Euler sums of generalized hyperharmonic numbers

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Abstract

This paper presents the evaluation of the Euler sums of generalized hyperharmonic numbers $H^{(p,q)}_n$

$$\zeta_{H^{(p,q)}} (r) = \sum_{n=1}^{\infty} \frac{H^{(p,q)}_n}{n^r}$$

in terms of the famous Euler sums of generalized harmonic numbers. Moreover, several infinite series, whose terms consist of certain harmonic numbers and reciprocal binomial coefficients, are evaluated in terms of Riemann zeta values.

Keywords: Harmonic numbers, hyperharmonic numbers, generalized harmonic numbers, Euler sums, Riemann zeta function, Stirling numbers.

MSC 2010: 11B83, 11M41, 11B73.

1 Introduction

The classical Euler sum $\zeta_H (r)$ is the following Dirichlet series

$$\zeta_H (r) = \sum_{n=1}^{\infty} \frac{H_n}{n^r},$$

where $H_n$ is the $n$th harmonic number. This series is also known as the harmonic zeta function. The famous Euler’s identity for this sum is [17, 25]

$$2\zeta_H (r) = (r + 2) \zeta (r + 1) - \sum_{j=1}^{r-2} \zeta (r - j) \zeta (j + 1), \quad r \in \mathbb{N} \setminus \{1\},$$

where $\zeta (r)$ is the classical Riemann zeta function. Many generalizations of Euler sums (the so called Euler-type sums) are given using generalizations of
harmonic numbers (see [3, 2, 6, 29, 30, 33, 35, 36, 37]). Evaluation of Euler-type sums and construction of closed forms are active fields of study in analytical number theory. Furthermore [5, 8, 11, 12] are some of the studies that make this area interesting in the sense that Euler sums have potential applications in quantum field theory and knot theory, especially in evaluation of Feynman diagrams.

Taking into account the \( n \)th partial sum of \( \zeta(p) \),

\[
H_n^{(p)} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p},
\]

which is called the \( n \)th generalized harmonic number, Euler considered general sums of the form \([9, 14, 17]\)

\[
\zeta_{H^{(p)}}(m) = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^m}. \tag{2}
\]

One of the most important issues here is to write Euler-type sums as combinations of the Riemann zeta function as in \((1)\). This problem has remained important for various Euler-type sums from the era of Euler to the present day. It’s shown by Euler himself that, the cases of \( p = 1, p = q, p + q \) odd, and for special pairs \((p, q) \in \{(2, 4), (4, 2)\}\), the sums of the form \((2)\) have evaluations in terms of the Riemann zeta function (see \([9, 14, 17]\)). There is a very comprehensive literature on this subject, both theoretical and numerical \((1, 4, 5, 14, 15, 16, 23, 30, 31, 32, 33, 34, 35, 36]\). One of these results; the Euler identity \((1)\) was further extended in the works of Borwein et al. \([4]\) and Huard et al. \([19]\). For odd weight \( N \geq 3 \) and \( p = 1, 2, \ldots, N - 2 \), we have \([19, \text{Theorem } 1]\) (or \([4, \text{p. } 278]\))

\[
\zeta_{H^{(p)}}(N - p) = (-1)^p \sum_{j=0}^{[(N-p-1)/2]} \binom{N-2j-1}{p-1} \zeta(N-2j) \zeta(2j)
\]

\[
+ (-1)^p \sum_{j=0}^{[p/2]} \binom{N-2j-1}{N-p-1} \zeta(N-2j) \zeta(2j) - \zeta(0) \zeta(N). \tag{3}
\]

Moreover, these so called "linear Euler sums" satisfy a simple reflection formula

\[
\zeta_{H^{(p)}}(r) + \zeta_{H^{(p)}}(p) = \zeta(p + r) + \zeta(p) \zeta(r). \tag{4}
\]

Considering nested partial sums of the harmonic numbers, Conway and Guy \([13]\) introduced hyperharmonic numbers for an integer \( r > 1 \) as

\[
h_n^{(r)} = \sum_{k=1}^{n} h_k^{(r-1)}, \tag{5}
\]

with \( h_n^{(1)} := H_n \). Hyperharmonic numbers are also important because they build a step in the transition to the multiple zeta functions (see \([21, 32]\)).
and Boyadzhiev [15] extended the Euler’s identity (1) to the Euler sums of the hyperharmonic numbers:

