Phenomenology of noncommutative field theories

C D Carone
Particle Theory Group, Department of Physics, College of William and Mary, Williamsburg, VA 23187-8795
E-mail: carone@phys.wm.edu

Abstract. Experimental limits on the violation of four-dimensional Lorentz invariance imply that noncommutativity among ordinary spacetime dimensions must be small. In this talk, I review the most stringent bounds on noncommutative field theories and suggest a possible means of evading them: noncommutativity may be restricted to extra, compactified spatial dimensions. Such theories have a number of interesting features, including Abelian gauge fields whose Kaluza-Klein excitations have self couplings. We consider six-dimensional QED in a noncommutative bulk, and discuss the collider signatures of the model.

1. Introduction

The possibility of extra compactified spatial dimensions at the TeV scale has led to serious consideration of other modifications of spacetime structure that may be experimentally accessible. One such possibility is that ordinary four-dimensional spacetime may become noncommutative [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14] at some scale $\Lambda_{\text{NC}}$:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}. \quad (1)$$

Here, the position four-vector $x^\mu$ has been promoted to an operator $\hat{x}^\mu$, and $\theta^{\mu\nu}$ is a real, constant matrix with elements of order $(\Lambda_{\text{NC}})^{-2}$. Noncommutative field theories defined in terms of commuting coordinates may be constructed by finding an appropriate mapping of the noncommutative algebra to the space of ordinary functions [15]. Given a classical function $f(x)$ with Fourier transform

$$\tilde{f}(k) = \frac{1}{(2\pi)^{n/2}} \int d^n x e^{ik\mu x^\mu} f(x), \quad (2)$$

one may associate the operator

$$W(f) = \frac{1}{(2\pi)^{n/2}} \int d^n k e^{-ik\mu \hat{x}^\mu} \tilde{f}(k) \quad (3)$$

in the noncommuting theory. Requiring that this correspondence holds for the product of functions,

$$W(f)W(g) = W(f \ast g) \quad (4)$$

one finds that

$$f \ast g = \lim_{y \to x} e^{i \frac{\partial}{\pi x^\mu \theta^{\mu\nu} \partial y^\nu}} f(x)g(y). \quad (5)$$
This is the Moyal-Weyl \( \star \)-product \[16\]. To understand its usefulness, notice that
\[
[x^\mu \star x^\nu] = x^\mu (1 + \frac{i}{2} \partial \cdot \theta \cdot \partial) x^\nu - (\nu \leftrightarrow \mu) = \frac{i}{2} \theta^{\mu\nu} - \frac{i}{2} \theta^{\nu\mu} = i \theta^{\mu\nu} .
\] (6)
The star product allows one to reproduce the original operator algebra while working exclusively with ordinary functions.

The star product represents the starting point for the phenomenological study of noncommutative field theories. For example, one can immediately write down the noncommutative generalization of \( \lambda \phi^4 \) theory,
\[
\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4} (\phi \star \phi)^2 ,
\] (7)
where we have used
\[
\int d^4 x f \star g = \int d^4 x f g ,
\] (8)
which follows from integration by parts. Noncommutative gauge theories require star multiplication as well as a modified form of the Lagrangian in order to preserve the desired local symmetries of the theory. For example, the Lagrangian of noncommutative QED (NCQED) is given by \[17\]
\[
\mathcal{L} = -\frac{1}{4} F_{\mu\nu} \star F^{\mu\nu} + \bar{\psi} \star (i \partial - m) \star \psi ,
\] (9)
\[
D_\mu = \partial_\mu - ie A_\mu ,
\] (10)
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ie [A_\mu \star A_\nu] ,
\] (11)
which is invariant under the noncommutative U(1) gauge transformation
\[
\psi(x) \rightarrow \psi'(x) = U \star \psi(x) ,
\] (12)
\[
A_\mu(x) \rightarrow A'_\mu(x) = U \star A_\mu(x) \star U^{-1} + \frac{i}{e} U \star \partial_\mu U^{-1} .
\] (13)
The gauge transformation matrix \( U \) is a star exponential, \( (e^{i\alpha})_\star \), in which each occurrence of ordinary multiplication in the the Taylor expansion of \( e^{i\alpha} \) is replaced by a star product. The Lagrangian in Eqs.\((9)-(11)\) is similar in form to that of an ordinary non-Abelian gauge theory. In particular, the gauge boson self interactions that originate from the noncommutativity of the group generators in a non-Abelian theory arise in NCQED due to the noncommutativity of ordinary functions under star multiplication.

