 - a_1 d_1)^{p_1-j_1} (b_2 c_2 - a_2 d_2)^{p_2-j_2} S_{c_1,d_1}^{c_2,d_2,j_2} (j_1,k)$$

$$= c_1^{-p_1} c_2^{-p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{j_1} a_2^{j_2} (b_1 c_1 - a_1 d_1)^{p_1-j_1} (b_2 c_2 - a_2 d_2)^{p_2-j_2} \times$$

$$\times \frac{1}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} c_1 (k-l) + d_1 \binom{k-1}{l} (c_2 (k-l) + d_2)^{j_2}$$

$$= \frac{1}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} c_1^{-p_1} \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} (b_1 c_1 - a_1 d_1)^{p_1-j_1} (a_1 c_1 (k-l) + a_1 d_1)^{j_1}$$

$$\times c_2^{-p_2} \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (b_2 c_2 - a_2 d_2)^{p_2-j_2} (a_2 c_2 (k-l) + a_2 d_2)^{j_2}$$

$$= \frac{1}{k!} \sum_{l=0}^{k} (-1)^l \binom{k}{l} (a_1 (k-l) + b_1)^{p_1} (a_2 (k-l) + b_2)^{p_2}$$

$$= S_{a_1,b_1}^{c_2,d_2,2} (p_1,k),$$

as desired. $\blacksquare$

In particular, from (32) (with $c_1 = c_2 = 1$, $d_1 = d_2 = 0$) we see that the GSN $S_{a_1,b_1}^{c_2,d_2,2} (p_1,k)$ can be written in terms of standard Stirling numbers as

$$S_{a_1,b_1}^{c_2,d_2,2} (p_1,k) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{j_1} a_2^{j_2} b_1^{p_1-j_1} b_2^{p_2-j_2} S (j_1+j_2,k). \tag{33}$$

If $a_1 = a_2$, $b_1 = b_2$ (or if $p_2 = 0$), expression (33) reduces to

$$S_a (p,k) = \sum_{j=0}^{p} \binom{p}{j} a^j b^{p-j} S (j,k). \tag{34}$$

(The particular case $a = b = 1$ of (34) is the known formula $S (p+1,k+1) = \sum_{j=0}^{p} \binom{p}{j} S (j,k)$.) Similarly, the standard Stirling numbers can be written in terms of GSN (from (32) with $a_1 = a_2 = 1$, $b_1 = b_2 = 0$) as

$$S (p_1+p_2,k) = c_1^{-p_1} c_2^{-p_2} \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-d_1)^{p_1-j_1} (-d_2)^{p_2-j_2} S_{c_1,d_1}^{c_2,d_2,j_2} (j_1,k). \tag{35}$$

If $c_1 = c_2$, $d_1 = d_2$ (or if $p_2 = 0$), expression (35) reduces to

$$S(p,k) = c_2^{-p} \sum_{j=0}^{p} \binom{p}{j} (-d)^{p-j} S_{c,d} (j,k). \tag{36}$$

(The particular case $c = d = 1$ of (36) is the known formula $S(p,k) = \sum_{j=0}^{p} \binom{p}{j} (-1)^{p-j} S (j+1,k+1)$.)
Lemma 3 We have
\[ S_{a_1, a_1 + b_1}^{a_2, a_2 + b_2, p_2} (p_1, k) = S_{a_1, b_1}^{a_2 + b_2, p_2} (p_1, k) + (k + 1) S_{a_1, b_1}^{a_2, b_2, p_2} (p_1, k + 1). \] (37)

Proof. By using (32) we can write the GSN \( S_{a_1, a_1 + b_1}^{a_2, a_2 + b_2, p_2} (p_1, k) \) (left-hand side of (37)) as
\[ S_{a_1, a_1 + b_1}^{a_2, a_2 + b_2, p_2} (p_1, k) = \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{p_1 - j_1} a_2^{p_2 - j_2} S_{a_1, b_1}^{a_2, b_2, j_2} (j_1, k), \]
that is
\[
S_{a_1, a_1 + b_1}^{a_2, a_2 + b_2, p_2} (p_1, k) = \sum_{j_1 = 0}^{p_1} \sum_{j_2 = 0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{p_1 - j_1} a_2^{p_2 - j_2} \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (a_1 (k - t) + b_1)^{j_1} (a_2 (k - t) + b_2)^{j_2} \]
\[
= \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (a_1 (k + 1 - t) + b_1)^{p_1} (a_2 (k + 1 - t) + b_2)^{p_2}. \] (38)

