Robust Multi-object Matching via Iterative Reweighting of the Graph Connection Laplacian

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Abstract

We propose an efficient and robust iterative solution to the multi-object matching problem. We first clarify serious limitations of current methods as well as the inappropriateness of the standard iteratively reweighted least squares procedure. In view of these limitations, we suggest a novel and more reliable iterative reweighting strategy that incorporates information from higher-order neighborhoods by exploiting the graph connection Laplacian. We demonstrate the superior performance of our procedure over state-of-the-art methods using both synthetic and real datasets.

1 Introduction

The problem of matching multiple objects is crucial in many data-oriented tasks, such as structure from motion (SfM) \cite{21}, simultaneous localization and mapping \cite{4}, multi-graph matching \cite{28, 32, 33}, community detection \cite{1} and solving jigsaw puzzles \cite{15}. One important instance of this problem is multi-image matching, where one is given a set of 2D images, whose 3D scenes include a fixed set of 3D points, and each image contains a set of 2D keypoints that correspond to the set of 3D points. The goal is to recover the correspondences between the keypoints of all images and the fixed 3D points, given measurements of keypoint matches between some pairs of images. Ideally, a keypoint match between two given images aligns pairs of keypoints that describe the same 3D point. In practice, measurements of keypoint matches can be corrupted. A solution of this problem thus requires the design and analysis of methods with provable robustness to corruption. The latter general task has become crucial in structure from motion \cite{21} and many other data-oriented tasks.

The multi-object matching problem can be cast as permutation synchronization (PS) \cite{22}. The latter problem assumes a connected graph $G(V,E)$, where each node has a hidden permutation of a fixed size. For example, in image matching it is the correspondence between indices of the keypoints of the image and the 3D points. These permutations on all graph nodes determine relative permutations between nodes. In the case of image matching, the relative permutations represent the keypoint matches between pairs of images. Permutation synchronization asks to recover the hidden permutations given measurements of the relative permutations.

The measured relative permutations can be highly corrupted. For example, in SfM, the pairwise keypoint matches are commonly derived by SIFT \cite{18} descriptors, whose accuracy is affected by the scene occlusion, change of illumination, viewing distance and perspective. Moreover, repetitive patterns and ambiguous symmetry in common objects of realistic scenarios result in malicious and self-consistent corruption of matches \cite{31}. More work is needed to address such nontrivial practical cases of inaccurate pairwise measurements. Existing guarantees for permutation synchronization often consider a “uniform” corruption model, which does not reflect real scenarios. Uniformity is pursued in two ways: 1) Using the “uniform” Haar distribution on the permutation group to generate corrupted relative permutations; 2) Choosing the corrupted edges in the graph in a uniform manner, such as randomly corrupting an edge with the same probability, while assuming graphs with uniform topology (e.g., generated by the Erdős-Rényi model). Here we try to carefully understand the drawbacks of previous approaches and develop instead a practically efficient method, with partial guarantees, for nonuniform corruption. We find a surprising relationship of our proposed method to a different method. We thus better clarify and improve the implementation of the other method in our setting.

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1.1 Relevant Works

The most common methods of permutation synchronization \([8,13,22]\) measure the closeness of the estimated relative permutations to the given ones by averaging the corresponding squared Frobenius norm. They aim to solve a convex relaxation of a non-convex least squares (LS) formulation. Due to the use of relaxation, the accuracy of the algorithms of \([8,13,22]\) is not competitive and the speed of the algorithm of \([8,13]\) is rather slow. A tighter approximation algorithm to the non-convex LS formulation is the projected power method (PPM). It was used earlier for solving the problems of angular synchronization \([3,25]\) and joint alignment \([7]\).

PPM for permutation synchronization was first briefly tested in \([7]\) and later more carefully studied in \([14]\).

MatchLift \([8,13]\) and MatchALS \([34]\) are multi-object matching algorithms that also apply to the setting of partial matching, where the number of keypoints vary among objects. MatchALS is faster than MatchLift, whereas MatchLift is theoretically guaranteed under a special probabilistic model. Wang et al. \([30]\) improves the matching accuracy of \([34]\) by incorporating a geometric constraint on the pixel coordinates of keypoints.

There are three additional frameworks for solving some types of group synchronization problems \([10,17,23]\). However, \([10]\) only handles Lie groups and \([23]\) only deals with Gaussian noise without outliers, thus neither \([10]\) or \([23]\) applies to the setting of this paper. Cycle-edge message passing (CEMP) \([17]\) handles all compact groups, in particular, the permutation group, with both small sub-Gaussian noise and adversarial outliers. However, its strong condition for recovery with adversarial outliers can restrict some interesting cases of nonuniform corruption. Furthermore, it does not directly estimate group elements, but corruption levels. The recent Message Passing Least Squares (MPLS) framework \([24]\) aims to resolve the latter problem in practice. However, this framework was only fully developed for the different problem of rotation synchronization.

1.2 This Work

These are the main contributions of this work:

- We clarify the serious limitations of the common least squares methods for the PS problem in handling nonuniform corruptions. A rigorous argument appears in the second part of Theorem 5.2. We also clarify why the standard iteratively reweighted least squares procedure is not a good solution for the PS problem.
- We propose (in §4.1) a simple method for estimating the corruption levels in PS. It directly uses the graph connection Laplacian (GCL). We establish the equivalence of this method with the recent CEMP framework \([17]\) (with a properly chosen metric). Unlike CEMP, our procedure is fully vectorized, its computational complexity linearly depends on the cycle size (see appendix) and is much simpler to explain.
- We establish new theory for our above procedure, and thus also for CEMP, under a special nonuniform setting (see Theorem 5.2). As far as we know, this setting has not been studied before.
- We propose an iteratively reweighted procedure for solving the PS problems, where the weights are obtained by the above simple method that uses the GCL. This procedure is similar to MPLS \([24]\), but has some different choices for the group of permutations (as opposed to rotations). We demonstrate the superior performance of our proposed method in comparison to state-of-the-art methods using nontrivial synthetic and real data.

In §2, we mathematically define the PS problem and introduce notation. In §3, we demonstrate the limitations of previous methods. We propose our method in §4, and provide some theoretical guarantees for a special nonuniform scenario in §5. We compare performance with previous methods in §6. At last, §7 concludes this work.

2 Preliminaries

We mathematically formulate our underlying problem, introduce notation and review the notion of GCL.

2.1 Permutation Synchronization

We formulate the permutation synchronization (PS) problem, while also establishing some notation. For \(m \in \mathbb{N}\), we denote by \([m]\) the set \(\{1, \ldots, m\}\) and by \(S_m\) the permutation group on \(m\) elements. For easier presentation, we equivalently represent each element of \(S_m\) by an \(m \times m\) doubly-stochastic binary matrix (which is orthogonal) and denote the set of these matrices by \(\mathcal{P}_m\). In particular, any permutation \(\sigma \in S_m\) can be represented as \(P \in \mathcal{P}_m\), where \(P(i,j) = 1\) if \(\sigma(i) = j\) and \(P(i,j) = 0\) otherwise. Clearly, \(\mathcal{P}_m\) with matrix multiplication is isomorphic to \(S_m\). The PS problem is thus formulated as follows: For \(n, m \in \mathbb{N}\), assume a graph \(G([n], E)\), where each node \(i\) is assigned an unknown ground-truth absolute permutation matrix \(P^*_i \in \mathcal{P}_m\) (the star superscript emphasizes ground-truth information). These absolute permutations determine the following set of ground-truth relative permutations \(\{X^*_i\}_{i \in [n]}\), where \(X^*_i := P^*_i P_j^{-1} = P^*_i P_j^+\). The goal of PS is to recover \(\{P^*_i\}_{i \in [n]}\) from given, possibly corrupted, measurements of the relative...
permutations \( \{ \tilde{X}_{ij} \}_{ij \in E} \). We use different letters \( P \) and \( X \) in order to distinguish between absolute and relative representative permutation matrices. We note that \( X^*_j = X^*_j \top \) and we similarly assume that \( \tilde{X}_{ji} = \tilde{X}_{ji} \top \) (where either \( \tilde{X}_{ji} \) or \( \tilde{X}_{ji} \top \) are provided). We may thus assume that \( G([n], E) \) is undirected.

More careful mathematical models assume that \( \tilde{X}_{ji} = X^*_j \top \) on good edges \( E_g \subset E \) and \( \tilde{X}_{ji} \neq X^*_j \top \) on bad edges \( E_b = E \setminus E_g \). Under some special corruption models, which require choices of \( E_g \) and \( \{ \tilde{X}_{ij} \}_{ij \in E_b} \), one may try to prove or disprove exact recovery of \( \{ X^*_j \}_{ij \in E} \) by a PS algorithm of interest. One can further assume a noise model for \( \{ \tilde{X}_{ij} \}_{ij \in E_g} \) and quantify the approximate recovery of \( \{ X^*_j \}_{ij \in E} \).

2.2 Further Notation and Conventions

PS solvers are often described using block matrices as follows. Denote by \( \tilde{X} \) the block matrix in \( \mathbb{R}^{nm \times nm} \) whose \([i,j] \)-th block is \( X_{ij} \), for \( ij \in E \), and zero otherwise. Denote by \( P \in \mathbb{R}^{nm \times m} \) the block matrix whose \( i \)-th block is \( P_i \in \mathcal{P}_m \). Let \( \mathcal{P}_m \) denote the space of block matrices in \( \mathbb{R}^{m \times m} \), whose blocks are in \( \mathcal{P}_m \). For a matrix \( A \), \( A(i,j) \) indicates its \((i,j) \)-th element, and for a block matrix \( B \), \( B[i,j] \) indicates its \([i,j] \)-th block. For \( A, B \in \mathbb{R}^{k \times m} \), we denote their Frobenius inner product by \( \langle A, B \rangle = \text{Tr}(A^\top B) \). We denote the blockwise inner product of \( A, B \in \mathbb{R}^{mk \times nl} \) by \( \langle A, B \rangle_{\text{block}} \in \mathbb{R}^{k \times l} \), where its \((i,j) \)-th element is \( \langle A[i,j], B[i,j] \rangle \).

We use \( I_n, 1_n, \) and \( \Theta_n \) to represent the \( n \times n \) identity, all-one and all-zero matrices, respectively. We also denote the Kronecker product, elementwise multiplication and elementwise division of matrices by \( \otimes, \odot \) and \( \oslash \), respectively. For \( i \in [n] \), let \( N(i) \) and \( N_g(i) \) denote the sets of neighboring nodes of \( i \) in \( G([n], E) \) and \( G([n], E_g) \) respectively. We use the shorthand notation \( \text{w.h.p.} \) to mean with high probability.

