The Magnetic Eisenhart lift
Part 1: point particle mechanics

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Abstract

We first unveil the origin of the Killing vector equation as an explicit symmetry of a geodesic. Then we demonstrate a modified Eisenhart lift to formulate stationary spacetimes from mechanical systems with magnetic fields, also showing how it lifts a Killing vector to higher dimensional space. The main result of our paper is to demonstrate both, Riemannian and Lorentzian versions of the Magnetic Eisenhart lift.

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1 Introduction

When studying a mechanical system, one has to take into account the existence of fictitious or imaginary forces that arise sometimes as a consequence of the frame of reference of the observer. Such forces include the centrifugal force that seems to exist to an observer with a cylindrical co-ordinate system view, and coriolis forces that seem to exist to an observer in a rotating frame of reference. Likewise, we can say that gravitational force is also a fictitious force that seems to exist due to the curvature of spacetime.

Gravity is one of the four fundamental forces of nature. If an observer underwent freefall close to a test particle, the local gravitational field would not seem to exist for the particle from the viewpoint of the observer, implying that it is a fictitious force that can exist or disappear depending on the observer’s reference frame. While the force of gravity is a consequence of the curvature of spacetime geometry, it can very much appear to be the result of a potential function acting on top of a flat spacetime background. Conversely, we can formulate curved spacetimes that simulate the effects of the other fundamental forces, such as the Jacobi-Maupertuis metric [1, 2]. Another such technique is the Eisenhart lift.

The Eisenhart lift was already formulated by Eisenhart [3], but rediscovered, revised and further developed by Gibbons, Duval, Horvathy and others [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. It was further generalized by Lichnerowicz and Teichmann [14] to a larger class of ambient spacetimes, discussed further by Morand and Bekaert in [15, 16], and extended by Galajinsky and others incorporating a cosmic scale factor [17, 18]. The Eisenhart lift allows us to lift a spacetime metric to higher dimension without altering its mechanics. The Riemannian version of this lift allowed us to convert a classical Lagrangian into a geodesic Lagrangian [6, 7, 8], which so far has only been performed on mechanical systems without magnetic fields, lifting them to static spacetimes. The Lorentzian version of this technique, also known as Eisenhart-Duval lift [6, 8, 9, 10], is employed to deal with mechanics on time dependent curved spaces. It involves defining a dummy variable as an extra co-ordinate, and involving it in the spacetime metric as a cross term with time.

Killing vectors fields, named after Wilhelm Killing, describe vector fields generating continuous isometries along which the metric remains invariant. Thus, Killing vectors generate explicit symmetries of a geodesic and are associated with a conserved quantity that is a first order polynomial in terms of momentum. Naturally, one may ask how lifting a spacetime via Eisenhart lift would affect the Killing vector fields of the original spacetime.

In this article, we shall study explicit mechanical symmetries in the form of Killing vectors and study them their properties. Then we shall study Eisenhart lifts, both the Riemannian and Lorentzian versions, especially demonstrating a modified lift for systems with magnetic fields to stationary spacetimes, showing how the equation of motion is preserved under such lifts. Furthermore, we will examine the lift of the Killing vector under the Eisenhart lift.

2 Preliminaries

When employing the Lagrangian formulation of mechanics, we must remember that there is definitely a geometric part to the formulation, and sometimes a physical part as well. It is important to understand this because actual forces produced by potentials described on the background spacetime are different from forces that seem to arise due to the curvature of the spacetime itself. Gravitational potentials deform spacetime geometry to produce curvature causing effects such as time-dilation and length contraction, while other potentials do not.

This can be seen in the classical Lagrangian familiar to us in classical mechanics, where the geometric part contributes the kinetic energy term, while the physical part takes the form of the...
potentials.

\[ L = \frac{m}{2} g_{ij}(x) \dot{x}^i \dot{x}^j - V(x), \quad 1 \leq i, j \leq n. \]

Here, the Lagrangian \( L \) describes a particle in \( n \)-dimensional curved space with the metric \( g_{ij}(x) \), with the potential \( V(x) \) generating forces in such space. Normally, the geodesic equation of motion in this case would be given by:

\[ \ddot{x} = -\Gamma^i_{jk} \dot{x}^j \dot{x}^k - \partial_i V(x). \]

The objective is to formulate a spacetime \((M, h)\) for which the geodesic equation describes dynamics that simulates the force arising due to the potential \( V(x) \). i.e.

\[ \mathcal{L} = \frac{1}{2} h_{\mu\nu}(x) x'^\mu x'^\nu, \quad x'^i = -\Omega^i_{\mu\nu} x'^\mu x'^\nu. \]

### 2.1 Maupertuis formulation of action

Given any Lagrangian \( L \), the mechanical action \( S \) along a path between two points parametrised by \( \tau \) can be written as:

\[ S = \int_1^2 d\tau \; L(x, \dot{x}). \tag{2.1} \]

Now, we shall introduce an important rule which will be necessary here:

\[ \delta \dot{x}^i = \frac{d}{dt} (\delta x^i(t)) \Rightarrow \delta x^j \partial_j \dot{x}^i = \dot{x}^j \partial_j (\delta x^i(t)) \tag{2.2} \]

Varying the action (2.1) and applying (2.2) gives us:

\[ \delta S \equiv \delta S = \int_1^2 d\tau \; \delta L = \int_1^2 d\tau \left[ \left\{ \frac{\partial L}{\partial x^i} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) \right\} \delta x^i + \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \delta x^i \right) \right] \tag{2.3} \]

We can say that along the geodesic, it shall satisfy the Euler-Lagrange equation:

\[ \frac{\partial L}{\partial x^i} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = 0 \quad , \quad p_i = \frac{\partial L}{\partial \dot{x}^i}. \tag{2.4} \]

Using (2.4) for the dynamical path allows us to write the variation of the action (2.1) as:

\[ \delta S = \int_1^2 d\tau \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \delta x^i \right) = [p_i \; \delta x^i]^2 = \left[ \frac{\partial S}{\partial \dot{x}^i} \delta x^i \right]^2. \]

Thus, we can clearly see that the action varies with the variation at the end points of the path, showing us that along the geodesic:

\[ \frac{\partial S}{\partial x^i} = \frac{\partial L}{\partial \dot{x}^i} = p_i \Rightarrow S = \int_1^2 d\tau \frac{dS}{d\tau} = \int_1^2 d\tau \frac{\partial S}{\partial \dot{x}^i} \dot{x}^i = \int_1^2 d\tau \; p_i \dot{x}^i. \tag{2.5} \]

Therefore, we can write the following theorem regarding the action integral:

**Theorem 1** Provided a Lagrangian \( L \), for which an action \( S \) is that of the Randers form of the Finsler metric:

\[ S = \int d\tau \; L, \quad L = -mc\sqrt{g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu} + A_{\lambda} \dot{x}^\lambda. \tag{2.6} \]
which is linear in dependence on velocity variables $\dot{x}^i$, and thus reparametrization invariant, which spans the entire set of velocity variables defining $L$, we can say that

$$S = \int p_\mu dx^\mu, \quad L = p_\mu \dot{x}^\mu. \quad (2.7)$$

which are the Maupertuis form of the action and Lagrangian, showing us that the overall Hamiltonian is a vanishing quantity, given by:

$$H = p_\mu \dot{x}^\mu - L = 0. \quad (2.8)$$

If we parametrize wrt any one of the co-ordinates $x^0$, then we will have:

$$S = \int d\tau \left( p_i \dot{x}^i + p_0 \dot{x}^0 \right) = \int dx^0 \left( p_i x^i + p_0 \right), \quad x^i = \frac{dx^i}{dx^0},$$

$$L' = p_a x^a + p_0 \Rightarrow -p_0 = p_a \dot{x}^a - L'. \quad (2.9)$$

Under the circumstances that a variable $x^0 = t$ is cyclical (ie. $\frac{\partial L}{\partial x^0} = 0$), we will have a conserved momentum $p_0$ conjugate to $x^0$ according to (2.4)

$$\frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^0} \right) = \frac{\partial L}{\partial x^0} = 0 \Rightarrow p_0 = \frac{\partial L}{\partial \dot{x}^0} = \text{const.}$$

