The Spectrum of the Hamiltonian with a PT-symmetric Periodic Optical Potential

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Abstract

We give a complete description, provided with a mathematical proof, of the shape of the spectrum of the Hill operator with potential $4 \cos^2 x + 4iV \sin 2x$, where $V \in (0, \infty)$. We prove that the second critical point $V_2$, after which the real parts of the first and second band disappear, is a number between 0.8884370025 and 0.8884370117. Moreover, we prove that $V_2$ is the degeneration point for the first periodic eigenvalue. Besides, we give a scheme by which one can find arbitrary precise value of the second critical point as well as the $k$-th critical points after which the real parts of the $(2k-3)$-th and $(2k-2)$-th bands disappear, where $k = 3, 4, ...$

Key Words: PT-symmetric operators, optical potentials, band structure.

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1 Introduction and Preliminary Facts

In this paper we investigate the one dimensional Schrödinger operator $L(q)$ generated in $L^2(-\infty, \infty)$ by the differential expression

$$-y''(x) + q(x)y(x),$$

where $q(x) = 4 \cos^2 x + 4iV \sin 2x$ is an optical potential which is the shift of

$$(1 + 2V)e^{i2x} + (1 - 2V)e^{-i2x}, \ V \geq 0.$$ (2)

Some physically interesting results have been obtained by considering the potential (2). The detailed investigation of the periodic optical potentials in the papers [7, 8] were illustrated on (2). For the first time, the mathematical explanation of the nonreality of the spectrum of $L(q)$ for $V > 0.5$ and finding the threshold 0.5 (first critical point $V_1$) was done by Makris et al [7,8]. Moreover, for $V = 0.85$ they sketch the real and imaginary parts of the first two bands by using the numerical methods. In [11] Midya et al reduce the operator $L(q)$ with potential (2) to the Mathieu operator and using the tabular values establish that there is second critical point $V_2 \sim 0.888437$ after which no part of the first and second bands remains real.

In this paper we give a complete description, provided with a mathematical proof, of the shape of the spectrum of the Hill operator $L(q)$ with potential (2), when $V$ changes from $1/2$ to $\sqrt{5}/2$. We prove that the second critical point $V_2$ is a number between 0.8884370025 and 0.8884370117. Moreover we prove that $V_2$ is the unique degeneration point for the first periodic eigenvalue, in the sense that the first periodic eigenvalue of the potential (2) is simple for all $V \in (1/2, \sqrt{5}/2) \backslash \{V_2\}$ and is double if $V = V_2$. Our approach give the
possibility to find the arbitrary close values of the $k$-th critical point $V_k$ and prove that no part of the $(2k - 3)$-th and $(2k - 2)$-th bands remains real for $V_k < V < V_k + \varepsilon$ for some positive $\varepsilon$, where $k = 2, 3, ...$

For the proofs we use the following results formulated below as summaries.

**Summary 1**

(a) The spectrum $\sigma(L(q))$ of $L(q)$ is the union of the spectra $\sigma(L_t(q))$ of the operators $L_t(q)$ for $t \in (-\pi, \pi]$ generated in $L_2[0, \pi]$ by (1) and the boundary conditions

$$y(\pi) = e^{it}y(0), \quad y'(\pi) = e^{it}y'(0),$$

where $t$ is the quasimomentum.

(b) The spectrum of $\sigma(L_t(q))$ consists of the Bloch eigenvalues $\mu_1(t), \mu_2(t), ..., $ that are the roots of the characteristic equation

$$F(\lambda) = 2\cos t,$$

where $F(\lambda) := \varphi'(\pi, \lambda) + \theta(\pi, \lambda)$ is the Hill discriminant, $\theta$ and $\varphi$ are the solutions of

$$-y''(x) + q(x)y(x) = \lambda y(x)$$

satisfying the initial conditions $\theta(0, \lambda) = \varphi'(0, \lambda) = 1, \theta'(0, \lambda) = \varphi(0, \lambda) = 0.$

(c) $\lambda \in \sigma(L(q))$ if and only if $F(\lambda) \in [-2, 2]$.

(d) The spectrum $\sigma(L)$ consists of the analytic arcs defined by (3) whose endpoints are the eigenvalues of $L_t$ for $t = 0, \pi$ and the multiple eigenvalues of $L_t$ for $t \in (0, \pi)$. Moreover $\sigma(L)$ does not contain the closed curves, that is, the resolvent set $\mathbb{C} \setminus \sigma(L)$ is connected.

(e) If the potential $q$ is PT-symmetric, that is, $q(-x) = \overline{q(x)}$, then the following implications hold:

$$\lambda \in \sigma(L_t(q)) \implies \overline{\lambda} \in \sigma(L_t(q)) \quad \& \quad \lambda \in \mathbb{R} \implies F(\lambda) \in \mathbb{R}. \quad (5)$$

For (a) - (c) see [3,9,10,13], for (d) see [13], for (e) see [7] and [14]. For the properties of the general PT-symmetric potentials see [1, 12 and references of them]. Here we only note that the investigations of PT-symmetric periodic potentials were begun by Bender et al [2].

As we noted above in [11] it was proved that the investigation of the operator $L(q)$ with potential (2) can be reduced to the investigation of the Mathieu operator. Besides in [15] (see Theorem 1 and (26) of [15]) we proved that if $ab = cd$, where $a, b, c,$ and $d$ are arbitrary complex numbers, then the operators $L(q)$ and $L(p)$ with potentials $q(x) = ae^{-i2x} + b e^{i2x}$ and $p(x) = ce^{-i2x} + d e^{i2x}$ have the same Hill discriminant $F(\lambda)$ and hence the same Bloch eigenvalues and spectrum. Therefore we have

$$\sigma(L(V)) = \sigma(H(a)), \quad \sigma(L_t(V)) = \sigma(H_t(a)), \quad \forall t \in [0, \pi], \quad a = \sqrt{1 - 4V^2}, \quad \sigma(L_t(q)) = \sigma(H_t(a)) \quad (6)$$

where, for brevity of notations, the operators $L_t(q)$ $(L(q))$ with potentials (2) and

$$q(x) = ae^{-i2x} + b e^{i2x} = 2a \cos 2x$$

are denoted by $L_t(V)$ $(L(V))$ and $H_t(a)$ $(H(a))$ respectively.

**Remark 1**

The potentials (2) and (7) are PT-symmetric and even functions respectively. The equalities in (6) show that to consider the spectrum we can use the properties of both cases. Namely, we use the properties (5) of the PT-symmetric potential (2) and the following properties of the even potential (7):
If \( a \neq 0 \), then the geometric multiplicity of of the eigenvalues of the operators \( H_0(a) \), \( H_\pm(a) \), \( D(a) \) and \( N(a) \), called as periodic, antiperiodic, Dirichlet and Neumann eigenvalues, is 1 and the following equalities hold

\[
\sigma(D(a)) \cap \sigma(N(a)) = \emptyset, \quad \sigma(H_0(a)) \cup \sigma(H_\pm(a)) = \sigma(D(a)) \cup \sigma(N(a)),
\]
where \( D(a) \) and \( N(a) \) denote the operators generated in \( L_2[0,\pi] \) by (1) with potential (7) and Dirichlet and Neumann boundary conditions respectively (see [6, 16]).

A great number of papers are devoted to the Mathieu operator \( H(a) \). Here we recall only the classical results and the results of [16] about Mathieu operator \( H(a) \) which are essentially used in this paper (see summaries 2 and 3). Moreover, in this paper we use the notations of the classical results and the results of [16] about Mathieu operator \( L \).

To easeify the readability of this paper in Section 2 we discuss the main results and give a brief and descriptive scheme of the proofs. Then in sections 3-5 we give the rigorous mathematical proofs of the results. In order to avoid eclipsing the essence by the technical details some calculations and estimations are given in the Appendix.
2 Discussion of the Main Results and Proofs

In this section we describe the main results and give a brief scheme of some proofs. Moreover, we describe the transfigurations of the spectra of $L(V)$ when $V$ changes from 0 to $\infty$. If $V$ changes from 0 to $\infty$ then $a = \sqrt{1 - 4V^2}$ moves from 1 to 0 over the real line and then moves from 0 to $\infty$ over the imaginary line. In the case $a \in (0, \infty)$ the spectrum is described in Summary 2. We consider in detail the case $a \in I(2)$, that is, $1/2 < V < \sqrt{3}/2$, where $I(c) = \{ix : x \in (0, c)\}$. The steps of the investigations are the followings.

Step 1. On the periodic and antiperiodic eigenvalues. In Section 3 we consider the periodic and antiperiodic eigenvalues. The main results are the followings.

(a) If $a \in I(2)$, then all antiperiodic eigenvalues are nonreal and simple. They consist of the numbers $\lambda_1^+ (a)$, $\lambda_2^+ (a)$, $\lambda_3^+ (a)$, ..., lying in the upper half plane and their conjugates denoted by $\lambda_1^- (a)$, $\lambda_2^- (a)$, $\lambda_3^- (a)$, .... respectively.

(b) If $a \in I(4/3)$, that is, if $1/2 < V < 5/6$ then all periodic eigenvalues are real and simple and hence can be numbered as in the self-adjoint case:

$$\lambda_0 (a) < \lambda_2^- (a) < \lambda_3^+ (a) < \lambda_4^- (a) < ..., \quad (9)$$

(see Summary 2). Furthermore, the eigenvalues $\lambda_1^+ (a)$, $\lambda_2^- (a)$, $\lambda_3^+ (a)$ are real and simple and satisfy (9) for all $a \in I(2)$. However for $\lambda_0 (a)$ and $\lambda_2^- (a)$ we prove that there exists a unique number $ir \in I(2)$ such that $\lambda_0 (ir) = \lambda_2^- (ir)$, that is, $\lambda_0 (ir)$ is the double eigenvalue of $H_0 (ir)$. Moreover, we prove that the number $\frac{1}{2} (1 + r^2)^{1/2}$ is the second critical number $V_2$. We say that the number $ir$, as well as $V_2$, is the degeneration point for the first periodic eigenvalue, since we prove that the first periodic eigenvalue of the potential (2) is simple for all $V \in (1/2, \sqrt{3}/2) \setminus \{ V_2 \}$ and is double if $V = V_2$. If $0 < a/i < r$, then both $\lambda_0 (a)$ and $\lambda_2^- (a)$ are real and simple and if $r < a/i < 2$ then $\lambda_0 (a)$ and $\lambda_2^- (a)$ are simple and nonreal and $\lambda_0 (a) = \lambda_2^- (a)$. Thus if $0 < a/i < r$ or equivalently if $1/2 < V < V_2$ then all periodic eigenvalues are real simple and satisfy (9).

Step 2. On the numerations of the Bloch eigenvalues and bands.

In the self-adjoint case the Bloch eigenvalues and bands can be numerated in increasing order, since they are real. It helps to describe all results for the self-adjoint Hamiltonian. Since, in the non-self-adjoint case the above listed quantities, in general, are not real we have the problems: (i) how numerate the Bloch eigenvalues and bands, (ii) how describe the real and nonreal parts of the bands in detail.

We prove that the Bloch eigenvalues corresponding to the quasimomentum $t$ can be numbered by $\mu_1 (t)$, $\mu_2 (t)$, ... such that $\mu_n (t)$ continuously depend on $t \in [0, \pi]$ and

$$\mu_1 (0) = \lambda_0 (a), \quad \mu_2 (0) = \lambda_2^- (a), \quad \mu_3 (0) = \lambda_3^+ (a), \quad \mu_4 (0) = \lambda_4^- (a), ..., \quad (10)$$

$$\mu_1 (\pi) = \lambda_1^- (a), \quad \mu_2 (\pi) = \lambda_1^+ (a), \quad \mu_3 (\pi) = \lambda_3^- (a), \quad \mu_4 (\pi) = \lambda_3^+ (a), ..., \quad (11)$$

Thus if $0 < a/i < r$ or if $0 < V < V_2$, then $\Gamma_n = \{ \mu_n (t) : t \in [0, \pi] \}$ is a continuous curve with periodic real endpoint $\mu_n (0)$ and antiperiodic nonreal endpoint $\mu_n (\pi)$. We say that $\Gamma_n$ is the $n$-th band of $\sigma (L)$. Then by (10) and (11) the first (second) band is the continuous curve joining the periodic real eigenvalue $\lambda_0 (a)$ ($\lambda_3^+ (a)$) and the antiperiodic nonreal eigenvalue $\lambda_1^- (a)$ ($\lambda_1^+ (a)$).

Step 3. On the shapes of the bands and components of the spectrum.

We prove that the first and second bands have different shapes in the following 3 cases:

Case 1: $0 < a/i < r$, Case 2: $a/i = r$, Case 3: $r < a/i < 2$ or equivalently:

Case 1: $1/2 < V < V_2$, Case 2: $V = V_2$, Case 3: $V_2 < V < \sqrt{3}/2$. In other words in the cases 1-3 we describe the bands before, at and after the second critical point.

