Horizon function in Landau gauge QCD revisited
–Free boundary case from viewpoint of network QCD–

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In Gribov-Zwanziger scenario of color confinement, Zwanziger proposed two types of horizon function in lattice Landau gauge, the first one and the second one of which foundation was discussed in detail. The second type horizon function is focussed on in the present study. Its derivation and the horizon condition are briefly reviewed along the line of Ref. 3), and it is also reviewed that this horizon condition and Kugo-Ojima color confinement criterion coincide in the continuum limit.

§1. Notations and generalities

We intend to define Landau gauge SU(N) QCD on general networks, and to analyze standard lattice Landau gauge QCD from general point of view. So we have sites \( x \) and links \( \ell = (x, x') \); pairs of sites in general networks. To each link \( \ell \) is assigned its intrinsic (positive) direction \( e_\mu \) as convention, say \( e_\mu = x' - x \), for specification of basic link variable \( U_{e_\mu} \) as a parallel transport \( U_{x, x'} \) in the direction of \( e_\mu \), and of link-field \( A_\mu \) (as “network current”), \( A_{x, \mu} \) denote a value on the link \( \ell = (x, x + e_\mu) \) called as \( e_\mu \)-component. Now links \( \ell \) with \( e_\mu \) are considered as vectors \( xx' = x x' \). We use notations \( U_{xx'} = U_{\ell} = U_{x, e_\mu} = U_{x, \mu} \) interchangeably, where \( x' - x = e_\mu \). Let \( \ell(x) \) denote a set of links,

\[
\ell(x) = \{ \ell | \ \ell = xx' = e_\mu \ \text{or} \ \ell = x'x = e_\mu \} \tag{1.1}
\]

and let \( \ell_+(x) \) be defined as a set of positive link at \( x \),

\[
\ell_+(x) = \{ \ell | \ \ell = xx' = e_\mu \} \tag{1.2}
\]

and similarly \( \ell_-(x) \) a set of negative link at \( x \),

\[
\ell_-(x) = \{ \ell | \ \ell = x'x = e_\mu \}. \tag{1.3}
\]

It is to be noted that number of elements of \( \ell_+(x) \) is not necessarily equal to that of \( \ell_-(x) \) in general networks in contrast to periodic regular lattice or infinite regular lattice.

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Thus gauge transformation by $g \in G$ is written for $\ell \in \ell_+(x)$ as

$$U^g_\ell = g_x^\dagger U_\ell g_{x+e_\mu}, \quad (1.4)$$

and for $\ell \in \ell_-(x)$ as

$$U^g_\ell = g_{x-e_\mu}^\dagger U_\ell g_x. \quad (1.5)$$

We denote normalized antihermitian matrices $\lambda_a$ as Lie algebra basis as

$$[\lambda_a, \lambda_b] = f_{abc} \lambda_c \quad (1.6)$$

and

$$(\lambda_a | \lambda_b) = \text{tr}(\lambda_a^\dagger \lambda_b) = \delta_{ab}. \quad (1.7)$$

All fields of the adjoint representation are often denoted as antihermitian fields in use of the above basis.

We use bracket notation for suitable innerproducts for scalar fields (site function) and vector fields (link function), respectively, as

$$\langle \psi | \phi \rangle = \sum_x \text{tr}(\psi_x^\dagger \phi_x), \quad (1.8)$$

$$\langle A_\mu | B_\mu \rangle = \sum_{x,\mu} \text{tr}(A_{x,\mu}^\dagger B_{x,\mu}), \quad (1.9)$$

where we use a simple notation for link-functions, e.g., $A_\mu$, and $\sum_{x,\mu}$ implies summation over $x$ and $\ell_+(x)$, or over $x$ and $\ell_-(x)$, which is equivalent to summation over all links $\sum_\ell$. From site-fields (scalars) $\phi$, two kinds of link-fields (vectors) are defined such that

$$\partial_\mu \phi = \phi_{x+e_\mu} - \phi_x, \quad (1.10)$$

according to the associated positive direction $e_\mu$ of the link $(x, x + e_\mu)$, and

$$\bar{\phi}_\mu = (\phi_{x+e_\mu} + \phi_x)/2, \quad (1.11)$$

where we use hereafter abbreviated notations $x+\mu$ for $x+e_\mu$.

The following relations hold on any networks as well as on regular lattice.

$$\langle A_\mu | \partial_\mu \phi \rangle = \langle -\partial_\mu A_\mu | \phi \rangle, \quad (1.12)$$

where the divergence of link-field is defined as

$$\langle (\partial_\mu A_\mu)_x = \sum_{\mu \in \ell_+(x)} A_{x,\mu} - \sum_{\mu \in \ell_-(x)} A_{x-\mu,\mu}. \quad (1.13)$$

Adjoint of commutator reads as

$$\langle A_\mu | [B_\mu C_\mu] \rangle = \langle -[B_\mu A_\mu] | C_\mu \rangle. \quad (1.14)$$
Adjoint of \( \tilde{\phi}^\mu \) reads as
\[
\langle A_\mu | \tilde{\phi}^\mu \rangle = \langle \tilde{A}_\mu^\mu | \phi \rangle. \tag{1.15}
\]
where a site function is defined as
\[
(\tilde{A}_\mu^\mu)_x = \frac{1}{2} \left\{ \sum_{\mu \in \ell_+(x)} A_{x,\mu} + \sum_{\mu \in \ell_-(x)} A_{x-\mu,\mu} \right\}. \tag{1.16}
\]

