GLOBAL EXISTENCE FOR SYSTEMS OF NONLINEAR WAVE AND KLEIN-GORDON EQUATIONS WITH COMPACTLY SUPPORTED INITIAL DATA

Dedicated to Professor Vladimir Georgiev on the occasion of his sixtieth birthday

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Abstract. We consider the Cauchy problem for coupled systems of nonlinear wave and Klein-Gordon equations in three space dimensions. The author previously proved the small data global existence for rapidly decreasing data under a certain condition on nonlinearity. In this paper, we show that we can weaken the condition, provided that the initial data are compactly supported.

1. Introduction. We consider the Cauchy problem for the following system of nonlinear wave and Klein-Gordon equations:

\[(\Box + m_j^2)u_j = F_j(u, \partial u) \quad \text{in } (0, \infty) \times \mathbb{R}^3, \]
\[u_j(0, x) = \varepsilon f_j(x), \quad (\partial_t u_j)(0, x) = \varepsilon g_j(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3 \]

for \(j = 1, 2, \ldots, N\), where \(u = (u_j)_{1 \leq j \leq N}\) is an \(\mathbb{R}^N\)-valued unknown function of \((t, x) \in (0, \infty) \times \mathbb{R}^3\), \(\Box := \partial_t^2 - \Delta\), and \(m_j \geq 0\). \(\partial u\) stands for the first derivatives of \(u\), that is to say, \(\partial u := (\partial_a u_j)_{1 \leq j \leq N, 0 \leq a \leq 3}\), where \(\partial_0 := \partial_t\) and \(\partial_k := \partial_{x_k}\) for \(1 \leq k \leq 3\). The nonlinear term \(F = (F_j)_{1 \leq j \leq N}\) is a smooth function of \((u, \partial u)\) with

\[F(u, \partial u) = O(|u|^2 + |\partial u|^2) \]

near \((u, \partial u) = (0, 0)\).

(1) is called a nonlinear wave equation when \(m_j = 0\), and a nonlinear Klein-Gordon equation when \(m_j > 0\). For simplicity, we assume that there is \(N_0 \in \{0, 1, \ldots, N\}\) such that

\[m_j > 0 \text{ for } 1 \leq j \leq N_0, \text{ and } m_j = 0 \text{ for } N_0 + 1 \leq j \leq N\]

(4)

(we understand this relation as \(m_j > 0\) for all \(j\) if \(N_0 = N\), and \(m_j = 0\) for all \(j\) if \(N_0 = 0\)). We set

\[v_j := u_j \text{ for } 1 \leq j \leq N_0, \text{ and } w_j := u_j \text{ for } N_0 + 1 \leq j \leq N.\]

We call \(v_j\) as the Klein-Gordon component, and \(w_j\) as the wave component. We write \(u = (v, w)\) with

\[v := (v_j)_{1 \leq j \leq N_0} \text{ and } w := (w_j)_{N_0+1 \leq j \leq N}.\]
For a while, we assume that \( f = (f_j)_{1 \leq j \leq N} \) and \( g = (g_j)_{1 \leq j \leq N} \) are smooth and decay sufficiently fast at spatial infinity, but are not necessarily of compact support. The parameter \( \varepsilon \) is always supposed to be sufficiently small.

For systems of nonlinear Klein-Gordon equations (namely the case where \( N_0 = N \)), we do not need any further assumption to get the small data global existence (see Klainerman [15] and Shatah [22]). On the other hand, for systems of nonlinear wave equations (namely the case where \( N_0 = 0 \)), we need an additional condition to obtain the small data global existence because the blow-up of the solution in finite time may occur even for small initial data; for example, the blow-up occurs when \( N = 1 \) and \( F = (\partial_t u)^2 \) (see John [8]). Klainerman [16] and Christodoulou [3] showed the small data global existence when the null condition is satisfied: We say that the null condition is satisfied if we have

\[
F^0(X, (\omega Y)) = 0, \quad X, Y \in \mathbb{R}^N, \quad \omega = (\omega_1, \omega_2, \omega_3) \in S^2
\]

with \( \omega_0 = -1 \). Here and hereafter \( F^k = (F^k_j)_{1 \leq j \leq N} \) denotes the quadratic part of \( F \), that is to say

\[
F^k(u, \partial u) := \lim_{\lambda \to 0} \lambda^{-2} F(\lambda u, \lambda \partial u).
\]

The left-hand side of (5) means that \( X \) and \( \omega Y \) are substituted in place of \( u \) and \( \partial_u u \), respectively.

The null condition for the wave equation is closely related to the null forms

\[
Q_a(\phi, \psi) := (\partial_t \phi) (\partial_t \psi) - (\nabla_x \phi) \cdot (\nabla_x \psi),
\]

(6)

\[
Q_{ab}(\phi, \psi) := (\partial_a \phi) (\partial_b \psi) - (\partial_b \phi) (\partial_a \psi), \quad 0 \leq a, b \leq 3.
\]

(7)

\( Q_{ab} \)'s are sometimes called the strong null forms. In fact, the null condition is satisfied if and only if each \( F^k \) is a linear combination of \( Q_0(u_k, u_l) \) and \( Q_{ab}(u_k, u_l) \) with \( 0 \leq a, b \leq 3 \) and \( 1 \leq k, l \leq N \).

Before we proceed further, we introduce some notation. For \( 1 \leq j \leq N \), we write

\[
F^k_j(u, \partial u) = \sum_{1 \leq k \leq l \leq N} F_{j}^{kl}[u_k, u_l]
\]

(8)

with

\[
F_{j}^{kl}[u_k, u_l] := \sum_{|\alpha|, |\beta| \leq 1} c_{j}^{\alpha \beta} (\partial^\alpha u_k)(\partial^\beta u_l),
\]

(9)

where \( c_{j}^{\alpha \beta} \) are real constants satisfying \( c_{j}^{k\alpha \beta} = c_{j}^{\alpha k \beta} \). For \( 1 \leq j \leq N \) we set

\[
F^{K}_j(v, \partial v) := \sum_{1 \leq k \leq l \leq N_0} F_{j}^{kl}[v_k, v_l],
\]

\[
F^{KW}_j(u, \partial u) := \sum_{1 \leq k \leq N_0, \ N_0+1 \leq l \leq N} F_{j}^{kl}[u_k, w_l],
\]

\[
F^{W}_j(w, \partial w) := \sum_{N_0+1 \leq k \leq l \leq N} F_{j}^{kl}[w_k, w_l].
\]

Then \( F^j \) is decomposed into three parts:

\[
F^q_j(u, \partial u) = F^K_j(v, \partial v) + F^{KW}_j(u, \partial u) + F^W_j(w, \partial w).
\]

We write

\[
F_K := (F_j)_{1 \leq j \leq N_0}, \quad F_W := (F_j)_{N_0+1 \leq j \leq N}.
\]
Similarly, for \( * = q, K, KW, W \), we put
\[
F^* := \left( F_j^* \right)_{1 \leq j \leq N}, \quad F_K^* := \left( F_j^* \right)_{1 \leq j \leq N_0}, \quad F_W^* := \left( F_j^* \right)_{N_0+1 \leq j \leq N}.
\]

For the case of coupled systems of nonlinear wave and Klein-Gordon equations (namely, the case where \( 1 \leq N_0 < N \)), Georgiev [4] proved the small data global existence under the strong null condition, that is to say,
\[
F^q(X, (\omega_a Y)) = 0, \quad X, Y \in \mathbb{R}^N, (\omega_0, \omega) \in \mathbb{R} \times \mathbb{R}^3.
\]

We can show that the strong null condition is satisfied if and only if each \( F_j^q \) is a linear combination of the strong null forms \( Q_{ab}(u_k, u_l) \) with \( 0 \leq a, b \leq 3 \) and \( 1 \leq k, l \leq N \).

The author [10] extremely relaxed the condition for the small data global existence in the following way. We assume the null condition for the interaction between wave components in the wave equations. To be more precise, we assume
\[
\text{(N)} \quad \text{The null condition for } F_W^W: \text{ We have}
\]
\[
F_W^W(X, (\omega_0 Y)) = 0, \quad X, Y \in \mathbb{R}^{N-N_0}, \quad \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2 \quad (10)
\]
with \( \omega_0 = -1 \).

We also assume that other quadratic parts are independent of the wave component \( w \) itself\(^1\). To be more precise, we assume the followings:
\[
\text{(A1) } F_W^K(u, \partial u) = (F_K^W(u, \partial u), F_W^K(u, \partial u)) \text{ is independent of } w \text{ itself; in short, we have } F_W^K = F_K^W(u, \partial u).
\]
\[
\text{(A2) } F_W^K(w, \partial w) \text{ is independent of } w \text{ itself; in short, we have } F_K^W = F_W^K(\partial w).
\]

Instead of (A1) and (A2), which restrict the dependence on \( w \), we have an alternative condition which is an additional restriction on the form of \( F_W^W \):
\[
\text{(B) } \text{There are functions } G_{j,a} = G_{j,a}(u) \text{ such that}
\]
\[
F_W = (F_j(u, \partial u))_{N_0+1 \leq j \leq N} = \left( \sum_{a=0}^3 \partial_a (G_{j,a}(u)) \right)_{N_0+1 \leq j \leq N}
\]
holds for any \( C^1 \)-function \( u \).