Let us describe briefly the shapes of all bands and then stress the shapes of the first and
second bands. In Section 4, we prove that the spectrum of \( L(V) \) or \( H(a) \) in Case 1 has the following properties (Pr. 1-Pr. 6):

**Pr. 1.** The real part \( \sigma(H(a)) \cap R \) of the spectrum of \( H(a) \) consist of the intervals

\[
I_1(a) = [\lambda_0(a), \lambda_2^-(a)], \quad I_2(a) = [\lambda_2^+(a), \lambda_4^1(a)], \ldots, \quad I_n(a) = [\lambda_{2n-2}^+(a), \lambda_{2n}^-(a)], \ldots \quad (12)
\]

**Pr. 2.** For each \( n = 1, 2, \ldots \), the interval \( I_n \) is the real part of \( \Omega_n := \Gamma_{2n-1} \cup \Gamma_{2n} \).

**Pr. 3.** The bands \( \Gamma_{2n-1} \) and \( \Gamma_{2n} \) have only one common point \( \lambda_n(a) \) which is interior point of \( I_n \). Moreover, \( \lambda_n(a) \) is a double eigenvalue of \( L_{t_n}(V) \) for some \( t_n \in (0, \pi) \) and a spectral singularity of \( L(V) \) and hence

\[
\Gamma_{2n-1} \cap \Gamma_{2n} = \lambda_n(a) = \mu_{2n-1}(t_n) = \mu_{2n}(t_n) \in \mathbb{R}. \quad (13)
\]

**Pr. 4.** The real parts of the bands \( \Gamma_{2n-1} \) and \( \Gamma_{2n} \) are respectively the intervals

\[
[\lambda_{2n-2}^+, \lambda_n] = \{\mu_{2n-1}(t) : t \in [0, t_n]\} \quad \& \quad [\lambda_n, \lambda_{2n}^-] = \{\mu_{2n}(t) : t \in [0, t_n]\} \quad (14)
\]

**Pr. 5.** The nonreal parts of \( \Gamma_{2n-1} \) and \( \Gamma_{2n} \) are respectively the analytic curves

\[
\gamma_{2n-1}(a) := \{\mu_{2n-1}(t) : t \in (t_n, \pi]\} \quad \& \quad \gamma_{2n}(a) := \{\mu_{2n}(t) : t \in (t_n, \pi]\} \quad (15)
\]

and \( \gamma_{2n}(a) = \{\lambda : \lambda \in \gamma_{2n-1}(a)\} \).

Thus the bands \( \Gamma_{2n-1} \) and \( \Gamma_{2n} \) are joined by \( \lambda_n \) and hence they form together the connected subset of the spectrum. The spectrum \( \sigma(L(a)) \) consist of the connected sets \( \Omega_1 := \Gamma_1 \cup \Gamma_2, \Omega_2 := \Gamma_3 \cup \Gamma_4, \ldots \) Moreover in Case 1 we prove that

**Pr. 6** The sets \( \Omega_1, \Omega_2, \ldots \) are connected separated subset of \( \sigma(H(a)) \).

By the last property \( \Omega_1, \Omega_2, \ldots \) are components of the spectrum.

In Case 1, by Pr. 2 the real part of the first component \( \Omega_1 = \Gamma_1 \cup \Gamma_2 \) is the closed interval \( I_1 := [\lambda_0(a), \lambda_2^-(a)] \). We prove that if \( V \) approaches \( V_2 \) from the left, that is, if \( a \) approaches to \( ir \) from below then the eigenvalues \( \lambda_0(a) \) and \( \lambda_2^-(a) \) get close to each other and the length of the interval \( I_1 \) approaches zero. As a result if \( V = V_2 \), that is, if \( a = ir \), then we get the equality \( \lambda_0(a) = \lambda_2^-(a) \) which means that the first and second bands \( \Gamma_1 \) and \( \Gamma_2 \) have only one real point which is their common point \( \lambda_0(a) = \lambda_2^-(a) = I_1 = \text{Re} \Omega_1 \). Thus, in Case 2 the real parts of \( \Gamma_1 \) and \( \Gamma_2 \) is a point \( \lambda_0(a) \). The other parts of the bands \( \Gamma_1 \) and \( \Gamma_2 \) are nonreal and symmetric with respect to the real line. Then we prove that if Case 3 occurs, then the eigenvalues \( \lambda_0(a) \) and \( \lambda_2^-(a) \) get off the real line and hence \( I_1 \) becomes the empty set. As a results, the first and second bands \( \Gamma_1 \) and \( \Gamma_2 \) became the curves symmetric with respect to the real line. Moreover, in all cases the intervals (12) are pairwise disjoint sets. Therefore they are called the real components of \( \sigma(L(V)) \) if \( V \in (1/2, \sqrt{5}/2) \).

Thus investigations in Step 1 and Step 3 show that, the following equivalent mathematical definitions of the second critical point are reasonable and it is natural to call the second critical point as the degeneration point for the first periodic eigenvalue.

**Definition 1** A real number \( V_2 \in (1/2, \sqrt{5}/2) \) is called the second critical point or the degeneration point for the first periodic eigenvalue if the first real eigenvalue of \( L_0(V_2) \) is a double eigenvalue.

**Definition 2** A real number \( V_2 \in (1/2, \sqrt{5}/2) \) is said to be the second critical point or the degeneration point for the first periodic eigenvalue if the first real component of \( \sigma(L(V_2)) \) is a point.

Note that in Case 2 and Case 3 the shapes of the components \( \Omega_2, \Omega_3, \ldots \) are as in Case 1. In this way one can prove that there exists \( k \)-th critical point, denoted by \( V_k \), such that
for $\frac{1}{2} < V < V_k$, $V = V_k$ and $V_k < V < V_k + \varepsilon$ the set $\Omega_{k-1} =: \Gamma_{2k-3} \cup \Gamma_{2k-2}$ have the shape as $\Omega_1$ in Case 1, Case 2 and Case 3 respectively.

**Step 4. Finding the approximate value of $V_2$.** By Summary 3(c) any periodic eigenvalue is either Dirichlet, called as periodic Dirichlet (briefly PD$(a)$ or PD) eigenvalue or Neumann, called as periodic Neumann (PN$(a)$ or PN) eigenvalue. Similarly antiperiodic eigenvalue is either antiperiodic Dirichlet (AD) or antiperiodic Neumann (AN) eigenvalues. Clearly, the eigenfunctions corresponding to PN, PD, AD and AN eigenvalues have the forms

$$\Psi_{PN}(x) = \frac{a_0}{\sqrt{2}} + \sum_{k=1}^{\infty} a_k \cos 2kx, \quad \Psi_{PD}(x) = \sum_{k=1}^{\infty} b_k \sin 2kx,$$

$$\Psi_{AD}(x) = \sum_{k=1}^{\infty} c_k \sin(2k-1)x \quad \& \quad \Psi_{AN}(x) = \sum_{k=1}^{\infty} d_k \sin 2kx,$$

respectively. Substituting these functions into equation (4) with potential (7) we obtain the following equalities for the PN, PD, AD and AN eigenvalues respectively

$$\lambda a_0 = \sqrt{2}aa_1, \quad (\lambda - 4)a_1 = a\sqrt{2}a_0 + aa_2, \quad (\lambda - (2k)^2)a_k = aak_{-1} + aak_{+1}, \quad (16)$$

$$\lambda b_1 = ab_2, \quad (\lambda - (2k)^2)b_k = ab_{k-1} + ab_{k+1}, \quad (17)$$

$$\lambda c_1 = ac_1 + ac_2, \quad (\lambda - (2k - 1)^2)c_k = ack_{-1} + ack_{+1}, \quad (18)$$

$$\lambda d_1 = ad_1 + ad_2, \quad (\lambda - (2k - 1)^2)d_k = ad_{k-1} + ad_{k+1} \quad (19)$$

for $k = 2, 3, \ldots$, where $a_0 \neq 0$, $b_1 \neq 0$, $c_1 \neq 0$, $d_1 \neq 0$ (see [3] and [16]).

As we noted in Step 1, the second critical point is a real number $V_2$ such that $\lambda_0(a) = \lambda_2^*(a)$, where $a = \sqrt{1 - 4V_2^2}$. In Section 3 we prove that $\lambda_0(a)$ and $\lambda_2^*(a)$ are the PN eigenvalues and hence satisfy (16). Therefore to find the approximate value of $V_2$ we use the following definition of $V_2$ which is equivalent to the definitions 1 and 2.

**Definition 3** A real number $V_2 \in (1/2, \sqrt{5}/2)$ is called the second critical point if the first real PN$(a)$ eigenvalue, where $a = \sqrt{1 - 4V_2^2}$, is a double eigenvalue.

Iterating (16) we obtain that the first PN eigenvalue satisfies the equality (55). To consider the value of $a$ and hence of $V$ for which the first periodic eigenvalue becomes double eigenvalue, we look for the double root of (55). To find approximately the double roots of (55) we use the second approximation (60) of (55). Using Mathematica 7 we calculate the roots of the second approximation and then estimating the remainder and using the Rouche's theorem we prove that: (i) both $\lambda_0$ and $\lambda_2^*$ are real if $V = 0.8884370025$, (ii) both $\lambda_0$ and $\lambda_2^*$ are nonreal and complex conjugate if $V = 0.8884370117$. Moreover, we prove that there exists unique $a$ from $I(2)$ and hence unique $V$ from $1/2 < V < \sqrt{5}/2$ such that the first eigenvalue $\lambda_0$ is double and this value of $V$, that is $V_2$, should be between 0.8884370025 and 0.8884370117.

Note that instead of the second approximation using the $m$-th approximation of (55) for $m > 2$ one can get more sharper estimation of $V_2$. In the same way we can get the arbitrary approximation of the $k$-th critical point $V_k$ for $k > 2$ (see Remark 5).

### 3 On the Periodic and Antiperiodic Eigenvalues

To define the numerations of the eigenvalues of $H_\pi(a)$ and $H_0(a)$ first we prove the following theorem. For this we use Summary 3 and take into account that the spectra of $H_0(a), H_\pi(a)$,
\[ D(a) \text{ and } N(a) \text{ for } a = 0 \text{ are}
\]
\[
\{(2k)^2 : k = 0, 1, \ldots\}, \{(2k+1)^2 : k = 0, 1, \ldots\}, \{k^2 : k = 1, 2, \ldots\}, \{k^2 : k = 0, 1, \ldots\}
\]

respectively. All eigenvalues of \(H_0(0)\), except 0, and \(H_\pi(0)\) are double, while the eigenvalues of \(D(0)\) and \(N(0)\) are simple.

**Theorem 1** (a) The number of eigenvalues (counting multiplicity) of operator \(H_\pi(a)\) lying in \(D_4((2n-1)^2)\) is 2 for all \(a \in I(2)\) and \(n = 1, 2, \ldots\). These eigenvalues are simple. They are either real numbers (first case) or nonreal conjugate numbers (second case). One of them is \(AD\) and the other is \(AN\) eigenvalue.

(b) The statements in (a) continue to hold if the operator \(H_\pi(a)\) is replaced by \(H_0(a)\) and the discs \(D_4((2n-1)^2)\) for \(n = 1, 2, \ldots\) are replaced by \(D_4((2n)^2)\) for \(n = 2, 3, \ldots\). The operator \(H_0(a)\) has 3 eigenvalues (counting multiplicity) in \(D_6(3)\) for all \(a \in I(2)\). Two of them are the PN eigenvalues and the other is the PD eigenvalue. The PN eigenvalues are either real numbers (first case) or nonreal conjugate numbers (second case). One of them is \(AD\) and the discs

\[
\text{Proof. (a) It readily follows from Summary 3(a) that the boundary of } D_4((2n-1)^2) \text{ lies in the resolvent sets of the operators } H_\pi(a) \text{ for all } a \in I(2) \cup \{0\}. \text{ Therefore the projection of } H_\pi(a) \text{ defined by contour integration over the boundary of } D_4((2n-1)^2) \text{ depend continuously on } a. \text{ It implies that the number of eigenvalues (counting the multiplicity) of } H_\pi(a) \text{ lying in } D_4((2n-1)^2) \text{ are the same for all } a = I(2) \cup \{0\}. \text{ Since } H_\pi(0) \text{ has two eigenvalues (counting the multiplicity) in } D_4((2n-1)^2), \text{ the operators } H_\pi(a) \text{ has also 2 eigenvalue. Moreover if } a \neq 0, \text{ then by Summary 3(a) these eigenvalues are simple and hence are different numbers. Therefore using (5) and taking into account that if } \lambda \text{ lies in } D_4((2n-1)^2), \text{ then } \overline{\lambda} \text{ lies also in } D_4((2n-1)^2) \text{ and does not lie in } D_4((2m-1)^2) \text{ for } m \neq n, \text{ we obtain that the eigenvalues lying in } D_4((2n-1)^2) \text{ are either two different real numbers or nonreal conjugate numbers. Instead of the operator } H_\pi(a) \text{ using the operators } N(a) \text{ and } D(a), \text{ taking into account that } N(0) \text{ and } D(0) \text{ have one eigenvalue in } D_4((2n-1)^2), \text{ and repeating the above arguments we get the proof of the last statement.}

(b) Instead of Summary 3(a) using Summary 3(b), repeating the proof of (a) and taking into account that \(H_0(0)\) has 2 eigenvalues in \(D_4((2n)^2)\) for \(n \geq 2\) we get the proof of (b).

