Method of Finite Bodies for Mathematical Modeling of the Stress-strain State of Cylindrical Orthotropic Shell with the Reinforced Rectangular Hole

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Abstract. An analytical-numerical method (finite bodies) for calculating the stress state of the cylindrical orthotropic shell with a large reinforced rectangular hole is proposed. The shell and reinforcement consist of various materials. The systems of equations that exactly satisfies the equilibrium equations of the orthotropic cylindrical shell and the orthotropic reinforced are used. The presentation of the solution is divided into basic and self-balanced states. The finite-body method uses the conditional partition of the three-connected surface of the shell and the reinforcement into simpler doubly-connected and simply connected parts. An algorithm is developed for analytic-numerical solution of boundary-value problems, based on the proposed universal method of reducing the satisfaction of all contact conditions of shell parts and boundary conditions to minimization of the generalized quadratic form. We establish criteria under which the construction of approximate solution coincides with the exact one.

1. Introduction
Cylindrical orthotropic shell stress strain state investigation is an important scientific and practical problem [1]–[10]. In [3, 6] the authors introduced the methods that allowed to describe the stress strain state for a cylindrical shell with a small hole, [7] helps us to consider the stress state of the shell weakened by an elliptic hole. In papers [8, 9] the finite element method was used to find the stress state for a shell with a rectangular hole, and [9, 10] experimental studies of the shells reinforced in the hole zone were carried out.

2. Statement of the problem.
Consider the stress-strain state of the orthotropic cylindrical shell: here $R$ is the radius of the central surface, $h$ is the cylinder thickness and $2H$ is its length. We also assume that the shell is weakened by the sizeable rectangular hole $\Pi$ with sides $2a$ and $2b$ directed along the forming and directing lines of the shell surface. The edges of the hole are reinforced by an orthotropic elastic strip (reinforcement): here $h_1$ is the reinforcement strip thickness, $c$ is its width (see Figure 1).

We denote the curve coordinate along the shell directrix by $\gamma$, $\gamma = R\varphi$ and that along the shell generatrix by $z$. The angle $\varphi$ is taken from the middle of the hole, and the coordinate $z$ is taken from the center of the cylinder.
In [11] the authors presented the equation system that identically satisfies all the equilibrium equations for the orthotropic cylindrical shell and can be written down in the cylindrical coordinates as follows:

\[ V_1 \Phi = \varepsilon E_1 \frac{\partial^2 w}{\partial z^2}, \quad V_2 w = -(1 - \nu_1 \nu_2) \frac{12 \varepsilon}{h^4 E_2} \frac{\partial^2 \Phi}{\partial z^2} - \frac{\varepsilon^2}{h^2 R^2} \frac{\partial^2 w}{\partial \varphi^2}, \] (1)

here \( \Phi(z, \varphi) \) is the stress function, \( w(z, \varphi) \) is the middle surface deflection function (these two functions are often called basic), \( \varepsilon = h/R \); and \( V_1, V_2 \) are fourth order differential operators [11]: \( E_1, E_2 \) are Young’s modules in the directions of the axial and circumferential variable, respectively; \( \nu_1, \nu_2 \) are Poisson’s coefficients; \( \nu_1 E_2 = \nu_2 E_1 \) [1]. The stress state of the reinforced strip (index 1 at the top) is described by functions \( \Phi^1(z, \varphi), w^1(z, \varphi) \), which satisfy the equation (1) with its orthotropy characteristics: \( E_1^1, E_2^1, \nu_1^1, \nu_2^1; \varepsilon_1 = h_1/R \), where \( h_1 \) is thickness.

We can find the forces \( T_1, T_2, S_{12}, S_{21}, \) moments \( M_1, M_2, H_{12} = H_{21} \) and shear forces \( N_1, N_2 \), which defined on the shell middle surface by formulas [11].

