CHARACTERIZATION AND ASYMPTOTIC ANALYSIS OF THE STATIONARY PROBABILITIES IN DISCRIMINATORY PROCESSOR SHARING SYSTEMS

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Abstract. In this paper, we establish two different results. The first result is a characterization theorem saying that if the stationary state probabilities for originally described Markovian discriminatory processor sharing (DPS) system have a closed product geometric form (the exact definition is given in the paper), then the system must only be Egalitarian, i.e. all flows in this system must have equal priorities. The second result is the tail asymptotics for the stationary probabilities. We provide a detailed asymptotic analysis of the system, and obtain the exact asymptotic form of the stationary probabilities in DPS systems when the number of flows in the system is large.

1. Introduction

Discriminatory processor sharing (DPS) policy was originally introduced and studied by Kleinrock [13] under the title priority processor sharing. It is an extension of usual (non-priority) processor sharing (PS) policy, which also was originally introduced by Kleinrock [14]. The DPS system is defined as follows. Suppose that there are \( I \) flow classes. All flows are served independently of each other. They share the service time as follows. If there are \( n_1, n_2, \ldots, n_I \) flows in the system of the classes 1, 2, \ldots, \( I \), respectively, then the rate of shared service of a class \( i \) flow is

\[
\frac{g_i}{\sum_{l=1}^{I} g_{nl}}
\]

where \( g_1, g_2, \ldots, g_I \) are ‘weights’ of flows belonging to the corresponding classes. Although the DPS policy was introduced long time ago, the progress in its investigation is very limited. The first substantial contribution to the theory of DPS systems was due to Fayolle, Mitrani and Iasnogorodski [9]. These authors derived the system of integro-differential equations for the conditional expectation of the response time of a flow (the time spent in the system by a flow of a given class arriving in the system) given that the required service time of the flow exceeds the level \( t \) for the \( M/G/1 \) DPS system with \( I \) flow classes, and provided a detailed study of that system of equations. Additional study of the system of integro-differential equations [9] is given in Avrachenkov et al. [4]. The stationary queue-length distributions and heavy-traffic behavior for Markovian DPS system have been studied by Rege and Sengupta [17]. The similar analysis for the phase-type service DPS system has been provided by Verloop, Ayesta and Núñez-Queija [18], who also established state-space collapse property for the heavy-traffic behavior. Bonald and
Proutière [5], [6] and [7] provided intensive study of a certain class of PS systems. They classified those systems and studied their important properties such as insensitivity and balance properties as well as established certain bounds for so-called monotonic PS networks that include DPS systems as a particular case. For other known results in the area of DPS systems see also the review papers by Altman, Avrachenkov and Ayesta [3] and Aalto et al. [1], and for recent results related to large deviation of monotonic PS networks that include DPS system see [11].

The present paper contains two important results. The first result is a simple characterization theorem telling us about the possibility to represent their stationary probabilities in closed form. Characterization of queueing system is an established area in queueing theory. Most of the results of this theory are associated with inverse problems (see e.g. the book by Kalashnikov and Rachev [12]; see also [2] for one of recent results).

The second one is an asymptotic theorem on the tail behavior of the stationary probabilities, when the numbers flows in the system is large. For review of the different approaches the light tail asymptotics see [10]. For a recent study of tail asymptotics in PS queueing systems see [19] and that in priority queueing systems see [20].

Up to this time, the important properties of the stationary distributions of DPS systems have been studied with the aid of the vector-valued $z$-transforms having a complicated form [17], [18]. Such an approach is straightforward, and it makes the analysis of the system characteristics cumbersome. Unlike many papers in this area (including aforementioned ones [17], [18]), the present paper does not use the traditional $z$-transform method. It is based on a direct study of the system of equations for this system.

Our approach uses the same bounds as those in Bonald and Proutièrè [7]. We prove that these bounds asymptotically dominate the stationary probabilities in the DPS system. Then, on the basis of these bounds we obtain the tail asymptotics for the stationary probabilities of the DPS system.

Throughout the paper, empty sums are assumed to be set to zero and empty products to one.

The rest of the paper is organized as follows. In Section 2, we describe the system, introduce notation and formulate the results of the paper. In Section 3, we introduce necessary concepts and prove the main results of the paper. In Section 4, we define the most likely direction of the process when the number of flows in the system is large and provide its numerical study. In Section 5, we conclude the paper and formulate an open problem.

2. DESCRIPTION OF THE SYSTEM, NOTATION AND MAIN RESULTS

Consider single server queueing system with $I$ classes of flows. Flows of the $i$th class ($i$-flows) arrive in the system according to an ordinary Poisson process with rate $\lambda_i$. The nominated service time distribution of an $i$-flow is exponential with parameter $\mu_i$. Denote the load parameter of $i$-flows by $\rho_i = \frac{\lambda_i}{\mu_i}$, and assume $\rho = \rho_1 + \rho_2 + \ldots + \rho_I < 1$. All flows presented in the system are served simultaneously, and share the service according to the DPS policy with the vector $g = (g_1, g_2, \ldots, g_I)$. The word nominated means that each single $i$-flow in the system, that does not share its service, is being served exponentially with parameter $\mu_i$ unless new arrival
in the system does not occur, and occasionally its service can be finished before a new arrival. The assumption \( \rho < 1 \) means that the system is stable.

Let \( Q(t) = (Q_1(t), Q_2(t), \ldots, Q_I(t)) \) denote the vector-valued queue-length process at time \( t \), where \( Q_i(t) \) denotes the number of \( i \)-flows in the system are being served at time \( t \), and let \( P_n = \lim_{t \to \infty} P\{Q(t) = n\} \), where \( n = (n_1, n_2, \ldots, n_I) \) is an integer-valued vector. For a stable system, the last limit exists.

Throughout the paper we also use the following notation:

\[
\begin{align*}
0 & = (0,0,\ldots,0) - I\text{-dimensional vector of zeros,} \\
1 & = (1,1,\ldots,1) - I\text{-dimensional vector of ones,} \\
1_i & = (0,0,\ldots,0,1,0,\ldots,0), \\
\langle n, g \rangle & = n_1g_1 + n_2g_2 + \ldots + n_Ig_I, \\
|n| & = \langle n, 1 \rangle = n_1 + n_2 + \ldots + n_I, \\
|n|_i & = n_1 + n_2 + \ldots + n_i; \\
(n|I = |n|, |n|_0 = 0).
\end{align*}
\]

The inequality between the vectors is understood as the componentwise inequalities. For example, \( n \geq 0 \) means that all components of a vector \( n \) are nonnegative; \( n > 0 \) means that in a nonnegative vector \( n \) there is at least one strictly positive component. A vector \( n \) is said to be separated from zero if \( n_i > 0 \) for all \( i = 1, 2, \ldots, I \). The set of all vectors that are separated from zero is denoted by \( \mathcal{N} \).

Let \( \Gamma = (\gamma_1, \gamma_2, \ldots, \gamma_I) \) be a vector of positive real numbers (vector of the direction). A vector \( \Gamma \) is called normalized if \( \gamma_1 + \gamma_2 + \ldots + \gamma_I = 1 \). In the sequel, all vectors \( \Gamma \) considered in the paper are assumed to be normalized.

For \( N = 0,1,\ldots \) and a given vector of the direction \( \Gamma \), the set of the vectors \((\lfloor N\gamma_1 \rfloor, \lfloor N\gamma_2 \rfloor, \ldots, \lfloor N\gamma_I \rfloor)\), where for any real \( a \), the symbol \( \lfloor a \rfloor \) denotes the integer part of \( a \), is denoted \( \mathcal{N}_\Gamma \). Let \( \mathcal{G} \) be an infinite set of directions \( \Gamma \) containing an interior. We define the cone \( \mathcal{C}(\mathcal{G}) = \cup_{\Gamma \in \mathcal{G}} \mathcal{N}_\Gamma \).

For a positive integer \( n \), denote \( \mathcal{N}_{\Gamma,n} = \{ n \in \mathcal{N}_\Gamma : n \geq (\lfloor n\gamma_1 \rfloor, \lfloor n\gamma_2 \rfloor, \ldots, \lfloor n\gamma_I \rfloor) \} \).

For a direction \( \Gamma \), let \( n\Gamma \) be an indexed integer number. Denote \( \mathcal{N}(\Gamma) = \{ n\Gamma : \Gamma \in \mathcal{G} \} \), and define the set \( \mathcal{C}(\mathcal{G}, \mathcal{N}(\Gamma)) = \cup_{\Gamma \in \mathcal{G}} n\mathcal{N}_{\Gamma,n\Gamma} \).

A vector \( n \in \mathcal{C}(\mathcal{G}, \mathcal{N}(\Gamma)) \) is called boundary vector of \( \mathcal{C}(\mathcal{G}, \mathcal{N}(\Gamma)) \), if there exists integer \( i, i = 1, 2, \ldots, I \), such that \( n - 1_i \) does not belong to the set \( \mathcal{C}(\mathcal{G}, \mathcal{N}(\Gamma)) \). The set of all pairs \( \{n,i\} \) where \( n \) is a boundary vector of \( \mathcal{C}(\mathcal{G}, \mathcal{N}(\Gamma)) \) and \( n - 1_i \) does not belong to the set \( \mathcal{C}(\mathcal{G}, \mathcal{N}(\Gamma)) \) is denoted by \( \mathcal{C}(\mathcal{G}, \mathcal{N}(\Gamma)) \).

In addition, for any integer \( N \) and vector of direction \( \Gamma \), the following notation \( \lfloor N\Gamma \rfloor = (\lfloor N\gamma_1 \rfloor, \lfloor N\gamma_2 \rfloor, \ldots, \lfloor N\gamma_I \rfloor) \) is used.

**Definition 2.1.** The stationary probabilities of the vector valued queueing process \( Q(t) \) are said to be presented in closed product geometric form if

\[
\begin{align*}
P_n & = (1-\rho)F(n, g) \prod_{i=1}^{I} \rho_i^{n_i}, \ n > 0, \quad (2.1) \\
P_0 & = 1-\rho, \quad (2.2)
\end{align*}
\]

for some function \( F(n, g) \) depending only on the vectors \( n \) and \( g \) (and hence independent of the vector-valued parameters \( (\lambda_1, \lambda_2, \ldots, \lambda_I) \) and \( (\mu_1, \mu_2, \ldots, \mu_I) \)).
Theorem 2.2. The stationary probabilities $P_n$ of the DPS queueing system can be represented in closed product geometric form if and only if the components of the vector $\mathbf{g}$ all are equal, that is, in the only case of Egalitarian PS system.