Phenomenological studies of noncommutative gauge theories have focused primarily on the collider \[1, 2, 3, 4, 5\] and low-energy signatures \[7, 8, 9, 10, 11, 12\] of NCQED. Provided that the scale \( \Lambda_{NC} \) is sufficiently low, the new photon self-interactions in Eq. (9) may be discerned at, for example, the Next Linear Collider (NLC) \[1\]. On the other hand, the most striking phenomenological feature of noncommutative theories is that they are Lorentz violating. The constant parameter \( \theta^{\mu\nu} \) defines the preferred directions \( \theta^0 \) and \( \epsilon^{ijk} \theta^j \) in a given Lorentz frame. Low-energy tests of Lorentz invariance \[18\] place bounds on \( \Lambda_{NC} \) of order 10 TeV, if one considers NCQED processes at tree-level \[8\]. However, there are operators generated at one- and two-loops that are more stringently bounded \[10, 11, 12\]. As pointed out by Anisimov, Banks, Dine and Graesser \[10\], effective interactions such as
\[
O_1 = m_e \theta^{\mu\nu} \bar{\psi} \sigma_{\mu\nu} \psi \quad O_2 = \theta^{\mu\nu} \bar{\psi} D_\mu \gamma_\nu \psi
\]
\[
O_3 = (\theta^2)^{\mu\nu} F_{\mu\rho} F_{\nu}^\rho \quad O_4 = \theta^{\mu\nu} \theta^{\rho\sigma} F_{\mu\nu} F_{\rho\sigma}
\] (14)
are constrained by a variety of low-energy and astrophysical processes. Notable, the operator \( O_1 \) will affect a spin-polarized torsion pendulum by providing a coupling between the net spin, and the fixed external “B field” defined by \( \theta^{ij} \). In NCQED, this operator is generated via the diagrams shown in Fig. 1.

Integrating over the full range of loop momenta, the diagrams in Fig. 1 yield the following finite amplitude \( A \):

\[
A = 24im_e e^4 \int \frac{d^4k}{(2\pi)^4} \frac{d^4l}{(2\pi)^4} \frac{e^{il\cdot k}}{k^2 l^2 (k+l)^4} = \frac{1}{8}m_e \alpha^2 \frac{\sigma_{\mu\nu} \theta^{\mu\nu}}{\sqrt{(-1/2)Tr \theta^2}} \tag{15}
\]

Notably, the peculiar form of this result implies that the coefficient of operator \( O_1 \) is \( \sim \frac{1}{8} m_e \alpha^2 \sim 10^{-5} \) MeV even when \( \Lambda_{\text{NC}} \) is taken to be arbitrarily large. If this result is taken at face value, then one concludes that noncommutative field theories are ruled out: the experimental upper bound on the same operator coefficient is \( O(10^{-25}) \) MeV [10]. However, the appearance of a \( \theta^{\mu\nu} \) in the denominator of Eq. (15) is a consequence of the high-momentum region of integration, and may not occur if the physics is altered at a high scale. In particular, the same integral may be evaluated with an ultraviolet cutoff \( \Lambda \), yielding

\[
A_\Lambda = \frac{3}{4} m_e \Lambda^2 \left( \frac{\alpha^2}{4\pi} \right)^2 \sigma_{\mu\nu} \theta^{\mu\nu}, \tag{16}
\]

which leads to the bound \( \theta \Lambda^2 < 10^{-19} \). The implications of this result are no less striking: \( \theta^{\mu\nu} \) must be 19 orders of magnitude smaller than the size one might expect based on naive dimensional arguments in order that the theory remain consistent with the experimental data. One may worry that this simple estimate involves a cut off that violates the underlying gauge invariance of the theory. However, studies of softly-broken supersymmetric noncommutative QED lead to qualitatively similar results, with the scale of supersymmetry breaking providing a natural gauge-invariant regulator [12].

