TWO-LOOP SUPERSTRINGS II

The Chiral Measure on Moduli Space

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Abstract

A detailed derivation from first principles is given for the unambiguous and slice-independent formula for the two-loop superstring chiral measure which was announced in the first paper of this series. Supergeometries are projected onto their super period matrices, and the integration over odd supermoduli is performed by integrating over the fibers of this projection. The subtleties associated with this procedure are identified. They require the inclusion of some new finite-dimensional Jacobian superdeterminants, a deformation of the worldsheet correlation functions using the stress tensor, and perhaps paradoxically, another additional gauge choice, “slice $\hat{\mu}$ choice”, whose independence also has to be established. This is done using an important correspondence between superholomorphic notions with respect to a supergeometry and holomorphic notions with respect to its super period matrix. Altogether, the subtleties produce precisely the corrective terms which restore the independence of the resulting gauge-fixed formula under infinitesimal changes of gauge-slice. This independence is a key criterion for any gauge-fixed formula and hence is verified in detail.

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1 Introduction

In string theory, Feynman rules correspond to a chiral measure on the moduli space of Riemann surfaces. At a given loop order $h$, the worldsheet is a surface of genus $h$, and string scattering amplitudes are given by integrals over all geometries on this surface. The chiral string measure results from factoring out all symmetries from these integrals.

For superstrings, the problem of determining the chiral string measure has remained intractable to this day. The main difficulty is the presence of supermoduli for worldsheets of non-trivial topology \cite{1, 2}. The usual gauge fixing methods express superstring amplitudes as measures on supermoduli space, which incorporates odd variables over moduli space \cite{3}. The odd variables have to be integrated out in order to produce the desired chiral string measure on moduli space. However, all such attempts so far have run into serious problems.

A recurrent problem is the occurrence of apparent ambiguities, which appear as total derivatives on local coordinate patches of moduli space. In the early Ansatz proposed by Friedan, Martinec, and Shenker \cite{1} based on BRST invariance, the chiral measure includes $2h - 2$ picture-changing operators, inserted at $2h - 2$ arbitrary points. Although the Ansatz should be invariant under changes of insertion points, it actually changes by total derivatives. In subsequent attempts to derive the chiral measure from first principles by gauge fixing the integrals over all geometries, similar total derivatives arise from changes of gauge slices in the gauge fixing process \cite{4}. These derivatives have been attributed to ambiguities in the general theory of fermionic integrals \cite{5}.

The total derivatives pose serious difficulties, because they are defined only on local coordinate patches, and cannot be reduced to boundary terms by Stokes’ theorem. There have been many attempts to overcome these difficulties. Since the total derivative ambiguities occur on local patches and are reminiscent of Cech cohomology, a program was begun in \cite{6} for the construction of counterterms by a series of descent equations similar to gauge anomalies. Another approach has been to assume that geometric conditions may exist under which the ambiguities are global exact differentials on moduli space, and analyze which contributions can arise in this way from the boundary of moduli space \cite{7}. Yet another approach is to put the insertion points at certain privileged points on the worldsheet, and hope that this produces the correct answer. A possible choice is the unitary gauge, where the insertion points are put at the $2h - 2$ zeroes of a holomorphic Abelian differential \cite{8}. The unitary gauge has the advantage of producing an explicit cancellation between ghost and longitudinal degrees of freedom, but introduces a new arbitrariness in the choice of Abelian differential. A related approach is to work directly in light-cone gauge, and put the insertion points at the branch points of the corresponding Mandelstam diagrams \cite{9}. In genus 2, several Ansätze have been proposed in the hyperelliptic representation, with a possible resolution left ultimately to factorization conditions \cite{10}. Operator methods \cite{11} as well as group theoretic constructions \cite{12} of string amplitudes have also been developed. In more radical departures, powerful tools from algebraic geometry have been brought to
bear, assuming relations between string amplitudes and deep geometric properties such as slopes of effective divisors on moduli space [13] or invoking formal constructions from super algebraic geometry [14]. Finally, there have been suggestions to resolve the ambiguities by shifting the superstring background and appealing to the Fischler-Susskind mechanism [15]. All these attempts have led to different, competing expressions for the string chiral measure, with none emerging as the more cogent choice. Worse still, at the most fundamental level, ambiguities are simply unacceptable, since they would signal a breakdown of local gauge invariance.

The purpose of this series of papers is to show that, at least in genus $h = 2$ and contrary to earlier worries, superstring scattering amplitudes do not suffer from any ambiguity, and in fact can be evaluated completely explicitly in terms of modular forms and sections of vector bundles over the moduli space of Riemann surfaces. The case of genus $h = 2$ is the simplest case when supermoduli difficulties must be addressed in all scattering amplitudes. Actually, our methods are quite general, and should apply to arbitrary genus $h$. It is the complexity of the actual calculations which restricts presently our implementation to the case of genus 2. In paper I of the series [16], we had provided a summary of the main formulas we obtained. In the present paper II, we provide the detailed derivation of the first step in our approach, namely a careful new gauge-fixing process. This gauge-fixing process results in the expression (1.11) below. This expression together with the proof of its invariance under infinitesimal changes $\delta(\chi_\alpha)_{\bar{z} +} = -2\partial_{\bar{z}}\zeta_\alpha^+ + \alpha \bar{z}$ of the gauge slice $\chi_{\bar{z} +} = \sum_{\alpha=1}^2 \zeta_\alpha^+ (\chi_\alpha)_{\bar{z} +}$ are the main results of II.

The source of all the earlier difficulties turns out to be an ill-defined projection from supermoduli space to moduli space. The local worldsheet supersymmetry of the string scattering amplitudes requires both a zweibein $e_m^a$ and a gravitino field $\chi_m^a$. Together, they correspond to a superzweibein $E_M^A$ in Wess-Zumino gauge. The earlier gauge fixing procedures had been implicitly based on the projection

$$E_M^A \rightarrow e_m^a \quad (1.1)$$

which seemed the obvious way of descending from superzweibeins to zweibeins. However, this projection is actually ill-defined from supermoduli space to moduli space, because superzweibeins equivalent under supersymmetry may project to zweibeins with distinct complex structures. In practice, gauge fixing procedures based on (1.1) resulted in a dependence on the gravitino component $\chi_m^\beta$ of the gauge slice chosen. We shall refer to this as slice $\chi$ dependence. The remedy proposed in this series of papers is to use instead the well-defined projection (once a canonical homology basis has been chosen)

$$E_M^A \rightarrow \hat{\Omega}_{IJ} \quad (1.2)$$

Here $\hat{\Omega}_{IJ}$ is the super period matrix, namely the modification of the period matrix of $e_m^a$ which is invariant under worldsheet supersymmetry

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int \int d^2z \ d^2w \ \omega_I(z) \chi_{\bar{z} +} \hat{S}_\delta(z,w) \chi_w^+ \chi_{\bar{w}} \omega_J(w) \quad (1.3)$$
The $\omega_I(z)$ are the dual basis of holomorphic differentials, and $\hat{S}_\delta(z,w)$ is a modified Szegö kernel, whose precise definition is given in (3.4). For genus 2, the modified Szegö kernel $\hat{S}_\delta(z,w)$ coincides with the ordinary Szegö kernel $S_\delta(z,w)$. Technically, the gauge fixing based on (1.2) introduces a number of significant additional complications which we explain next; but all these can and will be resolved, and the payoff will be that the resulting formula for the chiral measure can be verified explicitly to be free of any ambiguity.

A first issue requiring some care is chiral splitting. The correct degrees of freedom of string theory require a worldsheet formulation with Minkowski signature and $\chi_m^\beta$ two-dimensional Majorana-Weyl spinors. In the present Euclidian worldsheet formulation of the worldsheet $\Sigma$, $\chi_m^\beta$ includes fields $\chi_{z^+}$ and $\chi_{z^-}$ of both chiralities. Thus we need chiral splitting, that is, a process for separating and retaining in correlation functions only the contributions of the chiral half $\chi_{z^+}$ of the field $\chi_m^\beta$. Chiral splitting is an essential step in the implementation of the Gliozzi-Scherk-Olive (GSO) projection [17], which projects out the tachyon and insures space-time supersymmetry. For Type II superstrings [18], the GSO projection must be enforced independently on left and right worldsheet chiralities, while in the heterotic string [19], only one of the two worldsheet chiralities is retained. Thus, for both Type II and heterotic strings, the key building blocks are the chiral measure and amplitudes, and these are common to both types of superstring theories.

The rules for chirally splitting superfields have been obtained in [20]. Applied to the superstring measure, they give us the following first formula for the chiral measure on supermoduli space (see (3.15) below for a detailed explanation of all the ingredients of this formula)

\[
A^{\text{chi}}[\delta](p^\mu_I) = \prod_A dm_A \exp(ip^\mu_I \hat{\Omega}_{IJ} p^\mu_J) A[\delta]
\]

\[
A[\delta] = \left\langle \prod_A \delta(\langle H_A | B \rangle) \exp\left(\frac{1}{2\pi} \int_{\Sigma} d^2 z \chi_{z^+} S(z)\right) \right\rangle_+
\] (1.4)

Here, $p^\mu_I$, $I = 1, \cdots, h$ are $h$ independent internal loop momenta; a $(3h - 3|2h - 2)$ dimensional slice $S$ for supermoduli space has been chosen, which is parametrized by supermoduli $m^A$ with the label ranging over $A = 1, \cdots, (3h - 3|2h - 2)$; the $H_A$ are the corresponding Beltrami superdifferentials; $S(z)$ is the total supercurrent; and $B$ is the ghost superfield. The expectation value $\langle \cdots \rangle_+$ is taken using effective rules for chiral worldsheet fields, in the background metric $g_{mn} = e_m^a e_n^b \delta_{ab}$ on the worldsheet. Nevertheless, we would like to stress that it is the super period matrix $\hat{\Omega}_{IJ}$ and not the period matrix $\Omega_{IJ}$ of the metric $g_{mn}$ which appears as the covariance matrix of the internal loop momenta $p^\mu_I$ in the correct chiral splitting prescription. This is the first clue that the projection (1.2) is on the right track [3].

The main problem in superstring perturbation theory is to descend from the preceding measure (1.4) over supermoduli space to a measure $d\mu[\delta]$ over moduli space. It is this deceptively simple step of integrating out the odd supermoduli $\zeta^a$, $1 \leq a \leq 2h - 2$, which
has caused problems in the past and which has to be carried out with particular care. In this series of papers, our main guiding principle is to view this integration as an integration along the fibers of the projection (1.2). With this principle, the basic even parameters are local parameters for the super period matrix $\hat{\Omega}_{IJ}$, the odd parameters $\zeta^{\alpha}$ are independent variables, and we can write for the chiral measure $d\mu[\delta]$ over moduli space

$$d\mu[\delta] = \prod_{a=1}^{3h-3} dm^a \int \prod_{a=1}^{2h-2} d\zeta^a A[\delta]$$

We come now to the additional complications in gauge-fixing inherent to the projection (1.2). There are essentially three of them:

- Since the basic moduli parameters are now $\hat{\Omega}_{IJ}$, the whole amplitude $A[\delta]$ has to be re-expressed first in terms of $\hat{\Omega}_{IJ}$ and $\zeta^{\alpha}$ before the fiber integration can be carried out. Now, as can be seen in (1.4), superstring amplitudes are built out of correlation functions in conformal field theory, with respect to the background complex structure corresponding to the period matrix $\Omega_{IJ}$. While all anomalies cancel in the full amplitudes, whose dependence is therefore only on the moduli $\Omega_{IJ}$, the contributions from the amplitudes’ individual building blocks, such as the matter, ghost and gauge fixing parts each require specifying a metric, and not just a complex structure. Thus, in practice, to re-express the amplitudes in terms of the super period matrix $\hat{\Omega}_{IJ}$, we require a choice of zweibein $\hat{e}^m_a$ (or equivalently a choice of metric $\hat{g}_{mn} = \hat{e}^m_a \hat{e}^n_b \delta_{ab}$) with $\hat{\Omega}_{IJ}$ as its period matrix. This should be viewed as an additional gauge choice, and ultimately, we have to show that it is immaterial. Another way of describing this new gauge choice is the following. Let $\hat{\mu}_z$ be the Beltrami differential corresponding to the variation of complex structure from $\Omega_{IJ}$ to $\hat{\Omega}_{IJ}$

$$\Omega_{IJ} - \hat{\Omega}_{IJ} = i \int d^2 z \hat{\mu} \omega_I(z) \omega_J(z)$$

The variation of complex structure is only a finite-dimensional constraint on $\hat{\mu}$, which identifies only the equivalence class $[\hat{\mu}]$, modulo gauge transformations of the form $\hat{\mu} \to \hat{\mu} + \partial_z v^z$. But the evaluation of the individual blocks in the superstring amplitudes involves the full $\hat{\mu}$. The final amplitudes have to be shown to be independent of $\hat{\mu}$, as long as it satisfies (1.6). We shall refer to this additional gauge slice independence as slice $\hat{\mu}$ independence.

- The second complication resides with the Beltrami superdifferentials $H_A$, which are the tangent vectors to the slice within the space of superzweibeins. In Wess-Zumino gauge, the Beltrami superdifferential $\delta H_\zeta \sim E_M^A \delta E^A_M$ corresponding to a variation $\delta E^A_M$ can be written as

$$\delta H_\zeta = \bar{\theta}(\delta \mu_\zeta - \theta \delta \chi^+)$$

with $\delta \mu_\zeta = -e^m_z \delta e_z^m$ and $\delta \chi^+$ controlling respectively the variations of moduli and of gravitino fields. With the naive projection (1.3), slices can be chosen with split Beltrami
superdifferentials, that is, superdifferentials for either a change of moduli or a change of gravitino field, but not both. This is no longer the case for the Beltrami superdifferentials associated with the projection \((\hat{\Omega})\), since
\[
\delta \hat{\Omega}_{IJ} = 0 \quad (1.8)
\]
for deformations along the fiber of this projection. In view of \((\ref{1.3})\), this can only hold if a variation of \(\Omega_{IJ}\) is accompanied with a compensating variation of \(\chi_{\bar{z}}\).

Finally, the correlation functions in \((\ref{1.4})\) have to be rewritten in terms of the background geometry \(\hat{e}_m^a\), representing moduli \(\hat{\Omega}_{IJ}\). This is a problem of deformation of complex structures, and has to be addressed in conformal field theory by adding terms involving repeated insertions of the stress tensor \(T(z)\).

With the above points taken into account, we obtain our first formula for the genus \(h = 2\) gauge-fixed amplitude which is the following
\[
\mathcal{A}[\delta] = \frac{\langle \Pi_{a_1} b(p_a) \Pi_{\alpha} \delta(\beta(q_\alpha)) \rangle}{\det \Phi_{IJ+}(p_a) \det \langle H_\alpha | \Phi_\beta \rangle} \left\{ 1 - \frac{1}{8\pi^2} \int d^2z \chi^+ \int d^2w \chi_{\bar{w}}^+ \langle S(z) S(w) \rangle + \frac{1}{2\pi} \int d^2z \hat{\mu}(z) \langle T(z) \rangle \right\} \quad (1.9)
\]
Here \(\hat{\mu}(z)\) is the Beltrami differential deforming the zweibein from \(\hat{e}_m^a\) to \(e_m^a\); \(\Phi_{IJ}\) and \(\Phi_\beta^*\) are a specific basis of odd and even superholomorphic 3/2 forms. The correlation functions on the worldsheet are evaluated with respect to the background geometry \(\hat{g}_{mn}\) corresponding to \(\hat{\Omega}_{IJ}\). The points \(p_a, q_\alpha\) are arbitrary generic points, introduced merely as a computational device. By construction, the measure is independent of these points, as may be checked explicitly. The term \(\langle S(z) S(w) \rangle\) is common to the earlier and the present approaches. The stress tensor correlator and the finite-dimensional determinants on the right hand side are the key new terms; each separately is slice dependent, but their combined effect is to restore slice independence to the entire expression. For example, under changes of slices, the short distance singularities of the supercurrent correlator \(\langle S(z) S(w) \rangle\) as \(z \to w\), whose significance had been obscure before, are now manifestly cancelled by the stress tensor correlator term \(\langle T(z) \rangle\). Similarly, it will turn out that the arbitrariness in \(\hat{\mu}\) will be compensated by a related arbitrariness in the Beltrami superdifferentials \(H_\alpha\).

The gauge fixed formula \((\ref{1.3})\) illustrates well the main features of the method of parametrizing the even supermoduli by the super period matrix. However, as a formula for a measure on moduli space, it is not yet satisfactory, since it still involves supergeometric

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\*All two-dimensional integrations will be over the compact orientable worldsheet \(\Sigma\), whose dependence will henceforth be suppressed: we let \(\int_{\Sigma} \to \int\). A collection of useful explicit formulas for holomorphic and meromorphic differentials and Green’s functions is given in Appendix A and their superspace counterparts are given in Appendix B.
notions such as $\Phi_{IJ}$ and $H_\alpha$, and its independence from the choice of Beltrami differential $\hat{\mu}$ is not manifest. The key to overcoming this difficulty is a deep relation between superholomorphic notions with respect to the supergeometry ($e_m^a, \chi_m^\alpha$) and holomorphic notions with respect to the super period matrix $\hat{\Omega}_{IJ}$. This is an important issue which we shall revisit in detail in the later papers of this series. For the present paper, we require only the simplest example of this correspondence, which says that the superholomorphic differentials $\hat{\omega}_I$ dual to the canonical $A$-homology basis are given by

$$\hat{\omega}_I = \theta \omega_I(\hat{\Omega}, \chi = 0) + \mathcal{D}_+ \Lambda_I$$  \hspace{1cm} (1.10)$$

where $\Lambda_I$ is a superscalar. Exploiting such relations, we can actually express everything in terms of $\hat{\Omega}_{IJ}$ moduli only, and make the $\hat{\mu}$ independence manifest. We arrive in this way at the following formula, which is the first main result of the present paper, and the starting point of the subsequent ones

$$\mathcal{A}[\delta] = i \frac{\langle \prod_a b(p_a) \prod_a \delta(\beta(q_a)) \rangle}{\det(\omega_I \omega_J(p_a)) \cdot \det(\chi_\alpha | \psi_\beta^*)} \left\{ 1 + \chi_1 + \chi_2 + \chi_3 + \chi_4 + \chi_5 + \chi_6 \right\}$$  \hspace{1cm} (1.11)$$

Here all correlation functions, $\theta$-functions, and complex variables are written with respect to the $\hat{\Omega}_{IJ}$ complex structure. The expressions $\psi_\beta^*$ are the holomorphic $3/2$ differentials normalized at the points $q_\alpha$ by $\psi_\beta^*(q_\alpha) = \delta_{\alpha \beta}$, and $\omega_I$ are holomorphic 1-forms canonically normalized on $A_I$ cycles, for a given choice of canonical homology basis. The terms $\chi_i$, $i = 1, \ldots, 6$ are defined as follows, with $p_a$, $a = 1, 2, 3$ and $q_\alpha$, $\alpha = 1, 2$ arbitrary generic points on the worldsheet $\Sigma$,

$$\chi_1 = - \frac{1}{8\pi^2} \int d^2z \tilde{\chi}_z^+ \int d^2w \chi_w^+ \langle S(z)S(w) \rangle$$

$$\chi_2 + \chi_3 = + \frac{1}{16\pi^2} \int d^2z \int d^2w \tilde{\chi}_z^+ \chi_w^+ T^{IJ} \omega_J(z)S_\delta(z, w)\omega_I(w)$$

$$\chi_4 = + \frac{1}{8\pi^2} \int d^2w \partial_{p_a} \partial_{q_\alpha} \ln E(p_a, w) \chi_w^+ \int d^2u \delta_\theta(w, u) \chi_u^+ \omega_\alpha(u)$$

$$\chi_5 = + \frac{1}{16\pi^2} \int d^2u \int d^2v \delta_\theta(p_a, u) \chi_u^+ \partial_{p_a} \delta_\theta(p_a, v) \chi_v^+ \omega_\alpha(u, v)$$

$$\chi_6 = + \frac{1}{16\pi^2} \int d^2z \chi_\alpha^*(z) \int d^2w G_{3/2}(z, w) \chi_w^+ \int d^2v \chi_v^+ \Lambda_\alpha(w, v)$$ \hspace{1cm} (1.12)$$

Here $\chi_\alpha^*(z)$ is the linear combination of the $\chi_\alpha(z)$ characterized by $\langle \chi_\beta^* | \psi_\alpha^* \rangle = \delta_{\beta \alpha}$, $S_\delta(z, w)$ is the Szegö kernel, and the Green’s functions $G_{3/2}(z, w)$ and $G_2(z, w)$ are tensors of type $(3/2, -1/2)$ and $(2, -1)$ respectively, normalized by $G_{3/2}(q_\alpha, w) = G_{2}(p_a, w) = 0$. The object $T^{IJ}$ may be defined in terms of the holomorphic quadratic differential

$$T^{IJ} \omega_I \omega_J(w) = \langle T(w) \prod_{a=1}^3 b(p_a) \prod_{a=1}^2 \delta(\beta(q_a)) \rangle / \langle \prod_{a=1}^3 b(p_a) \prod_{a=1}^2 \delta(\beta(q_a)) \rangle$$

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\[-2 \sum_{a=1}^{3} \partial_{p_a} \partial_{u} \ln E(p_a, w) \omega_{a}(w) \]
\[+ \int d^2 z \chi_{a}^{\ast}(z) \left( -\frac{3}{2} \partial_{w} G_{3/2}(z, w) \psi_{a}(w) - \frac{1}{2} G_{3/2}(z, w) (\partial \psi_{a})^{\ast}(w) \right) + G_{2}(w, z) \partial_{z} \psi_{a}(z) + \frac{3}{2} \partial_{z} G_{2}(w, z) \psi_{a}^{\ast}(z) \right) \]  (1.13)

Furthermore, \( \Lambda_{a} \) is defined by
\[\Lambda_{a}(w, v) = 2G_{2}(w, v) \partial_{v} \psi_{a}^{\ast} + 3 \partial_{v} G_{2}(w, v) \psi_{a}^{\ast}(v) \]  (1.14)

Finally, \( \omega_{a}(u, v) \) and \( \omega_{a}^{\ast}(u) \) are holomorphic 1-forms constructed as ratios of finite dimensional determinants and may be defined by \( \omega_{a}^{\ast}(u) \equiv \omega_{a}(u, p_{a}) \) and
\[
\omega_{a}(u, v) = \frac{\det \omega_{I}(p_{b}\{u, v; a\})}{\det \omega_{I}(p_{b})} \ \ \ \ \omega_{I}(p_{b}\{u, v; a\}) = \begin{cases} \omega_{I}(p_{b}) & b \neq a \\ \frac{1}{2} (\omega_{I}(u) \omega_{J}(v) + \omega_{J}(u) \omega_{I}(v)) & b = a \end{cases} \]  (1.15)

The second main result of this paper is to show that the gauge fixed amplitude (1.11) satisfies the crucial requirement of invariance under infinitesimal deformations of the gauge slice
\[\delta_{\xi}(\chi_{a})^{+} = -2 \partial_{z} \xi_{a}^{+} \]  (1.16)
where \( \xi_{a}^{+} \) are spinor fields, generators of local supersymmetry transformations. In (1.11), the term which is familiar from the early literature in superstring perturbation theory and leads essentially to the picture changing operator is \( \chi_{1} \). It was known to generate ambiguities under changes of slices. In the present formula however, these ambiguities will be cancelled out by the combination of all the remaining terms. Since the issue of slice dependence has led to much confusion in the past, we present the proof in some detail.

