Informed Principal Problems in Bilateral Trading

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Abstract

We study bilateral trade with interdependent values as an informed-principal problem. The mechanism-selection game has multiple equilibria that differ with respect to principal’s payoff and trading surplus. We characterize the equilibrium that is worst for every type of principal, and characterize the conditions under which there are no equilibria with different payoffs for the principal. We also show that this is the unique equilibrium that survives the intuitive criterion.

Keywords: Informed principal; Bilateral trade; Interdependent values; Rothschild–Stiglitz–Wilson allocation; Intuitive criterion

JEL Classification C72 · D82 · D86

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1 Introduction

In this study, we examine bilateral trade with interdependent values as an informed-principal problem. Examples of this situation are found in decentralized markets, including used-cars, housing, and labor markets. This bilateral trade can also occur as aftermarket transactions (Dworczak, 2020). For example, after telecommunications companies won a spectrum auction held by the United Kingdom, they resold their spectrum licenses through the sale of companies (Hafalir and Krishna, 2008). Our analysis can be applied to understand how an auction winner trades with a third party in “continuation” games. We then pose the following fundamental questions: Which trading mechanism does an informed principal select in equilibrium? What efficiency properties do equilibrium allocations have?

These and related issues have been addressed in only a few studies. In independent private values (IPV) environments à la Myerson and Satterthwaite (1983), Yilankaya (1999) and Mylovanov and Tröger (2014) showed that the “full-information” optimal mechanism is an equilibrium of the mechanism-selection game and is ex-ante optimal. Hence, the privacy of the principal’s information is irrelevant to equilibrium outcomes in these IPV environments. Koessler and Skreta (2016) obtained a number of important results in an interdependent-values environment. In their model, the seller’s valuation is zero, while the buyer’s valuation depends, in an arbitrary way, on the two parties’ types. Using these features, Koessler and Skreta (2016) characterized the set of equilibrium outcomes as revenue-ranked allocations and showed that some practical selling procedures (e.g., book building in sales of companies) yield the ex-ante optimal revenue.

An implication of these results is that the interdependency of valuations gives rise to a multiplicity of equilibrium allocations in the mechanism-selection game. Therefore, we need some criteria for equilibrium selection or refinement to provide definite answers to our questions.

Several solution concepts for informed-principal problems are proposed by previous studies. Myerson (1983) and Maskin and Tirole (1990, 1992) developed the general theories of mechanism selection by an informed principal, while Mylovanov

\footnote{Hence, their model can capture an economically interesting scenario wherein the seller’s type determines the horizontal characteristic of a good, while the buyer’s type determines how these different varieties are evaluated. Our model cannot describe this scenario. See Balestrieri and Izmalkov (2016) for a related Hotelling model with an informed seller.}

\footnote{Koessler and Skreta (2019) studied an informed-seller problem with interdependent values and certifiable information. They showed that when the certifiability structure is rich enough, strong (unconstrained) Pareto optimal (SPO) allocations (Maskin and Tirole, 1990) are equilibria and are ex-ante optimal. Further, they provided an example in which the SPO profit vector is the unique equilibrium profit vector. This is in contrast to the case without certifiable information.}
and Tröger (2012, 2014) advanced them by focusing on general IPV environments. We follow the approach developed by Maskin and Tirole (1992, henceforth MT), who thoroughly studied these problems in an environment with interdependent (or common) values. The Rothschild–Stiglitz–Wilson (RSW) allocation plays a crucial role in their analysis. This is the best safe mechanism (Myerson, 1983) for every type of principal. Their main result is the equilibrium characterization: If an RSW allocation is interim efficient for some interior beliefs, then the equilibrium set coincides with the set of feasible allocations that weakly dominate the RSW. The characterization implies that the RSW is the worst equilibrium for every type of principal. Moreover, MT characterized the RSW allocation itself and proved that the RSW is the unique allocation that passes the intuitive criterion (Cho and Kreps, 1987), under the assumption that only the principal has private information.

Our aim is to extend MT’s results to our trading environment with bilateral asymmetric information. For expositional simplicity, we assume that the principal is a seller and the agent is a buyer. Our first theorem provides a simple characterization of RSW. By definition, the RSW is “belief-free” for the buyer. As a result, his ex-post payments are pinned down by the allocation rule. Hence, the seller’s interim revenues are determined by the buyer’s virtual valuation with the allocation rule. A novel feature of the theorem is that it characterizes the allocation rule via posterior beliefs about the seller’s type. This posterior derives from the prior by shifting probability mass toward lower types. This posterior with the prior defines the seller’s virtual cost. A key finding is that the RSW maximizes the expected virtual surplus, thereby reducing the seller’s overstatement incentives.

This characterization is closely related to our second theorem. It provides a necessary and sufficient condition under which the RSW allocation is undominated for a given belief. This new result has some implications. First, this condition holds for the RSW with a belief given by our first theorem. Thus, the RSW is undominated for at least one posterior belief. This belief generates posteriors that support the RSW as an intuitive equilibrium. Further, the second theorem

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3Other contributions include Cella (2008), Severinov (2008), Skreta (2011), and Bedard (2017). See Mylovanov and Tröger (2014) for a more detailed review of the literature. See also Wagner et al. (2015) for studies on informed-principal problems with moral hazard.

4Recently, Dosis (2022) found that, in some cases, the condition of interim efficiency is insufficient for this characterization result. To address this problem, he introduced a stronger condition called interim optimality. To obtain the same characterization as in MT, we use the condition of undominatedness, which is also stronger than interim efficiency for RSW allocations, and apply a proof method developed by Myerson (1983). See Remark 4 in Section 5.2 for details.

5Following MT, we use feminine pronouns for the principal and masculine ones for the agent.
characterizes the set of prior beliefs for which the RSW is undominated. In other
words, the theorem provides a necessary and sufficient condition for the existence
of a \textit{strong solution} introduced by Myerson (1983). As noted by MT, its existence
is equivalent to the uniqueness of the principal’s equilibrium payoff vector.

If the RSW allocation is dominated for the prior, the mechanism-selection
game has a continuum of equilibria. However, the extension of MT’s refinement
result to our environment is a nontrivial task. This is related to one of the primary
concerns in the literature on informed-principal problems: that is, the principal’s
gains from concealing her information. To clarify this point, let us assume that
the RSW is dominated by another equilibrium. From the definition of RSW, the
dominating allocation violates either ex-post incentive compatibility (EPIC) or ex-
post individual rationality (EPIR) for the agent. If some principal types mutually
benefit from “exchanging slack variables” on these constraints, these types lose by
making revealing deviations from the equilibrium.\footnote{This concept of slack exchange is central to the analysis of Maskin and Tirole (1990). They showed that the principal obtains no gain from slack exchange in an IPV environment with quasi-linear preferences. See Cella (2008), Mylovanov and Tröger (2012), and Koessler and Skreta (2019) for further analysis.} Therefore, the benefit from
privacy can reduce the cutting power of the intuitive criterion.

We overcome this difficulty by adapting the methods developed by Gershkov
et al. (2013) to our informed-principal problem with interdependent values. We
prove the result of \textit{interim-payoff equivalence} assuming that the valuation func-
tions are additively separable in type variables. Again, let us assume that the RSW
is dominated by another equilibrium. Then, the equivalence result guarantees that
the dominating allocation has an interim-payoff-equivalent feasible allocation that
is EPIC for the agent. Hence, there is no exchange of slack variables in the equiv-
alent allocation. Moreover, the equivalent allocation also dominates the RSW. It
follows that the agent’s EPIR constraints are slack for at least one type of prin-
cipal. We demonstrate that this type can convincingly deviate from the original
equilibrium to another mechanism, and thus, the equilibrium is unintuitive.

In summary, by focusing on the context of bilateral trading, we provide a
simpler characterization of RSW, identify a necessary and sufficient condition
under which the RSW allocation is undominated, and show that the equilibrium
allocations passing the intuitive criterion are interim-payoff-equivalent to the RSW
allocations. We thus contribute to the important but small body of literature on
informed-principal problems in interdependent-values environments.

In contrast to the intuitive criterion, some important solution concepts select
undominated equilibrium allocations. Representative concepts are \textit{core mecha-}
nism” and “neutral optimum” proposed by Myerson (1983). The former is the allocation with no blocking coalition of types and the latter is axiomatically defined as the smallest possible set of unblocked mechanisms. Balkenborg and Makris (2015) introduced a concept called “assured allocation” that provides an easily interpreted algorithm to find a neutral optimum. Balkenborg and Makris (2015) as well as Myerson (1983) argued that, if the principal can effectively communicate with the agents, then she should be expected to offer only undominated mechanisms. In our study, we do not oppose this argument. This is different from an argument in favor of the intuitive criterion, and each argument is based on an implicit “speech” made by the principal. It is challenging to examine which argument is more convincing, but such an analysis is beyond our objective.

The rest of the paper is organized as follows: Section 2 describes the model. Section 3 characterizes the RSW allocations. Section 4 illustrates how the RSW allocation is derived and how non-RSW allocations are eliminated as unintuitive. Section 5 proves that the intuitive equilibrium allocations are interim-payoff-equivalent to the RSW allocations for both parties. Section 6 concludes the paper. Several technical lemmas and proofs are presented either in Appendix A or in the Supplementary material.

2 The model

Environment. Consider a seller (i = 1) who owns a good that a buyer (i = 2) wants to buy. Each party i has a type $x_i \in X_i \equiv \{1, 2, ..., \bar{x}_i\}$. We assume that $x \equiv (x_1, x_2)$ are independently distributed according to interior (i.e., full-support) probability distributions $(p_1, p_2)$ on $X \equiv X_1 \times X_2$. The cumulative distribution function (cdf) of $x_i$ is denoted by $P_i$. Their types affect each party’s valuation $v_i(x) \in \mathbb{R}$ of the good. Let $(q, t) \in A \equiv [0, 1] \times \mathbb{R}$ denote an outcome. Here, $q$ is the probability that the buyer obtains the good, and $t$ is the payment from the buyer to the seller. Both parties are risk-neutral. The seller’s ex-post payoff is $u_1((q, t), x) \equiv t - v_1(x)q$, while the buyer’s is $u_2((q, t), x) \equiv v_2(x)q - t$.\footnote{If $v_1 \leq 0$ and $v_2 \geq 0$, we can regard parties 1 and 2 as a buyer and a seller, respectively.}

To apply the technique of Gershkov et al. (2013), we assume that each $v_i$ is additively separable in $x$ (i.e., $v_i(x) \equiv v^1_i(x_1) + v^2_i(x_2)$). We then make the generic assumption that each $v^1_i$ is strictly increasing in $x_i$ and $v_1(x) \neq v_2(x)$ for each $x$.\footnote{We use the assumption of non-zero social surplus to prove Lemma 5. See Section 5.3 for the lemma.} We also assume that the seller’s type positively affects the buyer’s value (i.e.,
$v_2^i$ is increasing in $x_1$). For future reference, let $dv_1(x_1) \equiv v_1^i(x_1) - v_1^i(x^-_1)$ and $dv_2(x_2) \equiv v_2^i(x^+_2) - v_2^i(x_2)$ be the differences between the valuations of two adjacent types, with the convention that $x^-_i \equiv x_i - 1$, $x^+_i \equiv x_i + 1$, and $dv_1(1) = dv_2(\bar{x}_2) = 1$.

**Allocation and mechanism.** An allocation $f \in A^X$ describes type-dependent outcomes that result from the parties' interaction. Further, the allocation $f = (q, t)$ can be interpreted as a direct mechanism with allocation rule $q \in [0, 1]^X$ and payment rule $t \in \mathbb{R}^X$. Each party's interim payoff from reporting $\hat{x}_i$ in $f$ (when the other party tells the truth) is denoted by

$$
U_i^f(\hat{x}_1 | x_1) \equiv E_{x_2}[u_1(f(\hat{x}_1, x_2))], \quad U_i^{f,\pi}(\hat{x}_2 | x_2) \equiv E_{x_1}[u_2(f(x_1, \hat{x}_2), x)],
$$

where $\pi \in \Delta(X_1)$ is the buyer's posterior belief about $x_1$. It is important to note that each expectation $E_{x_i}$ is based on the prior $p_i$, while $E_{x_1}^{\pi}$ is formulated according to the posterior $\pi$ because the buyer may update his belief through the interaction. Following Mylovanov and Tröger (2014), we call an allocation $f$ $\pi$-feasible if it satisfies the IC and IR constraints for the seller and the buyer with the belief $\pi$, as follows: For each $x_1, \hat{x}_1 \in X_1$ and $x_2, \hat{x}_2 \in X_2$,

$$
U_i^f(x_1) \geq U_i^f(\hat{x}_1 | x_1), \quad (S-IC) \\
U_i^f(x_1) \geq 0, \quad (S-IR) \\
U_2^{f,\pi}(x_2) \geq U_2^{f,\pi}(\hat{x}_2 | x_2), \quad (B-\pi-IC) \\
U_2^{f,\pi}(x_2) \geq 0, \quad (B-\pi-IR)
$$

where $U_i^f(x_1) \equiv U_i^f(x_1 | x_1)$ and $U_2^{f,\pi}(x_2) \equiv U_2^{f,\pi}(x_2 | x_2)$. Further, $f$ is called feasible if it is $p_1$-feasible. Let $U_2^f(\cdot) \equiv U_2^{f,\pi}(\cdot)$. With some abuse of notation, we denote by $U_i^f(\cdot) \equiv (U_i^f(1), ..., U_i^f(\bar{x}_i))$ party $i$'s interim payoff vector in $f$. We observe that the IC constraints require the allocation rule $q$ to be interim monotone. That is, (S-IC) requires $Q_1(\cdot) \equiv E_{x_2}[q(\cdot, x_2)]$ to be decreasing in $x_1$, and (B-\pi-IC) requires $Q_2^\pi(\cdot) \equiv E_{x_1}^{\pi}[q(x_1, \cdot)]$ to be increasing in $x_2$.

We define a mechanism $G = (M, g)$ as a finite strategic game form. Here, $M$ is the product of the parties' finite message spaces $M_1$ and $M_2 \cup \{0\}$, and $g \in A^M$ is an outcome function. A typical message profile is denoted by $m = (m_1, m_2) \in M$. The buyer's message $m_2 = 0$ is his opt-out option such that $g(\cdot, 0)$ always specifies the no-trade outcome $(0, 0)$. Let $G$ be the set of all mechanisms. The set includes every direct mechanism $f \in A^{X_1 \times (X_2 \cup \{0\})}$ with $f(\cdot, 0) = (0, 0)$.

\footnote{To simplify the notation, we identify a direct mechanism with its outcome function $f$.}
A mechanism $G$ with a belief $\pi \in \Delta(X_1)$ induces the static Bayesian game $(G, (X_1, X_2), (u_1, u_2), (\pi, p_2))$. We denote it by $(G, \pi)$. Let $BN(G, \pi) \neq \emptyset$ be the set of allocations generated by Bayesian Nash equilibria (BNE) in the game.

**Mechanism-selection game and equilibrium.** The mechanism-selection game proceeds as follows: First, $(x_1, x_2)$ are realized according to $(p_1, p_2)$, and each party $i$ privately observes $x_i$. Second, the seller offers the buyer a mechanism $(M, g) \in G$. Third, the seller and buyer simultaneously choose $m_1 \in M_1$ and $m_2 \in M_2 \cup \{0\}$, respectively. Finally, each party $i$ obtains $u_i(g(m, x))$.

For the mechanism-selection game, we consider strong perfect Bayesian equilibria (Maskin and Tirole, 1992), which would be equivalent to sequential equilibria if the set $G$ was finite. The inscrutability principle (Myerson, 1983) allows us to focus on a pooling equilibrium wherein all seller types offer the same feasible allocation. Formally, a feasible allocation $f$ is called an equilibrium (or equilibrium allocation) if for each mechanism $G \in \mathcal{G}$, there exists a posterior $\pi \in \Delta(X_1)$ with a BNE allocation $f' \in BN(G, \pi)$ such that $U^f_1(x_1) \geq U^{f'}_1(x_1)$ for each $x_1$. This equilibrium concept is equivalent to expectational equilibrium of Myerson (1983).

**Intuitive equilibrium.** To eliminate equilibria supported by “unreasonable” posterior beliefs, we focus on equilibria that pass the intuitive criterion. Following MT, we establish the criterion in our mechanism-selection game as follows: For each equilibrium $f$ and mechanism $G$, let $X^f_1(G)$ be the set of types $x_1$ such that

$$U^f_1(x_1) > U^{f'}_1(x_1) \text{ for each } \pi \in \Delta(X_1) \text{ and each } f' \in BN(G, \pi).$$

An equilibrium $f$ is called *intuitive* if there exists no mechanism $G$ such that, for some $x_1 \in X_1 \setminus X^f_1(G)$,

$$U^f_1(x_1) < U^{f'}_1(x_1) \text{ for each } \pi \in \Delta(X_1 \setminus X^f_1(G)) \text{ and each } f' \in BN(G, \pi).$$

Here, $\Delta(X_1 \setminus X^f_1(G))$ is called the set of *reasonable* posteriors. In other words, condition (1) indicates that the losing type $x_1 \in X^f_1(G)$ always loses by deviating from $f$ to $G$, while condition (2) indicates that the non-losing type $x_1 \notin X^f_1(G)$ gains from the same deviation, provided it induces the buyer to have a reasonable posterior. In particular, condition (2) requires that the deviator $x_1$ should benefit from the deviation if it convinces the buyer of her true type (i.e., $\pi(x_1) = 1$). We say that type $x_1$ can *convincingly (and profitably) deviate* from $f$ to $G$ if (2) is satisfied. We then discard the equilibrium $f$ as unintuitive.

Condition (1) is stringent because it requires that the losing type should suffer
from the deviation for arbitrary posterior beliefs $\pi$. If the buyer, observing the deviation, believes that the seller’s type is extremely high (i.e., the good is much valuable for the buyer), then the seller of lower types, expecting high revenues, may have incentives to deviate. Hence, the existence of higher types prevents middle types convincingly deviating to some mechanisms. This kind of problem motivates Cho and Kreps (1987) to introduce stronger criteria including D1. In sum, the stringency of condition (1) potentially weakens the cutting power of the intuitive criterion, but makes a convincing deviation more “credible.”

3 RSW allocation

We introduce an important concept for informed-principal problems with interdependent values: the RSW allocation or, equivalently, the best safe mechanism.

3.1 Characterization

In our single-agent model, a feasible allocation is called safe (Myerson, 1983) if it is EPIC and EPIR for the buyer: For each $x \in X$ and $\hat{x}_2 \in X_2$,

$$u^f_2(x) \geq u^f_2(\hat{x}_2 | x),$$  \hspace{5em} (B-EPIC)  

$$u^f_2(x) \geq 0,$$ \hspace{5em} (B-EPIR)

where $u^f_2(\hat{x}_2 | x) \equiv u_2(f(x_1, \hat{x}_2), x)$ and $u^f_2(x) \equiv u_2(f(x_1, x_2), x)$.

**Definition 1.** An allocation $f^*$ is RSW if it is a solution to the following problem for each $x_1$:

$$\max_{f \in A^X} U^f_1(x_1) \quad \text{s.t. } f \text{ satisfies (S-IC), (B-EPIC), and (B-EPIR)}. \quad (3)$$

The allocation possesses three basic properties. First, at least one RSW allocation exists. In particular, if $f^{x_1}$ denotes a solution to problem (3), then the combination $(f^{x_1}(x_1, \cdot))_{x_1 \in X_1}$ of their components satisfies all constraints in (3), and hence, it is an RSW allocation. Second, although multiple RSW allocations may exist, the seller’s RSW payoff vector $U^f_1$ is unique among all RSW allocations. Third, every RSW allocation satisfies (S-IR) because the no-trade allocation is safe.

We characterize the RSW allocations in three steps. First, we observe that any RSW allocation is the best safe allocation for every type of seller. In other words,
there is no conflict of interest over safe allocations among different seller types. Hence, $f^*$ is an RSW allocation if and only if it solves the following problem:

$$\max_{f \in A} \mathbb{E}_{x} [U_f^T(x_1)] \quad \text{s.t. } f \text{ satisfies (S-IC), (B-EPIC), and (B-EPIR)}. \quad (4)$$

Indeed, the same equivalence holds even if the objective function in (4) is replaced by $\sum_{x_1} w(x_1) U_f^T(x_1)$, provided the weights are all positive. We focus on the single problem (4) instead of the set of problems (3).

