Stable P-symmetric closed characteristics on partially symmetric compact convex hypersurfaces

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Dedicate to Professor Rou-Huai Wang’s 90th birth anniversary

Abstract

In this paper, let $n \geq 2$ be an integer, $P = \text{diag}(-I_{n-\kappa}, I_{\kappa}, -I_{n-\kappa}, I_{\kappa})$ for some integer $\kappa \in [0, n-1)$, and $\Sigma \subset \mathbb{R}^{2n}$ be a partially symmetric compact convex hypersurface, i.e., $x \in \Sigma$ implies $Px \in \Sigma$. We prove that if $\Sigma$ is $(r, R)$-pinched with $\frac{r}{R} < \sqrt{\frac{5}{3}}$, then $\Sigma$ carries at least two geometrically distinct P-symmetric closed characteristics which possess at least $2n - 4\kappa$ Floquet multipliers on the unit circle of the complex plane.

Key words: Compact convex hypersurfaces, stable P-symmetric closed characteristics, P-index iteration, Hamiltonian system.

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1 Introduction and main results

In this paper, we consider the stability of P-symmetric closed characteristics on partially symmetric hypersurfaces in $\mathbb{R}^{2n}$. Let $\Sigma$ be a $C^3$ compact hypersurface in $\mathbb{R}^{2n}$, bounding a strictly convex
Let \( J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix} \), \( I_n \) is the identity matrix in \( \mathbb{R}^n \) and \( N_\Sigma(y) \) is the outward normal unit vector of \( \Sigma \) at \( y \) normalized by the condition \( N_\Sigma(y) \cdot y = 1 \). Here \( a \cdot b \) denotes the standard inner product of \( a, b \in \mathbb{R}^{2n} \). A closed characteristic \((\tau, y)\) is prime if \( \tau \) is the minimal period of \( y \). Two closed characteristics \((\tau, x)\) and \((\sigma, y)\) are geometrically distinct, if \( x(\mathbb{R}) \neq y(\mathbb{R}) \). We denote by \( \mathcal{J}(\Sigma) \) the set of all closed characteristics \((\tau, y)\) on \( \Sigma \) with \( \tau \) being the minimal period of \( y \). For any \( s_i, t_i \in \mathbb{R}^{k_i} \) with \( i = 1, 2 \), we denote by \((s_1, t_1) \circ (s_2, t_2) = (s_1, s_2, t_1, t_2)\). Fixing an integer \( \kappa \) with \( 0 \leq \kappa < n - 1 \), let \( P = \text{diag}(-I_{n-\kappa}, I_{\kappa}, -I_{n-\kappa}, I_{\kappa}) \) and \( \mathcal{H}_\kappa(2n) = \{ \Sigma \in \mathcal{H}(2n) \mid x \in \Sigma \text{ implies } Px \in \Sigma \} \). For \( \Sigma \in \mathcal{H}_\kappa(2n) \), let \( \Sigma(\kappa) = \{ z \in \mathbb{R}^{2\kappa} \mid 0 \leq z \in \Sigma \} \), where 0 is the origin in \( \mathbb{R}^{2n-2\kappa} \). As in [DoLe], a closed characteristic \((\tau, y)\) on \( \Sigma \in \mathcal{H}_\kappa(2n) \) is \( P \)-asymmetric if \( y(\mathbb{R}) \cap Py(\mathbb{R}) = \emptyset \), it is \( P \)-symmetric if \( y(\mathbb{R}) = Py(\mathbb{R}) \) with \( y = y_1 \circ y_2 \) and \( y_1 \neq 0 \), or it is \( P \)-fixed if \( y(\mathbb{R}) = Py(\mathbb{R}) \) and \( y = 0 \circ y_2 \), where \( y_1 \in \mathbb{R}^{2(n-\kappa)} \), \( y_2 \in \mathbb{R}^{2\kappa} \). We call a closed characteristic \((\tau, y)\) is \( P \)-invariant if \( y(\mathbb{R}) = Py(\mathbb{R}) \). Then a \( P \)-invariant closed characteristic is \( P \)-symmetric or \( P \)-fixed.

Let \( j : \mathbb{R}^{2n} \rightarrow \mathbb{R} \) be the gauge function of \( \Sigma \), i.e., \( j(\lambda x) = \lambda \) for \( x \in \Sigma \) and \( \lambda \geq 0 \), then \( j \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^0(\mathbb{R}^{2n}, \mathbb{R}) \) and \( \Sigma = j^{-1}(1) \). Fix a constant \( \alpha \in (1, +\infty) \) and define the Hamiltonian \( H_\alpha : \mathbb{R}^{2n} \rightarrow [0, +\infty) \) by

\[
H_\alpha(x) := j(x)^\alpha
\]

Then \( H_\alpha \in C^3(\mathbb{R}^{2n} \setminus \{0\}, \mathbb{R}) \cap C^0(\mathbb{R}^{2n}, \mathbb{R}) \) is convex and \( \Sigma = H_\alpha^{-1}(1) \). It is well known that the problem (1.1) is equivalent to the following given energy problem of the Hamiltonian system

\[
\begin{cases}
\dot{y}(t) = JH_\alpha'(y(t)), H_\alpha(y(t)) = 1, \forall t \in \mathbb{R}, \\
y(\tau) = y(0),
\end{cases}
\tag{1.2}
\]

Denote by \( \mathcal{J}(\Sigma, \alpha) \) the set of all solutions \((\tau, y)\) of the problem (1.2), where \( \tau \) is the minimal period of \( y \). Note that elements in \( \mathcal{J}(\Sigma) \) and \( \mathcal{J}(\Sigma, \alpha) \) are in one to one correspondence with each other.

Let \((\tau, y) \in \mathcal{J}(\Sigma, \alpha) \). We call the fundamental solution \( \gamma_y : [0, \tau] \rightarrow Sp(2n) \) with \( \gamma_y(0) = I_{2n} \) of the linearized Hamiltonian system

\[
\dot{z}(t) = JH_\alpha''(y(t))z(t), \forall t \in \mathbb{R}.
\]
the associated symplectic path of \((\tau, y)\). The eigenvalue of \(\gamma_y(\tau)\) are called Floquet multipliers of \((\tau, y)\). By Proposition 1.6.13 of [Eke1], the Floquet multipliers with their multiplicities and Krein type numbers of \((\tau, y) \in J(\Sigma, \alpha)\) do not depend on the particular choice of the Hamiltonian function in (1.2). As in Chapter 15 of [Lon1], for any symplectic matrix \(M\), we define the elliptic height \(e(M)\) of \(M\) by the total algebraic multiplicity of all eigenvalues of \(M\) on the unit circle \(U\) in the complex plane \(C\). And for any \((\tau, y) \in J(\Sigma, \alpha)\) we define \(e(\tau, y) = e(\gamma_y(\tau))\), and call \((\tau, y)\) elliptic or hyperbolic if \(e(\tau, y) = 2n\) or \(e(\tau, y) = 2\), respectively.