On the other hand, the right-hand side of (37) is
\[
S_{a_1, a_1 + b_1}^{a_2, a_2 + b_2, p_2} (p_1, k) + (k + 1) S_{a_1, b_1}^{a_2, b_2, p_2} (p_1, k + 1) =
\]
\[
= \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (a_1 (k - t) + b_1)^{p_1} (a_2 (k - t) + b_2)^{p_2}
\]
\[
+ (k + 1) \frac{1}{(k + 1)!} \sum_{t=0}^{k+1} (-1)^t \binom{k+1}{t} (a_1 (k + 1 - t) + b_1)^{p_1} (a_2 (k + 1 - t) + b_2)^{p_2}
\]
\[
= \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (a_1 (k - t) + b_1)^{p_1} (a_2 (k - t) + b_2)^{p_2}
\]
\[
+ \frac{1}{k!} \sum_{t=0}^{k+1} (-1)^t \binom{k}{t} + \binom{k}{t-1} (a_1 (k + 1 - t) + b_1)^{p_1} (a_2 (k + 1 - t) + b_2)^{p_2}
\]
\[
= \frac{1}{k!} \sum_{t=0}^{k} (-1)^t + (-1)^{t+1} \binom{k}{t} (a_1 (k - t) + b_1)^{p_1} (a_2 (k - t) + b_2)^{p_2}
\]
\[
+ \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (a_1 (k + 1 - t) + b_1)^{p_1} (a_2 (k + 1 - t) + b_2)^{p_2}
\]
\[
= \frac{1}{k!} \sum_{t=0}^{k} (-1)^t \binom{k}{t} (a_1 (k + 1 - t) + b_1)^{p_1} (a_2 (k + 1 - t) + b_2)^{p_2},
\]
which is equal to (38), as desired. ■

(The case \( p_2 = 0, a_1 = 1, b_1 = 0 \) of (37) is the known recurrence for standard Stirling numbers.)

Formula (32) is just a particular case \((m = 0)\) of the following more general result.
Proposition 4 For $0 \leq k \leq p_1 + p_2$, and any non-negative integer $m$ we have

$$k! S_{a_1, b_1}^{p_1, p_2} (p_1, k)$$

\[= c_1^{-p_1} c_2^{-p_2} \sum_{j=0}^{p_1} \sum_{j=0}^{p_2} \left( \begin{array}{l} p_1 \\ j_1 \end{array} \right) \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) a_1^{i_1} a_2^{i_2} (b_1 c_1 - a_1 c_1 m - a_1 d_1)^{p_1 - j_1} (b_2 c_2 - a_2 c_2 m - a_2 d_2)^{p_2 - j_2} \times \]

\[
\times \sum_{t=0}^{m} \left( \begin{array}{c} m \\ t \end{array} \right) (k + t)! S_{c_1, d_1}^{p_2, d_2} (j_1, k + t). \]

Proof. If $k = p_1 + p_2$, the left-hand side of (39) is $(p_1 + p_2)! a_1^{p_1} a_2^{p_2}$, and the right-hand side of (39) reduces to the only term $c_1^{-p_1} c_2^{-p_2} a_1^{p_1} a_2^{p_2} (p_1 + p_2)! c_1^{p_1} c_2^{p_2} = (p_1 + p_2)! a_1^{p_1} a_2^{p_2}$, so we can suppose that $0 \leq k < p_1 + p_2$. Observe that the right-hand side of (39) can be written as

\[
\sum_{j=0}^{p_1} \sum_{j=0}^{p_2} \left( \begin{array}{l} p_1 \\ j_1 \end{array} \right) \left( \begin{array}{c} p_2 \\ j_2 \end{array} \right) a_1^{i_1} a_2^{i_2} (b_1 c_1 - a_1 c_1 m - a_1 d_1)^{p_1 - j_1} (b_2 c_2 - a_2 c_2 m - a_2 d_2)^{p_2 - j_2} \times \]

\[
\times \sum_{t=0}^{m} \left( \begin{array}{c} m \\ t \end{array} \right) \sum_{l=0}^{k+t} (-1)^l \binom{k+t}{l} (c_1 (k + t - l) + d_1)^{j_1} (c_2 (k + t - l) + d_2)^{j_2} \]

\[
= \sum_{t=0}^{m} \left( \begin{array}{c} m \\ t \end{array} \right) \sum_{l=0}^{k+t} (-1)^l \binom{k+t}{l} (a_1 (k + t - l) + b_1 - a_1 m)^{p_1} (a_2 (k + t - l) + b_2 - a_2 m)^{p_2}. \]

That is, we have to show that for $0 \leq k < p_1 + p_2$, and any non-negative integer $m$, one has

\[
\sum_{t=0}^{m} \left( \begin{array}{c} m \\ t \end{array} \right) \sum_{l=0}^{k+t} (-1)^l \binom{k+t}{l} (a_1 (k + t - l) + b_1 - a_1 m)^{p_1} (a_2 (k + t - l) + b_2 - a_2 m)^{p_2} \]

\[
= k! S_{a_1, b_1}^{p_2, p_2} (p_1, k). \]

We proceed by induction on $m$. For $m = 0$ formula (40) is trivial. If we suppose (40) is true for a given $m \in \mathbb{N}$, then

\[
\sum_{t=0}^{m+1} \left( \begin{array}{c} m+1 \\ t \end{array} \right) \sum_{l=0}^{k+t} (-1)^l \binom{k+t}{l} \times \]

\[
\times (a_1 (k + t - l) + b_1 - a_1 (m + 1))^{p_1} (a_2 (k + t - l) + b_2 - a_2 (m + 1))^{p_2} \]

\[
= \sum_{t=0}^{m} \left( \begin{array}{c} m \\ t \end{array} \right) \sum_{l=0}^{k+t} (-1)^l \binom{k+t}{l} \times \]

\[
\times (a_1 (k + t - l) + b_1 - a_1 (m + 1))^{p_1} (a_2 (k + t - l) + b_2 - a_2 (m + 1))^{p_2} \]

