We propose a whole family of physical states that yield a violation of the Bell CHSH inequality arbitrarily close to its maximum value, when using quadrature phase homodyne detection. This result is based on a new binning process called root binning, that is used to transform the continuous variables measurements into binary results needed for the tests of quantum mechanics versus local realistic theories. A physical process in order to produce such states is also suggested. The use of high-efficiency spacelike separated homodyne detections with these states and this binning process would result in a conclusive loophole-free test of quantum mechanics.

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I. INTRODUCTION

Non-separability, or entanglement, has emerged as one of the most striking feature of quantum mechanics. In 1935, it led Einstein, Podolsky and Rosen to suggest that quantum mechanics is incomplete, on the premise that any physical theory of nature must be both “local” and “realistic”. To quantify the debate between quantum mechanics and local realistic (classical) theories, Bell introduced a set of inequalities that must be obeyed by any local realistic theory whereas they are violated by quantum mechanics and local realistic (classical) theories. A physical process in order to produce such states is also suggested. The use of strong local oscillators detected by highly efficient photodiodes. Up to date, a few theoretical proposals that use quadrature-phase homodyne detections have been made but for these set-ups the Bell inequality violation is a few percents only, that lies far away of the maximal violation attainable. Gilchrist et al. use a circle or pair coherent state produced by non-degenerate parametric oscillation with the pump adiabatically removed. This state leads to a theoretical violation of about 1.015 (> 1) of the CH Bell inequality. Munro considers correlated photon number states of the form

\[ |\Psi\rangle = \sum_{n=0}^{N} c_n |n\rangle |n\rangle \] (1)

where \( N \) is truncated at \( N = 10 \). He then performs a numerical optimization on each \( c_n \) coefficient to maximize the violation of the CHSH Bell inequality when an homodyne measurement is performed. For this specific state, the CHSH inequality is violated by 2.076 (> 2) and the CH inequality by 1.019 (> 1). From a different phase space approach, Auberson et al. derive phase space Bell inequalities and propose a state that yields a maximal violation of up to 2\( \sqrt{2} \) (> 2). This state can be expressed in the position space by

\[ \Psi_{\pm}(q_1, q_2) = \frac{1}{2\sqrt{2}} \left[ 1 \pm e^{i\frac{\pi}{4}} \text{sgn}(q_1) \text{sgn}(q_2) \right] f(|q_1|) f(|q_2|) \] (2)

where \( f(q) \) is a regularized form of \( \frac{1}{\sqrt{q}} \) with

\[ \int_{-\infty}^{+\infty} dq f(q)^2 = 1 \]. The main problem with this wave function lies in its singularities and phase switches. Therefore, it requires nontrivial regularization proce-
dures to be considered as a suitable physical state. Following these various attempts, we are thus looking for a “simple” physical state that would lead to a maximal violation of a Bell inequality.

In this paper, we consider the Clauser-Horn-Shimony-Holt (CHSH) Bell inequality \( E \) (sometimes referred to as the spin inequality). Fig. 1 depicts an idealized setup for a general Bell inequality measurement. Two entangled sub-states are viewed by two analyzers and detectors at locations A and B, where a and b denote any adjustable parameter at A and B. In our particular case, we will use quadrature-phase homodyne measurements, which could have an efficiency high enough to close the “detection-efficiency” loophole. Moreover, the apparatuses A and B can be in principle spacelike separated, thereby excluding action at distance, and closing also the “locality” loophole. We point out that in the present approach all the detected light has to be taken into account, i.e. the relevant signal is the photocurrent generated by the interferometric mixing and photodetection of the local oscillator and input quantum state. Therefore, no “supplementary assumption” \( E \) will be needed to interpret the data. Under these conditions, the CHSH Bell inequality can be written:

\[
S = |E(a', b') + E(a', b) + E(a, b') - E(a, b)| \leq 2 \quad (3)
\]

where the correlation function \( E(a, b) \) is given by:

\[
E(a, b) = P_{++}(a, b) + P_{--}(a, b) - P_{+-}(a, b) - P_{-+}(a, b) \quad (4)
\]

with \( P_{+\pm}(a, b) \) the probability that a “+” occurs at both A and B, given \( a \) and \( b \).