As in Definition 5.1.6 of [Eke1], a \(C^3\) hypersurface \(\Sigma\) bounding a compact convex set \(U\), containing 0 in its interior is \((r, R)-\text{pinched}\), with \(0 < r \leq R\), if:

\[
|y|^2R^{-2} \leq \frac{1}{2}(H''_2(x)y, y) \leq |y|^2r^{-2}, \forall x \in \Sigma.
\]

For the existence, multiplicity and stability of closed characteristics on convex compact hypersurfaces in \(\mathbb{R}^{2n}\) we refer to [Rab1, Wel1, Ekl1, Ekl2, Gir1, EkH1, Szu1, LLZ1, LoZ1, WHL1] and references therein. It is very interesting to consider closed characteristics on hypersurfaces with special symmetries. [Wan1, Lin2, Zha1] studied the multiplicity of closed characteristics with symmetries on convex compact hypersurfaces without pinching conditions. For the stability problem of closed characteristics with symmetries, in [HuS1] of 2009, Hu and Sun studied the index theory and stability of periodic solutions in Hamiltonian systems with symmetries. As application they studied the stability of figure-eight orbit due to Chenciner and Montgomery in the planar three-body problems with equal masses. In [Liu1], Liu studied the stability of symmetric closed characteristics on central symmetric compact convex hypersurfaces under a pinching condition. In [DoL2], Dong and Long proved that there exists at least one \(P\)-invariant closed characteristic which possesses at least \(2n - 4\kappa\) Floquet multipliers on the unit circle of the complex plane. In this paper, we can obtain two such closed characteristics under a pinching condition:

**Theorem 1.1.** Assume \(\Sigma \in \mathcal{H}_\kappa(2n)\) and \(0 < r \leq |x| \leq R, \forall x \in \Sigma\) with \(\frac{R}{r} < \sqrt{\frac{3}{\kappa}}\). Then there exist at least two geometrically distinct \(P\)-symmetric closed characteristics which possess at least \(2n - 4\kappa\) Floquet multipliers on the unit circle of the complex plane.

**Remark 1.2.** In the above Theorem 1.1, let \(\kappa = 0\), the \(P\)-symmetric closed characteristic is just symmetric and the \(P\)-fixed closed characteristics vanish, so Theorem 1.1 covers Theorem 1.1 of [Liu1].

In this paper, let \(\mathbb{N}, \mathbb{N}_0, \mathbb{Z}, \mathbb{Q}, \mathbb{R}\) and \(\mathbb{C}\) denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers and complex numbers respectively. Denote by \(a \cdot b\) and \(|a|\) the standard inner product and norm in \(\mathbb{R}^{2n}\). Denote by \(\langle \cdot, \cdot \rangle\) and \(\|\cdot\|\) the standard \(L^2\)-inner
product and $L^2$-norm. For an $S^1$-space $X$, we denote by $X_{S^1}$ the homotopy quotient of $X$ by $S^1$, i.e., $X_{S^1} = S^\infty \times_{S^1} X$, where $S^\infty$ is the unit sphere in an infinite dimensional complex Hilbert space. We define the functions

$$[a] = \max \{ k \in \mathbb{Z} \mid k \leq a \}, \quad \{a\} = a - [a], \quad E(a) = \min \{ k \in \mathbb{Z} \mid k \geq a \}, \quad \phi(a) = E(a) - [a].$$

Specially, $\phi(a) = 0$ if $a \in \mathbb{Z}$, and $\phi(a) = 1$ if $a \notin \mathbb{Z}$. We use $Q$ coefficients for all homological modules.

2 A variational structure for P-invariant closed characteristics

In the rest of this paper, we fix a $\Sigma \in \mathcal{H}_\kappa(2n)$. In this section, we review a variational structure for P-invariant closed characteristics established in [Liu2].

As in [Liu2], we associate with $U$ a convex function $H_a$. Consider the fixed period problem

$$\begin{cases}
\dot{x}(t) = JH'_a(x(t)), \\
x(1/2) = Px(0).
\end{cases} \tag{2.1}$$

Then by Proposition 2.2 of [Liu2], nonzero solutions of (2.1) are in one to one correspondence with P-symmetric closed characteristics with period $\tau < a$ and P-fixed closed characteristics with period $\frac{\tau}{2} < \frac{a}{2}$. Let

$$L^2_{\kappa}\left(0, \frac{1}{2}\right) = \{ u = u_1 \diamond u_2 \in L^2((0, \frac{1}{2}), \mathbb{R}^{2n}) \mid u_1 \in L^2((0, \frac{1}{2}), \mathbb{R}^{2n-2\kappa}), \\
u_2 \in L^2((0, \frac{1}{2}), \mathbb{R}^{2\kappa}), u(\frac{1}{2}) = Pu(0), \int_0^{\frac{1}{2}} u_2(t)dt = 0 \} \tag{2.2}$$

Define a linear operator $\Pi_\kappa : L^2_{\kappa}(0, \frac{1}{2}) \to L^2_{\kappa}(0, \frac{1}{2})$ by

$$(\Pi_\kappa u)(t) = x_1(t) \diamond x_2(t),$$

$$x_1(t) = \int_0^t u_1(\tau)d\tau - \frac{1}{2} \int_0^{\frac{1}{2}} u_1(\tau)d\tau,$$

$$x_2(t) = \int_0^t u_2(\tau)d\tau - 2 \int_0^t dt \int_0^t u_2(\tau)d\tau,$$

for any $u = u_1 \diamond u_2 \in L^2_{\kappa}(0, \frac{1}{2})$.

The corresponding Clarke-Ekeland dual action functional is defined by

$$\Psi_a(u) = \int_0^{\frac{1}{2}} \left(\frac{1}{2} Ju \cdot \Pi_\kappa u + G_a(-Ju)\right)dt, \tag{2.3}$$

where $G_a$ is the Fenchel transform of $H_a$ defined by $G_a(x) = \sup \{ x \cdot y - H_a(y) \mid y \in \mathbb{R}^{2n} \}$. By Proposition 2.6 of [Liu2], $\Psi_a$ is $C^{1,1}$ on $L^2_{\kappa}(0, \frac{1}{2})$ and satisfies the Palais-Smale condition. Suppose
for every critical point \( u \) of \( \Psi \). Conversely, suppose \( u \) is a critical point of \( \Psi \). Then there exists a unique \( \xi \in \mathbb{R}^{2n} \) such that \( \Pi_u u - \xi \) is a solution of (2.1). In particular, solutions of (2.1) are in one to one correspondence with critical points of \( \Psi \). Moreover, \( \Psi_a(u) < 0 \) for every critical point \( u \neq 0 \) of \( \Psi \).

Suppose \( u \) is a nonzero critical point of \( \Psi \). Then the formal Hessian of \( \Psi \) at \( u \) is defined by

\[
Q_a(v, v) = \int_0^1 (Jv \cdot \Pi_k v + G_a''(-Ju)Jv \cdot Jv) \, dt,
\]

which defines an orthogonal splitting \( L^2_k(0, \frac{1}{2}) = E_- \oplus E_0 \oplus E_+ \) of \( L^2_k(0, \frac{1}{2}) \) into negative, zero and positive subspaces. The index of \( u \) is defined by \( i(u) = \text{dim} E_- \) and the nullity of \( u \) is defined by \( \nu(u) = \text{dim} E_0 \). cf. Definition 2.10 of \( \text{[Liu2]} \).

Note that we can identify \( x \) is a solution of (2.1). Then

\[
\Psi(x) = \int_0^1 \left( \frac{1}{2}Ju \cdot \Pi_k u + H^*(Ju) \right) \, dt, \quad \forall u \in \mathbb{R}^{2n}.
\]

The corresponding Clarke-Ekeland dual action functional on \( L^2_k(0, \frac{1}{2}) \) is defined by

\[
\Psi(u) = \int_0^1 \left( \frac{1}{2}Ju \cdot \Pi_k u + H^*(-Ju) \right) \, dt, \quad \forall u \in L^2_k(0, \frac{1}{2}),
\]

where \( H^* \) is the Fenchel transform of \( H \).

