SOME ALGEBRAIC INVARIANTS OF EDGE IDEAL OF CIRCULANT GRAPHS

GIANCARLO RINALDO

Abstract. Let $G$ be the circulant graph $C_n(S)$ with $S \subseteq \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}$ and let $I(G)$ be its edge ideal in the ring $K[x_0, \ldots, x_{n-1}]$. Under the hypothesis that $n$ is prime we: 1) compute the regularity index of $R/I(G)$; 2) compute the Castelnuovo-Mumford regularity when $R/I(G)$ is Cohen-Macaulay; 3) prove that the circulant graphs with $S = \{1, \ldots, s\}$ are sequentially $S_2$. We end characterizing the Cohen-Macaulay circulant graphs of Krull dimension 2 and computing their Cohen-Macaulay type and Castelnuovo-Mumford regularity.

Introduction

Let $S \subseteq \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \}$. The circulant graph $G := C_n(S)$ is a graph with vertex set $\mathbb{Z}_n = \{0, \ldots, n-1\}$ and edge set $E(G) := \{\{i, j\} \mid |j-i|_n \in S\}$ where $|k|_n = \min\{|k|, n-|k|\}$.

Let $R = K[x_0, \ldots, x_{n-1}]$ be the polynomial ring on $n$ variables over a field $K$. The edge ideal of $G$, denoted by $I(G)$, is the ideal of $R$ generated by all square-free monomials $x_i x_j$ such that $\{i, j\} \in E(G)$. Edge ideals of a graph have been introduced by Villarreal [11] in 1990, where he studied the Cohen–Macaulay property of such ideals. Many authors have focused their attention on such ideals (see [8], [6]). A known fact about Cohen-Macaulay edge ideals is that they are well-covered.

A graph $G$ is said well-covered if all the maximal independent sets of $G$ have the same cardinality. Recently well-covered circulant graphs have been studied (see [1], [2], [9]). In [14] and [4] the authors studied well-covered circulant graphs that are Cohen-Macaulay.

In this article we put in relation the values $n$, $S$ of a circulant graph $C_n(S)$ and algebraic invariants of $R/I(G)$. In particular we study the regularity index, the Castelnuovo-Mumford regularity, the Cohen-Macaulayness and Serre’s condition of $R/I(G)$.

In the first section we recall some concepts and notations and preliminary notions.

In the second section under the hypothesis that $n$ is prime we observe that the regularity index of $R/I(G)$ is 1 obtaining as a by-product the Castelnuovo-Mumford regularity of the ring when it is Cohen-Macaulay.
In the third section we prove that each $k$-skeleton of the simplicial complex of the independent set of $G = C_n(S)$ is connected when $n$ is prime. As an application we prove that the circulant graphs $C_n(\{1, \ldots, s\})$ (studied in [1], [2], [4], [9], [11], [14]) are sequentially $S_2$ (see [7]).

In the last section we characterize the Cohen-Macaulay circulant graphs of Krull dimension 2 and compute their Cohen-Macaulay type and Castelnuovo–Mumford regularity.

1. Preliminaries

In this section we recall some concepts and notations on graphs and on simplicial complexes that we will use in the article. Let $G$ be a simple graph with vertex set $V(G)$ and the edge set $E(G)$. A subset $C$ of $V(G)$ is called a clique of $G$ if for all $i$ and $j$ belonging to $C$ with $i \neq j$ one has $\{i, j\} \in E(G)$. A subset $A$ of $V(G)$ is called an independent set of $G$ if no two vertices of $A$ are adjacent. The complement graph $\bar{G}$ of $G$ is the graph with vertex set $V(\bar{G}) = V(G)$ and edge set $E(\bar{G}) = \{\{u, v\} \in V(G)^2 | \{u, v\} \notin E(G)\}$.

Set $V = \{x_1, \ldots, x_n\}$. A simplicial complex $\Delta$ on the vertex set $V$ is a collection of subsets of $V$ such that

(i) $\{x_i\} \in \Delta$ for all $x_i \in V$;
(ii) $F \in \Delta$ and $G \subseteq F$ imply $G \in \Delta$.

An element $F \in \Delta$ is called a face of $\Delta$. A maximal face of $\Delta$ with respect to inclusion is called a facet of $\Delta$.