This paper is organized as follows. The starting point is the well-known gauge fixed measure on supermoduli space \[3\]. This measure is recalled in Section 2, together with a brief review of the ambiguity difficulties encountered in earlier approaches. Section 3 is devoted to the general gauge fixing procedure based on the projection (1.2) on super period matrices. It is presented for genus \( h = 2 \), but much can be extended to general genus. This includes the chiral splitting of the partition function and measure giving (1.4), and a discussion of gauge slices and Beltrami superdifferentials giving (1.9). The evaluation of the genus 2 chiral string measure is begun in Section 4; for this, the supercurrent and stress tensor correlators are determined explicitly, with the finite-dimensional determinants left to the later sections. Section 5 is devoted to slice independence. Since the slice independence of the chiral string measure (and amplitudes) constitutes a key test of the validity of the proposed expressions, we present a careful and detailed proof of both the
slice $\hat{\mu}$ independence and the slice $\chi$ independence under infinitesimal variations. For this, the variations of the finite-dimensional determinants are required, and they are derived using relations between superholomorphic superdifferentials and holomorphic differentials with respect to the super period matrix. Finally, in Section 6, the formulas (1.11) and (1.12) are obtained by expressing all superspace quantities in components and arranging the result in a manifestly slice $\hat{\mu}$ independent way. Technical tools are collected in the Appendices for the reader’s convenience. Appendix A presents a summary of formulas on ordinary Riemann surfaces for holomorphic differentials, Green’s functions and $\vartheta$-functions as well as for their variations under changes in the worldsheet metric. Appendix B collects analogous formulas for $\mathcal{N} = 1$ supergeometry. Appendix C gives useful formulas for superdeterminants. Appendix D gives a summary of forms, vector fields and Beltrami differentials associated with a slice.
2 The Gauge-Fixed Measure on Supermoduli Space

Our starting point is the gauge-fixed measure for superstring perturbation theory, expressed as a measure on supermoduli space \[3\]. We recall it briefly in this section, with particular emphasis on the issues which will play an important role in the sequel.

2.1 Superstring Propagation

The worldsheet for superstring propagation in 10-dimensional space-time at loop order \( h \) is a closed orientable compact surface \( \Sigma \) of genus \( h \). In the Ramond-Neveu-Schwarz formulation \[21\], the scattering amplitudes are constructed out of a two-dimensional superconformal field theory on the worldsheet, consisting of 10 “matter” scalar superfields \( X_\mu, \mu = 0, 1, \cdots, 9 \) coupled to two-dimensional supergravity,

\[
I_\text{m} = \frac{1}{4\pi} \int d^2z \, E^D_+ X_\mu D_- X^\mu \quad E \equiv \text{sdet} E_M^A
\]  

(2.1)

Here the two-dimensional supergravity fields are given by a superzweibein \( E_M^A \) and a \( U(1) \) superconnection \( \Omega_M \) with Wess-Zumino constraints \[22\], and the worldsheet \( \Sigma \) has been equipped with a spin structure \( \delta \). The theory is invariant under super Weyl rescalings \( s\text{Weyl}(\Sigma) \), super \( U(1) \) Lorentz transformations \( sU(1) \), and local super reparametrizations or \( s\text{Diff}(M) \). Local super reparametrizations are particularly important to our considerations. Infinitesimal super-reparametrizations are generated by vector fields \( \delta V^M \)

\[
E_A^M \delta E_M^B = D_A \delta V^B - \delta V^C T_{CA}^B + \delta V^C \Omega_C E_A^B.
\]  

(2.2)

In components, the supercoordinates \( z \) and \( \bar{z} \) decompose into \( z = (z, \theta) \) and \( \bar{z} = (\bar{z}, \bar{\theta}) \), where \( \theta^a = (\theta, \bar{\theta}) \) are Grassmann coordinates. The superfields \( X_\mu \) and supergeometries \( E_M^A \) in Wess-Zumino gauge decompose accordingly into

\[
X_\mu(z, \bar{z}) = x_\mu(z, \bar{z}) + \theta \psi_+^\mu(z, \bar{z}) + \bar{\theta} \psi_-^\mu(z, \bar{z}) + i \theta \bar{\theta} F^\mu(z, \bar{z}),
\]

\[
E_m^a(z, \bar{z}) = e_m^a(z, \bar{z}) + \theta \gamma^a \chi_m(z, \bar{z}) - \frac{i}{2} \theta \bar{\theta} e_m^a A(z, \bar{z}),
\]  

(2.3)

where \( e_m^a \) and \( \chi_m^\beta \) are the zweibein and gravitino fields on the worldsheet \( \Sigma \), and \( F^\mu \) and \( A \) are auxiliary scalar fields. Super vector fields \( \delta V^M \) decompose accordingly as

\[
\delta V^m = \delta v^m - \theta \gamma^m \xi - \frac{1}{2} \partial \bar{\theta} \chi_n \gamma^m \gamma^n \xi
\]  

(2.4)

\[\dagger\] A summary of worldsheet supergeometry, two-dimensional supergravity, and some useful formulas is given in Appendix B. We use the following conventions for spinors. The Clifford algebra is generated by \( \{\gamma^a, \gamma^b\} = -\delta^{ab} \). Spinor indices \( \alpha = (+) \) are raised and lowered according to \( \psi_+ = -\psi_- \), \( \psi_- = \psi_+ \) and all contractions where indices are not exhibited explicitly are taken as \( \phi \psi \equiv \phi^a \psi_a = -\phi_a \psi^a \). We have \( \gamma^+_+ = \gamma^-_- = 1 \), all other components of \( \gamma^a \) being zero, and \( (\gamma_5)_+^+ = -(\gamma_5)_-^- = i \). Finally, we use the following convention for the measure \( d^2 z = d^2 z d\theta d\bar{\theta} \).
The vector field $\delta v^m$ generates reparametrizations, while the spinor $\xi$ generates local supersymmetry transformations

$$\text{SUSY : } \delta \xi e^m_a = \xi \gamma^a \chi_m$$

(2.5)

The zweibein $e^m_a$ equips the worldsheet $\Sigma$ with a complex structure. Let $z, \bar{z}$ be local holomorphic coordinates on $\Sigma$ with respect to this complex structure. Then the relevant components of $\chi^\mu_{m\beta}$ are given by $\chi_{\bar{z}}^+, \chi_{\bar{z}}^-$, while the others decouple by super Weyl invariance. These components can be viewed as the Euclidian counterparts of chiral Majorana-Weyl spinors on a two-dimensional worldsheet with Minkowski signature. Using the component expressions of the covariant derivatives $D_\pm$ and of $E$, given in Appendix B, and after elimination of the auxiliary field $F^\mu$ (carried out for example in [3]), the action $I_m$ may be expressed in components in the local coordinates $z, \bar{z}$ as

$$I_m = I_m^0 + \frac{1}{2\pi} \int d^2z \left( -\chi_{\bar{z}}^+ S_m(z) - \chi_{\bar{z}}^- S_m(\bar{z}) - \frac{1}{4 \chi_{\bar{z}}^+ \chi_{\bar{z}}^- \psi_+^\mu \psi_+^\mu} \right)$$

$$I_m^0 = \frac{1}{4\pi} \int d^2z \left( \partial_z x^\mu \partial_{\bar{z}} x^\mu - \psi_+^\mu \partial_z \psi_+^\mu - \psi_-^\mu \partial_{\bar{z}} \psi_-^\mu \right)$$

(2.6)

Here, $I_m^0$ is the "free matter action" and $S_m$ is the matter worldsheet supercurrent,

$$S_m = -\frac{1}{2} \psi_+^\mu \partial_z x^\mu .$$

(2.7)

2.2 The Non-Chiral Supermoduli Measure

The partition function of the worldsheet superconformal field theory for the scalar superfields $X^\mu$ at given spin structure $\delta$ is given by

$$\mathbf{A}[\delta] = \int DE_M^A D\Omega_M \delta(T) \int DX^\mu e^{-I_m}$$

(2.8)

To obtain the correct degrees of freedom of the superstring, we have to extract out of this expression the contributions of the chiral half $\chi_{\bar{z}}^+$ of the gravitino field $\chi^\mu_{m\beta}$ and ultimately carry out the GSO projection, which involves a summation over spin structures $\delta$. In this paper, we discuss only the partition function (2.8) in order to construct the chiral measure, leaving the case of scattering amplitudes and vertex operators to a later paper.

Let $s\mathcal{M}_h$ be the supermoduli space of supergeometries on surfaces of genus $h$, that is, the quotient space

$$s\mathcal{M}_h = \{E_M^A, \Omega_M \text{ obeying torsion constraints}\}/s\text{Weyl} \times sU(1) \times s\text{Diff}.$$ 

(2.9)

In view of its invariance under super Weyl rescalings, super $U(1)$ gauge transformations, and super-reparametrizations, the partition function $\mathbf{A}[\delta]$ reduces to an integral over $s\mathcal{M}_h$. 

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For genus $h \geq 2$, $s\mathcal{M}_h$ is a superspace of dimension $(3h - 3|2h - 2)$. Since it is a quotient space, there are no canonical coordinates which we may use. However, locally, we can always parametrize it by a $(3h - 3|2h - 2)$-dimensional slice $\mathcal{S}$ of supergeometries which is transversal to the orbit of all the symmetry transformations. Let $m^A = (m^a|\zeta^\alpha)$, with $a = 1, \cdots, 3h - 3$ and $\alpha = 1, \cdots, 2h - 2$ be parameters for such a slice $\mathcal{S}$. Let $H_A$ be the Beltrami superdifferentials tangent to the gauge slice. They are given by

$$(H_A)^z = (-)^{A(M+1)}E^{-M}_M \frac{\partial E_M}{\partial m^A}$$  \hspace{1cm} (2.10)$$

Then the symmetry groups can be factored out of the non-chiral partition function (2.8), giving the following gauge-fixed expression (see [3], p. 967)

$$A[\delta] = \int_\mathcal{S} \prod_A |dm^A|^2 \int D(XB\bar{B}C\bar{C}) \prod_A \left|\langle \delta(H_A|B) \rangle \right|^2 e^{-I_m - I_{gh}}$$  \hspace{1cm} (2.11)$$

Here, $B$ and $C$ are the ghost superfields of $sU(1)$ weight $3/2$ and $-1$ respectively, and given in components by

$$B(z, \bar{z}) = \beta(z, \bar{z}) + \theta b(z, \bar{z}) + \bar{\theta} B_2(z, \bar{z}) + i \theta \bar{\theta} B_3(z, \bar{z})$$
$$C(z, \bar{z}) = c(z, \bar{z}) + \theta \gamma(z, \bar{z}) + \bar{\theta} C_2(z, \bar{z}) + i \theta \bar{\theta} C_3(z, \bar{z})$$  \hspace{1cm} (2.12)$$

and the ghost action is given by

$$I_{gh} = \frac{1}{2\pi} \int d^2z E \left( B D_- C + \bar{B} D_+ \bar{C} \right)$$  \hspace{1cm} (2.13)$$

Here, $B_2$, $B_3$, $C_2$ and $C_3$ are auxiliary fields, which may be eliminated following [3]. The ghost action in components then becomes

$$I_{gh} = I_{gh}^0 + \frac{1}{2\pi} \int d^2z (-\chi^{\bar{z}} + \chi_z \bar{S}_{gh} - \bar{\chi}_{\bar{z}} S_{gh})$$
$$I_{gh}^0 = \frac{1}{2\pi} \int d^2z (b\partial_z c + \beta \partial_z \gamma + \bar{b} \partial_{\bar{z}} \bar{c} + \bar{\beta} \partial_{\bar{z}} \bar{\gamma})$$  \hspace{1cm} (2.14)$$

Here, $I_{gh}^0$ is the “free superghost action” and $S_{gh}$ is the ghost worldsheet supercurrent,

$$S_{gh} = \frac{1}{2} b\gamma - \frac{3}{2} \beta \partial_z c - (\partial_z \beta)c.$$  \hspace{1cm} (2.15)$$

The gauge-fixed expression (2.11) is independent of the choice of gauge slice $\mathcal{S}$. A change of gauge slice $\mathcal{S}$ results in compensating changes in the measure $\prod_A |dm^A|^2$ and in the insertion $\prod_A \langle \delta(H_A|B) \rangle^2$.

\footnote{The sign factor in front is + unless $A$ is odd and $M$ is even, when it is -. This factor arises from using (1.7) for the variation and pulling out the differential $\delta m^A$ to obtain the superderivative. It was omitted in [3] but correctly included in [23].}
2.3 The Main Difficulties

The main difficulties in superstring perturbation theory are how to chirally split the gauge-fixed amplitudes, and how to descend from a measure on supermoduli space to a measure on moduli space. These two difficulties turn out to be related. However, chiral splitting has been considerably clarified in [20], and we shall discuss it in detail for the partition function in the next section, and for general scattering amplitudes in a later paper of this series. We focus now on the specific problems in descending from supermoduli space to moduli space.

2.3.1 The Picture-Changing Ansatz

Early on, an Ansatz for chiral superstring amplitudes as integrals over moduli space had been proposed by Friedan, Martinec, and Shenker [1], based on BRST invariance

\[ A_{\text{BRST}}^{\chi} [\delta] = \frac{1}{(2\pi)^{2h-2}} \int \prod_{a=1}^{3h-3} dm^a \int D(xbc) \epsilon^{-P_m} \prod_{a=1}^{3h-3} \langle \mu_a | b \rangle \prod_{a=1}^{2h-2} Y(z_\alpha) \]  

Here, the integration is over a 3h - 3 dimensional slice of metrics g_{mn} on Σ, parametrized by 3h - 3 complex parameters m^a, and 2μ_a = g_{zz} ∂g^{zz} / ∂m^a are the Beltrami differentials tangent to the slice. The operator Y(z) is the picture-changing operator, which is formally BRST invariant. It can be expressed as

\[ Y(z) = \oint_{C_z} dw j_{\text{BRST}}(w) H(\beta(z)) \]  

C_z is a small contour surrounding the point z, H is the Heaviside function, T(w) is the stress tensor, and j_{\text{BRST}}(w) = c(w)(T(w) + b∂c) + \frac{1}{2}γ∂x^\mu ψ^a_+ + \frac{1}{4}γ^2 b is the BRST current.

The main problem with the above Ansatz is that \( A_{\text{BRST}}^{\chi} [\delta] \) should be independent of the insertion points z_α. However, this property does not hold for the right hand side of (2.16), although it is tantalizingly close. Indeed, under a change of insertion points, Y(z) changes by a BRST transform. Deforming the contour C_z away, we pick up the BRST transform of the b field insertions, which is the stress tensor. This produces derivatives with respect to the moduli parameters m^a. Such derivatives could be harmless, if they were defined globally over moduli space, so that the invariance under changes of z_α can be restored when the boundary of moduli space does not contribute. But, as noted already in [4], this is not the case: the derivatives are defined only over local coordinate patches in moduli space, where it makes sense to pick 2h - 2 points, i.e., 2h - 2 sections of the universal curve.

2.3.2 Integrating Odd Supermoduli

The natural way of reducing an integral over supermoduli space to an integral over moduli space is to integrate out the odd supermoduli ζ^α. This approach was tried by many authors [4], and ran as follows.
Choose again a $3h - 3$ slice of metrics $g_{mn}$ on $\Sigma$, parametrized by local coordinates $m^a$. Let $\zeta^a$ be $2h - 2$ anticommuting parameters, choose for each metric $2h - 2$ generic gravitino fields $\chi_\alpha = (\chi_\alpha)_m^\beta$, $\alpha = 1, \ldots, 2h - 2$, and set $\chi = \sum_{\alpha=1}^{2h-2} \zeta^\alpha \chi_\alpha$. Then

$$m^A \rightarrow (g_{mn}, \chi) \quad (2.18)$$

can be viewed as a $(3h-3|2h-2)$ space $S$ of supergeometries in Wess-Zumino gauge, which is a slice for supermoduli space. The Beltrami superdifferentials $H_A$ which are tangent to $S$ are readily evaluated

$$H_a = \bar{\theta} \mu_a, \quad H_a = -\bar{\theta} \chi_\alpha, \quad (2.19)$$

where $\mu_a$ is as before the Beltrami differential tangent to the slice of metrics $g_{mn}$. The gauge-fixed formula (2.11) becomes

$$A_{BRST}[\delta] = \int_S \prod_{a=1}^{3h-3} dm^a |^2 \int \prod_{a=1}^{2h-2} d\zeta^a |^2 \int D(x\psi_\pm b^\beta \bar{c}^\gamma) \prod_{a=1}^{3h-3} \langle \mu_a | b \prod_{a=1}^{2h-2} \delta(\chi_\alpha | \beta) \right|^2$$

$$\times \exp\left( -I_m^0 - I_{gh}^0 + \frac{1}{2\pi} \int (\chi S + \bar{\chi} \bar{S}) - \frac{1}{8\pi} \int \chi \bar{\chi} \psi_+ \psi_- \right) \quad (2.20)$$

where $S(z) = S_m(z) + S_{gh}(z)$ is the total supercurrent. The naive Ansatz to chirally split this expression is just to drop the term $\chi \bar{\chi} \psi_+ \psi_-$ in the action, and to keep as the chiral contribution of $\chi^z$ at each point $m^a$ on moduli space

$$A^{\text{ch}}_{BRST}[\delta](m^a) = \prod_{a=1}^{3h-3} dm^a \int \prod_{a=1}^{2h-2} d\zeta^a \int D(x_+ \psi_+ b^\alpha c^\beta) \prod_{a=1}^{3h-3} \langle \mu_a | b \prod_{a=1}^{2h-2} \delta(\chi_\alpha | \beta) \right|^2$$

$$\times \exp\left( -I_m^0 - I_{gh}^0 + \frac{1}{2\pi} \sum_{a=1}^{2h-2} \zeta^\alpha \int \chi_\alpha S \right) \quad (2.21)$$

The integral over the anticommuting supermoduli $\zeta^\alpha$ can now be carried out

$$A^{\text{ch}}_{BRST}[\delta] = \prod_{a=1}^{3h-3} dm^a \int D(x_+ \psi_+ b^\alpha c^\beta) \prod_{a=1}^{3h-3} \langle \mu_a | b \prod_{a=1}^{2h-2} \delta(\chi_\alpha | \beta) \right|^2 e^{-I_m^0 - I_{gh}^0} \quad (2.22)$$

Choosing the gravitino fields $\chi_\alpha(z)$ to be Dirac $\delta$-functions, $\chi_\alpha(z) = \delta(z, z_\alpha)$, the $\chi_\alpha$-dependent parts of the preceding formula become

$$\delta(\chi_\alpha | \beta) \langle \chi_\alpha | S \rangle \rightarrow \delta(\beta(z_\alpha)) S(z_\alpha) \equiv Y(z_\alpha) \quad (2.23)$$

and we recapture in this way the formula (2.16) proposed in [1].

This derivation is attractive, since it seems to derive the BRST invariant formula (2.16) from first principles. However, the outcome suffers then from the same ambiguities as (2.16). This is surprising, since we started from a manifestly slice-invariant gauge-fixed formula (2.11) on supermoduli space. The problem has been traced back to subtleties in integration over supermanifolds [3]. But as we shall see later, it lies in the present case with both the naive chiral splitting used in the derivation, as well as with the slice parametrization (2.18), which is not compatible with local supersymmetry.
2.3.3 Unitary Gauge

If one admits the inherent ambiguity in the preceding gauge-fixed formulas (2.22), a possible strategy would be to try and guess a choice of insertion points which would lead to physically acceptable string amplitudes. There are some natural choices: for example, the points \( z_\alpha \) can be taken to be the \( 2h - 2 \) zeroes of a holomorphic abelian differential \( \omega(z) \). This is called the unitary gauge, and has many practical advantages, including a concrete cancellation between longitudinal and ghost degrees of freedom. However, an arbitrariness in the choice of the differential \( \omega(z) \) has been introduced, together with a host of other delicate issues such as modular invariance and factorization properties. Although there can be consistency checks such as a vanishing cosmological constant, there is no answer to the basic question of why the gauge-fixed amplitudes (2.22) should be ambiguous, and if they do turn out to be so, of why a unitary gauge should be the correct one.
3 The Super Period Matrix Gauge Fixing Method

We describe now a new procedure for descending from supermoduli to moduli space, which will not produce the ambiguities encountered in the previous section.

3.1 Projection on the Super Period Matrix

Our starting point is the gauge fixed formula (2.11) which gives an integral over supermoduli space, parametrized locally by a slice $S$ with arbitrary local coordinates $m^a$. We consider again the procedure outlined in Section 2.3.2, where $m^a$ are coordinates for the moduli of the geometry $e_m$, and $\zeta^\alpha$ are the fermionic coordinates of the gravitino field $\chi(z) = \sum_{\alpha=1}^{2h-2} \zeta^\alpha \chi_\alpha(z)$. Ignoring for the moment the subtleties of chiral splitting, the main problem with this procedure is the fact that the integral over the odd supermoduli corresponds then to the integral along the fibers of the projection (1.1). However, this projection does not descend to a projection from supermoduli space, since a supersymmetry transformation (2.5) on $E_M$ will change the complex structure of $e_m$:

$$E_M^A \sim E_M^A + \delta E_M^A$$

Thus the fiber of the projection (1.1) is not well-defined, and the complex moduli of $e_m$ itself is not an intrinsic notion in supersymmetry.

Although there is no known modification of the zweibein $e_m$ invariant under supersymmetry, there is a natural supersymmetric invariant generalization of the period matrix $\Omega_{IJ}$. This is the super period matrix $\hat{\Omega}_{IJ}$, which can be constructed as follows. The first construction is in terms of supergeometry, and hence manifestly supersymmetric. Fix a canonical homology basis $(A_I, B_J)$, $I = 1, \cdots, h$ for $\Sigma$, $\#(A_I \cap B_J) = \delta_{IJ}$. Then there exists a unique basis of superholomorphic odd differentials $\hat{\omega}_I(z)$ of U(1) weight 1/2 which satisfy

$$D_- \hat{\omega}_I = 0, \quad \oint_{A_I} d\theta_0 \hat{\omega}_I = \delta_{IJ}$$

Here $\hat{\omega}_I(z) = \hat{\omega}_{I0} + \theta \hat{\omega}_{Iz}$, and $D_-$ is the covariant derivative on forms of weight 1/2 (cf. Appendix B for explicit formulas). The super period matrix $\hat{\Omega}_{IJ}$ is defined then by

$$\hat{\Omega}_{IJ} = \oint_{B_J} d\theta_0 \hat{\omega}_I$$

The second construction of $\hat{\Omega}_{IJ}$ is in components, and relates it to the period matrix $\Omega_{IJ}$ for the metric $g_{mn} = e_m e_n \delta_{ab}$. Recall that $z, \bar{z}$ are conformal coordinates for $g_{mn}$, and let $\omega_I$ be the basis of holomorphic differentials with respect to $g_{mn}$ which is dual to the homology cycles $A_I$. Then

$$\oint_{A_I} \omega_J(z) dz = \delta_{IJ}$$

$$\oint_{B_I} \omega_J(z) dz = \Omega_{IJ}$$

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The super period matrix can be equivalently defined by

\[
\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int \int d^2 z \: d^2 w \: \omega_{I} \chi_{z}^{+} \hat{S}_{\delta}(z, w) \chi_{w}^{+} \omega_{J}
\] (3.5)

The modified Szegö kernel \( \hat{S}_{\delta}(z, w) \) is a \(-1/2\) form in both \( z \) and \( w \), and is defined as the unique solution to the integral-differential equation

\[
\partial_{\bar{z}} \hat{S}_{\delta}(z, w) + \frac{1}{8\pi} \chi_{z}^{+} \int d^2 x \chi_{z}^{+} \partial_{\bar{z}} E(z, x) \hat{S}_{\delta}(x, w) = 2\pi \delta(z, w)
\] (3.6)

and may clearly be generated explicitly from the standard Szegö kernel \( S_{\delta}(z, w) \) in a perturbative series in \( \chi_{z}^{+} \), which terminates since \( \chi_{z}^{+} \) is anticommuting and contains only a finite number of odd Grassmann variables. (For genus 2, \( \hat{S}_{\delta}(z, w) = S_{\delta}(z, w) \).) Thus \( \Omega_{IJ} \) can be also be generated explicitly from \( \hat{\Omega}_{IJ} \) in a finite series in \( \chi_{z}^{+} \). With the help of the component representation of the super line integral,

\[
\int_{w}^{z} dz d\theta \hat{\omega}_{I} = \int_{w}^{z} \left( dz \hat{\omega}_{I_{+}} - \frac{1}{2} d\bar{z} \chi_{z}^{+} \hat{\omega}_{I_{0}} \right) + \theta_{z} \hat{\omega}_{I_{0}}(z) - \theta_{w} \hat{\omega}_{I_{0}}(w),
\] (3.7)

the component result (1.3) may be recovered from (3.3).

