EXPLICIT RELATIONS BETWEEN PRIMES IN SHORT INTERVALS AND EXPONENTIAL SUMS OVER PRIMES

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Abstract: Under the assumption of the Riemann Hypothesis, we prove explicit quantitative relations between hypothetical error terms in the asymptotic formulae for truncated mean-square average of exponential sums over primes and in the mean-square of primes in short intervals. We also remark that such relations are connected with a more precise form of Montgomery’s pair-correlation conjecture.

Keywords: exponential sum over primes, primes in short intervals, pair-correlation conjecture.

1. Introduction

In many circle-method applications a key role is played by the asymptotic behavior as $X \to \infty$ of the truncated mean square of the exponential sum over primes, i.e. by

$$ R(X, \xi) = \int_{-\xi}^{\xi} |S(\alpha) - T(\alpha)|^2 \, d\alpha, \quad \frac{1}{2X} \leq \xi \leq \frac{1}{2}, $$

where $S(\alpha) = \sum_{n \leq X} \Lambda(n)e(n\alpha)$, $T(\alpha) = \sum_{n \leq X} e(n\alpha)$, $e(x) = e^{2\pi ix}$ and $\Lambda(n)$ is the von Mangoldt function. In 2000 the first author and Perelli [6] studied how to connect, under the assumption of the Riemann Hypothesis (RH) and of Montgomery’s pair-correlation conjecture, the behavior as $X \to \infty$ of $R(X, \xi)$ with the one of the mean-square of primes in short intervals, i.e., with

$$ J(X, h) = \int_{1}^{X} (\psi(x + h) - \psi(x) - h)^2 \, dx, \quad 1 \leq h \leq X, $$

where $\psi(x) = \sum_{n \leq x} \Lambda(n)$. Recalling that Goldston and Montgomery [2] proved that the asymptotic behavior of $J(X, h)$ as $X \to \infty$ is related with Montgomery’s pair-correlation function

$$ F(X, T) = 4 \sum_{0 < \gamma, \gamma' \leq T} \frac{X^{i(\gamma - \gamma')}}{4 + (\gamma - \gamma')^2}, $$

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where $\gamma, \gamma'$ run over the imaginary part of the non-trivial zeros of the Riemann zeta-function, the following result was proved in [6].

**Theorem.** Assume RH. As $X \to \infty$, the following statements are equivalent:

(i) for every $\varepsilon > 0$, $R(X, \xi) \sim 2X\xi \log X\xi$ uniformly for $X^{-1/2+\varepsilon} \leq \xi \leq 1/2$;

(ii) for every $\varepsilon > 0$, $J(X, h) \sim hX \log(X/h)$ uniformly for $1 \leq h \leq X^{1/2-\varepsilon}$;

(iii) for every $\varepsilon > 0$ and $A \geq 1$, $F(X, T) \sim (T/2\pi) \log \min(X, T)$ uniformly for $X^{1/2+\varepsilon} \leq T \leq X^A$.

We remark that the uniformity ranges in the previous statement are smaller than the ones in [2]: this is due to the presence of a term $E(X, h)$ which arises from the estimation of some very short integrals naturally arising in applying Gallagher’s lemma. In particular in [6] it is proved, for every fixed $\varepsilon > 0$, that

$$E(X, h) \ll \begin{cases} (h + 1)^3 (\log X)^2 & \text{(uncond.) for } 0 < h \leq X^\varepsilon \\ h^3 & \text{(uncond.) for } X^\varepsilon \leq h \leq X \\ (h + 1)X (\log X)^4 & \text{(under RH) for } 0 < h \leq X. \end{cases} \ (1)$$

Hence it is clear that the above-mentioned limitation in the uniformity ranges comes from the fact that for $h > X^{1/2-\varepsilon}$ the estimates in (1) are too large if compared with the expected main term for $J(X, h)$. In Theorem 1 below we will see that $E(X, h)$ plays an important role here too.