If $\Delta$ is a simplicial complex with facets $F_1, \ldots, F_q$, we call $\{F_1, \ldots, F_q\}$ the facet set of $\Delta$ and we denote it by $\mathcal{F}(\Delta)$. The dimension of a face $F \in \Delta$ is $\dim F = |F| - 1$, and the dimension of $\Delta$ is the maximum of the dimensions of all facets in $\mathcal{F}(\Delta)$. If all facets of $\Delta$ have the same dimension, then $\Delta$ is called pure. Let $d - 1$ the dimension of $\Delta$ and let $f_i$ be the number of faces of $\Delta$ of dimension $i$ with the convention that $f_{-1} = 1$. Then the $f$-vector of $\Delta$ is $f(\Delta) = (f_{-1}, f_0, \ldots, f_{d-1})$. The $h$-vector of $\Delta$ is $h(\Delta) = (h_0, h_1, \ldots, h_d)$ with

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d-i}{k-i} f_{i-1}.$$ 

The sum

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{d-1} (-1)^i f_i$$

is called the reduced Euler characteristic of $\Delta$ and $h_d = (-1)^{d-1} \tilde{\chi}(\Delta)$. Given any simplicial complex $\Delta$ on $V$, we can associate a monomial ideal $I_\Delta$ in the polynomial ring $R$ as follows:

$$I_\Delta = \langle \{x_{j_1}x_{j_2}\cdots x_{j_r} : \{x_{j_1}, x_{j_2}, \ldots, x_{j_r}\} \notin \Delta\} \rangle.$$
Let $R/I$ be the quotient ring $R/I(G)$, and its Krull dimension is $d$. If $G$ is a graph, we call the independent complex of $G$ by
\[
\Delta(G) = \{ A \subseteq V(G) : A \text{ is an independent set of } G \}.
\]
The clique complex of a graph $G$ is the simplicial complex whose faces are the cliques of $G$. Let $F$ be the minimal free resolution of the quotient ring $R/I(G)$. Then
\[
F : 0 \to F_p \to \cdots \to F_1 \to F_0 \to R/I(G) \to 0
\]
with $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$. The numbers $\beta_{ij}$ are called the Betti numbers of $F$.

The Castelnuovo–Mumford regularity of $R/I(G)$, denoted by $\text{reg } R/I(G)$, is defined by
\[
\text{reg } R/I(G) = \max \{ j - i : \beta_{ij} \neq 0 \}.
\]
A graph $G$ is said Cohen-Macaulay if the ring $R/I(G)$, or equivalently $R/I_{\Delta(G)}$ is Cohen-Macaulay (over the field $K$) (see [3], [10], [17]). The Cohen-Macaulay type of $R/I(G)$ is equal to the last total Betti number in the minimal free resolution $F$.

We end this section with the following

**Remark 1.1.** Let $T = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor \}$ and $G$ be a circulant graph on $S \subseteq T$ with $s = |S|$, then:
1. $\bar{G}$ is a circulant graph on $\bar{S} = T \setminus S$;
2. The clique complex of $\bar{G}$ is the independent complex of $G$, $\Delta(G)$;
3. $|E(G)| = \begin{cases} ns - \frac{n}{2} & \text{if } n \text{ is even and } \frac{n}{2} \in S \\ ns & \text{otherwise}. \end{cases}$

### 2. Regularity and Connectedness of the Independent Complex of Circulant Graphs of Prime Order

We recall some basic facts about the regularity index (see also [15]). Let $R$ be standard graded ring and $I$ be a homogeneous ideal. The Hilbert function $H_{R/I} : \mathbb{N} \to \mathbb{N}$ is defined by
\[
H_{R/I}(k) := \dim_K (R/I)_k
\]
and the Hilbert-Poincaré series of $R/I$ is given by
\[
HP_{R/I}(t) := \sum_{k \in \mathbb{N}} H_{R/I}(k)t^k.
\]
By Hilbert-Serre theorem, the Hilbert-Poincaré series of $R/I$ is a rational function, that is
\[
HP_{R/I}(t) = \frac{h(t)}{(1-t)^n}.
\]
There exists a unique polynomial such that $H_{R/I}(k) = P_{R/I}(k)$ for all $k \gg 0$. The minimum integer $k_0 \in \mathbb{N}$ such that $H_{R/I}(k) = P_{R/I}(k) \forall k \geq k_0$ is called regularity index and we denote it by $\text{ri}(R/I)$.

