Improvement of generalization of Larman-Rogers-Seidel’s theorem

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Abstract

A finite set $X$ in the $d$-dimensional Euclidean space is called an $s$-distance set if the set of distances between any two distinct points of $X$ has size $s$. In 1977, Larman–Rogers–Seidel proved that if the cardinality of an two-distance set is large enough, then there exists an integer $k$ such that the two distances $\alpha, \beta$ ($\alpha < \beta$) having the integer condition, namely, $\frac{\alpha^2}{\beta^2} = \frac{k-1}{k}$. In 2011, Nozaki generalized Larman–Rogers–Seidel’s theorem to the case of $s$-distance sets, i.e. if the cardinality of an $s$-distance set $|X| \geq 2^s N$ with distances $\alpha_1, \alpha_2, \cdots, \alpha_s$, where $N = \binom{d+s-1}{s-1} + \binom{d+s-2}{s-2}$, then the numbers $k_i = \prod_{j=1,2, \cdots, s, j \neq i} \frac{\alpha_j^2}{\alpha_j^2 - \alpha_i^2}$ are integers. In this note, we reduce the lower bound of the requirement of integer condition of $s$-distance sets in $\mathbb{R}^d$. Furthermore, we can show that there are only finitely many $s$-distance sets $X$ in $\mathbb{R}^d$ with $|X| \geq 2^{(d+s-1)}$.

1 Introduction

Let $\mathbb{R}^d$ be the $d$-dimensional Euclidean space, and $|\ast|$ denote the cardinality. A finite set $X$ in $\mathbb{R}^d$ is said to be an $s$-distance set if there are exactly $s$ distinct distances between the points in $X$, i.e. $|A(X)| = s$, where $A(X) = \{\|x - y\| : x, y \in X\}$ be the set of the distances on $X$. If the size of an $s$-distance set is large enough, we are curious about whether there are any relations between the $s$ distinct distances. In 1977, Larman, Rogers, and Seidel (LRS) \cite{1} discovered the following theorem:

**Theorem 1.1.** \cite{1} Let $X$ be a set in $\mathbb{R}^d$ that only realises the two distances $\alpha$ and $\beta$ with $\alpha < \beta$. If $|X| > 2d + 3$, then $\frac{\alpha^2}{\beta^2} = \frac{k-1}{k}$, with $k \geq 2$ a positive integer, and $k \leq \frac{1}{2} + \sqrt{\frac{1}{2}d}$.

In 1981, Neumaier \cite{2} reduced the requirement of $|X|$ from $2d + 3$ in Theorem 1.1 to $2d + 1$ and found counterexamples with $2d+1$ points in $\mathbb{R}^d$. The counterexamples are the spherical embedding of conference graphs. In 2011, Nozaki \cite{3} generalized the LRS Theorem to the case of $s$-distance sets, and found that the ratios of the $s$ distances also have integer conditions if the cardinality of $s$-distance sets is large enough:

**Theorem 1.2.** \cite{3} Let $X$ be an $s$-distance set in $\mathbb{R}^d$ with $s \geq 2$ and $A(X) = \{\alpha_1, \cdots, \alpha_s\}$. Let $N = \binom{d+s-1}{s-1} + \binom{d+s-2}{s-2}$. If $|X| \geq 2N$, then

$$\prod_{j=1,2, \cdots, s, j \neq i} \frac{\alpha_j^2}{\alpha_j^2 - \alpha_i^2}$$

is an integer $k_i$ for each $i = 1, 2, \cdots, s$. Furthermore, $|k_i| \leq \lfloor \frac{1}{2} + \sqrt{\frac{N^2}{2N-2} + \frac{1}{4}} \rfloor$. 

However, in the case of $s = 2$, the condition in Theorem 1.2 gives $|X| > 2d + 3$, which is larger than the condition $|X| > 2d + 1$ improved by Neumaier. In this note, we prove that the latter term of the bound $|X| \geq 2(d+s-1) + 2(d+s-2)$ can be removed, and hence the theorem becomes as follows:

**Theorem 1.3.** Let $X$ be an $s$-distance set in $\mathbb{R}^d$ with $s \geq 2$ and $A(X) = \{\alpha_1, \ldots, \alpha_s\}$. Let $N = \binom{d+s-1}{s-1}$. If $|X| \geq 2N$, then

$$\prod_{j=1,2,\ldots,s, j \neq i} \frac{\alpha_j^2}{\alpha_j^2 - \alpha_i^2}$$

is an integer $k_i$ for each $i = 1, 2, \cdots, s$. Furthermore, $|k_i| \leq \lfloor \frac{1}{2} + \sqrt{\frac{N^2}{2(d-2)} + \frac{1}{4}} \rfloor$.

