On the solvability of 3-source 3-terminal sum-networks

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Abstract

We consider a directed acyclic network with three sources and three terminals such that each source independently generates one symbol from a given field $F$ and each terminal wants to receive the sum (over $F$) of the source symbols. Each link in the network is considered to be error-free and delay-free and can carry one symbol from the field in each use. We call such a network a 3-source 3-terminal (3s/3t) sum-network. In this paper, we give a necessary and sufficient condition for a 3s/3t sum-network to allow all the terminals to receive the sum of the source symbols over any field. Some lemmas provide interesting simpler sufficient conditions for the same. We show that linear codes are sufficient for this problem for 3s/3t though they are known to be insufficient for arbitrary number of sources and terminals. We further show that in most cases, such networks are solvable by simple XOR coding. We also prove a recent conjecture that if fractional coding is allowed, then the coding capacity of a 3s/3t sum-network is either 0, 2/3 or $\geq 1$.

Index Terms

Network coding, function computation, multicast, multiple unicast

I. INTRODUCTION

It was shown by Ahlswede et. al. [1] that mixing/coding of incoming information at the intermediate nodes, called network coding, could result in throughput advantages. In particular, it was shown that the coding capacity of a directed multicast network is equal to the minimum of the min-cuts between the source and the individual terminals. Further, linear network coding was shown to be sufficient to achieve this capacity [2], [3]. A polynomial time algorithm for linear multicast code construction was given in [4], whereas distributed random network codes were shown to achieve capacity for multicast networks in [5]. Network coding has since evolved into a rich field of study with connection to many other areas [6].

In this paper, we consider the problem of communicating the sum of messages at some sources to a set of terminals in a directed acyclic network of unit-capacity edges. The problem is a subclass of the problem of distributed computation over a network. Due to the immense complexity of the problem in its full generality with all its model-variations, the problem has been studied in various simplified forms by researchers from diverse fields. We list some known approaches to the problem below.

1) Simple and small networks: Early work in the area of information theory considered the distributed function computation problem as a generalization to the Slepian-Wolf problem. Here the network has multiple sources with separate encoders connected to a receiver which wants to compute a function of the symbols generated at the sources, possibly with a limited allowed distortion [7], [8], [9], [10]. Another variation is where the receiver has access to correlated side-information, and it wants to compute a function of the source symbol encoded by an encoder and the side-information [11], [12]. There are two features in this approach which make the problem complex. First, the sources are correlated with a known arbitrary joint distribution. Second, the aim is to compute the region of encoding rates which allow the recovery of the function at the receiver under the allowed distortion.

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2) **Large networks:** Gallager [13] first posed the problem of computing the parity or modulo-2 sum of a large number of binary sources in a broadcast network. Here all the nodes are independent sources and terminals. The question addressed is how the number of required communications scale with the number of nodes. This line of work became more popular in the context of wireless networks with scheduling constraints arising due to interference [14], [15], [16]. A large body of work now exists with many variations in various aspects.

3) **Distributed Detection:** The problem of distributed detection/estimation of some underlying parameter from the signals generated at different nodes is a problem which gained renewed impetus from the widespread interest in sensor networks [17], [18], [19]. The main aim is to find optimum or near-optimum algorithms. This problem also has the essence of a distributed function computation problem.

4) **Network Coding:** Before network coding acquired its recent level of maturity, the problem of distributed function computation was addressed in a rigorous way by information theorists only for small or simple networks. The techniques of network coding were used in some recent efforts to get some results of elementary nature for larger networks [20], [21], [22], [23], [24], [25], [26], [27]. Our present work is along this line.

In the first, second, and the fourth category of approaches listed above, the particular function “sum” received special interest [2], [9], [13], [20], [21], [22], [23], [24] because (i) it is a simple illustrative example function which is easier to work with, (ii) it reveals many interesting intricacies of the general problem, and (iii) it may reveal techniques for addressing the problem for more general functions (for example, [9] makes direct use of encoding for linear functions for other functions) or other network coding problems (the equivalence with other network coding problems is shown in [28]). In particular, linear multicast coding and linear coding for computing “sum” at one terminal are equivalent problems [22], [28], [29]. Both “modulo sum” as in a finite field, and more generally a finite abelian group; and “arithmetic sum” as in a characteristic-0 field are of interest. We consider the function “modulo-sum” in this paper. Arithmetic sum, though important for many practical applications, is more difficult to analyze because of the unbounded alphabet size. However, the techniques for modulo-sum has also been found useful for getting bounds for the capacity of computing arithmetic-sum [27].

We consider networks where there are multiple sources which generate independent i.i.d. random processes over an alphabet finite field \( F \), or more generally an abelian group \( G \). The edges are assumed to carry one symbol from the alphabet per use without delay or error, i.e., they are delay-free, error-free and unit-capacity. There are multiple terminals which want to recover the sum of the source symbols in each symbol-interval. We specifically consider the case of 3 sources and 3 terminals. This has been known to be the first, i.e. with the smallest number of sources and terminals, nontrivial and highly intriguing case for sum-networks [20], [21], [22], [23], [24].

A. **Standard definitions and review of known results**

We first define some standard terms which will be used in this paper. Our network is represented by a directed acyclic multigraph \( \mathcal{N} = (V, E) \). A network with source nodes \( \{s_1, s_2, s_3, \ldots, s_l\} \subseteq V \) and terminal nodes \( \{t_1, t_2, t_3, \ldots, t_j\} \subseteq V \), so that each terminal wants to recover \( \sum_{i=1}^{l} x_{it} \) for every \( t \), where \( (x_{it})_i \) is the source process of the \( i \)-th source, is called a sum-network with \( l \) sources and \( j \) terminals. A 3-source 3-terminal sum-network will be called a 3s/3t sum-network in short.

**Definition 1:** A sum-network where there is a path from every source to every terminal will be called a connected sum-network.

**Definition 2:** For a sum-network or a multiple-unicast network [30], the reverse network [30], [22], [28], [31] is the network obtained by reversing the direction of every edge, and interchanging the roles of the sources and the terminals.
Definition 3: If one can satisfy the demands of all the terminals over a finite field \(F\) using each edge of \(\mathcal{N}\) once, we say that \(\mathcal{N}\) is solvable over \(F\). In particular, for a sum-network, it means that all the terminals can recover one sum (for one \(t\)) by using the network once. If a linear network code over \(F\) is sufficient for this purpose, we say that \(\mathcal{N}\) is linearly solvable over \(F\). We say that \(\mathcal{N}\) is solvable if it is solvable over at least one field. If \(\mathcal{N}\) is not solvable over any field, we say that \(\mathcal{N}\) is non-solvable. In terms of another well-known term, solvability here refers to scalar solvability, i.e., solvability using a scalar network code \([6],[32]\).

Clearly, for solvability of a sum-network, it is necessary that every source-terminal pair is connected. For a single source, the sum-network reduces to the well-investigated multicast network, and the source-terminal connectivity is also a sufficient condition for solvability if the edges are unit-capacity.

We now define a simple form of linear network code.

Definition 4: A scalar linear network code is called an XOR network code if all the nodes in the network, including the terminal nodes, require to perform only addition and subtraction. In other words, all the local coding coefficients \([6],[32]\) are \(\pm 1\). For the binary alphabet, this means that the nodes only need to perform XOR operation. A network which is solvable by a XOR network code is said to be XOR solvable.

Such a network code is computationally much simpler. Further note that, if a sum-network is XOR solvable then only the group structure in the alphabet is relevant, and the multiplicative structure in the alphabet field is not relevant. Though for simplicity, we will restrict to a finite field alphabet from now onward, it can be checked that whenever a network is XOR solvable over all fields, it is also XOR solvable over any abelian group.

Definition 5: A \((k,n)\) fractional network code is a network code where the source processes are blocked into packets of length \(k\), and encoded into vectors/packets of length \(n\). The edges carry \(n\)-length vectors and nodes operate on incoming \(n\)-length vectors to construct \(n\)-length message vectors on outgoing edges. The terminals recover their demanded function (specifically their demanded source symbols for a traditional communication problem) for \(k\) consecutive symbols of the sources. Thus the rate of computing/communication achieved by using such a code is \(k/n\) per use of the network. Such a code can be linear or non-linear.

For example, a \((k,n)\) fractional network code for a sum-network will enable the terminals to recover \(\sum_{i=1}^{l} x_{it}\) for \(t = k\tau, k\tau + 1, \ldots, k\tau + k - 1\) using the links of the network \(n\) times. Here \(\tau\) denotes the block index.

Definition 6: The rate \(r\) is said to be achievable if there exists a \((k,n)\) fractional (possibly non-linear) network code such that \(k/n \geq r\).

Definition 7: The supremum of all achievable rates is called the capacity of the network. Clearly the capacity of a solvable sum-network is \(\geq 1\).

It can be easily argued that the minimum of the min-cuts for all source-terminal pairs is an upper bound on the capacity of a sum-network \([24]\).

For the most part of the paper, we will consider the question of solvability of a sum-network, and so will consider a single symbol interval and a single usage of the network. So, we will omit the index \(t\) in \(x_{it}\) and use \(x_i\) to mean the symbol generated by the \(i\)th source in one representative symbol-interval.

In the following, we list some results known till date which are related to our present work.

• Ramamoorthy \([20]\) showed that when there are at most two sources or at most two terminals, a sum-network is solvable over any field if and only if every source-terminal pair is connected. Their algorithm also used an XOR code as per our definition.
• The source-terminal connectivity condition is known to be insufficient when both the number of sources and the number of terminals are more than two [21]. In particular, a 3s/3t sum-network (see Fig. 2(a)) was presented in [21], [23] which is not solvable. Further, it was proved in [24] that the capacity of this network is 2/3. On examination of a variety of 3s/3t sum-networks, it was conjectured in [24] that the capacity of any non-solvable but connected sum-network is 2/3. This conjecture is proved in this paper.

• It was proved in [23] that a 3s/3t sum-network which is two-connected, i.e. has two edge-disjoint paths from every source to every terminal, is solvable over fields of odd characteristic. This condition is clearly not a necessary condition for solvability. For instance, the sum-networks shown in Fig. 1 do not satisfy the condition but are clearly solvable.

• It was shown in [22] that a sum-network has a \((k,n)\) fractional linear code if and only if the reverse network has a \((k,n)\) fractional linear code. This implies that the linear coding capacity of a sum-network is the same as that of its reverse network [24]. Since linear codes achieve capacity of a multicast network, this gives that the capacity of a one-terminal sum-network is the minimum of the min-cuts between the source-terminal pairs [24].

• The problem of communicating the sum was shown to be equivalent to the problem of multiple unicasts and more generally the arbitrary network communication problem by showing explicit constructions in [28]. This implied several interesting consequences like (i) existence of a solvably equivalent sum-network for every system of integer polynomial equations, (ii) unachievability of capacity of some sum-networks, and (iii) insufficiency of linear network coding for sum-networks.

• The communication of more general functions was considered in [25], [26], [27] over networks with one terminal. Specifically, some cut-based bounds on the capacity of such networks are presented in [27].

