Gravitational focusing of Imperfect Dark Matter

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Abstract

Motivated by the projectable Horava-Lifshitz model/mimetic matter scenario, we consider a particular modification of standard gravity, which manifests as an imperfect low pressure fluid. While practically indistinguishable from collection of non-relativistic weakly interacting particles on cosmological scales, it leaves drastically different signatures in the Solar system. The main effect stems from gravitational focusing of the flow of Imperfect Dark Matter passing near the Sun. This entails the strong amplification of Imperfect Dark Matter energy density compared to its average value in the surrounding halo. The enhancement is many orders of magnitude larger than in the case of Cold Dark Matter, provoking deviations of the metric in the second order in the Newtonian potential. Effects of gravitational focusing are prominent enough to substantially affect the planetary dynamics. Using the existing bound on the PPN parameter $\beta_{PPN}$, we deduce the stringent constraint on the unique constant of the model.

1 Introduction and Summary

Despite successes of General Relativity (GR) \cite{1, 2}, it appears to be incomplete in both high and low energy limits. First, GR is not perturbatively renormalizable and, consequently, loses its predictive power at distances of the order of the Planckian size, $l_{Pl} \sim 10^{-33}$ cm. To retain predictivity at those and smaller scales, one should replace GR by some ultraviolet complete theory \textit{a la} superstrings. The `infrared' problem stems from the existence of Dark

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Energy,—we yet do not know the physics behind the small Λ-constant. Another low energy
phenomenon, which is less commonly viewed as challenge for GR, is Dark Matter (DM).
While it could be relatively simply explained in some extensions of the Standard Model,
only gravitational manifestations of DM have been identified so far. Therefore, it may be
well a product of gravity modification. We will entertain this possibility in the present paper.

We will be interested in the model of gravity described by the following action [3, 4, 5],
\[ S = -\frac{1}{16\pi G} \int d^4x \sqrt{-g} R + \int d^4x \sqrt{-g} \left[ \frac{\Sigma}{2} (g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - 1) + \frac{\chi}{2} (\Box \phi)^2 \right], \tag{1} \]
(hereafter, we assume the mostly negative signature of the metric). The first term on the
r.h.s. here describes the Einstein–Hilbert action; \( G \equiv \frac{1}{M_{Pl}^2} \) is the gravitational constant, and
\( M_{Pl} \) is the Planck mass. The other two stand for what one calls Imperfect Dark Matter
(IDM). In the limit of vanishing dimensionful constant \( \chi \), IDM reduces to a pressureless
perfect fluid, with the fields \( \Sigma \) and \( \phi \) playing the role of its energy density and velocity
potential, respectively. Switching on the higher derivative term renders the fluid slightly
imperfect \( \chi \) (hence, the name) and equips it with a non-zero sound speed
\( c_s \sim \sqrt{\frac{\chi}{M_{Pl}}} \).

In the synchronous gauge, a homogeneous solution for the field \( \phi \) takes a simple form,
\[ \bar{\phi} = t. \tag{2} \]
So, it serves as the time parametrization. Existence of the preferred frame, i.e., the one,
where the background value of the field \( \phi \) is given by Eq. (2), implies dynamical Lorentz
violation in the model (1). In this regard, it is akin to the Einstein–Aether theory \( \chi \). The
latter, however, deals with a unit 4-vector field \( u_\mu \) rather than with the 4-gradient of a scalar.
This distinction produces drastically different dynamics in the two models.

Particularities of the cosmological evolution in the scenario (1) have been considered in
Refs. [4, 5, 8]. The main effect stems from non-zero sound speed [3]. Namely, the formation
of objects with the size smaller than the sound speed horizon is suppressed compared to the
predictions of Cold DM (CDM). Consequently, one risks to strongly affect the bottom-up
picture of the structure formation for sufficiently large values of the parameter \( \chi \). In Ref. [4],
this observation was used to set the bound,
\[ \frac{\chi}{M_{Pl}^2} \lesssim 10^{-10}. \tag{3} \]
For much smaller values of the parameter \( \chi \), IDM is indistinguishable from CDM at the
cosmological level. On the other hand, given values saturating the bound in Eq. (3), the
behaviour of IDM shares some similarities with the Warm DM.

\footnote{\textsuperscript{1}In Ref. [3], the parameter \( \chi \) was promoted to a function of the field \( \phi \). This was proved crucial for
generating the required amount of IDM in the early Universe (the issue analogous to getting the correct
relic abundance of DM in the CDM framework). On the other hand, the slight dependence on the field \( \phi \) is
completely irrelevant for the discussion of the present paper, where we focus on the Solar system scales.}

\footnote{\textsuperscript{2}For this reason, the model (1) is referred to as the 'scalar Einstein–Aether' in Ref. [7].}
Let us briefly discuss the quantum features of the model (1). Compared to GR, it propagates three degrees of freedom: the standard ones associated with two polarizations of the helicity-2 graviton, and the scalar potential $\Psi$, which is now a dynamical field \cite{9, 10, 11, 12, 13, 14}. The extra degree of freedom $\Psi$ exhibits gradient/ghost instabilities for negative/positive values of the parameter $\chi$. This, however, does not invalidate the model immediately. Indeed, those pathologies are not particularly dangerous provided that there is a sufficiently low Lorentz-violating cutoff on the spatial momenta of the modes of the field $\Psi$ \cite{15, 16}. That cutoff is associated with the scale of yet unknown UV completion of the model or the strong coupling scale.

In the version with gradient instabilities ($\chi < 0$), however, this cutoff turns out to be extremely low \cite{13}, $\Lambda \ll (0.1 \text{ mm})^{-1}$. The latter contradicts the tests of gravity extending from the sub-mm distances to the Solar system scales. On the other hand, the ghost unstable branch of the model ($\chi > 0$) allows for the cutoff scale $\Lambda$ as large as $\Lambda \sim 10 \text{ TeV}$. Assuming that the strong coupling and UV scales are of the same order, this bound implies the constraint on the parameter $\chi$ \cite{14},

$$\frac{\chi}{M_{Pl}^2} \lesssim 10^{-20}.$$  

(4)

For larger values of the constant $\chi$, the vacuum decay with photons and ghosts in the final state is too fast, what leads to the conflict with measured fluxes of the gamma- and X-ray emission \cite{17}.

One interesting way of UV completing the action (1) is suggested in the context of the projectable Horava-Lifshitz model \cite{9}. The latter postulates a non-uniform transformation of time and spatial coordinates under the scaling. This has a dramatic effect on the ultraviolet behaviour of gravitons resulting into the strong distortion of their dispersion relation, $\omega^2 \sim p^6$. Consequently, there are less divergencies in the graviton loop integrals, what eventually leads to the (power counting) renormalizability of gravity \cite{9, 18}. While the Horava-Lifshitz model is manifestly non-relativistic, it allows for the covariant description by introducing the Stuckelberg field $\varphi$ dubbed khronon. Then, its infrared limit exactly takes the form (1) \cite{13, 20}. It is thus not a surprise that DM has been identified in this context \cite{19}. In particular, the term with the Lagrange multiplier ensures the projectability condition, which eliminates the pathological mode otherwise present at low momenta \cite{20, 21}. The parameter $\chi$ is generically non-zero\footnote{The notation $\chi$ may be inconvenient for those familiar with Horava–Lifshitz gravity. There one deals with the parameter $\lambda$, which appears in front of the term $\sim K^2$ (the trace of the extrinsic curvature tensor squared) in the ADM formulation of the model. The two are related to each other by $\chi = \frac{1 - \lambda}{8\pi G}$.}, and is supposed to follow the renormalization group flow towards the...
'GR point’ $\chi = 0$.

Embedding the model (1) into the Horava-Lifshitz gravity is not without problems, though. The reason is that UV operators modifying the dispersion relation of the gravitons are not capable of curing ghost instabilities. Hence, the only way to stabilize the catastrophic vacuum decay is to assume that the model enters a putative strong coupling phase. This severely obstructs the main objective of the Horava’s proposal—perturbative renormalization of gravity.

Recently, the action (1) has been rediscovered in a completely different framework of the mimetic matter [3, 24]. There one deals with a non-invertible conformal transformation of the metric [25],

$$\tilde{g}_{\mu\nu} = A(\varphi, X)g_{\mu\nu} + B(\varphi, X)\partial_\mu\varphi\partial_\nu\varphi,$$

where $A$ and $B$ are the arbitrary functions of the scalar $\varphi$ and $X \equiv g_{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$. Performing this transformation on the standard Einstein–Hilbert action, one does not reproduce GR, but rather GR supplemented by a perfect pressureless fluid dubbed mimetic DM. Equivalently, the latter can be introduced by making use of the Lagrange multiplier as in Eq. (1), i.e., without directly referring to the disformally transformed metric [26, 27]. The higher derivative term is absent in the original formulation of the mimetic matter scenario. Nevertheless, that extension does not affect the main idea underlying the scenario, and even appears to be the only viable option for a number of phenomenological issues [3, 4].

In the present paper, we do not assume any particular gravitational framework behind the model (1). Neither, we are interested in its cosmological or microscopic manifestations. Our main focus here is the intermediate range of scales: Solar system. Surprisingly, this yields limits, which are many orders of magnitude stronger than those obtained from the structure formation considerations. Moreover, our discussion does not assume that IDM gives the dominant contribution to the invisible matter in the Universe (namely, IDM may constitute only its tiny fraction).

The behaviour of IDM in the Solar system shares some features (but not all) with CDM. Let us briefly summarize the main effect. The Sun moves relative to the cosmic microwave background and Milky Way rest frames with the speed $v \approx 10^{-3}$. Naturally, the preferred frame is associated with one of those. An observer in the Solar system sees a flow of IDM. Affected by the gravitational potential of the Sun, the flow focuses downstream from the Sun forming a caustic. This part of the story parallels to that of CDM.

