A Survey on Newmann’s proof of the prime number theorem

Zhentao Ma*
Chengdu Foreign Languages School, Chengdu, Sichuan, 610000, China
*Corresponding author’s e-mail: guanghua.ren@gecacademy.cn

Abstract. The prime number theorem is one of the most important theorems in number theory, which tells the asymptotic behavior of positive prime numbers. It was first proved independently by Jacques Hadamard and Charles-Jean de la Valée Poussin in 1896, by using the Riemann zeta function as a tool. We summarize Newmann’s proof in this survey.

1. Introduction
Denote \( \pi(x) \) as the function of counting the number of positive primes no bigger than \( x \), where \( x \) is any real number bigger than zero. For example, \( \pi(12) = 5 \) because only five numbers (2, 3, 5, 7, 11) are no bigger than 12. Taming infinity is an eternal topic for mathematicians. The prime number theorem is about asymptotic property of the number of primes. The statement of prime number theorem is
\[
\pi(x) \sim \frac{x}{\log x}.
\]

In 1797 or 1798, Anton Felkel and Jurij Vega, Adrien-Marie Legendre [16] made a bold guess: there exist two constants \( A \) and \( B \)
\[
\pi(x) \sim \frac{x}{A \log x + B}
\]
This conjecture marked the beginning of era for mathematicians to work on the asymptotic property of the number of primes.
In 1808, a more precise conjecture was made by Legendre[17]:
\[
\pi(x) \sim \frac{x}{\log x - 1.08366}
\]
In 1823, Abel gave great compliment to the prime number theorem in his letter to his friend Holmboe. Abel commented that no theorem is more remarkable than prime number theorem in all branches of mathematics.
In 1849, Gauss wrote a letter to his friends Encke. In the letter, Gauss expressed his fascination to the research on the prime number theorem. Through mathematical observation, Gauss made a conjecture that the number of primes has the same order with logarithmic integral function:
\[
\pi(x) \sim Li(x) = \int_2^x \frac{dt}{\log t}
\]
Inspired by approximation made by Legende that the number of prime numbers has same order as
following function,
\[ \pi(x) \sim \frac{x}{\log x - A(x)} \]
where \( A(x) \) has a limit close to 1.0836), Gauss had an intense interest in investigating whether \( A(x) \) has an asymptotic limit to 1 or not.

The first work on asymptotic property of the quantity of prime numbers was completed by Tchebychef.

Tchebychef first proved in his memoir [23] that if \( \pi(x) \) approaches to \( \frac{x}{\log(x)^N} \) where \( N \) is a large positive integer, then the number of primes shares the same magnitude with logarithmic integral function, i.e.

\[ \pi(x) \sim \text{Li}(x) = \int_2^x \frac{dt}{\log t}. \]

This work means that the conjecture made by Lengdre that \( A(x) \) has a limit close to 1.0836 was false. And if \( \lim_{n \to \infty} A(x) \) exists, the limit must be 1.

Tchebychef also was the first to prove the number of prime numbers has the asymptotic property that \( \pi(x) \sim \frac{x}{\log x} \). His proof is readable and the mathematical tool he used is the technique of factorials. The key point in his proof is to characterize the highest power that a prime \( p \) can have that could divide factorial of \( x \), where \( x \) is an integer.

Tchebychef [23] also gave an approximation that when \( x \) goes into infinity,
\[ B < \frac{\pi(x)}{x} < \frac{6B}{5}, \]
where
\[ B = \frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 5}{5} - \frac{\log 30}{30} \approx 0.92129 \]