Let us find the stress state of the shell loaded at its ends by the balanced symmetric forces and moments:

\[ T_1(\pm H, \varphi) = P_1(\varphi), \quad M_1(\pm H, \varphi) = P_2(\varphi), \quad N_1(\pm H, \varphi) = P_3(\varphi), \]
\[ S_{12}(\pm H, \varphi) = P_4(\varphi), \quad H_{12}(\pm H, y) = P_5(\varphi), \] (2)

here \( P_j(\varphi), j = 1,3 \) are even functions, \( P_4(\varphi) \) is an odd function and \( P_5(\varphi) = 0 \). The contour of the reinforced hole \( \{ z, \varphi \} \in \Pi \) is not loaded:

\[ T_n(z, \varphi) = 0, \quad S_{12}(z, \varphi) = 0, \quad M_n(z, \varphi) = 0, \quad N_n(z, \varphi) = 0, \quad H_{12}(z, \varphi) = 0. \] (3)

Paper [11] shows us that in order to satisfy simultaneously the boundary conditions both at the ends and at the hole contour one should mentally divide the three connected shell surface into simpler simply connected and doubly connected parts. Let us mentally divide the shell along the circles \( z = \pm a \) relative to the lines \( L_2, L_3 \) into three parts. Cut out the reinforcement along the intercepts \( A_1D_1, B_1C_1 \) and cut it along the intercepts \( AD, BC \).
We get seven separate parts. The ideal contact conditions should be met along these cuts. Due to the symmetry of the loads, it is sufficient to consider only the points of the shell $H \geq z \geq 0$. The part of the shell is $H \geq z \geq a$, denoted $D_1$, and the segment of the shell is $a \geq z \geq -a$, $\varphi \in [\theta_1, 2\pi - \theta_1]$, where $\theta_1 = b/R$, we denote by $D_2$ (figure 1).

The line $L_2$ is divided into four arcs: I is $L_{2,1}$, where $\varphi \in [\theta_1, 2\pi - \theta_1]$ it corresponds to the contact line of the shell segments $D_1$ and $D_2$, the rest corresponds to the contact line of the shell segment $D_1$ and the reinforcement: II is $L_{2,2}$, where $\varphi \in [-\theta_1, -\theta_2]$, III is $L_{2,3}$, where $\varphi \in [-\theta_2, \theta_2]$, IV is $L_{2,4}$, where $\varphi \in [\theta_2, \theta_1]$, $\theta_2 = (b - c)/R$. The parts of the reinforcement that contain the arc $L_{2,4}$ we denote $D_3$ and accordingly: $L_{2,3}$ we denote $D_5$; $L_{2,2}$ we denote $D_4$.

We denote the free straight edges of the reinforcement, respectively: $\Pi_1$, where $\varphi = \theta_1 - \theta_2$; $\Pi_2$, where $\varphi = -\theta_1 + \theta_2$, here $z \in [-a + c, a - c]$; $\Pi_3$, where $\varphi \in [-\theta_2, \theta_2]$, $z = a - c$.

To solve the problem, it is necessary to satisfy the conditions of ideal contact along the circle $L_2$, intercepts $A_1D_1$, $B_1C_1$, and parts of intercepts $AD$ where $(\varphi = \theta_1 - \theta_2)$, $BC$ where $(\varphi = -\theta_1 + \theta_2)$, here, $z \in [a - c, a]$, which we denote $S_1$, $S_2$. And also satisfy boundary conditions along the contour of the hole II and the circle $L_1$.

The stress state of separate parts of the shell will be divided into two components: the basic stress state, which is determined by the main vectors of forces and moments; and a self-balanced state (as a series), which decrease with distance from the edges of the shell segments. The method of calculating the stress state by identifying and finding only the basic state was proposed by de Saint-Venant and has received the wide practical application. Note that the basic and self-balanced stress states have different physical nature and accordingly must be described by different functions. The main stress state will assume known [11] and mark with the index 0 at the top.

