U-projectors and fields of U-invariants

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Abstract
We present the general construction of the U-projector (the homomorphism of the algebra into its field of U-invariants identical on the subalgebra of U-invariants). It is shown how to apply U-projector to find the systems of free generators of the fields of U-invariants for representations of reductive groups.

1 Introduction

It is well known that the fields of invariants of the triangular transformation groups of affine varieties are rational (see. [1, 2, 4]). In this paper, for the special instances, we are going to present the systems of free generators. For this purpose we introduce the notion of U-projector (it is a homomorphism of the algebra $K[X]$ to the field of invariants $K(X)^U$ identical on $K[X]^U$). In the section 2, we verify existence of U-projector and present its general construction. Furthermore, we show that applying the U-projector to the suitable system of functions $b_1, \ldots, b_m$ one can obtain the system of functions $P(b_1), \ldots, P(b_m)$ that freely generate the field of U-invariants.

Notice that the general U-projector construction method is realized by the induction method on the length of ideals chain. Since the length might by rather great, this method don’t provide the exact formula for U-projector. However, a priory the U-projector is not unique. In the next sections 3,4,5, we return to the problem of U-projector construction in the special cases. Our goal is to improve the above U-projector construction to make it more precise. We also plan to choose the system of functions $b_1, \ldots, b_m$ such that the system $P(b_1), \ldots, P(b_m)$ freely generate the field of U-invariants. The main results are formulated in the theorems 2.2, 2.3, 3.2, 4.2, 5.1 Notice that in the paper [3] the other approach is presented for the description of generators of the field of U-invariants for the adjoint representation.

2 The general construction of U-projector

Let $u$ be an nilpotent Lie algebra over a field $K$ of zero characteristic, $U = \exp(u)$ be the corresponding group, $\mathcal{A}$ a commutative associative finitely

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generated algebra defined over the field $K$ and without zero devisors, $\mathcal{F}$ be the field $\text{Frac}(\mathcal{A})$. Let $D$ be a homomorphism of the Lie algebra $\mathfrak{u}$ into the Lie algebra of locally nilpotent derivations of the algebra $\mathcal{A}$. Then the group $U$ acts on $\mathcal{A}$ by the formula $g(a) = \exp D_x(a)$, $g = \exp(x)$. The ring (the field) of $U$-invariants coincides with the ring (the field) of $\mathfrak{u}$-invariants. The field of $U$-invariants $\mathcal{F}^U$ is a field of fractions of the ring of $U$-invariants $\mathcal{A}^U$ [5, Theorem 3.3]. It is known that the field $\mathcal{F}^U$ is a pure transcendental extension of the main field $K$ (see [1, 2, 4]).

By definition, an $U$-projector is an arbitrary homomorphism $P : \mathcal{A} \rightarrow \mathcal{F}^U$ identical on $\mathcal{A}^U$. We are going to present the general construction of $U$-projector. One can construct a free generator system of generators of the field $\mathcal{F}^U$ in terms of $U$-projector.

Fix the chain of ideal $\mathfrak{u} = \mathfrak{u}_n \supset \mathfrak{u}_{n-1} \supset \ldots \supset \mathfrak{u}_1 \supset \mathfrak{u}_0 = 0$, where $\text{codim}(\mathfrak{u}_i, \mathfrak{u}_{i+1}) = 1$. For each pair $\mathfrak{u}_i \supset \mathfrak{u}_{i-1}$, the subalgebra of invariants $\mathcal{A}^u_i$ is contained in $\mathcal{A}^{u_{i-1}} \mathcal{A}^{u_0} = \mathcal{A}$. Let $i_1$ be the least number such that $\mathcal{A}^{u_{i_1}} \neq \mathcal{A}$. Fix $x_i \in \mathfrak{u}_i \setminus \mathfrak{u}_{i-1}$.

Lemma 2.1. There exist the elements $a_{i,1} \in \mathcal{A}$, $a_{1,0} \in \mathcal{A}^u$, $a_{1,0} \neq 0$ obeying

$$D_{x_i}(a_{1,1}) = a_{1,0}. \tag{1}$$

Proof. Since $D_{x_i}$ is a local nilpotent derivation, there exist $a_{i,1} \in \mathcal{A}_i$, $a_{1,0} \in \mathcal{A}^{u_{i_1}}$, $a_{1,0} \neq 0$. It is sufficient to prove $a_{1,0} \in \mathcal{A}^u$. Really, for any $y \in \mathfrak{u}$ the element $[y, x_i] \in \mathfrak{u}_{i-1}$. Therefore, $D_{[y, x_i]}(a) = 0$ for all $a \in \mathcal{A}$. Then

$$D_y(a_{1,0}) = D_y D_{x_i}(a_{1,1}) = D_{x_i} D_y(a_{1,1}) + D_{[y, x_i]}(a_{1,1}) = D_{x_i} D_y(a_{1,1}). \tag{2}$$

