ON THE STRONG CONVERGENCE OF PARTIAL SUMS WITH RESPECT TO BOUNDED VILENKIN SYSTEMS

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Abstract. In this paper we investigate some strong convergence theorems for partial sums with respect to Vilenkin system.

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1. Introduction

It is well-known (for details see e.g. [10] and [14]) that Vilenkin system does not form basis in the space \( L_1(G_m) \). Moreover, there is a function in the Hardy space \( H_1(G_m) \), such that the partial sums of \( f \) are not bounded in \( L_1 \)-norm. However, subsequence \( S_{M_n} \) of partial sums are bounded from the martingale Hardy space \( H_1(G_m) \) to the Lebesgue space \( L_1(G_m) \):

\[
\|S_{M_k}f\|_{H_1} \leq c\|f\|_{H_1} \quad (k \in \mathbb{N}).
\]

Moreover, we have the following norm equivalence:

\[
\|f\|_{H_1} \equiv \sup_n \|S_{M_n}f\|_1.
\]

Moreover, Gát [8] proved the following strong convergence result for all \( f \in H_1 \):

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \|S_k f - f\|_1 = 0,
\]

where \( S_k f \) denotes the \( k \)-th partial sum of the Vilenkin-Fourier series of \( f \).

It follows that there exists an absolute constant \( c \), such that

\[
\frac{1}{\log n} \sum_{k=1}^{n} \|S_k f\|_k \leq c\|f\|_{H_1} \quad (n = 2, 3, \ldots)
\]

and

\[
\lim_{n \to \infty} \frac{1}{\log n} \sum_{k=1}^{n} \|S_k f\|_k = \|f\|_{H_1},
\]

for all \( f \in H_1 \).

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Analogical result for the trigonometric system was proved by Smith [20], for the Walsh-Paley system by Simon [18].

If partial sums of Vilenkin-Fourier series was bounded from $H^1$ to $L^1$ we also would have:

$$\sup_{n \in \mathbb{N}_+} \frac{1}{n} \sum_{m=1}^{n} \|S_m f\|_1 \leq c \|f\|_{H^1}.$$  

but as it was present above that boundedness of partial sums does not hold from $H_1$ to $L_1$, However, we have inequality (3).

On the other hand, in one-dimensional, Fujii [6] and Simon [17] proved that maximal operator Fejér means is bounded from $H^1$ to $L^1$. It follows that

$$\sup_{n \in \mathbb{N}_+} \left\| \frac{1}{n} \sum_{m=1}^{n} S_m f \right\|_1 < c \|f\|_{H^1}.$$  

So, natural question has arisen that if inequality (4) holds true, which would be generalization of inequality (5) or we have negative answer on this problem.

In this paper we prove that there exists a function $f \in H^1$ such that

$$\sup_{n \in \mathbb{N}_+} \frac{1}{n} \sum_{m=1}^{n} \|S_m f\|_1 = \infty.$$  

This paper is organized as follows: in order not to disturb our discussions later on some definitions and notations are presented in Section 2. For the proofs of the main results we need some auxiliary Lemmas. These results are presented in Section 3. The formulation and detailed proof of main results can be found in Section 4.

2. Definitions and Notations

Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$.

Let $m := (m_0, m_1, \ldots)$ denote a sequence of the positive integers not less than 2.

Denote by

$$Z_{m_k} := \{0, 1, \ldots, m_k - 1\}$$

the additive group of integers modulo $m_k$.

Define the group $G_m$ as the complete direct product of the group $Z_{m_j}$ with the product of the discrete topologies of $Z_{m_j}$’s.

The direct product $\mu$ of the measures

$$\mu_k(\{j\}) := 1/m_k \quad (j \in Z_{m_k})$$

is the Haar measure on $G_m$ with $\mu(G_m) = 1$.

If $\sup_{n \in \mathbb{N}} m_n < \infty$, then we call $G_m$ a bounded Vilenkin group. If the generating sequence $m$ is not bounded then $G_m$ is said to be an unbounded
Vilenkin group. In this paper we discuss bounded Vilenkin groups only.

The elements of $G_m$ are represented by sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots) \quad (x_k \in \mathbb{Z}_{m_k}).$$

It is easy to give a base for the neighbourhood of $G_m$

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G_m, n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ for $n \in \mathbb{N}$ and $\mathbb{T}_n := G_m \setminus I_n$.

