DISTRIBUTION OF ANGLES IN HYPERBOLIC LATTICES

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Abstract. We prove an effective equidistribution result about angles in a hyperbolic lattice. We use this to generalize a result due to F. P. Boca.

1. Introduction

Consider the group $G = \text{SL}_2(\mathbb{R})$ that acts on the upper halfplane $\mathbb{H}$ by linear fractional transformations. Let $\Gamma \subset G$ be a cofinite discrete group, and let $d : \mathbb{H} \times \mathbb{H} \rightarrow \mathbb{R}_+$ denote the hyperbolic distance. Consider the counting function

$$N_\Gamma(R, z_0, z_1) = \# \{ \gamma \in \Gamma | d(z_0, \gamma z_1) \leq R \}.$$ 

The hyperbolic lattice point problem is the problem of estimating this function as $R \rightarrow \infty$. A typical result would be an asymptotic expansion of the form

$$N_\Gamma(R, z_0, z_1) = \frac{\kappa_\Gamma \pi}{\text{vol}(\Gamma \backslash \mathbb{H})} e^R + O(e^{R(\alpha + \varepsilon)})$$

for some $\alpha < 1$, where $\kappa_\Gamma = 2$ if $-I \in \Gamma$ and $\kappa_\Gamma = 1$ otherwise. The problem has been considered by numerous people including Delsarte [3], Huber [8, 9, 10] ($\Gamma$ cocompact), Patterson [20] ($\alpha = 3/4$ if there are no small eigenvalues), Selberg (unpublished) and Good [6] ($\alpha = 2/3$ if there are no small eigenvalues). Higher dimensional analogues have also been considered (see e.g. [14, 15, 4]), as well as the analogous problem for manifolds with non-constant curvature [16, 7]. For a discussion of the optimal choice of $\alpha$ we refer to [21], where the authors prove that $\alpha$ must be at least $1/2$ and they indicate that in many cases we should maybe expect (1) to hold with $\alpha = 1/2$.

Let $\varphi_{z_0, z_1}(\gamma)$ be $(2\pi)^{-1}$ times the angle between the vertical geodesic from $z_0$ to $\infty$ and the geodesic between $z_0$ and $\gamma z_1$. 

![Figure 1](image-url)
These normalized angles are equidistributed modulo one, i.e. for every interval $I \subset \mathbb{R}/\mathbb{Z}$ we have

$$ \frac{N_I^1(R, z_0, z_1)}{N_I(R, z_0, z_1)} \to |I| \text{ as } R \to \infty, $$

where

$$ N_I^1(R, z_0, z_1) = \#\{\gamma \in \Gamma | d(z_0, \gamma z_1) \leq R, \varphi_{z_0, z_1}(\gamma) \in I\}, $$

and $|I|$ is the length of the interval. This has been proved by Selberg (unpublished, see comment in [3, p. 120]), Nicholls [19] and Good [6].

In this paper we start by proving (2) with an error term:

**Theorem 1.** Let $K \subset \mathbb{H}$ be a compact set. There exists a constant $\alpha < 1$ possibly depending on $\Gamma$ and $K$ such that for all $z_0, z_1 \in K$ and all intervals $I$ in $\mathbb{R}/\mathbb{Z}$

$$ \frac{N_I^1(R, z_0, z_1)}{N_I(R, z_0, z_1)} = |I| + O(e^{R(\alpha-1+\varepsilon)}). $$

If we assume that the automorphic Laplacian on $\Gamma \setminus \mathbb{H}$ has no exceptional eigenvalues, i.e. eigenvalues in $]0, 1/4[$, we prove that we can take

$$ \alpha = 11/12. $$

If there are exceptional eigenvalues the exponent could become larger, depending on how close to zero they are. We prove Theorem 1 by proving asymptotic expansions for the exponential sums

$$ \sum_{\gamma \in \Gamma \atop d(z_0, \gamma z_1) \leq R} e(n \varphi_{z_0, z_1}(\gamma)), $$

where $n \in \mathbb{Z}$ and $e(x) = \exp(2\pi i x)$. The exponent 11/12 can certainly be improved. In fact our proof uses a variant of Huber’s method [8] which does not give the optimal bound even for the expansion (4). In principle Theorem 1 could be proved by using the method of Good from [6], which gives the best known error term in the hyperbolic lattice point problem [11]. The one missing point in [6] to prove Theorem 1 is the dependence of $n$ in the expansion of the exponential sum (4). Rather than patiently tracking down the $n$-dependence, we found it more to the point – albeit at the expense of poor error terms – to provide an alternative and more direct proof inspired by [8].

Recently Boca [2] considered a related problem: What happens if we order the elements according to $d(z_1, \gamma z_1)$ instead of $d(z_0, \gamma z_1)$? Let $\Gamma(N)$ be the principal congruence group of level $N$ i.e. the set of $2 \times 2$ matrices $\gamma$ satisfying $\gamma \equiv I \mod N$. Boca identified for these groups the limiting distribution using non-trivial bounds for Kloosterman sums. He proved the following [1]. Let $z_0, z_1 \in \mathbb{H}$ and let $\omega_{z_0, z_1}(\gamma)$ denote the angle in $[-\pi/2, \pi/2]$ between the vertical geodesic through $z_0$ and the geodesic containing $z_0$ and $\gamma z_1$ (if $z_0 = \gamma z_1$ you can assign $\omega_{z_0, z_1}(\gamma)$ the value 0 – it does not matter what you choose, since there are only a finite number of such $\gamma$’s).

