ASYMPTOTICS OF COMMUTING PROBABILITIES IN REDUCTIVE ALGEBRAIC GROUPS

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Abstract. Let $G$ be an algebraic group. For $d \geq 1$, we define the commuting probabilities $\text{cp}_d(G) = \frac{\dim(C_d(G))}{\dim(G)^d}$, where $C_d(G)$ is the variety of commuting $d$-tuples in $G$. We prove that for a reductive group $G$ when $d$ is large, $\text{cp}_d(G) \sim \frac{\alpha}{n}$ where $n = \dim(G)$, and $\alpha$ is the maximal dimension of an Abelian subgroup of $G$. For a finite reductive group $G$ defined over the field $\mathbb{F}_q$, we show that $\text{cp}_{d+1}(G(\mathbb{F}_q)) \sim q^{(\alpha - n)d}$, and give several examples.

1. Introduction

Commuting probability, also referred as commuting degree, is the probability of finding a commuting tuple in a group. The question of determining this for a given group is well studied for finite groups and compact groups (see for example [BFM, ET, FF, GR, HR]). To an interested reader we recommend the survey article [SS2] and the references therein for further reading. Let $G$ be a group and $d \geq 1$. Let $C_d(G) = \{(g_1, \ldots, g_d) \in G^d \mid g_i g_j = g_j g_i, \forall 1 \leq i, j \leq d\}$ (also denoted as $G^{(d)}$ sometimes). The elements of $C_d(G)$ are called commuting $d$-tuples. The commuting probability for a finite group $G$, for $d \geq 1$, is defined as

$$\text{cp}_d(G) = \frac{|C_d(G)|}{|G|^d}.$$ 

Since $C_1(G) = G$, and $\text{cp}_1(G) = 1$, we usually take $d \geq 2$. The commuting probability $\text{cp}_d(G)$ measures the probability of finding a $d$-tuple of elements of $G$ which commute pairwise (we will simply call it a $d$-tuple whereas we mean commuting $d$-tuple). While studying the commuting probabilities for compact groups, instead of size, one considers the measure of the sets involved. In this article, we would like to study the asymptotic value of commuting probabilities for algebraic groups and finite groups of Lie type. The notion of commuting probability in algebraic groups is introduced by the first-named author in [Ga] where $\text{cp}_2$ is defined using the dimension of the subsets involved. We generalise that to define $\text{cp}_d$ here.

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Let $K$ be an algebraically closed field. Let $G$ be an algebraic group over $K$. The set $\mathcal{C}_d(G)$ is an algebraic variety, often called commuting variety in the literature. For $d \geq 1$, we define the commuting probabilities as follows,

$$ cp_d(G) = \frac{\dim(\mathcal{C}_d(G))}{\dim(G^d) = \frac{\dim(\mathcal{C}_d(G))}{d \cdot \dim(G)}. $$

Clearly, $cp_1(G) = 1$, thus, in what follows we take $d \geq 2$. The questions such as if $\mathcal{C}_d(G)$ is an irreducible variety, is an intense topic of study. Richardson [Ri, Theorem C] proved that $\mathcal{C}_2(G)$ is an irreducible variety when $G$ is a simply connected semisimple algebraic group. The commuting varieties for matrices and Lie algebras are well studied (see, for example [FG, GuSe]). However, our concern here is its dimension. In Section 3, we get a bound on this using the idea of the branching matrix developed in Section 2.

Recall that we say $\{a_n\}$ is asymptotic to $\{b_n\}$, as $n$ gets large, if $\lim_{n \to \infty} \frac{a_n}{b_n} = 1$, and we write $a_n \sim b_n$. Let $G$ be a reductive algebraic group of dimension $n$, and maximal dimension of an Abelian subgroup is $\alpha$. In Section 4, we prove that for a reductive algebraic group $G$, when $d$ gets large, $cp_d(G) \sim \frac{\alpha n}{n}$ (see Theorem 4.1). In Section 5, using [KPP, Theorem 3.1] where simultaneous conjugacy classes are studied for finite groups, we note that for a finite group $G$ the commuting probabilities $cp_d(G) \sim m \left( \frac{a}{|G|} \right)^{d-1}$, where $m$ is a constant, and $a$ is the maximal size of an Abelian subgroup of $G$. We give an alternate proof of this result using the ideas developed in this paper. We apply this result on a finite reductive group $G$ defined over the field $\mathbb{F}_q$ to get $cp_d(G(\mathbb{F}_q)) \sim \left( \frac{1}{q^{a-\alpha}} \right)^{d-1}$ up to a constant (see Theorem 5.4). The maximal Abelian subgroups are known for finite groups of Lie type (see, for example [Vd1, Vd2, Wo1, Wo2, Br]). For several examples of classical groups, we use this to compute the asymptotic value of commuting probabilities, and find tuples of which common centralizer is a maximal Abelian subgroup.

2. BRANCHING MATRIX FOR ALGEBRAIC GROUPS

Let $K$ be an algebraically closed field, and $G$ be an algebraic group over $K$. To study the commuting probabilities $cp_d(G)$, we introduce branching matrix $B_G$ for $G$. This concept will be generalized from that of finite groups given in [SS, SS2]. The size of branching matrix $B_G$ will turn out to be the number of $z$-classes of commuting tuples, and entries of $B_G$ will be a measure of different conjugacy classes, which are in the same $z$-class. In this section, we define these notions for algebraic groups and prove the relation between $B_G$ and the commuting probabilities.

2.1. $z$-CLASSES OF TUPLES. This notion is a generalization of the similar concept studied for finite groups and algebraic groups (see for example [BS, GS]). We define an equivalence relation, namely $z$-equivalence, on the set of commuting $d$-tuples $\mathcal{C}_d(G)$, for $d \geq 1$, as follows.
Definition 2.1. The tuples \((g_1, \ldots, g_d)\) and \((h_1, \ldots, h_d)\) \(\in \mathfrak{C}_d(G)\) are said to be \(z\)-equivalent if \(Z_G(g_1, \ldots, g_d)\) and \(Z_G(h_1, \ldots, h_d)\) are conjugate in \(G\), where \(Z_G(g_1, \ldots, g_d) = \bigcap_{i=1}^d Z_G(g_i)\) denotes the intersection of centralizers of \(g_1, \ldots, g_d\) in \(G\). We call the corresponding equivalence classes the \(z\)-classes of \(d\)-tuples.

Notice that, we can make \(G\) act on \(\mathfrak{C}_d(G)\) by conjugation component wise, thus giving rise to the conjugacy classes of \(d\)-tuples. Hence, a \(z\)-class of \(d\)-tuple is a union of those conjugacy classes of \(d\)-tuples for which the corresponding common centralizers are conjugate within \(G\). For \(d = 1\), this definition coincides with the usual notion of \(z\)-classes in \(G\), thus \(z\)-classes of 1-tuples are simply the \(z\)-classes. The number of \(z\)-classes is known to be finite for a reductive algebraic group. This was proved by Steinberg (see Section 3.6 Corollary 1 to Theorem 2 \([St]\)) and is further explored over fields of type \((F)\) in \([GS]\). However, this number could be infinite for a more general algebraic group, for example upper triangular matrix group (see \([Bl\), Theorem 1.2]). For more on \(z\) classes, we refer an interested reader to the survey article \([BS]\). The notion of \(z\)-classes can be defined among all tuples.

