Free energies and critical exponents of the $A_1^{(1)}, B_n^{(1)}, C_n^{(1)}$ and $D_n^{(1)}$ face models

M. T. Batchelor$^a$, V. Fridkin$^a$, A. Kuniba$^b$, K. Sakai$^b$ and Y.-K. Zhou$^a$

$^a$ Department of Mathematics, School of Mathematical Sciences, Australian National University, Canberra ACT 0200, Australia
$^b$ Institute of Physics, University of Tokyo, Komaba, Meguro-ku, Tokyo 153, Japan

March 23, 2022

Abstract

We obtain the free energies and critical exponents of models associated with elliptic solutions of the star-triangle relation and reflection equation. The models considered are related to the affine Lie algebras $A_1^{(1)}, B_n^{(1)}, C_n^{(1)}$ and $D_n^{(1)}$. The bulk and surface specific heat exponents are seen to satisfy the scaling relation $2\alpha_s = \alpha_b + 2$. It follows from scaling relations that in regime III the correlation length exponent $\nu$ is given by $\nu = (l + g)/2g$, where $l$ is the level and $g$ is the dual Coxeter number. In regime II we find $\nu = (l + g)/2l$.

KEYWORDS: Exactly solved models, inversion relations, free energy, critical exponents, scaling relations.

The formulation of the boundary version of the Yang-Baxter equation has provided a systematic framework for the investigation of integrable models with a boundary [1-3]. Recent attention has turned to exploiting boundary integrability to derive off-critical surface phenomena, such as the surface magnetization [4, 5], the surface free energy and related critical exponents [6-11]. For face models, integrability in the bulk is assured by solutions of the star-triangle relation (STR) [12]

$$\begin{align*}
\sum_g W(f \quad e \quad c \quad a \quad g \quad u \quad v) W(a \quad g \quad b \quad c \quad v) W(e \quad d \quad c \quad u \quad v) &= \\
\sum_g W(f \quad g \quad a \quad b \quad d \quad u \quad v) W(g \quad d \quad b \quad c \quad u) W(f \quad e \quad g \quad d \quad v).
\end{align*}$$

(1)

Integrability at a boundary requires the additional relation,

$$\begin{align*}
\sum_{fg} W(c \quad b \quad a \quad g \quad u \quad v) K(g \quad a \quad f \quad u) W(c \quad g \quad d \quad f \quad u \quad v) K(d \quad f \quad e \quad v) &= \\
\sum_{fg} K(b \quad a \quad f \quad v) W(c \quad b \quad a \quad g \quad u \quad v) K(g \quad f \quad e \quad u) W(c \quad g \quad d \quad e \quad u \quad v).
\end{align*}$$

(2)
which is the face formulation of the reflection equation (RE) [13, 14, 6, 15, 16].

Solutions to the RE, defining boundary weights, have recently been found for the $A^{(1)}_n$, $A^{(2)}_n$ and $X^{(1)}_n = B^{(1)}_n, C^{(1)}_n, D^{(1)}_n$ models [17], for which the solutions of the STR have been known for some time [18, 19]. Our interest here lies in the critical behaviour of the bulk and surface free energies of the elliptic face models associated with the algebras $A^{(1)}_n$ and $X^{(1)}_n$. These models include some well known models as special cases. For example, the Andrews-Baxter-Forrester (ABF) model [20] is related to $A^{(1)}_1 = C^{(1)}_1$.

Two inversion relations,

\[
\sum_{g} W\left(\begin{array}{ccc} a & g & b \\ c & d & u \end{array}\right) W\left(\begin{array}{ccc} a & d & c \\ g & u & -u \end{array}\right) = \delta_{bd}\theta(u),
\]

(3)

\[
\sum_{g} \left(\frac{G_g G_h}{G_a G_c}\right) W\left(\begin{array}{ccc} g & a & b \\ c & d & \lambda - u \end{array}\right) W\left(\begin{array}{ccc} g & b & a \\ c & u & \lambda + u \end{array}\right) = \delta_{bd}\theta(u),
\]

(4)

are satisfied by the bulk weights of the models under consideration. On the other hand, the diagonal solutions of the RE found in [17] fulfill the boundary crossing relation

\[
\sum_{g} \left(\frac{G_g G_h}{2 G_a G_c}\right)^{1/2} W\left(\begin{array}{ccc} a & b & c \\ c & u & 2u + \lambda \end{array}\right) K\left(\begin{array}{ccc} a & b & c \\ c & u & u + \lambda \end{array}\right) = \theta_s(u) \theta_s(-u).
\]

(5)

The $G_a$ are crossing factors [18]. The crossing parameter is given by $\lambda = -tg/2$ where the parameters $t, g$ are given in Table 1. The key ingredients are the inversion function $\theta(u)$ and the boundary crossing function $\theta_s(u)$.

