Abstract

Our aim is to study existence, uniqueness and the limit, as $m \to \infty$, of the solution of reaction-diffusion porous medium equation with linear drift $\partial_t u - \Delta u^m + \nabla \cdot (u \mathbf{V}) = g(t, x, u)$ in bounded domain with Dirichlet boundary condition. We treat the problem without any sign restriction on the solution with an outpointing vector field $\mathbf{V}$ on the boundary and a general source term $g$ (including the continuous Lipschitz case). By means of new $BV_{\text{loc}}$ estimates in bounded domain, under reasonably sharp Sobolev assumptions on $\mathbf{V}$, we show uniform $L^1$-convergence towards the solution of reaction-diffusion Hele-Shaw flow with linear drift.

MSC2020 database: 35A01, 35A02, 35B20, 35B35

1 Introduction and main results

1.1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with regular boundary $\partial \Omega =: \Gamma$. Our aim here is to study the limit, as $m \to \infty$, of the equation

\begin{equation}
\frac{\partial u}{\partial t} - \Delta u^m + \nabla \cdot (u \mathbf{V}) = g(t, x, u) \quad \text{in} \quad Q := (0, T) \times \Omega,
\end{equation}

where the expression $r^m$ denotes $|r|^{m-1} r$, for any $r \in \mathbb{R}$, $1 < m < \infty$, $V : \Omega \to \mathbb{R}^N$ is a given vector field and $g : Q \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory application.

There is a huge literature on qualitative and quantitative studies of (1.1) in the case where $V \equiv 0$. We refer the reader to the book [10] for a thoroughgoing survey of results as well as corresponding literature. In the case $V \neq 0$, the PDE is a nonlinear version of Fokker-Planck equation of porous-media type. This kind of evolutionary problem has gained a significant attention in recent years. It arises mainly in biological applications and in the theory of population dynamics. Here $u$ represents density of agents traveling following a vector field $V$ and subject to some local random motion; i.e. random exchanges at a microscopic level between the agent at a given position and neighborhoods positions.
(on can see for instance \[15, 40, 41, 42, 42, 45, 17, 26\] and the refs therein for more complement on the applications and motivations).

Despite the broad results on nonlinear diffusion-transport PDE (cf. \[2, 3, 4, 17, 35, 44\], see also the expository paper \[5\] for a complete list), the structure of (1.1) where the drift depends linearly on the density excludes this class definitely out the scope of the current literature. As far as we know, the study of existence and uniqueness of weak solution (1.1) has been investigated only for the case of one population in the conservative case leading to one phase problem in \(\mathbb{R}^N\) or else in bounded domain with Neumann boundary condition (cf. \[15, 26\] in the case \(V = \nabla \varphi\) with reasonable assumptions on the potential \(\varphi\) and \(g \equiv 0\)). One can see also \[36\] for the study in the framework of viscosity solutions.

Asymptotic convergence to equilibrium is shown in \[15\] and \[19\] when \(\varphi\) is convex. For the regularity of the solution one can see \[22\] and the references therein.

In this paper, we focus chiefly on the case of bounded domain with Dirichlet boundary condition to study existence and uniqueness of a weak solution of the general formulation (1.1) as well as its limit, as \(m \to \infty\), under general reasonable assumptions on \(g\) and \(V\). See that the diffusion term \(-\Delta u^m\) may be written as \(-\nabla \cdot \left( u \frac{m}{m-1} u^{m-1} \right)\), so that the exponent \(m > 1\) manage in some sense the mobility of the agent through a "mobility potential" given by \(\frac{m}{m-1} u^{m-1}\). For large \(m\), this term becomes

\[
\frac{m}{m-1} u^{m-1} \approx \begin{cases} 
0 & \text{if } |u| < 1 \\
+\infty & \text{if } |u| > 1.
\end{cases}
\]

This formal analysis setup that the limiting density \(u\) is restrained to satisfy \(|u| \leq 1\) within two main phases: the so called congestion phase which corresponds to \(|u| = 1\) and a free one corresponding to \(|u| > 0\). More precisely, the limiting PDE system coincides at least formally with a density constrained diffusion equation with a linear drift

\[
\frac{\partial u}{\partial t} - \nabla \cdot (u \nabla |p|) + \nabla \cdot (u V) = g(t, x, u) \quad \text{in } Q,
\]

where we denote by \(\text{sign}\) the maximal monotone graph given by

\[
\text{sign}(r) = \begin{cases} 
1 & \text{for } r > 0 \\
[-1, 1] & \text{for } r = 0 \\
-1 & \text{for } r < 0.
\end{cases}
\]

See that \(\nabla \cdot (u \nabla |p|) = \Delta p\), so that (1.1) is a reaction-diffusion system of Hele-Shaw type with a linear drift. The emergence of density constraint \(|u| \leq 1\) is closely connected to the microscopic non-overlapping constraint between the agents in the limiting case. The complementary condition between the density \(u\) and the limiting mobility potential \(p\) typically allows to describe the motion of congested zones definitely characterized by \(|p| \neq 0\). This equation appears in pedestrian flow (cf. \[40\]) and in biological applications (cf. \[18\] and the references therein). The study of the problem without any sign restriction on the solution enables especially to cover mathematical models of two-species in interactions and occupying the same habitat, like diffusion-aggregation models. In this cases, \(\rho\) represents through its positive and negative parts the densities of each specie respectively. The source term \(g\) models reaction phenomena connected to agent supply in biological models. This happens in particular.
when one deals with reaction diffusion system coupling the equations (1.1) or (1.1) with other PDE. As to the boundary condition, homogeneous Dirichlet one is connected to the possibility of mobility through the boundary (exits) without any charge. One can see [28] for other possibilities of boundary condition and their interpretation.

In this paper, we give the proofs of existence, uniqueness and convergence process to Hele-Shaw flow with linear drift in the general context of $L^1$–theory for nonlinear PDE. The approach differs quite significantly from other recent papers which treats the problem in $\mathbb{R}^N$ (cf. [23, 25, 38]) in the one phase case by using mainly classical Aronson-Bénilan estimate (cf. [6]) for nonnegative solutions of porous medium equation.

In particular, our approach enables to give answers and evidence to many questions left open in many papers dedicated to this subject. Actually, we treat the problem without any sign restriction on the solution in a bounded domain with Dirichlet boundary condition, low regularity on $V$, and general source term $g$. Moreover, the approach offers many supply for the treatment of the challenging case of non compatible initial data; i.e. the case where $\|u_0\| > 1$. This will be treated separately in the forthcoming work [34].

1.2 Historical notes

The study of the incompressible limit of (1.2) received a lot of attention since its interest for the applications and for the description of constrained nonlinear flow. The problem is well understood by now in the case where $V$ and $g$ vanish (see for instance [10] and [9]). One can see also [33] for non-homogeneous Neumann boundary condition and [29] for non-homogeneous Dirichlet one. In the case where $V \equiv 0$ and $g \not\equiv 0$, it is know (see [12] for Dirichlet boundary condition and [13] for Neuman boundary condition) that the solution of the problem

$$\frac{\partial u}{\partial t} - \Delta u^m = g(\cdot, u) \quad \text{in } Q,$$

converges, as $m \to \infty$, to the solution of the so called Hele-Shaw problem

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} - \Delta p = g(\cdot, u) \\ u \in \text{sign}(p) \end{array} \right\} \quad \text{in } Q.$$

The convergence holds to be true in $C([0,T), L^1(\Omega))$ in the case where $|u_0| \leq 1$, a.e. in $\Omega$, otherwise it holds in $C((0,T), L^1(\Omega))$ and a boundary layer appears for $t = 0$. This boundary layer is given by the plateau-like function refereed to as ‘mesa’, and it is given by the limit, as $m \to \infty$, of the solution of homogeneous porous medium equation

$$\frac{\partial u}{\partial t} = \Delta u^m \quad \text{in } Q.$$

Yet, one needs to be careful with the special case of Neumann boundary condition since, in this case the limiting problem (1.2) could be ill posed. With respect to the assumptions on $g$, the limiting problem exhibits an extra phase to be mixed with the Hele-Shaw phase (see [13] for more details). Other variations of reaction term have been proposed in recent years together with the analysis of their incompressible limit (see for instance [45, 24, 43, 37] and the references therein). The recent work...
[30] treats the particular case of linear reaction term with a special focus on the limit of the so called associated pressure \( p := \frac{m}{m+1} u^{m-1} \), furthermore the authors seem to be altogether not aware of the general works [12, 13].

The treatment of the case where \( V \not\equiv 0 \), leads to the formal reaction-diffusion dynamic of Hele-Shaw type with a linear drift; i.e.

\[
\begin{align*}
\frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) &= g(t, x, u) \\
u &\in \text{sign}(p)
\end{align*}
\]

(1.5) \quad \text{in} \ (0, T) \times \Omega.

The problem was studied first in [14] when \( g \equiv 0 \) and the drift term is of the type \( \nabla \cdot F(u) \), with \( F : \mathbb{R} \to \mathbb{R}^N \) a Lipschitz continuous function (this corresponds particularly to space-independent drift). In [14], it is proven that \( L^1(\Omega) \)-compactness result remains to be true uniformly in \( t \). Moreover, the limiting problem here is simply the transport equation

\[
\partial_t u + \nabla \cdot F(u) = 0.
\]

The Hele-Shaw flow desappears since the nature of the transport term (incompressible) in [12] compel the solution to be less than 1, and then \( p \equiv 0 \). Then, in [38] the authors studied the case of space dependent drift and reaction terms both linear and regular in \( \Omega = \mathbb{R}^N \). Assuming a monotonicity on \( V \), and using the notion of viscosity solutions, the authors study the limit of nonnegative solution, as \( m \to \infty \). The benefit of this approach is its ability to cover accurately the free boundary view of the limiting problem (particularly the dynamic of the so called congestion region \( p > 0 \)), as well as the rate of convergence. Using a weak (distributional) interpretation of the solution the same problem was studied recently in [23] with a variant of reaction term \( g \) in \( \Omega = \mathbb{R}^N \). Using a blend of recently developed tools on Aronson-Bénilan regularizing effect as well as sophisticated \( L^p \)-regularity of the pressure gradient the authors studied the incompressible limit in the case of nonnegative compatible initial data and regular drift (one can see also [25] for some convergence rate in a negative Sobolev norm).

Here, we study the incompressible limit of (1.2) subject to Dirichlet boundary condition and compatible initial data (even changing sign data). The reaction term satisfies general conditions, including Lipschitz continuous assumptions, and the given velocity field enjoys Sobolev regularity and an outward pointing condition on the boundary that we’ll precise after. To this aim we use \( L^1 \)-nonlinear semigroup theory, more or less, in the same spirit of the approach of Bénilan and Crandall [10]. This consists in performing first the \( L^1 \)-strong compactness for the stationary problem and work with the general theory of nonlinear semigroup to pass to the limit in the evolution problem. The \( L^1 \)-compactness enroll a new \( BV_{loc} \)-estimate we perform for the stationary problem in bounded domain with reasonable assumptions on \( V \) in the neighborhood of the boundary.

### 1.3 Existence and uniqueness results

We assume that \( \Omega \subset \mathbb{R}^N \) is a bounded open set, with regular boundary \( \partial \Omega \) (say, piecewise \( C^2 \)). Throughout the paper, we assume that \( V \in W^{1,2}(\Omega) \), \( \nabla \cdot V \in L^\infty(\Omega) \) and satisfies the following outward pointing condition on the boundary:

\[
V \cdot \nu \geq 0 \quad \text{on} \ \partial \Omega,
\]

(1.6)
where \( \nu \) represents the outward unitary normal to the boundary \( \partial \Omega \). Notice here that this condition is fundamental in the case of Dirichlet boundary condition. This assumption is natural in many applications. Even if it looks alike to be stronger, it is fundamental for the uniqueness of weak solution for the limiting problem. A counter example for uniqueness of weak solutions for a Hele-Shaw problem is given in [33] whenever this condition is not fulfilled.

See here that \( V \cdot \nu \in H^{-\frac{1}{2}}(\partial \Omega) \), so that (1.3) needs to be understood a priori in a weak sense; i.e.

\[
\int_{\Omega} V \cdot \nabla \xi \, dx + \int_{\Omega} \nabla \cdot V \xi \, dx \geq 0, \quad \text{for any } 0 \leq \xi \in H^1(\Omega).
\]

To deal with this assumption, we operate technically with the euclidean distance-to-the-boundary function \( d(\cdot, \partial \Omega) \). For any \( h > 0 \), we denote by

\[
\xi_h(t, x) = \frac{1}{h} \min \left\{ h, d(x, \partial \Omega) \right\} \quad \text{and} \quad \nu_h(x) = -\nabla \xi_h, \quad \text{for any } x \in \Omega.
\]

We see that \( \xi_h \in H^1_0(\Omega) \), \( 0 \leq \xi_h \leq 1 \) in \( \Omega \) and

\[
\nu_h(x) = -\frac{1}{h} \nabla d(\cdot, \partial \Omega), \quad \text{for any } x \in \Omega \setminus \Omega_h =: D_h \text{ and } 0 < h \leq h_0 \text{ (small enough)},
\]

where

\[
\Omega_h = \left\{ x \in \Omega : d(x, \partial \Omega) > h \right\}, \quad \text{for small } h > 0.
\]

In particular, for any \( x \in \Omega_h \), we have

\[
\nu_h(x) = \frac{1}{h} \nu(\pi(x)),
\]

where \( \pi(x) \) design the projection of \( x \) on the boundary \( \partial \Omega \). Thanks to (1.3), we see that

\[
\liminf_{h \to 0} \int_{\Omega \setminus \Omega_h} \xi V(x) \cdot \nu_h(x) \, dx \geq 0, \quad \text{for any } 0 \leq \xi \in H^1(\Omega).
\]

Nevertheless, to avoid much more technicality in the proof of of uniqueness, we’ll assume that \( V \) satisfies (1.3) for any \( 0 \leq \xi \in L^2(\Omega) \) (cf. Remark [5]). We do not know if this is a consequence of the assumption (1.3). Anyway, this condition remains be to true for a large class of practical situations and implies necessarily (1.3).

**Remark 1.** Remember that, thanks to the local \( C^2 \)-boundary regularity assumption on \( \Omega \), for any \( 0 \leq \Phi \in H^1_0(\Omega) \), we have

\[
\liminf_{h \to 0} \int_{\Omega} \nabla \Phi \cdot \nabla \xi_h \, dx \leq 0.
\]

For the case of Lipschitz boundary domain, one needs to work with more sophisticated test functions in the spirit of \( \xi_h \) to fill (1) like property (one can see Lemma 4.4 and Remark 6.5 of [4]). So, typical examples of vector fields \( V \) which satisfies (1.3) may be given by

\[
V = -\nabla \Phi \quad \text{and} \quad 0 \leq \Phi \in H^1_0(\Omega) \cap W^{2,2}(\Omega).
\]

Here \( H^1_0(\Omega) \) denotes the usual Sobolev space

\[
H^1_0(\Omega) = \left\{ u \in H^1(\Omega) : u = 0, \ \mathcal{L}^{N-1}.a.e. \text{ in } \partial \Omega \right\}.
\]
We consider the evolution problem

\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u + \nabla \cdot (uV) = f & \text{in } Q := (0, T) \times \Omega \\
u = 0 & \text{on } \Sigma := (0, T) \times \partial \Omega \\
u(0) = u_0 & \text{in } \Omega.
\end{cases}
\] (1.11)

**Definition 1.1** (Notion of solution). A function \( u \) is said to be a weak solution of (1.3) if \( u \in L^2(Q), \ p := u^m \in L^2 \left(0, T; H^1_0(\Omega)\right) \) and

\[
d\frac{d}{dt} \int_\Omega u \xi + \int_\Omega (\nabla p - uV) \cdot \nabla \xi = \int_\Omega f \xi, \quad \forall \xi \in H^1_0(\Omega).
\]

We'll say plainly that \( u \) is a solution of (1.3) if \( u \in C([0, T], L^1(\Omega)) \), \( u(0) = u_0 \) and \( u \) is a weak solution of (1.3).

We denote by \( \text{sign}^+ \) (resp. \( \text{sign}^- \)) the maximal monotone graph given by

\[
\text{sign}^+(r) = \begin{cases}
1 & \text{for } r > 0 \\
[0, 1] & \text{for } r = 0 \\
0 & \text{for } r < 0.
\end{cases}
\]

(resp. \( \text{sign}^-(r) = \text{sign}^+( -r ) \), for \( r \in \mathbb{R} \)).

