PATTERN-EQUIVARIANT HOMOLOGY

JAMES J. WALTON

Abstract. Pattern-equivariant (PE) cohomology is a well established tool with which to interpret the Čech cohomology groups of a tiling space in a highly geometric way. We consider here homology groups of locally finite but non-compactly supported PE chains. For FLC tilings with respect to translations, we show that these groups are Poincaré dual to the PE cohomology groups. For tilings with FLC with respect to rigid motions, the PE chains exhibit a singular behaviour at points of rotational symmetry which often adds extra torsion to the calculated invariants. We present an efficient method for computation of these groups for hierarchical tilings.

Introduction

The study of aperiodic tilings, it could be argued, first came to the fore following Berger’s discovery that the domino problem is undecidable [5]. Since then, an interesting and varied collection of aperiodic tilings has been produced. Penrose’s famous “kite and dart” tilings, for example, have an interesting geometry exhibiting 5-fold symmetry whereas the Kari-Culik tilings [10] are notable for their somewhat ungeometric but rather combinatorial nature. However, it is post the discovery of quasicrystals in 1982 that there has been a marked increase in the interest and activity of research of aperiodic tilings, which can be seen as good models for the geometry of these quasi-crystalline structures.

Given some tiling of \( \mathbb{R}^d \), one can construct a metric space known as the tiling space or continuous hull of that tiling and hope to gain information about the original tiling by understanding properties of the tiling space or a dynamical system associated to it (an introduction to these ideas can be found in Sadun [29]). Much is known, for example, about the Čech cohomology of the tiling space, which can be explicitly computed in many cases (when one has a substitution tiling, as first studied in Anderson and Putnam’s work [2] or more recently Barge, Diamond Hunton and Sadun [3], or a projection tiling; see Forest, Gähler, Hunton, Kellendonk [12, 14], for example).

Beyond simply distinguishing different tilings, these invariants also have useful interpretations directly relevant to understanding the properties of the original tiling (or potential physical system associated to it). For example, the degree one Čech cohomology group (with \( \mathbb{R}^d \)-coefficients) says something about shape deformations of the tiling as seen in Clark, Sadun [9], and the K-theory of the tiling space is linked to the physics of quasicrystals via Bellissard’s gap labelling theorem [4].

A highly geometric way of viewing the Čech cohomology of the continuous hull of a tiling is through its pattern-equivariant (PE) cohomology. The ideas first surfaced in [18], where Kellendonk identifies the Čech cohomology groups of a tiling (over \( \mathbb{R} \)-coefficients) with PE forms on the tiling. When the tiling has a CW-decomposition, one may define PE cellular cochains on it, defining PE cohomology for arbitrary (abelian) coefficients (see [28]).

In this paper, we essentially consider the question of what one gets by taking chains instead of cochains. Singular chains will not be useful – a singular chain contained in some compact region will not be able to capture any information about the large scale geometry of the tiling.
One can, however, consider locally finite chains with non-compact support. Analogously to PE cohomology, particular such chains will be identified as being pattern-equivariant with respect to the tiling. One obtains a chain complex whose homology will be called the pattern-equivariant homology of the tiling. It should be remarked that much of this work is making precise this rather intuitive idea, and so we would recommend the reader to first examine section 5 and the examples there.

An essential ingredient for PE cohomology is the description of the tiling space as an inverse limit of “approximants” (as in [3]). So, in section 1 we define what will be called a “pattern” on a metric space. For each pattern, there will be an associated inverse limit of approximants. One can generalise the tiling metric to such patterns, and in theorem 1.6 we establish that the completion of this metric space is homeomorphic to the inverse limit of approximants. Patterns of Euclidean space will be general enough to handle the FLC (up to rigid motion) tilings which motivate them, as well as some other examples of interest. For example, the $d$-dimensional dyadic solenoid, which is not the tiling space of any FLC tiling of $\mathbb{R}^d$, may nevertheless be realised from a pattern of $\mathbb{R}^d$ in a relatively natural way (as a “hierarchical tiling”; see section 2).

The pattern-equivariant homology groups are defined in section 3. We formulate a cellular version and show that it agrees with the singular theory (given conditions on the CW-decomposition).

