Integral Transforms with $H$-Function Kernels on $\mathcal{L}_{\nu,r}$-Spaces

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Abstract

Integral transforms

$$(Hf)(x) = \int_0^\infty H_{p,q}^{m,n} \left[ \begin{array}{c} xt \\ \left( a_i, \alpha_i \right)_{1,p} \\ \left( b_j, \beta_j \right)_{1,q} \end{array} \right] f(t) \, dt$$

involving Fox’s $H$-functions as kernels are studied in the spaces $\mathcal{L}_{\nu,r}$ of functions $f$ such that

$$\int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} < \infty \quad (1 \leq r < \infty, \, \nu \in \mathbb{R}).$$

Mapping properties such as the boundedness, the representation and the range of the transforms $H$ are given.

1. Introduction

This paper deals with the integral transforms of the form

$$(Hf)(x) = \int_0^\infty H_{p,q}^{m,n} \left[ \begin{array}{c} xt \\ \left( a_i, \alpha_i \right)_{1,p} \\ \left( b_j, \beta_j \right)_{1,q} \end{array} \right] f(t) \, dt,$$

(1.1)
where \( H_{p,q}^{m,n} \left[ \begin{array}{l} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \) is the Fox \( H \)-function. This function of general hypergeometric type was introduced by Fox [8]. For integers \( m, n, p, q \) such that \( 0 \leq m \leq q, 0 \leq n \leq p, \alpha_i, \beta_j \in \mathbb{C} \) with \( \mathbb{C} \) of the field of complex numbers, and \( \alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty) \) \( (1 \leq i \leq p, 1 \leq j \leq q) \), it is defined by

\[
\begin{split}
H_{p,q}^{m,n} \left[ \begin{array}{l} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] &= H_{p,q}^{m,n} \left[ \begin{array}{l} (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} \right] \\
&= \frac{1}{2\pi i} \int_{\mathcal{L}} \mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{l} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] z^{-s} ds,
\end{split}
\]

where

\[
\mathcal{H}_{p,q}^{m,n} \left[ \begin{array}{l} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] s = \prod_{j=1}^{m} \Gamma(b_j + \beta_j s) \prod_{i=1}^{n} \Gamma(1 - a_i - \alpha_i s)
\prod_{i=n+1}^{p} \Gamma(a_i + \alpha_i s) \prod_{j=m+1}^{q} \Gamma(1 - b_j - \beta_j s).
\]

the contour \( \mathcal{L} \) is specially chosen and an empty product, if it occurs, is taken to be one. The theory of this function may be found in [2], [29, Chapter 1], [36, §8.3] and [52, Chapter 2]. We abbreviate the Fox \( H \)-function (1.2) and the function in (1.3) to \( H_{p,q}^{m,n}(x), \mathcal{H}_{p,q}^{m,n}(s) \) or \( H(x), \mathcal{H}(s) \) when no confusion occurs.

Most of the known integral transforms can be put into the form (1.1). When \( \alpha_1 = \cdots = \alpha_p = \beta_1 = \cdots = \beta_q = 1 \), then (1.2) is the Meijer \( G \)-function [7, Chapter 5.3] and (1.1) is reduced to the so-called integral transforms with \( G \)-function kernels or \( G \)-transforms. The classical Laplace and Hankel transforms, the Riemann-Liouville fractional integrals, the even and odd Hilbert transforms, the integral transforms with the Gauss hypergeometric function, etc. belong to these \( G \)-transforms, for whose theory and historical notices see [44, §§36, 39]. There are other transforms which can not be reduced to \( G \)-transforms but can be put into the transform \( H \) given in (1.1). Such kinds of transforms are the modified Laplace and Hankel transforms [50], [52], [40], [44, §§18, 23, 39], the Erdélyi-Kober type fractional integration operators [26], [6], [50], [44, §18], the transforms with the Gauss hypergeometric function as kernels [35], [30], [41], [42], [44, §§23, 39], the Bessel-type integral transforms [27], [38], [21], [22], etc.

The integral transforms (1.1) with \( H \)-function kernels or \( H \)-transforms were first considered by Fox [8] while investigating \( G \)- and \( H \)-functions as symmetrical Fourier kernels. This paper together with the ones [17], [45], [9], [10], [49], [14], [4], [28] and [34] were devoted to find the inversion formulae for the \( H \)-transforms (1.1) is the spaces \( L_1(0, \infty) \) and \( L_2(0, \infty) \).

Some properties of \( H \)-transforms such as their Mellin transform, the relation of fractional integration by parts, compositional formulae, etc. were considered in [11], [12], [13], [51], [46] and [15]. In [47], [5] and [1] the operators \( H \) in (1.1) were represented as the compositions of the Erdélyi-Kober type operators and the integral operators of the form (1.1) with the \( H \)-function of the less order. Factorization properties of (1.1) in special functional spaces

\[
\begin{split}
&\text{where } H_{p,q}^{m,n} \left[ \begin{array}{l} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \right] \\
&\text{is the Fox } H \text{-function. This function of general hypergeometric type was introduced by Fox [8]. For integers } m, n, p, q \text{ such that } 0 \leq m \leq q, 0 \leq n \leq p, \\
&\alpha_i, \beta_j \in \mathbb{C} \text{ with } \mathbb{C} \text{ of the field of complex numbers, and } \alpha_i, \beta_j \in \mathbb{R}_+ = (0, \infty) \text{ } (1 \leq i \leq p, 1 \leq j \leq q), \text{ it is defined by}
\end{split}
\]
L_2^0 were investigated in [53]. The properties of generalized fractional integration operators, being some modifications of the operators (1.1), with H_{m,0}^m-function as the kernel were investigated in the space L_p(0,\infty) in [24], [16], [25] and on McBride spaces F_{p,\mu} and F_{p,\mu}^\prime (see [31] and [44, §8]) in [37], [43]. We also note that some facts about multidimensional transforms H of the form (1.1) were given in [3, §4.4].

The papers [32], [33] were devoted to the range and the invertibility of the operators (1.1) in the special cases when H(s) = Γ(b + s/m), Γ(1 - a - s/m) and Γ(b + s/m)/Γ(1 - a - s/m) with m > 0, in the spaces L_ν,r and its subspaces F_{p,\mu}. The former space with ν ∈ \mathbb{R} = (-\infty, \infty) and 1 ≤ r < \infty is defined as the space of those Lebesgue measurable complex functions f for which

\[ \| f \|_{\nu,r} = \int_0^\infty |t^\nu f(t)|^r \frac{dt}{t} < \infty \]  

(1.4)

(see [40]). Our previous papers [18], [23] were dealt with the study of the mapping properties such as the boundedness, the representation and the range of the general transforms H defined by (1.1) in the space L_\nu,2. The results in [18], [23] were extended to the space L_\nu,r with any 1 ≤ r < \infty in [19], [20], [48]. It is proved that the obtained results are different in nine cases:

1) a* = \Delta = \text{Re}(\mu) = 0; 2) a* = \Delta = 0, \text{Re}(\mu) < 0; 3) a* = 0, \Delta > 0;
4) a* = 0, \Delta < 0; 5) a_1* > 0, a_2* > 0; 6) a_1* > 0, a_2* = 0; 7) a_1* = 0, a_2* > 0; 8) a* > 0, a_1* > 0, a_2* < 0; 9) a* > 0, a_1* < 0, a_2* > 0.

Here

\[ a* = \sum_{i=1}^{n} \alpha_i - \sum_{i=n+1}^{p} \alpha_i + \sum_{j=1}^{m} \beta_j - \sum_{j=m+1}^{q} \beta_j, \]  

(1.5)

\[ \Delta = \sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i, \]  

(1.6)

\[ \mu = \sum_{j=1}^{q} b_j - \sum_{i=1}^{p} a_i + \frac{p - q}{2}, \]  

(1.7)

\[ a_1* = \sum_{j=1}^{m} \beta_j - \sum_{i=n+1}^{p} \alpha_i, \quad a_2* = \sum_{i=1}^{n} \alpha_i - \sum_{j=m+1}^{q} \beta_j. \]  

(1.8)

We note that

\[ a* = a_1* + a_2*, \quad \Delta = a_1* - a_2*. \]  

(1.9)

The results in [19], [20] and [48] being the extensions of those by Rooney [40] from G-transforms to H-transforms were proved under the additional condition δ = 1, where

\[ \delta = \prod_{i=1}^{p} \alpha_i^{\alpha_i} \prod_{j=1}^{q} \beta_j^{\beta_j}. \]  

(1.10)
The present paper is devoted to extend the results in [19], [20], [48] from $\delta = 1$ to any $\delta > 0$. Section 2 contains preliminary information from [40] concerning the mapping properties on $L_{\nu,r}$ of the Mellin transform, Erdélyi-Kober type fractional integral operators and generalized Hankel and Laplace transforms. In Section 3 we summarize the results from [18] on the asymptotic properties of the function $H(s)$ defined in (1.3) and its derivative and on $L_{\nu,2}$-theory of the transform $H$. Sections 4 and 5 deal with the one-to-one boundedness, representation and range of $H$-transform (1.1) on the space $L_{\nu,r}$ ($1 \leq r < \infty$) in the cases when $a^* = \Delta = 0$ and $a^* = 0$, $\Delta \neq 0$, respectively. Sections 6 and 7 are devoted to consider the cases $a^*_1 \geq 0$, $a^*_2 \geq 0$ and $a^* > 0$, $a^*_1 < 0$ or $a^*_2 < 0$, respectively.

2. Some Auxiliary Results

In this section we collect a variety of facts concerning multipliers for the Mellin transform and well known integral operators (see [39], [40]) which we need in next sections. For $f \in L_{\nu,r}$ with $1 \leq r \leq 2$, the Mellin transform of $f$ is defined [40] by

$$\mathcal{M}f(\nu + it) = \int_{-\infty}^{\infty} e^{(\nu+it)\tau} f(e^\tau) d\tau$$

for $\nu, t \in \mathbb{R}$. We also write $\mathcal{M}f(s)$ for $\text{Re}(s) = \nu$ as $\mathcal{M}f(\nu + it)$. In particular, if $f \in L_{\nu,r} \cap L_{\nu,1}$, then $\mathcal{M}f(s)$ is given by the usual expression

$$\mathcal{M}f(s) = \int_{0}^{\infty} f(t) t^{s-1} dt.$$

First we give the definition of the set $A$ and formulate the multiplier theorem for the Mellin transform $\mathcal{M}$.

**Definition 1.** [39, Definition 3.1] We say that a function $m$ belongs to $A$ if there are extended real numbers $\alpha(m)$ and $\beta(m)$ with $\alpha(m) < \beta(m)$ such that:

a) $m(s)$ is analytic in the strip $\alpha(m) < \text{Re}(s) < \beta(m)$;

b) $m(s)$ is bounded in every closed substrip $\sigma_1 \leq \text{Re}(s) \leq \sigma_2$, where $\alpha(m) < \sigma_1 \leq \sigma_2 < \beta(m)$;

c) $|m'(\sigma + it)| = O(|t|^{-1})$ as $|t| \to \infty$ for $\alpha(m) < \sigma < \beta(m)$.

For two Banach spaces $X$ and $Y$ we use the notation $[X,Y]$ denoting the collection of bounded linear operators from $X$ to $Y$, and $[X,X]$ is abbreviated to $[X]$.

**Theorem 1.** [39, Theorem 1] Let $m \in A$, $\alpha(m) < \nu < \beta(m)$ and $1 < r < \infty$. Then there is a transform $T_m \in [L_{\nu,r}]$ such that for every $f \in L_{\nu,r}$ with $1 < r \leq 2$, the relation

$$\mathcal{M}T_m f(s) = m(s) \mathcal{M}f(s) \quad (\text{Re}(s) = \nu)$$
holds valid. For \( \alpha(m) < \nu < \beta(m) \) and \( 1 < r \leq 2 \) the transform \( T_m \) is one-to-one on \( \mathfrak{L}_{\nu,r} \) except when \( m = 0 \). If \( 1/m \in \mathcal{A} \), then for \( \max[\alpha(m), \alpha(1/m)] < \nu < \min[\beta(m), \beta(1/m)] \) and for \( 1 < r < \infty \), \( T_m \) maps \( \mathfrak{L}_{\nu,r} \) one-to-one onto itself, and there holds

\[
T_m^{-1} = T_{1/m}.
\]

In the discussion of this article we use the special integral operators as follows: The Erdélyi-Kober type fractional integrals (see [44, §18.1]) for \( \alpha, \eta \in \mathbb{C} \) with \( \text{Re}(\alpha) > 0 \) and \( \sigma, x \in \mathbb{R}_+ \):

\[
(I_{0+;\sigma,\eta}^{\alpha}) (f)(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x (x^\sigma - t^\sigma t^\eta + 1 t^\eta - 1 f(t) dt;
\]

(2.4)

\[
(I_{-;\sigma,\eta}^{\alpha}) (f)(x) = \frac{\sigma x^\eta}{\Gamma(\alpha)} \int_x^\infty (t^\sigma - x^\sigma) t^\sigma(1-\alpha-\eta) - 1 t^\eta - 1 f(t) dt;
\]

(2.5)

the modified Hankel transform for \( \kappa \in \mathbb{R}\{0\}; \text{Re}(\eta) > -1 \) and \( x \in \mathbb{R}_+ \):

\[
(H_{\kappa,\eta} f)(x) = \int_0^\infty (xt)^{1/\kappa - 1/2} J_\eta \left( |\kappa| (xt)^{1/\kappa} \right) f(t) dt;
\]

(2.6)

and the modified Laplace transform for \( \kappa \in \mathbb{R}\{0\}, \alpha \in \mathbb{C} \) and \( x \in \mathbb{R}_+ \):

\[
(L_{\kappa,\alpha} f)(x) = \int_0^\infty (xt)^{-\alpha} e^{-|\kappa| (xt)^{1/\kappa}} f(t) dt.
\]

(2.7)

All these transforms are defined for continuous functions \( f \) with compact support on \( \mathbb{R}_+ \) for the range of parameters indicated.

