Abstract. We introduce a simple model of deterministic particles in weakly disordered media which exhibits a transition from normal to anomalous diffusion. The model consists of a set of non-interacting overdamped particles moving on a disordered potential. The disordered potential can be thought as a substrate having some ‘defects’ scattered along a one-dimensional line. The distance between two contiguous defects is assumed to have a heavy-tailed distribution with a given exponent $\alpha$, which means that the defects along the substrate are scarce if $\alpha$ is small. We prove that this system exhibits a transition from normal to anomalous diffusion when the distribution exponent $\alpha$ decreases, i.e. when the defects become scarcer. Thus we identify three distinct scenarios: a normal diffusive phase for $\alpha > 2$, a superdiffusive phase for $1/2 < \alpha \leq 2$, and a subdiffusive phase for $\alpha < 1/2$. We also prove that the particle current is finite for all the values of $\alpha$, which means that the transport is normal independently of the diffusion regime (normal, subdiffusive, or superdiffusive). We give analytical expressions for the effective diffusion coefficient for the normal diffusive phase and analytical expressions for the diffusion exponent in the case of anomalous diffusion. We test all these predictions by means of numerical simulations.

Keywords: diffusion in random media, exact results, diffusion, transport properties

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1. Introduction

In recent years there has been an increasing interest in a singular phenomenon which is enhancement of diffusion by weak disorder [1–5]. This phenomenon has been shown to occur in a system consisting of an ensemble of non-interacting overdamped particles moving on a weakly disordered periodic potential with a constant driving force in presence of Gaussian white noise [1]. Thereafter, it was found that the diffusion of particles in such kind of systems can become anomalous, both, superdiffusive and subdiffusive in a wide range of the parameter space [2]. On the other hand, in purely deterministic systems, i.e. in systems without noise fluctuations, the anomalous diffusion has also been found [6–9]. However, contrary to the systems with noise in which the anomalous phase is robust with respect to other parameters, in deterministic systems the anomalous phase emerge as a critical property [6–9]. This means that, for deterministic systems, one of the parameters should have a critical value (the driving constant force) in order for the system to exhibits the asymptotic anomalous behavior. These findings suggest that the origin of the anomalous diffusion in the models presented in [2] could be due, besides to the long-range correlation of the disordered potential [2], to the presence of noise. In this work we show that this is not necessarily the case. Indeed we introduce a simple model for deterministic diffusion which exhibits a transition from normal to anomalous diffusion as a function of a parameter. This model is similar to previously proposed models such as the ‘gas of scatters’ model studied in [6] or the Lévy–Lorentz model for light particles introduced in [10]. Particularly, we found that in our model the anomalous phase
Scarce defects induce anomalous diffusion

does not emerge as a critical property, which means that we do not require a fine-tuning of the parameters to obtain anomalous diffusion. Moreover, we show that the anomalous behavior is robust with respect to a driving constant force. These findings thus provide a different mechanism leading to anomalous dispersion of particles in disordered media.

The paper is organized as follows. In section 2 we state the model for the disordered potential. In appendix A we perform the calculations to obtain the diffusion coefficient when the normal diffusion occurs. In appendix B we obtain the diffusion exponent for the anomalous phase and we prove that the systems transits from anomalous superdiffusion to subdiffusion. In section 3 we test our findings with numerical simulations. Finally in section 4 we give the conclusions of our work.

2. Model

Let us consider an ensemble of non-interacting overdamped particles moving on a one-dimensional substrate. The equation of motion of each particle is given by

$$\gamma \frac{dx}{dt} = f(x) + F,$$

where $f(x)$ is minus the gradient of a potential $V(x)$ and $F$ is constant driving force. We assume that $V(x)$ is a weakly disordered potential in the sense that it consists of some ‘defects’ scattered along the substrate. In order to write an analytical expression for $V(x)$ let us introduce a function defined on a finite interval that will play the role of defect. Let $\varphi : [0, L] \rightarrow \mathbb{R}$ be a real-valued function to which we will refer to as the ‘potential defect’. Here $L \in \mathbb{R}^+$ stands for the width of the defect. Let $\{\ell_j \in \mathbb{R}^+\}_{j \in \mathbb{Z}}$ be a sequence of non-negative numbers defined as follows,

$$\ell_j = \begin{cases} \delta_{j/2} & \text{if } j \text{ is even} \\ L & \text{if } j \text{ is odd,} \end{cases}$$

where $\{\delta_j \in \mathbb{R}^+\}_{j \in \mathbb{Z}}$ is a set of independent and identically distributed (i.i.d.) random variables. Additionally let $L_n$ be defined as the partial sum of the $\ell_j$’s up to $n$ (we set $L_0 = 0$),

$$L_n = \begin{cases} \sum_{j=0}^{n} \ell_j & \text{if } n > 0 \\ -\sum_{j=1}^{\lvert n \rvert} \ell_j & \text{if } n < 0. \end{cases}$$

Then, in terms of the above-defined quantities, the disordered potential $V(x)$ is defined as follows

$$V(x) = \begin{cases} \varphi(x - L_{2n}) & \text{if } L_{2n} \leq x < L_{2n+1} \\ 0 & \text{if } L_{2n+1} \leq x < L_{2n+2}. \end{cases}$$
This potential can be thought as a constant potential, $V(x) = 0$, that has been ‘contaminated’ with some defects, which are modeled through the potential profile $\phi$. The distance between two consecutive defects is $\delta_j$, which is randomly chosen from a prescribed distribution, while the width of the defects is a constant $L$. In figure 1 we can appreciate an schematic representation of a realization of this potential.

