On the existence of short trajectories of quadratic differentials related to generalized Jacobi polynomials with non real varying parameters.

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Abstract

The study of the asymptotic distributions of zeros of generalized Jacobi polynomials with non real varying parameters, leads with quadratic differentials. In fact, the support of the limit measure of the root-counting measures sits on the finite critical trajectories of a related quadratic differential.

In this paper, we study the trajectories of this quadratic differential, more precisely, we give a necessary and sufficient condition on the complex numbers $a, b$, and $\lambda$ for the existence of at list one finite critical trajectory of the quadratic differential $\frac{\lambda^2(z-a)(z-b)}{(z^2-1)^2}dz^2$.

1 Introduction

This paper is a continuation of oldest works motivated by the large-degree analysis of the behavior of the Jacobi polynomials $P_n^{(\alpha,\beta)}$, when the parameters $\alpha, \beta \in \mathbb{C}$ depend on the degree $n$ linearly. Recall that these polynomials
can be given explicitly by (see [19])

\[ P_n^{(\alpha, \beta)}(z) = 2^{-n} \sum_{k=0}^{n} \binom{n + \alpha}{n - k} \binom{n + \beta}{k} (z - 1)^k (z + 1)^{n-k}, \quad (1) \]

where \( \binom{\gamma}{k} \) = \( \frac{\gamma(\gamma-1)\ldots(\gamma-k+1)}{k!} \) for \((\gamma, k) \in \mathbb{C} \times \mathbb{N}\). Equivalently, the Jacobi polynomials can be defined by the well-known Rodrigues formula

\[ P_n^{(\alpha, \beta)}(z) = \frac{1}{2^n n!} \left( z - 1 \right)^{-\alpha} \left( z + 1 \right)^{-\beta} \left( \frac{d}{dz} \right)^n \left[ (z - 1)^{n+\alpha} (z + 1)^{n+\beta} \right]. \]

Clearly, polynomials \( P_n^{(\alpha, \beta)} \) are entire functions of the complex parameters \( \alpha, \beta \).

The classical case is when \( \alpha, \beta > -1 \); for these values of the parameters, the Jacobi polynomials are orthogonal on \([-1, 1]\) with respect to the weight function \((1-x)\alpha(1+x)^{\beta}\).

We consider the sequence \( P_n^{nA,nB} \), as \( n \to \infty \). The case \( A, B \geq 0 \) can be tackled using the standard tools related to varying orthogonality and equilibrium measures in an external field on \( \mathbb{R} \), see e.g. [6], or [5]. The general situation \( A, B \in \mathbb{R} \) was analyzed in [9], [8], [10]. The situation \( A / B \in \mathbb{R} \), \( B > 0 \) was analyzed in [3]. In this paper, we are interested in the situation when

\[ A / B \in \mathbb{R}, A / B \in \mathbb{R}, A + B + 1 \neq 0, A + B + 2 \neq 0. \quad (2) \]

**Definition 1** For a compactly supported finite complex-valued Borel measure \( \mu \), we define its Cauchy transform \( C_{\mu} \) as

\[ C_{\mu}(z) = \int_{\mathbb{C}} \frac{d\mu(t)}{z - t}, \quad z \in \mathbb{C} \setminus \text{supp}(\mu). \]

For instance, if \( P \) is a polynomial of degree \( n \), then the Cauchy transform \( C_{P} \) of its normalized root-counting measure \( \frac{1}{n} \sum_{p(a)=0} \delta_a \) (where \( \delta_a \) is the Dirac measure supported at \( a \)) is given by

\[ C_{P}(z) = \frac{1}{n} \sum_{p(a)=0} \frac{1}{z - a} = \frac{P'(z)}{nP(z)}. \]

The Cauchy transform of a compactly supported finite complex-valued Borel measure \( \mu \) defines an analytic function in \( \mathbb{C} \setminus \text{supp}(\mu) \) satisfying the properties

\[ C_{\mu}(z) \sim \frac{\mu(\mathbb{C})}{z}, \quad z \to \infty; \quad \mu = \frac{1}{\pi} \frac{\partial C_{\mu}}{\partial z}. \]
The Cauchy transform of a non compactly supported finite complex-valued Borel measure $\mu$ can be defined in the distribution sense.

