COSMOLOGICAL STRUCTURES IN GENERALIZED GRAVITY

J. HWANG

Department of Astronomy and Atmospheric Sciences
Kyungpook National University, Taegu, Korea

In a class of generalized gravity theories with general couplings between the scalar field and the scalar curvature in the Lagrangian, we describe the quantum generation and the classical evolution processes of both the scalar and tensor structures in a simple and unified manner.

1 Generalized Gravity

We consider a class of generalized gravity theories with an action

\[ S = \int d^4x \sqrt{-g} \left[ \frac{1}{2} f(\phi, R) - \frac{1}{2} \omega(\phi) \phi^\alpha \phi_\alpha - V(\phi) + L_m \right], \]

where \( f \) is a general algebraic function of the scalar (dilaton) field \( \phi \) and the scalar curvature \( R \), and \( \omega \) and \( V \) are general functions of \( \phi \); \( L_m \) is an additional matter part of the Lagrangian. Eq. (1) includes the following generalized gravity theories as subsets: (a) generally coupled scalar field, (b) generalized scalar tensor theory, (c) induced gravity, (d) the low energy effective action of string theory, (e) \( f(R) \) gravity, etc. Einstein gravity is a case with \( f = R \) and \( \omega = 1 \).

2 Cosmological Structures

As a background world model we consider a spatially homogeneous and isotropic metric with \( K = 0 = \Lambda \). As the structures in this world model we consider the most general scalar, vector, and tensor perturbations

\[ ds^2 = -(1 + 2\alpha) dt^2 - a^2 (\beta_\alpha + B_\alpha) dt dx^\alpha + a^2 \left[ \delta_{\alpha\beta} (1 + 2\varphi) + 2\gamma_{\alpha\beta} + 2C_{(\alpha|\beta)} + 2C_{\alpha\beta} \right] dx^\alpha dx^\beta. \]

The perturbed order variables \( \alpha(x, t), \beta(x, t), \varphi(x, t), \) and \( \gamma(x, t) \) describe the scalar type structure; the transverse vectors \( B_\alpha(x, t) \) and \( C_\alpha(x, t) \), and the transverse-tracefree tensor \( C_{\alpha\beta}(x, t) \) describe the vector and tensor structures, respectively. Due to the high symmetry of the background model the three types of perturbations decouple from each other to the linear order and evolve independently. We will deal with only the gauge invariant combination of variables for the scalar and vector perturbations; the tensor perturbation variables are naturally gauge invariant.

The vector perturbation is trivially described by a conservation of the angular momentum: For a vanishing anisotropic stress we have

\[ a^3 (\mu + p) \cdot a \cdot v_\omega \sim \text{constant in time}, \]

(3)
where $\mu(t)$, $p(t)$, and $v_\omega(x,t)$ are the background energy density and the pressure, and the vorticity part of the fluid velocity in $L_m$. The generalized nature of the gravity does not affect this result. In the following we will concentrate on the scalar and tensor type structures.

### 3 Classical Evolutions of the Scalar and Tensor Structures

For the scalar field we let $\phi(x,t) = \phi(t) + \delta \phi(x,t)$. When we consider a scalar perturbation the following gauge invariant combination plays an important role

$$\delta \phi_{\varphi} \equiv \delta \phi - \frac{\dot{\varphi}}{H} \varphi \equiv - \frac{\dot{\varphi}}{H} \varphi_{\delta \phi}. \tag{4}$$

$\delta \phi_{\varphi}$ is the same as $\delta \phi$ in the uniform-curvature gauge ($\varphi \equiv 0$), and $\varphi_{\delta \phi}$ is the same as $\varphi$ in the uniform-field gauge ($\delta \phi \equiv 0$). The perturbed action to the second order in the perturbation variables can be arranged in a simple and unified form

$$\delta^2 S = \frac{1}{2} \int a^3 Q \left( \dot{\Phi}^2 - \frac{1}{a^2} \Phi_{\gamma\gamma} \right) dtd^3x, \tag{5}$$

where for scalar and tensor perturbations, respectively, we have ($F \equiv \partial f/\partial R$)

$$\Phi = \varphi_{\delta \phi}, \quad Q = \frac{\omega \dot{\varphi}^2 + 3 \dot{F}^2/2F}{(H + \dot{F}/2F)}; \quad \Phi = C_{\alpha\beta}^0, \quad Q = F. \tag{6}$$