Moreover, we denote by \( \text{sign}_0^+ \) the discontinuous applications defined from \( \mathbb{R} \) to \( \mathbb{R} \) by

\[
\text{sign}_0^+(r) = \begin{cases}
1 & \text{for } r > 0 \\
0 & \text{for } r \leq 0
\end{cases}
\]

and \( \text{sign}_0^-(r) = \text{sign}_0^+( -r ) \), for \( r \in \mathbb{R} \).

As we said above, to avoid more technicality of the proofs of existence and uniqueness of a weak solution, we assume throughout this section that \( V \) satisfies the following outpointing condition on the boundary:

\[
\liminf_{h \to 0} \frac{1}{h} \int_{\Omega \setminus \Omega_h} \xi V(x) \cdot \nu(\pi(x)) \, dx \geq 0, \quad \text{for any } 0 \leq \xi \in L^2(\Omega).
\] (1.12)

**Theorem 1.1.** If \( u_1 \) and \( u_2 \) are two weak solutions of (1.3) associated with \( f_1, f_2 \in L^1(Q) \) respectively, then there exists \( \kappa \in L^\infty(Q) \) such that \( \kappa \in \text{sign}^+(u_1 - u_2) \) a.e. in \( Q \) and

\[
\frac{d}{dt} \int_\Omega (u_1 - u_2)^+ \, dx \leq \int_\Omega \kappa (f_1 - f_2) \, dx, \quad \text{in } \mathcal{D}'(0, T).
\] (1.13)

In particular, we have

1. \( \frac{d}{dt} \| u_1 - u_2 \|_1 \leq \| f_1 - f_2 \|_1, \text{ in } \mathcal{D}'(0, T). \)
2. If \( f_1 \leq f_2 \), a.e. in \( Q \), and \( u_1(0) \leq u_2(0) \) a.e. in \( \Omega \), then
\[
\begin{align*}
u_1 \leq u_2, \quad \text{a.e. in } Q.
\end{align*}
\]

**Theorem 1.2.** For any \( u_0 \in L^2(\Omega) \) and \( f \in L^2(Q) \), the problem (1.3) has a solution \( u \). Moreover, \( u \) satisfies the following:

1. For any \( q \in [1, \infty] \), we have
\[
\begin{align*}
\|u(t)\|_q \leq M_q := \begin{cases}
e^{(q-1)T \|\nabla \cdot V\|_{\infty}} \left( \|u_0\|_q + \int_0^T \|f(t)\|_q \, dt \right), & \text{if } q < \infty, \\
e^T \|\nabla \cdot V\|_{\infty} \left( \|u_0\|_{\infty} + \int_0^T \|f(t)\|_{\infty} \, dt \right), & \text{if } q = \infty.
\end{cases}
\end{align*}
\]

2. For any \( t \in [0, T) \), we have
\[
\begin{align*}
\frac{1}{m+1} \frac{d}{dt} \int_\Omega |u|^{m+1} \, dx + \int_\Omega |\nabla p|^2 \, dx \leq \int_\Omega f \, dx + \int_\Omega p \, (\nabla \cdot V)^+ \, dx, \quad \text{in } D'(0,T).
\end{align*}
\]

**Corollary 1.1.** For any \( 0 \leq u_0 \in L^2(\Omega) \) and \( 0 \leq f \in L^2(Q) \), the problem (1.3) has a unique nonnegative solution \( u \).

**Remark 2.** The assumption (1.3) is technical for the the proof of Theorem (1.1). This assumption is fulfilled for a large class of vector field \( V \), like for instance the case where \( V \) is outward pointing in a neighborhood of the boundary. We think that this condition could be removed if favor of merely (1.3) for instance if the solution \( \rho \) has a trace on the boundary (for instance if one works with BV solution). We postpone the technicality of this assumption in Remark 2 after the proof of Theorem 1.1.

### 1.4 Incompressible limit results

As we said in the introduction, as \( m \to \infty \), the problem (1.3) converges formally to so called Hele-Shaw problem
\[
\begin{align*}
\begin{cases} \\
\frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = f \\
u \in \text{sign}(p) \\
p = 0
\end{cases} \quad \text{in } Q \\
\text{on } \Sigma
\end{align*}
\]

Existence, \( L^1 \)–comparison and uniqueness of weak solution for the problem (1.4), with mixed boundary conditions, has been studied recently in [31] (see also [32]) under the assumption (1.3). Thanks to [31], we know that for any \( f \in L^2(Q) \) and \( u_0 \in L^\infty(\Omega) \), s.t. \( 0 \leq |u_0| \leq 1 \), a.e. in \( \Omega \), (1.4) has a unique weak solution (see the following Theorem for the precise sense) satisfying \( u(0) = u_0 \).

To prove rigorously the convergence of \( u_m \) to the solution of (1.4), we assume moreover that \( V \) satisfies the following assumption : there exists \( h_0 > 0 \), such that for any \( 0 < h < h_0 \), we have
(1.17) \[ V(x) \cdot \nu(\pi(x)) \geq 0, \quad \text{for a.e. } x \in D_h. \]

This may be written also as \( V(x) \cdot \nabla d(x, \partial \Omega) \leq 0 \), for any \( x \in \Omega \) being such that \( d(x, \partial \Omega) < h_0 \). See that the condition (1.4) implies definitely (1.3). In fact, with (1.4), we are assuming that (1.17) \[ V \]

This may be written also as (1.17) \[ V \]

outpointing along the paths given by the distance function in a neighborhood of \( \partial \Omega \). As we will see, this assumption can be weaken into an outpointing vector field condition along a given arbitrary paths in the neighborhood of \( \partial \Omega \) (cf. Remark 7). Nevertheless, a control of the outpointing orientation of \( V \) in the neighborhood of the boundary seems to be important in order to handle the oscillation of \( u_m \) and establish BV_{loc}-estimate.

**Theorem 1.3.** Under the assumption (1.4), for each \( m = 1, 2, \ldots \), we consider \( u_{0m} \in L^2(\Omega), f_m \in L^2(Q) \) and \( u_m \) be the corresponding solution of (1.3). If, as \( m \to \infty, f_m \to f \) in \( L^1(Q), u_{0m} \to u_0 \) in \( L^1(\Omega) \), and \( |u_0| \leq 1 \), then

\[ u_m \to u, \quad \text{in } C([0, T); L^1(\Omega)), \]

\[ u_m^m \to p, \quad \text{in } L^2([0, T); H_0^1(\Omega))-\text{weak}, \]

and \((u, p)\) is the unique solution of (1.4) satisfying \( u(0) = u_0 \). That is \( u \in C([0, T), L^1(\Omega)), u(0) = u_0, u \in \text{sign}(p) \), a.e. in \( Q \), and

\[ \frac{d}{dt} \int_\Omega u \xi + \int_\Omega \nabla p \cdot \nabla \xi \, dx - \int_\Omega u V \cdot \nabla \xi \, dx = \int_\Omega f \xi \, dx, \quad \text{in } \mathcal{D}'([0, T)), \quad \text{for any } \xi \in H_0^1(\Omega). \]

The results for the case where \( f \) is given by a reaction term \( g(., u) \), are develop in Section 5.

**Remark 3.** Thanks to [31], we can deduce that \( u \), the limit of \( u_m \), satisfies the following:

1. If there exists \( \omega_1, \in W^{1,1}(0, T) \) (resp. \( \omega_2 \in W^{1,1}(0, T) \)) such that \( u_0 \leq \omega_2(0) \) (resp. \( \omega_1(0) \leq u_0 \)) and, for any \( t \in (0, T) \),

\[ \omega_2(t) + \omega_2(t) \nabla \cdot V \geq f(t, .) \quad \text{a.e. in } \Omega \]

(rep. \( \omega_1(t) + \omega_1(t) \nabla \cdot V \leq f(t, .) \), a.e. in \( \Omega \), then we have

\[ u \leq \omega_2 \quad \text{(resp. } \omega_1 \leq u) \quad \text{a.e. in } Q. \]

2. If \( f \) and \( V \) satisfies

\[ 0 \leq f \leq \nabla \cdot V, \quad \text{a.e. in } Q \]

then \( p \equiv 0 \), and \( u \) is the unique solution of the reaction-transport equation

\[
\begin{aligned}
&\frac{\partial u}{\partial t} + \nabla \cdot (u V) = f \\
\quad \text{in } Q \\
&0 \leq u \leq 1 \\
&u V \cdot \nu = 0 \\
&u(0) = u_0
\end{aligned}
\]

on \( \Sigma_N \)

in the sense that \( u \in C([0, T), L^1(\Omega)), 0 \leq u \leq 1 \) a.e. in \( Q \) and

\[ \frac{d}{dt} \int_\Omega u \xi - \int_\Omega u V \cdot \nabla \xi = \int_\Omega f \xi, \quad \text{in } \mathcal{D}'(0, T), \quad \forall \xi \in H_0^1(\Omega). \]

8
Remark 4. One sees that the assumption (1.4) is fulfilled for instance in the following cases:

1. $V$ satisfies (1.3), and there exists $h_0 > 0$ such that, for any $0 < h < h_0$, we have

\[ V(x) = V(\pi(x)), \quad \text{for any } x \in \Omega_h. \]

Indeed, since $\nu_h(x) = \nu(\pi(x))$, we have $V(x) \cdot \nabla \xi_h(x) = V(\pi(x)) \cdot \nabla \xi(\pi(x))$ which is nonnegative by the assumption (1.3).

2. $V$ compactly supported; i.e. $V$ vanishes on a neighbor of the boundary $\partial \Omega$.

Lemma 1.1. Under the assumption (1.4), there exists $h_0 > 0$, such that for any $0 < h < h_0$, there exists $0 \leq \omega_h \in C^2(\Omega_h)$ compactly supported in $\Omega$, such that $\omega_h \equiv 1$ in $\Omega_h$ and

\[ \int_{\Omega \setminus \Omega_h} \varphi V \cdot \nabla \omega_h \, dx \leq 0, \quad \text{for any } 0 \leq \varphi \in L^1(\Omega). \]

1.5 Plan of the paper

The next section is devoted to the proof of $L^1$–comparison principle for weak solutions of (1.3). To this aim, we use doubling and dedoubling variables techniques. This enables us to deduce the uniqueness and lay out the study plan of the equation in the framework of $L^1$–nonlinear semi-group theory. Section 3 concerns the study of existence of a solution. To set the problem in the framework of nonlinear semi group theory, we begin with stationary problem to operate the Euler-implicit discretization and construct an $\varepsilon$–approximate solution $u_\varepsilon$. Then, using mainly a Crandall-Ligget type theorem, $L^2(\Omega)$ and $H^1(\Omega)$ estimates on $u_\varepsilon$ and $u_m^\varepsilon$ respectively, we pass to the limit as $\varepsilon \to 0$, to built $u_m$ the solution of the evolution problem (1.3). Section 4 is devoted to the study of the limit as $m \to \infty$. Using the outpointing vector filed condition (1.4), we study first the limit for the stationary problem connecting it to the Hele Shaw flow with linear drift. To this aim, we establish $BV_{\text{loc}}$ new estimates for weak solutions in bounded domain. Then, using regular perturbation results for nonlinear semi group we establish the convergence results for the evolution problem. Section 6 is devoted to the study of the limit of the solution $u$ and $u_m^\varepsilon$ in the of the presence of a reaction term with linear drift. We prove the convergence of reaction diffusion problem of a Hele-Shaw flow with linear drift At last, in Section 7 (Appendix), we provide for the unaccustomed reader a short recap on the main tools from $L^1$–nonlinear semi-group theory.

2 $L^1$–comparison principle and uniqueness proofs

As usual for parabolic-hyperbolic and elliptic-hyperbolic problems, the main tool to prove the uniqueness is doubling and de-doubling variables. To this aim, we prove first that a weak solution satisfies the following version of entropic inequality:

We assume throughout this section that $V \in W^{1,2}(\Omega), (\nabla \cdot V) \in L^\infty(\Omega)$ and $V$ satisfies the outpointing condition (1.3).
Proposition 2.1. Let \( f \in L^1(Q) \) and \( u \) be a weak solution of (1.3). Then, for any \( k \in \mathbb{R} \), and \( 0 \leq \xi \in H^1_0(\Omega) \cap L^\infty(\Omega) \), we have

\[
\frac{d}{dt} \int_{\Omega} (u - k)^+ \xi \, dx + \int_{\Omega} (\nabla (u^m - k^m)^+ - (u - k)^+V) \cdot \nabla \xi \, dx \\
+ \int_{\Omega} (k \nabla \cdot V - f) \xi \, dx \leq -\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{[0 \leq u^m - k^m \leq \varepsilon]} |\nabla u^m|^2 \xi \, dx,
\]

and

\[
\frac{d}{dt} \int_{\Omega} (k - u)^+ \xi \, dx + \int_{\Omega} (\nabla (k^m - u^m)^+ - (k - u)^+V) \cdot \nabla \xi \, dx \\
+ \int_{\Omega} (f - k \nabla \cdot V) \xi \, dx \leq -\limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{[0 \leq k^m - u^m \leq \varepsilon]} |\nabla u^m|^2 \xi \, dx,
\]

in \( \mathcal{D}'(0, T) \).

Proof. We extend \( u \) onto \( \mathbb{R} \times \Omega \) by 0 for any \( t \notin (0, T) \). Then, for any \( h > 0 \) and nonnegative \( \xi \in H^1_0(\Omega) \) and \( \psi \in \mathcal{D}(0, T) \), we consider

\[
\Phi_h(t, x) = \xi(x) \frac{1}{h} \int_t^{t+h} \mathcal{H}_\sigma^+(u^m(s, x)) \psi(s) \, ds,
\]

for a.e. \( x \in \Omega \), where we extend \( \psi \) onto \( \mathbb{R} \) by 0, and \( \mathcal{H}_\sigma^+ \) is given by

\[
\mathcal{H}_\sigma^+(r) = \min \left( \frac{(r - k^m)^+}{\varepsilon}, 1 \right), \quad \text{for any } r \in \mathbb{R},
\]

for arbitrary \( \sigma > 0 \). It is clear that \( \Phi_h \in W^{1,2}(0, T; H^1_0(\Omega)) \cap L^\infty(Q) \) is an admissible test function for the weak formulation, so that

\[
-\int_Q u \partial_t \Phi_h \, dt \, dx + \int_Q (\nabla u^m - V u) \cdot \nabla \Phi_h \, dt \, dx = \int_Q f \Phi_h \, dt \, dx.
\]

See that

\[
\int_Q u \partial_t \Phi_h \, dt \, dx = \int_Q \psi(t) \mathcal{H}_\sigma^+(u^m(t)) \frac{u(t-h) - u(t)}{h} \xi \, dt \, dx \\
\leq \frac{1}{h} \int_Q \psi(t) \left( \int_{u(t)}^{u(t-h)} \mathcal{H}_\sigma^+(r^m) \, dr \right) \xi \, dt \, dx \\
\leq \frac{1}{h} \int_Q \left( \int_k^{u(t)} \mathcal{H}_\sigma^+(r^m) \, dr \right) (\psi(t+h) - \psi(t)) \xi \, dt \, dx.
\]

Letting \( h \to 0 \), we have

\[
\limsup_{h \to 0} \int_Q u \partial_t \Phi_h \, dt \, dx \leq \int_Q \left( \int_k^{u(t)} \mathcal{H}_\sigma^+(r^m) \, dr \right) \partial_t \psi \xi \, dt \, dx.
\]
So, by letting $h \to 0$ in (2), we get

\[
- \int_{Q} \left\{ \left( \int_{k}^{u(t)} \mathcal{H}_{\sigma}^{+}(r^{m})dr \right) \partial_{t}\psi \xi + \psi \nabla u^{m} \cdot \nabla \mathcal{H}_{\sigma}^{+}(u^{m})\xi - \mathcal{H}_{\sigma}^{+}(u^{m})(u - k) V \cdot \nabla \xi \right\} dtdx
\]

\[
\leq \int_{Q} \left\{ \psi (f + k \nabla \cdot V) \mathcal{H}_{\sigma}^{+}(u^{m})\xi + \psi (u - k) V \cdot \nabla \mathcal{H}_{\sigma}^{+}(u^{m}) \right\} dtdx
\]

\[
- \frac{1}{\sigma} \int_{\{0 \leq u^{m} - km \leq \sigma\}} |\nabla u^{m}|^{2} \xi dtdx,
\]

where we use the fact that $\nabla u^{m} \cdot \nabla \mathcal{H}_{\sigma}^{+}(u^{m}) = \frac{1}{\sigma} |\nabla u^{m}|^{2} \chi_{\{0 \leq u^{m} - km \leq \sigma\}} \text{ a.e. in } Q$. Setting

\[
\Psi_{\sigma} := \frac{1}{\sigma} \int_{\min(u^{m}, km)} \xi \left( \frac{r^{1/m}}{m} - k \right) dr,
\]

we see that

\[
(u - k) \mathcal{H}_{\sigma}'(u^{m} - km) \cdot \nabla u^{m} = \nabla \Psi_{\sigma}.
\]

This implies that the last term of (2) satisfies

\[
\int_{Q} \psi \xi (u - k) V \cdot \nabla \mathcal{H}_{\sigma}^{+}(u^{m} - km) = \int_{Q} \psi (u - k) \mathcal{H}_{\sigma}^{+}(u^{m} - km) V \cdot \nabla u^{m}
\]

\[
= \int_{Q} \psi \xi V \cdot \nabla \Psi_{\sigma} dx
\]

\[
= - \int_{Q} \psi \nabla \cdot (\xi V) \Psi_{\sigma} dx
\]

\[
\to 0, \quad \text{as } \sigma \to 0.
\]

See also that, by using Lebesgue’s dominated convergence Theorem, we have

\[
\limsup_{\varepsilon \to 0} \int_{Q} \left( \int_{k}^{u(t)} \mathcal{H}_{\sigma}^{+}(r^{m})dr \right) \partial_{t}\psi \xi = \int_{Q} (u(t) - k)^{+} \partial_{t}\psi \xi.
\]

Then, letting $\varepsilon \to 0$ in (2) and using the fact that $\text{sign}_{0}^{+}(u^{m} - km) = \text{sign}_{0}^{+}(u - k)$, for any $k \in \mathbb{R}$, we get (2.1). As to (2.1), it follows by using the fact that $-u$ is also a solution of (1.3) with $f$ replaced by $-f$, and applying (2.1) to $-u$.

