Some analytical merits of Kummer-Type function associated with Mittag-Leffler parameters

Firas Ghanim\textsuperscript{a} and Hiba Fawzi Al-Janaby\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, College of Science, University of Sharjah, Sharjah, United Arab Emirates; \textsuperscript{b}Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

ABSTRACT
As of late, the study of Fractional Calculus (FC) and Special Functions (SFs) has been interestingly prompted in various realms of mathematics, engineering and sciences. This is due to the considerable demonstrated potential of their applications. Among these SFs, Gamma function and Mittag-Leffler functions are the most renowned and distinguished. Numerous authors continue to study this line. The current analysis attempts to introduce and further examine new modifications of Gamma and Kummer function in terms of Mittag-Leffler functions, respectively. Several attributes and formulations of this new Kummer-type function that include integral representations, Beta transform, Laplace transform, derivative formulas, and recurrence relation are investigated. Furthermore, outcomes of Riemann-Liouville fractional integral and fractional derivative in relation to this newly established Kummer function are also investigated.

ARTICLE HISTORY
Received 24 January 2021
Revised 4 May 2021
Accepted 9 May 2021

KEYWORDS
Confluent hypergeometric function; Riemann-Liouville fractional derivatives and integrals; hypergeometric functions; Mittag-Leffler functions

2010 MATHEMATICS SUBJECT CLASSIFICATIONS
26A33; 33C60; 33E12
Secondary 33E20; 45J05

1. Introduction
During the 20th century, theories of fractional differ-integrals (fractional calculus (FC)) and the related special functions (SFs) have become marvellous tools in developing complex analysis. These theories also appear widely in a variety of disciplines of engineering, mathematics, and physics. FC is one of the outstanding disciplines of applied mathematics discussing the merits and implementations of non-integer (real or complex), order integrals and derivatives. It is a generalized version of classical (integer-order) calculus. This discipline involves left and right differ-integrals (correspondingly to left and right derivatives). The most frequently utilized fractional operators are right and left Caputo fractional derivative operators and right and left Riemann-Liouville integral and derivative operators. Actually, the differ-integral is an operator that involves both integer-order derivatives and integrals as particular cases, which is why several implementations have become popular in the current FC. Particularly, it includes the principles and techniques of resolving differential equations with fractional derivatives of the unknown function, called fractional differential equations (FDEs). The history of FC was initiated at around the same time as when classical calculus was created by Newton and Leibniz in the 17th century. It was first stated in Leibniz’s letter to L’Hospital in 1695, where the idea of semi-derivative was proposed. In other words, the idea of generalizing the derivative principle to non-integer order, especially to the order $1/2$, is included in the correspondence of Leibniz and L’Hospital. FC was structured on original systematic bases by a multitude of mathematicians, for instance, Euler, Lagrange, Abel, Liouville, Riemann, Grünwald, Laplace, Fourier, and others. For more details in chronological order, see (Oldham & Spanier, 1974) and (Podlubny, 1999).

In this context, SFs are specific mathematical functions that are crucial gadgets in advanced calculus and in almost all areas of mathematics. Several famed types of SFs are useful in solving various problems of FDEs. In fact, SFs have a considerable role in FC. As early as 1729, Euler provided the first fundamental function of FC, namely Gamma function, which is a generalized factorial formula that ranges from positive integers to complex values. This function is formulated by associating with a certain
special function called the exponential function $e^z$ (Podlubny, 1999), as:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \, dt, \quad (z \in \mathbb{C}, \Re(z) > 0). \quad (1)$$

Afterwards, Legendre and Euler introduced another important function closely related to the Gamma function, the Beta function (Podlubny, 1999) as:

$$B(x, \omega) = \int_0^1 t^{x-1} (1 - t)^{\omega-1} \, dt, \quad (x, \omega \in \mathbb{C}, \Re(\omega) > 0). \quad (2)$$

Further,

$$B(x, \omega) = \frac{\Gamma(x) \Gamma(\omega)}{\Gamma(x + \omega)}, \quad (x, \omega \in \mathbb{C}, \Im(\omega) > 0). \quad (3)$$

On the other side, corresponding to $\Gamma(z)$, the Pochhammer symbol (rising factorial), denotes $(z)_n$, and is defined (Podlubny, 1999) by

$$(z)_n = \frac{\Gamma(z + n)}{\Gamma(z)}. \quad (4)$$

Since then, the interest in utilizing Gamma function in all of SFs has vastly increased till the present day. The remarkable generalizations of the exponential function in terms of Gamma function, the so-called Mittag-Leffler type functions (M-LTFS), performs an appealing role in the study of FC. More precisely, these functions are principally used to discuss solutions for FDEs by means of the Laplace transform technique. Such generalization motivates the upcoming research to provide more innovative ideas that yield various formulations of M-LTFS and fractional operators, (Bansal, Jolly, Jain, & Kumar, 2019; Choi, Parmar, & Chopra, 2020; Kumar, Singh, & Baleanu, 2018; Parmar, 2015; Rahman, Mubeen, & Nisar, 2020 and Rahman et al., 2019). Furthermore, derivations of physical phenomena of exponential nature could be determined by the physical laws via the M-LTF (power-law), (Bhattar, Mathur, Kumar, Nisar, & Singh, 2020; Djida, Mophou, & Area, 2019 and Saqib, Khan, & Shafie, 2019). Due to successful diverse applications for M-LTFS, correlated with FC, in physics and mathematic allied problems, several researchers prompted a lot of attention to the behavior of M-LTFS and extended their outcomes to the complex domain, (for instance, see Al-Janaby, 2018; Al-Janaby & Ahmad, 2018; Al-Janaby & Darus, 2019 and Nisar, 2019).