**Remark 2.1.** Let $R/I_\Delta$ be a Stanley-Reisner ring. Then

$$\text{ri}(R/I_\Delta) = \begin{cases} 0 & \text{if } h_d = 0 \\ 1 & \text{if } h_d \neq 0 \end{cases}$$

*Proof.* By the hypothesis the Hilbert series can be represented by the reduced rational function

$$\frac{h(t)}{(1-t)^d}$$

where $d$ is the Krull dimension of $R/I_\Delta$ and $h(t) = \sum_{i=0}^{d} h_i t^i$ where $h_i$ are the entries of the $h$-vector of $\Delta$. We observe that $\text{ri}(R/I) = \max(0, \deg h(t) - d + 1)$. If $\text{ri}(R/I_\Delta) > 0$ then $\deg h(t) > d - 1$. But since $\deg h(t) \leq d$ we have $\deg h(t) = d$. Therefore $h_d \neq 0$ and $\text{ri}(R/I_\Delta) = 1$. The other case follows by the same argument. \qed

**Lemma 2.2.** Let $G$ be a circulant graph on $S$ with $n$ prime. Then the entries of the $f$-vector of $\Delta(G)$ are

$$f_i = nf'_i$$

with $0 \leq i \leq d - 1$ and $f'_i = f_{i,0}/(i+1) \in \mathbb{N}$ where $f_{i,0}$ is the number of faces of dimension $i$ containing the vertex $0$.

*Proof.* Call $F_i \subset \Delta$ the set of faces of dimension $i$, that is

$$F_i = \{F_1, \ldots, F_{f_i}\}.$$

Let $f_{i,j}$, number of faces in $F_i$ containing a given vertex $j = 0, \ldots, n - 1$. Since $G$ is circulant

$$f_{i,j} = f_{i,0} \quad \text{for all } j \in \{0, \ldots, n - 1\}.$$ 

Let $A \in \mathbb{F}_2^{f_i \times n} = (a_{jk})$ be the incidence matrix with $a_{jk} = 1$ if the vertex $k - 1$ belongs to the facet $F_j$ and 0 otherwise. We observe that each row has exactly $i + 1$ 1-entries. Hence summing the entries of the matrix we have $(i+1)f_i$. Moreover each column has exactly $f_{i,j}$ non zero entries. That is

$$nf_{i,0} = (i+1)f_i.$$ 

Since $n$ is prime the assertion follows. \qed

**Theorem 2.3.** Let $G$ be a circulant graph on $S$ with $n$ prime. Then

$$\text{ri}(R/I(G)) = 1.$$
Proof. By Remark 2.1 it is sufficient to show that $h_d$ is different from 0. Since

$$|h_d| = \left| \sum_{i=0}^{d} (-1)^i f_{i-1} \right| \neq 0,$$

it is sufficient to show that the reduced Euler formula is different from 0, that is

$$\sum_{i=1}^{d} (-1)^i f_{i-1} \neq 1.$$

By Lemma 2.2 we obtain

$$\sum_{i=1}^{d} (-1)^i f_{i-1} = n \sum_{i=1}^{d} (-1)^i f'_{i-1}$$

since $n$ is prime and the assertion follows. \(\square\)

Remark 2.4. In the proof of Theorem 2.3 we are giving a partial positive answer to the Conjecture 5.38 of \cite{9} that states that for all circulant graphs $\tilde{\chi}(\Delta) \neq 0$. In the article \cite{12} we found other families of circulant graphs satisfying the previous property. In the same article we found a counterexample that disprove the conjecture in general.

Corollary 2.5. Let $G$ be a circulant graph on $S$ with $n$ prime that is Cohen-Macaulay. Then $\text{reg} R/I(G) = \text{depth} R/I(G)$.

Proof. By Corollary 4.8 of \cite{5} since $\text{ri}(R/I) = 1$ the assertion follows. \(\square\)

3. Sequentially $S_2$ circulant graphs of prime order and connectedness

In this section we study good properties of the independent complex $\Delta(G)$ of a circulant graph $G$ that have prime order. We start by the following

Definition 3.1. Let $\Delta$ be a simplicial complex then we define the pure simplicial complexes $\Delta^{[k]}$ whose facets are

$$\mathcal{F}(\Delta^{[k]}) = \{ F \in \Delta : \dim(F) = k \}, \quad 0 \leq k \leq \dim(\Delta).$$

One interesting property of Cohen-Macaulay ring $R/I_\Delta$ is that the each simplicial complex $\Delta^{[k]}$ is connected. Hence the following Lemma is of interest.

