Geodesic Flow on the $n$-Dimensional Ellipsoid as a Liouville Integrable System

Petre Diţă

National Institute of Physics & Nuclear Engineering
Bucharest, PO Box MG6, Romania

Abstract

We show that the motion on the $n$-dimensional ellipsoid is complete integrable by exhibiting $n$ integrals in involution. The system is separable at classical and quantum level, the separation of classical variables being realized by the inverse of the momentum map. This system is a generic one in a new class of $n$-dimensional complete integrable Hamiltonians defined by an arbitrary function $f(q,p)$ invertible with respect to momentum $p$ and rational in the coordinate $q$.

Complete integrable Hamiltonian systems in the Liouville sense are a main subject of interest in the last decades both at classical as well as quantum level. Classical examples of such systems are the geodesic flow on the triaxial ellipsoid, Neumannn’s dynamical system and the integrable cases of the rigid body motion. The number of interesting examples increased after the seminal papers by Lax [1] and by Olshanetski and Peremolov [2], especially in connection with the inverse scattering method and the relation with classical simple Lie algebras.

The aim of this paper is the study of the geodesic motion on the $n$-dimensional ellipsoid, which is a direct generalization of the 2-dimensional case studied by Jacobi [3], since it seems that no solution is known for this system in the case $n > 2$. We obtain a complete description of the problem at the classical level by finding $n$ prime integrals in involution, the separation of variables, the explicit solution of the Hamilton-Jacobi equation and the equations of geodesics. We also show that the Schrödinger equation separates. This system is to a great extent universal among the various integrable systems and generates a new class of completely integrable models.

The Lagrangean for a particle of unit mass constrained to move on the $n$-dimensional ellipsoid

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n+1} \dot{x}_i^2$$

where $a_i$, $i = 1, 2 \ldots, n + 1$ are positive numbers $a_i \in \mathbb{R}_{+}^{n+1}$, is

$$\frac{x_1^2}{a_1} + \frac{x_2^2}{a_2} + \ldots + \frac{x_{n+1}^2}{a_{n+1}} = 1$$

The preceding equations define a constrained system and the model can be formulated in terms of constrained dynamical variables with Dirac brackets, or in an

1email: dita@hera.theory.nipne.ro
unconstrained form with canonical Poisson brackets. In the following we will use a third way, the classical one, which consists in reducing the Lagrangean by eliminating one degree of freedom \[ \text{by using the equation of constraint (1). The simplest way to do that would be to resolve Eq.}(1) \text{ with respect to the last coordinate and use it in the free Lagrangean, Eq.}(2), \text{but the drawback is that we obtain a non-diagonal metric. We follow here the Jacobi idea which was to find a clever parametrization such that the corresponding metric should be diagonal [3, 6].} \]

For what follows it is useful to define two polynomials

\[ P(x) = \prod_{i=1}^{n+1} (x - a_i) \quad Q(x) = \prod_{i=1}^{n} (x - u_i) \]

where \( a_i \) and \( u_i, i = 1, \ldots, n \) are the positive numbers entering the parametrization of the ellipsoid and the ellipsoidal coordinates, respectively. The orthogonal parametrization of the quadric (1) is given by [7]

\[ x_j^2 = \frac{a_j Q(a_j)}{P'(a_j)}, \quad j = 1, 2, \ldots, n + 1 \]

where \( P'(a_j) = d P(x)/d x |_{x=a_j} \), and the ellipsoidal coordinates \( u_1, \ldots, u_n \) satisfy \( a_1 < u_1 < a_2 < \ldots < u_n < a_{n+1} \). Using this parametrization in Eq.(2) we find

\[ \mathcal{L} = -\frac{1}{8} \sum_{i=1}^{i=n} g_{ii} \dot{u}_i^2 \]

where the (diagonal) metric is given by \( g_{ii} = u_i Q'(u_i)/P(u_i) \), \( i = 1, 2, \ldots, n \), and \( Q'(u_i) = d Q(x)/d x |_{x=u_i} \). Defining as usual the generalized momenta by \( p_i = \partial \mathcal{L}/\partial \dot{u}_i \) and using the Legendre transform we find the Hamiltonian of the problem