$$\zeta_{h(q)}(r) = \sum_{n=1}^{\infty} \frac{h^{(q)}_{n}}{n^r},$$

as

$$\zeta_{h(q)}(r) = \frac{1}{(q-1)!} \sum_{k=1}^{q} \left[ \frac{q}{k} \right]$$

$$\times \left\{ \zeta_H(r-k+1) - H_{q-1} \zeta(r-k+1) + \sum_{j=1}^{q-1} \mu(r-k+1, j) \right\},$$  \hspace{1cm} (6)

where $\left[ \frac{q}{k} \right]$ is the Stirling number of the first kind and

$$\mu(r, j) = \sum_{n=1}^{\infty} \frac{1}{n^r (n+j)} = \sum_{k=1}^{r-1} \frac{(-1)^{k-1}}{j^k} \zeta(r+1-k) + (-1)^{r-1} \frac{H_j}{j^r}.$$  \hspace{1cm} (7)

Formula (6) was the general form of the results obtained for some special values of $q$ and $r$ in the study of [23].

In this work we mainly concentrate on generalized hyperharmonic numbers defined as (see [16])

$$H^{(p,r)}_n = \sum_{k=1}^{n} H^{(p,r-1)}_k.$$  \hspace{1cm} (8)

These are a unified extension of generalized harmonic numbers and hyperharmonic numbers:

$$H^{(p,1)}_n = H^{(p)}_n \text{ and } H^{(1,r)}_n = h^{(r)}_n.$$  

Note that [31] shares the same title with the present study, however, generalized hyperharmonic numbers in the sense of Xu are different from $H^{(p,r)}_n$.

The main objective of this study is the evaluation of Euler sums of generalized hyperharmonic numbers

$$\zeta_{H(p,q)}(r) = \sum_{n=1}^{\infty} \frac{H^{(p,q)}_{n}}{n^r}.$$  \hspace{1cm} (9)

A step towards the solution of this problem is taken in [16]. However, the recurrence used by the authors did not return a closed formula. Without an available closed formula, they listed only the following few special cases

$$\zeta_{H^{(p,2)}}(r) = \zeta_{H^{(r-1)}}(p) - \zeta_{H^{(r)}}(p-1) + \zeta_{H^{(r)}}(p),$$

$$2\zeta_{H^{(p,3)}}(r) = 2\zeta_{H^{(r)}}(p) + 3\zeta_{H^{(r-1)}}(p) + \zeta_{H^{(r-2)}}(p) - 3\zeta_{H^{(r)}}(p-1)$$
Later, Göral and Sertbaş [18] showed that the Euler sums of generalized hyperharmonic numbers can be evaluated in terms of the Euler sums of generalized harmonic numbers and special values of the Riemann zeta function. However, their method does not determine the coefficients explicitly. This gap is filled in this study. The following recurrence relation for $H_{n}^{(p,q)}$ depending on the index $q$,

$$(q - 1) H_{n}^{(p,q)} = (n + q - 1) H_{n}^{(p,q-1)} - H_{n}^{(p-1,q-1)}$$

is obtained. Thanks to this recurrence relation, it is managed to obtain a closed formula for $H_{n}^{(p,q)}$ in terms of $H_{n}^{(p)}$ in Theorem 2. This enables the evaluation of Euler sums of generalized hyperharmonic numbers in terms of the Euler sums of generalized harmonic numbers as

$$
\zeta_{H_{(p,q+1)}}(r) = \frac{1}{q!} \sum_{m=0}^{q} \sum_{k=0}^{m} (-1)^{k} \left[ \frac{q + 1}{m + 1} \right] \left( \frac{m}{k} \right) \zeta_{H_{(p-k)}}(r + k - m).
$$

A demonstration of this formula is the following example

$$
\zeta_{H_{(6,5)}}(6) = \frac{25}{12} \zeta(11) + \frac{5}{3} \zeta(9) + \left( \frac{5\pi^{6}}{2268} + \frac{\pi^{4}}{72} \right) \zeta(5) - \frac{35}{24} (\zeta(5))^{2} - \frac{1}{6} \zeta(3) \zeta(5) + \frac{2 \pi^{6}}{2268} \zeta(3) + \frac{31 \pi^{8}}{1360800} + \frac{\pi^{10}}{58320} + \frac{703 \pi^{12}}{638512875}.
$$

In addition, a counterpart of the reflection formula (4) is obtained in the following form:

$$
\zeta_{H_{(p,q+1)}}(r) + \zeta_{H_{(r,q+1)}}(p).
$$

This formula serves to calculate sums similar to the foregoing example with less computational cost. Another contribution of this work is the evaluation of several infinite series, whose terms are generalizations of harmonic numbers and reciprocal binomial coefficients. For instance

$$
\sum_{n=1}^{\infty} \frac{H_{n}^{(p,q)}}{(n + m) \left( \frac{n + m + q}{q} \right)}
$$

is evaluated in terms of Riemann zeta values. These type of sums are in a close relation with hypergeometric series and are investigated in many researches such as [26, 27, 28, 34, 35].

