Thermodynamics of spin-orbit-coupled Bose-Einstein condensates

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In this paper we develop a quantum field approach to reveal the thermodynamic properties of the trapped BEC with the equal Rashba and Dresselhaus spin-orbit couplings. In the experimentally-feasible regime, the phase transition from the separate phase to the single minimum phase can be well driven by the tunable temperature. Moreover, the critical temperature, which is independent of the trapped potential, can be derived exactly. At the critical point, the specific heat has a large jump and can be thus regarded as a promising candidate to detect this temperature-driven phase transition. In addition, we obtain the analytical expressions for the specific heat and the entropy in the different phases. In the single minimum phase, the specific heat as well as the entropy are governed only by the Rubi frequency. However, in the separate phase with lower temperature, we find that they are determined only by the strength of spin-orbit coupling. Finally, the effect of the effective atom interaction is also addressed. In the separate phase, this effective atom interaction affects dramatically on the critical temperature and the corresponding thermodynamic properties.

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I. INTRODUCTION

The spin orbit coupling (SOC), which describes the interaction between the spin and orbit degrees of freedom of a particle, has not only generated many interesting quantum phenomena in modern physics ranging from the nuclear physics to condensed-matter physics, and but also become an important resource for realizing fault-tolerant topological quantum computing [1]. By controlling the external lasers, the different kinds of SOCs have been proposed to be simulated in the trapped Bose-Einstein condensates (BECs) with the neutral atoms [2]. Especially, in recent experiment at NIST, the equal Rashba and Dresselhaus SOCs has been realized successfully in the ultracold 85Rb atoms by a couple of Raman lasers [3]. Attributed to this pioneer experiment, the investigation of SOC-driven BECs has attracted much attention. Moreover, rich many-body phenomena with no analogy in condensed-matter physics (in BECs, all ultracold atoms can occupy the same quantum state) have been predicted by considering the ground-state properties [4–23]. For example, in the presence of the equal Rashba and Dresselhaus SOCs, the BEC is made up of two non-orthogonal dressed atom spin states carrying different momenta. Furthermore, the interaction between these spin states are modified, driving a quantum phase transition from a spin-mixed state to a phase-separated state [24]. In fact, even if the effective atom interaction governed by both the inter- and intra- spin interactions is not taken into account, a quantum phase transition from a separate phase (SP) to a single minimum phase (SMP) can also occur [25, 26]. In very recent experiment, this new quantum phase transition has been observed by measuring the amplitude ratio of spin and momentum oscillation [27].

It has been known that quantum phase transitions governed by the ground-state energies occur at absolute zero temperature [28]. However, it is unattainable experimentally due to the third law of thermodynamics, i.e., any system must work at a finite temperature. Thus, it is crucially important to investigate the thermodynamic properties to fully understand the fundamental physics for a given system. For instance, in the framework of finite-temperature theory, the system’s real evaluation can be described more accurately and some important physical quantities such as the specific heat, the entropy and the free energy, which have no zero temperature correspondence, can be explored. More importantly, some exotic phenomena driven only by thermal fluctuations can be revealed [29].

Motivated by the experimental developments and the third law of thermodynamics, we, for the first time, develop a quantum field approach to reveal the thermodynamic properties of the trapped BEC with the equal Rashba and Dresselhaus SOCs. Our main results are given as follows: (I) In the experimentally-feasible regime, the phase transition from the SP to the SMP can be driven by the tunable temperature. Moreover, the corresponding critical temperature is derived exactly and is independent of the trapped potential. (II) We find that the specific heat has a large jump at the critical temperature. This step behavior is quite different from that of the atom population, which varies smoothly when crossing the critical point. It implies that the temperature-driven phase transition can be well detected by measuring the specific heat. (III) In the different phases, the analytical expressions for the specific heat and the entropy are also given. In the SMP, the specific heat as
well as the entropy are governed only by the Rabi frequency. However, in the SP, the strong SOC modifies the energy structure and thus the thermodynamic statistics. At lower temperature, we find that the specific heat and the entropy in such phase are determined only by the SOC strength. (IV) Finally, the effect of the effective atom interaction is also addressed. In the SMP, no collective excitations can be found in SOC-driven BEC and thus the effective atom interaction does not affect the thermodynamic properties. However, in the SP with SOC-induced macroscopic excitations, this effective atom interaction affects dramatically on the critical temperature as well as the other thermodynamic quantities. For example, for the repulsive atom interaction, the critical temperature decreases, and vice versa.