§2. Definitions of gauge field and covariant derivative

There are two possible options of \( A_\mu(U) \),

- **U-linear definition;**
  \[
  A_{x,\mu} = (U_{x,\mu} - U_{x,\mu}^\dagger)/2|_{\text{traceless part}}, \tag{2.1}
  \]

- **log \( U \) definition;**
  \[
  U_{x,\mu} = e^{A_{x,\mu}}. \tag{2.2}
  \]

Next we define a covariant derivative \( D_\mu(U) \) which appears under an infinitesimal gauge transformation \( e^\varepsilon \) as \( \delta A_{x,\mu} = D_\mu(U)\varepsilon; \)
\[
D_\mu(U)\phi = G(U_\mu)\partial_\mu\phi + [A_\mu, \tilde{\phi}^\mu], \tag{2.3}
\]
where the operation \( G(U_\mu) \) on an antihermitian link variable, \( B_\mu \), is given by,

- **U-linear definition;**
  \[
  G(U_\mu)B_\mu = \frac{1}{2} \left\{ \frac{U_\mu + U_\mu^\dagger}{2}, B_\mu \right\}_{\text{traceless part}}, \tag{2.4}
  \]

- **log \( U \) definition;**
  \[
  G(U_\mu)B_\mu \equiv S(A_\mu)B_\mu = \{(A_\mu/2)/\text{th}(A_\mu/2)\}B_\mu \tag{2.5}
  \]

with
\[
A_\mu = \text{adj}_{A_\mu} = [A_\mu, \cdot]. \tag{2.6}
\]
It is to be noted that in both definitions \( G(U_\mu)_{ab} = G(U_\mu)_{ba} \) from
\[
G(U_\mu)_{ab} = \text{tr} \left( \lambda^a_n \frac{1}{2} \left\{ \frac{U_\mu + U_\mu^\dagger}{2}, \lambda^b \right\} \right) = G(U_\mu)_{ba}. \tag{2.7}
\]
and
\[
(A_\mu)_2^{ab} = \text{tr} \left( \lambda^a_n [A_\mu [A_\mu, \lambda^b]] \right) = \text{tr} \left( [A_\mu, \lambda^a_n [A_\mu, \lambda^b]] \right) = (A_\mu)_2^{ba}, \tag{2.8}
\]
respectively.

Adjoint of the covariant derivative is defined as
\[
\langle B_\mu | D_\mu(U)\phi \rangle = \langle -D_\mu(U)B_\mu | \phi \rangle, \tag{2.9}
\]
where a site function \( D_\mu(U)B_\mu \), with \( \mu \) summation understood, is given as
\[
D_\mu(U)B_\mu = \partial_\mu(G(U_\mu)B_\mu) + [A_\mu, B_\mu]'. \tag{2.10}
\]
§3. Optimization function and the Landau gauge

The Landau gauge $\partial A = 0$ can be characterized\(^5\) such that

$$\delta F_U(g) = 0 \text{ for } \delta g,$$

(3.1)

in use of the optimizing functions $F_U(g)$ for each option of $A_{\mu}(U)$ as $U$-linear definition;

$$F_U(g) = \sum_{x,\mu} \text{tr} \left\{ 2 - \left( U^g_{x,\mu} + U^g_{x,\mu} \right) \right\},$$

(3.2)

$log U$ definition;

$$F_U(g) = \sum_{x,\mu} \text{tr} \left( A^g_{x,\mu} D_{\mu}(U^g) \right) \equiv \langle A^g_{\mu} | A^g_{\mu} \rangle.$$

(3.3)

It is seen that in case of infinitesimal gauge transformations $g^{-1} \delta g = \varepsilon$, variation $\delta F_U(g)$ is given in either definition as

$$\delta F_U(g) = 2 \langle A^g_{\mu} | \partial_{\mu} \varepsilon \rangle,$$

(3.4)

that is, by putting $g = g(t)$, and $g^{-1} g_t = \omega_t$,

$$\frac{d}{dt} F_U(g(t)) = 2 \langle A^g_{\mu} | \partial_{\mu} \omega_t \rangle.$$

(3.5)

It holds on the arbitrary networks that

$$\delta F_U(g) = -2 \langle \partial_{\mu} A^g_{\mu} | \varepsilon \rangle,$$

(3.6)

or

$$\frac{d}{dt} F_U(g(t)) = -2 \langle \partial_{\mu} A^g_{\mu} | \omega_t \rangle,$$

(3.7)

which verifies the statement that a stationarity point of the optimizing function on a gauge orbit yields the Landau gauge. If we proceed further to a higher derivative in general, then we obtain that

$$\frac{d^2 F_U(g(t))}{dt^2} = -2 \langle \partial_{\mu} D_{\mu}(U^g) \omega_t | \omega_t \rangle - 2 \langle \partial_{\mu} A^g_{\mu} | \omega_{tt} \rangle.$$

(3.8)

Thus if $U^g_{\mu}(0) = U_{\mu}$, then the variation of the optimizing function, $\Delta F = F_U(\varepsilon) - F_U(1)$, is given up to the second order as

$$\Delta F = -2 \langle \partial_{\mu} A_{\mu} | \varepsilon \rangle + \langle \varepsilon | - \partial_{\mu} D_{\mu}(U)| \varepsilon \rangle,$$

(3.9)

where it is put that $\varepsilon = \eta \Delta t$ with any site function $\eta$ constant with respect to $t$, i.e., $\omega_{tt} = 0$ may be assumed in this case.

