Scaling is one of the cornerstone concepts in modern physics and plays a key role in understanding critical phenomena, particularly quantum phase transitions (QPTs) [1-4]. Near criticality all physical properties are described by a set of critical exponents and scaling functions. These are universal and only depend on the symmetry and dimensionality of the problem (the universality class of the transition), but not on the microscopic details of the Hamiltonian [5]. Unfortunately, the actual computation of critical exponents and particularly the scaling functions is rarely possible analytically [6]. When it is though, these exact results can be quantitatively applied to real word phase transitions, without knowing almost anything about the microscopic details of the experimental system.

One notable analytically solvable case are field-induced quantum phase transitions in one-dimensional ($d = 1$) quantum Heisenberg antiferromagnets (HAFs) between their various gapped and gapless phases. These transitions are understood in terms of a “condensation” of bosonic quasiparticles with a hard-core repulsion, where the magnetic field plays the role of a chemical potential [7-8]. The dynamical critical exponent is $z = 2$. All other critical exponents $\beta, \gamma, \delta$ and scaling functions are also known exactly [9-11]. A series of recent experiments on the $S = 1/2$ chain compound $\text{Cu}(\text{pz})(\text{NO}_3)_2$ [12-17] and the spin ladder system $(\text{CuH}_{12}\text{N}_2)\text{CuBr}_4$ (BPCB) [18-19] have provided a spectacular verification of the predicted scaling behavior of thermodynamic quantities and quasistatic local fluctuations. However, the scaling functions for spatial and temporal correlations at this transition are also fully universal. Calculating them exactly is a formidable task, but has been accomplished [10, 20, 21]. Can they be also measured experimentally?

One recent attempt was made in inelastic neutron scattering experiments on the spin chain compound $\text{K}_2\text{CuSO}_4\text{Cl}_2$ near magnetic saturation [24]. The measurement of quantum critical correlations largely failed due to an unexpected problem: for spin chains, the relevant scattering from critical spin fluctuations transverse to the applied magnetic field overlaps with that from strong non-critical longitudinal fluctuations. In the present work we overcome this seemingly insurmountable hurdle by performing similar measurements on a spin ladder material, and exploiting the ladder’s intrinsic rung-exchange symmetry to separate the two types of scattering. We measure the scaling function for the local dynamic structure factor over one and a half decades in $\hbar\omega/k_BT$ and find a good agreement with exact results for hard core bosons. We also measure the scaling of specific heat, and discuss some peculiar non-universal features of the excitation spectrum.

Our target compound is the well known strong-rung HAF $S = 1/2$ ladder system $(\text{CuH}_{12}\text{N}_2)\text{CuBr}_4$ (BPCB) [24]. The spin ladders are formed by magnetic Cu$^{2+}$ cations linked by super-exchange bridges via Br$^{-}$ anions [25]. They run along the $a$ axis of the monoclinic crystal structure (P2$_1$/c, $a = 8.49$, $b = 17.22$, $c = 12.38$ Å, $\beta = 99.3^\circ$ [24]), and are well separated by non-magnetic organic piperidinium molecules (Fig. 1). The ground state is a quantum spin singlet [26] with a gap $\Delta \approx 0.8$ meV in the magnetic excitation spectrum [27]. In an applied magnetic field the degeneracy of the triplet band is lifted by the Zeeman effect [27]. The present work is focused on the $z = 2$ quantum phase transition that occurs at a critical magnetic field $H_c = 6.66(6)$ T applied along the crystallographic $b$ axis [28], at which the gap for the lowest magnon branch closes and the ladder starts to get magnetized [18, 27]. For all our measurements we use fully deuterated single crystal BPCB samples grown from a saturated ethanol solution by slow evaporation.

