The problem of analytical calculation of barrier crossing characteristics for Lévy flights

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Abstract. By using the backward fractional Fokker–Planck equation we investigate the barrier crossing event in the presence of Lévy noise. After briefly reviewing recent results obtained with different approaches on the time characteristics of the barrier crossing, we derive a general differential equation useful to calculate the nonlinear relaxation time. We obtain analytically the nonlinear relaxation time for free Lévy flights and a closed expression in quadrature of the same characteristics for the cubic potential.

Keywords: exact results, stochastic particle dynamics (theory)
1. Introduction

Lévy flights are stochastic processes characterized by the occurrence of extremely long jumps, so that their trajectories are not continuous anymore. The length of these jumps is distributed according to a Lévy stable statistics with a power-law tail and divergence of the second moment. This peculiar property strongly contradicts the ordinary Brownian motion, for which all the moments of the particle coordinate are finite. The presence of anomalous diffusion can be explained as a deviation of the real statistics of fluctuations from the Gaussian law, giving rise to the generalization of the central limit theorem by Lévy and Gnedenko [1,2]. The divergence of the Lévy flights variance poses some problems as regards to the physical meaning of these processes. However, recently the relevance of the Lévy motion appeared in many physical, natural and social complex systems. The Lévy type statistics, in fact, is observed in various scientific areas, where scale-invariance phenomena take place or can be suspected (see [3]–[6] and references therein). Lévy flights are a special class of Markovian processes; therefore the Markovian analysis can be used to derive the generalized Kolmogorov equation directly from the Langevin equation with Lévy noise [7].

The problem of escape from a metastable state, first investigated by Kramers [8], is ubiquitous in almost all scientific areas (see, for example, the reviews [9,10] and [11]). Since many stochastic processes do not obey the central limit theorem, the corresponding Kramers escape behavior will differ. An interesting example is given by the α-stable noise-induced barrier crossing in long paleoclimatic time series [12]. Another new application is the escape from traps in optical or plasma systems [13]. The main tool to investigate the barrier crossing problem remains the first passage times technique. But for anomalous diffusion in the form of Lévy flights this procedure meets with some difficulties. First of all, the fractional Fokker–Planck equation describing the Lévy flights is integro-differential, and the conditions at absorbing and reflecting boundaries differ from those used for ordinary diffusion. Lévy flights are characterized by the presence of long jumps, and, as a result, a particle can reach instantaneously a boundary from an arbitrary position. One can mention some erroneous results obtained in [14] (see also the
Barrier crossing with Lévy flights

related correspondence [15]), because the author used the traditional conditions at two absorbing boundaries. There are a lot of numerical results regarding the different time characteristics of Lévy flights, but obtaining the exact analytical results remains an open problem (see [3]).

In this work, starting from the backward fractional Fokker–Planck equation we investigate the barrier crossing event in the presence of Lévy noise by focusing on the nonlinear relaxation time. This paper is organized as follows. In section 2 we briefly review some recent results on barrier crossing problems with different approaches. In section 3.1, the generalized equations useful to calculate the nonlinear relaxation time (NRLT) are derived. In section 3.2 we give the exact expressions of NRLT for free Lévy flights and for a cubic potential profile. Finally we draw conclusions.

2. Barrier crossing

The particle escape from a metastable state, and the first passage time probability density have been recently analyzed for Lévy flights in [3, 12], [16]–[26]. The main focus in these papers is to understand how the barrier crossing behavior, according to the Kramers law [8], is modified by the presence of the Lévy noise. Here we discuss briefly some results on the barrier crossing events with Lévy flights, recently obtained with different approaches.