For any \( \iota \in \mathbb{R} \), we denote by

\[
\Psi^{-\iota} = \left\{ w \in L^2_k(0, \frac{1}{2}) \mid \Psi(w) < \iota \right\}.
\]

As in Section 2 of \( \text{[Lin2]} \), we define

\[
c_i = \inf \{ \delta \in \mathbb{R} \mid \bar{I}(\Psi^\delta) \geq \iota \}.
\]
where $\hat{I}$ is the Fadell-Rabinowitz index defined above. Then

$$c_1 \leq c_2 \leq \cdots \leq c_i \leq c_{i+1} \leq \cdots < 0.$$  

By Propositions 2.15 and 2.16 of [Liu2], we have

**Proposition 2.1.** Every $c_i$ is a critical value of $\Psi$. If $c_i = c_j$ for some $i < j$, then there are infinitely many geometrically distinct $P$-invariant closed characteristics on $\Sigma$.

**Proposition 2.2.** For every $i \in \mathbb{N}$, there is a critical point $u_\alpha$ of $\Psi$ found in Proposition 2.1 such that

$$\Psi(u_\alpha) = c_i, \quad C_{S^1, 2i-2}(\Psi_a, S^1 \cdot u) \neq 0$$  

where $u$ is a critical point of $\Psi_a$ corresponding to $u_\alpha$ in the natural sense. In particular, we have

$$i(u) \leq 2(i-1) \leq i(u) + \nu(u) - 1.$$  

### 3 Index iteration theory for $P$-symmetric closed characteristics

In this section, we review the index iteration theory for $P$-symmetric closed characteristics which was studied in Section 3 of [Liu2].

Note that if $(\tau, y)$ is $P$-symmetric, then $((2m-1)\tau, y)$ is $P$-symmetric for any $m \in \mathbb{N}$. Thus $((2m-1)\tau, y)$ corresponds to a critical point of $\Psi_a$ via Propositions 2.2 and 2.6 of [Liu2], we denote it by $u^{2m-1}$. Recall that the action of a closed characteristic $(\tau, y)$ is defined by (cf. P.190 of [Eke1])

$$A(\tau, y) = \frac{1}{2} \int_0^\tau (Jy \cdot \dot{y}) dt.$$  

**Lemma 3.1.** (cf. Lemma 3.1 of [Liu2]) Suppose $u^{2m-1}$ is a nonzero critical point of $\Psi_a$ such that $u$ corresponds to $P$-symmetric closed characteristic $(\tau, y)$. Let $H_2(x) = j^2(x)$, where $j$ is the gauge function of $\Sigma$. And by (21) in P.191 of [Eke1], $\tau_2 = A(\tau, y)$. Then $i(u^{2m-1})$ equals the index of the following quadratic form

$$q_{(2m-1)\tau_2/2, \kappa}(v, v) := \int_0^{(2m-1)\tau_2/2} [(Jv, \Pi_{(2m-1)\tau_2/2, \kappa} v) + (H_2^\kappa(y(t))^{-1} Jv, Jv)] dt.$$  

where $v \in L^2_\kappa(0, (2m-1)\tau_2/2)$, the definitions of $q_{(2m-1)\tau_2/2, \kappa}$, $\Pi_{(2m-1)\tau_2/2, \kappa}$, $L^2_\kappa(0, (2m-1)\tau_2/2)$ are as in section 3 of [DoL2]. Moreover, we have $\nu(u^{2m-1}) = \nu(q_{(2m-1)\tau_2/2, \kappa}) - 1$.

Now we consider the linear Hamiltonian system

$$\begin{cases}
\dot{\xi}(t) = JA(t)\xi, \\
A(t + \tau_2/2) = PA(t)P.
\end{cases}$$  

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where $A(t) = H_2^0(y(t))$. Denote by $i_E^k(A, k)$ and $\nu_E^k(A, k)$ the index and nullity of the $k$-th iteration of the system (3.2) defined by Dong and Long (cf. Definition 3.4 of [DoL2]). Denote by $i_P^k(\gamma_A^k)$ and $\nu_P^k(\gamma_A^k)$ the P-index and P-nullity of the $k$-th iteration of the system (3.2) defined by Dong and Long (cf. Section 3 of [DoL1]), where $\gamma_A$ is the fundamental solution of (3.2) with $\gamma_A(0) = I_{2n}$.

Then we have

**Theorem 3.2.** (cf. Theorem 3.2 of [Liu2]) If $u^{2m-1}$ is a nonzero critical point of $\Psi_a$ such that $u$ corresponds to $P$-symmetric closed characteristic $((\tau, y))$. Then we have

\[ i(u^{2m-1}) = i_E^P(A, 2m - 1) = i_P^1(\gamma_{A}^{2m-1,P}) - \kappa, \]
\[ \nu(u^{2m-1}) = \nu_E^P(A, 2m - 1) - 1 = \nu_P^1(\gamma_{A}^{2m-1,P}) - 1. \]  

(3.3)

Now we compute $i(u^{2m-1})$ via the index iteration method in [Lon1] and [DoL1]. First we recall briefly an index theory for symplectic paths. All the details can be found in [Lon1], [DoL1] and [Liu2].

In the following of this section, we assume $P$ is some matrix of pattern $(-I_{2s-2t} \circ I_{2t})$, where $0 \leq t \leq s$.

As usual, the symplectic group $Sp(2n)$ is defined by

\[ Sp(2n) = \{ M \in GL(2n, \mathbb{R}) \mid M^TJM = J \}, \]

whose topology is induced from that of $\mathbb{R}^{4n^2}$. For $\tau > 0$ we are interested in paths in $Sp(2n)$:

\[ P_{\tau}(2n) = \{ \gamma \in C([0, \tau], Sp(2n)) \mid \gamma(0) = I_{2n} \}, \]

which is equipped with the topology induced from that of $Sp(2n)$. The following real function was introduced in [DoL1]:

\[ D_{P, \omega}(M) = (-1)^{n-1} \omega^n det(M - \omega P), \forall \omega \in U, M \in Sp(2n). \]

where $U$ is the unit circle in the complex plane. Thus for any $\omega \in U$ the following codimension 1 hypersurface in $Sp(2n)$ is defined in [DoL1]:

\[ Sp(2n)_{P, \omega}^0 = \{ M \in Sp(2n) \mid D_{P, \omega}(M) = 0 \}. \]

For any $M \in Sp(2n)_{P, \omega}^0$, we define a co-orientation of $Sp(2n)_{P, \omega}^0$ at $M$ by the positive direction $\frac{d}{dt}M e^{tJ}$ of the path $M e^{tJ}$ with $0 \leq t \leq 1$ and $\epsilon > 0$ being sufficiently small. Let

\[ Sp(2n)_{P, \omega}^* = Sp(2n) \setminus Sp(2n)_{P, \omega}^0, \]
\[ P_{P, \tau, \omega}^*(2n) = \{ \gamma \in P_{\tau}(2n) \mid \gamma(0) \in Sp(2n)_{P, \omega}^* \}, \]
\[ P_{P, \tau, \omega}^0(2n) = P_{\tau}(2n) \setminus P_{P, \tau, \omega}^*(2n). \]
For any two continuous arcs $\xi$ and $\eta : [0, \tau] \rightarrow Sp(2n)$ with $\xi(\tau) = \eta(0)$, it is defined as usual:

$$\eta * \xi(t) = \begin{cases} 
\xi(2t), & \text{if } 0 \leq t \leq \tau/2, \\
\eta(2t - 1), & \text{if } \tau/2 \leq t \leq \tau.
\end{cases}$$