II. MODEL AND HAMILTONIAN

Figure 1 shows the experimental scheme about how to create SOC in the trapped BEC with the ultracold $^{87}$Rb atoms at NIST [3]. In their experiment, the BEC is trapped in the $xy$ plane through a strong confinement with frequency $\omega_z$ along the $z$ direction. In the large detuning $\Delta$, the momentum-sensitive coupling between two hyperfine ground states $|F = 1, m_F = -1\rangle (|↑\rangle)$ and $|F = 1, m_F = 0\rangle (|↓\rangle)$ is constructed by a pair of Raman lasers with Rabi frequencies $\Omega_1$ and $\Omega_2$ incident at a $\pi/4$ angle from the $x$ axis, as illustrated in Fig. 1(a). In the dressed-state basis $|↑\rangle = \exp (i k_1 \cdot r) |↑\rangle$ and $|↓\rangle = \exp (i k_2 \cdot r) |↓\rangle$, where $k_1$ and $k_2$ are the wavevectors of the Raman lasers, an effective SOC, which is identical to the one-dimensional equal Rashba and Dresselhaus SOCs in condensed-matter physics, can be achieved. Moreover, the corresponding Hamiltonian with the atom-atom collision interaction can be written from the coupled Gross-Pitaevskii equations as [22]

$$H_0 = \hbar \omega_N a^\dagger a + \hbar \Omega S_x - \gamma_0 \sqrt{m a \omega_z / (\hbar - a)} S_z + \frac{\hbar q}{N} S_z^2. \tag{1}$$

Here, $a^\dagger a$ is a harmonic trap mode with $a = \sqrt{m a \omega_z / (\hbar - i \omega_z)}$ and $m$ being the atom mass. $S_z = (\Phi_{↑↑} - \Phi_{↓↓}) / 2$ reflects the experimentally-measurable population between the different spin components. $\omega_z$ is the trapped frequency in the $x$ direction. $\Omega = \Omega_1 \Omega_2 / \Delta$ is the effective Rabi frequency. $\gamma_0 = \sqrt{2 \hbar k_L / m}$ with $\hbar k_L = \sqrt{2 \pi \hbar / \lambda}$ being the SOC strength, where $\lambda$ is the wavelength of the Raman laser. The effective atom interaction $q$ is proportional to $N (g_{↑↑} + g_{↓↓} - 2 g_{↑↓})$, where $g_{↑↑} = g_{↓↓} = 4 \pi \hbar^2 N (c_0 + c_2) / (m a z)$ and $g_{↑↓} = 4 \pi \hbar^2 N c_0 / m a z$ are the inter- and intra-spin interaction constants with $c_0$ and $c_2$ being the $s$-wave scattering lengths and $a_z = \sqrt{2 \pi \hbar / m a z}$. $N$ is the total atom number.

If defining the number-dependent trapped frequency $\omega = N \omega_L$ and the effective SOC strength $\gamma = \sqrt{m} \gamma_0$, Hamiltonian (1) can be rewritten in the rotating frame as ($\hbar = 1$ henceforth)

$$H = \omega a^\dagger a + \Omega S_z + \frac{\sqrt{\omega}}{\sqrt{N}} (a^\dagger + a) S_x + \frac{q}{N} S_z^2. \tag{2}$$

In the following discussion, we focus mainly on Hamiltonian (2), in which $\langle S_z \rangle$ stands for the atom population. Before proceeding, we estimate the relative parameters under current experimental conditions [3, 27, 30]. In the experiment of NIST, the tunable trapped frequency $\omega_L$ is of the order of 10 Hz, and correspondingly, the number-dependent trapped frequency $\omega$ is of the order of MHz for $N = 1.8 \times 10^5$. Parameter $\gamma^2$ is of the order of kHz for $\lambda = 804.1$ nm. The effective Rabi frequency $\Omega$ can range from zero to the order of MHz. In addition, since $c_0 = 100.86 a B$ and $c_2 = -0.46 a B$ with $a B$ being the Bohr radius, we have $g_{↑↑} \approx g_{↓↓} \approx g_{↑↓}$ and thus $q \approx 0$. It means that the effective atom interaction need not be taken into account in the NIST’s experiment. It should be pointed out that this effective atom interaction can be well controlled through Feshbach resonance [31]. Moreover, its magnitude can reach the order of MHz near the Feshbach resonant point. Finally, we will take $E_L = \hbar^2 k_L^2 / 2m$, which is of the order of kHz, as the natural unit of the energy for simplicity.