The following Theorem gives the connection between the limit behavior of the zeros of the sequence $1, \ldots, n$ and the structure of trajectories of a particular quadratic differential.

**Theorem 2** Suppose that a sequence of Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}$ satisfies conditions:

(i) $\lim_{n \to \infty} \frac{\alpha_n}{n} = A, \lim_{n \to \infty} \frac{\beta_n}{n} = B; \ A, B$ satisfy conditions 2.

(ii) the sequence $\{\mu_n\}$ of the corresponding root-counting measures converges weakly to a compactly supported probability measure $\mu$ in $\mathbb{C}$.

Then the Cauchy transform $C_\mu$ satisfies almost everywhere in $\mathbb{C}$ the following quadratic equation:

$$(1 - z^2)C_\mu^2 - ((A + B)z + A - B)C_\mu + A + B + 1 = 0.$$ 

Moreover, the support of $\mu$ consists of finitely many critical horizontal trajectories of the quadratic differential

$$\varpi_{A,B} = - \frac{R_{A,B}(z)}{(z^2 - 1)^2} \, dz^2,$$

where

$$R_{A,B}(z) = (A + B + 2)^2 z^2 + 2(A^2 - B^2) z + (A - B)^2 - 4(A + B + 1). \quad (3)$$

**Proof.** See e.g [4], [14]; or [15], and references therein. \hfill \blacksquare

The critical graph $\Gamma_{A,B}$ of the quadratic differential $\varpi_{A,B}$, with $A, B \in \mathbb{R}$ depends on the sign of

$$\Delta = (A + 1)(B + 1)(A + B + 1).$$

If $\Delta > 0$, then $\Gamma_{A,B}$ is formed by two loops around $-1$ and $1$, joined by a segment included in $(-1,1)$. If $\Delta < 0$, then $\Gamma_{A,B}$ is formed by two loops around $-1$ and $1$, with common edge crossing $(-1,1)$. See Figure 1.

For the case $A \notin \mathbb{R}, B > 0, \Gamma_{A,B}$ is formed by a short trajectory, a loop around $-1$, and two critical trajectories diverging to 1 and $\infty$. See Figure 2.

The main result of this paper is the following theorem
Theorem 3 Let $A, B$ satisfy assumptions (2). Then, the structure of the critical graph $\Gamma_{a,b}$ of $\varpi_{A,B}$ is as follows:

- If $A + B \notin \mathbb{R}$, then there exist:
  - one short trajectory $\gamma_{A,B}$ of $\varpi_{A,B}$, joining the zeros $\zeta_-$ and $\zeta_+$ of $R_{A,B}$,
  - two infinite critical trajectories emanating from each zero diverging differently to $+1$, $-1$, or $\infty$.

- If $A + B \in \mathbb{R}$, then there exist:
  - two short trajectories of $\varpi_{A,B}$, joining $\zeta_-$ and $\zeta_+$, and forming a Jordan curve encircling $\pm 1$,
  - a trajectory emanating from each zero diverging differently to $+1$ or $-1$.

It was shown in [3] that the existence of the short trajectory $\gamma$ joining the zeros $\zeta_-$ and $\zeta_+$, is the cornerstone in the study of the asymptotic zero distribution of the Jacobi polynomials; it satisfies

$$\int_{\gamma} \frac{\sqrt{R_{A,B}(t)}}{t^2 - 1} dt = \pm 2\pi i.$$

The short trajectory $\gamma$ is the support of the measure $\mu$ limit in the weak-* topology of the sequence $\{\mu_n\}$ in Theorem [2]. Following the pioneering
works of Gonchar-Rakhmanov [6], and Stahl [16], the weak asymptotic of the polynomials measure $\mu$ is absolutely continuous with respect to the linear Lebesgue measure on $\gamma$ and given by the formula

$$d\mu(z) = \frac{1}{2\pi} \frac{\left(\sqrt{R_{A,B}(z)}\right)_+}{z^2 - 1} dz.$$ 

The strong uniform asymptotic can be tackled via the Riemann-Hilbert steepest descent method of Deift-Zhou [13].