Given any two $2m_k \times 2m_k$ matrices of square block form $M_k = \begin{pmatrix} A_k & B_k \\
C_k & D_k \end{pmatrix}$ with $k = 1, 2$, as in [Lon1], the $\diamond$-product of $M_1$ and $M_2$ is defined by the following $2(m_1 + m_2) \times 2(m_1 + m_2)$ matrix $M_1 \diamond M_2$:

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\
0 & A_2 & 0 & B_2 \\
C_1 & 0 & D_1 & 0 \\
0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\otimes k}$ the $k$-fold $\diamond$-product $M \diamond \cdots \diamond M$. Note that the $\diamond$-product of any two symplectic matrices is symplectic. For any two paths $\gamma_j \in P_{\tau}(2n_j)$ with $j = 0$ and 1, let $\gamma_1 \diamond \gamma_2(t) = \gamma_1(t) \diamond \gamma_2(t)$ for all $t \in [0, \tau]$.

A special path $\xi_n \in P_{\tau}(2n)$ is defined by

$$\xi_n(t) = \begin{pmatrix} 2 - \frac{t}{\tau} & 0 \\
0 & (2 - \frac{t}{\tau})^{-1} \end{pmatrix}^{\otimes n}, \text{ for } 0 \leq t \leq \tau.$$

**Definition 3.3.** (cf. [DoL1], also Definition 3.3 of [Liu2]) For any $\omega \in U$ and $M \in Sp(2n)$, via Definition 5.4.4 in [Lon1], we define

$$\nu_{P,\omega}(M) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(M - \omega P).$$

(3.4)

For any $\tau > 0$ and $\gamma \in P_{\tau}(2n)$, define

$$\nu_{P,\omega}(\gamma) = \nu_{\omega}(\gamma P) = \nu_{P,\omega}(\gamma(\tau)).$$

(3.5)

If $\gamma \in P_{P,\tau,\omega}(2n)$, define

$$i_{P,\omega}(\gamma) = [Sp(2n)^0_{P,\omega} : \gamma * \xi_n],$$

(3.6)

where the right hand side of (3.6) is the usual homotopy intersection number, and the orientation of $\gamma * \xi_n$ is its positive time direction under homotopy with fixed end points.
If \( \gamma \in \mathcal{P}_{P,\tau,\omega}^0(2n) \), we let \( \mathcal{F}(\gamma) \) be the set of all open neighborhoods of \( \gamma \) in \( \mathcal{P}_{\tau}(2n) \), and define
\[
i_{P,\omega}(\gamma) = \sup_{U \in \mathcal{F}(\gamma)} \inf \{ i_{P,\omega}(\beta) | \beta \in U \cap \mathcal{P}_{P,\tau,\omega}^0(2n) \}.
\]
(3.7)

Then
\[
(i_{P,\omega}(\gamma), \nu_{P,\omega}(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\},
\]
(3.8)
is called the P-index function of \( \gamma \) at \( \omega \).

For any \( M \in \text{Sp}(2n) \) and \( \omega \in \textbf{U} \), the splitting numbers \( S_M^\pm(P, \omega) \) of \( M \) at \( (P, \omega) \) are defined by
\[
S_M^\pm(P, \omega) = \lim_{\epsilon \to 0^+} i_{P,\omega} \exp(\pm \sqrt{-1} \epsilon)(\gamma) - i_{P,\omega}(\gamma),
\]
(3.9)
for any path \( \gamma \in \mathcal{P}_{\tau}(2n) \) satisfying \( \gamma(\tau) = M \).

Let \( \Omega^0(M) \) be the path connected component containing \( M = \gamma(\tau) \) of the set
\[
\Omega(M) = \{ N \in \text{Sp}(2n) | \sigma(N) \cap \textbf{U} = \sigma(M) \cap \textbf{U} \text{ and } \nu_\lambda(N) = \nu_\lambda(M), \forall \lambda \in \sigma(M) \cap \textbf{U} \}
\]
(3.10)
(3.11)
Here \( \Omega^0(M) \) is called the homotopy component of \( M \) in \( \text{Sp}(2n) \).

In [Lon1], the following symplectic matrices were introduced as basic normal forms:
\[
D(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \lambda = \pm 2,
\]
(3.12)
\[
N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}, \lambda = \pm 1, b = \pm 1, 0,
\]
(3.13)
\[
R(\theta) = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi),
\]
(3.14)
\[
N_2(\omega, B) = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}, \theta \in (0, \pi) \cup (\pi, 2\pi),
\]
(3.15)
where \( B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \) with \( b_i \in \mathbb{R} \) and \( b_2 \neq b_3 \).

Splitting numbers possess the following properties:

**Lemma 3.4.** (cf. Proposition 3.8 of [DoL1]) Let \( (p_\omega(MP), q_\omega(MP)) \) denote the Krein type of \( MP \) at \( \omega \). For any \( M \in \text{Sp}(2n) \) and \( \omega \in \textbf{U} \), the splitting numbers \( S_M^\pm(P, \omega) \) are well defined and satisfy the following properties.
(i) $S_{MP}^+(P, \omega) = S_{MP}^+(\omega)$, where the right-hand side is the splitting numbers given by Definition 9.1.4 of [Lon1].

(ii) $S_{MP}^+(P, \omega) - S_{MP}^-(P, \omega) = p_{\omega}(MP) - q_{\omega}(MP)$.

(iii) $S_{MP}^+(P, \omega) = S_{MP}^-(P, \omega)$ if $NP \in \Omega^0(MP)$.

(iv) $S_{MP)}^+(P, \omega) = S_{M_1}^+(P, \omega) + S_{M_2}^+(P, \omega)$ for $M_j, P_j \in Sp(2n_j)$ with $n_j \in \{1, \ldots, n\}$ satisfying $P = P_1 \circ P_2$ and $n_1 + n_2 = n$.

(v) $S_{MP}^+(P, \omega) = 0$ if $\omega \notin \sigma(MP)$.

We have the following

**Lemma 3.5.** (cf. Theorem 1.8.10 of [Lon1]) For any $M \in Sp(2n)$, there is a path $f : [0, 1] \to \Omega^0(M)$ such that $f(0) = M$ and

$$f(1) = M_1 \circ \cdots \circ M_l,$$

where each $M_i$ is a basic normal form listed in (3.12)-(3.15) for $1 \leq i \leq l$.

By Proposition 3.10 of [DoL1], we have the Bott-type formula for $(P, \omega)$-index:

**Lemma 3.6.** For any $\gamma \in P_\tau(2n)$, $z \in U$ and $m \in N$, we have

$$i_{P, m, z}^\gamma(\gamma^{m, P}) = \sum_{\omega^m = z} i_{P, \omega}^\gamma(\gamma),$$

$$\nu_{P, m, z}^\gamma(\gamma^{m, P}) = \sum_{\omega^m = z} \nu_{P, \omega}^\gamma(\gamma).$$

Now we deduce the index iteration formula for each case in (3.12)-(3.15). Note that the splitting numbers are computed in List 9.1.12 of [Lon1]. Let $M = \gamma(\tau)$.

**Case 1.** $MP$ is conjugate to a matrix

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}$$

for some $b > 0$.

