Fractionalized Excitations Revealed by Entanglement Entropy

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Abstract: Fractionalized excitations are key to many unusual many-body states such as quantum spin liquids (QSLs), disordered phases that cannot be described in terms of any local order parameter. Because these exotic excitations correspond to emergent degrees of freedom, how to probe them and establish their existence is a long-standing challenge. We present a rare mechanism to reveal the fractionalized excitations using real-space entanglement entropy in critical spin liquids that are particularly relevant to experiments. Moreover, we show how to use the entanglement entropy to construct an emergent spinon Fermi surface. Our work opens up a new pathway to establish and characterize exotic excitations in novel quantum phases of matter.
Introduction

A hallmark of strongly correlated systems is that degrees of freedom emerge at low energies out of strong correlations. A prototype case is fractionalized excitations, which are fundamentally different from the excitations in the weakly interacting limit. A particularly intriguing possibility arises in quantum insulating spin systems, when their emergent fermionic excitations form a critical surface in the momentum space, which make the properties of these insulators akin to those of conventional metals. The two-dimensional (2D) triangular lattice-based organic compounds EtMe$_3$Sb[Pd(dmit)$_2$]$_2$ and κ-(ET)$_2$Cu$_2$(CN)$_3$ (1–4) are among the famous candidate materials to host such a critical spin liquid state with an emergent spinon Fermi surface (5, 6) formed by the fractionalized excitations. An outstanding challenge is how to demonstrate and reveal the presence of these fractionalized fermionic/bosonic excitations.

Entanglement entropy (EE) (7), such as the von Neumann entanglement entropy (νEE) and the Renyi entanglement entropy (REE), is obtained from the reduced density matrix of a subsystem by tracing out the degrees of freedom outside it. It plays an important role in several fields, ranging from quantum information to condensed matter physics (8), and has been measured experimentally (9). It is believed that the EE of the ground state in most local Hamiltonians satisfies the so-called “entanglement entropy area-law” (10): When a system is divided into subsystems, the EE is proportional to the area of the boundary between the two subsystems. For gapped topologically ordered phases in 2D (11, 12), besides the leading “area-law” term, the EE contains a universal topological term known as the topological entanglement entropy, which characterizes the topological phases (13, 14).

Violations of the entanglement entropy area-law do exist in various systems. In one dimension (1D), they are found in many quantum critical systems (15). In higher dimensions, the violations are associated with the presence of a critical surface in the momentum space. The
most well known examples are the ground states of free fermions with Fermi surfaces (16, 17). In both cases, the violation is logarithmic; in other words, the EE is proportional to the surface area multiplied by a factor that grows logarithmically with the subsystem size. Intriguingly, the EE in these noninteracting systems takes an explicit formula referred to as the Widom formula (16–18), according to which the coefficient of the leading term in the dependence of EE on the subsystem size captures the geometric information of the Fermi surface and that of the subsystem. In gauge-gravity duality, this type of violation in the EE has also been considered for gauge-charged fermions (19, 20). For gapless electron systems, calculations perturbative in interactions (21) show that such a violation retains the same form as that of a free Fermi gas. Recently, it has been suggested that the EE associated with the composite Fermi liquid phase of the half-filled Landau level ($\nu = 1/2$) is also described by the Widom formula (22). By contrast, for strongly correlated electron or spin systems with a possible emergent Fermi surface, the EE has not been well explored (23, 24).

In this work, we introduce a direct probe of emergent fractionalized excitations using the real-space EE. We demonstrate their existence by probing an emergent Fermi surface formed by these excitations. Specifically, in a quantum spin model we show that the leading order of the EE has the form of an explicit formula, the aforementioned Widom formula multiplied by an additional factor of 2 compared with that of the spinless free fermions. The numerical factor captures the presence of two free gapless modes associated with two flavors of spinons. This formula, in turn, provides the basis (25) to explicitly reconstruct the geometry of the emergent Fermi surface.

**Entanglement Entropy and Widom Formula**

We first provide strong numerical evidence for the validity of the Widom formula in a critical spin liquid with emergent spinon Fermi surface. We begin by considering an isotropic triangular
lattice system. Based on the Widom formula, the REE associated with a subsystem consisting of $L_A \times \alpha L_A$ sites along $\mathbf{a}_1 \equiv (1, 0)$ and $\mathbf{a}_2 \equiv (1/2, \sqrt{3}/2)$ directions (lattice constant $a \equiv 1$), as illustrated in the bottom left part of Fig. 1(a), can be concisely expressed as follows (for further details, see Supplementary Materials)

$$S_2 \approx \frac{c_{\text{eff}}}{8\pi} (1 + \alpha) A_{sf} L_A \ln L_A = \frac{c_{\text{eff}}}{8\pi} (1 + \alpha) |\mathbf{k}_{FR}^\mathbf{n} - \mathbf{k}_{FL}^\mathbf{n}| L_A \ln L_A, \quad \text{(1)}$$

where $\approx$ means the leading logarithmic contribution in REE, $\alpha$ represents the ratio between the linear length of the subsystem ($L_A$) and that of the whole system ($L$), i.e., $\alpha = L_A/L$, $c_{\text{eff}}$ is effectively the number of free gapless modes in the low-energy limit. Additionally, $A_{sf}$ refers to the cross-section of the spinon Fermi surface, which is determined by the span in the momenta between Right/Left moving patches ($\mathbf{k}_{FR/FL}^\mathbf{n}$) of the Fermi surface along any particular observation direction $\mathbf{n}$; this is illustrated in the top right part of Fig. 1(a), where the emergent spinon Fermi surface is expected to be circular.

