AN $L^2$ DECOUPLING INTERPRETATION OF EFFICIENT CONGRUENCING IN 2D

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Abstract. We give a new proof of $L^2$ decoupling for the parabola inspired from efficient congruencing. Making quantitative this proof matches a bound obtained by Bourgain in [1] for the discrete restriction problem in two dimensions. We illustrate similarities and differences between this new proof and efficient congruencing and the proof of decoupling by Bourgain and Demeter. We also show where tools from decoupling such as $L^2$ decoupling, Bernstein, and bilinear Kakeya come into play.

1. Introduction

For an interval $J \subset [0, 1]$ and $g : [0, 1] \to \mathbb{C}$, let

$$(\mathcal{E}_J g)(x) := \int_J g(\xi)e(\xi x_1 + \xi^2 x_2) d\xi$$

where $e(a) := e^{2\pi i a}$. For an interval $I$, let $P_\ell(I)$ be the partition of $I$ into intervals of length $\ell$. By writing $P_\ell(I)$, we are assuming that $|I| = \ell \in \mathbb{N}$. We will also similarly define $P_\ell(B)$ for squares $B$ in $\mathbb{R}^2$. Next if $B = B(c, R)$ is a square in $\mathbb{R}^2$ centered at $c$ of side length $R$, let

$$w_B(x) := (1 + \frac{|x - c|}{R})^{-100}.$$ 

We will always assume that our squares have sides parallel to the $x$ and $y$-axis. We observe that $1_B \leq 2^{100}w_B$. For a function $w$, we define

$$\|f\|_{L^p(w)} := (\int_{\mathbb{R}^2} |f(x)|^p w(x) dx)^{1/p}.$$ 

For $\delta \in \mathbb{N}^{-1}$, let $D(\delta)$ be the best constant such that

$$\|\mathcal{E}_{[0,1]} g\|_{L^p(B)} \leq D(\delta)(\sum_{J \in P_\delta([0,1])} \|\mathcal{E}_J g\|_{L^p(w_B)})^{1/2}$$

for all $g : [0, 1] \to \mathbb{C}$ and all squares $B$ in $\mathbb{R}^2$ of side length $\delta^{-2}$. Let $D_p(\delta)$ be the decoupling constant where the $L^6$ in (1) is replaced with $L^p$. Since $1_B \leq w_B$, the triangle inequality combined with Cauchy-Schwarz shows that $D(\delta) \leq \delta^{-1/2}$. The $L^2$ decoupling inequality for the paraboloid proven by Bourgain and Demeter in [3] implies that for the parabola we have $D_p(\delta) \leq \delta^{-\epsilon}$ for $2 \leq p \leq 6$ and this range of $p$ is sharp.

Decoupling-type inequalities were first studied by Wolff in [14]. Following the proof of $L^2$ decoupling for the paraboloid by Bourgain and Demeter in [3], decoupling has found numerous applications to analytic number theory and dispersive PDE (see for example [2, 5, 7, 8, 9]). Most notably is the proof of Vinogradov’s mean value theorem using decoupling for the moment curve $t \mapsto (t, t^2, \ldots, t^n)$ in [5].
Wooley in [15] was also able to prove Vinogradov’s mean value theorem using his nested efficient congruencing method.

This paper attempts to probe the connections between efficient congruencing and \( l^2 \) decoupling in the simplest case of the parabola. Our proof of \( l^2 \) decoupling for the parabola is inspired by the exposition of efficient congruencing in Pierce’s Bourbaki seminar exposition [12]. This proof will give the following result.

**Theorem 1.1.** For \( \delta \in \mathbb{N}^{-1} \) such that \( 0 < \delta < e^{-200 \log \epsilon} \), we have

\[
D(\delta) \leq \exp\left(30 \frac{\log \frac{1}{\delta}}{\log \log \frac{1}{\delta}}\right).
\]

This improves upon a previous result of the author [11] where it was obtained that for \( \delta \) sufficiently small, we have

\[
D(\delta) \leq \exp(O(\frac{\log N}{\log \log N})), \tag{2}
\]

This result was obtained by carefully working through the proof in [4] and optimizing in \( \epsilon \).

In the context of discrete Fourier restriction, Theorem 1.1 implies that for all \( N \) sufficiently large and arbitrary sequence \( a_n \subset l^2 \), we have

\[
\| \sum_{|n| \leq N} a_n e^{2\pi i(nx+n^2t)} \|_{L^6(T^2)} \leq \exp(O(\frac{\log N}{\log \log N}))(\sum_{|n| \leq N} |a_n|^2)^{1/2}
\]

which rederives (up to constants) the upper bound obtained by Bourgain in [1, Proposition 2.36] but without resorting to using a divisor bound. It is an open problem whether the \( \exp(O(\frac{\log N}{\log \log N})) \) can be improved.

1.1. More notation. We will let \( \eta \) be a Schwartz function such that \( \eta \geq 1_{B(0,1)} \) and \( \text{supp}(\hat{\eta}) \subset B(0,1) \). For \( B = B(c,R) \) we also define \( \eta_B(x) := \eta(\frac{x-c}{R}) \). Since we care about explicit constants in Section 2, we will use the explicit \( \eta \) constructed in [11, Corollary 2.9]. In particular, for this \( \eta \), \( \eta_B \leq 10^{2400} w_B \). For the remaining sections, we will ignore this constant. We refer the reader to [4, Section 4] and [11, Section 2] for some useful properties of the weight \( w_B \) and \( \eta_B \).

Finally we define

\[
\|f\|_{L^p_p(B)} := \left( \frac{1}{|B|} \int_B |f(x)|^p \, dx \right)^{1/p}
\]

and given a collection \( C \) of squares, we let

\[
\text{Avg}_C f(\Delta) := \frac{1}{|C|} \sum_{\Delta \in C} f(\Delta).
\]

1.2. Outline of proof of Theorem 1.1. Our argument is inspired by the discussion of efficient congruencing in [12, Section 4] which in turn is based off Heath-Brown’s simplification [10] of Wooley’s proof of the cubic case of Vinogradov’s mean value theorem [16].

Our first step, much like the first step in both 2D efficient congruencing and decoupling, is to bilinearize the problem. Throughout we will assume \( \delta^{-1} \in \mathbb{N} \) and \( \nu \in \mathbb{N}^{-1} \cap (0, 1/100) \).

Fix arbitrary integers \( a, b \geq 1 \). Suppose \( \delta \) and \( \nu \) were such that \( \nu^a \delta^{-1}, \nu^b \delta^{-1} \in \mathbb{N} \). This implies that \( \delta \leq \min(\nu^a, \nu^b) \) and the requirement that \( \nu^\max(a,b) \delta^{-1} \in \mathbb{N} \) is
Lemma 1.5. Suppose more details.

\[ \int_B |E_1g|^2 |E_2g|^4 \leq M_{a,b}(\delta, \nu) \left( \sum_{j \in \mathbb{P}_a(I)} ||E_1g||_{L^3(w_B)}^3 \right) \left( \sum_{j' \in \mathbb{P}_b(I')} ||E_2g||_{L^3(w_B)}^3 \right) \]

for all squares \( B \) of side length \( \delta^{-2} \), \( g : [0, 1] \to \mathbb{C} \), and all intervals \( I \in \mathbb{P}_\nu([0,1]) \), \( I' \in \mathbb{P}_\nu([0,1]) \) with \( d(I, I') \geq 3\nu \). We will say that such \( I \) and \( I' \) are \( 3\nu \)-separated.

Applying Hölder followed by the triangle inequality and Cauchy-Schwarz shows that \( M_{a,b}(\delta, \nu) \) is finite. This is not the only bilinear decoupling constant we can use (see (28) and (32) in Sections 4 and 5, respectively), but in this outline we will use (3) because it is closest to the one used in [12] and the one we will use in Section 2.

Our proof of Theorem 1.1 is broken into the following four lemmas. We state them below ignoring explicit constants for now.

**Lemma 1.2 (Parabolic rescaling).** Let \( 0 < \delta < \sigma < 1 \) be such that \( \sigma, \delta, \sigma/\delta \in \mathbb{N}^{-1} \). Let \( I \) be an arbitrary interval in \([0, 1]\) of length \( \sigma \). Then

\[ ||E_Ig||_{L^3(B)} \leq D(\frac{\delta}{\sigma}) \left( \sum_{j \in \mathbb{P}_\nu(I)} ||E_jg||_{L^3(w_B)}^3 \right)^{1/2} \]

for every \( g : [0, 1] \to \mathbb{C} \) and every square \( B \) of side length \( \delta^{-2} \).

**Lemma 1.3 (Bilinear reduction).** Suppose \( \delta \) and \( \nu \) were such that \( \nu \delta^{-1} \in \mathbb{N} \). Then

\[ D(\delta) \leq D(\frac{\delta}{\nu}) + \nu^{-1} M_{1,1}(\delta, \nu). \]

**Lemma 1.4.** Let \( a \) and \( b \) be integers such that \( 1 \leq a \leq 2b \). Suppose \( \delta \) and \( \nu \) were such that \( \nu^{2b} \delta^{-1} \in \mathbb{N} \). Then

\[ M_{a,b}(\delta, \nu) \leq \nu^{-1/6} M_{2b,b}(\delta, \nu). \]

**Lemma 1.5.** Suppose \( b \) is an integer and \( \delta \) and \( \nu \) were such that \( \nu^{2b} \delta^{-1} \in \mathbb{N} \). Then

\[ M_{2b,b}(\delta, \nu) \leq M_{b,2b}(\delta, \nu)^{1/2} D(\frac{\delta}{\nu^b})^{1/2}. \]

Applying Lemma 1.4, we can move from \( M_{1,1} \) to \( M_{2,1} \) and then Lemma 1.5 allows us to move from \( M_{2,1} \) to \( M_{1,2} \) at the cost of a square root of \( D(\delta/\nu) \). Applying Lemma 1.4 again moves us to \( M_{2,4} \). Repeating this we can eventually reach \( M_{2N-1,2N} \) paying some \( O(1) \) power of \( \nu^{-1} \) and the value of the linear decoupling constants at various scales. This combined with Lemma 1.3 and the choice of \( \nu = \delta^{1/2N} \) leads to the following result.

**Lemma 1.6.** Let \( N \in \mathbb{N} \) and suppose \( \delta \) was such that \( \delta^{-1/2N} \in \mathbb{N} \) and \( 0 < \delta < 100^{-2N} \). Then

\[ D(\delta) \leq D(\delta^{1-\frac{1}{2N}}) + \delta^{-\frac{1}{2N}} D(\delta^{1/2}) \frac{1}{2N} \prod_{j=0}^{N-1} D(\delta^{1-\frac{j}{2N}}) \frac{1}{2N}. \]

This then gives a recursion which shows that \( D(\delta) \leq \epsilon \delta^{-\delta} \) (see Section 2.3 for more details).

The proof of Lemma 1.2 is essentially a change of variables and applying the definition of the linear decoupling constant (some technical issues arise because of the weight \( w_B \), see [11, Section 4]). The idea is that a cap on the paraboloid can be
stretched to the whole paraboloid without changing any geometric properties. The bilinear reduction Lemma 1.3 follows from Hölder’s inequality. The argument we use is from Tao’s exposition on the Bourgain-Demeter-Guth proof of Vinogradov [13]. In general dimension, the multilinear reduction follows from a Bourgain-Guth argument (see [6] and [4, Section 8]).

Lemma 1.4 is the most technical of the four lemmas and is where we use a Fefferman-Cordoba argument in Section 2. It turns out we can still close the iteration with Lemma 1.4 replaced by $M_{a,b} \lesssim M_{b,b}$ for $1 \leq a < b$ and $M_{b,b} \lesssim \nu^{-1/6} M_{2b,b}$. Both these estimates come from the same proof of Lemma 1.4 and is how we approach the iteration in Sections 3 and 4 (see Lemmas 3.3 and 3.5 and their rigorous counterparts Lemmas 4.10 and 4.11). The proof of these lemmas is a consequence of $\ell^2 L^2$ decoupling and bilinear Kakeya.

We note that as $a$ and $b$ get larger and larger the estimate in Lemma 1.4 gets better and better than the trivial bound of $M_{a,b} \lesssim \nu^{-(2b-a)/6} M_{2b,b}$. The $\nu^{-1/6}$ comes from the $\nu$-transversality of $I_1$ and $I_2$ in the definition of $M_{a,b}$. In particular, should be viewed as $(\nu^{-(2-1)})^{1/6}$ where the $1/6$ comes from that we are working in $L^6$ and the $\nu^{-(2-1)}$ comes from $\nu^{-(d-1)}$ with $d = 2$ which is the power of $\nu$ arising from multilinear Kakeya. Finally, Lemma 1.5 is an application of Hölder and parabolic rescaling.

1.3. Comparison with 2D efficient congruencing as in [12, Section 4]. The main object of iteration in [12, Section 4] is the following bilinear object

$$I_1(X; a, b) = \max_{\xi \not\equiv \eta \pmod{p}} \int_{(0,1)^4} | \sum_{\substack{1 \leq x, y \leq X \atop x \equiv \xi \pmod{p^n}}} e(\alpha_1 x + \alpha_2 x^2)|^2 \sum_{\substack{1 \leq y \leq X \atop y \equiv \eta \pmod{p^n}}} e((\alpha_1 y + \alpha_2 y^2))^{4/3} d\alpha.$$  

Lemma 1.2-1.5 correspond directly to Lemmas 4.2-4.5 of [12, Section 4]. The observation that Lemmas 4.2 and 4.3 of [12] correspond to parabolic rescaling and bilinear reduction, respectively was already observed by Pierce in [12, Section 8].

We can think of $p$ as $\nu^{-1}$, $J(X)$ as $D(\delta)$, and $I_1(X; a, b)$ as $M_{a,b}(\delta, \nu)^6$. In the definition of $I_1$, the max $\max_{\xi \not\equiv \eta \pmod{p}}$ condition can be thought of as corresponding to the transversality condition that $I_1$ and $I_2$ are $\nu$-transverse (or since we are in 2D, $\nu$-separated) intervals of length $\nu$. The integral over $(0,1)^2$ corresponds to an integral over $B$. Finally the expression

$$| \sum_{\substack{1 \leq x \leq X \atop x \equiv \xi \pmod{p^n}}} e(\alpha_1 x + \alpha_2 x^2),$$

can be thought of as corresponding to $|\mathcal{E}_I g|$ for $I$ an interval of length $\nu^a$ and so the whole of $I_1(X; a, b)$ can be thought of as $\int_B |\mathcal{E}_I g|^2 |\mathcal{E}_{I'} g|^2$ where $\ell(I_1) = \nu^b$ and $\ell(I_2) = \nu^b$ with $I_1$ and $I_2$ are $O(\nu)$-separated. This will be our interpretation in Section 2.