Under the conditions (N), (A1), and (A2), or under the conditions (N) and (B), the author proved the small data global existence for (1)–(2)\(^2\). Observe that this result under the conditions (N), (A1), and (A2) naturally covers the known results for the Klein-Gordon equations and the wave equations in [15], [22], [16], and [3]. See also LeFloch-Ma [20] for an alternative proof of the small data global existence under the conditions (N), (A1), and (A2), where much less regularity for initial data is necessary, though the compactness of the support of initial data is required.

In this paper, we would like to show that the conditions (N) and (A1) are sufficient for the small data global existence, if we assume the compactness of the support of the initial data; in other words, \( F_K^W \) can depend also on \( w \) itself under the compactness assumption.

\(^1\) If we say that “a function \( G \) is independent of the wave component \( w \) itself”, it means that \( G \) is independent of \( w = (w_j) \), but can depend on the derivatives \( \partial w = (\partial_w w_j) \).

\(^2\) To be more precise, in [10], the small data global existence is obtained for the case where the (A1, 2)-type condition is satisfied by some wave components, and the (B)-type condition by others, in addition to (N).
**Theorem 1.1.** Let (3) and (4) be fulfilled. We assume the conditions (N) and (A1). Then, for any $f = (f_j)_{1 \leq j \leq N}, g = (g_j)_{1 \leq j \leq N} \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^N)$, there is a positive constant $\varepsilon_0$ such that the Cauchy problem (1)–(2) admits a unique global solution $u \in C^\infty(0, \infty) \times \mathbb{R}^3; \mathbb{R}^N)$, provided that $0 < \varepsilon \leq \varepsilon_0$. Moreover, the global solution $u$ is asymptotically free.

Here we say that $u = (u_j)_{1 \leq j \leq N}$ is asymptotically free, if there are $(f_j^+, g_j^+) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for $1 \leq j \leq N_0$ and $(f_j^+, g_j^+) \in \dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for $N_0 + 1 \leq j \leq N$ such that

$$\lim_{t \to \infty} \left( \sum_{j=1}^N \|u_j^+(t) - u_j(t)\|_{E,m}^2 \right)^{1/2} = 0,$$

where each $u_j^+$ is the solution to $(\Box + m_j^2)u_j^+ = 0$ with $(u_j^+, \partial_t u_j^+) = (f_j^+, g_j^+)$ at $t = 0$, and

$$\|\phi(t)\|_{E,m} := \left( \int_{\mathbb{R}^3} \left( \sum_{a=0}^3 |\partial_a \phi(t, x)|^2 + m^2 |\phi(t, x)|^2 \right) dx \right)^{1/2}, \quad (11)$$

while $\dot{H}^1$ and $H^1$ denote the homogeneous and inhomogeneous Sobolev spaces, respectively. More precisely, $H^1(\mathbb{R}^3)$ is the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the norm $\|\psi\|_{H^1(\mathbb{R}^3)} = \|\nabla_x \psi\|_{L^2(\mathbb{R}^3)}$, and $H^1(\mathbb{R}^3)$ is the set of $L^2$-functions whose first derivatives (in the sense of distributions) also belong to $L^2(\mathbb{R}^3)$.

We emphasize that in the previous result of [10], the compactness of the support was not required. In Theorem 1.1 here, the condition (A2) is removed, but the compactness of the support is essentially used in the proof. We do not know whether Theorem 1.1 holds true or not for the initial data with non-compact support.

Our theorem can be extended to the quasi-linear systems. See Section 5 below for the details.

2. Preliminaries. For $z \in \mathbb{R}^d$, we define $\langle z \rangle = \sqrt{1 + |z|^2}$. In what follows, $C$ stands for positive constants whose actual values may change line by line.

First we recall Hardy’s inequality.

**Lemma 2.1** (Hardy’s inequality). For $\phi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\left( \int_{\mathbb{R}^3} \frac{|\phi(x)|^2}{|x|^2} dx \right)^{1/2} \leq 2 \left( \int_{\mathbb{R}^3} |\nabla_x \phi(x)|^2 dx \right)^{1/2}.$$  

**Proof.** Let $\phi \in C_0^\infty(\mathbb{R}^3)$. For $\omega \in \mathbb{S}^2$, we have

$$\int_0^\infty |\phi(r\omega)|^2 dr = \left[ |\phi(r\omega)|^2 \right]_{r=0}^\infty - 2 \int_0^\infty \phi(r\omega)(\partial_r \phi)(r\omega) r dr$$

$$\leq 2 \left( \int_0^\infty |\phi(r\omega)|^2 dr \right)^{1/2} \left( \int_0^\infty |\partial_r \phi(r\omega)|^2 r^2 dr \right)^{1/2},$$

which leads to

$$\int_0^\infty |\phi(r\omega)|^2 dr \leq 4 \int_0^\infty |\partial_r \phi(r\omega)|^2 r^2 dr.$$  

Integrating this inequality over $\mathbb{S}^2$, we obtain the desired result. \[ \Box \]

The next Hardy-type inequality is due to Lindblad [21].
Lemma 2.2. For $R > 0$, there is a positive constant $C_R$ such that we have
\[
 \left( \int_{\mathbb{R}^3} \frac{|\phi(t,x)|^2}{(t-|x|)^2} \, dx \right)^{1/2} \leq C_R \left( \int_{\mathbb{R}^3} |\nabla \phi(t,x)|^2 \, dx \right)^{1/2}, \quad t \geq 0
\] (12)
for any smooth function $\phi = \phi(t,x)$ satisfying
\[
 \phi(t,x) = 0, \quad |x| \geq t + R, \quad t \geq 0.
\]

By replacing $(t-|x|)$ with $(2R+t-|x|)$ on the left-hand side of (12), this lemma is proved by performing integration by parts in a similar manner to the proof of Hardy’s inequality. This replacement requires the support condition.

The following corollary is a key estimate for the treatment of nonlinear terms $w_k(\partial_a w_l)$ in the energy estimate, and this is one of the points where we need the compactness of the support of the initial data in our theorem.

Lemma 2.3. For $R > 0$, there is a positive constant $C_R$ such that we have
\[
 \left( \int_{\mathbb{R}^3} \frac{|\phi(t,x)|^2}{(t-|x|)^2} \, dx \right)^{1/2} \leq C_R (1+t)^{-1} \left( \int_{\mathbb{R}^3} |\nabla \phi(t,x)|^2 \, dx \right)^{1/2}, \quad t \geq 0
\]
for any smooth function $\phi = \phi(t,x)$ satisfying
\[
 \phi(t,x) = 0, \quad |x| \geq t + R, \quad t \geq 0.
\]

Proof. We have
\[
 \langle r \rangle^{-1} (t-r)^{-1} \leq C (|t+r|^{-1} (t-r)^{-1} + (t+r)^{-1} (r)^{-1}) \leq C (1+t)^{-1} (t-r)^{-1} + r^{-1}.
\]
Therefore, using Lemmas 2.1 and 2.2, we obtain
\[
 \int_{\mathbb{R}^3} \frac{|\phi(t,x)|^2}{(t-|x|)^2} \, dx \leq C(1+t)^{-2} \left( \int_{\mathbb{R}^3} \frac{|\phi(t,x)|^2}{(t-|x|)^2} \, dx + \int_{\mathbb{R}^3} \frac{|\phi(t,x)|^2}{|x|^2} \, dx \right)
\]
\[
\leq C_R (1+t)^{-2} \int_{\mathbb{R}^3} |\nabla \phi(t,x)|^2 \, dx.
\]
This completes the proof.