In the latter case, however, gravitational focusing is a rather moderate effect introducing a few percent correction to the annual modulation of the flux of DM particles passing through the Earth [30, 31, 32, 33, 34, 35]. The things are different in the IDM scenario. The

6Different extensions have been considered in Refs. [3, 28, 29]. The former equips the field $\varphi$ with some potential $V(\varphi)$. In this way, one manages to mimick fairly arbitrary cosmological evolution. In Refs. [28, 29], the mimetic matter scenario has been extended by means of the Horndeski higher derivative terms.
reason is the higher derivative structure of its action (1). The term $\sim \chi(\Box \varphi)^2$ serves as a powerful source of the IDM energy density $\Sigma$. The latter backreacts on the space-time geometry causing distortion of the metric. This is the main effect identified in the present paper. As in the case of CDM, it is particularly prominent in the direction opposite to the velocity of the Sun $\mathbf{v}$, where IDM is mainly accumulated. Borrowing terminology of Ref. [36], astrophysical objects moving relative to the preferred frame (2) leave 'star tracks' behind them. Because in our setup deviations from GR do not fit the standard PPN approach, we cannot directly use the PPN formalism. Instead, to quantify properly this effect, we estimate the value of the would-be parameter $\beta_{PPN}$. This deviates from its GR counterpart, unity, by $\beta_{PPN} - 1 \simeq \frac{4\pi \chi}{M_{Pl}^4} \cdot \frac{1}{\theta^4}$, where $\theta$ is the angle between the line of sight of the observer on the Sun and the direction $-\mathbf{v}$. We then convert the existing bounds on $\beta_{PPN}$ into the limit on the model constant $\chi$, ignoring the dependence on the angle $\theta$. The resulting constraint is quite stringent: $\chi/M_{Pl}^2 \lesssim 10^{-18}$, which is only two orders of magnitude weaker than the limit (4) inferred from the microscopic physics considerations.

The remainder of the paper is as follows. In Section 2, we deduce equations of motion following from the action (1). We discuss main assumptions and approximations used to study the dynamics of IDM in Section 3. For the sake of convenience, there we also outline the main formulae describing the IDM profile in the Solar system, as well as the induced metric corrections. Derivation of those results are explained in Sections 4 and 5, where we restrict to the linear and quadratic order analysis in the Newtonian potential, respectively. In Section 4, we also identify the narrow region of space, where the perturbative description of IDM breaks down. We assess metric perturbations in this region in Section 6. The reader interested in the final results, may skip Sections 4-6 and go directly to Section 7, where we contrast our predictions for metric perturbations to the observational bounds, and derive the constraint on the model constant $\chi$.

2 Setup

Let us write down equations of motion following from the action (1). Variation with respect to the Lagrange multiplier $\Sigma$ yields

$$g_{\mu \nu} \partial^\mu \varphi \partial^\nu \varphi = 1 .$$

Applying the covariant derivative to both parts of the constraint, one reproduces the geodesics equation followed by test particles in the gravitational field [20] [39],

$$\frac{du^\mu}{ds} + \Gamma^\mu_{\nu \lambda} u^\nu u^\lambda = 0 ,$$

where $u^\mu \equiv \partial^\mu \varphi$ is the 4-velocity. In the following, we will repeatedly make use of this analogy with the case of dust particles.
Varying the action (1) with respect to the field $\phi$, we obtain
\[ \nabla_\mu (\Sigma \nabla^\mu \phi) = \chi \Box^2 \phi. \] (7)

For the choice $\chi = 0$, Eq. (7) takes the form of the energy density conservation for a collection of particles. In this limit, there is no source of DM: if its density is zero everywhere at some moment, it remains so at later times. This behaviour changes upon switching on a non-zero $\chi$. Then, there is a source of DM for any non-trivial configuration of the field $\phi$. This difference between IDM and CDM will be important for our further discussions.

Finally, there are Einstein equations, which we write in the form,
\[ R_{\mu\nu} = 8\pi G \left( T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu} \right), \] (8)
where
\[ T_{\mu\nu} = T_{\mu\nu}^{\text{matter}} + T_{\mu\nu}^{IDM}. \]

Here $T_{\mu\nu}^{\text{matter}}$ and $T_{\mu\nu}^{IDM}$ are the stress-energy tensors corresponding to the standard matter and IDM. The latter is given by
\[ T_{\mu\nu}^{IDM} = \Sigma \nabla_\mu \phi \nabla_\nu \phi + \chi \left( \nabla_\alpha \phi \nabla^\alpha \Box \phi + \frac{1}{2} (\Box \phi)^2 \right) g_{\mu\nu} - \chi (\nabla_\nu \phi \nabla_\mu \Box \phi + \nabla_\nu \Box \phi \nabla_\mu \phi). \] (9)

In the limit $\chi \to 0$, this reduces to the stress-energy tensor of a pressureless perfect fluid. In what follows, we will refer to the field $\Sigma$ as the energy density of IDM even for non-zero values of the parameter $\chi$. The discrepancy between the physical and so defined energy density is small and irrelevant for our study. This can be verified rigorously at each step of calculations.

3 Flow of IDM past the Sun: assumptions and main formulae

Our goal in the present paper is to study the imprint of IDM on the metric created by the standard matter—the Sun. For this purpose, we employ the test field approximation. Namely, we assume that the Sun is the main source of the space-time curvature, while IDM produces small corrections on the background metric. This assumption will be checked by the structure of the final result.

It is convenient to perform calculations in the rest frame of the source. In that coordinate system, there is a flow of IDM moving towards the Sun with the velocity $v \equiv v^i$. Performing the Lorentz transformation on the background profile (2) of the field $\phi$, we get
\[ \phi = \sqrt{1 + v^2 t} - v^i x^i. \] (10)
One may naively expect the speed \( v \equiv |\textbf{v}| \) to play a role of a small expansion parameter, when evaluating metric corrections. This is indeed the case of the Einstein–Aether or khronometric theories \([13, 44]\). The situation is different in the IDM scenario: at least sufficiently far from the Sun, the velocity enters metric corrections as \( 1/v^n \), where \( n \) is some positive number. This peculiar dependence on the quantity \( v \) has a clear physical explanation. If IDM passes fast near the Sun, the latter does not have enough time to affect the flow appreciably. Hence, the effects of gravitational focusing milden in that case.

The perturbative treatment of the metric is still possible in terms of the Newtonian potential, provided that the following inequality is obeyed,

\[
|\Phi| \ll v^2. \tag{11}
\]

In other words, the actual expansion parameter is \( |\Phi|/v^2 \), and we anticipate linear and quadratic metric corrections to be of the order \( \chi M_{\text{Pl}}^2 \cdot \Phi/v^2 \) and \( \chi M_{\text{Pl}}^2 \cdot \Phi^2/v^4 \), respectively. This fact places limitations on the analysis of the present paper: it is not applied for sufficiently strong gravitational fields/small speeds. On the other hand, for the choice

\[
v \simeq 10^{-3}, \tag{12}
\]

the inequality \((11)\) represents a reasonable approximation within the Solar system. So, one has \( |\Phi|/v^2 \approx v_{\text{Mercury}}^2/v^2 \approx 1/36 \) at the Mercury distance, where \( v_{\text{Mercury}} \approx 47 \text{ km/s} \) denotes its average orbital speed. For more remote planets such a ratio decreases gradually, increasing the accuracy of the inequality \((21)\). The only region of space, where Eq. \((11)\) is not valid, is very close to the surface of the Sun, see Fig. 1. Being primarily interested in the planetary dynamics, we will ignore this subtlety in the bulk of the paper. The issue, however, comes into play, when calculating metric corrections induced by IDM. To fill in the gap, in Appendix A we provide a simplified analysis of the relevant fields in this region.

Before going into the details of the analysis, let us briefly comment on the origin of the estimate \((12)\). It is natural to associate the preferred frame with the rest frame either of the cosmic microwave background or the Milky Way halo. In the former case, the speed is inferred from the observed dipole anisotropy \([37]\) and is given by \( v \approx 369 \text{ km/s} \). In the latter case, \( v \approx 220 \text{ km/s} \). We see that both match the estimate \((12)\) very well. Still the two choices are not degenerate, as they imply different inclinations of the velocity \( \textbf{v} \) relative to the Solar system plane. This distinction will be relevant, when contrasting our theoretical predictions to the observational data.

When defining metric corrections, our strategy will be as follows. First, we will solve the constraint equation \((5)\) perturbatively expanding the field \( \phi \) in the powers of the Newtonian potential. Hereafter, we assume a \textit{stationary} flow of IDM and, henceforth, restrict to a static configuration of the velocity potential \( \phi \). In this approximation, the field \( \phi \) is uniquely
defined by the boundary condition
\[ \nabla \varphi \rightarrow -v \quad \text{at} \quad z \rightarrow -\infty. \quad (13) \]

We choose the z-axis of the reference frame to be aligned with the velocity \( v = v^i \), and set its origin at the location of the Sun, as shown in Fig. 1. Downstream from the Sun \( (z > 0) \), the perturbative series of interest is schematically given by
\[
\delta \varphi = \eta_1 \frac{M}{v M_p^2} \ln \frac{z}{\rho} + \eta_2 \frac{M^2}{v^3 M_p^4} \frac{z^2}{\rho^2} + \mathcal{O} \left( \frac{M^3}{v^5 M_p^6} \cdot \frac{z^3}{\rho^4} \right), \quad (14)
\]
where \( \delta \varphi \equiv \varphi - \bar{\varphi} \). Here \( M \) is the mass of the Sun; \( \eta_1 \) and \( \eta_2 \) are some order one coefficients. Now, we see explicitly the relevance of the condition (11)—had it not been obeyed, the series (14) would not converge. Still, the terms entering the series (14) formally blow up in the limit \( \rho / z \rightarrow 0 \), what eventually invalidates the perturbative treatment of the velocity potential \( \varphi \), see Fig. 1. Fortunately, there is a way to find a solution for the field \( \varphi \) in this narrow region. As we will see in Section 6, the role of non-perturbative effects is to smoothen the singularities present in Eq. (14), so that the actual field \( \varphi \) is finite everywhere. On the other hand, there is no such an issue for negative \( z \) (the upstream region). In that case, the perturbative series analogous to Eq. (14) remains well-defined for arbitrarily small values of \( \rho \).

