A Note on Islands in Schwarzschild Black Holes

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ABSTRACT: We consider evaporation of the Schwarzschild black hole and note that island configurations do not provide a bounded entanglement entropy. The same remark is valid also for some other static black holes. A proposal for improving the situation is discussed.
1 Introduction

The entropy of Hawking radiation of black holes grows up to infinity during evaporation and it is a manifestation of the information paradox [1, 2]. This increase contrasts with Page's hypothetical behavior, in which entropy decreases after the so-called Page time [3–5] and which ensures the unitarity of quantum mechanics.

An approach to treating the black hole information problem was suggested in [6–10], where the “island formula” for the entanglement entropy of Hawking radiation based on quantum extremal surfaces [11] has been proposed and it has been argued that the entanglement entropy is limited during the evaporation of black holes. According to the prescription of quantum extremal surfaces the entanglement entropy is given, after renormalization\(^1\), by the formula

\[
S(R) = \min_{\mathcal{I}} \left\{ \frac{\text{Area}(\partial\mathcal{I})}{4G} + S_{\text{matter}}(R \cup \mathcal{I}) \right\}.
\]

(1.1)

Here \(\mathcal{I}\) is a region, called the ”island”, whose boundary area is denoted by \(\text{Area}(\partial\mathcal{I})\) and \(S_{\text{matter}}\) is the von Neumann entropy \(S_{\text{VN}}(R \cup \mathcal{I})\) of union of the island and the region \(R\). \(G\) is Newton’s constant. An extremization on any possible island and then taking of the configuration with the minimum entropy is assumed. As the system evolves, the form of island can change dynamically and different extreme surfaces can

\(^1\)See [12, 13] for renormalization of the entanglement entropy in four-dimensional case.
dominate at different times. In several models this change of dominance provides the bound on $S(R)$ as time evolves [6, 7, 9, 10, 14–22].

Although the prescription of the quantum extremal surface was proposed in the framework of holography [23, 24], the island rule is applicable to black holes in more general theories. The formula was confirmed for some two dimensional models [8, 9]. For two dimensional gravity the island rule has been derived by making use of replica trick [25–27] and the island contribution has been associated with replica wormholes [20, 21]. The Page curve for evaporating black holes in JT gravity has also been studied in [28]. The wormhole configurations in JT gravity have been discussed in [29]. For a further development in low dimensional case see [30–37], and references therein, in particular, for asymptotically flat two-dimensional spacetime see [38–40]. The black holes in four and higher dimensional asymptotically flat spacetime have been considered in [14, 41–44].

Using the island formula (1.1) for asymptotically flat eternal Schwarzschild black holes in four spacetime dimension the saturation at late time of the entanglement entropy to the value

$$S_I = \frac{2\pi r_h^2}{G} + \frac{c}{6} \frac{b - r_h}{r_h} \log \frac{16 r_h^2 (b - r_h)^2}{G^2 b}$$

has been found in [41]. Here $r_h$ is the gravitational radius of the Schwarzschild black hole, $r_h = 2GM$, $M$ is the black hole mass, $c$ is the number of massless matter fields, $b$ defines the boundaries of the entanglement regions in the right and the left wedges of the Schwarzschild geometry. The following inequalities are assumed: $\sqrt{G} \ll r_h \ll b$. The entropy corresponding to non-island configuration dominates at small time and increases with time linearly

$$S_{nI} \approx \frac{c}{6} \frac{t}{r_h}.$$  

Equalizing the entropy without island with the entropy with island under condition of dominating the first term in (1.2), one estimates the Page time [41]

$$t_{Page} \sim \frac{6\pi r_h^3}{cG}.$$  

The above considerations, as noted in [41], do not resort neither to holographic correspondences, i.e. embedding into higher-dimensional AdS spacetime, nor to coupling with an auxiliary system to absorb the radiation.

In this paper we note that while applying the above estimates to the evaporation of a black hole, the second term $c b/6r_h$ in (1.2) plays an important role and becomes dominant for small $r_h$. Thus, the entropy starts to increase with decreasing mass.
of the black hole. Or in other words, although the inclusion of the island and the extremization of its location leads to the time-independent entropy of the eternal black hole at late times, when the black hole evaporates to the Schwarzschild radius \( r_h \ll (cGb/24\pi)^{1/3} \) entropy grows explosively over time.