The first appearance of Mittag-Leffler function (M-LF) of 1-parameter dates back to Gösta Magnus Mittag-Leffler (Mittag-Leffler, 1903) in 1903. Such series is proposed as:

$$E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + 1)}, \quad (z, \beta \in \mathbb{C}, \Re(\beta) > 0). \quad (5)$$

It is a generalization of $e^z$. For $\beta = 1$ in (5), it coincides with $e^z$. Two years later, Wiman gave the following generalization of M-LF of 2-parameters as:

$$E_{\beta,\eta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta + \eta)}, \quad \Re(\beta) > 0, \Re(\eta) > 0, \Re(\rho) > 0. \quad (6)$$

which is the so-called Wiman function (or MLT-F) (Wiman, 1905a), and (Wiman, 1905b). The initial and lengthy studies focussed on the base merits of the MLT-Fs as entire functions and extended to the theoretical field of pure mathematics. After around three decades, the MLT-Fs implementation period was mostly achieved. In 1930, the authors Hille and Tamarkin, (1930) employed them in resolving the integral equations, namely Abel integral equations. Meanwhile, in 1947, Gross (1947) made use of the MLT-Fs to discuss the creep and relaxation functions. In 1954, by utilizing them, Barrett (1952) became a prominent pioneer in solving fractional differential equations. Then, in the year 1971, Caputo and Mainardi (1971) examined the fractional viscoelasticity using the MLT-Fs. During the continuous research in 1971, Prabhakar (1971) presented the MLT-F of 3-parameters, which is a more general formula of power series (6) and commonly used among FDEs with three or more terms, as:

$$E_{\beta,\eta,\rho}(z) = \sum_{k=0}^{\infty} \frac{(\rho)_k}{\Gamma(k\beta + \eta + 1)} \frac{z^k}{k!}, \quad (z, \beta, \eta, \rho \in \mathbb{C}, \Re(\beta) > 0, \Re(\eta) > 0, \Re(\rho) > 0). \quad (7)$$

In 1995, Kilbas and Saigo (1995) proposed a generalized MLT-F to another 3-parameters, involving a special entire function as:

$$E_{\beta,\eta,\rho}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\beta + \eta + 1)} \frac{z^k}{k!}, \quad (z, \beta, \eta, \rho \in \mathbb{C}, \Re(\beta) > 0, \Re(\eta) > 0, \Re(\rho) > 0). \quad (8)$$

Later, in 2009, Srivastava and Z. Tomovski (2009) introduced a more general MLT-F of 4-parameters as:

$$E_{\beta,\eta,\rho,\eta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\beta + \eta + 1)} \frac{z^k}{k!}, \quad (z, \beta, \eta, \rho \in \mathbb{C}, \Re(\beta) > 0, \Re(\eta) > 0, \Re(\rho) > 0). \quad (9)$$

Other SFs, such as Wright and Kummer (confluent hypergeometric) functions, constructed by Gamma functions, are important in developing this regard. These functions are proposed, respectively, as:

$$\mathcal{W}^\varphi(z,\eta) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\varphi + \eta)} \frac{1}{k!}, \quad (z, \varphi \in \mathbb{C}, \varphi > -1), \quad (10)$$
\[
{_1F_1}(\varphi ; Z) = \sum_{k=0}^{\infty} \frac{(\varphi)_k}{k!} z^k, \quad (Z, \varphi \in \mathbb{C}, \varphi \in \mathbb{C}, Z \in \mathbb{C}) \tag{11}
\]

This Wright function, correlating with the Mittag-Leffler function (6) was first formulated by Wright in 1933 (Wright, 1933). The importance of the Wright function was demonstrated in solving a linear partial fractional differential equation, for instance, fractional diffusion-wave equation (Mainardi, 1996). Furthermore, there are several studies dedicated to employing this function in resolving the partial differential equation of the fractional-order extending the classical diffusion and wave equations (Luchko & Gorenflo, 1998).

Whilst, the Kummer function was presented by Kummer (1837), it was presented as a solution to the second-order linear homogeneous differential equation:
\[
z \frac{d^2 \vartheta}{dz^2} + (q - z) \frac{d \vartheta}{dz} - \vartheta = 0, \tag{12}
\]
where \( \vartheta, q, z \) are unrestricted. This function has a fruitful role in diverse problems in physics. Particularly, it is utilized as a solution to the differential equation for the velocity distribution function of electrons in a high-frequency gas discharge, see (MacDonald, 1949).

In a recent time, Ghanim and Al-Janably (2021a) imposed a new extension of generalized MLT-F and Kummer function of 4-parameters in another formula as:
\[
E_{j, \beta}^{p, q}(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k + \rho + \eta \varpi)}{\Gamma(k + \eta \varpi) k!} \Gamma(j + \rho) \Gamma(j + \beta + \eta \varpi) \varpi^k z^k, \quad (j, \beta, z, \eta \in \mathbb{C}, \min \{ \Re(\rho), \Re(\eta) \} > 0, \ j, \beta > 0, \ \text{and} \ j \leq 1 + \beta). \tag{13}
\]

This function (13) is named Mittag-Leffler-Confluent hypergeometric functions (MLCHF). Note that \( E_{j, \beta}^{p, q}(z) = e^z \) and \( E_{j, \beta}^{1, 0}(z) = E_{j, \beta}(z) \) were written by (5). Further \( E_{j, \beta}^{1, 0}(z) = _1F_1(\varphi; \varphi; z) \) is given by (11).

More recent investigations and outcomes have been provided in the study of MLT-Fs by several mathematicians, for instance, Srivastava, Frasin, and Pescar (2017), Srivastava and Bansal (2017), Nisar (2019), Ghanim and Al-Janably (2020), (2021b), Al-Janably and Darus, (2019), and others.

On the other hand, in recent times, various generalizations and extensions of Gamma and Beta functions have been fruitfully put forward and presented. In 1994, Chaudhry and Zubair (1994) presented the extension of \( \Gamma(\varphi) \) as follows:
\[
\Gamma^\varphi(\varphi) = \int_{1}^{\infty} t^{\varphi - 1} e^{-\sigma t} \, dt, \quad (\Re(\varphi) > 0, \Re(\sigma) > 0). \tag{14}
\]

After that, in 1997, Chaudhry, Qadir, Rafique, and Zubair (1997) investigated the following extension of Beta functions
\[
B^\psi(\psi, \psi) = \int_{0}^{1} t^{\psi - 1} (1 - t)^{\psi - 1} e^{-\tau t} \, dt, \tag{15}
\]
\[
(\Re(\psi) > 0, \Re(\psi) > 0, \Re(\psi) > 0).
\]

