An upper bound for the critical probability on the Cartesian product graph of a regular tree and a line

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Abstract
We study Bernoulli bond percolation on the Cartesian product graph of a regular tree and a line. We give an upper bound for the critical probability, which improves previous upper bound. We use a method which is similar to Golton-Watson process. Our result leads that there exists a non-empty phase in which there are infinitely many infinite clusters when a degree of a tree is 4.

1 Introduction
Let $G = (V, E, o)$ be a rooted, connected, locally finite, and infinite graph, where $V$ is the set of vertices, $E$ is the set of edges, and $o$ is a special vertex called a root. In Bernoulli bond percolation, each edge will be open with probability $p$, and closed with probability $1 - p$ independently, where $p \in [0, 1]$ is a fixed parameter. Let $\Omega = 2^E$ be the set of samples, where $\omega \subset E$ is the set of all open edges. Each $\omega \in \Omega$ is regarded as a subgraph of $G$ consisting of all open edges. The connected components of $\omega$ are referred to as clusters. Let $p_c = p_c(G)$ be the critical probability for Bernoulli bond percolation on $G$, that is,

$$p_c = \inf \{ p \in [0, 1] \mid \text{there exists an infinite cluster almost surely} \},$$

and let $p_u = p_u(G)$ be the uniqueness threshold for Bernoulli bond percolation on $G$, that is,

$$p_u = \inf \{ p \in [0, 1] \mid \text{there exists a unique infinite cluster almost surely} \}.$$

One of the most popular graphs in the theory of percolation is the Euclidean lattice $\mathbb{Z}^d$. In 1980 Kesten\cite{Kesten1980} proved that $p_c = 1/2$ in the case of two dimensions. But in the case of three dimensions or more, as a numerical value, the critical probability is not quite clear. Regarding the uniqueness threshold of the Euclidean lattice, in 1987 Aizenman, Kesten, and Newman\cite{Aizenman1987} proved that there exists at most one infinite cluster almost surely for all $d \geq 1$, that is, they showed that $p_c = p_u$ for all $d \geq 1$. The Cartesian product graph of a $d$-regular tree and a line $T_d \square \mathbb{Z}$ was presented as a first example of a graph with $p_c < p_u$ by Grimmett and Newman\cite{Grimmett1990} in 1990. They showed that $p_c < p_u$ holds when $d$ is sufficiently large. After this article had appeared, percolation on $T_d \square \mathbb{Z}$ has been a popular topic. However, the critical probability of $T_d \square \mathbb{Z}$ is, as a value, also not quite clear. In recent years, Lyons and Peres\cite{Lyons2010} gave the following upper bound of $p_c$ and lower bound of $p_u$. 

\begin{footnotesize}

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\end{footnotesize}
Theorem 1.1. For all $d \geq 3$, we have

\begin{align}
\text{(1.1)} & \quad p_c(T_d \square Z) \leq \frac{d - \sqrt{d^2 - 4}}{2}, \\
\text{(1.2)} & \quad p_u(T_d \square Z) \geq \left(\sqrt{d - 1} + 1 + \sqrt{2\sqrt{d - 1} - 1}\right)^{-1}.
\end{align}

From this result, $p_c < p_u$ holds for all $d \geq 5$. The main result of this paper is to give a new upper bound which improves the inequality (1.1).

Theorem A. Let $d \geq 3$. Then we have

$$p_c(T_d \square Z) \leq \frac{1}{d}.$$  

In the case of $d = 4$, we further improve this upper bound.

Theorem B. Let $d = 4$. Then we have

$$p_c < 0.225.$$ 

Then using the inequality (1.2) and (1.3), we have $p_c < 0.225 < 0.232 < p_u$ when $d = 4$.

Remark 1.2. A preprint of this paper first appeared in June 2017. At that time, it was not known whether $p_c < p_u$ holds when $d = 3$. In November 2017, Hutchcroft showed that $p_c < p_u$ holds for all $d \geq 3$. The method of Hutchcroft is different from ours.

## 2 Probability generating function

The critical probability of $T_d$ is found by only consider Galton-Watson process. In this process, we can know whether the tree is infinite or not by only consider the first step. Lyons and Peres used the natural projection from $T_d \square Z$ to $T_d$, and focus on the first step. They gave an upper bound by using the method like Galton-Watson process. We also use the projection and essentially the same method. But our strategy is to consider each step not just the first step to get a better estimate.