B. Our contribution

We assume that the sources generate symbols from a field \(F\), the edges can carry one symbol from \(F\) per use without error and delay, and the terminals want to recover the sum (defined in \(F\)) of the source symbols. The contribution of this paper is the following.

1. We find a set of necessary and sufficient conditions for solvability of a 3-source 3-terminal sum-network over any field \(F\) (Theorems 1 and 2).

2. We prove a conjecture made in [24] that the capacity of any non-solvable connected 3s/3t sum-network is 2/3.

3. The proof of the necessary and sufficient conditions also lead us to some interesting results and insights like sufficiency of linear codes. We also identify a significant class of solvable networks (\(\kappa \neq 2, 3\) in Lemma 5) which are XOR solvable over any field. In particular, it implies that networks with \(\kappa = 0\) (equivalently, where every source-terminal pair is two-connected) are \textit{XOR solvable over any field}, thus significantly strengthening the result of [23]. In contrast, it was shown in [28] that linear
codes are not sufficient in general for sum-networks with arbitrary number of sources and terminals.

4. As intermediate results, we prove some lemmas which give simpler sufficient conditions for solvability of a $3s/3t$ sum-network.

The paper is organized as follows. In Section II we introduce some notations and define some new terminology which will be used in this paper. We present our new results in Section III and prove them in Section IV. The paper is concluded in Section V.

II. Notations and new definitions

Recall that our network is represented by a directed acyclic multigraph $\mathcal{N} = (V, E)$ with source nodes $\{s_1, s_2, s_3, \ldots, s_t\} \subset V$ and terminal nodes $\{t_1, t_2, t_3, \ldots, t_j\} \subset V$. Each source node $s_i$ independently generates a symbol sequence $x_{it}$ from the alphabet finite field $F$ and each terminal wants to recover $\sum_{i=1}^{t} x_{it}$ defined over $F$ for every $t$. Each edge represents an error-free, delay-free link of unit-capacity. We specifically consider a $3s/3t$ sum-network. As the sum of sources can not be communicated to the terminals at any non-zero rate if a network is not connected, we consider only connected networks in this paper.

For any edge $e = (v_i, v_j) \in E$, the node $v_j$ is called its head and the node $v_i$ its tail and are denoted as $h(e)$ and $t(e)$ respectively. A path $P$ from $v_i$ to $v_j$ - also called a $(v_i, v_j)$ path - is a sequence of nodes $v_1, v_2, \ldots, v_l$ and edges $e_1, e_2, \ldots, e_{l-1}$ such that $v_i = t(e_1)$ and $v_{l+1} = h(e_{l-1})$ for $1 \leq i \leq l - 1$. For any path $P$, $P(v_j : v_k)$ denotes its section starting from the node $v_j$ and ending at $v_k$. If $P_1$ is a $(v_i, v_j)$ path and $P_2$ is a $(v_j, v_k)$ path, then $P_1P_2$ denotes the $(v_i, v_k)$ path obtained by concatenating $P_1$ and $P_2$.

**Definition 8:** For any $A, B \subset V, A \cap B = \emptyset$, we write $A \to B$ if there is a path from every node in $A$ to every node in $B$, and we write $A \nrightarrow B$ if there is no path from any node in $A$ to any node in $B$. Note that $\nrightarrow$ is not the negation of $\to$. If $A = \{v_i\}$ and $B = \{v_j\}$ are singletons, we simply write $v_i \to v_j$ and $v_i \nrightarrow v_j$. For any edges $e_1, e_2 \in E$, we write $e_1 \to e_2, e_1 \nrightarrow e_2$ and $v_i \to e_2$ to mean respectively $h(e_1) \to t(e_2), h(e_1) \nrightarrow v_j$ and $v_i \nrightarrow t(e_2)$. If for two nodes $m$ and $n$, $m \nrightarrow n$, $m$ is called an **ancestor** of $n$, and $n$ a **descendant** of $m$. We assume that a node is not its own ancestor or descendant. For any $A, B \subset V, A \cap B = \emptyset$, we define $\Gamma^A_B = \{v \in V : A \to v, v \to B\}, \Gamma^A = \{v \in V : A \to v\}, \Gamma_B = \{v \in V : v \to B\}$ and **mincut**$(A, B)$ to be the least number of edges whose removal causes $A \nrightarrow B$ in the remaining network.

We represent the network formed by removing the edges $\{e_1, e_2, \ldots, e_i\}$ from the original network $\mathcal{N}$ by $\mathcal{N} - \{e_1, e_2, \ldots, e_i\}$. An edge $e$ is said to **disconnect** an ordered pair of nodes $(v_i, v_j)$, if $v_i \to v_j$ in $\mathcal{N}$ but $v_i \nrightarrow v_j$ in $\mathcal{N} - \{e\}$.

**Definition 9:** For a connected sum-network $\mathcal{N}$ the maximum number of source-terminal pairs that can be disconnected by removing a single edge is called the **maximum-disconnectivity** of the network and denoted by $\kappa(\mathcal{N})$. We call any edge whose removal disconnects $\kappa(\mathcal{N})$ source-terminal pairs as a **maximum-disconnecting** edge. All edges are maximum-disconnecting edges if $\kappa(\mathcal{N}) = 0$.

For example, if in a $3s/3t$ sum-network every source-terminal pair is two-connected then removing any single edge can not disconnect any source-terminal pair; and so the network has $\kappa = 0$. On the other hand, the network shown in Fig. [I(b)] has a single bottleneck link whose removal disconnects all the source-terminal pairs; and so the network has $\kappa = 9$.

We classify the set of all maximum-disconnecting edges into the following three sets: (Recall that all edges are maximum-disconnecting edges if $\kappa(\mathcal{N}) = 0$.)

$\mathcal{A}$: the set of all maximum-disconnecting edges such that there is a path from its head to only one terminal.
\[ B \] : the set of all maximum-disconnecting edges such that there is a path from only one source to its tail.

\[ C \] : the set of all maximum-disconnecting edges such that there is a path from at least two sources to its tail and to at least two terminals from its head.

Clearly every maximum-disconnecting edge is in at least one of \[ A, B \] and \[ C \]. Also, \[ C \] is disjoint from \[ A \] and \[ B \]. If \( \kappa(N) = 0 \) or \( 1 \), a maximum-disconnecting edge may belong to both \[ A \] and \[ B \], however \[ A \] and \[ B \] are disjoint if \( \kappa(N) \geq 2 \) because then any maximum-disconnecting edge is connected to either at least two sources or at least two terminals.

### III. Results

In this section, first we present our main results as theorems, and then we present some lemmas which, on one hand, are used to prove the theorems and which, on the other hand, also provide simpler sufficient conditions for solvability. Recall that a sum-network is nonsolvable if it is not solvable over any field.

**Theorem 1:** [**Necessary and Sufficient condition for Solvability**] A. A 3s/3t connected sum-network \( N \) is nonsolvable if and only if there exist two edges \( e_1 \) and \( e_2 \) and some labeling of the sources and the terminals such that

1. \( \text{mincut}(\{s_1\}, \{t_3\}) = 0 \) in \( \{ N - \{e_1\} \} \)
2. \( \text{mincut}(\{s_3\}, \{t_1\}) = 0 \) in \( \{ N - \{e_1\} \} \)
3. \( \text{mincut}(\{s_2\}, \{t_3\}) = 0 \) in \( \{ N - \{e_2\} \} \)
4. \( \text{mincut}(\{s_2, s_3\}, \{t_2\}) = 0 \) in \( \{ N - \{e_2\} \} \)
5. \( \text{mincut}(\{s_3\}, \{t_3\}) = 0 \) in \( \{ N - \{e_1, e_2\} \} \)
6. \( e_1 \leftrightarrow e_2 \) and \( e_2 \leftrightarrow e_1 \)

B. Whenever a network is solvable, it is linearly solvable over all fields except possibly \( F_2 \).

C. Whenever a network is solvable over \( F_2 \), it is XOR solvable over any field.

D. [24 Conjecture 7] The capacity of a connected non-solvable network is 2/3.

Fig. 2 shows two networks ([21], [23], [24]) which are nonsolvable. It can be verified that for the given labeling of sources, terminals and edges, they satisfy Theorem 1.

In [24] it was conjectured that the capacity of a 3s/3t sum-network is either 0, 2/3 or \( \geq 1 \). Theorem 1 part D states that the capacity of a nonsolvable connected 3s/3t sum-network is 2/3 and thus proves this conjecture.

A network that does not satisfy the conditions in Theorem 1 is solvable over all fields except possibly \( F_2 \). So the conditions in Theorem 1 are necessary and sufficient for nonsolvability over any field other than \( F_2 \). For \( F_2 \), the violation of these conditions does not imply solvability. For example, Fig. 3(a) shows a network which does not satisfy the hypothesis of Theorem 1 but which is not solvable over \( F_2 \) as was shown in [22]. Theorem 2 below identifies the conditions under which a 3s/3t network is solvable over any field except \( F_2 \).

**Theorem 2:** [**Necessary and Sufficient condition for Non-solvability over \( F_2 \)**] A connected 3s/3t sum-network \( N \) is not solvable over \( F_2 \) but linearly solvable over any other field if and only if there exist two edges \( e_1 \) and \( e_2 \) and some labeling of the sources and the terminals such that

1. \( e_1 \) disconnects exactly \( (s_1, t_3) \) and \( (s_3, t_1) \)
2. \( e_2 \) disconnects exactly \( (s_2, t_3) \) and \( (s_3, t_2) \)
3. \( \text{mincut}(\{s_3\}, \{t_3\}) = 0 \) in \( \{ N - \{e_1, e_2\} \} \)
4. \( e_1 \leftrightarrow e_2 \) and \( e_2 \leftrightarrow e_1 \).

It can be verified that for the given labeling of the sources, terminals and edges, the networks in Fig. 3 satisfy Theorem 2.
The following lemma is applicable to sum-networks with arbitrary number of sources and terminals, and may be of independent interest for sum-networks in general.

**Lemma 1:** A connected $l$-source $j$-terminal sum network $\mathcal{N}$ with $\kappa(\mathcal{N}) = k$, $k > 0$, and $\mathcal{E} = \emptyset$ is linearly solvable (respectively XOR solvable) over a field $F$ if all $l$-source $j$-terminal sum networks with $\kappa < k$ are linearly solvable (respectively XOR solvable) over $F$.

In what follows, we present some lemmas which give simpler sufficient conditions for a connected $3s/3t$ network to be solvable. These lemmas will be used to prove the necessity parts of the main theorems.

**Lemma 2:** A connected $3s/3t$ sum-network where there is no edge which is connected to at least two sources and at least two terminals is linearly solvable by XOR coding over any field.

**Lemma 3:** Suppose a connected $3s/3t$ sum-network satisfies the following conditions. For some labeling of the sources and the terminals,

(a) there is an edge $e$ such that $\{s_1, s_2\} \rightarrow e \rightarrow \{t_1, t_2\}$.

