Higher order terms in the inflaton potential and the lower bound on the tensor to scalar ratio $r$

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The MCMC analysis of the CMB+LSS data in the context of the Ginsburg-Landau approach to inflation indicated that the fourth degree double–well inflaton potential best fits the present CMB and LSS data. This provided a lower bound for the ratio $r$ of the tensor to scalar fluctuations and as most probable value $r \approx 0.05$, within reach of the forthcoming CMB observations. We systematically analyze here the effects of arbitrary higher order terms in the inflaton potential on the CMB observables: spectral index $n_s$ and ratio $r$. Furthermore, we compute in close form the inflaton potential dynamically generated when the inflaton field is a fermion condensate in the inflationary universe. This inflaton potential turns to belong to the Ginsburg-Landau class too. The theoretical values in the $(n_s, r)$ plane for all double well inflaton potentials in the Ginsburg-Landau approach (including the potential generated by fermions) turn to be inside a universal banana-shaped region $B$. The upper border of the banana-shaped region $B$ is given by the fourth order double–well potential and provides an upper bound for the ratio $r$. The lower border of $B$ is defined by the quadratic plus an infinite barrier inflaton potential and provides a lower bound for the ratio $r$. For example, the current best value of the spectral index $n_s = 0.964$, implies $r$ is in the interval: $0.021 < r < 0.053$. Interestingly enough, this range is within reach of forthcoming CMB observations.

Contents

I. Introduction

II. Physical parametrization for inflaton potentials
   A. The fourth degree double–well inflaton potential
   B. The sixth–order double–well inflaton potential

III. Higher–order even polynomial double-well inflaton potentials

IV. The quadratic plus the $2^n$th order double-well inflaton potential
   A. The $n \to \infty$ limit at fixed $u$.
   B. The double limit $n \to \infty$ and $u \to 1$.

V. The quadratic plus the exponential potential.
   A. The limit $b \to \infty$ at fixed $u$.
   B. The double limit $b \to \infty$ and $u \to 1$.

VI. Dynamically generated inflaton potential from a fermion condensate in the inflationary stage.

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I. INTRODUCTION

The current WMAP data are validating the single field slow-roll scenario [1]. Single field slow-roll models provide an appealing, simple and fairly generic description of inflation [2, 3]. This inflationary scenario can be implemented using a scalar field, the *inflaton* with a Lagrangian density

\[ \mathcal{L} = a^3(t) \left[ \frac{\dot{\varphi}^2}{2} - \frac{(\nabla \varphi)^2}{2 a^2(t)} - V(\varphi) \right], \quad (1.1) \]

where \( V(\varphi) \) is the inflaton potential. Since the universe expands exponentially fast during inflation, gradient terms are exponentially suppressed and can be neglected. At the same time, the exponential stretching of spatial lengths classicalize the physics and permits a classical treatment. One can therefore consider an homogeneous and classical inflaton field \( \varphi(t) \) which obeys the evolution equation

\[ \ddot{\varphi} + 3 H(t) \dot{\varphi} + V'(\varphi) = 0, \quad (1.2) \]

in the isotropic and homogeneous Friedmann-Robertson-Walker (FRW) metric which is sourced by the inflaton

\[ ds^2 = dt^2 - a^2(t) d\vec{x}^2. \quad (1.3) \]

Here \( H(t) \equiv \dot{a}(t)/a(t) \) stands for the Hubble parameter. The energy density and the pressure for a spatially homogeneous inflaton are given by

\[ \rho = \frac{\dot{\varphi}^2}{2} + V(\varphi), \quad p = \frac{\dot{\varphi}^2}{2} - V(\varphi). \quad (1.4) \]

The scale factor \( a(t) \) obeys the Friedmann equation,

\[ H^2(t) = \frac{1}{3M_{Pl}^2} \left[ \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right]. \quad (1.5) \]

In the Ginsburg-Landau spirit the potential is a polynomial in the field starting by a constant term [4]. Linear terms can always be eliminated by a constant shift of the inflaton field. The quadratic term can have a positive or a negative sign associated to unbroken symmetry (chaotic inflation) or to broken symmetry (new inflation), respectively. The request of renormalizability restricts the degree of the inflaton potential to four. However, since the theory of inflation is an effective theory, potentials of degrees higher than four are in principle acceptable.

In order to have a finite number of inflation efolds, the inflaton potential \( V(\varphi) \) must vanish at its absolute minimum

\[ V'(\varphi_{\text{min}}) = V(\varphi_{\text{min}}) = 0 \quad (1.6) \]

Otherwise, inflation continues forever.

Since the inflaton potential must be bounded from below \( V(\varphi) \geq 0 \), the higher degree term must be even and with a positive coefficient. Hence, we consider polynomial potentials of degree \( n \) where \( 1 < n \leq \infty \).

A given Ginsburg-Landau potential will be **reliable** provided it is **stable** under the addition to the potential of terms of higher order. Namely, adding to the \( 2n \)th order potential further terms of order \( 2n + 1 \) and \( 2n + 2 \) should only produce **small** changes in the observables. Otherwise, the description obtained could not be trusted. Since, the highest degree term must be even and positive, this implies that all even terms of order higher or equal than four must be positive.

Moreover, when expressed in terms of the appropriate dimensionless variables, a relevant dimensionless coupling constant \( g \) can be defined by rescaling the inflaton field. This coupling \( g \) turns to be of order \( 1/N \) where \( N \sim 60 \) is the number of efolds since the cosmologically relevant modes exit the horizon till the end of inflation showing that the slow-roll approximation is in fact an expansion in \( 1/N \) [5]. It is then natural to introduce as coupling constant \( y \equiv 8N/g = \mathcal{O}(N^0) \). This is consistent with the stability of the results in the above sense. Generally speaking, the Ginsburg-Landau approach makes sense for small or moderate coupling.

As shown in refs. [6, 7] the double well (broken symmetric) fourth order potential

\[ V(\varphi) = \frac{1}{4} \lambda \left( \varphi^2 - \frac{m^2}{\lambda} \right)^2 = \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4 + \frac{m^4}{4\lambda} \quad (1.7) \]
provides a very good description of the CMB+LSS data. The mass term $m^2$ and the coupling $\lambda$ are naturally expressed in terms of the **two** energy scales which are relevant in this context: the energy scale of inflation $M$ and the Planck mass $M_{Pl} = 2.43534 \times 10^{18}$ GeV,

$$m = \frac{M^2}{M_{Pl}}, \quad \lambda = \frac{y}{8N} \left(\frac{M}{M_{Pl}}\right)^4.$$  \hfill (1.8)

Here $y = O(1)$ is the quartic coupling.

The MCMC analysis of the CMB+LSS data combined with the theoretical input above yields the value $y \approx 1.26$ for the coupling $[6, 7]$. $y$ turns to be **order one** consistent with the Ginsburg-Landau formulation of the theory of inflation $[7]$.

According to the current CMB+LSS data, this fourth order double–well potential of new inflation yields as most probable values: $n_s \approx 0.964$, $r \approx 0.051$ $[6, 7]$. This value for $r$ is within reach of forthcoming CMB observations. For the best fit value $y \approx 1.26$, the inflaton field exits the horizon in the negative concavity region $V''(\varphi) < 0$ intrinsic to new inflation. We find for the best fit $[6, 7]$,

$$M = 0.543 \times 10^{16} \text{ GeV for the scale of inflation and } m = 1.21 \times 10^{13} \text{ GeV for the inflaton mass.} \hfill (1.9)$$

Therefore, the energy scale of inflation turns out to coincide with the Grand Unification energy scale, well below the Planck energy scale. $M \ll M_{Pl}$ guarantees the validity of the effective theory approach to inflation. The fact that the inflaton mass is $m \ll M$ implies the appearance of infrared phenomena as the quasi-scale invariance of the primordial power.

Odd terms in the inflaton field $\varphi$ are allowed in $V(\varphi)$ in the effective theory of inflation. Choosing $V(\varphi)$ an even function of $\varphi$ implies that $\varphi \to -\varphi$ is a symmetry of the inflaton potential. At the moment, as stated in $[7, 12]$, we do not see reasons based on fundamental physics to choose a zero or a nonzero cubic term, which is the first non-trivial odd term. Only the phenomenology, that is the fit to the CMB+LSS data, decides on the value of the cubic and the higher order odd terms. The MCMC analysis of the WMAP plus LSS data shows that the cubic term is negligible and therefore can be ignored for new inflation $[6, 7]$. CMB data have also been analyzed at the light of slow-roll inflation in refs. $[8]$.

In the present paper we systematically study the effects produced by higher order terms ($n > 4$) in the inflationary potential on the observables $n_s$ and $r$.

In this approach, we show that all $r = r(n_s)$ curves for a large class of double–well potentials of arbitrary high order fall **inside** the **universal** banana region $B$ depicted in fig. $[10]$.

Moreover, the $r = r(n_s)$ curves for even double–well potentials with arbitrary positive higher order terms lie **inside** the universal banana region $B$ $[fig. \, 10]$. This is true for arbitrary large values of the coefficients in the potential.

