Delay-dependent Stability Criterion for a Class of Neutral Systems with Multiple Mixed Delays

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Abstract. In this paper, a new stability criterion for a class of neutral systems with multiple discrete and distributed time delays is provided. Based on the Lyapunov second method, a new delay-dependent stability criterion in terms of linear matrix inequality (LMI) is derived. Numerical examples are given to illustrate that the proposed result is less conservative.

Introduction

Delay is a kind of widespread physical phenomena in the nature [1-3]. Research of neutral delay system is controlled in recent decades the rise in the field of a hot spot, and is becoming more and more attention by people, such a time-delay systems not only contains the last motion, motion differential information also includes the past, this kind of neutral differential equation system can be used to describe the [1-6]. In the literature [6-9], some time-delay independent stability criteria were proposed. Literature [10-13] proposed the sufficient criterion of time-delay correlation stability by using the conversion method. In the literature [14], the variable delay neutral system was studied, but the stability criterion given by the literature was conservative. This paper presents delay-dependent criteria in the form of linear matrix inequality (LMI) based on Lyapunov second method. The paper compares the numerical examples with the existing ones, and illustrate the advantages of the conclusions obtained in this paper.

Before giving the main theorem of this chapter, the following lemmas are given.

Lemma 1: (Park P.[11]) Assume that for all given $\alpha \in \Omega$, there are $a(\alpha) \in R^n$ and $b(\alpha) \in R^n$.

For any positive definite matrix $N \in R^{n \times n}$, and matrix $M \in R^{n \times n}$, Satisfies the following matrix inequality:

$$-2 \int_{\Omega} b^T(\alpha)a(\alpha)d\alpha \leq \int_{\Omega} \begin{bmatrix} N & NM \end{bmatrix} \begin{bmatrix} a(\alpha) \\ b(\alpha) \end{bmatrix} d\alpha$$  \hspace{1cm} (1)

Here, $\phi$ represents $(M^TN + I)N^{-1}(NM + I)$.

Lemma 2[15]: For arbitrary constant matrices $M \in R^{n \times n}$, $M^T = M > 0$, scalar $\gamma > 0$ and Vector function $\omega : [0, \gamma] \rightarrow R^{n}$, satisfying (2):

$$\gamma \int_0^\gamma \omega^T(\beta)M\omega(\beta)d\beta \geq (\int_0^\gamma \omega(\beta)d\beta)^T M (\int_0^\gamma \omega(\beta)d\beta)$$  \hspace{1cm} (2)

Based on the above lemma, the main conclusions of this paper are given below.

Stability Theorem

Consider the following neutral system (3a) with time delay.

$$\dot{x}(t) = A_0x(t) + \sum_{i=1}^{m} B_i x(t-h_i) + \sum_{i=1}^{m} \int_{t-\tau_i}^{t} C_i x(s) ds + \sum_{i=1}^{m} D_i \dot{x}(t-\eta_i)$$  \hspace{1cm} (3a)
Its initial state is (3b).

\[ x(t_0 + \theta) = \phi(\theta), \forall \theta \in [-H, 0] \quad (3b) \]

Among them, \( x(t) \in \mathbb{R}^n \) represents state vector, \( A_0, B_i \) and \( C_i \in \mathbb{R}^{n \times m} \) are Constant coefficient matrix. \( H = \max_{i,m} \left\{ h_i, \tau_i, \eta_i \right\} \geq 0 \), \( h_i, \eta_i \) and \( \tau_i, i \in m, m = \{1, 2, \ldots, m\} \) respectively represents normal time delay. \( \phi(\cdot) \) is a continuous differentiable function on a given interval \([-\rho, 0]\).

**Theorem 1:** If there are matrix \( P > 0, R_i > 0, Q_i > 0, G_i > 0, U_i > 0 \) and \( W_i (i = 1, \ldots, m) \) satisfying the matrix inequality (4), then the system (3) is asymptotically stable.

\[
\begin{bmatrix}
X_{0,0} & X_{0,1} & \cdots & X_{0,3m} & h_1(W^T_1 + P) & \cdots & h_m(W^T_m + P) \\
X^T_{0,1} & X^T_{1,1} & \cdots & X^T_{1,3m} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\
X^T_{0,3m} & X^T_{1,3m} & \cdots & X^T_{3m,3m} & 0 & \cdots & 0 \\
h_1(W^T_1 + P) & 0 & \cdots & 0 & -U_1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
h_m(W^T_m + P) & 0 & \cdots & 0 & 0 & \cdots & -U_m
\end{bmatrix} < 0 \tag{4}
\]

