ON WEIGHTED NORM INEQUALITIES FOR THE CARLESON OPERATOR

ANDREI K. LERNER

Abstract. We obtain $L^p(w)$ bounds for the Carleson operator $\mathcal{C}$ in terms of the $A_q$ constants $[w]_{A_q}$ for $1 \leq q \leq p$. In particular, we show that, exactly as for the Hilbert transform, $\|\mathcal{C}\|_{L^p(w)}$ is bounded linearly by $[w]_{A_q}$ for $1 \leq q < p$. Our approach works in the general context of maximally modulated Calderón-Zygmund operators.

1. Introduction

For $f \in L^p(\mathbb{R})$, $1 < p < \infty$, define the Carleson operator $\mathcal{C}$ by
\[
\mathcal{C}(f)(x) = \sup_{\xi \in \mathbb{R}} |H(M^{\xi}f)(x)|,
\]
where $H$ is the Hilbert transform, and $M^{\xi}f(x) = e^{2\pi i \xi x} f(x)$.

A famous Carleson-Hunt theorem on a.e. convergence of Fourier series in one of its equivalent statements says that $\mathcal{C}$ is bounded on $L^p$ for any $1 < p < \infty$. The crucial step was done by Carleson [5] who established that $\mathcal{C}$ maps $L^2$ into weak-$L^2$. After that Hunt [15] extended this result to any $1 < p < \infty$. Alternative proofs of this theorem were obtained by Fefferman [10] and by Lacey-Thiele [22]. We refer also to [2], [13, Ch. 11] and [30, Ch. 7].

By a weight we mean a non-negative locally integrable function. The weighted boundedness of $\mathcal{C}$ also is well known. Hunt-Young [15] showed that $\mathcal{C}$ is bounded on $L^p(w)$, $1 < p < \infty$, if $w$ satisfies the $A_p$ condition (see also [13, p. 475]). In [14], Grafakos-Martell-Soria extended this result to a more general class of maximally modulated singular integrals. A different approach (as well as a kind of strengthening) to the Hunt-Young result was recently obtained by Do-Lacey [8].

In the past decade a lot of attention was devoted to sharp $L^p(w)$ estimates in terms of the $A_p$ constants $[w]_{A_p}$. Recall that these constants

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are defined as follows:

\[ [w]_{A_p} = \sup_{Q} \left( \frac{1}{|Q|} \int_{Q} w \, dx \right) \left( \frac{1}{|Q|} \int_{Q} w^{-\frac{1}{p-1}} \, dx \right)^{p-1}, \quad 1 < p < \infty, \]

and

\[ [w]_{A_1} = \sup_{Q} \left( \frac{1}{|Q|} \int_{Q} w \, dx \right) (\inf_{Q} w)^{-1}, \]

where the supremum is taken over all cubes \( Q \subset \mathbb{R}^n \). Sharp bounds for \( L^p(w) \) operator norms in terms of \([w]_{A_p}\) have been recently found for many central operators in Harmonic Analysis (see, e.g., [4, 7, 17, 24, 25, 32]). A relatively simple approach to such bounds based on local mean oscillation estimates was developed in [7, 18, 23, 24, 25].

In this paper we apply the “local mean oscillation estimate” approach to the Carleson operator \( C \). In particular, we obtain sharp linear bounds for \( \|C\|_{L^p(w)} \) in terms of \([w]_{A_q}\) for any \( 1 \leq q < p < \infty \).

Our main results can be described in the framework of maximally modulated singular integrals studied by Grafakos-Martell-Soria [14].

We give several main definitions. A Calderón-Zygmund operator on \( \mathbb{R}^n \) is an \( L^2 \) bounded integral operator represented as

\[ Tf(x) = \int_{\mathbb{R}^n} K(x, y) f(y) \, dy, \quad x \notin \text{supp} \, f, \]

with kernel \( K \) satisfying the following growth and smoothness conditions:

(i) \(|K(x, y)| \leq \frac{c}{|x-y|^n} \) for all \( x \neq y \);

(ii) \(|K(x, y) - K(x', y)| + |K(y, x) - K(y, x')| \leq c \frac{|x-x'|^\delta}{|x-y|^{n+\delta}} \) for some \( 0 < \delta \leq 1 \) when \( |x-x'| < |x-y|/2 \).

Let \( F = \{ \phi_\alpha \}_{\alpha \in A} \) be a family of real-valued measurable functions indexed by some set \( A \), and let \( T \) be a Calderón-Zygmund operator. Then the maximally modulated Calderón-Zygmund operator \( T^F \) is defined by

\[ T^F f(x) = \sup_{\alpha \in A} |T(M^{\phi_\alpha} f)(x)|, \]

where \( M^{\phi_\alpha} f(x) = e^{2\pi i \phi_\alpha(x)} f(x) \).

As it was shown in [14], the weighted theory of such operators can be developed under a single \textit{a priori} assumption on \( T^F \). We state this assumption as follows. Let \( \Phi \) be a Young function, that is, \( \Phi : [0, \infty) \to [0, \infty), \Phi \) is continuous, convex, increasing, \( \Phi(0) = 0 \) and \( \Phi(t) \to \infty \) as \( t \to \infty \). Define the mean Luxemburg norm of \( f \) on a cube \( Q \subset \mathbb{R}^n \) by

\[ \|f\|_{\Phi, Q} = \inf \left\{ \lambda > 0 : \frac{1}{|Q|} \int_{Q} \Phi \left( \frac{|f(x)|}{\lambda} \right) \, dx \leq 1 \right\}. \]
Our basic assumption on $T^F$ is the following: for any cube $Q \subset \mathbb{R}^n$,

$$\|T^F(f\chi_Q)\|_{L^{1,\infty}(Q)} \lesssim |Q|\|f\|_{\Phi,Q}. \quad (1.1)$$

If $\phi_\alpha(x) = 0$, then $T^F = T$ is the usual Calderón-Zygmund operator, and in this case (1.1) holds with $\Phi(t) = t$, which corresponds to the weak type $(1, 1)$ of $T$. Suppose that $n = 1$, $\phi_\alpha(x) = \alpha x$ and $A = \mathbb{R}$. Then $T^F = C$ is the Carleson operator, and the currently best known result is that (1.1) holds with $\Phi(t) = t \log(e + t) \log \log(e^{e^t} + t)$, see [14, Th. 5.1]. This represents an elaborated version of Antonov’s theorem [1] on a.e. convergence of Fourier series for $f \in L \log L \log \log \log L$ (see also [33]). For other examples concerning (1.1) we refer to [14].