2.3 The Graph Connection Weight and Laplacian

Throughout the paper we will need to estimate edge weights that express the similarity of the measured and ground-truth relative permutations. We thus assume in this section a weighted graph \( G([n], E) \) with arbitrary edge weights \( \{ w_{ij} \}_{ij \in E} \) and review relevant definitions and notation. We form the weight matrix \( W \in \mathbb{R}^{n \times n} \) such that \( W(i,j) = w_{ij} \) for \( ij \in E \) and \( W(i,j) = 0 \) otherwise. We recall that the degree of vertex \( i \in [n] \) is \( d_i = \sum_{j \in N(i)} w_{ij} \) and form the diagonal degree matrix \( D \in \mathbb{R}^{n \times n} \) such that \( D(i,i) = d_i \). When \( W(i,j) = 1 \) for all \( ij \in E \) we denote this weight matrix by \( E \) and refer to it as the adjacency matrix. The graph connection weight matrix (GCW), \( S \), and the graph connection Laplacian matrix (GCL), \( L \), are defined as follows:

\[
S = W \otimes 1_m \odot \tilde{X} \quad \text{and} \quad L = D \otimes I_m - S. \quad (1)
\]

Note that \( S_{ij} = w_{ij} \tilde{X}_{ij} \) for all \( i,j \in [n] \) and \( L_{ij} = -w_{ij} \tilde{X}_{ij} \) for \( i \neq j \in [n] \), and \( L_{ii} = d_i I_m \) for \( i \in [n] \). The normalized GCW is respectively defined as \( \tilde{S} = (D^{-1} W) \otimes 1_m \odot \tilde{X} \), so that \( \tilde{S}_{ij} = w_{ij} \tilde{X}_{ij} / d_i \) for \( i,j \in [n] \). Throughout the paper we iteratively estimate the graph weight matrix, GCW and normalized GCW and denote their estimated values at iteration \( t \) by \( W_t(i,j), S_t(i,j) \) and \( \tilde{S}_t(i,j) \). In practice, we work with the top eigenvectors of the GCW matrix (or the normalized one). Clearly, this is equivalent to using the bottom eigenvectors of the GCL matrix, and we thus use the term GCL when referring to and naming our method.

3 Drawbacks of Existing and Possible Solutions

Most established methods for permutation synchronization, such as \([22, 13, 8, 14]\), are based on least squares optimization. That is, they aim to find the set of absolute permutations whose relative permutations are “closest”, in least squares sense, to the measured ones. More specifically, they minimize the following objective function with \( q = 2 \) (we formulate this problem with general \( q > 0 \) for future reference):

\[
\min_{\{ P_i \}_{i \in [n]} \subset \mathcal{P}_m} \sum_{ij \in E} \| P_i P_j^\top - \tilde{X}_{ij} \|^q. \quad (2)
\]

We note that the optimization problem in \( \text{(2)} \) with \( q = 2 \) is equivalent to the following one:

\[
\max_{P \in \mathcal{P}_m} \langle PP^\top, \tilde{X} \rangle. \quad (3)
\]

Pachauri et al. \([22]\) approximates the solution of \( \text{(3)} \) by stacking the top \( m \) eigenvectors of the block matrix \( \tilde{X} \) and then projecting each block of the resulting matrix on \( \mathcal{P}_m \) by the Hungarian algorithm \([19]\). The state-of-the-art method for solving \( \text{(3)} \) is the projected power method (PPM) \([7, 14]\). It first initializes
\[ P_{(1)} \in \mathcal{P}_m \] following \([22]\), and then iteratively computes \( P_{(t+1)} \) for \( t \geq 1 \), as follows:

\[
P_{(t+1)} = \text{Proj}(X P_{(t)}) = \text{argmin}_{P \in \mathcal{P}_m} \| P - X P_{(t)} \|_F^2.
\]

The operator Proj is the blockwise projection onto \( \mathcal{P}_m \) that is computed by the Hungarian algorithm.

Both PPM \([7]\) and other least squares methods \([13, 8, 22]\) may tolerate uniform corruption. However, applications give rise to nonuniform corruption. For example, in image matching tasks that appear in 3D reconstruction datasets, the images of the same object may come from different sources of different qualities \([27]\). Matches of low quality images with other images are often erroneous. That is, their neighboring edges in the corresponding graph \( G([n], E) \) are more likely to be corrupted. This heterogeneity of images results in nonuniform topological structure of the bad subgraph \( G([n], E_b) \). Unfortunately, none of the previous methods can handle well such structure. We later try to quantify such a structure using a special nonuniform model. We also aim to clarify the failure of these methods in handling it in the last part of Theorem 5.2.

In principle, the above problem with PPM and other least squares methods can be addressed by a proper reweighting procedure that focuses only on good edges \( ij \in E_g \). A common global weighting method is iteratively reweighted least squares (IRLS). It has been successfully applied for synchronization-type problems with special continuous groups, such as \( SO(d) \) synchronization \([5, 12, 29]\) and camera location estimation \([11, 20]\). However, we claim that common IRLS methods for special synchronization problems with Lie groups do not directly generalize to synchronization problems with discrete groups, such as \( S_m \). Indeed, for our setting, standard IRLS aims to solve (3) with \( q = 1 \), that is, with least absolute deviations. The common hope is that the minimization of least absolute deviations instead of least squares deviations, which corresponds to \( q = 2 \), is more robust to adversarial corruption. The standard IRLS solution to (3) with \( q = 1 \) assumes an initial choice of \( \{ P_i \}_{i \in [n]} \subset \mathcal{P}_m \) and iteratively computes for \( t \geq 1 \):

\[
w^{\text{IRLS}}_{ij, (t+1)} = \frac{1}{\max_{P_i} \{ \| P_i P_j^T - \tilde{X}_{ij} \|_F^2 \} + \delta}
\]

\[
\{ P_i \}_{i \in [n]} = \text{argmin}_{\{ P_i \}_{i \in [n]} \subset \mathcal{P}_m} \| P_i P_j^T - \tilde{X}_{ij} \|_F^2,
\]

where \( \delta \) is a small regularization constant used to avoid a zero denominator. Unlike Lie groups, the discrete nature of permutations may result in exactly zero residuals, \( \| P_i P_j^T - \tilde{X}_{ij} \|_F \) on a few edges in the first few iterations of IRLS. These few edges with zero residuals, including the corrupted ones, will be extremely overweighed by (4). As a result, most of the “good” information on other edges are ignored and thus IRLS can produce poor solutions that are even worse than the given corrupted pairwise matches \( \tilde{X} \). Moreover, the solution of (3) typically involves convex relaxation, which may not be tight given poor edge weights. One may use less aggressive reweighting functions for heavy-tailed noise \([6]\). However, they are not expected to work well in our discrete scenario. One reason is that their weights are updated from the residuals. Since these residuals lie in a discrete finite space with size \( m \), there are very limited choices for the weights and the solution of the weighted least squares problem at each iteration can easily get stuck.

### 4 Our Proposed Method

Our idea is to iteratively and alternately estimate weights that emphasize uncorrupted edges and thus the underlying relative permutations. Similarly to IRLS, at each iteration the estimate of the absolute permutations is improved by solving (6) with the new estimated weights, instead of those computed by (5). Unlike IRLS, each edge weight is no longer determined by information obtained from only two nodes, but by information obtained from cycles containing the two nodes. The information on cycles is easily obtained by direct matrix multiplication. In §4.1 we explain how to initialize such weights. We establish a mathematical proposition that clarifies this simple approach. We also prove the equivalence of our simple method with the more involved CEMP framework \([17]\). In §4.2 we assume given weights and discuss weighted least squares (WLS) formulations and solutions. Using the ideas of §4.1 and §4.2 we formulate our complete procedure in §4.3.

#### 4.1 Weight Initialization

We estimate a good “similarity measure” and use it to estimate good edge weights. We remark that initial good weights that concentrate around the good edges is crucial for our whole procedure. Indeed we use a tight convex relaxation of a weighted least square formulation at each iteration and wrong weights can have a bad effect on its solution. For each \( ij \in E \), we use the following (correlation) affinity, or similarity measure, \( a_{ij} := \langle X_{ij}, X_{ij} \rangle / m \in [0, 1] \). We note that \( ij \in E_g \) if and only if \( a_{ij} = 1 \). One can choose the edge weight \( w_{ij} \) as the estimated \( a_{ij} \) or an increasing function of it (we clarify our choice below).

The following property of the GCW matrix \( S \) motivates our procedure for choosing weights.
Algorithm 1, then it is equivalent to CEMP with this metric and 3-cycles is equivalent to Algorithm 1. If one uses measured relative permutations the group has an orthogonal matrix representation. Assume further the following semimetric on the level used by CEMP for synchronization. Consequently, the established theory for CEMP in [17] extends to our procedure (Proposition 4.1). By taking Our third step uses the exponential function. In order to explain this choice, we note that if of (8), we estimate Proposition 4.1 provides a condition for making the unexplained approximation in(8) an equality. In view (of which is metric for the permutation group). Then approximations, \(X^{\text{apprx}}_{ij,t}\) for the \((i,j)\)th block of \(X^*, X^*[i,j]\), or equivalently, \(S^2_{\text{init},(t)} \odot (W^2_{\text{init},(t)} \odot 1_m)[i,j] \) or \(S^2_{\text{init},(t)}[i,j]/W^2_{\text{init},(t)}(i,j)\), can be written as

\[
X^{\text{apprx}}_{ij,t} = \sum_{k \in N(i,j)} W_{\text{init},(t)}(i,k) W_{\text{init},(t)}(k,j) X_{ik} X_{kj} = \frac{1}{|N_g(i,j)|} \sum_{k \in N_g(i,j)} X_{ik}^* X_{kj}^* = X_{ij}^*. \tag{8}
\]

Proposition 4.1 provides a condition for making the unexplained approximation in (8) an equality. In view of (8), we estimate \(a_{ij}^*\) by \(\langle \tilde{X}_{ij}, X^{\text{apprx}}_{ij,t} \rangle/m\).

Our third step uses the exponential function. In order to explain this choice, we note that if \(A_{\text{init},(t)} \rightarrow A^*\), or equivalently \(X^{\text{apprx}}_{ij,t} \rightarrow X_{ij}^*\), as \(t \rightarrow \infty\), then

\[
W_{\text{init},(t+1)}(i,j) \rightarrow \exp \left( \frac{-\beta_t}{2m} \| \tilde{X}_{ij} - X_{ij}^* \|_F^2 \right) \cdot \exp(\beta_t). \tag{9}
\]

Due to the arbitrary normalization of \(W_{\text{init},(t+1)}(i,j)\), which is evident from (8), we can ignore the term \(\exp(\beta_t)\). We note that this update rule is mathematically equivalent to the heat kernel used in Vector Diffusion Maps [20]. By taking \(\beta_t \rightarrow \infty\), we obtain that \(\exp(\beta_t)|X_{ij}^* - \tilde{X}_{ij}|_F^2/2m\rightarrow 1_{\{i,j \in E_g\}}\). This will clearly result in equality in (8) (that is, \(W_{\text{init},(t)}\) satisfies the requirements of Proposition 4.1) and consequently \(A_{\text{init},(t)} \rightarrow A^*\). Therefore, when \(t \rightarrow \infty\) and \(\beta_t \rightarrow \infty\), \(A^*\) is a fixed point of Algorithm 1.

Finally, we formulate the mentioned equivalence with CEMP [17] in the more general setting of group synchronization. Consequently, the established theory for CEMP in [17] extends to our procedure (Proposition 4.1) only motivates our procedure but does not justify it. We recall that CEMP directly estimates the corruption levels \(\{d(\tilde{X}_{ij}, X_{ij}^*)\}_{ij \in E}\) for some metric \(d\). It coincides with our approach when using the metric \(d(\tilde{X}, Y) = \|X - Y\|_F^2/(2m)\) for \(X, Y \in \mathbb{R}^{m \times n}\). We note that since \(\|X_{ij}^* - \tilde{X}_{ij}\|_F^2 = 2m - 2\|X_{ij}^* - X_{ij}\|_F^2\), the corruption level used by CEMP for \(i,j \in E\) is \(1 - a_{ij}^*\), where \(a_{ij}^*\) is the affinity of our procedure. Because the details of CEMP for estimating the corruption levels are more involved than our ideas, we prefer not to review them here.