In this case, we have $(S_{MP}^+(P, 1), S_{M}^-(P, 1)) = (1, 1)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

$$i_{P, 1}^\gamma(\gamma^{2m-1, P}) = \sum_{\omega^{2m-1} = 1} i_{P, \omega}^\gamma(\gamma) = \sum_{k=0}^{2m-2} i_{P, e^{2k\pi i/(2m-1)}}^\gamma(\gamma) = (2m-1)(i_{P, 1}^\gamma(\gamma) + 1) - 1,$$

$$\nu_{P, 1}^\gamma(\gamma^{2m-1, P}) = 1.$$  

(3.17)

**Case 2.** $MP = I_2$, the $2 \times 2$ identity matrix.

In this case, we have $(S_{M}^+(P, 1), S_{M}^-(P, 1)) = (1, 1)$ by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus as in Case 1, we have

$$i_{P, 1}^\gamma(\gamma^{2m-1, P}) = (2m-1)(i_{P, 1}^\gamma(\gamma) + 1) - 1,$$

$$\nu_{P, 1}^\gamma(\gamma^{2m-1, P}) = 2.$$  

(3.18)
**Case 3.** $MP$ is conjugate to a matrix \( \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \) for some $b < 0$.

In this case, we have \( (S^+_M(P,1),S^-_M(P,1)) = (0,0) \) by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

\[
i_P,1(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_P,\omega(\gamma) = \sum_{k=0}^{2m-2} i_P,e^{2k\pi i/(2m-1)}(\gamma) = (2m-1)i_P,1(\gamma),
\]

\[
\nu_P,1(\gamma^{2m-1,P}) = 1.
\]

(3.19)

**Case 4.** $MP$ is conjugate to a matrix \( \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \) for some $b < 0$.

In this case, we have \( (S^+_M(P,-1),S^-_M(P,-1)) = (1,1) \) by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

\[
i_P,1(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_P,\omega(\gamma) = \sum_{k=0}^{2m-2} i_P,e^{2k\pi i/(2m-1)}(\gamma) = (2m-1)i_P,1(\gamma),
\]

\[
\nu_P,1(\gamma^{2m-1,P}) = 0.
\]

(3.20)

**Case 5.** $MP = -I_2$.

In this case, we have \( (S^+_M(P,-1),S^-_M(P,-1)) = (1,1) \) by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus as in Case 4, we have

\[
i_P,1(\gamma^{2m-1,P}) = (2m-1)i_P,1(\gamma), \quad \nu_P,1(\gamma^{2m-1,P}) = 0.
\]

(3.21)

**Case 6.** $MP$ is conjugate to a matrix \( \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \) for some $b > 0$.

In this case, we have \( (S^+_M(P,-1),S^-_M(P,-1)) = (0,0) \) by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

\[
i_P,1(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_P,\omega(\gamma) = \sum_{k=0}^{2m-2} i_P,e^{2k\pi i/(2m-1)}(\gamma) = (2m-1)i_P,1(\gamma),
\]

\[
\nu_P,1(\gamma^{2m-1,P}) = 0.
\]

(3.22)

**Case 7.** $MP = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}$ with some $\theta \in (0, \pi) \cup (\pi, 2\pi)$.

In this case, we have \( (S^+_M(P,e^{\sqrt{-1}\theta}),S^-_M(P,e^{-\sqrt{-1}\theta})) = (0,1) \) by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

\[
i_P,1(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_P,\omega(\gamma) = \sum_{k=1}^{2m-1} i_P,e^{2k\pi i/(2m-1)}(\gamma)
\]
\[\begin{align*}
&= \sum_{0 < k < \frac{(2m-1)\theta}{\pi}} i_{P,1}(\gamma) + \sum_{\frac{(2m-1)\theta}{\pi} \leq k < \frac{(2m-1)(2\pi-\theta)}{\pi}} (i_{P,1}(\gamma) - 1) \\
&+ \sum_{\frac{(2m-1)(2\pi-\theta)}{\pi} < k \leq 4m-2} i_{P,1}(\gamma) \\
&= (2m-1)(i_{P,1}(\gamma) - 1) + 2E\left(\frac{(2m-1)\theta}{2\pi}\right) - 1,
\end{align*}\]
\[\nu_{P,1}(\gamma^{2m-1,P}) = 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right), \quad (3.24)\]

where the function \(E(\cdot)\) is defined as in Section 1.

Provided \(\theta \in (0, \pi)\). When \(\theta \in (\pi, 2\pi)\), we have

\[i_{P,1}(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2\pi i/(2m-1)}(\gamma)} \]
\[= \sum_{0 < k < \frac{(2m-1)(2\pi-\theta)}{\pi}} i_{P,1}(\gamma) + \sum_{\frac{(2m-1)(2\pi-\theta)}{\pi} < k < \frac{(2m-1)\theta}{\pi}} (i_{P,1}(\gamma) + 1) \\
+ \sum_{\frac{(2m-1)\theta}{\pi} < k \leq 4m-2} i_{P,1}(\gamma) \\
= (2m-1)(i_{P,1}(\gamma) - 1) + 2E\left(\frac{(2m-1)\theta}{2\pi}\right) - 1,
\]
\[\nu_{P,1}(\gamma^{2m-1,P}) = 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right). \quad (3.25)\]

**Case 8.** \(MP = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}\) with some \(\theta \in (0, \pi) \cup (\pi, 2\pi)\) and \(B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\) \(\in \mathbb{R}^{2 \times 2}\), such that \((b_2 - b_3)\sin \theta < 0\).

In this case, we have \((S^+_M(P, e^{\sqrt{-1}\theta}), S^-_M(P, e^{\sqrt{-1}\theta})) = (1, 1)\) by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

\[i_{P,1}(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2\pi i/(2m-1)}(\gamma)} \]
\[= (2m-1)i_{P,1}(\gamma) + 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right) - 2,
\]
\[\nu_{P,1}(\gamma^{2m-1,P}) = 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right). \quad (3.26)\]

Here from the first line to the second line of Equation (3.26), we used that if \(\frac{(2m-1)\theta}{2\pi} \notin \mathbb{N}\), then \(i_{P,e^{2\pi i/(2m-1)}(\gamma)} = i_{P,1}(\gamma)\) for all \(1 \leq k \leq 2m-1\) and if \(\frac{(2m-1)\theta}{2\pi} \in \mathbb{N}\), then exactly two of the \(i_{P,e^{2\pi i/(2m-1)}(\gamma)}\)'s equal to \(i_{P,1}(\gamma)\) and the other ones equal to \(i_{P,1}(\gamma)\).