\[
+ \sum_{t=0}^{m} \left( \begin{array}{c} m \\ t \end{array} \right) \sum_{l=0}^{k+t+1} (-1)^l \binom{k+t+1}{l} \times \]

\[
\times (a_1 (k + t + 1 - l) + b_1 - a_1 (m + 1))^{p_1} (a_2 (k + t + 1 - l) + b_2 - a_2 (m + 1))^{p_2} \]

\[
= k! S_{a_1, b_1}^{p_2, p_2} (p_1, k + (k + 1)! S_{a_1, b_1}^{p_2, p_2} (p_1, k + 1) \]

\[
= k! \left( S_{a_1, b_1}^{p_2, p_2} (p_1, k) + (k + 1)! S_{a_1, b_1}^{p_2, p_2} (p_1, k + 1) \right) \]

\[
= k! S_{a_1, b_1}^{p_2, p_2} (p_1, k), \]

as desired (in the last step we used (37)).
In particular, expression (39) gives us the following infinite family of formulas for the GSN \( S_{a_1, b_1}^{a_2, b_2, p_2} (p_1, k) \) in terms of standard Stirling numbers (where \( m \) is any non-negative integer)

\[
k! S_{a_1, b_1}^{a_2, b_2, p_2} (p_1, k) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{j_1} a_2^{j_2} (b_1 - a_1 m)^{p_1-j_1} (b_2 - a_2 m)^{p_2-j_2} \times
\]

\[
\sum_{t=0}^{m} \binom{m}{t} (k+t)! S (j_1 + j_2, k + t).
\]

We will see now that, for \( m > 0 \), there is a nicer form to write (41).

**Lemma 5** For non-negative integers \( j, k, 0 \leq k \leq j \leq p \), and any positive integer \( m \), we have the identity

\[
\sum_{t=0}^{m} \binom{m}{t} (k+t)! S (j, k + t) = k! \sum_{t=0}^{m-1} (-1)^t s (m, m-t) S (j + m-t, k + m),
\]

where \( s (\cdot, \cdot) \) are the Stirling numbers of the first kind.

**Proof.** We will use the recurrence for the Stirling numbers of the first kind \( s (p, k) = s (p-1, k-1) + (p-1) s (p-1, k) \), and the recurrence for the Stirling numbers of the second kind \( S (p, k) = S (p-1, k-1) + k S (p-1, k) \). We proceed by induction on \( m \): for \( m = 1 \) one can see easily that both sides of (42) are equal to \( k! S (j+1, k+1) \). Let us assume that (42) is true for a given \( m \in \mathbb{N} \). Thus, we begin our argument with

\[
\sum_{t=0}^{m+1} \binom{m+1}{t} (k+t)! S (j, k + t).
\]

After some easy algebraic work, we use the induction hypothesis to get

\[
\sum_{t=0}^{m+1} \binom{m+1}{t} (k+t)! S (j, k + t)
\]

\[
= \sum_{t=0}^{m} \binom{m}{t} (k+t)! S (j, k + t) + \sum_{t=0}^{m} \binom{m}{t} (k+t+1)! S (j, k + t+1)
\]

\[
= k! \sum_{t=0}^{m-1} (-1)^t s (m, m-t) (S (j + m-t, k+m) + (k+1) S (j + m-t, k+1+m)).
\]

Now use the recurrence for the Stirling numbers of the second kind, then again some elementary algebraic steps, and then use the recurrence for the Stirling numbers of the first kind, to get

\[
\sum_{t=0}^{m+1} \binom{m+1}{t} (k+t)! S (j, k + t)
\]

\[
= k! \sum_{t=0}^{m-1} (-1)^t s (m, m-t) (S (j + m+1-t, k+m+1) - m S (j + m-t, k+m+1))
\]

\[
= k! \sum_{t=0}^{m-1} (-1)^t s (m, m-t) S (j + m+1-t, k+m+1)
\]

\[
- k! \binom{m}{1} (-1)^t S (j + m, k+m+1)
\]

\[
= k! \sum_{t=0}^{m} (-1)^t s (m, m-t) (S (j + m-t, k+m+1) + m S (j + m-t, k+m+1))
\]

\[
= k! \sum_{t=0}^{m} (-1)^t s (m+1, m+1-t) S (j + m+1-t, k+m+1),
\]

10
which is the desired conclusion.

Corollary 6  The GSN $S_{a_1,b_1}^{a_2,b_2,p_2}$ $(p_1,k)$ can be written in terms of standard Stirling numbers as

$$S_{a_1,b_1}^{a_2,b_2,p_2} (p_1,k) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{j_1} a_2^{j_2} (b_1 - a_1 m)^{p_1-j_1} (b_2 - a_2 m)^{p_2-j_2} \times$$

$$\times \sum_{t=0}^{m-1} (-1)^t s(m,m-t) S(j_1 + j_2 + m - t,k + m),$$

(43)

where $m$ is any positive integer.

**Proof.** Formula (43) comes directly from (11) and (12). In particular, if $b_1$ is a positive integer, we have from (43) with $p_2 = 0, a_1 = 1$ and $m = b_1$ that

$$S_{1,m} (p,k) = \sum_{t=0}^{m-1} (-1)^t s(m,m-t) S(p+m-t,k+m).$$

(44)

For $m = 1, 2$, formula (44) gives (29). For $m = 3, 4$ we have

$$S_{1,3} (p,k) = S(p+3,k+3) - 3S(p+2,k+3) + 2S(p+1,k+3),$$

(45)

$$S_{1,4} (p,k) = S(p+4,k+4) - 6S(p+3,k+4) + 11S(p+2,k+4) - 6S(p+1,k+4).$$

An interesting consequence of (44) is the following.