Assuming (1.1), it is easy to show that $T^F$ is controlled (either via a good-$\lambda$ inequality or by a sharp function estimate) by the Orlicz maximal function $M_\Phi$ defined by

$$M_\Phi f(x) = \sup_{Q \ni x} \|f\|_{\Phi,Q}.$$  

Since we are interested in $L^p(w)$ estimates for $T^F$ with $w \in A_p$, it is assumed implicitly (by the Rubio de Francia extrapolation theorem) that $M_\Phi$ (and so $T^F$) is bounded on the unweighted $L^p$ for any $p > 1$. It was shown by Pérez [31] that $M_\Phi$ is bounded on $L^p$ if and only if $\Phi$ satisfies the $B_p$ condition: $\int_1^\infty \Phi(t)t^{-p-1}dt < \infty$. Therefore, throughout the paper, we assume that for any $r > 1$,

$$t \leq \Phi(t) \leq c_r t^r \quad (t \geq 0).$$

This condition includes all main cases of interest. Also we introduce the following notation for the $B_p$ constant of $\Phi$:

$$C_\Phi(p) = \left( \int_1^\infty \frac{\Phi(t) dt}{t} \right)^{1/p}. $$

Before stating our main results about $T^F$, we summarize below sharp weighted bounds for standard Calderón-Zygmund operators.

**Theorem A.** Let $T$ be a Calderón-Zygmund operator on $\mathbb{R}^n$.

(i) For any $1 \leq q < p < \infty$,

$$\|T\|_{L^p(w)} \leq c(n, T, q, p)[w]_{A_q},$$

and in the case $q = 1$, $c(n, T, 1, p) = c(n, T)pp'$;

(ii) for any $1 < p < \infty$,

$$\|T\|_{L^p(w)} \leq c(n, T, p)[w]_{A_p}^{\max \left(1, \frac{1}{p-1}\right)}.$$
Part (i) for $q = 1$ was obtained by Lerner-Ombrosi-Perez [27, 28], and later Duoandikoetxea [9] showed that the result for $q = 1$ can be self-improved by extrapolation to any $1 < q < p$. The sharp dependence of $c(n, T, 1, p)$ on $p$ is important for a weighted weak $L^1$ bound of $T$ in terms of $[w]_{A_1}$ [28]. Part (ii) (known as the $A_2$ conjecture) is a more difficult result. First it was proved by Petermichl [32] for the Hilbert transform, and recently Hytönen [17] obtained (ii) for general Calderón-Zygmund operators. A proof of Theorem A based on local mean oscillation estimates was found in [25, 26]. Observe that for $p \geq 2$, (i) follows from (ii) but for $1 < p < 2$, (i) and (ii) are independent results.

Denote by $S_0(\mathbb{R}^n)$ the class of measurable functions on $\mathbb{R}^n$ such that
\[
\mu_f(\lambda) = \left| \{ x \in \mathbb{R}^n : |f(x)| > \lambda \} \right| < \infty
\]
for all $\lambda > 0$. Our main results are the following.

**Theorem 1.1.** Let $T^\mathcal{F}$ be a maximally modulated Calderón-Zygmund operator satisfying (1.1).

(i) For any $1 \leq q < p < \infty$,
\[
\|T^\mathcal{F}f\|_{L^p(w)} \leq c(n, T, q, p)[w]_{A_q} \|f\|_{L^p(w)},
\]
and in the case $q = 1$, $c(n, T, 1, p) = c(n, T)pC_\Phi\left(\frac{p+1}{2}\right)$.

(ii) Assume that $\Phi(t) \simeq t \int_1^t \frac{\Psi(u)}{u^2} du$ for $t \geq c_0$, where $\Psi$ is a Young function. Then for any $1 < p < \infty$,
\[
\|T^\mathcal{F}f\|_{L^p(w)} \leq c(n, T, p)[w]_{A_p}^{\max\left(1, \frac{1}{p-1}\right) + \frac{1}{p} + \frac{1}{p'}} C_{\Psi_{p-\varepsilon}}(p) \|f\|_{L^p(w)},
\]
where $\varepsilon \simeq [w]_{A_p}^{-1/p'}$.

Both estimates in (i) and (ii) are understood in the sense that they hold for any $f \in L^p(w)$ for which $T^\mathcal{F}f \in S_0$.

Several remarks about Theorem 1.1 are in order.

**Remark 1.2.** The last sentence in Theorem 1.1 can be removed if it is additionally known that $T^\mathcal{F}f \in S_0$ for some dense subset in $L^p(w)$, for instance, for Schwartz functions. In particular, this obviously holds if $T^\mathcal{F}$ is of weak type $(r_0, r_0)$ for some $r_0 > 1$. Hence, there is no need in the last sentence in Theorem 1.1 for the Carleson operator.

**Remark 1.3.** It is easy to see that if $\Phi(t) = t$, then $C_\Phi\left(\frac{p+1}{2}\right) \simeq p'$, and hence part (i) of Theorem 1.1 contains part (i) of Theorem A as a particular case. On the other hand, part (ii) of Theorem 1.1 does not
contain Theorem A, since the assumption $\Phi(t) \simeq t \int_1^t \frac{\Psi(u)}{u^2} du$ implies
$t \log(e + t) \lesssim \Phi(t)$.

