Nonlinear non-Hermitian higher-order topological laser

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We investigate topological lasers in combination of nonlinear, non-Hermitian and topological lattice systems based on a quench dynamics starting from one site. We consider explicitly the topological laser in the Su-Schrieffer-Heeger (SSH) model with two topological edge states and the second-order topological laser in the breathing Kagome lattice with three topological corner states. Once we stimulate any one site, after a delay, all sites belonging to the topological edge or corner states begin to emit stable laser light depending on the density of states, although no wave propagation is observed from the stimulated site. It is intriguing that the profile of topological edge or corner states is observable by measuring the intensity of lasing. The phenomenon occurs due to a combinational effect of linear non-Hermitian loss terms and nonlinear non-Hermitian gain terms in the presence of the topological edge or corner states.

I. INTRODUCTION

Topological physics is one of the most essential concepts found in recent fundamental physics\cite{33–37}. Recently, it is ubiquitously found in various systems in photonic\cite{33–37, 55–63}, acoustic\cite{17, 19}, mechanical\cite{12, 13, 46–54} and electric circuit\cite{3–22} systems. Among them, topological photonics is most extensively studied theoretically and experimentally. One of the reasons is that it is possible to observe real-time and real-space dynamics. Another merit is that topological photonics has opened a new field of topological physics, i.e. non-Hermitian topology and nonlinear topology and their combination. Non-Hermitian effects are introduced by a loss and a gain of photons. On the other hand, nonlinear effects are introduced by the Kerr effect or a stimulated emission effect.

A topological laser is a prominent application of topological physics\cite{18, 19, 25, 26}. It utilizes topological edge states for the coherent laser emission. Thanks to the topological protection, the topological laser is robust against the randomness and the defects of the sample, which is favorable for future laser applications. A topological laser is an ideal play ground to investigate nonlinear non-Hermitian topological physics. The loss of photons and gain from stimulated emissions constitute the non-Hermitian terms. The nonlinear effect is included in the gain term, which represents the saturation of the gain.

Higher-order topological phases are extension of topological phases\cite{25, 26}. There emerge topological corner states in the second-order topological phase. A simplest example is given by the breathing Kagome lattice, where three topological corner states appear in triangle geometry. This model is a natural generalization of the Su-Schrieffer-Heeger (SSH) model to higher-order topological phases. Among them, topological photonics is most extensively studied theoretically and experimentally. One of the reasons is that it is possible to observe real-time and real-space dynamics. Another merit is that topological photonics has opened a new field of topological physics, i.e. non-Hermitian topology and nonlinear topology and their combination. Non-Hermitian effects are introduced by a loss and a gain of photons. On the other hand, nonlinear effects are introduced by the Kerr effect or a stimulated emission effect.

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In this paper, we analyze a quench dynamics of a nonlinear non-Hermitian topological laser and a higher-order topological laser by stimulating any one site. We study explicitly the SSH model for a topological laser and the breathing Kagome model for a higher-order topological laser. In the SSH model with two topological edge states, once we stimulate any one site, after a delay, all sites belonging to the two topological edge states begin to emit stable laser light depending on the density of states (DOS), although no wave propagation is observed from the stimulated site. Thus, the profile of topological edge state together with the DOS is observable by laser intensity. The strength of the laser light is identical for the two edges because of reflection symmetry. This is also the case for the second-order topological laser in the breathing Kagome lattice with three topological corner states. Here, the system has trigonal symmetry. The phenomenon occurs due to a combinational effect of linear non-Hermitian loss terms and nonlinear non-Hermitian gain terms in the presence of the topological edge or corner states.