Notice that the equation of motion (1) allows two types of solutions referred to as running and locked trajectories. The condition to have running solutions is that the driving constant force $F$ in equation (1) be larger than the critical value $F_c \triangleq \sup_2 \{ |f(x)| \} = \sup_2 \{ -\phi'(x) \}$. On the other hand, to have locked trajectories we require that $F \leq F_c$. In the following we will assume that the driving force $F$ is strictly above the critical value $F_c$, i.e. $F > F_c$, which means that every particle in the disordered potential moves always to the right and never gets stuck.

In the following we will assume that every random variable $\delta_j$ has a heavy tailed distribution. Particularly, we will chose a probability density function $\rho(x)$ given by,

$$\rho_\alpha(x) = \begin{cases} \alpha x^{-\alpha-1} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1. \end{cases}$$

Notice that the larger distance between defects the fewer defects are present in the substrate. For $\alpha$ small, the distance between two successive defects is typically larger than the distance between two contiguous defects when $\alpha$ is large. Thus, we can interpret $\alpha$ as a parameter controlling the ‘density’ of defects present in the substrate.

We are interested in the asymptotic behavior of typical trajectories $X_t$ for large $t$. For this purpose we will made use of generalized limit theorems [11–13] which have been shown to be useful in estimating the asymptotic behavior of typical trajectories in disordered media [5, 9, 14–17].

Notice that the equation of motion (1) can be solved analytically on every ‘piece’ of the potential. Indeed we can calculate the time that the particle spend in crossing every piece. First let us consider the time $\tau_D$ that the particle takes to go across the defect. This quantity is given by,

$$\tau_D = \int_0^L \frac{\gamma dx}{-\phi'(x) + F}.$$  \hfill (6)

On the other hand, the time $\tau_j$ that the particle takes to go from the $j$th defect to the $(j + 1)$th one is given by

$$\tau_j = \frac{\gamma}{F} \delta_j.$$  \hfill (7)

Clearly, the time $\tau_j$ is a random variable that depends linearly on the (random) distance between two defects.
Let us call ‘unit cell’ the piece of the potential which contains a defect followed by the flat potential between such a defect and the next one. Then the total time $T_n$ that the particle takes to cross throughout the first $n$ unit cells is given by

$$T_n = \sum_{j=0}^{n-1} \tau_j + n\tau_D = \frac{\gamma}{F} \sum_{j=0}^{n-1} \delta_j + n\tau_D. \quad (8)$$

Moreover, since the unit cells have a random length (given by $L + \delta_j$), the total displacement of the particle during a time $T_n$ is given by,

$$X_n = \sum_{j=0}^{n-1} \delta_j + nL. \quad (9)$$

The variables $T_n$ and $X_n$ define implicitly the (random) trajectory of a particle. However, we cannot extract directly from these expressions the behavior of the mean displacement and the mean square displacement as a function of time. First we need to perform an intermediate step. In order to have explicitly the dependence of $X_n$ in terms of the time $T_n$ we will use the classical limit theorems for sums of random variables. We should notice that both, $X_n$ and $T_n$ are expressed in terms of the sum of independent and identically distributed random variables,

$$S_n = \sum_{j=0}^{n-1} \delta_j. \quad (10)$$

As we stated above, the random variables $\{\delta_j\}_{j \in \mathbb{N}}$ have heavy-tailed distributions, and therefore, the asymptotic properties of $S_n$ for large $n$ strongly depends on the exponent $\alpha$ of the distribution $\rho_\alpha$. As we will see below, we have several scenarios depending on the values of $\alpha$.

### 3. Analytical results and numerical simulations

In appendices A and B we use the classical limit theorems for infinite sums of independent and identically distributed random variables [11–13, 17] to obtain the asymptotic properties of the transport process. First we show that for values of $\alpha$ belonging to the range $(2, \infty)$, both, the transport and the diffusion are normal. This means that the particle current and the effective diffusion coefficient,

$$J_{\text{eff}} := \lim_{t \to \infty} \frac{\mathbb{E}[X_t]}{t},$$

$$D_{\text{eff}} := \lim_{t \to \infty} \frac{\text{Var}(X_t)}{2t},$$

are well defined. Indeed, in appendix A we calculate closed expressions for these quantities, given by,

$$J_{\text{eff}} = \frac{L + \delta}{\tau_D + \frac{2\delta}{F}}, \quad (12)$$
Scarce defects induce anomalous diffusion

\[
D_{\text{eff}} = \left( \frac{1}{\sqrt{\tau_d + \frac{\gamma^2}{F}}} - \frac{\gamma((L + \delta))}{(\tau_d + \frac{\gamma^2}{F})^{3/2}} \right)^2 \frac{\sigma^2_\delta}{2},
\]  

where we defined \( \delta \) and \( \sigma_\delta \) as the expected value and standard deviation of \( \delta \), respectively, i.e.

\[
\delta := \mathbb{E}[\delta],
\]

\[
\sigma_\delta := \sqrt{\text{Var}(\delta)}.
\]

These quantities can be exactly computed for our model, giving,

\[
\delta := \frac{\alpha}{\alpha - 1},
\]

\[
\sigma^2_\delta := \frac{\alpha}{\alpha - 2} - \frac{\alpha^2}{(\alpha - 1)^2}.
\]