In 1892, Sylvester showed [20], [21] that if \( x \) is large enough, then \( 0.956 < \frac{\pi(x)}{\log(x)} < 1.045 \). But, the method of Sylvester was really computationally complicated and it offered no hope of leading to a proof of the prime number theorem. In 1896, Jacques Hadamard [10], [11] and Charles-Jean de la Vallée Poussin [24] independently gave the first proof of prime number theorem. The proof was lengthy and not elementary. Their proof consists of two components. The first component is to apply Hadamard’s theory of integral functions to the Riemann zeta functions \( \zeta(s) \). The second part was to prove the zeta function has no zero on the line \( Re(s) = 1 \) through a doubling identity for the trigonometric function. Afterwards, Landau [14] and Wiener [24], [25] gave proofs of prime number theory without the Hadamard theory. In 1948, Paul Erdos and Atle Selberg [8] announced an elementary proof of the prime number theorem. In 1980, American mathematician Donald J. Newman gave the simplest known proof of the prime number theorem. In this article, we'll summarize the Newmann’s proof [18] of prime number theorem.

2. Theorem 1
The number of primes is countably infinite.
Proof: Suppose that
\[ p_1, p_2, \ldots, p_n \]
are the first $n$ prime numbers. i.e.

$$p_1 < p_2 < \cdots < p_n$$

Then a new number $P$ could be generated. $P$ is constructed in the following way.

$$P = p_1 p_2 \cdots p_n + 1$$

Notice that any integer is either a prime number nor a composite number. Thus, $P$ is either a prime number or a composite number. Next, we show that no matter $P$ is a composite number or a prime number, we can construct a new prime that is different from $p_1, p_2, \cdots, p_n$.

If $P$ is a prime number, then we find a new prime number.

If $P$ is a composite number, then we claim that there is a new prime number $p_{n+1} | P$.

It is easy to see that $P$ is unable to be divided by any prime number $p_1, p_2, \cdots, p_n$. If these prime number divides $P$, then these numbers must divide 1, which is a contradiction.

Since $P$ is a composite number, there must exist a new prime number $p_{n+1}$ satisfying $p_{n+1} | P$.

It follows that no matter whether $P$ is a prime number or composite number, we can find a new prime number. Inductively, we know there exist countably infinite prime number.

3. Definition 2
The Riemann zeta function $\zeta(x)$ is defined as:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

4. Theorem 3
$\zeta(s)$ converges for $\text{Re}(s) > 1$ and is holomorphic function on the half-plane $\text{Re}(s) > 1$

Proof:
Suppose $s = iy$ where $x, y$ are two real numbers. By the usual computation, it is not difficulty to get:

$$\frac{1}{n^s} = e^{-x \log n} = n^{-x}$$

Thus, we can see that the module of the Riemann zeta function, $n^{-\text{Re}(s)} \leq \int_{m}^{\infty} x^{-1-\delta} \, dx \leq \frac{1}{\delta} m^{-\delta}$, is bounded by a sum of real numbers, $\sum_{n=1}^{\infty} \frac{1}{n^x}$.

Notice that $x$ is a real number strictly bigger than one. Then by knowledge of Calculus, it is clear that the sum converge.

As a sequence of weierstrass M-test theorem, the Riemann zeta function, $\sum_{n=1}^{\infty} \frac{1}{n^x}$ uniformly converges on the half plane $\text{Re}(s) > 1$ and $\zeta(s)$ is a holomorphic function in the half-plane $\text{Re}(s) > 1$.

5. Euler product theorem
$\zeta(s) = \prod_{p} \frac{1}{1-p^{-s}}$

The symbol of product on the right-side means that the product runs over all of positive primes.

Proof:
Denote $S_m$ as the collection of all positive integers with prime divisors $p$ no bigger than $m$. 

\begin{align*}
\text{Re} \{ 1 - p^{-s} \} & = \text{Re} \{ 1 + \frac{x}{p^{x-s}} \} \\
& = 1 + \frac{\text{Re} \{ x \}}{p^{x-s}} > 1
\end{align*}

\begin{align*}
\frac{1}{1-p^{-s}} & = \frac{1}{1-x} \\
& = 1 + \frac{x}{1-x} + \frac{x^2}{(1-x)^2} + \frac{x^3}{(1-x)^3} + \cdots
\end{align*}
Define $\zeta_m(s)$ as: $\zeta_m(s) = \sum_{n \leq m} n^{-s}$

