Turbulent $\alpha$-effect in twisted magnetic flux tubes dynamos in Riemannian space

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Abstract

Analytical solution of first order torsion $\alpha$-effect in twisted magnetic flux tubes representing a flux tube dynamo in Riemannian space is presented. Toroidal and poloidal component of the magnetic field decays as $r^{-1}$, while grow exponentially in time. The rate of speed of the helical dynamo depends upon the value of Frenet curvature of the tube. The $\alpha$ factor possesses a fundamental contribution from constant torsion tube approximation. It is also assumed that the curvature of the magnetic axis of the tube is constant. Though $\alpha$-effect dynamo equations are rather more complex in Riemann flux tube coordinates, a simple solution assuming force-free magnetic fields is shown to be possible. Dynamo solutions are possible if the dynamo action is able to change the signs of torsion and curvature of the dynamo flux tube simultaneously. \textbf{PACSnumbers:02.40.Hw-Riemann geometries}

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I Introduction

It is well-known that though, the astrophysical jets [1] chaotic magnetic fields components decay as $r^{-1}$ and $r^{-2}$, unbounded magnetic fields [1] may decay up to $r^{-3}$ at least. In this paper, the α-effect in bounded system of a twisted magnetic flux tube, is shown to lead to a simple dynamo solution if one considers magnetic force-free fields [1]. Despite the fact the dynamo equations are rather more complex than the cylindrical ones developed by Zeldovich, Ruzmaikin and Sokoloff [2] rather simple solutions can be obtained on simple few cases. Despite of the success of the application of the numerical simulations to the dynamo problem [3] in plasma astrophysics [3] and in the stretch-twist-fold (STF) Vainshtein-Zeldovich [2] mechanism, recently new dynamo analytical solutions have been found [4] by using the conformal mapping technique in Riemannian manifolds from old Arnold cat dynamo metric [5]. In this paper, the helical dynamo [6] solution of the system of nonlinear system composed by the self-induction, solenoidal and force-free equations are found in the Riemann curved and torsioned flux tube coordinates. Dynamo solutions are obtained vorticity is stronger than torsion, and fast rate of the dynamo depends purely on the curvature of the thin twisted magnetic flux tubes in Riemannian, where MHD equations are linear in the magnetic field and nonlinear in the velocity flow. Recently, Hanasz and Lesch [7] have also considered a conformal Riemannian metric in $E^3$ to the galactic dynamo magnetic flux tubes. Pioneering work on the magnetic flux tubes as dynamos was done earlier by Schussler [8], however in his work tubes were untwisted and straight. The main advantage of the investigation of the isolated flux tube dynamo is that one is able to investigate the curvature and twist contributions of the tube to the dynamo action. Twist is actually related to the torsion of the magnetic axis of the tube, which makes the words strong torsion equivalent to strong twist, which physically is important to the twist-kink relation investigated by Alfven [9]. Chaotic flows in magnetic fast dynamos [10] mentioned above are however, kinematic dynamos in which velocity appears linearly in the dynamo equations, given a priori which makes their applications somewhat limited, this is one of the motivations which led us here to considered turbulent dynamo in magnetic flux tubes where non-linear velocity flows appear in the magnetic dynamo equations. Helical dynamo here is understood as the one where the flow describes a circular helix where torsion and curvature are constants. The paper
is organized as follows: In section II the $\alpha$-dynamo equations in the Riemann metric representing flux rope (twisted tubes) are obtained along with the in section III the approximate solution. In section III conclusions are given.

II $\alpha$-effect dynamo equations in Riemannian flux tubes

In this section we shall be concerned with solving the MHD equations in the curved coordinates of a thin twisted magnetic flux tube of Riemann metric

$$ds^2 = dr^2 + r^2d\theta^2 + K^2(s)ds^2$$

which represents a Riemann line element

$$ds^2 = g_{ij}dx^idx^j$$

if the tube coordinates are $(r, \theta, s)$ [1] where $\theta(s) = \theta_R - \int \tau ds$ and $\tau$ is the Frenet torsion of the tube axis, $K(s)$ is given by

$$K^2(s) = [1 - \tau\kappa(s)\cos\theta(s)]^2$$

Since we are considered thin magnetic flux tubes, this expression shall be taken as $K \approx 1$ in future computations. In curvilinear coordinates the Riemannian Laplacian operator $\nabla^2$ is

$$\nabla^2 = \frac{1}{\sqrt{g}}\partial_i[\sqrt{g}g^{ij}\partial_j]$$

where $\partial_j := \frac{\partial}{\partial x^j}$ and $g := detg_{ij}$ where $g_{ij}$ are the covariant components of the Riemann metric of flux rope. Let us now start by considering the MHD field equations