They noted that \( \Gamma^\psi(\psi) = \Gamma(\psi) \) and \( B^\psi(\psi, \psi) = B(\psi, \psi) \). In 2005, following a different methodology, Diaz and Teruel (2009) posed k-Gamma and k-Beta functions as a general formula for Gamma and Beta functions, respectively. They also studied a number of their merits. While, in 2007 and 2010, they discussed k-hypergeometric functions in terms of Pochhammer k-symbols for factorial functions (Diaz, Ortiz, & Pariguan, 2010), and (Diaz & Pariguan, 2007), respectively. Since then, these studies have attracted great interest by various investigators, Kokologiannaki (2010), Krasniqi (2010), Mansour (2009) and Merovci (2010) introduced and posed the scope of k-Gamma and k-Beta functions. Pursuing this type of study, in 2011, Özergin, Özarslan, and Altun (2011) presented and examined a new extension of Gamma, Beta associated with hypergeometric functions as:
\[
\Gamma^\varphi(\psi, \varphi) = \int_{0}^{\infty} t^{\psi - 1} F_1 \left( \psi; \psi; -t + \frac{\sigma t}{1 + t} \right) \, dt \tag{16}
\]
\[
(\Re(\psi) > 0, \Re(\psi) > 0, \Re(\psi) > 0, \Re(\psi) > 0).
\]

They also studied new generalized Gauss hypergeometric function and confluent hypergeometric function. Moreover, they discussed some integral representations, differentiation properties, recurrence relations and summation formulas for these new generalized functions. In 2014, Srivastava, Çetinkaya, and Onur Kıymaz (2014) examined and introduced a certain generalized Pochhammer symbol along with its implementations based on hypergeometric function. These studies were followed by other complex analysts, who contributed to highlighting numerous new facets of this theme such as, Agarwal, Nieto, and Luo, (2017), Özarslan and Ustaoglu (2019) and Rahman et al. (2020). The implementation of fractional calculus in the physical model has succeeded in recent decades, the generalized M-LTFs were also used in mathematical and physical issues, as the solutions of the fractional integral and differential equations were naturally presented. Fractional order calculus is associated with practical endeavours, and it is widely used in nanotechnology (Baleanu, Guvenc, & Tenreiro Machado, 2010), chaos theory (Baleanu, Wu, & Zeng, 2017), optics (Esen, Sulaiman, Bulut, & Baskonus, 2018), human diseases (Veeresha, Prakash, & Baskonus, 2019) and other fields (Prakash, Veeresha, & Baskonus, 2019 and Taneco-Hernández et al., 2019). In fact, the authors are collaborating with a
group of college of engineering researchers on several recent engineering applications involving generalized multi-parameter Mittag-Leffler functions and their extended types, such as noise measurement and heat transfer in asphalt concrete.

Consequently, considering the aforementioned earlier works, this paper imposes new modifications of Gamma and Kummer functions based on Mittag-Leffler function, respectively. Several merits and formulations for this new generalized Kummer-type function which include integral representation, Beta transform, Laplace transform, derivative formulas, and recurrence relation are investigated. Moreover, outcomes associated with Riemann-Liouville fractional integral and fractional derivative for this considered generalization are also discussed.

2. Modified Gamma function

This section presents a new Gamma-type function based on MLT-F given by (13), namely the Gamma Mittag-Leffler function. It is an extension of the classical Gamma function written by (1). This function plays an important role in introducing the new Pochhammer symbol as well as the Kummer-type function in the next section.

In terms of the MLT-F given in (13), we define a new Gamma function as:

\[
\Gamma_{\beta, \eta}^{\rho}(x) = \int_{0}^{\infty} t^{x-1} e^{-t} (-\beta)^{-x} dt,
\]

\[
(\beta, \eta \in \mathbb{C}, \Re(x) > 0, \Re(j) > 0, \Re(\eta) > 0).
\]

(17)

It is called Gamma Mittag-Leffler function. Notice that \( \Gamma_{1,1,1}^{\rho}(x) = \Gamma(x) \).

Theorem 2.1. Let \( j, \beta, \rho, \eta \in \mathbb{C}, \Re(x) > 0, \Re(j) > 0, \Re(\beta) > 0, \Re(\rho) > 0, \Re(\eta) > 0 \). Then the integral representation for the product of two Gamma Mittag-Leffler functions (17) is:

\[
\Gamma_{\beta, \eta}^{\rho}(x) \Gamma_{\beta', \eta'}^{\rho'}(y) = 4 \int_{0}^{\infty} 2^{(x+y)} \cos^{2x-1} \theta \sin^{2\beta-1} \theta \times \mathcal{P}_{\beta, \eta}^{\rho}(-\theta^2) dt \times \mathcal{P}_{\beta', \eta'}^{\rho'}(-\theta^2) dt.
\]

(18)

Proof. Considering \( t = \lambda^2 \) and \( dt = 2\lambda \ d\lambda \) in (17), we gain

\[
\Gamma_{\beta, \eta}^{\rho}(x) = 2 \int_{0}^{\infty} \lambda^{2x-1} e^{-\lambda^2} (-\beta) \ d\lambda.
\]

(19)

Again, letting \( t = \zeta^2 \) and \( dt = 2\zeta \ d\zeta \) in (17), we get

\[
\Gamma_{\beta, \eta}^{\rho}(x) = 2 \int_{0}^{\infty} \zeta^{2x-1} e^{-\zeta^2} (-\beta) \ d\zeta.
\]

(20)

Therefore,

\[
\Gamma_{\beta, \eta}^{\rho}(x) \Gamma_{\beta', \eta'}^{\rho'}(y) = 4 \int_{0}^{\infty} 2^{2x-1} \zeta^{2x-1} e^{-\zeta^2} (-\beta) \ d\lambda \times \mathcal{P}_{\beta, \eta}^{\rho}(-\zeta^2) \ d\zeta.
\]

(21)

Setting \( \lambda = r \cos \theta \) and \( \zeta = r \sin \theta \) in (21), we acquire the desired outcome (18).

3. Proposed Kummer-type function

\( \mathcal{Q}_{\rho, \eta, \xi}^{\mu} (z) \)

This section imposes a fractional Pochhammer-Mittag-Leffler symbol, which is a general Pochhammer symbol (4). Moreover, it poses and discusses a significant special function called Kummer-type function in relation to this new Pochhammer symbol.