Second, we formulate a relaxed problem following the suggestion of MT (footnote 33). Note that (B-EPIC) requires the allocation rule $q$ to be ex-post monotone for the buyer, that is, $q \in Q \equiv \{ q' \in [0,1]^X \mid q'$ is increasing in $x_2 \}$. We then define the relaxed problem as follows:

$$\max_{f=(q,t) \in Q \times \mathbb{R}^X} \mathbb{E}_{x} [U_f^T(x_1)]$$

s.t. $U_f^T(x_1) \geq U_f^T(x_1^+ \mid x_1) \quad \forall x_1 < \bar{x}_1$. \quad \text{(S-IC-U)}

$u_f^2(x) \geq u_f^2(x_1^+ \mid x) \quad \forall x$ with $x_2 > 1$. \quad \text{(B-EPIC-D)}

$u_f^2(x_1, 1) \geq 0 \quad \forall x_1$. \quad \text{(B-EPIR-B)}

This relaxed-problem approach seems quite natural. The buyer’s local downward EPIC constraints, together with his EPIR at the bottom, immediately imply (B-EPIR). Further, recall that the buyer’s valuation $v_2$ is increasing in $x_1$ and strictly increasing in $x_2$. The sorting assumptions imply that, as in standard signaling and screening models, the seller wishes to overstate her type to increase revenues and the buyer wishes to understate his type to decrease payments.

Finally, we characterize the solutions to problem (5) by taking a Lagrangian approach. We then check that they satisfy the omitted constraints and thus are equivalent to the RSW allocations. A simple observation is that the constraints (B-EPIC-D) and (B-EPIR-B) are binding. Intuitively, each type $x_1$ can lower the buyer’s information rents as much as possible by reducing allocation probabilities in such a way that the adjacent type $x_1^-$ no longer mimics $x_1$. The ex-post payments are pinned down by the allocation rule $q$ as follows: For each $x$,

$$t(x) = v_2(x) q(x) - \sum_{\hat{x}_2 < x_2} d v_2(\hat{x}_2) q(x_1, \hat{x}_2). \quad (6)$$
Then, a standard argument implies that the seller’s interim revenue is given by
\[ E_{x_2}[t(x)] = E_{x_2} \left[ \left( v_2(x) - \frac{1 - P_2(x_2)}{p_2(x_2)} dv_2(x_2) \right) q(x) \right] = E_{x_2}[\psi_2(x)q(x)] \tag{7} \]
for each \( x_1 \), where \( \psi_2 \equiv v_2 - \frac{1 - P_2}{p_2} dv_2 \) is the buyer’s virtual valuation. If the seller’s type \( x_1 \) was common knowledge, it would be optimal for her to set a full-information monopoly price to maximize her profit \( E_{x_2}[(\psi_2 - v_1(x))q(x)] \). In the RSW, the lowest-type seller obtains her full-information profit, while a higher type \( x_1^+ \) maximizes her interim profit subject to the local upward IC constraint
\[ E_{x_2}[(\psi_2(x) - v_1(x))q(x)] \geq E_{x_2}[(\psi_2(x_1^+, x_2) - v_1(x))q(x_1^+, x_2)] \tag{8} \]

Through these steps, we characterize the RSW allocations as follows:

**Theorem 1.** An allocation \( f = (q, t) \) is RSW if and only if the payment rule \( t \) is determined by the formula (6) given \( q \) and there exists a belief \( \pi \in \Delta(X_1) \) with the cdf \( \Pi \) such that (i) the prior \( P_1 \) weakly first-order stochastically dominates the posterior \( \Pi \); (ii) the allocation rule \( q \) maximizes
\[ E_\pi \left[ \left( \psi_2(x) - v_1(x) - \frac{\Pi(x_1^-) - P_1(x_1^-)}{\pi(x_1^-)} dv_1(x_1) \right) q'(x) \right] \tag{9} \]
among all \( q' \in Q \); and (iii) for each \( x_1 < \bar{x}_1 \), \( q \) satisfies the seller’s local upward IC constraint (8), and either the equality of (8) or \( \Pi(x_1) = P_1(x_1) \) (or both) holds.

Theorem 1 extends the “inductive” characterization given by MT (Proposition 2) to our trading model with bilateral asymmetric information. The essential ingredient is the posterior belief \( \pi \) that characterizes the RSW allocation rule \( q \). The prior weakly first-order stochastically dominates the posterior (i.e., \( P_1(x_1) \leq \Pi(x_1) \) for each \( x_1 \)). In fact, the proof of Theorem 1 shows that the cdf is defined by \( \Pi \equiv P_1 + \kappa \) given a vector of nonnegative Lagrange multipliers \( \kappa \) for the seller’s IC constraints (8). Hence, the latter condition in (iii) is a complementary slackness condition. We can interpret the terms in parentheses in (9) as the virtual surplus, that is, the buyer’s virtual valuation minus the seller’s virtual cost. The virtual cost is greater than her opportunity cost \( v_1(x) \) unless she has the lowest type or the adjacent type has no overstatement incentive (i.e., the multiplier \( \kappa(x_1^-) = \Pi(x_1^-) - P_1(x_1^-) \) is zero).

To glean some intuition behind Theorem 1, we further examine the implications

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10See, for example, Börgers et al. (2015) for this benchmark result.
of the virtual surplus. As a benchmark, let us consider the case of private values (i.e., \( v_i(x) = x_i \) for each \( i \)). Then, the buyer’s virtual valuation \( \psi_2 \) no longer depends on \( x_1 \), so that the seller’s IC constraints (8) are nonbinding as in Maskin and Tirole (1990). The virtual surplus reduces to

\[
\left( x_2 - \frac{1 - P_2(x_2)}{p_2(x_2)} \right) - \left( x_1 + \frac{P_1(\kappa_1)}{p_1(\kappa_1)} \right) + \frac{P_1(\kappa_1)}{p_1(\kappa_1)}
\]

because \( \kappa = 0 \), \( \Pi = P_1 \), and \( dv_i = 1 \). The two bracketed terms in (10) appear in an outsider’s problem of Myerson and Satterthwaite (1983). Here, \( 1 - P_2(x_2) \) and \( P_1(\kappa_1) \) are the expected information rents paid by this outsider (e.g., social planner or broker) to the buyer and the seller, respectively.\(^{11}\) However, unlike in their model, the latter information rents are “repaid” to each type of principal in our model. This repayment corresponds to the last term in (10).

Next, let us consider the case of interdependent values. In general, the buyer’s virtual valuation \( \psi_2 \) is strictly increasing in \( x_1 \), so that the seller’s local upward IC constraints (8) bind. If this is the case, the adjustment term

\[
\frac{\Pi(\kappa_1) - P_1(\kappa_1)}{\pi(\kappa_1)} dv_1(\kappa_1)
\]

in (9) is positive for some \( x_1 \) with \( \kappa(\kappa_1) > 0 \). This adjustment is “due to possible desirable distortions arising from redistribution of income” in the phrase of Ledyard and Palfrey (2007), who characterized interim (incentive) efficient allocations in linear-IPV environments. In particular, the prior \( P_1 \) puts weights on high types more heavily than the posterior \( \Pi \) does as if it is desirable for the seller from the ex-ante point of view to “redistribute” rents from low to high types. However, this difference in weight endogenously arises from multipliers for the seller’s IC constraints (8), unlike in Ledyard and Palfrey (2007).

An important implication of Theorem 1 is that this redistribution is achieved through undersupply of the good. Specifically, each type \( x_1 \) chooses probabilities \( q(x_1, \cdot) \) to maximize the expected virtual surplus (9) given \( x_1 \). The seller distorts allocation probabilities away from efficient (i.e., social-surplus-maximizing) ones to reduce not only the buyer’s rents, but also her own overstatement incentives. This is an important consequence of interdependent values.

**Remark 1.** Let us consider the regular case wherein the virtual surplus is strictly increasing in the buyer’s type (this is equivalent to the primitive condition that

\(^{11}\)In the case of continuum type spaces, \( P_1(\kappa_1) \) is replaced by \( P_1(x_1) \).
\( \psi_2 - v_1 \) is strictly increasing in \( x_2 \). In this case, the RSW allocation \((q^*, t^*)\) takes a simple form because the ex-post monotonicity constraints \( q^* \in Q \) are nonbinding. Specifically, the allocation rule \( q^* \) maximizes the virtual surplus pointwise. Hence, each \( x_1 \) has a threshold type \( x_2^*(x_1) \) such that \( q^*(x) = 0 \) if \( x_2 < x_2^*(x_1) \) and \( q^*(x) = 1 \) if \( x_2 > x_2^*(x_1) \). If the buyer had a continuum type space with a continuous valuation function, the payment rule would determine prices for the good as \( t^*(x) = v_2(x_1, x_2^*(x_1)) \) for each \( x \) with \( q^*(x) = 1 \). For each \( x_1 \), this price is (weakly) higher than her full-information monopoly price, worsening undersupply.

**Remark 2.** Myerson (1985) derived a neutral optimum for an informed seller in a bilateral-trade model with an uninformed buyer. This derivation is based on the seller’s “virtual cost” (or “virtual valuation”) for some welfare weights on seller types. Balkenborg and Makris (2015) also characterized the assured allocation using the “virtual surplus” in a nonlinear-common-values environment with one-sided asymmetric information. The assured allocation is inductively defined via a sequence of optimization problems that include the principal’s “assured claim” constraints and the agent’s non-ex-post participation constraints. Some essential differences between these allocations and the RSW allocation are discussed in Section 5.4.

### 3.2 Undominatedness

Now, following Myerson (1983) and MT, we introduce the Pareto dominance relation on allocations among all seller types:

**Definition 2.** Let \( f \) and \( f' \) be a pair of allocations. We say that (i) \( f' \) weakly dominates \( f \) if \( U''_1(x_1) \geq U''_1(x_1) \) for each \( x_1 \); (ii) \( f' \) dominates \( f \) if \( U''_1(x_1) > U''_1(x_1) \) for each \( x_1 \), with strict inequality for at least one \( x_1 \); (iii) \( f \) is undominated for \( \pi \in \Delta(X_1) \) if \( f \) is \( \pi \)-feasible and \( f \) is not dominated by any other \( \pi \)-feasible allocation; and (iv) \( f \) is undominated if it is undominated for the prior \( p_1 \).

Given any belief \( \pi \), the following theorem provides a necessary and sufficient condition under which the RSW allocation is undominated for \( \pi \).

**Theorem 2.** Let \( f = (q, t) \) be an RSW allocation and \( \pi \in \Delta(X_1) \) be a belief with the cdf \( \Pi \). Then, \( f \) is undominated for \( \pi \) if and only if there exists an interior belief \( w \in \Delta(X_1) \) with the cdf \( W \) such that (i) \( W \) weakly first-order stochastically dominates \( \Pi \); (ii) the allocation rule \( q \) maximizes

\[
E_x^\pi \left[ \left( \psi_2(x) - v_1(x) - \frac{\Pi(x_1^{-}) - W(x_1^{-})}{\pi(x_1)} dv_1(x_1) \right) q'(x) \right]
\]  

(12)
among all \( q' \in Q \); and (iii) for each \( x_1 < \bar{x}_1 \), either the equality of the seller’s local upward IC constraint (8) or \( W(x_1) = \Pi(x_1) \) (or both) holds.

The idea behind the proof is as follows: A standard argument based on the supporting hyperplane theorem establishes that an allocation \( f \) is undominated for the given belief \( \pi \) if and only if there exists an interior belief \( w \in \Delta(X_1) \) such that \( f \) is a solution to the following problem:

\[
\max_{f \in A} E_{x_1}^w[U_f(x_1)] \quad \text{s.t.} \quad f \text{ is \( \pi \)-feasible.} \tag{13}
\]

Hence, it is sufficient to show that, given any interior \( w \), the RSW allocation \( f = (q,t) \) is a solution to (13) if and only if the belief \( w \) with the rule \( q \) satisfies conditions (i)–(iii) in Theorem 2. The intuition behind this characterization is similar to that for Theorem 1. Importantly, the cdf \( W \) must put weights on high types more heavily than the given cdf \( \Pi \) by Lagrange multipliers for the seller’s local upward IC (8). Otherwise, the redistribution of rents among seller types could increase her weighted average payoff in (13).

As a technical step, we must show that relaxing the buyer’s ex-post monotonicity constraints to the interim ones cannot increase the expected virtual surplus (12). Note that the virtual surplus is additively separable in \( x \). This leads us to apply a transformation method developed by Gershkov et al. (2013). This method ensures that each interim-monotone allocation rule has an ex-post-monotone rule with the same interim probabilities as the original rule.

We now apply Theorem 2 to obtain two important results. First, we denote the prior by \( w = p_1 \). Theorem 1 then guarantees the existence of a belief \( \pi \) such that the interior prior \( w \) with the RSW allocation rule \( q \) satisfies conditions (i)–(iii) in Theorem 2 given \( \pi \). This belief \( \pi \) plays a key role in proving that the RSW is an intuitive equilibrium. Additionally, using Theorem 2 twice, we can show that the set of beliefs \( \pi \) for which the RSW is undominated is convex. We thus obtain the following result, which corresponds to the corollary in Section 3.B of MT.\(^{12}\)

**Corollary 1.** For any RSW allocation, the set of beliefs \( \pi \) for which this allocation is undominated is nonempty and convex.

Second, we denote the prior by \( \pi = p_1 \). Theorem 2 then provides a necessary and sufficient condition for the RSW allocation to be undominated for the prior \( \pi = p_1 \). As noted by MT, the undominatedness of the RSW is equivalent to the

\(^{12}\)Unlike MT, we consider the condition of undominatedness, which is stronger than interim efficiency for RSW allocations. See footnote 4 for why we need the stronger condition.
existence of a strong solution (i.e., safe and undominated mechanism) introduced by Myerson (1983). Moreover, we show in Section 5.2 that the equilibrium characterization of MT holds in our model. That is, a feasible allocation is an equilibrium if and only if it weakly dominates the RSW. Together with this result, Theorem 2 allows us to characterize the set of prior beliefs for which the seller’s equilibrium payoff vector is uniquely determined by her RSW payoffs. Roughly, if the interior prior $p_1$ assigns a higher probability to the lowest type, then it is easier to find an interior belief $w$ with which the RSW allocation rule $q$ satisfies conditions (i)–(iii) in Theorem 2. The next section illustrates this characterization.

4 Illustrative example

We now consider a regular-case example to illustrate how the RSW allocation is derived and how non-RSW equilibrium allocations are eliminated as unintuitive.

Assume that $X_i = \{1, 2\}$ for each $i$ and $p_2$ is uniform, while $p_1$ is an arbitrary interior belief. The valuations are $v_1(x) = 100(x_1 - 1)$ and $v_2(x) = 300x_1 + 100x_2$. It is efficient to always allocate the good to the buyer. For each type $x_1$, her full-information monopoly price is $300x_1 + 100$. However, these prices are infeasible for the informed seller because the low-type seller mimics the high type.

From Theorem 1, the RSW $f^* = (q^*, t^*)$ is uniquely determined by Table 1, showing both $q^*(x)$ and $t^*(x)$ in each cell. The low-type seller selects her full-information price, while the high-type seller sells the good less often by raising the price to make the low type indifferent between truth-telling and lying.

From Theorem 2, $f^*$ is undominated (for the prior $p_1$) if and only if there exists an interior belief $w \in \Delta(X_1)$ such that $W \leq P_1$ and $q^*$ maximizes the virtual surplus in (12) pointwise, given $\pi = p_1$. Note that condition (iii) (i.e., complementary slackness condition) in Theorem 2 is trivially satisfied because the low-type seller’s IC binds in the RSW $f^*$. The virtual-surplus maximization is achieved if and only if the virtual surplus is nonpositive at $x = (2, 1)$ and nonnegative at $x = (2, 2)$. This implies that the undominatedness of $f^*$ is equivalent to $p_1(1) \in (5/6, 1)$.\(^{13}\)

The equilibrium characterization in Section 5.2 implies that the seller’s equilibrium payoff vector is unique if and only if the RSW is undominated (i.e., $p_1(1) \in (5/6, 1)$). Let us assume that $p_1$ is uniform. Then, the set of equilibria...

\(^{13}\)If $p_1(1) = 5/6$, there exists no interior $w$ such that the virtual surplus is nonpositive at $x = (2, 1)$. Then, $f^*$ is dominated by the constant feasible allocation $(1,450)$. Further, if $\pi(1) = 1$, the virtual surplus given any interior $w$ is negative infinity at $x = (2, 2)$. Then, $f^*$ is dominated by the $\pi$-feasible allocation $f$ such that $f(1, \cdot) = (1, 400)$ and $f(2, \cdot) = (0, 400)$.
Table 1: RSW allocation $f^*$

\[
\begin{array}{cc|cc}
  x_1 & 1 & x_2 & 2 \\
  \hline
  x_1 & 1 & 0 & 1 \\
  x_2 & 1 & 400 & 800 \\
\end{array}
\]

Table 2: Equilibrium allocation $f'$

\[
\begin{array}{cc|cc}
  x_1 & 1 & x_2 & 2 \\
  \hline
  x_1 & 1 & 4/5 & 30 \\
  x_2 & 1 & 1 & 960 \\
  x_1 & 2 & 1 & 0 \\
  x_2 & 2 & 4/5 & 0 \\
\end{array}
\]

Figure 1: Seller’s equilibrium payoff vectors

Equilibrium payoff vectors for the seller is characterized as the red triangle in Figure 1.\(^{14}\)

For example, consider the allocation $f' = (q', t')$ in Table 2. Simple computations show that $f'$ is feasible and $U_1^{f'} = (495, 405) > (400, 350) = U_1^{f^*}$. Hence, $f'$ is an equilibrium. We show that the high-type seller can convincingly deviate from the equilibrium $f'$ to another mechanism, so that $f'$ is unintuitive.

We prove this in two steps. First, we transform $f'$ into a well-behaved allocation. Note that $f'$ is neither EPIR nor EPIC for the buyer. In particular, the decreasing function $q'(2, \cdot)$ violates the ex-post monotonicity. In the equilibrium $f'$, if the high-type seller gains from exchanging slack variables on the EPIC and EPIR constraints with the low-type seller, the high type is reluctant to make a revealing deviation. However, she has no such benefit. To see why, consider the constant feasible allocation $f$ in Table 3. It still violates (B-EPIR), but does satisfy (B-EPIC). Moreover, $f$ is interim-payoff-equivalent to the original allocation $f'$ (i.e., $U_i^f = U_i^{f'}$ for each $i$).

Next, we construct a “less-trading” mechanism from $f$. Because the buyer’s EPIR constraints are slack for the high-type seller in $f$, she can deliver the good less often.\(^{15}\) However, this requires her to lower the buyer’s payments to prevent

\(^{14}\)See Appendix B in the Supplementary material for this characterization.

\(^{15}\)Indeed, the high-type seller must lower interim allocation probabilities to make a convincing and profitable deviation. This is because the low-type seller’s IC binds in the allocation $f$.\]
the low-type seller’s deviation. Specifically, we define a convex combination

\[ \tilde{f} \equiv (1 - \delta)f + \delta(0, \tau) \]  \hspace{1cm} (14)

of the two (constant) allocations, where \( \tau \in (405, 495) = (U_f(2), U_f(1)) \) is a fixed fee and \( \delta \in (0, 1) \) is a small weight with \( u_{f2}(2, 1) > 0 \). The interim-payoff equivalence implies that the low-type seller prefers the equilibrium \( f' \) to the new allocation \( \tilde{f} \), while the high-type seller has the opposite preference:

\[ U_{f'}(1) = U_f(1) > (1 - \delta)U_f(1) + \delta\tau = U_{\tilde{f}}(1), \] \hspace{1cm} (15)

\[ U_{f'}(2) = U_f(2) < (1 - \delta)U_f(2) + \delta\tau = U_{\tilde{f}}(2). \] \hspace{1cm} (16)

In fact, the low type loses by deviating to the less-trading mechanism \( \tilde{f} \) regardless of the buyer’s posterior and response. The buyer, believing that \( \tilde{f} \) is offered by the non-losing high-type seller, is willing to accept it. Thus, the high-type seller can convincingly deviate from the equilibrium \( f' \) to the direct mechanism \( \tilde{f} \).