We have carried out variational Mote Carlo (VMC) simulations, in which the whole system size is fixed to be $L \times L$ with $L$ up to 20. We calculate the REE associated with the subsystem of $L_A \times \alpha L_A$ sites, where both $L_A$ and $\alpha L_A$ are less than or equal to $L/2$. The resulting REE as a function of $L_A$ is illustrated in Fig. 1(b), which shows that $S_2/((1 + \alpha) L_A)$ vs. $\ln L_A$ has the same slope for different choices of $\alpha$ within the error bars. The proportionality $S_2 \sim (1 + \alpha) L_A \ln L_A$ is a direct evidence that the REE of the critical spin liquids satisfies the Widom formula Eq. (1). The slope in Fig. 1(b) gives the value of the combined variable $c_{\text{eff}} A_{sf} = c_{\text{eff}} |\mathbf{k}_{FR}^\mathbf{n} - \mathbf{k}_{FL}^\mathbf{n}|$. In order to pin down the explicit formula for the REE of the critical spin liquids, an additional means is needed for determining the values of $c_{\text{eff}}$ and $A_{sf}$ separately.
Spin Structure Factor

To proceed we calculate the spin structure factor $D_q$, where $D_q \equiv \sum_j \chi^s_j e^{-i\mathbf{q} \cdot \mathbf{r}_j}$ with $\chi^s_j$ being the real-space spin correlation function $\chi^s_j \equiv \sum_{\mu=x,y,z} \langle S^\mu_j S^\mu_i \rangle$. It is known that for an observation direction $\hat{n}$, $D_q$ should show singular peaks at $\mathbf{q} = 0$ and $\mathbf{k}^{\hat{n}}_{FR} - \mathbf{k}^{\hat{n}}_{FL}$, which are associated with forward and backward scattering processes. The information of $D_q$ can be used to determine the cross section of the emergent spinon Fermi surface whose surface unit vector is perpendicular to $\hat{n}$, i.e., $A^{\hat{n}}_sf = |\mathbf{k}^{\hat{n}}_{FR} - \mathbf{k}^{\hat{n}}_{FL}|$. In the isotropic case such as here, $A^{\hat{n}}_sf = A_{sf}$ is independent of direction.

In Fig. 2 we show the numerical data of the spin structure factor obtained on a triangular lattice with $30 \times 30$ sites. Fig. 2(a) gives a three dimensional (3D) side view of the spin structure factor in the Brillouin zone (BZ) illustrated by the black hexagon, where we can see a sharp singular point at $\mathbf{q} = 0$ and the weaker (compared with the sharp singular point at the center) singular lines on the surface whose locations are theoretically suggested to be $\mathbf{q} = \mathbf{k}^{\hat{n}}_{FR} - \mathbf{k}^{\hat{n}}_{FL}$. Fig. 2(b) shows the 3D top view of $D_q$. In the present finite-size calculations, the singular lines on the three-dimensional $D_q$ surface are more clearly revealed near the BZ boundary, while the weaker singular lines inside the BZ are masked by the sharper singular point at $\mathbf{q} = 0$. From Fig. 2(b), we can determine the location of the full singular lines by fitting $\mathbf{k}^{\hat{n}}_{FR} - \mathbf{k}^{\hat{n}}_{FL}$ (25), which, in turn, allows us to extract the cross sections of the emergent spinon Fermi surface to be $5.24 \pm 0.05$. When this value for the cross section is combined with the slopes of the normalized REE vs. $\ln(L_A)$ shown in Fig. 1(b), we obtain $c_{eff} \simeq 2.01 \pm 0.02$. This value indicates the presence of two free gapless modes in the low-energy limit. Further evidence of the validity of the explicit formula for REE of the critical spin liquids is illustrated in the Supplementary Materials. There, we show the results of REE calculations associated with two different subsystem geometries, which again lead us to the same formula. In addition,
in Supplementary Materials we present the results for νEE of the free fermions, whose Fermi surface size is fixed to be equal to that of the spinon Fermi surface obtained above, using the correlation function method (26). Since the sizes of the two critical surfaces are equal, the quantitative difference in EE indicates the different number of the gapless free modes in the low-energy limit.

We have thus demonstrated the existence of two free gapless modes for each pair of patches of the emergent spinon Fermi surface, and provided evidence that the REE of critical spin liquids with an emergent spinon Fermi surface obeys an explicit formula Eq. (1) with $c_{eff} = 2$. Because $c_{eff}$ corresponds to the number of free gapless modes for each “independent” 1D patch in the low-energy limit (27), it should be universal for all shapes of convex critical surface. If we introduce anisotropy into the system, $c_{eff}$ should remain the same.

Visualizing Emergent Spinon Fermi Surface

The explicit formula for the EE we obtain above can be used to directly reveal the emergent spinon Fermi surface. For an isotropic system, it is relatively straightforward, since the shape of an emergent spinon Fermi surface is circular, and its diameter can be extracted once we calculate the REE. To be more general, we here focus on a triangular lattice system with anisotropy. To be concrete, we consider a Gutzwiller-projected wave function with hopping amplitudes $t$ along each ladder ($±a_1$ directions) and $t'$ along the zigzag directions ($±a_2$ and $±(a_1 − a_2)$) that couple different ladders as illustrated in the bottom right part in Fig. 1(a). For an illustration, we take $t'/t = 0.7$ to obtain REE associated with the subsystems. Due to numerical and computational time limitations, below we choose three subsystem geometries to obtain the REE and thereby construct the anisotropic spinon Fermi surface.

We calculate the REE for a subsystem with $L_A × αL_A$ sites, where we set the aspect ratio $α = 1/2, 1, 2$. The REE results in the anisotropic systems are shown in Fig. 3(a). For the

6
present anisotropic system, the formula for REE becomes (we have set \( c_{eff} = 2 \)):

\[
S_2 = \frac{1}{4\pi} (\alpha A_{a_2} + A_{a_1}) L_A \ln(L_A),
\]

where \( A_{a_{1/2}} \) represents the cross-sections of the spinon Fermi surface projected onto \( a_{1/2} \) axis.