In this article, we propose explicitly a set of physical states that yield a violation of the CHSH inequality arbitrarily close to its maximum value, when measured by an ideal quadrature-phase homodyne detection. In section II, we describe how we convert the continuous quadrature amplitude into a binary result “+” or “-” for each apparatus \( A, B \) using a process called Root Binning. In section III, a specific state that yields a large violation of the CHSH inequality is presented. This state is generalized in section IV to derive a whole family of states that violate this Bell inequality. The issue of preparing such states is addressed in section V, and various other theoretical and practical issues are briefly discussed in the conclusion.

![FIG. 1: Schematic of a generalized Bell experiment. The source generates correlated states that are directed to the A and B devices used to perform the measurements, with adjustable parameters a and b. Each measurement provides a binary result “+” or “-” individually.](image)

### II. ROOT BINNING

To begin our study, we consider a state of the form of a superposition of two two-particles wave functions with a relative phase.

\[
|\Psi\rangle = \frac{1}{\sqrt{2}} (|f f\rangle + e^{i\theta} |g g\rangle) \quad 0 \leq \theta \leq 2\pi \quad (5)
\]

with \( f \) real, even and normalized to unity while \( g \) is assumed real, odd and normalized to unity. This kind of state looks similar to the one used by Auberson et al. [14], but the \( f \) and \( g \) functions will be quite different as well as the binning of the continuous variables being measured.

The quadrature-phase homodyne measurement outputs a continuous variable, and yet the majority of tests of local realism versus quantum mechanics require a binary result. Hence for a given quadrature measurement \( q_i \) \((i = 1, 2)\) at either location A or B, we need to classify the result as either “+” or “-”. In ref. 14, the “positive-negative” binning is used, that is the result is classified as “+” if \( q_i \geq 0 \) and “-” if \( q_i < 0 \).

The choice of binning is quite arbitrary. For the state wee consider another type of binning, we call root binning, that depends on the roots of the functions \( f \) and \( g \) (that are known in advance to the experimenters). We assign “+” when the result \( q \) lies in an interval where \( f(q) \) and \( g(q) \) have the same sign, and “-” if \( q \) is in an interval where \( f \) and \( g \) have opposite signs. We define \( D^+ \) as the union of the intervals in which \( f(q) \) and \( g(q) \) have the same sign and \( D^- \) as the union of the intervals in which \( f(q) \) and \( g(q) \) have opposite signs.

\[
D^+ = \{ \forall q \in R \mid f(q) g(q) \geq 0 \} \quad (6a)
\]
\[
D^- = \{ \forall q \in R \mid f(q) g(q) < 0 \} \quad (6b)
\]

Let us first consider the case when quadrature measurements in position space have been performed on both sides. So the binary probabilities we need for the CHSH type of Bell inequality will be

\[
P_{++} = \int_{D^+} \int_{D^+} dq_1 dq_2 P(q_1, q_2) \quad (7a)
\]
\[
P_{+-} = \int_{D^+} \int_{D^-} dq_1 dq_2 P(q_1, q_2) \quad (7b)
\]
\[
P_{-+} = \int_{D^-} \int_{D^+} dq_1 dq_2 P(q_1, q_2) \quad (7c)
\]
\[
P_{--} = \int_{D^-} \int_{D^-} dq_1 dq_2 P(q_1, q_2) \quad (7d)
\]

with

\[
P(q_1, q_2) = |\langle q_1 | q_2 | \Psi \rangle|^2 = \frac{1}{2} \left[ |f(q_1)|^2 |f(q_2)|^2 + |g(q_1)|^2 |g(q_2)|^2 \right] + 2 \cos \theta \ f(q_1) g(q_1) f(q_2) g(q_2) \quad (8)
\]
Hereafter, we calculate the correlation function for $E_{q_1,q_2}$:

