Gauge Invariant Lagrangians for Free and Interacting Higher Spin Fields. A Review of the BRST formulation.

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Abstract

We give a detailed review of the construction of gauge invariant Lagrangians for free and interacting higher spin fields using the BRST approach developed over the past few years.

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Higher spin gauge theories have attracted considerable interest during the last decade. Other than being a fascinating topic by itself, Higher Spin field theory has attracted a significant amount of attention due to its close relation with string- and M- theory.

The study of higher spin gauge theories is notoriously difficult and demanding. Even for free higher spin gauge fields it is highly nontrivial to construct Lagrangians that yield higher spin field equations with enough gauge invariance to remove non-physical polarizations - ghosts - from the spectrum. Moreover, the requirement of gauge invariance severely restricts the possible gravitational backgrounds where free fields with spin greater than two can consistently propagate. To date, only constant curvature backgrounds - Minkowski, de Sitter (dS) and Anti-de Sitter (AdS) spaces - are known to support consistent propagation of higher spin gauge fields.

Interacting higher spin gauge fields are much harder to deal with. An important landmark was reached with the understanding of [1] – [2] that the AdS background can accommodate consistent self-interactions of massless higher spin fields. An important property of this construction is that the coupling constants of massless higher spin interactions are proportional to positive powers of the AdS radius and
therefore the naive flat space–time limit cannot be defined. This picture has two crucial features: the presence of an infinite tower of massless higher spin fields and nonlocality.

The results of [1] – [2] have been obtained in the “frame–like” formulation of higher spin fields, which is a highly nontrivial generalization of the MacDowell–Mansouri–Stelle–West [3]–[4] formulation of Anti de Sitter gravity and the higher spin fields are encoded in generalized vierbeins and spin connections. It seems therefore to be of extreme importance to understand the results of [1] – [2] in terms of the so called “metric–like” formulation of the higher spin fields, where the basic objects are “customary” tensor fields of arbitrary rank and symmetry.

In the present review we concentrate mainly on the gauge invariant “metric–like” formulation of the free and interacting higher spin fields [5]–[29] (see [30]–[47] for earlier work). Particular emphasis will be made on the method of BRST constructions [5]–[21], which is analogous to the BRST approach in Open String field Theory [48]–[49]. The BRST method is based upon the principle of gauge invariance. Namely, the free Lagrangians must possess enough gauge invariance to remove nonphysical states-ghosts– from the spectrum, while interactions are constructed via consistent deformations of the “free” abelian gauge transformations.

There is however a crucial difference from string theory. In string theory the infinite dimensional conformal symmetry of the two dimensional world–sheet is translated into space–time gauge invariance, by building the corresponding BRST charge in terms of Virasoro generators and constructing the corresponding field theory Lagrangian. In the BRST approach for higher spin gauge fields the only constraint we impose is that of gauge invariance, since the corresponding world sheet-description is not known yet. One can try, however, to find a connection with the “high energy limit” [50]–[53] of string theory, at least as a formal tool for a better understanding of interacting higher spin fields. The studies in this direction are far from complete, and the best we can do here is to present a nontrivial model of interacting higher spin fields, derived from Open String Field Theory (OSFT).

We will start our discussion from the construction of various free Lagrangians for massless and massive bosonic higher spin fields, which belong to reducible and irreducible representations of an arbitrary dimension Poincare and Anti de–Sitter groups. In the second part of the review we turn to the interactions and discuss the general method as well as particular examples of the interactions between higher spin fields. In the last part of the paper we shall briefly describe the case of half-integer higher spin fields.

Until now several excellent reviews on the subject are available [54] . We believe that the present one along with [55] will provide a useful completion to the already existing ones. As we have already mentioned this is not a comprehensive review and therefore many important and interesting topics have not been included here such as (apart from the “frame–like” formulation of higher spin fields mentioned above [1] – [2], see also [56])

- The nonlocal formulation [8]–[9],
2 Free Massless Fields

2.1 Reducible representations of the Poincare group

Let us start with the simplest example of massless reducible representations of the Poincare group. The field we would like to describe is a tensor field of arbitrary rank \( s \), completely symmetric in its indexes or in other words, a field with spin \( s \). Since the field is massless it has to satisfy the Klein–Gordon equation

\[
\Box \varphi_{\mu_1 \mu_2 \ldots \mu_s}(x) = 0. \tag{2.1}
\]

Further, in order not to have propagation of states with negative norm—ghosts, the higher spin field should satisfy the transversality condition

\[
\partial^{\mu_1} \varphi_{\mu_1 \mu_2 \ldots \mu_s}(x) = 0. \tag{2.2}
\]

Our aim is therefore to construct a free Lagrangian which gives the mass-shell and transversality conditions as a result of its equations of motion.

The simplest example of this construction is the Maxwell field in arbitrary \( D \) dimensional Minkowski space–time described by the Lagrangian

\[
\mathcal{L} = A_\mu \Box A^\mu - A_\mu \partial^\mu \partial^\nu A_\nu \tag{2.3}
\]

which is invariant under the gauge transformations

\[
\delta A_\mu(x) = \partial_\mu \lambda. \tag{2.4}
\]

One can further impose the Lorentz gauge condition

\[
\partial^\mu A_\mu(x) = 0 \tag{2.5}
\]
which is consistent with on-shell gauge invariance, provided the parameter of gauge
transformations is constrained as $\Box \Lambda (x) = 0$. After imposing the Lorentz gauge
condition the Maxwell field satisfies the usual Klein–Gordon equation

$$\Box A_\mu (x) = 0.$$  \hfill (2.6)

The residual gauge transformation with the restricted parameter $\Lambda$ and the transver-
sality condition remove nonphysical polarization and one is left with only physical
polarizations which satisfy massless Klein–Gordon equations.

Our aim is to generalize this construction to the case of arbitrary spin. In order
to achieve this let us introduce an auxiliary Fock space spanned by oscillators $\alpha_\mu$
and $\alpha_\mu^+$, which satisfy commutation relations

$$[\alpha_\mu, \alpha_\nu^+] = g_{\mu \nu}, \quad g_{\mu \nu} = \text{diag}(-1, 1, \ldots, 1)$$  \hfill (2.7)

and consider a state in this space

$$|\Phi\rangle = \frac{1}{s!} \Phi_{\mu_1 \mu_2 \ldots \mu_s} (x) \alpha_{\mu_1}^+ \alpha_{\mu_2}^+ \ldots \alpha_{\mu_s}^+ |0\rangle_\alpha$$  \hfill (2.8)

with

$$\alpha_\mu |0\rangle_\alpha = 0.$$  \hfill (2.9)

In this auxiliary Fock space differentiation and trace operations are realized via the
operators

$$l_0 = p^\mu p_\mu, \quad l_1^+ = \alpha_\mu^+ p^\mu, \quad l_1 = \alpha_\mu p^\mu, \quad p_\mu = -i \partial_\mu,$$ \hfill (2.10)

$$M^+ = \frac{1}{2} \alpha_\mu^+ \alpha_\mu^+, \quad M = \frac{1}{2} \alpha_\mu \alpha_\mu.$$  \hfill (2.11)

$$(l_1)^\dagger = l_1^+, \quad (M)^\dagger = M^+.$$  \hfill (2.12)

It is straightforward to check that the action of operators (2.10)–(2.11) on the state
(2.8) is translated to the action on a symmetric tensor field $\Phi_{\mu_1 \mu_2 \ldots \mu_s} (x)$ as follows

$$l_0 |\Phi\rangle \rightarrow -\Box \Phi_{\mu_1 \mu_2 \ldots \mu_s},$$ \hfill (2.13)

$$l_1 |\Phi\rangle \rightarrow -i \partial^{\mu_1} \Phi_{\mu_1 \mu_2 \ldots \mu_s}, \quad l_1^+ |\Phi\rangle \rightarrow -i \partial_{(\mu_{s+1}} \Phi_{\mu_1 \mu_2 \ldots \mu_s)}$$ \hfill (2.14)

$$M |\Phi\rangle \rightarrow \frac{1}{2} \Phi_{\mu_1 \mu_3 \ldots \mu_s}, \quad M^+ |\Phi\rangle \rightarrow g_{(\mu_1 \mu_2} \Phi_{\mu_3 \ldots \mu_{s+2})}.$$ \hfill (2.15)

In order to describe reducible massless higher spin modes it is enough to consider the
operators $l_0, l_1, l_1^+$. The reason behind this is that was mentioned above at the end
the physical field should satisfy massless Klein–Gordon and transversality conditions.
These conditions are described by the equations

$$l_0 |\Phi\rangle = l_1 |\Phi\rangle = 0,$$ \hfill (2.16)
while the operator $l_1^+$ should be included because it is hermitian conjugate to $l_1$ and we would like to have a hermitian Lagrangian. We can compute the algebra between $l_0, l_1$ and $l_1^+$ which is
\[ [l_1, l_1^+] = l_0, \quad [l_1^+, l_0] = 0. \] (2.17)
The commutation relations with the operators $M^\pm$ are
\[ [M, l_1^+] = l_1, \quad [l_1, M^+] = l_1^+ \] (2.18)
The next step is to construct a nilpotent BRST charge for this system of operators. The reason is that having obtained a nilpotent BRST charge $Q^2 = 0$ it is straightforward to construct the gauge invariant hermitian Lagrangian of the form $\langle \Phi | Q | \Phi \rangle$ which is invariant under the gauge transformations $\delta | \Phi \rangle = Q | \Lambda \rangle$. Following the standard method we introduce Grassman - odd ghost variables $c_0, c_1^+, c_1$ with ghost number one and corresponding momenta $b_0, b_1, b_1^+$ with ghost number $-1$ with the only nonzero anti commutation relations
\[ \{ c_0, b_0 \} = \{ c_1, b_1^+ \} = \{ c_1^+, b_1 \} = 1. \] (2.19)
For the system at hand the BRST charge has the simple form
\[ Q = c_0 l_0 + c_1 l_1^+ + c_1^+ l_1 - c_1^+ c_1 b_0, \] (2.20)
which obviously satisfies the nilpotency property $Q^2 = 0$. Finally, before obtaining an explicit form of the Lagrangian one needs to define a ghost vacuum, which is taken conventionally as
\[ b_0 | 0 \rangle_{gh.} = c_1 | 0 \rangle_{gh.} = b_1 | 0 \rangle_{gh.} = 0 \] (2.21)
while the other ghost variables act as creators. This choice is clear from the form of terms linear in ghosts in the BRST charge (2.20). Indeed we would like the operators $l_0$ and $l_1$ to annihilate the physical states, when considering $Q | \Phi \rangle = 0$ as the equations of motion and therefore the choice of $c_0$ and $c_1^+$ as creators is natural. Therefore $b_0$ and $b_1$ are annihilators. We choose $b_1^+$ as creator and therefore $c_1$ as annihilator because one needs one creation operator with ghost number $-1$ to assign proper ghost numbers to the basic field and gauge transformation parameters as we shall see in a moment. We define a Fock vacuum in our enlarged space as
\[ | 0 \rangle = | 0 \rangle_\alpha \otimes | 0 \rangle_{gh.} \] (2.22)
Then one can write a gauge invariant Lagrangian
\[ L = \int dc_0 \langle \Phi | Q | \Phi \rangle \] (2.23)

\[ ^1\text{If Grassman even operators } G_i \text{ form a Lie algebra } [G_i, G_j] = U_{ij}^k G_k, \text{ with } U_{ij}^k \text{ being structure constants then for each operator } G_i \text{ one introduces a pair of Grassman – odd variables } c_i \text{ and } b_i \text{ with anti-commutation relations } \{ c_i, b_j \} = \delta_{ij} \text{ and then constructs a nilpotent BRST charge } Q = c_i G_i + \frac{1}{2} U_{ij}^k c_j c_k b_k. \]
which leads to the equations of motion
\[ Q|\Phi\rangle = 0, \quad (2.24) \]
and is invariant under gauge transformations
\[ \delta|\Phi\rangle = Q|\Lambda\rangle. \quad (2.25) \]
From the previous equations it is simple to obtain the general form of \(|\Phi\rangle\) and \(|\Lambda\rangle\). Indeed, the Lagrangian must have ghost number zero. The Grassmannian integration (i.e., differentiation) over the variable \(c_0\) has ghost number \(-1\), while the BRST charge has ghost number \(+1\) as can be seen from (2.20). Therefore, the field \(|\Phi\rangle\) must have ghost number zero. Since \(|\Phi\rangle\) has ghost number zero and \(Q\) has ghost number \(+1\), then \(|\Lambda\rangle\) must have ghost number \(-1\). Finally, one has
\[ |\Phi\rangle = |\varphi\rangle + c_1^+ b_1^+ |D\rangle + c_0 b_1^+ |C\rangle \quad (2.26) \]
\[ |\Lambda\rangle = b_1^+ |\lambda\rangle \quad (2.27) \]
where the fields \(|\varphi\rangle\), \(|D\rangle\), \(|C\rangle\) so named triplet in [9], [11], (see also [40], [76] for earlier work) depend only on the oscillators \(\alpha_+^\mu\) and have ghost number zero. Putting the expansion (2.26) into the Lagrangian (2.23), integrating over the bosonic ghost zero mode \(c_0\), according to the rules
\[ \int dc_0 \langle 0|c_0|0 \rangle = 1, \quad \int dc_0 \langle 0||0 \rangle = 0 \quad (2.28) \]
and performing normal ordering for the rest of the ghost variables, thus effectively integrating them out, (for example \(\langle A|b_1 c_1^+|B\rangle = \langle A||B\rangle\), \(\langle A|c_1 b_1^+|B\rangle = \langle A||B\rangle\), \(\langle A|b_1 c_1^+ b_1^+ |B\rangle = -\langle A||B\rangle\) ) one obtains the Lagrangian
\[ \mathcal{L} = \langle \varphi|l_0|\varphi\rangle - \langle D|l_0|D\rangle + \langle C||C\rangle 
- \langle \varphi|l_1^+|C\rangle + \langle D|l_1|C\rangle - \langle C|l_1|\varphi\rangle + \langle C|l_1^+|D\rangle \quad (2.29) \]
and equations of motion
\[ l_0|\varphi\rangle = l_1^+ |C\rangle \quad (2.30) \]
\[ l_0|D\rangle = l_1|C\rangle \quad (2.31) \]
\[ |C\rangle = l_1^+ |D\rangle - l_1|\varphi\rangle \quad (2.32) \]
while the gauge transformation rule (2.25) gives
\[ \delta|\varphi\rangle = l_1^+ |\lambda\rangle, \quad \delta|D\rangle = l_1|\lambda\rangle, \quad \delta|C\rangle = l_0|\lambda\rangle. \quad (2.33) \]
These equations are simple to derive. For example the gauge transformation rules (2.33) can be easily obtained from the explicit form of the BRST charge (2.20) acting on (2.27) namely
\[ Q|\Lambda\rangle = (l_1^+ + c_0 b_1^+ l_0 + c_1^+ b_1^+ l_1)|\lambda\rangle. \quad (2.34) \]
Comparing this with (2.26) one gets (2.33).

From the equations of motion (2.30)–(2.32) and the gauge transformation rules (2.33) one can easily obtain the $\alpha_\mu^+$ oscillator content of $|\varphi\rangle$, $|C\rangle$, $|D\rangle$ and $|\Lambda\rangle$. Since the operators $l_1^\dagger$, $l_0$, $l_1$ have $+1, 0, -1$ oscillator numbers respectively one can conclude that

$$|\varphi\rangle = \frac{1}{s!}\varphi_{\mu_1...\mu_s}(x)\alpha^{+,\mu_1}...\alpha^{+,\mu_s}|0\rangle$$  \hspace{1cm} (2.35)

$$|D\rangle = \frac{1}{(s-2)!}D_{\mu_1...\mu_{s-2}}(x)\alpha^{+,\mu_1}...\alpha^{+,\mu_{s-2}}c^+_1b^+_1|0\rangle$$  \hspace{1cm} (2.36)

$$|C\rangle = \frac{-i}{(s-1)!}C_{\mu_1...\mu_{s-1}}(x)\alpha^{+,\mu_1}...\alpha^{+,\mu_{s-1}}b^+_1|0\rangle,$$  \hspace{1cm} (2.37)

while the corresponding gauge transformation parameter $|\Lambda\rangle$ is,

$$|\Lambda\rangle = \frac{i}{(s-1)!}\Lambda_{\mu_1\mu_2...\mu_{s-1}}(x)\alpha^{+,\mu_1}...\alpha^{+,\mu_{s-1}}b^+_1|0\rangle.$$  \hspace{1cm} (2.38)

One can easily rewrite the gauge transformation rules, the Lagrangian and the equations of motion in tensorial notation. For example to obtain the Lagrangian one has to put expressions (2.35)–(2.37) into the Lagrangian (2.29) and perform the normal ordering with respect to the oscillators $\alpha_\mu^+$ and $\alpha_\mu$. The result is

$$\mathcal{L} = -\frac{1}{2} (\partial_\mu \varphi_{\mu_1...\mu_s})(\partial^{\mu_1...\mu_s}) + s (\partial^{\mu_1...\mu_{s-1}}C_{\mu_1...\mu_{s-1}}) + s(s-1) (\partial^{\mu_1...\mu_{s-2}}D_{\mu_1...\mu_{s-2}}) + \frac{s(s-1)}{2} (\partial_\mu D_{\mu_1...\mu_{s-2}})(\partial^{\mu_1...\mu_{s-2}}) - \frac{s}{2} C_{\mu_1\mu_2...\mu_{s-1}}C_{\mu_1\mu_2...\mu_{s-1}}.$$  \hspace{1cm} (2.39)

In a similar manner $^\S$

$$\Box \varphi_{\mu_1\mu_2...\mu_s} = \partial_{(\mu_1}C_{\mu_2...\mu_s)};$$

$$\partial^{\mu_1...\mu_{s-2}}\varphi_{\mu_1\mu_2...\mu_{s-2}} - \partial_{(\mu_{s-1}}D_{\mu_1\mu_2...\mu_{s-2})} = C_{\mu_1\mu_2...\mu_{s-1}};$$

$$\Box D_{\mu_1\mu_2...\mu_{s-2}} = \partial^{\mu_1...\mu_{s-2}}C_{\mu_1\mu_2...\mu_{s-1}};$$  \hspace{1cm} (2.40)

and

$$\delta \varphi_{\mu_1\mu_2...\mu_s} = \partial_{(\mu_1}A_{\mu_2...\mu_s)};$$

$$\delta C_{\mu_1\mu_2...\mu_{s-1}} = \Box A_{\mu_1\mu_2...\mu_{s-1}};$$

$$\delta D_{\mu_1\mu_2...\mu_{s-2}} = \partial^{\mu_{s-1}}A_{\mu_1\mu_2...\mu_{s-1}}.$$  \hspace{1cm} (2.41)

Finally let us discuss the spectrum of the system. Obviously the Lagrangian contains auxiliary fields alongside with physical ones. The field $C$ is obviously auxiliary

$^\S$In our notations the symmetrization is without a factorial in the denominator, for example $\partial_{(\mu}A_{\nu)} = \partial_\mu A_\nu + \partial_\nu A_\mu$.  

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since it can be integrated out via its equations of motion. The field $D$ has the \textit{wrong sign} kinetic term. However it does not propagate since it is \textquoteenum{pure gauge}. As a result one can show that the only physical polarizations are transverse components of spin $s, s-2, \ldots, 1/0$, depending on whether the rank of the tensor $\varphi$ is even or odd, described by the traceless parts of $\varphi$, $\varphi' - 2D \ldots$ respectively.

It is instructive to demonstrate the gauge fixing procedure with a simple example for a field $\varphi$ of rank 2. The case of an arbitrary spin can be treated in a completely analogous way. For a spin–2 triplet there should exist a field $\varphi_{\mu\nu}$ of rank 2, a field $C_{\mu}(x)$ of rank 1 and a field $D(x)$ of rank zero. The triplet equations take a rather simple form in this case [9]:

$$\Box \varphi_{\mu\nu} = \partial_{\mu} C_{\nu} + \partial_{\nu} C_{\mu} \quad (2.42)$$
$$C_{\mu} = \partial_{\nu} \varphi_{\mu}^{\nu} - \partial_{\mu} D \quad (2.43)$$
$$\Box D = \partial_{\mu} C^{\mu}. \quad (2.44)$$

The system is invariant under the gauge transformations

$$\delta \varphi_{\mu\nu} = \partial_{\mu} \Lambda_{\nu} + \partial_{\nu} \Lambda_{\mu}, \quad \delta C_{\mu} = \Box \Lambda_{\mu}, \quad \delta D = \partial_{\mu} \Lambda^{\mu}. \quad (2.45)$$

Let us also introduce a traceless field $\tilde{\varphi}_{\mu\nu}$

$$\tilde{\varphi}_{\mu\nu} = \varphi_{\mu\nu} - \frac{1}{D} g_{\mu\nu} \varphi', \quad \varphi' = g^{\mu\nu} \varphi_{\mu\nu}. \quad (2.46)$$

In order to see the physical polarizations described by these equations one can use the light-cone gauge fixing procedure i.e., eliminate $\tilde{\varphi}_{++}, \tilde{\varphi}_{+i}$ and $\tilde{\varphi}_{--}$ of the field $\tilde{\varphi}_{\mu\nu}$ using the gauge transformation parameter $\Lambda_{\mu}$. The other nonphysical polarizations $\tilde{\varphi}_{--}, \tilde{\varphi}_{-i}$ as well as the field $C_{\mu}$ are eliminated by the field equations. Therefore, one is left with the physical degrees of freedom $\tilde{\varphi}_{ij} (i, j = 1, \ldots, D-2)$ which correspond to the spin 2 field and a gauge invariant scalar

$$\tilde{D} = \varphi' - 2D. \quad (2.47)$$

The \textquoteenum{orthogonal} scalar $\tilde{D} = D - \frac{1}{D} \varphi'$ is \textquoteenum{pure gauge} and does not propagate. Therefore we have obtained a gauge invariant Lagrangian description of two free fields with spins 2 and 0.