Definition 2.2. We define \(z\)-equivalence on \(\mathfrak{C}(G) = \bigcup_{d \geq 1} \mathfrak{C}_d(G)\) as follows. The tuples \((g_1, \ldots, g_e) \in \mathfrak{C}_e(G)\) and \((h_1, \ldots, h_f) \in \mathfrak{C}_f(G)\) are said to be \(z\)-equivalent if \(Z_G(g_1, \ldots, g_e)\) and \(Z_G(h_1, \ldots, h_f)\) are conjugate in \(G\). We call the equivalence classes in \(\mathfrak{C}(G)\) the \(z\)-classes of tuples.

The number of \(z\)-classes of tuples is obviously finite when \(G\) is a finite group, but its finiteness for a reductive algebraic group requires some work. We begin with the following,

Proposition 2.3. Let \(G\) be a reductive algebraic group. Then, there are finitely many \(z\)-classes of \(d\)-tuples (i.e., \(z\)-classes in \(\mathfrak{C}_d(G)\)), for any \(d \geq 1\).

Proof. For \(d = 1\), this is a result due to Steinberg as mentioned earlier in this section. We prove this for \(d = 2\). Let \((g_1, g_2) \in \mathfrak{C}_2(G)\), write \(g_1 = s_1u_1\) and \(g_2 = s_2u_2\), its Jordan decomposition. First, we consider the case when \(s_1 = s_2 = s\). In this case,

\[ Z_G(g_1, g_2) = Z_G(g_1) \cap Z_G(g_2) = Z_G(s)(u_1) \cap Z_G(s)(u_2) = Z_G(s)(u_1, u_2). \]

Since, \(Z_G(s)\) is a reductive group, and there are only finitely many unipotent classes in such groups, this number is finite.

Now, we need to deal with the general case. Note that since \(g_1\) and \(g_2\) commute, the elements \(s_1, s_2, u_1\) and \(u_2\) commute pairwise. This is because of Jordan decomposition which also gives us that \(s_i, u_i\) are polynomials in \(g_i\). Hence \(u_1, u_2, s_2 \in Z_G(s_1)\), further,
$u_1, u_2 \in Z_{Z_G(s_1)}(s_2)$. Now,

$$Z_G(g_1, g_2) = Z_G(s_1) \cap Z_G(u_1) \cap Z_G(s_2) \cap Z_G(u_2) = Z_{Z_G(s_1)}(s_2)(u_1, u_2).$$

Once again the group, $Z_G(s_1)$ is reductive and $Z_{Z_G(s_1)}(s_2)$ as well. Since there are only finitely many conjugacy classes of unipotents in a reductive group, we get the finiteness of $z$-classes. The proof for $d$-tuples can be done similarly by looking at repeated centralizers of the semisimple components. \hfill \square

Now we prove,

**Proposition 2.4.** Let $G$ be a reductive algebraic group. Then, the number of $z$-classes of tuples in $G$ (i.e., $z$-classes in $\mathcal{C}(G)$) is finite.

**Proof.** Let $(g_1, \ldots, g_d) \in \mathcal{C}(G)$ and $Z_G(g_1, \ldots, g_d) = \bigcap_{i=1}^d Z_G(g_i)$. We can write,

$$Z_G(g_1) \supset Z_G(g_1, g_2) = Z_{Z_G(g_1)}(g_2) \supset \cdots \supset Z_G(g_1, \ldots, g_d) = Z_{Z_G(g_1, \ldots, g_{d-1})}(g_d) \supset \cdots \supset Z_G(g_1, \ldots, g_d) = Z_{Z_G(g_1, \ldots, g_{d-1})}(g_d).$$

Note that for large enough $d$, this series will end in an Abelian group. From Proposition 2.3 each step in the above chain has finitely many choices. Further, the length of such a chain is finite. Since, a strict inclusion in the above chain can come for one of the following two reasons: either the subgroup is connected then dimension of the subgroup decreases or if the subgroup is not connected then it is of finite index (being algebraic subgroup). Thus, we have finitely many $z$-classes of tuples. \hfill \square

The number of $z$-classes of $\mathcal{C}(G)$ will turn out to be the size of “branching matrix” of $G$ which we will define next.

### 2.2. Branching matrix.

Now, we define the **branching matrix** $B_G$ for an algebraic group $G$. The notion of branching matrix, and its relation with the commuting tuples for finite groups, has been explored in [Sh, SS, SS2]. The rows and columns of this matrix correspond to $z$-classes of tuples in $G$ (i.e. $z$-classes in $\mathcal{C}(G)$). We begin with fixing a convention where the $z$-classes of the group, i.e., for $d = 1$, will be written first. Furthermore, we take the first entry to be the $z$-class of identity (equivalently, any central element) of $G$. Then, we take the $z$-classes of 2-tuples, 3-tuples and so on. Fix an indeterminate $\psi$. The entries of the matrix $B_G$ are monomials in the variable $\psi$, and are defined as follows. For a $z$-class of a commuting $d$-tuple $(g_1, \ldots, g_d)$ we look at the group $Z_G(g_1, \ldots, g_d) := H$, and compute its $z$ classes (i.e., of 1-tuples). Notice that $\mathcal{C}(H) \subset \mathcal{C}(G)$. Suppose an $r$-tuple $(x_1, \ldots, x_r) \in \mathcal{C}_r(G)$ appears as a $z$-class of $H = Z_G(g_1, \ldots, g_d)$. Then, in the column corresponding to the $z$-class of $(g_1, \ldots, g_d)$, we put the entry

$$\psi^{\dim zcl(x_1, \ldots, x_r) - \dim cl(x_1, \ldots, x_r)}$$
in $B_G$ where $zcl(x_1,\ldots,x_r)$ and $cl(x_1,\ldots,x_r)$ denote the $z$-class and conjugacy class in $H = Z_G(g_1,\ldots,g_d)$ of the tuple $(x_1,\ldots,x_r)$, respectively. Equivalently, there exists an element $y \in H$ such that $Z_H(y) = Z_G(x_1,\ldots,x_r)$, and $zcl(x_1,\ldots,x_r) := zcl(y)$ and $cl(x_1,\ldots,x_r) := cl(y)$. If an $r$-tuple $(x_1,\ldots,x_r) \in \mathcal{C}_r(G)$ does not appear as a $z$-class of $H = Z_G(g_1,\ldots,g_d)$, then we enter 0 in $B_G$. Thus, to compute the branching matrix of an algebraic group we need to follow the steps mentioned below:

1. To begin with, we compute the $z$-classes in $G$, say the representatives for these classes are $\{z_1 = e, z_2,\ldots,z_r\}$. The first column corresponds to the identity (as per our convention). Now, to obtain the entries in first column, we compute the $z$-classes in $Z_G(1) = G$, and enter $\psi^{\dim zcl(z_i)−\dim cl(z_i)}$ as entries.