Returning to supermoduli space, we can now define a projection which is invariant under supersymmetry and more generally, infinitesimal super reparametrizations

\[
E_{M}^{A} \rightarrow \hat{\Omega}_{IJ}
\] (3.8)

This projection is well-defined for any genus \( h \). The prescription for descending from supermoduli to moduli is then to integrate along the fibers of (3.8). In the Introduction, we had described in detail the subtleties associated with using the projection (3.8). If we compare with the earlier faulty way of descending in Section 2.3.2, they can be summarized as follows.

- The even coordinates \( m^{a} \) should be taken as moduli for \( \hat{\Omega}_{IJ} \) instead of \( \Omega_{IJ} \) as in (2.18);

- Now individual correlation functions in conformal field theory cannot be written just with respect to a complex structure: they require a metric. Thus, we must choose a metric \( \hat{g}_{mn} \) whose period matrix is \( \hat{\Omega}_{IJ} \). This is similar to a choice of slice, and introduces an arbitrariness which must be shown to be immaterial in the final expressions for the measure and amplitudes;

- The Beltrami superdifferentials \( H_{A} \) of (2.10) are changed accordingly. In particular, \( H_{A} \) will no longer be split as in (2.19). Instead, in Wess-Zumino gauge, we have

\[
H_{A} = \bar{\theta}(\mu_{A} - \theta \chi_{A})
\] (3.9)

where all components of \( \mu_{A} \) and \( \chi_{A} \) will be non-vanishing;
• All correlation functions in the worldsheet supergeometry have to be expressed in terms of the $\Omega_{IJ}$ moduli instead of $\Omega_{IJ}$. This is a deformation of the background geometry, and will require an appropriate insertion of the stress tensor.

We shall see how this procedure generates compensating terms which eliminate the ambiguities of the original picture-changing formula.

3.2 Chiral Splitting

Our first step in carrying out the super period matrix gauge fixing procedure outlined above is a careful chiral splitting of the contribution of each chiral half $\chi_z^+$ or $\chi_z^-$ in the correlation functions (2.11).

The contributions of the superghosts $B, C$ are manifestly chiral, so the main difficulty in chiral splitting resides with the scalar superfields $X^\mu$. The scalar fields $x^\mu$ are not split because of zero modes, and fields of both chiralities couple in the action $I_m$ and the super covariant derivatives $D_\pm$. Nevertheless, as shown in [20], the chiral contributions of $\chi_z^+$ in the $X^\mu$ scalar superfield theory can be identified by a simple effective prescription. We provide it for even spin structure $\delta$, which is all we need in the present paper. The modifications required when $\delta$ is odd can also be found in [20].

For general vertex operators of the form $5$

$$V(z; k, \epsilon) = \exp \left( ik_\mu X^\mu + \epsilon_\mu D_+ X^\mu(z) + \bar{\epsilon}_\bar{\mu} D_- X^{\bar{\mu}}(z) \right)$$

(3.10)

the non-chiral amplitudes can be decomposed as follows into chiral amplitudes

$$\langle \prod_{i=1}^N V(z_i; k_i, \epsilon_i) X^\mu \rangle = \int dp_\mu I \left| \langle \prod_{i=1}^N V^{\text{chi}}(z_i, \theta_i; k_i, \epsilon_i, p_\mu) \rangle \right|^2$$

(3.11)

where the effective chiral vertex operators $V^{\text{chi}}(z_i, \theta_i; k_i, \epsilon_i, p_\mu)$ are contracted with the help of the effective rules of Table 1.

In Table 1, all correlators in the effective chiral formulation are computed in terms of the effective chiral fields $x^\mu_+$ and $\psi^\mu_+$. There, the $S_m$-dependent effective action is to be inserted, as is the internal loop momentum dependent exponential, and the corresponding expectation values will be indicated by $\langle \cdots \rangle_+$. As suggested in [3] for bosonic scalar fields, the parameters $p_\mu$ can be interpreted as internal loop momenta.

In this paper, we shall focus on the partition function, in order to construct the chiral measure. For scalar superfields, the preceding chiral splitting prescription yields

$$\langle 1 \rangle_{X^\mu} = \int dp_\mu I \left| \left\langle \exp \left( \frac{1}{2\pi} \int d^2 z \chi_z^+ S_m(z) + p_\mu \oint_{B_\mu} dz \partial z x^\mu_+ \right) \right\rangle_+ \right|^2$$

(3.12)

$5$It is understood that the actual vertex operators for the NS-NS part of the supergraviton multiplet are recovered by expanding each vertex operator to linear order in $\epsilon_\mu$ and linear order in $\bar{\epsilon}_{\bar{\mu}}$. The present form is especially useful since the derivatives occur in the exponential as linear sources, see [3].

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| $X^\mu$ Superfields | Non-Chiral | Effective Chiral |
|----------------------|------------|-----------------|
| Bosons               | $x^\mu(z)$ | $x^I_\mu(z)$    |
| Fermions             | $\psi^I_\mu(z)$ | $\psi^I_\mu(z)$ |
| Action               | $I_m$      | $-\frac{1}{2\pi} \int d^2 z \chi^+ S_m$ |
| Internal Loop momenta| None       | $\exp(p_I^\mu \oint_{B_I} d z \partial_x x^I_\mu)$ |
| $x$-propagators     | $\langle x^\mu(z) x^\nu(w) \rangle$ | $-\delta^{\mu\nu} \ln E(z, w)$ |
| $\psi_\pm$-propagators | $\langle \psi^I_\mu(z) \psi^I_\nu(w) \rangle$ | $-\delta^{\mu\nu} S_3(z, w)$ |
| Covariant Derivatives | $\mathcal{D}_+$, $\mathcal{D}_-$ | $\partial_0 + \theta \partial_z$, $\partial_0 + \theta \partial_{\bar{z}}$ |

Table 1: Effective Rules for Chiral Splitting

The chiral blocks on the right hand side are easily evaluated using the identities (A.11) for the prime form. We obtain in this way the following basic formula for the chiral block of the partition function

$$\left\langle \exp\left(\frac{1}{2\pi} \int d^2 z \chi^+ S_m + p_I^\mu \oint_{B_I} d z \partial_x x^I_\mu\right)\right\rangle_+ = e^{i\pi p_I^\mu \hat{\Omega}_{IJ} p_J^\mu} \left\langle \exp\left(\frac{1}{2\pi} \int d^2 z \chi^+ S_m\right)\right\rangle_+$$

(3.13)

where $\hat{\Omega}_{IJ}$ is the super period matrix introduced in the previous subsection. Integrating in $p_I^\mu$, this implies immediately the following formula for the scalar partition function

$$\langle 1 \rangle_{X^\mu} = (\det \text{Im} \hat{\Omega})^{-\frac{5}{2}} \left| \left\langle \exp\left(\frac{1}{2\pi} \int d^2 z \chi^+ S_m(z)\right)\right\rangle_+ \right|^2$$

(3.14)

With the basic formula (3.13) for chiral blocks, we now return to the construction of the chiral superstring amplitude. Assembling the chiral blocks of matter and ghost fields, we define the chirally split partition function $A^{\text{chi}}[\delta](p_I^\mu)$ in terms of a basic correlator $A[\delta]$,

$$A^{\text{chi}}[\delta](p_I^\mu) = \prod_A dm^A \exp(i\pi p_I^\mu \hat{\Omega}_{IJ} p_J^\mu) A[\delta]$$

$$A[\delta] = \left\langle \prod_A \delta(H_A|B) \exp\left(\frac{1}{2\pi} \int d^2 z \chi^+ S(z)\right)\right\rangle_+$$

(3.15)

Earlier prescriptions for chiral splitting had missed the appearance of the super period matrix $\hat{\Omega}_{IJ}$. But perhaps more important, its appearance in chiral splitting is a confirmation that it is the variable which should be used in gauge fixing the superstring.

### 3.3 Parametrizations of Supermoduli as Fiber Space

The next step in our gauge fixing procedure is to provide suitable coordinates $m^A$ in which the fiber of the supersymmetric projection (3.8) is conveniently parametrized. A
detailed discussion of the choices of slices and associated Beltrami differentials is given in Appendix D.

Henceforth, we shall restrict our discussion to the case of genus \( h = 2 \). This case is simpler, because all matrices in the Siegel upper half-space are then period matrices of a metric, and because the construction below of gravitino fields \( \chi \) does not require any iteration. In genus \( h = 2 \), it is convenient (although by no means necessary for our arguments) to take \( \{ \hat{\Omega}_{IJ} \}_{I \leq J} \) as local holomorphic coordinates for the space of matrices \( \hat{\Omega}_{IJ} \). Choose a corresponding 3-dimensional slice \( \hat{S} \) of zweibeins \( \hat{e} \) whose period matrices are the matrices \( \hat{\Omega}_{IJ} \). For each point on the slice \( \hat{S} \), choose 2 generic gravitino sections \( \hat{\chi}_{\alpha}, \alpha = 1, 2 \), and set \( \hat{\chi} = \sum_{\alpha=1}^{2h-2} \zeta^\alpha \chi_{\alpha} \), for \( 2h - 2 \) anticommuting parameters \( \zeta^\alpha \). We may choose then as follows a (3|2)-dimensional slice of supergeometries \( (\hat{\Omega}_{IJ}, \zeta^\alpha) = m^A \longrightarrow (e^a_m, \chi = \sum_{\alpha=1}^{2} \zeta^\alpha \chi_{\alpha}) \) (3.16) with the fiber of the projection (3.8) given precisely by \( m^a = \text{constant} \). It is convenient to introduce the Beltrami differential \( \hat{\mu}_w \) which deforms the metric \( \hat{g}_{mn} \) to the metric \( g_{mn} \)

\[
\hat{\mu}_w = \frac{1}{2} \hat{g}_{\bar{w}w} \left( g^{\bar{w}w} - \hat{g}^{\bar{w}w} \right)
\] (3.17)

If \( w, \bar{w} \) are conformal coordinates for the metric \( \hat{g}_{mn} \), then we may set \( \hat{g}_{ww} = \hat{g}_{\bar{w}\bar{w}} = 0 \).

The main consequence of a choice of a slice is the corresponding Beltrami superdifferentials \( H_A \). We have already stressed that, unlike the slice used in Section 2.3.3 for the derivation of the picture-changing formula (2.10), a generic slice \( \hat{S} \) based on \( \hat{\Omega}_{IJ} \) will lead in general to Beltrami superdifferentials which in Wess-Zumino gauge assume the simplified form (3.9), \( H_A = \bar{\theta}(\mu_A - \theta \chi_A) \), which have both non-vanishing \( \mu_A \) and \( \chi_A \) components. For the slice we have just constructed using \( \hat{\Omega}_{IJ} \), the Beltrami superdifferentials \( H_A \) have the following properties:

- Let \( \Phi_{IJ} \) be the following basis of odd superholomorphic 3/2 superdifferentials (see Appendix B)

\[
\Phi_{IJ} = -\frac{i}{2} \left( \hat{\omega}_J D_+ \hat{\omega}_I + \hat{\omega}_I D_+ \hat{\omega}_J \right)
\] (3.18)

Then the matrix \( \langle H_a | \Phi_{IJ} \rangle \) has maximal rank. In fact, since we have chosen the even supermoduli to be \( m^a = \hat{\Omega}_{IJ}, I \leq J \), we have \( \langle H_a | \Phi_{IJ} \rangle = \delta_{a,IJ} \), as shown in (B.29).
Geometrically, the tangent vectors to supermoduli space corresponding to $H_\alpha$ modulo super reparametrizations are dual to the cotangent vectors $\Phi_{IJ}$.

- The even component of $H_\alpha = \bar{\theta}(\mu_\alpha - \theta \chi_\alpha)$ is given by
  \[
  \chi_\alpha(z) = \frac{\partial \chi^{+}_\alpha(z)}{\partial \zeta^\alpha}, \quad \alpha = 1, 2
  \]  (3.19)

The odd Beltrami differential $\mu_\alpha$ is associated with the dependence of the metric on $\zeta^\alpha$ and is related to the Beltrami differential $\hat{\mu}$ corresponding to the deformation of the metric $\hat{g}_{mn}$ to the metric $g_{mn}$ by the following formula

\[
\mu_\alpha = \frac{\partial \hat{\mu}}{\partial \zeta^\alpha} \quad \text{(3.20)}
\]

where $\hat{\mu}$ is the Beltrami differential deforming $\hat{g}_{mn}$ to $g_{mn}$.

- The $H_\alpha$ obey an orthogonality condition with $\Phi_{IJ}$ which guarantees that $H_\alpha$ produce no variations in $\hat{\Omega}_{IJ}$,
  \[
  \langle H_\alpha | \Phi_{I,J} \rangle = 0.
  \]  (3.21)

This relation determines the conformal class $[\mu_\alpha]$ of $\mu_\alpha$ but leaves the precise form of $\mu_\alpha$ subject to the same choice of slice that exists for $\hat{\mu}$ itself. Similarly, the exact values of $\mu_\alpha$, $\chi_\alpha$ depend on the many choices which entered the construction of the slice $S$.

### 3.4 Auxiliary Dirac $\delta$ Beltrami Superdifferentials

The insertion of the superghost $\delta(\langle H_A | B \rangle)$ functions in Wess-Zumino gauge for generic Beltrami differentials $H_A$ produces an enormous complication of the superstring measure,

\[
\prod_A \delta(\langle H_A | B \rangle) = 3 \prod_{a=1}^2 \left( \langle \mu_a | b \rangle - \langle \chi_a | \beta \rangle \right) \prod_{\alpha=1}^2 \delta \left( \langle \mu_a | b \rangle + \langle \chi_a | \beta \rangle \right)
\]  (3.22)

The product over the bosonic index $a$ will produce already 8 terms (and $2^{3h-3}$ terms in genus $h$). It is then much more convenient basis to work with a basis $H^*_A$ where this proliferation does not take place. First, we derive the formulas for changing bases.

It is simplest to return to the chirally symmetric expression for the superghost contributions. For any given set of Beltrami superdifferentials $H_A$, we have (for genus $h \geq 2$)

\[
\int \mathcal{D}(B \bar{B} C \bar{C}) \prod_A \delta(\langle H_A | B \rangle) |^{2} e^{-I_{gh}} = |\text{sdet} \mathcal{D}_+ \mathcal{D}_-^{(3/2)}|^2 \frac{|\text{sdet} \langle H_A | \Phi_C \rangle|^2}{\text{sdet} \langle \Phi_A | \Phi_C \rangle}
\]  (3.23)

Here the upper index $3/2$ indicates that the super Laplacian $\mathcal{D}_+ \mathcal{D}_-^{(3/2)}$ acts on fields of $U(1)$ weight $3/2$, and $\Phi_C$ is any $(3h - 3|2h - 2)$ dimensional basis of superholomorphic $3/2$ superdifferentials. As an immediate consequence, we see that the inner product of $H_A$ effectively is always taken with a holomorphic form. Using the behavior of the $\delta$ function
under a change of basis, we readily obtain an expression involving the $\delta$ function for any other set of Beltrami superdifferentials $H_A^*$,

$$\prod_A \delta(\langle H_A | B \rangle) = \frac{s\text{det} \langle H_A | \Phi_B \rangle}{s\text{det} \langle H_A^* | \Phi_B \rangle} \prod_A \delta(\langle H_A^* | B \rangle)$$  \hspace{1cm} (3.24)

Thus we can exchange the correlation functions of $\prod_A \delta(\langle H_A | B \rangle)$ for the potentially much simpler correlation functions of $\prod_A \delta(\langle H_A^* | \Phi_B \rangle)$, at the cost of introducing a ratio of finite dimensional superdeterminants.

A convenient choice for the new basis $H_A^*$ of Beltrami superdifferentials is generic $\delta$-functions. Let $p_a$ and $q_\alpha$ be generic distinct points on the surface. By setting

$$H_a^*(z, \theta) = \bar{\theta} \delta(z, p_a) \quad a = 1, \cdots, 3$$
$$H_\alpha^*(z, \theta) = \bar{\theta} \delta(z, q_\alpha) \quad \alpha = 1, 2, \cdots$$  \hspace{1cm} (3.25)

we obtain the following simpler effective insertion formula

$$\prod_A \delta(\langle H_A | B \rangle) = \frac{s\text{det} \langle H_A | \Phi_B \rangle}{s\text{det} \langle H_A^* | \Phi_B \rangle} \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)).$$  \hspace{1cm} (3.26)

It is important to realize that with above insertions of $b(p_a)$ and $\delta(\beta(q_\alpha))$, the ghost and superghost Green’s functions $G_2$ and $G_{3/2}$ will automatically be normalized at the points $p_a$ and $q_\alpha$ by $G_2(p_a, w) = G_{3/2}(q_\alpha, w) = 0$, a fact that represents a very considerable simplification as compared with the general insertions of (3.22).

### 3.5 Evaluation of the Finite Dimensional Superdeterminant

With the previous formula (3.26), all the dependence on the choice of slice $S$ for supermoduli space is concentrated in the background geometry of the effective correlation functions and in the finite dimensional superdeterminant

$$\frac{s\text{det} \langle H_A | \Phi_B \rangle}{s\text{det} \langle H_A^* | \Phi_B \rangle}$$  \hspace{1cm} (3.27)

where $\Phi_B$ is an arbitrary basis of superholomorphic $3/2$ superdifferentials. We shall make use of two such bases, $\Phi_A$ and $\Phi_A^*$, which are dual respectively to the Beltrami superdifferentials $H_A$ and $H_A^*$.

- The first basis $\Phi_A$ of superholomorphic $3/2$ superdifferentials is defined by duality

$$\langle H_A | \Phi_B \rangle = \delta_{AB},$$  \hspace{1cm} (3.28)

Henceforth, we always choose, locally, the bosonic coordinates $m^a$ of the slice $S$ as a subset of the variables $\hat{\Omega}_{IJ}$. In this case, the odd superholomorphic superdifferentials $\Phi_a$...
are given by the 3/2 superdifferentials $\Phi_{IJ}$ of (3.18), $\Phi_a = \Phi_{IJ}$, $a = 1, \ldots, 3h - 3$. There is no such simple expression for the even superdifferentials $\Phi^\alpha$, since they depend on the gauge slice $S$.

- The second basis $\Phi^*_A$ of superholomorphic 3/2 differentials is defined instead by normalization conditions at the points $p_a$, $q_\alpha$. If we write $\Phi^*_A = \Phi_{A0} + \theta \Phi^*_A$, these normalization conditions are

$$
\begin{align*}
\Phi^*_{a0}(q_\beta) &= \delta_{ab} \\
\Phi^*_{a+}(p_b) &= \delta_{a+} \\
\Phi^*_{a+}(p_b) &= \delta_{ab} \\
\Phi^*_{a0}(q_\beta) &= 0.
\end{align*}
\tag{3.29}
$$

In particular, we have

$$
\langle H^*_A | \Phi^*_B \rangle = \delta_{AB} \tag{3.30}
$$

Returning to the superdeterminant (3.27), it suffices to evaluate $\text{s det} \langle H^*_A | \Phi_B \rangle$. Since both $\Phi_A$ and $\Phi^*_A$ are bases of superholomorphic 3/2 superdifferentials, $\Phi_{ij}$ can be expressed as a linear combination of the basis of even holomorphic differentials $\Phi^*_A$ as well as the odd holomorphic differentials $\Phi_{IJ}$

$$
\Phi_{ij}(z) = \Phi^*_A(z) C_{ij} + \Phi_{IJ}(z) D_{ij},
\tag{3.31}
$$

where the coefficients $C$ and $D$ are independent of $z$, $C$ even and $D$ odd. The superdeterminant $\text{s det} \langle H^*_A | \Phi_B \rangle$ may now be evaluated as follows

$$
\text{s det} \langle H^*_A | \Phi_B \rangle = \text{s det} \begin{pmatrix} \langle H^*_a | \Phi_{IJ} \rangle & \langle H^*_a | \Phi_{ij} \rangle \\ \langle H^*_a | \Phi_{ij} \rangle & \langle H^*_a | \Phi_{ij} \rangle \end{pmatrix} = \text{s det} \begin{pmatrix} \langle H^*_a | \Phi_{ij} \rangle & \langle H^*_a | \Phi^*_{ij} \rangle C_{ij} + \langle H^*_a | \Phi_{ij} \rangle D_{ij} \\ \langle H^*_a | \Phi^*_{ij} \rangle C_{ij} + \langle H^*_a | \Phi_{ij} \rangle D_{ij} & \langle H^*_a | \Phi^*_{ij} \rangle C_{ij} + \langle H^*_a | \Phi_{ij} \rangle D_{ij} \end{pmatrix}. \tag{3.32}
$$

Now we make use of the fact that the addition of multiples of columns in the superdeterminant is immaterial (shown in Appendix C). As a result, the shift by $D$ in (3.32) is immaterial, just as for ordinary determinants. We can simplify the resulting formula further by using the above duality relations between $H^*_A$ and $\Phi^*_B$

$$
\text{s det} \langle H^*_A | \Phi_B \rangle = \text{s det} \begin{pmatrix} \langle H^*_a | \Phi_{ij} \rangle & 0 \\ \langle H^*_a | \Phi_{ij} \rangle & C_{ij} \end{pmatrix} = \frac{\text{det} \langle H^*_a | \Phi_{ij} \rangle}{\text{det } C}. \tag{3.33}
$$

Finally, taking the inner product of (3.31) against $H_a$, we obtain

$$
\langle H_a | \Phi_{ij} \rangle = \delta_{ij} = \langle H_a | \Phi^*_{ij} \rangle C_{ij} \tag{3.34}
$$

Explicit formulas for $\Phi^*_A$ in terms of $\vartheta$-functions and Green’s functions can be found in Appendix B.
and conclude with the following final formula in terms of finite dimensional determinants

$$\frac{s\text{det} \langle H_A | \Phi_B \rangle}{s\text{det} \langle H_A^* | \Phi_B \rangle} = \left( \det \Phi_{IJ}(p_a) \cdot \det \langle H_\alpha | \Phi_\beta^* \rangle \right)^{-1}. \quad (3.35)$$

3.6 First Summary

It is convenient to summarize here our formula for the chiral superstring measure, for fixed spin structure $\delta$

$$A^{\text{chi}}[\delta](p_\mu^a) = \prod_a dm^a \prod_\alpha d\zeta^\alpha \exp(i\pi p_\mu^a \hat{\Omega}_{IJ} p_\mu^b) A[\delta]$$

$$A[\delta] = \left\langle \prod_{a} b(p_a) \prod_{\alpha} \delta(\beta(q_\alpha)) \exp \left( \frac{1}{2\pi} \int d^2z \chi_{\bar{z}+} S(z) \right) \right\rangle_+ (g)$$

$$= \frac{\det \Phi_{IJ}(p_a) \cdot \det \langle H_\alpha | \Phi_\beta^* \rangle}{\det \Phi_{IJ}(p_a) \cdot \det \langle H_\alpha | \Phi_\beta^* \rangle} \quad (3.36)$$

In this formula, we should stress that $p_a, q_\alpha$ are arbitrary generic points, unrelated to the slice $S$. As we saw in the above derivation, they are a computational device, and the amplitude (3.36) manifestly does not depend on them. The dependence on $g = g_{mn}$ is made explicit as a reminder that the correlation function is with respect to the metric $g_{mn}$. In our approach, $\hat{\Omega}_{IJ}$ is the only intrinsic notion, and thus the metric $g_{mn}$ is slice dependent. So is $\chi_{\bar{z}+}$. As pointed out before, we can change metric backgrounds from $g_{mn}$ to $\hat{g}_{mn}$ by using the stress tensor. After this is properly done, the slice dependence of the correlation functions will have to cancel out with the slice dependence of the finite dimensional determinants.
4 The Genus 2 Chiral Superstring Measure

In this section, we show how to evaluate the gauge-fixed formula (3.36) explicitly. Our method is quite general, but the calculations are much simpler in genus $h = 2$, since there are then only 2 supermoduli. We shall obtain in this case a formula which can be proven independently to be invariant under infinitesimal changes of the gauge slice $S$. Thus there are no ambiguities as had occurred earlier in the picture-changing formula (2.16).