As a first step, we verify that the thermodynamics of the transition is quantitatively consistent with exact results for the $d = 1$, $z = 2$ quantum critical point. The scaling form of specific heat is given by $C_p T^{-\alpha} = \mathcal{E}(r)$, where $r = g\mu_B \mu_0 (H - H_c)(k_BT)^{-z\nu}$ is the scaled magnetic field. The spin stiffness $\gamma = \hbar^2 / J_{\text{Min}}$, $J_{\text{Min}}/k_B = 5.07(10)$ K is directly determined from the measured magnon dispersion (see supplement, [29]). Due to universality, the scaling exponents $\alpha = 1/2$ and $\nu = 1$ and
the scaling function
\[
\mathcal{E}(r) = \frac{N_A k_B}{\pi \hbar} \sqrt{2m k_B} \int_0^\infty dx \frac{e^{x^2-r(x^2-r)^2}}{(e^{x^2-r}+1)^2}.
\]
are exactly as for a gas of non-interacting Fermions with a quadratic dispersion [8], Note that Eq. contains no arbitrary scaling factors [8]. It is plotted in Fig. 2a).

To compare this prediction to experiment, we carried out relaxation measurements of the magnetic specific heat of BPCB in a wide range of magnetic fields and temperatures around the transition. The data were collected on a 0.72(4) mg single crystal using a Quantum Design PPMS equipped with a 3He-4He dilution refrigerator insert. A calculated small nuclear spin contribution and a phonon contribution extrapolated from high temperatures were subtracted from the measured data. Typical data are shown in Fig. 2a. In agreement with expectation, at $H < H_c$ where the spectrum is gapped, the heat capacity shows characteristic activation behavior. At $H > H_c$, on the other hand, the observed temperature dependence is linear, characteristic of the gapless Tomogana-Luttinger liquid (TLL) regime [30]. To independently determine whether the data obey scaling, we adopted the approach described in Ref. 13. For every pair of critical exponents $(a, z\nu)$ the scaled specific heat $C_p T^{-a}$ was fit to a 5th degree polynomial of the scaled field $r = g\mu_B H/(k_B T)^{z\nu}$. The mean squared error (MSE) of the fit then serves as an empirical measure for the quality of the data collapse. Only data close to the critical point, with $0.17 < T < 0.5$ K and $|r| < 4$ were included. The resulting MSE landscape is plotted vs. $a$ and $z\nu$ in the inset in Figure 2b). Optimal scaling is found for $a = 0.57(10)$ and $b = 1.01(10)$, in agreement with the theoretical values $a = 1/2$ and $z\nu = 1$. In fact, the latter produce an excellent data collapse, as plotted in symbols in Fig. 2b). Where all data points in the range $0.17 < T < 0.5$ K were considered. A spectacular agreement with the exact theoretical scaling function is obtained with no adjustable parameters, not even an overall scale factors. This validates BPCB and the corresponding field-induced transition as a suitable realization of the $d = 1$, $z = 2$ QCP. At $T < 0.17$ K small 3D interactions are relevant [31, 32], whilst for $T > 0.5$ K we also begin to observe deviations from scaling.

The main purpose of this work is a direct measurement of the scaling function for the dynamic spin structure factor. For a spin ladder, the total spin structure factor (spatial and temporal Fourier transform of the spin correlation function) $S^\alpha(q, \omega)$ can be decomposed into its symmetric and antisymmetric parts, $S^\alpha_s(q||, \omega)$ and $S^\alpha_a(q||, \omega)$, respectively [33, 34]. Here $q|| = Q \cdot a$ is the wave vector transfer along the leg $a$ of the spin ladder, which for BPCB coincides with the crystallographic $a$ axis. These two structure factors represent correlations between the sums and differences of the two spins on each ladder rung, correspondingly. In the transition at hand, due to antiferromagnetic interactions on the ladder rungs in BPCB, it is the antisymmetric fluctuations that become critical. The dynamic structure factor is a tensor quantity, but it is the spin components that are transverse to the applied field that become critical. Thus, in regard to universal critical dynamics, the quantity of interest is $S^{\perp}(q||, \omega)$. Here, we focus on the $q||$,-integrated local dynamic structure factor.