For the formulation of the next main theorem, we introduce the following notation. For $i = 1, 2, \ldots, I - 1$ and positive real numbers $\gamma_1, \gamma_2, \ldots, \gamma_I$ ($\sum_{i=1}^{I} \gamma_i = 1$) set

$$\theta_i^{(1)} = \exp \left\{ \sum_{j=1}^{I-1} \left( \frac{g_i}{g_{i+j}} - 1 \right) \ln \left( 1 + \frac{\gamma_{i+j} g_{i+j}}{\sum_{k=1}^{I+j+1} \gamma_k g_k} \right) \right\}$$

and for $i = I$ set $\theta_I^{(1)} = 1$. Similarly, for $i = 1$ set $\theta_i^{(2)} = 1$ and for $i = 2, 3, \ldots, I$ and the same positive real numbers $\gamma_1, \gamma_2, \ldots, \gamma_I$ set

$$\theta_i^{(2)} = \exp \left\{ \sum_{j=1}^{I} \left( \frac{g_i}{g_j} - 1 \right) \ln \left( 1 + \frac{\gamma_j g_j}{\sum_{k=j+1}^{I} \gamma_k g_k} \right) \right\}.$$

For $i = 1, 2, \ldots, I$, introduce the following values:

$$\Delta_i^{(1)} = \frac{\gamma_1 g_1 + \gamma_2 g_2 + \ldots + \gamma_I g_I}{\gamma_i g_i} \cdot \theta_i^{(1)} \cdot \rho_i,$$

$$\Delta_i^{(2)} = \frac{\gamma_1 g_1 + \gamma_2 g_2 + \ldots + \gamma_I g_I}{\gamma_i g_i} \cdot \theta_i^{(2)} \cdot \rho_i.$$

For integer parameter $N$, we set $n_1 = \lfloor N \gamma_1 \rfloor$, $n_2 = \lfloor N \gamma_2 \rfloor$, \ldots, $n_I = \lfloor N \gamma_I \rfloor$. Then $n(N) = \lfloor N \Gamma \rfloor = \{\lfloor N \gamma_1 \rfloor, \lfloor N \gamma_2 \rfloor, \ldots, \lfloor N \gamma_I \rfloor\}$.

The following theorem describes the asymptotic behavior of the stationary probabilities.

Theorem 2.3. Assume that $g_1 < g_2 < \ldots < g_I$, and

$$\Delta_i^{(2)} < 1, \quad i = 1, 2, \ldots, I.$$

Then, as $N \to \infty$,

$$\lim_{N \to \infty} \frac{P_{n(N)+1}}{P_{n(N)}} = \Delta_i = \frac{\Delta_i^{(1)} - c \Delta_i^{(2)}}{1 - c},$$

$$c = \frac{\sum_{i=1}^{I} \mu_i \gamma_1 g_i \left( \sum_{i=1}^{I} \gamma_i g_i \right)^{-1} \left( \Delta_i^{(1)} - 1 \right) - \lambda_i \left( 1 - \left( \Delta_i^{(1)} \right)^{-1} \right)}{\sum_{i=1}^{I} \mu_i \gamma_1 g_i \left( \sum_{i=1}^{I} \gamma_i g_i \right)^{-1} \left( \Delta_i^{(2)} - 1 \right) - \lambda_i \left( 1 - \left( \Delta_i^{(2)} \right)^{-1} \right)}.$$

Corollary 2.4. Under the assumptions of Theorem 2.3

$$\lim_{N \to \infty} \frac{\ln P_{n(N)}}{N} = \sum_{i=1}^{I} \gamma_i \ln \Delta_i.$$

3. Proofs of the main results

3.1. Preliminaries. In the following, the fractions in which both the numerator and denominator are equal to zero, are set to zero. Specifically, the fractions $\frac{n_i}{n \cdot g}$ in which $n = 0$ are set to zero. The system of linear equations for the stationary
probabilities $P_n$, $n \geq 0$, which follows from the system of the Chapman-Kolmogorov equations, is

$$
(3.1) \quad \sum_{i=1}^{l} \left[ \frac{\mu_i g_i (n_i + 1)}{\langle n + 1, i, g \rangle} P_{n+1} - \left( \lambda_i + \frac{\mu_i g_i n_i}{\langle n, g \rangle} \right) P_n + \lambda_i P_{n-1} \right] = 0
$$

(see [17]), where $P_{n-1} = 0$ in the case where the vector $n - 1_i$ is not nonnegative.

For the further study, it is convenient to introduce the operators

$$
\mathcal{J}_{i,n}(X,Y) = \frac{\mu_i g_i n_i}{\langle n, g \rangle} X - \lambda_i Y,
$$

where $X$ and $Y$ are real numbers. Then, (3.1) can be rewritten in the form

$$
(3.2) \quad \sum_{i=1}^{l} [\mathcal{J}_{i,n+1}(P_{n+1}, P_n) - \mathcal{J}_{i,n}(P_n, P_{n-1})] = 0.
$$

3.2. Proof of Theorem 2.2. We first obtain properties that the function $F(n,g)$ must satisfy to be a solution of (2.1). Substituting (2.1) into (3.1) and canceling the (non-zero) factor $\prod_{i=1}^{l} P_i^{\mu_i}$ yields

$$
\sum_{i=1}^{l} \frac{\mu_i g_i n_i}{\langle n, g \rangle} F(n, g) - \sum_{i=1}^{l} \frac{\mu_i g_i (n_i + 1)}{\langle n + 1, i, g \rangle} F(n + 1, i, g) \rho_i
$$

$$
= \sum_{i=1}^{l} \lambda_i F(n - 1, i, g) \rho_i - \sum_{i=1}^{l} \lambda_i F(n, g),
$$

(3.3)

where $F(n - 1, i, g) = 0$ in the case where the vector $n - 1_i$ is not nonnegative. Equation (3.3) is equivalent to

$$
\sum_{i=1}^{l} \frac{\mu_i g_i n_i}{\langle n, g \rangle} F(n, g) - \sum_{i=1}^{l} \lambda_i \frac{g_i (n_i + 1)}{\langle n + 1, i, g \rangle} F(n + 1, i, g)
$$

$$
= \sum_{i=1}^{l} \mu_i F(n - 1, i, g) - \sum_{i=1}^{l} \lambda_i F(n, g).
$$

Since this must hold for all values of $\lambda_i$ and all values of $\mu_i$, we can equate the coefficients of $\lambda_i$ and $\mu_i$ to obtain that $F(n,g)$ must, for all $i$ and all nonnegative vectors $n$, satisfy the recurrence relation

$$
(3.4) \quad F(n + 1, i, g) = \frac{\langle n + 1, i, g \rangle}{\langle n, i, g \rangle} F(n, g),
$$

where $F(0, g)$ is the initial positive value for the recurrence relation of (3.4).

We prove now that representation (2.1) is correct if and only if the components of the vector $g$ all are equal. Set $F(0, g) = C$, where $C$ is a positive constant depending on the vector $g$, and assume, to obtain a contradiction, that there exist $i$ and $l$ such that $g_i \neq g_l$. By (3.4), for $F(n, g)$ to satisfy (2.1), we require $F(1, g) = F(0, g) g_i = C$, and $F(i + 1, i, g) = F(1, g) g_i^{i + g_i} = C g_i^{i + g_i}$. On the other hand, by the similar way we obtain $F(1 + i, i, g) = C g_i^{i + g_i}$ if we first find $F(1, g) = C$, and then $F(i + 1, i, g) = F(1, g) g_i^{i + g_i} = C g_i^{i + g_i}$. This is only correct when $g_i = g_l$ and, hence, it contradicts to the assumption that $g_i \neq g_l$. Hence, the function $F(0, g)$ is not uniquely defined if the vector $g$ has distinct components.
3.3.1. Concepts and notation. For any vector \( \mathbf{n} > \mathbf{0} \), let us present the elements of the vector \( \mathbf{g} \) in the following two orders

\[
S_{n,1}^{(1)} = g_1; S_{n,2}^{(1)} = g_2; \ldots; S_{n,n-1}^{(1)} = g_{n-1}; S_{n,n}^{(1)} = g_n
\]

and for the backward order, the sequence of partial sums is denoted by

\[
S_{n,1}^{(2)} = g_K; S_{n,2}^{(2)} = g_{K-1}; \ldots; S_{n,n}^{(2)} = g_{1}\]

The order in (3.6) is called forward, and the order in (3.7) is called backward.

For the forward order, denote the sequence of the partial sums by

\[
P_{n}^{(1)} = C^{(1)} \prod_{l=1}^{n} \frac{|S_{n,l}^{(1)}|}{n_l!} \prod_{i=1}^{l} \left( \frac{\rho_i}{g_i} \right)^{n_i}
\]

and

\[
P_{n}^{(2)} = C^{(2)} \prod_{l=1}^{n} \frac{|S_{n,l}^{(2)}|}{n_l!} \prod_{i=1}^{l} \left( \frac{\rho_i}{g_i} \right)^{n_i}
\]

where the normalization constants \( C^{(1)} \) and \( C^{(2)} \) are

\[
C^{(1)} = \left[ \sum_{n \geq 0} \prod_{l=1}^{n} \frac{|S_{n,l}^{(1)}|}{n_l!} \prod_{i=1}^{l} \left( \frac{\rho_i}{g_i} \right)^{n_i} \right]^{-1}
\]
and

\[ C^{(2)} = \left[ \sum_{n \geq 0} \frac{\Pi_{l=1}^{[n]} S_{n,l}^{(2)}}{n_1! n_2! \ldots n_l!} \prod_{i=1}^{l} \left( \frac{\rho_i}{g_i} \right)^{n_i} \right]^{-1}. \]

Apparently,

\[ \frac{\Pi_{l=1}^{[n]} S_{n,l}^{(1)}}{n_1! n_2! \ldots n_l!} \prod_{i=1}^{l} \left( \frac{\rho_i}{g_i} \right)^{n_i} \leq \frac{|n|!}{n_1! n_2! \ldots n_l!} \prod_{i=1}^{l} \rho_i^{n_i}, \]

for all \( n \geq 0 \). So, if \( \rho < 1 \), then \( P_{n}^{(1)} \) is a proper probability mass function with the normalization constant \( C^{(1)} \) satisfying the inequality \( C^{(1)} \geq 1 - \rho \). However, \( P_{n}^{(2)} \) is a proper probability mass function only in the case when the series

\[ \sum_{n \geq 0} \frac{\Pi_{l=1}^{[n]} S_{n,l}^{(2)}}{n_1! n_2! \ldots n_l!} \prod_{i=1}^{l} \left( \frac{\rho_i}{g_i} \right)^{n_i} \]

converges. We cannot claim the convergence of \( (3.12) \) in general.