2 The quadratic differential $\frac{\lambda^2(z-a)(z-b)}{(z^2-1)^2} dz^2$

In this section we focus on the quadratic differential on the Riemann sphere $\hat{\mathbb{C}}$ defined by

$$\varpi_{a,b,\lambda} = \frac{\varphi_{a,b,\lambda}(z)}{(z^2 - 1)^2} dz^2 = \frac{\lambda^2 (z - a)}{(z^2 - 1)^2} (z - b) dz^2,$$

where $a, b,$ and $\lambda$ are three complex numbers such that

$$a \neq b, a, b \notin \{-1, 1\}, \lambda \neq 0.$$  

(4)
Definition 4 We say that Property $P_{a,b,\lambda}$ is satisfied, if the imaginary part of at least one of the following four numbers (see Proposition 6) vanishes

$$\pm \pi i \lambda \left( \sqrt{(a-1)(b-1)} \pm \sqrt{(a+1)(b+1) - 2} \right). \quad (5)$$

The main result of this section is the following

Proposition 5 Let $a$, $b$, and $\lambda$ satisfying (4). Then, the quadratic differential $\varpi_{a,b,\lambda}$ has a short trajectory, if and only if, Property $P_{a,b,\lambda}$ is satisfied.

The horizontal trajectories (or just trajectories) of $\varpi_{a,b,\lambda}$ are the loci of the equation

$$\Im \int_{\gamma} \sqrt{\varphi_{a,b,\lambda}(t)} \frac{dt}{t^2 - 1} \equiv \text{const}, \quad (\Re \int_{\gamma} \sqrt{\varphi_{a,b,\lambda}(t)} \frac{dt}{t^2 - 1} \text{ monotonic}) \quad (6)$$

while the vertical or orthogonal trajectories are obtained by replacing $\Im$ by $\Re$ in the equation above. It is easy to check that equation (6) is equivalent to

$$\frac{\varphi_{a,b,\lambda}(z)}{(z^2 - 1)^2} \frac{dz^2}{z^2} > 0.$$

The trajectories and the orthogonal trajectories of $\varpi_{a,b,\lambda}$ produce a transversal foliation of the Riemann sphere $\hat{\mathbb{C}}$. The only critical points of $\varpi_{a,b,\lambda}$ are its zeros (the roots $a$ and $b$ of $\varphi_{a,b,\lambda}$) and its poles, located at $\pm 1$ and at infinity; all others points of $\mathbb{C}$ are regular.

A trajectory $\gamma$ of $\varpi_{a,b,\lambda}$ starting and ending at zeros $a$ and $b$ (if exists) is called finite critical or short; if it starts at one of the zeros $a$ or $b$ but tends to either pole, we call it infinite critical trajectory of $\varpi_{a,b,\lambda}$. In particular, if $\gamma$ is a short trajectory joining $a$ and $b$, then, necessarily

$$\Im \int_{\gamma} \frac{\sqrt{\varphi_{a,b,\lambda}(t)}}{t^2 - 1} \frac{dt}{\pm} = 0. \quad (7)$$

Notice that any critical trajectory is either finite or infinite; any non critical trajectory is either a loop, or it must diverge to infinite critical points in its two directions.

The set of both finite and infinite critical trajectories of $\varpi_{a,b,\lambda}$ together with their limit points (critical points of $\varpi_{a,b,\lambda}$) is the critical graph $\Gamma_{a,b}$ of $\varpi_{a,b,\lambda}$. (See [7], or [17] for further details on quadratic differentials)

In order to prove Theorem 3 we start by analyzing the local structure of the trajectories of $\varpi_{a,b,\lambda}$ at its critical points (see e.g. [7],[17]). Recall
that at any regular point trajectories look locally as simple analytic arcs passing through this point, and through every regular point of \( \varpi_{a,b,\lambda} \) passes a uniquely determined horizontal and uniquely determined vertical trajectory of \( \varpi_{a,b,\lambda} \), that are locally orthogonal at this point. Furthermore, there are 3 trajectories emanating from \( a \) and from \( b \) under equal angles \( 2\pi/3 \). The local structure of the trajectories near a double pole depends on the vanishing of the real and imaginary parts of the residues of the quadratic differential near this point.