Lemma 3.2. Let $G$ be a circulant graph on $S$ with $n$ prime. Then the $k$-skeleton of the simplicial complex $\Delta$, $\Delta^{[k]}$ is connected for every $k \geq 1$. 

Proof. To prove the claim we find a Hamiltonian cycle connecting all the vertices in $V = \{0, \ldots, n-1\}$ of the 1-skeleton of $\Delta^{[k]}$. Then it follows that since the 1-skeleton is connected then $\Delta^{[k]}$ is connected, too.

We assume without loss of generality that $F_0 = \{v_0, v_1, \ldots, v_k\} \in \Delta^{[k]}$ such that $v_0 = 0$, $v_1 = s \in S$. We define the set

$$F_j = \{v_{0,j}, v_{1,j}, \ldots, v_{k,j}\}$$

with $v_{i,j} = v_i + js \mod n$. It is easy to observe that since $F_0$ is in $\Delta^{[k]}$ and $G$ is circulant, $F_j$ is in $\Delta^{[k]}$, too.

Moreover if we focus on the first two vertices of $F_j$ we obtain that

$$v_{1,j} = v_{0,j-1} \text{ for all } j = 1, \ldots, n-1,$$

and $v_{0,n-1} = v_{1,0}$. Since

$$v_{0,j} = js \mod n$$

the set $\{v_0, \ldots, v_{n-1}\}$, by the primality of $n$, is equal to $V$. Hence the cycle with vertices

$$v_{0,0}, v_{0,1}, \ldots, v_{0,n-1}$$

and edges

$$\{v_{0,0}, v_{0,1}\}, \ldots, \{v_{0,n-2}, v_{0,n-1}\}, \{v_{0,n-1}, v_{0,0}\}$$

is a Hamiltonian cycle and the assertion follows. □

Recall that a finitely generated graded module $M$ over a Noetherian graded $K$-algebra $R$ is said to satisfy the Serre’s condition $S_r$ if

$$\text{depth } M_p \geq \min(r, \text{dim } M_p),$$

for all $p \in \text{Spec}(R)$.

Definition 3.3. Let $M$ be a finitely generated $\mathbb{Z}$-graded module over a standard graded $K$-algebra $R$ where $K$ is a field. For a positive integer $r$ we say that $M$ is sequentially $S_r$ if there exists a finite filtration of graded $R$-modules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each $M_i/M_{i-1}$ satisfies the $S_r$ condition and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

A nice characterization of sequentially $S_2$ simplicial complexes is the following:

Theorem 3.4 ([7]). Let $\Delta$ be a simplicial complex with vertex set $V$. Then $\Delta$ is sequentially $S_2$ if and only if the following conditions hold:

1. $\Delta^{[i]}$ is connected for all $i \geq 1$;
2. $\text{link}_\Delta(x)$ is sequentially $S_2$ for all $x \in V$. 

Example 3.5. Let $G$ be the circulant graph $C_6(\{1\})$. Then its simplicial complex $\Delta$ is connected, but $\Delta[2]$ is not (see Figure 1).

Sequentially Cohen-Macaulay cycles have been characterized in [6], that are in our notation are just $C_3(\{1\})$ and $C_5(\{1\})$. In [7] the authors proved that the only sequentially $S_2$ are the odd cycles. The following is related to these results.

Theorem 3.6. Let $G$ be the circulant graph $C_n(\{1, \ldots, s\})$ with $n$ prime. Then $G$ is sequentially $S_2$.

Proof. By Lemma 3.2 the first condition of Theorem 3.4 is satisfied. To check the second condition of Theorem 3.4 we prove that $K[\text{link}_\Delta(x_0)]$, is sequentially Cohen Macaulay. We observe that

$$K[\text{link}_\Delta(x_0)] \cong (R/I(G))_{x_0} \cong K[x_0^{s+1}][x_1, \ldots, x_{n-1}]/I(G)'$$

where $I(G)'$ is obtained by the $K$-algebra homomorphism induced by the mapping $x_0 \to 1$. Since the vertices adjacent to 0 are $\{1, \ldots, s\} \cup \{n-s, \ldots, n-1\}$ we have that