The non-Einstein nature of the theory is present in the parameter $Q(t)$. The equation of motion becomes

$$\frac{1}{a^3 Q}(a^3 Q \dot{\Phi}) - \frac{1}{a^2} \nabla^2 \Phi = 0. \tag{7}$$

This has a general large scale solution

$$\Phi(x,t) = C(x) - D(x) \int_0^t \frac{dt}{a^3 Q}, \tag{8}$$

where $C(x)$ and $D(x)$ are the integration constants for the growing and decaying modes, respectively. This solution is valid for general $V(\phi)$, $\omega(\phi)$, and $f(\phi, R)$, and expresses the large scale evolution in a remarkably simple and unified form; results for the scalar structures are valid for the single-component subclass of gravity theories in (a)-(c) without $L_m$, whereas results for the tensor structures are valid for the general theory in eq. (1) including the additional matter contributions in $L_m$ (except for the transverse-tracefree stresses).

Notice that the growing mode of $\Phi$ (thus, $\varphi_{\delta \phi}$ and $C_{\alpha\beta}$) is conserved in the large scale limit independently of the specific gravity theory under consideration. Thus, the classical evolutions of very large scale perturbations are characterized by conserved quantities. [This conserved behavior also applies for sufficiently large scale perturbations during the fluid eras based on Einstein gravity; in the fluid]
era the defining criteria for considering a perturbation to be large scale are the Jeans scale (sound horizon) for a scalar structure, and the visual horizon for a gravitational wave. The integration constant $C(x)$ encodes the information about the spatial structure of the growing mode. Thus, in order to obtain the information on large scale structures, we need the information on $\Phi = C(x)$ which must have been generated in some early evolutionary stage of the universe.

### 4 Quantum Generations

An acceleration phase in the early evolution stage of the universe provides a mechanism which can magnify the ever existing microscopic quantum fluctuations to macroscopic classical structures in the spacetime metric. In order to handle the quantum mechanical generations of the scalar and tensor structures, we regard the perturbed parts of the metric and matter variables as Hilbert space operators, $\hat{\Phi}$. The correct normalization of the equal time commutation relation follows from eq. (9) as [in the quantization of the gravitational wave we need to take into account of the two polarization states properly, see (3)]

$$[\hat{\Phi}(x, t), \hat{\Phi}(x', t')] = \frac{i}{a^3Q} \delta^3(x - x').$$

(9)

For $a\sqrt{Q} \propto \eta^q$ ($d\eta \equiv dt/a$) we have an exact solution for the mode function

$$\Phi_k(\eta) = \frac{\sqrt{\pi|\eta|}}{2a\sqrt{Q}} \left[ c_1(k) H_\nu^{(1)}(k|\eta|) + c_2(k) H_\nu^{(2)}(k|\eta|) \right], \quad \nu \equiv \frac{1}{2} - q,$$

(10)

where according to eq. (3) we have $|c_2(k)|^2 - |c_1(k)|^2 = 1$; the freedom in $c_1$ and $c_2$ indicates the dependence on the vacuum state. Although the condition used to get eq. (10) may look special, it actually includes most of the proto-type inflation models investigated in the literature: The exponential ($a \propto e^{Ht}$) and the power-law ($a \propto t^p$) expansions realized in Einstein gravity with a minimally coupled scalar field lead to $\nu = \frac{3}{2}$ and $\nu = \frac{1-3p}{2(1-p)}$, respectively. The pole-like inflations ($a \propto |t_0 - t|^{-s}$) realized in the generalized gravities in (b)-(d) with the vanishing potential lead to $\nu = 0$; these include the pre-big bang scenario based on the low energy effective action of the string theory.

After their generations from the vacuum fluctuations during the acceleration era the relevant scale becomes the superhorizon size and the later classical evolution can be traced using the powerful conservation properties in eq. (8); for the final observational spectrums of both structures generated in various inflation scenarios, see (3).
References

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