Penalty function method for imposing nonlinear multi freedom and multi node constraints in finite element analysis of frame systems

Vu Thi Bich Quyen¹, Dao Ngoc Tien¹, Nguyen Nhu Dung² and Cao Quoc Khanh ³

¹ Hanoi Architectural University
² Vietnam University of Fire
³ Mien Tay Construction University

E-mail: bquyen1312@gmail.com

Abstract. This paper focuses on the treatment of nonlinear multi-freedom and multi-point boundary condition in finite element analysis of frame systems. The treatment of boundary constraints is required to produce modified system of equations based on master stiffness equations considering nonlinear constraints. For imposing nonlinear multi freedom and multi node constraints, the Penalty Augmentation method and Lagrange Multiplier Adjunction method are better in many applications. In present paper, the Penalty Augmentation method is using for implementation of nonlinear boundary constraints. The nonlinear relationship considerably increases the difficulty in constructing and solving the modified system of equations. The Newton Raphson method, have been the most powerful and popular method for solution of nonlinear equations, is used for solving studied problem. Using the Newton Raphson technique, this paper develops the incremental-iterative algorithm to solve the nonlinear modified system of equations. Based on the presented algorithm, the paper proposed calculation procedure and established programs for determining internal forces and displacements of frames having nonlinear multi freedom and multi point constrains boundary. The numerical test results using the proposed method show the efficiency and reliability of proposed algorithm.

1. Introduction

The solution of any finite element method problem is highly dependent on the boundary conditions implemented. A boundary condition is a set of constraints imposed on nodal coordinates located at the boundaries of a virtual domain. In analyzing the finite element models with multi freedom and multi node constraints, the implementing boundary constraints is done by changing the assembled master stiffness equations to produce a modified system equation based on the master stiffness equation. Generally, the operation of imposing multi freedom constraints develops by master-slave elimination method, penalty augmentation method or Lagrange multiplier adjunction methods [1,2]. The master-slave method is useful only for simple cases but exhibits serious shortcomings for treating arbitrary constraints. The penalty augmentation method and Lagrange multiplier adjunction are better in many applications, whether linear and nonlinear [3,4]. There are not free of disadvantages. The penalty method has difficulty of choice of weight values that balance solution accuracy with the violation of constraint conditions. The multiplier method is sensitive to the degree of linear independence of the constraints, and the bordered stiffness is singular in the case of the dependent constrains. This research
employed the penalty function method for imposing nonlinear boundary constraints because of its straightforward computer implementation and easily extendible to nonlinear constraints. The nonlinear boundary constraints increases the difficulty in solving the nonlinear modified system of equations and requests the method of solving a nonlinear equation. In recent years, a number of solution procedures for solving the nonlinear equilibrium equation have been discussed in many research papers. The Newton Raphson technique is one of the most powerful and widely used techniques for the solution of nonlinear problems. This paper is intended to construct the incremental-iterative algorithm for solving the nonlinear modified system of equations based on the Newton Raphson method. Based on proposed solving algorithm, the calculation procedure and programs for determining internal forces and displacements of frames are established.

2. Imposing nonlinear multi freedom and multi point constraints

2.1. Nonlinear multi freedom and multi point constraint
Multifreedom constraint is defined as type of the constraint where two or more displacement components are combined to equate to a prescribed value. The multifreedom constraint is described as the canonical form of the constraint, when a multifreedom constraint is defined such that all the interacting nodal displacement components can be assembled to the right side of the equation and the left-hand side is either a zero or a non-zero prescribed value. The multifreedom constraint is nonlinear when the interacting nodal displacements combine in a nonlinear manner. The multifreedom constraint may require that the equation involves interaction of nodal displacements at multiple nodes, in which case, the multifreedom constraint is said to be multi-point or multi node. Unlike the single freedom constrain, using hand (or computer) oriented techniques cannot incorporate multifreedom constraints into the master stiffness equations. Accounting for multifreedom constraints is done by changing the assembled master equations to produce a modified system of equations.

2.2. Penalty function method for incorporating nonlinear multi freedom constrains into the master stiffness equations
The penalty method is the easiest, and perhaps earliest method which was and still is employed in the solution of contact problems. The original penalty method is employed in the solution of equality constrained optimization problems. An extensive discussion of this type of method can be found in in a wide variety of articles.