Let us denote after Zwanziger, the Landau gauge space as

$$\Gamma \equiv \{ U | \partial A = 0 \},$$

(3.10)
and the Gribov region as
\[ \Omega \equiv \{ U \mid M(U) \geq 0, \ U \in \Gamma \}, \tag{3.11} \]
where \( M(U) \) is a Faddeev-Popov operator, \( M = -\partial D(U) \), detailed properties of which will be investigated below.

Now one can define a **fundamental modular region** \( \Lambda \) as a set of global minima of the optimizing function.
\[ \Lambda = \{ U \mid F_U(1) \leq F_U(g) \text{ for } \forall g \} \tag{3.12} \]
It holds the following inclusion as
\[ \Lambda \subset \Omega \subset \Gamma. \tag{3.13} \]

So far, *all notions and formula are valid on any networks, i.e., in any topology, and/or boundary conditions.*

For a while from now on, we assume \( d \)-dimensional regular lattice with \( L \)-periodic boundary conditions, and a set of link-field \( U \) on this lattice is denoted as \( \Pi_L \), and a set of gauge transformation on this lattice, \( G_L \). We define the **Nth partial core of the fundamental region** as
\[ A_L^N \equiv \{ U \mid U \in \Pi_L \text{ and } F_U(1) \leq F_U(g), \ \forall g \in G_{NL} \}. \tag{3.14} \]
From \( G_L \subset G_{NL} \), inclusion \( A_L^N \subset A_L \) is easily understood, and one may write
\[ A_L^N = A_{LN} \cap \Pi_L, \tag{3.15} \]
and if \( N \) is chosen to be a power of 2, say, then the partial cores are nested,
\[ A_L^{N'} \subset A_L^{N} \subset A_L \text{ for } N' = 2^{M'} > N = 2^M. \tag{3.16} \]
The **core region** is defined by the limiting set,
\[ \Xi_L \equiv A_L^\infty = \lim_{N \to \infty} A_L^N = A_{\infty} \cap \Pi_L. \tag{3.17} \]
In connection to the core region, the following theorem holds.

**Theorem 1**
Let \( U \in \Pi_L \), where \( \Pi_L \) is a set of \( L \)-periodic configurations. The gauge transformation \( g \) which brings \( U \) to \( A_{NL} \) has a form such that
\[ g = he^{\theta x} \tag{3.18} \]
where
\[ h \in G_L, \ [\theta_\mu, \theta_\nu] = 0 \text{ and } e^{\theta_\mu L^N} = 1. \tag{3.19} \]

**Proof**
In the following, \( U \) is commonly used as denoting a configuration either being \( L \)-periodic or being \( NL \)-periodic. Let \( U^g \in A_{NL} \) where \( g(x) \in G_{NL} \). Since the optimizing function \( F \) is an extensive quantity, it holds that the shifted configuration
in any direction by lattice unit have the same value of $F$. Particularly, the shifted configuration of $U^g$ in the negative $\mu$-direction by $L$ units is also a gauge transform of $U$ into $\Lambda_{NL}$, and it can differ from $U^g$ only by constant gauge transformation, say, $g_\mu$. Now since the shifted configuration $U(x+L\mu)$ is identical to $U(x)$ itself, it follows that $g$ has a structure such that the $L$-unit shifted $g$, i.e., $g(x+L\mu)$ in the negative $\mu$-direction is given by $g(x)g_\mu$. It is obvious that $[g_\mu,g_\nu] = 0$, since translations in any directions commute each other, and $g_\mu^N = 1$ from $NL$-periodicity of $g$. We may write $g_\mu = e^{\theta_\mu L}$ where $\theta_\mu$'s belong to the same Cartan subalgebra, and $e^{\eta_\mu NL} = 1$.

Let $h(x) = g(x)e^{-\theta x}$, then it follows that
\[
h(x+L\mu) = g(x+L\mu)e^{-\theta_\mu L}e^{-\theta x} = g(x)g_\mu^{-1}e^{-\theta x} = h(x),
\]
and thus $g = he^{\theta x}$ and $h \in G_L$. q.e.d.

Remark 2
Let $U \in \Lambda^N_L$. It holds that
\[
F_U(1) \leq F_U(g) \text{ for } \forall g = he^{\theta x}, \ (h \in G_L)
\]
where $\theta_\mu = (M/NL)\eta_\mu$ with the nonzero smallest elements $\eta_\mu$ of the Cartan subalgebra such that $e^{\eta_\mu} = 1$, and $M$ is an arbitrary integer. It is to be noted here that $U^g \in \Pi^N_L$, but $U^g \notin \Pi_L$ in general.

Proof is selfevident from the definition of $\Lambda^N_L$.

Remark 3
Let $U \in \Xi_L$. It holds that
\[
F_U(1) \leq F_U(g) \text{ for } \forall g = he^{\theta x}, \ (h \in G_L)
\]
where $\theta_\mu = t\eta_\mu$ with the nonzero smallest elements $\eta_\mu$ of the Cartan subalgebra such that $e^{\eta_\mu} = 1$, and $t$ is an arbitrary real. Similarly to Remark 2, it is noted that $U^g \in \Pi^L_\infty$, but $U^g \notin \Pi_L$ in general.

Proof is selfevident from the definition of $\Xi_L$.