We use the so-called vector field method to obtain decay estimates of solutions. We introduce
\[
 L_j := x_j \partial_t + t \partial_j, \quad 1 \leq j \leq 3,
\]
\[
 \Omega_{jk} := x_j \partial_k - x_k \partial_j, \quad 1 \leq j, k \leq 3,
\]
and we set
\[
 \Gamma = (\Gamma_1, \ldots, \Gamma_{10}) := (L, \Omega, \partial) = (L_j)_{1 \leq j \leq 3}, (\Omega_{jk})_{1 \leq j, k \leq 3}, (\partial_a)_{0 \leq a \leq 3}.
\]
For a multi-index $\alpha = (a_1, \ldots, a_{10})$, we write $\Gamma^\alpha = \Gamma^{a_1} \cdots \Gamma^{a_{10}}$. For a smooth function $\phi = \phi(t,x)$ and a non-negative integer $s$, we define
\[
 |\phi(t,x)|_s = \sum_{|\alpha| \leq s} |\Gamma^\alpha \phi(t,x)|, \quad \|\phi(t)\|_s = \|\phi(t, \cdot)|_s \|_{L^2(\mathbb{R}^3)}.
\] (13)

It is easy to check the following:
\[
 [\Box + m^2, L_j] = [\Box + m^2, \Omega_{jk}] = [\Box + m^2, \partial_a] = 0
\]
Let $\chi_j$ with $j \geq 0$ be non-negative $C_0^\infty(\mathbb{R})$-functions satisfying
$$\sum_{j=0}^{\infty} \chi_j(\tau) = 1 \text{ for } \tau \geq 0,$$
and $\text{supp } \chi_j \subset [2^{j-1}, 2^{j+1}]$ for $j \geq 1$, and $\text{supp } \chi_0 \cap [0, \infty) \subset [0, 2]$. Let $m > 0$, and $v$ be a smooth solution to
$$(\Box + m^2) v(t, x) = \Phi(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^3.$$
Then there exists a positive constant $C = C(m)$ such that we have
$$\langle |x| \rangle^{3/2} |v(t, x)| \leq C \sum_{j=0}^{\infty} \sum_{|\alpha| \leq 4} \sup_{\tau \in [0, t]} \chi_j(\tau) \|\langle \tau + |\cdot| \rangle^{\alpha} \Phi(\tau, \cdot)\|_{L^2(\mathbb{R}^3)}$$
$$+ C \sum_{j=0}^{\infty} \sum_{|\alpha| \leq 5} \|\langle |\cdot| \rangle^{3/2} \chi_j(\cdot) \langle |\cdot| \rangle^{\alpha} v(0, \cdot)\|_{L^2(\mathbb{R}^3)}$$
for $(t, x) \in (0, \infty) \times \mathbb{R}^3$, provided that the right-hand side of $(15)$ is finite.

**Remark 1.** If $\Phi(t, x) = 0$ for $|x| \geq t + R$ with some $R > 0$, and
$$\sum_{|\alpha| \leq 4} \|\langle |x| \rangle^{\alpha} \Phi(t)\|_{L^2} \leq M(t)^\rho$$
with some $M > 0$ and $\rho \neq -1$, then there is a positive constant $C = C(\rho, R)$ such that
$$\sum_{j=0}^{\infty} \sum_{|\alpha| \leq 4} \sup_{\tau \in [0, t]} \chi_j(\tau) \|\langle \tau + |\cdot| \rangle^{\alpha} \Phi(\tau, \cdot)\|_{L^2} \leq CM(t)^{\rho+1},$$
where $(a)_+ a$ for $a > 0$, and $(a)_+ 0$ for $a < 0$. In Section 4 below, we will use Lemma 2.4 in combination with $(16)$, which requires the support condition for $\Phi$. But this usage of $(16)$ is just for the simplification of the calculation, and not essential.
We use the weighted $L^\infty-L^\infty$ estimates for wave equations. To state the estimates, we define
\[
W_\rho(t, r) := \begin{cases} 
\left\{ \log \left( 2 + (t + r)(t - r)^{-1} \right) \right\}^{-1} & \text{if } \rho = 0, \\
(t - r)^\rho & \text{if } \rho > 0.
\end{cases}
\]
We also introduce
\[
W_-(t, r) := \min \{ \langle r \rangle, \langle t - r \rangle \}.
\]

**Lemma 2.5.** Let $w$ be a smooth solution to
\[
\Box w(t, x) = 0, \quad (t, x) \in (0, \infty) \times \mathbb{R}^3
\]
with initial data $(w(0), (\partial_tw)(0)) = (w^{(0)}, w^{(1)})$.

Let $\kappa \geq 0$. Then, there exists a positive constant $C = C(\kappa)$ such that
\[
\langle t + |x| \rangle \sup_{|y - x| \leq t} \langle y \rangle^{1+\kappa} \sum_{|\alpha| \leq 1} \sum_{\nu = 1} \langle y \rangle^{1+\kappa+\nu-\rho} W_-(\tau, |y|)^{1-\nu} |\Psi(\tau, y)|,
\]
for $(t, x) \in (0, \infty) \times \mathbb{R}^3$. Here $\partial_x := (\partial_1, \partial_2, \partial_3)$, and we have used the standard notation of multi-indices.

For the proof, see Asakura [1] (see also [14] for the above expression).

The next estimates for the inhomogeneous wave equations are due to Kubota-Yokoyama [19] (see also [12] for the expression below):

**Lemma 2.6.** Let $w$ be a smooth solution to
\[
\Box w(t, x) = \Psi(t, x), \quad (t, x) \in (0, \infty) \times \mathbb{R}^3
\]
with initial data $w = \partial_tw = 0$ at $t = 0$.

Suppose that $\rho \geq 0$, $\kappa \geq 0$, and $\nu > 0$. Then there exists a positive constant $C = C(\rho, \kappa, \nu)$ such that
\[
\langle t + |x| \rangle^{1-\rho} W_\kappa(t, |x|) |w(t, x)|
\leq C \sup_{\tau \in [0, t]} \sup_{|y - x| \leq t - \tau} |y|^{(\tau + |y|)^{1+\kappa+\nu-\rho}} W_-(\tau, |y|^{1-\nu}) |\Psi(\tau, y)|,
\]
for $(t, x) \in (0, \infty) \times \mathbb{R}^3$.

Because of the terms in $F^K_{W_\kappa}$, whose decay rate is $\langle t + |x| \rangle^{-3}$ at best, we need to choose $\kappa = 0$ when we apply Lemma 2.6 in the proof of Theorem 1.1, but then, as $w$ is included in $F^K_{W_\kappa}$, the factor $W_0(t, |x|)$ in (18) causes some trouble in the energy estimates. It is known that some logarithmic factor like $W_0(t, |x|)$ is unremovable from (18) with $\kappa = 0$ in general (see the author [11]); however, as shown in Lemma 2.7 below, we can remove $W_0(t, |x|)$ if we assume a certain support condition on $\Psi$. This is another point where we need the compactness of the support of initial data in our proof.
Lemma 2.7. Let \( w \) be as in Lemma 2.6. Suppose that there is a positive constant \( R \) such that
\[
\Psi(t, x) = 0, \quad |x| \geq t + R, \quad t \geq 0.
\]
Then, for \( \rho \geq 0 \) and \( \nu > 0 \), there is a positive constant \( C_R = C_R(\rho, \nu) \) such that
\[
\langle t + |x| \rangle^{1-\rho} |w(t, x)| \leq C_R \sup_{\tau \in [0, t]} \sup_{|y-x| \leq t-\tau} |y| \langle \tau + |y| \rangle^{1+\nu-\rho} W_-(\tau, |y|) \langle |y| \rangle^{1-\nu} |\Psi(\tau, y)|
\] (20)
for \((t, x) \in (0, \infty) \times \mathbb{R}^3\).

Proof. Since the right-hand side of (20) becomes greater as \( \nu \) becomes greater, we may assume \( 0 < \nu < 1 \).

It is known that there is an \( \mathbb{S}^2 \)-valued function \( \Theta = \Theta(\lambda, \eta; t, x) \) such that
\[
w(t, x) = \frac{1}{4\pi r} \int_0^t \int_{|r-t+r|}^{r+r-t} 2\pi \lambda \Psi(\tau, \lambda \Theta(\lambda, \eta; t-r, x)) d\eta d\lambda dt,
\]
where \( r = |x| \) (see John [7] for this expression). If we put
\[
\chi_R(\tau, \lambda) := \begin{cases} 1, & \lambda \leq \tau + R, \\ 0, & \lambda > \tau + R, \end{cases}
\]
we get
\[
|w(t, x)| \leq CI_{\nu, p, R}(t, r) \sup_{\tau \in [0, t]} \sup_{|y-x| \leq t-\tau} |y| \langle \tau + |y| \rangle^{1+\nu-\rho} W_-(\tau, |y|) \langle |y| \rangle^{1-\nu} |\Psi(\tau, y)|,
\]
where
\[
I_{\nu, p, R}(t, r) = \frac{1}{r} \int_0^t \int_{|r-t+r|}^{r+r-t} (\tau + \lambda)^{-1-\nu+\rho} \chi_R(\tau, \lambda) d\lambda d\tau.
\]
It is easy to see that \( I_{\nu, p, R}(t, r) \leq C(t + r)^{\rho} \sum_{c=0}^1 I^c_{\nu, R}(t, r) \) with
\[
I^c_{\nu, R}(t, r) = \frac{1}{r} \int_0^t \int_{|r-t+r|}^{r+r-t} (1 + \tau + \lambda)^{-1-\nu} \chi_R(\tau, \lambda) d\lambda d\tau.
\]
Hence our task is to show that \( I^c_{\nu, R}(t, r) \leq C_R(t + r)^{-1} \) with some \( C_R > 0 \).