In 2003 Chan [1] formulated a more precise pair-correlation hypothesis and gave explicit results for the connections between the error terms in the asymptotic formulae for $F(X, T)$ and $J(X, h)$. Such results were recently extended and improved by the authors of this paper in a joint work with Perelli [7]: writing

$$F(X, T) = \frac{T}{2\pi} \left( \log \frac{T}{2\pi} - 1 \right) + R_F(X, T), \quad (2)$$

$$J(X, h) = hX \left( \log \frac{X}{h} + c' \right) + R_J(X, h) \quad (3)$$

and

$$c' = -\gamma - \log(2\pi) \quad (4)$$

($\gamma$ is Euler’s constant), they gave explicit relations between (2), (3) and error terms essentially of type

$$R_F(X, T) \ll \frac{T^{1-a}}{(\log T)^b} \quad \text{and} \quad R_J(X, h) \ll \frac{hX}{(\log X)^b} \left( \frac{h}{X} \right)^a,$$

with $X$, $T$ and $h$ in suitable ranges and $a, b \geq 0$. According to the heuristics in Montgomery-Soundararajan [9] (see p.511) it appears that such bounds are both reasonable ones if $0 \leq a \leq 1/2 - \varepsilon$, $b \geq 0 \; \text{and, respectively, uniformly for } T^{1+\varepsilon} \leq X \leq T^A \quad \text{and} \quad X^\varepsilon \leq h \leq X^{1-\varepsilon}$.

Our aim here is to investigate the connections between (2)-(3) with an asymptotic formula of the type

$$R(X, \xi) = 2X\xi \log X\xi + cX\xi + W(X, \xi), \quad (5)$$
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say, where the expected value for $c$ is given by

$$c = 2(c' - 2 + \gamma + \log(2\pi))$$  \hspace{1cm} (6)$$

(which, by (4), gives $c = -4$), and to prove explicit connections between the error terms involved. The heuristics in [9] suggests that a reasonable estimate should be

$$W(X, \xi) \ll \frac{(X\xi)^{1-a}}{\log X \xi^b},$$  \hspace{1cm} (7)$$

with $0 \leq a \leq 1/2 - \varepsilon$, $b \geq 0$, uniformly for $X^{-1+\varepsilon} \leq \xi \leq X^{-\varepsilon}$. Unfortunately the presence of the above-mentioned term $E(X, h)$ forces us, as in [6], to restrict our attention to the range $X^{-1/2+\varepsilon} \leq \xi \leq 1/2$ (or, equivalently, to $1 \leq h \leq X^{1/2-\varepsilon}$).

In what follows the implicit constants may depend on $a, b$. Our first result is

**Theorem 1.** Assume RH and let $1 \leq h \leq X^{1/2-\varepsilon}$, $X^{-1/2+\varepsilon} \leq \xi \leq 1/2$. Let further $0 \leq a < 1$, $b \geq 0$, $(a, b) \neq (0, 0)$ be fixed. If (7) holds uniformly for

$$\frac{1}{h} \left( \frac{h}{X} \right)^a (\log X)^{b-4} \leq \xi \leq \frac{1}{h} \left( \frac{X}{h} \right)^a (\log X)^{b+4},$$  \hspace{1cm} (8)$$

then

$$R_J(X, h) \ll X + E(X, h) + R_{a,b}(X, h)$$

uniformly for

$$X \left( \frac{1}{X\xi(\log X)^{b+4}} \right)^{1/(1-a)} \leq h \leq X \left( \frac{(\log X)^{b+4}}{X\xi} \right)^{1/(a+1)},$$

where $E(X, h)$ is defined in (1), and

$$R_{a,b}(X, h) = \begin{cases} hX \log \log X (\log X)^{-b} & \text{if } a = 0 \\ hX \left( \frac{h}{X} \right)^a (\log X)^{-b} & \text{if } a > 0. \end{cases}$$  \hspace{1cm} (9)$$

We explicitly remark that the conditions $\xi \leq 1/2$ and (8) imply

$$h \gg X^{a/(a+1)}(\log X)^{(b+4)/(a+1)}$$

which also leads to $R_{a,b}(X, h) \gg X$. It is also useful to remark that $E(X, h) \ll R_{a,b}(X, h)$ only for $h \ll X^{(1-a)/(2+a)}(\log X)^{-b/(2+a)}$.