Now we investigate the behavior of the optimizing functions $F_U(g)$ under the gauge transformation relaxed so as to include the gauge transformation of the Bloch wave type.

Let $g$ be as $g = he^{\theta x} = e^\omega e^{\theta x} \equiv e^\xi$, and let us consider that $\theta_\mu$ and $\omega$ are some functions of $t$ such that
\[
\theta_\mu = t\eta_\mu \text{ and } \omega = \omega(t)
\]
where $\eta_\mu$'s are suitably normalized constant elements of Cartan subalgebra, and $\omega(t)$ is a $L$-periodic scalar field with $\omega(0) = 0$. Then putting
\[
h_t \equiv \frac{dh}{dt} \equiv h\omega_t \text{ and } g_t \equiv \frac{dg}{dt} \equiv g\xi_t,
\]
we obtain
\[ \xi_t = e^{-\theta_x} \omega t e^{\theta_x} + \eta x \equiv \omega'_t + \eta x, \tag{3.25} \]
where it is to be noted that neither \( \frac{d\omega}{dt} = \omega_t \) nor \( \frac{d\xi}{dt} = \xi_t \) hold in general. Another point to be emphasized here is that \( U^g \) is considered as gauge transform of \( U \in \Pi_L \subset \Pi_{\infty L} \), and is not of \( L \)-periodicity in general. It follows from general derivation before that
\[
\frac{d}{dt} F_U(g(t)) = 2 \langle A_{\mu}^g \mid \partial_{\mu} \xi_t \rangle. \tag{3.26}
\]
From \( \xi_t \), we have
\[
\frac{d}{dt} F_U(g(t)) = 2(\langle A_{\mu}^g \mid \partial_{\mu} \omega'_t \rangle + \langle A_{\mu}^g \mid \eta_{\mu} \rangle), \tag{3.27}
\]
and
\[
\frac{d}{dt} F_U(g(t)) = 2(-\langle \partial_{\mu} A_{\mu}^g \mid \omega'_t \rangle + \langle A_{\mu}^g \mid \eta_{\mu} \rangle). \tag{3.28}
\]
The above equation \( \xi_t \) is trivial as one on \( \infty \) \( L \)-periodic lattice, but it has more implication than that. Although the periodicity of \( \omega'_t \) and \( A_{\mu}^g \) can not be demonstrated to be \( L \), actual contribution from each link in \( \xi_t \) appears \( L \)-periodic, and the same holds in \( \xi_t \), and thus the derivation of \( \xi_t \) can be seen as such.

Reasoning of this fact can be seen easily by noting that the inner product of link variables is invariant under the constant gauge transformation given by \( e^{\theta_x + \theta_{\mu}/2} \) i.e., the gauge transformation at the midpoint, and then there appear \( L \)-periodicity in the equation, and the subtraction can be inverted to the other side of the inner product with a minus sign. Explicit proof of this fact goes as follows.
\[
\langle A_{\mu}^g \mid \partial_{\mu} \omega'_t \rangle = \langle (A_{\mu}^h)^{e^{\theta_x}} \mid \partial_{\mu} (e^{-\theta_x} \omega_t e^{\theta_x}) \rangle \tag{3.29}
\]
Let \( A_{x,\mu} = A(U_{x,\mu}) \) denote the gauge field at a link \((x, x + \mu)\). Then under the gauge transformation of Bloch type, \( e^{\theta_x} \), it holds at each link that
\[
A_{x,\mu}^{e^{\theta_x}} = A(U_{x,\mu}^{e^{\theta_x}}) = A(e^{-(\theta_x + \theta_{\mu}/2)} e^{\theta_{\mu}/2} U_{x,\mu} e^{\theta_{\mu}/2} e^{\theta_x + \theta_{\mu}/2}) \tag{3.30}
\]
Then corresponding to the situation of constant gauge transformation, it reads that
\[
A(e^{-(\theta_x + \theta_{\mu}/2)} e^{\theta_{\mu}/2} U_{x,\mu} e^{\theta_{\mu}/2} e^{\theta_x + \theta_{\mu}/2}) = e^{-(\theta_x + \theta_{\mu}/2)} A(U_{x,\mu}^{e^{\theta_{\mu}/2}} e^{\theta_x + \theta_{\mu}/2}) \tag{3.31}
\]
where \( U_{x,\mu} = e^{\theta_{\mu}/2} U_{x,\mu} e^{\theta_{\mu}/2} \). And it holds that
\[
\partial_{\mu} (e^{-\theta_x \omega_t e^{\theta_x}}) = e^{-(\theta_x + \theta_{\mu}/2)} \omega_t \mu e^{\theta_{\mu}/2} e^{\theta_{\mu}/2} \omega_{t+\mu} e^{-\theta_{\mu}/2} e^{\theta_x + \theta_{\mu}/2}, \tag{3.32}
\]
where \( \omega_t = \omega_t(x) \) and \( \omega_{t+\mu} = \omega_t(x + \mu) \). Thus we obtain that
\[
\langle (A_{\mu}^h)^{e^{\theta_x}} \mid \partial_{\mu} (e^{-\theta_x \omega_t e^{\theta_x}}) \rangle = \langle e^{\theta_{\mu}/2} (A_{\mu}^h)^{e^{\theta_{\mu}/2}} \omega_{t+\mu} e^{\theta_{\mu}/2} \rangle - \langle e^{-\theta_{\mu}/2} (A_{\mu}^h)^{e^{\theta_{\mu}/2}} \omega_t \rangle \tag{3.33}
\]
where \((A^h)^{\theta\mu}_{x,\mu} = A((U^h)^{\theta\mu}_{x,\mu})\) and it is to be noted that gauge fields appearing in the inner products are \(L\)-periodic. Thus we can shift safely the expression as
\[
\langle e^{\theta t/2} (A^h)^{\theta\mu}_{\mu,-\mu} e^{-\theta t/2} | \omega_t, + \mu \rangle = \langle e^{\theta t/2} (A^h)^{\theta\mu}_{\mu,-\mu} e^{-\theta t/2} | \omega_t \rangle
\]
where
\[
\langle e^{\theta t/2} (A^h)^{\theta\mu}_{\mu,-\mu} e^{-\theta t/2} | \omega_t, + \mu \rangle = A(e^{\theta x} U^h_{x,-\mu,\mu}) = e^{\theta x} A(e^{-(\theta x - \theta \eta)} U^h_{x,-\mu,\mu} e^{\theta x}) = e^{\theta x} A(U^h_{x,-\mu,\mu} e^{-\theta x} \rangle. \quad (3.35)
\]
Thus it holds that
\[
\langle e^{\theta t/2} (A^h)^{\theta\mu}_{\mu,-\mu} e^{-\theta t/2} | \omega_t, + \mu \rangle = \langle e^{\theta t} A^g_{\mu,-\mu} e^{-\theta t} | \omega_t \rangle = \langle A^g_{\mu,-\mu} | \omega_t \rangle \quad (3.36)
\]
and thus we obtain that
\[
\langle A^g_{\mu} | \partial_{\mu} \omega' \rangle = \langle \partial_{\mu} A^g_{\mu} | \omega_t \rangle \quad (3.37)
\]