Proposition 4.2. Assume that \(\tilde{X}\) represents the measured relative permutations in permutation synchronization, or more generally, the measured relative groups ratios in compact group synchronization, where the group has an orthogonal matrix representation. Assume further the following semimetric on the matrix-represented elements \(X, Y\): \(\|X - Y\|_F^2/(2m)\) which is metric for the permutation group. Then CEMP with this metric and 3-cycles is equivalent to Algorithm 1. If one uses \(l\)-th powers with \(l \geq 2\) in Algorithm 1 then it is equivalent to CEMP with \((l+1)\)-cycles and the same metric.
4.2 Weighted Least Squares Approximation of Permutations

We assume the approximated weights \( \{w_{ij(t)}\}_{ij \in E} \) at iteration \( t \geq 0 \), where at \( t = 0 \) the weights are obtained by Algorithm 1. The GCW and normalized GCW matrices are \( S_{(t)} \) and \( \overline{S}_{(t)} \). Using these weights, one may approximate the absolute permutations as solutions of two different WLS problems. The first one, which we advocate for, aims to solve the following weighted power iterations, which is a weighted analog of (4):

\[
P_{i(t+1)} = \arg\min_{P_i \in \mathcal{P}_m} \left\| \sum_{j \in N(i)} w_{ij(t)}(P_i - \overline{X}_{ij} P_{j(t)}) \right\|^2_F \quad \text{for all } i \in [n], \tag{10}
\]

We note that the solution of (10) is \( P_{(t+1)} = \text{Proj}(S_{(t)} P_{(t)}) \). The second WLS formulation aims to solve

\[
\{P_i(t+1)\}_{i \in [n]} = \arg\min_{\{P_i\}_{i \in [n]} \subseteq \mathcal{P}_m} \sum_{j \in N(i)} \frac{1}{d_{i(t)}} \sum_{j \in N(i)} w_{ij(t)}(P_i - \overline{X}_{ij} P_j)^2, \tag{11}
\]

where \( d_{i(t)} \) is the degree of node \( i \). To approximately solve (11), one can relax its constraint by requiring that \( P^TP = I_m \), and after projection onto \( \mathcal{P}_m \) obtain that

\[
P_{(t+1)} = \text{Proj}_{P^TP = I_m} \left( \arg\min_{P^TP = I_m} \|P - \overline{S}_{(t)} P\|^2_F \right). \tag{12}
\]

The columns of the solution of the minimization in (12) are exactly the top \( m \) eigenvectors of \( \overline{S}_{(t)} \). An analogue of (11) for \( SE(3) \) synchronization appears in (2). We recommend using (10) over (11) as it is faster and often more accurate in practice, but we report results with both of them. Furthermore, since (12) does not require a prior estimate for \( P \), we use it for the initial estimate of the block matrix of absolute permutations, \( P_{(1)} \).

4.3 Iteratively Reweighted Graph Connection Laplacian

We combine together ideas of §4.1 and §4.2 to formulate the Iteratively Reweighted Graph Connected Laplacian (IRGCL) procedure in Algorithm 2. The initial affinity matrix is computed by CEMP and the initial

Algorithm 2 Iteratively Reweighted Graph Connection Laplacian (IRGCL)

\[\text{Input: } X, \{\beta_j\}_{j=0}^{t_0}, \{\alpha_k\}_{k=1}^{t_1}, \{\lambda_t\}_{t=1}^{t_{max}}, \mathcal{F}: \mathbb{R}^{n \times n} \rightarrow \mathbb{R}^{n \times n} \text{ (default: } F(A) = A)\]

\[\begin{align*}
A_0 &= \text{CEMP}(X, \{\beta_j\}_{j=0}^{t_0}) \\
W_0 &= F(A_0) \\
P_0 &= \text{WLS}(X, W_0) \text{ by solving (12)}
\end{align*}\]

\[\text{for } t = 1:t_{max} \text{ do}\]

\[X(t) = \text{Proj}_{P^TP = I_m} \left( \arg\min_{P^TP = I_m} \|P - \overline{S}_{(t)} P\|^2_F \right)\]

\[\begin{align*}
A_1(t) &= \frac{1}{2}(X(t) - X)_{\text{block}} \\
W_1(t) &= \exp(\alpha_t A_1(t)) \\
S(t) &= (W_1(t) \otimes 1_m) \odot X \\
A_2(t) &= \frac{1}{2} \left( S_{(t)} \otimes (W_1(t) \otimes 1_m) \right)_{\text{block}} \\
A(t) &= (1 - \lambda_t) A_{1(t)} + \lambda_t A_{2(t)} \\
W(t) &= F(A(t))
\end{align*}\]

\[P_{(t+1)} = \text{WLS}(X, W(t)) \text{ by solving (10) (or possibly (12))}\]

\[\text{end for}\]

\[\text{Output: } \text{estimated absolute permutations } P_{(t+1)}\]

block of absolute permutations \( P_{(1)} \) is obtained by solving (12) (we explained above why (10) cannot be used for initialization). At next iterations, the affinity matrix is obtained as a convex combination of two affinity matrices as follows \((1 - \lambda_t) A_{1(t)} + \lambda_t A_{2(t)}\). The matrix \( A_{1(t)} \) is directly obtained by the newly estimated absolute permutations. Its use is similar to that of standard IRLS. Indeed, in IRLS the residuals are updated, but here we work with dot products instead of residuals (squared norms). It is easy to note that the matrix \( A_{2(t)} \) is updated in a similar way to the procedure described in Algorithm 1. The permutations in \( P_{(t+1)} \) are updated by a WLS procedure. When using (12) for this purpose, we name the algorithm IRGCL-S (S for spectral; this is our recommended choice). When using (10) instead, we name the algorithm IRGCL-P (P for power iterations).

The main difference between IRLS and our IRGCL is that IRGCL uses both the 1st and 2nd order edge affinities: \( A_{1(t)} \) and \( A_{2(t)} \), defined in the algorithm, to approximate \( A^* \) and computes the WLS
weights $W_{(t)}$. However, standard IRLS computes the WLS weights from only the 1st order affinities (or equivalently residuals; see (5)), which are unreliable under high corruption. Indeed, we can write $A_{1,(t)}(i,j) = 1 - \|X_{ij,(t)} - X_{ij}\|_F/2m$ and note that the approximation of $A^*$ by $A_{1,(t)}$ can be poor when $X_{ij}$ deviates from $X^*$. Therefore, we address this issue by gradually incorporating the 2nd order affinities (in $A_{2,(t)}$), which encode the 3-cycle consistency information and their use is justified by Proposition 4.1. That is, we increase $\lambda_t$ towards 1 as $t$ increases. However, incorporating the information from $A_{1,(t)}$ during the first few iterations can accelerate the convergence; we thus start with $\lambda_1 = 0.5$.

Few more technical comments on Algorithm 2 are as follows. It contains two types of edge weights: $W_{1,(t)}$ for the CEMP-like reweighting for estimating $A_{2,(t)}$ and $W_{(t)}$ for the WLS formulation that is used to solve the absolute permutations. The latter weights are estimated using the affinity $(1-\lambda_t)A_{1,(t)} + \lambda_t A_{2,(t)}$. For simplicity and to avoid additional parameters, we assume that $F(A_{(t)}) = A_{(t)}$. For the same reason, we only incorporate second order affinities and avoid higher order ones. The default parameters are described in Section 6.

We further illustrate IRGCCL in Figure 2 in the appendix. As is evident from this Figure, its basic idea is similar to the MPLS [24] algorithm that was pursued for the different problem of rotation synchronization. Nevertheless, there are two main differences between the two implementations. First, the reweighting function $F$ of [24] is sensitive to zero residuals and requires iterative truncation that introduces additional parameters. Second, [24] enforces $\lambda_t \to 0$ (as opposed to $\lambda_t \to 1$) and thus emphasizes the standard IRLS procedure, except for the first few iterations. The latter choice is mainly due to numerical experience with real data, but we can justify in two different ways.

We do not have guarantees for Algorithm 2 but we believe it is successful due to the following properties. First, it utilizes information from 3-cycles (reflected in the powers of the GCW matrix), in addition to that of edges. We believe that this decreases the sensitivity to its initialization; some degree of insensitivity is evident in numerical experiments. We further remark that such global estimate is more robust to corruption with nonuniform graph topology. Indeed, using the cycle information allows messages from $G([n], E_0)$ to propagate through the entire graph more easily and consequently correct severely corrupted subgraphs with nonuniform topology. This claim is supported by the experiments with nonuniform corruption. Second, since the 3-cycle consistency information helps more faithfully recover the underlying corruption, it provides more accurate weights and consequently a better approximation to the convex relaxation of the WLS problem. At last, the elements of $A_{2,(t)}$ for each $ij \in E$ are essentially weighted averages (ideally, expectations) of the 3-cycle consistency. The corresponding expectations are continuous and thus our reweighting scheme smooths the space of edge weights. This may make the algorithm less likely to get stuck.

### 5 Theoretical Guarantees for Nonuniform Corruption

As we mentioned in Section 1, previous work mainly addressed uniform corruption models, where the degrees of the corrupted graph $G([n], E_0)$ have little variation and the corruption probabilities are uniform (see e.g., [8]). The theory of [17] considers arbitrarily corrupted relative permutations, however, it restricts the maximal ratio of corrupted cycles and consequently restricts the degree of nodes in $E_0$. Here we consider a toy model with non-uniform graph topology (with large variations in the degrees of the subgraph $G([n], E_0)$) and with a relatively general class of distributions for the absolute and corrupted relative permutations.

We refer to our model as the superspreader corruption model. In this model, the bad edges are connected to a single node $i_0$, so the set $E_b$ has a “star-shaped” topology. Moreover, we assume that most of the neighboring edges of $i_0$ are corrupted. The distributions for $\{P^{*}_i\}_{i \in [n]}$ and $X_{ij} : ij \in E_b$ are general. We show that under this model and an additional mild generic condition on the latter distribution, least-squares type methods, including PPM, may fail, whereas CEMP (equivalently, Algorithm 1) is able to achieve accurate estimation of $A^*$ in one iteration as long as $n$ and its parameter $\beta_0$ are sufficiently large. We remark that the generic theory established for CEMP in [17] does not apply to the superspreader model. Our ideas of proof are also different from those of [17].

We first formulate this model. We then formulate the theorem, which is proved in the appendix.

**Definition 5.1.** The superspreader corruption model with parameters $n \in \mathbb{N}$, $m \in \mathbb{N}$ and $0 < \varepsilon, p \leq 1 \in \mathbb{R}$; distributions $D_P$ and $D_X$ on $\mathcal{P}_m$; and superspreader node $i_0 \in [n]$ is a probabilistic model with the following components: an Erdős–Rényi graph $G(n, p)$ where $p$ is probability of connection; ground-truth absolute permutations $\{P^{*}_i\}_{i \in [n]}$ i.i.d. sampled from $D_P$; a set $E_b$ whose edges are of the form $i_0 j$, where $j$ is randomly sampled from $N(i_0)$ with probability $1-\varepsilon$; and corrupted measurements of relative permutations $\{\tilde{X}_{ij}\}_{ij \in E_b}$, such that $\tilde{X}_{ij}$ is i.i.d sampled from $D_X$ and for $ij \in E_0$, $\tilde{X}_{ij} = X^{\ast}_{ij} = P^{*}_i P^{\top}_j$. 

Theorem 5.2. Assume data generated by the superspreader corruption model with node \( i_0 \), parameters \( n, m \) and \( 0 < \varepsilon, p \leq 1 \), distributions \( D_P \) and \( D_X \), and ground-truth and measured relative permutations \( \{ X^*_ij \} \) and \( \{ \tilde{X}_{ij} \} \) respectively. Let \( A^* = (X^*, \tilde{X})_{\text{block}}/m \), \( \mu = \mathbb{E} \left( \| X_{ij} \|_F^2 \right) / (2m) \), and assume that for all \( k \in N_b(i_0) \)

\[
\mathbb{E} \left( \| \tilde{X}_{ij} - X^*_ij \|_F^2 \right) \leq \mathbb{E} \left( \| X_{ij} - X^*_ij \|_F^2 \right).
\]

Then, for \( n = \Omega(1/(\mu^2 \varepsilon^2 p^2)) \), and \( \beta_0 \) of CEMP, \( A_{\text{init}}(1) \) obtained by CEMP with one iteration satisfies w.h.p.

\[
\| A_{\text{init}}(1) - A^* \|_\infty \leq (2 - \varepsilon) \left( 2 - \varepsilon \beta_0 \mu / 2 \right)^{-1}.
\]

On the other hand, for sufficiently small \( \varepsilon \), large \( n \) and some choices for \( D_X \), any least squares method, in particular PPM, does not result in good approximation of \( \{ X^*_ij \} \) and subsequent good estimation of \( A^* \).