III. THERMODYNAMIC EQUILIBRIUM EQUATION

A key step to extract the thermodynamic properties of the SOC-driven BECs is to obtain the partition function of Hamiltonian (2) [22]. Here we develop a quantum field approach, i.e., an imaginary-time ($\tau = it$) functional path-integral technique, to arrive at the target.
We first rewrite the collective spin operators in the representation of the Grassmann Fermi fields, namely, \( S_z = \sum_{i=1}^N (\mu^+_i \mu_i - \nu^+_i \nu_i) \), \( S_+ = \sum_{i=1}^N \mu^+_i \nu_i \), and \( S_- = S_+^\dagger \), where the Fermi operators \( \mu^+_i (\mu_i) \) and \( \nu^+_i (\nu_i) \) satisfy the anticommutator relations \( \{ \mu^+_i, \mu_j \} = \{ \nu^+_i, \nu_j \} = \delta_{ij} \). Furthermore, we transform the harmonic trap mode and the corresponding integral is Gaussian. It means that in this case we can integrate over the Grassmann Fermi fields and then obtain the partition function of the bosonic mode. However, for nonzero \( q (q \neq 0) \), the integral in the Euclidean action \( A \) is not Gaussian and thus the corresponding integral is hard to be solved directly. Here we introduce an auxiliary field \( x \) to circumvent this difficult. Based on this auxiliary field \( x \), we have

\[ Z = \int [d\eta(\tau)] \exp[-A(\tau)]. \] (3)

In analogy of the mean field approximation, the value of auxiliary field \( x \) determines \( \langle S_z \rangle \), as will be shown. Substituting the formula about the auxiliary field \( x \) into the Euclidean action \( A \) yields

\[ A(\psi, x) = A_0(\psi, x) + \sum_i \int_0^\beta d\tau \Phi^*_i G(\psi, x) \Phi_i, \] (7)

where

\[ \Phi_i = (\mu^*_i, \nu^*_i)^T, \] (8)

\[ A_0(\psi, x) = \int_0^\beta d\tau \left[ \psi^* (\omega + \partial_{\tau}) \psi - x^* x \right], \] (9)

and

\[ G(\psi, x) = \left[ \begin{array}{c} \partial_{\tau} + \Omega \\ \frac{F(\psi, x)}{F(\psi, x) \partial_{\tau} - \Omega} \end{array} \right], \] (10)

In Eq. (3), \([d\eta(\tau)] = d[\psi, \psi^*, \mu, \mu^*, \nu, \nu^*] \) is the path integral measure. The Euclidean action is given by

\[ A = \int_0^\beta d\tau [\psi^* \partial_{\tau} \psi + \sum_{i=1}^N (\mu^*_i \partial_{\tau} \mu_i + \nu^*_i \partial_{\tau} \nu_i) + H_F], \] (4)

where \( \partial_{\tau} = \partial / \partial \tau, \beta = 1/(k_B T) \) with \( k_B \) being the Boltzmann constant and \( T \) being the system’s temperature, and

\[ H_F = \omega \psi^* \psi + \Omega \sum_{i=1}^N (\mu^*_i \mu_i - \nu^*_i \nu_i) + \sum_{i=1}^N \gamma \sqrt{\omega \gamma \mu \nu \beta} \left( \psi + \psi^* \right) (\mu^*_i \nu_i + \nu^*_i \mu_i) + q \left( \mu^*_i \mu_i - \nu^*_i \nu_i \right)^2. \] (5)

Then, we have

\[ \exp[-\frac{q}{N} \sum_{i=1}^N (\mu^*_i \mu_i - \nu^*_i \nu_i)^2] \propto \int [d\eta] \exp\{ \int_0^\beta d\tau \left[ \frac{1}{q} x^* x - \frac{1}{N} \sum_{i=1}^N (x + x^*) (\mu^*_i \mu_i - \nu^*_i \nu_i) \right] \}. \] (6)

with

\[ F(\psi, x) = \gamma \sqrt{\omega / N} (\psi^* + \psi) - \sqrt{1 / N} (x^* + x). \] (11)