It is an even simpler exercise to show that for a triplet with spin 1 one obtains the usual Maxwell Lagrangian, after elimination of the only auxiliary field $C$. Indeed, the Lagrangian (2.39) takes the form

$$\mathcal{L} = -\frac{1}{2} \left( \partial_{\mu} \varphi_{\nu} \right) \left( \partial^{\mu} \varphi^{\nu} \right) + \left( \partial^{\mu} \varphi_{\mu} \right) C - \frac{1}{2} C^{2}. \quad (2.48)$$

and after elimination of the field $C$ via its equation of motion $C = \partial_{\mu} \varphi^{\mu}$ (compare with (2.5)) one obtains the usual Maxwell Lagrangian.
String theory derivation

A formal way [41] to construct the nilpotent BRST charge of the previous subsection is to start with the BRST charge for the open bosonic string

\[ Q = \sum_{k,l=-\infty}^{+\infty} (C_{-k}L_k - \frac{1}{2}(k-l) : C_{-k}C_{-l}B_{k+l} :) - C_0, \]

perform the rescaling of oscillator variables

\[ c_k = \sqrt{2\alpha'} C_k, \quad b_k = \frac{1}{\sqrt{2\alpha'}} B_k, \quad c_0 = \alpha' C_0, \quad b_0 = \frac{1}{\alpha'} B_0, \]

and then take \( \alpha' \to \infty \). In this way one obtains a BRST charge

\[ Q = c_0 l_0 + \tilde{Q} - b_0 M \]

which is nilpotent in any space-time dimension. The oscillator variables obey the usual (anti)commutation relations

\[ [\alpha^k_{\mu}, \alpha^{l,+}_{\nu}] = \delta^{kl} \eta_{\mu\nu}, \quad \{ c^{k,+}, b^l \} = \{ c^k, b^{l,+} \} = \{ c_0, b^l_0 \} = \delta^{kl}, \]

and the vacuum in the Fock space is defined as

\[ \alpha^k_{\mu}|0\rangle = 0, \quad c_k|0\rangle = 0 \quad k > 0, \quad b_k|0\rangle = 0 \quad k \geq 0. \]

Let us note that one can take the value of \( k \) to be any fixed number without affecting the nilpotency of the BRST charge (2.51). Fixing the value of \( k \) to be \( k = 1 \) one obtains the description of totally symmetric massless higher spin fields, with spins \( s, s - 2, \ldots, 1 \) of the previous subsection, whereas for an arbitrary value of \( k \) one has the so called ”generalized triplet”

\[ |\Phi\rangle = \frac{c^+_{k_1} \cdots c^+_{k_p} b^+_{l_1} \cdots b^+_{l_p}}{(p!)^2} |D_{k_1, \ldots, k_p}^{l_1, \ldots, l_p} \rangle + \frac{c_0 c^+_{k_1} \cdots c^+_{k_{p-1}} b^+_{l_1} \cdots b^+_{l_p}}{(p-1)!p!} |C_{k_1, \ldots, k_{p-1}}^{l_1, \ldots, l_p} \rangle, \]

where the vectors \( |D_{k_1, \ldots, k_p}^{l_1, \ldots, l_p} \rangle \) and \( |C_{k_1, \ldots, k_p}^{l_1, \ldots, l_p} \rangle \) are expanded only in terms of oscillators \( \alpha^k_{\mu} \), and the first term in the ghost expansion of (2.55) with \( p = 0 \) corresponds to the state \( |\varphi\rangle \) in (2.26). One can show that the whole spectrum of the open bosonic string decomposes into an infinite number of generalized triplets, each of them describing a finite number of fields with mixed symmetries [11]. In other words we can actually justify the way the BRST charge for generalized triplets was obtained from the BRST charge of the open bosonic string since its cohomology classes correctly describe the degrees of freedom of massless bosonic fields belonging to mixed symmetry representations of the Poincare group (see e.g. [11]). So taking the point of view that, in the high energy limit the whole spectrum of the bosonic string collapses to zero mass, which becomes infinitely degenerate, one can take the BRST charge (2.51) as the one which correctly describes this spectrum.
2.2 Reducible representations of the AdS group

The description of the reducible massless representations of an arbitrary $D$ dimensional Anti de Sitter group follows the same lines as for the case of reducible massless representations of the Poincare group [36], [11], [12].

Here we give some basic definitions concerning $D$ dimensional Anti de Sitter space. More detailed treatment can be found in [75] or in reviews [87]– [88].

AdS space is a vacuum solution of Einstein equations with a negative cosmological constant. Its Riemann tensor has the form

$$R_{\mu\nu\rho\sigma} = \frac{1}{L^2} \left(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}\right),$$ (2.56)

where $L$ is the AdS radius, $L \to \infty$ corresponds to the flat space–time limit and $g_{\mu\nu}$ is the metric of AdS $D$. It is convenient to represent $D$ dimensional AdS space with coordinates $x^\mu (\mu = 0, \ldots, D - 1)$ and signature $(1, D - 1)$ as a hyperboloid in $D + 1$ dimensional flat space with signature $(2, D - 1)$, parameterized by coordinates $y^A (A = 0, \ldots, D)$. The coordinates in this ambient space obey the condition

$$\eta^{AB} y_A y_B = -L^2, \quad \eta^{AB} \eta_{AB} = D + 1, \quad \eta^{AB} = (-, +, +, \ldots, -).$$ (2.57)

Therefore, the isometry group is a pseudo-orthogonal group of rotations $SO(D - 1, 2)$ and the AdS space itself is isomorphic to the coset $SO(D - 1, 2)/SO(D - 1, 1)$. In order to simplify the equations we set the radius of the AdS space to unity and restore it when writing down the field equations.

The AdS isometry group is noncompact and therefore its unitary representations are infinite dimensional. In order to build them it is convenient to rewrite the $SO(D - 1, 2)$ algebra

$$[J_{AB}, J_{CD}] = \eta_{BC} J_{AD} - \eta_{AC} J_{BD} - \eta_{BD} J_{AC} + \eta_{AD} J_{BC},$$ (2.58)

$$J_{AB} = -J_{BA}, \quad (J_{AB})^+ = -J_{AB},$$

in a different form. Namely, after taking the following linear combinations

$$J^+_a = (-i J_{0a} \pm J_{Da}), \quad a = 1, \ldots, D - 1$$ (2.59)

$$H = i J_{0D},$$ (2.60)

one obtains the commutation relations

$$[H, J^+_a] = \pm J^+_a$$

$$\left[J^-_a, J^+_b\right] = 2(H \delta_{ab} + J_{ab})$$

$$[J_{ab}, J_{\pm c}] = \delta_{bc} J_{\pm a} - \delta_{ac} J_{\pm b}.$$ (2.61)

as well as

$$[J_{ab}, J_{cd}] = \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac} + \eta_{ad} J_{bc}.$$ (2.62)
From these commutation relations one can conclude that the AdS isometry group has a maximal compact subgroup $SO(2) \otimes SO(D - 1)$, spanned by generators $H$ and $J_{ab}$ respectively. These operators correspond to one dimensional and $D - 1$ dimensional rotations. The operator $H$ is the energy operator on AdS while time on AdS is defined as the variable conjugate to $H$. Therefore, the time variable is compact and energy eigenvalues are quantized having integer values in order for a wave function to be single valued. The quadratic Casimir operator in this basis has the form

$$C_2 = -\frac{1}{2} J^{AB} J_{AB} = H(H - D + 1) - \frac{1}{2} J^{ab} J_{ab} - J^+_a J^-_a$$  \hspace{1cm} (2.63)$$

Infinite dimensional unitary representations of the AdS group are obtained from the “lowest weight states” $|E_0, s\rangle$, which is a representation of $SO(2) \otimes SO(D - 1)$. The latter therefore is characterized by its energy eigenvalue and a Young tableaux with labels $s = (s_1, s_2, ..., s_k), \; k = \lfloor \frac{D-1}{2} \rfloor$. A lowest weight state is annihilated by all operators $J^-_a$

$$J^-_a |E_0, s\rangle = 0.$$  \hspace{1cm} (2.64)$$

Then the other states of each representation are obtained by successively applying operators $J^+_a$ on the lowest weight state

$$J^+_a J^+_a ... J^+_a |E_0, s\rangle$$  \hspace{1cm} (2.65)$$

The crucial point is that representations obtained in this way do not always have a positive norm. Therefore, when building new states with the help of operators $J^+_a$ one has to check their norm. For some special values of $E_0$ and $s$ the norm is equal to zero. There is a unitarity bound on the energy $E_0$ below which the states get negative norms and should be excluded from the physical spectrum. The unitarity bound is saturated (norm of states becomes zero) for states with $E_0$ and $s$ related via

$$E_0 = s_1 + D - t_1 - 2$$  \hspace{1cm} (2.66)$$

where $t_1$ is the number of rows of maximal length $s_1$ in the corresponding Young tableaux. The states which saturate the unitarity bound are identified with massless fields on AdS space – time. These states decouple from the original multiplet along with their descendants since their scalar product with the other states is zero. This effect is known as a ‘multiplet shortening’ and it is interpreted as an enhancement of gauge symmetry.

However, for physical applications one usually considers the covering space of AdS, where time is uncompactified [87]. There might be some extra states which saturate the unitarity bound. For example in the case of $D = 4$ there are two states for scalar massless fields with $E_0 = 1$ and $E_0 = 2$. These states have the same quadratic Casimir operator but correspond to different asymptotic behaviour on the AdS boundary [89].
Fields whose energy is above the unitarity bound are massive representations of AdS space. Both massive and massless fields on AdS have flat space-time counterparts i.e., one can take the usual flat space limit to obtain massless and massive fields propagating through Minkowski space-time. However there is one more type of field on AdS, which have no flat space-time analogue. These are called singletons. For example the unitarity bound for a spinless singleton is \( E_0 = \frac{1}{2}(D-3) \). Singletons do not admit a proper field theoretical description in AdS bulk, rather they are described as boundary degrees of freedom.

Let us turn to a field theoretical description of massless fields on AdS\(_D\). In order to obtain wave equations describing massless fields with an arbitrary integer value of spin on an AdS background one has to find a relation between the quadratic Casimir operator and the D'Alembertian. The result for totally symmetric representations of an AdS group i.e., when \( s = (s,0,..0) \), is [23]

\[
(\nabla^2 - \frac{(s-2)(s+D-3)}{L^2} ) F_{A_1A_2...A_s}(y) = 0. \tag{2.67}
\]

where

\[
\nabla^A = \theta^{AB} \frac{\partial}{\partial y^B}, \quad \theta^{AB} = \eta^{AB} + \frac{y^A y^B}{L^2}, \quad \nabla^2 = \nabla^A \nabla_A. \tag{2.68}
\]

A possible way to see where the condition (2.67) comes from, is to introduce a auxiliary Fock space spanned by a set of oscillators

\[
[\alpha^A, \alpha^{B+}] = \eta^{AB}, \tag{2.69}
\]

and consider a state in this Fock space

\[
|\Phi\rangle = \frac{1}{s!} F_{A_1A_2...A_s} \alpha^{A_1+} \alpha^{A_2+} ... \alpha^{A_s+} |0\rangle. \tag{2.70}
\]

The generators of SO(2,\(D-1\)) can be represented as

\[
J^{AB} = L^{AB} + M^{AB}, \tag{2.71}
\]

where the orbital part \( L^{AB} \) and spin part \( M^{AB} \) have the form

\[
L^{AB} = y^A \nabla^B - y^B \nabla^A, \quad M^{AB} = \alpha^{A+} \alpha^B - \alpha^{B+} \alpha^A. \tag{2.72}
\]

A field \( |\Phi\rangle \) in this Fock space is required to satisfy the mass-shell condition

\[
(\nabla^2 - m^2)|\Phi\rangle = 0, \tag{2.73}
\]

where \( m^2 \) is a “mass-like” parameter to be determined, divergencelessness condition

\[
\alpha^A \nabla_A |\Phi\rangle = 0, \tag{2.74}
\]

and transversality condition

\[
y^A \alpha_A |\Phi\rangle = 0. \tag{2.75}
\]
The requirement of invariance of these equations under gauge transformations

\[ \delta |\Phi\rangle = \alpha^{A+} \nabla_{A} |\Lambda_{1}\rangle + y^{A} \alpha^{A}_{\lambda} |\Lambda_{2}\rangle \]  

(2.76)
leads to the mass–shell equation (2.67). If one computes the explicit form of the quadratic Casimir operator in terms of realization (2.71)–(2.72), one finds its eigenvalues \(< C_{2} >\). Comparing equation

\[ (C_{2} - < C_{2} >)|\Phi\rangle = 0 \]  

(2.77)
with (2.67) one obtains the expression for the unitarity bound in an alternative way.

Using the formulas given in the Appendix and the relations

\[ \nabla^{\mu} \Phi_{\mu_{1}...\mu_{s}} = \frac{\partial y^{A_{1}}}{\partial x^{\mu_{1}}} ... \frac{\partial y^{A_{s}}}{\partial x^{\mu_{s}}} (\nabla^{A} + (D + s)y^{A}) \Phi_{A_{1}...A_{s}}, \]  

(2.78)

\[ \nabla_{(\mu_{1}} \Phi_{\mu_{2}...\mu_{s})} = \frac{\partial y^{A_{1}}}{\partial x^{\mu_{1}}} ... \frac{\partial y^{A_{s}}}{\partial x^{\mu_{s}}} (\partial_{(A_{1}} \Phi_{A_{2}...A_{s})} + (s - 1)y^{A} \eta_{(A_{1}A_{2}} \Phi_{A_{3}...A_{s})}), \]  

(2.79)

\[ \Box \Phi_{\mu_{1}...\mu_{s}} = \nabla^{2} \Phi_{A_{1}...A_{s}}, \]  

(2.80)

one can relate equations in the \(x\) and \(y\) spaces. For example, the massless Klein–Gordon equation in the “\(y\)-space” (2.67) when written in the “\(x\)-space” is

\[ (\Box - \frac{(s - 2)(s + D - 3) - s}{L^{2}}) F_{\mu_{1}\mu_{2}...\mu_{s}}(x) = 0. \]  

(2.82)

where \(\Box\) is the D’Alembertian of the AdS space–time. Below we shall work in the “\(x\)-space” [11], the corresponding equations in the “\(y\)-space” can be found in [36], [12].

In order to describe a triplet on \(D\) dimensional Anti de–Sitter space it is convenient to introduce the set of oscillators \((\alpha^{\mu^{+}}, \alpha^{\mu})\), which can be obtained from the ones in (2.69) and the AdS vielbein

\[ [\alpha_{\mu}, \alpha_{\nu}^{+}] = g_{\mu\nu}, \quad \alpha^{\mu} = e_{a}^{\mu} \alpha^{a}, \]  

(2.83)

where \(g_{\mu\nu}\) denotes the AdS metric. The ordinary partial derivative is replaced by the operator

\[ p_{\mu} = -i \left( \partial_{\mu} + \omega^{ab}_{\mu} \alpha_{a}^{+} \alpha_{b} \right). \]  

(2.84)

Acting with \(p_{\mu}\) on a state in Fock space

\[ |\Phi\rangle = \frac{1}{(s)!} \varphi_{\mu_{1}\mu_{2}...\mu_{s}}(x) \alpha^{\mu_{1}+} ... \alpha^{\mu_{s}+} |0\rangle, \]  

(2.85)
produces the proper covariant derivative
\[
p_\mu |\Phi \rangle = -\frac{i}{(s)!} \alpha^{\mu_1+} \ldots \alpha^{\mu_s+} \nabla_\mu \varphi_{\mu_1\mu_2\ldots\mu_s}(x) |0\rangle,
\] (2.86)
\[
\langle \Phi | p_\mu = \langle 0 | \alpha^{\mu_1+} \ldots \alpha^{\mu_s+} \nabla_\mu \varphi_{\mu_1\mu_2\ldots\mu_s}(x) \frac{i}{(s)!},
\] (2.87)
where in (2.84) \(\omega^{ab}_\mu\) denotes the spin – connection on AdS and \(\nabla_\mu\) is the AdS covariant derivative. These operators satisfy commutation relations
\[
[p_\mu, p_\nu] = \frac{1}{L^2} (\alpha^+_{\mu} \alpha_{\nu} - \alpha^+_\nu \alpha_\mu),
\] (2.88)
due to the expression (2.56) for the Riemann tensor.

Further, let us introduce the following operators

D'Alembertian operator
\[
l_0 = g^{\mu\nu} (p_\mu p_\nu + i \Gamma^\lambda_{\mu\nu} p_\lambda) = p^a p_a - i \omega^{ab}_\mu p_b
\] (2.89)
which acts on Fock-space states as the proper D'Alembertian operator
\[
l_0 |\Phi \rangle = -\frac{1}{(s)!} \alpha^{\mu_1+} \ldots \alpha^{\mu_s+} \Box \varphi_{\mu_1\mu_2\ldots\mu_s}(x) |0\rangle
\] (2.90)

Divergence operator
\[
l_1 = \alpha^\mu p_\mu
\] (2.91)
which acts on a state in the Fock space as divergence
\[
l_1 |\Phi \rangle = -\frac{i}{(s - 1)!} \alpha^{\mu_2+} \ldots \alpha^{\mu_s+} \nabla_\mu \varphi^{\mu_1}_{\mu_2\mu_3\ldots\mu_s}(x) |0\rangle
\] (2.92)

Symmetrized exterior derivative operator,
\[
l^+_1 = \alpha^\mu p_\mu
\] (2.93)
\[
l^+_1 |\Phi \rangle = -\frac{i}{(s)!} \alpha^{\mu_1+} \ldots \alpha^{\mu_s+} \nabla_\mu \varphi_{\mu_1\mu_2\mu_3\ldots\mu_s}(x) |0\rangle
\] (2.94)
which is the hermitian conjugate to the operator \(l_1\) with respect to the scalar product
\[
\int d^Dx \sqrt{-g} \langle \Phi_1 | \Phi_2 \rangle.
\] (2.95)

It is straightforward to obtain the commutation relations of the algebra generated by these operators. The commutator between \(l^+_1\) and \(l_1\) becomes
\[
[l_1, l^+_1] = \tilde{l}_0,
\] (2.96)
where the modified D’Alembertian is
\[
\tilde{l}_0 = l_0 - \frac{1}{L^2} \left( -\mathcal{D} + \frac{\mathcal{D}^2}{4} + 4 M^\dagger M - N^2 + 2N \right).
\tag{2.97}
\]
Here
\[
N = \alpha^{+\mu} \alpha_{\mu} + \mathcal{D} \tag{2.98}
\]
counts the number of indices of the Fock-space fields, up to the space-time dimension \(\mathcal{D}\), while
\[
M = \frac{1}{2} \alpha^{\mu} \alpha_{\mu} \tag{2.99}
\]
takes traces of the Fock-space fields.

The emergence of these new operators enlarges the algebra, that now includes the additional commutators
\[
\begin{align*}
[M^\dagger, l_1] &= -l_1^+ , \\
[\tilde{l}_0, l_1] &= \frac{2}{L^2} l_1 - \frac{4}{L^2} N l_1 + \frac{8}{L^2} l_1^+ M , \\
[N, l_1] &= -l_1 ,
\end{align*}
\tag{2.100}
\]
and their hermitian conjugates, together with
\[
\begin{align*}
[N, M] &= -2 M , \\
[M^\dagger, N] &= -2 M^\dagger , \\
[M^\dagger, M] &= -N ,
\end{align*}
\tag{2.101}
\]
that define an \(SO(1,2)\) subalgebra.