Let

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G_m \quad (n \in \mathbb{N}).$$

If we define the so-called generalized number system based on $m$ in the following way:

$$M_0 := 1, \quad M_{k+1} := m_k M_k \quad (k \in \mathbb{N})$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j M_j$ where $n_j \in \mathbb{Z}_{m_j}$ $(j \in \mathbb{N})$ and only a finite number of $n_j$’s differ from zero. Let $|n| := \max \{j \in \mathbb{N} \mid n_j \neq 0\}$.

For the natural number $n = \sum_{j=1}^{\infty} n_j M_j$, we define

$$\delta_j = \sign n_j = \sign (\ominus n_j), \quad \delta_j^* = |\ominus n_j - 1| \delta_j,$$

where $\ominus$ is the inverse operation for $a_k \oplus b_k = (a_k + b_k) \mod m_k$.

We define functions $v$ and $v^*$ by

$$v(n) = \sum_{j=0}^{\infty} |\delta_{j+1} - \delta_j| + \delta_0, \quad v^*(n) = \sum_{j=0}^{\infty} \delta_j^*,$$

Next, we introduce on $G_m$ an orthonormal system which is called the Vilenkin system.

At first define the complex valued function $r_k(x) : G_m \to \mathbb{C}$, the generalized Rademacher functions as

$$r_k(x) := \exp (2 \pi i x_k / m_k) \quad (i^2 = -1, \quad x \in G_m, \quad k \in \mathbb{N}).$$

Now define the Vilenkin system $\psi := (\psi_n : n \in \mathbb{N})$ on $G_m$ as:

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^n(x) \quad (n \in \mathbb{N}).$$

Specially, we call this system the Walsh-Paley one if $m \equiv 2$.

The norm (or quasi norm) of the space $L_p(G_m)$ is defined by

$$\|f\|_p := \left( \int_{G_m} |f(x)|^p \, d\mu(x) \right)^{1/p} \quad (0 < p < \infty).$$

The Vilenkin system is orthonormal and complete in $L_2(G_m)$ (for details see e.g. [1, 26]).
If \( f \in L^1(G_m) \) we can establish Fourier coefficients, partial sums of the Fourier series, Fejér means, Dirichlet kernels with respect to the Vilenkin system in the usual manner:

\[
\hat{f}(k) := \int_{G_m} f \overline{\psi_k} d\mu \quad (k \in \mathbb{N})
\]

\[
S_n f := \sum_{k=0}^{n-1} \hat{f}(k) \psi_k \quad (n \in \mathbb{N}_+, \ S_0 f := 0)
\]

\[
\sigma_n f := \frac{1}{n} \sum_{k=0}^{n-1} S_k f \quad (n \in \mathbb{N}_+)
\]

\[
D_n := \sum_{k=0}^{n-1} \psi_k \quad (n \in \mathbb{N}_+).
\]

Recall that

\[
D_{M_n}(x) = \begin{cases} 
  M_n & x \in I_n \\
  0 & x \notin I_n
\end{cases}
\]

and

\[
D_{s_nM_n} = D_{M_n} \sum_{k=0}^{s_n-1} \psi_{kM_n} = D_{M_n} \sum_{k=0}^{s_n-1} r_n^k \quad 1 \leq s_n \leq m_n - 1.
\]

The \( n \)-th Lebesgue constant is defined in the following way

\[
L_n = \| D_n \|_1.
\]

If \( f \in L_1(G_m) \), the maximal functions are also be given by

\[
f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|
\]

Hardy martingale space \( H_1(G_m) \) consist of all martingales for which (for details see e.g. [27, 28])

\[
\| f \|_{H_1} := \| f^* \|_1 < \infty.
\]

3. Auxiliary results

**Lemma 1.** [11] Let \( n \in \mathbb{N} \). Then

\[
\frac{1}{4\lambda} \nu(n) + \frac{1}{\lambda} \nu^*(n) + \frac{1}{2\lambda} \leq L_n \leq \frac{3}{2} \nu(n) + 4\nu^*(n) - 1,
\]

where \( \lambda = \sup_{n \in \mathbb{N}} m_n \).

**Lemma 2.** [12] Let \( n \in \mathbb{N} \). Then there exists an absolute constant \( c \), such that

\[
\frac{1}{nM_n} \sum_{k=1}^{M_n-1} v(k) \geq c > 0.
\]
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4. Main Result

Theorem 1. There exists a martingale \( f \in H_1 \), such that

\[
\sup_{n \in \mathbb{N}} \frac{1}{n} \sum_{k=1}^{n} \|S_k f\|_1 = \infty.
\]

5. Proof of the Theorem

Proof of Theorem 1. Let \( \{\alpha_k : k \in \mathbb{N}\} \) be an increasing sequence of the positive integers such that

\[
\sum_{k=0}^{\infty} \frac{1}{\alpha_k^{1/2}} < c < \infty.
\]

