Compact complete proper minimal immersions 

in strictly convex bounded regular domains of $\mathbb{R}^3$

Antonio Alarcón

Abstract Consider a strictly convex bounded regular domain $C$ of $\mathbb{R}^3$. For any arbitrary finite topological type we find a compact Riemann surface $\mathcal{M}$, an open domain $M \subset \mathcal{M}$ with the fixed topological type, and a conformal complete proper minimal immersion $X : M \to C$ which can be extended to a continuous map $X : \overline{M} \to \overline{C}$.

2000 Mathematics Subject Classification 53A10 · 53C42 · 49Q10 · 49Q05

Keywords Complete minimal surface · Proper immersion · Plateau problem · Limit set

1 Introduction

The global theory of complete minimal surfaces in $\mathbb{R}^3$ has been developed for almost two and one-half centuries. One of the central questions in this theory has been the Calabi-Yau problem, which dates back to the 1960s. Calabi [C] asked whether or not it is possible for a complete minimal surface in $\mathbb{R}^3$ to be bounded. The most important result in this line is due to Nadirashvili [N1], who constructed a complete minimal disk in a ball. After Nadirashvili’s answer, Yau [Y] stated new questions related to the embeddedness and properness of surfaces of this type.

Concerning the embedded question, Colding and Minicozzi [CM] proved that a complete embedded minimal surface with finite topology in $\mathbb{R}^3$ must be properly embedded. In particular, it must be unbounded. This result was generalized in two different directions. On the one hand, Meeks, Pérez and Ros [MPR] proved that any complete embedded minimal surface in $\mathbb{R}^3$ with finite genus and countably many ends must be proper in the space. On the other hand, Meeks and Rosenberg [MR] showed that a complete embedded minimal surface with positive injectivity radius is proper in $\mathbb{R}^3$.

Regarding the properness of the examples, Martín and Morales [MM1] introduced an additional ingredient into Nadirashvili’s technique to prove that every convex domain and every regular bounded domain admits a complete properly immersed minimal disk. Moreover, they showed that the limit set of such surfaces can be chosen close to a prescribed smooth Jordan curve on the boundary of the domain [MM2]. These arguments were generalized by Ferrer,
Martín and the author [AFM] to prove that any open Riemann surface of finite topology can be properly and minimally immersed in any convex domain of $\mathbb{R}^3$ or any bounded and smooth one. The same conclusion for any open Riemann surface was proved by Ferrer, Martín and Meeks [FMM]. In contrast to these existence results, Martín, Meeks and Nadirashvili [MMN] proved the existence of bounded open regions of $\mathbb{R}^3$ which do not contain a complete properly immersed minimal surface with finite topology.

The study of the Calabi-Yau problem gave rise to new lines of work and techniques. Among other things, these new ideas established a surprising relationship between the theory of complete minimal surfaces in $\mathbb{R}^3$ and the Plateau problem. This problem consists of finding a minimal surface spanning a given family of closed curves in $\mathbb{R}^3$, and it was solved independently by Douglas [D] and Radó [R], for any Jordan curve. The link between complete minimal surfaces and the Plateau problem is the existence of compact complete minimal immersions in $\mathbb{R}^3$, according to the following definition (see [AN, A]).

**Definition 1.** By a compact minimal immersion we mean a minimal immersion $X : M \to \mathbb{R}^3$, where $M$ is an open region of a compact Riemann surface $M$, and such that $X$ can be extended to a continuous map $X : M \to \mathbb{R}^3$.

Martín and Nadirashvili [MN] constructed compact complete conformal minimal immersions $X : \mathbb{D} \to \mathbb{R}^3$ such that $X|_{\partial \mathbb{D}}$ is an embedding and $X(S^1)$ is a Jordan curve with Hausdorff dimension 1. Furthermore, they showed that the set of Jordan curves $X(S^1)$ constructed by the above procedure is dense in the space of Jordan curves of $\mathbb{R}^3$ with the Hausdorff distance. After this, the author constructed compact complete minimal immersions of Riemann surfaces of arbitrary finite topology [A]. As in the simply connected case, the set of closed curves given by the limit sets of these immersions is dense in the space of finite families of closed curves in $\mathbb{R}^3$ which admit a solution to the Plateau problem. In spite of these density theorems, there are some requirements for the limit set of a compact complete minimal immersion. Nadirashvili and the author [AN] proved that there is no compact complete proper minimal immersion of the disk into a polyhedron of $\mathbb{R}^3$ (see [N2] for the case of a cube). In fact, given $D \subset \mathbb{R}^3$ a regular domain and $X : \mathbb{D} \to D$ a compact complete proper minimal immersion, then the second fundamental form of the surface $\partial D$ at any point of the limit set of $X$ must be nonnegatively definite [AN].

The aim of the present paper is to join the techniques used in the construction of complete proper minimal surfaces with finite topology in convex domains of $\mathbb{R}^3$, and those used to construct compact complete minimal immersions, in order to prove the following result.