Interpreting the proof of Lemma 1.4 using the uncertainty principle, we reinterpret $I_1(X; a, b)$ as (ignoring weight functions)

$$\text{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_I g\|_{L^2_B(\Delta)}^2 \|\mathcal{E}_{I'} g\|_{L^4_B(\Delta)}^4$$

(4)

where $I$ and $I'$ are length $\nu^a$ and $\nu^b$, respectively and are $\nu$-separated. The uncertainty principle says that (4) is essentially equal to $\frac{1}{|B|} \int_B |\mathcal{E}_I g|^2 |\mathcal{E}_{I'} g|^4$. 


Finally in Section 5 we replace (4) with
\[
\text{Avg}_{\Delta \in P_{\nu^{-1}}(B)} \left( \sum_{J \in P_{\sigma}(I)} \|E_J g\|_{L_2^2(\Delta)} \right)^2 \text{Avg}_{\Delta' \in P_{\nu^{-1}}(I')} \left( \sum_{J' \in P_{\sigma}(I')} \|E_{J'} g\|_{L_2^2(\Delta')} \right)^2
\]
where \(I\) and \(I'\) are length \(\nu\) and \(\nu\)-separated. Note that when \(b = 1\) this then is exactly equal to \(\frac{1}{|B|} \int_B |E_J g|^2 |E_{J'} g|^4\). The interpretation given above is now similar to the \(A_p\) object studied by Bourgain-Demeter in [4].

These comparisons should just be taken as an association of ideas. For example if one were to take the comparisons made above as literal, then our Lemma 1.4 should have read \(M_{a,b} \lesssim \nu^{-(-2b-a)/6}M_{2b,b}\) which is the trivial bound. However the same ideas as in the proof of [12, Lemma 4.4] show that \(M_{a,b} \lesssim \nu^{-1/6}M_{2b,b}\).

1.4. Comparison with 2D \(l^2\) decoupling as in [4]. Let \(M^{(2,4)}_{a,b}(\delta, \nu)\) be the bilinear constant defined in (3). Let \(M^{(3,3)}_{1,1}(\delta, \nu)\) be the bilinear constant defined as in (3) with \(a = b = 1\) except instead we use the true geometric mean. This latter bilinear decoupling constant is the one used by Bourgain and Demeter in [4].

The largest difference between our proof and the Bourgain-Demeter proof is how we iterate. Both proofs obtain that
\[
D(\delta) \lesssim D\left(\frac{\delta}{\nu}\right) + \nu^{-1}M^{(s,6-s)}_{1,1}(\delta, \nu)
\]
where \(s = 3\) corresponds to the Bourgain-Demeter proof while \(s = 2\) corresponds to our proof. However we proceed to analyze the iteration slightly differently. Bourgain-Demeter applies (5) to \(D(\delta/\nu)\) and \(D(\delta/\nu^2)\) to obtain
\[
D(\delta) \lesssim D\left(\frac{\delta}{\nu^2}\right) + \nu^{-1}(M^{(3,3)}_{1,1}\left(\frac{\delta}{\nu}, \nu\right) + M^{(3,3)}_{1,1}(\delta, \nu))
\]
\[
\lesssim D\left(\frac{\delta}{\nu^2}\right) + \nu^{-1}(M^{(3,3)}_{1,1}\left(\frac{\delta}{\nu^2}, \nu\right) + M^{(3,3)}_{1,1}(\delta, \nu) + M^{(3,3)}_{1,1}(\delta, \nu))
\]
and we continue to iterate until \(\delta/\nu^{2^n}\) is of size 1. It now remains to analyze \(M^{(3,3)}_{1,1}(\delta, \nu)\) for various scales \(\delta\) which is done by the \(A_p\) expressions that are used in [4].

For our proof, in two steps (of applying Lemmas 1.4 and 1.5) we obtain
\[
D(\delta) \lesssim D\left(\frac{\delta}{\nu^2}\right) + \nu^{-7/6}M^{(2,4)}_{1,2}(\delta, \nu)^{1/2}D\left(\frac{\delta}{\nu^2}\right)^{1/2}
\]
\[
\lesssim D\left(\frac{\delta}{\nu^2}\right) + \nu^{-5/4}M^{(2,4)}_{2,4}(\delta, \nu)^{1/4}D\left(\frac{\delta}{\nu^2}\right)^{1/4}D\left(\frac{\delta}{\nu}\right)^{1/2}
\]
and we continue to iterate \(\delta/\nu^{2^n}\) is of size 1. Of note in our iteration, we do not lose the \(\delta^{-1/2^{m+1}}\) which occurs when passing from the multilinear constant to the \(A_p\) objects as in Bourgain-Demeter (see [4, Page 199] and [11, Lemma 7.4]). This allows us to tackle the endpoint directly and obtain a slightly better bound at the endpoint (compare (2) with Theorem 1.1).

1.5. Comparison of the iteration in Section 2 and 4. The way we iterate in Section 2 will be slightly different than how we iterate in Section 4. In Section 2, we first apply the trivial bound \(M_{1,1} \lesssim \nu^{-1/6}M_{1,2}\). Then Lemmas 1.4 and 1.5 imply that for integer \(b \geq 2\),
\[
M_{b/2,b}(\delta, \nu) \lesssim \nu^{-1/6}M_{b,2b}(\delta, \nu)^{1/2}D\left(\frac{\delta}{\nu^2}\right)^{1/2}.
\]
Thus from this we can access $M_{2^{N-1}, 2^N}$ for arbitrary large $N$ but lose only $\nu^{-O(1)}$. In contrast, for Section 4, we use that $M_{a,b} \lesssim M_{b,b}$ for $1 \leq a < b$ (from $l^2L^2$ decoupling) and $M_{b,b} \lesssim \nu^{-1/6}M_{2b,b}$ (from bilinear Kakeya). Combining these two inequalities with Lemma 1.5 gives that for integer $b \geq 1$,

$$M_{b,b}(\delta, \nu) \lesssim \nu^{-1/6}M_{2b,2b}(\delta, \nu)^{1/2}D\left(\frac{\delta}{\nu}\right)^{1/2}.$$ 

Now we can access the constant $M_{2^N, 2^N}$ for arbitrary large $N$ but lose only $\nu^{-O(1)}$. Both iterations give similar quantitative estimates.

1.6. Overview of paper. Theorem 1.1 will be proved in Section 2 via a Fefferman-Cordoba argument. This argument does not generalize to proving that $D_p(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ except for $p = 4, 6$. However in Section 3, by the uncertainty principle we reinterpret a key lemma from Section 2 (Lemma 2.7) which allows us to generalize the argument in Section 2 so that it can work for all $2 \leq p \leq 6$. We make this completely rigorous in Section 4 by defining a slightly different (but morally equivalent) bilinear decoupling constant. This will make use of $l^2L^2$ decoupling, Bernstein’s inequality, and bilinear Kakeya. A basic version of the ball inflation inequality similar to that used in [4, Theorem 9.2] and [5, Theorem 6.6] makes an appearance. Finally in Section 5, we reinterpret the argument made in Section 4 and write an argument that is more like that given in [4]. We create a 1-parameter family of bilinear constants which in some sense “interpolate” between the Bourgain-Demeter argument and our argument here.

The three arguments in Sections 2-5 are similar but will use slightly different bilinear decoupling constants. We will only mention explicit constants in Section 2. In Sections 4 and 5, for simplicity, we will only prove that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$. The estimates in those sections can be made explicit by using explicit constants obtained from [11]. Because the structure of the iteration in Sections 4 and 5 is the same as that in Section 2, we obtain essentially the same quantitative bounds as in Theorem 1.1 when making explicit the bounds in Sections 4 and 5.

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2. Proof of Theorem 1.1

We recall the definition of the bilinear decoupling constant $M_{a,b}$ as in (3). The arguments in this section will rely strongly on that the exponents in the definition of $M_{a,b}$ are 2 and 4, though we will only essentially use this in Lemma 2.7.

Given two expressions $x_1$ and $x_2$, let

$$\text{geom}_{2,4} x_i := x_1^{2/6} x_2^{4/6}.$$

Hölder gives $\|\text{geom}_{2,4} x_i\|_p \leq \text{geom}_{2,4} \|x_i\|_p.$
2.1. Parabolic rescaling and consequences. The linear decoupling constant $D(\delta)$ obeys the following important property.

**Lemma 2.1** (Parabolic rescaling). Let $0 < \delta < \sigma < 1$ be such that $\sigma, \delta, \delta/\sigma \in \mathbb{N}^{-1}$. Let $I$ be an arbitrary interval in $[0, 1]$ of length $\sigma$. Then

$$\|\mathcal{E}_I g\|_{L^p(B)} \leq 10^{20000} D(\delta) \left( \sum_{J \in P_{\delta}(I)} \|\mathcal{E}_J g\|_{L^p(w_B)}^2 \right)^{1/2}$$

for every $g : [0, 1] \to \mathbb{C}$ and every square $B$ of side length $\delta^{-2}$.

**Proof.** See [4, Proposition 7.1] for the proof without explicit constants and [11, Section 4] with $E = 100$ for a proof with explicit constants (and a clarification of parabolic rescaling with weight $w_B$). \hfill \Box

As an immediate application of parabolic rescaling we have almost multiplicativity of the decoupling constant.

**Lemma 2.2** (Almost multiplicativity). Let $0 < \delta < \sigma < 1$ be such that $\sigma, \delta, \delta/\sigma \in \mathbb{N}^{-1}$, then

$$D(\delta) \leq 10^{20000} D(\sigma) D(\delta/\sigma).$$

**Proof.** See [11, Proposition 4.1] with $E = 100$. \hfill \Box

The trivial bound of $O(\nu^{(a+2b)/6} \delta^{-1/2})$ for $M_{a,b}(\delta, \nu)$ is too weak for applications. We instead give another trivial bound that follows from parabolic rescaling.

**Lemma 2.3.** If $\delta$ and $\nu$ were such that $\nu^a \delta^{-1}, \nu^b \delta^{-1} \in \mathbb{N}$, then

$$M_{a,b}(\delta, \nu) \leq 10^{20000} D(\delta) \nu^{1/3} D(\nu)^{2/3}.$$

**Proof.** Fix arbitrary $I_1 \in P_{\nu^{-1}}([0, 1])$ and $I_2 \in P_{\nu^{-1}}([0, 1])$ which are $3\nu$-separated. Hölder’s inequality gives that

$$\|g_{\text{geom}}_{2,4} |\mathcal{E}_{I_1} g|\|_{L^6(B)}^6 \leq \|\mathcal{E}_{I_1} g\|_{L^p(w_B)}^2 \|\mathcal{E}_{I_2} g\|_{L^p(w_B)}^4.$$

Parabolic rescaling bounds this by

$$10^{120000} D(\delta)^2 D(\nu)^4 \left( \sum_{J \in P_{\delta}(I_1)} \|\mathcal{E}_J g\|_{L^p(w_B)}^2 \right)^2 \left( \sum_{J' \in P_{\nu}(I_2)} \|\mathcal{E}_{J'} g\|_{L^p(w_B)}^2 \right)^2.$$

Taking sixth roots then completes the proof of Lemma 2.3. \hfill \Box

Hölder and parabolic rescaling allows us to interchange the $a$ and $b$ in $M_{a,b}$.

**Lemma 2.4.** Suppose $b \geq 1$ and $\delta$ and $\nu$ were such that $\nu^b \delta^{-1} \in \mathbb{N}$. Then

$$M_{b,\delta}(\delta, \nu) \leq 10^{10000} M_{b,\delta}(\delta, \nu)^{1/2} D(\delta) \nu^{1/2}.$$

**Proof.** Fix arbitrary $I_1$ and $I_2$ intervals of length $\nu^b$ and $\nu^b$, respectively which are $\nu$-separated. Hölder’s inequality then gives

$$\|\mathcal{E}_{I_1} g\|_{L^p(w_B)}^{1/3} \|\mathcal{E}_{I_2} g\|_{L^p(w_B)}^{2/3} \leq \left( \int_B |\mathcal{E}_{I_1} g|^4 |\mathcal{E}_{I_2} g|^2 \right)^{1/2} \left( \int_B |\mathcal{E}_{I_2} g|^6 \right)^{1/2}.$$

Applying the definition of $M_{b,\delta}$ and parabolic rescaling bounds the above by

$$(10^{20000})^3 M_{b,\delta}(\delta, \nu)^3 D(\delta)^3 \left( \sum_{J \in P_{\delta}(I_1)} \|\mathcal{E}_J g\|_{L^p(w_B)}^2 \right)^{1/2} \left( \sum_{J' \in P_{\nu}(I_2)} \|\mathcal{E}_{J'} g\|_{L^p(w_B)}^2 \right)^{2/3},$$

which completes the proof of Lemma 2.4. \hfill \Box
Lemma 2.5 (Bilinear reduction). Suppose $\delta$ and $\nu$ were such that $\nu\delta^{-1} \in \mathbb{N}$. Then

$$D(\delta) \leq 10^{30000} (D(\frac{\delta}{\nu}) + \nu^{-1} M_{1,1}(\delta, \nu)).$$

Proof. Let $\{I_i\}_{i=1}^{\nu^{-1}} = P_{\nu}([0, 1])$. We have

$$\|\mathcal{E}_{[0,1]}g\|_{L^6(B)} = \left\| \sum_{1 \leq i \leq \nu^{-1}} \mathcal{E}_{I_i}g\|_{L^6(B)} \right\| \leq \sum_{1 \leq i \leq \nu^{-1}} \|\mathcal{E}_{I_i}g\|_{L^6(B)}^{1/2} \|\mathcal{E}_{I_j}g\|_{L^6(B)}^{1/2}.$$

Parabolic rescaling bounds this by

$$10^{40000} D\left(\frac{\delta}{\nu}\right)^2 \sum_{1 \leq i, j \leq \nu^{-1}} \|\mathcal{E}_{I_i}g\|_{L^6(B)}^{1/2} \|\mathcal{E}_{I_j}g\|_{L^6(B)}^{1/2} \left( \sum_{j \in P_{\nu}(I_i)} \|\mathcal{E}_{J_j}g\|_{L^6(w_B)}^{1/2} \right)^{1/2} \leq 10^{40000} D\left(\frac{\delta}{\nu}\right)^2 \sum_{1 \leq i, j \leq \nu^{-1}} \left( \sum_{j \in P_{\nu}(I_i)} \|\mathcal{E}_{J_j}g\|_{L^6(w_B)}^{1/2} \right)^{1/2} \leq 10^{40010} D\left(\frac{\delta}{\nu}\right)^2 \sum_{j \in P_{\nu}([0,1])} \|\mathcal{E}_{J_j}g\|_{L^6(w_B)}^{1/2}.$$

Therefore the first term in (6) is bounded above by

$$10^{30000} D\left(\frac{\delta}{\nu}\right) \left( \sum_{j \in P_{\nu}([0,1])} \|\mathcal{E}_{J_j}g\|_{L^6(w_B)}^{1/2} \right)^{1/2}. \quad (7)$$

Next we consider the off-diagonal terms. We have

$$\|\mathcal{E}_{I_i}g\|_{L^3(B)}^{1/2} \leq \nu^{-1} \max_{1 \leq i, j \leq \nu^{-1}} \|\mathcal{E}_{I_i}g\|_{L^3(B)}^{1/2}$$

Hölder’s inequality gives that

$$\|\mathcal{E}_{I_i}g\|_{L^3(B)}^{1/2} \leq \|\mathcal{E}_{I_i}g\|_{L^6(B)}^{1/2} \left( \sum_{j \in P_{\nu}([0,1])} \|\mathcal{E}_{J_j}g\|_{L^6(w_B)}^{1/2} \right)^{1/2}. \quad (8)$$

and therefore from (3) (and using that $\nu\delta^{-1} \in \mathbb{N}$), the second term in (6) is bounded by

$$\sqrt{2\nu^{-1}} M_{1,1}(\delta, \nu) \left( \sum_{j \in P_{\nu}([0,1])} \|\mathcal{E}_{J_j}g\|_{L^6(w_B)}^{1/2} \right)^{1/2}.$$

Combining this with (7) and applying the definition of $D(\delta)$ then completes the proof of Lemma 2.5. \qed
2.2. A Fefferman-Cordoba argument. In the proof of Lemma 2.7 we need a version of \( M_{a,b} \) with both sides being \( L^6(w_B) \). The following lemma shows that these two constants are equivalent.

