Multiparticle Interference, GHZ Entanglement, and Full Counting Statistics

H.-S. Sim$^1$ and E. V. Sukhorukov$^2$

$^1$ Department of Physics, Korea Advanced Institute of Science and Technology, Daejeon 305-701, Korea
$^2$ Département de Physique Théorique, Université de Genève, CH-1211 Genève 4, Switzerland

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We investigate the quantum transport in a generalized $N$-particle Hanbury Brown–Twiss setup enclosing magnetic flux, and demonstrate that the $N$th-order cumulant of current cross correlations exhibits Aharonov-Bohm oscillations, while there is no such oscillation in all the lower-order cumulants. The multiparticle interference results from the orbital Greenberger-Horne-Zeilinger entanglement of $N$ indistinguishable particles. For sufficiently strong Aharonov-Bohm oscillations the generalized Bell inequalities may be violated, proving the $N$-particle quantum nonlocality.

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The Aharonov-Bohm (AB) effect [1], being a most remarkable manifestation of quantum coherence, is at the heart of quantum mechanics. It is essentially a topological effect, because it requires a multiple-connected physical system, e.g. a quantum ring, and consists in a periodic variation of physical observables as a function of the magnetic field threading the loop. It is also nonlocal effect, since no local physical observable is sensitive to the field. Originally introduced for a single particle [1], it can be generalized to a two-particle AB effect in the average current [2], or in the noise power [3], if only two particles are able to enclose the loop.

The last effect [3], being implemented in the Hanbury Brown and Twiss geometry (HBT) [4], shows the two-particle character in a most dramatic way [5], because single-particle observables, such as the average current, do not contain AB oscillations. In contrast to the single-particle AB effect, which may have a classical analog [6], the two-particle AB effect is essentially quantum, because it originates from quantum indistinguishability of particles. Moreover, it is strongly related to the orbital entanglement in HBT setup [6], and to the quantum nonlocality as expressed via the violation of Bell inequalities [7,8].

In this Letter, we investigate the generalized HBT setup, exemplified in Fig. 1 and introduce the $N$-particle AB effect for $N \geq 3$. We start by analyzing the full counting statistics (FCS) [9,10] and demonstrating that in the $N$-particle HBT setup the $N$th-order cumulant of current cross-correlations may exhibit AB oscillations, while there is no such oscillation in all the lower-order cumulants. Next we show that the $N$-particle AB effect originates from the orbital $N$-particle entanglement. Namely, we prove that a many-particle state injected into the HBT setup contains the Greenberger-Horne-Zeilinger-type (GHZ) state [11,12], which via the post-selection [12] contributes to the particle transport. We remark here that the concept of the FCS has been used in earlier works in the context of a two- [13] and three-particle [14] entanglement. We further demonstrate that the Svetlichny inequality [15] may be formulated in terms of the joint detection probability and then violated, if the visibility of the AB oscillations exceeds the value $1/\sqrt{2}$. This inequality, being most restrictive in the whole family of generalized $N$-particle Bell inequalities [15,16,17,18], discriminates quantum mechanics and all hybrid local-nonlocal theories [14], proving the $N$-particle quantum nonlocality in the generalized HBT setup. We note that in contrast to the two-particle case, the overall sign of the $N$th-order cumulant cannot be uniquely associated with statistics of particles in the cases of $N \geq 3$.

**Generalized HBT interferometer.**— We consider the coherent electron transport in the mesoscopic multiter-
minal conductor consisting of $4N$ electron reservoirs, $i = 1, 2, \ldots, 4N$ (enumerated clockwise), $2N$ beam splitters $M_i$, $i = 1, 2, \ldots, 2N$, and $2N$ chiral current paths enclosing magnetic flux $\Phi$ (see Fig. 1). The $(2i-1)$-th and the $2i$-th reservoirs couple to the splitter $M_i$, at which electrons can be transmitted with probability $T_i$ or reflected with probability $R_i = 1 - T_i$. Among $4N$ reservoirs, $N$ reservoirs, $2, 6, \cdots, 4N - 2$, behave as independent electron sources, and $N$ pairs of reservoirs with indices $\alpha = \{3, 4\}$, $\alpha = \{7, 8\}$, $\cdots$, $\alpha_N = \{4N - 1, 4N\}$, act as detectors. All sources are biased with voltage $eV$, while the rest of $3N$ reservoirs are grounded. Neighboring beam splitters $M_i$ and $M_{i+1}$ are connected via a one-dimensional spinless current path, where phase difference $\phi_i$ is accumulated.

