BRIDGELAND STABILITY CONDITIONS ON SOME THREEFOLDS OF GENERAL TYPE

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ABSTRACT. We prove the Bogomolov-Gieseker type inequality conjectured by Bayer, Macrì and Toda for some products of three curves. This gives the first examples of Bridgeland stability conditions on some threefolds of general type. The key ingredients are the spreading out technique, Frobenius morphism and Bogomolov’s inequality for product type varieties in positive characteristic, proved by the author recently.

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1. INTRODUCTION

Since Bridgeland’s introduction in [6], stability conditions for triangulated categories have drawn a lot of attentions, and have been investigated intensively. The existence of stability conditions on three-dimensional varieties is often considered the biggest open problem in the theory of Bridgeland stability conditions.

In [3], Bayer, Macrì and Toda introduced a conjectural construction of Bridgeland stability conditions for any projective threefold. Here the problem was reduced to proving a Bogomolov-Gieseker type inequality for the third Chern character of tilt-stable objects. It has been shown to hold for some Fano 3-folds [20, 22, 15, 5, 21], abelian 3-folds [18, 19, 3], étale quotients of abelian 3-folds [3], toric threefolds [5], product threefolds of projective spaces and abelian varieties [11] and quintic threefolds [10]. However, counterexamples of the original Bogomolov-Gieseker type inequality are found (see [23]). The modification of the original inequality for any Fano threefolds is proved in [5, 21], and it still implies the existence of stability conditions on such threefolds.

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All these known examples of Bridgeland stability conditions are on threefolds with non-positive Kodaira dimension. In this paper, we prove the original Bogomolov-Gieseker type inequality for some products of curves with product type polarizations. This gives the first examples of Bridgeland stability conditions on some threefolds of general type.

**Theorem 1.1.** Let $X = C_1 \times C_2 \times C_3$ be a product of three complex smooth projective curves with projections $f_i : X \to C_i$ for $1 \leq i \leq 3$. Let $H = f_1^*A_1 + f_2^*A_2 + f_3^*A_3$ be an ample divisor on $X$, where $A_i$ is an ample divisor on $C_i$ for $1 \leq i \leq 3$. Assume that $g(C_1) = g(C_2) = g(C_3)$ and $\deg A_1 = \deg A_2 = \deg A_3$. Then for any $\nu_{a,\beta}$-stable object $E$ with $\nu_{a,\beta}(E) = 0$, we have

$$\chi^3_1(E) \leq \frac{a^2}{6} H^2 \chi^3_1(E).$$

One notices that $X$ in the above theorem is of general type when $g(C_1) = g(C_2) = g(C_3) \geq 2$. The strategy of the proof is the following. We consider a spreading out $X \to S = \text{Spec} R$, where $R \subset \mathbb{C}$ is a finitely generated ring over $\mathbb{Z}$. By Bogomolov’s inequality for product type varieties in positive characteristic, proved by the author in [25], the tilt-stability is well defined on the fibers $X_s$, $s \in S$. [2] Theorem 12.17] gives the tilt-stability of $\mathcal{E}_s$ for a general point $s \in S$, where $\mathcal{E}$ is an extension of $E$ over $S$. We then compute the Euler characteristic $\chi(\mathcal{O}, F_s^* \mathcal{E}_s)$ of the Frobenius pullback of $\mathcal{E}_s$. By the Riemann-Roch theorem, one sees that $\chi(\mathcal{O}, F_s^* \mathcal{E}_s)$ is a polynomial of degree 3 with respect to $p_s$ and its leading coefficient is $\chi_3(E)$, where $p_s$ is the characteristic of the residue field of $s$.

On the other hand, using the tilt-stability of the Frobenius pushforward of some line bundles (see Proposition 3), we can show that $\text{ext}^i(\mathcal{O}, F^*_s \mathcal{E}_s) = O(p_s^i)$, for even $i$. Taking $p_s \to +\infty$, we obtain an inequality for the third Chern characters of $E$.

**Organized of the paper.** Our paper is organized as follows. In Section 2 we review basic notions and properties of some classical stabilities for coherent sheaves, tilt-stability, the conjectural inequality proposed in [3]. Then in Section 3 we show the tilt-stability of the Frobenius pushforward of some line bundles (see Proposition 3). Then in Section 4 will be proved in Section 4.

**Notation.** Let $X$ be a smooth projective variety defined over an algebraically closed field $k$ of arbitrary characteristic. We denote by $\Omega_X^1$ the sheaf of differentials of $X$ over $k$. $K_X$ and $\omega_X$ denote the canonical divisor and canonical sheaf of $X$, respectively. When $\text{dim } X = 1$, we write $g(X)$ for the genus of the curve $X$. For a triangulated category $\mathcal{D}$, we write $K(\mathcal{D})$ for the Grothendieck group of $\mathcal{D}$.

Let $\pi : \mathcal{X} \to S$ be a flat morphism of Noetherian schemes and $s \in S$ be a point. We denote by $\mathcal{X}_s = \mathcal{X} \times_S \text{Spec } k(s)$ the fibre of $\pi$ over $s$, where $k(s)$ is residue field of $s$. We write $X_s = \mathcal{X}_s \times_S \text{Spec } \overline{k}(s)$ for the geometric fibre of $\pi$ over $s$, here $\overline{k}(s)$ is the algebraic closure of $k(s)$. We denote by $\mathcal{D}^b(\mathcal{X})$ the bounded derived category of coherent sheaves on $\mathcal{X}$. Given $E \in \mathcal{D}^b(\mathcal{X})$, we write $E_s$ (resp., $E_{\overline{s}}$) for the pullback to the field $k(s)$ (resp., $\overline{k}(s)$).

We write $H^j(E)$ ($j \in \mathbb{Z}$) for the cohomology sheaves of a complex $E \in \mathcal{D}^b(X)$. We also write $H^j(F)$ ($j \in \mathbb{Z}_{\geq 0}$) for the cohomology groups of a sheaf $F \in \text{Coh}(X)$. Given a complex number $z \in \mathbb{C}$, we denote its real and imaginary part by $\Re z$ and $\Im z$, respectively.
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2. Preliminaries

Throughout this section, we let $X$ be a smooth projective variety of dimension $n \geq 2$ defined over an algebraically closed field $k$ of arbitrary characteristic and $H$ be a fixed ample divisor on $X$. We will review some basic notions of stability for coherent sheaves, the weak Bridgeland stability conditions and Bogomolov-Gieseker type inequalities.

2.1. Stability for sheaves. For any $\mathbb{R}$-divisor $D$ on $X$, we define the twisted Chern character $\text{ch}^D = e^{-D} \text{ch}$. More explicitly, we have

$$
\begin{align*}
\text{ch}_0^D &= \text{rk} \\
\text{ch}_2^D &= \text{ch}_2 - D \cdot \text{ch}_1 + \frac{D^2}{2} \cdot \text{ch}_0 \\
\text{ch}_1^D &= \text{ch}_1 - D \cdot \text{ch}_0 \\
\text{ch}_3^D &= \text{ch}_3 - D \cdot \text{ch}_2 + \frac{D^2}{2} \cdot \text{ch}_1 - \frac{D^3}{6} \cdot \text{ch}_0.
\end{align*}
$$

The first important notion of stability for a sheaf is slope stability, also known as Mumford stability. We define the slope $\mu_{H,D}$ of a coherent sheaf $E \in \text{Coh}(X)$ by

$$
\mu_{H,D}(E) = \begin{cases} 
+\infty, & \text{if } \text{ch}_0^D(E) = 0, \\
\frac{H^{n-1} \text{ch}_0^H(E)}{H^n \text{ch}_0^H(E)}, & \text{otherwise}.
\end{cases}
$$

Definition 2.1. A coherent sheaf $E$ on $X$ is $\mu_{H,D}$-(semi)stable (or slope-(semi)stable) if, for all non-zero subsheaves $F \hookrightarrow E$, we have

$$
\mu_{H,D}(F) < (\leq) \mu_{H,D}(E/F).
$$

Note that $\mu_{H,D}$ only differs from $\mu_H := \mu_{H,0}$ by a constant, thus $\mu_{H,D}$-stability and $\mu_H$-stability coincide. Harder-Narasimhan filtrations (HN-filtrations, for short) with respect to $\mu_{H,D}$-stability exist in $\text{Coh}(X)$: given a non-zero sheaf $E \in \text{Coh}(X)$, there is a filtration

$$
0 = E_0 \subset E_1 \subset \cdots \subset E_m = E
$$

such that: $G_i := E_i/E_{i-1}$ is $\mu_{H,D}$-semistable, and $\mu_{H,D}(G_1) > \cdots > \mu_{H,D}(G_m)$. We set $\mu_{H,D}^+(E) := \mu_{H,D}(G_1)$ and $\mu_{H,D}^-(E) := \mu_{H,D}(G_m)$.