Note that (2.100) and (2.101) actually define a non-linear algebra, and therefore the associated BRST charge should be naively constructed with the recipe of [90] (see also [7], [91]–[92]). As in [7], however, this would introduce a larger set of ghosts and corresponding fields, going beyond the triplet structure. The latter case which will lead to the description of irreducible higher spin modes will be discussed in the next subsections. Now in the spirit of the flat limit for the triplet, let us retain only the \((l_1^+, \tilde{l}_0)\) constraints, treating (2.100) as an ordinary algebra where \(M, M^\dagger\) and \(N\) play the role of “structure constants”. In other words, the first step is to construct the BRST charge using the standard formula adopted for the case of the constraints forming a Lie algebra i.e. ignore the fact that we have structure functions rather than structure constants. The second step is to compute the square of the BRST operator but we now take into account the fact that we have structure functions rather than constants. And the third step is to add compensating terms to restore the nilpotency of the BRST charge. Remarkably, this is possible and guarantees the Lagrangian nature of corresponding field equations. With this proviso, one can write the identically nilpotent BRST charge
\[
Q = c_0 \left( \tilde{l}_0 - \frac{4}{L^2} N + \frac{6}{L^2} \right) + c_1 l_1^+ + c_1^+ l_1 - c_1^- c_1^- b_0 \\
- \frac{6}{L^2} c_0 c_1^- b_1 - \frac{6}{L^2} c_0 b_1^+ c_1 + \frac{4}{L^2} c_0 c_1^+ b_1 N + \frac{4}{L^2} c_0 b_1^+ c_1 N \\
- \frac{8}{L^2} c_0 c_1^- b_1^+ M + \frac{8}{L^2} c_0 c_1 b_1 M^\dagger + \frac{12}{L^2} c_0 c_1^+ b_1^+ c_1 b_1 .
\tag{2.102}
\]
The nilpotency of $Q$ ensures the consistency of the construction, and as usual determines a BRST invariant Lagrangian of the form (2.23), and thus a Lagrangian set of equations as in (2.24). In component notation

$$
\mathcal{L} = -\frac{1}{2} (\nabla_\mu \varphi)^2 + s \nabla \cdot \varphi C + s(s-1) \nabla \cdot D \nabla C + \frac{s(s-1)}{2} \left( \nabla_\mu D \right)^2 - \frac{s}{2} C^2 + \frac{s(s-1)}{2L^2} \left( \varphi' \right)^2 - \frac{s(s-1)(s-2)(s-3)}{2L^2} \left( D' \right)^2 - \frac{4s(s-1)}{L^2} D \varphi' - \frac{1}{2L^2} \left[ (s-2)(D+s-3) - s \right] \varphi^2 + \frac{s(s-1)}{2L^2} \left[ s(D+s-2) + 6 \right] D^2.
$$

(2.103)

Here we introduced short-hand notation. The symbol $\nabla \cdot$ means divergence, while $\nabla$ is symmetrized action of $\nabla_\mu$ on a tensor. The symbol $'$ means that we take the trace of a field. Multiplication of a tensor by the metric $g$ implies the symmetrized multiplication, i.e., if $A$ is a vector $A_\mu$ we have $gA_\mu = g_{\mu\rho} A_\rho + g_{\mu\rho} A_\nu + g_{\nu\rho} A_\mu$. The Lagrangian leads to the corresponding field equations

$$
\Box \varphi = \nabla C + \frac{1}{L^2} \left\{ 8 g D - 2 g \varphi' + [(2-s)(3-D-s) - s] \varphi \right\},
$$

$$
C = \nabla \cdot \varphi - \nabla D,
$$

$$
\Box D = \nabla \cdot C + \frac{1}{L^2} \left\{ (s(D+s-2) + 6)D - 4 \varphi' - 2gD' \right\}
$$

(2.104)

which are invariant under the gauge transformations

$$
\delta \varphi = \nabla \Lambda,
$$

$$
\delta C = \Box \Lambda + \frac{(s-1)(3-s-D)}{L^2} \Lambda + \frac{2}{L^2} g \Lambda',
$$

$$
\delta D = \nabla \cdot \Lambda.
$$

(2.105)

Let us note that the invariance of the Lagrangian (2.103) and of the equations of motion (2.104) under the gauge transformations (2.105) can be checked directly using the action on a vector $V_\mu$ of the AdS covariant derivatives commutator

$$
[\nabla_\mu, \nabla_\nu] V_\rho = \frac{1}{L^2} (g_{\nu\rho} V_\mu - g_{\mu\rho} V_\nu).
$$

(2.106)

The gauge fixing procedure can be carried out as in the case of flat space–time. Therefore, one obtains massless spin $s$ field which is contained in the traceless part of $\varphi$, $\varphi' - 2D$, $\varphi'' - 4D'$... e.t.c. describing massless fields of spin $s$, $s-4$, ..., $1/0$, as was for the case of flat space–time. This completes our discussion of the irreducible higher spin modes on AdS.

### 2.3 Irreducible representations of the Poincare group

As we saw in the previous subsections, the inclusion of the operators $l_0$, $l_1$ and $l_1^+$ leads to the description of reducible massless higher spin fields which are subject to
mass-shell and transversality conditions (2.1)–(2.2). In order to describe irreducible higher spin mode one has to add the trace operator \( M = \frac{1}{2} \alpha^\mu \alpha_\mu \) (and its hermitian conjugate \( M^+ = \frac{1}{2} \alpha^{\mu+} \alpha^{\mu+} \)) which is needed for the hermiticity of the corresponding BRST charge) to the initial set of operators \( l_0, l_1 \) and \( l_{-1} \). The resulting algebra now has the form (2.17), (2.18), (2.101). The crucial point here is the appearance of an extra operator \( N = \alpha^{\mu+} \alpha^{\mu+} \) in the right hand side of the commutator \([M, M^+]\). This operator is strictly positive because its eigenvalues are \( s + \frac{D}{2} \) and therefore it can not annihilate any state in the Fock space whether it is physical or not. On the other hand, an inclusion of \( N \) into the total set of operators seems to be unavoidable as long as we include operators \( M^\pm \) into the total set of constraints since otherwise one will deal with operators whose algebra does not close. This in turn will cause problems with the construction of the nilpotent BRST charge.

A way out is that although the operator \( N \) can not annihilate a state in a Fock space, we can modify the operator with some new operator \( h \) whose eigenvalues can cancel those of \( N \) i.e., consider \( \tilde{N} = N + h \) in such a way that the operator \( \tilde{N} \) will be able to annihilate a state in the Fock space. However if one tries naively to modify the operator \( N \) by simply adding an operator \( h \) to it the algebra (2.101) will not close. To solve the problem [6], [11] we therefore use the method of BRST reduction which in fact originated from the dimensional reduction procedure described in the next section. To understand it better let us rephrase the problem. If we declare the operator \( N \) to be a constraint then we will have to introduce a real (since \( N \) is hermitian) ghost and therefore we shall have a term of the form \( c_N N \) in the BRST charge. Our task is to eliminate the \( c_N \) rather than the operator \( N \), since eliminating the ghost \( c_N \) effectively prevents \( N \) from imposing a condition on the Fock space. Combining this with the previous discussion we have to choose the parameter \( h \) to eliminate the dependence on \( c_N \) of the BRST charge, but the BRST charge should still be nilpotent.

As a first step to implement the procedure outlined above, we introduce extra oscillators with the commutation relations

\[ [d, \hat{d}] = -1 . \tag{2.107} \]

The states in the enlarged Fock space are expanded, as usual, in (anti)ghost modes, and each of the resulting terms have the form,

\[ |\phi^s\rangle = \sum_k |\phi^k_s\rangle \equiv \sum_k \varphi^k_{\mu_1 \mu_2 ... \mu_{s-2k+1 + s-2k-2n}} \alpha^{\mu_1+} \alpha^{\mu_2+} ... \alpha^{\mu_{s-2k}+} (d^\dagger)^k |0\rangle , \tag{2.108} \]

where \( s \) is the rank of the \( k = 0 \) component tensor in the expansion. With these variables one can build new “auxiliary” representations of the algebra \( SO(2,1) \) formed by operators \( N, M^\pm \), which have the form

\[
M_{(\text{aux})} = \sqrt{h + \hat{d} \hat{d} d} , \quad M^\dagger_{(\text{aux})} = d^\dagger \sqrt{h + \hat{d} \hat{d} d} , \quad N_{(\text{aux})} = -2 \hat{d} \hat{d} d - h . \tag{2.109}
\]

**One can check that the conversion procedure of [93], or use of the Dirac brackets does not simplify the problem at hand.**
Further, define new operators,

$$\tilde{M}_\pm = M^\pm + M^\pm_{(\text{aux})}, \quad \tilde{N} = N + N_{(\text{aux})},$$  \hspace{1cm} (2.110)

that realize again the SO(2,1) algebra (2.101). The nilpotent BRST charge for the resulting system is then formally constructed, treating all operators under consideration using the standard recipe,

$$\tilde{Q} = c_0 l_0 + c_1 l^+_1 + c_M \tilde{M}^+_1 + c_M^+ \tilde{M} + c_N \tilde{N}$$

$$- c_1 c_1 b_0 + c_1^+ b_1^+ c_M - c_M^+ c_1 b_1$$

$$+ c_N (2c_M^+ b_M + 2b_M^+ c_M + c_1^+ b_1 + b_1^+ c_1 - 3) - c_M^+ c_M b_N .$$  \hspace{1cm} (2.111)

The final step is the elimination of the term proportional to $c_N$ while maintaining the nilpotency of the BRST charge. This can be done performing the unitary transformation on the BRST charge

$$Q_1 = e^{-i \pi x_h} \tilde{Q} e^{i \pi x_h} ,$$  \hspace{1cm} (2.112)

where $x_h$ is the phase-space coordinate conjugate to $h$, so that

$$[x_h, h] = i ,$$  \hspace{1cm} (2.113)

and

$$\pi = N - 2 d^d d + 2 c_M^+ b_M + 2 b_M^+ c_M + c_1^+ b_1 + b_1^+ c_1 - 3$$  \hspace{1cm} (2.114)

is essentially a number operator. Note that this transformation removes all terms depending on $c_N$ from the BRST charge, while obviously preserving its nilpotency. Finally, the term containing $b_N$ can be also dropped without any effect on the nilpotency. This can be checked by direct computations but one can see it easily by the following argument. Let us write $Q_1$ in the form

$$Q_1 = Q - c_M^+ c_M b_N$$  \hspace{1cm} (2.115)

and take its square. By construction $Q_1$ is nilpotent, i.e., $Q_1^2 = 0$. On the right hand side the term $c_M^+ c_M b_N$ is nilpotent too, so we are left with terms $Q^2$ and $\{Q, c_M^+ c_M b_N\}$. Since $Q$ does not contain the $c_N$ ghost the anticomutator will be proportional to $b_N$. The operator $Q^2$ does not contain neither $c_N$ and $b_N$. Therefore each of these terms are separately nilpotent.

Finally, the BRST charge for this system takes the form

$$Q = Q_1 + Q_2 ,$$  \hspace{1cm} (2.116)

with

$$\{Q_1, Q_2\} = 0 , \quad Q_1^2 = - Q_2^2 ,$$  \hspace{1cm} (2.117)

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\[
Q_1 = c_0 l_0 + c_1 l_1^+ + c_M M^+ + c_1^+ l_1 + c_M^+ M
\]
\[
- c_1^+ c_1^+ b_0 + c_1^+ b_1^+ c_M - c_M^+ c_1^+ b_1
\]
and
\[
Q_2 = c_M^+ \sqrt{-1 + N - d^i d + 2 b_M^+ c_M + 2 c_M^+ b_M + b_1^+ c_1 + c_1^+ b_1 d}
\]
\[
+ d^i \sqrt{-1 + N - d^i d + 2 b_M^+ c_M + 2 c_M^+ b_M + b_1^+ c_1 + c_1^+ b_1 c_M}
\]
Again, this determines a BRST invariant Lagrangian of the type (2.23) and now the most general expansions of the state vector \(|\Phi\rangle\) and of the gauge parameter \(|\Lambda\rangle\) in ghost variables are
\[
|\Phi\rangle = |\varphi_1\rangle + c_1^+ b_1^+ |\varphi_2\rangle + c_M^+ b_M^+ |\varphi_3\rangle + c_1^+ b_M^+ |\varphi_4\rangle
\]
\[
+ c_M^+ b_1^+ |\varphi_5\rangle + c_1^+ c_M^+ b_M^+ |\varphi_6\rangle + c_0 b_1^+ |\Lambda_1\rangle
\]
\[
+ c_0 b_M^+ |\Lambda_2\rangle + c_0 c_1^+ b_1^+ b_M^+ |\Lambda_3\rangle + c_0 c_M^+ b_1^+ b_M^+ |\Lambda_4\rangle
\]
and
\[
|\Lambda\rangle = b_1^+ |\Lambda_1\rangle + b_M^+ |\Lambda_2\rangle + c_1^+ b_1^+ b_M^+ |\Lambda_3\rangle + c_1^+ b_1^+ b_M^+ |\Lambda_4\rangle
\]
\[
+ c_0 b_1^+ b_M^+ |\Lambda_5\rangle
\]
where \(|\varphi_i\rangle\) and \(|\Lambda_i\rangle\) have ghost number zero and depend only on the bosonic creation operators \(\alpha^{\mu+}\) and \(d^i\) as in (2.108). Let us also note that both the Lagrangian and the gauge transformations are not affected by redefinitions of the gauge parameters of the type
\[
\delta |\Lambda\rangle = Q |\omega\rangle
\]
and in particular with
\[
|\omega\rangle = b_1^+ b_M^+ |\omega_1\rangle
\]
As a result, one of the gauge parameters, \(|\Lambda_5\rangle\), is inessential and can be ignored.

With this proviso, the resulting Lagrangian in the bosonic Fock-space notation is
\[
\mathcal{L} = - \langle C_1 | C_1 \rangle - \langle C_2 | \varphi_2 \rangle + \langle C_3 | \varphi_3 \rangle + \langle C_4 | C_4 \rangle - \langle \varphi_2 | C_2 \rangle + \langle \varphi_3 | C_3 \rangle
\]
\[
- \langle C_1 | M_1^\dagger \varphi_4 \rangle - \langle C_1 | l_1^+ \varphi_2 \rangle + \langle C_1 | l_1 \varphi_1 \rangle - \langle C_2 | M_2^\dagger \varphi_3 \rangle - \langle C_2 | l_2^+ \varphi_5 \rangle
\]
\[
+ \langle C_2 | M_1 \varphi_1 \rangle - \langle C_3 | M_1^\dagger \varphi_6 \rangle + \langle C_3 | l_1 \varphi_6 \rangle - \langle C_3 | M_2 \varphi_2 \rangle + \langle C_4 | l_2^+ \varphi_6 \rangle
\]
\[
+ \langle C_4 | l_1 \varphi_5 \rangle - \langle C_4 | M_2 \varphi_4 \rangle + \langle \varphi_1 | M_2^\dagger C_2 \rangle + \langle \varphi_1 | l_2^+ C_1 \rangle - \langle \varphi_1 | l_0 \varphi_1 \rangle
\]
\[
- \langle \varphi_2 | M_1^\dagger C_3 \rangle + \langle \varphi_2 | l_0 \varphi_2 \rangle - \langle \varphi_2 | l_1 C_1 \rangle + \langle \varphi_3 | l_2^+ C_4 \rangle + \langle \varphi_3 | l_0 \varphi_3 \rangle
\]
\[
- \langle \varphi_3 | M_2^\dagger C_2 \rangle - \langle \varphi_4 | M_1^\dagger C_4 \rangle + \langle \varphi_4 | l_0 \varphi_5 \rangle - \langle \varphi_4 | M_2^\dagger C_1 \rangle + \langle \varphi_5 | l_2^+ C_3 \rangle
\]
\[
+ \langle \varphi_5 | l_0 \varphi_4 \rangle - \langle \varphi_5 | l_1 C_2 \rangle - \langle \varphi_5 | l_0 \varphi_6 \rangle + \langle \varphi_6 | l_1 C_4 \rangle - \langle \varphi_6 | M_3 \rangle
\]
\[
- \langle C_1 | d_1^X_1 \varphi_4 \rangle - \langle C_2 | d_2^X_2 \varphi_3 \rangle + \langle C_2 | X_0 d_1 \varphi_1 \rangle - \langle C_3 | d_1^X_4 \varphi_6 \rangle
\]
\[
- \langle C_3 | X_2 d_2 \varphi_2 \rangle - \langle C_4 | X_3 d_4 \varphi_4 \rangle + \langle \varphi_1 | d_1^X_0 | C_2 \rangle - \langle \varphi_2 | d_2^X_2 | C_3 \rangle
\]
\[
- \langle \varphi_3 | X_2 d_2 | C_2 \rangle - \langle \varphi_4 | d_1^X_3 | C_4 \rangle + \langle \varphi_4 | X_1 d_1 | C_1 \rangle - \langle \varphi_6 | X_4 d_4 | C_3 \rangle
\]
From this Lagrangian one can derive equations of motion

\[-M^+|C_2\rangle - l_1^+|C_1\rangle + l_0|\varphi_1\rangle - b^+X_0|C_2\rangle = 0, \tag{2.124}\]
\[|C_2\rangle + M^+|C_3\rangle - l_0|\varphi_2\rangle + l_1|C_1\rangle + d^+X_2|C_3\rangle = 0,\]
\[-|C_3\rangle - l_1^+|C_4\rangle - l_0|\varphi_3\rangle + M|c_2\rangle + X_2d|C_2\rangle = 0,\]
\[M^+|C_4\rangle - l_0|\varphi_5\rangle + M|C_1\rangle + d^+X_3|C_4\rangle + X_1d|C_1\rangle = 0,\]
\[-l_1^+|C_3\rangle - l_0|\varphi_4\rangle + l_1|C_2\rangle = 0,\]
\[l_0|\varphi_6\rangle - l_1|C_4\rangle + M|C_3\rangle + X_4d|C_3\rangle = 0,\]
\[|C_1\rangle + M^+|\varphi_4\rangle + l_1^+|\varphi_2\rangle - l_1|\varphi_1\rangle + d^+X_1|\varphi_4\rangle = 0,\]
\[M^+|\varphi_3\rangle + l_1^+|\varphi_6\rangle - M|\varphi_1\rangle + d^+X_2|\varphi_3\rangle - X_0d|\varphi_1\rangle + |\varphi_2\rangle = 0,\]
\[M^+|\varphi_6\rangle - l_1|\varphi_5\rangle + M|\varphi_2\rangle + d^+X_4|\varphi_6\rangle + X_2d|\varphi_2\rangle - |\varphi_3\rangle = 0,\]
\[-|C_4\rangle - l_1^+|\varphi_6\rangle - l_1|\varphi_3\rangle + M|\varphi_4\rangle + X_3d|\varphi_4\rangle = 0.\]

Both the Lagrangian (2.122) and gauge transformations (2.125) are invariant under gauge transformations

\[
\begin{align*}
\delta|\varphi_1\rangle &= l_1^+|\Lambda_1\rangle + M^+|\Lambda_2\rangle + d^+X_0|\Lambda_2\rangle, \tag{2.125} \\
\delta|\varphi_2\rangle &= |\Lambda_2\rangle + l_1|\Lambda_1\rangle + M^+|\Lambda_3\rangle + d^+X_2|\Lambda_3\rangle, \\
\delta|\varphi_3\rangle &= -|\Lambda_3\rangle + M|\Lambda_2\rangle - l_1^+|\Lambda_4\rangle + X_2d|\Lambda_2\rangle, \\
\delta|\varphi_4\rangle &= l_1|\Lambda_2\rangle - l_1^+|\Lambda_3\rangle, \\
\delta|\varphi_5\rangle &= M^+|\Lambda_4\rangle + M|\Lambda_1\rangle + d^+X_3|\Lambda_4\rangle + X_1d|\Lambda_1\rangle, \\
\delta|\varphi_6\rangle &= -M|\Lambda_3\rangle - X_4d|\Lambda_3\rangle + l_1|\Lambda_4\rangle, \\
\delta|C_1\rangle &= l_0|\Lambda_1\rangle, \\
\delta|C_2\rangle &= l_0|\Lambda_2\rangle, \\
\delta|C_3\rangle &= l_0|\Lambda_3\rangle, \\
\delta|C_4\rangle &= l_0|\Lambda_4\rangle.
\end{align*}
\]

From the field equations and the gauge transformations one can unambiguously read the oscillator content of the vectors $|\varphi_i\rangle$, $|C_i\rangle$ and $|\Lambda_i\rangle$. In order to describe a spin-$s$ field, let us fix the number of oscillators $\alpha^{\mu^+}$ in the zeroth-order term of the expansion of $|\varphi_1\rangle$ in the oscillator $d^\dagger$, that we shall denote by $\varphi_1^{0}$, to be equal to $s$. This is actually the field $\varphi$ of the previous subsections, while all other terms describe auxiliary fields. The zeroth-order components in the $d^\dagger$ oscillators for the other fields have thus the following $\alpha^{\mu^+}$ content, here summarised in terms of the resulting total spin, displayed within brackets: $\varphi_2^0 [s = 2]$, $\varphi_3^0 [s = 4]$, $\varphi_4^0 [s = 3]$, $\varphi_5^0 [s = 3]$, $\varphi_6^0 [s = 6]$, $C_2^0 [s = 1]$, $C_3^0 [s = 2]$, $C_4^0 [s = 4]$, $C_5^0 [s = 5]$, $C_6^0 [s = 5]$. Moreover, the field equations and the gauge transformations show that each power of the $d^\dagger$ oscillator reduces the number of $\alpha^{\mu^+}$ oscillators by two units,

\[X_r = \sqrt{-1 + N - d^\dagger d + r}. \quad (2.123)\]
so that, for instance, the $\varphi^k_i$ component field has $s - 2k$ oscillators of this type, and thus spin $(s - 2k)$. Therefore, as anticipated, this off-shell formulation of a spin-$s$ field requires finite number of auxiliary fields and gauge transformation parameters, although their total number grows linearly with $s$.