As for the next step, we calculate the energy density \( \Sigma \). The latter is mainly fixed by the configuration of the field \( \varphi \), while the metric perturbations by themselves give a negligible contribution at this level. Just like the velocity potential \( \varphi \), the density field \( \Sigma \) can be treated perturbatively almost in the whole region of space. Cutting the series at the quadratic term, one has for \( z > 0 \)
\[
\Sigma \simeq \frac{\chi M^2}{v^4 M_p^4} \cdot \frac{z^2}{\rho^6} + \mathcal{O} \left( \frac{\chi M^3}{v^6 M_p^6} \cdot \frac{z^3}{\rho^8} \right). \quad (15)
\]
We neglected here the background value of the field \( \Sigma \). This can be easily justified. Indeed, the amount of IDM generated by the gravitational field of the Sun turns out to be much larger than its expectation value in the surrounding halo. Furthermore, IDM is not produced at the linear level. We will show this explicitly in Section 4. Once again, the sharp \( \rho \)-dependence of the energy density in the limit \( \rho \rightarrow 0 \) is an artefact of the perturbative approach, which breaks down near the \( z \)-axis. At the same time, the non-perturbative effects convert the strong growth of the field \( \Sigma \sim \frac{1}{\rho^6} \) as in Eq. (15) into a milder one \( \Sigma \sim \frac{1}{\rho^2} \). See Section 6 for details.

Knowing the distribution of the fields \( \Sigma \) and \( \varphi \), we solve the Einstein equations and obtain metric perturbations defined as
\[
h_{00} \equiv g_{00} - 1 \quad h_{0i} \equiv g_{0i} \quad h_{ij} \equiv g_{ij} + \delta_{ij}. \quad (16)
\]
Figure 1: The ecliptic plane is shown relative to the direction of the IDM flow aligned with the $z$-axis. The shaded region corresponds to the part of the space, where the perturbative calculation of the IDM profile and the induced metric corrections breaks down. This covers not only the region downstream from the Sun, but also some small region near its surface.

There is a subtlety at this point. While the fields $\Sigma$ and $\varphi$ can be calculated unambiguously in the perturbative region, to find the metric perturbations one has to know the IDM profile everywhere in space, including non-perturbative region. To tackle this issue, we extrapolate results of the perturbative analysis for the fields $\Sigma$ and $\varphi$ from the region, where this calculation can be trusted, to the whole space. We also approximate the Sun by a pressureless perfect fluid and set to zero relative motion of its constituents. It is then straightforward to
evaluate the metric. To the quadratic order, the correction to its 00-component reads

$$\delta h_{00} \simeq \frac{\chi}{M_{Pl}^2} \cdot \frac{\Phi^2}{v^4} \cdot \frac{1}{\theta^4} ,$$  \hfill (17)

where $\theta$ is the angle between the line of sight of the observer sitting on the Sun and the direction $v$. The absence of linear order corrections here reflects the fact that the perturbative expansion (15) for the Lagrange multiplier field $\Sigma$ starts from the quadratic term. The same conclusion holds for the $0i$- and $ij$-metric components. Strictly speaking, metric perturbations calculated in this way, at best may serve as the estimate for the true ones. Therefore, we should critically assess the discrepancy caused by the non-perturbative effects. This is done in Section 6, where we discuss the IDM profile downstream from the Sun. In Appendix A, we estimate non-perturbative effects due to the breakdown of the inequality (21), which occurs near its surface. In both cases the discrepancy is not larger than the quadratic order metric correction (17) induced by IDM. This will serve as the justification of our perturbative calculation.

Finally, we contrast the resulting metric corrections to the existing experimental bounds on the deviations from GR manifested as the constraints on the PPN parameters. There is subtlety, however. One of the prescriptions underlying the PPN approach is the smooth distribution of the fields, which source the space-time curvature \[2\]. This is why it cannot be applied directly to the model under study. First, the fields $\varphi$ and $\Sigma$ vary fast in the narrow region downstream from the Sun, what eventually leads to the breakdown of the perturbative approach. Even away from this part of space, the $\theta$-dependence of the metric correction (17) does not allow for the direct comparison with its PPN counterpart governed by the constant parameter $\beta_{PPN}$. It is, however, still possible to impose a conservative limit on the model constant $\chi$. Indeed, the metric correction (17) has a non-vanishing minimal value with respect to the angle $\theta$. Note that for a fixed angle $\theta$, Eq. (17) exactly takes the form as in the PPN formalism. Then, the model can be constrained using the existing bounds on the parameter $\beta_{PPN}$. See Section 7 for more details.

Our last, but not the least important, assumption concerns the behaviour of IDM in the deeply non-linear regime, where a caustic is supposed to be formed. This issue is also closely related with one of our basic approximations—the stationarity of IDM flow. While the aforementioned non-perturbative effects substantially smoothen the IDM profile downstream from the Sun, the caustic singularity is not fully cured. The reason for this is the presence of the constraint (5), which effectively describes the potential flow of dust particles. Note that the solution for $\varphi$ is not affected at all by the presence of the higher-order term in the action. Employing for an instant this analogy with the CDM case, particles following the geodesics equation (6) eventually cross at what is called caustic. In the CDM framework, however, the appearance of caustics merely hints the breakdown of a single flow approach for the collection of particles. One should switch to the multi-flow description in the case of
CDM. On the contrary, IDM is fundamentally described by a single flow, parametrized in terms of the unique field $\varphi$. Therefore, an appearance of a caustic may imply a fundamental problem of the theory. On the other hand, formation of caustics is generic in all scalar-tensor theories, even in models which do not behave as dust [40].

Mechanisms for avoiding the caustic singularity in the model at hand have been discussed in Refs. 4, 38. So, Ref. 4 suggests that the test field approximation is exceeded at some point, and the field $\Sigma$ becomes the main source of gravity. It is then crucial that the energy density $\Sigma$ may take negative values, so that the gravitational force becomes repulsive. In this situation, there is the chance to avoid the caustic singularity. Alternatively, it may be resolved for a particular choice of UV operators completing the model [1]. This has been argued in Ref. 38 in the Horava-Lifshitz framework. Which mechanism is actually realized depends on the particularities of IDM dynamics. At the level of the Solar system, IDM always gives a negligible contribution to the total space-time curvature. Therefore, the test field approximation remains trustworthy in the whole space. Hence, the issue of caustic singularities should be attributed to the microscopic physics and interpreted as the problem of finding the UV completion for IDM. We assume in what follows that not only such a UV completion exists, but it can also maintain the stationarity of the IDM flow [4], at least at the scales and time intervals relevant for the discussion of the present paper.

4 Linear level analysis

In the bulk of the paper, we will evaluate the field $\delta \varphi \equiv \varphi - \bar{\varphi}$ in the straightforward manner—by resolving the constraint (5). Alternatively, one observes that Eq. (5) leads to the geodesics equation (6) followed by test particles in the gravitational field. Thus, not resorting to Eq. (5), we could equivalently consider a parallel flow of dust particles falling onto the Sun with the velocity $v$. The velocity distribution, which characterizes the flow, should match the one obtained directly from Eq. (5). This will serve as the cross-check of our computations. We relegate the further details to Appendix B.

In the stationary flow approximation, the constraint (5) linearized reads

$$\partial_i \delta \varphi^{(1)} v_i = (1 + 2v^2) \Phi . \quad (18)$$

Recall that we work in the rest frame of the Sun, where the background value of the field $\varphi$ is given by Eq. (10). We used the Newtonian gauge to fix metric potentials. The general solution of Eq. (18) is given by

$$\delta \varphi^{(1)} = -(1 + 2v^2) \frac{M}{v \cdot M_{Pl}} \arcsinh \frac{z}{\rho} + f(\rho) , \quad (19)$$

\[7\text{See Refs. 41, 42 for the debates on the similar issues.}\]
where \( f(\rho) \) is some function of integration. We rotated the reference frame in a way that its \( z \)-axis is aligned with the velocity \( \mathbf{v} \), and the origin is placed at the location of the Sun, as shown in Fig. 1. The function \( f(\rho) \) is uniquely defined from the boundary condition chosen by the analogy with the particle case,

\[
\nabla \delta \varphi \to 0 \quad \text{at} \quad z \to -\infty ,
\]

which reflects Eq. (13). We write down the first order solution for the field \( \delta \varphi \),

\[
\delta \varphi^{(1)} = - (1 + 2v^2) \frac{M}{vM_{Pl}^2} \left( \text{arcsinh} \frac{z}{\rho} - \ln \frac{\rho}{\rho_0} \right) ,
\]

where \( \rho_0 \) is an irrelevant dimensionful constant. Both terms on the r.h.s. here become singular in the limit \( \rho \to 0 \). However, these singularities cancel out in the upstream region \( (z < 0) \), and the overall solution (20) exhibits a regular behaviour,

\[
\delta \varphi^{(1)} = - (1 + 2v^2) \cdot \frac{M}{vM_{Pl}^2} \ln \frac{2\rho_0}{|z|} \left( \frac{\rho}{|z|} \ll 1, \quad z < 0 \right) ,
\]

as it should be. A different story occurs for \( z > 0 \), where the solution grows infinitely in the limit \( \rho \to 0 \) eventually exceeding the regime of the applicability of the linear approximation (see the discussion below),

\[
\delta \varphi^{(1)} = - (1 + 2v^2) \cdot \frac{M}{vM_{Pl}^2} \cdot \ln \frac{2z\rho_0}{\rho^2} \left( \frac{\rho}{z} \ll 1, \quad z > 0 \right) .
\]

That configuration of the field \( \delta \varphi \) has a clear physical meaning: it reflects the gravitational focusing experienced by IDM passing in the vicinity of the Sun. In the first order, however, this amplification of the field \( \delta \varphi \) produces essentially no effect on the metric perturbations, as we will see below.