The inflaton field may be a condensate of fermion-antifermion pairs in a grand unified theory (GUT) in the inflationary background. We explicitly compute in close the inflaton potential dynamically generated as the effective potential of fermions in the inflationary universe. This inflaton potential turns to belong to the Ginsburg-Landau class of potentials considered in this paper. We find that the corresponding $r = r(n_s)$ curves lie inside the **universal** banana region $B$ provided the one-loop part of the inflaton potential is at most of the same order as the tree level piece.

Therefore, there is **lower bound** for the ratio tensor/scalar fluctuations $r$ for **all** potentials just mentioned. This lower bound turns to be $r > 0.021$ for the current best value of the spectral index $n_s = 0.964$ $[11, 12]$.

The upper border of the universal region $B$ tells us that $r < 0.053$ for $n_s = 0.964$.

Therefore, we have within the large class of potentials **inside** the region $B$

$$0.021 < r < 0.053 \quad \text{for} \quad n_s = 0.964.$$  \hfill (1.10)

Anyhow, among the simplest potentials, the one that best reproduces the present CMB/LSS data, is just the fourth order double–well potential eq. $(1.7)$, yielding as most probable values: $n_s \approx 0.964$, $r \approx 0.051$. Our work here shows that adding higher order terms to the inflaton potential does not really improve the data description in spite of the addition of new free parameters.

Therefore, the fourth order double–well potential gives a robust and stable description of the present CMB/LSS data.

This paper is organized as follows: in section II we present in general inflaton potentials of arbitrary high degree, specializing them to fourth and sixth–order polynomial potentials and displaying their corresponding $r = r(n_s)$ curves. Sec. III contains the $2^n$th order double–well polynomial inflaton potentials with arbitrary random coefficients and their $r = r(n_s)$ curves. Sec. IV presents the $n \to \infty$ limits of these polynomial potentials and we present in sec. V the exponential potential and its infinite coupling limit. In sec. V we compute the inflaton potential from dynamically generated fermion condensates in a de Sitter space-time displaying their $r = r(n_s)$ curves. Finally, we present and discuss the universal banana region in sec. VII together with our conclusions.
II. PHYSICAL PARAMETRIZATION FOR INFLATON POTENTIALS

We start by writing the inflaton potential in dimensionless variables as

$$V(\phi) = M^4 v \left( \frac{\phi}{M_{Pl}} \right), \quad (2.1)$$

where $M$ is the energy scale of inflation and $v(\phi)$ is a dimensionless function of the dimensionless field argument $\phi = \phi/M_{Pl}$. Without loss of generality we can set $v'(0) = 0$. Moreover, provided $V''(0) \neq 0$ we can choose without loss of generality $|v''(0)| = 1/2$.

In the slow-roll regime, higher time derivatives in the equations of motion can be neglected with the final well known result for the number of efolds

$$N = -\int_{\phi_{exit}}^{\phi_{end}} d\phi \frac{v(\phi)}{v'(\phi)}, \quad (2.2)$$

where $\phi_{exit}$ is the inflaton field at horizon exit. To leading order in $1/N$ we can take $\phi_{end}$ to be the value $\phi_{min}$ at which $v(\phi)$ attains its absolute minimum $v(\phi_{min})$, which must be zero since inflation must stop after a finite number of efolds [7].

Then, in chaotic inflation we have $\phi_{min} = 0$, with $v'(\phi) > 0$ for $\phi > 0$, while in new inflation we have $\phi_{min} > 0$ with $v'(\phi) < 0$ for $0 < \phi < \phi_{min}$. We consider potentials $v(\phi)$ that can be expanded in Taylor series around $\phi = \phi_{min}$, with a non-vanishing quadratic (mass) term.

It is convenient to rescale the inflaton field in order to conveniently parametrize the higher order potential. We define a coupling parameter $g > 0$ by rescaling the inflaton and its potential keeping invariant the quadratic term, that is

$$v(\phi) = \frac{1}{g} v_1(\sqrt{g} \phi) \quad (2.3)$$

For a potential $v_1(u)$ expanded in power series around $u = 0$ we write:

$$v_1(u) = c_0 + \frac{1}{2} u^2 + \sum_{k \geq 3} c_k u^k \quad (2.4)$$

Then, replacing

$$u = \phi \sqrt{g}, \quad (2.5)$$

we find

$$v(\phi) = \frac{c_0}{g} + \frac{1}{2} \phi^2 + \sum_{k \geq 3} \frac{g^{k/2-1}}{k} c_k \phi^k. \quad (2.6)$$

The positive sign in the quadratic term corresponds to chaotic inflation (in which case $c_0 = 0$), while the negative sign corresponds to new inflation (in which case $c_0$ is chosen such that $v_1(u)$ vanishes at its absolute minimum).

Clearly $g$ plus the set of coefficients $c_k$ provide an overcomplete parametrization of the inflaton potential which we will now reduce. In the case of chaotic inflation a convenient choice is $c_4 = 1$, so that

$$v(\phi) = \frac{1}{2} \phi^2 + \sqrt{g} \frac{c_3}{3} \phi^3 + \frac{g}{4} \phi^4 + \sum_{k \geq 5} \frac{g^{k/2-1}}{k} c_k \phi^k \quad \text{[chaotic inflation]} \quad (2.7)$$

which represents a generic higher order perturbation of the trinomial chaotic inflation studied in refs. [6].

In the case of new inflation, where $\phi_{min} > 0$, it is more convenient to set without loss of generality that $u_{min} = 1$, $\phi_{min} = 1/\sqrt{g}$. In order to have appropriate inflation, $u_{min} = 1$ must be the absolute minimum of $v_1(u)$ and the closest one to the origin on the positive semi-axis. That is,

$$v_1'(1) = -1 + \sum_{k \geq 3} c_k = 0 \quad (2.8)$$

and then $v_1(1) = 0$ fixes from eq. the constant term $c_0$ in the potential

$$c_0 = \frac{1}{2} - \sum_{k \geq 3} \frac{c_k}{k} \quad (2.9)$$
We thus get for the inflaton potential
\[ v_1(u) = \frac{1}{2}(1 - u^2) + \sum_{k \geq 3} \frac{c_k}{k} (u^k - 1) \] [new inflation],
(2.10)
corresponding to
\[ v(\phi) = \frac{1}{2} \left( \frac{1}{g} - \phi^2 \right) + \sum_{k \geq 3} \frac{c_k}{k} \left( g^{k/2 - 1} \phi^k - \frac{1}{g} \right) \] [new inflation]
(2.11)
For the coupling \( g \) and the field \( \phi \) using eq. (2.5),
\[ g = \frac{1}{\phi_{\text{min}}^2} = \frac{M_{\text{Pl}}^2}{\varphi_{\text{min}}^2}, \quad u = \frac{\phi}{\phi_{\text{min}}} = \frac{\varphi}{\varphi_{\text{min}}}. \]
(2.12)
It follows, from eq. (2.2) it now follows that the parameter \( g \) can be expressed as the integral
\[ y(u) = 8 \int_{u_{\text{min}}}^{u} dx \frac{v_1(x)}{v_1'(x)}, \quad u = \sqrt{g} \phi_{\text{exit}} \]
(2.13)
where,
\[ g = \frac{y(u)}{8N}, \]
(2.14)
with \( u_{\text{min}} = 0 \) for chaotic inflation and \( u_{\text{min}} = 1 \) for new inflation. Eq. (2.13) can be regarded as a parametrization of \( g \) and \( y(u) \) in terms of the rescaled exit field \( u \). Clearly, as a function of \( u \), \( g \) is uniformly of order \( 1/N \). \( g \) is numerically of order \( 1/N \) as long as \( y(u) \) is of order one. As we shall see below, the coupling \( y \) or \( u \) is of order one. We have \( 0 < u < 1 \) for new inflation and \( 0 < u < +\infty \) for chaotic inflation.

In what follows we therefore use \( y(u) \) instead of \( g \) as a coupling constant and make contact with eq. (2.1) by setting
\[ \varphi = M_{\text{Pl}} \sqrt{\frac{8N}{y}} u, \quad V(\varphi) = \frac{8N M^4}{y} v_1 \left( \sqrt{\frac{y}{8N}} \frac{\varphi}{M_{\text{Pl}}} \right). \]
(2.15)
We can easily read from this equation the order of magnitude of \( \varphi \) and \( V(\varphi) \) since \( N \sim 60 \), \( M \) is given by eq. (1.9) and \( u \) and \( y \) are of order one. Hence, \( \varphi \sim M_{\text{Pl}} \) and \( V(\varphi) \sim N M^4 \).

As we will see below, the coupling \( y \) (or \( u \)) is the most relevant coupling since it is related to the inflaton rescaling: the tensor–scalar ratio \( r \) and the spectral index \( n_s \) vary in a more relevant manner with \( y \) than with the rest of the parameters \( c_k, \ k \geq 3 \) in the potential eq. (2.6).

By construction the function \( y(u) \) has the following properties
- \( y(u) > 0 \);
- \( y'(u) > 0 \) for \( u > 0 \) in chaotic inflation;
- \( y'(u) < 0 \) for \( 0 < u < u_{\text{min}} = 1 \) in new inflation;
- \( y(u) = 2(u - u_{\text{min}})^2 + O(u - u_{\text{min}})^3 \to 0 \) as \( u \to u_{\text{min}} \);
- \( y(u) \to \infty \) as \( u \to \infty \) in chaotic inflation;
- \( y(u) \approx -8v_1(0) \log u \to +\infty \) as \( u \to 0^+ \) in new inflation.