**Remark 1.4.** Consider the case corresponding to the Carleson operator, namely, assume that $\Phi(t) = t \log(e + t) \log \log \log(e^{e^e} + t)$. Simple computations show that in this case,

$$C_\Phi \left( \frac{p + 1}{2} \right) \simeq \frac{p^2}{(p-1)^2} \log \log \left( e^e + \frac{1}{p-1} \right).$$

Concerning part (ii), it is easy to see that $\Psi(t) \simeq t \log \log \log(e^{e^e} + t)$ and $C_{\Psi^{p-\varepsilon}}(p) \simeq \frac{1}{\varepsilon^{1/p}} \log \log(e^e + 1/\varepsilon)$. Therefore, if $\varepsilon \simeq [w]_{A_p}$, then

$$C_{\Psi^{p-\varepsilon}}(p) \simeq [w]_{A_p}^{1-p} \log \log(e^e + [w]_{A_p}).$$

Thus, we obtain the following corollary from Theorem 1.1.

**Corollary 1.5.** Let $\mathcal{C}$ be the Carleson operator.

(i) For any $1 \leq q < p < \infty$,

$$\| \mathcal{C} \|_{L^p(w)} \leq c(q,p) [w]_{A_q},$$

and in the case $q = 1$, $c(1,p) \simeq \frac{p^3}{(p-1)^2} \log \log \left( e^e + \frac{1}{p-1} \right)$;

(ii) for any $1 < p < \infty$,

$$\| \mathcal{C} \|_{L^p(w)} \leq c(p) [w]_{A_p}^{\max \left( p', \frac{2}{p-1} \right)} \log \log(e^e + [w]_{A_p}).$$

We make several additional remarks.

**Remark 1.6.** Since the linear $[w]_{A_q}, 1 \leq q < p$, bound is sharp for the Hilbert transform, it is obviously sharp also for $\mathcal{C}$. Further, observe that, as soon as we know, even in the unweighted case $C(1,p)$ from (i) is the currently best known bound for $\| \mathcal{C} \|_{L^p}$ when $p \to 1$. We could not find in the literature this bound written explicitly but it is apparently well known. In particular, it can be easily deduced from a good-$\lambda$ inequality related $\mathcal{C}$ and $M_\Phi$ with $\Phi(t) = t \log(e + t) \log \log \log(e^{e^e} + t)$ obtained in [14].

Concerning the bound for $\| \mathcal{C} \|_{L^p(w)}$ in terms of $[w]_{A_p}$ in (ii), most probably it is not sharp. We discuss this point in Section 4 below.

**Remark 1.7.** The key ingredient in the proof of the linear $[w]_{A_1}$, bound for usual Calderón-Zygmund operators $T$ in [27, 28] is a Coifman type estimate relating the adjoint operator $T^*$ and the Hardy-Littlewood maximal operator $M$. It was crucial that $T^*$ is essentially the same
operator as $T$. However, this is not the case with the Carleson operator $C$. Indeed, taking an arbitrary measurable function $\xi(\cdot)$, we can consider the standard linearization of $C$ given by

$$C_{\xi(\cdot)}(f)(x) = H(\mathcal{M}_{\xi}f)(x).$$

It is difficult to expect that its adjoint $C_{\xi(\cdot)}^*$ can be related (uniformly in $\xi(\cdot)$) with $M$ (or even with a bigger maximal operator) either via good-$\lambda$ or by a sharp function estimate. Indeed, such a relation would imply that $\|C_{\xi(\cdot)}^*\|_{L^p} \lesssim p$ as $p \to \infty$ (since $\|f\|_{L^p} \lesssim p\|f^\#\|_{L^p}$ as $p \to \infty$, where $f^\#$ is the sharp function), which in turn means that $\|C\|_{L^p} \lesssim \frac{1}{p-1}$ as $p \to 1$. But due to the previous remark, the currently known behavior of $\|C\|_{L^p}$ is far from $\frac{1}{p-1}$ for $p$ is close to 1 (in fact, it is reasonable to conjecture that the best possible bound for $\|C\|_{L^p}$ when $p$ is close to 1 is $\frac{1}{p-1}$, see a relevant discussion in Section 4).

In order to prove the linear $[w]_{A_1}$ bound for $C$, we use a modified approach based partially on ideas from [23] and [27].

The paper is organized as follows. In Section 2, we obtain a local mean oscillation estimate of $T^F$, and the corresponding bound by dyadic sparse operators. Using this result, we prove Theorem 1.1 in Section 3. In Section 4, we discuss a connection between the $L \log L$ conjecture about a.e. convergence of Fourier series and sharp $L^p(w)$ bounds for $C$ in terms of $[w]_{A_p}$.

Throughout the paper, we use the notation $A \lesssim B$ to indicate that there is a constant $c$, independent of the important parameters, such that $A \leq cB$. We write $A \simeq B$ when $A \lesssim B$ and $B \lesssim A$.

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2. An estimate of $T^F$ by dyadic sparse operators

2.1. A local mean oscillation estimate. By a general dyadic grid $\mathcal{D}$ we mean a collection of cubes with the following properties: (i) for any $Q \in \mathcal{D}$ its sidelength $\ell_Q$ is of the form $2^k$, $k \in \mathbb{Z}$; (ii) $Q \cap R \in \{Q, R, \emptyset\}$ for any $Q, R \in \mathcal{D}$; (iii) the cubes of a fixed sidelength $2^k$ form a partition of $\mathbb{R}^n$.

Denote the standard dyadic grid $\{2^{-k}([0,1]^n + j), k \in \mathbb{Z}, j \in \mathbb{Z}^n\}$ by $\mathcal{D}$. Given a cube $Q_0$, denote by $\mathcal{D}(Q_0)$ the set of all dyadic cubes with respect to $Q_0$, that is, the cubes from $\mathcal{D}(Q_0)$ are formed by repeated subdivision of $Q_0$ and each of its descendants into $2^n$ congruent subcubes.
We say that a family of cubes $S$ is sparse if for any cube $Q \in S$ there is a measurable subset $E(Q) \subset Q$ such that $|Q| \leq 2|E(Q)|$, and the sets $\{ E(Q) \}_{Q \in S}$ are pairwise disjoint.