Since \( \varpi_{a,b,\lambda} = \left( \frac{\lambda^2 (1 - a)(1 - b)}{(z - 1)^2} + \mathcal{O}\left( \frac{1}{z - 1} \right) \right) dz^2, \quad z \to 1, \)
\( \varpi_{a,b,\lambda} = \left( \frac{\lambda^2 (1 + a)(1 + b)}{(z + 1)^2} + \mathcal{O}\left( \frac{1}{z + 1} \right) \right) dz^2, \quad z \to -1, \)
\( \varpi_{a,b,\lambda} = \left( \frac{\lambda^2}{u^2} + \mathcal{O}\left( \frac{1}{u^3} \right) \right) du^2, \quad u \to 0, \quad z = 1/u, \)
we conclude that the residues of \( \varpi_{a,b,\lambda} \) at \( -1, 1, \) and \( \infty \) are respectively \( \lambda^2 (1 - a)(1 - b) \), \( \lambda^2 (1 + a)(1 + b) \), and \( \lambda^2 \). Recall that the local behavior of the trajectories near a double pole has the circle, the radial, or the log-spiral forms respectively if the residue there is negative, positive, or non real. The existence of a short trajectory joining \( a \) or \( b \) to itself implies that at list, one the residues above is negative.

Since \( \varpi_{a,b,\lambda} \) has only three poles, Jenkins’ three pole Theorem asserts that it cannot have any recurrent trajectory.

We denote \( J_{a,b} \) the set of all Jordan arcs joining \( a \) and \( b \) in \( \mathbb{C} \setminus \{-1,1\} \). Two arcs \( \alpha, \beta : [0, 1] \to \mathbb{C} \setminus \{-1,1\} \) from \( J_{a,b} \) are homotopic if there exists a continuous function \( H : [0, 1] \times [0, 1] \to \mathbb{C} \setminus \{-1,1\} \) such that
\[
\begin{cases}
H(t, 0) = \alpha(t), \\
H(t, 1) = \beta(t),
\end{cases}
\quad t \in [0, 1].
\]
It is an equivalence relation on \( J_{a,b} \). It is well known that \( \mathbb{C} \setminus \{-1,1\} \) and the wedged two circles have the same type of homotopy; in particular, there are four classes of equivalence of the relation ”homotopic” on \( J_{a,b} \), see Figure 3.

For \( \gamma_0 \in J_{a,b} \) fixed, we consider the single-valued branch in \( \mathbb{C} \setminus \gamma_0 \) of \( \sqrt{\varphi_{a,b,\lambda}} \) fixed by the condition
\[
\sqrt{\varphi_{a,b,\lambda}}(z) \sim \lambda z, \quad z \to \infty.
\]
For $t \in \gamma_0 \setminus \{a, b\}$, we denote by $(\sqrt[\gamma_0]{\varphi_{a,b,\lambda}(t)})_+$ and $(\sqrt[\gamma_0]{\varphi_{a,b,\lambda}(t)})_-$ the limits from the $+$-side and $-$-side respectively. (As usual, the $+$-side of an oriented curve lies to the left, and the $-$-side lies to the right, if one traverses the curve according to its orientation). Observe that

$$(\sqrt[\gamma_0]{\varphi_{a,b,\lambda}(t)})_+ = - (\sqrt[\gamma_0]{\varphi_{a,b,\lambda}(t)})_-, t \in \gamma_0 \setminus \{a, b\}.$$ 

We have,

**Proposition 6** Let $a, b,$ and $\lambda$ satisfy assumptions (2), and let $\gamma_0$ be a Jordan arc in $\mathbb{C} \setminus \{-1, 1\}$ joining $a$ and $b$, and $\sqrt[\gamma_0]{\varphi_{a,b,\lambda}}$ is its single-valued branch in $\mathbb{C} \setminus \gamma_0$ fixed by the condition (3). Then,