In general, if $H$ is a subgraph of $G$ containing $o$, then we have $p_c(G) \leq p_c(H)$. There exists a $(d - 1)$-ary tree as a subgraph of a $d$-regular tree, where $(d - 1)$-ary tree is a tree such that $\deg v = d$ except the root and $\deg o = d - 1$. Then we can find an upper bound of $p_c(T_d \square Z)$ by estimating the critical probability of the Cartesian product graph of a $(d - 1)$-ary tree and a line. Therefore, we may assume that $T_d$ is a $(d - 1)$-ary tree in the following. We denote the probability measure associated with this process by $P_p$ or $P_p^C$ and the expectation operator by $E_p$ or $E_p^C$. The definition of $p_c$ can be rewritten using $P_p$. Let $C$ be a cluster containing $o$ and $|C|$ be a order of the vertex set of $C$. Then we can rewrite

$$p_c = \inf \{p \in [0, 1] \mid P_p(|C| = \infty) > 0\}.$$ 

In Bernoulli percolation on $T_d$, we can show that $p_c = 1/(d - 1)$ by Galton-Watson process. In Galton-Watson process, let $X_n$ be the number of vertices such that it has distance $n$ from the root. It is well known that following equation holds.

$$E[X_n] = E[X_1]^n.$$ 

Therefore if $E[X_1] > 1$, then the probability that the tree is infinite, is positive. Let $B_n$ be a subgraph of $T_d$ defined by an $n$-ball centered at the root. Because we can decompose $T_d$ into
If these vertices as new roots. That is, we consider percolation on $H_{\pi}$ at least one vertex on and there exists an open path from $o$ of which is isomorphic to percolation on $H$ of (2.3) We compute composite function of $f$ where the notation (2.2) $X$ variable $X$ $H$ several pieces each of which is isomorphic to $B_n$, the equation (2.1) holds. The graph $T_d \Box \mathbb{Z}$ has a similar structure. To explain it, let $H_n$ be a subgraph of $T_d \Box \mathbb{Z}$ defined by

\[ V(H_n) = \{(v, k) \mid d_{T_d}(v, 0_{T_d}) \leq n, k \in \mathbb{Z} \}, \]

\[ E(H_n) = \{(v, k), (u, k) \mid (v, k), (u, k) \in V(H_n), (v, u) \in E(T_d), k \in \mathbb{Z} \} \]

\[ \cup \{((v, k), (v, k + 1)) \mid d_{T_d}(v, 0_{T_d}) \leq n - 1, k \in \mathbb{Z} \}, \]

where $d_{T_d}$ is the graph metric on $T_d$. Then $T_d \Box \mathbb{Z}$ is clearly decomposed into infinitely many copies of $H_n$. Let $\pi$ be the natural projection from $T_d \Box \mathbb{Z}$ to $T_d$. In percolation on $H_n$, we define a random variable $X_n(\omega)$ by the following formula.

\[ X_n(\omega) = \# \{ v \in T_d \mid d_{T_d}(v, 0_{T_d}) = n, o \leftrightarrow \pi^{-1}(v) \text{ on } H_n \}, \]

where the notation $A \leftrightarrow B$ means that there exists an open path between $A$ and $B$. Similar to Galton-Watson process, we have the following lemma.

**Lemma 2.1.** For any $p$, if there exists $n$ such that $\mathbb{E}_p[X_n] > 1$, then $\mathbb{P}_p(|C| = \infty) > 0$.

**Proof.** Let $s \in [0, 1]$ be a parameter. The probability generating function is defined by

\[ f_n(s) = \mathbb{E}_p[s^{X_n}] = \sum_{k=0}^{(d-1)^n} \mathbb{P}_p(X_n = k)s^k. \]

In Galton-Watson process, the equation $f_{nm}(s) = f_n(s)^{(m)}$ holds, where $f_n^{(m)}$ is the $m$ times composite function of $f_n$. Similarly, we will show the following inequality.

\[ f_{nm}(s) \leq f_n(s)^{(m)}. \]

We compute $f_{nm}(s)$ by rewriting probabilities involving $X_{nm}$ in terms of $X_{n(m-1)}$ as follows.

\[ f_{nm}(s) = \sum_{k \geq 0} \mathbb{P}_p(X_{nm} = k)s^k \]

\[ = \mathbb{P}_p(X_{nm} \geq 0) - \sum_{k \geq 1} \mathbb{P}_p(X_{nm} \geq k)s^{k-1}(1 - s) \]

\[ = 1 - \sum_{k \geq 1} \sum_{l \geq 1} \mathbb{P}_p(X_{nm} \geq k|X_{n(m-1)} = l) \cdot \mathbb{P}_p(X_{n(m-1)} = l)s^{k-1}(1 - s). \]

If $(X_{n(m-1)} = l)$ occurs, there exists $l$ vertices $v_1, \ldots, v_l$ on $T_d$ such that $d_{T_d}(v_i, 0_{T_d}) = n(m-1)$, and there exists an open path from $o$ to $\pi^{-1}(v_i)$ on $H_{n(m-1)}$. Therefore, for each $i$, there exists at least one vertex on $\pi^{-1}(v_i)$ which is connected with the root by an open path. We regard these vertices as new roots. That is, we consider percolation on $H_{n(m-1)}$ first, then we consider percolation on $H_{nm} \setminus H_{n(m-1)}$ next. Since $H_{nm} \setminus H_{n(m-1)}$ is a union of $(d-1)^n(m-1)$ pieces each of which is isomorphic to $H_n$, we can estimate a lower bound of $\mathbb{P}_p(X_{nm} \geq k|X_{n(m-1)} = l)$.