(b) there is no edge which disconnects $(s_2, t_3)$ and $(s_3, t_1)$; or $(s_1, t_3)$ and $(s_3, t_2)$.

Then the network is XOR solvable over any field.

**Lemma 4:** Suppose a $3s/3t$ connected sum-network $\mathcal{N}$ satisfies the following conditions. (1) There does not exist an edge-pair which satisfies all the four conditions of Theorem 2.

(2) For some labelling of the sources and the terminals, there exist two edges $e_1, e_2$ such that

(a) $e_1$ disconnects $(s_1, t_3)$ and $(s_3, t_1)$

(b) $e_2$ disconnects $(s_2, t_3)$ and $(s_3, t_2)$

(c) $e_1 \not\rightarrow e_2$ and $e_2 \not\rightarrow e_1$

(d) Removing both $e_1$ and $e_2$ simultaneously does not disconnect $(s_3, t_3)$.

Then the network is solvable over any field by XOR coding.

**Lemma 5:** If for a connected $3s/3t$ sum-network $\mathcal{N}$, $\kappa(\mathcal{N}) \neq 3$ then A. $\mathcal{N}$ is linearly solvable over all fields except possibly $F_2$, B. whenever $\mathcal{N}$ is solvable over $F_2$, it is XOR solvable over all fields, and C. if $\kappa(\mathcal{N}) \neq 2$; then $\mathcal{N}$ is XOR solvable over all fields.

The network shown in Fig. 3(a) (originally presented in [22]) and the network shown in Fig. 3(b) are examples of networks with $\kappa = 2$ which are not solvable over $F_2$ but are linearly solvable over other fields.
Lemma 6: Let $\mathcal{N}$ be a connected $3s/3t$ sum-network with $\kappa(\mathcal{N}) = 3$ and $\mathcal{C} = \emptyset$. A. If $\mathcal{N}$ has an edge pair satisfying the conditions 1-4 of Theorem 2 then it is not solvable over $F_2$ but linearly solvable over other fields. B. If $\mathcal{N}$ does not have an edge pair satisfying conditions 1-4 of Theorem 2 then it is XOR solvable over all fields.

Lemma 7: Given a connected $3s/3t$ sum-network $\mathcal{N}$ with $\kappa(\mathcal{N}) = 3$, if for some labeling of its sources and terminals, there exists an edge $e_2$ satisfying conditions 3 and 4 of Theorem 1 then $\mathcal{N}$ is nonsolvable only if another edge $e_1$ exists such that $e_1$ and $e_2$ satisfy all the six conditions of Theorem 1 else $\mathcal{N}$ is XOR solvable over all fields.

IV. PROOFS

We start by presenting some known results which will be used in the proofs of our results. Considering the complexity of the proof of the main results and their dependence on so many lemmas, a dependency graph of the results is shown in Fig. 4 for clarity.

Lemma 8: [20] A sum-network for which either the number of sources or the number of terminals is at most two is solvable.
if and only if the network is connected. Moreover, such a connected network is XOR solvable over any field.

In [23], the authors proved the following as a side-result:

**Lemma 9:** [23] If in a connected 3s/3t sum-network there exists a node \( v \) such that there is a path from all the sources (resp. at least two sources) to \( v \) and there is a path from \( v \) to at least two terminals (resp. all the terminals) then the network is XOR solvable over any field.

**Corollary 1:** In a 3s/3t connected sum-network, if there is a path from one source (or terminal) to another, then the network is XOR solvable over any field.

**Proof:** If \( s_i \rightarrow s_j \), then \( s_j \) satisfies the hypothesis of Lemma 9 and thus the corollary follows.

So w.l.o.g., we assume that the sources have no incoming edges and the terminals have no outgoing edges.

**Lemma 10:** [22, Theorem 5] If a sum-network \( \mathcal{N} \) is linearly solvable over a field \( F \), then so is its reverse sum-network \( \tilde{\mathcal{N}} \). Further, if \( \mathcal{N} \) has a XOR solution over \( F \), then so does the reverse network.

The second part of the above lemma was not explicitly mentioned in [22], but can be easily seen to follow from the reverse code construction proposed therein.

The next two lemmas are in relation to the double-unicast problem [33], where there are two source-terminal pairs \((s_1, t_1)\) and \((s_2, t_2)\), and each terminal wants to recover the symbol generated at the corresponding source over a directed acyclic network with unit capacity edges. In [33], a simple necessary and sufficient condition was given for such a “double-unicast” network to support two such simultaneous unicasts. The following lemma is a sufficient condition for supporting two simultaneous unicasts and was proved in Case IIB of [33, Proof of Theorem 1].

**Lemma 11:** [33] Suppose in a double-unicast network with connected source terminal pairs \((s_1, t_1)\) and \((s_2, t_2)\), removing all the edges of any \((s_1, t_1)\) path disconnects \((s_2, t_2)\) and there is no single edge in the network whose removal disconnects both \((s_1, t_1)\) and \((s_2, t_2)\). Then there exists a XOR code which allows the communication of \( x_1 \) to \( t_1 \) and \( x_2 \) to \( t_2 \).

The proof in [33] argued that the network is essentially a “grail” with possibly multiple (even or odd number of) “handles” as shown in Fig. 5. Explicit coding schemes, as shown in the figure, were given to achieve the double-unicast.

The next lemma follows by simple modifications in the coding schemes under case IIB of [33, Proof of Theorem 1].

**Lemma 12:** Suppose in a double-unicast network with connected source terminal pairs \((s_1, t_1)\) and \((s_2, t_2)\), removing all...
the edges of any \((s_1, t_1)\) path disconnects \((s_2, t_2)\) and there is no single edge in the network whose removal disconnects both \((s_1, t_1)\) and \((s_2, t_2)\). Then there exists a XOR code which allows the communication of \(x_1\) to \(t_1\) and \(x_1 + x_2\) to \(t_2\).

**Proof:** The proof is achieved by changing the coding on the grail networks as shown in Fig. 6.

Now we start proving our results. Because of Lemma 9 whenever we need to prove solvability under some conditions, we make the following assumption without loss of generality.

*Assumption 1:* \(\mathcal{N}\) does not contain a node that satisfies the hypothesis of Lemma 9

**Proof of Lemma 1**

Consider the new network \(\mathcal{N}^*\) formed by adding an edge \(e_i^*\) in parallel with the edge \(e_i\) for each edge \(e_i \in \mathcal{A} \cup \mathcal{B}\) (Adding \(e_i^*\) in parallel with \(e_i\) means that the head and the tail of \(e_i^*\) are the same as those of \(e_i\)). Clearly \(\kappa(\mathcal{N}^*) < k\), and so by the hypothesis of the lemma, \(\mathcal{N}^*\) is linearly solvable over \(F\). But in any linear code for \(\mathcal{N}^*\), for every edge in \(\mathcal{B}\), the edge and its added parallel edge carry essentially the same data since there is a path from only one source to the tail of these edges. So we can remove the edges we added in parallel to the edges of \(\mathcal{B}\) and the new resulting network \(\mathcal{N}^{**}\) will still be linearly solvable over \(F\). Then by Lemma 10 its reverse network is also linearly solvable over \(F\). But by the same argument, this reverse network remains linearly solvable over \(F\) even after removing the remaining extra edges in parallel to the edges in \(\mathcal{A}\). So, again by Lemma 10 the original network \(\mathcal{N}\) itself is linearly solvable over \(F\). The above arguments also hold word by word if “linearly solvable” is replaced by “XOR solvable”. This completes the proof.

**Proof of Lemma 2**

First communicate \(x_1 + x_2 + x_3\) to \(t_1, t_2\) by XOR coding, which is possible by Lemma 8. Let \(\mathcal{N}_1\) be the sub-network used for this code. Now let \(P_1, P_2, P_3\) be some \((s_1, t_3)\), \((s_2, t_3)\), \((s_3, t_3)\) paths respectively and let \(\mathcal{N}_2\) be the sub-network consisting of them. We can simultaneously communicate \(x_1 + x_2 + x_3\) to \(t_3\) by XOR coding over \(\mathcal{N}_2\) for the following reason. Any edge \(e \in \mathcal{N}_1 \cap \mathcal{N}_2\) has paths to at least two terminals: \(t_3\) and at least one of \(t_1, t_2\). By the hypothesis of the lemma, there is a path from exactly one source, say \(s_1\) (w.l.o.g.), to \(t_1\). Thus \(e\) essentially carries only \(x_1\) in the coding scheme over \(\mathcal{N}_1\), as well as in the coding scheme over \(\mathcal{N}_2\). Hence there is no conflict between the coding schemes over \(\mathcal{N}_1\) and \(\mathcal{N}_2\) and both the codes can be simultaneously implemented. Thus the network is linearly solvable over any field in this case.

**Proof of Lemma 3**

W.l.o.g. we assume that the network satisfies Assumption 1.
Observation 1: (i) By Assumption \( \square \) \( s_3 \rightarrow \Gamma_{h(c)} \cup \Gamma^t_{t_1,t_2} \).
(ii) Similarly, \( \Gamma^t_{t(e)} \cup \Gamma^{s_1,s_2}_{h(c)} \rightarrow t_3 \).

Observation \( \square \) implies the following.

Observation 2: (i) Observation \( \square ii \) implies that no \( (s_1,t_3) \) path contains any node from \( \Gamma^t_{l(e)} \cup \Gamma^t_{t(e)} \cup \Gamma^t_{t(e)} \).
(ii) Observation \( \square ii \) implies that no \( (s_2,t_3) \) path contains any node from \( \Gamma^{s_i}_{h(c)} \cup \Gamma^{s_i}_{h(c)} \cup \Gamma^{s_i}_{h(c)} \).
(iii) From Observation \( \square i \) and (ii), we have \( s_3 \rightarrow \Gamma_{s_3} \cup \Gamma^{s_2}_{l(e)} \) and \( \{t(e)\} \cup \Gamma^t_{t(e)} \rightarrow t_3 \). Together, they imply that no \( (s_3,t_3) \) path contains any node from \( \Gamma^{s_1}_{h(c)} \cup \Gamma^{s_2}_{l(e)} \cup \{t(e)\} \cup \Gamma^t_{t(e)} \).
(iv) Observation \( \square i \) implies that no \( (s_3,t_1) \) or \( (s_3,t_2) \) path contains any node from \( \Gamma^{s_1}_{h(c)} \cup \Gamma^{s_2}_{l(e)} \) and \( \{t(e)\} \cup \Gamma^t_{t(e)} \).

Let us denote the subnetwork obtained by taking all \( (s_1,t_3), (s_2,t_3), (s_3,t_3), (s_3,t_1) \) and \( (s_3,t_2) \) paths by \( \mathcal{N} \). This subnetwork contains all edges \( e' \) such that either \( e' \rightarrow t_3 \) or \( s_3 \rightarrow e' \).