2.2. Weak Bridgeland stability conditions. The notion of “weak Bridgeland stability condition” and its variant “very weak Bridgeland stability condition” have been introduced in [25Section 2] and [3Definition 12.1], respectively. We will use a slightly different notion in order to adapt our situation. The main difference is the rotation of the half-plane in $\mathbb{C}$.

Definition 2.2. A weak Bridgeland stability condition on $X$ is a pair $\sigma = (Z, \mathcal{A})$, where $\mathcal{A}$ is the heart of a bounded $t$-structure on $\text{D}^b(X)$, and $Z : K(\text{D}^b(X)) \to \mathbb{C}$ is a group homomorphism (called central charge) such that

- $Z$ satisfies the following positivity property for any $E \in \mathcal{A}$:

$$
Z(E) \in \{ re^{i\pi\phi} : r \geq 0, 0 < \phi \leq 1 \}.
$$
• Every non-zero object in $\mathcal{A}$ has a Harder-Narasimhan filtration in $\mathcal{A}$ with respect to $\nu_Z$-stability, here the slope $\nu_Z$ of an object $E \in \mathcal{A}$ is defined by

$$\nu_Z(E) = \begin{cases} +\infty, & \text{if } \exists Z(E) = 0, \\ \frac{\mathbb{R}Z(E)}{\mathbb{Z}Z(E)}, & \text{otherwise.} \end{cases}$$

Let $\alpha > 0$ and $\beta$ be two real numbers. We will construct a family of weak Bridgeland stability conditions on $X$ that depends on these two parameters. For brevity, we write $\text{ch}^\beta$ for the twisted Chern character $\text{ch}^{\beta\mathbb{H}}$.

There exists a torsion pair $(\mathcal{T}_{\beta\mathbb{H}}, \mathcal{F}_{\beta\mathbb{H}})$ in $\text{Coh}(X)$ defined as follows:

\[
\begin{align*}
\mathcal{T}_{\beta\mathbb{H}} &= \{ E \in \text{Coh}(X) : \mu^\mathbb{H}(E) > \beta \} \\
\mathcal{F}_{\beta\mathbb{H}} &= \{ E \in \text{Coh}(X) : \mu^\mathbb{H}(E) \leq \beta \}
\end{align*}
\]

Equivalently, $\mathcal{T}_{\beta\mathbb{H}}$ and $\mathcal{F}_{\beta\mathbb{H}}$ are the extension-closed subcategories of $\text{Coh}(X)$ generated by $\mu_{\mathbb{H},\beta\mathbb{H}}$-stable sheaves of positive and non-positive slope, respectively.

**Definition 2.3.** We let $\text{Coh}^{\beta\mathbb{H}}(X) \subset \mathbb{D}^b(X)$ be the extension-closure

$$\text{Coh}^{\beta\mathbb{H}}(X) = \langle \mathcal{T}_{\beta\mathbb{H}}, \mathcal{F}_{\beta\mathbb{H}}[1] \rangle.$$

By the general theory of torsion pairs and tilting [5], $\text{Coh}^{\beta\mathbb{H}}(X)$ is the heart of a bounded t-structure on $\mathbb{D}^b(X)$; in particular, it is an abelian category. Consider the following central charge

$$Z_{\alpha, \beta}(E) = H^{n-2} \left( \frac{\alpha^2 H^2}{2} \text{ch}^\beta_0(E) - \text{ch}^\beta_2(E) + iH \text{ch}^\beta_1(E) \right).$$

We think of it as the composition

$$Z_{\alpha, \beta} : K(\mathbb{D}^b(X)) \xrightarrow{\text{ch}_H} \mathbb{Q}^3 \xrightarrow{z_{\alpha, \beta}} \mathbb{C},$$

where the first map is given by

$$\text{ch}_H(E) = (H^n \text{ch}_0(E), H^{n-1} \text{ch}_1(E), H^{n-2} \text{ch}_2(E)),$$

and the second map is defined by

$$z_{\alpha, \beta}(e_0, e_1, e_2) = \frac{1}{2}(\alpha^2 - \beta^2)e_0 + \beta e_1 - e_2 + i(e_1 - \beta e_0).$$

**Definition 2.4.** We say $(X, H)$ satisfies Bogomolov’s inequality, if

$$H^{n-2}\Delta(E) := H^{n-2}(\text{ch}_1^2(E) - 2 \text{ch}_0(E) \text{ch}_2(E)) \geq 0$$

for any $\mu_H$-semistable sheaf $E$ on $X$.

**Theorem 2.5.** If $(X, H)$ satisfies Bogomolov’s inequality, then for any $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, $\sigma_{\alpha, \beta} = (Z_{\alpha, \beta}, \text{Coh}^{\beta\mathbb{H}}(X))$ is a weak Bridgeland stability condition.

**Proof.** The required assertion is proved in [7] [1] for the surface case. For the threefold case, the conclusion is showed in [4] [8]. But the proof in [3] Appendix 2 still works for the general case. \qed

**Corollary 2.6.** Assume that either $\text{char}(k) = 0$ or $X = C_1 \times \cdots \times C_n$ and $H = f_1^*A_1 + \cdots + f_n^*A_n$, where $C_i$ is a smooth projective curve over $k$, $A_i$ is an ample divisor on $C_i$ and $f_i : X \to C_i$ is the projection for $1 \leq i \leq n$. Then for any $(\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R}$, $\sigma_{\alpha, \beta} = (Z_{\alpha, \beta}, \text{Coh}^{\beta\mathbb{H}}(X))$ is a weak Bridgeland stability condition.
Proof. It is well known that Bogomolov’s inequality holds in characteristic zero (see [10, Theorem 3.4.1]). By [25, Theorem 1.2], Bogomolov’s inequality still holds for the second case.

We now suppose the assumption in the above Corollary holds. We write $\nu_{\alpha,\beta}$ for the slope function on $\text{Coh}^{\beta H}(X)$ induced by $Z_{\alpha,\beta}$. Explicitly, for any $E \in \text{Coh}^{\beta H}(X)$, one has

$$
\nu_{\alpha,\beta}(E) = \begin{cases} 
+\infty, & \text{if } H^{n-1}ch_1^\beta(E) = 0, \\
\frac{H^{n-2}ch_2^\beta(E) - \frac{1}{2}\alpha^2H^nch_0^\beta(E)}{H^{n-1}ch_1^\beta(E)}, & \text{otherwise.}
\end{cases}
$$

Corollary 2.6 gives the notion of tilt-stability:

**Definition 2.7.** An object $E \in \text{Coh}^{\beta H}(X)$ is tilt-(semi)stable (or $\nu_{\alpha,\beta}$-(semi)stable) if, for all non-trivial subobjects $F \hookrightarrow E$, we have $\nu_{\alpha,\beta}(F) < (\leq) \nu_{\alpha,\beta}(E/F)$.

For any $E \in \text{Coh}^{\beta H}(X)$, the Harder-Narasimhan property gives a filtration in $\text{Coh}^{\beta H}(X)$

$$
0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_m = \mathcal{E}
$$

such that: $\mathcal{F}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ is $\nu_{\alpha,\beta}$-semistable with $\nu_{\alpha,\beta}(\mathcal{F}_1) > \cdots > \nu_{\alpha,\beta}(\mathcal{F}_m)$.

2.3. **Bogomolov-Gieseker type inequality.** We now recall the Bogomolov-Gieseker type inequality for tilt-stable complexes proposed in [4, 3].

**Definition 2.8.** We define the generalized discriminant

$$
\Delta_H := (H^{n-1}ch_1^\beta)^2 - 2H^nch_0^\beta \cdot (H^{n-2}ch_2^\beta).
$$

A short calculation shows

$$
\Delta_H^{\beta H} = (H^{n-1}ch_1)^2 - 2H^nch_0 \cdot (H^{n-2}ch_2) = \Delta_H.
$$

Hence the generalized discriminant is independent of $\beta$.