4.1 Formulation in terms of $\hat{g}_{mn}$

We begin by making more explicit the deformation of background metric from $g_{mn}$ to $\hat{g}_{mn}$. First, in genus $h = 2$, the gravitino field $\hat{\chi}$ is identical with $\chi$, since $\hat{\chi} - \chi$ is of order $\zeta\zeta\zeta$, and must thus vanish. Next, the Beltrami differential of (3.17), given by $\hat{\mu}(z) = \frac{1}{2}\hat{g}_{zz}\hat{g}^{zz}$ is of order $\zeta\zeta\zeta$, so that deformations can be obtained exactly by a single insertion of the stress tensor

$$\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_a)) \exp\left(\frac{1}{2\pi} \int d^2z \hat{\chi}_z^+ S(z)\right) \rangle_+ (g)$$

$$= \langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_a)) \exp\left(\frac{1}{2\pi} \int d^2z \hat{\chi}_z^+ S(z)\right) \rangle_+ (\hat{g})$$

$$+ \int \Sigma d^2z \hat{\mu}(z) \langle T(z) \prod_a b(p_a) \prod_\alpha \delta(\beta(q_a)) \rangle_+ (\hat{g}) \tag{4.1}$$

In the second term on the right hand side, the supercurrent contribution has been dropped, since the remaining factors are already of order $\zeta\zeta\zeta$. Henceforth, we shall consider only correlation functions with respect to the background metric $\hat{g}_{mn}$, and denote them by $\langle \cdots \rangle$, dropping the subscript + and the dependence $\hat{g}$. Similarly, $z$ will denote henceforth a holomorphic coordinate for $\hat{g}_{mn}$ (and no longer for $g_{mn}$, as had been the case up to this point). The chiral superstring measure can then be expressed as

$$A[\delta] = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_a)) \rangle}{\text{det} \Theta_J(p_a) \cdot \text{det} \langle H_\alpha | \Phi^*_\beta \rangle} \left\{ 1 - \frac{1}{2(2\pi)^2} \int d^2z \hat{\chi}_z^+ \int d^2w \hat{\chi}_w^+ \langle S(z)S(w) \rangle + \frac{1}{2\pi} \int d^2z \hat{\mu}(z) \langle T(z) \rangle \right\} \tag{4.2}$$

where the supercurrent and stress tensor correlators are defined as usual

$$\langle S(z)S(w) \rangle = \frac{\langle S(z)S(w) \prod_a b(p_a) \prod_\alpha \delta(\beta(q_a)) \rangle}{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_a)) \rangle}$$

$$\langle T(z) \rangle = \frac{\langle T(z) \prod_a b(p_a) \prod_\alpha \delta(\beta(q_a)) \rangle}{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_a)) \rangle} \tag{4.3}$$

and all moduli are $\hat{\Omega}_{IJ}$. However, the finite-dimensional determinant prefactors $\text{det} \Phi_J(p_a)$ and $\text{sdet} \langle H_\alpha | \Phi^*_\beta \rangle$ are supergeometric notions, and as such, are still formulated in the supergeometry $(g_{mn}, \chi_z^+)$. 25
4.2 Evaluation of the Correlators

The chiral partition function may be expressed as \[24\]

\[ Z ≡ \langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \rangle = \frac{\vartheta[\delta](0)^5 \vartheta(D_b) \prod_{a<b} E(p_a, p_b) \prod_\alpha \sigma(p_\alpha)^3}{Z^{15} \vartheta[\delta](D_\beta) \prod_{a<\beta} E(q_a, q_\beta) \prod_\alpha \sigma(q_\alpha)^2} \]

where the ghost and superghost divisors are defined by

\[ D_b = \sum_a p_a - 3\Delta \quad D_\beta = \sum_\alpha q_\alpha - 2\Delta, \quad (4.4) \]

and the scalar partition function \( Z_3 \) is given by

\[ Z_3 = \vartheta(\sum_I z_I - w - \Delta) \prod_{I<J} E(z_I, z_J) \prod_I \sigma(z_I) \]

\[ \frac{\partial}{\partial w} \ln E(z, w) \]

\[ \langle b(z)c(w) \rangle = +G_2(z, w) \]

\[ \langle \beta(z)\gamma(w) \rangle = -G_3/2[\delta](z, w) \quad (4.8) \]

The Green’s functions are given by (see [25] and [24])

\[ G_0[\delta](z, w) = \frac{\vartheta[\delta](z - w + D_h) \prod_i E(z, z_i) \sigma(z)^{2n-1}}{\vartheta[\delta](z - w) \prod_i E(w, z_i) \sigma(w)^{2n-1}} \]

where the divisor \( D_h = \sum_i z_i - (2n-1)\Delta \) and \( \Delta \) is the Riemann vector. When no confusion is possible, the dependence on \( [\delta] \) will not be exhibited.
Since \( \langle S_m(z)S_{gh}(w) \rangle = 0 \), we may split the calculation of \( C(z, w) \) into matter and ghost parts \( C(z, w) = C_m(z, w) + C_{gh}(z, w) \). With the help of the above propagators, the matter contribution is found to be

\[
4C_m(z, w) = 10S_0(z, w) \partial_z \partial_w \ln E(z, w) \tag{4.10}
\]

while the ghost contribution is obtained from the following manipulations

\[
4C_{gh}(z, w) = \langle b\gamma(z)(-3\partial_w c(w) - 2(\partial_w \beta)c(w)) \rangle - (z \leftrightarrow w) \tag{4.11}
\]

\[
= -3\langle b(z)\partial_w c(w) \rangle \langle \gamma(z)\beta(w) \rangle - 2\langle b(z)c(w) \rangle \langle \gamma(z)\partial_w \beta(w) \rangle - (z \leftrightarrow w)
\]

\[
= 3\partial_w G_2(z, w)G_{3/2}(w, z) + 2G_2(z, w)\partial_w G_{3/2}(w, z) - (z \leftrightarrow w)
\]

### 4.2.2 The Stress Tensor Correlators

In view of the \( \mathcal{N} = 1 \) superconformal structure of both the matter and ghost parts of the superstring, we have the following operator product relation involving the supercurrent \( S(z) \) and the stress tensor \( T(z) \),

\[
S(z)S(w) = \frac{2c/3}{(z - w)^3} + \frac{1}{2z - w} + \text{regular}. \tag{4.12}
\]

For the superstring, the total central charge \( c \) vanishes (as may be checked explicitly by adding the cubic poles of \( C_m \) and \( C_{gh} \)) and the stress tensor term produces the leading and only singularity as \( z \to w \). It is convenient to calculate the stress tensor correlator \( \langle T(z) \rangle \) from \( C(z, w) \) by picking up the limit as \( z \to w \). To extract \( T(w) \), we need the expansion of the Green’s functions \( G_n(z, w) \) up to order \( \mathcal{O}(z - w) \) included. Denoting the coefficients as follows

\[
G_n(z, w) = \frac{1}{z - w} + f_n(w) + (z - w)\{g_n(w) - T_1(w)\} + \mathcal{O}(z - w)^2 \tag{4.13}
\]

where the chiral scalar boson stress tensor \(-T_1 \) is defined by

\[
T_1(z) = \lim_{w \to z} \frac{1}{2} \left( \partial_z x(z) \partial_w x(w) + \frac{1}{(z - w)^2} \right)
\]

\[
E(z, w) = (z - w) + (z - w)^3 T_1(w) + \mathcal{O}(z - w)^4 \tag{4.14}
\]

Using the explicit formulas for \( G_n(z, w) \), we find

\[
f_n(w) = \omega_1(w) \partial_I \ln \vartheta[\delta](D_n) + \partial_w \ln \left( \sigma(w) Q \prod_i E(w, z_i) \right)
\]

\[
g_n(w) = \frac{1}{2} \omega_1(w) \omega_f(w) \partial_I \partial_I \ln \vartheta[\delta](D_n) + \frac{1}{2} f_n(w)^2 + \frac{1}{2} \partial_w f_n(w) \tag{4.15}
\]

and from this the full stress tensor

\[
T_n(w) = g_n(w) - n \partial_w f_n(w) - T_1(w) \tag{4.16}
\]
Note that \( f_n(w) \) is the same as \( \langle j_w \rangle \) in Verlinde and Verlinde [24], formula (7.12).

The full stress tensor may be extracted as the residue of the \( z = w \) pole and we find

\[
T = 5T_{1/2} - 10T_1 - T_{3/2} + T_2
\]

(4.17)

where each of these tensors is given by

\[
T_{1/2}(w) = g_{1/2}(w) - T_1(w)
\]

\[
= \frac{1}{2} \omega I \omega J(w) \partial_I \partial_J \ln \vartheta[\delta](0) - T_1(w)
\]

\[
T_{3/2}(w) = g_{3/2}(w) - \frac{3}{2} \partial_w f_{3/2}(w) - T_1(w)
\]

\[
= \frac{1}{2} \omega I \omega J(w) \partial_I \partial_J \ln \vartheta[\delta](D_\beta) + \frac{1}{2} f_{3/2}(w)^2 - \partial_w f_{3/2}(w) - T_1(w)
\]

\[
T_2(w) = g_2(w) - 2\partial_w f_2(w) - T_1
\]

\[
= \frac{1}{2} \omega I \omega J(w) \partial_I \partial_J \ln \vartheta[\delta](D_b) + \frac{1}{2} f_2(w)^2 - \frac{3}{2} \partial_w f_2(w) - T_1(w).
\]

(4.18)

Combining all of the above, we find for the full stress tensor

\[
T(w) = -15T_1(w) + \frac{1}{2} f_2(w)^2 - \frac{3}{2} \partial_w f_2(w) - \frac{1}{2} f_{3/2}(w)^2 + \partial_w f_{3/2}(w)
\]

\[
+ \frac{1}{2} \omega I \omega J(w) \left( 5\partial_I \partial_J \ln \vartheta[\delta](0) - \partial_I \partial_J \ln \vartheta[\delta](D_\beta) + \partial_I \partial_J \ln \vartheta[\delta](D_b) \right)
\]

(4.19)
5 Slice Independence

One of the most fundamental criteria for our gauge fixed formulas is their independence of the choices of gauge slices. infinitesimally, this independence is equivalent to the invariance of the formulas under local diffeomorphisms (which would vary the choice of metrics $g_{mn}$, that is, the choice of the Beltrami differential $\hat{\mu}_z^z$ and under local supersymmetry transformations (which would vary the choice of $\chi_\alpha$). As the issue of slice independence has caused much confusion in previous approaches, we shall provide here careful and detailed accounts of both proofs.

One key ingredient is a deep relation between superholomorphic notions with respect to the supergeometry $(e^a_m, \chi^a_m)$ and holomorphic notions with respect to the super period matrix $\hat{\Omega}_{IJ}$. This is an important issue which we shall revisit in detail in the later papers of this series. For the present paper, we require only the simplest example of this correspondence, which we presently discuss.

5.1 Superholomorphicity and Holomorphicity

First kind Abelian superdifferentials $\hat{\omega}_I$ and ordinary first kind Abelian differentials on a surface with period matrix $\hat{\Omega}_{IJ}$ have the same homology integrals and thus they must differ by an exact form. Here, we work out this result in detail and compute the exact form for the simplest case of genus 2 and even spin structure.

The differential equations defining $\hat{\omega}_I$ are (see Appendix B, (B.9) for $n = \frac{1}{2}$),

$$\nabla_{\bar{z}} \hat{\omega}_I + \frac{1}{2} \nabla_z (\chi^{+}_z \hat{\omega}_{I0}) = 0$$

$$\nabla_{\bar{z}} \hat{\omega}_{I0} + \frac{1}{2} \chi^{+}_z \hat{\omega}_{I+} = 0 \quad (5.1)$$

Now, let $\hat{\mu}(z) \equiv \hat{\mu}_{z}^z = \frac{1}{2} g_{zz} g^{zz}$ be a Beltrami differential that accounts for the variation of the metric $\hat{g}$ to the metric $g$, so that

$$\Omega_{IJ} - \hat{\Omega}_{IJ} = i \int d^2 z \hat{\mu}(z) \omega_I \omega_J(z) \quad (5.2)$$

then the covariant derivatives $\nabla$ with respect to $g$ may be expressed in terms of the covariant derivatives $\nabla$ with respect to $\hat{g}$ as follows

$$\nabla^{(n)}_{\bar{z}} = \hat{\nabla}^{(n)}_{\bar{z}} + \hat{\mu} \nabla^{(n)}_{\bar{z}} + n(\nabla_{\bar{z}} \hat{\mu}) \quad (5.3)$$

The equation written with respect to the metric $\hat{g}$ is now

$$\hat{\nabla}^{(1)}_{\bar{z}} \hat{\omega}_{I+} + \nabla^{(-1)}_{z} (\hat{\mu} \hat{\omega}_{I+} + \frac{1}{2} \chi^{+}_z \hat{\omega}_{I0}) = 0 \quad (5.4)$$
The form $\hat{\mu} \hat{\omega}_I + \frac{1}{2} \chi \hat{\omega}_0$ has vanishing inner product with every holomorphic 1-form $\omega_J$, as can be seen by

$$\int \omega_J (\hat{\mu} \hat{\omega}_I + \frac{1}{2} \chi \hat{\omega}_0) = \int \hat{\mu} \omega_J + \frac{1}{2} \int \omega_J \chi \hat{\omega}_0 = 0 \quad (5.5)$$

and thus there exists a (well-defined, single valued) scalar function $\lambda_I(z)$ such that

$$\hat{\mu} \hat{\omega}_I + \frac{1}{2} \chi \hat{\omega}_0 = -\partial \bar{z} \lambda_I \quad (5.6)$$

The function $\lambda_I(z)$ itself may be recovered up to an additive constant

$$\lambda_I(z) = \lambda_I(z_0) + \frac{1}{2\pi} \int d^2 w \partial_w \ln \frac{E(w,z)}{E(w,z_0)} \left( \hat{\mu} w \omega_I(w) + \frac{1}{2} \chi_w \hat{\omega}_0(w) \right). \quad (5.7)$$

The full relation may now be written as follows

$$\hat{\omega}_I = \theta \omega_I(\hat{\Omega}, \chi = 0) + \mathcal{D}_+ \Lambda_I \quad \Lambda_I(z, \theta) = \lambda_I(z) + \theta \omega_I(\hat{\Omega}, \chi = 0). \quad (5.8)$$

Here, $\omega_I(\hat{\Omega}, \chi = 0)$ stands for the ordinary first kind Abelian differential on a surface with period matrix $\hat{\Omega}$. Of the Beltrami differential $\hat{\mu}$, only the class is known. The effect of a change of Beltrami differential within the class produces a simple transformation on $\lambda_I$ by

$$\hat{\mu} \rightarrow \hat{\mu} + \partial \bar{z} v \quad \lambda_I(z) \rightarrow \lambda_I(z) - v \omega_I(z), \quad (5.9)$$

a formula that will be very useful later on.

The effect of this reformulation on the holomorphic (odd) 3/2 superdifferentials is also easily worked out and we have

$$\Phi_{I,I} = -\frac{i}{2} \left( \hat{\omega}_I \mathcal{D}_+ \hat{\omega}_I + \hat{\omega}_I \mathcal{D}_+ \hat{\omega}_I \right) \quad (5.10)$$

and the components are given by

$$i \Phi_{I,J} = \frac{1}{2} (\hat{\omega}_J \omega_I + \hat{\omega}_J \omega_I) \quad i \Phi_{I,J} = \omega_I \omega_J + \omega_I \partial \bar{z} \lambda_J + \omega_J \partial \bar{z} \lambda_I - \frac{1}{2} \hat{\omega}_J \partial \bar{z} \hat{\omega}_J - \frac{1}{2} \hat{\omega}_J \partial \bar{z} \hat{\omega}_I \quad (5.11)$$

where $\omega_I = \omega_I(\hat{\Omega}, \chi = 0)$.  

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5.2 Slice \( \hat{\mu} \) Independence: Diffeomorphism Invariance

The change of metric \( \hat{g} \to g \) associated with the change of moduli \( \hat{\Omega}_{IJ} \to \Omega_{IJ} \) (at fixed choice of \( \chi_\alpha \)) is governed by the Beltrami differential \( \hat{\mu} \) defined by (3.17), namely

\[
\hat{\mu}(z) = \hat{\mu}_z = \frac{1}{2} \hat{g}_{zz}(g_{zz} - \hat{g}_{zz}) . \tag{5.12}
\]

The Beltrami differential \( \mu_\alpha \), which is part of the super Beltrami differential \( H_\alpha = \bar{\theta}(\mu_\alpha - \theta \chi_\alpha) \), is related to \( \hat{\mu} \) by \( \delta_v \theta \). While the conformal classes of \( \hat{\mu} \) and \( \mu_\alpha \) are determined by the associated moduli deformation \( \Omega_{IJ} - \hat{\Omega}_{IJ} \), the representative in the class is not determined and depends upon the choices of the metrics \( g \) and \( \hat{g} \). A change of representative is given by a diffeomorphism vector field \( v^z \),

\[
\delta_v(\hat{\mu}_z) = \partial_z v^z \quad \delta_v(\mu_\alpha z) = \partial_z v^\alpha_\alpha = \frac{\partial v^z}{\partial \zeta^\alpha} . \tag{5.13}
\]

Since \( \hat{\mu} \) is of order \( \zeta \zeta \), consistency requires that \( v^z \) itself be also of order \( \zeta \zeta \).

In this subsection, we shall show that the superstring measure \( A[\delta] \), expressed in terms of the moduli \( \hat{\Omega}_{IJ} \), as given in (4.2) is independent of the slice \( \hat{\mu} \) provided \( \hat{\mu} \) and \( \mu_\alpha \) transform as above and \( \chi_\alpha \) is kept fixed. Since \( v \) is already of order \( \zeta \zeta \), the supercurrent correlator term is invariant by itself. Thus, the variation of \( A[\delta] \) under a change in slice by \( \delta_v \) reduces to

\[
\delta_v \ln A[\delta] = \frac{1}{2\pi} \int d^2z \partial_z v^z \langle T(z) \rangle - \delta_v \ln \det \Phi_{IJ}(\Phi^*_a) - \delta_v \ln \det \langle H_\alpha | \Phi^*_\beta \rangle \tag{5.14}
\]

and we now compute each of these terms in turn.

5.2.1 Rank 2 Differential Contribution

The variation under \( \delta_v \) of \( \Phi_{IJ+} \) is deduced from the variation of \( \hat{\omega}_I \). We express \( \hat{\omega}_I \) in terms of the form \( \theta \omega_I(\hat{\Omega}) \) at the period matrix \( \hat{\Omega}_{IJ} \) plus an exact form as follows

\[
\hat{\omega}_I = \theta \omega_I(\hat{\Omega}) + D_+ \Lambda_I \\
\Lambda_I = \lambda_I + \theta \omega_{I0} . \tag{5.15}
\]

Since \( v^z \) is of order \( \zeta \zeta \), the transformation properties are simple

\[
\delta_v \omega_I(\hat{\Omega}) = \delta_v \omega_{I0} = 0 \quad \delta_v \lambda_I = -v^z \omega_I . \tag{5.16}
\]

Using the expression \( 2i \Phi_{IJ} = \hat{\omega}_I D_+ \hat{\omega}_J + \hat{\omega}_J D_+ \hat{\omega}_I \), the \( + \) component is readily evaluated

\[
i \Phi_{IJ+} = \omega_I(\hat{\Omega}) \omega_J(\hat{\Omega}) + \omega_I(\hat{\Omega}) \partial_z \lambda_J + \omega_J(\hat{\Omega}) \partial_z \lambda_I - \frac{1}{2} (\hat{\omega}_{I0} \partial_z \hat{\omega}_{J0} + \hat{\omega}_{J0} \partial_z \hat{\omega}_{I0}) . \tag{5.17}
\]
and its variation is given by
\[ \delta_v \Phi_{IJ} = -v^z \partial_z \Phi_{IJ} - 2(\partial_z v^z) \Phi_{IJ} \]  
(5.18)
as would be expected for a two-form under a diffeomorphism. The variation of the logarithm of the determinant is given by
\[ \delta_v \ln \det \Phi_{IJ}(p_a) = \sum_b \left( -v^p \partial_p \ln \det \Phi_{IJ}(p_a) - 2(\partial_p v^p) \right). \]  
(5.19)

We clarify the derivation of this expression. Since \( v \) is already of order \( \zeta \), \( \Phi_{IJ} \) on the rhs of (5.18) reduces to \( \Phi_{IJ} + (z) = -i \omega_I \omega_J(z) \), and is thus effectively a holomorphic 2-form.

Consider the following ratio of 3 \( \times \) 3 determinants,
\[ \frac{\det \Phi_{IJ}(p_a[w; b])}{\det \Phi_{IJ}(p_a)} = \phi^{(2)*}_b(w) \]  
(5.20)
where \( p_a[w; b] = p_a \) when \( a \neq b \) and \( p_a[w; a] = w \). The form \( \phi^{(2)*}_b(w) \) is clearly a holomorphic 2-form in \( w \), and satisfies \( \phi^{(2)*}_b(p_a) = \delta_{ab} \), so it is the normalized holomorphic 2-form introduced in (A.15). We thus obtain our final expression
\[ -\delta_v \ln \det \Phi_{IJ}(p_a) = \sum_b \left( v^p \partial_p \phi^{(2)*}_b(p_b) + 2(\partial_p v^p)(p_b) \right). \]  
(5.21)

5.2.2 Rank 3/2 Differential Contribution

Expressing the inner product \( \langle H_{\alpha} | \Phi^*_{\beta} \rangle \) in components, we have
\[ -\langle H_{\alpha} | \Phi^*_{\beta} \rangle = \langle \mu_{\alpha} | \Phi^*_{\beta+} \rangle + \langle \chi_{\alpha} | \Phi^*_{\beta0} \rangle . \]  
(5.22)
Its variation under \( v \) is given by
\[ -\delta_v \langle H_{\alpha} | \Phi^*_{\beta} \rangle = \langle \partial_z v^z_{\alpha} | \Phi^*_{\beta+} \rangle + \langle \chi_{\alpha} | \delta_v \Phi^*_{\beta0} \rangle . \]  
(5.23)
The first term is easily computed using the differential equation (3.39) with \( n = 3/2 \) for \( \Phi^*_{\beta+} \) and the fact that \( v^z \) is of order \( \zeta \),
\[ \langle \partial_z v^z_{\alpha} | \Phi^*_{\beta+} \rangle = \int d^2z v^z_{\alpha} \left( \frac{1}{2} \chi^+ \partial_z \psi^*_{\beta} + \frac{3}{2} (\partial_z \chi^+) \psi^*_{\beta} \right) \]  
(5.24)
The second term must be computed with some care and we use the variational formulas (A.30) for holomorphic forms
\[ \delta_v \Phi^*_{\beta0}(z) = \delta_v \psi^*_{\beta}(z) = \frac{1}{2\pi} \int d^2w \partial_w v^w \delta_{ww} \psi^*_{\beta}(z), \]  
(5.25)\footnote{The \( \chi \) - dependent corrections in \( \Phi^*_{\beta0} \) are immaterial since \( v \) is already of order \( \zeta \).}
with the variations given by
\[ \delta_{ww} \psi_\beta^* (z) = \frac{3}{2} \partial_w G_{3/2}(z, w) \psi_\beta^*(w) + \frac{1}{2} G_{3/2}(z, w) \partial_w \psi_\beta^*(w). \] (5.26)

Taking the \( \partial_w \) derivative
\[
\partial_w \delta_{ww} \psi_\beta^*(z) = \frac{3}{2} \partial_w \left( -2 \pi \delta(z, w) + 2 \pi \sum_\alpha \psi_\alpha^*(z) \delta(w, q_\alpha) \right) \psi_\beta^*(w) \\
+ \frac{1}{2} \left( -2 \pi \delta(z, w) + 2 \pi \sum_\alpha \psi_\alpha^*(z) \delta(w, q_\alpha) \right) \partial_w \psi_\beta^*(w),
\] (5.27)

integrating by part and regrouping terms, we obtain
\[
\delta_v \Phi_{\beta 0}^*(z) = \delta_v \psi_\beta^*(z) = -v^z \partial_z \psi_\beta^*(z) - \frac{3}{2} (\partial_z v^z) \psi_\beta^*(z) \\
+ \sum_\alpha \psi_\alpha^*(z) \left( v^{\alpha \alpha} \partial_\alpha \psi_\beta^*(q_\alpha) + \frac{3}{2} (\partial_\alpha v^{\alpha \alpha}) \psi_\beta^*(q_\alpha) \right).
\] (5.28)

This expression should have been expected: it states that \( \psi_\beta^*(z) \) transforms as a form of rank \( 3/2 \) in \( z \), \( -3/2 \) in \( q_\beta \) and of rank 0 in the remaining points \( q_\alpha, \alpha \neq \beta \).