The proof easily clarifies that condition (13) means that when the number of corrupted edges in a 3-cycle is enlarged from 1 to 2, then the cycle consistency decreases on average. A more precise statement of the second part of the theorem is that if \( \| (X_{ij} - P^*_ij) - P_{\text{opt}} \|_F < \varepsilon_0 / 2 \) for \( \varepsilon_0 > 0 \), such that \( 2\varepsilon / \sqrt{m + (1 - 2\varepsilon)\varepsilon_0} < 1 \), and \( P_{\text{opt}} \neq P^*_ij \), then PPM (and similarly any least squares method) cannot recover the ground truth w.h.p. for \( n \) sufficiently large.

6 Numerical Experiments

Using synthetic and real data, we compared IRGCL-S&P with the following methods for permutation synchronization: Spectral [22]; PPM [7]; IRLS-Cauchy-S&P: two methods that adapt the idea of [2] to permutation synchronization, while solving the WLS problem by either [10] for IRLS-Cauchy-P or [12] for IRLS-Cauchy-S; MatchLift [8] and MatchALS [34]. For the last two methods we used the codes from https://github.com/zju-3dv/multiway and their default choices. We implemented the rest of the methods using the default choices in the corresponding papers. We use the following parameters for IRGCL-S&P: \( t_0 = 5 \), \( t_{\text{max}} = 100 \), \( \beta_i = \min(2^i, 40) \), \( \alpha_i = \min(1.2^{i-1}, 40) \), \( \lambda_t = t/(t+1) \) and \( F(A) = A \). We stop the algorithm whenever \( P_{t+1}(t) = P_t(t) \).

In our paper we report results on synthetic data with a nonuniform corruption model, where the appendix further includes results with uniform corruption. In §6.2 we include results for real data.

6.1 Nonuniform Corruption Models

The following two models involve nonuniform corruption. For both models, we choose \( n = 100, m = 10 \) and assume an underlying complete graph \( G(n,E) \). Experiments with a more general Erdős-Rényi graph are reported in the appendix. We independently sample \( n_c \) nodes and for each sampled node we independently corrupt its \( m_c \) incident edges. We remark that \( n_c = 1 \) corresponds to our superspreader corruption model. We let \( \{ P^*_{ij} \} \) be i.i.d. sampled from the Haar measure on \( \mathcal{P}_m \), Haar(\( \mathcal{P}_m \)). We next describe the generation of \( X_{ij} \), where \( ij \in E_b \), in the two models. It is maliciously designed so that the distribution of \( X_{ij} P^*_j \) is no longer concentrated around \( P^*_j \), but biased towards some other permutation matrix.

1. Local Biased Corruption Model (LBC): For each \( ij \in E_b \),

\[
X_{ij} = \begin{cases} \{ P^*_j P^*_i \} & \text{if } (P^*_j P^*_i P^*_j)^T \leq 1, \\ X_{ij} \sim \text{Haar}(\mathcal{P}_m) & \text{otherwise.} \end{cases}
\]

Note that \( P^*_j P^*_i \) are self-consistent and since \( (P^*_j P^*_i P^*_j)^T \leq 1 \), they tend to be far away from the ground-truth \( P^*_j P^*_i \), and therefore the overall distribution of \( X_{ij} P^*_j \) is far away from \( P^*_j \).

2. Local Adversarial Corruption Model (LAC): For each \( ij \in E_b \), \( X_{ij} = Q^*_ij P^*_j \), where \( Q^*_ij \) is sampled by randomly permuting 3 columns of the \( m \times m \) identity matrix. We remark that the LAC model is even more malicious, since \( X_{ij} P^*_j \) explicitly concentrates around the identity matrix.

We fix \( m_c = 90 \) for LBC and \( m_c = 60 \) for LAC. We use the error \( \sum_{ij \in E_b} \| X_{ij}^* - X_{ij} \|_F^2 / \sum_{ij \in E_b} \| X_{ij}^* \|_F^2 \) to compare the different methods. We created 20 random samples from each model and we computed average errors and standard deviations for \( n_c = 1,\ldots,6 \). Figure [1] reports these average errors for the two different models, while designating standard deviations by error bars.

We note that both methods are able to achieve near exact recovery under all tested values of \( n_c \). In particular, they can exactly recover the ground truth under the super malicious LAC model, and outperform all other methods. We remark that both IRLS-Cauchy-S&P performs better than Spectral and PPM. However, their
improvement is limited and cannot achieve exact recovery. MatchLift and MatchALS are better than other least squares methods. However, they require hundreds of iterations and are thus slow.

![Graph showing average matching error under the LBC model (left) and LAC model (right).](image)

**Figure 1:** Average matching error under the LBC model (left) and LAC model (right).

### 6.2 Real Dataset

We compare the performance of the different methods on the Willow database [9], which consists of 5 image datasets. Each image dataset contains 40-108 images of the same object. We use the same method suggested by [30] to extract CNN features from 10 annotated keypoints for each image through AlexNet [16]. The candidate for the initial matching is obtained by applying the Hungarian algorithm on the feature similarity matrix, following the same procedure as [30]. However, the obtained initial matching is ill-posed for permutation synchronization. For example, given the initial matching obtained using the car dataset, there are 8 out of 40 nodes whose all neighboring edges are severely corrupted. That is, there is no chance to recover correct information of those nodes. To make those datasets well-posed to permutation synchronization solvers, we only use the relative permutation $\tilde{X}_{ij}$ between the nodes $i$ and $j$ whose incident edges are not completely corrupted. IRLS-Cauchy-P&S were comparable and we thus only report IRLS-Cauchy-S, while referring to it as IRLS. We also report the estimation error for $P_{(1)}$ in Algorithm 2 (we call it IRGCL-init and further test it in the appendix). We do not compare with [30] since it requires additional geometric information from the pixel coordinates of keypoints.

We report the relative estimation error $\sum_{i \neq j} \frac{\| \hat{X}_{ij} - X_{ij}^* \|^2_F}{\sum_{i \neq j} \| X_{ij}^* \|^2_F}$ of different methods in Table 1. We note that the four data sets which exclude FACE are highly corrupted (in view of their “Input” parameter).

| Datasets  | N  | Input | Spectral | M Lift | MALS | PPM | IRLS | IRLS-cauchy-S | IRLS-cauchy-P | IRGCL-init | IRGCL-S | IRGCL-P |
|-----------|----|-------|----------|--------|------|-----|------|---------------|---------------|------------|--------|--------|
| Car       | 32 | 0.41  | 0.23     | 0.17   | 0.14 | 0.14| 0.14 | 0.14          | 0.091         |
| Duck      | 30 | 0.46  | 0.20     | 0.26   | 0.22 | 0.20| 0.19 | 0.19          | 0.19          | 0.19      | 0.21   |
| Face      | 108| 0.14  | 0.042    | 0.091  | 0.057| 0.039| 0.042| 0.039         | 0.039         | 0.039     | 0.051  |
| Motorcycle| 14 | 0.53  | 0.46     | 0.39   | 0.38 | 0.46| 0.41 | 0.42          | 0.33          |
| Winebottle| 56 | 0.43  | 0.27     | 0.24   | 0.22 | 0.24| 0.24 | 0.24          | 0.24          |

**Table 1:** Matching performance comparison using the Willow datasets.

Our methods IRGCL-S, IRGCL-init and IRGCL-P are still able to achieve reasonable improvement over Spectral and PPM respectively. Among the least squares methods, Spectral and MatchLift perform the worst on average, and PPM performs the rest. We remark that IRLS with Cauchy weights does not have a significant advantage over the least squares methods. We note that IRGCL-S and IRGCL-init perform similarly. Furthermore, on average IRGCL-P performs the best, especially for the highly corrupted datasets (excluding FACE).

### 7 Conclusion

We proposed an iterative method for robustly solving multi-object matching. It overcomes the limitations of both IRLS and common least squares methods under nonuniform corruption models. We demonstrated through both experiments and theory the advantage of directly exploiting cycle-consistency information to guide the convergence of our non-convex optimization algorithm. There are several interesting future directions. First of all, although our work focuses on permutation synchronization, its ideas can be generalized to the setting of partial matching, which has more applications in structure from motion. Second, we believe that one can borrow ideas from the theory of graph connection Laplacian and vector diffusion maps in order to establish exact recovery guarantees for our method under different corruption models.
8 Broader Impact

Our proposed algorithms and ideas can be integrated in common 3D reconstruction software. Three-dimensional reconstruction has important applications in autonomous driving, virtual reality and augmented reality. In the past decade, the 3D reconstruction community has been switching from incremental reconstruction procedures to global optimization schemes [21]. We thus globally estimate correlations to provide consistent image matches as initial data for common global reconstruction pipelines. In order to address real applied scenarios of high corruption, it is important to further develop and utilize robust estimation methods within real-time 3D reconstruction. In addition to developing robust methods, we also provide some theoretical guarantees for a special setting of nonuniform corruption. Another important reason for detecting abnormal data in an unsupervised and interpretable way is to alleviate the vulnerability of deep learning based methods to adversarial attacks. Our work takes a step towards this aim through robust extraction of image or camera correspondence information without pre-training. This work is of interest to a broad community of machine learners that care about and use robustness, discrete optimization methods and iteratively reweighted least squares (IRLS). In fact, we show that the common IRLS method does not work well in our setting and explain how to carefully modify it. We use core and well-established testing methods and prove various mathematical propositions.

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Appendix

In §A we provide proofs of Propositions 4.2 and 4.1 and Theorem 5.2. In §B we provide an illustration that explains IRGCL and results of additional experiments on synthetic data generated by both uniform and nonuniform corruption models.

A Proofs

In this section we provide the proofs to all theoretical statements in the main manuscript. We find it more convenient to first establish Proposition 4.1 and then use the simple idea of the proof when establishing Proposition 4.2. At last, we prove Theorem 5.2.