For the effective Euclidean action \( A \) in Eq. (11), we can integrate over the Grassmann Fermi fields, i.e., the degree of freedom for the spin, and then obtain

\[ A = N \int_0^\beta d\tau [\psi^* (\omega + \partial_{\tau}) \psi - X^* X - \text{Tr} \ln G], \] (12)

where \( \Psi = \psi / \sqrt{N} \) and \( X = x / \sqrt{N} \). Finally, by means of the standard stationary phase approximation, namely, \( \delta A / \delta \Psi = \delta A / \delta \Psi^* = 0 \) and \( \delta A / \delta \bar{X} = \delta A / \delta \bar{X}^* = 0 \), the required \( \Psi \) and \( X \), which play a crucial role in determining thermodynamic properties of Hamiltonian \( [2] \), can be obtained by

\[ \begin{cases} \Psi = \Psi^* = \frac{\bar{Q}}{\bar{Q} + \sqrt{\omega \Psi} - q X} \tanh(\frac{\bar{Q}}{2}) \\ X = X^* = \frac{\bar{Q}}{\bar{Q} + \sqrt{\omega \Psi} - q X} \tanh(\frac{\bar{Q}}{2}) \end{cases}, \] (13)
We first address the case of $q = 0$, which has been realized at NIST [2]. At zero temperature ($T = 0$), \( \tanh(\beta \zeta) = 1 \) and thus, Eq. (13) becomes $\Psi = 2\gamma^2 \Psi / \zeta$, which leads to solutions of $\Psi = \langle S_x \rangle = 0$ for $\gamma \leq \sqrt{\Omega/2}$ and $\Psi = \pm \sqrt{(4\gamma^4 - \Omega^2)/(4\omega^2)}$ for $\gamma \geq \sqrt{\Omega/2}$. These zero-temperature solutions agree well with the direct numerical simulation of the SOC-driven Gross-Pitaevskii equations [23]. The nontrivial variations of both $\Psi$ (atom momentum) and $\langle S_x \rangle$ (atom population) show that a quantum phase transition occurs by adjusting the effective SOC strength $\gamma$. Moreover, we can call $\Psi = \langle S_x \rangle = 0$ as the single minimum phase (SMP) without collective excitations, and $\Psi \neq 0$ and $\langle S_x \rangle \neq 0$ as the separate phase (SP) with the macroscopic excitations [25]. With the increasing of the temperature $T$, the order parameter $\Psi$ or $\langle S_x \rangle$ will be destroyed by thermal fluctuation. In particular, when $\Psi(T_c) = \langle S_x \rangle(T_c) = 0$, the system enters into the SMP from the SP. By means of $\Psi(T_c) = 0$, the critical temperature can be obtained exactly by

$$T_c = \frac{\Omega}{2k_B \arctanh(\frac{\Omega}{2\gamma^2})}. \quad (16)$$

Eq. (16) shows that the phase transition from the SP to the SMP can be driven by the tunable temperature. Moreover, the corresponding critical temperature obtained exactly is independent of the trapped potential $\omega$. When $\Omega = 0.2E_L$ and $\gamma^2 = E_L$, the critical temperature is evaluated as $T_c = 84.9$ nK, which is feasible in experiments about SOC-driven BECs.

Having obtaining the critical temperature, we discuss the experimentally-measurable atom population at finite temperature. In terms of the thermodynamic equilibrium equation (15), the partition function is given in the SMP with $\Psi(T) = 0$ by

$$Z_{\text{SMP}} = \exp\{-N\beta[-\frac{2}{\beta} \ln(2 \cosh(\frac{\beta \Omega}{2}))]\}, \quad (17)$$
FIG. 4: (Color online) The critical temperature $T_c$ as a function of the effective SOC strength $\gamma$ for the different effective atom interactions $q = -0.3 E_L$ (Black solid line), $q = 0.0 E_L$ (Red dashed line) and $q = 0.3 E_L$ (Blue dashed-dotted line). Inset: The critical temperature versus the effective atom interaction $q$ with the different effective SOC strength $\gamma^2 = 2.6 E_L$ (Black solid line) and $\gamma^2 = 5.0 E_L$ (Red dotted line). In these figures, the other plotted parameters are given by $\omega \simeq 5.1 \times 10^4 E_L$ and $\Omega = 5.0 E_L$.