For any interval $I \subset [-\pi/2, \pi/2]$ we consider the counting function

$$ \mathfrak{N}_I^1(R, z_0, z_1) = \#\{\gamma \in \Gamma | d(z_1, \gamma z_1) \leq R, \omega_{z_0, z_1}(\gamma) \in I\}. $$

We emphasize that the elements are ordered according to $d(z_1, \gamma z_1)$ instead of $d(z_0, \gamma z_1)$. We shall write $\mathfrak{N}_I(R, z_0, z_1)$ instead of $\mathfrak{N}_I^1(-\pi/2, \pi/2)(R, z_0, z_1)$. Following

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1 Readers consulting [2] should be warned that our notation differs slightly from Boca’s.
Boca we define
\[
\eta_{z_0, z_1}(t) = \frac{2y_0 y_1 (y_0^2 + y_1^2 + (x_0 - x_1)^2)}{(y_0^2 + y_1^2 + (x_0 - x_1)^2)^2 - ((y_1^2 - y_0^2 + (x_0 - x_1)^2) \cos(t) + 2y_0(x_0 - x_1) \sin(t))^2}.
\]

Then Boca proves the following result:

**Theorem 2.** Let \( \Gamma = \Gamma(N) \). For any interval \( I \subset [-\pi/2, \pi/2] \)
\[
\frac{\mathcal{N}_I^\gamma(R, z_0, z_1)}{\#\Gamma(R, z_0, z_1)} = \frac{1}{\pi} \int_I \eta_{z_0, z_1}(t) \, dt + O(e^{(7/8 - 1 + \epsilon)R})
\]
for any \( \epsilon > 0 \).

In the view of [1] Theorem [2] is equivalent to an expansion of \( \mathcal{N}^\gamma(R, z_0, z_1) \).

We generalize and refine Boca’s result: With data as above, \( I \subset \mathbb{R}/\mathbb{Z} \) and \( w \in \mathbb{H} \) we consider the counting function
\[
\mathcal{M}_I^\gamma(R, z_0, z_1, w) = \# \{ \gamma \in \Gamma \mid d(z_1, \gamma w) \leq R, \varphi_{z_0, w}(\gamma) \in I \}.
\]

We emphasize that we order according to \( d(z_1, \gamma w) \). As before we shall write \( \mathcal{M}(R, z_0, z_1, w) \) instead of \( \mathcal{M}^{[-1/2, 1/2]}(R, z_0, z_1, w) \). Besides the more general ordering our result is more refined in the sense that we can distinguish between angles that differ by \( \pi \). Consider
\[
\rho_{z_0, z_1}(\omega) = \frac{2y_0 y_1 ((x_0 - x_1)^2 + y_0^2 + y_1^2)(1 - \cos(2\pi \omega)) + 2y_0^2 \cos(2\pi \omega) + 2(x_1 - x_0)y_0 \sin(2\pi \omega))}{(y_0^2 + y_1^2 + (x_0 - x_1)^2)^2}.
\]

Then we prove the following result:

**Theorem 3.** Let \( \Gamma \) be any cofinite Fuchsian group. There exists \( \alpha < 1 \) such that for any \( I \subset \mathbb{R}/\mathbb{Z} \) we have
\[
\frac{\mathcal{M}_I^\gamma(R, z_0, z_1, w)}{\mathcal{M}(R, z_0, z_1, w)} = \int_I \rho_{z_0, z_1}(\omega) \, d\omega + O(e^{(\alpha - 1 + \epsilon)R})
\]
for any \( \epsilon > 0 \).

Note that in the special case of \( \Gamma = \Gamma(N) \) and \( w = z_1 \) this implies Theorem [2] (with a poorer error term though), since
\[
\eta_{z_0, z_1}(2\pi t) = \rho_{z_0, z_1}(t) + \rho_{z_0, z_1}(t + 1/2).
\]

We will prove that Theorem [3] follows from Theorem [1].

Whereas Boca is using a non-trivial bound for Kloosterman sums, we are utilizing the fact that for any group there is a spectral gap between the zero eigenvalue of the Laplacian and the first non-zero eigenvalue. As in Theorem [1] the \( \alpha \) in Theorem [3] generally depends on the size of the first non-zero eigenvalue.
We remark that all the results presented here, can easily be phrased in terms of points in the orbit $\Gamma z_1$, rather than elements in $\Gamma$, since
\[ \# \{ z \in \Gamma z_1 \mid d(z_0, z) \leq R \} = \frac{N_{\Gamma}(R, z_0, z_1)}{\mid \Gamma z_1 \mid}, \]
where $\Gamma z_1$ denotes the stabilizer of $z_1$.

2. Effective equidistribution of angles

Let $G = \text{SL}_2(\mathbb{R})$. The group $G$ acts on the upper halfplane $\mathbb{H}$ by linear fractional transformations
\[ g z = \frac{az + b}{cz + d}, \quad g = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in G, \quad z \in \mathbb{H}. \]
Let $\Gamma \subset \text{SL}_2(\mathbb{R})$ be discrete and cofinite. For simplicity we assume that $-I \notin \Gamma$. If $-I \in \Gamma$ we need to multiply all main terms by 2.

