Polynomial Depth, Highness and Lowness for E

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Abstract. We study the relations between the notions of highness, lowness and logical depth in the setting of complexity theory. We introduce a new notion of polynomial depth based on time bounded Kolmogorov complexity. We show our polynomial depth notion satisfies all basic logical depth properties, namely neither sets in P nor sets random for EXP are polynomial deep, and only polynomial deep sets can polynomially Turing compute a polynomial deep set. We prove all EXP-complete sets are poly-deep, and under the assumption that NP does not have \( p \)-measure zero, then NP contains a polynomial deep set. We show that every high set for E contains a polynomial deep set in its polynomial Turing degree, and that there exists low for E polynomial deep sets.

1 Introduction

The concept of logical depth was introduced by C. Bennett [5] to differentiate useful information (such as DNA) from the rest, with the key observation that non-useful information pertains in both very simple structures (for example, a crystal) and completely unstructured data (for example, a random sequence, a gas). Bennett calls data containing useful information logically deep data, whereas both trivial structures and fully random data are called shallow. A sequence is Bennett deep [5] if every computable approximation of the Kolmogorov complexity of its initial segments satisfies that the difference between the approximation and the actual value of the Kolmogorov complexity of the initial segments dominates every constant function. This difference is called the depth magnitude of the sequence [16].

Moser and Stephan [16] studied the differences in computational power of sequences of different depth magnitudes; within the context of computability theory. They related logical depth to standard computability notions (e.g. highness, diagonally-non-computability and lowness). Highness and lowness are important characterisations of the computational power of sets used in computability theory [18]. Informally a set is high (resp. low) if it is useful (resp. not useful) given as an oracle. Among others, they showed that a Turing degree is high iff it contains a deep set of large depth magnitude [16]. They found that not all deep sets need be high, by constructing a low deep set.

In this paper, we revisit their results [16] within the context of computational complexity theory. Adapting Bennett’s logical depth to the computational complexity setting is an elusive task, and several authors have proposed polynomial versions of logical depth [2, 7, 15]; see [15] for a summary of most notions. Here we study a polynomial version of depth as close as possible to the original notion by Bennett, namely the difference of two Kolmogorov complexities with different time bounds. Informally Bennett’s depth measures some aspect of the difference in power between \( \Delta^0_2 \) and the computable sets. In our polynomial setting,
this becomes EXP vs P. This corresponds to polylog vs quasipolynomial in the setting of Kolmogorov complexity, because the size of the characteristic sequence is exponentially larger than the size of the strings it encodes (i.e. sets in P have their polynomial time complexity measured relative to the size of the input, which corresponds to polylog with respect to the size of the characteristic sequence). To allow for Kolmogorov complexity with polylog time bounds, i.e. where there is not enough time to read the whole program, we use the oracle Kolmogorov complexity model of [1]. This model is equivalent to standard Kolmogorov complexity for time bounds linear or greater, but allows for sublinear time bounds.

We show that similarly to logical depth, our polynomial depth notion has all the properties any reasonable depth notion should possess; namely both sets with low complexity (here: in P) and random enough sequences (here: EXP-random) are shallow, and polynomial depth satisfies a slow-growth law, i.e. no shallow sequence can quickly compute a deep one. As a consequence of our slow growth law, we obtain that all EXP-complete sets are poly-deep, which corresponds to Bennett’s result that the halting problem is deep. We investigate whether NP contains any polynomial deep sets. Since one cannot exclude P = NP (in which case all NP-sets are shallow), one can only hope for a conditional result. We prove that if NP does not have p-measure zero, then it contains a polynomial deep set. The assumption NP does not have p-measure zero is a reasonable assumption based on Lutz p-measure [13], which has implications not known to follow from P ≠ NP. Examples include separating many-one and Turing reductions [12] and derandomization of AM [9], see [11] for more. We fall short of proving all NP complete sets are deep, because the magnitude functions in our results do not overlap.

Next we study the relation between highness and depth, and show that all sets that are high for E (i.e. sets A such that E^E ⊆ E^A) contain a polynomial deep set in its polynomial Turing degree. This confirms the results of [16] at the polynomial level. The idea of the proof is that highness enables the set to compute polynomially random strings of small sizes, but large enough to guarantee depth of the whole sequence.

Our main result investigates whether all polynomial deep sets need be high. We find a negative answer, by constructing a polynomial deep set low for E. Lowness for E (i.e. sets A such that E^A ⊆ E) was first studied in [6], where a low set in E − P was constructed. The set constructed in [6] is a very sparse set of strings that are random at the polynomial time level but not at the exponential time level. The sparseness of the set guarantees that large queries can be answered with “no”, hence only small queries need be computed, which guarantees lowness. However the set in [6] is too sparse to be polynomial deep. We construct a new set with blocks of subexponential size each containing a random string. The size of the blocks are measured by a tower of subexponential functions, in order to be able to satisfy two conflicting requirements, namely the sequence need to be sparse enough to stay low, but the relative size of blocks need to be small enough so that the poly-depth of each block is preserved over the whole sequence.

As shown in [16], the depth magnitude has consequences on the computational power of the corresponding set. This seems to be the case in the complexity setting also, where different results hold for different depth magnitudes.
2 Preliminaries

Logarithms are taken in base 2 and rounded down. For simplicity of notation we write \( \log n \) for \( \lfloor \log_2 n \rfloor \) and \( \log^{(2)} n \) stands for \( \log \log n \). By convention, whenever a real number is considered to be an integer, we take the floor of the real number; we omit the floor notation for simplicity of notation.

We use standard complexity/computability/algorithmic randomness theory notations see [4, 3, 18, 8, 17]. We write \( 2^n \) for the set of strings of size \( n \). We denote by \( s_0, s_1, \ldots, s_n \) the standard enumeration of strings in lexicographic order. For a string \( x \), its length is denoted by \( |x| \). The empty string is \( s_0 = \epsilon \). The index of string \( x \) is the integer \( \text{ind}(x) \) s.t. \( x = s_{\text{ind}(x)} \). For every natural number \( n \), it holds \( |s_n| = \log(n + 1) \), and the index of strings of length \( n \) are \( [2^n - 1, 2^{n+1} - 2] \). We identify \( n \) with \( s_n \), in particular \( |n| = |s_n| = \log n + 1 \).