### 2. Noncommutative Bulk

A possible way of avoiding the stringent bounds on the violation of four-dimensional (4D) Lorentz invariance is to restrict the noncommutativity to extra spatial dimensions [19, 20]. (For another approach, see Ref. [13].) The simplest possibility is six-dimensional (6D) QED with noncommutativity restricted to the fifth and sixth dimensions [19]. While both the compactification of the extra dimensions and the noncommutativity break 6D Lorentz invariance, 4D Lorentz invariance remains intact.

We consider a model with gauge fields defined on the full space and fermion fields restricted to a brane [19]. The Lagrangian is

\[
\mathcal{L}_6 = -\frac{1}{4} \mathcal{F}_{MN} \ast \mathcal{F}^{MN} + \mathcal{L}_{\text{gauge fixing}} + \delta^{(2)}(\vec{y}) \left\{ \bar{\psi} (i \not{\partial} - m) \psi + \hat{e} \bar{\psi} \ast \mathcal{A} \ast \psi \right\}, \tag{17}
\]
\[ F_{MN} = \partial_M A_N - \partial_N A_M - i \hat{e} [A_M, A_N], \]  
(18)

and where \( \hat{e} \) is the 6D gauge coupling. Our notation for the position six-vector is
\[ X^M = (x^0, x^1, x^2, x^3, y^5, y^6), \]  
(19)

with \( \vec{y} \equiv (y^5, y^6) \).

We compactify the extra dimensions on the orbifold \( T^2/\mathbb{Z}_2 \), where \( T^2 \) is a general 2-torus. We take into account the possibility of two different radii \( R_5 \) and \( R_6 \) and a relative shift angle \( \phi \) between the two directions of compactification [21]. This allows us to avoid zeros in the scattering amplitudes of interest to us, that appear only in the limit \( R_5 = R_6 \) and \( \phi = \pi/2 \). The coordinates along the torus are \( \zeta^i \), related to \( y^i \) by
\[ y^5 = \zeta^5 + \zeta^6 \cos \phi \quad y^6 = \zeta^6 \sin \phi. \]  
(20)

The periodicity requirements on a function of orbifold coordinates \( f(\zeta^5, \zeta^6) \) are
\[ f(\zeta^5, \zeta^6) = f(\zeta^5 + 2\pi R_5, \zeta^6) = f(\zeta^5, \zeta^6 + 2\pi R_6). \]  
(21)

Without orbifolding, Eq. (21) implies that bulk fields have 6D wave functions proportional to
\[ \exp \left\{ i \frac{n^5 \zeta^5}{R_5} + i \frac{n^6 \zeta^6}{R_6} \right\} = \exp \left\{ i \frac{n^5 y^5}{R_5} + i \frac{n^6 y^6}{R_6} \left[ \frac{n^6}{R_6} - \frac{n^5}{R_5} \cos \phi \right] \right\}, \]  
(22)

where \( n^5 \) and \( n^6 \) are integers. The masses of the KK modes are eigenvalues of the mass operator
\[ -\partial^2_{y^5} - \partial^2_{y^6} \]  
and are given by
\[ m_{\vec{n}}^2 = \frac{1}{\sin^2 \phi} \left( \frac{n^2_5}{R_5^2} + \frac{n^2_6}{R_6^2} - \frac{2n_5 n_6}{R_5 R_6} \cos \phi \right), \]  
(23)

where \( \vec{n} \equiv (n^5, n^6) \).