2. Then, we fill the columns 2 to $r$ corresponding to the non-identity $z$-classes, i.e., for $z_2, z_3,\ldots,z_r$. For example, to get the second column we need to compute the $z$-classes within $Z_G(z_2)$. We fill the entries in $B_G$, as per the formula, if the $z$-classes match with the ones from that of $G$ obtained in the previous step, else we create a new row and a new column as this would correspond to a 2-tuple of $G$ (i.e., it would give rise to some $z$-class of $\mathcal{C}_2(G)$). We do this process for all $Z_G(z_i)$, and whenever we find a new type of 2 tuple, we add a new row and column at the end.

3. After finishing the previous step for all $z$-classes (of 1-tuples), we look at the new ones obtained in those steps. These new ones correspond to 2-tuples which will give rise to new centralizer subgroups, namely, the intersection of centralizers. We compute the $z$-classes in these new centralizer subgroups to fill the corresponding column as explained in the previous step, and possibly obtain some new types of 3-tuples. We continue this process till we get no more new tuples. The process ends when we get to the Abelian centralizers.

Notice that there is no guarantee that $B_G$ is a finite size matrix at the moment. To understand these steps better, we work out some examples with the help of following.

**Remark 2.5.** To understand the dimension of $cl(g)$, for $g \in G$, we can look at the dimension of $G/Z_G(g)$. However, to understand the dimension of $zcl(g)$ we need to look at the set $zcl(g) = \bigcup_t cl(t)$ where $Z_G(t)$ is conjugate to $Z_G(g)$. By associating $Z_G(t)$ to each $t$ we need to understand various conjugates of $Z_G(g)$, and those $t$ (up to conjugacy) for which $Z_G(t) = Z_G(g)$. This amounts to understanding the set $G/N_G(Z_G(g)) \cup \{x \in cl(G) \mid Z_G(x) = Z_G(g)\}$. Thus,

$$\dim zcl(g)−\dim cl(g) = −\dim(N_G(Z_G(g)))+\dim\{x \in cl(G) \mid Z_G(x) = Z_G(g)\}+\dim Z_G(g).$$

We will take help of this equation in the following computations.
Example 2.6. For the algebraic group $GL_2$ over $K$ the number of $z$-classes is 3 given by $I, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$ where $\lambda_1 \neq \lambda_2$. Notice that there are no $z$-classes for $d \geq 2$ tuples as all non-trivial centralizers are Abelian. The branching matrix is as follows:

$$B_{GL_2} = \begin{pmatrix} \psi & 0 & 0 \\ \psi & \psi^2 & 0 \\ \psi^2 & 0 & \psi^2 \end{pmatrix}.$$ 

To get the first column we compute $z$ by $I$, and, for 2 and 3-tuples $z$ has its centralizer, an Abelian subgroup of maximal dimension. The representative of $Z_{GL_2}(I) = GL_2$, to get the second column we compute the $z$-classes in $Z_{GL_2}(1 1 0 1) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a \in K^*, b \in K \right\}$, and to get the third column we compute the $z$-classes in $Z_{GL_2}(\lambda_1 \lambda_2)$, which is the diagonal group.

Example 2.7. Let us look at $GL_3$. The number of $z$-classes in $GL_3(K)$ is 6 and there are no higher tuples. We write them in the following order:

$$\left\{ aI_3, \begin{pmatrix} a & 1 \\ a & a \end{pmatrix}, \begin{pmatrix} a & a \\ a & b \end{pmatrix}, \begin{pmatrix} a & 1 \\ a & 1 \end{pmatrix}, \begin{pmatrix} a & 1 \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & c \end{pmatrix} \right\}$$

where $a, b$ are distinct and non-zero. The branching matrix $B_{GL_3}$ is:

$$B_{GL_3} = \begin{pmatrix} \psi & 0 & 0 & 0 & 0 & 0 \\ \psi & \psi^2 & 0 & 0 & 0 & 0 \\ \psi^2 & 0 & \psi^2 & 0 & 0 & 0 \\ \psi^2 & \psi^3 & 0 & \psi^3 & 0 & 0 \\ \psi^3 & 0 & \psi^3 & 0 & 0 & \psi^3 \end{pmatrix}.$$ 

Example 2.8. For the group $GL_4$, we have 14 $z$-classes of 1-tuples. In addition to these, there are four more new $z$-classes of 2-tuples (indicated in blue colour in the branching matrix) and one more new $z$-class of triples (indicated in red colour). The $z$-class of triple has its centralizer, an Abelian subgroup of maximal dimension. The representative of $z$-classes are as follows:

$$aI_4, \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \begin{pmatrix} a & a \\ a & a \end{pmatrix}, \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \begin{pmatrix} a & b \\ b & d \end{pmatrix}, \begin{pmatrix} a & b \\ b & b \end{pmatrix},$$

and, for 2 and 3-tuples

$$\begin{pmatrix} a & 1 \\ a & a \end{pmatrix}, \begin{pmatrix} b & c \\ b & c \end{pmatrix}, \begin{pmatrix} a & 1 \\ a & a \end{pmatrix}, \begin{pmatrix} b & c \\ b & d \end{pmatrix}, \begin{pmatrix} a & 1 \\ a & a \end{pmatrix}, \begin{pmatrix} b & 1 \\ b & b \end{pmatrix}.$$
The branching matrix, with the row and column indexing as above, is

\[
\begin{pmatrix}
\begin{pmatrix}
a & 1 \\
a & a
\end{pmatrix}, & \begin{pmatrix}
b & 1 \\
b & b
\end{pmatrix}
\end{pmatrix},
\begin{pmatrix}
a & 1 \\
a & a
\end{pmatrix},
\begin{pmatrix}
b & c \\
b & b
\end{pmatrix},
\begin{pmatrix}
u & 1 \\
u & u
\end{pmatrix}
\end{pmatrix}.
\]

The branching matrix has the following properties:

1. The (1,1)th entry of \( B_G \) is \( \psi^{\dim(Z(G))} \). In fact, the diagonal entries of \( B_G \) are \( \psi^{\dim(Z(Z_G(g_1,\ldots,g_r)))} \) where \( Z(Z_G(g_1,\ldots,g_r)) \) is the center of \( Z_G(g_1,\ldots,g_r) \).
2. The entries in the first column are \( \psi^{\dim(zcl(g))-\dim(cl(g))} \) for various z-classes in \( G \), and 0 corresponding to 2-tuples onwards.
3. All entries in the first row, except first one, are 0.
4. Every row (second onwards) has a non-zero entry before the diagonal, i.e., for all \( i > 1 \) there exists \( i_0 < i \) such that \( (B_G)_{i,i_0} \neq 0 \).
5. If \( Z_G(g_1,\ldots,g_r) \) is Abelian, then the corresponding column has all entries 0 except at the diagonal which is \( \psi^{\dim(Z_G(g_1,\ldots,g_r))} \).
6. When \( G \) is a reductive group, the branching matrix \( B_G \) is a finite size matrix.