**Proposition 2.2** (Kato’s inequality). If $u_{1}$ and $u_{2}$ satisfy (2.1) and (2.1) corresponding to $f_{1} \in L^{1}(Q)$ and $f_{2} \in L^{1}(Q)$ respectively, then

\[
\partial_{t}(u_{1} - u_{2})^{+} - \Delta(u_{1}^{m} - u_{2}^{m})^{+} + \nabla \cdot ((u_{1} - u_{2})^{+} V) \leq (f_{1} - f_{2}) \text{sign}_{0}^{+}(u_{1} - u_{2}) \text{ in } D'(Q).
\]
Proof. The proof of this lemma is based on doubling and de-doubling variable techniques. Let us give here briefly the arguments. To double the variables, we use first the fact that $u_1 = u_1(t, x)$ satisfies (2.1) with $k = u_1(s, y)$, we have

\[
\frac{d}{dt} \int (u_1(t, x) - u_2(s, y))^+ \zeta \, dx + \int (\nabla_x (u_1^m(t, x) - u_2^m(s, y))^+ - (u_1(t, x) - u_2(s, y))^+ \cdot V(x) \cdot \nabla_x \zeta \, dx
\]

\[
+ \int \nabla_x \cdot V \cdot u_2(s, y) \zeta \text{sign}_0^+(u_1(t, x) - u_2(s, y)) \, dx \leq \int f_1(t, x) \text{sign}_0^+(u_1(t, x) - u_2(s, y)) \zeta \, dx
\]

\[
- \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} |\nabla_x u_1^m(t, x)|^2 \zeta \, dx.
\]

Integrating with respect to $y$, we get

\[
\frac{d}{dt} \int \int (u_1(t, x) - u_2(s, y))^+ \zeta + \int (\nabla_x (u_1^m(t, x) - u_2^m(s, y))^+ - (u_1(t, x) - u_2(s, y))^+ \cdot V(x) \cdot \nabla_x \zeta
\]

\[
+ \int \nabla_x \cdot V \cdot u_2(s, y) \zeta \text{sign}_0^+(u_1(t, x) - u_2(s, y)) \leq \int \int f_1(t, x) \text{sign}_0^+(u_1(t, x) - u_2(s, y)) \zeta
\]

\[
- \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} |\nabla_x u_1^m(t, x)|^2 \zeta \, dx.
\]

See that

\[
\int \int \nabla_y (u_1^m(t, x) - u_2^m(s, y))^+ \cdot \nabla_x \zeta \, dxdy = - \lim_{\varepsilon \to 0} \int \int \nabla_y u_1^m(s, y) \cdot \nabla_x \zeta \cdot H_\varepsilon(u_1^m(t, x) - u_2^m(s, y)) \, dxdy
\]

\[
= - \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} \nabla_x u_1^m(t, x) \cdot \nabla_y u_2^m(s, y) \zeta \, dxdy,
\]

so that, denoting by

\[
u(t, s, x, y) = u_1(t, x) - u_2(s, y), \quad \text{and} \quad p(t, s, x, y) = u_1^m(t, x) - u_2^m(s, y),
\]

we obtain

\[
\frac{d}{dt} \int \int u(t, s, x, y)^+ \zeta \, dxdy + \int \int \left\{ (\nabla_x + \nabla_y) p(t, s, x, y) - u(t, s, x, y)^+ \cdot V(x) \right\} \cdot \nabla_x \zeta \, dxdy
\]

\[
+ \int \int \nabla_x \cdot V \cdot u_2(s, y) \zeta \text{sign}_0^+ u(t, s, x, y) \, dxdy \leq \int \int f_1(t, x) \text{sign}_0^+ u(t, s, x, y) \zeta \, dxdy
\]

\[
- \lim_{\varepsilon \to 0} \int \int \nabla_x u_1^m(t, x) \cdot \nabla_y u_2^m(s, y) \zeta \cdot H_\varepsilon(u_1^m(t, x) - u_2^m(s, y)) \, dxdy
\]

\[
- \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} |\nabla_x u_1^m(t, x)|^2 \zeta \, dxdy.
\]

On the other hand, using the fact that $u_2 = u_2(s, y)$ satisfies (2.1) with $k = u_1(t, x)$, we have

\[
\frac{d}{ds} \int u(t, s, x, y)^+ \zeta \, dy + \int \nabla_y p(t, s, x, y) - u(t, s, x, y)^+ \cdot V(y) \cdot \nabla_y \zeta
\]

\[
- \int \nabla y \cdot V \cdot u_1(t, x) \zeta \text{sign}_0^+ (u(t, s, x, y)) \, dy \leq - \int f_2(s, y) \text{sign}_0^+ (u(t, s, x, y)) \zeta \, dy
\]

\[
- \limsup_{\varepsilon \to 0} \frac{1}{\varepsilon} \int \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} ||\nabla_y u_2^m(s, y)||^2 \zeta \, dy.
\]
Working in the same way for (2), we get
\[
\begin{align*}
\frac{d}{ds} & \int \int u(t, s, x, y) + \zeta \, dx \, dy + \int \int (\nabla_x + \nabla_y)p(t, s, x, y) - u(t, s, x, y) + V(y) \right) \cdot \nabla_y \zeta \, dx \, dy \\
- \int \int \nabla_y \cdot V(y) u_1(t, x) \zeta \, dx \, dy & \leq - \int \int f_2(s, y) \text{sign}^+(u(t, s, x, y)) \zeta \, dx \, dy \\
- \lim_{\varepsilon \to 0} \int \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} \nabla_x u^m(t, x) \cdot \nabla_y u^m(s, y) \zeta \, dx \, dy \\
- \lim_{\varepsilon \to 0} \int \int_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} |\nabla_y u^m(s, y)|^2 \zeta \, dy \, dx.
\end{align*}
\]
Adding both inequalities, and using the fact that 
\[-(||\nabla u_1(t, x)||^2 + ||\nabla u_2(s, y)||^2 + 2 \nabla u_1(t, x) \cdot \nabla u_2(s, y)) \chi_{[0 \leq u_1^m - u_2^m \leq \varepsilon]} \leq 0, \text{ a.e. in } Q^2,\]
we obtain
\[
\begin{align*}
\left( \frac{d}{dt} + \frac{d}{ds} \right) & \int \int u(t, s, x, y) + \zeta \, dx \, dy + \int \int (\nabla_x + \nabla_y)p(t, s, x, y) \cdot (\nabla_x + \nabla_y)\zeta \, dx \, dy \\
- \int \int u(t, s, x, y) + V(x) \cdot \nabla_x \zeta + V(y) \cdot \nabla_y \zeta \, dx \, dy & + \int \int (\nabla_x \cdot V(x) u_2(s, y) - \nabla_y \cdot V(y) u_1(t, x) \right) \zeta \, dx \, dy \\
& \leq \int \int (f_1(t, x) - f_2(s, y)) \text{sign}^+(u(t, s, x, y)) \zeta \, dx \, dy
\end{align*}
\]
and then,
\[
\begin{align*}
\left( \frac{d}{dt} + \frac{d}{ds} \right) & \int \int u(t, s, x, y) + \zeta \, dx \, dy + \int \int (\nabla_x + \nabla_y)p(t, s, x, y) \cdot (\nabla_x + \nabla_y)\zeta \, dx \, dy \\
- \int \int u(t, s, x, y) + V(x) \cdot \nabla_x \zeta + V(y) \cdot \nabla_y \zeta \, dx \, dy & + \int \int (\nabla_x \cdot V(x) u_2(s, y) - \nabla_y \cdot V(y) u_1(t, x) \right) \zeta \, dx \, dy \\
& \leq \int \int (f_1(t, x) - f_2(s, y)) \text{sign}^+(u(t, s, x, y)) \zeta \, dx \, dy.
\end{align*}
\]
Now, we can de-double the variables \( t \) and \( s \), as well as \( x \) and \( y \), by taking as usual the sequence of test functions
\[
\psi_\varepsilon(t, s) = \psi \left( \frac{t + s}{2} \right) \rho_\varepsilon \left( \frac{t - s}{2} \right) \text{ and } \zeta_\lambda(x, y) = \xi \left( \frac{x + y}{2} \right) \delta_\lambda \left( \frac{x - y}{2} \right),
\]
for any \( t, s \in (0, T) \) and \( x, y \in \Omega \). Here \( \psi \in \mathcal{D}(0, T), \xi \in \mathcal{D}(\Omega), \rho_\varepsilon \text{ and } \delta_\lambda \text{ are sequences of usual mollifiers in } \mathbb{R} \text{ and } \mathbb{R}^N \text{ respectively. See that}
\[
\left( \frac{d}{dt} + \frac{d}{ds} \right) \psi_\varepsilon(t, s) = \rho_\varepsilon \left( \frac{t - s}{2} \right) \psi \left( \frac{t + s}{2} \right)
\]
and
\[
(\nabla_x + \nabla_y)\zeta_\lambda(x, y) = \delta_\lambda \left( \frac{x - y}{2} \right) \nabla \xi \left( \frac{x + y}{2} \right)
\]
Moreover, for any \( h \in L^1((0, T)^2 \times \Omega^2) \) and \( \Phi \in L^1((0, T)^2 \times \Omega^2)^N \), we know that
Thus the result of the proposition.

Moreover, we also know that

$$
\lim_{\lambda \to 0} \int_0^T \int_{\Omega} h(t, x, y) (V(x) - V(y)) \cdot \nabla_y \zeta_\lambda(x, y) \psi(t) \, dt \, dx
$$

The proof of this result is more or less well known by now (one can see for instance a detailed proof in [3]). So replacing $\zeta$ and $\psi$ in (2) by $\zeta_\lambda$ and $\psi_\varepsilon$ resp., and, letting $\varepsilon, \lambda \to 0$, we get

$$
\int_0^T \int_{\Omega} \left\{ -(u_1 - u_2)^+ \xi \dot{\psi} + \nabla(p_1 - p_2)^+ \cdot \nabla \xi \psi - (u_1 - u_2)^+ V \cdot \nabla \xi \psi \right\} \, dt \, dx
\leq \int_0^T \int_{\Omega} (f_1 - f_2) \operatorname{sign}_0^+(u_1 - u_2) \xi \psi \, dt \, dx.
$$

Thus

$$
\frac{d}{dt} \int (u_1 - u_2)^+ \xi \, dx + \int \nabla(u_1^m(t, x) - u_2^m(t, x))^+ \cdot \nabla \xi \, dx - \int (u_1 - u_2)^+ V \cdot \nabla \xi \, dx
\leq \int \kappa(x)(f_1 - f_2) \xi \, dx.
$$

Thus the result of the proposition.

The aim now is to process with the sequence of test function $\xi_h$ given by (1.3) in Kato’s inequality and let $h \to 0$, to cover (1.1).

**Proof of Theorem (1.1)** Let $(u_1, p_1)$ and $(u_2, p_2)$ be two couples of $L^\infty(Q) \times L^2(0, T; H^1_0(\Omega))$ satisfying (2.1) and (2.1) corresponding to $f_1 \in L^1(Q)$ and $f_2 \in L^1(Q)$ respectively, to prove (1.1) we see that

$$
\frac{d}{dt} \int (u_1 - u_2)^+ \xi \, dx - \int (f_1 - f_2) \operatorname{sign}_0^+(u_1 - u_2) \, dx
= \lim_{h \to 0} \frac{d}{dt} \int (u_1 - u_2)^+ \xi_h \, dx - \int (f_1 - f_2) \operatorname{sign}_0^+(u_1 - u_2) \xi_h \, dx,
$$

14
in the sense of distribution in \([0, T]\). Taking \(\xi_h\) as a test function in (2.2) and using (1), we have
\[
I(h) \leq - \int (\nabla (u_1^m - u_2^m)^+ - (u_1 - u_2)^+ V) \cdot \nabla \xi_h \, dx \\
\leq \int (u_1 - u_2)^+ V \cdot \nabla \xi_h \, dx.
\]
Then, using the outpointing velocity vector field assumption (1.3), we get
\[
\lim_{h \to 0} I(h) \leq - \lim_{h \to 0} \int (u_1 - u_2)^+ V \cdot \nu_h(x) \, dx \leq 0.
\]
Thus (1.1). The rest of the theorem is a straightforward consequence of (1.1).

Remark 5. See that (2) is the lonely step of the proof of Theorem 1.1 where we use assumption (1.3). Working so enables to avoid all the technicality related to doubling and de-doubling variable à la Carillo (17) by using test functions which do not vanish on the boundary. We do believe that the result of Theorem 1.1 remains to be true under the general assumption (1.3). One sees also that, if the solutions have a trace (like for \(BV\) solution), one can weaken this condition by handling (2) otherwise.

3 Main estimates and existence proofs

3.1 Stationary problem

To prove Theorem 1.2 we consider the stationary problem associated with Euler-implicit discretization of (1.3). That is
\[
\begin{aligned}
v - \lambda \Delta v^m + \lambda \nabla \cdot (v V) &= f &\text{in } \Omega \\
v &= 0 &\text{on } \partial \Omega,
\end{aligned}
\]
where \(f \in L^2(\Omega)\) and \(\lambda > 0\). Following Definition 1.1 a function \(v \in L^1(\Omega)\) is said to be a weak solution of (3.2) if \(v^m \in H^1_0(\Omega)\) and
\[
\int \Omega v \xi + \lambda \int \Omega \nabla v^m \cdot \nabla \xi - \lambda \int \Omega v V \cdot \nabla \xi = \int \Omega f \xi, \quad \text{for all } \xi \in H^1_0(\Omega).
\]

Theorem 3.4. Assume \(V \in W^{1,2}(\Omega)\) and \((\nabla \cdot V)^- \in L^\infty(\Omega)\). For \(f \in L^2(\Omega)\) and \(\lambda\) satisfying
\[
0 < \lambda < \lambda_0 := 1/\|\nabla \cdot V^\|_\infty,
\]
the problem (3.1) has a solution \(v\) that we denote by \(v_m\). Moreover, for any \(1 \leq q \leq \infty\), we have
\[
\|v_m\|_q \leq \begin{cases} 
(1 - (q-1)\lambda/\lambda_0)^{-1}\|f\|_q, & \text{if } 1 \leq q < \infty \\
(1 - \lambda/\lambda_0)^{-1}\|f\|_\infty, & \text{if } q = \infty
\end{cases}
\]
and
\[
(1 - \lambda/\lambda_0) \int |v_m|^{m+1} \, dx + \lambda \int |\nabla v^m|^2 \, dx \leq \int f v^m \, dx.
\]
Moreover, thanks to Theorem 1.1, we have

**Corollary 3.2.** Under the assumption of Theorem 3.4, if moreover \( V \) satisfies the outpointing condition (1.3), the problem (3.2) has a unique solution. Moreover, if \( v_1 \) and \( v_2 \) are two solutions associated with \( f_1 \in L^1(\Omega) \) and \( f_2 \in L^1(\Omega) \) respectively, then

\[
\|(v_1 - v_2)^+\|_1 \leq \|(f_1 - f_2)^+\|_1
\]

and

\[
\|v_1 - v_2\|_1 \leq \|f_1 - f_2\|_1.
\]

**Proof.** This is a simple consequence of the fact that if \((v, p)\) (which is independent of \( t \)) is a solution of (3.2), then it can be assimilated to a time-independent solution of the evolution problem (1.3) with \( f \) replaced by \( f - v \) (which is also independent of \( t \)). \(\square\)

To prove Theorem 3.4, we proceed by regularization and compactness. For each \( \varepsilon > 0 \), we consider \( \beta_\varepsilon \) a regular Lipschitz continuous function strictly increasing satisfying \( \beta_\varepsilon(0) = 0 \) and, as \( \varepsilon \to 0 \),

\[
\beta_\varepsilon(r) \to r^{1/m}, \quad \text{for any } r \in \mathbb{R}.
\]

One can take, for instance, \( \beta_\varepsilon \) the regularization by convolution of the application \( r \in \mathbb{R} \to r^{1/m} \).