We perform change of variables \( \alpha = \lambda + \tau \) and \( \beta = \lambda - \tau \). Then we get
\[
I^c_{\nu, R}(t, r) = \frac{1}{(1+c)^2} \int_{|t-r|}^{t+r} \frac{1}{2} (1+\alpha)^{-1-\nu} \int_{\beta^c_0(\alpha, t, r)}^\alpha (1+|\beta|)^{\nu-1} \bar{\chi}_R(\alpha, \beta) d\beta d\alpha
\]
with
\[
\beta^c_0(\alpha, t, r) := \frac{(1-c)\alpha + (1+c)(r-t)}{2}, \quad \bar{\chi}_R(\alpha, \beta) := \chi_R(\frac{\alpha - \beta}{c+1}, \frac{ca + \beta}{c+1}).
\]
We also put
\[
\beta^c(\alpha, R) := \frac{(1-c)\alpha + (1+c)R}{2}.
\]
Observing that \( \bar{\chi}_R(\alpha, \beta) = 0 \) for \( \beta \geq \beta^c(\alpha, R) \), we get
\[
I^c_{\nu, R}(t, r) \leq C \frac{1}{r} \int_{|t-r|}^{t+r} \frac{1}{2} (1+\alpha)^{-1-\nu} \int_{\beta^c(\alpha, t, r)}^{\min\{\beta^c(\alpha, R), \alpha\}} (1+|\beta|)^{\nu-1} d\beta d\alpha
\]
for \( r \leq t + R \), and \( I^c_{\nu, R}(t, r) = 0 \) for \( r > t + R \).
Let $r \leq (1+t)/2$. In this case, we have $(1+|t-r|)^{-1} \leq (1+t-r)^{-1} \leq C(t+r)^{-1}$, and we obtain
\[
I_{\nu,R}(t,r) \leq C r \int_{|t-r|}^{t+r} (1+\alpha)^{-1} d\alpha \leq C(1+|t-r|)^{-1} \frac{t+r-|t-r|}{r} \leq C(t+r)^{-1}
\]
as $\int_{\beta_0}^{\alpha} (1+|\beta|)^{-\nu} d\beta \leq C(1+\alpha)^\nu$.

Let $(1+t)/2 < r \leq t + R$. Then we have $r^{-1} \leq C(t+r)^{-1}$. If $c = 1$, observing that $\beta_0^1 = r - t$ and $\beta_1^1 = R$, we get
\[
\int_{\beta_0^1}^{\beta_1^1} (1+|\beta|)^{-\nu} d\beta \leq C(1+\max\{R,|t-r|\})^\nu \leq C_R(1+|t-r|)^\nu,
\]
which yields
\[
I_{\nu,R}(t,r) \leq C_R(t+r)^{-1} (1+|t-r|)^\nu \int_{|t-r|}^{t+r} (1+\alpha)^{-1-\nu} d\alpha \leq C_R(t+r)^{-1}
\]
If $c = 0$, observing that $\beta_0^0 = (\alpha + r - t)/2 \geq 0$ and $\beta_1^0 = (\alpha + R)/2$, we obtain
\[
\int_{\beta_0^0}^{\beta_1^0} (1+|\beta|)^{-\nu} d\beta \leq (1+\beta_0^0)^{-\nu} (\beta_1^0 - \beta_0^0) \leq C(1+\alpha + r - t)^{-\nu} (R + t - r).
\]
(21)
For $\alpha_0 \in \mathbb{R}$, we have
\[
J := \int_{|\alpha_0|}^{\infty} (1+\alpha)^{-1-\nu} (1+\alpha - \alpha_0)^{-\nu} d\alpha \leq C(1+|\alpha_0|)^{-1}.
\]
(22)
Indeed, this is trivial when $\alpha_0 \leq 0$, as the integrand is bounded from above by $(1+\alpha)^{-2}$. If $\alpha_0 > 0$, then
\[
J \leq (1+\alpha_0)^{-1-\nu} \int_{\alpha_0}^{2\alpha_0} (1+\alpha - \alpha_0)^{-\nu} d\alpha + C \int_{2\alpha_0}^{\infty} (1+\alpha)^{-2} d\alpha \leq C(1+\alpha_0)^{-1}.
\]
By (21) and (22), we obtain
\[
I_{\nu,R}^0(t,r) \leq C_R(t+r)^{-1} (1+|t-r|) \int_{|t-r|}^{t+r} (1+\alpha)^{-1-\nu} (1+\alpha + r - t)^{-\nu} d\alpha
\]
\[
\leq C_R(t+r)^{-1}
\]
as desired. This completes the proof.

The following Sobolev type inequality will be used to combine decay estimates with the energy estimates (see Klainerman [17] for the proof):

**Lemma 2.8.** There is a positive constant $C$ such that we have
\[
\sup_{x \in \mathbb{R}^3} (x)|\varphi(x)| \leq C \sum_{|\alpha| + |\beta| \leq 2} \|\partial_x^\alpha \Omega^\beta \varphi\|_{L^2(\mathbb{R}^3)}
\]
(23)
for any smooth function $\varphi$ on $\mathbb{R}^3$, provided that the right-hand side of (23) is finite.
3. Algebraic normal forms and the null condition. In this section, we summarize the method of algebraic normal forms.

Roughly speaking, the method of algebraic normal forms enables us to remove the undesirable nonlinear terms in the decay estimates through a certain algebraic transformations of the unknowns. See Sunagawa [23] and Katayama-Ozawa-Sunagawa [13] for the application of the algebraic normal form technique to systems of nonlinear Klein-Gordon equations with mass resonance in one and two space dimensions. Similar ideas were previously used in Bachelot [2], the author [9], Kosecki [18], and Tsutsumi [24] for example, but the method here is more systematic.

We start this section with the enhanced decay estimates for the strong null forms. As before, the null forms \( Q_0 \) and \( Q_{ab} \) are defined by (6) and (7). For \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \), we write \( r := |x| \) and \( \omega = (\omega_1, \omega_2, \omega_3) := x/|x| \), so that \( x = r\omega \). We put \( \partial_r := \sum_{k=1}^3 \omega_k \partial_k \) and \( L_r := \sum_{k=1}^3 \omega_k L_k = r \partial_r + t \partial_r \). We also set \( \omega_0 = -1 \). Using the vector fields defined in the previous section, we can easily check the following identities:

\[
\partial_j = \omega_j \partial_r - \frac{1}{t + r} \sum_{k=1}^3 \omega_k (\omega_j L_k - \omega_k L_j + \Omega_{jk}), \quad j = 1, 2, 3.
\] (24)

Since \((t + r)^{-1} (|L\phi| + |\Omega\phi|) \leq C|\partial\phi|\), (24) implies

\[
|Q_{jk}(\phi, \psi)| \leq C(t + r)^{-1} (|\Gamma \phi| |\partial \psi| + |\partial \phi| |\Gamma \psi|), \quad 1 \leq j < k \leq 3.
\] (25)

To estimate \( Q_{0j}(\phi, \psi) \) with \( 1 \leq j \leq 3 \), we use the following identity:

\[
(\partial_t \phi)(\partial_t \psi) - (\partial_r \phi)(\partial_r \psi) = \frac{1}{t + r} ((L_r \phi)(\partial_r \psi) - (\partial_r \phi)(L_r \psi))
\]

\[
+ \frac{1}{t + r} ((\partial_t \phi)(L_r \psi) - (L_r \phi)(\partial_t \psi)).
\] (26)

It follows from (24) and (26) that

\[
|Q_{0j}(\phi, \psi)| \leq C(t + r)^{-1} (|\Gamma \phi| |\partial \psi| + |\partial \phi| |\Gamma \psi|), \quad 1 \leq j \leq 3.
\] (27)

Let \( \varphi = (\varphi_j) \) be a smooth function. Since \( \Gamma^\alpha Q_{ab}(\varphi_k, \varphi_l) \) can be written in terms of \( Q_{cd}(\Gamma^{\beta}\varphi_k, \Gamma^{\gamma}\varphi_l) \) with \( 0 \leq c, d \leq 3 \) and \( |\beta| + |\gamma| \leq |\alpha| \), it follows from (25) and (27) that

\[
|Q_{ab}(\varphi_k, \varphi_l)| \leq C_s (t + r)^{-1} (|\varphi|_{s/2} + |\partial \varphi|_{s/2} + |\partial^2 \varphi|_{s/2})
\] (28)

for any non-negative integer \( s \), where \( C_s \) is a positive constant depending only on \( s \).