Since $S_m \subset N$ and $\zeta(s)$ converges absolutely on the half-plane $\text{Re}(s) > 1$, then $S_m$ also converges. We rewrite $\zeta_m(s)$ as:

$$\zeta_m(s) = \sum_{e_1, e_2, \ldots, e_k \geq 0} P_1^{e_1} P_2^{e_2} \cdots P_k^{e_k} s^{-s} = \sum_{e_1 \geq 0} p_1^{-e_1 s} \sum_{e_2 \geq 0} p_2^{-e_2 s} \cdots \sum_{e_k \geq 0} p_k^{-e_k s}$$

where $p_1, \ldots, p_k$ are the primes no bigger than $m$.

Notice that it holds for any prime that: $\sum p^{-k s} = 1 + p^{-s} + \cdots = (1 - p^{-1})^{-1}$.

Hence, it follows that $\zeta_m(s) = \prod_{p \leq m} (1 - p^{-s})^{-1}$.

Next, we show $\zeta_m(s)$ converges to $\zeta(s)$.

$$|\zeta_m(s) - \zeta(s)| \leq \sum_{n \leq m} |n^{-s}| \leq \sum_{n \leq m} |n^{-s}| = \sum_{n \leq m} n^{-\text{Re}(s)}.$$

Suppose $\text{Re}(s) \geq 1 + \delta$, where $\delta > 0$.

Then, $n^{-\text{Re}(s)} \leq \int_0^m x^{-1-\delta} \, \text{d}x \leq \frac{1}{\delta} m^{-\delta} n^{-\text{Re}(s)} \leq \int_0^m x^{-1-\delta} \, \text{d}x \leq \frac{1}{\delta} m^{-\delta}$

It follows that $\lim_{m \to \infty} \zeta_m(s) = \zeta(s)$ for the half-plane $\text{Re}(s) \geq 1 + \delta$. Furthermore, $p_m(s) = \prod_{p \leq m} (1 - p^{-s})^{-1}$ locally converges to the function $\log(\prod_p (1 - p^{-s})^{-1})$.

Notice that:

$$\sum_{p} \log(1 - p^{-s})^{-1} = \sum_{p} \sum_{e \geq 1} \frac{1}{e} p^{-e s} \leq \sum_{p} (|p|^s - 1)^{-1} \leq \infty.$$ 

It follows that $\prod_{p}(1 - p^{-s})^{-1}$ converges.

Finally, we have $\sum_{n \geq 1} \frac{1}{n^s} = \sum_{k_1, k_2, \ldots, k_n \geq 0} z^{k_1} z^{k_2} \cdots z^{-s} = \prod_{k \geq 0} p^{k s} = \prod_p (1 - p^{-s})^{-1}$

### 6. Theorem 5

$\zeta(s) = \frac{1}{s-1} + \phi(s)$ for $\text{Re}(s) > 1$, where $\phi(s)$ is holomorphic on $\text{Re}(s) = 1$.

Notice that $\frac{1}{s-1} = \int_1^s x^{-s} \, \text{d}x = \sum_{n \geq 1} \int_n^{n+1} x^{-s} \, \text{d}x$.

Thus, $\zeta(s) - \frac{1}{s-1} = \sum_{n \geq 1} (n^{-s} - \int_n^{n+1} x^{-s} \, \text{d}x) = \sum_{n \geq 1} \int_n^{n+1} (n^{-s} - x^{-s}) \, \text{d}x$.

Define $\phi_n(s)$ as $\int_n^{n+1} (n^{-s} - x^{-s}) \, \text{d}x$. It is clear that on the half-plane $\text{Re}(s) > 0$,

$\phi_n(s)$ is holomorphic function for every positive integer $n$.