Let us discuss the origin for the appearance of the term \( cb/6r_h \) in (1.2) which is singular for small \( r_h = 2G\mathcal{M} \). We will see that the origin of this term lies in using of the Kruskal coordinates which are singular for small \( G \). The island formula (1.1) has been derived in \([9, 10]\) by using the replica method and expanding the quantum gravity path integral over gravitational constant \( G \). The semi-classical expansion of the quantum gravity path integral over \( G \) is different from the standard semi-classical expansion over the Planck constant because in the case of the \( G \)-expansion the background metric itself depends on \( G \). Moreover, the Schwarzschild solution in the Kruskal coordinates is singular for small \( G \) (or small \( \mathcal{M} \)), see a discussion of this question in [48]. As a result, the term \( S_{\text{matter}}(\mathcal{R} \cup \mathcal{I}) \) is of the order \( 1/G \) for small \( G \).

Before extremization the first term is of the same order and it is given by \( 2\pi a^2/G \), here \( a \) is a location of the island. However the extremization procedure makes \( a \approx r_h \) and \( 2\pi a^2/G = 8\pi \mathcal{M}^2 \), i.e. this term becomes subdominant for small \( G \) yielding leadership to the term \( cb/(12G\mathcal{M}) \).²

This is just an opposite behavior as compared with the estimation based on taking only the first term \( 2\pi r_h^2/G \) into account in (1.2) and leading to decreasing of entropy to zero with decreasing of the black hole mass after the Page time \( t_{\text{Page}} \) given by (1.4). By taking these two terms into account different scenarios depending on the initial parameters of the evaporating black hole can be realized.

- In first scenario, before the Page time \( t_{\text{Page}} \) the entropy increases. Then the entropy decreases for some time, but then at the moment \( t_{\text{expl}} \) an explosive growth of the entropy begins. One can say that time evolution follows the anti-Page curve. This anti-Page part of evolution ends with a blow up at the point \( t_{\text{blow}} \), where the mass of the black hole is completely lost, see schematic plot in Fig.1.A.

- For another scenario, the period of decreasing entropy disappears, see Fig.1.B. In this case the initial increase of the entropy, inherent in the configuration without an island, at moment \( t_{\text{expl}} \) is replaced by explosive behavior up to

²In more detail, the entanglement entropy before extremization as a function of \( a \) has the form

\[
S(a) = \frac{2\pi a^2}{G} - \frac{c}{6} \left( \frac{a}{r_h} + \log \frac{a}{r_h} \right) + \text{const.}
\]

One can see that \( S(a) \) is an increasing function for \( a \geq r_h \) if \( r_h^2 > cG/12\pi \). Therefore, the function has its minimum at \( a = r_h \).
Figure 1. The entropy with the island increases in the end of evaporation and becomes infinite at $t_{\text{blow}}$. A) Increasing starts after some time of decreasing. B) There is no decreasing period. The entropy increases after the intersection point $t_{\text{Anti-Page}}$. C) All the time $S_{\text{nI}} < S_I$.

$t_{\text{blow}}$, typical for an island configuration with a low black hole mass. This blow up continues until complete evaporation at the moment $t_{\text{blow}}$. In this case, the Page point, where the increase of the entropy changes to decrease disappears, altogether, and the anti-Page point, where the rate of growth of entropy changes, appears.

- For more special case, Fig.1.C. the no-island configuration contribution dominates all the time till total evaporation.

We also consider the question when the effect of an explosive growth of the entropy is behind the Planck scale. A regularization of equation (1.2) at $r_h \to 0$ is discussed.

The paper is organized as follows. In the setup Section 2 we remind the formula for the entanglement entropy for configurations without and with an island for two sided eternal 4-dimensional Schwarzschild black hole obtained in [41]. In Section 3 we study the exchange of dominance between different configurations with an island and without an island for the evaporating black hole in details, especially in the end of evaporation. Special attention is paid to the localization of the occurrence of quantum effects in this process. In Section 4 we study a regularization of previous calculations that permits to consider the total evaporation of the black hole.

2 Setup

2.1 Two-sided black hole

One of the simple examples explicitly demonstrated how an island can help to make the bounded entanglement entropy of the Hawking radiation is the two-sided black
The island configuration for two-sided black hole considered in [41].

