SERIAL COALGEBRAS AND THEIR VALUED GABRIEL QUIVERS

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Abstract. We study serial coalgebras by means of their valued Gabriel quivers. In particular, Hom-computable and representation-directed coalgebras are characterized. The Auslander-Reiten quiver of a serial coalgebra is described. Finally, a version of Eisenbud-Griffith theorem is proved, namely, every subcoalgebra of a prime, hereditary and strictly quasi-finite coalgebra is serial.

Introduction

A systematic study of serial coalgebras was initiated in [4], where, in particular, it was shown that any serial indecomposable coalgebra over an algebraically closed field is Morita-Takeuchi equivalent to a subcoalgebra of a path coalgebra of a quiver which is either a cycle or a chain (finite or infinite) [4, Theorem 2.10]. In this paper, we take advantage of the valued Gabriel quivers associated to a coalgebra to characterize indecomposable serial coalgebras over any field (Theorem 1.5). In conjunction with localization techniques (see Section 2), this combinatorial tool allows to complete the study made in [4] in more remarkable aspects. Thus, in Section 3 we characterize Hom-computable serial coalgebras in the sense of [27] (Proposition 3.3), and representation-directed coalgebras (Proposition 3.4). Section 4 is devoted to describe the Auslander-Reiten quiver of the category of finite dimensional (right) comodules of a serial coalgebra.

It was observed in [4] that a consequence of [7, Corollary 3.2] is that the finite dual coalgebra of a hereditary noetherian prime algebra over a field is serial. In Section 5 we reconsider this result of Eisenbud and Griffith from the coalgebraic point of view: we prove, using the results developed in the previous sections, that any subcoalgebra of a prime, hereditary and strictly quasi-finite coalgebra is serial (Corollary 5.3).

Throughout we fix a field \( K \) and we assume \( C \) is a \( K \)-coalgebra. We refer the reader to the books [1], [17] and [29] for notions and notations about coalgebras. Unless otherwise stated, all \( C \)-comodules are right \( C \)-comodules. It is well-known that \( C \) has a decomposition, as right \( C \)-comodule,

\[
C_C = \bigoplus_{i \in I_C} E_i^{t_i},
\]

where \( \{E_i\}_{i \in I_C} \) is a complete set of pairwise non-isomorphic indecomposable injective right \( C \)-comodules and \( t_i \) is a positive integer for any \( i \in I_C \). This produces a decomposition

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of the socle of \( C \) (the sum of all its simple subcomodules), \( \text{soc} \ C \), as follows:

\[
\text{soc} \ C = \bigoplus_{i \in I_C} S_i^t_i,
\]

where \( \{S_i\}_{i \in I_C} \) is a complete set of pairwise non-isomorphic simple right \( C \)-comodules. It is easy to prove that

\[
t_i = \frac{\dim_K S_i}{\dim_K(\text{End}_C(S_i))}
\]

for any \( i \in I_C \), see [25].

For any right \( C \)-comodule \( M \), we denote by \( \text{soc} \ M \) the socle of \( M \) and by \( E(M) \) its injective envelope. We assume that \( \text{soc} \ E_i = S_i \), for each \( i \in I_C \), and consequently, \( E(S_i) = E_i \).

Throughout we denote by \( G_i \) the division \( K \)-algebra of endomorphism \( \text{End}_C(S_i) \) for each \( i \in I_C \). The coalgebra \( C \) is said to be basic if \( t_i = 1 \) for any \( i \in I_C \), or, equivalently, if \( \dim_K S_i = \dim_K G_i \) for any \( i \in I_C \), or, equivalently, if \( S_i \) is simple subcoalgebra of \( C \) for any \( i \in I_C \), see for instance [28]. In particular, \( C \) is called pointed if \( \dim_K S_i = 1 \) for any \( i \in I_C \).

If the field \( K \) is algebraically closed then \( C \) is pointed if and only if \( C \) is basic (cf. [25, Corollary 2.7]).

Since every coalgebra is Morita-Takeuchi equivalent (that is, their categories of comodules are equivalent) to a basic one (cf. [6]), throughout we assume that \( C \) is basic and there are decompositions

\[
C = \bigoplus_{i \in I_C} E_i \quad \text{and} \quad \text{soc} \ C = \bigoplus_{i \in I_C} S_i,
\]

where \( E_i \not\cong E_j \) and \( S_i \not\cong S_j \) for \( i \neq j \). Symmetrically, there exists the left-hand version of all the facts explained above. In particular, \( C \) admits a decomposition as left \( C \)-comodule

\[
cC = \bigoplus_{i \in I_C} F_i \quad \text{and} \quad \text{soc} \ C = \bigoplus_{i \in I_C} S'_i.
\]

**Remark 0.1.** Observe that \( S_i = S'_i \) for any \( i \in I_C \), since \( C \) is basic, and therefore each simple (left or right) \( C \)-comodule is a simple subcoalgebra. Nevertheless, the right injective envelope \( E_i \) and the left injective envelope \( F_i \) of \( S_i \) could be different.

We recall from [4] that a right \( C \)-comodule \( M \) is said to be uniserial if its lattice of subcomodules is a chain. This property can be characterized through the socle filtration, namely, \( M \) has a filtration

\[
0 \subset \text{soc} \ M \subset \text{soc}^2 M \subset \cdots \subset M
\]

called the Loewy series, where, for \( n > 1 \), \( \text{soc}^n M \) is the unique subcomodule of \( M \) satisfying that \( \text{soc}^{n-1} M \subset \text{soc}^n M \) and

\[
\frac{\text{soc}^n M}{\text{soc}^{n-1} M} = \text{soc} \left( \frac{M}{\text{soc}^{n-1} M} \right),
\]

see [10] and [20] for some properties of the Loewy series.

**Lemma 0.2.** [4] The following statements are equivalent:
(a) $M$ is uniserial.
(b) The Loewy series is a composition series.
(c) Each finite dimensional subcomodule of $M$ is uniserial.

The coalgebra $C$ is said to be right (left) serial if any indecomposable injective right (left) $C$-comodule is uniserial. $C$ is called serial if it is both right and left serial.

Throughout we denote by $\mathcal{M}^C_f$, $\mathcal{M}^C_{qf}$ and $\mathcal{M}^C$ the category of finite dimensional, quasi-finite and all right $C$-comodules, respectively. Dually, $\mathcal{C}^M_f$, $\mathcal{C}^M_{qf}$ and $\mathcal{C}^M$ denote the corresponding categories of left $C$-comodules.

A full subcategory $T$ of $\mathcal{M}^C$ is said to be dense (or a Serre class) if each exact sequence

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$$

in $\mathcal{M}^C$ satisfies that $M$ belongs to $T$ if and only if $M_1$ and $M_2$ belong to $T$. Following [S] and [22], for any dense subcategory $T$ of $\mathcal{M}^C$, there exists an abelian category $\mathcal{M}^C/T$ and an exact functor $T : \mathcal{M}^C \rightarrow \mathcal{M}^C/T$, such that $T(M) = 0$ for each $M \in T$, satisfying the following universal property: for any exact functor $F : \mathcal{M}^C \rightarrow \mathcal{C}$ such that $F(M) = 0$ for each $M \in T$, there exists a unique functor $\overline{F} : \mathcal{M}^C/T \rightarrow \mathcal{C}$ verifying that $F = \overline{F}T$. The category $\mathcal{M}^C/T$ is called the quotient category of $\mathcal{M}^C$ with respect to $T$, and $T$ is known as the quotient functor.

Let now $T$ be a dense subcategory of the category $\mathcal{M}^C$, $T$ is said to be localizing (cf. [S]) if the quotient functor $T : \mathcal{M}^C \rightarrow \mathcal{M}^C/T$ has a right adjoint functor $S$, called the section functor. If the section functor is exact, $T$ is called perfect localizing. Let us list some properties of the localizing functors (cf. [S] Chapter III).

**Lemma 0.3.** Let $T$ be a dense subcategory of the category of right comodules $\mathcal{M}^C$ over a coalgebra $C$. The following statements hold:

(a) $T$ is exact.
(b) If $T$ is localizing, then the section functor $S$ is left exact and the equivalence $TS \simeq 1_{\mathcal{M}^C/T}$ holds.