On the other hand, for values of \( \alpha \) belonging to the interval (0,2], we found different behaviors for the asymptotic properties of the transport. First, as we prove in appendix B, the transport remains normal in the sense that the mean displacement grows linearly in time. This means that the particle current is well defined. Interestingly, we show that the particle current of the system undergoes a kind of ‘second-order-like’ phase transition. This is because \( J_{\text{eff}} \) as a function of \( \alpha \) is a continuous and piece-wise differentiable function. Actually, the particle current can be written in a closed form giving,

\[
J_{\text{eff}} = \begin{cases} 
F/\gamma & \text{if } 0 < \alpha \leq 1 \\
L + \delta/\tau_d + \frac{\gamma^2}{F} & \text{if } \alpha > 1.
\end{cases}
\]

Finally, the square displacement of an ensemble of particles on the disordered potential no longer grows linearly in time. This fact implies that the diffusion is not normal. In appendix B we also prove that the square displacement, \( (X_t - \mathbb{E}[X_t])^2 \), grows in time as follows,

\[
(X_t - \mathbb{E}[X_t])^2 \sim \begin{cases} 
 t^{2\alpha} & \text{if } 0 < \alpha < 1 \\
 t^{2/\ln^2(t)} & \text{if } \alpha = 1 \\
 t^{2/\alpha} & \text{if } 1 < \alpha < 2 \\
 t \ln(t) & \text{if } \alpha = 2 \\
 t & \text{if } \alpha > 2.
\end{cases}
\]

The above result means that for \( 0 < \alpha < 1/2 \) the system exhibits anomalous subdiffusion, which means that the particles ‘spread’ slower in comparison to the case of normal diffusion. In other words, in the case of subdiffusion, the square displacement of particles, \( (X_t - \mathbb{E}[X_t])^2 \), grows in time as \( t^\beta \), with \( \beta := 2\alpha < 1 \). On the other hand, we can
also appreciate that, for $1/2 < \alpha < 2$, the system exhibits a superdiffusive behavior. This regime is characterized by the fact that the square displacement grows faster than linear in time, i.e. $(X_t - \mathbb{E}[X_t])^2 \propto t^\beta$, with $\beta > 1$.

In order to test the theoretical results we perform numerical simulations of the model. First we numerically solve the equation of motion given in equation (1) for an ensemble of non-interacting particles. Different particles are placed in different realizations of the random potential described by equations (4) and (5). Thus, once we have obtained the time series for the particle position, we compute the mean displacement and the square displacement by averaging over all the time series obtained. This actually corresponds to average over the ensemble of random potentials. For the sake of simplicity we model the defects of the random potential by means of symmetric peaks with constant height and width. Thus the potential profile modeling the defects is defined as

$$
\varphi(x) = \begin{cases} 
2ax/L & \text{if } 0 < x < \frac{L}{2} \\
2a(L - x)/L & \text{if } \frac{L}{2} < x < L,
\end{cases}
$$

where $a$ and $L$ stand for the height and width of the ‘potential peak’. The corresponding random force field is given by (see figure 2),

$$
\varphi'(x) = \begin{cases} 
-2a/L & \text{if } 0 < x < \frac{L}{2} \\
2a/L & \text{if } \frac{L}{2} < x < L.
\end{cases}
$$

In the numerical simulations we fixed the parameter values $L = 1$ and $a = 1/2$. We have also taken the driving force $F$ above the critical one, which, according to our random potential model, is given by

$$
F_c = \max_x |\varphi'(x)| = 2aL = 1.
$$

This choice for the driving force ensures the absence of locked trajectories.

In figure 3 we show the particle current as a function of the parameter $\alpha$ obtained by using the exact formula (18). We plot the particle current for two different values of the driving force, namely $F = 3$ and $F = 4$ (solid lines). We also plot the particle current obtained from numerical simulation for the same values of the parameter, i.e. for $F = 3$ and $F = 4$ (filled circles). The numerical simulations were performed as follows. We solved the equation of motion, equation (1), for 500 particles placed on random potentials during a time of $10^5$ arb. units. According to our simulation scheme, different particles move on different random potentials. After that, we obtained the mean displacement by averaging over the ensemble of trajectories obtained from the simulations.

It is important to emphasize that the system undergoes a kind of second order ‘phase transition’ in the sense that the particle current curve changes continuously with the parameter $\alpha$, but its derivative does not. Explicitly we observe that the particle current remains constant in the range $0 < \alpha \leq 1$, which seems to be a counterintuitive
Scarce defects induce anomalous diffusion.

This phenomenon arises because the presence of the defects has the effect of delaying the particles. Thus we would expect that if \( \alpha \) increases, then the particle current diminishes. As \( \alpha \) increases beyond the critical value \( \alpha = 1 \), the particle current starts decreasing due to the presence of the defects as expected from the above reasoning.

In figure 4 we show the behavior of \( D_{\text{eff}} \) as a function of \( \alpha \) for two different values of the driving force. We use the the exact result given in equation (13). It is clear that this expression is only valid for \( \alpha > 2 \) because in this case the diffusion is normal. We plot the theoretical curves for the cases \( F = 3 \) (solid line) and \( F = 4 \) (dashed line), which are compared with the corresponding coefficients obtained from numerical simulations.

