THE HODGE CONJECTURE FOR RATIONALLY CONNECTED FIVEFOLDS

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It has been pointed out to me, by A. Collino and H. Esnault, that the road taken here has been well travelled. The papers of Bloch-Srivinas [BS], Esnault-Levine [EL], Fakhruddin-Rajan [FR] – especially the first – contain a number of related ideas and applications, if not precisely the results proven here. That said, I will let the original text stand as it was written. Since the flatness of its prose may not quite have hidden the pleasure of writing it, I hope that this note can at least serve as introduction to this beautiful technique.

The purpose of this note is to prove the Hodge conjecture for a five dimensional smooth projective complex variety \( X \) possessing sufficiently many rational curves. To make this precise, recall [Ko] that there is a rational map \( f: X \rightarrow Y \) called the maximal rationally connected fibration. The fibers of \( f \) are rationally connected, and \( \dim Y \) is minimal among all such maps. We show that the Hodge conjecture holds provided that \( \dim Y \leq 3 \). In particular, it holds if \( X \) is already rationally connected as would be the case if it were Fano.

As to why we state the result only in dimension 5, we should recall that the Hodge conjecture is only an issue in dimensions greater than 3, but for fourfolds covered by rational curves the result is well known and elementary.

**Lemma 1** (Conte-Murre). *The Hodge conjecture holds for a smooth projective uniruled fourfold over \( \mathbb{C} \).*

The idea is that since a uniruled fourfold is dominated by a blow up of a product of \( \mathbb{P}^1 \) with a threefold \( Y \), its interesting cohomology “comes from” the lower dimensional varieties \( Y \) and the centers of the blow ups. We will give a more precise explanation later. A similar argument would be sufficient to prove the main theorem when \( f: X \rightarrow Y \) is fiberwise unirational i.e. if the map is dominated by a blow up of \( \mathbb{P}^{5-\dim Y} \times Y \). In general, however, a more sophisticated argument is needed. The key trick is the following lemma of Bloch that we learned from the papers of Esnault [E] and Kim [Ki].

**Lemma 2** (Bloch). *Let \( X \) be a smooth rationally connected variety over an algebraically closed field \( K \), then a positive multiple of the diagonal \( \Delta \subset X \times X \) is rationally equivalent to a sum \( \xi \times X + \Gamma \), where \( \xi \in Z_0(X) \) is a zero cycle, and \( \Gamma \) is supported on \( X \times Z \) for some proper closed subset \( Z \subset X \).*

The original statement is a bit stronger [B, E], but we won’t need it. In order to keep our presentation reasonably self contained, we give a geometric proof of the result in its present form.

**Proof.** Rational connectedness means that any two general points can be joined by an irreducible rational curve. More precisely [Ko] thm II 2.8, IV def 3.2, there is

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a morphism $R : \mathbb{P}^1 \times F \to X$, such that

$$R^{(2)} : \mathbb{P}^1 \times \mathbb{P}^1 \times F \to X \times X, \quad R^{(2)}(t_1, t_2, f) = (R(t_1, f), R(t_2, f))$$

is dominant. Fix a general point $x_0 \in X$. Consider the maps $F \to X \times X$ given by the composition of $0 \times \infty \times id_F$ and $R$, and $X \to X \times X$ given by $x_0 \times id_X$. Let $U = F \times_X X$ be the fiber product. The restriction of $R$ gives a morphism $r : \mathbb{P}^1 \times U \to X$, with $r(0, u) = x_0$ such that $g : U \to X$ given by $g(u) = r(\infty, u)$ is dominant. After replacing $U$ by the normalization of $K(X)$ in $K(U)$, we can assume that $\dim U = \dim X$. Thus we have a commutative diagram

$\begin{array}{ccc}
P^1 \times U & \xrightarrow{q} & X \times X \\
p_2 & & \downarrow \pi_2 \\
U & \xrightarrow{g} & X
\end{array}$

where $\pi_2$ are projections to the second factor, and $q(t, u) = (r(t, u), g(u))$. We see that $q(\infty \times U) = \Delta \cap (X \times g(U))$ and $q(0 \times U) = x_0 \times g(U)$ as sets. Therefore the difference $\deg(g)(\Delta - x_0 \times X) \in CH_2(X \times X)$ is supported on $X \times (X - g(U))$. □

**Corollary 1.** Suppose that $f : X \to Y$ is a morphism of varieties such that the fibers of $f$ are rationally connected. Then there exists a nonempty open subset $V \subset Y$, a relative zero cycle $\xi \in Z_0(X \times_Y V/V)$, a proper closed subset $Z \subset X V = X \times_Y V$, and a cycle $\Gamma$ supported on $X \times_Y Z$ such that a multiple of the relative diagonal $\Delta \subset X V \times_Y X V$ is rationally equivalent to $\xi \times X + \Gamma$ after restriction to the (closed) fibers of $X V \to V$.