The main tools to investigate the barrier crossing problem for Lévy flights are the first passage times, crossing times, arrival times and residence times. We should emphasize that the problem of mean first passage time (MFPT) meets with some difficulties because free Lévy flights represent a special class of discontinuous Markovian processes with infinite mean squared displacement. As already mentioned, the anomalous diffusion in the form of Lévy flights, for a particle moving in a potential profile $U(x)$, is described by the fractional Fokker–Planck equation [6] for the probability density function $W(x,t)$:

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x} [U'(x) W] + D \frac{\partial^\alpha W}{\partial |x|^\alpha}, \quad (1)$$

where the Riesz fractional derivative is defined as

$$\frac{\partial^\alpha f(x)}{\partial |x|^\alpha} = \frac{1}{K(\alpha)} \int_{-\infty}^{+\infty} \frac{f(\xi) - f(x)}{|x - \xi|^{1+\alpha}} d\xi$$

$$= \frac{1}{K(\alpha)} \int_{0}^{+\infty} \frac{f(x + \xi) + f(x - \xi) - 2f(x)}{\xi^{1+\alpha}} d\xi, \quad (2)$$

and

$$K(\alpha) = \frac{\pi}{\Gamma(\alpha + 1) \sin(\pi \alpha/2)}, \quad (3)$$

with $\Gamma(z)$ the gamma function and $0 < \alpha < 2$. Due to the integro-differential nature of equation (1), we cannot apply the usual boundary conditions at the reflecting and absorbing barriers of the system investigated. The particle, in fact, can reach instantaneously the boundaries from any position.

Numerical results for the first passage time of free Lévy flights confined in a finite interval were presented in [3]. There, the complexity of the first passage time statistics
Barrier crossing with Lévy flights

(mean first passage time and cumulative first passage time distribution) was elucidated together with a discussion of the proper set-up of corresponding boundary conditions, that correctly yield the statistics of first passages for these non-Gaussian noises. In particular, it has been demonstrated by numerical studies that the use of the local boundary condition of vanishing probability flux in the case of reflection, and vanishing probability in the case of absorption, valid for normal Brownian motion, no longer apply for Lévy flights. This in turn requires the use of nonlocal boundary conditions. Dybiec with coauthors found a nonmonotonic behavior of the MFPT as a function of the Lévy index $\alpha$ for two absorbing boundaries, with the maximum being assumed for $\alpha = 1$, in contrast with a monotonic increase for reflecting and absorbing boundaries.

According to the Kramers law, the probability distribution of the escape times from a potential well with a barrier of height $U_0$, has the exponential form

$$p(t) = \frac{1}{T_c} \exp \left\{ -\frac{t}{T_c} \right\},$$

(4)

with mean crossing time

$$T_c = C \exp \left\{ \frac{U_0}{D} \right\},$$

(5)

where $C$ is some positive prefactor and $D$ is the noise intensity. The barrier crossing behavior of the classical Kramers problem was investigated, both numerically and analytically, in [3], [19]–[21], where the role of the stable nature of Lévy flight processes on the barrier crossing event was analyzed. The authors considered Lévy flights in a bistable potential $U(x)$ by numerical solution of the Langevin equation associated with the fractional Fokker–Planck equation (1):

$$\dot{x} = -U'(x) + \xi^{(\alpha)}(t),$$

(6)

where $\xi^{(\alpha)}(t)$ is the symmetric Lévy $\alpha$-stable noise. It was shown that, although the survival probability decays again exponentially as in equation (4), the mean escape time $T_c$ has a power-law dependence on the noise intensity $D$:

$$T_c \simeq \frac{C(\alpha)}{D^{\mu(\alpha)}},$$

(7)

where the prefactor $C$ and the exponent $\mu$ depend on the Lévy index $\alpha$. Using the Fourier transform of equation (1):

$$\frac{\partial \tilde{W}}{\partial t} = -i k U'(\cdot) \left( -i \frac{\partial}{\partial k} \right) \tilde{W} - D |k|^\alpha \tilde{W},$$

(8)

the authors derived the mean escape rate for large values of $1/D$ in the case of Cauchy stable noise ($\alpha = 1$) in the framework of the constant flux approximation across the barrier. The probability law and the mean value of the escape time from a potential well for all values of the Lévy index $\alpha \in (0, 2)$, in the limit of small Lévy driving noise, were also determined in the paper [24] by purely probabilistic methods. The escape times have the same exponential distribution (4). The mean value depends on the noise intensity $D$, in accordance with equation (7) with $\mu(\alpha) = 1$, and the prefactor $C$ depends on $\alpha$ and the distance between the local extrema of the potential.