Observation 3: Observation \( \square \) above implies that irrespective of the coding used on \( \mathcal{N} \), we can still communicate \( x_1 + x_2 \) over \( e \) by passing \( x_1 \) and \( x_2 \) through any chosen \( (s_1,t(e)) \) and \( (s_2,t(e)) \) paths respectively. That is, \( x_1 + x_2 \) can be passed on edge \( e \) without putting any constraint on the coding on the subnetwork \( \mathcal{N} \).

Let \( \mathcal{P}(s_3,t_1) = \{P_1,P_2,\ldots\} \) be the set of all \( (s_3,t_1) \) paths. For any \( P_i \in \mathcal{P}(s_3,t_1) \), let \( z_i \) denote the first descendant of \( h(c) \) on this path. The existence of \( z_i \) is ensured by the fact that \( t_1 \) is a descendant of \( h(c) \), and is on \( P_i \). Similarly, let \( \mathcal{Q}(s_3,t_2) = \{Q_1,Q_2,\ldots\} \) be the set of all \( (s_3,t_2) \) paths, and for any \( Q_j \in \mathcal{Q}(s_3,t_2) \), let \( y_j \) denote the first descendant of \( h(c) \) on this path.

Observation 4: By Assumption \( \square \) \( \forall i,j, z_i \Rightarrow \{t_2,t_3\} \) and \( y_j \Rightarrow \{t_1,t_3\} \).

We consider the following two cases:

Case 1: For any \( P_i \in \mathcal{P}(s_3,t_1) \), removing all the edges on \( P_i(s_3 : z_i) \) disconnects \( (s_1,t_3) \) and/or \( (s_2,t_3) \).

Removing all the edges of any single \( (s_3,t_1) \) path cannot disconnect both \( (s_1,t_3) \) and \( (s_2,t_3) \), since otherwise this \( (s_3,t_1) \) path will contain a node \( v \) such that \( (s_1,s_2,s_3) \rightarrow v \rightarrow (t_1,t_3) \), thus contradicting Assumption \( \square \). So we consider the following three cases under Case 1: Case 1.1: the removal of any path \( P_i \) disconnects only \( (s_2,t_3) \), Case 1.2: the removal of any path \( P_i \) disconnects only \( (s_1,t_3) \), and Case 1.3: the removal of some of the paths disconnects only \( (s_1,t_3) \) and the removal of any of the others disconnects only \( (s_2,t_3) \).

Case 1.1: For any \( P_i \in \mathcal{P}(s_3,t_1) \), removing all the edges on \( P_i(s_3 : z_i) \) disconnects \( (s_2,t_3) \) but not \( (s_1,t_3) \).

Observation 5: By Assumption \( \square \) we have for this case,

(i) Any \( (s_1,t_3) \) path is node-disjoint from any \( P_i \in \mathcal{P}(s_3,t_1) \), since otherwise \( \Gamma^{s_1,s_2,s_3}_{t_1,t_3} \neq \emptyset \).
(ii) Any \( Q_j \in \mathcal{Q}(s_3,t_2) \) shares only those nodes with \( P_i \in \mathcal{P}(s_3,t_1) \) which are not descendants of \( s_2 \), since otherwise \( \Gamma^{s_2,s_3}_{t_1,t_2,t_3} \neq \emptyset \).
(iii) Any \( Q_j \in \mathcal{Q}(s_3,t_2) \) is node-disjoint from any \( (s_2,t_3) \) path, since otherwise \( \Gamma^{s_2,s_3}_{t_1,t_2,t_3} \neq \emptyset \).

Since for any \( i \), removing all the edges on \( P_i \) disconnects \( (s_2,t_3) \), and no single edge in the network disconnects both \( (s_2,t_3) \) and \( (s_3,t_1) \) (by hypothesis (b) of the lemma), by Lemma \( \square \) we can transmit \( x_2 + x_3 \) to \( t_3 \) and \( x_3 \) to \( t_1 \). The sub-network (say \( \mathcal{N}_1 \)) used for this purpose is a grail with either even or odd number of handles like those in Fig. \( \square \) and w.l.o.g., let us assume that the \( (s_3,t_1) \) path taking part in the grail is \( P_1 \). By this coding (on the grail), \( z_1 \) receives \( x_3 \) and \( t_3 \) receives \( x_2 + x_3 \). We now have the following two sub-cases under Case 1.1:

Case 1.1.1: For some path in \( \mathcal{Q}(s_3,t_2) \), say \( Q_1 \), removing all the edges on \( Q_1(s_3 : y_1) \) does not disconnect \( (s_1,t_3) \).

By Observations \( \square ii \) and \( \square iii \), \( Q_1 \) is node-disjoint from the grail \( \mathcal{N}_1 \) except at the dark-shaded part shown in Fig. \( \square \). Since this dark-shaded part also carries \( x_3 \), we can transmit \( x_3 \) on \( Q_1(s_3 : y_1) \) without affecting the coding on grail \( \mathcal{N}_1 \). Further, by
Observations 2(ii), (iii), (iv) and Observation 3 we can simultaneously communicate $x_1 + x_2$ to $z_1$ and $y_1$ via $e$. Then $t_1$ and $t_2$ get $x_1 + x_2 + x_3$ from $z_1$ and $y_1$ respectively. By the hypothesis of Case 1.1.1, there exists a $(s_1, t_3)$ path, say $P(s_1, t_3)$, which is edge-disjoint from $Q_1$. By Observation 5(i), $P(s_1, t_3)$ is also node-disjoint from the grail $N_1$ except at the light-shaded part shown in Fig. 7 which carries $x_2 + x_3$. This and Observations 2(i), and 3 imply that we can now simultaneously transmit $x_3$ along $P(s_1, t_3)$ till it meets the grail $N_1$ without conflicts in the existing coding. The first node in the light-shaded part of $N_1$ which is also on $P(s_1, t_3)$ can clearly compute $x_1 + x_2 + x_3$ and communicate this to $t_3$. This completes the proof for Case 1.1.1. As an illustrative example, in Fig. 8 we show the complete XOR coding solution for the case where the grail $N_1$ is the one in Fig. 7(b).

**Fig. 7.** The coding on grail $N_1$ for Case 1.1 of Lemma 3

**Fig. 8.** An illustration of the coding scheme for Case 1.1.1 of Lemma 3

**Case 1.1.2:** For any $Q_j \in \mathcal{Q}(s_3, t_2)$, removing all the edges on $Q_j(s_3 : y_j)$ disconnects $(s_1, t_3)$.

Since for any $j$, removing all the edges on $Q_j$ disconnects $(s_1, t_3)$, and no single edge in the network disconnects both $(s_1, t_3)$ and $(s_3, t_2)$ (by hypothesis (b) of the lemma), by Lemma 11 we can transmit $x_1$ to $t_3$ and $x_3$ to $t_2$. The sub-network
Because of Observation 5, the grails $N_1$ and $N_2$ can only intersect in the following possible ways: (i) The dark-shaded part of $N_1$ intersects with the dark-shaded part of $N_2$ and/or (ii) The light-shaded part of $N_1$ intersects with the light-shaded part of $N_2$. The remaining parts of the grails are node-disjoint. Now the dark-shaded parts of both the grails carry $x_3$, so such an intersection does not cause any conflict. As for the intersection between the light-shaded parts of the two grails, it can be easily worked out that in all possible cases, $t_3$ can easily recover $x_1 + x_2 + x_3$ by XOR coding. (One can check that the sub-network formed by the intersection in the light-shaded parts enables communication of $x_1 + x_2 + x_3$ to $t_3$ basically as the sum of some of the inputs to that part - and this is always feasible in a connected 1-terminal network.) By the inferences in Observations 2,3, we can simultaneously communicate $x_1 + x_2$ to $z_1$ and $y_2$ via $e$. Then $t_1$ and $t_2$ get $x_1 + x_2 + x_3$ from $z_1$ and $y_2$ respectively. This completes the proof for Case 1.1.2. As an illustrative example, in Fig. 10, we show the complete XOR coding solution for the case where the grail $N_1$ is the one in Fig. 7(b) and the grail $N_2$ is the one in Fig. 9(b) and their light-shaded parts and dark-shaded parts intersect as shown in the figure.

Fig. 10. An illustration of the coding scheme for Case 1.1.2 of Lemma 3
Case 1.2: For any $P_i \in \mathcal{P}(s_3, t_1)$, removing all the edges on $P_i(s_3 : z_i)$ disconnects $(s_1, t_3)$ but not $(s_2, t_3)$.

This is the symmetric counterpart of Case 1.1, and the proof is skipped.

Case 1.3: There exist paths $P_1, P_2 \in \mathcal{P}(s_3, t_1)$ such that removing all the edges on $P_1(s_3 : z_1)$ disconnects $(s_1, t_3)$ and removing all the edges on $P_2(s_3 : z_2)$ disconnects $(s_2, t_3)$.

By the hypothesis of the case, any $(s_1, t_3)$ (resp. $(s_2, t_3)$) path shares common edges with $P_1$ (resp. $P_2$) (the reader may like to keep Fig. 11 in mind). If any such $(s_1, t_3)$ (resp. $(s_2, t_3)$) path shares nodes with $P_2$ (resp. $P_1$), then $P_2$ (resp. $P_1$) has a node $v$ s.t. $\{s_1, s_2, s_3\} \rightarrow v \rightarrow \{t_1, t_3\}$, i.e., $\Gamma_{t_1, t_3}^{s_1, s_2, s_3} \neq \emptyset$, which contradicts Assumption 1. So this is not the case.

Since the network is connected, there is a $(s_3, t_2)$ path, say $Q_1$. So, under this case, we have a subnetwork as shown in Fig. 11. Now, again by Assumption 1, one can easily verify that $Q_1$ does not share a node with the rest of the subnetwork except on the path-segments $P_1(s_3 : v_1)$ above $v_1$, $P_2(s_2 : v_2)$ above $v_2$ and the $(h(e), t_2)$ path-segment below $h(e)$. So the coding scheme shown in Fig. 11 completes the proof of this case. In particular, $t_1$ and $t_3$ use $x_1$ obtained from $P_1$ and $x_2 + x_3$ obtained from $P_2$ to get $x_1 + x_2 + x_3$, while $t_2$ uses $x_3$ obtained from $Q_1$ and $x_1 + x_2$ obtained from $e$ to get $x_1 + x_2 + x_3$.

**Fig. 11.** The sub-network and code for Case 1.3 of Lemma 3

Case 2: There exists some path in $\mathcal{P}(s_3, t_1)$, say $P_1$, such that removing all the edges on $P_1(s_3 : z_1)$ does not disconnect either $(s_1, t_3)$ or $(s_2, t_3)$.

We can have the following sub-cases under Case 2:

Case 2.1: For any $Q_j \in \mathcal{Q}(s_3, t_2)$, removing all the edges on $Q_j(s_3 : y_j)$ disconnects $(s_1, t_3)$ and/or $(s_2, t_3)$.

This subcase statement is the symmetric counterpart of the Case 1 statement. Case 2.1 is thus a special case (because of the additional constraint in the Case 2 statement) of that symmetric counterpart and so the proof follows in a similar way.