\[
S^{\perp}(\omega) = \int S^{\perp}(q||, \omega) dq||.
\]
the predicted scaling form \[ S_\perp(\omega) = T^{-b}\Phi\left(\frac{\hbar\omega}{k_B T}\right), \quad b = 1/2, \] (3)
with a completely universal scaling function \( \Phi(x) \) that is known exactly \[ 8, 10, 20, 21, 23. \]

For BPCB we measured the dynamic structure factor in inelastic neutron scattering experiments \[ 33. \] For this we employed 4 fully deuterated single crystal samples of total mass 2.07 g, co-aligned to better than 1° effective mosaic spread. The measurements were carried out at the LET cold neutron time of flight spectrometer \[ 36. \] at the ISIS facility, using neutrons of a fixed 2.2 meV incident energy. The sample was mounted on a \(^3\)He-\(^4\)He dilution refrigerator in a 9 T cryomagnet, with the field applied vertical along the crystallographic \( b \) direction (\( z \) axis in our notation). Since all experimental scattering vectors lie close to the horizontal \( (x, y) \) plane, for unpolarized neutrons, to a good approximation, the measured scattered intensity is given by

\[
\frac{d\sigma}{d\Omega d\omega} \propto s^+(Q)\left[S_+^z(q_{||},\omega) + S_\perp^z(q_{||},\omega)\right] + s^-(Q)\left[S_-^z(q_{||},\omega) + S_\perp^z(q_{||},\omega)\right].
\] (4)

Here the rung structure factors are given by the spatial orientation of the rungs of the ladders \[ 33, 34. \]

\[
4s^\pm(Q) = 2 \pm \cos(Q \cdot d_1) \pm \cos(Q \cdot d_2),
\] (5)
where \( d_{1,2} \) are the rung vectors of the two inequivalent ladders in BPCB as shown in Fig. 1 \[ 37. \] An overview of our neutron scattering data collected near the critical field is given by the false color intensity plots in Figs. 3a-f). These intensities were integrated along the non-dispersive \( b^* \) and \( c^* \) directions, and therefore contain scattering from both symmetric and antisymmetric excitations. The background for all data shown in this work was collected at zero applied field at base temperature, where we assumed there to be no magnetic scattering other than in a single resolution-limited magnon band between 0.8 and 1.5 meV energy transfer \[ 25. \]

The three Zeeman-split magnon branches are clearly visible in our experiments (the upper one is beyond the range of the plots shown). The critical fluctuations that are of primary interest here are descendants of the lower branch that undergoes softening. From the data shown it is obvious that at \( H > H_c \) (Fig. 3c and d), but also at the critical field at elevated temperatures (Fig. 3e and f) this part of the spectrum additionally contains a vague
“inverted” band of excitations. This feature closely resembles what is observed in Heisenberg $S = 1/2$ spin chains near the field of saturation [23]. A direct connection is established by the spin-ladder to spin-chain mapping described in Ref. [33], and identifies the additional signal as due to non-critical longitudinal spin fluctuations [23]. As in the case of spin chains, these longitudinal excitations rapidly gain spectral weight as the ladder is magnetized (be it by increasing the magnetic field beyond $H_c$ or by thermally populating the low-energy magnon band). Eventually they become impossible to separate from the transverse universal spin fluctuations that we are interested in.

