Topology

Steenrod squares on conjugation spaces

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Abstract

We prove that the coefficients of the so-called conjugation equation for conjugation spaces in the sense of Hausmann–Holm–Puppe are completely determined by Steenrod squares. This generalises a result of V.A. Krasnov for certain complex algebraic varieties. It also leads to a generalisation of a formula given by Borel and Haefliger, thereby largely answering an old question of theirs in the affirmative.

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1. Statement of the results

Let \( X \) be a topological space with an involution \( \tau \). We look at \( X \) as a space with an action of the group \( G = \{1, \tau\} \).

We take cohomology with coefficients in \( \mathbb{F}_2 \) and consider the restriction map \( r: H^*(X) \to H^*(X^\tau) \), its equivariant counterpart \( r_G: H^*_G(X) \to H^*_G(X^\tau) = H^*(X^\tau) \otimes H^*(BG) \) and the canonical projection \( p: H^*_G(X) \to H^*(X) \). Recall that \( H^*(BG) = H^*(\mathbb{RP}^\infty) = \mathbb{F}_2[u] \) with \( \text{deg}(u) = 1 \).

According to Hausmann–Holm–Puppe [2], \((X, \tau)\) is called a conjugation space if \( H^{\text{odd}}(X) = 0 \) and if there exists a section \( \sigma: H^*(X) \to H^*_G(X) \) of \( p \) and a degree-halving isomorphism \( \kappa: H^{2n}(X) \to H^*(X^\tau) \) with the following property: for every \( x \in H^{2n}(X), n \in \mathbb{N} \), there exists elements \( y_1, \ldots, y_n \in H^*(X^\tau) \) such that the so-called conjugation equation holds:

\[
r_G(\sigma(x)) = \kappa(x)u^n + y_1u^{n-1} + \cdots + y_{n-1}u + y_n.
\]
A priori, $\sigma$ and $\kappa$ are only assumed to be additive, but the conjugation equation implies that they are in fact multiplicative and unique. There are many examples of conjugation spaces, including flag manifolds, co-adjoint orbits of compact Lie groups and (compact) toric manifolds, see [2].

For the conjugation space $\mathbb{C}P^k$, $k \leq \infty$, Hausmann–Holm–Puppe prove the formula $r_G(\sigma(v^n)) = (wu + w^2)^n$, where $v \in H^2(\mathbb{C}P^k)$ and $w \in H^1(\mathbb{R}P^k)$ denote the generators [2, Example 3.7]. In other words, $r_G(\sigma(v^n)) = (wu + \text{Sq}^1(w))^n$. The following result generalises this to an arbitrary conjugation space $X$.

**Theorem 1.1.** For every $x \in H^{2n}(X)$, $n \in \mathbb{N}$, one has

$$r_G(\sigma(x)) = \sum_{i=0}^{n} \text{Sq}^i(\kappa(x)) u^{n-i} =: \text{SQ}(\kappa(x)).$$

**Corollary 1.2.** For every $x \in H^{n}(X)$, one has

$$r(x) = \kappa(x)^2.$$  

We also show that the isomorphism $\kappa$ commutes with total Steenrod squares.

**Theorem 1.3.** For every $x \in H^*(X)$ one has

$$\kappa(\text{Sq}(x)) = \text{Sq}(\kappa(x)).$$

Note that the odd Steenrod squares of $x$ vanish since $H^*(X)$ is concentrated in even degrees. Hence, the above identity is equivalent to

$$\kappa(\text{Sq}^{2k}(x)) = \text{Sq}^{k}(\kappa(x)) \quad \text{for all } k \in \mathbb{N}. \quad (2)$$

2. Proofs

We denote the Steenrod algebra for the prime 2 by $A$.

**Lemma 2.1.** For every $n$ there exist universal elements $a_0, \ldots, a_n$, $b \in A$ such that for every conjugation space $X$ and every $x \in H^{2n}(X)$ one has

$$r_G(\sigma(x)) = \sum_{i=0}^{n} a_i(\kappa(x)) u^{n-i} \quad \text{and} \quad \kappa(\text{Sq}(x)) = b(\kappa(x)).$$

Moreover, $a_0 = 1$ and $a_1 = \text{Sq}^1$.

**Proof.** Since $\kappa$ is bijective, one can define, for every $X$, functions $a_i, b : H^*(X) \to H^*(X^\tau)$ such that the above identities hold. We show that they are (or, more precisely, come from) Steenrod squares, using that the restriction map $r_G$ commutes with all Steenrod squares. We write $\kappa(x) = z$.