Corresponding to (17), we introduce a new general Pochhammer-type symbol as:

\[
(\mu; j, \rho, \beta, \eta)_{x} = \frac{\Gamma_{\beta, \eta}^{\rho}(x)}{\Gamma(\mu)},
\]

\[
(\mu, j, \rho, \beta, \eta, \nu, \zeta) \in \mathbb{C}, \Re(j) > 0, \Re(\rho) > 0, \Re(\beta) > 0, \Re(\eta) > 0, \Re(x) > 0, \Re(\mu + \nu) > 0).
\]

(22)

Evidently, \((\mu; j, \rho, \beta, \eta)_{x} = (\mu; 1, 1, 1)_{x} = (\mu)_{x}\) and \((\mu; j, \rho, \beta, \eta)_{x} = (\mu; 1, 1, 1, 1)_{x} = (\mu)_{x}^{\prime}\).

In addition, the formula (22) achieves the following relations:

\[
(\mu + \nu; j, \rho, \beta, \eta)_{x+1} = (\mu + \nu)(\mu + \nu + 1; j, \rho, \beta, \eta)_{x},
\]

(23)

and

\[
(\mu + \ell; j, \rho, \beta, \eta)_{x+1} = (\mu + \ell)(\mu + \ell + 1; j, \rho, \beta, \eta)_{x}, \quad (\ell \in \mathbb{N}).
\]

(24)

Subsequently, we propose a modified identity special function as:

\[
I_{\beta, \eta}^{\rho, \psi}(z) = \sum_{k=0}^{\infty} \frac{(\mu; j, \rho, \beta, \eta)_{2k}}{k!} z^{2k},
\]

\[
(\mu, j, \rho, \beta, \eta, \zeta) \in \mathbb{C}, \Re(j) > 0, \Re(\rho) > 0, \Re(\beta) > 0, \Re(\eta) > 0).
\]

(25)

Clearly, we may verify that

\[
I_{\beta, \eta}^{\rho, \psi}(z) = I_{1,1,1}^{\mu, \nu}(z) = \sum_{k=0}^{\infty} \frac{(\mu)_{2k}}{k!} z^{2k} = (1 - z)^{-\mu}, \quad (\mu, \zeta) \in \mathbb{C}, \quad |z| < 1,
\]

\[
I_{\beta, \eta}^{\rho, \psi}(z) = I_{1,1,1}^{\mu, \nu}(z) = \sum_{k=0}^{\infty} \frac{(\mu)_{2k}}{k!} z^{2k} = (1 - z)^{-\mu}, \quad (\mu, \zeta) \in \mathbb{C}, \quad |z| < 1
\]

and
\[ I_{\mu, \beta}^1(z) = I_{\mu, \beta}^{1,1}(z) = \sum_{k=0}^{\infty} z^k = (1 - z)^{-1}, \quad (|z| < 1). \]

By employing (22), we present a new general special function and the well-known Kummer-type function as:

\[
Q_{\mu, \beta}^{1} (z) = \sum_{k=0}^{\infty} \frac{(\mu; \beta, \eta)_{2k+1}}{(\eta)_{k+1}(\beta)_{k+1}} z^k, \quad (\mu, \beta, \eta, \xi, \zeta \in \mathbb{C}, \Re(\beta) > 0, \\
\Re(\eta) > 0, \Re(\xi) > 0, \varphi \in \mathbb{C} Z_0^\ast). \tag{26}
\]

The absence of parameters \(\mu, \beta, \eta\) and \(\xi\) has yielded the following analytic function:

\[
Q_{\mu, \beta}^{1} (z) = \Gamma(\eta) \sum_{k=0}^{\infty} \frac{z^k}{(\eta + \xi k)_{k+1}} = \Gamma(\eta) \mathcal{W}(\eta, \xi; z), \quad (\xi \in \mathbb{C}, \Re(\xi) > 0, \varphi \in \mathbb{C} Z_0^\ast) \tag{27}
\]

and \(\mathcal{W}(\eta, \xi; z)\) is the Wright function defined in (Wright, 1933).

**Remark 3.1.** Notice the following interesting specific cases:

- \(Q_{\mu, \beta}^{1,1,1}(z) = Q_{\mu, \beta}^{1,1,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\eta} = e^z\),
- \(Q_{\mu, \beta}^{1,1,1}(z) = Q_{\mu, \beta}^{1,1,1}(z) = \sum_{k=0}^{\infty} \frac{(\mu)_k z^k}{(\eta)_k} = _1F_1(\mu; \eta; z)\), which is the familiar Kummer function given in (11),
- \(Q_{\mu, \beta}^{1,1,1}(z) = Q_{\mu, \beta}^{1,1,1}(z) = \sum_{k=0}^{\infty} \frac{(\mu)_k z^k}{(\eta)_k} = \mathcal{E}_1(\mu; \eta; z)\), which is the renowned generalized Kummer function,
- \(Q_{\mu, \beta}^{1,1,1}(z) = Q_{\mu, \beta}^{1,1,1}(z) = \sum_{k=0}^{\infty} \frac{(\eta)_{k+1} z^k}{(\xi k + \eta)_{k+1}} = \mathcal{E}_1(\mu; \eta; z).

**4. Analytical merits of** \(Q_{\mu, \beta}^{1} (z)\)

In this section, various analytic properties for the generalized Kummer-type function given by (26) that include integral representation, Beta transform, Laplace transform, derivative formulas, and recurrence relation are examined. In addition, some outcomes related to Riemann-Liouville fractional integral and fractional derivative for this new generalization are also studied.

The following outcome presents the integral formula for the function introduced in (26).

**Theorem 4.2.** Let \(\mu, \beta, \eta, \xi, \zeta \in \mathbb{C}, \Re(\beta) > 0, \Re(\eta) > 0, \Re(\xi) > 0, \varphi \in \mathbb{C} Z_0^\ast\). Then, the integral representation formula of (26) is:

\[
Q_{\mu, \beta}^{1} (z) = \frac{\Gamma(\eta)}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} \mathcal{E}_{\beta, \eta}^\alpha (\xi t^\alpha) \mathcal{W}(\eta, \xi; t^\alpha z) \, dt. \tag{28}
\]

**Proof.** Utilizing (22) in (26), we gain

\[
Q_{\mu, \beta}^{1} (z) = \sum_{k=0}^{\infty} \left\{ \frac{\Gamma(\mu)}{\Gamma(\eta)} \int_0^\infty t^{\mu-1} e^{\beta t^\alpha} \, dt \right\} \frac{(t^\alpha z)^k}{k!}.
\]

Interchanging the order of summation and integration, we yield

\[
Q_{\mu, \beta}^{1} (z) = \frac{\Gamma(\eta)}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} e^{\beta t^\alpha} \left( \sum_{k=0}^{\infty} \frac{(t^\alpha z)^k}{k!} \right) dt.
\]

From (26), we attain the desired outcome.