There are 2 alternatives for the form of the Lagrangian that can be derived from the form provided in (2.6), to which such rules will apply. Either we can modify the action to parametrization wrt:

1. **Proper time:** Proper time $\sigma$ is observed in the object’s own frame, where it is at rest wrt itself. Under this parametrisation, the action and Lagrangian are:

$$S = d\tau L = \int d\sigma L^* \quad , \quad \text{where} \quad d\sigma = d\tau \sqrt{g_{\alpha\beta}(x)\dot{x}^\alpha \dot{x}^\beta}$$

$$L^* = -mc g_{\mu\nu}(x) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} + A_\lambda \frac{dx^\lambda}{d\sigma}, \quad (2.10)$$

where the velocity vector $\{\frac{dx^\mu}{d\sigma}, \mu = 0, 1, \ldots, n\}$ obeys the constraint: $g_{\mu\nu}(x) \frac{dx^\mu}{d\sigma} \frac{dx^\nu}{d\sigma} = 1$.

2. **Observed time**

In this case, the (00) component for time is given by a perturbation around the flat space as $g_{00}(x) = 1 + 2\Phi(x)$. This is followed by a binomial expansion up to the first order, as discussed in [19].

$$S = \int d\tau \left( c dt \sqrt{1 - 2 \frac{1}{c^2} g_{ij}(x) x^i x^j - \Phi(x)} + A_\lambda(x) \dot{x}^\lambda \right), \quad x^i = \frac{dx^i}{dt}$$

$$\approx \int dt \left[ \frac{1}{2} g_{ij}(x) x^i x^j - A_k(x) x^k - (A_0(x) + \Phi(x)) - c \right]$$

$$L = \frac{1}{2} g_{ij}(x) x^i x^j - A_k(x) x^k - U(x) \quad , \quad \text{where} \quad U(x) = A_0(x) + \Phi(x). \quad (2.11)$$
Furthermore, under the circumstances of a time-dependent or non-autonomous system, parametris-
ing the action wrt observed time by setting \[ \dot{t} = 1 \]
\[ S = -mc \int d\tau \left( \sqrt{g_{00}(x,t)\dot{t}^2 + g_{ij}(x,t)\dot{x}^i\dot{x}^j} + A_\lambda(x,t)\dot{x}^\lambda \right) \]
\[ \approx \int dt \left[ \frac{1}{2}g_{ij}(x,t)x^i\dot{x}^j - A_k(x,t)x^{tk} - (A_0(x,t) + \Phi(x,t)) - c \right] \]
helps describe the Lagrangian of a time-dependent mechanical system
\[ L = \frac{1}{2}g_{ij}(x,t)x^i\dot{x}^j - A_k(x,t)x^{tk} - U(x,t) \]
where \( U(x,t) = A_0(x,t) + \Phi(x,t) \) (2.12)
as a Lagrangian dependent on time \( t \), but apparently independent of \( \dot{t} \).

Since in the 2nd case the velocity component for time is lost after it is given the status of a parameter, its conjugate momentum is provided according to the Legendre theorem (2.9), according to which, the conserved quantity for autonomous systems is given by:
\[ \frac{dL}{dt} = \frac{\partial L}{\partial x^i} \dot{x}^i + \frac{\partial L}{\partial \dot{x}^i} \ddot{x}^i = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \right) \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \right) = 0 \]

Thus, the Hamiltonian is given by:
\[ H = p_i \dot{x}^i - L. \] (2.13)

Since \( H \) is a function of \( x \) and \( p \), the variation of \( H \) gives the Hamilton’s equation of motion. Furthermore, the Hamilton’s equations of motion are given by:
\[ \frac{dx^i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial x^i} \] (2.14)

Now, we shall proceed to examine the Eisenhart lift for such systems.

### 2.2 Killing equation: Explicit Symmetry of a geodesic

Explicit symmetries are born from isometries and generated by Killing vectors. Thus, if we vary the Lagrangian for a curved space given by:
\[ \mathcal{L} = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j + \alpha_i(x)\dot{x}^i - \varphi(x), \] (2.15)
Under the circumstances that we are dealing with a geodesic Lagrangian given by setting \( \alpha_i(x) = \varphi(x) = 0 \) in (2.15)
\[ L = \frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j, \] (2.16)
From (2.2), we can introduce a new rule necessary here:
\[ \Delta \dot{x}^i = \delta x^j \left( \nabla_j \dot{x}^i \right) = \delta x^j \left( \partial_j \dot{x}^i + \Gamma^i_{jk} \dot{x}^k \right) \]
\[ = \dot{x}^j \left( \partial_j (\delta x^i(t)) + \Gamma^i_{jk} \delta x^k \right) = \dot{x}^j \nabla_j (\delta x^i) = \frac{D}{Dt} (\delta x^i) \]
\[ \therefore \Delta \dot{x}^i = \delta x^j \left( \nabla_j \dot{x}^i \right) = \dot{x}^j \nabla_j (\delta x^i) = \frac{D}{Dt} (\delta x^i), \] (2.17)
where the $\Delta$ operator is a covariant variation operator, while $\delta$ is the conventional variation operator. Using (2.17), we will have the variation of (2.16) given by:

$$\delta L = \Delta L = \frac{1}{2} \delta x^k \nabla_k \left( g_{ij}(x) \dot{x}^i \dot{x}^j \right)$$

$$= g_{ij}(x) \dot{x}^j \delta x^k \left( \nabla_k \dot{x}^i \right) = g_{ij}(x) \dot{x}^j \dot{x}^k \nabla_k \delta x^i$$

where we have written $\delta L = \Delta L$ since $L$ is a scalar. If we define the variation $\delta x^k = K^k$ as the Killing vector, and define it as a direction of symmetry for the Lagrangian, then we will have the Killing Vector equation:

$$g_{ij}(x) \dot{x}^j \dot{x}^k \nabla_k K^i = \frac{1}{2} (\nabla_j K_k + \nabla_k K_j) = 0$$

(2.18)

Alternatively, defining $\delta x^k = K^k$ as the Killing vector in (2.3), and defining it as a direction of symmetry for the action will lead us to the following equations as condition for symmetry:

$$\left\{ \frac{\partial L}{\partial x^i} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) \right\} K^i = 0 , \quad \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} K^i \right) = 0. \quad (2.19)$$

For the geodesic Lagrangian (2.16), the Euler-Lagrange equation of motion (2.4) can alternatively be written as:

$$p_i = \frac{\partial L}{\partial \dot{x}^i} = g_{ij}(x) \dot{x}^j$$

$$\frac{\partial L}{\partial x^i} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) \equiv -\frac{D}{D\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = -g_{ij}(x) \frac{D \dot{x}^j}{D\tau} = 0. \quad (2.20)$$

and the symmetry equations (2.19) will respectively become:

$$\frac{D}{D\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) K^i = 0 , \quad \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} K^i \right) = \frac{D}{D\tau} \left( \frac{\partial L}{\partial \dot{x}^i} K^i \right) = 0. \quad (2.21)$$

There are 2 situations under which the equations of (2.21) will hold, leading to an interesting property of the Killing vector:

### 2.2.1 On the classical path

Demanding isometry means demanding $\delta S = 0$, i.e. if we write $K^i = \delta x^i$, we will have the conserved quantity

$$\frac{D}{D\tau} (p_i K^i) = 0 \quad \Rightarrow \quad Q = p_i K^i. \quad (2.22)$$

and by applying the geodesic Lagrangian (2.16), the Killing vector equation is given by

$$\frac{D}{D\tau} (p_i K^i) = \frac{D}{D\tau} (\dot{x}^i K_i) = \dot{x}^i \dot{x}^j \nabla_j K_i = 0$$

$$\therefore \quad \nabla_i K_j + \nabla_j K_i = 0 \quad (2.23)$$

where $\frac{D}{D\tau}$ is the covariant time-derivative operator, and $\frac{D \dot{x}^i}{D\tau} = 0$, for geodesics.
2.2.2 Off the classical path