\[ \mathbb{P}_p(X_{nm} \geq k|X_{n(m-1)} = l) \geq \sum_{j_1, \ldots, j_l \geq 0} \left( \prod_{u=1}^{l} \mathbb{P}_p(X_n = j_u) \right). \]
Therefore, we have

\[ f_{nm}(s) \leq 1 - \sum_{l \geq 1} \sum_{j_1, \ldots, j_l \geq 0} \left( \prod_{u=1}^l \mathbb{P}_p(X_u = j_u) \right) \cdot \mathbb{P}_p(X_{n(m-1)} = l) \sum_{k=1}^{j_1 + \cdots + j_l} s^{k-1}(1-s) \]

\[ = 1 - \sum_{l \geq 1} \mathbb{P}(X_{n(m-1)} = l) \sum_{j_1, \ldots, j_l \geq 0} \left( \prod_{u=1}^l \mathbb{P}_p(X_u = j_u) \right) \]

\[ + \sum_{l \geq 1} \mathbb{P}(X_{n(m-1)} = l) \sum_{j_1, \ldots, j_l \geq 0} \left( \prod_{u=1}^l \mathbb{P}_p(X_u = j_u)s^{j_u} \right) \]

\[ = \mathbb{P}_p(X_{n(m-1)} = 0) + \sum_{l \geq 1} \mathbb{P}(X_{n(m-1)} = l) \left( \mathbb{E}_p[s^{X_n}] \right)^l \]

\[ = \mathbb{E}_p[f_n(s)^{X_{n(m-1)}}] \leq \cdots \leq f_n^{(m)}(s), \]

which completes the proof of the inequality (2.3). By the definition of the probability generating function, if \( \mathbb{E}[X_n] > 1 \), then we have \( \lim_{n \to \infty} f_n^{(m)}(0) < 1 \). Using the inequality (2.3), we have \( \lim_{n \to \infty} f_n(0) < 1 \). Hence we have \( \mathbb{P}_p(|C| = \infty) > 0 \).

Let \( v_n \) be a vertex in \( T_d \) such that \( d_{T_d}(v_n, \partial T_d) = n \). We have

\[ \mathbb{E}[X_n] = (d-1)^n \mathbb{P}_p^{H_n}(o \leftrightarrow \pi^{-1}(v_n)). \]

Using this equation and Lemma 2.1 if \( \limsup_{n \to \infty} \mathbb{P}_p^{H_n}(o \leftrightarrow \pi^{-1}(v_n))^{1/n} > 1/(d-1) \), then we have \( \mathbb{P}_p(|C| = \infty) > 0 \). It is difficult to estimate \( \limsup_{n \to \infty} \mathbb{P}_p^{H_n}(o \leftrightarrow \pi^{-1}(v_n))^{1/n} \) exactly. Therefore, we take a subgraph \( L_{n-1} \subset H_n \), where \( L_n \) is a segment of length \( n \) in \( T_d \) emanating from \( \partial T_d \).

Then we define a function \( \alpha : [0, 1] \to [0, 1] \) by

\[ \alpha(p) = \lim_{n \to \infty} \sup \mathbb{P}_p^{L_n \square Z}(o \leftrightarrow \pi^{-1}(v_n))^{1/n}. \]

Because \( L_{n-1} \subset H_n \), if \( \alpha(p) > 1/(d-1) \), then \( \limsup_{n \to \infty} \mathbb{P}_p^{H_n}(o \leftrightarrow \pi^{-1}(v_n))^{1/n} \geq \alpha(p) > 1/(d-1) \). Hence, we have the following lemma.

**Lemma 2.2.** Let \( p_0 = \inf \{ p \in [0, 1] \mid \alpha(p) > 1/(d-1) \} \), then we have \( p_c(T_d \square Z) \leq p_0 \).