For \( \nu \in \mathbb{R}_+ \) we denote by \( \mathfrak{L}_{\nu,\infty} \) the collection of functions \( f \), measurable on \( \mathbb{R}_+ \), such that

\[
\|f\|_{\nu,\infty} = \text{ess sup}_{x>0} |x^\nu f(x)| < \infty.
\]

The boundedness properties of the transforms (2.4) - (2.7) and their Mellin transforms are given by the following statement.

**Theorem 2.** [40, Theorem 5.1] (a) If \( 1 \leq r \leq \infty, \text{Re}(\alpha) > 0 \) and \( \nu < \sigma(1 + \text{Re}(\eta)) \), then for all \( s \geq r \) such that \( 1/s > 1/r - \text{Re}(\alpha) \), the operator \( I_{0+;\sigma,\eta}^{\alpha} \) belongs to \( \mathfrak{L}_{\nu,r} \), \( \mathfrak{L}_{\nu,s} \) and is a one-to-one transform from \( \mathfrak{L}_{\nu,r} \) onto \( \mathfrak{L}_{\nu,s} \). For \( 1 \leq r \leq 2 \) and \( f \in \mathfrak{L}_{\nu,r} \)

\[
(\mathfrak{M} I_{0+;\sigma,\eta}^{\alpha} f)(s) = \frac{\Gamma(1 + \eta - s/\sigma)}{\Gamma(1 + \eta + \alpha - s/\sigma)} (\mathfrak{M} f)(s) \quad (\text{Re}(s) = \nu).
\]

(2.8)

(b) If \( 1 \leq r \leq \infty, \text{Re}(\alpha) > 0 \) and \( \nu > -\sigma \text{Re}(\eta) \), then for all \( s \geq r \) such that \( 1/s > 1/r - \text{Re}(\alpha) \), the operator \( I_{-;\sigma,\eta}^{\alpha} \) belongs to \( \mathfrak{L}_{\nu,r} \), \( \mathfrak{L}_{\nu,s} \) and is a one-to-one transform from \( \mathfrak{L}_{\nu,r} \) onto \( \mathfrak{L}_{\nu,s} \). For \( 1 \leq r \leq 2 \) and \( f \in \mathfrak{L}_{\nu,r} \)

\[
(\mathfrak{M} I_{-;\sigma,\eta}^{\alpha} f)(s) = \frac{\Gamma(\eta + s/\sigma)}{\Gamma(\eta + \alpha + s/\sigma)} (\mathfrak{M} f)(s) \quad (\text{Re}(s) = \nu).
\]

(2.9)
If \( 1 < r < \infty \) and \( \gamma(r) \leq \kappa(\nu - 1/2) + 1/2 < \text{Re}(\eta) + 3/2 \), where

\[
\gamma(r) = \max \left[ \frac{1}{r}, \frac{1}{r'} \right] \quad \text{for} \quad \frac{1}{r} + \frac{1}{r'} = 1,
\]

then for all \( s \geq r \) such that \( s' \geq (\kappa(\nu - 1/2) + 1/2)^{-1} \) and \( 1/s + 1/s' = 1 \), the operator \( H_{\kappa,\eta} \) belongs to \([L_{\nu,r}, L_{1-\nu,s}]\) and is a one-to-one transform from \( L_{\nu,r} \) onto \( L_{1-\nu,s} \). If \( 1 < r \leq 2 \) and \( f \in L_{\nu,r} \), then

\[
(\mathcal{M}H_{\kappa,\eta}f)(s) = \left( \frac{2}{|\kappa|} \right)^{\kappa(s-1/2)} \frac{\Gamma([\eta + \kappa(s-1/2) + 1]/2)}{\Gamma([\eta - \kappa(s-1/2) + 1]/2)} (\mathcal{M}f)(1-s)
\]

\((\text{Re}(s) = 1 - \nu)\).

If \( 1 \leq r \leq s \leq \infty \), and if \( \nu < 1 - \text{Re}(\alpha) \) for \( \kappa > 0 \) and \( \nu > 1 - \text{Re}(\alpha) \) for \( \kappa < 0 \), then the operator \( L_{\kappa,\alpha} \) belongs to \([L_{\nu,r}, L_{1-\nu,s}]\) and is a one-to-one transform from \( L_{\nu,r} \) onto \( L_{1-\nu,s} \). If \( 1 \leq r \leq 2 \) and \( f \in L_{\nu,r} \), then

\[
(\mathcal{M}L_{\kappa,\alpha}f)(s) = \Gamma(\kappa[s - \alpha])|\kappa|^{1-\kappa(s-\alpha)} (\mathcal{M}f)(1-s) \quad (\text{Re}(s) = 1 - \nu).
\]

For further investigation we also need certain elementary operators. For a function \( f \) being defined almost everywhere on \( \mathbb{R}_+ \) we denote the operators \( M_\zeta, W_\delta \) and \( R \) as follows:

\[
(M_\zeta f)(s) = x^{\zeta} f(x) \quad \text{for} \quad \zeta \in \mathbb{C};
\]

\[
(W_\delta f)(s) = f \left( \frac{x}{\delta} \right) \quad \text{for} \quad \delta \in \mathbb{R}_+;
\]

\[
(Rf)(x) = \frac{1}{x} f \left( \frac{1}{x} \right).
\]

These operators have the following properties for \( \nu \in \mathbb{R} \) and \( 1 \leq r < \infty \) (see [40]):

(P1) \( M_\zeta \) is an isometric isomorphism of \( L_{\nu,r} \) onto \( L_{\nu-\text{Re}(\zeta),r} \), and if \( f \in L_{\nu,r} \) (\( 1 \leq r \leq 2 \)), then

\[
(\mathcal{M}M_\zeta f)(s) = (\mathcal{M}f)(s + \zeta) \quad (\text{Re}(s) = \nu - \text{Re}(\zeta));
\]

(P2) \( W_\delta \) is an isometric isomorphism of \( L_{\nu,r} \), and if \( f \in L_{\nu,r} \) (\( 1 \leq r \leq 2 \)), then

\[
(\mathcal{M}W_\delta f)(s) = \delta^s (\mathcal{M}f)(s) \quad (\text{Re}(s) = \nu);
\]

(P3) \( R \) is an isometric isomorphism of \( L_{\nu,r} \) onto \( L_{1-\nu,r} \), and if \( f \in L_{\nu,r} \) (\( 1 \leq r \leq 2 \)), then

\[
(\mathcal{M}Rf)(s) = (\mathcal{M}f)(1-s) \quad (\text{Re}(s) = 1 - \nu).
\]
3. Asymptotic Properties of $\mathcal{H}_{p,q}^{m,n}(s)$ and $\mathcal{L}_{\nu,2}$-Theory of $H$-Transforms

Let $a^*, \Delta, \mu, a_1^*, a_2^*$ and $\delta$ be given by (1.5) - (1.8) and (1.10), respectively. For integers $m, n, p, q$ such that $0 \leq m \leq q, 0 \leq n \leq p$ and for $a_1, \ldots, a_p, b_1, \ldots, b_q \in \mathbb{C}$ and for $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q \in \mathbb{R}^+$ we define $\alpha$ and $\beta$ by

$$\alpha = \begin{cases} \max \left[ \frac{-\Re(b_1)}{\beta_1}, \ldots, \frac{-\Re(b_m)}{\beta_m} \right] & \text{if } m > 0, \\ -\infty & \text{if } m = 0; \end{cases}$$

$$\beta = \begin{cases} \min \left[ \frac{1 - \Re(a_1)}{\alpha_1}, \ldots, \frac{1 - \Re(a_n)}{\alpha_1} \right] & \text{if } n > 0, \\ \infty & \text{if } n = 0. \end{cases}$$

**Lemma 1.** [18, Lemma 1] If $\sigma, t \in \mathbb{R}$, then the estimate

$$|\mathcal{H}_{p,q}^{m,n}(\sigma + it)| \sim (2\pi)^{c^*} \delta^{\sigma} \prod_{i=1}^{p} \alpha_i^{1/2-\Re(a_i)} \prod_{j=1}^{q} \beta_j^{\Re(b_j)-1/2}$$

$$\times |t|^{\Delta + \Re(\mu)} \exp \left\{ -\frac{\pi |t| a^*}{2} - \frac{\pi \Im(\xi) \text{sign}(t)}{2} \right\}$$

with

$$c^* = m + n - \frac{p + q}{2}, \quad \xi = \sum_{i=1}^{n} a_i - \sum_{i=n+1}^{p} a_i + \sum_{j=1}^{m} b_j - \sum_{i=m+1}^{q} b_j$$

holds as $|t| \to \infty$ uniformly in $\sigma$ for $\sigma$ in any bounded interval in $\mathbb{R}$. Further,

$$\left\{ \mathcal{H}_{p,q}^{m,n}(\sigma + it) \right\}' = \mathcal{H}_{p,q}^{m,n}(\sigma + it) \left[ \log \delta + a_1^* \log(it) - a_2^* \log(-it) + \frac{\mu + \Delta \sigma}{it} + O \left( \frac{1}{t^2} \right) \right] \quad (|t| \to \infty).$$

To present $\mathcal{L}_{\nu,2}$-theory for $H$-transform we have to define a certain set of real numbers.

**Definition 2.** For the function $\mathcal{H}(s)$ given in (1.3) we call the exceptional set $\mathcal{E}_{\mathcal{H}}$ of $\mathcal{H}$ the set of real numbers $\nu$ such that $\alpha < 1 - \nu < \beta$ and $\mathcal{H}(s)$ has a zero on the line $\Re(s) = 1 - \nu$.

**Theorem 3.** [18, Theorem 3] Suppose that

(a) $\alpha < 1 - \nu < \beta$
and that either of the conditions
(b) \( a^* > 0; \)
(c) \( a^* = 0, \Delta(1 - \nu) + \text{Re}(\mu) \leq 0. \)

holds, then we have:
(i) There is a one-to-one transform \( H \in [\mathfrak{L}_v, \mathfrak{L}_{1-\nu}] \) so that the relation

\[
(\mathfrak{M}H f)(s) = \mathcal{J}^{m,n}_{p,q} \begin{bmatrix} (a_p, \alpha_p) \\ (b_q, \beta_q) \end{bmatrix} (\mathfrak{M}f)(1 - s)
\]

holds for \( f \in \mathfrak{L}_r \) and \( \text{Re}(s) = 1 - \nu. \) If \( a^* = 0, \Delta(1 - \nu) + \text{Re}(\mu) = 0 \) and \( \nu \notin \mathfrak{E}_\mathfrak{M}, \) then the operator \( H \) maps \( \mathfrak{L}_v \) onto \( \mathfrak{L}_{1-\nu}. \)
(ii) For \( f, g \in \mathfrak{L}_v \) the relation

\[
\int_0^\infty f(x)(Hg)(x)dx = \int_0^\infty g(x)(Hf)(x)dx
\]

holds.
(iii) Let \( f \in \mathfrak{L}_v, \lambda \in \mathbb{C} \) and \( h > 0. \) If \( \text{Re}(\lambda) > (1 - \nu)h - 1, \) then \( Hf \) is given by

\[
(Hf)(x) = hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h}
\]

\[
\times \int_0^\infty \mathcal{J}^{m,n+1}_{p+1,q+1} \begin{bmatrix} xt \\ (-\lambda, h), (a_1, \alpha_1), \ldots, (a_p, \alpha_p) \\ (b_1, \beta_1) \ldots, (b_q, \beta_q), (-\lambda - 1, h) \end{bmatrix} f(t)dt.
\]

If \( \text{Re}(\lambda) < (1 - \nu)h - 1, \) then

\[
(Hf)(x) = -hx^{1-(\lambda+1)/h} \frac{d}{dx} x^{(\lambda+1)/h}
\]

\[
\times \int_0^\infty \mathcal{J}^{m+1,n}_{p+1,q+1} \begin{bmatrix} xt \\ (a_1, \alpha_1), \ldots, (a_p, \alpha_p), (-\lambda, h) \\ (-\lambda - 1, h), (b_1, \beta_1) \ldots, (b_q, \beta_q) \end{bmatrix} f(t)dt.
\]

(iv) \( H \) is independent on \( \nu \) in the sense that if \( \nu_1 \) and \( \nu_2 \) satisfy (a), and (b) or (c), and if the transforms \( H_1 \) and \( H_2 \) are given by (3.6), then \( H_1 f = H_2 f \) for \( f \in \mathfrak{L}_{\nu_1} \cap \mathfrak{L}_{\nu_2}. \)

**Corollary.** Let \( \alpha < \beta \) and one of the following conditions holds:
(b) \( a^* > 0; \)
(e) \( a^* = 0, \Delta > 0 \) and \( \alpha < -\frac{\text{Re}(\mu)}{\Delta}; \)
(f) \( a^* = 0, \Delta < 0 \) and \( \beta > -\frac{\text{Re}(\mu)}{\Delta}; \)
(g) \( a^* = 0, \Delta = 0 \text{ and } \text{Re}(\mu) \leq 0 \).