We say string \( y \) is a prefix of string \( x \), denoted \( y \prec x \), if there exists a string \( a \) such that \( x = ya \).

A sequence is an infinite binary string, i.e. an element of \( 2^\omega \). For string or sequence \( S \) and \( i, j \in \mathbb{N} \), we write \( S[i, j] \) for the string consisting of the \( i \)th through \( j \)th bits of \( S \), with the conventions that \( S[i, j] = \epsilon \) if \( i > j \), \( S[i] = S[i, i] \), and \( S[0] \) is the leftmost bit of \( S \). We write \( S \upharpoonright i \) for \( S[0, i-1] \) (the first \( i \) bits of \( S \)). The characteristic sequence of a set of strings \( L \) is the sequence \( \chi_L \in 2^\omega \), whose \( n \)th bit is one iff \( s_n \in L \). We abuse the notation and use \( L \) and \( \chi_L \) interchangeably. Note that for any string \( x \), \( |L \upharpoonright x| = 2|x| - 1 \).

A time bound is a monotone time constructible function \( t : \mathbb{N} \to \mathbb{N} \), i.e. there is a TM (Turing machine) that on input any string of length \( n \) halts in exactly \( t(n) \) steps (we write \( M(x)[t(n)] \downarrow \) for \( M(x) \) halts within \( t(n) \) steps). We consider the following standard time bound families: \( \mathbb{P} = \{kn^k | k \in \mathbb{N}\} \), \( \text{PolyLog} = \{k \log^k n | k \in \mathbb{N}\} \), \( \mathbb{E} = \{2^{kn} | k \in \mathbb{N}\} \), and \( \mathbb{EXP} = \{2^{kn} | k \in \mathbb{N}\} \). We abuse notations by using time bound families for complexity classes interchangeably, e.g. \( \mathbb{E} = \cup_{c \in \mathbb{N}} \text{DTIME}(2^{cn}) \).

We use \( \leq^+ \) (resp. \( =^+ \)) to denote less or equal (resp. equal) up to a constant term. We fix a poly-computable 1-1 pairing function \( \langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N} \). An order function is an unbounded non-decreasing function from \( \mathbb{N} \) to \( \mathbb{N} \), computable in polynomial time.

We consider standard polynomial Turing reductions \( \leq^p_T \). Two sets \( A, B \) are polynomial Turing equivalent \( (A \equiv^p_T B) \) if \( A \leq^p_T B \) and \( B \leq^p_T A \). The polynomial Turing degree of a set \( A \) is the set of sets polynomial Turing equivalent to \( A \).

Fix a universal prefix free Turing machine \( U \), i.e., such that no halting program of \( U \) is a prefix of another halting program. The prefix-free Kolmogorov complexity of string \( x \), denoted \( K_U(x) \), is the length of the lexicographically first program \( x^* \) such that \( U \) on input \( x^* \) outputs \( x \). It can be shown that the value of \( K_U(x) \) does not depend on the choice of \( U \) up to an additive constant, therefore we drop \( U \) from the notation and write \( K(x) \). \( K(x, y) \) is the length of a shortest program that outputs the pair \( (x, y) \), and \( K(x | y) \) is the length of a shortest program such that \( U \) outputs \( x \) when given \( y \) as an advice. \( C \) denotes the plain Kolmogorov complexity, i.e. where the universal machine is not prefix free.

We also consider time bounded Kolmogorov complexity. To allow logarithmic time bounds i.e. shorter than the time required to read the full program, we use the oracle model of [1]. In this model, the universal machine is provided with program \( p \) as an oracle.
(written $U^p$), and can query any bit of it. As noticed in [1], the definition coincides with the standard time bounded Kolmogorov complexity, for time bounds greater than $O(n)$. Given time bound $t \geq \log n$, define

$$K^t(x) = \min \{ |p| : \forall b \in \{0, 1, \epsilon\} \forall i \leq n, U^p(i, b)[t(n)] \downarrow \text{accepts iff } x[i] = b \}$$

where $n = |x|$, $x[0], x[1], \ldots, x[n-1]$ are the $n$ bits of $x$, and $x[m] = \epsilon$ for all $m \geq n$. $C^t$ denotes the plain version (non prefix free). Time bounded Kolmogorov complexity yields time-bounded version of Martin-Löf random sequences e.g., sequence $A$ is EXP-random if there exists $c \in \mathbb{N}$ s.t. for every $k \in \mathbb{N}$ and for almost every $n$,

$$K^{2\log^k n}(A | n) > n - c.$$

The symmetry of information holds for exponential time bounds.

**Theorem 1 (Symmetry of information [10], page 391).** Let $t_0 \in E$ be a time bound. Then there exists time bound $t' \in E$ such that for any strings $x, y$, we have $C^t(x, y) \geq C^{t'}(x) + C^{t'}(y|x) - O(\log(|x| + |y|)).$

In [14], Lutz used Lebesgue measure to define a measure notion on complexity classes e.g. $E, EXP$. Lutz [13] showed that sets with small circuit complexity have $p_2$-measure zero, which is implied by the following result.

**Theorem 2 (Lutz).** [13] Let $c \in \mathbb{N}$. The set $\{A \in 2^\omega | \exists \infty n \ K^n(A | n) < \log^c n \}$ has $p_2$-measure zero.

It is known from [19] that for any closed under symmetric difference class $C$, if $C$ does not have $p$-measure zero, then $C$ does not have $p_2$-measure zero.

**Lemma 3.** [19] Let $C$ is a class of languages closed under symmetric difference (or finite unions and intersections). If $C$ has $p_2$-measure zero, then $C$ has $p$-measure zero.