Case 2.2: For some path in $\mathcal{Q}(s_3, t_2)$, say $Q_1$, removing all the edges on $Q_1(s_3 : y_1)$ does not disconnect $(s_1, t_3)$ or $(s_2, t_3)$.

This is considered in two further sub-cases:

Case 2.2.1: Removing all the edges on the pair of paths $P_1(s_3 : z_1)$ and $Q_1(s_3 : y_1)$ simultaneously does not disconnect $(s_1, t_3)$ or $(s_2, t_3)$.

By the hypothesis of Case 2.2.1, there exist $(s_1, t_3)$ and $(s_2, t_3)$ paths, called respectively $P(s_1, t_3)$ and $P(s_2, t_3)$, which are edge-disjoint from both $P_1$ and $Q_1$. Consider a lowest node $v$ in ancestral order in the set $\Gamma_{t_3}(P_1 \cup Q_1)$. Such a node is above $z_1$ or $y_1$ by Observation 1. W.l.o.g., let us assume that $v$ is on $P_1$, and $P'$ is a $(v, t_3)$ path. Now, $P(s_3, t_3) = P_1(s_3 : v)P'$ is a $(s_3, t_3)$ path. By the hypothesis of Case 2.2.1, one can communicate $x_1 + x_2 + x_3$ to $t_3$ via XOR coding on $P(s_1, t_3), P(s_2, t_3)$ and $P(s_3, t_3)$, while simultaneously communicating $x_3$ on $P_1(s_3 : z_1)$ and $Q_1(s_3 : y_1)$. Further, by Observations 2 and 3 one
can also communicate $x_1 + x_2$ to $z_1$ and $y_1$ via $e$. Nodes $z_1$ and $y_1$ can then recover $x_1 + x_2 + x_3$ and transmit this to $t_1$ and $t_2$ respectively.

**Case 2.2.2:** Removing all the edges on the pair of paths $P_1(s_3 : z_1)$ and $Q_1(s_3 : y_1)$ simultaneously disconnects $(s_1, t_3)$ or $(s_2, t_3)$ or both.

![Diagram](image.png)

**Fig. 12.** The sub-network and code for Case 2.2.2 Lemma 4

W.l.o.g., let us assume that removing all the edges on $P_1(s_3 : z_1)$ and $Q_1(s_3 : y_1)$ simultaneously disconnects $(s_1, t_3)$. By the hypothesis of the case, in the network formed by removing all the edges on $P_1(s_3 : z_1)$, removing all the edges on $Q_1(s_3 : y_1)$ disconnects $(s_1, t_3)$. Hence there exists a $(s_1, t_3)$ path $P'$ which shares edges with $Q_1(s_3 : y_1)$. By Assumption 4, $P'$ is node-disjoint from $P_1$ and $Q_1(y_1 : t_2)$ since otherwise, $\Gamma^1_{e_1, t_2, t_3} \neq \emptyset$. By similar reasoning, there exists a $(s_1, t_3)$ path $P''$ which shares edges with $P_1(s_3 : z_1)$ but is node-disjoint from $Q_1$ and $P_1(z_1 : t_1)$. This and Observation 2(ii), (iii), (iv) imply that there exists a subnetwork as shown in Fig. 12. Here $P'''$ is any $(s_2, t_3)$ path. By Assumption 4 and Observation 2(i), it can be verified that $P'''$ does not share any node with the rest of the sub-network except on the path segments $P'(v' : t_3)$ below $v'$, $P''(v'' : t_3)$ below $v''$, and the $(s_2, t(e))$ path segment above $t(e)$. So the coding scheme shown in Fig. 12 completes the proof of this case. (The reader may note that the subnetwork in Fig. 12 is actually the reverse network of the one in Fig. 11)

**Proof of Lemma 4**

Let $\mathscr{N}$ satisfy the hypotheses of Lemma 4 and consider the edges $e_1, e_2$ and the labeling of the sources and terminals with which hypothesis (2) of Lemma 4 is satisfied. By hypothesis 2(d), the set $\mathcal{R}(s_3, t_3) = \{R_1, R_2, \ldots\}$ of all $(s_3, t_3)$ paths that do not contain either $e_1$ or $e_2$ is not empty. W.l.o.g., let $\mathcal{N}$ satisfy Assumption 4. Because $e_1, e_2$ satisfy hypothesis (2), and because no path in $\mathcal{R}(s_3, t_3)$ contains either of them, we have

**Observation 6:** No $(s_3, t_3)$ path in $\mathcal{R}(s_3, t_3)$ contains a node $v$ such that $s_1 \rightarrow v$ or $v \rightarrow t_j, j \in \{1, 2\}$ i.e. $s_1, s_2, t_1, t_2$ are not connected to any path in $\mathcal{R}(s_3, t_3)$.

This means that as shown in Fig. 13 any $R_1 \in \mathcal{R}(s_3, t_3)$ does not share any node with the rest of the sub-network except on the $(s_3, t(e_1))$ path-segment above $t(e_1)$, the $(s_3, t(e_2))$ path segment above $t(e_2)$, the $(h(e_1), t_3)$ path segment below $h(e_1)$ and the $(h(e_2), t_3)$ path segment below $h(e_2)$. Let $\mathcal{P}(s_1, t_2) = \{P_1, P_2, \ldots\}$ be the set of all $(s_1, t_2)$ paths and let $\mathcal{Q}(s_2, t_1) = \{Q_1, Q_2, \ldots\}$ be the set of all $(s_2, t_1)$ paths.

**Observation 7:** By Assumption 4

(i) No $(s_1, t_2)$ path contains any node from $\Gamma^s_{t(e_1)} \cup \Gamma^s_{t(e_2)} \cup \Gamma^s_{t(e_3)} \cup \Gamma^t_{t(e_4)} \cup \{t(e_1)\} \cup \{t(e_2)\} \cup \Gamma^t_{t(e_4)} \cup \Gamma^t_{t(e_5)}$.

(ii) No $(s_2, t_1)$ path contains any node from $\Gamma^s_{t(e_1)} \cup \Gamma^s_{t(e_2)} \cup \Gamma^s_{t(e_3)} \cup \Gamma^s_{t(e_4)} \cup \{t(e_1)\} \cup \{t(e_2)\} \cup \Gamma^t_{t(e_4)} \cup \Gamma^t_{t(e_5)}$.

Observation 6 gives,
For any \( P \) is as shown in Fig. 5 but with \( x \) the three incoming paths meet, this terminal can always recover \( s_1 \) node with the rest of the subnetwork except on the

Fig. 13. The sub-network and code for Case 1 of Lemma 4

(iii) any path in \( \mathcal{P}(s_3, t_3) \) is node-disjoint from any path in \( \mathcal{P}(s_1, t_2) \) or \( \mathcal{P}(s_2, t_1) \).

For any \( P_i \in \mathcal{P}(s_1, t_2) \) let \( z_i \) be the first descendant of \( e_2 \) on \( P_i \). Similarly, for any \( Q_j \in \mathcal{Q}(s_2, t_1) \) let \( y_j \) be the first descendant of \( e_1 \) on \( Q_j \). We consider two cases:

\[ \text{Case 1: There exist } P_1 \in \mathcal{P}(s_1, t_2) \text{ and } Q_1 \in \mathcal{Q}(s_2, t_1) \text{ such that } P_1 \text{ and } Q_1 \text{ are edge-disjoint.} \]

In this case there exists a subnetwork shown in Fig. 14(a) because, by Observation 7, we have that (i) \( P_1 \) does not share any node with the rest of the subnetwork except on the \((s_1, t(e_1))\) path segment above \( t(e_1) \) and the \((h(e_2), t_2)\) path segment below \( h(e_2) \), and (ii) \( Q_1 \) does not share any node with the rest of the subnetwork except on the \((s_2, t(e_2))\) path segment above \( t(e_2) \) and the \((h(e_1), t_1)\) path segment below \( h(e_1) \). The XOR coding scheme shown in Fig. 14(a) completes the proof. The shaded rectangle containing \( t_3 \) is drawn to mean that irrespective of the order in which the three incoming paths (carrying \( x_1 + x_3 \), \( x_3 \) and \( x_2 + x_3 \)) meet, \( t_3 \) can always recover \( x_1 + x_2 + x_3 \) by XOR coding.

\[ \text{Case 2: For any } P_i \in \mathcal{P}(s_1, t_2) \text{ removing all the edges of } P_i(s_1, z_i) \text{ disconnects } (s_2, t_1). \]

\[ \text{Case 2.1: There does not exist a single edge which disconnects both } (s_1, t_2) \text{ and } (s_2, t_1). \]

By Lemma 11 we can use a grail subnetwork \( N_1 \) to transmit \( x_1 \) to \( t_2 \) and \( x_2 \) to \( t_1 \). This grail network \( N_1 \) and its coding is as shown in Fig. 5 but with \( t_1 \) and \( t_2 \) interchanged. The situation then is as shown in Fig. 14(b) Here the details of the grail \( N_1 \) are suppressed for clarity and it is represented by a shaded region. By Observation 6, \( N_1 \) is node-disjoint from \( e_1, e_2, R_1 \) and the \((s_3, t(e_1)), (s_3, t(e_2)), (h(e_1), t_3) \) and \((h(e_2), t_3)\) path segments (which are shown with thick edges in the figure). The coding scheme (shown in the figure) where \( e_1 \) is used to communicate \( x_1 + x_3 \) to \( t_1 \), \( t_3 \), \( e_2 \) is used to communicate \( x_2 + x_3 \) to

Fig. 14. The coding for the different cases under Lemma 4. The shaded rectangle containing a terminal is drawn to mean that irrespective of the order in which the three incoming paths meet, this terminal can always recover \( x_1 + x_2 + x_3 \) by XOR coding.
$t_2, t_3$, $R_1$ is used to communicate $x_3$ to $t_3$ and the grail $N_1$ used to communicate $x_1$ to $t_2$ and $x_2$ to $t_1$ completes the proof for this case.

**Case 2.2:** There exists an edge $e_3$ which disconnects both $(s_1, t_2)$ and $(s_2, t_1)$.

In this case we will show the existence of a subnetwork as shown in Fig. 14(c). It is easy to see by Assumption 1 that, $e_1 \not\leftrightarrow e_3, e_2 \not\leftrightarrow e_3, e_3 \not\leftrightarrow e_1, e_3 \not\leftrightarrow e_2$. Now, removing the pair $e_1, e_3$ does not disconnect $(s_1, t_1)$ since otherwise $N$ would satisfy the hypothesis of Theorem 2 for some labelling of the sources and terminals. For the same reason, removing the pair $e_2, e_3$ does not disconnect $(s_2, t_2)$. Hence there exists a $(s_1, t_1)$ path $R_1^*$ not containing $e_1$ or $e_3$ and a $(s_2, t_2)$ path $R_2^*$ not containing $e_2$ or $e_3$.