Consider a point away from the original geodesic \(x(t)\), on a deviated trajectory \(y(t)\) which is the solution of the Lagrangian:
\[
L' = L + L_1, \quad L_1 = A_i(x) \dot{x}^i - \Phi(x),
\]
where \(L_1\) is the Finsler deformation of the original geodesic Lagrangian \(L\) given by (2.16). The Euler-Lagrange equation would thus be:
\[
\frac{\partial L}{\partial x^i} - \frac{d}{d\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) + \frac{d}{d\tau} \left( \frac{\partial L_1}{\partial \dot{x}^i} \right) = 0,
\]
where according to (2.20), we can write:
\[
- \frac{D}{D\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = - \left[ \frac{\partial L_1}{\partial x^i} - \frac{d}{d\tau} \left( \frac{\partial L_1}{\partial \dot{x}^i} \right) \right] = F_i = \partial_i \Phi(x) - \dot{x}^j \left( \partial_i A_j - \partial_j A_i \right). \tag{2.24}
\]
Applying (2.24), the first Killing symmetry equation of (2.21) will become:
\[
\frac{D}{D\tau} \left( p_i K^i \right) = 0 \Rightarrow Q = p_i K^i \quad \text{where} \quad p_i = \frac{\partial L}{\partial \dot{x}^i}. \tag{2.25}
\]

Thus, we can say that the Killing Vector \(K\) is orthogonal to the force \(F = \nabla \Phi(x) - \dot{x} \times (\nabla \times A)\) of the potential functions \(\Phi(x), A(x)\) that deforms the dynamical path from \(x(\tau)\) to \(y(\tau)\). This is a step further than \([20]\), where it was shown to be orthogonal only to the gradient of the scalar field only. Thus, the 2nd equation of (2.21) holding true implies:
\[
\frac{D}{D\tau} \left( p_i K^i \right) = 0 \Rightarrow Q = p_i K^i \quad \text{where} \quad p_i = \frac{\partial L}{\partial \dot{x}^i}. \tag{2.26}
\]

and by applying the geodesic Lagrangian (2.16) and the orthogonality condition (2.25), the Killing vector equation is given by
\[
\frac{D}{D\tau} \left( p_i K^i \right) = \frac{D}{D\tau} \left( \dot{x}^i K_i \right) = \frac{D}{D\tau} \left( \dot{x}^i K_i + \dot{x}^j \frac{D K_i}{D\tau} \right) = \dot{x}^i \dot{x}^j \nabla_j K_i = 0
\]
\[
\therefore \quad \nabla_i K_j + \nabla_j K_i = 0, \tag{2.27}
\]
thus confirming the Killing vector equation off the classical path.

3 The Magnetic Eisenhart lift

While the actual force due to potentials is not the same as apparent force due to the curvature of spacetime, we can always formulate another spacetime that imitates the effects of such potentials. We can either formulate the Jacobi metric \([1, 2]\), which is another spacetime of same dimensions as the previous, or lift the spacetime to higher dimension via the Eisenhart lift.

The Eisenhart lift is a technique developed by Eisenhart \([3]\) and developed by Gibbons, Duval, Horvathy and others \([4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]\) to formulate an expanded spacetime (hence, it is called a lift), where the curvature can dynamically imitate effects of a force due to a potential function.
3.1 The Riemannian lift for time independent systems

Here, we shall first demonstrate that version of the Eisenhart lift known as the Eisenhart Riemannian lift [6], by lifting a mechanical system with magnetic fields into a stationary spacetime. The Eisenhart Lorentzian lift alternatively deals with time dependent mechanical systems, by adding another dimension and including it in the metric as a cross term between itself and time.

Suppose that we have a stationary spacetime metric given by:

\[ ds^2 = g_{\mu\nu}(x)dx^\mu dx^\nu = g_{00}(x)c^2dt^2 + 2g_{0i}(x)c\,dt\,dx^i + g_{ij}(x)dx^idx^j. \]  

(3.1)

Then a geodesic Lagrangian defined on the tangent bundle \( TM \) derived directly from its metric:

\[ L = -mc\sqrt{\frac{ds}{d\tau}}^2 = -mc\sqrt{g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu}, \quad p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = -mc\frac{g_{\mu\nu}(x)\dot{\nu}}{\sqrt{g_{\alpha\beta}(x)\dot{\alpha}\dot{\beta}}}. \]

Now, if the dimensions of the space were increased by one, expanding the metric to read as:

\[ ds^2 = c^2dy^2 + g_{\mu\nu}(x)dx^\mu dx^\nu, \]

then upon parametrization wrt \( y \), we would have the Lagrangian under relativistic approximations \( ds^2 \ll c^2dy^2 \):

\[ S = \int dy \, \mathcal{L}', \quad \mathcal{L}' = -mc^2\sqrt{1 + \frac{g_{\mu\nu}(x)x'^\mu x'^\nu}{c^2}}, \quad x'^\mu = \frac{dx^\mu}{dy}, \]

\[ \mathcal{L}' \approx -mc^2 - \mathcal{L} = -mc^2 - \frac{m}{2} \left( g_{ij}(x)x^i x^j + 2g_{i0}(x)x^i ct' + g_{00}(x)c^2t'^2 \right), \]

\[ \mathcal{L} = \frac{m}{2} \left[ \gamma_{ij}(x)x^i x^j + g_{00}(x) \left( ct' + \frac{g_{0i}(x)}{g_{00}(x)} x^i \right)^2 \right], \quad \gamma_{ij}(x) = g_{ij}(x) - \frac{g_{0i}(x)g_{0j}(x)}{g_{00}(x)} \]

(3.2)

for which the canonical momenta would be:

\[ p_0 = \frac{1}{c} \frac{\partial \mathcal{L}}{\partial ct'} = mg_{00}(x) \left( ct' + \frac{g_{0i}(x)}{g_{00}(x)} x^i \right), \]

\[ p_i = \frac{\partial \mathcal{L}}{\partial x'^i} = m\gamma_{ij}(x)x'^j + mg_{0i}(x) \left( ct' + \frac{g_{0j}(x)}{g_{00}(x)} x^j \right) = m\gamma_{ij}(x)x'^j + \frac{g_{0i}(x)}{g_{00}(x)} p_0 \]

to which we will have a Hamiltonian

\[ \mathcal{H} = p_y = p_i x'^i + p_0 ct' - \mathcal{L} \equiv \mathcal{L} \]

\[ = \frac{1}{2m} \left( \gamma^{ij}(x)\Pi_i \Pi_j + \frac{p_0^2}{g_{00}(x)} \right), \quad \left[ \text{where} \quad \Pi_i = p_i - p_0 \frac{g_{0i}(x)}{g_{00}(x)} \right] \]

\[ = \frac{1}{2m} \gamma^{ij}(x)p_i p_j - \frac{1}{m} \gamma^{ij}(x)g_{0i}(x)p_j p_0 + \frac{1}{2mg_{00}(x)} \left( 1 + \frac{\gamma^{ij}(x)g_{0i}(x)g_{0j}(x)}{g_{00}(x)} \right) p_0^2, \]  

(3.3)

\[ \equiv \frac{1}{2m} (g^{ij}(x)p_i p_j + 2g^{0i}(x)p_i p_0 + g^{00}(x)p_0^2) \]
which is its dual on the cotangent bundle $T^*M$ as in [6]. Either by comparing the last two lines of (3.3), or deduction, the inverse metric components shall be similar to those in [21]:

$$g^{ij}(x) = g^{ik}(x)\gamma_{km}(x)\gamma^{mj}(x)$$

$$= g^{ik}(x) \left( g_{km}(x) - \frac{g_{ok}(x)g_{om}(x)}{g_{oo}(x)} \right) \gamma^{mj}(x) = \delta^i_m \gamma^{mj}(x) = \gamma^{ij}(x)$$

$$g^{0i}(x) = g^{0m}(x)\gamma_{mk}(x)\gamma^{ki}(x)$$

$$= \left[ g^{0m}(x)g_{mk}(x) - (1 - g^{00}(x)g_{00}(x)) \frac{g_{ok}(x)}{g_{00}(x)} \right] \gamma^{ki}(x) = -\frac{g_{0j}(x)}{g_{00}(x)} \gamma^{ji}(x)$$

$$g^{00}(x)g_{00}(x) + g^{0i}(x)g_{i0}(x) = 1 \implies g^{00}(x) = \frac{1}{g_{00}(x)} \left( 1 + \frac{g_{0k}(x)g_{0j}(x)}{g_{00}(x)} \gamma^{ji}(x) \right)$$

Since $t$ is a cyclical co-ordinate, $p_0$ is a conserved quantity. Thus, when we have a Hamiltonian in the above form, when can reverse the steps from (3.3) to (3.2) to produce a spacetime that simulates the force arising from a potential function.

