Enumerating cuspidal curves on toric surfaces

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Abstract

Enumerative algebraic geometry deals with problems of counting geometric objects defined algebraically. An important class of enumerative problems is that of counting curves: given a class of curves in some projective variety defined by fixing some algebraic or geometric invariants (such as degree, genus and types of singularities), the problem usually takes the form of ”how many curves of that class pass through a configuration of $n$ points in general position?”

Tropical Geometry deals with certain piecewise-linear complexes, which arise as degeneration of families of complex algebraic varieties, and can also be described algebraically using ”max-plus” algebra, (The tropical semi-field). The problem we solve is that of counting rational curves with one cusp and certain number of nodes on toric surfaces, passing through a configuration of sufficient points in general position.

We show that that this number equals the number of certain tropical curves counted with multiplicities and we describe these curves and their multiplicities.

The main tools are tropicalization and patchworking. In tropicalization we pass from an equisingular family of curves to a special limit fiber which can be described in terms of tropical data and analytic data. We then classify these possible limits, and use the patchworking theorem to reconstruct the families that correspond to them.
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Introduction

0.1 History

Enumerative algebraic geometry deals with problems of counting geometric objects defined algebraically over fields, usually over the complex numbers or the real numbers. An important class of enumerative problems is that of counting curves: given a class of curves in some projective variety defined by fixing some algebraic or geometric invariants (such as degree genus and types of singularities), the problem usually takes the form of "how many curves of that class pass through a configuration of \( n \) points in general position?". A classical problem was that of counting complex nodal curves of given genus \( g \) and degree \( d \) in the complex projective plane, passing through a configuration of \( 3d + g - 1 \) points in general position. A recursive calculation this number for rational curves (i.e. of genus 0) was given by Kontsevich in 1994 \[12\], and for any genus by Caporaso and Harris in 1998 \[3\]. This problem of curve counting is related to Gromov-Witten invariants, in Theoretical Physics and Symplectic Geometry. As for counting singular curves, In 2005, Alexei Zinger published a paper providing a method to count rational singular curves in projective spaces \( \mathbb{P}^n \), in particular giving a formula for rational curves with one cusp in any projective space \( \mathbb{P}^n \), using methods of algebraic geometry and moduli spaces. The result in this thesis differs in both the type generality (We count curves on any projective toric surface) and in the method involved, that of establishing a correspondence theorem between the curves in count and certain plane tropical curves.

Tropical geometry deals with a class of piecewise-linear objects, arising algebraically from an analogue of algebraic geometry over the Tropical Semi-Field \( (\mathbb{R} \cup \{\infty\}, +, ', \cdot) \) where \( a' + b' = \min\{a, b\} \) and \( a' \cdot b' = a + b \) where here + is the normal addition. It can also be seen as a degeneration of the complex structure in complex algebraic geometry, and can be thought of as what is left after looking at complex geometry in a logarithmic scale and passing to a limit.

In 2002, Grigory Mikhalkin proved his correspondence theorem, establishing a remarkable connection between enumerative complex geometry and tropical geometry. After establishing analogues of genus and degree for tropical curves in the plane, he proved that if counted with appropriate weights, the number of nodal tropical curves of degree \( d \) and genus \( g \) via \( 3d + g - 1 \) points in tropical general position in the tropical plane, is the same as the number of complex nodal curves of degree \( d \) and genus \( g \) via generic \( 3d + g - 1 \) points on the projective plane (more generally, a similar result holds for curves on any projective toric surface). Moreover, Mikhalkin’s correspondence theorem, establishes direct correspondence between given complex curves and their tropical counterparts. Since tropical curves can be described in terms of subdivisions of lattice polygons and piecewise linear convex functions, this gives a method of calculating this number via counting subdivisions of lattice polygons, essentially reducing the problem to the realm of combinatorics. Mikhalkin used
symplectic and pseudo-holomorphic techniques for his proof.

In 2006, Eugenii Shustin, provided a new proof for the correspondence theorem, using methods of complex algebraic geometry and basic algebraic topology, thus providing an algebro-geometric proof for the result. Moreover, Shustin’s proof allows for possible generalizations to correspondence theorems for singular curves.

In this thesis we prove a correspondence theorem for rational curves with nodes and one cusp, (1-cuspidal rational curves), on projective toric surfaces. Let $\mathbb{K}$ be the field of convergent Puiseux series in one variable over $\mathbb{C}$, equipped with the the non-archimedean valuation: $\text{Val}(b(t)) = -\min\{\tau \in \mathbb{R} : c_\tau \neq 0\}$. Let $\Sigma = \text{Tor}(\Delta)$ be a toric surface over $\mathbb{K}$ corresponding to a polygon $\Delta$, any rational 1-cuspidal curve on $\Sigma$ has $n = |\text{Int}(\Delta) \cap \mathbb{Z}^2| - 1$ nodes, (by genus formula). Our result states that the number of rational curves on $\Sigma$ with one cusp and $n$ nodes, passing through a configuration of $s = |\mathbb{Z}^2 \cap \Delta| - n - 3$ points in general position on $\Sigma$ equals the number of certain tropical curves through $s$ points in tropical general position in the tropical torus $\mathbb{R}^2$.

Let $x_1, \ldots, x_s \in \Sigma$ where $s = |\mathbb{Z}^2 \cap \Delta| - n - 3$, points in general position, such that they tropicalize to points $p_1, \ldots, p_s$ in tropical general position (In the sense that $p_i = \text{Val}(x_i)$). Denote by $N_\Delta(nA_1, 1 \cdot A_2)$ the number of rational curves on $\text{Tor}(\Delta)$ with $n$ nodes and one cusp, that pass through $x_1, \ldots, x_s$.

Denote by $N_{\Delta}^{\text{trop}}(nA_1, 1 \cdot A_2)$ the number of rational tropical curves in the plane, counted with weights (that will be defined soon), passing trough $p_1, \ldots, p_s$ such that:

1. Their Newton polygon is $\Delta$.
2. In the dual subdivision, every point in $\partial \Delta \cap \mathbb{Z}^2$ is a subdivision vertex.
3. In the dual subdivision only the following polygons appear: triangles, parallelograms, and a single quadrilateral with no pair of parallel edges.
4. The genus of the tropical curve is 0.

Each curve is counted with weight as defined in Definition 3.1. The weights can be calculated from the data of the subdivision and the edges lying on the marked points.

In this thesis we prove that $N_\Delta(nA_1, 1 \cdot A_2) = N_{\Delta}^{\text{trop}}(nA_1, 1 \cdot A_2)$

0.2 Strategy and Method of Proof

The strategy of the proof is composed of two stages: tropicalization (degeneration) and patchworking (deformation).

We work over $\mathbb{K}$, this will still give us the correct count for curves over $\mathbb{C}$ since all schemes, varieties and conditions involved in the proof are defined using algebraic numbers (in fact, most equations that appear, for example in Severi varieties have integer coefficients!). Both $\mathbb{C}$ and $\mathbb{K}$ are
algebraically closed fields of characteristic 0, containing the algebraic numbers. This is an example of the known Lefschetz Principle. ([1])

The goal of tropicalization is to assign to every 1-cuspidal curve over \( K \) a tropical limit, which consists of two kinds of data:

- **Combinatorial Data:** a tropical curve in the plane, or equivalently, a subdivision of the Newton polygon \( \Delta \) and a convex piece-wise linear function supported on \( \Delta \) with integer values at lattice points.

- **Algebraic Data:**
  - Limit algebraic curves on toric surfaces corresponding to subdivision polygons, and
  - Refinements, which are too, curves on surfaces, assigned to edges of the subdivision (which correspond to intersection of adjacent toric surfaces containing limit curves).

Given an algebraic curve over \( K = \mathbb{C}\{t\} \), after suitable changes of coordinates and other technical assumptions, we obtain an analytic family of complex curves on \( \Sigma \), parameterized over a small punctured disk in the complex plane \( D^* = \{ z \in \mathbb{C}\setminus\{0\} | |z| < \varepsilon \} \). Embedding the family in an appropriately chosen ambient space we can extend the family to the full disk \( D \), adding a limit fiber over 0. This fiber comprises a collection of curves lying on a collection of toric surfaces, this data is called the tropical limit of the family and can be described as a collection of combinatorial and algebraic data as above. The process of going from a family to a 'special' fiber, is called degeneration. Our first goal is to classify all possible tropical limits of the families in the class of curves in interest, in our case: 1-cuspidal rational curves. This is done mainly via topological consideration, with some algebraic geometry involved.

The second stage is patchworking (or deformation): given the data of a tropical limit in one of the classes we described in the first stage, construct a family or families of algebraic curves over \( K \) degenerating into the given tropical limit. This is done invoking a powerful theorem called the patchworking theorem, describing how families arise from the collection of tropical limit, and giving control on the types of singularities involved.

In summary, to calculate the number of 1-cuspidal curves on a toric surface, we start with a configuration of points in general position on the surface, tropicalizing to tropical general points in \( \mathbb{R}^2 \). We then construct the appropriate tropical curves, which correspond to the possible tropical limits, passing through a tropical general configuration of points, and for each such tropical curve we calculate it’s weight, or multiplicity. Summing all weights we arrive at the desired result.

### 0.3 Organization of the Work

In section[1] ”Polar Curves and Newton Polygons” we describe the construction of the polar curve for a given curve on a surface, and use the calculation of the intersection number of a curve and
it’s polar to show non-existence of certain curves on some surfaces.

In section 2 "Tropicalizations", subsections 2.1 - 2.3 we first give a short account of tropical curves, and then describe the tropicalization procedure, assigning a tropical limit to an algebraic family of complex curves. In subsections 2.4 - 2.6 we classify and describe what tropical limits can arise as the limit of a 1-cuspidal family of rational curves, both in terms of combinatorial data and analytic data.

In section 3 "Restoring 1-cuspidal rational algebraic curves out of a given tropical limit", we describe the procedure of how to reconstruct the data of possible tropical limits corresponding to a given tropical curve, counting how many such tropical limits can occur, and how many algebraic families tropicalize to those limits, solving the enumerative problem we set for.

In section 4 "Patchworking singular algebraic curves" we describe the equisingular patchworking theorem, allowing us to go in deformation from a tropical limit to a family of algebraic curves. We also show that in our case we can indeed apply the theorem by verifying some technical transversality criterion.
Body

1 Polar Curves and Newton Polygons

1.1 The Polar Curve and Intersection Formulas

Let $C \in \mathbb{P}^2$ a projective plane curve, given as the zero set of a homogeneous polynomial $F(x, y, z)$.

**Definition 1.1.** Define the Polar Curve associated to $C$ relative to a point $p = (a_1; a_2; a_3) \in \mathbb{P}^2$ (the polar of $p$ relative to $C$), denoted by $P_C(p)$, to be the projective curve defined by the equation

$$a_1 \frac{\partial F}{\partial x} + a_2 \frac{\partial F}{\partial y} + a_3 \frac{\partial F}{\partial z} = 0$$

Similarly, we can define the polar of a germ (in affine coordinates): let $f \in \mathbb{C}[x, y]$ and $(\alpha : \beta) \in \mathbb{P}^1$, we then call

$$P_{(\alpha, \beta)}(f) := \alpha \cdot \frac{\partial f}{\partial x} + \beta \cdot \frac{\partial f}{\partial y} \in \mathbb{C}[x, y]$$

the polar of $f$ with respect to $(\alpha : \beta)$ It is defined only up to a non-zero constant. One may check that this definition is invariant under multiplying the equation of $f$ by a function non-vanishing at $p$, and it is also invariant under linear (projective) coordinate change.

**Claim 1.1.** The only points of intersection of $C$ and $P_C(p)$: $q \in C \cap P_C(p)$ are either:

1. Singular points of $C$
2. Smooth points $q \in C$ such that $p$ lies on $T_q C$, the tangent line of $C$ at $q$

**Proof.** (2) Is obvious from the definition, and (1) follows because at singular points the Zariski tangent space is the whole plane. See for example, [6][Section 1.2]

Denote by $i_p(F, G)$ the local intersection multiplicity at point $p$, and denote by $i(F, G) = \sum_{p \in \mathbb{P}^2} i_p(F, G)$ the total intersection multiplicity of $F$ and $G$

**Proposition 1.1.** Let $C$ be a plane curve of degree $d$. then $i(C, P_C(p)) = d(d - 1)$

**Proof.** $P_C(p)$ is given by a polynomial of degree $d - 1$, hence the result follows from Bézout theorem.

We will start by recalling the definition of two invariants of a singular germ.

**Definition 1.2.** Let $f \in \mathbb{C}[x, y]$ be a reduced power series and let

$$\bar{\mathcal{O}} = \mathbb{C}[x, y]/\langle f \rangle \hookrightarrow \mathbb{C}[t_1] \oplus \ldots \oplus \mathbb{C}[t_r] = \bar{\mathcal{O}}$$

(2) denote the normalization. then we shall call $\delta(f) := \dim_{\mathbb{C}} \bar{\mathcal{O}}/\mathcal{O}$ the $\delta$ invariant of $f$. In essence, the $\delta$ invariant counts the number of double points concentrated at a point.
Remark 1. For a power series of a germ of an ordinary cusp or an ordinary node we have: \( \delta = 1 \). One first checks that \( \delta \) does not depend on analytic isomorphism, and then computes for the special cases \( x^3 - y^2 = 0 \) for a cusp and \( xy = 0 \) for a node.

Proposition 1.2. (Degree-Genus formula) Let \( C \) a projective plane curve (possibly singular) of degree \( d \) and genus \( g \). then:
\[
g = \frac{(d-1)(d-2)}{2} - \sum_{p \in C} \delta(p)
\]

Proof. See [16, Corollary 7.1.3 pp. 158]

An important version of this formula is relevant to toric surfaces:

Proposition 1.3. Let \( C \) be a curve on a toric surface \( \text{Tor}(\Delta) \) defined by the vanishing of a function with Newton polygon \( \Delta \) in the open torus \( (\mathbb{C}^*)^2 \) then:
\[
g = \left| \text{Int}(\Delta) \cap Z^2 \right| - \sum_{p \in C} \delta(p)
\]

Proof. See discussion at [13, Section 5.2].

The second invariant we will deal with is the kappa invariant, defined as:

Definition 1.3. Let \( f \in \mathbb{C}[x,y] \) be a reduced power series. the \( \kappa \)-invariant of \( f \) is defined to be the intersection multiplicity with a generic polar, that is,
\[
\kappa(f) := i \left( f, \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} \right)
\]

where \((\alpha : \beta) \in \mathbb{P}^1\) is a generic point in \( \mathbb{P}^1 \)

Proposition 1.4. The polar curve intersects the curve \( C \) at the following points: singular points and points whose tangent line passes through point \( P \)

Proof. follows from the form of the equation and definition of tangent space.

Definition 1.4. Let \( s = \sum_{i \geq r} a_i x^{\frac{r}{n}} \), a Puiseux series.
We define the order in \( x \) of \( s \) to be \( \text{ord}_x(s) = \infty \) if \( s = 0 \) and otherwise:
\[
\text{ord}_x(s) = \frac{\min\{i|a_i \neq 0\}}{n}
\]

Assuming that \( n \) is coprime with the collection of nominators appearing in the sum, we define the polydromy order of \( s \) to be \( n \). We will denote it by \( \nu(s) \)

In view of Claim [14] The intersection number with the polar, \( i(C,P_C(p)) \) can be calculated locally:
**Theorem 1.1.** For a reduced curve $C$ the contribution of a point $q \in C$ to $i(C, P_C(p))$ is as follows:

1. If $q$ is a singular point:
   
   (a) $q \neq p$:
   
   $$2\delta(q) + i_q(\ell, C) - \#\text{branches}(q)$$
   
   where $\ell$ is the line passing through $p$ and $q$ and $\#\text{branches}(q)$ is the number of local branches at the point $q$.

   (b) $q = p$:
   
   $$2\delta(q) + \text{mult}(p) - \#\text{branches}(p) + \sum_{\ell \text{ is tangent to a branch of } C \text{ at } q} i_p(\ell, C) - (\#\text{tangents} - 1) \cdot \text{mult}(p)$$
   
   where $\#\text{tangents}$ is the number of different tangents to the branches of $C$ at $q$.

2. If $q$ is a smooth point:

   $$i_q(\ell, C) \quad \text{(again } \ell \text{ is the line passing through } p \text{ and } q)$$

**Proof.** 1.a) We will not detail the proof, but instead refer to some results:

We will conduct a local calculation in affine coordinates. WLOG, choose affine coordinates such that $q = (0,0), f \in \mathbb{C}[x,y]$ a reduced power series and $(\alpha : \beta) \in \mathbb{P}^1$. From [9] Lemma 3.37 pp. 209 we know that the local intersection number:

$$i_q \left( f, \alpha \frac{\partial f}{\partial x} + \beta \frac{\partial f}{\partial y} \right) = \mu(f) + i(-\beta x + \alpha y, f) - 1$$

Note that in our terminology $-\beta x + \alpha y = 0$ is exactly $\ell$. Here $\mu(f)$ is the Milnor number, another invariant of singular points, which we will not elaborate upon, because of the following result which eliminates it from the formula: [9] Lemma 3.35 pp. 208]

$$\mu(f) = 2\delta(f) - r(f) + 1 \quad (4)$$

$r(f)$ denotes in the number of irreducible factors of $f$, which, since $f$ is reduced, equals the number of branches at $q$. Substituting we get

$$2\delta(q) + i_q(\ell, C) - \#\text{branches}(q)$$

as expected.