Figure 2. Schematic representation of the random potential and the corresponding random force field. The height and width of the potential peak are \( a \) and \( L \) respectively. The distance between \( j \)th and the \((j+1)\)th peaks is a random variable \( \delta_j \) whose distribution is given in equation (5).

Figure 3. Particle current as a function of \( \alpha \). (a) We plot the particle current, by using the exact result given in equation (18), for \( F = 3 \) (solid line). We also show the particle current obtained from the numerical simulation of 500 particles during a time of \( 10^5 \) arbitrary units, for \( F = 3 \) (filled circles). (b) As in (a) but using \( F = 4 \). This graph allows us to appreciate the abrupt change in the behavior of \( J_{\text{eff}} \) as \( \alpha \) decreases.
The simulations were performed by numerically solving the equation of motion (1) for 500 particles placed on random potentials during a time of $10^5$ arb. units.

In figures 5(a) and (b) we can appreciate the behavior of the particle current $J_{\text{eff}}$ versus the driving force $F$. We plot the theoretical prediction given in equation (12) for two different values of $\alpha$: for $\alpha = 3$ and $\alpha = 4$ (solid lines). We also plot the particle current obtained by means of numerical simulations for the same cases: for $\alpha = 3$ (filled circles) and $\alpha = 4$ (open circles) for several values of $\alpha$. We used the parameter values $\alpha = 3$ (filled circles) and $\alpha = 4$ (open circles) to compare against the theoretical counterpart, given a good agreement within the accuracy of our numerical experiments.
and $\alpha = 4$ (filled circles). To estimate the particle current (and the effective diffusion coefficient) we simulated 500 particles under the dynamics defined in equation (1), each particle placed on a different realization of the random potential. The total simulation time was $10^5$ arb. units. Then, we obtained the particle current and the diffusion coefficient estimating the mean position of the particles and its variance by averaging over the 500 trajectories obtained from the simulations. In figure 5(c) we show the effective diffusion coefficient $D_{\text{eff}}$ as a function of the driving force $F$. We plot the theoretical prediction for $D_{\text{eff}}$ given in equation (13) for $\alpha = 3$ (solid line) and $\alpha = 4$ (dashed line). We also display the diffusion coefficient obtained from the numerical simulations described above for the estimation of the particle current. In all cases we observe good agreement with the theoretical predictions within the accuracy of our numerical simulations.

In figure 6 we observe the behavior of the diffusion exponent $\beta$ versus $\alpha$. We plot the theoretical prediction for $\beta$ (solid line), given through equation 19, versus $\alpha$, which is independent of the driving force $F$. We also show the diffusion exponent versus $\alpha$ obtained from numerical simulations for $F = 1.2$ (open circles) and $F = 1.5$ (filled circles). It is interesting to note that in the range $0 < \alpha < 1$ we observe that the diffusion exponent fits better to the theoretical curve than in the range $1 < \alpha < 2$. This phenomenon is actually a manifestation of the nature of the random variable resulting from the limit theorems. As we see from equations (B.11) and (B.21) the square displacement $\Delta X_t^2 := (X_t - \mathbb{E}[X_t])^2$ behaves as

$$\Delta X_t^2 \approx \left( \frac{t^\alpha}{(\gamma F)^\alpha} \right)^2 (W^{-\alpha} - \mathbb{E}[W^{-\alpha}])^2, \text{ for } 0 < \alpha < 1,$$

$$\Delta X_t^2 \approx \left( \frac{2\gamma(L + \delta \tau_c)}{\tau_c^{1+1/\alpha} + \frac{1}{\tau_c^{1/\alpha}}} \right)^2 t^{2\gamma \alpha} W^2, \text{ for } 1 < \alpha < 2.$$

The main difference between these expressions for $\Delta X_t^2$ is that the corresponding mean value $\mathbb{E}[\Delta X_t^2]$ is finite for $0 < \alpha < 1$, but it does not exists in the range $1 < \alpha < 2$. Indeed, we have that the square displacement $\Delta X_t^2$ seen as a random variable for fixed $t$ has a distribution having a heavy tail for $1 < \alpha < 2$ whose mean value does not exists. Actually, the properties of the distribution of the square displacement are given by the random variable $W^2$, where $W$ has a $\alpha$-stable distribution. These facts imply that every realization of $\Delta X_t^2$ have large fluctuations impeding the convergence of the estimator of the mean value $\mathbb{E}[\Delta X_t^2]$. This phenomenon is of course absent in the range $0 < \alpha < 1$ since the random variable $(W^{-\alpha} - \mathbb{E}[W^{-\alpha}])^2$ has a distribution with a well-defined mean value.

It is also important to stress the fact that there is a transition from anomalous subdiffusion to superdiffusion at $\alpha = 1/2$. The occurrence of this transition can be seen as the result of a competition between two phenomena. First we should notice that a defect makes the particles slower than a particle that do not cross throughout a defect. This implies that the presence of defects along the substrate increases the dispersion of particles. On the other hand, when the particles move along a region in which there are no defects, the distance between them is preserved, thus lowering the dispersion of
particles and, consequently, decreasing the square displacement in time. According to these observations (which are valid only when the defects are very scarce) we can see that, if the density of defects diminishes the square displacement will also decrease. Thus, recalling that the ‘density’ of defects diminishes as $\alpha$ decreases, we observe that the diffusivity decreases dramatically with $\alpha$ in the region $0 < \alpha < 1$ because the system transits from super diffusion to subdiffusion.