When deriving Eq. (20), we implied a point source approximation for the standard matter. It is straightforward to generalize it to the realistic case of a finite size source. The 'true' solution coincides with the one of Eq. (20) apart from the cylindric region of the size \( \rho = R \) along the direction \( \mathbf{v} \), where \( R \) is the radius of the Sun. This is, however, a minor issue. That is, the point source approximation is applied wherever the perturbative description of the field \( \delta \varphi \) can be trusted.\(^8\)

The inequality (11)—one of the prerequisite for the perturbative expansion—breaks down near the surface of the Sun. Furthermore, the linear approximation holds, only if the following conditions are obeyed,

\[
|\partial_z \delta \varphi|, |\partial_\rho \delta \varphi| \ll v .
\]

\(^8\)This is true for the gravitational field of the Sun, or more generally, a sufficiently compact object. On the other hand, for the light source like the Earth, the linear approximation is essentially never exceeded. We discuss this case in Appendix C.
Otherwise, Eq. (18) gets modified due to the appearance of quadratic terms \( \sim \partial_i \delta \varphi \partial_i \delta \varphi \). These translate into the inequality

\[
\frac{\rho^2}{z^2} \gg \frac{|\Phi|}{v^2} \quad z > 0. \tag{21}
\]

Several remarks are in order here. First, it is worth pointing out that the restriction (21) concerns only the downstream region \((z > 0)\), while for \(z < 0\) the linear approximation is retained for arbitrary \(\rho\). Second, for very small ratios \(\rho/z\) not only the linear analysis, but the perturbative description of the field \(\delta \varphi\) is questionable. We further clarify this point in Section 6. In particular, at \(\rho/z \sim \sqrt{|\Phi|/v^2}\), the higher order corrections to the field \(\delta \varphi\) all become of the same order, \(|\delta \varphi^{(1)}| \sim |\delta \varphi^{(2)}| \sim ... \sim |\delta \varphi^{(n)}|\). Instead, for larger ratios \(\rho/z\) they are arranged in a well-defined perturbative series over the powers of the potential \(\Phi\). This comment is not trivial, since only the first order quantity \(\delta \varphi^{(1)}\) was used to derive the condition (21). Therefore, one should check rigorously, if Eq. (21) is robust against including the higher order corrections into the analysis. For a while, we take this statement for granted, and give an explicit proof in Section 6.

Now, let us switch to the calculation of the Lagrange multiplier field \(\Sigma\). We assume that it vanishes at the background level, i.e., \(\bar{\Sigma} = 0\), what is compatible with the configuration (10) of the field \(\varphi\). In the linear order, the relevant part of the source term for the field \(\Sigma\) (the r.h.s. of Eq. (7)) can be written as follows,

\[
\left(\Box^2 \varphi\right)^{(1)} = -\Delta \left(\Box \varphi\right)^{(1)}. \tag{22}
\]

The generic expression for the quantity \(\Box \varphi\) in the static field approximation is given by

\[
\Box \varphi = \frac{1}{\sqrt{-g}} \partial_i \left( -\sqrt{-g} g^{ij} \partial_j \delta \varphi + \sqrt{-g} g^{ij} \partial_j \delta \varphi + \sqrt{-g} g^{0i} \sqrt{1 + v^2} \right). \tag{22}
\]

The above equation is simplified in the harmonic gauge,

\[
\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h = 0, \tag{23}
\]

to which we stick in what follows. Keeping only linear terms, we obtain

\[
\left(\Box \varphi\right)^{(1)} = -\Delta \delta \varphi^{(1)}. \tag{24}
\]

The shortcut way of calculating the quantity \(\Delta \delta \varphi^{(1)}\) is to apply the Laplace operator to both parts of Eq. (18). The result is trivially zero wherever the perturbative analysis can be trusted, i.e., the inequalities (11) and (21) are satisfied,

\[
\Delta \delta \varphi^{(1)} = 0. \tag{24}
\]
Hence, there is no source of IDM energy density $\Sigma$ to the linear order, namely,

$$\Sigma^{(1)} = 0.$$  \hspace{1cm} (25)

Indeed, in the stationary flow approximation, the field $\Sigma$ is defined from the following equation,

$$\partial_i \left( \Sigma^{(1)} v^i \right) = \chi \Delta^2 \delta \varphi^{(1)} .$$  \hspace{1cm} (26)

We assume vanishing of the field $\Sigma$ at $z \to -\infty$ as for the boundary condition. In other words, we neglect the fact that the Solar system is surrounded by the DM halo fully or partially made of IDM. This uniquely fixes the solution (25) for the energy density $\Sigma^{(1)}$.

As it follows, IDM does not source metric perturbations to the linear order. Let us show this explicitly for the 00-metric component. In the harmonic gauge, the 00-component of the Ricci tensor is given by

$$R_{00}^{(1)} = \frac{1}{2} \Delta h_{00}^{(1)} .$$  \hspace{1cm} (27)

The standard matter contributes to the r.h.s. of Einstein equations by

$$\left( T_{00}^{\text{matter}} - \frac{1}{2} T_{\text{matter}} g_{00} \right)^{(1)} = \frac{1}{2} A(r) ,$$

where $A(r)$ is the mass density of the Sun, $\int dr A(r) = M$. The analogous expression for the contribution of IDM is described by the generic expression,

$$T_{00}^{\text{IDM}} - \frac{1}{2} T_{\text{IDM}} g_{00} = \Sigma \left( 1 + v^2 - \frac{1}{2} g_{00} \right) - \frac{\chi}{2} (\Box \varphi)^2 .$$  \hspace{1cm} (28)

In the order of interest, this reduces to

$$\left( T_{00}^{\text{IDM}} - \frac{1}{2} T_{\text{IDM}} g_{00} \right)^{(1)} = \Sigma^{(1)} \left( \frac{1}{2} + v^2 \right) .$$

The latter equals to zero according to Eq. (25). Note that the term $\sim (\Box \varphi)^2$ does not contribute at this level: the quantity $\Box \varphi$ vanishes, when calculated on the background (10). Thus, at best it may affect the second order metric corrections. Consequently, the 00-metric component remains intact in the presence of IDM,

$$h_{00}^{(1)} = 2\Phi .$$  \hspace{1cm} (29)

Following the similar steps, one can prove the same for the other metric components,

$$h_{0i}^{(1)} = 0 \quad h_{ij}^{(1)} = 2\Phi \delta_{ij} .$$  \hspace{1cm} (30)
These results, we reiterate, were obtained upon extrapolating the expressions (24) and (25) to the whole space. Strictly speaking the ‘true’ solution is rather described by

\[ h = h^{\text{pert}} + h^{\text{nonpert}}. \]  

(31)

Here \( h^{\text{pert}} \) is the perturbatively calculated part of the metric given by Eqs. (29) and (30). The contribution \( h^{\text{nonpert}} \) is the correction which accounts for the non-perturbative effects. It should obey the equation \( \Delta h^{\text{nonpert}} = 0 \) in the part of space, where the inequalities (11) and (21) hold (and, hence, the expressions (24) and (25) make sense). Otherwise, it is an unknown function. We will assess the quantity \( h^{\text{nonpert}} \) in Section 6, when knowing the full IDM profile. We will see that it is at most of the order of quadratic corrections induced by IDM,—thanks to a sufficiently smooth distribution of the fields \( \Sigma \) and \( \varphi \) in the non-perturbative regime. Thus, it looks plausible to neglect this term at the level of the linear analysis.

Our last comment is in order here. One could be interested in the situation, when the linear approximation holds essentially in the whole space. This is a realistic situation for a sufficiently light object, like the Earth. We relegate details of calculations to Appendix C. Here let us quote the main result: modulo a gauge transformation, metric corrections are still zero in that case. Though this conclusion is not directly applied to the case of the Sun, the indication is clear: one should not expect any substantial deviations from GR to the linear order.

5 Second order analysis

In the previous Section, we observed that gravitational focusing of IDM is irrelevant to the first order. Namely, it does not lead to the production of the field \( \Sigma \). A different story occurs in the second perturbation order,—the main subject of the present Section. To simplify calculations, hereafter we keep only the terms with the largest power of the quantity \( 1/v \).

In this approximation, the equation defining the quadratic correction to the field \( \varphi \) is given by

\[-2\partial_i\delta\varphi^{(2)}v^i + \partial_i\delta\varphi^{(1)}\partial_i\delta\varphi^{(1)} = 0,\]  

(32)

where we again assumed the stationarity of IDM flow. Here we omitted the terms arising from the quadratic order metric expansion, \( \sim \Phi^2 \), as well as the quantities \( \sim v\Phi\partial\delta\varphi^{(1)} \). These are suppressed by the factor \( \sim v^2 \) compared to those present in Eq. (32). On top of that, we neglected metric corrections induced in the presence of IDM. That assumption is not as trivial, as it may appear to be, and will be justified a posteriori. As it follows from Eq. (32), metric perturbations do not directly affect the velocity potential in the second order.
order. The former enter only via the quantity $\delta \varphi^{(1)}$. This is a generic fact, which takes place for all the higher order corrections to the field $\varphi$.

Integrating out Eq. (32), one obtains

$$ \delta \varphi^{(2)} = \frac{M^2}{v^3 M_{Pl}^4} \cdot \frac{z}{\rho^2} + \frac{M^2}{v^3 M_{Pl}^4} \cdot \frac{\sqrt{z^2 + \rho^2}}{\rho^2}. $$

(33)

It is easy to check that the second order velocity perturbation, $-\nabla \delta \varphi^{(2)}$, goes to zero in the limit $z \to -\infty$, as it should be. In particular, for $z < 0$, the field $\delta \varphi^{(2)}$ is independent of the variable $\rho$ in the limit $\rho \to 0$,

$$ \delta \varphi^{(2)} = - \frac{M^2}{2v^3 M_{Pl}^4 z} \left( \frac{\rho}{|z|} \ll 1, \ z < 0 \right), $$

(34)

and thus exhibits the regular behaviour upstream from the Sun. To the contrast, in the downstream region, the asymptotic behaviour of the field $\delta \varphi^{(2)}$ is described by

$$ \delta \varphi^{(2)} = \frac{2M^2}{v^3 M_{Pl}^4} \cdot \frac{z}{\rho^2} \left( \frac{\rho}{z} \ll 1, \ z > 0 \right). $$

(35)

Namely, we gained the large factor $\sim \frac{z^2}{\rho^2}$ compared to Eq. (34). This amplification is exactly the manifestation of gravitational focusing. Moving towards very small values of the variable $\rho$, however, one subsequently breaks down the validity of the inequality (21). In the non-perturbative phase, the result (33) is not applied, and one should find another way to treat the field $\varphi$. We postpone this task until Section 6. On the other hand, wherever the inequality (21) is obeyed, the second order quantity (33) is consistently smaller than the linear one (20), $|\delta \varphi^{(2)}| \ll |\delta \varphi^{(1)}|$. This justifies the statement made in Section 4.