In terms of this parametrization and to leading order in \( 1/N \), the tensor to scalar ratio \( r \) and the spectral index \( n_s \) read:
\[ r = \frac{y(u)}{N} \left[ \frac{v_1'(u)}{v_1(u)} \right]^{2}, \quad n_s - 1 = -\frac{3}{8} r + \frac{y(u)}{4N} \frac{v_1''(u)}{v_1(u)} \]
(2.16)
Since \( y = y(u) \) can be inverted for any \( 0 < u < u_{\text{min}} \), these two relations can also be regarded as parametrizations \( r = r(y) \) and \( n_s = n_s(y) \) in terms of the coupling constant \( y \).

We are interested in the region of the \((n_s, r)\) plane obtained from eq. (2.16) by varying \( y \) (or \( u \)) and the other parameters in the inflaton potential. We call \( \mathcal{B} \) this region.

From now on, we will restrict to new inflation.
For a generic $v_1(u)$ [with the required global properties described above] we can determine the asymptotics of $B$, since they follow from the weak coupling limit $y \to 0$ and from the strong coupling limit $y \to \infty$. When $y \to 0$, then $u \to u_{\text{min}} = 1$ and from the property above,

$$r = \frac{8}{N} + O(u - 1) = 0.1333 \ldots + O(u - 1)$$

(2.17)

and

$$n_s = 1 - \frac{2}{N} + O(u - 1) = 0.9666 \ldots + O(u - 1).$$

(2.18)

When $y \to \infty$ we have in new inflation $u \to 0$ and then,

$$n_s \simeq 1 + \frac{2}{N} \log u \to -\infty , \quad r \simeq \frac{8}{N} \frac{u^2 \log u}{v_1(0)} \to 0^+ .$$

(2.19)

We see that in the strong coupling regime $r$ becomes very small and $n_s$ becomes well below unity. However, the slow-roll approximation is valid for $|n_s - 1| < 1$ and in any case, the WMAP+LSS results exclude $n_s < 0.9 \ [1]$. Therefore, the strong coupling limit is ruled out.

Eq. (2.16) for $r$ can be rewritten using eq. (2.13) in the suggestive form,

$$r = \frac{64}{N y(u)} \left[ \frac{d \ln y(u)}{du} \right]^{-2}$$

(2.20)

Since $64/N \sim 1$, $r$ only may be small in case $y(u)$ is large (the logarithmic derivative of $y(u)$ has a milder effect for large $y(u)$). Therefore, we only find $r \ll 1$ in a strong coupling regime.

Let us now study large classes of physically meaningful inflaton potentials in order to provide generic bounds on the region $B$ of the $(n_s, r)$ plane within an interval of $n_s$, surely compatible with the WMAP+LSS data for $n_s$, namely $0.93 < n_s < 0.99$. To gain insight into the problem, we consider first the cases amenable to an analytic treatment, leaving the generic cases to a numerical investigation. As we will see below, the boundaries of the region $B$ turn to be described parametrically by the analytic formulas (2.23) and (4.4).

A. The fourth degree double-well inflaton potential

The case when the $V(\phi)$ is the standard double-well quartic polynomial

$$V(\phi) = \frac{1}{4} \lambda \left( \phi^2 - \frac{m^2}{\lambda} \right)^2$$

has been studied in refs. [6, 7]. In the general framework outlined above we have for this case,

$$v_1(u) = \frac{1}{4} (u^2 - 1)^2 = \frac{1}{4} - \frac{1}{2} u^2 + \frac{1}{4} u^4 , \quad \lambda = \frac{y}{8 N} \left( \frac{M^2}{M_{Pl}} \right)^4 , \quad m = \frac{M^2}{M_{Pl}} .$$

(2.21)

By explicitly evaluating the integral in eq. (2.13) one obtains

$$y(u) = u^2 - 1 - \log u^2 ,$$

(2.22)

and then, from eq. (2.16)

$$n_s = 1 - \frac{1}{N} \frac{3 u^2 + 1}{(1 - u^2)^2} (u^2 - 1 - \log u^2) , \quad r = \frac{1}{N} \frac{16 u^2}{(1 - u^2)^2} (u^2 - 1 - \log u^2)$$

(2.23)

where $0 \leq u \leq u_{\text{min}} = 1$. As required by the general arguments above, $u$ is a monotonically decreasing function of $y$, ranging from $u = 1$ till $u = 0$ when $y$ increases from $y = 0$ till $y = +\infty$. In particular, when $u \to 1^-$, $y$ vanishes quadratically as,

$$y(u) \to 1 - \frac{1}{2} (1 - u^2)^2 .$$

The concavity of the potential eq. (2.21) for the inflaton field at horizon crossing takes the value

$$v''_1(u) = 3 u^2 - 1 .$$
We see that $v''_1(u)$ vanishes at $u = 1/\sqrt{3}$, that is at $y = \ln 3 - 2/3 = 0.431946 \ldots$ (This is usually called the spinodal point $[13]$). Therefore,

$$v''_1(u) > 0 \quad \text{for} \quad y < 0.431946 \ldots \quad \text{and} \quad v''_1(u) < 0 \quad \text{for} \quad y > 0.431946 \ldots \quad (2.24)$$

Our MCMC analysis of the CMB+LSS data combined with the theoretical model eq. (2.21) yields $y \simeq 1.26 [6, 7]$ deep in the negative concavity region $v''_1(u) < 0$.

The negative concavity case $v''_1(u) < 0$ for $y > 0.431946 \ldots$ is specific to new inflation eq. (2.21). $v''_1(u)$ can be expressed as a linear combination of the observables $n_s$ and $r$ as

$$n_s - 1 + \frac{3}{8} r = \frac{y(u)}{4N} \frac{v''_1(u)}{v_1(u)}$$

As expected in the general framework presented above, the limit $u \to 1^-$ implies weak coupling $y \to 0^+$; that is, the potential is quadratic around the absolute minimum $u_{\min} = 1$ and we find,

$$n_s \left. y \right|_{u=0} - 1 = \frac{2}{N} , \quad r \left. y \right|_{u=0} = \frac{8}{N} , \quad u \left. y \right|_{u=0} = 1 , \quad (2.25)$$

which coincide with $n_s$ and $r$ for the monomial quadratic potential in chaotic inflation.

In the limit $u \to 0^+$ which implies $y \to +\infty$ (strong coupling), we have

$$u \left. y \right|_{u=0} = e^{-b/2} \to 0^+$$

and

$$n_s \left. y \right|_{u=0} = 1 - \frac{y}{N} , \quad r \left. y \right|_{u=0} = \frac{16 y}{N} e^{-y-1} . \quad (2.26)$$

Notice that the slow-roll approximation is no longer valid when the coefficient of $1/N$ becomes much larger than unity. Hence, the results in eq. (2.26) are valid for $y \lesssim N$. We see that in this strong coupling regime (see fig. 1), $r$ becomes very small and $n_s$ becomes well below unity. However, the WMAP+LSS results exclude $n_s \lesssim 0.9 [1]$. Therefore, this strong coupling limit $y \gg 1$ is ruled out.

For the fourth order double–well inflaton potential, the relation $r = r(n_s)$ defined by eq. (2.23) is a single curve depicted with dotted lines in fig. 1. It represents the upper border of the banana shaped region $\mathcal{B}$ in fig. 1.

Notice that there is here a maximum value for $n_s$, namely $n_s^{\text{max}} = 0.96782 \ldots$ with $r(n_s^{\text{max}}) = 0.1192 \ldots [7]$. The curve $r = r(n_s)$ has here two branches: the lower branch $r < r(n_s^{\text{max}})$ in which $r$ increases with increasing $n_s$ and the upper branch $r > r(n_s^{\text{max}})$ in which $r$ decreases with increasing $n_s$.

### B. The sixth–order double–well inflaton potential

We consider here new inflation described by a six degree even polynomial potential with broken symmetry. According to eq. (2.11) and eq. (2.3) we then have

$$V(\varphi) = \frac{M^4}{g} \left( \frac{\sqrt{g}}{M} \right) \varphi v_1 \left( \frac{\sqrt{g}}{M} \varphi \right) , \quad v_1(u) = c_0 - \frac{1}{2} u^2 + \frac{c_4}{4} u^4 + \frac{c_6}{6} u^6 . \quad (2.27)$$

where for stability we assume $c_6 \geq 0$. Moreover, if we regard this case as a higher order correction to the quartic double–well potential, then $c_4$ is positive.

The inflaton potential eq. (2.27) is a particular case of eq. (2.4). The conditions eqs. (2.8) and (2.9) that the absolute minimum of $v_1(u)$ be at $u_{\min} = 1$ yields

$$c_4 + c_6 = 1 , \quad c_0 = \frac{1}{2} - \frac{1}{4} c_4 - \frac{1}{6} c_6 \quad (2.28)$$

It is convenient to use $b \equiv c_6$ as free parameter so that $b \geq 0$ and $c_4 = 1 - b$. Thus,

$$v_1(u) = \frac{1}{2} (1 - u^2) - \frac{b}{4} (1 - u^4) - \frac{b}{6} (1 - u^6) = \frac{1}{12} (1 - u^2)^2 (3 + 2 b) u^2 \quad (2.29)$$

where $b \leq 1$ in order to ensure that $c_4 \geq 0$.