After all, the equilibrium \( f' \) is supported by unreasonable posteriors. To support it, the buyer must believe that \( \tilde{f} \) is sometimes offered by the low-type seller, who always suffers from the deviation. Similarly, the undominated equilibrium \( f^{**} \) in Table 4 is also eliminated as unintuitive. It should be noted that the unintuitive equilibrium \( f^{**} \) is efficient, while the RSW allocation \( f^* \) is not.

## 5 Intuitive equilibrium allocation

In this section, we provide a justification for the RSW allocation using the intuitive criterion. We show that every RSW allocation is an intuitive equilibrium and every intuitive equilibrium is interim-payoff-equivalent to an RSW allocation.
5.1 Existence

The following theorem ensures that every RSW allocation is an intuitive equilibrium. Therefore, our mechanism-selection game has at least one intuitive equilibrium.

**Theorem 3.** Every RSW allocation is an intuitive equilibrium.

To obtain the intuition behind the proof, let us assume that the RSW allocation is an equilibrium supported by reasonable posteriors. We then observe that no type of seller can convincingly deviate from the RSW to any other mechanism, and hence, the RSW is intuitive. The reason is simple: If some type had a convincing deviation, the reasonable posterior could no longer support the RSW as an equilibrium. Thus, to prove Theorem 3, it is sufficient to find reasonable posteriors that support the RSW as an equilibrium. Here, a belief $\pi$ for which the RSW is undominated plays a crucial role. Note that the existence of such a belief is guaranteed by Corollary 1. The idea is to derive, for each off-path mechanism $G$, a reasonable posterior $\pi^G$ from this belief $\pi$.

We outline the arguments as follows: In the first step, given any off-path mechanism $G$, we formulate an auxiliary mechanism-selection game wherein $(x_1, x_2)$ are realized according to *not* $(p_1, p_2)$, *but* $(\pi, p_2)$, and subsequently the seller selects either offering $G$ or ending up with the status quo $f^\star$. We find a trembling-hand perfect equilibrium of the auxiliary game. The trick is to choose a sequence of perturbed games wherein the losing types $x_1 \in X^f_1(G)$ “mistakenly” select $G$ less often than the non-losing types $x'_1 \not\in X^f_1(G)$ do. Naturally, this difference in the mistake likelihood ensures that any limit $\pi^G$ of equilibrium posteriors is reasonable, that is, the buyer believes that $G$ is never offered by losing types.

In the second step, we show that this reasonable posterior $\pi^G$ supports the RSW allocation as an equilibrium of the original mechanism-selection game. Here, we follow the argument of Myerson (1983, Theorem 2). To clarify this argument, we denote the perfect equilibrium allocation by

$$f'(x) \equiv \gamma^G(x_1)f(x) + (1 - \gamma^G(x_1))f^\star(x) \quad (17)$$

for each $x$, where $\gamma^G(x_1)$ is the probability with which type $x_1$ selects $G$ and $f \in BN(G, \pi^G)$ is the continuation BNE in the perfect equilibrium. Because each type of seller selects her favorable allocation in equilibrium, the allocation $f'$

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16This observation immediately follows from the definitions of reasonable posterior and convincing deviation at the end of Section 2. See Cho (1987) for a more detailed analysis.
weakly dominates \( f^* \). Moreover, \( f' \) is \( \pi \)-feasible because the parties use equilibrium strategies in the auxiliary game given the belief \( \pi \) and the RSW is safe. At this point, we use the undominatedness property: The RSW is not dominated by any \( \pi \)-feasible allocation. Hence, \( U_f^{f^*} = U_f^{f'} \). This implies that \( f^* \) weakly dominates \( f \). Thus, the reasonable posterior \( \pi^G \) with the continuation BNE \( f \) prevents the seller’s deviation from the RSW to the off-path mechanism.

**Remark 3.** We cannot apply these arguments to non-RSW allocations. To clarify this point, let us consider the equilibrium \( f \) in Table 3. We have already shown that \( f \) is unintuitive. We take \( \tilde{f} \) defined by (14) as an off-path mechanism, and consider an auxiliary game wherein the seller selects either \( \tilde{f} \) or \( f \). If a perfect equilibrium in this auxiliary game generates the reasonable posterior (i.e., \( \pi^{\tilde{f}}(2) = 1 \)) as in the first step, it dominates \( f \) because the high-type seller selects the new \( \tilde{f} \) rather than the status quo \( f \). Hence, the argument in the second step fails for this particular perfect equilibrium. However, another perfect equilibrium generates an unreasonable posterior that supports \( f \) as an equilibrium.\(^{17}\)

### 5.2 Equilibrium characterization

To highlight the effectiveness of the intuitive criterion as a refinement concept, we characterize the set of equilibrium allocations including unintuitive ones. Due to Corollary 1, the following proposition yields the same conclusion as that of Theorem 1* in MT.

**Proposition 1.** Suppose that an RSW allocation is undominated for some \( \pi \in \Delta(X_1) \). Then, the set of equilibrium allocations of the mechanism-selection game is the set of feasible allocations that weakly dominate the RSW allocations.

As a corollary of Theorem 3, every feasible allocation that weakly dominates the RSW allocations is an equilibrium. Thus, to prove Proposition 1, we need only show that each type of seller obtains at least her RSW payoff in equilibrium. Although every RSW allocation is by definition safe, simply selecting it may not guarantee the RSW payoff. This is due to the multiplicity of continuation equilibria. For example, the direct mechanism \( f^* \) in Table 1 has an untruthful BNE wherein each party always reports the low type. This is an ex-post equilibrium, and thus remains a BNE regardless of the buyer’s posterior. In the untruthful BNE, the high-type seller obtains \( 400 - 100 = 300 \). This is less than her RSW payoff.

\(^{17}\)By contrast, in the case of the RSW, the argument in the second step is valid for every perfect equilibrium. This property is closely related to strategic stability (Kohlberg and Mertens, 1986).
payoff $U^*_f(2) = 350$. We address the problem of multiple continuation equilibria by constructing an Abreu–Matsushima (AM) mechanism (Abreu and Matsushima, 1992a,b). This methodology is described in Section 5.3.

**Remark 4.** The hypothesis of Theorem 1* of MT is that the RSW allocation is interim efficient for some interior beliefs. The hypothesis corresponds to that of Proposition 1, but there are two differences. First, we strengthen the condition of interim efficiency to undominatedness (for some beliefs). As argued by Dosis (2022), the interim efficiency is insufficient for the characterization result of MT in some cases. Second, we prove Proposition 1 without the interior-belief requirement. In effect, we prove in the second step of Theorem 3 that, if a safe mechanism is undominated for some posteriors $\pi$, the mechanism (i.e., RSW) is an equilibrium. This proof is based on that of Myerson (1983, Theorem 2). He shows that, if a safe mechanism is undominated for the prior, the mechanism (i.e., strong solution) is an equilibrium. The essential difference from Myerson’s original proof is that we formulate an auxiliary mechanism-selection game by using the posterior $\pi$, not the prior $p_1$.

As emphasized by MT, the equilibrium characterization implies that the principal’s equilibrium payoff vector is unique if and only if the RSW allocation is undominated. This condition is characterized by Theorem 2. In the example of Section 4, the RSW is undominated if and only if the prior assigns a sufficiently high probability to the low type (i.e., $p_1(1) \in (5/6, 1)$). If the RSW is dominated, the mechanism-selection game has infinitely many equilibrium allocations. Therefore, we need to refine equilibria to obtain a stronger prediction.

### 5.3 Uniqueness

The intuitive criterion is weaker than well-known concepts including (universal) divinity (Banks and Sobel, 1987), perfect sequential equilibrium (Grossman and Perry, 1986), neologism-proofness (Farrell, 1993), and strong neologism-proofness (Mylovanov and Tröger, 2012, 2014). The following theorem, together with Theorem 3, shows that this reasonable criterion selects only a feasible allocation that yields the seller’s RSW payoffs.

**Theorem 4.** For every intuitive equilibrium $f$, $U^f_1$ is equal to the RSW payoff vector $U^*_f$.

The idea of the proof is illustrated in Section 4 and outlined as follows: From the equilibrium characterization (Proposition 1), we need only show that an equi-
librium $f'$ dominating the RSW is unintuitive. First, we prove the interim-payoff equivalence. This result provides a new feasible allocation $f$ that satisfies (B-EPIC) and is interim-payoff-equivalent to $f'$. As $f'$ dominates the RSW, so does the equivalent $f$. Hence, $f$ must violate (B-EPIR). This implies that the lowest-type buyer loses in $f$ for some seller types. However, $f$ satisfies the constraints (B-$p_1$-IR). This in turn implies that the lowest-type buyer benefits from $f$ for another type $\tilde{x}_1$. We call this type $\tilde{x}_1$ a candidate deviator. Second, we construct a less-trading allocation $\tilde{f}$ nearby $f$, as in (14). Further, we design an indirect mechanism $G$ that “virtually” implements $\tilde{f}$. This implementation is necessary only if the candidate should offer a mechanism wherein both parties make reports. Finally, we show that the candidate $\tilde{x}_1$ can convincingly deviate from the original equilibrium $f'$ to the mechanism $G$.

These construction procedures are summarized in Figure 2. Each step is described in detail below, with reasons as to why these approaches are needed.

**Payoff-equivalent allocation.** First, we prove the following result:

**Proposition 2.** Suppose that an allocation $f'$ satisfies (S-IC) and (B-$p_1$-IC). Then, there exists an allocation $f$ that satisfies (S-IC), (B-EPIC), and $U_i^f = U_i'^f$ for each $i = 1, 2$.

Such results are essential to eliminate all equilibria dominating the RSW as unintuitive. As discussed in Sections 1 and 4, if some seller types mutually benefit from exchanging slack variables on the buyer’s EPIC and EPIR constraints, these types cannot make a convincing deviation. This is because the separation of types hurts them. Proposition 2 ensures that seller types have no benefit from slack exchange in any equilibrium.

The intuition behind Proposition 2 is as follows: As in Theorem 2, the transformation method of Gershkov et al. (2013) plays a key role. In the original allocation $f' = (q', t')$, the rule $q'$ may violate the ex-post monotonicity for the buyer. Then, their lemma ensures that there exists an ex-post-monotone allocation rule $q$ that has the same interim probabilities as the original rule $q'$. The equivalence of the interim probabilities immediately implies the equivalence of ex-ante social surpluses.
(i.e., $E_x[(v_2(x) - v_1(x))q(x)] = E_x[(v_2(x) - v_1(x))q'(x)]$), under the assumption that each valuation $v_i$ is additively separable in $x$. This allows us to construct a payment rule $t$ such that the new allocation $f \equiv (q, t)$ satisfies both (S-IC) and (B-EPIC) and is interim-payoff-equivalent to the original $f'$.

**Remark 5.** Gershkov et al. (2013) considered a linear-IPV environment wherein an uninformed principal designs a mechanism. They showed that each IC mechanism has a dominant strategy IC mechanism that yields the same interim payoffs for each agent and the same ex-ante social surplus. They also provided a counterexample to the equivalence result in an interdependent-values auction environment by showing that there exists no EPIC mechanism that yields the same ex-ante social surplus as an IC mechanism. Nevertheless, Gershkov et al. (2013, p. 213) stated that each IC mechanism has an EPIC mechanism that yields the same interim payoffs for each agent. In our model, the informed principal herself participates in a mechanism and no third party balances the budget. Because of this feature, we cannot extend Proposition 2 in such a way that the interim-payoff-equivalent allocation $f$ is EPIC for both parties.

**Less-trading allocation.** Next, we construct a less-trading allocation $\tilde{f}$ from the equivalent $f$. This step is simple if the seller’s type is binary. In this case, the candidate deviator $\tilde{x}_1$ must be the high type. As in Section 4, the high-type seller can offer the buyer a “menu” of less-trading outcomes, reporting nothing after the offer. If this menu is appropriately designed, the low-type seller prefers $f$ to the menu regardless of the buyer’s posterior. The high-type seller can convincingly deviate from the equilibrium $f'$ (and the equivalent $f$) to this kind of menu.

Unfortunately, this argument fails if the seller has more than two types. In some cases, the highest-type seller is not a candidate deviator. Then, recall the stringency of condition (1) for the losing types. A deviation to a simple menu can be profitable for lower types $x_1 < \tilde{x}_1$ if the buyer, believing that the menu is offered by higher types $x_1 > \tilde{x}_1$, purchases the good more often. Hence, to prevent the deviation of the lower types, the candidate $\tilde{x}_1$ should offer a mechanism in which the higher types choose their own outcomes by making reports.

Then, we construct a less-trading mechanism $\tilde{f}$ in the same manner as in Section 4. As in (14), this allocation $\tilde{f}$ is defined as a convex combination of

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18 If the buyer’s EPIR at the bottom was slack for the low-type seller, she would obtain less than her full-information payoff because the allocation $f$ is EPIC for the buyer.

19 Appendix C in the Supplementary material provides an example of an equilibrium wherein the seller cannot make a convincing deviation to any menu. See Cho and Kreps (1987) for a related example in the Spence signaling game with three worker types.
the original \( f \) and a fixed fee \( \tau \). This fee does not depend on reports. Hence, the allocation \( \tilde{f} \) satisfies the desired property (S-IC). Further, as \( f \) satisfies (B-EPIC), so does \( \tilde{f} \). If this fixed fee is appropriate (i.e., \( U_1^f(\tilde{x}_1) < \tau < U_1^f(\tilde{x}_1) \)), then all lower types \( x_1 < \tilde{x}_1 \) prefer the original allocation \( f \) to the less-trading \( \tilde{f} \), while all higher types \( x_1 \geq \tilde{x}_1 \) have the opposite preference, as shown by (15) and (16). This difference in the preference follows from the general result that higher seller types obtain lower interim payoffs in the IC allocation \( f \). The candidate \( \tilde{x}_1 \) is better off deviating from the equilibrium \( f' \) to the nearby mechanism \( \tilde{f} \), provided that the buyer participates in \( \tilde{f} \) and both parties tell the truth in \( \tilde{f} \).

**Remaining steps.** We have two technical problems. First, in the construction of \( \tilde{f} \), we implicitly assumed that the constraint (B-EPIR-B) is slack in the original \( f \) for each type \( x_1 > \tilde{x}_1 \) as well as for the candidate \( \tilde{x}_1 \). Without this assumption, the buyer may opt out of the mechanism \( \tilde{f} \) in the case that he believes that the seller has a non-losing type \( x_1 > \tilde{x}_1 \). We address this problem by reconstructing both allocations \( f \) and \( \tilde{f} \). Roughly, we construct an allocation \( f \) wherein the candidate forms a “coalition” \( \{ \tilde{x}_1, ..., \bar{x}_1 \} \) of the types for which the buyer’s EPIR constraints are slack. This construction is possible because the buyer’s valuation is increasing in \( x_1 \).\(^{20}\) We also reconstruct the nearby \( \tilde{f} \) as a convex combination of this modified \( f \) and a fixed fee. As before, \( \tilde{f} \) satisfies both (S-IC) and (B-EPIC). The buyer has incentives to participate in this mechanism \( \tilde{f} \) as long as he believes that the seller’s type is a coalition member. Although some members \( x_1 > \tilde{x}_1 \) may be worse off in this allocation \( \tilde{f} \) than in the equilibrium \( f' \), the candidate’s interim payoff is higher in \( \tilde{f} \). See Lemmas 3 and 4 in Appendix A for details.

Second, the seller’s reporting opportunities cause the problem of multiple continuation equilibria in the direct mechanism \( \tilde{f} \).\(^{21}\) To address this problem, we apply the methodology of Abreu and Matsushima (1992a,b). They showed that, in a complete (incomplete) information environment, every allocation (every IC allocation satisfying a weak condition) is virtually implementable in iteratively undominated strategies. Note that their mechanisms, which are finite strategic game forms, are admissible in our model. Our idea is simple: Using the equilibrium hypothesis—that the buyer’s off-path beliefs are common knowledge—we design an AM mechanism \( G \in G \) that elicits his posterior \( \pi^G \in \Delta(X_1) \) after it is offered. Then, the elicited belief is used to decide which allocation is virtually implemented.\(^{22}\) Roughly, if the posterior \( \pi^G \) assigns a high probability to the

\(^{20}\)However, some high types \( x_1 > \tilde{x}_1 \) whose valuations are so high that the no-trade outcome is favorable for them should be removed from this coalition.

\(^{21}\)Appendix D in the Supplementary material illustrates this problem.

\(^{22}\)This idea of using AM mechanisms is inspired by a discussion with Takuro Yamashita.
coalition \{\tilde{x}_1, ..., \tilde{x}_1\}, then the allocation \tilde{f} is implemented, and otherwise the no-trade outcome is implemented. The former implementation is possible because \tilde{f} is both IC and IR for the buyer given coalition members. Thus, the non-coalition members are losing types, and the candidate \tilde{x}_1 can convincingly deviate from the original equilibrium \(f'\) to the AM mechanism \(G\). Although these ideas are simple, a formal analysis is complicated. See Lemma 5 in Appendix A for details.

**Remark 6.** MT (Proposition 7) showed the same refinement result as Theorems 3 and 4 in the case of one-sided asymmetric information. They also proved that, under sorting assumptions, (a) the RSW allocation passes the Farrell–Grossman–Perry (FGP) criterion if and only if it is interim efficient for the prior; and (b) no other allocation passes the FGP. Moreover, MT (Proposition 11) regarded their model as a renegotiation game and used the intuitive criterion with a renegotiation proofness to refine the equilibria of the game. Interestingly, they showed that the cutting power of the intuitive criterion is weak in the renegotiation game. An analysis of the renegotiation proofness with the intuitive criterion is beyond the scope of this study.

Our last theorem shows that every intuitive equilibrium is also equivalent to an RSW allocation with respect to the buyer’s interim payoffs.

**Theorem 5.** For every intuitive equilibrium \(f\), there exists an RSW allocation \(f^*\) with \(U_f^2 = U_f^*\).

The intuition behind the result is simple: From the interim-payoff equivalence (Proposition 2), every intuitive equilibrium has an interim-payoff-equivalent feasible allocation \(f\) that satisfies (B-EPIC). From the uniqueness result for the seller (Theorem 4), the equivalent \(f\) yields the RSW payoffs to her. If the constraint (B-EPIR-B) is slack for a seller type \(x_1\) in the equivalent \(f\), she must deliver the good more often in \(f\) than in the RSW to earn her RSW payoff \(U_f^*(x_1)\). However, the adjacent \(x_\sim\) then mimics \(x_1\) to obtain higher than her RSW payoff \(U_f^*(x_\sim)\). Hence, the constraints (B-EPIR-B) in \(f\) must bind, and \(f\) is an RSW allocation.

This posterior-elicitation mechanism is unrealistic, but it is used only to upset an unintuitive equilibrium. The idea that a well-designed mechanism can elicit the agent’s posterior is suggested by Maskin and Tirole (1990, Proposition 7). Using the idea, they showed that any equilibrium of the mechanism-selection game is “strongly unconstrained Pareto optimal.”

\(^{23}\)Hence, the FGP implies the intuitive criterion. The relation holds in standard signaling games, as shown by Grossman and Perry (1986). See van Damme (1991) for other criteria.
5.4 Inefficiency

Finally, we discuss the economic implications of Theorems 1–5. In particular, we investigate the efficiency properties of intuitive equilibrium allocations.

The ex-ante social surpluses in intuitive equilibria are fully characterized by those in RSW allocations as follows: First, Theorem 3 shows that every RSW allocation is an intuitive equilibrium. Second, Theorems 4 and 5 imply that every intuitive equilibrium $f = (q, t)$ has an RSW allocation $f^* = (q^*, t^*)$ with the same ex-ante social surplus:

$$E_x[(v_2(x) - v_1(x))q(x)] = E_x[(v_2(x) - v_1(x))q^*(x)].$$

With this equivalence result, the characterization of RSW (Theorem 1) implies that, in general, the intuitive equilibrium allocations are inefficient due to undersupply. Moreover, even if the mechanism-selection game has an efficient equilibrium allocation dominating the RSW, it is eliminated as unintuitive. These inefficiency results are clearly illustrated by the example in Section 4. The unique RSW allocation $f^*$ in Table 1 is inefficient, and thus, no intuitive equilibrium is efficient. The equilibrium $f^{**}$ in Table 4 is efficient, but it is unintuitive.