1.b) Here we will detail the calculation. Recall Euler’s identity for homogenous polynomial of degree $d$:

$$x \frac{\partial F}{\partial x} + y \frac{\partial F}{\partial y} + z \frac{\partial F}{\partial z} = dF$$
WLOG, using translation and change of coordinates we can assume \( p = (0,0,1) \in \mathbb{P}^2 \) in homogeneous coordinates, and that \( p \in C \); WLOG we can also assume that the \( y \)-axis is transversal to all branches at \( p \) (otherwise we change coordinates by rotation). The polar curve is the vanishing locus of the following polynomial: \( 0 \frac{\partial F}{\partial x} + 0 \frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} = \frac{\partial F}{\partial z} \). Now \( i_p(F, \frac{\partial F}{\partial z}) = i_p(F, z \frac{\partial F}{\partial z}) \) since \( z \) is not zero on \( p \).

Using Euler’s identity we see that \( i_p(F, z \frac{\partial F}{\partial z}) = i_p(F, dF - x \frac{\partial F}{\partial x} - y \frac{\partial F}{\partial y}) \) and we can now calculate this intersection in an affine chart. we will choose a chart such that \( \{z = 0\} \) is the line at infinity. Denote \( f(x, y) = F(x, y, 1) \), then by Puiseux theorem, we have a factorization in the field of Puiseux series \( \mathbb{C}\{x\} = \bigcup_{k=1}^{\infty} \mathbb{C}(x^{1/n}) \):

\[
f(x, y) = \prod_{i=1}^{m} (y - \xi_i(x))
\]

where each \( \xi_i(x) \) is a Puiseux series corresponding to a branch passing through \( p = (0,0,1) \). Intersection multiplicity is additive with respect to branches, so we can sum over the intersection multiplicity of each branch independently.

We will Denote by \( n_i = \nu(\xi_i) \) the polydromy order of the \( i \)-th Puiseux series in the decomposition. First we note:

\[
F(x, \xi_i(x)) = 0
\]

\[
x \frac{\partial}{\partial x} F(x, \xi_i(x)) = -x \xi_i'(x) \prod_{j \neq i} (\xi_i(x) - \xi_j(x))
\]

\[
y \frac{\partial}{\partial y} F(x, \xi_i(x)) = \xi_i(x) \prod_{j \neq i} (\xi_i(x) - \xi_j(x))
\]

Summing:

\[
i_p \left( f, df - x \frac{\partial f}{\partial x} - y \frac{\partial f}{\partial y} \right) = \sum_{i=1}^{m} n_i \cdot \text{ord}_x \left( (\xi_i(x) - x \xi_i'(x)) \prod_{j \neq i} (\xi_i(x) - \xi_j(x)) \right)
\]

First we show that \( n_i \cdot \text{ord}_x (\xi_i(x) - x \xi_i'(x)) = i_p(\xi_i, \xi_i) \), the intersection multiplicity of the branch corresponding to \( \xi_i(x) \) with it’s tangent line. This can be seen as follows:

Let \( \xi_i(x) = \lambda_0 x + \lambda_1 x^{r_1} + \cdots \) (starts from \( \lambda_0 x \) because the \( y \)-axis \( \{x = 0\} \) is not tangent to the branch. thus the tangent is \( y = \lambda_0 x \).

Then: \( i_0(\xi_i(x), y - \lambda_0 x) = n_i \cdot \text{ord}_x (\xi_i(x) - \lambda_0 x) = n_i \cdot r_1 \)

And: \( n_i \cdot \text{ord}_x (\xi_i(x) - x \xi_i'(x)) = n_i \cdot \text{ord}_x (\lambda_0 x + \lambda_1 x^{r_1} + \cdots - (\lambda_0 x + r_1 \lambda_1 x^{r_1} + \cdots)) = n_i \cdot \text{ord}_x ((r_1 - 1) \lambda_1 x^{r_1}) = n_i \cdot r_1 \)
The other term, $\sum_{i=1}^{m} n_i \cdot \text{ord}_x \left( \prod_{j \neq i} (x_i(x) - \xi_j(x)) \right)$ equals $\kappa$, the kappa invariant of $C$ at point $p = (0, 0)$, that is because this is the intersection number with $\frac{\partial f}{\partial y}$ which is the polar with respect to $(0, 1) \in \mathbb{P}^1$. This polar intersects all branches of $C$ transversally (as its tangent is the $y$-axis), thus the intersection is generic. Summing:

$$\sum_{i=1}^{m} i_p(\ell_i, \xi_i) = \sum_{i=1}^{m} i_p(\ell_i, C) - (\text{#tangents} - 1) \cdot \text{mult}(p)$$

Lastly by [9, Proposition 3.38 pp. 212]: $\kappa(f) = \mu(f) + \text{mult}(f) - 1$ and by equation (4) from part 1 of the proof the result follows. 2) follows as a special case of (1.b) \hfill \square

1.2 Applying polar curves to show non existence of certain curves

Our main application for this technique is to show non-existence of certain curves on given toric surfaces.

Let $P$ be a trapezoid given by the vertices: $(0, 0), (r, q), (p, q), (0, k), (0, 0)$ where:

1. $0 \leq r < p$
2. $0 < q$
3. $0 < k \leq p + q$

as in Figure [II]

We want to show that there are no rational cuspidal curves on Tor($P$) meeting each boundary divisor with a single branch, more precisely we will show:

**Proposition 1.5.** Let Tor($P$) the toric surface corresponding to $P$, and $(\mathbb{C}^*)^2$ the embedded torus compatible with the lattice. There are no curves $C$ satisfying the following conditions:

1. $C \cap (\mathbb{C}^*)^2$ is defined by a polynomial with Newton polygon Tor($P$)
2. $C$ meets every boundary divisor of Tor($P$) with a single branch, except maybe Tor($[[0, 0], (0, k)]$).
3. $C$ is a rational curve (genus 0)
4. $C$ has a cusp in the complex torus $(\mathbb{C}^*)^2 \in \text{Tor}(P)$ or a cusp on the line $X = 0$ (corresponds to Tor($[[0, 0], (0, k)]$))
Figure 1: Trapezoid and Corresponding Triangle of $\mathbb{P}^2$ Together With the Curve

**Proof.** Denote by $Tor(\Delta) \equiv \mathbb{P}^2$, The projective plane given by the right triangle: $(0,0), (p+q,0), (0,p+q)$ (Denote these vertices by $A, B$ and $C$ accordingly). Consider the curve $\bar{C}$ given as the closure of $C \cap (\mathbb{C}^*)^2$ in $\mathbb{P}^2 = Tor(\Delta)$

$\xymatrix{ Tor(\Delta) \ar@{-}[r]^{(\mathbb{C}^*)^2} & \mathbb{P}^2 \ar@{-}[r] & C \ar@{-}[r] & C \cap (\mathbb{C}^*)^2 \ar@{-}[r] & \bar{C}}$

Denote by $\ell_{AB}$ the toric divisor corresponding to the edge $AB$ (corresponds to $x = 0$ in coordinates), and similarly $\ell_{BC}$ and $\ell_{AC}$ corresponds to $BC$ and $AC$ ($z = 0$ and $y = 0$ accordingly) $C$ passes through the points $p_1 = \{x = 0\} \cap \{z = 0\}$ and $p_2 = \{y = 0\} \cap \{z = 0\}$ See Figure[1]

We calculate intersection with polar given by base point $p_1$: $mult(p_1) = p$ and $mult(p_2) = q$ as can be seen from the Newton polygon: If we change coordinates such that $p_1$ goes to 0, we see that the first monomials of degree less than $p$ vanish there.

We verify that the two branches have different tangents at $p_1$. After change of coordinates that takes $p_1 \rightarrow (0,0), -y \rightarrow y$ and $x-y \rightarrow x$ we see that the Newton diagram at $p_1$ has the form of the lower convex hull of $(0,t), (a,b), (c,b)$ where $t > b > 0$ and $a < c$. This decomposes as the lower convex hull of the Minkowsky sum of tho segments: $L_1 = [(0,t-b), (a,0)]$ and $L_2 = [(0,b), (c-a,b)]$, each corresponds to a branch. Thus, by looking on the order of vanishing monomials we see that either $x$-axis or $y$-axis are tangent to the branch corresponding to $L_1$ but both are not tangent to the branch corresponding to $L_2$ (remark: this holds unless $t-b = a$ but this case can be avoided by first applying an appropriate $SL_2(\mathbb{Z})$ action). Now $i_{p_1}(\ell_{AB}, C) = p+q$ by Bézout theorem and the fact we have one point of intersection.

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For the other tangent we know that the intersection multiplicity is at least $p + 1$ (one more than the multiplicity of the point), thus the contribution of $p_1$:

$$2\delta(p_1) + p - 2 + (p + q) + (p + 1) - p = 2\delta(p_1) + 2p + q - 1$$

Contribution of $p_2$:

$$2\delta(p_2) + i_{p_2}(\ell_{AC}, C) - \#\text{branches}(p_2) = 2\delta(p_2) + q - 1$$

The cusp contributes another $+1$ to the sum. Summing:

$$(p + q)(p + q - 1) = i(C, P_C(P_1)) \geq \sum_{p \in C} 2\delta(p) + 2p + 2q - 2 + 1$$

$$= (p + q - 1)(p + q - 2) + 2(p + q) - 1$$

(\ast) follows from Degree-Genus Formula (Proposition 1.2), recall that here $g = 0$. We get:

$$2(p + q) - 2 \geq 2(p + q) - 1$$

a contradiction.

We summarize what has been obtained in a corollary:

**Corollary 1.1.** Let $P$ be a lattice quadrilateral with a pair of parallel edges. (A lattice trapezoid). Then there are no rational curves with Newton polygon $P$, intersecting each boundary divisor at one branch, admitting a singular branch at either $(C^*)^2$ or one of the boundary divisors corresponding to the parallel edges.

**Proof.** Up to an affine lattice transformation we can assume WLOG that $P$ meets the conditions of Proposition 1.5. Then the result follows. □

Similarly, we want to eliminate the following case as well: Let $P$ a triangle given by the vertices (counter-clockwise): $(0, 0), (0, r), (p, q)$ where:

1. $0 < r < p + q$
2. $0 < q, p$

Similarly to figure 1 (only with a triangle)

**Proposition 1.6.** Let $\text{Tor}(P)$ be the toric surface corresponding to $P$, and $(C^*)^2$ the embedded torus compatible with the lattice. There are no curves $C$ satisfying the following conditions:

1. $C \cap (C^*)^2$ is defined by a polynomial with Newton polygon $\text{Tor}(P)$
2. $C$ meets every boundary divisor of $\text{Tor}(P)$ with a single branch, except maybe $\text{Tor}([(0,0),(0,r)])$.

3. $C$ is a rational curve (genus 0)

4. $C$ has a singular branch in the complex torus $(\mathbb{C}^*)^2 \in \text{Tor}(P)$ or a singular branch on the line $X = 0$ (corresponds to $\text{Tor}([(0,0),(0,r)])$)

**Proof.** Denote by $\ell_{AB}$ the toric divisor corresponding to the edge $AB$ (corresponds to $x = 0$ in coordinates), and similarly $\ell_{BC}$ and $\ell_{AC}$ corresponds to $BC$ and $AC$ ($y = 0$ and $z = 0$ accordingly) $C$ passes through the points $p_1 = \{x = 0\} \cap \{z = 0\}$ and $p_2 = \{y = 0\} \cap \{z = 0\}$. Similarly to the trapezoid case, we calculate intersection with polar given by base point $p_1$: $\text{mult}(p_1) = p$ and $\text{mult}(p_2) = q$ as can be seen from the Newton polygon. $i_{p_1}(\ell_{AB}, C) = p + q$ By Bézout theorem and the fact we have one point of intersection. Thus the contribution of $p_1$:

$$2\delta(p_1) + p - 1 + (p + q) = 2\delta(p_1) + 2p + q - 1$$

Contribution of $p_2$:

$$2\delta(p_2) + i_{p_2}(\ell_{AC}, C) - \#\text{branches}(p_2) = 2\delta(p_2) + q - 1$$

The cusp contributes another $+1$ to the sum. Summing:

$$(p + q)(p + q - 1) = i(C, P_C(P_1)) \geq \sum_{p \in C} 2\delta(p) + 2p + 2q - 2 + 1$$

$$= (p + q - 1)(p + q - 2) + 2(p + q) - 1$$

(*) follows from Degree-Genus Formula (Proposition 1.2), recall that here $g = 0$. We get:

$$2(p + q) - 2 \geq 2(p + q) - 1$$

a contradiction.

Corollary 1.2. Let $P$ be a lattice triangle. Then there are no rational curves with Newton polygon $P$ intersecting two boundary divisor at one branch each, and admitting a singular branch at either $(\mathbb{C}^*)^2$ or on the third boundary divisor.

**Proof.** Up to an affine lattice transformation we can assume WLOG that $P$ meets the conditions of 1.6. Then the result follows. ∎
2 Tropicalizations

2.1 Tropical curves and subdivisions of Newton polytope

The presentation of material in this section summarizes the presentation in [15], for more details see [15] and the corresponding references given there.

2.1.1 Basic Definitions

In this thesis we will use the following definition for a tropical curve (note that there exist other equivalent definitions): Let $K$ be the field of convergent Puiseux series over $\mathbb{C}$, with the non-archimedean valuation: $\text{Val}(b(t)) = -\min\{\tau \in \mathbb{R} : c_\tau \neq 0\}$ where $b(t) = \sum_{\tau \in R} c_\tau t^\tau$ where $R \subset Q$ is contained in bounded below arithmetic progression. $\text{Val}$ takes $K$ onto $Q$ and satisfies:

$$\text{Val}(ab) = \text{Val}(a) + \text{Val}(b), \quad \text{Val}(a + b) \leq \max\{\text{Val}(a), \text{Val}(b)\}, \quad a, b \in K^*$$

For a non-empty finite set $I \subset \mathbb{Z}^k$, denote by $F_K(I)$ the set of Laurent polynomials

$$f(z) = \sum_{\omega \in I} c_\omega z^\omega, \quad z = (z_1, ..., z_k), \quad c_\omega \in \mathbb{K}^*, \quad \omega \in I.$$ 

Put $Z_f = \{f = 0\} \subset (\mathbb{K}^*)^k$ and define the tropical variety of $f$ as

$$A_f = \text{Val}(Z_f) \subset \mathbb{R}^k, \quad \text{where } \text{Val}(z_1, ..., z_k) = (\text{Val}(z_1), ..., \text{Val}(z_n)).$$

We denote the set of tropical varieties $A_f, f \in F_K(I)$ by $\mathcal{A}(I)$. If $I$ is the set of all integral points in a convex lattice polygon $\Delta$, we write $\mathcal{A}(\Delta)$.

The following theorem is due to Kapranov [5]:

**Theorem 2.1.** The tropical curve $A_f$ coincides with the corner locus of the piece-wise linear convex function

$$N_f(x) = \max_{\omega \in I} (\omega \cdot x + \text{Val}(c_\omega)), \quad x \in \mathbb{R}^k$$

where the product of vectors is the standard scalar product.

For a polynomial $f \in F_K(I)$, one can define a subdivision of the Newton polytope $\Delta = \text{conv}(I)$ into convex polytopes with vertices from $I$. Namely, take the convex hull $\Delta_v(F)$ of the set $\{(\omega, -\text{Val}(c_\omega)) \in \mathbb{R}^{k+1} : \omega \in I\}$ and define the function

$$\nu_f : \Delta \to \mathbb{R}, \quad \nu_f(\omega) = \min\{x : (\omega, x) \in \Delta_v(f)\}.$$

This is a convex piece-wise linear function, whose linearity domains are convex polytopes with vertices in $I$, which form a subdivision $S_f$ of $\Delta$. It is easy to see (for example, from the fact that the functions $N_f$ and $\nu_f$ are dual by the Legendre transform) that
Lemma 2.1. The subdivision $S_f$ of $\Delta$ is combinatorially dual to the pair $(\mathbb{R}^k, A_f)$.

Notice that, in general, the geometry of an tropical curve $A \in \mathcal{A}(\Delta)$ determines a dual subdivision $S$ of $\Delta$ not uniquely, but up to a combinatorial isotopy, in which all edges remain orthogonal to the corresponding edges of $A$, and vice versa. Combinatorially isotopic tropical curves form a subset in $\mathcal{A}(I)$, whose dimension we call the rank of tropical curve (or the rank of subdivision) and denote $\text{rk}(A_f) = \text{rk}(S_f)$.

Lemma 2.2. For the case $k = 2$, and $S_f : \Delta = \Delta_1 \cup \ldots \cup \Delta_N$,

$$\text{rk}(S_f) \geq \text{rk}_{\exp}(S_f) \overset{\text{def}}{=} |V(S_f)| - 1 - \sum_{i=1}^{N}(|V(\Delta_i)| - 3) \tag{5}$$

where $V(S_f)$ is the set of vertices of $S_f$, $V(\Delta_i)$ is the set of vertices of the polygon $\Delta_i$, $i = 1, \ldots, N$. More precisely,

$$\text{rk}(S_f) = \text{rk}_{\exp}(S_f) + d(S_f) \tag{6}$$

where

- $d(S_f) = 0$ if all the polygons $\Delta_1, \ldots, \Delta_N$ are triangles or parallelograms,
- otherwise,

$$0 \leq 2d(S_f) \leq \sum_{m \geq 2} ((2m - 3)N_{2m} - N'_{2m}) + \sum_{m \geq 2} (2m - 2)N_{2m+1} - 1 \tag{7}$$

where $N_m, m \geq 3$, is the number of $m$-gons in $S_f$, $N'_{2m}$ is the number of $2m$-gons in $S_f$, whose opposite edges are parallel, $m \geq 2$.

Proof. See [15, Lemma 2.2] \qed

2.1.2 The Tropicalization Process

Let $\Delta \subset \mathbb{R}^2$ be a non-degenerate convex lattice polygon, and let $C \in \Lambda(\Delta)$ be a curve with only isolated singularities, which is defined by a polynomial $f(x, y)$ as in the previous section.