### 3 Polynomial depth

Informally a sequence $S$ is poly-deep if the difference of time bounded Kolmogorov complexities of the prefixes of $S$ exceeds some order function $h$. We call the order $h$ the depth magnitude of $S$. As shown in [16], the choice of $h$ can have consequences on the computational complexity of $S$. We will consider families $M$ of depth magnitudes, e.g. $O(1), O(\log n)$ and PolyLog.

**Definition 4.** Let $M$ be a family of order functions. A set $S$ is $M$-deep$_{P, EXP}$ if for every $m \in M$ and $t \in \text{PolyLog}$ there exists $t \in 2^{\text{PolyLog}}$ such that for almost every $n$, $K^t(S | n) - K^t(S | n) \geq m(n)$.

It is easy to check that depth is preserved if one takes a subset of $M$.

**Lemma 5.** Let $M, M'$ be a family of order functions. Let $\Delta_i, \Delta'_i$, with $i = 1, 2$ be a family of time bounds.

1. If $M \subseteq M'$ then every $M'$-deep$_{P, EXP}$ set is $M$-deep$_{P, EXP}$. 

2. If $\Delta_1 \subseteq \Delta'_1$ then every $M$-deep$_{(\Delta'_1, \Delta_2)}$ set is $M$-deep$_{(\Delta_1, \Delta_2)}$.
3. If $\Delta_2 \subseteq \Delta'_2$ then every $M$-deep$_{(\Delta_1, \Delta_2)}$ set is $M$-deep$_{(\Delta_1, \Delta'_2)}$.

**Proof.** Follows from the definition. □

As noticed in [15], for most depth notions it can be shown that easy and random sequences are not deep. The following two results show that this is also the case for deep$_{(P, \text{EXP})}$.

**Theorem 6.** Let $A$ be in $P$, then $A$ is not $O(1)$-deep$_{(P, \text{EXP})}$.

**Proof.** We need the following lemmas.

**Lemma 7.** Let $A \in \text{DTIME}(n^c)$ (resp. $\text{DTIME}(2^{n^c})$). For every $t \in \text{PolyLog}$ (resp. $t \in 2^{\text{PolyLog}}$), and $n \in \mathbb{N}$, $K^t(A \mid n) \leq + K^t(1^n)$, with $t'(n) = O(t(n) + \log^c n)$ (resp. $t'(n) = O(t(n) + 2\log^c n)$).

**Proof.** Let $A, t$ be as above and let $n \in \mathbb{N}$ and $M$ be a TM deciding $A$. Let $p$ denote a minimal $t$-program for $1^n$, i.e. $|p| = K^t(1^n)$. Consider prefix-free program $p' = qp$ (where instructions $q$ are encoded with every bit doubled followed by a 01 flag), such that $U^p(i, b)$ simulates $U^p(i, 1)$. If $U^p(i, 1)$ accepts (i.e. $i < n$), simulate $M(s_i)$ and accept iff $M(s_i) = b$. If $U^p(i, 1)$ rejects, then accept iff $b = \epsilon$. The first step takes at most $t(n)$ steps, the simulation of $M$ on an input of size at most $\log n$ takes at most $\log^c n$ steps, for a total of $O(t(n) + \log^c n)$ steps. The exponential case is similar. □

**Lemma 8.** For every set $A$ and time bound $t \geq \log n$, $K^t(A \mid n) \geq + K^{t'}(1^n)$, with $t' = + t$.

**Proof.** Let $A, t$ be as above and let $n \in \mathbb{N}$. Let $p$ denote a minimal $t$-program for $A \mid n$, i.e. $|p| = K^t(A \mid n)$. Consider prefix-free program $p' = qp$ (where instructions $q$ are encoded with every bit doubled followed by a 01 flag), such that $U^{p'}(i, b)$ simulates $U^p(i, \epsilon)$. If $U^p(i, \epsilon)$ rejects, accept iff $b = 1$; If $U^p(i, \epsilon)$ accepts, accept iff $b = \epsilon$. □

**Lemma 9.** For every $n$, $K^{t'}(1^n) \leq + K^t(n)$, where $t(n) = n$ and $t'(n) = 2 \log n$.

**Proof.** Let $n \in \mathbb{N}$, $t, t'$ be as above. Let $p$ denote a minimal $t$-program for $n$, i.e. $U(p)[|n|] \downarrow = n$ with $|n| = \log(n + 1)$. Consider program $p' = qp$ (where instructions $q$ are encoded with every bit doubled followed by a 01 flag), such that $U^{p'}(i, b)$ simulates $U(p) = n$. If $i < n$, accept iff $b = 1$, if $i \geq n$ accept iff $b = \epsilon$. □

**Lemma 10.** For every time bound $t \geq \log n$, and every $m \in \mathbb{N}$, $K^t(1^m) \geq + K(m)$.

**Proof.** It is straightforward to transform any time bounded program for $1^m$, into a program for $m$, of the same size up to a constant term. □

It is known (see e.g. [8]) that $K$ and time bounded $K$ coincide infinitely often.

**Lemma 11.** [8] There is a constant $c$ and infinitely many $n$ such that $K^t(n) - K(n) \leq c$ with $t(n) = n$. 

Let us prove the theorem. Let $A \in \mathbb{P}$, decidable in time $n^k$, let $t(n) = 2 \log n + \log^k n$, and let $t' \in 2^{\text{PolyLog}}$. Let $c$ be the constant of Lemma 11, $N$ be the infinite set for which Lemma 11 holds and let $n \in N$. We have

$$K^t(A \upharpoonright n) \leq^+ K^{2 \log n}(1^n)$$

$$\leq^+ K^t(n)$$

with $I(n) = n$. Also

$$K^{t'}(A \upharpoonright n) \geq^+ K^{2t'}(1^n)$$

$$\geq^+ K(n)$$

Thus

$$K^t(A \upharpoonright n) - K^{t'}(A \upharpoonright n) \leq^+ K^t(n) - K(n) \leq c$$

by Lemma 11. Since $t' \in 2^{\text{PolyLog}}$ is arbitrary, $A$ is not $O(1)$-deep$_{P,\text{EXP}}$. □

Martin-Löf random sequences are not deep as shown by Bennett [5]. A similar result holds in our setting, as the following shows.