$$\nabla.\vec{B} = 0$$

$$\frac{\partial}{\partial t}\vec{B} = \nabla \times [\alpha \vec{B}] = \alpha \lambda \vec{B} + \nabla \alpha \times \vec{B}$$

called the $\alpha$-dynamo equation [2]. Where we have used the force-free magnetic field equation

$$\nabla \times \vec{B} = \lambda \vec{B}$$
in (6), where $\lambda := \frac{2\kappa}{\tau}$, defined in terms of the constant curvature and torsion, for future computation convenience. To these equations one adds the sometimes the $\alpha := \langle \vec{\omega} \cdot \vec{\omega} \rangle$ parameter is constant but here we shall be considering the more general case where it depends on the radial and poloidal coordinate. Here $\omega := \nabla \times \vec{v}$ is the vorticity of the dynamo flow. Equation (3) represents the self-induction equation. The vectors $\vec{t}$ and $\vec{n}$ along with binormal vector $\vec{b}$ form the Frenet holonomic frame, which obeys the Frenet-Serret equations

\begin{align*}
\vec{t}' &= \kappa \vec{n} \\
\vec{n}' &= -\kappa \vec{t} + \tau \vec{b} \\
\vec{b}' &= -\tau \vec{n}
\end{align*}

where the dash represents the ordinary derivation with respect to coordinate $s$, and $\kappa(s, t)$ is the curvature of the curve, where $\kappa = R^{-1}$. Here $\tau$ represents the Frenet torsion. The gradient operator is

\begin{equation}
\nabla = \vec{t} \frac{\partial}{\partial s} + \vec{e}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \vec{e}_r \frac{\partial}{\partial r}
\end{equation}

Now we shall consider the analytical solution of the self-induction magnetic equation which represents a non-dynamo thin magnetic flux tube. Before the derivation of this result is obtained, we would like to point it out that it is not trivial, since the Zeldovich antidynamo theorem states that the two dimensional magnetic fields do not support dynamo action. Here, as is shown below, the flux tube axis possesses not only Frenet curvature, but torsion as well, and this last one vanishes in planar curves. The magnetic field does not possess a radial component and the magnetic field can be split into its toroidal and poloidal components as

\begin{equation}
\vec{B}(r, s, t) = B_\theta(t, r, \theta(s)) + B_s(r)\vec{t}
\end{equation}

Now let us substitute the definition of the poloidal plus toroidal magnetic fields into the self-induction equation, along with expressions

\begin{equation}
\vec{e}_\theta = -\vec{n} \sin \theta + \vec{b} \cos \theta
\end{equation}

and

\begin{equation}
\vec{e}_r = \vec{n} \cos \theta + \vec{b} \sin \theta
\end{equation}
\[ \partial_t \epsilon^\theta = \omega^\theta \epsilon_r - \partial_t \tilde{n} \sin \theta + \partial_t \tilde{b} \cos \theta \]  

(15)

Considering the equations for the time derivative of the Frenet frame given by the hydrodynamical absolute derivative

\[ \dot{\mathbf{X}} = \partial_t \mathbf{X} + [\mathbf{v} \cdot \nabla] \mathbf{X} \]  

(16)

where \( \mathbf{X} = (\mathbf{t}, \mathbf{n}, \mathbf{b}) \) is used into the expressions for the total derivative of each Frenet frame vector

\[ \dot{\mathbf{t}} = \partial_t \mathbf{t} + [\kappa' \mathbf{b} - \kappa \tau \mathbf{n}] \]  

(17)

\[ \dot{\mathbf{n}} = \kappa \tau \dot{\mathbf{t}} \]  

(18)

\[ \dot{\mathbf{b}} = -\kappa' \dot{\mathbf{t}} \]  

(19)

therefore leading to the following values of respective partial derivatives of the Frenet frame

\[ \partial_t \dot{\mathbf{t}} = -\tau \kappa [1 - \kappa \tau^{-2} \frac{v^\theta}{r}] \mathbf{n} \]  

(20)

\[ \partial_t \dot{\mathbf{n}} = \tau \kappa [1 - \kappa \tau^{-2} \frac{v^\theta}{r}] \mathbf{t} + \frac{v^\theta}{r} \mathbf{b} \]  

(21)

\[ \partial_t \dot{\mathbf{b}} = \kappa \tau^{-1} \frac{v^\theta}{r} \mathbf{n} \]  

(22)

where use has been made of the hypothesis that \( \dot{\mathbf{b}} = 0 \) or \( \kappa'(t, s) = 0 \), which means that the curvature only depends on time. An important vectorial expressions is

\[ \partial_t \epsilon^\theta = \tau_0 \sin \theta \mathbf{t} - \theta_0 \cos \theta \mathbf{n} - \omega_0 \sin \theta \mathbf{b} \]  