Combining the gauge transformations with the field equations, it is possible to choose a gauge where all fields aside from $\varphi^0_1$, $\varphi^0_2$, $\varphi^0_5$ and $C^0_3$ are eliminated, so that one is left with a reduced set of equations invariant under an unconstrained gauge symmetry with parameter $\Lambda^0_1$. To this end, one first gauges away all fields but $\varphi^0_1$, and the residual gauge transformations are restricted by the conditions

$$l_0 \Lambda^k_1 = 0 \ (k \neq 0) \quad \text{and} \quad l_0 \Lambda^k_i = 0 \ (i = 2, 3, 4) \quad \text{and} \quad k \geq 0.$$  \hspace{1cm} (2.126)

The parameters $\Lambda^k_1 (k \neq 0)$ and $\Lambda^k_4$ gauge away $\varphi^k_5 (k \neq 0)$, while the parameters $\Lambda^k_3$ gauge away $\varphi^k_6$. The conclusion is that one is finally left with gauge transformation parameters restricted by the additional condition

$$(M + X_4 d) |\Lambda_3\rangle = 0,$$  \hspace{1cm} (2.127)

and with the help of parameters $|\Lambda_2\rangle$ and $|\Lambda_3\rangle$ one can also gauge away $\varphi^k_1$, $\varphi^k_2 (k \neq 0)$ and $\varphi^k_3$, while $\varphi^k_4$ vanishes as a result of the field equations.

If one further eliminates the field $\varphi^0_5$ with the help of the gauge transformation parameter $\Lambda^0_1$, the system of equations of motion (2.14), apart from the dynamical equations for fields $|\varphi^0_1\rangle$, and $|\varphi^0_2\rangle$, becomes

\begin{align*}
M |C^0_1\rangle &= 0, \\
|C^0_1\rangle + l_1^+ |\varphi^0_2\rangle - l_1 |\varphi^0_1\rangle &= 0, \\
M |\varphi^0_1\rangle - |\varphi^0_2\rangle &= 0, \\
M |\varphi^0_2\rangle &= 0,
\end{align*}

(2.128-2.131)

with residual gauge invariance

$$\delta |\varphi^0_1\rangle = l_1^+ |\Lambda^0_1\rangle, \quad \delta |\varphi^0_2\rangle = l_1 |\Lambda^0_1\rangle, \quad \delta |C^0_1\rangle = l_0 |\Lambda^0_1\rangle,$$  \hspace{1cm} (2.132)

where the parameter $|\Lambda^0_1\rangle$ is restricted by the condition

$$M |\Lambda^0_1\rangle = 0.$$  \hspace{1cm} (2.133)

Using the equations (2.129) and (2.130) one can express $|\varphi^0_2\rangle$ and $|C^0_1\rangle$ through $|\varphi^0_1\rangle$ and insert them into (2.122). The Lagrangian now depends only on the field $|\varphi^0_1\rangle$ and takes the following form

$$\mathcal{L} = \langle \varphi^0_1 | l_0 - l_1^+ l_1 - l_1^+ M - M^+ l_1 l_1 - 2l_0 M^+ M - M^+ l_1^+ l_1 M | \varphi^0_1 \rangle$$  \hspace{1cm} (2.134)

The field $|\varphi^0_1\rangle$, as a consequence of the equations (2.130) and (2.131), is restricted by the condition

$$M^2 |\varphi^0_1\rangle = 0.$$  \hspace{1cm} (2.135)
making (2.128) become an identity. After taking the expansion

\[ |\varphi_1^0 = \varphi_{\mu_1\mu_2...\mu_s}(x)\alpha^{\mu_1+\alpha^{\mu_2+\alpha^{\mu_3+...\alpha^{\mu_s+}}}|0) \] (2.136)

we find that the Lagrangian (2.134) in terms of the fields \( \varphi_{\mu_1\mu_2...\mu_s}(x) \equiv \varphi_{\mu_1\mu_2...\mu_s}(x) \) coincides with the one given by Fronsdal [30]

\[ \mathcal{L} = \varphi_{\mu_1\mu_2...\mu_s}(x)\square\varphi_{\mu_1\mu_2...\mu_s}(x) - \frac{s(s-1)}{2}\varphi_{\mu_1\mu_3...\mu_s}(x)\varphi_{\nu_1\mu_3...\mu_s}(x) \] (2.137)

As a consequence of the condition (2.135) the field \( \varphi_{\mu_1\mu_2...\mu_n}(x) \) has a vanishing second trace \( \varphi_{\mu_\nu\mu_5\mu_6...\mu_n}(x) \) and the Lagrangian is invariant under the gauge transformation

\[ \delta \varphi_{\mu_1\mu_2...\mu_n}(x) = \partial_{(\mu_1}\Lambda_{\mu_2\mu_3...\mu_n)}(x) \] (2.138)

with constrained parameter \( \Lambda_{\mu_1...\mu_{n-1}} \equiv \Lambda_{\mu_1...\mu_{n-1}}, \Lambda_{\mu_1...\mu_{n-1}} \equiv 0 \).

To summarize, we have derived the Lagrangian for a single field with an arbitrary integer spin \( s \) without any off-shell constraint either on the field or on the parameter of gauge transformations. This Lagrangian contains however a finite set of auxiliary fields, and the number of these fields depends on the value of the spin under consideration. The Fronsdal Lagrangian can be obtained from the one we are considering after the particular choice of the gauge. The unconstrained Lagrangian description for the fermionic higher spin fields, which is analogous to the one described above, has been given in [19].

Let us note that here we use the “hermitian” auxiliary representations as in [6], [11], when the operators \( M_{(\text{aux})} \) and \( M_{(\text{aux})} \) are hermitian conjugate to each other. Alternatively, one can use “non-hermitian” auxiliary representations as in [7], [19]–[22], [94]. The hermiticity of the Lagrangian is maintained by introducing an extra kernel operator in the definition of the scalar product in the Fock space.

Example \( s = 3 \).

To illustrate the procedure let us consider in detail the simple example of the irreducible massless higher spin field with spin \( s = 3 \). Let us first determine the field content. As explained after equation (2.124) one has

\[ |\varphi_1^0 = \varphi_{\mu_1\mu_2\mu_3}(x)\alpha^{\mu_1+\alpha^{\mu_2+\alpha^{\mu_3+}}}|0) \] (2.139)

\[ |\varphi_2^0 = \varphi_{2,\mu_1}(x)\alpha^{\mu_1+}|0), \quad |\varphi_4^0 = -i\varphi_3^0(x)|0), \quad |\varphi_5^0 = -i\varphi_3^0(x)|0) \] (2.140)

\[ |C_1^0 = -\frac{i}{2!}C_{1,\mu_1\mu_2}(x)\alpha^{\mu_1+\alpha^{\mu_2+}} \] (2.141)

\[ |C_2^0 = -C_2^0(x)|0) \] (2.142)
the other fields being zero.

The gauge transformation parameters have the form

$$|\Lambda_1\rangle = (i/2! \Lambda^0_{1,\mu_1\mu_2}(x)\alpha^{\mu_1+}\alpha^{\mu_2+} + i\Lambda_1(x)d^+)|0\rangle, \quad |\Lambda_2\rangle = \Lambda^0_{2,\mu_1}(x)\alpha^{\mu_1+}|0\rangle \quad (2.143)$$

with all other gauge parameters being zero. The gauge transformation rules for these fields can be written down from (2.125)

$$\delta\varphi^0_{1,\mu\nu\rho} = \partial_{(\mu}\Lambda^0_{1,\nu\rho)} + g_{(\mu\nu}\Lambda^0_{2,\rho)}, \quad \delta\varphi^1_{1,\mu} = \partial_{\mu}\Lambda^1_1 + \sqrt{D/2}\Lambda^0_{2,\mu}, \quad (2.144)$$

$$\delta\varphi^0_{2,\mu} = \partial^\nu\Lambda^0_{1,\mu\nu} + \Lambda^0_{1,\mu}, \quad \delta\varphi^0_{4} = \partial^\mu\Lambda^0_{2,\mu}, \quad \delta\varphi^0 = \sqrt{D/2}\Lambda^1_1 - 1/2\Lambda^0_{1,\mu}. \quad (2.145)$$

$$\delta C^0_{1,\mu\nu} = \Box\Lambda^0_{1,\mu\nu}, \quad \delta C^1_1 = \Box\Lambda^1_1, \quad \delta C^0_{2,\mu} = \Box\Lambda^0_{2,\mu} \quad (2.146)$$

From (2.124) one can write down the equations of motion

$$\Box\varphi^0_{1,\mu\nu\rho} = \partial_{(\mu}\Lambda^0_{1,\nu\rho)} + g_{(\mu\nu}\Lambda^0_{2,\rho)} \quad (2.147)$$

$$\Box\varphi^1_{1,\mu} = \partial_{\mu}C^1_1 + \sqrt{D/2}C^0_{2,\mu}, \quad \Box\varphi^0_{2,\mu} = \partial^\nu C^0_{1,\mu\nu} + C^0_{2,\mu} \quad (2.148)$$

$$\Box\varphi^0_5 = \sqrt{D/2}C^1_1 - 1/2 C^0_{1,\mu}, \quad \Box\varphi^0_4 = \partial^\mu C^0_{2,\mu} \quad (2.149)$$

$$\partial^\nu \varphi^0_{1,\mu\rho} - C^0_{1,\mu\nu} - \partial_{(\mu}\varphi^0_{2,\nu)} - g_{\mu\nu}\varphi^0_4 = 0 \quad (2.150)$$

$$\partial^\mu \varphi^1_{1,\mu} - C^1_1 - \sqrt{D/2}\varphi^0_4 = 0 \quad (2.151)$$

$$\varphi^0_{2,\mu} - \partial_{\mu}\varphi^0_5 - 1/2 \varphi^0_{1,\mu} + \sqrt{D/2}\varphi^1_{1,\mu} = 0. \quad (2.152)$$

The field equations (2.147)–(2.152) are Lagrangian equations and can be obtained from the Lagrangian in (2.122). One can show combining the field equations and gauge transformations (2.144)–(2.146) that after complete gauge fixing the only propagating components are physical components of the tensor $\varphi^0_{1,\mu\nu\rho}$ i.e., physical components of the spin 3 field.

**Compensator equations** Another way to describe irreducible higher spin fields is to take triplet equations (2.40) and add manually the extra condition

$$\varphi' - 2D = \partial\alpha, \quad (2.153)$$

where the field $\alpha$ which has rank $s - 3$ is called the compensator field [8], [11]. In order to maintain gauge invariance the transformation law for the compensator has to be

$$\delta\alpha = \Lambda'. \quad (2.154)$$

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If one gauges away the compensator via the trace of the parameter $\Lambda$ one obtains the description of a single irreducible higher spin field as one can check using (2.40) and (2.154). One can check that these fields only (the triplet fields and the compensator) are not enough for the Lagrangian description of the system. On the other hand one can identify the field $-2\varphi^0_5$ with the compensator and thus the Lagrangian derived in this chapter is precisely the Lagrangian for the compensator field. After elimination of the fields $C$ and $D$ the compensator equations can be written in the form

$$
\Box \varphi - \partial \partial \cdot \varphi + \partial^2 \varphi' = 3 \partial^3 \alpha, \\
\varphi'' = 4 \partial \cdot \alpha + \partial \alpha',
$$

(2.155)

and are invariant under the **unconstrained** gauge transformations

$$
\delta \varphi = \partial \Lambda, \\
\delta \alpha = \Lambda'.
$$

(2.156)

(2.157)

Let us note that a Lagrangian for the compensator fields can be obtained in terms of a smaller (minimal) number auxiliary fields [10]. This Lagrangian (see [14] for a generalization to the case of an AdS space) contains higher derivatives, however the number of derivatives can be reduced to the ordinary one at the expense of introducing two more auxiliary fields [15].

One can write compensator equations for the fermionic higher spin field as well and also deform both fermionic and bosonic compensator equations to the AdS space [11], thus describing irreducible massless integer or half integer spin modes.

### 2.4 Irreducible representations of the AdS group

The description of irreducible massless representations of the $AdS_D$ group can be performed in a similar way [7]. All we have to do is to combine the results of subsections 2.2 and 2.3. Namely, start with the nonlinear algebra (2.100) – (2.101), but now introduce the ghost variables $(c_M, b^{-}_M)$ and $(c^+_M, b_M)$ for the operators $M$ and $M^+$ as well. Then construct the nilpotent BRST charge again formally including the operator $N$ along with its corresponding ghost-antighost pair $(c_N, b_N)$.

This procedure leads to the family of “bare” nilpotent BRST charges, which turn out to depend on three free parameters $k_1, k_2, k_3$, i.e, the expressions of terms which contain higher degrees in the ghost variables turn out not to be uniquely defined. The possible solution of the problem (not necessarily the most general one) has the form

$$
\tilde{Q}^i = \tilde{Q}^i_0 + k_1 \tilde{Q}^i_{k_1} + k_2 \tilde{Q}^i_{k_2} + k_3 \tilde{Q}^i_{k_3},
$$

(2.158)

where

$$
\tilde{Q}^i_0 = c_0(\tilde{l}_0 + \frac{6}{L^2}) + c_1 l^+_1 + c_M M^+ + c^+_1 l_1 + c^+_M M - c_N (N - 3) \\
+ \frac{2}{L^2} c_0 c^+_1 b_1 - \frac{8}{L^2} c_0 c^+_M b_M + \frac{2}{L^2} c_0 b^+_1 c_1
$$
the BRST charge is not unique, one can show along the lines of [7] that all these operators are nilpotent as well. Let us note that the particular choice of parameters $k_1 = 0, k_2 = 0, k_3 = 8$ leads to the BRST charge constructed in [7], which includes terms up to the fifth order in ghosts.

Further, the procedure goes on in complete analogy with the case of the flat space-time background. Namely, we first build auxiliary representations for $M^\pm$ and $N$, then define the new operators as the sum of old and auxiliary ones and finally after the transformation (2.112) we construct the Lagrangian (2.23). Though the BRST charge is not unique, one can show along the lines of [7] that all these BRST charges, after making a partial gauge fixing in the Lagrangian (2.134), lead to a unique final form for the Lagrangian, which contains only one double traceless physical field $|\varphi \rangle$ ($M^2|\varphi \rangle = 0$)

$$\mathcal{L} = \langle \varphi | \bar{\mathcal{L}}_0 - l_1 I_0 + 2M^+ I_0M + M^+ l_1 l_1 + l_1 l_1 M - M^+ l_1 l_1 M - \frac{1}{L^2}(6 - 4N + 10M^+ M - 4M^+ NM)|\varphi \rangle$$

or equivalently

$$S = \int d^Dx \sqrt{-g}(\varphi^{\mu_1 \mu_2 \cdots \mu_s}(x)(\nabla^2 - \frac{1}{L^2}(s^2 + sD - 6s - 2D + 6))\varphi_{\mu_1 \mu_2 \cdots \mu_s}(x)$$

$$- \frac{s(s - 1)}{2} \varphi_{\mu_1 \mu_3 \cdots \mu_s}(x)(\nabla^2 - \frac{1}{L^2}(s^2 + sD - 4s - D + 1))\varphi^{\nu_1 \nu_3 \cdots \mu_s}(x)$$

$$- s \varphi^{\mu_2 \cdots \mu_s}(x)\nabla_{\mu_1} \nabla_{\nu_1} \varphi^{\nu_1 \mu_2 \cdots \mu_s}(x)$$

$$+ s(s - 1) \varphi_{\mu_1 \mu_3 \cdots \mu_s}(x)\nabla_{\nu_1} \nabla_{\nu_2} \varphi^{\nu_1 \nu_2 \mu_3 \cdots \mu_s}(x)$$

$$- \frac{s(s - 1)(s - 2)}{4} \varphi_{\mu_1 \mu_3 \mu_4 \cdots \mu_s}(x)\nabla_{\mu_3} \nabla_{\nu_1} \varphi^{\nu_1 \nu_2 \mu_4 \cdots \mu_s}(x))$$

(2.164)
which is invariant under the gauge transformations

\[ \delta \varphi_{\mu_1 \mu_2 \cdots \mu_s}(x) = \nabla_{(\mu_s} \Lambda_{\mu_1 \cdots \mu_{s-1})}(x) \]  

(2.165)

with totally symmetric and traceless \( \Lambda^{\mu_1 \mu_2 \cdots \mu_{s-1}}(x) = 0 \) parameter.

3 Free Massive Fields

3.1 Reducible representations of the Poincare group

The description of massive reducible higher spin modes can be carried out in a similar way to that of reducible massless higher spin modes [77]. Namely, we need to consider the following set of operators

\[ L_0 = p_\mu^2 + m^2, \]  
\[ L_1 = \alpha^\mu p_\mu, \quad L_1^+ = \alpha^{\mu+} p_\mu. \]  

(3.1)

(3.2)

However, there is a complication since due to the nonzero value of the mass parameter\(^††\) the operators \( l_1, l_1^+ \) and \( l_0 \) no longer form a closed algebra and therefore BRST construction is more complicated.

Let us therefore consider the massless case (where, as we know, this problem does not exist) in one dimension higher and define the usual set of operators \( L_0, L_1, L_1^+ \) which satisfy the algebra (2.17) i.e., consider in \( D + 1 \) dimensions the following set of operators

\[ L_0 = p_\mu^2 + p_D^2 = l_0 + p_D^2, \quad \mu = 0, 1, ..., D - 1, \]  
\[ L_1 = \alpha^\mu p_\mu + \alpha^D p_D = l_1 + \alpha^D p_D, \]  
\[ L_1^+ = \alpha^{\mu+} p_\mu + \alpha^{D+} p_D = l_1^+ + \alpha^{D+} p_D. \]  

(3.3)

(3.4)

(3.5)

The BRST charge for this system is given by (2.20)

\[ Q = c_0 L_0 + c_1 L_1^+ + c_1^+ L_1 - c_1^+ c_1 b_0. \]  

(3.6)

In order to describe the massive fields we fix the following \( x_D \) dependence of the Fock space vector \( |\Phi\rangle \) in (2.23)

\[ |\Phi\rangle = U|\Phi'\rangle = e^{ix_D m} |\Phi'\rangle. \]  

(3.7)

The result of the substitution of (3.7) in the expression (2.23) is

\[ \mathcal{L} = \int dc_0 \langle \Phi' | \bar{Q} | \Phi' \rangle. \]  

(3.8)

\(^††\)One can make the mass parameter \( m \) to depend on spin thus describing a “Regge trajectory” [5]
The new BRST - charge $\tilde{Q}$ is nilpotent due to unitarity of the transformation $\tilde{Q} = \mathcal{U}^{-1}QU$. Our choice of $x_D$ dependence in the exponent in (3.7) leads to the presence in the BRST charge of the correct operator $l_0 + m^2$ for the massive case

$$\tilde{Q} = c_0(l_0 + m^2) + c_1(l_1^+ + m\alpha^{D^+}) + c_1^+(l_1 + m\alpha^D) - c_1^+c_1b_0. \quad (3.9)$$

Let us note that the unitarity transformation (2.112) has the same form as the transformation we are considering now. It is exactly in the heart of the approach and makes it possible to construct the nilpotent BRST charge in the presence of operators which do not form a closed algebra (second class constraints).

The expansion of the field $|\Phi\rangle$ and of the parameter of gauge transformations $|\Lambda\rangle$ in terms of ghost variables is again (2.26) and (2.27) but now one deals with a massive triplet. After the substitution of equation (3.7) into the Lagrangian (3.8) and integration over the ghost variables one gets

$$L = \langle \varphi | l_0 + m^2 | \varphi \rangle - \langle D | l_0 + m^2 | D \rangle + \langle C | | C \rangle - \langle \varphi | l_1^+ + m\alpha^{D^+} | C \rangle + \langle D | l_1 + m\alpha^D | C \rangle - \langle C | l_1 + m\alpha^D | \varphi \rangle + \langle C | l_1^+ + m\alpha^{D^+} | D \rangle. \quad (3.10)$$

The equations of motion for the massive triplet are:

$$\begin{align*}
(l_0 + m^2)|\varphi\rangle &= (l_1^+ + m\alpha^{D^+})|C\rangle \quad (3.11) \\
(l_0 + m^2)|D\rangle &= (l_1 + m\alpha^D)|C\rangle \quad (3.12) \\
|C\rangle &= (l_1^+ + m\alpha^{D^+})|D\rangle - (l_1 + m\alpha^D)|\varphi\rangle \quad (3.13)
\end{align*}$$

while the gauge transformation rule (2.25) gives

$$\delta|\varphi\rangle = (l_1^+ + m\alpha^{D^+})|\lambda\rangle, \quad \delta|D\rangle = (l_1 + m\alpha^D)|\lambda\rangle, \quad \delta|C\rangle = (l_0 + m^2)|\lambda\rangle. \quad (3.14)$$

Then, using the gauge transformations, one can show [43], [5], that the fields $|C\rangle$ and $|D\rangle$ as well as $\alpha^{D^+}$ dependence in $|\varphi\rangle$ can be gauged away. Finally, one obtains conditions

$$(l_0 + m^2)|\varphi\rangle = l_1|\varphi\rangle = 0 \quad (3.15)$$

as the result of the equations of motion. In the simplest example of a spin one massive triplet which describes a massive vector field one has the expansion

$$|\varphi\rangle = A_\mu(x)\alpha^{\mu+}|0\rangle + iA_D(x)\alpha^{D^+}|0\rangle, \quad |C\rangle = -iC(x)|0\rangle \quad (3.16)$$

and

$$|\Lambda\rangle = i\lambda|0\rangle. \quad (3.17)$$

Therefore, for gauge transformations we get

$$\delta A_\mu = \partial_\mu\lambda, \quad \delta A_D = m\lambda, \quad \delta C = (\Box - m^2)\lambda. \quad (3.18)$$

After the elimination of field $C$ via its equations of motion one obtains the Lagrangian which contains the physical field $A_\mu$ and the Stueckelberg field $A_D$. After gauging
away the latter field one obtains the usual Lagrangian for a massive spin 1 field (see also [67] for a discussion in the context of AdS/CFT correspondence).

