Universal low-temperature magnetic properties of the classical and quantum
dimerized ferromagnetic spin chain

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Low-temperature magnetic properties of both classical and quantum dimerized ferromagnetic
spin chains are studied. It is shown that at low temperatures the classical dimerized model re-
duces to the classical uniform model with the effective excha-

\[ J_0 = J(1 - \delta^2), \]

n

n+n+1

and spin correlation function are calculated

\[ H = H_{\text{spin}} + E_{\text{elastic}} \]

where

\[ H_{\text{spin}} = J \sum_{n=1}^{N} (1 - (-1)^n \delta) \mathbf{S}_n \cdot \mathbf{S}_{n+1} \]

\[ E_{\text{elastic}} = \frac{N \kappa \delta^2}{2} \]

where \( \mathbf{S} \) is the spin operator, \( J \) is the exchange integral, \( \delta \) is the dimerization parameter characterizing lattice
distortion, \( \kappa \) is the effective elastic constant.

I. INTRODUCTION

The Paierls instability plays an important role in quasi-one-dimensional materials. The Paierls metal-dielectric
transition originated from the coupling between electrons and phonons occurs, as a rule, in organic solids[1]. Such
transition can take place in the quantum spin chains coupled to phonons as well (so-called the spin-Paierls transition
(SP)). In particular, the SP transition has been observed in the antiferromagnetic spin chain \( \text{CuGeO}_{3} \)[2]. Currently
there is a growing interest to the quasi one-dimensional magnets with ferromagnetic exchange interactions[3-5] and
the possibility of the Paierls instability in them is widely discussed[6]. It is argued that the possible real system
where this instability takes place is monatomic chains of \( \text{Co} \) on the \( \text{Pt} \) surface[7]. As it is proposed[7], these chains
have the effectively ferromagnetic spin-spin interaction and very weak elastic constants. Recently, another mechanism
of the Paierls instability in spin systems has been proposed[8,9]. It is based on the coupling of the spins with an
electronic subsystem (spin-orbital mechanism). It is expected that this mechanism is relevant to the transition metal
oxide \( \text{YVO}_3 \). The properties of this compound is described by the spin-orbital model[10]. The mean-field treatment
of this model leads to the 1D dimerized spin model with the ferromagnetic sign of the interaction[4].

Therefore, the study of the dimerized ferromagnetic (FM) chains is important from both theoretical and experimen-
tal points of view. The spin chains with the spin-phonon interaction are often described in the adiabatic approximation
which is valid if the phonon energy is smaller than the Paierls gap. The Hamiltonian of this 1D spin model has the
form

\[ H = H_{\text{spin}} + E_{\text{elastic}} \]
There is essential difference between the antiferromagnetic (AF) and ferromagnetic (FM) chains with respect to the coupling to the lattice distortions. For the AF model \((J > 0)\) the ground state energy of model \([2]\) is \(\sim -N\delta^{4/3}\) and the gain in this energy exceeds the loss in \(E_{\text{elastic}}\) and the SP transition takes place. Contrary to the AF model the ground state of \(H_{\text{spin}}\) at \(J < 0\) does not depend on \(\delta\) and the ground state of model \([1]\) has the uniform lattice, \(\delta = 0\). However, as was shown in Ref.[6] the thermal fluctuations can activate the dimerization. It was shown in Ref.[6] that the free energy of the FM chain at \(T > 0\) is \(-NT^{3/2}\delta^{2}\) and the dimerized phase is stable at finite temperature for small enough elastic constant \(\kappa\). Such situation can occur in the system Co chains on the Pt surface where the Co atoms can be easily moved on the surface.

The dimerized FM chain can not been solved by the Bethe ansatz in contrast with the uniform model with \(\delta = 0\). The thermodynamics of this model has been studied in Ref.[11] using both numerical TMRG simulations and the analytical modified spin-wave theory \([13]\). In particular, the phase diagram of the model has been determined and the behavior of spin correlation functions and the zero-field susceptibility are studied.

It is interesting to consider the influence of the external magnetic field on the thermodynamics of the dimerized FM chain. The aim of the present paper is to study the low-temperature magnetic properties of model \([2]\) independent of the dimerization mechanism. Therefore, we will consider the model with the Hamiltonian having a form

\[
H = -J \sum (1 - (-1)^n) \mathbf{S}_n \cdot \mathbf{S}_{n+1} - h \sum S_n^z
\]

where \(h\) is the dimensionless magnetic field and \(J > 0\).

Before we study this model it is instructive to note the remarkable fact related to the uniform FM chain, \(\delta = 0\). It was claimed in Ref.[14] that the normalized magnetization \(M = \langle S^z \rangle / s\) of this model at \(T \to 0\) is the universal function of the scaling variable \(g = s^3 J h / T^2\), i.e. the universal function \(M(g)\) is valid for any spin value \(s\) and all dependence on \(s\) is captured in the scaling variable \(g\). In other words, this function is valid for both quantum and classical ferromagnetic chains.

According to Ref.[14] the form of the function \(M(g)\) can be determined by the computation of the magnetization of the classical FM chain in so-called scaling limit, when \(T \to 0\) and \(h \to 0\) but the value of \(g\) is finite. As was shown in Ref.[14, 15] the calculation of the partition function of the classical model reduces to an eigenvalue problem of a single quantum rotator. It allows to use the efficient method to compute the function \(M(g)\). As a result the explicit expansions of this function in small and large \(g\) were obtained \([14]\).

We will show that the universal dependence of the magnetization holds for dimerized model \([1]\) as well. The corresponding universal function coincides with that found in Ref.[14] but the scaling variable \(g\) is renormalized by a simple way to include the dimerization parameter \(\delta\), namely \(g\) is replaced by \(\gamma = g(1 - \delta^2)\).

