Two–fluid description of two–band superconductors

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Abstract

We present a systematic study of the response properties of two–band (multi–gap) superconductors with spin–singlet (s–wave) pairing correlations, which are assumed to be caused by both intraband ($\lambda_{ii}$, $i = 1, 2$) and interband ($\lambda_{12}$) pairing interactions. In this first of three planned publications we concentrate on the properties of such superconducting systems in global and local thermodynamic equilibrium, the latter including weak perturbations in the stationary long–wavelength limit. The discussion of global thermodynamic equilibrium must include the solution (analytical in the Ginzburg–Landau and the low temperature limit) of the coupled self–consistency equations for the two energy gaps $\Delta_i(T)$, $i = 1, 2$. These solutions allow to study non–universal behavior of the two relevant BCS–Mülschlegel parameters, namely the specific heat discontinuity $\Delta C/C_N$ and the zero temperature gaps $\Delta_i(0)/\pi k_B T_c$, $i = 1, 2$. The discussion of a local equilibrium situation includes the calculation of the supercurrent density as a property of the condensate, and the calculation of both the specific heat capacity and the spin susceptibility as properties of the gas of thermal excitations in the spirit of a microscopic two–fluid description. Non–monotonic behavior in the temperature dependences of the gaps and all these local response functions is predicted to occur particularly for very small values of the interband pair–coupling constant $\lambda_{12}$. 
1 Introduction

This work is devoted to a comprehensive investigation of general two–band superconductors with Cooper pairs in a relative spin–singlet state. This investigation is based on the celebrated BCS theory of superconductivity [1] and its modifications due to Bogoliubov [2], Valatin [3], Gor’kov [4] and Nambu [5].

In a series of previous publications, one of the authors investigated universal, so–called BCS–Mühl–schlegel [6] parameters in unconventional superconductors [7], as well as the possibility of establishing a microscopic two–fluid description applied to conventional [8] and unconventional [9] superconductors. These considerations were limited to superconductors in which the paired electrons reside on a single band. Very early after the publication of the BCS theory extensions of the description to more than one band have been investigated [10]. More recently, multi–band superconductors like MgB$_2$ [11], pnictides [12] and non–centrosymmetric superconductors [13] have attracted great attention. On the theory side recent calculations of the superfluid density and the specific heat [14] as well as a Ginzburg–Landau analysis [15, 16, 17] of two–band superconductors are worth mentioning. It turns out that the theoretical treatment of pnictides requires in particular an extension to a three–band description [18]. The purpose of this paper is therefore an extension of the discussion of both BCS–Mühlschlegel parameters and the local response functions and the associated two–fluid description to two–band (two–gap) superconductors, to begin with. In this paper, which is the first of a series of three publications [19, 20], we provide a comprehensive discussion of the behavior of general two–band superconductors first in the global equilibrium and second in the local equilibrium state, respectively. In the first part we consider the general extension of the weak coupling BCS theory to two bands and two gaps and the consequent (analytical and numerical) solution of the coupled gap equations. The second part is devoted to the local response of these superconductors to weak perturbations such as a local superflow velocity, a local temperature change and a local magnetic field. The associated local response functions, namely the supercurrent density (or equivalently, the magnetic penetration depth), the specific heat and the spin susceptibility, respectively, are detectable experimentally and therefore allow for a comparison of theory with experiment. Experimental activities often include studies of the electromagnetic response and the electronic Raman response, which are performed in a regime of frequencies and wavenumbers far beyond the local equilibrium. This is why we decided to shift the discussion of the kinetic theory of two–band superconductors to a second publication [19]. This discussion will include aspects as important as the Nambu–Goldstone or gauge mode [21], a new massive collective mode characterizing the phase dynamics of the order parameter, the so–called Leggett–mode [22], the condensate plasma mode and the shift of the gauge mode to the plasma frequency, sometimes referred to as the Higgs mechanism [23]. Finally, a comprehensive treatment of the electronic Raman response, to which Leggett’s collective mode as well as a two–peak structure similar to non–centrosymmetric superconductors [24] is found to contribute, will be published in a third paper [20].

This paper is organized as follows: In section 2 we establish our notation by generalizing the BCS Hamilton operator to the case of a weakly–coupled two–band (multi–gap) superconductor and elucidate its structure in Nambu space. Next we discuss the coupled self–consistency equations for the two gaps $\Delta_1(T)$ and $\Delta_2(T)$ in the Ginzburg–Landau (GL) and the low temperature limit and present numerical results for intermediate temperatures. In section 3 we derive expressions for all relevant local response functions for two–gap superconductors, namely the magnetic penetration depth, the specific heat discontinuity and the spin susceptibility and discuss numerical results for the tempera-
ture dependence of these quantities for various values of the intra- and inter-band pairing interaction parameters. Section 4 is devoted to our discussion and conclusions.

2 Two–band superconductors with singlet pairing correlations

2.1 BCS–Leggett Hamiltonian

Let us consider a superconductor, which is characterized by electrons, that occupy two bands \( i = 1, 2 \). The electrons on these bands are assumed to be created by the operators \( \hat{c}_{k\sigma}^\dagger \), \( i = 1, 2 \). In terms of these operators, the BCS–Hamiltonian can be written as [22]

\[
\hat{H} = \sum_{k_\sigma i} \xi_{k\sigma i} \hat{c}_{k\sigma i}^\dagger \hat{c}_{k\sigma i} + \sum_{kpi} \Gamma_{kp}^{(ij)} \hat{c}_{k\uparrow i}^\dagger \hat{c}_{-k\downarrow i}^\dagger \hat{c}_{-p\downarrow i} \hat{c}_{p\uparrow i} + \sum_{kp} \Gamma_{kp}^{(12)} \left\{ \hat{c}_{k\uparrow 1}^\dagger \hat{c}_{-k\downarrow 1}^\dagger \hat{c}_{-p\downarrow 2} \hat{c}_{p\uparrow 2} + \text{H.c.} \right\}. \tag{1}
\]

Here \( \xi_{k\sigma i} = \epsilon_{k\sigma i} - \mu_i \) is the energy measured from the chemical potential \( \mu_i \) in the \( i \)–th band, whereas the quantities \( \Gamma_{kp}^{(ij)} \), \( i, j = 1, 2 \) represent the pairing interactions. Like in the one–band case, we shall denote the Fermion occupation number operators on the bands \( i = 1, 2 \)

\[
\hat{n}_{k\sigma i} = \hat{c}_{k\sigma i}^\dagger \hat{c}_{k\sigma i} ; \quad \hat{n}_i = \frac{1}{V} \sum_{k\sigma} \hat{n}_{k\sigma i} = \frac{N_i}{V}.
\]

In the same way, we may introduce BCS pair operators for the two–band case:

\[
\hat{g}_{ki} = \hat{c}_{-k\downarrow i} \hat{c}_{k\uparrow i}
\]

with the aid of which the Hamiltonian assumes the form

\[
\hat{H} = \sum_{k\sigma i} \xi_{k\sigma i} \hat{n}_{k\sigma i} + \sum_{k_\sigma i} \hat{g}_{ki} \sum_{pj} \Gamma_{kp}^{(ij)} \hat{g}_{pj}.
\]