The island formula for the generalized entropy consists of two parts

$$S_{gen} = S_{gr} + S_{vN}. \quad (2.1)$$

the gravity part $S_{gr}$ that is associated with a nontrivial quantum extremal surface, island, and the matter (radiation) von Neumann entropy $S_{vN}$.

One supposes that the radiation is located at the union of two regions $R_+$ and $R_-$, which are located in the right and left wedges in the Penrose diagram (see Fig.2) near the null infinities, where the gravity is negligible.

The states that one considers are maximally symmetric, so the location of a possible island is fixed by its position in $(t, r)$ coordinates and effectively one deals with two-dimensional models with $(t, r)$ coordinates (or deals only with $s$-modes) and computes the corresponding entanglement entropy between several entangling regions using two dimensional answers [27].

It is convenient to work within the Kruskal coordinates [45] related with the Schwarzschild coordinates $t, r$ as

$$U = -\sqrt{\frac{r - r_h}{r_h}} e^{-\frac{t - (r - r_h)}{2r_h}}, \quad V = \sqrt{\frac{r - r_h}{r_h}} e^{\frac{i(r - r_h)}{2r_h}} \quad (2.2)$$

and by which the corresponding two-dimensional part of the Schwarzschild metric is

$$ds_{2-dim\, part\, Schw}^2 = -W^{-2} dUdV, \quad W = \sqrt{\frac{r}{4r_h^3}} e^{\frac{r - r_h}{2r_h}}. \quad (2.3)$$

For the configuration without islands the entanglement entropy $S_{nI}$ of the Hawking radiation is identified with that in the region $R = R_+ \cup R_-$ and

$$S_{nI} = \frac{c}{6} \log d(\ell_1, \ell_2), \quad (2.4)$$

$d(\ell_1, \ell_2)$ is the geodesic distance between points $\ell_1$ and $\ell_2$ given by

$$d(\ell_1, \ell_2) = \sqrt{\frac{(U(\ell_2) - U(\ell_1))(V(\ell_1) - V(\ell_2))}{W(\ell_1)W(\ell_2)}}. \quad (2.5)$$
Here points \( \ell_1 \) and \( \ell_2 \) are located at \((t_b, b)\) and \((-t_b + i2\pi r_h, b)\). The total entanglement entropy for this configuration is given by

\[
S_{n\mathcal{I}} = \frac{c}{6} \log \left[ \frac{16r_h^2(b - r_h)}{b} \cosh^2 \frac{t_b}{2r_h} \right].
\]  

(2.6)

At \( t_b \gg b(\gg r_h) \), the above result is approximated as

\[
S_{n\mathcal{I}} \simeq \frac{c}{6} \frac{t_b}{r_h} \quad r_h = 2GM
\]

(2.7)

and grows linearly in time. At the late times \( t \gg \frac{r_h^3}{\alpha c} \) this entropy becomes much larger than the black hole entropy, and this contradicts with the finiteness of the von Neumann entropy for a finite-dimensional black hole system. In such a case an island is expected to emerge [41].

For the configuration with the island, presented in Fig.2, the entanglement entropy for the conformal matter is given by

\[
S_{\text{matter}} = \frac{c}{3} \log \frac{d(a_+, a_-)d(b_+, b_-)d(a_+, b_+)d(a_-, b_-)}{d(a_+, b_-)d(a_-, b_+)}.
\]

(2.8)

Locations of \( a_\pm \) and \( b_\pm \) are indicated in Fig.2 and \( d(\ell_1, \ell_2) \) is given by (2.5). Supposing that \( \sqrt{G} < r_h \ll b \) and the extremizing of the total entropy about the location of the island, one gets [41] a unique island location, that is \( t_a = t_b \) and \( a = r_h + r_hX^2(b, G, c) \), where \( X^2 \ll 1 \). At this configuration the total entanglement entropy is given by (1.2). The main claim of [41], see also [43], is that although at early times one has a linear growth, the island comes to rescue the unitarity at late times in agreement with the Page curve. Equalizing the entropy without island \( S_{n\mathcal{I}} \) with the entropy \( S_{\mathcal{I}} \) with island one estimates the Page time given by (1.4). The mutual information between the island and the region of collecting the Hawking radiation has been considered in [46] and the subsystem volume complexity in this context has been considered in [47].