Evaluating the coefficients of $f(x, y)$ at a complex non-zero $t$ close enough to zero we obtain an equisingular family of curves $C_t \in \Lambda(\Delta)$. [15, Lemma 2.3].

We define the tropical limit of $C(t)$ when $t \to 0$ as follows: Let $\nu_f : \Delta \to \mathbb{R}$ be a convex function and $\Delta = \Delta_1 \cup \ldots \cup \Delta_N$, the corresponding subdivision as defined in the previous section. The restriction $\nu_f|_{\Delta_i}$ coincides with a linear (affine) function $\lambda_i : \Delta \to \mathbb{R}$, $\lambda_i(x) = \omega_i x + \gamma_i$, $\omega_i = (\alpha_1, \alpha_2) \in \mathbb{R}^2$, $\gamma_i \in \mathbb{R}$, $i = 1, \ldots, N$. Then the polynomial

$$t^{-\gamma_i} f(z_1 t^{-\alpha_1}, z_2 t^{-\alpha_2}) = \sum_{\omega \in \Delta \cap \mathbb{Z}^2} \tilde{c}_\omega z^\omega \tag{8}$$

1It is, in fact, the interior of a convex polyhedron in $\mathcal{A}(I)$. 

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satisfies the following condition:

\[
\text{Val}(\tilde{c}_\omega) = \begin{cases} 
0, & \text{if } \omega \text{ is a vertex of } \Delta, \\
\leq 0, & \text{if } \omega \in \Delta, \\
< 0, & \text{if } \omega \notin \Delta.
\end{cases}
\]

In other words, letting \( t = 0 \) on the right-hand side of (8), we obtain a complex polynomial \( f \) with Newton polygon \( \Delta_i \), which in turn defines a complex curve \( C_i \in \Lambda(\Delta_i) \), \( i = 1, \ldots, N \). Notice that multiplying \( f(x, y) \) by a constant from \( \mathbb{K}^* \) does not change \( S_f \) and \( C_1, \ldots, C_N \), but adds a linear function to \( \nu_f \). The collection \( (\nu_f, S_f; C_1, \ldots, C_N) \) is called the tropicalization (or dequantization) of the curve \( C \), and denoted by \( T(C) \). We also call \( f_i \) and \( C_i \) the tropicalizations of the polynomial \( f \) and the curve \( C \) on the polygon \( \Delta_i \), \( 1 \leq i \leq N \).

To relate the seemingly algebraic process of tropicalization of \( C \) to an actual limit of the family \( C^{(t)} \) we describe the following construction: Assume that the exponents of \( t \) in the coefficients \( a_{ij}(t) \) of \( f(x, y) \) are rational. By a change of parameter \( t \mapsto t^m \), we can make all these exponents integral and the function \( \nu_f \) integral-valued at integral points. Introduce the polyhedron

\[
\tilde{\Delta} = \{ (\alpha, \beta, \gamma) \in \mathbb{R}^3 : (\alpha, \beta) \in \Delta, \gamma \geq \nu_f(\alpha, \beta) \}.
\]

It defines a toric variety \( Y = \text{Tor}(\tilde{\Delta}) \), which naturally fibers over \( \mathbb{C} \) so that the fibres \( Y_t \) over \( t \neq 0 \) are isomorphic to \( \text{Tor}(\Delta) \), and \( Y_0 \) is the union of toric surfaces \( \text{Tor}(\tilde{\Delta}_i), i = 1, \ldots, N \), with \( \tilde{\Delta}_1, \ldots, \tilde{\Delta}_N \) being the faces of the graph of \( \nu_f \). By the choice of \( \nu_f \), \( \text{Tor}(\tilde{\Delta}_i) \simeq \text{Tor}(\Delta_i) \), and we shall simply write that \( Y_0 = \bigcup_i \text{Tor}(\Delta_i) \). Then the curve \( C \) can be interpreted as an analytic surface in a neighborhood of \( Y_0 \), which fibers into the complex curves \( C^{(t)} \subset Y_t \simeq \text{Tor}(\Delta) \), and whose closure intersects \( Y_0 \) along the curve \( C^{(0)} \) which can be identified with \( \bigcup_i C_i \subset \bigcup_i \text{Tor}(\Delta_i) \).

The complex curves \( C^{(t)} \) can be naturally projected onto \( C^{(0)} \) along the \( t \)-axis.

The singular points of the curves \( C^{(t)} \) define sections \( s : D \setminus \{0\} \to \text{Tor}(\tilde{\Delta}), D \subset \mathbb{C} \) being a small disc centered at 0. The limit points \( z = \lim_{t \to 0} s(t) \) are singular points of \( C^{(0)} \). We say that such a point \( z \in C^{(0)} \) bears the corresponding singular points of \( C^{(t)} \). If \( z \in C^{(0)} \) does not belong to the intersection lines \( \bigcup_{i \neq j} \text{Tor}(\Delta_i \cap \Delta_j) \) and bears just one singular point of \( C^{(t)} \), which is topologically equivalent to \( z \), we call \( z \) a regular singular point, otherwise it is irregular. If \( C^{(0)} \) has irregular singular points, we can define a refinement of the tropicalization in the following way: transform the polynomial \( f(x, y) \) into \( f(x + a, y + b) \) with \( a, b \in \mathbb{K} \) such that the irregular singular point of \( C^{(0)} \) goes to the origin, and consider the tropicalization of the curve defined by the new polynomial \( f(x + a, y + b) \). This provides additional information on the behavior of singular points of \( C^{(t)} \) tending to irregular singular points of \( C^{(0)} \), and correspond, in a sense, to blowing-up the threefold \( Y \) at the irregular singular points of \( C^{(0)} \) (cf. [15 Section 3.5]).
2.1.3 Topology of the degeneration

Definition 2.1. We will call a polygon which is not a triangle or a parallelogram, by the name singular polygon or irregular polygon.

Lemma 2.3. Let a complex threefold $Y$ be smooth at a point $z$, $U \subset Y$ a small ball centered at $z$. Assume that $\pi : U \to (\mathbb{C}, 0)$ is a flat family of reduced surfaces such that $U_0 = \pi^{-1}(0)$ consists of two smooth components $U'_0, U''_0$ which intersect transversally along a line $L \supset \{z\}$, and $U_t = \pi^{-1}(t)$ are nonsingular as $t \neq 0$. Let $C'_0 \subset U'_0$, $C''_0 \subset U''_0$ be reduced algebraic curves, which cross $L$ only at $z$ and with the same multiplicity $m \geq 2$. Assume also that $U'_0$, $U''_0$ are regular neighborhoods for the (possibly singular) point $z$ of $C'_0$ and $C''_0$, respectively. Let $\delta' = \delta(C'_0, z)$, $\delta'' = \delta(C''_0, z)$ be the $\delta$-invariants, $r'$, $r''$ the numbers of local branches of $C'_0$, $C''_0$ at $z$, respectively. Then in any flat deformation $C_t$, $t \in (\mathbb{C}, 0)$, of $C_0 = C'_0 \cup C''_0$ such that $C_t \subset U_t$, the total $\delta$-invariant of $C_t$, $t \neq 0$, in $U_t$ does not exceed

$$\delta' + \delta'' + m - \max\{r', r''\}.$$

Proof. See [15, Lemma 3.2] \qed

Remark 2. In the notations of Lemma 2.3 accept all the hypotheses but assume that $C'_0, C''_0$ are not necessarily reduced. Furthermore, let $C'_0$ (resp., $C''_0$) have $r'$ (resp., $r''$) reduced local branches at $z$ of multiplicities $\rho'_1, \ldots, \rho'_{r'}$ (resp., $\rho''_1, \ldots, \rho''_{r''}$). Then the argument used in the proof of Lemma 2.3 shows that, if $C_t$ is reduced in $U$, then

$$\chi(C_t \cap U) \leq -\min |m'_1 + \ldots + m'_{r'} - m''_1 - \ldots - m''_{r''}|,$$

where integers $m'_1, \ldots, m''_{r''}$ run over the range $1 \leq m'_1 \leq \rho'_1$, $\ldots$, $1 \leq m''_{r''} \leq \rho''_{r''}$.

Proof. We refer again to [15, Lemma 3.2], but also remark that the formula follows from close examination of the ways the branches can glue together. We quote from [15]: Topologically, the curves $C'_0$ and $C''_0$ (in $U$) are bouquets of $r'$ and $r''$ discs, respectively. Notice that the circles of $C'_0 \cap \partial U$ and $C''_0 \cap \partial U$ move slightly when $t$ changes, and they are not contractible in $U_t$ for $t \neq 0$. For instance, a circle of $C'_0 \cap \partial U$ is (positively) linked with the line $L$ in $U'_0$, and hence remains (positively) linked with the surface $U''_0$ in $U$; thus, it cannot be contracted in $U_t$, $t \neq 0$, which does not intersect $U'_0$. This means that the curve $C_t \subset U_t$, $t \neq 0$, is the union of a few immersed surfaces with a total of $r' + r''$ holes and at least $\max\{r', r''\}$ handles. \qed

2.2 Tropical curves and tropicalizations of 1-cuspidal curves passing through generic points

We will follow closely the analysis in [15, Section 3]:

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Let \( x_1, \ldots, x_r \in \mathbb{R}^2 \) be generic distinct points with rational coordinates, and let \( p_1, \ldots, p_r \in (\mathbb{K}^*)^2 \) be generic points satisfying \( \text{Val}(p_i) = x_i, \) \( i = 1, \ldots, r, \) and having only rational exponents of the parameter \( t. \)

Observe that the coefficients of a polynomial \( f \in \mathbb{K}[x, y], \) which defines a curve \( C \in \Sigma_\Delta(nA_1, 1A_2), \) are Puiseux series with rational exponents of \( t. \) \( (C \in \Sigma_\Delta(nA_1, 1A_2) \) is the Severi variety of curves with Newton polygon \( \Delta, n \) nodes and one cusp, which will be discussed later on). A parameter change \( t \mapsto t^n \) with a suitable natural \( m \) makes all these exponents integral, and the convex piecewise linear function \( \nu_f : \Delta \rightarrow \mathbb{R} \) integral-valued at integral points. We keep these assumptions throughout the rest of the section.

Let \( S_f : \Delta = \Delta_1 \cup \cdots \cup \Delta_N \) be the subdivision defined by \( \nu_f, \) and let \( (C_1, \ldots, C_N) \) be the tropicalization of the curve \( C = \{ f = 0 \} \in \Lambda_K(\Delta). \) The union of the divisors \( \text{Tor}(\sigma) \subset \text{Tor}(\Delta_k), \) where \( \sigma \) runs over all edges of \( \Delta_k, \) we shall denote by \( \text{Tor}(\partial \Delta_k), k = 1, \ldots, N. \) For any \( i = 1, \ldots, N, \) denote by \( C_{ij}, j = 1, \ldots, m_i, \) the distinct irreducible components of the curve \( C_i \subset \text{Tor}(\Delta_i) \) and by \( r_{ij}, j = 1, \ldots, m_i, \) their multiplicities. Denote by \( s_{ij} \) the number of local branches of \( C_{ij} \) centered on \( \text{Tor}(\partial \Delta_i), j = 1, \ldots, m_i. \)

**Claim 2.1.** Denote by \( \text{rk Alg}(C) \) the dimension of the Severi variety on which \( C \) lies, and denote by \( \text{rk Trop}(C) \) the dimension of the space of deformations of \( C \) of the same combinatorial type. Then

\[
\text{rk Alg}(C) \leq \text{rk Trop}(C)
\]

**Proof.** Fix a lattice polygon \( \Delta. \) First note that any configuration of \( r \) points \( p_1, \ldots, p_r \) in the tropical plane \( R^2 \) can be lifted to a configuration on \( r \) points on \( (\mathbb{K}^*)^2, x_1, \ldots, x_r \) in general position, such that they tropicalize to \( p_1, \ldots, p_r. \) (This follows since we can freely choose coefficients of \( t \)).

Also note that the condition to pass through \( k \) points in general position on \( (\mathbb{K}^*)^2, \) decreases the dimension of the deformations of a curve (say, the dimension of the appropriate Severi variety) by exactly \( k. \)

Since \( \Delta \) is fixed, there are only finite number of combinatorial types of tropical curves with Newton polygon \( \Delta \) (as there are finitely many possible subdivisions).

Let \( X \) be the configuration space of \( r \)-tuples of points in \( R^2. \) For each combinatorial type, consider the set \( Y_i \subset X \) of configurations in \( X \) such that the condition for a tropical curve to pass through them, decreased the dimension of deformations by less than \( r. \) These sets \( Y_i \) have an open dense complement, and since there is a finite number of them, so does their union. So we can define that \( r \) points are \( \Delta \)-generic points in \( \mathbb{R}^2 \) if the condition for a tropical curve of Newton polygon \( \Delta \) to pass through them, decreases the dimension of the space of deformations (preserving combinatorial type) by exactly \( r. \) (or, if this dimension is less than \( r, \) then no curve of such combinatorial type can pass through them)

By this definition of \( \Delta \)-generic points, the claim follows immediately:
Let $C$ a curve such that $\text{rk Alg}(C) = r$, (in our case $r$ is the dimension of the Severi variety for curves with one cusp and some $n$ nodes) Let $x_1, \ldots, x_r \in \mathbb{R}^2$ be points in general position such that they tropicalize to points $p_1, \ldots, p_r \in R^2$, in $\Delta$-general position, and such that $C$ passes through $x_1, \ldots, x_r$. As we have seen $C$ tropicalizes to a tropical curve $A$ passing through $\Delta$-generic points $p_1, \ldots, p_r$, thus $\text{rk Trop}(C) \geq r = \text{rk Alg}(C)$.

We intend to estimate $\tilde{\chi}(C(t))$ from above and from below and to compare these bounds.

Note that we give labels to certain inequalities by assigning them a greek letter, this is done for easier reference later.

Let $U$ be the union of small open balls $U_z$ in the three-fold $Y$ (see the definition in section 2.1) centered at all the points $z \in \bigcup(C_i \cap \text{Tor}(\partial \Delta_i))$. If $z \in C_i \cap \text{Tor}(\sigma)$, where $\sigma$ is an edge of $\Delta_i$ lying on $\partial \Delta$, then the local picture of the limit curve: $(C(0) \cap U_z)$, is topologically a bouquet of discs, each corresponds to a branch, joined in the point $z$, when $t \neq 0$ discs can only glue by attaching a ‘tube’ (homeomorphic to $S^1 \times [0,1]$), hence $\tilde{\chi}(C(t) \cap U_z)$ does not exceed the number of local branches of $C_i$ at the points of $C_i \cap \text{Tor}(\sigma)$, which is less than or equal the sum of multiplicities of the points $z \in C_i \cap \text{Tor}(\sigma)$, which, in turn is bounded by $|\sigma \cap \mathbb{Z}^2|$.

If $z \in \text{Tor}(\sigma) \cap C_i \cap C_k$, where $\sigma = \Delta_i \cap \Delta_k$ is a common edge, then $\tilde{\chi}(C(t) \cap U_z) \leq 0$ by Remark 2. Hence

$$\tilde{\chi}(C(t) \cap U) \leq |\partial \Delta \cap \mathbb{Z}^2| \quad (9)$$

with an equality if and only if, for any edge $\sigma \subset \Delta_i \cap \partial \Delta$, the reduction of the curve $C_i$ is non-singular along $\text{Tor}(\sigma)$ and meets $\text{Tor}(\sigma)$ transversally.