Now, let us evaluate the energy density $\Sigma$ in the second order. Again keeping only the leading powers of the quantity $1/v$, we write the relevant equation as

$$ \partial_i \left( \Sigma^{(2)} v^i \right) = \chi \Delta^2 \delta \varphi^{(2)}. $$

(36)

To the linear order, we remind, the energy density $\Sigma$ equals to zero. This explains the absence of the term $\sim \partial_i \left( \Sigma^{(1)} \partial_i \delta \varphi^{(1)} \right)$. Integrating out Eq. (36) with the boundary condition $\Sigma^{(2)} \to 0$ at $z \to -\infty$, one gets

$$ \Sigma^{(2)} = \frac{8\chi \cdot M^2}{v^4 \cdot M_{Pl}^4} \cdot \frac{1}{\rho^4} \cdot \left[ \frac{z^2}{\rho^2} + \frac{4z\sqrt{z^2 + \rho^2}}{\rho^2} - \frac{z}{\sqrt{z^2 + \rho^2}} + 1 \right]. $$

(37)

The latter exhibits a regular behaviour for all $z < 0$, while for $z > 0$ there is a singularity in the formal limit $\rho \to 0$,

$$ \Sigma^{(2)} = \frac{64\chi M^2 z^2}{v^4 M_{Pl}^4 \rho^6} \left( \frac{\rho}{z} \ll 1, \ z > 0 \right). $$

(38)
This enhancement translates into the strong amplification of metric perturbations. Once again, such a strong dependence on the variable \( \rho \) is trustworthy to the extent that the inequality (21) holds. Otherwise, non-perturbative effects cannot be ignored. These will be considered in Section 6.

In the quadratic order, the 00-component of the Ricci tensor \( R_{00} \) is given by

\[
R^{(2)}_{00} = \frac{1}{2} \Delta h^{(2)}_{00} + \frac{1}{2} h^{(1)}_{ij} h^{(1)}_{00,ij} - \frac{1}{2} h^{(1)}_{00,i} h^{(1)}_{00,i},
\]

where we made use of the harmonic gauge (23) and neglected terms with time derivatives. In the same order, the standard matter contributes to the r.h.s. of Einstein equations by

\[
\left( T_0^\text{matter} - \frac{1}{2} T^\text{matter}_{00} g_{00} \right)^{(2)} = \frac{1}{2} A(r) h^{(1)}_{00}.
\]

Here we explicitly neglect the pressure inside the Sun as well as the relative motion of its components. The analogous contribution due to IDM is inferred from Eq. (28),

\[
\left( T_{00}^{\text{IDM}} - \frac{1}{2} T^{\text{IDM}}_{00} g_{00} \right)^{(2)} = \frac{1}{2} \Sigma^{(2)}.
\]

Note that we ignored the term \( \sim \left( \Box \varphi \right)^2 \) present in Eq. (28). According to Eq. (24), it may be non-zero only in the non-perturbative region. To write the resultant expression for the second order metric perturbation \( h^{(2)}_{00} \), we switch to the spheric coordinates, where it takes the elegant form,

\[
h^{(2)}_{00} = 2 \Phi^2 + \frac{8 \pi \chi}{M_{Pl}^2} \cdot \frac{\Phi^2}{v^3} \cdot \frac{(1 + \cos \theta)^2}{\sin^4 \theta} - 4 \Phi;
\]

\( \theta \) is the angle between the z-axis and the line of sight. The potential \( \tilde{\Phi} \) is defined by

\[
\tilde{\Phi} = -\frac{1}{M_{Pl}^2} \int \frac{A(y) \Phi(y) dy}{|x - y|}.
\]

Notably, this part of the metric perturbation caused by ‘self-gravity’ of the Sun carries no imprint of IDM.

In the limit \( \theta \approx \frac{\pi}{2} \rightarrow 0 \), the correction to the quantity \( h^{(2)}_{00} \) grows fast and quickly becomes of the order of the main terms in the metric expansion. For very small values of the angle \( \theta \), however, the perturbative description employed in the present Section is not valid, and one may expect mildening of the singularity in the non-perturbative phase. This is indeed the case, as we will see in Section 6. On the other hand, the expression (40) remains finite upstream from the Sun (\( \theta \approx \pi \)). Namely, the seeming singularity \( \sim \frac{1}{(\theta - \pi)^2} \) cancels out due to the presence of the factor \( \sim (1 + \cos \theta)^2 \) in the numerator of the second term on the r.h.s. of Eq. (40). Still, IDM induced metric corrections do not vanish completely in the limit \( \theta \rightarrow \pi \).
but tend to the constant value (with respect to the angle \( \theta \)) \( \delta h_{00} \simeq \frac{\chi}{M_{Pl}^2} \cdot \frac{\Phi^2}{v^4} \). Therefore, gravitational focusing has a marginal impact on the planetary dynamics, even if they never pass through the downstream region.

In the remainder of the Section, we would like to check rigorously several approximations implied in the course of computations. The sharp \( \theta \)-dependence of the metric correction \( \delta h_{00} \) may compromise the present analysis. In particular, this may lead to a considerable modification of Eq. (33) due to the appearance of the terms growing as \( \frac{1}{\rho^4} \simeq \frac{z^4}{\rho^2} \) towards the positive \( z \)-axis. That is, the actual solution for the field \( \delta \varphi^{(2)} \) is rather given by

\[
\delta \varphi^{(2)} = \frac{2M^2}{v^2 M_{Pl}^4} \cdot \frac{z}{\rho^2} \left[ 1 + \mathcal{O} \left( \frac{\chi}{M_{Pl}^2 v^2} \cdot \frac{z^2}{\rho^2} \right) \right].
\]

In the formal limit \( \rho \to 0 \), the second term in the brackets becomes larger than the main one invalidating the solution (33). Recall, however, that we are not allowed to consider infinitely small values of the variable \( \rho \), but only those, which fulfill the condition (21). Consequently, results of the current Section are self-consistent, if the following inequality is obeyed,

\[
\frac{\chi}{M_{Pl}^2 v^2} \cdot \frac{v^2}{|\Phi|} \ll 1.
\]

This can be phrased as the limitation on the possible values of the potential \( \Phi \). In fact, Eq. (42) represents a very moderate assumption. In view of the results of Section 7, the condition (42) is satisfied for the potentials as small as \( |\Phi| \sim 10^{-18} \), many orders of magnitude below the values characteristic of the Solar system objects. On the other hand, for laboratory scale objects and not extremely small values of the parameter \( \chi \), the condition (42) may be substantially violated. Hence, this case should be treated more carefully. This task, however, is out of the scope of the present work.

Let us evaluate also the metric components \( h_{0i} \). Recall that they are zero in the linear order. While the PPN tests are generically not sensitive to the higher order corrections, we should be careful with the amplification occurring in the limit \( \frac{\rho}{z} \to 0 \). This may eventually compromise the computation of the fields \( \Sigma \) and \( \varphi \). Omitting details of calculations, we readily write down the solution,

\[
h_{0i}^{(2)} = -\frac{64\pi \chi}{M_{Pl}^2 v^4} \frac{\Phi^2}{\tilde{g}^4} v^i,
\]

where we again switched to the spherical coordinates; the case of small angles \( \theta \ll 1 \) is implied here. We see that the 0\( i \)-components of the metric exhibit the same singular behaviour in the limit \( \theta \simeq \frac{\rho}{z} \to 0 \) as the 00-one. They enter Eq. (32) via the combination \( \sim h_{0i} v^i \). This contribution can be safely neglected provided that the inequality \( \frac{\chi}{M_{Pl}^2} \cdot \frac{v^2}{|\Phi|} \ll 1 \) holds. In turn, the latter is satisfied automatically, once the condition (42) is obeyed. The same analysis can be done with the \( ij \)-metric components.
Finally, to evaluate the metric corrections, we again extrapolated the perturbative expression (37) to the whole space. In this regard, the solution (40) serves as a qualitative estimate rather than an exact result encoded in the formal expression (31). Still, Eq. (40) matches well the metric correction assessed in the non-perturbative phase. We will show it in the next Section. Therefore, we treat Eq. (40) as a good estimate for the 'true' metric perturbation. Would not that expectation meet the reality, we were still on the safe side in view of the ultimate goal—constraining the parameter $\chi$. For this purpose, it is enough that the correcting term $h^{\text{nonpert}}$ in Eq. (31) has a different shape compared to ones entering Eq. (40). To paraphrase, we assume no cancellations between the quantities $h^{\text{pert}}$ and $h^{\text{nonpert}}$. This is the minimal requirement ensuring that the (would-be) PPN parameter $\beta_{PPN}$ can be reconstructed in an unambiguous way out of Eq. (40), and the resultant constraint on the constant $\chi$ is trustworthy.