Given a measurable function $f$ on $\mathbb{R}^n$ and a cube $Q$, the local mean oscillation of $f$ on $Q$ is defined by

$$\omega_\lambda(f; Q) = \inf_{c \in \mathbb{R}} ((f - c) \chi_Q)^*(\lambda |Q|) \quad (0 < \lambda < 1),$$

where $f^*$ denotes the non-increasing rearrangement of $f$.

By a median value of $f$ over $Q$ we mean a possibly nonunique, real number $m_f(Q)$ such that

$$\max \left(\{x \in Q : f(x) > m_f(Q)\}, \{x \in Q : f(x) < m_f(Q)\}\right) \leq |Q|/2.$$

The following result was proved in [23]; in its current refined version given below it can be found in [18].

**Theorem 2.1.** Let $f$ be a measurable function on $\mathbb{R}^n$ and let $Q_0$ be a fixed cube. Then there exists a (possibly empty) sparse family $S$ of cubes from $D(Q_0)$ such that for a.e. $x \in Q_0$,

$$|f(x) - m_f(Q_0)| \leq 2 \sum_{Q \in S} \omega_{\frac{1}{2^n}}(f; Q) \chi_Q(x).$$

2.2. **An application to $T^F$.** We now apply Theorem 2.1 to $T^F$. Given a cube $Q$, we denote $\bar{Q} = 2\sqrt{n}Q$.

**Lemma 2.2.** Suppose $T^F$ satisfies (1.1). Then for any cube $Q \subset \mathbb{R}^n$,

$$\omega_\lambda(T^F f; Q) \lesssim \|f\|_{\Phi, Q} + \sum_{m=0}^{\infty} \frac{1}{2^{mb}} \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)| dy \right).$$

**Proof.** This result is a minor modification of [23, Prop. 2.3], and it is essentially contained in [14, Prop. 4.1]. We outline briefly main details.

Observe that (1.1) can be written in an equivalent form:

$$(T^F(f\chi_Q)\chi_Q)^*(t) \lesssim \frac{1}{t} |Q| \|f\|_{\Phi, Q},$$

which implies

$$(T^F(f\chi_Q)\chi_Q)^*(\lambda |Q|) \lesssim \|f\|_{\Phi, Q}. $$


Set \( f_1 = f\chi_Q \) and \( f_2 = f - f_1 \). Let \( x \in Q \) and let \( x_0 \) be the center of \( Q \). Then
\[
|T_F(f)(x) - T_F(f_2)(x_0)| = \left| \sup_{\alpha \in A} |T(M^{\phi_\alpha}f)(x)| - \sup_{\alpha \in A} |T(M^{\phi_\alpha}f_2)(x_0)| \right| \\
\leq \sup_{\alpha \in A} |T(M^{\phi_\alpha}f)(x) - T(M^{\phi_\alpha}f_2)(x_0)| \\
\leq T_F(f_1)(x) + \sup_{\alpha \in A} \|T(M^{\phi_\alpha} f_2)(\cdot) - T(M^{\phi_\alpha} f_2)(x_0)\|_{L^\infty(Q)}.
\]

Exactly as in [25, Prop. 2.3], by the kernel assumption,
\[
\sup_{\alpha \in A} \|T(M^{\phi_\alpha} f_2)(\cdot) - T(M^{\phi_\alpha} f_2)(x_0)\|_{L^\infty(Q)} \\
\leq \int_{\mathbb{R}^n \setminus \overline{Q}} |f(y)||K(\cdot, y) - K(x_0, y)||_{L^\infty(Q)}dy \\
\lesssim \sum_{m=0}^{\infty} \frac{1}{2^{m\delta}} \left( \frac{1}{|2^m Q|} \int_{2^m Q} |f(y)|dy \right).
\]

For the local part, by (2.2),
\[
(T_F(f_1)\chi_Q)^*(\lambda|Q|) \lesssim \|f\|_{\Phi,\overline{Q}}.
\]

Combining this estimate with the two previous ones, and taking \( c = T_F(f_2)(x_0) \) in the definition of \( \omega_\lambda(T_F f; Q) \) proves (2.1). \( \square \)

Given a sparse family \( S \), define the operators \( A_{\Phi,S} \) and \( T_{S,m} \) respectively by
\[
A_{\Phi,S}f(x) = \sum_{Q \in S} \|f\|_{\Phi,\overline{Q}} \chi_Q(x)
\]
and
\[
T_{S,m}f(x) = \sum_{Q \in S} |f|_{2^m Q} \chi_Q(x)
\]
(we use a standard notation \( f_Q = \frac{1}{|Q|} \int_Q f \)).

**Lemma 2.3.** Suppose \( T_F \) satisfies (1.1). Let \( 1 < p < \infty \) and let \( w \) be an arbitrary weight. Then
\[
\|T_F f\|_{L^p(w)} \lesssim \sup_{\mathcal{D},S} \|A_{\Phi,S}f\|_{L^p(w)}
\]
for any \( f \) for which \( T_F f \in S_0 \), where the supremum is taken over all dyadic grids \( \mathcal{D} \) and all sparse families \( S \subset \mathcal{D} \).
Proof. Let \( Q_0 \in \mathcal{D} \). Combining Theorem 2.1 with Lemma 2.2, we obtain that there exists a sparse family \( S \subset \mathcal{D} \) such that for a.e. \( x \in Q_0 \),

\[
|T^F f(x) - m_{T^F f}(Q_0)| \lesssim A_{\Phi, S} f(x) + \sum_{m=0}^{\infty} \frac{1}{2^m \delta} T_{S,m} f(x).
\]

If \( T^F f \in S_0 \), then \( m_{T^F f}(Q) \to 0 \) as \( |Q| \to \infty \). Hence, letting \( Q_0 \) to anyone of \( 2^n \) quadrants and using (2.4) along with Fatou’s lemma, we get

\[
\|T^F f\|_{L^p(w)} \lesssim \sup_{S \subset \mathcal{D}} \|A_{\Phi, S} f\|_{L^p(w)} + \sum_{m=0}^{\infty} \frac{1}{2^m \delta} \sup_{S \subset \mathcal{D}} \|T_{S,m} f\|_{L^p(w)}.
\]