Among them

\[
X_{0,0} = A^T P + PA + A^T_0 \bar{F} A_0 + \sum_{i=1}^{m} Q_i + \sum_{i=1}^{m} \eta_i G_i + \sum_{i=1}^{m} W^T_i B_i + \sum_{i=1}^{m} B_i W_i ;
\]

\[
X_{0,i} = -W^T_i B_i + A^T_0 \bar{F} B_i \quad (i = 1, 2, \ldots, m) ;
\]

\[
X_{0,i+m} = PD_i + A^T_0 \bar{F} D_i \quad (i = 1, 2, \ldots, m) ;
\]

\[
X_{0,i+2m} = PC_i + A^T_0 \bar{F} C_i \quad (i = 1, 2, \ldots, m) ;
\]

\[
X_{0,i+2m} = PC_i + A^T_0 \bar{F} C_i \quad (i = 1, 2, \ldots, m) ;
\]

\[
X_{i, i + 2m} = D^T_i \bar{F} D_i \quad (i = 1, 2, \ldots, m) ;
\]

\[
X_{i, i + 2m} = D^T_i \bar{F} C_j \quad (i = 1, 2, \ldots, m) ;
\]

\[
X_{i, i + 2m} = D^T_i \bar{F} C_j \quad (i = 1, 2, \ldots, m) ;
\]

\[
X_{i, i + 2m} = D^T_i \bar{F} C_j \quad (i = 1, 2, \ldots, m) ;
\]

\[
X_{i, i + 2m} = D^T_i \bar{F} C_j \quad (i = 1, 2, \ldots, m) ;
\]

and \( A = A_0 + \sum_{i=1}^{m} B_i \), \( \bar{F} = \sum_{i=1}^{m} (B^T_i U_i B_i + R_i) \).

**Proof:** Construct the following Lyapunov functional

\[
V = \sum_{i=1}^{m} V_i \tag{5}
\]

Among them
\[ V_1 = x(t)^T P x(t) \]  
\[ V_2 = \sum_{i=1}^{m} \int_{t-h_i}^{t} d\eta \int_{t-h_i}^{t} x^T(\alpha) B_i^T N_i B_i x(\alpha) d\alpha \]  
\[ V_3 = \sum_{i=1}^{m} \int_{t-h_i}^{t} x^T(\alpha) Q_i x(\alpha) d\alpha \]  
\[ V_4 = \sum_{i=1}^{m} \tau_i \int_{t-h_i}^{t} \dot{x}(\alpha) R_i \dot{x}(\alpha) d\alpha \]  
\[ V_5 = \sum_{i=1}^{m} \tau_i \int_{t-h_i}^{t} ds \int_{t-h_i}^{t} x^T(\alpha) G_i x(\alpha) d\alpha \]  

Derivatives \( V \) can obtain

\[ \dot{V} = \dot{V}_1 + \dot{V}_2 + \dot{V}_3 + \dot{V}_4 + \dot{V}_5 \]

Because the following (11) is true:

\[ x(t) - x(t-h_i) = \int_{t-h_i}^{t} \dot{x}(\sigma) d\sigma \]  
(11)

System (3) is rewritten as (12).

\[
\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{m} B_i x(t-h_i) + \sum_{i=1}^{m} \int_{t-h_i}^{t} C_i x(s) ds + \sum_{i=1}^{m} D_i \dot{x}(t-\eta_i)
\]

\[
= A_0 x(t) + \sum_{i=1}^{m} B_i x(t) - \sum_{i=1}^{m} B_i \int_{t-h_i}^{t} \dot{x}(\alpha) d\alpha + \sum_{i=1}^{m} \int_{t-h_i}^{t} C_i(s) ds + \sum_{i=1}^{m} D_i \dot{x}(t-\eta_i)
\]

\[
= A x(t) - \sum_{i=1}^{m} B_i \int_{t-h_i}^{t} \dot{x}(\alpha) d\alpha + \sum_{i=1}^{m} \int_{t-h_i}^{t} C_i(s) ds + \sum_{i=1}^{m} D_i \dot{x}(t-\eta_i)
\]  
(12)

So \( V_1 \) meet (13)

\[ \dot{V}_1 = \dot{x}^T(t) P x(t) + x^T(t) P \dot{x}(t) = 2 \dot{x}^T(t) P x(t) \]

\[ = 2 x^T(t) P A x(t) - 2 x^T(t) P \sum_{i=1}^{m} B_i \int_{t-h_i}^{t} \dot{x}(\alpha) d\alpha + 2 x^T(t) P \sum_{i=1}^{m} D_i \dot{x}(t-\eta_i) \]  
(13)

\[ + 2 x^T(t) P \sum_{i=1}^{m} \int_{t-h_i}^{t} C_i(s) ds \]