(c) It is clear that if \(a \in I(2) \cup \{0\}\), then the discs \(D_{\sqrt{n} |a|}(0)\) and \(D_{(1+ \sqrt{n}) |a|}(4)\) are contained in \(D_6(3)\). If \(a < 4/(1+2 \sqrt{2})\), then \(D_{\sqrt{n} |a|}(0)\) and \(D_{(1+ \sqrt{n}) |a|}(4)\) are pairwise disjoint discs and first disc contains one eigenvalue of \(N(0)\) and the second disc contains one eigenvalue of \(N(0)\) and one eigenvalue of \(D(0)\). Finally note that in \(D_6(3)\) there exist respectively 3, 2 and 1 eigenvalues of the operators \(H_0(0), N(0)\) and \(D(0)\). Therefore arguing as in the proof of (a) and using Summary 3(c) we get the proof of (c).

**Notation 1** By Theorem 1 (a) and (b) if \(a = I(2)\), then the operators \(H_\pi(a)\) and \(H_0(a)\) have 2 eigenvalues in \(D_4((2n-1)^2)\) for \(n = 1, 2, \ldots\) and \(D_4((2n)^2)\) for \(n = 2, 3, \ldots\) respectively. Let us denote the eigenvalues lying in \(D_4(n^2)\) by \(\lambda^-_n(a)\) and \(\lambda^+_n(a)\). Moreover, in the first case (see Theorem 1(a) for the first and second cases) due to the indexing of Summary 2 we put \(\lambda^-_n(a) < \lambda^+_n(a)\). In the second case, without loss of generality, the indexing can be done by the rule \(\text{Im } \lambda^-_n(a) < 0 \text{ and } \text{Im } \lambda^+_n(a) > 0\). Then \(\lambda^-_n(a) = \overline{\lambda^-_n(a)}\). Three eigenvalues of the operator \(H_0(a)\) lying in \(D_6(3)\) are denoted by \(\lambda_0(a), \lambda^-_2(a)\) and \(\lambda^+_2(a)\). Moreover \(\lambda_0(a)\) and \(\lambda^+_2(a)\) denote the PN eigenvalues and \(\lambda^-_2(a)\) denotes the PD eigenvalue.

Theorem 1 and Notation 1 imply the following.
Corollary 1 (a) If \( n \not= 2 \), and \( a \in I(2) \), then one of the following two cases occurs

\[
\lambda_n^+ (a) \in \mathbb{R}, \lambda_n^- (a) < \lambda_n^+ (a) \quad \& \\
\lambda_n^\pm (a) \not\in \mathbb{R}, \quad \text{Im} \lambda_n^\pm (a) > 0, \quad \lambda_n^- (a) = \lambda_n^\pm (a).
\]

(20)

Corollary 1 implies that if \( n \not= 2 \) and \( a \in I(2) \), then either \((\lambda_n^+ (a) - \lambda_n^- (a)) \in (0, \infty)\) or \((\lambda_n^+ (a) - \lambda_n^- (a)) \in \{ix : x \in (0, \infty)\}\), that is,

\[
(\lambda_n^+ (a) - \lambda_n^- (a)) \in (0, \infty) \cup \{ix : x \in (0, \infty)\}, \forall a \in I(2), \forall n \not= 2.
\]

(22)

Using this we prove the following.

Theorem 2 Let \( n \not= 2 \). Then \( \lambda_n^- (a) \) is real (nonreal) for all \( a \in I(2) \) if and only if there exists \( b \in I(2) \) such that \( \lambda_n^- (b) \) is a real (nonreal) number. The statement continues to hold if \( \lambda_n^- (a) \) is replaced by \( \lambda_n^+ (a) \).

**Proof.** First let us prove that the set

\[
G_n(2) = \{\lambda_n^+ (a) - \lambda_n^- (a) : a \in I(2)\}
\]

is a subinterval of either \((0, \infty)\) or \(\{ix : x \in (0, \infty)\}\). Suppose to the contrary that there exist \( c \in I(2) \) and \( d \in I(2) \) such that

\[
(\lambda_n^+(c) - \lambda_n^-(c)) \in (0, \infty) \quad \& \quad (\lambda_n^+(d) - \lambda_n^-(d)) \in \{ix : x \in (0, \infty)\}.
\]

(23)

By Theorem 1, \( \lambda_n^+ (a) \) and \( \lambda_n^- (a) \) are the simple eigenvalues for all \( a \in I(2) \). Therefore \( \lambda_n^+ - \lambda_n^- \) is a continuous function on \( I(2) \). Thus \( \{\lambda_n^+ (a) - \lambda_n^- (a) : a \in [b, d]\} \) is a continuous curve lying in \((0, \infty)\cup \{ix : x \in (0, \infty)\}\) (see (22)) and joining the points of \((0, \infty)\) and \(\{ix : x \in (0, \infty)\}\) (see (23)) which is impossible. Hence. either \( G_n(2) \subset (0, \infty) \) or \( G_n(2) \subset \{ix : x \in (0, \infty)\} \).

Now suppose that there exists \( c \in I(2) \) such that \( \lambda_n^- (c) \) is a real number. Then by Corollary 1 \( \lambda_n^+ (c) \) is also a real number and \( \lambda_n^+ (c) - \lambda_n^- (c) \in (0, \infty) \) and hence \( G_n(2) \subset (0, \infty) \).

It means that \((\lambda_n^+ (a) - \lambda_n^- (a)) \in (0, \infty)\) for all \( a \in I(2) \), that is, both \( \lambda_n^+ (a) \) and \( \lambda_n^- (a) \) are real numbers. In the same way we prove the other parts of the theorem.

By Theorem 2 if \( \lambda_n^- (a) \) is real (nonreal) for small \( a \) then it is real (nonreal) for all \( a \in I(2) \).

Remark 2 A lot of papers (see for example [5]) are devoted to the small perturbation and asymptotic formulas when \( a \to 0 \) (especially for real \( a \)) for eigenvalues of \( H_\varepsilon (a) \) and \( H_\delta (a) \) which imply that if \( a \) is a small number then \( \lambda_n^\pm (a) \) is real and nonreal respectively if \( n \) is even and odd integer.

However, in order to do the paper self-contained we prove these statements in Appendix (see Estimation 1 and Estimation 2). Namely, in Estimation 1 we prove that

\[
\lambda_n^+ (a) = 1 + a + O(a^2), \quad \lambda_n^- (a) = 1 - a + O(a^2),
\]

(24)

\( \lambda_n^+ (a) \) and \( \lambda_n^- (a) \) are \( AD \) and \( AN \) eigenvalues respectively. Then using Theorem 2 we prove that (see Proposition 2) \( \lambda_n^\pm (a) \) is a nonreal number if \( a \) is small and pure imaginary number. Thus if \( a \in I(2) \) is small, then for all odd \( n \) the case (21) occurs.

In Estimation 2 we prove that

\[
\lambda_0 (a) = -\frac{1}{4} a^2 + O(a^3), \quad \lambda_2 (a) = 4 + \frac{5}{12} a^2 + O(a^3), \quad \lambda_2^+ (a) = 4 - \frac{1}{12} a^2 + O(a^3),
\]

(25)

where \( \lambda_0 (a) \) and \( \lambda_2 (a) \) are \( PN \) and \( \lambda_1^+ (a) \) is \( PD \) eigenvalue for small \( a \). It implies that if \( a \in I(2) \) is small, then \( \lambda_0 (a), \lambda_2^- (a) \) and \( \lambda_2^+ (a) \) are the real numbers, \( \lambda_0 (a) < \lambda_2^- (a) < \lambda_2^+ (a) \).
and this notation agree with the notations of Summary 2. Then using Theorem 2 (see Proposition 3) we prove that \( \lambda_{2n}^\pm(a) \), where \( n > 1 \), are real numbers if \( a \in I(2) \) is a small number. Thus if \( a \) is small and pure imaginary number then for even \( n \) the case (20) occurs.

Now using Remark 2 and Theorem 2 we consider the reality and nonreality of the periodic and antiperiodic eigenvalues for all \( a \in I(2) \).

**Theorem 3** Let \( a \in I(2) \). Then \( \lambda_{2n}^-(a) \) and \( \lambda_{2n}^+(a) \) for all \( n = 2, 3, \ldots \), are real and

\[
\lambda_4^-(a) < \lambda_4^+(a) < \lambda_6^-(a) < \lambda_6^+(a) < \ldots. \tag{26}
\]

The eigenvalues \( \lambda_{2n-1}^-(a) \) and \( \lambda_{2n-1}^+(a) \) are nonreal for all \( n = 1, 2, \ldots \) and

\[
\text{Im} \lambda_{2n-1}^+(a) > 0, \quad \lambda_{2n-1}^-(a) = \overline{\lambda_{2n-1}^+(a)}. \tag{27}
\]

**Proof.** By Proposition 3 (see Remark 2), \( \lambda_{2n}^\pm(a) \), where \( n > 1 \), are real number if \( a \) is small. Therefore it follows from Theorem 2 that \( \lambda_{2n}^\pm(a) \) is real for all \( a \in I(2) \). Similarly, Proposition 2 and Theorem 2 imply that \( \lambda_{2n-1}^\pm(a) \) is nonreal for all \( a \in I(2) \). The inequalities in (26) and (27) follows from Notation 1. ■

To consider the remaining part of the periodic eigenvalue, that is, the eigenvalues \( \lambda_2^+(a) \), \( \lambda_2^-(a) \) and \( \lambda_0(a) \) we use the following.

**Proposition 1** Let \( d \) be a positive number. If \( \lambda(a) \) is a simple eigenvalue of \( H_0(a) \) for all \( a \in I(d) \), then it is real eigenvalue for all \( a \in I(d) \).

**Proof.** In Proposition 3 we prove that \( \lambda(a) \) is a real for small \( a \in I(2) \). Let \( c \) be greatest number such that \( \lambda(a) \) is real for \( a \in I(c) \) and \( c < d \). Then by assumption of the proposition \( \lambda(ic) \) is a simple eigenvalue. Therefore, by general perturbation theory, \( \lambda \) is analytic function in some neighborhood of \( ic \) and there exist positive constants \( \varepsilon \) and \( \delta \) such that the operator \( H_0(a) \) has only one eigenvalue in \( D_\varepsilon(\lambda(ic)) \) whenever \( |a - ic| < \delta \). On the other hand, by the definition of \( c \) for each \( k \in \mathbb{N} \) there exists \( c_k \in (c, c + \frac{1}{k}) \) such that \( \lambda(ic_k) \) is nonreal. Then by (5) \( \lambda(ic_k) \) is also periodic eigenvalue and for large value of \( k \) both \( \lambda(ic_k) \) and \( \overline{\lambda(ic_k)} \) lie in \( D_\varepsilon(\lambda(ic)) \) and \( |a - ic_k| < \frac{1}{k} < \delta \), which is a contradiction. ■

**Theorem 4** The eigenvalue \( \lambda_2^+(a) \) are real and simple for all \( a \in I(2) \) and

\[
\lambda_2^+(a) < \lambda_4^-(a) < \lambda_4^+(a). \tag{28}
\]

**Proof.** Since \( \lambda_2^+(a) \) is PD eigenvalue (see Remark 2) by summaries 3(a) and 3(c), if \( a \in I(2) \), then the eigenvalue \( \lambda_2^+(a) \) is simple. Therefore the propositions 1 and 3 imply that it is real for all \( a \in I(2) \). The inequalities in (28) follows from Notation 1. ■

It remains to consider the eigenvalues \( \lambda_0(a) \) and \( \lambda_2^-(a) \). Now we find the upper bounds for their simplicities and realities.

**Theorem 5** (a) In \( I(2) \) there exists a unique number \( ir \), called as the degeneration point for the first periodic eigenvalue, such that \( \lambda_0(ir) \) is a double periodic eigenvalue and

\[
\lambda_0(ir) = \lambda_2^-(ir). \tag{29}
\]

(b) If \( 0 < a/i < r \), then both \( \lambda_0(a) \) and \( \lambda_2^-(a) \) are real and simple

\[
\lambda_0(a) < \lambda_2^-(a) < \lambda_2^+(a). \tag{30}
\]

(c) The eigenvalues \( \lambda_0(a) \) and \( \lambda_2^-(a) \) are the real numbers.

(d) If \( r < a/i < 2 \) then \( \lambda_0(a) \) and \( \lambda_2^-(a) \) are nonreal and \( \lambda_0(a) = \lambda_2^+(a) \).
Proof. (a) In Theorem 10 we prove that there exists a unique number \( a \in I(2) \) such that \( \lambda_0(a) = \lambda_2^-(a) \). In other word \( \lambda_0(ir) \) is a multiple eigenvalue, where \( r = a/i, \) and (29) holds. On the other hand, there are only three periodic eigenvalues \( \lambda_0(a), \lambda_2^-(a), \lambda_2^+(a) \) in \( D_0(3) \), where \( \lambda_0(a) \) and \( \lambda_2^-(a) \) are Neumann eigenvalues and \( \lambda_2^+(a) \) is a Dirichlet eigenvalue (see Notation 1), and by the first equality of (8) \( \lambda_0(a) \neq \lambda_2^+(a) \) and \( \lambda_2^-(a) \neq \lambda_2^+(a) \) for all \( a \in I(2) \). Therefore \( \lambda_0(ir) \) and \( \lambda_2^-(ir) \) are the double eigenvalue, (29) holds and both \( \lambda_2^-(a) \) and \( \lambda_0(a) \) are simple for all \( a \in I(2) \setminus \{ir\} \).