**Theorem.** For any $C$ strictly convex bounded regular domain of $\mathbb{R}^3$, there exist compact complete proper minimal immersions $X : M \to C$ of arbitrary finite topological type. Moreover, for any finite family $\Sigma$ of closed curves in $\partial C$ which admits a solution to the Plateau problem, and for any $\xi > 0$, there exists a minimal immersion $X : M \to C$ in the above conditions and such that $\delta^H(\Sigma, X(\partial M)) < \xi$, where $\delta^H$ means the Hausdorff distance.

Since the existence of solution of the Plateau problem for any Jordan curve, it is deduced that the above result has specially interesting consequences for disks. Given $C$ a strictly convex bounded regular domain of $\mathbb{R}^3$, then any Jordan curve in $\partial C$ can be approximated in terms of the Hausdorff distance by the limit set of a compact complete proper minimal immersion of the disk into $C$ (Subsection 3.3).

Finally, we would like to point out that our result is sharp in the following sense. If we remove the hypothesis of $C$ being strictly convex, then the theorem fails [MMN, AN].
2 Preliminaries and background

Here we briefly summarize the notation and results that we use in the paper.

2.1 Riemann surfaces

Throughout the paper we work on a compact Riemann surface endowed with a Riemannian metric. We consider that the following data are fixed.

**Definition 2.** Let $M'$ be a compact Riemann surface of genus $\sigma \in \mathbb{N} \cup \{0\}$, and $ds^2$ a Riemannian metric in $M'$.

Consider a subset $W \subset M'$, and a Riemannian metric $d\tau^2$ in $W$. Given a curve $\alpha$ in $W$, by $\text{length}_{d\tau}(\alpha)$ we mean the length of $\alpha$ with respect to the metric $d\tau^2$. Moreover, we define:

- $\text{dist}_{(W,d\tau)}(p,q) = \inf\{\text{length}_{d\tau}(\alpha) : [0,1] \to W, \alpha(0) = p, \alpha(1) = q\}$, for any $p, q \in W$.
- $\text{dist}_{(W,d\tau)}(T_1, T_2) = \inf\{\text{dist}_{(W,d\tau)}(p,q) \mid p \in T_1, q \in T_2\}$, for any $T_1, T_2 \subset W$.

We usually work with a domain $W$ in $M'$ and a conformal minimal immersion $Y : \overline{W} \to \mathbb{R}^3$. Then, by $ds^2_Y$ we mean the Riemannian metric induced by $Y$ in $\overline{W}$. Moreover, we write $\text{dist}_{(\overline{W},Y)}(T_1, T_2)$ instead of $\text{dist}_{(\overline{W},ds_Y)}(T_1, T_2)$, for any sets $T_1$ and $T_2$ in $\overline{W}$.

Given $E \in \mathbb{N}$, consider $\mathbb{D}_1, \ldots, \mathbb{D}_E \subset M'$ open disks such that $\{\gamma_i := \partial \mathbb{D}_i\}_{i=1}^E$ are analytic Jordan curves and $\overline{\mathbb{D}_i} \cap \overline{\mathbb{D}_j} = \emptyset$ for all $i \neq j$.

**Definition 3.** Each curve $\gamma_i$ is called a cycle on $M'$ and the family $\mathcal{J} = \{\gamma_1, \ldots, \gamma_E\}$ is called a multicycle on $M'$. We denote by $\text{Int}(\gamma_i)$ the disk $\mathbb{D}_i$, for $i = 1, \ldots, E$. We also define $M(\mathcal{J}) = M' \setminus (\cup_{i=1}^E \text{Int}(\gamma_i))$.

In this setting $M(\mathcal{J})$ is a hyperbolic Riemann surface with genus $\sigma$ and $E$ ends. Most of our immersions will be defined over surfaces constructed in this way.

Given $\mathcal{J} = \{\gamma_1, \ldots, \gamma_E\}$ and $\mathcal{J}' = \{\gamma'_1, \ldots, \gamma'_E\}$ two multicyles on $M'$ we write $\mathcal{J}' \subset \mathcal{J}$ if $\text{Int}(\gamma_i) \subset \text{Int}(\gamma'_i)$ for $i = 1, \ldots, E$. Notice that $\mathcal{J}' \subset \mathcal{J}$ implies $M(\mathcal{J}') \subset M(\mathcal{J})$.

Let $\mathcal{J} = \{\gamma_1, \ldots, \gamma_E\}$ be a multicycle on $M'$. If $\epsilon > 0$ is small enough, we can consider the multicycle $\mathcal{J}' = \{\gamma'_1, \ldots, \gamma'_E\}$, where by $\gamma'_i$ we mean the cycle satisfying $\text{Int}(\gamma'_i) \subset \text{Int}(\gamma_i)$ and $\text{dist}_{(M',ds)}(q, \gamma_i) = \epsilon$ for all $q \in \gamma'_i$. Notice that $\mathcal{J}' \subset \mathcal{J}$.

2.2 Convex domains and Hausdorff distance

Given $E$ a bounded regular convex domain of $\mathbb{R}^3$, and $p \in \partial E$, we let $\kappa_2(p) \geq \kappa_1(p) \geq 0$ denote the principal curvatures of $\partial E$ at $p$ associated to the inward pointing unit normal. Moreover, we write

$$\kappa_1(\partial E) := \min\{\kappa_1(p) \mid p \in \partial E\} \geq 0.$$ 

If $E$ is in addition strictly convex, then $\kappa_1(\partial E) > 0$. Recall that $E$ is strictly convex if, and only if, the principal curvatures of $\partial E$ associated to the inward pointing unit normal are positive everywhere.