The formula (2) implies that the least containing $a_{1,0}$ and $D_u$-invariant subspace $< a_{1,0} >$ is contained in $\text{Im}D_{x_i}$. Since $\mathfrak{u}$ is a finite dimensional nilpotent Lie algebra and every derivation $\mathfrak{u}$ is locally nilpotent, the subspace $< a_{1,0} >$ is finite dimensional. The representation $D_u$ has a triangular form in $< a_{1,0} >$. There exists a nonzero vector annihilated by all $D_y$, $y \in \mathfrak{u}$. We may assume that it is $a_{1,0}$. $\square$

Let $a_{1,0}$ and $a_{1,0}$ be as in lemma 2.1. The element $Q_1 = a_{1,1} a_{1,0}^{-1}$ belongs to the localization $\mathcal{A}_1$ of the algebra $\mathcal{A}$ with respect to the denominator system generated by $a_{1,0}$, it obeys $D_{x_i}(Q_1) = 1$. Consider the linear mapping $S_1 : \mathcal{A} \rightarrow \mathcal{A}^{u_{i_1}}$ defined by

$$S_1(a) = a - D_{x_i}(a)Q_1 + D_{x_i}^2(a)\frac{Q_1^2}{2!} + \ldots + D_{x_i}^k(a)\frac{Q_1^k}{k!} + \ldots. \tag{3}$$

The mapping $S_1$ is an algebra homomorphism identical on $\mathcal{A}^{u_{i_1}}$. One can extend the representation $D$ of the Lie algebra $\mathfrak{u}$ to $\mathcal{A}_1$, and each derivation
$D_y$ remains locally nilpotent on $A_1$. The mapping $S_1$ to an homomorphism $A_1$ to $A_1^{u_1}$ identical on $A_1^{u_1}$.

The subalgebra $A_1^{u_1}$ is invariant with respect to all $D_x$, $x \in u$. Substitute $A$ by $A_1^{u_1}$, and proceed as above. Let $i_2$ be least number obeying $A_1^{u_2} \neq A_1^{u_1}$. Choose $x_2 \in u_{i_2} \setminus u_{i_2-1}$. As above there exist the elements $a_{2,1} \in A_1^{u_1}$ and $a_{2,0} \in A_1^u$, $a_{2,0} \neq 0$ such that

$$D_{x_{i_2}}(a_{2,1}) = a_{2,0}.$$ 

Let $A_2$ stands for localization of the algebra $A$ with respect to the denominator system generated by $a_{1,0}, a_{2,0}$. Consider the element $Q_2 = a_{2,1}a_{2,0}^{-1}$ of the algebra $A_2$. Similarly (3), we construct the homomorphism $S_2 : A_1^{u_1} \to A_2^{u_2}$.

Proceeding further we obtain the chain $n \geq i_m > \ldots > i_2 > i_1 \geq 1$, the systems of elements $a_{k,0} \in A^u$ and $a_{k,1} \in A$ obeying (1), and the mappings $S_1, S_2, \ldots S_m$, where each $S_k$ is a homomorphism $A_1^{u_{k-1}} \to A_2^{u_k}$ identical on $A_1^{u_{k-1}}$. Denote by $A_*$ the localization of the algebra $A$ with respect to the denominator system generated by $a_{1,0}, a_{2,0}, \ldots, a_{m,0}$. Consider the mapping

$$P = S_m \circ \ldots \circ S_2 \circ S_1.$$ 

From all above we conclude.

**Theorem 2.2.** The mapping $P$ is a homomorphism of the algebra $A$ into $A_1^u$ identical on $A_1^u$. That is $P$ is a $U$-projector.

**Remark.** Since $\{a_{k,0}\} \subset A^u$, the projector $P$ can be extended to a projector $A_* \to A_*^u$ identical on $A_*^u$.

Let the group $U$ rationally act on the irreducible affine algebraic variety $X$ defined over the field $K$. Then the group $U$ naturally acts in the algebra of regular functions $K[X]$ be the formula

$$T_g f(x) = f(g^{-1}x).$$

Any regular function on $X$ is contained in some finite dimensional invariant subspace [5, Lemma 1.4]. The representation $D = d_e T$ of the Lie algebra $u$ is nilpotent in this subspace. Therefore, for every $x \in u$ the operator $D_x$ is a locally nilpotent derivation of the algebra $A = K[X]$. As above there exist rational functions $a_{k,0}(x)$, $a_{k,1}(x)$, $1 \leq k \leq m$. Notice that $\{a_{k,0}(x)\}$ are $U$-invariant, moreover $a_{1,0}$ and $a_{1,1}$ are regular, and $a_{k,0}(x)$, $a_{k,1}(x)$ belong to the localization of the algebra of regular functions with respect to the denominator subset generated by $a_{i,0}(x)$, $1 \leq i \leq k - 1$. Consider the $U$-invariant open subset $X_0 = \{x \in X_0 : a_{k,0}(x) \neq 0, 1 \leq k \leq m\}$ and its subset

$$\mathcal{G} = \{x \in X_0 : a_{k,1}(x) = 0, 1 \leq k \leq m\}.$$
There is the restriction mapping $\mathrm{Res} : K[X_0] \to K[\mathcal{G}]$.