(b) In the proof of (a) we have proved that if \( 0 < a/i < r \), then both \( \lambda_0(a) \) and \( \lambda_2^-(a) \) are simple eigenvalues. It with Proposition 1 implies that both of them are real.

Now we prove the first inequality of (30). It follows from (25) that it holds for small \( a \). Let \( c \) be greatest number such that \( c < r \) and the first inequality in (30) holds for \( a \in I(c) \). Then \( \lambda_0(ic) = \lambda_2^-(ic) \), since \( \lambda_0(ic) \) and \( \lambda_2^-(ic) \) are real numbers and continuously depend on \( c \). It contradicts the simplicity of \( \lambda_0(ic) \).

Let us prove the second inequality in (29). By (25) it holds for small \( a \). On the other hand \( \lambda_2^-(a) \neq \lambda_2^+(a) \) for all \( a \), since one of them is Dirichlet and the other is Neumann eigenvalue and (8) holds. Therefore taking into account that the eigenvalues \( \lambda_2^+(ic) \) and \( \lambda_2^-(ic) \) are real numbers and continuously depend on \( c \) we get the proof.

(c) Since \( \lambda_0(a) \) and \( \lambda_2^-(a) \) are real for all \( a \in I(r) \), letting \( c \) tend to \( r \) from the left and taking into account that the eigenvalues \( \lambda_0(ic) \) and \( \lambda_2^-(a) \) continuously depend on \( c \) we get the proof.

(d) Let \( R =: R(\lambda_0) \) be the greatest positive number such that \( \lambda_0(a) \) is real for \( a \in I(R) \). It follows from (b) and (c) and Theorem 11(b) that \( r \leq R \) and \( R^2 < 2.16 \). If \( R > r \) then repeating the proof of the Proposition 1 we obtain that \( \lambda_0(iR) \) is a double eigenvalue that contradict to (a). Thus \( R = r \). Using the definition of \( R(\lambda_0) \) we see that if a number \( c \) lies in the small right neighborhood of \( R \) then \( \lambda_0(ic) \) is nonreal. Let \( d \) be largest number from \( (R, \infty) \) such that \( \lambda_0(ic) \) is nonreal for \( R < c < d \). Suppose that \( d < 2 \). Using the Summary 1 (e) and taking into account that \( \lambda_2^+(ic) \) is real (see Theorem 4) we conclude that \( \lambda_2^-(ic) = \lambda_0(ic) \), for all \( c \in (r, d) \). Now in the last equality letting \( c \) tend to \( d \), and using the continuity of \( \lambda_0 \) and \( \lambda_2^- \) we get \( \lambda_2^-(id) = \lambda_0(id) \) which contradicts (a).

### 4 On the Bands and Components of the Spectrum

In previous section we considered, in detail, the periodic eigenvalues that will be used essentially in this section. The results of the theorems 3-5 can be summarized as follows:

**Summary 4** Let \( ir \) be the degeneration point for the first periodic eigenvalue defined in Theorem 5. Then the followings hold:

(a) If \( 0 < a/i < r \), then all eigenvalues of \( H_0(a) \) are real simple and (9) holds.

(b) If \( a = ir \), then all eigenvalues of \( H_0(a) \) are real and other from \( \lambda_0(a) \) and \( \lambda_2^-(a) \) are simple and \( \lambda_0(a) = \lambda_2^-(a) < \lambda_2^+(a) < \lambda_3^-(a) < \lambda_4^+(a) < \ldots \).

(c) If \( r < a/i < 2 \), then all eigenvalues of \( H_0(a) \) are simple and other from \( \lambda_0(a) \) and \( \lambda_2^-(a) \) are real, \( \lambda_0(a) = \lambda_2^-(a) \) and \( \lambda_2^+(a) < \lambda_4^-(a) < \lambda_4^+(a) < \ldots \).

(d) The statements (a)–(c) continue to hold if \( 0, 2, ir, a \) and \( H_0(a) \) are replaced respectively by \( 1/2, \sqrt{5}/2, V_2 =: \frac{4}{3} (1 + r^2)^{1/2}, V =: \frac{4}{3} (1 - a^2)^{1/2} \) and \( L_0(V) \).

To investigate the bands and components of the spectrum we need to consider all Bloch eigenvalues for all values of quasimomentum \( t \in [0, \pi] \). In [17] and [18] we obtained the following results formulated below as **Summary 5 and Summary 6** (see (16) of [17] and Proposition 1, Remark 1 and Theorem 1 of [18]).
Remark 3 (a) Bloch eigenvalues $\mu_1(t), \mu_2(t), \ldots$ can be numbered so that $\mu_n(t)$ continuously depend on $t \in [0, \pi]$. Therefore $\Gamma_n = \{\mu_n(t) : t \in [0, \pi]\}$ is a continuous curve and is called the $n$-th band of the spectrum.

(b) $\mu_n(t)$ is a multiple eigenvalue of $L_i(q)$ of multiplicity $p$ if and only if $p$ bands of the spectrum have common point $\mu_n(t)$. In particular, $\mu_n(t)$ is a simple eigenvalue if and only if it belong only to one band $\Gamma_n$.

(c) $\Gamma_n$ is a single open curve with the end points $\mu_n(0)$ and $\mu_n(\pi)$.

Summary 6 Let $q$ be PT-symmetric potential. Then

(a) If $\mu_n(t_1)$ and $\mu_n(t_2)$ are real numbers, where $0 \leq t_1 < t_2 \leq \pi$ then

$$\gamma := \{\mu_n(t) : t \in [t_1, t_2]\}$$

is an interval of the real line with end points $\mu_n(t_1)$ and $\mu_n(t_2)$. In other words, if two real numbers $c_1 < c_2$ belong to the band $\Gamma_n$, then $[c_1, c_2] \subseteq \Gamma_n$.

(b) Two bands $\Gamma_n$ and $\Gamma_m$ may have at most one common point.

Notation 2 If $0 < a/i < r$, then, by Summary 4(a) and Summary 5 any eigenvalue in (9) is an end point of only one component $\Gamma_n$ and for any component $\Gamma_n$ there exists unique eigenvalue from (9) which is the end point of $\Gamma_n$. Thus there are one to one correspondence between bands of the spectrum and the periodic eigenvalues (9). Without loss of generality, it can be assumed that (10) holds. In other words, for all $a \in I(ir)$ we have

$$\lambda_0(a) = \Gamma_1 \cap \sigma(H_0(a)), \quad \lambda_n^+(a) = \Gamma_2 \cap \sigma(H_0(a)), \quad \lambda_n^-(a) = \Gamma_3 \cap \sigma(H_0(a)), \quad \lambda_n^+(a) = \Gamma_5 \cap \sigma(H_0(a)), \quad \lambda_n^+(a) = \Gamma_5 \cap \sigma(H_0(a)), \quad \lambda_n^+(a) = \Gamma_5 \cap \sigma(H_0(a)), \quad \lambda_n^+(a) = \Gamma_5 \cap \sigma(H_0(a)).$$

Thus the equalities (31) and (32) constitute one to correspondence between periodic eigenvalues and bands.

If $a = ir$ or $r < a/i < 2$, then by Summary 4(b) and Summary 5 equality (32) constitute one to correspondence between periodic eigenvalues $\lambda_2^-(a), \lambda_2^+(a), \lambda_2^+(a), \ldots$ and bands $\Gamma_3, \Gamma_4, \Gamma_5, \ldots$. If $a = ir$ then the first and second bands $\Gamma_1$ and $\Gamma_2$ have common end point $\lambda_0(a) = \lambda_2^-(a)$. If $r < a/i < 2$ then the first and second bands $\Gamma_1$ and $\Gamma_2$ are the bands whose one endpoints are nonreal periodic eigenvalues $\lambda_0(a)$ and $\lambda_2^-(a)$ respectively. Thus in any case the equalities (31) and (32) constitute one to correspondence between periodic eigenvalues and bands.

Note that Notation 2 with Summary 4 implies the followings.

Remark 3 If $0 < a/i \leq r$, then by Summary 4(a) and Summary 4(b) all periodic eigenvalues are real, and hence by Notation 2, all bands of the spectrum have a real part. If $r < a/i < 2$, then by Summary 4(c) all periodic eigenvalues except $\lambda_0(a)$ and $\lambda_2^-(a)$ and hence all bands except may be $\Gamma_1$ and $\Gamma_2$ have a real part.

To describe the shapes of the bands, in detail, we study the Hill discriminant $F(\lambda)$ defined in Summary 1. First of all recall that the eigenvalues of $H_0(a)$ and $H_r(a)$ are respectively the roots of $F(\lambda) = 2$ and $F(\lambda) = -2$. It is well known [3, 14] that $F$ is an entire function and

$$F(\lambda) \in \mathbb{R}, \quad \forall \lambda \in \mathbb{R}, \quad \lim_{\lambda \to -\infty} F(\lambda) = \infty.$$  \hspace{1cm} (33)

Since $\sigma(H(a)) = \{\lambda \in \mathbb{C} : -2 \leq F(\lambda) \leq 2\}$ (see Summary 1(c)), it is clear that the real part of the spectrum of $H(a)$ is

$$\text{Re}(\sigma(H(a))) =: \sigma(H(a)) \cap \mathbb{R} = \{\lambda \in \mathbb{R} : -2 \leq F(\lambda) \leq 2\}. \hspace{1cm} (34)$$

By (33), the set $G(F) = \{(\lambda, F(\lambda)) : \lambda \in \mathbb{R}\}$ is a continuous curve in $\mathbb{C}$ called as a graph of $F$. Therefore, $\text{Re}(\sigma(H))$ is the set of $\lambda \in \mathbb{R}$ such that the graph lies in the strip.
\{ \lambda \in \mathbb{C} : -2 \leq \text{Im} \lambda \leq 2 \}. Since all antiperiodic eigenvalues are nonreal \( G(F) \) never intersect the line \( y = -2 \). It with (33) implies that

\[
F(\lambda) > -2, \forall \lambda \in \mathbb{R}.
\]  
(35)

Now using (33)-(35) we prove **Pr. 1** of Section 2.

**Theorem 6** If \( 0 < a/i < r \), then the real part of the spectrum of \( H(a) \) consist of the intervals (12). In cases \( a = ir \) and \( r < a/i < 2 \), the real parts of \( \sigma(H) \) are \( \{ \lambda_0(a) \} \cup I_2 \cup I_3 \cup ... \) and \( I_2 \cup I_3 \cup ... \) respectively.

**Proof.** First we prove the theorem for \( 0 < a/i < r \). Since all periodic eigenvalues (9) are real and simple the intersection of \( G(F) \) and the line \( y = 2 \) are the points

\[
(\lambda_0, 2), (\lambda_2^-, 2), (\lambda_2^+, 2), (\lambda_4^-, 2), (\lambda_4^+, 2), ...
\]  
(36)

This with (35) implies that the graph \( G(H) \) may get in and out of the strip \( \{ \lambda \in \mathbb{C} : -2 \leq \text{Im} \lambda \leq 2 \} \) at the points (36). By (9) the leftmost intersection point of the graph \( G(F) \) and the line \( y = 2 \) is \( (\lambda_0(a), 2) \). Therefore it follows from (33) that

\[
\sigma(H(a)) \cap (-\infty, \lambda_0(a)) = \varnothing \quad \& \quad F(\lambda) > 2, \forall \lambda \in (-\infty, \lambda_0(a)).
\]  
(37)

Since \( \lambda_0(a) \) is a simple eigenvalue we have \( F'(\lambda_0(a)) \neq 0 \). Then the equality \( F(\lambda_0(a)) = 2 \) with the inequality in (37) implies that \( F'(\lambda_0(a)) < 0 \), that is, \( F(\lambda) \) decreases in some neighborhood of \( \lambda_0(a) \), and hence \( F(\lambda) < 2 \), on some right neighborhood of \( \lambda_0(a) \). Thus the graph \( G(F) \) get in of the strip \( \{ \lambda \in \mathbb{C} : -2 \leq \text{Im} \lambda \leq 2 \} \) for the first time at the point \( (\lambda_0, 2) \). Using this and taking into account that the second intersection point of the graph \( G(F) \) and the line \( y = 2 \) is \( (\lambda_2^-(a), 2) \) we see that \( F(\lambda) \leq 2 \) for all \( \lambda \) from the interval \( [\lambda_0(a), \lambda_2^-(a)] \). This with (35) implies that

\[
-2 < F(\lambda) < 2
\]  
(38)

for all \( \lambda \in [\lambda_0(a), \lambda_2^-(a)] \) and \( I_1 \subseteq \sigma(H(a)) \).