Let

\[
f = \sum_{k=1}^{\infty} \frac{a_k}{\alpha_k^{1/2}},
\]

where

\[
a_k = D_{M_{\alpha_k+1}} - D_{M_{\alpha_k}}.
\]

It is evident that

\[
S_{M_n} f = \sum_{\{k : \alpha_k < n\}} \frac{a_k}{\alpha_k^{1/2}},
\]

and

\[
|S_{M_n} f| \leq \sum_{\{k : \alpha_k < n\}} \frac{|a_k|}{\alpha_k^{1/2}} \leq \sum_{k=1}^{\infty} \frac{|a_k|}{\alpha_k^{1/2}}.
\]

It follows that

\[
\sup_{n \in \mathbb{N}} |S_{M_n} f| \leq \sum_{k=1}^{\infty} \frac{|a_k|}{\alpha_k^{1/2}}.
\]

Since (see equality (6))

\[
\|a_k\| \leq 2, \; \text{for all} \; k \in \mathbb{N},
\]

by combining (2) and (8) we get that

\[
\|f\|_{H_1} \leq c \sup_{k \in \mathbb{N}} |S_{M_k} f| \leq \sum_{k=1}^{\infty} \frac{|a_k|}{\alpha_k^{1/2}} \leq 2c \sum_{k=1}^{\infty} \frac{1}{\alpha_k^{1/2}} \leq c < \infty.
\]

Moreover,

\[
\hat{f}(j) = \begin{cases} \frac{1}{\alpha_k^{1/2}}, & j \in \{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1\}, \; k \in \mathbb{N} \\ 0, & j \notin \bigcup_{k=1}^{\infty} \{M_{\alpha_k}, \ldots, M_{\alpha_k+1} - 1\} \end{cases}
\]
Let \( M_{\alpha k} \leq j < M_{\alpha k+1} \). Since
\[
D_{j+M_{\alpha k}} = D_{M_{\alpha k}} + \psi_{M_{\alpha k}} D_j,
\]
when \( j < M_{\alpha k} \), if we apply (9) we obtain that
\[
S_j f = S_{M_{\alpha k}} f + \sum_{v=M_{\alpha k}}^{j-1} \hat{f}(v) \psi_v
\]
\[
= S_{M_{\alpha k}} f + \sum_{v=M_{\alpha k}}^{j-1} \hat{f}(v) \psi_v
\]
\[
= S_{M_{\alpha k}} f + \frac{M_{\alpha k}^{1/p-1}}{\alpha_k^{1/2}} \sum_{v=M_{\alpha k}}^{j-1} \psi_v
\]
\[
= S_{M_{\alpha k}} f + \frac{M_{\alpha k}^{1/p-1}}{\alpha_k^{1/2}} \left( D_j - D_{M_{\alpha k}} \right)
\]
\[
= S_{M_{\alpha k}} f + \frac{M_{\alpha k}^{1/p-1}}{\alpha_k^{1/2}} \psi_{M_{\alpha k}} D_j - M_{\alpha k}
\]
\[
= III_1 + III_2.
\]
In view of (11) we can write that
\[
\|III_1\|_1 \leq \left\| S_{M_{\alpha k}} f \right\|_1 \leq c \|f\|_{H_1}.
\]
By combining Lemma 1 and (15) we get that
\[
\|S_n f\|_1 \geq \|III_2\|_1 - \|III_1\|_1 \geq \frac{cv(n - M_{\alpha k})}{\alpha_k^{1/2}} - c \|f\|_{H_1}.
\]
Hence, according to Lemma 2 we can conclude that
\[
\sup_{n \in \mathbb{N}_+} \frac{1}{n} \sum_{k=1}^{n} \|S_k f\|_1
\]
\[
\geq \frac{1}{M_{\alpha k+1}} \sum_{M_{\alpha k} \leq l \leq 2M_{\alpha k}} \|S_l f\|_1
\]
\[
\geq \frac{1}{M_{\alpha k+1}} \sum_{M_{\alpha k} \leq l \leq 2M_{\alpha k}} \left( \frac{v(l - M_{\alpha k})}{\alpha_k^{1/2}} - c \|f\|_{H_1} \right)
\]
\[
\geq \frac{c}{\alpha_k^{1/2} M_{\alpha k}} \sum_{l=1}^{M_{\alpha k}-1} v(l) - c \|f\|_{H_1/2}^{1/2}
\]
\[
\geq c \alpha_k^{1/2} \to \infty, \text{ as } k \to \infty.
\]
The proof is complete. \(\square\)
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