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The barrier crossing of a particle driven by symmetric Lévy noise of index $\alpha$ and intensity $D$ for three different generic types of potentials was numerically investigated in [21]. Specifically: (i) a bistable potential, (ii) a metastable potential and (iii) a truncated harmonic potential were considered. For the low noise intensity regime, the result of equation (7) was recovered. As was shown, the exponent $\mu(\alpha)$ remains approximately constant, $\mu \approx 1$, for $0 < \alpha < 2$; at $\alpha = 2$ the power-law form of $T_c$ changes into the exponential dependence (5). It exhibits a divergence-like behavior as $\alpha$ approaches 2. In this regime a monotonic increase of the escape time $T_c$ with increasing $\alpha$ (keeping the noise intensity $D$ constant) was observed. For low noise intensities the escape time process corresponds to the barrier crossing by multiple Lévy steps. For high noise intensities, the average escape time curves collapse into a single curve for all values of $\alpha$. At intermediate noise intensities, the escape time exhibits nonmonotonic dependence on the index $\alpha$, while still retaining the exponential form of the escape time density.

The first arrival time is an appropriate parameter to analyze the barrier crossing problem for Lévy flights. If we insert in the fractional Fokker–Planck equation (1) a $\delta$ sink of strength $q(t)$ in the origin, we obtain the following equation for the non-normalized probability density function $W(x, t)$:

$$\frac{\partial W}{\partial t} = \frac{\partial}{\partial x} [U'(x) W] + D \frac{\partial^\alpha W}{\partial |x|^\alpha} - q(t) \delta(x),$$

from which by integration over all space we may define the quantity

$$q(t) = -\frac{d}{dt} \int_{-\infty}^{+\infty} W(x, t) dx,$$

which is the negative time derivative of the survival probability. According to definition (10), $q(t)$ represents the probability density function of the first arrival time: once a random walker arrives at the sink it is annihilated. As was shown in the paper [19] for free Lévy flights ($U(x) = 0$), the first arrival time distribution has a heavy tail

$$q(t) \sim t^{1/\alpha - 2},$$

with exponent depending on Lévy index $\alpha$ ($1 < \alpha < 2$) and differing from the universal Sparre Andersen result [27,28] for the probability density function of first passage time for arbitrary Markovian process:

$$p(t) \sim t^{-3/2}.$$
3. Nonlinear relaxation time with Lévy flights

3.1. General equations

The nonlinear relaxation time technique is more suitable for analytical investigations of Lévy flights’ temporal characteristics, because it does not request a constraint on the boundary conditions. According to the definition, the nonlinear relaxation time (NLRT) is

\[ T(x_0) = \int_0^\infty \frac{[P(t, x_0) - P(\infty, x_0)]}{P(0, x_0) - P(\infty, x_0)} \, dt, \]  

(13)

where \( P(\infty, x_0) = \lim_{t \to \infty} P(t, x_0) \) and

\[ P(t, x_0) = \int_{L_1}^{L_2} W(x, t|x_0, 0) \, dx \]  

(14)

represents the probability to find a particle in the interval \((L_1, L_2)\) at time \(t\), if it starts from point \(x = x_0\). Let us use the same procedure as for calculating the first passage time probability density (see [29]). If the random process \(x(t)\) is Markovian, the probability density of transitions obeys the following backward Kolmogorov equation [30]:

\[ \frac{\partial W(x, t|x_0, 0)}{\partial t} = \hat{L}^+(x_0) W(x, t|x_0, 0), \]  

(15)

with the initial condition

\[ W(x, 0|x_0, 0) = \delta(x - x_0). \]  

(16)

Here \( \hat{L}^+(x_0) \) is the adjoint kinetic operator. After integration with respect to \(x\) from \(L_1\) to \(L_2\) directly in equation (15) and taking into account equation (14) we arrive at

\[ \frac{\partial P(t, x_0)}{\partial t} = \hat{L}^+(x_0) P(t, x_0). \]  

(17)

Equation (17) should be solved with the initial condition following from equation (16):

\[ P(0, x_0) = 1_{(L_1, L_2)}(x_0), \]  

(18)

where \(1_{(L_1, L_2)}(x)\) is an indicator of the set \((L_1, L_2)\).