### 3.3.2. Auxiliary lemmas.

In Lemmas 3.3 and 3.7 given below, it is assumed that \( P_{n}^{(2)} \) is a proper probability mass function.

**Lemma 3.3.** For all \( n \in \mathbb{N} \) we have the following relations:

\[ J_{i,n}(P_{n}^{(1)}, P_{n-1}^{(1)}) = 0, \]

\[ J_{i,n}(P_{n}^{(1)}, P_{n-1}^{(1)}) < 0, \ i = 1, 2, \ldots, I - 1, \]

\[ J_{i,1}(P_{n}^{(2)}, P_{n-1}^{(1)}) = 0, \]

\[ J_{i,n}(P_{n}^{(2)}, P_{n-1}^{(2)}) > 0, \ i = 2, 3, \ldots, I, \]

**Proof.** For better readability, we provide the proof of this lemma for shifted indices by replacing \( J_{i,n}(P_{n}^{(1)}, P_{n-1}^{(1)}) \) with \( J_{i,n+1}(P_{n+1}^{(1)}, P_{n}^{(1)}) \) \( (i = 1, 2, \ldots, I) \). For instance, instead of \((3.13)\) we prove \( J_{i,n+1}(P_{n+1}^{(1)}, P_{n}^{(1)}) = 0. \)

Relations \((3.13)\) and \((3.15)\) follow by the direct substitution, since for the first \( |n| \) partial sums we have \( S_{n+1,1}^{(1)} = S_{n,1}^{(1)} \) and, respectively, \( S_{n+1,1}^{(2)} = S_{n,1}^{(2)} \ (l = 1, 2, \ldots, |n|) \), and hence,

\[
\frac{1}{\langle n + 1, g \rangle} \prod_{i=1}^{[n]+1} S_{n+1,1,i}^{(1,1)} = \prod_{i=1}^{[n]} S_{n,1,i}^{(1,1)},
\]

and

\[
\frac{1}{\langle n + 1, g \rangle} \prod_{i=1}^{[n]+1} S_{n+1,1,i}^{(2,1)} = \prod_{i=1}^{[n]} S_{n,1,i}^{(2,1)}.
\]

To prove the strong inequality of \((3.14)\) note, that in the relation

\[
P_{n+1}^{(1)} = C^{(1)} \frac{\prod_{i=1}^{[n]+1} S_{n+1,1,i}^{(1)}}{n_1! n_2! \ldots n_{i-1}!(n_i + 1)! n_{i+1}! \ldots n_l!} \left( \frac{\rho_i}{g_i} \right) \prod_{i=1}^{l} \rho_i^{n_i},
\]

where

\[
C^{(1)} = \sum_{n \geq 0} \frac{\Pi_{l=1}^{[n]} S_{n,l}^{(1)}}{n_1! n_2! \ldots n_l!} \prod_{i=1}^{l} \left( \frac{\rho_i}{g_i} \right)^{n_i}.
\]
the product term $\prod_{l=1}^{[n]+1} S_{n+1,l}^{(1)}$ contains the following $|n| + 1$ terms:

(3.17)

\[
S_{n+1,1}^{(1)} = g_1, \quad S_{n+1,2}^{(1)} = 2g_1, \ldots, \quad S_{n+1,n_1}^{(1)} = n_1 g_1, \quad S_{n+1,n_1+1}^{(1)} = n_1 g_1 + g_2, \ldots,
\]

\[
S_{n+1,n_i}^{(1)} = n_1 g_1 + \ldots + n_i g_i,
\]

\[
S_{n+1,n_i+1}^{(1)} = n_1 g_1 + \ldots + (n_i + 1)g_i;
\]

\[
S_{n+1,n_i+2}^{(1)} = n_1 g_1 + \ldots + (n_i + 1)g_i + g_{i+1},
\]

\[
\ldots,
\]

\[
S_{n+1,[n]}^{(1)} = (n + 1, g) \text{ minus the last element in sequence (3.16)},
\]

\[
S_{n+1,[n]+1}^{(1)} = (n + 1, g),
\]

and after dividing the term $\prod_{l=1}^{[n]+1} S_{n+1,l}^{(1)}$ by $(n + 1, g)$, the last term in (3.17) disappears.

Let us compare the product terms in (3.8) and the first $|n|$ terms in (3.17). The first $n_1 + n_2 + \ldots + n_i$ product terms in (3.8) and (3.17) coincide. However, for all of the following terms we have $S_{n,l}^{(1)} > S_{n+1,l}^{(1)}$, $l = n_1 + n_2 + \ldots + n_i + 1, \ldots, [n]$. For example,

\[
S_{n,n_1+n_2+\ldots+n_i+1}^{(1)} = n_1 g_1 + \ldots + n_i g_i + g_{i+1}
\]

\[
> n_1 g_1 + \ldots + n_i g_i + g_i
\]

\[
= S_{n+1,n_1+n_2+\ldots+n_i+1}^{(1)};
\]

since by the assumption of the theorem $g_{i+1} > g_i$. Henceforth, after algebraic reductions we obtain (3.14). The proof of the strong inequality of (3.16) is similar. Lemma 3.3 is proved.

\[\square\]

**Lemma 3.4.** Let $\gamma_1, \gamma_2, \ldots, \gamma_l$ be positive numbers ($\sum_{i=1}^{l} \gamma_i = 1$), let $\theta_i^{(1)}$ and $\theta_i^{(2)}$ be the values that are defined by (2.3) and (2.4), let $\Delta_i^{(1)}$ and $\Delta_i^{(2)}$ be the values that are defined by (2.5) and (2.6), and let Condition (2.7) be satisfied.

Then the limiting, as $N \to \infty$, stationary probabilities $P_{[N\Gamma]}^{(2)}$ are well-defined, and

(3.18)

\[
\lim_{N \to \infty} \frac{P_{[N\Gamma]}^{(1)} + 1}{P_{[N\Gamma]}^{(1)}} = \Delta_i^{(1)},
\]

(3.19)

\[
\lim_{N \to \infty} \frac{P_{[N\Gamma]}^{(2)} + 1}{P_{[N\Gamma]}^{(2)}} = \Delta_i^{(2)}.
\]

**Proof.** Indeed, for $P_{n}^{(1)}$ we have as follows:

(3.20)

\[
\frac{P_{n+1}^{(1)}}{P_{n}^{(1)}} = \frac{\rho_i}{(n_i + 1)g_i} \cdot \frac{\prod_{l=1}^{[n]+1} S_{n+1,l}^{(1)}}{\prod_{l=1}^{[n]} S_{n,l}^{(1)}} = \frac{\rho_i (n + 1, g)}{(n_i + 1)g_i} \cdot \frac{\prod_{l=1}^{[n]} S_{n+1,l}^{(1)}}{S_{n,l}^{(1)}}.
\]

where
Assuming that $N$ tends to infinity in (3.20), then for $i = 1, 2, \ldots, I$ we have the expansion

$$
\rho_i ([N \gamma_1] + 1, g_i) = \rho_i ([N \gamma_1] g_i + [N \gamma_2] g_i + \cdots + [N \gamma_I] g_i + \rho_i g_i
$$

(3.21)

Next, with the aid of (3.8) and (3.17) we prove

$$
\lim_{N \to \infty} \prod_{l=1}^{\left|\left[N \gamma_i\right]\right|} \frac{S_{\left[N \gamma_i\right]+1, l}}{S_{\left[N \gamma_i\right], l}} = \theta_i^{(1)}.
$$

(3.22)

Indeed, for $l = 1, 2, \ldots, [N \gamma_1] + [N \gamma_2] + \cdots + [N \gamma_i]$, we have $S_{\left[N \gamma_i\right]+1, l} = S_{\left[N \gamma_i\right], l}$ and hence

$$
\lim_{N \to \infty} \prod_{l=1}^{\left|\left[N \gamma_i\right]\right|} \frac{S_{\left[N \gamma_i\right]+1, l}}{S_{\left[N \gamma_i\right], l}} = 1.
$$

For further simplifications, we use the conventional notation

$$
|\left[N \gamma_i\right]|_i = [N \gamma_1] + [N \gamma_2] + \cdots + [N \gamma_i].
$$

Let us first find $\lim_{N \to \infty} \prod_{l=1}^{\left|\left[N \gamma_i\right]\right|+1} \frac{S_{\left[N \gamma_i\right]+1, l}}{S_{\left[N \gamma_i\right], l}}$

Notice, that for any $1 \leq m \leq [N \gamma_{k+1}]$, we have

$$
\frac{S_{\left[N \gamma_i\right]+1, l}}{S_{\left[N \gamma_i\right], l}} = \frac{g_1 [N \gamma_1] + g_2 [N \gamma_2] + \cdots + g_i [N \gamma_i] + g_i + (m-1)g_{i+1}}{g_1 [N \gamma_1] + g_2 [N \gamma_2] + \cdots + g_i [N \gamma_i] + g_i + (m-1)g_{i+1} + (g_{i+1} - g_i) = 1 - \frac{g_{i+1} - g_i}{g_1 [N \gamma_1] + g_2 [N \gamma_2] + \cdots + g_i [N \gamma_i] + (m-1)g_{i+1} + g_i}
$$

Hence,

$$
\lim_{N \to \infty} \prod_{l=\left|\left[N \gamma_i\right]\right|+1} S_{\left[N \gamma_i\right]+1, l} = \exp \left(- \int_0^1 \frac{(g_{i+1} - g_i) \gamma_{i+1}}{\gamma_{i+1} + \gamma_{i+2} + \cdots + \gamma_{i+I+1} + x} dx \right)
$$

(3.24)

$$
= \exp \left[ \left( \frac{g_i}{g_{i+1}} - 1 \right) \ln \left( 1 + \frac{\gamma_{i+1} g_{i+1}}{\sum_{k=1}^{I} \gamma_k g_k} \right) \right].
$$
Similarly, for any $j = 0, 1, \ldots, I - i$ we have

\[
\lim_{N \to \infty} \prod_{l = ||N\Gamma||_{i+j}+1}^{||N\Gamma||_{i+j+1}} S^{(1)}_{||N\Gamma||_{i+j+1}, l} - S^{(1)}_{||N\Gamma||_{i+j}, l} \\
\exp\left(-\int_{0}^{1} \frac{(g_{i+1+j} - g_{i})\gamma_{i+1+j}}{\gamma_{1}g_{1} + \gamma_{2}g_{2} + \cdots + \gamma_{i+j}g_{i+j} + \gamma_{i+j+1}g_{i+j+1} + x} \, dx\right) \\
\exp\left[\left(\frac{g_{i}}{g_{i+1+j}} - 1\right) \ln\left(1 + \frac{\gamma_{i+1+j}g_{i+1+j}}{\sum_{k=1}^{i+j} \gamma_{k}g_{k}}\right)\right].
\]

(3.25)

So, (3.22) follows.