The total scattering matrix of the generalized HBT setup consists of two $2N$-dimensional unitary chiral blocks $\hat{U}$ and $\hat{V}$. The block $\hat{U}$ is determined as follows: The electron propagation from source $4i + 2$ to detectors $4i - 1$, $4i$, $4i + 1$, $4i + 2$, or $4i + 4$ gives the matrix elements $\exp(-i\phi_{2i})\sqrt{T_{2i+1}T_{2i}}$, $\exp(-i\phi_{2i})\sqrt{T_{2i+1}R_{2i}}$, $-\exp(i\phi_{2i+1})\sqrt{R_{2i+1}T_{2i+2}}$, $\exp(i\phi_{2i+1})\sqrt{R_{2i+1}R_{2i+2}}$, respectively. The block $\hat{V}$ describes noiseless outgoing currents from detector reservoirs. It is not relevant for the following discussion, therefore the requirement of the coherence in this sector may be relaxed. All the other matrix elements are zero, thus no single electron can enclose the flux $\Phi$ in the generalized HBT setup. However, $N$ electrons can do so, and this leads to the $N$-particle interference in the FCS.

$N$-particle Aharonov-Bohm effect.— In the long measurement time limit $t \gg \tau_C$, where $\tau_C = 2\pi\hbar/eV$ is the correlation time, the electron transport is a Markovian random process. The characteristic property of such a process is that the irreducible correlators (cumulants) of the number of electrons arriving at detectors are proportional to the number of transmission attempts, $t/\tau_C$. Therefore, we normalize the cumulants to $t/\tau_C$, so that the cumulant generating function of the FCS at the detector reservoirs takes the form:

$$ S(\chi) = \frac{\tau_C}{2\pi\hbar} \int dE \text{Tr} \ln [\mathbb{1} - \hat{f} + \hat{f} \hat{U} \hat{\Lambda} \hat{U}^\dagger]. $$

(1)

Here $\hat{f}$ is the diagonal matrix with elements $\hat{f}_{ij} = \delta_{ij} f_i(E)$ being Fermi-Dirac occupations: $f_i(E) = f_F(E - eV)$ at the sources, and $f_i(E) = f_F(E)$ in the rest of reservoirs. The matrix $\hat{\Lambda}$ is diagonal with elements $\Lambda_{ij} = \delta_{ij} \exp(i\chi_i)$, where $\chi = \{\chi_i\}$ is the set of counting variables in the detector reservoirs. The cumulants may be obtained by evaluating the derivatives of $S(\chi)$ and setting $\chi = 0$.

The generating function (1) is simplified by introducing matrices $(\hat{A}_\alpha)_{nm} = f_n U^{\dagger}_{\alpha,m} U_{\alpha,n}$ [20]:

$$ S(\chi) = \frac{\tau_C}{2\pi\hbar} \int dE \text{Tr} \ln \left[ \mathbb{1} + \sum_{\alpha_i} (e^{i\chi_{\alpha_i}} - 1) \hat{A}_{\alpha_i} \right], $$

(2)

where the sum runs over all the detectors $\alpha_i$ and $n, m = 1, 2, 5, 6, \ldots, 4N - 3, 4N - 2$. In order to generalize the two-particle interference to the cases of $N \geq 3$, we consider the lowest-order cross-correlation functions with all detectors being different. It turns out that the number of such cross-correlators is limited, because between all the possible lowest-order products of the form $\prod_{i} \hat{A}_{\alpha_i}$, where all $\alpha_i$ are different, only few products give nonvanishing traces: $\text{Tr} \hat{A}_{\alpha_1} \hat{A}_{\alpha_2} \hat{A}_{\alpha_{\alpha+1}}$ for all $i$, and $\text{Tr} \hat{A}_{\alpha_1} \hat{A}_{\alpha_2} \cdots \hat{A}_{\alpha_n}$ and $\text{Tr} \hat{A}_{\alpha_1} \hat{A}_{\alpha_n} \cdots \hat{A}_{\alpha_1}$. Expanding the logarithm in the r.h.s. of Eq. (2) and collecting nonzero traces we obtain the first-order cumulant $Q_{\alpha_i} = \tau_C f_i$, (i.e. the normalized average current) and the second-order cumulant $Q_{\alpha_1, \alpha_{\alpha+1}}$ (the normalized zero-frequency noise power),