$$\int_{\gamma_0} \frac{(\sqrt[\gamma_0]{\varphi_{a,b,\lambda}(t)})_+}{t^2 - 1} dt = \pm i \pi \frac{\lambda}{2} \left( \sqrt{(1 - a)(1 - b)} - \sqrt{(1 + a)(1 + b)} - 2 \right).$$
**Proof.** Let \( \gamma \) be a closed contour encircling the curve \( \gamma_0 \) and not encircling \( z = \pm 1 \). Thus,

\[
\int_{\gamma_0} \frac{\sqrt{\varphi_{a,b,\lambda}(t)}}{t^2 - 1} dt + \left( - \int_{\gamma_0} \frac{\sqrt{\varphi_{a,b,\lambda}(t)}}{t^2 - 1} dt \right) = \frac{1}{2} \int_{\gamma} \sqrt{\varphi_{a,b,\lambda}(t)} \frac{dt}{t^2 - 1} = \pm i\pi \left( \text{res}_{-1} + \text{res}_{1} + \text{res}_{\infty} \right) \left( \frac{\sqrt{\varphi_{a,b,\lambda}(t)}}{t^2 - 1} \right).
\]

The values of \( \int_{\gamma} \sqrt{\varphi_{a,b,\lambda}(t)} \frac{dt}{t^2 - 1} \) for \( \gamma \in \Gamma_{a,b} \) can be deduced by the knowledge of the homotopic class of \( \gamma \), and applying Cauchy residue Theorem; without consideration of the signs, these values are those numbers defined in (5).

As an immediate consequence is the following

**Lemma 7** If the quadratic differential \( \varpi_{a,b,\lambda} \) has a short trajectory joining \( a \) and \( b \), then, Property \( P_{a,b,\lambda} \) is satisfied.

The next tool we need to finish the proof of Theorem 3 is the so-called Teichmüller lemma (see [17, Theorem 14.1]), following the idea already used in [1,3]. Recall that a \( \varpi_{a,b,\lambda} \)-polygon is any domain bounded only by trajectories or orthogonal trajectories of \( \varpi_{a,b,\lambda} \). If \( z_j \) are its corners, \( n_j \) is the multiplicity of \( z_j \) as a singularity of \( \varpi_{a,b,\lambda} \) (taking \( n_j = 1 \) if \( z_j \in \{a,b\} \), \( n_j = 0 \) if it is a regular point, and \( n_j = -2 \) if it is a double pole), and \( \theta_j \in [0, 2\pi] \) is the corresponding inner angle at \( z_j \), then

\[
\sum_j \beta_j = 2 + \sum_i n_i, \quad \text{where } \beta_j = 1 - \theta_j \frac{n_j + 2}{2\pi}, \tag{9}
\]

and the summation in the right hand side goes along all zeroes of \( \varpi_{a,b,\lambda} \) inside the \( \varpi_{a,b,\lambda} \)-polygon. As an immediate consequence, we have

**Lemma 8** There cannot exist two short trajectories that are homotopic in the punctured plane \( \mathbb{C} \setminus \{-1, 1\} \).
Proof. If such two short trajectories exist, then they will form an \( \varpi_{a,b,\lambda} \)-polygon splitting \( \hat{C} \) into two connected domains; let \( D \) be the bounded one. Clearly, \( D \cap \{-1, 1\} = \emptyset \), and then, the interior angles of this \( \varpi_{a,b,\lambda} \)-polygon equal \( \frac{2\pi}{3} \), therefore, the left-hand side of (9) equals 0, whereas the right-hand is 2, a contradiction. \( \blacksquare \)

Proposition 9 Suppose that Property \( \mathcal{P}_{a,b,\lambda} \) is satisfied. Then, there cannot exist two infinite critical trajectories emanating from the same zero \( a \) or \( b \), and diverging to the same pole.