Again, we have (3.21), and similarly to (3.22) we obtain

\[
\lim_{N \to \infty} \prod_{l = 1}^{||N\Gamma||_{i}+1} S^{(2)}_{||N\Gamma||_{i}+1, l} - S^{(2)}_{||N\Gamma||_{i}, l} = \theta^{(2)}_{i}.
\]

(3.26)

Then Relations (3.21), (3.26) and Condition (2.7) make the stationary probabilities $P_{n}^{(2)}$ well-defined, since according to these relations, $\lim_{N \to \infty} P_{n}^{(2)}_{||N\Gamma||_{i}+1} - P_{n}^{(2)}_{||N\Gamma||_{i}} < 1$, $i = 1, 2, \ldots, I$. The last also means that the series in (3.12) converges, if the infinite sum is taken on the set of indices specified by the vectors of $N\Gamma$. Hence, relations (3.18) and (3.19) follow. The lemma is proved. \hfill \Box

For the purpose of this paper, we need in stronger results than those are given by Lemma 3.4. The following lemma is an extension of Lemma 3.4.

**Lemma 3.5.** Under the assumptions of Lemma 3.4 as $N \to \infty$, for $i = 1, 2, \ldots, I$ we have:

\[
P_{||N\Gamma||_{i}+1}^{(1)} = \Delta_{i}^{(1)} P_{||N\Gamma||}^{(1)} \left[1 - \frac{\alpha_{i}^{(1)}}{N} + o\left(\frac{1}{N}\right)\right],
\]

(3.27)

and

\[
P_{||N\Gamma||_{i}+1}^{(2)} = \Delta_{i}^{(2)} P_{||N\Gamma||}^{(2)} \left[1 - \frac{\alpha_{i}^{(2)}}{N} + o\left(\frac{1}{N}\right)\right],
\]

(3.28)

where

\[
\alpha_{i}^{(1)} = \left(\frac{\sum_{j=1}^{I} \gamma_{j}g_{j} - \gamma_{i}g_{i}}{\gamma_{i} \sum_{j=1}^{I} \gamma_{j}g_{j}}\right) + \sum_{j=1}^{I-i} \frac{1}{2\gamma_{i+j}} \left(\frac{g_{i}}{g_{i+j}} - 1\right)^{2} \left[\ln\left(1 + \frac{\gamma_{i+j}g_{i+j}}{\sum_{k=1}^{i+j} \gamma_{k}g_{k}}\right)\right]^{2} \\
\times \exp\left[\left(\frac{g_{i}}{g_{i+1+j}} - 1\right) \ln\left(1 + \frac{\gamma_{i+1+j}g_{i+1+j}}{\sum_{k=1}^{i+j} \gamma_{k}g_{k}}\right)\right], \quad i = 1, 2, \ldots, I - 1,
\]

(3.29)

\[
\alpha_{i}^{(1)} = \left(\frac{\sum_{j=1}^{I} \gamma_{j}g_{j} - \gamma_{i}g_{i}}{\gamma_{i} \sum_{j=1}^{I} \gamma_{j}g_{j}}\right),
\]

(3.30)

\[
\alpha_{i}^{(2)} = \left(\frac{\sum_{j=1}^{I} \gamma_{j}g_{j} - \gamma_{i}g_{i}}{\gamma_{i} \sum_{j=1}^{I} \gamma_{j}g_{j}}\right),
\]

(3.31)
and
\begin{equation}
(3.32) \quad \alpha_i^{(2)} = \left( \frac{\sum_{j=1}^I \gamma_j g_j - \gamma_i g_i}{\gamma_i \sum_{j=1}^I \gamma_j g_j} \right) + \frac{1}{2} \sum_{j=1}^{i-1} \frac{1}{\gamma_j g_j - 1} \left[ \ln \left( 1 + \frac{\gamma_j g_j}{\sum_{k=j+1}^I \gamma_k g_k} \right) \right]^2 \times \exp \left[ \frac{\gamma_i g_i}{g_i} \ln \left( 1 + \frac{\gamma_i g_i}{\sum_{k=j+1}^I \gamma_k g_k} \right) \right], \quad i = 2, 3, \ldots, I.
\end{equation}

Proof. We are to establish the exact values of constants \( c_i, c_{\theta_i}^{(1)}, \) and \( c_{\theta_i}^{(2)} \) \((i = 1, 2, \ldots, I)\) in the expansions
\[ \rho_i \frac{\{N \Gamma + 1, g\}}{\{N \gamma_i + 1, g\}} = \rho_i \frac{\sum_{j=1}^I \gamma_j g_j}{\gamma_i g_i} \left[ 1 + \frac{c_i}{N} + O \left( \frac{1}{N^2} \right) \right], \]
\[ \prod_{i=1}^N \frac{S_{(N \Gamma + 1, l)}^{(1)}}{S_{(N \Gamma, l)}^{(1)}} = \theta_i^{(1)} \left[ 1 + \frac{c_{\theta_i}^{(1)}}{N} + o \left( \frac{1}{N} \right) \right] \]
and
\[ \prod_{i=1}^N \frac{S_{(N \Gamma + 1, l)}^{(2)}}{S_{(N \Gamma, l)}^{(2)}} = \theta_i^{(2)} \left[ 1 + \frac{c_{\theta_i}^{(2)}}{N} + o \left( \frac{1}{N} \right) \right] \]
for large \( N \). Then, we will arrive at necessary expansions
\[ P_{[N \Gamma + 1, l]}^{(1)} = \Delta_i^{(1)} P_{[N \Gamma]}^{(1)} \left[ 1 + \frac{c_i}{N} + \frac{c_{\theta_i}^{(1)}}{N} + o \left( \frac{1}{N} \right) \right] \]
and
\[ P_{[N \Gamma + 1, l]}^{(2)} = \Delta_i^{(2)} P_{[N \Gamma]}^{(2)} \left[ 1 + \frac{c_i}{N} + \frac{c_{\theta_i}^{(2)}}{N} + o \left( \frac{1}{N} \right) \right]. \]

From (3.21) one can obtain a more precise expansion than that is given by the right-hand side of (3.21). Namely, after some algebra,
\begin{equation}
(3.33) \quad \rho_i \frac{\{N \Gamma + 1, g\}}{\{N \gamma_i + 1, g\}} = \rho_i \frac{\sum_{j=1}^I \gamma_j g_j}{\gamma_i g_i} \left[ 1 + \frac{1}{N} \left( \frac{\gamma_i g_i - \sum_{j=1}^I \gamma_j g_j}{\gamma_i \sum_{j=1}^I \gamma_j g_j} \right) + o \left( \frac{1}{N} \right) \right].
\end{equation}

So, the constant \( c_i \) is found, and from this estimate we immediately arrive at the estimates (3.27) and (3.28) for \( i = I \) (containing the constant \( \alpha_i^{(1)} \)) and \( i = 1 \) (containing the constant \( \alpha_i^{(2)} \)), respectively.

Find now the constants \( c_{\theta_i}^{(1)}, i = 1, 2, \ldots, I - 1 \), and thus prove the estimate (3.27) for \( i = 1, 2, \ldots, I - 1 \). From (3.29), for large \( N \) using the mean value theorem, for some value \( \eta_N \in (0, 1) \) we obtain
\begin{equation}
(3.34) \quad \prod_{i=1}^{[N \Gamma] + 1} \frac{S_{[N \Gamma + 1, l]}^{(1)}}{S_{[N \Gamma, l]}^{(1)}} = \left( 1 - \frac{g_{i+1} - g_i}{\sum_{j=1}^I g_j [N \gamma_j] + \eta_N g_{i+1} [N \gamma_{i+1}]} \right)^{[N \gamma_{i+1}]}.
\end{equation}

Then, the integral given by (3.32) can be written in the form
\begin{equation}
(3.35) \quad \lim_{N \to \infty} \prod_{i=1}^{[N \Gamma] + 1} \frac{S_{[N \Gamma + 1, l]}^{(1)}}{S_{[N \Gamma, l]}^{(1)}} = \exp \left( -\frac{(g_{i+1} - g_i) \gamma_{i+1}}{\sum_{j=1}^I g_j \gamma_j + \eta_{i+1} \gamma_{j+1}} \right).
\end{equation}
where \( \eta = \lim_{N \to \infty} \eta_N \). Let us find the limit

\[
(3.36) \quad \lim_{N \to \infty} N \left[ \prod_{t=1}^{[N\gamma]_{i+1}} \frac{S_{[N\gamma],1,t}^{(1)}}{S_{[N\gamma],1,t}^{(1)}} - \exp \left( -\frac{(g_{i+1} - g_i)\gamma_{i+1}}{\sum_{j=1}^{i} g_j\gamma_j + \eta g_{i+1}\gamma_{j+1}} \right) \right].
\]

Expanding the right-hand side of (3.34) we obtain

\[
(3.37) \quad \left( 1 - \frac{g_{i+1} - g_i}{\sum_{j=1}^{i} g_j\gamma_j + \eta g_{i+1}\gamma_{j+1}} \right)^{[N\gamma_{i+1}]} = \exp \left( -\frac{(g_{i+1} - g_i)\gamma_{i+1}}{\sum_{j=1}^{i} g_j\gamma_j + \eta g_{i+1}\gamma_{j+1}} \right)
\times \left[ 1 - \frac{\gamma_{j+1}}{2N} \left( \frac{g_{i+1} - g_i}{\sum_{j=1}^{i} g_j\gamma_j + \eta g_{i+1}\gamma_{j+1}} \right)^2 + o \left( \frac{1}{N} \right) \right].
\]

