Integral solutions of $q$-difference equations of the hypergeometric type with $|q| = 1$. *

Michitomo Nishizawa† and Kimio Ueno‡
Department of Mathematics,
School of Science and Engineering,
Waseda University.

Abstract

Two integral solutions of $q$-difference equations of the hypergeometric type with $|q| = 1$ are constructed by using the double sine function. One is an integral of the Barnes type and the other is of the Euler type.

1 Introduction

The hypergeometric $q$-difference equation is one of the most important examples among $q$-difference systems and many studies have been achieved [2]. However, these are concerned with the case that $0 < q < 1$. In the case of $|q| = 1$, studies on $q$-difference systems are not sufficiently explored. The difficulty comes from the facts that fundamental functions such as “$q$-gamma function” are not known in the case of $|q| = 1$.

Recently, Jimbo and Miwa [3] have constructed an integral solution of the quantized Kniznik-Zamolodotikov equation with $|q| = 1$. Inspired by the result of Lukyanov [5], they have given an integral solution by means of Kurokawa’s double sine function [4]. From a point of view of $q$-analysis, their work is very significant because it is thought of a first step of the study of $q$-difference system with $|q| = 1$.

In this article, we give two integral solutions of $q$-difference equations of the hypergeometric type with $|q| = 1$. One is an integral of the Barnes type and the other is of the Euler type. Once we obtain the $q$-gamma function with $|q| = 1$, we can construct these integral representations in the same way as in the case

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†694m5035@cfi.waseda.ac.jp
‡uenoki@cfi.waseda.ac.jp
that $0 < q < 1$. Furthermore we can show that they are solutions of $q$-difference equations of the hypergeometric type with $|q| = 1$.

This article is organized as follows: In section 2, we give a survey of integral representations of the hypergeometric series and the basic hypergeometric series with $0 < q < 1$. In section 3, we define the “$q$-gamma function” with $|q| = 1$ by using the double sine function. In section 4, an integral of the Barnes type is introduced in the case of $|q| = 1$ and this function is shown to satisfy the hypergeometric $q$-difference equation. In section 5, we consider an analogue of Euler’s integral representation. On this consideration, we must regard $q$-shifted factorials as the “$q$-gamma function” with $|q| = 1$, so it is needed to transform a multiplicative variable to an additive variable. This integral gives a solution of the difference equation which is obtained by writing the hypergeometric $q$-difference equation by using an additive variable.

We would like to mention that our studies is significant when one considers the representation of the quantum group $SL_q(2, R)$. It is known that $q$ must be $|q| = 1$ in $SL_q(2, R)$ (Masuda et.al. [6]), therefore, the harmonic analysis on this quantum group should be closely linked to the hypergeometric $q$-difference equation with $|q| = 1$.

2 Preliminaries

In this section, we give a brief survey of integral representations of the hypergeometric series

$$F(a, b, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k \quad \text{(for } |z| < 1), \quad (1)$$

where $(a)_k := a(a-1) \cdots (a-k+1)$, and of the basic hypergeometric series with $0 < q < 1$

$$\phi(q^a, q^b; q^c; q, z) = \sum_{k=0}^{\infty} \frac{(q^a; q)_k(q^b; q)_k}{(q^c; q)_k(q; q)_k} q^k \quad \text{(for } |z| < 1), \quad (2)$$

where $(a; q)_k := \prod_{l=0}^{k} (1 - a q^l)$.

2.1 Barnes’ contour integral representation

Barnes’ contour integral representation is so defined that sum of residue of the integrand is equal to the hypergeometric series.

Let us define $(-z)^* := \exp(s \log(-z))$, where we choose such a branch of logarithm that this logarithm takes real value when $z$ is on negative real line. To define this integral, the following lemma is important.
Lemma 2.1 (1) The function $\pi(-z)^s/\sin \pi s$ has simple poles at $s = k$ ($k \in \mathbb{Z}$), and the residue there is $z^k$.

(2). We have, for $|z| < 1$, that
\[
\frac{\pi(-z)^s}{\sin \pi s} = O \left[ \exp \{-|\Im s| \arg(-z)\} \right]
\]
as $|\Im s| \to \infty$ preserving $|\Re s| < \infty$.