Corollary 7  We have the following recurrence for Stirling numbers

$$S(p_1 + p_2, l) = \sum_{k=1}^{p_2-1} (-1)^{p_2+1+k} s(p_2,k) S(p_1 + k, l) + \sum_{j=0}^{p_1} \binom{p_1}{j} p_2^{p_1-j} S(j, l-p_2).$$

(46)

**Proof.** From (44) and (44) we have that

$$S_{1,p_2} (p_1, l) = \sum_{j=0}^{p_1} \binom{p_1}{j} p_2^{p_1-j} S(j, l) = \sum_{t=1}^{p_2} (-1)^{p_2+t} s(p_2,t) S(p_1 + t, l + p_2).$$

(47)

Substitute $l$ by $l - p_2$ to obtain from (47) the desired conclusion (46).

For example, if $p_2 = 2, 3$, we have

$$S(p_1 + 2, l) = S(p_1 + 1, l) + \sum_{j=0}^{p_1} \binom{p_1}{j} 2^{p_1-j} S(j, l-2),$$

$$S(p_1 + 3, l) = -2S(p_1 + 1, l) + 3S(p_1 + 2, l) + \sum_{j=0}^{p_1} \binom{p_1}{j} 3^{p_1-j} S(j, l-3).$$

For $p_1$ given and $l = p_2$, (46) gives us formulas expressing Stirling numbers of the second kind in terms of Stirling numbers of the first kind. For example, for $p_1 = 1, 2$ we have

$$S(p_2 + 1, p_2) = s(p_2, p_2 - 1) + p_2,$$

$$S(p_2 + 2, p_2) = (s(p_2, p_2 - 1))^2 + p_2 s(p_2, p_2 - 1) - s(p_2, p_2 - 2) + p_2^2.$$
The main ideas involved in (48), (41) and (43) can be summarized as follows: we can write the GSN $S_{a_1,b_1}^{a_2,b_2,p_2}(p_1,k)$ in terms of standard Stirling numbers by using (49) (which comes from (41) with $m = 0$), but also for $m > 0$ we can use (43). For example, with $m = 0, 1, 2$ we have

$$S_{a_1,b_1}^{a_2,b_2,p_2}(p_1,k) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} a_1^{p_1} a_2^{j_2} b_1^{p_1-j_1} b_2^{p_2-j_2} S(j_1 + j_2, k)$$

Some concrete examples of (48) are the following,

$$S_{1,0}^{1,0,p_2}(p_1,k) = \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} S(j_1 + p_2, k)$$

$$= \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (-1)^{p_2-j_2} S(p_1 + j_2 + 1, k + 1)$$

$$= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{p_1-j_1} (-2)^{p_2-j_2} S(j_1 + j_2 + 2, k + 2) - S(j_1 + j_2 + 1, k + 2).$$

$$S_{1,2}^{1,0,p_2}(p_1,k) = \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} 2^{p_1-j_1} S(j_1 + p_2, k)$$

$$= \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_1}{j_1} \binom{p_2}{j_2} (-1)^{p_2-j_2} S(j_1 + j_2 + 1, k + 1)$$

$$= \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (-2)^{p_2-j_2} (S(p_1 + j_2 + 2, k + 2) - S(p_1 + j_2 + 1, k + 2)).$$
(In passing: from (51) we see that \( S^{1,2,p}_{1,1} (1, k) = \sum_{j=0}^{p} \binom{p}{j} S(j + 2, k + 1) \), and from (49) we see that \( S^{1,0,2}_{1,1} (p, k) = \sum_{j=0}^{p} \binom{p}{j} S(j + 2, k) \). So we have \( S^{1,1,1}_{1,1} (p, k) = S^{1,0,2}_{1,1} (p, k + 1) \) (see GSNT2 and GSNT3).)

Formula (52) gives us the following relations involving the GSN (49), (50), (51):

\[
S^{1,0,p_2}_{1,1} (p_1, k) = \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} (-1)^{p_1-j_1} S^{1,0,p_2}_{1,2} (j_1, k) \quad (52)
\]

\[
S^{1,2,p_2}_{1,2} (p_1, k) = \sum_{j_1=0}^{p_1} \sum_{j_2=0}^{p_2} \binom{p_2}{j_2} (-2)^{p_2-j_2} S^{1,2,p_2}_{1,1} (p_1, k) ,
\]

which, in terms of standard Stirling numbers, can be written as

\[
\sum_{j=0}^{p} \binom{p}{j} S(q + j, k) = \sum_{l=0}^{p} \sum_{j=0}^{l} \binom{l}{j} (-1)^{p-l-j} 2^{l-j} S(q + j, k) \quad (53)
\]

\[
\sum_{j=0}^{p} \binom{p}{j} 2^{p-j} S(q + j, k) = \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{p}{j} \binom{q}{i} (-2)^{q-i} S(j + l + 1, k + 1) .
\]

(Of course, (52) is just one of the infinite family of possibilities to write (52) in terms of standard Stirling numbers.)