This form gives rise to the following definition of pair potential operators, as generalized to the two–band case:

\[
\hat{\Delta}_{ki} = \sum_{pj} \Gamma_{kp}^{(ij)} \hat{g}_{pj} \tag{2}
\]

by means of which the Hamiltonian assumes the compact form:

\[
\hat{H} = \sum_{k\sigma i} \xi_{k\sigma i} \hat{n}_{k\sigma i} + \sum_{k_\sigma i} \hat{\Delta}_{ki} \hat{g}_{ki}^\dagger.
\]

We have now a convenient starting point for performing a mean field approximation by introducing the Gor’kov pair amplitudes \( g_{ki} \), \( i = 1, 2 \) [4]:

\[
g_{ki} \rightarrow \hat{g}_{ki} = \langle \hat{c}_{-k\downarrow i} \hat{c}_{k\uparrow i} \rangle \neq 0 \quad \text{only for} \quad T \leq T_c \tag{3}
\]
from which the following form of the two mean pair potentials $\Delta_{ki} = \langle \hat{\Delta}_{ki} \rangle$, $i = 1, 2$, may be deduced:

$$\Delta_{ki} = \sum_{pj} \Gamma_{kp}^{(ij)} g_{pj}$$

(4)

The BCS Hamiltonian reads finally

$$\hat{H}_{BCS} = \sum_{k\sigma i} \xi_{k\sigma i} \hat{n}_{k\sigma i} + \sum_{k i} \left\{ \Delta_{ki} \hat{g}^\dagger_{ki} + \Delta^*_{ki} \hat{g}_{ki} \right\}.$$  

(5)

2.2 Two–band superconductors in global thermodynamic equilibrium

Let us proceed by combining the energies $\xi_{ki}$ and $\Delta_{ki}$ into energy matrices in Nambu space appropriate for two–band superconductors (two–band Nambu space) in the form

$$\begin{pmatrix} \xi_{ki} & \Delta_{ki} \\ \Delta^*_{ki} & -\xi_{-ki} \end{pmatrix}.$$  

(6)

Using the energy matrix $\xi^0_{2ki}$, the BCS Hamilton operator can be written in a way reminiscent of the normal state

$$\hat{H}_{BCS} = \sum_{ki} \hat{C}^\dagger_{ki} \xi^0_{2ki} \hat{C}_{ki}$$

where we have defined spinor creation and annihilation operators in the two–band Nambu space

$$\begin{pmatrix} \hat{c}_{ki} \dagger & \hat{c}_{ki} \end{pmatrix} \begin{pmatrix} \hat{c}_{k\uparrow i} \dagger \\ \hat{c}_{k\downarrow i} \dagger \end{pmatrix} = \begin{pmatrix} \hat{c}_{ki} \dagger & \hat{c}_{ki} \end{pmatrix} \begin{pmatrix} \hat{c}_{ki} \dagger & \hat{c}_{ki} \end{pmatrix}.$$  

(7)

We show next that the energy matrix (6) can be diagonalized by a Bogoliubov–Valatin matrix, readily generalized to the two–band case. The latter can be written in the form:

$$U_{ki} \equiv \begin{pmatrix} u_{ki} & v_{ki} \\ -v^*_{ki} & u^*_{ki} \end{pmatrix}$$

with $u_{ki}$ and $v_{ki}$ denoting the usual BCS coherence factors

$$u^2_{ki} = \frac{1}{2} \left( 1 + \frac{\xi_{ki}}{E_{ki}} \right) = 1 - v^2_{ki}$$

for the two bands $i = 1, 2$. One can easily demonstrate, that this definition of the Bogoliubov–Valatin matrix leads to the following result:

$$U_{ki} \cdot \xi^0_{2ki} \cdot U_{ki} \equiv \begin{pmatrix} E_{ki} & 0 \\ 0 & -E_{ki} \end{pmatrix}; \quad E_{ki} = \sqrt{\xi^2_{ki} + |\Delta_{ki}|^2}$$

After the Bogoliubov–Valatin diagonalization with

$$\hat{C}_{ki} = U_{ki} \cdot \hat{\alpha}_{ki}; \quad \hat{\alpha}_{ki} = \begin{pmatrix} \hat{\alpha}_{k\uparrow i} \dagger \\ \hat{\alpha}_{k\downarrow i} \dagger \end{pmatrix}$$

4
the BCS Hamiltonian is of the desired diagonal form

\[ \hat{H}_{\text{BCS}} = \frac{U_{\text{BCS}}(0)}{T=0} + \sum_{k,\sigma i} E_k \alpha_k^\dagger \alpha_{k\sigma i} \]

which allows for the interpretation of the operators \( \alpha_{k\sigma i}^\dagger \) and \( \alpha_{k\sigma i} \) as ones to create and annihilate a fermionic elementary (thermal) excitation in a quantum state \( |k, \sigma, i⟩ \), a so-called Bogoliubov–Valatin quasiparticle (BVQP). The quantity \( U_{\text{BCS}}(0) \) denotes the ground state energy, whereas \( E_k \) represents the energy spectrum of the BVQP on the two bands \( i = 1, 2 \). The statistical physics of the excitation gas is exclusively describable by the BVQP Fermi–Dirac distribution

\[ \nu_{ki} = \nu(E_{ki}) = \langle \alpha_{k\sigma i}^\dagger \alpha_{k\sigma i} \rangle = \frac{1}{\exp \left( \frac{E_{ki}}{2k_B T} \right) + 1} \] (8)

and its derivative

\[ y_{ki} \equiv -\frac{\partial \nu_{ki}}{\partial E_{ki}} = \frac{1}{4k_B T} \frac{1}{\cosh^2 \left( \frac{E_{ki}}{2k_B T} \right)} \] (9)

which is known as the so-called Yosida kernel, since it generates the band–selected Yosida functions \[25\]

\[ Y_i(\hat{p}, T) = \int_{-\infty}^{\infty} d\xi \Delta^p=\Delta y_{pi} \theta_{ki} \] (10)

In global equilibrium, the electronic distribution functions \( n_{ki}^0 \) and \( g_{ki} \) can be evaluated by expressing the ordinary electron operators \( \hat{c}_{ki} \) and \( \hat{c}_{ki}^\dagger \) through the BVQP operators \( \alpha_{ki} \) and \( \alpha_{ki}^\dagger \) using the Bogoliubov–Valatin transformation method with the result

\[ n_{ki} = \langle \hat{c}_{k\sigma i}^\dagger \hat{c}_{k\sigma i} \rangle = u_{ki}^2 \nu_{ki} + v_{ki}^2 |1 - \nu_{ki}| = \frac{1}{2} - \xi_{ki} \theta_{ki} \] (11)

\[ g_{ki} = \langle \hat{c}_{-k\sigma i} \hat{c}_{k\sigma i} \rangle = -\Delta_{ki} \theta_{ki} \] ; \( \theta_{ki} \equiv \frac{1}{2E_{ki}} \tanh \frac{E_{ki}}{2k_B T} \) (12)

It is convenient to combine \( n_k \) and \( g_k \) in an equilibrium density matrix in two–band Nambu space:

\[ \underline{n}_{ki}^0 = \begin{pmatrix} n_{ki} & g_{ki} \\ g_{ki}^* & 1 - n_{-ki} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} \xi_{ki} & \Delta_{ki} \\ \Delta^*_{ki} & -\xi_{ki} \end{pmatrix} \theta_{ki} = \frac{1}{2} \begin{pmatrix} 1 - \xi_{ki} \theta_{ki} \\ \xi_{ki} \theta_{ki} \end{pmatrix} \]