Then the transform \( H \) can be defined on \( \mathfrak{L}_{\nu,2} \) with \( 1 - \beta < \nu < 1 - \alpha \).

**Theorem 4.** [18, Theorem 4] Let \( \alpha < 1 - \nu < \beta \) and either of the following conditions holds:

(b) \( a^* > 0 \);

(d) \( a^* = 0, \Delta(1 - \nu) + \text{Re}(\mu) < -1 \).

Then for \( x \in \mathbb{R}_+ \), \( (Hf)(x) \) is given by (1.1) for \( f \in \mathfrak{L}_{\nu,2} \).

**Corollary.** Let \( \alpha < \beta \) and one of the following conditions holds:

(b) \( a^* > 0 \);

(h) \( a^* = 0, \Delta > 0 \) and \( \alpha < -1 + \text{Re}(\mu) + \Delta \);

(i) \( a^* = 0, \Delta < 0 \) and \( \beta > -1 + \text{Re}(\mu) + \Delta \);

(j) \( a^* = 0, \Delta = 0 \) and \( \text{Re}(\mu) < -1 \).

Then the transform \( H \) can be defined by (1.1) on \( \mathfrak{L}_{\nu,2} \) with \( 1 - \beta < \nu < 1 - \alpha \).

4. \( \mathfrak{L}_{\nu,r} \)-Theory of the Transform \( H \) \( (a^* = \Delta = 0) \)

In this section, basing on the existence of the transform \( H \) on the space \( \mathfrak{L}_{\nu,2} \) which is guaranteed in Theorem 3 for some \( \nu \in \mathbb{R} \) and \( a^* = \Delta = 0 \), we prove that such a transform can be extended to \( \mathfrak{L}_{\nu,r} \) for \( 1 < r < \infty \) such that \( H \in [\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,r}] \) for a certain range of the value \( s \). We also characterize the range of \( H \) on \( \mathfrak{L}_{\nu,r} \) in terms of the Erdélyi-Kober type fractional integral operators \( I_{0+;\sigma,\eta}^{\alpha} \) and \( I_{-\sigma;\sigma,\eta}^{\alpha} \) given in (2.4) and (2.5) except for its isolated values \( \nu \in \mathfrak{E}_H \). The results will be different in the cases \( \text{Re}(\mu) = 0 \) and \( \text{Re}(\mu) \neq 0 \), where \( \mu \) is defined by (1.7). First we consider the former case.

**Theorem 5.** Let \( a^* = \Delta = 0, \text{Re}(\mu) = 0 \) and \( \alpha < 1 - \nu < \beta \).

(a) The transform \( H \) is defined on \( \mathfrak{L}_{\nu,2} \) and it can be extended to \( \mathfrak{L}_{\nu,r} \) as an element of \( [\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,r}] \) for \( 1 < r < \infty \).

(b) If \( 1 < r \leq 2 \), the transform \( H \) is one-to-one on \( \mathfrak{L}_{\nu,r} \) and there holds the equality

\[
(4.1) \quad (\mathfrak{M}Hf)(s) = \mathcal{H}(s)(\mathfrak{M}f)(1-s) \quad (\text{Re}(s) = 1 - \nu).
\]

(c) If \( \nu \notin \mathfrak{E}_H \), then \( H \) is one-to-one on \( \mathfrak{L}_{\nu,r} \) and there holds

\[
(4.2) \quad H(\mathfrak{L}_{\nu,r}) = \mathfrak{L}_{1-\nu,r}.
\]
(d) If \( f \in \mathcal{L}_{\nu,r} \) and \( g \in \mathcal{L}_{\nu,r'} \) with \( 1 < r < \infty \) and \( r' = r/(r-1) \), then the relation (3.7) holds.

(e) If \( f \in \mathcal{L}_{\nu,r} \) with \( 1 < r < \infty \) and \( \lambda \in \mathbb{C}, h > 0 \), then \( Hf \) is given by (3.8) for \( \text{Re}(\lambda) > (1-\nu)h-1 \), while \( Hf \) is given by (3.9) for \( \text{Re}(\lambda) < (1-\nu)h-1 \).

Proof. Since \( \alpha < 1-\nu < \beta \) and \( \Delta(1-\nu) + \text{Re}(\mu) \leq 0 \), then according to Theorem 3 the transform \( H \) is defined on \( f \in \mathcal{L}_{\nu,2} \). We denote by \( \mathcal{H}_0(s) \) the function

\[
\mathcal{H}_0(s) = \delta^{-s} \mathcal{H}(s),
\]

where \( \delta \) is defined in (1.10). It follows from (3.3) that

\[
|\mathcal{H}_0(\sigma + it)| \sim \prod_{i=1}^{p} \alpha_i^{\nu - \text{Re}(a_i)} \prod_{j=1}^{q} \beta_j^{\text{Re}(b_j)} (2\pi)^{\nu} e^{-\pi \text{Im}(\xi) \text{sign}(t)/2} (|t| \to \infty)
\]

is uniformly in \( \sigma \) for \( \sigma \) in any bounded interval in \( \mathbb{R} \). Therefore \( \mathcal{H}_0(s) \) is analytic in the strip \( \alpha < \text{Re}(s) < \beta \), and if \( \alpha < \sigma_1 \leq \sigma_2 < \beta \), then \( |\mathcal{H}_0(s)| \) is bounded in the strip \( \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \). Since \( a^* = \Delta = 0 \), then in accordance with (1.9) \( a_1^* = -a_2^* = \Delta/2 = 0 \). Then from (4.3) and (3.5) we have

\[
\mathcal{H}_0'(\sigma + it) = \mathcal{H}_0(\sigma + it) \left[ -\log(\delta) + \mathcal{H}'(\sigma + it) \right]
\]

\[
= \mathcal{H}_0(\sigma + it) \left[ -\log(\delta) + \log(\delta) + \frac{\text{Im}(\mu)}{it} + O \left( \frac{1}{t^2} \right) \right]
\]

\[
= O \left( \frac{1}{t} \right) \quad (|t| \to \infty)
\]

for \( \alpha < \sigma < \beta \). Thus \( \mathcal{H}_0(s) \) belongs to the class \( \mathcal{A} \) (see Definition 1) with \( \alpha(\mathcal{H}_0) = \alpha \) and \( \beta(\mathcal{H}_0) = \beta \). Therefore by virtue of Theorem 1, there is a transform \( T \in [\mathcal{L}_{\nu,2}] \) with \( 1 < r < \infty \) and \( \alpha < \nu < \beta \). When \( 1 < r \leq 2 \), then \( T \) is one-to-one on \( \mathcal{L}_{\nu,2} \) and the relation

\[
(\mathcal{M}Tf)(s) = \mathcal{H}_0(s) (\mathcal{M}f)(s) \quad (\text{Re}(s) = \nu)
\]

holds for \( f \in \mathcal{L}_{\nu,r} \). Let

\[
H_0 = W_0 TR,
\]

where \( W_0 \) and \( R \) are given by (2.14) and (2.15). According to the properties (P2) and (P3) of the operators \( W_0 \) and \( R \), we find \( R \in [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}] \), \( W_0 \in [\mathcal{L}_{1-\nu,r}, \mathcal{L}_{1-\nu,r}] \) and hence \( H_0 \in [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}] \) for \( \alpha < 1-\nu < \beta \) and \( 1 < r < \infty \), too. When \( \alpha < 1-\nu < \beta \), \( 1 < r \leq 2 \) and \( f \in \mathcal{L}_{\nu,r} \), it follows from (4.7), (2.17), (4.6), (2.18) and (4.3) that

\[
(\mathcal{M}H_0f)(s) = (\mathcal{M}W_0TRf)(s) = \delta^s (\mathcal{M}TRf)(s)
\]

\[
= \delta^s \mathcal{H}_0(s) (\mathcal{M}f)(s) = \delta^s \mathcal{H}_0(s) (\mathcal{M}f)(1-s) = \mathcal{H}(s)(\mathcal{M}f)(1-s)
\]
for Re(s) = 1 − ν. In particular, for f ∈ ℒν,2 Theorem 3 (i), (3.6) and (4.8) imply the equality

\[(4.9) \quad (\mathfrak{M}H_0 f)(s) = (\mathfrak{M}H f)(s) \quad (\text{Re}(s) = 1 − \nu).\]

Thus \(H_0 f = H f\) for \(f \in \mathcal{L}_{\nu,2}\) and therefore if \(\alpha < 1 - \nu < \beta\), \(H = H_0\) on \(\mathcal{L}_{\nu,2}\) by Theorem 3 (iv). Since \(\mathcal{L}_{\nu,2} \cap \mathcal{C}_{\nu, \alpha} = \emptyset\) [39, Theorem 2.2], \(H\) can be extended to \(\mathcal{C}_{\nu, \alpha}\) if we define it there by \(H_0\), and then \(H \in \{\mathcal{L}_{\nu,2}, \mathcal{L}_{1-\nu,2}\}\). This completes the proof of assertion (a) of the theorem.

The assertions (b) - (e) are proved similarly to those in [19, Theorem 4.1] where the case \(\delta = 1\) was considered, if we take into account that \(W_\delta\) is a one-to-one transform on \(\mathcal{L}_{1-\nu,2}\)

\[W_\delta(\mathcal{L}_{1-\nu,2}) = \mathcal{L}_{1-\nu,2} \quad \text{for} \quad \alpha < 1 - \nu < \beta \quad \text{and} \quad 1 < r < \infty.\]

The theorem is proved.

**Theorem 6.** Let \(a^* = \Delta = 0\), \(\text{Re}(\mu) < 0\) and \(\alpha < 1 - \nu < \beta\), and let either \(m > 0\) or \(n > 0\).

(a) The transform \(H\) defined on \(\mathcal{L}_{\nu,2}\) can be extended to \(\mathcal{L}_{\nu, r}\) for \(1 < r < \infty\) as an element of \([\mathcal{L}_{\nu, r}, \mathcal{L}_{1-\nu, s}]\) for all \(s \geq r\) such that \(1/s > 1/r + \text{Re}(\mu)\).

(b) If \(1 < r \leq 2\), then \(H\) is a one-to-one transform on \(\mathcal{L}_{\nu, r}\) and there holds the equality

\[(4.1) \quad H(\mathcal{L}_{\nu, r}) = I^{-\mu}_{-\nu, -\nu/k}(\mathcal{L}_{1-\nu, r})\]

for \(k \geq 1\) and \(m > 0\), and

\[(4.11) \quad H(\mathcal{L}_{\nu, r}) = I^{-\mu}_{0+\nu, -\nu/k-1}(\mathcal{L}_{1-\nu, r})\]

for \(0 < k \leq 1\) and \(n > 0\). If \(\nu \in \mathcal{E}_{\beta/k}, H(\mathcal{L}_{\nu, r})\) is a subset of \(I^{-\mu}_{-\nu, -\nu/k}(\mathcal{L}_{1-\nu, r})\) or \(I^{-\mu}_{0+\nu, -\nu/(k-1)}(\mathcal{L}_{1-\nu, r})\), when \(m > 0\) or \(n > 0\), respectively.