Proof. To prove (1) we note that a group \( H \) is centralizer of its central elements. The central elements form a single z-class but distinct conjugacy classes. Thus, \( (g_1,\ldots,g_r) \) appearing as a z-class in the group \( Z_G(g_1,\ldots,g_r) \) gives the following: \( zcl((g_1,\ldots,g_r)) = Z(Z_G(g_1,\ldots,g_r)), \) and \( cl((g_1,\ldots,g_r)) = (g_1,\ldots,g_r) \). Hence the required result.

Proof of (2) is clear from the process to obtain \( B_G \). Proof of (3) follows as the group \( G \) itself can’t appear as a subgroup of its proper centralizer. Proof of (4) follows from the process to obtain \( B_G \), as a new row (and column) is added when a new centralizer type appears. Proof of (5) is clear.

The proof of (7) follows form Proposition 2.4. \( \square \)
We require certain properties of the branching matrix $B_H$ when $H = Z_G(g_1, \ldots, g_r)$ with respect to $B_G$. Recall the process of constructing $B_G$ as mentioned in the beginning of this subsection. All of the $z$-classes of $m$-tuples of $H$ are nothing but $z$-classes of $r+m$-tuples of $G$. Thus, to get $B_H$ we mark these rows and columns in $B_G$ and collect these entries in a new matrix. We warn here that the submatrix simply obtained from $B_G$ may not be the $B_H$, if we simply compute branching matrix for $H$ as per definition, since we have not fixed any strict order on the tuples. The matrix $B_H$ is a submatrix of $B_G$ consisting of those entries $(B_G)_{a,b}$, where $a$ and $b$ occur in the list of branching of the class $\tau$ for $z$-classes $a, b$ of $Z_G(\tau)$ of tuples. We have the following.

**Proposition 2.10.** Let $(g_1, \ldots, g_r)$ be an $r$-tuple representing a $z$-class in a reductive group $G$. Then,

1. the branching matrix $B_H$ of $H := Z_G(g_1, \ldots, g_r)$ is a submatrix of $B_G$.
2. Let $\tau$ be a branch of $H$. Then, $(B_H^{d})_{a\tau} = B_{Z_H(\tau)}^{d}$.

**Proof.** The part (1) is clear from the explanation given above.

Now to prove (2) we note that given $\tau$, a branch of $H$, the branching submatrix $B_{Z_H(\tau)}$ of $B_H$ consists of $\tau$, branches of $\tau$, and the branches of those branches of $\tau$ and so on. When $a$ is a branch of $\tau$, or a branch of a branch of $\tau$, we see that, $(B_H^{2})_{a\tau} = \sum_{\eta}(B_H)_{a\eta}(B_H)_{\eta\tau}$. Now, $(B_H)_{a\eta}(B_H)_{\eta\tau} \neq 0$ if and only if $\eta$ is a branch of $\tau$, and $a$ is a branch of $\eta$. Hence $(B_H)_{a\eta}(B_H)_{\eta\tau}$ is non-zero only if both $a$ and $\eta$ are in the branching submatrix, $B_{Z_H(\tau)}$ of $\tau$. Hence, $(B_H^{2})_{a\tau} = \sum_{\eta}(B_{Z_H(\tau)})_{a\eta}(B_{Z_H(\tau)})_{\eta\tau} = (B_{Z_H(\tau)}^{2})_{a\tau}$.

Now, we complete the proof by induction. Let us assume the equation is true up to $d$, and prove it for $d+1$.

$$(B_H^{d+1})_{a\tau} = \sum_{\eta}(B_H)_{a\eta}(B_H^{d})_{\eta\tau} = \sum_{\eta}(B_H)_{a\eta}(B_{Z_H(\tau)}^{d})_{\eta\tau} \text{ by induction}$$

$$= \sum_{\eta}(B_{Z_H(\tau)})_{a\eta}(B_{Z_H(\tau)}^{d})_{\eta\tau} = (B_{Z_H(\tau)}^{d+1})_{a\tau}. $$

This completes the proof. 

Note that, when $G$ is reductive, $H := Z_G(g_1, \ldots, g_d)$ has finite size branching matrix even though $H$ may not be reductive.

3. Dimension of Commuting Tuples

Whether the variety of commuting $d$-tuples, $\mathcal{C}_d(G)$, is an irreducible variety is an active topic of research. For a simply connected semisimple algebraic group $G$, Richardson [R3] Theorem C] proved that $\mathcal{C}_d(G)$ is an irreducible variety. However, for our work we need to only understand the dimension of this variety. Clearly, when $G$ is Abelian, $\dim \mathcal{C}_d(G) = d \dim(G)$. We relate the dimension of $\mathcal{C}_d(G)$ with computation of $d$-th
power of the branching matrix $B_G$ here. Recall, that the entries of $B_G$ are monomials in $\psi$ and hence the entries of $B_G^d$ will be polynomials in $\psi$.

**Proposition 3.1.** Let $G$ be a reductive algebraic group, and let $H = Z_G(g_1, \ldots, g_r)$ be the centralizer of an $r$-tuple. Then for $d \geq 1$,

$$\deg(1_B^d H, e_1) \leq \dim(C_d(H)) \leq \deg(1_B^d H, e_1) + \dim(H)$$

where $1$ is a row matrix with all 1’s and $e_1$ is a column matrix with first entry 1 and all others 0.

**Proof.** First, we prove $\dim(C_d(H)) \leq \deg(1_B^d H, e_1) + \dim(H)$. We will use double induction on $r$ and $d$. First we prove this for $d = 1$. In this case, the required statement would be

$$\dim(H) \leq \deg(1_B H, e_1) + \dim(H)$$

which is trivially true for any $H$. Let us assume induction up to $d$. Before going ahead, we recall the following from Proposition 2.10. From the branching matrix $B_G$, we can obtain the branching matrix for any of the centralizer subgroup $Z_G(\tau)$ where $\tau$ is a $z$-class of tuples. Further, the branching matrix $B_{Z_G}(\tau)$ is a submatrix of $B_G$ consisting of those entries $(B_G)_{ab}$ where $a$ and $b$ occur in the list of branching of the class $\tau$ for $z$-classes $a, b$ of $Z_G(\tau)$ of tuples. Also, from Proposition 2.10 we note that when $B_G$ is multiplied with itself, this submatrix multiplies only with itself. Now, we write,