From the general theory of localization in Grothendieck categories [S], it is well-known that there exists a one-to-one correspondence between localizing subcategories of $\mathcal{M}^C$ and sets of indecomposable injective right $C$-comodules, and, as a consequence, sets of simple right $C$-comodules. More precisely, a localizing subcategory is determined by an injective right $C$-comodule $E = \bigoplus_{j \in J} E_j$, where $J \subseteq I_C$ (therefore the associated set of indecomposable injective comodules is $\{E_j\}_{j \in J}$). Then $\mathcal{M}^C/T \simeq \mathcal{M}^D$, where $D$ is the coalgebra of coendomorphism $\text{Cohom}_C(E, E)$ (cf. [30] for definitions), and the quotient and section functors are $\text{Cohom}_C(E, -)$ and $-\Box^D E$, respectively.

In [3], [13] and [31], localizing subcategories are described by means of idempotents in the dual algebra $C^*$. In particular, it is proved that the quotient category $\mathcal{M}^C/T$ is the category of right comodules over the coalgebra $eCe$, where $e \in C^*$ is an idempotent associated to the localizing subcategory $T$ (that is, $E = Ce$, where $E$ is the injective right $C$-comodule associated to the localizing subcategory $T$). The coalgebra structure of $eCe$ (cf. [23]) is given by

$$\Delta_{eCe}(exe) = \sum_{(x)} ex_{(1)}e \otimes ex_{(2)}e \quad \text{and} \quad \epsilon_{eCe}(exe) = \epsilon_C(x)$$
for any $x \in C$, where $\Delta_C(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ using the sigma-notation of [29]. Throughout we denote by $T_e$ the localizing subcategory associated to the idempotent $e$. For completeness, we recall from [3] (see also [13]) the following description of the localizing functors. We recall that, given an idempotent $e \in C^*$, for each right $C$-comodule $M$, the vector space $eM$ is endowed with a structure of right $eCe$-comodule given by

$$\rho_eM(ex) = \sum_{(x)} ex_{(1)} \otimes ex_{(0)}$$

where $\rho_M(x) = \sum_{(x)} x_{(1)} \otimes x_{(0)}$ using the sigma-notation of [29].

**Lemma 0.4.** Let $C$ be a coalgebra and $e$ be an idempotent in $C^*$. Then the following statements hold:

(a) The quotient functor $T : M^C \to M^{eCe}$ is naturally equivalent to the functor $e(-)$. $T$ is also naturally equivalent to the cotensor functor $- \square eCe$ and the Cohom functor $T_e = \text{Cohom}_C(e, -)$.

(b) The section functor $S : M^{eCe} \to M^C$ is naturally equivalent to the cotensor functor $S_e = - \square eCe$.

(c) $T_e$ is perfect localizing if and only if $Ce$ is injective as right $eCe$-comodule.

We refer the reader to [11], [12] and [13] for basic definitions, notations and properties about quivers and path coalgebras. The localization in categories of comodules over path coalgebras is described in detail in [13].

1. **The valued Gabriel quiver**

Associating a graphical structure to a certain mathematical object is a very common strategy. Sometimes, it provides us a nice method for replacing the object with a simpler one and improving our intuition about its properties. In our case, when dealing with representation theory of coalgebras, the quivers associated to a coalgebra play a prominent rôle in order to study their structure in depth. This section is devoted to analyze the shape of the so-called valued Gabriel quiver of a serial coalgebra carrying on with the results obtained in [4]. Throughout we assume that $C$ is a basic coalgebra with decompositions (0.1) and (0.2). Following [15], let us recall the notion of right valued Gabriel quiver $(Q_C, d_C)$ of the coalgebra $C$ as follows: the set of vertices of $(Q_C, d_C)$ is the set of simple right $C$-comodules $\{S_i\}_{i \in I_C}$, and there exists a unique valued arrow

$$S_i \xrightarrow{(d'_{ij}, d''_{ij})} S_j$$

if and only if $\text{Ext}_C^1(S_i, S_j) \neq 0$ and,

$$d'_{ij} = \dim_{G_i} \text{Ext}_C^1(S_i, S_j) \text{ and } d''_{ij} = \dim_{G_j} \text{Ext}_C^1(S_i, S_j),$$

as a right $G_i$-module and as a left $G_j$-module, respectively.

The (non-valued) Gabriel quiver of $C$ is obtained by taking the same set of vertices and the number of arrows from a vertex $S_i$ to a vertex $S_j$ is given by the integer $\dim_{G_i} \text{Ext}_C^1(S_i, S_j)$, where $\text{Ext}_C^1(S_i, S_j)$ is viewed as right $G_i$-module. If $C$ is pointed (or $K$ is algebraically closed) then it is isomorphic to the one used by Montgomery in [18].
and Woodcock in [31] in order to prove that C is a subcoalgebra of the path coalgebra of its (non-valued) Gabriel quiver.

In [26], the valued Gabriel quiver of C is described through the notion of irreducible morphisms between indecomposable injective right C-comodules. Let us denote by injC (respect. C-inj) the full subcategory of MC (respect. C-M) formed by socle-finite (i.e., comodules whose socle is finite-dimensional) injective right (respect. left) C-comodules. Let E and E' be two comodules in injC. A morphism f : E → E' is said to be irreducible if f is not an isomorphism and given a factorization

$$\xymatrix{ E \ar[r]^f \ar[d]_g & E' \ar[d]_h \ar[l]_Z & \ar[l]_h }$$

de Z is in injC, g is a section, or h is a retraction. Analogously to the case of finite-dimensional algebras, there it is proven that the set of irreducible morphisms IrrC(Ei, Ej) between two indecomposable injective right C-comodules Ei and Ej is isomorphic, as Gj-Gi-bimodule, to the quotient radC(Ei, Ej)/rad2C(Ei, Ej). We recall that, for each two indecomposable injective right C-comodules Ei and Ej, the radical of HomC(Ei, Ej) is the K-subspace radC(Ei, Ej) of HomC(Ei, Ej) generated by all non-isomorphisms. Observe that if i ≠ j, then radC(Ei, Ej) = HomC(Ei, Ej). The square of radC(Ei, Ej) is defined to be the K-subspace

$$\text{rad}^2_C(E_i, E_j) \subseteq \text{rad}_C(E_i, E_j) \subseteq \text{Hom}_C(E_i, E_j)$$

generated by all composite homomorphisms of the form

$$E_i \xrightarrow{f} E_k \xrightarrow{g} E_j,$$

where f ∈ radC(Ei, Ek) and g ∈ radC(Ek, Ej). The mth power radCm(Ei, Ej) of radC(Ei, Ej) is defined analogously, for each m > 2.

**Lemma 1.1.** [26, Theorem 2.3(a)] Let C be a basic coalgebra and set Gi = EndC(Si) for each i ∈ IC. There is an arrow

$$\xymatrix{ S_i \ar[r]^{(d^l_{ij}, d^m_{ij})} & S_j }$$

in the right valued Gabriel quiver (QC, dC) of C if and only if IrrC(Ej, Ei) ≠ 0 and

$$d^l_{ij} = \dim_{G_j} \text{Irr}_C(E_j, E_i) \quad \text{and} \quad d^m_{ij} = \dim_{Gi} \text{Irr}_C(E_j, E_i),$$

as right Gj-module and as left Gi-module, respectively.

Let us see that right serial coalgebras are easy to distinguish from its valued Gabriel quiver. The following lemma is not new, it appears in [4, Proposition 1.7]; anyhow, for the convenience of the reader, we give a new proof only by means of “coalgebraic” arguments.