(d) If \(f \in \mathcal{L}_{\nu, r}\) and \(g \in \mathcal{L}_{\nu, s}\) with \(1 < r < \infty\), \(1 < s < \infty\) and \(1 \leq 1/r + 1/s < 1 - \text{Re}(\mu)\), then the relation (3.7) holds.

(e) If \(f \in \mathcal{L}_{\nu, r}\) with \(1 < r < \infty\) and \(\lambda \in \mathbb{C}, h > 0\), then \(H f\) is given by (1.8) for \(\text{Re}(\lambda) > (1 - \nu)h - 1\), while \(H f\) is given by (3.9) for \(\text{Re}(\lambda) < (1 - \nu)h - 1\). If furthermore \(\text{Re}(\mu) < -1\), then \(H f\) is given by (1.1).

**Proof.** Since \(\alpha < 1 - \nu < \beta\), \(a^* = 0\) and \(\Delta(1 - \nu) + \text{Re}(\mu) = \text{Re}(\mu) < 0\), then from Theorem 3 the transform \(H\) is defined on \(\mathcal{L}_{\nu, 2}\).

If \(m > 0\) or \(n > 0\), then \(\alpha\) or \(\beta\) are finite in view of (3.1) and (3.2). We set

\[(4.12) \quad \mathcal{K}_1(s) = \frac{\Gamma([s - \alpha]/k - \mu)}{\Gamma([s - \alpha]/k)} \mathcal{K}(s)\]

\[= \mathcal{K}_{p+1,q}^{m+1,q} \left[ \begin{array}{c} (a_1, \alpha_1), \ldots, (a_p, \alpha_p), (-\mu/k, 1/k) \\ (-\mu - \alpha/k, 1/k), (b_1, \beta_1), \ldots, (b_q, \beta_q) \end{array} \right] \quad \text{for} \quad m > 0 \quad \text{and} \quad k \geq 1,\]

and

\[(4.13) \quad \mathcal{K}_2(s) = \frac{\Gamma([\beta - s]/k - \mu)}{\Gamma([\beta - s]/k)} \mathcal{K}(s)\]
\[ H^{m,n+1}_{p+1,q+1} \left[ \begin{array}{c} (1 + \mu - \beta/k, 1/k), (a_1, \alpha_1), \cdots, (a_p, \alpha_p) \\ (b_1, \beta_1), \cdots, (b_q, \beta_q), (1 - \beta/k, 1/k) \end{array} \right] \]

for \( n > 0 \) and \( 0 < k \leq 1 \). We denote by \( \alpha_1, \beta_1, \alpha^*_1, \Delta_1, \delta_1 \) and \( \mu_1 \) for \( \mathcal{H}_1 \), and by \( \alpha_2, \beta_2, \alpha^*_2, \Delta_2, \delta_2 \) and \( \mu_2 \) for \( \mathcal{H}_2 \) instead of that for \( \mathcal{H} \). Then we find that

\[
\alpha_1 = \max[\alpha, \alpha + \text{Re}(\mu)] = \alpha, \quad \beta_1 = \beta, \quad \alpha^*_1 = a^* = 0, \quad \Delta_1 = \Delta = 0, \quad \delta_1 = \delta, \quad \mu_1 = 0, \\
\alpha_2 = \alpha, \quad \beta_2 = \min[\beta, \beta - k\text{Re}(\mu)] = \beta, \quad \alpha^*_2 = a^* = 0, \quad \Delta_2 = \Delta = 0, \quad \delta_2 = \delta, \quad \mu_2 = 0,
\]

and the exceptional sets \( \mathcal{E}_{3\mathcal{H}_1} \) and \( \mathcal{E}_{3\mathcal{H}_2} \) of \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) coincide with that \( \mathcal{E}_{3\mathcal{H}} \) of \( \mathcal{H} \). Then according to Theorem 5, if \( \alpha < 1 - \nu < \beta \) and \( 1 < r < \infty \), there are transforms \( \widetilde{H}_1 \in [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}] \) for \( m > 0, k \geq 1 \) and \( \widetilde{H}_2 \in [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}] \) for \( n > 0, 0 < k \leq 1 \); and if \( f \in \mathcal{L}_{\nu,r} \) with \( 1 < r \leq 2 \), then by (4.1)

\[
(4.14) \quad (\mathfrak{M}\widetilde{H}_if)(s) = \mathcal{H}_i(s)(\mathfrak{M}f)(1-s) \quad (\text{Re}(s) = 1 - \nu; \ i = 1, 2).
\]

We set

\[
(4.15) \quad H_1 = I^{-\mu}_{-k, -\alpha/k} \widetilde{H}_1 \quad \text{for} \quad m > 0, k \geq 1,
\]

and

\[
(4.16) \quad H_2 = I^{-\mu}_{0 + k, \beta/k} \widetilde{H}_2 \quad \text{for} \quad n > 0, 0 < k \leq 1.
\]

Using Theorem 2 (b) in the first case and Theorem 2 (a) in the second, and taking the same arguments as in the proof of Theorem 5, we arrive at the assertion (a) of the theorem. The assertions (b) - (e) are proved similarly to that in [19, Theorem 4.2] (while considering the cases \( \delta = 1 \)) on the basis of Theorem 5 and Theorem 4 in the case \( \text{Re}(\mu) < -1 \).

5. \( \mathcal{L}_{\nu,r} \)-Theory of the Transform \( H \) \((a^* = 0, \Delta \neq 0)\)

In this section we discuss that, if \( a^* = 0 \) and \( \Delta \neq 0 \), then the transform \( H \) defined on \( \mathcal{L}_{\nu,2} \) can be extended to \( \mathcal{L}_{\nu,r} \) such as \( H \in [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}] \) for some range of values \( s \). Then we characterize the range \( H \) on \( \mathcal{L}_{\nu,r} \), except for its isolated values \( \nu \in \mathcal{E}_{3\mathcal{H}} \) in terms of the Hankel modified transform \( H_{k,q} \) and the elementary transform \( M_\zeta \) given in (2.6) and (2.13). The result will be different in the cases \( \Delta > 0 \) and \( \Delta < 0 \). First we consider the case \( \Delta > 0 \).

Theorem 7. Let \( a^* = 0, \Delta > 0, -\infty < \alpha < 1 - \nu < \beta, 1 < r < \infty \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \), where \( \gamma(r) \) is defined in (2.10).

(a) The transform \( H \) defined on \( \mathcal{L}_{\nu,2} \) can be extended to \( \mathcal{L}_{\nu,r} \) to be an element of \([\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}]\) for all \( s \) with \( r \leq s < \infty \) such that \( s' \geq [1/2 - \Delta(1 - \nu) - \text{Re}(\mu)]^{-1} \) with \( 1/s + 1/s' = 1 \).

(b) If \( 1 < r \leq 2 \), the transform \( H \) is one-to-one on \( \mathcal{L}_{\nu,r} \) and there holds the equality

\[
(5.1) \quad (\mathfrak{M}Hf)(s) = \mathcal{H}(s)(\mathfrak{M}f)(1-s) \quad (\text{Re}(s) = 1 - \nu).
\]
(c) If \( \nu \notin \mathcal{E}_3 \), then \( H \) is one-to-one transform on \( \mathcal{L}_{\nu,r} \). If we set \( \eta = -\Delta \alpha - \mu - 1 \), then Re(\( \eta \)) < -1 and there holds

\[
H(\mathcal{L}_{\nu,r}) = \left( M_{\mu/\Delta+1/2}H_{\Delta,\eta}M_{\mu/\Delta+1/2} \right) \left( \mathcal{L}_{\nu,r} \right) = \left( M_{\mu/\Delta+1/2}H_{\Delta,\eta} \right) \left( \mathcal{L}_{\nu,-\text{Re}(\mu)/\Delta-1/2,r} \right).
\]

When \( \nu \in \mathcal{E}_3 \), then \( H(\mathcal{L}_{\nu,r}) \) is a subset of the right hand side of (5.2).

(d) If \( f \in \mathcal{L}_{\nu,r} \) and \( g \in \mathcal{L}_{\nu,s} \) with \( 1 < r < \infty, 1 < s < \infty, 1/r + 1/s \geq 1 \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \max[\gamma(r), \gamma(s)] \), then the relation (3.7) holds.

(e) If \( f \in \mathcal{L}_{\nu,r} \) with \( 1 < r < \infty, \lambda \in \mathbb{C}, h > 0 \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \), then \( Hf \) is given by (3.8) for \( \text{Re}(\lambda) > (1 - \nu)h - 1 \), while \( Hf \) is given by (3.9) for \( \text{Re}(\lambda) < (1 - \nu)h - 1 \).

If \( \Delta(1 - \nu) + \text{Re}(\mu) < -1 \), \( Hf \) is given by (1.1).

Proof. Since \( \gamma(r) \geq 1/2 \), we have \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 0 \) by the assumption, and hence from Theorem 3 the transform \( H \) is defined on \( \mathcal{L}_{\nu,2} \). The condition \( \Delta > 0 \) and the relation \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 0 \) imply \( \nu \geq 1 + \text{Re}(\mu)/\Delta \) and \( \alpha < -\text{Re}(\mu)/\Delta \).

Since \( a^* = 0 \), then by (1.9)

\[
a^*_1 = -a^*_2 = \frac{\Delta}{2} > 0.
\]

We denote by \( \mathcal{H}_3(s) \) the function

\[
\mathcal{H}_3(s) = \delta^{s-1}(a^*_1)^{(1-s)\Delta+\mu} \frac{\Gamma(-\mu + a^*_1[s - 1 - \alpha])}{\Gamma(a^*_1[1 - \alpha - s])} \mathcal{H}(1 - s).
\]

As already known, the function \( \mathcal{H}(1 - s) \) is analytic in the strip \( 1 - \beta < \text{Re}(s) < 1 - \alpha \) and the function \( \Gamma(-\mu + a^*_1[s - 1 - \alpha]) \) is analytic in the half-plane \( \text{Re}(s) > \alpha + 1 + \text{Re}(\mu)/a^*_1 = \alpha + 1 + 2\text{Re}(\mu)/\Delta \). Since \( \alpha < -\text{Re}(\mu)/\Delta \), then \( 1 - \alpha > \alpha + 1 + 2\text{Re}(\mu)/\Delta \). Therefore if we take \( \alpha_1 = \max[1 - \beta, \alpha + 1 + 2\text{Re}(\mu)/\Delta] \) and \( \beta_1 = 1 - \alpha \), then \( \alpha_1 < \beta_1 \) and \( \mathcal{H}_3(s) \) is analytic in the strip \( \alpha_1 < \text{Re}(s) < \beta_1 \).

Setting \( s = \sigma + it \) and a complex constant \( k = c + id \), we have the behavior

\[
\Gamma(s + k) = \Gamma(c + \sigma + i[d + t]) \sim \sqrt{2\pi} |t|^{c+\sigma-1/2} e^{-\pi|t|/2 - \pi d \text{sign}(t)/2}
\]

as \( |t| \to \infty \) (see [18, (2.12)]). Then by taking \( a^* = 0, \Delta = 2a^*_1 \) and (3.3) into account we have from (5.4) that

\[
|\mathcal{H}_3(\sigma + it)| \sim (2\pi)^{c'} \prod_{i=1}^{p} \alpha^*_1^{1/2 - \text{Re}(a_i)} \prod_{j=1}^{q} \beta_j^{\text{Re}(b_j) - 1/2} (a^*_1)^{(1-\sigma)\Delta + \text{Re}(\mu)}
\]

\[
\times |a^*_1|^2 |2\pi - 2a^*_1 - \text{Re}(\mu)|^{1/2} \text{sign}(t)/2 = \kappa \neq 0
\]

as \( |t| \to \infty \), uniformly in \( \sigma \) for \( \sigma \) in any bounded interval. Therefore, if \( \alpha_1 < \sigma_1 \leq \sigma_2 < \beta_1 \), then \( \mathcal{H}_3(s) \) is bounded in \( \sigma_1 \leq \text{Re}(s) \leq \sigma_2 \).