**Theorem 2.9.** Under the assumption in Corollary 2.6, if $E \in \text{Coh}^{\beta H}(X)$ is $\nu_{\alpha,\beta}$-semistable, then $\Delta_H(E) \geq 0$.

**Proof.** This inequality was proved in [4, Theorem 7.3.1] and [3, Theorem 3.5] on threefolds, but their proof works for the general case.

**Conjecture 2.10** ([4, Conjecture 1.3.1]). Assume that $n = 3$, char$(k) = 0$ and $E \in \text{Coh}^{\beta H}(X)$ is $\nu_{\alpha,\beta}$-semistable with $\nu_{\alpha,\beta}(E) = 0$. Then we have

$$
(2.1) \quad ch_3^\beta(E) \leq \frac{\alpha^2}{6}H^2ch_1^\beta(E).
$$

Such an inequality provides a way to construct Bridgeland stability conditions on threefolds. Recently, Schmidt [23] found a counterexample to Conjecture 2.10 when $X$ is the blowup at a point of $\mathbb{P}^3$. Therefore, the inequality (2.1) needs some modifications in general setting. See [21] and [5] for the recent progress.
Definition 2.11. Assume that $n = 3$ and $(X, H)$ satisfies the assumption in Corollary 2.6. For any object $E \in \text{Coh}^{\beta H}(X)$, we define
\[
\overline{\beta}(E) = \begin{cases} 
\frac{H^2 ch_1(E) - \sqrt{\Delta_{H}(E)}}{H^2 ch_0(E)}, & \text{if } ch_0(E) \neq 0, \\
\frac{H ch_2(E)}{H^2 ch_1(E)}, & \text{otherwise}.
\end{cases}
\]
Moreover, we say that $E$ is $\overline{\beta}$-(semi)stable, if it is $\nu_{\alpha, \beta}$-(semi)stable in an open neighborhood of $(0, \overline{\beta}(E))$ in $(\alpha, \beta)$-plane.

Conjecture 2.10 can be reduced as follows:

Theorem 2.12 ([3, Theorem 5.4]). Assume that $n = 3$, char$(k) = 0$ and for any $\overline{\beta}$-stable object $E \in \text{Coh}^{\beta H}(X)$ with $\overline{\beta}(E) \in [0, 1)$ and $ch_0(E) \geq 0$ the inequality
\[
ch_3(E)(E) \leq 0
\]
holds. Then Conjecture 2.10 holds.

3. Tilt-stability of Frobenius direct images

Throughout this section, we let $k$ be an algebraically closed field of characteristic $p > 0$ and $X = C_1 \times \cdots \times C_n$ be the product of $n$ smooth projective curves defined over $k$ with projections $f_i : X \to C_i$, $1 \leq i \leq n$. We fix an ample divisor $H = f_1^*A_1 + \cdots + f_n^*A_n$ on $X$, where $A_i$ is an ample divisor on $C_i$. Let $F : X \to X$ denote the relative Frobenius morphism over $k$. We will investigate the tilt-stability of $F_\ast \mathcal{L}$ for a line bundle $\mathcal{L}$ on $X$.

Lemma 3.1. Let $\mathcal{L}$ be a line bundle on $X$. Then we have

1. $ch_1(F_\ast \mathcal{L}) = p^{n-1} \left( ch_1(\mathcal{L}) + \frac{c_1}{2} K_X \right)$;
2. $ch_2(F_\ast \mathcal{L}) = \frac{1}{2} p^{n-2} \left( ch_1(\mathcal{L}) + \frac{p-1}{2} K_X \right)^2$.

In particular, when $\mathcal{L} = \omega_X^{-\frac{n-1}{2}}$, we have $ch_1(F_\ast \mathcal{L}) = 0$ and $ch_2(F_\ast \mathcal{L}) = 0$.

Proof. From the Grothendieck-Riemann-Roch theorem, it follows that
\[
ch(F_\ast \mathcal{L}) \text{td}(X) = F_\ast \left( ch(\mathcal{L}) \text{td}(X) \right).
\]
Since $\text{td}(X) = 1 + \frac{1}{12}c_1 + \frac{1}{12} (c_1^2 + c_2) + \cdots$, the above equation implies
\[
\frac{1}{2} ch_0(F_\ast \mathcal{L}) c_1 + ch_1(F_\ast \mathcal{L}) = F_\ast \left( \frac{c_1}{2} + c_1(\mathcal{L}) \right) = p^{n-1} \left( \frac{c_1}{2} + c_1(\mathcal{L}) \right)
\]
and
\[
\frac{c_1^2 + c_2}{12} ch_0(F_\ast \mathcal{L}) + \frac{1}{2} ch_1(F_\ast \mathcal{L}) + ch_2(F_\ast \mathcal{L}) = p^{n-2} \left( \frac{c_1^2 + c_2}{12} + \frac{c_1}{2} c_1(\mathcal{L}) + ch_2(\mathcal{L}) \right).
\]
A simple computation shows $ch_0(F_\ast \mathcal{L}) = p^n$,
\[
ch_1(F_\ast \mathcal{L}) = \frac{p^n - p^{n-1}}{2} K_X + p^{n-1} c_1(\mathcal{L})
\]
and
\[
ch_2(F_\ast \mathcal{L}) = \frac{p^{n-2} - p^n}{12} (K_X^2 + c_2) + \frac{p^n - p^{n-1}}{4} K_X^2 + \frac{p^{n-1} - p^{n-2}}{2} K_X c_1(\mathcal{L}) + \frac{p^{n-2} c_1^2(\mathcal{L})}{2}.
\]
β is a quotient sheaf of the symmetric power $S^l$.

Theorem 4.1. If we can show the following:

By Lemma 3.1 and 3.2, one sees that $\mu_H(S^l(\Omega_X^1))$ is a quotient sheaf of the symmetric power $S^l(\Omega_X^1)$ with

$$\mu_H(S^l(\Omega_X^1)) = \frac{l}{n} K_{X}^{n-1} = \mu_H(S^l(\Omega_X^1)).$$

Since $\Omega_X^l = f^*\omega_{C_1} \oplus \cdots \oplus f^*\omega_{C_n}$, by our assumption, we conclude that $S^l(\Omega_X^1)$ and $T^l(\Omega_X^1)$ is $\mu_H$-semistable and $H^{n-1}K_X > 0$. From [27 Theorem 4.9], it follows that $F_*\mathcal{L}$ is $\mu_H$-stable.

Lemma 3.2. Let $\mathcal{L}$ be a line bundle on $X$. Assume that $g(C_1) = \cdots = g(C_n) \geq 2$ and $\deg A_1 = \cdots = \deg A_n$. Then $F_*\mathcal{L}$ is $\mu_H$-stable.

Proof. Xiaotao Sun [26, 27] proved that the stability of $F_*\mathcal{L}$ depends on the stability of $T^l(\Omega_X^1)$, $0 \leq l \leq n(p-1)$. By [26] Theorem 3.7, Lemma 4.3, one sees that $T^l(\Omega_X^1)$ is a quotient sheaf of the symmetric power $S^l(\Omega_X^1)$ with

$$\mu_H(T^l(\Omega_X^1)) = \frac{l}{n} H^{n-1}K_X = \mu_H(S^l(\Omega_X^1)).$$

Since $\Omega_X^l = f^*\omega_{C_1} \oplus \cdots \oplus f^*\omega_{C_n}$, by our assumption, we conclude that $S^l(\Omega_X^1)$ and $T^l(\Omega_X^1)$ is $\mu_H$-semistable and $H^{n-1}K_X > 0$. From [27 Theorem 4.9], it follows that $F_*\mathcal{L}$ is $\mu_H$-stable.

Proposition 3.3. Let $m$ be an integer and $\mathcal{L} = \mathcal{O}_X(mH - \frac{2n}{p}K_X)$. Assume that $g(C_1) = \cdots = g(C_n) \geq 2$ and $\deg A_1 = \cdots = \deg A_n$. Then

1. $F_*\mathcal{L}$ is $\nu_{\alpha,\beta}$-stable for any $\alpha > 0$ and $\beta < \frac{m}{p}$.
2. $F_*\mathcal{L}[1]$ is $\nu_{\alpha,\beta}$-stable for any $\alpha > 0$ and $\beta \geq \frac{m}{p}$.