2 Euler sums of generalized hyperharmonic numbers

In this section we present an evaluation formula for Euler sums $\zeta_{H_{(p,q)}}(r)$ under certain conditions. To state and prove our result we need some preliminaries.
Firstly, recall the polylogarithm defined by
\[ \text{Li}_p (t) = \sum_{k=1}^{\infty} \frac{t^k}{k^p}. \]

The generating function of the numbers \( H_n^{(p,q)} \) in terms of the polylogarithm is [16]
\[ \sum_{n=0}^{\infty} H_n^{(p,q)} t^n = \frac{\text{Li}_p (t)}{(1-t)^q}. \] (9)

Our first result presents the following reduction formula for \( H_n^{(p,q)} \).

**Lemma 1** Reduction relation for \( H_n^{(p,q)} \) in the index \( q \) is
\[ (q - 1) H_n^{(p,q)} = (n + q - 1) H_n^{(p,q-1)} - H_n^{(p-1,q-1)}. \] (10)

**Proof.** We define the polynomial \( H_n^{(p,q)} (z) \) as
\[ H_n^{(p,q)} (z) = \sum_{k=0}^{n} H_k^{(p,q)} z^k. \]

Considering (9), we obtain the ordinary generating function of \( H_n^{(p,q)} (z) \) as
\[ \sum_{n=0}^{\infty} H_n^{(p,q)} (z) t^n = \frac{\text{Li}_p (zt)}{(1-t)(1-zt)^q}. \] (11)

From (11), it can be seen that
\[ z \frac{d}{dz} H_n^{(p,q)} (z) = H_n^{(p-1,q)} (z) + qz H_n^{(p,q+1)} (z). \] (12)

On the other hand, we utilize (8) twice to find that
\[ H_n^{(p,q)} = \sum_{k=1}^{n} H_k^{(p,q-1)} = \sum_{k=1}^{n} \sum_{j=1}^{k} H_j^{(p,q-2)} \]
\[ = (n + 1) H_n^{(p,q-1)} - \sum_{j=1}^{n} j H_j^{(p,q-2)} \]
\[ = (n + 1) H_n^{(p,q-1)} - \frac{d}{dz} H_n^{(p,q-2)} (z) \bigg|_{z=1}, \]

or equivalently
\[ \frac{d}{dz} H_n^{(p,q-2)} (z) \bigg|_{z=1} = (n + 1) H_n^{(p,q-1)} - H_n^{(p,q)} \bigg|_{z=1} = n H_n^{(p,q-1)} - H_n^{(p,q)} - H_{n-1}^{(p,q)}. \] (13)

Therefore, (12) and (13) yield the desired formula. ■
The objective here is to express \(H_n^{(p,q)}\) in terms of \(H_n^{(p)}\). In [16] this relation is listed for at most \(q = 4\) due to the complexity of the process. However, the next result provides a general solution to this problem where the numbers \(H_n^{(p,q)}\) are expressed in terms of the numbers \(H_n^{(p)}\) and \(\left[ \begin{array}{c} q \\end{array} \right]_r\). Here \(\left[ \begin{array}{c} q \\end{array} \right]_r\) denotes the \(r\)-Stirling number of the first kind defined by the “horizontal” generating function [7, Theorem 21]

\[
(x + r) (x + r + 1) \cdots (x + r + q - 1) = \sum_{j=0}^{q} \left[ \begin{array}{c} q \\end{array} \right]_j x^j.
\]

(14)

**Theorem 2** Let \(p\) and \(q\) be integers with \(q \geq 0\). Then,

\[
q!H_n^{(p,q+1)} = \sum_{k=0}^{q} (-1)^k \left[ \begin{array}{c} q \\end{array} \right]_k H_n^{(p-k)}.
\]

(15)

**Proof.** We utilize (10) on the right-hand side of

\[
(q - 1) qH_n^{(p,q+1)} = (n + q) (q - 1) H_n^{(p,q)} - (q - 1) H_n^{(p-1,q)},
\]

and see that

\[
(q - 1) qH_n^{(p,q+1)} = H_n^{(p,q-1)} \{(n + q)(n + q - 1)\}
- H_n^{(p-1,q-1)} \{(n + q) + (n + q - 1)\} + H_n^{(p-2,q-1)}.
\]