Proof. Assume that \( \gamma^- \) and \( \delta^- \) are two infinite critical trajectories emanating from the same zero (for example \( a \)), spacing with angle \( \theta_- \), diverging to the same pole, for example, \( z = -1 \). We treat the case when the residue \( \lambda^2 (1 + a) (1 + b) \) of the quadratic differential \( \varpi_{a,b,\lambda} \) at the pole \(-1\) is not real. Let \( \sigma \) be an orthogonal trajectory (not necessary critical) diverging to \( z = -1 \). Clearly, \( \sigma \) intersects \( \gamma^- \) and \( \delta^- \) alternatively infinitely many times; let \( A \) and \( B \) be two consecutive intersections. Let \( \Omega \) be an \( \varpi_{a,b,\lambda} \)-polygon \( D \), with vertices \( a, A, B, A \), and with edges, the arcs of \( \gamma^-, \delta^- \) and \( \sigma \) connecting respectively \( a \) and \( A, A \), and \( B \) and \( a \). The interior angles of \( \Omega \) at \( A \) and \( B \) are equal to \( \frac{\pi}{2} \). Direct calculation shows that for \( \Omega \), the right hand side of (9) equals 1 if \( \theta_- = \frac{2\pi}{3} \), or 0 if \( \theta_- = \frac{4\pi}{3} \); it follows that:

If \( \theta_- = \frac{2\pi}{3} \), then \( \Omega \) must contain \( b \) and another pole, necessarily, \( z = 1 \).

We conclude that the third trajectory emanating from \( a \) diverges to \( \infty \), and that all trajectories emanating from the other zero \( b \) stay inside \( D \); the same reasoning applied on trajectories emanating from \( b \) shows that two of them, say \( \gamma_1^+ \) and \( \gamma_2^+ \), diverge to \(-1 \) (with angle \( \frac{4\pi}{3} \) at \( b \) ), and the third one diverges alone to \( 1 \). We may assume without loss of generalities, that \( \sigma \) intersects \( \gamma^-, \gamma_1^+, \gamma_2^+, \delta^- \), and again \( \gamma^- \), respectively in \( A, B, C, D, \) and \( E \). We construct 4 paths connecting \( a \) and \( b \) as follows (see Figure 4):

- \( \gamma_0 \): formed by the part of \( \gamma_1^+ \) joining \( b \) to \( B \), the part of \( \sigma \) joining \( B \) to \( A \), and the part of \( \gamma_- \) joining \( A \) to \( a \).
- \( \gamma_{-1} \): formed by the part of \( \gamma_1^+ \) joining \( b \) to \( B \), the part of \( \sigma \) joining \( B \) to \( E \), and the part of \( \gamma_- \) joining \( E \) to \( a \).
- \( \gamma_1 \): formed by the part of \( \gamma_2^+ \) joining \( b \) to \( C \), the part of \( \sigma \) joining \( C \) to \( A \), and the part of \( \gamma_- \) joining \( A \) to \( a \).
- \( \gamma_{+1} \): formed by the part of \( \gamma_2^+ \) joining \( b \) to \( C \), the part of \( \sigma \) joining \( C \) to \( E \), and the part of \( \gamma_- \) joining \( A \) to \( a \).
Clearly, these paths are not homotopic in $\mathbb{C} \setminus \{−1, 1\}$, and we have, by definition of trajectories and orthogonal trajectories

$$\Im \int_{\gamma_0} (\frac{\sqrt{\varphi_{a,b,\lambda}}(t)}{t^2 - 1}) + dt = \Im \int_{B} (\frac{\sqrt{\varphi_{a,b,\lambda}}(t)}{t^2 - 1}) + dt$$

$$+ \Im \int_{A} (\frac{\sqrt{\varphi_{a,b,\lambda}}(t)}{t^2 - 1}) + dt$$

$$+ \Im \int_{a} (\frac{\sqrt{\varphi_{a,b,\lambda}}(t)}{t^2 - 1}) + dt$$

$$= \Im \int_{B} (\frac{\sqrt{\varphi_{a,b,\lambda}}(t)}{t^2 - 1}) + dt$$

$$\neq 0.$$  

With the same way, we prove that the imaginary part of the integral of $$\frac{(\sqrt{\varphi_{a,b,\lambda}})(z)}{z^2 - 1}$$ along each one of the paths $\gamma_1, \gamma_{−1}, \text{ and } \gamma_{±1}$ cannot vanish, which contradicts Proposition 6.

Figure 4: Construction of 4 paths not homotopic in $\mathbb{C} \setminus \{−1, 1\}$. Here, $a = 1 - i, b = 1 + 2i,$ and $\lambda = 1 + 0.5i$.