The concept of penalty function method, for imposing multifreedom constraint, is that each multifreedom constraint is viewed as the presence of a fictitious elastic structural element called penalty element that enforces it approximately. This element is parameterized by a numerical weight \( w \). The exact constraint is recovered if the weight goes to infinity. The multifreedom constraints are imposed by augmenting the finite element model with the penalty elements \([5]\).

Consider the frame system having \( n \) degree of freedom

The global nodal displacement vector of frame system is \( \mathbf{u} = [u_1, u_2, \ldots, u_n]^T, \mathbf{u} \in \mathbb{R}^n \)

The augmented potential energy of the unconstrained finite element model is \( \Pi(\mathbf{u}) \)

Equation of nonlinear multifreedom constrains is expressed as

\[
\mathbf{g}_k(\mathbf{u}) = 0, k = 1..m
\]

The penalty augmented system can be developed by minimization of the augmented potential energy function by incorporating the nonlinear multifreedom constraints (1) as

\[
\min \left\{ \Pi(\mathbf{u}) : \mathbf{g}_k(\mathbf{u}) = 0, k = 1..m, \mathbf{u} \in \mathbb{R}^n \right\}
\]

The augmented potential energy can be defined as

\[
\Pi(\mathbf{u}) = U(\mathbf{u}) - u^T \mathbf{P} = \frac{1}{2} u^T Ku - u^T \mathbf{P}
\]

Where:
- \( U(\mathbf{u}) \) is strain energy of system and \((-\mathbf{u}^T \mathbf{P})\) is external work
- \( \mathbf{P} = \{P_1, P_2, \ldots, P_n\}^T \) is global nodal force vector
- \( \mathbf{K} \) is global stiffness matrix

The concept of the method is to change from the constrained optimization problem to non-constrained optimization problem using the penalty objective function \( Q(\mathbf{u}, \mathbf{w}), [6,7] \).

\[
Q(\mathbf{u}, \mathbf{w}) = \frac{1}{2} \mathbf{u}^T \mathbf{K} \mathbf{u} - \mathbf{u}^T \mathbf{P} + \frac{1}{2} \mathbf{w}^T \sum_{k=1}^{\mathbf{m}} g_k^2(\mathbf{u})
\]  

(4)

Where \( \mathbf{w} \) is the penalty weight or penalty element.

Solving the followed non-constrained optimization

\[
\min \{Q(\mathbf{u}, \mathbf{w}) : \mathbf{u} \in \mathbb{R}^n\}
\]  

(5)

First, derivative of \( Q(\mathbf{u}, \mathbf{w}) \) with respect to \( \mathbf{u} \), setting equal to zero as below

\[
\mathbf{K} \mathbf{u} - \mathbf{P} + \mathbf{w} \sum_{k=1}^{\mathbf{m}} g_k(\mathbf{u}) \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} = 0
\]  

(6)

Where \( \delta \mathbf{u} = \{\delta u_1, \delta u_2, \ldots, \delta u_n\}^T \) is nodal incremental displacement vector.

Equation (6) can be written as

\[
\mathbf{K} \mathbf{u} + \mathbf{w} \sum_{k=1}^{\mathbf{m}} g_k(\mathbf{u}) \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} = \mathbf{P}
\]  

(7)

Where

\[
\frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix}
\frac{\partial g_k(\mathbf{u})}{\partial u_1} \\
\frac{\partial g_k(\mathbf{u})}{\partial u_2} \\
\vdots \\
\frac{\partial g_k(\mathbf{u})}{\partial u_n}
\end{bmatrix}; \quad \mathbf{P} = \begin{bmatrix}
P_1 \\
P_2 \\
\vdots \\
P_0
\end{bmatrix}
\]

Produced system of equation (7) is nonlinear. The basic concept of nonlinear analysis using finite element method is to divide total load into many steps \( \Delta \mathbf{P} \), carry out linear analysis in each loading step, and sum the results to obtain final.