A.1 Proof of Proposition 4.1

We first note that

\[
W^t(i,j) = \sum_{k_1 \in [n]} \sum_{k_2 \in [n]} \cdots \sum_{k_{l-1} \in [n]} \prod_{k=1}^{l-1} w_{ik_1} w_{k_1 k_2} \cdots w_{k_{l-1} j} \tag{16}
\]

and

\[
S^t[i,j] = \sum_{k_1 \in [n]} \sum_{k_2 \in [n]} \cdots \sum_{k_{l-1} \in [n]} \prod_{k=1}^{l-1} X_{ik_1} X_{k_1 k_2} \cdots X_{k_{l-1} j}. \tag{17}
\]

We note that for any pair of nodes \(i, j \in [n]\), \(w_{ij} X_{ij} = w_{ij} X^*_ij\). Indeed, if \(ij \notin E\), then \(w_{ij} = 0\); if \(ij \in E\), then \(X_{ij} = X^*_ij\); and if \(ij \in E\), then \(W(i,j) = 0\) by the assumption of the proposition. Combining this observation with (16) and (17) and then applying the cycle-consistency of \(\{X^*_i\}_{ij \in E}\), we obtain that for \(ij \in E\),

\[
(S^t \circ (W^t \otimes 1_m))[i,j] = \sum_{k_1 \in [n]} \sum_{k_2 \in [n]} \cdots \sum_{k_{l-1} \in [n]} \prod_{k=1}^{l-1} w_{ik_1} X^*_i X^*_k X^*_j = X^*_ij.
\]

A.2 Proof of Proposition 4.2

We first introduce the following definitions and notation for describing the original version of CEMP [17]. Let \(N_{ij}\) be the set of all \((l+1)\)-cycles that contain \(ij\). Any cycle in \(N_{ij}\) can be represented as \(L := \{ik_1, k_1k_2, \ldots, k_{l-1}j\}\). Using the metric stated in the proposition (while squaring and normalizing it), the cycle inconsistency (proposed in [17]) for each \(L \in N_{ij}\) is defined as

\[
d_L := ||X_{ik_1} X_{k_1 k_2} \cdots X_{k_{l-1} j} X_{ji} - I_m||_F^2/2m. \tag{18}
\]

The original version of CEMP with \((l+1)\)-cycles is iterated over \(t \geq 0\) using the following message passing procedure (see (10) and (35) of [17]):

\[
s_{ij,(t)} = \sum_{L \in N_{ij}} d_L |N_{ij}|. \tag{19}
\]

and

\[
s_{ij,(t+1)} = \sum_{L \in N_{ij}} \prod_{ab \in L \setminus \{ij\}} \exp(-\beta t s_{ab,(t)}) \frac{d_L}{\sum_{L \in N_{ij}} \prod_{ab \in L \setminus \{ij\}} \exp(-\beta t s_{ab,(t)})} \quad \text{for } ij \in E. \tag{20}
\]

We use here a generalized version of Algorithm 1 with power \(l \geq 2\). That is, we replace the step

\[
A_{init,(t)} = \frac{1}{m} (S^2_{init,(t)} \circ (W^2_{init,(t)} \otimes 1_m)).X_{block} \text{ in Algorithm 1 with}
\]

\[
A_{init,(t)} = \frac{1}{m} (S^l_{init,(t)} \circ (W^l_{init,(t)} \otimes 1_m)).X_{block} \quad \text{for } l \geq 2. \tag{21}
\]

To prove the equivalence between Algorithm 1 with \(l \geq 2\) and CEMP with the chosen metric, we show that at each iteration \(t \geq 0\), \(A_{init,(t)}(i,j) = 1 - s_{ij,(t)}\), where \(A_{init,(t)}(i,j)\) is obtained by Algorithm 1 and \(s_{ij,(t)}\) by the original CEMP. We verify this by induction. For simplicity, we denote \(W_{init,(t)}(i,j)\) by \(q_{ij,(t)}\). For \(t = 0\), we
first apply (21), we then combine (16) and (17) and at last apply basic algebraic manipulations to obtain that
\[ A_{\text{init.}(t)}(i,j) = \frac{1}{m} \left( S_{\text{init.}(t)} \otimes (W_{\text{init.}(t)} \otimes 1) \right) [i,j] \tilde{X}_{ij} \]

\[ = \frac{1}{m} \left( \sum_{k,t \in [n]} \sum_{k_{t-1} \in [n]} q_{ik_1(0)} \ldots q_{k_{t-1},j(0)} \tilde{X}_{ik_1} \tilde{X}_{k_1 k_2} \ldots \tilde{X}_{k_{t-1},j} \tilde{X}_{ij} \right) \]

\[ = \frac{1}{m} \sum_{k,t \in [n]} \sum_{k_{t-1} \in [n]} q_{ik_1(0)} \ldots q_{k_{t-1},j(0)} \left( \tilde{X}_{ik_1} \tilde{X}_{k_1 k_2} \ldots \tilde{X}_{k_{t-1},j} \tilde{X}_{ij} \right) \]

\[ = - \frac{1}{m} \sum_{k,t \in [n]} \sum_{k_{t-1} \in [n]} q_{ik_1(0)} \ldots q_{k_{t-1},j(0)} \left( m - \left( \tilde{X}_{ik_1} \tilde{X}_{k_1 k_2} \ldots \tilde{X}_{k_{t-1},j} \tilde{X}_{ij} \right) \right). \]

We further simplify this equation as follows. We first use the fact that
\[ m - (X, Y) = \|X - Y\|_F^2 / 2 \text{ for } X, Y \in O_m. \] (22)

We then use the facts that for \( X, Y, Z \in O_m \), \( \|X - Y\|_F = \|XZ - YZ\|_F \) and \( \tilde{X}_{ij} \tilde{X}_{ji} = I_m \). Next, we apply (18) and the fact that due to the initialization of \( W_{\text{init.}(0)} \) by Algorithm \[ \[ d_{ij}(0) = 1 \] for \( i, j \in E \). At last, we apply (19) and result in the desired relationship when \( t = 0 \):

\[ A_{\text{init.}(0)}(i,j) = - s_{ij}(0). \] (23)

Next, assuming that for all \( ij \in E \) \( A_{\text{init.}(t)}(i,j) = 1 - s_{ij}(t) \), we show that all \( ij \in E \) \( A_{\text{init.}(t+1)}(i,j) = 1 - s_{ij}(t+1) \). We first derive an identity for \( A_{\text{init.}(t+1)}(i,j) \) which is similar to the second equality of (23) by following the same arguments. We then apply (18) with compact notation for the multiplication of the different weights. Next, we use the weights assigned by Algorithm (1) for \( t+1 \), that is, \( W_{\text{init.}(t+1)} = \exp(\beta_t A_{\text{init.}(t)}) \). Next, we use the induction assumption \( A_{\text{init.}(t)}(i,j) = 1 - s_{ij}(t) \). We then apply basic algebraic manipulations and, at last, use (20) to conclude the induction argument as follows:

\[ A_{\text{init.}(t+1)}(i,j) = - \sum_{L \in N_j} d_L \]

\[ = - \sum_{L \in N_j} \prod_{a \in L \setminus \{i,j\}} q_{ab(t+1)} d_L \]

\[ = - \sum_{L \in N_j} \prod_{a \in L \setminus \{i,j\}} \exp(\beta_t A_{\text{init.}(t)}(a,b)) d_L \]

\[ = - \sum_{L \in N_j} \prod_{a \in L \setminus \{i,j\}} \exp(\beta_t (1 - s_{ab}(t))) d_L \]

\[ = - \sum_{L \in N_j} \prod_{a \in L \setminus \{i,j\}} \exp((-\beta_t) s_{ab}(t)) d_L \]

\[ = - \sum_{L \in N_j} \prod_{a \in L \setminus \{i,j\}} \exp(-\beta_t s_{ab}(t)) d_L \]

\[ = - \sum_{L \in N_j} \prod_{a \in L \setminus \{i,j\}} \exp(-\beta_t s_{ab}(t)) d_L \]

Consequently, for all \( ij \in E \) and \( t \geq 0 \), \( A_{\text{init.}(t)}(i,j) = 1 - s_{ij}(t) \), and thus Algorithm (1) is equivalent to the original CEMP.

**Remark A.1.** The equivalence between CEMP and Algorithm (1) is not restricted to permutations. Indeed, our arguments apply to any group that can be represented as a subgroup of the orthogonal group \( O_m \), where
we assign to any the representations of the elements $X$ and $Y$ the semimetric $d(X,Y) = \|X - Y\|_2^2/2m$. For permutation synchronization, this is a metric and thus the theory of CEMP directly extends. For other groups, this semimetric satisfies a relaxed triangle inequality (with constant $2$). Thus one may still extend the theory of CEMP, but with weaker estimates.

### A.3 Proof of Theorem 5.2

We describe the proof in two different sections. In §A.3.1 we prove the first part of the theorem that guarantees sufficiently near recovery by CEMP under the suppersreader model. In §A.3.2 we verify that other least squares methods generally do not succeed with recovery under our nonuniform setting.

#### A.3.1 A theoretical guarantee for CEMP in the nonuniform case.

For each $i \in N$ and $j \in E$, we define $N(i) = \{j \in [n]: ij \in E\}$ and $N(ii) = \{k \in [n]: ik, jk \in E\}$. For any $ij \in E$, let $N_y(ij) = \{k \in N(ij): k \neq i, j \text{ and } ik, jk \in E_y\}$ and $N_y(ij) = N(ij) \setminus N_y(ij)$. We will use the following Chernoff bound. For i.i.d. Bernoulli random variables $\{X_i\}_{i=1}^M$ with means $\mu$ and any $0 < \eta < 1$,

$$\Pr\left(\sum_{i=1}^M X_i - M \mu \geq \eta M \mu \right) < 2e^{-\frac{\eta^2 \mu}{2} M \mu}.$$  

(24)

We first show that the following deterministic conditions hold with high probability under the assumptions of our model:

$$\frac{1}{2} np \leq N(i) \leq 2np \quad \text{for any } i \in [n],$$

(25)

$$\frac{1}{2} np^2 \leq N(ij) \leq 2np^2 \quad \text{for any } ij \in E,$$

(26)

$$\frac{1}{2} |N(i_0j)| \geq \sigma^{-2} \eta \geq 2|N(i_0j)| \geq \sigma^{-2} \eta \quad \text{for any } j \in N(i_0).$$

(27)

Indeed, since $1_{(j \in N(i))}, 1_{(k \in N_y(ij))}, 1_{(k \in N_y(i_0j))}$ are all Bernoulli random variables with mean $p, p^2, \varepsilon$, respectively, by first applying the Chernoff bound (24) to the above Bernoulli random variables with $M = n, n, |N(i_0j)|$ and then the union bound over $i \in [n], ij \in E$ and $j \in N(i_0)$ we obtain that (25)-(27) hold with probability at least

$$1 - 2\exp(\Omega(np)) - 2\exp(\Omega(np^2)) - 2|N(i_0)|\exp(\Omega(np^2\varepsilon))$$

(28)

$$= 1 - 2\exp(\Omega(np)) - 4np^2\exp(\Omega(np^2)) - 4nexp(\Omega(np^2\varepsilon)).$$

(29)

Indeed, the probability in (29) is high given our assumption $n = \Omega(1/(p^2\mu^2\varepsilon^2))$. We next show that the theorem holds with high probability given (25)-(27).