If we consider $\mathcal{N} : \partial E \to S^2$ the outward pointing unit normal or Gauss map of $\partial E$, then there exists a constant $a > 0$ (depending on $E$) such that $\partial E_t = \{ p + t \cdot \mathcal{N}(p) \mid p \in \partial E \}$ is a regular (convex) surface $\forall t \in [-a, +\infty[$. We label $E_t$ as the convex domain bounded by $\partial E_t$.  


The set $C^n$ of convex bodies of $\mathbb{R}^n$, i.e., convex compact sets of $\mathbb{R}^n$ with nonempty interior, can be made into a metric space in several geometrically reasonable ways. The Hausdorff metric is particularly convenient and applicable. The natural domain for this metric is the set $K^n$ of the nonempty compact subsets of $\mathbb{R}^n$. Given $C, D \in K^n$, the Hausdorff distance between $C$ and $D$ is defined by

$$\delta^H(C, D) = \max \left\{ \sup_{x \in C} \inf_{y \in D} \| x - y \|, \sup_{y \in D} \inf_{x \in C} \| x - y \| \right\}.$$ 

2.3 Preliminary lemma

Next lemma was proved by Ferrer, Martín and the author [AFM, Lemma 5]. However, its usefulness in the construction of compact complete minimal immersions has been entirely developed and exploited in this paper.

Lemma 1. Let $J$ be a multicyle on $M'$, $X : M(J) \to \mathbb{R}^3$ a conformal minimal immersion, and $p_0 \in M(J)$ with $X(p_0) = 0$. Consider $E$ a strictly convex bounded regular domain, and $E'$ a convex bounded regular domain, with $0 \in E \subset E' \subset E'$. Let $a$ and $\epsilon$ be positive constants satisfying that $p_0 \in M(J_\epsilon)$ and

$$M(J_\epsilon) \cap (E \setminus E' - a).$$

Then, for any $b > 0$ there exist a multicyle $\hat{J}$ and a conformal minimal immersion $\hat{X} : M(\hat{J}) \to \mathbb{R}^3$ with the following properties:

(L1) $\hat{X}(p_0) = 0$.

(L2) $J_\epsilon < \hat{J} < J$.

(L3) $1/\epsilon < \text{dist}_{M(J_\epsilon), \hat{X}}(p, J_\epsilon)$, $\forall p \in \hat{J}$.

(L4) $\hat{X}(\hat{J}) \subset E' \setminus E'_{-b}$.

(L5) $\hat{X}(M(J) \setminus M(J_\epsilon)) \subset \mathbb{R}^3 \setminus E_{-2b - a}$.

(L6) $\| \hat{X} - X \| < \epsilon$ in $M(J_\epsilon)$.

(L7) $\| \hat{X} - X \| < m(a, b, \epsilon, E, E')$ in $M(J)$, where

$$m(a, b, \epsilon, E, E') := \epsilon + \sqrt{\frac{2(\delta^H(E, E') + a + 2b)}{\kappa_1(\partial E)} + (\delta^H(E, E') + a)^2}.$$ 

The above lemma essentially asserts that a minimal surface of finite topology with boundary can be perturbed outside a compact set (see (L6)) in such a way that the intrinsic diameter of the surface grows (see (L3)), and the boundary of the resulting surface achieves the boundary of a prescribed convex domain (see (L4)). Moreover, the deformation keeps the perturbed part of the surface outside another prescribed convex domain (see (L1) and (L5)). Finally, the domain of definition of the perturbed immersion is contained in the one of the original immersion (see
and there is an upper bound for the difference between both immersions in the whole domain of definition of the deformed one (see (L7)). Let us point out that the assumption of $E$ being strictly convex is essential in this lemma, otherwise statement (L7) has no sense.

The proof of Lemma [II] is based on the use of a Runge’s type theorem for compact Riemann surfaces and the López-Ros transformation for minimal surfaces. For a detailed proof of this lemma we refer the reader to [AFM]. We have stated it here just to make this paper self-contained.

3 The main theorem

In this section we prove the main result of this paper and derive some corollaries. From now on $C$ represents a strictly convex bounded regular domain of $\mathbb{R}^3$.

The theorem stated in the introduction trivially follows from the following one.

Theorem 1. Let $C$ be a strictly convex bounded regular domain of $\mathbb{R}^3$. Consider $\mathcal{J}$ a multicycle on the Riemann surface $M'$ and $\phi : \overline{M(\mathcal{J})} \to \overline{C}$ a conformal minimal immersion satisfying $\phi(\mathcal{J}) \subset \partial C$.

Then, for any $\mu > 0$, there exist a domain $M_\mu$ and a complete proper conformal minimal immersion $\phi_\mu : M_\mu \to C$ such that:

(i) $\overline{M(\mathcal{J}_\mu)} \subset M_\mu \subset \overline{M(\mathcal{J})}$, and $M_\mu$ has the topological type of $M(\mathcal{J})$.