The Lagrangian (3.10) describes a chain of massive states with mass equal to \( m \). Due to the absence of the zero trace constraint, each state \( |\phi\rangle \) describes a chain of massive states with spins \( s, s-2, s-4 \ldots 1/0 \) as was in the case of massless triplet. This is to be expected, since we are describing the Kaluza-Klein reduction on a circle \( S^1 \) of a massless triplet, which as we know describes a chain of massless states, in \( D + 1 \) dimensions.

### 3.2 Irreducible representations of the Poincare group

A Lagrangian for irreducible massive representations of the Poincare group was given in [46]. In this description the Lagrangian contains a massive spin \( s \) physical field and auxiliary fields with spins \( s-1, s-2, \ldots, 1, 0 \) all of them having zero traces. Un “unconstrained” BRST formulation for massive irreducible higher spin fields is given in [20]. Apart from these descriptions there is an alternative one [47] which contains the physical field with spin \( s \) and only three auxiliary fields with spin \( s-1, s-2, s-3 \). This approach is based again on the method of dimensional reduction.

Since for this computation “the mostly minus” signature is slightly more convenient we shall use this signature in this subsection. The operators are again \( L_0 = p^\mu p_\mu \), \( L_1 = \alpha^\mu p_\mu \), \( M = \frac{1}{2} \alpha^\mu \alpha_\mu \), and their hermitian conjugates but \( N = -\alpha^{\mu+} \alpha_\mu + \frac{D}{2} \), \([\alpha_\mu, \alpha^\nu_+] = -g_{\mu\nu} \), with \( g_{\mu\nu} = (1, -1, \ldots -1) \).

Let us start with Fronsdal Lagrangian in \( D + 1 \) dimensions

\[
\mathcal{L} = \langle \varphi_{D+1} | L_0 + L_1^+ L_1 - 2\overline{M}^+ \tilde{L}_0 \overline{M} \\
+ \overline{M}^+ L_1 L_1 + L_1^+ L_1^+ \overline{M} + \overline{M}^+ L_1^+ L_1 \overline{M} | \varphi_{D+1} \rangle.
\]

where

\[
\overline{M} = M - \frac{1}{2} \alpha_D \alpha_D
\]

The basic field satisfies once more the condition

\[
\overline{M}^2 |\varphi\rangle = 0,
\]

the gauge transformation rule is

\[
\delta |\varphi\rangle = L_1^+ |\Lambda\rangle,
\]

and the parameter of the gauge transformations is traceless

\[
\overline{M} |\Lambda\rangle = 0.
\]

The next step is to perform the dimensional reduction procedure in a similar way to the previous subsection. Namely we take \( |\varphi_{D+1}\rangle = e^{ix_{D+1}m} |\varphi\rangle \). One can solve
the condition (3.21) explicitly. The solution is given in terms of four completely unrestricted vectors
\[
|\varphi\rangle = \sum_{k=0}^{\infty} [(\alpha_D^+)^{2k}(\frac{2^k}{(2k)!}M^k|\varphi_0\rangle + \frac{2^{k-1}}{(2k-1)!}M^{k-1}|\varphi_2\rangle) + [(\alpha_D^+)^{2k+1}(\frac{2^k}{(2k+1)!}M^k|\varphi_1\rangle + \frac{k2^k}{(2k+1)!}M^{k-1}|\varphi_3\rangle)].
\] (3.24)

In a similar manner the equation (3.23) can be solved to give
\[
|\Lambda\rangle = \sum_{k=0}^{\infty} [(\alpha_D^+)^{2k}(\frac{2M^k}{(2k)!}|\Lambda_0\rangle + (\alpha_D^+)^{2k+1}(\frac{2M^k}{(2k+1)!}|\Lambda_1\rangle].
\] (3.25)

Putting (3.24) back into the Lagrangian (3.19) and performing the normal ordering with respect to the oscillators \(\alpha_D^+\), thus integrating them out, one arrives at the Lagrangian
\[
\mathcal{L} = \sum_{k=0}^{\infty} [\langle\varphi_0|(M^+)^{2k}2^k + 2k\langle\varphi_2|(2M^+)^{k-1})(\frac{2M^k}{(2k)!}|T_0\rangle + \frac{(2M)^{k-1}}{(2k-1)!}(\frac{1}{2} - N)|T_2\rangle) + \langle\varphi_1|(2M^+)^k + 2k\langle\varphi_3|(2M^+)^{k-1})(\frac{2M^k}{(2k+1)!}|T_1\rangle + \frac{2k(2M)^{k-1}}{(2k+1)!}(\frac{1}{2} + N)|T_3\rangle]
\] (3.26)

where
\[
|T_0\rangle = |l_0 - m^2 + l_1^+l_1 + M^+l_1l_1 + 2m^2M^+M||\varphi_0\rangle - m[l_1^+ + 2M^+M]|\varphi_1\rangle + [2l_0M^+ - l_1^+l_1^+ - M^+l_1l_1]|\varphi_2\rangle + mM^+l_1^+|\varphi_3\rangle
\] (3.27)
\[
|T_1\rangle = -m[l_1 + 2l_1^+M + 4M^+l_1M]|\varphi_0\rangle + [l_0 + l_1^+l_1 + M^+l_1l_1 + 2m^2M^+M]|\varphi_1\rangle + m[-3M^+l_1 + 2M^+l_1^+]|\varphi_2\rangle + [(2l_0 - m^2)M^+ - l_1^+l_1^+ - l_1^+M^+l_1]|\varphi_3\rangle
\] (3.28)
\[
|T_2\rangle = |l_1l_1 + 2m^2M||\varphi_0\rangle - 2ml_1|\varphi_1\rangle + [2l_0 - l_1^+l_1]|\varphi_2\rangle + ml_1^+|\varphi_3\rangle
\] (3.29)
\[
|T_3\rangle = 4ml_1M|\varphi_0\rangle - [l_1l_1 + 2m^2M]|\varphi_1\rangle - m[2l_1^+M - 3l_1]|\varphi_2\rangle + [l_1^+l_1 + m^2 - 2l_0]|\varphi_3\rangle
\] (3.30)

The equations of motion which can be obtained from the Lagrangian (3.26) are
\[
|T_i\rangle = 0, \quad i = 0, 1, 2, 3.
\] (3.31)
In addition to this there are two Jacobi identities

\[
l_1 |T_0\rangle - M^+ l_1 |T_2\rangle - m M^+ |T_3\rangle - m |T_1\rangle = 0 \quad (3.32)
\]

\[
2 m M |T_0\rangle + l_1 |T_1\rangle - m (1 - 2 M^+ M) |T_2\rangle + M^+ l_1 |T_3\rangle = 0. \quad (3.33)
\]

The gauge transformation rules are

\[
\delta |\varphi_0\rangle = l_1^+ |\lambda_0\rangle, \quad \delta |\varphi_1\rangle = l_1^+ |\lambda_1\rangle - m |\lambda_0\rangle, \quad (3.34)
\]

\[
\delta |\varphi_2\rangle = - l_1 |\lambda_0\rangle + m |\lambda_1\rangle, \quad \delta |\varphi_3\rangle = - l_1 |\lambda_1\rangle + 2 m M |\lambda_0\rangle. \quad (3.35)
\]

One can see that the system decouples into quartets. For each physical field with mass \(m\) and spin \(s\) contained in the vector \(|\varphi_0\rangle\), there are three auxiliary fields contained in vectors \(|\varphi_1\rangle, |\varphi_2\rangle, |\varphi_3\rangle\) which have ranks \(s - 1, s - 2,\) and \(s - 3\). After gauge fixing one can prove that in each quartet there is only one physical polarization with mass \(m\) and spin \(s\).

### 3.3 Irreducible representations of the AdS group

As we noted before the Lagrangian description of a higher spin field with spin \(s\) and nonzero mass contains a set of auxiliary fields with spins \(s - 1, s - 2,\ldots, 1, 0\) [46]. The generalization of this result to the (A)dS case was given in [25] while the unconstrained Lagrangian formulation for massive higher spin fields on AdS has been carried out in [21] using the BRST formalism.

### 4 Interactions

#### 4.1 General method

In this section we discuss the general construction [13] of the cubic vertex for massless higher spin fields on flat and AdS spaces which is based on a generalization of the BRST method. This approach is analogous in some aspects to the cubic vertex construction in string field theory, however, in our case there is no analog of the overlap conditions on the three-string interaction vertex that would strongly restrict its form. In the case of interacting massless higher spin fields the only guiding principle is gauge invariance which manifests itself as the requirement of BRST invariance of the vertex (see also [24]).

There is one crucial point regarding interacting higher spin fields. It appears that a length parameter is necessary for the construction of the interaction vertex, so that the latter has the right dimensions. For higher spin fields in flat space there is no obvious candidate for this length parameter. One possibility would be to consider higher spin gauge fields emerging in the tensionless limit of string theory, in which case the role of the above mentioned parameter is played by the inverse
of the string tension $\alpha'$. On the other hand, for higher spin fields in curved space-times such a dimensionful parameter is naturally given by the inverse curvature. In particular, in the case of higher spin gauge fields on AdS space-times this parameter is naturally associated with the AdS radius $L$. Note that the zero radius limit of such a construction is the large-curvature limit.

After these remarks we will proceed along the lines of [37], [45]. We wish to construct the most general cubic vertex; for that we use three copies of the triplet defined in (2.26) as $|\Phi_i\rangle$, $i = 1, 2, 3$. If we studied the quartic vertex we would use four copies of the Higher Spin functional $|\Phi\rangle$ etc [27]. The tensor fields in $|\Phi_i\rangle$ are all at the same space-time point. Then, the $|\Phi_i\rangle$ interacting among each other are expanded in terms of the set of oscillators $\alpha^{i+}, c^i$ and $b^i$.

\[
[a^{i+}_\mu, a^{j+}_\nu] = \delta^{ij} g_{\mu\nu}, \quad \{c^i, b^j\} = \{c^i, b^{j+}\} = \{c^i_0, b^j_0\} = \delta^{ij},
\]

in complete analogy to the free field case. The BRST charge of our construction consists of three copies of the free BRST change $\tilde{Q} = Q_1 + Q_2 + Q_3$. The full interacting Lagrangian can be written as [48]–[49]

\[
L = \sum_i \int dc_i^0 \langle \Phi_i | Q_i | \Phi_i \rangle + g \int dc_i^0 dc_j^0 dc_k^0 \langle \Phi_1 | \langle \Phi_2 | \langle \Phi_3 | V \rangle + h.c) ,
\]

where $|V\rangle$ is the cubic vertex and $g$ is a dimensionless coupling constant\(^\dagger\).

It is straightforward to show that the Lagrangian (4.2) is invariant up to terms of order $g^2$ under the nonabelian gauge transformations

\[
\delta|\Phi_1\rangle = Q_1|\Lambda_1\rangle - g \int dc_1^0 dc_2^0 dc_3^0 \langle \Phi_2 | \langle \Lambda_3 | + \langle \Phi_3 | \langle \Lambda_2 | \rangle) \langle V \rangle + O(g^2),
\]

\[
\delta|\Phi_2\rangle = Q_2|\Lambda_2\rangle - g \int dc_2^0 dc_3^0 dc_1^0 \langle \Phi_3 | \langle \Lambda_1 | + \langle \Phi_1 | \langle \Lambda_3 | \rangle) \langle V \rangle + O(g^2),
\]

\[
\delta|\Phi_3\rangle = Q_3|\Lambda_3\rangle - g \int dc_3^0 dc_1^0 dc_2^0 \langle \Phi_1 | \langle \Lambda_2 | + \langle \Phi_2 | \langle \Lambda_1 | \rangle) \langle V \rangle + O(g^2),
\]

provided that the vertex $V$ satisfies the BRST invariance condition

\[
\sum_i Q_i |V\rangle = 0.
\]

Indeed the invariance for the terms of zeroth order in $g$ is guaranteed by the nilpotence of the BRST charges $Q_i$ and the invariance for the terms of first order in $g$ is guaranteed by the BRST invariance of the vertex. In a similar way the closure of the algebra of gauge transformations for the terms linear in $g$ is guaranteed by the BRST invariance condition of the vertex. The gauge transformations (4.3)–(4.5) are nonlinear deformations of the previously considered abelian gauge transformations.

\(^\dagger\)Each term in the Lagrangian (4.2) should have length dimension $-D$. This requirement holds true for each space-time vertex contained in (4.2) after multiplication by an appropriate power of the length scale of the theory, as discussed before.
We assume here that the tensor fields obtained after the expansion of the $|\Phi_i\rangle$ functionals in terms of the oscillators $\alpha_i^\pm$ are different from each other. One can also consider cases when two or all three higher spin functionals contain the same tensor fields, as we show at the end of this chapter.

In order to ensure zero ghost number for the Lagrangian, the cubic vertex must have ghost number 3. We make the following ansatz for the cubic vertex

$$|V\rangle = V|\rangle_{123}$$

where the vacuum $|\rangle$, with ghost number 3, is defined as the product of the individual Fock space ghost vacua

$$|\rangle_{123} = c_0^1 c_0^2 c_0^3 |0\rangle_1 \otimes |0\rangle_2 \otimes |0\rangle_3 .$$

The function $V$ has ghost number 0 and it is a function of the rest of the creation operators as well as of the operators $p_i^\mu$. In Open String Field Theory the r.h.s. of (4.8) is multiplied by $\delta^D(\sum_i p_i)$ which imposes momentum conservation on the three string vertex. In our case the analogous constraint is to discard total derivative terms of the Lagrangian which is certainly true for flat and AdS space-times. So in what follows we will impose "momentum" conservation in the sense described above.

The condition of BRST invariance (4.6) does not completely fix the cubic vertex. There is an enormous freedom due to Field Redefinitions (FR) just like in any field theory Langrangian. It is clear in the free theory case that any FR of the form

$$\delta \Phi_i = F(\Phi_i) ,$$

(4.9)

gives a gauge equivalent set of equations of motion for the fields $\Phi_i$. Lagrangians obtained from the free one after the field redefinition (4.9) yield additional "fake interactions" and should be discarded. For the interacting case at hand we see from (4.6), that the modified gauge variation (4.3) – (4.5) can only determine the cubic vertex up to $\tilde{Q}$-exact cohomology terms:

$$\delta |V\rangle = \tilde{Q}|W\rangle ,$$

where $|W\rangle$ is a state with total ghost charge 2. We will see in what follows that this FR freedom can lead to major simplifications for the functional form of the vertex.

Next, we expand the vertex operator $|V\rangle$ and the function $|W\rangle$ in terms of ghost variables or equivalently in terms of the following two ghost quantities with ghost number zero

$$\gamma^{ij,+} = c^{i,+} b^{j,+} , \quad \beta^{ij,+} = c^{i,+} b_0^j .$$

These are $3 \times 3$ matrices of fields with no symmetry properties. For the cubic vertex we have the expansion

$$|V\rangle = \left\{ X^1 + X_2^2 \gamma^{ij,+} + X_3^3 \beta^{ij,+} + X_4^4 (ij; (kl)) \gamma^{ij,+} + \gamma^{kl,+} + X_5^5 (ij; (mn)) \gamma^{ij,+} + \gamma^{kl,+} + \gamma^{mn,+} + X_6^6 (ij; (kl)) \beta^{ij,+} + \beta^{kl,+} + \beta^{mn,+} + X_7^7 (ij; (kl); (mn)) \gamma^{ij,+} + \gamma^{kl,+} + \gamma^{mn,+} + X_8^8 (ij; (kl); (mn)) \gamma^{ij,+} + \gamma^{kl,+} + \gamma^{mn,+} + X_9^9 (ij; (kl); (mn)) \beta^{ij,+} + \beta^{kl,+} + \beta^{mn,+} \right\} |\rangle_{123} .$$

(4.12)
since the function $V$ in (4.7) has ghost number zero. In our notation we put in parentheses pairs of indices which are symmetric under mutual exchange. For example, $X_{(ij):(kl)}^4$ is symmetric under $(ij) \leftrightarrow (kl)$. The coefficient $X_{(ij):(kl)}^4$ is also antisymmetric under $i \rightarrow k$ since $\{c_i^+, c_k^+\} = 0$ but we have not indicated these symmetries in order to avoid clustering notation.

In a similar manner we have the following expansion:

$$|W\rangle_{123} = \left\{ W_1^1 b_{i}^{i:+} + W_2^1 b_{j}^{j:+} + W_3^1 b_{i}^{i:+} + W_4^1 b_{j}^{j:+} + W_5^1 b_{i}^{i:+} + b_{j}^{j:+} + W_6^1 b_{i}^{i:+} + \gamma_{j}^{j:+} + \gamma_{l}^{l:+} + W_7^2 b_{i}^{i:+} + \beta_{j}^{j:+} + \beta_{l}^{l:+} + W_8^2 b_{i}^{i:+} + \beta_{j}^{j:+} + \beta_{l}^{l:+} + W_9^2 b_{i}^{i:+} + \beta_{j}^{j:+} + \beta_{l}^{l:+} + W_{10}^2 b_{i}^{i:+} + \gamma_{j}^{j:+} + \gamma_{l}^{l:+} + \beta_{m}^{m:+} + W_{11}^2 b_{i}^{i:+} + \beta_{j}^{j:+} + \beta_{l}^{l:+} + \beta_{m}^{m:+} \right\}|\rangle_{123}(4.13)$$

for the FR functional $W$.

We consider higher spin fields in flat space-time first. Each component of the vertex in (4.12) has an oscillator expansion in terms of matter oscillators $\alpha_{\mu}^{i:+}$ and derivatives $p_{\mu}^{i}$, where the latter act to the left. We will restrict our study to the case of totally symmetric massless higher spin fields and therefore we have only to consider three different sets of oscillators and momenta. The generalization to the case of the fields that belong to the reducible mixed symmetry representations of the Poincaré group will be given in the last subsection.

The interaction vertex glues together three Fock spaces and for this reason it is convenient to define, in complete analogy to the free case, the following generators

$$l^{ij} = \alpha_{\mu}^{i} p_{\mu}^{j}, \quad l^{ij+} = \alpha_{\mu}^{i} + p_{\mu}^{j}, \quad l^{i 0} = p_{\mu}^{i} p_{\mu}^{0},$$

$$M^{ij} = \frac{1}{2} \alpha_{\mu}^{i} \alpha_{\mu}^{j}, \quad M^{ij+} = \frac{1}{2} \alpha_{\mu}^{i} + \alpha_{\mu}^{j+},$$

$$N^{ij} = \alpha_{\mu}^{i} + \alpha_{\mu}^{j} + \delta^{ij} \frac{D}{2}.$$  \hspace{1cm} (4.14)

We see that generators (4.14) are indexed by integers $i, j = 1, 2, 3$. The three values for $i$ and $j$ originate from the fact that we consider a three field interaction. In the general case of an n-field interaction, we should take the same generators with $i, j = 1, 2, ..., n$. Using the generators above one can build all possible interaction terms between symmetric higher spin fields. Therefore, our ansatz for the vertex is that of the most general polynomial made out from the operators $l^{ij}_{0}$, $l^{ij+}$ and $M^{ij+}$. This corresponds to the usual derivative expansion for the vertex, since the operators $l_{0}^{ij}$ have dimensions $[\text{Length}]^{-2}$ and the operators $l^{ij+}$ have dimension $[\text{Length}]^{-1}$. To make sense of such an expansion one needs to introduce a physical length parameter. In flat space-times it is not clear where such a length scale may come from, nevertheless the hope is that it would be connected to the length scale of a fundamental theory such as string or M-theory.