**Lemma 2.6.** Suppose \( \delta \) and \( \nu \) were such that \( \nu^\delta \delta^{-1}, \nu^\delta \delta^{-1} \in \mathbb{N} \). Let \( M'_{a,b}(\delta, \nu) \) be the best constant such that

\[
\int |\mathcal{E}_I g|^2 |\mathcal{E}_I g|^4 w_B \leq M'_{a,b}(\delta, \nu)^6 \left( \sum_{I \in P(I)} |\mathcal{E}_J g|_{L^\delta(w_B)}^2 \right) \left( \sum_{I' \in P(I')} |\mathcal{E}_{I'} g|_{L^\delta(w_B)}^2 \right)^2
\]

for all squares \( B \) of side length \( \delta^{-2} \), \( g : [0, 1] \to \mathbb{C} \), and all \( 3 \nu \)-separated intervals \( I \in P_{\nu^\delta}([0, 1]) \) and \( I' \in P_{\nu^\delta}([0, 1]) \). Then

\[
M'_{a,b}(\delta, \nu) \leq 12^{100/6} M_{a,b}(\delta, \nu).
\]

**Remark 1.** Since \( 1_B \preceq w_B \), \( M_{a,b}(\delta, \nu) \preceq M'_{a,b}(\delta, \nu) \) and hence Lemma 2.6 implies \( M_{a,b} \sim M'_{a,b} \).

**Proof.** Fix arbitrary \( 3 \nu \)-separated intervals \( I_1 \in P_{\nu^\delta}([0, 1]) \) and \( I_2 \in P_{\nu^\delta}([0, 1]) \). It suffices to assume that \( B \) is centered at the origin.

Corollary 2.4 of [11] gives

\[
\| \text{geom}_{2,4} (|\mathcal{E}_{I} g|) \|_{L^6(w_B)} \leq 3^{100} \int_{\mathbb{R}^2} \| \text{geom}_{2,4} (|\mathcal{E}_{I} g|) \|_{L^6(B(y, \delta^{-2}))}^6 w_B(y) dy.
\]

Applying the definition of \( M_{2b,b} \) gives that the above is

\[
\leq 3^{100} \delta^4 M_{2b,b}(\delta, \nu)^6 \int_{\mathbb{R}^2} \text{geom}_{2,4} \left( \sum_{I \in P_{\nu^\delta}(I)} |\mathcal{E}_J g|_{L^\delta(w_B(y, \delta^{-2}))}^2 \right)^3 w_B(y) dy
\]

\[
\leq 3^{100} \delta^4 M_{2b,b}(\delta, \nu)^6 \text{geom}_{2,4} \left( \sum_{I \in P_{\nu^\delta}(I)} |\mathcal{E}_J g|_{L^\delta(w_B(y, \delta^{-2}))}^2 \right)^{3/2} w_B(y) dy
\]

\[
\leq 3^{100} \delta^4 M_{2b,b}(\delta, \nu)^6 \text{geom}_{2,4} \left( \sum_{I \in P_{\nu^\delta}(I)} \int_{\mathbb{R}^2} |\mathcal{E}_J g|_{L^\delta(w_B(y, \delta^{-2}))}^6 w_B(y) dy \right)^{1/3}
\]

where the second inequality is by Hölder and the third inequality is by Minkowski. Since \( B \) is centered at the origin, \( w_B * w_B \preceq 4^{100} \delta^{-4} w_B \) ([11, Lemma 2.1]) and hence

\[
\delta^4 \int_{\mathbb{R}^2} |\mathcal{E}_J g|_{L^\delta(w_B(y, \delta^{-2}))}^6 w_B(y) dy \preceq 4^{100} \| \mathcal{E}_J g \|_{L^\delta(w_B)}^6.
\]

This then immediately implies that \( M'_{2b,b}(\delta, \nu) \leq 12^{100/6} M_{2b,b}(\delta, \nu) \) which completes the proof of Lemma 2.6.

We have the following key technical lemma of this paper. We encourage the reader to compare the argument with that of [12, Lemma 4.4]. This lemma is a large improvement over the trivial bound of \( M_{a,b} \preceq \nu^{-2(2b-a)/6} M_{2b,b} \) especially at very small scales (large \( a, b \)).

**Lemma 2.7.** Let \( a \) and \( b \) be integers such that \( 1 \leq a \leq 2b \). Suppose \( \delta \) and \( \nu \) was such that \( \nu^b \delta^{-1} \in \mathbb{N} \). Then

\[
M_{a,b}(\delta, \nu) \leq 10^{1000} \nu^{-1/6} M_{2b,b}(\delta, \nu).
\]
Proof. It suffices to assume that $B$ is centered at the origin with side length $\delta^{-2}$. The integrality conditions on $\delta$ and $\nu$ imply that $\delta \leq \nu^{2b}$ and $\nu^b \delta^{-1} \in \mathbb{N}$. Fix arbitrary intervals $I_1 = [\alpha, \alpha + \nu^a] \in P_{\nu^a}([0, 1])$ and $I_2 = [\beta, \beta + \nu^b] \in P_{\nu^b}([0, 1])$ which are $3\nu$-separated.

Let $g_\beta(x) := g(x + \beta)$, $T_\beta = \left(\frac{1}{\nu^a}, \frac{2\beta}{\nu^b}\right)$, and $d := \alpha - \beta$. Shifting $I_2$ to $[0, \nu^b]$ gives that

$$\int_B |(\mathcal{E}_{I_1} g)(x)|^2 |(\mathcal{E}_{I_2} g)(x)|^4 \, dx = \int_B |(\mathcal{E}_{[d, d + \nu^b]} g_\beta)(T_\beta x)|^2 |(\mathcal{E}_{[0, \nu^a]} g_\beta)(T_\beta x)|^4 \, dx$$

$$= \int_{T_\beta(B)} |(\mathcal{E}_{[d, d + \nu^b]} g_\beta)(x)|^2 |(\mathcal{E}_{[0, \nu^a]} g_\beta)(x)|^4 \, dx. \quad (9)$$

Note that $d$ can be negative, however since $g : [0, 1] \to \mathbb{C}$ and $d = \alpha - \beta$, $\mathcal{E}_{[d, d + \nu^b]} g_\beta$ is defined. Since $|\beta| \leq 1$, $T_\beta(B) \subset 100B$. Combining this with $1_{100B} \leq \eta_{100B}$ gives that (9) is

$$\leq \int_{\mathbb{R}^2} |(\mathcal{E}_{[d, d + \nu^b]} g_\beta)(x)|^2 |(\mathcal{E}_{[0, \nu^a]} g_\beta)(x)|^4 \eta_{100B}(x) \, dx$$

$$= \sum_{J_1, J_2 \in P_{\nu^a}([d, d + \nu^b])} \int_{\mathbb{R}^2} (\mathcal{E}_{J_1} g_\beta)(x)(\mathcal{E}_{J_2} g_\beta)(x) |(\mathcal{E}_{[0, \nu^a]} g_\beta)(x)|^4 \eta_{100B}(x) \, dx. \quad (10)$$

We claim that if $d(J_1, J_2) > 10\nu^{2b-1}$, the integral in (10) is equal to 0.

Suppose $J_1, J_2 \in P_{\nu^a}([d, d + \nu^b])$ such that $d(J_1, J_2) > 10\nu^{2b-1}$. Expanding the integral in (10) for this pair of $J_1, J_2$ gives that it is equal to

$$\int_{\mathbb{R}^2} \left(\int_{J_1 \times [0, \nu^a]^2} \int_{J_2 \times [0, \nu^b]^2} \sum_{i=1}^{3} g_\beta(\xi_i) g_\beta(\xi_i + 3) e(\cdot \cdot \cdot) \prod_{i=1}^{6} d\xi_i \right) \eta_{100B}(x) \, dx \quad (11)$$

where the expression inside the $e(\cdot \cdot \cdot)$ is

$$((\xi_1 - \xi_4) x_1 + (\xi_1^2 - \xi_4^2) x_2) + ((\xi_2 + \xi_3 - \xi_5 - \xi_6) x_1 + (\xi_2^2 + \xi_3^2 - \xi_5^2 - \xi_6^2) x_2).$$

Interchanging the integrals in $\xi$ and $x$ shows that the integral in $x$ is equal to the Fourier inverse of $\eta_{100B}$ evaluated at

$$\left(\sum_{i=1}^{3} (\xi_i - \xi_{i+3}), \sum_{i=1}^{3} (\xi_i^2 - \xi_{i+3}^2)\right).$$

Since the Fourier inverse of $\eta_{100B}$ is supported in $B(0, \delta^2/100)$, (11) is equal to 0 unless

$$|\sum_{i=1}^{3} (\xi_i - \xi_{i+3})| \leq \delta^2/200$$

$$|\sum_{i=1}^{3} (\xi_i^2 - \xi_{i+3}^2)| \leq \delta^2/200. \quad (12)$$

Since $\delta \leq \nu^{2b}$ and $\xi_i \in [0, \nu^b]$ for $i = 2, 3, 5, 6$, (12) implies

$$|\xi_1 - \xi_4| |\xi_1 + \xi_4| = |\xi_1^2 - \xi_4^2| \leq 5\nu^{2b}. \quad (13)$$

Since $I_1, I_2$ are $3\nu$-separated, $|d| \geq 3\nu$. Recall that $\xi_1 \in J_1$, $\xi_4 \in J_2$ and $I_1, I_2$ are subsets of $[d, d + \nu^a]$. Write $\xi_1 = d + r$ and $\xi_4 = d + s$ with $r, s \in [0, \nu^a]$. Then

$$|\xi_1 + \xi_4| = |2d + (r + s)| \geq 6\nu - |r + s| \geq 6\nu - 2\nu^b \geq 4\nu. \quad (14)$$
Since \( d(J_1, J_2) > 10\nu^{2b-1} \), \( |\xi_1 - \xi_4| > 10\nu^{2b-1} \). Therefore the left hand side of (13) is \( > 40\nu^{2b} \), a contradiction. Thus the integral in (10) is equal to 0 when \( d(J_1, J_2) > 10\nu^{2b-1} \).

The above analysis implies that (10) is

\[
\sum_{J_1, J_2 \in P_{\nu^{2b}}(I_1)} \int_{d(J_1, J_2) \leq 10\nu^{2b-1}} \left| (\mathcal{E}_{J_1} g_3)(x) \right| |(\mathcal{E}_{J_2} g_3)(x)|||\mathcal{E}(0, \nu^a) g_3)(x)||^4 \eta_{100B}(x) \, dx.
\]

Undoing the change of variables as in (9) gives that the above is equal to

\[
\sum_{J_1, J_2 \in P_{\nu^{2b}}(I_1)} \int_{d(J_1, J_2) \leq 10\nu^{2b-1}} \left| (\mathcal{E}_{J_1} g)(x) \right| |(\mathcal{E}_{J_2} g)(x)|||\mathcal{E}_{I_2} g)(x)||^4 \eta_{100B}(T_{\beta} x) \, dx. \quad (15)
\]

Observe that

\[
\eta_{100B}(T_{\beta} x) \leq 10^{2400} w_{100B}(T_{\beta} x) \leq 10^{2600} w_{100B}(x) \leq 10^{2800} w_B(x)
\]

where the second inequality is by Lemma 2.16 of [11] and the last inequality is because \( w_B(x)^{-1} w_{100B}(x) \leq 10^{200} \). An application of Cauchy-Schwarz shows that (15) is

\[
\leq 10^{2800} \sum_{J_1, J_2 \in P_{\nu^{2b}}(I_1)} \left( \int_{\mathbb{R}^2} |\mathcal{E}_{J_1} g|^2 |\mathcal{E}_{I_2} g|^4 w_B \right)^{1/2} \left( \int_{\mathbb{R}^2} |\mathcal{E}_{J_2} g|^2 |\mathcal{E}_{I_2} g|^4 w_B \right)^{1/2}.
\]

Note that for each \( J_1 \in P_{\nu^{2b}}(I_1) \), there are \( \leq 10000\nu^{-1} \) intervals \( J_2 \in P_{\nu^{2b}}(I_1) \) such that \( d(J_1, J_2) \leq 10\nu^{2b-1} \). Thus two applications of Cauchy-Schwarz bounds the above by

\[
10^{2802} \nu^{-1/2} \left( \sum_{J_1 \in P_{\nu^{2b}}(I_1)} \int_{\mathbb{R}^2} |\mathcal{E}_{J_1} g|^2 |\mathcal{E}_{I_2} g|^4 w_B \right)^{1/2} \times
\]

\[
\left( \sum_{J_1 \in P_{\nu^{2b}}(I_1)} \sum_{J_2 \in P_{\nu^{2b}}(I_2)} \int_{\mathbb{R}^2} |\mathcal{E}_{J_1} g|^2 |\mathcal{E}_{I_2} g|^4 w_B \right)^{1/2}.
\]

Since there are \( \leq 10000\nu^{-1} \) relevant \( J_2 \) for each \( J_1 \), the above is

\[
\leq 10^{3000} \nu^{-1} \sum_{J \in P_{\nu^{2b}}(I_1)} \int_{\mathbb{R}^2} |\mathcal{E}_{J} g|^2 |\mathcal{E}_{I_2} g|^4 w_B
\]

\[
\leq 10^{3000} 12^{100} M_{2b}(\delta, \nu)^6 \left( \sum_{J \in P_{\nu^{2b}}(I_1)} \|\mathcal{E}_J g\|_{L^2(w_B)}^2 \right) \left( \sum_{J \in P_{\nu^{2b}}(I_2)} \|\mathcal{E}_J g\|_{L^2(w_B)}^2 \right)^2
\]

where the last inequality is an application of Lemma 2.6. This completes the proof of Lemma 2.7.