For an oriented $d$-dimensional (not necessarily compact) manifold $M$, one has Poincaré duality $H^\bullet(M) \cong H_{BM}^{d-\bullet}(M)$, the Borel-Moore homology of $M$. This isomorphism may be induced by capping with the fundamental class of $M$. In many cases this construction may be restricted to the pattern-equivariant setting. We examine this in section 4 and show duality for the translational hull of a tiling of $\mathbb{R}^d$. Although this means that one gets no new information from this approach, another interpretation is that one has yet another way of visualising these groups of invariants (which we illustrate with some examples). Indeed, one of the upshots is that the higher dimensional cohomological information, which is often the more interesting, is converted into lower dimensional homological information. For example, an element of the top dimensional cohomology group can be thought of as a pattern-equivariant point charge of the tiling, modulo moving that charge around in a pattern-equivariant way and, for a CW tiling, it can be computed using just the PE chains on the 1-skeleton of the tiling. The cup product of two cochains can be viewed instead as the intersection product in homology.

For a certain class of examples, that we do not obtain duality is also interesting. Given a tiling with FLC up to rigid motion, there is an associated pattern which takes rotations into account. Its associated topological space is named $\Omega^0$ in [3] and is the quotient of $\Omega^{rot}$, the continuous hull of a tiling with respect to rigid motions of $\mathbb{R}^d$, by a natural $SO(d)$-action. We find extra torsion present to the cohomology of $\Omega^0$ in the PE homology groups, seemingly retaining extra information about the rotational symmetry of the tiling. We show how one may produce a modified PE chain complex, which restricts the allowed chains at points of rotational symmetry, for which the homology groups do exhibit Poincaré duality. However, we consider the extra torsion in the PE homology groups as a feature of potential interest.

In section 5, we present a method for computation of the PE homology groups of hierarchical cellular tilings (as well as some explicit examples of such computations). Because of Poincaré duality, this then gives a new way to compute the Čech cohomology of a substitution tiling space. Moreover, there appear to be some advantages to this approach. The computations not only calculate the groups of invariants, but also directly provide highly geometric descriptions of each of the elements of these groups in the process (see figure 5.3 for example). The
“approximant” chain complex has an intuitive description in terms of the original cells of the tiling; in degree $i$, the chain group consists of classes of $i$-cells from the tiling. The example calculations seem to indicate that this approach is also computationally efficient. We note that the approximant chain complexes (or more precisely, their duals) have been used before, in the work of Gonçalves [16] who provides a method for computing the $K$-theory of the $C^*$-algebra of the stable equivalence relation of a substitution tiling. At least in low dimensions, therefore, we have an explanation of the duality seen between the $K$-theory of the stable and unstable equivalence relations.

Acknowledgements. The author would like to thank his supervisors, John Hunton and Alex Clark, for their helpful advise and comments. The work of this author is supported by the EPSRC.

1. Patterns on Metric Spaces

1.1. Patterns of partial symmetries. Given a periodic tiling of $\mathbb{R}^d$, it is natural to consider its group of global symmetries, that is, the isometries of $\mathbb{R}^d$ preserving the tiling. Two points of the tiling “look similar locally” at $x$ and $y$ if and only if there is a global symmetry preserving the tiling and taking $x$ near to $y$. This is no longer true for aperiodic tilings: two points $x$ and $y$ of the tiling may look similar but only by a partial isometry preserving some large but possibly finite patch taking $x$ near to $y$. Hence for aperiodic (but nevertheless reasonably ordered) structures it makes sense to consider instead of groups of global symmetries, semigroups of partial symmetries.

For the following definition (and throughout this paper), we use the convention that partial isometries are composed on the largest domain for which the composition makes sense, that is, $\Phi_2 \circ \Phi_1 : \Phi_1^{-1}(\text{ran}(\Phi_1) \cap \text{dom}(\Phi_2)) \rightarrow \Phi_2(\text{ran}(\Phi_1) \cap \text{dom}(\Phi_2))$, where $\text{dom}(\Phi)$ and $\text{ran}(\Phi)$ denote the domain and codomain of $\Phi$, respectively. Isometries and partial isometries will always be assumed to be surjective.

Definition 1.1. Let $(X,d_X)$ be a metric space. A pattern $\mathcal{P}$ (on $(X,d_X)$) is a collection of isometries $\Phi : U \rightarrow V$ between the open sets of $(X,d_X)$ such that:

1. for every open set $U$ the identity map $Id_U \in \mathcal{P}$.
2. If $\Phi \in \mathcal{P}$ its inverse $\Phi^{-1} \in \mathcal{P}$.
3. If $\Phi_1, \Phi_2 \in \mathcal{P}$ then $\Phi_2 \circ \Phi_1 \in \mathcal{P}$.

Note that, by (1), we should always have the “empty morphism” $Id_\emptyset : \emptyset \rightarrow \emptyset$ as an element of $\mathcal{P}$, which is the unique such morphism with domain and range equal to the empty set in the category of sets.