3. Representation of functions describing the self-balanced shell parts state.

We write down the basic functions of shell parts $D_1$, $D_2$ are found in [11]

$$\Phi_j(\alpha, \varphi) = h^2 E_1 \sum_{k=1}^{M_j} b_k^j \varphi^j(k, \alpha, \varphi), \quad w_j(\alpha, \varphi) = h \sum_{k=1}^{M_j} b_k^j w^j(k, \alpha, \varphi),$$

where $j = 1, 2$, $\alpha = z/R$, $M_1 = 8N$, $M_2 = 12N$, $N$ is the number of harmonics decomposition.

For parts of the reinforcement $D_j$, $j = 3, 5$, we define the boundaries of the change in dimensionless coordinates $\alpha_j = z/R$, $\varphi_j = \varphi$:

$$\alpha_j \in [J_{1j}^j, J_{2j}^j], \quad j = 3, 5, \quad J_{1j}^3 = J_{1j}^5 = (-1)^j \frac{a}{R}, \quad j = 1, 2, \quad J_{1j}^1 = \frac{a - c}{R}, \quad J_{2j}^2 = \frac{a}{R},$$

$\varphi_3 \in [\theta_2, \theta_1]$, $\varphi_4 \in [-\theta_1, -\theta_2]$, $\varphi_5 \in [-\theta_2, \theta_2]$. Similarly to [11], for each part of the lining basic functions are built that allow to satisfy the conditions specified on separate arcs $L_{2j}, j = 2, 4$.

Construction of the shell functions that satisfy the boundary conditions on intercepts parallel to the axis $Oz$. For example, consider the part of the reinforcement $D_3$, where we will find for them in this form:

$$\Phi_3(\alpha_3, \varphi_3) = h^2 E_1 \text{Re}[a_3 \exp((-1)^n \omega \eta (\varphi - \theta_n))] \cos(\omega \alpha),$$

$$w_3(\alpha_3, \varphi_3) = h \text{Re}[(\eta)b_3 \exp((-1)^n \omega \eta (\varphi - \theta_n))] \cos(\omega \alpha), \quad n = 1, 2,$$

here $\omega = k_1 R/\alpha$, $4 > k_1 > \pi$, the choice of the parameter $k_1$ allows to improve the convergence, $a_3$ is the complex coefficient, $\eta$ is the complex spectral parameter, which is determined from the following characteristic equation

$$[\eta^4 + (1 - k^2 q_2) \eta^2 + k^4 E_2^4/2] [\eta^4 - q_1 k^2 \eta^2 + E_2^4/k^4] - (1 - \nu_1 \nu_2) \frac{12 a^4}{h_1^2 R^2 k_1^4} E_2^4 k^4 = 0,$$
Using the first equation (1) and the representation (5), we find the connection between

\[ b_3 = \delta_3(\eta) \alpha_3, \quad \delta_3(\eta) = -\frac{h \omega^2}{R k^2} [\eta^4 - q^2 k^2 \eta^2 + \frac{E_1}{E_2} k^4]. \]

An analysis of the roots of the equation (6) is given in [11].

Considering the above and formulas (4), (6) we write the functions of the parts of the reinforcement \( D_j \), which are determined by the roots \( \mu_{k,m}^1, \eta_{k,m}^1; m = 1, 2 \), in the following compact form:

\[ \Phi_j(\alpha, \varphi) = h^3 E_1^1 \sum_{k=1}^{M_j} b_k^j \varphi^j(k, \alpha, \varphi), \quad w_j(\alpha, \varphi) = h_1 \sum_{k=1}^{M_j} b_k^j w^j(k, \alpha, \varphi), \tag{7} \]

Here \( j = 3, 5 \), \( M_3 = 12N \), \( M_5 = 16N \), \( b_k^j \) are real unknowns.