In particular, if $s = 2$, the condition gives $|X| > 2d + 1$, which is same as the best situation in the case of two-distance sets.

In section 2, we will recall some lemmas that will be used later. In section 3, we will prove our main result Theorem 1.3. In section 4, we will reduce the requirement of the size of $s$-distance sets to guarantee that there are only finitely many $s$-distance sets $X$ in $\mathbb{R}^d$ with $|X| \geq 2(d+s-1)$, due to our improvement of Theorem 1.2.

## 2 Preliminaries

Here, we denote $P_\ell(\mathbb{R}^d)$ for the space of all the polynomials with $d$ variables of degree $\leq \ell$. Let $x_1, \cdots, x_d$ be independent variables, and set $x_0 = x_1^2 + \cdots + x_d^2$. We denote $W_\ell(\mathbb{R}^d)$ for the linear space spanned by monomials $x_0^{\lambda_0}x_1^{\lambda_1}\cdots x_d^{\lambda_d}$ with $\lambda_0 + \lambda_1 + \cdots + \lambda_d \leq \ell$ and $\lambda_i \geq 0$. First, we recall that Theorem 1.2 in [6] is proved by using the following lemma:

**Lemma 2.1.** [6] Let $X$ be a finite subset of $\Omega \subset \mathbb{R}^d$, and $\mathcal{P}(\mathbb{R}^d)$ a linear subspace of $P_\ell(\mathbb{R}^d)$. Let $N = \dim(\mathcal{P}(\Omega))$. Suppose that there exists $F_x \in \mathcal{P}(\Omega)$ for each $x \in X$ such that $F_x(x) = k$ where $k$ is a constant, $F_x(y)$ are 0 or 1 for all $y \in X$ with $y \neq x$, and $F_x(y) = F_y(x)$ for all $x, y \in X$. If $|X| \geq 2N$, then $k$ is an integer and $|k| \leq \lfloor \frac{1}{2} + \sqrt{\frac{N^2}{2(d-2)} + \frac{1}{4}} \rfloor$.

From Lemma 2.1 we can see that the lower bound

$$|X| \geq 2\binom{d+s-1}{s-1} + 2\binom{d+s-2}{s-2}$$

in Theorem 1.2 is coming from the dimension of the function space where the set of functions

$$\{ F_y(x) = \prod_{j=1,2,\ldots,s, j \neq i} \frac{\alpha_j^2 - \|y - x\|^2}{\alpha_j^2 - \alpha_i^2} : y \in X \} \subset W_{s-1}(\mathbb{R}^d)$$

lives in. In fact, a similar set of functions

$$\{ G_y(x) = \prod_{j=1,2,\ldots,s} \frac{\alpha_j^2 - \|y - x\|^2}{\alpha_j^2} : y \in X \} \subset W_s(\mathbb{R}^d)$$

had been considered in [2] by Bannai, Bannai, and Stanton in 1983 for getting new upper bounds of the cardinality of $s$-distance sets in $\mathbb{R}^d$. They found that $\{G_y(x) : y \in X\}$ and $\{x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_d^{\lambda_d} :$
0 \leq \lambda_1 + \lambda_2 + \cdots + \lambda_d \leq s - 1 \}\) are linearly independent sets in \(W_s(\mathbb{R}^d)\), and therefore improved the upper bounds of \(s\)-distance sets to
\[
|X| \leq \left(\frac{d+s}{s}\right).
\]

In section 3, we show that the ideal in \([2]\) can be used to prove that \(\{F_y(x) : y \in X\}\) and \(\{x_1^\lambda x_2^{\lambda_2} \cdots x_d^{\lambda_d} : 0 \leq \lambda_1 + \lambda_2 + \cdots + \lambda_d \leq s - 2\}\) are linearly independent sets in \(W_{s-1}(\mathbb{R}^d)\), and thus the lower bound in Theorem \([1,2]\) can be improved.