Note that the Bogoliubov–Valatin matrix \( U_k \) also diagonalizes \( \underline{n}_{ki}^0 \) with the result

\[ U_k^\dagger \cdot \underline{n}_{ki}^0 \cdot U_k = \begin{pmatrix} \nu_{ki} & 0 \\ 0 & 1 - \nu_{-ki} \end{pmatrix} = \begin{pmatrix} \nu(E_{ki}) & 0 \\ 0 & \nu(-E_{ki}) \end{pmatrix} \] .
2.3 Equilibrium gap equations in two–band superconductors

The energy gaps $\Delta_{ki}$ are related to the Gor’kov amplitudes $g_{ki}$ through the set of coupled self–consistency equations (4). In strict analogy to the one–band case, we now have to choose three weak coupling BCS model pairing interactions, which are introduced in the standard factorizable way:

$$\Gamma_{kp}^{(ij)} = \begin{cases} -\Gamma_{ij} & \text{for } |\xi_{ki}|, |\xi_{pj}| < \epsilon_0, \ i, j = 1, 2 \\ 0 & \text{otherwise} \end{cases} \quad (13)$$

Here $\epsilon_0$ is a characteristic cut–off energy. Inserting this weak coupling BCS model form [13] for the pairing interactions $\Gamma_{kp}^{(ij)}$, one may identify $\Delta_{ki} = \Delta_i$ and write

$$\Delta_i = -\sum_j \Gamma_{ij} \sum_p g_{pj}$$

where we have defined

$$\sum_p' A_{pi} = N_i(0) \int_{-\epsilon_0}^{\epsilon_0} d\xi_{pi} A_{pi}$$

with $N_i(0)$ the density of states for one spin projection on the $i$–th band. Now using the result of the Bogoliubov–Valatin transformation from equation (12) we arrive at

$$\Delta_i = \sum_j \Gamma_{ij} \sum_p' \theta_{pj} \Delta_j = \sum_j \Gamma_{ij} N_j(0) \int_{\equiv \lambda_{ij}}^{\epsilon_0} d\xi_{pj} \theta_{pj} \Delta_j \equiv \Xi_j.$$ 

It is convenient to define here dimensionless pairing interactions

$$\lambda_{ij} \equiv N_j(0) \Gamma_{ij} \quad \text{and} \quad \lambda \equiv \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{12} & \lambda_{22} \end{pmatrix}$$

with $\lambda$ the symmetric pairing interaction matrix and integrals

$$\Xi_i \equiv \int_{-\epsilon_0}^{\epsilon_0} d\xi_{pi} \theta_{pi} = L(T) - P_i(T) \quad (14)$$

with

$$L(T) = \ln \frac{2\epsilon_0 e^\gamma}{\pi k_B T} \quad ; \quad \gamma = 0.5777(2) \quad (\text{Euler’s constant})$$

$$P_i(T) = \int_{-\epsilon_0}^{\epsilon_0} d\xi_{pi} \left( \frac{\tanh \frac{E_{pi}}{2k_B T}}{2\xi_{pi}} - \frac{\tanh \frac{E_{pi}}{2k_B T}}{2E_{pi}} \right).$$
Note, that the integrals (14) take very simple forms in two different limiting cases:

\[ \Xi_i = \begin{cases} 
\ln \frac{2\epsilon_0}{\Delta_i(0)} & ; \quad T \to 0 \quad \text{(low-} T \text{ regime)} \\
L(T) - \frac{7\zeta(3)}{8} \frac{\Delta_i^2(T)}{(\pi k_B T)^2} & ; \quad T \to T_c \quad \text{(GL regime)}
\end{cases} \]  

(15)

Now using the definition (14) we are able to write the equilibrium gap equation in a way reminiscent of the one–band case:

\[ 0 = \sum_j \{-\delta_{ij} + \lambda_{ij} \Xi_j\} \Delta_j \iff 0 = \left\{ -\lambda^{-1} + \begin{pmatrix} \Xi_1 & 0 \\ 0 & \Xi_2 \end{pmatrix} \right\} \cdot \begin{pmatrix} \Delta_1 \\ \Delta_2 \end{pmatrix}. \]  

(16)

The coupled gap equations can be rewritten in the form:

\[ \Xi_1 = \frac{1}{\lambda} [\lambda_{22} - \lambda_{12} r(T)] \quad ; \quad r(T) \equiv \frac{\Delta_2(T)}{\Delta_1(T)} \]

\[ \Xi_2 = \frac{1}{\lambda} [\lambda_{11} - \lambda_{12} t(T)] \quad ; \quad t(T) \equiv \frac{\Delta_1(T)}{\Delta_2(T)} = \frac{1}{r(T)} \]

Taking the difference of these coupled equations turns out to be a convenient starting point for the calculation of the unknown ratio \( r(T) = 1/t(T) \). Defining

\[ a \equiv \frac{\lambda_{22} - \lambda_{11}}{\lambda} \]

\[ \gamma_L \equiv \frac{\lambda_{12}}{\lambda} \]

\[ \lambda \equiv \text{det} \lambda = \lambda_{11} \lambda_{22} - \lambda_{12}^2 \]  

(17)

with the quantity \( \gamma_L \) being referred to as the *Leggett coupling* [22], we obtain

\[ \Xi_1 - \Xi_2 = a - \gamma_L [r(T) - t(T)] . \]

In what follows we assume without restricting generality that \( \Delta_2(T) > \Delta_1(T) \), or alternatively \( r(T) > 1 \).

### 2.3.1 Determination of the transition temperature

At the transition temperature \( T_c \) we may state that

\[ \Xi_1(T_c) = \Xi_2(T_c) = L(T_c) = \frac{1}{\lambda} [\lambda_{22} - \lambda_{12} r(T_c)] = \frac{1}{\lambda} [\lambda_{11} - \lambda_{12} t(T_c)] . \]

This condition yields quadratic equations in \( r(T_c) \) and \( t(T_c) \),

\[ r^2(T_c) - \mu r(T_c) - 1 = 0 \]

\[ t^2(T_c) + \mu t(T_c) - 1 = 0 \]

which have the solutions

\[ r(T_c) = \frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} + 1} \quad ; \quad \mu = \frac{a}{\gamma_L} \]

\[ t(T_c) = -\frac{\mu}{2} + \sqrt{\frac{\mu^2}{4} + 1} \]  

(18)
and the obvious condition \( r(T_c) \cdot t(T_c) = 1 \) holds. From this, the transition temperature \( T_c \) is obtained in the following form

\[
\pi k_B T_c = 2\epsilon_0 e^\gamma e^{-\frac{1}{2} \left( \lambda_{22} + \lambda_{11} - \sqrt{\lambda_{22} - \lambda_{11}}^2 + 4\lambda_{12}^2 \right)},
\]

which coincides with the result of Suhl, Matthias and Walker [10] derived as early as 1959. Note, that in case of two decoupled gaps (\( \lambda_{12} = 0 \)) one is left with two different BCS transition temperatures:

\[
\pi k_B T_{c(i)} = 2\epsilon_0 e^\gamma e^{-\frac{1}{2} \lambda_{ii}}, \quad i = 1, 2
\]

A careful analysis of Eq. (19) shows, that the transition temperature \( T_c(\lambda_{12}) \) is always larger than the largest of the values \( T_c(\lambda_{12} = 0) \). This fact has actually been discussed in previous publications [26, 27]. In order to visualize this statement, we have plotted the quantity \( T_c(\lambda_{12})/T_{c2} \) vs. \( \lambda_{12} \) in Fig. 1. For calculations we used the model with the intra–band coupling constants \( \lambda_{11} = 0.12 \) and \( \lambda_{22} = 0.19 \), which is suitable for MgB\(_2\)–like two–band superconductors. The figure clearly shows the monotonic increase of \( T_c(\lambda_{12}) \) away from its value \( T_{c2} \) at \( \lambda_{12} = 0 \).