If we proceed further to a higher derivative in general, then we obtain as
\[
\frac{d^2 F_U(g(t))}{dt^2} = 2(-\langle \partial_{\mu} D_{\mu}(U^g) \xi_t | \omega'_t \rangle - \langle \partial_{\mu} A^g_{\mu} | (\omega'_t)_{\xi} \rangle + \langle D_{\mu}(U^g) | \eta_{\xi} \rangle). \quad (3.39)
\]
Now we consider a situation \(U^g(0) \in \Sigma_L\) and \(A^g(0) = A_{\mu}\), and then since
\[
\left| \frac{d}{dt} A^g_{\mu} \right|_{t=0} = D_{\mu}(U)|\omega_t + \eta x) = D_{\mu}(U)|\omega_t + G(U_{\mu}) \eta_{\mu} + [A_{x,\mu}, \eta_{\mu} x_{\mu}], \quad (3.40)
\]
the \(L\)-periodicity of \(A_{\mu}\) is easily violated due to the third term. Now we have
\[
\left| \frac{d}{dt} F_U(g(t)) \right|_{t=0} = 2(-\langle \partial_{\mu} A_{\mu} | \omega_t \rangle + \langle A_{\mu} | \eta_{\mu} \rangle), \quad (3.41)
\]
\[
\left| \frac{d^2 F_U(g(t))}{dt^2} \right|_{t=0} = 2(-\langle \partial_{\mu} D_{\mu}(U) \xi_t | \omega_t \rangle - \langle \partial_{\mu} A_{\mu} | (\omega'_t)_{\xi} \rangle + \langle D_{\mu}(U) | \xi_t | \eta_{\mu} \rangle) \quad (3.42)
\]
where \(\xi_t = \omega_t + \eta x\), and then noting \(\partial A = 0\),
\[
\left| \frac{d}{dt} F_U(g(t)) \right|_{t=0} = 2 \langle A_{\mu} | \eta_{\mu} \rangle, \quad (3.43)
\]
\[
\left| \frac{d^2 F_U(g(t))}{dt^2} \right|_{t=0} = 2(\omega_t - \partial_{\mu} D_{\mu}(U) \xi_t + \langle \eta_{\mu} | D_{\mu}(U) \xi_t \rangle). \quad (3.44)
\]
It should follow from the fact $U_{\mu} \in A_{\infty}$ that
\[
\frac{d}{dt} F_{U}(g(t)) \bigg|_{t=0} = 0, 
\]
\[
\frac{d^{2} F_{U}(g(t))}{dt^{2}} \bigg|_{t=0} \geq 0, 
\] (3.46)
for $\forall h \in G_{L}$ and $\forall \eta_{\mu}$. For the estimate,
\[
\langle \eta_{\mu} | D_{\mu}(U) \xi_{t} \rangle = \langle \eta_{\mu} | D_{\mu}(U) \omega_{t} + G(U_{\mu}) \eta_{\mu} + [A_{x,\mu}, \overline{\eta}] \rangle, 
\]
we have
\[
\langle \eta_{\mu} | D_{\mu}(U) \xi_{t} \rangle = \langle \eta_{\mu} | D_{\mu}(U) \omega_{t} \rangle + \langle \eta_{\mu} | G(U_{\mu}) \eta_{\mu} \rangle 
\]
\[-\langle D_{\mu}(U) \eta_{\mu} | \omega_{t} \rangle + \langle \eta_{\mu} | G(U_{\mu}) \eta_{\mu} \rangle, 
\]
where
\[
\langle \eta_{\mu} | [A_{x,\mu}, \overline{\eta}] \rangle = \langle A_{x,\mu} | [\overline{\eta}, \eta_{\mu}] \rangle = 0 
\] (3.49)
is used. From derivation,
\[
\partial_{\mu} D_{\mu}(U) \xi_{t} = \partial_{\mu} (D_{\mu}(U) \omega_{t} + G(U_{\mu}) \eta_{\mu} + [A_{x,\mu}, \overline{\eta}] ) 
\]
\[-\partial_{\mu} D_{\mu}(U) \omega_{t} + \partial_{\mu} G(U_{\mu}) \eta_{\mu} + [\partial_{\mu} A_{\mu}, \eta_{\mu}] 
\]
\[-\partial_{\mu} D_{\mu}(U) \omega_{t} + \partial_{\mu} G(U_{\mu}) \eta_{\mu} + [A_{\mu}, \eta_{\mu}] 
\]
\[-\partial_{\mu} D_{\mu}(U) \omega_{t} + D_{\mu}(U) \eta_{\mu}, 
\] (3.50)
we obtain for (3.44) that
\[
\frac{1}{2} \frac{d^{2} F_{U}(g(t))}{dt^{2}} \bigg|_{t=0} = \langle \omega_{t} \rangle - \partial_{\mu} D_{\mu}(U) \omega_{t} - \langle \omega_{t} | D_{\mu}(U) \eta_{\mu} \rangle 
\]
\[-\langle D_{\mu}(U) \eta_{\mu} | \omega_{t} \rangle + \langle \eta_{\mu} | G(U_{\mu}) \eta_{\mu} \rangle, 
\] (3.51)
\[
\frac{1}{2} \frac{d^{2} F_{U}(g(t))}{dt^{2}} \bigg|_{t=0} = \langle \omega_{t} \rangle - \frac{1}{\partial D} \partial D \eta \rangle - \partial D | \omega_{t} - \frac{1}{\partial D} \partial D \eta \rangle 
\]
\[-\langle D \eta | \frac{1}{\partial D} | D \eta \rangle + \langle \eta_{\mu} | G(U_{\mu}) \eta_{\mu} \rangle 
\] (3.52)
Now we draw some necessary conditions for $U_{\mu} \in \Xi_{L}$ from (3.43), (3.45), (3.46) and (3.52).