Note that the consideration in [41] and presented above is relied on the assumptions, that Hawking radiation has no gravitational interaction and the main contribution to the matter von Neumann entropy comes from the entanglement between s-wave modes of the quantum field, so one can reduce the theory into two dimensional conformal field theory.

### 2.2 Time dependence of the mass of the evaporating black hole

In four dimensions due to radiation the mass \( M \) of the black hole is reduced as [4]

\[
M(t) = \frac{r_0}{2G} \left( 1 - \frac{24\alpha c Gt}{r_0^3} \right)^{1/3}
\]

(2.9)
where \( \alpha \) is a constant dependent on the spin of the radiating particle, \( c \) is the number of massless matter fields and \( r_0 \) is the Schwarzschild radius at \( t = 0 \). The semiclassical estimate of the black hole lifetime is

\[
t_{\text{evaporate}} = \frac{r_0^3}{24c \alpha G}.
\] (2.10)

3 Time dependence of the entanglement entropy of the evaporating black hole

In this section using equation (1.2) we analyze what happens, when the black hole is losing its mass. We consider this process adiabatically just supposing, that the dependence of the entanglement entropy on time is defined by dependence of mass of black hole on time \( M(t) \) considered in [41]. From formula (1.2) one sees, that for small \( r_h \), the term \( cb/6r_h \) in (1.2) dominates and in this case the entropy increases when mass of the black hole goes to zero. We will show that just this increasing leads to so called anti-Page time dependence of entropy of the system.

**Figure 3.** Dependence of \( S_{\text{Island}} \) on \( M \) (green lines). The dashed lines show \( S_{\text{Island},M}' \). The red dashed line shows the location of the Planck mass.

The typical dependence of the entanglement entropy (1.2) on mass is presented in Fig.3. We see that this dependence has a minimum located for large \( b \), \( b > r_h \) at

\[
M_{\text{min}} = \frac{1}{4} \left( \frac{bc}{3\pi G^2} \right)^{1/3}.
\] (3.1)

This minimum can be realized outside of the Planck domain, i.e.

\[
M_{\text{min}} > M_{\text{Planck}} \approx 1/\sqrt{G},
\] (3.2)
that corresponds in the used approximation to

$$\sqrt{G} < \frac{1}{64} \frac{bc}{3\pi}. \quad (3.3)$$

In all our plots $c = 3$, and therefore should be $\sqrt{G} < \frac{b}{63\pi}$.

One can compare the entropy with an island and the entropy for a configuration without an island, see Fig.4. We see, that for a given mass of the black hole, after some time, depending on the mass of the black hole, the entropy without island (red lines with increasing thickness for increasing time) reaches the generalized entropy for the configuration with the island (the green line). This is true for both increasing and decreasing branches of the entanglement entropy with the island. This observation gives the Page time, where two entropies are equalized. In the cases of cyan points indicated in Fig.4 the time dependence of the entropy for the eternal black has a desirable form [41]. In these cases dependence of the Page time on mass of the black hole is given by

$$t_{Page} = \frac{48\pi M^3}{cG^2}. \quad (3.4)$$

However, if the red line intersects the green line at $M$, that is less then corresponding $M_{cr}$ (two red points at Fig.4), the increasing is preserved, only the slopes of the increase change. We call this point the anti-Page one, see the schematic plot presented in Fig.1.B.

Now we would like to consider the modification of the Page curve for the black hole evaporation following equation (2.9). In plots Fig.5 and Fig.6 the competition between two entropies is shown. The position of the Page time is indicated by the cyan line. A new feature arises at the end of evaporation. As was mentioned in the Introduction, due to the presence of the term inverse of $r_h$ the island entropy starts to increase and blows up in the end of evaporation. This period of evolution may obscure behind the Planck scale or may not, depending on the parameters of the theory.

It may happen, that the growth of the entropy of the configuration with the island begins earlier than the two curves intersect, see Fig.6 and the schematic picture presented at Fig.1. B.

Moreover, the equalizing of two types of entropy may not appear before the complete evaporation of the black hole, see Fig.7 and the schematic picture presented at Fig.1. C. We see at Fig.7 that entropy for the island configuration begins to increase at some time and continues to exceed the entropy of the non-island configuration for all the time until the end of evaporation.