Assembling all contributions, we have
\[
\delta_v \langle H_\alpha | \Phi_\beta^* \rangle = - \int d^2 z v^\alpha \left( \frac{1}{2} \chi^z + \partial_z \psi_\beta^* + \frac{3}{2} (\partial_z \chi^z) \psi_\beta^* \right) + \int d^2 z \chi^\alpha \left( v^z \partial_z \psi_\beta^* + \frac{3}{2} (\partial_z v^z) \psi_\beta^* \right) \\
- \sum_\gamma \langle \chi_\alpha | \psi_\gamma^* \rangle \left( v^{\alpha \gamma} \partial_\gamma \psi_\beta^*(q_\gamma) + \frac{3}{2} (\partial_\gamma v^{\alpha \gamma}) \psi_\beta^*(q_\gamma) \right).
\] (5.29)

The first two terms on the rhs will cancel upon using the fact that \( v^z \) is of order \( \zeta^2 \), a fact that allows us to introduce a unique \( \zeta \)-independent quantity \( \bar{v}^z \), such that
\[
v^z \equiv \zeta^1 \zeta^2 \bar{v}^z \quad \Rightarrow \quad \frac{\partial v^z}{\partial \zeta^\alpha} = \epsilon_{\alpha \beta} \zeta^\beta \bar{v}^z, \quad \epsilon_{12} = 1.
\] (5.30)

Indeed, pulling out their \( \zeta \)-dependence, the first two terms become
\[
- \epsilon_{\alpha \gamma} \zeta^\gamma \zeta^\delta \int d^2 z \bar{v}^z \left( \frac{1}{2} \chi^z \partial_z \psi_\beta^* + \frac{3}{2} (\partial_z \chi^z) \psi_\beta^* \right) + \zeta^1 \zeta^2 \int d^2 z \chi^\alpha \left( \bar{v}^z \partial_z \psi_\beta^* + \frac{3}{2} (\partial_z \bar{v}^z) \psi_\beta^* \right)
\]

Using the elementary relation \( -\epsilon_{\alpha \gamma} \zeta^\gamma \zeta^\delta = \delta_{\alpha \delta} \zeta^1 \zeta^2 \), and integrating by parts in \( z \) so as to leave \( \chi_\alpha \) without derivatives acting, the above two terms cancel as announced,
\[
\zeta^1 \zeta^2 \int d^2 z \left( \frac{1}{2} \bar{v}^z \chi^\alpha \partial_z \psi_\beta^* - \frac{3}{2} \chi^\alpha \partial_z (\bar{v}^z) \psi_\beta^* + \chi_\alpha \bar{v}^z \partial_z \psi_\beta^* + \frac{3}{2} \chi_\alpha (\partial_z \bar{v}^z) \psi_\beta^* \right) = 0
\]

The remaining change is given by
\[
\delta_v \langle H_\alpha | \Phi_\beta^* \rangle = - \sum_\gamma \langle \chi_\alpha | \psi_\gamma^* \rangle \left( v^{\alpha \gamma} \partial_\gamma \psi_\beta^*(q_\gamma) + \frac{3}{2} (\partial_\gamma v^{\alpha \gamma}) \psi_\beta^*(q_\gamma) \right).
\] (5.31)
The change in the determinant is computed with the standard formula

\[ \delta_v \ln \det \langle H_\alpha |\Phi^*_\beta \rangle = \text{tr} \left( \langle H_\alpha |\Phi^*_\beta \rangle^{-1} \delta_v \langle H_\alpha |\Phi^*_\beta \rangle \right). \]  \hfill (5.32)

Using the fact that the variation \( \delta_v \langle H_\alpha |\Phi^*_\beta \rangle \) is already of order \( \zeta \zeta \), we see that the inverse matrix \( \langle H_\alpha |\Phi^*_\beta \rangle^{-1} \) may be taken at \( \zeta = 0 \) and thus reduces \( -\langle \chi_\alpha |\psi^*_\beta \rangle^{-1} \). But this factor now cancels the matrix \( \langle \chi_\alpha |\psi^*_\beta \rangle \) in (5.31) and we are left with

\[ -\delta_v \ln \det \langle H_\alpha |\Phi^*_\beta \rangle = -\sum_\beta \left( v^{q\beta} \partial \psi^*_\beta(q_\beta) + \frac{3}{2} (\partial v^{q\beta})(q_\beta) \right) \]  \hfill (5.33)

### 5.2.3 Stress Tensor Contribution

The stress tensor correlator was computed in (4.19) and is given by

\[ T(w) = -15 T_1(w) + \frac{1}{2} f_2(w)^2 - \frac{3}{2} \partial_w f_2(w) - \frac{1}{2} f_{3/2}(w)^2 + \partial_w f_{3/2}(w) \]
\[ + \frac{1}{2} \omega_J \partial_J \ln \partial[\delta](D_n) + \partial_J \ln \partial[\delta] \ln \partial[\delta](D_b) \]

where all ingredients in the above formula were discussed in Section 4.2.3. The singularities of \( T(w) \) are derived from the knowledge of the singularities of \( f_n(w) \), and the part of the expansion needed here is given by

\[ f_n(w) = \frac{1}{w - z_i} + \partial \phi_i^{(n)*}(z_i) + \mathcal{O}(w - z_i) \]
\[ \partial \phi_i^{(n)*}(z_i) = \omega_I(z_i) \partial_I \ln \partial[\delta](D_n) + \partial_{z_i} \ln \partial[\delta] \ln \partial[\delta](D_b) \]

plus terms holomorphic at \( w \sim z_i \). Here, \( \phi_i^{(n)*}(w) \) are the holomorphic \( n \)-forms normalized on the points \( z_i \) as usual \( \phi_i^{(n)*}(z_j) = \delta_{ij} \). The singular terms in \( T(w) \) are now readily identified

\[ T(w) = \sum_a \left( \frac{2}{(w - p_a)^2} + \frac{\partial \phi_a^{(2)*}(p_a)}{w - p_a} \right) - \sum_a \left( \frac{3/2}{(w - q_a)^2} + \frac{\partial \phi_a^{(3/2)*}(q_a)}{w - q_a} \right) + \text{reg.} \]  \hfill (5.36)

As a result, and using the previous notation \( \psi_\alpha^*(w) = \phi_\alpha^{(3/2)*}(w) \), we have

\[ \frac{1}{2\pi} \int d^2 z \partial_z \psi^* \langle T(z) \rangle = \sum_\beta \left( v^{q\beta} \partial \psi^*_\beta(q_\beta) + \frac{3}{2} (\partial v^{q\beta})(q_\beta) \right) \]
\[ - \sum_\beta \left( v^{p_b} \partial \phi_b^{(2)*}(p_b) + 2 (\partial v^{p_b})(p_b) \right). \]  \hfill (5.37)

Assembling the partial results (5.21), (5.33) and (5.37), we see that \( \delta_v \mathcal{A}[\delta] = 0 \), and thus the chiral measure is invariant under infinitesimal changes of \( \hat{\mu} \)-slice.
5.3 Slice $\chi_\alpha$ Independence: Worldsheet supersymmetry

Local supersymmetries act as follows

$$\delta_\xi \chi^+ = -2 \partial_z \xi^+ \quad \delta_\xi \hat{\chi}^+ = \xi^+ \chi^+$$  \hspace{1cm} (5.38)$$

where $\hat{\mu}$ is the shift in metric accompanying the shift in complex structure from $\Omega$ to $\hat{\Omega}$. As this supersymmetry should correspond to a change in $\chi_\alpha$ - slice, the supersymmetry parameter $\xi^+$ should be viewed as being of order $\zeta$. It is convenient to separate the $\xi$-variations of $\ln A[\delta]$ into those arising from the correlators $\ln A_{\text{corr}}[\delta]$ and those from the rank 2 and rank 3/2 differentials.

5.3.1 Correlator contributions

The supersymmetry transformation of the correlator terms is given by

$$\delta_\xi \ln A_{\text{corr}}[\delta] = \frac{1}{2\pi^2} \int d^2 z \partial_\xi \xi^+ \int d^2 w \chi^+ w \langle S(z)S(w) \rangle + \frac{1}{2\pi} \int d^2 z \xi^+ \chi^+ \langle T(z) \rangle .$$  \hspace{1cm} (5.39)$$

The pole at $z = w$ in the $S(z)S(w)$ correlator is precisely cancelled by the corresponding contribution from the stress tensor term, using the fact that

$$\partial_z S(z)S(w) = 2\pi \delta(z, w) \frac{1}{2} T(z) + \text{other than } z = w$$  \hspace{1cm} (5.40)$$

and an integration by parts in $\partial_z$. The remaining poles now arise only from the ghost contributions, given through (I.11),

$$\delta_\xi \ln A_{\text{corr}}[\delta] = -\frac{1}{2\pi} \int d^2 z \xi^+ \langle z \rangle \int d^2 w \chi^+ \partial_z C_{gh}(z, w) \bigg|_{z \neq w}$$  \hspace{1cm} (5.41)$$

The calculation of $\partial_z C_{gh}(z, w)$ is simplified by the fact that we are instructed to ignore the $z = w$ pole, so that effectively $\partial_z G_2(z, w) = \partial_z G_{3/2}(z, w) = 0$ and

$$\partial_z G_{3/2}(w, z) = 2\pi \sum_\alpha \delta(z, q_\alpha) \phi^{(3/2)*}_\alpha (w)$$

$$\partial_z G_2(w, z) = 2\pi \sum_\alpha \delta(z, p_\alpha) \phi^{(2)*}_\alpha (w)$$  \hspace{1cm} (5.42)$$

Thus, we find

$$4\partial_z C_{gh}(z, w) \bigg|_{z \neq w} = +3 \partial_w G_2(z, w) \partial_z G_{3/2}(w, z) + 2G_2(z, w) \partial_z \partial_w G_{3/2}(w, z)$$

$$-3 \partial_z \partial_z G_2(w, z) G_{3/2}(z, w) - 2 \partial_z G_2(w, z) \partial_z G_{3/2}(z, w)$$  \hspace{1cm} (5.43)$$

and using the $\partial_z G$ formulas above, we have

$$4\partial_z C_{gh}(z, w) \bigg|_{z \neq w} = -2\pi \sum_\alpha \phi^{(2)*}_\alpha (w) \left(3 \partial_\delta(z, p_\alpha) G_{3/2}(z, w) + 2 \delta(z, p_\alpha) \partial_z G_{3/2}(z, w) \right)$$

$$+ 2\pi \sum_\alpha \delta(z, q_\alpha) \left(3 \phi^{(3/2)*}_\alpha (w) \partial_w G_2(z, w) + 2 \partial_w \phi^{(3/2)*}_\alpha (w) G_2(z, w) \right)$$
Substituting this result into $\delta \ln A_{\text{corr}}[\delta]$, we find first for the $\alpha$-sum that

$$
\frac{1}{2\pi} \sum_{\alpha} \xi^+(q_\alpha) \int d^2 w \chi_w^+ \left( G_2(q_\alpha, w) \partial_w \phi^{(3/2)*}_\alpha(w) + \frac{3}{2} \partial_w G_2(q_\alpha, w) \phi^{(3/2)*}_\alpha(w) \right)
$$

$$
= \frac{1}{2\pi} \sum_{\alpha} \xi^+(q_\alpha) \int d^2 w G_2(q_\alpha, w) \left( \frac{1}{2} \partial_w \phi^{(3/2)*}_\alpha(w) \chi_w^+ + \frac{3}{2} \partial_w \chi_w^+ \phi^{(3/2)*}_\alpha(w) \right)
$$

We now recall the definitions of the components of the (even) superholomorphic 3/2 forms (B.11) and recognize that these are precisely the combinations that occur here. Putting all together, we have

$$
\delta \ln A_{\text{corr}} = - \sum_{\alpha} \xi^+(q_\alpha) \Phi^*_\alpha \Phi^{*}_\alpha + \sum_a \left( \xi^+(p_a) \partial \Phi^*_a(p_a) + 3(\partial \xi^+)(p_a) \Phi^*_a(p_a) \right)
$$

a form that is suggestive of the effect of supersymmetry Ward identities.

### 5.3.2 Rank 2 Differentials Contribution

To evaluate $\delta \ln \det \Phi_{IJ+}(p_a)$, we compute the transformation properties of $\Phi_{IJ+}$,

$$
i \Phi_{IJ+} = \omega_I \omega_J + \omega_I \partial_z \lambda_I + \omega_J \partial_z \lambda_J - \frac{1}{2} \hat{\omega}_{I0} \partial_z \hat{\omega}_{J0} - \frac{1}{2} \hat{\omega}_{J0} \partial_z \hat{\omega}_{I0},
$$

where the transformation laws of each of the ingredients is given by

$$
\delta \xi \omega_I(\hat{\Omega}) = 0
$$

$$
\delta \xi \hat{\omega}_{I0}(z) = \xi^+(z) \hat{\omega}_I(z)
$$

$$
\delta \lambda_I(z) = \xi^+(z) \hat{\omega}_{I0}(z).
$$

Collecting all terms and using the fact that $\xi$ is of order $\zeta$, the total transformation law may be recast in the following form,

$$
\delta \xi \Phi_{IJ+}(z) = \xi^+(z) \partial_z \Phi_{IJ0}(z) + 3(\partial_z \xi^+) \Phi_{IJ0}(z).
$$

The variation of the determinant is computed as follows

$$
\delta \xi \ln \det \Phi_{IJ+}(p_a) = \sum_b \left( \xi^+ \partial_w F_b(w) + 3(\partial_w \xi^+) F_b(w) \right)_{w=p_b}
$$

where the quantity $F_b(w)$ is defined as follows

$$
F_b(w) = \frac{\det \Phi_{IJ}(p_a[w, b])}{\det \Phi_{IJ+}(p_a)} \Phi_{IJ+}(w) = \begin{cases} 
\Phi_{IJ+}(p_a) & a \neq b \\
\Phi_{IJ0}(w) & a = b
\end{cases}
$$

It remains to compute $F_b(w)$. In view of the differential equation (B.9), needed here for $n = 3/2$,

$$
\partial_w \Phi_{IJ0}(w) = \frac{i}{2} \chi_w^+ \omega_I \omega_J(w),
$$

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the 3/2 form $F_b(w)$ obeys a simple differential equation
\[ \partial_w F_b(w) = -\frac{1}{2} \chi_w + \phi_b^{(2)*}(w) \]
\[ \phi_b^{(2)*}(w) = \frac{\text{det} \omega_I \omega_J(p_a[w,b])}{\text{det} \omega_I \omega_J(p_a)} \quad p_a[w,b] = \begin{cases} p_a & a \neq b \\ w & a = b \end{cases} \quad (5.51) \]

The form $\phi_b^{(2)*}(w)$ is readily recognized to be holomorphic of rank 2 in $w$ and to be normalized so that $\phi_b^{(2)*}(p_a) = \delta_{ab}$. Hence, $F_b(w)$ obeys the same differential equation as the quantity $\Phi^{*}_{b0}(w)$, and must thus differ from it only by a holomorphic 3/2 form:
\[ F_p(w) = \Phi^{*}_{b0}(w) + \sum_{\alpha} \psi_{\alpha}^{*}(w) R_{\alpha b}^{*}. \quad (5.52) \]

The coefficients $R_{\alpha b}^{*}$ are independent of $w$, first order in $\zeta$ and may be determined by evaluating the equation at $w = q_\alpha$, using the fact that $\Phi^{*}_{b0}(q_\alpha) = 0$, so that
\[ R_{\alpha b}^{*} = F_b(q_\alpha) = \frac{\text{det} \Phi_{IJ}(p_a[q_\alpha,b])}{\text{det} \Phi_{IJ}(p_a)}. \quad (5.53) \]

In terms of the matrices
\[ M_{\alpha IJ}^{*} \equiv \Phi_{IJ}(p_a) \quad N_{\alpha IJ}^{*} \equiv \Phi_{IJ}(0_{\alpha}), \quad (5.54) \]

we simply have $R_{\alpha b}^{*} = (NM^{-1})_{\alpha b}$. Assembling the entire contribution, we have
\[ - \delta_\xi \ln \text{det} \Phi_{IJ}(p_a) = -\sum_b \left( \xi^+(p_b) \partial \Phi^{*}_{b0}(p_b) + 3(\partial \xi^+)(p_b) \Phi^{*}_{b0}(p_b) \right) \]
\[ -\sum_{b,\alpha} \left( \xi^+(p_b) \partial \psi_{\alpha}^{*}(p_b) + 3(\partial \xi^+)(p_b) \psi_{\alpha}^{*}(p_b) \right) (NM^{-1})_{\alpha b} \quad (5.55) \]

We shall not need to make the form of the matrices $M$ and $N$ explicit because their contribution will be cancelled later on by expressions manifestly of the same form.

### 5.3.3 Supersymmetry variations of $H_{\alpha}$

The supersymmetry variation $\delta_\xi H_{\alpha}$ will be needed for the calculations of the $\xi$-variation of the finite-dimensional determinant involving $H_{\alpha}$. The calculation of this variation is subtle and interesting. Furthermore, the variation evaluated with respect to the metric $\hat{g}$ will present further delicate new features, the thorough understanding of which will be crucial later on. Therefore, we devote this subsection to these issues before working out the $\xi$-variation of the rank 3/2 differentials.

The starting point is the supersymmetry transformations of (5.38) which, by construction, are written with respect to the metric $g$. It is assumed that the metric $\hat{g}$ is independent
of $\zeta^\alpha$ and that it transforms under $\xi$ at most by a diffeormorphism (whose effects have been shown to be immaterial in the preceding section and may thus be safely dropped) and that derivation with respect to coordinates like $\zeta^\alpha$ commutes with $\delta_\xi$,

$$\frac{\partial}{\partial \zeta^\alpha} \delta_\xi = \delta_\xi \frac{\partial}{\partial \zeta^\alpha}.$$  \hfill (5.56)

¿From these facts, we shall now prove the following transformation formulas,

$$\delta_\xi \chi^\alpha = -2\partial_z \zeta^{\alpha+} - 2\mu_\alpha \partial_z \xi^{\alpha+} + (\partial_z \mu_\alpha) \xi^{\alpha+},$$

$$\delta_\xi \mu_\alpha = \xi^{\alpha+} \chi^{\alpha+} - \xi^{\alpha+} \chi_\alpha.$$  \hfill (5.57)

for the variation under a local susy $\xi$ of the slice function $\chi_\alpha$, defined by

$$\chi^\alpha = \partial \chi + \bar{\zeta} \partial \zeta^\alpha \quad \mu_\alpha = \partial \hat{\mu} = \partial \zeta^\alpha \quad \xi^{\alpha+} = \partial \zeta^\alpha$$  \hfill (5.58)

To do this, start from (5.38) and differentiate with respect to $\zeta^\alpha$, using (5.56),

$$\delta_\xi \chi^\alpha = \frac{\partial}{\partial \zeta^\alpha} \delta_\xi \chi^{\alpha+} = \frac{\partial}{\partial \zeta^\alpha} \left( -2\nabla_{\bar{z}}^{(-1/2)} \xi^{\alpha+} \right) = -2\partial_z \zeta^{\alpha+} - 2 \left( \frac{\partial}{\partial \zeta^\alpha} \nabla_{\bar{z}}^{(-1/2)} \right) \xi^{\alpha+},$$

$$\delta_\xi \mu_\alpha = \frac{\partial}{\partial \zeta^\alpha} \delta_\xi \hat{\mu} = \frac{\partial}{\partial \zeta^\alpha} \left( \xi^{\alpha+} \chi^{\alpha+} \right) = + \xi^{\alpha+} \chi^{\alpha+} - \xi^{\alpha+} \chi_\alpha.$$  \hfill (5.59)

The second line thus establishes the desired formula for the variation of $\mu_\alpha$. The formula for $\delta_\xi \chi^\alpha$ needs more work and involves interesting subtleties. First, the operator $\nabla_{\bar{z}}^{(-1/2)}$ depends on $\zeta^\alpha$ through its dependence on the metric $g$ which itself depends on $\zeta^\alpha$. (This is in contrast with the metric $\hat{g}$ which is $\zeta^\alpha$-independent.) Second, the $g$-dependence of $\nabla_{\bar{z}}^{(-1/2)}$ is known and may be expressed in terms of the differential $\hat{g}$ as follows,

$$\nabla_{\bar{z}}^{(-1/2)} = \hat{\nabla}_{\bar{z}}^{(-1/2)} + \hat{\mu} \partial_z - \frac{1}{2} \partial_z \hat{\mu}$$  \hfill (5.60)

where $\hat{\nabla}$ is now evaluated with respect to the metric $\hat{g}$. Differentiating with respect to $\zeta^\alpha$ and using the second equation in (5.58), we find

$$\frac{\partial}{\partial \zeta^\alpha} \nabla_{\bar{z}}^{(-1/2)} = \mu_\alpha \partial_z - \frac{1}{2} \partial_z \mu_\alpha$$  \hfill (5.61)

Combining this result with (5.59), we recover the proposed formula for $\delta_\xi \mu_\alpha$.

We now seek to re-express the transformation laws (5.59) with respect to the metric $\hat{g}$ instead of $g$, as written in (5.59). The entire transformation law of $\mu_\alpha$ is of order $\zeta$ already and thus a change from $g$ to $\hat{g}$ is immaterial. Next we deal with the case of $\delta_\xi \chi^\alpha$. ¿From the definitions of $\mu_\alpha$ and the fact that $\hat{\mu}$ is of order $\zeta \bar{\zeta}$, we have

$$\mu_\alpha = \epsilon_{\alpha\beta} \zeta^\beta \hat{\mu}, \quad \hat{\mu} = \zeta^1 \zeta^2 \mu, \quad \xi^{\alpha+} = \zeta^\alpha \zeta^{\alpha+}.$$  \hfill (5.62)
We now perform a Fierz type rearrangement on the products \( \mu_\alpha \xi^+ \) occurring in (5.59). Omitting irrelevant derivatives, and using \( \zeta^\beta \zeta^\gamma = e^{\beta \gamma} \zeta^1 \zeta^2 \), we have

\[
\mu_\alpha \xi^+ = \epsilon_{\alpha\beta} \bar{\mu}^\gamma \zeta^\gamma = -\bar{\mu} \xi^+_\alpha
\]  
(5.63)

The expression for the variation becomes (restoring now derivatives either on \( \mu_\alpha \) or on \( \xi^+ \))

\[
\delta \xi \chi_\alpha = -2\partial_z \xi^+_\alpha + 2\bar{\mu} \partial_z \xi^+_\alpha - (\partial_z \bar{\mu}) \xi^+_\alpha
\]  
(5.64)

Using the fact that \( g = \hat{g} + \mu \), equation (5.60), and expressing all quantities with respect to \( \hat{g} \), we recognize that a remarkable simplification occurs in the transformation rule,

\[
\delta \xi \chi_\alpha = -2\hat{\partial}_z \xi^+_\alpha
\]  
(5.65)

### 5.3.4 Rank 3/2 Differential Contribution

To evaluate \( \delta \xi \ln \det \langle H_\alpha | \Phi^*_\beta \rangle \), we start again from the component expression

\[
- \langle H_\alpha | \Phi^*_\beta \rangle = \langle \mu_\alpha | \Phi^*_\beta+ \rangle + \langle \chi_\alpha | \Phi^*_0 \rangle
\]  
(5.66)

and deduce its variation

\[
- \delta \xi \langle H_\alpha | \Phi^*_\beta \rangle = \langle \delta \xi \mu_\alpha | \Phi^*_\beta+ \rangle + \langle \delta \xi \chi_\alpha | \Phi^*_\beta0 \rangle + \langle \mu_\alpha | \delta \xi \Phi^*_\beta+ \rangle + \langle \chi_\alpha | \delta \xi \Phi^*_\beta0 \rangle.
\]  
(5.67)

We shall now evaluate each of these inner products in turn. We begin by computing the variations of the components of \( \Phi^*_\alpha \).

