Quantum Cauchy surfaces in canonical quantum gravity

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Abstract

For a Dirac theory of quantum gravity obtained from the refined algebraic quantization procedure, we propose a quantum notion of Cauchy surfaces. In such a theory, there is a kernel projector for the quantized scalar and momentum constraints, which maps the kinematic Hilbert space $\mathcal{K}$ into the physical Hilbert space $\mathcal{H}$. Under this projection, a quantum Cauchy surface isomorphically represents a physical subspace $\mathcal{D} \subset \mathcal{H}$ with a kinematic subspace $\mathcal{V} \subset \mathcal{K}$. The isomorphism induces the complete sets of Dirac observables in $\mathcal{D}$, which faithfully represent the corresponding complete sets of self-adjoint operators in $\mathcal{V}$. Due to the constraints, a specific subset of the observables would be ‘frozen’ as number operators, providing a background physical time for the rest of the observables. Therefore, a proper foliation with the quantum Cauchy surfaces may provide an observer frame describing the physical states of spacetimes in a Schrödinger picture, with the evolutions under a specific physical background. A simple model will be supplied as an initiative trial.

Keywords: canonical quantum gravity, refined algebraic quantization, Dirac observables, Schrödinger picture

1. Introduction

The Schrödinger picture is crucial for our understanding and experimental testing of the existing quantum theories. In this picture, the dynamics is the evolution of the wave functions over the spectrum of a certain complete set of observables, happening under a physical background that provides the notion of time. As is well known, a Schrödinger theory is a quantum representation of a Hamiltonian dynamics.

In quantum gravity, finding the Schrödinger representation poses profound challenges [1, 2]. Canonical general relativity had succeeded in providing an initial data formulation for...
the globally hyperbolic spacetimes. However, since the theory has no fixed physical background, formulating it as a Hamiltonian dynamics in a physical time demands a splitting of the phase space coordinates, into two subsets of the background fields and the dynamical fields \([3, 4]\). Particularly, the background has to yield a monotonic physical time, and the dynamical fields have to define a phase space of complete observables. Under gravity, the background and the dynamical fields must be coupled, and are constrained by one another through the scalar and momentum constraints. Therefore, the proper splitting becomes a highly nontrivial and state dependent issue \([1, 2]\). At the quantum level, a direct Schrödinger representation is thus hindered for a quantum theory of general relativity.

In addressing this problem, the existing theories of canonical quantum gravity mainly follow either one of the two approaches we now briefly describe. Throughout the paper, we will allow matter fields in a theory of our concern. To focus on the gravitational issues, we will assume a kinematic phase space which already solves all the constraints except for the scalar and momentum constraints \(\{C_i^N\}\), respectively labeled by \(N = 0\) and \(N = 1, 2, 3\). The index \(i\) labels the spatial modes in a compact space.

The first type of theories follows the reduced phase space quantization \([3, 5–7]\), and they are Schrödinger theories by construction. Starting from the classical level, one first identifies a physically significant subset \(\mathbb{D}^d\) of the physical states, which allows the proper splitting of the phase space coordinates mentioned above. We denote the chosen background fields as \(\{\phi_i^N, P_i^N\}\), and the dynamical fields as \(\{\psi_i^K, P_i^K\}\). For the full theory, the background consists of four clock fields \(\{\phi_i^N\}\) which coordinatize each of the constraint orbits on the constraint surface belonging to \(\mathbb{D}^d\). These background fields then provide a set of physical spacetime coordinates to each of the spacetimes represented by the orbits. We may then define the set of dynamical observables \(\{\psi_i^K(\phi_i^N), P_i^K(\phi_i^N)\}\) as functions in \(\mathbb{D}^d\) taking the values of the dynamical fields at where \(\phi_i^N = \tilde{\phi}_i^N\) in each of the spacetimes. Meanwhile, the observables \(\{\phi_i^N(\tilde{\phi}_i^N)\}\) are trivially defined, and the values of \(\{P_i^N(\tilde{\phi}_i^N)\}\) are determined by solving the constraints. Remarkably, for each allowed value of \(\tilde{\phi}_i^N\) the dynamical observables \(\{\psi_i^K(\tilde{\phi}_i^N), P_i^K(\tilde{\phi}_i^N)\}\) represent \(\{\psi_i^K, P_i^K\}\) through a Poisson isomorphism \([5]\), and thus coordinatize a reduced phase space of the physical states \(\mathbb{D}^{d'}\). Subsequently, one may choose a one parameter family of the background field values \(\{\tilde{\phi}_i^N(\tau)\}\) with a monotonic time function \(T(\tilde{\phi}_i^N(\tau))\), which defines a specific foliation of the spacetimes in \(\mathbb{D}^d\). This way, we obtain an unconstrained Hamiltonian theory in the reduced phase space, with the evolutions happening in the physical time \(T(\tilde{\phi}_i^N(\tau))\) under the physical background \(\{\tilde{\phi}_i^N(\tau)\}\). Finally, the Hamiltonian dynamics may be quantized into a Schrödinger theory with the wave functions of the form \(\tilde{\Psi}(\tau)|\psi^K_i\rangle\).

In this approach, the obtained Schrödinger theory is built from the classical Cauchy surfaces labeled by the background fields \(\tilde{\phi}_i^N(\tau)\), which are clearly specialized in the theory to remain classical. Also, since a reduced phase space covers only the region \(\mathbb{D}^{d'}\), the Schrödinger theory could not recover the full solutions of canonical general relativity in the classical limits. These issues arise from reconciling the relative and local notion of time in the classical theory, with the absolute and global time required by the quantum theory. People considering the general covariance as fundamental tend to regard this approach as effective or phenomenological. In canonical loop quantum gravity the approach is applied \([5]\) as an effective method, assuming negligible quantum fluctuations of the background fields; in the dust model \([3]\) proposed by Brown and Kuchař the four privileged background dust fields with special coupling to gravity are introduced as phenomenological objects. On the other
hand, people regarding the principle of quantum mechanics as fundamental [6, 7] tend to regard the approach as fundamental, selecting a truly physical sector of states accompanied with a privileged absolute time, which then leads to a familiar well-defined quantum field theory. For the context of this paper, we observe that the physical time in these theories, be it absolute or not, is very different from the Newtonian time. It is given by the non-dynamical background fields which are experimentally observable just like the dynamical fields. Thus one may always ask whether the theory may still encode the possibly observable quantum nature of the background fields in some alternative notion of time.

The second type of theories [8, 9] seeks to quantize the full canonical general relativity following the Dirac quantization program, such that the general covariance remains in the quantum level. Currently, the most concrete way to implement the program is given by a procedure known as the refined algebraic quantization [10–12].

In these theories, the canonical quantization is first applied to the unconstrained phase space, such that the mentioned fields are equally represented by the operators \{\hat{\phi}_N, \hat{P}_N^X, \psi_i, \hat{P}_i^K\} defined in a kinematic Hilbert space \(\mathbb{K}\). The constraints should be also properly quantized as self-adjoint operators \{\hat{G}_N^N\} in \(\mathbb{K}\). The physical Hilbert space \(\mathbb{H}\) is then given by the (generalized) kernel of the constraint operators. Note that \(\mathbb{H}\) demands a definition of Hermitian inner product without an absolute notion of time. Remarkably, this inner product can be naturally defined through the refined algebraic quantization [10–12], such that the self-adjoint operators in \(\mathbb{K}\) commuting with \{\hat{G}_N^N\} become the Dirac observables in \(\mathbb{H}\). This way, each state \(\Psi \in \mathbb{H}\) is a quantum representation of a spacetime.

This type of theories faces many challenges from its unconventional timeless nature. For the dynamics, one has to identify the Dirac observables carrying the information of physical evolutions. A widely studied method is to identify the specific Dirac observables \{\hat{\psi}_i^K (\hat{\phi}_j^N), \hat{P}_i^K (\hat{\phi}_j^N)\} representing the classical observables \{\psi_i^K (\phi_j^N), P_i^K (\phi_j^N)\} mentioned above. These observables are called the quantum relational observables [30–32], and they differ from their counterpart in the reduced phase space theories in two crucial aspects. First, one set of quantum relational observables \{\hat{\psi}_i^K (\hat{\phi}_j^N), \hat{P}_i^K (\hat{\phi}_j^N)\} is at the same footing of another set \{\psi_i^K (\phi_j^N), P_i^K (\phi_j^N)\}; they are just two different sets of observables using different clocks. Therefore, the notion of time here is truly relative and compatible with the general covariance. Second, the observables \{\hat{\psi}_i^K (\hat{\phi}_j^N), \hat{P}_i^K (\hat{\phi}_j^N)\} are composite operators utilizing the clock fields as the quantum field operators \(\hat{\phi}_j^N\) rather than a classical background. This means that the algebra among \{\hat{\psi}_i^K (\hat{\phi}_j^N), \hat{P}_i^K (\hat{\phi}_j^N)\} would contain the complicated corrections from the quantum fluctuations of the clock fields. The corrections can give the quantum effects of the instruments in a realistic measurement [30], but they also obscure the notion of a complete set of observables for the quantum theory. In contrast to the case of the reduced phase space approach, the Cauchy surfaces identified by the quantum relational observables fluctuate with the quantized clock fields, and they are unsuitable for defining a Schrödinger theory.

The goal of this paper is to propose an exact notion of Cauchy surfaces in the Dirac theories, which is analogous to that in the reduced phase space theory and suitable for defining a Schrödinger theory. By demands of the Dirac theories, these Cauchy surfaces must be defined at the quantum level in a relative way. In a general prescription, we will provide a natural definition for the quantum Cauchy surfaces in the context of the refined algebraic quantization. Then, we will develop a local construction of quantum Cauchy surfaces with a specified physical background for a physical subspace \(\mathcal{D} \subset \mathbb{H}\). Then, we will extrapolate the
conditions for a set of quantum Cauchy surfaces to form a foliation for a Schrödinger theory, under a specified background appearing to be without quantum fluctuations. Also, we will relate our formulation to the path integral transition amplitudes from the covariant theories of quantum gravity. Lastly, as an initial trial, we will apply the idea to a simple model with a single scalar constraint, and demonstrate that the Schrödinger theories of the different foliations could emerge from the same underlying Dirac theory, as perceived from the observer frames with the different backgrounds.