Let us fix a real number $\delta$ such that $0 < \delta < \pi$ and suppose $z$ to be in a sector $S_1 := \{z \in \mathbb{C} | -\pi + \delta < \arg(-z) < \pi - \delta, |z| < 1\}$. Barnes’ contour integral of the hypergeometric series is given as follows:
\[
F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \left( \frac{-1}{2\pi i} \right) \int_{c - \infty}^{c + \infty} \Gamma(a + s)\Gamma(b + s) \pi(-z)^s \Gamma(c + s)\Gamma(s + 1) \sin \pi s ds
\]
where the contour lies on the right of poles
\[
s = -an + n \quad s = -b + n \quad (n \in \mathbb{Z}_{\leq 0})
\]
and on the left of poles $s = m \quad (m \in \mathbb{Z}_{\geq 0})$.

Thanks to the Stirling formula of the gamma function and Lemma 2.1 (2), we can see that the integral (3) converges uniformly in $S_1$. Furthermore, by using deformation of the integral contour and residue calculus based on Lemma 2.1 (1), one can show that the integral (3) is the hypergeometric series. For the details, see [10].

Next, we consider a $q$-analogue of (3) in the case that $0 < q < 1$. Let us put $q = e^{-2\pi \tau}$ ($\tau > 0$). The counterpart of (4), which is known as Watson’s contour integral, is given as follows:
\[
\phi(q^a, q^b, q^c; q, z) = \frac{\Gamma(c; q)}{\Gamma(a; q)\Gamma(b; q)} \left( \frac{-1}{2\pi i} \right) \int_{c - \infty}^{c + \infty} \Gamma(a + s; q)\Gamma(b + s; q) \pi(-z)^s \Gamma(c + s; q)\Gamma(s + 1; q) \sin \pi s ds
\]
where $\Gamma(z; q)$ is the $q$-gamma function defined by
\[
\Gamma(z; q) := \frac{(q; q)_\infty}{(q^z; q)_\infty} (1 - q)^{1-z},
\]
and the contour lies on the right of poles
\[
s = -a + n_1 + \frac{n_2}{\tau}, \quad s = -b + n_1 + \frac{n_2}{\tau}, \quad (n_1 \in \mathbb{Z}_{\leq 0}, \quad n_2 \in \mathbb{Z})
\]
and on the left of poles $s = m \quad (m \in \mathbb{Z}_{\geq 0})$.

From Lemma 2.1 (2) and the fact that
\[
\frac{\Gamma(a + s; q)\Gamma(b + s; q)}{\Gamma(c + s; q)\Gamma(1 + s; q)} \leq \text{Const.} \prod_{k=1}^{\infty} \frac{1 + e^{-(c+k+n\tau)}(1 + e^{-(1+k+n\tau)}(1 - e^{-(b+k+n\tau)}) \leq \text{Const.} \prod_{k=1}^{\infty} \frac{1 + e^{-(c+k+n\tau)}(1 + e^{-(1+k+n\tau)}(1 - e^{-(b+k+n\tau)})
\]
it follows that the integral (3) converges uniformly in $S_1$. By using the same technique, one can show that the integral (3) is equal to the basic hypergeometric series [11].
2.2 Euler’s integral representation

Euler’s integral representation for the hypergeometric series is

\[ F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-b)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-zt)^{-b} dt. \]  

(8)

From the binomial theorem and an integral representation of the beta function, it follows that the integral (8) gives the hypergeometric series.

A \(q\)-analogue of this representation is given, by using the Jackson integral, as follows:

\[ \phi(q^a, q^b, q^c; q, z) = \frac{\Gamma(c; q)}{\Gamma(b; q)\Gamma(c-b; q)} \int_0^1 t^b \frac{(tzq^{a}; q)_\infty(tq; q)_\infty}{(tzq^{c-b}; q)_\infty} \frac{dz}{t}. \]  

(9)

In the same way as the classical case, by using the \(q\)-binomial theorem and the Jackson integral representation of the \(q\)-beta function, we can prove that (9) is equal to the basic hypergeometric series.