**Proof.** The geometric general fiber $X_{\bar{\eta}} = X \times_Y Spec \overline{K(Y)}$ is rationally connected by (the proof of) [Ko] thm IV 3.11. The previous lemma applied to the $X_{\bar{\eta}}$, produces cycles, $\xi', Z', \Gamma' \subset X \times Z'$, on $X_{\bar{\eta}} \times X_{\bar{\eta}}$, such that $\xi' \times X_{\bar{\eta}} + \Gamma'$ is rationally equivalent to the diagonal $\Delta'$ times a positive integer $N$. These cycles and the data defining the rational equivalence are defined over a finite extension of $K(Y)$. Therefore they can be spread out to relative cycles $\xi''$, $Z''$, $\Gamma''$ and data on $X_{\bar{\eta}}$, for some generically finite map $\bar{V} \to Y$. We then have $N$ times the relative diagonal of $X_{\bar{\eta}}$ is rationally equivalent to $\xi'' \times X_{\bar{\eta}} + \Gamma''$ on the fibers of $X_{\bar{\eta}}$. Now let $\xi, Z, \Gamma$ be the pushforward of $\xi''$, $Z''$, $\Gamma''$ onto $X_{\bar{\eta}}$ where $V = im(\bar{V})$. Then $(\deg(\bar{V}/V)) N \Delta$ will be rationally equivalent to $\xi \times X + \Gamma$ on the fibers, and the support of $\Gamma$ will lie in $X \times_Y Z$. □

From now on, we assume our varieties are complex. Cohomology is singular cohomology with rational coefficients. These groups carry mixed Hodge structures, but we will omit Tate twists, unless they seem essential. Given a smooth projective variety $X$ and closed subset $Z$, we want to be able to reduce the verification of the Hodge conjecture on $X$ into a separate verifications on $Z$ and $X - Z$. Here we use Jannsen’s formulation [J] for the Hodge conjecture for singular or nonproper varieties, which says that the space of rational $(-p, -p)$ cycles in the pure part of Borel-Moore homology $Gr_{-2p}W^pH_2p(X)$ consists of algebraic cycles. In other words, this space is spanned by fundamental classes of $p$-dimensional subvarieties of $X$. When $X$ is smooth, we can formulate this in compactly supported cohomology thanks to the duality isomorphism

$$H^{2(dim X - p)}_c(X, \mathbb{Q}) \cong H_{2p}(X, \mathbb{Q})(-dim X).$$
It is worth noting that this form of the Hodge conjecture for general $X$ would follow from the usual Hodge conjecture on a desingularization of a compactification of $X$ [J, thm 7.9].

**Lemma 3.**

1. If $X$ is an open subset of a smooth projective variety $X$, then the Hodge conjecture holds for $H^2_p(X)$, i.e. rational $(p, p)$ cycles in it are all algebraic if the conjecture holds for $H^2_p(X)$.

2. If $X$ is projective with a desingularization $\tilde{X} \to X$, then the Hodge conjecture holds for $H^2_p(X)$ i.e. rational $(-p, -p)$ cycles in it are all algebraic if the usual form of the conjecture holds for $H^2_p(\tilde{X})$ i.e. if the usual form of the conjecture holds for $H^{2(\dim X - p)}(\tilde{X})$.

3. Suppose that $X$ is a projective variety with a closed subset $Z \subset X$. If the Hodge conjecture holds for $H^2_p(Z)$ and $H^2_p(X - Z)$, then it holds for $H^2_p(X)$.

**Proof.** The proof of the first two statements are essentially contained in the arguments on [J, pp. 113-114]. In summary, for (1), we use the injectivity of $Gr^W_2 H^2_p(X) \to H^2_p(X)$ to reduce the Hodge conjecture for $H^2_p(X)$ to the corresponding statement $H^2_p(\tilde{X})$. For (2), we use the surjectivity of $H^2_p(\tilde{X}) \to Gr^W_{-2p} H^2_p(X)$ to achieve a similar reduction.

For the third statement, we use the exact sequence of mixed Hodge structures

$$H^2_p(X - Z) \to H^2_p(X) \to H^2_p(Z)$$

to obtain an exact sequence of pure Hodge structures

$$Gr^W_{-2p} H^2_p(Z) \to Gr^W_{-2p} H^2_p(X) \to Gr^W_{-2p} H^2_p(X - Z),$$

and then a similar sequence for the spaces of Hodge i.e. rational $(-p, -p)$-cycles. Given a Hodge cycle $\alpha$ in $Gr^W_{-2p} H^2_p(X)$, its image $\beta$ in $Gr^W_{-2p} H^2_p(X - Z)$ is algebraic by hypothesis. Thus $\beta = \sum_i n_i [V_i]$. Taking closures of the components allows us to lift $\beta$ to an algebraic cycle $\bar{\beta} = \sum_i n_i [\bar{V}_i]$ on $X$. Then the difference $\beta - \bar{\beta}$ would be represented by a Hodge and hence algebraic cycle supported on $Z$. □

As a prelude to the proof of the theorem, let us sketch a proof of lemma 4.