Repeating this procedure we find that

\[
(q - 2) (q - 1) qH_n^{(p,q+1)}
= H_n^{(p,q-2)} \{(n + q)(n + q - 1)(n + q - 2)\}
- H_n^{(p-1,q-2)} \{(n + q)(n + q - 1) + (n + q)(n + q - 2) + (n + q - 1)(n + q - 2)\}
+ H_n^{(p-2,q-2)} \{(n + q) + (n + q - 1) + (n + q - 2)\}
- H_n^{(p-3,q-2)},
\]

and

\[
(q - 3) (q - 2) (q - 1) qH_n^{(p,q+1)}
= H_n^{(p,q-3)} \{(n + q)(n + q - 1)(n + q - 2)(n + q - 3)\}
- H_n^{(p-1,q-3)} \{(n + q)(n + q - 1)(n + q - 2) + (n + q)(n + q - 1)(n + q - 3)
+ (n + q)(n + q - 2)(n + q - 3) + (n + q - 1)(n + q - 2)(n + q - 3)\}
+ H_n^{(p-2,q-3)} \{(n + q)(n + q - 1) + (n + q)(n + q - 2) + (n + q)(n + q - 3)
+ (n + q - 1)(n + q - 2) + (n + q - 1)(n + q - 3) + (n + q - 2)(n + q - 3)\}
- H_n^{(p-3,q-3)} \{(n + q) + (n + q - 1) + (n + q - 2) + (n + q - 3)\}
\[ + H_n^{(p-4,q-3)}, \]

and finally

\[ q!H_n^{(p,q+1)} = \sum_{k=0}^{q} (-1)^k e_{q-k} (n + 1, \ldots, n + q) H_n^{(p-k)}. \]  

(16)

Here \( e_k (X_1, \ldots, X_q) \) denotes the \( k \)th elementary symmetric polynomial in variables \( X_1, \ldots, X_q \). These polynomials are defined by (see for example [22])

\[ e_0 (X_1, \ldots, X_q) = 1, \]

\[ e_k (X_1, \ldots, X_q) = \sum_{1 \leq j_1 < j_2 < \cdots < j_k \leq q} \prod_{i=1}^{k} X_{j_i}, \quad 1 \leq k \leq q, \]

\[ e_k (X_1, \ldots, X_q) = 0, \quad k > q, \]

and possess the identity

\[ \prod_{j=1}^{q} (x - X_j) = \sum_{j=0}^{q} (-1)^{q-j} e_{q-j} (X_1, \ldots, X_q) x^j. \]

Comparing this identity with (14) gives

\[ e_{q-j} (n + 1, n + 2, \ldots, n + q) = \left[ \begin{array}{c} q \\ j \end{array} \right]_{n+1}. \]  

(17)

Hence, (15) follows from (16) and (17). □

Now we give an expression for the \( r \)-Stirling numbers of the first kind in terms of the well-known Stirling numbers of the first kind and binomial coefficients.

**Lemma 3** We have

\[ \left[ \begin{array}{c} n \\ k \end{array} \right]_{r+1} = \sum_{m=k}^{n} \left[ \begin{array}{c} n + 1 \\ m + 1 \end{array} \right] \left( \begin{array}{c} m \\ k \end{array} \right) r^{m-k}. \]  

(18)

**Proof.** From [7, Theorem 15]

\[ \sum_{n=k}^{\infty} \left[ \begin{array}{c} n \\ k \end{array} \right]_{r+1} t^n = \left( -\ln (1 - t) \right)^k \frac{1}{k! (1 - t)^r} \frac{1}{(1 - t)} \]

\[ = \sum_{j=k}^{\infty} \left[ \begin{array}{c} j \\ k \end{array} \right] \frac{t^j}{r!} \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} \left( \sum_{n=0}^{\infty} \left[ \begin{array}{c} j \\ k \end{array} \right] \frac{1}{r!} \right) t^n, \]

we have

\[ \frac{1}{n!} \left[ \begin{array}{c} n \\ k \end{array} \right]_{r+1} = \sum_{j=k}^{n} \left[ \begin{array}{c} j \\ k \end{array} \right] \frac{1}{r!}. \]  

(19)
We now use \[24, \text{Theorem 2.6}\]

\[
\begin{bmatrix} n \\ k \end{bmatrix}_r = \sum_{m=k}^{n} \binom{m}{k} r^{m-k},
\]

in (19) and deduce that

\[
\frac{1}{n!} \begin{bmatrix} n \\ k \end{bmatrix}_{r+1} = \sum_{m=k}^{n} \binom{m}{k} r^{m-k} \sum_{j=m}^{n} \frac{1}{j!} \begin{bmatrix} j \\ m \end{bmatrix}.
\]

Utilizing (19) for \( r = 0 \) gives (18). □

Now, we are ready to state and prove our evaluation formula for \( \zeta_H(p,q) (r) \).