2.3 The hypergeometric \(q\)-difference equations

When \(|q| < 1\), the basic hypergeometric series \(\phi(q^a, q^b, q^c; q, z)\) is convergent for \(|z| < 1\), and satisfies the hypergeometric \(q\)-difference equation:

\[ (L_q\phi)(z) = 0, \]  

(10)

where

\[ [z] := \frac{1-q^z}{1-q}, \quad (T_qf)(z) := f(qz), \]

\[ D_q := \frac{1-T_q}{1-q}z, \quad [\theta + a] := \frac{1-q^aT_q}{1-q}, \]

\[ L_q := z^{-1}[\theta][\theta + c - 1] - [\theta + a][\theta + b] \]

\[ = z(q^c - q^{a+b+1})D_q^2 \]

\[ - \left\{ [c] - (1-q^a)(1-q^b) - (1-q^{a+b+1}) \right\} \frac{1}{1-q} D_q - [a][b]. \]  

(11)

We should note that the basic hypergeometric series with \(|q| = 1\) is not convergent (so it gives only a formal solution to the hypergeometric \(q\)-difference equation).

3 “\(q\)-gamma function” with \(|q| = 1\)

Let us define a function \(\tilde{\Gamma}(z; q)\) which satisfies

\[ \tilde{\Gamma}(z + 1; q) = [z]\tilde{\Gamma}(z; q) \]  

(12)
in the case of $|q| = 1$.

For this end, we need the double zeta function $\zeta_2(s, z|\omega)$, the double gamma function $\Gamma_2(z|\omega)$ and the double sine function $S_2(z|\omega)$ (cf. [3], [8], [4], [8]).

**Definition 3.1** For $\omega := (\omega_1, \omega_2) \in C^2$, we define $\zeta_2(s, z|\omega)$, $\Gamma_2(z|\omega)$ and $S_2(z|\omega)$ by

\[
\zeta_2(s, z|\omega) := \sum_{m_1, m_2 \in \mathbb{Z}_{\geq 0}} (z + m_1 \omega_1 + m_2 \omega_2)^{-s},
\]

\[
\Gamma_2(z|\omega) := \exp\left(\frac{\partial}{\partial s} \zeta_2(s, z|\omega)|_{s=0}\right),
\]

\[
S_2(z|\omega) := \Gamma_2(z|\omega)^{-1} \Gamma_2(\omega_1 + \omega_2 - z|\omega).
\]

It is known that the double sine function satisfies the functional relation

\[
\frac{S_2(z + \omega_1|\omega)}{S_2(z|\omega)} = \frac{1}{2 \sin \frac{\pi z}{\omega}}.
\]

Thus, we can construct a function satisfying (12) by using $S_2(z|\omega)$. We suppose that $|q| = 1$ and that $q$ is not a root of unity. Let us put $q = e^{2\pi i \omega} (0 < \omega < 1, \omega \notin \mathbb{Q})$.

**Definition 3.2** We set

\[
\bar{\Gamma}(z; q) := (q - 1)^{1-z^{-1}} q^{\frac{z(1-z)}{2}} S_2(z|(1, \frac{1}{\omega})^{-1},
\]

which has the following properties.

**Proposition 3.3**

1. $\bar{\Gamma}(z; q)$ has simple zeros at $z = n_1 + \frac{n_2}{\omega}$ $(n_1, n_2 \in \mathbb{Z}_{>0})$, and has simple poles at $z = n_1 + \frac{n_2}{\omega}$ $(n_1, n_2 \in \mathbb{Z}_{\leq 0})$.

2. $\bar{\Gamma}(z; q)$ satisfies the functional relation (12).

3. If we take $z \to \infty$ as $z$ is in any sector not containing real line then $\bar{\Gamma}(z; q)$ has the following asymptotic behavior.

\[
\bar{\Gamma}(z; q) = \exp\left[(1 - z) \log(q - 1) + (z - 1) \log i \right.
\]

\[
+ \frac{z(z - 1)}{4} \log q \mp \pi i \left\{\frac{\omega z^2 - \omega + 1}{2} z \right\} + O(1) \right) \quad (\text{for } \pm \Im z > 0).
\]

These properties follow from the facts in the papers [3], [8].