**Proof of lemma 1.** Given a uniruled fourfold $X$, it is dominated by a blow up $\tilde{X}$ of $\mathbb{P}^1 \times Y$ with $\dim Y = 3$. Since any Hodge cycle can be lifted to $\tilde{X}$, it suffices to prove the conjecture for this. Now $\tilde{X}$ can be decomposed into a disjoint union of strata of the form $C^1 \times S$ with $\dim S \leq 3$. Part 3 of the previous lemma shows that it suffices to check the Hodge conjecture for these strata. By part 1 of the lemma, it suffices to do this for compactifications $\mathbb{P}^1 \times S$. But this follows immediately from Künneth’s formula. □

**Theorem 1.** Let $X$ be a smooth projective five dimensional variety over $\mathbb{C}$, such that the base of the maximal rationally connected fibration is at most three dimensional. Then the Hodge conjecture holds for $X$. 
Proof. It is enough to check the Hodge conjecture for $H^n(X)$ for $n = 4$, since the other cases would follow from the Lefschetz $(1,1)$ theorem when $n = 2$, or by previous cases and Hard Lefschetz when $n \geq 6$. Let $f : X \to Y$ be maximal rationally connected fibration. Since the validity of the Hodge conjecture for $X$ is implied by its validity on any smooth blow up $\tilde{X} \to X$, we can assume that $f$ is a morphism. Note that all the fibers of $f_D$ decomposes as $\xi$ implied by its validity on any smooth blow up $\tilde{m}$ morphism. Let $D \subset X$ be a divisor containing $Z$, and let $\tilde{D} \to D$ be a desingularization. Note that we can assume that $f_V : X_V \to V$ and $g : \tilde{D}_V \to V$ are smooth, after shrinking $V$. Let $W = Y - V$. By lemma [4] it is enough to check the Hodge conjecture for $H^i_c(X_V) \cong H_0(X_V)$ and $H_0(X_W)$ separately. For $X_W$, note that it is a uniruled variety of dimension at most 4. The same goes for any desingularization of it. Therefore $X_W$ satisfies the Hodge conjecture in all degrees by lemma [1]

Now we turn to $X_V$. The identity on $H^i(X_V, \mathbb{Q})$, with $y \in V$, is given by the action of the restriction of the correspondence $\Delta$. This decomposes into a sum of the action of the restriction of $\xi$, which vanishes for $i > 0$, and the action of the restriction of $\Gamma$. Note that the image under $\Gamma |_{X}$ lies in the kernel of $H^i(X_V, \mathbb{Q}) \to H^i(X_V - D_y, \mathbb{Q})$, and this coincides with the image of the Gysin map $H^{i-2}(D_y, \mathbb{Q})(-1) \to H^i(X_V, \mathbb{Q})$ by [D2 Prop 8.2.8]. It follows that the map of local systems $R^{i-2}g_* \mathbb{Q} \to R^if_{V*} \mathbb{Q}$ is surjective; it is necessarily split surjective by Deligne’s theorem on semisimplicity of monodromy [D2 thm 4.2.6]. Therefore

$$H^i_c(V, R^{i-2}g_* \mathbb{Q}) \to H^i_c(V, R^if_{V*} \mathbb{Q})$$

is surjective for $i > 0$. This together with the fact that the Leray spectral sequence degenerates [D1] implies that $H^i_c(X_V, \mathbb{Q})$ is spanned by the sum of the images of $H^2_c(\tilde{D}_V, \mathbb{Q})(-1)$ and $H^3_c(V, \mathbb{Q})$ as mixed Hodge structures. In particular, it suffices to check the Hodge conjecture for $H^2_c(\tilde{D}_V, \mathbb{Q})$ and $H^3_c(V, \mathbb{Q})$. By lemma [3] we can check the corresponding statements after replacing $\tilde{D}_V$ and $V$ by their compactifications $\tilde{D}$ and $\tilde{V}$. The Hodge conjecture holds for $H^2(\tilde{D}, \mathbb{Q})$ by the Lefschetz $(1,1)$ theorem, and it holds for $Y$ since dim $Y \leq 3$. Now we are done. \hfill \Box

Recall that smooth variety is Fano if its anticanonical bundle $\omega^{-1}$ is ample. Such varieties are rationally connected by the work of Campana, Kollár, Mori and Miyaoka [Ko cor. V 2.14]. Thus:

**Corollary 2.** The Hodge conjecture holds for a Fano fivefold.

**Corollary 3.** The Hodge conjecture holds for a smooth hypersurface in $\mathbb{P}^n \times \mathbb{P}^{6-n}$ of bidegree $(a, b)$ (or degree $a$ when $n = 6$) with $3 \leq n \leq 6$ and $a \leq n$.

Proof. Projection of the hypersurface to $\mathbb{P}^{6-n}$ yields a family of Fano varieties on it of dimension two or more. This forces the base of the maximal rationally connected family to be less than or equal to 3. \hfill \Box

By similar arguments, it follows that a rationally connected $n$-fold is motivated by an $(n - 2)$-fold in the sense of [AI]. In particular, the Lefschetz standard conjecture holds for a rationally connected fourfold.
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