**Lemma 1.2.** A basic coalgebra C is right serial if and only if the right C-comodule soc2E/soc E is zero or simple for each indecomposable injective right C-comodule E.
Proof. Let $E$ be an indecomposable injective right $C$-comodule. Let us prove that the quotient $\text{soc}^i E / \text{soc}^{i-1} E$ is simple or zero for any $i \geq 2$. We proceed by induction on $i$. The case $i = 2$ is a consequent of the hypothesis. Let us now assume that the statement holds for some integer $k \geq 2$, that is,

$$
\text{soc} \left( \frac{E}{\text{soc}^k E} \right) = \frac{\text{soc}^k E}{\text{soc}^k - 1 E}
$$

is a simple comodule (if it was zero, then $E = \text{soc}^k E$ and the result would follow) and hence the right injective envelope of $E / \text{soc}^k - 1 E$ is an indecomposable injective right $C$-comodule $E'$. Therefore, by [20, Lemma 1.4],

$$
\frac{\text{soc}^{k+1} E}{\text{soc}^k E} \cong \frac{\text{soc}^{k+1} E}{\text{soc}^k E} \cong \frac{\text{soc}^2 \left( \frac{E}{\text{soc}^k E} \right)}{\text{soc} \left( \frac{E}{\text{soc}^k E} \right)} \leq \frac{\text{soc}^2 E'}{\text{soc} E'}
$$

which is simple or zero by hypothesis. The converse implication is trivial. \( \square \)

**Proposition 1.3.** A basic coalgebra $C$ is right serial if and only if each vertex $S_i$ of the right valued Gabriel quiver $(Q_C, d_C)$ is at most the sink of one arrow and, if such an arrow exists, it is of the form

$$
S_j \xrightarrow{(1,d)} S_i,
$$

for some vertex $S_j$ and some positive integer $d$. In particular, if $C$ is pointed, $C$ is right serial if and only if each vertex in the (non-valued) Gabriel quiver of $C$ is the sink of at most one arrow.

**Proof.** Recall that, for any simple right $C$-comodule $S_i$,

$$
\text{Ext}^1_C(S_j, S_i) \cong \text{Hom}_C(S_j, E_i/S_i)
$$

as right $G_j$-modules for all simple right $C$-comodule $S_j$, see for instance [15] and [20, Lemma 1.2].

Assume now that $C$ is right serial and $E_i/S_i \neq 0$ (otherwise there is no arrow ending at $S_i$) then $E_i/S_i$ is a subcomodule of an indecomposable injective right comodule $E_j$, and then

$$
\text{Ext}^1_C(S, S_i) \cong \text{Hom}_C(S, E_i/S_i) \cong \begin{cases} G_j, & \text{if } S_j = S; \\ 0, & \text{otherwise}. \end{cases}
$$

as right $\text{End}_C(S)$-modules. Hence, there is a unique arrow ending at $S_i$ of the form

$$
S_j \xrightarrow{(1,d)} S_i.
$$

Conversely, the immediate predecessors of $S_i$ correspond to the simple comodules contained in $\text{soc} (E_i/S_i)$. Since, by hypothesis, there is only one arrow ending at $S_i$, $\text{soc} (E_i/S_i) = (S_j)^t$ for some simple right comodule $S_j$ and some positive integer $t$. Now, since $t$ is the first component of the label of the arrow, $\text{soc}^3 E_i / \text{soc} E_i = \text{soc} (E_i/S_i) = S_j$ is a simple comodule. By the previous lemma, $C$ is right serial. \( \square \)
Symmetrically, we prove that $C$ is left serial if and only if each vertex $S$ of the left valued Gabriel quiver $(cQ, cd)$ is at most the sink of one arrow and, if such an arrow exists, it is of the form 

$$S' \xrightarrow{(1,d)} S,$$

for some vertex $S'$ and some positive integer $d$.

The following simple result is very useful, see also [16, Corollary 2.26].

**Proposition 1.4.** Let $C$ be a basic coalgebra. The right valued Gabriel quiver $(Q_C, d_C)$ of $C$ is the opposite valued quiver of the left valued Gabriel quiver $(cQ, cd)$ of $C$. Consequently, the left (non-valued) Gabriel quiver of $C$ is the opposite of the right (non-valued) Gabriel quiver of $C$.

**Proof.** We recall from [5] that there exists a duality $D : \text{inj}^C \to \text{c-inj}$ given by $D = \text{Cohom}_C(\cdot, C)$ whose inverse (which we also denote by $D$) is given by $D = \text{Cohom}_{C^{op}}(\cdot, C^{op})$. Let us denote $D(E_i) = F_i$ (and then $D(F_i) = E_i$) for each indecomposable injective right $C$-comodule $E_i$. Therefore, $F_i$ is an indecomposable injective left $C$-comodule. Moreover, following [4], if $S_i$ is the socle of $E_i$ and $E_i = Ce_i$ for some idempotent $e_i \in C^*$, then $F_i = D(E_i) = \text{Cohom}_C(Ce_i, C) = e_iC$, then $soc F_i = S_i$. Summarizing, $E_i$ and $F_i$ are the right and the left injective envelopes of $S_i$, respectively.

Now, since $\text{Hom}_C(E_i, E_j) \cong \text{Hom}_C(F_j, F_i)$, for each two indecomposable injective right comodules $E_i$ and $E_j$, it is easy to see that also $\text{Irr}_C(E_i, E_j) \cong \text{Irr}_C(F_j, F_i)$ and then

$$\dim_K \text{Irr}_C(E_i, E_j) = \dim_K \text{Irr}_C(F_j, F_i).$$

Thus

$$\dim_{G_i} \text{Irr}_C(E_i, E_j) = \frac{\dim_K \text{Irr}_C(E_i, E_j)}{\dim_K S_i} = \frac{\dim_K \text{Irr}_C(F_j, F_i)}{\dim_K S_i} = \dim_{G_i} \text{Irr}_C(F_j, F_i).$$

Analogously, $\dim_{G_j} \text{Irr}_C(E_i, E_j) = \dim_{G_j} \text{Irr}_C(F_j, F_i)$. Therefore, there exists an arrow

$$S_i \xrightarrow{(d'_{ij}, d''_{ij})} S_j$$

in $(Q_C, d_C)$ if and only if there exists an arrow

$$S_j \xrightarrow{(d'_{ij}, d''_{ij})} S_i$$

in $(cQ, cd)$ and the result follows. □

Let us now prove the main result of this section that generalizes [4, Theorem 2.10]. In what follows we denote the labeled arrows $\xrightarrow{(1,1)} \circ$ simply by $\circ \rightarrow \circ$. As well we denote a valued quiver $(Q, d)$ simply by $Q$ if $(d_{ij}^1, d_{ij}^2) = (1, 1)$ for any $i$ and $j$. 
Theorem 1.5. Let $C$ be a indecomposable basic coalgebra over an arbitrary field $K$. Then $C$ is serial if and only if the right (and then also the left) valued Gabriel quiver of $C$ is one of the following valued quivers:

- $(a) \quad \overrightarrow{\infty A}_{\infty}$
- $(b) \quad \infty A_{\infty}$
- $(c) \quad \infty \tilde{A}$
- $(d) \quad A_n$ for $n \geq 1$
- $(e) \quad \tilde{A}_n$ for $n \geq 1$

Proof. Assume that $C$ is serial and let $S$ be a simple right (and left) $C$-comodule. Since $C$ is right serial, there exists at most one arrow in $(Q_C, d_C)$ ending at $S$. Analogously, since $C$ is left serial there exists at most one arrow in $(CQ, Cd_C)$ ending at $S$ and then, by Proposition 1.4, we may deduce that there is at most one arrow in $(Q_C, d_C)$ starting at $S$. Also, by Proposition 1.3 and its left-hand version and Proposition 1.4, any of these possible arrows are labelled by $(1, 1)$. Taking into account the above discussion, let $S_0$ be a simple right comodule. If there is no arrow neither ending nor starting at $S_0$, i.e., $S_0$ is an isolated vertex, since $C$ is indecomposable (and therefore $Q_C$ is connected, cf. [26]) then $Q_C = \infty A_{\infty}$. Similarly, if there is a loop at $S_0$, then $Q_C = \tilde{A}_1$. Therefore we may assume that there is no loop in $Q_C$ and then $S_0$ is inside a (maybe infinite) path

$$
\cdots \rightarrow S_{-n} \cdots \rightarrow S_{-1} \rightarrow S_0 \rightarrow S_1 \rightarrow \cdots \rightarrow S_m \rightarrow \cdots
$$

If there exist two non-negative integers $n$ and $m$ such that $S_{-n} = S_m$, then $Q_C$ must be a crown, i.e., $Q_C = \tilde{A}_p$ for some integer $p$. If not, $Q_C$ must be a line, that is, it is a quiver as showed in $(a), (b), (c)$ or $(d)$ depending on the finiteness of the two branches. Clearly, the converse holds. \qed

Corollary 1.6. A basic coalgebra $C$ is serial if and only if each of the connected component of its right (or left) valued Gabriel quiver is either $\infty A_{\infty}$, or $A_{\infty}$, or $\infty \tilde{A}$; or $A_n$ or $\tilde{A}_n$ for some $n \geq 1$.