The paper is organized as follows. In Section II we consider the partition function of the dimerized classical ferromagnetic chain in the magnetic field. We show that the field-dependent part of the free energy coincides in the scaling limit with that for the uniform model with the renormalized scaling parameter. In Section III it is demonstrated that the spin-wave expansion of the dimerized quantum ferromagnetic chain reproduces the large \(\gamma\) expansion of the magnetization of the classical model. In Section IV the summary of the results is given and the generalization to the model with the next-nearest neighbor antiferromagnetic interaction is discussed.

## II. CLASSICAL DIMERIZED SPIN CHAIN IN THE SCALING LIMIT

In this section we show that the low-temperature magnetic properties of the classical dimerized spin model are reduced to that of the FM spin chain with the renormalized exchange integral.

It is convenient to represent the Hamiltonian function of the considered classical model \([1]\) in a form

\[
H = -J_1 \sum \mathbf{S}_{2i-1} \cdot \mathbf{S}_{2i} - J_2 \sum \mathbf{S}_{2i} \cdot \mathbf{S}_{2i+1} - \sum \mathbf{h} \cdot \mathbf{S}_i
\]

where \(J_1 = J(1 + \delta),\) \(J_2 = J(1 - \delta),\) and \(\mathbf{S}_i\) are spin vectors of the fixed length \(s\) and the magnetic field is directed along the Z axis: \(\mathbf{h} = (0, 0, h)\).

We represent spin vectors on odd and even sites as follows:

\[
\mathbf{S}_{2i-1} = s \mathbf{n}_i \\
\mathbf{S}_{2i} = s \mathbf{n}_i + s \mathbf{\tilde{n}}_i
\]

where \(\mathbf{n}_i\) are unit vectors and \(\mathbf{\tilde{n}}_i\) are vector differences between neighbor spins (we will assume \(|\mathbf{\tilde{n}}_i|\) to be small at low temperatures).
Then, the scalar products of spins on odd and even bonds become:

\[ \vec{S}_{2i-1} \cdot \vec{S}_{2i} = s^2 - \frac{1}{2}(\vec{S}_{2i-1} - \vec{S}_{2i})^2 = s^2 - \frac{s^2}{2} \vec{n}_i^2 \]
\[ \vec{S}_{2i} \cdot \vec{S}_{2i+1} = s^2 - \frac{s^2}{2}(\vec{n}_i' - \vec{n}_i)^2 \]

where we denoted

\[ \vec{n}_i' = \vec{n}_{i+1} - \vec{n}_i \]

After simple algebra the Hamilton function can be transformed to the form

\[ H = -\frac{\hbar^2 N}{2(J_1 + J_2)} + s^2 \frac{J_1 + J_2}{2} \sum \left( \vec{m}_i - \frac{sJ_0 \vec{n}_i'}{s} \right)^2 + \frac{s^2 J_1 J_2}{2(J_1 + J_2)} \sum \vec{n}_i^2 - 2s \sum \vec{h} \cdot \vec{n}_i \]

Then, the partition function of the system reads:

\[ Z = \int \ldots \int \prod_{i=1}^{N/2} d\vec{m}_i d\vec{n}_i \exp \left( -\frac{H \{\vec{m}_i, \vec{n}_i\}}{T} \right) \]

We stress that up to here we did not do any assumption and Eq.(9) and Eq.(10) are the exact expressions. Now we assume that for low temperature \( T \ll (J_1 + J_2) \) all vectors \( \vec{m}_i \) are small and directed in plane perpendicular to the corresponding vectors \( \vec{n}_i \). Then, we can integrate over vectors \( \vec{m}_i \) in the infinite limits. That gives

\[ Z = e^{\frac{\hbar^2 N^2}{2s^2(J_1 + J_2)} \left( \frac{2\pi T}{J_1 + J_2} \right)^{N/2}} \int \ldots \int \prod_{i=1}^{N/2} d\vec{n}_i \exp \left( \frac{s^2 J_1 J_2}{T(J_1 + J_2)} \sum (\vec{n}_i \cdot \vec{n}_{i+1} - 1) + \frac{2s}{T} \sum \vec{h} \cdot \vec{n}_i \right) \]

The first factor in Eq.(11) gives a constant contribution to the magnetic susceptibility \( \sim 1/J \). As will be shown below, in the low-temperature limit the main contribution is given by the integral in Eq.(11) and it is much higher \( (\sim J/T^2) \). Therefore, we neglect the first factor in Eq.(11). The second factor does not influence on the magnetic properties of the system, and will be omitted. Thus, we reduced the partition function of the dimerized chain to that of the uniform ferromagnetic chain with the effective exchange integral

\[ J_0 = \frac{2J_1 J_2}{J_1 + J_2} = J(1 - \delta^2) \]

The partition function of the classical FM chain in the low-temperature limit was calculated in Ref.[14] by taking the continuum limit of the model. We will follow this method. Partition function (11) in the continuum approximation takes the form:

\[ Z \propto \int D[\vec{n}(x)] \exp \left( -\int_0^L dx \left[ \frac{s^2 J_0 a^2}{2a} \left( \frac{d\vec{n}}{dx} \right)^2 - 2\frac{\hbar s}{T} \gamma z \right] \right) \]

where \( L = Na \) and we notice that the distance between neighbor vectors \( \vec{n}_i \) and \( \vec{n}_{i+1} \) is two lattice spaces \( 2a \), so that the vector \( \vec{n}_i \) corresponds to the vector field \( \vec{n}(x) \) at the point \( x = 2ia \) in the continuum limit.