The \( \mathbb{Z}_2 \) orbifolding consists of identifying points connected by \( \vec{y} \rightarrow -\vec{y} \). Different components of the gauge field may be Fourier expanded with different \( \mathbb{Z}_2 \)-parities so that zero modes are only present for the first four components,
\[ A_M(X) = \sum_{\vec{n}} A_M^{(\vec{n})}(x) f_{\vec{n}}(\zeta^5, \zeta^6), \]  
(24)

where
\[ f_{\vec{n}}(\zeta^5, \zeta^6) = \begin{cases} 
\cos \left( \frac{n^5 \zeta^5 + n^6 \xi \zeta^6}{R} \right), & M = \mu \\
\sin \left( \frac{n^5 \zeta^5 + n^6 \xi \zeta^6}{R} \right), & M = 5, 6
\end{cases} \]  
(25)

with \( \mu = 0, 1, 2, 3 \). Here, \( \xi \) is the ratio of the radii, with
\[ R_5 \equiv R = \xi R_6. \]  
(26)
Since orbifolding has provided wave functions with distinct parities, the value of $\vec{n}$ is now restricted to a half plane including the origin,

$$\{\vec{n}_+\} = \begin{cases} \vec{n} = \vec{0}; & \text{or} \\ n^6 = 0, n^5 > 0; & \text{or} \\ n^5 > 0, n^6 = \text{any integer} \end{cases}$$, \hspace{1cm} (27)

Within the set $\{\vec{n}_+\}$, masses are unique for most values of $\xi$ and $\phi$.

One obtains the 4D Lagrangian by integrating over the extra dimensions,

$$\mathcal{L}_4 = \int d^2y \mathcal{L}_6 = \frac{1}{\xi_0} \int_0^{2\pi R} d\zeta^5 \int_0^{2\pi R} d\zeta^6 \mathcal{L}_6$$, \hspace{1cm} (28)

where $\xi_0 \equiv \xi / \sin \phi$ and $\zeta^6 \equiv \xi \zeta^6$.

The gauge fixing Lagrangian is chosen as

$$\mathcal{L}_{\text{gauge fixing}} = -\frac{1}{2\eta} \left( \partial_\mu A^\mu + \frac{1}{\eta} \partial_k A^k \right)^2$$ \hspace{1cm} (29)

with $k = 5,6$. Terms in the Lagrangian quadratic in the gauge field become,

$$\left( -\frac{1}{4} \mathcal{F}_{MN} \mathcal{F}^{MN} + \mathcal{L}_{\text{gauge fixing}} \right)_{\text{free, 4d}} = -\frac{1}{4} F^{\mu\nu}_{(0)} F^{\mu\nu}_{(0)} - \frac{1}{2\eta} \left( \partial_\mu A^\mu_{(0)} \right)^2 + \sum \left\{ \frac{1}{4} F_{\mu\nu}^{(\vec{n})} F^{\mu\nu}_{(\vec{n})} - \frac{1}{2\eta} \left( \partial_\mu A^\mu_{(\vec{n})} \right)^2 + \frac{1}{2} \partial_\mu A^L_{(\vec{n})} \partial_\mu A^{L*}_{(\vec{n})} + \frac{1}{2} \partial_\mu A^H_{(\vec{n})} \partial_\mu A^{H*}_{(\vec{n})} \right\}. \hspace{1cm} (30)$$

The primed sum is over the Kaluza-Klein (KK) modes, i.e., over $\{\vec{n}_+\}$ excluding $\vec{0}$. The 6D fields $A$ have been rescaled,

$$A^M_{(0)} = \frac{\sqrt{\xi_0}}{2\pi R} A^M_{(0)}$$, \hspace{1cm} A^M_{(\vec{n})} = \frac{\sqrt{2\xi_0}}{2\pi R} A^M_{(\vec{n})} \quad [\vec{n} \neq \vec{0}], \hspace{1cm} (31)

where the fields $A$ have their canonical 4D mass dimensions. The fifth and sixth components have been combined into