Thanks to this formula the evaluation of Euler sums of generalized hyperharmonic numbers reduces to the evaluation of Euler sums of generalized harmonic numbers.

**Theorem 4** For \( p, q \geq 1 \) and \( r > q + 1 \), we have

\[
\zeta_H(p,q+1) (r) = \frac{1}{q!} \sum_{m=0}^{q} \sum_{k=0}^{m} (-1)^k \binom{q+1}{m+1} \binom{m}{k} r^{m-k} \zeta_H(p-k) (r + k - m).
\]

**Proof.** From (15) and (18), we have

\[
H_n^{(p,q+1)} = \frac{1}{q!} \sum_{k=0}^{q} \binom{q}{k} r_{n+1}^{(p-k)} = \frac{1}{q!} \sum_{k=0}^{q} \sum_{m=k}^{q} (-1)^k \binom{q+1}{m+1} \binom{m}{k} r^{m-k} H_n^{(p-k)}.
\]

Multiplying both sides with \( n^{-r} \) and summing over \( n \) from 1 to \( \infty \), we deduce the desired result. □

As mentioned introductory the sums \( \zeta_H(p,q) (r) \) were listed up to \( q = 3 \) in [16]. With the help of Theorem 4 these sums can be evaluated for further choices of \( q \). For instance for \( q = 4 \) one can obtain:

\[
\zeta_H(p,4) (r) = \zeta_H(p) (r) + \frac{11}{6} \zeta_H(p) (r-1) + \zeta_H(p) (r-2) + \frac{1}{6} \zeta_H(p) (r-3)
\]

\[
- \frac{11}{6} \zeta_H(p-1) (r) - 2 \zeta_H(p-1) (r-1) - \frac{1}{2} \zeta_H(p-1) (r-2) + \zeta_H(p-2) (r)
\]

\[
+ \frac{1}{2} \zeta_H(p-2) (r-1) + \frac{1}{6} \zeta_H(p-3) (r).
\]

Hence, with the use of some values of \( \zeta_H(p) (r) \) listed in forthcoming Remark 1, a few concrete expressions of \( \zeta_H(p,q) (r) \) are:

- \( \zeta_H(1,4) (5) = \frac{11}{2} \zeta (5) - \left( 1 - \frac{11}{36} \pi^2 \right) \zeta (3) - \frac{1}{2} (\zeta (3))^2 - \frac{11}{216} \pi^2 - \frac{\pi^4}{810} + \frac{\pi^6}{540} \),
\[ \zeta_{H^{(2,4)}}(5) = -10 \zeta(7) + \left( \frac{5}{6} \pi^2 - \frac{21}{2} \right) \zeta(5) + \left( \frac{\pi^4}{45} + \frac{5}{6} \pi^2 + \frac{5}{12} \right) \zeta(3) + \frac{11}{4} \left( \zeta(3) \right)^2 + \frac{7\pi^4}{1080} - \frac{55\pi^6}{13608}, \]

\[ \zeta_{H^{(3,4)}}(5) = \zeta_{H^{(3)}}(5) + \frac{154}{3} \zeta(7) - 2 \left( \zeta(3) \right)^2 - \left( \frac{7\pi^2}{18} + \frac{11\pi^4}{270} \right) \zeta(3) - \left( \frac{55}{12} \pi^2 - \frac{51}{12} \right) \zeta(5) - \frac{\pi^4}{540} + \frac{\pi^6}{324}, \]

\[ \zeta_{H^{(4,4)}}(5) = -\frac{11}{6} \zeta_{H^{(3)}}(5) - \frac{125}{2} \zeta(9) + \left( \frac{35}{6} \pi^2 - 63 \right) \zeta(7) + \left( \frac{35}{6} \pi^2 + \frac{\pi^4}{18} \right) \zeta(5) + \frac{\pi^4}{30} \zeta(3) - \frac{\pi^6}{1944} + \frac{143\pi^8}{680400}, \]

\[ \zeta_{H^{(5,4)}}(5) = 231 \zeta(9) + \left( 21 - \frac{385}{18} \pi^2 \right) \zeta(7) - \left( \frac{11}{60} \pi^4 + \frac{23}{12} \pi^2 \right) \zeta(5) + \frac{1}{2} \left( \zeta(5) \right)^2 + \zeta(3) \zeta(5) - \frac{7\pi^4}{540} \zeta(3) - \frac{\pi^6}{8100} + \frac{\pi^{10}}{187110}. \]

The following corollary gives the reflection formula for Euler sums of generalized hyperharmonic numbers. Combined with (3), this corollary shows that \( \zeta_{H^{(p,q+1)}}(r) + \zeta_{H^{(r,q+1)}}(p) \) can be written as a combination of Riemann zeta values. In this way, particular Euler sums of type \( \zeta_{H^{(p,q)}}(p) \) can be evaluated with less computation.