We start by proving the claim about the $a_i$ by induction on $i$, beginning at $a_0(z) = z$. If $i > 0$ is even, we apply $\text{Sq}^{2k}$, where $k \leq i/2$ will be chosen later. By the Leray–Hirsch theorem, we can write

$$\text{Sq}^{2k}(\sigma(x)) = \sum_{l=-n}^{k} \sigma(x_l) u^{2(k-l)} \quad (3)$$

for some $x_l \in H^{2(n+l)}(X)$. Write $z_l = \kappa(x_l)$. The restriction $r_G(\sigma(x_l) u^{2(k-l)})$ has leading term $z_l u^{n+2k-l}$, while, by (1) and the Cartan formula, the leading power of $u$ in $\text{Sq}^{2k}(r_G(\sigma(x)))$ is at most $u^{n+2k}$. Hence, the summation in (3) is in fact only over $0 \leq l \leq k$.

We first compare coefficients of $u^{n+2k-l}$ in $r_G(\text{Sq}^{2k}(\sigma(x))) = \text{Sq}^{2k}(r_G(\sigma(x)))$. Using Eq. (3) and the formula [5, Lemma 2.4]

$$\text{Sq}^j(u^i) = \binom{i}{j} u^{i+j},$$
we get for $0 \leq l \leq k$
\[ z_l = \sum_{j=0}^{l} \binom{n-l+j}{2k-j} Sq^l(a_{l-j}(z)) + \sum_{j=1}^{l} a_j(z_{l-j}), \tag{4} \]
in particular
\[ z_0 = \binom{n}{2k} z. \]

Since $l < i$, this inductively shows $z_l = b_l(z)$ for some $b_l \in \mathcal{A}$. Comparing coefficients of $u^{n+2k-i}$ then gives
\[ \sum_{l=0}^{k} a_{i-l}(z_l) = \binom{n}{2k} a_i(z) + \sum_{l=1}^{k} a_{i-l}(b_l(z)) = \binom{n-i}{2k} a_i(z) + \sum_{j=1}^{2k} \binom{n-i+j}{2k} Sq^j(a_{i-j}(z)). \tag{5} \]

Now suppose that $k \leq i/2$ is such that
\[ \binom{n-i}{2k} \neq \binom{n}{2k}. \]

For instance, this is true if $2k$ is the largest power of 2 dividing $i$. (Recall that a binomial coefficient mod 2 is the product of the binomial coefficients taken for each pair of binary digits, cf. [5, Lemma I.2.6].) Then Eq. (5) can be solved for $a_i(z)$ and shows that $a_i(z)$ can be expressed in terms of repeated Steenrod squares of $z$.

For odd $i$, a similar (but simpler) reasoning based on commutativity with respect to $Sq^1$ gives $a_i(z) = Sq^1(a_{i-1}(z))$, in particular $a_1(z) = Sq^1(z)$.

Now that all $a_i(z)$ are known, we apply $Sq^{2k}$ for any $k$. Using the same notation as above, we have $Sq^{2k}(x) = p(Sq^{2k}(\sigma(x))) = x_k$. Comparing coefficients as before gives a formula for $b_l(z)$ similar to Eq. (4), but where the summation index $j$ starts at $l-n$ if $l > n$. Still, the equations can be recursively solved for $z_l$. Hence,
\[ \kappa(Sq(x)) = \kappa(x_0) + \cdots + \kappa(x_n) = b_0(z) + \cdots + b_n(z) = b(z). \] \qed

In principle, the preceding proof could be used to determine the coefficients of the conjugation equation completely (as well as those of $Sq^i(\sigma(x))$ for any $k$). We will take a less tedious approach which relies on the fact that suitable products of infinite-dimensional real projective space can “detect” Steenrod squares, cf. [5, Corollary I.3.3].

**Fact 2.2.** The restricted evaluation map $\mathcal{A}_{\leq n} \to H^*(\mathbb{RP}^\infty)^n$, $a \mapsto a(w \times \cdots \times w)$ is injective for any $n \in \mathbb{N}$.

**Proof of Theorem 1.1.** We want to show $r_G(\sigma(x)) = SQ(\kappa(x))$ for all cohomology classes of all conjugation spaces.