According to equation (17) \( \hat{L}^+(x_0) P(\infty, x_0) = 0 \), and after integration of this equation with respect to \(t\) from 0 to \(\infty\) we obtain (see equation (18))

\[ \hat{L}^+(x_0) Q(x_0) = P(\infty, x_0) - 1_{(L_1, L_2)}(x_0), \]  

(19)

where \(Q(x_0)\) is the numerator of the expression (13), i.e.

\[ Q(x_0) = \int_0^\infty [P(t, x_0) - P(\infty, x_0)] \, dt. \]  

(20)

Finally, in accordance with equations (13) and (20) the nonlinear relaxation time can be calculated as

\[ T(x_0) = \frac{Q(x_0)}{1 - P(\infty, x_0)}, \]  

(21)
with \( x_0 \in (L_1, L_2) \). Although equations (19) and (21) are useful tools to analyze the temporal characteristics of Lévy flights in different potential profiles \( U(x) \), obtaining the exact analytical results for the generic \( \alpha \) parameter, characterizing the anomalous diffusion, is one of the unsolved problems in this research area. Even for some particular potential profile, like the cubic one, to derive a general expression of the NLRT as a function of the Lévy index \( \alpha \) is a nontrivial problem. In section 3.2 we derive a general differential equation useful to calculate the NLRT for arbitrary Lévy index and we find a closed expression for the case of Cauchy stable noise excitation (\( \alpha = 1 \)).

### 3.2. Lévy flights in a cubic potential

The forward fractional Fokker–Planck equation for Lévy flights in the potential profile \( U(x) \) is

\[
\frac{\partial W(x, t|x_0, 0)}{\partial t} = \frac{\partial}{\partial x} \left[ U'(x) W(x, t|x_0, 0) \right] + D \frac{\partial^\alpha W(x, t|x_0, 0)}{\partial |x|^{\alpha}},
\]

where \( 0 < \alpha < 2 \). It is easy to find from equation (22) the expression for the adjoint kinetic operator:

\[
\hat{L}^+(x_0) = -U'(x_0) \frac{\partial}{\partial x_0} + D \frac{\partial^\alpha}{\partial |x_0|^\alpha}.
\]

Substituting equations (23) in (19) we arrive at

\[
D \frac{d^\alpha Q(x_0)}{d|x_0|^\alpha} - U'(x_0) \frac{dQ(x_0)}{dx_0} = P(\infty) - 1_{(L_1, L_2)}(x_0),
\]

because the probability \( P(\infty, x_0) \) does not depend on the initial position of the particles.

The Fourier transform of equation (24) gives

\[
\left[ U'' \left( \frac{1}{i \frac{d}{dk}} \right) - ikU' \left( \frac{1}{i \frac{d}{dk}} \right) \right] \tilde{Q}(k) - D |k|^\alpha \tilde{Q}(k) = P(\infty) \delta(k) + \frac{e^{-ikL_2} - e^{-ikL_1}}{2\pi i k},
\]

where

\[
\tilde{Q}(k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Q(x_0) e^{-ikx_0} dx_0,
\]

and we took into account that, in accordance with equations (14) and (20), \( Q(\pm \infty) = 0 \).