**Case 9.** \(MP = \begin{pmatrix} R(\theta) & B \\ 0 & R(\theta) \end{pmatrix}\) with some \(\theta \in (0, \pi) \cup (\pi, 2\pi)\) and \(B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}\) \(\in \mathbb{R}^{2 \times 2}\), such that \((b_2 - b_3)\sin \theta > 0\).
In this case, we have \((S^+_M(P,e^{\sqrt{-1}\theta}),S^-_M(P,e^{\sqrt{-1}\theta})) = (0,0)\) by List 9.1.12 of [Lon1] and Lemma 3.4 (i), (iii). Thus by Lemma 3.4 (v) and Lemma 3.6, we have

\[
i_{P,1}(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) = (2m-1)i_{P,1}(\gamma),
\]

\[
\nu_{P,1}(\gamma^{2m-1,P}) = 2 - 2\phi\left(\frac{(2m-1)\theta}{2\pi}\right),
\]

(3.27)

**Case 10.** MP is hyperbolic, i.e., \(U \cap \sigma(MP) = \emptyset\).

In this case, by Lemma 3.4 (v) and Lemma 3.6, we have

\[
i_{P,1}(\gamma^{2m-1,P}) = \sum_{\omega^{2m-1}=1} i_{P,\omega}(\gamma) = \sum_{k=1}^{2m-1} i_{P,e^{2k\pi i/(2m-1)}}(\gamma) = (2m-1)i_{P,1}(\gamma),
\]

\[
\nu_{P,1}(\gamma^{2m-1,P}) = 0.
\]

(3.28)

The following new theorem will play a crucial role in our proof of Theorem 1.1:

**Theorem 3.7.** Suppose \(u^{2m-1}\) is a nonzero critical point of \(\Psi_a\) such that \(u\) corresponds to a \(P\)-symmetric closed characteristic \((\tau,y)\). Then we have \(i(u^3) - 3i(u) \leq 2\kappa + 2n\) and \(i(u^3) + \nu(u^3) - 3(i(u) + \nu(u)) \geq 2\kappa + 2 - 2n\). In particular, we have the following

(i) if \(i(u^3) - 3i(u) \geq 2n - 2\kappa\).

(ii) if \(i(u^3) + \nu(u^3) - 3(i(u) + \nu(u)) \leq 6\kappa + 2 - 2n\), then \(e(\tau,y) \geq 2n - 4\kappa\).

**Proof.** Firstly, we compute \(i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma)\) and \(i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma))\) for any symplectic path \(\gamma \in P_\tau(2n)\) satisfying \(\gamma(\tau) = M\). We consider each of the above cases.

**Case 1.** MP is conjugate to a matrix \(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\) for some \(b > 0\).

In this case, we have

\[
i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) = 3(i_{P,1}(\gamma) + 1) - 1 - 3i_{P,1}(\gamma) = 2,
\]

\[
i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) = 3(i_{P,1}(\gamma) + 1) - 3(i_{P,1}(\gamma) + 1) = 0.
\]

**Case 2.** MP = \(I_2\), the \(2 \times 2\) identity matrix.

In this case, we have

\[
i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) = 3(i_{P,1}(\gamma) + 1) - 1 - 3i_{P,1}(\gamma) = 2,
\]

\[
i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) = 3(i_{P,1}(\gamma) + 1) + 1 - 3(i_{P,1}(\gamma) + 2) = -2.
\]

**Case 3.** MP is conjugate to a matrix \(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\) for some \(b < 0\).
In this case, we have

\[ i_{P,1}(γ^{3,P}) - 3i_{P,1}(γ) = 3i_{P,1}(γ) - 3i_{P,1}(γ) = 0, \]
\[ i_{P,1}(γ^{3,P}) + ν_{P,1}(γ^{3,P}) - 3(i_{P,1}(γ) + ν_{P,1}(γ)) = 3i_{P,1}(γ) + 1 - 3(i_{P,1}(γ) + 1) = -2. \]

**Case 4.** \( MP \) is conjugate to a matrix \( \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \) for some \( b < 0 \).

In this case, we have

\[ i_{P,1}(γ^{3,P}) - 3i_{P,1}(γ) = 3i_{P,1}(γ) - 3i_{P,1}(γ) = 0, \]
\[ i_{P,1}(γ^{3,P}) + ν_{P,1}(γ^{3,P}) - 3(i_{P,1}(γ) + ν_{P,1}(γ)) = 3i_{P,1}(γ) - 3i_{P,1}(γ) = 0. \]

**Case 5.** \( MP = -I_2 \).

In this case, we have

\[ i_{P,1}(γ^{3,P}) - 3i_{P,1}(γ) = 3i_{P,1}(γ) - 3i_{P,1}(γ) = 0, \]
\[ i_{P,1}(γ^{3,P}) + ν_{P,1}(γ^{3,P}) - 3(i_{P,1}(γ) + ν_{P,1}(γ)) = 3i_{P,1}(γ) - 3i_{P,1}(γ) = 0. \]

**Case 6.** \( MP \) is conjugate to a matrix \( \begin{pmatrix} -1 & b \\ 0 & -1 \end{pmatrix} \) for some \( b > 0 \).

In this case, we have

\[ i_{P,1}(γ^{3,P}) - 3i_{P,1}(γ) = 3i_{P,1}(γ) - 3i_{P,1}(γ) = 0, \]
\[ i_{P,1}(γ^{3,P}) + ν_{P,1}(γ^{3,P}) - 3(i_{P,1}(γ) + ν_{P,1}(γ)) = 3i_{P,1}(γ) - 3i_{P,1}(γ) = 0. \]

**Case 7.** \( MP = \begin{pmatrix} \cos θ & -\sin θ \\ \sin θ & \cos θ \end{pmatrix} \) with some \( θ \in (0, π) \cup (π, 2π) \).

In this case, we have

\[ i_{P,1}(γ^{3,P}) - 3i_{P,1}(γ) = 3(i_{P,1}(γ) - 1) + 2E\left(\frac{3θ}{2π}\right) - 1 - 3i_{P,1}(γ) \leq 2, \]
\[ i_{P,1}(γ^{3,P}) + ν_{P,1}(γ^{3,P}) - 3(i_{P,1}(γ) + ν_{P,1}(γ)) \]
\[ = \left(3(i_{P,1}(γ) - 1) + 2E\left(\frac{3θ}{2π}\right) - 1 + 2 - 2\phi\left(\frac{3θ}{2π}\right)\right) - 3i_{P,1}(γ) \]
\[ = -2 + 2E\left(\frac{3θ}{2π}\right) - 2\phi\left(\frac{3θ}{2π}\right) \geq -2. \]

**Case 8.** \( MP = \begin{pmatrix} R(θ) & B \\ 0 & R(θ) \end{pmatrix} \) with some \( θ \in (0, π) \cup (π, 2π) \) and \( B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in \mathbb{R}^{2×2}, \)

such that \((b_2 - b_3)\sin θ < 0\).
In this case, we have
\[ i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) = 3i_{P,1}(\gamma) + 2\phi\left(\frac{3\theta}{2\pi}\right) - 2 - 3i_{P,1}(\gamma) \leq 0, \]
\[ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) = 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0. \]

**Case 9.** \( MP = \left( \begin{array}{cc} R(\theta) & B \\ 0 & R(\theta) \end{array} \right) \) with some \( \theta \in (0, \pi) \cup (\pi, 2\pi) \) and \( B = \left( \begin{array}{cc} b_1 & b_2 \\ b_3 & b_4 \end{array} \right) \in \mathbb{R}^{2 \times 2} \), such that \((b_2 - b_3) \sin \theta > 0\).

