Cosmological Models in Lyra Geometry: Kinematics Tests

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Abstract

In this paper the observational consequence of the cosmological models and the expression for the neoclassical tests, luminosity distance, angular diameter distance and look back time are analyzed in the framework of Lyra geometry. It is interesting to note that the space time of the universe is not only free of Big Bang singularity but also exhibits acceleration during its evolution.

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1 Introduction

Einstein geometrized gravitation. Weyl\cite{1} was inspired by it and he was the first to unify gravitation and electromagnetism in a single spacetime geometry. He showed how one can introduce a vector field in the Riemannian spacetime with an intrinsic geometrical significance. But this theory was not accepted as it was based on non-integrability of length transfer. Lyra\cite{2} introduced a gauge function, i.e., a displacement vector in Riemannian spacetime which removes the non-integrability condition of a vector under parallel transport. In this way Riemannian geometry was given a new modification by him and the modified geometry was named as Lyra’s geometry.

Sen\cite{3} and Sen and Dunn\cite{4} proposed a new scalar-tensor theory of gravitation and constructed the field equations analogous to the Einstein's field

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equations, based on Lyra’s geometry which in normal gauge may be written in the form
\[ R_{ij} - \frac{1}{2} g_{ij} R + \frac{3}{2} \phi_i \phi_j - \frac{3}{4} g_{ij} \phi_k \phi^k = -8\pi G T_{ij}, \]

where \( \phi_i \) is the displacement vector and other symbols have their usual meanings.

Halford\[5\] has pointed out that the constant vector displacement field \( \phi_i \) in Lyra’s geometry plays the role of cosmological constant \( \Lambda \) in the normal general relativistic treatment. It is shown by Halford\[6\] that the scalar-tensor treatment based on Lyra’s geometry predicts the same effects, within observational limits as the Einstein’s theory. Several investigators\[7\]–\[15\] have studied cosmological models based on Lyra geometry in different context. Soleng\[8\] has pointed out that the cosmologies based on Lyra’s manifold with constant gauge vector \( \phi \) will either include a creation field and be equal to Hoyle’s creation field cosmology \[16\]–\[18\] or contain a special vacuum field which together with the gauge vector term may be considered as a cosmological term. In the latter case the solutions are equal to the general relativistic cosmologies with a cosmological term.

The purpose of this work is to analyze general features of Bianchi type-I cosmological model with time dependent displacement vector in the framework of Lyra geometry.

2 Field equations

We consider LRS Bianchi Type-I space time
\[ ds^2 = dt^2 - A^2 dx^2 - B^2(dy^2 + dz^2), \]

where, \( A \) and \( B \) are functions of \( x \) and \( t \). We take a perfect fluid form for the energy momentum tensor
\[ T_{ij} = (\rho + p)u_i u_j - pg_{ij}, \]

together with comoving coordinates \( u^i u_i = 1 \), where \( u_i = (0, 0, 0, 1) \). Let us consider a time-like displacement field vector defined by
\[ \phi_i = (0, 0, 0, \beta(t)). \]

The energy momentum tensor \( T^{ij} \) is not conserved in Lyra’s geometry. The essential difference between the cosmological theories based on Lyra geometry and the Riemannian geometry lies in the fact that the constant vector displacement field \( \beta \) arises naturally from the concept of gauge in Lyra geometry whereas the cosmological constant \( \Lambda \) was introduced in \textit{adhoc} fashion in the usual treatment.

The field equations (1) with the equations (2), (3) and (4) take the form
\[ \frac{2\dot{B}}{B} + \frac{\dot{B}^2}{B^2} - \frac{B^2}{A^2 B^2} + \frac{3}{4} \beta^2 = -\chi p, \]
\[ \dot{B}' - \frac{B' \dot{A}}{A} = 0, \]  
\[ \frac{\ddot{A}}{A} + \frac{\dot{B}}{B} + \frac{\dot{A} \dot{B}}{AB} = \frac{B''}{A^2 B} + \frac{A'B'}{A^3 B} + \frac{3}{4} \beta^2 = -\chi p, \]  
\[ \frac{2B''}{A^2 B} - \frac{2A'B'}{A^3 B} + \frac{B'^2}{A^2 B^2} \frac{B''}{B^2} - \frac{2\dot{A}B}{AB} - \frac{\dot{B}^2}{B^2} + \frac{3}{4} \beta^2 = \chi p. \]  

The energy conservation equation is
\[ \chi \dot{\rho} + \frac{3}{2} \beta \dot{\beta} + \left[ \chi (\rho + p) + \frac{3}{2} \beta^2 \right] \left( \frac{\dot{A}}{A} + \frac{2 \dot{B}}{B} \right) = 0, \]  
where \( \chi = 8\pi G \). Here and in what follows, a prime and a dot indicate partial differentiation with respect to \( x \) and \( t \), respectively. We assume that the fluid obeys a barotropic equation of state
\[ p = \gamma \rho, \]  
where \( \gamma (0 \leq \gamma \leq 1) \) is a constant.