**Theorem 12.** Let $A$ be EXP-random, then $A$ is not $O(1)$-deep$_{P,\text{EXP}}$.

**Proof.** Let $A$ be as above i.e., there exists $c_1 \in \mathbb{N}$, for every $k \in \mathbb{N}$, for almost every $n$

$$K^{2 \log n}(A \upharpoonright n) > n - c_1. \quad (1)$$

Let $t' \in 2^{\text{PolyLog}}$, and $\{k_i\}_i$ be an increasing sequence of integers such that for all $i$ and all $n \geq k_i$,

$$K^{t'}(A \upharpoonright k_i) - k_i \leq K^{t'}(A \upharpoonright n) - n \quad (2)$$

which exists by Equation 1. Let $t(n) = \log^2 n$. Let $\sigma_i$ denote a minimal $t'$-program for $A \upharpoonright k_i$, i.e. $|\sigma_i| = K^{t'}(A \upharpoonright k_i)$, and let $n_i = 2^{t'(2k_i)}$. Let $p' = q_0\sigma_i A[k_i, n_i - 1]$, where instructions $q$ are encoded with every bit doubled followed by a 01 flag, and where $U^{p'}(j, b)$ starts with $k = 0$, sets $p_k = p'[c_2, c_2 + k]$ (with $c_2 = |q|$), simulates $U^{p_k}$ for $t'(1 + k + c_1)$ steps; If the simulation does not halt, increment $k$ and redo the procedure, until the simulation halts. Note the simulation halts exactly when $k = |\sigma_i| - 1$, since at that stage $p_k = \sigma_i$ and $U^{p_k}$ will be simulated for $t'(|\sigma_i| + c_1) \geq t'(k_i)$ steps and will output $A \upharpoonright k_i$ (recall $|\sigma_i| \geq k_i - c_1$ by randomness of $A$). No previous simulations with $k < |\sigma_i| - 1$ will halt, because $U$ is prefix free. Thus we can recover $A \upharpoonright k_i$, hence $n_i$ and $|\sigma_i|$. Check that $|p'| = c_2 + |\sigma_i| + n_i - k_i$, if not then loop (this guarantees prefix-freeness). If $0 \leq j < k_i$, accept iff $b = A(j)$. If $k_i \leq j < n_i$, accept iff $b = p'[j + c_2 + |\sigma_i| - k_i]$. If $n_i \leq j$, accept iff $b = \varepsilon$. This ends the description of $U^{p'}(j, b)$.

Note that $p'$ is prefix free because only one prefix $p_k$ will halt since $U$ is prefix free, and the check that $|p'| = c_2 + |\sigma_i| + n_i - k_i$. For time, there are at most $|\sigma_i| \leq 2k_i$ simulations, each taking at most $t'(|\sigma_i| + c_1) \leq t'(2k_i)$ steps, thus a total of $2k_i t'(2k_i) \leq \log n_i \log n_i = \log^2 n_i$. Thus, $K^{\log^2 n}(A \upharpoonright n_i) \leq c_2 + |\sigma_i| + (n_i - k_i)$. Therefore

$$K^{\log^2 n}(A \upharpoonright n_i) - K^{t'}(A \upharpoonright n_i) \leq c_2 + K^{t'}(A \upharpoonright k_i) + (n_i - k_i) - K^{t'}(A \upharpoonright n_i)$$

$$\leq c_2 \quad \text{by Equation 2.}$$
Since the above inequality holds for every $i$ and $t' \in 2^{\text{PolyLog}}$ is arbitrary, $A$ is not $O(1)$-deep$_{(P,\text{EXP})}$. \hfill \Box

Bennett proved a slow growth law, namely if set $A$ truth-table computes a deep set $B$, then $A$ is deep. A similar result holds for deep$_{(P,\text{EXP})}$, for polynomial time Turing reductions.

**Theorem 13.** Let $A, B \in \text{EXP}$, $A \leq^p_T B$ and $A$ is PolyLog-deep$_{(P,\text{EXP})}$, then $B$ is PolyLog-deep$_{(P,\text{EXP})}$.

**Proof.** We need the following lemma.

**Lemma 14.** Let $A \leq^p_T B$, where $n^c$ is the running time of the reduction. For every $s' \in \text{PolyLog}$ (with $s' = \log^k n$) and $n \in \mathbb{N}$ we have

$$K^s(A \upharpoonright 2^{\log^{1/2} n}) \leq^+ K^{s'}(B \upharpoonright n) + 2 \log n$$

where $s(n) = 3 \log^{4kd} n$

**Proof.** Let $A, B, c, s' = \log^k n$ be as above, $M$ be the TM computing the reduction, and $n \in \mathbb{N}$. Let $p$ be a minimal $s'$-program for $B \upharpoonright n$. Let $p' = qq'p$, where $q, q'$ are encoded with every bit doubled followed by a 01 flag, $q$ is a set of instructions, $q'$ encodes $n$ (i.e. $|q'| \leq^+ 2 \log n$), where $U^p(i, b)$ does the following: Recover $n$. If $i \geq 2^{\log^{1/2} n}$, accept iff $b = \epsilon$. If $i < 2^{\log^{1/2} n}$, simulate $M^B(s_i)$, answering each query $s_i$ to $B$ by simulating $U^p(q, 0)$ and $U^p(q, 1)$. Notice that all queries are within $B \upharpoonright n$, because $A \upharpoonright 2^{\log^{1/2} n}$ codes for string of length at most $\log^{1/2} n$. Therefore the largest query has size at most $(\log^{1/2} n)^c \leq \log^{1/2} n$, i.e. a string with index at most $2^{1+\log^{1/2} n} - 2 < n$. The simulation of $M^B(s_i)$ takes at most $|s_i|^c \leq \log n$ steps, thus there are at most $\log n$ queries to $B \upharpoonright n$, each requiring at most $2s'(n)$ steps, thus a total of $2s'(n) \log n + \log n \leq 3s'(n) \log n$. Let us express this time bound as a function of the input size $m = 2^{\log^{1/2} n}$. We have $\log^{2c} m = \log n$, i.e. $2^{\log^{2c} m} = n$ hence the total number of steps becomes, $3s'(2^{\log^{2c} m}) \log^{2c} m \leq 3 \log^{4kd} m$ i.e. a PolyLog time bound.