Summarising this considerations, we get the plots schematically presented in Introduction, Fig.1. Note that it is interesting to know if explosion regime is behind the Planck scale. For this purpose we indicate the Planck scale on all our plots. There are several cases:
Figure 4. Dependence of the entropy for the configuration with the island, $S_I$ (green line), and the configuration without the island, $S_{nI}$ (red lines), on $M$. Red lines of different thickness correspond to different $t_b$ for the case of $S_{nI}$. Green lines of different thickness correspond to different $b$ (here $b = 150$ and 500) for the case of $S_I$. The gray lines show the derivative of corresponding $S_I$ with respect to $M$. These derivatives are equal to zero at $M_{cr,1}$ and $M_{cr,2}$ for $b = 150$ and 500, respectively. We see that $M_{cr,2} > M_{cr,1}$. The points show the intersections of the green lines with the red ones: darker cyan points show that at these points the increasing of entropy with time changes to decreasing, while in the red points the increasing is preserved, only the slopes of the increase change. The magenta dot shows the intersection at masses less than the Planck mass.

- by the time $t_{expl}$ the black hole has not yet evaporated to the Planck mass, see Fig.5. B;
- by the time $t_{expl}$ the black hole has already evaporated to a mass less than the Planck mass, see Fig.6.

In the first case we see the increasing region and therefore the island does not solve the information problem. In the second case one can say, that in this increasing entropy region one has to modify the semi-classical approximation and include the quantum gravity corrections, since the point of the increasing $t_{expl}$ is out of the semiclassical approximation. The same concerns also to the case where the period of decreasing is absent. In this case we have found the situation presented in Fig.8. C and D, i.e. the increasing point is obscured by quantum effects.
Figure 5. A) The green line shows the evolution of the configuration with the island, and the red one – the configuration without island. The cyan line indicates the Page time. B) Plot near the end of evaporation. The darker red line shows the decreasing of mass of evaporating BH, the darker magenta line shows the Planck mass, corresponding to $G = 0.01$, i.e. $M_{\text{Planck}} = 1/\sqrt{G} = 10$, $t_P$ shows the time when $M(t) = M_P$, and $t_{\text{exp}}$ and $t_{\text{blow}}$ show the times when the entropy starts to increase and increases to infinity, respectively.

Figure 6. The green line shows the evolution of the configuration with the island and the red one shows evolution of the configuration without island. The thick red line indicates the anti-Page time. The darker red line shows the decreasing of mass of evaporating BH, the darker magenta line shows the Planck mass, $M_{\text{Planck}} = 1/\sqrt{G}$. Here $G = 0.1$ and $M_{\text{Planck}} = 3.16$.

4 Time dependence of the entanglement entropy of the evaporating black hole in thermal coordinates

As we have seen in the previous section, time dependence of the entanglement entropy of the evaporating black hole in the end of evaporation has singular behaviour. This
Figure 7. The green line shows the evolution of the configuration with the island and the red one configuration without island. We see that there is no Page time at all, since the entropy for the configuration without island lies below the entropy of the configuration with the island in all periods of evaporation. The darker red line shows the decrease of mass of the evaporating black hole, the magenta line shows the Planck mass, $M_{\text{Planck}} = 1/\sqrt{G}$. The red line lies below the green one. Here $G = 0.1$ and $M_{\text{Planck}} = 3.16$.

Behaviour is related with singular behaviour of the Kruskal coordinates in the limit of the black hole mass $M \to 0$, see discussion of this problem in [48]. One can remove this singularity using thermal coordinates [48], that provide a regularization of the Kruskal coordinates near $M = 0$. These coordinates are introduced as

$$
\mathcal{U} = -e^{\frac{t +(r-r_h)}{\mu}} \left( \frac{r-r_h}{r_h} \right)^{\frac{r}{r_h}},
$$

$$
\mathcal{V} = e^{\frac{t +(r-r_h)}{\mu}} \left( \frac{r-r_h}{r_h} \right)^{\frac{r}{r_h}},
$$

where

$$
B = (4M + \mu)G,
$$

and $\mu$ is a positive constant. In these coordinates the standard Schwarzschild solution has the temperature $T = 1/2\pi B$. Two dimensional part of the Schwarzschild metric is