For constructing the incremental equation, utilizing Taylor series formula for a short of \( \delta \mathbf{u} \) to expand function of (7) around of variable point, keeping only linear term in \( \delta \mathbf{u} \), getting

\[
\mathbf{K} \delta \mathbf{u} + \mathbf{w} \sum_{k=1}^{\mathbf{m}} \frac{\partial}{\partial \mathbf{u}} \left( g_k(\mathbf{u}) \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} \right) \delta \mathbf{u} = \mathbf{P} + \Delta \mathbf{P} - \left\{ \mathbf{K} \mathbf{u} + \mathbf{w} \sum_{k=1}^{\mathbf{m}} g_k(\mathbf{u}) \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} \right\}
\]  

(8)

Equation (8) can be written as below

\[
\mathbf{K} \delta \mathbf{u} + \mathbf{w} \sum_{k=1}^{\mathbf{m}} \frac{\partial}{\partial \mathbf{u}} \left( g_k(\mathbf{u}) \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} \right) \delta \mathbf{u} = \mathbf{P} + \Delta \mathbf{P} - \left\{ \mathbf{K} \mathbf{u} + \mathbf{w} \sum_{k=1}^{\mathbf{m}} g_k(\mathbf{u}) \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} \right\}
\]  

(9)

To expand the second component in left hand side of (9)

\[
\mathbf{w} \sum_{k=1}^{\mathbf{m}} \frac{\partial}{\partial \mathbf{u}} \left( g_k(\mathbf{u}) \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} \right) \delta \mathbf{u} = \mathbf{w} \sum_{k=1}^{\mathbf{m}} \left\{ \frac{\partial}{\partial \mathbf{u}} \left( \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} \right) g_k(\mathbf{u}) \delta \mathbf{u} + \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} \right\} \left( \frac{\partial g_k(\mathbf{u})}{\partial \mathbf{u}} \right)^T \delta \mathbf{u}
\]  

(10)

Utilizing Taylor series formula for a short of \( \delta \mathbf{u} \) to expand function of (1) around of variable point, keeping only linear term in \( \delta \mathbf{u} \), getting
\[
g_k(u) + \left( \frac{\partial g_k(u)}{\partial u} \right)^T \delta u = 0
\] 

From (10) and (11), having

\[
w \sum_{k=1}^{m} \frac{\partial}{\partial u} \left( g_k(u), \frac{\partial g_k(u)}{\partial u} \right) \delta u = w \sum_{k=1}^{m} \left\{ -\delta \left( \frac{\partial g_k(u)}{\partial u} \right) \left( \frac{\partial g_k(u)}{\partial u} \right)^T \delta u + \frac{\partial g_k(u)}{\partial u} \left( \frac{\partial g_k(u)}{\partial u} \right)^T \delta u \right\}
\]

Second order infinitesimal \( \delta u, \delta u \) right hand side of (12), getting

\[
w \sum_{k=1}^{m} \frac{\partial}{\partial u} \left( g_k(u), \frac{\partial g_k(u)}{\partial u} \right) \delta u = w \sum_{k=1}^{m} \left\{ -\delta \left( \frac{\partial g_k(u)}{\partial u} \right) \left( \frac{\partial g_k(u)}{\partial u} \right)^T \delta u \right\}
\]

Incorporating (13) into (9), the incremental equation can be rewritten as

\[
\left[ K + w \sum_{k=1}^{m} \frac{\partial g_k(u)}{\partial u} \left( \frac{\partial g_k(u)}{\partial u} \right)^T \right] \delta u = P + \Delta P - \left\{ Ku + w \sum_{k=1}^{m} \frac{\partial g_k(u)}{\partial u} \right\}
\]

Obviously, \( \frac{\partial}{\partial u} \left( \frac{\partial g_k(u)}{\partial u} \right) = \frac{\partial^2 g_k(u)}{\partial u \partial u} \) \( i, j = 1..n \) is the Hessian matrix of the \( g_k(u) \)

Incremental equation (14) can be written compactly as

\[
\underbrace{K(\bar{u}), \delta \bar{u}}_{\text{K(\bar{u})}} = P + \Delta P - q(\bar{u})
\]

Where

\[
\bar{u} = \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \text{ is vector consist of displacement unknowns; } \bar{P} = \begin{bmatrix} \bar{P}_1 \\ \bar{P}_2 \\ \vdots \\ \bar{P}_n \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ \vdots \\ P_n \end{bmatrix}
\]

\[
\delta \bar{u} = \begin{bmatrix} \delta \bar{u}_1 \\ \delta \bar{u}_2 \\ \vdots \\ \delta \bar{u}_n \end{bmatrix} ; \quad K(\bar{u}) = \left[ K + w \sum_{k=1}^{m} \frac{\partial g_k(u)}{\partial u} \left( \frac{\partial g_k(u)}{\partial u} \right)^T \right] ; \quad q(\bar{u}) = \left\{ Ku + w \sum_{k=1}^{m} g_k(u), \frac{\partial g_k(u)}{\partial u} \right\}
\]