### 3 Lower bound of \( \alpha(p) \)

We have defined a function \( \alpha(p) \) in (2.5) which is useful to give, as in Lemma 2.2, an upper bound for \( p_c(T_d \square Z) \). However, it is still difficult to handle. Thus, we shall prepare another lower bound which depends on both \( p \) and \( n \). Let \( L_\infty = Z_{\geq 0} \) be a ray. Let \( H \) be a subgraph of \( L_\infty \square Z \) defined by

\[ V(H) = \{(n, k) \mid n \in \{0, 1\}, k \in \mathbb{Z}\}, \]

\[ E(H) = \{\{(0, k), (0, k+1)\} \mid k \in \mathbb{Z}\} \]

\[ \cup \{\{(0, k), (1, k)\} \mid k \in \mathbb{Z}\}. \]
$L_\infty \Box Z$ is decomposed into infinitely many pieces each of which is isomorphic to $H$. We denote this decomposition as $L_\infty \Box Z = \cup_{i=1}^\infty H_i$ where $H_i$ is a copy of $H$. We set $G_n = \cup_{i=1}^n H_i$. Then we have

$$\alpha(p) = \limsup_{n \to \infty} P^{G_n}_p (o \leftrightarrow \pi^{-1}(n))^{1/n}.$$  

We will make a lower bound of $\alpha(p)$. We define the sequence of numbers $\{\alpha_p(n)\}$ by

$$\alpha_p(n) = \sum_{m_i \geq 0} \sum_{1 \leq i \leq n} \prod_{j=1}^n (m_j + 1) \left( \frac{m_j + l_{j-1}}{l_j} \right),$$

where $l_0 = 1$.

**Lemma 3.1.** For all $n \geq 1$, we have

\[(3.1) \quad P^{G_n}_\pi (o \leftrightarrow \pi^{-1}(n)) \geq \alpha_p(n).\]

**Proof.** When $n = 1$, $P^{H_1}_\pi (o \leftrightarrow \pi^{-1}(1))$ can be computed exactly as follows. First, we consider percolation on $\pi^{-1}(0)$, which is isomorphic to $Z$. Let $m$ be a nonnegative integer. Then we have

$$P^Z_m (|C| = m + 1) = (m + 1)p^m(1-p)^2.$$  

Second, we consider percolation on the remaining edges, that means on $\{(0,k), (1,k) \mid k \in Z\}$. We are now thinking on the case where the event $|C| = m + 1$ occurs. Thus, we only consider percolation on $\{(0,k), (1,k) \mid (0,k) \in C\}$. If at least one of the $m + 1$ edges is open, then $(o \leftrightarrow \pi^{-1}(1))$ occurs. So, we have

$$P^{H_1}_\pi (o \leftrightarrow \pi^{-1}(1)) = \sum_{m \geq 0} (m + 1)p^m(1-p)^2(1 - (1-p)^{m+1})$$

\[(3.2) \quad = \sum_{m \geq 0} (m + 1)p^m(1-p)^2 \sum_{l=1}^{m+1} \binom{m+1}{l} p^l(1-p)^{m+1-l}.$$  

Next, we would like to show general case. If $(o \leftrightarrow \pi^{-1}(1))$ occurs, then there exists a non-empty subset $A_1 \subseteq \pi^{-1}(1)$ such that $o \leftrightarrow v$ on $H_i$ for all $v \in A_1$. We would like to know the probability $P^{H_2}_\pi (A_1 \leftrightarrow \pi^{-1}(2))$ with $|A_1| = l_1$. It depends on a configuration of $A_1$, but we can obtain a lower bound which does not depend on a configuration of $A_1$. Since $\pi^{-1}(1)$ and $Z$ are isomorphic, we replace $\pi^{-1}(1)$ with $Z$. The case where $l_1 = 1$ is explain as above. So, we assume $l_1 \geq 2$. Let $A_1 = \{v_1, \ldots, v_{l_1}\}$ such that $v_1 < v_2 < \cdots < v_{l_1}$. First, we consider percolation on $Z \setminus [v_1, v_{l_1}]$, and all edges of $[v_1, v_{l_1}]$ are assumed to be closed. We divide computation into several cases according to the size of the cluster containing $A_1$. In other words, the cluster containing $A_1$ is $C = \bigcup_{i=2}^{l_1} \{v_i\} \cup C(v_1) \cup C(v_{l_1})$, where $C(v_i)$ is the intersection of the cluster containing $v_i$ and $Z \setminus [v_1, v_{l_1}]$. Then we have

$$P^Z_m (|C'| = m_2 + l_1) = (m_2 + 1)p^{m_2}(1-p)^2.$$  

Second, we consider percolation on the remaining edges. We are now thinking about the case where the event $(|C'| = m_2 + l_1)$ occurs. Thus, we only consider percolation on $\{(0,k), (1,k) \mid (0,k) \in C'\}$. If at least one of the $m_2 + l_1$ edges is open, then $(A_1 \leftrightarrow \pi^{-1}(2))$ occurs. So, we have

$$P^{H_1}_\pi (A_1 \leftrightarrow \pi^{-1}(2)) \geq \sum_{m_2 \geq 0} (m_2 + 1)p_2^{m_2}(1-p)^2(1 - (1-p)^{m_2 + l_1})$$