We recall that $A_{init,(0)} = \frac{1}{m}(S_{init,(0)} \otimes (W_{init,(0)} \otimes 1_m), \tilde{X})$ block, where $S_{init,(t)} = (W_{init,(t)} \otimes 1_m) \otimes \tilde{X}$. We note that for $ij \in E$

$$A_{init,(0)}(i,j) = \langle \tilde{X}_{ij}, \sum_{k \in N(ij)} \tilde{X}_{ik} \tilde{X}_{kj} \rangle / (m|N(ij)|).$$

(30)

We further note that for $i_0j \in E_b$ and $i_0k \in E_b$

$$a^{bb}_{i_0j} := \langle \tilde{X}_{i_0j}, \tilde{X}_{i_0k} \tilde{X}_{kj} \rangle / m = \langle \tilde{X}_{i_0j}, \tilde{X}_{i_0k} P_{i_0}^* P_{j}^T \rangle / m;$$

(31)

for $i_0j \in E_b$ and $i_0k \in E_g$

$$a^{bg}_{i_0j} := \langle \tilde{X}_{i_0j}, \tilde{X}_{i_0k} \tilde{X}_{kj} \rangle / m = \langle \tilde{X}_{i_0j}, \tilde{X}_{i_0k} P_{i_0}^* P_{j}^T \tilde{X}_{ij} \rangle / m;$$

(32)

for $i_0j \in E_g$, and $i_0k \in E_b$

$$a^{gb}_{i_0j} := \langle \tilde{X}_{i_0j}, \tilde{X}_{i_0k} \tilde{X}_{kj} \rangle / m = \langle \tilde{X}_{i_0j}, \tilde{X}_{i_0k} P_{i_0}^* P_{j}^T \tilde{X}_{ij} \rangle / m;$$

(33)

and for $i_0j \in E_g$, and $i_0k \in E_g$

$$a^{gg}_{i_0j} = 1.$$  

(34)

We denote the expectations of $a^{bb}_{i_0j}, a^{bg}_{i_0j}, a^{gb}_{i_0j}$ by $\mu_{bb}, \mu_{bg}$ and $\mu_{gb}$, respectively. We use this notation and the above equations to estimate $\mathbb{E} (A_{init,(0)}(i_0j), i_0j \in E_b)$ and $\mathbb{E} (A_{init,(0)}(i_0j), i_0j \in E_g)$. For this purpose, we note that for $i_0j \in E_b$ and $k \neq j, i_0$, there are $(1 - \varepsilon)|N(i_0)| - 1$ edges $i_0k \in E_b$ and $\varepsilon|N(i_0)|$ edges $i_0k \in E_g$. Similarly, for $i_0j \in E_g$ and $k \neq j, i_0$, there are $(1 - \varepsilon)|N(i_0)| - 1$ edges $i_0k \in E_b$ and $\varepsilon|N(i_0)|$ edges $i_0k \in E_g$. Combining (31)-(34) with these observations, we conclude that
\[ \mathbb{E}(A_{\text{init},(0)}(i_0,j)|i_0,j \in E_g) = \mathbb{E}\left( \frac{1}{|N(i_0)|} \left( \sum_{k \in N(i_0)} a^g_{i_0,k} + \sum_{k \in N(i_0)} a^{g_0}_{i_0,k} \right) \right| i_0,j \in E_g \]

\[ = (1 - \varepsilon)\mu_{gb} + \varepsilon \]  

(35)

and

\[ \mathbb{E}(A_{\text{init},(0)}(i_0,j)|i_0,j \in E_b) = \mathbb{E}\left( \frac{1}{|N(i_0)|} \left( \sum_{k \in N(i_0)} a^{bb}_{i_0,k} + \sum_{k \in N(i_0)} a^{b}_{i_0,k} \right) \right| i_0,j \in E_b \]

\[ = (1 - \varepsilon)\mu_{bb} + \varepsilon \mu_{bg}. \]  

(36)

Next we prove that

\[ \mu_{bb} \leq \mu_{gb} = \mu_{bg} = 1 - \mu. \]  

(37)

Note that (13) and (22) imply that

\[ \mu = \mathbb{E}(m - (X_{i_0},X^*_{i_0})) | j \in N_b(i_0)/m \leq \mathbb{E}(m - (\tilde{X}_{i_0},X^*_{i_0},X^*_k) | j,k \in N_b(i_0))/m. \]

Applying this equation, we conclude (37) as follows:

\[ \mu_{gb} = \mathbb{E}(\langle \tilde{X}_{i_0} - \tilde{X}_{i_0} \rangle | j \in N_b(i_0))/m \leq \mathbb{E}(\langle \tilde{X}_{i_0} - \tilde{X}_{i_0} \rangle | j \in N_b(i_0))/m \]

\[ = \mathbb{E}(\langle \tilde{X}_{i_0} - \tilde{X}_{i_0} \rangle | j \in N_b(i_0))/m \leq \mathbb{E}(\langle \tilde{X}_{i_0} - \tilde{X}_{i_0} \rangle | j \in N_b(i_0))/m = \mu_{gb}. \]

Note that the combination of (35), (36) and (37) yields

\[ \mathbb{E}(A_{\text{init},(0)}(i_0,j)|i_0,j \in E_g) - \mathbb{E}(A_{\text{init},(0)}(i_0,k)|i_0,k \in E_b) \geq (1 - \mu_{bg}) \varepsilon = \mu \varepsilon. \]  

(38)

We also note that for any \( j \in E \), where \( i_0 \in N(j,k) \), the cycle \( j,k,i_0 \) is the only cycle that contain \( j,k \) whose edges may belong to \( E \). We use (38), then the fact that for \( X,Y \in \mathcal{P}_n \), \( \langle X,Y \rangle \geq 0 \) together with the latter observation. At last, we use the fact that \( \langle X_{jk},X^*_{jk} \rangle = \langle X^*_{jk},X^*_{jk} \rangle = m \) to conclude that for any \( j \in E \)

\[ A_{\text{init},(0)}(j,k) = \frac{1}{m|N(j,k)|} \left( \langle \tilde{X}_{jk} - \tilde{X}_{jk}, \tilde{X}_{jk} \rangle + \sum_{i \in N(j,k) \setminus i_0} \langle \tilde{X}_{jk} - \tilde{X}_{jk}, \tilde{X}_{jk} \rangle \right) \]

\[ \geq \frac{1}{m|N(j,k)|} \sum_{i \in N(j,k) \setminus i_0} \langle X^*_{jk} - X^*_{jk}, X^*_{jk} \rangle = \frac{1}{m|N(j,k)|}(|N(j,k)| - 1)m = 1 - \frac{1}{|N(j,k)|} \geq 1 - \frac{2}{\mu \varepsilon^2}, \]

and consequently

\[ \max_{j,k' \in E} |A_{\text{init},(0)}(j,k) - A_{\text{init},(0)}(j',k')| \leq \frac{2}{\mu \varepsilon^2}. \]  

(39)

We note that for the given \( i_0 \in [n], j \in N(i_0) \) and \( k \in N(i_0) \): \( a_{i_0,j,k}^{gb}, a_{i_0,j,k}^{b}, a_{i_0,j,k}^{g} \) are all independent random variables \( \in [0,1] \). Therefore, application of Hoeffding’s inequality and the assumption that \( n = \Omega(1/(\mu^2 \varepsilon^2)) \) yields for \( j \in N_b(i_0) \)

\[ \Pr \left( A_{\text{init},(0)}(i_0,j) \leq \mathbb{E} A_{\text{init},(0)}(i_0,j) - \frac{\mu \varepsilon}{4} + \frac{1}{\mu \varepsilon^2} \right) \]

\[ < \exp \left( -\Omega \left( \frac{\mu \varepsilon}{4} - \frac{1}{\mu \varepsilon^2} \right)^2 \right) = \exp(-\Omega(\mu_\varepsilon^2 \mu \varepsilon^2)) \]  

(40)

and for \( k \in N_b(i_0) \)

\[ \Pr \left( A_{\text{init},(0)}(i_0,k) \geq \mathbb{E} A_{\text{init},(0)}(i_0,k) + \frac{\mu \varepsilon}{4} - \frac{1}{\mu \varepsilon^2} \right) \]

\[ < \exp \left( -\Omega \left( \frac{\mu \varepsilon}{4} - \frac{1}{\mu \varepsilon^2} \right)^2 \right) = \exp(-\Omega(\mu_\varepsilon^2 \mu \varepsilon^2)). \]  

(41)

Taking a union bound over \( j \in N_b(i_0) \), while using (40), and another union bound over \( k \in N_b(i_0) \), while using (41), result in

\[ \Pr \left( \min_{j \in N_b(i_0)} A_{\text{init},(0)}(i_0,j) \geq \mathbb{E} A_{\text{init},(0)}(i_0,j)|i_0,j \in E_g) - \frac{\mu \varepsilon}{4} + \frac{1}{\mu \varepsilon^2} \right) > 1 - 2n \exp(-\Omega(\mu_\varepsilon^2 \mu \varepsilon^2)) \]

and

\[ \Pr \left( \max_{k \in N_b(i_0)} A_{\text{init},(0)}(i_0,k) \leq \mathbb{E} A_{\text{init},(0)}(i_0,k)|i_0,k \in E_b) + \frac{\mu \varepsilon}{4} - \frac{1}{\mu \varepsilon^2} \right) > 1 - 2n \exp(-\Omega(\mu_\varepsilon^2 \mu \varepsilon^2)). \]
Combining the above two equations and then applying (38) we obtain that
\[
\Pr \left( \min_{k \in N_{\beta}(i_0)} A_{\text{init}(0)}(i_0, k) > \max_{k \in N_{\beta}(i_0)} A_{\text{init}(0)}(i_0, k) + \frac{\mu \varepsilon}{2} + \frac{2}{np^2} \right) \\
\geq \Pr \left( \min_{k \in N_{\beta}(i_0)} A_{\text{init}(0)}(i_0, k) > \max_{k \in N_{\beta}(i_0)} A_{\text{init}(0)}(i_0, k) + \frac{\mu \varepsilon}{2} + \frac{2}{np^2} \right)
\]

The combination of (39) and (42) yields for any \( j \neq i_0 \)
\[
\Pr \left( \min_{k \in N_{\beta}(i_0)} \left( A_{\text{init}(0)}(i_0, k) + A_{\text{init}(0)}(k, j) \right) > \max_{k \in N_{\beta}(i_0)} \left( A_{\text{init}(0)}(i_0, k) + A_{\text{init}(0)}(k, j) \right) + \frac{\mu \varepsilon}{2} \right) > 1 - 4np \exp(-\Omega(np^2 \mu^2 \varepsilon^2)).
\]
Recall that for \( i j \in E \), \( W_{\text{init}(1)}(i, j) = \exp(\beta_i A_{\text{init}(0)}(i, j)) \). In view of this equality and the above equation, we conclude that for any \( j \neq i_0 \)
\[
\min_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j) \geq \max_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j) e^{\beta_i \mu \varepsilon/2}
\]
with probability at least \( 1 - 4np \exp(-\Omega(np^2 \mu^2 \varepsilon^2)) \). (43)

Using this inequality, we establish the desired upper bound of \( \| A_{\text{init}(1)} - A^* \|_\infty \) in the following three complementary cases.

Case 1: Edge \( ij \in E \) is incident to node \( i_0 \). That is, without loss of generality, the edge \( ij \) is of the form \( i_0 j \) for \( j \in [n] \setminus \{i_0\} \). In this case, by assumption (27),
\[
\frac{|N_{\beta}(i_0)|}{|N(i_0)|} \geq \frac{|N(i_0)| \varepsilon / 2}{|N(i_0)| (1 - \varepsilon / 2)} = 2 - \varepsilon.
\]
(44)

Combining the definition of \( A_{\text{init}(1)} \), the fact that \( \| X_{i_0 j} X_{i_0 k} X_{k j} \| / m - A^*(i_0, j) \| \leq 1 \) (as it is an absolute value of a difference of two numbers in \([0, 1]\)) as well as (43) and (44), we obtain that with the probability indicated in (43)
\[
A_{\text{init}(1)}(i_0, j) - A^*(i_0, j)
\]
\[
= \frac{\sum_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j) \left( (\bar{X}_{i_0 j} X_{i_0 k} X_{k j}) / m \right) - A^*(i_0, j)}{\sum_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j)}
\]
\[
\leq \frac{\sum_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j) \left( (\bar{X}_{i_0 j} X_{i_0 k} X_{k j}) / m - A^*(i_0, j) \right)}{\sum_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j)}
\]
\[
\leq \frac{1}{1 + \sum_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j)} \leq \frac{1}{1 + \sum_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j)} e^{\beta_i \mu \varepsilon/2} \leq \frac{1}{1 + \sum_{k \in N_{\beta}(i_0)} W_{\text{init}(1)}(i_0, k) W_{\text{init}(1)}(k, j)} e^{\beta_i \mu \varepsilon/2}
\]

Case 2: Edge \( jk \) is not incident to \( i_0 \) and \( i_0 \in N_{\beta}(j k) \). That is, we assume that \( j k \) is in \([n] \setminus \{i_0\}\) and at least one of them is in \( N_{\beta}(i_0) \). In this case, and \( N_{\beta}(j k) = N(j k) \setminus \{i_0\} \)

and consequently
\[
|N_{\beta}(j k)| = 1 \quad \text{and} \quad |N_{\beta}(j k)| = |N(j k)| - 1.
\]

Following the same arguments deriving (45), but using the above two equations (instead of (44)), we obtain that the probability indicated in (43)
\[
|A_{\text{init}(1)}(j, k) - A^*(j, k)| \leq W_{\text{init}(1)}(j, i_0) W_{\text{init}(1)}(i_0, k) + \sum_{l \in N(j k) \setminus \{i_0\}} W_{\text{init}(1)}(j, l) W_{\text{init}(1)}(l, k)
\]
\[
= \frac{1}{1 + \sum_{l \in N(j k) \setminus \{i_0\}} W_{\text{init}(1)}(j, l) W_{\text{init}(1)}(l, k)} \leq \frac{1}{1 + (|N(j k)| - 1)} e^{\beta_i \mu \varepsilon/2}
\]

Case 3: Edge \( jk \) is not incident to \( i_0 \) and \( i_0 \in N_{\beta}(j k) \). That is, we assume that both \( j \) and \( k \) are in \( N_{\beta}(i_0) \).