$$I(G)' = I(G') + (x_1, \ldots, x_s) + (x_{n-s}, \ldots, x_{n-1}).$$

with $G'$ be the subgraph of $G$ induced by the vertices $\{s+1, \ldots, n-(s+1)\}$. That is

$$(R/I(G))_{x_0} \cong K[x_{s+1}, \ldots, x_{n-s-1}]/I(G').$$

We claim that $I(G')$ is chordal, hence it is sequentially Cohen-Macaulay by Theorem 3.2 of [6]. To prove the claim we observe that the labelling on the vertices of $G'$

$$s+1, s+2, \ldots, n-s-1$$

induces a perfect elimination ordering, that is $N^+(i) = \{j : \{i, j\} \in E(G'), i < j\}$ is a clique. Let $j, k \in N^+(i)$. That is $\{i, j\}$ and $\{i, k\}$ are two edges with $i < j$ and $i < k$ and assume $j < k$. Then $|j-i|_n = j-i \leq s$ and $|k-i|_n = k-i \leq s$. Moreover

$$0 < j-i < k-i \leq s.$$
Hence it follows $|k - j| = k - j < s$. Therefore $\{j, k\} \in E(G')$ and $N^+(i)$ is a clique. □

Example 3.7. If a ring is Cohen-Macaulay it is pure and sequentially $S_n$ for all $n$. The circulant graph of prime order with minimum number of vertices that is Cohen-Macaulay and has Krull dimension greater than 2 is $C_{13}(\{1, 5\})$ (see [4]).

4. COHEN-MACAUVAL CIRCULANT GRAPHS OF DIMENSION 2 AND THEIR CASTELNUOVO-MUMFORD REGULARITY

We start this section by the following

Theorem 4.1. Let $G$ be the circulant graph $C_n(S)$ with $S \subset \{1, \ldots, \lfloor \frac{n}{2} \rfloor \}$. The following conditions are equivalent:

1. $G$ is Cohen-Macaulay of dimension 2;
2. $\Delta(G)$ is connected of dimension 1;
3. $\gcd(n, \bar{S}) = 1$ and $\forall a, b \in \bar{S}$ we have $b - a \notin \bar{S}$ and $n - (b + a) \notin \bar{S}$.

Proof. (1) $\iff$ (2). Known fact. See also [9] Corollary 4.54.

(2) $\implies$ (3). If $\Delta(G)$ is connected then there is a path in $\bar{G} \cong \Delta(G)$ connecting the vertices 0 and 1 (see Remark 1.1) whose vertices are

$$0 = v_0, v_1, \ldots, v_r = 1$$

and edges

$$\{0, s_1\}, \{s_1, s_1 + s_2\}, \ldots, \{\sum_{i=1}^{r-1} s_i, \sum_{i=1}^{r} s_i \equiv 1 \mod n\}$$

with $s_i \in \bar{S}$. Hence there exists a relation

$$\sum a_i s_i \equiv 1 \mod n, \quad \text{with } a_i \in \mathbb{N}, s_i \in \bar{S}.$$ 

By the Euclidean algorithm we have that $\gcd(n, \bar{S}) = 1$. Suppose there exist $a, b \in \bar{S}$ with $b - a \in \bar{S}$. This implies $a \neq b$. We observe that $\{0, a, b\}$ is a clique in $\Delta(G)$, that is $\dim \Delta(G) \geq 2$. In fact since $\bar{G}$ is circulant $\{0, a\}, \{0, b\}$ and $\{a, a + (b - a) = b\}$ are edges in $\bar{G}$. Now suppose that $n - (b + a) \in \bar{S}$. We observe that $\{0, a, a + b\}$ is a clique in $\Delta(G)$. In fact since $\bar{G}$ is circulant $\{0, a\}, \{a, a + b\}$ and $\{a + b, a + b + n - (a + b) \equiv 0\}$ are edges in $\bar{G}$. The implication (3) $\implies$ (2) follows by similar arguments.

Theorem 4.2. Let $G$ be a Cohen-Macaulay circulant graph $C_n(S)$ of dimension 2. Then $\text{reg } R/I(G) = 2$.