2. General prescription

Refined algebraic quantization [10–12] is a concrete procedure to realize the Dirac quantization of a system with first-class constraints. The application of the procedure to canonical general relativity has been a subject under intense studies [8, 13, 14]. In quantum cosmology, refined algebraic quantization has been completed in many symmetrically reduced models [15, 16], yielding rich implications of the quantum gravitational effects. As for the full theory, the most promising on-going program lies in loop quantum gravity [8, 13, 14]. In the following, we summarize the steps of the procedure. Throughout this paper we should choose our units such that \( c \equiv \hbar \equiv 1 \).

Taking both cosmological models and the full theory into consideration, we set the ranges of the indices in \( \{ C_i^N \} \) to vary with the assumed symmetries. For instance, a homogeneous and isotropic model has only the scalar constraint \( N = 0 \) with a global spatial mode \( i = 0 \). The first step is the canonical quantization of the unconstrained phase space, leading to the kinematic Hilbert space \( \mathcal{H} \), where the constraints are represented by a set of self-adjoint operators \( \{ \hat{C}_i^N \} \). Then, one must impose the quantum constraints to obtain the physical Hilbert space. This crucial step in the procedure is through finding a constraint kernel projection map \( \hat{P} : \mathbb{K} \to \mathbb{K}^* \), called a rigging map [10–12]. The map has to properly implement the form

\[
\hat{P} \sim \prod_{N,i} \delta(\hat{C}_i^N),
\]

such that its image satisfies \( \hat{C}_i^N \lvert_{\text{Image} \hat{P}} = 0 \) in a proper sense. When this map is found, it naturally endows \( \text{Image} \hat{P} \) with a Hermitian inner product. Such an inner product between two physical states \( \lvert \Psi_1 \rangle = \hat{P} \lvert \psi_1 \rangle \) and \( \lvert \Psi_2 \rangle = \hat{P} \lvert \psi_2 \rangle \), would be given by [10–12]

\[
\langle \Psi_1 | \Psi_2 \rangle = \langle \psi_1 | \hat{P}^* | \psi_2 \rangle.
\]

The physical Hilbert space \( \mathbb{H} \equiv \text{Image} \hat{P} \) is thus defined through the rigging map, and a physical state \( \lvert \Psi \rangle \in \mathbb{H} \) is a quantum representation of a spacetime. Furthermore, with this inner product, a self-adjoint operator in \( \mathbb{K} \) commuting with \( \{ \hat{C}_i^N \} \) automatically becomes a Dirac observable [10–12] in \( \mathbb{H} \).

For the simplest cases of the FLRW quantum cosmological models with a massless scalar matter field \( T \) [16–18], the kinematic phase space coordinates consist of the scalar-field conjugate pair \( (\tilde{T}, \tilde{P}_T) \) and the gravitational conjugate pair \( (\tilde{\alpha}, \tilde{P}_\alpha) \) for the scale factor and the extrinsic curvature. The only constraint operator is the global scalar constraint operator \( \hat{C}_{i=0}^0 \equiv \hat{C} \), generating a one-parameter group of unitary transformations. In this case, \( \hat{P} \) is shown [18] to be given by the group averaging expression

\[
\hat{P} \equiv \int d\lambda \, \hat{U}(\lambda); \quad \hat{U}(\lambda) \equiv e^{-i\lambda \hat{C}}.
\]
From the resulted physical Hilbert space, various important implications about the cosmic evolution have been derived using the quantum relational observables of the form \[ [18] \]

\[ \hat{O}(\hat{T}) = \int d\lambda \ \hat{U}(\lambda) \text{sym} \{\hat{O} f(\hat{P}_T) \delta(\hat{T} - \hat{T})\} \hat{U}^{-1}(\lambda); \ \hat{O} \equiv O(\hat{\alpha}, \hat{\beta}) , \quad (2.3) \]

where sym denotes a proper self-adjoint symmetrization, and the factor \( f(\hat{P}_T) \) contains both the absolute value of the Jacobian between \( dT \) and \( d\lambda \), and the factor needed to resolve possible Gribov ambiguity. These Dirac observables represent the value of the gravitational variables at the clock time \( \hat{T} = \hat{T} \) in the Universe.

Constructing a rigging map for the full theory requires the detailed knowledge of the group of transformations generated by the scalar and momentum constraints, which have a complicated algebra with structure functions. Despite the challenge, remarkable progresses have been achieved especially in the framework of loop quantum gravity \([13, 14]\).

As a remarkable triumph, a rigging map solving the momentum constraints has been rigorously implemented in loop quantum gravity, leading to a spatial-diffeomorphism invariant kinematic Hilbert space \([33–35]\) that concretely realizes the spatial quantum geometry. Recently, there are also significant advances in the pursuit of anomaly-free quantization of both the scalar and momentum constraints \([20, 21]\). Therefore, it is hopeful that a full rigging map for both the quantum scalar and momentum constraints may be defined, once the quantum constraints’ algebra could be controlled and simplified. The method of the master constraint \([22, 23]\) provides another possible direction. The classical master constraint is given by the weighted sum over the square of each of the original constraints, which combines the infinitely many original constraints into a single one. The master constraint operator in loop quantum gravity \([22, 23]\) has been successfully constructed. Also, it has been shown \([24]\) that the master constraint may have the proper spectrum for a rigging map. In this method with only one constraint operator, it is even hopeful that the rigging map may be constructed with the group averaging method \((2.2)\).

In a broader context, the matrix elements of a rigging map defined in \((2.1)\) have been studied as the transition amplitudes in many path integral formalisms of quantum gravity \([25–28]\). Conversely, one of such consistent path integral theory may define a rigging map for a corresponding Dirac theory \([25, 26]\). Through the progress of these path integral formalisms, it is hopeful that we may understand and calculate the rigging maps via the path integral transition amplitudes. This is especially the case in loop quantum gravity, which has a path integral formalism called the spinfoam models \([8, 25, 26]\). The models are originated to calculate a rigging map for the canonical loop quantum gravity, with the path integrals defined according to the actions of the constraint operators. The transition amplitudes in the spinfoam models are given by summing over the discretized history of the quantum geometry \([8, 25, 26]\), which may be computed perturbatively. With the remarkable progresses in the spinfoam models, it is hopeful that the models would effectively define the desired rigging map for canonical loop quantum gravity.

Having mentioned the above, we should assume that the refined algebraic quantization can be applied to quantize a theory of our concern. The resulted quantum theory is thus equipped with the triplet \((\hat{H}, \hat{K}, \hat{P})\) specified above.

### 2.1. Definition

We start by making an observation in the FRW cosmology, where \(\hat{P}\) can be constructed by the group averaging method as in \((2.2)\). Recall in these cases we have the standard quantum relational observables \((2.3)\), whose actions on a quantum state of spacetime \(\Psi \in \mathcal{H}\) would be
\[ \hat{O}(\hat{T})|\Psi\rangle \equiv \int d\lambda \; \hat{U}(\lambda) \text{sym}\{\hat{O} f(\hat{P}_T) \delta(\hat{T} - \hat{T})\} \hat{U}^{-1}(\lambda) |\Psi\rangle \]
\[
= \hat{P} \cdot \text{sym}\{f(\hat{P}_T) \delta(\hat{T} - \hat{T})\} |\Psi\rangle
\]  

(2.4)

where the invariance \(\hat{U}^{-1}(\lambda)|\Psi\rangle = |\Psi\rangle\) holds for the physical state. A classical Cauchy surface in a spacetime represents a causal instance, which contains complete but non-redundant information about the spacetime. Thus If the clock \(T\) is to specify a unique Cauchy surface with \(T = \hat{T}\) for every physical solution, the condition \(I(T) = 1\) should be satisfied on the constraint surface. The analogous condition at the quantum level is
\[ \hat{I}(\hat{T})|\Psi\rangle = \hat{P} \cdot \text{sym}\{f(\hat{P}_T) \delta(\hat{T} - \hat{T})\} |\Psi\rangle = |\Psi\rangle \]

(2.5)

for any \(\Psi \in \mathbb{H}\). That is, the operator defining the Cauchy surface is a right inverse operator to \(\hat{P}\).

In light of this observation, we now take (2.5) to be the fundamental definition of a quantum Cauchy surface. That is, for any given rigging map \(\hat{P}\), we define a quantum Cauchy surface \(\hat{\Pi}_b\), labeled by \(b\), to be a linear map satisfying
\[ \hat{\Pi}_b : \mathbb{H} \to \mathbb{K} ; \hat{P} \cdot \hat{\Pi}_b = \hat{I} . \]

(2.6)

Essentially, \(\hat{\Pi}_b\) is a specific isomorphism between \(\mathbb{H}\) and \(\text{Image}[\hat{\Pi}_b] \equiv \mathcal{V}^b \subset \mathbb{K}\), so that each state \(\Psi \in \mathbb{H}\) is uniquely represented by an element \(\hat{\Pi}_b \Psi \equiv \psi_b \in \mathcal{V}^b\).