Now let us prove that the interval \( (\lambda_2^-(a), \lambda_2^+(a)) \) has no common point with the spectrum. Since \( \lambda_2^-(a) \) is a simple eigenvalue we have \( F'(\lambda_2^-(a)) \neq 0 \). On the other hand, (38) with \( F(\lambda_2^-(a)) = 2 \) shows that \( F(\lambda) \) increases, that is, \( F(\lambda) > 2 \) in some right neighborhood of \( \lambda_2^-(a) \). Thus the graph \( G(F) \) goes out of the strip \( \{ \lambda \in \mathbb{C} : -2 \leq \text{Im} \lambda \leq 2 \} \) for the first time at point \( (\lambda_2^-(a), 2) \) and come back to the strip at \( (\lambda_2^+(a), 2) \), since the last is the third intersection point of \( G(F) \) and the line \( y = 2 \) (see (36)) and \( F'(\lambda_2^+(a)) \neq 0 \). Thus we have proved that \( (\lambda_2^-(a), \lambda_2^+(a)) \cap \sigma(H(a)) = \varnothing \). Repeating these proofs we see that \( [\lambda_2^-(a), \lambda_2^+(a)] \subseteq \sigma(H(a)) \) and \( (\lambda_2^-(a), \lambda_2^+(a)) \cap \sigma(H(a)) = \varnothing \). Continuing this process we get the proof of the theorem for \( 0 < a/i < r \).

To prove the theorem for \( a = ir \) we repeat the above proof and take into account that the line \( y = 2 \) is the tangent to \( G(F) \) at the the point \( (\lambda_0(a), 2) \) and the graph \( G(H) \) does not get in of the strip \( \{ \lambda \in \mathbb{C} : -2 \leq \text{Im} \lambda \leq 2 \} \) at this point. To prove the theorem for \( r < a/i < 2 \) we also repeat the above proof and take into account that the graph \( G(H) \) get in and out of the strip \( \{ \lambda \in \mathbb{C} : -2 \leq \text{Im} \lambda \leq 2 \} \) at the points \( (\lambda_2^-(a), 2), (\lambda_2^+(a), 2), (\lambda_4^-(a), 2), ... \).

**Theorem 7** If \( 0 < a/i < r \), then for each \( n = 1, 2, ... \) **Pr. 2-Pr. 6** hold.

**Proof.** The proof of **Pr. 2.** By Theorem 6 to prove **Pr. 2** it is enough to show that

\[
I_n \cap \Gamma_m = \varnothing, \forall m \neq 2n - 1, 2n
\]  
(39)
If (39) is not true then by Notation 2 there exists \( c \in (\lambda_{2n-2}^+(a), \lambda_{2n}^-(a)) \) such that \( c \in \Gamma_m \). Then by Summary 6(a) the interval with end points \( c \) and \( \mu_m(0) \) is subset of \( \Gamma_m \). It implies that either \( \lambda_{2n-2}^+(a) \) or \( \lambda_{2n}^-(a) \) belong to \( \Gamma_m \) which contradicts to Notation 2.

**The proof of Pr. 3.** By Theorem 6 we have

\[
F(\lambda_{2n-2}^+(a)) = F(\lambda_{2n}^-(a)) = 2 \quad \& \quad -2 < F(\lambda) < 2, \forall \lambda \in (\lambda_{2n-2}^+(a), \lambda_{2n}^-(a)).
\]

Since \( F \) is differentiable function by the Roll’s theorem there exists

\[ \Lambda_n(a) \in (\lambda_{2n-2}^+(a), \lambda_{2n}^-(a)) \subset I_n \] such that \( F'(\Lambda_n(a)) = 0 \). It with the inequality in (40) implies that \( \Lambda_n(a) \) is a multiple eigenvalue of \( H_{t_n}(a) \) for some \( t_n \in (0, \pi) \). On the other hand, by (39), \( \Lambda_n(a) \notin \Gamma_m \) for all \( m \neq 2n - 1, 2n \). Therefore \( \Lambda_n(a) \) is a double eigenvalue and by Summary 5(b) we have \( \Lambda_n(a) = \mu_{2n-1}(t_n) = \mu_{2n}(t_n) \subset \Gamma_{2n-1} \cap \Gamma_{2n} \). This with Summary 6(b) implies (13). By Proposition 2 of [17] the double eigenvalue \( \Lambda_n(a) \) of the operator \( L(q) \) for \( t \in (0, \pi) \) is the spectral singularities of \( L(q) \), where \( q \) is an arbitrary periodic potential.

**The proof of Pr. 4.** By Notation 2 and (13) \( \lambda_{2n-2}^+(a) \) and \( \lambda_{2n}^-(a) \) belong to \( \Gamma_{2n-1} \) and \( \lambda_{2n-2}^+(a) = \mu_{2n-1}(0) \in \mathbb{R}, \lambda_{2n}^-(a) = \mu_{2n-1}(t_n) \in \mathbb{R} \). Therefore using Summary 6(a) we obtain \( [\lambda_{2n-2}^+(a), \lambda_{2n}^-(a)] = \{ \mu_{2n-1}(t) : t \in [0, t_n] \} \subset \Gamma_{2n-1} \). Similarly \( [\lambda_{2n}^-(a), \lambda_{2n}^+(a)] \subset \Gamma_{2n-1} \). Now to prove Pr. 4 it is enough to show that the curves \( \gamma_{2n-1}(a) \) and \( \gamma_{2n}(a) \) defined in (15) lie in \( \mathbb{C} \setminus \mathbb{R} \). We prove it in the proof of Pr. 5.

**The proof of Pr. 5.** We need to show that \( \gamma_{2n-1}(a) \) and \( \gamma_{2n}(a) \) have no real points. We prove it for \( \gamma_{2n}(a) \). The proof for \( \gamma_{2n-1}(a) \) is the same. Suppose to the contrary that there exists \( c \in (t_n, \pi] \) such that \( \mu_{2n}(c) \) is real. Then the interval joining \( \mu_{2n}(t_n) \) and \( \mu_{2n}(c) \) has overlapping subintervals with either \( [\lambda_{2n}^-(a), \lambda_{2n}^+(a)] \) or \( [\Lambda_n(a), \lambda_{2n}^-(a)] \) which contradicts either Summary 6(b) or Summary 5(c). Thus \( \mu_{2n}(t) \) is nonreal for all \( t \in (t_n, \pi] \). Let us prove that it is simple eigenvalue. Suppose that \( \mu_{2n}(t) \) is multiple for some \( t \in (t_n, \pi] \). Then there exists \( m \neq 2n \) such that \( \mu_{2n}(t) = \mu_m(t) \). Since \( \mu_{2n}(t_n) = \mu_{2n-1}(t_n) \) (see (13)) by Summary 6(b) \( m \neq 2n - 1, 2n \). Then the spectrum contains the continuous curve \( \gamma \) joining the real numbers \( \mu_{2n}(0) \) and \( \mu_m(0) \) and passing though nonreal \( \mu_{2n}(t_n) \). It with Summary 1(e) implies that the closed curve \( \gamma \cup \pi \) is a subset of the spectrum which contradicts Summary 1(d).

It remains to prove that \( \gamma_{2n}(a) = \{ \lambda : \lambda \in \gamma_{2n-1}(a) \} \). Since \( \mu_{2n}(t_n) \) is a double eigenvalue, \( (\mu_{2n}(t_n) = \mu_{2n-1}(t_n)) \) and the functions \( \mu_{2n} \) and \( \mu_{2n-1} \) are continuous, it follows from Summary 1(c) that there exists \( \varepsilon > 0 \) such that \( \mu_{2n}(t) = \mu_{2n-1}(t) \) for \( t \in (t_n, t_n + \varepsilon) \). On the other hand, \( \mu_{2n}(t) \) and \( \mu_{2n-1}(t) \) are simple and hence analytically depend on \( t \in (t_n, \pi] \). Therefore using the uniqueness of analytic continuation we complete the proof.

**The proof of Pr. 6.** By (13) and Summary 4(a) \( \Omega_n \) is a connected set. To prove the separability suppose to the contrary that there exists \( \lambda \in (\Omega_n \cap \Omega_m) \) for some \( m \neq n \). Since the real parts of \( \Omega_n \) and \( \Omega_m \) are disjoint intervals (see Pr. 2 and (9)), \( \lambda \) is a nonreal number. Then repeating the proof of simplicity of \( \mu_{2n}(t) \) which was done in the proof of Pr. 5 we get a contradiction with Summary 1(d) ■

Repeating the proof of Theorem 7 we get the following results for \( r < a/i < 2 \).

**Theorem 8** If \( a = ir \), then \( I_1 = \Lambda_1(a) = \mu_1(t_1) = \mu_2(t_1) \in \mathbb{R} \) and all statements of Theorem 7 continue to hold. If \( r < a/i < 2 \), then \( I_1 = \emptyset \) and all statements of Theorem 7 continue to hold for \( n = 2, 3, \ldots \).

The arguments of these chapter give as the following result for the operator \( L(q) \) with general PT-symmetric periodic potential \( q \).

**Theorem 9** Suppose that

\[
q \in L_1[0, \pi], \quad \int_0^\pi q(x)dx = 0, \quad q(x + \pi) = q(x), \quad q(-x) = q(x) \quad (a.e.).
\]
If there exists \( m > 0 \) such that the periodic and antiperiodic eigenvalues \( \mu_n(0) \) and \( \mu_n(\pi) \) for \( n > m \) are nonreal numbers then there exists \( R \) such that \( [R, \infty) \subset \sigma(L(q)) \) and the number of the gaps in the real part \( \Re(\sigma(L(q))) \) of \( \sigma(L(q)) \) is finite.

**Proof.** It is well-known that \( \mu_n(0) \) and \( \mu_n(\pi) \) are the zeros of \( F(\lambda) = 2 \) and \( F(\lambda) = -2 \) respectively (see (3)), where the Hill discriminant \( F \) is continuous on \( \mathbb{R} \), \( F(\lambda) \in \mathbb{R} \) for \( \lambda \in \mathbb{R} \) (see (5)), the asymptotic formula \( F(\lambda) = 2 \cos \sqrt{\lambda} + O(1/\sqrt{\lambda}) \) as \( \lambda \to \infty \) holds (see [3]) and \( \lambda \in \sigma(L(q)) \) if and only if \( F(\lambda) \in [-2, 2] \) (see Summary 1 (c)). Thus it is enough to show that there exists a large number \( R \) such that \( F(\lambda) \in [-2, 2] \) for all \( \lambda \geq R \). Suppose to the contrary that for any large positive number \( R \) there exists \( \lambda_1 \geq R \) such that \( F(\lambda_1) \notin [-2, 2] \). Without loss of generality, assume that \( F(\lambda_1) > 2 \). On the other hand, it follows from the above asymptotic formula for \( F(\lambda) \) that there exists \( \lambda_2 > \lambda_1 \) such that \( F(\lambda_2) < 2 \). Since \( F \) is a continuous real-valued function on \([\lambda_1, \lambda_2]\), there exists \( \lambda \in [\lambda_1, \lambda_2] \) such that \( F(\lambda) = 2 \), that is, \( \lambda = \mu_n(0) \) and hence \( \mu_n(0) \in \mathbb{R} \). It contradicts the conditions of the Theorem.

**Remark 4** There exist a lot of asymptotic formulas for \( \mu_n(0) \) and \( \mu_n(\pi) \). Using these formulas one can find the conditions on \( q \) such that \( \mu_n(0) \) and \( \mu_n(\pi) \) are nonreal number which implies that the number of gaps in \( \Re(\sigma(L(q))) \) is finite. Suppose that we have the formulas \( \mu_n(0) = a_n + o(n^{-\alpha}) \) and \( \mu_n(\pi) = b_n + o(n^{-\alpha}) \). If there exist \( m > 0 \) and \( c > 0 \) such that \( |\Im a_n| > cn^{-\alpha} \) and \( |\Im b_n| > cn^{-\alpha} \) for all \( n > m \), then \( [R, \infty) \subset \sigma(L(q)) \) for some \( R \). Besides, there exist a lot of asymptotic formulas for the distances between neighboring periodic and antiperiodic eigenvalues and it readily follows from (5) that the distances are either \( |2\Im \mu_n(0)| \) or \( |2\Im \mu_n(\pi)| \) if \( \mu_n(0) \) and \( \mu_n(\pi) \) are nonreal and \( n \) is a large number. Using these relations and asymptotic formulas one can construct a large class of the potentials \( q \) for which the number of gaps in \( \Re(\sigma(L(q))) \) is finite.