We can now write down three equations, corresponding to \( \alpha = 1/2, 1, \) and \( 2 \), respectively:

(i) \( A_{a_1} + A_{a_2} = 4\pi \times 0.86(1) \), (ii) \( A_{a_1} + A_{a_2}/2 = 4\pi \times 0.61(2) \), and (iii) \( A_{a_1} + 2A_{a_2} = 4\pi \times 1.31(3) \).

We can choose any two out of the three equations to obtain the values of \( A_{a_{1/2}} \). Since there are three choices, we can obtain three numerical approximations for \( A_{a_{1/2}} \), which we average over to reduce the statistical error. We find that \( A_{a_1} \simeq 1.53\pi \) and \( A_{a_2} \simeq 1.89\pi \), based on which the shape of the emergent spinon Fermi surface is constructed as illustrated in Fig. 3(b). The green and the blue lines represent the cross-sections of \( A_{a_1} \) and \( A_{a_2} \). Since there is an inversion center for the emergent surface in the momentum space (25), once we know \( A_{a_2} \), we can draw its inverted partner denoted as \( \tilde{A}_{a_2} \) (brown line) in Fig. 3(b). The dashed lines are perpendicular to \( A_{a_{1/2}} \) and \( \tilde{A}_{a_2} \), respectively. Connecting all the intersections of the dashed lines results in the red hexagonal shape, which gives the leading-order approximation to the shape of the emergent spinon Fermi surface. In principle, we can improve the accuracy of the shape if we perform more (time consuming) REE calculations using different subsystem geometries (25).

For comparison, we also show the shape of the spinon Fermi surface in Fig. 3(b) as the light-gray ellipse, which is obtained by extracting the \( k_{F_R}^\hat{h} - k_{F_L}^\hat{h} \) of the spin structure factor. The exact numerical results of the spin structure factor are shown in Supplementary Materials. The emergent spinon Fermi surface reconstructed from the REE results is well consistent with the light-gray ellipse in Fig. 3(b), which provides an added support for our procedure. We remark that in the cases where it may be difficult (or sometimes impossible) to determine the locations of the \( k_{F_R}^\hat{h} - k_{F_L}^\hat{h} \), the EE probe exhibits particular virtue. From this point of view, the present work builds a foundation for using the EE to probe an emergent Fermi surfaces.
Conclusion and Outlook

We have examined the entanglement property of a critical spin liquid state with an emergent spinon Fermi surface. We have proved a generalized Widom formula Eq. (1) (for further details, see Supplementary Materials) for the strongly correlated systems with an emergent Fermi surface, such as the critical spin liquids. Based on this formula, we can reveal and construct the shape/size of the emergent Fermi surface. This has allowed, for the first time, a direct visualization of emergent fractionalized excitations.

The current work can be straightforwardly generalized to critical spin liquid states of higher-spin ($S \geq 1$) systems. Of particular interest is the 6H-B phase of $S = 1$ Ba$_3$NiSb$_2$O$_9$ (28, 29) that was recently suggested to realize the critical spin liquid with three flavors of fermionic spinons, forming a large spinon Fermi surface (30). Although the energetics of the $S = 1$ critical spin liquid state have not been addressed, the state can be straightforwardly written down as the Gutzwiller-projected wave function of three flavors of fermions. Based on the results of the current work, we conjecture that the leading entanglement entropy in this case also satisfies the Widom formula, but with $c_{eff} = 3$. More generally, our work points to the exciting prospect of probing emergent excitations of systems characterized by interplay of multiplied degrees of freedom. Heavy fermion systems is one such example, involving strongly interacting $f$-electrons, which appear in the form of localized quantum spins, and more weakly correlated $s/p/d$ conduction electrons. Here, the nature of the Fermi surface has been advanced to play a crucial role in the understanding of beyond-Landau quantum criticality (31). For this and a variety of other strongly correlated systems, examining the entanglement entropy promises to be a conceptually new way of uncovering and visualizing the emergent low-energy degrees of freedom, which are the building blocks crucial for such spectacular phenomena as high-temperature superconductivity.
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Supplementary Materials