\[(3.3) \quad = \sum_{m_2 \geq 0} (m_2 + 1)p_2^{m_2}(1-p)^2 \sum_{l_2=1}^{m_2 + l_1} \binom{m_2 + l_1}{l_2} p_2^l(1-p)^{m_2 + l_1 - l_2}.$$
If \((o \leftrightarrow A_1)\) on \(H_1\) and \((A_1 \leftrightarrow \pi^{-1}(2))\) on \(H_2\) occur, then \((o \leftrightarrow \pi^{-1}(2))\) on \(G_2\) occur. Therefore, using \((3.2)\) and \((3.3)\), we have

\[
P^G_2(o \leftrightarrow \pi^{-1}(2)) \geq \sum_{m_j \geq 0} \sum_{l_i=1}^{m_j \leq 1} \sum_{1 \leq i \leq 2} p^{\sum_{j=1}^{m_j} (m_j + l_j)} (1-p)^{2+2n+\sum_{j=1}^{m_j} m_j} \left( \prod_{j=1}^{m_j} \left( m_j + \frac{l_j-1}{l_j} \right) \right),
\]

where \(l_0 = 1\). If \((A_1 \leftrightarrow \pi^{-1}(2))\) occurs, then there exists a non-empty subset \(A_2 \subset \pi^{-1}(2)\) such that \(A_1 \nleftrightarrow v\) on \(H_2\) for all \(v \in A_2\). Repeating this process, If there exists non-empty subset \(A_i \subset \pi^{-1}(i)\) and \((A_{i-1} \leftrightarrow A_i)\) on \(H_i\) occurs for \(1 \leq i \leq n\) where \(A_0 = o\), then \((o \leftrightarrow \pi^{-1}(n))\) on \(G_n\) occurs. It completes the proof.

By Lemma \(3.1\) we have

\[
\alpha(p) \geq \limsup_{n \to \infty} \alpha_p(n)^{1/n}. \tag{3.4}
\]

### 4 Generating function and radius of convergence

Since it is not quite easy to handle \(\alpha_p(n)\), we introduce another sequence of numbers which is easier to handle than \(\alpha_p(n)\). The sequence of numbers \(\beta_p(n)\) is defined by

\[
\beta_p(n) = \sum_{m_j \geq 0} \sum_{l_i=1}^{m_j \leq 1} \sum_{1 \leq i \leq n} p^{\sum_{j=1}^{m_j} (m_j + l_j)} (1-p)^{2+2n+\sum_{j=1}^{m_j} m_j} \left( \prod_{j=1}^{m_j} \left( m_j + \frac{l_j-1}{l_j} \right) \right),
\]

where \(l_0 = 1\). Since \(\alpha_p(n) \geq \beta_p(n)\) for all \(n\), we have \(\limsup_{n \to \infty} \alpha_p(n)^{1/n} \geq \limsup_{n \to \infty} \beta_p(n)^{1/n}\). We know \(\limsup_{n \to \infty} \beta_p(n)^{1/n}\) equals the inverse of the radius of convergence of the generating function \(F_p(z) = \sum_{l \geq 1} \beta_p(l) z^l\). Therefore, we focus on the function \(F_p(z)\). Since \(1 \geq \alpha(p) \geq \limsup_{n \to \infty} \beta_p(n)^{1/n}\), we have that \(F_p(z)\) is finite for all \(|z| < 1\). When \(p < p_c(Z^2) = 1/2\), we know that \(\limsup_{n \to \infty} \alpha_p(n) = 0\) holds. In the case of \(z = 1\), since

\[
\alpha_p(n) = \alpha_p(n-1) - \left( \frac{1-p}{1-p+p^2} \right)^2 \beta_p(n-1)
\]

\[
= \alpha_p(1) - \left( \frac{1-p}{1-p+p^2} \right)^2 \sum_{k=1}^{n-1} \beta_p(k),
\]

we have that

\[
F_p(1) = \left( \frac{1-p+p^2}{1-p} \right)^2 \alpha_p(1).
\]

Thus, we would like to consider whether \(F_p(z)\) converges or not in \(|z| \geq 1\), \(z \neq 1\). We set

\[
\Phi_p(l) = \prod_{i=1}^{l} \frac{1-p}{1-p+p^i+1},
\]

\[
H_p(z) = \sum_{l \geq 1} \Phi_p(l) z^l.
\]
It is easy to show that the radius of convergence of $H_p(z)$ equals 1. But we would like to consider $|z| \geq 1, z \neq 1$. Therefore, we consider an analytic continuation of $H_p(z)$. We have

$$\Phi_p(l)^2 = \left(1 - \frac{p + p^{(l+1)+1}}{1 - p}\right)^2 \Phi_p(l + 1)^2 = \left(1 + \frac{2p^2}{1 - p}p^l + \frac{p^4}{(1 - p)^2}p^{2l}\right) \Phi_p(l + 1)^2.$$  