The integral in eq. (2.13) can be explicitly evaluated with the result

$$y(u) = 8 \int_1^u dx \frac{v_1(x) }{v_1(x)} = \frac{2}{3} (u^2 - 1) - \frac{1}{3} (3 + b) \log u^2 + \frac{(1 + b)^2}{3 b} \log \frac{1 + b u^2}{1 + b} \quad (2.30)$$
According to the general arguments presented above [see the lines below eq. (2.13)] one can verify that $y(u)$ is a monotonically decreasing function of $u$ for $0 < u < 1$, where $+\infty > y > 0$.

The scalar index $n_s$ and the tensor–scalar ratio $r$ are evaluated from eq. (2.16) as

$$r = \frac{y}{N} \left[ \frac{12 u (1 + b u^2)}{(1 - u^2)(3 + 2 b u^2)} \right]^2, \quad n_s = 1 - \frac{3}{8} r + \frac{3 y(u)}{N} \frac{5 b u^4 + 3 (1 - b) u^2 - 1}{(1 - u^2)^2 (3 + 2 b u^2)}$$

(2.31)

Various curves $r = r(n_s)$ are plotted in fig. 1 for several values of $b$ in the interval $[0, 1]$ sweeping the region $B$. We see that for increasing $b$ [namely, for increasing sextic coupling and decreasing quartic coupling, see eq. (2.29)] the curves move down and right, sweeping the banana-shape region $B$ depicted on fig. 1.

Clearly, $y$ is a variable more relevant than $b$. Changing $y$ moves $n_s$ and $r$ in the whole available range of values, while changing $b$ only amounts to displacements transverse to the banana region $B$ in the $n_s, r$ plane. In particular, for a given $n_s$, $r$ becomes smaller for increasing $b$.

![Fig. 1: We plot here $r$ vs. $n_s$ for the broken-symmetry sixth-order inflaton potential eq. (2.29) setting $N = 60$. The curves are obtained from eq. (2.31) with the sextic coefficient $b \equiv c_6$ fixed to the values indicated in the figure. We see that $y$ is the relevant coupling while $b$ only varies $r$ and $n_s$ transversally to the narrow banana-shape region. The two important limiting curves are shown: $b \to 0$ corresponding to the fourth degree potential eq.(2.21) and $b \to 1$ corresponding to the sixth degree potential eq.(2.32). The upmost point where all curves coalesce corresponds to the monomial quadratic potential $n_s = 0.9666 \ldots$, $r = 0.1333 \ldots$ for $N = 60$ [see eqs.(2.17)-(2.18)].](image)

We see in fig. 1 two important limiting curves: the $b \to 0$ and the $b \to 1$ curves. When $b = 0$ the function $v_1(u)$ reduces to the fourth order double-well potential eq.(2.21) and we recover its characteristic curve $r = r(n_s)$. When $b = 1$ the potential has no quartic term and reduces to the quadratic plus sixth order potential:

$$v_1(u) \overset{b=1}{=} \frac{1}{6} (1 - u^2)^2 (2 + u^2) = \frac{1}{3} - \frac{1}{2} u^2 + \frac{1}{6} u^6. \quad (2.32)$$

In summary, the quadratic plus quartic broken-symmetry potential describes the upper/left border of the banana-shaped region $B$ of fig. 1, while the quadratic plus sextic broken-symmetry potential describes its lower/right border.

### III. HIGHER–ORDER EVEN POLYNOMIAL DOUBLE-WELL INFLATON POTENTIALS

The generalization of the sixth order inflaton potential with broken symmetry to arbitrarily higher orders is now straightforward:

$$V(\varphi) = \frac{M^4}{g} v_1 \left( \frac{\sqrt{g} \varphi}{M_P} \right), \quad v_1(u) = \frac{1}{2} (1 - u^2) + \sum_{k=2}^n \frac{c_{2k}}{2k} (u^{2k} - 1), \quad (3.1)$$
with the constraint eq. (2.8)

$$\sum_{k=2}^{n} c_{2k} = 1$$  \hspace{1cm} (3.2)$$

which guarantees that $u = 1$ is an extremum of $v_1(u)$.

We consider here the case when all higher coefficients $c_{2k}$ are positive or zero:

$$c_{2k} \geq 0, \quad k = 2, \ldots, n$$

such that $u_{\text{min}} = 1$ is the unique positive minimum.

We determine the shape of the $B$ region for arbitrary positive or zero values of the coefficients $c_{2k}$ subject to the constraint (3.2), performing a large number of simulations with different setups. After producing coefficients $c_{2k}$ we numerically computed the function $y(u)$ following eq. (2.13)

$$y(u) = 4 \int_{u}^{1} \frac{dx}{x} \frac{1 - x^2 + \sum_{k=2}^{n} \frac{c_{2k}}{k} (x^{2k} - 1)}{1 - \sum_{k=2}^{n} c_{2k} x^{2k-2}}$$

and obtain the $r = r(n_s)$ curves from eq. (2.16) by plotting directly $r$ vs. $n_s$.

Uniform distributions of coefficients are obtained by setting

$$c_{2k} = \left( \sum_{j=1}^{n} \log \xi_j \right)^{-1} \log \xi_k, \quad k = 1, 2, \ldots, n$$  \hspace{1cm} (3.3)$$

where the numbers $\xi_k$ are independently and uniformly distributed in the unit interval. We used the parametrization eq. (3.3) also when the $\xi_k$ are chosen according to other rules.

For example, in figs. 2-3 we plot the results when $n = 5$, that is for the ten degree polynomial. In this case we let $\xi_4$, $\xi_6$, $\xi_8$ and $\xi_{10}$ take independently the values 0.001, 0.5 or 0.999, for a total of 78 distinct configurations of

FIG. 2: $r$ vs. $n_s$ for the 10th. order even polynomial potential eq. (3.1) with $n = 5$ and the coefficients $c_{2k}$ taking independently the values indicated. The relation to the numbers $\xi_k$ is given in eq. (3.3). The upper/left border curve $c_4 = 1, c_6 = c_8 = c_{10} = 0$ corresponds to the fourth order potential eq. (2.21). The lower/right border curve $c_{10} = 1, c_4 = c_6 = c_8 = 0$ corresponds to the quadratic plus 10th order term potential eq. (4.1) for $n = 5$. These are the limiting curves of the banana $B$ region.
FIG. 3: A detail The banana region $B$ in the $(n_s, r)$ plane for the quadratic plus 10th order polynomial as in fig. 2, but with the curves split in two parts by the value $r(n_s^{max})$. The upper panel shows the upper branches $r > r(n_s^{max})$ in which $r$ decreases with $n_s$, while the lower panel shows the lower branches $r < r(n_s^{max})$ in which $r$ increases with $n_s$. The quadratic plus 10th order polynomial thus provides the lower border of the banana region $B$ setting the lower bound on $r$. This bound is here $r >$ for the observed allowed range $< n_s <$.

coefficients. For better clarity, in figs. 2 we also include the two border cases $c_4 = 1, c_6 = c_8 = c_{10} = 0$ and $c_{10} = 1, c_4 = c_6 = c_8 = 0$.

For higher values of $n$ we extracted the numbers $\xi_k$ at random within the unit interval. In particular, for the highest case considered, $n = 50$, we used three distributions: in the first, the $\xi_k$ were all extracted independently and uniformly over unit interval; in the second we set $\log \xi_k = 2^{-k} \log \xi_k$ and extracted the $\xi_k$ independently and uniformly; in the third we picked at random four $\xi_k$ freely varying and fixed to 1 the remaining 45 ones (that is we picked at random four possibly non–zero $c_{2k}$, setting the rest to zero); the values of the four free $\xi_k$ were chosen at random in the same set of values ($0.001, 0.5, 0.999$) of the $n = 5$ case. The results of these simulations are shown in fig. 3.

As evident from fig. 3 where the $r = r(n_s)$ curves are split in upper/lower branches with growing/decreasing $r = r(n_s)$ and especially from fig. 5 the case of the quadratic plus 2nth order polynomial provides a bound to the banana region $B$ from below. That is, for any fixed value of $n_s$, the quadratic plus 2nth order polynomial provides the lowest value for $r$.

One sees from fig. 5 that some blue curves $r = r(n_s)$ go beyond the slashed red curve $r = r(n_s)$ for the quadratic plus $u^{100}$ potential on the right upper border of the banana region $B$. Namely, the right upper border of the $B$ region is not given by the quadratic plus $u^{100}$ potential while this potential provides the lower border of the $B$ region.

We performed many other tests with intermediate values of $n$ and several other distributions, including other $k$–dependent distributions, with characteristic values for $c_{2k}$ growing linearly with $k$ or decreasing in a power–like or exponential way. In all cases, the results were consistent with those given above.