If we have a classical Lagrangian with scalar potential field $V(x)$, and magnetic potential $A(x)$ given as:

$$L = \frac{m}{2} G_{ij}(x)x'^i x'^j + A_i(x) x'^i - V(x),$$

such that we have the Euler-Lagrange equation of motion:

$$x'^m + \Gamma^i_{jk} x'^j x'^k = \frac{G^{ik}(x)}{m} \left[ x'^i \left( \partial_k A_i(x) - \partial_i A_k(x) \right) - \partial_k V(x) \right].$$

and the Hamiltonian:

$$H = \frac{1}{2m} G^{ij}(x) \pi_i \pi_j + V(x), \quad \pi_i = m G_{ij}(x) x'^j = p_i - A_i(x),$$

with the Hamilton’s equations of motion according to (2.14):

$$x'^i = \frac{\partial H}{\partial \pi_i} = \frac{1}{m} G^{ij}(x) \pi_j$$

$$p'_i = -\frac{\partial H}{\partial x'^i} = \frac{1}{m} G^{cd}(x) \pi_d \frac{\partial A_c}{\partial x^i} - \frac{\partial V}{\partial x^i},$$

then we can lift the Hamiltonian into the form of (3.3) by inserting $p_0 = \text{const}$ as shown below:

$$A_i(x) = p_0 \alpha_i(x), \quad V(x) = \Phi(x) p_0^2$$

$$\mathcal{H} = \frac{1}{2m} G^{ij}(x) \Pi_i \Pi_j + \Phi(x) p_0^2, \quad \Pi_i = p_i - p_0 \alpha_i(x)$$

$$= \frac{1}{2m} G^{ij}(x) p_i p_j - \frac{1}{m} G^{ij}(x) \alpha_j(x) p_i p_0 + \left( \Phi(x) + \frac{1}{2m} G^{ij}(x) \alpha_i(x) \alpha_j(x) \right) p_0^2.$$  \hspace{1cm} (3.9)

The principle behind this lifting process is to ensure that the equations of motion remain undisturbed by preserving the Hamilton’s equations of motion. The insertion of the conserved momentum $p_0$ into the Hamiltonian (3.7) will not alter the derived Hamilton’s equations:

$$x'^i = \frac{\partial \mathcal{H}}{\partial \pi_i} = \frac{1}{m} G^{ij}(x) \Pi_j$$

$$p'_i = -\frac{\partial \mathcal{H}}{\partial x'^i} = \frac{p_0}{m} G^{cd}(x) \Pi_d \frac{\partial \alpha_c}{\partial x^i} - (p_0)^2 \frac{\partial \Phi}{\partial x^i}. \hspace{1cm} (3.10)$$
seen from how \( (3.10) \) matches \( (3.8) \) upon setting \( p_0 = 1 \), ensuring a proper lift. If we compare \( (3.9) \) to \( (3.3) \), then we can deduce that:

\[
\gamma_{ij}(x) = G_{ij}(x) \quad \Rightarrow \quad \gamma_{ij}(x) = G_{ij}(x) = g_{ij}(x) - \frac{g_{0i}(x)g_{0j}(x)}{g_{00}(x)}
\]

\( \Phi(x) = \frac{1}{2m g_{00}(x)} \), \quad \alpha_i(x) = \frac{g_{0i}(x)}{g_{00}(x)}.

(3.11)

and the lifted metric components, by applying \( (3.11) \) to \( (3.4) \) are:

\[
g_{00}(x) = \frac{1}{2m \Phi(x)}
\]

\[
g_{0i}(x) = g_{00}(x) \alpha_i(x) = \frac{\alpha_i(x)}{2m \Phi(x)}
\]

\[
g_{ij}(x) = \gamma_{ij}(x) + \frac{g_{0i}(x)g_{0j}(x)}{g_{00}(x)} = G_{ij}(x) + \frac{\alpha_i(x) \alpha_j(x)}{2m \Phi(x)}
\]

(3.12)

\[
g^{ij}(x) = G^{ij}(x)
\]

\[
g^{0i}(x) = -\frac{g_{0k}(x)}{g_{00}(x)} G^{ki}(x) = -G^{ik}(x) \alpha_k(x)
\]

\[
g^{00}(x) = \frac{1}{g_{00}(x)} \left( 1 + \frac{g_{0i}(x)g_{0j}(x)}{g_{00}(x)} G^{ij}(x) \right) = 2m \Phi(x) + \alpha_i(x) \alpha_j(x) G^{ij}(x)
\]

the Eisenhart lifted Lagrangian according to \( (3.2) \) and \( (3.12) \) is:

\[
\mathcal{L} = \frac{m}{2} \left( G_{ij}(x) + \frac{\alpha_i(x) \alpha_j(x)}{2m \Phi(x)} \right) x^i x^j + \frac{\alpha_i(x)}{2 \Phi(x)} x^i x^j t' + \frac{1}{4 \Phi(x)} t'^2,
\]

(3.13)

so the Eisenhart lifted metric is given by:

\[
ds^2 = \left( G_{ij}(x) + \frac{\alpha_i(x) \alpha_j(x)}{2m \Phi(x)} \right) dx^i dx^j + \frac{\alpha_i(x)}{m \Phi(x)} dx^i dt + \frac{1}{2m \Phi(x)} dt'^2.
\]

(3.14)

with the geodesic equation of motion:

\[
x^m + \Omega^i_{\mu \nu} x^\mu x^\nu = 0, \quad \mu, \nu = i, 0.
\]

(3.15)

where we must keep in mind that according to \( (3.12) \),

\[
g^{0i}(x) = -G^{im}(x) \alpha_m(x) \quad , \quad g^{00}(x) = 2m \Phi(x) - g^{0m}(x) \alpha_m(x)
\]

\[
\Omega^0_{\mu \nu} = \frac{1}{2} g^{0m} (\partial_{\mu} g_{m \nu} + \partial_{\nu} g_{m \mu} - \partial_m g_{\mu \nu}) + \frac{1}{2} g^{00} (\partial_{\mu} g_{0 \nu} + \partial_{\nu} g_{0 \mu} - \partial_0 g_{\mu \nu})
\]

\[
= -\frac{\alpha_i(x)}{2} G^{im}(x) \left[ \partial_{\mu} g_{m \nu} + \partial_{\nu} g_{m \mu} - \partial_m g_{\mu \nu} + \alpha_m(x) \left( \partial_{\mu} g_{0 \nu} + \partial_{\nu} g_{0 \mu} - \partial_0 g_{\mu \nu} \right) \right]
\]

\[
+ \alpha_i(x) (\partial_0 g_{m \nu} + \partial_{\nu} g_{0 m} - \partial_0 g_{m \mu})
\]