Because of the conditions that $e_1, e_2, e_3$ satisfy, we have (i) there is no node $v$ on $R_1^*$ satisfying either $s_2 \rightarrow v, s_3 \rightarrow v, v \rightarrow t_2$ or $v \rightarrow t_3$ and (ii) there is no node $v$ on $R_2^*$ satisfying either $s_1 \rightarrow v, s_3 \rightarrow v, v \rightarrow t_1$ or $v \rightarrow t_3$.

This and Observation 7 imply the existence of the subnetwork subnetwork shown in Fig. 14(c) such that

(i) $R_1^*$ does not share any node with the rest of the subnetwork except on the $(s_1, t(e_1))$ path segment above $t(e_1)$, the $(s_1, t(e_3))$ path segment above $t(e_3)$, the $(h(e_1), t_1)$ path segment below $h(e_1)$, and the $(h(e_3), t_1)$ path segment below $h(e_3)$; and

(ii) $R_2^*$ does not share any node with the rest of the subnetwork except on the $(s_2, t(e_2))$ path segment above $t(e_2)$, the $(s_2, t(e_3))$ path segment above $t(e_3)$, the $(h(e_2), t_2)$ path segment below $h(e_2)$ and the $(h(e_3), t_2)$ path segment below $h(e_3)$.

The XOR code shown in Fig. 14(c) completes the proof.

**Proof of Lemma 5**

We will only prove part A of Lemma 5. Since our proof is constructive, the proof of the other parts will follow from the coding solutions offered in the proof of part A.

If $\kappa(N) \geq 5$, $N$ is XOR solvable over any field by Lemma 9. Hence in the remaining part of this proof, we only consider networks with $\kappa(N) = 0, 1, 2$ and 4 and prove Lemma 5 for each value. In the light of Lemma 9 it is enough to prove Lemma 5 for networks satisfying Assumption 1. Then if $\mathcal{C} \neq \emptyset$, it only contains maximum-disconnecting edges such that there is a path from exactly two sources to its tail and there is a path from its head to exactly two terminals.

- $\kappa(N) = 0$: In this case, there exist two edge-disjoint paths between each source-terminal pair. The main result of [23] is that such a sum-network is solvable over fields of odd characteristic. In the following, we present a significantly different proof which also gives a stronger result that such a network is solvable over any field by a XOR code.

  We consider two cases depending on whether or not $\mathcal{C} = \emptyset$:

  **Claim 1:** Sum-networks with $\kappa = 0$ and $\mathcal{C} = \emptyset$ are XOR solvable over any field.

  **Proof:** The proof follows by Lemma 2.

  **Claim 2:** Sum-networks with $\kappa = 0$ and $\mathcal{C} \neq \emptyset$ are XOR solvable over any field.

  **Proof:** The proof follows by Lemma 3 since hypothesis (a) of the lemma follows from $\mathcal{C} \neq \emptyset$ with suitable labeling of the sources and the terminals and hypothesis (b) follows from $\kappa = 0$.

- $\kappa(N) = 1$: We consider two cases depending on whether or not $\mathcal{C} = \emptyset$:

  **Claim 3:** Sum-networks with $\kappa = 1$ and $\mathcal{C} = \emptyset$ are XOR solvable over any field.

  **Proof:** The proof follows from Lemma 4 since we have proved that networks with $\kappa = 0$ are XOR solvable over any field.

  **Claim 4:** Sum-networks with $\kappa = 1$ and $\mathcal{C} \neq \emptyset$ are XOR solvable over any field.

  **Proof:** The proof follows by Lemma 5 since hypothesis (a) of the lemma follows from $\mathcal{C} \neq \emptyset$ with suitable labeling of the sources and the terminals and hypothesis (b) follows from $\kappa = 1$. 

Proof of \(\kappa(N) = 2\): We will prove that networks with \(\kappa = 2\) are linearly solvable. But solvability in this case is not necessarily over \(F_2\), and even over other fields, the solvability may not be by XOR coding. Specifically, this happens only in Case 1.2 under \(\mathcal{C} \neq \emptyset\) (Claim 6).

We consider two cases depending on whether or not \(\mathcal{C} = \emptyset\):

**Claim 5:** Sum-networks with \(\kappa = 2\) and \(\mathcal{C} = \emptyset\) are XOR solvable over any field.

**Proof:** The proof follows from Lemma 1 since we have proved that networks with \(\kappa = 0, 1\) are XOR solvable over any field.

**Claim 6:** Sum-networks with \(\kappa = 2\) and \(\mathcal{C} \neq \emptyset\) are linearly solvable over all fields except possibly \(F_2\).

**Proof:** Let us assume that \(e \in \mathcal{C}\) and that \(\{s_1, s_2\} \rightarrow e \rightarrow \{t_1, t_2\}\).

Now \(e\) can disconnect two source-terminal pairs in essentially three different ways. It can disconnect either \((s_1, t_1)\) and \((s_2, t_2)\), \((s_1, t_1)\) and \((s_2, t_1)\), or \((s_1, t_1)\) and \((s_1, t_2)\). We consider each case in turn.

**Case 1:** Edge \(e\) disconnects \((s_1, t_1)\) and \((s_2, t_2)\).

**Case 1.1:** There does not exist an edge disconnecting either \((s_2, t_3)\) and \((s_3, t_1)\); or \((s_1, t_3)\) and \((s_3, t_2)\).

The network is easily seen to satisfy the hypotheses of Lemma 3 and is thus XOR solvable over any field.

**Case 1.2:** There exists an edge \(e'\) disconnecting either \((s_2, t_3)\) and \((s_3, t_1)\); or \((s_1, t_3)\) and \((s_3, t_2)\).

**Note:** The networks shown in Fig. 3 fall under this case, and they are not solvable over \(F_2\) but linearly solvable over any other field though not by XOR coding [22].

In this case, \(e, e'\) satisfy conditions 1, 2 in Theorem 2 for a suitable relabeling of the sources since \(\kappa = 2\), and condition 4 by Assumption 1. Thus, if there does not exist an edge pair satisfying all the four conditions in Theorem 2 then by taking \(e, e'\) as \(e_1, e_2\) under a suitable relabeling of the sources and terminals, the hypotheses of Lemma 4 are satisfied, and the network is XOR solvable over any field. If on the other hand, there exists an edge pair satisfying all the four conditions in Theorem 2 then by the sufficiency part of Theorem 2 (proved independently later, see the dependency graph in Fig. 4), the network is not solvable over \(F_2\) but linearly solvable over all other fields.

**Case 2:** Edge \(e\) disconnects \((s_1, t_1)\) and \((s_2, t_1)\)

We assume that there is no maximum-disconnecting edge disconnecting \((s_i, t_j)\) and \((s_i', t_{j'})\), \(i \neq i', j \neq j'\) for any labeling of the sources and terminals, since otherwise such an edge satisfies Case 1 and the proof follows from the proof of that case.

So there does not exist another edge \(e'\) which disconnects \((s_2, t_3)\) and \((s_3, t_1)\); or \((s_1, t_3)\) and \((s_3, t_2)\). Thus the hypotheses of Lemma 3 are satisfied, and the network is linearly solvable over any field using XOR coding in this case.

**Case 3:** Edge \(e\) disconnects \((s_1, t_1)\) and \((s_1, t_2)\)

In this case, the reverse network falls under Case 2, and is thus XOR solvable over any field. The proof then follows by Lemma 10.

**\(\kappa(N) = 4\):** We consider three cases:

**Case 1:** A maximum-disconnecting edge \(e\) disconnects one terminal from all the sources, and another terminal from a single source.

This case can not occur under Assumption 1 since then that edge would be connected to all the sources and two terminals.

**Case 2:** A maximum-disconnecting edge \(e\) disconnects one source from all the terminals, and another source from a single terminal.

This case can not occur under Assumption 1 since then that edge would be connected to two sources and all the three terminals.

**Case 3:** A maximum-disconnecting edge \(e\) disconnects both \(s_1\) and \(s_2\) from both \(t_1\) and \(t_2\).
Under the given labeling of the sources and terminals, hypothesis (a) of Lemma 5 is satisfied by \( e \). We now show that hypothesis (b) is also satisfied. If this was not so, let there exist an edge \( e' \) which disconnects (w.l.o.g.) \((s_2, t_3)\) and \((s_3, t_1)\). This implies that \( s_2 \to e' \to t_1 \) i.e. \( s_2 \to t_1 \). But since \( e \) disconnects \((s_2, t_1)\), we must have \( e \to e' \) (or \( e' \to e \)). Then it is easy to check that \( s_1, s_2 \to h(e) \to t_1, t_2, t_3 \) (or resp. \( s_2, s_3 \to h(e') \to t_1, t_2, t_3 \)), which violates Assumption 1. Hence hypothesis (b) of Lemma 5 is also satisfied. Then the network is XOR solvable over any field by the lemma.

**Proof of Lemma 6**

Part A of the lemma follows from the sufficiency part of Theorem 2 (proved later independently, see Fig. 4).

Now we prove part B of the lemma. The following observation sums up some of the things we have already proved, and which we will draw upon.

Observation 8: (i) If \( \kappa(\mathcal{N}) = 0, 1, \) or \( \geq 4 \), then by Lemma 5 the network is XOR solvable over any field.

(ii) If \( \kappa(\mathcal{N}) = 2 \), only under Case 1.2 of \( \kappa = 2 \) in the proof of Lemma 5 the network may not be solvable over \( F_2 \). In all the other cases, the network is XOR solvable over any field. Further, networks under Case 1.2 were shown to be of two types, namely, they either satisfied the hypothesis of Theorem 2 and were not solvable over \( F_2 \) but linearly solvable over other fields, or, they did not satisfy the hypothesis of Theorem 2 and were XOR solvable over any field.

For networks in part B of the lemma, consider the network \( \mathcal{N}^* \) obtained by adding parallel edges to the edges in \( \mathcal{C} \) as in the proof of Lemma 1. Now, \( \kappa(\mathcal{N}^*) \leq 2 \), so, as inferred in Observation 8, \( \mathcal{N}^* \) is either XOR solvable over all fields or \( \mathcal{N}^* \) is a \( \kappa = 2 \) network having an edge pair satisfying conditions 1-4 of Theorem 2 and thus is not solvable over \( F_2 \) but linearly solvable over other fields. We will show that \( \mathcal{N}^* \) does not contain such an edge pair and, thus, is XOR solvable over all fields. Then, as was shown in the proof of Lemma 1, \( \mathcal{N} \) too will be XOR solvable over all fields, thereby proving the lemma.

Suppose this was not true, and \( \mathcal{N}^* \) has \( e_1, e_2 \) satisfying conditions 1-4 of Theorem 2. Since we are only adding parallel edges in the process of constructing \( \mathcal{N}^* \) from \( \mathcal{N} \), \( e_1, e_2 \) satisfy conditions 3 and 4 of Theorem 2 in \( \mathcal{N} \) itself. Further, the only way \( e_1 \) (resp. \( e_2 \)) could dissatisfy condition 1 (resp. 2) of Theorem 2 in \( \mathcal{N} \) itself would be if it also disconnected an additional source-terminal pair in \( \mathcal{N} \), which will mean that \( e_1 \) (resp. \( e_2 \)) is in \( \mathcal{C} \), thus contradicting \( \mathcal{C} = \emptyset \). Thus \( e_1, e_2 \) satisfy conditions 1-4 of Theorem 2 in \( \mathcal{N} \) itself. This gives a contradiction.