The technique used to prove Theorem 1 is similar to the one in Lemma 2 in [7]; the main difference is in the presence of the terms $E(X, h)$ (which comes from Lemma 3) and $O(X)$ (which comes from the term $O(1)$ in (12)). We further remark that eq. (12) of Lemma 1 is directly connected to the ability of detecting the second order term in (5) and to establish the relation (4), which leads to the expected value of $c$ in (5)-(6).
Concerning the opposite direction, we have

**Theorem 2.** Assume RH and let \(1 \leq h \leq X^{1/2-\varepsilon}, X^{-1/2+\varepsilon} \leq \xi \leq 1/2\). Let further \(0 \leq a < 1, b \geq 0, (a, b) \neq (0, 0)\) be fixed. If we have

\[
R_J(X, h) \ll \frac{hX}{(\log X)^{b}} \left(\frac{h}{X}\right)^{a}
\]

uniformly for

\[
h \leq \frac{1}{\xi} \frac{(X\xi)^{-a/(2a+6)}}{\log X} \leq h \leq \frac{1}{\xi} (X\xi)^{4a/(a+3)} (\log X)^{(3a+4b+13)/(a+3)},
\]

then

\[
W(X, \xi) \ll \frac{(X\xi)^{3/(3+a)}}{\log X}^{(b-a-2)/(3+a)}, \tag{10}
\]

uniformly for

\[
\frac{1}{h} \left(\frac{h}{X}\right)^{a/(3a+6)} (\log X)^{-(a+b+4)/(3a+6)} \leq \xi \leq \frac{1}{h} \left(\frac{X}{h}\right)^{4a/(3-3a)} (\log X)^{(3a+4b+13)/(3-3a)}.
\]

Note that for \(a = 0\) we have to take \(b > 2\) to get that the error term in (10) is \(o(X\xi)\). The technique used to prove Theorem 2 is similar to the one in Lemma 5 of [7]; the main difference is in the use of Lemma 4 which is needed to provide pair-correlation independent estimates of the involved quantities.

We remark that results similar to Theorems 1-2 can be proved for the weighted quantities

\[
\tilde{S}(\alpha) = \sum_{n=1}^{\infty} \Lambda(n) e^{-n/X} e(n\alpha),
\]

\[
\tilde{T}(\alpha) = \sum_{n=1}^{\infty} e^{-n/X} e(n\alpha),
\]

\[
\tilde{R}(X, \xi) = \int_{-\xi}^{\xi} |\tilde{S}(\alpha) - \tilde{T}(\alpha)|^2 d\alpha,
\]

\[
\tilde{J}(X, h) = \int_{0}^{\infty} (\psi(x + h) - \psi(x) - h)^2 e^{-2x/X} dx.
\]

The proofs are similar; in the analogue of Theorem 1 the main difference is in using the second part of Lemma 3 thus replacing \(E(X, h)\) with the sharper quantity \(\tilde{E}(X, h)\) defined in (15). Concerning the analogue of Theorem 2, the key point is in Eq. (33): in this case we will be able to extend its range of validity to \(\xi \leq x \leq \xi X^{1-\varepsilon}\) and to get rid of the term \((x^3/\xi)(\log X)^2\). These remarks lead to results which hold in wider ranges: \(1 \leq h \leq X^{1-\varepsilon}\) and \(X^{-1+\varepsilon} \leq \xi \leq 1/2\).
The order of magnitude of $\tilde{J}(X,h)$ can be directly deduced from the one of $J(X,h)$ via partial integration, see e.g. eq. (18). Unfortunately, the vice-versa seems to be very hard to achieve; this depends on the fact that we do not have sufficiently strong Tauberian theorems to get rid of the exponential weight in the definition of $J(X,h)$. Such a phenomenon is well known in the literature, see, e.g., Heath-Brown’s remark on pages 385-386 of [4].

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2. Some lemmas

In the following we will need two weight functions and, in particular, precise information on their total mass and size of the derivatives. For $h > 0$ we let

\[
K(\alpha, h) = \sum_{-h \leq n \leq h} (h - |n|) \cdot e(n\alpha) \quad \text{and} \quad U(\alpha, h) = \left(\frac{\sin(\pi h\alpha)}{\pi\alpha}\right)^{2}.
\] (11)

Lemma 1. For $h > 0$, we have $\int_{0}^{1/2} K(\alpha, h) \, d\alpha = h/2$ and $\int_{0}^{+\infty} U(\alpha, h) \, d\alpha = h/2$. Moreover we also have

\[
\int_{0}^{1/2} \log(h\alpha) K(\alpha, h) \, d\alpha = -\frac{h}{2} (\log(2\pi) + \gamma - 1) + \mathcal{O}(1),
\] (12)

\[
\int_{0}^{+\infty} \log(h\alpha) U(\alpha, h) \, d\alpha = -\frac{h}{2} (\log(2\pi) + \gamma - 1).
\]

Before the proof, we remark that this lemma is consistent with the constant in Lemma 2 of Languasco, Perelli and Zaccagnini [7], taking into account the fact that our variable $h$ here corresponds to $\pi\kappa$ there.