Materials and Methods
Supplementary Text
Figs. S1 to S4
References (S1-S12)
Figure 1: The triangular lattice with subsystem geometries and the REE for isotropic case. (a) Illustration of the subsystem geometries on a triangular lattice that we consider for obtaining the REE. In the simplest case, we consider the subsystem, illustrated in the left bottom part, consisting of $L_A \times \alpha L_A$ sites along $\mathbf{a}_1 = (1, 0)$ and $\mathbf{a}_2 = (1/2, \sqrt{3}/2)$ directions. The top right part illustrates the emergent spinon Fermi surface. For an observation $\hat{n}$, there are two Fermi patches perpendicular to $\hat{n}$ that define momenta $k_{FR/L}^n$ for the Right/Left Fermi patches. For an emergent surface with an inversion center, which is natural for a system with time-reversal symmetry or inversion symmetry, the length of the difference between $k_{FR}^n$ and $k_{FL}^n$ gives its cross-section along $\hat{n}$. (b) REE, $S_2$, for different subsystem geometries for the isotropic Gutzwiller-projected wave function. Based on Eq. (1), we plot $S_2/((1 + \alpha)L_A)$ vs $\ln(L_A)$. The slopes of the lines give the prefactor of the leading term of the REE. We fix the whole system size and choose the subsystem size to be $L_A \times \alpha L_A$ with $\alpha = 1$ and $1/2$. The data on $L = 18$ with $\alpha = 1$ (red squares) are consistent with Ref. (24). We can clearly see the proportionality $S_2 \propto (1 + \alpha)L_A \ln L_A$. 
Figure 2: The 3D plot of spin structure factor for the isotropic case. (a) Side view of the spin structure factor within BZ for the isotropic case, where black hexagon represents the BZ. There is a sharp and singular peak at $q = 0$, which corresponds to the uniform real-space power-law decaying behavior. The much weaker singular lines near the boundary of the BZ correspond to the oscillating real-space behavior caused by the presence of the spinon Fermi surface. (b) Top view of the spin structure factor. The blue lines are obtained by examining the spin structure factor of the mean-field spinon Fermi-surface wavefunction. The theoretical blue lines are seen to match well with the singular lines obtained by exact numerical calculations on a triangular lattice consisting of $30 \times 30$ sites.
Figure 3: **The REE and emergent spinon surface for anisotropic case.** (a) Entanglement entropy in the presence of anisotropy along the zigzag bonds. We fix the whole system size to be $18 \times 18$ and choose the subsystem size to be $L_A \times \alpha L_A$ with $\alpha = 1/2$ (blue squares), $1$ (red circles), and $2$ (green diamonds) to extract the REE. (b) Reconstruction of the spinon Fermi surface with an inversion center. Based on the REE results in (a), we can obtain the cross-sections of the spinon Fermi surface projected onto $a_{1/2}$ axis, $A_{a_{1/2}}$ (green and blue lines). Due to the presence of an inversion center, we can draw an inverted partner of $A_{a_2}$ denoted as $\tilde{A}_{a_2}$ (brown line). The dashed lines are perpendicular to $A_{a_{1/2}}$ and $A_{a_2}$, and connecting all the intersections of the dashed lines gives the shape of the spinon Fermi surface in the lowest order. The light-gray ellipse is the spinon Fermi surface obtained through extracting the $k_{FR}^{\hat{n}} - k_{FL}^{\hat{n}}$ of the spin structure factor.
Supplementary Materials for Fractionalized Excitations Revealed by Entanglement Entropy

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1 Heuristic Derivation of the Generalized Widom Formula

In $d$ dimensions we consider a specific real-space partition in which the boundary between the two subsystems is a plane whose normal direction is $\hat{n}_d$. This partition preserves the translational symmetries in $d-1$ dimensions perpendicular to $\hat{n}_d$, and one can perform partial Fourier transformation for all the physical degrees of freedom along these $d-1$ axes, since the momenta $k_1, k_2, \cdots, k_{d-1}$ are good quantum numbers. We thus view the momentum space as consisting of arrays of parallel 1D chains with spacings $\delta k_1, \delta k_2, \cdots, \delta k_{d-1} = 2\pi/L_\perp$, where $L_\perp$ is the linear size of these transverse directions.

Compared with the well-established results in free fermions and coupled harmonic lattice systems with critical surfaces ($S1$–$S4$), we assume that each 1D chain in the momentum space intersecting the critical surfaces (critical points) contributes a 2nd Renyi EE (REE) $(c_{eff}/4) \ln L_\parallel$ [or a von Neumann EE ($\nu$EE) $(c_{eff}/3) \ln L_\parallel$] to the total leading REE ($S5, S6$), where $L_\parallel$ is the linear size of the (smaller) subsystem along $\hat{n}_d$ and $c_{eff}$ represents the effective number of free gapless modes for each 1D chain.

We note that we only consider the universal part of the leading terms in REE ($S5, S6$), i.e., for each 1D chain the leading REE explicitly should be $c_{eff}/4 \ln L_\parallel + c_2$ ($S6$), where $c_2$ is a non-universal constant that we ignore. For free fermions or Fermi liquids (FL) ($S2, S3, S7, S8$), $c_{eff} = c_F = 1$; for coupled harmonic lattice models realizing the lattice version of the Exciton Bose liquid phase (EBL) ($S9$), $c_{EBL}^{eff} = 2$ ($S4$).
For the Gutzwiller-projected wave function in 2D, the \( c_{\text{eff}} \) has not been known. Nevertheless, the leading REE can be obtained by counting total number of chains (in momentum space) intersecting the critical surface, which corresponds to the critical surface cross-sectional area divided by the \((d-1)\) dimensional area spacing between the chains, \( i.e., \left( \frac{2\pi}{L_{\perp}} \right)^{d-1} \). Explicitly, the leading universal part of REE is

\[
S_{DD} = \frac{c_{\text{eff}}}{4} \ln \left( \frac{L_{\parallel}}{2} \right) \left( \frac{L_{\perp}}{2\pi} \right)^{d-1} \int_{\partial\Gamma} \int_{\partial A} \left| d\vec{S}_{\Gamma} \cdot d\vec{S}_{A} \right|, \tag{1}
\]

where \( \hat{=} \) represents the leading contribution. \( L_{\parallel} \) is the linear size of the (smaller) subsystem along \( \hat{n}_d \). The factor of \( \frac{1}{2} \) in the first line is due to the over counting of the cross-section. In second line, we rewrite \( \hat{n}_d \) as real-space partition surface integral (with \( d\vec{S}_{A} \) being the corresponding oriented area element whose direction is along the local normal direction) divided by the partition surface area in \( d-1 \) dimensions, \( 2L_{\perp}^{d-1} \). \( \int_{\partial\Gamma} \) represents the surface integral along the critical surface in momentum space (with \( d\vec{S}_{\Gamma} \) being the corresponding oriented area element). While we arrived at Eq. (2) by considering a special partition, it is actually the correct formula for free fermion state for arbitrary cuts (S2) if we set \( c_{\text{eff}} = 1 \) and replace \( L_{\parallel} \) by the generic linear size of the smaller subsystem. For the Gutzwiller-projected wave function, if we consider an \( L_y \)-legged chain system with infinite length along \( x \)-axis, the total number of gapless modes is \( 2L_y - 1 \). In real 2D system, taking a periodic boundary condition along \( y \) direction, we expect \( c_{\text{eff}} \) for each line in the momentum space to be \( c_{\text{eff}} = 2 - 1/L_y \big|_{L_y \to \infty} \to 2 \).