Now we begin to explore a new relation involving GSN.

**Lemma 8** For \( 0 \leq l \leq p_2 + q_1 + q_2 \) we have

\[
S^{a_2,b_2,q_1+q_2}_{a_1,b_1} (p_2, l) = \sum_{m=0}^{p_2+q_2} S^{a_2,b_2,q_2}_{a_1,b_1} (p_2, m) S^{a_2,a_2,m+b_2}_{a_2,a_2,m+b_2} (q_1, l - m) . \quad (54)
\]

**Proof.** We proceed by induction on \( q_1 \). If \( q_1 = 0 \) formula (54) is trivial. If we suppose it is true for \( q_1 \in \mathbb{N} \), we have

\[
\sum_{m=0}^{p_2+q_2} S^{a_2,b_2,q_2}_{a_1,b_1} (p_2, m) S^{a_2,a_2,m+b_2}_{a_2,a_2,m+b_2} (q_1 + 1, l - m)
\]

\[
= \sum_{m=0}^{p_2+q_2} S^{a_2,b_2,q_2}_{a_1,b_1} (p_2, m) \left( a_2 S^{a_2,a_2,m+b_2}_{a_2,a_2,m+b_2} (q_1, l - m - 1) + (a_2 l + b_2) S^{a_2,a_2,m+b_2}_{a_2,a_2,m+b_2} (q_1, l - m) \right)
\]

\[
= a_2 \sum_{m=0}^{p_2+q_2} S^{a_2,b_2,q_2}_{a_1,b_1} (p_2, m) S^{a_2,a_2,m+b_2}_{a_2,a_2,m+b_2} (q_1, l - m - 1)
\]

\[
+ (a_2 l + b_2) \sum_{m=0}^{p_2+q_2} S^{a_2,b_2,q_2}_{a_1,b_1} (p_2, m) S^{a_2,a_2,m+b_2}_{a_2,a_2,m+b_2} (q_1, l - m)
\]

\[
= a_2 S^{a_2,b_2,q_1+q_2}_{a_1,b_1} (p_2, l - 1) + (a_2 l + b_2) S^{a_2,b_2,q_1+q_2}_{a_1,b_1} (p_2, l)
\]

\[
= S^{a_2,b_2,q_1+q_2}_{a_1,b_1} (p_2, l) ,
\]

as desired. ■
**Proposition 9** For $0 \leq l \leq p_1 + p_2 + q_1 + q_2$ we have

$$S_{a_1, b_1}^{a_2, b_2, q_1 + q_2} (p_1 + p_2, l) = \sum_{m=0}^{p_2 + q_2} S_{a_1, b_1}^{a_2, b_2, q_2} (p_2, m) S_{a_1, a_2, m + b_2, q_1}^{a_1, q_1} (p_1, l - m). \quad (55)$$

**Proof.** We proceed by induction on $p_1$. If $p_1 = 0$, formula (55) is (51). If formula (55) is true for $p_1 \in \mathbb{N}$, then (by using the recurrence (31))

$$\begin{align*}
p_2 + q_2 & \sum_{m=0}^{p_2 + q_2} S_{a_1, b_1}^{a_2, b_2, q_2} (p_2, m) S_{a_1, a_2, m + b_2, q_1}^{a_1, q_1} (p_1 + 1, l - m) \\
= & \sum_{m=0}^{p_2 + q_2} S_{a_1, b_1}^{a_2, b_2, q_2} (p_2, m) \left( a_1 S_{a_1, a_2, m + b_2, q_1}^{a_1, q_1} (p_1, l - m - 1) + (a_1 l - m) + a_2 b_2 q_2 \right) S_{a_1, a_2, m + b_2, q_1}^{a_1, q_1} (p_1, l - m) \\
= & \sum_{m=0}^{p_2 + q_2} S_{a_1, b_1}^{a_2, b_2, q_2} (p_2, m) \left( a_1 S_{a_1, a_2, m + b_2, q_1}^{a_1, q_1} (p_1, l - m - 1) + (a_1 l + b_2) \right) S_{a_1, a_2, m + b_2, q_1}^{a_1, q_1} (p_1, l - m) \\
= & \sum_{m=0}^{p_2 + q_2} S_{a_1, b_1}^{a_2, b_2, q_2} (p_2, m) S_{a_1, a_2, m + b_2, q_1}^{a_1, q_1} (p_1, l - m - 1) \\
+ & (a_1 l + b_2) \sum_{m=0}^{p_2} S_{a_1, b_1}^{a_2, b_2, q_2} (p_2, m) S_{a_1, a_2, m + b_2, q_1}^{a_1, q_1} (p_1, l - m) \\
= & S_{a_1, b_1}^{a_2, b_2, q_1 + q_2} (p_1 + p_2, l - 1) + (a_1 l + b_2) S_{a_1, b_1}^{a_2, b_2, q_1 + q_2} (p_1 + p_2, l) \\
= & S_{a_1, b_1}^{a_2, b_2, q_1 + q_2} (p_1 + p_2 + 1, l),
\end{align*}$$

as desired. \Hfill \blacksquare

The case $q_1 = q_2 = 0$ of (55) is

$$S_{a_1, b_1} (p_1 + p_2, l) = \sum_{m=0}^{p_2} S_{a_1, b_1} (p_2, m) S_{a_1, a_2, m + b_1} (p_1, l - m), \quad (56)$$

and the standard case $a_1 = 1, b_1 = 0$ of (55) can be written (by using (51)) as

$$S (p_1 + p_2, l) = \sum_{m=0}^{p_2} \sum_{j=0}^{p_1} \binom{p_1}{j} m^{p_1 - j} S (j, l - m) S (p_2, m) \quad (57)$$