The inefficiency of intuitive equilibria might increase the relative attractiveness of undominated equilibria. Then, alternative solution concepts should be the core mechanism and neutral optimum proposed by Myerson (1983), and the assured allocation introduced by Balkenborg and Makris (2015). In the example of Section 4, the efficient equilibrium $f^{**}$ is selected by the core mechanism and the neutral optimum, and other dominated allocations are eliminated by the two concepts. Further, following Balkenborg and Makris (2015), we can naturally extend their assured allocation to our environment with bilateral asymmetric information. This assured allocation is undominated, and hence, it also selects the efficient equilibrium $f^{**}$.

Nevertheless, the inefficiency itself should not be a reason to discard intuitive equilibria. Whenever an unintuitive equilibrium is common knowledge, some convincing deviations upset the equilibrium. While Cho and Kreps (1987) established the intuitive criterion using introspective arguments, Fudenberg and He (2020) introduced a learning-based equilibrium selection criterion that is stronger than the

\footnote{From the equilibrium characterization, an efficient feasible allocation is an equilibrium if it weakly dominates the RSW. See Fieseler et al. (2003) for a necessary and sufficient condition for the existence of an efficient feasible allocation in an interdependent-values environment with continuous type spaces.}
intuitive criterion and weaker than divinity.\textsuperscript{25} If the intuitive equilibrium allocations also describe long-run outcomes in our bilateral-trade environment, highly inefficient trading outcomes will emerge in the long run.

6 Concluding remarks

We provided a simple characterization of the RSW allocation in our bilateral-trade model with interdependent values. The RSW allocation is derived from the maximization of the expected virtual surplus given some posterior beliefs. We also identified a necessary and sufficient condition under which the RSW is undominated. If this condition does not hold, the RSW is dominated by infinitely many equilibrium allocations. However, the RSW allocations are interim-payoff-equivalent to the intuitive equilibrium allocations. From a normative viewpoint, the inefficient RSW allocation should not be selected. Nevertheless, we can expect that the intuitive RSW allocation describes well actual trading outcomes.

Bilateral trade is a fundamental economic activity, and therefore, our results can be applied to many problems. For example, in the growing literature on aftermarkets, a combination of mechanism and information design is an important problem (Dworczak, 2020). Then, let us consider a post-auction resale with a third party. Our results demonstrate how information disclosure rules in the auction affect resale outcomes. In particular, if an auction winner has full bargaining power in the resale and the resale always results in the belief-free RSW allocation, then disclosure rules are irrelevant to parties’ payoffs and resale surplus.

We have obtained clear results by focusing on the simple bilateral-trade model. While the assumption of additively separable values excludes some situations, it is useful for establishing the result of interim-payoff equivalence. This separability with sorting assumptions allows us to extend many of our results to other single-agent environments. However, the extension to multiple-agents environments (e.g., auction, collusion, and subcontracting) involves conceptual issues. Specifically, in the definition of the best safe mechanism, the agents’ EPIC and EPIR constraints should be replaced by their IC and IR constraints given each type of principal. An interesting question is how this best safe mechanism is characterized in multiple-agents environments. This analysis is left for future work.

\textsuperscript{25}By extending Theorem 3, we can show that every RSW allocation is a D1 (and hence divine) equilibrium. See Theorem 1 of Nishimura (2019) for this extension.
A Appendix

The following lemma shows that the IC constraints are characterized by the local upward and local downward ones. See, for example, Lemma 1 of Balkenborg and Makris (2015) for the proof.

**Lemma 1.** (i) An allocation $f$ satisfies (S-IC) if and only if for each $x_1 < \bar{x}_1$,

$$U_1^f(x_1) \geq U_1^f(x_1^+ | x_1), \quad \text{(S-IC-U)}$$

$$U_1^f(x_1^+) \geq U_1^f(x_1^+ | x_1^+). \quad \text{(S-IC-D)}$$

(ii) An allocation $f$ satisfies (B-EPIC) if and only if for each $x$ with $x_2 > 1$,

$$u_2^f(x_1, x_2^-) \geq u_2^f(x_2 | x_1, x_2^-), \quad \text{(B-EPIC-U)}$$

$$u_2^f(x) \geq u_2^f(x_2^- | x). \quad \text{(B-EPIC-D)}$$

We present a transformation method developed by Gershkov et al. (2013). While their method is general, the following result is sufficient for our analysis. See Lemmas 1–3 of their article (or Lemma 4 of Nishimura (2019)) for the proof.

**Lemma 2.** Let $q' \in [0, 1]^X$ be an allocation rule and fix any belief $\pi \in \Delta(X_1)$. Suppose that $Q_2^\pi$ is increasing in $x_2$. Let $q$ denote a solution to the problem:

$$\min_{q \in [0, 1]^X} E_x^\pi [(q(x))^2] \quad \text{s.t.} \quad Q_1 = Q_1', \ Q_2 = Q_2^\pi. \quad \text{(A.1)}$$

Then, $q(x_1, \cdot)$ is increasing in $x_2$ for each $x_1$ with $\pi(x_1) > 0$.\(^{26}\)

**Proof of Theorem 1.** By taking a Lagrangian approach, we show that the solutions to problem (5) in Section 3.1 are characterized by the conditions in the statement. Then, we prove that an allocation is a solution to problem (4) (i.e., RSW) if and only if it is a solution to the relaxed problem (5).

**Step 1.** We define the Lagrangian function $L$ for problem (5) as

$$L(f, \kappa, \lambda) \equiv E_x E_{x_1} \left[U_1^f(x_1) + \sum_{x_1 < x_1^+} \kappa(x_1) \left[U_1^f(x_1) - U_1^f(x_1^+ | x_1)\right]ight]$$

$$+ \sum_{x_1} \sum_{x_2 > 1} \lambda(x) \left[u_2^f(x) - u_2^f(x_2^+ | x)\right] + \sum_{x_1} \lambda(x_1, 1)u_2^f(x_1, 1), \quad \text{(A.2)}$$

\(^{26}\)Under the additional hypothesis that $Q_1'$ is decreasing, the solution $q$ to problem (A.1) is decreasing in $x_1$. However, the ex-post monotonicity for the seller is unnecessary for our results.
where \((\kappa, \lambda) \in \mathbb{R}_+^X \times \mathbb{R}_+^X\) is a vector of Lagrange multipliers (with \(\kappa(\bar{x}_1) \equiv 0\)).

The domain \(Q \times \mathbb{R}^X\) of \((5)\) is convex and contains an allocation that satisfies the constraints in \((5)\) with strict inequality.\(^{27}\) The functionals \(U_1^f(\cdot), U_1^f(\cdot | \cdot), u_2^f(\cdot | \cdot)\), and \(u_2^2(\cdot)\) are linear in \(f\). It then follows from the “saddle-point theorem” (see, for example, Luenberger (1969, Sections 8.3 and 8.4)) that \((q, t)\) is a solution to \((5)\) if and only if there exists a nonnegative vector \((\kappa, \lambda)\) such that the Lagrangian \(L\) has a saddle point at \(((q, t), (\kappa, \lambda))\):

\[
L(q', t', \kappa, \lambda) \leq L(q, t, \kappa, \lambda) \leq L(q, t, \kappa', \lambda')
\]  

(A.3)

for each \((q', t') \in Q \times \mathbb{R}^X\) and \((\kappa', \lambda') \in \mathbb{R}_+^X \times \mathbb{R}_+^X\).

Because payments can be any real number, the saddle-point condition \((A.3)\) requires that no payment rule should affect the Lagrangian \(L\) at \((\kappa, \lambda)\). This is equivalent to the condition that

\[
(p_1(x_1) + \kappa(x_1) - \kappa(x_1^-)) \cdot p_2(x_2) = \lambda(x_1, x_2) - \lambda(x_1, x_2^+)\]

for each \(x\), where \(\kappa(0) \equiv 0\) and \(\lambda(x_1, x_2^+) \equiv 0\). This condition pins down the multipliers \(\lambda\) as \(\lambda(x) = (p_1(x_1) + \kappa(x_1) - \kappa(x_1^-))(1 - P_2(x_2^-))\) for each \(x\).

In Steps 2–6, we derive necessary conditions for a solution \(f = (q, t)\) to problem \((5)\). Let \((\kappa, \lambda)\) be a nonnegative vector with which the allocation \(f\) satisfies the saddle-point condition \((A.3)\). We define \(\pi(x_1) \equiv p_1(x_1) + \kappa(x_1) - \kappa(x_1^-)\) for each \(x_1\).

**Step 2.** We show that the vector \(\pi\) is a belief satisfying condition (i). By definition, \(\sum_{x_1} \pi(x_1) = \sum_{x_1} p_1(x_1) = 1\). From Step 1, the multipliers \(\lambda\) are given by \(\lambda(x) = \pi(x_1)(1 - P_2(x_2^-))\) for each \(x\). As the vector \(\lambda\) is nonnegative, so is the vector \(\pi\). Hence, \(\pi \in \Delta(X_1)\). Its cdf is given by \(\Pi(x_1) \equiv \sum_{\hat{x}_1 \leq x_1} \pi(\hat{x}_1) = P_1(x_1) + \kappa(x_1) \geq P_1(x_1)\) for each \(x_1\).

**Step 3.** We claim that the allocation rule \(q\) with the belief \(\pi\) satisfies condition (ii). By substituting the multipliers \((\kappa, \lambda)\) into the Lagrangian and interchanging summations, we observe that, for each \(q'\), \(L(q', t, \kappa, \lambda)\) is equal to the expected virtual surplus \((9)\). Then, the saddle-point condition \((A.3)\) verifies our claim, that is, \(L(q, t, \kappa, \lambda) = \max_{q' \in Q} L(q', t, \kappa, \lambda)\).

**Step 4.** We claim that the payment rule \(t\) is determined by the formula \((6)\) given \(q\). Fix any \(x\). We have to prove that \(u_2^2(x) = u_2^2(x_2^- | x)\) if \(x_2 > 1\), and

\(^{27}\)For example, \((q, t)\) defined by \(q(x) = 0\) and \(t(x) = -(x_1 + x_2)\) for each \(x\) works well.
we observe that interim revenue is given by (7). Substituting these revenues into (S-IC-U) in (5), solves (5). Thus, \((q,t)\) satisfies (S-IC-U), \(\epsilon\) satisfies (iii), is equivalent to the complementary slackness condition given \(\kappa\). We show that Step 5. We show that \(q\) with \(\pi\) satisfies condition (iii). From Step 4, the seller’s interim revenue is given by (7). Substituting these revenues into (S-IC-U) in (5), we observe that \(q\) satisfies the inequality (8). Moreover, the latter condition in (iii) is equivalent to the complementary slackness condition given \(\kappa = \Pi - P_1\).

Step 6. We show that \(f\) is both EPIC and EPIR for the buyer. The allocation \(f\) satisfies both (B-EPIR-B) and (B-EPIC-D), and hence, it is EPIR for the buyer. From Step 4, \(f\) satisfies (B-EPIC-D) with equality. This, together with the ex-post monotonicity \(q \in Q\) for the buyer, implies that \(f\) is EPIC for the buyer.

We also show that \(f\) is IC for the seller. Because \(f\) satisfies (S-IC-U), it is sufficient to show that it satisfies (S-IC-D). Fix any \(x_1 < x_1\). To derive a contradiction, suppose \(U_1^{f}(x_1^+) < U_1^{f}(x_1^-)\). Let us define a new allocation \(f' = (q', t')\) as \(f'(x_1^+, \cdot) \equiv f(x_1, \cdot)\) and \(f'(x_1^+, \cdot) \equiv f(x_1^+, \cdot)\) for each \(x_1^+ \neq x_1^+\). By definition, \(f'\) satisfies (S-IC-U). It also satisfies (B-EPIC-D) because

\[
  t'(x_1^+, x_2) - t'(x_1^+, x_2^-) = (q(x_1, x_2) - q(x_1, x_2^-)) v_2(x_1, x_2) 
  \leq (q'(x_1^+, x_2) - q'(x_1^+, x_2^-)) v_2(x_1^+, x_2)
\]

for each \(x_2 > 1\), where the equality follows from the binding (B-EPIC-D) for \((q, t)\) and the inequality from the monotonicity of \(q\) in \(x_2\) and that of \(v_2\) in \(x_1\). Similarly, \(f'\) satisfies (B-EPIR-B). Thus, \(f'\) satisfies all the constraints in problem (5). This contradicts the hypothesis that \(f\) is a solution to (5).

From Steps 2–5, if \(f = (q,t)\) solves (5), the payment rule \(t\) is determined by the formula (6) and there exists a belief \(\pi \in \Delta(X_1)\) with which \(q\) satisfies conditions (i)–(iii). Conversely, if \(f\) satisfies these conditions, then the Lagrangian \(L\) for (5) has a saddle point at \((f, (\kappa, \lambda))\) given the nonnegative vector \((\kappa, \lambda)\) defined by \(\kappa \equiv \Pi - P_1\) and \(\lambda(x) \equiv \pi(x_1)(1 - P_2(x^-_2))\) for each \(x\). From Step 6, every solution to problem (5) satisfies all the constraints in problem (4). Hence, \(f\) solves (5) if and only if it solves (4) (i.e., \(f\) is RSW). This completes the proof.

Proof of Theorem 2. Step 1. We claim that an allocation \(f\) is undominated for \(\pi\) if and only if there exists an interior belief \(w \in \Delta(X_1)\) such that \(f\) solves problem
As the set of all \( \pi \)-feasible allocations is a convex polyhedron, so is the set of all interim payoff vectors \( U^f_1 \) in \( \pi \)-feasible allocations \( f \). Further, the latter set is bounded due to (B-\( \pi \)-IR) and (S-IC). With these facts, the supporting hyperplane theorem verifies our claim, as in Myerson (1983).

In Steps 2 and 3, fix any RSW allocation \( f = (q, t) \), any belief \( \pi \in \Delta(X_1) \), and any interior belief \( w \in \Delta(X_1) \). We prove that the RSW \( f \) solves problem \( (13) \) if and only if the interior belief \( w \) with the rule \( q \) satisfies conditions (i)–(iii).

**Step 2.** Denote by \( Q^\pi \) the set of interim-monotone allocation rules for the buyer:

\[
Q^\pi \equiv \{ q' \in [0, 1]^X \mid Q^\pi_2 \text{ is increasing in } x_2 \}\.
\]

By definition, the RSW \( f \) is safe. In particular, it is \( \pi \)-feasible. Hence, \( f \) solves problem \( (13) \) if and only if it solves the following relaxed problem:

\[
\begin{aligned}
\max_{f' \in Q^\pi \times \mathbb{R}^X} & \quad E_{x_1}^w \left[ U^f_1(x_1) \right] \\
\text{s.t.} & \quad U^f_1(x_1) \geq U^f_1(x_1^+ \mid x_1) \quad \forall x_1 < \bar{x}_1, \\
& \quad U^f_{2,\pi}(x_2) \geq U^f_{2,\pi}(x_2^- \mid x_2) \quad \forall x_2 > 1, \\
& \quad U^f_{2,\pi}(1) \geq 0.
\end{aligned}
\]

(A.4)

We define the Lagrangian function \( L \) for (A.4) as

\[
L(f', \kappa', \lambda') \equiv E_{x_1}^w \left[ U^f_1(x_1) \right] + \sum_{x_1 < x_1} \kappa'(x_1) \left[ U^f_1(x_1) - U^f_1(x_1^+ \mid x_1) \right] + \sum_{x_2 > 1} \lambda'(x_2) \left[ U^f_{2,\pi}(x_2) - U^f_{2,\pi}(x_2^- \mid x_2) \right] + \lambda'(1)U^f_{2,\pi}(1),
\]

(A.5)

where \( (\kappa', \lambda') \in \mathbb{R}^{X_1}_+ \times \mathbb{R}^{X_2}_+ \) (with \( \kappa'(\bar{x}_1) \equiv 0 \)). As in Theorem 1, the saddle-point theorem implies that \( f = (q, t) \) solves problem (A.4) if and only if there exists a nonnegative vector \( (\kappa, \lambda) \) such that \( L \) has a saddle point at \( ((q, t), (\kappa, \lambda)) \):

\[
L(q', t', \kappa, \lambda) \leq L(q, t, \kappa, \lambda) \leq L(q, t, \kappa', \lambda')
\]

(A.6)

for each \( (q', t') \in Q^\pi \times \mathbb{R}^X \) and \( (\kappa', \lambda') \in \mathbb{R}^{X_1}_+ \times \mathbb{R}^{X_2}_+ \).

As in Theorem 1, the saddle-point condition (A.6) requires that no payment rule should affect \( L \) at \( (\kappa, \lambda) \). This is equivalent to the condition that

\[
(w(x_1) + \kappa(x_1) - \kappa(x_1^+)) p_2(x_2) = \pi(x_1) (\lambda(x_2) - \lambda(x_2^+))
\]
for each $x$, where $\kappa(0) \equiv 0$ and $\lambda(\bar{x}_2) \equiv 0$. This condition pins down the multipliers as $\kappa = \Pi - W$ and $\lambda(x_2) = 1 - P_2(x_2)$ for each $x_2$.

**Step 3.** To complete the proof, we show that, given the interior belief $w$, the RSW $f$ satisfies the saddle-point condition (A.6) for some nonnegative multipliers $(\kappa, \lambda)$ if and only if $w$ with $q$ satisfies conditions (i)–(iii) in the statement.

First, we assume that $w$ with $q$ satisfies conditions (i)–(iii). Define $\kappa = \Pi - W$ and $\lambda(x_2) = 1 - P_2(x_2)$ for each $x_2$. Condition (i) implies that $\kappa$ is nonnegative. By substituting $(\kappa, \lambda)$ into $L$ and interchanging summations, we obtain

$$L(q', t, \kappa, \lambda) = \sum_{x_1} \pi(x_1) \left( v_2^1(x_1) - v_1^1(x_1) - \frac{\Pi(x_1^-) - W(x_1^-)}{\pi(x_1)} dv_1(x_1) \right) Q_1^t(x_1)$$

$$+ \sum_{x_2} p_2(x_2) \left( v_2^2(x_2) - v_1^2(x_2) - \frac{1 - P_2(x_2)}{p_2(x_2)} dv_2(x_2) \right) Q_2^t(x_2)$$

for each $q' \in Q^\pi$. It then holds that

$$L(q, t, \kappa, \lambda) = \max_{q' \in Q} L(q', t, \kappa, \lambda) = \max_{q' \in Q^\pi} L(q', t, \kappa, \lambda),$$

where the first equality follows from condition (ii) and the second from the transformation method of Gershkov et al. (2013) (Lemma 2). Now, the RSW satisfies the constraints for the buyer in problem (A.4) with equality, and the belief $w$ with the rule $q$ satisfies condition (iii) (i.e., the complementary slackness condition for the seller’s local upward IC (8)). Hence, the right inequality of the saddle-point condition (A.6) is satisfied. Thus, $((q, t), (\kappa, \lambda))$ satisfies the condition (A.6).

Second, we assume that the RSW $f$ satisfies (A.6) for some nonnegative $(\kappa, \lambda)$. Step 2 then implies that $\kappa = \Pi - W$ and $\lambda(x_2) = 1 - P_2(x_2)$ for each $x_2$. We thus obtain condition (i). The optimality condition $L(q, t, \kappa, \lambda) = \max_{q' \in Q^\pi} L(q', t, \kappa, \lambda)$ with the ex-post monotonicity $q \in Q$ implies condition (ii). Finally, condition (iii) follows from the right inequality of the saddle-point condition (A.6).

**Proof of Corollary 1.** Fix any RSW allocation $f = (q, t)$. Given the belief $\pi$ in Theorem 1, the interior prior $w = p_1$ with the rule $q$ satisfies conditions (i)–(iii) in Theorem 2. Thus, the set of beliefs for which $f$ is undominated is nonempty.