![Figure 1](image)

**Figure 1**: Normalized transition temperature as a function of inter–band coupling constant \( \lambda_{12} \). The solid line represents the transition temperature \( T_c \) of a two–band superconductor with intra–band coupling constants \( \lambda_{11} = 0.12 \) and \( \lambda_{22} = 0.19 \). The dashed lines show the case of decoupled bands with two distinct transition temperatures \( T_{c1} \) and \( T_{c2} \) for \( \lambda_{11} = 0.12 \) and \( \lambda_{22} = 0.19 \) respectively.

### 2.3.2 Determination of the gaps in the Ginzburg–Landau regime

Inserting the GL expansions (15) into the eqs.(16) for \( \Xi_i(T) \) and solving for \( \Delta_i(T) \), we arrive at

\[
\begin{align*}
\frac{\Delta_2^2(T)}{(\pi k_B T)^2} &= \frac{8}{\zeta(3)} \left\{ \ln \frac{T_c}{T} + \gamma_L \left[ r(T) - r(T_c) \right] \right\} \\
\frac{\Delta_2^2(T)}{(\pi k_B T)^2} &= \frac{8}{\zeta(3)} \left\{ \ln \frac{T_c}{T} + \gamma_L \left[ t(T) - t(T_c) \right] \right\}.
\end{align*}
\]

(20)
The result (21) can be iteratively improved as follows:

\[ \ln \frac{T_c}{T} + \gamma_L [t(T) - t(T_c)] = r^2(T) \left\{ \ln \frac{T_c}{T} + \gamma_L [r(T) - r(T_c)] \right\} \]

This can easily be rearranged to the form of a pair of fourth order equations for \( r(T) \) and \( t(T) \), respectively:

\[
\begin{align*}
    r^4(T) + \alpha r^3(T) - \beta r(T) - 1 &= 0 \\
    t^4(T) + \beta t^3(T) - \alpha t(T) - 1 &= 0
\end{align*}
\]

where we have introduced

\[
\alpha = \frac{1}{\gamma_L} \left[ \ln \frac{T_c}{T} - \gamma_L r(T_c) \right] \\
\beta = \frac{1}{\gamma_L} \left[ \ln \frac{T_c}{T} - \gamma_L \right]
\]

In order to solve these equations, we used Newton’s method. We started from the known ratios \( r_0(T) = r(T_c), \ t_0(T) = t(T_c) \) and obtained by means of this procedure the first order approximation \( r_1(T), t_1(T) \) in the following obvious way:

\[
\begin{align*}
    r_1(T) &= r(T_c) - \frac{r^4(T_c) + \alpha r^3(T_c) - \beta r(T_c) - 1}{4r^3(T_c) + 3\alpha r^2(T_c) - \beta} \\
    &= r(T_c) - \frac{r^2(T_c)[r^2(T_c) - 1] \ln \frac{T_c}{T}}{\gamma_L[1 + r^4(T_c)] + r(T_c)[3r^2(T_c) - 1] \ln \frac{T_c}{T}} \\
    t_1(T) &= t(T_c) - \frac{t^4(T_c) + \beta t^3(T_c) - \alpha t(T_c) - 1}{4t^3(T_c) + 3\beta t^2(T_c) - \alpha} \\
    &= t(T_c) - \frac{t^2(T_c)[t^2(T_c) - 1] \ln \frac{T_c}{T}}{\gamma_L[1 + t^4(T_c)] + t(T_c)[3t^2(T_c) - 1] \ln \frac{T_c}{T}}
\end{align*}
\]

(21)

Note that the temperature dependence of the gap ratios \( r(T), t(T) \) differs from \( r(T_c), t(T_c) \) only in the case \( r(T_c) \neq 1, t(T_c) \neq 1 \).

The result (21) can be iteratively improved as follows:

\[
\begin{align*}
    r_{n+1}(T) &= r_n(T) - \frac{r_n^4(T) + \alpha r_n^3(T) - \beta r_n(T) - 1}{4r_n^3(T) + 3\alpha r_n^2(T) - \beta} ; \ n \geq 1 \\
    t_{n+1}(T) &= t_n(T) - \frac{t_n^4(T) + \beta t_n^3(T) - \alpha t_n(T) - 1}{4t_n^3(T) + 3\beta t_n^2(T) - \alpha} ; \ n \geq 1
\end{align*}
\]

Eventually, one arrives at the exact result for the ratio \( r(T) \) and \( t(T) \) in the limiting form

\[
\begin{align*}
    r(T) &= \lim_{n \to \infty} r_n(T) \\
    t(T) &= \lim_{n \to \infty} t_n(T)
\end{align*}
\]
Now the known results for \( r(T) \) and \( t(T) \) can be inserted back into (20) to yield the final results for the two gaps \( \Delta_1(T) \) and \( \Delta_2(T) \). Using the first order approximations (21), i.e. \( r(T) \approx r_1(T) \) and \( t(T) \approx t_1(T) \), we obtain for the temperature dependence of the gaps in the Ginzburg–Landau regime

\[
\begin{align*}
\frac{\Delta_1^2(T)}{(\pi k_B T)^2} &= \frac{\ln \frac{T_c}{T} + \frac{\gamma L}{1 + \frac{\gamma L}{T(1/4T - 1)}}}{\ln \frac{T_c}{T} + \frac{\gamma L}{1 + \frac{\gamma L}{T(1/4T - 1)}}} \frac{8}{7\zeta(3)} \ln \frac{T_c}{T} \\
\frac{\Delta_2^2(T)}{(\pi k_B T)^2} &= \frac{\ln \frac{T_c}{T} + \frac{\gamma L}{1 + \frac{\gamma L}{T(1/4T - 1)}}}{\ln \frac{T_c}{T} + \frac{\gamma L}{1 + \frac{\gamma L}{T(1/4T - 1)}}} \frac{8}{7\zeta(3)} \ln \frac{T_c}{T}
\end{align*}
\]

(22)

As expected in the limit of vanishing Leggett coupling \( \gamma_L \to 0 \), the two gaps \( \Delta_1(T) \) and \( \Delta_2(T) \) show BCS temperature dependence in the Ginzburg–Landau regime

\[
\lim_{\gamma_L \to 0} \frac{\Delta_i^2(T)}{(\pi k_B T)^2} = \frac{8}{7\zeta(3)} \ln \frac{T_{ci}}{T}; \quad i = 1, 2.
\]

### 2.3.3 Determination of the gaps at zero temperature

At low temperature, we may use equation (15) for \( \Xi_i(T) \) and immediately obtain from (16) a pair of transcendental equations for the ratios \( r(0) \) and \( t(0) \):

\[
\begin{align*}
\ln r(0) &= a - \gamma_L \left[ r(0) - \frac{1}{r(0)} \right] \\
\ln t(0) &= -a - \gamma_L \left[ t(0) - \frac{1}{t(0)} \right].
\end{align*}
\]

(23)