Notice that if $\text{Re}(s) > 0$ and $x \in [n, n+1]$, then through the following complicated computations we have

$$|n^{-s} - x^{-s}| \leq \int_{n}^{n+1} |s|^{-s} \, \text{d}t \leq \int_{n}^{n+1} \frac{|s|}{|t|^{s+1}} \, \text{d}t = \int_{n}^{n+1} \frac{|s|}{|t|^{\text{Re}(s)+1}} \, \text{d}t \leq \frac{|s|}{n^{\text{Re}(s)}}.$$ 

Thus $|\phi_n(s)| \leq \frac{|s|}{n^{\text{Re}(s)}}$.

Denote $r = \frac{\text{Re}(s)}{2}$.

$|\phi_n(s)| \leq \frac{|s|}{n^{\text{Re}(s)}}$. Denote $r = \frac{\text{Re}(s)}{2}$.

$\forall s_0$ satisfying $\text{Re}(s) > 0$, we have $\sup_{s \in B(s_0)} |\phi_n(s)| \leq \frac{|s_0| + r}{n^{\text{Re}(s)}}$.

By the Weierstrass M-test, we know $\sum_{n \geq 1} \phi_n$ converges to $\zeta(s) - \frac{1}{s-1}$ on the half-plane $\text{Re}(s) > 0$.

From the above, it is clear that $\zeta(s)$ has a meromorphic extension on the half-plane $\text{Re}(s) > 0$. 
7. Theorem 6 (Merten’s theorem)
\[
\forall x, y \in \mathbb{R} \text{ and } x > 1, \text{then } \left| \zeta(x)^3 \zeta(x + iy)^4 \zeta(x + 2iy) \right| \geq 1
\]

By using Euler product formula and Taylor expansion of \(\log(1 - z) = -\sum \frac{z^n}{n}\), we have
\[
\log \left| \zeta(s) \right| = -\sum \log \left| 1 - p^{-s} \right| = -\sum \Re(\log(1 - p^{-s})) = \sum \frac{\Re(p^{-ns})}{n}.
\]

Let \( s = x + iy \), then
\[
\log \left| \zeta(x + iy) \right| = \sum \sum \frac{\cos(ny \cdot \log(p))}{np^{nx}}.
\]

Thus, by computation, we further have
\[
\log(\left| \zeta(x) \zeta(x + iy)^4 \zeta(x + 2iy) \right|) = \sum \frac{s + 4 \cos(ny \cdot \log(p)) + \cos(2ny \cdot \log(p))}{np^{nx}}.
\]

8. Theorem 7
The Riemann zeta function has no zeros on the half plane \( \Re(s) > 1 \)

Proof: By Euler production formula, it is easy to see that the Riemann Zeta function \( \zeta(s) \) could not vanish on the half-plane \( \Re(s) > 1 \). Suppose there exist a real number \( y \) such that \( 1 + iy \) is a zero of the Riemann zeta function. \( \zeta(s) \). Since \( \zeta(s) - \frac{1}{s-1} \) is holomorphic function on the half plane \( \Re(s) > 1 \), and \( 1 + 2iy \) is not a pole of \( \zeta(s) \). Notice that \( 1 \) is a simple pole of \( \zeta(s) \), it is easy to get that \( \lim_{x \to 1} \zeta(x)^3 \zeta(x + iy)^4 \zeta(x + 2iy) = 0 \)
which is contradiction with Mertens' theorem.

9. Definition 8
Define Chebyshev function as:
\( \Theta(x) := \sum_{p \leq x} \log p \)

10. Theorem 9
\( \pi(x) \sim \frac{x}{\log x} \) if and only if \( \Theta(x) \sim x \)

By observation, it is clear that \( 0 \leq \Theta(x) \leq \pi(x) \log x \).