It is also important to observe that the class of potentials considered, that is arbitrary even polynomials with positive or zero couplings, is a class of weakly coupled models. This is evident from fig. 4 were $n_s$ is plotted vs. the coupling $y$, which remains of order one when $n_s$ decreases well below the current experimental limits. This weak coupling is the reason why the addition of higher even monomials to these potentials causes only minor quantitative changes to the shape of the $r = r(n_s)$ curves.
IV. THE QUADRATIC PLUS THE $2^n$TH ORDER DOUBLE-WELL INFLATON POTENTIAL

In order to find the observationally interesting right and down border of the banana we consider the quadratic plus the $2^n$th order potential for new inflation [10],

$$v_1(u) = \frac{1}{2} \left( 1 - u^2 \right) + \frac{1}{2^n} \left( u^{2n} - 1 \right). \quad (4.1)$$

As in the general case eq.(2.10), we choose the absolute minimum at $u = 1$. The customary relation eq.(2.13) takes here the form [10],

$$y(u) = \frac{4}{n} \int_u^1 dx \frac{n (1 - x^2) + x^{2n} - 1}{1 - x^{2n-2}} \quad \text{where} \quad 0 < u < 1. \quad (4.2)$$

This integral can be expressed as a sum of $n$ terms including logarithms and arctangents [9].

In the weak coupling limit $y \to 0$, $n_s$ and $r$ take the values of the quadratic monomial potential eqs.(2.17)-(2.18) [7, 10]:

$$n_s - 1 \ y \to 0 \ = \ - \frac{2}{N} = -0.0333 \ldots, \quad r \ y \to 0 \ = \ \frac{8}{N} = 0.1333 \ldots, \quad (4.3)$$

while in the strong coupling limit $y \to \infty$ at fixed $n$, $n_s$ and $r$ take the values

$$n_s \simeq 1 + \frac{2}{N} \log u \to -\infty, \quad r \simeq -\frac{16}{N} \frac{n}{n - 1} u^2 \log u \to 0^+, \quad$$

in accordance with the general formula eq.(2.19). In fig. 6 we plot $r$ vs. $n_s$ for the potential eq.(4.1) and the exponents $n = 5, 10, 20, 100, 500$ and 5000. We see that for $n \to \infty$, $r$ vs. $n_s$ tends towards a limiting curve. For $y \to 0$ we reach the upper end of the curve [the monomial quadratic potential eq.(4.3)] while for large $y$ the left and lower end of the curve is reached. However, the current CMB–LSS data rule out this strong coupling part of the curve for $n_s < 0.95$.

A. The $n \to \infty$ limit at fixed $u$.

Let us first compute $y(u)$ eq.(4.2) for $n \to \infty$ at fixed $u$. Since $0 < x < 1$ in the integrand of eq.(4.2),

$$\lim_{n \to \infty} x^2 = 0,$$

and eq.(4.2) reduces to

$$y(u) \ n \to \infty \ = \ \frac{4}{n} \int_u^1 dx \left( n (1 - x^2) - 1 \right) = 2 \left[ u^2 - 1 - \ln u^2 + O \left( \frac{1}{n} \right) \right].$$
FIG. 5: \( r \) vs. \( n_s \) for the 100th order polynomial potential eq. (3.1) for \( n = 50 \). The coefficients \( c_{2k} \) were chosen or extracted at random as indicated in the two panels. The two border curves of the banana region \( B \) are clearly indicated. The upper border is the fourth order potential eq. (2.21) and the lower border is the quadratic plus the 2nth order potential eq. (4.1). The quadratic plus the 2nth order potential always provides the lowest value for \( r \) at any fixed \( n_s \) in its lower branch. In the upper panel, all the coefficients \( c_{2k} \) were extracted independently from a flat distribution ranging from 0 to 100; in this case the curves accumulate near the quadratic plus quartic potential eq. (2.21). The upper panel is the generic case. In the lower panel, we picked at random four possibly non-zero \( c_{2k} \) and fixed to zero the remaining 44 ones; in this case the curves accumulate near the quadratic plus 2nth order potential eq. (4.1) with \( 2n = 100 \).
FIG. 6: $r$ vs. $n_s$ for the quadratic plus $u^2n$ potential eq. (4.7) setting $N = 60$. The curves for the exponents $n = 2$, 3, 5, 10, 20 and 100 are displayed as well as the limiting curves obtained in the $n = \infty$ limits eqs. (4.6) and (4.11). Eq. (4.6) describes the lower bordering curve while eq. (4.11) describes the upper-right bordering curve. We see that for growing $n$ the curves $r$ vs. $n_s$ tend towards the limiting curves. The upmost point where all curves coalesce corresponds to the monomial quadratic potential $n_s = 0.9666 \ldots$, $r = 0.13333 \ldots$ [see eq. (2.25)].

Hence, eq. (4.2) becomes

$$y(u)_{n \to \infty} = \frac{1}{2} \frac{(1 - u^2)}{- \ln u^2 - 1 + u^2}$$

where $0 < u < 1$ and $0 < y < +\infty$. (4.4)

which is just twice the result found in the quartic double–well potential, eq. (2.22). Notice that $v_1(u)$ eq. (4.1) in the $n \to \infty$ limit becomes

$$\lim_{n \to \infty} v_1(u) = \begin{cases} \frac{1}{2} (1 - u^2) & \text{for } u < 1 \\ +\infty & \text{for } u > 1 \end{cases}.$$ (4.5)

From eqs. (2.16), (4.1) and (4.4) we find for $r$ and $n_s$ in the $n \to \infty$ limit

$$n_s - 1 \to \infty - \frac{1}{N} \frac{2 u^2 + 1}{(1 - u^2)^2} (-\ln u^2 - 1 + u^2),$$

$$r \to \infty \frac{8}{N} \frac{u^2}{(1 - u^2)^2} (-\ln u^2 - 1 + u^2).$$ (4.6)

Now, in the limiting cases $u \to 0$ and $u \to 1$ (at $n = \infty$), that is, the strong coupling limit $y \to \infty$ and the weak coupling limit $y \to 0$, respectively, we obtain from eqs. (4.6)

$$\lim_{u \to 1} n_s(n = \infty) - 1 = -\frac{3}{2N} = -\frac{1}{40} = -0.025, \quad \lim_{u \to 1} r(n = \infty) = \frac{4}{N} = \frac{1}{15} = 0.0666 \ldots,$$ (4.7)

$$\lim_{u \to 0} n_s(n = \infty) = -\infty, \quad \lim_{u \to 0} r(n = \infty) = 0.$$

However, as explained in sec. III the slow-roll expansion is no more valid when $|n_s - 1| \gtrsim 1$. Moreover, the WMAP+LSS results exclude $n_s \lesssim 0.9$ [1]. Therefore, the limit $u \to 0$ is ruled out.

Eqs. (4.6) describe the rightmost (limiting) curve in fig. 6 in its lower part, namely $0 < r < 4/N = 0.0666 \ldots$. The upper part is obtained in the double limit $n \to \infty$ and $u \to 1$ (or, equivalently $n \to \infty$ and $y \to 0$), as we show in the next section.
B. The double limit $n \to \infty$ and $u \to 1$.

As we can see from fig. 6 when $y$ varies from zero to infinity at fixed $n$, the potential eq. (4.1) covers the region

$$0 < r < \frac{8}{N},$$

the point $r = 8/N$ corresponding to the small coupling limit $y = 0$.

Notice however that the $n \to \infty$ limit eqs. (4.4) and (4.6) only describe the region $0 < r < 4/N$. In order to also describe the small coupling region $8/N > r > 4/N$ for $n \to \infty$, we have to take in eq. (4.2) the double limit $u \to 1$ and $n \to \infty$. This can be achieved by changing the integration variable in eq. (4.2) as $x = t^{\frac{1}{2}}$

$$y(u) = \frac{2}{n^2} \int_{\tau}^{1} \frac{dt}{\tau} \frac{n(1-t^{\frac{1}{2}})+t-1}{1-t^{\frac{1}{2}}-\frac{n}{2}} \quad \text{where} \quad \tau = u^{2n}, \quad 0 < \tau < 1.

Letting $n \to \infty$ at fixed $\tau$ yields,

$$\frac{n^2}{2} y(u) \xrightarrow{n \to \infty} n \int_{\tau}^{1} \frac{dt}{\tau} \frac{1-\ln t}{1-t} = \ln \tau + \frac{1}{2} \ln^2 \tau + \text{Li}_2(1-\tau), \quad (4.8)$$

where

$$\text{Li}_2(s) = - \int_0^1 \frac{dt}{t} \ln(1-t),$$

is the dilogarithmic function [6].