**Theorem 2.3.** Assume that the system $\{a_{k,1} : 1 \leq k \leq m\}$ generate the defining ideal of the subset $\mathcal{G}$ in the algebra $K[X_0]$. Let $b_1, \ldots, b_s \in \mathcal{A}$ be the system of elements such that

$$\mathrm{Res}(b_1), \ldots, \mathrm{Res}(b_s), \mathrm{Res}(a_{0,0})^{\pm 1}, \ldots, \mathrm{Res}(a_{m,0})^{\pm 1}$$

generate the algebra $K[\mathcal{G}]$. Then $P(b_1), \ldots, P(b_s), a_{0,0}^{\pm 1}, \ldots, a_{m,0}^{\pm 1}$ generate the algebra of invariants $K[X_0]^U$. In particular, $P(b_1), \ldots, P(b_s), a_{0,0}^{\pm 1}, \ldots, a_{m,0}^{\pm 1}$ generate the field of invariants $K(X)^U$.

**Proof.** Let $\mathrm{Res}_U$ stands for the restriction of Res on the subalgebra of $U$-invariants. Since all $\mathrm{Res}(Q_i)$ annihilate on $\mathcal{G}$, we have

$$\mathrm{Res} = \mathrm{Res}_U \circ P.$$ 

By the assumption,

$$\mathrm{Res}_U P(b_1), \ldots, \mathrm{Res}_U P(b_s), \mathrm{Res}_U(a_{0,0})^{\pm 1}, \ldots, \mathrm{Res}_U(a_{m,0})^{\pm 1}$$

generate the algebra $K[\mathcal{G}]$. To conclude the proof it is sufficient to prove that $\mathrm{Res}_U$ is an isomorphism of the algebra $K[X_0]^U$ onto $K[\mathcal{G}]$. Actually, the image $\text{Im}(\mathrm{Res}_U)$ coincides with $K[\mathcal{G}]$. Let us show that $\text{Ker}(\mathrm{Res}_U) = 0$. If $f$ is an $U$-invariant and $\mathrm{Res}_U(f) = 0$, then $\mathrm{Res}(f) = 0$. Hence $f = \phi_1 a_{1,1} + \ldots + \phi_m a_{m,1}$ for some $\phi_1, \ldots, \phi_m \in K[X_0]$ and

$$f = P(f) = P(\phi_1)P(a_{1,1}) + \ldots + P(\phi_m)P(a_{m,1}). \quad (4)$$

Let us prove that $P(a_{k,1}) = 0$ for each $1 \leq k \leq m$. Since the function $a_{k,1}$ is $u_{i_k}$-invariant, $S_i(a_{k,1}) = a_{k,1}$ for all $1 \leq i < k$. The formula (3) implies $S_k(a_{k,1}) = 0$. Then

$$S_kS_{k-1} \cdots S_1(a_{k,1}) = S_k(a_{k,1}) = 0$$

for all $1 \leq k \leq m$. Therefore $P(a_{k,1}) = 0$ for all $k$. By the formula (4), we conclude $f = 0$. $\square$

### 3 $U$-projectors for the adjoint representations

The goal of this section is to present an exact construction of the $U$-projector for the adjoint representation reductive split group. Let $G$ be a connected reductive split group over a field $K$ of zero characteristic, $\mathfrak{g}$ be its Lie algebra, $\Delta$ be a root system with respect to the Cartan subalgebra $\mathfrak{h}$ (respectively, $\Delta^+$ be a set of positive roots),

$$u = \sum_{\alpha \in \Delta^+} KE_\alpha$$
be the standard maximal nilpotent subalgebra in $\mathfrak{g}$. Via the Killing form we identify $\mathfrak{g}$ with $\mathfrak{g}^*$, and the algebra of regular functions $K[\mathfrak{g}]$ with the symmetric algebra $\mathcal{S}(\mathfrak{g}) = K[\mathfrak{g}^*]$. We extend the adjoint representation $\text{ad}_x$ of the Lie algebra $\mathfrak{g}$ to the representation in $\mathcal{S}(\mathfrak{g})$ by derivations $D_x(a) = \{ x, a \}$, where $x \in \mathfrak{g}, a \in \mathcal{S}(\mathfrak{g})$ and $\{ \cdot, \cdot \}$ is natural Poisson bracket in $\mathcal{S}(\mathfrak{g})$. If $x = E_\alpha$, then we denote $D_\alpha = D_{\alpha}$. Let $\xi = \xi_1$ be one of maximal roots in $\Delta^+$. Consider the subset $\Delta_2$ of the root system that consists of all $\alpha \in \Delta$ obeying $(\alpha, \xi) = 0$. The subset $\Delta_2$ is a root system for the reductive subalgebra $\mathfrak{g}_2 = \{ x \in \mathfrak{g} : [x, E_\xi] = 0 \}$. The subalgebra $\mathfrak{g}_2$ contains the maximal nilpotent subalgebra $\mathfrak{u}_2$ spanned by $E_\alpha$, where $\alpha$ runs through the set of positive roots $\Delta_2$. The set $\Delta^+$ splits into two subsets $\Delta_2 = \Gamma \cup \Delta_2$, where $\Gamma$ consists of all roots $(\alpha, \xi) > 0$. The subset $\Gamma$ contains $\xi$; denote $\Gamma_0 = \Gamma \setminus \{ \xi \}$. For each $\alpha \in \Gamma_0$, there exists a unique $\alpha' \in \Gamma_0$ such that $\alpha + \alpha' = \xi$ (see [3]). The subalgebra $\mathfrak{n} = \mathfrak{n}_1$ spanned by $\{ E_\alpha, \alpha \in \Gamma \}$ is isomorphic to the Heisenberg algebra, and it is an ideal in $\mathfrak{u}$. The element $E_\xi$ is annihilated by all derivations $D_x, x \in \mathfrak{u}$. The derivations $D_x$ can be extended to the derivations of the localization $\mathcal{S}'(\mathfrak{g})$ of the algebra $\mathcal{S}(\mathfrak{g})$ with respect to the denominator subset generated by $E_\xi$. The element