**Theorem 2**

Let $U_{\mu} \in \Xi_{L} = A_{\infty} \cap \Pi_{L}$. Then it follows that for the gauge transformation $g = e^{\omega t} e^{\eta_{\mu} t}$ where $\gamma \omega$ belongs to $L$-periodic scalar and $\gamma \eta_{\mu}$ belongs to the same Cartan subalgebra, the optimizing function $F_{U}(g)$ behaves as
\[
\frac{d}{dt} F_{U}(g(t)) \bigg|_{t=0} = 2 \langle A_{\mu} | \eta_{\mu} \rangle = 0, 
\] (3.53)
\[ \frac{1}{2} \frac{d^2 F_U(g(t))}{dt^2} \bigg|_{t=0} = \langle \omega - \frac{1}{-\partial D} D\eta| - \partial D|\omega - \frac{1}{-\partial D} D\eta \rangle - \langle D\eta| \frac{1}{-\partial D} |D\eta \rangle + \langle \eta_\mu | G(U_\mu) |\eta_\mu \rangle \geq 0. \] 

(3.54)

Thus it is concluded that

\[ \bar{A}_\mu \equiv \sum_x A_{x,\mu} = 0 \]  

(3.55)

and

\[ \langle D\eta| \frac{1}{-\partial D} |D\eta \rangle - \langle \eta_\mu | G(U_\mu) |\eta_\mu \rangle \leq 0. \]  

(3.56)

**Proof** is self-evident from the fact that since \( D\eta \) is \( L \)-periodic for \( \gamma \eta \), one can choose \( \omega \) such that

\[ \omega - \frac{1}{-\partial D} D\eta = 0. \]  

(3.57)

§4. Horizon function, horizon condition and Kugo-Ojima color confinement criterion

**Horizon tensor** is defined as

\[ H_{\mu\nu} = -D_\mu (-\partial D)^{-1} D_\nu - \delta_{\mu\nu} G(U_\mu). \]  

(4.1)

Taking the trace of the operator \( H_{\mu\nu} \) with respect to the normalized constant colored vectors \( \eta_{\mu}^{\nu,a} = \delta_{\mu\nu} \lambda_a \) with \( \text{tr} \lambda_a^\dagger \lambda_b = \delta_{ab} \), one defines the **horizon function** \( H(U) \) as

\[ H(U) = \sum_{\nu,a} \langle \eta_{\mu}^{\nu,a} | H_{\mu\rho} |\eta_{\rho}^{\nu,a} \rangle \]

\[ = \sum_{\nu,a} \langle \eta_{\mu}^{\nu,a} | - D_\mu (-\partial D)^{-1} D_\rho |\eta_{\rho}^{\nu,a} \rangle - (N^2 - 1) E(U) \]

\[ \equiv h(U)V \]  

(4.2)

where

\[ (N^2 - 1) E(U) = \sum_{x,\mu,a} \text{tr}(\lambda_a^\dagger G(U_{x,\mu}) \lambda_a). \]  

(4.3)

From Theorem 2, one has for \( U \in \Xi_L \) that

\[ \bar{A}_\mu = V^{-1} \sum_x A_{x,\mu} = 0 \]  

(4.4)

and

\[ H(U) \leq 0, \]  