Then, we consider the problem

\[
\begin{cases}
\begin{align*}
v - \lambda \Delta p + \lambda \nabla \cdot (v \nabla p) &= f \\
v &= \beta_\varepsilon(p) \\
p &= 0
\end{align*}
\end{cases}
\text{ in } \Omega,
\]

\[
\text{on } \partial \Omega.
\]

**Lemma 3.2.** For any \( f \in L^2(\Omega) \) and \( \varepsilon > 0 \), the problem (3.1) has a solution \( v_\varepsilon \), in the sense that \( v_\varepsilon \in L^2(\Omega) \), \( p_\varepsilon := \beta_\varepsilon^{-1}(u_\varepsilon) \in H^1_0(\Omega) \), and

\[
\int v_\varepsilon \xi \, dx + \lambda \int \nabla p_\varepsilon \cdot \nabla \xi \, dx - \lambda \int v_\varepsilon V \cdot \nabla \xi \, dx = \int f \xi \, dx,
\]

for any \( \xi \in H^1_0(\Omega) \). Moreover, for any \( \lambda \) satisfying (3.4) the solution \( v_\varepsilon \) satisfies the estimates

\[
\|v_\varepsilon\|_q \leq \begin{cases}
(1 - (q - 1)\lambda/\lambda_0)^{-1}\|f\|_q, & \text{if } 1 \leq q < \infty \\
(1 - \lambda/\lambda_0)^{-1}\|f\|_\infty, & \text{if } q = \infty
\end{cases}
\]

and

\[
(1 - \lambda/\lambda_0) \int v_\varepsilon p_\varepsilon \, dx + \lambda \int |\nabla p_\varepsilon|^2 \, dx \leq \int f p_\varepsilon \, dx.
\]
Proof. We can assume without loss of generality throughout the proof that $\lambda = 1$ and remove the script $\varepsilon$ in the notations of $(v_{\varepsilon}, p_{\varepsilon})$ and $\beta_\varepsilon$. We consider $H^{-1}(\Omega)$ the usual topological dual space of $H_0^1(\Omega)$ and $\langle ., . \rangle$ the associate dual bracket. See that the operator $A : H_0^1(\Omega) \to H^{-1}(\Omega)$, given by

$$\langle Ap, \xi \rangle = \int \beta(p) \xi \, dx + \int \nabla p \cdot \nabla \xi \, dx - \int \beta(p) V \cdot \nabla \xi \, dx,$$

for any $\xi, p \in H_0^1(\Omega)$, is a bounded weakly continuous operator. Moreover, $A$ is coercive. Indeed, for any $p \in H_0^1(\Omega)$, we have

$$\langle Ap, p \rangle = \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx - \int \beta(p) V \cdot \nabla p \, dx$$

$$= \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx - \int V \cdot \nabla \left( \int_0^p \beta(r) \, dr \right) \, dx$$

$$= \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx + \int \nabla \cdot V \left( \int_0^p \beta(r) \, dr \right) \, dx$$

$$\geq \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx - \int (\nabla \cdot V)^- \, p \beta(p) \, dx$$

$$\geq \frac{1}{2} \int \beta(p) p \, dx + \int |\nabla p|^2 \, dx - \frac{1}{2} \int (\nabla \cdot V)^- \, dx$$

$$\geq \int |\nabla p|^2 \, dx - \frac{1}{2} \int (\nabla \cdot V)^- \, dx,$$

where we use Young inequality. So, for any $f \in H^{-1}(\Omega)$ the problem $Ap = f$ has a solution $p \in H_0^1(\Omega)$. Now, for each $1 < q < \infty$, taking $v^{q-1}$ as a test function, and using the fact that

$$v \nabla (v^{q-1}) = \frac{q-1}{q} \nabla |v|^q, \quad \text{a.e. in } \Omega$$

and

$$\nabla p \cdot \nabla (v^{q-1}) \geq 0,$$

we get

$$\int |v|^q \, dx \leq \int f v^{q-1} \, dx + \lambda \frac{q-1}{q} \int V \cdot \nabla |v|^q \, dx$$

$$\leq \int f v^{q-1} \, dx - \lambda \frac{q-1}{q} \int \nabla \cdot V |v|^q \, dx$$

$$\leq \int f v^{q-1} \, dx + \lambda \frac{q-1}{q} \int (\nabla \cdot V)^- |v|^q \, dx$$

$$\leq \frac{1}{q} \int |f|^q \, dx + \frac{q-1}{q} \int |v|^q \, dx + \lambda \frac{q-1}{q} \| (\nabla \cdot V)^- \|_{\infty} \int |v|^q \, dx,$$

where we use again Young inequality. This implies that

$$(1 - \lambda(q - 1) \| (\nabla \cdot V)^- \|_{\infty}) \int |v|^q \, dx \leq \int |f|^q \, dx.$$
Thus (3.4). To prove (3.4), we take \( p \) as a test function, we obtain

\[
\lambda \int \left| \nabla p \right|^2 \, dx = \int fp \, dx - \int vp \, dx + \lambda \int \beta(p) V \cdot \nabla p \, dx
\]

\[
= \int fp \, dx - \int vp \, dx + \lambda \int V \cdot \nabla \left( \int_0^p \beta(r) \, dr \right) \, dx
\]

\[
= \int fp \, dx - \int vp \, dx - \lambda \int \nabla \cdot V \left( \int_0^p \beta(r) \, dr \right) \, dx
\]

\[
\leq \int fp \, dx - \int vp \, dx + \lambda \int \nabla \cdot V \left( \int_0^p \beta(r) \, dr \right) \, dx
\]

\[
\leq \int fp \, dx - \int vp \, dx + \lambda \int \nabla \cdot V \left( \int_0^p \beta(r) \, dr \right) \, dx
\]

where we use the fact that \( \int_0^p \beta(r) \, dr \leq p \beta(p) = vp \). Thus (3.4) for \( 1 < q < \infty \).

For the case \( q \in \{1, \infty\} \), we take \( H_\sigma(v-k) \in H^0_0(\Omega) \), for a given \( k \geq 0 \) and \( \sigma > 0 \), as a test function in (3.1), where

\[
H_\sigma(r) = \begin{cases} 
1 & \text{if } r \geq 1 \\
\frac{r}{\sigma} & \text{if } |r| < \sigma \\
-1 & \text{if } r \leq -1.
\end{cases}
\]

(3.25)

Then, letting \( \sigma \to 0 \) and using the fact that \( \nabla p \cdot \nabla H_\sigma(u-k) \geq 0 \) a.e. in \( \Omega \), it is not difficult to see that

\[
\int (v-k)^+ \, dx \leq \int (f-k(1+\lambda \nabla \cdot V)) \, dx + \lambda \lim_{\sigma \to 0} \int (v-k) \nabla H_\sigma^+(v-k) \, dx
\]

\[
\leq \int (f-k(1+\lambda \nabla \cdot V)) \, dx,
\]

where we use the fact that \( \lim_{\sigma \to 0} \int (v-k) \nabla H_\sigma(v-k) \, dx = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_{0 \leq v-k \leq \sigma} (v-k) \nabla (v-k) \, dx = 0 \).

Thus

\[
\int (v-k)^+ \, dx \leq \int (f-k(1-\lambda \nabla \cdot V)^+)) \, dx.
\]

In particular, taking

\[
k = \frac{\|f\|_\infty}{1 - \lambda/\lambda_0},
\]

we deduce that \( v \leq \frac{\|f\|_\infty}{1 - \lambda/\lambda_0} \). Working in the same way with \( H_\sigma(-v+k) \) as a test function, we obtain

\[
v \geq -\frac{\|f\|_\infty}{1 - \lambda/\lambda_0}.
\]

Thus the result of the lemma for \( q = \infty \). The case \( q = 1 \) follows by Corollary 3.2. \( \square \)
Lemma 3.3. Under the assumption of Theorem 3.4, by taking a subsequence \( \varepsilon \to 0 \) if necessary, we have

\[ v_\varepsilon \to v, \quad \text{in } L^2(\Omega) \text{-weak} \]

and

\[ p_\varepsilon \to v^m, \quad \text{in } H^1_0(\Omega). \]

Moreover, \( v \) is a weak solution of (3.2).

Proof. Using Lemma 3.2 as well as Young and Poincaré inequalities, we see that the sequences \( v_\varepsilon \) and \( p_\varepsilon \) are bounded in \( L^2(\Omega) \) and \( H^1_0(\Omega) \), respectively. So, there exists a subsequence that we denote again by \( v_\varepsilon \) and \( p_\varepsilon \) such that (3.3) is fulfilled and

\[ p_\varepsilon \to v^m, \quad \text{in } H^1_0(\Omega) \text{-weak}. \]

Letting \( \varepsilon \to 0 \) in (3.2), we obtain that \( v \) is a weak solution of (3.2). Let us prove that actually (3.1) holds to be true strongly in \( H^1_0(\Omega) \).

Indeed, taking \( p_\varepsilon \) as a test function, we have

\[
\lambda \int |\nabla p_\varepsilon|^2 \, dx = \int (f - v_\varepsilon) \, p_\varepsilon \, dx + \lambda \int V \cdot \nabla \left( \int_0^{p_\varepsilon} \beta_\varepsilon(r) \, dr \right) \, dx
= \int (f - v_\varepsilon) \, p_\varepsilon \, dx - \lambda \int \nabla \cdot V \int_0^{p_\varepsilon} \beta_\varepsilon(r) \, dr \, dx.
\]

Since \( \int_0^{p_\varepsilon} \beta_\varepsilon(s) \, ds \) converges to \( \int_0^r \beta(s) \, ds \), for any \( r \in \mathbb{R}, p_\varepsilon \to v^m \) a.e. in \( \Omega \) and \( \left| \int_0^{p_\varepsilon} \beta_\varepsilon(s) \, ds \right| \leq v_\varepsilon \, p_\varepsilon \) which is bounded in \( L^1(\Omega) \) by (3.2), we have

\[
\int_0^{p_\varepsilon} \beta_\varepsilon(s) \, ds \to \int_0^r s^{1/m} \, ds = \frac{m}{m+1} |v|^{m+1}, \quad \text{in } L^1(\Omega).
\]

So, in one hand we have

\[
\lim_{\varepsilon \to 0} \lambda \int |\nabla p_\varepsilon|^2 \, dx = \int (f - v) \, p \, dx - \lambda \frac{m}{m+1} \int \nabla \cdot V |v|^{m+1} \, dx.
\]

On the other, since \( v \) is a weak solution of (3.2), one sees easily that

\[
\lambda \int |\nabla p|^2 \, dx = \int (f - v) \, p \, dx - \lambda \frac{m}{m+1} \int \nabla \cdot V |v|^{m+1} \, dx;
\]

which implies that \( \lim_{\varepsilon \to 0} \int |\nabla p_\varepsilon|^2 \, dx = \int |\nabla p|^2 \, dx \). Combing this with the weak convergence of \( \nabla p_\varepsilon \), we deduce the strong convergence (3.3).

Remark 6. One sees in the proof that the results of Lemma 3.3 remain to be true if one replace \( f \) in (3.1) by a sequence of \( f_\varepsilon \in L^2(\Omega) \) and assumes that, as \( \varepsilon \to 0 \),

\[ f_\varepsilon \to f, \quad \text{in } L^2(\Omega). \]

Proof of Theorem 3.4. The proof follows by Lemma 3.3. Moreover, the estimates hold to be true by letting \( \varepsilon \to 0 \), in the estimate (3.2) and (3.2) for \( v_\varepsilon \) and \( p_\varepsilon \).
3.2 Existence for the evolution problem

To study the evolution problem, we use Euler-implicit discretization scheme. For any $n \in \mathbb{N}^*$ being such that $0 < \varepsilon := T/n \leq \varepsilon_0$, we consider the sequence $(u_i, p_i)_{i=0, ... N}$ given by the $\varepsilon-$Euler implicit scheme associated with (1.13):

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
  u_{i+1} - \varepsilon \Delta p_{i+1} + \varepsilon \nabla \cdot (u_{i+1} V) &= u_i + \varepsilon f_i \\
  p_{i+1} &= u_{i+1}^m \\
  p_{i+1} &= 0
\end{array}
\right. \\
\text{in } \Omega, \\
\text{on } \partial \Omega,
\end{align*}
$$

(3.29)

where, $f_i$ is given by

$$
f_i = \frac{1}{\varepsilon} \int_{i\varepsilon}^{(i+1)\varepsilon} f(s) \, ds, \quad \text{a.e. in } \Omega, \quad i = 0, ... n-1.
$$

Now, for a given $\varepsilon-$time discretization $t_i = i\varepsilon$, $i = 0, 1, ... n$, we define the $\varepsilon-$approximate solution by

$$
u_\varepsilon := \sum_{i=0}^{n-1} u_i \chi_{[t_i, t_{i+1})}, \quad \text{and} \quad p_\varepsilon := \sum_{i=1}^{n-1} p_i \chi_{[t_i, t_{i+1})}.
$$

(3.30)

In order to use the results of the previous section and the general theory of evolution problem governed by accretive operator (see for instance [8, 7]), we define the operator $A_m$ in $L^1(\Omega)$, by $\mu \in A_m(z)$ if and only if $\mu, z \in L^1(\Omega)$ and $z$ is a solution of the problem

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
  -\Delta z^m + \nabla \cdot (z V) &= \mu \\
  z &= 0
\end{array}
\right. \\
\text{in } \Omega, \\
\text{on } \partial \Omega,
\end{align*}
$$

in the sense that $z \in L^2(\Omega)$, $z^m \in H^1_0(\Omega)$ and

$$
\int_{\Omega} \nabla z^m \cdot \nabla \xi - \int_{\Omega} z V \cdot \nabla \xi = \int_{\Omega} \mu \xi, \quad \forall \xi \in H^1_0(\Omega) \cap L^\infty(\Omega).
$$

As a consequence of Theorem [3.1] we see that the operator $A_m$ is accretive in $L^1(\Omega)$; i.e. $(I + \lambda A_m)^{-1}$ is a contraction in $L^1(\Omega)$, for small $\lambda > 0$ (cf. Appendix section). Moreover, thanks to Theorem [3.4] we have

**Lemma 3.4.** For $0 < \lambda < \lambda_0$, $\overline{R(I + \lambda A_m)} = L^1(\Omega)$ and $\overline{D(A_m)} = L^1(\Omega)$.