4. Conclusions

We have introduced a simple model for deterministic diffusion which exhibit a transition from normal to anomalous diffusion. The model consists of an ensemble of non-interacting overdamped particles on a random potential under the influence of a constant driving force. The random potential can be seen as a one-dimensional medium with scarce defects which are responsible of ‘delaying’ the particles. We have shown that this model is able to exhibit anomalous diffusion if the distance between defects has a heavy tailed distribution with the distribution exponent $\alpha < 2$. The system also exhibits normal diffusion when the distribution exponent $\alpha > 2$. In the anomalous diffusive phase we observed both superdiffusion (for $1/2 < \alpha < 2$) and subdiffusion (for $0 < \alpha < 1/2$). Moreover, we proved that the transport is normal (which means that the particle current is well-defined) for all the values of $\alpha$. However, we observed that the particle current versus $\alpha$ exhibits a second-order-like ‘phase transition’. Explicitly, we showed that the particle current is continuous and piecewise smooth: it is a strictly decreasing function of $\alpha$ for $\alpha > 1$ and a constant function in the range $0 < \alpha < 1$. Particularly the
fact that the particle current remains constant in the interval $0 < \alpha < 1$ seems to be a counterintuitive phenomenon. This is because we intuitively expect that the more defects in the medium the lower particle current we have. This is not the case for $\alpha < 1$ because the particle current remains constant independently of the ‘quantity’ of defects (or, equivalently, the value of $\alpha$). Finally, another phenomenon that it is worth mentioning is the fact that the square displacement $\Delta X_t^2$, seen as a random variable, has a heavy tailed distribution in the range $1 < \alpha < 2$. Such a distribution is such that the mean value $E[\Delta X_t^2]$ does not exists. This implies that the average of realizations (through numerical experiments) of $\Delta X_t^2$ does not converge. On the contrary, in the interval $0 < \alpha < 1$ the mean value $E[\Delta X_t^2]$ is finite, and by this reason, the numerical simulations fits better to the theoretical prediction for diffusion exponent.

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Appendix A. Normal diffusion

We should remind that the distribution of $\delta_j$ has first and second moments finite if $\alpha > 2$. Since all the random variables $\{\delta_j\}_{j \in \mathbb{Z}}$ are assumed to be independent, we can apply the central limit theorem [11, 12]. Then, for sufficiently large $n$, this theorem implies the sum of i.i.d. random variables $\sum_{j=0}^{n-1} \delta_j$ can be approximated by a normal random variable [9, 11, 12],

$$\sum_{j=0}^{n-1} \delta_j \approx n\bar{\delta} + \sqrt{n}\sigma_{\delta}Z,$$

where we defined $\bar{\delta}$ and $\sigma_{\delta}$ as the expected value and standard deviation of $\delta_j$ respectively,

$$\bar{\delta} := E[\delta],$$

$$\sigma_{\delta} := \sqrt{\text{Var}(\delta)},$$

which in our case are explicitly given by

$$\bar{\delta} := \frac{\alpha}{\alpha - 1},$$

$$\sigma_{\delta}^2 := \frac{\alpha}{\alpha - 2} - \frac{\alpha^2}{(\alpha - 1)^2}.$$  

Within this approximation we can rewrite the time $T_n$ and the displacement $X_n$, for asymptotically large $n$, as

$$T_n \approx n\tau_D + n\frac{\gamma\bar{\delta}}{F} + \frac{\gamma\sigma_{\delta}}{F} \sqrt{n} Z,$$

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X_n \approx nL + n\delta + \sqrt{n}\sigma_{\delta}Z. \quad (A.7)

Calling \(N_t\) the number of unit cells that the particle has crossed during a time \(t\), we can define implicitly \(N_t\) by the equation \(T_{N_t} = t\), as it has been done in [9]. From equation (A.6) we can observe that the random variable \(N_t\) is related to the random variable \(Z\) as follows,

\[
\frac{t - N_t(\tau_D + \gamma_{D}/F)}{\sqrt{\frac{\tau_D}{F^2}}\sigma_{\delta}N_t} \approx Z. \quad (A.8)
\]

The most probable values for \(Z\) are around zero, which implies that the distribution of \(N_t\) should be centered around the root of a function,

\[
\psi(N_t) := \frac{t - N_t(\tau_D + \gamma_{D}/F)}{\sqrt{\frac{\tau_D}{F^2}}\sigma_{\delta}N_t}.
\]

We can see that \(t/(\tau_D + \gamma_{D}/F)\) is the unique root of \(\psi(N_t)\). To find an expression of \(N_t\) in terms of \(Z\) we proceed to make a linear expansion of \(\psi(N_t)\) around its root. We have

\[
\psi(N_t) \approx -\left(\frac{\tau_D + \gamma_{D}/F}{\gamma_{D}/F}\right)^{3/2} \left(N_t - \frac{t}{\tau_D + \gamma_{D}/F}\right) + O(t^{-3/2}), \quad (A.9)
\]

which leads to

\[
-\left(\frac{\tau_D + \gamma_{D}/F}{\gamma_{D}/F}\right)^{3/2} \left(N_t - \frac{t}{\tau_D + \gamma_{D}/F}\right) \approx Z. \quad (A.10)
\]