(3.16)
The Christoffel connection components computed from the lifted metric (3.12) and (3.16) are:

\[
\Omega_{jk}^i = \frac{1}{2} g^{im} \left( \partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk} \right) + \frac{1}{2} g^{i0} \left( \partial_j g_{k0} + \partial_k g_{0j} - \partial_0 g_{jk} \right) \\
= \Gamma_{jk}^i + \frac{G_{im}(x)}{4m} \left( \frac{\partial_m \Phi(x)}{\Phi^2(x)} \alpha_j(x) \alpha_k(x) \right) + \frac{\alpha_k(x) (\partial_j \alpha_m(x) - \partial_m \alpha_j(x)) + (j \leftrightarrow k)}{\Phi(x)} \\
= \frac{G_{im}(x)}{4m} \left( \frac{\partial_j \alpha_m(x) - \partial_m \alpha_j(x)}{\Phi(x)} + \alpha_j(x) \frac{\partial_m \Phi}{\Phi^2} \right) \\
\Omega_{0j}^i = \frac{1}{2} g^{im} \left( \partial_j g_{0m} + \partial_0 g_{mj} - \partial_m g_{0j} \right) + \frac{1}{2} g^{i0} \left( \partial_j g_{00} + \partial_0 g_{0j} - \partial_0 g_{0j} \right) \\
= \frac{G_{im}(x)}{4m} \left( \frac{\partial_j \alpha_m(x)}{\Phi(x)} - \alpha_j(x) \frac{\partial_m \Phi}{\Phi^2} \right) \\
\Omega_{00}^i = \frac{1}{2} g^{im} \left( \partial_0 g_{0m} + \partial_0 g_{0m} - \partial_m g_{00} \right) + \frac{1}{2} g^{i0} \left( \partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00} \right) \\
= \frac{G_{im}(x)}{4m} \partial_m \Phi(x) \\
(3.17) \\
\Omega_{jk}^0 = \frac{1}{2} g^{0m} \left( \partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk} \right) + \frac{1}{2} g^{00} \left( \partial_j g_{k0} + \partial_k g_{0j} - \partial_0 g_{jk} \right) \\
= -\alpha_i(x) \Omega_{jk}^i + \frac{\Phi(x)}{2} \left( \partial_j \left( \frac{\alpha_k(x)}{\Phi(x)} \right) + \partial_k \left( \frac{\alpha_j(x)}{\Phi(x)} \right) \right) \\
\Omega_{j0}^0 = \frac{1}{2} g^{0m} \left( \partial_j g_{0m} + \partial_0 g_{mj} - \partial_m g_{j0} \right) + \frac{1}{2} g^{00} \left( \partial_j g_{00} + \partial_0 g_{0j} - \partial_0 g_{j0} \right) \\
= -\alpha_i(x) \Omega_{j0}^i - \frac{\partial_0 \Phi(x)}{2} \\
\Omega_{00}^0 = \frac{1}{2} g^{0m} \left( \partial_0 g_{0m} + \partial_0 g_{0m} - \partial_m g_{00} \right) + \frac{1}{2} g^{00} \left( \partial_0 g_{00} + \partial_0 g_{00} - \partial_0 g_{00} \right) \\
= -\alpha_i(x) \Omega_{00}^i \\
\Omega_{\mu\nu}^i x'^\mu x'^\nu = \Gamma_{jk}^i x'^j x'^k + 2 \Omega_{j0}^i x'^j x'^0 + \Omega_{00}^i (c t')^2 \\
= \Gamma_{jk}^i x'^j x'^k + 2 \Omega_{j0}^i x'^j \left( 2 \Phi(x) p_0 - \alpha_k(x) x'^k \right) + \Omega_{00}^i \left( 2 \Phi(x) p_0 - \alpha_k(x) x'^k \right)^2 \\
= \Gamma_{jk}^i x'^j x'^k + \frac{G_{im}(x)}{m} \left[ p_0 (\partial_j \alpha_m(x) - \partial_m \alpha_j(x)) x'^j + p_0^2 \partial_m \Phi(x) \right] \\
\Omega_{\mu\nu}^i x'^\mu x'^\nu = \Gamma_{jk}^i x'^j x'^k + \frac{G_{im}(x)}{m} \left[ p_0 (\partial_j \alpha_m(x) - \partial_m \alpha_j(x)) x'^j + p_0^2 \partial_m \Phi(x) \right]. \\
\text{So, upon setting } p_0 = 1 \text{ in (3.15), we shall reproduce the equation of motion (3.6):} \\
x'^m + \Gamma_{jk}^i x'^j x'^k = -G_{im}(x) \left[ x'^j (\partial_j A_m(x) - \partial_m A_j(x)) - \partial_m V(x) \right]. \\
\text{showing that the equation of motion is preserved in geodesic equation form.}

Thus, the Eisenhart-Riemannian lift is effectively an inverse of to the Kaluza-Klein reduction. The Kaluza-Klein reduction isolates out a reduced metric from the main stationary metric, while the Eisenhart Riemannian lift elevates a lower dimensional Hamiltonian system to a higher dimensional static space-time.
3.2 Lifting the Killing vector

So far, we have only considered the Killing vector for a geodesic Lagrangian \((2.16)\). However, we should also consider the Killing vector for a general Lagrangian introduced in \((3.5)\), i.e.:

\[
L = \frac{m}{2} G_{ij}(x) \dot{x}^i \dot{x}^j + A_i(x) \dot{x}^i - V(x).
\]

This is a concern since from a dynamical approach, the Killing vector equation \((2.23)\) arises from fulfilment of \(\frac{D}{D\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = 0\) in the covariant Euler-Lagrange equation for the geodesic Lagrangian, as seen in \((2.20)\), or \(\frac{D}{D\tau} \left( \frac{\partial L}{\partial \dot{x}^i} \right) K^i = 0\) in \((2.23)\). Such equations won’t be available for \((3.5)\), thus, it would be helpful if we could convert the Lagrangian into a geodesic Lagrangian of the form \((3.13)\) via Eisenhart-Riemannian lift.

\[
\mathcal{L} = \frac{m}{2} \left( G_{ij}(x) + \frac{\alpha_i(x) \alpha_j(x)}{2m \Phi(x)} \right) \dot{x}^i \dot{x}^j + \frac{\alpha_i(x)}{2 \Phi(x)} \dot{x}^i \dot{y}^i + \frac{1}{4 \Phi(x)} \dot{y}^2,
\]

\[
\Phi(x) = \frac{V(x)}{\dot{y}^2}.
\]

The symmetry equation \((2.21)\) for this lifted Lagrangian is given by:

\[
\frac{D}{D\tau} \left( \frac{\partial L}{\partial \dot{x}^\mu} \right) K^\mu = 0 \quad , \quad \frac{D}{D\tau} \left( \frac{\partial L}{\partial \dot{\chi}^\mu} K^\mu \right) = 0
\]

which results in the Killing equations with the lifted metric \((3.12)\):

\[
\bar{\nabla}_\mu K_\nu + \bar{\nabla}_\nu K_\mu = 0 \quad \Rightarrow \quad \begin{cases} \bar{\nabla}_i K_j + \bar{\nabla}_j K_i = 0 \\ \bar{\nabla}_0 K_j + \bar{\nabla}_j K_0 = 0 \\ \bar{\nabla}_0 K_0 = 0 \end{cases} (3.18)
\]