It was shown in \([25]\) that

\[
\sup_{S \subset \mathcal{D}} \|T_{S,m} f\|_{L^p(w)} \lesssim m \sup_{g \in S} \|T_{S,0} f\|_{L^p(w)}.
\]

Since \( t \leq \Phi(t) \), we have \( |f|_Q \lesssim \|f\|_{\Phi, \bar{Q}} \), and hence

\[
\|T_{S,0} f\|_{L^p(w)} \lesssim \|A_{\Phi, S} f\|_{L^p(w)}.
\]

Combining this with the two previous estimates completes the proof. \( \square \)

Remark 2.4. Observe that the implicit constant in (2.3) depends only on \( T^F \) and \( n \). In fact, (2.3) holds with an arbitrary Banach function space \( X \) instead of \( L^p(w) \) exactly as for standard Calderón-Zygmund operators (see \([25]\)).

3. Proof of Theorem 1.1

3.1. Proof of Theorem 1.1, part (i). We start with some preliminaries. Given a Young function \( \Phi \), the complementary Young function \( \bar{\Phi} \) is defined by

\[
\bar{\Phi}(t) = \sup_{s > 0} \{st - \Phi(s)\}.
\]

A well known result about the equivalence of Orlicz and Luxemburg norms (see, e.g., \([3\), Th. 8.14]) says that

\[
\|f\|_{\Phi, Q} \leq \sup_{g : \|g\|_{\Phi, Q} \leq 1} \left| \frac{1}{|Q|} \int_Q fg dx \right| \leq 2\|f\|_{\Phi, Q}.
\]

For \( r > 0 \) let \( M_r f(x) = M(|f|^r)(x)^{1/r} \), where \( M \) is the Hardy-Littlewood maximal operator. We summarize below several results from \([27]\) (notice that part (ii) is contained in the proof of \([27\), Lemma 3.3]).
Proposition 3.1. The following estimates hold:

(i) if \( w \in A_1 \) and \( r_w = 1 + \frac{1}{2^n + [w]_{A_1}} \), then
\[
M_{r_w}w(x) \leq 2[w]_{A_1}w(x);
\]

(ii) for any \( p > 1 \) and \( 1 < r < 2 \),
\[
\|Mf\|_{L^p((M_{r_w})^{-\frac{1}{r-1}})} \leq c(n)p\left(\frac{1}{r-1}\right)^{1-1/pr}\|f\|_{L^p((M_{r_w})^{-\frac{1}{r-1}})}.
\]

Also we use the following generalization of the classical Fefferman-Stein inequality [11] obtained by Pérez [31]: if \( p > 1 \) and \( \Phi \in B_p \), then
\[
\|M\Phi f\|_{L^p(w)} \leq c(n)[w]_{A_1}C_{\Phi}(p)\|f\|_{L^p(Mw)}.
\]

Proof of Theorem 1.1, part (i). By extrapolation ([9, Cor. 4.3.]), it suffices to consider only the case \( q = 1 \). Hence, our aim is to show that for any \( 1 < p < \infty \),
\[
\|T^Ff\|_{L^p(w)} \leq c(n,T)pC_{\Phi}\left(\frac{p+1}{2}\right)[w]_{A_1}\|f\|_{L^p(w)}.
\]

By Lemma 2.3 (see also Remark 2.4), this would follow from
\[
\sup_{Q \in S} \|A_{\Phi,S}f\|_{L^p(w)} \leq c(n)pC_{\Phi}\left(\frac{p+1}{2}\right)[w]_{A_1}\|f\|_{L^p(w)}.
\]

Fix a dyadic grid \( \mathcal{D} \) and a sparse family \( S \subset \mathcal{D} \). Using \([3.1]\), we linearize the operator \( A_{\Phi,S} \). One can assume that \( f \geq 0 \). For any \( Q \in S \) there exists \( g(Q) \) supported in \( \bar{Q} \) such that \( \|g(Q)\|_{\Phi,\bar{Q}} \leq 1 \) and
\[
\|f\|_{\Phi,\bar{Q}} \leq (fg(Q))_Q.
\]

Define now a linear operator
\[
L(h)(x) = \sum_{Q \in \mathcal{S}} (hg(Q))_Q \chi_Q(x).
\]

Then in order to prove (3.3), it suffices to show that
\[
\|L(h)\|_{L^p(w)} \leq c(n)pC_{\Phi}\left(\frac{p+1}{2}\right)[w]_{A_1}\|h\|_{L^p(w)},
\]
uniformly in \( g(Q) \).

Exactly as it was done in [27], we have that (3.4) will follow from
\[
\|L(h)\|_{L^p(w)} \leq c(n)pC_{\Phi}\left(\frac{p+1}{2}\right)\left(\frac{1}{r-1}\right)^{1-1/pr}\|h\|_{L^p(M_{r_w}w)},
\]
where $1 < r < 2$. Indeed, taking here $r = r_w = 1 + \frac{1}{2^{n+1}|w|A_1}$, by (i) of Proposition 3.1,
\[
\left(\frac{1}{r_w - 1}\right)^{1 - 1/p_w} \|h\|_{L^p(M_r,w)} \leq c(n)|w|_{A_1} \|h\|_{L^p(w)},
\]
which yields (3.4).