For $z \in \mathbb{H}$ we let $r = r(z)$ and $\varphi = \varphi(z)$ be the geodesic polar coordinates of $z$. These are related to the rectangular coordinates by
\[ z = \left( \begin{array}{cc} \cos \varphi(z) & \sin \varphi(z) \\ -\sin \varphi(z) & \cos \varphi(z) \end{array} \right) \exp(-r(z))i. \]
We note that if $z_0 = x_0 + iy_0$ and we let
\[ \gamma_0 = \left( \begin{array}{cc} 1/\sqrt{y_0} & -x_0/\sqrt{y_0} \\ 0 & \sqrt{y_0} \end{array} \right) \]
then it is straightforward to check that $\gamma_0 z_0 = i$. We see that
\[ \varphi_{z_0, z_1}(\gamma) = \varphi_{i, \gamma_0 z_1}(\gamma_0 \gamma_0^{-1}) = \varphi(\gamma_0 \gamma_0^{-1}(\gamma_0 z_1))/\pi \]
and
\[ d(z_0, \gamma z_1) = d(i, \gamma_0 z_0^{-1}(\gamma_0 z_1)) = r(\gamma_0 \gamma_0^{-1}(\gamma_0 z_1)). \]
Therefore after conjugation of the group $\Gamma$ the counting problems in the introduction may be formulated in terms of $r(\gamma z)$ and $\varphi(\gamma z)$ with $z = \gamma_0 z_1$.

The Laplacian for the $G$-invariant measure $d\mu(z) = dxdy/y^2$ on $\mathbb{H}$ is given in Cartesian coordinates by
\[ \Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right). \]

In geodesic polar coordinates the Laplace operator is given by
\[ \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\tan \varphi} \frac{\partial}{\partial \varphi} + \frac{1}{4 \sinh^2(r)} \frac{\partial^2}{\partial \varphi^2}. \]

Consider $L^2(\Gamma \backslash \mathbb{H}, d\mu(z))$ with inner product $\langle f, g \rangle = \int_{\Gamma \backslash \mathbb{H}} f \overline{g} d\mu(z)$ and norm $\|f\|_2 = \sqrt{\langle f, f \rangle}$. The Laplacian induces an operator on $L^2(\Gamma \backslash \mathbb{H}, d\mu(z))$ called the automorphic Laplacian defined as follows: Consider the operator defined by $-\Delta f$ on smooth, bounded, $\Gamma$-invariant functions satisfying that $-\Delta f$ is also bounded. This operator is densely defined in $L^2(\Gamma \backslash \mathbb{H})$ and is in fact essentially selfadjoint. The closure of this operator is called the automorphic Laplacian. By standard abuse of notation we also denote this operator by $-\Delta$.

The automorphic Laplacian is selfadjoint and non-negative and has eigenvalues
\[ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \lambda_i \leq \ldots \]
with the number of eigenvalues being finite or $\lambda_i \to \infty$. It has a continuous spectrum $[1/4, \infty]$ with multiplicity equal to the number of inequivalent cusps.

By standard operator theory for selfadjoint operators (See e.g. [L3]) the resolvent $R(s) = (-\Delta - s(1 - s))^{-1}$ is a bounded operator which is meromorphic in $s$ for
s(1 − s) off the spectrum of −Δ. For an eigenvalue λi outside the continuous spectrum the operator \( R(s) = P_i/(\lambda_i - s(1-s)) \) is analytic at s satisfying s(1−s) = λi
where \( P_i \) is the projection to the \( \lambda_i \)-eigenspace. In particular for \( \lambda = 0 \) we note
that
\[
R(s) - \frac{P_0}{s(1-s)}
\]
is analytic for \( \Re(s) > 1 - \delta \) for some \( \delta \) where \( P_0 f = \int f(z) d\mu(z)/\text{vol}(\Gamma \setminus \mathbb{H}) \) is the projection to the 0-eigenspace. (Alternatively one may quote [11, Theorem 7.5] to obtain the same result.)

We define for \( \Re(s) > 1 \)
\[
G_n(z, s) = \sum_{\gamma \in \Gamma} \frac{e(n\varphi(\gamma z)/\pi)}{(\cosh(r(\gamma z)))^s}.
\]
We recall that
\[
cosh(r(\gamma z)) = 1 + 2u(\gamma z, i),
\]
where \( u(z, w) \) is the point pair invariant defined by
\[
u(z, w) = \frac{|z - w|^2}{4\Im(z)\Im(w)}.
\]
Hence
\[
|\frac{e(n\varphi(z)/\pi)}{(\cosh(r(z)))^s}| \leq \frac{1}{(1 + 2u(z, i))^{\Re(s)}}.
\]

It therefore follows from [22, Theorem 6.1] and the discussion leading up to it that the sum (10) converges absolutely and uniformly on compact sets and the limit is \( \Gamma \)-automorphic, and bounded in \( z \) – in particular square integrable on \( \Gamma \setminus \mathbb{H} \).

By applying the Laplace operator to \( G_n(z, s) \) a straightforward calculation shows that
\[
(-\Delta - s(1-s))G_n(z, s) = s(s + 1)G_n(z, s + 2) + \sum_{\gamma \in \Gamma} \frac{n^2e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z)) (\cosh(r(\gamma z)))^s}.
\]
The sum on the right converges absolutely and uniformly on compacta for \( \Re(s) > -1 \). Since \( G_n(z, s) \) is square integrable, we may invert (11) using the resolvent
\[
R(s) = (-\Delta - s(1-s))^{-1},
\]
so we have
\[
G_n(z, s) = R(s) \left( s(s + 1)G_n(z, s + 2) + \sum_{\gamma \in \Gamma} \frac{n^2e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z)) (\cosh(r(\gamma z)))^s} \right).
\]
The right-hand-side is meromorphic in s for \( \Re(s) > 1/2 \) since the resolvent is holomorphic for \( s(1-s) \) not in the spectrum of the automorphic Laplacian. This gives the meromorphic continuation of \( G_n(z, s) \) to \( \Re(s) > 1/2 \). The only potential poles are at \( s = 1 \) and \( s = s_j \) where \( s_j(1-s_j) \) is a small eigenvalue for the automorphic Laplacian. Using the analyticity of (7) we see that the pole at \( s = 1 \) has residue
\[
\frac{1}{\text{vol}(\Gamma \setminus \mathbb{H})} \int_{\Gamma \setminus \mathbb{H}} \left( 2G_n(z, 3) + \sum_{\gamma \in \Gamma} \frac{n^2e(n\varphi(\gamma z)/\pi)}{\sinh^2(r(\gamma z)) \cosh(r(\gamma z))} \right) d\mu(z).
\]
By unfolding the integral we find that this equals
\[
\frac{1}{\text{vol}(\Gamma \setminus \mathbb{H})} \int_{\mathbb{H}} \left( 2\pi(n\varphi(z)/\pi) \cosh^3(r(z)) + \frac{n^2e(n\varphi(z)/\pi)}{\sinh^2(r(z)) \cosh(r(z))} \right) d\mu(z).
\]
Changing to polar coordinates we find