Using the Christoffel connection components \((3.17)\), we can write:

\[
\bar{\nabla}_i K_j = \partial_i K_j + \Omega^m_{ij} K_m + \Omega^0_{ij} K_0
\]

\[
= \nabla_i K_j + (\Omega^m_{ij} - \Gamma^m_{ij}) K_m - \left[ \alpha_m(x) \Omega^m_{ij} - \frac{\Phi(x)}{2} \left( \partial_i \left( \frac{\alpha_j(x)}{\Phi(x)} \right) + \partial_j \left( \frac{\alpha_i(x)}{\Phi(x)} \right) \right) \right] K_0
\]

\[
\bar{\nabla}_0 K_i = \partial_0 K_i + \Omega^m_{0i} K_m + \Omega^0_{0i} K_0 = \partial_0 K_i + \Omega^m_{0i} K_m - \left[ \alpha_m(x) \Omega^m_{0i} + \frac{\partial_i \Phi(x)}{2 \Phi(x)} \right] K_0
\]

\[
\bar{\nabla}_i K_0 = \partial_i K_0 + \Omega^m_{0i} K_m + \Omega^0_{0i} K_0 = \partial_i K_0 - \left[ \alpha_m(x) \Omega^m_{0i} + \frac{\partial_i \Phi(x)}{2 \Phi(x)} \right] K_0 + \Omega^m_{0i} K_m
\]

\[
\bar{\nabla}_0 K_0 = \partial_0 K_0 + \Omega^m_{00} K_m + \Omega^0_{00} K_0 = \partial_0 K_0 + \Omega^m_{00} K_m - \alpha_m(x) \Omega^m_{00} K_0
\]

Thus, we can see that the original Killing vector equation \((2.23)\) is deformed under the Magnetic Eisenhart Riemannian lift:

\[
\nabla_i K_j + \nabla_j K_i = 2 \left( \Omega^m_{ij} - \Gamma^m_{ij} \right) K_m + 2 \alpha_m(x) \Omega^m_{ij} K_0 + \Phi(x) \left[ \partial_i \left( \frac{\alpha_j(x)}{\Phi(x)} \right) + \partial_j \left( \frac{\alpha_i(x)}{\Phi(x)} \right) \right] K_0 \quad (3.19)
\]

and we get the following equations with the lifted component of the Killing vector:

\[
\partial_i K_0 + \partial_0 K_i = -2 \Omega^m_{0i} K_m + 2 \alpha_m(x) \Omega^m_{0i} K_0 + \frac{\partial_i \Phi(x)}{\Phi(x)} K_0
\]

\[
\partial_0 K_0 = -2 \Omega^m_{00} K_m + 2 \alpha_m(x) \Omega^m_{00} K_0
\]
where we can see that upon applying the condition $A_i(x) = \alpha_i(x) = 0$ to (3.19) to lift a system without magnetic fields to a static spacetime, the Killing vector equation (2.23) will be preserved, and (3.20) will match the results in [22]. Furthermore, for all points off the classical path, we shall have the orthogonality condition according to (2.25):

$$K_{\mu} \frac{Dx^\mu}{D\tau} = K_i \frac{Dx^i}{D\tau} + K_0 \frac{Dy'}{D\tau} = K_i \frac{Dx^i}{D\tau} + K_0 \left( y'' + \Omega_{jk}x^j x^k + 2\Omega_{j0}x^j y' + \Omega_{00}y'^2 \right) = 0.$$ (3.21)

Now, we shall consider the lift for time-dependent Lagrangians.

### 3.3 The Lorentzian lift for time dependent systems

A conserved momentum conjugate to a cyclical co-ordinate (usually time for autonomous mechanical systems), is indispensable when performing an Eisenhart lift. Under the circumstances that such a luxury is unavailable (as in the case of a time-dependent system), we must modify the geometry to include a cyclical auxiliary co-ordinate without disturbing the equation of motion. This approach is called the Eisenhart-Lorentzian lift or the Eisenhart-Duval lift.

Let us define a time-dependent spacetime metric as:

$$ds^2 = g_{ij}(x, t)dx^i dx^j + 2g_{0i}(x, t)cdt \, dx^i + g_{00}(x, t)c^2 dt^2 + c \, dt \, dv.$$ (3.22)

As in section [3.1] if we modify the metric by adding another auxiliary co-ordinate to write $ds'^2 = c^2 dy'^2 + ds^2$, and expand binomially, parametrising wrt $y$ and assuming $ds'^2 \ll c^2 dy'^2$ for relativistic speeds and writing $x'^i = \frac{dx^i}{dy}$, then we will have the geodesic Lagrangian:

$$\mathcal{L} = \frac{m}{2} \left( g_{ij}(x, t)x'^i x'^j + 2g_{0i}(x, t)x'^i ct' + g_{00}(x, t)c^2 t'^2 + 2v' ct' \right).$$ (3.23)

where since $g_{0v}(x, t) = 1$, the geodesic equation will be unaffected. The momenta, where one is a conserved quantity, are given by:

$$p_v = \frac{\partial \mathcal{L}}{\partial \dot{v}} = mct'$$

$$p_0 = \frac{1}{c} \frac{\partial \mathcal{L}}{\partial \dot{v}} = m \left[ v' + g_{00}(x, t)ct' + g_{0k}(x, t)x'^k \right] = m \left( v' + g_{0i}(x, t)x'^i \right) + g_{00}(x, t)p_v$$

$$\Rightarrow \quad m \left( v' + g_{0i}(x, t)x'^i \right) = p_0 - g_{00}(x, t)p_v$$ (3.24)

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{x}^i} = m \left[ g_{ik}(x, t)x'^k + g_{0k}(x, t)ct' \right] = mc \, g_{ik}(x, t)x'^k + g_{0i}(x, t)p_v,$$

$$\Rightarrow \quad \Pi_i = m \, g_{ik}(x, t)x'^k = p_i - g_{0i}(x, t)p_v$$

If we define an inverse for $g_{ij}(x, t)$ as $\gamma^{ij}(x, t)$ such that:

$$\gamma^{ij}(x, t)g_{jk}(x, t) = \delta^i_j$$ (3.25)

and the geodesic Hamiltonian according to (3.23) is given by:

$$\mathcal{H} = -p_y = p_v x'^i + p_v v' + p_0 c t' - \mathcal{L} \equiv \mathcal{L}$$

$$= \frac{1}{2m} g^{\mu\nu}(x, t)p_{\mu}p_{\nu} = \frac{1}{2mc} \left( \gamma^{ij}(x, t)\Pi_i \Pi_j - g_{00}(x, t)p_v^2 + 2p_0 p_v \right)$$

$$= \frac{1}{2m} \left[ \gamma^{ij}(x, t)p_i p_j - 2\gamma^{ij}(x, t)g_{0i}(x, t)p_j p_v \right.$$  

$$\left. + (\gamma^{ij}(x, t)g_{0i}(x, t)g_{0j}(x, t) - g_{00}(x, t)) p_v^2 + 2p_0 p_v \right].$$ (3.26)
Now, let us consider a time-dependent mechanical system defined by the Lagrangian:

\[ L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{m}{2} G_{ij}(\mathbf{x}, t) \dot{x}^i \dot{x}^j + A_i(\mathbf{x}, t) \dot{x}^i - V(\mathbf{x}, t), \]  

(3.27)

for which the canonical momenta are given by:

\[ p_i = \frac{\partial L}{\partial \dot{x}^i} = mG_{ij}(\mathbf{x}, t) \dot{x}^j + A_i(\mathbf{x}, t), \]  

(3.28)

and the Euler-Lagrange equation of motion is:

\[ \ddot{x}^i + \Gamma^i_{jk} \dot{x}^j \dot{x}^k = G^{im}(\mathbf{x}, t) \left[ \frac{1}{m} (\partial_m A_k - \partial_k A_m) \dot{x}^k - \frac{1}{m} \partial_m V(\mathbf{x}, t) \right. \]

\[ \left. - \left( \partial_0 G_{mj}(\mathbf{x}, t) \dot{x}^j + \frac{1}{m} \partial_0 A_m(\mathbf{x}, t) \right) \right] \]  

(3.29)

Now if we have a time-dependent Lagrangian \( L(\mathbf{x}, \dot{\mathbf{x}}, t) \), then we can see that the Hamiltonian given by the Legendre transform will not be conserved:

\[ \frac{dL}{dt} = \frac{\partial L}{\partial x^i} \dot{x}^i + \frac{\partial L}{\partial \dot{x}^i} \ddot{x}^i + \frac{\partial L}{\partial t} = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \right) + \frac{\partial L}{\partial t} \]

\[ \Rightarrow \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \right) = \frac{dH}{dt} = - \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i \neq 0. \]  