This is where the advantage of a spin ladder over a spin chain comes into play. It is well understood that the low-energy longitudinal excitations are due to scattering within the lower magnon band, conserve the number of magnons and therefore lie in the symmetric structure factor [33]. This distinguishes them from the universal fluctuations which are antisymmetric. To separate the two, at every fixed $q_\parallel$, we obtain two intensities by integrating over a restricted range in $q_\perp$ where $s^+(Q) > 0.7$ or $s^-(Q) > 0.7$, respectively (recall that $s^+(Q) + s^-(Q) = 1$). These two intensities are predominantly due to symmetric or anti-symmetric scattering, respectively, but also contain small contributions of the alternative components. The coefficients are easily calculated and by solving a system of two linear equations we determine the symmetric and antisymmetric structure factor contributions at every point in $(q_\parallel, \omega)$. In application to BPCB, the antisymmetric rung structure factor $s^-(Q)$ is shown in a false color plot in Fig. 3g. Fig. 3h shows the low energy part of the spectrum collected at $H = 6.75$ T, $T = 0.35$ K, as projected over all of reciprocal space, with the two contributions overlapping. The separated symmetric and anti-symmetric contributions are shown in Fig. 3i and j, respectively.

Using this approach and integrating the data over $q_\parallel$, we obtain the local dynamic structure factor $S^\parallel(\omega)$. The results for three temperatures $T = 0.35, 0.75, 2.5$ K obtained at $H = 6.5$ T (almost exactly at $H_c$) are plotted in symbols in the scaling plot Fig. 4, using three different values of the scaling exponent $b$. The value of $b$ that produces the best data collapse was obtained as in the analysis of specific heat, using a 5th degree polynomial fit. The resulting magnitude of the data mismatch $\chi^2$ is plotted in the inset of Fig. 4. From its minimum we determine $b = 0.57(10)$, which is within the error bar of the theoretical value $b = 1/2$. Disregarding this difference we can consider the scaling plot with $b = 1/2$ to be our experimental measurement of the scaling function $\Phi(x)$ in Eq. 3. We note, that the experimental data both covers the range $\hbar \omega > k_B T$ which essentially reflects zero temperature properties of a dilute bose gas, as well as the quantum relaxational regime $\hbar \omega < k_B T$ where magnons strongly interact with thermally excited partners [8].

The main result of this work is that the measured scaling function is in excellent agreement with the exact result for this class of phase transitions (solid line in Fig. 4) that was evaluated numerically following Refs. [10, 20, 23]. The comparison to the arbitrarily normalized neutron scattering data was obtained by only fitting an overall scale factor (vertical shift along the logarithmic ordinate). In this, our result differs from previous studies of scaling of the dynamic structure factor in spin chains [33, 49] and ladders [47], which all focused on the $z = 1$ Tomonaga-Luttinger liquid regime. For that situation the scaling function is also known exactly, but explicitly depends on the Luttinger parameter, which in turn depends on the applied magnetic field and the magnitude of XXY anisotropy in the system. In contrast, in the present case of $z = 2$ quantum criticality, to apply the exact theoretical result, we did not require any information about BPCB other than it being one-dimensional, free of magnetic anisotropy and having a quadratic magnon dispersion relation at the transition.

Our work is entirely devoted to the universal low-energy critical excitations in the system. Nevertheless,
as a side note, we would like to draw the reader’s attention to a peculiar non-universal spectral feature that emerges in a slightly magnetized ladder, i.e., at elevated fields or temperatures. Specifically, we observe a distinct splitting of the middle triplet band at its maxima and a “sharpening” of the dispersion in one of the emerging modes. Further work, particularly numerical simulations, will be required to understand this behavior. At the present stage we may only suggest that the appropriate language to describe it may be found in the mapping of an insulating spin ladder to the one-dimensional $t-J$ model \cite{33}. In this construct the singlet and lower triplet states correspond to effective spin states, while the middle triplet excitations are treated as holes. The emergent additional branch may then be a bound state in this model, but clearly only additional in-depth studies will fully clarify this issue.