Iterating Lemmas 2.4 and 2.7 repeatedly gives the following estimate.

**Lemma 2.8.** Let \( N \in \mathbb{N} \) and suppose \( \delta \) and \( \nu \) were such that \( \nu^N \delta^{-1} \in \mathbb{N} \). Then

\[
M_{1,1}(\delta, \nu) \leq 10^{60000} \nu^{-1/3} D(\frac{\delta}{\nu^{2N-1}})^{\frac{1}{3} + \frac{2N-1}{2}} D(\frac{\delta}{\nu^{2N}})^{\frac{2N-1}{2}} \prod_{j=0}^{N-1} D(\frac{\delta}{\nu^{2j+1}})^{1/2 + 1}.
\]
2.3. The iteration and in particular Lemma 2.7 this way in Sections 3-5. 

Corollary 2.9. Let \( M_{a,b} \) be such that \( M_{a,b} \) were such that \( \nu^2 \delta^{-1} = \in N \), then

\[
M_{a,b}(\delta, \nu) \leq 10^{20000} \nu^{-1/6} M_{b,2b}(\delta, \nu)^{1/2} D(\frac{\delta}{\nu^2})^{1/2}. \tag{16}
\]

Since \( \nu^2 \delta^{-1} = \in N \), \( \nu^2 \delta^{-1} = \in N \) for \( i = 0, 1, 2, \ldots, 2^N \). Applying (16) repeatedly gives

\[
M_{1,1}(\delta, \nu) \leq 10^{10000} \nu^{-1/3} M_{2N-1,2N}(\delta, \nu)^{\frac{N-1}{2}} \prod_{j=0}^{N-1} D(\frac{\delta}{\nu^2})^{1/2j+1}. \]

Bounding \( M_{2N-1,2N} \) using Lemma 2.3 then completes the proof of Lemma 2.8. \( \square \)

Remark 2. A similar analysis as in (12)-(14) shows that if \( 1 \leq a < b \) and \( \delta \) and \( \nu \) were such that \( \nu^2 \delta^{-1} = \in N \), then \( M_{a,b}(\delta, \nu) \leq M_{b,b} \). Though we do not iterate this way in this section, it is enough to close the iteration with \( M_{a,b} \leq M_{b,b} \) for \( 1 \leq a < b \), and \( M_{b,b} \leq \nu^{-1/6} M_{2b,2b} \) and Lemma 2.4. This gives \( M_{b,b} \leq \nu^{-1/6} M_{2b,2b}^2 D(\delta/\nu^2)^{1/2} \) which is much better than the trivial bound. We interpret the iteration and in particular Lemma 2.7 this way in Sections 3-5.

2.3. The \( O_\epsilon(\delta^{-\epsilon}) \) bound. Combining Lemma 2.8 with Lemma 2.5 gives the following.

**Corollary 2.9.** Let \( N \in \mathbb{N} \) and suppose \( \delta \) and \( \nu \) were such that \( \nu^2 \delta^{-1} = \in N \). Then

\[
D(\delta) \leq 10^{10^5} \left( D(\frac{\delta}{\nu}) + \nu^{-4/3} D(\frac{\delta}{\nu^2}) \frac{1}{2 \nu} D(\frac{\delta}{\nu^{2N-1}})^{\frac{1}{2 \nu}} \prod_{j=0}^{N-1} D(\frac{\delta}{\nu^2})^{1/2j+1} \right).
\]

Choosing \( \nu = \delta^{1/2} \) in Corollary 2.9 and requiring that \( \nu = \delta^{1/2} \in \mathbb{N}^{-1} \cap (0, 1/100) \) gives the following result.

**Corollary 2.10.** Let \( N \in \mathbb{N} \) and suppose \( \delta \) was such that \( \delta^{-1/2} = \in N \) and \( \delta < 100^{-2^N} \). Then

\[
D(\delta) \leq 10^{10^5} \left( D(\delta^{1/2}) + \delta^{-\frac{1}{2 \nu}} D(\delta^{1/2}) \frac{1}{2 \nu} \prod_{j=0}^{N-1} D(\delta^{1/2})^{1/2j+1} \right).
\]

Corollary 2.10 allows us to conclude that \( D(\delta) \leq \epsilon \delta^{-\epsilon} \). To see this, the trivial bounds for \( D(\delta) \) are \( 1 \leq D(\delta) \leq \delta^{-1/2} \) for all \( \delta \in \mathbb{N}^{-1} \). Let \( \lambda \) be the smallest real number such that \( D(\delta) \leq \epsilon \delta^{-\lambda - \epsilon} \) for all \( \delta \in \mathbb{N}^{-1} \). From the trivial bounds, \( \lambda \in [0, 1/2] \). We claim that \( \lambda = 0 \). Suppose \( \lambda > 0 \).

Choose \( N \) to be an integer such that

\[
\frac{5}{6} + \frac{N}{2} - \frac{4}{3\lambda} \geq 1. \tag{17}
\]

Then by Corollary 2.10, for \( \delta^{-1/2} \in \mathbb{N} \) with \( \delta < 100^{-2^N} \),

\[
D(\delta) \leq \epsilon \delta^{-\lambda(1-\frac{1}{2\nu})-\epsilon} + \delta^{-\frac{1}{2 \nu}} \delta^{1-\lambda(1-\frac{1}{2\nu})-\epsilon} \sum_{j=0}^{N-1} D(\delta^{1/2})^{1/2j+1} - \epsilon \\
\leq \epsilon \delta^{-\lambda(1-\frac{1}{2\nu})-\epsilon} + \delta^{-\lambda(1-\frac{1}{2\nu})-\epsilon} \leq \epsilon \delta^{-\lambda(1-\frac{1}{2\nu})-\epsilon}
\]
where in the last inequality we have used (17). Applying almost multiplicativity of
the linear decoupling constant (similar to [11, Section 10] or the proof of Lemma
2.12 later) then shows that for all \( \delta \in \mathbb{N}^{-1} \),
\[
D(\delta) \lesssim_{N, \varepsilon} \delta^{-(1 - \frac{1}{3\varepsilon})}.
\]
This then contradicts minimality of \( \lambda \). Therefore \( \lambda = 0 \) and thus we have shown
that \( D(\delta) \lesssim_{\varepsilon} \delta^{-\varepsilon} \) for all \( \delta \in \mathbb{N}^{-1} \).

2.4. An explicit bound. Having shown that \( D(\delta) \lesssim_{\varepsilon} \delta^{-\varepsilon} \), we now make this
dependence on \( \varepsilon \) explicit. Fix arbitrary \( 0 < \varepsilon < 1/100 \). Then \( D(\delta) \leq C_\varepsilon \delta^{-\varepsilon} \) for all
\( \delta \in \mathbb{N}^{-1} \).

**Lemma 2.11.** Fix arbitrary \( 0 < \varepsilon < 1/100 \) and suppose \( D(\delta) \leq C_\varepsilon \delta^{-\varepsilon} \) for all
\( \delta \in \mathbb{N}^{-1} \). Let integer \( N \geq 1 \) be such that
\[
\frac{5}{6} + \frac{N}{2} - \frac{4}{3\varepsilon} > 0.
\]
Then for \( \delta \) such that \( \delta^{-1/2^N} \in \mathbb{N} \) and \( \delta < 100^{-2^N} \), we have
\[
D(\delta) \leq 2 \cdot 10^{10^5} C_\varepsilon^{1 - \frac{1}{2^N}} \delta^{-\varepsilon}.
\]
**Proof.** Inserting \( D(\delta) \leq C_\varepsilon \delta^{-\varepsilon} \) into Corollary 2.10 gives that for all integers \( N \geq 1 \)
and \( \delta \) such that \( \delta^{-1/2^N} \in \mathbb{N} \), \( \delta < 100^{-2^N} \), we have
\[
D(\delta) \leq 10^{10^5} (C_\varepsilon^{\frac{1}{2^N}} + C_\varepsilon^{1 - \frac{2}{3\varepsilon} \delta^{\frac{1}{2^N}}}) \delta^{-\varepsilon}.
\]
Thus by our choice of \( N \),
\[
D(\delta) \leq 10^{10^5} (C_\varepsilon^{\frac{1}{2^N}} + C_\varepsilon^{1 - \frac{2}{3\varepsilon} \delta^{\frac{1}{2^N}}}) \delta^{-\varepsilon}. \tag{18}
\]
There are two possibilities. If \( \delta < C_\varepsilon^{-1} \), then since \( 0 < \varepsilon < 1/100 \), (18) becomes
\[
D(\delta) \leq 10^{10^5} (C_\varepsilon^{\frac{1}{2^N}} + C_\varepsilon^{1 - \frac{2}{3\varepsilon} \delta^{\frac{1}{2^N}}}) \delta^{-\varepsilon} \leq 2 \cdot 10^{10^5} C_\varepsilon^{1 - \frac{1}{2^N}} \delta^{-\varepsilon}. \tag{19}
\]
On the other hand if \( \delta \geq C_\varepsilon^{-1} \), the trivial bound gives
\[
D(\delta) \leq 2^{100/6} \delta^{-1/2} \leq 2^{100/6} C_\varepsilon^{1/2}
\]
which is bounded above by the right hand side of (19). This completes the proof
of Lemma 2.11.

Note that Lemma 2.11 is only true for \( \delta \) satisfying \( \delta^{-1/2^N} \in \mathbb{N} \) and \( \delta < 100^{-2^N} \).
We now use almost multiplicativity to upgrade the result of Lemma 2.11 to all
\( \delta \in \mathbb{N}^{-1} \).

**Lemma 2.12.** Fix arbitrary \( 0 < \varepsilon < 1/100 \) and suppose \( D(\delta) \leq C_\varepsilon \delta^{-\varepsilon} \) for all
\( \delta \in \mathbb{N}^{-1} \). Then
\[
D(\delta) \leq 10^{10^5} C_\varepsilon^{1 - \frac{1}{2^N}} \delta^{-\varepsilon}
\]
for all \( \delta \in \mathbb{N}^{-1} \).
**Proof.** Choose
\[
N := \left\lceil \frac{8}{3\varepsilon} - \frac{5}{3} \right\rceil \tag{20}
\]
and \( \delta \in \{2^{-2^N} \} \cap [0,1] \). Then for these \( \delta \), \( \delta^{-1/2} \in \mathbb{N} \) and \( \delta < 100^{-2} \). If \( \delta \in (\delta, 1] \cap \mathbb{N}^{-1} \), then
\[
D(\delta) \leq 2^{100/6} \delta^{-1/2} \leq 2^{100/6} 2^{-N-1/2}.
\]
If \( \delta \in (\delta, 1] \) for some \( n \geq 7 \), then almost multiplicativity and Lemma 2.11 gives that
\[
D(\delta) \leq 10^{20000} D(\delta) D(\delta^{-1}) \leq 10^{20000} (2 \cdot 10^{10^5} C_{\varepsilon}^{-1} \frac{2}{\varepsilon} \delta^{-\varepsilon}) (2^{1000/6} \delta) \leq 10^{10^5} 2^{2^N-1} C_{\varepsilon}^{-1} \frac{2}{\varepsilon} \delta^{-\varepsilon}
\]
where \( N \) is as in (20) and the second inequality we have used the trivial bound for \( D(\delta/\delta_n) \).

Combining both cases above then shows that if \( N \) is chosen as in (20), then
\[
D(\delta) \leq 10^{10^5} 2^N 2^{-N-1} C_{\varepsilon}^{-1} \frac{2}{\varepsilon} \delta^{-\varepsilon}
\]
for all \( \delta \in \mathbb{N}^{-1} \). Since we are no longer constrained by having \( N \in \mathbb{N} \), we can increase \( N \) to be \( 3/\varepsilon \) and so we have that
\[
D(\delta) \leq 10^{10^5} 2^{4.8^{1/\varepsilon}} C_{\varepsilon}^{-1} \frac{2}{\varepsilon} \delta^{-\varepsilon}
\]
for all \( \delta \in \mathbb{N}^{-1} \). This completes the proof of Lemma 2.12.

**Lemma 2.13.** For all \( 0 < \varepsilon < 1/100 \) and all \( \delta \in \mathbb{N}^{-1} \), we have
\[
D(\delta) \leq 2^{2001/\varepsilon} \delta^{-\varepsilon}.
\]

**Proof.** Let \( P(C, \lambda) \) be the statement that \( D(\delta) \leq C \delta^{-\lambda} \) for all \( \delta \in \mathbb{N}^{-1} \). Lemma 2.12 implies that for \( \varepsilon \in (0, 1/100) \),
\[
P(C, \varepsilon) \implies P(10^{10^5} 2^{4.8^{1/\varepsilon}} C_{\varepsilon}^{-1} \frac{2}{\varepsilon} \delta^{-\varepsilon}, \varepsilon).
\]

Iterating this \( M \) times gives that
\[
P(C, \varepsilon) \implies P(\left[10^{10^5} 2^{4.8^{1/\varepsilon}} \right] \sum_{j=0}^{M-1} (1 - \frac{1}{s^{1/\varepsilon}})^j C_{\varepsilon}^{-1} \frac{2}{\varepsilon} \delta^{-\varepsilon} \varepsilon, \varepsilon).
\]

Letting \( M \to \infty \) thus gives that for all \( 0 < \varepsilon < 1/100 \),
\[
D(\delta) \leq (10^{10^5} 2^{4.8^{1/\varepsilon}})^{s^{1/\varepsilon}/\varepsilon} \delta^{-\varepsilon} \leq 2^{1001/\varepsilon} \delta^{-\varepsilon} \leq 2^{2001/\varepsilon} \delta^{-\varepsilon}
\]
for all \( \delta \in \mathbb{N}^{-1} \). This completes the proof of Lemma 2.13.

Optimizing in \( \varepsilon \) then gives the proof of our main result.