\[ \mathcal{H} = \sum_{i=1}^{i=n} p_i \dot{u}_i - \mathcal{L} = -\sum_{i=1}^{i=n} g^{ii} p_i^2 \]

where \( g^{ii} = P(u_i)/u_i Q'(u_i) \). We define now the symmetric functions of the polynomials \( Q^{(j)}(x) = Q(x)/(x - u_j) \)

\[ Q^{(j)}(x) = \sum_{k=0}^{n-1} x^k S^{(j)}_{n-k-1}, \quad j = 1, 2, \ldots, n \]

where \( S^{(0)}_0 = 1, \quad S^{(j)}_1 = - (u_1 + \ldots + u_{j-1} + u_{j+1} + \ldots + u_n) \), etc. The upper index means that the coordinate \( u_j \) does not enter the symmetric sum \( S^{(j)}_k, k = 1, \ldots, n-1 \). We define the following functions

\[ H_k = \sum_{i=1}^{n} S^{(i)}_{k-1} g^{ii} p_i^2 = \sum_{i=1}^{n} S^{(i)}_{k-1} \frac{P(u_i)}{u_i Q'(u_i)} p_i^2, \quad k = 1, 2, \ldots, n \]

where \( H_1 \) differs from \( \mathcal{H} \) by a numerical factor. A careful inspection of the Eqs.(8) shows that for each degree of freedom the contribution to the Hamiltonian \( H_k \) is given by the product of two different factors. The first one depends on the "Vandermonde" structure \( f_1 = S^{(i)}_{k-1}/Q'(u_i) \) and the second one \( f_2 = (P(u_i)/u_i)p_i^2 \)
depends on the "singularities", i.e. the hyperellitic curve defined by the parameters \(a_i\).

Let \(g(p, u) = \mathcal{H}(p, u)\) be an arbitrary function depending on the canonical variables \(p\) and \(u\) which is invertible with respect to the momentum \(p\). As we will see later the invertibility condition is necessary for the separation of variables in the Hamilton-Jacobi equation. In particular we may suppose that \(\mathcal{H}(p, u)\) is an one-dimensional Hamiltonian. For each \(n \in \mathbb{N}\) we define an \(n\)-dimensional integrable model by giving \(n\) integrals in involution

\[
\mathbb{H}_k(p, u) = \sum_{i=1}^{n} \frac{S_{k-1}^{(i)}}{Q'(u_i)} g(p_i, u_i), \quad k = 1, 2, \ldots, n
\]  

(8')

Our main result is contained in the following proposition.

**Proposition.** Let \(M^{2n} \simeq T^*(\mathbb{R}^n)\) be the canonically symplectic phase space of the dynamical system defined by the Hamilton function Eq.(6). Then

i) the functions \(H_i, \ i = 1, 2, \ldots, n\) are in involution

\[
\{H_i, H_j\} = 0, \quad i, j = 1, 2, \ldots, n
\]

ii) the momentum map is given by

\[
\mathcal{E} : M^{2n} \to \mathbb{R}^n : M = \{(u_i, p_i) : H_i = h_i, \quad i = 1, 2, \ldots, n\}, \ h_i \in \mathbb{R}
\]

then \(\mathcal{E}^{-1}(M_h)\) realizes the separation of variables giving an explicit factorisation of Liouville’s tori into one-dimensional ovals

iii) canonical equations are integrable by quadratures

In the following we sketch a proof of the above proposition. As it will be easily seen the same proof is also true for the Hamiltonians defined in Eqs.(8').

**Proof.**

i) We calculate the Poisson bracket

\[
\{H_k, H_l\} = \sum_{j=1}^{n} \left( \frac{\partial H_k}{\partial u_j} \frac{\partial H_l}{\partial p_j} - \frac{\partial H_k}{\partial p_j} \frac{\partial H_l}{\partial u_j} \right) = 2 \sum_{j=1}^{n} p_j P(u_j) \left( S_{k-1}^{(j)} \frac{\partial H_k}{\partial u_j} - S_{k-1}^{(j)} \frac{\partial H_l}{\partial u_j} \right) =
\]