$$Q_\xi = -\frac{1}{2} H_\xi E^{-1}_\xi \in \mathcal{S}'(\mathfrak{g})$$

obeys the equality $D_\xi(Q_\xi) = 1$. Following the formula (3) we construct the projector $S_\xi$ of the algebra $\mathcal{S}'(\mathfrak{g})$ onto the subalgebra of $D_\xi$-invariants. For each $\alpha \in \Gamma_0$, the element

$$Q_\alpha = -\frac{1}{N_{\alpha, \alpha'}} E_\alpha E^{-1}_\xi \in \mathcal{S}'(\mathfrak{g})$$

obeys $D_\alpha(Q_\alpha) = 1$. As above, for each $\alpha \in \Gamma_0$, we construct the projector $S_\alpha$. The mapping

$$P_1 = \left( \prod_{\alpha \in \Gamma_0} S_\alpha \right) \circ S_\xi,$$

is a homomorphism of the algebra $\mathcal{S}(\mathfrak{g})$ to the subalgebra of invariants $\mathcal{S}'(\mathfrak{g})^{n_1}$ identical on $\mathcal{S}(\mathfrak{g})^{n_1}$. One can extend the mapping $P_1$ to the projector of $\mathcal{S}'(\mathfrak{g})$ onto the subalgebra $\mathcal{S}'(\mathfrak{g})^{n_1}$. Notice that the product in the formula (7) does not depend on the ordering of factors.

Lemma 3.1. Let $\mathfrak{m}$ be a Heisenberg algebra, $[\mathfrak{m}, \mathfrak{m}] = Kz$, $\mathcal{S}(\mathfrak{m})_z$ be the localization of $\mathcal{S}(\mathfrak{m})$ with respect to the denominator subset generated by $z$. Let $\mathfrak{m}_0$ stand for the complimentary subspace for $Kz$ in $\mathfrak{m}$. Then if is a derivation $D$ of the algebra $\mathfrak{m}$ obeys $D(z) = 0$ and $D(\mathfrak{m}_0) \subseteq \mathfrak{m}_0$, then there
exists a unique element $b_D \in (m_0)^2 z^{-1} \in S(m)_z$ such that $D(a) = \{b_D, a\}$ for any $a \in m$.

**Proof.** The proof is similar [7, Lemma 4.6.8]. $\Box$

The element $x \in g_2$ provides the derivation $D_x(a) = [x, a]$ of the Heisenberg algebra $n$ with $D_x(E_\xi) = 0$ and $D$ preserves the subspace

$$n_0 = \text{span}\{E_{\alpha} : \alpha \in \Gamma_0\}.$$  

By the lemma (3.1), there exists a unique $b_x \in n_0^2 E_\xi^{-1}$ such that $D_x(a) = \{b_x, a\}$. Then for any element $\tilde{x} = x - b_x$ from $S'(g)$, we obtain $[\tilde{x}, n] = 0$. For any $x, y \in g_2$, we have

$$\{\tilde{x}, y\} = \{x - b_x, y - b_y\} = \{x, y - b_y\} = \{x, y\} - \{x, b_y\}.$$  

Since $\{x, y\} = [x, y] \in g_2 \in \{x, b_y\} \in n_0^2 E_\xi^{-1}$, the lemma 3.1 implies $\{x, b_y\} = b_{[x, y]}$. Hence

$$[\tilde{x}, y] = \{\tilde{x}, y\}$$  

for any $x, y \in g_2$. The subset $\tilde{g}_2 = \{\tilde{x} : x \in g_2\}$ is a Lie algebra with respect to the Poisson bracket in $S'(g)$, and it is isomorphic to $g_2$. Applying (8), we obtain

$$\tilde{D}_x(y) = [\tilde{x}, y] = \{\tilde{x}, y\} = \{x - b_x, y - b_y\} = D_x(\tilde{y})$$  

for any $x, y \in g_2$.

Further we proceed analogically to what have been done for $g$; choose a maximal root $\xi_2$ in $\Delta_2^+$, then we have got the subset of positive roots $\Gamma_2$, the subalgebras $n_2$, $g_3$ in $g_2$. The $Q_{\xi_2}$ is defined similarly as for $\xi$, substituting $H_\xi$ for $\tilde{H}_{\xi_2}$ and $E_\xi$ for $E_{\xi_2}$ in the formula (5). The element $Q_{\xi_2}$ belongs to the subalgebra $S''(g)$ that is the localization of $S(g)$ with respect to the denominator system generated by $E_{\xi_1}$ and $E_{\xi_2}$. The both elements $E_{\xi_1}$ and $E_{\xi_2}$ are $u$-invariants.