(ii) $\phi_\mu$ admits a continuous extension $\Phi_\mu : \overline{M_\mu} \to \overline{C}$ and $\Phi_\mu(\partial M_\mu) \subset \partial C$.

(iii) $\|\phi - \Phi_\mu\| < \mu$ in $\overline{M_\mu}$.

(iv) $\delta H(\phi(\overline{M(\mathcal{J})}), \Phi_\mu(\overline{M_\mu})) < \mu$.

(v) $\delta H(\phi(\mathcal{J}), \Phi_\mu(\partial M_\mu)) < \mu$.

The proof of the above theorem consists roughly of the following. First we look for an exhaustion sequence $\{E^n\}_{n \in \mathbb{N}}$ of strictly convex bounded regular domains covering $C$. Then we use Lemma [I] in a recursive way in order to construct a sequence of minimal immersions $\{X_n\}_{n \in \mathbb{N}}$, starting at $X_1 = \phi$. To construct the immersion $X_{n+1}$ we apply the lemma to the data $X = X_n$, $E = E^n$, $E' = E^{n+1}$ and constants $a = b_n$, $b = b_{n+1}$ and $\epsilon = \epsilon_{n+1}$. These constants and the convex domains $\{E^n\}_{n \in \mathbb{N}}$ are suitably chosen so that the sequence $\{X_n\}_{n \in \mathbb{N}}$ has a limit immersion $\phi_\mu$ which satisfies the conclusion of Theorem [I]. The completeness of $\phi_\mu$ is derived from property (L3) of Lemma [I]. The properness in $C$ follows from (L4) and (L5).

Finally, to guarantee that the immersion $\phi_\mu$ is compact we use property (L7).

3.1 Proof of Theorem [I]

Assume $\mathcal{J} = \{\gamma_1, \ldots, \gamma_6\}$. First of all, we define a positive constant $\varepsilon < \mu/2$. In order to do it, consider $T(\gamma_i)$ a tubular neighborhood of $\gamma_i$ in $\overline{M(\mathcal{J})}$, and denote by $P_i : T(\gamma_i) \to \gamma_i$ the natural projection, $i = 1, \ldots, 6$. Choose $\varepsilon > 0$ small enough so that $\overline{M(\mathcal{J}) \setminus M(\mathcal{J}^\varepsilon)} \subset \cup_{i=1}^6 T(\gamma_i)$, and

$$\|\phi(p) - \phi(P_i(p))\| < \frac{\mu}{2},$$

(2)
for any \( p \in (\overline{M(\mathcal{J}) \setminus M(\mathcal{J}^\varepsilon)}) \cap T(\gamma_i), i = 1, \ldots, E \). This choice is possible since the uniform continuity of \( \phi \). The definition of \( \varepsilon \) is nothing but a trick to obtain statements (iv) and (v) from statement (iii).

Now, let us describe how to define the family \( \{E^n\}_{n\in \mathbb{N}} \) of convex sets. Consider \( t_0 > 0 \) small enough so that, for any \( t \in ]0, t_0[ \),

- \( C_t \) is a well defined strictly convex bounded regular domain.
- \( \Gamma_t := \phi^{-1}((\partial C_t) \cap \phi(M(\mathcal{J}))) \) is a multicycle on \( M' \).

Let \( c_1 \) be a positive constant (which will be specified later) small enough so that

\[
\frac{c_1^2}{n^4} < \min\{t_0, \varepsilon\},
\]

and define, for any natural \( n \),

\[
t_n := c_1^2 \cdot \sum_{k \geq n} \frac{1}{k^4}.
\]  

(3)

Then, \( \forall n \in \mathbb{N} \), we consider the strictly convex bounded regular domain

\[
E^n := C - t_n.
\]  

(4)

Notice that \( E^n \subset E^{n+1}, \forall n \in \mathbb{N} \), and \( \cup_{n\in \mathbb{N}} E^n = C \). Furthermore, from (3) and (4), the Hausdorff distance between \( E^n \) and \( E^{n+1} \) is known. In fact

\[
\delta(H(E^{n-1}, E^n)) = \frac{c_1^2}{n^4}, \quad \forall n \in \mathbb{N}.
\]  

(5)

Finally, consider a decreasing sequence of positives \( \{b_n\}_{n\in \mathbb{N}} \) satisfying

\[
b_1 < 2(t_0 - t_1), \quad \text{and} \quad b_n < \frac{c_1^2}{n^4}.
\]  

(6)

These numbers will take the role of the constants \( a \) and \( b \) of Lemma 1 in the recursive process.