## 5 Finding the Second Critical Point

In this section we find the approximate value of the second critical point. For this we find the approximate value of the degeneration point for the first periodic critical. Recall that it is the value of \( a \in H(2) \) for which \( \lambda_0(a) = \lambda_2^{-}(a) \). Since \( \lambda_0(a) \) and \( \lambda_2^{-}(a) \) are the \( PN(a) \) eigenvalues lying in \( D_0(3) \) (see Notation 1) we consider (16) for \( |a| < 2 \) and \( |\lambda| \leq 9 \). The third formula in (16) can be written as

\[
ak = \frac{aa_{k-1} + aa_{k+1}}{\lambda - (2k)^2}, \quad \forall k > 1. \tag{41}
\]

Using it for \( a_2 \), that is, for \( k = 2 \) in the second formula of (16) we get

\[
(\lambda - 4)a_1 = a\sqrt{2}a_0 + \frac{a^2a_1}{(\lambda - 16)} + \frac{a^2a_3}{(\lambda - 16)}. \tag{42}
\]

Now we use (41) in (42) as follows. In the right hand side of (42), we isolate the term with multiplicand \( a_1 \) and use (41) for \( a_3 \). Then in the obtained formula we replace everywhere \( a_k \) by the right side of (41) if \( k > 1 \). In other word, the rule of usage of (41) is the following. Every time we isolate the terms with multiplicand \( a_1 \) and do not change it, while use (41) for \( a_2, a_3, \ldots \). One can readily see that the second, fourth,..., \( 2m \)-th usages (41) in (42) give the terms, denoted by \( A_1(a, \lambda)a_1, A_2(a, \lambda)a_1, \ldots, A_m(a, \lambda)a_1 \), with multiplicands \( a_1 \). Thus after \( 2m \) times usages we get

\[
(\lambda - 4)a_1 = a\sqrt{2}a_0 + \frac{a^2a_1}{(\lambda - 16)} + \left( \sum_{k=1}^{m} A_k(a, \lambda) \right)a_1 + R_m(a, \lambda), \tag{43}
\]
where $R_m(\lambda)$ is the sum of the terms without multiplicand $a_1$. To explain (43) and write the formulas for $A_k(a, \lambda)$ and $R_m(a, \lambda)$ we use the indices $n_1, n_2, \ldots$ whose values are either $-1$ or $1$. The formula (41) for $k = 3$ and $k = 3 + n_1$ can be written as follows

$$a_3 = \sum_{n_1 = -1, 1} \frac{aa_{3+n_1}}{\lambda - 36} \quad \text{and} \quad a_{3+n_1} = \sum_{n_2 = -1, 1} \frac{aa_{3+n_1+n_2}}{\lambda - (6 + 2n_1)^2}.$$

These two formulas give

$$a_3 = \sum_{n_2 = -1, 1} \left( \sum_{n_1 = -1, 1} \frac{a^2a_{3+n_1+n_2}}{(\lambda - 36)(\lambda - (6 + 2n_1)^2)} \right). \quad (44)$$

If $3 + n_1 + n_2 = 1$, that is, if $n_1 = -1$ and $n_2 = -1$, then we get the term with multiplicand $a_1$. In (44) isolating the term with multiplicand $a_1$ and using (41) for $a_{3+n_1+n_2}$ when $3 + n_1 + n_2 > 1$, and then for $a_{3+n_1+n_2+n_3}$, we get

$$a_3 = \frac{a^2a_1}{(\lambda - 36)(\lambda - 16)} +$$

$$\sum_{n_1, n_2, n_3, n_4} (\lambda - (6 + 2n_1)^2) (\lambda - (6 + 2n_1 + 2n_2)^2) (\lambda - (6 + 2n_1 + 2n_2 + 2n_3)^2)^{-1} a^4a_{3+n_1+n_2+n_3+n_4},$$

where the summation is taken under condition $3 + n_1 + n_2 > 1$. Repeating these usage of (41) $2m$ times and then using the formula obtained for $a_3$ in (42) we get (43) and see $A_k(a, \lambda)$ and $R_m(a, \lambda)$ has the form

$$A_1(a, \lambda) = \frac{a^4}{(\lambda - 16)(\lambda - 36)}, \quad (46)$$

$$A_k(a, \lambda) = \sum_{n_1, n_2, \ldots, n_{2k-1}} (\lambda - 16)^{-1} (\lambda - 36)^{-1} a^{2k+2} \prod_{s=1,2,\ldots,2k-1} [\lambda - (6 + 2n_1 + 2n_2 + \ldots + 2n_s)^2], \quad (47)$$

for $k = 2, 3, \ldots$, and

$$R_m = \sum_{n_1, n_2, \ldots, n_{2m}} (\lambda - 16)^{-1} (\lambda - 36)^{-1} a^{2m+2} \prod_{s=1,2,\ldots,2m-1} [\lambda - (6 + 2n_1 + 2n_2 + \ldots + 2n_s)^2], \quad (48)$$

Note that $A_1(a, \lambda)$ is obtained due to the first term in the right side of (45), that is, obtained in the second usage of (41) by taking $3 + n_1 + n_2 = 1$. The term $A_2$ is obtained in the fourth usage of (41) if $3 + n_1 + n_2 + n_3 + n_4 = 1$ and $3 + n_1 + n_2 > 1$. The last inequality is necessary, for doing the third and fourth usages by the rule of usage. The term $A_k$ is obtained in the $2k$th usage of (41) if

$$3 + n_1 + n_2 + \ldots + n_{2k} = 1, \quad (49)$$

$$3 + n_1 + n_2 + \ldots + n_{2s} > 1, \quad (50)$$

for $s = 1, 2, \ldots, k - 1$, since the last inequalities are necessary for doing the $(2k - 1)$th and $2k$th usages of (41) by the rule of usage. In other word, the summation in (47) is taken under conditions (49) and (50). Moreover, using (50) for $s = k - 1$ and taking into account
that $n_1 + n_2 + \ldots + n_{2k-2}$ is an even number from (49) we obtain $n_{2k} = n_{2k-1} = -1$,
\[ n_1 + n_2 + \ldots + n_{2k-2} = 0. \quad (51) \]

These equalities imply that
\[ \lambda - (6 + 2n_1 + 2n_2 + \ldots + 2n_{2k-2})^2 = \lambda - 36, \quad \lambda - (6 + 2n_1 + 2n_2 + \ldots + 2n_{2k-1})^2 = \lambda - 16. \]

Using it in (47) we get
\[ A_k(a, \lambda) = \sum_{n_1, n_2, \ldots, n_{2k-3}} \frac{(\lambda - 16)^{-2} (\lambda - 36)^{-2} \left(a^{2k+2}\right)^2}{\prod_{s=1,2,\ldots,2k-3} \left[ \lambda - (6 + 2n_1 + 2n_2 + \ldots + 2n_s)^2 \right]} \quad (52) \]
for $k \geq 2$, where the summation is taken under conditions (50) for $s = 1, 2, \ldots, k - 2$ and (51). Since $n_{2k-2}$ is $\pm 1$ and does not take part in (52) the equality (51) can be written as
\[ n_1 + n_2 + \ldots + n_{2k-2} = \pm 1, \quad 3 + n_1 + n_2 + \ldots + n_{2s} > 1, \quad \forall s = 1, 2, \ldots, k - 2 \quad (53) \]

Then for $k = 2$ we have
\[ A_2(a, \lambda) = \frac{a^6}{(\lambda - 16)^3(\lambda - 36)^2} + \frac{a^6}{(\lambda - 16)^2(\lambda - 36)^2(\lambda - 64)}. \quad (54) \]

In Appendix (see (78)) we prove that $R_m(a, \lambda) \to 0$ as $m \to \infty$. Therefore in (43) letting $m$ tend to infinity, using $a_1 = \frac{\lambda a_0}{\sqrt{2}}$ (see (16)) and then dividing by $a_0$ we get
\[ \lambda^2 - 4\lambda - 2a^2 - \frac{a^2\lambda}{(\lambda - 16)} - \sum_{k=1}^{\infty} \lambda A_k(a, \lambda) = 0, \quad (55) \]
where the series in (55) converges to some analytic function (see Remark 6).

**Theorem 10** (a) If $a \in I(2)$, then equation (55) has 2 roots (counting multiplicity) inside the circle $C_9(0) = \{ \lambda : |\lambda| = 9 \}$. These roots coincide with the $PN(a)$ eigenvalues $\lambda_0(a)$ and $\lambda_2(a)$ defined in Notation 1.

(b) For each $a \in I(2)$ the number $\lambda_0(a) - \lambda_2(a)$ is either real or pure imaginary.

c) There exists a unique number $a \in I(2)$, such that $\lambda_0(a) = \lambda_2(a)$.

**Proof.** (a) The equation $\lambda^2 - 4\lambda - 2a^2 = 0$ has 2 roots in the disc $D_9(0)$ and
\[ |\lambda^2 - 4\lambda - 2a^2| > 37, \quad \forall \lambda \in C_9(0). \]
Therefore the estimation $|a^2\lambda/(\lambda - 16)| < 36/7$ and Remark 6 imply that (55) has two roots in $D_9(0)$ due to the Rouche’s theorem. Since the $PN(a)$ eigenvalues $\lambda_0(a)$ and $\lambda_2(a)$ are also the roots of (55) lying in $D_9(0)$ they coincides with those roots.

(b) If $\lambda$ is a root of (55) lying in $D_9(0)$ then $\overline{\lambda}$ is also a root of (55) lying in $D_9(0)$ due to reality of $a^2$. Therefore the proof of (b) follows from (a).

(c) Denote by $N(a, \lambda)$ the right hand side of (55). Double root of (55) satisfies $N(a, \lambda) = 0$ and $N'(a, \lambda) = 0$, where $N'(a, \lambda)$ is the derivative of $N(a, \lambda)$ with respect to $\lambda$. From (79)-(81) we see that the series in (55) can be differentiated term by term and
\[ N'(a, \lambda) = 2\lambda - 4 - \frac{a^2}{\lambda - 16} + \frac{\lambda a^2}{(\lambda - 16)^2} - \sum_{k=1}^{\infty} \left( A_k(a, \lambda) + \lambda A_k(a, \lambda) \right), \quad (56) \]
\[
N''(a, \lambda) = 2 + \frac{2a^2}{(\lambda - 16)^2} - \frac{2\lambda a^2}{(\lambda - 16)^3} - \sum_{k=1}^{\infty} (2A_k(a, \lambda) + \lambda A_k''(a, \lambda)). \tag{57}
\]

If \( |\lambda| < 9 \) and \(|a| < 2 \), then from \( N'(a, \lambda) = 0 \), by using (82), we obtain that \( |\lambda| < 3 \) and \( N''(a, \lambda) \neq 0 \). Thus, by the implicit function theorem, \( \lambda(a) \) is an analytic function satisfying

\[
\lambda(a) = 2 + \frac{a^2}{2(\lambda(a) - 16)} - \frac{\lambda(a)a^2}{2(\lambda(a) - 16)^2} + \frac{1}{2} \sum_{k=1}^{\infty} (A_k(a, \lambda(a)) + \lambda(a)A_k''(a, \lambda(a))) \tag{58}
\]

from which we obtain \( \lambda(a) = 2 + f(a) \) and \(|f(a)| < 1/2 \), where \( f \) is an analytic function. Using it in (55) we get

\[
2 + a^2 = g(a), \quad |g(a)| < 1, \tag{59}
\]

where \( g \) is an analytic function. Now using the Rouche’s theorem for functions \( 2 + a^2 \) and \( 2 + a^2 - g(a) \) on the circle \( C_{1/2}(\sqrt{2}) \) we obtain that the equation (55) has a unique double root inside the circle. It implies that there exists unique value of \( a \in I(2) \) such that \( \lambda_0(a) \) is a double eigenvalue. \( \blacksquare \)

**Remark 5** Solving (59) by the numerical methods one can find an arbitrary approximation for the second critical point \( V_2 \). It is the value of \( V \in (1/2, \sqrt{5}/2) \) for which (55) for \( a = \sqrt{1 - 4V^2} \) has a double eigenvalue in \( D_0(0) \). In other words, we find the degeneration point for the first periodic eigenvalue, that is, the value of \( a \) for which \( \lambda_0(a) = \lambda_2(a) \). These investigations show that \( (n + 1) \text{th} \) critical point \( \nu_{n+1} \) is the value of \( V \in (1/2, \infty) \) for which \( \lambda_{2n}^{(a)}(a) = \lambda_{2n}^{(a)}(a) \) for \( a = \sqrt{1 - 4V^2} \). Therefore we say that, \( a \) is the degeneration point for the \((2n - 1)\text{th} \) periodic eigenvalue. Note also that then \( I_0(a) \) consists of one point (see (12)) and after \( a \) the real parts of \((2n - 1)\text{th} \) and \( 2n\text{th} \) bands disappear. The eigenvalues \( \lambda_{2n-1}^{(a)}(a) \) and \( \lambda_{2n}^{(a)}(a) \) are either \( PN(a) \) or \( PD(a) \) eigenvalues and hence satisfy either (16) or (17). Therefore considering these equations in the corresponding regions and iterating the formulas (16) or (17) for \( k = n \) and arguing as in the proof of (43) (each time isolate the terms with multiplicant \( a_n \) or \( b_n \) and do not change they and use (16) or (17) if \( k \neq n \) we get a formula similar to (55). The value of \( V \in (1/2, \infty) \) for which the obtained equation for \( a = \sqrt{1 - 4V^2} \) has a double root is the \( n \text{th} \) critical point or the degeneration point for the \((2n - 1)\text{th} \) periodic eigenvalue.