Then, in this double limit $n \to \infty$, $u \to 1$ eq. (4.2) becomes

$$\gamma^2(\tau) = \frac{n^2}{2} y(u) \xrightarrow{n \to \infty} \ln \tau + \frac{1}{2} \ln^2 \tau + \text{Li}_2(1-\tau). \quad (4.9)$$

That is, $\tau$ and $\gamma^2$ are fixed in this $n \to \infty$, $u \to 1$ limit. Notice that $0 < \tau < 1$, $0 < \gamma < \infty$ while $y \to 0$

$$y(u) \xrightarrow{n \to \infty} n \int_{\tau}^{1} \frac{dt}{\tau} \frac{1}{2} \gamma^2(\tau) \to 0 \quad \text{and} \quad u = \tau \xrightarrow{n \to \infty} 1 + \mathcal{O} \left( \frac{1}{n} \right). \quad (4.10)$$

From eq. (2.16) the spectral index $n_s$, and the ratio of tensor to scalar fluctuations $r$ for fixed $\gamma$ and $\tau$ take here ($n = \infty$, $y = 0$ and $u = 1$) the following form,

$$n_s - 1 \xrightarrow{n \to \infty} n \int_{\tau}^{1} \frac{dt}{\tau} \frac{2 \gamma^2(\tau) (1-\tau^2)}{N (\tau - 1 - \ln \tau)^2} \equiv \frac{2 \gamma^2(\tau)}{N} \frac{\tau}{(\tau - 1 - \ln \tau)^2}. \quad (4.11)$$

We obtain from eqs. (4.11) in the limiting cases $\tau \to 0$ and $\tau \to 1$,

$$\lim_{\tau \to 0} n_s - 1 = - \frac{3}{2N} = - \frac{1}{40} = -0.025, \quad \lim_{\tau \to 0} r = \frac{4}{N} = \frac{1}{15} = 0.0666\ldots,$$

$$\lim_{\tau \to 1} n_s - 1 = - \frac{2}{N} = - \frac{1}{30} = -0.0333\ldots, \quad \lim_{\tau \to 1} r = \frac{8}{N} = \frac{2}{15} = 0.1333\ldots \quad (4.12)$$

Notice that $n_s$ and $r$ for $n \to \infty$ and then $u \to 1$ eq. (4.7) coincides with $r$ and $n_s$ in the double limit $n \to \infty$, $u \to 1$ for $\tau = u^{2n} \to 0$ eq. (4.12). Namely, eqs. (4.3) and (4.11) match to each other as eqs. (4.1) and (4.12).

Eqs. (4.11) describe the rightmost (limiting) curve in fig. 6 in its upper part, namely $8/N = 0.1333\ldots > r > 4/N = 0.0666\ldots$. The lower part, $0 < r < 4/N = 0.0666\ldots$, is described by eqs. (4.5). $r$ and $n_s$ given by eqs. (4.10) and (4.11) continuously match at $n_s = 0.975$, $r = 0.0666\ldots$. However, the derivative $dr/dn_s$ is discontinuous at this point.

There is here a quadratic relation between $n_s$ and $r$ for $r \to 4^-/N$ valid in the $n = \infty$ limit:

$$\left( r - \frac{4}{N} \right)^2 = - \frac{64}{3N} \left( n_s - 1 \right) + \frac{3}{2N} \left[ 1 + \mathcal{O} \left( \sqrt{n_s - 1} + \frac{3}{2N} \right) \right]. \quad (4.13)$$

From eqs. (4.13) and (4.11) we get respectively

$$\lim_{r \to 4^-/N} \frac{dr}{dn_s} = +\infty, \quad \lim_{r \to 4^+/N} \frac{dr}{dn_s} = - \frac{8}{3},$$

as we can see in fig. 6.
V. THE QUADRATIC PLUS THE EXPONENTIAL POTENTIAL.

Since the exponential function contains all powers of the variable, it is worthwhile to consider it. As before, we restrict ourselves to potentials even in \( u \):

\[
v(\phi) = \frac{c_0}{g} - \frac{1}{2} \phi^2 + \frac{1}{2} \frac{c}{g} \left( \phi^2 - 1 - \hat{g} \phi^2 \right),
\]

where \( \hat{g} > 0 \) and \( c > 0 \) are free parameters, while as usual \( c_0 \) ensures that \( v(\phi) \) vanishes at its absolute minimum \( \phi = \phi_{\text{min}} = 1/\sqrt{\hat{g}} \). We find

\[
\phi_{\text{min}} = \frac{1}{\sqrt{\hat{g}}} = \sqrt{\frac{1}{\hat{g}} \log(1 + c)} , \quad b \equiv \frac{1}{2} \log(1 + c) > 0 , \quad g = \frac{\hat{g}}{2b}
\]

and

\[
c_0 = \frac{1}{2} \left[ \left( 1 + \frac{1}{c} \right) \log(1 + c) - 1 \right]
\]

In terms of the variable \( u = \phi/\phi_{\text{min}} \) the potential \( v_1(u) \) defined in general by eq. (5.1) takes here the form,

\[
v_1(u) = \frac{e^{-2b(1-u^2)} - 1 + 2b (1-u^2)}{4b (1-e^{-2b})} .
\]

Expanding the potential eq. (5.2) in powers of \( u \) yields

\[
v_1(u) \overset{u \to 0}{=} \frac{1 + e^{2b} (2b - 1)}{4b (e^{2b} - 1)} - \frac{1}{2} u^2 + \frac{b}{2 (e^{2b} - 1)} u^4 + O(u^6) .
\]

It is interesting to expand the potential in powers of \( b \) in order to make contact with the polynomial potentials of sec. II A II B. We get from eq. (5.2)

\[
v_1(u) \overset{b \to 0}{=} \frac{1}{12} (1 - u^2)^2 (3 + 2b u^2) + O(b^2)
\]

which is exactly the fourth order double–well potential eq. (2.21) to zeroth order in \( b \) and the sixth–order double–well potential eq. (2.29) to first order in \( b \).

The field \( u \) at horizon exit follows from the customary eq. (2.13) which takes here the form:

\[
y(u) = \frac{2}{b} \int_1^u \frac{dx}{x} \frac{e^{-2b(1-x^2)} + 2b (1-x^2) - 1}{e^{-2b(1-x^2)} - 1} ,
\]

Changing the integration variable to \( w \equiv 1 - e^{-2b(1-x^2)} \), eq. (5.3) becomes

\[
y(u) = 2 \left( - \ln u^2 - 1 + u^2 \right) + \frac{2}{b} \ln u - \frac{1}{b} \int_0^{1-e^{-2b(1-u^2)}} \frac{dw}{w} \frac{\log(1-w)}{2b + \log(1-w)} .
\]

The spectral index \( n_s \), and the ratio \( r \) are expressed from eq. (2.16) as,

\[
n_s - 1 = \frac{3}{8} r + \frac{b y(u)}{N} \left( \frac{4b u^2 + 1}{e^{-2b(1-u^2)} + 2b (1-u^2) - 1} - 1 \right),
\]

\[
r = \frac{16 b^2}{N} u^2 y(u) \left[ \frac{e^{-2b(1-u^2)} - 1}{e^{-2b(1-u^2)} + 2b (1-u^2) - 1} \right]^2 .
\]

We study below eqs. (5.1)–(5.5) in the \( b \to \infty \) limit in the two regimes: \( b \to \infty \) with \( u \) fixed and \( b \to \infty \) with \( u \to u_{\text{min}} = 1 \). These are the limits investigated in secs. IV A and IV B for the quadratic plus \( u^{2n} \) potential, respectively.

In fig. 7 we plot \( r \) vs. \( n_s \) for the quadratic plus exponential potential eq. (5.1) and the values of the coefficient \( c = 0.1, 0.5, 1, 5, \) and \( 10 \). We see that for growing \( c \), \( r \) vs. \( n_s \) tends towards a limiting curve. This curve is the lower border of the banana shaped region \( \mathcal{B} \). The upper border is determined by the fourth order potential eq. (2.21).
FIG. 7: \( r \) vs. \( n_s \) for the quadratic plus exponential potential eq.(5.1) with the coefficient \( 0 \leq b \leq \infty \) and setting \( N = 60 \). We see that for growing \( b \gg 1 \), \( r \) vs. \( n_s \) tends towards a limiting curve to the right and down of the banana shaped region \( B \). This curve is the lower border of the region \( B \). The upper border is determined by the fourth order potential eq.(2.21). The upmost point where all curves coalesce corresponds to the monomial quadratic potential \( n_s = 0.9666 \ldots , r = 0.13333 \ldots \) [see eq.(2.25)].

### A. The limit \( b \to \infty \) at fixed \( u \).

For large \( b \) we have in eqs.(5.4)-(5.5),

\[
e^{-2b(1-u^2)} \ll 1 \quad \text{since} \quad b \gg 1 \quad \text{and} \quad u < 1 ,
\]

and we find

\[
\begin{align*}
v_1(u) &\xrightarrow{b \to \infty} \frac{1}{2} (1 - u^2) + \mathcal{O} \left( \frac{1}{b} \right) \quad \text{for} \quad u < 1 , \\
\lim_{n \to \infty} v_1(u) &\xrightarrow{u > 1} +\infty , \\
y(u) &\xrightarrow{b \to \infty} 2 \left( -\ln u^2 - 1 + u^2 \right) + \mathcal{O} \left( \frac{1}{b} \right) , \\
n_s - 1 &\xrightarrow{b \to \infty} -\frac{1}{N} \frac{2u^2 + 1}{(1 - u^2)^2} \left( -\ln u^2 - 1 + u^2 \right) + \mathcal{O} \left( \frac{1}{b} \right) , \\
r &\xrightarrow{b \to \infty} 8 \frac{u^2}{N} \frac{(1 - u^2)^2}{(1 - u^2)^2} \left( -\ln u^2 - 1 + u^2 \right) + \mathcal{O} \left( \frac{1}{b} \right) .
\end{align*}
\]

These equations for \( r \) vs. \( n_s \) exactly coincide with eqs.(4.5)-4.6) for the quadratic plus 2\( n \)th order potential. We have therefore proven that the quadratic plus the \( u^2 n \) potential and the quadratic plus exponential potential have identical limits letting \( n \to \infty \) in the former and \( b \to \infty \) in the latter, keeping always \( u \) fixed.