We denote the set of all linear operators in \(\mathbb{K}\) as \(\{\hat{O}_b : \mathbb{K} \to \mathbb{K}\}\), and the set of all Dirac observables as \(\{\hat{O}_b = \hat{O}_N : \mathbb{H} \to \mathbb{H}\}\). For a given \(\hat{\Pi}_b\), we denote the set of all \(\mathcal{V}^b\)-preserving operators as \(\{\hat{O}^b = \hat{O}_N\} \subset \{\hat{O}_N\}\). Through the isomorphism, the Dirac observables \(\{\hat{O}_N\}\) can be represented by a corresponding set of operators \(\{\hat{O}^b = \hat{O}_N\} \subset \{\hat{O}_N\}\) satisfying
\[ \hat{O}_N = \hat{P} \cdot \hat{O}_N^b \hat{\Pi}_b \equiv \hat{O}_N^b (b) . \]

(2.7)

Clearly from the definition, the restriction \(\hat{O}_N^b \mid_{\mathcal{V}^b}\) is determined by \(\hat{O}_N^b\), while its extension outside of the subspace \(\mathcal{V}^b\) is arbitrary. Also, we have introduced the notation \(\hat{O}_N^b (b)\) with the interpretation that \(\hat{O}_N^b\) represents the value of \(\hat{O}_N\) at the Cauchy surface \(b\). This interpretation is supported by the algebraic isomorphism
\[ \hat{O}^b (b) \hat{O}^b (b) = (\hat{O}^b \hat{O}^b) (b) \text{ and } [\hat{O}^b (b), \hat{O}^b (b)] = [\hat{O}^b, \hat{O}^b] (b) . \]

(2.8)

To understand the local limits of the observables \(\{\hat{O}^b (b)\}\), it is necessary to relate them to the self-adjoint subset \(\{\hat{O}^b (b) = \hat{O}^b (b)\} \subset \{\hat{O}^b\}\). Note that the operator \(\hat{O}_N^b\) in (2.7) generally cannot be chosen as self-adjoint in \(\mathbb{K}\), since \(\hat{\Pi}_b\) is generally not an isometry between the two Hilbert spaces \(\mathcal{V}^b\) and \(\mathbb{H}\). However, \(\hat{\Pi}_b\) is always related to an isometry through a similar transformation in \(\mathcal{V}^b\). Setting \([|E_b\rangle\}\) to be an orthonormal basis of \(\mathbb{H}\), we introduce a similar map \(\hat{\Lambda}_b \in \{\hat{O}^b\}\) such that \(|e_b\rangle \equiv \hat{\Lambda}_b \hat{\Pi}_b |E_b\rangle\). The isometry \(\hat{\Lambda}_b \hat{\Pi}_b\) then leads to a new expression of (2.7) with
\[ \hat{O}^b (b) = \hat{\Lambda}^b \hat{O} (b) \hat{\Lambda}^b ; \hat{O}^b (b) = (\hat{\Lambda}^b \hat{O} \hat{\Lambda}^b) (b) . \]

(2.9)

Here, \(\hat{O}_N^b \mid_{\mathcal{V}^b}\) is self-adjoint and determined up to an unitary ambiguity in choosing the \(\hat{\Lambda}_b\). The algebraic isomorphism above is then carried over as...
\[
(\hat{\Lambda}^{-1}_b \hat{O}_N \hat{\Lambda}_b)(t_0) \hat{\Lambda}^{-1}_b \hat{O}_N \hat{\Lambda}_b)(t_0) = (\hat{\Lambda}^{-1}_b \hat{O}_N \hat{\Lambda}_b)(t_0), \quad \text{and}
\]
\[
[(\hat{\Lambda}^{-1}_b \hat{O}_N \hat{\Lambda}_b)(t_0), (\hat{\Lambda}^{-1}_b \hat{O}_N \hat{\Lambda}_b)(t_0)] = (\hat{\Lambda}^{-1}_b [\hat{O}_N, \hat{O}_N] \hat{\Lambda}_b)(t_0).
\]

In the reverse direction, we have the following important conclusion. Given a quantum Cauchy surface \( \Pi_{n_{\text{f}}} \), each complete set of self-adjoint of operators in \( \mathcal{V}^n \), namely the set \( \{\hat{O}_N^{\text{op}}\} \), determines a complete set of Dirac observables \( \{(\hat{\Lambda}^{-1}_b \hat{O}_N^{\text{op}} \hat{\Lambda}_b)(t_0)\} \) up to an unitary ambiguity in \( \mathbb{H} \) associated to the choice of \( \hat{\Lambda}_b \).

Let us now observe the relation between the quantum relational observables and the Dirac observables we just defined. We look into the simple case of the FLRW quantum cosmology by comparing the expression (2.4) with the expression (2.7) when applied to the setting. We see that the two types of observables may be closely related if the pair \((f(\hat{P}_T) \delta(\hat{T} - \hat{T}), \mathcal{O}(\hat{\alpha}, \hat{\beta}))\) is identifiable with the pair \((\hat{\Pi}_{n_{\text{f}}} \hat{\Lambda}^{-1}_b \hat{O}_N \hat{\Lambda}_b)\) under certain approximations. In many interesting models where the \( T \) field has a good asymptotic behavior, the condition (2.5) would hold exactly [18, 29] with a proper \( f(\hat{P}_T) \). In these models we may choose \( \Pi_{n_{\text{f}}} \equiv \text{sym}(f(\hat{P}_T) \delta(\hat{T} - \hat{T})) \) as a quantum Cauchy surface. The corresponding subspace \( \mathcal{V}^n \) would be an eigenspace of \( \hat{T} \) with \( \hat{T}_{|\mathcal{V}^n} = \hat{T} \), and it would also have a complete set of self-conjugate operators \( \{\mathcal{O}_N^{\text{op}}(\hat{\alpha}, \hat{\beta})\} \). With each given \( \hat{\Lambda}_b \), the set gives a complete set of Dirac observables \( \{(\hat{\Lambda}^{-1}_b \mathcal{O}_N^{\text{op}}(\hat{\alpha}, \hat{\beta}) \hat{\Lambda}_b)(t_0)\} \). When the given \( \hat{\Lambda}_b \) is such that
\[
(\hat{\Lambda}^{-1}_b \mathcal{O}_N^{\text{op}}(\hat{\alpha}, \hat{\beta}) \hat{\Lambda}_b)(t_0) = \mathcal{O}_N^{\text{op}}(\hat{\alpha}, \hat{\beta})(0) + \mathcal{O}(\hbar),
\]
we have
\[
(\hat{\Lambda}^{-1}_b \hat{\mathcal{O}}^{\text{op}}_N \hat{\Lambda}_b)(t_0) = \hat{\mathcal{O}}^{\text{op}}_N(\hat{T}) + \mathcal{O}(\hbar); \quad (\hat{\Lambda}^{-1}_b \hat{T} \hat{\Lambda}_b)(t_0) = \hat{T}.
\]

Therefore, in these two sets of observables are two distinct quantum representations of the same classical observables.

Here, the difference term of \( \mathcal{O}(\hbar) \) represents a crucial distinction between the two types of observables. The self adjointness of the quantum relational observables is established through the self-adjoint extensions, which symmetrize the operator orderings. On the other hand, our observables are isomorphically induced from the set of kinematic self-adjoint operators in \( \mathcal{V}^n \). Therefore, while the relational quantum observables’ algebra contains the quantum corrections from the quantum clocks, the quantum Cauchy surfaces lead to the Dirac observables faithfully representing their kinematic counterparts.

2.2. Local construction of a quantum cauchy surface

The purpose of introducing the quantum Cauchy surfaces is to derive a Schrödinger theory from a Dirac theory \((\mathbb{H}, \mathbb{K}, \hat{\mathbb{P}})\). To do that, we first identify the instantaneous wave functions at a quantum Cauchy surface with a given physical background.

Let us continue using the previous notations with \( \mathbb{K} \equiv \text{Span}\{\ket{\phi_i} \otimes \ket{\psi_f}\} \), in which the field operators \([\hat{\phi}^X, \hat{P}^X, \hat{\psi}^X, \hat{\mathbb{P}}^X]\) are defined. Although all the fields are quantized in the Dirac theory, the physical quantum degrees of freedom are ‘fewer’ than those of \( \mathbb{K} \) due to the quantum constraints \( \{\mathcal{C}_N^X\} \). By our construction, these physical degrees of freedom are exactly the degrees of freedom in \( \mathcal{V}^n \). Our goal is to further show that the degrees of freedom in \( \mathcal{V}^n \) can provide the physical spectra to certain Schrödinger wave functions, at the moment of time given by the quantum Cauchy surface \( \hat{\Pi}_{n_{\text{f}}} \). Such a moment could be labeled by a certain set of background fields, whose quantum fluctuations are absent in \( \mathcal{V}^n \) due to the constraints. They would be as many as the number of the constraints, and taking a set of specific eigenvalues in \( \mathcal{V}^n \). For a set of chosen background fields, we expect such a description to be valid only
locally in a sub-Hilbert space $\mathbb{D} \subset \mathbb{H}$. Accordingly, we would also localize our construction in the following way.

We now set the instantaneous background to be $\phi^N_t = \phi^N(\tau_0)$. This restricts $\mathbb{K}$ to its maximal subspace consistent with the background, which is just the corresponding eigenspace given by $S^g_\mathbb{K} = \text{Span}\{ |\phi^N_j(\tau_0)\rangle \otimes |\psi^K_j\rangle \}$. This also defines the domain of the Heisenberg states for our Schrödinger theory, which would be $\mathbb{D}_g = \hat{\mathbb{P}} S^g_\mathbb{K} \subset \mathbb{H}$. In general, the spectra of the wave functions cannot be found in $S^g_\mathbb{K}$, as it may have a nontrivial kernel of $\hat{\mathbb{P}}$ and thus fail to be isomorphic to $\mathbb{D}_g$. Instead, we should look into a quantum Cauchy surface $\hat{\Pi}_g = \hat{\mathbb{P}} S^g_\mathbb{K}$. Being consistent with the background and isomorphic to $\mathbb{D}_g$, the subspace $S^g_\mathbb{K}$ would satisfy

$$S^g_\mathbb{K} \supset \text{Ker}[\hat{\mathbb{P}}|_{S^g_\mathbb{K}}] = S^g_. \quad (2.12)$$

Reversely, each $S^g_\mathbb{K}$ satisfying the above defines a quantum Cauchy surface in the domain $\mathbb{D}_g$ which is consistent with the background $\phi^N_\mathbb{K} = \phi^N(\tau_0)$.