Hence, the limit in (3.36) is

\[
(3.38) \quad -\frac{1}{2\gamma_{i+1}} \left( \frac{g_{i+1} - g_i}{\sum_{j=1}^{i} g_j\gamma_j + \eta g_{i+1}\gamma_{j+1}} \right)^2 \exp \left( -\frac{(g_{i+1} - g_i)\gamma_{i+1}}{\sum_{j=1}^{i} g_j\gamma_j + \eta g_{i+1}\gamma_{j+1}} \right).
\]

On the other hand, from (3.24) and (3.35) we find

\[
(3.39) \quad -\frac{(g_{i+1} - g_i)\gamma_{i+1}}{\sum_{j=1}^{i} g_j\gamma_j + \eta g_{i+1}\gamma_{j+1}} = \left( \frac{g_{i+1}}{g_i} - 1 \right) \ln \left( 1 + \frac{\gamma_{i+1}g_{i+1}}{\sum_{k=1}^{i}\gamma_kg_k} \right).
\]

Hence, it follows from (3.38) and (3.39) that the limit in (3.36) is

\[
-\frac{1}{2\gamma_{i+1}} \left( \frac{g_i}{g_{i+1}} - 1 \right)^2 \left[ \ln \left( 1 + \frac{\gamma_{i+1}g_{i+1}}{\sum_{k=1}^{i}\gamma_kg_k} \right) \right]^2 \times \exp \left[ \left( \frac{g_i}{g_{i+1}} - 1 \right) \ln \left( 1 + \frac{\gamma_{i+1}g_{i+1}}{\sum_{k=1}^{i}\gamma_kg_k} \right) \right].
\]

Similarly, for \( j = 0, 1, \ldots, I - i \), we obtain the limit

\[
(3.40) \quad \lim_{N \to \infty} N \left\{ \prod_{t=1}^{[N\gamma]_{i+j+1}} \frac{S_{[N\gamma],1+j,t}^{(1)}}{S_{[N\gamma],1+j,t}^{(1)}} - \exp \left[ \left( \frac{g_i}{g_{i+1+j}} - 1 \right) \ln \left( 1 + \frac{\gamma_{i+1+j}g_{i+1+j}}{\sum_{k=1}^{i+j}\gamma_kg_k} \right) \right] \right\}
\]

\[
= -\frac{1}{2\gamma_{i+1+j}} \left( \frac{g_i}{g_{i+1+j}} - 1 \right)^2 \left[ \ln \left( 1 + \frac{\gamma_{i+1+j}g_{i+1+j}}{\sum_{k=1}^{i+j}\gamma_kg_k} \right) \right]^2 \times \exp \left[ \left( \frac{g_i}{g_{i+1+j}} - 1 \right) \ln \left( 1 + \frac{\gamma_{i+1+j}g_{i+1+j}}{\sum_{k=1}^{i+j}\gamma_kg_k} \right) \right].
\]
Then, the limit in (3.40) enables us to obtain the estimate for \( \prod_{i=1}^{|[N\Gamma]|} \frac{S_{i}^{(1)}}{S_{i}^{(N\Gamma),i}} \) as \( N \to \infty \):

(3.41)

\[
\prod_{i=1}^{|[N\Gamma]|} \frac{S_{i}^{(1)}}{S_{i}^{(N\Gamma),i}} =: g_{i}^{(1)} \left\{ 1 - \frac{1}{N} \sum_{j=1}^{l-i} \frac{1}{2\gamma_{i,j}} \left( \frac{g_{i_1} - 1}{g_{i_1} + 1} \right)^{2} \ln \left[ 1 + \frac{\gamma_{i,j} g_{i_1} + 1}{\sum_{k=1}^{l-i} \gamma_{k} g_{k}} \right] \right\}^{2} \times \exp \left[ \frac{g_{i_1} - 1}{\gamma_{i_1} g_{i_1} + 1} \ln \left( 1 + \frac{\gamma_{i_1} g_{i_1} + 1}{\sum_{k=1}^{l-i} \gamma_{k} g_{k}} \right) \right] + o \left( \frac{1}{N} \right) \right),
\]

from which we arrive at relation (3.27) for \( i = 1, 2, \ldots, I - 1 \). The proof of (3.28) for \( i = 2, 3, \ldots, I \) is similar.

Lemma 3.6. Under the assumptions of Lemma 3.4, we have the asymptotic expansions:

(3.42)

\[
P_{[N\Gamma]}^{(1)} = C^{(1)} (2\pi N)^{-\frac{1}{2}} (l - 1) \prod_{i=1}^{l} \exp \left( -\alpha^{(1)}_{i} \gamma_{i} + 1 - \gamma_{i} \right) \left( \Delta^{(1)}_{i} \right)^{[N\gamma_{i}]} \times [1 + o(1)],
\]

and

(3.43)

\[
P_{[N\Gamma]}^{(2)} = C^{(2)} (2\pi N)^{-\frac{1}{2}} (l - 1) \prod_{i=1}^{l} \exp \left( -\alpha^{(2)}_{i} \gamma_{i} + 1 - \gamma_{i} \right) \left( \Delta^{(2)}_{i} \right)^{[N\gamma_{i}]} \times [1 + o(1)],
\]

where \( \alpha^{(1)}_{i} \) and \( \alpha^{(2)}_{i} \), \( i = 1, 2, \ldots, I \), are defined by (3.29) and (3.30) and (3.31) and (3.32).

Proof. The proofs of (3.42) and (3.43) are similar. Therefore, we prove (3.42) only. Notice that for \( \prod_{i=1}^{n} s_{i}^{(1)} \) we have the following obvious inequalities:

(3.44)

\[
g_{i}^{(1)} |n| n! \leq \prod_{i=1}^{n} S_{i}^{(1)} \leq g_{i}^{(1)} |n| n!
\]

Hence, keeping in mind that for any \( l, 1 \leq l < |n| \), we have

\[
S_{n,l}^{(1)} \leq \frac{S_{n,l+1}}{l + 1},
\]

then one can arrive at the conclusion that, as \( N \to \infty \), there exists the limit

(3.45)

\[
\lim_{N \to \infty} \prod_{l=1}^{|[N\Gamma]|} \frac{S_{l}^{(1)}}{S_{l}^{(N\Gamma),l}} \right) \right] = g^{(1)}.
\]

One can find the constant \( g^{(1)} \) in (3.45) as follows. We have

(3.46)

\[
\lim_{N \to \infty} \prod_{l=1}^{|[N\Gamma]|} \frac{S_{l}^{(1)}}{S_{l}^{(N\Gamma),l}} = \lim_{N \to \infty} \frac{S_{l}^{(1)}}{S_{l}^{(N\Gamma),l}} = \lim_{N \to \infty} \sum_{l=1}^{I} g_{l} \left( N \gamma_{l} \right) = \sum_{l=1}^{I} \gamma_{l} g_{l}.
\]
This enables us to conclude that \( g^{(1)} \) must be equal to the right-hand side of (3.46), i.e. \( g^{(1)} = \sum_{i=1}^{I} \gamma_i g_i \).

Let us find an asymptotic expansion for \( \prod_{i=1}^{||N\Gamma||} \frac{S_{(N\Gamma),i}}{I} \) as \( N \to \infty. \)

Denote \( C_n = \frac{|n|!}{n_1!n_2! \cdots n_I!}. \) Then,
\[
\frac{C_{n+1,i}}{C_n} = \frac{|n|+1}{n_i+1},
\]
and as \( N \to \infty, \) we obtain the expansion
\[
(3.47) \quad \frac{C_{[N\Gamma]+1,i}}{C_{[N\Gamma]}} = \frac{1}{\gamma_i} \left[ 1 - \frac{1}{N} \frac{1}{\gamma_i} + O \left( \frac{1}{N^2} \right) \right].
\]

Hence, taking into account Lemma 3.5, we arrive at the conclusion that an asymptotic expansion for \( \prod_{i=1}^{||N\Gamma||} \frac{S_{(N\Gamma),i}}{I} \), as \( N \to \infty, \) is
\[
\prod_{i=1}^{||N\Gamma||} \frac{S_{(N\Gamma),i}}{I} = \left( \sum_{i=1}^{I} \gamma_i g_i \right)^N \exp \left[ -\sum_{i=1}^{I} \left( \alpha_i^{(1)} \gamma_i - (1 - \gamma_i) \right) \right] \left[ 1 + o(1) \right],
\]
where \( \alpha_i^{(1)}, \) \( i = 1, 2, \ldots, I, \) are given by (3.29) and (3.30). Now, using Stirling's formula for \( \Gamma \) as \( N \to \infty, \) one can write the expansion
\[
P_{[N\Gamma]}^{(4)} = C^{(1)} (2\pi N)^{-\frac{1}{2}(I-1)} \sqrt{\frac{1}{\gamma_i \gamma_2 \cdots \gamma_I}} \prod_{i=1}^{I} \left( -\alpha_i^{(1)} \gamma_i + 1 - \gamma_i \right) \left( \Delta_i^{(1)} \right)^{[N\gamma_i]} \times [1 + o(1)],
\]
and the statement of the lemma follows. \( \square \)

**Lemma 3.7.** There exists a positive integer \( n \) such that for any vector \( n \in N_{I,n} \) and \( i = 1, 2, \ldots, I, \) we have the inequalities
\[
(3.48) \quad J_{i,n+1}(P_{n+1}^{(1)}, P_n^{(1)}) - J_{i,n}(P_n^{(1)}, P_{n-1}^{(1)}) \geq 0,
\]
\[
(3.49) \quad J_{i,n+1}(P_{n+1}^{(2)}, P_n^{(2)}) - J_{i,n}(P_n^{(2)}, P_{n-1}^{(2)}) \leq 0,
\]
In the case \( i \neq I, \) the inequalities in (3.48) are strong and, respectively, in the case \( i \neq 1 \) the inequalities in (3.49) are strong.

**Proof.** Note first, that in the case \( i = I \) relation (3.48) and, respectively, in the case \( i = 1 \) relation (3.49) in Lemma 3.7 are automatically satisfied for all \( n \in N, \) since in the case \( i = I \) according to relation (3.13) in Lemma 3.3 we have
\[
J_{i,n+1}(P_{n+1}^{(1)}, P_n^{(1)}) = J_{i,n}(P_n^{(1)}, P_{n-1}^{(1)}) = 0,
\]
and in the case \( i = 1 \) according to relation (3.15) in the same lemma we have
\[
J_{1,n+1}(P_{n+1}^{(2)}, P_n^{(2)}) = J_{1,n}(P_n^{(2)}, P_{n-1}^{(2)}) = 0.
\]

We prove now that there exists a positive integer \( n, \) \( n \in N_{I,n} \) such that the strong inequalities of (3.48) hold for \( i = 1, 2, \ldots, I - 1. \) The proof of the strong inequalities of (3.49) for \( i = 2, 3, \ldots, I \) is similar.