If $\theta_− = \frac{4\pi}{3}$, then the right hand side of 9 equals 0, and then, $\Omega$ must contain only the pole 1. We conclude that the third trajectory emanating from $a$ diverges to 1, and no trajectory emanating from $b$ diverges to 1. The same reasoning applied on trajectories emanating from $b$ shows that at list, one of them diverges to $−1$. By changing the roles of $b$ and $a$ in the previous case, we get the same contradiction.

The case $\lambda^2 (1 + a)(1 + b) > 0$ can be treated in the same vein.
Proposition 10 If Property $P_{a,b,\lambda}$ is satisfied, then it cannot happen that to each pole, $+1, -1,$ and $\infty$, there diverge two infinite critical trajectories.

Proof. If such a situation happens, then the critical graph $\Gamma_{a,b}$ splits $\mathbb{C}$ into three connected domains of the strip types $\Omega_0, \Omega_1$, and $\Omega_2$; see Figure 5. Let $\gamma_0, \gamma_1$, and $\gamma_{-1}$ be three Jordan arc joining $a$ and $b$ respectively in the domains $\Omega_0, \Omega_1$, and $\Omega_2$. We define the square root $\sqrt{\varphi_{a,b,\lambda}}$ in $\mathbb{C} \setminus \gamma_0$ with condition 8. Clearly, the paths $\gamma_0, \gamma_1$, and $\gamma_{-1}$ belong to three different classes of homotopy in $\mathbb{C} \setminus \{-1, 1\}$. Consider an orthogonal trajectory (not necessary critical) $\sigma_1$ diverging to $z = 1$. With the same way of the proof of Proposition 9 we construct two paths connecting $a$ and $b$, that are respectively homotopic in $\mathbb{C} \setminus \{-1, 1\}$ to $\gamma_0$ and $\gamma_1$, and we get

$$\Im \int_{\gamma_0} \frac{\sqrt{\varphi_{a,b,\lambda}(t)}}{t^2 - 1} dt \neq 0, \quad \Im \int_{\gamma_1} \frac{\sqrt{\varphi_{a,b,\lambda}(t)}}{t^2 - 1} dt \neq 0.$$  

With another orthogonal trajectory $\sigma_{-1}$ diverging to $z = -1$, we get

$$\Im \int_{\gamma_2} \frac{\sqrt{\varphi_{a,b,\lambda}(t)}}{t^2 - 1} dt \neq 0.$$  

Let us denote by $\gamma_a$ and $\gamma_b$ the critical trajectories that emanate respectively from $a$ and $b$, and diverge respectively to $-1$ and $\infty$. Let $\sigma$ be an orthogonal trajectory that diverges to $-1$ and $\infty$. $\sigma$ intersects $\gamma_a$ and $\gamma_b$ infinitely many times; let $A, B$, and $C$ be respectively its first and second intersection with $\gamma_a$, and its first intersection with $\gamma_b$. We consider the paths $\gamma$ and $\gamma'$

- $\gamma$: formed by the part of $\gamma_a$ joining $a$ to $B$, the part of $\sigma$ joining $B$ to $A$, and the part of $\gamma_b$ joining $A$ to $b$.

- $\gamma'$: formed by the part of $\gamma_a$ joining $a$ to $C$, the part of $\sigma$ joining $C$ to $A$, and the part of $\gamma_b$ joining $A$ to $b$.

Clearly, $\gamma$ and $\gamma'$ are not homotopic in $\mathbb{C} \setminus \{-1, 1\}$, moreover, one of them, we denote it by $\gamma_3$ is not homotopic in $\mathbb{C} \setminus \{-1, 1\}$ to $\gamma_0, \gamma_1$, and $\gamma_2$; besides, we have

$$\Im \int_{\gamma_3} \frac{\sqrt{\varphi_{a,b,\lambda}(t)}}{t^2 - 1} dt \neq 0.$$  

Finally, each one of the paths $\gamma_0, \gamma_1, \gamma_2$, and $\gamma_3$ belong to different homotopy class in $\mathbb{C} \setminus \{-1, 1\}$, and Property $P_{a,b,\lambda}$ is violated.