We show that this set is convex. Suppose that the RSW is undominated for beliefs $\pi^0$ and $\pi^1$. From Theorem 2, for each $k \in \{0, 1\}$, there exists an interior belief $w^k$ with which the allocation rule $q$ satisfies conditions (i)–(iii) given $\pi^k$. For any $\alpha \in (0, 1)$, the convex combination $\alpha w^0 + (1 - \alpha) w^1$ is an interior belief with which $q$ satisfies conditions (i)–(iii) given $\alpha \pi^0 + (1 - \alpha) \pi^1$. This, together with
Theorem 2, implies that the RSW is undominated for $\alpha \pi^0 + (1 - \alpha) \pi^1$. \hfill \qed

**Proof of Theorem 3.** By using Corollary 1, we assume that the RSW allocation $f^*$ is undominated for a belief $\pi$. Suppose that all seller types select the same mechanism $f^*$. Fix any off-path mechanism $G = (M, g) \in \mathcal{G}$. The proof consists of two steps. In Step 1, we formulate an auxiliary game given $G$ with $\pi$ and find a perfect equilibrium that generates a reasonable posterior. In Step 2, we show that this reasonable posterior supports the RSW as an equilibrium of the mechanism-selection game.

**Step 1.** We define the auxiliary game as follows: First, $x = (x_1, x_2)$ are realized according to $(\pi, p_2)$, and each party $i$ privately observes $x_i$. Second, the seller selects either $f^*$ or $G$. If the seller selects the status quo $f^*$, each party $i$ obtains $u_i(f^*(x), x)$. If the seller selects the new mechanism $G$, then the seller and buyer simultaneously choose $m_1 \in M_1$ and $m_2 \in M_2 \cup \{0\}$, respectively, and each party $i$ obtains $u_i(g(m), x)$. Note that $x_1$ is realized not according to $p_1$, but $\pi$.

For each $k \in \mathbb{N}$, we define perturbed game $k$ of the “agent” strategic form of the auxiliary game as follows: Each type of seller has two agents, and each type of buyer has one agent. An agent of $x_1$ chooses a probability $\gamma^k(x_1) \in [\delta^k(x_1), 1 - \delta^k(x_1)]$ of selecting the mechanism $G$, where $\delta^k(x_1) \equiv 1/(2k^3)$ if $x_1 \in X^f_1(G)$, and $\delta^k(x_1) \equiv 1/(2k)$ otherwise. Note that the losing types $x_1 \in X^f_1(G)$ mistakenly select $G$ less often than the non-losing types. The other agent of $x_1$ chooses a distribution $\sigma^k_1(\cdot \mid x_1) \in \Delta(M_1)$ such that $\sigma^k_1(\cdot \mid x_1) \geq 1/(|M_1|k)$. The agent of $x_2$ chooses a distribution $\sigma^k_2(\cdot \mid x_2) \in \Delta(M_2 \cup \{0\})$ such that $\sigma^k_2(\cdot \mid x_2) \geq 1/(|M_2| + 1)k)$. Each $\sigma^k_1$ represents a perturbed reporting strategy in $G$. We denote $\sigma^k \equiv (\sigma^k_1, \sigma^k_2)$ and $\sigma^k(m \mid x) \equiv \sigma^k_1(m_1 \mid x_1) \sigma^k_2(m_2 \mid x_2)$ for each $m$ and $x$.

Given an action profile $(\gamma^k, \sigma^k)$, the two agents of $x_1$ obtain the same payoff

$$\gamma^k(x_1) \mathbb{E}_{x_2} \left[ \sum_m \sigma^k(m \mid x) u_1(g(m), x) \right] + (1 - \gamma^k(x_1)) U^f_1(x_1),$$

and the agent of $x_2$ obtains the payoff

$$\mathbb{E}_{x_1} \left[ \gamma^k(x_1) \sum_m \sigma^k(m \mid x) u_2(g(m), x) + (1 - \gamma^k(x_1)) u^f_2(x) \right],$$

where $(\rho^k)_{k=1}^\infty$ is an arbitrary sequence of interior beliefs in $\Delta(X_1)$ that converges to the given belief $\pi$ as $k \to \infty$ and satisfies $\rho^k(\cdot) \geq 1/(|X_1|k)$ for each $k$.\textsuperscript{28}

\textsuperscript{28}The purpose of perturbing $\pi$ is to prove that limit posteriors are reasonable. This perturbation is unnecessary if we prove only that the RSW allocation is an equilibrium.
Each perturbed game $k$ has at least one Nash equilibrium. For each $k$, fix any Nash equilibrium $(\gamma^k, \sigma^k)$ and define the buyer’s interior belief $\pi^k \in \Delta(X_1)$ as

$$\pi^k(x_1) \equiv \frac{\rho^k(x_1)\gamma^k(x_1)}{\sum_{x_1'} \rho^k(x_1')\gamma^k(x_1')} .$$  \hspace{1cm} (A.7)

With some abuse of notation, let $(\gamma^k, \sigma^k, \pi^k)_{k=1}^\infty$ denote a subsequence of the mother sequence that converges in the Euclidean space. Let $(\gamma^G, \sigma^G, \pi^G) \equiv \lim_{k \to \infty}(\gamma^k, \sigma^k, \pi^k)$. Then, $(\gamma^G, \sigma^G)$ is a perfect equilibrium of the agent strategic form of the auxiliary game. Hence, $\sigma^G$ is a BNE of the continuation game $(G, \pi^G)$.

Define the continuation BNE allocation $f$ as $f(x) \equiv \sum_m \sigma^G(m \mid x)g(m)$.

Now, we prove that the limit posterior $\pi^G$ is reasonable, that is, $\pi^G \in \Delta(X_1 \setminus X_1^I(G))$ whenever $X_1^I(G) \subseteq X_1$. If the set of losing types is empty, there is nothing to prove. Fix any losing type $x_1 \in X_1^I(G)$. Then, $U_1'(x_1) < U_1^{f^*}(x_1)$. The strict inequality implies that, if $k$ is sufficiently large, then the losing type $x_1$ selects $G$ with the lowest probability $\gamma^k(x_1) = \delta^k(x_1) = 1/(2k^3)$, and hence,

$$\pi^k(x_1) = \frac{\rho^k(x_1)\gamma^k(x_1)}{\sum_{x_1'} \rho^k(x_1')\gamma^k(x_1')} \leq \frac{\frac{1}{2k^3}}{\sum_{x_1' \not\in X_1^I(G)} \rho^k(x_1')\gamma^k(x_1')} \leq \frac{\frac{1}{2k^3}}{|X_1 \setminus X_1^I(G)| \cdot \frac{1}{2k^3}} ,$$

where the second inequality follows from the fact that $\rho^k(\cdot) \geq 1/(|X_1|k)$ and $\gamma^k(x_1') \geq 1/(2k)$ for each $x_1' \not\in X_1^I(G)$. If $|X_1 \setminus X_1^I(G)| > 0$, then we obtain $\pi^G(x_1) = \lim_{k \to \infty} \pi^k(x_1) = 0$, and hence, $\pi^G \in \Delta(X_1 \setminus X_1^I(G))$.

**Step 2.** Given the two allocations $f$ and $f^*$, we show that $U_1'(x_1) \leq U_1^{f^*}(x_1)$ for each $x_1$. We define the perfect equilibrium allocation $f'$ as

$$f'(x) \equiv \gamma^G(x_1)f(x) + (1 - \gamma^G(x_1))f^*(x) .$$

Because $\gamma^G$ is the seller’s best response to $\sigma^G$, $\gamma^G(x_1) = 0$ if $U_1'(x_1) < U_1^{f^*}(x_1)$, and $\gamma^G(x_1) = 1$ if $U_1'(x_1) > U_1^{f^*}(x_1)$. This implies that $f'$ weakly dominates $f^*$.

We claim that $f'$ is $\pi$-feasible. Because $\sigma^G_2$ is the buyer’s best response to $(\gamma^G, \sigma^G_1)$, we obtain

$$E_{x_1}^\pi \left[ \gamma^G(x_1)u_2^f(x) \right] \geq E_{x_1}^\pi \left[ \gamma^G(x_1)u_2^f(\hat{x}_2 \mid x) \right] \hspace{1cm} (A.8)$$

for each $x_2 \in X_2$ and $\hat{x}_2 \in X_2 \cup \{0\}$. The RSW satisfies both (B-EPIC) and (B-EPIR). Multiplying both sides of each constraint by $1 - \gamma^G(x_1)$ and taking the
expectation $E^*_t$, we obtain
\[ E^*_t \left[ (1 - \gamma^G(x_1))u^*_2(x) \right] \geq E^*_t \left[ (1 - \gamma^G(x_1))u^*_2(\hat{x}_2 | x) \right] \tag{A.9} \]
for each $x_2 \in X_2$ and $\hat{x}_2 \in X_2 \cup \{0\}$. Summing inequalities (A.8) and (A.9), we observe that $f'$ satisfies both (B-$\pi$-IC) and (B-$\pi$-IR). Because $(\gamma^G, \sigma^G_1)$ is the seller’s best response to $\sigma^G_1$, we obtain
\[ \gamma^G(x_1)U^*_1(x_1) + (1 - \gamma^G(x_1))U^*_1(x_1) \geq \gamma^G(\hat{x}_1)U^*_1(\hat{x}_1 | x_1) + (1 - \gamma^G(\hat{x}_1))U^*_1(\hat{x}_1) \]
for each $x_1$ and $\hat{x}_1$. Because $f^*$ satisfies (S-IC), $U^*_1(x_1) \geq U^*_1(\hat{x}_1 | x_1)$ for each $x_1$ and $\hat{x}_1$. Thus, $f'$ satisfies (S-IC). As $f^*$ satisfies (S-IR), so does $f'$. These arguments establish that $f'$ is $\pi$-feasible. The $\pi$-feasible $f'$ cannot dominate $f^*$, because $f^*$ is undominated for the given belief $\pi$. Hence, $U^*_1 = U^*_1$, and the RSW $f^*$ weakly dominates $f$.

From Steps 1 and 2, $f^*$ is an equilibrium of the mechanism-selection game that is supported by the reasonable posterior $\pi^G$ with the continuation BNE allocation $f \in BN(G, \pi^G)$ for each off-path $G$. This implies that $f^*$ is intuitive. \qed

Proof of Proposition 2. Fix any allocation $f' = (q', t')$ that satisfies (S-IC) and (B-$p_1$-IC). Given $q'$, let $q$ be a solution to problem (A.1) in Lemma 2 for $\pi = p_1$. Denote $Q_2 \equiv Q'^{p_1}_2$ and $Q'_2 \equiv Q'^{p_1}_2$. The new rule $q$ satisfies $Q_i = Q'_i$ for each $i$. Further, $q$ is increasing in $x_2$. Let $T'(\cdot) \equiv E_{x_1} [t'(x_1, \cdot)]$ denote the buyer’s interim payments. We define an “adjusted” private-value component $\alpha \in \mathbb{R}^{X_2}$ such that $\alpha(1) \equiv v^2_2(1)$, and for each $x_2 > 1$,
\[ \alpha(x_2) \equiv \frac{T'(x_2) - T'(x^-_2) - E_{x_1} [v^2_2(x_1) (q'(x) - q'(x, x^-_2))]}{Q'_2(x_2) - Q^2_2(x^-_2)} \]
if $Q'_2(x_2) > Q^2_2(x^-_2)$, and $\alpha(x_2) \equiv v^2_2(x_2)$ if $Q'_2(x_2) = Q^2_2(x^-_2)$. We denote $d\alpha(x_2) \equiv \alpha(x_2^+) - \alpha(x_2)$ for each $x_2 < x_2$, and let $d\alpha(x_2)$ be an arbitrary number. We then inductively define a new payment rule $t$ as follows:
\[ t(x_1, 1) \equiv v_2(x_1, 1)q(x_1, 1) + U^*_1(x_1) \]
\[ - E_{x_2} \left[ \left( v^2_2(x_1) + \alpha(x_2) - \frac{1 - P_2(x_2)}{P_2(x_2)} d\alpha(x_2) - v_1(x) \right) q(x) \right], \]
\[ t(x_1, x_2) \equiv t(x_1, x^-_2) + \left( v^2_2(x_1) + \alpha(x_2) \right) (q(x_1, x_2) - q(x_1, x^-_2)) \tag{A.11} \]
for each $x_1$ and $x_2 > 1$. 

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Because \( f' \) satisfies (B-\( p_1 \)-IC), it follows from the definition of the function \( \alpha \) that \( v^2_2(x_2^-) \leq \alpha(x_2) \leq v^2_2(x_2) \) for each \( x_2 > 1 \). These inequalities, together with (A.11) and the monotonicity of \( q \) in \( x_2 \), imply that \( f \) satisfies (B-EPIC).

Each type \( x_1 \) of seller obtains the following interim payoff from \( f \):

\[
U^I_f(x_1) = E_{x_2} \left[ \left( v^1_2(x_1) + \alpha(x_2) - \frac{1 - P_2(x_2)}{p_2(x_2)} d\alpha(x_2) - v_1(x) \right) q(x) \right] - u^I_2(x_1, 1) \\
= U^I_{f'}(x_1),
\]

(A.12)

where the second equality follows from (A.10). The seller’s interim-payoff equivalence (A.12) implies that \( f \) satisfies (S-IC) because for each \( x_1, \hat{x}_1 \in X_1 \),

\[
U^I_f(x_1) = U^I_{f'}(x_1) \geq U^I_f(\hat{x}_1 \mid x_1) = U^I_{f'}(\hat{x}_1) + (v^1_1(\hat{x}_1) - v^1_1(x_1)) Q_1(\hat{x}_1) \\
= U^I_f(\hat{x}_1) + (v^1_1(\hat{x}_1) - v^1_1(x_1)) Q_1(\hat{x}_1) \\
= U^I_f(\hat{x}_1 \mid x_1),
\]

where the inequality follows from the hypothesis that \( f' \) satisfies (S-IC).

We obtain the equivalence of the ex-ante expected social surpluses as follows:

\[
E_x [(v_2(x) - v_1(x)) q(x)] \\
= E_{x_1} [(v^1_2(x_1) - v^1_1(x_1)) Q_1(x_1)] + E_{x_2} [(v^2_2(x_2) - v^2_1(x_2)) Q_2(x_2)] \\
= E_{x_1} [(v^1_2(x_1) - v^1_1(x_1)) Q'_1(x_1)] + E_{x_2} [(v^2_2(x_2) - v^2_1(x_2)) Q'_2(x_2)] \\
= E_x [(v_2(x) - v_1(x)) q'(x)].
\]

The equivalence implies \( E_{x_1}[U^I_f(x_1)] + E_{x_2}[U^I_f'(x_2)] = E_{x_1}[U^I_{f'}(x_1)] + E_{x_2}[U^I_{f'}(x_2)] \). From the seller’s interim-payoff equivalence (A.12), \( E_{x_2}[U^I_f'(x_2)] = E_{x_2}[U^I_{f'}(x_2)] \).

Finally, we prove the buyer’s interim-payoff equivalence. It follows from the definitions of \( t \) and \( \alpha \) that, for each \( x_2 > 1 \),

\[
U^I_2(x_2) - U^I_2(x_2^-) = v^2_2(x_2)Q_2(x_2) - v^2_2(x_2^-)Q_2(x_2^-) - \alpha(x_2) (Q_2(x_2) - Q_2(x_2^-)) \\
= v^2_2(x_2)Q'_2(x_2) - v^2_2(x_2^-)Q'_2(x_2^-) - \alpha(x_2) (Q'_2(x_2) - Q'_2(x_2^-)) \\
= U^I_{f'}(x_2) - U^I_{f'}(x_2^-).
\]

This, together with \( E_{x_2}[U^I_2(x_2)] = E_{x_2}[U^I_{f'}(x_2)] \), implies \( U^I_f = U^I_{f'} \).

As explained in Section 5.3, we reconstruct the interim-payoff-equivalent allocation \( f \) in Figure 2. Given any candidate deviator \( \tilde{x}_1 \) from the equilibrium, the following lemma modifies the equivalent allocation by using a coalition \( \{\tilde{x}_1, ..., \tilde{x}_1'\} \)
of types such that the constraints (B-EPI-R-B) are slack for these types and the constraint (S-IC-D) is slack for $\tilde{x}_1^+$.

**Lemma 3.** Let $\hat{f} = (\hat{q}, \hat{t})$ be an allocation that satisfies (S-IC), (S-IR), and (B-EPIC). Suppose that $u_2^f(\tilde{x}_1, 1) > 0$ for some $\tilde{x}_1 \in X_1$. Then, there exists an allocation $f$ and a type $\tilde{x}_1' \geq \tilde{x}_1$ with the following three properties:

(i) $f(x_1, \cdot) = \hat{f}(x_1, \cdot)$ if $x_1 \leq \tilde{x}_1$.

(ii) $u_2^f(x_1, 1) > 0$ if $\tilde{x}_1 \leq x_1 \leq \tilde{x}_1'$, and $U_1^f(\tilde{x}_1') = 0 > U_1^f(\tilde{x}_1' \mid \tilde{x}_1^+)$ if $\tilde{x}_1' < \tilde{x}_1$.

(iii) $f$ satisfies (S-IC), (S-IR), and (B-EPIC).

**Proof of Lemma 3.** Assume, without loss of generality, that $\tilde{x}_1 < \tilde{x}_1$. For notational simplicity, we denote $\tilde{q}(\cdot) = \tilde{q}(\tilde{x}_1, \cdot)$. As $U_1^f(\tilde{x}_1) \geq 0$, $u_2^f(\tilde{x}_1', 1) > 0$, and $\hat{f}$ satisfies (B-EPIC), the candidate $\tilde{x}_1$ delivers the good with positive probability (i.e., $\tilde{q}(\cdot) \neq 0$). Define an allocation $f' \equiv (q', t')$ as follows: for each $x_1 \leq \tilde{x}_1$, $f'(x_1, \cdot) \equiv \hat{f}(x_1, \cdot)$; for each $x_1 > \tilde{x}_1$, $q'(x_1, \cdot) \equiv \tilde{q}(\cdot)$ and

$$t'(x) \equiv v_2(x)\tilde{q}(x_2) - \sum_{\tilde{x}_2 < x_2} dv_2(\tilde{x}_2)\tilde{q}(\tilde{x}_2) - u_2^f(\tilde{x}_1, 1)$$

for each $x_2$. Because $\tilde{q}$ is increasing in $x_2$ from the ex-post monotonicity and $u_2^f(x) = u_2^f(x_2 \mid x)$ for each $x$ with $x_1 > \tilde{x}_1$ and $x_2 > 1$, $f'$ satisfies (B-EPIC).

However, the allocation $f'$ may violate (S-IC-U). This is because $v_2$ is increasing in $x_1$, and hence, for each $x_1', U_1^f(\cdot \mid x_1')$ is increasing in $x_1$ on $\{\tilde{x}_1, ..., \tilde{x}_1\}$.

Then, we “shrink” $f'$ to prevent each seller type from lying. Define $\tilde{x}_1' \equiv \min\{x_1 > \tilde{x}_1 \mid U_1^f(x_1 \mid x_1) < 0\} - 1$; we denote $\tilde{x}_1' \equiv \tilde{x}_1$ if $U_1^f(x_1 \mid x_1) \geq 0$ for each $x_1 > \tilde{x}_1$. Note that $\tilde{x}_1 < x_1 \leq \tilde{x}_1'$ implies $U_1^f(x_1) > U_1^f(x_1 \mid x_1) \geq 0$.

Here, the strict inequality follows from $q'(x_1, \cdot) = \tilde{q}(\cdot) \neq 0$. We define a decreasing function $\alpha \in [0, 1]^{X_1}$ as follows: $\alpha(x_1) \equiv 1$ if $x_1 \leq \tilde{x}_1$,

$$\alpha(x_1) \equiv \alpha(x_1^+) \frac{U_1^f(x_1)}{U_1^f(x_1 \mid x_1^+)}$$

if $\tilde{x}_1 < x_1 \leq \tilde{x}_1'$, and $\alpha(x_1) \equiv 0$ if $x_1 > \tilde{x}_1'$. The function $\alpha$ is well-defined because $U_1^f(x_1 \mid x_1^+) \geq U_1^f(x_1) > 0$ if $\tilde{x}_1 < x_1 \leq \tilde{x}_1'$.

Using this function, we define an allocation $f = (q, t)$ as $f(x) \equiv \alpha(x_1)f'(x)$ for each $x$. It satisfies property (i). We also obtain two facts: (a) $\tilde{x}_1 \leq x_1 \leq \tilde{x}_1'$ implies that $\alpha(x_1) > 0$ and $u_2^f(x_1, 1) = \alpha(x_1)u_2^f(x_1, 1) = \alpha(x_1)u_2^f(\tilde{x}_1, 1) > 0$; and (b) $\tilde{x}_1' < \tilde{x}_1$ implies $U_1^f(\tilde{x}_1'^+) > 0 > \alpha(\tilde{x}_1')U_1^f(\tilde{x}_1' \mid \tilde{x}_1^+)$. Thus, $f$ has property (ii).