4. Representation of boundary conditions, conditions of ideal contact and the method of their satisfaction.

On the circle \( L_1 \) and the arc \( L_{2,1} \), the boundary conditions are considered in [11]. On arcs \( L_{2,j} \), \( j = 3, 5 \), which are the connection line of the shell \( D_1 \) and the parts of the reinforcement \( D_j \), \( j = 3, 5 \), ideal contact conditions are met, which can be submitted to as follows:

\[ T_1 + T_1^0 = T_{1,j}^1, \quad S_{12} + S_{12}^0 = S_{12,j}^1, \quad N_1 + N_1^0 = N_{1,j}^1, \quad M_1 + M_1^0 = M_{1,j}^1, \tag{8} \]

\[ w_1 + w_1^0 = w_1^j, \quad \frac{\partial w_1 + w_1^0}{\partial z} = \frac{\partial w_1^j}{\partial z}, \quad u_1 + u_1^0 = u_1^j, \quad v_1 + v_1^0 = v_1^j, \quad j = 3, 5. \]

Express the conditions of ideal contact on the intercepts \( A_1 D_1, B_1 C_1 \)

\[ T_2 + T_2^0 = T_{2,j}^1, \quad S_{21} + S_{21}^0 = S_{21,j}^1, \quad N_2 + N_2^0 = N_{2,j}^1, \quad M_2 + M_2^0 = M_{2,j}^1, \tag{9} \]

\[ w_2 + w_2^0 = w_2^j, \quad \frac{\partial w_2 + w_2^0}{\partial \varphi} = \frac{\partial w_2^j}{\partial \varphi}, \quad u_2 + u_2^0 = u_2^j, \quad v_2 + v_2^0 = v_2^j, \quad j = 3, 4; \]

and intercepts \( S_1, S_2 \)

\[ T_{2,5}^1 = T_{2,5}^1, \quad S_{21,5}^1 = S_{21,5}^1, \quad N_{2,5}^1 = N_{2,5}^1, \quad M_{2,5}^1 = M_{2,5}^1, \tag{10} \]

\[ w_5^1 = w_5^j, \quad \frac{\partial w_5^1}{\partial \varphi} = \frac{\partial w_5^j}{\partial \varphi}, \quad u_5^1 = u_5^j, \quad v_5^1 = v_5^j, \quad j = 3, 4. \]

Let us write together the boundary conditions on the arcs \( L_1, L_{2,1} \), systems of equations (8)–(10), and also the relations (3), after transformations of which we get

\[ \sum_{k=1}^{M} c_k A_{m,k}(\gamma_m) = P_m(\gamma_m), \quad m = 1, K, \quad \gamma_m \in [\beta_m, \alpha_m], \tag{11} \]

Here

\[ K = 84; \beta_m = 0, \alpha_m = 2\pi, m = 1, 5; \beta_m = \theta_1, \alpha_m = 2\pi - \theta_1, m = 6, 13; \beta_m = \theta_2, \alpha_m = \theta_1, m = 14, 21; \beta_m = -\theta_1, \alpha_m = -\theta_2, m = 22, 29; \beta_m = -\theta_2, \alpha_m = \theta_2, m = 30, 37; \beta_m = 0, \alpha_m = J_1^3, m = 38, 53; \beta_m = J_1^3, \alpha_m = J_2^3, m = 54, 69; \beta_m = 0, \alpha_m = J_0^3, m = 70, 79; \beta_m = J_0^3, \alpha_m = J_1^3, m = 80, 84; \]

functions \( P_m(\gamma_m) \) are determined by the the basic stress state of shell.
In order to meet conditions (11) we apply analytical and numerical method that was proposed in [12] for plane problem solving for the rectangular domain, in [13] for boundary problems in the homogeneous cylinders, and in [14] in a multilayer cylinder using homogeneous solutions. The given in [11]–[14] method allows us reduce to numerical solution of the system eighty-four equations (11) to finding the minimum

$$ F(N) $$

of such a generalized quadratic form:

$$ \sum_{m=1}^{K} \left\| f_{m,N}(\gamma) - P_{m}(\gamma) \right\|_{m}^{2} = \sum_{k,j=1}^{M} c_{k}c_{j}W_{kj} - 2 \sum_{k=1}^{M} c_{k}V_{k} + P^{2}, $$