Proof. By Lemma 3.1 and 3.2, one sees that $F_*\mathcal{L}$ is $\mu_H$-stable, $\mu_H(F_*\mathcal{L}) = \frac{m}{p}$ and

$$\nabla_H(F_*\mathcal{L}) = (p^{n-1}mH^n)^2 - 2p^nH^{n-1} \frac{1}{2} p^{n-2}m^2 H^n = 0.$$

Hence we obtain our conclusion by [24 Corollary 3.11] or [24 Theorem 1.3, 1.4].

4. The proof of the main theorem

In this section, we will proof Theorem 1.1. By Theorem 2.12, this will be done, if we can show the following:

Theorem 4.1. Under the situation of Theorem 1.1, let $E \in \text{Coh}^H(X)$ be a $\mathfrak{g}$-stable object with $\mathfrak{g}(E) \in [0, 1)$ and $\chi_0(E) \geq 0$. Then we have $\chi_{\mathfrak{g}}(E)(E) \leq 0$.

Since the statement of Theorem 4.1 is independent of scaling $H$, we will assume throughout this section that $H$ is very ample. In order to prove Theorem 4.1, we use the standard spreading out technique and Frobenius morphism. There is a subring $R \subset \mathbb{C}$, finitely generated over $\mathbb{Z}$, and a scheme

$$\pi : X = C_1 \times C_2 \times C_3 \to S = \text{Spec } R$$

so that $\pi$ is smooth, projective and $C_i = C_i \times_R \mathbb{C}$. We also have an object $\mathcal{E} \in D^b(X)$ and a divisor $\mathcal{H} = f_1^*A_1 + f_2^*A_2 + f_3^*A_3$ ($1 \leq i \leq 3$) on $X$ such that $E = \mathcal{E} \times_R \mathbb{C}$ and $A_i = A_i \times_R \mathbb{C}$, where $f_i : X \to C_i$ is the projection.
Since the semistability of sheaves is preserved by field extensions, by [25, Theorem 1.2], one sees that Bogomolov’s inequality holds for any $\mu_{H_s}$-semistable sheaves on the fiber of $\pi$ over a general point $s \in S$. Thus from [2, Proposition 25.3], it follows that for a general closed point $s \in S$, $E_s \in \text{Coh}^{[\beta]}(X_s)$ is $\beta$-stable. By [2, Theorem 12.17], we have:

**Lemma 4.2.** The object $E_s \in \text{Coh}^{[\beta]}(X_s)$ is $\beta$-stable for a general point $s \in S$.

Let $g(C_1) = g(C_2) = g(C_3) = g$ and $\text{deg} A_1 = \text{deg} A_2 = \text{deg} A_3 = a > 0$. Then $K_X$ and $2a - 2H$ are numerically equivalent. The case of $g \leq 1$ has been proved in [3] and [5]. Hence we suppose that $g \geq 2$.

### 4.1. Proof of Theorem 4.1, integral case

Assume that $\overline{\beta}(E) = 0$, i.e.,

$$H \text{ch}_2(E) = 0 = K_X \text{ch}_2(E).$$

We want to show that $\text{ch}_3(E) \leq 0$.

We assume the contrary $\text{ch}_3(E) > 0$, and so $\text{ch}_3(E) \geq 1$. For a general closed point $s \in S$, let $F_s : X_s \to X_s$ be the relative Frobenius morphism and $p_\ast$ the characteristic of the residue field $k(s)$. Since $H^2 \text{ch}_1(E) = H^2 \text{ch}_1(E) \geq 0$ and $\text{ch}_0(E) \geq 0$, by using the Riemann-Roch theorem we can compute

$$\chi(O_{X_s}, F_s^* E_s \otimes \omega_{X_s}^{\frac{p_\ast + 1}{2}}) = p_\ast^3 \left( \text{ch}_3(E) + \frac{K_X^2 \text{ch}_1(E)}{8} + \frac{\text{ch}_0(E)}{48} K_X^3 \right) + O(p_\ast^2).$$

On the other hand, since $E_s$ is a two term complex concentrated in degree $-1$ and $0$, one sees

$$\chi(O_{X_s}, F_s^* E_s \otimes \omega_{X_s}^{\frac{p_\ast + 1}{2}}) \leq \text{hom}(O_{X_s}, F_s^* E_s \otimes \omega_{X_s}^{\frac{p_\ast + 1}{2}}) + \text{ext}^2(O_{X_s}, F_s^* E_s \otimes \omega_{X_s}^{\frac{p_\ast + 1}{2}}).$$

Our goal is to bound from above the right hand side of this inequality with a lower order in $p_\ast$.

By Proposition 3.3, $F_s^* O_{X_s}(m H_s - \frac{p_\ast - 1}{2} K_{X_s})$ and $F_s^* O_{X_s}(-m H_s - \frac{p_\ast - 1}{2} K_{X_s})[1]$ are $\nu_{\alpha, \beta}$-stable for any positive integer $m$, $\alpha > 0$ and $\beta$ close to $0$. For $(\alpha, \beta) \to (0, 0)$, by Lemma 3.1 one sees

$$\nu_{\alpha, \beta} \left( F_s^* O_{X_s}(m H_s - \frac{p_\ast - 1}{2} K_{X_s}) \right) \to \frac{m}{2p_\ast} > 0;$$

$$\nu_{\alpha, \beta} \left( F_s^* O_{X_s}(-m H_s - \frac{p_\ast - 1}{2} K_{X_s})[1] \right) \to -\frac{m}{2p_\ast} < 0;$$

$$\nu_{\alpha, \beta}(E_s) \to 0.$$

These imply

$$\text{Hom} \left( F_s^* O_{X_s}(m H_s - \frac{p_\ast - 1}{2} K_{X_s}), E_s \right) = 0$$

and

$$\text{Hom} \left( E_s, F_s^* O_{X_s}(-m H_s - \frac{p_\ast - 1}{2} K_{X_s})[1] \right) = 0.$$

**Bound on** hom $(O_{X_s}, F_s^* E_s \otimes \omega_{X_s}^{\frac{p_\ast + 1}{2}})$
We want to show \( \text{hom}((\mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1})) = O(p_s^2) \). By Lemma 7.1, we have the exact triangle in \( D^b(X_s) \)
\[
F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1}(-H_s) \rightarrow F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1} \rightarrow \big( F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1} \big) \otimes \mathcal{O}_{Y_s},
\]
where \( Y \) is a divisor on \( X \) so that \( Y := Y \times_R \mathbb{C} \) is a general smooth surface in \( |H| \).
It follows that
\[
\text{hom}\left( \mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1} \right) \leq \text{hom}\left( \mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1}(-H_s) \right) + \text{hom}\left( \mathcal{O}_{X_s}(F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1}) \otimes \mathcal{O}_{Y_s} \right).
\]
We consider the cohomology sheaves of \( \mathcal{E}_s \) and the exact triangle in \( D^b(X_s) \)
\[
\mathcal{H}^{-1}(\mathcal{E}_s)[1] \rightarrow \mathcal{E}_s \rightarrow \mathcal{H}^0(\mathcal{E}_s).
\]
Since \( Y \) is general, Lemma 7.1 gives
\[
\text{hom}\left( \mathcal{O}_{X_s}, (F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1}) \otimes \mathcal{O}_{Y_s} \right) \leq h^0\left( (F_s^* \mathcal{H}^0(\mathcal{E}_s) \otimes \omega_{X_s}^{p+1})|_{Y_s} \right) + h^1\left( (F_s^* \mathcal{H}^{-1}(\mathcal{E}_s) \otimes \omega_{X_s}^{p+1})|_{Y_s} \right).
\]
One concludes that
\[
\text{hom}\left( \mathcal{O}_{X_s}, (F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{p+1}) \otimes \mathcal{O}_{Y_s} \right) = O(p_s^2)
\]
by the following lemma.