Comparing with (49), we conclude the identity (for positive $p_2$)

$$\sum_{m=0}^{p_2 - 1} \sum_{j=0}^{p_1} \binom{p_1}{j} m^{p_1 - j} S (j, l - m) S (p_2, m) = \sum_{k=1}^{p_2 - 1} (-1)^{p_2 + k + 1} s (p_2, k) S (p_1 + k, l). \quad (58)$$

(It is possible to prove (58) by induction on the non-negative integer $p_1$, by using only the recurrence for standard Stirling numbers of the second kind. It is a nice exercise left to the reader.)

It is interesting to note that, for $p_1$ given, formula (57) gives us an explicit formula for the Stirling number $S (p + p_1, l)$ in terms of the Stirling numbers $S (p, l), S (p, l - 1), \ldots, S (p, l - p_1)$ (the case $p_1 = 0$ is trivial). If $p_1 = 1$, formula (57) is just the known standard recurrence $S (p + 1, l) = l S (p, l) + S (p, l - 1)$. For example, if $p_1 = 2, 3$, formula (58) gives us

$$\begin{align*}
S (p + 2, l) & = l^2 S (p, l) + (2l - 1) S (p, l - 1) + S (p, l - 2), \quad (59) \\
S (p + 3, l) & = l^3 S (p, l) + (3l^2 - 3l + 1) S (p, l - 1) + 3 (l - 1) S (p, l - 2) + S (p, l - 3),
\end{align*}$$

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respectively (iterations of the standard recurrence).

A final comment about (55): we can use (43) to write (55) as

\[ S_{a_1,b_1}(p_1 + p_2, l) = \sum_{m=0}^{p_2} \sum_{j_1=0}^{p_1} \left( \frac{p_1}{j_1} \right) a_1^{j_1} (b_1 - a_1 (n - m))^{p_1-j_1} \sum_{t=0}^{n-1} (-1)^t s(n, n-t) S(j_1 + n - t, l - m + n) S_{a_1,b_1}(p_2, m), \]

where \( n \) is an arbitrary positive integer. Some particular cases of (60) are the following (relatives of (57))

\[ S(p_1 + p_2, l) = \sum_{m=0}^{p_2} \sum_{j_1=0}^{p_1} \left( \frac{p_1}{j_1} \right) (m - 1)^{p_1-j_1} S(j_1 + 1, l - m + 1) S(p_2, m) \]

\[ = \sum_{m=0}^{p_2} \sum_{j_1=0}^{p_1} \left( \frac{p_1}{j_1} \right) (m - 2)^{p_1-j_1} (S(j_1 + 2, l - m + 2) - S(j_1 + 1, l - m + 2)) S(p_2, m). \]

Formula (55) can be seen as an identity of two polynomials in the variables \( a_1, b_1, a_2, b_2 \). This fact produces some natural corollaries. We show next one of them.

**Corollary 10** For \( 0 \leq l, t \leq p_1 + p_2 + q_1 + q_2 \), we have

\[ \sum_{r=0}^{q_1+q_2} \left( \frac{p_1 + p_2 + r}{t} \right) \left( \frac{q_1 + q_2}{r} \right) S(p_1 + p_2 + r - t, l) \]

\[ = \sum_{m=0}^{p_2+q_2} \sum_{r_1=0}^{q_1} \sum_{r_2=0}^{q_2} \sum_{p_1+r_1=0}^{p_1} \sum_{k=0}^{p_1} \sum_{s=0}^{r_1} \left( \frac{q_1}{r_1} \right) \left( \frac{q_2}{r_2} \right) \left( \frac{p_1 + r_1}{t - s} \right) \left( \frac{p_2 + r_2}{s} \right) \]

\[ \times m^{k-s} S(p_2 + r_2 - t + s, m) S(p_1 + r_1 - k, l - m). \]

**Proof.** We consider the case \( a_1 = a_2(= a), b_2 = b_1 + 1(= b + 1) \) of (55), that is

\[ \sum_{m=0}^{p_2+q_2} S_{a,b}^{a+b+1,q_2}(p_2, m) S_{a,am+b+1}^{a+q_1}(p_1, l - m) = S_{a,b}^{a+b+1,q_1+q_2}(p_1 + p_2, l). \]

We claim that

\[ S_{a,b}^{a+b+1,q}(p, k) = \sum_{r=0}^{q} \left( \frac{q}{r} \right) S_{a,b}(p + r, k). \]
In fact, we have
\[ S_{a,b}^{p+1,q} (p, k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (a (k - j) + b)^p (a (k - j) + b + 1)^q \]
\[ = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (a (k - j) + b)^p \sum_{r=0}^{q} \binom{q}{r} (a (k - j) + b)^r \]
\[ = \sum_{r=0}^{q} \binom{q}{r} \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (a (k - j) + b)^{p+r} \]
\[ = \sum_{r=0}^{q} \binom{q}{r} S_{a,b} (p + r, k), \]
which proves our claim. Thus, by using (63) we can write (62) as
\[ \sum_{m=0}^{p_2+q_2} \sum_{r_1=0}^{q_1} \sum_{r_2=0}^{q_2} \binom{q_1}{r_1} \binom{q_2}{r_2} S_{a,b} (p_2 + r_2, m) S_{a,am+b} (p_1 + r_1, l - m) \]
\[ = \sum_{r=0}^{q_1+q_2} \binom{q_1+q_2}{r} S_{a,b} (p_1 + p_2 + r, l). \]