**Proof of Theorem 1.1.** Note that if \( \eta = \log A - \log \log A \), then \( \eta \exp(\eta) = A(1 - \log \log A / \log A) \leq A \). Choose \( \varepsilon \) such that \( A = (\log_2 200)(\log \frac{1}{\varepsilon}) \), \( \eta = \frac{1}{\varepsilon} \log 200 \), and \( \eta = \log A - \log \log A \). Then
\[
200^{1/\varepsilon} \log 2 \leq \varepsilon \log \frac{1}{\delta}
\]
and hence
\[
2^{200^{1/\varepsilon} \delta^{-\varepsilon}} \leq \exp(2\varepsilon \log \frac{1}{\delta}).
\]

(21)
prove (heuristically under the uncertainty principle) the following two statements:

(II) For arbitrary \( \nu \in (0, 1] \) such that \( d(I_1, I_2) \geq \nu \). From Lemma 2.8, we only need (23) to be true for \( 1 \leq a \leq b \). Our goal of this section is to prove (heuristically under the uncertainty principle) the following two statements:

I. For \( 1 \leq a < b \), \( d(I_1, I_2) \geq \nu \). From Lemma 2.8, we only need (23) to be true for \( 1 \leq a < b \). Our goal of this section is to prove (heuristically under the uncertainty principle) the following two statements:

I. For \( 1 \leq a < b \), \( d(I_1, I_2) \geq \nu \). From Lemma 2.8, we only need (23) to be true for \( 1 \leq a < b \). Our goal of this section is to prove (heuristically under the uncertainty principle) the following two statements:

(II) \( d(I_1, I_2) \geq \nu, \) in other words

\[
\int_B |E_{I_1}g|^2 |E_{I_2}g|^4 \leq \nu^{-1} \sum_{J \in P_{\nu b}(I_1)} \int_B |E_Jg|^2 |E_{I_2}g|^4
\]

(24)

(II) \( d(I_1, I_2) \geq \nu, \) in other words

\[
\int_B |E_{I_1}g|^2 |E_{I_2}g|^4 \leq \nu^{-1} \sum_{J \in P_{\nu b}(I_1)} \int_B |E_Jg|^2 |E_{I_2}g|^4
\]

(25)

Repeating 4 with \( p = 2 \) then allows us to generalize to \( 2 \leq p < 6 \). We leave the rest of the argument of the iteration for \( 2 \leq p < 6 \) to the interested reader and concentrate only on the case when \( p = 6 \). Note that all results in this section are only heuristically true. In this section we will pretend all weight functions are just indicator functions and will make these heuristics rigorous in the next section.
The particular instance of the uncertainty principle we will use is the following. Let $I$ be an interval of length $1/R$ with center $c$. Fix an arbitrary $R \times R^2$ rectangle $T$ oriented in the direction $(-2c, 1)$. Heuristically for $x \in T$, $(\mathcal{E}_Ig)(x)$ behaves like $a_{T,I}e^{2\pi i \omega_T \cdot x}1_T(x)$. Here the amplitude $a_{T,I}$ depends on $q, T$, and $I$ and the phase $\omega_T$ depends on $T$ and $I$. In particular, $|(\mathcal{E}_Ig)(x)|$ is essentially constant on every $R \times R^2$ rectangle oriented in the direction $(-2c, 1)$. This also implies that if $\Delta$ is a square of side length $R$, then $|(\mathcal{E}_Ig)(x)|$ is essentially constant on $\Delta$ (with constant depending on $\Delta$) and $\|\mathcal{E}_Ig\|_{L^p_\omega(\Delta)}$ is essentially constant with the same constant independent of $p$.

We introduce two standard tools from [4, 5].

**Lemma 3.1** (Bernstein’s inequality). Let $I$ be an interval of length $1/R$ and $\Delta$ a square of side length $R$. If $1 \leq p \leq q < \infty$, then

$$
\|\mathcal{E}_Ig\|_{L^q_\omega(\Delta)} \lesssim \|\mathcal{E}_Ig\|_{L^p_\omega(\Delta)}.
$$

We also have

$$
\|\mathcal{E}_Ig\|_{L^{p/2}(\Delta)} \lesssim \|\mathcal{E}_Ig\|_{L^p_\omega(\Delta)}.
$$

**Proof.** See [4, Corollary 4.3] for a rigorous proof.

The reverse inequality in the above lemma is just an application of Hölder.

**Lemma 3.2** ($l^2L^2$ decoupling). Let $I$ be an interval of length $\geq 1/R$ such that $R|I| \in \mathbb{N}$ and $\Delta$ a square of side length $R$. Then

$$
\|\mathcal{E}_Ig\|_{L^2_\omega(\Delta)} \lesssim \left( \sum_{J \in \mathcal{P}_{\nu/2}(I)} \|\mathcal{E}_Jg\|_{L^2_\omega(\Delta)}^2 \right)^{1/2}.
$$

**Proof.** See [4, Proposition 6.1] for a rigorous proof.

The first inequality (24) is an immediate application of the uncertainty principle and $l^2L^2$ decoupling.

**Lemma 3.3.** Suppose $1 \leq a < b$ and $\delta$ and $\nu$ were such that $\nu^a\delta^{-1} \in \mathbb{N}$. Then

$$
\int_B |\mathcal{E}_{I_1}g|^2|\mathcal{E}_{I_2}g|^4 \lesssim \sum_{J \in \mathcal{P}_{\nu}(I_1)} \int_B |\mathcal{E}_Jg|^2|\mathcal{E}_{I_2}g|^4
$$

for arbitrary $I_1 \in P_{\nu,\nu}([0,1])$ and $I_2 \in P_{\nu^a}(\nu,\nu)$ such that $d(I_1, I_2) \gtrsim \nu$. In other words, $M_{a,\nu}(\delta,\nu) \lesssim M_{b,\nu}(\delta,\nu)$.

**Proof.** It suffices to show that for each $\Delta' \in P_{\nu^{-b}}(B)$, we have

$$
\int_{\Delta'} |\mathcal{E}_{I_1}g|^2|\mathcal{E}_{I_2}g|^4 \lesssim \sum_{J \in \mathcal{P}_{\nu^a}(I_1)} \int_{\Delta'} |\mathcal{E}_Jg|^2|\mathcal{E}_{I_2}g|^4.
$$

Since $I_2$ is an interval of length $\nu^b$, $|\mathcal{E}_{I_2}g|$ is essentially constant on $\Delta'$. Therefore the above reduces to showing

$$
\int_{\Delta'} |\mathcal{E}_{I_1}g|^2 \lesssim \sum_{J \in \mathcal{P}_{\nu^a}(I_1)} \int_{\Delta'} |\mathcal{E}_Jg|^2
$$

which since $a < b$ and $I_1$ is of length $\nu^a$ is just an application of $l^2L^2$ decoupling. This completes the proof of Lemma 3.3.

□
Inequality (25) is a consequence of the following ball inflation lemma which
is reminiscent of the ball inflation in the Bourgain-Demeter-Guth proof of Vinogradov’s mean value theorem. The main point of this lemma is to increase the
spatial scale so we can apply $l^2L^2$ decoupling while keep the frequency scales constant.

**Lemma 3.4 (Ball inflation).** Let $b \geq 1$ be a positive integer. Suppose $I_1$ and $I_2$
are $\nu$-separated intervals of length $\nu^b$. Then for any square $\Delta'$ of side length $\nu^{-2b}$, we have

$$\text{Avg}_{\Delta \in P_{\nu^{-1b}}(\Delta')} \|E_{I_1}g\|_{L^2(\Delta)}^2 \|E_{I_2}g\|_{L^2(\Delta')}^2 \nu^{-1} \|E_{I_1}g\|_{L^2(\Delta)}^2 \|E_{I_2}g\|_{L^2(\Delta')}^2.$$  

**Proof.** The uncertainty principle implies that $|E_{I_1}g|$ and $|E_{I_2}g|$ are essentially constant on $\Delta$. Therefore we essentially have

$$\text{Avg}_{\Delta \in P_{\nu^{-1b}}(\Delta')} \|E_{I_1}g\|_{L^2(\Delta)}^2 \|E_{I_2}g\|_{L^2(\Delta')}^2 \sim \frac{1}{|P_{\nu^{-1b}}(\Delta')} \sum_{\Delta \in P_{\nu^{-1b}}(\Delta')} \frac{1}{|\Delta|} \int_{\Delta} |E_{I_1}g|^2 |E_{I_2}g|^2 \Delta' = \frac{1}{|\Delta'|} \int_{\Delta'} |E_{I_1}g|^2 |E_{I_2}g|^2 \Delta'.$$

On $\Delta'$, note that $|E_{I_1}g| \sim \sum_{T_1} |c_{T_1}|_T$ and similarly for $I_2$ where $\{T_i\}$ are the $\nu^{-b} \times \nu^{-2b}$ rectangles covering $\Delta'$ and pointing in the normal direction of the cap
on the parabola living above $I_i$. Since $I_1$ and $I_2$ are $\nu$-separated, for any two tubes $T_1, T_2$ corresponding to $I_1, I_2$, we have $|T_1 \cap T_2| \leq \nu^{-1-2b}$. Therefore

$$\frac{1}{|\Delta'|} \int_{\Delta'} |E_{I_1}g|^2 |E_{I_2}g|^2 \Delta' \sim \nu^{-1} \frac{\nu^{-2b}}{|\Delta'|} \sum_{T_1, T_2} |c_{T_1}|^2 |c_{T_2}|^2.$$  

Since

$$|E_{I_1}g|_{L^2(\Delta')}^2 \|E_{I_2}g|_{L^2(\Delta')}^2 \nu^{-6b} \sum_{T_1, T_2} |c_{T_1}|^2 |c_{T_2}|^2$$

and $|\Delta'| = \nu^{-4b}$, this completes the proof of Lemma 3.4. \hfill $\square$

We now prove inequality (25).

**Lemma 3.5.** Suppose $\delta$ and $\nu$ were such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$. Then

$$\int_B |E_{I_1}g|^2 |E_{I_2}g|^2 \nu^{-1} \sum_{j \in P_{\nu^b}(I_1)} \int_B |E_{I_j}g|^2 |E_{I_2}g|^2$$

for arbitrary $I_1 \in P_{\nu^b}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$ such that $d(I_1, I_2) \gtrsim \nu$. In other words, $M_{b,\delta}(\nu) \lesssim \nu^{-1/6} M_{2b,\delta}(\nu)$.

**Proof.** This is an application of ball inflation, $l^2L^2$ decoupling, Bernstein, and the uncertainty principle. Since $\nu^{2b}\delta^{-1} \in \mathbb{N}$, $\nu^{b}\delta^{-1} \in \mathbb{N}$ and $\delta \leq \nu^{2b}$. Fix arbitrary
Let $I_1, I_2 \in P_{\nu^b}([0,1])$. We have
\[
\frac{1}{|B|} \int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 = \frac{1}{|B|} \sum_{\Delta \in P_{\nu^b}(B)} \int_{\Delta} |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \\
\leq \frac{1}{|B|} \sum_{\Delta \in P_{\nu^b}(B)} (\int_{\Delta} |\mathcal{E}_{I_1} g|^2) |\mathcal{E}_{I_2} g|_{L^\infty(\Delta)}^4 \\
\leq \frac{1}{|P_{\nu^b}(B)|} \sum_{\Delta \in P_{\nu^b}(B)} \left( \frac{1}{|\Delta|} \int_{\Delta} |\mathcal{E}_{I_1} g|^2 \right) |\mathcal{E}_{I_2} g|_{L^\infty(\Delta)}^4 \\
= \frac{\text{Avg}}{\Delta \in P_{\nu^b}(B)} |\mathcal{E}_{I_1} g|_{L^2(\Delta)}^2 |\mathcal{E}_{I_2} g|_{L^4(\Delta)}^4
\] (26)
where the second inequality is because of Bernstein. From ball inflation we know that for each $\Delta' \in P_{\nu^b}(B)$,
\[
\text{Avg}_{\Delta \in P_{\nu^b}(\Delta')} |\mathcal{E}_{I_1} g|_{L^2(\Delta')}^2 |\mathcal{E}_{I_2} g|_{L^4(\Delta')}^4 \leq \nu^{-1} |\mathcal{E}_{I_1} g|_{L^2(\Delta)}^2 |\mathcal{E}_{I_2} g|_{L^4(\Delta)}^4.
\]
Averaging the above over all $\Delta' \in P_{\nu^b}(B)$ shows that (26) is
\[
\leq \nu^{-1} \text{Avg}_{\Delta \in P_{\nu^b}(B)} |\mathcal{E}_{I_1} g|_{L^2(\Delta)}^2 |\mathcal{E}_{I_2} g|_{L^4(\Delta)}^4.
\]
Since $I_1$ is of length $\nu^b$, $t^2L^2$ decoupling gives that the above is
\[
\leq \nu^{-1} \sum_{J \in P_{\nu^b}(I_1)} \text{Avg}_{\Delta \in P_{\nu^b}(B)} |\mathcal{E}_{J} g|_{L^2(\Delta)}^2 |\mathcal{E}_{I_2} g|_{L^4(\Delta)}^4 \\
= \nu^{-1} \frac{1}{|B|} \sum_{J \in P_{\nu^b}(I_1)} \sum_{\Delta \in P_{\nu^b}(B)} |\mathcal{E}_{I_2} g|_{L^4(\Delta)}^4 |\mathcal{E}_{J} g|_{L^2(\Delta')}^2 \\
= \nu^{-1} \frac{1}{|B|} \sum_{J \in P_{\nu^b}(I_1)} \sum_{\Delta \in P_{\nu^b}(B)} \left( \int_{\Delta'} |\mathcal{E}_{I_2} g|^4 \right) |\mathcal{E}_{J} g|_{L^2(\Delta')}^2.
\]
Since $|\mathcal{E}_{J} g|$ is essentially constant on $\Delta'$, the uncertainty principle gives that essentially we have
\[
\left( \int_{\Delta'} |\mathcal{E}_{I_2} g|^4 \right) |\mathcal{E}_{J} g|_{L^2(\Delta')}^2 \sim \int_{\Delta'} |\mathcal{E}_{J} g|^2 |\mathcal{E}_{I_2} g|^4.
\]
Combining the above two centered equations then completes the proof of Lemma 3.5.

**Remark 3.** The proof of Lemma 3.5 is reminiscent of our proof of Lemma 2.7. The $\|\mathcal{E}_{I_2} g\|_{L^\infty(\Delta)}$ can be thought as using the trivial bound for $\xi_1, i = 2, 3, 5, 6$ to obtain (13). Then we apply some data about separation, much like in ball inflation here to get large amounts of cancelation.