Before going into the proof of Theorem \([1,3]\) we recall some Lemmas in \([1,2]\) which will be used later:

**Lemma 2.2.** \([1]\)
\[
\dim(W_s(\mathbb{R}^d)) \leq \left(\frac{d + \ell}{\ell}\right) + \left(\frac{d + \ell - 1}{\ell - 1}\right)
\]

**Lemma 2.3.** \([2]\) Fix \(s' \in \mathbb{N}\). Let \(x_1, \ldots, x_d\) be independent variables, and denote \(\partial_i = \frac{\partial}{\partial x_i}\). Let \(2 \leq \ell \leq s' + 2\), then we have

the space spanned by \(\{\partial_1^{b_1} \cdots \partial_\ell^{b_{d}}(x_1^2 + \cdots + x_d^2)^{s'} : b_1 + \cdots + b_d = 2s' - \ell + 2\}\)

= the space spanned by \(\{x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d} : a_1 + \cdots + a_d = \ell - 2\}\)

The following lemma is a modification of lemma in Bannai-Bannai-Stanton \([2]\). Since some of the parameters are different to the original theorem, we give it a prove.

**Lemma 2.4.** For \(i = 1, \ldots, N\), let \(m_i \in \mathbb{R}\) and \(y^{(i)} = (y_1^{(i)}, \ldots, y_d^{(i)}) \in \mathbb{R}^d\). For fixed integers \(2 \leq \ell \leq s + 1\), suppose
\[
\sum_{i=1}^{N} m_i \|x - y^{(i)}\|^{2(s-1)}
\]
is a polynomial in \(x\) of degree \(\leq 2s - \ell - 1\), then
\[
\sum_{i=1}^{N} m_i (y_1^{(i)})^{a_1} \cdots (y_d^{(i)})^{a_d} = 0, \quad \forall 0 \leq a_1 + \cdots + a_d \leq \ell - 2.
\]

**Proof.** Note that \(\sum_{i=1}^{N} m_i \|x - y^{(i)}\|^{2(s-1)}\) is a polynomial of degree \(\leq 2s - \ell - 1\) in \(x\) if and only if
\[
\partial_1^{b_1} \partial_2^{b_2} \cdots \partial_\ell^{b_{d}} \left(\sum_{i=1}^{N} m_i \|x - y^{(i)}\|^{2(s-1)}\right) = 0, \quad \forall 2s - \ell \leq b_1 + \cdots + b_d \leq 2(s - 1). \tag{3}
\]

By lemma \([2,3]\) (choose \(s' = s - 1\))
\[
(x_1 - y_1^{(i)})^{a_1} \cdots (x_d - y_d^{(i)})^{a_d} = \sum_{b_1 + \cdots + b_d = 2s' - \ell + 2 = 2s - \ell} \sum_{b_1 + \cdots + b_d = 2s - \ell} C_{b_1 \cdots b_d} \partial_1^{b_1} \cdots \partial_\ell^{b_{d}} [(x_1 - y_1^{(i)})^2 + \cdots + (x_d - y_d^{(i)})^2]^{s-1}
\]
for some real \(C_{b_1 \cdots b_d}\). So, as polynomials in \(x\), we have
\[
\sum_{i=1}^{N} m_i (x_1 - y_1^{(i)})^{a_1} \cdots (x_d - y_d^{(i)})^{a_d}
\]
\[
= \sum_{i=1}^{N} m_i \sum_{b_1 + \cdots + b_d = 2s - \ell} C_{b_1 \cdots b_d} \partial_1^{b_1} \cdots \partial_\ell^{b_{d}} [(x_1 - y_1^{(i)})^2 + \cdots + (x_d - y_d^{(i)})^2]^{s-1}
\]
\[
= \sum_{b_1 + \cdots + b_d = 2s - \ell} C_{b_1 \cdots b_d} \sum_{i=1}^{N} m_i \partial_1^{b_1} \cdots \partial_\ell^{b_{d}} [(x_1 - y_1^{(i)})^2 + \cdots + (x_d - y_d^{(i)})^2]^{s-1}
\]
\[
= 0. \quad \text{(by (3))}
\]
By putting \( x_1 = \cdots = x_d = 0 \), we have done. \( \square \)