For \( 1 \leq j \leq N \), we put

\[
\Box_j := \Box + m_j^2.
\]

We also set

\[
\mathcal{M} = (\mathcal{M}_{ab})_{0 \leq a, b \leq 3} := \text{diag} (1, -1, -1, -1).
\]

Since we have \((\partial_t \phi)(\partial_t \psi) = (\partial_0 \phi)(\partial^2 \psi) + Q_{ba}(\phi, \partial_b \psi)\), we get

\[
Q_0(\phi, \partial_a \psi) = (\partial_a \phi)(\Box \psi) + \sum_{b=0}^3 \mathcal{M}_{ab} Q_{ba}(\phi, \partial_b \psi)
\] (29)

for \( 0 \leq a \leq 3 \) and smooth functions \( \phi, \psi \).

Let \( 1 \leq j, k, l \leq N \) and \( |\alpha|, |\beta| \leq 1 \). We put

\[
U_{kl}^{\alpha \beta} := ((\partial^\alpha u_k)(\partial^\beta u_l) - Q_0(\partial^\alpha u_k, \partial^\beta u_l)).
\]
It follows from simple calculation and (29) that
\[ \Box_j U^{\alpha \beta}_{kl} = U^{\alpha \beta}_{kl} A^{ijkl} + (R^{kl \alpha \beta}_1 \ R^{kl \alpha \beta}_2) \]
with
\[ A^{ijkl} = \begin{pmatrix} m_j^2 - m_k^2 & -m_j m_k \n m_k^2 - m_j^2 \n m_j^2 & m_k^2 \end{pmatrix}, \]
\[ R^{kl \alpha \beta}_1 = (\partial^\alpha \Box_k u_k)(\partial^\beta u_l) + (\partial^\alpha u_k)(\partial^\beta \Box_l u_l), \]
\[ R^{kl \alpha \beta}_2 = Q_0(\partial^\alpha \Box_k u_k, \partial^\beta u_l) + Q_0(\partial^\alpha u_k, \partial^\beta \Box_l u_l) + 2(\partial^\alpha \Box_k u_k)(\partial^\beta \Box_l u_l) - 2m_j^2(\partial^\alpha \Box_k u_k)(\partial^\beta \Box_l u_l) + \sum_{a,b=0}^3 \mathcal{M}_{aa}M_{ab}Q_{ba}(\partial_a \partial^\alpha u_k, \partial_b \partial^\beta u_l). \]

Using these formulas, we can show the following key lemma in the method of algebraic normal forms:

**Lemma 3.1.** Let \( u \) be the solution to (1). We assume that \( |\alpha|, |\beta| \leq 1 \), and \( 1 \leq j, k, l \leq N \). If \( A^{ijkl} \) is invertible, then there exist two constants \( c_{jkl} \) and \( d_{jkl} \) such that, writing\[ (\partial^\alpha u_k)(\partial^\beta u_l) = \Box_j \left( c_{jkl}(\partial^\alpha u_k)(\partial^\beta u_l) + d_{jkl}Q_0(\partial^\alpha u_k, \partial^\beta u_l) \right) + R^{\alpha \beta}_{jkl}, \]
we have
\[ |R^{\alpha \beta}_{jkl}| \leq C_s(|u|_{s/2}+2|F|_{s/2}+2(|u|_{s/2}+|F|_{s/2}) \]
\[ + C_s(t+r)^{-1}|\partial u|_{s/2}+2|\partial u|_{s/2} \]
for any non-negative integer \( s \), where \( C_s \) is a positive constant.

**Proof.** By (30), we have
\[ U^{\alpha \beta}_{kl} = \Box_j U^{\alpha \beta}_{kl}(A^{ijkl})^{-1} - (R^{kl \alpha \beta}_1 \ R^{kl \alpha \beta}_2)(A^{ijkl})^{-1}. \]
Writing
\[ (A^{ijkl})^{-1} = \begin{pmatrix} c_{jkl} & * \\ d_{jkl} & * \end{pmatrix}, \]
we get
\[ (\partial^\alpha u_k)(\partial^\beta u_l) = \Box_j \left( c_{jkl}(\partial^\alpha u_k)(\partial^\beta u_l) + d_{jkl}Q_0(\partial^\alpha u_k, \partial^\beta u_l) + R^{\alpha \beta}_{jkl} \right) \]
with \( R^{\alpha \beta}_{jkl} = -c_{jkl}R^{kl \alpha \beta}_1 - d_{jkl}R^{kl \alpha \beta}_2 \). If we replace \( \Box_k u_k \) and \( \Box_l u_l \) with \( F_k(u, \partial u) \) and \( F_l(u, \partial u) \), respectively, in (31) and (32), we obtain the desired estimate for \( |R^{\alpha \beta}_{jkl}| \) with the help of (28). \( \square \)

When we derive decay estimates for local solutions, Lemma 3.1 can be used to replace terms like \( (\partial^\alpha u_k)(\partial^\beta u_l) \) in \( F_j \) with harmless terms, provided that \( \det A^{ijkl} \neq 0 \). For example, if \( m_j > 0 \) and \( m_k = 0 \), we have \( \det A^{ijkl} = m_j^4 > 0 \). Thus we can replace \( F^W_k \) with harmless terms in the decay estimates below.

When \( m_j = 0 \), we have \( \det A^{ijkl} = (m_k^2 - m_j^2)^2 \). Hence, if \( m_j = 0 \) and \( m_k \neq m_l \), we can replace \( (\partial^\alpha u_k)(\partial^\beta u_l) \) in \( F^W_k \) with harmless terms. Especially, \( F^W_k \) can be replaced. When \( m_j = 0 \) and \( m_k = m_l \), we cannot apply Lemma 3.1 in general, since \( \det A^{ijkl} = 0 \). However, it turns out that the null form \( Q_0(u_k, u_l) \) in \( F^W_k \) can be similarly replaced with better terms.
Lemma 3.2. Let $u$ be the solution to (1). For $N_0 + 1 \leq k, l \leq N$, if we write
\[
Q_0(w_k, w_l) = \frac{1}{2} \Box (w_k w_l) + R_{kl},
\]
then, for any non-negative integer $s$ we have
\[
|R_{kl}|_s \leq C_s \left( |w|_{[s/2]} |F|_s + |F|_{[s/2]} |w|_s \right)
\]
with a positive constant $C_s$ depending only on $s$.

Proof. Since we have
\[
\Box (w_k w_l) = (\Box w_k) w_l + w_k (\Box w_l) + 2Q_0(w_k, w_l),
\]
we can show the result in a similar fashion to the proof of Lemma 3.1. \qed

4. Proof of the main theorem. Let $u$ be a solution to (1)–(2). If $f(x) = g(x) = 0$ for $|x| \geq R$ with some $R > 0$, then we have
\[
u(t, x) = 0 \quad \text{for} \quad |x| \geq t + R \quad \text{and} \quad t \geq 0,
\]
which is known as finite speed of propagation. This enables us to use Lemmas 2.3 and 2.7 in the following proof.

Let $c_j^{kl\alpha\beta}$ be from (9). In the proof of the theorem, without loss of generality, we may assume the following condition in addition to (N) and (A1):
\[
(A3) \quad F^W_k(w, \partial w)
\]
does not contain terms like $w_k w_l$ for $N_0 + 1 \leq k, l \leq N$.

Indeed, we can reduce the system to a new one satisfying (A3) by a kind of algebraic normal form as follows. If we set
\[
p(w) = (p_j(w))_{1 \leq j \leq N_0} := \left( \frac{1}{m_j} \sum_{N_0 + 1 \leq k \leq l \leq N} \sum_{|\alpha| = |\beta| = 0} c_j^{kl\alpha\beta} w_k w_l \right)_{1 \leq j \leq N_0},
\]
and if we define $\tilde{u} = (\tilde{u}_j)_{1 \leq j \leq N}$, $\tilde{v} = (\tilde{v}_j)_{1 \leq j \leq N}$, and $\tilde{w} = (\tilde{w}_j)_{N_0 + 1 \leq j \leq N}$ by
\[
\begin{aligned}
\tilde{u}_j &= \tilde{v}_j := v_j - p_j(w), \quad 1 \leq j \leq N_0, \\
\tilde{u}_j &= \tilde{w}_j := w_j, \quad N_0 + 1 \leq j \leq N,
\end{aligned}
\]
then we see that
\[
(\Box + m_j^2) \tilde{u}_j = \tilde{F}_j(\tilde{u}, \partial \tilde{u}), \quad 1 \leq j \leq N,
\]
where
\[
\tilde{F}_j(\tilde{u}, \partial \tilde{u}) := F_j(u, \partial u) - \sum_{N_0 + 1 \leq k \leq l \leq N} \sum_{|\alpha| = |\beta| = 0} c_j^{kl\alpha\beta} \left( w_k w_l + \frac{2}{m_j} Q_0(w_k, w_l) \right)
\]
for $1 \leq j \leq N_0$, and $\tilde{F}_j(\tilde{u}, \partial \tilde{u}) := F_j(u, \partial u)$ for $N_0 + 1 \leq j \leq N$, with the substitution $u = (v, w) = (\tilde{v} + p(\tilde{w}), \tilde{w})$ on the right-hand sides. We can easily check that $\tilde{F}^W_k(\tilde{u}, \partial \tilde{u}) = F^W_k(\tilde{u}, \partial \tilde{u})$ and $\tilde{F}^W(\tilde{u}, \partial \tilde{u}) = F^W(\tilde{w}, \partial \tilde{w})$. Hence the conditions (N) and (A1) are preserved by this reduction, while we can see that $\tilde{F}_K^W$ does not contain $\tilde{w}_k \tilde{w}_l (= w_k w_l)$ for $N_0 + 1 \leq k, l \leq N$. In conclusion, by the reduction (33), we can obtain a system satisfying the condition (A3), without affecting the assumptions (N) and (A1).
Recall that $|\cdot|_s$ and $\|\cdot\|_s$ are defined by (13). We put
$$\|\phi(t)\|_{s,\infty} := \|\phi(t, \cdot)|_s\|_{L^\infty(\mathbb{R}^3)}$$
for any smooth function $\phi = \phi(t, x)$.