Solving for \(N_t\) we obtain

\[
N_t \approx \frac{t}{\tau_D + \gamma_{D}/F} - \frac{\gamma_{D}/F}{(\tau_D + \gamma_{D}/F)^{3/2}} \sqrt{L\tau_D + \gamma_{D}/F} Z. \quad (A.11)
\]

Now we can substitute equation (A.11) into equation (A.7) to find the expected value and the variance for the particle position \(X_t\). This gives

\[
\mathbb{E}[X_t] = \frac{L + \delta}{\tau_D + \gamma_{D}/F} t, \quad (A.12)
\]

\[
\text{Var}(X_t) = \left(\frac{1}{\sqrt{\tau_D + \gamma_{D}/F}} - \frac{\gamma_{D}/F}{(\tau_D + \gamma_{D}/F)^{3/2}}\right)^2 \sigma_{\delta}^2 t. \quad (A.13)
\]
Recalling the usual definitions of the particle current, $J_{\text{eff}}$, and the effective diffusion coefficient, $D_{\text{eff}}$, we obtain the following expressions,

$$J_{\text{eff}} = \lim_{t \to \infty} \frac{\mathbb{E}[X_t]}{t} = \frac{L + \delta}{\tau_D + \gamma \delta / F},$$

$$D_{\text{eff}} = \lim_{t \to \infty} \frac{\text{Var}(X_t)}{2t} = \left( \frac{1}{\sqrt{\tau_D + \gamma \delta / F}} - \frac{\gamma \delta (L + \delta)}{(\tau_D + \gamma \delta / F)^{3/2}} \right)^2 \frac{\sigma_\delta^2}{2}$$

(A.14) (A.15)

Appendix B. Anomalous diffusion

B.1. The case $1 < \alpha < 2$

If the exponent $\alpha$ lies in the interval $1 < \alpha < 2$, the variance of $\delta_j$ is no longer finite. Conversely, the mean value $\mathbb{E}[\delta_j]$ remains finite for $1 < \alpha < 2$. In that case, the sum $\sum_{j=0}^{n-1} \delta_j$ tends to a stable law if it is normalized according to [11, 12],

$$\frac{\sum_{j=0}^{n-1} \delta_j - n \bar{\delta}}{n^{1/\alpha}} \to W,$$

(B.1)

where $W$ is a $\alpha$-stable random variable. This means that we can approximate the sum $\sum_{j=0}^{n-1} \delta_j$ by

$$\sum_{j=0}^{n-1} \delta_j \approx n \bar{\delta} + n^{1/\alpha} W.$$  

(B.2)

Now, we proceed to define the random variable $N_t$ by using the relationship $T_{N_t} = t$. Recalling the definition for $T_n$ given in equation (8) we obtain that $N_t$ satisfies the equation,

$$\frac{\gamma}{F} (N_t \bar{\delta} + N_t^{1/\alpha} W) + N_t \gamma \delta / \tau_D \approx t.$$  

(B.3)

The above equation implicitly defines a transformation from the random variable $W$ to $N_t$. Notice that the above equation can be rewritten as

$$\psi(N_t) := \frac{t - \tau_e N_t}{\gamma / F N_t^{1/\alpha}} \approx W,$$

(B.4)

where we have denoted by $\tau_e$ the quantity

$$\tau_e := \frac{\gamma \delta}{F} + \tau_D.$$  

(B.5)

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Equation (B.4) establishes the transformation from $W$ to $N_t$, which is mediated by the inverse function $\psi^{-1}$. As it has been shown in [9] we have that the asymptotic behavior of $N_t$ for $t \to \infty$ is given by,

$$
N_t \approx 1 - t + \frac{\gamma}{F} \frac{t^{1/\alpha}}{\tau_c^{1+1/\alpha}} W. \tag{B.6}
$$

In order to have an expression for the displacement of the particle as a function of $t$, we will use the approximation given in equation (B.2) to obtain an approximation of $X_n$ for $n \to \infty$. Recalling that $X_n$ is given by

$$
X_n = \sum_{j=0}^{n-1} \delta_j + nL,
$$

we can observe that

$$
X_n \approx n\delta + n^{1/\alpha}W + nL. \tag{B.7}
$$

Next, if we substitute $n = N_t$, given in equation (B.6) into the above expression for $X_n$ we obtain,

$$
X_t \approx (L + \bar{\delta})N_t + N_t^{1/\alpha}W,
$$

$$
\approx (L + \bar{\delta}) \left( \frac{t}{\tau_c} + \frac{\gamma}{F} \frac{t^{1/\alpha}}{\tau_c^{1+1/\alpha}} W \right) + \left( \frac{t}{\tau_c} + \frac{\gamma}{F} \frac{t^{1/\alpha}}{\tau_c^{1+1/\alpha}} W \right)^{1/\alpha} W. \tag{B.8}
$$

Retaining the leading terms in the above expression we have that,

$$
X_t \approx \left( \frac{L + \bar{\delta}}{\tau_c} \right) t + \left( \frac{\gamma}{F} \frac{(L + \bar{\delta})}{\tau_c^{1+1/\alpha}} + \frac{1}{\tau_c^{1/\alpha}} \right) t^{1/\alpha} W + O(t^{2/\alpha-1}). \tag{B.9}
$$

With the above result we can see that the mean displacement of an ensemble of particles is given by

$$
E[X_t] \approx \frac{L + \bar{\delta}}{F + \tau_D} t, \quad \text{for } t \to \infty. \tag{B.10}
$$