If \( \alpha_1 < \sigma < \beta_1 \), then

\[
\mathcal{H}_3'(\sigma + it) = \mathcal{H}_3(\sigma + it) \left\{ \log(\delta) - \Delta \log(a^*_1) + a^*_1 \psi(a^*_1[s - \alpha - 1] - \mu)
\]

\[
+ a^*_1 \psi(a^*_1[1 - \alpha - s]) - \frac{\mathcal{H}'(1 - s)}{\mathcal{H}(1 - s)} \right\},
\]

13
where $\psi$ is the psi-function $\psi(z) = \Gamma'(z)/\Gamma(z)$. Applying the estimate

$$
(5.8) \quad \psi(c + \sigma + it) = \log(it) + \frac{c + \sigma - 1/2}{it} + O\left(\frac{1}{t^2}\right) \quad (|t| \to \infty)
$$

with $c \in \mathbb{C}$ (see [40, (3.9a)]) and using (3.5) with $\alpha^* = 0$, (5.3), (5.6) and (5.8), we have from (5.7), as $|t| \to \infty$,

$$
(5.9) \mathcal{H}_3^\alpha(\sigma + it) = \mathcal{H}_3(\sigma + it)
$$

$$
= \mathcal{H}_3(\sigma + it) \left\{ \log(\delta) - 2a_1^* \log(a_1^*) + a_1^* \left[ \log(i a_1^* t) + \frac{a_1^*(\sigma - \alpha - 1) - \mu - 1/2}{i a_1^* t} \right] 
\right.
$$

$$
+ a_1^* \left[ \log(-i a_1^* t) - \frac{a_1^*(1 - \alpha - \sigma) - 1/2}{i a_1^* t} \right] \right.

- \left[ \log(\delta) + a_1^* \log(-it) + a_1^* \log(it) - \frac{\mu + \Delta(1 - \sigma)}{it} \right] + O\left(\frac{1}{t^2}\right) \right\} = O\left(\frac{1}{t^2}\right).
$$

So $\mathcal{H}_3 \in \mathfrak{A}$ with $\alpha(\mathcal{H}_3) = \alpha_1$ and $\beta(\mathcal{H}_3) = \beta_1$. Hence due to Theorem 1 there is a transform $T_3 \in [\mathfrak{L}_{\nu,r}]$ corresponding to $\mathcal{H}_3$ with $1 < r < \infty$ and $\alpha_1 < \nu < \beta_1$, and if $1 < r \leq 2$, then $T_3$ is a one-to-one on $\mathfrak{L}_{\nu,r}$ and

$$
(5.10) \quad (\mathfrak{M} T_3 f)(s) = \mathcal{H}_3(s)(\mathfrak{M} f)(s) \quad (\Re(s) = \nu).
$$

In particular, if $\nu$ and $r$ satisfy the hypothesis of this theorem, it is directly verified that $\alpha_1 < \nu < \beta_1$ and hence the relation (5.10) is true.

For $\eta = -\Delta \alpha - \mu - 1$ let $H_3$ be the operator

$$
(5.11) \quad H_3 = W_\delta M_{\mu/\Delta + 1/2} H_{\Delta,\eta} M_{\mu/\Delta + 1/2} T_3
$$

composed by the operator $W_\delta$ in (2.14), the operator $M_\xi$ in (2.13), the modified Hankel transform (2.6) and the transform $T_3$ above, where $\alpha < -\Re(\mu)/\Delta$ so that $\Re(\eta) > -1$. For $1 < r < \infty$ and $\alpha_1 < \nu < \beta_1$ the properties (P1), (P2) and Theorem 2(c) yield $H_3 \in [\mathfrak{L}_{r,v}, \mathfrak{L}_{1-v,r}]$ for all $s \geq r$ such that $s' \geq [1/2 - \Delta(1 - \nu) - \Re(\mu)]^{-1}$, and in particular, $H_3 \in [\mathfrak{L}_{r,v}, \mathfrak{L}_{1-v,r}]$.

If $f \in \mathfrak{L}_{\nu,r}$ with $1 < r \leq 2$, then applying (2.17), (2.16), (2.11), (5.10) and (5.4) and using the relation $\eta = -\Delta \alpha - \mu - 1$, we have for $\Re(s) = 1 - \nu$

$$
(5.12) \quad (\mathfrak{M} H_3 f)(s)
$$

$$
= \left( \mathfrak{M} W_\delta M_{\mu/\Delta + 1/2} H_{\Delta,\eta} M_{\mu/\Delta + 1/2} T_3 f \right)(s)
$$

$$
= \delta^s \left( \mathfrak{M} M_{\mu/\Delta + 1/2} H_{\Delta,\eta} M_{\mu/\Delta + 1/2} T_3 f \right)(s)
$$

$$
= \delta^s \left( \mathfrak{M} H_{\Delta,\eta} M_{\mu/\Delta + 1/2} T_3 f \right) \left( s + \frac{\mu}{\Delta} + \frac{1}{2} \right)
$$

$$
= \delta^s \left( \frac{2}{\Delta} \right)^{\Delta + \mu} \Gamma(\eta + \Delta s + \mu + 1/2) \Gamma(\eta - \Delta s - \mu + 1/2) \left( \mathfrak{M} M_{\mu/\Delta + 1/2} T_3 f \right) \left( 1 - s - \frac{\mu}{\Delta} - \frac{1}{2} \right)
$$

14
\[\delta^s \left( \frac{2}{\Delta} \right)^{\Delta s + \mu} \frac{\Gamma(\Delta[s - \alpha]/2)}{\Gamma(-\mu - \Delta[s + \alpha]/2)} (\mathcal{M} T_3 f) (1 - s)\]
\[= \delta^s \left( a_1^* \right)^{-\mu - \Delta s} \frac{\Gamma(a_1^*[s - \alpha])}{\Gamma(-\mu - a_1^*[s + \alpha])} \mathcal{H}(1 - s) (\mathcal{M} f) (1 - s)\]
\[= \mathcal{H}(s) (\mathcal{M} f) (1 - s).\]

In particular, if we take \( r = 2 \) and \( f \in \mathcal{L}_{\nu,2} \), then \((\mathcal{M} H_3 f) (s) = (\mathcal{M} H f) (s)\) for \( \text{Re}(s) = 1 - \nu \), and hence \( H_3 = H \) on \( \mathcal{L}_{\nu,2} \). Thus, for all \( \nu \) and \( r \) satisfying the hypotheses of this theorem, \( H \) can be extended from \( \mathcal{L}_{\nu,2} \) to \( \mathcal{L}_{\nu,r} \) if we define it by \( H_3 \) given in (5.11) as an operator on \( [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}] \). This completes the proof of the statement (a) of the theorem.

The assertions (b) - (e) are proved similarly to those in [19, Theorem 5.1], when the case \( \delta = 1 \) is considered, and the theorem is proved.

**Theorem 8.** Let \( a^* = 0, \Delta < 0, \alpha < 1 - \nu < \beta < \infty, 1 < r < \infty \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \).

(a) The transform \( H \) defined on \( \mathcal{L}_{\nu,2} \) can be extended to \( \mathcal{L}_{\nu,r} \) to be an element of \( [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,r}] \) for all \( s \) with \( r \leq s < \infty \) such that \( s' \geq [1/2 - \Delta(1 - \nu) - \text{Re}(\mu)]^{-1} \) with \( 1/s + 1/s' = 1 \).

(b) If \( 1 < r \leq 2 \), then the transform \( H \) is one-to-one on \( \mathcal{L}_{\nu,r} \) and there holds the equality (5.1).

(c) If \( \nu \notin \mathcal{E}_{\beta} \), then \( H \) is a one-to-one transform on \( \mathcal{L}_{\nu,r} \). If we set \( \eta = -\Delta \beta - \mu - 1 \), then \( \text{Re}(\eta) > -1 \) and the relation (5.2) holds. When \( \nu \in \mathcal{E}_{\beta} \), then \( H(\mathcal{L}_{\nu,r}) \) is a subset of the right hand side of (5.2).

(d) If \( f \in \mathcal{L}_{\nu,r} \) and \( g \in \mathcal{L}_{\nu,s} \) with \( 1 < r < \infty, 1 < s < \infty, 1/r + 1/s \geq 1 \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \max[\gamma(r), \gamma(s)] \), then the relation (3.7) holds.

(e) If \( f \in \mathcal{L}_{\nu,r} \) with \( 1 < r < \infty, \lambda \in \mathbb{C}, h > 0 \) and \( \Delta(1 - \nu) + \text{Re}(\mu) \leq 1/2 - \gamma(r) \), then \( Hf \) is given by (3.8) for \( \text{Re}(\lambda) > (1 - \nu) h - 1 \), while \( H f \) is given by (3.9) for \( \text{Re}(\lambda) < (1 - \nu) h - 1 \). If \( \Delta(1 - \nu) + \text{Re}(\mu) < -1 \), then \( H f \) is given by (1.1).

The proof of this theorem is based on Theorem 7 and it is similarly to that in the case \( \delta = 1 \) in [19, Theorem 5.2].

**Corollary.** Let \( 1 < r < \infty, \alpha < \beta, a^* = 0 \) and one of the following conditions holds

(a) \( \Delta > 0, \alpha < \frac{1/2 - \text{Re}(\mu) - \gamma(r)}{\Delta} \);
(b) \( \Delta < 0, \beta > \frac{1/2 - \text{Re}(\mu) - \gamma(r)}{\Delta} \);
(c) \( \Delta = 0, \text{Re}(\mu) \leq 0 \).

Then the transform \( H \) can be defined on \( \mathcal{L}_{\nu,r} \) with \( \alpha < 1 - \nu < \beta \).

6. \( \mathcal{L}_{\nu,r} \)-Theory of the Transform \( H \) \((a^* > 0, a_1^* \geq 0 \) and \( a_2^* \geq 0 \))
When \( a^* > 0 \) and \( \alpha < 1 - \nu < \beta \), then according to Theorem 4 the \( H \)-transform is defined on \( \mathfrak{L}_{\nu,2} \) and given by (1.1). Let us show that it can be extended to \( \mathfrak{L}_{\nu,r} \) for any \( 1 \leq r \leq \infty \). The next statement is proved similarly to that in case \( \delta = 1 \) in [20, Theorem 2.3].

**Theorem 9.** Let \( a^* > 0, \alpha < 1 - \nu < \beta \) and \( 1 \leq r \leq s \leq \infty \).

- (a) The \( H \)-transform given in (1.1) is defined on \( \mathfrak{L}_{\nu,2} \) and can be extended to \( \mathfrak{L}_{\nu,r} \) as an element of \( [\mathfrak{L}_{\nu,r}, \mathfrak{L}_{1-\nu,s}] \). If \( 1 \leq r \leq 2 \), then \( H \) is a one-to-one transform from \( \mathfrak{L}_{\nu,r} \) onto \( \mathfrak{L}_{1-\nu,s} \).

- (b) If \( f \in \mathfrak{L}_{\nu,r} \) and \( g \in \mathfrak{L}_{\nu,s'} \) with \( 1/s + 1/s' = 1 \), then the relation (3.7) holds.

We give conditions for the transform \( H \) to be one-to-one on \( \mathfrak{L}_{\nu,r} \) and to characterize its range on \( \mathfrak{L}_{\nu,r} \) except for its isolated values \( \nu \in \mathcal{E}_{3l}, \) in terms of the modified Laplace transform \( L_{\kappa,\alpha} \) and the Erdélyi-Kober type fractional integration operators \( I_{0+;\sigma,\eta}^\alpha \) and \( I_{1-;\sigma,\eta}^\alpha \) given in (2.7), (2.4) and (2.5). In this section we consider the case when \( a^* > 0, a_1^* \geq 0 \) and \( a_2^* \geq 0 \). The results will be different along combinations of signs of \( a_1^* \) and \( a_2^* \). First we consider the case \( a_1^* > 0 \) and \( a_2^* > 0 \).

**Theorem 10.** Let \( a_1^* > 0, a_2^* > 0, -\infty < \alpha < 1 - \nu < \beta < \infty \) and \( \omega = \mu + a_1^*\alpha - a_2^*\beta + 1 \) and let \( 1 < r < \infty \).

- (a) If \( \nu \notin \mathcal{E}_{3l} \), or if \( 1 \leq r \leq 2 \), then \( H \) is a one-to-one transform on \( \mathfrak{L}_{\nu,r} \).

- (b) If \( \text{Re}(\omega) \geq 0 \) and \( \nu \notin \mathcal{E}_{3l} \), then

\[
(6.1) \quad H(\mathfrak{L}_{\nu,r}) = \left( L_{a_1^*,\alpha} L_{a_2^*,1-\beta-\omega/a_2^*} \right) (\mathfrak{L}_{1-\nu,r}).
\]

When \( \nu \in \mathcal{E}_{3l} \), \( H(\mathfrak{L}_{\nu,r}) \) is a subset of the right hand side of (6.1).

- (c) If \( \text{Re}(\omega) < 0 \) and \( \nu \notin \mathcal{E}_{3l} \), then

\[
(6.2) \quad H(\mathfrak{L}_{\nu,r}) = \left( I_{-;1/a_1^*,-a_1^*\alpha} L_{a_1^*,\alpha} L_{a_2^*,1-\beta} \right) (\mathfrak{L}_{1-\nu,r}).
\]

When \( \nu \in \mathcal{E}_{3l} \), \( H(\mathfrak{L}_{\nu,r}) \) is a subset of the right hand side of (6.2).