$\blacksquare$
Figure 5: The path $\gamma_0$ (red), and the domains $\Omega_0, \Omega_1,$ and $\Omega_2$; here, $a = 1 + i, b = 0.5 - 0.5i,$ and $\lambda = 0.1 + i$.

Proof of Proposition 5

The necessary condition is done in Lemma 7.

From each zero $a$ and $b$ there emanate three critical trajectories; if Property $P_{a,b,\lambda}$ is satisfied, then, by Propositions 9 and 10, these six critical trajectories cannot diverge all to the poles $-1, 1,$ and $\infty$. We conclude that necessarily we have at list one short trajectory. If Property $P_{a,b,\lambda}$ is satisfied for exactly one value from the four possible values that can take, then, Lemma 8 insure the uniqueness of the short trajectory. ■

3 Proof of Theorem 3

The zeros of $R_{A,B}$ defined in theorem 2 are

$$\zeta_{\pm} = \frac{-A^2 + B^2 \pm 4\sqrt{(A + 1)(B + 1)(A + B + 1)}}{(A + B + 2)^2},$$

(10)

respectively, in a way that

$$R_{A,B}(z) = (A + B + 2)^2 \left( z - \zeta_+ \right) \left( z - \zeta_- \right).$$

Since $R_{A,B}(-1) = 4B^2$ and $R_{A,B}(1) = 4A^2$, it is obvious that for $A$ and $B$ satisfying (2), $\zeta_-$ and $\zeta_+$ are simple and different from $\pm 1$. The quadratic differential $\varpi_{A,B}$ can be written $\varpi_{A,B} = \varpi_{\zeta_-, \zeta_+, \lambda}$ with $\lambda = i(A + B + 2)$. The residues of $\varpi_{\zeta_-, \zeta_+, \lambda}$ at the poles $1, -1,$ and $\infty$ are respectively $-A^2, -B^2,$ and $-(A + B + 2)^2$; we conclude that the trajectories
have the radial or the log-spiral form in a neighborhood of ±1, respectively if \( A, B \in i\mathbb{R} \) or \( A^2, B^2 \notin \mathbb{R} \); and the circular, the radial, or the log-spiral form at \( \infty \), respectively if \((A + B + 2) \in \mathbb{R}\), or \((A + B + 2) \notin i\mathbb{R}\), or \((A + B + 2) \notin \mathbb{R}\).

Straightforward calculations shows that

\[
(A + B + 2) \sqrt{(\zeta_+ - 1)(\zeta_- - 1)} = \pm 2A,
\]
\[
(A + B + 2) \sqrt{(\zeta_+ + 1)(\zeta_- + 1)} = \pm 2B,
\]

and we have then, from Proposition 6

**Proposition 11** Let \( A, B \) satisfy assumptions (2), and let \( \gamma \) be a Jordan arc in \( \mathbb{C} \setminus \{-1, 1\} \) joining \( \zeta_- \) and \( \zeta_+ \), and \( \sqrt{R_{A,B}} \) is its single-valued branch in \( \mathbb{C} \setminus \gamma \) fixed by the condition (8). Then

\[
\int_{\gamma} \frac{\sqrt{R_{A,B}(t)}}{t^2 - 1} \, dt \in \pm 2\pi i \{1, (A + 1), (B + 1), (A + B + 1)\}.
\]

Taking into account assumptions (2), it follows from Lemma (7) and Proposition 11, that, if \( A + B \notin \mathbb{R} \), then there is always exactly one short trajectory of the quadratic differential \( \wp_{A,B} \) connecting \( \zeta_- \) and \( \zeta_+ \).

If \( A + B \in \mathbb{R} \), then the critical graph is bounded; with the same idea of the proof of Proposition 9, one can show that there are at most two critical trajectories diverging to the poles ±1. We conclude the existence of two two short trajectories, the case of short trajectory connecting a zero to itself can be discarded easily by Lemma 9. See Figure 6.

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Figure 6: Critical graphs when, $A + B \notin \mathbb{R}$, here $A = 1 + 0.1i$ and $B = -1 + 0.1i$ (right); $A + B \in \mathbb{R}$, here $A = 1 + 0.1i$ and $B = -1 - 0.1i$ (left).

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