Indeed, after canceling the multiplier \( C^{(1)} \sum_{n_1!n_2! \cdots n_I!} \) in the relations for
\[
J_{i,n+1}(P_{n+1}^{(1)}, P_n^{(1)}) - J_{i,n}(P_n^{(1)}, P_{n-1}^{(1)}),
\]
and a small algebra the problem reduces to prove that there exists a positive integer \( n \) such that the inequality

\[
(3.50) \quad \frac{\rho_i}{n_i g_i} \left[ \prod_{i=1}^{n} S_{n+1,1}^{(1)} - \prod_{i=1}^{n} S_{n,1}^{(1)} \right] = \frac{1}{\sum l=1^{n-1} S_{n,l}^{(1)} + \prod l=1^{n-1} S_{n-1,l}^{(1)}} > 0
\]

is true for all \( n \in N_{\Gamma,n} \). Denote the left-hand side of (3.50) by \( f(n, \rho_i) \). It follows from Lemma 3.3 that \( f(n,0) > 0 \) for all \( n \in N \). From same Lemma 3.3 the derivative of \( f(n, \rho_i) \) satisfies the property \( \frac{df(n,\rho_i)}{d\rho_i} < 0 \). Hence, the lemma will be proved if we show that there exists a positive integer \( n \), such that for \( n \in N_{\Gamma,n} \) the function \( f(n, \rho_i) \) is positive for all \( \rho_i \), under which the probability mass function \( P_n^{(1)} \) is proper. From (3.50) we have

\[
(3.51) \quad f(\rho_i) = \frac{1}{\prod l=1^{n} S_{n,l}^{(1)}} = \frac{\rho_i}{n_i g_i} \left[ \prod_{i=1}^{n} S_{n+1,1}^{(1)} - 1 \right] - \frac{1}{\langle n, g \rangle} \left[ 1 - \prod l=1^{n-1} S_{n,l}^{(1)} \right].
\]

Hence, as \( N \to \infty \), according to Lemma 3.4 from (3.51) we obtain

\[
(3.52) \quad \lim_{N \to \infty} N f(\sum l=1^{N} \rho_i) = \prod l=1^{n} S_{n,l}^{(1)} = \frac{\rho_i}{\gamma_i g_i} \left( \theta_i^{(1)} - 1 \right) - \frac{1}{\sum l=1 \gamma_i g_i} \left( 1 - \frac{1}{\theta_i^{(1)}} \right).
\]

The right-hand side of (3.52) is positive, since

\[
(3.53) \quad \frac{\gamma_i g_i + \gamma_2 g_2 + \ldots + \gamma_l g_l}{\gamma_i g_i} \cdot \theta_i^{(1)}, \rho_i = \Delta_i^{(1)} < 1.
\]

Hence, for any \( \rho_i \) satisfying (3.53), there exists a large value \( n \) for which \( f(n, \rho_i) > 0 \) for any \( n \in N_{\Gamma,n} \). The lemma is proved.

3.3.3. **Final part of the proof of Theorem 2.3** Let us define the set \( G \) as the set of all directions \( \Gamma \) for which the condition \( \Delta_i^{(2)} < 1 \), \( i = 1, 2, \ldots, I \), is satisfied. According to Lemma 3.7 there exists a set of positive integer numbers \( n_\Gamma \) denoted \( N(G) \), and we define the set \( C(G,N(G)) = \cup_{\Gamma \in G} N_{\Gamma,n_\Gamma} \). Note, that the set of positive integer numbers \( n_\Gamma \) can be chosen such that \( n_\Gamma \geq N_0 \), where \( N_0 \) is a sufficiently large integer number.

For all \( n \in C(G,N(G)) \) from Lemma 3.7 we have:

\[
(3.54) \quad \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}^{(1)}(P_n^{(1)}, P_n^{(1)}) - \mathcal{J}_{i,n}^{(1)}(P_n^{(1)}, P_n^{(1)}) \right] > 0,
\]

\[
(3.55) \quad \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n}^{(2)}(P_n^{(2)}, P_n^{(2)}) - \mathcal{J}_{i,n}^{(2)}(P_n^{(2)}, P_n^{(2)}) \right] < 0.
\]

Hence, taking into account (3.54) together with (3.54) and (3.55), we can conclude that there exists the sequence of constants \( \beta_n \), \( 0 < \beta_n < 1 \), \( n \in C(G,N(G)) \), such
that
\[
\beta_n \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}(P_{n+1}^{(1)}, P_n^{(1)}) - \mathcal{J}_{i,n}(P_n^{(1)}, P_{n-1}^{(1)}) \right] \\
+ (1 - \beta_n) \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}(P_{n+1}^{(2)}, P_n^{(2)}) - \mathcal{J}_{i,n}(P_n^{(2)}, P_{n-1}^{(2)}) \right] 
\]
(3.56)
\[
= \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}(P_{n+1}, P_n) - \mathcal{J}_{i,n}(P_n, P_{n-1}) \right] = 0.
\]

The system of equations (3.56) is basic for our following study. Notice, that for the left-hand side of (3.56) we have
\[
\beta_n \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}(P_{n+1}^{(1)}, P_n^{(1)}) - \mathcal{J}_{i,n}(P_n^{(1)}, P_{n-1}^{(1)}) \right] \\
+ (1 - \beta_n) \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}(P_{n+1}^{(2)}, P_n^{(2)}) - \mathcal{J}_{i,n}(P_n^{(2)}, P_{n-1}^{(2)}) \right] 
\]
\[
= \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}(\beta_n P_n^{(1)} + (1 - \beta_n)P_{n+1}^{(1)}, \beta_n P_n^{(2)} + (1 - \beta_n)P_{n+1}^{(2)}) \right. \\
- \mathcal{J}_{i,n}(\beta_n P_n^{(1)} + (1 - \beta_n)P_{n+1}^{(1)}, \beta_n P_n^{(2)} + (1 - \beta_n)P_{n+1}^{(2)}) \right],
\]
and hence, (3.56) can be rewritten
\[
\sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}(P_{n+1}, P_n) - \mathcal{J}_{i,n}(P_n, P_{n-1}) \right]
\]
(3.57)
\[
= \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}(\beta_n P_n^{(1)} + (1 - \beta_n)P_{n+1}^{(1)}, \beta_n P_n^{(2)} + (1 - \beta_n)P_{n+1}^{(2)}) \right. \\
- \mathcal{J}_{i,n}(\beta_n P_n^{(1)} + (1 - \beta_n)P_{n+1}^{(1)}, \beta_n P_n^{(2)} + (1 - \beta_n)P_{n+1}^{(2)}) \right].
\]

The left-hand side of (3.57) equated to zero defines the system of linear equations for $P_n$, and the right-hand of (3.57) equated to zero defines the system of equations for the convex combination $\beta_n P_n^{(1)} + (1 - \beta_n)P_n^{(2)}$. For $\mathbf{n} \in \mathcal{C}(\mathcal{G}, \mathcal{N}(\mathcal{G}))$ these systems of equations are identical. However, they are expressed via the values $P_n$, $P_{n-1}$, and $\beta_n P_n^{(1)} + (1 - \beta_n)P_n^{(2)}$, respectively, in the first and second equations, in which if $\mathbf{n}$ is a boundary vector of $\mathcal{C}(\mathcal{G}, \mathcal{N}(\mathcal{G}))$, the vector $\mathbf{n} - \mathbf{1}$ may not belong to the set $\mathcal{C}(\mathcal{G}, \mathcal{N}(\mathcal{G}))$.

For $\{\mathbf{n}, i\} \in \mathcal{C}^0(\mathcal{G}, \mathcal{N}(\mathcal{G}))$, let $P_{n-1} = d_{n,i}$ and let
\[
\sum_{\mathbf{n} \in \mathcal{C}(\mathcal{G}, \mathcal{N}(\mathcal{G}))} P_n = p > 0,
\]
and
\[
\sum_{\mathbf{n} \in \mathcal{C}(\mathcal{G}, \mathcal{N}(\mathcal{G}))} [\beta_n P_n^{(1)} + (1 - \beta_n)P_n^{(2)}] = p^* > 0,
\]
where the constants $p$ and $p^*$ are the normalizing constants, $p$ depends on the values $d_{n,i}$. Notice, that for the left-hand side of (3.57) equated to zero, $\mathbf{n} \in \mathcal{C}(\mathcal{G}, \mathcal{N}(\mathcal{G}))$,
we have the following system of equations:

\[(3.58) \quad \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}, \left( \frac{P_{n+1}}{P_n}, 1 \right) - \mathcal{J}_{i,n} \left( 1, \frac{P_{n-1}}{P_n} \right) \right] = 0.\]

For the right-hand side of \((3.57)\) equated to zero, \(n \in \mathcal{C}(\mathcal{G}, \mathcal{N}(\mathcal{G}))\), the system of equations is similar:

\[(3.59) \quad \sum_{i=1}^{I} \left[ \mathcal{J}_{i,n+1}, \left( \frac{\beta_n P_{n+1}}{P_n} + (1 - \beta_n) P_{n+1}^{(2)}, 1 \right) - \mathcal{J}_{i,n} \left( 1, \frac{\beta_n P_{n-1}}{P_n} + (1 - \beta_n) P_{n-1}^{(2)} \right) \right] = 0.\]

With the aforementioned initial conditions \(P_{n-i} = d_{n,i}, \{n, i\} \in \mathcal{C}(\mathcal{G}, \mathcal{N}(\mathcal{G}))\), the system of equations \((3.58)\) is uniquely defined. For any \(\epsilon > 0\) the value \(N_0\) can be chosen so large, that for \(N \geq N_0\) and all \(i = 1, 2, \ldots, I\),

\[(3.60) \quad \left| \frac{P_{[N\Gamma]-1}}{P_{[N\Gamma]}} - \frac{P_{[N\Gamma]-1}}{P_{[N\Gamma]+1}} \right| < \epsilon,\]

and similarly we have

\[(3.61) \quad \left| \frac{\beta_{[N\Gamma]} P_{[N\Gamma]-1}^{(1)} + (1 - \beta_{[N\Gamma]}) P_{[N\Gamma]-1}^{(2)}}{\beta_{[N\Gamma]} P_{[N\Gamma]}^{(1)} + (1 - \beta_{[N\Gamma]}) P_{[N\Gamma]}^{(2)}} - \frac{\beta_{[N\Gamma]+1,1} P_{[N\Gamma]+1}^{(1)} + (1 - \beta_{[N\Gamma]+1,1}) P_{[N\Gamma]+1}^{(2)}}{\beta_{[N\Gamma]+1,1} P_{[N\Gamma]+1}^{(1)} + (1 - \beta_{[N\Gamma]+1,1}) P_{[N\Gamma]+1}^{(2)}} \right| < \epsilon.\]