The two gaps \( \Delta_i(0) \) at \( T = 0 \) can then be expressed through \( r(0) \) and \( t(0) \) in the form

\[
\begin{align*}
\frac{\Delta_1(0)}{k_B T_c} &= \frac{\pi}{e^\gamma} e^{-\gamma_L[r(T_c) - r(0)]} \\
\frac{\Delta_2(0)}{k_B T_c} &= \frac{\pi}{e^\gamma} e^{-\gamma_L[t(T_c) - t(0)]}.
\end{align*}
\]

(24)

Clearly, the modification of the BCS result for the zero temperature gap consists in enlargement factor involving \( r(0) \) and a reduction factor involving \( t(0) \) for the two gaps \( \Delta_1(0) \) and \( \Delta_2(0) \), respectively. As a consequence, the zero–temperature gaps (24) of two–band (two–gap) superconductors are seen to represent the first non–universal BCS–Mühlischlegel parameter (the second being the specific heat discontinuity, to be discussed in section 3.2.1). It remains to calculate \( r(0) \). The transcendental equation (23) for the determination of the ratio \( r(0) \) can be solved, if \( r(0) \) is close to one. Then one may expand the logarithm \( \ln r(0) \approx -[1 - r(0)] \) and obtains the approximate result

\[
r_1(0) = \frac{1}{2(1 + \gamma L)} \left\{ 1 + a + \sqrt{(1 + a)^2 + 4 \gamma_L(1 + \gamma L)} \right\}
\]

\[
a \ll 1; \quad 1 + \frac{a}{1 + 2 \gamma L} + \ldots
\]

This result can be iteratively improved using Newton’s procedure

\[
r_{n+1}(0) = r_n(0) - \frac{\ln r_n(0) + \gamma L \left[ r_n(0) - \frac{1}{r_n(0)} \right] - a}{\frac{1}{r_n(0)} + \gamma L \left[ 1 + \frac{1}{r_n(0)} \right]}; \quad n \geq 1
\]
and we may state that we have found an exact result for $r(0)$:

$$r(0) = \lim_{n \to \infty} r_n(0) \quad (25)$$

2.3.4 Numerical results for the gaps

This section is devoted to a discussion of the numerical solutions of the coupled gap equations at arbitrary temperatures, as compared with available analytical solutions in both the Ginzburg–Landau and the zero temperature limit. The intra–band pair coupling constants were again chosen to be $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$. In Figs. 2a and 2b, we have plotted the normalized gap functions $\Delta_i^2(T)/\Delta_i^2(0)$, $i = 1$ (right axis) and 2 (left axis) as a function of reduced temperature $T/T_c$ for values of $\lambda_{12} = 0.005$ (2a) and $\lambda_{12} = 0.07$ (2b). As can be immediately seen from these figures, the large gap $\Delta_2^2(T)$ shows a BCS–like behavior when plotted against the reduced temperature, whereas the small gap $\Delta_1^2(T)$ displays a non–monotonic temperature dependence, when $\lambda_{12}$ is very small. The latter behavior can be explained by the fact, that the gaps are nearly independent in this case. This is reflected also in the low–$T$ behavior of the local response functions, which will be discussed in section 3.

Also shown in Figs. 2a and 2b as dashed lines are the analytical results obtained in the Ginzburg–Landau and the low–$T$ limit, respectively. Fig. 2c shows a comparison of $\Delta_i^2(T)/\Delta_i^2(0)$, $i = 1, 2$ for the case $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$ and various interband coupling parameters $\lambda_{12}$, as indicated in the figure caption. Since the temperature dependence of the large gap remains nearly unaffected by the variation of the interband coupling, we present in Fig. 2c only the case of $\lambda_{12} = 0.07$. The small gap shows, however, a transition to the nearly independent BCS behavior for a sufficiently small value of $\lambda_{12}$.

![Figure 2a: Two–axis plot for the evolution of the normalized superconducting gaps $\Delta_i^2(T)/\Delta_i^2(0)$ (small gap, right axis) and $\Delta_2^2(T)/\Delta_2^2(0)$ (large gap, left axis) with reduced temperature $T/T_c$ for intra–band coupling constants $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$ and inter–band coupling constant $\lambda_{12} = 0.005$. Analytical solutions in low temperature limit and in the Ginzburg–Landau limit are illustrated by dashed lines.](image-url)
Figure 2b: Evolution of the normalized superconducting gaps $\Delta_i^2(T)/\Delta_i^2(0)$ (small gap) and $\Delta_j^2(T)/\Delta_j^2(0)$ (large gap) with reduced temperature $T/T_c$ for intra–band coupling constants $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$ and inter–band coupling constant $\lambda_{12} = 0.07$. Analytical solutions in low temperature limit and in the Ginzburg–Landau limit are shown as dashed lines.

Figure 2c: Evolution of the normalized superconducting gaps $\Delta_i^2(T)/\Delta_i^2(0)$ (small gap) and $\Delta_j^2(T)/\Delta_j^2(0)$ (large gap) with reduced temperature $T/T_c$ for intra–band coupling constants $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$ and various inter–band coupling constants:

(a) $\lambda_{12} = 0.07$ (for large gap),
(b) $\lambda_{12} = 0.07$, (c) $\lambda_{12} = 0.03$, (d) $\lambda_{12} = 0.005$ (for small gap).
3 Two–band superconductors in local thermodynamic equilibrium

In what follows, we consider deviations from global thermodynamic equilibrium. To keep things on the simplest nontrivial level, we consider perturbation potentials $\delta \xi_{\text{ext}}^{\text{p}_{\sigma\sigma'}}$, which are slowly varying in time and space. Such a situation, which corresponds to the long wavelength ($q \to 0$) low frequency or stationary ($\omega \to 0$) limit, is referred to as local equilibrium, and can be described by the perturbation Hamiltonian

$$\hat{H}_{\text{ext}} = \frac{1}{V} \sum_{p\sigma\sigma'i} \hat{c}_{p\sigma'i}^{\dagger} \delta \xi_{\text{ext}}^{\text{p}_{\sigma\sigma'}} \hat{c}_{p\sigma'i}$$

(26)

For the following discussion, we shall show that the perturbation potentials $\delta \xi_{\text{ext}}^{\text{p}_{\sigma\sigma'}}$ can be written in the general form

$$\delta \xi_{\text{ext}}^{\text{p}_{\sigma\sigma'}} = \left[ p_{i} \cdot v_{i}^{s} - \left( \frac{E_{k_{i}}}{T} - \frac{\partial E_{k_{i}}}{\partial T} \right) \delta T \right] \delta_{\sigma\sigma'} - \frac{\gamma \hbar}{2} \tau_{\sigma\sigma'} \cdot \mathbf{H}_{\text{ext}}$$

(27)

The three terms above describe the coupling of the quasiparticle system to (i) the condensate velocity $v_{i}^{s}$, to be derived below, (ii) to a local temperature change $\delta T$ and (iii) to an external magnetic field (Zeeman coupling). In the latter term $\gamma$ denotes the gyromagnetic ratio of electrons.