II. TOPOLOGICAL LASER

A. Model

We consider a coupled-ring system made of active resonators\cite{46}. The dynamics of a laser system is governed by

\[ i \frac{d \psi_n}{dt} = \sum_{m} M_{nm} \psi_m - i \gamma \left( 1 - \frac{\xi}{\eta} \right) \frac{P_n}{|\psi_n|^2 / \eta} \psi_n, \]  

(1)

where \( \psi_n \) is the amplitudes of the site \( n \), where \( n = 1, 2, 3, \cdots, N \) in the system composed of \( N \) sites; \( M_{nm} \) describes a hopping matrix; \( \gamma \) represents the loss in each resonator; \( \xi \) represents the amplitude of the optical gain via stimulated emission; \( \eta \) represents the nonlinearity; \( P_n \) stands for the spatial profile of the pump. The system turns into the linear model in the limit \( \eta \to \infty \). On the other hand, \( \gamma \) controls the non-Hermiticity. The system turns into a Hermitian model for \( \gamma = 0 \). We call the term proportional to \( \gamma \) the loss term and the term proportional to \( \gamma \xi \) the nonlinear gain term.

We take

\[ P_n = \sum_{\bar{n}} \delta_{n, \bar{n}}, \]  

(2)

where \( \bar{n} \) runs over the edge or corner sites. Namely, optical gains are introduced only at the edge or corner sites. We are interested in the case where \( M_{nm} \) represents a tight-binding model possessing a topological phase. We explicitly consider the SSH model illustrated in Figs. (a1)~(a3), where \( \bar{n} \) takes values at the left and right edges, and the breathing
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Appendix for the topological charges in these models.

Kagome model illustrated in Fig 1(b1)~(b3), where \( \bar{n} \) takes values at the top, bottom-left and bottom-right corners. See Appendix for the topological charges in these models.

It is possible to solve Eq. (1) numerically for explicit system parameters, as we do later. However, to reach a deeper understanding of the phenomena, an analytical study is indispensable. Since this is impossible for general system parameters, we make an analytical study for special cases.

### B. Edge or corner dynamics

We first consider the dynamics of an edge or corner site when it is perfectly isolated as in Fig 1(a1) or (b1). This is the case where \( M_{nm} = 0 \) for the edge or corner sites. For instance, this is realized by setting \( \kappa_A = 0 \) in Eq. (12) or \( \lambda = -1 \) in Eq. (15) for the SSH model.

The dynamics is governed by isolated equations,

\[
\frac{d\psi_n}{dt} = -\gamma \left( 1 - \xi \frac{1}{1 + |\psi_n|^2/\eta} \right) \psi_n.
\]

Solving this equation numerically with the initial conditions \( \psi_n(0) = 1 \) and \( \psi_n(0) = 0.1 \), we show the results in Fig 2(a) and (b), respectively. It is intriguing that the dynamics of the topological edge or corner state does not depend on the initial condition of \( \psi_n \). The saturated value of \( |\psi_n| \) as \( t \to \infty \) is identical for all isolated edge or corner states. This can be understood analytically as follows.

We solve Eq. (3) for nontrivial stationary solutions. The stationary solution for an edge or corner site \( \bar{n} \) is given by

\[
\frac{1}{\frac{1}{1 + |\psi_n|^2/\eta}} = 1.
\]

Hence, the nontrivial solution reads

\[
\lim_{t \to \infty} |\psi_n|^2 = \eta(\xi - 1),
\]

where it is necessary that \( \xi > 1 \). We have the trivial solution, \( \lim_{t \to \infty} |\psi_n|^2 = 0 \), for \( \xi \leq 1 \).

As a result, there are only two stable solutions in Eq. (3). One is the ground state mode \( \psi_n = 0 \), and the other is the stimulated mode \( |\psi_n|^2 = \eta(\xi - 1) \) for \( \xi > 1 \). The initial condition determines which state is realized.

The nonlinear term is essential to have a stationary solution. Indeed, we obtain a linear theory in the limit \( \eta \to \infty \), where the amplitude \( |\psi_n|^2 \) diverges.