Applying (9), we have $D_{\xi_2}(Q_{\xi_2}) = 1$. Following the formula (3), we define the operator $S_{\xi_2}$. Easy to show that if $a \in S'(g)^{n_1}$, then $S_{\xi_2}(a)$ also belongs to $S'(g)^{n_1}$.

Likewise $S_{\xi_2}$ we define $S_\alpha$ for each $\alpha \in (\Gamma_2)_0 = \Gamma_2 \setminus \xi_2$. The mapping

$$P_2 = \left( \prod_{\alpha \in (\Gamma_2)_0} S_{\alpha} \right) \circ S_{\xi_2},$$  

is a homomorphism of the algebra $S(g)$ onto the subalgebra of invariants $S''(g)^{n_2}$, and it is identical in $S(g)^{n_2}$. The operator $P_2$ extends to a projector $S''(g)$ into the subalgebra $S''(g)^{n_2}$. If $a \in S''(g)^{n_1}$, then $P_2(a)$ is invariant.
with respect to all \( D_x, \ x \in n_1 \oplus n_2 \). Therefore, the mapping \( P_2 \circ P_1 \) is a homomorphism of the algebras

\[
S(g) \to S''(g)^{n_1 \oplus n_2},
\]

and it is identical on \( n_1 \oplus n_2 \)-invariants in \( S(g) \).

Continuing the process, we obtain the chain of positive roots \( \xi = \xi_1, \xi_2, \ldots, \xi_m \), that is referred to as a \textit{Kostant cascade}. The maximal nilpotent subalgebra \( u \) decomposes into the sum of Heisenberg subalgebras

\[
u = n_1 \oplus n_2 \oplus \cdots \oplus n_m
\]

with \([n_i, n_j] \subset n_i\) for all \( i < j \).

We define the chain of subalgebras \( g = g_1 \supset g_2 \supset \cdots \supset g_m \) with the systems of positive roots \( \Delta = \Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_m \); each \( \xi_i \) is a maximal root in \( \Delta_i \). By induction method with respect to \( i \), we define the projectors \( P_1, P_2, \ldots, P_m \) such that each product \( P_i \circ \cdots \circ P_1 \) is a projector of the algebra \( S(g) \) into the subalgebra of \( n_1 \oplus \cdots \oplus n_i \)-invariants in the localization of the algebra \( S(g) \) with respect to the denominator system generated by

\[
\Xi = \{E_{\xi_1}, \tilde{E}_{\xi_2}, \ldots, \tilde{E}_{\xi_i}\}.
\]

Take \( \Xi = \Xi_m \). Let \( S(g)_* \) stand for the localization of \( S(g) \) with respect to the denominator system generated by \( \Xi \).

**Theorem 3.2.**

1) The mapping \( P = P_m \circ \cdots \circ P_1 \) is a homomorphism of the algebra \( S(g) \) into \( S(g)^U_* \) identical on \( S(g)^U \), i.e. \( P \) is an \( U \)-projector of the adjoint representation of the group \( G \). 

2) Let \( \{H_i\} \) be a basis orthogonal complement to the Kostant cascade in \( h \). Then the system of elements

\[
\{P(E_{-\alpha}) : \alpha \in \Delta^+\} \cup \{P(H_i)\} \cup \Xi
\]

freely generates \( S(g)^U_* \) (it also freely generates the field of \( U \)-invariants).

**4 U-projectors for an arbitrary representations**

In this section, we present the general scheme of construction of the \( U \)-projector for an arbitrary finite dimensional representation. As in the previous section \( K \) is a field of characteristic zero, \( g \) is a reductive split Lie algebra over the field \( K \), \( G \) is a connected reductive group with the Lie algebra \( g \), \( B \) is a Borel subgroup in \( G \) that contains the Cartan subgroup \( H \) and the maximal unipotent subgroup \( U = B' \), their Lie algebras are \( b, h, u \).
Let \( V \) be an arbitrary finite dimensional representation of the group \( G \). This representation defines the representation \( f(v) \to f(g^{-1}v) \) in the algebra \( \mathcal{A} = K[V] \). The corresponding representation of the Lie algebra \( \mathfrak{g} \) is realized in this space by the formula

\[
D_x f(v) = -f(xv), \quad x \in \mathfrak{g}.
\]

For any \( x \in \mathfrak{u} \), the operator \( D_x \) is a locally nilpotent derivation of the algebra \( \mathcal{A} \).

Decompose \( V \) into a direct sum \( V = W_0 \oplus W_1 \), where \( W_0 \) is an irreducible representation, and \( W_1 \) is its invariant complement. Suppose that \( \dim W_0 > 1 \). Choose a lowest vector \( v_0 \) in \( W_0 \). The stabilizer \( p^- \) of the one dimensional subspace \( \langle v_0 \rangle \) is a parabolic subalgebra in \( \mathfrak{g} \) containing \( \mathfrak{h} \). Suppose that \( p^- \neq \mathfrak{g} \). Acting by the Cartan involution \( \theta \) on \( p^- \), we obtain the parabolic subalgebra \( p^- \). The intersection \( g_1 = p \cup p^- \) is a Levi subalgebra in \( p \) (and in \( p^- \)). Let \( m = \text{rad}(p) \).