The next step consists of using Lemma 1 to construct, for any \( n \in \mathbb{N} \), a family \( \chi_n = \{\mathcal{J}_n, X_n, \varepsilon_n, \xi_n\} \), where

- \( \mathcal{J}_n \) is a multicycle on \( M' \).
- \( X_n : \overline{M(\mathcal{J}_n)} \to C \) is a conformal minimal immersion.
- \( \{\varepsilon_n\}_{n\in \mathbb{N}} \) and \( \{\xi_n\}_{n\in \mathbb{N}} \) are decreasing sequences of positive real numbers with

\[
\xi_n < \varepsilon_n < \frac{c_1}{n^2}.
\]  

(7)

Moreover, the sequence of families \( \{\chi_n\}_{n\in \mathbb{N}} \) must satisfy the following list of properties:

(A\(_n\)) \( \mathcal{J}^\varepsilon < \mathcal{J}^{\varepsilon_{n-1}}_n < \mathcal{J}^{\varepsilon_n}_{n-1} < \mathcal{J}^{\xi_n}_n < \mathcal{J}_n < \mathcal{J}_{n-1} \).

(B\(_n\)) \( \frac{1}{\varepsilon_n} < \text{dist}_{(M(\mathcal{J}^{\xi_n}_n), X_n)}(\mathcal{J}^{\xi_n}_{n-1}, \mathcal{J}^{\xi_n}_n) \).
(C_n) \[ \| X_n - X_{n-1} \| < \epsilon_n \text{ in } M(J_{n-1}^{\epsilon_n}). \]

(D_n) \[ ds_{X_n} \geq \alpha_n \cdot ds_{X_{n-1}} \text{ in } M(J_{n-1}^{\epsilon_{n-1}}), \]
where the sequence \( \{ \alpha_k \}_{k \in \mathbb{N}} \) is given by
\[ \alpha_1 := \frac{1}{2} e^{1/2}, \quad \alpha_k := e^{-1/2^k} \text{ for } k > 1. \]

Notice that \( 0 < \alpha_k < 1 \) and \( \prod_{m=1}^{k} \alpha_m \) converges to \( 1/2 \).

(E_n) \[ X_n(p) \in E^n \setminus (E^n)_{-b_n}, \text{ for any } p \in J_n. \]

(F_n) \[ X_n(p) \in \mathbb{R}^3 \setminus (E^{n-1})_{-b_{n-1}-2b_n}, \text{ for any } p \in M(J_n) \setminus M(J_{n-1}^{\epsilon_n}). \]

(G_n) \[ \| X_n - X_{n-1} \| < m(b_{n-1}, b_n, \epsilon_n, E^{n-1}, E^n) \text{ in } M(J_n), \]
where \( m \) is the map defined in Lemma 1.

The sequence \( \{ X_n \}_{n \in \mathbb{N}} \) is constructed in a recursive way. To define \( X_1 \), we choose \( X_1 = \phi \) and \( J_1 = \Gamma_{-t_1-b_1/2} \). The first inequality of (6) guarantees that \( J_1 \) is well defined. From this choice we conclude that \( X_1(J_1) \subset \partial C_{-t_1-b_1/2} \subset E^1 \setminus (E^1)_{-b_1}, \) and so property (E_1) holds.

Then, we take \( \epsilon_1 \) and \( \xi_1 \) satisfying (7) and being \( \xi_1 \) small enough so that \( J^\xi < J_1^{\xi_1} \). The remainder properties of the family \( \chi_1 \) do not make sense. The definition of \( \chi_1 \) is done.

Now, assume that we have constructed the families \( \chi_1, \ldots, \chi_n \) satisfying the desired properties. Let us show how to construct \( \chi_{n+1} \). First of all, notice that property (E_n) guarantees the existence of a positive constant \( l \) such that \( X_n(M(J_n) \setminus M(J_{n-1}^{\epsilon_n})) \subset E^n \setminus (E^n)_{-b_n} \). Then, Lemma 1 can be applied to the data
\[ J = J_n, \quad X = X_n, \quad E = E^n, \quad E' = E^{n+1}, \quad a = b_n, \quad \epsilon, \quad b = b_{n+1}, \]
for any \( 0 < \epsilon < l \). Now, consider a sequence of positives \( \{ \epsilon_m \}_{m \in \mathbb{N}} \) decreasing to zero and such that
\[ \xi_m < \min \{ l, \xi_n, \epsilon_1/(n+1)^2 \}, \quad \text{for any } m \in \mathbb{N}. \]

Consider \( I_m \) and \( Y_m : M(I_m) \to \mathbb{R}^3 \) the multicycle and the conformal minimal immersion given by Lemma 1 for the above data and \( \epsilon = \xi_m \). Statement (L2) in Lemma 1 and (8) imply that
\[ J_{n}^{\xi_m} < J_{n+1}^{\xi_m} < I_m, \quad \text{for any } m \in \mathbb{N}. \]

Taking (9) into account, (L6) guarantees that the sequence \( \{ Y_m \}_{m \in \mathbb{N}} \) converges to \( X_n \) uniformly in \( M(J_n^{\xi_n}) \). In particular, the sequence of metrics \( \{ ds_{Y_m} \}_{m \in \mathbb{N}} \) converges to \( ds_{X_n} \) uniformly in \( M(J_n^{\xi_n}) \). Therefore, there exists \( m_0 \in \mathbb{N} \) large enough so that
\[ ds_{Y_{m_0}} \geq \alpha_{n+1} \cdot ds_{X_n} \text{ in } M(J_n^{\xi_n}). \]