6 Going beyond perturbative regime

To understand better the region of the applicability of the perturbative analysis, let us calculate the third order correction to the field $\varphi$. Using the same arguments as in the beginning of Section 5, we write down the simplified version of the relevant equation,

$$- \partial_i \delta \varphi^{(3)} v^i + \partial_i \delta \varphi^{(1)} \partial_i \delta \varphi^{(2)} = 0 .$$

Once again, higher order corrections to the field $\varphi$ are defined by the lower order ones, while metric perturbations do not directly affect them. The solution for this equation is given by

$$\delta \varphi^{(3)} = \frac{8M^3}{v^5 M_P^6} \rho^4 ,$$

where we restrict to the case of small variables $\rho$, still obeying the inequality (21). Despite the strong amplification in the limit $\rho \to 0$, the hierarchy $|\delta \varphi^{(3)}| \ll |\delta \varphi^{(2)}| \ll |\delta \varphi^{(1)}|$ holds everywhere in the region (21). On the other hand, for $\frac{v}{z} \sim \sqrt{\frac{|\Phi|}{v^2}}$, one has $|\delta \varphi^{(1)}| \sim |\delta \varphi^{(2)}| \sim |\delta \varphi^{(3)}|$, what implies the breakdown of the perturbation approach. By recursion, that statement can be extended to an arbitrary order. We conclude that Eq. (21) remains intact upon including higher order corrections.

Now let us compare the leading order asymptotic behaviour of the corrections $\nabla \delta \varphi^{(1)}$, $\nabla \delta \varphi^{(2)}$ and $\nabla \delta \varphi^{(3)}$ in the region (21). We obtain from Eqs. (35) and (44),

$$\nabla \delta \varphi^{(1)}, \nabla \delta \varphi^{(2)}, \nabla \delta \varphi^{(3)} \propto \left( \frac{z}{\rho} e_\rho - \frac{e_z}{2} \right).$$

By recursion, one can check that the same holds to an arbitrary order,

$$\nabla \delta \varphi^{(n)} = F_n \cdot \left( \frac{z}{\rho} e_\rho - \frac{e_z}{2} \right).$$
That is, in the limit $\rho \to 0$, corrections to the velocity $v = -\nabla \varphi$ possess the same directional dependence being different only in the magnitude.

This simple observation allows to assess the relevant fields in the deeply non-linear phase,

$$ \frac{\rho^2}{z^2} \ll \frac{|\Phi|}{v^2}. $$

The following ansatz for the perturbation $\nabla \delta \varphi$ should work,

$$ \nabla \delta \varphi = \nabla \delta \varphi^{(1)} + F \cdot \left( \frac{z}{\rho} e_\rho - \frac{e_z}{2} \right). \tag{46} $$

For the sake of future convenience, we explicitly separated the linear order term. In the perturbative phase, the function $F$ is defined as a series $F = \sum_{n=2}^{\infty} F_n$. Our task is to find the function $F$ suitable for both regimes (21) and (45). For this purpose, we substitute the ansatz (46) into the equation

$$ -2 \partial_i \delta \varphi v^i + \partial_i \delta \varphi \partial_i \delta \varphi = -2\Phi. $$

We again keep only linear metric perturbations. This gives for the function $F$,

$$ F = -\frac{2M}{vM_\text{Pl}^2} \frac{1}{z} - \frac{v}{2} \frac{\rho^2}{z^2} + \rho \left( \frac{2M}{v^2M_\text{Pl}^2} \cdot \frac{\rho^2}{4z^2} \right). \tag{47} $$

Combining everything altogether, we obtain

$$ \nabla \delta \varphi = -\left( \frac{v}{2} \frac{\rho^2}{z^2} - \rho \left( \frac{2M}{v^2M_\text{Pl}^2} \cdot \frac{\rho^2}{4z^2} \right) \right) \left( \frac{z}{\rho} e_\rho - \frac{e_z}{2} \right). \tag{48} $$

Expanding this expression in powers of the potential $\Phi$, one reproduces our perturbative results, as it should be. Furthermore, it matches the velocity distribution, which characterizes a hypothetic flow of dust particles past the point source. See Appendix B for the exact formulae. Here let us point out the cancellation between the term $\nabla \delta \varphi^{(1)}$ in Eq. (46) and the first term on the r.h.s. of Eq. (47). Each of them taken separately would lead to the singular behaviour of the resulting velocity perturbation in the limit $\rho \to 0$. Due to their cancellation, the expression (48) remains finite. This is the smoothening of the field $\varphi$ (and, consequently, $\Sigma$) quoted in the previous Sections. Such a smoothening of the solution for $\varphi$ is akin to the Vainshtein screening due to the non-linear effects (see e.g. the review on the Vainshtein mechanism [43]). However, the effect observed here has a different nature.

The expression (48) is enough in order to calculate higher derivatives of the field $\varphi$ and the energy density $\Sigma$ in the non-perturbative regime. In particular, the field $\Sigma$ is defined by
the equation analogous to Eqs. (26) and (36), but with one important modification. Since
derivatives with respect to the variable $\rho$ are more relevant now, one makes the replacement
$$
\partial_i (\Sigma v^i) \rightarrow -\frac{1}{\rho} \partial_\rho (\rho \Sigma \partial_\rho \delta \varphi) \ .
$$
Using Eq. (48) and performing integration, we find the leading order asymptotics in the limit
$\rho \rightarrow 0$ for the density field $\Sigma$,
$$
\Sigma = \frac{\chi}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho}\right) \ .
$$
(49)
Note a milder dependence on the variable $\rho$ compared to Eq. (38). Interestingly, the Lagrange
multiplier field is independent of the mass of the source. This appears to be a generic feature
of the static solutions in the highly non-linear regime. See Appendix A. From Eq. (49) it is
straightforward to evaluate the correction to the Newtonian potential induced by IDM,
$$
\delta h_{00} = \frac{4\pi\chi}{M^2_{Pl}} \ln^2 \theta + \mathcal{O}(\ln \theta) \ .
$$
(50)
At the same time, in the region (45), the $0i$-metric components are given by
$$
h_{0i} = -\frac{8\pi\chi}{M^2_{Pl}} \ln^2 \theta \cdot v^i + \mathcal{O}(\ln \theta) \ .
$$
Now, let us contrast the estimate (50) to our perturbative result (40). These are to be
compared at the boundary values of the angle $\theta \sim \sqrt{\frac{\left|\Phi\right|}{v^2}}$, which separate the perturbative
and non-linear regions. Ignoring the logarithmic amplification, we see that the two match
each other very well modulo an order one factor. This justifies our perturbative calculation
of the metric corrections performed in previous Sections.
The energy density $\Sigma$ and the induced metric corrections still diverge downstream from
the Sun, though not so strongly compared to the expectations of Section 5. The singularity
can be traced back to the $\rho$-component of the velocity perturbation $\nabla \delta \varphi$, which does not
vanish in the limit $\rho \rightarrow 0$. Therefore, a caustic is formed nearby the positive $z$-axis. This
result is not particularly surprising, and could be anticipated readily from the existence
of the constraint (5). While our estimates assume the point source approximation for the
standard matter, this argument implies the generic nature of the divergence.
In the end of Section 3, it was pointed out that the caustic singularity may be resolved
apart from the test field approximation. The latter, however, is never exceeded in practice.
Let us show this explicitly. The contribution due to IDM is sub-dominant, if the metric
correction (50) is not larger than the Newtonian potential. This condition translates into an
inequality for the angle $\theta$,
$$
|\ln \theta| \lesssim |\Phi|^{1/2} \left[\frac{4\pi\chi}{M^2_{Pl}}\right]^{-1/2} \ .
$$
(51)
For the viable choice of the parameter $\chi$, the bound here is saturated only for non-realistically small values of $\theta$. Hence, IDM always gives a negligible contribution to the total space-time curvature. We will substantiate this statement shortly, when having the limit on the parameter $\chi$ at hand.

We see that the IDM scenario taken at its face value leads to the caustic singularity. Rather than invalidating the model, this rather puts the restriction on the possible choice of UV completing operators. On the other hand, the existence of the UV cutoff by itself is not a new assumption. Indeed, the model [1] is plagued with the ghost instabilities, which lead to a rapid vacuum destabilization, unless there is a sufficiently low cutoff/strong coupling scale.

7 Solar system constraints

Our purpose in the present Section is to constrain the model parameter $\chi$ starting from the non-observation of deviations from GR in the Solar system. As we have seen in the Section 4, to the linear order, gravitational focusing experienced by the flow of IDM produces no effect on the metric perturbations. Thus, no constraints on the constant $\chi$ proceed at this level. On the contrary, vector/scalar-tensor scenarios with the preferred frame, e.g., Einstein-Aether/khronometric theories [13, 44], generically predict non-zero PPN parameters $\alpha_1$ and $\alpha_2$. Such a difference can be explained by the fact, that in the IDM scenario preferred frame effects associated with the field $\varphi$ source metric perturbations only indirectly via the energy density $\Sigma$.

Non-trivial constraints on the parameter $\chi$ follow from the second order metric perturbations studied in Section 5. For the sake of convenience, we rewrite Eq. (40) in such a form, that the deviation of the 00-component of the metric from its GR counterpart $h^{GR}_{00}$ is manifest,

$$h_{00} - h^{GR}_{00} = 2(\beta - 1)\Phi^2.$$  \hspace{1cm} (52)

Here we introduced the parameter $\beta$ (not to be confused with the correct PPN parameter $\beta_{PPN}$) given by

$$\beta = 1 + \frac{4\pi\chi}{M_{Pl}^2v^4} \cdot \frac{(1 + \cos \theta)^2}{\sin^4 \theta}.$$  \hspace{1cm} (53)

The above defined quantity $\beta$ is similar to the standard PPN parameter $\beta_{PPN}$. Therefore, we expect the metric correction (52) to influence the precession of the Mercury perihelion, the subject of high precision measurements nowadays. Non-observation of any deviations from GR at this level implies a stringent bound on the model constant $\chi$ inferred from the existing limits on the PPN parameter $\beta_{PPN}$. We would like to stress, however, that the parameters $\beta$ and $\beta_{PPN}$ are not identical, as the former explicitly depends on the spatial coordinate $\theta$, while the latter is assumed to be constant. To be on the safe side, we formulate our goal as
to conservatively bound the model constant $\chi$. First, the quantity $(\beta - 1)$ is always positively defined. This ensures that the effect encoded in Eq. (52) does not average to zero, when considered on large time scales. Furthermore, the quantity $(\beta - 1)$ exhibits a monotonous growth towards the smaller angles $\theta$ starting from the minimal value
\[ \beta - 1 \simeq \frac{4\pi \chi}{M_{Pl}^2 v^4}, \]  
which sets the lower bound on the deviation from GR in the IDM scenario. This ‘minimal’ deviation has manifestly a PPN form, and thus the bounds on the parameter $\beta_{PPN}$ may be used for our estimate.