For a smooth solution $u = (v, w)$ to (1)–(2) in $[0, T) \times \mathbb{R}^3$, we define
$$E[u](T) := \sup_{t \in [0, T)} (t)^{-\lambda}(\|v(t)\|_{2t} + \|\partial u(t)\|_{2t})$$
$$+ \sup_{(t, x) \in [0, T) \times \mathbb{R}^3} (t + |x|)^{3/2}|v(t, x)|_{I+1}$$
$$+ \sup_{(t, x) \in [0, T) \times \mathbb{R}^3} ((t + |x|)|w(t, x)|_{I+1} + (t - |x|)|\partial w(t, x)|_I,$$
where $\lambda \in (0, 1/100)$, and $I$ is an integer with $I \geq 15$. Observe that there is a positive constant $C$ such that
$$C^{-1}(t + |x|)W_-(t, |x|) \leq (t - |x|)W_-(t, |x|) \leq C(t + |x|)W_-(t, |x|)$$
for $(t, x) \in [0, \infty) \times \mathbb{R}^3$, where $W_-$ is given by (17). Note also that we have
$$|u(t, x)|_{I+1} \leq C(t + |x|)^{-1}E[u](T),$$
$$|(v(t, x), \partial u(t, x))|_I \leq C(t + |x|)^{-1}W_-(t, |x|)^{-1/2}E[u](T)$$
for $(t, x) \in [0, T) \times \mathbb{R}^3$ with a positive constant $C$ which is independent of $T$.

Let $M$ be a sufficiently large constant, and let $\varepsilon$ be sufficiently small compared to $M$, so that we have $M\varepsilon \leq 1$ and $M^2\varepsilon \leq 1$. We are going to prove that $E[u](T) \leq M\varepsilon$ implies $E[u](T) \leq M\varepsilon/2$, provided that $\varepsilon$ is sufficiently small. Then, by the so-called bootstrap argument, we obtain a priori estimates for $u$, and we can show the global existence of the solution $u$.

In what follows, $C$ stands for positive constants which are independent of $M, \varepsilon$, and $T$.

**Step 1: Energy estimates.** Suppose that $E[u](T) \leq M\varepsilon$. By applying the standard energy inequality for wave and Klein-Gordon equations to
$$(\Box + m^2)\Gamma^a u_j = \Gamma^a F_j(u, \partial u)$$
with $|\alpha| \leq 2I$, we obtain
$$\|v(t)\|_{2t} + \|\partial u(t)\|_{2t} \leq C\left(\varepsilon + \int_0^t \|F(u, \partial u(\tau))\|_{2I} d\tau\right). \quad (34)$$

We are going to estimate $\|F(u, \partial u(t))\|_{2I}$.

Let $|\beta| + |\gamma| \leq 2I$. If $|\beta| \leq |\gamma|$, then we have
$$\|\Gamma^a w_j(\Gamma^\gamma \partial_a w_k)\|_{L^2} \leq C\|w_j\|_{L,\infty}\|\partial_a w_k\|_{2t} \leq CM^2\varepsilon^2(t)^{\lambda-1}.$$

If $|\gamma| \leq |\beta|$, it follows from Lemma 2.3 and (14) that
$$\|\Gamma^a w_j(\Gamma^\gamma \partial_a w_k)\|_{L^2} \leq \|\langle \cdot \rangle(t - |\cdot|)\Gamma^a \partial_a w_k\|_{L^\infty} \|\langle \cdot \rangle^{-1}(t - |\cdot|)^{-1}\Gamma^a w_j\|_{L^2} \leq CM\varepsilon(t)^{-1}\|\partial w_j\|_{2t} \leq CM^2\varepsilon^2(t)^{\lambda-1}.$$

Hence we obtain
$$\|w_j(\partial_a w_k)\|_{2t} \leq CM^2\varepsilon^2(t)^{\lambda-1}. \quad (35)$$

Since
$$|w|_{2t} \leq C(|w| + |\Gamma w|_{2t-1}) \leq C(|w| + (t + |x|)|\partial w|_{2t-1}), \quad (36)$$
we get
\[ \|u_{I+1}^7 w_{2I}\|_{L^2} \leq C(\|u_{I+1}^7 w\|_{L^2} + \langle t \rangle \|w\|_{L^2} + \|\partial w\|_{L^2} + \|\partial w\|_{L^2} - 1) \]
\[ \leq C(M^3 \varepsilon^3 \langle t + \cdot \rangle^{-3})_{L^2} + M^3 \varepsilon^3 \langle t \rangle^\lambda - 1) \]
\[ \leq CM^3 \varepsilon^3 \langle t \rangle^\lambda - 1. \] (37)

Other terms in $F^\alpha$ with $|\alpha| \leq 2I$ can be easily treated and we get
\[ \|F(u, \partial u)(t)\|_{2I} \leq C(\|u\|_{t+1,\infty} + \|u\|_{I+1,\infty}^2 \|\partial u\|_{2I} + CM^2 \varepsilon^2 \langle t \rangle^\lambda - 1) \]
\[ \leq CM^2 \varepsilon^2 \langle t \rangle^\lambda - 1. \] (38)

Now (34) yields
\[ \|v(t)\|_{2I} + \|\partial u(t)\|_{2I} \leq C(\varepsilon + M^2 \varepsilon^2) \langle t \rangle^\lambda \leq C\varepsilon \langle t \rangle^\lambda, \] (39)
since $M^2 \varepsilon \leq 1$.

**Step 2: Rough decay estimates.** By Lemma 2.4 and (38), we obtain
\[ \langle t + |x| \rangle^{3/2} \|v(t, x)\|_{2I} \leq C\varepsilon + CM^2 \varepsilon^2 \langle t \rangle^\lambda \leq C\varepsilon \langle t \rangle^\lambda \] (40)
in view of Remark 1.

For $\rho \geq 0$, $\nu > 0$, a non-negative integer $s$, and a smooth function $\phi = \phi(t, x)$, we put
\[ A_{p,s}[\phi] := \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \langle t + |x| \rangle^{1-\rho} \phi(t, x), \]
\[ B_{p,s}[\phi] := \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} |x| \langle t + |x| \rangle^{1+\nu-\rho} W(t, |x|) \langle t + |x| \rangle^{-1-\nu} \phi(t, x). \]

Suppose that $\Box \phi(t, x) = 0$ for $|x| \geq t + R$ and $t \geq 0$ with some $R > 0$. Then, for a positive integer $s$, Lemmas 2.5 and 2.7 yield
\[ A_{p,s}[\phi] \leq C(\varepsilon + B_{p,s}[\Box \phi]) \] (41)
for $\rho \geq 0$ and $\nu > 0$, provided that $\phi(0)$ and $(\partial_t \phi)(0)$ are compactly supported, and their amplitude is of order $\varepsilon$. By Lemma 2.6, we also have
\[ \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \langle t + |x| \rangle^{-\rho} \langle t - |x| \rangle \partial \phi(t, x) \|_{s-1} \leq C(\varepsilon + B_{p,s}[\Box \phi]). \] (42)

We put
\[ F_{h}^u(u, \partial u) = (F_{h}^u(u, \partial u))_{1 \leq j \leq N} := F(u, \partial u) - F^h(u, \partial u). \]

We also use the notation $F_{h} = (F_{h}^h, F_{h}^w)$. For $s \leq 2I$, we have
\[ |F_{h}^u|_s \leq C|u|_{I+1}^7 \|w\|_s + \|(v, \partial u)\|_s \]
\[ \leq CM^2 \varepsilon^2 \langle t + |x| \rangle^{-3} \|v, \partial u\|_s + CM^2 \varepsilon^2 \langle t + |x| \rangle^{\rho - 3} A_{p,s}[w]. \] (43)

By (N) and (A1), $F_{w}^h$ is independent of $w$ itself, and we get
\[ |F_{w}^h|_s \leq C|\|(v, \partial u)\|_{s/2}|(v, \partial u)|_s \]
\[ \leq CM\varepsilon \langle t + |x| \rangle^{-1} W(t, |x|)^{-1/2} \|\partial \phi(t, x)\|_s. \]