\[
2 \sum_{j=1}^{n} \sum_{i=1}^{n} p_i^2 p_j \frac{P(u_j)}{u_j Q'(u_j)} \left[ S_{l-1}^{(j)} \frac{\partial}{\partial u_j} \left( S_{k-1}^{(j)} \frac{P(u_i)}{u_i Q'(u_i)} \right) - S_{k-1}^{(j)} \frac{\partial}{\partial u_j} \left( S_{k-1}^{(j)} \frac{P(u_i)}{u_i Q'(u_i)} \right) \right] =
\]

\[
\sum_{j=1}^{n} \sum_{i=1}^{n} p_i^2 p_j \frac{P(u_j)}{u_j Q'(u_j)} \left[ \frac{\partial}{\partial u_j} \left( S_{l-1}^{(j)} S_{k-1}^{(j)} \frac{P(u_i)}{u_i Q'(u_i)} \right) - \frac{\partial}{\partial u_j} \left( S_{k-1}^{(j)} S_{l-1}^{(j)} \frac{P(u_i)}{u_i Q'(u_i)} \right) \right] =
\]

The last step was possible because the symmetric functions \(S_{k}^{(j)}\) and \(S_{l}^{(j)}\) depend on all \(u_1, u_2, \ldots, u_n\), but \(u_j\). Looking at the last expression it is easily seen that the partial derivative with respect to \(u_j\) vanishes for \(i = j\). For \(i \neq j\) we have to show that

\[
\frac{\partial}{\partial u_j} \frac{S_{l-1}^{(j)} S_{k-1}^{(i)}}{u_i - u_j} = 0
\]

3
but this is a direct consequence of the following identities

\[ \frac{\partial}{\partial u_j} S_{k-1}^{(i)} = -S_{k-2}^{(i,j)} \quad \text{and} \quad S_{k-1}^{(i)} - S_{k-1}^{(j)} = (u_i - u_j) S_{k-2}^{(i,j)} \]

where the upper index \((i, j)\) means that the corresponding expression does not depend on both \(u_i\) and \(u_j\). In this way we have shown that \(\{H_k, H_l\} = 0\).

ii) The surface \(M_h = \{(u_i, p_i) : H_i = h_i, \ i = 1, 2, \ldots, n\}, h_i \in \mathbb{R}\) is given by the system of equations

\[ \sum_{i=1}^{n} S_{k-1}^{(i)} \frac{P(u_i)}{u_i Q'(u_i)} p_i^2 = h_k, \quad k = 1, 2, \ldots, n \quad (9) \]

and resolving it with respect to \(p_i\) is equivalent to the calculation of the following determinant

\[ V_n = \begin{vmatrix} 1 & 1 & \ldots & \ldots & 1 \\ S_1^{(1)} & S_1^{(2)} & \ldots & \ldots & S_1^{(n)} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ S_{n-1}^{(1)} & S_{n-1}^{(2)} & \ldots & \ldots & S_{n-1}^{(n)} \end{vmatrix} \]

which is equal to the Vandermonde determinant, i.e.

\[ V_n = \prod_{1 \leq i < j \leq n} (u_j - u_i) \]

Let \(V_{n-1}^{(j)}\) be the determinant obtained by removing the \(j\)th column and the last row in \(V_n\) and \(W_{n,j}\) the determinant obtained by replacing the \(j\)th column of \(V_n\) by \((h_1, \ldots, h_n)^t\). It is easily seen that

\[ V_{n-1}^{(j)} = \prod_{1 \leq k < l \leq n \atop k \neq j \neq l} (u_l - u_k) \]

i.e. \(V_{n-1}^{(j)}\) is the Vandermonde determinant of the variables \(u_1, \ldots, u_n\), but \(u_j\). We have the identities

\[ \prod_{j=1}^{n} V_{n-1}^{(j)} = (V_n)^{n-2} \]

\[ \frac{V_n}{V_{n-1}^{(j)}} = (-1)^{n-j} Q'(u_j), \quad j = 1, \ldots, n \]

\[ W_{n,j} = (-1)^{n-j} V_{n-1}^{(j)} \sum_{i=0}^{n-1} h_{n-i} u_j^i, \quad j = 1, \ldots, n \]

Using these identities it is easily seen that \(E^{-1}(M_h)\) is equivalent to the relations

\[ \frac{P(u_i)p_i^2}{u_i} = \sum_{k=0}^{n-1} h_{n-k} u_i^k \quad i = 1, 2, \ldots, n \quad (10) \]

which shows that \(E^{-1}(M_h)\) is a \(n\)-dimensional submanifold of \(M^{2n}\) and more important \(p_i^2\) are functions which depend only on the variable \(u_i\). The last relation shows
that the application $E^{-1}(M_h)$ realizes the separation of variables for the geodesic motion on the ellipsoid.