In fact, the choice $\theta \simeq 1$ implied in Eq. (54) appears to be the most reasonable one. Namely, it corresponds to the case, when the preferred and halo rest frames coincide. The latter moves relative to the Sun with the speed $v \approx 220$ km/s in the direction inclined towards the ecliptic plane by the angle $\theta \simeq \frac{\pi}{3}$. Hence, the use of the estimate (54) is fully justified also from the viewpoint of the physically realistic conditions. According to the discussion above, we identify the parameter $\beta$ at its minimal value to the PPN parameter $\beta_{PPN}$. We then make use of the observational bounds deduced from the study of the precession of the Mercury perihelion [45],
\[ \beta_{PPN} - 1 = (-4.1 \pm 7.8) \times 10^{-5}. \]  
We arrive at the following limit for the constant $\chi$,
\[ \frac{\chi}{M_{Pl}^2} \lesssim 10^{-18}. \]  
This constraint carries a certain advantage over the limits of Eqs. (3) and (4). In particular, our constraint is valid independently on whether IDM gives the main contribution to the overall DM (as it is assumed in Eq. (3)), or it constitutes only a tiny fraction. Besides, it is not sensitive to details of microscopic physics.

Having the constraint (56), we can justify our assumptions made above. First, Eq. (56) implies validity of our approximations used at the level of the second order analysis. In particular, the inequality (42) ensuring the self-consistency of the discussion in Section 5 is indeed fulfilled for all relevant values of the potential $\Phi$. Furthermore, using Eq. (56) we can check Eq. (51), which defines the region of space, where the test field approximation can be trusted. Taking the upper bound of Eq. (56) and $|\Phi| \sim 10^{-8}$, which correspond to the gravitational potential of the Sun at the Earth-Sun distance, we find $|\ln \theta| \gtrsim 3000$ from Eq. (51). The latter implies tiny angles $\theta$. Though our analysis is not solid in that region, the message is clear: IDM always gives a sub-dominant contribution to the total space-time curvature.

\[ \text{The actual minimal value for the quantity } \beta - 1 \text{ is the quarter of the r.h.s. of Eq. (54). Our sloppy estimate, however, better fits to the realistic physical situation discussed below.} \]
Stronger constraints on the parameter $\chi$ would follow, if the velocity $v$ was inclined towards the ecliptic plane by the small angle $\tilde{\theta} \ll 1$, see Fig. 1. For example, this is the case, if the preferred frame is at rest with respect to the cosmic microwave background. Note that the Sun moves relative to the cosmic frame with the speed $v \approx 369 \text{ km/s}$ in the direction characterized by the angle of inclination $\tilde{\theta} \approx \frac{\pi}{18}$ \cite{37}. In that case, the effects of gravitational focusing are more pronounced. For simplicity, let us fix the angle $\theta$ entering (53) at the value $\tilde{\theta} = \frac{\pi}{18}$. This is by no means a realistic assumption, since the parameter $\beta$ undergoes the large modulation during the period of the Mercury orbiting in the situation of interest. We make it merely to probe the sensitivity of the Mercury perihilion observations to the model constant $\chi$. The result reads

$$\frac{\chi}{M_{Pl}^2} \lesssim 10^{-21}.$$ \hfill (57)

To be more specific, one should consider the peculiar motion of Mercury with respect to the direction of the velocity $v$. This task is out of the scope of the paper.

The constraint (57) is not trustworthy also for another reason. It is unlikely that the preferred and cosmic frames coincide. Indeed, IDM residing in the halo is decoupled from the cosmological background. Still, this discussion may be of some interest for the following considerations. The limit (56) assumes that IDM has a zero velocity everywhere in the halo. The real picture is somewhat more complicated, as the Galaxy may be permeated by the local flows \cite{33}, which sum up to give the zero net momentum of the overall DM distribution. This our ignorance about the halo physics in the IDM scenario leads to the uncertainty, when we attempt to quantify an IDM flow past the Sun. While barring the fine-tuning, the velocity $v$ is estimated by $v \approx 10^{-3}$, its inclination towards the ecliptic plane is less constrained. Therefore, there is a room left for strengthening the bound (56) on the parameter $\chi$ with the better understood halo physics.

As for concluding remarks, let us discuss our result (56) in relation to other constraints. The constraint (56) is many orders of magnitude stronger than the one concluded from the structure formation considerations \cite{4} given in Eq. (3). As a result, when (56) is satisfied, IDM behaves identically to CDM at the cosmological scales. On the other hand, IDM leaves drastically different signatures at the level of the Solar system, compared to CDM. In particular, using the expression (38), we can estimate the energy density of IDM. Taking $\frac{\chi}{M_{Pl}^2} \sim 10^{-18}$ and $\theta \simeq 1$, we obtain at the Earth distance

$$\Sigma \simeq 1.5 \times 10^5 \text{ GeV cm}^{-3}.$$ 

This is to be contrasted to the background energy density of DM near the Earth $\Sigma \sim 0.3 \text{ GeV cm}^{-3}$. Downstream from the Sun, where the effects of gravitational focusing are particularly prominent, the energy density is further enhanced by the factor $1/\theta^6$. These estimates justify neglecting the 'inevitable' part of IDM related to its presence in the halo.
Finally, we would like to elucidate some implications of the constraint (56) for the microscopic physics. The strong coupling in the model at hand is given by \( \Lambda_{\text{str}} \sim \frac{\chi^3}{M_{\text{Pl}}^2} \). Consequently, with the limit (56), we expect the UV physics to come into play at the scales \( \Lambda \lesssim \Lambda_{\text{str}} \lesssim 300 \text{ TeV} \). Interestingly, within one or two orders of magnitude, this expectation meets the microscopic physics requirement \([14]\): \( \Lambda \lesssim 10 \text{ TeV} \).

As we showed, the Solar system is a useful playground for testing IDM. We leave for future studies astrophysical systems with compact objects, e.g., pulsars and black holes. These are believed to provide alternative (hopefully, even more stringent) limits on the parameter \( \chi \).

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**Appendix A: Spherically symmetric solutions**

The present Appendix serves to assess the modification of gravity produced in the regime, where the inequality (11) is not obeyed. For the sake of simplicity, we will consider the limit of vanishing speed \( v \to 0 \). In this limit, the model possesses the spherical symmetry. A similar setup has been employed for study of spherically symmetric objects in Horndeski/Galileon theory \([46, 47, 52, 48, 49, 50, 51]\). There is an important difference, however: in the case of the Galileon scenario, the time-dependent ansatz is caused by the requirement that the local solutions matches the cosmologically evolving scalar field. There are also scenarios where the Galileon does not evolve with time. On the contrary, the nature of the theory \([1]\) requires that the field \( \varphi \) is time-dependent.

The constraint equation (5) then takes the form

\[
(\partial_r \delta \varphi)^2 = -2\Phi .
\]  

This gives for the field \( \delta \varphi(r) \),

\[
\delta \varphi(r) = \frac{\sqrt{2Mr}}{M_{\text{Pl}}} ,
\]  

modulo the irrelevant constant of integration.

In the spherically symmetric static case, the l.h.s. and r.h.s. of the (non)conservation equation (7) can be written as

\[
\nabla_\mu (\Sigma \nabla^\mu \varphi) \to -\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \cdot \Sigma \partial_r \varphi)
\]
and 
\[ \Box^2 \varphi \to \Delta^2 \varphi , \]
respectively; \( \Delta = \frac{1}{r^2} \frac{d}{dr} r^2 \frac{d}{dr} \). Combining altogether, we get for the energy density \( \Sigma \),
\[ \Sigma = -\chi \cdot \int \frac{r^2 \Delta^2 \delta \varphi (r) dr}{r^2 \partial_r \delta \varphi (r)} . \quad (60) \]
Performing the integration with respect to the variable \( r \), we obtain
\[ \Sigma (r) = \frac{9}{8} \cdot \frac{\chi}{r^2} + \frac{C}{r^{3/2}} . \quad (61) \]
Here \( C \) is some constant of integration. In the Newton’s limit, only the 00-component of the IDM stress energy tensor is relevant,
\[ T_{00}^{IDM} = \Sigma (r) . \]
Here we omitted the \( \chi \)-terms in the expression for the IDM stress-energy tensor \[\text{[9]}\]: they are suppressed by the value of the potential \( \Phi \). The only relevant part of the Ricci tensor is given by \( R_{00} = \frac{1}{2} \Delta h_{00} \). Consequently, Einstein equations reduce to
\[ \frac{1}{2} \Delta h_{00} = 4\pi G (M \delta (r) + \Sigma (r)) . \]
The first term on the r.h.s. results into the standard Newton’s law, while the second one encodes the correction to the gravitational potential,
\[ \delta h_{00} = \frac{9\pi \chi}{M_{Pl}^2} \ln \frac{r}{r_0} + C' \sqrt{r} , \quad (62) \]
where \( r_0 \) is the irrelevant dimensionful parameter. To assess the value of the constant \( C' \), one should consider the problem with properly defined boundary conditions, i.e., at the infinity and in the center of the Sun. This task is out of the scope of the paper. We will derive our further conclusions starting from the first term on the r.h.s. here.

As it follows, the correction to the Newtonian potential \( \Phi \sim 1/r \) grows as \( \sim r \ln r \) (relative to the potential \( \Phi \)). The different picture occurs in the regime \( v^2 \gg |\Phi| \)—in that case, IDM induced metric corrections do not exhibit any amplification with the radius \( r \) (relative to the background terms in the metric expansion). See Eq. \[\text{[40]}\]. To paraphrase, upon switching from the regime \( v^2 \ll |\Phi| \) to \( v^2 \gg |\Phi| \), the effects due to gravitational focusing become substantially milder. The same conclusion only strengthens, once the second term on the r.h.s. of Eq. \[\text{[62]}\] is included.