Let $L^*$ denote the formal adjoint of $L$. By duality, (3.5) is equivalent to
\[
\|L^*(h)\|_{L^p(M_r,w)^{-1/p}} \leq c(n)pC\Phi\left(\frac{p + 1}{2}\right)\left(\frac{1}{r - 1}\right)^{1-1/p} \|h\|_{L^p(w)^{-1/p}},
\]
which, by (ii) of Proposition 3.1, is an immediate corollary of

\[
(3.6) \quad \|L^*(h)\|_{L^p((M_r,w)^{-1/p})} \leq c(n)C\Phi\left(\frac{p + 1}{2}\right)\|Mh\|_{L^p((M_r,w)^{-1/p})};
\]

We now prove (3.6). By duality, pick $\eta \geq 0$ such that $\|\eta\|_{L^p(M_r,w)} = 1$ and
\[
\|L^*(h)\|_{L^p((M_r,w)^{-1/p})} = \int_{\mathbb{R}^n} L^*(h)\eta dx = \int_{\mathbb{R}^n} hL(\eta) dx.
\]
Applying (3.1) again, we get
\[
\int_{\mathbb{R}^n} hL(\eta) dx = \sum_{Q \in S} (\eta g(Q)) \int_{Q} h \leq 2 \sum_{Q \in S} \|\eta\|_{\Phi,Q} \int_{Q} h
\]
\[
\leq 2(2\sqrt{n})^n \sum_{Q \in S} \|\eta\|_{\Phi,Q}(h_Q)|Q|
\leq 4(2\sqrt{n})^n \sum_{Q \in S} \|(Mh)^{\frac{1}{p+1}}\eta\|_{L^p(Q)}(h_Q)^{\frac{p}{p+1}}|E(Q)|
\leq 4(2\sqrt{n})^n \sum_{Q \in S} \int_{E(Q)} M\Phi((Mh)^{\frac{1}{p+1}}\eta)(Mh)^{\frac{p}{p+1}} dx
\leq 4(2\sqrt{n})^n \int_{\mathbb{R}^n} M\Phi((Mh)^{\frac{1}{p+1}}\eta)(Mh)^{\frac{p}{p+1}} dx.
\]
Next, by Hölder’s inequality with the exponents $s = \frac{p+1}{2}$ and $s' = \frac{p+1}{p-1}$,
\[
\int_{\mathbb{R}^n} M\Phi((Mh)^{\frac{1}{p+1}}\eta)(Mh)^{\frac{p}{p+1}} dx = \int_{\mathbb{R}^n} M\Phi((Mh)^{\frac{1}{p+1}}\eta)(M,w)^{\frac{1}{p+1}}(Mh)^{\frac{p}{p+1}}(M,w)^{-\frac{1}{p+1}} dx
\leq \|M\Phi((Mh)^{\frac{1}{p+1}}\eta)\|_{L^p((M,w)^{1/2})} \|Mh\|^{\frac{p}{p+1}}_{L^p((M,w)^{-1/p})}.
\]
Further, we apply \((3.2)\) along with Coifman’s inequality [6] saying that 
\[
M(Mr^w)^{1/2} \leq c(n)(Mr^w)^{1/2}.
\]
We obtain
\[
\|M_{\Phi}((Mh)^{\frac{1}{p+1}}\eta)\|_{L^{\frac{p+1}{2}}(M(Mr^w)^{1/2})} \\
\leq c(n)C_{\Phi}\left(\frac{p+1}{2}\right)\|\eta\|_{L^{\frac{p+1}{2}}(M(Mr^w)^{1/2})}.
\]

Using again Hölder’s inequality with \(s = 2p'\) and \(s' = \frac{2p}{p+1}\) gives
\[
\|(Mh)^{\frac{1}{p+1}}\eta\|_{L^{\frac{p+1}{p}}(M(Mr^w)^{1/2})} = \left(\int_{\mathbb{R}^n} (Mh)^{\frac{1}{p+1}}(Mr^w)^{\frac{1}{2p}}(\eta^{p+1}(Mr^w)^{\frac{p+1}{2p}}) dx\right)^{\frac{2}{p+1}}.
\]

Combining this estimate with the three previous ones yields \((3.6)\), and therefore the theorem is proved.

**Remark 3.2.** Inequality \((3.6)\) looks exactly as a Coifman type estimate relating \(L^*\) and \(M\). However, we do not know whether there is a good-\(\lambda\) inequality related \(L^*\) and \(M\) by the reasons described in Remark 1.7.

### 3.2. A Buckley type result for \(M_{\Phi}\).

In order to prove the second part of Theorem 1.1, we need an extension of Buckley’s bound [4]:

\[
(3.7) \quad \|M\|_{L^p(w)} \leq c(p, n)[w]_{A_p}^{\frac{1}{p-1}} (1 < p < \infty)
\]
to Orlicz maximal functions \(M_{\Phi}\) with general \(\Phi\). In the recent work [29], the case \(\Phi(t) = t \log^\lambda(e + t), \lambda \geq 0\), was considered:

\[
\|M_{L(\log L)\lambda}\|_{L^p(w)} \leq c(p, n)[w]_{A_p}^{\frac{1}{p-1}} (1 < p < \infty).
\]

Observe that the proof in [29] essentially contains an estimate for general \(\Phi\) as stated below in Theorem 3.3. For the sake of completeness we give a somewhat different proof avoiding certain details in [29] (such as extrapolation). As we will see below, our proof is a direct generalization of Buckley’s proof of \((3.7)\).

**Theorem 3.3.** For all \(p > 1\) and any \(w \in A_p\),

\[
\|M_{\Phi}\|_{L^p(w)} \leq c(p, n)[w]_{A_p}^{\frac{1}{p-1}} C_{\Phi^\varnothing}(p),
\]

where \(\varnothing \simeq [w]_{A_p}^{1-p'}\).
Proof. Given a cube $Q$, define the weighted mean Luxemburg norm

$$\|f\|_{\Phi^w, Q}^w = \inf \left\{ \alpha > 0 : \frac{1}{w(Q)} \int_Q \Phi \left( \frac{|f(x)|}{\alpha} \right) w \, dx \leq 1 \right\},$$

and consider the weighted centered Orlicz maximal function $M_{\Phi^w, f}^c$ defined by

$$M_{\Phi^w, f}^c(x) = \sup_{Q \ni x} \|f\|_{\Phi^w, Q}^w,$$

where the supremum is taken over all cubes $Q$ centered at $x$ (similarly we denote by $M_{\Phi}^c f$ the unweighted centered maximal function). Then we have the following version of (3.2): for any weight $w$ and all $p > 1$,

$$(3.8) \quad \|M_{\Phi^w, f}^c\|_{L^p(w)} \leq c(n)C_{\Phi}(p) \|f\|_{L^p(w)}.$$

The proof follows exactly the same lines as the proof of the unweighted version in [31] (only one should apply the Besicovitch covering theorem to get a weak type bound) and hence we omit details.