**Lemma 4.3.** Let \( Q \) be a sheaf and \( \mathcal{L} \) and \( \mathcal{M} \) be line bundles on \( X \). Let \( Z \) be a divisor on \( X \) such that \( X_s := Z \times_R \mathbb{C} \) is a general smooth surface in the very ample linear system \( |H| \). Then for \( i = 0, 1, 2 \) and for a general closed point \( s \in S \), we have
\[
h^i(Z_s, (F_s^* Q_s \otimes L_s^{p+1} \otimes \mathcal{M})|_{Z_s}) \leq a_1 p_s^2 + a_2 \mu_H(M)p_s + a_3 p_s + a_4 \mu_H(M)^2 + a_5 \mu_H(M) + a_6,
\]
where \( M = \mathcal{M} \times_R \mathbb{C} \) and the constants \( a_j \)'s are independent of \( M \) and \( s \).

**Proof.** Let \( Q = Q \times_R \mathbb{C} \). We assume first that \( Q \) is torsion-free. Since
\[
h^i(Z_s, (F_s^* Q_s \otimes L_s^{p+1} \otimes \mathcal{M})|_{Z_s}) \leq \sum_{j=1}^t h^i(Z_s, (F_s^* (Q_{j,s}/Q_{j-1,s}) \otimes L_s^{p+1} \otimes \mathcal{M})|_{Z_s}),
\]
for any filtration \( 0 = Q_0 \subset Q_1 \subset \cdots \subset Q_t = Q \) of \( Q \). Thus we can assume that \( Q \) is \( \mu_H \)-semistable. By the Grauert-Mülich Theorem (see, e.g., [10] Corollary 3.1.6)), one deduces that
\[
\mu_{H|z}^+(Q_{|Z_s}) \leq \mu_H(Q) + \frac{\text{rk}Q - 1}{2}.
\]
Hence \( \mu_{H|Z_s}^+(Q_{|Z_s}) \leq \mu_H(Q) + \frac{\text{rk}Q - 1}{2} \), for a general closed point \( s \in S \). Let \( m \) be an positive integer such that \( T_X(mH) \) is globally generated. Then by Langer's
[12] Corollary 2.5, one concludes that
\[
\mu^+_s |_{Z_s} (F^*_s Q_s | Z_s) = \mu^+_s |_{Z_s} (F^*_s | Z_s)
\]
\[
\leq p_* \mu^+_s (Q_s | Z_s) + \frac{m p_s (\text{rk} Q - 1)}{p_s - 1}
\]
\[
\leq p_* (\mu_H (Q) + \frac{\text{rk} Q - 1}{2}) + 2m (\text{rk} Q - 1),
\]
here \(F_{Z_s}\) is the Frobenius morphism of \(Z_s\). This implies
\[
\mu^+_s |_{Z_s} \left( (F^*_s Q_s \otimes \mathcal{L}_{s}^{\mu+1} \otimes \mathcal{M}) | Z_s \right) \leq c_1 p_s + \mu_H (M) + c_2,
\]
where \(c_i\)'s are independent of \(s\) and \(M\). Therefore, from Langer’s estimation [13, Theorem 3.3], it follows that
\[
h^0 (Z_s, (F^*_s Q_s \otimes \mathcal{L}_{s}^{\mu+1} \otimes \mathcal{M}) | Z_s) \leq \]
\[
b_1 p_s^2 + b_2 \mu_H (M) p_s + b_3 p_s + b_4 \mu_H (M)^2 + b_5 \mu_H (M) + b_6,
\]
where \(b_i\)'s are independent of \(s\) and \(M\). The \(h^2\)-estimate follows similarly, by using Serre Duality. For \(h^1\), the Riemann-Roch theorem gives
\[
h^1 (Z_s, (F^*_s Q_s \otimes \mathcal{L}_{s}^{\mu+1} \otimes \mathcal{M}) | Z_s) = h^0 (Z_s, (F^*_s Q_s \otimes \mathcal{L}_{s}^{\mu+1} \otimes \mathcal{M}) | Z_s) + h^2 (Z_s, (F^*_s Q_s \otimes \mathcal{L}_{s}^{\mu+1} \otimes \mathcal{M}) | Z_s) - \chi (Z_s, (F^*_s Q_s \otimes \mathcal{L}_{s}^{\mu+1} \otimes \mathcal{M}) | Z_s).
\]
It follows that the upper bound of \(h^1\) has the same form as that of \(h^0\). This finishes the proof in the torsion-free case. The proof for a general sheaf \(Q\) is the same as that of [3, Lemma 7.3].

On the other hand, using Serre duality and the adjointness (see, e.g., [11, Lemma 4.2]), we have
\[
(4.5) \quad \text{hom} (\mathcal{O}_{X_s}, F^*_s \mathcal{E}_s \otimes \omega_{X_s}^{-\mu-1} \otimes (-\mathcal{H}_s)) = \text{hom} (F^*_s \mathcal{E}_s (\omega_{X_s}^{-\mu-1} \otimes \mathcal{H}_s), \mathcal{E}_s (K_{X_s})).
\]
The assumption \(g \geq 2\) implies the linear system \(|3K_X|\) is very ample. By [3, Lemma 7.1], one has the following exact triangle:
\[
\mathcal{E}_s \otimes \omega_{X_s}^{-2} \rightarrow \mathcal{E}_s \otimes \omega_{X_s} \rightarrow \mathcal{E}_s \otimes \mathcal{O}_{D_s} (K_{X_s}),
\]
where \(D\) is a divisor on \(X\) so that \(D := D \times_R \mathbb{C}\) is a general smooth surface in \(|3K_X|\). Since \(K_X\) is numerically proportional to \(H\), one sees by Proposition 3.3 that \(\omega_{X_s}^\alpha \otimes F^*_s (\omega_{X_s}^{-\mu-1} \otimes \mathcal{H}_s)\) is \(\nu_{\alpha,\beta}\)-stable for any \(\alpha > 0\) and \(\beta\) close to zero. A simple computation shows that the object has positive \(\nu_{\alpha,\beta}\)-slope when \((\alpha, \beta) \to 0\). Therefore, similar to [11, 4.1], we obtain
\[
\text{hom} (F^*_s (\omega_{X_s}^{-\mu-1} \otimes \mathcal{H}_s), \mathcal{E}_s (-2K_{X_s})) = \text{hom} (\omega_{X_s}^\alpha \otimes F^*_s (\omega_{X_s}^{-\mu-1} \otimes \mathcal{H}_s), \mathcal{E}_s) = 0.
\]
Hence
\[
\text{hom} (F^*_s (\omega_{X_s}^{-\mu-1} \otimes \mathcal{H}_s), \mathcal{E}_s (K_{X_s})) \leq \text{hom} (F^*_s (\omega_{X_s}^{-\mu-1} \otimes \mathcal{H}_s), \mathcal{E}_s \otimes \mathcal{O}_{D_s} (K_{X_s}))
\]
\[
= \text{hom} (\mathcal{O}_{X_s}, F^*_s (\mathcal{E}_s \otimes \mathcal{O}_{D_s}) \otimes \omega_{X_s}^{-\mu-1} \otimes (-\mathcal{H}_s)).
\]
Since the Frobenius morphism \(F_s\) is flat, [9, Proposition 9.3] gives
\[
F^*_s (\mathcal{E}_s \otimes \mathcal{O}_{D_s}) = F^*_s (\mathcal{E}_s |_{D_s}),
\]
where \( F_D^* \) is the Frobenius map of \( D_s \). Therefore, by Lemma 4.3 one sees
\[
\text{hom} \left( F_{s,*}(\omega_{X_s}^{\frac{p-1}{2}}(H_s)), \mathcal{E}_s(K_{X_s}) \right) \leq \text{hom} \left( \mathcal{O}_{X_s}, F_D^*(\mathcal{E}_s|_{D_s}) \otimes \omega_{X_s}^{\frac{p-1}{2}}(-H_s) \right) \\
\leq h^0 \left( F_D^*(\mathcal{H}^0|_{D_s}) \otimes \omega_{X_s}^{\frac{p-1}{2}}(-H_s)|_{D_s} \right) \\
+ h^1 \left( F_D^*(\mathcal{H}^{-1}|_{D_s}) \otimes \omega_{X_s}^{\frac{p-1}{2}}(-H_s)|_{D_s} \right) \\
= O(p_s^2).
\]
Combining the above inequality, (4.3), (4.4) and (4.5), we conclude that
\[
\text{hom} \left( \mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}} \right) = O(p_s^2).
\]