$$ Q_{\alpha_i} = P_{\alpha_i} R_{2i-1} + (1 - P_{\alpha_i}) T_{2i+1} \quad (3a) $$

$$ Q_{\alpha_i, \alpha_{\alpha+1}} = -P_{\alpha_i + 1} (1 - P_{\alpha_i}) T_{2i+1} R_{2i+1}. \quad (3b) $$

where $P_{\alpha_i} = R_{2i} (P_{\alpha_i} = T_{2i})$ for $\alpha_i$ being odd (even).

The next nonzero cumulant is the $N$th-order cross-correlation function

$$ Q_{\alpha_1, \alpha_2, \cdots, \alpha_N} = 2 \text{sign}(\alpha_1, \alpha_2, \cdots, \alpha_N) \Gamma_{BS} \cos(\phi_{tot}), \quad (4) $$

where the sign depends on the choice of the detectors, $\text{sign}(\alpha_1, \alpha_2, \cdots, \alpha_N) = (-1)^{N-1} \Sigma \alpha_i$, the factor $\Gamma_{BS} = \prod_{i=1}^{2N} \sqrt{T_i R_i}$ characterizes the transmission of beam splitters, and the total phase accumulated around the HBT loop is $\phi_{tot} = 2\pi\Phi/\Phi_0 + \sum_{i=1}^{2N} \phi_i$, with $\Phi_0 = 2\pi\hbar/e$ being the flux quantum. Below we will use a notation $\{\alpha\} = \{\alpha_1, \alpha_2, \cdots, \alpha_N\}$ for an arbitrary set of $N$ detectors, so that $Q_{\alpha_1, \alpha_2, \cdots, \alpha_N} \equiv Q(\{\alpha\})$.

We note several important points. First, the result (4) holds under the usual condition [3, 4] of the “cancellation of paths”, which prevents dephasing due to the energy averaging: The difference of the total lengths of the clockwise [see Fig. 1(a)] and counter-clockwise [Fig. 1(b)] paths should not exceed $\tau_C/\sqrt{T_{tot}}$, where $v_F$ is Fermi velocity. Second, every cumulant in Eqs. (3a, 3b), and (4) has the prefactor $(\tau_C/2\pi\hbar) \int dE (f - f_0)^k$, where $k = 1, 2, N$. We consider the zero temperature limit and set this prefactor to 1. Next, the sign function in the equation (4) has two contributions: the term $(-1)^{N-1} \Sigma \alpha_i$ comes from the detector phases, while the term $(-1)^{N-1}$ originates from the fermionic exchange effect. Thus, in contrast to the case of the second cumulant [19], the overall sign of high-order cumulants is not universal. Finally, only the $N$th-order cross-correlator shows oscillations as a function of the magnetic flux threading the HBT loop. We stress that these oscillations appear in the FCS of electrons injected from $N$ uncorrelated sources, and thus they can be regarded as a $N$-particle AB effect. Below we connect this effect with $N$-particle GHZ entanglement.

GHZ entanglement.— To clarify the origin of the AB oscillations in the cumulant $Q(\{\alpha\})$ in Eq. (4), we analyze
the multiparticle state injected from the sources, $|\Psi\rangle = \prod_{0<E_e<V} \prod_{i=1}^{N} c_{4i-2}^a(E)|0\rangle$. Here, $|0\rangle$ denotes the filled Fermi sea below $E=0$, and the operator $c_{4i-2}^a(E)$ creates an electron with the energy $E$ in the source reservoir $4i-2$. Using the scattering matrix $U$, we write $c_{4i-2}^a = -i\sqrt{R_{2i-1}} a_{4i-1}^a + \sqrt{T_{2i-1}} b_i^a$, where $a_{4i-1}^a$ creates an electron moving clockwise from the beam splitter $M_{2i-1}$ to $M_{2i}$, and $b_i^a$ creates an electron moving counter-clockwise from $M_{2i+1}$ to $M_{2i}$. Then up to the overall constant prefactor the total state becomes