The problem of describing explicitly right (and not left) serial coalgebras by means of its valued Gabriel quiver turns out much more difficult. Using a reasoning similar to the proof of Theorem 1.5 we now state, without proof, an approximation to this question. By a diagram
we mean a quiver which is a subquiver of an (infinite) tree with descendent orientation, that is, a subquiver of a quiver of the following shape:

![Quiver Diagram]

**Proposition 1.7.** A basic coalgebra $C$ is right serial if and only if its right valued Gabriel quiver is one of the following:

(a) If $(Q_C, d_C)$ is acyclic, then $Q_C$ has the form

![Acyclic Quiver Diagram]

where each branch of the line may be finite or infinite.

(b) If $(Q_C, d_C)$ is not acyclic, then there exists a unique cycle in it, and $Q_C$ is of the form

![Non-acyclic Quiver Diagram]

Moreover, each arrow $S_i \rightarrow S_j$ in $(Q_C, d_C)$ is labelled by $(1, d_{ij})$, where $d_{ij}$ is a positive integer for any $i$ and $j$ in $I_C$.

2. **Localization in serial coalgebras**

Let us now apply the localization techniques developed in [12], [13], [20] and [28] to (right) serial coalgebras. In particular, we give a characterization of serial coalgebras by means of its “local structure”, that is, by means of its localized coalgebras.
The following proposition shows that the localization process preserves the uniseriality of comodules and the seriality of coalgebras. For each idempotent $e \in C^*$ we denote by $T_e$ the quotient functor $\text{Cohom}_C(Ce, -) : \mathcal{M}^C \to \mathcal{M}^{Ce}$.

**Proposition 2.1.** Let $E = Ce$ be a quasi-finite injective right $C$-comodule and $M$ a uniserial right $C$-comodule. Then $T_e(M) = \text{Cohom}_C(E, M) = eM$ is a uniserial right $eCe$-comodule.

**Proof.** Let us consider the (composition) Loewy series of $M$ as right $C$-comodule,
\[
soc M = S_1 \subset soc^2 M \subset soc^3 M \subset \cdots \subset M
\]
whose composition factors are $S_1$ and $S_k = M[k] = soc^k M / soc^{k-1} M$ for $k \geq 2$. Since a simple $C$-comodule is either torsion or torsion-free, let us suppose that $S_{i_1}$, $S_{i_2}$, $S_{i_3}$, $\ldots$ are the torsion-free composition factors of $M$, where $i_1 < i_2 < i_3 < \cdots$.

For each $k < i_1$, $T_e(M[k]) = T_e(S_{i_1}) = 0$ and then $T_e(soc^k M) = T_e(S_{i_1}) = 0$. As a consequence, by \cite[Remark 2.3]{13}, $T_e(soc^{i_1} M) = T_e(S_{i_1}) = S_{i_1}$. Moreover, since $M[i_1] = soc(M / soc^{i_1-1} M)$ is infinitesimal, then $M / soc^{i_1-1} M$ is torsion-free and, by \cite[Proposition 3.2(c)]{20},
\[
S_{i_1} = T_e(S_{i_1}) = T_e \left( \text{soc} \left( \frac{M}{soc^{i_1-1} M} \right) \right) = \text{soc} \left( T_e \left( \frac{M}{soc^{i_1-1} M} \right) \right) = \text{soc} T_e(M).
\]

Applying the same arguments, we may obtain that $T_e(soc^k M) = S_{i_k}$ for each $i_1 \leq k < i_2$, and $M / soc^{i_2-1} M$ is a torsion-free right $C$-comodule. Then
\[
T_e(M)[2] = \frac{soc^2 T_e(M)}{soc T_e(M)} = \text{soc} \left( \frac{T_e(M)}{soc T_e(M)} \right) = \text{soc} \left( \frac{T_e(M)}{soc^{i_1} M} \right) = \text{soc} \left( \frac{M}{soc^{i_2-1} M} \right) = S_{i_2}
\]
Thus $soc^2 T_e(M) = T_e(soc^{i_2} M)$. If we continue in this fashion, we may prove that
\[
T_e(soc^{i_1} M) \subset T_e(soc^{i_2} M) \subset T_e(soc^{i_3} M) \subset \cdots \subset T_e(M)
\]
is the Loewy series of $T_e(M)$. Hence $T_e(M)$ is uniserial as a right $eCe$-comodule. \hfill \Box

**Corollary 2.2.** Let $C$ be a right (left) serial coalgebra and $e \in C^*$ an idempotent. Then the localized coalgebra $eCe$ is right (left) serial.

**Proof.** Let $E_i$ be an indecomposable injective right $eCe$-comodule. By \cite[Proposition 3.2]{20}, $T_e(E_i) = \overline{E_i}$, where $E_i$ is the indecomposable injective right $C$-comodule such that $soc E_i = soc \overline{E_i}$. Since $E_i$ is uniserial, by Proposition 2.1 so is $\overline{E_i}$. \hfill \Box

**Lemma 2.3.** Let $C$ be a coalgebra. If the localized coalgebra $eCe$ is right (left) serial for each idempotent $e \in C^*$ associated to a subset of simple comodules with cardinal less or equal than three, then $C$ is right (left) serial.

**Proof.** Let us suppose that $C$ is not right serial. By Lemma \cite[2.2]{12} there exists an indecomposable injective right $C$-comodule $E$ such that $S_1 \oplus S_2 \subseteq soc(E/S)$, where $S_1$ and $S_2$ are simple comodules. Consider the idempotent $e \in C^*$ associated to the set $\{S, S_1, S_2\}$. Then, by \cite[Lemma 2.1]{20},
\[
T_e(S_1 \oplus S_2) = S_1 \oplus S_2 \subseteq T_e(soc(E/S)) \subseteq soc(T(E/S)) = soc(\overline{E}/S),
\]
where $T(e(E) = \overline{E}$ is an indecomposable injective $eCe$-comodule. Thus $eCe$ is not right serial.

**Proposition 2.4.** Let $C$ be a coalgebra. $C$ is right (left) serial if and only if each socle-finite localized coalgebra of $C$ is right (left) serial.

**Proof.** Apply Corollary 2.2 and Lemma 2.3.

**Remark 2.5.** The subsets of simple comodules mentioned in Lemma 2.3 cannot have cardinal bounded by less than three. For instance, if $C$ is the path coalgebra $KQ$ of the quiver

```
1  
|  |
|  |
2 3
```

then each localized coalgebra of $C$ at subsets of two or one simple comodules is right (and left) serial but clearly $C$ is not.

The former results are quite surprising since, as the following proposition shows, the localization process increases the label of an arrow (if exists) between two torsion-free vertices.

**Proposition 2.6.** Let $C$ be a coalgebra and $e \in C^*$ idempotent. Let $S_1$ and $S_2$ be two torsion-free simple $C$-comodules in the torsion theory associated to the localizing subcategory $T_e$. If there exists an arrow $S_1 \to S_2$ in $(Q_C, d_C)$ labelled by $(d'_1, d''_1)$, then there exists an arrow $S_1 \to S_2$ in $(Q_{eCe}, d_{eCe})$ labelled by $(t'_1, t''_1)$, where $t'_1 \geq d'_1$ and $t''_1 \geq d''_1$.

**Proof.** Let us suppose that $\text{soc}(E_2/S_2) = \bigoplus_{i \in I_C} S_1^{n_i}$ for some non-negative integers $n_i$. Then there exists an arrow $S_i \to S_2$ if and only if $n_i \neq 0$ and, furthermore, in such a case, it is labelled by $(n_i, m_i)$ for some positive integer $m_i$. Now, if there exists an arrow $S_1 \to S_2$ in $(Q_C, d_C)$ labelled by $(d'_1, d''_1)$, then

$$S_1^{n_1} \subseteq \bigoplus_{i \in I_C} S_1^{n_i} = T(e(\text{soc}(E_2/S_2)) \subseteq \text{soc}(E_2/S_2) \subseteq \overline{E_2}/S_2,$$

since $S_1$ and $S_2$ are torsion-free. Hence there is an arrow

$$S_1 \xrightarrow{(t'_1, t''_1)} S_2$$

in $(Q_{eCe}, d_{eCe})$ and $t'_1 = \text{dim}_{C_1} \text{Hom}_{eCe}(S_1, \overline{E_2}/S_2) \geq n_1 = d'_1$. By the left-hand version of this reasoning and Proposition 1.4, also $t''_1 \geq d''_1$.

