Excision technique in constrained formulations of Einstein equations: collapse scenario.

I Cordero-Carrión¹,², N Vasset², J Novak² and J L Jaramillo³

¹ Department of Applied Mathematics, University of Valencia, C/ Dr. Moliner 50, E-46100 Burjassot, Spain.
² Laboratoire Univers et Théories (LUTH), Observatoire de Paris/CNRS/Université Paris Diderot, 5 place Jules Janssen, F-92190 Meudon, France.
³ Université de Bretagne Occidentale, 6 ave. Le Gorgeu, CS 93837, 29238 Brest Cedex 3, France

E-mail: isabel.cordero@uv.es

Abstract. We present a new excision technique used in constrained formulations of Einstein equations to deal with black hole in numerical simulations. We show the applicability of this scheme in several scenarios. In particular, we present the dynamical evolution of the collapse of a neutron star to a black hole, using the CoCoNuT code and this excision technique.

1. Introduction

When black hole (BH) are present in numerical simulations, one has to handle the physical singularity, where some fields may diverge. One option is the excision technique, where the singularity, together with its neighborhood, is removed from the computational domain. In the constrained formulations of Einstein equations, as in the case of the so-called Fully Constrained formulation (FCF) [1], the elliptic nature of the resulting partial differential equations (PDEs) require boundary conditions at the excision surface which have to be defined and tested.

A geometric approach has been undertaken on the basis of the isolated horizon paradigm [2] or the dynamical “trapping horizon” concept [3]. They have been successfully applied to stationary spacetimes [4], which can then be used as initial data for further dynamical evolutions. When no symmetry is assumed, the apparent horizon (AH) does not have a simple shape in general and we suggest using an arbitrary but nearby sphere inside the AH to define simple and appropriate boundary conditions. We consider here spherically symmetric spacetimes, but we plan to follow similar ideas in more general spacetimes.

We use geometrical units. Greek (latin) indices run from 0 (1) to 3. We adopt the standard convention for the summation over repeated indices. \( \partial_\alpha \) denotes partial derivatives.

2. Fully Constrained Formulation

The metric line element in the 3+1 formalism can be written as

\[
g_{\mu\nu}dx^\mu dx^\nu = -N^2dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),
\]

where \( N \) is the lapse function, \( \beta^i \) is the shift vector, \( \gamma_{ij} = \psi^4 \tilde{\gamma}_{ij} \) is the 3-metric onto the spatial hypersurfaces, being \( \psi \) the conformal factor and \( \tilde{\gamma}_{ij} \) the conformal 3-metric. The difference
between the conformal and flat metrics is denoted by \( h_{ij} \). \( K_{\mu\nu} = -\frac{1}{2} \mathcal{L}_n \gamma_{\mu\nu} \) is the extrinsic curvature. As gauge conditions, we impose the maximal slicing and the generalized Dirac gauge (see [5] for more details), for which the resulting Einstein equations are formed by a set of elliptic equations for \( N, \beta, \psi \), and a set of hyperbolic equations for \( h_{ij} \) and the components of a conformal traceless extrinsic curvature \( \tilde{A}^{ij} = \psi^{10} K^{ij} \). In the case of spherical symmetry and with topologically \( \mathbb{R}^3 \) spatial hypersurfaces, the FCF reduces to the isotropic gauge, where \( h_{ij} = 0 \). This is not true anymore for more general topologies. In a nonconvex one such as \( \mathbb{R}^3 - B \), where \( B \) is a ball, the expression for \( \tilde{\gamma}_{ij} \) involves an extra real function \( \omega(t) \); in \( \mathbb{R}^3, \omega = 0 \), but in general this function can be chosen arbitrarily and depends on the given value of the metric at the excision surface. Setting \( \omega = 0 \) implies \( h_{ij} = 0 \) and provides a time-independent prescription for \( \tilde{A}^{ij} \) and a redundant condition in the bulk for the hyperbolic equations; then, we only have to solve the elliptic equations.

3. Excision method
Excision consists in removing from each spatial hypersurface the open interior of a topological sphere. We choose this sphere to be located strictly inside the AH of the modeled BH region, encoded as boundary conditions of the elliptic equations to be solved for. We denote by \( s^i \) the unit outward-directed spacelike vector normal to the spheres, by \( b \) the contraction \( \beta^i s_i \), and by \( \theta \) the scalar outward expansion. \( \theta \) can be expressed in terms of \( s^i, \psi \), and the Levi-Civita connection associated with the conformal 3-metric and could be used as a boundary condition on \( \psi \) to compute initial data. \( \theta \) can be prescribed to be negative to place the excision surface inside the AH. One can choose freely the value of \( N \) at the excision surface (differential gauge condition) for the initial data; in a dynamical evolution it has to be consistent with the assumption \( h_{ij} = 0 \).