**Remark 3.4** In the case that $0 < q < 1$, we can also define $\bar{\Gamma}(z, q)$ by Definition 3.2 (in this case, $\omega = it$, $t > 0$). Of course, we can see that $\bar{\Gamma}(z, q) = C(z, q) \Gamma(z, q)$, where $C(z, q)$ is a function satisfying $C(z + 1, q) = C(z, q)$ (cf. [8]).
4 An integral representation of the Barnes type with $|q| = 1$

4.1 Definition of the integral

In order that the integral makes sense, we impose some conditions on parameters $a, b$ and $c$.

**Conditions on the parameters**  
(B1) If we define sets $A_1$ and $A_2$ of the parameters by $A_1 := \{a, b\}, A_2 := \{c, 1\}$, then we suppose that

\[ \Re \alpha > \Re \beta \quad \text{for} \quad \forall \alpha \in A_1, \quad \forall \beta \in A_2 \]

or

\[ \Im \alpha \neq \Im \beta \quad \text{for} \quad \forall \alpha \in A_1, \quad \forall \beta \in A_2 \]

(B2) We suppose that

\[ \omega \Re (a + b - c + 1) < 1. \]

Under these conditions, we can define a function $\Phi(a, b, c; q, z)$ in the same way as Bernes’ contour integral by using $\tilde{\Gamma}(z, q)$.

**Definition 4.1** Let us fix a real number $\delta$ such that $0 < \delta < \pi - \pi \omega \Re (a + b - c + 1)$. For $z$ in the sector $S := \{z \in \mathbb{C} | -\pi + \delta < \arg(-z) < -2\pi \omega \Re (a + b - c + 1), |z| < 1\}$, we define $\varphi(a, b, c; q; s, z)$ and $\Phi(a, b, c; q, z)$ by

\[
\varphi(a, b, c; q; s, z) := \frac{\Gamma(a + s; q)\Gamma(b + s; q)\pi(-z)^s}{\Gamma(c + s; q)\Gamma(1 + s; q) \sin \pi s} \sin \pi s
\]

\[
\Phi(a, b, c; q, z) := \frac{\tilde{\Gamma}(c; q)\Gamma(b; q)}{\Gamma(a; q)\Gamma(b; q)} \left( \frac{-1}{2\pi i} \right) \int_{-i\infty}^{i\infty} \varphi(a, b, c; q; s, z) ds \quad (15)
\]

where the contour lies on the right of poles

\[ s = -a + n_1 + \frac{n_2}{\omega}, \quad s = -b + n_1 + \frac{n_2}{\omega} \quad (n_1, n_2 \in \mathbb{Z}_{\leq 0}) \]

and on the left of poles

\[ s = -c + n_1 + \frac{n_2}{\omega}, \quad s = -1 + n_1 + \frac{n_2}{\omega} \quad (n_1, n_2 \in \mathbb{Z}_{> 0}), \]

\[ s = m \quad (m \in \mathbb{Z}_{\geq 0}). \]

By using Lemma 2.1 (2) and Proposition 3.3 (2), it is shown that

\[ \varphi(a, b, c; q; s, z) = O[\exp(-\delta|s|)] \quad \text{as} \quad s \to \pm i\infty \]

under the condition (B1). Thus the integral (15) converges uniformly in $S$, and the analytic continuation (also denote $\Phi(a, b, c; q, z)$) defines a many-valued analytic function of $z$. 

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4.2 The hypergeometric $q$-difference equation with $|q| = 1$

We prove that $\Phi(a, b, c; q, z)$ is a solution of the hypergeometric $q$-difference equation with $|q| = 1$. We also use the notation (10) in the case that $|q| = 1$.