Proof. The results on $U(\alpha, h)$ can be immediately obtained by integrals n.3.821.9 and n.4.423.3, respectively on pages 460 and 594 of Gradshteyn and Ryzhik [3]. The first identity for $K(\alpha, h)$ immediately follows by isolating the contribution of $n = 0$ in its definition and making a trivial computation. Now we prove (12). Separating again the contribution of the term $n = 0$, a straightforward computation gives

\[
I(h) := 2 \int_{0}^{1/2} \log(h\alpha) K(\alpha, h) \, d\alpha
\]

\[
= h \log h - h(\log 2 + 1) + 2 \sum_{1 \leq n \leq h} (h-n) \int_{0}^{1} \log\left(\frac{h\beta}{2}\right) \cos(\pi n\beta) \, d\beta.
\]
A standard argument lets us write

\[
I(h) = h \log h - h(\log 2 + 1) + 2 \sum_{1 \leq n \leq h} (h - n) \int_0^1 \log \beta \cos(\pi n \beta) \, d\beta
\]

\[
= h \log h - h(\log 2 + 1) - \sum_{1 \leq n \leq h} \frac{h - n}{n} - 2 \sum_{1 \leq n \leq h} (h - n) \frac{\sin(\pi n)}{\pi n},
\]

by Formula 4.381.2 on page 581 of [3], where the sine integral function is defined by

\[
\sin(x) = - \int_{\infty}^{+\infty} \frac{\sin t}{t} \, dt \quad (13)
\]

for \( x > 0 \). The elementary relation \( \sum_{1 \leq n \leq h} 1/n = \log h + \gamma + \mathcal{O}(h^{-1}) \) shows that

\[
I(h) = -h(\log 2 + \gamma) + \mathcal{O}(1) - \frac{2h}{\pi} \sum_{1 \leq n \leq h} \frac{\sin(\pi n)}{n} + \frac{2}{\pi} \sum_{1 \leq n \leq h} \sin(\pi n).
\]

Finally we remark that Eq. (13) implies, by means of a simple integration by parts, that \( \sin(x) \ll x^{-1} \) as \( x \to +\infty \). Hence

\[
\sum_{1 \leq n \leq h} \frac{\sin(\pi n)}{n} = \sum_{n \geq 1} \frac{\sin(\pi n)}{n} + \mathcal{O}(h^{-1}) = \frac{\pi}{2} (\log \pi - 1) + \mathcal{O}(h^{-1}),
\]

by Formula 6.15.2 on page 154 of [10]. Moreover, by a double partial integration in (13) we get

\[
\sum_{1 \leq n \leq h} \sin(\pi n) = \sum_{1 \leq n \leq h} \frac{(-1)^{n+1}}{\pi n} + \mathcal{O}\left(\sum_{1 \leq n \leq h} \frac{1}{n^2}\right) \ll 1.
\]

In conclusion

\[
I(h) = -h(\log 2 + \gamma) - \frac{2h}{\pi} \left(\frac{\pi}{2} (\log \pi - 1) + \mathcal{O}(h^{-1})\right) + \mathcal{O}(1),
\]

and Lemma 1 is proved. \( \blacksquare \)

**Lemma 2.** For \( h \geq 1 \) we have

\[
K(\alpha, h) \ll \min\left(h^2, \|\alpha\|^{-2}\right),
\]

and

\[
\frac{d}{d\alpha} K(\alpha, h) \ll h\|\alpha\| \min\left(h^3, \|\alpha\|^{-3}\right).
\]

The proof of Lemma 2 is standard and hence we omit it. We also remark that estimates similar to the ones in Lemma 2 hold for \( U(\alpha, h) \) too; since they immediately follow from the definition we omit their proofs too.
We need the following auxiliary result which is based on Gallagher’s lemma.