The inversion relation method [12] has recently been applied to a number of models to obtain the off-critical surface free energy [7-11]. The unitarity relation

\[
T(u)T(u + \lambda) = \frac{\theta_s(u)\theta_s(-u)}{\theta(2u)} \theta^{2N}(u),
\]

(6)

for the transfer matrix eigenvalues $T(u)$ follow from the crossing unitarity relation and disregarding finite-size corrections. Define $T_b(u) = \kappa_b^{2N}$ and $T_s(u) = \kappa_s$, then the bulk and the surface free energies per site can be defined by $f_b(u) = -\log \kappa_b(u)$ and $f_s(u) = -\log \kappa_s(u)$, respectively.

The restricted-solid-on-solid (RSOS) models follow in a natural way from the unrestricted models that we have discussed so far. One introduces a positive integer $l$ and sets $L$ as specified in Table 1. Local state $a$ in the Boltzmann weights is taken as a level $l$ dominant integral weight of $A^{(1)}_n$ and $X^{(1)}_n$. In Table 1 we have also listed the levels under consideration. We do not treat $l = 1$ for the $B^{(1)}_n, D^{(1)}_n$ RSOS models as they are then completely frozen.

We begin with the $A^{(1)}_1$ model, for which

\[
\theta(u) = \frac{[1 + u][1 - u]}{[1]^{2}},
\]

(7)

and

\[
\theta_s(u) = \frac{[2 - 2u]}{[1]}.
\]

(8)

Here we define

\[
[u] = [u, p] = \vartheta_1(\pi u/L, p),
\]

(9)
where
\[ \vartheta_1(u, p) = 2p^{1/8} \sin u \prod_{n=1}^{\infty} (1 - 2p^n \cos 2u + p^{2n})(1 - p^n) \] (10)
is a standard elliptic theta-function of nome \( p = e^{2\pi i \tau} \). We first consider the model in regime III \((-1 < u < 0 \text{ and } 0 < p < 1)\).

After taking a convenient normalisation in (6) we have
\[ \kappa_b(u)\kappa_b(-1 + u) = \frac{[1 + u][1 - u]}{[1]^2} \] (11)
for the bulk and
\[ \kappa_s(u)\kappa_s(-1 + u) = \frac{[2 + 2u][2 - 2u]}{[2]^2} \] (12)
for the surface. To proceed, we introduce the new variables
\[ x = e^{-4\pi^2 \lambda/\epsilon}, \quad w = e^{-4\pi^2 u/\epsilon}, \quad q = e^{-4\pi^2 L/\epsilon}, \] (13)
where \( \lambda = -1 \) for the \( A_{1}^{(1)} \) model. The conjugate modulus transformation of the theta-function,
\[ \vartheta_1(\pi u/L, p) \sim E(w, q), \] (14)
is also required, where \( p = e^{-\epsilon/L} \) and
\[ E(z, y) = \prod_{n=1}^{\infty} (1 - y^{n-1}z)(1 - y^n z^{-1})(1 - y^n). \] (15)