The variation of the components \( \Phi^*_\beta+ \) of the holomorphic superdifferentials is

\[
\delta \xi \Phi^*_\beta+(z) = \xi^+(z) \partial \Phi^*_\beta0(z) + 3(\partial \xi^+)(z) \Phi^*_\beta0(z)
\]

\[
- \sum_a \left( \xi^+(p_a) \partial \Phi^*_\beta0(p_a) + 3\partial \xi^+(p_a) \Phi^*_\beta0(p_a) \right) \Phi^*_\alpha+(z),
\]  
(5.68)

where no variation of the metric is required since the differential is already of order \( \zeta \).

The variation of the component \( \Phi^*_\beta0 \) contains a contribution due to the variation of the metric for its leading term which is of order zero in \( \zeta \). We have

\[
\delta \xi \Phi^*_\beta0(z) = \delta \xi \psi^*_\beta(z) - \frac{1}{2\pi} \int d^2 w G_{3/2}^3(z, w) \frac{1}{2}(\delta \xi \chi^+_w) \Phi^*_\beta+(w)
\]

\[
- \frac{1}{2\pi} \int d^2 w G_{3/2}^3(z, w) \frac{1}{2} \chi^+_w (\delta \xi \Phi^*_\beta+(w)),
\]  
(5.69)

where the first term on the rhs of (5.69), due to the variation of the metric, is given by

\[
\delta \xi \psi^*_\beta(z) = \frac{1}{2\pi} \int d^2 w \xi^+(w) \chi^+_w \left( \frac{3}{2} \partial_w G_{3/2}^3(z, w) \psi^*_\beta(w) + \frac{1}{2} G_{3/2}^3(z, w) \partial_w \psi^*_\beta(w) \right).
\]  
(5.70)
The second term on the rhs of (5.69) may be worked out more explicitly and is given by
\[ \xi^+(z) \Phi^*_\alpha + (z) - \sum_\alpha \xi^+(q_\alpha) \Phi^*_\alpha + (q_\alpha) \psi^*_\alpha (z) \]
\[ + \frac{1}{2\pi} \int d^2 w G_{3/2}(z, w) \xi^+(w) \left( \frac{1}{2} \partial_w \Phi^*_\alpha (w) + \frac{3}{2} \partial_w \chi^+(w) \Phi^*_\beta (w) \right), \]
while the third term on the rhs of (5.69) becomes
\[ - \frac{1}{2\pi} \int d^2 w G_{3/2}(z, w) \xi^+(w) \left( \xi^+(w) \partial_w \psi^*_\beta (w) + 3 \partial_w \xi^+(w) \psi^*_\beta (w) \right) \]
\[ + \sum_\alpha \left( \xi^+(p_\alpha) \partial \psi^*_\beta (p_\alpha) + 3 \partial \xi^+(p_\alpha) \psi^*_\beta (p_\alpha) \right) \Phi^*_\alpha (z). \]
Combining all contributions, we have
\[ \delta \xi \Phi^*_\beta (z) = \xi^+(z) \Phi^*_\beta + (z) - \sum_\alpha \xi^+(q_\alpha) \Phi^*_\alpha + (q_\alpha) \Phi^*_\alpha (z) \]
\[ + \sum_\alpha \left( \xi^+(p_\alpha) \partial \psi^*_\beta (p_\alpha) + 3 \partial \xi^+(p_\alpha) \psi^*_\beta (p_\alpha) \right) \Phi^*_\alpha (z) \quad (5.71) \]
Assembling \( \langle \delta \xi \chi_\alpha | \Phi^*_\beta \rangle + \langle \chi_\alpha | \delta \xi \Phi^*_\beta \rangle \), all terms involving the Green’s function \( G_{3/2}(z, w) \) cancel one another, while upon further addition of the term \( \langle \delta \xi \mu_\alpha | \Phi^*_\beta \rangle \), all terms involving the Green’s function \( G_2(z, w) \) also cancel one another. One is thus left with (care is needed, while arranging these formulas, to the precise order of the anti-commuting entries)
\[ \langle \delta \xi \mu_\alpha | \Phi^*_\beta \rangle + \langle \delta \xi \chi_\alpha | \Phi^*_\beta \rangle + \langle \chi_\alpha | \delta \xi \Phi^*_\beta \rangle \]
\[ = - \sum_\gamma \xi^+(q_\gamma) \Phi^*_\gamma (q_\gamma) \langle \chi_\alpha | \psi^*_\gamma \rangle - \int d^2 z \mu_\alpha (\xi^+ \partial \psi^*_\beta + 3 \partial \xi^+ \psi^*_\beta) \]
\[ + \sum_\alpha \left( \xi^+(p_\alpha) \partial \psi^*_\beta (p_\alpha) + 3 \partial \xi^+(p_\alpha) \psi^*_\beta (p_\alpha) \right) \langle \chi_\alpha | \Phi^*_\alpha \rangle. \quad (5.72) \]
On the other hand, one has
\[ \langle \mu_\alpha | \delta \xi \Phi^*_\beta \rangle = \int d^2 z \mu_\alpha (\xi^+ \partial \psi^*_\beta + 3 \partial \xi^+ \psi^*_\beta) \]
\[ + \sum_\alpha \left( \xi^+(p_\alpha) \partial \psi^*_\beta (p_\alpha) + 3 \partial \xi^+(p_\alpha) \psi^*_\beta (p_\alpha) \right) \langle \mu_\alpha | \Phi^*_\alpha \rangle. \quad (5.73) \]
Combining this with preceding contributions as well, and reassembling the terms belonging to \(- \langle H_\alpha | \Phi^*_\beta \rangle = \langle \mu_\alpha | \Phi^*_\beta \rangle + \langle \chi_\alpha | \Phi^*_\beta \rangle\), we obtain a fairly simple formula
\[ \delta \xi \langle H_\alpha | \Phi^*_\beta \rangle = - \sum_\gamma \xi^+(q_\gamma) \Phi^*_\gamma + (q_\gamma) \langle H_\alpha | \Phi^*_\gamma \rangle \]
\[ + \sum_\alpha \left( \xi^+(p_\alpha) \partial \psi^*_\beta (p_\alpha) + 3 \partial \xi^+(p_\alpha) \psi^*_\beta (p_\alpha) \right) \langle H_\alpha | \Phi^*_\alpha \rangle. \quad (5.74) \]
The variation of the determinant follows directly from this result and is given by

\[- \delta \xi \ln \det \langle H_\alpha | \Phi^*_\beta \rangle = + \sum_{\gamma} \xi^+(q_\gamma) \Phi^*_{\gamma+}(q_\gamma) \]

\[\quad - \sum_{a\alpha\beta} \left( \xi^+(p_a) \partial \bar{\psi}^*_\beta(p_a) + 3 \partial \xi^+(p_a) \psi^*_\beta(p_a) \right) \right(\langle H_\alpha | \Phi^*_\beta \rangle)^{-1} \langle H_\alpha | \Phi^*_\alpha \rangle. \] (5.75)

To compare this result with the one for the other finite dimensional determinant, it is necessary to reformulate the last term as follows. We use the decomposition of \( \Phi_{IJ} \) onto \( \Phi^*_a \) and \( \Phi^*_\alpha \)

\[\Phi_{IJ}(z) = \sum_a \Phi^*_a(z) M^a_{IJ} + \sum_{\alpha} \Phi^*_\alpha(z) N^\alpha_{IJ}, \] (5.76)

and from projections onto \( H^*_a \) and \( H^*_\alpha \), we recognize that the matrices \( M \) and \( N \) coincide with the ones introduced in (5.54). Taking now the inner product with \( H_\gamma \), we find

\[\langle H_\gamma | \Phi^*_\alpha \rangle = - \sum_{\alpha} \langle H_\gamma | \Phi^*_\alpha \rangle (NM^{-1})^\alpha_a \] (5.77)

Using this result, we have

\[- \delta \xi \ln \det \langle H_\alpha | \Phi^*_\beta \rangle = + \sum_{\gamma} \xi^+(q_\gamma) \Phi^*_{\gamma+}(q_\gamma) \]

\[\quad + \sum_{a\beta} \left( \xi^+(p_a) \partial \bar{\psi}^*_\beta(p_a) + 3 \partial \xi^+(p_a) \psi^*_\beta(p_a) \right) \right)(NM^{-1})^\beta_a. \] (5.78)

### 5.3.5 Summary of slice \( \chi \) independence

Assembling all contributions (5.44), (5.55), (5.78) to \( \delta \xi \ln A_{\text{corr}}[\delta] \), we easily see that they all cancel one another, and thus the chiral string measure is completely independent of all choices of slice.
6 Manifestly Reparametrization Invariant Formulas

We have established in the previous sections the invariance of our gauge fixed formulas under infinitesimal changes of gauge slice $S$. In this section, we shall obtain a last formula in which we make the invariance under infinitesimal diffeomorphisms manifest, by eliminating completely the dependence on the Beltrami differential $\hat{\mu}$. The key fact is that, unlike $\hat{\mu}$ itself, the conformal class of $\hat{\mu}$ is known. Thus it suffices to show that the entire dependence on $\hat{\mu}$ of the gauge fixed formulas resides in its pairing with a holomorphic quadratic differential. We begin with a more detailed discussion of the conformal class of $\mu_\alpha$ in subsection 6.1, and then work out the various contributions to the final formula.

6.1 The conformal class of $\mu_\alpha$ for genus 2

Having fixed the gravitino slice $\chi_\alpha$, the orthogonality relation $\langle H_\alpha | \Phi_{IJ} \rangle = 0$ fixes the conformal class of $\mu_\alpha$, though not the differential itself. To see this, we work in WZ gauge, where $H_\alpha = \bar{\theta} (\mu_\alpha - \theta \chi_\alpha)$, recall the expression for $\Phi_{IJ}$ of (3.18),

$$\Phi_{IJ} = -\frac{i}{2} (\hat{\omega}_I D_+ \hat{\omega}_J + \hat{\omega}_J D_+ \hat{\omega}_I)$$  \hspace{1cm} (6.1)

and use the expression for $\hat{\omega}_I = \hat{\omega}_{10} + \theta \hat{\omega}_{I+}$, neglecting the contribution from the auxiliary field $A$, which cancels out,

$$\int d^2z \mu_\alpha (\hat{\omega}_{I+} \hat{\omega}_{J+} - \hat{\omega}_{10} \partial z \hat{\omega}_{J0} + I \leftrightarrow J) = -\int d^2z \chi_\alpha (\hat{\omega}_{10} \hat{\omega}_{J+} + \hat{\omega}_{J0} \hat{\omega}_{I+})$$  \hspace{1cm} (6.2)

For genus 2, both sides are odd and of order $\zeta$, so that $\mu_\alpha$ is odd and first order in $\zeta$, while the term $\hat{\omega}_{10} \partial z \hat{\omega}_{J0}$ is of order $\zeta \zeta$ and may be dropped, leading to the following simplified form involving the ordinary holomorphic differentials $\omega_I$,

$$2 \int d^2z \mu_\alpha \omega_I \omega_J = -\int d^2z \chi_\alpha \left( \hat{\omega}_{10} \omega_J + \hat{\omega}_{J0} \omega_I \right)$$  \hspace{1cm} (6.3)

This equation uniquely determines the conformal class of $\mu_\alpha$.

Next, we show that this conformal class coincides with the one predicted by the relation $\mu_\alpha = \partial \hat{\mu} / \partial \zeta^\alpha$, thus further reinforcing the general validity of our approach. To this end, we compute in two different ways the derivative of $\partial \Omega_{IJ} / \partial \zeta^\alpha$, keeping $\hat{\Omega}_{IJ}$ fixed. The first way uses the fact that $\hat{\mu}$ is the Beltrami differential that takes us from metric $\hat{g}$ to $g$,

$$\Omega_{IJ} - \hat{\Omega}_{IJ} = i \int d^2z \hat{\mu}_I \omega_J$$  \hspace{1cm} \implies \hspace{1cm} \frac{\partial \Omega_{IJ}}{\partial \zeta^\alpha} = i \int d^2u \mu_\alpha \omega_J \omega_I$$  \hspace{1cm} (6.4)

The second way makes use of the explicit relation between $\Omega_{IJ}$ and $\hat{\Omega}_{IJ}$, given by

$$\Omega_{IJ} = \hat{\Omega}_{IJ} + \frac{i}{8\pi} \int d^2u \int d^2v \omega_I(u) \chi_u^+ S_\delta(u,v) \chi_v^+ \omega_J(v)$$  \hspace{1cm} (6.5)
and the constancy of $\Omega_{IJ}$, so that
\begin{align*}
\frac{\partial \Omega_{IJ}}{\partial \zeta^\alpha} &= \frac{i}{8\pi} \int d^2 w \int d^2 v \omega_I(w) \chi_\alpha(w) S_\delta(w, v) \chi_\delta^+ \omega_J(v) + (I \leftrightarrow J) \\
&= -\frac{i}{2} \int d^2 w \chi_\alpha(w) \left( \hat{\omega}_{I0}(w) \omega_J(w) + \hat{\omega}_{J0}(w) \omega_I(w) \right) \tag{6.6}
\end{align*}
where we have used the following relation in passing from the first to the second line above
\[ \hat{\omega}_{I0}(z) = -\frac{1}{4\pi} \int d^2 w \ S_\delta(z, w) \chi_\delta^+ \omega_I(w). \tag{6.7} \]
The agreement of both calculations of (6.3) confirms our determination of the class of $\mu_\alpha$.

### 6.2 Contributions from $\hat{\mu}$: spin 2 part

The contribution from the first finite dimensional determinant involves $\Phi_{IJ}$ of (3.18),
\[ i \Phi_{IJ+} = \omega_I \omega_J + \omega_I \partial_z \lambda_J + \omega_J \partial_z \lambda_I - \frac{1}{2} \hat{\omega}_{I0} \partial_z \hat{\omega}_{J0} - \frac{1}{2} \hat{\omega}_{J0} \partial_z \hat{\omega}_{I0}, \tag{6.8} \]
whose entire $\hat{\mu}$ dependence is through $\lambda_I$, and is given by
\[ \partial_z \lambda_I(z) \bigg|_{\hat{\mu}} = \frac{1}{2\pi} \int d^2 w \partial_z \omega \ln E(z, w) \hat{\mu}(w) \omega_I(w). \tag{6.9} \]
Recalling the definition of the holomorphic 1-form $\varpi_a^*(w)$, introduced in (1.15) via $\varpi_a^*(w) = \varpi_a(p_a, w)$ and noting that this object satisfies $\varpi_a^*(p_a) = 1$, we have the following expansion in $\hat{\mu}$, which terminates to first order,
\[ -i \left. \frac{\det \omega_I \omega_J(p_a)}{\det \Phi_{IJ+}(p_a)} \right|_{\hat{\mu}} = -\frac{1}{2\pi} \sum_a \int d^2 w \partial_{p_a} \omega \ln E(p_a, w) \hat{\mu}(w) \varpi_a^*(w). \tag{6.10} \]
Clearly, the 2-form integrated versus $\hat{\mu}$ has poles at $p_a$, but is holomorphic everywhere else. This contribution goes naturally together with the one from the stress tensor insertion $T_2$ involving $f_2$. We add here the suitable multiple of $T_1(w)$ to make this contribution into a well-defined and single valued 2-form. Assembling all parts, we find
\begin{align*}
-\left. i \frac{\det \omega_I \omega_J(p_a)}{\det \Phi_{IJ+}(p_a)} \right|_{\hat{\mu}} &+ \frac{1}{2\pi} \int d^2 w \hat{\mu}(w) \left\{ -27 T_1(w) + \frac{1}{2} f_2(w)^2 - \frac{3}{2} \partial_w f_2(w) \right\} \\
&= \frac{1}{2\pi} \int d^2 w \hat{\mu}(w) B_2(w), \tag{6.11}
\end{align*}
where the meromorphic 2-form is given by
\[ B_2(w) = -27 T_1(w) + \frac{1}{2} f_2(w)^2 - \frac{3}{2} \partial_w f_2(w) - 2 \sum_a \partial_{p_a} \partial_w \ln E(p_a, w) \varpi_a^*(w). \tag{6.12} \]
Actually, even though this is only a partial result from the spin 2 sector only, \( B_2(w) \) by itself is a holomorphic and single-valued 2-form. To see this, apply \( \partial_w \), and use the asymptotic expansion for \( w \sim p_a \),

\[
f_2(w) = \frac{1}{w - p_a} + \partial \phi^{(2)*}_a(p_a) + O(w - p_a). \tag{6.13}
\]

One finds

\[
\partial_w B_2(w) = \sum_a \partial_w \left( -2 \partial_{p_a} \partial_w \ln E(w, p_a) \bar{w}_a^*(w) - 2 \partial_w \frac{1}{w - p_a} + \frac{1}{w - p_a} \partial \phi^{(2)*}_a(p_a) \right) = 2 \pi \sum_a \left( 2 \partial_w \delta(w, p_a) \bar{w}_a^*(w) - 2 \partial_w \delta(w, p_a) + \delta(w, p_a) \partial \phi^{(2)*}_a(p_a) \right). \tag{6.14}
\]

This expression vanishes using \( \bar{w}_a^*(p_a) = 1 \) as well as \( \partial \phi^{(2)*}_a(p_a) = 2 \partial \bar{w}_a^*(p_a) \).

### 6.3 Contributions from \( \hat{\mu} \): spin 3/2 part

This finite dimensional determinant receives contributions from \( \langle \chi_\alpha | \Phi^*_\beta \rangle \) and \( \langle \mu_\alpha | \Phi^*_\beta \rangle \). The latter is given by

\[
\langle \mu_\alpha | \Phi^*_\beta \rangle = -\frac{1}{4 \pi} \int d^2 w \mu_\alpha(w) \int d^2 z G_2(w, z) \left( \chi_\beta^* \partial_z \psi^*_\beta(z) + 3(\partial_z \chi_\beta^*) \psi^*_\beta(z) \right). \tag{6.15}
\]

The expression may be simplified by using the \( \rho \)-dependence \( \hat{\mu} = \rho^1 \rho^2 \hat{\mu} \) and

\[
\mu_\alpha(w) \chi_\beta^*(z) = \epsilon_{\alpha \beta} \rho^3 \bar{\hat{\mu}} \chi^*_\gamma(z) = -\delta_{\alpha \beta} \rho^3 \rho^1 \bar{\hat{\mu}}(w) \chi_\gamma(z) = -\bar{\hat{\mu}}(w) \chi_\gamma(z). \tag{6.16}
\]

Using this simplification, we may assemble the \( \hat{\mu} \) contribution to the finite dimensional determinant as follows

\[
- \langle H_\alpha | \Phi^*_\beta \rangle = \langle \chi_\alpha | \Phi^*_\beta \rangle + \langle \mu_\alpha | \Phi^*_\beta \rangle = \langle \chi_\alpha | \psi^*_\beta \rangle \left( \frac{3}{2} \partial_w G_{3/2}(z, w) \psi^*_\beta(w) + \frac{1}{2} G_{3/2}(z, w) \partial_w \psi^*_\beta(w) - \frac{3}{2} G_{3/2}(w, z) \partial_z \psi^*_\beta(z) - \frac{3}{2} G_{3/2}(w, z) \partial_z \psi^*_\beta(z) \right). \tag{6.17}
\]

Note that in the first term, \( \langle \chi_\alpha | \psi^*_\beta \rangle \), the differential \( \psi^*_\beta \) is evaluated with respect to the metric \( \hat{g} \), while \( \chi_\alpha \) is still considered with respect to the metric \( g \).

To compute the determinant, it is useful to change basis for the \( \chi_\alpha \), as follows

\[
\chi_\alpha(z) = \sum_\beta \langle \chi_\alpha | \psi^*_\beta \rangle \chi_\beta(z) \quad \langle \chi_\alpha | \psi^*_\beta \rangle = \delta_{\alpha \beta}. \tag{6.18}
\]
As a result, the determinant takes on a simple form in this basis, as the $\hat{\mu}$-dependent correction amounts to taking the trace.

$$\det(H_\alpha|\Phi_{\hat{\mu}}^\alpha) = \frac{\det(\chi_\alpha|\psi^\alpha_\hat{\mu})}{\det(H_\alpha|\Phi_{\hat{\mu}}^\alpha)} \left(1 + \frac{1}{2\pi} \int d^2w \hat{\mu}(w) \int d^2z \chi^{*}_\alpha(z) \left[\frac{3}{2} \partial_w G_{3/2}(z, w) \psi^{*}_\alpha(w) + \frac{1}{2} G_{3/2}(z, w) \partial_w \psi^{*}_\alpha(w) - G_2(w, z) \partial_z \psi^{*}_\alpha(z) - \frac{3}{2} \partial_z G_2(w, z) \psi^{*}_\alpha(z)\right]\right).$$

It is natural to combine this contribution with the part of the stress tensor insertion $T_{3/2}$ that only depends upon the form $f_{3/2}$, suitably augmented by a multiple of $T_1(w)$ to make this contribution into a well-defined form. One obtains

$$\frac{\det(\chi_\alpha|\psi^\alpha_\hat{\mu})}{\det(H_\alpha|\Phi_{\hat{\mu}}^\alpha)} = \frac{1}{2\pi} \int d^2w \hat{\mu}(w) B_{3/2}(w)$$

where

$$B_{3/2}(w) = 12T_1(w) - \frac{1}{2} f_{3/2}(w)^2 + \partial_w f_{3/2}(w) + \int d^2z \chi^{*}_\alpha(z) \left(-\frac{3}{2} \partial_w G_{3/2}(z, w) \psi^{*}_\alpha(w) - \frac{1}{2} G_{3/2}(z, w) \partial_w \psi^{*}_\alpha(w) + G_2(w, z) \partial_z \psi^{*}_\alpha(z) + \frac{3}{2} \partial_z G_2(w, z) \psi^{*}_\alpha(z)\right).$$

This 2-form is also holomorphic, as can be seen from applying $p_w$,

$$\partial_w B_{3/2}(w) = 3\pi \sum_\gamma \partial_w \delta(w, q_\gamma) - 2\pi \sum_\gamma \delta(w, q_\gamma) \partial \psi^{*}_\gamma(q_\gamma) + \pi \sum_\gamma \left(-3\psi^{*}_\gamma(w) \partial_w \delta(w, q_\gamma) - \delta(w, q_\gamma) \partial_w \psi^{*}_\gamma(q_\gamma)\right)$$

which vanishes in view of the fact that $\psi^{*}_\gamma(q_\gamma) = 1$.

### 6.4 Remaining contributions from spin 2 part

The remaining contributions from the finite-dimensional determinant $\det \Phi_{I,I+}(p_a)$ that do not involve $\hat{\mu}$ arise from the $\lambda_I$ terms and from the terms in $\hat{\omega}_{I0}$ through their $\chi$-dependence.

The first terms arise from

$$\left.\partial_z \lambda_I\right|_{\chi} = \frac{1}{4\pi} \int d^2w \partial_z \partial_w \ln E(z, w) \chi^{+}_w \hat{\omega}_{I0}(w)$$

$$= -\frac{1}{16\pi^2} \int d^2w \partial_z \partial_w \ln E(z, w) \chi^{+}_w + \int d^2u S_\delta(w, u) \chi^{+}_w + \omega_I(u)$$

(6.22)
Using the same object $\omega^*_a$ as defined above, we find that the $\chi$-dependence in $\lambda$ produces the contribution,

$$-i \frac{\det \Phi_{IJ}(p_a)}{\det \omega_I \omega_J(p_a)} \chi_{\lambda} = -\frac{1}{8\pi^2} \int d^2 w \partial_{p_a} \partial_w \ln E(p_a, w) \chi_w^+ \int d^2 u S_b(w, u) \chi^+_u \omega^*_a(u)$$ (6.23)

This contribution gives rise to the term $X_4$ below.