**Corollary 4.3.** Let \(\mu, \beta, \eta, \xi, \zeta \in \mathbb{C}, \Re(\beta) > 0, \Re(\eta) > 0, \Re(\xi) > 0, \varphi \in \mathbb{C} Z_0^\ast\). Then

\[
Q_{\mu, \beta}^{1} (z) = \frac{\Gamma(\eta)}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} \mathcal{W}(\eta, \xi; t^\alpha z) \, dt.
\]

The next outcome yields the property that includes usual integration for function \(Q_{\mu, \beta}^{1} (z)\) given in (26).

**Theorem 4.4.** Let \(\omega, \mu, \beta, \eta, \xi, \zeta \in \mathbb{C}, \Re(\beta) > 0, \Re(\eta) > 0, \Re(\xi) > 0, \varphi \in \mathbb{C} Z_0^\ast\). Then

\[
\int_0^\infty t^{\mu-1} Q_{\mu, \beta}^{1} (z) (\omega t^\alpha) \, dt = \frac{\zeta^\alpha}{\eta} Q_{\mu, \beta}^{1,1,1} (\omega z^\alpha).
\]

**Proof.** Consider

\[
\int_0^\infty t^{\mu-1} Q_{\mu, \beta}^{1} (z) \, dt = \sum_{k=0}^{\infty} \frac{(\mu)_k (\beta, \eta)_{2k+1}}{(\eta)_{k+1} (\beta)_{k+1}} t^{\mu+k-1} \frac{\Gamma(\mu+k)}{\Gamma(\mu)} \int_0^\infty e^{\beta t^\alpha} \frac{(t^\alpha z)^k}{k!} \, dt.
\]

Theorem 4.5. Let \(\mu, \beta, \eta, \xi, \zeta \in \mathbb{C}, \Re(\beta) > 0, \Re(\eta) > 0, \Re(\xi) > 0, \varphi \in \mathbb{C} Z_0^\ast\). Then the Beta transform of (26) is:

\[
B(Q_{\mu, \beta}^{1} (z); \varphi, \eta) = B(\varphi, \eta) Q_{\mu, \beta}^{1} (x),
\]

where \(B(\varphi, \eta)\) is given in (4).
Proof. Applying the principle of Beta transform (Sneddon, 1979) and the function (26), we attain

\[
\begin{align*}
B\{Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(x^2); \psi, \omega \} &= \int_0^1 (1-z)^{\nu-1} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(x^2) \, dz \\
&= \sum_{k=0}^{\infty} \frac{(\mu;\ell, \beta, \eta)_{\omega \xi}}{(\psi)_{\xi k}} \left[ \int_0^1 z^{\nu+k-1} (1-z)^{\nu-1} \, dz \right] \frac{x^k}{k!} \\
&= \frac{\Gamma(\psi) \Gamma(\omega + \alpha)}{\Gamma(\psi + \xi \alpha) \Gamma(\omega + \xi \alpha + \omega) \Gamma(\omega + \xi \alpha + \omega)} \sum_{k=0}^{\infty} \frac{(\mu;\ell, \beta, \eta)_{\omega \xi}}{(\psi)_{\xi k}} x^k \frac{1}{k!}. \\
\end{align*}
\]

(35)

which yields to the desired outcome (35).

Theorem 4.6. Let \( x, \mu, \rho, \beta, \eta, \alpha, \xi \in \mathbb{C}, \mathbb{R}(j) > 0, \mathbb{R}(\rho) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\eta) > 0, \mathbb{R}(x) > 0, \mathbb{R}(\xi) > 0, \psi \in \mathbb{C} Z_0. \) Then, the Laplace transform of (26) is:

\[
\mathcal{L}\{Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(x^2)\} = \Gamma(\psi) \xi^{-\psi} I_{\beta,\mu,\alpha,\beta,\zeta}^{(x)}(x/\xi^2).
\]  

(36)

Proof. Employing the Laplace transform (Sneddon, 1979) and utilizing (26), we acquire

\[
\begin{align*}
B\{Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(x^2)\} &= \int_0^{\infty} e^{-\xi^2} \sum_{k=0}^{\infty} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(x^2) \frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(\mu;\ell, \beta, \eta)_{\omega \xi}}{(\psi)_{\xi k}} \left[ \int_0^{\infty} e^{-\xi^2} z^{\nu+k-1} \, dz \right] \frac{x^k}{k!} \\
&= \frac{\Gamma(\psi) \Gamma(\omega + \alpha)}{\Gamma(\psi + \xi \alpha) \Gamma(\omega + \xi \alpha + \omega)} \sum_{k=0}^{\infty} \frac{(\mu;\ell, \beta, \eta)_{\omega \xi}}{(\psi)_{\xi k}} \frac{x^k}{k!}, \\
(\xi, j, \mu, \rho, \beta, \eta, \alpha, \xi \in \mathbb{C}, \mathbb{R}(j) > 0, \mathbb{R}(\rho) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\eta) > 0, \mathbb{R}(x) > 0, \mathbb{R}(\xi) > 0),
\end{align*}
\]

and we gain the outcome.

The next outcome provides a formal derivative for the new Kummer-type function coined in (26).

Theorem 4.7. Let \( x, \mu, \rho, \beta, \eta, \alpha, \xi \in \mathbb{C}, \mathbb{R}(j) > 0, \mathbb{R}(\rho) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\eta) > 0, \mathbb{R}(x) > 0, \mathbb{R}(\xi) > 0, \psi \in \mathbb{C} Z_0, \tau \in \mathbb{N}_0. \) Then the derivative formula of (26) is:

\[
\frac{d^\tau}{dz^\tau} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z) = \frac{(\mu)_{\tau} \Gamma(\psi)}{\Gamma(\psi + \xi \alpha + \omega)} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z). 
\]  

(38)