On the other hand, the square fluctuations of $X_t$ grow as

$$
(X_t - E[X_t])^2 \approx \left( \frac{\gamma}{F} \frac{(L + \bar{\delta})}{\tau_c^{1+1/\alpha}} + \frac{1}{\tau_c^{1/\alpha}} \right)^2 t^{2/\alpha} W^2, \tag{B.11}
$$

which means that the diffusion exponent $\beta$ is given by,

$$
\beta = \frac{2}{\alpha}. \tag{B.12}
$$

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for $1 < \alpha < 2$. In this case we see that the system undergoes a transition from normal to anomalous superdiffusion when the parameter $\alpha$ diminishes. In this anomalous phase, the mean displacement is still finite, and therefore the particle current can be written as

$$J_{\text{eff}} = \frac{L + \delta}{\tau_c}. \quad \text{(B.13)}$$

**B.2. The case $0 < \alpha < 1$**

Now we will explore the case in which the parameter $\alpha$ is in the range $0 < \alpha < 1$. In this case we have that the mean value of $\delta_j$ diverge, which means that the approximation for the sum of $\delta_j$ given in equation (B.2) cannot be applied. However the sum $\sum_{j=0}^{n-1} \delta_j$ still converge to a stable law if it is normalized appropriately. Explicitly we have that $\sum_{j=0}^{n-1} \delta_j/n^{1/\alpha} \to W$, \quad \text{(B.14)}

where $W$ has an $\alpha$-stable distribution [12]. In this case we can approximate the sum of random variables by

$$\sum_{j=0}^{n-1} \delta_j \approx n^{1/\alpha} W, \quad \text{(B.15)}$$

allowing us to obtain an asymptotic expression for $N_t$ by using the relationship $T_{\kappa_t} = t$,

$$\frac{\tau}{F} N_t^{1/\alpha} W + N_t \gamma_D \approx t. \quad \text{(B.16)}$$

We proceed as in the above cases, i.e. we will obtain an asymptotic expression for $N_t$ for $t \to \infty$. For such purpose we first write the above equation as follows

$$N_t \approx \frac{t^\alpha}{(\frac{\tau}{F} W)^\alpha} \left( 1 - \frac{\gamma_D N_t}{t} \right) \left( 1 - \frac{\gamma_D N_t}{t} \right)^\alpha,$$

and then we use this expression recursively in order to obtain an asymptotic expression for $t \to \infty$. We obtain

$$N_t \approx \frac{t^\alpha}{(\frac{\tau}{F} W)^\alpha} + O(t^{2\alpha-1}), \quad \text{for } t \to \infty. \quad \text{(B.17)}$$

Now, in order to see how $X_t$ behaves in time, we use the approximation (B.15) to obtain an asymptotic expression for $X_n$ for large $n$,

$$X_n = \sum_{j=0}^{n-1} \delta_j + nL \approx n^{1/\alpha} W + nL. \quad \text{(B.18)}$$
Thus, if we substitute $n$ by $N_t$ into the above equation we obtain

$$X_t \approx \left( \frac{t^\alpha}{\langle \frac{F}{W} \rangle^\alpha} \right)^{1/\alpha} W + \frac{Lt^\alpha}{\langle \frac{F}{W} \rangle^\alpha}$$

(1)

This result for $X_t$ allows us to obtain an expression for the particle current. First notice that the expected value of $X_t$ is given by,

$$E[X_t] = \frac{F}{\gamma} t + \frac{t^\alpha}{\langle \frac{F}{W} \rangle^\alpha} E[W^{-\alpha}],$$

from which, after noticing that $E[W^{-\alpha}]$ is finite and recalling that $0 < \alpha < 1$, we obtain,

$$J_{\text{eff}} = \lim_{t \to \infty} \frac{E[X_t]}{t} = \frac{F}{\gamma}. \tag{20}$$

The expression for $X_t$ that we obtained in equation (1) also allows us to calculate the asymptotic behavior of the mean square displacement,

$$\text{Var}(X_t) := \mathbb{E}[(X_t - \mathbb{E}[X_t])^2] = \left( \frac{t^\alpha}{\langle \frac{F}{W} \rangle^\alpha} \right)^2 \mathbb{E}[(W^{-\alpha} - \mathbb{E}[W^{-\alpha}])^2]. \tag{21}$$

The last result states that the mean square displacement of the particle distribution grows in time as $t^{2\alpha}$, giving a diffusion exponent

$$\beta = 2\alpha. \tag{22}$$

Equation (22) establishes that for $0 < \alpha < 1$ the system exhibits two different behaviors, namely, an anomalous superdiffusive phase for $1/2 < \alpha < 1$, and an anomalous subdiffusive phase for $0 < \alpha < 1/2$. Then, a transition from superdiffusion to subdiffusion occurs at the critical value $\alpha = 1/2$.