Let:

\[ a(\alpha) = B_i \dot{x}(\alpha) , \ b(\alpha) = P x(t), \ \forall i \ \text{and} \ \alpha \in [t-h_i, t]. \]

By using lemma 1, (13) satisfies the following:

\[ \dot{V}_1 \leq x^T(t) \left[ A^T P + PA + \sum_{i=1}^{m} h_i P (M_i^T N_i + I) N_i^{-1} (N_i M_i + I) P \right] x(t) \]

\[ + 2 x^T(t) P \sum_{i=1}^{m} M_i^T N_i B_i \int_{t-h_i}^{t} \dot{x}(\alpha) d\alpha + \sum_{i=1}^{m} \int_{t-h_i}^{t} \dot{x}(\alpha) B_i^T N_i B_i \dot{x}(\alpha) d\alpha \]
+ 2x^T(t)P \sum_{i=1}^{m} D_i \dot{x}(t - \eta_i) + 2x^T(t)P \sum_{j=1}^{i} C_i(s)ds$

From (7)-(10), we can obtain the following:

\[
\dot{V}_2 = \sum_{i=1}^{m} h_i \dot{x}^T(t)B_i^T N_i B_i \dot{x}(t) - \sum_{i=1}^{m} \int_{t-h_i}^{t} \dot{x}^T(\alpha)B_i^T N_i B_i \dot{x}(\alpha) d\alpha
\]

\[
\dot{V}_3 = \sum_{i=1}^{m} x^T(t)Q_i x(t) - \sum_{i=1}^{m} x^T(t - h_i)Q_i x(t - h_i)
\]

\[
\dot{V}_4 = \sum_{i=1}^{m} \dot{x}^T(t)R_i \dot{x}(t) - \sum_{i=1}^{m} \dot{x}^T(t - \eta_i)R_i \dot{x}(t - \eta_i)
\]

\[
\dot{V}_5 = \sum_{i=1}^{m} \tau_i x^T(t)G_i x(t) - \sum_{i=1}^{m} \tau_i \int_{t-t_i}^{t} x^T(s)G_i x(s)ds
\]

Let

\[
W_i = N_i M_i P, \quad U_i = h_i N_i \quad (i = 1, 2, ..., m) \text{ and } \tilde{F} = \sum_{i=1}^{m} (B_i^T U_i B_i + R_i), \text{ and the following formula is established:}
\]

\[
-\tau_i \int_{t-t_i}^{t} x^T(s)G_i x(s)ds \leq -[\int_{t-t_i}^{t} x(s)ds]^T G_i [\int_{t-t_i}^{t} x(s)ds]
\]

The following inequality (14) can be obtained.

\[
\dot{V} \leq x^T(t)\{A^T P + PA + \sum_{i=1}^{m} h_i^2 (W_i^T + P)U_i^{-1}(W_i^T + P) + \sum_{i=1}^{m} Q_i + \sum_{i=1}^{m} \tau_i^2 G_i\}x(t)
\]

\[
+ 2x^T(t)\sum_{i=1}^{m} W_i B_i \int_{t-h_i}^{t} \dot{x}(\alpha)d\alpha + \dot{x}^T(t)\tilde{F}\dot{x}(t) - \sum_{i=1}^{m} x^T(t - h_i)Q_i x(t - h_i)
\]

\[
+ 2x^T(t)P \sum_{i=1}^{m} D_i \dot{x}(t - \eta_i) + 2x^T(t)P \sum_{j=1}^{i} C_i(s)ds - \sum_{i=1}^{m} \dot{x}^T(t - \eta_i)R_i \dot{x}(t - \eta_i)
\]

\[
-\sum_{i=1}^{m} [\int_{t-t_i}^{t} x(s)ds]^T G_i [\int_{t-t_i}^{t} x(s)ds] = \xi^T G \xi
\]

(14)

In (14)

\[
\xi = \text{col}\{x(t), x(t - h_1), ..., x(t - h_m), \dot{x}(t - \eta_1), ..., \dot{x}(t - \eta_m), \int_{t-t_i}^{t} x(s)ds, ..., \int_{t-t_m}^{t} x(s)ds\}
\]

\[
G = \begin{bmatrix}
X_{0,0} & X_{0,1} & \cdots & X_{0,3m} \\
X_{0,1} & X_{1,1} & \cdots & X_{1,3m} \\
\vdots & \vdots & \ddots & \vdots \\
X_{0,3m} & X_{1,3m} & \cdots & X_{3m,3m}
\end{bmatrix}
\]

(15)

Among $X_{0,0} = X_{0,0} + \sum_{i=1}^{m} h_i^2 (W_i^T + P)U_i^{-1}(W_i^T + P)$, the other parameters are the same as theorem 1.
Therefore, when the following inequality (16) is established, \( \dot{V} \) is guaranteed to be negative.

\[
G < 0 \tag{16}
\]

Applying the Schur complement theorem, \( G < 0 \) is guaranteed when the linear matrix inequality (4) is established. And then the system (3) is asymptotically stable.