3.1 Normal and superfluid density

In the presence of a vector potential there will be an Amp`ere coupling induced quasiparticle energy shift

$$\delta \xi_{\text{ext}}^{k_{i}\sigma\sigma'} = \left\{ \frac{1}{2m} \left( p_{i} - \frac{e}{c} A \right)^{2} - \frac{p_{i}^{2}}{2m} \right\} \delta_{\sigma\sigma'} = -v_{k_{i}} \cdot \frac{e}{c} A \delta_{\sigma\sigma'}$$

In order to arrive at a gauge–invariant form of the Amp`ere coupling, we perform a gauge transformation to the vector potential of the form

$$-v_{k_{i}} \cdot \frac{e}{c} A \to -v_{k_{i}} \cdot \left\{ \frac{e}{c} A - \frac{\hbar}{2} \nabla \chi_{i} \right\} \equiv p_{i} \cdot v_{i}^{s}$$

$$v_{i}^{s} = \frac{1}{m} \left( \frac{\hbar}{2} \nabla \chi_{i} - \frac{e}{c} A \right)$$

where the scalar functions $\chi_{i}$ denote the phases of the superconducting order parameters on the bands $i = 1, 2$, which therefore acquire the physical meaning of velocity potentials for the superflow described by $v_{i}^{s}$. Hence we may rewrite $\delta \xi_{\text{ext}}^{k_{i}\sigma\sigma'}$ in the convenient form of Doppler shifts:

$$\delta \xi_{\text{ext}}^{k_{i}\sigma\sigma'} = p_{i} \cdot v_{i}^{s} \delta_{\sigma\sigma'}$$

(28)
Now the supercurrent density can be written in the standard gauge–invariant quantum–mechanical form, which can be Taylor–expanded w.r.t. the small superflow velocities $v^s_i$:

$$j^s = \sum_{i=1}^{2} \left[ n_i v^s_i + \frac{1}{V} \sum_{p\sigma} v_{p\sigma} (E_{p\sigma} + p_i \cdot v^s_i) \right]$$

$$= \sum_{i=1}^{2} \left[ n_i v^s_i + \frac{1}{V} \sum_{p\sigma} v_{p\sigma} \left\{ \nu (E_{p\sigma}) + \frac{\partial \nu (E_{p\sigma})}{\partial E_{p\sigma}} (p_i \cdot v^s_i) \right\} \right]$$

$$= \sum_{i=1}^{2} \left[ n_i v^s_i - \frac{1}{V} \sum_{p\sigma} y_{p\sigma} v_{p\sigma} (p_i \cdot v^s_i) \right]$$

This immediately implies the definition of the band–selected normal fluid density tensor in the form

$$n^n_{\mu\nu i} = \sum_{p\sigma} \frac{p_{\mu p\sigma i} v_{p\sigma}}{m} = N_{Fi} \left\langle \int_{-\mu_i}^{\infty} d\xi_{p\mu i} \frac{p_{\mu p\sigma i}}{m} \right\rangle_{FS_i}$$

$$= N_{Fi} \frac{p_{\mu i}^2}{m} \langle Y_i(\hat{p}, T) \hat{p}_\mu \hat{p}_{\nu} \rangle_{FS_i} = n_i 3 \langle Y_i(\hat{p}, T) \hat{p}_\mu \hat{p}_{\nu} \rangle_{FS_i} \Delta_{\mu=\nu} = n_i Y_i(T) \delta_{\mu\nu}$$

Note that $\langle \ldots \rangle_{FS_i}$ denotes the average over the $i$–th Fermi surface and $n = n_1 + n_2$. Correspondingly, the band–selected superfluid density tensor can be defined as

$$n^s_{\mu\nu i} = n_i \delta_{\mu\nu} - n^n_{\mu\nu i} = n_i \left[ \delta_{\mu\nu} - 3 \langle Y_i(\hat{p}) \hat{p}_\mu \hat{p}_{\nu} \rangle_{FS_i} \right] \Delta_{\mu=\nu} = n_i [1 - Y_i(T)] \delta_{\mu\nu}$$

Finally, we may summarize our result for the supercurrent response in two–band superconductors as follows:

$$j^s_\mu = \sum_{i=1}^{2} \left( n_i \delta_{\mu\nu} - n^n_{\mu\nu i} \right) v^s_{\nu i} = \sum_{i=1}^{2} n^s_{\mu\nu i} v^s_{\nu i} \Delta_{\mu=\nu} = \sum_{i=1}^{2} n^s_{\mu\nu} v^s_{\nu i}$$

3.1.1 The London–BCS magnetic penetration depth

In order to derive an expression for the London–BCS magnetic penetration depth $\lambda_L(T)$ we start from [27] and write

$$\mathbf{j}^s = e \mathbf{j}^s = en^s v^s; \quad n^s = \sum_{i=1}^{2} n^s_i$$

As a next step we use Ampère’s law $\nabla \times \mathbf{B} = 4\pi \mathbf{j}^s / c$ to derive the screening differential equation $\nabla^2 \mathbf{B} = \mathbf{B} / \lambda_L^2$, with

$$\lambda_L^2(T) = \frac{mc^2}{4\pi n^s(T)e^2}$$

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representing the London–BCS magnetic penetration depth. In Fig. 3 we have plotted the dependence of the normalized magnetic penetration depth $\lambda_L(T)/\lambda_L(0)$ on reduced temperature $T/T_c$ for the set of intra–band pair coupling constants $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$ and two values for $\lambda_{12} = 0.005$ (i) and $\lambda_{12} = 0.07$ (ii). For the lower value of $\lambda_{12}$, the magnetic penetration depth shows non–monotonic behavior, which can be traced back to thermal activation processes associated with the smaller gap $\Delta_1(T)$.

**Figure 3:** Temperature dependence of the normalized magnetic penetration depth $\lambda_L(T)/\lambda_L(0)$ from equation (32) for intra–band coupling constants $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$ and inter–band coupling constant (i) $\lambda_{12} = 0.005$ and (ii) $\lambda_{12} = 0.07$.

### 3.2 Specific heat capacity in two–band superconductors

The energy change $\delta E^\text{ext}_{ki}$ that is related to a local temperature change $\delta T$ in the $i$–th band can be derived as follows:

$$
\frac{E_{ki} + \delta E^\text{ext}_{ki}}{k_B T} = \frac{E_{ki} + \frac{\partial E_{ki}}{\partial T} \delta T}{k_B (T + \delta T)} = \frac{1}{k_B T} \left\{ E_{ki} - \left( \frac{E_{ki}}{T} - \frac{\partial E_{ki}}{\partial T} \right) \delta T \right\}
$$

$$
\delta E^\text{ext}_{ki} = - \left( \frac{E_{ki}}{T} - \frac{\partial E_{ki}}{\partial T} \right) \delta T
$$

The change of the BVQP entropy density $\delta \sigma$ due to the temperature change $\delta T$ can be written as

$$
T \delta \sigma = \frac{1}{V} \sum_{p\sigma i} E_{pi} \nu(E_{pi}) \left( E_{pi} + \delta E^\text{ext}_{pi} \right)
$$

$$
= \frac{1}{V} \sum_{p\sigma i} E_{pi} \left\{ \nu(E_{pi}) + \frac{\partial \nu(E_{pi})}{\partial E_{pi}} \delta E^\text{ext}_{pi} \right\}
$$

$$
= \frac{1}{V} \sum_{p\sigma i} y_{pi} \left( \frac{E_{pi}^2}{T} - E_{pi} \frac{\partial E_{pi}}{\partial T} \right) \delta T
$$
The local thermodynamic relation

\[ T \delta \sigma = C_V(T) \delta T \]

allows for the identification of the specific heat capacity \( C_V(T) \) of a two–band superconductor in the form

\[
C_V(T) = \sum_{i=1}^{2} C_{Vi}(T) \\
C_{Vi}(T) = \sum_{p\sigma} y_{pi} \left( \frac{E_{pi}^2}{T} - E_{pi} \frac{\partial E_{pi}}{\partial T} \right)
\]

In Fig. 4a we have plotted the dependence of the normalized specific heat capacity \( C_V(T)/C_V(T_c^+) \) on reduced temperature \( T/T_c \) for the set of intra–band pair coupling constants \( \lambda_{11} = 0.12 \) and \( \lambda_{22} = 0.19 \) and two values for \( \lambda_{12} = 0.005 \) (i) and \( \lambda_{12} = 0.07 \) (ii). For the lower value of \( \lambda_{12} \) the specific heat capacity shows non–monotonic behavior, which can be explained by thermal activation processes associated with the smaller gap \( \Delta_1(T) \). Note that the discontinuity of \( C_V(T)/C_V(T_c^+) \) at the transition temperature depends on \( \lambda_{12} \) in a way, which will be investigated analytically in more detail in the following section 3.2.1.