Proof. For \( \tau = 0, \) the formula in (38) is gained. The proof of (38) is based on mathematical induction \( \tau \in \mathbb{N}_0. \) For \( \tau = 1, \) in view of (26), we attain

\[
\begin{align*}
\frac{d}{dz} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z) &= \sum_{k=0}^{\infty} \frac{(\mu;\ell, \beta, \eta)_{\omega \xi}}{(\psi)_{\xi k}} \frac{z^k}{k!} \\
&= \sum_{k=0}^{\infty} \frac{(\mu;\ell, \beta, \eta)_{\omega \xi}}{(\psi)_{\xi k}} \frac{z^k}{k!} \\
&= \frac{\mu \Gamma(\psi)}{\Gamma(\psi + \xi \alpha + \omega)} \sum_{k=0}^{\infty} \frac{(\mu + 1;\ell, \beta, \eta)_{\omega \xi}}{(\psi)_{\xi k}} \frac{z^k}{k!} \\
&= \frac{\mu \Gamma(\psi)}{\Gamma(\psi + \xi \alpha + \omega)} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z).
\end{align*}
\]

(39)

By applying (26) in an analogous manner, we conclude

\[
\begin{align*}
\frac{d^2}{dz^2} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z) &= \frac{\mu + 1}{\Gamma(\psi + 2\xi \alpha + \omega)} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z). 
\end{align*}
\]

(40)

Inductively, the desired formula in (38) is gained.

Theorem 4.8. Let \( x, \mu, \rho, \beta, \eta, \alpha, \xi \in \mathbb{C}, \mathbb{R}(j) > 0, \mathbb{R}(\rho) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\eta) > 0, \mathbb{R}(x) > 0, \mathbb{R}(\xi) > 0, \psi \in \mathbb{C} Z_0, \tau \in \mathbb{N}_0. \) Then the derivative formula of (26) is:

\[
\begin{align*}
\frac{d^\tau}{dz^\tau} \left\{ Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z) \right\} &= (\psi - 1) \Gamma(\psi - \tau) (\psi + \xi \alpha + \omega)^{-\tau} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z), \\
\mathbb{R}(\psi - \tau) > 0.
\end{align*}
\]  

(41)

Proof. Utilizing (26) and \( \tau \)-term differentiation, we yield

\[
\begin{align*}
\frac{d^\tau}{dz^\tau} \left\{ Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z) \right\} &= (\psi - 1) \Gamma(\psi - \tau) (\psi + \xi \alpha + \omega)^{-\tau} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z), \\
(\psi - 1) \Gamma(\psi - \tau) (\psi + \xi \alpha + \omega)^{-\tau} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z).
\end{align*}
\]

(42)

By means of the considered definition (26), the following recurrence relation is achieved.

Theorem 4.9. Let \( x, j, \mu, \rho, \beta, \eta, \alpha, \xi \in \mathbb{C}, \mathbb{R}(j) > 0, \mathbb{R}(\rho) > 0, \mathbb{R}(\beta) > 0, \mathbb{R}(\eta) > 0, \mathbb{R}(x) > 0, \mathbb{R}(\xi) > 0, \psi \in \mathbb{C} Z_0, \tau \in \mathbb{N}_0. \) Then the recurrence relation of (26) in:

\[
\begin{align*}
Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z) &= Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z) + \frac{z^\tau}{\psi} \frac{d}{dz} Q_{\nu,\mu,\alpha,\beta,\zeta}^{(x)}(z). 
\end{align*}
\]  

(43)
Proof. By employing (26), we yield
\[
Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(z) + \frac{d}{dz} Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(z) = \sum_{n=0}^{\infty} \left( \frac{\mu}{\gamma} \right)_{n}^{\alpha,\beta,\gamma} z^n \frac{\Gamma(n+\lambda+\beta)}{\Gamma(n+\beta)} = Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(z).
\]
(44)

As a consequence of the outcome 6 and 8, we attain the following interesting recurrence relation for the considered function (26).

Corollary 4.10. Let \( z, \mu, \lambda, \nu, \kappa, \alpha, \beta, \eta \in \mathbb{C}, \mathcal{R}(\mu) > 0, \mathcal{R}(\beta) > 0, \mathcal{R}(\eta) > 0, \varphi \in \mathbb{C} \setminus 0, \tau \in \mathbb{N}_0. \) Then, the recurrence relations of (26) will be:
\[
Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(z) = Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(z; p) + \left( \frac{\mu}{\gamma} \right)_{n}^{\alpha,\beta,\gamma} z^n Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(z).
\]
(45)

The following outcome pertain to the Riemann-Liouville fractional integral along-with the fractional derivative of the Kummer-type function introduced in (22). For this sake, we start by recalling the term of Riemann-Liouville fractional integral and derivative, \( I_{x}^{\alpha} \) and \( D_{x}^{\alpha} \), respectively.
\[
(I_{x}^{\alpha} \phi)(z) = \frac{1}{\Gamma(\alpha)} \int_{0}^{z} (z-t)^{\alpha-1} \phi(t) dt,
\]
(46)
\[
(D_{x}^{\alpha} \phi)(z) = \left( \frac{d}{dz} \right)^{m} (I_{x}^{\alpha-m} \phi)(z)
\]
(47)
where \( \mu \in \mathbb{C}, \mathcal{R}(\mu) > 0, m = \lfloor \mathcal{R}(\mu) \rfloor + 1 \) (Kilbas, Srivastava, & Trujillo, 2006). Furthermore, in (Samko, Kilbas, & Marichev, 1993), the fractional integral achieves the following formula
\[
(I_{x}^{\alpha} \{ (t-\gamma)^{\nu-1} \} ) = \frac{\Gamma(wp)}{\Gamma(\nu+wp)} (z-\gamma)^{wp-1},
\]
(48)

Theorem 4.11. Let \( z, \mu, \lambda, \nu, \kappa, \alpha, \beta, \eta, \varphi, \xi, \in \mathbb{C}, \mathcal{R}(\mu) > 0, \mathcal{R}(\beta) > 0, \mathcal{R}(\eta) > 0, \mathcal{R}(\xi) > 0, \mathcal{R}(\varphi) > 0, \varphi \in \mathbb{C} \setminus 0, \gamma \in \mathbb{N}_0. \) Then
\[
\left( I_{x}^{\alpha} \{ (t-\gamma)^{\nu-1} Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(\eta(t-\gamma)) \} \right)(z) = \frac{\Gamma(\nu)}{\Gamma(\nu+wp)} Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(\eta(z-\gamma)).
\]
(49)