The state remains to be real for a real initial condition, and Eq. (3) is simplified

\[
\frac{d\psi_n}{dt} = \gamma dt,
\]

which is solved as

\[
\frac{2 \log |\psi_n| - \xi \log \left[ \frac{\gamma (|\psi_n|^2/\eta + 1 - \xi)}{2(\xi - 1)} \right]}{2(\xi - 1)} = \gamma (t + t_0).
\]

The state evolution \( \psi_n(t) \) is given by the inverse of this equation.

### C. Bulk dynamics

We next analyze the dynamics of the bulk site, where there is no nonlinear non-Hermitian term because \( P_{nm} = 0 \). Then, the dynamics is governed by the linear equation

\[
i \frac{d\psi_n}{dt} = \sum_{nm} M_{nm} \psi_m - i \gamma \psi_n.
\]

With the use of a solution of the linear equation

\[
i \frac{d\psi_n^0}{dt} = \sum_{nm} M_{nm} \psi_m^0,
\]

the solution of Eq. (8) is written in the form

\[
\psi_n = e^{-\gamma t} \psi_n^0.
\]

It means that the amplitude exponentially decays as a function of time in the bulk.
FIG. 3: Saturated amplitude $|\psi_1|$ at (a1) the edge site in the SSH model, and (b1) the corner site in the breathing Kagome model, where $\gamma = 1/2$ for magenta, $\gamma = 1/4$ for cyan and $\gamma = 1/8$ for orange colors. (a2) The energy spectrum in the SSH model made of a finite chain with topological edge states in red. (b2) The energy spectrum of the breathing Kagome model in triangle geometry with topological corner states in red. The horizontal axis is $\lambda$. We have set $\kappa = 1$, $\eta = 1$ and $\xi = 2$.

D. Quench dynamics

Quench dynamics starting from one site is a good signal to detect whether the system is topological or trivial. It is also applicable to various nonlinear systems. Let us study a quench dynamics starting from one site indexed by $m$,

$$\psi_n(t) = \delta_{n,m} \quad \text{at} \quad t = 0,$$

where $m = 1$ represents the left-edge site or the top-corner site as in Fig. 1(a2) and (b2).

III. NONLINEAR NON-HERMITIAN SSH MODEL

A. Model

A topological laser based on the SSH model has been discussed. We consider the case where the hopping matrix is governed by the SSH matrix in Eq. (1). The matrix is explicitly given by

$$M_{nm} = - (\kappa_A + \kappa_B) \delta_{n,m} + \kappa_A (\delta_{m,2n-1} + \delta_{m-1,2n}) + \kappa_B (\delta_{2n,2m} + \delta_{2m-1,2n-1}).$$

(12)

The explicit equations for a finite chain with length $N$ are given by

$$i \frac{d\psi_{2n-1}}{dt} = \kappa_A (\psi_{2n} - \psi_{2n-1}) + \kappa_B (\psi_{2n-2} - \psi_{2n-1}) - i\gamma \left(1 - \frac{\delta_{n,1}}{1 + |\psi_{2n-1}|^2/\eta}\right) \psi_{2n-1},$$

(13)

$$i \frac{d\psi_{2n}}{dt} = \kappa_B (\psi_{2n+1} - \psi_{2n}) + \kappa_A (\psi_{2n-1} - \psi_{2n}) - i\gamma \left(1 - \frac{\delta_{n,N}}{1 + |\psi_{2n}|^2/\eta}\right) \psi_{2n}.$$  

(14)

It is convenient to introduce the coupling strength $\kappa$ and the dimerization parameter $\lambda$ by

$$\kappa_A = \kappa (1 + \lambda), \quad \kappa_B = \kappa (1 - \lambda).$$

(15)
with $|\lambda|\leq1$. The isolated edge limit is realized at $\lambda=-1$ in Fig. 4(a1).

The present SSH model (1) has the same topological structure as in the original SSH model by the following reasoning. First, the contribution of the nonlinear gain term ($\propto\gamma$) is negligible in the bulk since it exists only at the edge site. Next, Eq. (1) is rewritten in the form of the linear model for the bulk with $N_{a}$ sample with $n=7$. We have set $\kappa=1$, $\eta=1$, $\gamma=1/2$ and $\xi=2$. We take a sample with $N=20$.