I summary, long standing exact theoretical results for the universal finite-temperature scaling behavior of both specific heat and the local dynamic structure factor at the universal finite-temperature scaling behavior of both will be required to understand this behavior. At the present stage we may only suggest that the appropriate language to describe it may be found in the mapping of an insulating spin ladder to the one-dimensional $t-J$ model \cite{33}. In this construct the singlet and lower triplet states correspond to effective spin states, while the middle triplet excitations are treated as holes. The emergent additional branch may then be a bound state in this model, but clearly only additional in-depth studies will fully clarify this issue.

Acknowledgement. We would like to thank Stanislaw Galeski and Severian Gvasaliya (ETHZ) as well as the sample environment team of the ISIS facility for their help with our experiments. This work is partially supported by the Swiss National Science Foundation under Division II.

References

\begin{thebibliography}{99}
\bibitem{1} S. Sachdev, \textit{Quantum Phase Transitions} (Cambridge University Press, 1999).
\bibitem{2} M. Vojta, \textit{Rep. Prog. Phys.} \textbf{66}, 2069 (2003) cond-mat/0309604
\bibitem{3} S. Sachdev and B. Keimer, \textit{Physics Today} \textbf{64}, 29 (2011) arXiv:1102.4628 [cond-mat.str-el]
\bibitem{4} S. Sachdev, \textit{Nature Physics} \textbf{4}, 173 EP (2008)
\bibitem{5} H. Stanley, \textit{Introduction to Phase Transitions and Critical Phenomena} International Series of Monogr (Oxford University Press, 1971).
\bibitem{6} H. E. Stanley, \textit{Rev. Mod. Phys.} \textbf{71}, 8358 (1999)
\bibitem{7} I. Affleck, \textit{Phys. Rev. B} \textbf{43}, 3215 (1991).
\bibitem{8} S. Sachdev, T. Senthil, and R. Shankar, \textit{Phys. Rev. B} \textbf{50}, 258 (1994)
\bibitem{9} S. Sachdev, Zeitschrift für Physik B Condensed Matter \textbf{94}, 469 (1994)
\bibitem{10} V. E. Korepin and N. A. Slavnov, \textit{Communications in Mathematical Physics} \textbf{129}, 103 (1990)
\bibitem{11} M. Girardeau, \textit{Journal of Mathematical Physics} \textbf{1}, 516 (1960)
\bibitem{12} O. Breunig, M. Garst, A. Klümper, J. Rohrkamp, M. M. Turnbull, and T. Lorenz, \textit{Science Advances} \textbf{3}, eaa03773 (2017) arXiv:1709.00274 [cond-mat.str-el]
\bibitem{13} M. Jeong and H. M. Rennow, \textit{Phys. Rev. B} \textbf{92}, 180409 (2015)
\bibitem{14} Y. Kono, T. Sakakibara, C. P. Aoyama, C. Hotta, M. M. Turnbull, C. P. Landee, and Y. Takano, \textit{Phys. Rev. Lett.} \textbf{114}, 037202 (2015)
\bibitem{15} J. Rohrkamp, M. D. Phillips, M. M. Turnbull, and T. Lorenz, \textit{Journal of Physics: Conference Series} \textbf{200}, 012169 (2010)
\bibitem{16} H. Kühne, H.-H. Klauss, S. Grossjohann, W. Brenig, F. J. Litterst, A. P. Reyes, P. L. Kuhns, M. M. Turnbull, and C. P. Landee, \textit{Phys. Rev. B} \textbf{80}, 045110 (2009)