For the Hamiltonians given by Eqs.(8') the preceding equations have the form

$$g(p_i, u_i) = \sum_{k=0}^{n-1} h_{n-k} u_i^k, \quad i = 1, 2, \ldots, n$$ (10')

The relations (10-10') have the classical form $[8]

$$\varphi(x_i, p_i, h_1, \ldots, h_n) = 0 \quad i = 1, 2, \ldots, n$$

which is an explicit factorization of Liouville’s tori into one-dimensional ovals. This is important for the quantization problem.

With the notation $R(u) = \sum_{k=0}^{n-1} h_{n-k} u^k$ the above relations can be written

$$p_i = \epsilon_i \sqrt{u_i R(u_i) / P(u_i)}, \quad i = 1, \ldots, n$$

$$p_i = g^{-1}(R(u_i)), \quad i = 1, \ldots, n$$

where $\epsilon_i = \pm 1$ and $g^{-1}$ is the inverse of the relation (10') with respect to the momentum $p$. The last relations allow immediately to resolve the Hamilton-Jacobi equation because in this case the action splits into a sum of terms

$$S(h, u_1, \ldots, u_n) = S_1(h, u_1) + \ldots + S_n(h, u_n)$$

each of them satisfying an ordinary differential equation. Only the solution of the Hamilton-Jacobi equation for the geodesic motion on the ellipsoid will be presented the other more general case being similar.

$$S(h, u_1, \ldots, u_n) = \sum_{i=1}^{n} \epsilon_i \int_{u_i^0}^{u_i} \sqrt{u R(w) / P(w)} \, dw$$

iii) The above formulae allows us to choose new canonical variables as follows $Q_1 = H_1, Q_k = H_k, k = 2, \ldots, n$ and the corresponding variables $P_i, i = 1, \ldots, n$. The Hamilton equations take the form

$$\dot{Q}_i = 0, \quad i = 1, \ldots, n$$

$$\dot{P}_1 = -1, \quad \dot{P}_i = 0, \quad i = 2, \ldots, n$$

and therefore $Q_i = h_i, i = 1, \ldots, n$ and $P_1 = -t + g_1, P_k = g_k, k = 2, \ldots, n$, with $g_i, h_i \in \mathbb{R}, i = 1, \ldots, n$.

Because

$$P_i = -\frac{\partial S}{\partial Q_i} = -\frac{\partial S}{\partial h_i} = \frac{\epsilon_i}{2} \int_{u_i^0}^{u_i} \frac{t^{n-i+1/2}}{\sqrt{P(t)R(t)}} \, dt$$

we obtain the system

$$-t \delta_{ij} + b_j = \frac{1}{2} \sum_{i=1}^{n} \epsilon_i \int_{u_i^0}^{u_i} \frac{t^{n-j+1/2}}{\sqrt{P(t)R(t)}} \, dt, \quad j = 1, \ldots, n$$
which gives the implicit equations of the geodesics. In this way the integration of the Hamilton equations was reduced to quadratures. On the last expressions one can see that all the subtleties of the geodesic motion on the n-dimensional ellipsoid are encoded by the hyperelliptic curve $y^2 = P(x) R(x)$ whose genus is $g = n$.

For quantization we use both the forms (9) and (10) and show first that the quantization of $H_1$ is equivalent to the quantization of Eq.(10). It is well known that because of the ambiguities concerning the ordering of $p$ and $u$ we must use the Laplace-Beltrami operator $\Delta$. Its general form is $\Delta_n = \frac{1}{\sqrt{g}} p_i (\sqrt{g} g^{ij} p_j), \, i,j = 1, \ldots, n$, where $g = det(g_{ij})$ and $g_{ij}$ is the metric tensor. Taking into account that

$$g_{ii} = u_i \Psi_i(u_i)/P(u_i) = (-1)^{n-i} \frac{V_n}{V_{n-1}} \frac{u_i}{P(u_i)}$$

after some simplifications the Schrödinger equation generated by the Hamiltonian $H_1$ is written in the following form

$$- \sum_{i=1}^{n} \frac{1}{V_n} \sqrt{\frac{P(u_i)}{u_i}} \frac{\partial}{\partial u_i} \left( (-1)^{n-i} V_{n-1}^{(i)} \frac{\partial}{\partial u_i} \right) = h_1 \Psi$$