(4.5)

where \( V = L^4 \).
Zwanziger hypothesized that the dynamics on $\Xi_L$ tends to that on $\Lambda_L$ in the infinite volume limit, and derived the horizon condition, statistical average

$$\langle h(U) \rangle = 0, \quad (4.6)$$

in the infinite volume limit. Taking the Fourier transform of the tensor propagator of the color point source,

$$\langle -D_\mu (-\partial D)^{-1} D_\nu \rangle_{xa,yb}, \quad (4.7)$$

one has, with assumption of the global color symmetry not broken,

$$G_{\mu\nu}(p)\delta^{ab} = \delta^{ab} \left[ (e/d)(p_\mu p_\nu/p^2) - \{\delta_{\mu\nu} - (p_\mu p_\nu/p^2)\} u(p^2) \right], \quad (4.8)$$

where

$$e = \langle E(U) \rangle / V$$

and dimension $d = 4$.

In local operator formalism of QCD, Kugo and Ojima proposed color confinement criterion based on the BRST (Becchi-Rouet-Stora-Tyutin) symmetry without Gribov's problem taken into account. Kugo-Ojima two-point function in the continuum theory is defined in the lattice Landau gauge QCD as

$$(\delta_{\mu\nu} - p_\mu p_\nu/p^2) u^{ab}(p^2) = \frac{1}{V} \sum_{x,y} e^{-ip(x-y)} \left\langle \text{tr}(\lambda^a D_\mu \frac{1}{-\partial D} [A_\nu, \lambda^b]_{xy}) \right\rangle \quad (4.10)$$

where $u^{ab}(p^2) = \delta^{ab} u(p^2)$ and it was shown that the sufficient condition of color confinement is given by

$$u(0) = -1. \quad (4.11)$$

Putting Kugo-Ojima parameter as

$$u(0) = -c \quad (4.12)$$

and comparing

$$\lim_{p_\mu \to +0} G_{\mu\mu}(p) \quad (4.13)$$

with

$$\langle h(U) \rangle = 0 \quad (4.14)$$

one finds that the horizon condition reduces to

$$\left\langle \frac{h(U)}{N^2 - 1} \right\rangle = \left( \frac{e}{d} \right) + (d - 1) c - e = (d - 1) \left( c - \frac{e}{d} \right) \equiv (d - 1) h = 0. \quad (4.15)$$

Kugo-Ojima's and Zwanziger's arguments emerge to be consistent with each other provided the lattice covariant derivative naturally meets with the continuum one

$$e/d = 1. \quad (4.16)$$

This fact was pointed out and some numerical data were presented.8), 9)
§5. Horizon condition in case of free boundary condition

Let us consider the following quantity

$$\langle \partial_\mu \phi | D_\mu \phi \rangle$$

(5.1)

on the arbitrary networks with assumptions that \( \partial A = 0 \) in the sense of (1.13), i.e.,

$$\partial_\mu A_\mu = \sum_{\mu \in \ell^+ (x)} (A_\mu)_+ - \sum_{\mu \in \ell^- (x)} (A_\mu)_-$$

(5.2)

where we put indices \( \pm \) for link-field \( A_\mu \) if \( \mu \in \ell^\pm (x) \), and with an assumption that a scalar function \( \phi \) does not have the zero-eigenvalue eigenvector component of \(-\partial D(U)\). It follows that

$$\langle \partial_\mu \phi | D_\mu \phi \rangle = \langle \phi | -\partial D | \phi \rangle$$

$$= \langle \phi | (-\partial D) \frac{1}{-\partial D} (-\partial D) | \phi \rangle$$

$$= \langle -\partial D \phi | \frac{1}{-\partial D} (-\partial D) | \phi \rangle.$$  

(5.3)

One finds the Faddeev-Popov operator is symmetric when \( \partial A = 0 \) which is seen below.

$$D_\mu \partial_\mu \phi = \partial_\mu G(U_\mu) \partial_\mu \phi + [A_\mu, \partial_\mu \phi]^\mu,$$

(5.4)

$$\partial_\mu D_\mu \phi = \partial_\mu G(U_\mu) \partial_\mu \phi + \partial_\mu [A_\mu, \tilde{\phi}^\mu],$$

(5.5)

$$\left( [A_\mu, \partial_\mu \phi]^\mu - \partial_\mu [A_\mu, \tilde{\phi}^\mu] \right) = \sum_{\mu \in \ell^+ (x)} \left( [A_\mu, \frac{1}{2} (\partial_\mu \phi)]_+ - [A_\mu, \tilde{\phi}^\mu]_+ \right)$$

$$+ \sum_{\mu \in \ell^- (x)} \left( [A_\mu, \frac{1}{2} (\partial_\mu \phi)]_- + [A_\mu, \tilde{\phi}^\mu]_- \right)$$

$$= - \left( \sum_{\mu \in \ell^+ (x)} [A_\mu, \tilde{\phi}^\mu - \frac{1}{2} (\partial_\mu \phi)]_+ $$

$$+ \sum_{\mu \in \ell^- (x)} [-A_\mu, \tilde{\phi}^\mu + \frac{1}{2} (\partial_\mu \phi)]_- \right)$$

$$= - \left[ \sum_{\mu \in \ell^+ (x)} A_\mu - \sum_{\mu \in \ell^- (x)} A_\mu, \phi_x \right]$$

$$= - [\partial_\mu A_\mu, \phi_x].$$

(5.6)

Thus it hold that if \( \partial A = 0 \), then

$$D_\mu \partial_\mu \phi = \partial_\mu D_\mu \phi,$$

(5.7)
and
\[ \langle D\partial\phi|\frac{1}{-\partial D}|D\partial\phi\rangle - \langle \partial_\mu\phi|D_\mu\phi\rangle = 0. \] (5.8)

This equation (5.8) is an identity which holds on any networks when \( \partial A = 0 \) and \( \phi \) is free from zero-eigenvalue eigenvector component of \( -\partial D \).