In this case, we have
\[ i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) = 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0, \]
\[ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) = 3i_{P,1}(\gamma) + 2 - 2\phi\left(\frac{3\theta}{2\pi}\right) - 3i_{P,1}(\gamma) \geq 0. \]

**Case 10.** \( MP \) is hyperbolic, i.e., \( \mathbb{U} \cap \sigma(MP) = \emptyset \).

In this case, we have
\[ i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) = 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0, \]
\[ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) = 3i_{P,1}(\gamma) - 3i_{P,1}(\gamma) = 0. \]

Now for a \( \text{P}\)-symmetric closed characteristic \( (\tau, y) \), \( H_2, \tau_2 \) is defined as in Lemma 3.1. Its fundamental solution \( \gamma = \gamma_y : [0, \tau_2/2] \to Sp(2n) \) with \( \gamma_y(0) = I_{2n} \) of the linearized Hamiltonian system
\[ \gamma_y(t) = JH_2^T(y(t))\gamma_y(t), \quad \forall \ t \in \mathbb{R}, \tag{3.29} \]
is called the associated symplectic path of \( \text{P}\)-symmetric closed characteristic \( (\tau, y) \). By Lemma 3.5, we suppose \( \gamma(\frac{\tau}{2})P \) is conjugate to \( N_1(1, 1)^{op-} \circ N_1(1, -1)^{op+} \circ (I_{2p_0}) \circ R(\theta_1) \circ \cdots \circ R(\theta_r) \circ N_2(\omega_1, B_1) \circ \cdots \circ N_2(\omega_s, B_s) \circ M_0 \) with \( \sigma(M_0) \cap (\mathbb{U} - \{-1\}) = \emptyset \). Combining the above cases 1-10, we obtain
\[ i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) \leq 2p_- + 2p_0 + 2r \leq 2n, \]
\[ i_{P,1}(\gamma^{3,P}) + \nu_{P,1}(\gamma^{3,P}) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) \geq -2p_0 - 2p_+ - 2r \geq -2n. \tag{3.30} \]
And by Theorem 3.2, we have \( \nu(u^3) = \nu_{P,1}(\gamma^{3,P}) - 1, \nu(u) = \nu_{P,1}(\gamma) - 1, i(u^3) = i_{P,1}(\gamma^{3,P}) - \kappa, \)
\( i(u) = i_{P,1}(\gamma) - \kappa. \) Then we obtain
\[ i(u^3) - 3i(u) = 2\kappa + i_{P,1}(\gamma^{3,P}) - 3i_{P,1}(\gamma) \leq 2\kappa + 2p_- + 2p_0 + 2r \leq 2\kappa + 2n, \tag{3.31} \]
and
\[
i(u^3) + \nu(u^3) - 3(i(u) + \nu(u)) \\
= 2\kappa + 2 + i_{P,1}(\gamma^3, P) + \nu_{P,1}(\gamma^3, P) - 3(i_{P,1}(\gamma) + \nu_{P,1}(\gamma)) \\
\geq 2\kappa + 2 - 2p_0 - 2p_+ - 2r \geq 2\kappa + 2 - 2n. \tag{3.32}
\]

If \(i(u^3) - 3i(u) \geq 2n\), then by (3.31), we have
\[
2p_- + 2p_0 + 2r \geq 2n - 2\kappa, \tag{3.33}
\]
i.e. \(e(\gamma(\frac{\tau_1}{2})P) \geq 2n - 2\kappa\).

If \(i(u^3) + \nu(u^3) - 3(i(u) + \nu(u)) \leq 6\kappa + 2 - 2n\), then by (3.32), we have
\[
2p_0 + 2p_+ + 2r \geq 2n - 4\kappa, \tag{3.34}
\]
i.e. \(e(\gamma(\frac{\tau}{2})P) \geq 2n - 4\kappa\). Noticing that \(e(\tau, y) = e(\gamma(\tau)) = e((\gamma(\frac{\tau}{2})P)^2)\), we complete our proof.

4 Proof of the main theorem

In this section, we give the proof of the main theorem.

Firstly, we point out a minor error in Example 6 on Page 278 of [DoL2] which is useful for us:

Lemma 4.1. For any \(c > 0\), we have
\[
i^E_P(cI2n|_{0, a}) = 2\kappa \left( E\left(\frac{c^2}{2\pi} \right) - 1 \right) + 2(n - \kappa) \left( E\left(\frac{c^2 + \frac{\pi}{2}}{2\pi} \right) - 1 \right). \tag{4.1}
\]

Proof. Note that the definition of \(E(a)\) in our paper is different from that in [DoL2] by 1. In the proof of Example 6 of [DoL2], \(t = s_k = (2k\pi + \frac{\pi}{2})/c\) should be changed into \(t = s_k = (2k\pi + \pi)/c\), and the expression in (3.14) of [DoL2] is wrong. One can easily verify our expression in (4.1) is right.

Note that since \(H_2(\cdot)\) is positive homogeneous of degree-two, by the \((r, R)\)-pinched condition we have
\[
|x|^2R^{-2} \leq H_2(x) \leq |x|^2r^{-2}, \forall x \in \Sigma \tag{4.2}
\]
Comparing with the theorem of Morse-Schoenberg in the study of geodesics, we have the following...
Proposition 4.2. Let $\Sigma \in \mathcal{H}_n(2n)$ which is $(r, R)$-pinched. Suppose $u^{2m-1}$ is a nonzero critical point of $\Psi_a$ such that $u$ corresponds to a $P$-symmetric closed characteristic $(\tau, y)$, and $\tau_2 = A(\tau, y)$ as in Lemma 3.1. Then we have the following

\[ i(u^{2m-1}) \geq 2nl, \quad \text{if} \quad \frac{(2m-1)\tau_2}{2} > l\pi R^2; \tag{4.3} \]

\[ i(u^{2m-1}) + \nu(u^{2m-1}) \leq 2n(l - 1) - 1, \quad \text{if} \quad \frac{(2m-1)\tau_2}{2} < (l - \frac{1}{2})\pi r^2, \tag{4.4} \]

for some $l \in \mathbb{N}$.

**Proof.** Consider the following three quadratic forms on $L^2_\kappa(0, (2m-1)\tau_2/2)$

\[ q^{R}_{(2m-1)\tau_2/2, \kappa}(v, v) := \int_0^{(2m-1)\tau_2/2} [(Jv, \Pi_{(2m-1)\tau_2/2, \kappa}v) + \left(\frac{R^2}{2} Jv, Jv\right)] dt \]

\[ q_{(2m-1)\tau_2/2, \kappa}(v, v) := \int_0^{(2m-1)\tau_2/2} [(Jv, \Pi_{(2m-1)\tau_2/2, \kappa}v) + (H^\kappa_2(y(t))^{-1} Jv, Jv)] dt \]

\[ q^r_{(2m-1)\tau_2/2, \kappa}(v, v) := \int_0^{(2m-1)\tau_2/2} [(Jv, \Pi_{(2m-1)\tau_2/2, \kappa}v) + \left(\frac{r^2}{2} Jv, Jv\right)] dt \]

By the $(r, R)$-pinched condition, we have

\[ q^{R}_{(2m-1)\tau_2/2, \kappa}(v, v) \geq q_{(2m-1)\tau_2/2, \kappa}(v, v) \geq q^r_{(2m-1)\tau_2/2, \kappa}(v, v). \]

Thus we have $i^{R}_{(2m-1)\tau_2/2} \leq i^{R}_{(2m-1)\tau_2/2} \leq i^r_{(2m-1)\tau_2/2}$, where $i^{R}_{(2m-1)\tau_2/2}$, $i^{R}_{(2m-1)\tau_2/2}$ and $i^r_{(2m-1)\tau_2/2}$ denote the indices of $q^{R}_{(2m-1)\tau_2/2, \kappa}$, $q_{(2m-1)\tau_2/2, \kappa}$ and $q^r_{(2m-1)\tau_2/2, \kappa}$ respectively. By Lemma 4.1 and the condition in (4.3), we obtain

\[ i^{R}_{(2m-1)\tau_2/2} = i^{R}_{(2m-1)\tau_2/2} \geq 2nl, \tag{4.5} \]

\[ i(u^{2m-1}) = i_{(2m-1)\tau_2/2} \geq i^{R}_{(2m-1)\tau_2/2} \geq 2nl. \tag{4.6} \]