For any $\alpha > 0$, by Hölder’s inequality,

$$\frac{1}{|Q|} \int_Q \Phi(|f|/\alpha) dx \leq [w]_{A_p}^{1/p} \left( \frac{1}{w(Q)} \int_Q \Phi(|f|/\alpha) w dx \right)^{1/p},$$

which implies

$$\|f\|_{\Phi, Q} \leq \|f\|_{[w], A_p, \Phi^w, Q}^w.$$

From this and from the standard estimate $M_{\Phi} f(x) \leq M_{\Phi^w, f}^c(x)$ we obtain

$$M_{\Phi} f(x) \leq M_{[w], A_p, (\Phi^w, f)}^c(x).$$

Now we use the fact that if $\varepsilon \simeq [w]_{A_p}^{1-p'}$, then $w \in A_{p-\varepsilon}$ and $[w]_{A_{p-\varepsilon}} \lesssim [w]_{A_p}$ (see [4]). Combining this with the previous estimate yields

$$M_{\Phi} f(x) \leq M_{c(n, p)[w], A_p, \Phi^{p-\varepsilon}, w}^c f(x).$$

Therefore, by (3.8),

$$\|M_{\Phi} f\|_{L^p(w)} \leq \|M_{c(n, p)[w], A_p, \Phi^{p-\varepsilon}, w}^c f\|_{L^p(w)} \leq c(n)C_c(n, p)[w]_{A_p} C_{\Phi^{p-\varepsilon}}(p) \|f\|_{L^p(w)} = c(n)C_c(n, p) \frac{1}{p} [w]_{A_p}^{\frac{1}{p}} C_{\Phi^{p-\varepsilon}}(p) \|f\|_{L^p(w)},$$

which completes the proof. \qed
3.3. Proof of Theorem 1.1, part (ii). We will need a generalization of the classical equivalence [34]

$$\frac{1}{|Q|} \int_Q M(f \chi_Q) dx \simeq \|f\|_{L^{\log L, Q}}$$

to general Young functions. This can be stated as follows.

Given a Young function $\Psi$, define

$$\Psi^*(t) = \begin{cases} 
    t, & 0 \leq t \leq 1 \\
    t + t \int_1^t \frac{\Psi(u)}{u^2} du, & t > 1.
\end{cases}$$

Then (see [35, Theorems 10.5,10.6])

$$\frac{1}{|Q|} \int_Q M_{\Psi}(f \chi_Q) dx \simeq \|f\|_{\Psi^*, Q}.$$  

Proof of Theorem 1.1, part (ii). This is just a combination of several previously established bounds. As in the proof of the first part of Theorem 1.1, by Lemma 2.3, it is enough to get a uniform estimate of $\|A_{\Phi,S}f\|_{L^p(w)}$.

By the assumption $\Phi(t) \simeq t \int_1^t \frac{\Psi(u)}{u^2} du$ for $t \geq c_0$ we have that $\Phi \simeq \Psi^*$, where $\Psi^*$ is defined by (3.9). Hence, using (3.10), we obtain

$$A_{\Phi,S}f(x) = \sum_{Q \in S} \|f\|_{\Phi,W} \chi_Q(x) \simeq \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q M_{\Psi}(f \chi_Q) dx \right) \chi_Q(x) \leq \mathcal{T}(M_{\Psi}f)(x),$$

where the operator $\mathcal{T}$ is defined by

$$\mathcal{T}f(x) = \sum_{Q \in S} \left( \frac{1}{|Q|} \int_Q f dx \right) \chi_Q(x).$$

Therefore, using that $\|\mathcal{T}\|_{L^p(w)} \lesssim [w]_{A_p}^{\max(1,1/p-1)}$ (see [7]) and applying Theorem 3.3, we obtain

$$\|A_{\Phi,S}\|_{L^p(w)} \lesssim \|\mathcal{T}\|_{L^p(w)} \|M_{\Psi}\|_{L^p(w)} \lesssim [w]_{A_p}^{\max(1,1/p-1)} [w]_{A_p}^{1/2} C_{\Psi^{-\varepsilon}}(p),$$

where $\varepsilon \simeq [w]_{A_p}^{1-p'}$, and this completes the proof. \hfill \square

4. Remarks and complements

4.1. More about $A_p$ bounds for $\|C\|_{L^p(w)}$. Let $\alpha_p$ be the best possible exponent in

$$\|C\|_{L^p(w)} \lesssim [w]_{A_p}^{\alpha_p},$$
As we have seen, our proof of Corollary 1.5 part (ii), is based essentially on (1.1) with
\[ \Phi(t) = t \log(e + t) \log \log \log(e^{e^t} + t), \]
which is intimately related to Antonov’s theorem [1] on a.e. convergence of Fourier series for functions in \( L \log L \log \log L \). A question whether the class \( L \log L \log \log L \) can be improved is still open. The main conjecture about this says that Fourier series converge a.e. for functions in \( L \log L \). A natural reformulation of this conjecture is that (1.1) for \( C \) holds with \( \Phi(t) = t \log(e + t) \). Let us check what can be done assuming that this result is true.