**Proof.** Since $R(I + \lambda A_m) \supset L^2(\Omega)$ (by Theorem [3.4]), it is clear that $\overline{R(I + \lambda A_m)} = L^1(\Omega)$. To prove the density of $D(A_m)$ in $L^1(\Omega)$, we prove that $L^\infty(\Omega) \subseteq D(A_m)$. To this aim, for a given $v \in L^\infty(\Omega)$, we consider the sequence $(v_\varepsilon)_{\varepsilon > 0}$ given by the solution of the problem

$$
\begin{align*}
\left\{ 
\begin{array}{ll}
  v_\varepsilon - \varepsilon \Delta v_\varepsilon^m + \varepsilon \nabla \cdot (v_\varepsilon V) &= v \\
  v_\varepsilon &= 0
\end{array}
\right. \\
\text{in } \Omega, \\
\text{on } \partial \Omega.
\end{align*}
$$

(3.31)

Thanks to (3.3) and (3.4) with $q = \infty$, one sees easily by letting $\varepsilon \to 0$ that $v_\varepsilon \to v$ in $L^1(\Omega)$-weak and then $\|v\|_1 \leq \liminf_{\varepsilon \to 0} \|v_\varepsilon\|_1$. Moreover, thanks to (3.3) with $q = 1$, we deduce that $\lim_{\varepsilon \to 0} \|v_\varepsilon\|_1 = \|v\|_1$. Thus $v_\varepsilon \to v$ in $L^1(\Omega)$. \hfill \Box
So, thanks to the general theory of nonlinear semi-group governed by accretive operator (see Appendix), for any \( u_0 \in L^1(\Omega) \), we have

\[
\tag{3.32} u_\varepsilon \to u, \quad \text{in } C([0,T),L^1(\Omega)), \quad \text{as } \varepsilon \to 0,
\]

where, \( u \) is the so called "mild solution" of the evolution problem

\[
\tag{3.33}
\begin{cases}
  u_t + A_m u \ni f & \text{in } (0,T) \\
  u(0) = u_0.
\end{cases}
\]

To accomplish the proof of existence (of weak solution) for the problem (1.3), we prove that the mild solution \( u \) satisfies all the conditions of Definition 1.1. More precisely, we prove the following result.

**Proposition 3.3.** Assume \( V \in W^{1,2}(\Omega) \), \( (\nabla \cdot V)^- \in L^\infty(\Omega) \) and \( V \) satisfies the outpointing condition (1.3). For any \( u_0 \in L^2(\Omega) \) and \( f \in L^2(Q) \), the mild solution \( u \) of the problem (3.2) is the unique solution of (1.3).

To prove this result, thanks to (3.2), it is enough to study moreover the limit of sequence \( p_\varepsilon \) given by the \( \varepsilon \)--approximate solution.

**Lemma 3.5.** Let \( u_\varepsilon \) and \( p_\varepsilon \) be the \( \varepsilon \)--approximate solution given by (3.2). We have

1. For any \( q \in [1, \infty) \), we have

\[
\tag{3.34} \|u_\varepsilon(t)\|_q \leq M_\varepsilon^q, \quad \text{for any } t \geq 0,
\]

where

\[
M_\varepsilon^q := \begin{cases}
  \left( \|u_0\|_q + \int_0^T \|f_\varepsilon(t)\|_q \, dt \right) e^{(q-1)T \|\nabla \cdot V\|^\infty} & \text{if } 1 \leq q < \infty \\
  \left( \|u_0\|_\infty + \int_0^T \|f_\varepsilon(t)\|_\infty \, dt \right) e^{T \|\nabla \cdot V\|^\infty} & \text{if } q = \infty.
\end{cases}
\]

2. For each \( \varepsilon > 0 \), we have

\[
\tag{3.35} \frac{1}{m+1} \int_\Omega |u_\varepsilon(t)|^{m+1} + \int_0^t \int_\Omega |\nabla p_\varepsilon|^2 \leq \int_0^t \int_\Omega f_\varepsilon p_\varepsilon \, dx + \int_0^t \int_\Omega (\nabla \cdot V)^- p_\varepsilon u_\varepsilon \, dx + \int_\Omega |u_0|^{m+1}.
\]

**Proof.** Thanks to Theorem 3.4 the sequence \( (u_i)_{i=1,...,n} \) of solutions of (3.2) is well defined in \( L^2(\Omega) \) and satisfies

\[
\int_\Omega u_{i+1} \xi + \varepsilon \int_\Omega \nabla p_{i+1} \cdot \nabla \xi - \varepsilon \int_\Omega u_{i+1} V \cdot \nabla \xi = \int_{i \varepsilon}^{(i+1) \varepsilon} \int_\Omega f_i \xi, \quad \text{for } i = 1, ..., n - 1,
\]

21
for any $\xi \in H^1_0(\Omega)$. Thanks to (3.3), for any $1 \leq q \leq \infty$, we have
\[
\|u_t\|_q \leq \|u_{i-1}\|_q + \varepsilon \|f_i\|_q + \varepsilon (q - 1) \|\nabla \cdot V\|_{\infty} \|u_i\|_q.
\]
By induction, this implies that, for any $t \in [0,T)$, we have
\[
\|u(t)\|_q \leq \|u_0\|_q + \int_0^T \|f(t)\|_q dt + (q - 1) \|\nabla \cdot V\|_{\infty} \int_0^T \|u(t)\|_q dt.
\]
Using Gronwall Lemma, we deduce (1), for any $1 \leq q < \infty$. The proof for the case $q = \infty$ follows in the same way by using (3.4) with $q = \infty$. Now, using the fact that
\[
(u_i - u_{i-1})p_i = (u_i - u_{i-1})u_i^m \geq \frac{1}{m+1} (u_{i+1}^m - u_{i-1}^m)
\]
and
\[
\int u_i V \cdot \nabla p_i \leq \int \int (\nabla \cdot V)^- p_i u_i,
\]
we get
\[
\frac{1}{m+1} \int |u_i|^{m+1} + \varepsilon \int |\nabla p_i|^2 \leq \varepsilon \int \int f_i p_i dx + \varepsilon \int (\nabla \cdot V)^- p_i u_i dx + \frac{1}{m+1} \int |u_{i-1}|^{m+1}.
\]
Summing this identity for $i = 1, \ldots, n$, and using the definition of $u_\varepsilon$, $p_\varepsilon$ and $f_\varepsilon$, we get (2).

Proof of Proposition 3.3
Recall that we already know that $u_\varepsilon \to u$ in $C([0,T]; L^1(\Omega))$, as $\varepsilon \to 0$. Now, combining (1) and (2) with Poincaré and Young inequalities, one sees that
\[
\frac{1}{m+1} \frac{d}{dt} \int |u_\varepsilon|^{m+1} dx + \int |\nabla p_\varepsilon|^2 dx \leq C(N, \Omega) \left( \int |f_\varepsilon|^2 dx + \|\nabla \cdot V\|_{\infty} (M^\varepsilon_2)\right), \text{ in } D'(0,T).
\]
This implies that $p_\varepsilon$ is bounded in $L^2(0,T; H^1_0(\Omega))$. This implies that
\[
p_\varepsilon \to u^m, \text{ in } L^2(0,T; H^1_0(\Omega)) - \text{weak, as } \varepsilon \to 0.
\]
Recall that taking
\[
\tilde{u}_\varepsilon(t) = \frac{(t-t_i)u_{i+1} - (t-t_{i+1})u_i}{\varepsilon}, \text{ for any } t \in [t_i, t_{i+1}), i = 1, \ldots n,
\]
we have
\[
(3.39) \quad \partial_t \tilde{u}_\varepsilon - \Delta p_\varepsilon + \nabla \cdot (u_\varepsilon V) = f_\varepsilon, \text{ in } D'(Q).
\]
Moreover, we know that $\tilde{u}_\varepsilon \to u$, in $C([0,T), L^1(\Omega))$. So letting $\varepsilon \to 0$ in (3.2), we deduce that $u$ is a solution of (1.3). Letting $\varepsilon \to 0$ in (1) and (2), we get respectively (1) and (2).

Proof of Theorem 1.2
The proof follows by Proposition 3.3.
4 The limit as $m \to \infty$.

Since the solution of the problem (3.3) is the mild solution associated with the operator $A_m$, we begin by studying the $L^1$ limit, as $m \to \infty$, of the solution of the stationary problem (3.2). Formally, this limiting problem is given by

$$
\begin{aligned}
\begin{cases}
v - \Delta p + \nabla \cdot (vV) = f & \text{in } \Omega \\
v \in \text{Sign}(p) & \\
p = 0 & \text{on } \partial \Omega.
\end{cases}
\end{aligned}
$$

(4.1)

This is the stationary problem associated with the so-called Hele-Shaw problem. Thanks to [33], for any $f \in L^2(\Omega)$, (4.1) has a unique solution $(v, p)$ in the sense that $(v, p) \in L^\infty(\Omega) \times H^1_0(\Omega)$, $v \in \text{sign}(p)$ a.e. in $\Omega$, and

$$
\int_\Omega v \xi + \int_\Omega \nabla p \cdot \nabla \xi - \int_\Omega vV \cdot \nabla \xi = \int_\Omega f \xi, \quad \text{for any } \xi \in H^1_0(\Omega).
$$

(4.2)

To begin with, we prove the following incompressible limit for stationary problem.

**Proposition 4.4.** Under the assumptions of Theorem 3.4, let $v_m$ be the solution of (3.2). As $m \to \infty$, we have

$$
v_m \to v, \quad \text{in } L^2(\Omega)-\text{weak},
$$

(4.3)

$$
v_m^m \to p, \quad \text{in } H^1_0(\Omega),
$$

(4.4)

and $(v, p)$ is the solution of (4.1).

**Proof.** Thanks to (3.4), there exists $v \in H^1_0(\Omega)$, such that (4.4) is fulfilled. Thanks to (3.4), we see that the sequence $p_m$ is bounded in $H^1_0(\Omega)$, which implies that, by taking a sub-sequence if necessary,

$$
v_m^m \to p, \quad \text{in } H^1_0(\Omega)-\text{weak}
$$

(4.5)

and

$$
v_m^m \to p, \quad \text{in } L^2(\Omega).
$$

(4.6)

Using monotonicity arguments (see for instance Proposition 2.5 of [16]) we get $v \in \text{sign}(p)$ a.e. in $\Omega$, and letting $m \to \infty$ in (3.2), we obtain that $(v, p)$ satisfies (4.1). To prove the strong convergence of $p_m$, we use the same argument of the proof of Lemma 3.3. Indeed, taking $p_m$ as a test function in (3.1), we have

$$
\lambda \int \nabla p_m \cdot \nabla p_m^m \, dx = \int (f - v_m) p_m \, dx + \lambda \int \nabla \cdot V \left( \int_0^{p_m} \frac{1}{r} \, dr \right) \, dx,
$$

$$
= \int (f - v_m) p_m \, dx + \lambda \frac{m}{m + 1} \int \nabla \cdot V v_m p_m \, dx.
$$
Letting \( m \to \infty \), and using (4.4) and (4), we see that

\[
\lim_{m \to \infty} \lambda \int |\nabla p_m|^2 \, dx = \int (f - u) \, p \, dx + \lambda \int u \, \nabla \cdot V \, dx
\]

\[
= \int (f - u) \, p \, dx + \lambda \int \nabla \cdot V |p| \, dx.
\]

We know that \((u, p)\) is a solution of (4), so one sees easily that

\[
\lambda \int |\nabla p|^2 \, dx = \int (f - u) \, p \, dx + \lambda \int \nabla \cdot V |p|,\]

so that

\[
\lim_{m \to \infty} \lambda \int |\nabla p_m|^2 \, dx = \lambda \int |\nabla p|^2 \, dx.
\]

Thus the strong convergence of \( \nabla p_m \).

\[\square\]

For the strong convergence of \( v_m \), under the assumption (1.3), we prove first the following convergence for the solution of the stationary problem.

**Theorem 4.5.** Under the assumptions of Theorem 4.6, i.e. \( V \in W^{1,2}(\Omega) \), \( \nabla \cdot V \in L^\infty(\Omega) \) and satisfies (1.4), for any \( 0 < \lambda < \lambda_1 \), the convergence (4.4) holds to be true strongly in \( L^1(\Omega) \). Here

\[
\lambda_1 := 1/ \sum_{i,k} \| \partial x_i V_k \|_{\infty}.
\]

**Corollary 4.3.** Under the assumptions of Theorem 4.5, the operator \( A_m \) converges to \( A \) in the sense of resolvent in \( L^1(\Omega) \), where \( A \) is defined by : \( \mu \in A(z) \) if and only if \( \mu, z \in L^1(\Omega) \) and \( z \) is a solution of the problem

\[
\begin{cases}
-\Delta p + \nabla \cdot (z \, V) = \mu & \text{in } \Omega \\
z \in \text{sign}(p) \\
p = 0 & \text{on } \partial \Omega,
\end{cases}
\]

in the sense that \( z \in L^\infty(\Omega) \), \( \exists \, p \in H_0^1(\Omega) \) such that \( p \in H_0^1(\Omega) \), \( u \in \text{sign}(p) \) a.e. in \( \Omega \) and

\[
\int_\Omega \nabla p \cdot \nabla \xi - \int_\Omega z \, V \cdot \nabla \xi = \int_\Omega \mu \, \xi, \quad \forall \, \xi \in H_0^1(\Omega) \cap L^\infty(\Omega).
\]

Moreover, we have

\[
\overline{D(A)} = \{ z \in L^\infty(\Omega) : |z| \leq 1 \text{ a.e. in } \Omega \}.
\]

The main element to prove Theorem 4.5 is \( BV_{loc} \)-estimates on \( v_m \). Recall that a given function \( u \in L^1(\Omega) \) is said to be of bounded variation if and only if, for each \( i = 1, \ldots, N \),

\[
TV_i(u, \Omega) := \sup \left\{ \int_\Omega u \, \partial x_i \xi \, dx : \xi \in C_c^1(\Omega) \text{ and } \| \xi \|_{\infty} \leq 1 \right\} < \infty,
\]
here $C^1_c(\Omega)$ denotes the set of $C^1$–function compactly supported in $\Omega$. More generally a function is locally of bounded variation in a domain $\Omega$ if and only if for any open set $\omega \subset \subset \Omega$, $TV_i(u,\omega) < \infty$ for any $i = 1, \ldots, N$. In general a function locally of bounded variation (as well as function of bounded variation) in $\Omega$, may not be differentiable, but by the Riesz representation theorem, their partial derivatives in the sense of distributions are Borel measure in $\Omega$. This gives rise to the definition of the vector space of functions of bounded variation in $\Omega$, usually denoted by $BV(\Omega)$, as the set of $u \in L^1(\Omega)$ for which there are Radon measures $\mu_1, \ldots, \mu_N$ with finite total mass in $\Omega$ such that

$$
\int_\Omega \nabla \xi \cdot \nabla v \, dx = - \int_\Omega \xi \, d\mu_i, \quad \text{for any } \xi \in C^1_c(\Omega), \quad \text{for } i = 1, \ldots, N.
$$

Without abuse of notation we continue to point out the measures $\mu_i$ by $\partial_x v$ anyway, and by $|\partial_x v|$ the total variation of $\mu_i$. Moreover, we’ll use as usual $Dv = (\partial_x v,...,\partial_x v)$ the vector valued Radon measure pointing out the gradient of any function $v \in BV(\Omega)$, and $|Dv|$ indicates the total variation measure of $v$. In particular, for any open set $\omega \subset \subset \Omega$, $TV_i(v,\omega) = |\partial_x v|(\omega) < \infty$, and the total variation of the function $v$ in $\omega$ is finite too; i.e.

$$
\|Dv\|(\omega) = \sup \left\{ \int_\Omega \nabla \xi \cdot \nabla v \, dx : \xi \in C^1_c(\omega) \text{ and } \|\xi\|_{\infty} \leq 1 \right\} < \infty.
$$

At last, let us remind the reader here the well known compactness result for functions of bounded variation: given a sequence $u_n$ of functions in $BV_{loc}(\Omega)$ such that, for any open set $\omega \subset \subset \Omega$, we have

$$
\sup_n \left\{ \int_{\Omega \setminus \omega} |v_n| \, dx + |Dv_n|(\omega) \right\} < \infty,
$$

there exists a subsequence that we denote again by $v_n$ which converges in $L^1_{loc}(\Omega)$ to a function $v \in BV_{loc}(\Omega)$. Moreover, for any compactly supported continuous function $0 \leq \xi$, the limit $u$ satisfies

$$
\int \xi |\partial_x v| \leq \liminf_{n \to \infty} \int \xi |\partial_x v_n|,
$$

for any $i = 1, \ldots, N$, and

$$
\int \xi |Dv| \leq \liminf_{n \to \infty} \int \xi |Dv_n|.
$$

Under the assumption (1.4), the following sequence of test functions plays an important role in the proof of $BV_{loc}$-estimates and convergence results of Theorem 1.3

**Lemma 4.6.** Under the assumption (1.4), there exists $0 \leq \omega_h \in H^2(\Omega_h)$ compactly supported in $\Omega$, such that $\omega_h \equiv 1$ in $\Omega_h$ and

$$
\int_{\Omega \setminus \Omega_h} \varphi \nabla \omega_h \, dx \geq 0, \quad \text{for any } 0 \leq \varphi \in L^2(\Omega),
$$