Define \( J_{n+1} := I_{m_0}, X_{n+1} := Y_{m_0}, \) and \( \epsilon_{n+1} := \xi_{m_0} \). From (9) and statement (L3) in Lemma 1 we deduce that
\[ 1/\epsilon_{n+1} < \text{dist}(M(J_{n+1}), M(J_{n+1}^{\xi_n})). \]

Then, the above equation and (9) imply the existence of a positive \( \xi_{n+1} \) small enough so that (7), (A_{n+1}) and (B_{n+1}) hold. Properties (C_{n+1}), (D_{n+1}), (E_{n+1}), (F_{n+1}) and (G_{n+1}) follow from (L6), (10), (L4), (L5) and (L7), respectively. The definition of \( \chi_{n+1} \) is done.
In this way the sequence \( \{ \chi_n \} \) satisfying the desired properties has been constructed.

The next step consists of defining the domain of definition of the immersion which satisfies the conclusion of the theorem. Consider

\[
M_\mu := \bigcup_{n \in \mathbb{N}} M(J_n^{\epsilon_n+1}) = \bigcup_{n \in \mathbb{N}} M(J_n^{\epsilon_n}).
\]

Since \((A_n), n \in \mathbb{N}\), the set \(M_\mu\) is an expansive union of domains with the same topological type as \(M(J)\). Therefore, elementary topological arguments give that \(M_\mu\) is a domain with the same topological type as \(M(J)\). Furthermore, \((A_n), n \in \mathbb{N}\), also imply that

\[
\overline{M_\mu} = \bigcap_{n \in \mathbb{N}} \overline{M(J_n)}. \tag{11}
\]

Now we specify the constant \(c_1\). We take \(c_1\) small enough so that

\[
\sum_{n=2}^{\infty} \min(b_{n-1}, b_n, \epsilon_n, E_n^{n-1}, E_n^n) < \epsilon. \tag{12}
\]

Let us show that this choice is possible. Indeed, the strictly convexity of \(C\) guarantees that

\[
\kappa_1(\partial E^{n-1}) > \kappa_1(\partial C), \quad \forall n \geq 2. \tag{13}
\]

Then, taking into account equation (5) and inequalities (6), (7) and (13) we conclude that

\[
\min(b_n, b_{n+1}, \epsilon_{n+1}, E_n^n, E_{n+1}^{n+1}) < \frac{c_1}{n^2} \left( 1 + 2 \sqrt{\frac{c_1^2}{n^4} + \frac{2}{\kappa_1(\partial C)}} \right) < c_1 \cdot \frac{c}{n^2}, \quad \forall n \in \mathbb{N},
\]

where \(c\) is a constant which depends only on \(C\). Therefore, it can be assumed that \(c_1\) was chosen small enough so that (12) holds.

Finally we define the desired immersion \(\phi_\mu\). Taking into account (11), (12) and properties \((G_n), n \in \mathbb{N}\), we infer that \(\{ X_n \} \) is a Cauchy sequence uniformly in \(M_\mu\) of continuous maps. Hence, it converges to a continuous map \(\Phi_\mu : \overline{M_\mu} \to \mathbb{R}^3\). Define \(\phi_\mu := (\Phi_\mu)|_{M_\mu} : M_\mu \to \mathbb{R}^3\).

Let us check that \(\phi_\mu\) satisfies the conclusion of the theorem.

- Properties \((D_n), n \in \mathbb{N}\), guarantee that \(\phi_\mu\) is a conformal minimal immersion.
- The completeness of \(\phi_\mu\) follows from properties \((B_n), (D_n), n \in \mathbb{N}\), and the fact that the sequence \(\{ 1/\epsilon_n \} \) diverges.
- The properness of \(\phi_\mu\) in \(C\) is equivalent to the fact that \(\Phi_\mu(\partial M_\mu) \subseteq \partial C\). Let us check it. Consider \(p \in \partial M_\mu\). For any \(n \in \mathbb{N}\), let \(p_n\) be a point in \(M(J_n^{\epsilon_n})\) such that the sequence \(\{ p_n \} \) converges to \(p\). This sequence of points trivially exists since the definition of \(M_\mu\). Fix \(k \in \mathbb{N}\). Then, the convex hull property for minimal surfaces and \((E_n)\) imply that \(X_n(p_k) \in E_n\), for any \(n \geq k\). Taking limits as \(n \to \infty\) we obtain that \(\Phi_\mu(p_k) \in \overline{C}\). Now, taking limits as \(k \to \infty\), we conclude that \(\Phi_\mu(p) \in \overline{C}\). On the other hand, \(p \in \partial M_\mu \subseteq \overline{M(J_n^{\epsilon_n})} \setminus M(J_n^{\epsilon_n}), \forall n \in \mathbb{N}\). Again, fix \(k \in \mathbb{N}\). Properties \((F_n), n \in \mathbb{N}\), imply that \(X_n(p) \in \mathbb{R}^3 \setminus (E^{k-1})_{-b_{k-1}-2b_k}\), for any \(n > k\). Taking limits as \(n \to \infty\) we have that \(\Phi_\mu(p) \in \overline{C} \setminus (E^{k-1})_{-b_{k-1}-2b_k}\). Therefore, \(\Phi_\mu(p) \in \overline{C} \setminus \overline{(E^{k-1})_{-b_{k-1}-2b_k}}\).
- Statement (i) follows from \((A_n), n \in \mathbb{N}\).
- Statement (ii) trivially holds.
Taking into account (11), (12) and properties (Gₙ), n ∈ \mathbb{N}, we conclude that

\[ \|\phi - \Phi\|_\mu < \varepsilon \text{ in } \overline{M\mu}. \]  

(14)

This inequality implies statement (iii).