$$A^L_{(\vec{n})} = \frac{1}{|\vec{n}|} \left( \vec{n}^5 A^6_{(\vec{n})} - \vec{n}^6 A^5_{(\vec{n})} \right)$$, \hspace{1cm} A^{H*}_{(\vec{n})} = \frac{1}{|\vec{n}|} \left( \vec{n}^5 A^{5*}_{(\vec{n})} + \vec{n}^6 A^6_{(\vec{n})} \right)$$, \hspace{1cm} (32)

where $\vec{n} = (\vec{n}^5, \vec{n}^6)$ with

$$\vec{n}^5 = n^5 \quad \vec{n}^6 = \frac{1}{\sin \phi} \left( \xi n^6 - n^5 \cos \phi \right),$$ \hspace{1cm} (33)

and $m_R = |\vec{n}|/R$. The fields $A_L$ and $A_H$ are physical and unphysical scalars in the 4D theory, respectively. As $\eta \to 0$, the field $A_H$ is removed from the theory, the extra-dimensional generalization of unitary gauge. We work in the $\eta \to 0$ limit henceforth. Thus, from the free gauge Lagrangian the physical states are the ordinary massless photon, the vector KK modes, and the scalar KK modes $A^L_{(\vec{n})}$. 
The fermion fields $\psi$ are defined only at the $\vec{y} = \vec{0}$ orbifold fixed point, and involve no rescaling. Since $A_L^{(n)}$ is odd under the $\mathbb{Z}_2$ parity it vanishes at $\vec{y} = \vec{0}$. Hence the fermions interact only with the photon and its vector KK excitations. The fermion Lagrangian is

$$L_{f,4d} = \bar{\psi}(i\not\!D - m)\psi + e\bar{\psi} \star A^{(0)} \star \psi + e\sqrt{2} \sum' \bar{\psi} \star A^{(n)} \star \psi.$$  

The 4D gauge coupling has been identified through the rescaling

$$\hat{e} = \frac{2\pi R}{\sqrt{\xi_0}} e.$$  

The pure gauge field interactions come from the terms

$$L_{int,6d} = i\hat{e} \partial_M A_N [A^M \star A^N] + \frac{1}{4} \hat{e}^2 [A_M \star A_N] \star [A^M \star A^N],$$  

in which the Moyal commutator may be written as

$$[A_M \star A_N] = 2i \lim_{X \to Y} \sin \left(\frac{1}{2} \frac{\partial}{\partial X^i} \theta_{ij} \frac{\partial}{\partial Y^j} \right) A_M(X) A_N(Y).$$  

One may now extract the three-photon coupling in the 4D Lagrangian,

$$L_{3\gamma,4d} = -e\sqrt{2} \sum' (\delta_{\vec{n}_a,\vec{n}_b + \vec{n}_c} + \delta_{\vec{n}_b,\vec{n}_c + \vec{n}_a} - \delta_{\vec{n}_c,\vec{n}_a + \vec{n}_b}) \times \partial_\alpha A^{(n_a)}_\beta A^{(n_b)} \alpha A^{(n_c)} \beta \sin \left(\frac{\vec{n}_a^i \theta_{ij} \vec{n}_b^j}{2R^2}\right),$$  

where $\vec{n}$ is defined in Eq. (33). The triple-photon couplings involve only the KK modes, and never any ordinary massless photons. The Feynman rule that corresponds to the $3\gamma$ term in the Lagrangian, for the momenta, Lorentz indices, and KK modes labeled in Fig. 2, is given by

$$V_{3\gamma} = -e\sqrt{2} \left(\delta_{\vec{n}_a,\vec{n}_b + \vec{n}_c} + \delta_{\vec{n}_b,\vec{n}_c + \vec{n}_a} - \delta_{\vec{n}_c,\vec{n}_a + \vec{n}_b}\right) \times \sin \left(\frac{\vec{n}_a^i \theta_{ij} \vec{n}_b^j}{2R^2}\right) [g_{\mu\nu}(p - q)_\rho + g_{\nu\rho}(q - r)_\mu + g_{\rho\mu}(r - p)_\nu].$$  