Therefore, the first important step is identifying the kernel $\text{Ker}[\hat{\mathbb{P}}|_{S^g_\mathbb{K}}]$. The kernel is given by the set of elements corresponding to the zero vector in $\mathbb{H}$ with zero norm under the rigging map. Since the rigging map also defines the inner product through (2.1), the basis of the kernel is specified by the linearly independent maximal set $\{ |e^K_\lambda\rangle \} \subset S^g_\mathbb{K}$ (with the members labeled by $\lambda$) satisfying

$$\langle e^K_\lambda | \hat{\mathbb{P}} | e^K_\mu \rangle = 0. \quad (2.13)$$

The space $\text{Span}\{ |e^K_\lambda\rangle \} = \text{Ker}[\hat{\mathbb{P}}|_{S^g_\mathbb{K}}]$ then specify the valid choices for $S^g_\mathbb{K}$ satisfying (2.12). Clearly whenever the kernel is non-trivial, the space $S^g_\mathbb{K}$ is non-unique.

Suppose the kernel is determined and a specific $S^g_\mathbb{K}$ is chosen correspondingly. This gives us a quantum Cauchy surface $\hat{\Pi}_g$ specified in $\mathbb{D}_g$. To define the wave functions we are looking for, we use (2.7) to construct the Dirac observables preserving the Heisenberg domain $\mathbb{D}_g$. By definition, they are the representations of the $S^g_\mathbb{K}$-preserving set $\{ \hat{O}^S_\mathbb{K} \} \subset \{ \hat{O}^V_\mathbb{K} \}$, whose restriction $\{ \hat{O}^S_\mathbb{K} |_{S^g_\mathbb{K}} \equiv \hat{O}^S_\mathbb{K} (\phi^N_\mathbb{K}, \hat{\psi}_1^K, \hat{\psi}_2^K) : S^g_\mathbb{K} \rightarrow S^g_\mathbb{K} \}$ is the set of all the self-adjoint operators in $S^g_\mathbb{K}$.

The observables also require a $S^g_\mathbb{K}$-preserving $\hat{\Lambda}_{\mathbb{K}}$, which will be denoted as $\hat{\Lambda}^S_\mathbb{K}$, whose restriction $\hat{\Lambda}^S_\mathbb{K} |_{S^g_\mathbb{K}} \equiv \hat{\Lambda}^S_\mathbb{K} : S^g_\mathbb{K} \rightarrow S^g_\mathbb{K}$ is a similar transformation in $S^g_\mathbb{K}$. Set $\{ |e^K_\lambda\rangle \}$ to be an orthonormal basis of the Hilbert space $S^g_\mathbb{K}$. In this paper, we assume integer values for the index $K$, for the convenience of our description. By definition, the basis $\{ \hat{\mathbb{P}} \hat{\Lambda}^S_\mathbb{K} | e^K_\lambda \rangle \equiv | E^K_\lambda \rangle \}$ of $\mathbb{D}_g$ should be orthonormal in the Hilbert space $\mathbb{H}$. This condition then defines $\hat{\Lambda}^S_\mathbb{K}$ through the equation

$$\langle \Lambda^{-1}_J L^K | \hat{\Lambda}^{-1}_J L^M | \hat{\mathbb{P}} | e^K_\lambda \rangle = (E^L_\lambda E^M_\lambda) = \delta_{LM}. \quad (2.14)$$

Obviously, the $\hat{\Lambda}^S_\mathbb{K}$ is defined up to a unitary transformation in $S^g_\mathbb{K}$.

So far we have argued that, given a background $\phi^N_\mathbb{K} = \phi^N(\tau_0)$, via (2.13) and (2.14) the rigging map elements in $S^g_\mathbb{K}$ define a quantum Cauchy surface $\hat{\Pi}_g$ and the corresponding Dirac observables in $\mathbb{D}_g$, up to the two ambiguities. Before addressing the ambiguities, let us write down the instantaneous wave functions.

Through (2.7), the set of $\mathbb{D}_g$ preserving Dirac observables is the set $\{ (\hat{\Lambda}^{-1}_J \hat{O}^S_\mathbb{K} \hat{\Lambda}^S_\mathbb{K}) (\tau_0) \}$, whose action on a state $\Psi_{\mathbb{D}_g} \in \mathbb{D}_g$ takes the form
To explore the dynamics of the theory, we specify a background $\tilde{\phi}^N_i(\tau)$ over various $\tau$, which gives a scalar time field $T_j(\tilde{\phi}^N_i(\tau))$ increasing monotonically with $\tau$. Following our construction in the previous section, we find a family of quantum Cauchy surfaces $\{\tilde{\Pi}_\tau\}$, whose each member $\tilde{\Pi}_\tau$ has the specified background $\tilde{\phi}^N_i(\tau)$ and the corresponding domain $\mathbb{D}_\tau$. Suppose $\mathbb{D}_0 = \mathbb{D}$, then each normalized state $\Phi_0 \in \mathbb{D}$ can be written as a normalized wave function $\Psi_{\mathbb{D}_0}[X_{S_0}^N; \tilde{\Lambda}_{S_0}^\tau]$. If we further have $(X_{S_0}, P_{S_0}) = (X_N, P_N)$, the family of wave functions would share a common spectrum basis. This allows a wave
function $\Psi_{\mathbb{D}(\tau)}|X_N; \hat{A}^\phi|$, describing an evolution over the spectrum of $\hat{A}^\phi_N$. Also, the unitary evolution of $\Psi_{\mathbb{D}(\tau)}$ in $\tau$ would be governed by an effective self-adjoint Hamiltonian operator $\hat{H} = H(\hat{X}_N, \hat{P}_N, \tau)$. Therefore, we see that the two stability conditions lead to a Schrödinger theory. In the following, we address the two stability conditions separately.

We first look into the stability of the observables $\{X_N^S, P_N^S\}$. Let us set $X_N^S = X_N$ and observe that the stability is equivalent to the validation of the choice with $S^r = \text{Span}\{|\phi_i^N(\tau)\rangle\otimes|X_N^S\rangle\}$. From the previous section, this specific form of $S^r$ is valid if and only if the equation $S^r \oplus \text{Ker}[\hat{P}_N^S] = S^r$ is satisfied. Here we know that $S^r = \text{Span}\{|\phi_i^N(\tau)\rangle\otimes|P_N^S\rangle\}$, and $\text{Ker}[\hat{P}_N^S] = \text{Span}\{|\eta_N(\tau)\rangle\}$ with $\{\eta_N(\tau)\}$ being a basis for a $\tau$ dependent subspace of the $\Psi^k$-space. Therefore, the equation is equivalent to

$$\text{Span}\{|X_N^S\rangle\} \oplus \text{Span}\{|\eta_N(\tau)\rangle\} = \text{Span}\{|\Psi^k\rangle\}. \quad (2.21)$$

Note that the only $\tau$ dependence lies in the second term, and that the equation is satisfied at $\tau = \tau_0$ by definition. Let $\tau$ increases from $\tau_0$, as long as the space $\text{Span}\{|\eta_N(\tau)\rangle\}$ varies continuously, we expect it to remain a complement to $\text{Span}\{|X_N^S\rangle\}$ in $\text{Span}\{|\Psi^k\rangle\}$ in a finite range $\tau_0 \leq \tau \leq \tau_1$. Within that range, we may set $X_N^S = X_N^S \equiv X_N$ and $P_N^S = P_N^S \equiv P_N$, thereby obtaining a stable set of observables in the range $\tau_0 \leq \tau \leq \tau_1$.

Now we look into the domain stability $\mathbb{D}_r = \mathbb{D}$, which says that the quantum Cauchy surfaces must define a foliation for a definite set of quantum spacetimes given by $\mathbb{D}$. Such a condition has to do with the relations between the different quantum Cauchy surfaces, which by construction must base on the algebra of the quantum constraints. From the physical point of view, the quantum constraints should generate the deformations of the quantum Cauchy surfaces. Indeed, since the rigging map by construction is invariant under the transformations generated by the quantum constraints, we have

$$\hat{P}_N^\phi \mathcal{N}_N^1 N \hat{P}_N = \hat{P}_N^\phi \mathcal{N}_N^1 N \hat{I} = \hat{I}, \quad (2.22)$$

where the lapse and shift operators $\mathcal{N}_N^1 N$ could be arbitrary kinematic operators, as long as it is arranged to the right of $\mathcal{N}_N^1 N$. Therefore, these transformations indeed deforms $\hat{P}_N^\phi$ into another quantum Cauchy surface, so we may write

$$e^{i \mathcal{N}_N^1 N} \hat{P}_N^\phi \mathcal{N}_N^1 N \hat{I} = \hat{P}_N^\phi \mathcal{N}_N^1 N \hat{I} = \hat{I}. \quad (2.23)$$

One may then define a one parameter family of quantum Cauchy surfaces $\hat{P}_N^\phi \mathcal{N}_N^1 N (\tau)$ deformed from the $\hat{P}_N^\phi$ (with $\mathcal{N}_N^1 N (\tau) = 0$). This family give a foliation to the set of quantum spacetimes $\mathbb{D}^N$. In terms of such deformations we may write down a sufficient condition for the domain stability, which is

$$\text{Image}[e^{i \mathcal{N}_N^1 N (\tau)}]\}_{\tau \in [\tau_0, \tau_1]} = S^r. \quad (2.24)$$

for some $\mathcal{N}_N^1 N (\tau)$ and over some finite interval $\tau_0 \leq \tau \leq \tau_1$. The sufficient condition is thus the existence of the specific deformations that translate the clock fields’ values according to $\phi_i^N = \mathcal{N}_N^1 N (\tau)$. When this condition is met, the domain stability would be granted in the interval with $\mathbb{D}_r = \mathbb{D} = \mathbb{D}$. In many homogeneous cosmological models with only one decoupled clock variable, the requirement (2.24) can be achieved easily through setting the lapse operator inverse to the clock momentum operator, which leads to a translation generator for the clock variable. In a generalized case, the validity of (2.24) depends greatly on the details of the theory, especially on the explicit quantization of the constraints and the resulted algebra in the quantum level.
Finally, whenever the two stability conditions are established in the respective intervals, we expect a Schrödinger theory defined in the overlapping interval $\tau_0 \leq \tau \leq \tau_1$ with $\min\{\tau_1', \tau_1''\} \equiv \eta$.