$$E_{q_1,q_2} = P_{++} + P_{--} - P_{+-} - P_{-+}$$

(9)

and as we have chosen $f$ even and $g$ odd, we get the remarkably simple expression:

$$E_{q_1,q_2} = V^2 \cos \theta$$

(10)

where

$$V = \int_{D^+} f(q)g(q) dq - \int_{D^-} f(q)g(q) dq$$

$$= \int_{-\infty}^{+\infty} |f(g(q))| dq$$

(11)

A similar binning will be applied for the momentum part. Since we suppose that $f(q)$ is a real and even function while $g(q)$ is real and odd, $f(q)$ has a real even Fourier transform $\tilde{f}(p)$ while $g(q)$ has an imaginary Fourier transform $i\hbar(p)$, where $\hbar(p)$ is a real and odd function. Using these properties, and taking care of the supplementary $i$ factor, the same reasoning applies for $\tilde{f}$ and $\hbar$ as for $f, g$. Denoting as $D^+$ and $D^-$ the intervals associated with $\tilde{f}$ and $\hbar$, we obtain

$$E_{p_1,p_2} = -W^2 \cos \theta$$

(12)

where

$$W = \int_{D^+} \tilde{f}(p)\hbar(p) dp - \int_{D^-} \tilde{f}(p)\hbar(p) dp$$

$$= \int_{-\infty}^{+\infty} |\tilde{f}(p)| \hbar(p) dp$$

(13)

and equivalently,

$$E_{q_1,q_2} = -V \ W \ \sin \ \theta$$

(14)

$$E_{p_1,p_2} = -V \ W \ \sin \ \theta$$

(15)

Hence by combining Eqs. (10), (12), (14) and (15) we can write the CHSH inequality [3]:

$$S = |\cos(\theta)(V^2 + W^2) - 2\sin(\theta) VW| \leq 2.$$  

(16)

The maximum of $S$ with respect to $\theta$ is obtained for $\tan \theta_m = -2WV/(V^2 + W^2)$, and we have $\theta_m \to -\pi/4$ as $V, W \to 1$. For this optimized $\theta_m$ we get the Bell inequality

$$S = |\sqrt{V^4 + W^4 + 6V^2W^2}| \leq 2.$$  

(17)

Using this really simple expression [17], the debate between quantum mechanics and local realistic theories boils down to find functions $f$ and $g$ whose integrals $V, W$ violate [17]. An interesting feature appears when the distributions are eigenstates of the Fourier transform, so that $V = W$ and equation (17) becomes

$$S = 2\sqrt{2} \ V^2 \leq 2$$

(18)

So if such functions have the right overlap needed to obtain $V = 1$, one will get the maximal violation of the above inequality, which is obtained for $S = 2\sqrt{2}$.

When compared to the positive-negative binning, root binning has the advantage of having two parameters $V$ and $W$ to play with while the positive-negative binning has only one [2]. Moreover, as we will show now, the above Bell inequality is violated by simple wave functions, that no longer have the singularities that appeared in ref. [2].

### III. FOUR PAWS SCHröDINGER CATS

In order to propose an explicit expression of a state that violates the Bell inequality [17], let us first consider the case of a Schrödinger cat state. The Schrödinger cats, that are a superposition of two coherent states of amplitudes $a$ and $-a$, involve intrinsic quantum features such as negative Wigner functions, which make them interesting candidates for our case [3]. We must choose an even cat for $f(q)$ and an odd cat for $g(q)$:

$$f(q) \propto e^{-(q+a)^2/2} + e^{-(q-a)^2/2}$$

(19b)

$$g(q) \propto -e^{-(q+a)^2/2} + e^{-(q-a)^2/2}$$

(19b)

Unfortunately for this simple state we get $V = 1$ and $W \simeq 0.64$ for $a \to \infty$, so that $S \simeq 1.90 < 2$. Therefore this state cannot be used for violating Bell inequality (note that Gilchrist et al. [10] also consider Schrödinger cats, but without getting a violation of Bell inequalities).