When the noncommutativity is only in the extra dimensions, the only independent non-zero component of the noncommutativity tensor is $\theta^{56} \equiv \theta$. (Theories with space-like
The argument of the sine simplifies using
\[ \tilde{n}_i^i \theta_{ij} \tilde{n}_j = \xi_0 \theta \left( n_5^5 n_6^6 - n_5^6 n_6^5 \right) . \]  
(40)

A four-photon vertex may be computed in a similar way, but will not be relevant to the physical processes studied in the sections that follow.

3. U(1) Charges

Let us now focus on the couplings of the gauge field to matter localized on the \( \vec{y} = 0 \) brane. Since the field \( \psi \equiv \psi(x^\mu, \vec{0}) \) is independent of the coordinates \( y^5 \) and \( y^6 \), the star products in Eq. (34) reduce trivially to ordinary multiplication. At this point, one might suspect that there is no restriction on the allowed U(1) charges for brane-localized matter. To check the consistency of this claim, consider the behavior of gauge transformations near the \( \vec{y} = 0 \) brane. Recall, that the full 6D noncommutative gauge transformation is given by
\[ A \rightarrow U \star A \star U^{-1} + i \frac{e}{\xi_0} \partial_\mu U^{-1} . \]  
(41)

Infinitesimally, this may be written
\[ \delta A_\mu = \frac{1}{e} \partial_\mu \alpha + i [\alpha \star A_\mu] + \cdots . \]  
(42)

Since \( A^\mu \), and hence \( \delta A^\mu \), are even under the \( Z_2 \) orbifold parity, it follows that the gauge parameter \( \alpha \) is also an even function. Thus,
\[ \frac{\partial}{\partial y^5} \alpha |_{\vec{y}=0} = \frac{\partial}{\partial y^6} \alpha |_{\vec{y}=0} = 0 , \]  
(43)
and Eq. (41) reduces to the familiar result
\[ A^\mu(x^\mu, \vec{0}) \rightarrow A^\mu(x^\mu, \vec{0}) + i \frac{e}{\xi_0} \partial_\mu . \]  
(44)

on the \( \vec{y} = 0 \) brane. Gauge invariance places no restriction on the U(1) charges of matter on the brane [19], unlike the case in 4D NCQED [17].
4. The Smoking Gun

Pair production of KK photons at colliders can occur through the Feynman diagrams shown in Fig. 3. Notice that the first diagram involves the noncommutative triple-photon vertex, Eq. (39).

We present the cross section for the case that the Kaluza-Klein states are the $n = (1,0)$ and $(1,1)$ modes. While noncommutativity also affects the production of $(0,1)$-$(1,0)$ pairs, the nonstandard diagram in this case involves contributions from different intermediate states that tend to cancel, suppressing the rate. At the parton level, for the final states we have chosen, the nonstandard cross section is

$$
\sigma_{NS}(\hat{s}) = \frac{\pi \alpha^2 \lambda}{3\hat{s}^2} \left[ \frac{(m_{11}^2 - m_{01}^2)/(m_{11}m_{10})}{(s-m_{11}^2)(s-m_{10}^2)} \right]^2 \frac{\sin^2 \left( \frac{\xi_0}{2R^2} \right)}{s^3 + 8(m_{11}^2 + m_{10}^2)s^3 - 18m_{11}^4 + 32m_{11}^2m_{10}^2 + 18m_{10}^4}s^2 + 8(m_{11}^6 - 4m_{11}^4m_{10}^2 - 4m_{11}^2m_{10}^4 + m_{10}^4)s + (m_{11}^2 - m_{10}^2)^2(m_{11}^4 + 10m_{11}^2m_{10}^2 + m_{10}^4) \right],
$$