$$C_{d+1}(H) = \bigcup_{\tau} zcl(\tau) \times C_d(Z_H(\tau)),$$

where the union runs over $z$-classes in $H$. Let us denote the dimension of $H$ by $n$, and that of $Z_H(\tau)$ by $n_{\tau}$. So, we have

$$\dim C_{d+1}(H) = \max_{\tau} \{ \dim zcl(\tau) + \dim C_d(Z_H(\tau)) \} \leq \max_{\tau} \left\{ (\deg(B_H)_{\tau}, \dim cl(\tau)) + \deg \left(1_B^d Z_H(\tau), e_\tau \right) + n_{\tau} \right\} \text{ by induction}$$

$$= \max_{\tau} \left\{ \deg(B_H)_{\tau}, n - n_{\tau} + \deg \left(1_B^d Z_H(\tau), e_\tau \right) + n_{\tau} \right\} \text{ types } u, v \text{ are branches of type } \tau$$

$$= n + \max_{\tau} \left\{ \deg(B_H)_{\tau}, + \deg \left(\sum_a (B_H^d)_{a\tau} \right) \right\} \text{ from Proposition 2.10}$$

$$= n + \max_{\tau} \left\{ \deg \left(\sum_a (B_H^d)_{a\tau} (B_H)_{\tau} \right) \right\} = n + \deg \sum_a (B_H^{d+1})_{a1}$$

$$= \deg(1_B^{d+1} H, e_1) + n.$$
Now, we need to prove \( \deg(1.B^d_H.e_1) \leq \dim(\mathcal{C}_d(H)) \). We follow the notation set above and prove it by induction. To begin with, for \( d = 1 \), we need to show \( \deg(1.B_H.e_1) \leq \dim(H) \). This follows as the left hand side is maximal possible \( \dim(zcl(g) - \dim(cl(g)) \), where \( g \in H \), and \( cl(g) \subset zcl(g) \subset H \). Now let us assume this for \( d \) and prove for \( d + 1 \).

\[
\dim \mathcal{C}_{d+1}(H) = \max_\tau \{ \dim zcl(\tau) + \dim \mathcal{C}_d(Z_H(\tau)) \} \geq \max_\tau \{ \deg(B_H)_{\tau_1} + \dim(cl(\tau)) + \deg(1.B^d_H(\tau),e_\tau) \} \text{ by induction} \geq \max_\tau \{ \deg(B_H)_{\tau_1} + \deg(1,((B_H)_{aw})^d.e_\tau) \} = \max_\tau \{ \deg(B_H)_{\tau_1} + \deg(\sum_a(B^d_H)_{a\tau}) \} = \max_\tau \{ \deg(\sum_a(B^d_H)_{a\tau}.(B_H)_{\tau_1}) \} = \deg(\sum_a(B^d_H)_{a\tau}) = \deg(1.B^{d+1}_H.e_1).
\]

This completes the proof. □

Thus, we can rewrite

\[
\frac{\deg(1.B^d_G.e_1)}{d \dim G} \leq cp_d(G) \leq \frac{\deg(1.B^d_G.e_1) + \dim(G)}{d \dim G} = \frac{\deg(1.B^d_G.e_1)}{d \dim G} + \frac{1}{d}.
\]

In the next section, we compute this for reductive algebraic groups.

4. Commuting probability for reductive algebraic groups

Let \( K \) be an algebraically closed field and \( G \) be a reductive algebraic group over \( K \) of dimension \( n \). In this section, we discuss the asymptotic value of the commuting probabilities for \( G \). In [Ga], it is proved that \( cp_2(G) = \frac{n+\rho}{2n} \) where \( \rho \) is the rank of \( G \). Using the argument there, one can show that \( cp_d(G) \geq \frac{n+(d-1)\rho}{dn} \). As noticed in [KPP] for finite groups while studying asymptotic behavior of \( cp_d(G) \) as \( d \) gets large, we see that the maximal dimension of an Abelian subgroup plays a role here. Henceforth, whenever we talk about maximal Abelian subgroup, we mean a subgroup of maximal size/dimension among Abelian subgroups. Our main theorem is as follows:

**Theorem 4.1.** Let \( G \) be a reductive algebraic group over an algebraically closed field \( K \). Let \( \dim(G) = n \), maximal dimension of an Abelian subgroup be \( \alpha \) (in general, \( \alpha \geq \text{rank}(G) \)), and the size of the branching matrix be \( \beta \). Then, for large enough \( d \),

\[
\left(1 - \frac{\beta}{d}\right) \frac{\alpha}{n} = \frac{(d-\beta)\alpha}{dn} \leq cp_d(G) \leq \frac{\alpha}{n} + \frac{1}{d}.
\]

Thus, as \( d \) gets large, the commuting probabilities \( cp_d(G) \sim \frac{\alpha}{n} \).

We need a couple of Lemmas before we prove this result.
**Lemma 4.2.** Let $G$ be a reductive algebraic group, and $B_G$ be its branching matrix. Then, the maximal entry of the branching matrix $B_G$ is $\psi^\alpha$.

**Proof.** Let $(g_1, \ldots, g_k)$ be a commuting $k$-tuple of $G$, and suppose the common centralizer $Z_G(g_1, \ldots, g_k)$ is Abelian of order $b$. For each $1 \leq i \leq k$, let $Z_i = Z_G(g_1, \ldots, g_i)$. Thus, we have $Z_{i+1} = Z_i(g_{i+1})$, for $1 \leq i \leq k-1$. We get a non-increasing sequence of centralizer subgroups $Z_1 \supset Z_2 \supset \cdots \supset Z_{k-1} \supset Z_k$. Now, we claim that the centres of these centralizers $Z(Z_i)$ form a non-decreasing sequence, i.e., $Z(Z_i) \subset Z(Z_{i+1})$. For this, let $z \in Z(Z_i)$. Now, as $z$ commutes with $g_i+1$, we have $z \in Z_{i+1}$. But, $Z_{i+1} \subset Z_i$ hence $z \in Z(Z_{i+1})$. This proves $Z(Z_{i+1}) \supset Z(Z_i)$.

Let $g \in H$, which is a common centralizer of a commuting tuple. Let $k$ be the smallest integer such that $(g = g_1, \ldots, g_k)$ be a commuting $k$-tuple of $H$, with the common centralizer $Z_H(g_1, \ldots, g_k)$, Abelian of dimension $b$. We claim that

$$\dim zcl(g) - \dim cl(g) \leq b.$$ 

Since $zcl(g) = \bigcup_t cl(t)$ where union is over $t \in H$ of which centralizer is conjugate to $Z_H(g)$ (see Remark 2.5). Thus,

$$\dim zcl(g) - \dim cl(g)$$

$$= - \dim N_H(Z_H(g)) + \dim \{ x \in cl(H) \mid Z_H(x) = Z_H(g) \} + \dim Z_H(g)$$

$$\leq \dim \{ x \in cl(H) \mid Z_H(x) = Z_H(g) \} \leq \dim Z(Z_H(g))$$

$$\leq \dim Z(Z_H(g_1, \ldots, g_k)) \text{ (from the first para of this proof)}$$

$$= b \leq \alpha.$$  

For the last but one line, we use the following: $\{ x \in H \mid Z_H(x) = Z_H(g) \} \subset Z(Z_H(g))$.