**Bound on** \( \text{ext}^2 \left( \mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}} \right) \)

This is similar to the previous case. We consider the exact triangle
\[
F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}} \rightarrow F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}}(H_s) \rightarrow \left( F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}} \right) \otimes \mathcal{O}_{Y_s}(H_s).
\]
By (4.2), Serre duality and the adjointness, one obtains
\[
\text{ext}^2 \left( \mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}}(H_s) \right) = \text{ext}^1 \left( F_s^* \mathcal{E}_s, \omega_{X_s}^{\frac{p-1}{2}}(-H_s) \right) \\
= \text{ext}^1 \left( \mathcal{E}_s, F_{s,*} \omega_{X_s}^{\frac{p-1}{2}}(-H_s) \right) \\
= \text{hom} \left( \mathcal{E}_s, F_{s,*} \omega_{X_s}^{\frac{p-1}{2}}(-H_s)|_1 \right) \\
= 0.
\]
Thus Lemma 4.3 gives
\[
\text{ext}^2 \left( \mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}} \right) \leq \text{ext}^1 \left( \mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}} \otimes \mathcal{O}_{Y_s}(H_s) \right) \\
\leq h^1 \left( F_s^* \mathcal{H}^0 \otimes \omega_{X_s}^{\frac{p-1}{2}} \otimes \mathcal{O}_{Y_s}(H_s) \right) \\
+ h^2 \left( F_s^* \mathcal{H}^{-1} \otimes \omega_{X_s}^{\frac{p-1}{2}} \otimes \mathcal{O}_{Y_s}(H_s) \right) \\
= O(p_s^2).
\]
In conclusion, we have
\[
p_s^3 + O(p_s^2) \leq \chi \left( \mathcal{O}_{X_s}, F_s^* \mathcal{E}_s \otimes \omega_{X_s}^{\frac{p-1}{2}} \right) \leq O(p_s^2),
\]
which gives the required contradiction for \( p_s \) sufficiently large.

**4.2. Proof of Theorem 4.1, rational case.** We assume that \( \beta(E) \in \mathbb{Q} \setminus \mathbb{Z} \) and write \( \beta(E) = \frac{v}{u} \) with \( v \) and \( u \) coprime and \( u > v > 0 \). By Dirichlet’s theorem on arithmetic progressions, there are infinitely many primes of the form \( ku + 1 \), where \( k \) is a positive integer. Hence for a very general closed point \( s \in S \), one has \( c_s := \frac{v}{u} \in \mathbb{Z} \). It follows that
\[
\frac{c_s v}{p_s} = (1 - \frac{1}{p_s}) \beta(E).
\]
By using the Riemann-Roch theorem we can compute
\[ \chi(O_{X_s}, F^{s}_{\alpha, \beta}(\mathcal{E}_s \otimes \omega_{X_s}^{\frac{p_1}{2}} (-c_s v \mathcal{H}_s))) = \chi_3 \left( F^{s}_{\alpha, \beta}(\mathcal{E}_s \otimes \omega_{X_s}^{\frac{p_1}{2}} (-c_s v \mathcal{H}_s)) \right) + O(p^2 s) \]
\[ = p^3 \left( \text{ch}^{1-p_1}_1 (\mathcal{E}_s)^2 \chi_2^3 \left( \chi_1 \left( \omega_{X_s}^{\frac{p_1}{2}} \mathcal{E}_s \right) \right) + \frac{1}{8} K_{X_s} \chi_1^2 \left( \mathcal{E}_s \right) + \frac{1}{48} K_{X_s}^3 \right) + O(p^2 s) \]
\[ \geq p^3 + O(p^2 s). \]

From (4.8), one obtains that
\[ K_{X_s}^{3-i} \text{ch}^{\frac{p_1}{2}}_i (\mathcal{E}_s) = K_{X_s}^{3-i} \text{ch}^{(1-p_1)}_i (\mathcal{E}_s) = K_{X_s}^{3-i} \text{ch}^{\frac{p_1}{2}}_i (\mathcal{E}_s) + \frac{O(1)}{p^3}, \]
for 0 \leq i \leq 3. From \( \text{ch}_0 (E) \geq 0 \), \( H^2 \text{ch}_1^2 (E) \geq 0 \), \( H \text{ch}_2^2 (E) = 0 \) and the above equality, we conclude that
\[ \chi(O_{X_s}, F^{s}_{\alpha, \beta}(\mathcal{E}_s \otimes \omega_{X_s}^{\frac{p_1}{2}} (-c_s v \mathcal{H}_s))) = p^3 \left( \text{ch}^{\frac{p_1}{2}}_3 (E) + \frac{1}{2} K_{X_s} \text{ch}^{\frac{p_1}{2}}_3 (E) + \frac{1}{8} K_{X_s}^3 \right) + O(p^2 s) \]
\[ \geq p^3 \text{ch}^{\frac{p_1}{2}}_3 (E) + O(p^2 s) \]
and
\[ \chi(O_{X_s}, F^{s}_{\alpha, \beta}(\mathcal{E}_s \otimes \omega_{X_s}^{\frac{p_1}{2}} (-c_s v \mathcal{H}_s))) \leq \text{hom} \left( O_{X_s}, F^{s}_{\alpha, \beta}(\mathcal{E}_s \otimes \omega_{X_s}^{\frac{p_1}{2}} (-c_s v \mathcal{H}_s)) \right) \]
\[ + \text{ext}^2 \left( O_{X_s}, F^{s}_{\alpha, \beta}(\mathcal{E}_s \otimes \omega_{X_s}^{\frac{p_1}{2}} (-c_s v \mathcal{H}_s)) \right). \]

From Proposition 5.23 and \( c_s v = (p_s - 1) (\mathcal{O}(E)) \), it follows that
\[ F_{\alpha, \beta}(\omega_{X_s}^{\frac{p_1}{2}} (m + c_s v \mathcal{H}_s)) \]
is \( \nu_{\alpha, \beta} \)-stable for any \( m > 0 \), \( \alpha > 0 \) and \( \beta \) close to \( \mathcal{O}(E) \). Similarly, one sees that
\[ F_{\alpha, \beta}(\omega_{X_s}^{\frac{p_1}{2}} (c_s v - l \mathcal{H}_s)) \]
is \( \nu_{\alpha, \beta} \)-stable for any \( l \geq 0 \), \( \alpha > 0 \) and \( \beta \) close to \( \mathcal{O}(E) \). For \( (\alpha, \beta) \to (0, \mathcal{O}(E)) \), by Lemma 3.1, one sees
\[ \nu_{\alpha, \beta} \left( F_{\alpha, \beta}(\omega_{X_s}^{\frac{p_1}{2}} (m + c_s v \mathcal{H}_s)) \right) \to \frac{m - \mathcal{O}(E)}{2 p_s} > 0; \]
\[ \nu_{\alpha, \beta} \left( F_{\alpha, \beta}(\omega_{X_s}^{\frac{p_1}{2}} (c_s v - l \mathcal{H}_s)) \right) \to \frac{1 + \mathcal{O}(E)}{2 p_s} < 0; \]
\[ \nu_{\alpha, \beta}(\mathcal{E}_s) \to 0. \]
These imply that
\[ \text{Hom} \left( F_{\alpha, \beta}(\omega_{X_s}^{\frac{p_1}{2}} (m + c_s v \mathcal{H}_s)), \mathcal{E}_s \right) = 0 \]
and
\[ \text{Ext}^1 \left( \mathcal{E}_s, F_{\alpha, \beta}(\omega_{X_s}^{\frac{p_1}{2}} (c_s v - l \mathcal{H}_s)) \right) = 0. \]
for any \( m > 0 \) and \( l \geq 0 \). Therefore, similar to the proof of (4.7), one obtains

\[
\text{hom} \left( O_{X_2}, F_{s}^* E_s \otimes \omega_{X_2}^{n+1} (-c_s v \mathcal{H}_s) \right) \\
= \text{hom} \left( O_{X_2} (\mathcal{H}_2), F_{s}^* E_s \otimes \omega_{X_2}^{n+1} (-c_s v \mathcal{H}_s) \right) + O(p_s^2) \\
= \text{hom} \left( F_{s}^* \omega_{X_2} \right) \left( (1 + c_s v) \mathcal{H}_s, E_s (K_{X_2}) \right) + O(p_s^2) \\
= \text{hom} \left( \omega_{X_2} \otimes F_{s}^* \omega_{X_2} \right) \left( (1 + c_s v) \mathcal{H}_s, E_s \right) + O(p_s^2) \\
= O(p_s^2).
\]

On the other hand

\[
\text{ext}^2 \left( O_{X_2}, F_{s}^* E_s \otimes \omega_{X_2}^{n+1} (-c_s v \mathcal{H}_s) \right) = \text{ext}^1 \left( F_{s}^* E_s, \omega_{X_2}^{n+1} (c_s v \mathcal{H}_s) \right) = \text{ext}^1 \left( E_s, F_{s}^* \omega_{X_2} \right) \left( (c_s v \mathcal{H}_s) \right) = 0.
\]

In conclusion, we have

\[
p_s^3 \text{ch}_3 (E) + O(p_s^2) \leq \chi \left( O_{X_2}, F_{s}^* E_s \otimes \omega_{X_2}^{n+1} (-c_s v \mathcal{H}_s) \right) \leq O(p_s^2).
\]

This gives \( \text{ch}_3^3 (E) \leq 0 \) by taking \( p_s \to +\infty \).