It is useful to transform Eq.(13) to dimensionless variables. We rescale the spatial coordinate \( x = ys^2aJ_0/T \) and obtain

\[ Z \propto \int D[\vec{n}(y)] \exp \left( -\int_0^\lambda dy \left[ \frac{1}{2} \left( \frac{d\vec{n}}{dy} \right)^2 - \gamma n_z \right] \right) \]

where \( \lambda = LT/as^2J_0 \) is the scaled system length and

\[ \gamma = \frac{s^3 J_0 \hbar}{T^2} = g(1 - \delta^2) \]
To calculate the partition function we utilize the well-known equivalence of the $n$-dimensional statistical field theory with the $(n - 1)$-dimensional quantum field theory. The transition amplitude (or propagator) of a particle located initially at $\vec{n}(0) = \vec{n}_0$, and finally at $\vec{n}(\lambda) = \vec{n}_\lambda$ takes the form of a path integral

$$
\langle \vec{n}_\lambda | e^{-\lambda \hat{H}} | \vec{n}_0 \rangle \propto \int_{\vec{n}_0}^{\vec{n}_\lambda} D[\vec{n}(y)] \exp \left\{ \int_0^\lambda L(\vec{n}', \vec{n}) dy \right\}
$$

(16)

where $\hat{H}(\vec{n})$ is the Hamiltonian operator obtained by quantization of the Hamilton function $H$ corresponding to the Lagrangian $L$.

Then, imposing the periodic boundary conditions $\vec{n}_\lambda = \vec{n}_0 = \vec{n}$ and integrating over $\vec{n}$, we represent the partition function (14) in a form

$$
Z \propto \int d\vec{n} \langle \vec{n} | e^{-\lambda \hat{H}} | \vec{n} \rangle
$$

(17)

The quantum Hamiltonian has the form [14]:

$$
\hat{H} = \frac{\hat{l}^2}{2} - \gamma n_z
$$

(18)

where $\hat{l}$ is an angular momentum operator. Hamiltonian (18) describes the quantum rotator in the field $\gamma n_z$ and coincides with the Hamiltonian for the uniform model [14] with $g$ replaced by $\gamma$.

The corresponding Schrödinger equation in the spherical coordinates has the form:

$$
\left( -\frac{1}{2} \frac{d^2}{d\theta^2} - \frac{\cot \theta}{2} \frac{d}{d\theta} - \frac{m^2}{2 \sin^2 \theta} - \gamma \cos \theta \right) \psi_{nm} = \epsilon_{nm} \psi_{nm}
$$

(19)

where we used an axial symmetry of the model and introduced the azimuthal quantum number $m$.

The exponent of the operator $\hat{H}(\vec{n})$ can be represented using the eigenvalues and the eigenfunctions of the Schrödinger equation as follows:

$$
e^{-\lambda \hat{H}(\vec{n})} = \sum_{nm} |\psi_{nm}(\vec{n})\rangle e^{-\lambda \epsilon_{nm}} \langle \psi_{nm}(\vec{n})|
$$

(20)

Then, the partition function becomes

$$
Z \propto \int d\vec{n} \langle \vec{n} | e^{-\lambda \hat{H}} | \vec{n} \rangle = \sum_{nm} e^{-\lambda \epsilon_{nm}}
$$

(21)

In the thermodynamic limit $\lambda \to \infty$ only the lowest eigenvalue $\epsilon_{00}$ (with $m = 0$) gives contribution to the partition function,

$$
Z \to e^{-\lambda \epsilon_{00}}
$$

(22)

The field-dependent part of the free energy per site is determined by the ground state energy of Eq.(19)

$$
F = \frac{T^2 \epsilon_{00}(\gamma)}{s^2 J (1 - \delta^2)}
$$

(23)

The normalized magnetization $M = \langle n^z \rangle$ is

$$
M = -\frac{\partial \epsilon_{00}(\gamma)}{\partial \gamma}
$$

(24)

The solution of the Schrödinger equation [19] has been found analytically for small and large scaling parameter in Ref.[14]. On the other hand, this equation can be solved numerically for all values of $\gamma$ and the magnetization curve can be found. It is shown on Fig.1. To demonstrate $\delta$-dependence of the magnetization we represent it as a function of the scaled magnetic field $g$ rather than $\gamma$. As follows from Fig.1 the increase of the dimerization leads to the decrease of the magnetization for all values of the magnetic field.
A. Spin correlation functions

The low-temperature magnetization of the dimerized FM chain coincides with that for the uniform model and only the scaling parameter $\gamma$ is renormalized. But the spin correlation functions of the dimerized and uniform models are different generally. Besides, the longitudinal and transverse correlators are different for the non-zero magnetic field and we will consider both types of correlators. At first we consider the correlators $\langle S^z_i S^z_{i+2r} \rangle$ and $\langle S^x_i S^x_{i+2r} \rangle = \langle S^y_i S^y_{i+2r} \rangle$ for distances of even number of lattice spacing $2r$. In this case the correlation functions do not depend on $i$ and are defined by the reduced uniform model (11):

$$\langle S^\alpha_i S^\alpha_{i+2r} \rangle = s^2 \langle n^\alpha_0 n^\alpha_r \rangle \quad (25)$$

The correlator $\langle n^\alpha_0 n^\alpha_r \rangle$ can be expressed by the eigenvalues and the eigenfunctions of the Schrödinger equation. We represent the brief derivation of these expressions on the example of the correlator $\langle n^z_0 n^z_r \rangle$. The correlator $\langle n^z_0 n^z_r \rangle$ is by definition

$$\langle n^z_0 n^z_r \rangle = \frac{1}{Z} \int D[\vec{\eta}(y)] n^z(0)n^z(\rho) \exp \left( \int_0^\lambda L[\vec{\eta}(y)]dy \right) \quad (26)$$

where

$$\rho = \frac{2Tr}{as^2J_0} \quad (27)$$

We divide the system on two parts $[0, \rho]$ and $[\rho, \lambda]$. Then the correlator is expressed as an integral of product of propagators:

$$\langle n^z_0 n^z_r \rangle = e^{\lambda\varepsilon_{00}} \int n^z(0)d\tilde{n}(0) \int n^z(\rho)d\tilde{n}(\rho)I(0, \rho)I(\rho, \lambda) \quad (28)$$