$$
ds_{\text{2-dim part Schw}}^2 = \frac{d\mathcal{U} d\mathcal{V}}{\mathcal{W}^2}, \quad \mathcal{W} = \frac{1}{B} \left( \frac{\mathcal{U} \mathcal{V}}{(1 - GM/r)} \right)^{1/2},
$$

compare with (2.3). For the configuration without islands, by analogy with (4.5), the entanglement entropy of the Hawking radiation is identified with that in the region $R = R_+ \cup R_-$ and it is

$$
S_{nI} = \frac{c}{6} \log \mathcal{D}(\ell_1, \ell_2),
$$
Figure 8. The entropy of configurations with the island increasing in the end of evaporation. A) Increasing starts after some decreasing period and by the time $t_{\text{expl}}$ the black hole has evaporated to a mass less than the Planck mass. B) Increasing starts after some decreasing period and at the moment $t_{\text{expl}}$ the black hole is still more that the Planck mass. C) There is no decreasing period and new increasing period starts at $t_{\text{Antip}}$ and the mass of the BH in this time is less than the Planck mass. D) There is no decreasing period and new increasing period starts before the black hole becomes of the Planck mass. In all of these plots, $t_{\text{Pl}}$ indicates the time when the mass of the black hole becomes equal to the Planck mass.

where $d(\ell_1, \ell_2)$ is the geodesic distance between points $\ell_1$ and $\ell_2$ given by

$$
D(\ell_1, \ell_2) = \sqrt{\frac{\mathcal{W} (\ell_2) - \mathcal{W} (\ell_1)) (\mathcal{V}(\ell_1) - \mathcal{V}(\ell_2))}{\mathcal{W}(\ell_1)\mathcal{W}(\ell_2)}}. \quad (4.6)
$$

Here points $\ell_1$ and $\ell_2$ are located at $(t_b, b_+)$ and $(-t_b + i\pi B, b_-)$. The regularized entanglement entropy for the configuration presented in Fig.2 is given by

$$
S_{\text{I,reg}} = \frac{2\pi a^2}{G} + \frac{c}{3} \log \mathcal{L}_{\text{reg}}(a, b, t_a, t, b, r_h, \mu), \quad (4.7)
$$

where

$$
\mathcal{L}_{\text{reg}}(a, b, t_a, t, b, r_h, \mu)^2 = \mathcal{R}_{\text{reg}}(a, b, t_a, t, b, r_h, \mu) = \frac{D(a_+, a_-)D(b_+, b_-)D(a_+, b_+)}{D(a_+, b_-)D(a_-, b_+)}.
$$
we have seen in the previous Section 3 does not admit the finite limit for $r$

For $\mu = 0$ this expression reproduces the corresponding expression from [41] and as we have seen in the previous Section 3 does not admit the finite limit for $r_h = 0$.

To get the limit $r_h \to 0$ of the regularized entanglement entropy $S_{I, reg}$, we take the limit $r_h \to 0$ in the expression (4.8), and then find the regularized entanglement entropy for zero mass by finding extremum of the expression $S_{reg,0}$ varying over $a$ and $t_a$. $S_{reg,0}$ is given as

$$S_{reg,0} = \lim_{r_h \to 0} S_{reg,r_h} = \frac{2\pi a^2}{G} + \frac{c}{6} \log R_{reg,0}, \quad (4.9)$$

where

$$R_{reg,0} = R_{reg} \bigg|_{r_h \to 0} = \frac{16B_0^4 \cosh^2 \left( \frac{t_a}{B_0} \right) \cosh^2 \left( \frac{t_b}{B_0} \right) \left( e^{\frac{t_a+t_b}{B_0}} + e^{\frac{t_a-t_b}{B_0}} - 2 \cosh \left( \frac{t_a-t_b}{B_0} \right) \right)^2}{G^2 \left( e^{\frac{t_a}{B_0}} + e^{\frac{t_b}{B_0}} + 2 \cosh \left( \frac{t_a+t_b}{B_0} \right) \right)^2},$$

$$B_0 = \mu G. \quad (4.10)$$

Extremizing on $t_a$ at large $t_b, t_b \to \infty$ we get that $t_a = t_b$ and at large $t_b, t_b$

$$R_{reg,0} \approx \frac{4B_0^4}{G^2} \left( \cosh \left( \frac{a-b}{B_0} \right) - 1 \right)^2. \quad (4.11)$$