For the upper bound to $\tilde{\chi}(C(t))$, we can assume that, for any $i = 1, \ldots, N$, and $1 \leq j < j' \leq m_i$, the components $C_{ij}$ and $C_{ij'}$ do not glue up in $Y \setminus U$ when $C(0)$ deforms into $C(t)$ as this would only decrease $\tilde{\chi}(C(t))$. Then the normalization of $C(t) \setminus U$ is the union of connected components, each of them tending to some curve $C_{ij} \setminus U$. Furthermore, the components which tend to a certain $C_{ij} \setminus U$ can be naturally projected onto $C_{ij} \setminus U$, and this projection is an $r_{ij}$-sheeted covering (possibly ramified at a finite set). Hence

$$\tilde{\chi}(C(t) \setminus U) \leq \sum_{i=1}^{N} \sum_{j=1}^{m_i} r_{ij} \tilde{\chi}(C_{ij} \setminus U) = \sum_{i=1}^{N} \sum_{j=1}^{m_i} r_{ij} (\tilde{\chi}(C_{ij}) - s_{ij})$$

$$\leq 2 \sum_{i=1}^{N} m_i - \sum_{i=1}^{N} \sum_{j=1}^{m_i} s_{ij}$$

with an equality in $\eta$ if and only if the covering is an unramified covering and equality in $\alpha$ if only if all $C_{ij}$ are rational, and $r_{ij} = 1$ as far as $s_{ij} > 2$. Next we notice that $s_{ij} \geq 2$ for any $C_{ij}$, and $s_{ij} \geq 3$ for at least one of the components $C_{ij}$ if $\Delta_i$ has an odd number of edges, or $\Delta_i$ has an even
number of edges, but not all pairs of opposite sides are parallel. Hence (in the notation of Lemma 2.2)

\[ \bar{\chi}(C^{(t)} \setminus U) \leq -N_3 - \sum_{j \geq 2} (N_{2j+1} + N_{2j} - N_{2j}^2) \]  

(10)

with an equality only if, for each triangle \( \Delta_i \), \( C_i \) is irreducible and satisfies \( s_{ij} = 3 \); for each \( \Delta_i \) with an odd \( r \geq 5 \) number of edges or with an even number of edges, but not all pairs of opposite sides parallel, exactly one component \( C_{ij} \) satisfies \( s_{ij} = 3 \) and the others satisfy \( s_{ij} = 2 \); and, finally, \( s_{ij} = 2 \) for all components \( C_{ij} \) in the remaining polygons \( \Delta_i \). Notice also that \( s_{ij} = 2 \) means that \( C_{ij} \) is defined by a binomial. On the other hand,

\[ \bar{\chi}(C^{(t)}) = 2 - 2g(C^{(t)}) = 2 - 2(|\text{Int}(\Delta) \cap \mathbb{Z}^2| - n - 1) = \]

Since the curve has \( n \) nodes and one cusp:

\[ = 2 - 2|\text{Int}(\Delta) \cap \mathbb{Z}^2| + 2(|\Delta \cap \mathbb{Z}^2| - 2 - r) = 2|\partial \Delta \cap \mathbb{Z}^2| - 2r - 2 \geq \gamma \]

And \( r \) is the dimension of the Severi variety of curves with \( n \) nodes, one cusp and Newton polygon \( \Delta \)

\[ \geq 2|\partial \Delta \cap \mathbb{Z}^2| - 2 \cdot \text{rk}(S_f) - 2 = 2|\partial \Delta \cap \mathbb{Z}^2| - 2 \cdot \text{rk}_{\text{exp}}(S_f) - 2d(S_f) - 2 \]

with an equality only if \( \text{rk}(S_f) = \text{rk}_{\text{exp}}(S_f) + d(S_f) = r \). Next, by (10) we have

\[ \bar{\chi}(C^{(t)}) \geq 2|\partial \Delta \cap \mathbb{Z}^2| - 2|V(S_f)| + 2 + 2 \sum_{i=1}^{N} (|V(\Delta_i)| - 3) - 2d(S_f) - 2 \]

\[ = 2|\partial \Delta \cap \mathbb{Z}^2| - 2|V(S_f)| + 2 - 2|V(S_f) \cap \partial \Delta| + 4|E(S_f)| - 6N - 2d(S_f) - 2, \]

where \( E(S_f) \) denotes the set of edges of \( S_f \). Since \( |V(S_f)| - |E(S_f)| + N = 1 \) (Euler’s formula), and \( 2|E(S_f)| = 3N_3 + 4N_4 + 5N_5 + ... + |V(S_f) \cap \partial \Delta| \) (By counting the edges according to the polygons they belong to), we finally obtain

\[ \bar{\chi}(C^{(t)}) \geq 2(|\partial \Delta \cap \mathbb{Z}^2| - |V(S_f) \cap \partial \Delta|) + |V(S_f) \cap \partial \Delta| - N_3 + N_5 + 2N_6 + ... - 2d(S_f) - 2 \]

\[ = 2(|\partial \Delta \cap \mathbb{Z}^2| - |V(S_f) \cap \partial \Delta|) + |V(S_f) \cap \partial \Delta| - 2d(S_f) - 2 + \sum_{m \geq 3} (m - 4)N_m . \]

Recalling (9) and (10), we obtain

An upper bound:

\[ \bar{\chi}(C^{(t)}) \leq |\partial \Delta \cap \mathbb{Z}^2| - N_3 - \sum_{j \geq 2} (N_{2j+1} + N_{2j} - N_{2j}^2) \]  

(11)

A lower bound:

\[ \bar{\chi}(C^{(t)}) \geq 2(|\partial \Delta \cap \mathbb{Z}^2| - |V(S_f) \cap \partial \Delta|) + |V(S_f) \cap \partial \Delta| - 2d(S_f) - 2 + \sum_{m \geq 3} (m - 4)N_m \]  

(12)
We now calculate the difference between the upper and lower bounds:

\[ 0 \leq (UB) - (LB) = \]

\[ = -(|\partial \Delta \cap \mathbb{Z}^2| - |V(S_f) \cap \partial \Delta|) - \sum_{m \geq 2} ((2m - 3)N_{2m} - N'_{2m}) - \sum_{m \geq 2} (2m - 2)N_{2m+1} + 2d(S_f) + 2 \]  

(13)

By lemma 2.2 If the dual subdivision is nodal (comprises only triangles and parallelograms) then \(2d(S_f) = 0\) and we get

\[ (UB) - (LB) = 2 - (|\partial \Delta \cap \mathbb{Z}^2| - |V(S_f) \cap \partial \Delta|) \geq 0 \]

Otherwise, If an irregular polygon is present, \(2d(S_f) \leq \sum_{m \geq 2} ((2m - 3)N_{2m} - N'_{2m}) + \sum_{m \geq 2} (2m - 2)N_{2m+1} - 1\) and we get:

\[ (UB) - (LB) = 1 - (|\partial \Delta \cap \mathbb{Z}^2| - |V(S_f) \cap \partial \Delta|) \geq 0 \]

(14)

In the nodal limit case, equality of \((UB)\) and \((LB)\) means we are in the same situation as in [15], and we summarize the resulting tropical limit: each integral point on \(\partial \Delta\) is a vertex of \(S_f\) and all the non-triangular \(\Delta_i\) are parallelograms.

Alright the equality conditions for the upper and lower bounds to \(\chi(C^{(l)})\) prove that the tropical curve \(A_f\) is nodal of rank \(r\). Furthermore,

- for each triangle \(\Delta_i\), the curve \(C_i\) is rational and meets \(\text{Tor}(\partial \Delta_i)\) at exactly three points, where it is unibranch;
- for each parallelogram \(\Delta_i\), the polynomial, defining \(C_i\), is of type \(x^a y^b (\alpha x^c + \beta y^d) p(\gamma x^e + \delta y^f) q\) with \((a, b) = (c, d) = 1\) and \((a : b) \neq (c : d)\).

Later we will see that for families with \(n\) nodes and one cusp, tropical limits with two or more irregular polygons are impossible, and in fact, the only possible irregular polygon is a quadrilateral. The difference between the upper bound and lower bound is 2, hence there can be total jump of at most 2 in the chain of inequalities.

By [15] Lemma 3.12] nodal tropical limits with rational curves in the triangles lift by patchworking to a family of nodal curves, since our family is cuspidal, the only places where we should look for jumps are those who determine the shape of the tropical limit or the properties of the limit curves. These are the inequalities previously marked as \((\alpha)\) and \((\beta)\).

**Lemma 2.4.** For any lattice triangle \(\Delta' \subset \mathbb{R}^2\), there exists a polynomial with Newton polygon \(\Delta'\) and prescribed coefficients at the vertices of \(\Delta'\), which defines a rational curve \(C \subset \text{Tor}(\Delta')\), meeting \(\text{Tor}(\partial \Delta')\) at exactly three points, where it is unibranch. Furthermore, the curves defined by these
polynomials are nodal, and nonsingular at the intersection with \( \text{Tor}(\partial \Delta') \). Moreover, the number of such polynomials is finite and equal to \(|\Delta'|\). An additional fixation of one or two intersection points of \( C \) with \( \text{Tor}(\partial \Delta') \) divides the number of polynomials under consideration by the product of the length of the corresponding edges of \( \Delta' \).

**Proof.** See [15, Lemma 3.5]  

**Lemma 2.5.** Given integers \( a, b, c, d \) such that \((a, b) = (c, d) = 1\), \((a : b) \neq (c : d)\), and any non-zero \( \alpha, \beta, \gamma, \delta \), the curve \((\alpha x^a + \beta y^b)(\gamma x^c + \delta y^d) = 0\) has \(|\Delta' \cap \mathbb{Z}^2| - 3\) nodes as its only singularities in \((\mathbb{C}^*)^2\), where \( \Delta' \) is the lattice parallelogram built on the vectors \((a, -b), (c, -d)\).

**Proof.** Straightforward.

A few words about terminology. What we do is to consider a flat family of curves parameterized by a disc \( \Delta \), such that the fibers over points \( y \in \Delta^* \) in the punctured disc are all ‘alike’ - the singular points vary in continuous families and are locally topologically equivalent, but the fiber over \( 0 \in \Delta \) can be different, and will be referred to as the special fiber. When we refer to going from the generic fiber to the special fiber over 0 we call this the degeneration of the family. When we refer to going from the special fiber over 0 to the generic fiber we call this the deformation.

A general principle is that in equisingular families, singularities become more complicated in the deformation. This loose generality can be formalized as the semicontinuity of many invariants of the singularity:

**Claim 2.2.** The Milnor number \( \mu \), the delta invariant \( \delta \), kappa invariant \( \kappa \) and local intersection multiplicity of branches are all semicontinuous.

**Proof.** For \( \mu \) see [9, Theorem 2.6, p. 114], for \( \delta \) see [9, Theorem 2.54, p. 347]  

Therefore, since we have a cusp in the family, we must attain a cusp of some more complicated singularity in the limit curves. we will refer it at "The Singularity". The singularity may appear in the open torus \((\mathbb{C}^*)^2\) of one of the polygons \( \Delta_i \) or on the intersection of two limit surfaces, e.g. on the \( \text{Tor}(\sigma) \) for a corresponding internal edge \( \sigma \) in the intersection of two polygons. In the latter case, since there is more than one branch at the singularity, each branch may not be ‘complicated enough’. Thus, we can gain further information by examining the refinement of this singular point on this edge, and in the refinement we will get a singular point ("at least a cusp") lying in the open torus of a toric surface (usually a triangle).

\(^2\)We define the length \(|\sigma|\) of a segment \( \sigma \) with integral endpoints as \(|\sigma \cap \mathbb{Z}^2| - 1\).
2.3 Hypersurfaces in Toric Varieties

We now cite an important property of hypersurfaces in toric varieties, from [11, pp:24]:

A hypersurface in $\text{Tor} \left( \Delta \right)$, which does not contain the boundary divisors, is uniquely determined by its intersection with the torus $(\mathbb{C}^*)^n$, and that intersection can be defined by an equation

$$f(z) := \sum_{i \in \Delta \cap \mathbb{Z}^n} a_i z^i = 0,$$

(15)

containing at least two monomials. We restore the original hypersurface by taking the closure $\overline{\{ f = 0 \} \cap (\mathbb{C}^*)^n} \subset \text{Tor}(\Delta)$. We can, if necessary, replace $\Delta$ by $N \cdot \Delta$ with $N \in \mathbb{N}$ since $\text{Tor}(\Delta) \approx \text{Tor}(N \cdot \Delta)$. Consider an algebraic hypersurface defined in $(\mathbb{C}^*)^n$ by equation (15). The closure $\overline{\{ f = 0 \} \cap (\mathbb{C}^*)^n} \subset \text{Tor}(\Delta)$ is an algebraic hypersurface in the toric variety $\text{Tor}(\Delta)$. The intersection of this hypersurface with the subvarieties $\text{Tor}(\sigma)$, $\sigma$ being a proper face of $\Delta$, can be described in the following way:

$$\overline{\{ f = 0 \} \cap \text{Tor}(\sigma)} = \{ f^\sigma = 0 \},$$

where $f^\sigma = \sum_{i \in \sigma \cap \mathbb{Z}^n} a_i \cdot z^i$ is the truncation of $f$ to $\sigma$. More generally, let $\Delta' \subset \Delta$ be the Newton polytope of $f$. To recover the intersection $\overline{\{ f = 0 \} \cap \text{Tor}(\sigma)}$, we cannot just take the restriction of $f$ to a face $\sigma$, since maybe no integer point of this face corresponds to a monomial of $f$, i.e. $\sigma \cap \Delta' = \emptyset$. Instead, assuming for simplicity that $\sigma \subset \Delta$ is a facet (face of codimension 1), we take the exterior normal vector $v \in \mathbb{Z}^n$ of $\sigma$ and then choose the face $\sigma' \subset \Delta'$ where the functional

$$(\mathbb{R}^n \ni u \leftarrow u \cdot v)|_{\Delta'}$$

attains its maximum. Then we get

$$\overline{\{ f = 0 \} \cap \text{Tor}(\sigma)} = \{ f^\sigma = 0 \}.$$

An important corollary is that we can determine which boundary divisors a hypersurface intersects just by examining the Newton polytope of it’s defining equation in $(\mathbb{C}^*)^n$.

Recall the definition of a geometric genus of reduced projective curve to be the arithmetic genus of the normalization. We cite a useful lemma by Harris and Diaz: [4, Lemma 2.4]

**Lemma 2.6.** Let

$$X \hookrightarrow \mathbb{P}^m \times Y \quad \xrightarrow{\pi} \quad Y$$
Be a flat family of projective curves with all fibers reduced. Also assume that $X$ and $Y$ are complex varieties. Define the function $\phi_\pi$ on $Y$ by letting $\phi_\pi(y)$ be the geometric genus of the fiber $\pi^{-1}(y)$. Then $\Phi_\pi$ is lower semi-continuous in the Zariski topology.

In other words, the lemma states the lower semi-continuity of geometric genus in flat family. Tropicalization is a limit of a flat family, and thus by the stable reduction theorem ([2 Chapter X, section 4], if we start with a family of rational curves, the limit curve must be rational as well.

### 2.4 Coarse Classification of Possible Tropical Limits

We now deal with the possible polygons in the tropical limit. We will use dimension counting to show that there can be only two options: either a nodal limit (triangles and parallelograms) or a single exceptional polygon which is a quadrilateral.

**Lemma 2.7.** In the tropicalization of 1-cuspidal rational curves, the cusp cannot tend to a limit point lying on the common boundary of two polygons meeting only a single branch in each surface, i.e let $\sigma$ be an edge common to two polygons in the subdivision: $\Delta_1$ and $\Delta_2$, such that $p \in \text{Tor}(\sigma)$ is met by a single branch both in $\text{Tor}(\Delta_2)$ and in $\text{Tor}(\Delta_1)$, then the point $p$ cannot bear a cusp.

**Proof.** In order to get more information about the points in the family which degenerate to a point on the intersection of two limit surfaces, a procedure called ‘refinement’ is introduced. In essence, this is a non-toric change of coordinates, resulting in a new limit for the family, describing local information around the limit point. Here, we use a modified version:

Locally the family can be thought as subset of $(\mathbb{C}^*)^3 \equiv (\mathbb{C}^*)^2 \times \mathbb{C}^*$ with coordinates $(x, y), t$. For each $t$ we get a curve in the torus. Suppose that $p$ does bear a cusp and consider the $x$-coordinate of the cusp $x_c(t)$. Instead of the coordinate change $X := X - \xi$ where $\xi$ is the local coordinate of the intersection of the curve with the toric divisor, we consider $X := X - x_c(t) = X - \xi - (x_c(t) - \xi)$ Since $(x_c(t) - \xi) = o(t)$, i.e. contains only positive powers of $t$ greater than 1, it does not affect the shape of the resulting Newton polygon, but now the singularity in the limit is on the $y$-axis. There is no further subdivision, since the only points for such subdivision are on the segment $[(0,0), (0, m)]$ (Except $(0, m - 1)$), which will result in a cycle (handle) around the subdivision point. This must lift to a cycle in the deformation and thus a contradiction. Hence we end up with a rational curve on the surface corresponding to a triangle, which two of it’s sides admit a single branch each, and the toric divisor corresponding to the third side, meets a singular branch. such a curve cannot exist, by Proposition 1.6

**Lemma 2.8.** The subdivision in the tropical limit of a family of rational curves with one cusp and $n$ nodes, is either a nodal limit or contains a single exceptional polygon which is a quadrilateral.

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Proof. Let $\Sigma_\Delta := \Sigma_\Delta(nA_1, 1A_2)$ be the Severi variety of curves with $n$ nodes and one cusp inside the linear system $A_\Sigma(\Delta)$ of curves with Newton polygon $\Delta$ on the surface $\text{Tor}(\Delta)$. Similarly, let $\Sigma_\Delta(nA_1)$ be the Severi variety of $n$-nodal curves in $A_\Sigma(\Delta)$.

It is well known that $\text{dim}(\Sigma_\Delta(nA_1)) = |\Delta \cap \mathbb{Z}^2| - 1 - n$. Since it can be defined by $3n$ independent equations in $(\mathbb{K})^{|\Delta \cap \mathbb{Z}^2| - 1} \times (\mathbb{K})^{2n}$, for details see \cite{10} Theorem 1.49a pp. 30. Similarly, $\text{dim}(\Sigma_\Delta(nA_1, 1A_2)) = |\Delta \cap \mathbb{Z}^2| - 1 - n - 2$ because defining the cusp requires one more equation: having the determinant of the hessian equal to zero at the point of cusp. For brevity we will write $\Sigma_\Delta$ when we refer to $\Sigma_\Delta(nA_1, 1A_2)$

For a curve $C$ and it’s tropicalization $C_{\text{trop}}$ we have that $\text{rk}(C_{\text{trop}}) \geq \text{rk}(C)$, meaning, the dimension of space of tropical deformation is greater or equal that of algebraic curve deformations (see claim \ref{2.2}). Now, assume that in the tropical limit we have $\dim X = 2$, and this also forces another jump, either in the second polygon. Thus gaining a total jump of 2 which exceeds the 1 allowed. Moreover, $\sigma$ cannot be met by more than one branch from the surface on the other side of $\sigma$ as this would force an increase of the genus in the deformation. Thus $s_{ij} = 2$ and $C_{ij}$ is a smooth curve (see for reference the case of limit curves in a parallelogram). If $C_{ij}$ meets 3 boundary divisors then $s_{ij} = 3$ (from the same reasoning as before it cannot meet a boundary divisor in two branches) and thus it is determined by rational curve in a triangle and is a smooth curve (as we have seen in the analysis for such curves in a triangle).
The exceptional component $\tilde{C} = C_{ij'}$, can meet the boundary in at most 4 branches as the jump $\leq 1$ and must meet 4 different boundary divisors from the same reasoning as before, thus it is given by an equation having a quadrilateral as it’s Newton polygon.