### 2.4. Path Integral and Schrödinger transition amplitudes

By construction, the matrix elements of the rigging map provide the full information of the Dirac theory. Thus, we should be able to formulate our construction in terms of the associated sectors of $\{\mathcal{S}^r_i\}$. As mentioned, these elements are expected to be the transition amplitudes given by a path integral theory [25–28], so the formulation may also provide an interpretation to the path integral amplitudes in terms of the Schrödinger picture. In the following, we describe a procedure of using the matrix elements of the associated sectors to obtain the Schrödinger theories under the given background.

Starting from the $\mathcal{S}^r_i$ specified by the background $\hat{\phi}^N_i(\tau)$, our first step is to identify the rigging map kernels $\{\text{Ker}[\hat{\mathcal{P}}^r_{\mathcal{L}C}]\}$ of various $\tau$ values. Recall that these kernels are specified by (2.13), which states that the kernels are given by the members in $\{\mathcal{S}^r_i\}$ with zero self-transition amplitudes. Therefore, the matrix elements $\{\left\langle \hat{\phi}^N_i(\tau), \psi_j^M \right| \hat{\mathcal{P}}^r_{\mathcal{L}C} \right| \hat{\phi}^N_i(\tau), \psi_j^M\}$ in $\mathcal{S}^r_i$ can determine Ker$[\hat{\mathcal{P}}^r_{\mathcal{L}C}]$ for each $\tau$.

After the kernels are determined, one may choose $\mathcal{S}^r = \text{Span}\{\left| \hat{\phi}^N_i(\tau), \mathbf{X}_N \right\rangle\}$ that is complement to the kernel at each $\tau$ value, over a finite interval $\tau_0 \leq \tau \leq \tau_1'$ of our concern. From the argument in the previous section, we expect this to be achievable quite generally. The complete set $\{\mathbf{X}_N\}$ in $\mathcal{S}^r$ then induces a complete set of Dirac observables in $\mathbb{D}_r$, with an orthonormal eigenbasis given by (2.16). Denoting the set of eigenvalues for $\mathbf{X}_N$ as $\{\mathbf{X}_N^{(j)}\}$, we can write down the inner product between any two of the basis members in $\tau_0 \leq \tau \leq \tau_1'$, as

$$(\mathbf{X}_N^{(j)}; \hat{\Lambda}_{\mathcal{S}^r}|\mathbf{X}_N^{(j)}; \hat{\Lambda}_{\mathcal{S}^r}) = (\mathbf{A}^{-1}_{\mathcal{S}^r})_{LM}^{KL} \langle \hat{\phi}^N_i(\tau), \mathbf{X}_N^{(j)} \right| \hat{\mathcal{P}}^r_{\mathcal{L}C} \left| \hat{\phi}^N_i(\tau), \mathbf{X}_N^{(K)}\rangle = \delta_{LM}. \quad (2.25)$$

Setting $\tau = \tau'$, the orthonormal condition for the eigenbasis leads to (2.14) in the form

$$(\mathbf{A}^{-1}_{\mathcal{S}^r})_{LM}^{KL} \langle \hat{\phi}^N_i(\tau), \mathbf{X}_N^{(j)} \right| \hat{\mathcal{P}}^r_{\mathcal{L}C} \left| \hat{\phi}^N_i(\tau), \mathbf{X}_N^{(K)}\rangle = \delta_{LM}. \quad (2.26)$$

One may then solve the above for $\hat{\Lambda}_{\mathcal{S}^r} = \mathbf{A}^{-1}_{\mathcal{S}^r}(\hat{\mathbf{X}}_N, \hat{\mathcal{P}}_N) = \mathbf{A}(\hat{\phi}^N_i, \hat{\mathbf{X}}_N, \hat{\mathcal{P}}_N)$ using the matrix elements $\{\langle \hat{\phi}^N_i(\tau), \mathbf{X}_N^{(j)} \right| \hat{\mathcal{P}}^r_{\mathcal{L}C} \left| \hat{\phi}^N_i(\tau), \mathbf{X}_N^{(K)}\rangle\}$. Finally, with the $\hat{\Lambda}_{\mathcal{S}^r}$ determined, we may use the matrix elements $\{\langle \hat{\phi}^N_i(\tau), \mathbf{X}_N^{(j)} \right| \hat{\mathcal{P}}^r_{\mathcal{L}C} \left| \hat{\phi}^N_i(\tau), \mathbf{X}_N^{(K)}\rangle\}$ to evaluate values of the inner products (2.25) with $\tau' = \tau$, and if one finds

$$\sum_j \langle \mathbf{X}_N^{(j)}; \hat{\Lambda}_{\mathcal{S}^r}|\mathbf{X}_N^{(j)}; \hat{\Lambda}_{\mathcal{S}^r}\rangle^2 = 1 \quad (2.27)$$

for $\tau_0 \leq \tau \leq \tau_1'$ then we have $\mathbb{D}_r = \mathbb{D} = \mathbb{D}$ in the interval. In this case, the matrix (2.25) gives the unitary evolution operator for the wave function

$$\mathbf{\Psi}_{\mathcal{D}(\tau)}|\mathbf{X}_N^{(j)}; \hat{\Lambda}_{\mathcal{S}^r} \rangle \equiv (\mathbf{X}_N^{(j)}; \hat{\Lambda}_{\mathcal{S}^r}|\mathbf{\Psi}_{\mathcal{D}}). \quad (2.28)$$

If (2.27) does not hold for any interval, then there is no domain stability and no Schrödinger theory is obtained. Thus, the transition amplitudes between $\mathcal{S}^r$ and $\mathcal{S}^d$ detect the domain stability, and in the presence of the stability they also govern the evolution of the wave functions.
In the above we have shown a procedure of using the path integral transition amplitudes \( \left\{ \langle \hat{\phi}^N_i (\tau) \rangle, \psi^f_i \mid \hat{P} \right| \hat{\phi}^N_i (\tau) \rangle, \psi^f_i \right\} \) to extract the Schrödinger theories with the given background.

We now address the possible transformations relating one quantum foliation to another. Suppose we have two Schrödinger theories with their own sets of background fields \( \hat{\phi}^N \), \( *\hat{\phi}^N \), respectively defined in the domains \( \mathbb{D} \) and \( *\mathbb{D} \). If \( \mathbb{D} \subset *\mathbb{D} \), then every wave function \( \Psi_{\mathbb{D}}[^N \mathbf{X}_N; \hat{\Lambda}_S^N] \) describing a state \( \Psi \in *\mathbb{D} \subset \mathbb{D} \) can be transformed into \( \Psi_{*\mathbb{D}}[^N \mathbf{X}_N; \hat{\Lambda}_S^N] \) through the matrix

\[
(\mathbf{X}_N^{(j)}; \hat{\Lambda}_S^N)^* \mathbf{X}_N^{(j)}; \hat{\Lambda}_S^N) = (\mathbf{A}^N_{\mathbb{D}})^* (\mathbf{A}^N_{*\mathbb{D}}) (\hat{\phi}^N_i (t), \mathbf{X}_N^{(K)}) \hat{P} \hat{\phi}^N_i (\tau), *\mathbf{X}_N^{(k)}).
\]

Physically, this transformation is to switch from the foliation in \( \tau \) to that in \( t \) for the quantum spacetimes in \( *\mathbb{D} \). Also, it is given by the path integral transition amplitudes between \( S^\tau \) and \( S^t \). Moreover, a state \( \Psi \in (\mathbb{D} - *\mathbb{D}) \) has a unitary Schrödinger representation only under the background \( \hat{\phi}^N_i (t) \), but not under \( *\hat{\phi}^N_i (\tau) \). The reason is clear—while \( \{ \mathbf{A}^N_{\mathbb{D}} \mathbf{X}_N^{(j)} \hat{\Lambda}_S^N (\tau) \} \) is a complete set of observables in \( *\mathbb{D} \), it is not a complete set in \( \mathbb{D} \). In the special cases of \( \mathbb{D} = *\mathbb{D} \), the two Schrödinger theories are dual to each other, as the descriptions from the two different foliations of the same set of quantum spacetimes. If \( \mathbb{D} \subset *\mathbb{D} \), the theory \( (\mathbb{D}, \{ \hat{\Pi}_K \}_{K \in \mathbb{E}_0 \subset \mathbb{C}} \) is more global since it applies to a broader range of measurements.

We’ve mentioned the theories of quantum gravity [6, 7] with an absolute notion of time defined under a privileged physical background \( \phi^N_{\text{Abs}} \), such that the Universe is described with a fundamental Schrödinger theory. We also raised the question about whether such a theory could account for the possibly detectable quantum behavior of the background fields \( \phi^N_{\text{Abs}} \). Here we offer a possible scenario in which the answer is positive. The physical subspace that can support a fundamental Schrödinger theory may turn out to be an exceptional domain \( \mathbb{D}_{\text{Abs}} \). In such consideration, this domain would be regarded as the true physical Hilbert space \( \mathbb{H} _{\text{phys}} = \mathbb{D}_{\text{Abs}} \). In the fundamental Schrödinger theory \( (\mathbb{D}_{\text{Abs}}, \{ \hat{\Pi}_{(K)} \}_{K \in \mathbb{E}_0 \subset \mathbb{C}} \), the fields \( \phi^N_{\text{Abs}} \) are without quantum fluctuations. Nevertheless, there can be a sub domain \( \mathbb{D} \subset \mathbb{D}_{\text{Abs}} \) that provides an effective Schrödinger theory \( (\mathbb{D}, \{ \hat{\Pi}_K \}_{K \in \mathbb{E}_0 \subset \mathbb{C}} \) in a different foliation, in which \( \phi^N_{\text{Abs}} \) are dynamical quantum fields.

3. Application to spatially flat FLRW quantum cosmology

To illustrate our points with a simple example, we now apply our proposal to a model of the spatially flat FLRW quantum cosmology [38–40]. Specifically, our model will be obtained from quantizing the homogeneous, isotropic sector of general relativity with zero cosmological constant, describing a Universe of the gravitational fields minimally coupled to a massless Klein–Gordon field.