**Figure 4a:** Temperature dependence of the normalized specific heat capacity \( C_V(T)/C_V(T_c^+) \) from equation (33) for intra–band coupling constants \( \lambda_{11} = 0.12 \) and \( \lambda_{22} = 0.19 \) and inter–band coupling constant (i) \( \lambda_{12} = 0.005 \) and (ii) \( \lambda_{12} = 0.07 \).
3.2.1 The specific heat discontinuity in two–band superconductors

In this section we wish to calculate analytically the discontinuity in the specific heat at the transition temperature and start from the equation (33):

\[ C_V(T) = 2 \sum_{p_i} y_{p_i} \left[ \frac{E_{p_i}^2}{T} - \frac{1}{2} \frac{\partial \Delta_2(T)}{\partial T} \right] \]

\[ \lim_{T \to T_c} \sum_{i=1}^{2} \left\{ \frac{N_i(0)(\pi k_B T_c)^2}{T_c} - \frac{N_i(0)}{2} \lim_{T \to T_c} \frac{\partial \Delta_2(T)}{\partial T} \right\} \]

\[ = \frac{N_F (\pi k_B T_c)^2}{T_c} - \frac{N_F}{4} \lim_{T \to T_c} \sum_{i=1}^{2} \frac{\partial \Delta_2(T)}{\partial T} \tag{34} \]

In order to make the following calculations more transparent, we have assumed in the latter equality [34], that \( N_1(0) = N_2(0) = N_F/2 \). The general form of the specific heat discontinuity will be given, however, at the end of this section. As a next step we compute the temperature derivatives of the two gap functions \( \Delta_1(T) \) and \( \Delta_2(T) \) as given by equation (20) near \( T_c \):

\[ \lim_{T \to T_c} \frac{\partial \Delta_1^2(T)}{\partial T} = \frac{8}{7 \zeta(3)} \frac{(\pi k_B T_c)^2}{T_c} \left[ -1 + \gamma_l T_c \lim_{T \to T_c} \frac{\partial r(T)}{\partial T} \right] \]

\[ \lim_{T \to T_c} \frac{\partial \Delta_2^2(T)}{\partial T} = \frac{8}{7 \zeta(3)} \frac{(\pi k_B T_c)^2}{T_c} \left[ -1 + \gamma_l T_c \lim_{T \to T_c} \frac{\partial t(T)}{\partial T} \right] \]

The temperature derivatives of the gap ratios \( r(T) \) and \( t(T) \) at the transition temperature are obtained from equation (21) and can be expressed as follows by using the relation \( r(T_c) = 1/t(T_c) \):

\[ \gamma_i T_c \lim_{T \to T_c} \frac{\partial r(T)}{\partial T} = \frac{1 - t^2(T_c)}{1 + t^4(T_c)} \]

\[ \gamma_i T_c \lim_{T \to T_c} \frac{\partial t(T)}{\partial T} = \frac{t^2(T_c)|t^2(T_c) - 1|}{1 + t^4(T_c)} \]

From this we immediately get

\[ \lim_{T \to T_c} \frac{\partial \Delta_1^2(T)}{\partial T} = -\frac{8}{7 \zeta(3)} \frac{(\pi k_B T_c)^2}{T_c} \left[ 1 - \frac{1 - t^2(T_c)}{1 + t^4(T_c)} \right] \]

\[ \lim_{T \to T_c} \frac{\partial \Delta_2^2(T)}{\partial T} = -\frac{8}{7 \zeta(3)} \frac{(\pi k_B T_c)^2}{T_c} \left[ 1 + \frac{t^2(T_c)(1 - t^2(T_c))}{1 + t^4(T_c)} \right] \]

This can finally be inserted into the expression for \( C_V(T_c^-) \) which leads to

\[ C_V(T_c^-) = \frac{N_F (\pi k_B T_c)^2}{3 T_c} - \frac{N_F}{4} \lim_{T \to T_c} \sum_{i=1}^{2} \frac{\partial \Delta_i(T)}{\partial T} \]

\[ = C_V(T_c^+) \left\{ 1 + \frac{12}{7 \zeta(3)} \left[ 1 - \frac{1}{2} \frac{(1 - t^2(T_c))^2}{1 + t^4(T_c)} \right] \right\} \]
The analytic result for the specific heat discontinuity of two–band superconductors can now be identified to read:

\[
\frac{\Delta C}{C_N} \equiv \frac{C_V(T_i^-) - C_V(T_i^+)}{C_V(T_i^+)} = \frac{12}{7\zeta(3)} \left[ 1 - \frac{1}{2} \left( 1 - \frac{1}{1 + t^4(T_c)} \right)^2 \right]
\]

\[
= \left( \frac{\Delta C}{C_N} \right)_{BCS}
\]

\[
= \left( \frac{\Delta C}{C_N} \right)_{BCS} \left[ 1 - \frac{1}{2} \left( 1 - \frac{1}{1 + r^4(T_c)} \right)^2 \right]
\]

Equation (35)

Accordingly we obtain a modification of the well–known BCS result by the term, which depends on the strength of the pairing interaction.

![Figure 4b](https://via.placeholder.com/150)

**Figure 4b:** Specific heat discontinuity $\Delta C/C_N$ as a function of inter–band coupling constant $\lambda_{12}$ for intra–band coupling constants $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$.

In order to study the expression (35), we have plotted the dependence of the specific heat discontinuity on the inter–band coupling constant $\lambda_{12}$ for a two–band superconductor in Fig. 4a. As one can see in the figure, the specific heat discontinuity comes close to the BCS–value in the case of large enough $\lambda_{12}$. However, in the opposite limit ($\lambda_{12} \to 0$) it is equal to half of the BCS–value, as can be seen from equation (35). This behavior can be explained by the fact, that in the latter case only excitations over the large gap give non–vanishing contributions to the specific heat capacity (compare Figs. 2a and 2b).