Therefore, we have
\[
\frac{\pi(x) \log x}{x} \leq \frac{\pi(x) \log x}{x} \leq \frac{\pi(x) \log x}{x} \Rightarrow \Theta(x) \geq \sum_{x^{1/2} < p \leq x} \log p \geq (1 - \epsilon)(\log x)(\pi(x) - \pi(x^{1/2})) \geq (1 - \epsilon)(\log x)(\pi(x) - x^{1/2})
\]

Thus, we have
\[
\pi(x) \leq \frac{1}{1 - \epsilon} \frac{\Theta(x)}{\log x} + x^{1/2}.
\]
It follows that
Notice that \( \log x \sim x \). Let \( \epsilon \to 0^+ \), we can have \( \frac{\vartheta(x) \log x}{x} \sim \frac{\varpi(x) \log x}{x} \).

11. Theorem 10

\( \vartheta(x) \leq 4 \log(2) \cdot x \) if \( x \geq 1 \)

Proof:
Notice that
\[
2^{2n} \geq \sum_{m=0}^{2n} \binom{2n}{m} \geq \binom{2n}{n} = \prod_{n < p \leq 2n} p = e^{\vartheta(2n) - \vartheta(n)}.
\]

It follows that
\[
\vartheta(2n) - \vartheta(n) \leq 2 n \log 2.
\]
Also, if a prime \( p \in (n, 2n] \), \( p \) is a factor of \( (2n)! \) but not a factor of \( n! \).
Therefore, \( \forall \text{ integer } N \),
\[
\vartheta(2^N) = \sum_{n=1}^{N} (\vartheta(2^n) - \vartheta(2^{n-1})) \leq \sum_{n=1}^{N} (\log 2) 2^n \leq 2^{N+1} \log 2.
\]
\( \forall x \in \mathbb{R}^+ \), there must exist an integer \( t \) such that \( 2^{t-1} \leq x < 2^t \). Hence, \( \vartheta(x) \leq 2^{t+1} \log 2 \leq (4 \log 2) x \)

12. Theorem 11

Suppose \( f \) is nondecreasing on the interval \( (1, +\infty) \). Then \( \int_{1}^{\infty} \frac{f(t) - t}{t^2} \) implies \( f(x) \sim x \)

Proof:
Denote \( I(x) = \int_{1}^{x} \frac{f(t) - t}{t^2} \). Notice that \( \lim_{x \to \infty} I(x) < \infty \).

It is clear that for any \( \lambda > 1 \)
\[
\lim_{x \to \infty} I(\lambda x) - I(x) = 0.
\]
The strategy used to finish this proof is by contradiction. Suppose there exist a sequence \( (x_n) \) satisfying \( \lim_{n \to +\infty} x_n = +\infty \) and \( f(x_n) \geq \lambda x_n \), where \( \lambda > 1 \). Then, we have:
\[
I(\lambda x_n) = \int_{x_n}^{\lambda x_n} \frac{f(t) - t}{t^2} dt \geq \int_{x_n}^{\lambda x_n} \frac{\lambda x_n - t}{t^2} dt = \int_{1}^{\lambda x_n} \frac{\lambda - t}{t^2} dt \neq 0.
\]
The equations above show the contradiction with the fact that limit of difference of function \( I \) is zero, i.e., \( \lim_{x \to \infty} I(\lambda x) - I(x) = 0 \).

Notice that \( \vartheta(x) \) is non-decreasing function. Thus, if the integration \( \int_{1}^{\infty} \frac{\vartheta(t) - t}{t^2} dt \) converges, then \( \vartheta(x) \sim x \).

In order to achieve such a goal, by the technique of change of variables \( t = e^s \), it is easy to see that
the integration \( \int_1^\infty \frac{\mathcal{G}(t) - t}{t^2} \, dt \) converges if and only if \( \int_1^\infty (\mathcal{G}(e^t)e^{-s} - 1) \, ds \) converges.