Here and further, \( \gamma_1 \leq \gamma_2 \) is an ordering on the set of all weights such that \( \gamma_2 - \gamma_1 \) is a sum of simple roots with nonnegative coefficients.

We extend \( v_0 \) to the basis \( v_0, v_1, \ldots, v_k, \ldots, v_n \) in \( V \) as follows

1) if \( v_i \in W_0 \) and \( v_j \in W_1 \), then \( i < j \);
2) let \( E_{\alpha_1}, \ldots, E_{\alpha_k} \) be the basis of \( m \) over \( K \), then \( v_i = E_{\alpha_i} v_0, \ 1 \leq i \leq k \), is a basis in \( m v_0 \), and \( v_{k+1}, \ldots, v_n \) is a basis in the \( g_1 \)-invariant complement for \( \langle v_0 \rangle \oplus m v_0 \) in \( V \);
3) each vector \( v_i \) is a weight vector with respect to \( \mathfrak{h} \), moreover if \( v_i, v_j \in W_0 \) and \( \text{wt}(v_i) < \text{wt}(v_j) \), then \( i < j \) (i.e. \( \alpha_i < \alpha_j \) implies \( i < j \)).

Let \( \omega_0, \omega_1, \ldots, \omega_k, \ldots, \omega_n \) be the dual basis. The linear form \( \omega_0 \) is invariant with respect to \( \mathfrak{u} \); we extend the operators \( D_x, \ x \in \mathfrak{u} \) to locally nilpotent derivations of the localization \( \mathcal{A}_{\omega_0} \) of the algebra \( \mathcal{A} \) with respect to the denominator system generated by \( \omega_0 \). Our first goal is to construct a projector \( \mathcal{A} \to \mathcal{A}_{\omega_0} \).

Notice that for each \( i \) the linear form \( D_{E_i}(\omega_j) \) belongs to the subspace \( \langle \omega_0, \ldots, \omega_{j-1} \rangle \). Moreover, if \( i > j \), then \( D_{E_i}(\omega_j) = 0 \). In the case \( i = j \), we have \( D_{E_i}(\omega_j) = -\omega_0 \). Then for the element \( Q_j = -\omega_j \omega_0^{-1} \) and each \( i > j \), we obtain

\[
D_{E_i}(Q_j) = \begin{cases} 
0, & \text{if } i > j, \\
1, & \text{if } i = j.
\end{cases}
\] (11)

For each \( 1 \leq i \leq k \), we construct the mapping \( S_{\alpha_i} \) according to (3). Define the mapping

\[
P_0 = S_{\alpha_1} \circ \cdots \circ S_{\alpha_k}.
\] (12)
Lemma 4.1. The mapping $P_0$ is a homomorphism $A$ to $A^m_w$ identical on $A^m$; its kernel is generated (as an ideal) by $\omega_1, \ldots, \omega_k$.

Proof. One can directly verify $S_{\alpha_j}(\omega_j) = 0$ for all $j \leq k$. Since $D_{E_i}(\omega_j) = 0$ for $i > j$, we have $S_{\alpha_i}(\omega_j) = \omega_j$ and

$$P_0(\omega_j) = S_{\alpha_1} \circ \cdots \circ S_{\alpha_k}(\omega_j) = S_{\alpha_1} \circ \cdots \circ S_{\alpha_j}(\omega_j) = 0.$$  

As $P_0$ is a homomorphism of algebras, the ideal $I$ that is generated by $\omega_1, \ldots, \omega_k$ belongs to the kernel of $P_0$. On the other hand, for $j = 0$ or $j > k$, the image $P_0(\omega_j)$ can be written in the form $P_0(\omega_j) = \omega_j + b$, where $b \in I$. Therefore $\text{Ker}(P_0) = I$.

Since $S_{\alpha_j}(a) = a$ for each $j$ and $m$-invariant $a$, we have $P_0(a) = a$. Let us show that for any $a \in \mathcal{A}_w$ the image $P_0(a)$ is $m$-invariant. For each $1 \leq s \leq k$ denote $P_0(s) = S_{\alpha_s} \circ \cdots \circ S_{\alpha_k}$. We shall prove by induction on $s$, beginning from $s = k$ that

$$D_{\alpha_s}(P_0^{(s)}(a)) = \cdots = D_{\alpha_k}(P_0^{(s)}(a)) = 0.$$  

Indeed, for $s = k$ we obtain $D_{\alpha_k}(P_0^{(k)}(a)) = D_{\alpha_k}(S^{(k)}(a)) = 0$. Suppose that the statement holds for $s + 1$; let us prove it for $s$. Easy to see that

$$D_{\alpha_s}(P_0^{(s)}(a)) = D_{\alpha_s}S_{\alpha_s}(P_0^{(s+1)}(a)) = 0.$$  

Let $t > s$. By induction on $n$, one can easily prove that for elements of an arbitrary Lie algebra the following equality holds

$$xy^n = (y - \text{ad}_y)^n(x). \quad (13)$$  

Applying (13), we verify that there exist the operators $L_1, \ldots, L_{k-t}$ obeying

$$D_{\alpha_s}S_{\alpha_s} = S_{\alpha_s}D_{\alpha_s} + L_1D_{\alpha_s+1} + \ldots + L_{k-t}D_{\alpha_k}.$$  