The classical phase space in this setting is coordinatized by two canonical conjugate pairs: the pair for the gravitation sector \((a, P_a)\) describing the scale factor and the extrinsic curvature of the homogeneous and isotropic space, and the pair for the Klein–Gordon sector \((\phi, P_{\phi})\) describing the value of the scalar field and its momentum. The theory is governed by one reduced scalar constraint \( C(a, P_a, \phi, P_{\phi}) \). The existing paradigmatic quantization procedure [38–40] on this model, called the Wheeler–DeWitt quantization, involves two steps.

First, one properly quantizes the constraint as an operator \( \hat{C} \) in the canonically quantized phase space \( \mathbb{K} \), and then finds the complete set of wave functions annihilated by \( \hat{C} \). This can be easily accomplished thanks to the symmetrically reduced setting. Second, one selects a specific wave function variable as the physical time for a corresponding sector of the solution.
space, in which the Schrödinger interpretation can be applied. This means that a conserved and positive definite inner product must be identified for these wave functions under the chosen physical time. Naturally, such a choice of the physical time and the corresponding sector must be guided by the physics one wish to describe.

As a result, this paradigmatic quantization procedure can lead to a Schrödinger theory of the gravity \[38, 39\] in the scalar field background, or a Schrödinger theory of the scalar field \[40\] in the gravitation background. That is, it is essentially a reduced phase space quantization procedure. Clearly, both of the two Schrödinger theories explore important physics. Particularly, the quantum dynamics of gravity can describe the effects of the quantum gravitational fluctuation near the big-bang singularity, while the scalar field quantum dynamics provides a model of QFT in a quantum spacetime background. It is then important to ask if the physics explored by both theories could be derived from the same underlying principle governing the state of the Universe.

As mentioned before, in the case of a self-adjoint constraint operator \(\hat{C}\), the Dirac quantization can be easily carried out for this model. In this way, the same complete set of wave function solutions can be generated by the group averaging procedure along with the physical inner product without involving physical time. Further, it is commonly acknowledged \[42\] that there can be a unitary equivalence between the Schrödinger and Dirac theories built from the solution space of this model. However, to our knowledge, the thereby suggested embedding of the paradigmatic Schrödinger theories of various physical times into the Dirac theory has never been formulated explicitly from a fundamental principle.

To demonstrate that our approach serves this purpose, we will apply the Dirac quantization to the model, and use the two families of quantum Cauchy surfaces having respectively the scalar field and the gravitation field as the background. From these we will derive the corresponding local Dirac observables which indeed describe the two Schrödinger theories of gravity and the scalar field.

### 3.1. Timeless physical Hilbert space \(\mathcal{H}\)

Applying the standard partial gauge fixing, we restrict to the comoving frames with a specified spatial coordinate system. Then, a chosen fiducial comoving spatial cell with the coordinate volume \(V\) is described by our kinematic phase space \(\{(a, P_a, \phi, P_\phi)\}\), with \(a \in \mathbb{R}^+\) and \(P_a, \phi, P_\phi \in \mathbb{R}\). The gravitational sector is described by the scale factor \(a\) and its conjugate momentum \(P_a\), which give the physical volume of the cell as \(a^3V\) and the constant extrinsic curvature as \(K = \frac{3}{a^2G}\). For the scalar field sector, \(\phi\) and \(P_\phi\) represent the value of the field and its total conjugate momentum in the cell. The only non-trivial Poisson brackets among all the variables are given by \(\{a, P_a\} = \{\phi, P_\phi\} = 1\). Due to the symmetry and the gauge fixing, the Hamiltonian for the theory is reduced to only one scalar constraint

\[
C \equiv C_\phi(a, P_\phi) + C_\phi(\phi, P_\phi) = -a^2P_a^2 + \kappa P_\phi^2,
\]

where \(\kappa = \frac{3}{4\pi G}\).

Next, we apply the canonical quantization to obtain a kinematic Hilbert space \(\mathcal{K}\). Because of the requirement \(a > 0\), \(P_a\) cannot be consistently quantized into a self-adjoint operator. For this reason, we follow a common scheme \[38–40\] of quantizing instead the conjugate pair \((a \equiv \ln \frac{a}{a_0}, P_a \equiv \frac{\dot{a}}{a_0}P_a)\), where we will set \(a_0 = 1\) in the following for convenience. Thus, the space \(\mathcal{K}\) is obtained as
\[ \mathbb{K} = \text{Span} \left\{ \int_{\mathbb{R} \times \mathbb{R}} dP_{\alpha} dP_{\phi} \Phi(P_{\alpha}, P_{\phi}) |P_{\alpha}, P_{\phi}\rangle \right\} \]

with

\[ \langle P'_{\alpha}, P'_{\phi}|P_{\alpha}, P_{\phi}\rangle = \delta (P'_{\alpha} - P_{\alpha}, \phi (P'_{\phi} - P_{\phi}), \phi). \]

where \( \Phi(P_{\alpha}, P_{\phi}) \) is any smooth function in \( L^2(\mathbb{R}^2, dP_{\alpha} dP_{\phi}) \). A complete set of self-adjoint operators in \( \mathbb{K} \) can be given by either \( \hat{a} \) and \( \hat{\phi} \) acting as \( i \frac{\partial}{\partial P_{\alpha}} \) and \( i \frac{\partial}{\partial P_{\phi}} \) on \( \Phi(P_{\alpha}, P_{\phi}) \).

Under this set up, we obtain the self-adjoint quantum constraint operator in \( \mathbb{K} \) as

\[ \hat{C} = \hat{P}^2_{\alpha} + \kappa \hat{P}^2_{\phi}. \]

Now we can easily construct our physical Hilbert space through defining the group averaging operator \( \mathbb{P} : \mathbb{K} \to \mathbb{K}^* \) as

\[ \hat{P} \equiv \delta (\hat{C}) = \int_{-\infty}^{\infty} d\lambda e^{i\lambda \hat{C}}. \]

The explicit mapping of the operator is given by \( (\beta \equiv \sqrt{\mathbb{P}}) \)

\[ \hat{P} \int dP_{\alpha} dP_{\phi} \Phi(P_{\alpha}, P_{\phi}) |P_{\alpha}, P_{\phi}\rangle = \int dP_{\alpha} \frac{1}{2|P_{\alpha}|} [\Phi(\beta|P_{\alpha}|, P_{\phi}) |\beta|P_{\phi}\rangle - \Phi(-\beta|P_{\alpha}|, P_{\phi}) |\beta|P_{\phi}\rangle]. \]

Treating these images of the map as the elements in \( \mathbb{H} \), we denote \( |\pm |\beta P_{\alpha}|, P_{\phi}\rangle \equiv \pm \frac{1}{\sqrt{2|P_{\alpha}|}} |\pm |\beta P_{\alpha}|, P_{\phi}\rangle. \) With the physical inner product given by \( (2.1) \) the set provides an orthonormal basis satisfying

\[ (|\pm |\beta P_{\alpha}|, P_{\phi} \pm |\beta P_{\alpha}|, P_{\phi}'\rangle = \delta (P_{\phi} - P_{\phi}'). \]

The physical Hilbert space is then obtained as

\[ \mathbb{H} \equiv \text{Span} \{ |+ |\beta P_{\alpha}|, P_{\phi}\rangle \} \oplus \text{Span} \{ |- |\beta P_{\alpha}|, P_{\phi}\rangle \}, \]

whose members are given by all the pairs (corresponding to the \( \pm \) branches) of smooth wave functions in \( L^2(\mathbb{R}, dP_{\alpha}) \oplus L^2(\mathbb{R}, dP_{\phi}) \).

This completes the construction of \( \mathbb{H} \). To find the desired Dirac observables, we now apply the quantum Cauchy surfaces with the backgrounds chosen to be either the scalar field or the gravitational fields. Particularly, we will follow the procedure described in section 2.4 using the relevant transition amplitudes.

3.2. Quantum Cauchy surfaces with \( \bar{\phi}_+(t) \) or \( \bar{\alpha}_+(\tau) \) as the background

Here we look for the two family of quantum Cauchy surfaces. One is to be labeled by \( t \in (-\infty, \infty) \) and is required to have the background \( \bar{\phi}(t) \equiv t \) with \( P_{\phi}(t) > 0 \). The other one is to be labeled by \( \tau \in (-\infty, \infty) \) and is required to have the background \( \bar{\alpha}(\tau) \equiv \tau \) with \( P_{\alpha} > 0 \). According to these chosen backgrounds, the relevant eigenspaces for each \( t \) or \( \tau \) is given by \( \mathbb{S}_t \equiv \text{Span} \{ |P_{\alpha}, \bar{\phi}(t)\rangle \} \) or \( \mathbb{S}_\tau \equiv \text{Span} \{ |\bar{\alpha}+(\tau), P_{\phi}\rangle \} \), with the required background states as

\[ |\bar{\phi}_+(t)\rangle \equiv \int_{0}^{\infty} dP_{\phi} e^{-iP_{\phi}\bar{\phi}(t)} |P_{\phi}\rangle \text{ and } |\bar{\alpha}_+(\tau)\rangle \equiv \int_{0}^{\infty} dP_{\alpha} e^{-iP_{\alpha}\bar{\alpha}(\tau)} |P_{\alpha}\rangle. \]
Following the prescription, we first calculate the transition amplitudes between $S'_\alpha$ and $S'_\beta$, and also that between $S''_\alpha$ and $S''_\beta$. The amplitudes can be obtained easily as

$$
\langle P_{\alpha}'(t'), \tilde{\omega}_\alpha(t') | \hat{P}_{\alpha}, \tilde{\omega}_\alpha(t) \rangle = \frac{\beta}{2|P_{\alpha}|} \frac{e^{i|P_{\alpha}|(t'-t)/\beta}}{\delta(P_{\alpha}' - P_{\alpha})} \delta \quad \text{and}
$$

$$
\langle \tilde{\alpha}_\beta(\tau'), P_{\beta}'(t') | \hat{P}_{\beta}, \tilde{\alpha}_\beta(\tau) \rangle = \frac{e^{i|P_{\beta}|(\tau'-\tau)}}{2|P_{\beta}|} \delta(P_{\beta}' - P_{\beta}).
$$

(3.9)

By setting $t' = t$ and $\tau = \tau'$ in the above, we see that the equation (2.13) in these cases has only the zero vector solution so the kernel is trivial for any $t$ or $\tau$. Thus for each value of $t$, we have identified the quantum Cauchy surface $\hat{\Pi}_t$, with Image $[\hat{\Pi}_t|_{S'_\beta}] = S'_\beta = \text{Span} \{|P_{\alpha}', \tilde{\omega}_\alpha(t') \}$ isomorphic to a certain $\mathbb{H}_t \subset \mathcal{H}$ under $\hat{P}$. Therefore we may set $(X_{S'_\beta}) \equiv (X_{S'_\beta}) \equiv \{ \hat{P}_{\beta} \}$, and $(P_{S'_\beta}) \equiv (P_{S'_\beta}) \equiv \{ \hat{\phi} \}$.