Then, the topological properties are determined by the matrix $\bar{M}_{nm}$, and they are identical to those in the original SSH model, as explained in Appendix: See Eq. (A4). We show the band structure for a finite chain in Fig. 3(a2). The system is topological for $\lambda<0$ with the emergence of the topological edge states marked in red, while it is trivial for $\lambda>0$.

B. Quench dynamics

We numerically solve Eqs. (13) and (14) under the initial condition (11). We show the time evolution of $|\psi_{n}|$ in Fig. 6. The quench dynamics is significantly different between the topological and trivial phases. In the topological phase, the amplitude $|\psi_{1}|$ at the left edge gradually decreases as shown in Fig. 6(a1). Furthermore, there is no excitation $|\psi_{N}|$ at the right edge.

Fig. 6(a3) shows a spatial profile of the amplitude $|\psi_{n}|$ after enough time, which is the DOS for a pair of topological edge states in nonlinear non-Hermitian system.

We find that there is no reflection of the propagating wave by the right edge. This is due to the loss term ($\propto\gamma$) in the bulk. It is highly contrasted with the case of the Hermitian model. In addition, the amplitudes $|\psi_{1}|$ and $|\psi_{N}|$ are always identical. It is due to reflection symmetry $x\leftrightarrow N-x$ in the right-hand side of Eqs. (13) and (14) because $d\psi_{m}/dt=0$ for the stationary solution. This is confirmed in Eq. (5) explicitly for the limit ($\lambda=-1$) of the isolated edge states, and numerically for any value of $\lambda$.

We show the saturated amplitude $|\psi_{1}|$ as a function of the dimerization $\lambda$ in Fig. 4(a1). It is finite for the topological phase although it deviates from 1 other than $\lambda=-1$ due to the hopping term. On the other hand, it is almost zero for the trivial phase. These features correspond to the emergence or the absence of the topological edge states as shown in Fig. 5(a2). Namely, the quench dynamics well signatures the topological phase transition although there are nonlinear non-Hermitian terms.

We also study the dynamics under the initial condition (11) by taking the site $m$ in the bulk. The result is shown in Fig. 5 by choosing $m=7$. All sites belonging to the two topological edge states are stimulated after a delay with the intensity depending on the DOS. The timing of the stimulation is determined by the distance from the initial site.

C. Dimer states

We study the trivial phase ($\lambda>1$). In particular, we may solve the equations of motion analytically in the dimer limit ($\lambda=1$). In this case, we obtain a closed set of equation for

\[ i\frac{d\psi_{m}}{dt} = \sum_{nm} \bar{M}_{nm} \psi_{m}. \]
FIG. 7: Time evolution of the amplitude $|\psi_n|$ of the breathing Kagome model in (a) the topological phase with $\lambda = -0.5$, (b) the metal phase with $\lambda = 0.2$ and (c) the trivial phase with $\lambda = 0.5$. The color density indicates the amplitude $|\psi_n|$. (a5) and (c5) give the profiles of the topological corner states and the trimer states together with their DOS, respectively. We have set $\kappa = 1$, $\eta = 1$, $\gamma = 1/2$ and $\xi = 2$. We take a triangle with $N = 108$.

The stationary solutions of (18) and (19) are either the trivial one

$$|\psi_1| = |\psi_2| = 0,$$

or a nontrivial one

$$\psi_1 = \sqrt{\eta (\xi - 1)}, \quad \psi_2 = \text{const.}$$

We note that the right-hand side of Eq. (19) is not necessary to be zero because only the phase rotates if it is not zero. It is a dynamical problem depending on the initial condition which stationary solution is actually chosen.

