Extension of Chronological Calculus for Dynamical Systems on Manifolds

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Abstract:
We present an extension of Chronological Calculus to the case of infinite-dimensional C^\infty-smooth manifolds. The original Chronological Calculus was developed by Agrachev and Gamkrelidze for the study of dynamical systems on C^\infty-smooth finite-dimensional manifolds. The extension of this calculus allows for the study of control systems with merely measurable controls and may be applied to C^\infty-smooth manifolds modeled over Banach spaces. We apply our extension to establish a formula of Mauhart and Michor for the generation of Lie brackets of vector fields and we present a proof of the Chow-Rashevskii theorem on C^\infty-smooth manifolds modeled over Banach spaces.

Classical Chronological Calculus:
The C^\infty(M) algebra: A central object of study in the Chronological Calculus of Agrachev and Gamkrelidze is the algebra C^\infty(M) of C^\infty-smooth functions f:M \to R. An important observation is that inherently nonlinear objects such as diffeomorphisms of manifolds give rise to inherently linear objects such as automorphisms of this algebra. Indeed, given a diffeomorphism \phi:M \to M, one obtains an automorphism \hat{\phi}: C^\infty(M) \to C^\infty(M) by \hat{\phi}(f) = f \circ \phi. There are similar correspondences for points in M, for tangent vectors, and for vector fields. These correspondences provide a means to study many of the nonlinear objects of control theory in a setting where they behave linearly.

The Whitney Topology: Agrachev and Gamkrelidze place a topology on C^\infty(M) in which \phi_k \to \phi if and only for any compact subset K of M, one has the uniform convergence over K of \phi_k and derivatives of all orders to \phi. The precise meaning of this statement can be formulated through the Whitney embedding theorem. Equipped with this topology, C^\infty(M) has the structure of a Fréchet space and the correspondences described above lead to the study of nonlinear objects as linear operators on this space.

Challenges: The classic chronological calculus is unable to handle control problems in which the dynamics are merely C^\infty-smooth, or are merely measurable in time, or which take place on a manifold whose local structure is infinite dimensional. In addition, the use of Fréchet space structure seems to complicate proofs for a number of important results.

Main Results

Extension of Chronological Calculus: Given a C^\infty-smooth manifold M modeled over a Banach space E, let C(M,E) denote the vector space of \infty-times differentiable functions f: M \to E. The principle setting for our extension is the study of families of operators on these vector spaces. In this way, we are able to develop results for C^\infty-smooth dynamics on manifolds modeled over Banach spaces. For example, the local flow P^t_0 of a nonautonomous vector field V_t gives rise under appropriate assumptions to a family of operators C(M,E) \to C(M,E).

Calculus of Little a’s: In order to facilitate use of the calculus, we have developed a calculus of remainder terms, so that one is able to refer to a family of operators Q_t as being differentiable with derivative V_t whenever one has Q_{t+h} = Q_t + hV_t + o(h). This rule is satisfied, for example, when Q_t is the flow of an autonomous vector field V. We establish the following useful properties for these operators:
1. \dot{a}(t) + a(t) = a(t)
2. a(t)\dot{\phi} = \dot{a}(t)\phi
3. For vector fields V_t and W_t with locally bounded derivatives, \dot{V}_t\phi = \dot{a}(t)\phi = a(t)
4. If \phi_t and Q_t are families of operators arising from flows of vector fields, \phi_{t+h}\phi_t = \phi_h\phi_t

These properties lead to simplified proofs of important results such as the bracket formula of Mauhart and Michor.

Product Rule for Composition of Operators: We say that a family of operators is differentiable at t with derivative A_t if \phi_{t+h} = \phi_t + hA_t + o(h). Using the above properties, one may check that if \phi_t and \psi_t are differentiable at t with derivatives A_t and B_t, respectively, then the composition \phi_t \circ \psi_t is differentiable with derivative A_t \circ B_t.

Flows of Perturbed Vector Fields: Given vector fields V_t and W_t, we derive a formula for the flow of their sum as a correction to the flow of V_t. This is done in a general setting which allows the study of perturbations to nonautonomous C^\infty-smooth vector fields on Banach manifolds which are merely measurable in time.

Bracket Formula: Mauhart and Michor define a bracket of flows as \{P_t, Q_t\} = P_tQ_t - Q_tP_t + Q_{-1}P_{-1}. The calculus of remainder terms gives us an algebraic proof of the following formula of Mauhart and Michor:
B(P_t, P_{t+h}, ..., P_{t+h+t}) = Id + h^2B(X_1, X_2, ..., X_n) + o(h^2)
where B is a bracket expression.

Chow-Rashevskii Theorem: We apply the bracket formula of Mauhart and Michor, along with some nonsmooth analysis for manifolds, to prove a variant of the Chow-Rashevskii Theorem on Banach Manifolds. In particular, we prove that if M is modeled over a smooth Banach space, then a smooth affine control system is globally approximately controllable when the Lie algebra of the associated vector fields spans T_M for any q.