2 Renyi Entanglement Entropy by Variational Monte Carlo

The wave function for a critical spin liquid with a spinon Fermi surface is defined as

\[
|\psi\rangle = \mathcal{P}_G |\psi_0\rangle, \tag{3}
\]

where \( \mathcal{P}_G = \prod_i (1 - n_i^+ \cdot n_i^-) \) is the Gutzwiller projector, which enforces no double occupation on each site. \( |\psi_0\rangle \) is the ground state of the following mean-field Hamiltonian on the triangular lattice:

\[
\mathcal{H}_{\text{MF}} = \sum_{\langle i,j \rangle, \sigma} t_{ij} c_{i,\sigma}^\dagger c_{j,\sigma} + h.c. \tag{4}
\]

In a system with subsystems \( A \) and \( B \), the Renyi entanglement entropy of order \( n \) in \( A \) is defined as

\[
S_n = \frac{1}{1-n} \log[Tr\rho_B^n], \tag{5}
\]
with reduced density matrix associated with $A$, $\rho_A = \text{Tr}_B |\psi\rangle\langle\psi|$. We will focus on $n = 2$ REE, $S_2$.

To compute $S_2$, as in Ref. (S7) we introduce an identical copy of the original system: We divide the original system into two subsystems $a$ and $b$ and, likewise, the replica into $a'$ and $b'$. The operator $\text{Swap}$ is introduced as $\text{Swap}|a, b\rangle|a', b'\rangle = |a, b\rangle|a', b\rangle$, and the REE is

$$e^{-S_2} = \frac{\langle \Psi | \text{Swap} | \Psi \rangle}{\langle \Psi | \Psi \rangle},$$

with $|\Psi\rangle$ the wave function of the product between the original system and its replica. The operator

$$\langle \text{Swap} \rangle = \sum_{a,b,a',b'} P(a, b, a', b') \frac{\psi(a', b)\psi(a, b')}{\psi(a, b)\psi(a', b')},$$

is calculated according to the weight $P(a, b, a', b') = \frac{|\psi(a, b)|^2 |\psi(a', b')|^2}{\sum_{a,b} |\psi(a, b)|^2 \sum_{a',b'} |\psi(a', b')|^2}$ by variational Monte Carlo (VMC). In this method, we generate the Markov chain for each configuration of both the original system and its replica, and use the Metropolis algorithm (S10) to update the configurations according to the probability distribution $P(a, b, a', b')$.

3 Renyi Entanglement Entropy of the Isotropic Case by Variational Monte Carlo

The results for REE in the isotropic case ($t = t'$), with $t/t'$ representing the fermion hopping amplitudes in the Hamiltonian Eq. (4) (prior to Gutzwiller projection), are shown in Figs. S1. Fig. S1(a) illustrates three different subsystems that we use for calculating the REE. The red and blue regions represent, in general, the subsystem with $L_A \times \alpha L_A$ sites, while the green region represents the subsystem with $L_A \times L$ sites with periodic boundary condition along $a_2$ direction, where $L$ represents the full system length. In Fig. S1(b), the slopes of the lines give the value of $c_{eff} A_{sf}$. The red squares are consistent with the data in Ref. (S7) on a $18 \times 18$ triangular lattice with periodic boundary condition along both $a_1$ and $a_2$ directions. The green diamonds represent our numerical data on a system with $20 \times 20$ sites along $a_1$ and $a_2$ directions. We remark that the consistency between these two results shows the convergence of REE from $L = 18$ to $L = 20$ clusters, and the $L = 18$ cluster is sufficient to illustrate our VMC results. The open purple triangles represent REE associated with the subsystem, which is half of the whole system in our setup, with a periodic boundary condition along $a_2$. Within the error bars, we can see that the slope of the latter is equal to the former obtained using different subsystem geometry. The equality can be explained using the formula, Eq. (2), which we elaborate below.
In the first setup with $\alpha = 1$, the REE is associated with the subsystem with four boundaries with equal surface area $L_A$ on a 2D triangular lattice with $L \times L$ sites. For each surface in the real space, the integral along the real-space surface contributes $2L_A a$ to REE leading to $S^I_2 = c_{eff} a/(4\pi) L_A \ln(L_A)$ based on Eq. (2). On the other hand, if the subsystem only preserves translational symmetry along $a_2$ direction, in the real space there are only two surfaces with surface area $L$. For each boundary the integral gives $2L a_{sf}$ that results in REE as $S^{II}_2 = c_{eff} a_{sf}/(8\pi)L \ln(L_A) = (2\gamma)^{-1} c_{eff} a_{sf}/(4\pi) L_A \ln(L_A)$, where we rewrite $L = L_A/\gamma$. Setting $\gamma = 1/2$ gives $S^I_2 = S^{II}_2$. The consistency between the leading terms of $S^I_2$ and $S^{II}_2$ as shown in Fig. S1(b) suggests the applicability of the Widom formula for the gapless spin liquid with a spinon Fermi surface described by the Gutzwiller-projected wave function.

Yo further illustrate the applicability of the Widom formula to the gapless spin liquid states, we calculate the REE associated with a subsystem with different lengths along $a_1$ and $a_2$ directions, which is chosen to be $L_A \times L_A/2$ in our work. If the Widom formula is applicable, the slopes of $S_2/((+\alpha) L_A)$ vs $\ln(L_A)$ should be the same for different values of $\alpha$. The blue open circles in Fig. S1(b) are the data obtained in $L_A \times 1/2L_A$ subsystems, and we can see that the slope is comparable to the previous results using different subsystem setups, which again suggest the validity of Widom formula in this system.