We suppose that \( \kappa_b(w) \) is analytic and nonzero in the annulus \( 1 \leq w \leq x \) and Laurent expand \( \log \kappa_b(w) \) in powers of \( w \). Then matching coefficients in (11) we obtain
\[ f_b(u) = \sum_{n=-\infty}^{\infty} \frac{\sinh(4\pi^2 nu/\epsilon) \sinh\left(\frac{4\pi^2 n(1+u)}{\epsilon}\right) \cosh\left(\frac{2\pi^2 n(L-2)}{\epsilon}\right)}{n \sinh(2\pi^2 n L/\epsilon) \cosh\left(\frac{2\pi^2 n}{\epsilon}\right)} \] (16)
for the bulk free energy. In a similar manner, we obtain the surface free energy
\[ f_s(u) = \sum_{n=-\infty}^{\infty} \frac{\sinh(4\pi^2 nu/\epsilon) \sinh\left(\frac{4\pi^2 n(1+u)}{\epsilon}\right) \cosh\left(\frac{2\pi^2 n(L-4)}{\epsilon}\right)}{n \sinh(2\pi^2 n L/\epsilon) \cosh\left(\frac{4\pi^2 n}{\epsilon}\right)}. \] (17)

Now consider the \( A_{1}^{(1)} \) model in regime II \((0 < u < -1 + L/2 \text{ and } 0 < p < 1)\). In this case we need to modify for the appropriate analyticity strip, with
\[ \kappa_b(u)\kappa_b(-u) = \frac{[1 + u][1 - u]}{[1]^2} \] (18)
\[ \kappa_b(u)\kappa_b(L - 2 - u) = \frac{[2 + u][u]}{[1]^2} \] (19)
for the bulk and
\[ \kappa_s(u)\kappa_s(-1 + L/2 + u) = \frac{[2 + 2u][2 - 2u]}{[2]^2} \] (20)
for the surface. We assume that \( \kappa_{b}(u) \) and \( \kappa_{s}(u) \) are analytic and nonzero in this regime, and in a similar manner obtain

\[
f_{b}(u) = -\sum_{n=-\infty}^{\infty} \frac{\sinh\left(\frac{2\pi^2 nu}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n(L-3)}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n(L-1-u)}{\varepsilon}\right)}{n \sinh\left(\frac{2\pi^2 n L}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n L-2}{\varepsilon}\right)} + \sum_{n=-\infty}^{\infty} \frac{\sinh\left(\frac{2\pi^2 nu}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n(1+u)}{\varepsilon}\right)}{n \sinh\left(\frac{2\pi^2 n L}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n L-2}{\varepsilon}\right)},
\]

(21)

\[
f_{s}(u) = -\sum_{n=-\infty}^{\infty} \frac{\sinh\left(\frac{4\pi^2 nu}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n(L-2-2u)}{\varepsilon}\right) \cosh\left(\frac{2\pi^2 n(L-4)}{\varepsilon}\right)}{n \sinh\left(\frac{2\pi^2 n L}{\varepsilon}\right) \cosh\left(\frac{2\pi^2 n L-2}{\varepsilon}\right)}
\]

(22)

for the bulk and surface free energy.

Now consider the \( B_{n}^{(1)} \) and \( D_{n}^{(1)} \) models in regime III \( (\lambda < u < 0 \text{ and } 0 < p < 1) \) with \( \lambda \) as given in Table 1. For these models the inversion and boundary crossing functions are given by

\[
\varphi(u) = \frac{[\lambda + u][\lambda - u][1 + u][1 - u]}{[\lambda]^2[1]^2},
\]

(23)

\[
\varphi_{s}(u) = \frac{[2\lambda + 2u][1 - \lambda - 2u]}{[\lambda][1]}
\]

(24)

After appropriate normalization, we have

\[
\kappa_{b}(u)\kappa_{b}(\lambda + u) = \frac{[-\lambda + u][-\lambda - u][1 + u][1 - u]}{[-\lambda]^2[1]^2}
\]

(25)

for the bulk and

\[
\kappa_{s}(u)\kappa_{s}(\lambda + u) = \frac{[-2\lambda + 2u][-2\lambda - 2u][1 - \lambda + 2u][1 - \lambda - 2u]}{[-2\lambda]^2[1 - \lambda]^2}
\]

(26)

for the surface. Under the appropriate analyticity assumptions we obtain the bulk and surface free energies

\[
f_{b}(u) = -2 \sum_{n=-\infty}^{\infty} \frac{\sinh\left(\frac{2\pi^2 nu}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n(L+\lambda-1)}{\varepsilon}\right) \cosh\left(\frac{2\pi^2 n(L+\lambda+1)}{\varepsilon}\right)}{n \sinh\left(\frac{2\pi^2 n L}{\varepsilon}\right) \cosh\left(\frac{2\pi^2 n L}{\varepsilon}\right)}
\]