\[
16 \frac{1}{\text{vol}(\Gamma \setminus \mathbb{H})} \int_0^\infty \int_0^\pi \left( \frac{2e(n\varphi/\pi)}{\cosh^3(r)} + \frac{n^2e(n\varphi/\pi)}{\sinh^2(r) \cosh(r)} \right) 2 \sinh(r) d\varphi dr,
\]

which equals

\[
17 \frac{2\pi \delta_{n=0}}{\text{vol}(\Gamma \setminus \mathbb{H})}.
\]

This follows since

\[
\int_0^\infty 2 \sinh(r) \cosh(r)^3 dr = 1.
\]

From a Wiener-Ikehara Tauberian theorem (see e.g. [18, Theorem 3.3.1 and Exercises 3.3.3+3.3.4]) we may conclude that

\[
18 \sum_{\gamma \in \Gamma} e(n\varphi(\gamma)/\pi) \leq 2\pi \frac{\delta_{n=0}}{\text{vol}(\Gamma \setminus \mathbb{H})} R + o(R).
\]

This implies immediately – via Weyl’s criterion – that the angles \(\varphi(\gamma)/\pi\) are equidistributed modulo 1.

Since we intend to obtain a power saving in the remainder term we investigate \(G_n(z,s)\) a bit more careful:

**Lemma 1.** Write \(s = \sigma + it\). For \(z\) in a fixed compact set \(K \subset \mathbb{H}\), \(|t| > 1\) and \(\sigma > \sigma_0 > 1/2\) we have

\[
G_n(z,s) = O(|t| (|t|^2 + n^2)),
\]

where the implied constant may depend on \(\Gamma\), \(K\), and \(\sigma_0\).

**Proof.** We recall that [13 V (3.16)]

\[
19 \|R(s)\|_{\infty} \leq \frac{1}{\text{dist}(s(1-s), \text{spec}(-\Delta))} \leq \frac{1}{|t| (2\sigma - 1)},
\]

where \(\|\cdot\|_{\infty}\) denotes the operator norm. For \(\sigma > 3/2\) we have

\[
20 \|G_n(z,s)\|_2 \leq \|G_0(z,3/2)\|_2 = O(1).
\]

For \(\sigma > \sigma_0\) we may use this and [13] to conclude that

\[
21 \|G_n(z,s)\|_2 \leq \|R(s)\|_{\infty} \left( \|s(s+1)G_n(z,s+2)\|_2 + \left\| \sum_{\gamma \in \Gamma} \frac{n^2e(n\varphi(\gamma)/\pi)}{\sinh^2(r(\gamma))(\cosh(r(\gamma)))^{1/2}} \right\|_2 \right)
\]

\[
\leq \frac{1}{|t| (2\sigma - 1)} \left( |t|^2 \|G_0(z,3/2)\|_2 + \left\| \sum_{\gamma \in \Gamma} \frac{n^2}{\sinh^2(r(\gamma))(\cosh(r(\gamma)))^{1/2}} \right\|_2 \right)
\]

\[
= O(|t|^{-1} (|t|^2 + n^2)).
\]

Using this and [11] we find

\[
22 \|\Delta G_n(z,s)\|_2 = O(|t| (|t|^2 + n^2)).
\]

We can now use the Sobolev embedding theorem and elliptic regularity theory to get a pointwise bound:

For any non-empty open set \(\Omega\) in \(\mathbb{R}^2\) we consider the classical Sobolev space \(W^{k,p}(\Omega)\) with corresponding norm \(\|\cdot\|_{W^{k,p}(\Omega)}\) (See [1 p. 59]). Whenever \(\Omega\) satisfies
the cone condition (See [11 p. 82]) the Sobolev embedding theorem [11 Thm 4.12]) gives an embedding
\[ W^{2,2}(\Omega) \rightarrow C_B(\Omega) \]
where \( C_B(\Omega) \) is the set of bounded continuous functions on \( \Omega \) equipped with the sup norm. In particular for \( f \in W^{2,2}(\Omega) \) we have
\[ \sup_{z \in \Omega} |f(z)| \leq C \|f\|_{W^{2,2}(\Omega)} \]
where \( C \) is a constant which depends only on \( \Omega \).

By elliptic regularity theory, if \( \Delta_E = \partial^2/\partial x^2 + \partial^2/\partial y^2 \) is the Euclidean Laplace operator we have also that if \( u \in W^{1,2}(\Omega) \) satisfies \( \Delta_E u \in L^2(\Omega) \) (weak derivative) then
\[ \|u\|_{W^{2,2}(\Omega')} \leq C' (\|u\|_{L^2(\Omega)} + \|\Delta_E u\|_{L^2(\Omega)}) \]
for all \( \Omega' \subset \Omega \) which satisfies that the closure of \( \Omega' \) is compact and contained in \( \Omega \). Here \( C' \) is a constant which depends only on \( \Omega \), and \( \Omega' \) (See [12 Theorem 8.2.1]).