- Inequality (2) implies that \( \delta^H(\phi(M(J^\varepsilon)), \phi(M(J))) < \mu/2 \). Then, to prove statement (iv) we use (14), the fact that \( M(J^\varepsilon) \subset \overline{M\mu} \subset M(J) \) and the above inequality in the following way:

\[
\delta^H(\Phi\mu(\overline{M\mu}), \phi(M(J))) < \varepsilon + \delta^H(\phi(M(J^\varepsilon)), \phi(M(J))) < \varepsilon + \frac{\mu}{2} < \mu.
\]

- Finally, let us check statement (v). Consider \( p \in \partial M\mu \). Let \( i \in \{1, \ldots, E\} \) such that \( p \in T(\gamma_i) \) and label \( q = P_i(p) \in J \). Then

\[
\|\Phi\mu(p) - \phi(q)\| < \|\Phi\mu(p) - \phi(p)\| + \|\phi(p) - \phi(q)\| < \varepsilon + \frac{\mu}{2} < \mu,
\]

where we have used (14) and (2). On the other hand, given \( q \in J \) we can find a point \( p \in \partial M\mu \) such that \( q = P_i(p) \) for some \( i \in \{1, \ldots, E\} \). The above computation gives \( \|\Phi\mu(p) - \phi(q)\| < \mu \).

In this way we have proved (v). The proof of Theorem 1 is done.

### 3.2 Some consequences of Theorem 1

In this subsection we remark some results that follow straightforwardly from Theorem 1. The first one is a density type theorem. Given \( C \) a strictly convex bounded regular domain of \( \mathbb{R}^3 \), the set of finite families of curves in \( \partial C \) spanned by complete (connected) minimal surfaces is dense in the set of finite families of curves in \( \partial C \) spanned by (connected) minimal surfaces, with the Hausdorff metric.

**Corollary 1.** Let \( \Sigma \) be a finite family of closed curves in \( \partial C \) so that the Plateau problem for \( \Sigma \) admits a solution. Then, for any \( \xi > 0 \), there exist a compact Riemann surface \( \mathcal{M} \), an open domain \( M \subset \mathcal{M} \) and a continuous map \( \Phi : \overline{M} \to \overline{C} \) such that

- \( \Phi|_M : M \to C \) is a conformal complete proper minimal immersion.
- \( \delta^H(\Sigma, \Phi(\partial M)) < \xi. \)

The following result shows that compact complete proper minimal immersions in \( C \) are not rare. Recall that any Riemann surface with finite topology and analytic boundary can be seen as the closure of an open region of a compact Riemann surface [AS].

**Corollary 2.** The family of complete minimal surfaces spanning a set of closed curves in \( \partial C \) is dense in the space of minimal surfaces spanning a finite set of closed curves in \( \partial C \), endowed with the topology of the Hausdorff distance.

Theorem 1 can be seen as an improvement of the following result [AFM, Theorem 3].
Corollary 3. Let $C$ be a strictly convex bounded regular domain of $\mathbb{R}^3$. Consider $\mathcal{J}$ a multicycle on the Riemann surface $M'$ and $\varphi : \overline{M(\mathcal{J})} \to \overline{C}$ a conformal minimal immersion satisfying $\varphi(\mathcal{J}) \subset \partial C$.

Then, for any $\epsilon > 0$, there exists a subdomain $M_\epsilon$ with the same topological type as $M(\mathcal{J})$, $\overline{M(\mathcal{J}^\epsilon)} \subset M_\epsilon \subset \overline{M_\epsilon} \subset M(\mathcal{J})$, and a complete proper conformal minimal immersion $\varphi_\epsilon : M_\epsilon \to C$ so that

$$\|\varphi - \varphi_\epsilon\| < \epsilon, \quad \text{in } M_\epsilon.$$

3.3 The case of the disk

As we remark in the introduction of this paper, any Jordan curve in $\mathbb{R}^3$ can be spanned by a minimal disk. This fact allows us to improve the preceding results in the simply-connected case.

Corollary 4. Let $C$ be a strictly convex bounded regular domain of $\mathbb{R}^3$. For any smooth Jordan curve $\gamma \subset \partial C$ and for any $\epsilon > 0$ there exists a compact complete proper minimal immersion $\phi(\gamma, \epsilon) : D \to C$ such that $\delta^H(\phi(\gamma, \epsilon)(\partial D), \gamma) < \epsilon$.

Now, we can prove the following existence result.

Corollary 5. Let $D$ be a domain of $\mathbb{R}^3$ with boundary. Assume $\partial D$ contains a regular open connected region $A$ such that the mean and Gauss curvatures are positive on $A$. Then, there exists a compact complete proper minimal immersion $\phi : D \to D$.