Using this equation, $H_p(z)$ is deformed into the form

$$H_p(z) = \sum_{l \geq 1} \Phi_p(l + 1)^2 z^l + \frac{2p^2}{1 - p} \sum_{l \geq 1} \Phi_p(l + 1)^2 (pz)^l + \frac{p^4}{(1 - p)^2} \sum_{l \geq 1} \Phi_p(l + 1)^2 (p^2 z)^l$$

$$= \frac{1}{z} H_p(z) + \frac{2p}{1 - p} z H_p(pz) + \left(\frac{p}{1 - p}\right)^2 \frac{1}{z} H_p(p^2 z) - 1.$$

Then we have

$$(4.1) \quad H_p(z) = \frac{2p}{1 - p} H_p(pz) + \left(\frac{p}{1 - p}\right)^2 H_p(p^2 z) - \frac{1}{z} H_p(z) - 1.$$

Therefore, the right-hand side is the analytic continuation of $H_p(z)$ defined in $|z| < 1/p, z \neq 1$.

**Lemma 4.1.** If there exists $x_0 \in (1, 1/p) \subset \mathbb{R}$ such that $H_p(x_0) = -1$, then $x_0$ is a pole of $F_p(z)$.

Lemma 4.1 means that the radius of convergence of $F_p(z)$ is less than or equal to $x_0$. Then we have $\alpha(p) \geq 1/x_0$.

**Proof.** First, we consider the relationship between $\beta_p$ and $\Phi_p$. We will show the following equations.

$$(4.2) \quad \beta_p(1) = \frac{p}{1 - p} \Phi_p(2)^2 (1 - p^2(1 - p)(1 - p^2)),$$

$$(4.3) \quad \beta_p(n) = \frac{p}{(1 - p)^2} \Phi_p(n + 1)^2 (1 - p^n)(1 - p^2(1 - p)(1 - p^{n+1})) - \sum_{k=1}^{n-1} \Phi_p(n - k)^2 \beta_p(k).$$

We define $B_p(i) = p^{m_i + l_i - 1}(1 - p)^{2 + m_i}(m_i + 1)(m_i + l_i - 1)$, then we can express $\beta_p(n)$ in terms of $B_p(i)$ as

$$\beta_p(n) = (1 - p) \sum_{m_i \geq 0} \sum_{1 \leq i \leq n} \left( \prod_{j=1}^{n} B_p(j) \right).$$

We set $\sum_{k=0}^{l} p^k$. Then the following equation holds.

$$\sum_{m_i \geq 0} \sum_{l_i = 1}^{m_i + l_i - 1} B_p(i) S_p(t)^{l_i} = (1 - p)^2 \sum_{m_i \geq 0} (m_i + 1) p^{m_i}(1 - p)^{m_i} \sum_{l_i = 1}^{m_i + l_i - 1} (p S_p(t))^{l_i} \left( \binom{m_i + l_i - 1}{l_i} \right)$$

$$= (1 - p)^2 \sum_{m_i \geq 0} (m_i + 1) p^{m_i}(1 - p)^{m_i} \left( (1 + p S_p(t))^{m_i + l_i - 1} - 1 \right)$$

$$= \left( \frac{1 - p}{1 - p + p^2 + 3} \right)^2 S_p(t + 1)^{l_i - 1} = \left( \frac{1 - p}{1 - p + p^2} \right)^2 .$$

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If \( i = 1, t = 0 \), then we have

\[
\beta_p(1) = \left( \frac{1 - p}{1 - p + p^3} \right)^2 S_p(1)^1 - \left( \frac{1 - p}{1 - p + p^2} \right)^2
= \frac{p}{1 - p} \Phi_p(2)(1 - p^2(1 - p)(1 - p^2)).
\]

In the case where \( n \geq 2 \), we see

\[
\beta_p(n) = (1 - p) \sum_{m_i \geq 0} \sum_{l_i = 1}^{m_i + l_i - 1} \left( \prod_{j=1}^{n-1} B_p(j) \right) \sum_{m_n \geq 0} \sum_{l_n = 1}^{m_n + l_n - 1} B_p(n) S_p(0)^n
= (1 - p) \sum_{m_i \geq 0} \sum_{l_i = 1}^{m_i + l_i - 1} \left( \prod_{j=1}^{n-1} B_p(j) \right) \left( \left( \frac{1 - p}{1 - p + p^3} \right)^2 S_p(1)^{n-1} - \left( \frac{1 - p}{1 - p + p^2} \right)^2 \right)
= (1 - p) \left( \frac{1 - p}{1 - p + p^2} \right)^2 \sum_{m_i \geq 0} \sum_{l_i = 1}^{m_i + l_i - 1} \left( \prod_{j=1}^{n-2} B_p(j) \right) \sum_{m_{n-1} \geq 0} \sum_{l_{n-1} = 1}^{m_{n-1} + l_{n-1} - 2} B_p(n - 2) S_p(1)^{n-2}
= (1 - p) \left( \frac{1 - p}{1 - p + p^2} \right)^2 \beta_p(n - 1).
\]