4 Von Neumann Entanglement Entropy and Renyi Entanglement Entropy for Spinless Free Electrons in the Isotropic Case

For free fermion systems, we use the correlation function method ($SII$) to obtain the von Neumann entanglement entropy ($\nu$EE) and Renyi entanglement entropy (REE) in Figs. S2 and S3. We use two kinds of subsystem setups illustrated in Fig. S1(a): Setup 1: $L_A \times L_A$ subsystem (red region in Fig. S1(a)); Setup 2: $L_A \times L$ subsystem with a periodic boundary along $a_2$ direction (green region in Fig. S1(a)). The results are illustrated in Figs. S2-S3. In Fig. S2, we plot $S^F_\nu/((1 + \alpha)L_A) [S^F_\nu/((1 + \alpha)L_A)]$ vs $\ln(L_A)$, with $\alpha = 1$. In Fig. S3, we plot $\gamma S^F_\nu/L_A (\gamma S^F_\nu/L_A)$ vs $\ln(L_A)$, with $\gamma$ being the ratio between the subsystem length and that of the whole system, $\gamma = L_A/L = 1/2$. The numerical calculations have been performed up to the triangular lattice with $L = 72$. In both cases, we observe stronger oscillating behaviors for the $S_2$ data in the free fermion systems, which makes it difficult to obtain the fitting lines. Focusing on the von Neumann entropy data, $S^F_\nu$, we find that the two different setups give comparable results. If we average the two slopes obtained in these two setups, we get the slope to be $\sim 0.28$. Utilizing the theoretical understandings that $S^F_\nu = 4/3 S^F_2$ (the universal part of the entanglement entropy), we get the theoretical slope for the REE data to be 0.21, which is comparable to the average of the slopes in the two REE data in Figs. S2(b) and S3(b). The slope we obtained in this spineless free fermion system
is half of the value obtained in the Gutzwiller-projected wave function system, which again suggests that $c_{\text{eff}} \simeq 2$.

5 Long Wavelength Analysis of the Spin Structure Factor

For pinning down the exact formula for the leading entanglement entropy of the Gutzwiller-projected wave function describing the gapless spin liquid with a spinon Fermi surface, we can numerically determine $c_{\text{eff}}$ by comparing the result of the spin structure factor $D_q$ and the REE results, where

$$D_q = \sum_j \chi_j^s e^{-iq \cdot r_j},$$

and $\chi_j^s$ is the real-space spin correlation functions defined as

$$\chi_j^s = \sum_{\mu=x,y,z} \langle S_\mu^s j S_\mu^s i \rangle_{\ell=0}.$$

The emergent spinon Fermi surface in the momentum space can be viewed as consisting of patches of critical surfaces which at the low-energy descriptions are independent to each other ($S12$). It is expected that these independent patches contribute equally to REE with the same $c_{\text{eff}}$. To determine $c_{\text{eff}}$, we focus on the isotropic critical surface case, i.e., a circular critical surface. For an arbitrary observing direction $\hat{n}$, we can determine two momenta corresponding to the right-moving patch, $k_{FR}^n$, and the left-moving patch, $k_{FL}^n$, respectively. For an isotropic convex critical surface with an inversion center, the vector $A_n^s \equiv k_{FR}^n - k_{FL}^n$ must pass through the center of the critical surface and, therefore, the cross-section of the isotropic convex critical surface can be determined by the length $A_n^s = |A_n^s| = |k_{FR}^n - k_{FL}^n|$.

The wave vectors $k_{FR}^n - k_{FL}^n$ can be extracted by examining the spin structure factor. The power-law correlations in real space correspond to singularities in momentum space that can be revealed in the spin structure factor. At mean-field level, the power-law behavior can be explicitly determined. In general, the low-energy description dictates that the spin structure factor should show singularities at $q = 0$, $k_{FR}^n - k_{FL}^n$. Identifying $k_{FR}^n - k_{FL}^n$ along different directions can determine the cross-sections of the spinon Fermi surface at different directions. The exact result for the isotropic case is illustrated in Fig. 2 in the main texts, which shows wiggling lines on the surface of the spin structure factor corresponding to the weak singularities of $k_{FR}^n - k_{FL}^n$. To numerically fit the exact $k_{FR}^n - k_{FL}^n$, we assume that the Gutzwiller projection does not dramatically change the locations of the singularities, while only the exponents of the power-law behaviors of the singularities are modified. We then adopt a mean-field fermionic state with a Fermi surface at $1/2$-filling. Extracting $k_{m,FR}^n - k_{m,FL}^n$ of the mean-field ansatz, we find that the $k_{m,FR}^n - k_{m,FL}^n$ can fit the exact $k_{FR}^n - k_{FL}^n$, which indeed suggests that...
the Gutzwiller projection does not change dramatically the geometric information of the spinon Fermi surface in this system.

At mean-field level, the Hamiltonian for an isotropic fermionic tight-binding model without enlargement of unit cells is

$$H_{mf} = \sum_{\alpha} \sum_{\langle \mathbf{r}, \mathbf{r}' \rangle} f_{\alpha}^\dagger (\mathbf{r}) A(\mathbf{r} - \mathbf{r}') f_{\alpha} (\mathbf{r}') ,$$

(10)

where $\langle ... \rangle$ represents nearest-neighbors and $\alpha = 1, 2$ represents the spin flavor of fermions. $f_{\alpha}^\dagger (\mathbf{r}) (f_{\alpha} (\mathbf{r}))$ represents the fermion creation (annihilation) operator with flavor $\alpha$ at location $\mathbf{r}$. We assume translational invariance so that $A_{\mathbf{r} \mathbf{r}'} = A(\mathbf{r} - \mathbf{r}')$, where the matrix $A(\mathbf{r} - \mathbf{r}')$ represents the hopping matrix between sites at $\mathbf{r}$ and $\mathbf{r}'$. The mean-field Hamiltonian can be diagonalized by the complex fermions in Fourier space as

$$f_{\alpha} (\mathbf{r}) = \sqrt{\frac{T}{N_s}} \sum_{\mathbf{k} \in BZ} e^{i \mathbf{k} \cdot \mathbf{r}} f_{\alpha} (\mathbf{k}) ,$$