(3.30)

One way to formulate a mechanical system dependent on a co-ordinate, but not it’s velocity is to parametrize the action wrt the co-ordinate, as shown with \( 2.12 \). As a result, it is not possible to deduce the momentum directly from the Lagrangian, but the Euler-Lagrange equation is still valid. Thus, we can say that:

\[ \frac{dp_i}{dt} = \frac{\partial L}{\partial \dot{x}^i} = - \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \right) \Rightarrow p_i(t) = -H(t) = - \left( \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L \right) + Q, \]

\[ Q = \frac{\partial L}{\partial \dot{x}^i} \dot{x}^i - L - H(t) \equiv p_i \dot{x}^i - L + p_t, \]  

(3.31)

where \( Q \) is a conserved quantity inserted as a constant of integration, showing that the traditional Legendre Hamiltonian is not conserved in this case. Naturally, the Hamilton’s equations of motion with this conserved quantity are:

\[ \frac{\partial Q}{\partial x^i} = \frac{\partial}{\partial x^i} \left( p_i \dot{x}^i - L \right) = -\dot{p}_i, \quad \frac{\partial Q}{\partial \dot{x}^i} = \frac{\partial}{\partial \dot{x}^i} \left( p_i \dot{x}^i - L \right) = \dot{x}^i, \quad \frac{\partial Q}{\partial p_i} = \frac{\partial}{\partial p_i} \left( p_i \dot{x}^i - L \right) = \dot{p}_i, \]  

(3.32)

which are easily preserved when we lift the conserved quantity \( Q \). Therefore, we will have the conserved quantity \( Q \) for \( 3.27 \) given by:

\[ Q = \frac{1}{2m} G^{ij}(\mathbf{x}, t) \pi_i \pi_j + V(\mathbf{x}, t) + p_0(t) \]  

where \( \pi_i = p_i - A_i(\mathbf{x}, t) \)

\[ = \frac{1}{2m} \left[ G^{ij}(\mathbf{x}, t)p_i p_j - 2G^{ij}(\mathbf{x}, t)A_i(\mathbf{x}, t)p_j \right. \]

\[ + \left. G^{ij}(\mathbf{x}, t)A_i(\mathbf{x}, t)A_j(\mathbf{x}, t) \right] + V(\mathbf{x}, t) - H(t) \]  

(3.33)
Under a lift, this conserved quantity will become

$$Q_l = \frac{1}{2m} \left[ G^{ij}(x, t)p_ip_j - 2G^{ij}(x, t)\alpha_i(x, t)p_jp_v \right. $$

$$\left. + \{ G^{ij}(x, t)\alpha_i(x, t)\alpha_j(x, t) + 2m\Lambda(x, t) \} p_v^2 + 2p_0(t)p_v \right] , \quad (3.34)$$

where

$$\alpha_i(x, t) = \frac{\dot{A}_i(x, t)}{p_v}, \quad \Lambda(x, t) = \frac{V(x, t)}{p_v^2}, \quad p_0(t) = -\frac{mH(t)}{p_v}. \quad (3.35)$$

Thus, by comparing (3.34) to (3.26) and remembering (3.25), we can deduce the metric components to be:

$$g^{ij}(x, t) = G^{ij}(x, t) \quad , \quad g^{iv}(x, t) = -G^{ij}(x, t)\alpha_j(x, t)$$

$$g^{vv}(x, t) = G^{ij}(x, t)\alpha_i(x, t)\alpha_j(x, t) + 2m\Lambda(x, t) \quad (3.35)$$

$$g_{ij}(x, t) = G_{ij}(x, t) \quad , \quad g_{0i}(x, t) = \alpha_i(x, t) \quad , \quad g_{00}(x, t) = -2m\Lambda(x, t)$$

Thus, we have successfully lifted a time-dependent mechanical system described by the Lagrangian given by:

$$ds^2 = G_{ij}(x, t)dx^i dx^j + 2\alpha_i(x, t)cdt dx^i - 2m\Lambda(x, t)c^2 dt^2 + c dt dv. \quad (3.36)$$

Furthermore, to verify if the lift we have performed is correct, we shall deduce the geodesic equation for this lifted spacetime given by:

$$x'^i + \Omega^i_{\mu\nu}x'^\mu x'^\nu + \Omega^i_{\mu\nu}x'^\nu + \Omega^i_{\nu\nu}v'^2 \quad \text{where} \quad \Omega^i_{\mu\nu}x'^\mu x'^\nu = \Omega^i_{ij}x'^i x'^j + \Omega^i_{i0}x'^i ct' + \Omega^i_{00}c^2 t'^2$$

$$\Omega^i_{\mu\nu}x'^\nu = \Omega^i_{ij}x'^i v' + \Omega^i_{0v}c t' v' \quad (3.37)$$

From the lifted metric components (3.35), we can compute the connection components:

$$\Omega^i_{jk} = \frac{1}{2} g^{im} (\partial_j g_{mk} + \partial_k g_{mj} - \partial_m g_{jk}) + \frac{1}{2} g^{iv} (\partial_j g_{vk} + \partial_k g_{vj} - \partial_v g_{jk})$$

$$= \frac{1}{2} G^{im} (\partial_j G_{mk} + \partial_k G_{mj} - \partial_m G_{jk}) = \Gamma^i_{jik} \quad (3.37)$$

$$\Omega^i_{00} = \frac{1}{2} g^{im} (\partial_0 g_{0m} + \partial_0 g_{0m} - \partial_m g_{00}) + \frac{1}{2} g^{iv} (\partial_0 g_{0v} + \partial_0 g_{0v} - \partial_v g_{00})$$

$$= G^{im} \partial_0 \alpha_m + m G^{im} \partial_m \Lambda \quad (3.37)$$

$$\Omega^i_{vv} = \frac{1}{2} g^{im} (\partial_v g_{mv} + \partial_v g_{mv} - \partial_m g_{vv}) + \frac{1}{2} g^{iv} (\partial_v g_{vv} + \partial_v g_{vv} - \partial_v g_{vv}) = 0 \quad (3.37)$$

$$\Omega^i_{j0} = \frac{1}{2} g^{im} (\partial_j g_{0m} + \partial_0 g_{mj} - \partial_m g_{j0}) + \frac{1}{2} g^{iv} (\partial_j g_{0v} + \partial_0 g_{vj} - \partial_v g_{j0})$$

$$= \frac{1}{2} G^{im} (\partial_j \alpha_m + \partial_0 G_{mj} - \partial_m \alpha_j) = \frac{1}{2} G^{im} \partial_0 G_{mj} + \frac{1}{2} G^{im} (\partial_j \alpha_m - \partial_m \alpha_j) \quad (3.37)$$

$$\Omega^i_{jv} = \frac{1}{2} g^{im} (\partial_j g_{mv} + \partial_v g_{mj} - \partial_m g_{jv}) + \frac{1}{2} g^{iv} (\partial_j g_{v} + \partial_v g_{v} - \partial_v g_{jv}) = 0 \quad (3.37)$$

$$\Omega^i_{0e} = \frac{1}{2} g^{im} (\partial_0 g_{me} + \partial_v g_{0m} - \partial_m g_{0e}) + \frac{1}{2} g^{iv} (\partial_0 g_{ve} + \partial_v g_{ve} - \partial_v g_{ve}) = 0 \quad (3.37)$$

$$\Omega^i_{\mu\nu}x'^\nu = \Omega^i_{jk} x'^j x'^k + 2\Omega^i_{j0} x'^j c t' + \Omega^i_{00} c^2 t'^2$$

$$= \Gamma^i_{jk} x'^j x'^k + G^{im}(x, t) \left[ (\partial_j \alpha_m - \partial_m \alpha_j) x'^j c t' + m \partial_m \Lambda(x, t) c^2 t'^2 \right]$$

$$+ G^{im}(x, t) \left[ \partial_0 G_{mj}(x, t) x'^j + \partial_0 \alpha_m(x, t) c t' \right] c t' \quad \text{ct'}$$
Thus, if we appropriately substitute \( c t' = \frac{p^i}{m} \) from (3.24) and \( \alpha_i(x, t) \) and \( \Lambda(x, t) \) from (3.34) into the geodesic equation above, we will reproduce the original equation of motion (3.29):

\[
x^i + \Gamma^i_{jk} x'^j x'^k = G^i_{\mu} (x, t) \left[ \frac{1}{m} (\partial_m A_k - \partial_k A_m) x'^k - \frac{1}{m} \partial_m V(x, t) \right.
\]

\[
- \left( \partial_0 G_{mj}(x, t) x'^j + \frac{1}{m} \partial_0 A_m(x, t) \right) c t'
\]

thus, validating the lifting procedure described.