**Proof.** We first consider the case \( \text{Re}(\omega) \geq 0 \). We define \( \mathcal{H}_4(s) \) by

\[
(6.3) \quad \mathcal{H}_4(s) = \frac{(a_1^*)^{\alpha(s-\alpha)-1}(a_2^*)^{\beta(s-\beta)+\omega-1}}{\Gamma(a_1^*[s-\alpha])\Gamma(a_2^*[-\beta-\omega] + \omega)} \delta^{-s}\mathcal{H}(s).
\]

Since \( \text{Re}(\omega) \geq 0 \), the function \( \mathcal{H}_4(s) \) is analytic in the strip \( \alpha < \text{Re}(s) < \beta \). According to (3.3) and (5.5) we have the estimate, as \( |t| \to \infty \),

\[
(6.4) |\mathcal{H}_4(\sigma + it)| \sim (a_1^*)^{\alpha(s-\alpha)-1}(a_2^*)^{\beta(s-\beta)+\text{Re}(\omega)-1} \times \frac{1}{2\pi} (a_1^*|t|)^{\alpha(s-\alpha)+1/2} (a_2^*|t|)^{\beta(s-\beta)-\text{Re}(\omega)+1/2} e^{-[\alpha a_1^* + \beta a_2^*]|t| - \text{Im}(\omega) \text{sign}(t) \pi/2}
\]

\[
\times \prod_{i=1}^p a_i^* \frac{1}{2\pi} (a_i^*|t|)^{\alpha(s-\alpha)+1/2} (a_j^*|t|)^{\beta(s-\beta)-\text{Re}(\omega)+1/2} e^{-[\alpha a_i^* + \beta a_j^*]|t| - \text{Im}(\omega) \text{sign}(t) \pi/2}
\]

\[
\times \prod_{j=1}^q b_j^* \frac{1}{2\pi} (b_j^*|t|)^{\alpha(s-\alpha)+1/2} (b_j^*|t|)^{\beta(s-\beta)-\text{Re}(\omega)+1/2} e^{-[\alpha b_i^* + \beta b_j^*]|t| - \text{Im}(\omega) \text{sign}(t) \pi/2}
\]

\[
\sim \prod_{i=1}^p a_i^* \frac{1}{2\pi} (a_i^*|t|)^{\alpha(s-\alpha)+1/2} (a_j^*|t|)^{\beta(s-\beta)-\text{Re}(\omega)+1/2} e^{-[\alpha a_i^* + \beta a_j^*]|t| - \text{Im}(\omega) \text{sign}(t) \pi/2}
\]
uniformly in $\sigma$ on any bounded interval in $\mathbb{R}$. Further, in accordance with (3.5) and (5.8)

$$(6.5) \mathcal{H}_4' (\sigma + it) = \mathcal{H}_4 (\sigma + it) \left\{ a_1^* \log (a_1^*) - a_2^* \log (a_2^*) - a_1^* \psi (a_1^*[s - \alpha]) \
+a_2^* \psi (a_2^*[\beta - s] + \omega) - \log (\delta) + \frac{\mathcal{H}' (\sigma + it)}{\mathcal{H} (\sigma + it)} \right\}$$

$$= \mathcal{H}_4 (\sigma + it) \left\{ a_1^* \log (a_1^*) - a_2^* \log (a_2^*) - a_1^* \left[ \log (ia_1^* t) + \frac{a_1^* (\sigma - \alpha) - 1/2}{ia_1^* t} \right] \
+a_2^* \left[ \log (-ia_2^* t) - \frac{a_2^* (\beta - \sigma) + \omega - 1/2}{ia_2^* t} \right] - \log (\delta) \
+ \left[ \log (\delta) + a_1^* \log (it) - a_2^* \log (-it) + \frac{\mu + \Delta \sigma}{it} \right] + O \left( \frac{1}{t^2} \right) \right\}$$

$$= O \left( \frac{1}{t^2} \right)$$

as $|t| \to \infty$. So $\mathcal{H}_4 \in \mathcal{A}$ with $\alpha (\mathcal{H}_4) = \alpha$ and $\beta (\mathcal{H}_4) = \beta$ and Theorem 1 implies that there is a transform $T_4 \in \mathcal{L}_{\nu,r}$ for $1 < r < \infty$ and $\alpha < \nu < \beta$ so that if $1 < r \leq 2$, then the relation

$$(6.6) (\mathcal{MT}_4 f) (s) = \mathcal{H}_4 (s) (\mathcal{M} f) (s) \quad (\text{Re} (s) = \nu)$$

holds. Let

$$(6.7) \quad H_4 = W_4 L_{a_1^*, \alpha} L_{a_2^*, 1 - \beta - \omega / a_2^*} T_4 R,$$

where $W_4$ and $R$ are defined by (2.14) and (2.15). Then it follows from the properties (P2) and (P3) in Section 2 and Theorem 2 (d) that if $1 < r \leq s < \infty$ and $\alpha < 1 - \nu < \beta$, then $H_4 \in \mathcal{L}_{\nu,r}, \mathcal{L}_{1 - \nu,s}$. For $f \in \mathcal{L}_{\nu,2}$ applying (6.7), (2.17), (2.12), (6.6) and (2.18), we have

$$(6.8) \quad (\mathcal{MH}_4 f) (s) = \left( \mathcal{M} W_4 L_{a_1^*, \alpha} L_{a_2^*, 1 - \beta - \omega / a_2^*} T_4 R f \right) (s)$$

$$= \delta^s \frac{\Gamma (a_1^*[s - \alpha])}{(a_1^*)^s (s - \alpha) - 1} \left( \mathcal{M} L_{a_2^*, 1 - \beta - \omega / a_2^*} T_4 R f \right) (1 - s)$$

$$= \delta^s \frac{\Gamma (a_1^*[s - \alpha]) \Gamma \{ a_2^*[1 + s - (1 - \beta - \omega / a_2^*)] \}}{(a_2^*)^s (s - \alpha) - 1} \left( \mathcal{MT}_4 R f \right) (s)$$

$$= \delta^s \frac{\Gamma (a_1^*[s - \alpha]) \Gamma (a_2^*[\beta - s] + \omega)}{(a_2^*)^s (s - \alpha) - 1} \mathcal{H}_4 (s) \left( \mathcal{M} R f \right) (s)$$

$$= \mathcal{H} (s) \left( \mathcal{M} f \right) (1 - s).$$

Thus we obtain that if $f \in \mathcal{L}_{\nu,2}$, then $(\mathcal{MH}_4 f) (s) = (\mathcal{MH} f) (s)$ with $\text{Re} (s) = 1 - \nu$. Hence $H_4 = H$ on $\mathcal{L}_{\nu,2}$ and $H$ can be extended from $\mathcal{L}_{\nu,2}$ to $\mathcal{L}_{\nu,r}$ if we define it by (6.7). Further
the proof of assertion (b) is carried out similarly to that in [20, Theorem 3.1] for the case $\delta = 1$.

Let $\Re(\omega) < 0$. We denote by $H_5(s)$ the function

$$H_5(s) = \frac{(a_1^*)^{\alpha_1(s-\alpha)-1}(a_2^*)^{\alpha_2(\beta-s)-1}\Gamma(a_1^*[s-\alpha] - \omega)}{\Gamma^2(a_1^*[s-\alpha])\Gamma(a_2^*[(\beta-s)])} \delta^{-s}H(s),$$

which is analytic in the strip $\alpha < \Re(s) < \beta$. Similar arguments to (6.4) and (6.5) show the estimates

$$|H_5(\sigma + it)| \sim \prod_{i=1}^{p} \alpha_i^{-\frac{1}{2} - \Re(a_i)} \prod_{j=1}^{q} \beta_j^{\Re(b_j)-1/2}(2\pi)e^{-1}(a_1^*)^{-\Re(\omega)-1/2}(a_2^*)^{-1/2}e^{-[\Im(\xi) + \Im(\omega)]\pi/2}$$

and

$$H'_5(\sigma + it) = H_5(\sigma + it) \left\{ a_1^* \log(a_1^*) - a_2^* \log(a_2^*) + a_1^* \psi(a_1^*[s-\alpha] - \omega) - 2a_1^* \psi(a_1^*[s-\alpha]) + a_2^* \psi(a_2^*[(\beta-s)] - \log(\delta) + \frac{H'(\sigma + it)}{H(\sigma + it)} \right\}$$

$$= O\left(\frac{1}{t^2}\right),$$

as $|t| \to \infty$ hold uniformly in $\sigma$ on any bounded interval in $\mathbb{R}$. So $H_5 \in \mathcal{A}$ with $\alpha(H_5) = \alpha$ and $\beta(H_5) = \beta$. By Theorem 1 there is a transform $T_5 \in [\Sigma_{\nu,2}, \Sigma_{\nu,s}]$ for $1 < r < \infty$ and $\alpha < \nu < \beta$ so that, if $1 < r \leq 2$,

$$(\mathcal{M}T_5f)(s) = H_5(s)(\mathcal{M}f)(s) \quad (\Re(s) = \nu).$$

Let

$$H_5 = W_\delta I_{-1/a_1^*,a_1^*}^\omega L_{a_2^*,a_1^*\alpha}L_{a_2^*,-1-\beta}T_5R.$$

Using again the properties (P2), (P3) in Section 2 and Theorem 2 (b), (d), we have that if $\alpha < 1 - \nu < \beta$ and $1 < r \leq s < \infty$, then $H_5 \in [\Sigma_{\nu,r}, \Sigma_{1-\nu,s}]$. For $f \in \Sigma_{\nu,2}$ and $\Re(s) = 1 - \nu$ applying (6.13), (2.17), (2.9), (2.12), (6.12), (2.18) and (6.9) we obtain similarly to (6.8) that

$$(\mathcal{M}H_5f)(s)$$

$$= \delta^s \left( \mathcal{M}I_{-1/a_1^*,-a_1^*\alpha}^\omega L_{a_2^*,a_1^*\alpha}L_{a_2^*,-1-\beta}T_5Rf \right)(s)$$

$$= \delta^s \frac{\Gamma(a_1^*[s-\alpha])}{\Gamma(a_1^*[s-\alpha] - \omega)} \left( \mathcal{M}L_{a_2^*,a_1^*\alpha}L_{a_2^*,-1-\beta}T_5Rf \right)(s)$$

$$= \delta^s \frac{\Gamma(a_1^*[s-\alpha])}{\Gamma(a_1^*[s-\alpha] - \omega)} (a_1^*)^{\alpha_1(s-\alpha)-1} \left( \mathcal{M}L_{a_2^*,-1-\beta}T_5Rf \right)(1-s).$$
in (6.4) and (6.5) lead to the estimates

\[
\delta^s \frac{\Gamma(a_1^*[s - \alpha]) \Gamma(a_1^*[1 - s] - [1 - \beta])}{\Gamma(a_1^*[s - \alpha] - \omega) (a_1^*)^{a_1^*[s - \alpha] - 1} (a_2^*)^{a_2^*[1 - s] - [1 - \beta] - 1}} (\mathcal{M}T_5Rf)(s)
\]

\[
= \delta^s \frac{\Gamma^2(a_1^*[s - \alpha]) (a_2^*)^{a_1^*[s - \alpha] - 1}}{\Gamma(a_1^*[s - \alpha] - \omega) (a_1^*)^{a_1^*[s - \alpha] - 1} (a_2^*)^{a_2^*[s - \beta] - 1}} \mathcal{H}_5(s) (\mathcal{M}Rf)(s)
\]

\[
= \mathcal{H}(s)(\mathcal{M}f)(1 - s).
\]

Applying this equality and using similar arguments to those in the case \(\text{Re}(\omega) \geq 0\), we complete the proof for \(\text{Re}(\omega) < 0\).

Now we proceed to the case \(a_1^* > 0\) and \(a_2^* = 0\).

**Theorem 11.** Let \(a_1^* > 0\), \(a_2^* = 0\), \(-\infty < \alpha < 1 - \nu < \beta\), \(\omega = \mu + a_1^* \alpha + 1/2\) and \(1 < r < \infty\).

- **(a)** If \(\nu \notin \mathcal{E}_\mathcal{H}\), or \(1 < r \leq 2\), then \(H\) is a one-to-one transform on \(\mathcal{L}_\nu, r\).
- **(b)** If \(\text{Re}(\omega) \geq 0\) and \(\nu \notin \mathcal{E}_\mathcal{H}\), then

\[
H(\mathcal{L}_\nu, r) = L_{a_1^*, \alpha - \omega / a_1^*}(\mathcal{L}_\nu, r).
\]

When \(\nu \in \mathcal{E}_\mathcal{H}\), \(H(\mathcal{L}_\nu, r)\) is a subset of the right hand side of (6.15).