Then

$$D_{\alpha_t}(P_0^{(s)}(a)) = S_{\alpha_s}D_{\alpha_t}(P_0^{(s+1)}(a)) + L_1D_{\alpha_t+1}(P_0^{(s+1)}(a)) + \ldots + L_{k-t}D_{\alpha_k}(P_0^{(s+1)}(a)).$$  

According to the induction assumption $D_{\alpha_t}(P_0^{(s)}(a)) = 0$. □

Let $G_1$ be a subgroup in $G$ which Lie algebra coincides with $\mathfrak{g}_1$. The projector $P_0$ is invariant with respect to $G_1$. Indeed, since $g_1m^{-1}_1 = m$ for any $g_1 \in G_1$, the projector $g_1P_0g_1^{-1}$ has the same kernel and image as $P_0$, and hence $g_1P_0g_1^{-1} = P_0$. This implies $g_1P_0(a) = P_0(g_1a)$.

The group $G_1$ acts on the space $V_1 = \langle v_0, v_{k+1}, \ldots, v_n \rangle$. The algebra $\mathcal{A}_1 = K[V_1]$ is the symmetric algebra

$$S(V_1^*) = K[\omega_0, \omega_{k+1}, \ldots, \omega_n].$$
The homomorphism \( P_0 \) is an isomorphism of \( K[\omega_0^{\pm 1}, \omega_{k+1}, \ldots, \omega_n] \) to the algebra \( A_\infty^m \). Since \( P_0 \) commutes with \( g_1 \), the operator \( P_0 \) is an isomorphism of the \( G_1 \)-representations \( V_1^* \) and \( P_0(V_1^*) \).

Choose the lowest (for \( g_1 \)) vector \( v_0' \) in \( V_1 \), and continue the process as above. Finally, we obtain the chain of subspaces \( V \supset V_1 \supset \ldots \supset V_s \), and that of reductive subalgebras \( g \supset g_1 \supset \ldots \supset g_s \), where \( g_s \)-action in \( V_s \) is diagonalizable. We obtain the chain of lowest vectors

\[
v_0, v_0', \ldots, v_0^{(s-1)} \subset V_s
\]

and the corresponding linear forms

\[
f_0 = \omega_0, f_1 = \omega', \ldots, f_{s-1} = \omega^{(s-1)}_0 \subset V_s^* \subset V^*.
\]

We extend \( \{f_1, \ldots, f_{s-1}\} \) to the basis \( \{f_1, \ldots, f_{s-1}, \ldots, f_m\} \) in \( V_s^* \). For each \( 1 \leq i \leq s-1 \), determine a homomorphism \( P_i \) defined on the localization \( A_{\Lambda_i} \) of the algebra \( A \) with respect to the denominator system generated by

\[
\Lambda_i = \{f_0, P_0(f_1), \ldots, P_{i-1} \circ \cdots \circ P_0(f_{i-1})\}.
\]

Denote \( P = P_{s-1} \), and \( A_* = A_{\Lambda_{s-1}} \).

**Theorem 4.2.**

1) The mapping \( P \) is a homomorphism of the algebra \( A \) to \( A_*^U \) identical on \( A^U \), i.e. \( P \) is a \( U \)-projector for the representation of \( G \) in \( V \).

2) The system of elements \( \Lambda \cup \{P(f_i) : 1 \leq i \leq s-1\} \) freely generate \( A_*^U \) (and it freely generate the field of \( U \)-invariants).

## 5 U-projector on reductive group

Let \( G \) be as above (i.e. it is a connected reductive split group defined over a field \( K \) of zero characteristic). We consider the representation of the group \( G \) in the space \( K[G] \) defined by the formula \( R_g f(s) = f(g^{-1} s g) \). Our goal is to construct the \( U \)-projector for this representation in the algebra \( K[G] \).

Let \( g \) be the Lie algebra of the group \( G \). Let \( \Pi = \{\alpha_1, \ldots, \alpha_n\} \) be the system of simple roots in \( \Delta^+ \), and \( \Phi = \{\phi_1, \ldots, \phi_n\} \) be the system of fundamental weights, \( \phi_i(\alpha_j) = \delta_{ij} \). In each fundamental representation \( V_i \) with the highest weight \( \phi_i \), we choose the highest vector \( v_i^+ \) and the lowest vector \( v_i^- \). In the dual space \( V_i^* \), we choose the highest and the lowest vectors \( l_i^+ \) and \( l_i^- \) such that \( (v_i^-, l_i^+) = (v_i^+, l_i^-) = 1 \). The matrix element \( d_i(g) = (gv_i^+, l_i^+) \), \( 1 \leq i \leq n \), is an \( U \)-invariant. Denote by \( K[G]_* \) the localization of the algebra \( K[G] \) with respect to the denominator system generated by \( \{d_i(g) : 1 \leq i \leq n\} \).
Order the positive roots $\Delta^+ = \{\beta_1, \ldots, \beta_m\}$ such that $\beta_t < \beta_s$ implies $t < s$. Each $\beta_s \in \Delta^+$ is either simple (i.e. $\beta_s$ coincides with some $\alpha_{\nu(s)} \in \Pi$), or $\beta_s = \alpha_{\nu(s)} + \beta'_s$ for some $\alpha_{\nu(s)} \in \Pi$ and $\beta'_s \in \Delta^+$. For each $\beta_s$, let us fix $\alpha_{\nu(s)} \in \Pi$. Let us correspond to each $\beta_s \in \Delta^+$ the matric element