Solving equation (25) and using the backward Fourier transform, we can calculate the nonlinear relaxation time as (see equation (21))

\[
T(x_0) = \frac{1}{1 - P(\infty)} \int_{-\infty}^{+\infty} \tilde{Q}(k) e^{ikx_0} dk,
\]

where \( x_0 \in (L_1, L_2) \). It is easy to check from equation (26) that \( \tilde{Q}(-k) = \tilde{Q}^*(k) \). Dividing the integral in equation (27) into two parts for negative and positive variables \( k \) and using this relation we easily arrive at

\[
T(x_0) = \frac{2}{1 - P(\infty)} \text{Re} \left\{ \int_{0}^{\infty} \tilde{Q}(k) e^{ikx_0} dk \right\}.
\]
As a result, it is sufficient to solve equation (25) only for positive values of $k$:

$$
\left[ U'' \left( \frac{d}{dk} \right) - ikU' \left( \frac{d}{dk} \right) \right] \tilde{Q}(k) - Dk^\alpha \tilde{Q}(k) = P(\infty) \delta(k) + \frac{e^{-ikL_2} - e^{-ikL_1}}{2\pi i k}, \quad (k > 0).
$$

(29)

This is one of the main results of this paper. By solving equation (29) for a particular potential profile $U(x)$, we are able to calculate the NLRT by using equation (28) for a particle moving in that potential. However, the general solution of this equation strictly depends on the functional form of the potential profile $U(x)$, and there is a solution of this equation not for all the potential profiles.

We now consider two cases: (a) a free anomalous diffusion and (b) a cubic potential.

(a) For free Lévy flights ($U(x) = 0$): $P(\infty) = 0$, and from equation (29) we have

$$
\tilde{Q}(k) = \frac{e^{-ikL_1} - e^{-ikL_2}}{2\pi i Dk^{\alpha+1}} \quad (k > 0).
$$

(30)

After substitution of equations (30) in (28) and evaluation of the integral we find finally for the case $0 < \alpha < 1$

$$
T(x_0) = \frac{(x_0 - L_1)^\alpha + (L_2 - x_0)^\alpha}{2\Gamma(\alpha+1)\cos(\pi\alpha/2)}.
$$

(31)

As is seen from equation (31), the nonlinear relaxation time decreases monotonically with increasing the noise intensity $D$ and has a maximum as a function of initial position $x_0$ in the middle point of the interval $(L_1, L_2)$. For Lévy index $1 \leq \alpha < 2$ the nonlinear relaxation time is infinite as for free Brownian motion ($\alpha = 2$).

(b) Lévy flights in a metastable cubic potential with a sink at $x = +\infty$:

$$
U(x) = -\frac{x^3}{3} + a^2 x.
$$

(32)

Substituting this potential in equation (29) and taking into account that $P(\infty) = 0$ we obtain

$$
\frac{d}{dk} \left[ k^2 \frac{d \tilde{Q}(k)}{dk} \right] + (k^2 a^2 - iDk^{\alpha+1}) \tilde{Q}(k) = \frac{e^{-ikL_2} - e^{-ikL_1}}{2\pi} \quad (k > 0).
$$

(33)

To solve equation (33) we introduce a new function $R(k) = k \tilde{Q}(k)$. After substitution of this new function, equation (33) can be rearranged as

$$
\frac{d^2 R(k)}{dk^2} + (a^2 - iDk^{\alpha-1}) R(k) = \frac{e^{-ikL_2} - e^{-ikL_1}}{2\pi k} \quad (k > 0).
$$

(34)

It is quite difficult to find an analytical solution of this equation for arbitrary Lévy index $\alpha$. Thus, we limit our further considerations to the case $\alpha = 1$. Substituting $\alpha = 1$ into equation (34) and representing its right-hand part in the form of an integral, we arrive at

$$
\frac{d^2 R(k)}{dk^2} + (a^2 - iD) R(k) = -\frac{i}{2\pi} \int_{L_1}^{L_2} e^{-iky} dy \quad (k > 0).
$$

(35)
This linear differential equation can be exactly solved and, as a result, we find (for any
$k > 0$) the finite solution in the form

$$
\tilde{Q}(k) = \frac{1}{k} \left\{ c_0 e^{-\beta k - i\gamma k} + \frac{i}{2\pi} \int_{L_1}^{L_2} \frac{e^{-iky}}{y^2 + iD - a^2} \, dy \right\} \quad (k > 0),
$$