The propagators in the regions $[0, \rho]$ and $[\rho, \lambda]$ are calculated with the use of Eqs. (16) and (20):

$$I(0, \rho) = \int_{\vec{n}_0}^{\vec{n}_\rho} D[\vec{\eta}(y)] \exp \left( \int_0^\rho L[\vec{\eta}(y)]dy \right) = \sum_{nm} \langle \vec{\eta}_\rho | \psi_{nm}(\vec{\eta}) \rangle e^{-\rho \varepsilon_{nm}} \langle \psi_{nm}(\vec{\eta}) | \vec{n}_0 \rangle$$

and, similarly,

$$I(\rho, \lambda) = \sum_{nm} \langle \vec{\eta}_\lambda | \psi_{nm}(\vec{\eta}) \rangle e^{-(\lambda-\rho)\varepsilon_{nm}} \langle \psi_{nm}(\vec{\eta}) | \vec{n}_\rho \rangle$$

\[ \text{FIG. 1: Dependence of the normalized magnetization } M \text{ on the scaled magnetic field } g = s^2 J h / T^2 \text{ for several values of the dimerization parameter } \delta. \]
where $\psi_{nm}$ and $\varepsilon_{nm}$ are the eigenfunctions and the eigenvalues of the Schrödinger equation (19).

Then, imposing the periodic boundary conditions $\tilde{n}_\lambda = \tilde{n}_0$ and using the definition $\tilde{n}_z = \int d\tilde{n} |\tilde{n}| n^z$ we obtain

$$\langle \tilde{n}_0^z \tilde{n}_r^z \rangle = \sum_{nmn'm'} e^{-\lambda(|\varepsilon_{nm} - \varepsilon_{00}| - \rho(|\varepsilon_{n'm'} - \varepsilon_{nm} - \varepsilon_{n'm'}|))} |\langle \psi_{n'm'} | \tilde{n}_z | \psi_{nm} \rangle|^2$$

(31)

In the thermodynamic limit $\lambda \to \infty$ only the lowest level in the region $[\rho, \lambda]$ survives, so that $n' = m' = 0$ and

$$\langle \tilde{n}_0^z \tilde{n}_r^z \rangle = \sum_{nm} |\langle \psi_{00} | \tilde{n}_z | \psi_{nm} \rangle|^2 \exp \left( -\frac{T(\varepsilon_{nm} - \varepsilon_{00})}{s^2 J(1 - \delta^2)} - 2r \right)$$

(32)

The expressions for other types of the correlation functions like $\langle \tilde{n}_0^z \tilde{n}_r^z \rangle$ has the same form as in Eq. (32) with replacing $\tilde{n}_z$ by $\tilde{n}_x$:

$$\langle \tilde{n}_0^x \tilde{n}_r^x \rangle = \sum_{nm} |\langle \psi_{00} | \tilde{n}_x | \psi_{nm} \rangle|^2 \exp \left( -\frac{T(\varepsilon_{nm} - \varepsilon_{00})}{s^2 J(1 - \delta^2)} - 2r \right)$$

(33)

We expect that the universality in the long distance behavior of the correlation functions holds, so we are interested in the asymptotic of the correlation function $r \gg 1$. In this limit only the lowest level(s) having non-zero matrix element makes contribution to Eqs. (32) and (33). The operator $n_x$ has non-zero expectation value over the ground state $\langle \psi_{00} | \tilde{n}_x | \psi_{00} \rangle$, which is the normalized magnetization of the system $M$. Therefore, the main contribution to sum (32) is given by $n = m = 0$ term and equals $M^2$. The decaying correction to this main term is given by the lowest excited state with the same azimuthal number $m = 0, \varepsilon_{10}$. Thus, the long-distance asymptotic for the correlation function $\langle \tilde{n}_0^x \tilde{n}_r^x \rangle$ is

$$\langle \tilde{n}_0^x \tilde{n}_r^x \rangle = M^2 + |\langle \psi_{00} | n_x | \psi_{10} \rangle|^2 \exp \left( -\frac{2r}{\xi} \right)$$

(34)

with the correlation length defined by the energy of the lowest excited states as

$$\xi = \frac{s^2 J(1 - \delta^2)}{T(\varepsilon_{10} - \varepsilon_{00})}$$

(35)

The operator $n_x$ changes the azimuthal number $m$, therefore the lowest level for correlator $\langle \tilde{n}_0^x \tilde{n}_r^x \rangle$ is $\varepsilon_{10}$. This implies that the transverse correlation function does not show the long range order and exponentially decays on large distances:

$$\langle \tilde{n}_0^x \tilde{n}_r^x \rangle = |\langle \psi_{00} | n_x | \psi_{10} \rangle|^2 \exp \left( -\frac{2r}{\xi} \right)$$

$$\xi = \frac{s^2 J(1 - \delta^2)}{T(\varepsilon_{10} - \varepsilon_{00})}$$

(36)

The correlation lengths $\xi_{||}$ and $\xi_{\perp}$ and the preexponential factors can be found analytically in the limits $\gamma \to 0$ and $\gamma \to \infty$. At $\gamma = 0$ the Schrödinger equation (19) reduces to the equation for the operator of angular momentum with well-known spherical eigenfunctions and the spectrum $l(l+1)/2$. So the correlation functions for large $r$ are

$$\langle \tilde{S}_l^z \tilde{S}_{l+2r}^z \rangle = \frac{s^2}{3} \exp(-2r/\xi)$$

(37)

where $\xi_{||} = \xi_{\perp} = \xi$ and

$$\xi = \frac{s^2 J(1 - \delta^2)}{T}$$

(38)