25
Proof. It is enough to take \( \omega_h(x) = \eta_h(d(., \partial \Omega)) \), for any \( x \in \Omega \), where \( \eta_h : [0, \infty) \to \mathbb{R}^+ \) is a nondecreasing \( C^2 \)-function compactly supported in \((0, \infty)\) such that \( \eta_h \equiv 1 \) in \([h, \infty)\). In this case,
\[
\nabla \omega_h = \eta_h'(d(., \partial \Omega)) \nabla d(., \partial \Omega),
\]
so that
\[
\int_{\Omega \setminus \Omega_h} \varphi V \cdot \nabla \omega_h \, dx = -\int_{\Omega \setminus \Omega_h} \eta_h'(d(., \partial \Omega)) \varphi V \cdot \nabla d(., \partial \Omega) \, dx
\]
\[
\geq 0, \quad \text{for any } 0 \leq \varphi \in L^2(\Omega),
\]
This function may be define
\[
\eta_h(r) = \begin{cases} 
0 & \text{if } 0 \leq r \leq c_1 h \\
\frac{-C_h}{e^{r^2-c_1 h^2}} & \text{if } c_1 h \leq r \leq c_2 h \\
1 - \frac{C_h}{e^{r^2-c_2 h^2}} & \text{if } c_2 h \leq r \leq h \\
1 & \text{if } h \leq r
\end{cases}
\]
with \( 0 < c_1 < c_2 < 1 \) and \( C_h > 0 \) given such that \( 2c_2^2 - c_1^2 = 1 \) and \( e^{-C_h} = 1 - e^{-M_h} \), where \( M_h := (c_2^2 - c_1^2) h^2 = (1 - c_2^2) h^2 \). For instance one can take \( c_1 = 1/2 \) and \( c_2 = \sqrt{5}/(2\sqrt{2}) \). See that
\[
\eta_h'(r) = \begin{cases} 
0 & \text{if } 0 \leq r \leq c_1 h \\
\frac{2rC_h}{(r^2-c_1 h^2)^2} e^{-C_h} & \text{if } c_1 h \leq r \leq c_2 h \\
\frac{2rC_h}{(h^2-r^2)^2} e^{-C_h} & \text{if } c_2 h \leq r \leq h \\
0 & \text{if } h \leq r
\end{cases}
\]
is continuous and derivable at least on \( \mathbb{R} \setminus \{c_2 h\} \). Thus \( \eta_h \in H^2(\Omega) \).
\[\]
\(\square\)

We have

**Theorem 4.6.** Assume \( f \in BV_{loc}(\Omega) \cap L^2(\Omega) \), \( V \in W^{1,\infty}(\Omega)^N \), \( \nabla \cdot V \in W^{1,2}_{loc}(\Omega) \) and let \( v_m \) be the solution of (3.2). Then, for any \( 0 < \lambda < 1/\lambda_1 \), \( v_m \in BV_{loc}(\Omega) \) and we have

\[
(1 - \lambda \lambda_1) \sum_{i=1}^N \int \omega_h \, d|\partial x_i v| \leq \lambda \sum_{i=1}^N \int (\Delta \omega_h)^+ \, |\partial x_i p| \, dx + \sum_{i=1}^N \int \omega_h \, d|\partial x_i f|\]
\[
+ \lambda \sum_{i=1}^N \int \omega_h \, |v| \, |\partial x_i (\nabla \cdot V)| \, dx,
\]
(4.11)
where \( \omega_h \) is given by Lemma 4.6.

To prove this result we use again the regularized problem (3.1) and we let \( \varepsilon \to 0 \). To begin with, we prove first the following lemma concerning any weak solutions of the general problem

\[
(4.12) \quad v - \Delta \beta^{-1}(v) + \nabla \cdot (vV) = f \text{ in } \Omega,
\]

\( \beta \) is a given a nondecreasing function assumed to be regular (at least \( C^2 \)).

**Lemma 4.7.** Assume \( f \in W^{1,2}_{loc}(\Omega) \), \( V \in W^{1,2}_{loc}(\Omega)^N \), \( \nabla \cdot V \in W^{1,\infty}_{loc}(\Omega) \) and \( v \in H^1_{loc}(\Omega) \cap L^\infty_{loc}(\Omega) \) satisfy (1) in \( \mathcal{D}'(\Omega) \). Then, for each \( i = 1, \ldots, N \), we have

\[
(4.13) \quad |\partial_{x_i}v| - \sum_{k=1}^{N} |\partial_{x_k}v| \sum_{k=1}^{N} |\partial_{x_i}V_k| - \Delta|\partial_{x_i}\beta^{-1}(v)| + \nabla \cdot (|\partial_{x_i}v| V) \leq |\partial_{x_i}f| + |v| |\partial_{x_i}(\nabla \cdot V)| \quad \text{in } \mathcal{D}'(\Omega).
\]

**Proof.** Set \( p := \beta^{-1}(v) \). Thanks to (1) and the regularity of \( f \) and \( V \), it is not difficult to see that \( v, p \in H^2_{loc}(\Omega) \cap L^\infty_{loc}(\Omega) \), and for each \( i = 1, \ldots, N \), the partial derivatives \( \partial_{x_i}v \) and \( \partial_{x_i}p \) satisfy the following equation

\[
(4.14) \quad \partial_{x_i}v - \Delta \partial_{x_i}p + \nabla \cdot (\partial_{x_i}v V) = \partial_{x_i}f - (\nabla v \cdot \partial_{x_i}V + v \partial_{x_i}(\nabla \cdot V)), \quad \text{in } \mathcal{D}'(\Omega).
\]

By density, we can take \( \xi H_{\sigma}(\partial_{x_i}v) \) as a test function in (1) where \( \xi \in H^2(\Omega) \) is compactly supported in \( \Omega \) and \( H_{\sigma} \) is given by (3.1). We obtain

\[
(4.15) \quad \int \left( \partial_{x_i}v \xi H_{\sigma}(\partial_{x_i}v) + \nabla \partial_{x_i}p \cdot \nabla (\xi H_{\sigma}(\partial_{x_i}v)) \right) dx - \int \partial_{x_i}v V \cdot \nabla (\xi H_{\sigma}(\partial_{x_i}v)) dx = \int \partial_{x_i}f \xi H_{\sigma}(\partial_{x_i}v) dx - \int (\nabla v \cdot \partial_{x_i}V + v \partial_{x_i}(\nabla \cdot V)) \xi H_{\sigma}(\partial_{x_i}v) dx.
\]

To pass to the limit as \( \sigma \to 0 \), we see first that

\[
(4.16) \quad H'_{\sigma}(\partial_{x_i}v) \partial_{x_i}v = \frac{1}{\sigma} \partial_{x_i}v \chi_{[|\partial_{x_i}v| \leq \sigma]} \to 0, \quad \text{in } L^\infty(\Omega)-\text{weak}^*.
\]

So, the last term of the first part of (1) satisfies

\[
\lim_{\sigma \to 0} \int \partial_{x_i}v V \cdot \nabla (\xi H_{\sigma}(\partial_{x_i}v)) dx = \int |\partial_{x_i}v| V \cdot \nabla \xi dx + \lim_{\sigma \to 0} \int \partial_{x_i}v \nabla \partial_{x_i}v \cdot V H'_{\sigma}(\partial_{x_i}v) \xi dx = \int |\partial_{x_i}v| V \cdot \nabla \xi,
\]

On the other hand, we see that

\[
\int \nabla \partial_{x_i}p \cdot \nabla (\xi H_{\sigma}(\partial_{x_i}v)) dx = \int H_{\sigma}(\partial_{x_i}v) \nabla \partial_{x_i}p \cdot \nabla \xi dx + \int \xi \nabla \partial_{x_i}p \cdot \nabla H_{\sigma}(\partial_{x_i}v) dx.
\]

27
Since $\text{sign}_0(\partial_x v) = \text{sign}_0(\partial_x p)$, the first term satisfies

$$
\lim_{\sigma \to 0} \int H_\sigma(\partial_x v) \nabla \partial_x p \cdot \nabla \xi \, dx = - \int |\partial_x p| \Delta \xi \, dx.
$$

As to the second term, we have

$$
\lim_{\sigma \to 0} \int \xi \nabla \partial_x p \cdot \nabla H_\sigma(\partial_x v) \, dx = \lim_{\sigma \to 0} \int \xi H'_\sigma(\partial_x v) \nabla (\beta'(v) \partial_x v) \cdot \nabla \partial_x v \, dx
$$

$$
= \lim_{\sigma \to 0} \int \xi H'_\sigma(\partial_x v) \beta'(v) \| \nabla \partial_x v \|^2 \, dx
+ \lim_{\sigma \to 0} \int \xi H'_\sigma(\partial_x v) \partial_x v \beta''(v) \nabla v \cdot \nabla \partial_x v \, dx
= \lim_{\sigma \to 0} \int \xi H'_\sigma(\partial_x v) \partial_x v \beta''(v) \nabla v \cdot \nabla \partial_x v \, dx
\geq 0,
$$

where we use again (4). So, letting $\sigma \to 0$ in (4) and using again the fact that $\text{sign}_0(\partial_x v) = \text{sign}_0(\partial_x p)$, we get

$$
|\partial_x v| - \Delta |\partial_x p| + \nabla \cdot (|\partial_x v| V) \leq \text{sign}_0(\partial_x v) \partial_x f - (\nabla v \cdot \partial_x V)
+ v \partial_x \beta(\nabla \cdot V) \text{sign}_0(\partial_x v) \quad \text{in } \mathcal{D}'(\Omega)
$$

At last, using the fact that

$$
|\nabla v \cdot \partial_x V| \leq \sum_k |\partial_x v_k| \sum_k |\partial_x V_k|,
$$

the result of the lemma follows. \hfill \Box

**Proof of Theorem 4.6.** Under the assumptions of Theorem 4.6, for any $\epsilon > 0$, let us consider $f_\epsilon$ a regularization of $f$ satisfying $f_\epsilon \to f$ in $L^1(\Omega)$ and

$$
\int \xi |\partial_x f_\epsilon| \, dx \to \int \xi d|\partial_x f|, \quad \text{for any } \xi \in C_c(\Omega) \text{ and } i = 1, \ldots, N.
$$

Thanks to Lemma 3.2, we consider $v_\epsilon$ be the solution of the problem (3.2), where we replace $f$ by the regularization $f_\epsilon$. Applying Lemma 4.1 by replacing $V$ by $\lambda V$ and $\beta^{-1}$ by $\lambda \beta^{-1}_\epsilon$, we obtain

$$
\int |\partial_x v_\epsilon| \xi \, dx - \lambda \int \sum_{k=1}^N |\partial_x V_k| \sum_{k=1}^N |\partial_x v_\epsilon| \xi \, dx \leq \lambda \sum_{k=1}^N \int |\partial_x v_\epsilon| (\Delta \xi)^+ \, dx + \int |\partial_x f_\epsilon| \xi \, dx
+ \lambda \int |v_\epsilon| |\partial_x (\nabla \cdot V)| \xi \, dx - \lambda \int |\partial_x v_\epsilon| V \cdot \nabla \xi \, dx, \quad \text{for any } i = 1, \ldots, N \text{ and } 0 \leq \xi \in D(\Omega).
$$

28
By density, we can take $\xi = \omega_h$ as given by Lemma 4.6 so that the last term is nonnegative, and we have
\[
\int |\partial_x v| \omega_h \, dx \leq \lambda \sum_k |\partial_x V_k| \int |\partial_x v| \omega_h \, dx \leq \lambda \sum_k \int |\partial_x p| (\Delta \omega_h)^+ \, dx
\]
\[+ \int |\partial_x f| \omega_h \, dx + \lambda \int |v| |\partial_x (\nabla \cdot V)| \xi_h \, dx, \quad \text{for any } i = 1, \ldots, N.
\]
Summing up, for $i = 1, \ldots, N$, and using the definition of $\lambda_1$, we deduce that
\[
\sum_i \int |\partial_x v| \omega_h \, dx - \lambda \sum_i \int |\partial_x v| \xi_h \, dx \leq \lambda \sum_i \int (\Delta \omega_h)^+ |\partial_x p| \, dx
\]
\[+ \int \sum_i |\partial_x f| \omega_h \, dx + \lambda \int \sum_i |\partial_x (\nabla \cdot V)| \omega_h \, dx,
\]
and then the corresponding property (4.6) follows for $v_\varepsilon$. Thanks to (3.4) and (3.4), we know that $v_\varepsilon$ and $\partial_x p_\varepsilon$ are bounded in $L^2(\Omega)$. This implies that, for any $\omega \subset \subset \Omega$, $\sum_i \int |\partial_x v| \, dx$ is bounded. So, $v_\varepsilon$ is bounded in $BV_{loc}(\Omega)$. Combining this with the $L^1-$bound (3.4), it implies in particular, taking a subsequence if necessary, the convergence in (3.3) holds to be true also in $L^1(\Omega)$ and then $v \in BV_{loc}(\Omega)$. At last, letting $\varepsilon \to \infty$ in (4.7) and, using moreover (3.3) and the lower semi-continuity of variation measures $|\partial_x v_\varepsilon|$, we deduce (4.6) for the limit $v$, which is the solution of the problem (3.2) by Lemma 3.3.

\[
\text{Proof of Theorem 4.5.} \quad \text{Recall that under the assumptions of the theorem, the } BV_{loc} \text{ estimate (4.6) is fulfilled for } v_m. \text{ Since the constant } C \text{ in (3.4) does not depend on } m, \text{ this implies that } v_m \text{ is bounded in } BV(\omega). \text{ Since } \omega \text{ is arbitrary, we deduce in particular that the convergence in (4.4) holds to be true also in } L^1(\Omega), \text{ } v \in BV_{loc}(\Omega), \text{ and (4.6) is fulfilled.}
\]

\[
\text{Proof of Theorem 1.3.} \quad \text{Thanks to Corollary 4.3 and Theorem 6.11, we have}
\]
\[u_m \to u, \quad \text{in } C([0,T];L^1(\Omega)).
\]
On the other hand, thanks to (3.4) and (3.4), it is clear that $p_m$ is bounded in $L^2(0,T;H^1_0(\Omega))$. So, there exists $p \in L^2(0,T;H^1_0(\Omega))$, such that, taking a subsequence if necessary, we have
\[u^m \to u, \quad \text{in } L^2(0,T;H^1_0(\Omega)) \quad \text{weak}.
\]
Then using monotonicity arguments we have $u \in \text{sign}(p)$ a.e. in $Q$, and letting $m \to \infty$, in the weak formulation we deduce that the couple $(u,p)$ satisfies (1.3). Thus the results of the theorem.
Remark 7. See that we use the condition \((1.4)\) for the proof of \(BV_{\text{loc}}\)-estimate through \(w_h\) we introduce in Lemma 4.6. Indeed, \(BV_{\text{loc}}\) estimates follows from Lemma 4.7 once the term \(\lambda \int_{\Omega} |\partial_x v| V \cdot \nabla \xi \, dx\) nonnegative. Clearly, the construction of \(\omega_h\) by using \(d(.,\partial\Omega)\) is basically connected to the condition \((1.4)\). Otherwise, this condition could be replaced definitely by the existence of \(0 \leq \omega_h \in H^2(\Omega_h)\) compactly supported in \(\Omega\), such that \(\omega_h \equiv 1\) in \(\Omega_h\) and
\[
\int_{\Omega\setminus\Omega_h} \varphi V \cdot \nabla \omega_h \, dx \geq 0, \quad \text{for any } 0 \leq \varphi \in L^2(\Omega).
\]

5 Reaction-diffusion case

Let us consider now the reaction-diffusion porous medium equation with linear drift
\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u^n + \nabla \cdot (u V) &= g(.,u) \quad \text{in } Q \\
u &= 0 \quad \text{on } \Sigma \\
 u(0) &= u_0 \quad \text{in } \Omega,
\end{aligned}
\tag{5.26}
\]

Thanks to Theorem 6.10 and Theorem 6.11 we assume that \(g : Q \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory application ; i.e. continuous in \(r \in \mathbb{R}\) and measurable in \((t,x) \in Q\), and satisfies moreover the following assumptions :

\((G_1)\) \(g(.,r) \in L^2(Q)\) for any \(r \in \mathbb{R}\).