On the other hand, the system of equations \((3.57)\) implies a continuous correspondence between the systems of equations \((3.58)\) and \((3.59)\). It means that for any \(\epsilon > 0\) there exists the value \(\epsilon > 0\) such that \((3.60)\) and \((3.61)\) imply

\[\left| \frac{P_{[N\Gamma]-1}}{P_{[N\Gamma]}} - \frac{\beta_{[N\Gamma]} P_{[N\Gamma]-1}^{(1)} + (1 - \beta_{[N\Gamma]}) P_{[N\Gamma]-1}^{(2)}}{\beta_{[N\Gamma]} P_{[N\Gamma]}^{(1)} + (1 - \beta_{[N\Gamma]}) P_{[N\Gamma]}^{(2)}} \right| < \epsilon,\]

which in turn means

\[(3.62) \quad \lim_{N \to \infty} \frac{P_{[N\Gamma]+1}}{P_{[N\Gamma]}} = \lim_{N \to \infty} \frac{\beta_{[N\Gamma]} P_{[N\Gamma]+1}^{(1)} + (1 - \beta_{[N\Gamma]}) P_{[N\Gamma]+1}^{(2)}}{\beta_{[N\Gamma]} P_{[N\Gamma]}^{(1)} + (1 - \beta_{[N\Gamma]}) P_{[N\Gamma]}^{(2)}}.\]

Applying asymptotic relations \((3.18)\) and \((3.19)\) of Lemma \(3.4\) to the right-hand side of \((3.62)\) we obtain

\[(3.63) \quad \lim_{N \to \infty} \frac{P_{[N\Gamma]+1}}{P_{[N\Gamma]}} = \lim_{N \to \infty} \frac{\beta_{[N\Gamma]} \Delta_{[N\Gamma]}^{(1)} P_{[N\Gamma]}^{(1)} + (1 - \beta_{[N\Gamma]}) \Delta_{[N\Gamma]}^{(2)} P_{[N\Gamma]}^{(2)}}{\beta_{[N\Gamma]} P_{[N\Gamma]}^{(1)} + (1 - \beta_{[N\Gamma]}) P_{[N\Gamma]}^{(2)}}.\]

The two propositions below enable us to establish the limit in \((3.63)\). In the following, for two sequences \(\{a_N\} \) and \(\{b_N\}\) vanishing as \(N \to \infty\), the writing \(a_N \sim b_N\) means that \(\lim_{N \to \infty} \frac{b_N}{a_N} = 1\).
Proposition 3.8. As $N \to \infty$,
\begin{align}
\sum_{i=1}^{l} \left[ \mathcal{J}_{i,[NT]+1} (P_{[NT]+1}^{(1)}, P_{[NT]}^{(1)}) - \mathcal{J}_{i,[NT]} (P_{[NT]}^{(1)}, P_{[NT]-1}^{(1)}) \right] \\
\sim P_{[NT]}^{(1)} \sum_{i=1}^{l} \mathcal{J}_{i,[NT]} \left( \Delta_{i}^{(1)} - 1, 1 - \frac{1}{\Delta_{i}^{(1)}} \right),
\end{align}
(3.64)
and
\begin{align}
\sum_{i=1}^{l} \left[ \mathcal{J}_{i,[NT]+1} (P_{[NT]+1}^{(2)}, P_{[NT]}^{(2)}) - \mathcal{J}_{i,[NT]} (P_{[NT]}^{(2)}, P_{[NT]-1}^{(2)}) \right] \\
\sim P_{[NT]}^{(2)} \sum_{i=1}^{l} \mathcal{J}_{i,[NT]} \left( \Delta_{i}^{(2)} - 1, 1 - \frac{1}{\Delta_{i}^{(2)}} \right).
\end{align}
(3.65)

Proof. From Lemma 3.4 we have $\lim_{N \to \infty} \frac{P_{[NT]+1}^{(1)}}{P_{[NT]}^{(1)}} = \Delta_{1}^{(1)}$ and $\lim_{N \to \infty} \frac{P_{[NT]+1}^{(2)}}{P_{[NT]}^{(2)}} = \Delta_{1}^{(2)}$. Assume that $N_0$ is chosen large enough such that for all $N > N_0$ the inequalities $|P_{[NT]}^{(1)} \Delta_{i}^{(1)} - P_{[NT]-1}^{(1)}| < \epsilon P_{[NT]}^{(1)}$, and $|P_{[NT]}^{(2)} \Delta_{i}^{(2)} - P_{[NT]-1}^{(2)}| < \epsilon P_{[NT]}^{(2)}$, and $i = 1, 2, \ldots, I$ are satisfied. Then, for the upper bound we obtain
\begin{align}
\sum_{i=1}^{l} \left[ \mathcal{J}_{i,[NT]+1} \left( P_{[NT]+1}^{(1)}, 1 \right) - \mathcal{J}_{i,[NT]} \left( 1, \frac{P_{[NT]-1}^{(1)}}{P_{[NT]}^{(1)}} \right) \right] \\
\leq \sum_{i=1}^{l} \left[ \mathcal{J}_{i,[NT]} \left( \Delta_{i}^{(1)}, 1 \right) - \mathcal{J}_{i,[NT]} \left( 1, \frac{1}{\Delta_{i}^{(1)}} \right) \right]
\end{align}
(3.66)

+ $\epsilon \sum_{i=1}^{l} \mu_i g_i \left[ \frac{[N \gamma_i] + 1}{\langle [NT] + 1, g \rangle} + \frac{\lambda_i}{\Delta_{i}^{(1)}} \right]$

+ $\epsilon \sum_{i=1}^{l} \mu_i g_i \left[ \frac{[N \gamma_i] + 1}{\langle [NT] + 1, g \rangle} + \frac{\lambda_i}{\Delta_{i}^{(1)}} \right]$

+ $\epsilon \sum_{i=1}^{l} \mu_i g_i \left[ \frac{[N \gamma_i] + 1}{\langle [NT] + 1, g \rangle} - \frac{[N \gamma_i]}{\langle [NT], g \rangle} \right]$

The estimate for the lower bound is similar.

Clearly, the terms
\begin{align}
\epsilon \sum_{i=1}^{l} \mu_i g_i \left[ \frac{[N \gamma_i] + 1}{\langle [NT] + 1, g \rangle} + \frac{\lambda_i}{\Delta_{i}^{(1)}} \right]
\end{align}
and
\begin{align}
\epsilon \sum_{i=1}^{l} \mu_i g_i \left[ \frac{[N \gamma_i] + 1}{\langle [NT] + 1, g \rangle} - \frac{[N \gamma_i]}{\langle [NT], g \rangle} \right]
\end{align}
in (3.66) vanish as $\epsilon \to 0$ and $N \to \infty$. Hence, (3.64) follows. The proof of (3.65) is similar. \qedsymbol
Proposition 3.9. As $N \to \infty$, the sequence of $\beta_{[NT]}$ tends to 1. Furthermore,

$$1 - \beta_{[NT]} = \frac{\sum_{i=1}^{\ell} \left[ \mu_i \gamma_i g_i \left( \sum_{l=1}^{\ell} \gamma_l g_l \right)^{-1} \left( \Delta_i^{(1)} - 1 \right) - \lambda_i \left( 1 - \left( \Delta_i^{(1)} \right)^{-1} \right) \right]}{\sum_{i=1}^{\ell} \left[ \mu_i \gamma_i g_i \left( \sum_{l=1}^{\ell} \gamma_l g_l \right)^{-1} \left( \Delta_i^{(2)} - 1 \right) - \lambda_i \left( 1 - \left( \Delta_i^{(2)} \right)^{-1} \right) \right]} \times \frac{C^{(1)}}{C^{(2)}} \prod_{i=1}^{\ell} e^{(\alpha_i^{(2)} - \alpha_i^{(1)}) \gamma_i} \left( \frac{\Delta_i^{(1)}}{\Delta_i^{(2)}} \right)^{[N\gamma_i]} \left[ 1 + o(1) \right],$$

(3.67)

where $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ are given by (3.29), (3.30), (3.31) and (3.32).

Proof. It follows from Lemma 3.6 that $P^{(1)}_{\lfloor NT \rfloor} = O \left( N^{-\frac{1}{2}(I-1)} \prod_{i=1}^{\ell} \left[ \Delta_i^{(1)} \right]^{[N\gamma_i]} \right)$ and $P^{(2)}_{\lfloor NT \rfloor} = O \left( N^{-\frac{1}{2}(I-1)} \prod_{i=1}^{\ell} \left[ \Delta_i^{(2)} \right]^{[N\gamma_i]} \right)$ as $N \to \infty$. Furthermore, it is readily seen from the explicit expressions of (2.3) and (2.4) that $\theta_i^{(1)} < 1$, $i = 1, 2, \ldots, I - 1$, $\theta_i^{(1)} = 1$, and $\theta_i^{(2)} > 1$, $i = 2, \ldots, I$, $\theta_1^{(2)} = 1$, i.e. $\theta_i^{(1)} < \theta_i^{(2)}$, $i = 1, 2, \ldots, I$. Consequently, $\Delta_i^{(1)} < \Delta_i^{(2)}$, $i = 1, 2, \ldots, I$. Recall also that $\Delta_i^{(2)} < 1$. Hence, from Proposition 3.8 we have:

$$\lim_{N \to \infty} \prod_{i=1}^{\ell} \left( \frac{1}{\Delta_i^{(1)}} \right)^{[N\gamma_i]} \sum_{i=1}^{\ell} \left[ J_i_{\lfloor NT \rfloor + 1}, (P_{\lfloor NT \rfloor + 1}^{(1)}, P_{\lfloor NT \rfloor}^{(1)}) \right] = -J_i_{\lfloor NT \rfloor} \left( P_{\lfloor NT \rfloor}^{(1)}, P_{\lfloor NT \rfloor - 1}^{(1)} \right) = 0,$$