We can use this general theory to bound \( |G_n(z, s)| \) in the following way: For every \( z \) in the compact set \( K \) we fix a small open (Euclidean) disc \( \Omega_z \) centered at \( z \) with some radius such that its closure \( \overline{\Omega_z} \) is contained in \( \mathbb{H} \). Let \( \Omega_i \) be the open disc with half the radius. By compactness of \( K \) the cover \( \{\Omega_i\} \) admits a finite subcover i.e. \( K \subset \bigcup_{i=1}^{n} \Omega_i \), for \( z_i \in K \). Choose as a fundamental domain for \( \Gamma \) a normal polygon \( F \). Since \( \Gamma \) is a discrete subgroup of \( SL_2(\mathbb{R}) \), \( \Omega_i \) intersects non-trivially with \( \gamma F \) for only finitely many (say \( n_i \)) \( \gamma \in \Gamma \) (See [14 1.6.2 (3)]).

Therefore, for any automorphic function \( f \),
\[ \|f\|^2_{L^2(\Omega_i)} := \int_{\Omega_i} |f(z)|^2 \, dx \, dy \]
\[ \leq n_i y_i^2 \int_F |f(z)|^2 \, d\mu(z) = n_i y_i^2 \|f\|^2 \]
and
\[ \|\Delta_E f\|^2_{L^2(\Omega_i)} := \int_{\Omega_i} |\Delta_E f(z)|^2 \, dx \, dy \]
\[ \leq n_i y_i^{-2} \int_F |\Delta f(z)|^2 \, d\mu(z) = n_i y_i^{-2} \|\Delta f\|^2 \]
where \( y_i < \infty \) and \( y_i > 0 \) are heights over and under \( \Omega_i \). It is straightforward to verify that \( G_n(z, s) \) is in \( W^{1,2}(\Omega_i) \) (since it is continuously differentiable) and that \( \Omega_i \) has the cone property, so we may use the above inequalities to conclude
\[ \sup_{z \in K} |G_n(z, t)| \leq \max_{i=1}^{n} \sup_{z \in \Omega_i} |G_n(z, s)| \]
\[ \leq \max_i C_i \|G_n(z, s)\|_{W^{2,2}(\Omega_i)} \text{ by (21)} \]
\[ \leq \max_i C_i C_i' \|G_n(z, s)\|_{L^2(\Omega_i)} + \|\Delta_E G_n(z, s)\|_{L^2(\Omega_i)} \text{ by (25)} \]
\[ \leq \max_i C_i C_i' \|G_n(z, s)\|_2 + \|\Delta G_n(z, s)\|_2 \text{ by (21) and (27)} \]
which concludes the proof. \( \square \)

We note that Lemma 1 implies that
\[ G_n(z, s) = O(|t|^3) \]
when \( |n| \leq |t| \), and by applying the Phragmén-Lindelöf theorem we may reduce the exponent to \( \max(6(1 - \sigma) + \varepsilon, 0) \) for any \( \varepsilon > 0 \).
We may now use the meromorphic continuation of $G_n(z, s)$ and Lemma 1 to get an asymptotic expansion with error term for the sum in (13). We will assume that there are no exceptional eigenvalues, which implies that $G_n(z, s)$ is regular in $\Re(s) > 1/2$. If this is not the case $G_n(z, s)$ will still be regular in $\Re(s) > h$ for some $h < 1$. In (33) below we then move the line of integration to $\Re(s) > h + \varepsilon$. Proceeding with the obvious changes still gives a nontrivial error term in the end. We shall not dwell on the details.

Let $\psi_U : \mathbb{R}_+ \to \mathbb{R}, U \geq U_0$, be a family of smooth non-increasing functions with

$$\psi_U(t) = \begin{cases} 1 & \text{if } t \leq 1 - 1/U \\ 0 & \text{if } t \geq 1 + 1/U, \end{cases}$$

and $\psi_U^{(j)}(t) = O(U^j)$ as $U \to \infty$. For $\Re(s) > 0$ we let

$$M_U(s) = \int_0^\infty \psi_U(t)t^{s-1}dt$$

be the Mellin transform of $\psi_U$. Then we have

$$M_U(s) = \frac{1}{s} + O\left(\frac{1}{U}\right)$$

as $U \to \infty$

and for any $c > 0$

$$M_U(s) = O\left(\frac{1}{|s|}\left(\frac{U}{1 + |s|}\right)^c\right)$$

as $|s| \to \infty$.

Both estimates are uniform for $\Re(s)$ bounded. The first is a mean value estimate while the second is successive partial integration and a mean value estimate. We use here the estimate $\psi_U^{(j)}(t) = O(U^j)$. The Mellin inversion formula now gives

$$\sum_{\gamma \in \Gamma} e(n\varphi(\gamma z)/\pi)\psi_U\left(\frac{\cosh(r(\gamma z))}{R}\right) = \frac{1}{2\pi i} \int_{\Re(s)=2} G_n(z, s)M_U(s)R^s ds.$$