**Corollary 5** Let \( p > q + 1, r > q + 1 \) and \( p + r \) be even. Then

\[ \zeta_{H^{(p,q+1)}}(r) + \zeta_{H^{(r,q+1)}}(p) = 2 \left( \frac{-1}{q!} \right)^p \sum_{m=0}^{q} \sum_{k=0}^{m} \left( \frac{q+1}{m+1} \right) \left( \begin{array}{c} m \\ k \end{array} \right) \zeta_{H^{(p-k)}}(r+k-m) + \zeta(2) + \frac{1}{q!} \sum_{m=0}^{q} \sum_{k=0}^{m} (-1)^{m+k} \left( \begin{array}{c} q+1 \\ m+1 \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) \zeta(p-k) \zeta(r+k-m). \]

**Proof.** Let \( (p+r) \) be even. It is obvious from Theorem 4 that

\[ \zeta_{H^{(p,q+1)}}(r) + \zeta_{H^{(r,q+1)}}(p) = \frac{1}{q!} \sum_{m=0}^{q} \sum_{k=0}^{m} (-1)^{k} \left( \begin{array}{c} q+1 \\ m+1 \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) \left\{ \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-m}} + (-1)^{m} \sum_{n=1}^{\infty} \frac{H_n^{(r-k)}}{n^{p-k}} \right\}. \]

We write the right-hand side as

\[ \sum_{m=0}^{q} \sum_{k=0}^{m} (-1)^{k} \left( \begin{array}{c} q+1 \\ m+1 \end{array} \right) \left( \begin{array}{c} m \\ k \end{array} \right) \left\{ \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-m}} - \sum_{n=1}^{\infty} \frac{H_n^{(r-k)}}{n^{p-k}} \right\}. \]
\[ + \sum_{0 \leq m \leq q/2} (-1)^k \left[ \frac{q + 1}{2m + 1} \right] \left( \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-2m}} + \sum_{n=1}^{\infty} \frac{H_n^{(r+k-2m)}}{n^{r-k}} \right) \cdot (2m)^k \].

By the reflection formula (4) we have
\[ \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-2m}} + \sum_{n=1}^{\infty} \frac{H_n^{(r+k-2m)}}{n^{r-k}} = \zeta(p + r - 2m) + \zeta(p - k) \zeta(r + k - 2m). \]

Moreover, for odd \( m \), it can be seen from (3) that
\[ \sum_{n=1}^{\infty} \frac{H_n^{(p-k)}}{n^{r+k-m}} - \sum_{n=1}^{\infty} \frac{H_n^{(r+k-m)}}{n^{r-k}} = 2(-1)^{p-k} \zeta(\zeta(p-k)) (r + k - m) - \zeta(p - k) \zeta(r + k - m). \]

Hence, we obtain the desired equation. 

**Remark 1** For interested readers we would like to list some values of \( \zeta_{H,(p)}(r) = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^r} \), used in the evaluations of \( \zeta_{H,(p,4)}(5) \), \( 1 \leq p \leq 5 \), and \( \zeta_{H,(6,5)}(6) \). These are calculated with the help of (1), (3) and (4).