Note that in this case, all 3-cycles containing \( jk \) are uncorrupted. That is
\[
N_{\beta}(j k) = \emptyset \quad \text{and} \quad N_{\beta}(j k) = N(j k).
\]
We believe that condition (13) for CEMP is less restrictive than (48). Nevertheless, we point out that in We start with considering PPM. In view of (4), the PPM iteration at node \(i_0\) with probability at least \(\epsilon \sum_{j \in N_b(i_0)}\) and thus PPM cannot recover \(i\). By law of large numbers, the condition this special deterministic example.

Combining all the above three cases, for all \(j \in E\)

\[|A_{\text{init},(1)}(j,k) - A^*(j,k)| = 0.\]

with probability at least \(1 - 4np \exp\left(-\Omega\left(np^2 \mu^2 \varepsilon^2\right)\right)\). Since \(n = \Omega\left(1/(p^2 \mu^2 \varepsilon^2)\right)\) this probability is sufficiently large. Note that the only free parameter in the right hand side of the above inequality is \(\beta_0\). Thus one can apply an aggressive reweighting with very large \(\beta_0\) and guarantee in this special case near exact recovery for CEMP. The only restriction of Theorem (5.2) on \(\{P^*_i\}_{i \in [n]}\) is the condition:

\[E(||X_{i_0j} - X^*_{i_0j}||^2_F | j \in N_b(i_0)) \leq E(||X_{i_0j} - X^*_{i_0j}||^2_F | j,k \in N_b(i_0))\].

We note that since \(||X^*_{i_0j} X^*_{i_0j} - I_m||^2_F\) and \(||X^*_{i_0j} X^*_{i_0j} - I_m||^2_F\) this condition is equivalent to

\[E\left(||X_{i_0j} X_{i_0j} - I_m||^2_F | j \in N_b(i_0)\right) \leq E\left(||X_{i_0j} X_{i_0j} - I_m||^2_F | j,k \in N_b(i_0)\right).

Both sides of the inequality contain conditional expectations of the cycle inconsistency of the 3 cycle \(i_0j,k\). In the LHS it is condition on the edge \(i_0j\) being corrupted, where in the RHS it is conditioned on both edges \(i_0j\) and \(i_0k\) being corrupted. That is, the above condition means that when the number of corrupted edges in a 3-cycle is enlarged from 1 to 2, then the cycle inconsistency increases on average.

A.3.2 Failure cases of least squares methods

We demonstrate some failure cases of least squares methods for permutation synchronization under the superspread model. In view of (27), we assume that \(1-\epsilon\)-fraction of \(i_0j \in E\) is corrupted.

We start with considering PPM. In view of (4), the PPM iteration at node \(i_0\) is

\[P_{i_0(t+1)} = \arg\max_{P_{i_0} \in \mathcal{P}_m} P_{i_0} \frac{1}{|N(i_0)|+1} \sum_{j \in [n]} X_{i_0j} P_{i_0j}(t)\].

The following proposition demonstrates failure cases of PPM. It uses the notation \(Q = \sum_{j \in N_b(i_0)} X_{i_0j} P^*_j / |N_b(i_0)|\).

Proposition A.2. If there exist \(Q < P_{i_0} < 1\) such that \(2 \varepsilon \sqrt{2m} + (1-2\varepsilon) \varepsilon < 1\) and

\[\|Q-P_{i_0}\|_{F} < \varepsilon_0,\]

then

\[P_{i_0} = \arg\max_{P_{i_0} \in \mathcal{P}_m} P_{i_0} \frac{1}{|N(i_0)|+1} \sum_{j \in [n]} X_{i_0j} P^*_j\]

and thus PPM cannot recover \(P^*_i\).

Before we prove this proposition, we clarify it. It states that if the average of \(\sum_{j \in N_b(i_0)} X_{i_0j} P^*_j\) over \(j \in N_b(i_0)\) concentrates around a certain permutation matrix, which is different than \(P^*_i\), and \(\varepsilon\) is sufficiently small, then PPM fails to recover the ground-truth permutations. By law of large numbers, the condition \(\|Q-P_{i_0}\|_{F} < \varepsilon_0\) is satisfied when

\[\left\|\text{E}(\sum_{j \in [n]} X_{i_0j} P^*_j) - P_{i_0}\right\|_{F} < \varepsilon_0/2\]

and \(|N(i_0)|\) is sufficiently large. The LAC model described in (6.1) represents this setting. Indeed, in this case \(P_{i_0}\) is the identity matrix \(I_m\) and \(\sum_{j \in [n]} X_{i_0j} P^*_j\) randomly permutes 3 columns of the identity. In this case, \(E(X_{i_0j} P^*_j(i,i) = (m-3)/m\) for \(i \in [m]\) and \(E(X_{i_0j} P^*_j(i,i) = 3/(m(m-1))\) for \(i,j \in [m]\), where \(i \neq j\). We have tested this case with \(m = 10\), where we have violated the condition \(2 \varepsilon \sqrt{2m} + (1-2\varepsilon) \varepsilon < 1\). Nevertheless, we have still seen a clear advantage of IRGCL, which uses CEMP, over PPM.

We believe that condition (13) for CEMP is less restrictive than (48). Nevertheless, we point out that in the deterministic case when \(X_{i_0j} P^*_j = P_{i_0}\) for all \(j \in N_b(i_0)\), then both CEMP and PPM fail. We first note that the RHS of (13) is 0, so the proposition does not hold for CEMP. We also note that in this scenario the problem of exact recovery is ill-posed as no algorithm can recover \(P_{i_0}\). Indeed, setting \(X_{i_0j} P^*_j = P_{i_0}\) is equivalent to corrupting \(X^*_{i_0j} = P^*_j P^*_j\) as follows: \(\tilde{X}_{ij} := P_{i_0} P^*_j P^*_j\) and thus replacing the underlying ground-truth permutation \(P^*_j\) by \(P_{i_0}\). Anyway, Proposition A.2 assumes a much broader scenario than this special deterministic example.
Proof. We note that for any $P_{i_0} \in \mathcal{P}_m$

$$\left\langle P_{i_0}, \frac{1}{|N(i_0)|+1} \sum_{j \in [n]} \hat{X}_{i_0j} P_j \right\rangle = \left\langle P_{i_0}, \frac{1}{|N(i_0)|+1} \left( P_{i_0}^* + \sum_{j \in N_g(i_0)} X_{i_0j}^* P_j^* + \sum_{j \in N_b(i_0)} \hat{X}_{i_0j} P_j^* \right) \right\rangle = \left\langle P_{i_0}, \frac{|N_g(i_0)|+1}{|N(i_0)|+1} P_{i_0}^* + |N_b(i_0)| Q \right\rangle.$$ 

Therefore, it is sufficient to prove that

$$P_{\text{crpt}} = \arg\max_{P_{i_0} \in \mathcal{P}_m} \left\langle P_{i_0}, \frac{1}{|N(i_0)|+1} \sum_{j \in [n]} \hat{X}_{i_0j} P_j \right\rangle = \arg\max_{P_{i_0} \in \mathcal{P}_m} \left\langle P_{i_0}, \hat{P}_{i_0} \right\rangle,$$

(49)

where $P_{\text{crpt}} \neq P_{i_0}^*$ and $\hat{P}_{i_0} = ((|N_g(i_0)|+1)P_{i_0}^* + |N_b(i_0)|Q)/(|N(i_0)|+1)$. Using basic algebraic relationships and at last applying together the conditions $2\varepsilon \sqrt{2m} + (1 - 2\varepsilon)\varepsilon_0 < 1$ and $\varepsilon_0 < 1$, the fact that $\|X - Y\|_F^2/2m \in [0, 1]$ for any $X, Y \in \mathcal{P}_m$ and (49), we obtain that for $|N(i_0)|$ sufficiently large

$$\|\hat{P}_{i_0} - P_{\text{crpt}}\|_F = \left\| \frac{|N_g(i_0)|+1}{|N(i_0)|+1} (P_{i_0}^* - P_{\text{crpt}}) + \frac{|N_b(i_0)|}{|N(i_0)|+1} (Q - P_{\text{crpt}}) \right\|_F$$

$$\le 2\varepsilon \sqrt{2m} + (1 - 2\varepsilon)\varepsilon_0 < 1.$$ 

(50)

By combining (50) and the fact that $\|X - Y\|_F \ge 2$ for $X \neq Y \in \mathcal{P}_m$, we obtain that

$$\|\hat{P}_{i_0} - P_{\text{crpt}}\|_F < \frac{1}{2} \min_{P' \in \mathcal{P}_m, P' \neq P_{\text{crpt}}} \|P' - P_{\text{crpt}}\|_F.$$ 

Consequently, we conclude (49) and thus the auxiliary proposition as follows

$$P_{\text{crpt}} = \arg\min_{P_{i_0} \in \mathcal{P}_m} \|P_{i_0} - \hat{P}_{i_0}\|_F = \arg\max_{P_{i_0} \in \mathcal{P}_m} \left\langle P_{i_0}, \hat{P}_{i_0} \right\rangle.$$ 

\(\square\)

The argument for failure of general least squares methods is more delicate. Using the above rigorous argument for PPM, we provide some intuition why least methods can fail. We note that such methods aim to solve

$$\max_{\{P_i\}_{i \in [n]} \subset \mathcal{P}_m} \sum_{j \in [n]} \sum_{k \in [n]} \left\langle P_j P_k^\top, \hat{X}_{jk} \right\rangle.$$ 

(51)

We rewrite the objective function of (51) as follows

$$\sum_{j \in [n]} \sum_{k \in [n]} \left\langle P_j P_k^\top, \hat{X}_{jk} \right\rangle = \left\langle P_{i_0} P_{i_0}^\top, I_m \right\rangle + \sum_{j \neq i_0} \left\langle P_j P_{i_0}^\top, \hat{X}_{j i_0} \right\rangle + \sum_{k \neq i_0} \left\langle P_{i_0} P_k^\top, \hat{X}_{i_0k} \right\rangle + \sum_{j \neq i_0} \sum_{k \neq i_0} \left\langle P_j P_k^\top, \hat{X}_{jk}^* \right\rangle.$$ 