We note that by Lemma 1 the integral is convergent as long as $G_n(z, s)$ has polynomial growth on vertical lines. We now move the line of integration to the line $\Re(s) = h$ with $h < 1$ by integrating along a box of some height and then letting this height go to infinity. Using Lemma 1 we find that the contribution from the horizontal sides goes to zero. Assume that $s = 1$ is the only pole of the integrand with $\Re(s) \geq 1/2 + \varepsilon$. Then using Cauchy’s residue theorem we obtain

$$\frac{1}{2\pi i} \int_{\Re(s)=2} G_n(z, s)M_U(s)R^s ds$$

$$= \text{Res}_{s=1} \left( G_n(z, s)M_U(s)R^s \right) + \frac{1}{2\pi i} \int_{\Re(s)=1/2+\varepsilon} G_n(z, s)M_U(s)R^s ds$$

$$= \frac{2\pi R}{\text{vol}(\Gamma \backslash \mathbb{H})} + O(R/U) + \frac{1}{2\pi i} \int_{\Re(s)=1/2+\varepsilon} G_n(z, s)M_U(s)R^s ds.$$

If there are other small eigenvalues there are additional main terms. In bypassing we note that their coefficients will depend on the $n$-th hyperbolic Fourier coefficients of the eigenfunctions corresponding to small eigenvalues. (See [1], Theorem 4 p. 116.) If we choose $c = 3 + \varepsilon$ and use Lemma 1 the last integral is $O(R^{1/2+\varepsilon}U^{3+\varepsilon}(n^2 + 1))$. The interval with $|\Im(s)| \leq 1$ can easily be dealt with using the bound

$$|R(s)||_{\infty} \leq \max_j \left| \frac{1}{\sigma(1 - \sigma)} - \frac{1}{\sigma_j(1 - \sigma_j)} \right|,$$

which in turn gives us an estimate for $G_n(z, s)$. 


If } n = 0 \text{ we see that by further requiring } \psi_U(t) = 0 \text{ if } t \geq 1 \text{ and } \psi_U(t) = 1 \text{ if } t \leq 1, \text{ we have}

\[
\sum_{\gamma \in \Gamma} \psi_U \left( \frac{\cosh(r(\gamma z))}{R} \right) \leq \sum_{\gamma \in \Gamma} \frac{1}{\cosh(r(\gamma z))} \leq \sum_{\gamma \in \Gamma} \frac{1}{\cosh(r(\gamma z))}.
\]

Choosing } U = R^{1/8} \text{ we therefore obtain:}

**Lemma 2.** With assumptions as above we have

\[
(34) \quad \#\{ \gamma \in \Gamma | \cosh(r(\gamma z)) \leq R \} = \frac{2\pi R}{\text{vol}(\Gamma \setminus \mathbb{H})} + O(R^{7/8+\varepsilon}).
\]

We note that this implies (1) with } \alpha = 7/8. \text{ Using this we can now deal with the general case. To get from a smooth cut-off to a sharp one we notice that if } \psi_U(t) = 1 \text{ for } t \leq 1 \text{ then we may bound the difference}

\[
\sum_{\gamma \in \Gamma} e(n\varphi(\gamma z)/\pi) \psi_U \left( \frac{\cosh(r(\gamma z))}{R} \right) - \sum_{\gamma \in \Gamma} e(n\varphi(\gamma z)/\pi) = O \left( \sum_{R<\cosh(r(\gamma z))\leq R(1+1/U)} 1 \right)
\]

which by Lemma 2 is } O(R/U + R^{7/8+\varepsilon}). \text{ Combining the above we find that for } n \neq 0

\[
\sum_{\gamma \in \Gamma} e(n\varphi(\gamma z)/\pi) \leq O(R^{1/2+\varepsilon}U^{3+\varepsilon}(n^2 + 1) + R/U + R^{7/8+\varepsilon}).
\]

Using the Erdős-Turán inequality [5, Theorem 3] we find that

\[
\frac{\#\{ \gamma \in \Gamma | \cosh(r(\gamma z)) \leq R, \varphi(\gamma z)/\pi \in I \}}{\#\{ \gamma \in \Gamma | \cosh(r(\gamma z)) \leq R \}} = |I| + O(1/M + R^{-1/2+\varepsilon}U^{3+\varepsilon}M^2 + \log M(1/U + R^{-1/8+\varepsilon}))
\]

for any } M. \text{ Letting } M = U = R^{1/12} \text{ we arrive at the following (still assuming that there are no small eigenvalues):}

**Theorem 4.** For all } \varepsilon > 0 \text{ and } I \subset \mathbb{R}/\mathbb{Z} \text{ we have}

\[
\frac{\#\{ \gamma \in \Gamma | \cosh(r(\gamma z)) \leq R, \varphi(\gamma z)/\pi \in I \}}{\#\{ \gamma \in \Gamma | \cosh(r(\gamma z)) \leq R \}} = |I| + O(R^{-1/12+\varepsilon}).
\]

Theorem 1 follows easily.

### 3. Proof of Theorem 3

We wish to find the limiting distribution of the number of lattice points in angular sectors defined from } z_0 \text{ when ordering the lattice points } \gamma w \text{ according to the distance to } z_1. \text{ More precisely we want to find the asymptotics of}

\[
A^l_\varepsilon(R, z_0, z_1, w) = \#\{ \gamma \in \Gamma | d(z_1, \gamma w) \leq R, \varphi_{z_0, w}(\gamma) \in I \}.
\]

Our strategy for finding the asymptotics is the following: We find the hyperbolic distance from } z_0 \text{ to the intersection(s) between the hyperbolic circle with center at } z_1 \text{ and radius } R \text{ and the geodesic through } z_0 \text{ determined by an angle } t \in [-\pi, \pi] \text{ relative to the vertical geodesic through } z_0. \text{ Once we have an asymptotic expression for this distance we can make a Riemann sum approximation of the counting function (35). The summands can be estimated via Theorem 1 leading to a proof of Theorem 3.}

We may safely assume that } z_0 = i \text{ – it is easy to extend our results to the general case. We would like to find the distance from } i \text{ to the relevant intersection point
which will be denoted by \( w' = x' + iy' \). There are 2 intersection points, but we choose the one which has negative real part for \( t > 0 \). This distance will be denoted \( Q(z_1, t, R) \).