## 3 Solutions of the field equations

On integrating the equation (6), we obtain
\[ A = \frac{B'}{l}, \]  
where \( l \) is an arbitrary function of \( x \). Using equation (11), equations (5) and (7) can be written as
\[ \frac{B'}{B'} \frac{d}{dx} \left( \frac{\dot{B}}{B} \right) + \frac{\dot{B}}{B'} \frac{d}{dt} \left( \frac{B'}{B} \right) + \frac{l^2}{B^2} \left( 1 - \frac{B'}{B'T} \right) = 0. \]  
Since \( A \) and \( B \) are separable functions of \( x \), so, \( \frac{B'}{B} \) is a function of \( x \). Consequently, equation (12) gives after integration
\[ B = lS(t), \]  
where \( S(t) \) is the scale factor which is an arbitrary function of \( t \). Using the equation (13), (11) becomes
\[ A = \frac{l'}{l} S. \]  

The metric (5) then takes the form
\[ ds^2 = dt^2 - S^2(t) \left[ dX^2 + e^{2X} (dy^2 + dz^2) \right], \]
where $X = \ln l$.

The expression for the density and pressure from the Equations (8) and (5) give

$$\frac{3\dot{S}^2}{S^2} - \frac{3}{4} \beta^2 = \rho_d + \rho_x, \quad (16)$$

$$\frac{2\ddot{S}}{S} + \dot{S}^2 - \frac{1}{4} \beta^2 = -p_x. \quad (17)$$

Eqs. (16) and (17) leads to the continuity equation

$$\dot{\rho}_d + \dot{\rho}_x + \frac{3}{2} \beta \dot{\beta} + 3 \left[ \rho_d + \rho_x + p_x + \frac{3}{2} \beta^2 \right] \left( \frac{\dot{S}}{S} \right) = 0, \quad (18)$$

where we assume $\chi = 8\pi G = 1$, $p = p_x$ and $\rho = \rho_d + \rho_d$, $\rho_d$ is the dust matter whereas $\rho_x$ refers to the missing component of the energy obeys an equation of state

$$p_x = -\gamma \rho_x, \quad (19)$$

where $\gamma (0 < \gamma < 1)$ is constant.

We assume that energies densities due to dust matter and missing component of the Universe are always less than to its critical value $\rho_{\text{crit}} = 3(\frac{\dot{S}}{S^2})$. Hence one can consider

$$\rho_d = a \rho_{\text{crit}} \quad \text{and} \quad \rho_x = b \rho_{\text{crit}}, \quad (20)$$

where the fractions $a$ (density parameter due to dust contribution), $b$ (density parameter due to missing mass contribution) are such that $a, b < 1$ and $a + b < 1$.

From Eqs. (16), (17) and (20), we get

$$\frac{3\dot{S}}{S} = \frac{3\gamma - 1}{2} \rho_x - \frac{1}{2} \rho_d - \frac{3}{2} \beta^2. \quad (21)$$

Now if

$$\frac{3\gamma - 1}{2} \rho_x > \frac{1}{2} \rho_d - \frac{3}{2} \beta^2, \quad (22)$$

then we have

$$\frac{\dot{S}}{S} > 0.$$

This shows that the deceleration parameter

$$q = -\frac{\ddot{S} S}{\dot{S}^2} < 0.$$

In other words, the Universe is accelerating because acceleration at a certain stage in the evolution of the Universe implies $q < 0$ for some ‘t’.

From Eqs. (16), (17) and (20), we get

$$\frac{\dot{S}}{S} + \frac{M S^2}{S^2} = \frac{2}{S^2}, \quad (23)$$
where

\[ 2M = 4 - 3a - 3b(1 + \gamma). \]  

(24)

Integrating Eq. (23), we obtain

\[
\int \left[ \frac{2}{M} + c_1 S^{-2M} \right]^{-1/2} dS = t - c_2,
\]

(25)

where \( c_1 \) and \( c_2 \) are constants of integration.