Set \( a = 1 \) and use (64) to obtain from (65) that
\[ \sum_{t=0}^{p_1+p_2+q_1+q_2} \sum_{r=0}^{q_1+q_2} \binom{p_1+p_2+r}{t} \binom{q_1+q_2}{r} b^t S (p_1 + p_2 + r - t, l) \]
\[ = \sum_{m=0}^{p_2+q_2} \sum_{r_1=0}^{q_1} \sum_{r_2=0}^{q_2} \binom{q_1}{r_1} \binom{q_2}{r_2} \sum_{k=0}^{p_1+r_1} \binom{p_1+r_1}{k} \times \]
\[ \sum_{j=0}^{p_2+r_2} \binom{p_2+r_2}{j} S (p_2 + r_2 - j, m) \sum_{s=0}^{k} \binom{k}{s} b^{s+j} m^{k-s} S (p_1 + r_1 - k, l - m). \]

In the right-hand side of (65) introduce the new summation index \( t = s + j \), to obtain
\[ \sum_{t=0}^{p_1+p_2+q_1+q_2} \sum_{r=0}^{q_1+q_2} \binom{p_1+p_2+r}{t} \binom{q_1+q_2}{r} b^t S (p_1 + p_2 + r - t, l) \]
\[ = \sum_{m=0}^{p_2+q_2} \sum_{r_1=0}^{q_1} \sum_{r_2=0}^{q_2} \binom{q_1}{r_1} \binom{q_2}{r_2} \sum_{k=0}^{p_1+r_1} \binom{p_1+r_1}{k} \sum_{s=0}^{k} \binom{k}{s} m^{k-s} \times \]
\[ \sum_{t=s}^{p_2+r_2+k} \binom{p_2+r_2}{t-s} S (p_2 + r_2 - t + s, m) b^t S (p_1 + r_1 - k, l - m), \]
from where we obtain the desired conclusion (61).

The case \( q_1 = q_2 = 0 \) of (61) says that for \( 0 \leq t, l \leq p_1 + p_2 \) we have
\[ \left( \frac{p_1 + p_2}{t} \right) S (p_1 + p_2 - t, l) \]
\[ = \sum_{m=0}^{p_2} \sum_{k=0}^{p_1} \sum_{s=0}^{k} \binom{p_1}{k} \left( \frac{p_2}{t-s} \right) \binom{k}{s} m^{k-s} S (p_2 - t + s, m) S (p_1 - k, l - m), \]
Proposition 11  For integers $m > 0$ and $0 \leq k \leq m - 1$, we have

$$
\sum_{t=0}^{m-k-1} \binom{m}{t+k+1} t! S^{a_2, b_2, a_2+2, p_2}_{a_1, b_1 + a_1 m, p_2} (p_1, t) = \sum_{t=k+1}^{m} \binom{t-1}{k} (b_1 + a_1 t)^{p_1} (b_2 + a_2 t)^{p_2}.
$$

Proof. We proceed by induction on $m$. If $m = 1$ (and then $k = 0$) we have

$$
S^{a_2, b_2, a_2, p_2}_{a_1, b_1 + a_1} (p_1, 0) = (b_1 + a_1)^{p_1} (b_2 + a_2)^{p_2}.
$$
If it is true for an \( m \), then
\[
\sum_{t=0}^{m-k} \binom{m+1}{t+k+1} t! S_{a_2,b_2+a_2+a_2 m,p_2}^{a_2,b_2} (p_1,t) + \sum_{t=0}^{m-k} \binom{m}{t+k+1} t! S_{a_1,b_1+a_1+a_1 m}^{a_1,b_1} (p_1,t) \]

\[
= \sum_{t=k+1}^{m} \binom{m}{t} (b_1 + a_1 + a_1 t)^{p_1} (b_2 + a_2 + a_2 t)^{p_2} + \sum_{t=k}^{m} \binom{m}{t} \binom{m}{t-1} (b_1 + a_1 + a_1 t)^{p_1} (b_2 + a_2 + a_2 t)^{p_2}
\]

\[
= \sum_{t=k+1}^{m} \binom{m}{t} (b_1 + a_1 + a_1 t)^{p_1} (b_2 + a_2 + a_2 t)^{p_2} + \sum_{t=k+1}^{m} \binom{m}{t} (b_1 + a_1 + a_1 t)^{p_1} (b_2 + a_2 + a_2 t)^{p_2}
\]

\[
= \sum_{t=k+1}^{m} \binom{m}{t} (b_1 + a_1 + a_1 t)^{p_1} (b_2 + a_2 + a_2 t)^{p_2},
\]
as desired. \( \blacksquare \)

Two examples of (70) are
\[
\sum_{t=k+1}^{m} \binom{m}{t} (m-t) (t-m-1)^2 = 4 \binom{m+1}{k+3} + 6 \binom{m+1}{k+4}.
\]
\[
\sum_{t=k+1}^{m} \binom{m}{t} (m-t)^3 = \binom{m}{k+2} + 6 \binom{m+1}{k+4}.
\]

To end this section we show a convolution formula involving the GSN.