(11)

where $N_s$ is the number of sites, and the complex fermion field $f$ satisfies the usual anti-commutation relations, $\{ f_{\alpha}^\dagger (\mathbf{k}), f_{\alpha'}^\dagger (\mathbf{k}') \} = \delta_{\alpha\alpha'} \delta_{\mathbf{k}\mathbf{k'}}$. The component of the spin-1/2 spin operator is

$$S_{\mu} (\mathbf{r}) = \sum_{\alpha, \beta = 1, 2} f_{\alpha}^\dagger (\mathbf{r}) \left( \frac{\sigma_{\alpha\beta}^\mu}{2} \right) f_{\beta} (\mathbf{r}) ,$$

(12)

which can be expressed in Fourier space as

$$S_{\mu} (\mathbf{r}) \simeq \sum_{\mathbf{k}, \mathbf{k}' \in B.Z.} \sum_{\alpha, \beta} \frac{\sigma_{\alpha\beta}^\mu}{2N_s} f_{\alpha}^\dagger (\mathbf{k}) f_{\beta} (\mathbf{k}') e^{-i(\mathbf{k} - \mathbf{k}') \cdot \mathbf{r}} .$$

(13)

In order to determine long-distance behavior at separation $\mathbf{r}$, we focus on patches near the Fermi surface where the group velocity is parallel or antiparallel to the observation direction $\hat{\mathbf{n}} = \mathbf{r} / |\mathbf{r}|$, because at large separation $|\mathbf{r}| \gg k_F^{-1}$, the main contributions to the correlations come precisely from such patches. Specifically, we introduce Right(R) and Left(L) Fermi patch fields and the corresponding energies

$$f_{\alpha}^{P, (\hat{\mathbf{n}})} (\delta \mathbf{k}) = f_{\alpha} (\mathbf{k}_F^{(\hat{\mathbf{n}})} + \delta \mathbf{k}) ,$$

(14)

$$\epsilon_{P}^{(\hat{\mathbf{n}})} (\delta \mathbf{k}) = |\mathbf{v}_F^{(\hat{\mathbf{n}})}| \left( P \delta k_\parallel \right) + \frac{C_{P=\pm R/L}^{(\hat{\mathbf{n}})}}{2} \delta k_\perp^2 ,$$

(15)

where the superscript $(\hat{\mathbf{n}})$ refers to the observation direction and $P = R/L = +/- \mathbf{v}_F^{(\hat{\mathbf{n}})}$ is the corresponding group velocity (parallel to $\hat{\mathbf{n}}$ for the Right patch and anti-parallel for the Left patch); $C_{P=\pm R/L}^{(\hat{\mathbf{n}})}$
is the curvature of the Fermi surface at the Right/Left patch; $\delta k_{||}$ and $\delta k_{\perp}$ are respectively components of $\delta k$ parallel and perpendicular to $\hat{n}$. It is convenient to define fields in real space

$$f_{P}^{\alpha}(\hat{n}) (r) \sim \sum_{\delta k \in \text{Fermi Patch}} f_{P}^{\alpha} (\delta k) e^{i\delta k \cdot r},$$

which vary slowly on the scale of the lattice spacing [from now on, we will drop the superscript $(\hat{n})$]. In this long-wavelength analysis, the relevant terms in the spin operator are

$$S^\mu (r) \sim \sum_{P, P', \alpha, \beta} \sigma^\mu_{\alpha\beta} f_{P}^{\alpha} (r) f_{P'}^{\beta} (r) e^{-i(k_{FP} - k_{FP'}) \cdot r}.$$  \hspace{1cm} (17)

The above long-wavelength expression for the $S^\mu$ operator implies that the corresponding correlation function defined in Eq. (9) contains contributions with $q = 0$ and $q = k_{FR} - k_{FL}$. More explicitly, for a patch specified by $\epsilon (\delta k)$ in Eqs. (14)-(15), we can derive the Green’s function for the continuum complex fermion fields as

$$\langle f_{R/L}^{\alpha \dagger} (0) f_{R/L}^{\alpha} (r) \rangle = \frac{\exp[-(\mp i^{3}\pi) \frac{3}{4}]}{2^{3/2} \pi^{3/2} c_{R/L}^{1/2} |r|^{3/2}}.$$ \hspace{1cm} (18)

Using this and Eq. (17), we can obtain the spin correlation

$$\chi_{s} (r) \sim -\frac{1}{c_{R} |r|^3} - \frac{1}{c_{L} |r|^3} + \frac{2 \sin[(k_{FR} - k_{FL}) \cdot r]}{c_{R}^{1/2} c_{L}^{1/2} |r|^3}.$$ \hspace{1cm} (19)

Focusing on the structure factors $D_{q}$ defined in Eq. (8), we expect that there should be a cone-shaped singularity at $q = 0$, based on Eq. (19):

$$D_{q \sim 0} \sim |q|,$$ \hspace{1cm} (21)

which can be seen straightforwardly by performing Fourier transform exactly or by scaling analysis with $q \sim r^{-1}$. Furthermore, the spin structure factor should also reveal the singular surface at $Q_{-}$, as expected from Eq. (20). At the long wavelength analysis at the mean-field level, we note that the singularities are expected to be one-sided,

$$D_{Q_{-} + \delta q} \sim |\delta q||^{3/2} \Theta(-\delta q||).$$ \hspace{1cm} (22)

Fitting the exact $q = k_{FR} - k_{FL}$ in the spin structure factor data illustrated in Fig. 2(b) in the main texts, we extract the cross sections of the emergent spinon Fermi surface in the isotropic case to be the $5.24 \pm 0.05$ (where we set the lattice constant $a$ to be 1 and $\hbar = 1$), which allows us to obtain $c_{eff} \simeq 2.01 \pm 0.02$. 