- \( \zeta_{H,(1)}(2) = 2 \zeta(3) \), \( \zeta_{H,(2)}(2) = \frac{7 \pi^4}{360} \),
- \( \zeta_{H,(3)}(3) = \frac{\pi^4}{2} \), \( \zeta_{H,(2)}(3) = -\frac{9}{2} \zeta(5) + \frac{\pi^2}{2} \zeta(3) \),
- \( \zeta_{H,(4)}(4) = 3 \zeta(5) - \frac{\pi^2}{6} \zeta(3) \), \( \zeta_{H,(2)}(4) = \zeta^2(3) - \frac{\pi^6}{2880} \),
- \( \zeta_{H,(5)}(5) = -\frac{9}{2} \zeta^2(3) + \frac{\pi^6}{180} \), \( \zeta_{H,(2)}(5) = -10 \zeta(7) + \frac{5 \pi^2}{6} \zeta(5) + \frac{\pi^4}{18} \zeta(3) \),
- \( \zeta_{H,(3)}(6) = 3 \zeta(5) - \frac{\pi^2}{3} \zeta(3) \), \( \zeta_{H,(3)}(3) = -17 \zeta(7) + \frac{5 \pi^2}{3} \zeta(5) + \frac{\pi^4}{90} \zeta(3) \),
- \( \zeta_{H,(4)}(3) = \frac{1}{2} \zeta^2(3) + \frac{\pi^6}{1800} \), \( \zeta_{H,(4)}(4) = \frac{13 \pi^8}{113400} \),
- \( \zeta_{H,(5)}(4) = 18 \zeta(7) - \frac{5 \pi^2}{3} \zeta(5) \), \( \zeta_{H,(4)}(5) = -\frac{125}{2} \zeta(9) + \frac{35 \pi^2}{6} \zeta(7) + \frac{\pi^4}{18} \zeta(5) \),
- \( \zeta_{H,(6)}(4) = -\frac{\pi^2}{3} \zeta(3) + \frac{37 \pi^6}{13440} \), \( \zeta_{H,(5)}(5) = 11 \zeta(7) - \frac{25 \pi^2}{6} \zeta(5) - \frac{\pi^4}{6} \zeta(3) \),
- \( \zeta_{H,(5)}(4) = \frac{127}{2} \zeta(9) - \frac{35 \pi^2}{6} \zeta(7) - \frac{2 \pi^4}{45} \zeta(5) \), \( \zeta_{H,(5)}(5) = \frac{1}{2} \zeta(5)^2 + \frac{\pi^6}{187110} \).

### 3 Evaluation of further series

In this section, we introduce evaluation formulas for some specific series involving the harmonic numbers and their generalizations.

**Proposition 6** For all integers \( p \geq 1 \) and \( q \geq 0 \)
\[ \sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{n^{(p,q)}} = \zeta(p + 1), \]
and for $m \geq 1$,
\[
\sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{(n+m)^{(n+m+q)}} = (-1)^{p-1} \frac{H_m}{m^p} + \sum_{i=1}^{p-1} (-1)^{i-1} \frac{\zeta (p+1-i)}{m^i}.
\]  

(22)

Note that the variable $q$ does not appear in the right-hand sides.

Proof. Using the formula (see [20])
\[
\int_0^1 t^{n+m-1} (1-t)^q \, dt = \frac{1}{(n+m)^{(n+m+q)}},
\]  

(23)
we can write
\[
\frac{H_n^{(p,q)}}{(n+m)^{(n+m+q)}} = \int_0^1 t^{n+m-1} (1-t)^q \, dt.
\]

With the help of (9), we get
\[
\sum_{n=1}^{\infty} \frac{H_n^{(p,q)}}{(n+m)^{(n+m+q)}} = \int_0^1 t^{m-1} L_p(t) \, dt = \sum_{n=1}^{\infty} \frac{1}{n^p (n+m)}.
\]

For $m = 0$, we have (21). If $m \geq 1$, then (22) follows from (7).

Now, we deal with some special cases of this proposition. For $p = 1$ in (21), we have Proposition 6 of [15]. (22) also yields [34, Eq. (2.30)] for $q = 1$. Additionally, when $p = 1$ in (22), we reach the following:

**Corollary 7** For all integers $m \geq 1$ and $q \geq 0$
\[
\sum_{n=1}^{\infty} \frac{h_n^{(q)}}{(n+m)^{(n+m+q)}} = \frac{H_m}{m}.
\]  

(24)

It is worth noting that for $m = 1$, (24) reduces to [15, Proposition 5]
\[
\sum_{n=1}^{\infty} \frac{h_n^{(q)}}{n^{(n+q)}} = 1.
\]

The companion of this identity involving generalized harmonic numbers is given by Sofo [27, Theorem 2.2]
\[
\sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^{(n+q)}} = \sum_{i=1}^{q} (-1)^{i+1} \begin{pmatrix} q \cr i \end{pmatrix}
\times \left\{ \zeta (p+1) + (-1)^{p+1} \sum_{j=1}^{i-1} \frac{H_j}{j^p} + \sum_{l=1}^{p-1} (-1)^{l+1} \zeta (p+1-l) H_{l-1}^{(l)} \right\}.
\]  

(25)
The former identity is generalized by Xu et al. [34, Theorem 4.1].

The next proposition extends the formula (25) by introducing a shift \( m \) in the denominator. This new form can be thought of as an analogues of (24).