Since the term $V_{n-1}^{(i)}$ does not depend on $u_i$ it can be pulled out of the bracket and the precedent equation takes the form

$$- \sum_{i=1}^{n} (-1)^{n-i} V_{n-1}^{(i)} \sqrt{\frac{P(u_i)}{u_i}} \frac{\partial}{\partial u_i} \left( \sqrt{\frac{P(u_i)}{u_i}} \frac{\partial \Psi}{\partial u_i} \right) = h_1 V_n \Psi$$

Now we make use of the Jacobi identity for the Vandermonde determinant. With the above notations the identity is

$$\sum_{j} (-1)^{n-j} V_{n-1}^{(j)} u_j^k = \delta_{n-1,k} V_n$$

and using it in the preceding relation we obtain

$$\sum_{i=1}^{n} (-1)^{n-i} V_{n-1}^{(j)} \left[ \sqrt{\frac{P(u_i)}{u_i}} \frac{\partial}{\partial u_i} \left( \sqrt{\frac{P(u_i)}{u_i}} \frac{\partial \Psi}{\partial u_i} \right) + \sum_{k=0}^{n-1} c_{n-k} u_i^k \Psi \right] = 0$$

which is equivalent with $n$ independent equations of the form

$$\sqrt{\frac{P(u_i)}{u_i}} \frac{\partial}{\partial u_i} \left( \sqrt{\frac{P(u_i)}{u_i}} \frac{\partial \Psi}{\partial u_i} \right) + \left( \sum_{k=0}^{n-1} c_{n-k} u_i^k \right) \Psi = 0, \quad i = 1 \ldots, n \quad (11)$$

Here $c_1 = h_1$ and the other $c_k$ are arbitrary. The direct approach, starting from Eq.(10), is simpler the problem being one-dimensional and one arrives at the same equation, Eq.(11). It has the advantage that the arbitrary coefficients $c_k$ are identified to $c_k = h_k$, i.e. $c_k$ are the eigenvalues of the Hamiltonians $H_k$.

The Eq.(11) has the general form of a Sturm-Liouville problem

$$- \frac{d}{dx} \left( p(x) \frac{df(x)}{dx} \right) + v(x) f(x) = \lambda r(x) f(x)$$
which has to be resolved on an interval \([a, b]\). It is well known that its eigenfunctions will live in a Hilbert space iff \(p(x) r(x) > 0\) on \([a, b]\). If \(p(x)\) has a continuous first derivative and \(p(x) r(x)\) a continuous second derivative, then by the following coordinate and function transforms

\[ \varphi = \int_{u_0}^{u} \left( \frac{r(x)}{p(x)} \right)^{1/2} dx, \quad \Phi = \left( r(u)p(u) \right)^{1/4} f(u) \]  

the preceding equation takes the standard form

\[ -\frac{d^2 \Phi}{d\varphi^2} + q(\varphi) \Phi = \lambda \Phi \]

where

\[ q(\varphi) = \frac{\mu''(\varphi)}{\mu(\varphi)} - \frac{v(u)}{r(u)}, \quad \mu(\varphi) = \left( p(u) r(u) \right)^{1/4} \]

and \(u = u(\varphi)\) is the solution of the Jacobi inverse problem (12).

In our case, Eq.(11), the transformation is

\[ \varphi = \int_{u_0}^{u} \left( \frac{u R(u)}{P(u)} \right)^{1/2} du \]

and the Schrödinger equation has the form

\[ -\frac{d^2 \Phi}{d\varphi^2} + \frac{\mu''(\varphi)}{\mu(\varphi)} \Phi = h_1 \Phi \]  

(13)

where \(\mu(\varphi) = \left( R(u(\varphi)) \right)^{1/4}\) and in \(R(u)\) we made the rescaling \(h_k \to h_k/h_1, \ k = 1, \ldots, n\). Thus we have obtained that the solving of the Schrödinger equation (11) is equivalent to the solving of the motion of one-dimensional particle in a potential generated by \(R(u(\varphi))\).