# 3 Improvement of Generalization of LRS Theorem

Now, we can prove Theorem \ref{1.3}

**Proof of Theorem \ref{1.3}** Fix \( i \in \{1, 2, \ldots, s\} \). For each \( y \in X \), we define the polynomial

\[
F_y(x) = \prod_{j=1,2,\ldots,s, j\neq i} \frac{\alpha_j^2 - \|x - y\|^2}{\alpha_j^2 - \alpha_i^2},
\]

then \( F_y \in W_{s-1}(\mathbb{R}^d) \) for each \( y \in X \). In order to prove Theorem \ref{1.3} we only need to show that the two sets

\[
\{F_y(x) : y \in X\} \quad \text{and} \quad \{x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_d^{\lambda_d} : 0 \leq \lambda_1 + \lambda_2 + \cdots + \lambda_d \leq s - 2\}
\]

are linearly independent functions on \( \mathbb{R}^d \), because

\[
\dim(\text{span}(\{x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_d^{\lambda_d} : 0 \leq \lambda_1 + \lambda_2 + \cdots + \lambda_d \leq s - 2\})) = \binom{d+s-2}{s-2}
\]

and Lemma \ref{2.2} gives

\[
\dim(W_{s-1}(\mathbb{R}^d)) \leq \binom{d+s-1}{s-1} + \binom{d+s-2}{s-2}.
\]

By the independence of \ref{4}, we have

\[
\dim(\text{span}\{F_y(x) : y \in X\}) \leq \binom{d+s-1}{s-1} + \binom{d+s-2}{s-2} - \binom{d+s-2}{s-2} = \binom{d+s-1}{s-1}.
\]

By the following two conditions:

1. \( \text{span}(\{F_y(x) : y \in X\}) \) is a linear subspace of \( P_{s-1}(\mathbb{R}^d) \) with dimension \( \leq \binom{d+s-1}{s-1} \).

2. \( F_y(y) = \prod_{j \neq i} \frac{\alpha_j^2}{\alpha_j^2 - \alpha_i^2} \) is a constant for all \( y \in X \), \( F_y(x) = 1 \) if \( d(x,y) = \alpha_i \), \( F_y(x) = 0 \) if \( d(x,y) \neq \alpha_i \), and \( F_y(x) = F_x(y) \) for all \( x, y \in X \).

in conjunction of Lemma \ref{2.1} the theorem follows. Now, we prove the independence of \ref{4}. Suppose

\[
\sum_{y \in X} C_y F_y(x) + \sum_{0 \leq \lambda_1 + \lambda_2 + \cdots + \lambda_d \leq s-2} C_{\lambda_1,\lambda_2,\ldots,\lambda_d} x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_d^{\lambda_d} = 0 \tag{5}
\]

for some \( C_y, C_{\lambda_1,\lambda_2,\ldots,\lambda_d} = C_\lambda \in \mathbb{R}^d \), with \( y \in X, 0 \leq \lambda_1 + \lambda_2 + \cdots + \lambda_d = \lambda \leq s - 2 \). Choosing \( x = u \in X \) in \ref{5}, we get

\[
\left(\prod_{j \neq i} \frac{\alpha_j^2}{\alpha_j^2 - \alpha_i^2}\right) C_u + \sum_\lambda C_\lambda u^\lambda = 0
\]

By multiplying \( C_u \) and summing over \( u \in X \), we get

\[
\left(\prod_{j \neq i} \frac{\alpha_j^2}{\alpha_j^2 - \alpha_i^2}\right) \sum_{y \in X} C_y^2 + \sum_\lambda C_\lambda \sum_{y \in X} C_y y_1^{\lambda_1}y_2^{\lambda_2}\cdots y_d^{\lambda_d} = 0. \tag{6}
\]
It is enough to show
\[ \sum_{y \in X} C_y y_1^{\lambda_1} y_2^{\lambda_2} \cdots y_d^{\lambda_d} = 0 \] (7)
for any \(0 \leq \lambda_1 + \lambda_2 + \cdots + \lambda_d \leq s - 2\), since this implies \(\sum_{y \in X} C_y^2 = 0\) in (5), and hence \(C_y = 0\) for all \(y \in X\).