Now we would like to give the more general form of the result (35) for the specific heat discontinuity, which depends on two different densities of states $N_i(0)$ for each band $i = 1, 2$, respectively:

\[
\frac{\Delta C}{C_N} = \left( \frac{\Delta C}{C_N} \right)_{BCS} \left\{ 1 - \left[ 1 - \frac{t^2(T_c)}{1 + t^4(T_c)} \right] \left[ \frac{N_1(0)}{N_F} - \frac{N_2(0)}{N_F} t^2(T_c) \right] \right\}
\]

As a consequence, the specific heat discontinuity of two–band (two–gap) superconductors is seen to represent the second non–universal BCS–Mühlschlegel parameter.
### 3.3 Spin susceptibility

The spin susceptibility describes the response of the quasiparticle magnetization $M$ to an external magnetic field $H^\text{ext}$:

$$
M = \frac{\gamma \hbar}{2} \sum_{p_i} \text{tr} \{ \tau \delta \nu \} \\
= \frac{\gamma \hbar}{2} \sum_{p_i} \text{tr} \left\{ \tau \left[ \nu(E_{p_i}) - \frac{\partial \nu(E_{p_i})}{\partial E_{p_i}} \frac{\gamma \hbar}{2} \tau \cdot H^\text{ext} \right] \right\} \\
= \left( \frac{\gamma \hbar}{2} \right)^2 \sum_{p_i} y_{p_i} H^\text{ext} = \left( \frac{\gamma \hbar}{2} \right)^2 \sum_{i=1}^2 Y_i(T) H^\text{ext} = \chi_s(T) H^\text{ext}
$$

Therefore one may identify the quasiparticle spin susceptibility as

$$
\chi_s(T) = \sum_{i=1}^2 \chi_{si}(T) \\
\chi_{si}(T) \equiv \left( \frac{\gamma \hbar}{2} \right)^2 Y_i(T) \quad (36)
$$

In Fig. 5 we have plotted the dependence of the normalized spin susceptibility $\chi_s(T)/\chi_s(T_c)$ on reduced temperature $T/T_c$ for the set of intra–band pair coupling constants $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$ and two values for $\lambda_{12} = 0.005 \ (i)$ and $\lambda_{12} = 0.07 \ (ii)$. For the lower value of $\lambda_{12}$ the spin susceptibility shows non–monotonic behavior, which can be traced back again to thermal activation processes associated with the smaller gap $\Delta_1(T)$, as it has been obtained in the previous local response functions: specific heat capacity and magnetic penetration depth.

Note that in the case of singlet $s$–wave pairing, the temperature dependence of both the spin susceptibility from Eq.(36) and the normal fluid density from Eq.(29) is characterized by the same form of the band–selected Yosida functions $Y_i(T)$. 

Figure 5: Temperature dependence of the normalized spin susceptibility $\chi_s(T)/\chi_s(T_c)$ from equation (36) for intra–band coupling constants $\lambda_{11} = 0.12$ and $\lambda_{22} = 0.19$ and inter–band coupling constant (i) $\lambda_{12} = 0.005$ and (ii) $\lambda_{12} = 0.07$.

4 Summary and conclusions

This paper is devoted to a comprehensive study of two–band (two–gap) superconductors with spin singlet $s$–wave pairing. The treatment of these systems requires a generalization of the microscopic BCS description of global equilibrium and weak response to the case that two bands, on which the gaps $\Delta_i(T)$, $i = 1, 2$ reside, cross the Fermi surface. From this generalized BCS description there emerge a couple of remarkable aspects, which deserve being spotlighted in what follows:

- Two–band superconductors are seen to display a common transition temperature only in the presence of a finite inter–band pair–coupling constant $\lambda_{12}$.
- In one–band superconductors there are two so–called BCS–Mühlschlegel [6] parameters, namely the gap at zero temperature $\Delta(0)/\pi k_B T_c$ and the specific heat discontinuity $\Delta C/C_N$ at the transition

$\left[ \frac{\Delta(0)}{\pi k_B T_c} \right]_{\text{BCS}} = \frac{1}{e^\gamma} ; \left[ \frac{\Delta C}{C_N} \right]_{\text{BCS}} = \frac{12}{7\zeta(3)}$

which are universal in the sense that they may depend on the nodal structure implied by the possible unconventionality of the pairing [7], but they do not depend on the characteristic frequency $\epsilon_0/\hbar$ of the exchange boson and the strength $\lambda = \tilde{N}(0)\Gamma$ of the pairing interaction. In two–band superconductors with small but finite inter–band coupling $\lambda_{12}$ this universality gets lost and both $\Delta_i(0)/\pi k_B T_c$, $i = 1, 2$ and $\Delta C/C_N$, which can both be evaluated analytically, are seen to depend on $\lambda_{12}$:

$\Delta_i(0)/\pi k_B T_c = \left[ \frac{\Delta(0)}{\pi k_B T_c} \right]_{\text{BCS}} e^{-\gamma \ln[r(T_c)−r(0)]} ; \Delta_i(0)/\pi k_B T_c = \left[ \frac{\Delta(0)}{\pi k_B T_c} \right]_{\text{BCS}} e^{-\gamma \ln[t(T_c)−t(0)]}$

$\Delta C/C_N = \left[ \frac{\Delta C}{C_N} \right]_{\text{BCS}} \left\{ 1 - \frac{1}{2} \frac{[1−r^2(T_c)]^2}{1+r^4(T_c)} \right\} = \left[ \frac{\Delta C}{C_N} \right]_{\text{BCS}} \left\{ 1 - \frac{1}{2} \frac{[1−t^2(T_c)]^2}{1+t^4(T_c)} \right\}$
with \( r(T_c) = 1/t(T_c) \) from (18) and \( r(0) = 1/t(0) \) from (25).

- While the temperature dependence of the energy gap in one–band superconductors is known to be strictly monotonic, the smaller of the gaps in two–band superconductors displays non–monotonic behavior in the limit where \( \lambda_{12} \) gets sufficiently small.

- The knowledge of the full temperature dependence of the two gaps \( \Delta_i(T) \) is a necessary prerequisite for the calculation of the relevant local response functions of the superconductor under consideration. In one–band superconductors these local response functions, namely the normal fluid density, the magnetic penetration depth, the specific heat capacity and the spin susceptibility, are known to decrease monotonically with decreasing temperature and display thermally activated behavior in the low temperature limit. In contrast, the local response functions of two–band superconductors may show a non–monotonic decrease with decreasing temperature particular in the case of very small inter–band pair–coupling constants \( \lambda_{12} \). In such a case all these response functions display a hump at the low temperature end, which can be associated with the activated behavior connected with the smaller gap.

All the results obtained in this paper form the basis for a general discussion of the dynamic response of two–gap superconductors. The dynamics of the phase of the order parameter will be characterized, besides the Nambu–Goldstone boson [21] or gauge mode, by a new massive collective mode, which was first discussed in the literature by A. J. Leggett [22]. This so–called Leggett–mode owes its very existence to the finiteness of the Leggett–parameter \( \gamma_L = \lambda_{12}/\det \lambda \) (17) and is seen to scale with the product of the two gaps \( \Delta_1(T) \) and \( \Delta_2(T) \), which we have calculated rigorously in section 2 of this paper. The frequency of the Leggett–mode reads [22]

\[
h^2 \omega^2 = 4\gamma_L \Delta_1(T) \Delta_2(T) \langle \lambda \rangle_1 + \langle \lambda \rangle_2 \over \langle \lambda \rangle_1 \langle \lambda \rangle_2
\]

with \( \langle \lambda \rangle_i, i = 1,2 \) the dimensionless condensate densities on the two bands. The Leggett–mode turns out to be unaffected by the long–range Coulomb interaction and therefore by the Higgs mechanism. Its existence leads to numerous consequences for the dynamic response of two–band superconductors, which we plan to publish in two forthcoming papers, namely on the electromagnetic response [19] and on the electronic Raman response [20].

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