In the limit of high magnetic field ($\gamma \gg 1$) the lowest eigenfunctions and the corresponding eigenvalues are

$$\psi_{00} = 2 \gamma^{1/4} e^{-\sqrt{\gamma} \theta/2}, \quad \varepsilon_{00} = -\gamma + \sqrt{\gamma}$$

$$\psi_{01} = \gamma^{1/4} \theta \psi_{00}, \quad \varepsilon_{01} = -\gamma + 2 \sqrt{\gamma}$$

$$\psi_{10} = (\sqrt{\gamma} \theta^2 - 1) \psi_{00}, \quad \varepsilon_{10} = -\gamma + 3 \sqrt{\gamma}$$

(39)
The correlation functions in the limit \( \gamma \to \infty \) are

\[
\langle S_z^i S_z^{i+2r} \rangle = s^2 M^2 + \frac{s^2}{4\gamma} \exp(-2r/\xi_{||}), \quad \xi_{||} = \frac{1}{2} \sqrt{\frac{sJ(1-\delta^2)}{h}}
\]

\[
\langle S_z^i S_z^{i+2r} \rangle = \frac{s^2}{\sqrt{\gamma}} \exp(-2r/\xi_{\perp}), \quad \xi_{\perp} = \sqrt{\frac{sJ(1-\delta^2)}{h}}
\]

(40)

According to Eqs. (37) and (40) the correlation lengths are changed from \( \xi \sim 1/T \) at \( \gamma = 0 \) to \( \xi \sim h^{-1/2} \) for \( \gamma \to \infty \). The crossover between two types of the behavior of \( \xi \) occurs at \( \gamma \approx 1 \). The dependencies of \( \xi_{\perp} \) on \( g \) for some values of \( \delta \) are shown on Fig. 2.

Now we study the spin correlation function on ‘odd’ distances. In this case the correlation function \( \langle S_{\alpha}^i S_{\alpha}^{i+r} \rangle \) is different for odd and even \( i \). Therefore, we distinguish two types of ‘odd’ correlators: \( \langle S_\alpha^i S_\alpha^{i+2r} \rangle \) and \( \langle S_\alpha^i S_\alpha^{i+r+1} \rangle \). The longitudinal correlators \( \langle S_z^i S_z^{i+r+1} \rangle \) has non-zero asymptotic \( M^2 \) at \( r \to \infty \) and the calculation of the small corrections caused by the dimerization to this value is not important. On the contrary, the transverse correlator \( \langle S_x^i S_x^{i+r+1} \rangle \) decays exponentially and it is more interesting object for the calculation of subtle effects like difference on odd and even distances.

According to Eq. (6), the transverse correlator on odd distances is represented as:

\[
\langle S_\alpha^i S_\alpha^{i+2r} \rangle = s^2 \langle n_0^x (n_r^x + m_r^x) \rangle
\]

(41)

The correlator \( \langle n_0^x n_r^x \rangle \) was found above in Eq. (36). For the correlator \( \langle n_0^x m_r^x \rangle \) we use the following identity:

\[
\int_{-\infty}^{\infty} xe^{-\alpha(x-y)^2} dx = y \int_{-\infty}^{\infty} e^{-\alpha(x-y)^2} dx
\]

(42)

Therefore, the integration over \( \vec{m}_r \) in the multiple integral

\[
\langle n_0^x m_r^x \rangle = \int \ldots \int d\vec{m}_r d\vec{n}_i n_0^x m_r^x \exp \left( -\frac{H \{ \vec{m}_i, \vec{n}_i \}}{T} \right)
\]

(43)

can be transformed as

\[
\int m_r^x \exp \left[ -\frac{J_1 + J_2}{2T} \left( \vec{m}_r - \frac{sJ_2 \vec{n}_r + \vec{h}}{s(J_1 + J_2)} \right)^2 \right] d\vec{m}_r = \frac{J_2 m_r^x}{J_1 + J_2} \int \exp \left[ -\frac{J_1 + J_2}{2T} \left( \vec{m}_r - \frac{sJ_2 \vec{n}_r + \vec{h}}{s(J_1 + J_2)} \right)^2 \right] d\vec{m}_r
\]

(44)
so that

\[ \langle n_0^z n_r^z \rangle = \frac{J_2}{J_1 + J_2} \langle n_0^z n_r^z \rangle \]  

(45)

Using the definition \( n_r^z = n_{r+1}^z - n_r^z \) and Eq. (30) we obtain

\[ \langle S_{2j-1}^z S_{2j+2r}^z \rangle = s^2 \langle \psi_{00} | n_x | \psi_{01} \rangle^2 e^{-2r/\xi_{\perp}} \left( 1 - \frac{J_2(1 - e^{-2/\xi_{\perp}})}{J_1 + J_2} \right) \]

(46)

Here we note that for \( \xi_{\perp} \gg 1 \), which is always assumed for low temperature limit, the last factor in Eq. (46) can be expanded and the correlator takes the form

\[ \langle S_{2j-1}^z S_{2j+2r}^z \rangle = s^2 \langle \psi_{00} | n_x | \psi_{01} \rangle^2 e^{-(2r+1)/\xi_{\perp}} \left( 1 - \frac{\delta}{\xi_{\perp}} \right) \]

(47)

Similar, for the correlator \( \langle S_{2j}^z S_{2j+2r+1}^z \rangle \) we need merely to exchange \( J_2 \leftrightarrow J_1 \), which gives

\[ \langle S_{2j}^z S_{2j+2r+1}^z \rangle = s^2 \langle \psi_{00} | n_x | \psi_{01} \rangle^2 e^{-(2r+1)/\xi_{\perp}} \left( 1 + \frac{\delta}{\xi_{\perp}} \right) \]

(48)

Let us consider an alternation correlation functions [12]:

\[ \Delta_{\perp}(r) = \left| \langle S_n^z S_{n+r}^z \rangle - \langle S_n^z S_{n-r}^z \rangle \right| \]

(49)