□

Now we prove a result regarding entries of power of a matrix which we will apply to the branching matrix.

**Lemma 4.3.** Let $B = (b_{i,j})$ be a non-negative $m \times m$ matrix with all diagonal entries non-zero. Suppose, $B$ has the property that every row has a non-zero entry before the diagonal, i.e., $\forall i > 1$ there exists $i_0 < i$ such that $b_{i,i_0} \neq 0$. Then, for $r \geq 2$, the $(l,1)$th entry of $B^r$ is a polynomial in $b_{l,1}$ of degree at most $r$, and at least $r - m$.

**Proof.** Let $B = (b_{i,j})$. Then,

$$(B^r)_{s,1} = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_{r-1}} b_{s,l_1} b_{l_1,l_2} \cdots b_{l_{r-1},1}$$

$$= b_{s,1} b_{l_1,1}^{r-1} + b_{s,2} b_{l_2,1}^{r-1} + \cdots$$
If \( l = 1 \) and \( b_{i,1} = 0 \) for all \( i > 1 \), then \( B = \begin{pmatrix} b_{1,1} & * \\ 0 & C \end{pmatrix} \), where \( C \) is an \((m - 1) \times (m - 1)\) matrix. For any \( r \), we can see that \((B^r)_{1,1} = b_{1,1}^r\).

Let \( b_{l,1} \neq 0 \). Then, \((B^2)_{l,1} = \sum_{i=1}^{m} b_{i,l}b_{i,1} = b_{l,1}b_{1,1} + \cdots + b_{l,l}b_{l,1} + \cdots\), which we rewrite as \( b_{l,1}b_{1,1} + d_2 \), where \( d_2 \) denotes the rest of the terms (which are constant in \( b_{l,l} \) and \( b_{1,1} \)). Now, \((B^3)_{l,1} = (B.B^2)_{l,1} = \sum b_{i,i}(B^2)_{i,1} = b_{l,1}b_{1,1} + d_2 + d_3 = b_{l,j}^2b_{1,1} + b_{l,j}b_{1,1}b_{1,1} + b_{l,1}b_{1,1}^2 + d_4\). The result follows by induction, as we can prove that \((B^r)_{l,1} = b_{l,1}b_{1,1}^{r-1} + c_1b_{l,1}b_{1,1}^{-2} + \cdots + c_{r-1}b_{1,1}^{r-1} + d_r\), where \( d_r \) denotes the rest of the terms in the sum.

Now, suppose \( b_{l,1} = 0 \) (obviously \( l > 1 \)). Let \( u \leq m \) be the smallest such that we have a sequence of numbers \( l = k_1, k_2, \ldots, k_u \) such that the entries \( b_{k_1,k_2}, \ldots, b_{k_{u-1},k_u}, b_{k_u,1} \) are all non-zero. Then, \((B^u)_{1,1} = b_{k_2,k_3}b_{k_3,k_4} \cdots b_{k_u,1} + \cdots \) is non-zero. Now, following the argument similar to the last para, we see that for \( r > u \), we have \((B^r)_{l,1}\) is a polynomial in \( b_{l,l}\) of degree \( r - u \). Finally, a sequence of numbers \( l = k_1, k_2, \ldots, k_u \) with the required property is guaranteed because of the given condition as follows. Begin with the \( l^{th} \) row, and find smallest \( k_2 < l \) such that \( b_{k_2} \neq 0 \). Next, look at the row \( k_2 \) and find smallest \( k_3 < k_2 \) such that \( b_{k_3,1} \neq 0 \). We will be done when we get \( k_u \) with \( b_{k_u,1} \neq 0 \) (and noting that \( b_{2,1} \neq 0 \)).

Now, we prove the theorem.

**Proof of the Theorem 4.1.** In view of Proposition 3.1 and Equation 3.1, we require to prove the following for large enough \( d \),

\[(d - \beta)\alpha \leq \deg(1.B^d_G,e_1) \leq d\alpha.\]

Note that entries of \( B_G \) are either 0 or powers of \( \psi \), which follow the condition required in the Lemma 4.3. Write \( B_G = (\psi^{x_{i,j}}) \). Then for \( d \geq 2 \),

\[(B^d)^{s,1} = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_{d-1}} b_{s,l_1}b_{l_1,l_2} \cdots b_{l_{d-1},1} = \sum_{l_1} \sum_{l_2} \cdots \sum_{l_{d-1}} \psi^{x_{s,l_1} + x_{l_1,l_2} + \cdots + x_{l_{d-1},1}}\]

gives us that \( \deg(B^d)^{s,1} \leq r\alpha \). Thus, \( \deg(1.B^d_G,e_1) \) which is the degree of sum of the first column \( \leq d\alpha \).

From Lemma 4.3, we note that \((B^d)^{s,1}\) is a polynomial in \( \psi^{x_{s,s}} \) of degree at least \( d - \beta \), i.e., \( \deg(B^d)^{s,1} \geq (d - \beta)x_{s,s} \). The largest degree on diagonal (in fact whole of \( B_G \)) is \( \alpha \). Thus, \( \deg(1.B^d_G,e_1) \geq (d - \beta)\alpha \).

Hence,

\[cp_d(G) \leq \frac{\deg(1.B^d_G,e_1)}{d \dim G} + \frac{1}{d} \leq \frac{\alpha}{n} + \frac{1}{d}.\]

Also,

\[cp_d(G) \geq \frac{\deg(1.B^d_G,e_1)}{d \dim G} \geq \frac{(d - \beta)\alpha}{dn} = \left(1 - \frac{\beta}{d}\right)\frac{\alpha}{n}.\]
This completes the proof.

5. Asymptotic value of commuting probabilities in finite groups

Let \( G \) be a finite group, and \( d \geq 2 \) be a positive integer. We begin with recalling the relation between commuting probabilities and simultaneous conjugacy classes of commuting tuples via branching matrix \( B_G \). The entries of this matrix represent the number of conjugacy classes of tuples which are in the same \( z \)-class, i.e., the size of \( z \)-class divided by the size of conjugacy class. This is done in [SS], and we refer to the same for various definitions and terminologies used in this section. Some of these ideas have been generalized in the earlier sections for algebraic groups. Since \( G \) acts on the set \( \mathcal{C}_d(G) \), by component-wise conjugation, we have simultaneous conjugacy classes (of commuting \( d \)-tuples). Let \( c_G(d) \) denote the number of orbits in the above action, also called simultaneous conjugacy classes of \( d \) tuples in \( G \). It has been proved in [SS, Theorem 1.1] that,

\[
\text{Theorem 5.1. Let } G \text{ be a finite group and } d \geq 2, \text{ an integer. Let } B_G \text{ be the branching matrix of } G. \text{ Then, }
\]

\[
cp_d(G) = \frac{c_G(d - 1)}{|G|^{d-1}} = \frac{1_B G^{d-1} e_1}{|G|^{d-1}}
\]

where \( 1 \) is a row matrix, with all 1’s, and \( e_1 \) is a column matrix with first entry 1, and 0 elsewhere.