(27)

\[
f_{s}(u) = -2 \sum_{n=-\infty}^{\infty} \frac{\sinh\left(\frac{4\pi^2 nu}{\varepsilon}\right) \sinh\left(\frac{4\pi^2 n(L+\lambda-1)}{\varepsilon}\right) \cosh\left(\frac{4\pi^2 n(L+\lambda+1)}{\varepsilon}\right)}{n \sinh\left(\frac{4\pi^2 n L}{\varepsilon}\right) \cosh\left(\frac{4\pi^2 n L}{\varepsilon}\right)}
\]

(28)

Now consider the \( B_{n}^{(1)}, C_{n}^{(1)} \) and \( D_{n}^{(1)} \) models in regime II \( (0 < u < \lambda + L/2 \text{ and } 0 < p < 1) \). Similar to the \( A_{1}^{(1)} \) model the inversion relations are modified to

\[
\kappa_{b}(u)\kappa_{b}(-u) = \frac{[-\lambda + u][-\lambda - u][1 + u][1 - u]}{[-\lambda]^2[1]^2},
\]

(29)

\[
\kappa_{b}(u)\kappa_{b}(L + 2\lambda - u) = \frac{[u][-2\lambda + u][1 - \lambda + u][-1 - \lambda + u]}{[-\lambda]^2[1]^2}
\]

(30)

\[
\kappa_{s}(u)\kappa_{s}(\lambda + L/2 + u) = \frac{[-2\lambda + 2u][-2\lambda - 2u][1 - \lambda + 2u][1 - \lambda - 2u]}{[-2\lambda]^2[1 - \lambda]^2}
\]

(31)
From these relations we obtain

\[
    f_b(u) = -2 \sum_{n=-\infty}^{\infty} \frac{\sinh\left(\frac{2\pi^2 n u}{\varepsilon}\right) \cosh\left(\frac{2\pi^2 n (\lambda+1)}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n (L+\lambda-u)}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n (L+2\lambda-1)}{\varepsilon}\right)}{n \sinh\left(\frac{2\pi^2 n L}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n (L+2\lambda)}{\varepsilon}\right)}
\]

\[
    -2 \sum_{n=-\infty}^{\infty} \frac{\sinh\left(\frac{2\pi^2 n u}{\varepsilon}\right) \cosh\left(\frac{2\pi^2 n (\lambda+1)}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n (L-\lambda-1)}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n}{\varepsilon}\right)}{n \sinh\left(\frac{2\pi^2 n L}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n (L+2\lambda)}{\varepsilon}\right)},
\]

(32)

\[
f_s(u) = -2 \sum_{n=-\infty}^{\infty} \frac{\sinh\left(\frac{4\pi^2 n u}{\varepsilon}\right) \sinh\left(\frac{4\pi^2 n (L+2\lambda-2u)}{\varepsilon}\right) \cosh\left(\frac{2\pi^2 n (L+3\lambda-1)}{\varepsilon}\right) \cosh\left(\frac{2\pi^2 n (\lambda+1)}{\varepsilon}\right)}{n \sinh\left(\frac{2\pi^2 n L}{\varepsilon}\right) \sinh\left(\frac{2\pi^2 n (L+2\lambda)}{\varepsilon}\right)}
\]

(33)

for the bulk and surface free energies. Note that in this case we also obtain the free energies of the level \( l A_{2n-1} \) model, which corresponds to the level \( n C_l \) model on changing the signs of \( u \) and \( \lambda \), as follows from level-rank duality [19].

We are particularly interested in the critical behavior of these models as the nome \( p = e^{-\varepsilon/L} \to 0 \). In each case the singular term in the free energies is obtained by making use of the Poisson summation formula [12].