Now, consider the cusp in the family, it must degenerate to some point in the limit. this point must be singular, and cannot lie on a boundary divisor (either by Lemma 2.7 for a common edge, or by increase in $K$ for an edge in $\Delta$) and thus must be a point in $\tilde{C} \cap (C^*)^2$. The analysis of such curves in a quadrilateral show that they cannot have more than one singular branch. Thus we conclude that all boundary divisors in the subdivision are met by smooth branches.

Since all curves in all polygons are rational, in the deformation, each polygon $\Delta_i$ contributes $\delta = \lfloor \text{Int}(\Delta_i) \cap \mathbb{Z}^2 \rfloor$ nodes, as evident by toric degree-genus formula. Note that a cusp appears in the family in the deformation, and thus we subtract 1 from the global count (since $\delta(\text{cusp}) = 1$).

For the parallelograms we can get a sharper count: since we know the curve there is defined by a product of binomials, we get that it contributes $\text{Area}(\Delta)$ nodes, which will conveniently be written as $|\Delta \cap \mathbb{Z}^2| - |I_1 \cap \mathbb{Z}^2| - |I_2 \cap \mathbb{Z}^2| + 1$ where $I_1, I_2$ are two non parallel edges. (That this equals the area, can be seen as follows: tile $\Delta$ with lattice parallelograms of unit area, and count their top right vertices).

As we have previously seen when considered refinement, another source for nodes are the points on the intersection of the surfaces in the tropical limit, corresponding to the edges. By [15, Lemma 3.9] each edge $I_i$ contributes at most $\text{Integer}_{\text{Length}}(I_i) = \lfloor \text{Int}(I_i) \cap \mathbb{Z}^2 \rfloor$ nodes (see Lemma 2.3). Summing up we get:

$$\text{rk}(C) \geq |\Delta \cap \mathbb{Z}^2| - 3 - \sum_{\Delta \text{ not a parallelogram}} \lfloor \text{Int}(\Delta_i) \cap \mathbb{Z}^2 \rfloor - \sum_{\Delta \text{ is a parallelogram}} (\lfloor \text{Int}(\Delta_i) \cap \mathbb{Z}^2 \rfloor + 1)$$

$$- \sum_{\text{all edges } I_i} (\lfloor \text{Int}(I_i) \cap \mathbb{Z}^2 \rfloor + 1) + 1 = V(S) - \#\text{Parallelograms} - 3 + 1$$

Now, because $\text{rk}(C_{\text{trop}}) \geq \text{rk}(C)$ it follows that $\sum_{i=1}^{s} \left( \frac{|V(\Delta_i)| - 3}{2} \right) + \frac{1}{2} \leq 2$, and thus, it is only possible to have a single exceptional polygon which is a quadrilateral.

2.5 Tropical limits of a family of 1-cuspidal curves and the limit curves

Denote by $\alpha$ the jump at the inequality $(\alpha)$. We will describe only the special curves inside the polygons, with the understanding that the rest of the components $C_{ij}$ have $s_{ij} = 2$ and $\chi_{ij} = 2$ (as a matter of fact, as consequence of intersection theory for hypersurface in toric varieties, components such that $s_{ij} = 2$ are impossible unless $\Delta_{ij}$ has a pair of parallel edges)

1. $\alpha = 1$:
(a) A polygon $\Delta_i$ with curve $C_{ij}$ such that $r_{ij} = 2, \chi_{ij} = 2, S_{ij} = 3$

2. $\alpha = 2$:

(a) A polygon $\Delta_i$ with curve $C_{ij}$ such that $r_{ij} = 1, \chi_{ij} = 0, S_{ij} = 3$
(b) A polygon $\Delta_i$ with curve $C_{ij}$ such that $r_{ij} = 2, \chi_{ij} = 2, S_{ij} = 4$
(c) Two polygons or a single polygon with two curves such as described by $\alpha = 1$

Proof. Consider the inequality:

$$\sum_{i=1}^{N} \sum_{j=1}^{m_i} r_{ij}(\chi(C_{ij}) - s_{ij}) \leq \sum_{i=1}^{N} \sum_{j=1}^{m_i} (2 - s_{ij})$$

And for given $i, j$ we let $\alpha_{i,j} = (2 - s_{ij}) - r_{ij}(\chi(C_{ij}) - s_{ij})$. Note that if $s_{ij} = 2$ then $C_{ij}$ is defined by a monomial and thus it is rational, $\chi(C_{ij}) = 2$ and $\alpha_{i,j} = 0$. Since $s_{ij} \geq 2$ and $\chi(C_{ij}) \leq 2$ we have $2 \geq \alpha_{i,j} \geq (2 - s_{ij}) - r_{ij}(2 - s_{ij}) = (1 - r_{ij})(2 - s_{ij})$. Let $s_{ij} > 2$, then $r_{ij} \leq 2$ and now we check case by case. Assume $\alpha_{i,j} = 1$. If $r_{ij} = 1$, then $1 = (2 - s_{ij}) - (\chi(C_{ij}) - s_{ij}) = 2 - \chi(C_{ij})$ but the right hand side is odd and the left hand side is even and thus a contradiction. If $r_{ij} = 2$, then $1 = (2 - s_{ij}) - 2(\chi(C_{ij}) - s_{ij}) = 2 + s_{ij} - \chi(C_{ij})$ and the only solution is $s_{ij} = 3, \chi(C_{ij}) = 2$

Now assume $\alpha_{i,j} = 1$. If $r_{ij} = 1$, then $2 = 2 - \chi(C_{ij})$, thus $\chi(C_{ij}) = 0$. If $r_{ij} = 2$, then $2 = (2 - s_{ij}) - 2(\chi(C_{ij}) - s_{ij}) = 2 + s_{ij} - \chi(C_{ij})$, so $s_{ij} = \chi(C_{ij}) = 0$ and since $\chi(C_{ij})$ takes only even values smaller or equal to 2, the only solution is $s_{ij} = 4, \chi(C_{ij}) = 2$.

We now analyze which of these limits are possible.

We begin with (2b). $s_{ij} = 4$ will give another jump at $(\beta)$ inequality, which is more than 2.

We will analyze next (1a). $\Delta_i$ cannot be a parallelogram, otherwise at least one of the branches would have to intersect boundary divisor at the vertices.

If $\Delta_i$ is a quadrilateral, the total jump allowed is 1. but, to have all boundary divisors meet the curve, we must have another component $C_{ij}$, such component meets at least two boundary divisors, by pigeonhole principle, one of the boundary divisors meets two branches. this divisor must correspond to an internal edge in the subdivision, because it intersects the curve at multiplicity at least 2. On the other polygon adjacent to that edge we have a single branch, thus by Remark 2 we have another jump, which is too much.

Thus, $\Delta_i$ Must be a triangle, and the curve $C_i$ is nodal, otherwise we obtain a contradiction by considering Proposition 1.6 with the reduced curve underlying $C_i$. Now, in the notation of 2.2 we discard small balls around the points on the boundary divisors, and Denote by $U$ the resulting open set in the threefold $Y$. Inside $U$, $C^{(t)}$ is a double cover of $C^0 = C_i$. $C_i$ is rational, and meets 3 boundary divisors, thus $C_i \cap U$ is homeomorphic to a 3-punctured sphere, $\chi(C_i \cap U) = -1$
the well known Riemann-Hurwitz formula for a d-fold ramified cover $\chi(C^{(t)} \cap U) = d \cdot \chi(C_i \cap U) - \sum_{P \in C_i \cap U} (e_P - 1)$ where $e_P$ is the ramification index of point $P$. In our case we get:

$$\chi(C^{(t)} \cap U) = -2 - \#\text{[Ramification Points]} < -1$$

Thus, one of the punctures in $C_i \cap U$ is covered by two punctures. If this happens on a boundary divisor corresponding to an internal edge of the subdivision, then we are in the case of Remark 3 but moreover, not only we get a jump, we see that we have created a nontrivial cycle (a handle), since we have to attach a piece of "pair of pants" to our punctured sphere, creating the cycle between those pieces. (See Figure 2) This is in contradiction to the assumption that the family was of rational curves. So we deduce this puncture must sit on a divisor corresponding to an edge on the boundary of the subdivision, the intersection multiplicity with this edge is at least 2, so we get $K=1$, thus exists an internal vertex on this edge, which gives us another jump of value 1.

Now, consider the cusp in the family, if it tends to a point in $U$ or on a boundary divisor which corresponds to an external edge of the subdivision (the covering map extends to there as well), we must have ramification, because $C^{(t)}$ are singular at the cusp, but $C^0$ is an immersed smooth curve (nodal) and a non ramified covering map is locally diffeomorphism. The ramification gives another jump in inequality ($\eta$). thus, a total jump of 3 and a contradiction. On the other hand, if the cusp tends to a point on a boundary divisor for an internal edge, this is a contradiction to Lemma 2.7.

We are thus left with 2a. Which is impossible in the setting of rational curves, but is a major obstacle to generalization to higher genus, due to the need to enumerate elliptic curves on a toric surface. In the rational setting we use the existence of a rational parametrization, perhaps parameterizing by elliptic functions could help.

We conclude that none of these cases are possible tropical limits in our setting.

As for $\beta$, at all cases below $\chi_{ij} = 0$ and $r_{ij} = 1$, as for the branches on $\partial \Delta$:

1. $\beta = 1$:

   (a) If $\Delta_i$ is a Triangle or a Quadrilateral:
i. a curve $C_{ij}$ such that $S_{ij} = 4$.

ii. two curves $C_{ij_1}, C_{ij_2}$ such that $S_{ij_1} = 3, S_{ij_2} = 3$.

(b) If $\Delta_i$ is a Parallelogram:

i. a curve $C_{ij}$ such that $S_{ij} = 3$.

2. $\beta = 2$:

(a) If $\Delta_i$ is a Triangle:

i. a curve $C_{ij}$ such that $S_{ij} = 5$.

ii. two curves $C_{ij_1}, C_{ij_2}$ such that $S_{ij_1} = 4, S_{ij_2} = 3$.

iii. three curves $C_{ij_1}, C_{ij_2}, C_{ij_3}$ such that $S_{ij_1} = 3, S_{ij_2} = 3, S_{ij_3} = 3$.

(b) If $\Delta_i$ is a Parallelogram:

i. a curve $C_{ij}$ such that $S_{ij} = 4$.

ii. two curves $C_{ij_1}, C_{ij_2}$ such that $S_{ij_1} = 3, S_{ij_2} = 3$.

Proof. By inequality $\sum_{i=1}^{N} \sum_{j=1}^{m} (2 - s_{ij}) \leq -N_3 - \sum_{j \geq 2} (N_{2j+1} + N_{2j} - N_{2j}'),$ and the fact that for parallelograms $s_{ij} \geq 2$ for all $j$ and for triangles and quadrilaterals $s_{i1} \geq 3$ and $s_{ij} \geq 2$ for $j \geq 2$, we deduce that each increase in one of these values gives rise to a jump of 1 in the inequality $(\beta)$.

Remark 3. Consider an edge $\sigma \in \Delta_i \cap \Delta_j$ common to two polygons $\Delta_i$ and $\Delta_j$. Denote by $\rho_i$ the total number of branches meeting Tor($\sigma$) in Tor($\Delta_i$) and $\rho_j$ the corresponding number for Tor($\Delta_j$). If $\rho_i \leq \rho_j$ then in (9), we have a jump of at least 1 (more precisely, in inequality ($\zeta$)).

Proof. This follows directly from Remark 2.

Proposition 2.1. Let $C$ be a curve over $\mathbb{K}[x, y]$, if $C$ is rational as a curve over $\mathbb{K}$ then for each $t$ in some neighborhood of 0, $C^{(t)}$ is a rational curve as a curve defined over $\mathbb{C}$.

Proof. The condition to be of genus 0 can be formulated in terms of equations with algebraic numbers as coefficients. Since both fields $\mathbb{K}$ and $\mathbb{C}$ are of characteristic 0 and contain the algebraic numbers, the result follows.

Now we start ruling out options.

We will use two important facts: In the toric surface corresponding to a triangle, a rational curve meeting each boundary divisor in one branch is nodal. (Lemma 2.4) In a refinement of the intersection of two smooth branches, we get a triangle, and a rational curve on the corresponding surface which is nodal. [15, Lemmas 3.9 and 3.10]

We will start first with topological considerations. We will see that many options require a jump on inequality (9), by gluing of branches. A point of terminology, we will refer to the actual
difference between the upper bound and the lower bound we calculated, as the "total jump". We Denote by $K := |\partial \Delta \cap \mathbb{Z}^2| - |V(S_f) \cap \partial \Delta|$ the number of integer points on the boundary of the Newton polygon $\Delta$ which are not subdivision vertices.

We start with the 1(a)ii. If $\Delta_i$ is a triangle: By the theory of intersection between boundary divisors and hypersurfaces in projective toric varieties, we deduce that both $C_{ij_1}$ and $C_{ij_2}$ have a triangle as Newton polygon, hence each edge meets exactly two branches. If $K = 0$ then all edges of $\Delta_i$ are internal edges of the subdivision, and since the total jump is at most 2, two edges of $\Delta_i$ are common to a regular polygon (as in the case of limit of nodal curves). In particular, on the other side of each such edge there is a single branch. thus by Remark 8 we have two jumps of 1 in inequality $(\zeta)$ amounting to 3, which is more than 2. hence contradiction. If $K = 1$ then the total jump is 1. then the remaining two internal edges meet regular polygons, again giving two jumps of 1 in $(\zeta)$ which amounts to 2 which is a contradiction. As for the case where $\Delta_i$ is a quadrilateral, the total jump is 1, because $\beta = 1$ already, we deduce $K = 0$ and all edges are internal to the subdivision. at least in one edge there is the same configuration of multiple branches meeting a single branch on the other side, which gives another jump and a contradiction.

1(b)i and 2(b)ii. Are not possible by the theory of curves in toric surfaces. in such case $C_{ij}$ must intersect the boundary divisors in the points which correspond to the vertices of the parallelogram, instead of in the 'interior' (the 1-dimensional open torus embedded in each boundary divisor).

2(a)i, 2(a)ii, 2(a)iii. Are all impossible: since $\beta = 2, K = 0$ the edge with more than one branch must be internal, and all neighboring polygons are regular, thus we get another jump in $(\zeta)$ and a contradiction.

In the case of 2(b)i we claim that either there is a singular branch for the curve $C_{ij}$ on $\text{Tor}(\Delta)$ or that in the refinement of one of the points on the boundary divisors we encounter a singular branch in the refinement curve. otherwise, by patchworking theorem this should lift to a nodal family, which contradicts the fact that we have a cusp in the family. Both cases are impossible: by Corollary 1.1, it is impossible for a parallelogram to admit a curve with singular branches, and the refinements of an intersection of two smooth branches (each from a different surface) does not admit a curve with singular branches, by (Proposition 1.6) (the picture is that of a triangle with two divisors met by a single smooth branch, each).

We now turn attention to 1(a)i. we claim that Since $s_{ij} = 4$ and $\Delta_i$ is a triangle, one boundary divisor meets two branches. Denote by $p_0 \in C_{ij}$ point the intersection of these two branches. We claim that $p_0$ must be a node, as well as all points of the curve in the open torus: $C_{ij} \cap (\mathbb{C}^\times)^2 \subset \text{Tor}(\Delta)$ This follows from proposition 1.6 $p_0$ Cannot lie on a divisor which corresponds to an internal edge of the subdivision. assuming it is, we look at the polygon on the other side, it is either a regular polygon or another triangle such that $s_{ij} = 4$ In either case, in the deformation, locally we must glue a two-punctured sphere to a either a one-punctured sphere or a two-punctured sphere, by gluing a ‘pair of pants’ or ‘two tubes’, either one causes an increase in genus, in contradiction
to the fact that the family was rational. Moreover, the cusp in the family cannot tend to \( p_0 \) since a cusp cannot degenerate into a node, which is simpler. Thus, the cusp degenerates into a point on one of the other boundary divisors, which correspond to internal edges. If on the other side the branch is met by two branches, again we get a contradiction to the genus. otherwise, we examine the refinement and get a contradiction by Lemma 2.7 We also claim that the two other boundary divisors that are met by a single branch, are met by a smooth branch. Let \( p' \) be such a point. if the divisor \( p' \) meets corresponds to an edge on the boundary of the subdivision, b We first claim that the curve \( C_{ij} \) must be nodal.

We thus have proven:

**Theorem 2.2.** In the limit of a rational 1-cuspidal family, the tropical limit consists of triangles and parallelograms with curves as described in Lemma 2.4 and Lemma 2.5 plus an additional polygon: a quadrilateral with a rational curve meeting the boundary divisors in four branches.

Note that we can further describe the curve in the quadrilateral: In the resulting tropical limit all boundary divisors are met by a single branch from either side, and since the cusp cannot tend to a boundary divisor (By Lemma 2.7), it must tend to the open torus \((C^*)^2\) in the corresponding toric surface, giving rise to a singular branch there.

By Claim 1.1 the quadrilateral cannot have a pair of parallel edges.

Moreover in the next part we will show that the curve must have a single singular branch and it is indeed a cusp.

### 2.6 The limit curve in a quadrilateral

We want to enumerate the 1-cuspidal curves in a quadrilateral, intersecting each boundary divisor in a single branch, fixing the intersection points on two of the boundary divisors. We start with the case of fixing the intersection points on adjacent edges, and thus we know the coefficients at three consecutive vertices up to multiplication by a constant. in fact we know more, since we know there must be only a single branch intersecting the divisor, we know the restriction of the polynomial defining the curve, to the monomials corresponding to those edges.