The commutator algebra of the operators in (4.14) is:

$$[l^{ij}, l^{kl+}] = 0, \quad [N^{ij}, l^{kl}] = -\delta^{ik} l^{jl},$$

$$[M^{ij+}, l^{kl}] = -\frac{1}{2} (\delta^{jl} l^{ij+} + \delta^{jl} l^{ij+}), \quad [N^{ij}, M^{kl}] = -\delta^{ik} M^{jl} + \delta^{jl} M^{ik},$$

$$[M^{ij}, M^{kl+}] = -\frac{1}{2} (\delta^{ik} N^{jl} - \delta^{jl} N^{ik} - \delta^{ik} N^{jl} - \delta^{jl} N^{ik}).$$  \hspace{1cm} (4.15)
Let us consider the constraints imposed by momentum conservation on the vertex. Clearly, not all generators in (4.14) are linearly independent once we consider the operatorial equation $\sum_i p^\mu_i = 0$, which means that we omit total derivatives. A convenient set of linearly independent generators is the following:

$$l_{ij}^0 = (l_{01}^{11}, l_{02}^{22}, l_{03}^{33}) = (l_0^1, l_0^2, l_0^3)$$

$$l_{ij, +} = (l_{ij}^{1+}, l_{ij}^{2+}, l_{ij}^{3+}, I^{3+})$$

$$l_{i, +} = l_{i, +}, I_i^{+} = \alpha^{+,1} (p_\mu^2 - p_\mu^3)$$

$$l_{i, +} = \alpha^{+,2} (p_\mu^3 - p_\mu^1) I_i^{3+} = \alpha^{+,3} (p_\nu^1 - p_\nu^2)$$

$$M_{ij, +} = (M_{ij}^{11+}, M_{ij}^{22+}, M_{ij}^{33+}, M_{ij}^{12+}, M_{ij}^{13+}, M_{ij}^{23+}) \quad (4.16)$$

Based on the above analysis we can write the most general form of the expansion coefficients $X_i^{l}(...)$:

$$X_i^{l} = X_i^{l} (l_{01}^{n1} \ldots (l_{0n}^{1+})^{m1} (I_{12}^{+1})^{k1} \ldots (M_{11}^{+1})^{p1} \ldots (M_{12}^{+1})^{r1} \ldots ) \quad (4.17)$$

Using the explicit form of the BRST charges:

$$Q = c_i^0 l_0 + c_i^{l, +} + c_i^{l, +} - c_i^{l, +} c_i^{l, 0} \quad (no \ sum) \quad (4.18)$$

and equations (4.6), (4.17) we arrive to the following set of equations:

$$c_i^{l, +}[l_i^{ }X^1 - l_s^{ +} X_{is}^2 - l_0^{ } X_{is}^3] = 0 \quad (4.19)$$

$$c_i^{l, +} \delta_{jk}^{ +} [l_i^{ }X_{j} - 2 l_s^{ +} X_{(is);(jk)} - l_0^{ } X_{jk;is}] = 0$$

$$c_i^{l, +} \delta_{jk}^{ +} X_{ij} + l_i^{ } X_{jk}^3 - l_s^{ +} X_{is;jk}^5 - 2 l_0^{ } X_{(is);(jk)} = 0$$

$$c_i^{l, +} \delta_{jk}^{ +} X_{ij}^4 + l_i^{ } X_{jk}^8 - l_s^{ +} X_{is;jk;kl}^9 - 2 l_0^{ } X_{(is);(jk);(lm);is} = 0$$

To simplify the analysis of these equations we define the operator:

$$\hat{N} = \alpha^{+,i} \alpha_i^+ + b_i^{l, +} a_i + c_i^{l, +} b_i^{l, +} \quad (4.20)$$

This operator commutes with the BRST charges $Q_i$ and its eigenvalues count the degree of the $X_i^{l}(...)$s in the $\alpha_i^{+,i}$ oscillator expansion. Namely, as can be seen from equation (4.12), if the degree of the coefficient $X^1$ in oscillators $\alpha_i^{+,i}$ is $K$, then the rest of the coefficients have the following degrees in the oscillators $\alpha_i^{+,i}$:**

$$X^1(K), X^2(K - 2), X^3(K - 1), X^4(K - 4), X^5(K - 3),$$

$$X^6(K - 2), X^7(K - 6), X^8(K - 5), X^9(K - 4), X^{10}(K - 3).$$
For example, the first equation has degree $K - 1$, since $l^{ij}$ reduces the value of $K$ by one, $l^{ij, +}$ increases it by one and $l^{ij}_0$ leaves it unchanged.

There is yet another number which can be used in a manner similar to $K$. Namely, if a term in the expansion of $V$ has powers of operators $l^{ij}_0, l^{ij, +}, M^{ij, +}, \gamma^{ij, +}$ and $\beta^{ij, +}$ equal to $s_1, s_2, s_3, s_4$ and $s_5$ respectively, then the total number $s = s_1 + s_2 + s_3 + s_4 + s_5$ is unchanged under the action of the BRST charge.

The above observations can be used to classify equations (4.18) according to their degree $K$ and the number $s$. This means that the vertex can be expanded in a sum of contribution with fixed degrees $K$ and $s$ as

$$|V\rangle = \sum_{K,s} |V(K, s)\rangle.$$  \hspace{1cm} (4.21)

Therefore, the equation (4.6) can be split into an infinite set of equations

$$\sum_i Q_i V(K, s) = 0$$  \hspace{1cm} (4.22)

for each value of $K$ and $s$.

To construct the vertex on AdS we use the same procedure as in the flat case, in particular we solve the same equation (4.6). In this case, however, care is needed when trying to extend the algebra (4.15) to a nontrivial background.

For the construction of the interaction vertex on AdS it is convenient to slightly modify the definition of the operator (2.84) \cite{21} as

$$p_{\mu} = -i \left( \nabla_{\mu} + \omega^{ab}_{\mu} \alpha^+_a \alpha_b \right),$$  \hspace{1cm} (4.23)

where $\nabla_{\mu}$ is AdS covariant derivative. The reason behind this modification is the following: in the free case, one is working with only one Fock space and all indexes of the tensor fields are contracted with the corresponding oscillators. Therefore, in the free case the last term in (4.23) is enough for $p_{\mu}$ to act covariantly on Fock space states. However, in the interacting case, where we have three different Fock spaces, expressions of the form $\varphi_\mu^1(x) \alpha^{\mu 3+}$ have a free index $\mu$ with respect to the first and third Fock spaces. Therefore, $p_{\mu}^1$ should act as a covariant derivative $\nabla_{\mu}$ on $\varphi_\mu^1(x)$ instead of a partial one. With this modification the operators of the type $l_0^i$ read simply

$$l_0^{ij} = p_{\mu}^i p_{\mu}^j,$$  \hspace{1cm} (4.24)

while the definition of the operators $l^{ij}, l^{ij, +}$ is given in (4.14). In order to compute the algebra it is useful to recall how various operators defined previously act on physical states. For example operator $l_0^{12} = p_{\mu}^1 p_{\mu}^2$, where $p_{\mu}$ is the operator (4.23), acts as follows

$$l_0^{12} |\Phi_1\rangle \otimes |\Phi_2\rangle = \frac{i}{(s_1)!} \alpha^{\mu_1, 1+} \ldots \alpha^{\mu_s, 1+} \nabla^\mu \varphi_{\mu_1 \mu_2 \ldots \mu_s}^1(x) |0\rangle_1 \otimes$$  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspace{1cm}  \hspe
In obtaining the above result it was crucial that 
\[ p^j_\mu p^j_\nu = \delta^{ij} (-[\nabla^i_\mu, \nabla^i_\nu] + \frac{1}{L^2} (\alpha^i_\mu + \alpha^i_\nu - \alpha^{i+}_\nu \alpha^i_\mu)) = \delta^{ij} D^i_{\mu\nu}. \] (4.25)

The other operators are defined in an analogous way. For example the operator 
\[ l^{12} = \alpha^{\mu \cdot 1} p^2_\mu \] acts as
\[ l^{12} |\Phi_1\rangle \otimes |\Phi_2\rangle = \frac{1}{(s_1 - 1)!} \alpha^{\mu_1 \cdot 1} \ldots \alpha^{\mu_{s_1} \cdot 1} \phi^{1\mu}_{\mu_2 \ldots \mu_{s_1}} (x)|0\rangle_1 \otimes \]
\[ - \frac{i}{(s_2)!} \alpha^{\nu_1 \cdot 2} \ldots \alpha^{\nu_{s_2} \cdot 2} \nabla^\mu \phi^{2\mu}_{\nu_1 \nu_2 \ldots \nu_{s_2}} (x)|0\rangle_2, \]

the operator \( l^{12+} = \alpha^{\mu \cdot 1} p^2_\mu \) acts as
\[ l^{12+} |\Phi_1\rangle \otimes |\Phi_2\rangle = \frac{1}{(s_1 - 1)!} \alpha^{\mu_1 \cdot 1} \ldots \alpha^{\mu_{s_1} \cdot 1} \phi^{1\mu}_{\mu_2 \ldots \mu_{s_1}} (x)|0\rangle_1 \otimes \]
\[ - \frac{i}{(s_2)!} \alpha^{\nu_1 \cdot 2} \ldots \alpha^{\nu_{s_2} \cdot 2} \nabla^\mu \phi^{2\mu}_{\nu_1 \nu_2 \ldots \nu_{s_2}} (x)|0\rangle_2, \]

and the operator \( M^{12} = \frac{1}{2} \alpha^{\mu \cdot 1} \alpha^2_\mu \) acts as
\[ M^{12} |\Phi_1\rangle \otimes |\Phi_2\rangle = \frac{1}{2 (s_1 - 1)!} \alpha^{\mu_1 \cdot 1} \ldots \alpha^{\mu_{s_1} \cdot 1} \phi^{1\mu}_{\mu_2 \ldots \mu_{s_1}} (x)|0\rangle_1 \otimes \]
\[ \frac{1}{(s_2 - 2)!} \alpha^{\nu_1 \cdot 2} \ldots \alpha^{\nu_{s_2} \cdot 2} \phi^{2\mu}_{\mu_2 \ldots \mu_{s_2}} (x)|0\rangle_2. \]

At this point it is instructive to present an explicit example of a computation. Let us compute the commutator between \( l^{11} \) and \( l^{12+} \) acting on \( |\Phi_1\rangle \otimes |\Phi_2\rangle \), where, for clarity, we take \( |\Phi_1\rangle \) to be a vector and \( |\Phi_2\rangle \) to be a scalar.

\[ [\alpha^{\mu \cdot 1} p^1_\mu, \alpha^{\nu \cdot 1} p^2_\nu] \phi^{1\mu}_{\mu_1 \cdots \mu_{s_1}} |0\rangle_1 \otimes \phi^{2\mu}_{0} |0\rangle_2 = \]
\[ = -i \left( \alpha^{\mu \cdot 1} p^1_\mu, \alpha^{\nu \cdot 1} \right) \phi^{1\mu}_{\mu_1 \cdots \mu_{s_1}} \alpha^{\mu_1 \cdot 1} \phi^{\mu_1 \cdot 1}_{\mu_2 \cdots \mu_{s_2}} (\nabla^\nu \phi^{2\mu}_{\nu_1 \nu_2 \cdots \nu_{s_2}}) |0\rangle_1 \otimes |0\rangle_2 \]
\[ = -\alpha^{\mu \cdot 1} \left( \nabla^\nu \phi^{\mu}_{\nu} \right) \phi^{2\mu}_{\nu_1 \nu_2 \cdots \nu_{s_2}} |0\rangle_1 \otimes |0\rangle_2. \] (4.26)

In obtaining the above result it was crucial that \( p^j_\mu \), as defined in (4.23), commutes with \( \alpha^{\nu \cdot j+} \).

Proceeding this way one obtains the algebra of operators
\[ l^i_0 = p^{\mu \cdot i} p^j_\mu \quad l^{ij} = \alpha^{\mu \cdot i} p^j_\mu \quad l^{ij+,+} = \alpha^{\mu \cdot i+} p^j_\mu \] (4.27)
on AdS for the interacting case
\[ [l^{ij}, l^{mn, +}] = \delta^{im} l^{jn}_0 - \delta^{jn} \alpha^{\mu \cdot m, +} D^j_{\mu \nu} \alpha^{\nu i} \] (4.28)
\[ [m^{mn}, l^{kl}] = \delta^{nl} \alpha^{\mu \cdot m} D^j_{\mu \nu} \alpha^{\nu k} \] (4.29)
supplemented by the part of the algebra (4.15) which involves commutators of $M^{ij}$, $M^{ij,+}$ and $N^{ij}$. We will call the algebra (4.28) – (4.31) the symmetry algebra of interacting higher spin theory in AdS space-time.

The commutation relations above differ from the corresponding flat space–time ones (4.15), in that they involve extra terms which when acting on states give $O(1/L^2)$ contributions. These terms are sub-leading in the $L \to \infty$ limit, hence the algebra (4.28) contracts to the flat space–time algebra (4.15) in the small curvature limit. This implies that free higher spin gauge fields in flat space-time can be viewed as the zero curvature limit of free higher spin gauge fields on AdS. However, interacting higher spin gauge fields on AdS do not have a smooth $L \to \infty$ limit since interaction vertices contain positive powers of $L$. Nevertheless, as we shall see below, the functional form of the cubic vertex of higher spin gauge fields on AdS differs from the cubic vertex in flat space-time by terms which are sub-leading as $L \to \infty$.

The action of the algebra (4.28-4.31) on Fock space states is more complicated than the free case one and appears in [13]. We find it again useful to demonstrate how calculations are done in the interacting case on AdS with an example:

\[
[l^{12}, l^{22,+}] l^{12,+} \varphi^{1}_\rho \alpha^{1}_\rho,1^{+} |0\rangle \otimes \varphi^{2} |0\rangle = \alpha^{2,+}_\mu D^{2}_\mu \alpha^{1}_\rho l^{12,+} \varphi^{1}_\rho \alpha^{1,1^+} |0\rangle \otimes \varphi^{2} |0\rangle = -\frac{1}{L^2} (l^{22,+} (N^{11} - 1 + \frac{D}{2}) - 2M^{12,+} l^{12}) \varphi^{1}_\rho \alpha^{1,1^+} |0\rangle \otimes \varphi^{2} |0\rangle = \frac{i}{L^2} \alpha^{2,+}_\mu \alpha^{1,1^+}_\nu (D \varphi^{\nu}_1 (\nabla^\mu \varphi^{2}) - g^{\mu \nu} \varphi^{\rho}_1 (\nabla^\rho \varphi^{2})) |0\rangle \otimes |0\rangle.
\]

There is only one $p^2_\lambda$ from the second Fock space involved in the example above. In the first equality we used (4.28). In the second equality we acted with $D^{2}_\mu$ on the $p^2_\lambda$ of the $l^{12,+}$ operator using (4.25) and (2.106). This was the only ”tensor” operator in the 2nd Fock space, since $\varphi^2$ is a scalar. Consequently, we commuted operators $\alpha^{1}_\rho$ and $p^2_\sigma$ past each other to bring the result to the second line of (4.32). Finally, we used the action of the operator (4.25) on scalar Fock space states to complete the calculations since no other ”vector” operator, in the the second Fock space, was left for $l^{22,+}$ or $l^{12}$ to act upon.

From the manipulations above we conclude the following: the algebra of constraints being obviously more complicated than in the case of flat space-time shares its main property – namely it preserves the polynomial form of (4.12), (4.13), (4.17). Therefore, we can proceed in an analogous manner as in the flat case.

The next step is to choose an expansion for the cubic vertex in terms of the AdS generators (4.27) and (4.14). In the AdS case the creation generators of (4.16)
do not commute among each other, unlike the flat case, as one can see from e.g. (4.30). Nevertheless, we can choose a standard ordering as in (4.17). All other possible orderings can be brought into the standard form (i.e., use an analogue of the Weyl ordering in quantum mechanics), using the algebra (4.28-4.31) and the manipulations described in the previous subsection, modulo $\frac{1}{L^2}$ terms. The action of $D^\mu_\nu$ on "tensors" produces terms proportional to $\frac{1}{L^2}$ as one can easily verify from (4.25) and (2.106) that affect lower dimension terms in the $L^2$ expansion of $X^n$.

These latter terms can again be brought into the standard form following the same procedure and finally can be absorbed into the definition of the matrix elements with lower dimension than the one we started from.

In addition, although naively we do not have momentum conservation in AdS space-time, we can still make use of the equation $\sum_i p^\mu_i = 0$, since it leads to total derivative terms in the Lagrangian.

To conclude, one can construct the same linearly independent set of generators as in (4.16). The expansion of the coefficients is exactly the same as in (4.17) with all generators the AdS equivalent of the flat ones. In order to write down the BRST invariance condition in a simpler form let us write the BRST charge on AdS in a compact form

$$Q = c_0 \hat{l}_0 + cl^+ + c^+l - \frac{8}{L^2} c_0 (\gamma^+ M + \gamma M^+) - c^+ cb_0$$ (4.33)

with

$$l_0 \rightarrow \hat{l}_0 = p^\mu p_\mu + \frac{1}{L^2} ((\alpha^{\mu+} \alpha_\mu)^2 + D\alpha^{\mu+} \alpha_\mu - 6 \alpha^{\mu+} \alpha_\mu - 2D + 6 - 4M^+ M + c^+ b(4\alpha^{\mu+} \alpha_\mu + 2D - 6) + b^+ c(4\alpha^{\mu+} \alpha_\mu + 2D - 6) + 12 c^+ bb^+ c).$$ (4.34)

Using the explicit form of (4.33) it is straightforward to write down the equations resulting from (4.6). They are the same as in flat case with the substitution $l_0 \rightarrow \hat{l}_0$ as in (4.34) along with some modifications which appear because of the explicit $\frac{1}{L^2}$ dependence of the BRST charge. The final result is:

$$c^{i,+}[l^i X^1 - l^{s,+} X^2_{is} - \hat{l}_0 X^3_{is} + \frac{16}{L^2} M^{s,+} X^5_{is;ss}] = 0$$

$$c^{i,+} \gamma^{jk,+}[l^i X^2_{jk} - 2l^{s,+} X^4_{(is);(jk)} - \hat{l}_0 X^5_{jk;is} + \frac{8}{L^2} (\delta_{jk} M_j X^3_{is} - 6 M^{s,+} X^8_{(ss);(jk);is})] = 0$$ (4.35)

$$c^{i,+} \beta^{jk,+}[-\delta_{jk} X_{ij} + l^i X^3_{jk} - l^{s,+} X^5_{is;jk} - 2\hat{l}_0 X^6_{(is);(jk)} - \frac{32}{L^2} M^{s,+} X^9_{ss;(jk);is}] = 0$$

$$c^{i,+} \gamma^{jk,+} \gamma^{lm,+}[l^i X^4_{(jk);(lm)} - 3l^{s,+} X^7_{(is);(jk);(lm)} - \hat{l}_0 X^8_{(jk);(lm);is} - \frac{8}{L^2} \delta_{jk} M^j X^5_{lm;ij}] = 0$$
\[
c^i + \gamma^j + b^{lm} + [\delta_{lm} X^4_{(ij);(jk)} + l^i X^{5}_{jk;lm} - 2l^{s+} X^8_{(is);(jk);lm} - 2l^i X^9_{jk;is};(lm)] + 16 \frac{\delta_{jk} M^j X^6_{(lm);(ij)}}{L^2} = 0
\]

\[
c^i + \gamma^j + b^{lm} + [\delta_{jk} X^5_{ji;lm} + l^i X^6_{(jk);lm} - l^{s+} X^9_{is;(jk);lm} - 3l^i X^{10}_{(is);(jk);(lm)}] = 0.
\]

Combinations involving the operator \( \hat{l}^i_0 \) should be understood as follows: for example the term in the first equation \( c^i + l^i X^3_{mn} \) is a result of an action of the operator \( c^i_0 \hat{l}^i_0 \) at \( X^3_{mn} \) and using the expression (4.34)

\[
c^i_0 + \hat{l}^i_0 X^3_{mn} \beta^{mn} = -c^i + (\mu^s p^s + \frac{1}{L^2} (\alpha^{\mu,s+} + \alpha^s_{\mu})^2 + D \alpha^{\mu,s+} + \alpha^{s}_{\mu} - 6 \alpha^{\mu,s+} \alpha^{s}_{\mu} - (2D - 6)^s - 4M^{s+}M^s)) X^3_{is} - \frac{1}{L^2} c^i + (4 \alpha^{\mu,i} \alpha^i_{\mu} + (2D - 6)^i) X^3_{is}.
\]

The equations in (4.35) are more difficult to analyze compared to the flat case despite their apparent similarity. The main reason is obvious from the algebra (4.28)–(4.31) which has nontrivial commutators containing \( D^i_{\mu \nu} \). This causes more of a technical difficulty rather than a conceptual one. It would be interesting to find a solution in a closed compact form (if such a solution exists of course) but at the present moment we are content to have a well defined iteration procedure and a system of equations which can be straightforwardly solved via this procedure as we shall demonstrate with a couple of examples below.