To solve the above integration, let us choose

\[ 2M = -1, \text{i.e., } 5 = 3a + 3b(1 + \gamma). \]

For this choice Eq. (25), after integration, gives

\[ S = \frac{4}{c_1} + \frac{c_1}{4} (t - c_2)^2. \]

(26)

Hence from the field equations the displacement vector takes the form

\[
\beta^2 = \frac{c_1^2}{3\left[\frac{4}{c_1} + \frac{4}{3}(t - c_2)^2\right]^2} \left[ \frac{1}{2}(7 - 6a - 6b)(t - c_1)^2 - 4 \right] - \frac{2c_1}{3\left[\frac{4}{c_1} + \frac{4}{3}(t - c_2)^2\right]}. \]

(27)

The deceleration parameter \( q \) and Hubble parameter \( H \) are given respectively

\[ q = -\frac{1}{2} - \frac{8}{c_1^2} \frac{1}{(t - c_2)}, \]

(28)

\[ H = \frac{c_1}{4} (t - c_2) \left[ \frac{4}{c_1} + \frac{c_1}{4} (t - c_2)^2 \right]^{-1}. \]

(29)

In this model, the particle horizon exist because

\[
\int_{-\infty}^{t} [S(T)]^{-1} dT
\]

is a convergent integral. As the model is singularity free, we consider \( T \to -\infty \).

4 Neoclassical Tests (Proper Distance \( d(z) \))

The proper distance between the source and observer is given by

\[ d(z) = S_0 \int_{S}^{S_0} \frac{dS}{SS}. \]

(30)

From Eq. (26), after integration, we get

\[ d(z) = S_0 \left[ \sin^{-1} \sqrt{\frac{4(1 + z)}{c_1 S_0}} - \sin^{-1} \sqrt{\frac{4}{c_1 S_0}} \right]. \]

(31)

where \( 1 + z = \frac{S_0}{S} \) = redshift. This is increasing function of ‘\( z \)’.
5 Luminosity Distance

Luminosity distance is another important concept of theoretical cosmology of a light source. The luminosity distance is a way of expanding the amount of light received from a distant object. It is the distance that the object appears to have, assuming the inverse square law for the reduction of light intensity with distance holds.

If $d_L$ is the luminosity distance to the object, then

$$d_L = \left( \frac{L}{4\pi l} \right)^{\frac{1}{2}},$$

(32)

where $L$ is the total energy emitted by the source per unit time, $l$ is the apparent luminosity of the object. Therefore one can write

$$d_L = (1 + z)d(z).$$

(33)

This is also an increasing function of ‘$z$’.

6 Angular Diameter Distance

The angular diameter distance is a measure of how large objects appear to be. As with the luminosity distance, it is defined as the distance that an object of known physical extent appears to be at, under the assumption of Euclidean geometry.

The angular diameter $d_A$ of a light source of proper distance $d$ is given by

$$d_A = d(z)(1 + z)^{-1} = d_L(1 + z)^{-2},$$

(34)

which is a decreasing function of ‘$z$’.

The angular diameter and luminosity distances have similar forms, but have a different dependence on redshift. As with the luminosity distance, for nearly objects the angular diameter distance closely matches the physical distance, so that objects appear smaller as they are put further away. However the angular diameter distance has a much more striking behaviour for distant objects. The luminosity distance effect dims the radiation and the angular diameter distance effect means the light is spread over a large angular area. This is so-called surface brightness dimming is therefore a particularly strong function of redshift.

7 Look Back Time

The time in the past at which the light we now receive from a distant object was emitted is called the look back time. How long ago the light was emitted (the look back time) depends on the dynamics of the universe.
The radiation travel time (or lookback time) \((t - t_0)\) for photon emitted by a source at instant \(t\) and received at \(t_0\) is given by
\[
t_0 - t = \int_S \frac{dS}{S}.
\] (35)

By using the value from Eq. 26, Eq. 35 after integration, can be expressed as
\[
t_0 - t = \frac{2\sqrt{S_0}}{c_1} \left[ 1 - (1 + z)^{-1/2} \right].
\] (36)

8 Discussions

In the present investigation, we examine the cosmological problems for considering a homogeneous and isotropic cosmological model in the framework of Lyra geometry. We have shown that space time of the universe is free from Big Bang singularity. Moreover, the displacement vector field \(\beta\) arises in Lyra geometry, we refer to as vacuum energy and missing energy, causes the universe to pass through an accelerating phase. In particular if we take \(b = 0.75\) and \(\gamma = 0.95\), then ‘a’ will be 0.204166. This result matches with the experiment (Today, all most all determination of matter density are consistent with \(\rho_d/\rho_{\text{crit}} = a = 0.2 - 0.3\)). In our model, \((a + b) < 1\), so to reach total energy density = 1, one can assume that the displacement vector plays the role of additional energy density, which we have referred to as vacuum energy.

In cosmology there are many ways to specify the distance between two points, because in the expanding universe, the distance between comoving objects are constantly changing and Earth-bound observers look back in time as they look out in distance. The exact expression for the proper distance, luminosity distance red-shift, angular diameter distance red-shift and look back time red-shift for the model are presented in sections 4 to 7, in the framework of Lyra geometry. Beside, the implication of Lyra’s geometry for astrophysical interesting bodies is still an open question. The problem of equations of motion and gravitational radiation need investigation.

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