(12)

Here

$$ W_{kj} = \sum_{m=1}^{K} (A_{m,k}, A_{m,j} m), \; W_{jk} = W_{kj}, \; V_{k} = \sum_{m=1}^{K} (A_{m,k}, P_{m} m), \; k, j = 1, M, $$

$$ P^{2} = \sum_{m=1}^{K} \left\| P_{m} \right\|_{m}^{2}, \; (f, g)_{m} = \int_{\beta_{m}}^{\alpha_{m}} f(\gamma)g(\gamma)d\gamma - \text{scalar product}. $$

The found variables $c_{k}$, on which the minimum $F(N)$ is reached, are denoted $c_{k}^{N}$, and the basic functions of the parts of the shell which defined by the coefficients $c_{k}^{N}$, we denote $\Phi_{j}^{N}(z, \varphi), \; w_{j}^{N}(z, \varphi)$. According to the known formulas we find the forces and moments in the shell and the reinforcement.

4.1. Criteria for convergence of the constructed solution.

Lemma ([13]). The function $F(N)$ is nonnegative and nonincreasing.

Theorem ([14]). If there exists $N$ such that $F(N) < \frac{\varepsilon^{2}}{4}$ for any $\varepsilon > 0$, then displacements, forces and moments expressed through functions $\Phi_{j} = \lim_{N \to \infty} \Phi_{j}^{N}, w_{j} = \lim_{N \to \infty} w_{j}^{N}$, will exactly satisfy meet conditions (11) in the metrics of $L_{2}[\beta_{m}, \alpha_{m}]$.

Note that for minimum $F(N)$ the equality is fulfilled

$$ F(N) = \sum_{m=1}^{K} \left\| f_{m,N}(\gamma) - P_{m}(\gamma) \right\|_{m}^{2}, $$

(13)

which serves as an estimate of the accuracy of the found solution.

The results review. Numerical calculations show us that the process of simultaneous boundary conditions satisfaction by the Fourier series for the triply connected cylindrical shell with a large rectangular hole is unstable. This is explained by the fact that the representation of the solution includes exponential functions attached to individual sides of the hole or the ends of the shell, whose performance increases linearly with increasing $N$. They cannot (without conditional cutting the shell into pieces) rationed so that they do not grow on some sides. In the works [12, 13] it is shown that the computational process of construction using the Fourier series method of solving a boundary value problem in a rectangular region is stable. Since the exponential functions can be normalized so that they will be equal unity the given sides of the rectangle and decrease with distance from them. On this basis, it has been proposed to divide the shell with a reinforced hole into simpler, rectangular segments in the plan.

5. Conclusions.

The finite-body method is substantiated, based on a special separation of the three-connected shell surface with the reinforced rectangular hole into simpler doubly connected or simply connected parts, at the edges of which the exponential functions entering the solution are normalized by one. This allowed us to construct a stable algorithm for solving the boundary
problem. An approach has been developed that allowed simultaneously to take into account a large number of boundary or contact conditions, both in the forces and in the displacements specified on different edges of the shell. The principle of de Saint-Venant is confirmed. The mathematical justification for constructing a representation of the self-balanced state (in the form of series) is given, which rapidly decreases when away from the edges of the shell segments. The technique has been developed to reduce satisfaction of all boundary conditions and conditions of contact of shell segments to finding the minimum of the generalized quadratic form.

The algorithm has been developed for the calculation of the shell with the hole and the reinforcement of various materials. Its advantage is that there are no restrictions on the basic systems of functions, in particular, non-orthogonal functions can be used. The additive generalized quadratic form was introduced, which is not linear, but its minimum tends to zero with increasing $N$, which determines the convergence and accuracy of the solution. The developed technique allowed generalizing the well-known Fourier decomposition method for a single equation to the case of the system of equations with non-orthogonal coefficients. The criteria for convergence of the proposed analytical-numerical method are established, which are based on the analysis of the value of one number - the minimum of the generalized quadratic form.

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