Now we find the approximate value of \( V_2 \) as follows. For the \( m \text{-th} \) approximations we use the equation obtained from (55) by replacing the summations from 1 to \( \infty \) with the summation from 1 to \( m \). For the estimation of the second critical point we use the second approximation which, by (46) and (54) has the form

\[
Q(a^2, \lambda) =: \lambda^2 - 4\lambda - \frac{a^2\lambda}{(\lambda - 16)} - \frac{a^4\lambda}{(\lambda - 16)^2(\lambda - 36)} - \frac{a^6\lambda}{(\lambda - 16)^3(\lambda - 36)^2(\lambda - 64)} - 2a^2 = 0. \tag{60}
\]

**Theorem 11** (a) If \( a^2 = -2.15728123 \) then \( \lambda_0(a) \) and \( \lambda_2(a) \) are the real eigenvalues of \( H_0(a) \) lying respectively inside the circles

\[
\gamma_1 = \{|\lambda - 2.088438808| = 0.00023\} \quad \& \quad \gamma_2 = \{|\lambda - 2.08895036| = 0.00023\}. \tag{61}
\]

(b) If \( a^2 = -2.157281295 \) then \( \lambda_0(a) \) and \( \lambda_2(a) \) are the nonreal eigenvalues of \( H_0(a) \) lying respectively inside the circles

\[
\gamma_{3,4} = \{|\lambda - 2.088698925 \pm 0.000232839i| = 0.00023\}. \tag{62}
\]
(c) The second critical number \( V_2 \) satisfies the inequalities

\[
0.8884370025 < V_2 < 0.8884370117
\]  

(63)

Proof. (a) It is clear that

\[
P(a^2, \lambda) =: (\lambda - 16)^3(\lambda - 36)^2(\lambda - 64)Q(a^2, \lambda),
\]

where \( Q(a^2, \lambda) \) is defined in (60), is a polynomial with respect to \( \lambda \) of order 8. Computing by Matematika 7 we see that the roots of \( P(-2.15728123, \lambda) \) are

\[
\lambda_1 = 2.088438808, \lambda_2 = 2.088959036, \lambda_3 = 15.85581654, \lambda_4 = 63.99999991,
\lambda_5 = 15.98321016 \pm 0.11878598i, \lambda_7 = 36.00018270 \pm 0.00333046i.
\]

Using the decomposition

\[
Q(a, \lambda) = \frac{(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_8)}{(\lambda - 16)^3(\lambda - 36)^2(\lambda - 64)}
\]

by direct calculations we obtain

\[
|Q(-2.15728123, \lambda)| > 5 \times 10^{-8}, \forall \lambda \in \gamma_1 \cup \gamma_2,
\]  

(64)

On the other hand, in the Estimation 4 of the Appendix (see (94)) we prove that

\[
\left| \sum_{k \geq 3} \lambda A_k(-2.15728123, \lambda) \right| < 4.7357 \times 10^{-8}, \forall \lambda \in \gamma_1 \cup \gamma_2\]

(65)

Hence by the Rouche’s theorem the equation (55) for \( a^2 = -2.15728123 \) has only one root inside of each circles \( \gamma_1 \) and \( \gamma_2 \). Therefore using Theorem 10 and taking into account that if \( \lambda \) lies inside \( \gamma_1 \) then \( \lambda \) does not lie inside \( \gamma_2 \), we obtain that \( \lambda_0(a) \) and \( \lambda_2(a) \) are the real eigenvalues lying respectively inside \( \gamma_1 \) and \( \gamma_2 \).

(b) The roots of \( P(a, \lambda) \) for the cases \( a^2 = -2.157281295 \) are

\[
2.088698925 \pm 0.000232839i, \quad 15.98321016 \pm 0.11878599i,
\]

\[
15.85581654, \quad 36.00018270 \pm 0.00333046i, \quad 63.99999991.
\]

Instead of \( \gamma_1, \gamma_2 \) using \( \gamma_3, \gamma_4 \) and repeating the proof of (a) we get the proof of (b).

(c) By Theorem 10 (b), \( \lambda_0(a) - \lambda_2(a) \) is either real number or pure imaginary number. Therefore it follows from (a) and (b) that when \( a^2 \) changes from \(-2.15728123 \) to \(-2.157281295 \) then \( \lambda_0(a) - \lambda_2(a) \) moving over \( x \) and \( y \) axes changes from real to pure imaginary number. Since \( \lambda_0(a) - \lambda_2(a) \) continuously depend on \( a \in I(2) \), there exists a real number \( (ir)^2 \) between \(-2.157281295 \) and \(-2.15728123 \) such that \( \lambda_0(ir) = \lambda_2(ir) \). Thus \( ir \) is the degeneration point for the first periodic eigenvalue defined in Theorem 5. Therefore \( V_2 = \frac{1}{2} (1 + r^2)^{1/2} \) is the second critical point and (63) holds. \( \blacksquare \)

6 Appendix: Estimations and Calculations

ESTIMATION 1. In this estimation we prove (24) by using (18) and (19). One of the eigenvalues \( \lambda_1(a) \) and \( \lambda_2(a) \) is AN and the other is AD eigenvalue lying in \( D_4(1) \). First let us consider the AD eigenvalue \( \lambda(a) \) lying in \( D_4(1) \) by using (18). By Summary 3, \( \lambda(a) \) is a
simple eigenvalue for small $a$ and $\lambda(a) = 1 + O(a)$. Then by the general perturbation theory, the corresponding eigenfunction is close to the eigenfunction corresponding to $\lambda(0)$. It means that $c_1 = 1 + o(1)$ as $a \to 0$. Therefore formula (18) for $k = 2$ implies that $c_2 = O(a)$. Using it in the first formula in (18) we get $(\lambda(a) - 1)c_1 = ac_1 + O(a^2)$. Now dividing both sides of the last equality by $c_1$ we get the formula $\lambda(a) = 1 + a + O(a^2)$. Instead (18) using (19) and repeating the above proof we obtain that the AN eigenvalue satisfy the formula $\lambda(a) = 1 - a + O(a^2)$. Thus (24) is proved. The obtained formulas with Notation 1 imply that $\lambda^+_1(a)$ and $\lambda^+_2(a)$ are AN and AD eigenvalues and $\lambda^+_1(a) = \lambda^+_1(a)$.

By this way we prove the following

**Proposition 2** The eigenvalues $\lambda^+_{2n-1}(a)$ and $\lambda^+_{2n-1}(a)$ for small values of $a \in I(2)$ are nonreal numbers and $\lambda^+_{2n-1}(a) = \lambda^-_{2n-1}(a)$.

**Proof.** Let us first consider the AD eigenvalue $\lambda(a)$ lying in $D_4((2n - 1)^2)$ by using (18). Suppose to the contrary that $\lambda(a)$ is real for some small $a \in I(2)$. Then by Theorem 2 it is real for all small $a$. It readily follows from Summary 3 and (18) that $\lambda(a)$ is a simple eigenvalue and

$$\lambda^+_{2n-1}(a) = (2n + 1)^2 + O(a) \quad \& \quad c_n = 1 + O(a), \ c_m = O(a), \forall m \neq n$$

To prove the proposition we iterate $(2n - 2)$-times the formula

$$(\lambda - (2n - 1)^2)c_n = ac_{n-1} + ac_{n+1} \quad (66)$$

(see (18) for $k = n$) as follows. Each time isolate the terms with multiplicand $c_n$ and do not change they and use the formulas

$$c_k = \frac{ac_{k-1} + ac_{k+1}}{\lambda - (2k - 1)^2} \quad \& \quad (67)$$

$$c_1 = \frac{ac_2}{\lambda(a) - 1 - a} \quad (68)$$

for the terms with multiplicand $c_k$ when $k \neq n$. After $(2n - 2)$ times usages of (67) or (68) in (66) we obtain

$$\lambda(a) = (2n + 1)^2 + G_n(\lambda(a)) + S_n(\lambda(a)) + R_n(\lambda(a)), \quad (69)$$

where the terms $G_n$, $S_n$ and $R_n$ are defined as follows. The term $R_n$ is the sum of terms containing the multiplicands $c_k$ for $k \neq n$ that is the sum of all nonisolated terms. Since $c_k$ for $k \neq n$ is $O(a)$ and $R_n$ is obtained after $(2n - 2)$ times iterations, we have $R_n(a, \lambda) = O(a^{2n})$.

Now consider the isolated terms, that is, the terms with multiplicand $c_n$. Without loss of generality it can be assumed that $c_n = 1$. It is clear that the isolated terms are obtained in first, third, ..., $(2n - 3)$th usage. Let $S_n$ be the special isolated term which is obtained by using the formulas (67) or (68) in the following order $k = n - 1, n - 2, ..., 2, 1, 2, ..., n - 1$. Then $S_n$ has the form

$$S_n = \frac{a^{2n-2}}{(\lambda(a) - 1 - a)F(\lambda(a))},$$

where $F(\lambda(a))$ is the products of $\lambda(a) - (2s - 1)^2$ for $s \neq n$ and $F(\lambda(a)) \sim 1$. Note that $f(a) \sim g(a)$ means that $f(a) = O(g(a))$ and $g(a) = O(f(a))$ as $a \to 0$. The multiplicand $\lambda(a) - 1 - a$ in $S_n$ is obtained due to application of (68). It is clear that only one isolated term, called special isolated term, contains $\lambda(a) - 1 - a$, since for the other isolated terms we do not apply (68). The sum of other isolated terms is denoted by $G_n(\lambda(a))$. Thus $G_n(\lambda(a))$
is the sum of fractions whose numerators are $a^{2k}$ for $k = 1, 2, \ldots, (n - 1)$ denominators are the products of $\lambda(a) - (2s - 1)^2$ for $s \neq n$ and hence are real number. Using the formula 
\[(1 - a)^{-1} = 1 + a + a^2 + \ldots\] and taking into account that $a^{2n}$ and $a^{2n+1}$ are real and nonreal number respectively, we see that the nonreal part of the special term $S_n$ is of order $a^{2n-1}$. Using this in (69) and taking into account that $G_n$ is a real number and $R_n(a, \lambda) = O(a^{2n})$ we get a contradiction $a^{2n-1} = O(a^{2n})$. Now the proof follows from Corollary 1. 

**ESTIMATION 2.** Here we prove (25). By Theorem 1, for the small value of $a \in I(2)$, the disc $D_{\sqrt{|a|}}(0)$ contains one PN eigenvalue denoted by $\lambda_0(a)$. Arguing as in the proof of (24) we see that $\lambda_0(a) = O(a)$, $a_0 = 1 + O(a)$. Using it in (16) and taking into account that $a_2 = O(a)$ we obtain

\[\lambda_0(a)a_0 = \frac{2a^2a_0}{\lambda_0(a) - 4} + O(a^3).\]

Dividing by $a_0$ and using $\lambda_0(a) = O(a)$ we get $\lambda_0(a) = O(a^2)$ and

\[\lambda_0(a) = \frac{2a^2}{O(a^2)} - 4 + O(a^3) = -\frac{1}{2}a^2 + O(a^3).\] (70)

Now we prove the second formula in (25), where $\lambda_2^{-}(a)$ and $\lambda_2^{+}(a)$ are the PN and PD eigenvalues lying in $D_{1+\sqrt{|a|}}(4)$ respectively, by using (16) and (17) and $a_1 = 1 + O(a)$, $b_1 = 1 + O(a)$, $\lambda_1^{-} = 4 + O(a)$. Using the first and third equalities of (16) in the second equality of (16) and taking into account that $a_2 = O(a)$ we get

\[(\lambda_2^{-}(a) - 4)a_1 = \frac{2a^2}{\lambda_2^{-}(a)}a_1 + \frac{a^2a_1}{(\lambda_2^{-}(a) - 16)} + O(a^3)\]

Dividing by $a_1$ and then iterating it we obtain

\[\lambda_2^{-}(a) = 4 + \frac{2a^2}{\lambda_2^{-}(a)} + \frac{a^2}{(\lambda_2^{-}(a) - 16)} + O(a^3) = 4 + \frac{5}{12}a^2 + O(a^3).\]

Now we prove the third formula in (25). Using the second formula of (17) for $k = 2$ in the first formula of (17) and taking into account that $b_3 = O(a)$ we get

\[(\lambda - 4)b_1 = \frac{a^2}{\lambda - 16}b_1 + O(a^3)\]

Dividing by $b_1$ we obtain

\[\lambda = 4 + \frac{a^2}{\lambda - 16} + O(a^3) = 4 - \frac{1}{12}a^2 + O(a^3).\]

Thus the formulas in (25) are proved.

To consider the eigenvalues $\lambda_{2n}^{-}(a)$ and $\lambda_{2n}^{+}(a)$ for $n > 2$ we use the formulas

\[(\lambda - 4 - \frac{2a^2}{\lambda})a_1 = aa_2, \quad (\lambda - (2k)^2)a_k = aa_{k-1} + aa_{k+1},\] (71)

\[(\lambda - 4)b_1 = ab_2, \quad (\lambda - (2k)^2)b_k = ab_{k-1} + ab_{k+1},\] (72)

where the first formula in (71) is obtained from the first and second formulas of (16).

**Proposition 3** The eigenvalues $\lambda_0(a)$ and $\lambda_{2n}^{\pm}(a)$, where $n = 1, 2, \ldots$, for all small $a \in I(2)$ are real numbers.
Proof. It follows from (25) that \( \lambda_0(a), \lambda^{-}_n(a) \) and \( \lambda^{+}_n(a) \) are real. Indeed if at least one of them is nonreal then by (5) its conjugate is also periodic eigenvalue lying in \( D_6(3) \) and by (25) differ from the other eigenvalues. It is contradiction, since by Theorem 1, \( D_6(3) \) contains 3 periodic eigenvalues.