It equals zero for even \( r \). But for large odd \( r \) it becomes

\[ \Delta_{\perp} = s^2 \langle \psi_{00} | n_x | \psi_{01} \rangle^2 \frac{2|\delta|}{\xi_{\perp}} e^{-r/\xi_{\perp}} \]

(50)

For the small and large \( \gamma \) we obtain

\[ \Delta_{\perp} = \frac{2T|\delta|}{3J(1 - \delta^2)} e^{-r/\xi_{\perp}}, \quad \gamma \ll 1 \]

\[ \Delta_{\perp} = \frac{2T|\delta|}{J(1 - \delta^2)} e^{-r/\xi_{\perp}}, \quad \gamma \gg 1 \]

(51)

The comparison of correlation functions [57] and [51] for \( \gamma = 0 \) with those for the quantum dimerized FM model obtained in [12] shows that they coincide in the leading terms in \( T \). Therefore, we claim that these correlation functions for \( r \gg 1 \) obtained for the classical model are valid in scaling limit for the quantum model as well.

Comparing Eqs. (47) and (48) one can see that the corrections \( \delta/\xi_{\perp} \) annihilate each other and gives no contribution to the spin structure factor

\[ S_{\perp}(q) = \frac{1}{N} \sum_{j,r} \langle S_j^z S_{j+r}^z \rangle e^{iqr} \]

(52)

which is the sum of these equations. This is valid in the linear in \( \delta/\xi_{\perp} \) terms in the low-temperature limit. The terms \( \sim (\delta/\xi_{\perp})^2 \) can introduce this dimerization effect into \( S_{\perp}(q) \), but this effect is out of the scope of the used continuum approximation. Thus, in the leading term in \( \delta T/J \) the spin structure factor for the dimerized model coincides with that of the FM model with the renormalized exchange coupling. Using Eq. (33) for the correlator \( \langle S_j^z S_{j+r}^z \rangle \) we obtain the spin structure factor in the form

\[ S_{\perp}(q) = \frac{s^4 J(1 - \delta^2)}{T} \sum_n \langle \psi_{00} | n_x | \psi_{n1} \rangle^2 \frac{(\varepsilon_{n1} - \varepsilon_{00})}{(\varepsilon_{n1} - \varepsilon_{00})^2 + \tilde{q}^2} \]

(53)

with \( \tilde{q} = q s^2 J_0/T \). The dependencies of the normalized spin structure factor \( \hat{S}(q) = S_{\perp}(q)T/s^4 J_0 \) on \( \tilde{q} \) for several values of \( \gamma \) is demonstrated in Fig.3.
FIG. 3: Dependencies of the normalized spin structure factor \( \tilde{S}(q) = S_\perp(q)T/s^4J_0 \) on the scaled wave vector \( \tilde{q} = qs^2J_0/T \) for \( \gamma = 0, 1, 3 \).

III. SPIN-WAVE EXPANSION OF THE QUANTUM MODEL

According to the results of Sec.II the magnetization of the classical dimerized model in the scaling limit \( M(\gamma) \) coincides with that found in Ref.[14]. In Ref.[14] a method of the computation of this function to an arbitrary accuracy was developed. We cite several leading terms of the expansion of \( M(\gamma) \) for small and large values of \( \gamma \) obtained in Ref.[14]:

\[
M(\gamma) = \frac{2}{3}\gamma - \frac{44}{135}\gamma^3 + \frac{752}{2835}\gamma^5 + \ldots, \quad \gamma \ll 1
\]

\[
M(\gamma) = 1 - \frac{1}{2\gamma^{1/2}} - \frac{1}{128\gamma^{3/2}} - \frac{3}{512\gamma^2} + \ldots, \quad \gamma \gg 1
\] (54)

However, it is not clear whether the function \( M(\gamma) \) is universal in the sense that it is valid for both the classical and the quantum dimerized FM chains. Below we will produce arguments in favor that such universality is the case.

At first, we compare the zero-field susceptibility given by

\[
\chi(0) = \frac{s^4J(1-\delta^2)}{T^2} \frac{dM}{d\gamma} \bigg|_{\gamma=0}
\] (55)

with the asymptotic of \( \chi(0) \) at \( T \to 0 \) obtained in Ref.[12] for the quantum dimerized FM chain. According to Ref.[12]

\[
\chi(0) = \frac{2}{3} \frac{s^4J(1-\delta^2)}{T^2}
\] (56)

As follows from Eqs.(54)-(56) both expressions for \( \chi(0) \) coincide and function \( M(\gamma) \) correctly describes the limit \( \gamma \to 0 \) for the magnetization of the quantum model. In connection with Eq.(55) we note that it is not applicable for the case of full dimerization, \( \delta = 1 \). In this case the system consists of decoupled dimers and the susceptibility follows the Curie law \( \chi(0) \sim 1/T \). Thus, our approach is valid when \( J(1-\delta^2) \gg T \).

Another check of the hypothesis of the universality is the comparison of the spin-wave expansion for the quantum model with the expansion of \( M(\gamma) \) for large \( \gamma \) given by Eq.(54).

The spin-wave expansion is usually carried out by expressing the spin operators using either the Holstein-Primakoff or the Dyson-Maleev transformations. Here we use the latter which is

\[
S^+_n = \sqrt{2s}(1 - \frac{1}{2s}a^+_n a_n) a_n
\]

\[
S^-_n = \sqrt{2s}a^+_n
\]

\[
S^z_n = s - a^+_n a_n
\] (57)
where \( a_+^\dagger \) and \( a_n \) are the Bose-operators.