Repeating this process, we have

\[
\beta_p(n) = (1 - p) \Phi_p(n)^2 \left( \frac{1 - p + p^2}{1 - p} \right)^2 \left( \left( \frac{1 - p}{1 - p + p(n+1)} \right)^2 S_p(n)^1 - \left( \frac{1 - p}{1 - p + p^2} \right)^2 \right)
- \sum_{k=1}^{n-1} \Phi_p(n - k) \beta_p(k)
= \frac{p}{(1 - p)^2} \Phi_p(n + 1)^2 \left( 1 - p^n \right) \left( 1 - p^2(1 - p)(1 - p^{n+1}) \right) - \sum_{k=1}^{n-1} \Phi_p(n - k)^2 \beta_p(k).
\]

By the equations \(4.2\) and \(4.3\), we obtain the following expression of \( F_p(z) \) in terms of \( \Phi_p(l) \).

\[
F_p(z) = \sum_{l \geq 2} \left( \frac{p}{(1 - p)^2} \Phi_p(l + 1) \left( 1 - p^l \right) \left( 1 - p^2(1 - p)(1 - p^{l+1}) \right) - \sum_{k=1}^{l-1} \Phi_p(l - k)^2 \beta_p(k) \right) z^l + \beta_p(1) z
= \frac{p}{(1 - p)^2} \left( 1 - p^2 + p^3 \right) \sum_{l \geq 1} \Phi_p(l + 1)^2 z^l - \frac{p}{(1 - p)^2} \left( 1 - p^2 + 2p^3 - p^4 \right) \sum_{l \geq 1} \Phi_p(l + 1)^2 (pz)^l
+ \frac{p}{(1 - p)^2} \left( p^3 - p^4 \right) \sum_{l \geq 1} \Phi_p(l + 1)^2 (p^2z)^l - \sum_{l \geq 2} \sum_{k=1}^{l-1} \Phi_p(l - k)^2 \beta_p(k) z^l.
\]

Each term in the above expression is rewritten as

\[
\sum_{l \geq 1} \Phi_p(l + 1)^2 z^l = \frac{1}{z} \sum_{l \geq 2} \Phi_p(l)^2 z^l = \frac{1}{z} (H_p(z) - \Phi(1)^2 z),
\]

\[
\sum_{l \geq 1} \Phi_p(l + 1)^2 (pz)^l = \frac{1}{pz} \sum_{l \geq 2} \Phi_p(l)^2 (pz)^l = \frac{1}{pz} (H_p(pz) - \Phi(1)^2 pz),
\]

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\[
\sum_{l \geq 1} \Phi_p(l+1)^2(p^2z)^l = \frac{1}{p^2z} \sum_{l \geq 2} \Phi_p(l)^2(p^2z)^l = \frac{1}{p^2z}(H_p(p^2z) - \Phi(1)^2p^2z),
\]
\[
\sum_{l \geq 2} \sum_{k=1}^{l-1} \Phi_p(l-k)^2\beta_p(k)z^l = \sum_{k \geq 1} \beta_p(k)z^l \sum_{l-k \geq 1} \Phi_p(l-k)^2z^{l-k} = F_p(z)H_p(z).
\]

Then we obtain
\[
(4.4) \quad (1 + H_p(z))F_p(z) = \frac{p}{(1-p)^2} \left( \frac{1-p^2(1-p)}{z}H_p(z) - \frac{1-p^2(1-p)}{pz}H_p(pz) - \frac{p^3(1-p)}{p^2z}H_p(p^2z) \right).
\]

By the equation (4.4), it is enough to show the right hand side of (4.4) is not equal to 0 when \(H_p(x_0) = -1\). By the equation (4.1), we have
\[
H_p(p^2x) = \left( \frac{1-p}{p} \right)^2 - 2\frac{1-p}{p}H_p(px).
\]

Using this equation, if the right hand side of (4.4) is equal to 0, then we have \(H_p(px_0) = -1\). This is contrary to \(H_p(px) \geq 0\) for all \(x \in [0, 1/p)\). \(\square\)

We set
\[
h_p(x) = 2H_p(px) + \frac{p}{1-p}H_p(p^2x).
\]

**Lemma 4.2.** Let \(p_0 = \inf \{p \in [0, 1] \mid \exists x \in (1, 1/p) \text{ s.t. } h_p(x) \geq (1-p)/p, x < (d-1)\} \). Then we have \(p_c(T_d \triangle Z) \leq p_0\).