whereas it becomes

$$Z_{\text{SP}} = 2 \exp\{-N\beta[\omega \Psi^2 - 2 \frac{\beta}{\beta} \ln(2 \cosh(\frac{\beta \zeta}{2}))]\}$$

(18)

in the SP with $\Psi(T) \neq 0$. Thus, the atom population can be derived from the formula

$$\langle S_x \rangle (T) = \frac{\partial \langle N \rangle}{\partial \beta} (2 \gamma \sqrt{\omega} \Psi)$$

(19)

by

$$\langle S_x \rangle_{\text{SMP}} (T) = 0$$

(20)

in the SMP and

$$\langle S_x \rangle_{\text{SP}} (T) = -\frac{\sqrt{\omega}}{\gamma} \Psi(T)$$

(21)

in the SP. In general, $\Psi(T)$ should be determined numerically by solving the nonlinear equation (15). However, when $T = 0$, Eqs. (20) and (21) reduce to the known analytical results [25, 26]. In Fig. 2, we plot the scaled atom population $\langle S_x \rangle / N$ as the functions of the effective SOC strength $\gamma$ and the temperature $T$. This figure shows that thermal fluctuations destroy the collective excitations. As a result, the system finally enters into the SMP from the SP.

For a full understanding of the temperature-driven phase transition, it is very important to discuss the thermodynamic quantities in the different phases. Here we consider the specific heat per atom and the entropy per atom. The other thermodynamic quantities can be calculated using the same procedure. By means of the formula

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V$$

(22)

with $U = -N \frac{\Omega^2}{2B} \ln Z$ being the total energy, the specific heat per atom in the SMP is obtained exactly by

$$C_{V_{\text{SMP}}} = \frac{\Omega^2}{2kB^2T^2}\cosh^2\left(\frac{\Omega}{2kB^2T}\right)$$

(23)

which is independent of both the trapped frequency $\omega$ and the effective SOC strength $\gamma$. In the SP, the specific heat per atom is evaluated as

$$C^S_{V} = \frac{1}{2kB^2T^2} \left[ (\zeta' + \frac{\zeta'}{kB^2T}) \frac{2}{\sqrt{\omega} \Psi' + \frac{2}{kBT}} + 2 \left( \frac{\zeta''}{kB^2T} + 2\zeta' \right) \tanh \left( \frac{\zeta}{kB^2T} \right) - 4\omega \left( \frac{\Psi'^2}{kBT} + 2\Psi' + \frac{\Psi''}{kB^2T} \right) \right]$$

(24)

where $\zeta = \sqrt{\Omega^2 + 4\omega^2 \Psi^2}$, $\zeta' = \partial \zeta / \partial \beta = 4\gamma^2 \omega \Psi \Psi' / \zeta$, and $\zeta'' = \partial^2 \zeta / \partial \beta^2 = 4\gamma^2 \omega \left( \Omega^2 \Psi^2 + \zeta^2 \Psi' / \zeta \right)$ with $\Psi' = \partial \Psi / \partial \beta = \zeta' / \left\{ 2 \omega \Psi \left[ 1 - 2 \beta \gamma^2 + \cosh (2 \beta \zeta) \right] \right\}$ and $\Psi'' = \partial^2 \Psi / \partial \beta^2 = \left\{ -16 \gamma^4 \omega^2 [1 - 2 \beta \gamma^2] \Psi^2 \Psi'' + 4 \gamma^2 \zeta \sech^2 (2 \beta \zeta) \tanh (2 \beta \zeta) \left[ (\zeta^2 + 4 \gamma^2 \omega \Psi')^2 + 4 \gamma^2 \zeta^2 \sech^2 (2 \beta \zeta) \Psi^2 - 8 \gamma^4 \omega^2 \sech^2 (2 \beta \zeta) \Psi' \Psi'' - \left\{ -4 \gamma^2 \omega \zeta^2 [1 - \beta \gamma^2 \sech^2 (2 \beta \zeta) \Psi] \right\} \right\}$. The specific heat $C^S_{V}$ implies that the strong SOC can modify the energy structure of Hamiltonian [2] and thus the thermodynamic statistics, as expected.