(3.68)

and

$$\lim_{N \to \infty} \prod_{i=1}^{\ell} \left( \frac{1}{\Delta_i^{(2)}} \right)^{[N\gamma_i]} \sum_{i=1}^{\ell} \left[ J_i_{\lfloor NT \rfloor + 1}, (P_{\lfloor NT \rfloor + 1}^{(2)}, P_{\lfloor NT \rfloor}^{(2)}) \right] = -J_i_{\lfloor NT \rfloor} \left( P_{\lfloor NT \rfloor}^{(2)}, P_{\lfloor NT \rfloor - 1}^{(2)} \right) = \infty.$$

(3.69)

It follows from (3.68) and (3.69) that $\beta_{[NT]}$ tends to 1 as $N \to \infty$. To obtain the exact expansion given by (3.67) we take into account (3.42) and (3.43) of Lemma 3.6 and (3.64) and (3.65) of Proposition 3.8. From these estimates we obtain

$$1 - \beta_{[NT]} \sim -\frac{\sum_{i=1}^{\ell} J_i_{\lfloor NT \rfloor} \left( \Delta_i^{(1)} - 1, 1 - \left( \Delta_i^{(1)} \right)^{-1} \right)}{\sum_{i=1}^{\ell} \left[ \mu_i \gamma_i g_i \left( \sum_{l=1}^{\ell} \gamma_l g_l \right)^{-1} \left( \Delta_i^{(2)} - 1 \right) - \lambda_i \left( 1 - \left( \Delta_i^{(2)} \right)^{-1} \right) \right]} \times \frac{C^{(1)}}{C^{(2)}} \prod_{i=1}^{\ell} e^{(\alpha_i^{(2)} - \alpha_i^{(1)}) \gamma_i} \left( \frac{\Delta_i^{(1)}}{\Delta_i^{(2)}} \right)^{[N\gamma_i]}.$$  

Proposition 3.9 is proved. \(\square\)

Let us now calculate the limit in the left-hand side of (3.63). Inserting (3.67) into the limit in the right-hand side of (3.63), with the aid of asymptotic expansions...
(3.32) and (3.33) of Lemma 3.6 we obtain

\[
\lim_{N \to \infty} \frac{P_{n+1}}{P_n} = \frac{\Delta_i^{(1)} - c\Delta_i^{(2)}}{1 - c},
\]

where

\[
c = \frac{\sum_{i=1}^{I} \left[ \mu_i \gamma_i \left( \sum_{l=1}^{I} \gamma_l g_l \right)^{-1} (\Delta_i^{(1)} - 1) - \lambda_i \left( 1 - (\Delta_i^{(1)})^{-1} \right) \right]}{\sum_{i=1}^{I} \left[ \mu_i \gamma_i \left( \sum_{l=1}^{I} \gamma_l g_l \right)^{-1} (\Delta_i^{(2)} - 1) - \lambda_i \left( 1 - (\Delta_i^{(2)})^{-1} \right) \right]}.
\]

Theorem 2.3 is proved.

3.3.4. The proof of Corollary 2.4. Assume first that \(\gamma_1, \gamma_2, \ldots, \gamma_I\) are rational numbers. Denote \(\delta = \inf \{x : \gamma_i x \in \mathbb{N}, i = 1, 2, \ldots, I\}\). Then, from Theorem 2.3 we obtain

\[
\lim_{N \to \infty} \frac{P_{\lfloor N \gamma \rfloor + \delta \gamma}}{P_{\lfloor N \gamma \rfloor}} = \prod_{i=1}^{I} (\Delta_i)^{\delta\gamma_i}.
\]

Then, denoting \(M = \max \{m : \delta m \leq N\}\) we obtain

\[
\delta \sum_{i=1}^{I} \gamma_i \ln \Delta_i = \lim_{N \to \infty} \left( \ln P_{\lfloor N \gamma \rfloor + \delta \gamma} - \ln P_{\lfloor N \gamma \rfloor} \right)
\]

\[
= \lim_{N \to \infty} \frac{\delta}{N} \left[ \sum_{m=1}^{M-1} (\ln P_{\delta(m+1)\gamma} - \ln P_{\delta m \gamma}) \right]
\]

\[
+ \lim_{N \to \infty} \frac{\delta}{N} \left[ \ln P_{\lfloor N \gamma \rfloor} - \ln P_{\delta M \gamma} \right] = 0
\]

\[
+ \lim_{N \to \infty} \frac{\delta}{N} \left[ \ln P_{\lfloor N \gamma \rfloor + \delta \gamma} - \ln P_{\lfloor N \gamma \rfloor} \right] = 0
\]

\[
= \lim_{N \to \infty} \frac{\delta}{N} \ln P_{\delta M \gamma} = \lim_{N \to \infty} \frac{\delta}{N} \ln P_{\lfloor N \gamma \rfloor}.
\]

Hence,

\[
\lim_{N \to \infty} \frac{\ln P_{\lfloor N \gamma \rfloor}}{N} = \sum_{i=1}^{I} \gamma_i \ln \Delta_i.
\]

Assume now, that there is at least one of \(\gamma_1, \gamma_2, \ldots, \gamma_I\) that is irrational. Then, there is a sequence of rational numbers \(\gamma_{1,n}, \gamma_{2,n}, \ldots, \gamma_{I,n}\) that converges to the limit \(\gamma_1, \gamma_2, \ldots, \gamma_I\). Denote \(\delta_n = \inf \{x : \gamma_{i,n} x \in \mathbb{N}, i = 1, 2, \ldots, I\}\). Then, for any \(\{\gamma_{1,n}, \gamma_{2,n}, \ldots, \gamma_{I,n}\}\), the limiting relation of (3.71) holds. Then, keeping in mind that \(\Delta_i = \Delta_i^{(1)} - c\Delta_i^{(2)}\) is continuous in \(\gamma_1, \gamma_2, \ldots, \gamma_I\), because each of \(\Delta_i^{(1)}, \Delta_i^{(2)}\) and \(c\) is continuous in \(\gamma_1, \gamma_2, \ldots, \gamma_i\), one can take a limit in (3.70) as \(\delta_n\) increases to infinity to arrive at (3.71). The corollary is proved.
Table 1. The table of optimal values of $\gamma_1$ and $\gamma_2$ for DPS systems with two classes of flow

| Variable parameter $g_2$ | Optimal value $\gamma_1$ | Optimal value $\gamma_2$ |
|--------------------------|--------------------------|--------------------------|
| 2                        | .401                     | .599                     |
| 2.5                      | .435                     | .565                     |
| 3                        | .474                     | .526                     |
| 3.5                      | .633                     | .367                     |
| 4                        | .622                     | .378                     |

4. MOST LIKELY ASYMPTOTIC DIRECTION

Asymptotic Theorem 2.3 is obtained under the assumption that $n_1 = \lfloor N \gamma_1 \rfloor$, $n_2 = \lfloor N \gamma_2 \rfloor$, ..., $n_I = \lfloor N \gamma_I \rfloor$ for large value $N$. By most likely direction we mean such values $\gamma_1, \gamma_2, \ldots, \gamma_I$ that minimize $\left( -\lim_{N \to \infty} \frac{1}{N} \ln P(n_{\lfloor N \gamma \rfloor}) \right)$. Then, the problem is to minimize

$$- \sum_{i=1}^{I} \gamma_i \ln \frac{\Delta_i^{(1)} - c\Delta_i^{(2)}}{1 - c},$$

subject to the constraints

$$\Delta_i^{(2)} < 1, \quad i = 1, 2, \ldots, I,$$

$$\sum_{i=1}^{I} \gamma_i g_i \Delta_i^{(2)} < \gamma_i g_i, \quad i = 1, 2, \ldots, I,$$

$$\sum_{i=1}^{I} \gamma_i = 1.$$

This is a convex optimization problem. It can be solved by the interior point method [8]. Some numerical examples for its solution are given in Table 1. For the numerical calculations, the following set of parameters is taken: $I = 2, \mu_1 = \mu_2 = 1$ and $\lambda_1 = 0.2, \lambda_2 = 0.3$. The value $g_1 = 2$ is taken fixed for all calculations in the table. The variable parameter $g_2$ takes the values 2, 2.5, 3, 3.5 and 4. The case where $g_2 = g_1 = 2$ (the first row in the table) is related to the Egalitarian PS system.

5. CONCLUDING REMARKS AND AN OPEN PROBLEM

In the present paper we established a characterization theorem on impossibility of presenting the stationary probabilities in closed geometric form. Implicitly we have shown that the stationary probability cannot have the form $F(n, g)G(n, \lambda, \mu)$, where $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_I\}$ and $\mu = \{\mu_1, \mu_2, \ldots, \mu_I\}$, since if $P_n$ can be represented in this form, then it can be shown that $G(n, \lambda, \mu)$ must be equal to $(1 - \rho) \prod_{i=1}^{I} \rho_i^{n_i}$.

While for Egalitarian PS systems the explicit formula for the stationary distribution is known and has a relatively simple closed geometric form, the analysis of the DPS system it very hard. We have provided a full asymptotic analysis of the tail probabilities that is based on an analysis of the system of the equations for the stationary probabilities. The method of asymptotic analysis uses technical assumption (2.7) that includes the constants $\theta_i^{(2)} > 1, i = 2, 3, \ldots, I$ and $\theta_1^{(2)} = 1$. 
Unfortunately, the methods of asymptotic analysis of the present paper enable us to merely obtain the asymptotes for \( P_{\lceil N\Gamma \rceil}^{(i)} \), \( i = 1, 2, \ldots, I \), for large \( N \), but do not permit to obtain an asymptotic expansion for \( P_{\lceil N\Gamma \rceil} \) itself. This type of asymptotic expansion requires more delicate methods of asymptotic analysis. Our conjecture is that under the assumptions made in Theorem 2.3, \( P_{\lceil N\Gamma \rceil} = O\left( N^{-\frac{1}{2}(I-1)} \prod_{i=1}^{I} \left[ \frac{\Delta_{i}^{(1)} - c\Delta_{i}^{(2)}}{1 - c} \right] \lceil N\gamma_i \rceil \right) \).

The probabilities \( P_{\lceil N\Gamma \rceil}^{(1)} \) and \( P_{\lceil N\Gamma \rceil}^{(2)} \) have the similar type of asymptotes (Lemma 3.6), and this is the reason for this conjecture.

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