By Lemma 2.2 and (39), we get
\[ \|(v, \partial u)(t, x)\|_{2I-2} \leq C\langle x \rangle^{-1} \|(v, \partial u)(t)\|_{2I} \leq C\varepsilon(x)^{-1} \langle t \rangle^\lambda. \]
Now, it follows from (47) that

$$|F_W(u, \partial u)|_{2I-2} \leq CM\varepsilon^2 (t + |x|)^{-1}(t + |x|)^\lambda W_\lambda(t, |x|)^{-1/2} + CM^2\varepsilon^2 (t + |x|)^{\lambda-(5/2)} A_{\lambda+(1/2, 2I-2)}[w],$$

which leads to

$$B_{\lambda+(1/2), 2I-2}[F_W] \leq CM\varepsilon^2 + CM^2\varepsilon^2 A_{\lambda+(1/2), 2I-2}[w].$$

It follows from (41) that

$$A_{\lambda+(1/2), 2I-2}[w] \leq C (\varepsilon + M^2\varepsilon^2 A_{\lambda+(1/2), 2I-2}[w]),$$

and if we choose sufficiently small $\varepsilon$ satisfying $CM^2\varepsilon^2 \leq 1/2$, we get

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} (t + |x|)^{(1/2)-\lambda}|w(t, x)|_{2I-2} = A_{\lambda+(1/2), 2I-2}[w] \leq C\varepsilon. \quad (45)$$

By (42), we also have

$$|\partial w(t, x)|_{2I-3} \leq C\varepsilon(t + |x|)^{(1/2)}(t + |x|)^{-1}(t - |x|)^{-1}. \quad (46)$$

**Step 3: Better decay estimates through the method of algebraic normal forms.** We use the method of the algebraic normal forms to replace some part of $F_W^0$ with harmless terms. $F_W^0$ can be treated by Lemma 3.1; we can use Lemma 3.2 to replace $Q_0(w_j, w_k)$ in $F_W^0$, and only the strong null forms are left in $F_W^0$. Because of the enhanced decay estimate (28), we can consider the strong null forms as harmless terms. In this way, we can find $P_W = (P_j(u, \partial u, \partial^2 u))_{N_0+1 \leq j \leq N}$, whose components are homogeneous polynomials of degree 2 in $(u, \partial u, \partial^2 u)$, such that

$$\Box(w - P_W) = F_K^0(v, \partial v) + F_K^1(u, \partial u) + R_W,$$

where, for $s \leq 2I - 4$, we have

$$|R_W|_s \leq C(t + |x|)^{-1}(|w|_{I+1}|\partial w|_s + |\partial w|_1|w|_{s+1} + |(v, \partial u)|_{I+1}|(v, \partial u)|_{s+2}) + C(|u|_{I+1}|F|_{s+2} + |F|_1(|u|_{s+2} + |F|_{s+2})).$$

By (A3), we have

$$|F_K^0|_s \leq C(|(v, \partial u)|_1(v, \partial u)|_s + |w|_{I}|\partial w|_s + |\partial w|_{I}|w|_s)$$

$$\leq CM\varepsilon(t + |x|)^{-1}W_\lambda(t, |x|)^{-1/2}|(v, \partial u)|_s$$

$$+ CM\varepsilon(t + |x|)^{-1}(W_\lambda(t, |x|)^{-1}|w|_s + |\partial w|_s)$$

for $s \leq 2I$, which, together with (43) and (44), yields

$$|F|_s \leq CM\varepsilon(t + |x|)^{-1}W_\lambda(t, |x|)^{-1/2}|(v, \partial u)|_s$$

$$+ CM\varepsilon(t + |x|)^{-1}(W_\lambda(t, |x|)^{-1}|w|_s + |\partial w|_s). \quad (47)$$

From (40), (45), and (46), we get

$$|(v, \partial u)|_{2I-5} + W_\lambda(t, |x|)^{-1}|w|_{2I-5} \leq C\varepsilon(t + |x|)^{\lambda-(1/2)}W_\lambda(t, |x|)^{-1}. \quad (48)$$

Now, it follows from (47) that

$$|R_W|_{2I-7} \leq CM\varepsilon^2(t + |x|)^{\lambda-(5/2)}W_\lambda(t, |x|)^{-1}. \quad (49)$$

On the other hand, (40) implies

$$|F_K^0|_{2I-7} \leq CM\varepsilon^2(t + |x|)^{\lambda-3}. \quad (50)$$
For $\rho \geq 0$ and $s \leq 2I - 7$, (43) and (48) lead to
\[
|F_{W}^h|_{s} \leq CM^2 \varepsilon^3 (t + |x|)^{\lambda - (5/2)} W_-(t, |x|)^{-1} \\
+ CM^2 \varepsilon^2 (t + |x|)^{\rho - 3} A_{\rho,s}[w].
\] (51)

Therefore, we get
\[
B_{\lambda,\lambda,2I-7}[\Box(w - P_{W})] \leq CM \varepsilon^2 \sup_{(t,x)} (t + |x|)^{\lambda - (1/2)} W_-(t, |x|)^{-\lambda} \\
+ CM^2 \varepsilon^2 (1 + A_{\lambda,2I-7}[w]) \sup_{(t,x)} (t + |x|)^{\lambda - 1} W_-(t, |x|)^{1-\lambda} \\
\leq C \varepsilon + CM^2 \varepsilon^2 A_{\lambda,2I-7}[w].
\]

As we have $A_{\lambda,2I-7}[P_{W}] \leq A_{0,2I-7}[P_{W}]$ and
\[
A_{0,2I-7}[P_{W}] = \sup_{(t,x)} (t + |x|)^{1-\lambda} |w(t,x)|_{2I-5} \leq CM \varepsilon^2,
\] (52)

(41) implies
\[
A_{\lambda,2I-7}[w] \leq A_{\lambda,2I-7}[P_{W}] + A_{\lambda,2I-7}[w - P_{W}] \leq C \varepsilon + CM^2 \varepsilon^2 A_{\lambda,2I-7}[w].
\]
If $\varepsilon$ is sufficiently small to satisfy $CM \varepsilon^2 \leq 1/2$, we obtain
\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}^3} (t + |x|)^{1-\lambda} |w(t,x)|_{2I-7} = A_{\lambda,2I-7}[w] \leq C \varepsilon.
\] (53)

Here we switch to the estimate for $v$. We use the method of algebraic normal forms to replace $F_K^W$ with harmless terms. By Lemma 3.1, we can find $P_K = (P_j(v, \partial v, \partial^2 w))_{1 \leq j \leq N_0}$, whose components are homogeneous polynomials of degree 2 in $(w, \partial w, \partial^2 w)$, such that
\[
(\Box + m_j^2)(v_j - P_j) = F_j^k(v, \partial v) + F_j^{KW}(u, \partial u) + F_j^h(u, \partial u) + R_j
\]
for $1 \leq j \leq N_0$, where
\[
|R_j|_{2I-9} \leq C (|u|^2_{I+1} |(u, \partial u)|_{2I-7} + (t + r)^{-1} |\partial u|_1 |\partial u|_{2I-7}).
\]

It follows from (53) that
\[
||u||^2_{I+1} |w|_{2I-7} \leq CM^2 \varepsilon^3 \|t + |\cdot|\|^{\lambda-3} \| \leq CM^2 \varepsilon^3 (t)^{\lambda - (3/2)}.
\]

Other terms in $R_j$ can be easily treated by (39), and we get
\[
|R_j||_{2I-9} \leq CM \varepsilon^2 (t)^{\lambda - (3/2)}.
\]

By (39), we also have
\[
\|F_k^{KV}(v, \partial v)||_{2I-9} \leq \|v||_{I+1, \infty} \|v||_{2I-8} \leq CM \varepsilon^2 (t)^{\lambda - (3/2)}.
\]

We estimate $F_k^{KW}$ as
\[
\|F_k^{KW}(u, \partial u)||_{2I-9} \leq C \|v||_{2I-8, \infty} \|\partial w||_{2I-9} \leq C \varepsilon^2 (t)^{2\lambda - (3/2)}
\]
by (39) and (40). Since $F_j^h$ enjoys the same estimate as $R_j$, we obtain
\[
(\Box + m_j^2)(v_j - P_j)||_{2I-9} \leq C \varepsilon (t)^{2\lambda - (3/2)},
\] (54)

and Lemma 2.4 yields
\[
(t + |x|)^{3/2} |v - P_K|(t,x)|_{2I-13} \leq C \varepsilon.
\]

By (53), we have
\[
(t + |x|)^{3/2} |P_K||_{2I-13} \leq C (t + |x|)^{3/2} |w|^2_{2I-11} \leq CM \varepsilon^2 (t + |x|)^{2\lambda - 1/2},
\]

or
and we obtain
\[ \langle t + |x| \rangle^{3/2}|v(t, x)|_{2I-13} \leq C\varepsilon. \] (55)