13. Definition 12.

\[ H(t) = \mathcal{G}(e^t)e^{-t} - 1 \]

14. Definition 13 [Laplace transform]

Suppose \( f \) is a continuous function. The transform of \( f, \mathcal{L}(f) \) is defined as:

\[ \mathcal{L}f(s) := \int_0^\infty e^{-st} f(t) \, dt \]

15. Definition 14

\[ \Phi(x) = \sum_p p^{-s}\log(p) \]

16. Theorem 15

\[ (\mathcal{L}(e^t))(s) = \frac{\Phi(s)}{s} \]

Since \( \mathcal{G} \sim x \), we have \( \mathcal{G}(e^t) \sim e^t \). Suppose that \( p_n \) is the \( n \)-th prime, and \( p_0 = 0 \). Notice that \( \mathcal{G}(e^t) \) does not change value on \( t \in (\log p_n, \log p_{n+1}) \).

It is clear that

Then, we have

\[ \mathcal{L}\mathcal{G}(e^t)(s) = \frac{1}{s} \sum_{n=1}^{+\infty} \mathcal{G}(p_n)(p_n^{-s} - p_{n+1}^{-s}) = \]

\[ = \frac{1}{s} \sum_{n=1}^{+\infty} \mathcal{G}(p_n)p_n^{-s} - \frac{1}{s} \sum_{n=1}^{+\infty} \mathcal{G}(p_{n-1})p_n^{-s} = \]

\[ = \frac{1}{s} \sum_{n=1}^{+\infty} (\mathcal{G}(p_n) - \mathcal{G}(p_{n-1}))p_n^{-s} = \frac{1}{s} \sum_{n=1}^{+\infty} p_n^{-s}\log(p_n) = \frac{\Phi(s)}{s} \]

17. Theorem 16

\[ \mathcal{L}H(s) = \frac{\Phi(s+1)}{s+1} - \frac{1}{s} \]

Proof:

It is easy to check the result by using linearity property of Laplace transform and its definition.

18. Theorem 17

\[ \Phi(s) - \frac{1}{s-1} \] has a meromorphic extension to the half-plane \( \text{Re}(s) > \frac{1}{2} \)

Since \( \zeta(s) \) has a meromorphic extension on the half-plane \( \text{Re}(s) > 0 \) and \( \zeta(s) \) has no zeros on the half-plane \( \text{Re}(s) \geq 1 \), it is clear that \( \frac{\zeta'(s)}{\zeta(s)} \) is meromorphic on the half-plane \( \text{Re}(s) > 0 \).

By Euler product formula, we have
\begin{align*}
-\frac{\zeta'(s)}{\zeta(s)} &= (-\log(\zeta(s)))' = (-\log(\prod_p (1 - p^{-s}))') \\
= \left(\sum_p \log(1 - p^{-s})\right)' &= \sum_p \frac{p^{-s}\log p}{1 - p^{-s}} \\
= \sum_p \left(\frac{1}{p^s} + \frac{1}{p^s(p^s - 1)}\right)\log(p) &= \Phi(s) + \sum_p \frac{\log(p)}{p^s(p^s - 1)}.
\end{align*}

Notice that the left-hand side is meromorphic on $\text{Re}(s) > 0$. Thus $\Phi(s) - \frac{1}{s - 1}$ has a meromorphic extension.

19. Theorem 18

Let $f$ be a bounded continuous function, and suppose the holomorphic extension of Laplacian transform $\mathcal{L}f$ is $g(s)$ on $\text{Re}(s) > 0$. Then $\int_0^\infty f(t)dt = g(0)$

Denote $g_\tau(s) = \int f(t)e^{-st}dt$. Thus, we have $\int_0^\infty f(t)dt = \lim_{\tau \to \infty} g_\tau(0)$.