### 4.2 Some explicit examples

#### Spin-1 with two scalars

Let us work out in detail the most trivial example of a vector field interacting with two scalars i.e., the case of scalar electrodynamics. Let us put the scalars in the first and the second Fock spaces respectively, and the vector field in the third Fock space. Since the oscillators \( \alpha^{i+}_\mu \), \( c^{i+} \) and \( b^{i+} \) occur only in the third Fock space we omit the index \( i \) for them in what follows. The fields we are using are

\[
|\Phi_1\rangle = \phi_1(x)|0\rangle, \quad |\Phi_2\rangle = \phi_2(x)|0\rangle, \quad (4.37)
\]

\[
|\Phi_3\rangle = (A_\mu(x) \alpha^{\mu+} - i \lambda b^+)|0\rangle, \quad (4.38)
\]

\[
|\Lambda\rangle = i \lambda b^+|0\rangle. \quad (4.39)
\]

Then, according to the discussion after equation (4.20), in order to saturate the last term in (4.2) we need the expansion of the vertex at \( K = 1 \) and ghost number zero. Obviously the unique possibility is

\[
V = a_i \alpha^{i+}_\mu + d_i c^{i+} b^+_0, \quad i = 1, 2, 3 \quad (4.40)
\]

where \( a_i \) and \( d_i \) are constants to be determined. However, one can show that some of these constants are redundant. Let us consider the cohomology of the BRST charge

\[
Q = Q_1 + Q_2 + Q_3 = c^1_0 l^1_0 + c^2_0 l^3_0 + c^3_0 l^3_0 + c^+ l^3 + c^{+;3} - c^+ c b^+_3. \quad (4.41)
\]

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Recalling that the vertex is determined modulo $QW$ and taking (the unique option)

$$W = b^+w, \quad QW = c^+b_0^3$$ (4.42)

one ‘gauges away’ the parameter $d_3$. Furthermore, since we have the momentum conservation law $p_\mu^1 + p_\mu^2 + p_\mu^3 = 0$, we can express $p_\mu^3$ in terms of the other two. This means that the parameter $a_3$ is redundant as well. So we have four parameters $a_1, a_2, d_1, d_2$. The condition for BRST invariance of the vertex gives:

$$c^+(d_1p_\mu^1p_\mu^1 - d_2p_\mu^2p_\mu^2 + a_1p_\mu^1p_\mu^3 + a_2p_\mu^2p_\mu^3) = 0.$$ (4.43)

Applying momentum conservation to the first two terms one arrives at the equations

$$d_1 + d_2 = 0, \quad d_1 + a_1 = 0, \quad a_2 - d_1 = 0,$$ (4.44)
i.e., we can choose

$$a_1 = ig, \quad a_2 = -ig, \quad d_1 = -ig, \quad d_2 = ig.$$ (4.45)

Let us write down the interaction vertex

\[
V = \int dc_0^1dc_0^2dc_0^3\langle 0|\phi_1(x)|0\rangle\langle 0|\phi_2(x)|0\rangle\langle A_\mu\alpha^\mu \\
+iCbc_0^3)(a_ip_i^1\alpha^{\mu+} + b_ic^+b_0^1)c_0^1c_0^2c_0^3|0\rangle|0\rangle|0\rangle + h.c. \\
= \int dc_0^1dc_0^2dc_0^3\langle 0|\phi_1(x)|0\rangle\langle 0|\phi_2(x)|0\rangle\langle A_\mu(a_ip_i^1c_0^1c_0^2c_0^3|0\rangle|0\rangle|0\rangle + h.c. \\
= -\phi_1\phi_2A_\mu a_ip_i^1 + h.c. = -g(\partial_\mu\phi_1)\phi_2A_\mu + g(\partial_\mu\phi_2)\phi_1A_\mu + h.c. (4.46)
\]

which is the standard vertex for scalar electrodynamics. For the gauge transformations we get

$$\delta\phi_1 = -\int dc_0^1dc_0^2dc_0^3\langle 0|\phi_2(x)|0\rangle\lambda b(-i)d_1c^+b_0^1c_0^1c_0^2c_0^3|0\rangle|0\rangle|0\rangle (4.47)$$

$$= \int dc_0^1dc_0^2dc_0^3\langle 0|\phi_2(x)|0\rangle\lambda id_1c_0^2c_0^3|0\rangle|0\rangle|0\rangle$$

$$= -id_1\phi_2(x)\lambda = -g\phi_2(x)\lambda$$

$$\delta\phi_2 = g\phi_1(x)\lambda, \quad \delta A_\mu = \partial_\mu\lambda. \quad (4.48)$$

The analysis for the case of $AdS_D$ is absolutely the same and gives the same result.

**Spin-2 triplet with two scalars**

We assign the field with spin two to the third Fock space, so we have [18]

$$|\Phi_3\rangle = \frac{1}{2!}\langle h_{\mu\nu}(x)\alpha^{\mu+}\alpha^{\nu+} + D(x)c^+b^+ - iC_\mu(x)\alpha^{\mu+}c_0^3b^+|0\rangle, \quad (4.49)$$

$$|\Lambda\rangle = i\lambda_\mu(x)\alpha^{\mu+}b^+|0\rangle. \quad (4.50)$$


In this case we need the expansion of the vertex at $K = 2$. Following the same procedure as in the previous case we get the Lagrangian

$$ L = L_{\text{free}} + L_{\text{int}}, $$

$$ L_{\text{free}} = (∂_μ φ_1)(∂^μ φ_1) + (∂_μ φ_2)(∂^μ φ_2) + m^2(φ_1^2 + φ_2^2) + (∂_μ h_μν)(∂^μ h^{μν}) - 4(∂_μ h_μν)C_ν - 4(∂_μ C^μ)D - 2(∂_μ D)(∂^μ D) + 2C_μ C^μ, $$

$$ L_{\text{int}} = C_{2,0}(h_μ^ν(∂_μ ∂_ν φ_1)φ_2 + h_μ^ν(∂_μ ∂_ν φ_2)φ_1 - 2h_μ^ν(∂_μ φ_1)(∂_ν φ_2)) $$

and the relevant gauge transformations

$$ δφ_1 = C_{2,0}(2λ_μ∂_μ φ_2 + φ_2 ∂_μ λ_μ), $$

$$ δφ_2 = C_{2,0}(2λ_μ∂_μ φ_1 + φ_1 ∂_μ λ_μ), $$

$$ δh_μν = ∂_μ λ_ν + ∂_ν λ_μ, \ δC_μ = ∂_μ λ_μ, \ δD = ∂_μ λ_μ, $$

where $C_{2,0}$ and $C_{2,1}$ are arbitrary real constants. Note that we have added a mass-term for the scalars in the Lagrangian. Curiously enough the Lagrangian describing the interaction of two massless scalars with a spin two triplet is still gauge invariant after the addition of the mass terms for the scalar. This opens the interesting possibility to start with the Lagrangian for the free massive scalars and gauge its symmetries. In this way one recovers the Lagrangian given above after gauging the symmetries generated by a parameter $λ_μ$. A similar result holds for the case of two scalars interacting with a spin 3 gauge field. In this case one gauges the symmetries of the free Lagrangian generated by a parameter $λ_{μν}$ [18] (see also [95], [96]).

According to our general construction we have obtained the cubic vertex which involves two different scalars and the triplet with higher spin 2. To obtain the interaction of a single scalar with the spin-2 field we need to set $φ_1 = φ_2$. Note that setting i.e. $φ_2 = 0$ is meaningless since in our formalism that would mean to consider two Fock spaces, hence no cubic interaction vertex. It should also be noted that for $φ_1 = φ_2$ (4.53) is equivalent to the linearized interaction of a scalar field with gravity. The generalization for the coupling of a spin-2 triplet with an arbitrary number of scalar fields $n$ goes in an analogous manner.

In $AdS_D$ we replace ordinary derivatives with covariant ones. There will be no other changes for the gauge transformation rules (4.56) (i.e., for all fields $δ_{AdS} = δ$) except for

$$ δ_{AdS}C_μ = δC_μ + \frac{1 - D}{L^2}λ_μ. $$

The free Lagrangian is modified to include the standard AdS “mass-terms” of order $1/L^2$

$$ ΔL_{\text{free}} = -\frac{1}{L^2}(2h_μ^νh_ν^μ - 16h_μ^νD + 2h_μνh_ν^μ + (4D + 12)D^2 + (2D - 6)(φ_1^2 + φ_2^2)). $$
The interaction part also changes and gets an additional piece

$$\Delta L_{\text{int.}} = C_{2,0} \frac{\mathcal{D} - 1}{L^2} D\phi_1 \phi_2. \quad (4.59)$$

This is an additional interaction of the $D$ scalar with a “spin-0” current.

**Spin-3 triplet with two scalars**

The spin-3 triplet is described by the field $[18]$

$$|\Phi_3\rangle = \left( \frac{1}{3!} h_{\mu \rho \sigma}(x) \alpha^\mu \alpha^\rho \alpha^\sigma + D_\mu(x) \alpha^\mu + b^\mu \right) |0\rangle, \quad (4.60)$$

and

$$|\Lambda\rangle = \frac{i}{2} \lambda_{\mu \nu \rho} (x) \alpha^\mu \alpha^\nu \alpha^\rho |0\rangle. \quad (4.61)$$

Again, solving the BRST invariance condition for the vertex at $K = 3$ we get the relevant gauge transformations

$$\delta \phi_1 = 3i \ C_{3,0} \left( 4 \lambda^\mu \partial_\mu \partial_\nu \phi_2 + \phi_2 \partial_\mu \partial_\nu \lambda^\mu + 4 (\partial_\mu \phi_2)(\partial_\nu \lambda^\mu) \right) + i \ C_{3,1} \ \phi_2 \lambda_\nu, \quad (4.62)$$

$$\delta \phi_2 = -3i \ C_{3,0} \left( 4 \lambda^\mu \partial_\mu \partial_\nu \phi_1 + \phi_1 \partial_\mu \partial_\nu \lambda^\mu + 4 (\partial_\mu \phi_1)(\partial_\nu \lambda^\mu) \right) - i \ C_{3,1} \ \phi_1 \lambda_\mu, \quad (4.63)$$

$$\delta h_{\mu \nu \rho} = \partial_\mu \lambda_\nu + \partial_\nu \lambda_\mu + \partial_\rho \lambda_\mu, \quad \delta C_\mu^\nu = \Box \lambda_\mu \nu, \quad \delta D_\mu = \partial_\nu \lambda_\mu. \quad (4.64)$$

and the free and interacting parts of the Lagrangian

$$L_{\text{free}} = \left( \partial_\mu \phi_1 \right) \left( \partial^\mu \phi_1 \right) + \left( \partial_\mu \phi_2 \right) \left( \partial^\mu \phi_2 \right) + m^2 \left( \phi_1^2 + \phi_2^2 \right) + \left( \partial_\mu h_{\nu \rho} \right) \left( \partial^\mu h_{\nu \rho} \right)$$

$$+ \left[ -6 (\partial_\mu h_{\nu \rho}) C_\mu^{\rho \nu} - 12 (\partial_\mu C_\nu^{\rho \mu}) D_\mu - 6 (\partial_\mu D_\nu) (\partial^\mu D^\nu) + 3 C_\mu C_\mu \right], \quad (4.65)$$

$$L_{\text{int.}} = i \ C_{3,0} \left( h_{\mu \nu \rho} \partial_\mu \partial_\nu \partial_\rho \phi_2 - h_{\mu \nu \rho} \partial_\mu \partial_\nu \partial_\rho \phi_1 - 3 h_{\mu \nu \rho} (\partial_\mu \partial_\nu \phi_2)(\partial_\rho \phi_1) \right)$$

$$+ 3 h_{\mu \nu \rho} (\partial_\mu \partial_\nu \phi_2)(\partial_\rho \phi_2) + i \ C_{3,1} \left( h_{\nu}^{\mu \rho} - 2 D^\rho \right) (\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1) + h.c., \quad (4.66)$$

where $C_{3,0}$ and $C_{3,1}$ are arbitrary pure imaginary constants. Note that in this case, had we set $\phi_1 = \phi_2$ the interaction would have vanished. Unlike the previous example for the case of an interacting triplet with higher spin 3 with two scalars one cannot put the scalars $\phi_1$ and $\phi_2$ to be equal to each other so one needs a complex scalar in analogy with scalar electrodynamics. There is one more difference with respect to the previous example, namely when doing the deformation to the $AdS_D$ case, apart from changing ordinary derivatives to covariant ones, both the Lagrangian and gauge transformation rules for scalars get deformed

$$\Delta L_{\text{free}} = -\frac{1}{L^2} (6 h^{\mu \rho} h_{\nu \rho} - 48 h^{\mu \nu} D_\nu - (\mathcal{D} - 3) h_{\mu \nu} h^{\mu \nu} +$$

$$+ 18 (\mathcal{D} + 3) D^\mu D_\mu + (2 \mathcal{D} - 6) (\phi_1^2 + \phi_2^2)) \quad (4.67)$$

$$\Delta L_{\text{int.}} = i \ C_{3,0} \frac{6 \mathcal{D}}{L^2} D^\mu (\phi_1 \nabla_\mu \phi_2 - \phi_2 \nabla_\mu \phi_1) + h.c. \quad (4.68)$$

$$\delta_{AdS} \phi_1 = \delta_0 \phi_1 - i \ C_{3,0} \frac{6}{L^2} \lambda_\mu^\nu \phi_2, \quad \delta_{AdS} \phi_2 = \delta_0 \phi_2 + i \ C_{3,0} \frac{6}{L^2} \lambda_\mu^\phi \phi_1, \quad (4.69)$$

$$\delta_{AdS} C_\mu^\nu = \delta C_\mu^\nu + \frac{2(1 - \mathcal{D})}{L^2} \lambda_\mu^\nu + \frac{2}{L^2} g_{\mu \nu} \lambda_\rho^\rho. \quad (4.70)$$
4.3 An exact vertex

In this subsection we will give a solution to the cubic vertex which is exact to all orders in the constant $g$ [17]. We begin first with the simple case of a vertex for totally symmetric fields. This means we consider only one set of oscillators as in (2.53). The form of the vertex can be deduced from the high energy limit of the corresponding vertex of OSFT. In bosonic OSFT the cubic vertex has the form

$$|V_3\rangle = \int dp_1 \ dp_2 \ dp_3 \ (2\pi)^d \ \delta^d(p_1 + p_2 + p_3) \times \exp \left( \frac{1}{2} \sum_{i,j=1}^{3} \sum_{n,m=0}^{\infty} \alpha_{n,\mu}^{+,i} N_{nm}^{ij} \alpha_{m,\nu}^{+,j} \eta^{\mu\nu} + \sum_{i,j=1}^{3} \sum_{n \geq 1, m \geq 0} c_{n}^{+,i} X_{nm}^{ij} b_{m}^{+,j} \right) |-\rangle_{123}, \tag{4.71}$$

where the solution is given in terms of the Neumann coefficients and all string modes contribute. The oscillators $\alpha_{n,\mu}^{+,i}$ are proportional to the momenta $p_{\mu}^{i}$. The vertex is invariant under the action of the BRST charge (2.49). In addition, the action (4.2) with the vertex (4.71) is invariant under the gauge transformations (4.3) to all orders in $g$.

Furthermore, since the BRST charge can be truncated to contain any finite number of oscillator variables [11], it is possible to look for the BRST invariant vertex that describes the interaction among only totally symmetric tensor fields of arbitrary rank, without the inclusion of modes with mixed symmetries. One possibility is to start from the SFT vertex (4.71) and keep in the exponential only terms proportional to at least one momentum $p_{\mu}^{i}$, therefore dropping all trace operators $(\alpha_{r}^{+,i} \eta^{\mu\nu} \alpha_{s}^{+,j})$, as one does when obtaining the BRST charge (2.51) from (2.49) since they are leading in the $\alpha' \rightarrow \infty$ limit. However, since these terms are exponentiated and the term $\alpha_{n,\mu}^{+,i} N_{\alpha\beta} R_{\mu}^{s} R_{\beta}^{s}$ is of the same order as $\alpha_{n,\mu}^{+,i} N_{\alpha\beta} R_{\mu}^{s} R_{\beta}^{s} (\alpha_{r}^{+,i} N_{\alpha\beta} R_{\mu}^{s} R_{\beta}^{s})^{p}$, $m, n \geq 1$, a priori one can keep them both. The same is true regarding the ghost part where, although the term $c_{n}^{+,i} b_{m}^{s}$ is leading compared to the term $c_{n}^{+,i} X_{nm}^{ij} b_{m}^{s}$, $n, m \geq 1$, one can not neglect the latter one in the exponential. Let us stress that all these terms will be essential to maintain the off shell closure of the algebra of gauge transformations and complete gauge invariance of the action.

Based on the discussion above one can make the following ansatz for the vertex which describes interactions between massless totally symmetric fields with an arbitrary spin

$$|V\rangle = V^1 \times V^{\text{mod}} |-\rangle_{123}, \tag{4.72}$$

where the vertex contains two parts: a part considered in [44]

$$V^1 = \exp \left( Y_{ij} l^{+,ij} + Z_{ij} \beta^{+,ij} \right), \tag{4.73}$$

and the part which ensures the closure of the nonabelian algebra

$$V^{\text{mod}} = \exp \left( S_{ij} \gamma^{+,ij} + P_{ij} M^{+,ij} \right), \tag{4.74}$$
where $P_{ij} = P_{ji}$. Putting this ansatz into the BRST invariance condition and using momentum conservation $p^{1}_{\mu} + p^{2}_{\mu} + p^{3}_{\mu} = 0$ one can obtain a solution for $Y^{rs}$ and $Z^{rs}$

\begin{equation}
Z_{i,i+1} + Z_{i,i+2} = 0 \tag{4.75}
\end{equation}

\begin{align*}
Y_{i,i+1} &= Y_{ii} - Z_{ii} - 1/2(Z_{i,i+1} - Z_{i,i+2}) \\
Y_{i,i+2} &= Y_{ii} - Z_{ii} + 1/2(Z_{i,i+1} - Z_{i,i+2}).
\end{align*}

\begin{align*}
S_{ij} &= P_{ij} = 0 \quad i \neq j \\
P_{ii} - S_{ii} &= 0 \quad i = 1, 2, 3. \tag{4.76}
\end{align*}

In what follows we will assume cyclic symmetry in the three Fock spaces which implies along with (4.75)

\begin{align*}
Z_{12} &= Z_{23} = Z_{31} = Z_a, \quad Z_{21} = Z_{13} = Z_{32} = Z_b = -Z_a \tag{4.77} \\
Y_{12} &= Y_{23} = Y_{31} = Y_a, \quad Y_{21} = Y_{13} = Y_{32} = Y_b \\
Y_{ii} &= Y, \quad Z_{ii} = Z, \quad P_{ii} = S_{ii} = S = P
\end{align*}

Having determined the form of the vertex from (4.75) and (4.76) we will proceed in computing the commutator of two gauge transformations with gauge parameters $|\Xi\rangle$ and $|\Lambda\rangle$. In general, closure of the algebra to order $O(g)$ implies

\begin{equation}
[\delta_\Lambda, \delta_\Xi]|\Phi_1\rangle = \delta_\Lambda|\Phi_1\rangle = Q_1|\Lambda_1\rangle - g[(\langle \Phi_2 | \langle \Lambda_3 | + \langle \Phi_3 | \langle \Lambda_2 |)|V\rangle] + O(g^2) \tag{4.78}
\end{equation}

where

\begin{equation}
|\Lambda_1\rangle = g(\langle \Lambda_2 | \langle \Xi_3 | + \langle \Lambda_3 | \langle \Xi_2 |)|V\rangle + O(g^2). \tag{4.79}
\end{equation}

It should be emphasized that unlike the case of free triplets where the total Lagrangian splits into an infinite sum of individual ones [11], for the case of interacting triplets the fields $|\Phi_i\rangle$ of (2.26) need to be composed of an infinite tower of higher spin triplet fields, at least when the vertex is defined via (4.72) or (4.73). In other words:

\begin{equation}
|\Phi_i\rangle \rightarrow \sum_{s=0}^\infty |\Phi_i^{(s)}\rangle. \tag{4.80}
\end{equation}

Let us give the full nonabelian gauge transformations based on (4.3)

\begin{align*}
\delta(|\varphi_1\rangle + c_1^+ b_1^+ |d_1\rangle + c_1^0 b_1^+ |c_1\rangle) &= (l_{11}^+ + l_{11}^- c_1^+ b_1^+ + c_1^0 b_1^+ l_{1}^0) |\lambda_1\rangle + \\
&+ g \epsilon^s c_i^+ b_i^+ \{ -Z_a(\langle \varphi_2 | \langle \lambda_3 | A ) - S \ (\langle d_2 | \langle \lambda_3 | A ) + \\
&+ (Z_a - Z_b^2) \langle c_2 | \langle \lambda_3 | A ) + (2 \leftrightarrow 3, \ Z_a \rightarrow Z_b) \}
\end{align*}

\begin{equation}
\text{where for convenience we have defined the matter part of the vertex}
\end{equation}

\begin{equation}
|A\rangle = exp \left( Y_{ij} t^{+;ij} + P \sum_{i=1}^3 M^{+,ii} \right) |0\rangle. \tag{4.82}
\end{equation}
From this transformation rule it is very simple to prove the exactness of the vertex. Namely, the crucial point is the presence of the term $e^{Sc_i b^+_i}$. Indeed, taking the variation with respect to say $|\Phi_1\rangle$ in the interaction term and considering the term of order $g^2$ one obtains a term $\langle 0_1 | e^{-Sc_i b^+_i}$. This term should be saturated by the term $e^{Sc_i b^+_i} | 0_1 \rangle$ from the vertex. Therefore, the whole expression vanishes if

$$|S|^2 = 1 \quad (4.83)$$

In a similar manner one can prove the closure of the algebra at order $g^2$. The commutator of two gauge transformations is

$$[\delta_{\Lambda}, \delta_{\Xi}] | \Phi_1 \rangle = Q_1 | \tilde{\Lambda}_1 \rangle \quad (4.84)$$

$$+ g^2 [(\langle V | (| \Phi_1 \rangle | \Lambda_3 \rangle + | \Lambda_1 \rangle | \Phi_3 \rangle ) \langle \Xi_3 | V \rangle + \langle V | (| \Phi_1 \rangle | \Lambda_2 \rangle + | \Lambda_1 \rangle | \Phi_2 \rangle ) \langle \Xi_2 | V \rangle$$

$$- \langle V | (| \Phi_1 \rangle | \Xi_3 \rangle + | \Xi_1 \rangle | \Phi_3 \rangle ) \langle \Lambda_3 | V \rangle - \langle V | (| \Phi_1 \rangle | \Xi_2 \rangle + | \Xi_1 \rangle | \Phi_2 \rangle ) \langle \Lambda_2 | V \rangle$$

where we have suppressed the integrations over the ghost fields of (4.3). One can show that the condition (4.83) leads to

$$[\delta_{\Lambda}, \delta_{\Xi}] | \Phi_1 \rangle = 0, \quad (4.85)$$

or rather to

$$\delta_{\Lambda} \delta_{\Xi} | \Phi_1 \rangle = 0. \quad (4.86)$$

In other words we can consider the vertex (4.72) as a field dependent deformation of the BRST charge in (2.51), which can be written schematically

$$Q' = Q + gV(\Phi) \quad (4.87)$$

with the nilpotency property

$$Q'^2 = Q^2 + 2gQV(\Phi) + g^2V(\Phi)^2 = 0. \quad (4.88)$$

Proceeding further in analogy with String Field Theory one can make both the string functional and gauge transformation parameters to be matrix valued (i.e., introduce Chan–Paton factors). The resulting theory will still satisfy (4.86).

