Shellability is NP-complete

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Arranging a jigsaw puzzle

Rules:

- triangular pieces
- add pieces one by one
- more pieces can share an edge
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allowed

boundary of tetrahedron

not allowed
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This is shellability in 2D (purely 2D object obtained by the rules).
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- \( f_i = \# \) of faces of dimension \( i \)
- \( h_i \) determines \( f_i \)

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... “almost” Euler formula!

(Why $h_3 = 1$?)
Works in higher dimensions as well

\[ \mathcal{K} = \text{a finite simplicial complex of dimension } d \geq 1 \]

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- \( \sigma \in \mathcal{K}, \tau \text{ is a face of } \sigma \Rightarrow \tau \in \mathcal{K} \)
- \( \sigma_1, \sigma_2 \in \mathcal{K} \Rightarrow \sigma_1 \cap \sigma_2 \text{ is a face of } \sigma_1 \text{ and } \sigma_2 \)
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- **Assume:** all facets (\( = \) inclusion maximal faces) have dim \( d \)

\( \mathcal{K} \) is shellable if \( \exists \) ordering \( F_1, \ldots, F_N \) of facets of \( \mathcal{K} \) s.t.
\( \forall 2 \leq i \leq N: F_i \cap (\bigcup_{j<i} F_j) \) is a non-empty union of \( (d - 1) \)-dim faces.

... “gluing facets along \( (d - 1) \)-dim faces“
boundaries of polytopes are shellable . . . Bruggesser, Mani ’72

used e.g. in Schläfli’s “proof” of Euler-Poincaré formula (1852):
\[ f_0 - f_1 + \cdots + (-1)^{d-1}f_{d-1} = 1 - (-1)^d \]
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  \[ \Rightarrow h_i = h_{d-i} \quad \text{Dehn-Sommerville relations ’27} \]
Polytope theory

- Boundaries of polytopes are shellable ... Bruggesser, Mani ’72

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  \[ \Rightarrow h_i = h_{d-i} \]  Dehn-Sommerville relations ’27

  \[ \Rightarrow \] Every planar triangulation has \( 2f_0 - 4 \) faces.

  **In higher dimensions:**
  - UBT: Cyclic polytopes maximize \# of faces  McMullen ’70
  - LBT: Stacked polytopes minimize \# of faces  Klee ’75
Shellability in various areas of mathematics

- **Polytope theory**: Inductive procedure for removing facets while all intermediate complexes are topological balls.

- **PL Topology**: Shelling of a PL-manifold (with boundary) keeps the homeomorphism type.

- **Shellability of posets**: Pioneered by Björner and Wachs in 80’s. Consequences, e.g., on enumerative properties of posets.

- **Shelling monoids**: Peeva, Reiner, Sturmfels ’98: Koszul property of monoidal algebra via shellability of monoids.
Shellability is in NP – try all possible orderings

Danaraj, Klee '78: Is shellability efficiently decidable?

Showed: shellability of 2-dim pseudomanifolds can be tested in linear time (pseudomn. = every edge in exactly two triangles)
Main result

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- **It is NP-complete to decide whether a given 2-dim simplicial complex is shellable.**

  ...reduction from 3-SAT
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- **It is NP-complete to decide whether a given 2-dim simplicial complex is shellable.**
  
  …reduction from **3-SAT**

- **It is NP-complete to decide whether a given \( \geq 3 \)-dim contractible simplicial complex is shellable.**
  
  .............easy corollary
High-level overview of the proof

- We go backwards and remove triangles.
- For a 3-CNF formula \( \phi \) build a 2-dim simpl complex \( K_\phi \) s.t.
  \[
  \phi \text{ satisfiable} \iff K_\phi \text{ collapsible \ after removing \# \ var \ triangles}
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$$\Phi = (u \lor \neg x \lor \neg y) \land (\neg u \lor y \lor \neg z)$$
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Open problems

- Is shellability of 2-dim complexes \textbf{FPT} in Euler characteristics?

- Is it NP-hard to decide whether a given 2-dim complex embedded in $\mathbb{R}^3$ (or even $\mathbb{R}^4$) is shellable?

- Can a greedy algorithm fail when shelling a $k$-skeleton of $n$-simplex? Simon’s conjecture: \textbf{NO}.

- How hard is to recognize shellable 3-balls? essentially asked by Danaraj, Klee ’78

- If a poset is given by its Hasse diagram, does recognition of shellable posets belong to NP? Shellability of a poset $P = \text{shellability of the order complex } \Delta(P)$. 
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Thank you!