(45)

where $\hat{s}$ is the partonic center-of-mass (CM) energy squared and $\lambda$ is defined in terms of the CM 3-momentum of either final state particle, $|\vec{p}| = \lambda/2\sqrt{s}$ with

$$
\lambda = \sqrt{s^2 - 2\hat{s}(m_{11}^2 + m_{10}^2) + (m_{11}^2 - m_{10}^2)^2}.
$$

(46)

The initial partons are treated as massless. The parton level cross section for the standard process is

$$
\sigma_S(\hat{s}) = \frac{16\pi \alpha^2}{\hat{s}^2} \left\{ -\frac{\lambda m_{11} + m_{10}}{m_{11}} + \frac{\hat{s}^2 + (m_{11}^2 + m_{10}^2)^2}{\hat{s} - m_{11}^2 - m_{10}^2} \ln \frac{\hat{s} - m_{11}^2 - m_{10}^2}{\hat{s} - m_{11}^2 - m_{10}^2 - \lambda} \right\}.
$$

(47)

The collider cross section is

$$
\sigma(s, AB \to \gamma_{11}\gamma_{10}X) = \int_{\hat{s}}^{1} dx_1 \int_{x_1}^{1} dx_2 \frac{1}{3} \sum_q \left[ f_{q/A}(x_1)f_{\bar{q}/B}(x_2) + f_{\bar{q}/A}(x_1)f_{q/B}(x_2) \right] 
$$

$$
\times \left\{ e_q^4 \sigma_S(\hat{s}) + e_{\bar{q}}^2 \sigma_{NS}(\hat{s}) \right\},
$$

(48)

where $\hat{s} = x_1x_2s$, $\tau = \hat{s}_{min}/s$, and $\hat{s}_{min}$ is the square of the sum of the KK excitation masses, or

$$
\hat{s}_{min} = (m_{10} + m_{11})^2.
$$

(49)

Also, $f_{q/A}(x) = f_{q/A}(x, \mu)$ are the parton distribution functions for quark $q$ in hadron $A$ evaluated at renormalization scale $\mu$, and the $1/3$ is from color averaging.

We evaluated the cross section for a proton-proton collider using the CTEQ5L parton distribution functions [23] at a fixed scale $\mu = 2$ TeV, with $\xi = 0.8$, $\phi = 1.5$ and $\sin^2(\xi_0/2R^2) = 1$. Fig. 4 shows the event rate over a range of center of mass energies for $1/R = 4$ TeV and 100 fb$^{-1}$ of integrated luminosity. At a 200 TeV VLHC, for example, we find 294 events where there is an expectation without noncommutativity of 165 $\pm$ 12.8, a 10.1 sigma effect. This is a significant signal for a choice of $1/R$ that is consistent with current indirect bounds on the compactification scale [24]. What is more significant is the relative effect of the noncommutative vertex Eq. (39) on the production of different KK mode pairs. For example, production of $(1,0)$-$(1,0)$ pairs receives no noncommutative corrections while production of $(1,0)$-$(1,1)$ pairs does. Comparison of these channels may help eliminate systematic uncertainty originating, for example, from parton distribution functions.
5. Conclusions

Four-dimensional noncommutative theories are tightly constrained by low-energy searches for the violation of Lorentz-invariance. By restricting noncommutativity to the bulk, we avoid conflict with the most stringent experimental limits, which otherwise force the magnitude of noncommutativity to be small. We presented an explicit example, based on the orbifold $T^2/Z_2$, to illustrate the effects of spatial noncommutativity in 6D QED with fermions confined to an orbifold fixed point. Notably, we find new three- and four-point couplings involving KK excitations of the photon, but not its zero mode. With extra dimensions at the TeV scale, the most promising way of detecting these interactions is through the pair production of KK modes at a very large hadron collider (VLHC), $f \bar{f} \to \gamma^{(m)}\gamma^{(n)}$, with $\vec{m} \neq \vec{n}$. Observing order 100% corrections to the production of certain pairs of KK modes at a VLHC while finding no corrections to others would provide a clear signal of noncommutativity in the bulk.

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