Using Eq. (57) we can write Hamiltonian (4) in terms of the Bose-operators. The Bose analog of the spin Hamiltonian contains terms which are quadratic and quartic in the Bose-operators. The Fourier transform to the momentum space operators leads to the Hamiltonian in the form

\[
H = H_0 + H_{int} - Ns J - Nsh
\]

where

\[
H_0 = \sum_k [2s J (1 - \cos k) + \hbar]a_+^\dagger a_k + 2i\delta s J \sum_k \sin k a_+^\dagger a_{k+\pi}
\]

\[
H_{int} = \sum V(k_1', k_2', k_1) a_+^\dagger_{k_1'} a_{k_2} a_{k_1} \delta(k_1' + k_2' - k_2 - k_1) - i\delta \sum W(k_1', k_2', k_2) a_+^\dagger_{k_2'} a_{k_2} a_{k_1} \delta(k_1' + k_2' - k_2 - k_1 - \pi)
\]

\[
V(k_1', k_2', k_1, k_2) = - \frac{J}{4} [\cos(k_1 - k_1') + \cos(k_2 - k_1') + \cos(k_1 - k_2') + 2 \cos(k_1') - 2 \cos(k_2')]
\]

\[
W(k_1', k_2', k_2, k_1) = - \frac{J}{4} [\sin(k_1 - k_1') + \sin(k_2 - k_1') + \sin(k_1 - k_2') + 2 \sin(k_1') + 2 \sin(k_2')]
\]

Hamiltonian (59) can be diagonalized by a standard way. Let us rewrite the Hamiltonian \( H_0 \) using the transformation of \( k \)-sums to those over the reduced Brillouin zone and introducing the new Bose-operators \( \alpha_k \) and \( \beta_k \) by the relation

\[
a_+^\dagger = u_k \alpha_+^\dagger + v_k \beta_+^\dagger
\]

\[
a_{k+\pi}^\dagger = -iv_k \alpha_+^\dagger + iu_k \beta_+^\dagger
\]

where \( |k| < \pi/2 \) and

\[
u_k^2 = \frac{1}{2} + \frac{\cos k}{2\varepsilon(k)}
\]

\[
v_k^2 = \frac{1}{2} - \frac{\cos k}{2\varepsilon(k)}
\]

\[
\varepsilon(k) = \sqrt{1 - (1 - \delta^2) \sin^2 k}
\]

Then, the Hamiltonian \( H_0 \) takes the form:

\[
H_0 = \sum_{|k| < \pi/2} [E_\alpha(k) \alpha_+^\dagger \alpha_k + E_\beta(k) \beta_+^\dagger \beta_k]
\]

where

\[
E_\alpha(k) = 2Js [1 - \varepsilon(k)] + \hbar
\]

\[
E_\beta(k) = 2Js [1 + \varepsilon(k)] + \hbar
\]

Hamiltonian (60) describes the non-interacting bosons. At \( T \to 0 \) the main contribution to the free energy from \( H_0 \) is given by the small \( k \) region. The expansion for \( k \to 0 \) results in

\[
E_\alpha(k) \approx s J (1 - \delta^2) k^2 + \hbar
\]

\[
E_\beta(k) \approx 4s J
\]

so that the thermal occupation numbers of \( \alpha \) and \( \beta \) particles are

\[
n_\alpha(k) = \frac{1}{e^{E_\alpha(k)/T} - 1} \approx \frac{T}{E_\alpha(k)}
\]

\[
n_\beta(k) \approx \exp \left( -\frac{4s J}{T} \right) \to 0
\]
According to Eq. (68), we can omit in Eq. (65) the $\beta$ terms giving the exponentially small contribution to the thermodynamics at $T \to 0$. Then the Hamiltonian $H_0$ takes a form

$$H_0 = \sum [8J(1-\delta^2)k^2 + h]\alpha_k^+\alpha_k$$  \hspace{1cm} (69)$$

Eq. (69) has a form of the Hamiltonian $H_0$ of the uniform FM model with the renormalized exchange integral $J_0 = J(1-\delta^2)$.

Now, let us consider the Hamiltonian $H_{int}$. First of all, we have to express the operators $a_k$ in Eq. (69) by the operators $\alpha_k$ and $\beta_k$ using Eq. (58). As was noted above, for sufficiently low temperatures we can neglect the terms in $H_{int}$ containing $\beta^+_k$ and $\beta_k$ operators. Besides, we can replace the Dyson-Maleev vertices $V$ and $W$ by their long-wavelength limits. Carrying out some algebra for both terms in Eq. (60) we obtain $H_{int}$ in the form

$$H_{int} = -\frac{1}{2}J(1-\delta^2)\sum k_1k_2\alpha_{k_1}^+\alpha_{k_2}^+\alpha_{k_1}\alpha_{k_2}\delta(k_{1}^2 + k_{2}^2 - k_{1} - k_{2})$$  \hspace{1cm} (70)$$

A remarkable fact is that Eq. (70) is nothing but the quartic in the Bose-operators part of the Hamiltonian of the uniform FM chain with the renormalized exchange integral $J_0 = J(1-\delta^2)$. In other words, the Dyson-Maleev vertex of the dimerized chain is renormalized one of the uniform model.

Thus, we established that in the long-wavelength limit, which is justified at low temperatures, the Hamiltonian of the quantum dimerized chain (58) reduces to that of the uniform FM chain with the effective exchange integral $J_0 = J(1-\delta^2)$. That is exactly as was found in Sec.II for the classical spin chains. This fact implies that the known equivalence of the low-temperature magnetic properties of classical and quantum FM chains remains for the dimerized chains. It means that if the universality relatively to the spin value holds for the uniform model then this property remains valid for the dimerized FM chain as well. Nevertheless, it is interesting to compare the large $\gamma$ expansion for the quantum and the classical models.