For the \( A_1 \) model in regime III we have

\[
f_b \sim \begin{cases} 
    p^{2-\alpha_b} \log p & \text{for } L = 2m \\
    \text{nsc} & \text{for } L = 2m+1 
\end{cases}
\]

(34)

for the bulk, where nsc denotes “no singular contribution” and \( m \) is some integer. The bulk specific heat exponent is given by

\[
    \alpha_b = 2 - \frac{L}{2}.
\]

(35)

For the surface free energy we find

\[
f_s \sim \begin{cases} 
    p^{2-\alpha_s} & \text{for } L = 2m+1 \\
    p^{2-\alpha_s} \log p & \text{for } L = 4m \\
    \text{nsc} & \text{for } L = 4m+2 
\end{cases}
\]

(36)

where the excess specific heat exponent is given by

\[
    \alpha_s = 2 - \frac{L}{4}.
\]

(37)

On the other hand, for the \( A_1 \) model in regime II we have

\[
f_b \sim \begin{cases} 
    p^{2-\alpha_b} & \text{for } L \neq 4 \\
    p^{2-\alpha_b} \log p & \text{for } L = 4 
\end{cases}
\]

(38)

for the bulk, with

\[
    \alpha_b = 2 - \frac{L}{L-2}.
\]

(39)

For the surface free energy,

\[
f_s \sim \begin{cases} 
    p^{2-\alpha_s} & \text{for } L \neq 4 \\
    p^{2-\alpha_s} \log p & \text{for } L = 4 
\end{cases}
\]

(40)
with
\[ \alpha_s = 2 - \frac{L}{2(L-2)}. \]  
(41)

The bulk results (34), (35), (38), (39) have been obtained for the ABF model [21], as have the surface results (36) and (37) [8, 11].

For the \( B_n^{(1)} \) and \( D_n^{(1)} \) models in regime III we have
\[ f_b \sim \begin{cases} 
  p^{2-\alpha_b} & \text{for } L \neq -2m\lambda, L \neq -2m\lambda + 1 \\
  p^{2-\alpha_b} \log p & \text{for } L = -2m\lambda \\
  \text{nsc} & \text{for } L = -2m\lambda + 1
\end{cases} \]  
(42)
with exponent
\[ \alpha_b = 2 + \frac{L}{2\lambda}. \]  
(43)

While for the surface energy,
\[ f_s \sim \begin{cases} 
  p^{2-\alpha_s} & \text{for } L \neq -4m\lambda, L \neq -4m\lambda + \lambda + 1 \\
  p^{2-\alpha_s} \log p & \text{for } L = -4m\lambda \\
  \text{nsc} & \text{for } L = -4m\lambda + \lambda + 1
\end{cases} \]  
(44)
with exponent
\[ \alpha_s = 2 + \frac{L}{4\lambda}. \]  
(45)

For the \( B_n^{(1)} \), \( C_n^{(1)} \) and \( D_n^{(1)} \) models in regime II we have
\[ f_b \sim \begin{cases} 
  p^{2-\alpha_b} & \text{for } \frac{L}{L+2\lambda} \neq m_1 \text{ and } \frac{2(\lambda+1)}{L+2\lambda} \neq 2m_2 - 1 \\
  p^{2-\alpha_b} \log p & \text{for } \frac{L}{L+2\lambda} = m_1 \text{ and } \frac{2(\lambda+1)}{L+2\lambda} \neq 2m_2 - 1 \\
  \text{nsc} & \text{for } \frac{2(\lambda+1)}{L+2\lambda} = 2m_2 - 1
\end{cases} \]  
(46)
with exponent
\[ \alpha_b = 2 - \frac{L}{L+2\lambda}. \]  
(47)

For the surface energy,
\[ f_s \sim \begin{cases} 
  p^{2-\alpha_s} & \text{for } \frac{L}{2(L+2\lambda)} \neq m_1 \text{ and } \frac{L+3\lambda-1}{L+2\lambda} \neq 2m_2 - 1 \text{ and } \frac{\lambda+1}{L+2\lambda} \neq 2m_3 - 1 \\
  p^{2-\alpha_s} \log p & \text{for } \frac{L}{2(L+2\lambda)} = m_1 \text{ and } \frac{L+3\lambda-1}{L+2\lambda} \neq 2m_2 - 1 \text{ and } \frac{\lambda+1}{L+2\lambda} \neq 2m_3 - 1 \\
  \text{nsc} & \text{for } \frac{L+3\lambda-1}{L+2\lambda} = 2m_2 - 1 \text{ or } \frac{\lambda+1}{L+2\lambda} = 2m_3 - 1
\end{cases} \]  
(48)
with exponent
\[ \alpha_s = 2 - \frac{L}{2(L+2\lambda)}. \]  
(49)

In the above \( m_1, m_2 \) and \( m_3 \) are arbitrary integers.
In this case the exponents of the level $l A_{2n-1}^{(2)}$ model follow under level-rank duality with the level $n C_i^{(1)}$ model on changing the sign of $\lambda$.