Since $\hat{X}_{ij} = \hat{X}_{ji}^\top$ and

$$\sum_{j \neq i_0} \left\langle P_j P_{i_0}^\top, \hat{X}_{j i_0} \right\rangle = \sum_{j \neq i_0} \left\langle (P_j P_{i_0}^\top)^\top, \hat{X}_{j i_0}^\top \right\rangle = \sum_{j \neq i_0} \left\langle P_{i_0} P_j^\top, \hat{X}_{i_0j} \right\rangle,$$

we have

$$\sum_{j \neq i_0} \left\langle P_j P_{i_0}^\top, \hat{X}_{j i_0} \right\rangle = \sum_{j \neq i_0} \left\langle (P_j P_{i_0}^\top)^\top, \hat{X}_{j i_0}^\top \right\rangle = \sum_{j \neq i_0} \left\langle P_{i_0} P_j^\top, \hat{X}_{i_0j} \right\rangle.$$
We reformulate the above maximization problem by plugging its constraint into its objective function as follows:

$$\sum_{j \in [n]} \sum_{k \in [n]} \langle P_j P_k^T, \tilde{X}_{jk} \rangle = \langle P_{i_0} P_{i_0}^T, I_m \rangle + 2 \sum_{j \neq i_0} \langle P_{i_0} P_j^T, \tilde{X}_{i_0 j} \rangle + \sum_{j \neq i_0 k \neq i_0} \langle P_j P_k^T, X_{j k}^* \rangle$$

$$= -\langle P_{i_0} P_{i_0}^T, I_m \rangle + 2 \sum_{j \neq i_0} \langle P_{i_0} P_j^T, \tilde{X}_{i_0 j} \rangle + \sum_{j \neq i_0 k \neq i_0} \langle P_j P_k^T, X_{j k}^* \rangle$$

$$= -m+2 \left( P_{i_0} \sum_{j \in [n]} \tilde{X}_{i_0 j} P_j \right) + \frac{1}{2} \sum_{j \neq i_0 k \neq i_0} \langle P_j P_k^T, X_{j k}^* \rangle$$

$$= -m+2 \left( P_{i_0} \sum_{j \in [n]} \tilde{X}_{i_0 j} P_j \right) + \frac{1}{2} \sum_{j \neq i_0 k \neq i_0} \left( m - \frac{1}{2} \| P_j P_k^T - X_{j k}^* \|_F^2 \right)$$

$$= C + 2 \left( P_{i_0} \sum_{j \in [n]} \tilde{X}_{i_0 j} P_j \right) - \frac{1}{4} \sum_{j \neq i_0 k \neq i_0} \| P_j P_k^T - X_{j k}^* \|_F^2$$

for some constant $C$. We note that the last term in the right hand side of (52) is a double sum of $(n - 1)^2$ terms, which are independent of $i_0$. The minimization of this double sum over the variables $\{ P_j \}_{j \in [n]} \setminus \{ i_0 \}$ results in the ground-truth solution $\{ P_j^* \}_{j \in [n]} \setminus \{ i_0 \}$ (since $j k \in E_0$ for $j, k \in [n] \setminus \{ i_0 \}$) with minimal value 0. Thus the right hand side of (52) can be viewed as a Langrangian with multiplier $\lambda = 1/4$ of the constrained optimization problem

$$\max_{\{ P_j \}_{j \in [n]} \setminus \{ i_0 \}} \left( P_{i_0} \sum_{j \in [n]} \tilde{X}_{i_0 j} P_j \right)$$

subject to $\sum_{j \neq i_0 k \neq i_0} \| P_j P_k^T - X_{j k}^* \|_F^2 = 0,$

which is equivalent to

$$\max_{P_{i_0} \in \mathcal{P}_m} \left( P_{i_0} \sum_{j \in [n]} \tilde{X}_{i_0 j} P_j \right)$$

subject to $P_j = P_j^*$ for $j \neq i_0.$

We reformulate the above maximization problem by plugging its constraint into its objective function as follows:

$$\max_{P_{i_0} \in \mathcal{P}_m} \left( \langle P_{i_0}, P_{i_0} \rangle + \left( P_{i_0} \sum_{j \neq i_0} \tilde{X}_{i_0 j} P_j^* \right) \right) = \max_{P_{i_0} \in \mathcal{P}_m} \left( m + \left( P_{i_0} \sum_{j \neq i_0} \tilde{X}_{i_0 j} P_j^* \right) \right).$$

The above problem is almost similar to the one in the RHS of (47). They only differ in the term of the sum that correspond to $j = i_0$. Therefore, under the superspread model, the least squares method is a regularized version of a similar energy function maximized on the RHS of (47). In a similar way to establishing (47), which results in wrongly estimating $P_i^*$ as $P_{crop}$ by PPM, one can prove that under similar conditions to the ones of Proposition A.2 a least squares solver may produce $P_{crop}$ instead of $P_i^*$.

**B Additional Demonstration and Numerical Results**

In §B.1 we provide a simple demonstration of the new idea in comparison to CEMP and IRLS. In §B.2 we briefly comment on the computational complexity of our methods. In §B.3 we present the experiments on a uniform corruption model. In §B.4 we provide additional results on the nonuniform corruption models.

**B.1 A Figure Demonstrating the IRGCL Algorithm**

The following figure tries to convey the basic idea of IRGCL in comparison to IRLS and CEMP. In this figure, the notation $X \rightarrow Y$ means that $Y$ is generated from $X$. We recall that $A$, $W$, $P$ and $S^2$ respectively represent the estimated matrices of (correlation) affinity, weight, permutation and squared GCW. We also recall that $A_1$ and $A_2$ respectively denote the first and second order affinities. The two merged lines on the top of the diagram for IRGCL (one is dashed and the other is full) designate the fact that $A$ is a weighted average of the first and second order affinities. We use a dashed line to remind the reader that the weights of $A_1$ diminish as the number of iterations increases. We note that the two merged components represent two different algorithms, IRLS and CEMP.
B.2 On the Computational Complexity

We remark that the complexity of Algorithm 1 (which uses only 3-cycles) is \(O(m^3 \times n^3)\). The complexity of the projected power iteration is \(O(m^3 \times n^2)\). The spectral decomposition of the graph connection Laplacian has complexity \(O(m^3 \times n^3)\). Thus, IRGCL-S&P, Spectral and PPM have the same complexity \(O(m^3 \times n^3)\), which is typically lower than that of the SDP method MatchLift.

We remark that Algorithm 1 can be easily generalized to exploit higher order cycles with length \(l\) by using the \(l\)-th power of the GCW matrix. In this case, its complexity is \(O(m^3 \times n^3 \times l)\). On the other hand, the complexity of the original CEMP with general \(l\)-cycles is \(O(m^3 \times n^l)\). Therefore, our idea significantly reduces the complexity of CEMP when using higher-order cycles and the specific metric discussed in this paper.

B.3 Experiments on Uniform Corruption Model

We test the different methods using data generated from a uniform corruption model. In this model, we independently sample corrupted edges with probability \(q\), and for each \(ij \in E_b\), \(\tilde{X}_{ij} \sim \text{Haar}(\mathcal{D}_m)\).

We plot the estimation error

\[
\sum_{i \neq j} \frac{||\hat{X}_{ij} - X^*_{ij}||_F^2}{\sum_{i \neq j} ||X^*_{ij}||_F^2}
\]

for each corruption probability \(q = 0.7, 0.8, 0.88, 0.9, 0.92\). We compare IRGCL-P and IRGCL-S with all methods described in §6. Since IRLS-Cauchy-S and IRLS-Cauchy-P performed similarly we report only one of them. We also tested the standard IRLS described in (5) and (6), which we refer to as IRLS-L1-S. The implementation of IRLS-L1-S approximately solves (6) using the spectral formulation of (12) at each iteration. It also initializes by the solution of (12) using the adjacency matrix for the weight matrix. For each method we run 100 trials and report the means and standard deviations of the estimation errors in Figure 3, where standard deviations are denoted by error bars. We note that IRGCL-S and IRGCL-P consistently achieve the lowest errors, and IRGCL-S seems to work slightly better (with lower mean errors and standard deviations).
deviations) under the highest corruption ratio, $q = 0.92$. The spectral method and MatchLift perform the worst. They are unable to recover the ground-truth permutations when $q = 0.8$. We also remark that PPM works better than the other least squares methods. However, it is not competitive with IRGCL-S and IRGCL-P in the high corruption range of $0.88 - 0.92$. In this range, IRLS-L1-S and IRLS-Cauchy-S have lower means than PPM, but they have large standard deviations, which indicate that they are unstable.

B.4 Additional Experiments on Nonuniform Corruption Models

We report additional results for the LBC and LAC models in §B.4.1 and §B.4.2 respectively. Numerical results for an Erdős-Rényi graph are included in §B.4.3.

B.4.1 Additional synthetic experiments using the LBC model

Figure 4 reports the estimation errors

$$\frac{\sum_{ij \in E_b} \| \hat{X}_{ij} - X^*_{ij} \|^2_F}{\sum_{ij \in E_b} \| X^*_{ij} \|^2_F}$$

of different methods under the LBC model with parameters $m_c = 90$ and $n_c = 10, 20, 30, 40$. For each method and each fixed value of $n_c$ we run 20 trials and present the mean and standard deviations of the estimation errors. We note that both IRGCL-S and IRGCL-P are able to achieve near exact recovery when $n_c \leq 30$. PPM performs the worst among the tested methods for all values of $n_c$. We note that in terms of the averaged errors, IRLS-Cauchy-S performs better than the other least squares methods. However, it has high standard deviations, so that it is unstable, and its averaged values are still not competitive when compared with IRGCL-S and IRGCL-P. We also note that the standard deviations of the latter two methods are nearly 0 when $n_c \leq 30$.

B.4.2 Additional synthetic experiments using the LAC Model

Figure 5 reports the estimation errors

$$\frac{\sum_{ij \in E_b} \| \hat{X}_{ij} - X^*_{ij} \|^2_F}{\sum_{ij \in E_b} \| X^*_{ij} \|^2_F}$$

of different methods under the LAC model with $m_c = 60$ and $n_c = 10, 20, 30, 40$. For each method and each value of $n_c$ we run 20 trials and report the mean and standard deviations of the errors. We note that both IRGCL-P and IRGCL-S are able to recover the ground-truth solution under the LAC model when $n_c \leq 40$, whereas other methods cannot.

B.4.3 Additional synthetic experiments with an Erdős-Rényi graph

We repeat the experiments in the main text with $G([n], E)$ as an Erdos-Renyi graph with probability 0.5 instead of a complete graph. Figure 6 reports the estimation errors

$$\frac{\sum_{ij \in E_b} \| \hat{X}_{ij} - X^*_{ij} \|^2_F}{\sum_{ij \in E_b} \| X^*_{ij} \|^2_F}$$

Figure 4: Average matching errors under the local biased corruption model.
of different methods under the LAC model with \( m_c = 30 \) and LBC model with \( m_c = 45 \). Both models have \( n_c = 1, 2, 3, 4, 5, 6 \). For each method and each value of \( n_c \) we run 20 trials and report the mean and standard deviations of the errors. We also report the final error of IRGCL-S and IRGCL-P compared with \( P(1) \) in Algorithm 2 (we call it IRGCL-init) in figure 7.

We note that both IRGCL-P and IRGCL-S are able to give exact recovery on LAC and almost exact recovery on LBC, while other methods cannot. Also, we find that on LBC both IRGCL iterations effectively decrease error compared to its initialized permutation, though the initialization is already quite good in this synthetic setting; On LAC the initialization of IRGCL already achieves exact recovery.

Figure 5: Average matching errors under the local adversarial corruption Model.

Figure 6: Average matching errors under the local biased corruption model (left) and local adversarial corruption model (right) with an Erdős-Rényi graph with \( p = 0.5 \).
Figure 7: Average matching errors of IRGCL-S and IRGCL-P compared with IRGCL initialization under the local biased corruption model (left) and local adversarial corruption model (right) with an Erdős-Rényi graph with $p = 0.5$. 