We have in mind the BH formation in stellar gravitational collapse simulations. Starting from a simulation for regular data, a trapped region forms at a given time; shortly after this moment, one performs excision inside the trapped region. We choose the excision surface to be located strictly inside the trapped region, where the metric components are determined by the previous evolution and are employed as initial values, and the scalar expansion \( \theta \) is generically (and on average) negative. Once the initial excised surface has been chosen, one needs to determine a geometrical prescription for the evolution of the excision surface in time. Imposing \( b \) to be constant in time at the excised surface provides us with a simple boundary condition for \( \beta^i \). The trace part of the evolution equations gives a consistent time evolution for \( \psi \), which provides an additional coherent boundary condition for this variable and allows to solve the corresponding elliptic equation. The value of \( N \) at the excision surface is given as a consequence of the choice \( h_{ij} = 0 \) from the evolution equations. During the numerical simulations, we check that \( (b - N) \geq 0 \). With these boundary conditions, all elliptic equations can be solved in the numerical domain for all times, and no evolution equations are solved in the bulk.

We found in our numerical simulations that the metric exponentially converges to a stationary solution for intrinsic stationary spacetimes. Under some reasonable assumptions (see [5] for more details), one can understand this numerical behavior and check that the coordinates adapt to the stationarity of the spacetime.

4. Numerical results
The excision method already presented for the FCF works well in practice in a numerical code, as we show here in two toy models, the evolution of a Schwarzschild BH and the accretion of a massless scalar field, and also in a full simulation of the collapse of a neutron star to a BH in spherical symmetry.
Figure 1. Evolution in terms of coordinate time $t$ of $N$ minus its time asymptotic value $N_{\text{lim}} \approx 0.549$ (left top panel, logarithmic scale); of $(b-N)$ (right top panel); of $\psi$ (left bottom panel), at the excision surface ($r = 0.916 \, M_{\text{ADM}}$); and of the AH coordinate radius $r_*$ (right bottom panel), for a spherically symmetric vacuum (i.e. Schwarzschild) BH spacetime.

4.1. Setup
We use spherical (polar) coordinates. Since we are imposing $h^{ij} = 0$ and considering spherically symmetric spacetimes, we only need to solve for $N$, $\beta^r$ and $\psi$.

In the first two scenarios we start from an existing BH, with the excision sphere located at the coordinate radius $r = 1$. Therefore, we need to construct initial data at $t = 0$ by solving the elliptic set of equations, with the stress-energy tensor being either zero (vacuum) in the Schwarzschild BH evolution, or given by corresponding expressions in the case of the scalar field accretion (see corresponding subsection below). The initial excision surface is chosen as a sphere with a given (arbitrary) value of the expansion $\theta$, prescribed to be initially negative. This guarantees that the initial excision surface is inside the AH, which then is located using an AH finder. Initial data boundary conditions are needed for the elliptical system and are taken as follows: setting an initial value for $\theta$, a boundary condition for $\psi$ is obtained; the boundary value of $N$ is fixed yielding a Dirichlet condition; a value is prescribed for $(b-N)$, from which one can get a Dirichlet boundary condition for the radial component of the shift vector $\beta^r$, the other two components being zero in spherical symmetry. The elliptic system is solved iteratively with the library LORENE [6].

In the third scenario (neutron star collapse to a BH), initial data are obtained solving Einstein equations, coupled to the fluid equilibrium equations in the isotropic gauge, in whole space. The numerical approach is very similar to the isolated BH case, but with no inner boundary condition imposed.
4.2. Evolution of a Schwarzschild black hole

A spherically symmetric spacetime is computed for \( r \geq 1 \), with the previously defined boundary conditions, and the specific values \( \theta|_{t=0} = -0.01 \), \( N|_{t=0, r=1} = 0.55 \) and \( (b - N)|_{t=0} = 0.01 \) for these initial data. Once having solved the corresponding elliptic system, one verifies that this setting induces a nonzero value for the Arnowitt-Deser-Misner (ADM) mass, \( M_{\text{ADM}} \approx 1.09 \). Numerically, it is obtained in isotropic gauge from the asymptotic behavior of the conformal factor (see e.g. [1]). This causes the excision sphere to be located at \( r \approx 0.916 M_{\text{ADM}} \). Using an AH finder, we find in the initial data an AH located at the coordinate radius \( r^* \approx 0.94 M_{\text{ADM}} \). This is evidence that the initial data represent a BH in spherical symmetry. This BH spacetime is then numerically evolved in time. This is obviously only a gauge evolution, since the spacetime is Schwarzschild by construction. The time evolution of the metric variables at the excision surface (\( r = 1 \)) as well as the coordinate radius \( r^* \) of the AH, in the interval \( 0 \leq t \leq 50 \), are displayed in Fig. 1. An exponential convergence toward stationary values is observed for all metric quantities, with an explicit behavior shown for the lapse \( N \). From the top right panel of Fig. 1, one can check that the difference \( (b - N) \) remains positive. In left panel of Fig. 2 the time evolution of the expansion \( \theta \) is displayed, which is decreasing during the simulation and always remains negative, ensuring that the excision surface always remains inside the AH.