The commuting probabilities \( cp_d(G) \), for \( d = 2, 3, 4, 5 \) have been explicitly calculated in [SS2] using the corresponding branching matrices with the help of SageMath [SA] for the following classical groups over a finite field \( \mathbb{F}_q \) (where \( q \) is odd): \( G = GL_2(\mathbb{F}_q), U_2(\mathbb{F}_q), GL_3(\mathbb{F}_q), U_3(\mathbb{F}_q), \) and \( Sp_4(\mathbb{F}_q) \). The data obtained from these groups led us to explore the asymptotic behavior of \( cp_d(G) \) as \( d \) gets large for a fixed \( G \). In other words, we would like to understand what is \( cp_d(G) \) asymptotic to, as a function of \( d \)? Interestingly, Kaur, Prajapati and Prasad [KPP, Theorem 3.1] have shown that for a finite group \( G \) and positive integer \( d \), the number \( c_G(d) \) is asymptotic to \( a^d \), up to multiplication by a positive constant, where \( a \) denotes the maximal size of an Abelian subgroup of \( G \). That is, there exist a positive integer \( m \) so that \( c_G(d) \sim ma^d \). Thus, from Theorem 5.1 it follows that,

\[
(5.1) \quad cp_d(G) \sim m \left( \frac{a}{|G|} \right)^{d-1}.
\]

Using the ideas in Section 4 we give an alternate proof of this.

We are going to make use of the branching matrix \( B_G \), for the finite group \( G \), as described in [SS, SS2] and the Theorem 5.1. For \( d \geq 2 \), the size of the set \( \mathcal{C}_d(G) \) of
commuting $d$-tuples of elements of $G$ is 

$$|\mathcal{C}_d(G)| = cp_d(G).|G|^d = |G| \left(1.B_G^{d-1}.e_1 \right).$$

Thus, to understand $\mathcal{C}_d(G)$ we need to understand the entries of the first column in the matrix $B_G^{d-1}$. We begin with a result for finite groups similar to Lemma 4.2 proved earlier for algebraic groups.

**Lemma 5.2.** With the notation as above, the maximal entry in the branching matrix $B_G$ of $G$ is the maximal size of an Abelian subgroup $a$.

**Proof.** Proceeding along the lines of the proof of Lemma 4.2, we note that the entries of the matrix $B_G$ will satisfy the following:

$$\frac{|\text{zcl}(g)|}{|\text{cl}(g)|} = \frac{|Z_H(g)|}{|N_H(Z_H(g))|}.|\{x \in \text{cl}(H) \mid Z_H(x) = Z_H(g)\}| \leq |Z(Z_H(g))| \leq a$$

where $H$ is a centralizer of some tuples. The result follows. □

Now, we give an alternate proof of the Equation 5.1. The proof is along the same lines as that of Theorem 4.1 and hence we keep it brief.

**Proposition 5.3.** Let $G$ be a finite group and $a$ be the size of maximal Abelian subgroup. Then, for large enough $d$, the size of commuting $d$-tuples, $|\mathcal{C}_d(G)| \sim m|G|^d$ where $m$ is a constant.

**Proof.** We begin with proving that there exists a constant $m$ such that $1.B_G^{d-1}.e_1 = ma^d + O(a^{d-1})$ when $d$ is large. Note that, the branching matrix $B_G = (b_{i,j})$ satisfies the properties required in the Lemma 4.3. Thus, $(B_G^d)_{s,1}$ is a polynomial in $b_{s,s}$ of degree at most $d$ and at least $d - \beta$ where $\beta$ is the size of $B_G$. From Lemma 5.2 the largest diagonal entry (in fact, the largest entry) of $B_G$ is $a$. Thus, for large $d$, we get $1.B_G^{d-1}.e_1 = ma^d + O(a^{d-1})$ where $m$ is a constant depending on $G$ only.

Now we have, $|\mathcal{C}_d(G)| = |G| \left(1.B_G^{d-1}.e_1 \right)$, thus,

$$|\mathcal{C}_d(G)| = m|G|^a^{d-1} + O(a^{d-2}).$$

This proves the required result. □

Next we look at some examples.

5.1. **Application to finite reductive groups.** Let $\mathbb{F}_q$ be a finite field and $K$ its algebraic closure. Let $G$ be a connected reductive group over $K$, with Frobenius map $F$ so that $G(\mathbb{F}_q) = G^F$ is a finite group of Lie type. Then, we have the following.

**Theorem 5.4.** Let $G$ be a connected reductive group defined over a finite field $\mathbb{F}_q$. Let us denote the $\mathbb{F}_q$ points of $G$ by $G(\mathbb{F}_q) = G^F$. Then, for large enough $q$,

$$|\mathcal{C}_d(G(\mathbb{F}_q))| \sim q^{n+(d-1)\alpha}.$$
up to a constant where \( n \) is the dimension of \( G \), and \( \alpha \) is the maximal dimension of an Abelian subgroup. Hence up to a constant, \( \text{cp}_d(G(\mathbb{F}_q)) \sim \left(\frac{1}{q^{n-\alpha}}\right)^{d-1} \).

Proof. From [St2 Theorem 11.16], it follows that \( |G(\mathbb{F}_q)| = q^n + \mathcal{O}(q^{n-1}) \). Thus, when \( q \) is large enough (to ensure that only its power is dominating), we get the result from Proposition 5.3. \( \square \)

Maximal size/dimension of Abelian subgroups are well studied for finite classical groups and more generally for finite simple groups (see [Vd1, Vd2, Wo1, Wo2, Ba]). It turns out that for \( G \), a finite simple group of Lie Type of large enough rank, an Abelian subgroup of maximal order is unipotent. If \( G \) is not simple, then an Abelian subgroup of maximal order is the product of the centre of the group and an Abelian unipotent group in \( G \) of maximal order. Now we look at some examples, mainly of finite groups of Lie type where maximal sized Abelian subgroups are known, and give asymptotic value of commuting probabilities as \( d \) gets large. In what follows, we take \( q \) large enough and use the formula \( \frac{a}{|G|} \) to compute the asymptotic value of \( \text{cp}_d(G) \).

**Example 5.5.** For the group \( GL_2(q) \), we have \( |GL_2(q)| = (q^2 - 1)(q^2 - q) \) and the maximal order of an Abelian subgroup is \( q^2 - 1 \) (given by an anisotropic torus). Then, \( \text{cp}_d(GL_2(q)) \sim \left(\frac{1}{q(q-1)}\right)^{d-1} \) up to a constant.