**Remark 2.7.** It is not possible to prove the equalities on the components of the labels in the statement of Proposition 2.7. The reader only have to consider the path coalgebra $KQ$ of the quiver

```
1  
|  |
|  |
2 3
```

and the idempotent $e \in (KQ)^*$ associated to the subset $\{1, 3\}$. 

3. Hom-computable and representation-directed serial coalgebras

The study of the directing modules of an artin algebra comes from different motivations. On the one hand, they are treated as a generalization of the modules lying in a postprojective or a preinjective component (or more generally, in an acyclic component) of the Auslander-Reiten quiver of this algebra. Hence, they possess common properties with these modules as, for instance, they are determined up to isomorphism by their composition factors. On the other hand, they have interesting properties of their own as, for example, that any algebra having a sincere and directing module is a tilted algebra (that is, the endomorphism algebra of a hereditary algebra). It is also well-known that a representation-directed algebra (all its modules are directing) is finite representation-type, see [2] for details. This section deals with representation-directed coalgebras as defined in [27]. In particular, we describe the representation-directed serial coalgebras following the ideas of the previous sections, i.e., by means of their valued Gabriel quiver and using the localization in categories of comodules. In order to do this, we shall make use of the so-called computable comodules and Hom-computable coalgebras.

Assume that \( C \) is a basic coalgebra with fixed decompositions (0.1) and (0.2). Following [27], a right \( C \)-comodule \( M \) is defined to be computable if, for each \( i \in I_C \), the sum \( \ell_i(M) = \sum_{n=1}^{\infty} \ell_i(M[n]) \), called the composition \( S_i \)-length of \( M \), is finite, where \( M[n] = \text{soc}^n M/\text{soc}^{n-1} M \) and \( \ell_i(M[n]) \) is the number of times that the simple comodule \( S_i \) appears as a summand in a semisimple decomposition of \( M[n] \). We denote by comp\( C \) (resp. by \( C \text{comp} \)) the full subcategory of \( M^C \) (resp. of \( C M \)) whose objects are computable comodules. The coalgebra \( C \) is said to be Hom-computable if every indecomposable injective right \( C \)-comodule is computable or, equivalently (cf. [27]), if \( \text{Hom}_C(E_i, E_j) \) has finite \( K \)-dimension for any two indecomposable injective right \( C \)-comodules \( E_i \) and \( E_j \). Therefore, by the duality \( \text{inj}^C \to \text{cijn} \) stated in [5], the notion of Hom-computability is left-right symmetric. We now describe Hom-computable serial coalgebras. For that purpose we give a version for coalgebras of the Periodicity Theorem proved by Eisenbud and Griffith in [7].

**Lemma 3.1.** Let \( C \) be a coalgebra and \( N \) a uniserial right \( C \)-comodule. If \( M \) is a subcomodule of \( N \) then \( M \) is uniserial and, moreover, \( \text{soc}^t M = \text{soc}^t N \) for any positive integer \( t \) such that \( \text{soc}^t M \neq \text{soc}^{t-1} M \). As a consequence, \( M \) is uniserial if and only if every subcomodule of \( M \) is uniserial.

**Proof.** Obviously, if \( M \leq N \) then \( \text{soc} M = \text{soc} N = S \) is a simple comodule. Let us now assume that \( \text{soc}^t M = \text{soc}^t N \) for each \( k \leq t - 1 \), and also \( \text{soc}^{t-1} M \neq \text{soc}^t M \). Then

\[
0 \neq \frac{\text{soc}^t M}{\text{soc}^{t-1} M} = \frac{\text{soc}^t N}{\text{soc}^{t-1} N} \cong S_t,
\]

where \( S_t \) is a simple comodule. That is, \( \text{soc}^t M/\text{soc}^{t-1} M \cong S_t \). Thus \( M \) is uniserial and, by its definition, \( \text{soc}^t M = \text{soc}^t N \).

**Proposition 3.2** (Periodicity Theorem). Let \( C \) be an indecomposable serial coalgebra and \( E_0 \) an indecomposable injective right \( C \)-comodule. Let the sequence of composition factors of \( E_0 \) be \( S_0 = E_0[1] = \text{soc} E \), \( S_1 = E_0[2] \), \( S_2 = E_0[3] \), . . . . Suppose that \( S_t \cong S_k \) for some \( k > 1 \), and let \( h \neq 1 \) be the smallest such integer. Then the valued Gabriel quiver of \( C \) is
\(\tilde{\mathcal{A}}_h\), and \(S_m \cong S_n\) if and only if \(m \equiv n (\text{mod } h)\). If there is no such an \(h\), then \(S_m \cong S_n\) implies \(m = n\).

**Proof.** Let us assume that \(S_0 \not\cong S_h\) for any \(h > 1\). Suppose also that \(S_n \cong S_m\) for some \(m \neq n\) and, furthermore, this is the first repetition, i.e., \(S_i \not\cong S_j\) for \(i, j < m\). Let us consider the injective right \(C\)-comodule \(E = E_0 \oplus E_n = Ce\), where \(e = e_0 + e_n\). Then, by Proposition 2.1 and its proof, \(eCe\) is serial, \(T_v(E_0) = \widetilde{E}_0\) is uniserial and its Loewy series is

\[
S_0 \subseteq T_v(\text{soc}^n E_0) \subseteq T_v(\text{soc}^m E_0) \subseteq \cdots \subseteq \widetilde{E}_0,
\]

where \(\text{soc}^2 \widetilde{E}_0 / S_0 \cong S_n\) and \(\text{soc}^2 \widetilde{E}_0 / \text{soc}^2 \widetilde{E}_0 \cong S_n\). Let now \(M = \widetilde{E}_0 / \text{soc}^2 \widetilde{E}_0 \leq \widetilde{E}_n / S_n\). Then

\[
S_n \cong \text{soc}^3 \widetilde{E}_0 / \text{soc}^2 \widetilde{E}_0 = \text{soc} M \subseteq \text{soc}(\widetilde{E}_n / S_n).
\]

Thus there is a loop in the vertex \(S_n\) of the valued Gabriel quiver of \(eCe\), \(Q_e\). In addition, \(\text{soc}^2 \widetilde{E}_0 / S_0 \cong S_n\), so \(Q_e\) contains the subquiver

\[
\begin{tikzpicture}
    \node (S_n) at (0,0) {$S_n$};
    \node (S_0) at (1,0) {$S_0$};
    \draw[->] (S_n) to (S_0);
\end{tikzpicture}
\]

By Theorem 1.5, \(eCe\) is not serial. Hence \(m\) must equal \(n\).

Suppose now that \(S_h \cong S_0\) for some \(h \neq 1\) and, moreover, it is the smallest such integer.

First, by [20, Theorem 1.9], there is a path in \((Q_C, d_C)\) of length \(h\) starting and ending at \(S_0\), i.e., there is a cycle in \((Q_C, d_C)\). By Theorem 1.5 \((Q_C, d_C) = \tilde{\mathcal{A}}_h\). By a reasoning similar to the one done above, we may prove that \(S_i \not\cong S_j\) for any \(i, j < h\) with \(i \neq j\). Therefore it remains to show that \(S_{i+h} \cong S_i\) for any \(i > 1\). We denote by \(M\) the right comodule \(E_0 / \text{soc}^h E_0 \leq E_0\). Then, for any \(l > 0\)

\[
S_{i+h} \cong \frac{\text{soc}^{l+h+1} E_0}{\text{soc}^{l+h} E_0} \cong \frac{\text{soc}^{l+h+1} E_0}{\text{soc}^{l+h} E_0} \cong \frac{\text{soc}^{l+1} M}{\text{soc}^{l} M} \cong \frac{\text{soc}^{l+1} E_0}{\text{soc}^{l} E_0} \cong S_l,
\]

where in \(\blacktriangleleft\) we use [20, Lemma 1.4] and in \(\blacktriangledown\) we use Lemma 3.1. \(\square\)

**Proposition 3.3.** Let \(C\) be an indecomposable serial coalgebra. \(C\) is Hom-computable if and only if one of the following conditions holds:

(a) The right valued Gabriel quiver of \(C\) are either \(\tilde{\mathcal{A}}_{\infty}\), or \(\tilde{\mathcal{A}}_{\infty}\), or \(\tilde{\mathcal{A}}_n\), or \(\tilde{\mathcal{A}}_n\) for some \(n \geq 1\).