**Theorem 2.3.** Given a quadrilateral \( \Delta \) having no pair of parallel opposite edges, The number of rational curves having nodes and 1 cusp in the open torus inside \( \text{Tor}(\Delta) \) such that each boundary divisor is met by a single branch, With a given intersection point on two edges \( \sigma_1, \sigma_2 \in \Delta \) is:

\[
\frac{\text{Area}(\Gamma)}{\ell_1 \ell_2}
\]

(16)

Where, \( \Gamma \) is the parallelogram spanned by the two vectors corresponding to the edges, and \( \ell_i \) is the integer length of \( \sigma_i \).
WLOG, after applying $\text{Affine}(\mathbb{Z}^2)$ transformation (as depicted in Figure 3) - we straighten one edge with Euclidean GCD algorithm, then apply a series of translations and shears) we can assume the quadrilateral is in the following position $p_0, m, q, p, q, p, q, p, q, p, q, p, q, p, q, p, q, p, q$ where $0 < p, q < m$ and $r, s > 0$, and that the two edges given are $[(0, m), p, q]$ and $[(p, q), (p + r, 0)]$.

Let $\bar{C} = Cl(C \cap (\mathbb{C}^*)^2)$ where the closure is taken inside $\mathbb{P}^2$ under an identification of the torus $\mathbb{P}^2 \cong \mathbb{C}^* \cong \text{Tor}(\Delta)$ since $\bar{C}$ is rational we parameterize it, denote the parameter by $t$, WLOG we can assume that intersection with $[(0, m), p, q]$ is at $t = 1$, with $[(p, q), (p + r, 0)]$ is at $t = 0$, and with $[(0, m), (p + r + s, 0)]$ at $t = \infty$. denote also the intersection with $[(p + r, 0), (p + r + s, 0)]$ to be at $t = \xi$ and the singular point at $t = \eta$, as also indicated figure 4.

Up to homogeneity and linear change of variables ($x := ax + by, y := cx + dy$) we can assume that the coefficients of $y^m, x^p y^q, x^{p+r}$ are all 1.

From intersections with the coordinate axes we deduce the parametrization of the curve, in coordinates is:

$$
X = \alpha(t - 1)^{m-q} t^q
$$

$$
Y = \beta(t - 1)^p t^r(r - \xi)^s
$$

$$
Z = 1
$$

At $(0,0)$ the orders of intersection are determined by decomposing the lower convex envelope of the Newton polygon as Minkowsky sum of two segments.

First and foremost - note that $\eta \neq \xi, 0, 1, \infty$. To see that, assume the contrary:

We are at the case of Quadrilateral so the total jump is 1, thus $K = 0$ (All integer points on external edges are vertices of the subdivision), and therefore the singularity at $t = \eta = \xi$ lies on an edge common to another polygon, (as it is not a transversal intersection). This polygon contains a smooth curve. An impossible situation according to lemma 2.7.

We want to find the value of $\eta$. as it is a singular branch, we must have then

$$
\frac{\partial X}{\partial t}(\eta) = \frac{\partial Y}{\partial t}(\eta) = 0.
$$

$$
\frac{\partial^2 X}{\partial t^2}(t) = \alpha(t - 1)^{m-q-1} t^{q-1} ((m - q)t + q(t - 1))
$$

but $\eta \neq 1, 0$, thus it follows that $(m - q)\eta + q(\eta - 1) = 0$ which implies $\eta = \frac{q}{m}$.
Figure 4: The quadrilateral, the branches of the curve, and values of the parametrization

\[ \frac{\partial Y}{\partial t}(t) = \beta(t-1)^{p-1} t^{r-1}(t-\xi)^{s-1} (pt(t-\xi) + r(t-1)(t-\xi) + s(t-1)t) \] but \( \eta \neq \xi, 0, 1, \)
and thus we deduce:

\[ \xi = \frac{(p+r+s)\eta^2 - (r+s)\eta}{(p+r)\eta - r} \]

Now, by restricting to edges of and using toric intersection theory (Which lets us know how
the curve intersects each boundary divisor) we deduce equations in the following way: We restrict
the equation for \( C \) to a boundary segment, and since we obtained a parametrization, the terms
of lowest order in \( t \) must vanish, accordingly. For segment \( [(0,m),(p,q)] \), \( t \to 1 \) we obtain:

\[ y^q \left( y^{\frac{m-q}{d_1}} + \epsilon_1 x^{\frac{q}{d_1}} \right)^{d_1} = 0 \]
where \( d_1 = \gcd(p,m-q) \) and \( \epsilon_1^{d_1} = 1 \) thus, after substitution of
parametrization we deduce that:

\[ ((1 - \xi)^{s} \beta)^{\frac{m-q}{d_1}} + \epsilon_1 x^{\frac{q}{d_1}} = 0 \]

For segment \( [(p,q),(p+r,0)] \) we obtain:

\[ x^p \left( y^{\frac{r-q}{d_2}} + \epsilon_2 x^{\frac{q}{d_2}} \right)^{d_2} = 0 \]
where \( d_2 = \gcd(q,r) \) and \( \epsilon_2^{d_2} = 1 \). thus:

\[ ((-1)^{p+s} \xi \beta)^{\frac{r-q}{d_2}} + \epsilon_2 ((-1)^{m-q} \alpha)^{\frac{q}{d_2}} = 0 \]
And since \( \eta \) and \( \xi \) are completely determined by the Newton
polygon, we have two equations \( \alpha \) and \( \beta \) of the following form:

\[
\begin{align*}
A\alpha^a + \epsilon_1 B\beta^b &= 0 \\
C\alpha^c + \epsilon_2 D\beta^d &= 0 \\
\epsilon_1^{d_1} &= 1 \\
\epsilon_2^{d_2} &= 1
\end{align*}
\]

\( A, .., D \) constants, \( a = \frac{p}{d_1}, b = \frac{m-q}{d_1}, c = \frac{r-q}{d_2}, d = \frac{q}{d_2} \) Each solution \( (\alpha_0, \beta_0) \) determines \( \epsilon_1 \) and \( \epsilon_2 \). We
have and \( ac - bd \) solutions for \( \alpha \) and \( \beta \):

\[
\begin{vmatrix}
\frac{p}{d_1} & \frac{m-q}{d_1} \\
\frac{r}{d_2} & \frac{q-m}{d_2}
\end{vmatrix}
\] (20)

If the intersection point with the boundary wasn’t fixed, We would have \( d_1d_2 \) solutions for \( \epsilon_1 \) and \( \epsilon_2 \),

\[
d_1d_2 \cdot \begin{vmatrix}
\frac{p}{d_1} & \frac{m-q}{d_1} \\
\frac{r}{d_2} & \frac{q-m}{d_2}
\end{vmatrix} = \begin{vmatrix}
p & m-q \\
r & q-m
\end{vmatrix} = Area(Parallelogram[(0, m), (p, q), (p + r, 0), (r, m - q)]) \] (21)

(22)

Now we turn to the case of a fixed intersection with a pair of non adjacent edges. Again, using Affine\( (\mathbb{Z}^2) \) transformations we can reduce to the following case. The polygon is \((0, m), (p, q), (p + r, 0), (p + r + s, 0)\) where \(0 < p, q < m\) and \(r, s > 0\), as before, and the two edges given are \([(0, m), (p, q)]\) and \([(p + r, 0), (p + r + s, 0)]\). Thus we are given the ratio of the coefficients of \( y^m \) and \( x^py^q \), so we denote them by \( t \) and \( Bt \) for a given \( B \) and an unknown \( t \). And the coefficients of \( x^{b+r} \) and \( x^{b+r+s} \) are \( l \) and \( lD \) for given \( D \) and unknown \( l \).

First by homogeneity, we divide by \( t \) getting coefficients: \( 1, B, \frac{1}{t}, \frac{1}{t}D \) listed counterclockwise from \( y^m \). Denoting \( \omega = \frac{1}{t} \neq 0 \), we derive equations from the restrictions to edges, as before:

From the segment \([(0, m), (p, q)]\), \( t \to 1 \) we obtain: \( ((1 - \xi)^s \beta) \frac{m-s}{\alpha} + \sqrt[\xiB]{\alpha} = 0 \) Where \( \sqrt[\xiB]{\alpha} \) is one of the \( d_1 \)-th roots of \( B \) determined by the point of intersection with Tor\([(0, m), (p, q)]\).

From the segment \([(p + r, 0), (p + r + s, 0)]\), \( t \to \xi \) we obtain: \( \xi^s = \sqrt[\xiD]{D} = 0 \) Where \( \sqrt[\xiD]{D} \) is a \( s \)-th roof of \( D \) determined by the intersection point with Tor\([(p + r, 0), (p + r + s, 0)]\). Again we have a set of binomial equations for \( \alpha \) and \( \beta \) which has \( \frac{1}{\alpha D} \) solutions. i.e. the area of the parallelogram given by the two edges, divided by the integer lengths of the edges.

We have to verify one more thing - how many values possible for \( \omega = \frac{1}{t} \)?

From the edge \([(p, q), (p + r, 0)]\), \( t \to 0 \) we get the equation:

\[
\sqrt[\xiB]{B}((-1)^{p+r+sp} \xi \beta) + \sqrt[\xiD]{((-1)^{m-q} \alpha)} = 0
\]

Beside \( \sqrt[\xiD]{\omega} \) all other terms are knows, so \( \omega \), which is the ratio between \( t \) and \( \xi \) is uniquely determined.

We now show that each solution we found is indeed a 1-cuspidal curve, with possible nodes. The strategy is to compare the dimension of the equisingular deformations of the solutions we found, to the dimension of such deformations of curves with more complicated singularities. In our calculation we assumed WLOG that the coefficient at three vertices is 1. we can deform these coefficients to obtain an equisingular deformation of the solution curve. These coefficients are determined up to homogeneous multiplication, thus the deformation space of the solutions is 2-dimensional. in fact,
as the coefficient multiply the parameterizations of $C$ (i.e. it is of the form $x = \alpha P(t), y = \beta Q(t)$ for polynomials $P, Q$) the deformation is equisingular.

We will show that if there exists more than one singular branch, or a more complicated singularity, the dimension drops below 2. Denote by $\Sigma = \text{Tor}(\Delta)$, the toric surface. our rational curve $C$ can be thought as a mapping $\mathbb{P}^1 \to \Sigma$. Consider, $\mathcal{N}$, the normal sheaf on $\mathbb{P}^1$ defined by the exact sequence of sheaves: $0 \to T\mathbb{P}^1 \to \nu^*T\Sigma \to \mathcal{N} \to 0$ We denote by $\mathcal{T} \text{Def}$ the space of deformations such that the tangency conditions at the boundary divisors are maintained. Let $\ell_i$ be the edges of $\partial \Delta$, of integer length $|\ell_i|$, then $C$ intersects $\sigma_i \equiv (\text{Tor}(\ell_i))$ with multiplicity $|\ell_i|$ in one point

It is known $\dim \mathcal{T} \text{Def} \leq h^0\left(\mathcal{N}/\mathcal{N}_{\text{tors}}(-\sum(|\ell_i| - 1)\right)$ (See details in [10] Section ”Dimension Counts For Plane Curves” p.115) The points $z_1, ..., z_k$ where $\nu$ fails to be immersion are isolated, hence:

$$\deg\left(\mathcal{N}/\mathcal{N}_{\text{tors}}\right) \leq \deg(c_1(\mathcal{N})) - \sum\left(\text{ord}(z_i) - 1\right)$$

By additivity of first Chern class

$$c_1(\mathcal{N}) = c_1(\nu^*T\Sigma) - c_1(T\mathbb{P}^1)$$

hence:

$$\deg c_1(\mathcal{N}) = \deg c_1(\nu^*T\Sigma) - c_1(T\mathbb{P}^1) = -K_\Sigma C - 2$$

Recall from toric geometry that $-K_\Sigma = \sum \ell_i$, Thus,

$$\deg\left(\mathcal{N}/\mathcal{N}_{\text{tors}}\right) = \sum(|\ell_i|) - 2 - \sum(\text{ord}(z_i) - 1)$$

Now Recall that on $\mathbb{P}^1$ every bundle $\mathcal{L}$ is of the form $\mathcal{O}_{\mathbb{P}^1}(\text{deg}(\mathcal{L}))$, note that the correspondence is given by the degree function. Hence we have:

$$h^0\left(\mathcal{N}/\mathcal{N}_{\text{tors}}(-\sum(|\ell_i| - 1)\right) = h^0\left(\mathcal{O}_{\mathbb{P}^1}(\sum(|\ell_i|) - 2 - \sum(\text{ord}(z_i) - 1) - \sum(|\ell_i| - 1))\right) =$$

$$= -2 - \sum(\text{ord}(z_i) - 1) + 4 + 1 = 3 - \sum(\text{ord}(z_i) - 1)$$

(The 4 term is because we have 4 boundary divisors) The last formula is to be understood as is when $3 - \sum(\text{ord}(z_i) - 1) \geq 0$ and $h^0$ is to be taken as equal to 0 otherwise. Now, if for some $i$, $\text{ord}(z_i) \geq 3$ or exist some $i_1 \neq i_2$ such that $\text{ord}(z_{i_1}) \geq 2$, $\text{ord}(z_{i_2}) \geq 2$, we get that the dimension of the deformation space is strictly less than 2, which leads to a contradiction. thus we have shown

**Corollary 2.1.** There exists exactly one singular point $z$. Moreover, $\text{ord}(z) = 2$ and a single branch of $C$ passes through $z$. thus, $z$ is analytically equivalent to a singularity of the form $x^2 + y^k = 0$ for $k \geq 3$

**Proof.** See [10] pp. 26. \qed
our goal now is to show that $z$ is a cusp, i.e. $k$ must equal $3$.

We will estimate the dimension of equisingular deformations, by estimating from above the dimension of the Zariski tangent space for the space of equisingular deformation satisfying given tangency conditions. First, we consider local equisingular deformations at a given singular point, when dealing with ADE singularities, as we have here, this space is given by the Tjurina ideal, thus $I^{es}(C, z_i) = \langle f_x, f_y, f \rangle \subset \mathcal{O}_{\Sigma, z_i}$. This ideal corresponds to a zero-dimensional scheme concentrated in the point $z_i$ that we denote by $X^{es}(C, z_i)$. We consider the following zero-dimensional scheme $X^{es} = X^{es}(C, z_1) \cup X^{es}(C, z_1) \cup \ldots \cup X^{es}(C, z_n)$, and with the corresponding structure sheaf $\mathcal{O}_{X^{es}} = \bigoplus_{z \in \text{Sing}(C)} \mathcal{O}_{\Sigma, z} / I^{es}(C, z)$. It’s ideal sheaf $\mathcal{J}_{X^{es}/C}$ defined as the kernel of the restriction map, by $0 \to \mathcal{J}_{X^{es}/C} \to \mathcal{O}_C \to \mathcal{O}_X \to 0$. We will be interested in deformations keeping tangency conditions, this introduces some twisting to the sheaves involved in the calculation, as we have seen before, but might be understood better using the language of zero-dimensional schemes.

Let $E = \text{Tor}(\sigma)$ be a boundary divisor and let $w_\sigma \in C \cap E$ the tangency point, we consider the ideal of all local deformations (in the analytic local ring $\mathcal{O}_{\Sigma, w_\sigma}$) such that the order of tangency to $E$ is fixed (i.e. the intersection number with $E$ is fixed). we denote it by $I^{tgt}(C, w_\sigma)$. In coordinates this can be described as follows, let $x, y$ local coordinates such that $(0, 0)$ corresponds to the point $w_\sigma$ and the $x$-axis, given by $y = 0$, is the tangent line to $E$, then $I^{tgt}(C, w_\sigma) = \langle y, x^{\ell-1} \rangle$, $\ell$ being the integer length of $\sigma$. because when we take a local equation $f$ and restrict to $y = 0$ we expect the least power of $x$ to be greater than $\ell - 1$. Again, to globalize this notion we consider the zero-dimensional scheme: $X^{tgt}(C, w_\sigma)$ given by the ideal in the local ring described above, and denote by $X^{tgt} = \bigcup_{\sigma \in \Delta} X^{tgt}(C, w_\sigma)$ the union over all tangency points. Similarly, let $\mathcal{O}_{X^{tgt}}$ and $\mathcal{J}_{X^{tgt}/C}$ the corresponding structure sheaf and ideal sheaf.

We are looking for the Zariski tangent space of equisingular deformations keeping the tangency conditions with the boundary divisors, inside the linear system given by curves on the surface defined by an equation with Newton polygon $\Delta$. This space is known to be given by: (see [4] Theorem 3.25, Proposition 4.10) where it is proven locally)

$$\text{TDef} = H^0(\mathfrak{i}_*\mathcal{J}_{X^{es}/C} \otimes \mathfrak{i}_*\mathcal{J}_{X^{tgt}/C} \otimes \mathcal{L}_\Delta)$$

First we focus attention on $\mathcal{L}_\Delta$, the sheaf corresponding to the linear systems of curves with Newton polygon $\Delta$. The curve $C$ is defined by such Newton polygon, and thus is an effective divisor corresponding to such a linear system, thus $\mathcal{L}_\Delta = \mathcal{O}_C(C)$

Recall that for any divisor $\mathcal{D} \subset \Sigma$, $\mathfrak{i}_*\mathcal{O}_C(C \cap \mathcal{D}) = \mathfrak{i}_*\mathcal{O}_C \otimes \mathcal{O}_C(\mathcal{D})$ (can be checked locally)

Also note that $\mathfrak{i}_*\mathcal{J}_{X^{es}/C} \otimes \mathfrak{i}_*\mathcal{J}_{X^{tgt}/C} = \mathfrak{i}_*\mathcal{J}_{X^{es}/C} \otimes \mathfrak{i}_*\mathcal{J}_{X^{tgt}/C} \otimes \mathfrak{i}_*\mathcal{O}_C$

we get:

$$\text{TDef} = H^0(\mathfrak{i}_*\mathcal{J}_{X^{es}/C} \otimes \mathfrak{i}_*\mathcal{J}_{X^{tgt}/C} \otimes \mathcal{L}_\Delta)$$

$$= H^0(\mathfrak{i}_* (\mathcal{J}_{X^{es}/C} \otimes \mathfrak{i}_*\mathcal{J}_{X^{tgt}/C} \otimes (\mathcal{O}_C \otimes \mathcal{L}_\Delta)))$$

(23) (24)
Where \( \mathcal{O}_C \otimes \mathcal{L}_\Delta \) is to be understood as the following sheaf on \( C \): 
\[ i^* (i_* \mathcal{O}_C \otimes \mathcal{L}_\Delta) \]
Also note that \( \text{deg}(\mathcal{O} \otimes \mathcal{L}_\Delta) = C.C \), Where \( C.C \) is the self intersection number of \( C \). (See [8, Remark 2 p.104])

While \( \mathcal{O}_C \otimes \mathcal{L}_\Delta \) and \( \mathcal{J}_{X^{\text{es}}/C} \) are invertible sheafs, \( \mathcal{J}_{X^{\text{es}}/C} \) is not. To remedy this we will pass to an ideal sheaf called the conductor sheaf, which is invertible.