Next, by using $t' = t$ and $\tau = \tau'$ in (3.9) again, we can solve the equation (2.26) and find the obvious solutions given by

$$
\hat{\Lambda}_{S'} = \sqrt{\beta/2|P_{\alpha}|} \quad \text{and} \quad \hat{\Lambda}_{S'} = \sqrt{1/2|P_{\alpha}|}.
$$

(3.10)

It then follows that a complete set of Dirac observables for $\mathbb{D}_t^+$ may be given by $(\hat{\Lambda}_{S'}^{-1} \hat{P}_{\beta} \hat{\Lambda}_{S'}(t))$ or $(\tilde{\alpha}_\beta \hat{\Lambda}_{S'}(t))$. Also, a complete set for $\hat{\mathbb{D}}_t^+$ may be provided as $(\tilde{\alpha}_\beta \hat{P}_{\beta} \hat{\Lambda}_{S'}(t))$ or $(\tilde{\alpha}_\beta \hat{\phi})$.

According to (2.10), the two pairs of operators faithfully represent $(\hat{P}_{\alpha}, \tilde{\alpha})$ and $(\hat{P}_{\beta}, \phi)$ through the explicit forms of

$$
\langle \tilde{\Lambda}_S^{-1} (X_{S'_\beta}, P_{S'_\beta}) \hat{\Lambda}_S(\tau)|_{S'_\beta} \equiv \hat{P}_{\beta} \frac{1}{\sqrt{|P_{\beta}|}} \hat{\Pi}_\tau \quad \text{and}
$$

$$
\langle \tilde{\Lambda}_S^{-1} (X_{S''_\beta}, P_{S''_\beta}) \hat{\Lambda}_S(\tau)|_{S''_\beta} \equiv \hat{P}_{\beta} \frac{1}{\sqrt{|P_{\beta}|}} \hat{\Pi}_\tau.
$$

(3.11)

Next, inserting the values of (3.9) and the given (3.10) into (2.25), we find that (2.27) is satisfied in both cases, thus we have $\mathbb{D}_t^+ = \hat{D}_t^+$ and $\hat{\mathbb{D}}_t^+ = \hat{D}_t^+$ for the full ranges of $t$ and $\tau$. Actually, one can easily see that $\mathbb{D}_t^+ = \text{Span} \{|P_{\alpha}, |P_{\alpha}|/\beta \}$ and $\hat{\mathbb{D}}_t^+ = \text{Span} \{||P_{\beta}| P_{\beta} \}$. It is important to see that the proper background states play a crucial role in this domain stability. Had we chosen the background state to be $|\tilde{\phi}(\eta)\rangle$ instead of $|\tilde{\phi}(\eta)\rangle$, without specifying the sign of $P_{\beta}$, we would have obtained $\mathbb{D}_t^+ = \text{Span} \{|P_{\alpha}, |P_{\alpha}|/\beta \}$ and $\hat{\mathbb{D}}_t^+ = \text{Span} \{|\beta P_{\beta}| P_{\beta} \}$ and thus $\mathbb{D}_t^+ = \mathbb{D}_{t+\epsilon}$. The same thing would happen had we chosen the background state to be $|\tilde{\alpha}(\eta)\rangle$ instead of $|\tilde{\alpha}(\eta)\rangle$.

Just by reversing the signs, one can find two additional families of Cauchy surfaces with the backgrounds $|\tilde{\phi}(s)\rangle$ and $|\tilde{\alpha}(s)\rangle$ foliating the domains $\mathbb{D}_t^- = \text{Span} \{|P_{\alpha}, |P_{\alpha}|/\beta \}$ and $\hat{\mathbb{D}}_t^- = \text{Span} \{|\beta P_{\beta}| P_{\beta} \}$. We had thus identified the four different Schrödinger domains constituting the full physical space as $\mathcal{H} = \mathbb{D}_t^+ \cup \mathbb{D}_t^- = \hat{\mathbb{D}}_t^+ \cup \hat{\mathbb{D}}_t^-$. 


3.3. Quantum dynamics under $\hat{\phi}_+(t)$ or $\hat{\alpha}_+(\tau)$ as the background

In our construction, the quantum Cauchy surfaces $\hat{\mathcal{H}}_t$ and $\hat{\mathcal{H}}_\tau$ describe $\mathbb{D}^+$ and $\mathbb{D}^+$ respectively in terms of $(\hat{\Lambda}^\dag_\mathcal{S}_s(\hat{P}_s, \hat{\alpha})\hat{\Lambda}_\mathcal{S}_s)(t)$ and $(\hat{\Lambda}^\dag_\mathcal{S}_s(\hat{P}_s, \hat{\phi})\hat{\Lambda}_\mathcal{S}_s)(\tau)$. These observables capture the quantum dynamics of gravity and the scalar field. Let us now write down the two Schrödinger theories explicitly.

We can use (2.16) to write down an orthonormal eigenbasis for $\mathbb{D}^+$, of each of the two complete sets at any $t$, and obtain $\{ |P_s; \hat{\Lambda}_\mathcal{S}_s\rangle = e^{iP_s1/\beta} |P_s; P_0\rangle / \beta \}$ and $\{ |\alpha; \hat{\Lambda}_\mathcal{S}_s\rangle = \int dp_s \langle \alpha |P_s\rangle |P_s; \hat{\Lambda}_\mathcal{S}_s\rangle\}$. We have thus identified a Schrödinger theory $(\mathbb{D}^+, \{ \hat{\mathcal{H}}_t \})_{t \in \mathbb{R}}$ describing a physical state $\Psi \in \mathbb{D}^+$ with the wave functions of the forms $\Psi_{\mathbb{D}^+}(|P_s; \hat{\Lambda}_\mathcal{S}_s\rangle$ and $\Psi_{\mathbb{D}^+}(|\alpha; \hat{\Lambda}_\mathcal{S}_s\rangle$, as defined in (2.28). Similarly, an orthonormal eigenbasis for $\mathbb{D}^+$, of each of the two complete sets at any $\tau$, is given by $\{ |P_s; \hat{\Lambda}_\mathcal{S}_s\rangle = e^{iP_s\alpha/\beta} |P_s; P_0\rangle \}$ and $\{ |\phi; \hat{\Lambda}_\mathcal{S}_s\rangle = \int dp_s \langle \phi |P_s\rangle |P_s; \hat{\Lambda}_\mathcal{S}_s\rangle\}$. The identified Schrödinger theory $(\mathbb{D}^+, \{ \hat{\mathcal{H}}_\tau \})_{\tau \in \mathbb{R}}$ describes a physical state $\hat{\Psi} \in \mathbb{D}^+$ with the wave functions of the forms $\hat{\Psi}_{\mathbb{D}^+}(|P_s; \hat{\Lambda}_\mathcal{S}_s\rangle$ and $\hat{\Psi}_{\mathbb{D}^+}(|\phi; \hat{\Lambda}_\mathcal{S}_s\rangle$.

To extract the physical Hamiltonians of the two theories, we simply refer to the transformations between the sets of eigenbasis of various $t$, or various $\tau$. These transformations imply that

\[
(\hat{\Lambda}^\dag_\mathcal{S}_s(\hat{P}_s, \hat{\alpha})\hat{\Lambda}_\mathcal{S}_s)(t) = e^{-i\hat{\mathcal{H}}_t} (\hat{\Lambda}^\dag_\mathcal{S}_s(\hat{P}_s, \hat{\alpha})\hat{\Lambda}_\mathcal{S}_s)(0) e^{i\hat{\mathcal{H}}_t}
\]

\[
(\hat{\Lambda}^\dag_\mathcal{S}_s(\hat{P}_s, \hat{\phi})\hat{\Lambda}_\mathcal{S}_s)(\tau) = e^{-i\hat{\mathcal{H}}_\tau} (\hat{\Lambda}^\dag_\mathcal{S}_s(\hat{P}_s, \hat{\phi})\hat{\Lambda}_\mathcal{S}_s)(0) e^{i\hat{\mathcal{H}}_\tau},
\]

where the physical Hamiltonians take the forms

\[
\hat{\mathcal{H}}_\phi = \beta^{-1} |(\hat{\Lambda}^\dag_\mathcal{S}_s(\hat{P}_s, \hat{\alpha})\hat{\Lambda}_\mathcal{S}_s)(0)| \text{ and } \hat{\mathcal{H}}_\alpha = \beta |(\hat{\Lambda}^\dag_\mathcal{S}_s(\hat{P}_s, \hat{\phi})\hat{\Lambda}_\mathcal{S}_s)(0)|.
\]

3.4. Unification under the Dirac theory

Starting from the Wheeler–DeWitt equation imposed by the same operator (3.3), one may instead apply the paradigmatic quantization on the same solution space in $\mathbb{H}^*$. To describe the quantum dynamics of gravity, one can choose $\phi$ as the physical time and restrict to the solutions with $P_0 > 0$, and then correspondingly introduce a conserved instantaneous inner product at an arbitrary $\phi$ value. The simplest choice of scheme [38, 39] in doing so leads to the Schrödinger theory with the observables $(\hat{\alpha}, \hat{P}_\alpha)$ and the physical Hamiltonian $\beta^{-1}\hat{P}_\alpha$. To describe the quantum dynamics of the scalar field, one can instead choose $\alpha$ as the physical time and restrict to the solutions with $P_0 > 0$, the same paradigmatic procedure in the simplest scheme [40] leads to another Schrödinger theory with the observables $(\hat{\phi}, \hat{P}_\phi)$ and the physical Hamiltonian $\beta\hat{P}_\phi$.