(b) The right valued Gabriel quiver of \(C\) is \(\tilde{\mathcal{A}}_n\) for some \(n \geq 1\), and \(C\) is finite dimensional.

**Proof.** Let us assume that \(C\) verifies either the condition (a) or the condition (b). First, if \(C\) is finite dimensional then \(\text{Hom}_C(E_i, E_j)\) is finite dimensional for each pair of indecomposable injective comodules. Now, if the valued Gabriel quiver of \(C\) is either \(\tilde{\mathcal{A}}_{\infty}\), or \(\tilde{\mathcal{A}}_{\infty}\), or \(\tilde{\mathcal{A}}_n\), or \(\tilde{\mathcal{A}}_n\) for some \(n \geq 1\); then, by the Periodicity Theorem, for each indecomposable injective \(E\) and each \(i \in I_C\), \(\ell_i(E)\) is one or zero. Then \(C\) is Hom-computable.

Conversely, it is enough to prove that if \((Q_C, d_C) = \tilde{\mathcal{A}}_n\) for some \(n \geq 1\), and \(C\) is infinite dimensional, then \(C\) is not Hom-computable. Now, since \(C\) is socle-finite, this is a consequence of [26 Corollary 2.10]. \(\square\)
Following [28], we say that a finitely copresented indecomposable comodule $M$ is said to be directing if there is no chain

$$M \xrightarrow{f_1} M_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} M_n \xrightarrow{f_{n+1}} M$$

where $M_i$ is finitely copresented and indecomposable for any $i = 1, \ldots, n$ and $f_j$ is a non-zero non-isomorphism for any $j = 1, \ldots, n + 1$. A coalgebra is said to be right (left) representation-directed if each finitely copresented indecomposable right (left) comodule is directing. Let us now classify serial representation-directed coalgebras in terms of their valued Gabriel quiver:

**Proposition 3.4.** Let $C$ be an indecomposable serial coalgebra. The following statements are equivalent:

(a) $C$ is right representation-directed.

(a’) $C$ is left representation-directed.

(b) The right valued Gabriel quiver of $C$ are either $\infty \mathbb{A}_\infty$, or $\mathbb{A}_\infty$, or $\mathbb{A}_n$ for some $n \geq 1$.

(b’) The left valued Gabriel quiver of $C$ are either $\infty \mathbb{A}_\infty$, or $\mathbb{A}_\infty$, or $\mathbb{A}_n$ for some $n \geq 1$.

**Proof.** Since, by Proposition 1.4, the left valued Gabriel quiver of $C$ is the opposite to the right one, it is enough to prove that $C$ is right representation-directed if and only if $(Q_C, d_C)$ is one of the above valued quivers.