\bibitem{17} H. Kühne, A. A. Zvyagin, M. Günther, A. P. Reyes, P. L. Kuhns, M. M. Turnbull, C. P. Landee, and H.-H. Klauss, \textit{Phys. Rev. B} \textbf{83}, 100407 (2011)
\bibitem{18} B. C. Watson, V. N. Kotov, M. W. Meisel, D. W. Hall, G. E. Granroth, W. T. Montfrooij, S. E. Nagler, D. A. Jensen, R. Backov, M. A. Petruska, G. E. Fanucci, and D. R. Talham, \textit{Phys. Rev. Lett.} \textbf{86}, 5168 (2001)
\bibitem{19} T. Lorenz, O. Heyer, M. Garst, F. Anfuso, A. Rosch, C. Rüegg, and K. Krämer, \textit{Phys. Rev. Lett.} \textbf{100}, 067208 (2008)
\bibitem{20} V. Korepin, N. Bogoliubov, and A. Izergin, \textit{Quantum Inverse Scattering Method and Correlation Functions} Cambridge Monographs on Mathem (Cambridge University Press, 1997).
\bibitem{21} T. Barthel, U. Schollwöck, and S. Sachdev, ArXiv e-prints (2012) arXiv:1212.3570 [cond-mat.str-el]
\bibitem{22} M. Panfil and J.-S. Caux, \textit{Phys. Rev. A} \textbf{89}, 033605 (2014)
\bibitem{23} D. Blesser, N. Kestin, K. Y. Povarov, R. Bewley, E. Coira, T. Giamarchi, and A. Zheludev, \textit{Phys. Rev. B} \textbf{96}, 134406 (2017)
\bibitem{24} B. R. Patyal, B. L. Scott, and R. D. Willett, \textit{Phys. Rev. B} \textbf{41}, 1657 (1990)
\bibitem{25} A. T. Savici, G. E. Granroth, C. L. Broholm, D. M. Pajerowski, C. M. Brown, D. R. Talham, M. W. Meisel, K. P. Schmidt, G. S. Uhrig, and S. E. Nagler, \textit{Phys. Rev. B} \textbf{80}, 094411 (2009)
\bibitem{26} C. Rüegg, K. Kiefer, B. Thielemann, D. F. McMorrow, V. Zapf, B. Normand, M. B. Zvonarev, P. Bouillot, C. Kollath, T. Giamarchi, S. Capponi, D. Poilblanc, D. Biner, and K. W. Krämer, \textit{Phys. Rev. Lett.} \textbf{101}, 247202 (2008)
\bibitem{27} B. Thielemann, C. Rüegg, H. M. Rønnow, A. M. Läuchli, J.-S. Caux, B. Normand, D. Biner, K. W. Krämer, H.-U. Güdel, J. Stahn, K. Habicht, K. Kiefer, M. Boehm, D. F. McMorrow, and J. Mesot, \textit{Phys. Rev. Lett.} \textbf{102}, 107204 (2009)
\bibitem{28} For our samples the exact value of the critical field was determined from inelastic neutron scattering data obtained at different magnetic fields as described in the supplemental material \cite{29}.
\bibitem{29} Supplemental material.
\bibitem{30} T. Giamarchi, \textit{Quantum Physics in One Dimension} International Series of Monogr (Clarendon Press, 2004).
\bibitem{31} M. Klajnišek, H. Mayaffre, C. Berthier, M. Horvatić, B. Chiari, O. Piovesana, P. Bouillot, C. Kollath, E. Origanc, R. Citro, and T. Giamarchi, \textit{Phys. Rev. Lett.} \textbf{101}, 137207 (2008)
\bibitem{32} B. Thielemann, C. Rüegg, K. Kiefer, H. M. Rønnow, B. Normand, P. Bouillot, C. Kollath, E. Origanc, R. Citro, T. Giamarchi, A. M. Läuchli, D. Biner, K. W. Krämer, F. Wolff-Fabris, V. S. Zapf, M. Jaime, J. Stahn, N. B. Christensen, B. Grenier, D. F. McMorrow, and J. Mesot, \textit{Phys. Rev. B} \textbf{79}, 020408 (2009)
\bibitem{33} P. Bouillot, C. Kollath, A. M. Läuchli, M. Zvonarev, B. Thielemann, C. Rüegg, E. Origanc, R. Citro, M. Klajnišek, C. Berthier, M. Horvatić, and T. Gia-
marchi, Phys. Rev. B 83, 054407 (2011)