Finally, we observe that the metric correction Eq. \[\text{[62]}\] matches the perturbative result \[\text{[40]}\] in the regime \( v^2 \sim |\Phi| \). Once again, this indicates that the expressions \[\text{[29]}, \text{[30]}\] and \[\text{[40]}\] give correct estimates for the linear and quadratic order metric perturbations.

26
Appendix B: Flow of dust particles past the point source

In Section 2, it was pointed out that Eq. (5) leads to the geodesics equation followed by test particles in the external gravitational field. We also observed in the main part of the paper, that the Newtonian limit works fairly well, when evaluating the corrections to the field $\varphi$. This suggests an alternative way to calculate the velocity potential $\varphi$, or, more precisely, its spatial derivatives. Namely, one tracks the flow of dust particles in the gravitational field created by the point source. Hereafter, we follow Ref. [33], and parametrize the trajectory of a single particle as

$$
t = \frac{a}{v} \left( e \cdot \sinh \Psi - \Psi \right),
$$

$$
z(b, \Psi) = a \left( e \cdot \sinh \Psi + 1 - \frac{1}{e} \exp \Psi \right)
$$

(63)

and

$$
\rho(b, \Psi) = b \left( 1 - \frac{1}{e} \exp \Psi \right).
$$

(64)

Here $b$ is the impact parameter, $\Psi$ is the eccentric anomaly parameter, $e = \sqrt{1 + \frac{b^2}{a^2}}$, and $a = \frac{GM}{v^2}$. In terms of the parameters $\Psi, b, e$ and $a$, the velocity of the flow is described by

$$
v_\rho = -v \frac{b \cdot \exp \Psi}{e \cdot a \left( e \cdot \cosh \Psi - 1 \right)},
$$

(65)

$$
v_z = v \frac{e \cdot \cosh \Psi - \frac{1}{e} \cdot \exp \Psi}{e \cdot \cosh \Psi - 1}.
$$

(66)

One then inverts Eqs. (63) and (64), so that to express the parameters $b$ and $\Psi$ via the variables $z$ and $\rho$,

$$
b_\pm = \frac{\rho}{z} \left( 1 \pm \sqrt{1 + y} \right)
$$

(67)

and

$$
\Psi_\pm = \ln \left( e_\pm \cdot \frac{1}{y} \left[ 1 \mp \sqrt{1 + y} \right]^2 \right),
$$

(68)

where $y$ and $e_\pm$ are given by

$$
y = \frac{4a \cdot (z + \sqrt{\rho^2 + z^2})}{\rho^2}
$$

(69)

and

$$
e_\pm = \sqrt{1 + \frac{b_\pm^2}{a^2}},
$$

respectively. Substituting Eqs. (67) and (68) into Eqs. (65) and (66), one finally obtains the velocity of the flow as the function of the spatial variables $z$ and $\rho$. This should be contrasted to the results obtained in the bulk of the paper. There is a subtlety. In the case of the particle
DM, two solutions labeled by the subscripts \( ' + \) and \( ' - \) correspond to the two flows produced at the shell-crossing. On the other hand, IDM is fundamentally single flow, as it has been pointed out in the end of Section 3. Therefore, out of two, we should pick one solution. The solution must satisfy our boundary conditions for the field \( \varphi \): \( \nabla \delta \varphi \to 0 \) at \( z \to -\infty \). Only the branch with the upper signs in the equations above fulfills this condition. We stick to this branch in what follows. For simplicity, we consider only the radial component of the velocity \( v_\rho \). This is given by

\[
v_\rho = \frac{4va}{\rho \cdot y \cdot F(\rho, y)} \cdot \left(1 - \sqrt{1 + y}\right),
\]

where

\[
F(\rho, y) = 1 + \frac{4a^2}{\rho^2} \cdot \frac{[1 - \frac{1}{y}(1 - \sqrt{1 + y})^2]}{[1 - \sqrt{1 + y}]^2}.
\]

In the limit \( \rho \to 0 \) and for positive \( z \), the argument \( y \) blows up, i.e., \( y \to \infty \). Simultaneously, \( F(\rho, y) \to 1 \), and one arrives at

\[
v_\rho \to -\sqrt{\frac{2M}{M_{Pl}z}}.
\]

This reproduces our result (48), where one should consider the limit \( \rho \to 0 \). At the same time, in the particle picture the perturbative regime corresponds to very small values of the variable \( y \), i.e., \( y \ll 1 \). Expanding Eq. (70) in a series over the powers of the parameter \( y \), we get in the linear and in the quadratic order

\[
v^{(1)}_\rho = -\frac{2av}{\rho}
\]

and

\[
v^{(2)}_\rho = \frac{v \cdot a \cdot y}{4\rho} \left(1 + \frac{z}{\sqrt{\rho^2 + z^2}}\right),
\]

respectively. These perfectly match Eqs. (20) and (33) from the bulk of the paper.

### Appendix C: Linear level analysis for the light source

In Section 4, we concluded that IDM leaves no trace on the metric to the linear order. The discussion there was provided for a sufficiently compact object,—the Sun. In that case, the conditions (11) and (21) break down near its surface and in the downstream region, respectively. To overcome this obstacle, we loosely extrapolated our perturbative calculation for the IDM fields \( \Sigma \) and \( \varphi \) to the whole space. One may inquire now: what is the structure of linear corrections to the metrics would be these inequalities trustworthy everywhere? That
question is not of the pure heuristic interest. Indeed, for a sufficiently light and not very compact object like the Earth, the condition (11) is obeyed down to its center. On the other hand, violation of the inequality (21) occurs only in a very narrow part of space measured by the angle $\theta \sim 10^{-2}$ near the surface of the Earth. We consider it a reasonable approximation to neglect possible effects, which stem from this region. Remarkably, despite the seeming differences between the cases of the Sun and the Earth, linear corrections to the metric are absent in the latter case either. This gives a further support for the main statement of Section 4: the effects due to gravitational focusing are completely irrelevant at the level of the linear analysis.

As for the first step, we find the field $\delta \varphi^{(1)}$, or, more precisely, $\Delta \varphi^{(1)}$. We apply the Laplace operator to both parts of Eq. (18), and then integrate it out with respect to the variable $z$,

$$
\Delta \delta \varphi^{(1)} = \frac{4\pi}{M_{Pl}^2} \cdot \frac{1 + 2v^2}{v} \int_{-\infty}^{z} A d\tilde{z},
$$

where $A(r)$ denotes the mass distribution inside the Earth. In the linear order, the equation defining the field $\Sigma$ is given by Eq. (26). Performing the integration with respect to the variable $z$ there and using Eq. (71), we obtain

$$
\Sigma^{(1)} = \frac{4\pi(1 + 2v^2)}{M_{Pl}^2 \cdot v^2} \Delta \int_{-\infty}^{z} dz' \int_{-\infty}^{z'} A dz''.
$$

We see that the energy density $\Sigma$ is zero everywhere, except for the tube-like region close to the positive $z$-axis. Away from it, the field $\Sigma^{(1)}$ does not affect metric perturbations, as it clearly follows from the structure of Eq. (72). Hence, we can safely ignore it.

As it follows from Eqs. (27) and (28), the 00-metric component remains intact in the presence of IDM. We then evaluate the $ij$-components of the metric. The relevant components of the Ricci tensor are given by

$$
R^{(1)}_{ij} = \frac{1}{2} \Delta h^{(1)}_{ij},
$$

where we made use of the gauge condition (23). The standard matter and IDM contribute to the r.h.s. of Einstein equations by

$$
\left( T^{\text{matter}}_{ij} - \frac{1}{2} T^{\text{matter}} g_{ij} \right)^{(1)} = \frac{1}{2} A(r) \delta_{ij}
$$

and

$$
\left( T^{\text{IDM}}_{ij} - \frac{1}{2} T^{\text{IDM}} g_{ij} \right)^{(1)} = \frac{1}{2} \Sigma^{(1)} \left( \delta_{ij} + v^i v^j \right) - \chi v^i \partial_j \Delta \delta \varphi^{(1)} - \chi v^j \partial_i \Delta \delta \varphi^{(1)},
$$

respectively. Neglecting the contribution from the field $\Sigma$, we obtain for the $ij$-metric components,

$$
h^{(1)}_{ij} = 2\Phi \delta_{ij} - \frac{16\pi \chi}{M_{Pl}^2} \left( v^i \partial_j \delta \varphi^{(1)} + v^j \partial_i \delta \varphi^{(1)} \right).
$$
Finally, we calculate the $0i$-metric components. The relevant part of the Ricci tensor is given by

$$R^{(1)}_{0i} = \frac{1}{2} \Delta h^{(1)}_{0i}.$$ 

IDM contributes to the r.h.s. of Einstein equations by

$$\left(T^{IDM}_{0i} - \frac{1}{2}T^{IDM}_{00}\right)^{(1)} = \Sigma^{(1)} \partial_0 \varphi v^i + \chi \partial_0 \varphi \partial_i \Delta \delta \varphi^{(1)},$$

while the standard matter does not contribute at this level. Combining altogether, we obtain

$$h^{(1)}_{0i} = \frac{16 \pi \chi}{M_{Pl}^2} \partial_i \delta \varphi^{(1)}.$$ 

Now let us perform the additional gauge transformation with

$$\xi_\mu = \frac{16 \pi \chi}{M_{Pl}^2} \partial_\mu \varphi \cdot \delta \varphi^{(1)}.$$ 

This serves to eliminate the anisotropic part in the metric components $h^{(1)}_{ij}$. Simultaneously, the components $h^{(1)}_{0i}$ turn into 0, while the 00-component remains intact. We conclude that the linear order metric perturbations are unaffected by IDM also in the case of sufficiently light astrophysical objects.

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