Now, we prove (7) by using induction on \(\lambda := \lambda_1 + \lambda_2 + \cdots + \lambda_d\). For the basic case \(\lambda = 0\), it is clear that
\[ \sum_{y \in X} C_y = 0 \]
by comparing the term of degree \(2s - 2\) in (5). Assume that (7) holds for \(0 \leq \lambda \leq \ell - 3\), and show that (7) holds for \(0 \leq \lambda \leq \ell - 2\) if \(3 \leq \ell \leq s\). Expand
\[ \sum_{y \in X} C_y F_y = \sum_{y \in X} C_y \sum_{t=1}^{s} A_t \|x - y\|^{2(s-t)} = \sum_{t=1}^{s} A_t \sum_{y \in X} C_y \|x - y\|^{2(s-t)} \]
(8)
for some \(A_t \in \mathbb{R}\). By induction hypothesis, the \(y^\lambda\) terms with degree \(0 \leq \lambda \leq \ell - 3\) vanish in \(\sum_{y \in X} C_y \|x - y\|^{2(s-t)}\), so \(\sum_{y \in X} C_y \|x - y\|^{2(s-t)}\) as polynomial in \(x\) has degree at most \(2(s-t) - (\ell - 2)\). Note that the only term in (8) which allows degree \(2s - \ell\) in \(x\) is the term \(t = 1\), so
\[ \sum_{y \in X} C_y \|x - y\|^{2(s-1)} \]
is a polynomial in \(x\) of degree \(\leq 2s - \ell\). Since there are no terms in \(\sum_{y \in X} C_y F_y\) with degree \(2s - \ell\) (by (4)), so \(\sum_{y \in X} C_y \|x - y\|^{2(s-1)}\) has degree \(\leq 2s - \ell - 1\). Using Lemma 2.4 (with \(N = \|X\|, m_i = C_y\)), we have (7) holding for \(0 \leq \lambda \leq \ell - 2\) if \(3 \leq \ell \leq s\). Thus, by induction, we have shown (7). This completes the proof of Theorem 1.3.

4 Finitely Many \(s\)-distance Sets

In 1966, Einhorn and Schoenberg [3] proved that there are only finitely many two-distance sets \(X\) in \(\mathbb{R}^d\) with \(\|X\| \geq d + 2\). In [6], Nozaki also generalized this theorem from the situation of two-distance sets to \(s\)-distance sets, which stated as below:

**Theorem 4.1. (6)** There are finitely many \(s\)-distance sets \(X\) in \(\mathbb{R}^d\) with
\[ \|X\| \geq 2 \binom{d + s - 1}{s - 1} + 2 \binom{d + s - 2}{s - 2}. \]

This theorem is proved by using Theorem 1.2 and the following lemma in [6].

**Lemma 4.1. (6)** Let \(X\) be an \(s\)-distance set with \(\|X\| \geq 2 \binom{d + s - 1}{s - 1} + 2 \binom{d + s - 2}{s - 2}\) and \(A(X) = \{\alpha_1, \alpha_2, \cdots, \alpha_s = 1\}\). Suppose \(k_i\) are the ratios in Theorem 1.2 Then the distances \(\alpha_i\) are uniquely determined from given integers \(k_i\).

The bound \(\|X\| \geq 2 \binom{d + s - 1}{s - 1} + 2 \binom{d + s - 2}{s - 2}\) in Lemma 4.1 is actually coming from the bound in Theorem 1.2. Since this bound have been improved by our Theorem 1.3, we can also reduce the requirement of Theorem 4.1 to the bound \(\|X\| \geq 2 \binom{d + s - 1}{s - 1}\) immediately. Therefore, we have the following theorem:
Theorem 4.2. There are finitely many $s$-distance sets $X$ in $\mathbb{R}^d$ with

$$|X| \geq 2 \binom{d+s-1}{s-1}.$$

However, for $s = 2$, Theorem 4.2 gives $|X| \geq 2^{\frac{d+2-1}{2-1}} = 2d + 2$, which is bigger than the bound $|X| \geq d + 2$ given by Einhorn and Schoenberg. Hence, the bound in Theorem 4.2 might be improved.

5 Discussion

If $s = 2$, Theorem 1.3 proves that any two-distance set $X$ in $\mathbb{R}^d$ with $|X| \geq 2^{\frac{d+2-1}{2-1}} = 2d + 2$ has the integer condition. This is the same as the result of Neumaier’s improvement of LRS theorem in 1981. Furthermore, if $X$ is a two-distance set in $\mathbb{R}^d$ with $|X| = 2d + 1$, Neumaier provided the counterexamples from the spherical embedding of conference graphs. Therefore, the bound of $|X|$ in Theorem 1.3 cannot be improved for any $s$-distance set in $\mathbb{R}^d$ in general. We are curious about whether the bounds can be improved or not for $s \geq 3$.

We think that those $s$-distance sets in $\mathbb{R}^d$ without integer condition can be obtained from the coherent configurations which contains a spherical embedding of conference graphs. However, such example may not have the size larger enough to attain the lower bounds $2^{\frac{d+s-1}{s-1}}$ in Theorem 1.3.

It is interesting to ask if there exists $s$-distance sets with size $2^{\frac{d+s-1}{s-1}} - 1$ in $\mathbb{R}^d$ without integer condition for some $s \geq 3$.

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