$$|\Psi\rangle = \prod_{0<E_e<V} \prod_{i=1}^{N} [\gamma_a c_{4i-2}^a(E) + \gamma_b c_{4i}^b(E) + D^i(E)]|0\rangle,$$

where $\gamma_a = \prod_{i=1}^{N} (-i\sqrt{R_{2i-1}})$, and $\gamma_b = -\prod_{i=1}^{N} \sqrt{T_{2i-1}}$. The operator $c_{4i-2}^a = \prod_{i=1}^{N} a_{4i-1}^a$ creates $N$ electrons moving clockwise [see Fig. 1(a)], and $c_{4i}^b = \prod_{i=1}^{N} b_i^a$ creates the $N$ electrons moving counter-clockwise [Fig. 1(b)], while $D^i$ creates the other possible states [Fig. 1(c-h)].

It follows from Eq. 5 that the total state $|\Psi\rangle$ is a product of $N$-particle states where all $N$ particles are taken at same energy. It is obvious that the zero-frequency $N$th-order cumulant $Q_{(N)}$ originates from the independent contribution of such states, more precisely, from the part $|\psi_N\rangle = (\gamma_a c_{4i-2}^a + \gamma_b c_{4i}^b)|0\rangle$. Introducing orbital pseudo-spin notations $|\uparrow\rangle = a_{4i-1}^a|0\rangle$ for electrons moving clockwise and $|\downarrow\rangle = b_i^a|0\rangle$ for electrons moving counter-clockwise, we rewrite the relevant $N$-particle state as

$$|\psi_N\rangle = p e^{i\varphi_\uparrow} |\uparrow\rangle |\uparrow\rangle |\uparrow\rangle \cdots |\uparrow\rangle + q e^{i\varphi_\downarrow} |\downarrow\rangle |\downarrow\rangle |\downarrow\rangle \cdots |\downarrow\rangle,$$

where $p^2 = 1 - q^2 = |\gamma_a|^2/(|\gamma_a|^2 + |\gamma_b|^2)$, $\theta_p = -\pi N/2$, and $\varphi_{\uparrow}$ and $\varphi_{\downarrow}$ are the pseudo-spin rotations, with $\theta_p = \pi$. This state is nothing but the $N$-particle GHZ-type entangled state [11]. We thus arrive at the important result that in fact GHZ entanglement is responsible for the $N$-particle AB effect, and that the measurement of the $N$th-order cumulant $Q_{(N)}$ effectively postselects the $N$-particle entangled state.

There exists an equivalent representation of the total state $|\Psi\rangle$ in the wavepacket basis [9], which obviously leads to the same physical results but allows slightly different interpretation that we will use below. In this representation, electrons fully occupy the stream of wave packets regularly approaching, with the rate $eV/(2\pi\hbar)$, the HBT interferometer from the sources $4i-2$, $i = 1, 2, \ldots, N$. After being injected to the interferometer, the wave packets form a $N$-particle state that contains the entangled state $|\psi_N\rangle$. The entangled state then moves towards the detectors $\alpha_i = \{4i-1, 4i\}$, where each electron accumulates the phase $\pm \phi_i$, it is rotated by the beam splitters $M_{2i}, a_{4i-1}^a = -i\sqrt{R_{2i}} c_{4i-1}^a + \sqrt{T_{2i}} c_{4i}^a$ and $b_{i}^a = \sqrt{T_{2i}} c_{4i}^a - i\sqrt{R_{2i}} c_{4i-1}^a$ and then $|\psi_N\rangle$ is detected in one of the states $\{\alpha\} = \{\alpha_1, \alpha_2, \ldots, \alpha_N\}$ of $N$ detectors.

Following Ref. [21] we introduce the probability $P_{(N)}$ of joint detection (JDP) of $N$ particles in the detector reservoirs defined as $P_{(N)} = \langle \Psi | I_{\alpha_1} I_{\alpha_2} \cdots I_{\alpha_N} | \Psi \rangle$, where $I_{\alpha_i} = (e/2\pi\hbar) \int dE dE' c_{\alpha_i}^a(E) c_{\alpha_i}^{a'}(E')$ are the current operators in reservoirs $\alpha_i$ taken at the same time $t = 0$. Then the straightforward calculation gives