The definition we use for the conductor ideal at a singular point \( z \) is as follows: (See [3, Section 2.4])
\[ I_z^{\text{cond}} = \{ \phi \in \mathcal{O}_{\Sigma,z} | (\phi \cdot C)_z \geq 2\delta(z) \} \]
Where \( (\phi \cdot C)_z \) is the intersection multiplicity at point \( z \), and \( \delta(z) \) is the delta-invariant at point \( z \).

In fact, we will define a slightly more general class of ideals in the local ring at \( z \):
\[ I_{\lambda,z}^{\text{cond}} = \{ \phi \in \mathcal{O}_{\Sigma,z} | (\phi \cdot C)_z \geq 2\delta(z) + \lambda \} \]
It is a known fact that \( \mathcal{J}_{X^{\text{es}}/C} \) is a subsheaf of \( \mathcal{J}_{X^{\text{cond}}} \) ([4, p.435]), but unfortunately this will not yield a tight enough bound. We take the following approach:

Consider \( z_0 \) to be the non nodal singular point of \( C \), it is a singularity of type \( A_{k-1} \) thus, analytically equivalent to \( x^2 + y^k = 0 \) in some coordinates. We now consider the deformations such that the singular points remains on the \( y \)-axis (in those coordinates), while keeping all previous conditions (equisingularity, tangency conditions, Newton polygon \( \Delta \) ) This gives rise to a codimension-1 subspace of the space of deformations we considered before. We denote these deformations by \( \mathcal{D} \text{ef}' \) thus
\[ \dim T \mathcal{D} \text{ef}' = \dim T \mathcal{D} \text{ef} - 1 \]

We have to replace the scheme \( X^{\text{es}} \) by a different scheme, \( X^{\text{es}'} \) taking into account the extra condition. We only change the ideal in the local ring of \( z_0 \), it’s elements are given by deformations which fix the \( x \) coordinate, ie:
\[ (x + \sum_{i+j>0} \alpha_{ij} x^i y^j)^2 + (y + \sum_{i+j\geq0} \beta_{ij} x^i y^j)^k) \]
The tangent cone is the ideal generated by the coefficients of linear part in \( \alpha_{ij} \) and \( \beta_{ij} \):
\[ < x^2, xy, y^{k-1} > \]

We now check for which value of \( \lambda \) this ideal is contained in \( I_{\lambda,z_0}^{\text{cond}} \). Consider the following parametrization of the singularity:
\[ x = t^k, y = t^2 \] for each \( \phi(x, y), \phi \cdot C \) equals the lowest power in which \( t \) appears in the expression \( \phi(t^k, t^2) \) recall that \( \delta(A_{k-1}) = (k-1)/2 \) (k is odd) Thus, assuming \( k > 3 \):
\[ (x^2 \cdot C) = 2k = 4\delta + 2 = 2\delta + (2\delta + 2) \geq 2\delta + 6 \tag{25} \]
\[ (xy \cdot C) = k + 2 = 2\delta + 3 \tag{26} \]
\[ (y^{k-1} \cdot C) = 2k - 2 = 4\delta = 2\delta + (2\delta) \geq 2\delta + 4 \tag{27} \]
Thus \( < x^2, xy, y^{k-1} > \subset I_{\lambda,z_0}^{\text{cond}} \)
Let us define a modified conductor scheme $X^{\text{cond}'}$ by the ideals $I_{0,z}^{\text{cond}}$ at all the nodal points $z$ of $C$, and by $I_{(3),z_0}^{\text{cond}}$ at $z_0$. Now $J_{X^{\text{ext}}/C}$ is a subsheaf of $J_{X^{\text{cond}'}/C}$ and we can let the calculation begin. We use adjunction formula and toric description for canonical divisor of $\Sigma$:

$$\dim TD_{e,f} \leq h^0 \left( J_{X^{\text{cond}'}/C} \otimes i_\ast J_{X^{\text{int}}/C} \otimes (\mathcal{O}_C \otimes L_\Delta) \right)$$

$$= h^0 \left( \mathcal{O}_{\mathbb{P}^1}(-2\delta_{\text{tot}} - 3) \otimes \mathcal{O}_{\mathbb{P}^1}(-\sum_{i=1}^{4}(\ell_i - 1)) \otimes \mathcal{O}_{\mathbb{P}^1}(-K_{\Sigma} \cdot C - 2 + 2\delta_{\text{tot}}) \right)$$

$$= h^0 \left( \mathcal{O}_{\mathbb{P}^1}(-2\delta_{\text{tot}} - 3) \otimes \mathcal{O}_{\mathbb{P}^1}(-\sum_{i=1}^{4}(\ell_i - 1)) \otimes \mathcal{O}_{\mathbb{P}^1} \left( \sum_{i=1}^{4}(\ell_i) + 2\delta_{\text{tot}} \right) \right)$$

$$= h^0 \left( \mathcal{O}_{\mathbb{P}^1}(-3 + 4 - 2) \right) = h^0 \left( \mathcal{O}_{\mathbb{P}^1}(-1) \right) = 0$$

Where $\delta_{\text{tot}}$ is the sum of $\delta$ invariant at all points of $C$.

We arrived at $\dim TD_{e,f} \leq 1 < 2$, a contradiction. (recall that the curves we found that satisfy our conditions varied in a two dimensional family) Thus, at $z_0$ $C$ has a cusp.
3 Restoring 1-cuspidal rational algebraic curves out of a given tropical limit

3.1 The resulting tropical limit curve minus the marked points

In [13, pp. 48, Lemma 4.20] the following is proven:

**Lemma 3.1.** Let $C$ be a simple tropical curve of genus $g$ and dual polygon $\Delta$ (simple means that the dual subdivision of $\Delta$ contains only triangles and parallelograms.) Suppose that $C$ is parameterized by $h : \Gamma \rightarrow \mathbb{R}^2$. Suppose that $C$ passes through a configuration $P$ of $s + g − 1$ points in general position. Then each component $K$ of $\Gamma \setminus h^{-1}(P)$ is a tree and the closure of $h(K) \subset \mathbb{R}^2$ has exactly one end of weight one at infinity.

Note that in our case, since $g = 0$, $\Gamma$ is a tree and we do not need the lemma to deduce that each component is a tree.

This allows, as further explained in [13] to obtain a partial order on the vertices of each connected component $K$ of $\Gamma \setminus h^{-1}(P)$ as follows: direct each edge of $K$ such that it points toward the infinite end. (i.e, there is a unique path between each given edge and the infinite edge, the edge is oriented to the direction of the path) This defines a direction on the edges of $h(K)$ as well, since each 4-valent vertex corresponds to a pair of edges crossing, so we can keep the original direction of the edges. (See figure 5)

![Figure 5: How to orient the edges on a node](image)

Two vertices $v, u$ satisfy $v \leq u$ if and only if there exist a directed path from $v$ to $u$.

This order satisfies two important properties we will incorporate later:

**Proposition 3.1.**

1. each vertex different from a marked point has exactly two edges 'going in’ (the indegree is 2).

2. the marked points are initial vertices in the sense that in each component single edge is going out of them, thus they have indegree 0 and outdegree 1 (in each component).

In the case of 1-cuspidal curves, one of the connected components $K$ contains a 4-valent vertex dual to a quadrilateral. We would like to direct the graph $h(K)$ accordingly such that still the indegree of each vertex is 2.
Remark 4. A point of terminology, we will use the term 1-cuspidal curves, to describe curves having one cusp, zero or more nodes and no other singularises. As for tropical curves, a tropical 1-cuspidal curve is a curve whose dual subdivision contains only triangles, parallelograms, and one quadrilateral which is not a trapezoid, and such that all the edges on the boundary of the Newton polygon are of integer length 1.

We first need a better description of the special component:

Lemma 3.2. Let $K$ be the connected component containing the 4-valent vertex $v'$, dual to a quadrilateral. Then $K$ is a tree, has exactly two infinite edges (without marked points), and the unique path between them passes through $v'$

Proof. If we consider $K$ (without the marked points) as a tropical curve it is of rank $x-2$ where $x$ is the number of infinite ends: had it been 3-valent it’s rank would have been $x-1$ [13, Corollary 2.24] and the 4-valent vertex adds an extra condition. Since the marked points are in general position, $K$ admits no deformations passing through the marked points (other than the constant deformation), and the conditions imposed by the points are independent (each subsequent point decreases the dimension by 1), thus there are $x-2$ marked points, each on a different infinite end, leaving two free ends. The (unique) path between the infinite ends must pass through $v'$. otherwise, it is a path between infinite ends whose all vertices are 3-valent, such a configuration is known as a string, in the terminology of [7] and gives rise to a one parametric family of deformation (by moving the string), which is absurd.

We now direct the edges. $K \backslash \{v'\}$ has four components, two of them contain an infinite edge. we orient the edges toward the infinite edge in those two components. on the other two we orient towards the vertex $v'$. we induce the directions on the edges of $h(K)$ and orient the crossings as before. this ordering satisfies the two properties in proposition 3.1. Other connected components of the curve minus the marked points are simple tropical curves and we direct the edges toward the infinite end in the component.

3.2 Restoring the tropical limit (incl. limit curves) and counting the corresponding families

For reconstruction of the tropical curve, we follow Shustin [15], with the appropriate modification for the 1-cuspidal case.

We denote by $r$ the dimension of the Severi variety of curves with $n$ nodes and one cusp, i.e.

$$r = \dim \Sigma_\Delta(nA_1, 1A_2)$$

Denote by $Q_\Delta(nA_1, 1A_2)$ the set of quadruples $(A,S,F,R)$, where $A \in A(\Delta)$ is a 1-cuspidal tropical curve of rank $r$, $S : \Delta = \Delta_1, \ldots, \Delta_N$ is a subdivision of $\Delta$ dual to $A$, and $F,R$ are
collections of the following polynomials in $\mathbb{C}[x,y]$ which together are defined up to multiplication by the same non-zero (complex) constant. $F = (f_1, \ldots, f_N)$, where each $f_i$ is a polynomial with Newton polygon $\Delta_i$, $i = 1, \ldots, N$, such that:

- If $\Delta_i$ is a triangle, then $f_i$ defines a rational curve in $\text{Tor}(\Delta_i)$ as described in Lemma 2.4.
- If $\Delta_i$ is a parallelogram, then $f_i$ defines a curve in $\text{Tor}(\Delta_i)$ as described in Lemma 2.5.
- And if $\Delta_i$ is a quadrilateral, then $f_i$ defines a rational curve with 1 cusp in $\text{Tor}(\Delta_i)$ as described in theorem 2.3.

Also, for any common edge $\sigma = \Delta_i \cap \Delta_j$, the truncations $f_i^\sigma$ and $f_j^\sigma$ coincide; $R$ is a collection of deformation patterns compatible with $F$ as defined in section 4.1 (cf. [15, ]). In short, a deformation pattern is a Newton polygon and a polynomial resulting from the process of refinement of the tropicalization.

We are given the points $x_1, \ldots, x_r \in \mathbb{Q}^2$ and $p_1, \ldots, p_r \in (\mathbb{K}^\ast)^2$ such that $\text{Val}(p_i) = x_i$, $i = 1, \ldots, r$, and we intend to find:

- How many elements $(A, S, F, R) \in \mathcal{Q}_\Delta(nA_1, 1A_2)$ correspond to a 1-cuspidal tropical curve $A \in \mathcal{A}(\Delta)$ of rank $r$ passing through $x_1, \ldots, x_r$ and,
- How many polynomials $f \in \mathbb{K}[x,y]$ (determined up to multiplication by a non-zero $\mathbb{K}$-constant) with Newton polygon $\Delta$, which define curves $C \in \Sigma_\Delta(nA_1, 1A_2)$ passing through $p_1, \ldots, p_r$, arise from a tropicalization $(A, S, F, R) \in \mathcal{Q}_\Delta(nA_1, 1A_2)$
Definition 3.1. Define the weight of a 1-cuspidal tropical curve $A$ of rank $r$, with $r$ marked points in general position, $\{x_1, \ldots, x_r\}$, by:

$$W(A, \{x_1, \ldots, x_r\}) = W(\tilde{\Delta}, \sigma', \sigma'') \cdot \left( \prod_{\Delta' \in P(S)} \frac{2|\Delta'|}{|V(\Delta')|=3} \right),$$

where $P(S)$ denotes the set of polygons of $S$, $|\Delta'|$ stands for the Euclidean area of $\Delta'$, $\tilde{\Delta}$ is the unique quadrilateral in $S$, $\sigma', \sigma''$ are the unique pair of edges in $\tilde{\Delta}$ dual to the two edges that do not lie on the unique path between infinite ends going through the 4-valent vertex in $h(A)\setminus\{x_1, \ldots, x_r\}$ as defined in lemma 3.2.

Step 1:
Let $A \in \mathcal{A}(\Delta)$ be a rational 1-cuspidal tropical curve of rank $r$ with Newton polygon $\Delta$, passing through the given points $x_1, \ldots, x_r \in \mathbb{Q}^2$. Observe, first, that $A$ uniquely determines a dual subdivision $S$ of $\Delta$. Indeed, the unbounded components of $\mathbb{R}^2\setminus A$ are in a natural one-to-one correspondence with $\partial \Delta \cap \mathbb{Z}^2$. The bounded edges of $A$ in the boundary of the above components define germs of the edges of $S$ starting at $\partial \Delta \cap \mathbb{Z}^2$. There is a pair of non-parallel neighboring germs which start at distinct points of $\Delta \cap \mathbb{Z}^2$, and their extension uniquely determines a triangle, a parallelogram or a quadrilateral in the subdivision $S$. Then we remove this polygon out of $A$ and continue the process.

Second, $A$ determines (uniquely up to a constant shift) a convex piece-wise linear function $\nu : \Delta \to \mathbb{R}$ whose graph projects onto the subdivision $S$. To see this - we consider each connected component of $A\setminus\{x_1, \ldots, x_r\}$ and build the function $\nu$ inductively. We order the vertices with respect to the partial order defined in the discussion after proposition 3.1. Then, in each iteration we take a minimal vertex from set of the vertices not yet processed.

The points $x_1, \ldots, x_r$ lie on $r$ distinct edges of $A$ which are dual to some $r$ edges of $S$. If $\sigma_i \in E(S)$ corresponds to a point $x_i$, and $\omega'_i, \omega''_i$ are the endpoints of $\sigma_i$ $1 \leq i \leq r$ then we have linear conditions on $\nu(\omega'_i), \nu(\omega''_i)$:

$$\nu(\omega'_i) - \nu(\omega''_i) = (\omega''_i - \omega'_i) x_i, \quad i = 1, \ldots, r$$  \hspace{1cm} (28)

Since $x_1, \ldots, x_r$ are generic, system (28) is independent.

These points will be processed first and will determine $\nu$ up to a multiplicative constant on each of the endpoints of edges dual to those passing through the marked points. The picture one should keep in mind is that of coloring edges on the graph induced by vertices and edges of the dual subdivision. at any stage, the values of $\nu$ on the vertices of each connected component, are determined up to a constant. when the next vertex of $A$ is processed, edges are colored along the
boundary of a subdivision polygon, either extending a colored component, or making two of these components connect (for example when we process a quadrilateral arriving from opposite edges) in the latter case the common vertices forces the two constants to be equal. In more detail, if the next vertex is dual to a triangle, an edge is colored, between two existing vertices in the same connected component (since in our ordering, always two edges are directed into a vertex) and no condition need be imposed. If the vertex is a 4-valent vertex of $A$, since two edges are directed in, two are directed out, and thus, either a new subdivision vertex is added to the colored subgraph, or two components of the colored graph connect (in the case of quadrilateral, two edges from different components may be opposite edges). For a parallelogram $\Delta_j \in P(S)$ the corresponding 4-valent vertex of $A$, imposes the following linear condition on the values of $\nu$ at the vertices $\omega_j^{(1)}, \omega_j^{(2)}, \omega_j^{(3)}, \omega_j^{(4)}$ of $\Delta_j$ (listed, say, clockwise):

$$\nu(\omega_j^{(1)}) + \nu(\omega_j^{(3)}) = \nu(\omega_j^{(2)}) + \nu(\omega_j^{(4)})$$

(29)

since it translates into the condition of four planes meeting in a common point. For a quadrilateral a similar condition is imposed, i.e, some linear combination with non-zero coefficients of $\omega_j^{(1)}, \ldots, \omega_j^{(4)}$ must equal 0.