Now fix \( z_1, t \) and \( R \). Let \( \alpha \in \mathbb{R} \) and \( \delta \in \mathbb{R}_+ \) denote the center and the radius respectively of the Euclidean half-circle which is the geodesic through \( i \) and \( w' \). From Figure 3 it is clear that

\[
\delta = 1/|\sin(t)|, \quad \alpha = -\cot(t)
\]

if \( t \neq 0, \pm \pi \). Thus we see that

\[
y' = \sqrt{\delta^2 - (x' - \alpha)^2} = \sqrt{1 - x'^2 + 2\alpha x'}.
\]

On the other hand it is well-known that the locus of points on the hyperbolic circle with center at \( x_1 + iy_1 \) and radius \( R \) is determined by the equation

\[
|x_1 + iy_1 \cosh(R) - z| = y_1 \sinh(R),
\]

which is equivalent to

\[
x^2 + y^2 + x_1^2 - 2xx_1 + y_1^2 = 2y_1 \cosh(R).
\]

Using the expression for \( y' \) given in (37) we obtain the equation

\[
\frac{\beta}{2} + (\alpha - x_1)x' = y_1 \cosh(R) \sqrt{\delta^2 - (x' - \alpha)^2}
\]

for \( x' \), where \( \beta = |z_1|^2 + 1 \). By squaring (38) we get the quadratic equation

\[
\left( \frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} + 1 \right) x'^2 + \left( \frac{\beta(\alpha - x_1)}{y_1^2 \cosh^2(R)} - 2\alpha \right) x' + \frac{\beta^2}{4y_1^2 \cosh^2(R)} - 1 = 0,
\]

with the solution

\[
x' = \frac{\alpha - \beta(\alpha - x_1)}{2y_1^2 \cosh^2(R) - \text{sign}(t) \sqrt{\delta^2 + \frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} - \beta^2 + 2\beta^2 \frac{\alpha(\alpha - x_1)}{y_1^2 \cosh^2(R)} - \frac{\alpha(\alpha - x_1)}{y_1^2 \cosh^2(R)}}}.
\]
Naturally, the quadratic equation has 2 solutions, but the solution above is the intersection point we are interested in. The distance \( Q(z_1, t, R) \) is

\[
Q(z_1, t, R) = \log \left( \frac{|w' + i| + |w' - i|}{|w' + i| - |w' - i|} \right) .
\]

(40)

We note that

\[
\frac{|w' + i| + |w' - i|}{|w' + i| - |w' - i|} = \frac{x'^2 + y'^2 + 1 + \sqrt{(x'^2 + y'^2 + 1)^2 - 4y'^2}}{2y'} = \frac{1 + \alpha x' + \delta |x'|}{y'} = \frac{1 + \alpha x' - \delta' x'}{y'},
\]

(41)

where \( \delta' = 1/\sin(t) \). Using Taylor’s formula with remainder we see that

\[
\text{sign}(t) \sqrt{\delta^2 + \frac{(\alpha - x_1)^2}{y_1^2 \cosh^2(R)} - \frac{\beta^2}{4y_1^2 \cosh^2(R)} - \frac{\alpha \beta (\alpha - x_1)}{y_1^2 \cosh^2(R)}} = \delta' + \frac{(\alpha - x_1)^2}{2y_1^2 \cosh^2(R)} - \frac{\beta^2}{4y_1^2 \cosh^2(R)} - \frac{\alpha \beta (\alpha - x_1)}{y_1^2 \cosh^2(R)} + O \left( \frac{\delta}{\cosh^4(R)} \right)
\]

as \( R \to \infty \), where the constant implied depends on \( z_1 \). From this and (41) we deduce that

\[
x' = \frac{\alpha - \delta'}{1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2} + \frac{O((1 + \delta)e^{-R})}{1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2}
\]

(42)

and hence

\[
1 + \alpha x' - \delta' x' = 1 + \frac{(\alpha - \delta')^2}{1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2} + \frac{O(\delta(1 + \delta)e^{-R})}{1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2}.
\]

(43)

This implies that

\[
\sin^2(t) \left( 1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2 \right) (1 + \alpha x' - \delta' x') = 2 + 2 \cos(t) + O(e^{-R}).
\]

(44)

Now we look at

\[
y'^2 \left( 1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2 \right).
\]
Using Taylor’s formula as before we get

\[
\begin{align*}
y^2 \left(1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2 \right)^2 &= \left(1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2 \right)^2 - \left( \alpha - \frac{\beta(\alpha - x_1)}{2y_1^2 \cosh^2(R)} \right) \\
\text{sign}(t) \sqrt{\delta^2 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2} &= \left( \alpha - \frac{\beta(\alpha - x_1)}{2y_1^2 \cosh^2(R)} \right)^2 + 2\alpha \left(1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2 \right) \\
\text{sign}(t) \sqrt{\delta^2 + \left( \frac{\alpha - x_1}{y_1 \cosh^2(R)} \right)^2} &= \left( \alpha - \frac{\beta(\alpha - x_1)}{2y_1^2 \cosh^2(R)} \right)^2 + \frac{1}{y_1^2 \cosh^2(R)} \left( \frac{\beta^2}{4} + (\alpha - x_1)^2 + \alpha \beta(\alpha - x_1) + 2\alpha^2(\alpha - x_1)^2 - \delta'(\alpha - x_1)(\beta + 2\alpha(\alpha - x_1)) \right) + O \left( \frac{\delta^4}{\cosh^4(R)} \right) \\
&= \frac{(\beta - (\beta - 2) \cos(t) + 2x_1 \sin(t))^2(1 + \cos(t))^2}{4y_1^2 \cosh^2(R) \sin^2(t)} + O \left( \frac{\delta^4}{\cosh^4(R)} \right)
\end{align*}
\]