Instead of Schrödinger cats which can be alive or dead, we consider acrobat cats which have 4 paws. Let us for instance consider:

$$f(q) \propto -e^{-(q+3a)^2/2} + e^{-(q+a)^2/2} - e^{-(q-3a)^2/2}$$

(20a)

$$g(q) \propto -e^{-(q+3a)^2/2} - e^{-(q+a)^2/2} + e^{-(q-a)^2/2}$$

(20a)

$$+ e^{-(q-3a)^2/2}$$

(20a)

$f(q)$ and $g(q)$ are depicted in Fig. 3 together with their Fourier Transform. Note that for this choice of functions, each peak is distant from its neighbours of $a = 2a$, this disposition yields an optimal overlap of $\tilde{f}$ and $\tilde{g}$ and thus a high value of $S$. The best appearance appears when the peaks move off as $a \to \infty$. In that case, $V = 1$, $W = \frac{3}{2}$ and thus we get the significant violation of $S = 2 \simeq 2.417$ (in facts, the condition $a \to \infty$ appeared to be not so strict numerically, as an amplitude $a = 5$ is enough to obtain $S \simeq 2.417$). Such a violation represents a large improvement compared to Munro’s best result of 0.076 [13], for a state with no singularity and at least as easy to produce as Munro’s $c_5$ optimized state [4]. However, we are still away from the maximal value $2\sqrt{2}$ of the CHSH Bell inequality [13]. In the following section, we will propose a set of states to get closer to the maximal violation.
IV. N PAWS SCHRODINGER CATS

The result obtained with the 4 paws cat suggests that a way to get a stronger violation is to increase the total number \( N \) of paws of the cat states \( |f_N\rangle \) and \( |g_N\rangle \), with the proper sign between the peaks. We thus define, for a given amplitude \( \alpha \)

\[
\begin{align}
    f_{N;\alpha}(q) &\propto \sum_{j=-\infty}^{\infty} \cos \left( \frac{\pi}{4} |2j+1| \right) e^{-\frac{i}{2}(q-(j+\frac{1}{2})\alpha)^2} \quad (21a) \\
g_{N;\alpha}(q) &\propto \sum_{j=-\infty}^{\infty} \sin \left( \frac{\pi}{4} |2j+1| \right) e^{-\frac{i}{2}(q-(j+\frac{1}{2})\alpha)^2} \quad (21b)
\end{align}
\]

Table I presents the results of the calculation of \( S \) according to the formula (17) for the state defined by (5). As expected, the quantity \( S \) increases with the number of paws and tends to \( 2 \) when \( \alpha \to \infty \). Wide segments denote Dirac Delta functions.

\[
\begin{array}{cccccc}
N & 2 & 4 & 6 & 8 & 10 \\
S & 1.895 & 2.417 & 2.529 & 2.608 & 2.649 & 2.681 \\
\end{array}
\]

TABLE I: \( S \) for N-paws cat defined by eqs. (21) and \( \alpha = 15 \), each peak having the same high.

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Another regularization of the wave functions (22) consists in a widening of the Dirac functions to Gaussians and taking a finite number of paws. Thus one can understand that \( S \to 2\sqrt{2} \) as \( N \to \infty \).

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the above functions to $N$ paws for given parameters $s$ and $\alpha$, but as such $f$ and $g$ can not directly be considered as truncated $f_{\alpha,s}$ and $g_{\alpha,s}$, the best amplitude will differ from $\sqrt{2}$ and $S$ will slightly move away from $2\sqrt{2}$.