We show numerical solution with the initial condition $\psi_1 = 1$ and $\psi_2 = 0$ in Fig.7(a). The amplitudes $|\psi_1|$ and $|\psi_2|$ exponentially decreases to zero with oscillations. Hence, the stationary solutions are $\psi_1 = \psi_2 = 0$. Furthermore, we have found numerically that $|\psi_1|$ is almost zero in the trivial phase as in Fig.3(a). As far as we have checked numerically, there is no nontrivial dimer solution in the trivial phase of the SSH model.

IV. NONLINEAR NON-HERMITIAN BREATHING KAGOME MODEL

A. Quench dynamics

We proceed to investigate the system where the hopping matrix $M_{nm}$ describes the breathing Kagome lattice, whose lattice structure is illustrated in Fig.1(b). The hopping matrix and the topological number are given by (B1) and (C2) in Appendix. There are topological, trivial and metal phases in the breathing Kagome model as shown in Fig.3(b2). By solving Eq. (1) under the initial condition (11) with the choice of $m = 1$, we show the time evolution of the amplitude $|\psi_n|$ in Fig.7 and in Fig.8, where $n = 1$ denotes the top-corner site. Fig.7 displays a global picture how the stimulated signal at site $n = 1$ propagates all over the sites as time evolves. Fig.8 displays a detailed evolution along the left side of the lattice. We also show the saturated amplitude as a function of $\lambda$ in Fig.3(b1). The saturated amplitudes are identical at the three corner sites due to trigonal symmetry of the breathing Kagome lattice, as found in Fig.7.

We comment that Fig.7(a5) shows a spatial profile of the saturated amplitude $|\psi_n|$, which is the DOS for three topological corner states in nonlinear non-Hermitian system.

These figures show typically different behaviors in the topological, trivial and metal phases. First, in the topological phase, the amplitudes at the other two corner sites increase after a delay to the saturated amplitude as shown in Fig.3(a2). Second, in the metal phase, the amplitude at the corner site rapidly decreases as in Fig.3(b2). These features are very much similar to those in the the SSH model.

On the other hand, in the trivial phase, there is a significant difference between the SSH model and breathing Kagome
Without loss of generality we may set $\psi$. The stationary solutions are either the trivial one or a nontrivial one

$$\psi_1 = \sqrt{\eta} (\xi - 1), \quad \psi_2 = \text{const},$$  \hspace{1cm} (28)

which is identical to the stationary solution (21) in the SSH model. There is no $\gamma$ dependence in the stationary solution (28), which agrees with in Fig.3(b1).

We have found analytically the trimer state in the limit of $\lambda = 1$ in the trivial phase. A numerical solution is given as a function of time $t$ in Fig.6(b), where $|\psi_1|$ and $|\psi_2|$ approach two saturated values (28), as is consistent with Fig.3(b1). Fig.3(b1) suggests that the trimer state is formed also away from the limit $\lambda = 1$ depending on the value of $\gamma$. Indeed, Fig.7(c5) shows a spatial profile of three trimer states at $\lambda = 0.5$. The formation of trimer states implies the presence of a nontopological laser in the breathing Kagome model. These behaviors are contrasted with the dimer state in the SSH model, where $|\psi_1| = 0$ in the trivial phase as in Fig.3(a1).

V. DISCUSSION

A topological laser provides a unique arena of topology, non-Hermicity and nonlinearity. We have studied topological lasing in the SSH lattice and the breathing Kagome lattice. They have topological edge or corner states in the topological phase. When any one site is stimulated, all sites belonging to the topological edge or corner states begin to emit stable laser light depending on the DOS. The results would be universal for the physics of topological lasing with the use of topological edge or corner states.

Non-Hermicity and nonlinearity play essential roles for the stabilization of lasing. Without the nonlinearity, there is no stable lasing because the amplitude exponentially grows or decays. The amplifier plays an essential role of the lasing. If there is no loss term, there should be reflection wave at the right edge site or the bottom corner sites, which are absent in a topological laser.