Let us prove the theorem. Let $A, B$ be as above, and $n^c$ be the running time of the $\leq^p_T$-reduction. Suppose $B$ is not PolyLog-deep$_{(P,\text{EXP})}$ i.e., there exists $k \in \mathbb{N}$ and $t = \log^d n$ such that for every $t' \in 2^{\text{PolyLog}}$ there exists an infinite set $N$ such that for every $n \in N$, $K^{t'}(B \upharpoonright n) < \log^k n + K^{t'}(B \upharpoonright n)$. Suppose $B$ is decidable in time $2^{n^b}$. Consider $t(n) = 3 \log^{4kd} n$, and $t'(n) = n + 2^\log n$, and let $n \in N$ where $N$ is the infinite set of lengths testifying the non depth of $B$ for this $t'$. We have

$$K^{t'}(A \upharpoonright 2^{\log^{1/2} n}) \leq^+ K^{t'}(B \upharpoonright n) + 2 \log n \quad \text{By Lemma 14}$$

$$\leq \log^k n + K^{t'}(B \upharpoonright n) + 2 \log n \quad \text{Because } n \in N$$

$$\leq^+ \log^k n + K^n(1^n) + 2 \log n \quad \text{By Lemma 7}$$

$$\leq^+ \log^k n + 2 \log n + 2 \log n < \log^{k+1} n.$$ 

Thus for all $n \in N$ and all $s \in 2^{\text{PolyLog}}$,

$$K^t(A \upharpoonright 2^{\log^{1/2} n}) - K^s(A \upharpoonright 2^{\log^{1/2} n}) \leq K^t(A \upharpoonright 2^{\log^{1/2} n}) - 0 < \log^{k+1} n = \log^{2c(k+1)}(2^{\log^{1/2} n})$$
i.e. \( A \) is not PolyLog-deep\(_{(P,\text{EXP})}\).

Bennett showed in [5] that the halting problem is deep. The slow growth law implies that EXP-complete sets are deep.

**Corollary 15.** If \( A \) is EXP-complete under \( \leq^p_T \), then \( A \) is PolyLog-deep\(_{(P,\text{EXP})}\).

**Proof.** By enumerating all short \( O(n) \) time programs, one can construct a PolyLog-random sequence in EXP, i.e. a sequence \( A \in \text{EXP} \) such that there exists \( c_1 \in \mathbb{N} \) s.t. for every \( n \in \mathbb{N}, K^n(A \upharpoonright n) > n - c_1 \), i.e. for every \( k \in \mathbb{N} \), for almost every \( n \) \( K^k n(A \upharpoonright n) > n - c_1 \). Since \( A \in \text{EXP} \), by Lemma 7 \( A \) is PolyLog-deep\(_{(P,\text{EXP})}\). By Theorem 13 any EXP-complete under \( \leq^p_T \) is PolyLog-deep\(_{(P,\text{EXP})}\).\[\] It is natural to ask whether some NP sets are deep\(_{(P,\text{EXP})}\). One cannot exclude the possibility that \( P = \text{NP} \), in which case the answer is negative, but if one assumes that \( \text{NP} \) is not a small subset of \( \text{EXP} \), then one can show that \( \text{NP} \) contains deep sets.

**Theorem 16.** If \( \mu_p(\text{NP}) \neq 0 \) then \( \text{NP} \) contains a \( O(\log n) \)-deep\(_{(P,\text{EXP})}\) set.

**Proof.** Suppose \( \mu_p(\text{NP}) \neq 0 \) and \( \text{NP} \) does not contain a \( O(\log n) \)-deep\(_{(P,\text{EXP})}\) set i.e., for every \( A \in \text{NP} \) there exist \( k \in \mathbb{N} \), \( t \in \text{PolyLog} \) s.t. for every \( t' \in 2^{\text{PolyLog}} \) there exists an infinite set \( N \) s.t. for every \( n \in N \), \( K^t(A \upharpoonright n) - K^{t'}(A \upharpoonright n) \leq k \log n \), i.e. \( K^t(A \upharpoonright n) \leq (k + 2) \log n \) by Lemma 7 since \( A \in \text{EXP} \). Thus for every \( A \in \text{NP} \) there exists an infinite set \( N_A \) s.t. for every \( n \in N_A \), \( K^n(A \upharpoonright n) < \log^2 n \). Thus by Theorem 2, \( \mu_{p_2}(\text{NP}) = 0 \); thus \( \mu_{p_2}(\text{NP}) = 0 \) by Lemma 3, i.e. a contradiction.

### 3.1 Higness and depth

Higness and lowness are important characterisations of the computational power of sets used in computability theory [18]. Informally a set is high (resp. low) if it is useful (resp. not useful) given as an oracle. The notions were generalised to considering two classes, with applications in randomness theory [8, 17]. Complexity versions of these notions were developed for NP [4, 3], and E [6].

**Definition 17.** Let \( C \subseteq D \) be two complexity classes.

1. Set \( A \) is Low\(_{(C, D)}\) if \( C^A \subseteq D \).
2. Set \( A \) is High\(_{(C, D)}\) if \( C^A \supseteq D^D \).

Set \( A \) is low for \( E \) if it is Low\(_{(E, E)}\). Set \( A \) is high for \( E \) if it is High\(_{(E, E)}\).

Lowness and highness for \( E \) are preserved under polynomial Turing reductions.

**Lemma 18.** Let \( A \) be high (resp. low) for \( E \). Then all sets in the polynomial Turing degree of \( A \) are high (resp. low) for \( E \).