### B. The double limit \( b \to \infty \) and \( u \to 1 \).

It is useful to introduce here the variable

\[
\tau \equiv e^{-2b(1-u^2)} \quad \text{hence} \quad u^2 = 1 + \frac{\log \tau}{2b} \to 1^- \quad \text{for} \quad b \to \infty \quad \text{at fixed} \quad \tau , \quad 0 < \tau < 1 .
\]
We then find from eq. (5.4) for \( b \to \infty \) and fixed \( \tau \),

\[
2 b^2 y(u) = 2 \left( -\ln u^2 - 1 + u^2 \right) + \frac{2}{b} \ln u - \frac{1}{2} b^2 \int_0^{1-\tau} \frac{dw}{w} \log(1-w) + \mathcal{O}\left(\frac{1}{b}\right)
\]

\[
= \ln \tau + \frac{1}{2} \ln^2 \tau + \text{Li}_2(1-\tau) + \mathcal{O}\left(\frac{1}{b}\right),
\]

(5.7)

We find in this limit from eq. (5.5) for \( r \) vs. \( n_s \),

\[
\begin{align*}
p &\to \infty, u \to 1, \quad 8 \gamma^2(\tau) = \frac{(1-\tau)^2}{N (1-\tau + \ln \tau)^2}, \\
\gamma^2(\tau) &\to \frac{3 \gamma^2(\tau)}{N (\tau - 1 - \ln \tau)^2} + \frac{2 \gamma^2(\tau)}{N \tau - 1 - \ln \tau}, \\
\gamma^2(\tau) &\equiv 2 e^2 y(u) \ln \tau + \frac{1}{2} \ln^2 \tau + \text{Li}_2(1-\tau),
\end{align*}
\]

(5.8)

where we keep fixed \( \gamma^2 \). Eqs. (5.8) coincide with eqs. (4.9)–(4.11) for the quadratic plus \( u^2 n \) potential.

These results plus those in sec. IV.A prove that the quadratic plus \( u^2 n \) potential and the quadratic plus exponential potential eq. (5.1) have identical limits letting \( n \to \infty \) in the former and \( b \to \infty \) in the latter.

VI. DYNAMICALLY GENERATED INFATON POTENTIAL FROM A FERMION CONDENSATE IN THE INFLATIONARY STAGE.

The inflaton may be a coarse-grained average of fundamental scalar fields, or a composite (bound state) or condensate of fields with spin, just as in superconductivity. Bosonic fields do not need to be fundamental fields, for example they may emerge as condensates of fermion-antifermion pairs \( < \bar{\Psi} \Psi > \) in a grand unified theory (GUT) in the cosmological background \( [6] \).

We investigate in this section an inflaton potential dynamically generated as the effective potential of fermions in the inflationary universe. We consider the inflaton field coupled to Dirac fermions \( \Psi \) through the interaction Lagrangian

\[
\mathcal{L} = \bar{\Psi} [i \gamma^\mu D_\mu - m_f - g_\Psi \varphi] \Psi.
\]

(6.1)

Here \( g_\Psi \) stands for a generic Yukawa coupling between the fermions and the inflaton \( \varphi \). The fermion mass \( m_f \) will be absorbed by a constant shift of the inflaton field. The Dirac matrices \( \gamma^\mu \) are the curved space-time \( \gamma \)-matrices and \( D_\mu \) stands for the fermionic covariant derivative.

For the purpose here, the inflationary stage can be approximated by a de Sitter space-time (that is, we neglect the slow decrease in time during inflation of the Hubble parameter \( H \)). In this way, the effective potential of fermions can be computed in close form with the result \( [2] [11] \),

\[
V_f(\varphi) = V_0 + \frac{1}{2} \mu^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4 + H^4 Q \left( \frac{g_\Psi \varphi}{H} \right),
\]

(6.2)

where,

\[
Q(x) = \frac{x^2}{8 \pi^2} \left( \left[ (1 + x^2) \left[ \gamma + \text{Re} \psi(1 + i x) \right] - \zeta(3) x^2 \right] \right), \quad x \equiv g_\Psi \frac{\varphi}{H},
\]

\[
= \frac{x^4}{8 \pi^2} \left[ (1 + x^2) \sum_{n=1}^{\infty} \frac{1}{n (n^2 + x^2)} - \zeta(3) \right],
\]

(6.3)

\[
= \frac{x^4}{8 \pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \zeta(2 n + 1) - \zeta(2 n + 3) \right] x^{2 n}.
\]

We included in \( V_f(\varphi) \) the renormalized mass \( \mu^2 \) and renormalized coupling constant \( \lambda \) which are free and finite parameters. \( \psi(x) \) stands for the digamma function, \( \gamma \) for the Euler-Mascheroni constant and \( \zeta(x) \) for the Riemann zeta function \( [9] \).

Eq. (6.2) is the energy density for an homogeneous inflaton field \( \varphi \) coupled to massless fermions through the Lagrangian eq. (6.1) in a de Sitter space-time.

The power series of the function \( Q(x) \) has coefficients with alternating signs, but it can be readily verified that \( Q(x) > 0 \) and \( Q'(x) > 0 \) for \( x > 0 \). Moreover, to leading order we have

\[
Q(x) \overset{x \to \infty}{=} - \frac{x^4}{8 \pi^2} \left[ \ln x + \gamma - \zeta(3) + \mathcal{O}\left(\frac{1}{x}\right) \right],
\]

(6.4)
The constant $V_0$ in eq. (6.2) must be such that the potential $V_f(\varphi)$ fulfills eq. (1.6) producing a finite number of inflaton efolds. We consider new inflation and choose $\mu^2 = -m^2 < 0$. Hence $V_f(\varphi)$ has a double–well shape with the absolute minimum at $\varphi = \varphi_{\text{min}}$, with $\varphi_{\text{min}}$ a function of the free parameters of the potential.

Expanding $V_f(\varphi)$ in powers of $\varphi$ gives

\[
V_f(\varphi) = V_0 - \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4 + \frac{1}{8 \pi^2} [\zeta(3) - \zeta(5)] \left( \frac{g_Y \varphi}{H^2} \right)^6 + O(g^8 Y_8^8) .
\]

where $\zeta(3) - \zeta(5) = 0.16513 \ldots > 0$.

In the $H \to 0$ limit, eq.(6.2) becomes the effective potential for fermions in Minkowski space-time $[7]$. Recall that

\[
H = \sqrt{\mathcal{H}} m \quad , \quad m = \frac{M^2}{M_{Pl}}
\]

where the dimensionless Hubble parameter $\mathcal{H}$ turns to be of order one $[7]$.

As in the general description of section $[11]$ eq.(2.12) we introduce the dimensionless coupling constant $g$ as

\[
g = \frac{M^2}{\varphi_{\text{min}}^2}
\]

Besides $g$ we can now form two other independent and positive dimensionless shape parameters, that is

\[
s \equiv g_Y \frac{\varphi_{\text{min}}}{H} = \frac{g_Y M_{Pl}}{\sqrt{g}} \quad , \quad q \equiv \frac{H^2}{m \varphi_{\text{min}}} = \frac{H^2}{m M_{Pl}} = \frac{H^2}{M_{Pl}^2} , \quad x = s u .
\]

We derive the dimensionless potential $v_1(u)$ from eq.(6.2) using the general transformation equations (2.15). We obtain,

\[
v_1(u) = \frac{g}{M^2} V_f(\varphi_{\text{min}} u) = c_0 - \frac{1}{2} u^2 + \frac{1}{4} c_4 u^4 + \frac{g}{M^4} H^4 Q(s u)
\]

The parameters $c_0$ and $c_4$ are determined by requiring that $v_1(1) = v'_1(1) = 0$ as in sec. $[11]$ We thus obtain from eq.(6.7)

\[
c_4 = 1 - q^2 s Q'(s) , \quad c_0 = \frac{1}{4} + \frac{1}{4} q^2 s Q'(s) - q^2 Q(s)
\]
FIG. 9: We plot here \( r \) vs. \( \eta_5 \) for the effective potential obtained from fermions in de Sitter stage eq. (6.7) for the physical value of the parameter \( q \) eq. (6.11). For weak Yukawa coupling \( s \ll 1 \) we recover the \( r = r(\eta_5) \) curve for the quadratic plus quartic potential eq. (2.21). The \( r = r(\eta_5) \) curves are inside the universal banana region [fig. 10] provided \( s \leq 850 \), slightly exceeding the bound eq. (6.12).

Inserting \( c_0 \) and \( c_4 \) into eq. (6.7) yields for the inflaton potential

\[
v_1(u) = \frac{g}{M^4} V_j(\varphi_{\text{min}} u) = \frac{1}{2} (1 - u^2) + \frac{1}{4} [1 - q^2 s Q'(s)] (u^4 - 1) + q^2 [Q(s u) - Q(s)] \\
= \frac{1}{2} (1 - u^2) + \frac{1}{4} (1 - b) (u^4 - 1) + b F(u, s) ,
\]

where

\[ b \equiv q^2 s Q'(s) \geq 0 , \quad F(u, s) \equiv \frac{Q(s u) - Q(s)}{s Q'(s)} \] (6.9)

Notice that \( v_1(u) \) reduces to the quartic double–well potential \( \frac{1}{2}(1 - u^2)^2 \) when \( s \to 0 \) at fixed \( q \) (that is, when \( g_Y \to 0 \)) as well as when \( b \to 0 \) at fixed \( s \). This last limit means \( g_Y \to 0 \) with \( g_Y, M_{Pl}/H \) fixed.