4.3. Proof of Theorem (4.11) irrational case. We now assume that \( \overline{\beta} (E) \in \mathbb{R} \setminus \mathbb{Q} \).

By assumption, there exists \( 0 < \varepsilon < \overline{\beta} (E) \) such that \( E \) is \( \nu_{\alpha, \beta} \)-stable for all \( (\alpha, \beta) \) in

\[
V_{\varepsilon} := \{ (\alpha, \beta) \in \mathbb{R}_{>0} \times \mathbb{R} : 0 < \alpha < \varepsilon, \overline{\beta} (E) - \varepsilon < \beta < \overline{\beta} (E) + \varepsilon \}.
\]

By the Dirichlet approximation theorem, there exists a sequence \( \{ \beta_n = \frac{u_n}{v_n} \}_{n \in \mathbb{N}} \) of rational numbers with \( u_n > 0, v_n > 0, u_n \) coprime and \( u_n \to +\infty \) as \( n \to +\infty \) such that

\[
\left| \overline{\beta} (E) - \frac{u_n}{v_n} \right| < \frac{1}{u_n^2} < \varepsilon
\]

for all \( n \).

As in the rational case, by Dirichlet’s theorem on arithmetic progressions, for any \( n \), there are infinitely many primes of the form \( au_n + 1 \), where \( a \) is a positive integer. Hence for any \( n \), there is a sequence \( \{ s_{n,k} \}_{k \in \mathbb{N}} \) of closed points in \( S \) so that

\[
c_{nk} := \frac{p_{nk} - 1}{u_n} \in \mathbb{Z} \quad \text{and} \quad \lim_{k \to +\infty} \frac{p_{nk}}{u_n} = +\infty,
\]

where \( p_{nk} \) is the characteristic of the residue field of \( s_{n,k} \). It turns out that

\[
(4.11) \quad \left( 1 - \frac{1}{p_{nk}} \right) \left( \overline{\beta} (E) - \frac{1}{u_n^2} \right) < \frac{c_{nk} v_n}{p_{nk}} = (1 - \frac{1}{p_{nk}}) \beta_n < (1 - \frac{1}{p_{nk}}) \left( \overline{\beta} (E) + \frac{1}{u_n^2} \right).
\]

For any scheme \( Z \) over \( S \) and object \( \mathcal{G} \in D^b(X) \), we shall write \( Z_{nk} = Z_{s_{n,k}} \) and \( \mathcal{G}_{nk} = \mathcal{G}_{s_{n,k}} \) for brevity. Let

\[
\mathcal{Q}_{nk} := F_{s_{nk}}^* E_{s_{nk}} \otimes \omega_{X_{nk}}^{p_{nk}+1} (-c_{nk} v_n \mathcal{H}_{nk}).
\]
We compute, for \( k \gg 0 \),
\[
\chi(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}) = \text{ch}_3(\mathcal{Q}_{nk}) + O(p_n^2)
\]
\[
= p_n^3 \left( \text{ch}_1^2 \mathcal{Q}_{nk} \mathcal{E}_{nk} + \frac{1}{2} K_{X_{nk}} \text{ch}_2 \mathcal{Q}_{nk} \mathcal{E}_{nk} \right)
\]
\[
+ \frac{K_{X_{nk}}^3}{8} \text{ch}_1 \mathcal{Q}_{nk} \mathcal{E}_{nk} + \frac{\text{ch}_2(\mathcal{E}_{nk}) K_{X_{nk}}^3}{48} + O(p_n^2)
\]
\[
= p_n^3 \left( \text{ch}_2^2 \mathcal{E}_{nk} + \frac{1}{2} K_{X_{nk}} \text{ch}_3 \mathcal{E}_{nk} \right)
\]
\[
+ \frac{K_{X_{nk}}^3}{8} \text{ch}_3 \mathcal{E}_{nk} + \frac{\text{ch}_2(\mathcal{E}_{nk}) K_{X_{nk}}^3}{48} + O(p_n^2)
\]
(4.12)
\[
\geq p_n^3 \left( \text{ch}_3^2(\mathcal{E}) + \frac{c_1}{u_n} + \frac{c_2}{u_n^2} \right) + O(p_n^2),
\]
where the constants \( c_1 \) and \( c_2 \) are independent of \( k \) and \( n \). The last inequality follows since, by definition, \( \text{ch}_3(\mathcal{E}) \) has a local minimum at \( \beta = \overline{\beta}(\mathcal{E}) \).

As in the previous case, we want to bound
(4.13)
\[
\chi(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}) \leq \text{hom}(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}) + \text{ext}^2(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk})
\]
for \( k \gg 0 \) and \( n \gg 0 \).

From Proposition \(
\mathbb{3.3}
\) it follows that \( F_{nk,*}(\omega_{X_{nk}}^{-\frac{p_n-1}{p_n}}((m + c_{nk} v_n) \mathcal{H}_{nk})) \) is \( \nu_{\alpha, \beta} \)-stable for any \( \alpha > 0 \) and \( \beta < \frac{m + c_{nk} v_n}{p_n} \). Assume that \( m \geq \frac{p_n-1}{u_n^2} + \overline{\beta}(\mathcal{E}) \). Then by (4.11) one has
\[
\frac{m + c_{nk} v_n}{p_n} > \overline{\beta}(\mathcal{E}) + \frac{m - \overline{\beta}(\mathcal{E})}{p_n} - (1 - \frac{1}{p_n}) \frac{1}{u_n^2} \geq \overline{\beta}(\mathcal{E}).
\]
For \((\alpha, \beta) \rightarrow (0, \overline{\beta}(\mathcal{E}))\), one sees
\[
\nu_{\alpha, \beta} \left( F_{nk,*}(\omega_{X_{nk}}^{-\frac{p_n-1}{p_n}}((m + c_{nk} v_n) \mathcal{H}_{nk})) \right)
\]
\[
= \frac{p_n^2}{2} (m + c_{nk} v_n)^2 - (m + c_{nk} v_n) p_n^2 \overline{\beta}(\mathcal{E}) + \frac{1}{2} p_n^3 \overline{\beta}(\mathcal{E})^2
\]
\[
= \frac{1}{2} \left( \frac{m + c_{nk} v_n}{p_n} - \overline{\beta}(\mathcal{E}) \right) > 0.
\]
This implies
(4.14)
\[
\text{Hom} \left( F_{nk,*}(\omega_{X_{nk}}^{-\frac{p_n-1}{p_n}}((m + c_{nk} v_n) \mathcal{H}_{nk}), \mathcal{E}_{nk}) \right) = 0,
\]
for \( m \geq \frac{p_n-1}{u_n^2} + \overline{\beta}(\mathcal{E}) \).