**Proof.** For any \(p > p_0\), there exists \(x \in (1, 1/p)\) such that \(h_p(x) \geq (1-p)/p, x < (d-1)\). By the equation (4.1), \(H_p(x_0) = -1\) if and only if \(h_p(x_0) = (1-p)/p\). The function \(h_p(x)\) is continuous, increasing on \([0, 1/p)\), and \(h_p(0) = 0\). Therefore, we have \(\alpha(p) \geq 1/x_0 \geq 1/x > 1/(d-1)\). Using Lemma 2.2, we have \(p_c(T_d \triangle Z) \leq p_0\). \(\square\)

**5 Proof of Theorem A**

As a candidate of the real number \(x\) which appeared in Lemma 4.2, we consider \(x = (1-p)/p\). If \(p > 1/d\), then \(x < (d-1)\) holds. Therefore, we must only show \(h_p(x) \geq (1-p)/p\). Now we let \(d \geq 3\) and assume \(1/d < p < 0.34\). Let \(x_1, \ldots, x_l > 0\) be real numbers. By the relation between the harmonic mean and the geometric mean, we have
\[
(5.1) \quad \left( \prod_{k=1}^{l} x_k \right)^{\frac{1}{l}} \geq \left( \frac{1}{l} \sum_{k=1}^{l} \frac{1}{x_k} \right)^{-1}.
\]
Using this inequality, we have

\[ \Phi_p(l) = \prod_{k=1}^{l} \frac{1}{1 - p + p^{k+1}} \]

\[ \geq \left( \frac{l}{l + \left( \frac{p}{1 - p} \right)^2} \right)^l \]

\[ \geq \left( 1 - \frac{\left( \frac{p}{1 - p} \right)^2}{l} \right)^l \]

\[ \geq e^{-\left( \frac{p}{1 - p} \right)^2}. \]

From this inequality, we obtain

\[ h_p \left( \frac{1 - p}{p} \right) \geq 2H_p(1 - p) = 2 \sum_{l \geq 1} \Phi_p(l)^2(1 - p)^l \]

\[ \geq 2e^{-2\left( \frac{p}{1 - p} \right)^2} \sum_{l \geq 1} (1 - p)^l \]

\[ = 2e^{-2\left( \frac{p}{1 - p} \right)^2} \frac{1 - p}{p}. \]

Since \( p \) is assumed to satisfy \( p < 0.34 \), then we have

\[ 2e^{-2\left( \frac{p}{1 - p} \right)^2} \geq 1. \]

Hence \( h_p(x_0) \geq (1 - p)/p \) holds.

6 Proof of Theorem B

In Section 5 we have

\[ \Phi_p(l)^2 \geq e^{-2\left( \frac{p}{1 - p} \right)^2} \geq 1 - 2 \left( \frac{p}{1 - p} \right)^2. \]

Using this inequality, we have

\[ h_p(x) \geq 2 \left( 1 - 2 \left( \frac{p}{1 - p} \right)^2 \right) \sum_{l \geq 1} (px)^l + \frac{p}{1 - p} \left( 1 - 2 \left( \frac{p}{1 - p} \right)^2 \right) \sum_{l \geq 1} (p^2x)^l \]

\[ = 2 \left( 1 - 2 \left( \frac{p}{1 - p} \right)^2 \right) \frac{px}{1 - px} + \frac{p}{1 - p} \left( 1 - 2 \left( \frac{p}{1 - p} \right)^2 \right) \frac{p^2x}{1 - p^2x}. \]

In the case of \( d = 4 \), let \( p = 0.225 \), \( x = 2.999 \). Then we have \( h_p(x) \geq (1 - p)/p, x < d - 1 \). Therefore, \( p_c < 0.225 \) holds. Using the inequality (1.2), we have \( p_u > 0.232 \). Hence, \( p_c < p_u \) holds when \( d = 4 \).
References

[1] M. Aizenman, H. Kesten and C. M. Newman. *Uniqueness of the infinite cluster and continuity of connectivity functions for short and long range percolation*. Comm. Math. Phys. 111, no.4, 505-531 (1987).

[2] G. R. Grimmett. *Percolation, 2nd edition*. Springer-Verlag, Berlin (1999).

[3] G. R. Grimmett and C. M. Newman. *Percolation in $\infty + 1$ dimensions*. In disorder in physical systems, Oxford Sci. Publ. 167190 Oxford Univ. Press, New York (1990).

[4] H. Hutchcroft. *Non-uniqueness and mean-field criticality for percolation on nonunimodular transitive graphs*. arXiv preprint arXiv:1711.02590 (2017).

[5] H. Kesten. *The critical probability of bond percolation on the square lattice equals 1/2*, Comm. Math. Phys. 74, no. 1, 41-59 (1980).

[6] Russell Lyons and Yuval Peres. *Probability on trees and networks*. Cambridge University Press, New York (2016).

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