It is very hard to directly extract the fundamental properties of the specific heat $C^S_{V}$ from the complicate expression (24). In Fig. 3, we plot the specific heat $C_{V}$ (Red lines) as a function of the temperature $T$. This figure shows two interesting features. (I) The specific heat $C_{V}$ has a large jump at the critical point $T_c$, separating the SP from the SMP. This step behavior is quite different from that of the atom population $\langle S_x \rangle$ in Fig. 2 which varies smoothly when crossing the critical point. It im-

\[ \text{FIG. 4: (Color online) The critical temperature } T_c \text{ as a function of the effective SOC strength } \gamma \text{ for the different effective atom interactions } q = -0.3 E_L \text{ (Black solid line), } q = 0.0 E_L \text{ (Red dashed line) and } q = 0.3 E_L \text{ (Blue dashed-dotted line). Inset: The critical temperature versus the effective atom interaction } q \text{ with the different effective SOC strength } \gamma^2 = 2.6 E_L \text{ (Black solid line) and } \gamma^2 = 5.0 E_L \text{ (Red dotted line). In these figures, the other plotted parameters are given by } \omega \simeq 5.1 \times 10^4 E_L \text{ and } \Omega = 5.0 E_L. \]
However, its step behavior at the critical point implies that the temperature-driven phase transition can be well detected by measuring the specific heat $C_V$. (II) At lower temperature, we have approximately $\Psi' \approx \Psi'' \approx 0$.

Thus, the specific heat in the SP can be obtained analytically by

$$C_V^{SP} \approx \frac{2\gamma^4}{k_B T^3} \frac{1}{\mathrm{sech}^2 \left( \frac{\gamma^2}{k_B T} \right)},$$

which agrees well with the direct numerical simulations, as shown the Red lines of Fig. 3. Eq. (25) shows, in contrast to the behavior of the specific heat $C_V$, the specific heat $C_V^{SP}$ is governed only by the effective SOC strength $\gamma$, i.e., it is independent of both the trapped potential $\omega$ and the effective Rabi frequency $\Omega$.

Another important thermodynamic quantity discussed in this paper is the entropy, which can be derived from the formula

$$S = -\frac{\partial G}{\partial T},$$

with $G = -k_B T \ln Z$ being the Gibbs function. In the SMP, the entropy per atom is obtained exactly by

$$S^{\text{SMP}} = 2k_B \ln \left[ 2 \cosh \left( \frac{\Omega}{2k_B T} \right) \right] - \frac{1}{T} \tanh \left( \frac{\Omega}{2k_B T} \right) \left( \frac{\Omega}{k_B T} - \frac{2}{k_B T^2} \omega \Psi' \right).$$

In Fig. 4 we plot the critical temperature $T_c$ as a function of the effective SOC strength $\gamma$ for the different effective atom interactions. This figure shows clearly that for the attractive interaction ($q < 0$), the critical temperature $T_c$ increases for a fixed SOC strength $\gamma$, and vice versa. It means that in experiments we can manipulate the effective atom interaction $q$ to arrive at the experimentally-required critical temperature $T_c$, where the thermodynamic phase transition from the SP to the SMP occurs.

On the other hand, in the SMP, no collective excitations can be found in SOC-driven BEC and thus the effective atom interaction does not change the energy structure. It implies that the partition function in the SMP is the same as Eq. (17), i.e., it is independent of the effective SOC strength $\gamma$, the trapped frequency $\omega$ and the effective atom interaction $q$. However, in the SP the strong SOC leads to the system’s collective excitations with nonzero atom population. As a result, the term $\frac{1}{N} \mathcal{S}_q^2$ in Hamiltonian (2) plays a crucial role in system’s energy structure and thus the thermodynamic statistics.

V. WITH EFFECTIVE ATOM INTERACTION

In this section, we illustrate the effect induced by the effective atom interaction, which is indeed controlled by the Feshbach resonant technique in experiments. With the effective atom interaction ($q \neq 0$), the critical temperature can be also obtained exactly by

$$T_c(q) = \frac{\Omega}{2k_B \arctanh \left( \frac{\Omega}{2\gamma^2 - q} \right)},$$

in the SP. At lower temperature, the entropy can be evaluated approximately as

$$S^{\text{SMP}} \approx 2k_B \ln \left[ \frac{2}{N} + \ln \left( 2 \cosh \left( \frac{\gamma^2}{k_B T} \right) \right) \right] - \frac{\gamma^2}{k_B T} \tanh \left( \frac{\gamma^2}{k_B T} \right).$$

Similar to the behaviors of the specific heat $C_V$, the entropy $S$ in the SMP is governed only by the effective Rabi frequency $\Omega$, whereas it is determined only by the effective SOC strength $\gamma$ in the SP with lower temperature. However, its step behavior at the critical point $T_c$ is very small, as shown the Blue lines of Fig. 3.
critical temperatures in the different cases.