### 4. An Alternate Proof of $D(\delta) \leq \delta^{-\varepsilon}$

The ball inflation lemma and our proof of Lemma 3.5 inspire us to define a new bilinear decoupling constant that can make our uncertainty principle heuristics from the previous section rigorous.

For $\delta \in \mathbb{N}^{-1}$, let $D(\delta, n)$ be the best constant such that
\[
\|\mathcal{E}_{[0,1]} g\|_{L^n(B)} \leq D(\delta, n) \left( \sum_{J \in P_{\nu^b}([0,1])} |\mathcal{E}_{J} g|_{L^2(\nu^b(w^n))}^2 \right)^{1/2}
\]
for all \( g : [0, 1] \to \mathbb{C} \) and all squares \( B \) of side length \( \delta^{-2} \). Note that \( D(\delta) = D(\delta, 1) \).

Since we lose some decay in the weights when applying Bernstein, we will need the extra \( n \) parameter (see Lemma 4.2).

The left hand side of the definition of \( D(\delta, n) \) is unweighted, however convolution properties of the weight \( w_B^n \) ([11, Proposition 2.11]) give that we have also have
\[
\| \mathcal{E}_{[0,1]} g \|_{L^\infty(w_B^n)} \lesssim_n D(\delta, n) \left( \sum_{J \in P_\nu([0,1])} \| \mathcal{E}_J g \|_{L^\infty(w_B^n)}^2 \right)^{1/2}.
\]

(27)

for all \( g : [0, 1] \to \mathbb{C} \) and squares \( B \) of side length \( \delta^{-2} \).

Next we define the bilinear decoupling constant. We will assume that \( \delta^{-1} \in \mathbb{N} \) and \( \nu \in \mathbb{N}^{-1} \cap (0, 1/100) \). Let \( M_{a,b}(\delta, \nu, n) \) be the best constant such that
\[
\begin{align*}
\text{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \| \mathcal{E}_{I_1} g \|_{L^2_n(w_B^n)}^2 \| \mathcal{E}_{I_2} g \|_{L^2_n(w_B^n)}^2 \\
\leq M_{a,b}(\delta, \nu, n)^6 \left( \sum_{J \in P_\nu(I_1)} \| \mathcal{E}_J g \|_{L^2_n(w_B^n)} \right) \left( \sum_{K \in P_\nu(I_2)} \| \mathcal{E}_K g \|_{L^2_n(w_B^n)} \right)
\end{align*}
\]

for all squares \( B \) of side length \( \delta^{-2} \), \( g : [0, 1] \to \mathbb{C} \) and all intervals \( I \in P_\nu([0,1]) \), \( I' \in P_\nu([0,1]) \) with \( d(I, I') \geq \nu \).

Suppose \( a > b \) (the proof when \( a \leq b \) is similar). The uncertainty principle implies that
\[
\begin{align*}
\text{Avg}_{\Delta \in P_{\nu^{-\min(a,b)}}(B)} \| \mathcal{E}_{I_1} g \|_{L^2_n(\Delta)} \| \mathcal{E}_{I_2} g \|_{L^2_n(\Delta)} \\
&\leq \frac{1}{|P_\nu^{-\min(a,b)}(B)|} \sum_{\Delta \in P_\nu^{-\min(a,b)}(B)} \frac{1}{|\Delta|} \int_{\Delta} |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^2 \\
&\sim \frac{1}{|B|} \int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4
\end{align*}
\]

where the last \( \sim \) is because \( |\mathcal{E}_{I_j} g| \) is essentially constant on \( \Delta \). Therefore our bilinear constant \( M_{a,b} \) is essentially the same as the bilinear constant \( M_{\max(a,b)} \) we defined in (3).

Our goal will be to prove that for \( \delta \in \mathbb{N}^{-1} \), \( D(\delta, 1) \lesssim_\varepsilon \delta^{-\varepsilon} \). Because we need to work with \( D(\delta, n) \), many implicit constants depend on \( n \) however since we will just prove \( D(\delta, 1) \lesssim_\varepsilon \delta^{-\varepsilon} \) this \( n \) dependence is harmless. Using [11], the \( n \) dependence is of order \( n^{O(n)} \).

4.1. Some tools from decoupling. Note that the decoupling constant obeys the following monotonicity expressions. Suppose \( n_2 \leq n_1 \), then since \( w_{n_1}^B \leq w_{n_2}^B \), we immediately have \( D(\delta, n_2) \leq D(\delta, n_1) \). The reverse inequality is also in fact true.

Lemma 4.1. If \( n_2 \leq n_1 \), then \( D(\delta, n_1) \leq D(\delta, n_2) \).

Proof. See [11, Proposition 3.11].

Lemma 4.2 (Bernstein). Let \( I \) be an interval of length \( 1/R \) and \( \Delta \) a square of side length \( R \). If \( 1 \leq p \leq q < \infty \), then
\[
\| \mathcal{E}_{I} g \|_{L^p_n(w_B^n)} \lesssim_n \| \mathcal{E}_{I} g \|_{L^q_n(w_B^n)}.
\]

We also have
\[
\| \mathcal{E}_{I} g \|_{L^\infty(\Delta)} \lesssim_n \| \mathcal{E}_{I} g \|_{L^p_n(w_B^n)}.
\]
Proof. See [4, Corollary 4.3] for a proof without explicit constants or [11, Lemma 2.20] for a version with explicit constants.

**Lemma 4.3** \((l^2L^2\text{ decoupling})\). Let \(I\) be an interval of length \(\geq 1/R\) such that \(R|I| \in \mathbb{N}\) and \(\Delta\) a square of side length \(R\). Then
\[
\|E_I g\|_{L^2(w^\Delta)} \lesssim_n \left( \sum_{J \in P_{\epsilon,\eta}(I)} \|E_J g\|_{L^2(w^\Delta)}^2 \right)^{1/2}.
\]

**Proof.** See [4, Proposition 6.1] for a proof without explicit constants or [11, Lemma 2.21] for a version with explicit constants.

4.2. **Parabolic rescaling and consequences.** We now run through some basic properties as we did in Section 2 except this time with the decoupling constants \(D(\delta, n)\) and \(\mathcal{M}_{a,b}(\delta, \nu)\).

**Lemma 4.4** (Parabolic rescaling). Let \(0 < \delta < \sigma < 1\) be such that \(\sigma, \delta, \delta/\sigma \in \mathbb{N}^{-1}\). Let \(I\) be an arbitrary interval in \([0, 1]\) of length \(\sigma\). Then
\[
\|E_I g\|_{L^6(B)} \lesssim_n D(\frac{\delta}{\sigma}, n) \left( \sum_{J \in P_{\delta}(I)} \|E_J g\|_{L^6(w^\Delta)}^2 \right)^{1/2}
\]
for every \(g : [0, 1] \to \mathbb{C}\) and every square \(B\) of side length \(\delta^{-2}\). We have the same estimate when the left hand side above is weighted with \(w^\Delta_B\).

**Proof.** See [4, Proposition 7.1] with \(E = 100n\).

**Lemma 4.5** (Almost multiplicativity). Let \(0 < \delta < \sigma < 1\) be such that \(\sigma, \delta, \delta/\sigma \in \mathbb{N}^{-1}\). Then
\[
D(\delta, n) \lesssim_n D(\sigma, n)D(\delta/\sigma, n).
\]

**Proof.** See [11, Proposition 4.1] with \(E = 100n\).

**Lemma 4.6.** Suppose \(\delta\) and \(\nu\) were such that \(\nu^\delta \delta^{-1}, \nu^\delta \delta^{-1} \in \mathbb{N}\). Then
\[
\mathcal{M}_{a,b}(\delta, \nu, n) \lesssim_n D(\frac{\delta}{\nu^\delta}, n)^{1/3}D(\frac{\delta}{\nu^\delta}, n)^{2/3}.
\]

**Proof.** Let \(I_1 \in P_{\nu\sigma}([0, 1])\) and \(I_2 \in P_{\nu\sigma}([0, 1])\). Hölder’s inequality gives that
\[
\text{Avg}_{\Delta \in P_{\nu\sigma}(\text{max}(a, b), (B))} \|E_{I_1} g\|_{L^2(w^\Delta)}^2 \|E_{I_2} g\|_{L^4(w^\Delta)}^4 \lesssim \text{Avg}_{\Delta \in P_{\nu\sigma}(\text{max}(a, b), (B))} \|E_{I_1} g\|_{L^6(w^\Delta)}^6 \|E_{I_2} g\|_{L^6(w^\Delta)}^6 \lesssim (\text{Avg}_{\Delta \in P_{\nu\sigma}(\text{max}(a, b), (B))} \|E_{I_1} g\|_{L^6(w^\Delta)}^6 \|E_{I_2} g\|_{L^6(w^\Delta)}^6)^{1/3}\left(\text{Avg}_{\Delta \in P_{\nu\sigma}(\text{max}(a, b), (B))} \|E_{I_2} g\|_{L^6(w^\Delta)}^6 \|E_{I_2} g\|_{L^6(w^\Delta)}^6\right)^{2/3} \lesssim_n \|E_{I_1} g\|_{L^6(w^\Delta)}^6 \|E_{I_2} g\|_{L^6(w^\Delta)}^6 \|E_{I_2} g\|_{L^6(w^\Delta)}^6
\]
where the last inequality we have used that \(\sum_{\Delta} w^\Delta_n \lesssim_n w^\Delta_B\) (see [11, Proposition 2.14] with \(E = 100n\)). Finally applying (27) with parabolic rescaling then completes the proof of Lemma 4.6.

The next lemma is the only place we decrease in \(n\) which is because of an application of Bernstein and is the only reason why we need the \(n\) parameter.
Lemma 4.7. Suppose $\nu^a \delta^{-1}, \nu^b \delta^{-1} \in \mathbb{N}$. Then

$$M_{a,b}(\delta, \nu, n) \lesssim_{\nu} M_{b,a}(\delta, \nu, n)^{1/2} \frac{n^{1/2}}{\nu^b}. $$

Proof. Let $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$. We have

$$\begin{align*}
\text{Avg}_{\Delta \in P_{\nu^a - \max(a,b)}(B)} \| \mathcal{E} I_1 g \|^2_{L^2_{\nu} (w^n_{\Delta})} & \leq \text{Avg}_{\Delta \in P_{\nu^a - \max(a,b)}(B)} \| \mathcal{E} I_1 g \|^4_{L^2_{\nu} (w^n_{\Delta})}^{1/2} \| \mathcal{E} I_2 g \|^4_{L^4_{\nu} (w^n_{\Delta})}^{1/2} \\
& \leq \text{Avg}_{\Delta \in P_{\nu^a - \max(a,b)}(B)} \| \mathcal{E} I_1 g \|^4_{L^4_{\nu} (w^n_{\Delta})} \| \mathcal{E} I_2 g \|^4_{L^4_{\nu} (w^n_{\Delta})}^{1/2} \| \mathcal{E} I_2 g \|^4_{L^4_{\nu} (w^n_{\Delta})}^{1/2} \\
& \lesssim_{\nu} \text{Avg}_{\Delta \in P_{\nu^a - \max(a,b)}(B)} \| \mathcal{E} I_1 g \|^2_{L^2_{\nu} (w^n_{\Delta})} \| \mathcal{E} I_2 g \|^2_{L^2_{\nu} (w^n_{\Delta})} \| \mathcal{E} I_2 g \|^2_{L^2_{\nu} (w^n_{\Delta})} \| \mathcal{E} I_2 g \|^2_{L^2_{\nu} (w^n_{\Delta})}
\end{align*}$$

where the first inequality is because of Hölder and the second inequality is an application of Hölder, Bernstein, and the estimate $\sum_{\Delta} w^n_{\Delta} \leq n w^n_B$. Applying $w^n_{\Delta} \leq w^n_B^{1/2}$, parabolic rescaling and the definition of $M_{b,a}$ then completes the proof of Lemma 4.7. 

Lemma 4.8 (Bilinear reduction). Suppose $\delta$ and $\nu$ were such that $\nu \delta^{-1} \in \mathbb{N}$. Then

$$D(\delta, 1) \lesssim_{\nu} D(\delta, 1) + \nu^{-1} M_{1,1}(\delta, \nu, n).$$

Proof. The proof is essentially the same as that of Lemma 2.5 except when analyzing (8) in the off-diagonal terms we use

$$\begin{align*}
\| \mathcal{E} I_1 g \|_{L^6_{\nu} (B)}^{1/3} \| \mathcal{E} I_2 g \|_{L^6_{\nu} (B)}^{2/3} & = \text{Avg}_{\Delta \in P_{\nu^4} (B)} \frac{1}{|\Delta|} \int_{\Delta} |\mathcal{E} I_1 g|^2 |\mathcal{E} I_1 g|^4 \\
& \leq \text{Avg}_{\Delta \in P_{\nu^4} (B)} \| \mathcal{E} I_1 g \|^2_{L^2_{\nu} (\Delta)} \| \mathcal{E} I_2 g \|^4_{L^4_{\nu} (\Delta)} \\
& \lesssim_{\nu} \text{Avg}_{\Delta \in P_{\nu^4} (B)} \| \mathcal{E} I_1 g \|^2_{L^2_{\nu} (w^n_{\Delta})} \| \mathcal{E} I_2 g \|^4_{L^4_{\nu} (w^n_{\Delta})}
\end{align*}$$

where the second inequality we have used Bernstein. 

4.3. Ball inflation. We now prove rigorously the ball inflation lemma we mentioned in the previous section.

Lemma 4.9 (Ball inflation). Let $b \geq 1$ be a positive integer. Suppose $I_1$ and $I_2$ are $\nu$-separated intervals of length $\nu^b$. Then for any square $\Delta'$ of side length $\nu^{-2b}$, we have

$$\begin{align*}
\text{Avg}_{\Delta \in P_{\nu^b - \nu}(\Delta')} \| \mathcal{E} I_1 g \|^2_{L^2_{\nu} (w^n_{\Delta})} \| \mathcal{E} I_2 g \|^4_{L^4_{\nu} (w^n_{\Delta})} & \lesssim_{\nu} \nu^{-1} \| \mathcal{E} I_1 g \|^2_{L^2_{\nu} (w^n_{\Delta})} \| \mathcal{E} I_2 g \|^4_{L^4_{\nu} (w^n_{\Delta})}.
\end{align*}$$

Proof. Without loss of generality we may assume that $\Delta'$ is centered at the origin. Fix intervals $I_1$ and $I_2$ intervals of length $\nu^b$ which are $\nu$-separated with centers $c_1$ and $c_2$, respectively.