Proof. By utilizing (26) and (46) and employing (48), we attain
\[
\left( I_{x}^{\alpha} \{ (t-\gamma)^{\nu-1} Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(\eta(t-\gamma)) \} \right)(z) = \frac{\Gamma(\nu)}{\Gamma(\nu+wp)} Q_{\alpha,\beta,\gamma}^{\mu,\lambda,\nu}(\eta(z-\gamma)).
\]
(50)

5. Conclusion

In light of the various employments of Fractional Calculus (FC) and Special Functions (SFs) in numerous areas of mathematics and applied science, here, a new extension of Gamma function by means of Mittag-Leffler type functions are introduced in a complex domain. Then, utilizing the extension of the Gamma function, a new formal formula of the Pochhammer symbol is obtained. In addition, an interesting special function named Kummer-type function is presented. Certain related properties connected to this new Gamma, Pochhammer and Kummer functions involving integral representations, Beta transform, Laplace transform, derivative formulas, recurrence relation, and Riemann-Liouville fractional integral and fractional derivative are also investigated. For future research, the new functions and techniques achieved in this paper can be employed to enrich several areas of mathematics that include operator theory, inequalities theory, solving fractional differential equations along with developing a lot of realms of physics immensely.

Disclosure statement

No potential conflict of interest was reported by the author(s).
References

Agarwal, P., Nieto, J. J., & Luo, M.-J. (2017). Extended Riemann-Liouville type fractional derivative operator with applications. Open Mathematics, 15(1), 1667–1681. doi:10.1515/math-2017-0137

Al-Janaby, H. F., & Ahmad, M. Z. (2018). Differential inequalities related to Salagean type integral operator involving extended generalized Mittag-Leffler function. J. Physics: Conference Series, 1132(1), 1–9.

Al-Janaby, H. F., & Darus, M. (2019). Differential subordination results for Mittag-Leffler type functions with bounded turning property. Mathematica Slovaca, 69(3), 573–582. doi:10.1015/ms-2017-0248

Al-Janaby, H. F. (2018). On certain subclass of complex harmonic functions involving a differential operator. J. Adv Research in Dynamical and Control Systems, 10(02), 27–36.

Baleanu, D., Gueven, Z.B., & Tenreiro Machado, J.A. (2010). New trends in nanotechnology and fractional calculus applications. Dordrecht Heidelberg: Springer.

Baleanu, D., Wu, G.C., & Zeng, S.D. (2017). Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations. Chaos, Solitons Fractals, 102, 99–105. doi:10.1016/j.chaos.2017.02.007

Bansal, M. K., Jolly, N., Jain, R., & Kumar, D. (2019). An integral operator involving generalized Mittag-Leffler function and associated fractional calculus results. The Journal of Analysis, 27(3), 727–740. doi:10.1007/s41478-018-0119-0

Barret, J. H. (1952). Differential equations of non-integer order. Canadian Journal of Mathematics, 31, 528–529.

Bhatte, S., Mathur, A., Kumar, D., Nisar, K. S., & Singh, J. (2020). Fractional modified Kawahara equation with Mittag-Leffler law. Chaos, Solitons Fractals, 131, 109508. 2019, doi:10.1016/j.chaos.2019.109508

Caputo, M., & Mainardi, F. (1971). Linear models of dissipation in anelastic solids. La Rivista Del Nuovo Cimento, 1(2), 161–198. doi:10.1007/BF02820620

Chaudhry, M. A., & Zubair, S. M. (1994). Generalized incomplete gamma functions with applications. The Journal of Computational and Applied Mathematics, 55(1), 99–124. doi:10.1016/0377-0427(94)90187-2

Chaudhry, M. A., Qadir, A., Rafique, M., & Zubair, S.M. (1997). Extension of Eulers beta function. The Journal of Computational and Applied Mathematics, 78(1), 19–32. doi:10.1016/S0377-0427(96)00102-1

Choi, J., Parmar, R. K., & Chopra, P. (2020). Extended Mittag-Leffler function and associated fractional calculus operators. Georgian Mathematical Journal, 27(2), 199–209. doi:10.1515/gmj-2019-2030

Diaz, R., & Pariguan, E. (2007). On hypergeometric functions and Pochhammer k-symbol. Divulgaciones Matemáticas, 15, 179–192.

Diaz, R., & Teruel, C. (2995). q, k-Generalized Gamma and Beta functions. Journal of Nonlinear Mathematical Physics, 12, 118–134.

Diaz, R., Ortiz, C., & Pariguan, E. (2010). On the k-Gamma q-distribution. Central European Journal of Mathematics, 8(3), 448–458. doi:10.2478/s11533-010-0029-0

Djida, J. D., Mophou, G., & Aerea, I. (2019). Optimal control of diffusion equation with fractional time derivative with nonlocal and nonsingular Mittag-Leffler kernel. Journal of Optimization Theory and Applications, 182(2), 540–557. doi:10.1007/s10957-018-1305-6

Esen, A., Sulaiman, T.A., Bulut, H., & Baskonus, H.M. (2018). Optical solitons and other solutions to the conformable space-time fractional Fokas-Lenells equation. Optik, 167, 150–156. doi:10.1016/j.ijleo.2018.04.015

Ghanim, F., & Al-Janaby, H. F. (2020). Inclusion and convolution features of univalent meromonic functions correlating with Mittag-Leffler function. Filomat, 34(7), 2141–2150. 202298/IFIL2007141G

Ghanim, F., & Al-Janaby, H. F. (2021a). An analytical study on Mittag-leffler–confluent hypergeometric functions with fractional integral operator. Mathematical Methods in the Applied Sciences, 44(5), 3605–3610. doi:10.1002/mma.6966

Ghanim, F., & Al-Janaby, H. F. (2021b). Fractional Calculus Connections on Mittag-Leffler-Confluent Hypergeometric Functions, preprint.

Gross, B. (1947). On creep and relaxation. Journal of Applied Physics, 18(2), 212–221. doi:10.1063/1.1697606

Hille, E., & Tamarkin, J. D. (1930). On the theory of linear integral equations. The Annals of Mathematics, 31(3), 479–528. doi:10.2307/1968241

Kilbas Ad, A., & Saigo, A. M. (1995). On solution of integral equations of Abel-Volterra type. Differential Integral Equations, 8, 993–1011.

Kilbas, A. A., Srivastava, H. M., & Trujillo, J.J. (2006). Theory and applications of fractional differential equations, North-Holland mathematical studies (vol. 204). Amsterdam: Elsevier (North-Holland) Science Publishers.