**Proposition 8** For all integers \( m, p, q \geq 1 \),

\[
\sum_{n=1}^{\infty} H^{(p)}_n \frac{1}{(n + m)(n + m + q)} = \sum_{j=0}^{q-1} \frac{(-1)^j}{j!(q - 1 - j)!} \left\{ \frac{(-1)^{p-1}}{(m + j)^p} H_{m+j} + \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{(m + j)^k} \zeta(p + 1 - k) \right\}, \tag{26}
\]

**Proof.** Using (23) and (9), we have

\[
\sum_{n=1}^{\infty} \frac{H^{(p)}_n}{(n + m)(n + m + q)} = \int_0^1 t^{m-1} Li_p(t)(1 - t)^{q-1} dt
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n^p (n + m)(n + m + 1) \cdots (n + m + q - 1)}
\]

\[
= \sum_{n=1}^{\infty} \frac{1}{n^p} \left( \frac{A_0(q)}{n + m} + \frac{A_1(q)}{n + m + 1} + \cdots + \frac{A_{q-1}(q)}{n + m + q - 1} \right)
\]

\[
= \sum_{j=0}^{q-1} A_j(q) \mu(p, m + j),
\]

where

\[
A_j(q) = \frac{(-1)^j}{j!(q - 1 - j)!}.
\]

Setting \( p = 1 \) in (26) and using [10, Eq.(18)]

\[
\sum_{k=0}^{m} (-1)^k \binom{m}{k} H_{n+k} \frac{1}{n+k} = H_{n+m} - H_m,
\]

we obtain that,

\[
\sum_{n=1}^{\infty} \frac{H^{(p)}_n}{(n + m)(n + m + q)} = \frac{(m - 1)!}{(m + q - 1)!} (H_{m+q-1} - H_{q-1}) = \frac{H^{(q)}_m}{m(q - 1)!(m + q - 1)^2}.
\]

In a similar fashion the following particular cases can be deduced.

\[
\sum_{n=1}^{\infty} \frac{H^{(2)}_n}{(n + m)(n + m + q)} = \frac{(m - 1)!\pi^2}{(m + q - 1)!6} - \frac{1}{(q - 1)!} \sum_{j=0}^{q-1} \frac{(-1)^j}{j!} \binom{q - 1}{j} H_{m+j} \frac{1}{(m + j)^2},
\]
\[ \sum_{n=1}^{\infty} \frac{H_n^{(3)}}{(n+m) \binom{n+m+q}{q}} = \frac{1}{(q-1)!} \sum_{j=0}^{q-1} (-1)^j \binom{q-1}{j} \frac{H_{m+j}}{(m+j)^3} \]

For our final results, we deal with the special case \( q = 2 \) of (22). By aid of (15), \( \frac{\zeta(1)}{r} = r (r + 1) \cdots (r + q - 1) \) and \( \frac{\zeta(1)}{r^q} = 1, \) (22) becomes

\[ \sum_{n=1}^{\infty} \frac{nH_n^{(p)}}{(n+m) \binom{n+m+2}{2}} = \sum_{n=1}^{\infty} \frac{H_n^{(p-1)}}{(n+m) \binom{n+m+2}{2}} - \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{(n+m) \binom{n+m+2}{2}} \]

Now, using (26) with \( q = 2 \) gives:

**Corollary 9** For all integers \( m, p \geq 1, \)

\[ \sum_{n=1}^{\infty} \frac{nH_n^{(p)}}{(n+m) (n+m+1) (n+m+2)} = \sum_{k=1}^{p-2} (-1)^{k-1} \binom{1}{2m+k} \frac{1}{2 (m+1)^{k+1}} \zeta(p-k) + \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{2 (m+1)^k} \zeta(p+1-k) \]

\[ - (-1)^p \cdot \frac{m+2}{2 (m+1)^p} H_{m+1} + (-1)^p \frac{H_m}{2m^{p-1}}. \]

Some special cases of the corollary are listed in the following:

\[ \sum_{n=1}^{\infty} \frac{nH_n}{(n+m) (n+m+1) (n+m+2)} = \frac{1}{2 (m+1)} (H_{m+1} + 1), \]

\[ \sum_{n=1}^{\infty} \frac{nH_n^{(2)}}{(n+m) (n+m+1) (n+m+2)} = - \frac{(m+2)}{2 (m+1)} H_{m+1} + \frac{1}{2m} H_m + \frac{\pi^2}{12 (m+1)^3}, \]

\[ \sum_{n=1}^{\infty} \frac{nH_n^{(3)}}{(n+m) (n+m+1) (n+m+2)} = \frac{\zeta(3)}{2 (m+1)} + \frac{m+2}{2 (m+1)^3} H_{m+1} + \frac{\pi^2}{12m (m+1)^2}. \]

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