- **(c)** If \(\text{Re}(\omega) < 0\) and \(\nu \notin \mathcal{E}_\mathcal{H}\), then

\[
H(\mathcal{L}_\nu, r) = \left( I_{-1 / a_1^*, -a_1^* \alpha} L_{a_1^*, \alpha} \right)(\mathcal{L}_\nu, r).
\]

When \(\nu \in \mathcal{E}_\mathcal{H}\), \(H(\mathcal{L}_\nu, r)\) is a subset of the right hand side of (6.16).

**Proof.** We first consider the case \(\text{Re}(\omega) \geq 0\). We define \(\mathcal{H}_6(s)\) by

\[
\mathcal{H}_6(s) = \frac{(a_1^*)^{a_1^*[s - \alpha] + \omega - 1}}{\Gamma(a_1^*[s - \alpha] + \omega)} \delta^{-s} \mathcal{H}(s).
\]

Since \(\text{Re}(\omega) \geq 0\), \(\mathcal{H}_6(s)\) is analytic in the strip \(\alpha < \text{Re}(s) < \beta\). Arguments similar to those in (6.4) and (6.5) lead to the estimates

\[
|\mathcal{H}_6(\sigma + it)| \sim \prod_{i=1}^{p} a_i^{1/2 - \text{Re}(a_i)} \prod_{j=1}^{q} \beta_j^{\text{Re}(b_j) - 1/2} (2\pi e^{1/2} (a_1^*)^{1/2} (a_1^*)^{-1} e^{-|\text{Im}(\xi)|} + \text{sign}(t) \pi/2,
\]

and

\[
\mathcal{H}_6'(\sigma + it) = \mathcal{H}_6(\sigma + it) \left\{ a_1^* \log(a_1^*) - a_1^* \psi(a_1^*[s - \alpha] + \omega) - \log(\delta) + \frac{\mathcal{H}'(\sigma + it)}{\mathcal{H}(\sigma + it)} \right\}
\]

\[
= O\left( \frac{1}{t^2} \right) \quad (|t| \to \infty)
\]

19
uniformly in $\sigma$ in any bounded interval in $\mathbb{R}$. Thus $\mathcal{H}_6 \in \mathcal{A}$ with $\alpha(\mathcal{H}_6) = \alpha$ and $\beta(\mathcal{H}_6) = \beta$ and Theorem 1 implies that there is a transform $T_6 \in \mathcal{L}_{\nu,r}$ for $1 < r < \infty$ and $\alpha < \nu < \beta$ and, if $1 < r \leq 2$,

$$(6.20) \quad (\mathcal{MT}_6 f)(s) = \mathcal{H}_6(s)(\mathcal{M} f)(s) \quad (\text{Re}(s) = \nu).$$

We set

$$(6.21) \quad H_6 = W_6 L_{a_1^*, \alpha - \omega/a_1^*} R T_6 R.$$

Then it follows from the properties (P2) and (P3) in Section 2 and Theorem 2 (d) that, if $\alpha < \nu < \beta$ and $1 < r \leq s < \infty$, then $H_6 \in \mathcal{L}_{\nu,r} \mathcal{L}_{1-\nu,s}$.

For $f \in \mathcal{L}_{\nu,2}$ applying (6.21), (2.17), (2.12), (6.20), (2.18) and (6.17), we obtain for $\text{Re}(s) = 1 - \nu$

$$(6.22) \quad (\mathcal{MH}_6 f)(s) = (\mathcal{MW}_6 L_{a_1^*, \alpha - \omega/a_1^*} R T_6 R f)(s) = \delta^*(\mathcal{ML}_{a_1^*, \alpha - \omega/a_1^*} R T_6 R f)(s)$$

$$= \delta^* \frac{\Gamma(a_1^*[s - \alpha + \omega/a_1^*])}{(a_1^*)^s[a_1^*[s - \omega/a_1^*]]} (\mathcal{MT}_6 R f)(1 - s)$$

$$= \delta^* \frac{\Gamma(a_1^*[s - \alpha] + \omega)}{(a_1^*)^s[a_1^*[s - \omega]]} (\mathcal{MT}_6 R f)(s)$$

$$= \delta^* \frac{\Gamma(a_1^*[s - \alpha] + \omega)}{(a_1^*)^s[a_1^*[s - \omega]]} \mathcal{H}_6(s)(\mathcal{M} R f)(s)$$

$$= \delta^* \frac{\Gamma(a_1^*[s - \alpha] + \omega)}{(a_1^*)^s[a_1^*[s - \omega]]} \mathcal{H}_6(s)(\mathcal{M} f)(1 - s) = \mathcal{H}(s)(\mathcal{M} f)(1 - s).$$

Applying this relation and using the arguments similarly to those in the case $\text{Re}(\omega) \geq 0$ of Theorem 10, we complete the proof of theorem for $\text{Re}(\omega) \geq 0$.

For the case $\text{Re}(\omega) < 0$. We define $\mathcal{H}_7(s)$ by

$$(6.23) \quad \mathcal{H}_7(s) = \frac{(a_1^*)^s[a_1^*[s - \omega]] - \Gamma(a_1^*[s - \omega] - \omega)}{\Gamma^2(a_1^*[s - \alpha])} \delta^{-s} \mathcal{H}(s),$$

which is analytic in the strip $\alpha < \text{Re}(s) < \beta$, and in accordance with (3.3), (3.5), (5.5) and (5.8), we find

$$(6.24) \quad |\mathcal{H}_7(\sigma + it)| \sim \prod_{i=1}^{p} \alpha_i^{1/2 - \text{Re}(a_i)} \prod_{j=1}^{q} \beta_j^{\text{Re}(b_j) - 1/2} (2\pi e^{-1/2}$$

$$\times (a_1^*)^{-\text{Re}(\omega) + 1/2} e^{-|\text{Im}(\xi) - \text{Im}(\omega)|\text{sign}(t)\pi/2},$$

$$(6.25) \quad \mathcal{H}_7'(\sigma + it) = \mathcal{H}_7(\sigma + it) \left[ a_1^* \log(a_1^*) + a_1^* \psi(a_1^*[s - \alpha] - \omega)ight.$$  

$$- 2a_1^* \psi(a_1^*[s - \alpha]) - \log(\delta) + \frac{\mathcal{H}'(\sigma + it)}{\mathcal{H}(\sigma + it)} = O \left( \frac{1}{t^2} \right)$$

20
as \(|t| \to \infty\) uniformly in \(\sigma\) on any bounded interval in \(\mathbb{R}\). Then \(\mathcal{H}_7 \in \mathcal{A}\) with \(\alpha(\mathcal{H}_7) = \alpha\) and \(\beta(\mathcal{H}_7) = \beta\). By Theorem 1, there is \(T_7 \in [\mathcal{L}_{\nu,r}]\) for \(1 < r < \infty\) and \(\alpha < \nu < \beta\) so that, if \(1 < r \leq 2\),

\[
(6.26) \quad (\mathfrak{H}T_7 f)(s) = \mathcal{H}_7(s)(\mathfrak{M}f)(s) \quad (\text{Re}(s) = \nu).
\]

Setting

\[
(6.27) \quad H_7 = W_\delta I_{-1/a_{1},-a_{1}^{*},\alpha} L_{a_1^{*},\alpha} R T_7 R
\]

according to the properties (P2) and (P3) in Section 2 and Theorem 2 (b), (d), we have that if \(\alpha < 1 - \nu < \beta\) and \(1 < r \leq s < \infty\), then \(H_7 \in [\mathcal{L}_{\nu,r}, \mathcal{L}_{1-\nu,s}]\). If \(f \in \mathcal{L}_{\nu,2}\) and \(\text{Re}(s) = 1 - \nu\), applying (6.27), (2.17), (2.9), (2.12), (2.18) and (6.26) we obtain similarly to (6.8) that

\[
(6.28) \quad (\mathfrak{M}H_7 f)(s) = \left(\mathfrak{M}W_\delta I_{-1/a_{1},-a_{1}^{*},\alpha} L_{a_1^{*},\alpha} R T_7 R f\right)(s)
\]

\[
= \delta^s \left(\mathfrak{M}I_{1/a_{1}^{*},-a_{1}^{*},\alpha} L_{a_1^{*},\alpha} R T_7 R f\right)(s)
\]

\[
= \delta^s \left(\Gamma(a_1^{*}[s - \alpha]) \over \Gamma(a_1^{*}[s - \alpha] - \omega)(a_1^{*})^{a_1^{*}(s - \alpha) - 1}(\mathfrak{M}RT_7 R f)(1 - s)
\]

\[
= \delta^s \frac{\Gamma^2(a_1^{*}[s - \alpha])}{\Gamma(a_1^{*}[s - \alpha] - \omega)(a_1^{*})^{a_1^{*}(s - \alpha) - 1}(\mathfrak{M}T_7 R f)(s)
\]

\[
= \delta^s \frac{\Gamma^2(a_1^{*}[s - \alpha])}{\Gamma(a_1^{*}[s - \alpha] - \omega)(a_1^{*})^{a_1^{*}(s - \alpha) - 1} \mathcal{H}_7(s)(\mathfrak{M}R f)(s)
\]

\[
= 3 \mathcal{H}(s)(\mathfrak{M}f)(1 - s).
\]

Using this relation and the arguments similar to those in the case \(\text{Re}(\omega) < 0\) of Theorem 10, we complete the proof for \(\text{Re}(\omega) < 0\).

In the case \(a_1^{*} = 0 \) and \(a_2^{*} > 0\) the following statement is proved on the basis of Theorem 11 similarly to that in the case \(\delta = 1\) in [20, Theorem 3.3].

**Theorem 12.** Let \(a_1^{*} = 0, a_2^{*} > 0, \alpha < 1 - \nu < \beta < \infty\) and \(\omega = \mu - a_2^{*}\beta + 1/2\) and let \(1 < r < \infty\).

(a) If \(\nu \notin \mathcal{E}_{\delta t}, \) or \(1 < r \leq 2\), then \(H\) is a one-to-one transform on \(\mathcal{L}_{\nu,r}\).

(b) If \(\text{Re}(\omega) \geq 0\) and \(\nu \notin \mathcal{E}_{\delta t}\), then

\[
(6.29) \quad H(\mathcal{L}_{\nu,r}) = L_{-a_2^{*},\beta + \omega/a_2^{*}}(\mathcal{L}_{\nu,r}).
\]

When \(\nu \in \mathcal{E}_{\delta t}, H(\mathcal{L}_{\nu,r})\) is a subset of the right hand side of (6.29).

(c) If \(\text{Re}(\omega) < 0\) and \(\nu \notin \mathcal{E}_{\delta t}\), then

\[
(6.30) \quad H(\mathcal{L}_{\nu,r}) = \left(L_{0^{*};1/a_2^{*},a_2^{*}\beta - 1} L_{-a_2^{*},\beta}\right)(\mathcal{L}_{\nu,r}).
\]
When $\nu \in \mathcal{E}_\zeta$, $H(\mathcal{L}_{\nu,r})$ is a subset of the right hand side of (6.30).

7. $\mathcal{L}_{\nu,r}$-Theory of $H$-transform ($a^*>0$ and $a_1^*<0$ or $a_2^*<0$)

In this section we give conditions to the transform $H$ to be one-to-one on $\mathcal{L}_{\nu,r}$ and characterize its range on $\mathcal{L}_{\nu,r}$ except for its isolated values $\nu \in \mathcal{E}_\zeta$ in terms of the modified Hankel transform $H_{\kappa,\eta}$ and modified Laplace transform $L_{\kappa,\alpha}$ given in (2.6) and (2.7). The results will be different in the cases $a_1^*>0, a_2^*<0$ and $a_1^*<0, a_2^*>0$. We first consider the former case.

**Theorem 13.** Let $a^*>0$, $a_1^*>0$, $a_2^*<0$, $\alpha<1-\nu<\beta$ and $1<r<\infty$.

(a) If $\nu \notin \mathcal{E}_\zeta$, or if $1<r \leq 2$, then $H$ is a one-to-one transform on $\mathcal{L}_{\nu,r}$.