**Proof of Lemma 7**

Let \( \mathcal{N} \) be a nonsolvable network with \( \kappa(\mathcal{N}) = 3 \) containing an edge \( e_2 \) satisfying conditions 3 and 4 of Theorem 1. We will show that the statement of Lemma 7 holds for \( \mathcal{N} \). We assume that \( \mathcal{N} \) satisfies Assumption 1 since otherwise the network is XOR solvable over all fields by Lemma 9.

We note that there can not exist an edge \( e' \) which disconnects \((s_1, t_2)\) and \((s_2, t_1)\) (or \((s_1, t_3)\) and \((s_2, t_1)\)) or \((s_3, t_1)\) - though we are not using these) since otherwise \( s_2 \to e' \to t_2 \) implies \( e' \to e_2 \) or \( e_2 \to e' \) (since \( e_2 \) disconnects \((s_2, t_2)\)), any of which contradicts Assumption 1.

Then, if there does not exist an edge \( e_1 \) that satisfies conditions 1 and 2 of Theorem 1, \( \mathcal{N} \) satisfies the hypotheses in Lemma 5, for a suitable relabeling of the sources and terminals and so is XOR solvable over any field. So an edge \( e_1 \) satisfying conditions 1 and 2 of Theorem 1 exists.

Further, \( s_1 \to h(e_2) \) & \( s_1 \to t(e_2) \) (by Assumption 1) & \( s_1 \to t(e_1) \Rightarrow e_1 \to e_2 \). Similarly, \( s_2 \to h(e_1) \) & \( s_2 \to t(e_1) \& s_2 \to t(e_2) \Rightarrow e_2 \to e_1 \).

Thus, we have so far proved that there exists \( e_1, e_2 \) satisfying conditions 1, 2, 3, 4, and 6 of Theorem 1.

Now, we argue that the only way an edge \( e' \) disconnecting exactly \((s_i, t_j)\) and \((s_{i'}, t_{j'})\), \( i \neq i', j \neq j' \) can exist in this network is if it disconnects exactly \((s_1, t_3)\) and \((s_3, t_1)\). We prove this using a sequence of four steps in the following.
(i) If \( \{i, i'\} = \{j, j'\} = \{1, 3\} \) is not true, then \( e_2 \rightarrow e' \) or \( e' \rightarrow e_2 \). This is because, if, w.l.o.g., \( i = 2 \), since one of \( j, j' \) has to be 2 or 3, there is a path from \( s_2 \) to \( t_2 \) or \( t_3 \) via \( e' \). Since \( e_2 \) disconnects \( (s_2, t_2) \) and \( (s_3, t_2) \), \( e' \) must be an ancestor or descendant of \( e_2 \).

(ii) If \( e' \) is a descendant or ancestor of \( e_2 \), then \( \{i, i'\} = \{j, j'\} = \{2, 3\} \) by Assumption I.

(iii) If \( e' \) is a descendant or ancestor of \( e_2 \), then \( e' \) can not disconnect exactly the source-terminal pairs \( (s_2, t_3) \) and \( (s_3, t_2) \) (or \( (s_2, t_2) \) and \( (s_3, t_3) \) -this case follows similarly, and will not be elaborated). Otherwise, after removing \( e' \), there exists a \( (s_2, t_2) \) path \( P \) containing \( e_2 \). If \( e' \) is an ancestor of \( e_2 \), then after removing \( e' \), \( P(s_2 : t(e_2)) \) concatenated with any \( (t(e_2), t_3) \) path gives a \( (s_2, t_3) \) path not containing \( e' \) and thus gives a contradiction. Similarly we can reach a contradiction if \( e' \) is a descendant of \( e_2 \).

(iv) If \( \{i, i'\} = \{j, j'\} = \{1, 3\} \), then \( e' \) can not disconnect exactly \( (s_1, t_1) \) and \( (s_3, t_3) \). This follows by similar arguments as in (iii) above by considering the edge \( e_1 \).

This proves that the only way an edge \( e' \) disconnecting exactly \( (s_1, t_2) \) and \( (s_i', t_{j'}), i \neq i', j \neq j' \) can exist in this network is if it disconnects exactly \( (s_1, t_3) \) and \( (s_3, t_1) \). Thus an edge pair satisfying conditions 1-4 of Theorem II does not exist in this network. So, the network satisfies condition 1 of Lemma IV. Now, if \( (e_1, e_2) \) do not satisfy condition 5 of Theorem II then since they satisfy conditions 1, 2, 3, 4, and 6 of Theorem II they also satisfy condition 2 in Lemma IV. Thus both the conditions in Lemma IV are satisfied and thus the network is XOR solvable over all fields. Hence for nonsolvability, \( (e_1, e_2) \) satisfy condition 5 of Theorem II as well.

**Proof of Theorem I:**

The major part of the proof is in two parts. In the Sufficiency part, we show that once conditions 1)-6) in part A are satisfied by two edges in a connected network, the network has the capacity 2/3 and is thus not solvable. This will prove part D as well as the sufficiency of part A of the theorem. In the necessity part of the proof, we will show that if a pair of edges satisfying conditions 1)-6) in part A does not exist, then the network is linearly solvable over any field. Parts B and C will be proved in parallel. The reader may find it useful to keep Fig. 2 in mind while going through the proof.

**Sufficiency:**

We will show that the capacity of a connected 3s/3t sum-network satisfying the hypothesis of Theorem I is 2/3 and thus is not solvable. It was proved in [24 Theorem 4] using time-sharing arguments that the coding capacity of any connected 3s/3t network is at least 2/3. Hence all we need to prove is that the capacity of a network satisfying conditions 1 – 6 of Theorem I is \( \leq 2/3 \). The idea of this proof is similar to that of [24 Theorem 6]. Suppose there is a \( (k, n) \) fractional coding solution for the network. That is, the messages at the sources are \( x_1, x_2, x_3 \in F^k \), the terminals recover the sum \( x_1 + x_2 + x_3 \in F^k \), and each edge in the network carries an element from \( F^n \). We allow non-linear coding. Let the symbols transmitted over \( e_1 \) and \( e_2 \) be denoted by \( Y_{e_1} \) and \( Y_{e_2} \) respectively. Let us add an edge \( e^*_2 \) from \( h_{(e_2)} \) to \( t_3 \) and an edge \( e^*_3 \) from \( h_{(e_3)} \) to \( t_3 \). Clearly this new network \( \mathcal{N}^* \) also satisfies the six conditions of Theorem I and is stronger than \( \mathcal{N} \). We show that the capacity of \( \mathcal{N}^* \) itself is bounded by 2/3.

Since \( \mathcal{N}^* \) is a connected sum-network and satisfies the hypothesis of Theorem I:

1. Conditions 1, 2 \( \Rightarrow \{s_1, s_3\} \rightarrow t_{(e_1)} \) and \( h_{(e_1)} \rightarrow \{t_1, t_3\} \).
2. Condition 3, 4 \( \Rightarrow \{s_2, s_3\} \rightarrow t_{(e_2)} \) and \( h_{(e_2)} \rightarrow \{t_2, t_3\} \).
3. Conditions 1, 3, 6 \( \Rightarrow s_1 \rightarrow t_{(e_2)}, s_2 \rightarrow t_{(e_1)} \).

By statement 3 above, \( Y_{e_1} \) is only a function of \( x_1 \) and \( x_3 \), but not of \( x_2 \); and \( Y_{e_2} \) is only a function of \( x_2 \) and \( x_3 \), but not of \( x_1 \). Let us denote them as \( Y_{e_1} = \phi(x_1, x_3) \) and \( Y_{e_2} = \psi(x_2, x_3) \).
Claim 7: (i) \( \phi(x_1, x_3) \) is a 1-1 function of \( x_3 \) for a fixed value of \( x_1 \) and a 1-1 function of \( x_1 \) for a fixed value of \( x_3 \). (ii) \( \psi(x_2, x_3) \) is a 1-1 function of \( x_2 \) for a fixed value of \( x_3 \) and a 1-1 function of \( x_3 \) for a fixed value of \( x_2 \).

Proof: We prove (i). The proof of (ii) is similar.

Since \( t_1 \) can recover \( x_1 + x_2 + x_3 \), for any fixed values of \( x_1 \) and \( x_2 \), the set of messages received by the terminal \( t_1 \) is a 1-1 function of \( x_3 \) as \( x_1 + x_2 + x_3 \) is a 1-1 function of \( x_3 \) for fixed \( x_1 \) and \( x_2 \). But by condition 2 of Theorem 1, all \((s_1, t_1)\) paths pass through \( e_1 \). Hence \( \phi(x_1, x_3) \) is a 1-1 function of \( x_3 \) for a fixed value of \( x_1 \).

Similarly, since \( t_3 \) can recover \( x_1 + x_2 + x_3 \), for any fixed values of \( x_2 \) and \( x_3 \), the set of messages received by the terminal \( t_3 \) is a 1-1 function of \( x_1 \) as \( x_1 + x_2 + x_3 \) is a 1-1 function of \( x_1 \) for fixed \( x_2 \) and \( x_3 \). But by condition 1 of Theorem 1, all \((s_1, t_3)\) paths pass through \( e_1 \). Hence \( \phi(x_1, x_3) \) is a 1-1 function of \( x_1 \) for a fixed value of \( x_3 \).

Claim 8: In \( \mathcal{N}^* \) the node \( t_3 \) can recover \( x_1, x_2 \) and \( x_3 \).

Proof: For a fixed \( x_1, x_1 + x_2 + x_3 \) is a 1 – 1 function of \( x_2 + x_3 \). Since \( t_2 \) recovers \( x_1 + x_2 + x_3 \), by condition 4, it implies that \( \psi(x_2, x_3) \) is a 1 – 1 function of \( x_2 + x_3 \). But since \( t_3 \) also gets \( \psi(x_2, x_3) \) via \( e^*_2 \), it can also recover \( x_2 + x_3 \).

Then by subtracting this from \( x_1 + x_2 + x_3 \), \( t_3 \) can get \( x_1 \). Then using \( x_1 \) and \( \phi(x_1, x_3) \), which it gets via \( e^*_1 \) and which is a 1-1 function of \( x_3 \) for fixed \( x_1 \), \( t_3 \) can recover \( x_3 \). As \( \psi(x_2, x_3) \) is a 1-1 function of \( x_2 \) for a fixed \( x_3 \), \( t_3 \) can recover \( x_2 \).