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29 If the candidate is the highest type, we can complete the proof by setting $f \equiv \hat{f}$ and $\tilde{x}_1' \equiv \tilde{x}_1$. 
Further, $f$ satisfies (S-IC). This is because $Q_1 = \alpha Q'_1$ is decreasing, $\hat{x}_1 < x_1 \leq \hat{x}'_1$ implies $U'_1(x_1 \mid x^-_1) = U'_1(x^-_1)$, and $x_1 > \hat{x}'_1$ implies both $f(x_1, \cdot) \equiv (0,0)$ and $0 > U'_1(\hat{x}'_1 \mid \hat{x}'_1)$. It is clear from the definition of $\alpha$ that $f$ satisfies (S-IR). As $f'$ satisfies (B-EPIC), so does $\hat{f}'$. Thus, $f$ has property (iii). \hfill \Box

As explained in Section 5.3, we construct a less-trading allocation $\tilde{f}$ from the modified allocation $f$ with the coalition $\{\hat{x}_1, \ldots, \hat{x}'_1\}$ in Lemma 3. This allocation $\tilde{f}$ is defined as a convex combination of $f$ and a fee, as in (14). However, no type $x_1 > \hat{x}'_1$ outside the coalition should charge this fee, because otherwise the buyer, believing that the seller’s type is the non-coalition member $x_1$, opts out of $\tilde{f}$. Nonetheless, $\tilde{f}$ can be IC for the seller because (S-IC-D) is slack for $\hat{x}'_1$ in the original $f$. The next lemma shows how this allocation $\tilde{f}$ is constructed.

**Lemma 4.** Let $f$ be an allocation that satisfies (S-IC), (S-IR), and (B-EPIC). Suppose that there exists a pair of types $\hat{x}_1, \hat{x}'_1 \in X_1$ such that $u'_2(x_1, 1) > 0$ if $\hat{x}_1 \leq x_1 \leq \hat{x}'_1$, and $U'_1(\hat{x}'_1) > U'_1(\hat{x}'_1 \mid \hat{x}'_1)$ if $\hat{x}'_1 < \hat{x}_1$. Then, there exists an allocation $\tilde{f}$ with the following three properties:

(i) $\tilde{f}$ satisfies (S-IC), (S-IR), and (B-EPIC).

(ii) $u'_2(x_1, 1) > 0$ if $\hat{x}_1 \leq x_1 \leq \hat{x}'_1$.

(iii) $U'_1(x_1) > U'_1(\hat{x}_1)$ if and only if $\hat{x}_1 \leq x_1 \leq \hat{x}'_1$.

**Proof of Lemma 4.** Denote by $f = (q,t)$ the original allocation. Fix any $\delta \in (0,1)$. Define $\tau \equiv (U'_1(\hat{x}_1) + U'_1(\hat{x}'_1))/2$ with $U'_1(0) = U'_1(1) + 1$. We also define an allocation $\tilde{f}$ as follows: $\tilde{f}(x_1, \cdot) \equiv (1 - \delta) f(x_1, \cdot) + \delta(0, \tau)$ if $x_1 \leq \hat{x}'_1$, and $\tilde{f}(x_1, \cdot) \equiv (1 - \delta) f(x_1, \cdot)$ if $x_1 > \hat{x}'_1$. Note that as $\delta \to 0$, $\tilde{f} \to f$.

**Properties (i) and (ii).** Using the hypotheses that $\hat{x}_1 \leq x_1 \leq \hat{x}'_1$ implies $u'_2(x_1, 1) > 0$ and that $\hat{x}'_1 < \hat{x}_1$ implies $U'_1(\hat{x}'_1) > U'_1(\hat{x}'_1 \mid \hat{x}'_1)$, we make $\delta > 0$ sufficiently small such that the nearby allocation $\tilde{f}$ satisfies property (ii) and the seller’s local IC constraints (which are equivalent to (S-IC) from Lemma 1 (ii)). As $f$ satisfies both (S-IR) and (B-EPIC), so does $\tilde{f}$. Thus, $\tilde{f}$ satisfies property (i).

**Property (iii).** Because $f$ satisfies (S-IR), $U'_1(\hat{x}'_1) \geq 0$. Then, the hypothesis that $u'_2(\hat{x}'_1, 1) > 0$ and $f$ satisfies (B-EPIC) implies $q(\hat{x}'_1, \cdot) \neq 0$. Because $q$ is interim monotone for the seller, $q(x_1, \cdot) \neq 0$ if $x_1 < \hat{x}'_1$. This, together with the hypothesis that $f$ satisfies (S-IC-U), implies that $U'_1$ is strictly decreasing in $x_1$ on $\{1, \ldots, \hat{x}'_1\}$. Hence, for each $x_1 \leq \hat{x}'_1$, the interim payoff $U'_1(x_1) = U'_1(x_1) + \delta(\tau - U'_1(x_1))$ is larger than $U'_1(x_1)$ if $x_1 \geq \hat{x}_1$ and smaller than $U'_1(x_1)$ otherwise. For each $x_1 > \hat{x}'_1$, $U'_1(x_1) = (1 - \delta)U'_1(x_1) \leq U'_1(x_1)$. Thus, $\tilde{f}$ has property (iii). \hfill \Box

The following lemma constructs an AM mechanism $G = (M,g)$ described in Section 5.3. Note that, for a pure strategy profile denoted by $s = (s_1, s_2) \in$
to the Supplementary material for the proof.

**Lemma 5.** Suppose that \((f, \tilde{x}_1, \tilde{x}_1')\) satisfies the hypotheses of Lemma 4. Then, there exists a mechanism \(G = (M, g) \in \mathcal{G}\) with the following three properties:

(i) \(X_f^1(G) = \{x_1 \in X_1 \mid x_1 < \tilde{x}_1 \text{ or } x_1 > \tilde{x}_1'\}\).

(ii) There exists a profile \(s\) such that, for each \(\pi \in \Delta(X_1 \setminus X_f^1(G))\), \(s\) is the unique profile that survives the iterative deletion of strictly dominated strategies in \((G, \pi)\).

(iii) The strategy profile \(s\) satisfies \(U_f^{\text{pos}}(x_1) > U_f^1(x_1)\) for each \(x_1 \in X_1 \setminus X_f^1(G)\).

**Proof of Proposition 1.** From Theorem 3, every feasible allocation that weakly dominates the RSW allocations is an equilibrium. Fix any equilibrium \(f'\) and RSW allocation \(f^*\). To derive a contradiction, suppose that \(U_f^1(x_1) < U_f^{\text{R}}(x_1)\) for some \(x_1\). We show that this type \(x_1\) can profitably deviate from \(f'\) to another mechanism. Take any \(\varepsilon \in (0, U_f^1(x_1) - U_f^{\text{R}}(x_1)]\). Define an allocation \(f \in A^X\) by \(f \equiv f^* - (0, \varepsilon)\). The triplet \((f, \tilde{x}_1, \tilde{x}_1')\) with \(\tilde{x}_1 \equiv 1\) and \(\tilde{x}_1' \equiv \tilde{x}_1\) satisfies the hypotheses of Lemma 4. Lemma 5 then provides a mechanism \(G = (M, g)\) with properties (i)–(iii). Property (i) implies \(X_f^1(G) = \emptyset\). From properties (ii) and (iii), we obtain a pure strategy profile \(s\) such that \(\cup_{\pi \in \Delta(X_1)} BN(G, \pi) = \{g \circ s\}\) and \(U_f^{\text{pos}}(x_1) > U_f^1(x_1) \geq U_f^{\text{R}}(x_1)\). This contradicts the hypothesis that \(f'\) is an equilibrium.

**Proof of Theorem 4.** Fix any intuitive equilibrium \(f'\). Let \(f^*\) be an RSW allocation. It follows from the equilibrium characterization (Proposition 1) that \(f'\) weakly dominates \(f^*\) (i.e., \(U_f^1 \geq U_f^{\text{R}}\)). To derive a contradiction, suppose that \(f'\) dominates \(f^*\) (i.e., \(U_f^1 \neq U_f^{\text{R}}\)). Using the interim-payoff equivalence (Proposition 2), we obtain a feasible allocation \(\hat{f}\) that satisfies (B-EPIC) and \(U_f^1 = U_f^{\text{R}}\) for each \(i\). Because the new \(\hat{f}\) dominates \(f^*\), it violates (B-EPIC). It follows that \(u_f^1(x_1, 1) < 0\) for some \(x_1\). However, \(\hat{f}\) satisfies (B-p1-IR). Hence, we can find some \(\tilde{x}_1\) with \(u_f^1(\tilde{x}_1, 1) > 0\).

We show that this type \(\tilde{x}_1\) can convincingly deviate from \(f'\). Using Lemma 3, we obtain \(f \in A^X\) and \(\tilde{x}_1' \geq \tilde{x}_1\) with the following properties: first, \(f(x_1, \cdot) = \hat{f}(x_1, \cdot)\) if \(x_1 \leq \tilde{x}_1\); second, \(u_f^1(x_1, 1) > 0\) if \(\tilde{x}_1 \leq x_1 \leq \tilde{x}_1'\), and \(U_f^1(\tilde{x}_1') = 0 \geq U_f^1(\tilde{x}^+_1)\) if \(\tilde{x}_1' < \tilde{x}_1\); and third, \(f\) satisfies (S-IC), (S-IR), and (B-EPIC). The triplet \((f, \tilde{x}_1, \tilde{x}_1')\) satisfies the hypotheses of Lemma 4. Lemma 5 then provides a mechanism \(G = (M, g)\) with properties (i)–(iii). Because \(U_f^1(x_1) = U_f^1(x_1)\) for each \(x_1 < \tilde{x}_1\) and \(U_f^1(x_1) \leq U_f^1(\tilde{x}_1') = 0 \leq U_f^1(x_1)\) for each \(x_1 > \tilde{x}_1'\), property (i) implies \(X_f^1(G) \subseteq X_f^1(G) = X_f^1(G)\). The last equality follows from the interim-payoff equivalence \(U_f^1 = U_f^{\text{R}}\). Further, from properties (ii) and (iii), we obtain a
pure strategy profile $s$ such that, if $\pi \in \Delta(X_1 \setminus X^f_1(G))$, $BN(G, \pi) = \{ g \circ s \}$ and $U^g_1(\bar{x}_1) > U^f_1(\bar{x}_1) = U^f_1(\tilde{x}_1)$. Hence, if the off-path belief $\pi^G$ given $G$ is reasonable (i.e., $\pi^G \in \Delta(X_1 \setminus X^f_1(G)) \subseteq \Delta(X_1 \setminus X^f_1(G))$, the type $\tilde{x}_1$ is better off deviating from $f'$ to $G$. This contradicts the hypothesis that $f'$ is intuitive.

Proof of Theorem 5. Fix any intuitive equilibrium $f'$. Theorem 4 implies that $U^f_1$ is the seller’s RSW payoff vector. Using the interim-payoff equivalence (Proposition 2), we obtain a feasible allocation $f$ that satisfies (B-EPIC) and $U^f_i = U'^f_i$ for each $i$. We complete the proof by showing that $u^g_2(x_1, 1) = 0$ for each $x_1$, and hence, $f$ is an RSW allocation. To derive a contradiction, suppose that $u^g_2(x_1, 1) \neq 0$ for some $x_1$. Because $f$ satisfies (B-p1-IR), we obtain a type $x'_1$ with $u^g_2(x'_1, 1) > 0$. Let $f^*$ be an RSW allocation. We define an allocation $f^{**}$ as follows: $f^{**}(x_1, \cdot) \equiv f(x_1, \cdot)$ if $x_1 = x'_1$, and $f^{**}(x_1, \cdot) \equiv f^*(x_1, \cdot)$ otherwise. Because $U^{f^{**}}_1$ is the RSW payoff vector and $f^{**}$ is safe, the new allocation $f^{**}$ is an RSW allocation. This, together with $u^{f^{**}}_2(x'_1, 1) > 0$, contradicts the characterization of RSW (Theorem 1).

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Appendix B provides the equilibrium characterization presented in Section 4 of the paper. Appendix C gives an example of an equilibrium in which the seller cannot make a convincing deviation to any “menu.” Appendix D constructs an Abreu–Matsushima (AM) mechanism that is described in Section 5.3 of the paper.

## B Equilibrium characterization in Section 4

In this appendix, we characterize the seller’s equilibrium payoff vectors in Section 4 of the paper. We assume that $X_1 = X_2 = \{1, 2\}$, $p_1 = p_2 = 1/2$, $v_1(x) = 100(x_1 - 1)$, and $v_2(x) = 300x_1 + 100x_2$. Let $f = (q, t)$ denote an equilibrium. We show that the set of equilibrium payoff vectors is characterized by the following conditions (i.e., the red triangle in Figure B.1):

\begin{align}
U_1^f(2) &\geq U_1^f(1) - 100, \\
U_1^f(2) &\leq \frac{2}{3}U_1^f(1) + \frac{250}{3}, \\
U_1^f(2) &\geq 350.
\end{align}

Denote by $f^*$ the RSW allocation in Table 1 of Section 4. The seller’s RSW payoff vector is given by $U_1^{f^*} = (400, 350)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figureB1.png}
\caption{Seller’s equilibrium payoff vectors}
\end{figure}

(i) We show that $U_1^f$ must satisfy (B.1)–(B.3). The equilibrium characteriza-
tion in the paper (Proposition 1 with Corollary 1) implies that $U_f^1$ satisfies

$$U_f^1(2) \geq U_f^{\ast}(2) = 350.$$  \hfill (B.4)

From (B.4), (B.3) follows immediately. The equilibrium $f$ is feasible, and hence,

$$U_f^1(1) - U_f^1(2) \leq 50(q(1,1) + q(1,2)), \hfill (B.5)$$

$$U_f^1(1) - U_f^1(2) \geq 50(q(2,1) + q(2,2)), \hfill (B.6)$$

$$U_f^1(1) + U_f^1(2) \leq 150q(1,1) + 250q(1,2) + 250q(2,1) + 350q(2,2), \hfill (B.7)$$

where the last inequality follows from

$$E_{x_1}[U_f^1(x_1)] = E_x[(v_2(x) - v_1(x))q(x)] - E_{x_2}[U_f^2(x_2)],$$

$$U_f^2(2) \geq U_f^2(1) + 50(q(1,1) + q(2,1)),$$

$$U_f^2(1) \geq 0.$$ 

Because $q \leq 1$, (B.5) implies (B.1). By substituting (B.6) into (B.7), we obtain

$$U_f^1(1) + U_f^1(2) \leq 5(U_f^1(1) - U_f^1(2)) + 150q(1,1) + 250q(1,2) + 100q(2,2).$$

This inequality with $q \leq 1$ implies (B.2).

(ii) We complete the proof by finding two feasible allocations $f^{**}$ and $f$ with $U_f^{**} = (550, 450)$ and $U_f^f = (450, 350)$. Then, the constant allocations defined by $f^{**} \equiv (1,550)$ and $f \equiv (1,450)$ are what we need.

C Deviation to menus

Using an example with three seller types, we construct an equilibrium in which no type can make a convincing deviation to any menu.

We assume that $X_1 = \{1,2,3\}$, $X_2 = \{1,2\}$, $p_1 = 1/3$, $p_2 = 1/2$, $v_1(x) = 100(x_1 - 1)$, and $v_2(x) = 300x_1 + 200x_2$. From Theorem 1 in the paper, the unique RSW allocation $f^*$ is given by Table C.1. We also consider the allocation $f$ in Table C.2. The seller’s interim payoff vectors in these allocations are given by

$$U_f^{**} = (500, 450, 825/2), \quad U_f^f = (525, 460, 425). \hfill (C.1)$$

Thus, $f$ dominates $f^*$. Simple computations show that $f$ is feasible. With these facts, the equilibrium characterization in the paper implies that $f$ is an equilib-
rium. Note that the low-type buyer benefits from $f$ only if the seller has the middle type.

$$x_1 = 1 \quad x_2 = 1 \quad x_2 = 2$$
$$x_1 = 2 \quad 1, 500 \quad 1, 1000$$
$$x_1 = 3 \quad 0, 0 \quad 3/4, 975$$

Table C.1: RSW allocation $f^*$

$$x_1 = 1 \quad x_2 = 1 \quad x_2 = 2$$
$$x_1 = 2 \quad 3/10, 175 \quad 1, 875$$
$$x_1 = 3 \quad 0, 40 \quad 7/10, 950$$

Table C.2: Equilibrium allocation $f$

We call a mechanism $G \in \mathcal{G}$ a menu (of outcomes) if the seller has no reporting opportunity in $G$ (i.e., $M_1 = \emptyset$). For each posterior $\pi \in \Delta(X_1)$, we denote by $(q, t) \in BN(G, \pi)$ an allocation generated by a “BNE” (i.e., possibly mixed optimal choices for the buyer) in the game $(G, \pi)$. Because this allocation does not depend on the seller’s type, we denote $(q, t) = (q(x_2), t(x_2))_{x_2 \in X_2}$.

Now, fix any menu $G$. In the following, we show that no type $x_1 \in \{1, 2, 3\}$ of seller can convincingly deviate from the equilibrium $f$ to the menu $G$.

**Low type’s deviation.** The low-type seller’s interim payoff $U_1^f(1) = 525$ in the equilibrium $f$ is higher than her full-information payoff $U_1^{f^*}(1) = 500$. From this fact, it is clear that the low type cannot convincingly deviate from $f$ to $G$.

**Middle type’s deviation.** To derive a contradiction, suppose that the middle type can convincingly deviate from the equilibrium $f$ to the menu $G$. Denote by $\pi$ the posterior with $\pi(2) = 1$. Fix any BNE allocation $(q, t) \in BN(G, \pi)$. Because the middle type benefits from the revealing deviation, we obtain

$$\frac{1}{2} (t(1) + t(2) - 100(q(1) + q(2))) > 460 = U_1^f(2).$$

The BNE allocation is IC and IR for the buyer with the posterior $\pi$, and hence,

$$1000(q(2) - q(1)) \geq t(2) - t(1), \quad \text{(C.3)}$$
$$800q(1) \geq t(1). \quad \text{(C.4)}$$

By substituting these two inequalities with $t(2) \geq t(1)$ into (C.2), we obtain

$$t(2) \geq \frac{3680}{7} > 525 = U_1^f(1). \quad \text{(C.5)}$$

Let $\pi'$ denote the most optimistic posterior (i.e., $\pi'(3) = 1$). From inequalities (C.3) and (C.4), it is optimal for each type of buyer with $\pi'$ to choose an outcome
with a payment that is equal to (or higher than) \( t(2) \). With this fact, the two inequalities in (C.5) imply that the deviation from \( f \) to \( G \) can be profitable for the low type (i.e., \( 1 \notin X^f_1(G) \)). Thus, the middle type must benefit from the deviation given the most pessimistic posterior (i.e., \( \pi''(1) = 1 \)). This is impossible.

**High type’s deviation.** To derive a contradiction, suppose that the high type can convincingly deviate from \( f \) to \( G \). First, we claim that the middle type is a non-lowing type (i.e., \( 2 \notin X^f_1(G) \)). Denote by \( \pi \) the most optimistic posterior (i.e., \( \pi(3) = 1 \)). Fix any BNE \( (q, t) \in BN(G, \pi) \). Denote \( Q \equiv E_{x_2}[q(x_2)] \) and \( T \equiv E_{x_2}[t(x_2)] \). The high type benefits from the revealing deviation, and hence,

\[
T - 200Q > 425 = U^f_1(3). \tag{C.6}
\]

The BNE allocation \( (q, t) \) is IR for the buyer with \( \pi \), and hence,

\[
Q \geq \frac{T}{1300} > \frac{425 + 200Q}{1300} \quad \Rightarrow \quad Q > \frac{17}{44}. \tag{C.7}
\]

From (C.6) and (C.7), the middle type is a non-losing type because

\[
T - 100Q = (T - 200Q) + 100Q > 460 = U^f_1(2). \tag{C.8}
\]

Second, we claim that the low type is a non-lowing type (i.e., \( 1 \notin X^f_1(G) \)). Denote by \( \pi \) the posterior with \( \pi(2) = 1 \). Note that \( \pi \) is reasonable. Fix any BNE \( (q, t) \in BN(G, \pi) \). Denote \( Q \equiv E_{x_2}[q(x_2)] \) and \( T \equiv E_{x_2}[t(x_2)] \). Because the high type benefits from the deviation given the reasonable \( \pi \), we obtain the same inequality as (C.6). The BNE \( (q, t) \) is IR for the buyer with \( \pi \), and hence,

\[
Q \geq \frac{T}{1000} > \frac{425 + 200Q}{1000} \quad \Rightarrow \quad Q > \frac{17}{32}. \tag{C.9}
\]

Hence, the low type is a non-losing type because

\[
T = (T - 200Q) + 200Q > 525 = U^f_1(1). \tag{C.10}
\]

The two claims imply \( X^f_1(G) = \emptyset \). Thus, the high type must benefit from the deviation given the most pessimistic posterior. This is impossible.
D AM mechanism

In this appendix, we construct an AM mechanism that is described in Section 5.3 of the paper. Because the formal proof is quite complicated, we begin with a simpler example to provide some intuition behind the proof.