[34] D. Schmidiger, S. Mühlbauer, A. Zheludev, P. Bouillot, T. Giamarchi, C. Kollath, G. Ehlers, and A. M. Tsvelik, Phys. Rev. B 88, 094411 (2013)

[35] D. Blosser, V. K. Bhartiya, and A. Zheludev, RB1720009, STFC ISIS Facility (2018), 10.5286/ISIS.E.87813753

[36] R. Bewley, J. Taylor, and S. Bennington., Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment 637, 128 (2011)

[37] The rung vectors are $d_{1,2} = [0.3904, \pm 0.1598, 0.4842]$ in relative lattice units for the two inequivalent ladders per unit cell in BPCB [41].

[38] C. D. F. Bella Lake, D. Alan Tennant and S. E. Nagler, Nature Materials 4, 329 (2005)

[39] M. Hälg, D. Huvonen, N. P. Butch, F. Demmel, and A. Zheludev, Phys. Rev. B 92, 104416 (2015)

[40] K. Y. Povarov, D. Schmidiger, N. Reynolds, R. Bewley, and A. Zheludev, Phys. Rev. B 91, 020406 (2015)

[41] B. Thielemann, Spin Ladder Physics, Ph.D. thesis, PSI and ETH Zurich, Switzerland (2009).

[42] M. Reigrotzki, H. Tsunetsugu, and T. M. Rice, Journal of Physics: Condensed Matter 6, 9235 (1994)
SUPPLEMENTAL MATERIAL

Determination of the critical field

Inelastic neutron scattering measurements of the magnon dispersion have been performed at various values of magnetic field. From these measurements the spin gap is determined. From a linear fit of the spin gap vs. applied magnetic field, we find \( H_c = 6.66(6) \) T and \( g = 2.16(3) \) in agreement with previously published estimates \[27\].

Triplet dispersion and quasi-particle mass

For the strong rung ladder with \( \gamma = J_{\text{Leg}}/J_{\text{Rung}} \ll 1 \), a high order expansion in \( \gamma \) yields an excellent approximation for the triplet dispersion relation. In Ref. [12] the following result is obtained up to third order in \( \gamma \):

\[
\epsilon(q_{\parallel}) = J_{\text{Rung}} \left( 1 + \gamma \cos(q_{\parallel}) + \frac{\gamma^2}{4} \left( 3 - \cos(2q_{\parallel}) \right) - \frac{\gamma^3}{8} \left( 2 \cos(q_{\parallel}) + 2 \cos(2q_{\parallel}) \cos(3q_{\parallel}) - 3 \right) \right).
\]

Fitting this dispersion to our neutron scattering data, we find good agreement for \( J_{\text{Leg}}/k_B = 3.54(3) \) K and \( J_{\text{Rung}}/k_B = 12.67(6) \) K (\( \gamma = 0.28 \)) as shown in Fig. 5. These estimates are fully consistent with previously published values \[27\]. In the vicinity of the parabolic minimum at \( q_{\parallel} = \pi \), we thus obtain

\[
\epsilon(q_{\parallel}) \approx \frac{1}{2} J_{\text{Min}}(q_{\parallel} - \pi)^2 = \frac{\hbar^2(q_{\parallel} - \pi)^2}{2m},
\]

where \( J_{\text{Min}}/k_B = 5.07(10) \) K is the band curvature and \( m = \hbar^2/J_{\text{Min}} \) the triplet quasi-particle mass.

---

FIG. 5. False color plot of the background subtracted inelastic neutron scattering intensity measured at \( H = 6.00 \) T. For the middle triplet the black solid line corresponds to eqn. 6 with \( J_{\text{Leg}}/k_B = 3.54 \) K and \( J_{\text{Rung}}/k_B = 12.67 \) K. The red dotted line shows the parabolic approximation (eqn. 7) at the dispersion minimum. For the low energy triplet the same curves are simply shifted by the Zeeman energy.