Cover $\Delta'$ by a set $\mathcal{T}_1$ of mutually parallel nonoverlapping rectangles $T_1$ of dimensions $\nu^{-b} \times \nu^{-2b}$ with longer side pointing in the direction of $(-2c_1, 1)$ (the normal direction of the piece of parabola above $I_1$). Note that any $\nu^{-b} \times \nu^{-2b}$ rectangle outside $4\Delta'$ cannot cover $\Delta'$ itself. Thus we may assume that all rectangles in $\mathcal{T}_1$ are contained in $4\Delta'$. Finally let $T_1(x)$ be the rectangle in $\mathcal{T}_1$ containing $x$. Similarly define $\mathcal{T}_2$ except this time we use $I_2$. 

For $x \in 4\Delta'$, define
\[
F_1(x) := \begin{cases} 
\sup_{y \in 2T_1(x)} \|E_1g\|_{L^2_{\mu}(w^{\mu}_{\nu})} & \text{if } x \in \bigcup_{T_1 \in \mathcal{T}_1} T_1 \\
0 & \text{if } x \in 4\Delta' \setminus \bigcup_{T_1 \in \mathcal{T}_1} T_1
\end{cases}
\]
and
\[
F_2(x) := \begin{cases} 
\sup_{y \in 2T_2(x)} \|E_2g\|_{L^4_{\mu}(w^{\mu}_{\nu})} & \text{if } x \in \bigcup_{T_2 \in \mathcal{T}_2} T_2 \\
0 & \text{if } x \in 4\Delta' \setminus \bigcup_{T_2 \in \mathcal{T}_2} T_2.
\end{cases}
\]

Given a $\Delta \in \mathcal{P}_{\nu-b}(\Delta')$, if $x \in \Delta$, then $\Delta \subset 2T_1(x)$. This implies that the center of $\Delta$, $c_\Delta \in 2T_1(x)$ for $x \in \Delta$ and hence for all $x \in \Delta$,
\[
\|E_1g\|_{L^2_{\mu}(w^{\mu}_{\nu})} \leq F_1(x)
\]
and
\[
\|E_2g\|_{L^4_{\mu}(w^{\mu}_{\nu})} \leq F_2(x).
\]

Therefore
\[
\|E_1g\|^2_{L^2_{\mu}(w^{\mu}_{\nu})} \|E_2g\|^4_{L^4_{\mu}(w^{\mu}_{\nu})} \leq \frac{1}{|\Delta|} \int_{\Delta} F_1(x)^2 F_2(x)^4 \, dx.
\]
(30)

By how $F_i$ is defined, $F_i$ is constant on each $T_i \in \mathcal{T}_i$. That is, for each $x \in \bigcup_{T_i \in \mathcal{T}_i} T_i$,
\[
F_i(x) = \sum_{T_i \in \mathcal{T}_i} c_{T_i} 1_{T_i}(x)
\]
for some constants $c_{T_i} > 0$.

Thus using (30) and that the $T_i$ are disjoint, the left hand side of (29) is bounded above by
\[
\frac{1}{|\Delta|} \int_{\Delta'} F_1(x)^2 F_2(x)^4 \, dx = \frac{1}{|\Delta'|} \sum_{T_1 \cap T_2} c_{T_1} c_{T_2} |T_1 \cap T_2| \lesssim \nu^{-1} \frac{L_{\nu-b}^{2b}}{|\Delta|} \sum_{T_1 \cap T_2} c_{T_1} c_{T_2} \quad (31)
\]
where the last inequality we have used that since $I_1$ and $I_2$ are $\nu$-separated, sine of the angle between $T_1$ and $T_2$ is $\geq \nu$ and hence $|T_1 \cap T_2| \lesssim \nu^{-1} L_{\nu-b}^{2b}$.

Note that
\[
\|F_1\|^2_{L^2_{\mu}(4\Delta')} = \frac{\nu^{-3b}}{|4\Delta'|} \sum_{T_1} c_{T_1}^2
\]
and
\[
\|F_2\|^4_{L^4_{\mu}(4\Delta')} = \frac{\nu^{-3b}}{|4\Delta'|} \sum_{T_2} c_{T_2}^4.
\]

Therefore (31) is
\[
\lesssim \nu^{-1} \|F_1\|^2_{L^2_{\mu}(4\Delta')} \|F_2\|^4_{L^4_{\mu}(4\Delta')}.
\]

Thus we are done if we can prove that
\[
\|F_1\|^2_{L^2_{\mu}(4\Delta')} \lesssim_n \|E_1g\|^2_{L^2_{\mu}(w^{\mu}_{\nu})}
\]
and
\[
\|F_2\|^4_{L^4_{\mu}(4\Delta')} \lesssim_n \|E_2g\|^4_{L^4_{\mu}(w^{\mu}_{\nu})}
\]
but this was exactly what was shown in [4, Eq. (29)] (and [11, Lemma 6.3] for the same inequality but with explicit constants).
Our choice of bilinear constant \((28)\) makes the rigorous proofs of Lemmas 3.3 and 3.5 immediate consequences of ball inflation and \(l^2 L^2\) decoupling.

**Lemma 4.10.** Suppose \(1 \leq a < b\) and \(\delta\) and \(\nu\) were such that \(\nu^b \delta^{-1} \in \mathbb{N}\). Then
\[
\mathcal{M}_{a,b}(\delta, \nu, n) \lesssim n \mathcal{M}_{b,b}(\delta, \nu, n).
\]

**Proof.** For arbitrary \(I_1 \in P_{\nu^b}([0, 1])\) and \(I_2 \in P_{\nu^b}([0, 1])\) which are \(\nu\)-separated, it suffices to show that
\[
\text{Avg}_{\Delta \in P_{\nu^b}(B)} \| \mathcal{E}_{I_1} g \|^2_{L^2(B, w_{\nu^b}^2)} \| \mathcal{E}_{I_2} g \|^4_{L^4(B, w_{\nu^b}^2)} \lesssim n \sum_{J \in P_{\nu^b}(I_1) \Delta \in P_{\nu^b}(B)} \text{Avg}_{\Delta' \in P_{\nu^b}(B)} \| \mathcal{E}_{J} g \|^2_{L^2(B, w_{\nu^b}^2)} \| \mathcal{E}_{I_2} g \|^4_{L^4(B, w_{\nu^b}^2)}.
\]
But this is immediate from \(l^2 L^2\) decoupling which completes the proof of Lemma 4.10. \(\square\)

**Lemma 4.11.** Let \(b \geq 1\) and suppose \(\delta\) and \(\nu\) were such that \(\nu^{2b} \delta^{-1} \in \mathbb{N}\). Then
\[
\mathcal{M}_{b,b}(\delta, \nu, n) \lesssim n \nu^{-1/6} \mathcal{M}_{2b,2b}(\delta, \nu, n).
\]

**Proof.** For arbitrary \(I_1 \in P_{\nu^b}([0, 1])\) and \(I_2 \in P_{\nu^b}([0, 1])\) which are \(\nu\)-separated, it suffices to prove that
\[
\text{Avg}_{\Delta \in P_{\nu^b}(B)} \| \mathcal{E}_{I_1} g \|^2_{L^2(B, w_{\nu^b}^2)} \| \mathcal{E}_{I_2} g \|^4_{L^4(B, w_{\nu^b}^2)} \lesssim n \nu^{-1} \sum_{J \in P_{\nu^b}(I_1) \Delta \in P_{\nu^b}(B)} \text{Avg}_{\Delta' \in P_{\nu^b}(B)} \| \mathcal{E}_{J} g \|^2_{L^2(B, w_{\nu^b}^2)} \| \mathcal{E}_{I_2} g \|^4_{L^4(B, w_{\nu^b}^2)}.
\]
But this is immediate from ball inflation followed by \(l^2 L^2\) decoupling which completes the proof of Lemma 4.11. \(\square\)

Combining Lemmas 4.7, 4.10, and 4.11 gives the following corollary.

**Corollary 4.12.** Suppose \(\delta\) and \(\nu\) were such that \(\nu^{2b} \delta^{-1} \in \mathbb{N}\). Then
\[
\mathcal{M}_{b,b}(\delta, \nu, n) \lesssim n \nu^{-1/6} \mathcal{M}_{2b,2b}(\delta, \nu, n)^{1/2} D\left(\frac{\delta}{\nu^{b}}, n\right)^{1/2}.
\]

This corollary should be compared to the trivial estimate obtained from Lemma 4.6 which implies \(\mathcal{M}_{b,b}(\delta, \nu, n) \lesssim D(\delta/\nu^b, n)\).

4.4. The \(O_\varepsilon(\delta^{-\varepsilon})\) bound. We now prove that \(D(\delta, 1) \lesssim \delta^{-\varepsilon}\). The structure of the argument is essentially the same as that in Section 2.3. Repeatedly iterating Corollary 4.12 gives the following result.

**Lemma 4.13.** Let \(N\) be an integer chosen sufficiently large later and let \(\delta\) be such that \(\delta^{-1/2^N} \in \mathbb{N}\) and \(0 < \delta < 100^{-2^N}\). Then
\[
D(\delta, 1) \lesssim N D(\delta^{1/2^N}, 1) + \delta^{-1/2^N} \prod_{j=0}^{N-1} D\left(\delta^{1/2^N}, 1\right)^{2^{j+1}}.
\]

**Proof.** Iterating Corollary 4.12 \(N\) times starting from \(\mathcal{M}_{1,1}(\delta, \nu, 2^N)\) gives that if \(\delta\) and \(\nu\) were such that \(\nu^{2^N} \delta^{-1} \in \mathbb{N}\), then
\[
\mathcal{M}_{1,1}(\delta, \nu, 2^N) \lesssim N \nu^{-1/3} \mathcal{M}_{2^N,2^N}(\delta, \nu, 1)^{1/2^N} \prod_{j=0}^{N-1} D\left(\frac{\delta}{\nu^{2^j}}, 2^N-1\right)^{2^{j+1}}.
\]
Applying Lemma 4.1 and the trivial bound for the bilinear constant bounds gives that the above is
\[ \leq N \nu^{-1/3} D(\delta \nu^{1/2N}, 1)^{1/2N} \prod_{j=0}^{N-1} D(\delta \nu^{1/2N}, 1)^{1/(2N)}. \]
Choosing \( \nu = \delta^{1/2N} \) shows that if \( \delta^{-1/2N} \in \mathbb{N} \) and \( 0 < \delta < 100^{-2N} \), then
\[ M_{1, 1}(\delta, \delta^{1/2N}, 2N) \leq N \delta^{-\frac{2}{3N}} \prod_{j=0}^{N-1} D(\delta^{1-\frac{1}{2N}}, 1)^{\frac{1}{2N}}. \]
By the bilinear reduction, if \( \delta \) was such that \( \delta^{-1/2N} \in \mathbb{N} \) and \( 0 < \delta < 100^{-2N} \), then
\[ D(\delta, 1) \leq N D(\delta^{1-\frac{1}{2N}}, 1) + \delta^{-\frac{2}{3N}} \prod_{j=0}^{N-1} D(\delta^{1-\frac{1}{2N}}, 1)^{\frac{1}{2N}}. \]
This completes the proof of Lemma 4.13. \( \square \)

Trivial bounds for \( D(\delta, 1) \) show that \( 1 \leq D(\delta, 1) \leq \delta^{-1/2} \) for all \( \delta \in \mathbb{N}^{-1} \). Let \( \lambda \) be the smallest real number such that \( D(\delta, 1) \leq \delta^{\lambda-\varepsilon} \) for all \( \delta \in \mathbb{N}^{-1} \). From the trivial bounds \( \lambda \in [0, 1/2] \). We claim \( \lambda = 0 \). Suppose \( \lambda > 0 \).

Let \( N \) be a sufficiently large integer \( \geq \frac{8}{3\lambda} \). This implies
\[ 1 + \frac{N}{2} - \frac{4}{3\lambda} \geq 1. \]
Lemma 4.13 then implies that for \( \delta \) such that \( \delta^{-1/2N} \in \mathbb{N} \) and \( 0 < \delta < 100^{-2N} \), we have
\[ D(\delta, 1) \leq N \varepsilon \delta^{-\lambda(1-\frac{1}{2N})-\varepsilon} + \delta^{-\lambda(1-\frac{1}{2N})(1+\frac{2}{3\lambda}))-\varepsilon} \leq N \varepsilon \delta^{-\lambda(1-\frac{1}{2N})-\varepsilon} \]
where the last inequality we have applied our choice of \( N \). By almost multiplicity we then have the same estimate for all \( \delta \in \mathbb{N}^{-1} \) (with a potentially larger constant depending on \( N \)). But this then contradicts minimality of \( \lambda \). Therefore \( \lambda = 0 \).

5. UNIFYING THE TWO STYLES OF PROOF

We now attempt to unify the Bourgain-Demeter style of decoupling and the style of decoupling mentioned in the previous section. In view of Corollary 4.12, instead of having two integer parameters \( a \) and \( b \) we just have one integer parameter.

Let \( b \) be an integer \( \geq 1 \) and choose \( s \in [2, 3] \) any real number. Suppose \( \delta \in \mathbb{N}^{-1} \) and \( \nu \in \mathbb{N}^{-1} \cap (0, 1/100) \) were such that \( \nu^b \delta^{-1} \in \mathbb{N} \). Let \( M_b^{(s)}(\delta, \nu) \) be the best constant such that
\[ \frac{\text{Avg}}{\Delta \in P_{\nu^{-a}(B)}(I)} \left( \sum_{J \in P_{\nu^{(b)}(1)}} \|E_Jg\|_{L^2\nu(I)}^{2s} \right) \leq M_b^{(s)}(\delta, \nu) \left( \sum_{J \in P_{\nu^{(b)}(1)}} \|E_Jg\|_{L^2\nu(I)}^{2s} \right) \left( \sum_{J' \in P_{\nu^{(b)}(1')}} \|E_{J'}g\|_{L^2\nu(I')}^{2s} \right)^{\frac{1}{2}} \]
for all squares \( B \) of side length \( \delta^{-2} \), \( g : [0, 1] \to \mathbb{C} \), and all intervals \( I, I' \in P_{\nu}([0, 1]) \) which are \( \nu \)-separated. Note that left hand side of the definition of \( M_b^{(s)}(\delta, \nu) \) is the same as \( A_b(\delta, B\nu, B\nu, \nu) \) defined in [4] and from the uncertainty principle, \( M_b^{(2)}(\delta, \nu) \) is morally the same as \( M_{1, 1} \) defined in (3) and \( M_{1, 1}(\delta, \nu, n) \) defined in (28).
The $l^2$ piece in the definition of $M_b^{(s)}(\delta, \nu)$ is so that we can make the most out of applying $l^2L^2$ decoupling.