Hence \( t_3 \) can recover \( x_1, x_2 \) and \( x_3 \). \( \square \)

Now \( (x_1, x_2, x_3) \) takes \( |F|^{3k} \) possible values. On the other hand, by conditions 1, 3 and 5 of Theorem 1, \( \{(e_1), (e_2)\} \) is a cut between \( \{s_1, s_2, s_3\} \) and \( t_3 \) (even in \( \mathcal{N}^* \)), and this cut can carry at most \( |F|^{2n} \) possible different message-pairs. So \( |F|^{2n} \geq |F|^{3k} \Rightarrow k/n \leq 2/3 \). Thus the capacity of \( \mathcal{N}^* \) and hence of \( \mathcal{N} \) is bounded by \( 2/3 \). As this rate is achievable in \( \mathcal{N} \), the capacity of \( \mathcal{N} \) is exactly \( 2/3 \).

Necessity:

In this part, we will show that if a network does not satisfy the conditions 1)-6) in part A of the theorem, then the network is solvable. Parts B and C of the theorem will also be proved in parallel. In light of Lemma 5 and Lemma 6, we only need to concern ourselves with networks \( \mathcal{N} \) having \( \kappa(\mathcal{N}) = 3 \) and \( \mathcal{C} \neq \emptyset \). In light of Lemma 5, we can also additionally assume that \( \mathcal{N} \) satisfies Assumption 1. Then \( \mathcal{N} \) contains an edge \( e_2 \) satisfying conditions 3, 4 of Theorem 1 for suitable labeling of the sources and the terminals. The desired result then follows from Lemma 7.

Proof of Theorem 2:

The reader may find it useful to keep Fig. 3 in mind while going through the proof.

Sufficiency:

Let \( \mathcal{N} \) satisfy the hypothesis of Theorem 2.

Part 1: Non-solvability of \( \mathcal{N} \) over \( F_2 \).

Let the symbols transmitted over \( e_1 \) and \( e_2 \) be denoted by \( Y_{e_1} \) and \( Y_{e_2} \) respectively. By arguments similar to those given in the proof of Sufficiency of Theorem 1, one can show that \( Y_{e_1} \) is a function of only \( x_1 \) and \( x_3 \), and \( Y_{e_2} \) is a function of only \( x_2 \) and \( x_3 \). Let us call them \( f(x_1, x_3) \) and \( g(x_2, x_3) \) respectively.

Claim 9: (i) \( f(x_1, x_3) \) is a 1-1 function of \( x_3 \) for a fixed value of \( x_1 \) and a 1-1 function of \( x_1 \) for a fixed value of \( x_3 \). (ii) \( g(x_2, x_3) \) is a 1-1 function of \( x_2 \) for a fixed value of \( x_3 \) and a 1-1 function of \( x_3 \) for a fixed value of \( x_2 \).

Proof: The proof is the same as the one given for Claim 7.

If \( \mathcal{N} \) is solvable over \( F_2 \), then \( f(x_1, x_3) \) is a function of \( F_2 \times F_2 \) into \( F_2 \). It is easy to verify that all such functions can be represented by polynomials of the form \( \alpha x_1 + \beta x_3 + \gamma x_1 x_3 + \delta \) for \( \alpha, \beta, \gamma, \delta \in F_2 \). It is also easy to verify that the only such functions that satisfy Claim 9(i) are of the form \( x_1 + x_3 + \delta \) for \( \delta \in F_2 \). Hence w.l.o.g., we assume that \( f = x_1 + x_3 \).
By similar arguments, we assume \( g = x_2 + x_3 \).

But by conditions (1-3) of Theorem 2 \( \{e_1, e_2\} \) is a cut between \( \{s_1, s_2, s_3\} \) and \( t_3 \). So \( t_3 \) can obtain \( x_1 + x_2 + x_3 \) only if for some \( \alpha, \beta, \gamma, \delta \in F_2 \), \( \alpha f(x_1, x_3) + \beta g(x_2, x_3) + \gamma f(x_1, x_3)g(x_2, x_3) + \delta = x_1 + x_2 + x_3 \Rightarrow \alpha(x_1 + x_3) + \beta(x_2 + x_3) + \gamma(x_1 + x_3)(x_2 + x_3) + \delta = x_1 + x_2 + x_3 \). Now, substituting \( x_1 = x_2 = x_3 = 0 \) in this equation gives \( \delta = 0 \) while substituting \( x_1 = x_2 = x_3 = 1 \) gives \( \delta = 1 \) — a contradiction since \( 1 \neq 0 \) in \( F_2 \). Hence \( \mathcal{N} \) is not solvable over \( F_2 \).

**Part 2: Solvability of \( \mathcal{N} \) over all other fields.**

For this part let \( F \) be any field except \( F_2 \).

Since \( e_1 \) does not disconnect \( (s_1, t_1) \) and \( e_2 \) does not disconnect \( (s_2, t_2) \), let

(a) \( Q_1 \) be a \( (s_1, t_1) \) path not containing \( e_1 \),

(b) \( Q_2 \) be a \( (s_2, t_2) \) path not containing \( e_2 \),

(c) \( R_1 \) be a \( (s_1, t_2) \) path and

(d) \( R_2 \) be a \( (s_2, t_1) \) path.

In this case there exists a subnetwork shown in Fig. 15(a) where the shaded circular region means that \( Q_1, Q_2, R_1, R_2 \) may share edges. They do not share edges with other parts of the network shown in the figure. This is because,

(i) By condition 1, 4 of Theorem 2 \( Q_1 \) does not contain \( e_2 \) or any node from the \( (s_3, t(e_1)) \), \( (s_3, t(e_2)) \), \( (s_2, t(e_2)) \), \( (h(e_1), t_3) \) or \( (h(e_2), t_3) \) path segments. It also does not contain \( e_1 \) by definition.

(ii) Similarly by condition 2, 4 of Theorem 2 \( Q_2 \) does not contain \( e_1 \) or any node from the \( (s_3, t(e_1)) \), \( (s_3, t(e_2)) \), \( (s_2, t(e_1)) \), \( (h(e_1), t_3) \) or \( (h(e_2), t_3) \) path segments. It also does not contain \( e_2 \) by definition.

(iii) By condition 4 of Theorem 2 \( R_1 \) or \( R_2 \) does not contain both \( e_1 \) and \( e_2 \). Then by condition 1, \( R_1 \) can not contain \( e_2 \) or a node from the \( (s_3, t(e_2)) \) or \( (s_2, t(e_2)) \) or \( (h(e_2), t_3) \), and by condition 2 it can not contain \( e_1 \) or a node from \( (s_3, t(e_1)) \) or \( (h(e_1), t_3) \). Similarly by condition 2, \( R_2 \) can not contain \( e_1 \) or a node from \( (s_3, t(e_1)) \) or \( (s_1, t(e_1)) \) or \( (h(e_1), t_3) \), and by condition 1 it can not contain \( e_2 \) or a node from \( (s_3, t(e_2)) \) or \( (h(e_2), t_3) \).

![Fig. 15. The sub-network and the code over other fields](image)

Now we give the coding scheme over any field \( F \neq F_2 \). Let \( \alpha \in F \setminus \{0, 1\} \), \( \beta = (1 - \alpha)^{-1} \) and \( \gamma = 1 - \alpha^{-1} \). Consider the sub-network \( \mathcal{N}^* \) formed by considering all the nodes of \( \mathcal{N} \), but only those edges from \( \mathcal{N} \) belonging to the paths \( Q_1, Q_2, R_1 \) or \( R_2 \). Due to the statements above, \( \mathcal{N}^* \) does not contain \( e_1, e_2 \) or edges from the \( (s_3, t(e_1)) \), \( (s_3, t(e_2)) \), \( (h(e_1), t_3) \) or \( (h(e_2), t_3) \) path segments; and further, in \( \mathcal{N}^* \), \( \{s_1, s_2\} \rightarrow \{t_1, t_2\} \). So using the edges in \( \mathcal{N}^* \), and by pre-multiplying \( x_1 \) by \( \gamma \) at \( s_1 \),
we can communicate $x_2 + \gamma x_1$ to $t_1$ and $t_2$ by Lemma 8. Then in $\mathscr{N}$ we can simultaneously transmit $x_1 + \alpha x_3$ on $e_1$ and $x_3 + \beta x_2$ on $e_2$. This is shown in Fig. [15(b)]. By obtaining $x_1 + \alpha x_3$ through $P(h_{e_1}, t_3)$ and $x_3 + \beta x_2$ through $P(h_{e_2}, t_3)$, $t_3$ can get $x_1 + x_2 + x_3 = (x_1 + \alpha x_3) + \beta^{-1}(x_3 + \beta x_2)$. Using $x_2 + \gamma x_1$ (received from $\mathscr{N}^*$) and $x_1 + \alpha x_3$ (received from $e_1$), $t_1$ can get $x_1 + x_2 + x_3 = (x_2 + \gamma x_1) + \alpha^{-1}(x_1 + \alpha x_3)$. Similarly, $t_2$ can combine $x_2 + \gamma x_1$ (received on $\mathscr{N}^*$) and $x_3 + \beta x_2$ (received from $e_2$) to get $x_1 + x_2 + x_3 = \gamma^{-1}(x_2 + \gamma x_1) + (x_3 + \beta x_2)$.

Necessity:

We wish to show that networks which are not solvable over $F_2$ but solvable over all other fields have an edge pair satisfying conditions 1-4 of Theorem 2.

From Lemma 5 we see that networks with $\kappa = 0, 1, \geq 4$ are XOR solvable over all fields. Lemma 7 shows that networks with $\kappa(\mathscr{N}) = 3$ and $\mathscr{C} \neq \emptyset$ are either XOR solvable over all fields or are nonsolvable. Lemma 6 shows that a network with $\kappa(\mathscr{N}) = 3$ and $\mathscr{C} = \emptyset$ either satisfies the four conditions in Theorem 2 and is nonsolvable over $F_2$ but solvable over other fields; or does not satisfy the conditions in Theorem 2 and is XOR solvable over all fields. The proof of Lemma 5 for networks with $\kappa = 2$ shows that networks with $\kappa = 2$ which are not solvable over $F_2$ but solvable over all other fields (some networks in Case 1.2) have an edge pair satisfying conditions 1-4 of Theorem 2. Thus the necessity of Theorem 2 holds for all $3s/3t$ networks.

V. CONCLUSION

We presented a set of necessary and sufficient conditions for solvability of a $3$-source $3$-terminal sum-network over any field $F$. The conditions are the same for all fields except $F_2$. This explains the existence of the networks in Fig. 3 which are not solvable over $F_2$ though they are solvable over any other field. The conditions present full insight into the case of $3$-sources and $3$-terminals - the smallest sum-networks with non-trivial characterization.

The complexity of the proofs for this very specific case makes it clear that stronger tools are needed to characterize the problem for higher number of sources and terminals. However, this is not surprising, considering that sum-networks have been proved to be equivalent to the multiple-unicast networks as a class of problems. Even for multiple-unicast networks, explicit characterization of solvable networks is not available. Except for the double-unicast problem [34], [33], only cut based necessary conditions [35] are known to the best of our knowledge. It is fair to expect that tools developed to analyze/characterize will have strong relation with each other for these two classes of problems.

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