(b) Let

\[(7.1)\quad \omega = a^*\eta - \mu - \frac{1}{2},\]

where $\mu$ is given by (1.7), and let $\eta$ be chosen as

\[(7.2)\quad a^*\text{Re}(\eta) \geq \gamma(r) + 2a_2^*(\nu - 1) + \text{Re}(\mu),\]

\[(7.3)\quad \text{Re}(\eta) > \nu - 1,\]

with $\gamma(r)$ being given by (2.10), and let $\zeta$ be chosen such that

\[(7.4)\quad \text{Re}(\zeta) < 1 - \nu.\]

If $\nu \notin \mathcal{E}_\zeta$, then

\[(7.5)\quad H(\mathcal{L}_{\nu,r}) = \left(M_{1/2+\omega/(2a_2^*)}H_{-2a_2^*;2a_2^*\zeta+\omega-1,L-a_2^*;1/2+\eta-\omega/(2a_2^*)}\right)\left(\mathcal{L}_{\delta/(2+\text{Re}(\omega)/(2a_2^*))-\nu,r}\right),\]

where $M_{\xi}$ is given by (2.13). If $\nu \in \mathcal{E}_\zeta$, then $H(\mathcal{L}_{\nu,r})$ is a subset of the right hand side of (7.5).

**Proof.** We consider the function

\[(7.6)\quad \mathcal{K}_\delta(s) = \frac{(a^*)^{a^*(s+\eta)-1}a_2^*\Gamma(a_2^*[s + \zeta] + \omega)}{\Gamma(a^*[s + \eta])\Gamma(a_2^*[\zeta - s])} \delta^{-s}\mathcal{K}(s).\]

For $\text{Re}(s) = 1 - \nu$ according to (7.1), (7.2), (7.4) and the relations $\gamma(r) \geq 1/2$ and $-a_2^*>0$, we have

\[\text{Re}[a_2^*[s + \zeta] + \omega] = a_2^*[1 - \nu + \text{Re}(\zeta)] + a^*\text{Re}(\eta) - \text{Re}(\mu) - \frac{1}{2}\]

\[\geq a_2^*[1 - \nu + \text{Re}(\zeta)] + [\gamma(r) + 2a_2^*(\nu - 1) + \text{Re}(\mu)] - \text{Re}(\mu) - \frac{1}{2}\]

\[= a_2^*\nu - 1 + \text{Re}(\zeta) + \gamma(r) - \frac{1}{2} \geq a_2^*\nu - 1 + \text{Re}(\zeta) > 0\]
and hence the function $\mathcal{H}_s(s)$ is analytic in the strip $\alpha < \text{Re}(s) < \beta$. Applying (3.3), (3.5), (5.5) and (5.8) we obtain the estimates

\begin{equation}
|\mathcal{H}_s(\sigma + it)| \sim \prod_{i=1}^{P} \alpha_i^{1/2 - \text{Re}(a_i)} \prod_{j=1}^{q} \beta_j^{\text{Re}(b_j) - 1/2} (2\pi)^{-1/2} (a_i^+)^{-1/2} e^{-[2a_i^+\text{Im}(\zeta) + \text{Im}(\xi) + \text{Im}(\omega) - a^+ \text{Im}(\eta)] \text{sign}(t)/2}
\end{equation}

and

\begin{equation}
\mathcal{H}'_s(\sigma + it) = \mathcal{H}_s(\sigma + it) \left\{ a^+ \log(a^+) - 2a_2^+ \log(|a_2^+|) + a_2^+ \psi(a_2^+[s + \zeta] + \omega) - a^+ \psi(a_2^+[s + \eta]) + a_2^+ \psi(a_2^+[-s]) - \log(\delta) + \frac{\mathcal{H}'(\sigma + it)}{\mathcal{H}(\sigma + it)} \right\} = O\left(\frac{1}{t^2}\right)
\end{equation}

as $|t| \to \infty$, uniformly in $\sigma$ on any bounded interval in $\mathbb{R}$.

Thus we have $\mathcal{H}_s \in \mathcal{A}$ with $\alpha(\mathcal{H}_s) = \alpha$ and $\beta(\mathcal{H}_s) = \beta$ and by Theorem 1, there is a transform $T_8 \in [\mathfrak{L}_{\nu,r}]$ for $1 < r < \infty$ and $\alpha < \nu < \beta$, so that if $1 < r \leq 2$,

\begin{equation}
(\mathcal{M}T_8 f)(s) = \mathcal{H}_s(s)(\mathfrak{M}f)(s) \quad (\text{Re}(s) = \nu).
\end{equation}

Let

\begin{equation}
H_8 = W_8 M_{1/2+\omega/(2a_2^+)} H_{-2a_2^+2a_2^+\zeta+\omega+1} L_{-a^+,1/2+\eta-\omega/(2a_2^+)} M_{-1/2-\omega/(2a_2^+)} T_8 R.
\end{equation}

It is directly verified that if the conditions of theorem satisfy, then in accordance with properties (P1) - (P3) in Section 2 and Theorem 2 (c), (d), $H_8 \in [\mathfrak{L}_{\nu,2}, \mathfrak{L}_{1-\nu,2}]$.

If $f \in \mathfrak{L}_{\nu,2}$, then applying (7.10), (2.17), (2.16), (2.11), (2.12), (2.16), (7.9), (2.18) and (7.6), we have for $\text{Re}(s) = 1 - \nu$

\begin{equation}
(\mathcal{M}H_8 f)(s) = \frac{\Gamma(a_2^+[-s])}{\Gamma(a_2^+[-s+\zeta] + \omega)} \times \left(\mathcal{ML}_{-a^+,1/2+\eta-\omega/(2a_2^+)} M_{-1/2-\omega/(2a_2^+)} T_8 R f\right) \left(\frac{1}{2} - s - \frac{\omega}{2a_2^+}\right)
\end{equation}

23
Using this relation and the arguments similar to those in the case \( \delta = 1 \) in [20, Theorem 5.1], we complete the proof of theorem.

**Corollary 1.** Let \( a^* > 0, a_1^* > 0, a_2^* < 0 \) and let \(-\infty \leq \alpha < 1 - \nu < \beta \leq \infty, 1 < r < \infty\).

(a) If \( \nu \notin \mathcal{E}_3, \) or if \( 1 < r \leq 2, \) then \( H \) is a one-to-one transform on \( \mathcal{L}_{\nu,r}. \)

(b) Let \( \omega \) be given by (7.1) and let \( \eta \) and \( \zeta \) be chosen such that either of the following conditions holds

(i) \( a^* \text{Re}(\eta) \geq \gamma(r) - 2a_2^*\beta + \text{Re}(\mu), \text{Re}(\eta) \geq -\alpha, \text{Re}(\zeta) \leq \alpha, \text{ if } -\infty < \alpha < \beta < \infty; \)

(ii) \( a^* \text{Re}(\eta) \geq \gamma(r) - 2a_2^*\beta + \text{Re}(\mu), \text{Re}(\eta) > \nu - 1, \text{Re}(\zeta) < 1 - \nu, \text{ if } -\infty = \alpha < \beta < \infty; \)

(iii) \( a^* \text{Re}(\eta) \geq \gamma(r) + 2a_2^*(\nu - 1) + \text{Re}(\mu), \text{Re}(\eta) \geq -\alpha, \text{Re}(\zeta) \leq \alpha, \text{ if } -\infty < \alpha < \beta = \infty. \)

Then if \( \nu \notin \mathcal{E}_3, H(\mathcal{L}_{\nu,r}) \) can be represented by the relation (7.5), and if \( \nu \in \mathcal{E}_3, H(\mathcal{L}_{\nu,r}) \) is a subset of the right hand side of (7.5).

**Corollary 2.** Let \( a^* > 0, a_1^* > 0, a_2^* < 0 \) and let \(-\infty < \alpha < 1 - \nu < \beta < \infty, 1 < r < \infty\).

(a) If \( \nu \notin \mathcal{E}_3, \) or if \( 1 < r \leq 2, \) then \( H \) is a one-to-one transform on \( \mathcal{L}_{\nu,r}. \)

(b) Let \( a^*\alpha - 2a_2^*\beta + \text{Re}(\mu) + \gamma(r) \leq 0, \) where \( \mu \) is given by (1.7), \( \omega = -a^*\alpha - \mu - 1/2 \) and \( \zeta \) be chosen such that \( \text{Re}(\zeta) \leq \alpha. \) Then if \( \nu \notin \mathcal{E}_3, H(\mathcal{L}_{\nu,r}) \) can be represented in the form (7.5), and if \( \nu \in \mathcal{E}_3, H(\mathcal{L}_{\nu,r}) \) is a subset of the right hand side of (7.5).

Finally we consider the case when \( a^* > 0, a_1^* < 0 \) and \( a_2^* > 0. \) On the basis of Theorem 13, the following statement is proved similarly to that in the case \( \delta = 1 \) in [20, Theorem 5.2].

**Theorem 14.** Let \( a^* > 0, a_1^* < 0, a_2^* > 0, \alpha < 1 - \nu < \beta \) and let \( 1 < r < \infty. \)

(a) If \( \nu \notin \mathcal{E}_3, \) or if \( 1 < r \leq 2, \) then \( H \) is a one-to-one transform on \( \mathcal{L}_{\nu,r}. \)

(b) Let

(7.12) \[
\omega = a^*\eta - \Delta - \mu - \frac{1}{2},
\]

where \( \Delta \) and \( \mu \) are given by (1.6) and (1.7) and let \( \eta \) be chosen such that

(7.13) \[
a^*\text{Re}(\eta) \geq \gamma(r) - 2a_1^*\nu + \Delta + \text{Re}(\mu), \text{Re}(\eta) > -\nu,
\]
and let \( \zeta \) be as

\[
\text{(7.14) \quad \text{Re}(\zeta) < \nu.}
\]

If \( \nu \notin \E_{\mathfrak{H}} \), then

\[
\text{(7.15) \quad H}(\mathcal{L}_{\nu,r}) = \left( M_{-1/2-\omega/(2a_1^*)} H_{2a_1^*,2a_1^*\zeta+\omega-1} L_{a^*,1/2-\eta+\omega/(2a_1^*)} \right) \left( \mathcal{L}_{1/2-\text{Re}(\omega)/(2a_1^*)-\nu,r} \right),
\]

where \( M_{\zeta} \) is given by (2.13). If \( \nu \in \E_{\mathfrak{H}} \), then \( H}(\mathcal{L}_{\nu,r}) \) is a subset of the right hand side of (7.15).

**Corollary 1.** Let \( a^* > 0, a_1^* < 0, a_2^* > 0 \), \(-\infty \leq \alpha < 1 - \nu < \beta \leq \infty \) and let \( 1 < r < \infty \).

(a) If \( \nu \notin \E_{\mathfrak{H}} \), or if \( 1 < r \leq 2 \), then \( H \) is a one-to-one transform on \( \mathcal{L}_{\nu,r} \).

(b) Let \( \omega \) be given by (7.12) and let \( \eta \) and \( \zeta \) be chosen such that either of the following conditions holds

(i) \( a^*\text{Re}(\eta) \geq \gamma(r) - 2a_1^*(1 - \nu) + \Delta + \text{Re}(\mu), \text{Re}(\eta) \geq \beta - 1, \text{Re}(\zeta) \leq 1 - \beta, \) if \(-\infty < \alpha < \beta < \infty; \)

(ii) \( a^*\text{Re}(\eta) \geq \gamma(r) - 2a_1^*\nu + \Delta + \text{Re}(\mu), \text{Re}(\eta) \geq \beta - 1, \text{Re}(\zeta) \leq 1 - \beta, \) if \(-\infty = \alpha < \beta < \infty; \)

(iii) \( a^*\text{Re}(\eta) \geq \gamma(r) - 2a_1^*(1 - \nu) + \Delta + \text{Re}(\mu), \text{Re}(\eta) > -\nu, \text{Re}(\zeta) < \nu, \) if \(-\infty < \alpha < \beta = \infty. \)

Then if \( \nu \notin \E_{\mathfrak{H}} \), \( H}(\mathcal{L}_{\nu,r}) \) can be represented by the relation (7.15), and if \( \nu \in \E_{\mathfrak{H}} \), \( H}(\mathcal{L}_{\nu,r}) \) is a subset of the right hand side of (7.15).

**Corollary 2.** Let \( a^* > 0, a_1^* < 0, a_2^* > 0 \) and let \(-\infty < \alpha < 1 - \nu < \beta < \infty, 1 < r < \infty \).

(a) If \( \nu \notin \E_{\mathfrak{H}} \), or if \( 1 < r \leq 2 \), then \( H \) is a one-to-one transform on \( \mathcal{L}_{\nu,r} \).

(b) Let \( 2a_1^*\alpha - a_2^*\beta + \text{Re}(\mu) + \gamma(r) \leq 0 \), where \( \mu \) is given by (1.7), and let \( \zeta \) be chosen such that \( \text{Re}(\zeta) \leq 1 - \beta. \) Then if \( \nu \notin \E_{\mathfrak{H}} \), \( H}(\mathcal{L}_{\nu,r}) \) can be represented by the relation (7.15), and if \( \nu \in \E_{\mathfrak{H}} \), then \( H}(\mathcal{L}_{\nu,r}) \) is a subset of the right hand side of (7.15).

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