For \( G = GL_3(q) \), we have the maximal size of an Abelian subgroup \( a = q^3 - 1 \) (again given by an anisotropic torus), and \( |G| = (q^3 - 1)(q^3 - q)(q^3 - q^2) \). Then, \( \text{cp}_d(GL_3(q)) \sim \left(\frac{1}{q^3(q^2 - 1)(q - 1)}\right)^{d-1} \).

In both of these cases, the maximal Abelian is obtained by centralizer of a regular semisimple element, that is, by a 1-tuple.

**Example 5.6.** Consider the group \( GL_{2l}(q) \) and \( l \geq 2 \). The following block diagonal matrices

\[
A = \left\{ \begin{pmatrix} \lambda I_l & X \\ \lambda I_l \end{pmatrix} \mid X \in M_l(q), \lambda \in \mathbb{F}_q^\times \right\}
\]

give a maximal sized Abelian subgroup with order \((q - 1)q^{2l} = q^{2l+1} + \mathcal{O}(q^{2l})\). Notice that a maximal torus is of size \( q^{2l} + \mathcal{O}(q^{2l-1}) \) and \( A \) is bigger than this. Further, we note that \( A \) can be obtained as a centralizer of commuting 3-tuple as follows:

\[
A = Z_{GL_{2l}(q)} \left( \begin{pmatrix} I & I \\ I & I \end{pmatrix}, \begin{pmatrix} I & A \\ I & I \end{pmatrix}, \begin{pmatrix} I & N \\ I & I \end{pmatrix} \right)
\]
Example 5.8. We notice matrices:

\[
\begin{pmatrix}
\lambda_1 & X \\
\lambda_{l+1} &
\end{pmatrix}
\]

with all distinct entries, and \( N = \begin{pmatrix} 0 & \cdots & 1 \\ 1 & 0 & \cdots \\ \vdots & \ddots & \ddots \\ 0 & 1 & 0 \end{pmatrix} \).

Now, consider the group \( GL_{2l+1}(q) \) for \( l \geq 2 \) and let \( A \) be the following block diagonal matrices:

\[
A = \left\{ \begin{pmatrix} \lambda I_l & X \\ \lambda I_{l+1} & \end{pmatrix} \mid X \in M_{l \times (l+1)}(q), \lambda \in \mathbb{F}_q^* \right\}.
\]

Then, \( A \) is a maximal size Abelian subgroup with order \((q - 1)q^{l(l+1)} = q^{(l+1)} + O(q^{2l+1})\). Once again this can be obtained as a centralizer of commuting tuple.

Thus, for all \( n \geq 4 \), the maximal cardinality of any Abelian subgroup of \( GL_n(q) \) is \( q^{[n^2/4]}(q - 1) \). Thus, up to a constant, the commuting probabilities

\[
cp_d(GL_{2l}(q)) \sim \left( \frac{1}{q^{l(l-1)} \prod_{i=2}^{2l}(q^i - 1)} \right)^{d-1}
\]

and

\[
cp_d(GL_{2l+1}(\mathbb{F}_q)) \sim \left( \frac{1}{q^{2l+1} \prod_{i=2}^{2l+1}(q^i - 1)} \right)^{d-1}.
\]

Example 5.7. For \( U_2(q) \), from [SS2, Proposition 3.3] we have \( cp_d(U_2(q)) = cp_d(GL_2(q)) \) for all \( d \geq 2 \), so the asymptoticity is the same as in Example 5.5.

For \( U_3(q) \), the maximal size for an abelian subgroup is \((q + 1)^3\). Thus, \( cp_d(U_3(q)) \) is asymptotic to

\[
\left( \frac{1}{q^3(q^2 - q + 1)(q - 1)} \right)^{d-1}.
\]

Now we take \( U_n(q) \) for \( n \geq 4 \), its centre is of size \( q + 1 \), and the maximal cardinality of its unipotent Abelian subgroup is \( q^{[n^2/4]} \), like it is with \( GL_n(q) \). Hence, the maximal Abelian cardinality is \( q^{[n^2/4]}(q + 1) \). Thus up to a constant,

\[
cp_d(U_{2l}(q)) \sim \left( \frac{1}{q^{l(l-1)} \prod_{i=2}^{2l}(q^i - (-1)^i)} \right)^{d-1}
\]

and

\[
cp_d(U_{2l+1}(q)) \sim \left( \frac{1}{q^{2l+1} \prod_{i=2}^{2l+1}(q^i - (-1)^i)} \right)^{d-1}.
\]

We notice \( q \leftrightarrow -q \), Ennola like duality, between the formula of \( GL \) and \( U \) for the asymptotic value.

Example 5.8. For \( l \geq 1 \) and \( q \) odd, let us consider the symplectic group \( Sp_{2l}(q) = \{ g \in GL_{2l}(q) \mid g \beta g = \beta \} \) where \( \beta = \begin{pmatrix} I_l & \\
-I_l & \end{pmatrix} \). For \( l = 1 \) the group \( Sp_2(q) \cong SL_2(q) \), and
the maximal abelian subgroup is of size $2q$. So, for $d \geq 2$, the commuting probabilities $c_{p_d}(Sp_2(q))$ is asymptotic (upto multiplication by some positive constant) to

$$\left(\frac{2}{q^2 - 1}\right)^{d-1}.$$ 

Now, for large enough $q$,

$$A = \left\{ \pm \begin{pmatrix} I_l & X \\ I_l & I_l \end{pmatrix} \mid \begin{pmatrix} I_l \\ X \end{pmatrix} = X \right\}$$

is a maximal size Abelian subgroup of $Sp_2(q)$ of order $2q^{(l+1)/2}$. Hence, up to a constant,

$$c_{p_d}(Sp_2(q)) \sim \left(\frac{2q^{(l+1)/2}}{q^l \prod_{i=1}^l (q^{2i} - 1)}\right)^{d-1} = \left(\frac{2}{q^{(l+1)/2} \prod_{i=1}^l (q^{2i} - 1)}\right)^{d-1}.$$ 

**Example 5.9.** Let us consider the orthogonal group $O_{2l}(q) = \{g \in GL_{2l}(q) \mid g\beta g = \beta\}$ where $\beta = \begin{pmatrix} I_l \\ I_l \end{pmatrix}$ and $q$ odd. Then,

$$A = \left\{ \pm \begin{pmatrix} I_l & X \\ I_l & I_l \end{pmatrix} \mid \begin{pmatrix} I_l \\ X \end{pmatrix} = -X \right\}$$

is a maximal size Abelian subgroup with order $2q^{(l+1)/2}$. Hence up to a constant,

$$c_{p_d}(O_{2l}(q)) \sim \left(\frac{2q^{(l+1)/2}}{2q^{l(l-1)/2} \prod_{i=1}^l (q^{2i} - 1)}\right)^{d-1} = \left(\frac{1}{q^{(l+1)/2} \prod_{i=1}^l (q^{2i} - 1)}\right)^{d-1}.$$ 

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