3. Incremental-iterative algorithm for solving the nonlinear modified system of equations based on the Newton Raphson method

Methods of finding roots of nonlinear equation mostly are bracketing (such as bisection and false position method) and open methods. Bracketing methods require bracketing of root by two guesses always convergent since they are based on reducing the interval between the two guesses so as to zero in on the root of the equation. Open methods differ from bracketing methods so that only one initial guess of the root is needed to get the iterative process started to find the root of an equation. Convergence in open methods is not guaranteed but if the method does converge, it does so much faster than the bracketing methods.

The Newton Raphson method is open method for solving nonlinear system of equations. The concept of this method based on estimating the new value of the root depending initial guess and Newton Raphson formula and repeating this process until one find the root within a desirable.

In structural analysis, the Newton Raphson type methods or load control methods have been the most powerful and popular for solving nonlinear system of equations. Furthermore, many advances in nonlinear solvers consist of variations of the basic Newton Raphson method.

Using Newton Raphson technique, external loads are computed at the first iteration of each incremental step and held constant throughout the remaining iterations in the step, as illustrated in Fig.1. This method sets the load increment parameter to 1, for first iteration \( (j = 1) \), and to zero for the
rest of the iterations \((j > 2)\). The first iteration in this process is identical to the linear incremental method except at the end of the first iteration, member forces are calculated and transformed into the global coordinates. Then, unbalanced load vector between the applied external load and the internal nodal forces are determined. Subsequent iterations are employed until a predefined convergence criterion is satisfied.

![Diagram](https://via.placeholder.com/150)

**Figure 1.** Newton Raphson technique

The block diagram of algorithm for solving the nonlinear modified system of equations is established (shown in Fig.2) based on the Newton Raphson technique.
Figure 2. Incremental-iterative procedure for solving nonlinear modified system of equations based on Newton Raphson technique

4. Test example

Based on proposed above incremental-iterative algorithm for solving the nonlinear modified system, the calculation program for static analysis of finite element truss is written in MathCAD software.

4.1. Example formulation

The system is composed of bars made of the same material and had the same geometrical properties (system is shown in fig 3), having nonlinear multi freedom and multi node constrain. The geometric parameters, material parameters and loading parameters are given:

\[ E = 2.10^4 \text{(kN/cm}^2\text{)}, \ A = 10\text{cm}^2, \ L = 300\text{cm}, \ H_1 = 300\text{cm}, \ H_2 = 100\text{cm}, \ \beta = 1. \]

Equation of nonlinear multi freedom and multi node constrains is expressed as
Multi node constraint

\[ g_1(u) = (L + u_3 - u_1)^2 + (H_2 + u_6 - u_4)^2 - (L^2 + H_2^2) = 0 \]

Multi freedom constraint

\[ g_2(u) = u_2 - \beta u_1^2 = 0 \]

For investigating the convergence speed of the proposed method, the problem was solve with different options of weight values \( w \), as following

\[ w_{(1)} = 10^3 = 10^{-2}EA; w_{(2)} = 10^4 = 10^{-3}EA; w_{(3)} = 10^5 = EA; w_{(4)} = 10^6 = 10EA; w_{(5)} = 10^7EA \]

4.2. Numerical results

The calculating results are nodal displacements and internal forces. The load - nodal displacement and load - internal force relationships shown in Fig. 4-14. The red, blue, black, pink and green diagrams are the results corresponding to the results in cases with weight values \( w_{(1)}, w_{(2)}, w_{(3)}, w_{(4)} \) and \( w_{(5)} \).
5. Conclusion
The proposed method is simple and effective method for imposing nonlinear multi freedom and multi node constraints in finite element analysis of frame systems.

The root convergence factor depends on the way to choose appropriate weights \( w \). Analysing test results show that, the solution will be converged faster in the case of increasing the weight value. It will be better for convergence speed when selecting the initial value of penalty element approximately (or about) value of element stiffness.

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