**Example 1.** Consider the example in Appendix C. Let \( f \) be the equilibrium in Table C.2. The middle-type seller is the candidate deviator from the equilibrium.

As in Lemma 4 of the paper, we define a nearby allocation \( \tilde{f} \) as follows:

\[
\tilde{f}(x_1, \cdot) \equiv (1 - \delta) f(x_1, \cdot) + \delta (0, \tau) \quad (D.1)
\]

if \( x_1 \neq 3 \), and \( \tilde{f}(3, \cdot) \equiv (1 - \delta) f(3, \cdot) \), where \( \tau \in (460, 525) = (U_1^f(2), U_1^f(1)) \) is a fixed fee and \( \delta \in (0, 1) \) is a small weight with \( u_2(2, 1) > 0 \) and \( U_1^f(3) > U_1^f(2 | 3) \).

The middle-type seller prefers the allocation \( \tilde{f} \) to the equilibrium \( f \), while the other types have the opposite preference (i.e., \( U_1^{\tilde{f}}(2) > U_1^f(2) \) and \( U_1^{\tilde{f}}(x_1) < U_1^f(x_1) \) for each \( x_1 \neq 2 \)). The direct mechanism \( \tilde{f} \) is IC for the seller and EPIC for the buyer. However, the middle type cannot convincingly deviate from the equilibrium \( f \) to the mechanism \( \tilde{f} \). This is because the game \( (\tilde{f}, \pi) \) given the posterior \( \pi(2) = 1 \) has BNE allocations for which the middle type suffers from the deviation.

Then, we construct an AM mechanism to eliminate \( f \) as unintuitive. For simplicity, we assume here that the set of posterior beliefs for the buyer is finite and does not depend on off-path mechanisms. Let \( \mathcal{P} \subset \Delta(X_1) \) denote a nonempty finite set of posterior beliefs \( \pi \) with which the buyer benefits from the allocation \( \tilde{f} \) (i.e., \( U_2^{\tilde{f}, \pi}(1) > 0 \)).

This AM mechanism is defined as a finite strategic game form \( G = (M, g) \in \mathcal{G} \). In this static mechanism, each party simultaneously makes \( (K+1) \) reports given a large integer \( K \geq 1 \). Each party reports both its type and the buyer’s belief about the seller’s type. A typical message profile is denoted by \( m = (m_1, m_2) \in M_1 \times M_2 \) with \( m_i = (m_i^0, ..., m_i^K) \), \( m_i^0 = x_i^0 \), \( m_i^0 = (x_i^0, \pi_2^0) \), and \( m_i^k = (x_i^k, \pi_2^k) \) for each \( i \) and \( k \geq 1 \). The outcome function \( g \) (with \( g(\cdot, 0) \equiv (0, 0) \)) is a convex combination of some functions defined as follows: For some small \( \varepsilon > 0 \),

\[
g(m) \equiv \varepsilon f_1(x_1^0) + \varepsilon^2 \left( f_2(x_1^0, x_2^0) + b(x_1^0, \pi_2^0) \right) + \varepsilon^3 d(m) + \frac{1 - \varepsilon - \varepsilon^2 - \varepsilon^3}{K} \sum_{k=1}^{K} \varphi(m^k)
\]

\[\text{30}\]For example, the game \( (\tilde{f}, \pi) \) has a BNE in which the seller always reports that she has the highest type and the buyer always opts out of \( \tilde{f} \). As discussed by Gresik (1991), this multiplicity of equilibria is typical of interdependent-values environments.
and $m^k \equiv (m_1^k, m_2^k)$. First, $f_1$ is a “dictatorial” allocation that is strictly IC for the seller. Second, $f_2$ is an allocation that is strictly EPIC and EPIR for the buyer. We can construct each $f_i$ using the strict monotonicity of $v_i$ in $x_i$ (see Lemmas 6 and 7 below). Third, $b$ is a strictly proper scoring rule in which the buyer reports his belief $\pi_2^0$ about the seller’s type report $x_1^0$. Fourth, $d$ is an AM penalty rule in which “the first to lie” pays a small amount (say, $\$1$) to the other party. Here, $i$’s lie at $k$ means $(x_i^k, \pi_i^k) \neq (x_i^0, \pi_i^0)$. Finally, $\varphi$ is defined by

$$
\varphi(x_1^k, \pi_1^k, x_2^k, \pi_2^k) = \begin{cases} 
\tilde{f}(x_1^k, x_2^k) & \text{if } \pi_1^k \in P, \ \pi_2^k \in P, \\
(0, 0) & \text{otherwise.}
\end{cases}
$$

That is, the direct mechanism $\tilde{f}$ is played if each party reports that the buyer’s posterior is in the set $P$, and no trade takes place otherwise.

Now, let $\pi$ be a posterior after this off-path mechanism $G$ is offered. We show that the static Bayesian game $(G, \pi)$ has a unique strategy profile that survives the iterative deletion of strictly dominated strategies. This is shown by using the following inductive argument. Note that the off-path belief $\pi$ is common knowledge. First, lying in $f_1$ (i.e., $x_1^0 \neq x_1$) is dominated for the seller because $\varepsilon \approx 0$. Second, lying in $f_2$ (i.e., $x_2^0 \neq x_2$) is dominated for the buyer because $\varepsilon \approx 0$. Third, lying in $b$ (i.e., $\pi_2^0 \neq \pi$) is dominated for the buyer because the scoring rule is strictly IC for him. Finally, mathematical induction implies that lying in $\varphi$ at each $k$ (i.e., $(x_i^k, \pi_i^k) \neq (x_i, \pi)$) is dominated for each $i$. This is because $\varphi(\cdot, \pi, \cdot, \pi)$ is IC and IR for both parties given $\pi$, and $1/K$ can be small enough that the small penalty imposed by $d$ outweighs all benefits from lying at each $k$.

Thus, each party tells the truth at each $k$. For each $x$, the ex-post outcome is

$$
g(m) = \varepsilon f_1(x_1) + \varepsilon^2 (f_2(x_1, x_2) + b(x_1, \pi)) + (1 - \varepsilon - \varepsilon^2 - \varepsilon^3) \varphi(x_1, \pi, x_2, \pi),
$$

where $\varphi(x_1, \pi, x_2, \pi) = \tilde{f}(x_1, x_2)$ if $\pi \in P$, and $\varphi(x_1, \pi, x_2, \pi) = (0, 0)$ otherwise. Because $\varepsilon \approx 0$, the low type and the high type are always worse off deviating from $f$ to $G$, while the middle type is better off deviating from $f$ to $G$ if the deviation convinces the buyer that the seller has the middle type. Thus, we can conclude

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31 See, for example, Selten (1998) for the scoring rule.

32 We construct this allocation $f_1$ in a manner such that the seller pays sufficient amounts to the buyer to induce his participation in $G$.

33 This penalty rule $d$ also requires a liar at each $k$ to make a small additional payment to the other party. Hence, lying is strictly dominated by truth-telling.
that the equilibrium $f$ is unintuitive.

In Example 1, we make the ad hoc assumption that the set of posterior beliefs is finite and does not depend on off-path mechanisms. Dropping the assumption, we redesign an AM mechanism more formally. As a preliminary step, we construct two allocations incorporated into this AM mechanism.

The first auxiliary result provides the seller’s “dictatorial” allocation that is strictly IC for her.

**Lemma 6.** There exists a function $\tilde{f}_1 \in A^{x_1}$ such that $x_1 \neq \hat{x}_1$ implies

$$U_1^{\tilde{f}_1}(x_1) - U_1^{\tilde{f}_1}(\hat{x}_1 \mid x_1) > 0.$$  

(D.2)

**Proof of Lemma 6.** We define the function $\tilde{f}_1 = (q, t) \in A^{x_1}$ as follows:

\[
q(x_1) = \frac{x_1 - x_1 + 1}{x_1}, \quad t(x_1) = \sum_{\hat{x}_1 = x_1} \frac{v_1^1(\hat{x}_1^+) + v_1^1(\hat{x}_1) + 2E_{x_2}v_2^2(x_2)}{2x_1},
\]

where $v_1^1(x_1^+) = v_1^1(x_1) + 1$. For each $x_1, \hat{x}_1 \in X_1$ with $\hat{x}_1 > 1$,

\[
E_{x_2} \left[ u_1(\tilde{f}_1(\hat{x}_1), x) - u_1(\tilde{f}_1(\hat{x}_1^{-}), x) \right] = \frac{1}{x_1} \left( v_1^1(x_1) - \frac{v_1^1(\hat{x}_1) + v_1^1(\hat{x}_1^{-})}{2} \right).
\]

This payoff difference is positive if $x_1 \geq \hat{x}_1$ and negative if $x_1 < \hat{x}_1$. Hence, $\tilde{f}_1$ is strictly IC for the seller. \qed

The second auxiliary result provides a direct mechanism that is strictly EPIC and strictly EPIR for the buyer.

**Lemma 7.** There exists a function $f_2 \in A^{x_1 \times (x_2 \cup \{0\})}$ such that

$$u_2^{f_2}(x) - u_2^{f_2}(\hat{x}_2 \mid x) > 0$$  

(D.3)

for each $x = (x_1, x_2) \in X$ and $\hat{x}_2 \in X_2 \cup \{0\}$ with $x_2 \neq \hat{x}_2$.

**Proof of Lemma 7.** We define the function $f_2 = (q, t) \in A^{x_1 \times (x_2 \cup \{0\})}$ as follows:

\[
q(x_1, x_2) = \frac{x_2}{x_2}, \quad t(x_1, x_2) = \sum_{\hat{x}_2 = x_2} \frac{v_2^2(\hat{x}_2) + v_2^2(\hat{x}_2^{-}) + 2v_2^1(x_1)}{2x_2},
\]
Let by \( m \in M \) and \( \eta > 0 \) the statement, let \( \tilde{x} \) be the nearby allocation given by Lemma 4 of the paper. Let \( f_1 \) and \( f_2 \) be the functions given by Lemmas 6 and 7, respectively. Denote by 

\[
\tilde{u}_i(x_1) = \max \{ u_i(a, x) \mid a \in f_1(X_1) \cup f_2(X) \} 
\]

for each \( i \in \{1, 2\} \) and \( k \in \{0, ..., K\} \). Denote typical elements of these sets by 

\[
m = (m_1, m_2) \in M, \quad m_i = (m_i^0, ..., m_i^K) \in M_i, \quad m^k = (m_1^k, m_2^k) \in M^k, \quad \text{and} \quad m^k_i = (x_i^k, h_i^k) \in M^k_i \text{ for each } i \in \{1, 2\} \text{ and } k \in \{0, ..., K\}. 
\]

We define \( \overline{\eta} > 0 \) as

\[
\overline{\eta} = \max \{|u_i(a, x)| \mid a \in f_1(X_1) \cup f_2(X) \cup \tilde{f}(X) \cup \{(1, 0)\}, x \in X \} 
\]

for each \( i \in \{1, 2\} \) and \( k \in \{0, ..., K\} \). Denote typical elements of these sets by 

\[
\bar{m} = (m_1, m_2) \in M, \quad m_i = (m_i^0, ..., m_i^K) \in M_i, \quad m^k = (m_1^k, m_2^k) \in M^k, \quad \text{and} \quad m^k_i = (x_i^k, h_i^k) \in M^k_i \text{ for each } i \in \{1, 2\} \text{ and } k \in \{0, ..., K\}. 
\]

We define \( \bar{\pi} > 0 \) as

\[
\bar{\pi} = \overline{\eta} - (0, 2\overline{\eta}). 
\]

Second, we construct a scoring rule to elicit the buyer’s posterior. Denote \( \bar{X}_1 \equiv \{\tilde{x}_1, ..., \tilde{x}_1'\} \). Let \( \pi^H \) be the center of \( \Delta(\bar{X}_1) \). Denote \( \rho \in [0, 1]^{X_1} \) be the vector

\[
\pi^H(x_1) = \frac{1}{|\bar{X}_1|} \text{ if } x_1 \in \bar{X}_1, \quad \text{and} \quad \pi^H(x_1) = 0 \text{ otherwise.} 
\]

\[\text{Lemma 5.}\] Suppose that \((f, \tilde{x}_1, \tilde{x}_1')\) satisfies the hypotheses of Lemma 4 in the paper. Then, there exists a mechanism \( G = (M, g) \in G \) with the following three properties:

(i) \( X_1^f(G) = \{x_1 \in X_1 \mid x_1 < \tilde{x}_1 \text{ or } x_1 > \tilde{x}_1'\} \).

(ii) There exists a profile \( s \) such that, for each \( \pi \in \Delta(X_1 \setminus X_1^f(G)) \), \( s \) is the unique profile that survives the iterative deletion of strictly dominated strategies in \((G, \pi)\).

(iii) The strategy profile \( s \) satisfies \( U_1^{\text{pos}}(x_1) > U_1^f(x_1) \) for each \( x_1 \in X_1 \setminus X_1^f(G) \).

\[\text{Proof of Lemma 5.}\] We define an AM mechanism \( G = (M, g) \in G \) as follows:

**Message spaces.** Let \( K, H \in \mathbb{N} \) be positive integers. The message spaces are defined by

\[
M \equiv M_1 \times (M_2 \cup \{0\}), \quad M_i \equiv M_i^0 \times \cdots \times M_i^K, \quad M^k \equiv M_1^k \times M_2^k, \quad M_i^k \equiv M_i^k \times M_i^k, \quad M_1^k \equiv X_1, \quad M_2^k \equiv X_2 \cup \{0\}, \quad M_i \equiv \{0, ..., H\} 
\]

for each \( i \in \{1, 2\} \) and \( k \in \{0, ..., K\} \). Denote typical elements of these sets by 

\[
m = (m_1, m_2) \in M, \quad m_i = (m_i^0, ..., m_i^K) \in M_i, \quad m^k = (m_1^k, m_2^k) \in M^k, \quad \text{and} \quad m^k_i = (x_i^k, h_i^k) \in M^k_i \text{ for each } i \in \{1, 2\} \text{ and } k \in \{0, ..., K\}. 
\]

**Outcome function.** First, we define some allocations. For the allocation \( f \) in the statement, let \( \tilde{f} \) denote the nearby allocation given by Lemma 4 of the paper. Let \( \eta > 0 \) the smallest number among the left sides of inequalities (D.2) and (D.3). We define \( \eta > 0 \) as

\[
\eta = \max \{|u_i(a, x)| \mid a \in f_1(X_1) \cup f_2(X) \cup \tilde{f}(X) \cup \{(1, 0)\}, x \in X \}. 
\]

Let \( f_1 \equiv \tilde{f}_1 - (0, 2\eta) \).
such that $\rho(x_1) \equiv 0$ if $x_1 \in \tilde{X}_1$, and $\rho(x_1) \equiv 1/|X_1 \setminus \tilde{X}_1|$ otherwise. Pick a small $\delta \in (0, 1)$ such that, if $\pi$ is in the $\sqrt{2}\delta$-neighborhood of $\Delta(\tilde{X}_1)$, then $U_{\delta}^f(1) > 0$. For each $h \in \{0, \ldots, H\}$, we set $\pi^h \equiv \delta(1 - h/H)(\rho - \pi^H) + \pi^H$ and define

$$\mathcal{P}^h \equiv \{\pi \in \Delta(X_1) \mid \|\pi - \pi^h\| \leq \|\pi - \pi^h\| \quad \forall h' \neq h\}$$

with $\mathcal{P}^h \equiv \mathcal{P}^h \setminus \mathcal{P}^{h-1}$ and $\mathcal{P}^{-1} \equiv \emptyset$. Here, $\|\cdot\|$ denotes the Euclidean norm. Figure D.1 illustrates how the choice set $\{\pi^h\}$ is constructed in the case that $|X_1| = 3$, $\tilde{X}_1 = \{2, 3\}$, and $H = 2$. We define a quadratic scoring rule $\tau \in [-1, 1]^{M_0 \times M_2}$ as

$$\tau(x_1, h) \equiv -2\pi^h(x_1) + \|\pi^h\|^2.$$ 

For each $\pi$ and $h$, the buyer’s expected payment is

$$E_x^\pi[\tau(x_1, h)] = -\sum_{x_1} 2\pi^h(x_1)\pi(x_1) + \|\pi^h\|^2 = -\|\pi\|^2 + \|\pi - \pi^h\|^2.$$ 

Denote $b \equiv (0, \tau)$. Given the scoring rule $b$, if the seller always reports her true type $x_1$, then it is optimal for the buyer to report some $h$ with $\pi \in \mathcal{P}^h$.

Third, we introduce some penalty (or reward) rules. Let $\overline{X} \equiv \{x \in X \mid v_2(x) > v_1(x)\}$ and $\underline{X} \equiv \{x \in X \mid v_2(x) < v_1(x)\}$. The assumption of non-zero social surplus in Section 2 of the paper implies $X = \overline{X} \cup \underline{X}$. Let $y$ and $z$ be the

\[\text{Figure D.1: Choice set } \{\pi^h\} \text{ in } \Delta(X_1)\]
given the mechanism \( \tilde{m} \) and \( c \) for each \( x \in X \). We define a function \( c \in A^{M^0} \) as

\[
c(x_1, h_1, x_2, h_2) \equiv \begin{cases} g(x_1, x_2) & \text{if } h_1 = h_2, \\ z(x_1, x_2) & \text{if } h_1 \neq h_2,
\end{cases}
\]

and \( c(\cdot, \cdot, 0, \cdot) \equiv (0, 0) \). Note that, for each \( h, h' \) with \( h \neq h' \) and \( x \in X \), it holds

\[
\begin{align*}
    u_1(c(x_1, h, x_2, h), x) - u_1(c(x_1, h', x_2, h), x) &= |v_2(x) - v_1(x)| > 0, \quad (D.5) \\
    u_2(c(x_1, h, x_2, h), x) &= u_2(c(x_1, h, x_2, h'), x) = 0. \quad (D.6)
\end{align*}
\]

Define an AM penalty rule \( t_0 \in \{0, 1\}^M \) as

\[
t_0(m) \equiv \begin{cases} 1 & \text{if } \exists k \in \{1, ..., K\} \text{ s.t. } m_1^k = m_1^0, \ m_2^k \not\in \{x_2^0\} \times \{h_1^0 = 1, h_1^0, h_1^0 + 1\}, \\
    & \forall l \in \{1, ..., k - 1\}, \ m_1^l = m_1^0, \ m_2^l \in \{x_2^0\} \times \{h_1^0 = 1, h_1^0, h_1^0 + 1\}, \\
    & 0 \quad \text{otherwise}.
\end{cases}
\]

We also define \( t_1 \in [0, 1]^{M_1} \) as \( t_1(m_1) \equiv \sum_{k=1}^K I(m_1^k \neq m_1^0) / K \) and \( t_2 \in [0, 1]^{M_1 \times M_2} \) as

\[
t_2(h_1^0, m_2) \equiv \sum_{k=1}^K (I(x_2^k \neq x_2^0) + |h_2^k - h_2^0| / H) / (2K).
\]

Denote \( t \equiv t_0 - t_1 + t_2 \) and \( d \equiv (0, t) \). Note that this penalty rule \( d \) is, by definition, independent of \( h_2^0 \).