In the classical theory of our model a general physical Hamiltonian is derived from the form $H = N' C(a, P_a, \phi, P_\phi)$, where each chosen $N' = N(a, \phi, P_a, P_\phi)$ determines a lapse function--and thus the temporal metric value--of the corresponding time frame. By referring to (3.3), we can see that the physical Hamiltonian $\beta\hat{P}_\phi$ is effectively given by setting $\hat{N'} = \beta^{-1}\hat{P}_\phi$. Therefore, in comparison with the standard Klein–Gordon quantum theory in a fixed classical FLRW spacetime where $N'$ is a real number, our scalar field dynamics is corrected by the quantum back-reaction on the temporal metric component. This effect of the ‘dressed metric’ is studied [43, 44] in the subject of QFT in quantum spacetimes using models.
with local degrees of freedom. The quantum dynamics of gravity governed by $\beta^{-1}|\hat{P}_\alpha|$ also receives the analogous quantum back reaction correction. Through the observables representing the geometric variables in the special form $(\alpha = \ln a, P_\alpha = \alpha P_\alpha)$, this effect gives specific quantum corrections \cite{38, 39} to the classical FLRW cosmology.

By looking at (3.13) and (3.12), we see that the two reduced phase space theories are truly unitarily equivalent to the Dirac theory restricted to their corresponding domains. Such equivalence is given by the application of $\hat{P}_t$ and $\hat{P}_t$. Under our approach, they are now derived from one underlying Dirac theory, described by the corresponding sets of Dirac observables which are really alternative forms of the quantum relational observables. Indeed, the observables can be explicitly written as

$$\langle \hat{\Lambda}_\Sigma^{-1}(\hat{P}_\alpha, \hat{\alpha})\hat{\Lambda}_\Sigma \rangle(t) = \int_{-\infty}^{\infty} d\lambda_e e^{i\lambda \hat{C}} \sqrt{2|\beta|} \hat{P}_t(\hat{P}_\alpha, \hat{\alpha}) \frac{1}{\sqrt{2|\beta|}} \delta(\hat{\phi} - t) |\phi\rangle |\theta(\hat{P}_\phi)\rangle e^{-i\lambda \hat{C}}$$

$$\langle \hat{\Lambda}_\Sigma^{\dagger}(\hat{P}_\alpha, \hat{\phi})\hat{\Lambda}_\Sigma \rangle(\tau) = \int_{-\infty}^{\infty} d\lambda_e e^{i\lambda \hat{C}} \sqrt{2|\beta|} \hat{P}_t(\hat{P}_\alpha, \hat{\phi}) \frac{1}{\sqrt{2|\beta|}} \delta(\hat{\phi} - \tau) |\phi\rangle |\theta(\hat{P}_\phi)\rangle e^{-i\lambda \hat{C}},$$

(3.14)

where we had used the notations following from the classical identities $\hat{\phi} \equiv 2\kappa \hat{P}_\phi$ and $\hat{\alpha} \equiv 2\hat{P}_\phi$. Therefore, they have the classical limits of the gauge invariant phase space functions $(P_\alpha, \alpha)(\hat{\phi}(t))$ and $(P_\alpha, \phi)(\hat{\alpha}(\tau))$.

Remarkably, a physical state $\Psi \in \mathbb{D}^+ \cap \mathbb{D}^+$ is described by both of the two theories with the wave functions $\Psi_{\mathbb{D}^+}^{\hat{\phi}(t)}(P_\alpha; \hat{\Lambda}_\Sigma)$ and $\Psi_{\mathbb{D}^+}^{\hat{\phi}(\tau)}(P_\alpha; \hat{\Lambda}_\Sigma^{\dagger})$. In this case, the two quantum dynamics provide dual descriptions for the same physical state, and in our terms they come from two distinct foliations of the same quantum spacetime $\hat{\Psi}$. The exact transformation between these two wavefunctions can be calculated through (2.27), and we obtain the matrix elements as

$$\langle P_{\alpha}^{\hat{\Lambda}_\Sigma^{\dagger}}; \hat{\Lambda}_\Sigma \rangle = 2|P_\alpha| e^{-iP_{\alpha}^{\hat{\Lambda}_\Sigma^{\dagger}}P_{\alpha^{\hat{\Lambda}_\Sigma}}^\dagger} \delta(P_\alpha^2 - \kappa P_\phi^2).$$

(3.15)

Essentially, this matrix represents the transformation between the two frames corresponding to $\hat{\Pi}_t$ and $\hat{\Pi}_\tau$.

Lastly, we should mention that the Brown–Kuchař dust can also be applied to this model \cite{41} through the introduction of one additional privileged time field. In comparison, our approach to this model shows how the Schrödinger theories can emerge from a Dirac theory containing only the ‘fundamental fields’, without introducing any additional phenomenological field.

4. Summary and conclusion

For a Dirac theory of quantum gravity $(\hat{H}, \mathbb{K}, \hat{\Pi})$, we have proposed an exact notion of Cauchy surfaces from the quantum level, which we have argued to be essential for obtaining an exact Schrödinger theory. They are generally defined as the right inverse maps of the rigging map $\hat{P}$.

Similar to its classical counterpart, a quantum Cauchy surface can represent $\mathbb{D} \subset \mathbb{H}$ with an instantaneous ‘quantum reduced phase space’ $\mathbb{S}' \subset \mathbb{K}$. A self-adjoint complete set of operators in $\mathbb{S}'$ provides a spectrum for the Schrödinger wave function describing $\Psi \in \mathbb{D}$, which is defined at the moment given by the quantum Cauchy surface. Through this
representation, a physical fundamental algebra in $\mathcal{D}$ is also induced by the fundamental algebra in $\mathcal{S}'$. Further, the quantum degrees of freedom absent in $\mathcal{S}'$ due to the constraints naturally yield a physical background without any quantum fluctuation, which may provide a notion of time for the wave function. This is very much in analogy to the classical reduced phase space theory. Under a specified background, we also deduced the two essential stabilities for a Schrödinger theory to emerge for a finite interval of time. The Heisenberg operators for such a Schrödinger theory are labeled by the background field values, and thus they are special Dirac observables in the Dirac theories that are closely related to the standard quantum relational observables. Moreover, we argued that this Schrödinger theory can be written in terms of the relevant path integral transition amplitudes, which are given by the rigging map matrix elements in the sectors with the specified background. Our formulation thus provides a fundamental notion of foliations in the Dirac theory, in the sense that it yields all the Schrödinger theories corresponding to all the legitimate backgrounds available as a choice of physical time.

When individually constructed through the reduced phase space quantization, these Schrödinger theories appear contradictory having different sets of fields in the system being promoted into quantum fields. Also, each of them as a theory unrelated to others generally has many quantization ambiguities of its own. Thus, as mentioned, given the abundance of well-studied reduced phase space theories specialized in describing different kinds of quantum dynamics, a fundamental approach to relate between them is highly desirable. Our approach is a significant step toward this goal. In our simple example, the gravitational quantum dynamics and the scalar QFT in the quantum spacetime are unified in the symmetrical reduced setting, under the same Dirac theory of the FLRW Universe. Under this unification, the physical Hamiltonians $\hat{H}_3$ corresponds to one specific form of $\hat{H}_3$.

In view of the method of quantum relational observables in the Dirac theories, our approach offers a reformulation of these relational observables based on the foliations with the quantum Cauchy surfaces, such that the new observables give rise to exact Schrödinger theories. In view of the approach of reduced phase space quantization, our construction offers a possible unification among the different Schrödinger theories quantized from the same classical system, through the transformations between the various foliations with the quantum Cauchy surfaces.

In the next step, it is natural to apply our proposal to the more realistic quantum cosmological models capturing more degrees of freedom. Specifically, there are important models with the quantum constraints whose solution space is known and given, such as the Bianchi IX midisuperspace quantum cosmology [45], and the spherically symmetric loop quantum gravity for the quantum black holes [46]. A rigging map can be readily constructed for these models, and so we can apply the quantum Cauchy surfaces to study and relate the various types of quantum dynamics emerges from them.

As mentioned, the currently most promising Dirac theory of quantum gravity is loop quantum gravity, which has a solid kinematic Hilbert space of the quantum geometry. In our previous works [36, 37], we had applied the Dirac observables of the form (2.7) to a model of loop quantum gravity, to derive the semi classical limits of the model. The model shares same kinematic Hilbert space with the full theory of loop quantum gravity, and is obtained by simplifying the standard scalar constraint operator in the full theory. Although the quantum Cauchy surfaces are applied only in the semi classical limits, the core idea is that our Dirac observables faithfully represent the loop algebra of the quantum geometry in the limits. As a result, the dynamics obtained from using these Dirac observables recovers a specific semi classical limit of general relativity, accompanied by the signature corrections from the quantum geometry. Along this line, our proposal in this paper serves to specify our method
right from the quantum level. Hence, it is also important to improve the previous model so that an implementation of the quantum Cauchy surfaces can be demonstrated in the quantum level.

There are many other interesting and important topics about quantum gravity we may discuss through the proposal. As what we have argued, the path-integral transition amplitudes may be translated to the components of our Schrödinger theories. Further studies in this direction could provide us insights relating the covariant and canonical formulations of quantum gravity, from a more physical point of view. Lastly, we should also address the issue of consistent probabilistic interpretations for the locally defined Schrödinger theories suggested by our proposal.

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