### B.3. The marginal cases $\alpha = 2$ and $\alpha = 1$

In this section we will explore the transport properties for the special values $\alpha = 2$ and $\alpha = 1$. We should emphasize that in these cases the asymptotic behavior of $S_n$ for large $n$ is rather different from in the above cases. First let us consider the marginal value $\alpha = 2$. In this case the variance of $\delta_j$ diverge. However, according to [12] we have that $\sum_{j=0}^{n-1} \delta_j$ still converges to a normal distribution if it is appropriately normalized. Indeed we have that [11, 12]
Scarcely defects induce anomalous diffusion

\[ \frac{\sum_{j=0}^{n-1} \delta_j - n\bar{\delta}}{\sqrt{n \ln n} \rightarrow W, \quad \text{for } n \to \infty,} \]

where \( W \) is a normal random variable. The above means that if \( n \) is large enough we can approximate the sum \( \sum_{j=0}^{n-1} \delta_j \) as follows,

\[ \sum_{j=0}^{n-1} \delta_j \approx n\bar{\delta} + \sqrt{n \ln n} \ W. \]  

Using the equation \( T_t = t \) to approximate \( N_t \) for large \( t \), we obtain that \( N_t \) satisfies the equation,

\[ \psi(N_t) := \frac{t - \tau_c N_t}{(N_t \ln(N_t))^{1/2}} \approx \frac{\gamma}{F} W. \]  

As we argued in preceding sections, the random variable \( \frac{\gamma}{F} W \) has zero mean value, which implies that the most probable values of \( N_t \) are around the (unique) root of \( \psi(N_t) \). If we expand \( \psi \) around \( N_t = t/\tau_c \) we obtain,

\[ \psi(N_t) \approx -\frac{\tau_c}{\left[ \frac{\tau_c}{\tau_c \ln\left(\frac{t}{\tau_c}\right)}\right]^{1/2}} \left( N_t - \frac{t}{\tau_c} \right) \]  

which allows us to write \( N_t \) in terms of \( W \) by means of equation (B.24),

\[ N_t \approx \frac{t}{\tau_c} - \frac{1}{\tau_c} \left[ \frac{t}{\tau_c} \ln\left(\frac{t}{\tau_c}\right)\right]^{1/2} W. \]  

Now, we use again the approximation for the sum given in (B.23) to obtain an asymptotic expression for \( X_n \). This gives,

\[ X_n \approx n\bar{\delta} + \sqrt{n \ln n} \ W + nL. \]  

Next we substitute \( n \) by \( N_t \) in the above expression, resulting in an expression for the displacement \( X_t \) given by,

\[ X_n \approx \left( \frac{L + \delta}{\tau_c} \right) t + \left( \frac{F}{\gamma} - \frac{L + \delta}{\tau_c} \right) \left[ \frac{t}{\tau_c} \ln\left(\frac{t}{\tau_c}\right)\right]^{1/2} W. \]

Thus, the last expression implies that mean displacement of the trajectory grows linearly in time. This means that the particle current is well defined and has the value

\[ J_{\text{eff}} = \frac{L + \delta}{\tau_c}. \]  

On the other hand, the mean square fluctuations of the trajectory can also be calculated, giving,

\[ \text{https://doi.org/10.1088/1742-5468/aa6505} \]
Scarce defects induce anomalous diffusion

\[ \text{Var}(X_t) = \left( \frac{F}{\gamma} + \frac{L + \delta}{\tau_c} \right)^2 \left( \frac{t}{\tau_c} \right) \ln \left( \frac{t}{\tau_c} \right) \mathbb{E}[W^2]. \] \hspace{1cm} (B.29)

The above result states that the mean square fluctuations do not grow linearly in time nor as a power law, but it still grows faster than linear by the presence of the logarithm term \( \ln(t/\tau_c) \). This kind of behavior of the mean square displacement is commonly called marginal superdiffusion.

In the case \( \alpha = 1 \), the sum \( S_n \) converge to a stable law, but the renormalizing factor is not of the form \( n^{1/\alpha} \). Actually we have that \[ 12 \],

\[ \sum_{j=0}^{n-1} \delta_j \to W, \quad \text{for } n \to \infty, \] \hspace{1cm} (B.30)

where \( W \) is a random variable with a \( \alpha \)-stable distribution with \( \alpha = 1 \). In this case we approximate the sum \( S_n \) by

\[ \sum_{j=0}^{n-1} \delta_j \approx n \ln(n) W, \]

which allows us to estimate the asymptotic behavior of \( N_t \) by the equation,

\[ \frac{\gamma}{F} N_t \ln(N_t) W + N_t \tau_D \approx t. \] \hspace{1cm} (B.31)

Following a similar reasoning as in preceding sections we use the above relation to obtain an asymptotic expression for \( N_t \). Indeed, if we notice that the leading term in equation (B.31) is \( N_t \ln(N_t) \) we obtain,

\[ N_t \approx \frac{t \left( \frac{\gamma}{F} W \right)}{\ln \left( t \left( \frac{\gamma}{F} W \right) \right)}. \] \hspace{1cm} (B.32)

Next, if we substitute \( n \) by \( N_t \) into the expression for \( X_n \) we can observe that,

\[ X_t \approx \frac{F}{\gamma} t + \left( L - \frac{F \tau_D}{\gamma} \right) \ln \left( t \left( \frac{\gamma}{F} W \right) \right). \] \hspace{1cm} (B.33)

The last result means that the mean displacement exists and grows linearly in time, which gives for the particle current,

\[ J_{\text{eff}} = \frac{F}{\gamma}. \]

Additionally we obtain this case that the square displacement grows as

\[ (X_t - \mathbb{E}[X_t])^2 \sim \left( \frac{t}{\ln(t)} \right)^2, \]

which can be considered as marginally ballistic since the square displacement grows nearly as \( t^2 \) but this growth is screened by the inverse logarithmic factor.

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Scarce defects induce anomalous diffusion

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