### 3.4 Killing vector under the Lorentzian lift

We previously showed the deformation of the Killing vector under the Riemannian lift. Now we shall describe the same for the Lorentzian lift for non-autonomous systems.

The Killing equations for the lifted metric (3.36) are:

\[
\tilde{\nabla}_\mu K_\nu + \tilde{\nabla}_\nu K_\mu = 0 \quad \Rightarrow \quad \begin{cases}
\tilde{\nabla}_i K_j + \tilde{\nabla}_j K_i = 0 \\
\tilde{\nabla}_0 K_i = 0 \\
\tilde{\nabla}_v K_j + \tilde{\nabla}_j K_v = 0 \\
\tilde{\nabla}_v K_0 = 0 \\
\tilde{\nabla}_v K_v = 0
\end{cases}
\]

(3.38)

One can see that \( \Omega^0_{\mu\nu} = 0 \), and \( \Omega^v_{\mu\nu} = -\alpha_i(x, t) \Omega^i_{\mu\nu} \). Thus, using such properties alongside the Christoffel connection components (3.37), we can write:

\[
\tilde{\nabla}_i K_j = \partial_i K_j + \Omega^m_{ij} K_m + \Omega^v_{ij} K_v = \partial_i K_j + \Omega^m_{ij} (K_m - \alpha_m(x, t) K_v)
\]

\[
\tilde{\nabla}_0 K_i = \partial_0 K_i + \Omega^m_{i0} K_m + \Omega^v_{i0} K_v = \partial_0 K_i + \Omega^m_{i0} K_m - \alpha_m(x, t) \Gamma^m_{ij} K_v
\]

\[
\tilde{\nabla}_v K_i = \partial_v K_i + \Omega^m_{iv} K_m + \Omega^v_{iv} K_v = \partial_v K_i
\]

\[
\tilde{\nabla}_v K_0 = \partial_v K_0 + \Omega^m_{0v} K_m + \Omega^v_{0v} K_v = \partial_v K_0
\]

\[
\tilde{\nabla}_0 K_i = \partial_0 K_i + \Omega^m_{0i} K_m + \Omega^v_{0i} K_v = \partial_0 K_i + \frac{1}{2} \left[ G^{mp} \partial_0 G_{pi} + G^{mp} (\partial_i \alpha_p - \partial_p \alpha_i) \right] (K_m - \alpha_m(x, t) K_v)
\]

\[
\tilde{\nabla}_0 K_0 = \partial_0 K_0 + \Omega^m_{00} K_m + \Omega^v_{00} K_v = \partial_0 K_0
\]

\[
\tilde{\nabla}_0 K_0 = \partial_0 K_0 + \Omega^m_{00} K_m + \Omega^v_{00} K_v = \partial_0 K_0 + \frac{1}{2} \left[ G^{mp} \partial_0 G_{pi} + G^{mp} (\partial_i \alpha_p - \partial_p \alpha_i) \right] (K_m - \alpha_m(x, t) K_v)
\]

\[
\tilde{\nabla}_0 K_0 = \partial_0 K_0 + \Omega^m_{00} K_m + \Omega^v_{00} K_v = \partial_0 K_0 + \frac{1}{2} \left[ G^{mp} \partial_0 G_{pi} + G^{mp} (\partial_i \alpha_p - \partial_p \alpha_i) \right] (K_m - \alpha_m(x, t) K_v)
\]

\[
\tilde{\nabla}_0 K_0 = \partial_0 K_0 + \Omega^m_{00} K_m + \Omega^v_{00} K_v = \partial_0 K_0 + \frac{1}{2} \left[ G^{mp} \partial_0 G_{pi} + G^{mp} (\partial_i \alpha_p - \partial_p \alpha_i) \right] (K_m - \alpha_m(x, t) K_v)
\]

\[
\tilde{\nabla}_v K_0 = \partial_v K_0 + \Omega^m_{0v} K_m + \Omega^v_{0v} K_v = \partial_v K_0
\]

\[
\tilde{\nabla}_v K_0 = \partial_v K_0 + \Omega^m_{0v} K_m + \Omega^v_{0v} K_v = \partial_v K_0
\]

\[
\tilde{\nabla}_v K_0 = \partial_v K_0 + \Omega^m_{0v} K_m + \Omega^v_{0v} K_v = \partial_v K_0
\]
Upon applying the above equations into (3.38), we can see that the original Killing vector equation (2.23) is deformed under the Magnetic Eisenhart Lorentzian lift into:

\[
\nabla_i K_j + \nabla_j K_i = 2\alpha_m(x, t)\Gamma^m_{ij} K_v, \tag{3.39}
\]

and we get the following equations with the lifted component of the Killing vector:

\[
\partial_i K_0 + \partial_0 K_i = -G^{mp}(x, t) \left[ \partial_0 G_{pi}(x, t) + (\partial_i \alpha_p(x, t) - \partial_p \alpha_i(x, t)) \right] (K_m - \alpha_m(x, t) K_v) \]
\[
\partial_0 K_0 = -2 \left[ G^{mp} \partial_0 \alpha_p + m G^{mp} \partial_p \Lambda \right] (K_m - \alpha_m(x, t) K_v) \tag{3.40}
\]
\[
\partial_v K_i + \partial_i K_v = 0 , \quad \partial_v K_0 + \partial_0 K_v = 0 , \quad \partial_v K_v = 0 .
\]

where we can see that upon applying the condition \( A_i(x) = \alpha_i(x) = 0 \) to (3.39) to lift a system without magnetic fields to a static spacetime, the Killing vector equation (2.23) will be preserved.

Furthermore, for all points off the classical path, we shall have the orthogonality condition according to (2.25):

\[
\frac{D}{Dy} \left( g_{\mu\nu}(x) x^\mu \right) K^\nu = K_\mu \frac{D x^\mu}{Dy} = K_i \frac{D x^i}{Dy} + K_0 \frac{D t'}{Dy} + K_v \frac{D v'}{Dy} = 0 ,
\]

\[
\Rightarrow \quad K_i x'^i + K_v v'^v + (K_m - \alpha_m(x, t)) \Gamma^m_{ij} x'^j x'^k = 0 . \tag{3.41}
\]

4 Conclusion and Discussion

We started by reviewing mechanics from Lagrangian viewpoint, followed by demonstrating how the Killing equation arises from the isometry of the metric, and from the symmetry of the action for a geometric Lagrangian. Furthermore, we have also shown that when a particle’s trajectory for a geometric Lagrangian is deformed from the classical path, the Killing vector is orthogonal to the force derived from the potentials for which the deformed path would be the classical one.

Then, we have reviewed how a modified Eisenhart-Riemannian lift can convert a classical autonomous Lagrangian with magnetic fields into a geometric one, essentially constructing a stationary spacetime while preserving the equation of motion in the form of a geodesic equation. Then we proceeded to show that the Killing vector also gets lifted to higher dimensions. After that, we reviewed the Eisenhart-Lorentzian lift for time-dependent mechanical systems, and examined the deformation of the Killing vector equations under it.

In this paper, we have restricted our attention to point particle mechanics. In the next article [23], we will show how to perform the Eisenhart lift in \( n \)-dimensional field theory.

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