The scheme appears stable (we have run it for \( t \sim 1000 \)) and, in order to check its accuracy, we have monitored the variation of the AH irreducible mass, \( M_{\text{AH}} \), determined by the AH finder. The conservation of \( M_{\text{AH}} \) is displayed in right panel of Fig. 2. The conservation of the ADM mass has also been checked, with quantitatively very similar results to the conservation of the AH surface. Finally, a second-order convergence of these conserved quantities has been obtained numerically while decreasing the time step, in agreement with the implemented second-order Adams-Bashforth scheme. The exponential convergence obtained for the chosen initial data, given by the specific values of \( \theta|_{t=0}, N|_{t=0, r=1} \) and \( (b - N)|_{t=0} \), is independent of these initial data. A similar behavior is found for different initial values.

4.3. Accretion of a massless scalar field

In this case we evolve a BH spacetime with energy content in the form of a minimally coupled massless scalar field in spherical symmetry. The massless Klein-Gordon equation, or (simply) the wave scalar equation, is given by \( \nabla^\mu \nabla_\mu \phi = 0 \), where \( \nabla \) is the Levi-Civita connection associated with the spacetime metric. The stress-energy tensor associated with this scalar field is given by

\[
T_{\mu \nu} = \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} \gamma_{\mu \nu} \nabla_\rho \phi \nabla^\rho \phi.
\]
The wave equation can be rewritten as a first-order system in space and time, by introducing auxiliary variables, and solved using a second-order Adams-Bashforth scheme (see more details in [5]). No boundary condition is needed for the field at the excision surface, as all the characteristics are directed out of the computational domain as long as the excision surface is spacelike, which is verified in our case with \((b - N) > 0\) in our numerical simulations.

Initial data are given by a Gaussian profile for the scalar field outside the excision surface. In the evolution of these initial data a fraction of the scalar field is radiated away, while the other part is accreted onto the BH; its time evolution at the excision surface is given in left panel of Fig. 3. The metric quantities follow a similar evolution with respect to the vacuum case and, once the scalar field has been accreted to the BH, they settle rapidly to stationary values. Right panel of Fig. 3 shows the evolution of the AH mass as a function of the coordinate time. As expected, the AH grows in time while accreting energy from the scalar field, before reaching a stationary limit. This limit does not represent all of the ADM mass of the spacetime, as part of this asymptotic mass is still contained in the scalar field traveling to higher radii. The accuracy of the code has been checked by monitoring the variation of the ADM mass, which conservation is obtained up to the level \(10^{-6}\). The excision boundary conditions allow us to study the growth of a spherically symmetric accreting BH in a stable and accurate way.

### 4.4. Collapse of a neutron star to a black hole

We here describe a simulation in spherical symmetry, starting from an unstable static neutron star, up to the formation of a BH and its accretion of all matter into the horizon. Excision is switched on during the simulation, after the AH is formed. We have modified the code CoCoNuT [7], so that it uses the excision technique described above. This code solves the general-relativistic Euler equations, with Einstein equations in isotropic gauge. There is no need for boundary conditions for the Euler equations at the excision surface, as all the characteristics point out of the numerical integration domain. The initial data consists of a static neutron star, computed in isotropic gauge with a polytropic equation of state.

The collapse proceeds until the formation of an AH. Excision is switched on when 85% of the total baryon mass has entered the horizon (changing this value has little influence). The excision radius is defined as a fraction of the AH radius \(r_*\) between 0.9 and 0.98 (changing this value does not influence the physical results). Density profiles at various time steps around the time when excision is started are given in left panel of Fig. 4. The evolution of the BH irreducible mass is shown in right panel of Fig. 4. The whole simulation is stable. The main error in the
conservation of the ADM mass comes from the numerical solution of the Euler equations. The relative change in the ADM mass after excision has been started is of the order of $10^{-8}$, similar to previous tests.

5. Conclusions
In this work we have presented a new excision technique for the dynamical evolution of spherically symmetric spacetimes in the FCF in order to numerically simulate systems forming a BH, like the stellar collapse to a BH. In this case, we have to deal with boundary conditions for elliptic-type PDEs. We have checked the practical applicability of this approach in several cases, showing stable and accurate numerical results. We plan to extend this approach to spacetimes with less symmetries in forthcoming studies.

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Figure 4. Radial profiles of the rest-mass density, for different moments around the start of the excision in the simulation of the collapse of a neutron star to a BH (left panel), where excision starts at $t = 0.43788$ ms, and evolution in terms of coordinate time $t$ of the AH irreducible mass for the collapse of a neutron star to a BH (right panel), where the vertical dotted line indicates the moment when excision is turned on.