(36)

where

$$
\beta = a \left[ 1 + \left( \frac{D}{a^2} \right)^2 \right]^{1/4} \sin \left[ \frac{1}{2} \arctan \left( \frac{D}{a^2} \right) \right],
$$

$$
\gamma = a \left[ 1 + \left( \frac{D}{a^2} \right)^2 \right]^{1/4} \cos \left[ \frac{1}{2} \arctan \left( \frac{D}{a^2} \right) \right].
$$

(37)

Because of $\tilde{Q}(0) < \infty$, the expression in curly brackets of equation (36) should be equal
to zero, and we easily find the unknown constant $c_0$:

$$
c_0 = \frac{1}{2\pi} \int_{L_1}^{L_2} \frac{dy}{y^2 + iD - a^2}.
$$

(38)

Substitution of equations (36) and (38) into equation (28) gives

$$
T(x_0) = \frac{1}{\pi} \text{Re} \left\{ \int_{0}^{\infty} \frac{\text{e}^{ikx_0}}{k} \, dk \int_{L_1}^{L_2} \frac{e^{-iky} - e^{-\beta k - i\gamma k}}{y^2 + iD - a^2} \, dy \right\}.
$$

(39)
Barrier crossing with Lévy flights

Figure 2. NLRT (au) as a function of the noise intensity $D$ for three values of the initial position of the particle, namely: $x_0 = -2.0, -1.0$ and 0. The parameter values are the same as in figure 1.

After changing the order of integration and evaluation of the integral on $k$ we arrive at

$$T(x_0) = \frac{1}{\pi} \int_{L_1}^{L_2} \left\{ \frac{D}{2} \ln[A(x_0, y)] + y_2 B(x_0, y) \right\} \frac{dy}{y_2^2 + D^2},$$

(40)

where

$$A(x_0, y) = \frac{\beta^2 + (x_0 - \gamma)^2}{(x_0 - y)^2}; \quad y_2 = y^2 - a^2;$$

(41)

and

$$B(x_0, y) = \arctan \left( \frac{x_0 - \gamma}{\beta} \right) - \frac{\pi}{2} \text{sgn}(x_0 - y).$$

(42)

In figure 1 we report the behavior of the nonlinear relaxation time $T(x_0)$, calculated by equation (40), as a function of the initial position of the particle for different values of the noise intensity $D$, namely $D = 0.07, 0.35, 1.0, 3.0$ and 5.0.

The potential parameter $a$ (see equation (32)) is $a = 1$ and the interval boundaries are $L_1 = -10$ and $L_2 = +10$. The integration step used to calculate $T(x_0)$ from equation (40) is $\Delta y = 10^{-4}$. For the initial position of the particle we focus on the range of values around the potential well, that is we consider $x_0 \in [-2, +1]$. A monotonic decreasing behavior of the nonlinear relaxation time is shown. The NLRT decreases with initial positions moving from the left of the minimum ($x_0 = -1$) towards the maximum ($x_0 = +1$) of the potential and with increasing noise intensity. An overlap of the different curves appears near the maximum of the potential. This behavior could be ascribed to the role of initial positions near the maximum. For initial positions that are close to the maximum of the potential


$(x_0 = 1)$ the height of the barrier to cross decreases considerably and the probability of the particle to fall back into the potential well increases. For the role of the initial conditions in barrier crossing, with Gaussian noise, see [10, 11].

In figure 2 we report the log–log plot of the behavior of the NLRT as a function of the noise intensity $D$, for three initial positions of the particle, namely: $x_0 = -2.0, -1.0$ and 0. As we can see, the decreasing behavior of the NLRT with increasing noise intensity is recovered (see [21]).

4. Conclusions

In this paper we obtain the general differential equation useful to calculate the nonlinear relaxation time for a particle moving in a cubic potential and with an arbitrary Lévy index $\alpha$. For Cauchy noise ($\alpha = 1$) we obtain the closed expression in quadrature of the NLRT as a function of the noise intensity, the initial position and the parameters of the potential. A monotonic behavior of the NLRT as a function of the initial position of the particle is obtained in this case. For free anomalous diffusion the NLRT decreases monotonically with the noise intensity as in the presence of the cubic potential.

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