Kokologiannaki, C. G. (2010). Properties and inequalities of generalized k-Gamma, Beta and Zeta functions. International Journal of Contemporary Mathematical Sciences, 5, 653–660.

Krasniqi, V. (2010). A limit for the k-Gamma and k-Beta function. International Mathematical Forum, 5, 1613–1617.

Kumar, D., Singh, J., & Baleanu, D. (2018). A new analysis of the Fornberg-Whitham equation pertaining to a fractional derivative with Mittag-Leffler-type kernel. The European Physical Journal Plus, 133, 70. doi:10.1140/epjp/i2018-11934-y

Kummer, E. E. (1837). De integrabilis quibusdam definitis et seriebus infinitis. Journal für die reine und angewandte Mathematik, 17, 228–242.

Luchko, Y., & Gorenflo, R. (1998). Scale-invariant solutions of a partial differential equation of fractional order. Fractional Calculus and Applied Analysis, 1, 63–78.

MacDonald, A. D. (1949). Properties of the confluent hypergeometric function. Journal of Mathematics and Physics, 28(1-4), 183–191. doi:10.2307/jmp.1949.281183

Mainardi, F. (1996). The fundamental solutions for the fractional diffusion-wave equation. Applied Mathematics Letters, 9(6), 23–28. doi:10.1016/0893-9659(96)00089-4

Mansour, M. (2009). Determining the k-Generalized Gamma function $\Gamma_k(x)$ by functional equations. International Journal of Contemporary Mathematical Sciences, 4, 1037–1042.

Merovci, F. (2010). Power product inequalities for the $T_k$ function. International Journal of Mathematical Analysis, 4, 1007–1012.

Mittag-Leffler, G. M. (1903). Sur la nouvelle fonction $E_a(x)$. Comptes Rendus de l’Academie des Sciences 137, 554–558.

Nisar, K. S. (2019). Fractional integrations of a generalized Mittag-Leffler type function and its application. Mathematics, 7(12), 1221–1230. doi:10.3390/math7121230

Oldham, K. B., & Spanier, J. (1974). The fractional calculus; theory and applications of differentiation and integration to arbitrary order. New York: Academic Press,
Çınar, M. A., & Ustaoglu, C. (2019). Some incomplete hypergeometric functions and incomplete Riemann-Liouville fractional integral operators. *Mathematics, 7*(5), 417–483. doi:10.3390/math7050483

Özergin, E., Özarslan, M. A., & Altın, A. (2011). Extention of gamma, beta and hypergeometric functions. *The Journal of Computational and Applied Mathematics, 235*(16), 4461–4601. doi:10.1016/j.cam.2010.04.019

Parmar, R. (2015). A class of extended Mittag-Leffler functions and their properties related to integral transforms and fractional calculus. *Mathematics, 3*(4), 1069–1082. doi:10.3390/math3041069

Podlubny, I. (1999). *Fractional differential equations*. San Diego: Academic Press.

Prabhakar, T. R. (1971). A singular integral equation with a generalized Mittag-Leffler function in the kernel. *Yokohama Mathematical Journal, 19*, 7–15.

Prakasha, D.G., Veeresha, P., & Baskonus, H.M. (2019). Analysis of the dynamics of hepatitis E virus using the Atangana-Baleanu fractional derivative. *The European Physical Journal Plus, 134*(5), 1–11. doi:10.1140/epjp/i2019-12590-5

Rahman, G., Mubeen, S., & Nisar, K. S. (2020). On generalized K-fractional derivative operator. *AIMS Mathematics, 5*(3), 1936–1945. doi:10.3934/math.2020129

Rahman, G., Nisar, K. S., Choi, J., Mubeen, S., & Arshad, M. (2019). Pathway fractional integral formulas involving extended Mittag-Leffler functions in the kernel. *Kyungpook Mathematical Journal, 59*, 125–134.

Saqib, M., Khan, I., & Shafie, S. (2019). New direction of Atangana-Baleanu fractional derivative with Mittag-Leffler kernel for non-Newtonian channel flow, fractional derivatives with Mittag-Leffler Kernel; Book chapter (pp. 253–268). Germany: Springer. doi:10.1007/978-3-030-11662-0

Sneddon, I. N. (1979). *The use of integral transforms*. New Delhi: Tata McGraw-Hill.

Srivastava, H. M., & Tomovski, Z. (2009). Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. *Applied Mathematics and Computation, 217*(1), 198–210. doi:10.1016/j.amc.2009.01.055

Srivastava, H. M., & Bansal, D. (2017). Closed-to-convexity of a certain family of q Mittag-Leffler functions. *Journal of Nonlinear and Variational Analysis, 1*, 6–69.

Srivastava, H.M., Çetinkaya, A., & Onur Kiyaz, İ. (2014). A certain generalized Pochhammer symbol and its applications to hypergeometric functions. *Journal of Applied Mathematics and Computing, 226*, 484–491. doi:10.1016/j.amc.2013.10.032

Srivastava, H. M., Frasin, B. A., & Pescar, V. (2017). Univalence of integral operators involving Mittag-Leffler functions. *Applied Mathematics & Information Sciences, 11*(3), 635–641. doi:10.18576/amis/110301

Taneco-Hernández, M.A., Morales-Delgado, V.F., & Gómez-Aguilar, J.F. (2019). Fractional Kuramoto–Sivashinsky equation with power law and stretched Mittag-Leffler kernel. *Physica A: Statistical Mechanics and Its Applications, 527*, 121085. doi:10.1016/j.physa.2019.121085

Veeresha, P., Prakasha, D.G., & Baskonus, H.M. (2019). Solving smoking epidemic model of fractional order using a modified homotopy analysis transform method. *Mathematical Sciences, 13*(2), 115–128. doi:10.1007/s40096-019-0284-6

Wiman, A. (1905a). Über den fundamentalsatz in der theorie der funktionen E_a(x). *Acta Mathematica, 29*(0), 191–201. doi:10.1007/BF02403202

Wiman, A. (1905b). Über die nullstellen der funktionen E_a(x). *Acta Mathematica, 29*(0), 217–234. doi:10.1007/BF02403204

Wright, E. M. (1933). On the coefficients of power series having exponential singularities. *Journal of the London Mathematical Society, s1-8*(1), 71–79. doi:10.1112/jlms/s1-8.1.71