This allows to determine the value of $\nu$ at the new subdivision vertex or to relate the constants between two components of the forest now joining together. When the process stops after iterating through all vertices of $A$, (performed separately in each component of $A$ minus the marked points), the values of $\nu$ at the vertices of $S$ are determined uniquely up to a constant shift.

**Step 2:**

Our end-goal is to use patchworking theorem to produce polynomials of the form:

$$f(x, y) = \sum_{(i,j) \in \Delta} \tilde{c}_{ij}(t)\nu^{(ij)}x^iy^j, \quad \tilde{c}_{ij}(0) = c_{ij}, \quad (i, j) \in \Delta,$$

(30)

such that:

$$f_k(x, y) = \sum_{(i,j) \in \Delta_i} c_{ij}x^iy^j, \quad k = 1, \ldots, N.$$  

(31)

We claim that the condition

$$f(p_1) = \ldots = f(p_r) = 0$$

(32)

uniquely determines both the coefficients of $f_1, \ldots, f_N$ at the vertices of $S$, and the truncations of $f_1, \ldots, f_N$ on the edges $\sigma_1, \ldots, \sigma_r$, dual to $x_1, \ldots, x_r$, up to multiplication by the same non-zero constant. Indeed, let

$$x_i = (-\alpha_i, -\beta_i), \quad p_i = (\xi, \eta), \quad \xi = \xi_i^0 t_i^\alpha + \text{h.o.t.,} \quad \eta = \eta_i^0 t_i^\beta + \text{h.o.t.,} \quad \xi_i^0, \eta_i^0 \in \mathbb{C}^*,$$

and let the endpoints of the edge $\sigma_i$ be $\omega_i' = (i_1, j_1)$, $\omega_i'' = (i_2, j_2)$. The conditions $f(p_i) = 0$, $i = 1, \ldots, r$, then transform into the following equations:

$$f(p_i) = t^{\nu^{(i_1,j_1)}+i_2\alpha_i+j_2\beta_i}(g_i(\xi_i^0, \eta_i^0) + O(t)) = 0$$

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where a quasihomogeneous polynomial $g_i(x, y) = c_{x^i}x^{i_1}y^{i_2} + \ldots + c_{x^j}x^{i_2}y^{i_2}$ is the tropicalization of $f(x)$. Since $g_i$ is the product of a monomial and a power of an irreducible binomial, (33) determines it uniquely up to a constant factor.

**Step 3:**

We now, again iterate through all the vertices of the tropical curve $A$, taking a minimal vertex out of those not yet processed. In every iteration for every vertex we examine the dual subdivision polygon $\Delta_i$ in $S$, and reconstruct the polynomial $f_i$ up to a multiplicative constant. By a property of our ordering (Each vertex has two edges going in) we are always given the restriction of $f_i$ to two edges $\sigma'$, $\sigma''$ of $\Delta_i$, corresponding to the incoming edges in the tropical curve $A$. We now examine the possible shapes for $\Delta_i$ that we may encounter:

1. $\Delta_i$ is a triangle: By Lemma 2.4, for a triangle $\Delta_i$, and given truncations to two edges $\sigma', \sigma''$ of $\Delta_i$, an admissible polynomial $f_i$ (i.e., defining a rational nodal curve with precisely three unibranch intersection points with Tor $\partial \Delta_k$) can be restored in $|\Delta_k|(|\sigma'\cdot|\sigma''|)^{-1}$ ways. ($|\sigma|$ denoting integer length)

2. $\Delta_i$ is a parallelogram: since quadrilaterals come from 'crossings' ("nodes" in $h(A)$) $\sigma', \sigma''$ are always adjacent edges, and the admissible polynomial is determined uniquely.

3. $\Delta_i$ is a quadrilateral. then, an admissible polynomial (i.e. defining a rational 1-cuspidal curve with four smooth unibranch intersection points with Tor $\partial \Delta_k$) can be restored in $W(\Delta, \{\sigma', \sigma''\})|\sigma'\cdot|\sigma''|)^{-1}$ ways, by 2.3. Where $W(\Delta, \{\sigma', \sigma''\})$ is the area of a parallelogram spanned by the vectors given by the edges $\sigma'$ and $\sigma''$ also note that if two components with different constants are adjoined by quadrilateral $\Delta_i$ (when $\sigma', \sigma''$ are opposite edges) a condition is imposed on the constants such that one determines the other uniquely.

**Step 4:**

By [15] Lemma 3.9, any collection $f_1, ..., f_N$ can be completed with any of $\prod_{\sigma \in E^*(A)} |\sigma|$ collections of deformation patterns (Note that in our case the same deformation patterns appear as in [15] Lemma 3.9); hence, we can find $W(A)\prod_{i=1}^r |\sigma_i|^{-1}$ elements $(A, S, F, R) \in \mathbb{Q}(nA_1)$ compatible with the given nodal tropical curve $A$ and the points $x_1, ..., x_r \in \mathbb{R}^2$, $p_1, ..., p_r \in (\mathbb{K}^*)^2$. (were $W(A)$ is as defined above in definition (3.1))

Note that any edge with a marked point was considered twice as an edge where intersection is fixed, hence they stay at the denominator after reconstructing the deformation patterns

**Step 5:**

The Patchworking theorem, described briefly in section 4 and in detail in [15] and [14], states that...
for every quadruple \((A, S, F, R)\) we are given a family of solutions to the deformation problem, i.e. a family of polynomials \(f \in \Lambda_{\mathbb{K}}(\Delta)\), tropicalizing to \((A, S, F, R)\), that depend smoothly on \(r\) parameters. (See remark 5 in the next section)

The conditions for the curve \(C\) over \(\mathbb{K}\), given by \(f\) to pass through the points \(p_1, \ldots, p_r \in (\mathbb{K}^*)^2\), i.e. \(f(p_1) = f(p_2) = \ldots = f(p_r) = 0\), provide us with a finite set of possible systems of equations for these \(r\) parameters, which, since they belong in \(\mathbb{K}\), the converging puiseux series, can be viewed as indeterminate smooth functions in a variable \(t\).

The linearization of this system of equations is equivalent to the linear system described in Step 1 for \(\nu\), which was shown to be independent.

Therefore, the implicit function theorem gives us a unique solution for each of these systems of \(r\) parameters, uniquely describing \(C\) in each case.

There are \(\prod_{i=1}^{r} |\sigma_i|\) different systems, hence we conclude that there are exactly \(W(A)\) curves in \(\Lambda_{\mathbb{K}}(\Delta)\) having \(n\) nodes and one cusp, passing through \(p_1, ..., p_n \in (\mathbb{K}^*)^2\), which tropicalize into \(A\).

**Remark 5.** For more details about this system of equations and its solutions see [13, Section 5.4 Proof of Lemma 3.12]

**Remark 6.** To invoke the patchworking theorem, a technical condition called \(S\)-Transversality has to be satisfied. In our case it holds. For discussion about it and why it holds, see the subsection 4.3 on transversality.
4 Patchworking singular algebraic curves

Following Shustin [15] we describe and quote the necessary definitions and lemmas necessary to use patchworking construction:

4.1 Initial data for patchworking

Let $\Delta \subset \mathbb{R}^2$ be a non-degenerate convex lattice polygon, $S : \Delta = \Delta_1 \cup \ldots \cup \Delta_N$ its subdivision into convex lattice polygons, defined by a convex piece-wise linear function $\nu : \Delta \to \mathbb{R}$ such that $\nu(\mathbb{Z}^2) \subset \mathbb{Z}$.

Let $a_{ij} \in \mathbb{C}$, $(i, j) \in \Delta \cap \mathbb{Z}^2$, be such that $a_{ij} \neq 0$ for each vertex $(i, j)$ of all the polygons $\Delta_1, \ldots, \Delta_N$. Then we define polynomials

$$f_k(x, y) = \sum_{(i, j) \in \Delta_k \cap \mathbb{Z}^2} a_{ij} x^i y^j, \quad k = 1, \ldots, N,$$

and curves $C_k = \{f_k = 0\} \subset \text{Tor}(\Delta_k), k = 1, \ldots, N$, on which we impose the following conditions.

(A) For any $k = 1, \ldots, N$, each multiple component of $C_k$ (if it exists) is defined by a binomial; it crosses any other component of $C_k$ transversally, only at non-singular points, and not on $\text{Tor}(\partial \Delta_k)$.

(B) For any edge $\sigma \subset \partial \Delta, \sigma \subset \Delta_k, 1 \leq k \leq N$, the curve $C_k$ is non-singular along $\text{Tor}(\sigma)$ and crosses $\text{Tor}(\sigma)$ transversally.

(C) If $\sigma$ is an edge of $\Delta_k, 1 \leq k \leq N$, and $z \in \text{Tor}(\sigma) \cap C_k$ is an isolated singular point of $C_k$, then the germ $(C_k, z)$ is topologically equivalent to $(y^n)^m(k, z) + (x^n)^m = 0$, in local coordinates $x^n, y^n$ with $y^n$-axis coinciding with $\text{Tor}(\sigma)$.

Now we introduce additional polynomials which will play the role of deformation patterns, arising from refinement of the tropicalization as defined in [15, Remarks 3.8 Remarks 3.11].

Consider all the triples $(k, \sigma, z)$, where $1 \leq k \leq N, \sigma \subset \partial \Delta$ is an edge of $\Delta_k, z \in \text{Tor}(\sigma) \cap C_k$ and $(C_k \cdot \text{Tor}(\sigma))z = m \geq 2$. Then introduce the equivalence of triples: (i) $(k, \sigma, z) \sim (l, \sigma, z)$ if $\sigma = \Delta_k \cap \Delta_l$, and (ii) $(k, \sigma, z) \sim (k, \sigma', z')$ if $\sigma, \sigma'$ are parallel sides of $\Delta_k$ and $z, z'$ belong to the same multiple component of $C_k$. The transitive extension of this equivalence distributes the triples into disjoint classes. We denote the set of equivalence classes by $\Pi$. In fact, a pair of points $z, z'$ from equivalent triples $(k, \sigma, z), (l, \sigma', z')$ determines an element of $\Pi$ uniquely, and we write simply $(z, z') \in \Pi$.

To any element of $\Pi$ we assign a deformation pattern. Namely, in any class there are exactly two triples $(k, \sigma, z), (l, \sigma', z')$ with coinciding or parallel edges $\sigma, \sigma'$, and isolated singular (or non-singular) points $z, z'$ of the curves $C_k, C_l$, respectively. In some local coordinates in neighborhoods
of $z$ and $z'$ as required in the above property (C), the curves $C_k$ and $C_l$ are defined by

$$\sum_{i-m(k,z)+jm\geq m(k,z)} \alpha_{ij}x^iy^j = 0, \quad \sum_{i-m(l,z') \geq jm - mm(l,z')} \beta_{ij}x^iy^j = 0,$$

respectively, with $\alpha_{m0} = \beta_{m0}$, and non-degenerate homogeneous polynomials

$$\varphi^{(k)}_z(x, y) = \sum_{i-m(k,z)+jm = m(k,z)} \alpha_{ij}x^iy^j, \quad \varphi^{(l)}_z(x, y) = \sum_{i-m(l,z') \geq jm - mm(l,z')} \beta_{ij}x^iy^j.$$

A deformation pattern attached to the chosen class of triples is a curve $C_{z,z'} \subset \text{Tor}(\Delta_{z,z'})$, $\Delta_{z,z'} = \text{conv}\{(m, 0), (0, m(k, z)), (0, -m(l, z'))\}$, defined by a polynomial $F_{z,z'}(x, y)$ with Newton triangle $\Delta_{z,z'}$ and truncations $\varphi^{(k)}_z(x, y)$, $\varphi^{(l)}_z(x, y^{-1})$ on the edges $[(m, 0), (0, m(k, z))], [(m, 0), (0, -m(l, z'))]$, respectively.

### 4.2 Patchworking theorem

Let us be given the data introduced in section 4.1, i.e., subdivision $S : \Delta = \Delta_1 \cup \cdots \cup \Delta_N$, induced by a function $\nu : \Delta \rightarrow \mathbb{R}$, tropical curve $A$, polynomials $f_1, \ldots, f_N$, and deformation patterns defined by polynomials $f_{z,z}$. Let $\mathcal{G}$ be the set of orientations of the tropical curve $A$ (as a graph), which have no oriented cycles and obey the following requirements. For $\Gamma \in \mathcal{G}$, denote by $\Delta_k^- (\Gamma)$ the union of those edges of $\Delta_k$ which correspond to arcs of $A$, which are $\Gamma$-oriented inside $\Delta_k$. We assume that $\Delta_k^- (\Gamma)$ is connected for any $k = 1, \ldots, N$, and any two arcs of $A$, having a common vertex and lying on a straight line, are cooriented. Denote by $\text{Arc}(\Gamma)$ the set of ordered pairs $(k, l)$, where $\Delta_k$, $\Delta_l$ have a common edge, and the corresponding arc of $A$ is $\Gamma$-oriented from $\Delta_k$ to $\Delta_l$.

**Theorem 4.1.** Under the assumptions of sections 4.1, suppose that all the given deformation patterns are $S$-transversal, and there is $\Gamma \in \mathcal{G}$ such that every triad $(\Delta_k, \Delta_k^- (\Gamma), C_k)$ is $S$-transversal, $k = 1, \ldots, N$. Then there exists a polynomial $f \in \mathbb{K}[x, y]$ with Newton polygon $\Delta$, whose refined tropicalization consists of the given data, $\nu$, $S$, $f_1, \ldots, f_N$, and the given deformation patterns, and which defines a family of reduced curves $C^{(t)} \subset \text{Tor}(\Delta)$, $t \neq 0$, such that there is an $S$-equivalent 1-to-1 correspondence between $\text{Sing}(C^{(t)})$ and the disjoint union of

- the sets $\text{Sing}^\text{iso}(C_k) \cap (\mathbb{C}^*)^2$, $k = 1, \ldots, N$,
- the sets $\text{Sing}(C_{z,\tilde{z}}) \cap \mathbb{C}^2$, $\{z, \tilde{z}\} \in \Pi$,
- the set of $\sum_{k=1}^N \sum_z \dim \mathcal{O}_{C_{z,\tilde{z}}}/I^{s\text{g}}(C_k, z)$ nodes, where $z$ runs over $\text{Sing}(C_k^\text{red}) \backslash \text{Sing}^\text{iso}(C_k)$, $k = 1, \ldots, N$.

Furthermore, Let $G \subset E(S)$ be subgraph given by the edges of the subdivision that are dual to edges of the tropical curve $A$ that pass through the marked points. Let $B \subset V(S)$ the set of vertices
of $G$ Then

$$f(x, y) = \sum_{(i, j) \in \Delta} (a_{ij} + c_{ij})x^iy^j t^{v(i, j)}, \quad (34)$$

$$\begin{align*}
c_{ij} &= c_{ij}(t) \in \mathbb{K}, \quad c_{ij}(0) = 0, \quad (i, j) \in \Delta, \\
c_{ij}(t) &= \Phi^B_{ij}(\{c_{kl}(t), (k, l) \in B\}, t), \quad (i, j) \in \Delta \cap \mathbb{Z}^2 \setminus B,
\end{align*} \quad (35)$$

with certain complex analytic functions $\Phi^B_{ij}, (i, j) \in \Delta \cap \mathbb{Z}^2 \setminus B$.

**Remark 7.** The theorem provides us with a family of deformations parameterized by the coefficients at the points of $B$. Note that in [13, Theorem 4.1] $B$ is described in a more complicated way, but the meaning that underlies that description is the definition of $B$ we gave above.

**Remark 8.** Also note that if $|B| = r + 1$ then the linear system of curves given by the theorem has $r$ degrees of freedom. This is due to homogeneity; multiplying a polynomial by a scalar gives the same zero-set, thus we can set one of the coefficients from $B$ to have the value 1.

### 4.3 Transversality

Transversality of equisingular strata provides sufficient conditions for the patchworking (cf. [14]). Here we quote a result that gives sufficient conditions for $S$-transversality:

We cite a lemma from [11] pp. 51

**Lemma 4.1.** Let $f_1, f_2, \ldots, f_N$ define curves on toric surfaces. There exists a non-negative integer topological invariant $b(w)$ of isolated planar curve singular points $w$ such that if $f_k$ is irreducible and

$$\sum_{w \in \text{Sing}(f_k)} b(w) < \sum_{\sigma \subset \partial \Delta_k} \text{length}(\sigma)$$

Then the triple $(\Delta_k, \partial \Delta_k, f_k)$ is $S$-transversal. The precise definition of the invariant $b(w)$ can be found, for example, [14].

Here we simply recall that:

- if $w$ is a node, then $b(w) = 0$,
- if $w$ is a cusp, then $b(w) = 1$.

Moreover, in [13, Shustin], there is a numerical criterion for $S$-transversality of deformation patterns. Since in our case the deformation patterns are the same as in that paper, we will just state that these deformation patterns are $S$-transversal and refer the interested reader to [13, Shustin].

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Summary and Conclusions

To conclude, we used the following approach:

- **Degeneration**: We built a tropical limit of families of complex curves.
- **Classification**: We found out what types of tropical limit can arise.
- **Deformation**: We used the patchworking to reconstruct a family of curves out of tropical data.

This thesis raises two main directions for further research:

1. **Enumeration 1-cuspidal curves of genus \( g \geq 1 \):**
   Their tropical limits encompass curves of genus 1. We strongly used rationality in the process of counting the curves on the limit surfaces. Curves of genus 0 admit parametrization by elliptic function, which might replace the role of the algebraic parametrization.

2. **Lattice-path algorithm** - unlike the nodal case, where a lattice path algorithm for constructing tropical curves is known (see [13] and [11]), it is not immediate to adapt it to the 4-valent case. The 4-valent vertex might send an edge going backwards, violating the strict order in which the curve is constructed using the lattice-path algorithm.
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