Now let us consider the case with free boundary condition of regular lattice \( L^d \).

and let us assume \( \phi = x\eta \) where \( \eta_\mu \)'s are suitably normalized antihermitian matrices such that
\[ [\eta_\mu, \eta_\nu] = 0. \] (5.9)

Then it holds from
\[ \partial_\mu\phi = \eta_\mu \] (5.10)

and from
\[ \langle \eta_\mu | [A_\mu, \overline{\phi}\overline{\phi}] \rangle = \langle A_\mu | [\overline{\eta}\overline{\phi}, \eta] \rangle = 0 \] (5.11)

that
\[ \langle D_\mu\eta_\mu|\frac{1}{-\partial D}|D_\nu\eta_\nu\rangle - \langle \eta_\mu | G(U_\mu) | \eta_\mu \rangle = 0. \] (5.12)

By putting
\[ \eta_\mu^{\rho,\alpha} = \delta_{\mu\rho}\lambda_\alpha, \] (5.13)

one obtains
\[ D_\mu\eta_\mu^{\rho,\alpha} = D_\nu\eta_\nu^{\rho,\alpha} = D_\rho\lambda_\alpha \] (5.14)

and then the vanishing horizon function
\[ H(U) = \sum_{\rho,\alpha} \left( \langle D_\rho\lambda_\alpha|\frac{1}{-\partial D}|D_\rho\lambda_\alpha\rangle - \langle \lambda_\alpha | G(U_\rho) | \lambda_\alpha \rangle \right) = 0. \] (5.15)

It is to be noted that \( \lambda_\alpha \) in equation (5.15) is located on links, and as seen from (5.12), \( D_\rho \) acts on \( \lambda_\alpha \) as defined in (2.10) with non-vanishing first term.

§6. Discussions and conclusions

In §1, notations and generalities are given in order to discuss the case of free boundary condition from more general point of view, i.e., network QCD. The main purposes are to discuss behavior of the optimizing function for Landau gauge, gauge non-invariant function, and we need extended definitions, e.g., \( \partial_\mu\phi \) and \( \partial_\mu B_\mu \), with clear distinction between site-functions and link-functions, and finally to obtain an identity which holds on arbitrary networks. As a matter of course, full formulation of network QCD is out of scope of the present study.

In §2, definitions of gauge field, \( U \)-liner type and log \( U \)-type, are given, together with covariant derivatives for each type, respectively, where difference between covariant derivative (2.3) and covariant divergence (2.10) should be noted.

In §3, following Ref. 3), various kinds of regions in Landau gauge are defined, i.e., Gribov region, \( \Omega_L \), fundamental modular region, \( A_L \), and core region, \( \Xi_L \), on regular lattice of period \( L \), where the following inclusions hold,
\[ \Xi_L = (A_\infty \cap \Pi_L) \subset A_L \subset \Omega_L. \] (6.1)
Theorem 2 states that for each $U_\mu \in \Xi_L = \Lambda_\infty \cap \Pi_L$, the horizon function defined in §4 (4.2) takes non-positive value, $H(U) \leq 0$.

In §4, although we have skipped the derivation of the horizon condition, (4.6), Zwanziger showed in Ref. 3) that it can be derived from statistical average on augmented core region, $\Psi_L$, in infinite volume limit, $L \to \infty$, where

$$\Xi_L \subset \Psi_L \equiv \{ U | H(U) \leq 0, \ U \in \Omega_L \} \subset \Omega_L. \quad (6.2)$$

It is reviewed that the horizon condition and the Kugo-Ojima criterion of the color confinement(5) coincide with each other in the continuum limit.(6), (7)

In §5, we focus on the fact that the horizon condition holds for each configuration in Landau gauge on finite regular lattice with the free boundary condition. (7) It is found that the fact can be derived from the equation (5.8),

$$\langle D\partial \phi | -1/\partial D | D\partial \phi \rangle - \langle \partial_\mu \phi | D_\mu \phi \rangle = 0,$$

that is an identity which holds on any networks when $\partial A = 0$ and $\phi$ is free from zero-eigenvalue eigenvector component of $-\partial D$. Obvious reason why vanishing horizon function (5.15) is not realized in the case of the periodic boundary condition is that (5.10) does not hold. In a special case of $U = 1$ with free boundary condition, (5.15) turns out to be

$$H(U) = \sum_{\rho, a} \left( \langle \partial_\rho \lambda_a | 1/\partial_\rho \partial_\rho \lambda_a \rangle - \langle \lambda_a | 1 | \lambda_a \rangle \right) = 0, \quad (6.3)$$

where there appear $\pm \delta_0$ 'charge density' only on the boundary surfaces. However, in the generic non-constant $U$ cases, 'charge density' $D_\rho \lambda_a$ in (5.15) spreads over $d$-dimensional volume. Thus one of the most important open questions is if the boundary condition affects the physics in the 4-dimensional bulk system in the thermodynamic limit, i.e., continuum limit.

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