Let us assume that $(Q_C, d_C) = \tilde{\mathbb{A}}_n$ for some $n \geq 1$. We use the following labels in the vertices (we omit the labels of the arrows):

```
5 6 7 8
↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓↓→
4. Finite dimensional comodules over serial coalgebras

This section is devoted to give a complete list of all indecomposable finite-dimensional right comodules over a serial coalgebra and a description of the Auslander-Reiten quiver of the category \( \mathcal{M}^f_\gamma \). We recall from [1] that any finite dimensional indecomposable comodule \( M \) over a serial coalgebra \( C \) is uniserial, and then, there exists an integer \( t \geq 1 \) such that \( \text{soc}^t M = M \). Thus \( M \) is a right \( D \)-comodule, where \( D \) is the subcoalgebra of \( C \), \( \text{soc}^t C = \oplus_{i \in I_C} \text{soc}^t E_i \) (and then it is serial, cf. [1]). We refer the reader to [5, 15, 21] and [24] for definitions and terminology concerning almost split sequence and the Auslander-Reiten quiver of a coalgebra, see also [2].

**Theorem 4.1.** Let \( C \) be a serial coalgebra. The following statements hold:

(a) Each finite dimensional indecomposable right \( C \)-comodule is isomorphic to \( \text{soc}^k E \) for some positive integer \( k \) and some indecomposable injective right \( C \)-comodule.

(b) The category of finite dimensional right \( C \)-comodules has almost split sequences. Furthermore, for each indecomposable non-injective right \( C \)-comodule \( \text{soc}^k E \), the almost split sequence starting on this comodule is

\[
0 \rightarrow \text{soc}^k E \xrightarrow{(i)_{p \geq 0}} \text{soc}^{k+1} E \oplus \frac{\text{soc}^k E}{\text{soc} E} \xrightarrow{(q-j)} \frac{\text{soc}^{k+1} E}{\text{soc} E} \rightarrow 0 ,
\]

where \( i \) and \( j \) are the standard inclusions and \( p \) and \( q \) are the standard projections.

**Proof.** (a). Let \( M \) be a finite dimensional indecomposable comodule, by [1], \( M \) is uniserial and then \( M = \text{soc}^t M \) for some \( t > 0 \) (we may consider the minimal one). Since \( M \) has simple socle, its injective envelope is an indecomposable injective comodule \( E \). By Lemma 3.1 the Loewy series of \( M \) and \( E \) are the same until the step \( t \).

(b). Here we essentially follow the proof of [2, Theorem 4.1]. The given sequence is easily seen to be exact. By Krull-Remak-Schmidt-Azumaya Theorem, it is not split and, since \( C \) is right serial, it has indecomposable end terms. Let us prove that the homomorphism \( f = (i)^p \) is left almost split. It is clear that \( f \) is not a section. Let \( N \) be an indecomposable finite dimensional \( C \)-comodule, and \( h : \text{soc}^k E \rightarrow N \) be a non-isomorphism. We have two cases. If \( h \) is not injective, it decomposes through \( \text{soc}^k E / \text{soc} E \), namely \( h = h' p \). But then the homomorphism \( g = (0 h') \) satisfies that \( gf = h \). If, on the other hand, \( h \) is injective, since it is not an isomorphism, by (a), \( N \cong \text{soc}^t E \) with \( t > k \). Then \( N \) is injective as right \( \text{soc}^t C \)-comodule and there exists a morphism \( h' : \text{soc}^{k+1} E \rightarrow N \) such that \( h = h'i \). Hence \( (h' 0) f = h \). Since the left term and the right term are indecomposable comodules and \( f \) is a left almost split morphism, the sequence is almost split in the category of finite dimensional \( C \)-comodules, see [2, Chapter IV, Theorem 1.13].

By applying Theorem 4.1, we can easily calculate the Auslander-Reiten quiver of a serial coalgebra. For example, we do it for hereditary serial path coalgebras.

**Type \( \infty A_{\infty} \).** Let \( Q \) be the quiver

\[
\ldots \xrightarrow{\alpha_{-2}} \xrightarrow{2} \xrightarrow{-1} \xrightarrow{0} \xrightarrow{\alpha_0} 0 \xrightarrow{\alpha_1} 1 \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_3} \ldots
\]

and let \( C = KQ \) be the path coalgebra of \( Q \). By Theorem 1.5 \( C \) is serial. Let \( E_i \) be the indecomposable injective right \( C \)-comodule associated to the vertex \( i \), that is,
$E_i = Ke_i \bigoplus \left( \bigoplus_{t \geq 0} Ka_i \cdots \alpha_i \right)$, where $e_i$ is the stationary path at $i$. Let

$$S_i^k = \text{soc}^k E_i = Ke_i \bigoplus \left( \bigoplus_{t=0}^{k-1} Ka_i \cdots \alpha_i \right).$$

Now, since $\text{soc}^k E_i / \text{soc} E_i \cong \text{soc}^{k-1} E_{i-1}$ for any $k$ and $i$, the almost split sequences are the following:

$$0 \rightarrow S_i^k \rightarrow S_i^{k+1} \oplus S_{i-1}^{k-1} \rightarrow S_{i-1}^k \rightarrow 0,$$

for each $k \geq 0$ and each $i \in \mathbb{Z}$. Therefore, the Auslander-Reiten quiver of $C$ is the following.

![Auslander-Reiten quiver](image)

where each dashed arrow $Y \leftarrow \cdots X$ means that $Y = \tau(X)$, where $\tau = \text{DTr}$ is the Auslander-Reiten translation, see [5] for definitions and details.

**Type $\infty A$.** Let $Q$ be the quiver

$$\cdots \xrightarrow{\alpha_4} 1 \xrightarrow{\alpha_3} 3 \xrightarrow{\alpha_2} 2 \xrightarrow{\alpha_1} 1 \xrightarrow{\alpha_0} 0$$

and let $C = KQ$ be the path coalgebra of $Q$. Again, by Theorem [1.5], $C$ is serial. Let $E_i$ be the indecomposable injective right $C$-comodule associated to the vertex $i$, that is, $E_i = Ke_i \bigoplus \left( \bigoplus_{t \geq 0} Ka_i \cdots \alpha_i \right)$, where $e_i$ is the stationary path at $i$. Let

$$S_i^k = \text{soc}^k E_i = Ke_i \bigoplus \left( \bigoplus_{t=0}^{k-1} Ka_i \cdots \alpha_i \right).$$

Since $\text{soc}^k E_i / \text{soc} E_i \cong \text{soc}^{k-1} E_{i+1}$ for any $k$ and $i$, the almost split sequences are the following:

$$0 \rightarrow S_i^k \rightarrow S_i^{k+1} \oplus S_{i+1}^{k-1} \rightarrow S_{i+1}^k \rightarrow 0,$$
for each \(i, k \geq 0\). Therefore, the Auslander-Reiten quiver of \(C\) is the following.

\[
\begin{array}{c}
S_0^1 \rightarrow \cdots \rightarrow S_1^4 \leftarrow S_2^4 \rightarrow \cdots \rightarrow S_3^4 \rightarrow S_3^0 \rightarrow \cdots \rightarrow S_1^0 \\
S_0^2 \rightarrow \cdots \rightarrow S_1^3 \leftarrow S_2^3 \rightarrow \cdots \rightarrow S_3^3 \rightarrow S_3^1 \rightarrow \cdots \rightarrow S_1^1 \\
\end{array}
\]

**Type** \(\tilde{A}_n\), \(n \geq 1\). Let \(Q\) be the quiver

\[
\begin{array}{c}
1 \alpha_n \alpha_1 \alpha_2 \alpha_3 \\
\alpha_6 \alpha_4 \\
5 \rightarrow 0 \alpha_5 \\
\end{array}
\]

and let \(C = KQ\) be the path coalgebra of \(Q\). Clearly, \(C\) is serial. Let \(E_i\) be the indecomposable injective right \(C\)-comodule associated to the vertex \(i\), that is, \(E_i = Ke_i \bigoplus (\bigoplus_{t \geq 0} K\alpha_{[i]} \cdots \alpha_{[i-t]})\) where \([p] \equiv p \pmod{n}\) for any \(p > 0\). Let

\[
S_i^k = \text{soc}^k E_i = Ke_i \bigoplus \left(\bigoplus_{t=0}^{k-1} K\alpha_{[i]} \cdots \alpha_{[i-t]}\right).
\]

Now, since \(\text{soc}^k E_n / \text{soc} E_n \cong \text{soc}^{k-1} E_1\) and \(\text{soc}^k E_i / \text{soc} E_i \cong \text{soc}^{k-1} E_{i+1}\) for each \(i = 1, \ldots, n - 1\), for any \(k \geq 0\), the almost split sequences are the following:

\[
0 \rightarrow S_i^k \rightarrow S_i^{k+1} \oplus S_{i+1}^{k-1} \rightarrow S_{i+1}^k \rightarrow 0,
\]

for each \(i = 1, \ldots, n - 1\) and \(k \geq 0\); and

\[
0 \rightarrow S_n^k \rightarrow S_n^{k+1} \oplus S_1^{k-1} \rightarrow S_1^k \rightarrow 0,
\]
for any $k \geq 0$. Therefore, the Auslander-Reiten quiver of $C$ is the following.

This structure is called a stable tube of rank $n$ since the shape of the quiver obtained is a tube if we identify the vertical double lines.

The Auslander-Reiten quiver of the remaining serial path coalgebras are described in [21].

**Remark 4.2.** Observe that, for each indecomposable finite-dimensional comodule $M = \text{soc}^n E$, $\tau(M) = \text{soc}^{k+1} E/\text{soc} E$, and then $\text{length } M = \text{length } \tau(M)$. That is, the comodules lying in the same $\tau$-orbit has the same length.

5. A theorem of Eisenbud and Griffith for coalgebras

We finish the paper by a version of the theorem of Eisenbud and Griffith [7, Corollary 3.2] for coalgebras. We recall that this theorem asserts that every proper quotient of a hereditary noetherian prime ring is serial. Obviously, first we need a translation of the concepts from ring terminology to the notions used in coalgebra theory. About hereditariness, the concept is well-known in coalgebras (cf. [19]) and it is not needed any explication. The “coalgebraic” version of noetherianess is the so-called co-noetherianess (cf. [9]). We recall that a comodule $M$ is said to be co-noetherian if every quotient of $M$ is embedded in a finite direct sum of copies of $C$. Nevertheless, we shall use a weaker concept: strictly quasi-finiteness [9], namely, $M$ is strictly quasi-finite if every quotient of $M$ is quasi-finite. This is due to fact that we may reduce the problem to socle-finite coalgebras and then, under this condition, both classes of comodules coincide [9, Proposition 1.6]. Finally, following [14], a coalgebra is called prime if for any subcoalgebras $A, B \subseteq C$ such that $A \wedge B = C$, then $A = C$ or $B = C$. For the convenience of the reader we present the following example:

**Example 5.1.** Let $C$ be a hereditary colocal coalgebra such that $C/S \cong C \oplus C$, where $S$ is the unique simple comodule (or subcoalgebra). We prove that $C$ is not co-noetherian.
Let us consider the subcomodule $N_2$ of $C$ which yields the following commutative diagram

\[
\begin{array}{c}
0 \\[-2ex] \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\begin{array}{cccccc}
S & N_2 & S' & 0 \\
0 & S & C & C \\
& (0,i) & (0,0,i) &
\end{array}
\]

Then, $N_2 \subseteq \text{soc}^2 C$ and $N_2/S \cong S$. Now,

\[
C/N_2 \cong (C/S)/(N_2/S) \cong (C \oplus C)/S \cong (C \oplus C \oplus C) = C^3.
\]

Analogously, let $N_3$ be the subcomodule of $C$ which yields the commutative diagram

\[
\begin{array}{c}
0 \\[-2ex] \downarrow \quad \downarrow \quad \downarrow \quad \downarrow
\end{array}
\begin{array}{cccccc}
0 & N_2 & N_3 & S & 0 \\
& C & C \oplus C \oplus C & 0
\end{array}
\]

Again, $N_3/N_2 \cong S$ and

\[
C/N_3 \cong (C/N_2)/(N_3/N_2) \cong (C \oplus C \oplus C)/S \cong (C \oplus C \oplus C \oplus C) = C^4.
\]

If we continue in this way, we obtain a increasing family of subcomodules $\{N_i\}_{i \geq 1}$, where $N_1 = S$, such that $C/N_t \cong C^t$ for any $t \geq 1$. Let us consider the uniserial subcomodule $N = \bigcup_{t \geq 1} N_t$ of $C$ whose composition series (or Loewy series) is given by

\[
0 \subset S \subset N_2 \subset N_3 \subset \cdots \subset N.
\]

The comodule $C/N$ has infinite dimensional socle. To see this, for each $i \geq 0$, consider the short exact sequence

\[
0 \longrightarrow N/N_i \longrightarrow C/N_i \longrightarrow C/N \longrightarrow 0
\]

which yields the exact sequence

\[
0 \longrightarrow \text{soc} (N/N_i) \longrightarrow \text{soc} (C/N_i) \longrightarrow \text{soc} (C/N),
\]

where $\text{soc} (N/N_i) \cong S$ and $\text{soc} (C/N_i) \cong S^{i+1}$. Thus $\dim_K \text{soc} (C/N) \geq i \cdot \dim_K S$ for any $i \geq 1$. Consequently, $C$ is not co-noetherian.

**Theorem 5.2.** Let $C$ be a basic socle-finite coalgebra over an arbitrary field. If $C$ is coprime, hereditary and co-noetherian then $C$ is serial.

**Proof.** If $C$ is colocal, then the valued Gabriel quiver of $C$ is either a single point or a vertex with a loop labeled by a pair $(d', d'')$. Now, if $d' \geq 2$ or $d'' \geq 2$, proceeding as in Example 5.1, $C$ is not co-noetherian. Thus $d' = d'' = 1$ and, by Theorem 1.5, $C$ is serial. Assume then that $C$ is not colocal. Let us first develop some properties about the valued Gabriel quiver of a localized coalgebra of $C$. These are inspired by the ones obtained in [13] for path coalgebras. Let us suppose that $S_x$, $S_y$ and $S_z$ are three simple $C$-comodules (where $S_z$ could equals $S_x$) such that there is path in $(Q_C, d_C)$

\[
S_x \xrightarrow{(d_1, d_2)} S_y \xrightarrow{(c_1, c_2)} S_z.
\]

Let $e \in C^*$ be an idempotent and $eCe$ the localized coalgebra associated to $e$ whose quotient functor we denote by $T_e$. Assume that $S_x$ and $S_z$ are torsion-free and $S_y$ is torsion. Since $C$ is hereditary, $E_z/S_z \cong \bigoplus_{j \in J} \bigcup_{j \in J} E_j^y \bigoplus_{t \in T} E_t$, where $S_j$ is torsion for all
j ∈ J and S_t is torsion-free for all t ∈ T, and r_α is a positive integer for any α ∈ J ∪ T.
Now, since there is an arrow from S_y to S_z, y ∈ J and E_y^x ⊆ E_z/S_z, where r_y = c_1. Then
T_e(E_y^x) ⊆ T_e(E_z/S_z). Finally, since S_x^d ⊆ T_e(E_y) ∩ T_e(E_y/S_y) then S_x^{d_1c_1} ⊆ T_e(E_z/S_z) =
E_z/S_z. That is, there exists an arrow

\[
S_x \overset{(h_1, h_2)}{\longrightarrow} S_z
\]

in \((Q eC_e, d eC_e)\) such that \(h_1 \geq d_1c_1\). By Proposition [14], it is easy to see that \(h_2 \geq d_2c_2\).
Note that the hereditariness is a left-right symmetric property.

By an easy induction one may prove that if there is a path

\[
S_x \overset{(a_0, b_0)}{\longrightarrow} S_1 \overset{(a_1, b_1)}{\longrightarrow} \cdots \overset{(a_{n-1}, b_{n-1})}{\longrightarrow} S_n \overset{(a_n, b_n)}{\longrightarrow} S_z
\]

such that \(S_i\) is torsion, for all \(i = 1, \ldots, n\). Then there is an arrow

\[
S_x \overset{(h_1, h_2)}{\longrightarrow} S_z
\]

in \((Q eC_e, d eC_e)\) such that \(h_1 \geq a_0a_1 \cdots a_n\) and \(h_2 \geq b_0b_1 \cdots b_n\). Furthermore, following
this procedure, one may prove that if \(\mathcal{P} = \{p^j\}_{i \in \Lambda}\) is non empty, where \(\mathcal{P}\) is the set of all
possible paths \(p^j\) in \((Q_C, d_C)\) as described in [5,3], i.e., starting at \(S_x\), ending at \(S_y\) and
whose intermediate vertices are torsion, then there is an arrow

\[
S_x \overset{(h_1, h_2)}{\longrightarrow} S_z
\]

in \((Q eC_e, d eC_e)\) such that \(h_1 = \sum_i a_i^i a_{i+1}^i \cdots a_n^i\) and \(h_2 = \sum_i b_i^i b_{i+1}^i \cdots b_n^i\). Here we have
denoted by \(a_i^0, \ldots, a_n^i\) and by \(b_i^0, \ldots, b_n^i\) the first and the second component, respectively,
of the labels of the arrows whose composition build the path \(p^j\). We refer the reader to [20] for more details about injective comodules and the localization functors.

Now we consider a primitive orthogonal idempotent \(e_x \in C^*\) and \(e_xC e_x\) the localized
coalgebra of \(C\) associated to \(e_x\). By [9, Proposition 1.8] and [8, p. 376, Corollary 5], \(e_xC e_x\)
is co-neotherian and hereditary, respectively. Therefore, following the colocal case, the
valued Gabriel quiver of \(e_xC e_x\) must be a single point or a vertex with a loop labeled by
\((1, 1)\). As a consequence, by the above considerations, each vertex of the valued Gabriel
quiver of \(C\) is inside of at most one cycle and, if exists, the arrows of that cycle are labeled by
\((1, 1)\).

Finally, we prove that for each pair of vertices of \(Q_C\) there is a cycle passing through
these two vertices. This yields the statement of the theorem since, together with the
above conditions, the only possible quiver is \((Q_C, d_C) = \tilde{A}_n\) for some \(n \geq 1\) and then \(C\) is
serial.

Fix two different simple comodules \(S_x\) and \(S_y\). Let \(e_x\) and \(e_y\) be the primitive orthogonal
idempotents in \(C^*\) associated to \(S_x\) and \(S_y\), respectively. We set \(e = e_x + e_y\). By [14, Proposition 4.1], \(eC e\) is prime. First, let us suppose that there is no path in \(Q_C\) from
\(S_x\) to \(S_y\) nor vice versa. Then \(eC e\) has two connected components and, by [20, Corollary
2.4(b)], \(eC e\) is not indecomposable. Thus \(eC e\) is not prime (cf. [14, Lemma 1.4]). Now,
suppose that there is a path from \(S_x\) to \(S_y\) but there is no path from \(S_y\) to \(S_x\). Then the
valued Gabriel quiver of $eCe$ is a subquiver of the following quiver:

$$\begin{array}{ccc}
S_x & \xrightarrow{(a,b)} & S_y \\
\end{array}$$

By [20, Lemma 3.7], $e_xCe_y = T_x(E_y) \neq 0$ and $e_yCe_x = T_y(E_x) = 0$. Therefore, there is a vector space direct sum decomposition $eCe = e_xCe_x \oplus e_xCe_y \oplus e_yCe_y$. A straightforward calculation shows that the linear map

$$\Psi : D = \begin{pmatrix} e_yCe_y & e_yCe_x \\ e_xCe_y & e_xCe_x \end{pmatrix} \rightarrow eCe,$$

between $eCe$ and the bipartite coalgebra $D$ (in the sense of [16]), given by

$$\Psi \left( \begin{array}{cc} a & b \\ 0 & c \end{array} \right) = a + b + c$$

is an isomorphism of coalgebras, see [3] for definitions and details about the structures of the spaces $e_xCe_x$, $e_yCe_y$ and $e_xCe_y$. We recall that the coalgebra structure of the bipartite coalgebra $D$ is given by the formulæ:

- $\Delta(a+b+c) = \Delta_y(a) + \rho_y(b) + \rho_x(b) + \Delta_x(c)$, where $\rho_y$ and $\rho_x$ are the $e_yCe_y$-$e_xCe_x$-bicomodule structure maps of $e_xCe_y$; and $\Delta_x$ and $\Delta_y$ are the comultiplication of the coalgebras $e_yCe_y$ and $e_xCe_x$, respectively.

- $\epsilon(a+b+c) = \epsilon_y(a) + \epsilon_x(b)$, where $\epsilon_y$ and $\epsilon_x$ are the counit of the coalgebras $e_yCe_y$ and $e_xCe_x$, respectively.

Then $eCe = e_yCe_y \land e_xCe_x$ and therefore $eCe$ is not prime.

Now we prove the Eisenbud-Griffith Theorem for coalgebras.

**Corollary 5.3.** If $C$ is a subcoalgebra of a prime, hereditary and strictly quasi-finite (left and right) coalgebra over an arbitrary field, then $C$ is serial.

**Proof.** By [41, Proposition 1.5], we may assume that $C$ is prime, hereditary and strictly quasi-finite itself. Let $e \in C^*$ be an idempotent such that the localized coalgebra $eCe$ is socle-finite. By [3, p. 376, Corollary 5], [44, Proposition 4.1] and [9, Proposition 1.8], $eCe$ is hereditary, prime and right strictly quasi-finite, respectively. Moreover, by [9, Proposition 1.5], $eCe$ is co-noetherian. Therefore, by Theorem 5.2, $eCe$ is serial. Thus the result follows from Proposition 2.4.

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