The second term arises from the $\chi$-dependence of

$$\hat{\omega}_{I0}(z) = -\frac{1}{4\pi} \int d^2 u S_b(z, u) \chi^+_u$$ (6.24)

and requires the use of $\omega_a(u, v)$ of (1.13). Recall that this object is a holomorphic 1-form in $u$ and in $v$ separately, it is a scalar in $p_b$, $b \neq a$ and a $-2$ form in $p_a$. It also satisfies $\omega_a(v, u) = \omega_a(u, v)$ as well as $\omega_a(p_a, v) = \omega^*_a(v)$. Using this definition, the contribution of the second term is

$$-i \frac{\det \Phi_{IJ}(p_a)}{\det \omega_I \omega_J(p_a)} \chi_{\omega_{I0}} = -\frac{1}{16\pi^2} \int d^2 u \int d^2 v S_b(p_a, u) \chi^+_u \chi^+_v + \chi^+_v \omega^*_a(u, v).$$ (6.25)

This contribution gives rise to the term $X_5$ below.

### 6.5 Remaining contributions from spin $3/2$ part

The remaining contribution from the spin $3/2$ part is due solely to the terms quadratic in $\chi$ appearing in the $\Phi^*_\beta$ differential, as integrated versus $\chi_\alpha$. It is given by

$$\det \langle H_\alpha | \Phi^*_\beta \rangle = 1 - \frac{1}{16\pi^2} \int d^2 z \chi^*_\alpha(z) \int d^2 w G_{3/2}(z, w) \chi_w^+ \int d^2 v \chi^+_v \Lambda_\alpha(w, v)$$ (6.26)

where $\Lambda_\alpha$ is defined by

$$\Lambda_\alpha(w, v) \equiv 2G_2(w, v) \partial_w \psi^*_\alpha + 3\partial_v G_2(w, v) \psi^*_\alpha(v).$$ (6.27)

This term contribution gives rise to the term $X_6$ below.

### 6.6 Summary

Using the results of the previous calculations of the finite-dimensional determinants and stress tensor insertions, we may write a more explicit result as follows.

$$\mathcal{A}[\delta] = \left\{ \frac{\Pi_a b(p_a) \Pi_\alpha \delta(q_a)}{\det (\omega_I \omega_J(p_a))} \cdot \det \langle \chi_\alpha | \psi^*_\beta \rangle \left\{ 1 + X_1 + X_2 + X_3 + X_4 + X_5 + X_6 \right\} \right\}$$

$$X_1 = -\frac{1}{8\pi^2} \int d^2 z \chi^+_z \int d^2 w \chi^+_w \langle S(z) S(w) \rangle$$
\[ X_2 = \frac{i}{4\pi} (\hat{\Omega}_{IJ} - \Omega_{IJ}) \left( 5\partial_I \partial_J \ln \vartheta[\delta](0) - \partial_I \partial_J \ln \vartheta[\delta](D_\beta) + \partial_I \partial_J \ln \vartheta(D_\delta) \right) \]

\[ X_3 = \frac{1}{2\pi} \int d^2 w \ln(\vartheta(w)B_2(w) + B_{3/2}(w)) \]

\[ X_4 = \frac{1}{8\pi^2} \int d^2 w \partial_p \partial_w \ln(\varrho(p_w,w)\chi_{a}^+ + \int d^2 w S_\delta(w,u)\chi_{a}^+ + \varpi_a(u)) \]

\[ X_5 = \frac{1}{16\pi^2} \int d^2 u \int d^2 v S_\delta(p_a,u)\chi_{a}^+ + \partial_p \partial_v S_\delta(p_a,v)\chi_{a}^+ + \varpi_a(u,v) \]

\[ X_6 = \frac{1}{16\pi^2} \int d^2 z \chi_{a}^*(z) \int d^2 w G_{3/2}(z,w)\chi_{a}^+ + \int d^2 v \chi_{a}^+ + \Lambda_{a}(w,v) \]

(6.28)

The various ingredients in the formula have been defined throughout the text.

It remains to re-express the above formula (6.28) in terms of the final result (1.11) and (1.12). This may be done by exhibiting a detailed correspondence between the \( X_i \), \( i = 1, \cdots, 6 \) of (1.12) and those of (6.28). The quantities \( X_1, X_4, X_5 \) and \( X_6 \) are identical in both cases already. Thus, it simply remains to regroup \( X_2 \) and \( X_3 \) and to combine them into the sum given in (1.11). Using (5.2), we represent \( X_2 \) of (6.28) in terms of an integral versus \( \hat{\omega} \), of a form similar to the integral in \( X_3 \). Next, using the expressions for \( B_2 \) and \( B_{3/2} \) in (6.12) and (6.20), and combining those with the calculation of the full stress tensor given in (4.19), we obtain a holomorphic two-form, given by \( T_{IJ}(w)\omega_I(w) \) integrated versus the Beltrami differential \( \hat{\omega} \). The final step consists in expressing the inner product \( \langle \hat{\omega}|\omega_I\omega_J \rangle \) in terms of \( \Omega_{IJ} - \hat{\Omega}_{IJ} \) with the help of (5.2) and using (B.18) to re-express the result solely in terms of \( \chi \),

\[ X_2 + X_3 = \frac{1}{2\pi} \int d^2 w \hat{\vartheta}(w)T_{IJ}(w)\omega_I(w) \]

\[ = \frac{i}{2\pi} T_{IJ} \left( \Omega_{IJ} - \hat{\Omega}_{IJ} \right) \]

\[ = \frac{1}{16\pi^2} \int d^2 u \int d^2 v T_{IJ}(u)\chi_{a}^+ + S_\delta(u,v)\chi_{a}^+ + \omega_J(v) \]

(6.29)

This result now precisely coincides with the final result given for \( X_2 + X_3 \) in (1.12), thereby completing the proof of this formula.
A Appendix: Bosonic Riemann Surface Formulas

In this section, we review basic formulas, holomorphic and meromorphic differentials and Green’s functions on an ordinary Riemann surface $\Sigma$ of genus $h$, as well as associated variational formulas. Standard references are [23], [24] and [8].

A.1 Basic Objects

The basic objects on a Riemann surface $\Sigma$, from which all others may be reconstructed, are the holomorphic Abelian differentials, the Jacobi $\vartheta$-function, and the prime form. We choose a canonical homology basis $A_I, B_I, I = 1, \cdots, h$, with canonical intersection matrix $\#(A_I, B_J) = \delta_{I,J}$. Modular transformations are defined to leave the intersection form invariant and form the group $Sp(2h, \mathbb{Z})$.

The holomorphic Abelian differentials $\omega_I$ are holomorphic 1-forms which may be normalized on $A_I$ cycles, and whose integrals on $B_I$ cycles produce the period matrix,

$$\oint_{A_I} \omega_J = \delta_{IJ} \oint_{B_I} \omega_J = \Omega_{IJ} \quad \text{(A.1)}$$

The Jacobian is then defined as $J(\Sigma) \equiv \mathbb{C}^h / \{ \mathbb{Z}^h + \Omega \mathbb{Z}^h \}$.

Given a base point $z_0$, the Abel map sends $d$ points $z_i$, with multiplicities $q_i \in \mathbb{Z}$, $i = 1, \cdots, d$ and divisor $D = q_1 z_1 + \cdots + q_d z_d$ of degree $q_1 + \cdots + q_d$ into $\mathbb{C}^h$ by

$$q_1 z_1 + \cdots + q_d z_d \equiv \sum_{i=1}^{d} q_i \int_{z_0}^{z_i} (\omega_1, \cdots, \omega_h) \quad \text{(A.2)}$$

The Abel map onto $\mathbb{C}^h$ is multiple valued, but it is single valued onto $J(\Sigma)$.

The Jacobi $\vartheta$-functions are defined on $\zeta = (\zeta_1, \cdots, \zeta_h)^t \in \mathbb{C}^h$ by

$$\vartheta[\delta](\zeta, \Omega) \equiv \sum_{n \in \mathbb{Z}^h} \exp\left(i\pi(n + \delta')^t \Omega (n + \delta') + 2\pi i(n + \delta')^t (\zeta + \delta'')\right) \quad \text{(A.3)}$$

Here, $\delta = (\delta'|\delta'')$ is a general characteristic, where $\delta', \delta'' \in \mathbb{C}^h$ are both written as a column vector. Henceforth, we shall assume that $\delta$ corresponds to a spin structure, and thus be valued in $\delta', \delta'' \in (\mathbb{Z}/2\mathbb{Z})^h$. The parity of the $\vartheta$-functions depends on $\delta$ and is defined by (for $\zeta$ and $\Omega$ such that $\vartheta[\delta](\zeta, \Omega) \neq 0$),

$$\vartheta[\delta](-\zeta, \Omega) = (-1)^{4\delta' \cdot \delta''} \vartheta[\delta](\zeta, \Omega) \quad \text{(A.4)}$$

According to whether $4\delta' \cdot \delta''$ is even or odd, $\delta$ is referred to as an even or odd spin structure. One often denotes $\vartheta(\zeta, \Omega) \equiv \vartheta[0](\zeta, \Omega)$. Upon shifting by full periods, $M, N \in \mathbb{Z}^h$,

$$\vartheta[\delta](\zeta + M + \Omega N, \Omega) = \exp\left(-i\pi N^t \Omega N - 2\pi i N^t (\zeta + \delta') + 2\pi i M^t \delta''\right) \vartheta[\delta](\zeta, \Omega) \quad \text{(A.5)}$$
Under a modular transformation \( U \in Sp(2h, \mathbb{Z}) \), the characteristic \( \delta = (\delta' \mid \delta'') \) transforms as (see for example [28, 25])

\[
\begin{pmatrix}
\tilde{\delta}' \\
\tilde{\delta}''
\end{pmatrix} =
\begin{pmatrix}
D & -C \\
-B & A
\end{pmatrix}
\begin{pmatrix}
\delta' \\
\delta''
\end{pmatrix}
+ \frac{1}{2} \text{diag} \begin{pmatrix}
CD^t \\
AB^t
\end{pmatrix}
= \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}
\begin{pmatrix}
\delta' \\
\delta''
\end{pmatrix}
+ \frac{1}{2} \text{diag} \begin{pmatrix}
CD^t \\
AB^t
\end{pmatrix}
U
\] (A.6)

The period matrix transforms as

\[
\tilde{\Omega} = (A\Omega + B)(C\Omega + D)^{-1}
\] (A.7)

while the \( \vartheta \)-function transforms as (see [25, 28]), with \( \epsilon^8 = 1 \),

\[
\vartheta[\tilde{\delta}](\{((C\Omega + D)^{-1})\zeta, \tilde{\Omega}\}) = \epsilon(\delta, U) \det(C\Omega + D)^{\frac{1}{8}} \vartheta[\delta](\zeta, \Omega)
\] (A.8)

The Riemann vector \( \Delta \in \mathbb{C}^h \), which depends on the base point \( z_0 \) of the Abel map, enters the Riemann vanishing Theorem, which states that \( \vartheta(\zeta, \Omega) = 0 \) if and only if there exist \( h - 1 \) points \( p_1, \cdots, p_{h-1} \) on \( \Sigma \), so that \( \zeta = \Delta - p_1 \cdots - p_{h-1} \). The explicit form of \( \Delta \) may be found in [3], formula (6.37) and will not be needed here.

The prime form is constructed as follows [27]. For any odd spin structure \( \nu \), all the \( 2h-2 \) zeros of the holomorphic 1-form \( \sum_I \partial_I \vartheta[\nu](0, \Omega)\omega_I(z) \) are double and the form admits a unique (up to an overall sign) square root \( h_\nu(z) \) which is a holomorphic 1/2 form. The prime form is a \(-\frac{1}{2}\) form in both variables \( z \) and \( w \), defined by

\[
E(z, w) \equiv \frac{\vartheta[\nu](z - w, \Omega)}{h_\nu(z)h_\nu(w)}
\] (A.9)

where the argument \( z - w \) of the \( \vartheta \)-functions stands for the Abel map of (A.2) with \( z_1 = z \), \( z_2 = w \) and \( q_1 = -q_2 = 1 \). The form \( E(z, w) \) defined this way is actually independent of \( \nu \). It is holomorphic in \( z \) and \( w \) and has a unique simple zero at \( z = w \). It is single valued when \( z \) is moved around \( A_I \) cycles, but has non-trivial monodromy when \( z \to z' \) is moved around \( B_I \) cycles,

\[
E(z', w) = -\exp\left(-i\pi I_{II} + 2\pi i \int_w^z \omega_I \right) E(z, w).
\] (A.10)

The combination \( \partial_z \partial_w \ln E(z, w) \) is a single valued meromorphic differential (Abelian of the second kind) with a single double pole at \( z = w \). Its integrals around homology cycles are given by

\[
\oint_{A_I} dz \partial_z \partial_w \ln E(z, w) = 0
\]

\[
\oint_{B_I} dz \partial_z \partial_w \ln E(z, w) = 2\pi i \omega_I(w)
\] (A.11)

and will be of use throughout.
A.2 Holomorphic differentials

The covariant derivatives on forms of rank $n$ will be denoted by $\nabla^{(n)}_{\bar{z}}$ and $\nabla^{(n)}{z}$. In complex coordinates adapted to the metric, we may simply use $\partial_z$ and $\partial_{\bar{z}}$ instead, whenever no confusion is possible. For $n \in \mathbb{Z}+1/2$, proper definition of these forms requires specification of a spin structure. The elliptic operators $\nabla^{(n)}_{\bar{z}}$ have kernels with the following dimensions $\Upsilon(n) \equiv \text{dim Ker} \nabla^{(n)}_{\bar{z}}$

$$\Upsilon(n) = \begin{cases} 0 & n < 0, \text{ and } n = 1/2 \text{ even spin structure} \\ 1 & n = 0, \text{ and } n = 1/2 \text{ odd spin structure} \\ (2n-1)(h-1) & n \geq 3/2 \end{cases} \quad (A.12)$$

while the cokernels have $\text{dim coKer} \nabla^{(n)}_{\bar{z}} = \Upsilon(1-n)$. (The dimensions listed for $n = 1/2$ are for generic moduli and are valid for exceptional moduli mod 2.) A set of basis holomorphic differentials are denoted by $\phi_a^{(n)}$, $a = 1, \cdots, \Upsilon(n)$, and are section of the line bundles $T^n$, ($n$-th power of the canonical bundle $T$), for which the number of zeros and poles are related by

$$(\# \text{ zeros} - \# \text{ poles}) \phi_a^{(n)}(z) = c_1(T^n) = 2n(h-1). \quad (A.13)$$

For $n = 0$, they are just constants, for $n = 1/2$ and $\nu$ odd they are denoted by $h_\nu(z)$, while for $n = 1$ they are the Abelian differentials usually denotes by $\omega_I$, $I = 1, \cdots, h$.

Given any set of $\Upsilon(n)$ points $z_1, \cdots, z_{\Upsilon(n)}$ on the surface, we may choose a basis $\phi_a^{(n)*}$ for the holomorphic $n$-differentials normalized at the points $z_b$ by

$$\phi_a^{(n)*}(z_b) = \delta_a^b. \quad (A.14)$$

The holomorphic differentials with this normalization may be exhibited explicitly in terms of the prime form $E(z, w)$, the $h/2$ differential $\sigma(z)$ and $\vartheta$-functions. For $n \geq 3/2$, we have

$$\phi_a^{(n)*}(z) = \frac{\vartheta[\delta](z - z_a + \sum z_b - (2n-1)\Delta) \prod_{b \neq a} E(z, z_b)}{\vartheta[\delta](\sum z_b - (2n-1)\Delta) \prod_{b \neq a} E(z_a, z_b)} \left(\frac{\sigma(z)}{\sigma(z_a)}\right)^{2n-1}. \quad (A.15)$$

To simplify notation, we shall not exhibit the spin structure dependence of these differentials. Here, $\sigma(z)$ is a tensor of rank $h/2$ without zeros or poles, and which may be defined up to a constant by the ratio

$$\frac{\sigma(z)}{\sigma(w)} = \frac{\vartheta(z - \sum p_i + \Delta) \prod_{i=1}^h E(w, p_i)}{\vartheta(w - \sum p_i + \Delta) \prod_{i=1}^h E(z, p_i)} \quad (A.16)$$

where $p_i$, $i = 1, \cdots, h$ are arbitrary points on the surface. Note that $\sigma(z)$ is single valued around $A_I$ cycles but multivalued around $B_I$ cycles in the following way

$$\sigma'(z) = \sigma(z) \exp\{-i\pi(h-1)\Omega_{II} + 2\pi i \Delta_{Iz}\} \quad (A.17)$$
Besides the $\Upsilon(n) - 1$ zeros $z_b$, $b \neq a$, the differential $\phi_a^{(n)*}(z)$ has $n$ additional zeros. The tensor $\phi_a^{(n)*}(z)$ is of rank $n$ in $z$, rank $-n$ in $z_a$ and rank $0$ in $z_b$ with $b \neq a$. For $n = 1$, we have

$$
\phi_a^{(1)*}(z) = \frac{\vartheta(z - z_a + \sum z_b - w_0 - \Delta)}{\vartheta(\sum z_b - w_0 - \Delta)} \prod_{b \neq a} \frac{E(z, z_b) E(z_a, w_0)}{E(z_a, z_b) E(z, w_0)} \frac{\sigma(z)}{\sigma(z_a)}.
$$

(A.18)

### A.3 Meromorphic differentials: Green’s functions

Meromorphic Green’s functions $G_n(z, w) = G_n(z, w; z_1, \cdots, z_{\Upsilon(n)})$ for the operators $\nabla^{(n)} 3 \geq n$ with $n \geq 3/2$ for general spin structure and $n = 1/2$ for even spin structure may be defined by the following relations

$$
\nabla^{(n)} G_n(z, w) = +2\pi \delta(z, w) \quad \text{(A.19)}
$$

$$
\nabla^{(1-n)} G_n(z, w) = -2\pi \delta(z, w) + 2\pi \sum_{a=1}^{\Upsilon(n)} \phi_a^{(n)*}(z) \delta(w, z_a).
$$

(A.20)

The properly normalized holomorphic $n$-differentials $\phi_a^{(n)*}(z)$ are defined in (A.14) and (A.13). Setting $z = z_a$, we have $\nabla^{(1-n)} G_n(z_a, w) = 0$, so that $G_n(z_a, w) = 0$. Explicit expressions for the Green’s function are

$$
G_n[\delta](z, w) = \frac{\vartheta[\delta](z - w + \sum z_b - (2n - 1)\Delta)}{\vartheta[\delta](\sum z_b - (2n - 1)\Delta)} \frac{\prod_a E(z, z_a)}{\prod_a E(w, z_a)} \left(\frac{\sigma(z)}{\sigma(w)}\right)^{2n-1}.
$$

(A.21)

In particular, for $n = 1/2$, this reduces to the standard form of the Szegő kernel, usually denoted by

$$
S_{1/2}(z, w) = \frac{\vartheta[\delta](z - w)}{\vartheta[\delta](0)} \frac{E(z, w)}{E(z_a, w_a)}.
$$

(A.22)

For $n = 1$, the Green’s function $G_1(z, w) = G_1(z, w; z_1, \cdots, z_h, w_0)$ is the Abelian differential of the third kind, satisfying

$$
\nabla^{(1)} G_1(z, w) = +2\pi \delta(z, w) - 2\pi \delta(z, w_0)
$$

(A.23)

$$
\nabla^{(0)} G_1(z, w) = -2\pi \delta(z, w) + 2\pi \sum_{a=1}^h \phi_a^{(1)*}(z) \delta(w, z_a),
$$

(A.24)

and explicitly given by the following expression

$$
G_1(z, w) = \frac{\vartheta(z - w + \sum z_b - \Delta)}{\vartheta(-w_0 + \sum z_b - \Delta)} \frac{\prod_a E(z, z_a) E(w, w_0)}{\prod_a E(w, z_a) E(z, w)} \frac{\sigma(z)}{\sigma(w)}.
$$

(A.25)
A.4 Variational Formulas

The variation $\delta_{ww}\phi$ of any object $\phi$ under a variation of the metric, parametrized by a Beltrami differential $\mu$ is defined as follows

$$\delta\phi \equiv \frac{1}{2\pi} \int_\Sigma d^2w \mu^w \delta_{ww}\phi$$

(A.26)

The variational formulas for the covariant derivatives on rank $n$ forms are

$$\delta \nabla^z \equiv + \frac{1}{2} \delta g^{zz} \nabla^z + \frac{n}{2} (\nabla^z \delta g^{zz})$$

$$\delta \nabla_z \equiv - \frac{1}{2} \delta g_{zz} \nabla^z + \frac{n}{2} (\nabla^z \delta g_{zz})$$

(A.27)

ignoring the Weyl variation part. From these formulas, one derives the following variational formulas for differentials and periods,

$$\delta_{ww} \omega_I(z) = \omega_I(w) \partial_z \partial_w \ln E(z, w)$$

$$\delta_{ww} \Omega_{IJ} = \frac{1}{2h-2} \left( (\partial_w \ln \psi(w, z))^2 - \partial^2_w \ln \psi(w, z) \right)$$

$$\delta_{ww} \ln \sigma(z) = - \frac{1}{2} \left( \partial_w \ln \frac{E(x, w)}{E(y, w)} \right)^2$$

(A.28)

Here, we have $\psi(w, z) = \sigma(w) E(w, z)^{h-1}$. Under an analytic coordinate change $u \to w(u)$ the variation $\delta_{uu} \ln \sigma(z)$ transforms with a Schwarzian derivative $\{u, w\}$,

$$\delta_{ww} \ln \sigma(z) = \left( \frac{du}{dw} \right)^2 \delta_{uu} \ln \sigma(z) + \frac{1}{2h-2} \{u, w\}.$$  

(A.29)

since $\sigma(z)$ is the carrier of the gravitational anomaly.

For the holomorphic differentials $\phi^{(n)*}_a$ introduced above with normalizations $\phi^{(n)*}_a(z_b) = \delta^b_a$ and the Green’s functions $G_n$ with normalization $G_n(z_a, w) = 0$, we have the following variational formulas

$$\delta_{ww} \phi^{(n)*}_a(z) = n \nabla^w G_n(z, w) \phi^{(n)*}_a(w) + (n-1) G_n(z, w) \nabla_w \phi^{(n)*}_a(w)$$

$$\delta_{ww} G_n(z, y) = n \nabla^w G_n(z, w) G_n(w, y) + (n-1) G_n(z, w) \nabla_w G_n(w, y)$$

(A.30)

Clearly, the variations preserve the normalization conditions.
B Appendix: $\mathcal{N} = 1$ Supergeometry Formulas

It is useful to define a $\mathcal{N} = 1$ supergeometry in 2 dimensions in the following way. A detailed account, including supercomplex structures and complete Wess-Zumino expressions may be found in [3]. We begin with a space of real dimension $2|2$, and local coordinates $v^M = (v, \bar{v}|\theta, \bar{\theta})$, where the index $M$ is a super-Einstein index. We also have a U(1) gauge group of (Euclidean) frame rotations, and we classify superfields according to their U(1) charge, which for our purposes is $n \in \mathbb{Z}/2$. The supergeometry data are a superframe $E_M^A$ and a U(1) connection $\Omega^A_M$. The inverse of $E_M^A$ will be denoted by $E^{AM}$. The frame index $A = (a|\alpha)$ runs over $2|2$ values, customarily denoted by $(z, \bar{z}|+, -)$.