For \(n = 1\) Eq.(13) is the equation of the one-dimensional rotator

\[ \frac{d^2 \Psi}{d\varphi^2} + \ell^2 \Psi = 0 \]

with the solution \(\Psi(\varphi) = \frac{1}{\sqrt{2\pi}} e^{i\ell \varphi}, \ l \in \mathbb{Z},\) etc. In all the other cases we have to make use of the theory of hyperelliptic curves, \(\theta\)-functions and/or hyperelliptic Abelian functions in order to obtain explicit solutions. This problem will be treated elsewhere. What is remarkable is that the solving of the classical problem, or the solving of the associated Schrödinger equation leads to the use of the same mathematical formalism, \(\theta\)-functions or hyperelliptic Kleinian functions.

However there is a simpler alternative to Eq.(13), namely by the change of variable

\[ \varphi = \int_{u_0}^{u} \sqrt{\frac{t}{P(t)}} dt \]

the Eq.(11) takes the form

\[ \frac{d^2 \Psi}{d\varphi^2} + \left( \sum_{k=0}^{n-1} h_{n-k} u(\varphi)^k \right) \Psi = 0 \]
which is the equation of a particle moving in a potential generated by the integrals in involution.

As concerns the quantization of the Hamiltonian $H_1$ it depends on its explicit form and we do not pursue it here.

From the proof of our results it follows that the hyperelliptic curve was only a tool in obtaining the separation of variables Eq.(10); in fact the separation was a direct consequence of the properties of the Vandermonde determinant.

In the following we exhibit a few examples of new $n$-dimensional integrable models. Two models which show that the dimension $n$ of the system has no direct connection with the number of zeros and/or poles of the function $g(p,u)$ could be: $g(p,u) = (\sin u/u) p^2$ and $g(p,u) = \tan u e^{\alpha p}$, the first example being a function with a denumerable number of zeros and the second one with a denumerable number of poles and zeros, in both cases the hyperelliptic curve being of infinite genus. Other examples are deduced for example from the many-body elliptic Calogero-Moser [10], or the elliptic Ruijenaars models [11]. Starting with the corresponding one-dimensional Hamiltonians $H_{CM}(p,u) = f(p,u) = p^2 / 2 + \nu^2 \wp(\tau)(u)$ and $f_R(p,u) = \cosh(\alpha p) \sqrt{1 - \alpha^2 \wp(\tau)(u)}$ respectively, where $\wp(\tau)(u)$ is the Weierstrass function, we obtain $n$-dimensional models.

In conclusion we discovered in this paper a new class of complete integrable systems which allows to uncover the origin of their integrability or solvability property. We have shown that there is a simple and general mechanism allowing us to construct complete integrable Hamiltonian systems with an arbitrary number of degree of freedom, and for all these systems the separation of classical variables is given by the inverse of the momentum map.

References

[1] P.D. Lax Commun.Pure Appl.Math. 21, 467 (1968)
[2] M. Olshanetsky and A. Peremolov, Lett.Math.Phys., 2, 7 (1977)
[3] C. Jacobi, Vorlesungen über Dynamik, Reimer, Berlin, 1866 (quoted apud the next reference)
[4] M. Olshanetsky and A. Peremolov, Phys.Rept., 71, 315 (1981)
[5] E.T. Whittaker, Analytical Dynamics, Cambridge Univ. Press, 1937
[6] A.M. Vinogradov and B.A. Kupershmidt Russian Math. Surveys 32, 177 (1977)
[7] E.G. Kalnins and W. Miller, Jr. J.Math.Phys. 27, 1721 (1986)
[8] E.K. Sklyanin, Progr.Theor.Phys.Suppl. 118 (1995) 35
[9] B. Podolsky, Phys.Rev. 32, 812 (1928)
[10] H.W. Braden, A. Marshakov, A. Mironov and A. Morozov, hep-th/9906240
[11] S.N.M. Ruijenaars, Commun.Math.Phys., 110, 191 (1987)