Now we go back to the estimate of \( w \). (55) implies
\[ |F^K_W|_{2I-14} \leq CM\varepsilon^3(t + |x|)^{-3} \] (56)
(cf. (50)). By (49), (56), and (51) with \( \rho = 0 \) and \( s = 2I - 14 \), we obtain
\[ \mathcal{B}_{0,\lambda,2I-14}[\square(w - P_W)] \leq CM\varepsilon^2 \sup_{(t, x)} \langle t + |x| \rangle^{2\lambda - (1/2)} W_-(t, |x|)^{-\lambda} \]
\[ + CM^2\varepsilon^2 \langle 1 + \mathcal{A}_{0,2I-14}[w] \rangle \sup_{(t, x)} \langle t + |x| \rangle^{\lambda-1} W_-(t, |x|)^{1-\lambda} \]
\[ \leq C\varepsilon + CM^2\varepsilon^2 \mathcal{A}_{0,2I-14}[w]. \] (57)

As before, because of (52), it follows from (41) and (57) that
\[ \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} (t + |x|)|w(t, x)|_{2I-14} = \mathcal{A}_{0,2I-14}[w] \leq C\varepsilon, \] (58)
provided that \( \varepsilon \) is sufficiently small.

Since we have
\[ |\partial P_W|_{2I-15} \leq C \left[ |t|_{L^1} |\partial u|_{2I-13} + |\partial u| \left( |t|_{2I-15} + |(v, \partial u)|_{2I-14} \right) \right] \]
\[ \leq CM\varepsilon^2 \langle t + |x| \rangle^{\lambda - (3/2)} W_-(t, |x|)^{-1} \leq CM\varepsilon^2 \langle t - |x| \rangle^{-1} \]
by (40), (46), and (58), it follows from (42), (57), and (58) that
\[ \sup_{(t, x) \in [0, T] \times \mathbb{R}^3} \langle t - |x| \rangle |\partial w(t, x)|_{2I-15} \leq C\varepsilon. \] (59)

**Step 4: Conclusion.** Since \( I + 1 \leq 2I - 14 \) for \( I \geq 15 \), by (39), (55), (58), and (59), we have the following: For any \( M > 0 \), there is a positive constant \( C_0 \), which is independent of \( (M, \varepsilon, T) \), and a positive constant \( \varepsilon_0(M) \), which depends on \( M \), but is independent of \( T \), such that we have
\[ E[u](T) \leq C_0\varepsilon \]
for \( 0 < \varepsilon \leq \varepsilon_0(M) \). If we choose \( M \) to satisfy \( M \geq 2C_0 \), then we have proved that \( E[u](T) \leq M\varepsilon \) implies \( E[u](T) \leq M\varepsilon/2 \) for \( 0 < \varepsilon \leq \varepsilon_0(M) \). This completes the proof for the global existence of the solution.

From (57) and (58), we get
\[ L_2(w_j - P_j) \leq C\varepsilon |x|^{-1} \langle t + |x| \rangle^{-1-\lambda} W_-(t, |x|)^{1+\lambda}, \]
from which we get \( \|L_2(w_j - P_j(t))\|_{L^2} \leq C\varepsilon |t|^{-1-\lambda} \). Hence, with the help of (54), we have
\[ \int_0^\infty \|L_2(\partial w_j(t) - \partial P_j(t))\|_{L^2} dt < \infty, \quad 1 \leq j \leq N. \]

From this, we see that there is a free solution \( u_j^+ \) to \( (\Box + m_j^2)u_j^+ = 0 \) such that \( \|u_j(t) - P_j(t)\|_{E, m_j} \to 0 \) as \( t \to \infty \), where \( \| \cdot \|_{E, m_j} \) is given by (11). Since it is easy to get \( \|P_j(t)\|_{E, m_j} \to 0 \) as \( t \to \infty \), we find that \( \|u_j(t) - u_j^+(t)\|_{E, m_j} \to 0 \) as \( t \to \infty \).

Finally, in the case where we use the reduction (33), the above argument implies \( \|u_j(t) - u_j^+(t)\|_{E, m_j} \to 0 \) as \( t \to \infty \). Since it is easily checked that \( \|P_j(\tilde{w})\|_{E, m_j} \leq C\varepsilon^2(1 + t)^{-1/2} \to 0 \) as \( t \to \infty \), we obtain \( \|u_j(t) - u_j^+(t)\|_{E, m_j} \to 0 \) as desired. This completes the proof for the asymptotic behavior.
5. Remarks on the quasi-linear case. In fact, the following quasi-linear system is considered in almost all the previous works mentioned in the introduction:

\[(\Box + m_J^2)u_j = F_j(u, \partial u, \partial^2 u) \quad \text{in} \ (0, \infty) \times \mathbb{R}^3 \quad (60)\]

for \(j = 1, 2, \ldots, N\), where \(F_j\) has the form

\[F_j(u, \partial u, \partial^2 u) = \sum_{k=1}^{N} \sum_{a,b=0}^{3} G^{ab}_{jk}(u, \partial u)(\partial_a \partial_b u_k) + H_j(u, \partial u)\]

with some \(G^{ab}\) and \(H_j\) satisfying \(G^{ab}_{jk}(u, \partial u) = O(|u| + |\partial u|)\) and \(H_j(u, \partial u) = O(|u|^2 + |\partial u|^2)\) near \((u, \partial u) = (0, 0)\). If we assume the symmetry condition

\[G^{ab}_{jk}(u, \partial u) = G^{ba}_{jk}(u, \partial u) = G^{ab}_{kj}(u, \partial u) \quad (61)\]

for \(1 \leq j, k \leq N\) and \(0 \leq a, b \leq 3\), and if we make some apparent modifications in the assumptions, such as addition of \(\partial^2 u\) to the arguments of the nonlinearity, and replacement of the null condition for the semilinear case with that for the quasi-linear case, then the global existence results in [15, 22, 16, 3, 4], and [10] hold true also for (60). To be more precise, as for the result in [10], we replace \(\sum_{|\alpha|, |\beta| \leq 1} \) in (9) with \(\sum_{|\alpha|, |\beta| \leq 2}\), and the null condition (10) for \(F^W_k\) is replaced by the quasi-linear null condition

\[F^W_k(X, (\omega_0 Y), (\omega_0 \omega_b Z)) = 0, \quad X, Y, Z \in \mathbb{R}^{N-3}, \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{S}^2,\]

whose left-hand side is obtained by substituting \(X, \omega_0 Y, \) and \(\omega_0 \omega_b Z\) in place of \(u, \partial_u,\) and \(\partial_u \partial_u\) in \(F^W_k(u, \partial u, \partial^2 u)\), respectively, with \(\omega_0 = -1\) as before.

We can also show that if (the modified version of) (A1), (A3), and (N), as well as (61), are assumed, then the conclusion of Theorem 1.1 holds true for the quasi-linear system (60). Here we need to assume (A3) explicitly, because the reduction (33) is not always compatible with the symmetry condition (61). In the proof, we replace the standard energy inequality with the energy inequality for hyperbolic systems

\[(\Box + m_J^2)\phi_j(t, x) - \sum_{k=1}^{N} \sum_{a,b=0}^{3} \gamma^{ab}_{jk}(t, x)\partial_a \partial_b \phi_k(t, x) = \Phi_j(t, x)\]

for \(1 \leq j \leq N\), whose coefficients satisfy \(\gamma^{ab}_{jk}(t, x) = \gamma^{ba}_{kj}(t, x) = \gamma^{ab}_{kj}(t, x)\), and

\[\left| \sum_{j,k=1}^{N} \gamma^{00}_{jk}(t, x)\xi_j \xi_k \right| \leq C_0 |\xi|^2, \quad \left| \sum_{j,k=1}^{N} \sum_{l,m=1}^{3} \gamma^{lm}_{jk}(t, x)\eta_j \eta_k \eta_l \eta_m \right| \leq C_0 |\eta|^2\]

for \(\xi = (\xi_j)\) and \(\eta = (\eta_j, i)\) with a positive constant \(C_0 \in (0, 1)\) (see [16] and [6] for such energy inequalities). This is the reason why we assume (61) when we consider the quasi-linear system (60). The modification for the other parts of the proof is straightforward.

Remark 2. We note that terms like \(w_k(\partial_a \partial_b w_l)\) seem to be allowed in \(F^W_k\) if we only look at (A3) for the quasi-linear system; however such terms are excluded by (61) because entry of such terms causes entry of terms like \(w_k(\partial_a \partial_b w_l)\) in \(F^W_k\), which is excluded by (A1). Therefore the quadratic part of \(G^{ab}_{jk}\) does not depend on \(w\) itself under the assumption of modified Theorem 1.1 for (60).
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