Only the interval \( 0 < u < 1 \) is relevant for the inflaton evolution. For any \( u \) in this interval, \( F(u, s) \) is negative definite and is monotonically decreasing as a function of \( s \). In particular,

\[
F(u, s) \xrightarrow{s \to 0} \frac{1}{6} (u^6 - 1) , \quad F(u, s) \xrightarrow{s \to \infty} \frac{1}{4} (u^4 - 1) + O \left( \frac{1}{\log s} \right) , \quad 0 < u < 1 \text{ fixed}
\]

Hence for \( s \to 0 \) at fixed \( b \) we obtain again the sixth–order double–well potential of eq. (2.29)

\[
v_1(u) \rightarrow \frac{1}{12} (1 - u^2)^2 (3 + 2b + 2b u^2) , \quad s \to 0 \text{ at fixed } b ,
\]

while \( b \) cancels out for large \( s \) and we get back the quartic double–well potential

\[
v_1(u) \rightarrow \frac{1}{4} (1 - u^2)^2 , \quad s \to \infty \text{ at fixed } b .
\]

The terms containing \( Q \) in the effective potential eqs. (6.2) and (6.9) represent the one–loop quantum contributions. They vanish when \( b = 0 \) while for \( b > 0 \) they should be sizably smaller than the tree level contribution, otherwise
all higher loops effects must also be taken into account. In particular, the quartic term in eq. (6.9) must have a nonnegative coefficient, that is $0 < b \leq 1$ as in sec. [111].

We see from eq. (6.9) that the one-loop $(Q)$ pieces are of the order $q^2 Q(su)$ compared with the tree-level pieces. We can compute $q$ using eqs. (2.14), (6.5) and (6.6) with the result

$$q = \sqrt{\frac{N s}{8} \left( \frac{\mathcal{H} M}{M_{Pl}} \right)^2} \simeq 0.854 \times 10^{-5} \ll 1,$$

where $y \simeq 1.3$ and $\mathcal{H} \simeq 0.5 \ [2]$. Hence, the one-loop pieces are negligible unless $s \gg 1$. We can therefore use the asymptotic behavior eq.(6.4) to estimate $Q(su)$ for large $s$. In order the one-loop part to be smaller or of the order of the tree level piece we must impose in the strong coupling regime $s \gg 1$,

$$\frac{1}{8 \pi^2} s^4 q^2 \ln s \lesssim 1 \quad \Rightarrow \quad s \lesssim 1020 [\ln 1020]^{-1/4} \simeq 616. \tag{6.12}$$

The one-loop potential eq.(6.9) is therefore reliable for $s \lesssim 600$. For larger values of $s$ the one-loop piece is larger than the tree level part and hence all higher order loops should be included too.

For $s \sim 1$ we recover the quadratic plus quartic potential eq.(2.21) since the terms in $s^4 q^2$ are negligible in eq.(6.9) and $n_s$ and $r$ are thus given by eqs.(2.23). We find from eqs. (6.9) and (6.11) in the case $s \sim 1$,

$$v_1(u) = \frac{1}{4} - \frac{1}{2} u^2 + \frac{1}{4} u^4 + \mathcal{O}\left(\frac{H^2}{M_{Pl}^2}\right)$$

That is, the terms beyond $u^4$ in the effective potential from the fermions are of the same order of magnitude as the loop corrections to inflation $\ [2, 14]$ and can be neglected since $(H/M_{Pl})^2 \sim 10^{-9}$.

We display in fig. 9 $r$ vs. $n_s$ for various values of the Yukawa coupling $s$. Therefore, the banana region $B$ in the $(n_s, r)$ plane for the effective potential eq.(6.9) is the region limited by the curves for the potential for $s \leq 500$ and for $s \to 0$ as displayed in fig. 9. Notice that the lower border of the region $B$ for the effective potential eq.(6.9) is well above the lower border of the universal region $B$ region displayed in fig. [10]

In summary, the $r = r(n_s)$ curves for the dynamically generated inflaton potential eq.(6.9) are inside the universal banana region $B$ for all values of the Yukawa coupling $g_Y$ that keep the result for this one-loop potential reliable. Namely, the one-loop piece is smaller or of the order of the tree level part.

VII. THE UNIVERSAL BANANA REGION $B$

In summary, we find that all $r = r(n_s)$ curves for double–well inflaton potentials in the Ginsburg-Landau spirit fall inside the universal banana region $B$ depicted in fig. [10] Namely,

- The fourth degree double–well potentials containing a cubic term studied in ref. [4]:

$$v_1(u) = \frac{1}{4} + \frac{\beta}{6} - \frac{1}{2} u^2 - \frac{2}{3} \beta u^3 + \frac{1}{4} (1 + 2 \beta) u^4,$$

where $\beta \geq 0$ is the asymmetry parameter. This potential reduces to eq.(2.21) for $\beta = 0$.

- The quadratic plus sixth-order potential eq.(2.20).

- The even polynomial potentials with arbitrarily higher–order degrees and positive coefficients (sec. [III]).

- The quadratic plus exponential potential (sec. [V]).

- The inflaton potential dynamically generated from fermions (sec. [VI]).

Potentials in the Ginsburg-Landau spirit have usually coefficients of order one when written in dimensionless variables. This is the case of the inflaton potentials $v_1(u)$. In that case, we found that all $r = r(n_s)$ curves for double–well potentials fall inside the universal banana region $B$ depicted in fig. [10] Moreover, for even double–well potentials with arbitrary large positive coefficients, their $r = r(n_s)$ curves lie inside the universal banana region $B$ [fig. [10]]

The study of the dynamically-generated inflaton potential in sec. [VII] leads to analogous conclusions. This one-loop inflaton potential is reliable as long as the one-loop piece is smaller or of the same order than the tree level part. In such regime all the curves $r = r(n_s)$ produced by this fermion-generated potential lie inside the universal banana region $B$.
More generally, we see from eq. (2.20) that $r \ll 1$ is generally linked to a large coupling $y \gg 1$. However, this strong coupling regime corresponds to $n_s$ values well below the current best observed value $n_s = 0.964$, and is therefore excluded by observations.

The lower border of the universal region $B$ corresponds to the limit binomial potential eq. (4.9)

$$v_1(u) = \frac{1}{2} (1 - u^2) \quad \text{for } u < 1, \quad v_1(u) = +\infty \quad \text{for } u > 1.$$ 

and is described parametrically by eq. (4.10). We obtain such potential and such parametrization of $r = r(n_s)$ both as the $n \to \infty$ limit of the quadratic plus $u^{2n}$ potential in sec. IV as well as the $b \to \infty$ limit of the $e^{2b} g \phi^2$ potential in sec. V.

The upper-right border of the universal banana-shaped region $B$ is not given by eqs. (4.9) and (4.11) corresponding to the double limit $n \to \infty$ and $u \to 1$ (or, alternatively $b \to \infty$ and $u \to 1$). This follows from the fact that some potentials of order 100 yield $r = r(n_s)$ curves above the limiting curves for the quadratic plus $u^{100}$ potential as depicted in fig. 10.

The upper-left border of the universal region $B$ depicted in fig. 10 is given by the fourth order double–well potential eq. (2.21) and it is described parametrically by eq. (2.23).

The lower border of the universal region $B$ is particularly relevant since it gives a lower bound for $r$ for each observationally allowed value of $n_s$. For example, the best $n_s$ value $n_s = 0.964$ implies from fig. 10 that $r > 0.021$.

The upper border of the universal region $B$ tells us the upper bound $r < 0.053$ for $n_s = 0.964$.

Therefore, we have within the large class of potentials inside the region $B$

$$0.021 < r < 0.053 \quad \text{for } n_s = 0.964.$$ 

Anyhow, the fourth order double–well potential eq. (1.7) best reproduces the present CMB/LSS data and yields as most probable values:

$$n_s \simeq 0.964, \quad r \simeq 0.051.$$ 

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FIG. 10: We plot here the borders of the universal banana region \( B \) in the \((n_s, r)\)-plane setting \( N = 60 \). The curves are computed with the quadratic plus quartic potential eq. (2.21) and with the \( n = \infty \) limit of the quadratic plus \( u^2n \) potential eq. (4.11) (or the \( b = \infty \) limit of the quadratic plus exponential potential eq. (5.1), which gives identical results) as given by eqs. (4.4)-(4.6) and eqs. (4.9) and (4.11). Notice that the lower part of the right border of \( B \), \( 0 < r < 4/N = 0.06666 \ldots \) corresponds to the limit \( n = \infty \) at fixed \( u \) eq. (4.6). The upper part \( 4/N < r < 8/N \), however, of the right border of \( B \) is not displayed here. We display in the vertical full line the LCDM+r value \( n_s = 0.968 \pm 0.015 \) using WMAP5+BAO+SN data \( n_s = 0.968 \pm 0.015 \). The broken vertical lines delimit the \( \pm 1 \sigma \) region.