Hence (4.3) holds. Denote by the orthogonal splitting $L^2_\kappa(0, (2m-1)\tau_2/2) = E_- \oplus E_0 \oplus E_+$ of $L^2_\kappa(0, (2m-1)\tau_2/2)$ into negative, zero and positive subspaces. Then we have the following observation: If $V$ is a subspace of $L^2_\kappa(0, (2m-1)\tau_2/2)$ such that $q_{(2m-1)\tau_2/2, \kappa}$ is negative semi-definite, i.e., $v \in V$ implies $q_{(2m-1)\tau_2/2, \kappa}(v, v) \leq 0$, then $\dim V \leq \dim E_- + \dim E_0$. In fact, this is a simple fact of linear algebra: Let

\[ \text{pr}_- : L^2_\kappa(0, (2m-1)\tau_2/2) = E_- \oplus E_0 \oplus E_+ \to E_- \]
be the orthogonal projection. Consider \( pr_-|V : V \to E_- \). Then \( v \in ker(pr_-|V) \) must belong to \( E_0 \). That is, since \( q_{(2m-1)\tau_2/2,\kappa}(v,v) > 0, 0 \neq v \in E_+ \). From

\[
\dim V = \dim \text{Im}(pr_-|V) + \dim \ker(pr_-|V)
\]

we prove our claim.

Let \( \epsilon > 0 \) be small enough such that \( \frac{(2m-1)\tau_2}{2} < (l - \frac{1}{2})\pi(r - \epsilon)^2 \). If \( V \) is a subspace of \( L^2_\kappa(0,(2m-1)\tau_2/2) \) such that \( q_{(2m-1)\tau_2/2,\kappa}
\mid V \leq 0, \) then \( q_{(2m-1)\tau_2/2,\kappa}(v,v) < 0, 0 \neq v \in V \). Thus we have \( \dim V \leq \iota_{(2m-1)\tau_2/2}^{r-\epsilon} \). In particular, by Lemma 3.1, we have

\[
i(u^{2m-1}) + \nu(u^{2m-1}) = i_{(2m-1)\tau_2/2} + \nu(q_{(2m-1)\tau_2/2,\kappa}) - 1 \leq \iota_{(2m-1)\tau_2/2}^{r-\epsilon} - 1. \tag{4.7}
\]

On the other hand, similarly to (4.5), by the condition in (4.4), we have

\[
\iota_{(2m-1)\tau_2/2}^{r-\epsilon} = 2\kappa \left( E(\frac{2}{\epsilon - \nu^2}(2m - 1)\tau_2/2 - 1) + 2(n - \kappa) \left( E(\frac{2}{\epsilon - \nu^2}(2m - 1)\tau_2/2 + \pi - 1) - 1 \right) \right)
\]

\[
\leq 2n(l - 1). \tag{4.8}
\]

Hence, \( i(u^{2m-1}) + \nu(u^{2m-1}) \leq 2n(l - 1) - 1 \). The proof is complete.

By Theorem 1.1 of \([\text{LiZ1}]\), we have:

**Lemma 4.3.** Assume \( \Sigma \in \mathcal{H}_\kappa(2n) \) and \( 0 < r \leq |x| \leq R, \ \forall \ x \in \Sigma \) with \( \frac{R}{r} < \sqrt{2} \). Then there exist at least \( n - \kappa \) geometrically distinct \( P \)-symmetric closed characteristics \( (\tau_i,y_i) \) on \( \Sigma \), where \( \tau_i \) is the minimal period of \( y_i \), and the actions \( A(\tau_i,y_i) \) satisfy:

\[
\pi r^2 \leq A(\tau_i,y_i) \leq \pi R^2, \ \forall 1 \leq i \leq n - \kappa. \tag{4.9}
\]

By the proof of the above Lemma and Proposition 2.2, we have

**Theorem 4.4.** Let \( \{(\tau_1,y_1), \cdots, (\tau_{n-\kappa},y_{n-\kappa})\} \) be the \( P \)-symmetric closed characteristics found in the above Lemma. Then we have

\[
\Psi'_a(u_i) = 0, \quad i(u_i) \leq 2(i - 1) \leq i(u_i) + \nu(u_i) - 1. \tag{4.10}
\]

for \( 1 \leq i \leq n - \kappa \), where \( u_i \) is the unique critical point of \( \Psi_a \) corresponding to \( (\tau_i,y_i) \).

Now we give the proof of the main theorem.

**Proof of Theorem 1.1.** Let \( \Sigma \in \mathcal{H}_\kappa(2n) \) which is \((r,R)\)-pinched with \( \frac{R}{r} < \sqrt{\frac{\kappa}{2}} \), then by (4.2)

we have

\[
r \leq |x| \leq R, \ \forall \ x \in \Sigma \tag{4.11}
\]
Thus by Theorem 4.4, we obtain $n - \kappa$ geometrically distinct prime P-symmetric closed characteristics $\{(\tau_1, y_1), \cdots, (\tau_{n-\kappa}, y_{n-\kappa})\}$ such that (4.10) hold.

In the following, we prove $e(\tau_i, y_i) \geq 2n - 4\kappa$ for $i = 1, n - \kappa$.

Note that we always have $e(\tau_1, y_1) \geq 2n - 4\kappa$ by the proof of Theorem 1 of [DoL2], $\sqrt{\frac{3}{2}}$-pinching is not necessary.

Note that we can prove $e(\tau_1, y_1) \geq 2n - 2\kappa$ under $\sqrt{\frac{3}{2}}$-pinching condition. In fact, by (4.10), we have $i(u_1) = 0$. On the other hand, from (4.9), we have

$$\frac{(4 - 1)}{2} A(\tau_1, y_1) \geq \frac{3}{2} \pi r^2 > \pi R^2,$$

where we used the pinching condition $\frac{R}{r} < \sqrt{\frac{3}{2}}$. By Proposition 4.2, we obtain

$$i(u_1^3) \geq 2n,$$

then

$$i(u_1^3) - 3i(u_1) \geq 2n.$$

By Theorem 3.7 (i), we get $e(\tau_1, y_1) \geq 2n - 2\kappa$.

Now we prove $e(\tau_{n-\kappa}, y_{n-\kappa}) \geq 2n - 4\kappa$. In fact, by (4.10), we have

$$2(n - \kappa) - 1 \leq i(u_{n-\kappa}) + \nu(u_{n-\kappa}). \tag{4.12}$$

On the other hand, from (4.9), we have

$$\frac{(4 - 1)}{2} A(\tau_{n-\kappa}, y_{n-\kappa}) \leq \frac{3}{2} \pi R^2 < \frac{5}{2} \pi r^2 = (3 - \frac{1}{2}) \pi r^2.$$

By Proposition 4.2, we obtain

$$i(u_{n-\kappa}^3) + \nu(u_{n-\kappa}^3) \leq 4n - 1, \tag{4.13}$$

Combining (4.12) with (4.13), we obtain

$$i(u_{n-\kappa}^3) + \nu(u_{n-\kappa}^3) - 3(i(u_{n-\kappa}) + \nu(u_{n-\kappa})) \leq 6\kappa + 2 - 2n, \tag{4.14}$$

then $e(\tau_{n-\kappa}, y_{n-\kappa}) \geq 2n - 4\kappa$ follows from Theorem 3.7 (ii). The proof is complete.

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