Assuming $a, b > B_0$ we have

$$S_{0, reg, apr} \approx \frac{2\pi a^2}{G} + \frac{c}{6} \log \left[ 4G^2 \mu^4 \cosh \left( \frac{a-b}{B_0} \right) - 1 \right]^2. \quad (4.12)$$

Extremizing (4.12) on $a$ we get

$$\frac{c \sinh \left( \frac{a-b}{\mu} \right)}{3\mu \left( \cosh \left( \frac{a-b}{\mu} \right) - 1 \right)} + 4\pi a = 0. \quad (4.13)$$

Finding solution of this equation numerically we get the dependence of $S_{I, reg,r_h=0}$ on regularization parameter $\mu$, Fig.9. We see, as should be, this expression is singular for $\mu \to 0$. 

– 13 –
Figure 9. A) Dependence of the regularized entropy $S_{I\text{Island}}^{(B)}$, $B = 2r_h + \mu$ on $M$, here $\mu = 0.01$. B) Dependence of the zero mass regularized entropy $S_{I\text{Island}}^{(B)}|_{B=\mu}$ on regularization parameter $\mu$.

5 Conclusion and discussion

We have considered the evaporation of the Schwarzschild black hole and note that, generally speaking, the island configurations do not provide a bounded entanglement entropy in the end of the black hole evaporation. Despite the fact that including the island and the extremization about its location results in saturation of the entropy for the eternal black hole, the evaporating of black hole ends up by unbounded increasing of the entropy.

One can argue that the formula (1.2) applies for a black hole held in equilibrium with radiation, and not for a freely evaporating black hole. The second term $cb/6r_h$ can be interpreted as the entropy of infalling radiation$^3$.

General arguments of possible inconsistency of islands in theories with long-range gravity have been presented in [49].

In [48] it was suggested that the origin for such singular behavior lies in the use of the Kruskal coordinates. The Kruskal coordinates are suitable to the analytical continuation of the Schwarzschild metric but they are singular in the limit of vanishing of the black hole mass $M \to 0$. Thus, the Kruskal coordinates are not appropriate to describe small black holes. Depending of the initial state parameters the unboundness of the entanglement entropy can occur inside or outside of the Planck length. Similar remarks about islands and Page/anti-Page curves are applied to other static black holes, in particular, to Reissner-Nordström black holes [50, 51], black holes in modified gravity [43] as well as black holes in dS space-time [52].

$^3$We thank Henry Maxfield for providing this interpretation.
We have considered the regularization of applying the quantum extremal surface prescription to gravity theory with matter fields to the Schwarzschild black hole metric in $r_h \to 0$ using thermal coordinates [48]. This consideration permits to consider the entanglement entropy of island configurations in the end of evaporation of the black hole, i.e. take the limit $r_h \to 0$. This regularization can be applied also to other static black holes. It would be also interesting to consider the modification of the Page curve in BCFT models of black hole [53–56] and holographic moving mirror in the end of evaporation [57, 58].

The consideration of small black hole presented here can be generalized to the Reissner-Nordström black holes [50, 51], charged dilaton black holes in non-asymptotically flat cases [59, 60] as well as to black holes in (A)dS [52]. The latter are of particular interest in the context of possible phenomenological applications in the condense matter physics [61, 62] and quark gluon plasma [63]. In particular, the entropy for the dilaton charged black hole contains the $\log b/r_h$ term\(^4\).

As a general remark, let us note that in [64] the black hole information problem has been considered as a particular example of the fundamental irreversibility problem in statistical physics. It was pointed out that similar problem occurs when we study ordinary gas or the formation of the ordinary black body and its thermal radiation. Actually, one has to give a quantum mechanical explanation for the emergence of the second law of thermodynamics in macroscopic systems. In this context one can say that the Boltzmann equation for the one-particle distribution function in the kinetic theory of gases is an analog of the extremization equation for the entanglement entropy with island.

It can be assumed that if we take into account more detailed information about the dynamic properties of matter in the radiation region (not limited to the $s$-mode only) and use a more structured model of islands, perhaps we will be able to more accurately reconstruct the fine grained entropy of quantum gravity and avoid the problem with entropy growth in the end of the black hole evaporation.

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