**Proof.** Let \( A \) be high for \( E \), \( B \) be in the \( \leq^p_T \)-degree of \( A \) and \( L \in E^E \). By highness for \( E \) of \( A \), there is an oracle TM \( M^A \) that decides \( L \) in time \( 2^m \). On an input \( x \) of size \( n \), \( M \) makes at most \( 2^m \) queries to \( A \), each of size less than \( 2^m \). Since \( A, B \) are in the same \( \leq^p_T \)-degree, there is a machine \( N^B \) deciding \( A \) in time \( n^b \). Thus each query \( q \) to \( A \) can be answered
by \( N^B \) in \(|q|^b \leq (2^n)^b = 2^{abn} \) steps. Thus \( L \) can be decided in time \( 2^{abn}2^{an} \leq 2^{2abn} \) with oracle access to \( B \), i.e. \( L \in E^B \).

The proof for lowness is similar. \( \square \)

The following result shows that any high set computes a deep set. The idea of the proof is that highness enables the set to compute polynomially random strings of small sizes, but large enough to guarantee depth of the whole sequence.

**Theorem 19.** Let \( A \) be high for \( E \). Then for every \( \epsilon > 0 \), the \( \leq^p_T \)-degree of \( A \) contains a set that is \((\log^\epsilon(n))^{1-\epsilon}\)-deep \( (P,\text{EXP}) \).

**Proof.** Let \( \epsilon > 0 \) let \( d \in \mathbb{N} \) and let \( c \in \mathbb{N} \) to be determined later. Consider the following set of random strings \( R = \{ x \mid C^{2^{2jn}}(x) \geq |x| \} \in E^A \subseteq E^A \). Let \( j \in \mathbb{N} \), \( j' = 2^{1+\log j^c} \) such that \( j^c \leq j' \leq 2j^c \) and \( \log j' = 1 + \log j^c \). Define

\[
B[2^j - 1, 2^{j+1} - 2] = R[j'-1, 2j'-2]A[2^{j-1} - 1, 2^j - 2][0^{2j'-2j-1}]
\]

i.e. \( B \cap 2^j \) codes for \( R \cap 2^{\log j'} \) and \( A \cap 2^{j-1} \).

**Claim.** \( B \) is in the \( \leq^p_T \)-degree of \( A \).

For each \( x \) of length \( \log j' \), deciding whether \( x \in R \) requires at most \( 2^{a|x|} \) queries of size at most \( 2^{a|x|} \) to \( A \) (for some \( a \in \mathbb{N} \)), i.e. \( 2^{a\log j'} \leq O(j^ac) \) queries of size \( 2^{a\log j'} = O(j^ac) \). Since there are \( j' \) such \( x \)'s, we have \( R[j', 2j' - 2] \) can be computed in at most \( O(j^{2ca}) \) steps, hence \( B \leq^p_T A \), hence \( B \leq^p_T A \). Since \( A \leq^p_T B \) the claim is proved.

Let us show that \( B \) is deep. Let \( t \in \text{PolyLog} \), \( j \in \mathbb{N} \) and \( v \in [2^{j+1} - 1, 2^{j+2} - 2] \). Since this guarantees all of \( R \cap 2^{\log j'} \) is available from \( B \mid v \), we have

\[
K^t(B \mid v) \geq K^n(B \mid v) \geq+ K^{2^n}(r, v)
\]

where \( r \) is the first string of size \( \log j' \) in \( R \). Let \( t_1 \in E \) be the time bound given by Theorem 1 with \( t_0(n) = 2^n \), we have

\[
K^{2^{2n}}(r, v) \geq C^{2^n}(r, v) \geq C^{t_1}(r) + C^{t_1}(v \mid r) - O(\log |v|)
\]

Let \( p \) be testifying \( C^{t_1}(v \mid r) \), i.e. \( U(p, r) = v \) in at most \( t_1(|v|) \) steps. Let \( p' = qq'p \) where \( q, q' \) are encoded with every bit doubled followed by a 01 flag, \( q \) are instructions, \( q' \) is an encoding of \( j \), and \( U \) on input \( p' \) recovers \( j \) then \( j' \); Computes \( r \), i.e. the first element of \( R[j' - 1, 2j' - 2] \) (for each bit of \( R \) there are \( 2^{\log j'} \) programs to be simulated for \( 2^{2\log j'} \) steps, i.e. a total of \( 2^{j'2^{j^c}} \) steps to compute \( R[j' - 1, 2j' - 2] \)); Then simulates \( U(p, r) = v \) (in \( t_1(|v|) \) steps) and outputs \( v \). The total running time is less than \( t_2(|v|) = t_1(|v|) + 2|v|^{\epsilon c} \), since \( |v| = 1 + j \). Therefore we have

\[
C^{t_2}(v) \leq C^{t_1}(v \mid r) + |q| + |q'| \leq C^{t_1}(v \mid r) + 2 \log j.
\]

Also,

\[
K^{2^{t_2}}(v) \leq+ C^{t_2}(v) + 2 \log |v|
\]
by encoding the length of $p$ at the start of $p$, where $p$ testifies $C^{t_2}(v)$.

Also $B \in \text{EXP}$ because $A \in E$ and $R[j' - 1, 2j' - 2]$ requires at most $2^{2^{o(c)}}$ steps to be computed, i.e. $B \in \text{EXP}$.