$\forall r > 0$, due to holomorphic property of $g$ on the half-plane $\text{Re}(s) \geq 0$, there must exist an open ball $B_{\delta(y)}(y)$ for $\forall y \in [-r, r]$ on which $g$ is holomorphic. And the compactness of the closed interval $[-r, r]$ leads to that $\delta_r := \inf \{\delta(y) : y \in [-r, r]\}$ must be bigger than zero. Let $\gamma_r$ be the $\partial \{z : |z| \leq r, \text{Re}(s) \leq \delta_r\}$.

Define $h_\tau(s) = (g(s) - g_\tau(s))e^{-st}(\frac{r^2 + s^2}{r^2})$. It is clear that $h_\tau(s)$ is holomorphic on an open region which contains the closed simple curve $\gamma_r$. Thus, by Cauchy theorem from complex analysis, we have

\[ g(0) - g_\tau(h(0)) = \frac{1}{2\pi i} \int_{\gamma_r} (g(z) - g_\tau(z))e^{zt}\left(\frac{r^2 + z^2}{sr^2}\right)dz. \]

Notice that for $s \in \gamma_r \cap \{s : \text{Re}(s) > 0\}$, we have

\[ |g(z) - g_\tau(z)| \cdot |e^{zt}\left(\frac{r^2 + z^2}{sr^2}\right)| = \left| \int_0^{+\infty} f(t)e^{-zt}dt\right| \cdot \left| \frac{e^{Re(z)t}}{r} \cdot \frac{r}{z + r} \right| \leq \int_0^{+\infty} e^{-Re(z)t}dt \cdot \frac{2Re(z)}{r} \cdot \frac{2}{r^2}. \]

Thus, $\left| \frac{1}{2\pi i} \int_{\gamma_r \cap \{\text{Re}(s) > 0\}} (g(z) - g_\tau(z))e^{zt}\left(\frac{r^2 + z^2}{sr^2}\right)dz \right| \leq \frac{1}{2\pi} \cdot \pi r \cdot \frac{2}{r^2} = \frac{1}{r}$

Similar computation for the part of curve $\gamma_r$ that lies in $\{z : \text{Re}(z) < 0\}$

For $|z| = r$ but $\text{Re}(z) < 0$, by computation, it follows that

\[ |g_\tau(e^{zt}\left(\frac{r^2 + z^2}{sr^2}\right)| = \left| \int_0^t f(t)e^{-zt}dt\right| \cdot \left| \frac{e^{Re(z)t}}{r} \cdot \frac{r^2 + z^2}{sr^2} \right| \leq \int_0^t e^{-Re(z)t}dt \cdot \frac{2Re(z)e^{Re(z)t}}{r^2} = (1 - \text{Re}(z)e^{Re(z)t}). \]
Notice that \( Re(z) < 0 \). Thus \( \lim_{z \to +\infty} Re(z) \tau = 0 \)

Hence let \( r \) and \( \tau \) approaches infinity. We have \( |g(0) - g_r(0)| \to 0 \)

(Prime number theorem)
\[
\pi(x) \sim \frac{x}{\log x}
\]

Proof:

Notice that \( H(z) = \Theta(e^z)e^{-z} \) is both bounded and continuous function and its Laplace transform has a holomorphic extension on \( \text{Re}(s) \geq 0 \). By theorem 18, it is clear that
\[
\int_0^\infty H(z)dz = \int_0^\infty (\Theta(e^z)e^{-z} - 1)dz \leq +\infty
\]

Taking \( z = \log x \), it follows that
\[
\int_1^\infty (\Theta(x) - 1) \frac{dx}{x} = \int_1^\infty \frac{\Theta(x)-x}{x^2}dx \leq +\infty.
\]
It implies that \( \Theta(x) \) has the same order of \( x \). Thus, by theorem 9, it follows that
\[
\pi(x) \sim \frac{x}{\log x}.
\]

20. Conclusion

We summarize their proofs based on the proof of the prime number theorem by Jacques Hadamard and Charles-Jean de la Vallee Poussin, Newman. After that, we carried out a series of computational derivations through the Riemann equation and finally succeeded in proving the prime number theorem.

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