The case of arbitrary mixed symmetry fields is completely analogous to the construction for totally symmetric fields. As in (4.72) we make the ansatz

$$V = \exp \left( \sum_{n=1}^{\infty} Y_{ij}^{(n)} b^{+,(n)}_{ij} + Z_{ij}^{(n)} b^{+,(n)}_{ij} \right) \times$$

$$\exp \left( \sum_{n,m=1}^{\infty} S_{ij}^{(nm)} \gamma^{+,(nm)}_{ij} + P_{ij}^{(nm)} M^{+,(nm)}_{ij} \right)$$

where in this case we are summing over $n, m$ as well. We put the oscillator level indices in parentheses in order to distinguish them from the Fock space ones. The oscillator algebra takes the form

$$[\alpha_{\mu}^{(m),i}, \alpha^{+,(n),j}] = \delta^{mn} \delta^{ij} g_{\mu\nu}, \quad \{ c^{+, (m), i}, b^{(n), j} \} = \{ c^{(m), i}, b^{+, (n), j} \} = \delta^{mn} \delta^{ij}. \quad (4.90)$$
BRST invariance with respect to (2.51) implies
\[ \sum_{i=1}^{3} \sum_{r=0}^{\infty} c^{+(r),i}(Y_{is}^{(r)}|0 - Z_{is}^{(r)}|0)\langle -|123 = 0 \] (4.91)
\[ \sum_{i=1}^{3} \sum_{r=0}^{\infty} \{ c^{+(r),i} \left( \frac{1}{2}(P_{il}^{(rs)} l^{+(s),li} + P_{ti}^{(sr)} l^{+(s),li}) + S_{ik}^{(rs)} l^{+(s),kk} \right) - b_{i}^{k} c^{+(p),i} e^{+(r),m} p S_{mi}^{(rp)} \}|-\rangle_{123} = 0 \] (4.92)
where the summation over repeated indexes is assumed. Solving (4.91) we get
\[ Z_{i,i+1}^{(r)} + Z_{i,i+2}^{(r)} = 0 \] (4.93)
\[ Y_{i,i+1}^{(r)} = Y_{ii}^{(r)} - Z_{ii}^{(r)} - \frac{1}{2}(Z_{i,i+1}^{(r)} - Z_{i,i+2}^{(r)}) \] (4.94)
\[ Y_{i,i+2}^{(r)} = Y_{ii}^{(r)} - Z_{ii}^{(r)} + \frac{1}{2}(Z_{i,i+1}^{(r)} - Z_{i,i+2}^{(r)}) \] (4.95)
Equation (4.92) gives
\[ S_{ij}^{(ps)} = P_{ij}^{(ps)} = 0 \quad i \neq j \text{ or } p \neq s \] (4.96)
\[ P_{ii}^{(ss)} - S_{ii}^{(ss)} = 0 \quad i = 1, 2, 3. \]
We can choose once more a cyclic solution in the three Fock spaces as in (4.77) and in this way get an obvious generalization of (4.93). Finally as in the case of only one oscillator, the complete invariance of the vertex requires.
\[ |S^{(r)}|^2 = 1, \] (4.97)
for all \( r \) being integer numbers.

Let us conclude with several remarks

- Dropping the cyclicity constraint does not seem to alter the conclusions. In this case we will have \( |S_{ii}|^2 = 1, \ i = 1, 2, 3. \)

- Despite the algebra being trivial it seems the vertex cannot be obtained from the free Lagrangian via some field redefinition. In other words the vertex (4.72) is not an exact cohomology state of the BRST charge (2.51): \( |V\rangle \neq Q|W\rangle \) for any \( |W\rangle \). One can show that only terms diagonal in the Fock spaces \( i, j \) can be removed from the exponent of (4.72) via a specific field redefinition scheme [13].

- The infinite tower of triplets is essential for the closure. The nonabelian part of the gauge transformation of each component of \( |\varphi\rangle \) is cancelled against the same rank tensor component of \( |d\rangle \). However, the two tensors belong to different triplets.
5 Fermions

In this section we briefly describe the generalization of the constructions given in the previous chapters to the case of the fermionic fields.

In order to describe massless reducible representations of the Poincare group with an arbitrary half-integer spin let us start with the open sector of the type-I superstring, (closed superstrings could be treated in a similar way). Let us first perform the \( \alpha' \to \infty \) limit in the BRST charge for the open superstring

\[
Q = \sum_{-\infty}^{+\infty} \left[ L_n C_n + G_r \Gamma_r - \frac{1}{2} (m-n) : C_m C_n B_{m+n} : 
+ \left( \frac{3n}{2} + m \right) : C_n B_m \Gamma_{m+n} : - \Gamma_n \Gamma_m B_{m+n} \right] - a C_0 ,
\]

where \( a \) is the intercept and the super-Virasoro generators

\[
L_k = \frac{1}{2} \sum_{l=-\infty}^{+\infty} \alpha_{k-l} \alpha_l + \frac{1}{4} \sum_r (2r-l) \psi_{l-r} \psi_r ,
\]

\[
G_r = \sum_{l=-\infty}^{+\infty} \alpha_l \psi_{r-l} ,
\]

obey the super-Virasoro algebra

\[
[L_k, L_l] = (k-l) L_{k+l} + \frac{D}{8} (k^3 - k) ,
\]

\[
[L_k, G_r] = \left( \frac{k}{2} - r \right) G_{k+r} ,
\]

\[
\{G_r, G_s\} = 2 L_{r+s} + \frac{D}{2} \left( r^2 - \frac{1}{4} \right) \delta_{rs} .
\]

Here \((k,l)\) are integers for both the Neveu-Schwarz (NS) and Ramond (R) sectors, while \((r,s)\) are integers for the R sector and half-odd integers for the NS sector, \(D\) denotes once more the space-time dimension \((D = 10 \text{ for the tensile string})\) and \(\alpha_0^\mu = \sqrt{2\alpha'} p^\mu\). The fermionic oscillators \(\psi_\mu^r\) and the ghosts \(\Gamma_r\) and antighosts \(B_r\) satisfy

\[
\{\psi_\mu^r, \psi_\nu^s\} = \delta_{r+s,0} \eta^{\mu\nu} , \quad [\Gamma_r, B_s] = i \delta_{r+s,0} ,
\]

and the intercept is \(a = 0\) in the R sector and \(a = \frac{1}{2}\) in the NS sector.

Rescaling the ghost variables as

\[
\gamma_{-r} = \sqrt{2\alpha'} \Gamma_{-r} , \quad \beta_r = \frac{1}{\sqrt{2\alpha'}} B_r
\]

and then taking the \(\alpha' \to \infty\) limit, one obtains the BRST charge for the NS sector

\[
Q_{NS} = c_0 \ell_0 + \tilde{Q}_{NS} - M_{NS} b_0 ,
\]
with
\[ \tilde{Q}_{NS} = \sum_{k \neq 0} \left[ c_{-k} \ell_k + \gamma_{-r} g_r \right], \]
\[ M_{NS} = \frac{1}{2} \sum_{-\infty}^{+\infty} \left[ k c_k c_k + \gamma_{-r} \gamma_r \right], \tag{5.7} \]
and
\[ g_r = p \cdot \psi_r. \tag{5.8} \]

In a similar fashion, the limiting BRST charge for the R sector reads
\[ \tilde{Q}_R = c_0 \ell_0 + \gamma_0 g_0 + \tilde{Q}_R - M_R b_0 - \frac{1}{2} \gamma_0^2 b_0, \tag{5.9} \]
where \( \tilde{Q}_R \) and \( M_R \) are again given by (5.7), the only difference being that their sums are over half-odd integer modes for fermionic Virasoro generators and bosonic (anti)ghosts. Both BRST charges are again identically nilpotent, independently of the space-time dimension \( D \).

For the type I superstring, the string field is invariant under the action of the BRST invariant GSO projection operators for the NS sector
\[ P_{NS} = \frac{1}{2} \left[ 1 - (-1)^{p^1} \psi_{p^1} + i \gamma_{-p^1} \beta_{-p^1} - i \gamma_{p^1} \beta_{p^1} \right] \tag{5.10} \]
and for the R sector
\[ P_R = \frac{1}{2} \left[ 1 + \gamma_{11} (-1)^{p^1} \psi_{p^1} + i \gamma_{p^1} \beta_{p^1} - i \gamma_{-p^1} \beta_{-p^1} + i \gamma_{0} \beta_{0} \right], \tag{5.11} \]
where \( \gamma_{11} \) is the ten-dimensional chirality matrix. Expanding the NS string field and gauge parameter in terms of the fermionic ghost zero mode as
\[ |\Phi_{NS}\rangle = |\Phi_{NS}^1\rangle + c_0 |\Phi_{NS}^2\rangle, \]
\[ |\Lambda_{NS}\rangle = |\Lambda_{NS}^1\rangle + c_0 |\Lambda_{NS}^2\rangle, \tag{5.12} \]
and making use of the BRST charge (5.6), one obtains the field equations
\[ \ell_0 |\Phi_{NS}^1\rangle - \tilde{Q}_{NS} |\Phi_{NS}^2\rangle = 0, \]
\[ \tilde{Q}_{NS} |\Phi_{NS}^1\rangle - M_{NS} |\Phi_{NS}^2\rangle = 0, \tag{5.13} \]
along with the gauge transformations
\[ \delta |\Phi_{NS}^1\rangle = \tilde{Q}_{NS} |\Lambda_{NS}^1\rangle - M_{NS} |\Lambda_{NS}^2\rangle, \]
\[ \delta |\Phi_{NS}^2\rangle = \ell_0 |\Lambda_{NS}^1\rangle - \tilde{Q}_{NS} |\Lambda_{NS}^2\rangle. \tag{5.14} \]

The R sector is more complicated, due to the presence of the bosonic ghost zero mode \( \gamma_0 \). However, one can work with the truncated string field
\[ |\Phi^R\rangle = |\Phi_1^R\rangle + \gamma_0 |\Phi_2^R\rangle + 2 c_0 g_0 |\Phi_2^R\rangle, \tag{5.15} \]
while still preserving the relevant portion of the gauge symmetry and, of course, not affecting the physical spectrum [97]. The resulting, consistently truncated, field equations

\begin{align*}
g_0 |\Phi_1^R\rangle + \tilde{Q}_R |\Phi_2^R\rangle &= 0 , \\
\tilde{Q}_R |\Phi_1^R\rangle - 2 M_R g_0 |\Phi_2^R\rangle &= 0 ,
\end{align*}

are then invariant under the gauge transformations

\begin{align*}
\delta |\Phi_1^R\rangle &= \tilde{Q}_R |\Lambda_1^R\rangle + 2 M_R g_0 |\Lambda_2^R\rangle , \\
\delta |\Phi_2^R\rangle &= g_0 |\Lambda_1^R\rangle - \tilde{Q}_R |\Lambda_2^R\rangle .
\end{align*}

(5.16)

\textbf{Symmetric spinor-tensors}

If, as for the bosonic string, one considers fields $|\Phi_{R,1}\rangle$ and $|\Phi_{R,2}\rangle$ depending only on the bosonic oscillator $\alpha^{\mu+}$ and on the fermionic ghost variables $c_{-1}$ and $b_{-1}$, the expansions

\begin{align*}
|\Phi_{1}^R\rangle &= \frac{1}{n!} \psi_{\mu_1 \mu_2 \ldots \mu_n} (x) \alpha^{\mu_1+} \alpha^{\mu_2+} \ldots \alpha^{\mu_n+} |0\rangle \\
&\quad + \frac{1}{(n-2)!} \lambda_{\mu_1 \mu_2 \ldots \mu_{n-2}} (x) \alpha^{\mu_1+} \alpha^{\mu_2+} \ldots \alpha^{\mu_{n-2}+} |0\rangle , \\
|\Phi_{2}^R\rangle &= - \frac{1}{\sqrt{2} (n-1)!} \chi_{\mu_1 \mu_2 \ldots \mu_{n-1}} (x) \alpha^{\mu_1+} \alpha^{\mu_2+} \ldots \alpha^{\mu_{n-1}+} |0\rangle
\end{align*}

(5.18)
define spinor-tensor fields $\psi$, $\chi$ and $\lambda$ totally symmetric in their tensor indices and of spin $(n+1/2)$, $(n-1/2)$ and $(n-3/2)$, respectively. Substituting these expressions in the field equations (5.16) then yields precisely the fermionic triplet equations of [9]:

\begin{align*}
\hat{\theta} \psi &= \partial \chi , \\
\partial \cdot \psi - \partial \lambda &= \hat{\theta} \chi , \\
\hat{\theta} \lambda &= \partial \cdot \chi ,
\end{align*}

(5.19)

where $\hat{\theta} = \gamma^\mu \partial_\mu$. The BRST gauge invariance involves an unconstrained parameter,

\begin{align*}
|\Lambda'_1\rangle &= \frac{1}{(n-1)!} \epsilon_{\mu_1 \mu_2 \ldots \mu_{n-1}} (x) \alpha^{\mu_1+} \alpha^{\mu_2+} \ldots \alpha^{\mu_{n-1}+} |0\rangle ,
\end{align*}

(5.20)

and determines the gauge transformations

\begin{align*}
\delta \psi &= \partial \epsilon , \\
\delta \Lambda &= \partial \cdot \epsilon , \\
\delta \chi &= \hat{\theta} \epsilon ,
\end{align*}

(5.21)

in agreement with [9].
Let us note, however, that the totally symmetric bosonic triplets do not arise directly in the NS sector of the open superstring, since all states containing only bosonic $\alpha^\mu$ oscillators and fermionic $b, c$ ghosts are eliminated by the GSO projection operator (5.10). However, they can emerge from tensors with mixed symmetry, or even directly if the GSO projection is modified to correspond to type-0 strings [98] (see [99] for a review). One can also consider generalized triplets for spinor-tensors in complete analogy to the case of the bosonic string [11].

**Space–time Supersymmetry**

Generalized triplets of mixed symmetry are actually the superpartners of symmetric fermionic triplets in the type-I superstring. Below we briefly outline how the supersymmetry for the triplets can be established.

We consider the case where the fields in the Ramond sector consist of totally symmetric fields, while the field in the NS sector, apart from the oscillators $\alpha^\mu, c$ and $b$, contain at most one creation operator $\psi_{1/2}^+$ along with the $\beta$ and $\gamma$ ghosts. This case corresponds to the $N = 1$ SUSY. Fixing the number of oscillator $\psi_{+}^{-\mu}$ and $\alpha^{+\mu}$, will determine the content of the generalized triplet. Namely, one can show that the bosonic ghosts and their conjugate momenta $\gamma_{1/2}^{+}$ and $\beta_{1/2}^{+}$ can appear only once in the expansion of $|\Phi^{NS}\rangle$. Then one can show that the total Lagrangian

$$L_{\text{tot.}} = \langle \Phi_{1}^{NS}| l_{0}| \Phi_{1}^{NS}\rangle - \langle \Phi_{2}^{NS}| \tilde{Q}_{NS}| \Phi_{1}^{NS}\rangle - \langle \Phi_{1}^{NS}| \tilde{Q}_{NS}| \Phi_{2}^{NS}\rangle + \langle \Phi_{2}^{NS}| M_{NS}| \Phi_{2}^{NS}\rangle + \langle \Phi_{1}^{R}| g_{0}| \Phi_{1}^{R}\rangle + \langle \Phi_{2}^{R}| \tilde{Q}_{R}| \Phi_{1}^{R}\rangle + \langle \Phi_{2}^{R}| \tilde{Q}_{R}| \Phi_{2}^{R}\rangle - 2\langle \Phi_{2}^{NS}| M_{R}g_{0}| \Phi_{2}^{R}\rangle$$

is invariant under the supersymmetry transformations

$$\delta|\Phi_{1}^{NS}\rangle = u^{+}|\Phi_{1}^{R}\rangle - \gamma_{1/2}^{+}u^{+}|\Phi_{2}^{R}\rangle, \quad \delta|\Phi_{2}^{NS}\rangle = 2u^{+}g_{0}|\Phi_{2}^{R}\rangle$$

$$\delta|\Phi_{1}^{R}\rangle = -2g_{0}u|\Phi_{1}^{NS}\rangle + \gamma_{1/2}u|\Phi_{2}^{NS}\rangle, \quad \delta|\Phi_{2}^{R}\rangle = u|\Phi_{2}^{NS}\rangle$$

with

$$u = \langle 0^{NS}| \exp \left(\psi_{0}^{\mu}\psi_{1/2}^{\mu} + \frac{i}{2} \gamma_{1/2}^{+}\beta_{1/2}\right)|0^{R}\rangle.$$  

(5.25)

In this manner the generalized triplet in the NS sector with physical field (top spin) which has one $\psi_{1/2}^{\mu}$ oscillator and $n$ bosonic $a^{+\mu}$ oscillators is superpartner of the R sector triplet in (5.18) with $n$ bosonic oscillators for the physical field (top spin). Therefore the expansion of the exponential in (5.25) will have a finite number of terms in its Taylor expansion.

One can possibly try to consider an arbitrary oscillator content in both sectors. For this reason one has to construct the operator which is similar to that of [97] (which is an analog of the fermion emission vertex operator of [100]) which transforms states of the NS sector to states of the R sector and vice versa. This operator should have the property

$$Q_{R}U = UQ_{NS}.$$  

(5.26)

However this problem along with the problem of finding irreducible supermultiplets for the generalized triplet is still open.

50
Compensator equations
Similarly to the case of bosonic fields, one can write the compensator equations for the fermionic fields as well. Namely, introducing the fermionic Fang-Fronsdal operator \[32\]
\[
S = i (\not{\partial} \psi - \partial \not{\psi})
\] (5.27)
and using the short-hand notations of section 1 one can write the compensator equations in the form
\[
S = -2i \partial^2 \xi, \\
\psi' = 2 \partial \cdot \xi + \partial \xi' + \not{\partial} \not{\xi},
\] (5.28)
where the field \(\xi\) is the compensator. These equations are then invariant under the gauge transformations
\[
\delta \psi = \partial \epsilon, \\
\delta \xi = \not{\epsilon},
\] (5.29)
involving an unconstrained gauge parameter, and are consistent, since the first implies the second via the Bianchi identity
\[
S - \frac{1}{2} \partial S' - \frac{1}{2} \not{\partial} S = i \partial^2 \psi'.
\] (5.30)
The corresponding Lagrangian description of compensator equations are derived in [19], while the AdS deformation of both fermionic and bosonic compensator equations can be found in [11].

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51
A Some Formulas in Ambient Space

For simplicity we shall put the radius of the AdS space equal to 1. One can check some useful relations for an ambient space

\[ \theta^{AB} \theta^{BC} = \theta^{AC}, \quad \nabla^C \theta^{AB} = \theta^{CA} y^B + \theta^{CB} y^A, \quad \nabla^A \theta^{AB} = \mathcal{D} y^B, \quad (A.1) \]

\[ [\nabla_A, \nabla_B] = -y_A \nabla_B + y_B \nabla_A, \quad (A.2) \]

\[ [\nabla^2, y^A] = 2 \nabla_A + \mathcal{D} y^A, \quad [\nabla^2, \nabla^A] = (2 - \mathcal{D}) \nabla^A + 2 y^A \nabla^2. \quad (A.3) \]

\[ y^A \nabla_A = 0, \quad y^A \theta_B^A = 0, \quad \nabla^A y_A = \mathcal{D}. \quad (A.4) \]

The induced metric, its inverse and Christoffel connection look as follows:

\[ g_{\mu\nu} = (\partial_\mu y^A) (\partial_\nu y^A), \quad g^{\mu\nu} = (\nabla^A x^\mu) (\nabla^A x^\nu), \quad \Gamma^\lambda_{\mu\nu} = \frac{\partial x^\lambda}{\partial y^A} \frac{\partial^2 y^A}{\partial x^\mu \partial x^\nu}. \quad (A.5) \]

We have also

\[ \theta^A_B = \frac{\partial y^A}{\partial x^\mu} \frac{\partial x^\mu}{\partial y^B}, \quad \delta^\mu_\nu = \frac{\partial y^A}{\partial x^\nu} \frac{\partial x^\mu}{\partial y^A}, \quad (A.6) \]

as well as

\[ \theta^{AB} = g^{\mu\nu} (\partial_\mu y^A) (\partial_\nu y^B), \quad (A.7) \]

which follow from the differentiation rules

\[ \theta^{AB} \frac{\partial}{\partial y^B} = \eta^{AC} \frac{\partial x^\mu}{\partial y^C} \frac{\partial}{\partial x^\mu}, \quad \frac{\partial}{\partial x^\mu} = \frac{\partial y^A}{\partial x^\mu} \frac{\partial}{\partial y^A}. \quad (A.8) \]

Finally, it is straightforward to derive the following relations

\[ \theta^{AB} \frac{\partial x^\mu}{\partial y^B} = g^{\mu\nu} \frac{\partial y^A}{\partial x^\nu}, \quad \nabla^2 x^\mu = -\Gamma^\mu_{\nu\rho} g^{\nu\rho}. \quad (A.9) \]

\[ \nabla^A \nabla_A \Phi_{C_1 C_2 \ldots C_s} = \nabla^\mu \nabla_\mu \Phi_{C_1 C_2 \ldots C_s}, \quad \nabla_\mu \frac{\partial y^A}{\partial x^\nu} = g_{\mu\nu} y^A. \quad (A.10) \]

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