$$P_{(N)} \propto \prod_{i=1}^{N} R_{2i-1} P_{\alpha_i} + \prod_{i=1}^{N} T_{2i-1} (1 - P_{\alpha_i}) + Q_{(N)}$$

with the prefactor determined by the normalization $\sum_{\{\alpha\}} P_{(N)} = 1$. This result can be easily understood after looking closely at Fig. 1. The wave packets moving as shown in Fig. 1(c) do not contribute to $P_{(N)}$, because one of the currents $I_{\alpha_i}$ is exactly zero. Thus, $P_{(N)}$ originates from the states (a) and (b), i.e., from the GHZ-type state [11]. In equation 7 the first and the second terms come from the direct contribution of the spin-up and spin-down parts of the state [9], respectively, while the third, $Q_{(N)}$, originates from the overlap of the two parts. Thus we conclude that the JDJ $P_{(N)}$ postselects from the Fermi sea the entangled state [9], which via the term $Q_{(N)}$ leads to the $N$-particle AB effect.

**Quantum nonlocality.**— The GHZ-type states [9] play a special role in quantum mechanics, because they violate generalized Bell inequalities [13, 14, 17, 18] and thus demonstrate quantum nonlocality of $N$-particle states. The inequalities may be formulated in terms of the $N$-spin correlation function $E_N = \langle \sigma_\uparrow \cdots \sigma_\uparrow \sigma_\downarrow \cdots \sigma_\downarrow \rangle$, where $\sigma_\uparrow = \hat{n}_i \cdot \hat{\sigma}$ is the spin operator along the direction $\hat{n}_i = (\cos \theta_i \sin \delta_i, \sin \theta_i \sin \delta_i, \cos \delta_i)$ at the $i$th spin detector. For the GHZ-type state [9] we have

$$E_N = g(p, q) \prod_{i=1}^{N} \cos \delta_i + 2pq \cos \theta \prod_{i=1}^{N} \sin \delta_i,$$

where $g(p, q) = p^2 + (-1)^N q^2$ and $\theta = \theta_p - \varphi_{\uparrow} + \sum_{i=1}^{N} \theta_i$. Turning now to the generalized HBT setup, we note that the detector beam splitters $M_{2i}$ implement the orbital pseudo-spin rotation, while the reservoirs $4i$ and $4i-1$ detect pseudo-spins in $z$-direction. The pseudo-spin correlation function can now be found as

$$E_N = \sum_{\{\alpha\}} (-1)^{\sum_{i=1}^{N} \alpha_i} P_{(N)},$$

generalizing two-particle cases [9]. After identifying $\delta_i = T_{2i-1} - R_{2i}$ and $\theta_i = \varphi_{\uparrow} - \varphi_{\downarrow}$ we find that $E_N$ takes the same form as [8] with $\theta$ being replaced with $\varphi_{\downarrow} - \pi (N-1)$.

For a particular choice of two sets of directions $\{\hat{n}_1, \ldots, \hat{n}_N\}$ and $\{\hat{n}_1', \ldots, \hat{n}_N'\}$, the expectation value of the $N$-spin operator introduced via iterations, $M_N = \frac{1}{2}(\sigma_\uparrow \cdots \sigma_\uparrow \sigma_\downarrow \cdots \sigma_\downarrow) \otimes M_{N-1} + \frac{1}{2}(\sigma_\uparrow \cdots \sigma_\uparrow \sigma_\downarrow \cdots \sigma_\downarrow) \otimes M_{N-1}^{-1}$ (with $M_{N}^{-1}$ obtained from $M_N$ by exchanging all $\hat{n}_i$ and $\hat{n}_i'$), violate the Mermin-Ardehali-Belinski-Klyshko (MABK) inequalities $\langle M_N \rangle \leq 1$ [19] that discriminate local variable theories and quantum mechanics. These inequalities generalize the well known Clauser-Horne-Shimony-Holt inequality [8], $\langle M_2 \rangle = (1/2) \langle E(\hat{n}_1, \hat{n}_2) + E(\hat{n}_1', \hat{n}_2') + E(\hat{n}_1, \hat{n}_2') + E(\hat{n}_1', \hat{n}_2) \rangle$.
the nonlocality condition (10) may require a more accurate determination of the visibility $V_{AB}$ via measuring the JDP. Such high-frequency “quantum noise” detection techniques have recently become available 26.

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