On the other hand, nontopological lasing is not universal. Indeed, the quench dynamics is different between the SSH model and the breathing Kagome model in the trivial phase. There is no stable laser emission in the SSH model. However, once one site is stimulated, all three corner states emit stable laser lights in the breathing Kagome model due to the formation of trimer states at the three corners.

In conclusion, we have found stable laser emission to occur in the topological phase. Our results show that topological lasers provide an ideal play ground of nonlinear non-Hermitian topological physics, where the profile of a topological edge or corner state is observable by measuring the intensity of lasing.

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Appendix A: Topological number in the SSH model

The hopping matrix of the SSH model \([12]\) is given by
\[
\mathbf{M}(k) = -(\kappa_A + \kappa_B + i\gamma) I_2 + M_0(k) \tag{A1}
\]
in the momentum space, with
\[
q(k) = \kappa_A + \kappa_B e^{-ik}, \tag{A2}
\]
and
\[
M_0(k) = \begin{pmatrix} 0 & q(k) \\ q^*(k) & 0 \end{pmatrix}. \tag{A3}
\]

The topological number in the original SSH model is given by the Berry phase
\[
\Gamma = \frac{1}{2\pi} \int_0^{2\pi} A(k) \, dk, \tag{A4}
\]
where \(A(k) = -i \langle \phi(k) | \partial_k | \phi(k) \rangle\) is the Berry connection with \(\phi(k)\) the eigenfunction of \(\mathbf{M}(k)\). Note that the diagonal term in Eq. (A1) with Eq. (A2) does not contribute to the topological charge because the wave function \(\phi(k)\) does not depend on the diagonal term. Hence, the present model \([1]\) has the same phases as the original SSH model. The system is topological for \(\lambda < 0\) and trivial for \(\lambda > 0\).

Appendix B: Breathing Kagome model

The hopping matrix of the breathing Kagome model is given by \([53]\)
\[
\mathbf{M}(k) = - \begin{pmatrix} 0 & h_{12} & h_{13} \\ h_{12} & 0 & h_{23} \\ h_{13} & h_{23} & 0 \end{pmatrix}. \tag{B1}
\]
with
\[
h_{12} = \kappa_A e^{i(k_x/2 + \sqrt{3}k_y/2)} + \kappa_B e^{-i(k_x/2 + \sqrt{3}k_y/2)}, \tag{B2}
\]
\[
h_{23} = \kappa_A e^{i(k_x/2 - \sqrt{3}k_y/2)} + \kappa_B e^{-i(k_x/2 - \sqrt{3}k_y/2)}, \tag{B3}
\]
\[
h_{13} = \kappa_A e^{ik_x} + \kappa_B e^{-ik_x} \tag{B4}
\]
in the momentum space, where we have introduced two hopping parameters \(\kappa_A\) and \(\kappa_B\) corresponding to upward and downward triangles in Fig. (B).

Appendix C: Topological number in the breathing Kagome model

The topological number in the breathing Kagome model is defined by \([53]\)
\[
\Gamma = 3 \left(p_x^2 + p_y^2\right), \tag{C1}
\]
with
\[
p_i = \frac{1}{3} \int_{BZ} A_i d^2 k, \tag{C2}
\]
where \(A_i = -i \langle \phi(k) | \partial_{k_i} | \phi(k) \rangle\) being the Berry connection with \(x_i = x, y\), and \(S = 8\pi^2 / \sqrt{3}\) being the area of the Brillouin zone; \(\phi(k)\) the eigenfunction of \(\mathbf{M}(k)\). We obtain \(\Gamma = 0\) for \(1 \geq \lambda > 1/3\), which is the trivial phase with no topological corner states. On the other hand, we obtain \(\Gamma = 1\) for \(-1 \leq \lambda < -1/3\), which is the topological phase with the emergence of three topological corner states. Finally, \(\Gamma\) is not quantized for \(-1/3 < \lambda < 1/3\), which is the metal phase.