\((G_2)\) There exists \(0 \leq \theta \in C(\mathbb{R})\), such that
\[
\frac{\partial g}{\partial r}(t,x,.) \leq \theta, \quad \text{in } D'(\mathbb{R}), \text{ for a.e. } (t,x) \in Q.
\]

\((G_3)\) There exists \(\omega_1, \omega_2 \in W^{1,\infty}(0,T)\) such that \(w_1(0) \leq u_0 \leq w_2(0)\) a.e. in \(\Omega\) and, for any \(t \in (0,T)\),
\[
\dot{\omega}_1(t) + \omega_1(t) \nabla \cdot V \leq g(t,.,\omega_1(t)) \quad \text{a.e. in } \Omega
\]

and
\[
\dot{\omega}_2(t) + \omega_2(t) \nabla \cdot V \geq g(t,.,\omega_2(t)) \quad \text{a.e. in } \Omega.
\]

Remark 8. 1. On sees that \((G2)\) implies that \(g(.,u) \in L^1(Q)\), for any \(u \in L^\infty(Q)\). Indeed, setting \(M = \int_{0}^{\|u\|_{\infty}} \theta(r) \, dr\), we have
\[
-g^-(.,M) - 2M \max_{[-M,M]} \theta \leq g(.,u(\cdot)) \leq g^+(.,-\|u\|_{\infty}) + 2M \max_{[-M,M]} \theta, \quad \text{a.e. in } Q.
\]
2. As we will see, the main achievement of the condition (G₃) is some kind of à priori $L^\infty$ estimates. These conditions are accomplish in many practical situations. For instance they are fulfilled in the case where $g(.,r) \in L^\infty(Q)$, for any $r \in \mathbb{R}$, and there exists $w \geq 0$ solution of the following autonomous ODE

$$
\dot{w} = w\| (\nabla \cdot V)^- \|_\infty + \| g(.,w) \|_\infty \quad \text{in } (0,T), \quad \text{and } \omega(0) = \| u_0 \|_\infty.
$$

Actually, in this case it is enough to take $w_2(t) = -w_1(t) = w(t)$, for any $t \in [0,T)$. This is fulfilled for instance in the case where the application $r \in \mathbb{R} \rightarrow \| g(.,r) \|_\infty$ is locally Lipschitz and $\| g(.,0) \|_\infty = 0$. However, one needs to be careful with the choice of $T$ to fit it on with the maximal time for the solution of the ODE above. This may generates local (and not necessary global) existence of a solution even if $g(t,x,r)$ is well defined for any $t \geq 0$.

3. Particular example for $g$ may be given as follows :

(a) If $g(.,r) = f(.,)$, a.e. in $Q$, and for any $r \in \mathbb{R}$, where $f \in L^\infty(Q)$, then it enough to take $w_2(t) = -w_1(t) = \left( \| u_0 \|_\infty + \int_0^t \| f(t) \|_\infty \right) e^{t\| (\nabla \cdot V)^- \|_\infty}$, for any $t \in (0,T)$.  

(b) If $g(.,r) = f(.,) r$, a.e. in $Q$, and for any $r \in \mathbb{R}$, where $f \in L^\infty(Q)$, then it enough to take $w_2(t) = -w_1(t) = \| u_0 \|_\infty e^{t\| (\nabla \cdot V)^{-} \|_\infty + \int_0^t \| f(t) \|_\infty}$, for any $t \in (0,T)$.  

(c) If $\nabla \cdot V \geq 0$ and $g(t,x,r) = r^2$, one can take $w_1(t) = \frac{\| u_0 \|}{1 - t \| u_0 \|}$ and $w_2(t) = \frac{-\| u_0 \|}{1 + t \| u_0 \|}$, for $i = 1, 2$. But, in this case $T$ needs to be taken such that $T \leq 1/\| u_0 \|_\infty$.  

4. Thanks to the remarks above, one sees that $\nabla \cdot V^+$ is less involved in the existence of a solution than $g$ and $\nabla \cdot V^-$. 

**Theorem 5.7.** Assume $u_0 \in L^2(\Omega)$ and $V \in W^{1,2}(\Omega)$ is such that $\nabla \cdot V \in L^\infty(\Omega)$ and satisfies the outpointing condition (1.3). Under the assumption (G₃), (G₂) and (G₃), the problem (5.2) has a unique weak solution $u_m$ in the sense of Definition 1.1 with $f = g(.,u)$. Moreover, we have

1. $u$ is the unique mild solution of the Cauchy problem (3.2) with $f(,) = g(.,u(\.))$ a.e. in $Q$.

2. For any $0 \leq t < T$, $\omega_1(t) \leq u(t) \leq \omega_2(t)$ a.e. in $\Omega$. 

**Remark 9.** See that in the case where, for any $r \in \mathbb{R}$, $g(.,t) = f$ a.e. in $Q$, with $f \in L^\infty(Q)$ we retrieve the $L^\infty$-estimate (1) for $q = \infty$. Indeed, thanks to Theorem 5.7 and Remark 8 (see the item 3-(a)), we see that

$$
\| u(t) \|_\infty \leq \left( \| u_0 \|_\infty + \int_0^t \| f(t) \|_\infty \right) e^{t\| (\nabla \cdot V)^- \|_\infty}, \quad \text{for any } t \in (0,T)
$$

$$
\leq \left( \| u_0 \|_\infty + \int_0^T \| f(t) \|_\infty \right) e^{T\| (\nabla \cdot V)^- \|_\infty} = M_\infty.
$$
Corollary 5.4. Assume \(0 \leq u_0 \in L^2(\Omega)\) and \(V \in W^{1,2}(\Omega)\) is such that \(\nabla \cdot V \in L^\infty(\Omega)\) and satisfies the outpointing condition \([1.3]\). If

\[
(\mathcal{G}_4) \quad 0 \leq g(\cdot, 0) \text{ a.e. in } Q.
\]

and, there exists \(\omega \in W^{1,\infty}(0, T)\) such that \(0 \leq u_0 \leq w(0)\) a.e. in \(\Omega\) and for any \(t \in (0, T)\),

\[
\dot{\omega}(t) + \omega(t) \nabla \cdot V \leq g(t, \cdot, \omega(t)) \quad \text{a.e. in } Q
\]

then the solution of \([5]\) satisfies

\[
0 \leq u(t) \leq \omega_2(t), \quad \text{a.e. in } \Omega, \text{ for any } t \in (0, T).
\]

Proof. This is a simple consequence of Theorem 5.7 where we take \(w_1 \equiv 0\).

Remark 10. A typical example of \(g\) satisfying the assumption of Corollary 5.4 may be given by \(g(\cdot, r) = \mu(r), \text{ a.e. in } Q, \text{ for any } r \geq 0, \text{ with } 0 \leq \mu \in C(\mathbb{R}^+)\), and there exists \(w \geq 0\) such that

\[
\|\nabla \cdot V\|_\infty \leq \frac{\mu(w)}{w}.
\]

Proof of Theorem 5.7. Let \(F : [0, T) \times L^1(\Omega) \to L^1(\Omega)\) be given by

\[
F(t, z(\cdot)) = g(t, \cdot, (z(\cdot) \vee (-M)) \wedge M) \quad \text{a.e. in } \Omega, \text{ for any } (t, z) \in [0, T) \times L^1(\Omega),
\]

where \(M := \max(\|\omega_1\|_\infty, \|\omega_2\|_\infty)\). Thanks to Remark 8 one sees that \(F\) satisfies all the assumptions of Theorem 5.10. Then, thanks to Theorem 5.9 we consider \(u \in C([0, T), L^1(\Omega))\) the mild solution of the evolution problem

\[
\begin{cases}
  u_t + A_m u \ni F(\cdot, u) & \text{in } (0, T) \\
  u(0) = u_0.
\end{cases}
\]

Thanks to \([11]\), it is clear that \(F(\cdot, u) \in L^2(Q)\), so that, using Proposition 3.3, we can deduce that \(u\) is a weak solution of \([1.3]\). The uniqueness follows from the equivalence between weak solution and mild solution as well as the uniqueness result of Theorem 6.10. To end up the proof, it is enough to show that \(\omega_1(t) \leq u(t) \leq \omega_2(t)\) a.e. in \(\Omega\), for any \(0 \leq t < T\). Indeed, in particular this implies that \(F(t, u(t)) = g(t, \cdot, u(t))\), and the proof of existence is complete. To this aim, we use Theorem 6.14 with the the fact that \(\omega_2\) is a weak solution of \([1.3]\) with \(f = \dot{\omega}_2 + \omega_2 \nabla \cdot V\), to see that

\[
\frac{d}{dt} \int (u - \omega_2)^+ \ dx \leq \int_{[u \geq \omega_2]} (g(\cdot, u) - \dot{\omega}_2 - \omega_2 \nabla \cdot V) \ dx
\]

\[
\leq \int_{[u \geq \omega_2]} (g(\cdot, u) - g(\cdot, \omega_2)) \ dx
\]

\[
\leq \max_{[\omega_1, \omega_2]} \theta \int (u - \omega_2)^+ \ dx.
\]
Applying Gronwall Lemma and using the fact that \( u(0) \leq \omega_2(0) \), we obtain \( u(t) \leq \omega_2 \) a.e. in \( Q \). The proof of \( u \geq \omega_1 \) in \( Q \) follows in the same way by proving that
\[
\frac{d}{dt} \int (\omega_1 - u)^+ \leq \max_{[\omega_1, \omega_2]} \theta \int (\omega_1 - u)^+.
\]
Thus the results of the theorem.

Now, for the limit of the solution of (5), thanks to Theorem 4.5 and Theorem 6.11, we have the following result.

**Theorem 5.8.** Assume \( V \in W^{1,2}(\Omega), \nabla \cdot V \in L^\infty(\Omega) \) and \( V \) satisfies the outpointing condition (1.4). Let \( g_m \) be a sequence of Carathéodory applications satisfying (G1), (G2) and (G3) with \( \theta \) independent of \( m \). For any \( u_{0m} \in L^2(\Omega) \) being a sequence of initial data let \( u_m \) be the sequence of corresponding solution of (5). If

\[
(5.31) \quad g_m(\cdot, r) \to g(\cdot, r), \quad \text{in } L^1(Q), \quad \text{for any } r \in \mathbb{R},
\]

and
\[
u_{0m} \to u_0, \quad \text{in } L^1(\Omega), \quad \text{and } |u_0| \leq 1 \text{ a.e. in } \Omega,
\]

then, we have

1. \( u_m \to u \) in \( C([0, T); L^1(\Omega)) \)
2. \( u_m \to \mu \) in \( L^2(0, T; H^1_0(\Omega)) \)-weak
3. \((u, \mu)\) is the solution of the Hele-Shaw problem

\[
(5.32) \quad \begin{cases}
\frac{\partial u}{\partial t} - \Delta p + \nabla \cdot (u V) = g(\cdot, u) & \text{in } Q \\
u = 0 & \text{on } \Sigma \\
u(0) = u_0 & \text{in } \Omega,
\end{cases}
\]

in the sense that \((u, \mu)\) is the solution of (1.4) with \( f(\cdot) = g(\cdot, u(\cdot)) \) a.e. in \( Q \) satisfying \( u(0) = u_0 \).

**Proof.** To begin with we prove compactness of \( u_m \) in \( C([0, T); L^1(\Omega)) \). We know that \( u_m \) is the mild solution of the sequence of Cauchy problems

\[
\begin{cases}
u_t + \mathcal{A}_m v \ni F_m(\cdot, u) & \text{in } (0, T) \\
u(0) = u_{0m}.
\end{cases}
\]
where, for a.e. \( t \in (0,T) \), \( F_m(t,z) = g_m(t,.,z(\cdot)) \vee (-M) \land M \), a.e. in \( \Omega \), for any \( z \in L^1(\Omega) \), and
\[
M := \max(\|\omega_1\|_\infty, \|\omega_2\|_\infty).
\]

Thanks to (1), one sees that \( F_m \) satisfies all the assumptions of Theorem \( 6.11 \). This implies, by Theorem \( 6.11 \), that
\[
(5.33) \quad u_m \to u, \quad \text{in } C([0,T];L^1(\Omega)), \text{ as } m \to \infty.
\]
Thus the compactness of \( u_m \). On the other hand, remember that \( u_m \) is a weak solution of
\[
\begin{cases}
\frac{\partial u}{\partial t} - \Delta u^m + \nabla \cdot (uV) = f_m & \text{in } Q \\
u = 0 & \text{on } \Sigma \\
u(0) = u_{0m} & \text{in } \Omega.
\end{cases}
\]
with \( f_m := g(.,u_m) \). Using again (1), (5.8) and (5), we see that
\[
f_m \to g(.,u) \quad \text{in } L^1(Q), \quad \text{as } m \to \infty.
\]
So, by Theorem \( 1.3 \) we deduce that \( u_m \to p \) in \( L^2(0,T;H^1_0(\Omega)) \)-weak and \((u,p)\) is the solution of the Hele-Shaw problem \( \mathcal{H} \). At last the uniqueness follows from the \( L^1 \)-comparison results of the solutions of the Hele-Shaw problem \( \mathcal{H} \) (cf. \( \mathcal{H} \)) as well as the assumption \((G_2)\) (one can see also \( \mathcal{H} \) for more details on Reaction-Diffusion Hele-Shaw flow with linear drift).

\section{Appendix}

\subsection{Reminder on evolution problem governed by accretive operator}

Our aim here is to remind the reader on some basic tools on \( L^1 \)-nonlinear semi-group theory we use in this paper. We are interested in PDE which can be be written in the following form

\[
(6.1) \quad \begin{cases}
\frac{du}{dt} + Bu \ni f & \text{in } (0,T) \\
u(0) = u_0,
\end{cases}
\]

where \( B \) is a possibly multivalued operator defined on \( L^1(\Omega) \) by its graph
\[
B = \{(x,y) \in L^1(\Omega) \times L^1(\Omega) : y \in Bx\},
\]
\( f \in L^1(0,T;L^1(\Omega)) \) and \( u_0 \in L^1(\Omega) \). An operator \( B \) is said to be accretif in \( L^1(\Omega) \) if and only if the operator \( J_\lambda := (I + \lambda B)^{-1} \) defines a contraction in \( L^1(\Omega) \), for any \( \lambda > 0 \); i.e. if for \( i = 1,2 \), \( (f_i - u_i) \in \lambda Bu_i \), then \( \|u_1 - u_2\| \leq \|f_1 - f_2\| \).

To study the evolution problem \( (6.1) \) in the framework of nonlinear semi-group theory in the Banach space \( L^1(\Omega) \), the main ingredient is to use the operator \( J_\lambda \), through the Euler-Implicit time
discretization scheme. For an arbitrary $n \in \mathbb{N^*}$ such that $0 < \varepsilon := T/n \leq \varepsilon_0$, we consider the sequence of $(u_t, p_t)_{t=0,...n}$ given by:

$$u_i + \varepsilon Bu_i \ni \varepsilon f_i + u_{i-1}, \quad \text{for } i = 1,...n,$$

where, for each $i = 0,...n - 1$, $f_i$ is given by

$$f_i = \frac{1}{\varepsilon} \int_{i\varepsilon}^{(i+1)\varepsilon} f(s) \, ds, \quad \text{a.e. in } \Omega.$$

Then, for a given $\varepsilon$–time discretization $t_i = i\varepsilon$, $i = 0,...n$, we define the $\varepsilon$–approximate solution

$$u_\varepsilon := \sum_{i=0}^{n-1} u_i \chi_{[t_i, t_{i+1})},$$

and its linear interpolate given by

$$\tilde{u}_\varepsilon(t) = \sum_{i=0}^{n-1} \frac{(t - t_i)u_{i+1} - (t - t_{i+1})u_i}{t_{i+1} - t_i} \chi_{[t_i, t_{i+1})}(t), \quad \text{for any } t \in [0, T).$$

In particular, one sees that $u_\varepsilon$, $\tilde{u}_\varepsilon$ and $f_\varepsilon$ satisfies the following $\varepsilon$–approximate dynamic

$$\frac{d\tilde{u}_\varepsilon}{dt} + Bu_\varepsilon \ni f_\varepsilon, \quad \text{in } (0, T).$$

The main goal afterwards is to let $\varepsilon \to 0$, to cover the “natural” solution of the Cauchy problem (6.1). The following theorem known as Crandall-Liggett theorem (at least in the case where $f \equiv 0$, cf. [21]) pictures the limit of $u_\varepsilon$ and $\tilde{u}_\varepsilon$.