Proof. It is sufficient to prove that $h_2 \neq 0$ (see Remark 2.1 and the proof of Corollary 2.5). We need to compute $h_2 = f_1 - f_0 + f_{-1}$. We observe

\[h_2 = f_1 - f_0 + f_{-1} = \]
that \( f_1 \) is the number of edges of \( \bar{G} \). By Remark 1.1 one of the two cases to study is
\[
\binom{n}{2} - ns,
\]
with \( h_2 = \binom{n}{2} - n(s+1) + 1 \). The only roots \( n \in \mathbb{N} \) of the quadratic equation
\[
\binom{n}{2} - n(s+1) + 1 = 0
\]
are 1 and 2 with \( s = 0 \). Absurd. The other case follows by the same argument.

**Theorem 4.3.** Let \( G \) be a Cohen-Macaulay circulant graph \( C_n(S) \) of dimension 2. Then its Cohen-Macaulay type is
\[
h_2 = \begin{cases} 
\binom{n}{2} - n\left(s + \frac{1}{2}\right) + 1 & \text{if } n \text{ is even and } \frac{n}{2} \in S \\
\binom{n}{2} - n(s+1) + 1 & \text{otherwise.}
\end{cases}
\]

**Proof.** By Auslander-Buchsbaum Theorem (Theorem 1.3.3, [3]) and since the depth \( R/I(G) = 2 \) we need to compute the Betti number in position \( \beta_{i,j} \) when \( i = n - 2 \). By Theorem 4.2 and the definition of Castelnuovo-Mumford regularity, the Betti numbers that are not trivially 0 are \( \beta_{n-2,j} \) in the degrees \( j \in \{n - 1, n\} \). We recall the Hochster’s formula (see [10], Corollary 5.1.2)
\[
\beta_{i,\sigma}(R/I_\Delta) = \dim_K \tilde{H}_{|\sigma|-i-1}(\Delta|\sigma; K)
\]
where \( \tilde{H}(\cdot) \) is the simplicial homology and \( \sigma \in \Delta \) is interpreted as squarefree degree in the minimal free resolution and it induces a restriction in \( \Delta \) defined by
\[
\Delta|\sigma = \{ F \in \Delta : F \subseteq \sigma \}.
\]
We observe that in the squarefree degree \( \sigma \) having total degree \( n - 1 \)
\[
\beta_{i,\sigma} = \dim_K \tilde{H}_0(\Delta|\sigma; K) = 0.
\]
In fact \( \Delta \cong \bar{G} \) is connected and the same happens removing one of the vertices of the circulant graph \( \bar{G} \) since circulant graphs are biconnected. Now, if we consider the squarefree degree \( \sigma \) having total degree \( n \), again, by Hochster formula, we obtain
\[
\beta_{i,\sigma} = \dim_K \tilde{H}_1(\Delta|\sigma; K).
\]
In this case \( \Delta|\sigma \cong \Delta \cong \bar{G} \) and the chain complex of \( \Delta \)
\[
\mathcal{C} : 0 \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \to 0,
\]
has the two homologies \( \tilde{H}_0 = \tilde{H}_{-1} = 0 \). Therefore
\[
\dim_K \tilde{H}_1(\bar{G}; K) = \beta_{i,\sigma} = f_1 - f_0 + f_{-1}
\]
and the assertion follows by Remark 1.1. □
Figure 2. \( G = C_8(\{2, 3\}) \) and \( C_8(\{1, 4\}) \cong \Delta(G) \).

**Example 4.4.** Let \( G = C_8(\{2, 3\}) \) that is \( \bar{S} = \{1, 4\} \) (see Figure 2). We observe that it satisfies conditions (3) of Theorem 4.1. Its Cohen-Macaulay type by Theorem 4.3 is

\[
\binom{8}{2} - 8(2 + 1) + 1 = 5.
\]

**Remark 4.5.** We observe that the rings satisfying Theorem 4.3 are level. For a description of level algebras see Chapter 5.4 and 5.7 of [3].

**Corollary 4.6.** Let \( G \) be the circulant graph \( C_n(S) \) with \( S \subset \{1, \ldots, \lfloor \frac{n}{2} \rfloor \} \) and \( s = |S| \). The following conditions are equivalent:

1. \( G \) is Gorenstein of dimension 2;
2. \( S = \{1, \ldots, \hat{i}, \ldots, n\} \) and \( \gcd(n, i) = 1 \) with \( n \geq 4 \);
3. \( \Delta(G) \cong \bar{G} \) is a \( n \)-gon with \( n \geq 4 \).