Some results obtained for $s = 0.3$ are presented on table IV for $N$ running from 4 to 12. The value of $\alpha$ used in this table is numerically calculated in order to maximize $S$. Here the violation is considerably improved compared to table I, with for instance $S = 2.764$ with $N = 4$, and $S = 2.828$ with $N = 10$. In figure 4 we display $f(q)$ and $g(q)$ for the case $N = 12$ showing that these functions are nearby self Fourier Transform.

In appendix A we rewrite these states on the Fock-states basis and express the $f$ and $g$ functions as combinations of Hermite polynomials.

$$ N \begin{array}{c|cccccc} \alpha_{opt} & 2.6 & 2.3 & \alpha_{opt} & 1.8 & 1.8 \\ S & 2.764 & 2.823 & 2.826 & 2.828 & 2.828 \end{array} $$

TABLE II: $S$ as a function of the number of peaks $N$ for a squeezing parameter $s=0.3$ and a gaussian envelope of width $1/s$, for an optimized amplitude $\alpha_{opt}$.

$N = 12$ cat states described by eqs(23) presented in position space (left) and in momentum space (right) for parameters $\alpha = 1.8$ and $s = 0.3$. Left axes are in arbitrary units and normalized to unity.

V. CONDITIONAL PREPARATION OF ENTANGLED $N$ PAWS SCHRODINGER CATS

Preparing entangled $N$ paws Schrödinger cats is a challenging task, we begin by focusing our attention on how one single $N$ paws Schrödinger cat as defined by eqs(24) could be generated. This state has strong similarities with the encoded states introduced by Gottesman and coworkers [10] to perform quantum error-correction codes. Recently, Travaglione and Milburn [17] presented a proposal to generate non-deterministically such encoded states. Following this study, we will first show how to generate the state $|g\rangle$ by applying a specific sequence of operations similar to (26) and then derive a setup to produce the whole state (27).

The preparation procedure begins with the quantum system in the vacuum state $|0\rangle$ and an ancilla qubit in the ground state $|0\rangle_a$. If some squeezing parameter is needed, one may take a squeezed vacuum as quantum system and follow the procedure we describe here. For simplicity reasons, we set $s = 1$.

Let us first apply:

$$ H e^{-i\alpha p\sigma_z} H $$

with $p$ is the momentum operator applied to the continuous variable state, $H$ the Hadamard gate and $\sigma_z$ the Pauli matrix applied to the qubit:

$$ H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} $$

We then have a probability $1/2$ of measuring the qubit either in the excited or in the ground state. If it is found in the $|1\rangle_a$ state, the continuous variable is left in the state $|\Upsilon_2\rangle \propto -|3\alpha\rangle + |\alpha\rangle + |0\rangle$. Otherwise the procedure is stopped and we try again. The qubit is then bit-flipped to $|0\rangle_a$ and we go on by applying the sequence

$$ H e^{-i2\alpha p\sigma_z} H $$

Measuring the qubit in the $|0\rangle_a$ results in the continuous variable left in the state $|\Upsilon_2\rangle \propto -|3\alpha\rangle + |\alpha\rangle + |0\rangle + |\Upsilon_1\rangle$. To increase the number of paws, we iterate the following procedure given $|\Upsilon_{n-1}\rangle$ and the qubit in $|0\rangle_a$:

- Apply the operators

$$ H e^{-i2^{n-1}\alpha p\sigma_z} H $$

- Measure the qubit

- If the qubit was in the state $|0\rangle_a$, we have created $|\Upsilon_n\rangle$

- If the qubit was in the state $|1\rangle_a$, discard and try again.

Once the number of paws is considered satisfactory, we stop the previous iteration. The last point to generate $|g\rangle$ is to apply the following sequence

$$ H e^{i\frac{\alpha}{\sqrt{2}} p\sigma_z} H $$

If the qubit is found in the state $|0\rangle_a$, we have created the state $|g\rangle$ as defined in eq.(26). If the protocol was stopped after $n$ iterations (corresponding to $|\Upsilon_n\rangle$), the state $|g\rangle$ is created with probability $1/2^{n+1}$ and shows $N = 2^{n+1}$ paws. In the last section of ref.[17], Travaglione and Milburn briefly consider the question of the physical implementation of this iteration process using a radio-frequency ion trap. We refer the reader to this article for further details.