By Lemma 2.1 and Fact 2.2, it suffices to do so for $X = (\mathbb{CP}^\infty)^n$ (which is a conjugation space by [2, Proposition 4.5]) and $x$, the $n$-fold cross product of the generator $v$ because $X^r = (\mathbb{RP}^\infty)^n$ and $\kappa(x) = w \times \cdots \times w$ in this case. For $n = 1$ the identity is true since we already know $a_1$. The general case reduces to the case $n = 1$ because of the multiplicativity of the maps $\kappa$, $\sigma$, $r_G$ and $SQ$: writing $v_1 \in H^2(X)$ for the pull-back of $v$ induced by the projection $X \to \mathbb{CP}^\infty$ onto the $i$-th factor, we get
\[ r_G(\sigma(x)) = r_G(\sigma(v_1 \cdots v_n)) = r_G(\sigma(v_1)) \cdots r_G(\sigma(v_n)) = SQ(\kappa(v_1)) \cdots SQ(\kappa(v_n)) = SQ(\kappa(v_1 \cdots v_n)) = SQ(\kappa(x)). \] \qed

**Proof of Corollary 1.2.** We have for $x \in H^{2n}(X)$
\[ r(x) = r(p(\sigma(x))) = p(r_G(\sigma(x))) = p(Sq^2(\kappa(x))) = \kappa(x)^2. \] \qed

**Proof of Theorem 1.3.** As in the proof of Theorem 1.1, it suffices to show the claimed identity for $X = (\mathbb{CP}^\infty)^n$ and $x = v \times \cdots \times v$. Again, the general case can be reduced to $n = 1$, where we find
\[ \kappa(Sq(v)) = \kappa(v^2) = \kappa(v) + \kappa(v)^2 = Sq(\kappa(v)). \] \qed
3. Remarks

Let $X$ be a non-singular complex projective variety defined over the reals such that its real locus $X^\tau$ is non-empty. In what follows, all algebraic cycles in $X$ are understood to be defined over the reals. Borel and Haefliger have shown that if $H_*(X)$ and $H_*(X^\tau)$ are generated by algebraic cycles, then the restriction $\lambda$ of cycles in $X$ to their real locus induces a degree-halving isomorphism $H_2(X) \to H_2(X^\tau)$ respecting intersection products [1, §5.15]. They also show that if $H_*(X^\tau)$ is generated by algebraic cycles and $x \in H^*(X)$ is Poincaré dual to a linear combination of non-singular subvarieties, then the identity in Theorem 1.3 holds, and they ask whether it holds more generally [1, §5.17].

Krasnov has proved that for a variety $X$ as above, Theorem 1.1 holds for cohomology classes Poincaré dual to algebraic cycles, where $\kappa$ is the Poincaré transpose of $\lambda$ and $\sigma$ the canonical section [3, Theorem 4.2]. This implies that if $H_*(X)$ is generated by algebraic cycles, then so is $H_*(X^\tau)$ [4, Theorem 0.1]. Moreover, $X$ is a conjugation spaces in the sense of [2].

In a topological framework van Hamel has recently shown that certain topological manifolds with involutions are conjugation spaces [6, Theorem]. The necessary assumptions are formulated in terms of topological cycles.

The following simple example shows that in general the existence of a degree-halving multiplicative isomorphism $\kappa : H^*(X) \to H^*(X^\tau)$ by itself does not imply that $(X, \tau)$ is a conjugation space.

Example 1. Let $X = S^2 \times S^4$ be equipped with the componentwise involution $\tau$ which is the identity for $S^2$ and for $S^4$ has fixed point set $S^1$. So $X^\tau = S^2 \times S^1$. Clearly there is a degree-halving multiplicative isomorphism $\kappa : H^*(X) \to H^*(X^\tau)$. It is easy to check there is also a multiplicative section $\sigma : H^*(X) \to H^*_G(X)$. But $(X, \tau)$ is not a conjugation space: the restriction map

$$r_G : H^n_G(S^2 \times S^4) \cong H^n(S^2 \times S^4) \otimes \mathbb{F}_2[u] \to H^n(S^2 \times S^1) \otimes \mathbb{F}_2[u]$$

is given by $s_2 \otimes 1 \mapsto s_2 \otimes 1$ and $s_4 \otimes 1 \mapsto s_1 \otimes u^3$, where $s_n \in H^n(S^n)$ denotes the generator. Hence the conjugation equation does not hold. Of course, $S^2 \times S^4$ with the different componentwise involution $\tilde{\tau}$ which has $S^1 \subset S^2$ and $S^2 \subset S^4$ as fixed point sets (and hence $X^{\tilde{\tau}} = S^1 \times S^2 \cong X^\tau$) is a conjugation space.

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