The calculation of the spin wave expansion for the free energy and the magnetization in all orders in $1/\gamma$ is a complicated problem. But the leading terms can be found analytically. In the zeroth order in $H_{int}$, which corresponds to the linear spin-wave approximation, the magnetization in the scaling limit is

$$M^{(0)} = -\frac{1}{2\sqrt{\gamma}}$$  \hspace{1cm} (71)$$

It is easy to check that the contribution of the first order, $M^{(1)}$, vanishes by a symmetry. The two-loop correction $M^{(2)}$ was calculated in [16] and it is given by

$$M^{(2)} = -\frac{1}{128\gamma^{3/2}}$$  \hspace{1cm} (72)$$

Thus, the spin-wave expansion of the quantum dimerized FM chain is

$$M = 1 - \frac{1}{2\sqrt{\gamma}} - \frac{1}{128\gamma^{3/2}} + O(\gamma^{-2})$$  \hspace{1cm} (73)$$

The comparison of Eq. (54) with Eq. (73) shows that both expansions are identical. Though we can not calculate the spin-wave expansion in all orders, coincidence of the non-trivial terms in Eq. (73) with those for $M(\gamma)$ is a strong argument that the function $M(\gamma)$ gives the low-temperature magnetization of both the classical and the quantum dimerized FM chains.

IV. DISCUSSION

We studied the low-temperature magnetic properties of the classical and quantum dimerized ferromagnetic spin chain. It is shown that at low temperatures the classical dimerized model reduces to the classical uniform model with the effective exchange integral $J_0 = J(1-\delta^2)$, where $\delta$ is the dimerization parameter. The partition function and spin correlation function of the classical model are calculated with use of the mapping to the continuum limit, which is justified at low temperatures. In the continuum limit the field-dependent thermodynamics depends on one scaling parameter $\gamma = \hbar s^3 J(1-\delta^2)/T^2$. The calculation of the partition function and spin correlation function reduces to the solution of the Schrödinger equation for the quantum rotator in the “gravitational” field $\gamma$.

We have studied the influence of the dimerization on the magnetic properties of the classical spin model. In particular, we have shown that the magnetization decreases with the increase of the dimerization. We found the
dependence of the spin correlation functions on both the magnetic field and the dimerization parameter. In contrast with the uniform model the correlation functions as a function of the distance \( r \) are different for the even and odd \( r \). Though the correlation lengths of the spin correlations are the same for even and odd \( r \), the preexponential factors are different.

It was argued in Ref.\[14\] that the magnetization \( M \) of the uniform classical FM chain at \( T \to 0 \) is the universal function of the scaling variable \( g = s^3Jh/T^2 \), i.e. the universal function \( M(g) \) is valid for any spin value \( s \) and the dependence on \( s \) is captured in the scaling variable \( g \) only. It implies that the magnetization curve \( M(g) \) is valid for both quantum and classical FM chains and can be determined by the computation of the magnetization of the classical FM chain in the so-called scaling limit, when \( T \to 0 \) and \( h \to 0 \) but the value of \( g \) is finite.

We have shown that this universality holds for the dimerized chain as well. To confirm this fact we studied the quantum dimerized spin model with the use of the Dyson-Maleev representation of spin operators. It is shown that in the long-wavelength limit, which is justified at low temperatures, the Hamiltonian of the quantum dimerized chain reduces to that of the uniform quantum FM chain with the effective exchange integral \( J_0 = J(1-\delta^2) \). That is exactly the same renormalization of the exchange coupling as was found for the classical dimerized spin chains. This fact implies that the known equivalence of the low-temperature magnetic properties of classical and quantum FM chains remains valid for the dimerized chains.

The physical reason of the equivalence of the low-temperature magnetic properties of quantum and classical models is that the de Broglie wavelength of spin waves \( \lambda_B \) is less than the ferromagnetic correlation length \( \xi \). Indeed, for the spectrum \( Jk^2 \) the de Broglie wavelength \( \lambda_B \sim 1/k \sim \sqrt{J/T} \), while \( \xi \sim J/T \). This implies that the physical properties defined by the long-distance asymptotics like magnetization are equal for quantum and classical models.

But on the short-distances the equivalence failed. For example, the short-distance correlation function behaves as \( \langle S_i^z S_{i+r}^z \rangle \sim s^2 - \alpha r^2 \) in quantum case \[12\], while the classical model has \( \langle S_i^z S_{i+r}^z \rangle \sim s^2 - br \).

The considered dimerized ferromagnetic model can be generalized by including in Hamiltonian \[4\] the next-nearest-neighbor antiferromagnetic exchange interaction

\[
J_{13} \sum S_n \cdot S_{n+2}, \quad J_{13} > 0
\]  

This term leads to the frustration. It is known \[15\] that the ground state of the quantum dimerized chain with this interaction has the ferromagnetic ground state for \( \alpha = 4J \) \( < \frac{1-\delta^2}{4} \) (\( \alpha \) is the frustration parameter) and the singlet ground state with the helical spin correlations for \( \alpha > \frac{1-\delta^2}{4} \). For the classical model the transition from the ferromagnetic to the helical phase occurs at the same value \( \alpha = \frac{1-\delta^2}{4} \). The classical dimerized FM chain with the frustration can be studied in full analogy with that for the pure ferromagnetic model. In particular, the low-temperature thermodynamics is defined by the solution of the Schrödinger equation \[19\] with the scaling parameter

\[
\gamma = \frac{h s^3 J(1-\delta^2 - 4\alpha)}{T^2}
\]  

It is easy to check that all presented results are valid for the frustrated model. In particular, the free energy and the magnetization are equal to those for the uniform FM chain with the renormalized exchange integral \( J_0 = J(1-\delta^2-4\alpha) \). Therefore, we believe that the magnetization is described by the universal function, which is valid for both classical and quantum model if the frustration parameter \( \alpha \) is not too close to \( \frac{1-\delta^2}{4} \). The behavior of the magnetization and the susceptibility in the the critical point \( \alpha = \frac{1-\delta^2}{4} \) it is radically different \[18\] \[19\]. For example, the zero-field susceptibility \( \chi(0) \sim T^{-4/3} \) in contrast with \( T^{-2} \) behavior for \( \alpha < \frac{1-\delta^2}{4} \).

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