For the creation process of the state $|\Psi\rangle = \frac{1}{\sqrt{2}} (|ff\rangle + e^{i\theta}|gg\rangle)$, we present on fig. 4 a global view of our scheme. The $\Gamma$-labelled box corresponds to the
We have also tested root binning on other interesting forms of Bell inequality that are the information-theoretic inequalities developed by Braunstein and Caves [15] and generalized by Cerf and Adami [19]. Using quadrature measurements, root binning and our state defined by eqs. (21), we unfortunately could find no violation for neither Braunstein’s nor Cerf’s form of information-theoretic Bell inequality. Our state in fact tends to the minimum limit for violation of these information inequalities when \( V \rightarrow 1 \) and \( W \rightarrow 1 \). As a matter of fact, the binning process discards a lot of information that lies within each interval of the binning. This information loss may prevent any violation of information-theoretic inequalities.

As a conclusion, let us point out that though the present idea sounds quite attractive, its practical implementation is very far-fetched. Though we do propose a theoretical scheme to prepare the required states, it relies on various features (coupling Hamiltonian, CNOT gates) that are not presently available with the required degree of efficiency. For going in more details into the implementation of the present scheme, inefficiencies and associated decoherence effects should be examined in detail for each required step, i.e., preparation, propagation, and detection. Also, various possible implementations of the proposed scheme should be considered [9, 17, 20, 21].

Such a study is out of the scope of the present paper, that has mostly the goal to show that arbitrarily high violations of Bell inequalities are in principle possible, by using continuous-variables measurements and physically meaningful - though hardly feasible - quantum states.

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**APPENDIX A: EXPRESSION IN THE FOCK BASIS**

An interesting point is to study the decomposition of states \( |f \rangle \) and \( |g \rangle \) on the Fock basis \( |n \rangle \), which are experimentally accessible. Starting from states (23) with \( \alpha = \sqrt{7} \), \( s \approx 0.4 \) and with \( N \) satisfying condition (24), we obtained after truncating at the 14th order:

\[
|f\rangle = \sqrt{0.459}|0\rangle - \sqrt{0.491}|4\rangle - \sqrt{0.008}|8\rangle - \sqrt{0.042}|12\rangle \tag{A1a}
\]

\[
|g\rangle = \sqrt{0.729}|1\rangle + \sqrt{0.155}|5\rangle - \sqrt{0.107}|9\rangle - \sqrt{0.009}|13\rangle \tag{A1b}
\]

These states allow to reach a violation \( S - 2 = 0.81 \). The state \( |f\rangle \) only involves orders \( n = 0 \) (mod 4), as we have \( n = 1 \) (mod 4) for \( |g\rangle \). This directly comes from the fact that \( \langle p|n\rangle = (-i)^n\langle q|n\rangle \) where \( q = p \) and \( (-i)^4 = 1 \), so that states of the form \( \sum_{n=0}^{\infty} (\text{mod } 4) c_n|n\rangle \) are eigenvectors of the Fourier transform. From these considerations we have
obtained $S = 2.68$ for:

$$|f\rangle = \sqrt{0.585}|0\rangle - \sqrt{0.415}|4\rangle \quad (A2a)$$

$$|g\rangle = \sqrt{0.848}|1\rangle + \sqrt{0.152}|5\rangle \quad (A2b)$$

and $S = 2.3$ for:

$$|f\rangle = \sqrt{0.67}|0\rangle - \sqrt{0.33}|4\rangle, \quad |g\rangle = |1\rangle \quad (A3)$$

These states are quite simple, and it is expected that a specific procedure to produce them might be designed.

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