The bulk and surface specific heat exponents are seen to satisfy the relation $2\alpha_s = 2 + \alpha_b$. More generally, this relation can be inferred directly from a comparison of the singular behaviour of the functional relations for $\kappa_b(u)$ and $\kappa_s(u)$. The known scaling relations $\alpha_b = 2 - 2\nu$, $\alpha_s = \alpha_b + \nu$ are consistent with this relation and can be used to infer the value of the correlation length exponent $\nu$. These relations have been confirmed explicitly for the eight-vertex [7] and the CSOS [9] models for which the exponent $\nu$ is known. For the present models, in regime III we thus expect

$$\nu = -\frac{L}{4\lambda} = \frac{l + g}{2g} = \begin{cases} \frac{l + 2}{4} & \text{for } A_1^{(1)} \\ \frac{l + 2n - 1}{2(2n - 1)} & \text{for } B_n^{(1)} \\ \frac{l + 2n - 2}{4(n - 1)} & \text{for } D_n^{(1)} \end{cases}$$

(50)

In regime II

$$\nu = \frac{L}{2(L + 2\lambda)} = \frac{l + g}{2l} = \begin{cases} \frac{l + 2}{2l} & \text{for } A_1^{(1)} \\ \frac{l + 2n - 1}{2l} & \text{for } B_n^{(1)} \\ \frac{l + n + 1}{2l} & \text{for } C_n^{(1)} \\ \frac{l + 2n - 2}{2l} & \text{for } D_n^{(1)} \end{cases}$$

(51)

and

$$\nu = \frac{L}{2(L - 2\lambda)} = \frac{l + n + 1}{2n} \quad \text{for } A_{2n-1}^{(2)}.$$  

(52)

These results remain to be confirmed via a direct calculation of the correlation length.

Our results are consistent with a number of partial checks:

- **Regime III**
  (i) There is an equivalence at level $l$ between the $B_1^{(1)}$ model and the degree 2 fusion $A_1^{(1)}$ model. Formally setting $n = 1$ in the $B_n^{(1)}$ model we see that the bulk free energy of the level $l B_1^{(1)}$ model agrees with the result obtained for the level $l$ degree 2 fusion $A_1^{(1)}$ model [22]. The bulk free energy of the $B_n^{(1)}$ model at the critical point ($p \to 0$) is consistent with the result obtained from the string hypothesis [23, 24].
  (ii) Formally setting $n = 2$ in the $D_n^{(1)}$ model, the $D_2^{(1)}$ bulk and surface free energies agree with twice those of the $A_1^{(1)}$ RSOS model. This is due to the fact that $D_2^{(1)} = A_1^{(1)} \oplus A_1^{(1)}$. The bulk free energy of the $D_n^{(1)}$ model at the critical point is also consistent with the result from the string hypothesis. Further we can check that the exponent $2/\nu$ of the $D_n^{(1)}$ model is consistent with the result from the thermal scaling relation if we identify the dimension of the generalized $(1, 3)$ operator in the conformal field theory [25] as playing the role of the thermal operator.
- **Regime II**
(i) The bulk free energy of the level \( l B_1^{(1)} \) model agrees with the degree 2 fusion \( A_1^{(1)} \) model with level \( l \).

(ii) The results of the level \( l C_1^{(1)} \) model are consistent with those of the \((l+1)\)-state \( A_1^{(1)} \) model. The results of the level \( l C_n^{(1)} \) model are also consistent with those of the \((n+1)\)-state \( A_1^{(1)} \) model in regime III with nome \( p^2 \). This latter correlation length exponent has been recently obtained directly for the ABF model in regime III \cite{26}.

(iii) The bulk and surface free energies of the \( D_2^{(1)} \) model agree with twice those of the \( A_1^{(1)} \) RSOS model.

Acknowledgements
MTB and YKZ are supported by the Australian Research Council. VF is supported by an Australian Postgraduate Research Award.