We will use $M_b^{(s)}$ as our bilinear constant in this section to show that $D(\delta) \leq \varepsilon \delta^{-\varepsilon}$. The bilinear constant $M_b^{(s)}$ obeys much the same lemmas as in the previous sections.

**Lemma 5.1** (cf. Lemmas 2.3 and 4.6). If $\delta$ and $\nu$ were such that $\nu^\delta \delta^{-1} \in \mathbb{N}$, then
\[
M_b^{(s)}(\delta, \nu) \leq D\left(\frac{\delta}{\nu}\right).
\]

**Proof.** Fix arbitrary $I_1, I_2 \in P_{\nu}([0, 1])$ which are $\nu$-separated. Moving up from $L^2_{\nu}$ to $L^6_{\#}$ followed by Hölder in the average over $\Delta$ bounds the left hand side of (32)
\[
\left( \text{Avg} \left( \sum_{\Delta \in P_{\nu}^{-1}(B)} \|E_{I_j}g\|_{L^2_{\nu}(w_{\Delta})}^2 \right) \right)^{\frac{2}{3}} \left( \text{Avg} \left( \sum_{\Delta \in P_{\nu}^{-1}(B)} \|E_{I_j}g\|_{L^6_{\#}(w_{\Delta})}^6 \right) \right)^{\frac{1}{3}}.
\]
Using Minkowski to switch the $l^2$ and $l^6$ sum followed by $\sum_{\Delta} w_{\Delta} \leq w_B$ shows that this is
\[
\leq \left( \sum_{J \in P_{\nu}^{-1}(I_1)} \|E_{I_j}g\|_{L^2_{\nu}(w_{B})}^2 \right)^{\frac{2}{3}} \left( \sum_{J' \in P_{\nu}^{-1}(I_2)} \|E_{I_j}g\|_{L^6_{\nu}(w_{B})}^6 \right)^{\frac{1}{3}}.
\]
Parabolic rescaling then completes the proof of Lemma 5.1. $\square$

**Lemma 5.2** (Bilinear reduction, cf. Lemmas 2.5 and 4.8). Suppose $\delta$ and $\nu$ were such that $\nu \delta^{-1} \in \mathbb{N}$. Then
\[
D(\delta) \leq D\left(\frac{\delta}{\nu}\right) + \nu^{-1}M_b^{(s)}(\delta, \nu).
\]

**Proof.** Note that the left hand side of the definition of $M_1^{(s)}(\delta, \nu)$ is
\[
\text{Avg}_{\Delta \in P_{\nu}^{-1}(B)} \|E_{I_1}g\|_{L^2_{\nu}(w_{\Delta})}^2 \|E_{I_2}g\|_{L^6_{\#}(w_{\Delta})}^{6-s}.
\]
Proceeding as in the proof of Lemmas 2.5 and 4.8, for $I_1, I_2 \in P_{\nu}([0, 1])$ which are $\nu$-separated, we have
\[
\|E_{I_1}g\|_{L^2_{\nu}(B)} \leq \|E_{I_1}g\|_{L^6_{\#}(w_{B})} \|E_{I_2}g\|_{L^6_{\nu}(w_{\Delta})}^{6-s} \|E_{I_2}g\|_{L^6_{\#}(B)}^{1-s} \|E_{I_1}g\|_{L^6_{\#}(B)}^{1-s}.
\] (33)
We have
\[
\|E_{I_1}g\|_{L^6_{\#}(B)}^{1-s} \leq \text{Avg}_{\Delta \in P_{\nu}^{-1}(B)} \left( \int_{\Delta} |E_{I_1}g|^s |E_{I_2}g|^{6-s} \right)^{\frac{1}{6}} \leq \text{Avg}_{\Delta \in P_{\nu}^{-1}(B)} \|E_{I_1}g\|_{L^6_{\#}(w_{\Delta})} \|E_{I_2}g\|_{L^6_{\#}(w_{\Delta})}^{6-s} \|E_{I_2}g\|_{L^6_{\#}(w_{\Delta})}^{1-s} \|E_{I_1}g\|_{L^6_{\#}(B)}^{1-s} \|E_{I_2}g\|_{L^6_{\#}(B)}^{1-s},
\]
where the last inequality we have used Bernstein. Inserting this into (33) and applying the definition of $M_1^{(s)}(\delta, \nu)$ then completes the proof of Lemma 5.2. $\square$

**Lemma 5.3** (Ball inflation, cf. Lemma 4.9). Let $b \geq 1$ be a positive integer. Suppose $I_1$ and $I_2$ are $\nu$-separated intervals of length $\nu$. Then for any square $\Delta'$ of
side length $\nu^{-2b}$ and any $\varepsilon > 0$, we have
\[
\frac{\text{Avg}}{\Delta \in P_{\nu^{-b}}(\Delta')} \left( \sum_{J \in P_{\nu^{-b}}(I_1)} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in P_{\nu^{-b}}(I_2)} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}} \lesssim \varepsilon \nu^{-1-bc} \left( \sum_{J \in P_{\nu^{-b}}(I_1)} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in P_{\nu^{-b}}(I_2)} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}} \lesssim \varepsilon \nu^{-1-bc} \left( \sum_{J \in P_{\nu^{-b}}(I_1)} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in P_{\nu^{-b}}(I_2)} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}}.
\]

Proof. The $s = 2$ case be proven directly using Lemma 4.9 without any loss in $\nu^{-b}$. The proof for $s \in [2, 3]$ proceeds as in the proof of ball inflation in [4, Section 9.2] (see also [11, Section 6] for more details and explicit constants).

From dyadic pigeonholing, since we can lose a $\nu^{-b}$, it suffices to restrict the sum over $J$ and $J'$ to families $\mathcal{F}_1$ and $\mathcal{F}_2$ such that for all $J \in \mathcal{F}_1$, $\|\mathcal{E}_J g\|_{L^2_\nu(w_{\Delta'})}$ are comparable up to a factor of 2 and similarly for all $J' \in \mathcal{F}_2$. Hölder gives
\[
\frac{\text{Avg}}{\Delta \in P_{\nu^{-b}}(\Delta')} \left( \sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}} \lesssim \left( \# \mathcal{F}_1 \right)^{\frac{1}{2}} \left( \# \mathcal{F}_2 \right)^{\frac{1}{2}} \nu^{-1-bc} \left( \sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}}.
\]
The proof of Lemma 4.9 shows that this is
\[
\lesssim \nu^{-1-bc} \left( \sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}}.
\]
Since for $J \in \mathcal{F}_1$ the values of $\|\mathcal{E}_J g\|_{L^2_\nu(w_{\Delta'})}$ are comparable and similarly for $J' \in \mathcal{F}_2$, the above is
\[
\lesssim \nu^{-1-bc}. \left( \sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}} \lesssim \nu^{-1-bc} \frac{\text{Avg}}{\Delta \in P_{\nu^{-b}}(\Delta')} \left( \sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}}.
\]
This completes the proof of Lemma 5.3.

\[\square\]

Lemma 5.4 (cf. Corollary 4.12). Suppose $\delta$ and $\nu$ were such that $\nu^{2b} \delta^{-1} \in \mathbb{N}$. Then for every $\varepsilon > 0$,
\[
M_{\delta}^{(s)}(\delta, \nu) \lesssim \nu^{-\frac{p}{2}+ \frac{1}{6}} \|f\|_{L^\infty_\delta \nu}^2 \|f\|_{L^\infty_\delta \nu}^{-\theta} \|f\|_{L^\infty_\delta \nu}^{-\theta} \|f\|_{L^\infty_\delta \nu}^{-\theta}.
\]

Proof. Let $\theta$ and $\varphi$ be such that $\frac{\theta}{2} + \frac{1}{6} = \frac{1}{s}$ and $\frac{\varphi}{2} + \frac{1}{6} = \frac{1}{6-s}$. Then Hölder gives $\|f\|_{L^\infty_\delta \nu} \leq \|f\|_{L^\infty_\delta \nu}^{\frac{1}{6}} \|f\|_{L^\infty_\delta \nu}^{-\theta}$. Then Hölder gives $\|f\|_{L^\infty_\delta \nu} \leq \|f\|_{L^\infty_\delta \nu}^{\frac{1}{6}} \|f\|_{L^\infty_\delta \nu}^{-\theta}$. Then Hölder gives $\|f\|_{L^\infty_\delta \nu} \leq \|f\|_{L^\infty_\delta \nu}^{\frac{1}{6}} \|f\|_{L^\infty_\delta \nu}^{-\theta}$.

Fix arbitrary $I_1, I_2 \in P_{\nu}([0, 1])$ which are $\nu$-separated. We have
\[
\frac{\text{Avg}}{\Delta \in P_{\nu^{-b}}(B) \Delta \in P_{\nu^{-b}}(\Delta')} \left( \sum_{J \in P_{\nu^{b}}(I_1)} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in P_{\nu^{b}}(I_2)} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}} \lesssim \varepsilon \nu^{-1-bc} \left( \sum_{J \in P_{\nu^{b}}(I_1)} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in P_{\nu^{b}}(I_2)} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}} \lesssim \varepsilon \nu^{-1-bc} \left( \sum_{J \in P_{\nu^{b}}(I_1)} \|\mathcal{E}_J g\|_{L^2_\nu(w_\Delta)}^2 \right)^{\frac{1}{2}} \left( \sum_{J' \in P_{\nu^{b}}(I_2)} \|\mathcal{E}_{J'} g\|_{L^6_{b\varepsilon}(w_\Delta)}^2 \right)^{\frac{1}{2}}.
\]
where the first inequality is from Hölder and the second inequality is from ball inflation. We now use how $\theta$ and $\varphi$ are defined to return to a piece which we
control by $L^2 L^2$ decoupling and a piece which we can control by parabolic rescaling. Hölder (as in the definition of $\theta$ and $\varphi$) gives that the average above is bounded by

$$\begin{align*}
\text{Avg}_{\Delta' \in P_{\nu-2s}(B)} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{2}{1-\theta}} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\theta}{2}} \times \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\varphi}{2}} \times \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1}{2}}.
\end{align*}$$

Hölder in the sum over $J$ and $J'$ shows that this is

$$\begin{align*}
\sum_{J \in P_{\nu s}(I_1)} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{2}{1-\theta}} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\theta}{2}} \times \left( \sum_{J' \in P_{\nu s}(I_2)} \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\varphi}{2}} \right) \times \left( \sum_{J' \in P_{\nu s}(I_2)} \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1}{2}} \right).
\end{align*}$$

Since $\theta s = 3 - \frac{3}{4}$ and $\varphi(6-s) = \frac{3}{4}$, rearranging the above gives

$$\begin{align*}
\sum_{J \in P_{\nu s}(I_1)} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{2}{1-\theta}} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\theta}{2}} \times \left( \sum_{J' \in P_{\nu s}(I_2)} \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1}{2}} \right).
\end{align*}$$

Cauchy-Schwarz in the average over $\Delta'$ then bounds the above by

$$\begin{align*}
\sum_{J \in P_{\nu s}(I_1)} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{2}{1-\theta}} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\theta}{2}} \times \left( \sum_{J' \in P_{\nu s}(I_2)} \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1}{2}} \right).
\end{align*}$$

After $L^2 L^2$ decoupling, the first term in (34) is

$$\begin{align*}
\sum_{J \in P_{\nu s}(I_1)} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{2}{1-\theta}} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\theta}{2}} \times \left( \sum_{J' \in P_{\nu s}(I_2)} \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1}{2}} \right).
\end{align*}$$

Hölder in the average over $\Delta'$ bounds the second term in (34) by

$$\begin{align*}
\sum_{J \in P_{\nu s}(I_1)} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{2}{1-\theta}} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\theta}{2}} \times \sum_{J' \in P_{\nu s}(I_2)} \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1}{2}}.
\end{align*}$$

Applying Minkowski to interchange the $\ell^2$ and $\ell^6$ norms shows that this is

$$\begin{align*}
\sum_{J \in P_{\nu s}(I_1)} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{2}{1-\theta}} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\theta}{2}} \times \sum_{J' \in P_{\nu s}(I_2)} \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1}{2}}.
\end{align*}$$

Parabolic rescaling bounds this by

$$\begin{align*}
D^2 \left( \sum_{J \in P_{\nu s}(I_1)} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{2}{1-\theta}} \left( \sum_{j \in P_{\nu s}(I_1)} \|E_j g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1-\theta}{2}} \times \left( \sum_{J' \in P_{\nu s}(I_2)} \left( \sum_{j' \in P_{\nu s}(I_2)} \|E_{j'} g\|^2_{L^2_\omega(w_{\Delta'})} \right)^{\frac{1}{2}} \right).
\end{align*}$$

Combining (35) and (36) then completes the proof of Lemma 5.4.

With Lemma 5.4, the same proof as Lemma 4.13 gives the following.
Lemma 5.5 (cf. Corollary 2.10 and Lemma 4.13). Let $N$ be an integer chosen sufficient large later and let $\delta$ be such that $\delta^{-1/2^N} \in \mathbb{N}$ and $0 < \delta < 100^{-2^N}$. Then

$$D(\delta) \lesssim_\varepsilon D(\delta^{1-\frac{1}{2^N}}) + \delta^{-\frac{4}{3}2^N - \frac{N\varepsilon}{6}2^N} \prod_{j=0}^{N-1} D(\delta^{1-\frac{1}{2^N-j}})2^{1+j}.$$

Proof. This follows from the proof of Lemma 4.13 and the observation that

$$M_1^{(s)}(\delta, \nu) \lesssim_\varepsilon \nu^{-\frac{1}{2} - \frac{\varepsilon}{4}N\varepsilon} M_{2N}^{(s)}(\delta, \nu)2^N \prod_{j=0}^{N-1} D(\delta^{\frac{\varepsilon}{\nu^{2^j}}}2^{1+j}).$$

along with Lemmas 5.1 and 5.2. \qed

To finish, we proceed as at the end of the previous section. Let $\lambda \in [0, 1/2]$ be the smallest real such that $D(\delta) \lesssim_\varepsilon \delta^{-\lambda - \varepsilon}$. Suppose $\lambda > 0$. Choose $N$ such that

$$1 + \frac{N}{2} - \frac{4}{3\lambda} \geq 1.$$

Then for $\delta$ such that $\delta^{-1/2^N} \in \mathbb{N}$ and $0 < \delta < 100^{-2^N}$, Lemma 5.5 gives

$$D(\delta) \lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon} + \delta^{-\lambda(1-\frac{1}{2^N}(1+\frac{1}{2^N}))-\varepsilon} \lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon}.$$

Almost multiplicativity gives that $D(\delta) \lesssim_{N, \varepsilon} \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$, contradicting the minimality of $\lambda$.

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