**Proof.** (1) \( \Rightarrow \) (2). \( G \) is Gorenstein if and only if \( G \) is Cohen-Macaulay of type 1. Hence by Theorem 4.3 \( \Delta(G) \) is connected that is \( \gcd(n, \bar{S}) = 1 \). Moreover by Theorem 4.3 \( h_2 = 1 \) and solving the two quadratic equations

\[
\binom{n}{2} - n(s + 1) + 1 = 1, \quad \binom{n}{2} - n(s + 2) + 1 = 1,
\]

we obtain respectively

\[
n = 2s + 2 \quad \text{and} \quad n = 2s + 3.
\]

In both cases \( s = \left\lfloor \frac{n}{2} \right\rfloor - 1 \). Hence \( \bar{S} = i \) with \( \gcd(i, n) = 1 \) and the assertion follows.

(2) \( \Rightarrow \) (3). Let \( \bar{S} = \{i\} \) with \( \gcd(n, i) = 1 \). We easily observe that the vertices

\[
0, i, \ldots, (n - 1)i \mod n
\]

and edges

\[
\{0, i\}, \{i, 2i\}, \ldots, \{(n - 1)i, (n)i \equiv 0 \mod n\}
\]

define a Hamiltonian cycle that is \( \bar{G} \) itself.

(3) \( \Rightarrow \) (1). Since \( \Delta(G) \) is a simplicial 1-sphere is Gorenstein of Krull dimension 2 (see Corollary 5.6.5 of [3]). \( \square \)
We observe that in Theorem 4.1 of [4] the Cohen-Macaulayness of the graphs described in Corollary 4.6 has been studied by a different point of view.

REFERENCES

[1] J. Brown, R. Hoshino, Independence polynomials of circulants with an application to music, Discrete Mathematics, 309, 2009, 2292–2304.
[2] J. Brown, R. Hoshino, Well-covered circulant graphs, Discrete Mathematics, 311, 2011, 244–251.
[3] W. Bruns, J. Herzog, Cohen-Macaulay rings, Cambridge Univ. Press, Cambridge, 1997.
[4] J. Earl, K. N. Vander Meulen, A. Van Tuyl, Independence Complexes of Well-Covered Circulant Graphs, Experimental Mathematics, 25, 2016, 441–451.
[5] D. Eisenbud, The Geometry of Syzygies, Graduate texts in Mathematics, Springer, 2005.
[6] C. A. Francisco, A. Van Tuyl, Sequentially Cohen-Macaulay edge ideals, Proc. Amer. Math. Soc., 135, 2007, 2327–2337.
[7] H. Haghighi, N. Terai, S. Yassemi, R. Zaare-Nahandi, Sequentially $S_r$ simplicial complexes and sequentially $S_2$ graphs, Proc. Amer. Math. Soc., 139, 2011, 1993–2005.
[8] J. Herzog, T. Hibi, Distributive lattices, bipartite graphs and Alexander duality, J. Alg. Combin., 22, 2005, 289–302.
[9] R. Hoshino, Independence polynomials of circulant graphs, PhD Thesis, Dalhousie University, 2008, 1–280.
[10] E. Miller, B. Sturmfels, Combinatorial Commutative Algebra, Springer-Verlag, Berlin, 2005.
[11] A. Mousivand, Circulant $S_2$ graphs, preprint arXiv:1512.08141v1, 2015, 1–11.
[12] Rinaldo, F. Romeo On the reduced Euler characteristic of independence complexes of circulant graphs, preprint arXiv:1706.00863, 2017, 1–12.
[13] R. P. Stanley, Combinatorics and Commutative Algebra, Second Edition, Birkhäuser, Boston/Basel/Stuttgart, 1996.
[14] K. N. Vander Meulen, A. Van Tuyl, C. Watt, Cohen-Macaulay circulant graphs, Communications in Algebra, 42, 2014, 1896–1910.
[15] W. Vasconcelos, Computational methods in commutative algebra and algebraic geometry, Springer Science & Business Media, 2, 2004.
[16] R. H. Villarreal, Cohen–Macaulay graphs, Manuscripta Math., 66, 1990, 277–293.
[17] R. Villarreal, Monomial algebras, Marcel Dekker, New-York, 2001.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TRENTO, VIA SOMMARIVE, 14, 38123 POVO (TRENTO), ITALY