References
[1] E. K. Sklyanin, J. Phys. A 21 (1988) 2375.
[2] L. Mezincescu and R. I. Nepomechie, Int. J. Mod. Phys. A 6 (1991) 5231.
[3] S. Ghoshal and A. Zamolodchikov, Int. J. Mod. Phys. A 9 (1994) 3841.
[4] M. Jimbo, R. Kedem, T. Kojima, H. Konno and T. Miwa, Nucl. Phys. B 441 (1995) 437.
[5] M. Jimbo, R. Kedem, H. Konno, T. Miwa and R. Weston, Nucl. Phys. B 448 (1995) 429.
[6] Y. K. Zhou, Nucl. Phys. B 458 (1996) 504.
[7] M. T. Batchelor and Y. K. Zhou, Phys. Rev. Lett. 76 (1996) 14; 2826 (E).
[8] Y. K. Zhou and M. T. Batchelor, Nucl. Phys. B 466 (1996) 488.
[9] Y. K. Zhou and M. T. Batchelor, J. Phys. A 29 (1996) 1987.
[10] M. T. Batchelor, V. Fridkin and Y. K. Zhou, J. Phys. A 29 (1996) L61.
[11] D. L. O'Brien, P. A. Pearce and R. E. Behrend, in Statistical Models, Yang-Baxter Equation and Related Topics, M. L. Ge and F. Y. Wu eds (Singapore, World Scientific, 1996) p 285
[12] R. J. Baxter, Exactly Solved Lattice Models in Statistical Mechanics (London, Academic, 1982).
[13] R. E. Behrend, P. A. Pearce and D. L. O'Brien, J. Stat. Phys. 84 (1996) 1.
[14] H. Fan, B. Y. Hou and K. J. Shi, J. Phys. A 28 (1995) 4743.
[15] P. P. Kulish, in Lecture Notes in Physics 466 (1996) 125.
[16] C. Ahn and W. M. Koo, Nucl. Phys. B 468 (1996) 461.
[17] M. T. Batchelor, V. Fridkin, A. Kuniba and Y. K. Zhou, Phys. Lett. B 376 (1996) 266.
[18] M. Jimbo, T. Miwa and M. Okado, Commun. Math. Phys. 116 (1988) 507.
[19] A. Kuniba, Nucl. Phys. B 355 (1991) 801.

[20] G. E. Andrews, R. J. Baxter and P. J. Forrester, J. Stat. Phys. 35 (1984) 193.

[21] K. Binder, in Phase Transitions and Critical Phenomena, Vol 8, ed C. Domb and J. L. Lebowitz (London, Academic, 1983).

[22] E. Date, M. Jimbo, A. Kuniba, T. Miwa, and M. Okado, Adv. Stud. Pure Math. 16 (1988) 17.

[23] A. Kuniba T. Nakanishi J. Suzuki, Int. J. Mod. Phys. A 9 (1994) 5215.

[24] A. Kuniba T. Nakanishi J. Suzuki, Int. J. Mod. Phys. A 9 (1994) 5267.

[25] T. Eguchi and S. K. Yang, Phys. Lett. B 224 (1989) 373.

[26] D. L. O’Brien and P. A. Pearce, cond-mat/9607033.

Table 1

| type | \( A_1^{(1)} \) | \( B_n^{(1)} \) (\( n \geq 2 \)) | \( C_n^{(1)} \) (\( n \geq 1 \)) | \( D_n^{(1)} \) (\( n \geq 3 \)) |
|------|-----------------|---------------------------------|---------------------------------|---------------------------------|
| level | \( l \geq 2 \)  | \( l \geq 2 \)                  | \( l \geq 1 \)                  | \( l \geq 2 \)                  |
| \( g \) | \( n + 1 \)     | \( 2n - 1 \)                   | \( n + 1 \)                     | \( 2n - 2 \)                    |
| \( t \) | 1               | 1                              | 2                              | 1                              |
| \( \lambda \) | -1              | \(-n + \frac{1}{2}\)          | \(-n - 1\)                     | \(-n + 1\)                     |
| \( L \) | \( l + 2 \)     | \( l + 2n - 1 \)              | \( 2(l + n + 1) \)             | \( l + 2n - 2 \)               |