**Numerical Simulation Example**

**Case 1.**

Consider the following system \([9,10]\).

\[
\dot{x} = Ax(t) + Bx(t - h) + Gx(t - \tau) + \int_{t-\tau}^{t} x(s) ds
\]

\[
A = \begin{bmatrix} -a_i & 0 \\ 0 & -a_2 \end{bmatrix}; \quad B = \begin{bmatrix} b_1 & b_2 \\ -b_2 & b_1 \end{bmatrix}; \quad C = \begin{bmatrix} c_1 & c_2 \\ -c_2 & c_1 \end{bmatrix}.
\]

\( a_i, b_i, c_i, i = 1,2 \) are constant coefficient, when \( a_i = a_2 = 1.25, b_2 = c_1 = c_2 = 0.12, b_1 = -1 \) in [9], the system (17) is stable for all delays \( h \) and \( \tau \leq 1.4308 \). By using the theorem 1, the stability condition of system (17) is \( \tau \leq 1.4308 \) and all \( 0 \leq h \). Let \( b_1 = -1.25 \) and the other parameters are not different, the conclusions given in [9] cannot guarantee the stability of the system (17). The conclusion of [10] also cannot guarantee the stability of system (17). Applying the theorem 1, we can get the system (17) stable when \( \tau \leq 1.019, h \leq 2.7 \). Take \( \tau = 1.019, h = 2.7 \), To solve the LMI (4), we can get the following matrix:

\[
P = \begin{bmatrix} 3.3966 & 1.9604e-10 \\ 1.9604e-10 & 3.3966 \end{bmatrix}; \quad R = \begin{bmatrix} 0.1796e-3 & 6.4875e-16 \\ 6.4875e-16 & 0.1796e-3 \end{bmatrix};
\]

\[
Q = \begin{bmatrix} 4.1978 & -5.3644e-12 \\ -5.3644e-12 & 4.1978 \end{bmatrix}; \quad G = \begin{bmatrix} 0.6283 & 2.6212e-10 \\ 2.6212e-10 & 0.6283 \end{bmatrix};
\]

\[
U = \begin{bmatrix} 1.5044 & 2.3436e-10 \\ 2.3436e-10 & 1.5044 \end{bmatrix}; \quad W = \begin{bmatrix} -2.9129 & -1.3867e-10 \\ -1.3867e-10 & -2.9129 \end{bmatrix};
\]

The above results show that the theorem given in this paper is more applicable than [9,10].

**Case 2.**

Consider the following neutral system in [13].

\[
\dot{x} = A_0x + B_1x(t - h_1) + B_2x(t - h_2) + D_1\dot{x}(t - \tau_1) + D_2\dot{x}(t - \tau_2)
\]

\[
A_0 = \begin{bmatrix} -2 & 0.5 \\ 0 & -1 \end{bmatrix}; \quad B_1 = \begin{bmatrix} 1 & 0.4 \\ 0.4 & -1 \end{bmatrix}; \quad B_2 = \begin{bmatrix} -0.3 & 0.2 \\ 0.2 & -0.3 \end{bmatrix};
\]

\[
D_1 = \begin{bmatrix} 0.2 & 0.1 \\ 0.1 & 0.2 \end{bmatrix}; \quad D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}.
\]

The system (18) stability condition is \( h_1 \leq 0.99 \) and \( h_2 \leq 2 \) in [13]. Apply the theorem 1 in this paper(let \( \eta_1 = h_1 \)). We can proof (18) is stable when \( 0 \leq h_1 \leq 1.088 \) and \( 0 \leq h_2 \).

Let \( h_1 = 1.088, h_2 = 10000 \), to solve LMI (4) can obtain:
Thus it can be seen that the conclusion of this chapter has a smaller conservatism and wider applicability.

**Conclusion**

In this chapter, we discuss the stability of multi-mixed delay neutral systems with multi-distributed delay and discrete time delay. By using the second method of Lyapunov, it is concluded that the time-delay correlation stability of LMI can be easily solved by convex optimization method. By comparing with the existing conclusions, the stability criterion given in this chapter has a small conservatism and wide applicability.

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