B.1 Supergeometry and super derivatives

On a superfield $V$ of U(1) weight $n$, the superderivatives are defined by

$$D_A^{(n)}V \equiv E_A^M(\partial_M V + in\Omega_M V) \quad (B.1)$$

and the torsion $T_{AB}^C$ and curvature $R_{AB}$ tensor are defined by

$$[D_A, D_B]V = T_{AB}^C D_C V + inR_{AB} V \quad (B.2)$$

and $[,]$ is a commutator unless both $A$ and $B$ are spinor indices, in which case it is an anti-commutator. The torsion constraints are

$$T_{\alpha\beta}^\gamma = T_{ab}^c = 0, \quad T_{\alpha\beta}^c = 2\gamma_{\alpha\beta}^c \quad (B.3)$$

We assume $\{\gamma^a, \gamma^b\} = -\delta^{ab}$, so that $\gamma^z_+ = \gamma^\bar{z}_- = 1$ and $\gamma^a_+ = \gamma^a_- = 0$. The torsion constraints imply that the odd superderivatives have very simple anti-commutation relations on a superfield of U(1) weight $n$,

$$D_+^2 V = D_2 V \quad D_2^2 V = D_2 V \quad \{D_+, D_-\} V = inR_{+-} V \quad (B.4)$$

In Wess-Zumino gauge, the algebraic components of sDiff($\Sigma$), sWeyl($\Sigma$) and U(1) are eliminated and the remaining independent fields are the ordinary frame $e_m^a$, the gravitino field $\chi^a_m$ and an auxiliary scalar $A$,

$$E_M^A = e_m^a + \theta \gamma^a_m \chi_m - \frac{i}{2} \theta \bar{\theta} A e_m^a \quad \text{sdet} E_M^A = e(1 + \frac{1}{2} \theta \gamma^m \chi_m - \frac{i}{2} A + \frac{1}{8} \theta \bar{\theta} \epsilon^{mn} \chi_m \gamma_5 \chi_n) \quad (B.5)$$

The superderivatives acting on a superfield $V$ of U(1) weight $n$,

$$V = V_0 + \theta V_+ + \bar{\theta} V_- + i\theta \bar{\theta} V_1 \quad (B.6)$$
are given by
\[
\mathcal{D}^{(n)}_{+}V = V_+ + \theta \left( \partial_\z V_0 + \frac{1}{2} \chi_\z V_- \right) + \bar{\theta} \left( i V_1 + \frac{i}{2} n A V_0 \right) + \theta \left( \frac{i}{4} (1 - 2n) A V_+ - \frac{1}{4} \chi_\z^+ \chi_\z V_+ + \frac{1}{2} \chi_\z^- \partial_\z V_0 + \partial_\z V_- - n \partial_\z \chi_\z^- V_0 \right)
\]
\[
\mathcal{D}^{(n)}_{-}V = V_- + \bar{\theta} \left( \partial_\z V_0 + \frac{1}{2} \chi_\z^+ V_+ \right) + \theta \left( -i V_1 + \frac{i}{2} n A V_0 \right) + \bar{\theta} \left( \frac{i}{4} (1 + 2n) A V_- - \frac{1}{4} \chi_\z^+ \chi_\z^- V_- - \frac{1}{2} \chi_\z^+ \partial_\z V_0 - \partial_\z V_+ - n \partial_\z \chi_\z^+ V_0 \right)
\]

\section*{B.2 Holomorphic superdifferentials}

We seek to solve the equation \( \mathcal{D}^{(n)} \Phi = 0 \) with \( n \geq 3/2 \) both for the cases of even and odd valued \( \phi \), in Wess-Zumino gauge. The starting point is the generic expression for \( \Phi \) as a superfunction
\[
\Phi(z, \theta, \bar{\theta}) = \Phi_0(z) + \theta \Phi_+(z) + \bar{\theta} \Phi_-(z) + i \theta \bar{\theta} \Phi_1(z)
\]
and the expression for the operator \( \mathcal{D}^{(n)}_{-} \) of (B.7). As a result, \( \Phi_- = 0 \) and \( \Phi_1 = \frac{n}{2} A \Phi_0 \), so that the remaining non-trivial equations are
\[
0 = \nabla^{(n)}_z \Phi_0 + \frac{1}{2} \chi^+_z \Phi_+
\]
\[
0 = \nabla^{(n+1/2)}_z \Phi_+ + \frac{1}{2} \chi^+_z \nabla_z \Phi_0 + n (\nabla_z \chi^+_z) \Phi_0.
\]

The general solution is recovered from iterating the equivalent set of integral equations
\[
\Phi_0(z) = \phi_0(z) - \frac{1}{2\pi} \int d^2 w \ G_n(z, w) \left( \frac{1}{2} \chi^+_w \Phi_+(w) \right)
\]
\[
\Phi_+(z) = \phi_+(z) - \frac{1}{2\pi} \int d^2 w \ G_{n+1/2}(z, w) \left( \frac{1}{2} \chi^+_w \nabla_w \Phi_0 + n (\nabla_w \chi^+_w) \Phi_0 \right)(w)
\]

where \( \phi_0(z) \) and \( \phi_+(z) \) satisfy the bosonic equations \( \nabla^*_z \phi_0 = 0 \) and \( \nabla^*_z (n+1/2) \phi_+ = 0 \), i.e. they are holomorphic \( n \) and \( n + 1/2 \) differentials respectively.

The Green’s functions \( G_n \) and \( G_{n+1/2} \) are not unique in general, as explained in the previous subsection. However, the arbitrariness may be absorbed by a shift in \( \phi_0 \) and \( \phi_+ \). Since these spaces are generated by \( \psi_\alpha \equiv \phi_\alpha^{(n)} \) with \( \alpha = 1, \cdots, \Upsilon(n) \) and \( \phi_\alpha \equiv \phi_\alpha^{(n+1/2)} \) with \( \alpha = 1, \cdots, \Upsilon(n + 1/2) \) respectively, we may define the following basis of all differentials
\[
\text{EVEN} \quad \Phi_{\alpha_0}(z) = \psi_\alpha(z) - \frac{1}{2\pi} \int d^2 w \ G_n(z, w) \left( \frac{1}{2} \chi^+_w \Phi_{\alpha_0}(w) \right)
\]
\[
\Phi_{\alpha_+}(z) = -\frac{1}{2\pi} \int d^2 w \ G_{n+1/2}(z, w) \left( \frac{1}{2} \chi^+_w \nabla_w \Phi_{\alpha_0} + n (\nabla_w \chi^+_w) \Phi_{\alpha_0} \right)(w)
\]
there are normalization on homology cycles, since they may be integrated. For even spin structure, the case of Abelian super-differentials is special, mainly because they have a natural

\[ B.3 \text{ Abelian Super-Differentials – Even Spin Structures} \]

Abelian differentials which we now spell out in more detail.

\[ \Phi_{a0}(z) = -\frac{1}{2\pi} \int d^2w \ G_n(z, w) \frac{1}{2} \chi_w^+ \Phi_{a+}(w) \]  

(\text{B.12})

\[ \Phi_{a+}(z) = \phi_a(z) - \frac{1}{2\pi} \int d^2w \ G_{n+1/2}(z, w) \left( \frac{1}{2} \chi_w^+ \nabla_w \Phi_{a0} + n(\nabla_w \chi_w^+) \Phi_{a0} \right)(w) \]

It will be convenient to normalize the differentials and to fix the Green’s functions in a consistent way as we did in the preceding subsection. To this end, we introduce arbitrary points \( q_\alpha, \alpha = 1, \cdots, \Upsilon(n) \) and arbitrary points \( p_a, a = 1, \cdots, \Upsilon(n+1/2) \), and denote the normalized holomorphic differentials with * superscripts:

\[ \psi^*_a(q_\beta) = \delta_{\alpha\beta} \quad \quad G_n(q_\alpha, w) = 0 \]

\[ \phi_a^*(p_b) = \delta_{ab} \quad \quad G_{n+1/2}(p_a, w) = 0. \]  

(\text{B.13})

\( \Phi^*_{a0}(q_\beta) = \delta_{\alpha\beta} \quad \quad \Phi^*_{a+}(p_b) = 0 \)

\( \Phi^*_a(p_b) = \delta_{ab} \quad \quad \Phi^*_{a0}(q_\beta) = 0. \)  

(\text{B.14})

\( \Phi^*_a(p_b) = \delta_{ab} \quad \quad \Phi^*_{a0}(q_\beta) = 0. \)

(\text{B.14})

As a result, the differential \( \psi^*_a(z) \) is a differential of degree \( n \) in \( z \) and of degree \( -n \) in \( q_\alpha \), while it is a scalar in \( q_\beta \) when \( \beta \neq \alpha \). Similarly for \( \phi_a(z) \) which is a differential of degree \( n + 1/2 \) in \( z \), \( -n - 1/2 \) in \( p_a \) and 0 in \( p_b \) when \( b \neq a \).

The cases of interest to string theory have \( n = 3/2 \), so that \( \alpha = 1, \cdots, 2h - 2 \) for even \( 3/2 \) superdifferentials and \( a = 1, \cdots, 3h - 3 \) for odd \( 3/2 \) superdifferentials, as well as Abelian differentials which we now spell out in more detail.

\[ \text{B.3 Abelian Super-Differentials – Even Spin Structures} \]

The case of Abelian super-differentials is special, mainly because they have a natural normalization on homology cycles, since they may be integrated. For even spin structure, there are \( h \) odd superholomorphic \( 1/2 \) differentials, denoted by \( \hat{\omega}_I, I = 1, \cdots, h \), and there are no even ones. The odd differentials are normalized by

\[ \oint_{A_I} \hat{\omega}_I = \delta_{IJ} \quad \quad \oint_{B_I} \hat{\omega}_I = \Omega_{IJ}. \]  

(\text{B.15})

Since the \( \chi \)-dependent corrections to the differential must integrate to 0 along \( A_I \)-cycles, the relevant Green’s function on 1-forms to be used is the one built out of the prime form. Thus, we have (including the auxiliary field \( A \) of the supergeometry (\text{B.5}),

\[ \hat{\omega}_I(z, \theta, \bar{\theta}) = \hat{\omega}_{I0}(z) + \theta \hat{\omega}_{I+}(z) + \frac{i}{4} \theta \bar{\theta} A \hat{\omega}_{I0}(z) \]  

(\text{B.16})

and the solution is given by the implicit equations,

\[ \hat{\omega}_{I0}(z) = -\frac{1}{4\pi} \int d^2w \ S_{\delta}(z, w) \chi_w^+ \hat{\omega}_{I+}(w) \]  

(\text{B.17})

\[ \hat{\omega}_{I+}(z) = \omega_I(z) - \frac{1}{4\pi} \int d^2w \ \partial_z \partial_w \ln E(z, w) \chi_w^+ \hat{\omega}_{I0}(w). \]
which may be solved by iterating precisely $h - 1$ times. Here, $S_\delta(z, w)$ is the Szego kernel for (even) spin structure $\delta$. The superperiod matrix is given by

$$
\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int d^2 u \int d^2 v \omega_I(u) \chi_u^+ S_\delta(u, v) \chi_v^+ \hat{\omega}_{J+}(v). \tag{B.18}
$$

which may also be solved by iterating $h - 1$ times. For genus 2, the full solution is simply obtained by setting $\hat{\omega}_{J+} = \omega_J$ on the rhs.

### B.4 Super-Beltrami differentials

We spell out the signs that arise when dealing with the superdifferential. First, using \textbf{(2.2)}, we have the following result for the super-Beltrami differentials in Wess-Zumino gauge \cite{3},

$$
H = H_\chi - \bar{\theta} \left( e^{\bar{\chi}_m \delta e_m z} - \theta \delta \chi^+ \right) \tag{B.19}
$$

When evaluating the derivatives $H_A$ in Wess-Zumino gauge, we use the definitions

$$
H_A = \bar{\theta} (\mu_A - \theta \nu_A). \tag{B.20}
$$

In terms of the differential $\hat{\mu}$, we have $\mu_a = \partial \hat{\mu}/\partial \zeta^a$.

The inner product with the superghost field $B = \beta + \theta b$ is given by

$$
\langle H_a | B \rangle = +\langle \mu_a | b \rangle - \langle \nu_a | \beta \rangle,
\quad
\langle H_\alpha | B \rangle = -\langle \mu_\alpha | b \rangle - \langle \nu_\alpha | \beta \rangle \tag{B.21}
$$

### B.5 Variational Formulas

The variation $\delta_{w+}\Phi$ of any object $\Phi$ under a variation of the supergeometry is defined by

$$
\delta \Phi = \frac{1}{2\pi} \int_{\Sigma} d^{2|2} w E \ H^w \ \delta_{w+} \Phi. \tag{B.22}
$$

The variation under $H$ of the covariant derivatives are given by $\delta D_+^{(n)} = 0$ and

$$
\delta D_-^{(n)} = -H D_-^{(n)} + \frac{1}{2} (D_+ H) D_+^{(n)} - n D z H. \tag{B.23}
$$

The variation $\delta \hat{\omega}_I$ of any holomorphic Abelian differential $\hat{\omega}_I$ satisfies

$$
D_- \delta \hat{\omega}_I + D_+ (H D_+ \hat{\omega}_I - \frac{1}{2} (D_+ H) \hat{\omega}_I) = 0. \tag{B.24}
$$
Using the fact that the super-prime form \([20, 23]\) satisfies
\[
\mathcal{D}_-^\mathbf{z} \left( \mathcal{D}_+^\mathbf{z} \mathcal{D}_+^\mathbf{w} \ln \mathcal{E}_\delta(\mathbf{z}, \mathbf{w}) \right) = -2\pi \mathcal{D}_+^\mathbf{z} \delta^{(2|2)}(\mathbf{z}, \mathbf{w}) \tag{B.25}
\]
as well as the canonical normalizations of \(\hat{\omega}_I\) and the following integrals
\[
\oint_{A_I} dz \mathcal{D}_+^\mathbf{z} \mathcal{D}_+^\mathbf{w} \ln \mathcal{E}(\mathbf{z}, \mathbf{w}) = 0
\]
\[
\oint_{B_I} dz \mathcal{D}_+^\mathbf{z} \mathcal{D}_+^\mathbf{w} \ln \mathcal{E}(\mathbf{z}, \mathbf{w}) = 2\pi i \hat{\omega}_I(\mathbf{w}) \tag{B.26}
\]
we obtain a unique expression for the variation of the holomorphic differentials
\[
\delta \hat{\omega}_I(\mathbf{z}) = \frac{1}{2\pi} \int d^{2|2} w \mathcal{D}_+^\mathbf{z} \mathcal{D}_+^\mathbf{w} \ln \mathcal{E}(\mathbf{z}, \mathbf{w}) \left( H \mathcal{D}_+ \hat{\omega}_I - \frac{1}{2}(\mathcal{D}_+ H) \hat{\omega}_I \right)(\mathbf{w}), \tag{B.27}
\]
which may alternatively be expressed as
\[
\delta w + \hat{\omega}_I(\mathbf{z}) = \frac{1}{2} \mathcal{D}_+^\mathbf{z} \mathcal{D}_+^\mathbf{w} \ln \mathcal{E}(\mathbf{z}, \mathbf{w}) \mathcal{D}_+ \hat{\omega}_I(\mathbf{w}) + \frac{1}{2} \mathcal{D}_+^\mathbf{z} \mathcal{D}_w \ln \mathcal{E}(\mathbf{z}, \mathbf{w}) \omega_I(\mathbf{w}). \tag{B.28}
\]
The variational formula for the super-period matrix automatically follows
\[
\delta \hat{\Omega}_{IJ} = \langle H | \Phi_{IJ} \rangle = \int d^{2|2} w H \Phi_{IJ}
\]
\[
\delta w + \hat{\Omega}_{IJ} = 2\pi \Phi_{IJ} = -\pi i \left( \hat{\omega}_J \mathcal{D}_+ \hat{\omega}_I + \hat{\omega}_I \mathcal{D}_+ \hat{\omega}_J \right)(\mathbf{w}). \tag{B.29}
\]

C Appendix: The superdeterminant

We have the following two equivalent forms of the superdeterminant
\[
\text{sdet} \begin{pmatrix} L & M \\ N & P \end{pmatrix} = \det(L - MP^{-1}N)(\det P)^{-1} = \det L \det(P - NL^{-1}M)^{-1} \tag{C.1}
\]
We prove both formulas by showing that the variations of both sides are equal, using
\[
\delta \ln \text{sdet} \begin{pmatrix} L & M \\ N & P \end{pmatrix} = \text{str} \begin{pmatrix} L & M \\ N & P \end{pmatrix}^{-1} \left( \frac{\delta L}{\delta N} \frac{\delta M}{\delta P} \right). \tag{C.2}
\]
The existence of these two seemingly different formulas is due to the existence of two seemingly different inverses. We use the abbreviations \(\tilde{L} = L - MP^{-1}N\) and \(\tilde{P} = P - NL^{-1}M\). The first form of the inverse is
\[
\begin{pmatrix} L & M \\ N & P \end{pmatrix}^{-1} = \begin{pmatrix} \tilde{L}^{-1} & -\tilde{L}^{-1}MP^{-1} \\ -P^{-1}NL^{-1} & \tilde{P}^{-1} + P^{-1}NL^{-1}MP^{-1} \end{pmatrix}. \tag{C.3}
\]
and the resulting variational formula is

\[ \delta \ln \text{sdet} \left( \begin{array}{cc} L & M \\ N & P \end{array} \right) = \text{tr} \tilde{L}^{-1} \delta L - \text{tr} \tilde{L}^{-1} MP^{-1} \delta N \]

\[ + \text{tr} P^{-1} N \tilde{L}^{-1} \delta M - \text{tr}(P^{-1} + P^{-1} N \tilde{L}^{-1} MP^{-1}) \delta P \]  

(C.4)

The second form of the inverse is

\[ \left( \begin{array}{cc} L & M \\ N & P \end{array} \right)^{-1} = \left( \begin{array}{cc} L^{-1} + L^{-1} M \tilde{P}^{-1} NL^{-1} & -L^{-1} M \tilde{P}^{-1} \\ -\tilde{P}^{-1} NL^{-1} & \tilde{P}^{-1} \end{array} \right) \]  

(C.5)

and the resulting variational formula is

\[ \delta \ln \text{sdet} \left( \begin{array}{cc} L & M \\ N & P \end{array} \right) = \text{tr}(L^{-1} + L^{-1} M \tilde{P}^{-1} NL^{-1}) \delta L - \text{tr} L^{-1} M \tilde{P}^{-1} \delta N \]

\[ + \text{tr} \tilde{P}^{-1} NL^{-1} \delta M - \text{tr} \tilde{P}^{-1} \delta P \]  

(C.6)

which proves both formulas. Thus, addition of multiples of columns or rows is immaterial

\[ \text{sdet} \left( \begin{array}{cc} L & M + LD \\ N & P + ND \end{array} \right) = \text{sdet} \left( \begin{array}{cc} L & M \\ N & P \end{array} \right) \]  

(C.7)
D Appendix: Slices, Forms and Vector Fields

If \( x^1, \ldots, x^n \) are local coordinates on an \( n \)-dimensional manifold \( M \), then the corresponding tangent vectors \( H_j, 1 \leq j \leq n \), are vector fields, normalized on 1-forms \( dx^j \),

\[
H_j = \frac{\partial}{\partial x^j}, \quad \langle H_k | dx^j \rangle = \delta^j_k.
\]  

(D.1)

Consider now the case of a slice for supermoduli at genus 2, parametrized by the coordinates \( (\hat{\Omega}_{IJ}, \zeta^\alpha) \), and let the corresponding tangent vectors be \( H_{IJ}, H^\alpha \) and the corresponding covectors be \( d\hat{\Omega}_{IJ}, d\zeta^\alpha, I \leq J = 1, 2, \alpha = 1, 2 \). The preceding duality relations become

\[
\langle H_{KL} | d\hat{\Omega}_{IJ} \rangle = \delta_{IK} \delta_{JL} \quad \langle H_{KL} | d\zeta^\alpha \rangle = 0 \quad \langle H^\alpha | d\zeta^\beta \rangle = \delta^\alpha_\beta.
\]

(D.2)

These duality relations have the following practical use. If the \( H_A \) are known, then they determine completely the forms \( dm^A \) within the class of superholomorphic 3/2-differentials. If the \( dm^A \) are known, then they determine completely the equivalence classes \([H_A]\) of the superBeltrami differentials \( H_A \). Note however that the \( H_A \) themselves are not determined, a distinction of importance in the sequel.

For the covectors \( d\hat{\Omega}_{IJ} \), we have a concrete description. Recall the deformation formula (B.29) for the superperiod matrix resulting from a variation \( H^{-z} \) of a supergeometry

\[
\delta \hat{\Omega}_{IJ} = -\frac{i}{2} \int d^{2|2}z \left( \hat{\omega}_I \partial_+ \hat{\omega}_J + \hat{\omega}_J \partial_+ \hat{\omega}_I \right)
\]

(D.3)

The left hand side should be interpreted as a pairing \( \langle H | d\hat{\Omega}_{IJ} \rangle \) between the covector \( d\hat{\Omega}_{IJ} \) and the vector \( H \) on supermoduli space. The right hand side is just the usual pairing between super Beltrami differentials and super quadratic differentials. Thus we may write

\[
d\hat{\Omega}_{IJ} = \Phi_{IJ} = -\frac{i}{2} (\hat{\omega}_I \partial_+ \hat{\omega}_J + \hat{\omega}_J \partial_+ \hat{\omega}_I)
\]

(D.4)

giving a realization of the covector \( d\hat{\Omega}_{IJ} \) in terms of superholomorphic 3/2 differentials.

D.1 Matrix Elements of \( H_A \)

Let \( H_A = \bar{\theta}(\mu_A - \theta \chi_A) \) be the expression of the super Beltrami differentials \( H_A \) in terms of a Beltrami differential \( \mu_A \) and a gravitino variation \( \chi_A \). As preparation for the component formalism, we seek now the matrix elements of \( \mu_A \) and \( \chi_A \) themselves.
D.1.1 The case of \( H_\alpha \)

The case of \( H_\alpha \) is easier, since \( \chi_\alpha \) is the gravitino slice, given as an input. The equivalence class \([\mu_\alpha]\) can then be determined from the condition \( \langle H_\alpha | \Phi_{IJ} \rangle = 0 \), which may be recast in components of \( \tilde{\omega}_I = \tilde{\omega}_{I0} + \theta \tilde{\omega}_{I+} \) as

\[
\langle \mu_\alpha | - \tilde{\omega}_{I0} \partial_z \tilde{\omega}_{I0} - \tilde{\omega}_{I0} \partial_w \tilde{\omega}_{I0} + 2 \tilde{\omega}_{I+} \tilde{\omega}_{I+} \rangle + \langle \chi_\alpha | \tilde{\omega}_{I0} \tilde{\omega}_{I+} + \tilde{\omega}_{I0} \tilde{\omega}_{I+} \rangle = 0 \quad (D.5)
\]

Since \( \tilde{\omega}_{I+} = \omega_I + \mathcal{O}(\zeta^1 \zeta^2) \), and \( \mu_\alpha \) and \( \tilde{\omega}_{I0} \) are both of order \( \zeta \), we have, in terms of the holomorphic forms \( \omega_I \) defined by \( \Omega_{IJ} \),

\[
\langle \mu_\alpha | \omega_I \omega_J \rangle = \frac{1}{8\pi} \int d^2 z d^2 w \chi_{\alpha \bar{z}}^+ S_\delta(z, w) \chi_{\bar{z}}^+ \left( \omega_J(z) \omega_I(w) + \omega_I(z) \omega_J(w) \right) \quad (D.6)
\]

which uniquely characterizes the class \([\mu_\alpha]\) of \( \mu_\alpha \).

We observe that \([\mu_\alpha]\) can also be obtained by the following different argument. For fixed \( \tilde{\Omega}_{IJ} \) and varying \( \zeta_\alpha \), \( \mu_\alpha \) can be recognized as the Beltrami differential which shifts the period matrix \( \Omega(\tilde{\Omega}, \zeta) \) to \( \Omega(\tilde{\Omega}, \zeta + \delta \zeta) \). From the formula (B.18) giving \( \Omega(\tilde{\Omega}, \zeta) \), we find readily

\[
\delta \Omega_{IJ} = i \frac{1}{8\pi} \int d^2 v d^2 w \chi_{\alpha \bar{v}}^+ S_\delta(v, w) \chi_{\bar{v}}^+ \left( \omega_J(v) \omega_I(w) + \omega_I(v) \omega_J(w) \right) \quad (D.7)
\]

Comparing with the standard variational formula \( \delta \Omega_{IJ} = i \int d^2 z \tilde{\mu}_{\bar{z}} \omega_I \omega_J \), we obtain again the previous relation (D.6).

D.1.2 The case of \( H_{IJ} \)

In components, the equations for \( H_{IJ} \) can be rewritten as

\[
\langle \mu_{KL} | \Phi_{IJ+} \rangle - \langle \chi_{KL} | \Phi_{IJ0} \rangle = \delta_{KI} \delta_{JL}
\]

\[
\langle \mu_{KL} | \Phi_{\beta+} \rangle - \langle \chi_{KL} | \Phi_{\beta0} \rangle = 0 \quad (D.8)
\]

Although these equations can be solved, they lead to complicated expressions involving Green’s functions, and it is not easy to see how they can be put to practical use.
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