Similarly, one sees that \( F_{nk,*}(\omega_{X_{nk}}^{-\frac{p_n-1}{p_n}}((c_{nk} v_n - l) \mathcal{H}_{nk}))[1] \) is \( \nu_{\alpha, \beta} \)-stable for any \( \alpha > 0 \) and \( \beta \geq \frac{c_{nk} v_n - l}{p_n} \). Assume that \( l \geq \frac{p_n-1}{u_n^2} - \overline{\beta}(\mathcal{E}) \). Then by (4.11) one has
\[
\frac{c_{nk} v_n - l}{p_n} < \overline{\beta}(\mathcal{E}) + (1 - \frac{1}{p_n}) \frac{1}{u_n^2} - \frac{1 + \overline{\beta}(\mathcal{E})}{p_n} \leq \overline{\beta}(\mathcal{E}).
\]
For \((\alpha, \beta) \to (0, \beta(E))\), one sees

\[
\nu_{\alpha, \beta} \left( F_{nk, \nu} \omega_{X_{nk}}^{\frac{p_{nk} - 1}{2}} ((c_{nk}v_n - l)H_{nk})[1] \right) \\
\to \frac{p_{nk}(c_{nk}v_n - l)^2 - (c_{nk}v_n - l)p_{nk}^{\beta(E)} + \frac{1}{2}p_{nk}^{\beta(E)^2}}{(c_{nk}v_n - l)p_{nk} - p_{nk}^{\beta(E)}} \\
= \frac{1}{2} \left( \frac{c_{nk}v_n - l}{p_{nk}} - \beta(E) \right) < 0.
\]

It follows that

\[
(4.15) \quad \text{Ext}^1 \left( E_{nk}, F_{nk, \nu} \omega_{X_{nk}}^{\frac{p_{nk} - 1}{2}} ((c_{nk}v_n - l)H_{nk}) \right) = 0,
\]

for \(l \geq \frac{p_{nk} - 1}{u_n^2} - \beta(E)\).

Let \(m_0 = \lceil \frac{p_{nk} - 1}{u_n^2} + \beta(E) \rceil \). Consider the exact triangle in \(D^b(X_{nk})\)

\[
Q_{nk}(-(j + 1)H_{nk}) \xrightarrow{\cdot a} Q_{nk}(-jH_{nk}) \rightarrow Q_{nk}(-jH_{nk}) \otimes \mathcal{O}_{Y_{nk}},
\]

where \(0 \leq j \leq m_0 - 1\) and \(Y\) is a divisor on \(X\) so that \(Y := Y \times_R \mathbb{C}\) is a general smooth surface in \(|H|\). It follows that

\[
\text{hom} \left( \mathcal{O}_{X_{nk}}, Q_{nk} \right) \\
\leq \text{hom} \left( \mathcal{O}_{X_{nk}}, Q_{nk}(-m_0H_{nk}) \right) + \sum_{j=0}^{m_0-1} \text{hom} \left( \mathcal{O}_{X_{nk}}, Q_{nk}(-jH_{nk}) \otimes \mathcal{O}_{Y_{nk}} \right).
\]

By \(4.14\) and the same proof of \(4.6\), for \(k \gg 0\) we have

\[
\text{hom} \left( \mathcal{O}_{X_{nk}}, Q_{nk}(-m_0H_{nk}) \right) = \text{hom} \left( F_{nk, \nu} \omega_{X_{nk}}^{\frac{p_{nk} - 1}{2}} ((c_{nk}v_n + m_0H_{nk}), E_{nk}(K_{X_{nk}})) \right) \leq O(p_{nk}^2).
\]

On the other hand, by Lemma \(4.3\) and the definition of \(c_{nk}\), one sees for \(k \gg 0\),

\[
\sum_{j=0}^{m_0-1} \text{hom} \left( \mathcal{O}_{X_{nk}}, Q_{nk}(-jH_{nk}) \otimes \mathcal{O}_{Y_{nk}} \right) \\
\leq \sum_{j=0}^{m_0-1} \left( a_1p_{nk}^2 + (a_2p_{nk} + a_3)(c_{nk}v_n + j) + a_3p_{nk} + a_4(c_{nk}v_n + j)^2 + a_0 \right) \\
= \sum_{j=0}^{m_0-1} \left( a_1p_{nk}^2 + a_2(c_{nk}v_n + j)p_{nk} + a_4(c_{nk}v_n + j)^2 \right) + O(p_{nk}^2) \\
= m_0(a_1p_{nk}^2 + a_2c_{nk}v_n^2p_{nk} + a_4c_{nk}v_n^2) + \frac{m_0(m_0 - 1)}{2}(a_2p_{nk} + 2a_4c_{nk}v_n) \\
+ \frac{a_4}{6}m_0(m_0 - 1)(2m_0 - 1) + O(p_{nk}^2) \\
= \frac{p_{nk}}{u_n^2} \left( a_1p_{nk}^2 + a_2b_n^2p_{nk} + a_4b_n^2 \right) + \frac{p_{nk}^2}{2u_n^2}(a_2p_{nk} + 2a_4b_np_{nk}) \\
+ \frac{a_4}{3}p_{nk}^3 + O(p_{nk}^2) \\
\leq \left( \frac{b_1}{u_n^2} + \frac{b_2}{u_n^3} + \frac{b_3}{u_n^6} \right) p_{nk}^3 + O(p_{nk}^2),
\]

where \(a_i, b_j\) are independent of \(k\).
where \(a_i\)'s and \(b_j\)'s are independent of \(k\) and \(n\). Therefore for \(k \gg 0\) we have

\[(4.16) \quad \text{hom}(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}) \leq \left(\frac{b_1}{u_n^1} + \frac{b_2}{u_n^2} + \frac{b_3}{u_n^3}\right)p_{nk}^3 + O(p_{nk}^2).\]

To bound \(\text{ext}^2(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk})\), we let \(l_0 = \left\lfloor \frac{p_{nk}^2 - 1}{u_n^2} - \beta(E) \right\rfloor\). As before, we consider the exact triangle in \(D^b(X_{nk})\)

\[Q_{nk}((j-1)\mathcal{H}_{nk}) \rightarrow Q_{nk}(j\mathcal{H}_{nk}) \rightarrow Q_{nk}(j\mathcal{H}_{nk}) \otimes \mathcal{O}_{Y_{nk}},\]

where \(1 \leq j \leq l_0\). It follows that

\[\text{ext}^2(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}) \leq \text{ext}^2(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}(l_0\mathcal{H}_{nk})) + \sum_{j=1}^{l_0} \text{ext}^1(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}(j\mathcal{H}_{nk}) \otimes \mathcal{O}_{Y_{nk}}).\]

By \([4.15]\), Serre duality and adjointness, we deduce

\[\text{ext}^2(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}(l_0\mathcal{H}_{nk})) = \text{ext}^1(\mathcal{E}_{nk}, F_{nk,E_n}, \omega_{X_{nk}})^{\frac{-\beta(E)-2}{\beta(E) + 2}}((c_n v_n - l_0)\mathcal{H}_{nk})) = 0.\]

Hence as the same proof of \((4.10)\), for \(k \gg 0\) one obtains,

\[\text{ext}^2(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}) \leq \sum_{j=1}^{l_0} \text{ext}^1(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk}(j\mathcal{H}_{nk}) \otimes \mathcal{O}_{Y_{nk}})\]

\[(4.17) \quad \leq \left(\frac{d_1}{u_n^1} + \frac{d_2}{u_n^2} + \frac{d_3}{u_n^3}\right)p_{nk}^3 + O(p_{nk}^2),\]

where the constants \(d_i\)'s are independent of \(k\) and \(n\).

In conclusion, by \((4.12)\), \((4.13)\), \((4.16)\) and \((4.17)\), we obtain, for \(k \gg 0\),

\[\left(\frac{b_1 + d_1}{u_n^1} + \frac{b_2 + d_2}{u_n^2} + \frac{b_3 + d_3}{u_n^3}\right)p_{nk}^3 + O(p_{nk}^2)\]

\[\geq \chi(\mathcal{O}_{X_{nk}}, \mathcal{Q}_{nk})\]

\[\geq p_{nk}^3 \left(\text{ch}_3^E(E) + \frac{c_1}{u_n^1} + \frac{c_2}{u_n^2}\right) + O(p_{nk}^2).\]

This implies

\[\text{ch}_3^E(E) + \frac{c_1}{u_n^1} + \frac{c_2}{u_n^2} \leq \frac{b_1 + d_1}{u_n^1} + \frac{b_2 + d_2}{u_n^2} + \frac{b_3 + d_3}{u_n^3}.\]

Taking \(n \rightarrow +\infty\), we conclude that \(\text{ch}_3^E(E) \leq 0\). This completes the proof of Theorem \([4.1]\).

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