**Lemma 3.** Let $1 \leq h \leq X$,

$$R(\alpha) = S(\alpha) - T(\alpha) \quad \text{and} \quad \tilde{R}(\alpha) = \tilde{S}(\alpha) - \tilde{T}(\alpha).$$

(14)

Then

$$\int_{-1/2}^{1/2} |R(\alpha)|^2 K(\alpha, h) \, d\alpha = \int_{-\infty}^{+\infty} |R(\alpha)|^2 U(\alpha, h) \, d\alpha = J(X, h) + \mathcal{O}(E(X, h)),$$

where $E(X, h)$ is defined in (1). Moreover we have,

$$\int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 K(\alpha, h) \, d\alpha = \int_{-\infty}^{+\infty} |\tilde{R}(\alpha)|^2 U(\alpha, h) \, d\alpha = \tilde{J}(X, h) + \mathcal{O}(\tilde{E}(X, h)),$$

where, for every fixed $\varepsilon > 0$, we define

$$\tilde{E}(X, h) = \begin{cases} (h + 1)^3(\log X)^2 & \text{(uncond.) for } 0 < h \leq X^\varepsilon \\
 h^3 & \text{(uncond.) for } X^\varepsilon < h \leq X \\
 (h + 1)^2(\log X)^4 & \text{(under RH) for } 0 < h \leq X. \end{cases}$$

(15)

**Proof.** The first part is Lemma 1 of [6], so we skip the proof. For the second part, we start arguing as in Lemma 1 of [6] thus getting

$$\int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 K(\alpha, h) \, d\alpha = \int_{-\infty}^{+\infty} |\tilde{R}(\alpha)|^2 U(\alpha, h) \, d\alpha$$

$$= \int_{-\infty}^{+\infty} \left| \sum_{|n-x|<h/2} (\Lambda(n) - 1)e^{-n/X} \right|^2 \, dx.$$

A standard computation hence gives

$$\int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 K(\alpha, h) \, d\alpha = \int_{0}^{+\infty} \left| \sum_{x<n\leq x+h} (\Lambda(n) - 1)e^{-n/X} \right|^2 \, dx$$

$$+ \mathcal{O}((h + 1)^2(\log(h + 1))^4),$$

(16)

where in the last estimate we assumed RH and we used the asymptotic formula

$$\psi(y) = y + \mathcal{O}\left(y^{1/2}(\log y)^2\right)$$

(17)

on a interval of length $\leq h$. Noting that

$$\sum_{x<n\leq x+h} (\Lambda(n) - 1)e^{-n/X} = e^{-x/X}(\psi(x + h) - \psi(x) - h)\left(1 + \mathcal{O}\left(\frac{h + 1}{X}\right)\right).$$
and recalling that $h \leq X$, from (16) we have

$$\int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 K(\alpha, h) \, d\alpha = \tilde{J}(X, h) \left( 1 + O\left( \frac{h + 1}{X} \right) \right) + O(\{(h + 1)^2 \log X\}^4).$$

To estimate the last error term we connect $\tilde{J}(X, h)$ to $J(X, h)$. A partial integration immediately gives

$$\tilde{J}(X, h) = 2X \int_0^\infty J(t, h) e^{-2t/X} \, dt. \tag{18}$$

Splitting the range of integration on the right-hand side of (18) into $[0, h] \cup [h, +\infty)$, a direct computation using (17) shows that

$$\int_0^h J(t, h) e^{-2t/X} \, dt \ll h^3 (\log h)^4$$

while, still assuming RH, in the remaining range we use the Selberg [11] estimate

$$J(t, h) \ll ht (\log t)^2$$

for $1 \leq h \leq t$, \tag{19}

and so we get

$$\int_h^{+\infty} J(t, h) e^{-2t/X} \, dt \ll h \int_h^{+\infty} t (\log t)^2 e^{-2t/X} \, dt \ll hX^2 (\log X)^2.$$

Summing up, under RH we have

$$\tilde{J}(X, h) \ll (h + 1)X (\log X)^4$$

and we can finally write

$$\int_{-1/2}^{1/2} |\tilde{R}(\alpha)|^2 K(\alpha, h) \, d\alpha = \tilde{J}(X, h) + O(\{(h + 1)^2 \log X\}^4).$$

The unconditional cases follow by replacing (17) with the Brun-Titchmarsh inequality and (19) with the estimate $J(t, h) \ll h^2 t + ht \log t$ (see the Lemma in [5]).