as \( R \to \infty \). From this we conclude that

\[
\begin{align*}
\frac{1 + \cos(t)}{2y^2 \left(1 + \left( \frac{\alpha - x_1}{y_1 \cosh(R)} \right)^2 \right)^2 \sin^2(t)} &= \frac{y_1 \cosh(R)}{\beta - (\beta - 2) \cos(t) + 2x_1 \sin(t)} + O(e^{-4R}) \\
&= \frac{y_1 e^R}{2(\beta - (\beta - 2) \cos(t) + 2x_1 \sin(t))} + O(e^{-R}).
\end{align*}
\]

We are interested in \( e^{Q(z, t, R)} \). Combining (11), (40), (44) and (45) we conclude that

\[
\begin{align*}
e^{Q(z, t, R)} &= \frac{1 + \alpha x' + \delta' x'}{y'} = \frac{2y_1 e^R}{\beta - (\beta - 2) \cos(t) + 2x_1 \sin(t)} + O(1).
\end{align*}
\]

To finish the proof we use the following elementary lemma which ‘integrates’ Theorem [1] over more general regions:

**Lemma 3.** Let \( D(R, \theta) : \mathbb{R}_+ \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}_+ \) be a function which satisfies \( e^{D(R, \theta)} = k(\theta) e^R + O(e^{\beta R}) \) for some \( \beta < 1 \) uniformly in \( \theta \). Assume that \( k(\theta) \in C^1(\mathbb{R}/\mathbb{Z}) \). Then as \( R \to \infty \)

\[
N_{\Gamma, D}(R, z_0, z_1) := \# \{ \gamma \in \Gamma \mid d(z_0, z_1, \gamma) = D(R, \varphi_{z_0, z_1}(\gamma)), \varphi_{z_0, z_1}(\gamma) \in I \} = \frac{K \pi}{\text{vol}(\Gamma \backslash \mathbb{H})} \int_{t} k(\theta) d\theta e^R + O(e^{\delta R})
\]

for some \( \delta < 1 \).

**Proof.** Let \( B = B(R) \) be a integer valued function of \( R \) to be determined later. For each integer \( j \leq B \) we choose \( \omega_j, \omega^j \in \left[a + \frac{(j - 1)(b - a)}{B}, a + \frac{j(b - a)}{B}\right] \) such that

\[
k(\omega_j) = \inf \left\{ k(\omega) \mid \omega \in \left[a + \frac{(j - 1)(b - a)}{B}, a + \frac{j(b - a)}{B}\right] \right\}
\]
and
\[ k(\omega) = \sup \left\{ k(\omega) \mid \omega \in \left[ a + \frac{(j-1)(b-a)}{B}, a + \frac{j(b-a)}{B} \right] \right\}. \]

We split the interval in \( B \) equal intervals (and compensate for counting the end-points twice) to get
\[ N^I_{\Gamma,D}(R, z_0, z_1) = \sum_{j=0}^{B} N^I_{\Gamma,D}(a + \frac{(j-1)(b-a)}{B}, a + \frac{j(b-a)}{B})(R, z_0, z_1) \]
\[ - \sum_{j=1}^{B-1} N^I_{\Gamma,D}(b-a, a + \frac{(j-1)(b-a)}{B})(R, z_0, z_1). \]

The last sum is \( O(Be^{\alpha R}) \) by Theorem 1 and the assumption on \( D(R, \theta) \). The first sum can be evaluated as follows. By using Theorem 1 again we have
\[ N^I_{\Gamma,D}(a + \frac{(j-1)(b-a)}{B}, a + \frac{j(b-a)}{B})(R, z_0, z_1) \leq \frac{k_{\Gamma,\pi}(b-a)Ce^R}{B \text{vol}(\Gamma \setminus \mathbb{H})} \omega_j e^{R} + Ce^{\alpha R}. \]

Summing this inequality we find the Riemann sums
\[ \sum_{j=0}^{B} \omega_j \frac{(b-a)}{B}, \sum_{j=0}^{B} \omega_j \frac{j(b-a)}{B}. \]

Since \( k \) is \( C^1 \) these converge to \( \int_I k(\theta)d\theta \) with rate \( O(1/B) \) as is seen using the mean value theorem. We therefore find that
\[ N^I_{\Gamma,D}(R, z_0, z_1) = \frac{k_{\Gamma,\pi}}{\text{vol}(\Gamma \setminus \mathbb{H})} \int_I k(\theta)d\theta e^{R} + O(e^R/B) + O(Be^{\alpha R}). \]

Balancing the error terms we get the result.

\[ \square \]

We can now finish the proof of Theorem 3. Let \( \rho_{z_0, z_1}(\omega) \) denote the fraction
\[ \frac{2y_0y_1}{((x_0 - x_1)^2 + y_0^2 + y_1^2)(1 - \cos(2\pi \omega)) + 2y_0^2 \cos(2\pi \omega) + 2(x_1 - x_0)y_0 \sin(2\pi \omega)}. \]

We start with the case \( z_0 = i \). Equation (46) allows us to use Lemma 3 which gives Theorem 3 immediately. The general case can easily be reduced to the case where \( z_0 = i \) by conjugation of \( \Gamma \) with the element \( \left( \sqrt{\frac{\pi}{2\omega}} x_0/\sqrt{\pi}, 0 \right) \). This finishes the proof of Theorem 3.

\[ \square \]

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