**Proposition 12** For arbitrary non-negative integers \( k \) and \( \mu \) we have
\[
\binom{k+\mu}{k} S_{a_2,b_2,p_2}^{a_2,b_2} (p_1,k+\mu) = \sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} \binom{p_2}{j_2} S_{a_1,0,p_2-j_2}^{a_2,0,p_2-j_2} (p_1-j_1, \mu) S_{a_1,0}^{a_2,b_2} (j_1,k).
\] (71)

**Proof.** We proceed by induction on \( k \). For \( k = 0 \) we have
\[
\sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} \binom{p_2}{j_2} S_{a_1,0,j_2}^{a_2,0,j_2} (p_1, j_1, \mu) S_{a_1,0}^{a_2,b_2} (j_1,0) \]

\[
= \sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} \binom{p_2}{j_2} b_1^j_1 b_2^j_2 S_{a_1,0}^{a_2,0} (p_1-j_1, \mu)
\]

\[
= \sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} \binom{p_1}{j_1} \binom{p_2}{j_2} b_1^j_1 b_2^j_2 \frac{\mu}{\mu} \sum_{s=0}^{\mu} (-1)^s \binom{\mu}{s} (a_1 (\mu-s))^{p_1-j_1} (a_2 (\mu-s))^{p_2-j_2}
\]

\[
= \frac{1}{\mu} \sum_{s=0}^{\mu} (-1)^s \binom{\mu}{s} (a_1 (\mu-s) + b_1)^{p_1} (a_2 (\mu-s) + b_2)^{p_2}
\]

\[
= S_{a_2,b_2,p_2}^{a_2,b_2} (p_1, \mu),
\]
as desired. If formula (71) is true for a given \( k \in \mathbb{N} \), we have (by using (67))

\[
\sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} \left( \frac{p_1}{j_1} \right) \left( \frac{p_2}{j_2} \right) S_{a_1,0}^{a_2,0,p_2-j_2} (p_1-j_1, \mu) S_{a_1,b_1}^{a_2,b_2,j_2} (j_1,k+1)
= \frac{1}{k+1} \sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} \left( \frac{p_1}{j_1} \right) \left( \frac{p_2}{j_2} \right) S_{a_1,0}^{a_2,0,p_2-j_2} (p_1-j_1, \mu) \left( S_{a_1,a_1+b_1}^{a_2,a_2+b_2,j_2} (j_1,k) - S_{a_1,b_1}^{a_2,b_2,j_2} (j_1,k) \right)
= \frac{1}{k+1} \left( \sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} \left( \frac{p_1}{j_1} \right) \left( \frac{p_2}{j_2} \right) S_{a_1,0}^{a_2,0,p_2-j_2} (p_1-j_1, \mu) \right) \left( S_{a_1,a_1+b_1}^{a_2,a_2+b_2,j_2} (j_1,k) \right)
= \frac{1}{k+1} \left( \sum_{j_2=0}^{p_2} \sum_{j_1=0}^{p_1} \left( \frac{p_1}{j_1} \right) \left( \frac{p_2}{j_2} \right) S_{a_1,0}^{a_2,0,p_2-j_2} (p_1-j_1, \mu) \right) \left( S_{a_1,b_1}^{a_2,b_2,j_2} (j_1,k) \right)
= \frac{1}{k+1} \left( k+\mu \right) S_{a_1,a_1+b_1}^{a_2,a_2+b_2,j_2} (p_1,k+\mu) - S_{a_1,b_1}^{a_2,b_2,j_2} (p_1,k+\mu)
= \frac{1}{k+1} \left( k+\mu \right) S_{a_1,a_1+b_1}^{a_2,a_2+b_2,j_2} (p_1,k+\mu+1)
= \left( k+\mu+1 \right) S_{a_1,b_1}^{a_2,b_2,j_2} (p_1,k+\mu+1),
\]

as wanted. ■

(The case \( a_1 = a_2 = 1, b_1 = b_2 = 0 \) of (71) is formula (6.28), p. 265 in [12].)

4 Further Results: Recurrence and Differential Operator

The GSN \( S_{1,b,r}^{a_1,b_1,r_1,p_r} (p, k) \) satisfy the recurrence

\[
S_{1,b,r}^{a_1,b_1,r_1,p_r} (p,k) = r! \sum_{t=0}^{r} \frac{1}{t!} \left( b + k - t \right) \frac{d^t}{dx^t} S_{1,b,r}^{a_1,b_1,r_1,p_r} (p-1,k-t). \tag{72}
\]

In terms of Differential Operators, the GSN \( S_{1,b,r} (p,k) \) can be defined by

\[
\left( \frac{r! \sum_{j=0}^{r} \frac{1}{j!} \frac{d^j}{dx^j}}{x^k} \right)^{p} = \sum_{k=0}^{\infty} S_{1,b,r} (p,k) x^k \frac{d^k}{dx^k}. \tag{73}
\]

Formulas (72) and (73) are (some of) the main results of the second part of this work [19].

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