$$d_{\beta_s}(g) = (gv^+, E_{-\beta_s}l^+).$$

Then for any $x \in \mathfrak{u}$, we obtain

$$D_x d_{\beta_s}(g) = -(xgv^+, E_{-\beta_s}l^+) + (gxv^+, E_{-\beta_s}l^+) = (gv^+, xE_{-\beta_s}l^+). \quad (14)$$

If $x = E_{\beta}$, then as above we simplify notation $D_\beta = D_{E_{\beta}}$. The formula (14) implies $D_{\beta_s}d_{\beta_t}(g) = 0$, if $s > t$, and

$$D_{\beta_s}d_{\beta_s}(g) = \phi(H_{\beta_s})d_{\nu(s)}(g).$$

Then for

$$Q_{\beta_s}(g) = d_{\beta_s}(g)(\phi(H_{\beta_s})d_{\nu(s)}(g))^{-1},$$

we have

$$D_{\beta_s}Q_{\beta_t}(g) = \begin{cases} 0, & \text{if } s > t, \\ 1, & \text{if } s = t. \end{cases} \quad (15)$$

For each $1 \leq s \leq m$ we construct the mapping $S_{\beta_s}$ by the formula (3). Define the operator

$$P = S_{\beta_1} \circ \cdots \circ S_{\beta_m}.$$

For each $\beta_s \in \Delta^+$, consider the matrix element

$$c_{\beta_s} = (gE_{-\beta_s}v^+, l^+).$$

**Theorem 5.1.**

1) The operator $P$ is a homomorphism $K[G]_*$ to $K[G]_U^*$ identical on $K[G]_U^*$, i.e. it is an $U$-projector.

2) The system of rational functions

$$\{d_i(g) : 1 \leq i \leq n\} \cup \{P(c_{\beta_s})(g) : 1 \leq s \leq m\}$$

freely generate the algebra $K[G]_U^*$ (it also freely generate the field $K(G)U$).

**Proof.** The statement 1) follows from above. Let us prove 2). It is sufficient to show that the selected system of functions satisfy the conditions of the theorem 2.3.

1) The inequalities $d_i \neq 0$, $1 \leq i \leq n$, define the open Bruhat cell $Bw_0B$. Let us show that the system $\{d_{\beta_s}(g) : 1 \leq s \leq m\}$ generate the defining ideal $I_{w_0B}$ of the subset $w_0B$ in $Bw_0B$. The element $g$ belongs to $Bw_0B$ if $g$ can
be written in the form $g = aw_0bh$, where $a = \exp(x) \in U$, $b = \exp(y) \in U$, $h \in H$, and
\[
x = \sum_{\alpha \in \Delta^+} x_\alpha E_\alpha, \quad y = \sum_{\alpha \in \Delta^+} y_\alpha E_\alpha.
\]
The ideal $I_{w_0B}$ is generated by $\{x_\beta : \beta \in \Delta^+\}$.

On the other hand,
\[
d_{\beta_s}(g) = (aw_0bhv^+, E^-_{-\beta_s}l^+) = \phi_{\nu(s)}(h)(v^-, a^{-1} E^-_{-\beta_s}l^+) = \phi_{\nu(s)}(h)f_s(x),
\]
where $f_s(x)$ is a polynomial in $\{x_\alpha\}$ of the form
\[
f_s(x) = cx_{\beta_s} + \text{polynomial in } \{x_{\beta_t} : t < s\}
\]
with $c \neq 0$. Therefore, the ideal $I_{w_0B}$ is generated by $\{d_{\beta_s}(g) : 1 \leq s \leq m\}$.

2) Let us show that $K[w_0B]$ is generated by the restrictions of $\{d_i(g), c_{\beta_s}(b)\}$ on $w_0B$. Indeed, $K[w_0B]$ is generated by $\{\phi_i(h), y_{\beta_s}\}$.

On the other hand, $d_i(w_0bh) = (w_0bhv^+, l^+) = \phi_i(h),$
\[
c_{\beta_s}(w_0bh) = (w_0bhE^-_{-\beta_s}v^+, l^+) = \phi_{\nu(s)}(h)(bE^-_{-\beta_s}v^+, l^-) = \phi_{\nu(s)}f'_s(y),
\]
where $f'_s(y)$ in a polynomial in $\{y_\alpha\}$ of the form
\[
f'_s(y) = cy_{\beta_s} + \text{polynomial in } \{y_{\beta_t} : t < s\}
\]
with $c \neq 0$. Hence $K[w_0B]$ is generated by the restrictions of $\{d_i(g), c_{\beta_s}(b)\}$ on $w_0B$. □

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