**Theorem 4.2** $\Phi(z) := \Phi(a, b, c; q, z)$ satisfies the hypergeometric $q$-difference equation

$$(L_q \Phi)(z) = 0.$$  

**Outline of Proof**: From the condition (B2), it follows that the action of $L_q$ commute with the integration. On the other hand, straightforward calculation shows that the integrand $\varphi(a, b, c; q; s, z)$ satisfies the relation

$$(L_q \varphi)(a, b, c; q; s, z) = \varphi(a + 1, b + 1, c; q; s - 1, z) - \varphi(a + 1, b + 1, c; q; s, z).$$  

(16)

By means of Cauchy’s theorem, one can verify that the integral of the right-hand side of (16) vanishes.

5 An integral representation of the Euler type with $|q| = 1$

5.1 Definition of the integral

First let us recall Euler’s integral (9) in the case that $0 < q < 1$. If we transform the variables $z$ and $t$ in the integrand of (9) to $q^z$ and $q^t$ respectively, then we have

$$(\text{the integrand of (9)}) = \text{Const.} \frac{\Gamma(s + x; q) \Gamma(s + c - b; q)}{\Gamma(s + x + a; q) \Gamma(s + 1; q)} q^b s.$$

Therefore, in the case that $|q| = 1$, we consider the integral

$$\int \frac{\Gamma(s + x; q) \Gamma(s + c - b; q)}{\Gamma(s + x + a; q) \Gamma(s + 1; q)} q^b s \, ds$$

(17)

as a counterpart of (9). In order that the integral makes sense, we impose the following conditions on the parameters $a$, $b$ and $c$.

**Conditions on the parameters.** (E1) $b - c \notin \mathcal{R}_{>0}$, (E2) $a \notin \mathcal{R}_{<0}$, (E3) $\Re b > 0$, $\Re(a - c - 1) > 0$.

Under these conditions, we can take such a suitable contour that the integral (17) makes sense.
Definition 5.1 For $x \notin \mathbb{R}_{<0}$, we define a function $\Psi(a, b, c; q, x)$ by
\[
\Psi(a, b, c; q, x) := \int_{-i\infty}^{i\infty} \frac{\Gamma(s + x; q)\Gamma(s + c - b; q)}{\Gamma(s + x + a; q)\Gamma(s + 1; q)} q^{bs} ds
\]
where the contour lies on the right of the poles
\[
s = -x + n_1 + \frac{n_2}{\omega}, \quad s = b - c + n_1 + \frac{n_2}{\omega}, \quad (n_1, n_2 \in \mathbb{Z}_{\leq 0}),
\]
and on the left of the poles
\[
s = -x - a + n_1 + \frac{n_2}{\omega}, \quad s = -1 + n_1 + \frac{n_2}{\omega}, \quad (n_1, n_2 \in \mathbb{Z}_{> 0}).
\]

Thanks to the conditions (E1) and (E2), we can take the contour of Definition 5.1. From the condition (E3), it follows that the integral (18) converges uniformly and defines a single-valued analytic function of $x$.

5.2 The difference equation for $\Psi(a, b, c; q, x)$

Let us present an equation which $\Psi(a, b, c; q, x)$ satisfies. For this end, we write the hypergeometric $q$-difference equation by using the “additive” variable $x$. We employ the following notations:
\[
(T_+ g)(x) := g(x + 1), \quad [\vartheta + a]_+ := \frac{1 - q^a T_+}{1 - q}
\]
\[
L_+ := q^{-x} [\vartheta]_+ [\vartheta + c - 1]_+ - [\vartheta + a]_+ [\vartheta + b]_+
\]
\[
= \frac{1}{(1 - q)} \{ (q^{c-1-x} - q^{a+b}) \{ T_+^2 - (1 + q)T_+ + q \}
- \{ (1 - q^a)q^{-x} + (1 - q^b)(1 - q^b) \} (T_+ - 1)
- (1 - q^a)(1 - q^b) \}
\]

Then the next theorem holds.

Theorem 5.2 $\Psi(x) := \Psi(a, b, c; q, x)$ satisfies the difference equation
\[
(L_+ \Psi)(x) = 0.
\]

This theorem can be proved just like Theorem 5.1.

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