In the next sections we will also need the following remark. Let $\xi > 0$ and $\delta \xi = 1/2$. In this case $U(\alpha, \delta) \gg \delta^2$ for $|\alpha| \leq \xi$; hence by the first equation in Lemma 3 we obtain

$$\int_{-\xi}^{\xi} |R(\alpha)|^2 \, d\alpha \ll \xi^2 \left( J(X, \frac{1}{2\xi}) + E\left( X, \frac{1}{2\xi} \right) \right).$$

By (19) and (1), under RH we immediately obtain, for every $1/(2X) \leq \xi \leq 1/2$, that

$$\int_{-\xi}^{\xi} |R(\alpha)|^2 \, d\alpha \ll X\xi (\log X)^4. \tag{20}$$
3. Proof of Theorem 1

We use Lemma 3 in the form

\[ J(X, h) = \int_{-1/2}^{1/2} |R(\alpha)|^2 K(\alpha, h) \, d\alpha + \mathcal{O}(E(X, h)), \]  

(21)

where \( R(\alpha) \) is defined in (14). Observe that both \(|R(\alpha)|^2\) and \(K(\alpha, h)\) are even functions of \(\alpha\), and hence we may restrict our attention to \(\alpha \in [0, 1/2]\). Recalling (6) and writing

\[ f(X, \alpha) = X \log(X\alpha) + \left(\frac{c}{2} + 1\right)X = X \log \frac{X}{h} + X \log(h\alpha) + \left(\frac{c}{2} + 1\right)X, \]  

(22)

we can approximate \(|R(\alpha)|^2\) as

\[ |R(\alpha)|^2 = f(X, \alpha) - f(X, \alpha) \]  

(23)

where \(c'\) is defined in (4).

Let now \(U_1 < 1/h < U_2 \leq 1\) be two parameters to be chosen later. By Lemma 2, (20) and a partial integration we immediately obtain

\[ \left( \int_0^{U_1} + \int_{U_2}^{1/2} \right) (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) \, d\alpha \ll h^2 U_1 X \log X^4 + \frac{X (\log X)^4}{U_2}. \]  

(24)

From (24) it is clear that the optimal choice is \(h^2 U_1 = 1/U_2\). We now evaluate the integral over \([U_1, U_2]\). A direct computation and the hypothesis show that

\[ \int_0^{\xi} (|R(\alpha)|^2 - f(X, \alpha)) \, d\alpha \ll \frac{(X\xi)^{1-a}}{(\log X\xi)^b}, \]

and hence, by partial integration and Lemma 2, we obtain

\[ \int_{U_1}^{U_2} (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) \, d\alpha \ll h^2 \frac{(XU_1)^{1-a}}{(\log X)^b} + \frac{X^{1-a}U_2^{-1-a}}{(\log X)^b} \]

\[ + \frac{hX^{1-a}}{(\log X)^b} \int_{U_1}^{U_2} \xi^{-a} \min(h^3, \xi^{-3}) \, d\xi. \]

Using the constraints \(h^2 U_1 = 1/U_2\) and \(U_1 < 1/h\), the right-hand side is

\[ \ll \frac{h^{1+a}X^{1-a}}{(\log X)^b} + \frac{hX^{1-a}}{(\log X)^b} \int_{1/h}^{U_2} \xi^{-a} \, d\xi \ll R_{a,b}(X, h, U_2), \]  

(25)

where

\[ R_{a,b}(X, h, U_2) = \begin{cases} 
    hX \log(hU_2)(\log X)^{-b} & \text{if } a = 0 \\
    h^{1+a}X^{1-a} (\log X)^{-b} & \text{if } a > 0. 
\end{cases} \]
Hence, by (24)-(25) and $h^2 U_1 = 1/U_2$ we get
\[
\int_0^{1/2} (|R(\alpha)|^2 - f(X, \alpha)) K(\alpha, h) \, d\alpha \ll \frac{X (\log X)^4}{U_2} + R_{a,b}(X, h, U_2). \tag{26}
\]
Choosing
\[
U_2 = \frac{X^a (\log X)^{b+4}}{h^{1+a}} \quad \text{and} \quad U_1 = \frac{h^{-1}}{X^a (\log X)^{b+4}},
\]
by (23) and (26) we finally get
\[
\int_0^{1/2} |R(\alpha)|^2 K(\alpha, h) \, d\alpha = \frac{h}{2} X \log \frac{X}{h} + c' \frac{h}{2} X + O(X + R_{a,b}(X, h))
\]
where $c'$ and $R_{a,b}(X, h)$ are defined in (4) and (9). Theorem 1 follows from (21).