Let $p$ testify $K^{2t_2}(v)$, i.e. $U(p) = v$ in $2t_2(|v|)$ steps. Let $p' = qp$ where $q$ are instructions encoded with every bit doubled followed by a 01 flag, and $U$ on input $p'$ simulates $U(p) = v$ (in $2t_2(|v|)$ steps), and outputs $B \mid v$ (in $v2^{|v|^b}$ steps, where $B \in \text{Dtime}(2^{|v|^b})$) with a total running time less than $t_3(v) = 2t_2(|v|) + v2^{|v|^b} \leq 2t_2(\log(v + 1)) + v2^{\log^3(v+1)} \in 2^{\text{PolyLog}}$. Thus

$$K^{2t_2}(v) \geq \ K^{t_3}(B \mid v)$$

hence

$$K^{t_3}(B \mid v) \leq \ C^{t_2}(v) + 2 \log |v| \tag{6}$$

Thus,

$$K^{t}(B \mid v) - K^{t_3}(B \mid v) \geq \ C^{t_1}(r) + C^{t_1}(v \mid r) - O(\log |v|) - C^{t_2}(v) - 2 \log |v| \geq \text{By Equation 4,6}$$

$$C^{t_1}(r) + C^{t_2}(v) - 2 \log j - C^{t_2}(v) - O(\log |v|) \geq \text{By Equation 5}$$

$$C^{t_1}(r) - O(\log j)$$

Since $t_1 \in E$, we have $C^{t_1}(r) \geq |r| = \log j' \geq \log j^c$ because $r \in R$. Thus,

$$K^{t}(B \mid v) - K^{t_3}(B \mid v) \geq c \log j - O(\log j) \geq 2 \log j \geq \log^{(2)}(v) > d(\log^{(2)}(v))^{1-\epsilon}$$

for an appropriate choice of $c$. Since $t \in \text{PolyLog}, v, d \in N$ are arbitrary and $t_3 \in 2^{\text{PolyLog}}$, $B$ is $(\log^{(2)} n)^{1-\epsilon}\text{-deep}_{(P,\text{EXP})}$. \[\square\]

### 3.2 Lowness and depth

The following result shows that some deep sets can be low. The idea of the proof is to construct a very sparse set of strings that are random at the polynomial time level but not at the exponential time level. The sparseness of the set guarantees that large queries can be answered with “no”, hence only small queries need be computed, this guarantees lowness. To make the set deep, one needs to cut the sequence in blocks of subexponential size each containing a random string. The blocks need be large enough to not hurt the sparseness property, but small enough to ensure that the depth of the blocks is preserved over the whole sequence.

**Theorem 20.** There exists a set in Low(E, EXP) which is $\log^{1/3} n\text{-deep}_{(P,\text{EXP})}$.

**Proof.** For all $n \geq 1$, define $T_{n+1} = 2^{2^{\log^2 T_n}}$ with $T_0 = 1$. Consider the following set $A$ where $A[T_n, T_{n+1} - 1]$ is constructed as follows: For all $k < 2 \log^{(2)}T_n$ with $2^{\log^{x_k+1} T_n} < T_{n+1}$, put the lex-first string $x^n_k$ of $R = \{ x : C^{2^{2^k}}(x) \geq |x| \}$ with $2^{\log^{x_k} T_n} \leq \text{ind}(x^n_k) < 2^{\log^{x_k+1} T_n}$ into $A$. Note such a string exists since all the strings of length $1 + \log^{x_k} T_n$ are included in this interval, and $\forall n \ R \cap 2^n \neq \emptyset$. Also add the lex-first string of $R$ of length $\log(T_{n+1}) - 1$. Note all strings of this length have their index within interval $[T_n, T_{n+1} - 1]$. 


Claim. \( A \in \text{EXP} \).

Because to decide whether \( x \in R \) one needs to simulate \( 2^{|v|} \) programs for \( 2^{|v|^2} \) steps i.e. less that \( t_A(|x|) = 2^{|x|^2} \) steps.

Claim. \( A \) is \( \text{Low}(E, \text{EXP}) \).

Let us prove the claim. Let \( L \in E^A \); we need to show that \( L \in \text{EXP} \). Let \( M^A \) be an oracle TM deciding \( L \) in time \( 2^{cn} \). Let \( n \in \mathbb{N} \) and \( v \in [T_n, T_{n+1} - 1] \). \( M^A(v) \) makes at most \( 2^{|v|} \) queries of size at most \( 2^{|v|} \) to \( A \). First note that all queries are within \( A \). Indeed since \( |v| \leq \log T_{n+1} \), the largest query to \( A \) has size at most \( 2^c \log T_{n+1} = T_c \), i.e. a string with index less than \( 2^{1+T_{n+1}} < 2^{2\log^2 T_{n+1}} = T_{n+2} \). Let us show the queries to \( A \) can be answered in exponential time.

Claim. Let \( q \) be a query to \( A \) by \( M^A(v) \) with \( |q| \geq |v|^4 \). Then \( C^{2cn}(q) < |q| \).

Let us prove the claim. Any such \( q \) can be specified by its index in \( M^A(v) \)'s queries list (i.e. \( O(c|v|) \) bits) with the index of the previous queries to \( A \) to be answered with yes (the other ones are answered with no), i.e. at most \( O(|v|3\log^2 T_n) < |v|^4 \) bits because the max number of 1s in \( A[T_0, T_{n+2} - 1] \) is less than

\[
2 \log(2) T_{n+1} + 2 \log(2) T_n + \ldots + 2 \log(2) T_1 \leq 2 \log(2) T_{n+1} + 2n \log(2) T_n
\]

\[
\leq 3 \log(2) T_{n+1} = 3 \log^2 T_n \leq 3|v|^2.
\]

Since \( |v| \geq \log T_n \). It takes at most \( 2^{|v|} \) steps to simulate \( M^A(v) \), plus \( |v|^4 \) steps to check the list of yes query answers, i.e. a total of less than \( 2^{2|v|} \). Thus \( C^{2cn}(q) < |v|^4 \leq |q| \) which proves the claim. Consequently, if \( |q| \geq |v|^4 \), then \( A(q) = 0 \) since \( R(q) = 0 \). Hence only queries with \( |q| < |v|^4 \) need be computed to simulate \( M^A(v) \). Such queries to \( A \) are answerable in \( 2^{2|v|^2} \leq 2^{|v|^4} \) steps, because \( A \in \text{Dtime}(2^{2n^2}) \). Thus \( M^A(v) \) can be simulated in \( 2^{O(|v|^8)} \) steps, which proves \( A \) is in \( \text{Low}(E, \text{EXP}) \).

Claim. \( A \) is \( \log^{1/5} n \)-deep_{(P, \text{EXP})}.