(23)

note that in the mean field dynamo case, where \( \dot{\mathbf{v}} = \mathbf{v}(\mathbf{B}) \), equation (6) is an eigenvalue problem equation. Dynamo operators and eigenvalue of dynamos in compact Riemannian manifolds have been previously investigated by Chicone and Latushkin \[11\]. Before presenting the field equations for the magnetic components, we assume to simplify matters that the ratio between them is

\[ \frac{B_\theta}{B_s} = \frac{\kappa_0}{\tau_0} \]  

(24)
Substitution of previous equations into equation (6) and splitting these equations along the components of the Frenet frame \((\vec{t}, \vec{n}, \vec{b})\) yields the following three scalar equations

\[
\partial_t B_s + \sin\theta \kappa_0 B_\theta = \lambda \alpha B_s + (\partial_r \alpha) B_\theta \tag{25}
\]
\[
\partial_t B_\theta = \lambda \alpha B_\theta + (\partial_r \alpha) B_s \tag{26}
\]

where we have used the approximation

\[
\Omega_0 + \tau_0 = 0 \tag{27}
\]

where \(\Omega_0\) is the vorticity along the magnetic tube axis. To obtain the solution one computes the dynamo factor \(\alpha\) one needs the equations for the dynamo flow vorticity

\[
\omega_r = -\partial_s v_\theta = \kappa_0 \tau_0 r v_\theta \sin \theta \tag{28}
\]

where we have used in this equation the physical assumption of the incompressibility of the dynamo flow. The remaining vorticity expressions are

\[
\omega_\theta = -\partial_r v_s \tag{29}
\]
\[
\omega_s = \partial_r v_\theta + \frac{1}{r} v_\theta \tag{30}
\]

This allows us to obtain the following value for \(\alpha\)

\[
\alpha(r) = \Omega_0 [\Omega_0 - 2r] \tag{31}
\]

Substitution of these equations into equations (25) and (26) yields the toroidal component

\[
B_s = B_0(r) \frac{\kappa_0}{\beta} \exp[\beta t] \tag{32}
\]

where \(\beta := \tau_0^2 \frac{\kappa_0}{\tau_0} - \lambda\). Thus the value for the poloidal component as

\[
B_\theta = \frac{\tau_0}{\kappa_0} B_0(r) \frac{\kappa_0}{\beta} \exp[\lambda \alpha + \frac{\tau_0^3}{\kappa_0} t] \tag{33}
\]

where we have already used the approximation that we are very close to the magnetic flux tube axis \((r=0)\). By equating the time exponentials, since the
ratio between poloidal and magnetic components does not depend on time one is able to determine $\lambda$ as

$$\lambda = \frac{1}{2} \frac{\kappa_0^2 - \tau_0^2}{\tau_0 \kappa_0}$$  \hspace{1cm} (34)$$

Toroidal component is finally given by

$$B_s = B_0(r) \frac{\kappa_0}{\beta} \exp[\frac{\kappa_0}{2\tau_0} t]$$  \hspace{1cm} (35)$$

Note that to pursue dynamo action we must have the curvature and torsion of the tube axis must possess the same sign or if we change the torsion sign dynamo action must change the curvature of the axis of the tube dynamo. To determine finally the function $B_0(r)$ one uses the expression for the force-free magnetic field equation (7) yields

$$\partial_r B_0 + \left(\frac{1}{r} - \frac{\lambda^2}{r}\right) B_0 = 0$$  \hspace{1cm} (36)$$

which yields

$$B_0 = \frac{c_0}{r}$$  \hspace{1cm} (37)$$

where $c_0$ is an integration constant. Thus the complete solution is

$$B_\theta = \tau_0 \frac{c_0}{r} \beta^{-1} \exp[\frac{\kappa_0}{2\tau_0} t]$$  \hspace{1cm} (38)$$

Note however that the while which decays slower as we go away from the magnetic axis of flux tube dynamo.

### III Conclusions

In conclusion, we have used an approximate method of first-order torsion to find near the magnetic axis of the dynamo flux tube to analytical (non-numerical) solutions of $\alpha$-dynamo equation. Since the tube we used here is twisted, it is also non-axisymmetric, and Cowling 1934 antidynamo theorem does not apply here. Previously Ruzmaikin et al [1] have found solutions of turbulent dynamos where unbounded magnetic field could decay at least as $r^{-3}$ which is distinct to the bounded case of twisted magnetic flux tube helical
dynamo we have found here. The general equations of the helical dynamo found in section II could be used to find out more general solutions, with less degree of approximation that we used here, for example by letting the magnetic poloidal component vary with time and also considered the case of untwisted tubes. One finally could say that one important consequence of the $\alpha$-effect in the flux tube dynamo is to change signs of torsion and curvature simultaneously. The radial component of the magnetic field was also dropped since we have assumed that the cross-section of the tube is constant. These considerations can be addressed elsewhere.

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