FIG. 6: (Color online) The specific heat $C_V$ as a function of the temperature $T$ for the different effective atom interactions $q = -0.25E_L$, $q = 0.00E_L$, and $q = 0.25E_L$. The other plotted parameters are given by $\omega \simeq 5.1 \times 10^4 E_1$, $\Omega = 3.0 E_1$, and $\gamma^2 = 1.8 E_1$, respectively. The three gray lines indicate the critical temperatures in the different cases.

In such case, the partition function becomes

$$Z_{SP}(q) = 2 \exp\{-N\beta \omega \delta \Psi^2 / \gamma^4 - 2 \beta \ln[2 \cosh(\beta \zeta / 2)]\},$$ (31)

where $\delta(q) = \gamma^2 - q/\gamma$. $\Psi$ and $\zeta$ can be obtained from the nonlinear equation \text{15} with the effective atom interaction $q$.

Based on the obtained partition function, the experimentally-measurable atom population is given by $\langle S_x \rangle_{SP}(q,T) = 0$ and $\langle S_x \rangle_{SP}(q,T) = -\delta(q) / \gamma$ (along the gray dotted line $\gamma^2 = 2.6E_1$ there). In addition, in the SMP, the specific heat $C_V$ and the entropy $S$ are also identical to the results of $q = 0$. Whereas, in the SP they become

$$C_{V,SP}(q) = k_B \beta \omega \delta \Psi^2 / \gamma^4 - 2 \beta \ln[2 \cosh(\beta \zeta / 2)]\},$$ (32)

and

$$S_{SP}(q) \approx 2 k_B \left\{ \ln \left( 2 \Omega / \gamma \right) + \ln \left[ \gamma \eta(q) \right] - \Lambda \tanh (\Lambda) \right\},$$ (33)

where $\Lambda(q) = \delta(q) / k_B T$. In Fig. 6 the specific heat $C_V$ as a function of the temperature $T$ for the different effective atom interactions is plotted. This figure shows that the fundamental properties of temperature-driven phase transition remain in the framework of the effective atom interaction. However, for the repulsive interaction ($q > 0$), the critical temperature $T_c$ can decrease. Moreover, the step amplitude at the critical point also decreases. For the attractive interaction ($q < 0$), the opposite result also exists. The similar behaviors of the entropy $S$ can also be found.

VI. CONCLUSIONS AND REMARKS

In summary, we have explored the thermodynamic properties of the trapped BEC with the equal Rashba and Dresselhaus SOCs, which has been realized in experiments. The thermodynamic phase transition from the SP to the SMP as well as the critical temperature has been revealed. We have also discussed the important thermodynamic quantities such as the specific heat and the entropy and obtained their analytical expressions in the different phases. At the critical point, the specific heat has a large jump and can be thus regarded as a promising physical quantity to detect this temperature-driven phase transition. Finally, we have illustrated the effect of the effective atom interaction, which can be well controlled by the experimentally-feasible Feshbach resonant technique. Especially, we have found that in the SP this effective atom interaction affects dramatically on the critical temperature and the corresponding thermodynamic properties for the SOC-driven BEC. Before ending up this paper, we briefly make two remarks. Firstly, our analysis is mainly based on the imaginary-time functional path-integral approach, in which the fluctuation of the space-dependent physical quantity (such as the density) is usually “hidden” or “averaged out” \text{29}. As a consequence, the important stripe phase predicted before \text{3, 24, 26} cannot be distinguished from the SP effectively. Secondly, without SOC ($\gamma = 0$), Hamiltonian \text{24} turns into $H = \Omega S_z + \frac{\gamma}{\gamma} S_2$. In this case, the relation $X_\delta = q \sqrt{\Omega} \Psi / \gamma$ becomes invalid. However, we can use the same procedure (introducing the auxiliary field in path-integral technique) to discuss the corresponding thermodynamics.