Proof. Consider $C$ a strictly convex bounded regular domain $C \subset D$ and such that $(\partial C) \cap (\partial D)$ is a non empty open subset of $A$ (see Figure 1). Then, apply Theorem 4 to a Jordan curve $\gamma \subset (\partial C) \cap (\partial D)$ and an $\epsilon > 0$ small enough. Hence, we obtain a compact complete proper minimal immersion $\phi(\gamma, \epsilon) : D \to C \subset D$ whose boundary lies in $A \subset \partial D$. This is the immersion we are looking for.

Figure 1: The domains $D$ and $C$ and the curve $\gamma$.

An interesting consequence follows from Corollary 4 and the following well-known result in convex geometry [MM2, Lemma 4].

Lemma 2. Let $K$ be a connected compact set in $\partial C$, then for every $\nu > 0$ there exists an smooth Jordan curve $\gamma \subset \partial C$ such that $\delta^H(K, \gamma) < \nu$.

Next result essentially asserts that the Jordan curve $\gamma$ can be substituted by an arbitrary compact set in the statement of Corollary 4.
Corollary 6. Let \( C \) be a strictly convex bounded regular domain, and consider a connected compact set \( K \subset \partial C \). Then, for any \( \epsilon > 0 \) there exists a compact complete proper minimal immersion \( \phi_{(K,\epsilon)} : \mathbb{D} \to C \) satisfying that the Hausdorff distance \( \delta_H(\phi_{(K,\epsilon)}(\partial \mathbb{D}), K) < \epsilon \).

Remark 1. If we do not take care of the compactness of the immersions, then the results stated in this subsection are those proved by Martín and Morales in [MM2].

References

[AS] L. V. Ahlfors and L. Sario, *Riemann Surfaces*. Princeton University Press, Princeton, New Jersey (1974).

[A] A. Alarcón, *Compact complete minimal immersions in \( \mathbb{R}^3 \)*. Trans. Amer. Math. Soc. (to appear).

[AFM] A. Alarcón, L. Ferrer and F. Martín, *Density theorems for complete minimal surfaces in \( \mathbb{R}^3 \)*. Geom. Funct. Anal. 18 (1), 1–49 (2008).

[AN] A. Alarcón and N. Nadirashvili, *Limit sets for complete minimal immersions*. Math. Z. 258 (1), 107–113 (2008).

[C] E. Calabi, *Problems in differential geometry*. Ed. S. Kobayashi and J. Ells, Jr., Proceedings of the United States-Japan Seminar in Differential Geometry, Kyoto, Japan, 1965. Nippon Hyoronsha Co., Ltd., Tokyo 170 (1966).

[CM] T. H. Colding and W. P. Minicozzi, *The Calabi-Yau conjectures for embedded surfaces*. Ann. of Math. 167 (1), 211–243 (2008).

[D] J. Douglas, *Solution of the problem of Plateau*. Trans. Amer. Math. Soc. 33 (1), 263–321 (1931).

[FMM] L. Ferrer, F. Martín and W. H. Meeks III, *The existence of proper minimal surfaces of arbitrary topological type*. Preprint.

[MMN] F. Martín, W. H. Meeks III and N. Nadirashvili, *Bounded domains which are universal for minimal surfaces*. Amer. J. Math. 129 (2), 455–461 (2007).

[MM1] F. Martín and S. Morales, *Complete proper minimal surfaces in convex bodies of \( \mathbb{R}^3 \)*. Duke Math. J. 128 (3), 559–593 (2005).

[MM2] F. Martín and S. Morales, *Complete proper minimal surfaces in convex bodies of \( \mathbb{R}^3 \) (II): The behavior of the limit set*. Comment. Math. Helv. 81 (3), 699–725 (2006).

[MN] F. Martín and N. Nadirashvili, *A Jordan curve spanned by a complete minimal surface*. Arch. Ration. Mech. Anal. 184 (2), 285–301 (2007).

[MPR] W. H. Meeks III, J. Pérez and A. Ros, *The embedded Calabi-Yau conjectures for finite genus*. Preprint.

[MR] W. H. Meeks and H. Rosenberg, *The minimal lamination closure theorem*. Duke Math. J. 133 (3), 467–497 (2006).
[N1] N. Nadirashvili, *Hadamard’s and Calabi-Yau’s conjectures on negatively curved and minimal surfaces*. Invent. Math. **126** (3), 457–465 (1996).

[N2] N. Nadirashvili, *An application of potential analysis to minimal surfaces*. Mosc. Math. J. **1** (4), 601–604 (2001).

[R] T. Radó, *On Plateau’s problem*. Ann. of Math. **31** (3), 457–469 (1930).

[Y] S.-T. Yau, *Review of geometry and analysis*. Mathematics: frontiers and perspectives. Amer. Math. Soc., Providence, RI, 353–401 (2000).

**Antonio Alarcón**
Departamento de Matemática Aplicada
Universidad de Murcia
E-30100 Espinardo, Murcia, Spain
e-mail: ant.alarcon@um.es