**Theorem 6.9.** Let $B$ be an accretive operator in $L^1(\Omega)$ and $u_0 \in \overline{D(B)}$. If for each $\varepsilon > 0$, the $\varepsilon$–approximate solution $u_\varepsilon$ is well defined, then there exists a unique $u \in C([0,T), L^1(\Omega))$ such that $u(0) = u_0$,

$$u_\varepsilon \to u \quad \text{and} \quad \tilde{u}_\varepsilon \to u \quad \text{in } C([0,T), L^1(\Omega)), \text{ as } \varepsilon \to 0.$$

The function $u$ is called the mild solution of the evolution problem (6.1). Moreover, if $u_1$ and $u_2$ are two mild solutions associated with $f_1$ and $f_2$, then there exists $\kappa \in L^\infty(\Omega)$, such that $\kappa \in \text{sign}(u_1 - u_2)$ a.e. in $Q$, and

$$\frac{d}{dt} \|u_1 - u_2\|_1 \leq \int_{[u_1 = u_2]} |f_1 - f_2| \, dx + \int_{[u_1 \neq u_2]} \kappa (f_1 - f_2) \, dx, \quad \text{in } \mathcal{D}'(0,T).$$

On sees that this theorem figures out in a natural way a solution to the Cauchy problem (6.1) to settle existence and uniqueness questions for the associate PDE. However, in general we do not know in which sense the limit $u$ satisfies the concluding PDE; this is connected to the regularity of $u$ as well as to the compactness of $\frac{d\tilde{u}_\varepsilon}{dt}$. We refer interested readers to [7] and [8] for more developments and
examples in this direction. One can see also the book \[16\] in the case of Hilbert space, for which the concept of accretive operator is appointed by monotone graph notion.

One sees that besides the accretivity (monotinicity in the case of Hilbert space) the well posedness for the "generic" associate stationary problem

\[ u + \lambda Bu \ni g, \quad \text{for a given } g \]

is first need. Thereby, a sufficient condition for the results of Theorem 6.9 is given by the so called range condition

\[ \mathcal{R}(I + \lambda B) = L^1(\Omega), \quad \text{for small } \lambda > 0. \]

Indeed, in this case Euler-Implicit time discretization scheme is well pose for any \(i = 0, \ldots, n - 1\), and the \(\varepsilon\)-approximate solution is well defined (for small \(\varepsilon > 0\)). Then the convergence to unique mild solution \(u\) follows by accretivity (monotinicity in the case of Hilbert space).

In particular, Theorem 6.9 enables to associate to each accretif operator \(B\) satisfying the range condition a nonlinear semi-group of contraction in \(L^1(\Omega)\). It is given by Crandall-Ligget exponential formula

\[ e^{-tB}u_0 = L^1 - \lim \left( I + \frac{t}{n}B \right)^{-n} u_0, \quad \text{for any } u_0 \in D(B). \]

In other words the mild solution of (6.1) with \(f \equiv 0\) is given by \(e^{-tB}u_0\).

The attendance of a reaction in nonlinear PDE hints to study evolution problem of the type

\begin{align*}
\begin{cases}
\frac{du}{dt} + Bu &\ni F(.,u) \quad \text{in } (0,T) \\
u(0) &= u_0,
\end{cases}
\end{align*}

\tag{6.2}

where \(F : (0, T) \times L^1(\Omega) \to L^1(\Omega)\), is assumed to be Carathéodory, i.e. \(F(t,z)\) is measurable in \(t \in (0, T)\) and continuous in \(z \in L^1(\Omega)\). To solve the evolution problem (6.1) in the framework of \(\varepsilon\)-approximate/mild solution, we say that \(u \in C([0, T); L^1(\Omega))\) is a mild solution of (6.1) if and only if \(u\) is a mild solution of (6.1) with \(f(t) = F(t, u(t))\) for a.e. \(t \in (0, T)\). Existence and uniqueness are more or less well known in the case where \(F(t, r) = f(t) + F_0(r)\), with \(f(t) \in L^1(\Omega)\), for a.e. \(t \in [0, T)\), and \(F_0\) a Lipschitz continuous function in \(\mathcal{R}\). The following theorems set up general assumptions on \(F\) to ensure existence and uniqueness of mild solution for (6.1), as well as continuous dependence with respect to \(u_0\) and \(F\). We refer the readers to \[12\] for the detailed of proofs in abstract Banach spaces.

To call back these results, we assume moreover that \(F\) satisfies the following assumptions:

\((F_1)\) There exists \(k \in L^1_{loc}(0, T)\) such that

\[ \int (F(t, z) - F(t, \hat{z})) \text{sign}_0(z - \hat{z}) \, dx \leq k(t) \| z - \hat{z} \|_1, \quad \text{a.e. } t \in (0, T), \]

for every \(z, \hat{z} \in \overline{D(B)}\).
(F2) There exists \( c \in L^1_{\text{loc}}(0, T) \) such that
\[
\|F(t, z)\|_1 \leq c(t), \quad \text{a.e. } t \in (0, T)
\]
for every \( z \in \overline{D(B)} \).

In particular, one sees that under these assumptions, \( F(\cdot, u) \in L^1_{\text{loc}}(0, T; L^1(\Omega)) \) for any \( u \in C([0, T); L^1(\Omega)) \).

**Theorem 6.10.** (cf. [12]) If \( B \) be an accretive operator in \( L^1(\Omega) \) such that \( J_\lambda \) well defined in a dense subset of \( L^1(\Omega) \), then, for any \( u_0 \in \overline{D(B)} \) there exists a unique mild solution \( u \) of (6.1); i.e. \( u \) is the unique function in \( C([0, T); X) \), s.t. \( u \) is the mild solution of
\[
\begin{aligned}
\frac{du}{dt} + Bu &\ni f \\
\quad &\text{in } (0, T) \\
\quad u(0) &\cdot u_0,
\end{aligned}
\]
with \( f(t) = F(t, u(t)) \) a.e. \( t \in (0, T) \).

Another important results concerns the continuous dependence of the solution with respect to the operator \( B \) as well to the data \( f_n \) and \( u_{0n} \) is given in the following theorem. The proof may be found in [12].

**Theorem 6.11.** (cf. [12]) For \( m = 1, 2, \ldots \), let \( B_m \) be an accretive operators in \( L^1(\Omega) \) satisfying the range condition and \( F_m : (0, T) \times \overline{D(B_m)} \to L^1(\Omega) \) a Carathéodory applications satisfying (F1) and (F2) with \( k \) and \( c \) independent of \( m \). For each \( m = 1, 2, \ldots \) we consider \( u_{0m} \in \overline{D(B_m)} \) and \( u_m \) the mild solution of the evolution problem
\[
\begin{aligned}
\frac{du}{dt} + B_m u &\ni f_m \\
\quad &\text{in } (0, T) \\
\quad u(0) &\cdot u_{0m},
\end{aligned}
\]
with \( f_m = F_m(\cdot, u) \). If, there exists an accretive operators \( B \) in \( L^1(\Omega) \) and a Carathéodory \( F : (0, T) \times \overline{D(B)} \to L^1(\Omega) \) such that
\begin{enumerate}
\item[(a)] \( (I + \lambda B_m)^{-1} \to (I + \lambda B)^{-1} \) in \( L^1(\Omega) \), for any \( 0 < \lambda < \lambda_0 \)
\item[(b)] \( F_m(t, z_m) \to F(t, z) \) in \( L^1(\Omega) \), for a.e. \( t \in (0, T) \), and for any \( z_m \in \overline{D(B_m)} \) such that \( \lim_{m \to \infty} z_m = z \in \overline{D(B)} \).
\item[(c)] there exists \( u_0 \in \overline{D(B)} \), such that \( u_{0m} \to u_0 \).
\end{enumerate}
then
\[
u_m \to u, \quad \text{in } C([0, T); L^1(\Omega)),
\]
and $u$ is the unique mild solution of

$$\begin{cases}
\frac{du}{dt} + Bu \ni F(\cdot, u) & \text{in } (0,T) \\
u(0) = u_0.
\end{cases}$$

References

[1] D. ALEXANDER, I. KIM and Y. YAO. Quasi-static evolution and congested crowd transport. Nonlinearity, 27 (2014), No.4, 823-858.

[2] H. W. ALT and H. W. LUCKHAUS. Quasilinear elliptic-parabolic differential equations. Math. Z., 183(1983), pp. 311-341.

[3] B. ANDREIANOV and N. IGBIDA. Revising Uniqueness for a Nonlinear Diffusion-Convection Equation. J. Diff. Eq., Vol. 227 (2006), no-1 69-79.

[4] B. ANDREIANOV and N. IGBIDA. Uniqueness for the Inhomogeneous Dirichlet Problem for Elliptic-Parabolic Equations. Proc. Edinburgh Math. Society, 137A (2007), 1119-1133.

[5] B. ANDREIANOV and N. IGBIDA. On Uniqueness techniques for degenerate convection-diffusion problems. Int. J. of Dynamical Systems and Differential Equations, Vol. 4(2012), No.1/2 pp. 3 - 34

[6] G. ARONSON and Ph. BÉNILAN. Régularité des solutions de l’équation des milieux poreux dans $\mathbb{R}^N$. C. R. Acad. Sci. Paris Sér. A-B, 288(2): A103-A105, 1979.

[7] V. BARBU. Nonlinear differential equations of monotone types in Banach spaces. Springer Monographs in Mathematics, Springer, New York, 2010.

[8] Ph. BÉNILAN. Opérateurs accrétifs et semi-groupes dans les espaces $L^p$ (1 ≤ $p$ ≤ ∞). Functional Analysis and Numerical Analysis, Japan-France seminar, H.Fujita (ed.), Japan Society for the Advancement of Science, 1978.

[9] Ph. BÉNILAN, L. BOCCARDO, and M. HERRERO. On the limit of solution of $u_t = \Delta u^m$ as $m \to \infty$. Some Topics in Nonlinear PDE, Proceedings Int. Conf. Torino, 1989.

[10] Ph. BÉNILAN and M.G. CRANDALL. The Continuous Dependence on $\varphi$ of Solutions of $u_t - \Delta \varphi(u) = 0$. Ind. Uni. Math. J., 2(30):162–177, 1981.

[11] Ph. BÉNILAN and N. IGBIDA. Singular Limit of the Changing Sign Solutions of the Porous Medium Equation. J. Evol. Equations. 3 (2003), no. 2, 215–224

[12] Ph. BÉNILAN and N. IGBIDA. Singular Limit of Perturbed Nonlinear semi-groups. Comm. Appl. Nonlinear Anal., 3 (1996), no. 4, 23-42.
[13] Ph. BÉNILAN and N. IGBIDA. The Mesa Problem for the Neumann Boundary Value Problem. J. Differential Equations, 196 (2004), no. 2, 301-315.

[14] Ph. BÉNILAN and N. IGBIDA. Limite de $u_t = \Delta u^m + \text{div}(F(u))$, lorsque $m \to \infty$. Rev. Mat. Complut. , 13 (2000), no. 1, 195-205.

[15] M. BERTSCH and D. HILHORST. A density dependent diffusion equation in population dynamics: stabilization to equilibrium. SIAM Journal on Mathematical Analysis, 17(4):863–883, 1986.

[16] H. BRÉZIS, Opérateurs maximaux monotones et semi-groups de contractions dans les espaces de Hilbert. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York, 1973.

[17] J. CARRILLO. Entropy solutions for nonlinear degenerate problems. Arch. Rational Mech. Anal., 147(1999), pp. 269-361.

[18] J.A CARRILLO, K. CRAIG and Y. YAO. Aggregation-Diffusion Equations: Dynamics, Asymptotics, and Singular Limits. Active Particles , 2(2019) pp 65-108.

[19] J.A CARRILLO, A. JUNGEL P. A. MARKOWICH, and G. TOSCANI Entropy dissipation methods for degenerate parabolic problems and generalized sobolev inequalities. A Monatshefte fur Mathematik, 133(1):1–82, 2001.

[20] L. A. CAFFErelli and A. FRIEDMAN. Asymptotic Behavior of Solution of $u_t = \Delta u^m$ as $m \to \infty$. Indiana Univ. Math. J., 711-728, 1987.

[21] M.-G. CRANDALL and T.M. LIGGETT. Generation of Semi-Groups of Nonlinear Transformations on General Banach Spaces . American Journal of Mathematics, 93(2), 265-298, 1971.

[22] I. KIM and Y. ZHANG. Porous Medium Equation with a Drift: Free boundary regularity. Arch. Ration. Mech. Anal., (2021), 1-52.

[23] N. DAVID and M. SCHMIDTCHEN. On the incompressible limit for a tumour growth model incorporating convective effects . Preprint. https://hal.archives-ouvertes.fr/hal-03162169.

[24] N. DAVID and B. PERTHAME. Free boundary limit of a tumor growth model with nutrient. Journal de Mathématiques Pures et Appliquées. 2021.

[25] N. DAVID T. DEBIEC and B. PERTHAME. Convergence rate for the incompressible limit of nonlinear diffusion-advection equations. https://arxiv.org/pdf/2108.00787.pdf. 2021.

[26] E. DiBENEDETTO. Continuity of weak solutions to a general porous medium equation Indiana University Mathematics Journal. 32(1):83–118, 1983.

[27] C.M. ELLIOT, M.A. HERRERO, J.R. KING, and J.R.OCKENDON. The Mesa Patterns for $u_t = \nabla(u^m\nabla u)$ as $m \to \infty$. IMA J.Appl. Math., 37:147-154, 1986.

[28] H. ENNAJII, N. IGBIDA and G. JRADI. Prediction-Correction Pedestrian Flow by Means of Minimum Flow Problem. https://arxiv.org/pdf/2302.11315.pdf.
[29] O. Gil and F. Quirós. Convergence of the porous media equation to Hele-Shaw. Nonlinear Anal. Ser. A: Theory Methods, 44 (2001), no. 8, 1111-1131.

[30] N. Guillen I. Kim and A. Mellet. A Hele-Shaw limit without monotonicity. Arch. Rational Mech. Anal., Vo243, 829–868 (2022)

[31] N. Igbida. $L^1$–Theory for Hele-Shaw flow with linear drift. To appear in Math. Mod. and Meth. Appl. Sci., 2023. (https://arxiv.org/pdf/2105.00182.pdf).

[32] N. Igbida. Reaction-Diffusion Hele-Shaw flow with linear drift. Preprint 2023.

[33] N. Igbida. The Mesa-Limit of the Porous Medium-Equation and the Hele-Shaw Problem. Differential Integral Equations, 15 (2002), no. 2, 129-146.

[34] N. Igbida. Singular incompressible limit for reaction-diffusion porous medium equation with linear drift. Preprint 2023.

[35] N. Igbida and J. M. Urbano. Uniqueness for nonlinear degenerate problems. NoDEA Nonlin. Diff. Eq. Appl. 10(2003), no.3, pp. 287-307.

[36] I. C. Kim and H. K. Lei. Degenerate diffusion with a drift potential: A viscosity solutions approach Dynamical Systems, 27(2):767-786, 2010.

[37] I. C. Kim and N. Požár. Porous medium equation to Hele-Shaw flow with general initial density. Trans. Amer. Math. Soc., 370(2):873–909, 2018.

[38] I. C. Kim N. Požár and B. WOODHOUSE. Singularlimit of the porous medium equation with a drift Adv. Math., 349:682–732, 2019.

[39] S.N. Kruzhkov. First order quasilinear equations with several space variables. Mat. USSR-Sbornik, 10(1970), pp. 217–242.

[40] B. Maury, A. Roudneff-Chupin, F. Santambrogio. A macroscopic crowd motion model of gradient flowtype,Math. Models and Meth. in Appl. Sci., 20 (2010), No. 10, 1787-1821.

[41] B. Maury, A. Roudneff-Chupin, F. Santambrogio. Congestion-driven dendritic growth. Discrete Con-tin. Dyn. Syst., 34 (2014), no. 4, 1575-1604.

[42] B. Maury, A. Roudneff-Chupin, F. Santambrogio and J. Venel. Handling congestion in crowd motion modeling,Netw. Heterog. Media, 6 (2011), No. 3, 485-519.

[43] T. Diebic and M. Schmidtchen. Incompressible limit for a two-species tumor model with coupling through brinkman’s law in one dimension. Acta Applicandae Mathematicae, 2020.

[44] F. Otto. $L^1$ contraction and uniqueness for quasilinear elliptic-parabolic equations. J. Diff. Eq., 131(1996), pp. 20-38.

[45] B. Perthame, F. Quirós, J.L. Vázquez. The Hele-Shaw asymptotics for mechanical models of tumor growth. Arch. Rational Mech. Anal., 212 (2014), 93-127.
[46] J. L. VÁZQUEZ. The porous medium equation: mathematical theory. Oxford University Press, 2007.