Now consider \( \lambda^{+}_n(a) \) for \( n > 1 \). One of \( \lambda^{-}_n(a) \) and \( \lambda^{+}_n(a) \) satisfies (71) and the other satisfies (72). For simplicity of notation suppose that \( \lambda^{-}_n(a) \) satisfies (71). To prove the proposition we iterate 2\( n \)-times the formulas (71) and (72) in the same manner as were iterated the formula (66) in the proof of Proposition 2. Here to iterate (71) we also each time isolate the terms with multiplicant \( a_n \) (isolated terms) and do not change them and use the formulas (71) for the term with multiplicant \( a_k \) for \( k \neq n \). Thus iterating the second formula in (71) (for \( k = 2n \) \( 2n \) times we see that the sum of nonisolated term is \( O(a^{2n+2}) \) and get the equality

\[
\lambda^{-}_n(a) = (2n)^2 + G_n(\lambda^{-}_n(a)) + S_{n-1}(\lambda^{-}_n(a)) + S_n(\lambda^{-}_n(a)) + O(a^{2n+2})
\] (73)

Here \( G_n(\lambda^{-}_n(a)) \) is the sum of isolated terms whose denominators does not contain the multiplicant \( \lambda^{-}_n - 4 - 2a_n \), \( S_{n-1}(\lambda^{-}_n) \) and \( S_n(\lambda^{-}_n) \) are the sum of isolated term (special isolated terms) whose denominator contain the multiplicant \( \lambda^{-}_n - 4 - 2a_n \) and are obtained in \( (2n-3) \)-th and \( (2n-1) \)-th iterations respectively. It is clear that \( S_{n-1}(\lambda^{-}_n) \sim a^{2n-2} \) and \( S_n(\lambda^{-}_n) \sim a^{2n} \).

Similarly iterating the formulas (72) \( 2n \) times and arguing as above we get

\[
\lambda^{+}_n(a) = (2n)^2 + G_n(\lambda^{+}_n(a)) + \tilde{S}_{n-1}(\lambda^{+}_n(a)) + \tilde{S}_n(\lambda^{+}_n(a)) + O(a^{2n+2})
\] (74)

Here \( \tilde{G}_n(\lambda^{+}_n(a)) \) is the sum of isolated terms whose denominators does not contain the multiplicant \( \lambda^{+}_n - 4 \), \( \tilde{S}_{n-1}(\lambda^{+}_n) \) and \( \tilde{S}_n(\lambda^{+}_n) \) are the sum of isolated terms whose denominator contain the multiplicant \( \lambda^{+}_n - 4 \) and are obtained in \( (2n-3) \)-th and \( (2n-1) \)-th iterations respectively.

Now suppose to the contrary that \( \lambda^{-}_n(a) \) is nonreal for some small \( a \). Then by Theorem 2 it is nonreal for all small \( a \) and \( \lambda^{+}_n(a) = \lambda^{+}_n(a) \). Then using the formulas (73) and (74) and taking into account that \( G_n(\lambda^{-}_n) = G_n(\lambda^{+}_n), S_{n-1}(\lambda^{-}_n) = S_{n-1}(\lambda^{+}_n), S_n(\lambda^{-}_n) = S_n(\lambda^{+}_n) \) we get

\[
G_n(\lambda^+_{2n}) + S_{n-1}(\lambda^+_{2n}) + S_n(\lambda^+_{2n}) = \tilde{G}_n(\lambda^+_{2n}) + \tilde{S}_{n-1}(\lambda^+_{2n}) + \tilde{S}_n(\lambda^+_{2n}) + O(a^{2n+2}).
\] (75)

Now using the definitions of \( G_n \) and \( \tilde{G}_n \) and the equality \( (1 - \alpha)^{-1} = 1 + \alpha + O(\alpha^2) \) we obtain

\[
G_n(\lambda^+_{2n}) = \tilde{G}_n(\lambda^+_{2n}), \quad S_n(\lambda^+_{2n}) = \tilde{S}_n(\lambda^+_{2n}) + O(a^{2n+2}).
\] (76)

Let us consider \( S_{n-1}(\lambda^+_{2n}) \). It is clear that both \( S_{n-1}(\lambda^+_{2n}) \) and \( \tilde{S}_{n-1}(\lambda^+_{2n}) \) contain only one term and \( S_{n-1}(\lambda^+_{2n}) \) can be obtained from \( \tilde{S}_{n-1}(\lambda^+_{2n}) \) by replacing the multiplicant \( \lambda^+_{2n} - 4 \) with \( \left( \lambda^+_{2n} - 4 - \frac{2a_n}{\lambda^+_{2n}} \right)^{-1} \). Therefore we have

\[
S_{n-1}(\lambda^+_{2n}) = \tilde{S}_{n-1}(\lambda^+_{2n}) + \tilde{S}_{n-1}(\lambda^+_{2n}) \left( \frac{2a^2}{\lambda^+_{2n} - 4} \right) + O(a^{2n+2})
\] (77)

Now using (77) and (76) in (75) we get a contradiction \( 2a^2 \tilde{S}_{n-1}(\lambda^+_{2n}) = O(a^{2n+2}) \) ■

**ESTIMATION 3.** Here we estimate \( R_m(a, \lambda) \) defined in (48) and consider the convergence of the series in (55) for \( |a| < 2 \) and \( |\lambda| \leq 9 \). First let us consider \( R_m(a, \lambda) \). Since the
indices \(n_1, n_2, \ldots, n_{2m}\) take only two values the number of the summands of \(R_m(a, \lambda)\) is not more than \(4^m\). On the other hand, the largest (by absolute value) summand is not greater than \(a^{2m+2} (\lambda - 16)^{-m-1} (\lambda - 36)^{-m}\), since the eigenfunction (16) can be normalized so that \(|a_s| \leq 1\) for all \(s\). Therefore we have

\[
|R_m(a, \lambda)| < \frac{a^2}{\lambda - 16} \left| \frac{4a^2}{(\lambda - 16)(\lambda - 36)} \right|^m < \frac{4}{7} \left( \frac{16}{189} \right)^m \tag{78}
\]

Now we consider \(A_k(a, \lambda)\) defined in (52) in a similar way. Since the summation in \(A_k(a, \lambda)\) is taken over \(n_1, n_2, \ldots, n_{2k-3}\) the number of the summands in \(A_k(a, \lambda)\) is not more than \(2^{2k-3}\). It is clear that, the largest (by absolute value) summand is not greater than \(a^{2k+2} (\lambda - 16)^{-k-1} (\lambda - 36)^{-k}\). Therefore we have

\[
|A_k(a, \lambda)| \leq \frac{|a^2|}{8(\lambda - 16)^2} \left| \frac{4a^2}{(\lambda - 16)(\lambda - 36)} \right|^k < \frac{1}{14} \left( \frac{16}{189} \right)^k \tag{79}
\]

\[
|A'_k(a, \lambda)| \leq \frac{(2k + 1)a^2}{8(\lambda - 16)^3} \left| \frac{4a^2}{(\lambda - 16)(\lambda - 36)} \right|^k < \frac{2k + 1}{98} \left( \frac{16}{189} \right)^k \tag{80}
\]

\[
|A''_k(a, \lambda)| \leq \frac{(2k + 2)(2k + 1)a^2}{8(\lambda - 16)^4} \left| \frac{4a^2}{(\lambda - 16)(\lambda - 36)} \right|^k < \frac{4k^2 + 6k + 2}{686} \left( \frac{16}{189} \right)^k \tag{81}
\]

**Remark 6** From (79) and (80) immediately follows that if \(|a| \leq 2\) and \(|\lambda| \leq 9\), then the series in (55) converges uniformly to some analytic function on the disc \(\{\lambda \in \mathbb{C} : |\lambda| \leq 9\}\). Moreover using (79)-(81) by direct calculations we get

\[
\sum_{k \geq 1} |A_k(a, \lambda)| < \frac{1}{100}, \quad \sum_{k \geq 1} |A'_k(a, \lambda)| < \frac{1}{200}, \quad \sum_{k \geq 1} |A''_k(a, \lambda)| < \frac{1}{300}, \tag{82}
\]

**ESTIMATION 4.** Here we estimate \(A_k(a, \lambda)\), in detail, when

\[
\lambda \in (\gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4) \quad \text{and} \quad 2.156 < -a^2 < 2.158. \tag{83}
\]

It is clear that if (83) holds then

\[
\frac{2.15}{2 - (2s)^2} < \frac{-a^2}{\lambda - (2s)^2} < \frac{2.16}{2.1 - (2s)^2} \tag{84}
\]

for \(s = 2, 3, \ldots\). It with (79) implies that

\[
|A_k(a, \lambda)| < \frac{(2.16)}{8(2.1 - 16)} \left| \frac{4(2.16)}{(2.1 - 16)(2.1 - 36)} \right|^k .
\]

Therefore, using the geometric series formula by direct calculations in SWP we obtain

\[
\sum_{k \geq 4} |A_k(a, \lambda)| < 4.101 \times 10^{-11}. \tag{85}
\]

Now we estimate \(A_3(a, \lambda)\) and \(A_4(a, \lambda)\). To easyify the application of (84) we redonete
where the summation is taken under conditions $3 + i + j > 1$ and $i + j + s = \pm 1$. Therefore $A_3(a^2, \lambda)$ consist of 5 summand and 2 of them are the same. Namely,

$$A_3(a^2, \lambda) = \frac{a^8}{(\lambda - 16)^4(\lambda - 36)^3} + \frac{a^8}{(\lambda - 16)^2(\lambda - 36)^3(\lambda - 64)^2} \sum_{i,j,s} \frac{a^8}{(\lambda - 16)^3(\lambda - 36)^3(\lambda - 64)} + \frac{a^8}{(\lambda - 16)^2(\lambda - 36)^3(\lambda - 100)^2}.$$ (86)

Using (84) we see that

$$|A_3(a^2, \lambda)| < -A_3(2.16, 2.1) < 2.2707 \times 10^{-8}$$ (87)

It remains to estimate $A_4(a^2, \lambda)$. Let $C_4(a^2, \lambda)$ and $D_4(a^2, \lambda)$ be respectively the sum of the terms of $A_4(a^2, \lambda)$ subject to the constraints $n_1 = -1, n_2 = 1$ and $n_1 = 1, n_2 = -1$. In (52) replacing $n_3, n_4$ and $n_5$ respectively by $i, j$ and $s$ one can readily see that

$$C_4(a^2, \lambda) = \frac{a^2}{(\lambda - 16)(\lambda - 36)} A_3(a^2, \lambda), \quad D_4(a^2, \lambda) = \frac{a^2}{(\lambda - 64)(\lambda - 36)} A_3(a^2, \lambda).$$ (88)

Now let us consider the remaining terms of $A_4(a^2, \lambda)$, that is, the terms of $A_4(a^2, \lambda) - C_4(a^2, \lambda) - D_4(a^2, \lambda) =: E_4(a^2, \lambda)$. Using the definition of $C_4(a^2, \lambda)$ and $D_4(a^2, \lambda)$ and taking into account (53) we see that the terms of $E_4(a^2, \lambda)$ are obtained subject to the constraint $n_1 = 1, n_2 = 1$ and $n_3 + n_4 + n_5$ is either $-1$ or $-3$. Therefore the triple $(n_3, n_4, n_5)$ is either $(1, -1, -1)$ or $(1, 1, -1)$ or $(-1, 1, 1)$ or $(-1, -1, -1)$. Thus

$$E_4(a^2, \lambda) = \frac{a^{10}}{(\lambda - 16)^2(\lambda - 36)^2(\lambda - 64)^2(\lambda - 100)^2(\lambda - 144)} + \frac{a^{10}}{(\lambda - 16)^2(\lambda - 36)^2(\lambda - 64)^2(\lambda - 100)^2} + \frac{a^{10}}{(\lambda - 16)^2(\lambda - 36)^2(\lambda - 64)^2(\lambda - 100)^2}.$$ (89)

and by (88) we have

$$A_3(a^2, \lambda) + A_4(a^2, \lambda) = E_4(a^2, \lambda) + (1 + F(a^2, \lambda)) A_3(a^2, \lambda),$$ (90)

where

$$F(a^2, \lambda) = \frac{a^2}{(\lambda - 16)(\lambda - 36)} + \frac{a^2}{(\lambda - 16)(\lambda - 36)}$$

Since the summands in $E_4(-2, 16, 2.1)$ are positive number we have

$$|E_4(a^2, \lambda)| < E_4(-2, 16, 2.1).$$ (91)

Using the first inequality of (84) and taking into account that the summands in $F(-2.15, 2)$
are negative numbers and $|F(a^2, \lambda)| < 1$ we obtain

$$|1 + F(a^2, \lambda)| < 1 + F(-2.15, 2).$$

(92)

Thus using (90)-(92) we conclude that

$$|A_3(a^2, \lambda) + A_4(a^2, \lambda)| < E_4(-2, 16, 2.1) + (1 + F(-2, 15, 2))|A_3(-2, 16, 2.1)|.$$  (93)

Calculating the right-hand side of (93) by SWP we get

$$|A_3(a^2, \lambda) + A_4(a^2, \lambda)| < 1.6009 \times 10^{-12} + (0.991)2.2707 \times 10^{-8} < 2.251 \times 10^{-8}$$

This with (85) implies that

$$\left| \sum_{k \geq 3} \lambda A_k(a, \lambda) \right| < 2.1 \left( 4.101 \times 10^{-11} + 2.251 \times 10^{-8} \right) < 4.7357 \times 10^{-8}. \quad (94)$$

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