7
6 Spin Structure Factor of the Anisotropic Case

Fig. S4 shows the 3D spin structure factor obtained on a triangular lattice with $30 \times 30$ sites for anisotropic case $t' = 0.7t$. The spin structure factor is defined as $D_q \equiv \sum_j \chi_j^x e^{-i q \cdot r_j}$ with the real-space spin correlation function $\chi_j^x \equiv \sum_{\mu=x,y,z} \langle S_j^\mu S_i^\mu \rangle$. It is known that for an observation direction $\hat{n}$, $D_q$ should show singular peaks at $q = 0$, $k_{FR}^n - k_{FL}^n$ associated with forward and backward scattering process. Fig. S4(a) gives a side view of the $D_q$ in the hexagonal Brillouin zone (B.Z.) where we can see a sharp singular point at $q = 0$ and the wiggle lines on the surface correspond to the singular lines located at $k_{FR}^n - k_{FL}^n$. Unlike the isotropic case, we can see that the structure factor breaks the $C_6$ rotation. Fig. S4(b) shows the top view of $D_q$. To determine the location of the $k_{FR}^n - k_{FL}^n$, we adopt a mean-field fermionic state with a Fermi surface at 1/2-filling. Extracting $k_{m,FR}^n - k_{m,FL}^n$ of the mean-field ansatz, where the subscript $m$ means mean-field, we find that the $k_{m,FR}^n - k_{m,FL}^n$ can fit the exact $k_{FR}^n - k_{FL}^n$ quite well for an arbitrary observation direction $\hat{n}$, which gives the locations of the weak singular lines on the surface of the 3D spin structure factor. In Fig. S4(b) the blue lines are obtained by examining the mean-field spinon Fermi surface spin liquid state, which overlap the weak singular lines on the surface of the 3D spin structure factor obtained by exact numerical calculations.

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Figure S1: The triangular lattice with subsystem geometries and the REE for isotropic case with two setups. (a) Illustration of the different subsystem geometries on the triangular lattice that we consider for REE calculations. We set $t \equiv 1$, and consider $t'/t = 1.0$ for the isotropic model. The colored regions represent the subsystems that we consider. The blue and red regions represent subsystems with size $L_A \times \alpha L_A$, where $\alpha = 1/2, 1$. The green region represents the subsystem preserving translation along $a_2$ direction. (b) The REE, $S_2$, for different subsystem geometries for the isotropic case. We choose two setups to extract REE. For red squares, green diamonds and blue circles, we fix the whole system size and choose the subsystem size to be $L_A \times \alpha L_A$ with $\alpha = 1/2$ (blue circles), 1 (red squares and green diamonds). For purple triangles, we fix the ratio of the linear size of the subsystem and that of the whole system to be $1/2$ ($L_A/L = 1/2$) and vary the value of $L$ to extract the REE. Based on Eq. (2), we plot $S_2/((1 + \alpha)L_A)$ or $\gamma S_2/L_A$ vs $\ln(L_A)$. The slopes of the lines give the prefactor of the leading REE.
Figure S2: The entanglement entropy of free fermions with the subsystem geometry of Setup 1. The Fermi surface size of free fermions is fixed to be equal to the one obtained from spin structure factor. Here we choose the subsystem with $L_A \times L_A$ lattice sites along $\alpha_1$ and $\alpha_2$ directions and calculate the von Neumann entanglement entropy (a) and the Renyi entanglement entropy (b). (a) We find that the fitting line is $y \simeq 0.278x + 0.355$. (b) We observe a stronger oscillating behavior in Renyi entropy, which makes it hard to give a conclusive fitting line. Based on the current data, we get the fitting line to be $y \simeq 0.194x + 0.331$. 
Figure S3: **The entanglement entropy of free fermions with the subsystem geometry of Setup 2.**

The Fermi surface size of free fermions is fixed to equal to the one determined from spin structure factor result. Here we choose the subsystem to preserve the translation symmetry along $a_2$ direction to obtain von Neumann entanglement entropy (a) and Renyi entanglement entropy (b). In both cases, we find stronger oscillating behaviors compared with those in Fig. S2, but we still can get a consistent result for the von Neumann entanglement entropy. (a) We find the fitting line to be $y = 0.282x + 0.424$, whose slope is comparable to that in Fig. S2(a) obtained in different subsystem geometry. (b) The Renyi entropy shows a very strong oscillating behavior, which makes it very difficult to find a fitting line. A linear fitting line based on the current data is $y = 0.260x + 0.199$. 
Figure S4: The 3D plot of spin structure factor for the anisotropic case. (a) Side view of the spin structure factor within BZ for the anisotropic case $t'/t = 0.7$, where black hexagon represents the BZ. There is a sharp and singular peak at $q = 0$ corresponding to the uniform real-space power-law decaying behavior. The much weaker singular lines near the boundary of the BZ correspond to the oscillating real-space behavior caused by the presence of the spinon Fermi surface. (b) Top view of the spin structure factor. The weak singular lines on the surface are due to the presence of the emergent spinon Fermi surface and the theoretical locations of these lines are at $k_{FR} - k_{FL}$, i.e., for an observation direction we can define a momentum $k_{FR/L}$ associated with right-moving (left-moving) patch, represented by the blue lines. The theoretical blue lines match excellently with the singular lines obtained by numerical calculations on a triangular lattice consisting of $30 \times 30$ sites.