Let \( t \in \text{PolyLog}, n, a \in \mathbb{N}, \) and \( v \in [T_n, T_{n+1} - 1] \). Let \( 0 < k < 2 \log(2) T_n \) such that \( v \in [2^{\log^k T_n}, 2^{\log^{k+1} T_n} - 1] \) \( (k = 0 \) will be done later). Note that such a \( k \) exists because when \( k = 2 \log(2) T_n - 1 \) then \( 2^{\log^{k+1} T_n} \geq T_{n+1} \) (applying log on both sides of the equation twice).

We have

\[
K^t(A \upharpoonright v) \geq C^n(A \upharpoonright v) \geq C^{2^n}(x^n_{k-1}, v)
\]

because given \( A \upharpoonright v \) one can find \( v \), and find \( x^n_{k-1} \) (corresponding to the last bit equal to 1 in \( A \upharpoonright 2^{\log^{k+1} T_n} \sim A \upharpoonright v \)) in \( v \leq 2^{|v|} \) steps. Let \( t_1 \in E \) be the time bound given by choosing \( t_0(n) = 2^n \) in Theorem 1. We have

\[
K^t(A \upharpoonright v) \geq C^{2^n}(x^n_{k-1}, v) \geq C^{t_1}(x^n_{k-1}) + C^{t_1}(v \upharpoonright x^n_{k-1}) - O(\log |v|)
\]

By Theorem 1.

(7)

(8)
Let \( p \) be a program testing \( C^{t_1}(v \mid x_{k-1}^n) \), i.e. \( U(p, x_{k-1}^n) = v \) in \( t_1(|v|) \) steps. Let \( p' = qq'p \) where \( q, q' \) are encoded with every bit doubled followed by a 01 flag, \( q \) are instructions, \( q' \) encodes \( k \), and such that \( U(p') \) recovers \( k \), finds \( x_{k-1}^n \) (i.e. the first string in \( R \) with \( 2^{log^k T_n} \leq ind(x_{k-1}^n) < 2^{log^k T_n} \), then simulates \( U(p, x_{k-1}^n) = v \) and outputs \( v \). For each bit of \( R \mid x_{k-1}^n + 1 \), there are less than \( 2^{2|x_{k-1}^n|} \) programs to simulate, each for at most \( 2|x_{k-1}^n| \) steps, i.e. at most \( 2^{2|x_{k-1}^n|} \) steps. The simulation of \( U(p, x_{k-1}^n) = v \) takes less than \( t_1(|v|) \) steps, thus a total of \( t_2(|v|) = t_1(|v|) + 2^{2|v|} \) steps.

\[
C^{t_2}(v) \leq C^{t_1}(v \mid x_{k-1}^n) + 2 \log k < C^{t_1}(v \mid x_{k-1}^n) + \log |v|.
\]

Let \( p' = qq'p \) where \( q, q' \) are encoded with every bit doubled followed by a 01 flag, \( q \) are instructions, \( q' \) encodes \( |p| \), and \( p \) is a minimal program testing \( C^{t_2}(v) \); where \( U \) on input \( p' \) recovers \( |p| \), simulates \( U(p) = v \) and outputs \( A \mid v \). The simulation of \( U(p) \) takes at most \( t_3(|v|) \) steps, the computation of the first \( v \) bits on \( A \) takes time \( v2^{2n^2} = v2^{2 \log^2 (v+1)} \leq 2 \log^2 v \) (because \( A \in Dtime(2^{2n^2}) \), i.e. a total of less than \( 2^{ \log^2 v + t_3(|v|)} = 2^{ \log^2 v + t_3(\log(v+1))} \)). Letting \( t_4(n) = 2^{ \log^2 n + t_3(\log(n+1))} \in 2^{ \text{PolyLog} } \), we have \( K^{t_3}(v) \leq K^{t_4}(A \mid v) \), hence

\[
K^{t_4}(A \mid v) \leq C^{t_2}(v) + 2 \log |v| \leq C^{t_1}(v \mid x_{k-1}^n) + O(\log |v|).
\]

Let \( d \in \mathbb{N} \) be arbitrary, we have

\[
K^{t_4}(A \mid v) - K^{t_4}(A \mid v) \geq C^{t_1}(x_{k-1}^n) + C^{t_1}(v \mid x_{k-1}^n) - C^{t_1}(v \mid x_{k-1}^n) - O(\log |v|) \geq \log^4 v - O(\log(2) v) > d \log^{1/5} v
\]

because \( |x_{k-1}^n| \geq \log^{k-1} T_n \) thus \( |x_{k-1}^n| \geq (\log^{k-1} T_n)^4 = \log^{2k+1} T_n > \log v \).

The case \( k = 0 \) is similar: the same proof applies except that \( x_{k-1}^n \) is replaced with the lex-last string \( x \) whose index bit is 1 in \( A[T_{n-1}, T_n-1] \). By construction of \( A \) the length of \( x \) is \( |x| = \log(T_n) - 1 \). Since \( k = 0 \), \( |x|^2 \geq |v| = \log(v+1) \), and the same argument as for the case \( k > 0 \) applies.

Since \( d, v \in \mathbb{N}, t \in \text{PolyLog} \) are arbitrary and \( t_4 \in 2^{ \text{PolyLog} } \), \( B \) is \( \log^{1/5} n \)-deep \( (P, \text{EXP}) \).

\section{Future work}

Highness and lowness have also been studied within the polynomial time hierarchy, i.e. for the class NP. Our polynomial depth notion can be adapted to yield a depth measure
between P and NP, however it is not obvious how to translate our highness and lowness results in that setting. Highness/lowness for E seem to mimic the domination properties of high sets in computability theory, whereas highness for NP is more like a complexity version of computable vs c.e. sets.

Our slow growth law falls short to prove that all NP sets are poly-deep. Can this be improved?

In [16], connections between DNC degrees and deep sets were shown. It is not clear what a meaningful notion of DNC degrees would be in the computational complexity setting would be.

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