4. Proof of Theorem 2

We adapt the proof of Lemma 5 of [7], which is an explicit form of Lemma 4 of [2].

We recall that $0 < \eta < 1/4$ is a parameter to be chosen later and
\[
K_\eta(x) = \frac{\sin(2\pi x) + \sin(2\pi (1+\eta)x)}{2\pi x (1-4\eta^2 x^2)},
\]
so that its Fourier transform becomes
\[
\hat{K}_\eta(t) = \begin{cases} 
1 & \text{if } |t| \leq 1 \\
\cos^2 \left( \frac{\pi (|t| - 1)}{2\eta} \right) & \text{if } 1 \leq |t| \leq 1 + \eta \\
0 & \text{if } |t| \geq 1 + \eta
\end{cases}
\]
and
\[
K''_\eta(x) \ll \min(1; (\eta x)^{-3}), \tag{27}
\]
see Eqs. (3.14)-(3.15) and Lemma 4 of [7]. Moreover, by Lemma 3 of [7], we also have
\[
\hat{K}_\eta(t) = \int_0^\infty K''_\eta(x) U(t, x) \, dx. \tag{28}
\]

Hence, again considering only positive values of $\alpha$, we have
\[
\int_0^\infty |R(\alpha)|^2 \hat{K}_\eta \left( \frac{\alpha}{\xi} (1+\eta) \right) \, d\alpha \leq \frac{R(X, \xi)}{2} \leq \int_0^\infty |R(\alpha)|^2 \hat{K}_\eta \left( \frac{\alpha}{\xi} \right) \, d\alpha \tag{29}
\]
where $R(\alpha)$ is defined in (14). Writing $f(X, \alpha)$ as in (22), we approximate $|R(\alpha)|^2$ as
\[
|R(\alpha)|^2 = f(X, \alpha) + (|R(\alpha)|^2 - f(X, \alpha)).
\]
Observing that $U(\alpha/\xi, x) = \xi^2 U(\alpha, x/\xi)$, letting
\[
g(x, \xi) = \xi^2 \int_0^\infty (|R(\alpha)|^2 - f(X, \alpha)) U \left( \frac{\alpha}{\xi}, x \right) \, d\alpha
\]
and using (28), we get
\[ \int_{0}^{\infty} |R(\alpha)|^2 \hat{K}_{\eta}(\frac{\alpha}{\xi}) \, d\alpha = \int_{0}^{\infty} f(X, \alpha) \hat{K}_{\eta}(\frac{\alpha}{\xi}) \, d\alpha + \int_{0}^{\infty} K''_{\eta}(x)g(x, \xi) \, dx = J_1 + J_2, \]
(30)
say. A direct computation and (6) show that
\[ J_1 = X\xi \log X\xi + \frac{6}{2}X\xi + O(\eta X\xi \log X\xi). \]
(31)
In order to estimate \( J_2 \) we first remark that by Lemma 1, (22) and (4), we have
\[ \xi^2 \int_{0}^{\infty} f(X, \alpha)U_0(\alpha, \frac{x}{\xi}) \, d\alpha = \frac{xX\xi}{2} \log \frac{X\xi}{x} + \frac{c'}{2}xX\xi. \]
(32)
Now we need the following Lemma whose proof follows the line of Lemma 2 of [6].