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[1] C. Nayak, S. H. Simon, A. Stern, M. Freedman, and S. Das Sarma, Rev. Mod. Phys. 80, 1083 (2008).
[2] J. Dalibard, F. Gerbier, G. Juzeliūnas, and P. Öhberg, Rev. Mod. Phys. 83, 1523 (2011).
[3] Y. -J. Lin, K. Jimenez-Garcia, and I. B. Spielman, Nature (London), 471, 83 (2011).
[4] J. Larson and E. Sjöqvist, Phys. Rev. A 79, 043627 (2009).
[5] C. Wang, C. Gao, C. -M. Jian, and H. Zhai, Phys. Rev. Lett. 105, 160403 (2010).
[6] C. Wu, I. Mondragon-Shem, and X. -F. Zhou, Chin. Phys. Lett. 28, 097102 (2011).
[7] S. -K. Yip, Phys. Rev. A 83, 043616 (2011).
[8] Y. Zhang, L. Mao, and C. Zhang, Phys. Rev. Lett. 108, 035302 (2012).
[9] Y. Zhang and C. Zhang, arXiv:1203.2389 (2012).
[10] S. Sinha, R. Nath, and L. Santos, Phys. Rev. Lett. 107, 270401 (2011).
[11] X. -Q. Xu and J. H. Han, Phys. Rev. Lett. 107, 200401 (2011); Phys. Rev. Lett. 108, 185301 (2012).
[12] Q. Zhu, C. Zhang, and B. Wu, arXiv:1109.5811 (2011).
[13] H. Hu, H. Pu, and X. -J. Liu, Phys. Rev. Lett. 108, 010402 (2012).
[14] D. -W. Zhang, Z. -Y. Xue, H. Yan, Z. D. Wang, and S. -L. Zhu, Phys. Rev. A 85, 013628 (2012).
[15] D. -W. Zhang, L. -B. Fu, Z. D. Wang, and S. -L. Zhu, Phys. Rev. A 85, 043609 (2012).
[16] T. Ozawa and G. Baym, Phys. Rev. A 85, 013612 (2012); Phys. Rev. Lett. 109, 025301 (2012).
[17] Y. Deng, J. Cheng, H. Jing, C.-P. Sun, and S. Yi, Phys. Rev. Lett. 108, 125301 (2012).
[18] W. Zheng and Z. Li, Phys. Rev. A 85, 053607 (2012).
[19] H. Zhai, Int. J. Mod. Phys. B 26, 1230001 (2012).
[20] J. P. Vyasanakere and V. B. Shenoy, arXiv:1201.5332 (2012).
[21] Z. F. Xu, Y. Kawaguchi, L. You, and M. Ueda, arXiv:1203.2005 (2012).
[22] O. Fialko, J. Brand, and U. Zülicke, Phys. Rev. A 85, 051605(R) (2012).
[23] J. Radič, A. D. Ciolo, K. Sun, and V. Galitski, Phys. Rev. Lett. 109, 085303 (2012).
[24] T. -L. Ho and S. Zhang, Phys. Rev. Lett. 107, 150403 (2011).
[25] Y. Zhang, G. Chen, and C. Zhang, arXiv:1111.4778 (2011).
[26] Y. Li, L. P. Pitaevskii, and S. Stringari, Phys. Rev. Lett. 108, 225301 (2012).
[27] J. -Y Zhang, S. -C. Ji, Z. Chen, L. Zhang, Z. -D. Du, B. Yan, G. -S. Pan, B. Zhao, Y. -J. Deng, H. Zhai, S. Chen, and J. -W. Pan, Phys. Rev. Lett. accepted for publication.
[28] S. Sachdev, *Quantum Phase Transitions* (Cambridge University Press, Cambridge, 1999).
[29] N. Nagaosa, *Quantum Field Theory in Condensed Matter Physics*, (Springer-Verlag, Berlin Heidelberg, 1999).
[30] Z. Fu, P. Wang, S. Chai, L. Huang, and J. Zhang, Phys. Rev. A 84, 043609 (2011).
[31] C. Chin, R. Grimm, P. Julienne, and E. Tiesinga, Rev. Mod. Phys. 82, 1225 (2010).
[32] M. Aparicio Alcalde, A. H. Cardenas, N. F. Svaiter, and V. B. Bezerra, Phys. Rev. A 81, 032335 (2010).