Finally, we define an outcome function \( g \in A^M \) as follows: \( g(\cdot, 0) \equiv (0, 0) \) and

\[
g(m) \equiv \varepsilon f_1(x_1^0) + \varepsilon^2 \left( f_2(x_1^0, x_2^0) + b(x_1^0, h_2^0) \right) + \varepsilon^3 c(m^0) + \varepsilon^4 d(m) \\
+ \frac{1 - \varepsilon - \varepsilon^2 - \varepsilon^3 - \varepsilon^4}{K} \sum_{k=1}^K \varphi(m^k)
\]

for each \( m \in M_1 \times M_2 \), where \( \varepsilon \in (0, 1/2) \) and \( \varphi \in A^{M^k} \) is defined by

\[
\varphi(x_1^k, h_1^k, x_2^k, h_2^k) \equiv \min \left\{ h_1^k, h_2^k \right\} \frac{\tilde{f}(x_1^k, x_2^k)}{H}
\]

given the mechanism \( \tilde{f} \) with \( \tilde{f}(\cdot, 0) \equiv (0, 0) \). We assume that \( \varepsilon, 1/K, \) and \( 1/H \)

\[\text{[36]}\]

Here, \( I \) is the indicator function (i.e., \( I(\cdot) = 1 \) if \( \cdot \) is true, and \( I(\cdot) = 0 \) otherwise).
are small enough that
\[ \bar{\eta} > \epsilon + 2\epsilon^3, \]  
\[ \bar{\eta} > (2\bar{\eta} + 2)\epsilon + 4\bar{\eta}\epsilon^2 + 2\epsilon^3, \]
\[ \min\{|v_2(x) - v_1(x)| \mid x \in X\} > 5\epsilon, \]  
\[ \epsilon^4 \min\{p_2(x_2) \mid x_2 \in X_2\} > 2\bar{\eta}/K, \]
\[ \epsilon^4 > 2\bar{\eta}/H. \]

**Properties of G.** In the rest of the proof, fix a posterior \( \pi \in \Delta(X_1) \). Let \( h \) be the integer for which \( \pi \in \mathcal{P}^h \). Denote by \( s = (s_1, s_2) \) a pure strategy profile in the Bayesian game \((G, \pi)\). For each \( k \in \{0, ..., K\} \), we denote by \( R(k) \) the statement:

“If \( s \) is iteratively undominated in \((G, \pi)\), then for each \( i, x_i, \) and \( l \in \{0, ..., k\} \), \( \bar{s}_i^l(x_i) = x_i, \bar{s}_i^l(x_1) = \bar{s}_i(x_1) \in \{h, h + 1\}, \) and \( \bar{s}_i^l(x_2) \in \{h, h + 1\} \).” Here, for each \( i \) and \( k \), we denote \( s_i = (s_i^0, ..., s_i^K) \) and \( s_i^k = (\bar{s}_i^k, \bar{s}_i^k) \in (M_i^k)^{X_i} \).

Suppose that \( s \) is iteratively undominated in \((G, \pi)\). First, we prove \( R(0) \).

**Step 1.** Fix any \( x_2 \in X_2 \). We show that \( s_2(x_2) \neq 0 \) (i.e., the opt-out option is strictly dominated). To derive a contradiction, suppose the opposite. Define \( m_2 \) by \( m_2^k \equiv (0, h) \) for each \( k \). Then, \( m_2 \) strictly dominates \( s_2(x_2) = 0 \) because it holds that, for each \( x_1 \),

\[ u_2(g(s_1(x_1), m_2), x) - u_2(g(s(x)), x) \]
\[ = \epsilon \left(u_2(f_1(s_1^0(x_1)), x) + 2\bar{\eta}\right) - \epsilon^2 \tau_1(s_1^0(x_1), h) - \epsilon^4 t(s_1(x_1), m_2) \]
\[ \geq \epsilon \left(\bar{\eta} - \epsilon - 2\epsilon^3\right) > 0, \]

where the last inequality follows from (D.8). Hence, \( s_2(x_2) \neq 0 \).

**Step 2.** Fix any \( x_1 \in X_1 \). We show that \( \bar{s}_i^0(x_1) = x_1 \). To derive a contradiction, suppose the opposite. Define \( m_1 \) by \( m_1^0 \equiv (x_1, \bar{s}_1^0(x_1)) \) and \( m_1^k \equiv s_1^k(x_1) \) for each \( k \). Then, \( m_1 \) strictly dominates \( s_1(x_1) \) because

\[ E_{x_2} [u_1(g(m_1, s_2(x_2)), x) - u_1(g(s(x)), x)] \]
\[ = \epsilon \left(U_1^h(x_1) - U_1^h(\bar{s}_1^0(x_1) \mid x_1)\right) \]
\[ + \epsilon^2 E_{x_2} \left[u_1(f_2(s_1^0(x_1)), x) - u_1(f_2(\bar{s}_1^0(x_1)), x) + \tau(x_1, \bar{s}_2^0(x_2)) - \tau(s_1^0(x_1), \bar{s}_2^0(x_2))\right] \]
\[ + \epsilon^3 E_{x_2} \left[u_1(c(m_1^0, s_2^0(x_2)), x) - u_1(c(s_1^0(x), x)) + \epsilon^4 E_{x_2} [t(m_1, s_2(x)) - t(s(x))] \right] \]
\[ \geq \epsilon \left(\bar{\eta} - (2\bar{\eta} + 2)\epsilon - 4\bar{\eta}\epsilon^2 - 2\epsilon^3\right) > 0, \]
where the last inequality follows from (D.9). Hence, $s_0^0(x_1) = x_1$.

**Step 3.** Fix any $x_2 \in X_2$. We show that $s_0^0(x_2) = x_2$. To derive a contradiction, suppose the opposite. Define $m_2$ by $m_2^0 \equiv (x_2, s_2^0(x_2))$ and $m_2^k \equiv s_2^k(x_2)$ for each $k > 0$. Then, $m_2$ strictly dominates $s_2(x_2)$ because it holds that, for each $x_1$,

$$u_2(g(s_1(x_1), m_2), x) - u_2(g(s(x)), x)$$

$$= \varepsilon^2 \left( u_2^f(x) - u_2^f(s_2^0(x_2) | x) \right) - \varepsilon^3 u_2(e(s_0(x)), x) - \varepsilon^4 \left( t(s_1(x_1), m_2) - t(s(x)) \right)$$

$$\geq \varepsilon^2 \left( \eta - 2\eta \varepsilon - (3/2)\varepsilon^2 \right) > 0,$$

where the last inequality follows from (D.9). Hence, $s_0^0(x_2) = x_2$.

**Step 4.** Fix any $x_2 \in X_2$. We show that $s_0^0(x_2) \in \{h, h + 1\}$. To derive a contradiction, suppose the opposite. Define $m_2$ by $m_2^0 \equiv (x_2, h)$ and $m_2^k \equiv s_2^k(x_2)$ for each $k > 0$. The $c$-scheme is irrelevant to the buyer’s ex-post payoffs, as shown by (D.6). The $d$-punishment is, by definition, independent of his report $h_0^0$. These facts imply

$$E_{x_2}^{\pi} \left[ u_2(g(s_1(x_1), m_2), x) - u_2(g(s(x)), x) \right] = \varepsilon^2 \left( \| \pi - \pi^{s_2^0(x_2)} \|^2 - \| \pi - \pi^{h} \|^2 \right).$$

This payoff difference is positive because $\pi^{h}$ is closer to $\pi$ than $\pi^{s_2^0(x_2)}$ is. Thus, $m_2$ strictly dominates $s_2(x_2)$. Hence, $s_0^0(x_2) \in \{h, h + 1\}$.

**Step 5.** Fix any $x_1 \in X_1$. We show that $s_0^0(x_1) \in \{h, h + 1\}$. To derive a contradiction, suppose the opposite. Step 4 then implies $s_0^0(x_1) \neq s_2^0(x_2)$ for each $x_2$. Let $m_1, n_1$ be the two messages such that $m_1^0 = (x_1, h)$, $n_1^0 = (x_1, \min \{h + 1, H\})$, and $m_1^k = n_1^k = s_2^k(x_1)$ for each $k > 0$. Let $\sigma_1(\cdot | x_1)$ be the mixed action randomizing $m_1$ and $n_1$ with equal probabilities (i.e., $\sigma_1(m_1 | x_1) = \sigma_1(n_1 | x_1) = 1/2$). Then, $\sigma_1(\cdot | x_1)$ strictly dominates $s_1(x_1)$ because it holds that, for each $x_2$,

$$(u_1(g(m_1, s_2(x_2)), x) + u_1(g(n_1, s_2(x_2)), x)) / 2 - u_1(g(s(x)), x)$$

$$= \varepsilon^3 \left( u_1(y(x), x) - u_1(z(x), x) \right) / 2 + \varepsilon^4 \left( t(m_1, s_2(x_2)) + t(n_1, s_2(x_2)) \right) / 2 - t(s(x))$$

$$\geq \varepsilon^3 \left( \| v_2(x) - v_1(x) \| / 2 - (5/2)\varepsilon \right) > 0,$$

where the last inequality follows from (D.10). Hence, $s_0^0(x_1) \in \{h, h + 1\}$.

Second, we prove $R(k)$ for each $k$. Fix any $k \in \{1, ..., K\}$ and suppose $R(k - 1)$.

**Step 6.** Fix any $x_1 \in X_1$. We show that $s_1^k(x_1) = (x_1, s_0^0(x_1))$. To derive a

Note that, if $\pi \in \mathcal{P}^h \setminus \mathcal{P}^{h+1}$, then $\| \pi - \pi^h \| < \| \pi - \pi^{h+1} \|$, and hence, $s_0^0(x_2) = h$. 

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contradiction, suppose the opposite. Define \( m_1 \) by \( m_1^k \equiv (x_1, h_1) \) with \( h_1 \equiv s_1^k(x_1) \) and \( m_1^l \equiv s_1^l(x_1) \) for each \( l \neq k \).

First, assume that \( s_2^k(x_2) \not\in \{x_2\} \times \{h_1 - 1, h_1, h_1 + 1\} \) for some \( x_2 \). By reporting \( m_1 \), the seller can make the buyer be “the first to lie” with a probability of at least \( \min\{p_2(x_2) \mid x_2 \in X_2\} \). This implies

\[
E_{x_2} \left[ u_1(g(m_1, s_2(x_2)), x) - u_1(g(s(x)), x) \right] \\
= \varepsilon^4 E_{x_2} \left[ t_0(m_1, s_2(x_2)) \right] + \frac{\varepsilon^4}{K} \mathbb{I}(s_1^k(x_1) \neq s_1^l(x_1)) \\
+ \frac{1 - \varepsilon - \varepsilon^2 - \varepsilon^3 - \varepsilon^4}{K} E_{x_2} \left[ u_1(\varphi(m_1^k, s_2(x_2)), x) - u_1(\varphi(s_1^k(x_1), s_2^k(x_2)), x) \right] \\
\geq \varepsilon^4 \min\{p_2(x_2) \mid x_2 \in X_2\} + \frac{\varepsilon^4}{K} - \frac{1 - \varepsilon - \varepsilon^2 - \varepsilon^3 - \varepsilon^4}{K} 2\eta > 0, \quad (D.13)
\]

where the last inequality follows from \( (D.11) \).

Second, assume that \( s_2^k(x_2) \not\in \{x_2\} \times \{h_1 - 1, h_1, h_1 + 1\} \) for each \( x_2 \). For notational simplicity, we denote \( h_2(\cdot) \equiv \min\{h_1, s_2^k(\cdot)\} \), \( h_2'(\cdot) \equiv \min\{s_1^l(x_1), s_2^k(\cdot)\} \), and \( \hat{x}_1 \equiv s_1^k(x_1) \). We then obtain

\[
E_{x_2} \left[ u_1(g(m_1, s_2(x_2)), x) - u_1(g(s(x)), x) \right] \\
\geq \frac{\varepsilon^4}{K} + \frac{1 - \varepsilon - \varepsilon^2 - \varepsilon^3 - \varepsilon^4}{K} E_{x_2} \left[ \frac{h_2(x_2)}{H} u_1^f(x) - \frac{h_2'(x_2)}{H} u_1^f(\hat{x}_1 \mid x) \right], \quad (D.14)
\]

where \( u_1^f(\hat{x}_1 \mid x) \equiv u_1(\hat{f}(\hat{x}_1, x_2), x) \) and \( u_1^f(x) \equiv u_1(\hat{f}(x), x) \). We claim that the right side of \( (D.14) \) is positive. Consider two cases. First, suppose \( s_1^k(x_1) \geq h_1 - 1 \). Then, the payoff difference in \( (D.14) \) is bounded from below as follows:

\[
E_{x_2} \left[ \frac{h_2(x_2)}{H} u_1^f(x) - \frac{h_2'(x_2)}{H} u_1^f(\hat{x}_1 \mid x) \right] \\
= \frac{h_1}{H} \left( U_1^f(x_1) - U_1^f(\hat{x}_1 \mid x_1) \right) + E_{x_2} \left[ \frac{h_2(x_2) - h_1}{H} u_1^f(x) + \frac{h_1 - h_2'(x_2)}{H} u_1^f(\hat{x}_1 \mid x) \right] \\
\geq -\frac{2\eta}{H}, \quad (D.15)
\]

where the inequality follows from the fact that \( |h_2(x_2) - h_1| \leq 1, |h_1 - h_2'(x_2)| \leq 1 \), and \( \hat{f} \) is IC for the seller from Lemma 4 in the paper. Inequalities \( (D.12) \) and \( (D.15) \) verify our claim. Next, suppose \( s_1^k(x_1) < h_1 - 1 \). Then, \( h_2'(\cdot) = \min\{s_1^l(x_1), s_2^k(\cdot)\} = s_1^k(x_1) \). Because \( \hat{f} \) is both IC and IR for the seller from
Lemma 4, we obtain

\[
E_{x_2} \left[ \frac{h_1 - 1}{H} u'_1(x) - \frac{h_2(x_2)}{H} u'_1(\hat{x}_1 | x) \right] = \frac{\pi^k_1(x_1)}{H} \left( U_{1}^f(x_1) - U_{1}^f(\hat{x}_1 | x_1) \right) + \frac{h_1 - 1 - \pi^k_1(x_1)}{H} U_{1}^f(x_1) \geq 0.
\]

This inequality, together with the first case, verifies our claim.

Thus, \( m_1 \) strictly dominates \( s_1(x_1) \). Hence, \( s^k_1(x_1) = s^0_1(x_1) \).

**Step 7.** Fix any \( x_2 \in X_2 \). We show that \( s^{k}_2(x_2) = x_2 \) and \( \pi^k_2(x_2) \in \{h, h + 1\} \). To derive a contradiction, suppose the opposite. First, suppose \( s^k_2(x_2) \in X_2 \setminus \{x_2\} \). This yields a contradiction because \( s^k_2(x_1) \equiv x_1 \) from Step 6, \( \tilde{f} \) is EPIC for the buyer from Lemma 4 in the paper, and \( t_2 \) imposes a penalty (i.e., \( \eta(s^k_2(x_2) \neq x_2) = 1 \)) on the buyer. Second, suppose \( s^k_2(x_2) = 0 \). This yields a contradiction because \( \pi \not\in \mathcal{P}^0 \setminus \mathcal{P}^1 \) implies \( U_{2}^{f, \pi}(x_2) > 0 \), and \( \pi \in \mathcal{P}^0 \setminus \mathcal{P}^1 \) implies \( \pi^k_1 = \pi^0_1 = 0 \) (and thus, \( \varphi(\cdot, \pi^k_1(\cdot), \cdot) \) always specifies the no-trade outcome) from Steps 4 and 5. Finally, suppose \( \pi^k_2(x_2) \not\in \{h, h + 1\} \). We suppose, without loss of generality, that \( \pi^k_2(x_2) \geq h + 2 \) because \( \pi^k_1(\cdot) \in \{h, h + 1\} \), and \( h \geq 1 \) implies \( U_{2}^{f, \pi}(x_2) > 0 \). The message \( s_2(x_2) \) is strictly dominated by \( m_2 \) such that \( m_2^k \equiv (x_2, h + 1) \) and \( m_2^l \equiv s^k_2(x_2) \) for each \( l \neq k \) because for each \( x_1 \), \( t_2(\pi^0_1(x_1), m_2) - t_2(\pi^0_1(x_1), s_2(x_2)) \geq 1/(2H)K > 0 \). This is a contradiction.

Third, we characterize the ex-post outcomes given the iteratively undominated strategy profile \( s = (s_1, s_2) \) in the game \((G, \pi)\).

**Step 8.** Fix any \( x \in X \). Steps 1–7 imply that, for each \( i \) and \( k \), \( s^k_i(x_i) = x_i \), \( \pi^0_1(x_1) = \pi^k_1(x_1) \in \{h, h + 1\} \), and \( \pi^k_2(x_2) \in \{h, h + 1\} \). The ex-post outcome is

\[
g(s(x)) = \varepsilon f_1(x_1) + \varepsilon^2 (f_2(x) + b(x_1, \pi^0_1(x_2))) + \varepsilon^3 c(x_1, \pi^0_1(x_1), x_2, \pi^0_2(x_2)) + \varepsilon^4 (0, t_2(\pi^0_1(x_1), s_2(x_2))) + \frac{1 - \varepsilon - \varepsilon^2 - \varepsilon^3 - \varepsilon^4}{K} \sum_{k=1}^{K} \min \{\pi^k_1(x_1), \pi^k_2(x_2)\} \tilde{f}(x).
\]

From this equation, we obtain the following inequality:

\[
U_{i}^{\text{g.s.}}(x_1) \leq -\eta \varepsilon + (\eta + 1) \varepsilon^2 + 2\eta \varepsilon^3 + \frac{\varepsilon^4}{2} + \frac{\eta}{H} + U_{i}^f(x_1).
\]

(D.16)

If \( \pi \in \mathcal{P}^H \) (i.e., \( h = H \)), then \( \pi^k_i(x_i) = H \) for each \( i \) and \( k \), and thus, \( s \) is the

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38The term \( \eta/H \) is needed if for some \( k \geq 1 \), \( \pi^k_1(x_1) = h + 1 \) and \( \pi^k_2(X_2) = \{h, h + 1\} \) (i.e., the seller is uncertain about the probability of implementing \( f \) at \( k \)).
unique iteratively undominated strategy profile in \((G, \pi)\) and
\[
g(s(x)) = \varepsilon f_1(x_1) + \varepsilon^2 (f_2(x) + b(x_1, H)) + \varepsilon^3 c(x_1, H, x_2, H) + (1 - \varepsilon - \varepsilon^2 - \varepsilon^3 - \varepsilon^4) \tilde{f}(x). \tag{D.17}
\]
Equation (D.17) implies that, if \(\pi \in \mathcal{P}^H\), then
\[
U^{\mathbb{Q}_s}(x_1) \geq -3\bar{\eta}\varepsilon - (\bar{\eta} + 1)\varepsilon^2 + (1 - \varepsilon - \varepsilon^2 - \varepsilon^3 - \varepsilon^4) U^{\tilde{f}}(x_1). \tag{D.18}
\]
Finally, we show that the mechanism \(G\) has properties (i)–(iii) in Lemma 5.

**Step 9.** Lemma 4 in the paper implies that \(U^{f}_1(x_1) > U^{f}_1(x_1)\) if and only if \(x_1 \in \tilde{X}_1\). Using this fact with (D.16) and (D.18), we assume, without loss of generality, that \(\varepsilon\) and \(1/H\) are small enough that, for each \(\pi' \in \Delta(X_1)\) and \(f' \in \text{BN}(G, \pi')\),
\[
U^{f'}_1(x_1) < U^{f}_1(x_1) \text{ if } x_1 \not\in \tilde{X}_1, \tag{D.19}
\]
\[
U^{f'}_1(x_1) > U^{f}_1(x_1) \text{ if } x_1 \in \tilde{X}_1 \text{ and } f' = g \circ s, \tag{D.20}
\]
where the allocation \(g \circ s\) is given by (D.17). It follows that \(X'_1(G) = X_1 \setminus \tilde{X}_1\), that is, \(G\) has property (i) in Lemma 5. Because \(\Delta(X_1 \setminus X'_1(G)) = \Delta(\tilde{X}_1) \subseteq \mathcal{P}^H\), Step 8 with (D.20) implies that \(G\) has properties (ii) and (iii) in Lemma 5. \qed

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