**Lemma 4.** Assume RH and let \( \varepsilon > 0 \). We have
\[ g(x, \xi) \ll \begin{cases} X\xi^2 \log X & \text{if } 0 < x \leq \xi \\ x\xi(\log X)^2 & \text{if } \xi \leq x \leq \xi X^{1/2-\varepsilon} \\ x\xi(\log X)^4 & \text{if } x \geq \xi X^{1/2-\varepsilon}. \end{cases} \]
Assume further the hypothesis of Theorem 2. We have
\[ g(x, \xi) \ll x^{1+a} \frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{x^3}{\xi} (\log X)^2 \quad \text{if } \xi \leq x \leq \xi X^{1/2-\varepsilon}. \]
(33)
Choosing now \( V_1, V_2 \) such that \( \xi < V_1 < 1/\eta < V_2 < \xi X^{1/2-\varepsilon} \), we split \( J_2 \)’s integration range into six subintervals. We obtain
\[ J_2 = \left( \int_{0}^{\xi} + \int_{\xi}^{V_1} + \int_{\xi}^{1/\eta} + \int_{1/\eta}^{V_2} + \int_{V_2}^{\xi X^{1/2-\varepsilon}} + \int_{\xi X^{1/2-\varepsilon}}^{+\infty} \right) K''_{\eta}(x)g(x, \xi) \, dx \]
\[ = M_1 + M_2 + M_3 + M_4 + M_5 + M_6, \]
(34)
say. By Lemma 4 and (27), we obtain
\[ M_1 \ll X\xi^2 \log X \int_{0}^{\xi} dx \ll X\xi^3 \log X, \]
\[ M_2 \ll X\xi(\log X)^2 \int_{\xi}^{V_1} x \, dx \ll X\xi V_1^2(\log X)^2, \]
\[ M_3 \ll \int_{V_1}^{1/\eta} \left( x^{1+a} \frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{x^3}{\xi} (\log X)^2 \right) \, dx \ll \frac{(X\xi)^{1-a}}{\eta^{2+a}(\log X)^b} + \frac{(\log X)^2}{\xi \eta^{4}}, \]
\[ M_4 \ll \frac{1}{\eta^3} \int_{1/\eta}^{V_2} \left( x^{a-2} \frac{(X\xi)^{1-a}}{(\log X)^b} + \frac{(\log X)^2}{\xi} \right) \, dx \ll \frac{(X\xi)^{1-a}}{\eta^{2+a}(\log X)^b} + \frac{V_2(\log X)^2}{\xi \eta^{3}}, \]
\[ M_5 \ll \frac{X\xi(\log X)^2}{\eta^3} \int_{V_2}^{\xi X^{1/2-\varepsilon}} \frac{dx}{x^2} \ll \frac{X\xi(\log X)^2}{V_2 \eta^3}, \]
and
\[ M_6 \ll \frac{X\xi(\log X)^4}{\eta^3} \int_0^{+\infty} \frac{dx}{x^2} \ll \frac{X^{1/2+\epsilon}(\log X)^4}{\eta^3}. \]

Hence, recalling \( \xi > X^{-1/2+\epsilon} \), by (34) and the definitions of \( V_1 \) and \( V_2 \) we get
\[ J_2 \ll X\xi (\log X)^2 \left( V_1^2 + \frac{(\log X)^2}{V_2 \eta^3} \right) + \frac{(X\xi)^{1-a}}{\eta^{2+a}(\log X)^b}. \quad (35) \]

Choosing \( V_1 = \eta^{1/2}/\log X \) and \( V_2 = \log^3 X/\eta^4 \), by (30)-(31) and (35), we obtain
\[ \int_0^\infty |R(\alpha)|^2 \hat{\alpha}(\frac{\alpha}{\xi}) \, d\alpha = X\xi \log X\xi + \frac{c}{2} X\xi + O\left( \eta X\xi \log X + \frac{(X\xi)^{1-a}}{\eta^{2+a}(\log X)^b} \right). \quad (36) \]

To optimize the error term we choose \( \eta^{3+a} = (X\xi)^{-a}(\log X)^{-b-1} \), so that (36) becomes
\[ \int_0^\infty |R(\alpha)|^2 \hat{\alpha}(\frac{\alpha}{\xi}) \, d\alpha = X\xi \log X\xi + \frac{c}{2} X\xi + O\left( \frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}} \right). \quad (37) \]

Finally, by (29) and (37), we obtain
\[ R(X,\xi) \leq 2X\xi \log X\xi + cX\xi + O\left( \frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}} \right). \]

In a similar way we also get that
\[ R(X,\xi) \geq 2X\xi \log X\xi + cX\xi + O\left( \frac{(X\xi)^{3/(3+a)}}{(\log X)^{(b-a-2)/(3+a)}} \right), \]

and Theorem 2 follows.

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