First, it is easy to see that following our approach we would obtain that for all \( p > 1 \),
\[ \|C\|_{L^p(w)} \leq c \frac{p^3}{(p - 1)^2} [w]_{A_1}, \]
and
\[ \|C\|_{L^p(w)} \leq c(p)[w]_{A_p}^{\max\left(\frac{r'}{p}, \frac{2}{p-1}\right)}. \]

In particular, the “\( L \log L \) conjecture” implies \( \|C\|_{L^p} \lesssim \frac{p^3}{(p-1)^2} \) and \( \alpha_p \leq \max\left(p', \frac{2}{p-1}\right) \). It is natural to conjecture further that the un-weighted bound for \( \|C\|_{L^p} \) is best possible, that is, \( \|C\|_{L^p} \simeq \frac{p^3}{(p-1)^2} \). Then one can easily get a lower bound for \( \alpha_p \) that coincides with the upper bound for \( 1 < p \leq 2 \).

Indeed, a well known argument given by Fefferman-Pipher [12] (see also [29] for an extension of this argument) says that if \( T \) satisfies \( \|T\|_{L^{p_0}(w)} \lesssim N([w]_{A_1}) \) for some \( p_0 \), then \( \|T\|_{L^r} \lesssim N(c r) \) as \( r \to \infty \). Hence, on one hand, since \( \|C\|_{L^r} \simeq r \) as \( r \to \infty \), we obtain that \( \alpha_p \geq 1 \) for all \( p > 1 \). On the other hand, let \( C_{\xi(\cdot)} \) be a linearization of \( C \) as in Remark 1.7. Then, by duality and by (4.1),
\[ \|C_{\xi(\cdot)}^*\|_{L^{p_0}(w)} = \|C_{\xi(\cdot)}\|_{L^{p_0}(w^{-p_0})} \lesssim [w^{-p_0}]_{A_{p'}}^{\alpha_p} = [w]_{A_{p'}}^{\alpha_p}, \]
and hence \( \|C_{\xi(\cdot)}^*\|_{L^r} \lesssim r^{\alpha_p(p_0-1)} \) as \( r \to \infty \), which implies
\[ \|C\|_{L^r} \lesssim \frac{1}{(r-1)^{\alpha_p(p-1)}} \]
as \( r \to 1 \). Conjecturing that \( \|C\|_{L^r} \simeq \frac{1}{(r-1)^{\alpha_p(p-1)}} \) as \( r \to 1 \), we obtain \( \alpha_p \geq \frac{2}{p-1} \). Therefore, \( \alpha_p \geq \max(1, \frac{2}{p-1}) \).

Concluding, we see that if the “\( L \log L \) conjecture” holds and if the best possible behavior of \( \|C\|_{L^p} \) is \( \frac{1}{(p-1)^{\alpha_p(p-1)}} \) when \( p \) is close to 1, then for
all $p > 1$,

$$\max\left(1, \frac{2}{p-1}\right) \leq \alpha_p \leq \max\left(p', \frac{2}{p-1}\right).$$

In particular, $\alpha_p = \frac{2}{p-1}$ for $1 < p \leq 2$.

It seems that a natural obstacle in our approach is that the “local mean oscillation estimate” essentially relies on the end-point information of a given operator, while a sharp end-point information of the Carleson operator is currently unknown. It is natural to ask whether there is an approach to sharp $L^p(w)$ estimates avoiding the information about end-point bounds. Observe that this is unknown even for Calderón-Zygmund operators.

4.2. On mixed $A_p-A_\infty$ bounds. Following recent works, where the $A_p$ bounds were improved by mixed $A_p-A_\infty$ bounds (see, e.g., [19] [20] [21]), we can give similar results for $T^\Phi$.

Given a weight $w$, define its $A_\infty$ constant by

$$[w]_{A_\infty} = \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q)dx.$$ 

It was shown in [20] that part (i) of Proposition 3.1 holds with the $[w]_{A_1}$ constant replaced by $[w]_{A_\infty}$. Changing only this point in the proof of Theorem 1.1, part (i), we get that for any $w \in A_1$ and for all $p > 1$,

$$\|T^\Phi f\|_{L^p(w)} \leq c(n,T)pC\Phi\left(\frac{p+1}{2}\right)\left[w\right]_{A_1}^{\frac{1}{p}}\left[w\right]_{A_\infty}^{\frac{1}{p'}}\|f\|_{L^p(w)}.$$

For Calderón-Zygmund operators this inequality was obtained in [20].

Further, it was shown in [21] that if $w \in A_p$ and $\varepsilon \simeq [\sigma]_{A_\infty}$, where, as usual, $\sigma = w^{1/p}$, then $w \in A_p-\varepsilon$ and $[w]_{A_p-\varepsilon} \lesssim [w]_{A_p}$. It is easy to see from this result that the condition $\varepsilon \simeq [w]_{A_p}^{1-p'}$ in Theorem 1.1 can be replaced by $\varepsilon \simeq [\sigma]_{A_\infty}$.

Then, in the case of the Carleson operator, by Remark 1.4

$$C_{\Psi_{p-\varepsilon}}(p) \simeq \frac{1}{\varepsilon^{1/p}} \log \log (e^\varepsilon + 1/\varepsilon) \simeq [\sigma]_{A_\infty}^{\frac{1}{p}} \log \log (e^\varepsilon + [\sigma]_{A_\infty}),$$

and hence

$$\|M_\Psi\|_{L^p(w)} \lesssim ([w]_{A_p}[\sigma]_{A_\infty})^{\frac{1}{p}} \log \log (e^\varepsilon + [\sigma]_{A_\infty}).$$

Also, observe that the operator $T$ defined in the proof of Theorem 1.1 satisfies (see [19])

$$\|T\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p}} ([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p'}}).$$
Therefore, combining this with the bound for $M_{\Psi}$, we obtain
\[
\|C\|_{L^p(w)} \lesssim [w]_{A_p}^{\frac{1}{p}} ([w]_{A_{\infty}}^{\frac{1}{p}} + [\sigma]_{A_{\infty}}^{\frac{1}{p}}) [\sigma]_{A_{\infty}}^{\frac{1}{p}} \log \log (e^\nu + [\sigma]_{A_{\infty}}). \]

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Department of Mathematics, Bar-Ilan University, 5290002 Ramat Gan, Israel.

E-mail address: aklerner@netvision.net.il