We shall briefly compare the definition of a pattern here to the notion of a pseudogroup. A pattern is a pseudogroup, in the sense of [1], where we restrict to metric spaces and partial isometries (and impose condition (1)). An interesting feature (although one we shall not pursue here) for such pseudogroups defined on locally compact spaces is that they may be topologised using an analogue of the compact-open topology for partial open maps and this topology on $\mathcal{P}$ makes composition $\mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ and inversion $\mathcal{P} \rightarrow \mathcal{P}$ continuous [1] (that is, with this topology $\mathcal{P}$ is in fact a topological pseudogroup).

For open sets $U$ we may restrict $\Phi \in \mathcal{P}$ to $U$ since $\Phi|_U = \Phi \circ Id_U \in \mathcal{P}$. We do not require the sheaf-like glueing condition given in the classical definition of a pseudogroup. Whilst this is an intuitive condition for a pattern associated to a tiling (see the next section) we shall not need it here and it will in fact rule out some examples of patterns that we see later (i.e., the pattern associated to a hierarchical tiling; see the next section).
The pattern $\mathcal{P}$ has the structure of an inverse semigroup $S(\mathcal{P})$ with multiplication given by (partial) composition of functions $\Phi_1 \cdot \Phi_2 \colon = \Phi_2 \circ \Phi_1$. Semigroups associated to aperiodic tilings have been considered before, see for example [19], although our approach here is quite different. Notice that the idempotents of $S(\mathcal{P})$ correspond to the morphisms $Id_U$. The inverse semigroup $S(\mathcal{P})$ has an identity (the morphism $Id_X$) and a zero (the morphism $Id_\emptyset$). Inverse semigroups have a natural partial order given by $s \leq t$ if and only if $s = et$ for some idempotent $e$. Hence, for the inverse semigroup $S(\mathcal{P})$, $\Phi_1 \leq \Phi_2$ if and only if $\Phi_1$ is a restriction of $\Phi_2$. Notice that this ordering on the idempotents corresponds to the ordering of the lattice of open sets of $(X, d_X)$ by inclusion, that is, the poset of idempotents are a locale (some authors refer to such inverse semigroups as abstract pseudogroups).

One may also view a pattern $\mathcal{P}$ as a groupoid $G(\mathcal{P})$: the objects of $G(\mathcal{P})$ are the open sets of $(X, d_X)$ with morphisms $U \to V$ being those partial isometries $\Phi \in \mathcal{P}$ with $\text{dom}(\Phi) = U$ and $\text{ran}(\Phi) = V$. The ordering of restriction on the morphisms makes $G(\mathcal{P})$ an inductive groupoid. There is an equivalence in the approaches of viewing $\mathcal{P}$ as the inverse semigroup $S(\mathcal{P})$ and inductive groupoid $G(\mathcal{P})$ made explicit by the Ehresmann-Schein-Nambooripad Theorem (see, for example, [21]).

1.2. The Pattern Metric and Pattern Space. Let $\mathcal{P}$ be a pattern. Define $\mathcal{P}^R_{x,y}$ to be the set of morphisms $\Phi \in \mathcal{P}$ such that $\Phi(x) = y$, $\text{dom}(\Phi) = B_{d_X}(x, R)$ and $\text{ran}(\Phi) = B_{d_X}(y, R)$.

For each $R \in \mathbb{R}_{>0}$ set $x \sim_R y$ if and only if $\mathcal{P}^R_{x,y} \neq \emptyset$. Then each $\sim_R$ is an equivalence relation on $X$. The results of this section will depend only on these induced equivalence relations, the required properties of which we collect in the following definition:

**Definition 1.2.** A **collage** $\hat{\mathcal{P}}$ (on $(X, d_X)$) is a set of equivalence relations $\sim_R$ on $X$, one for each $R \in \mathbb{R}_{>0}$, satisfying the following properties:

1. if $x \sim_R y$ then $x \sim_r y$ for all $0 < r < R$ (one may like to say that $\sim_R \subset \sim_r$).
2. For all $0 < r < R$, if $x \sim_R y$ and $x' \in B_{d_X}(x, r)$ then $x' \sim_{R-r} y'$ for some $y' \in B_{d_X}(y, r)$.

We say that $\hat{\mathcal{P}}$:

- **Has finite local complexity** (or is FLC, for short) if the quotient spaces $X/\sim_R$ are compact for all $R > 0$.
- **Is repetitive** if for all $R > 0$, there exists some $R' > 0$ such that given any two points $x, y \in X$ there exists some $y' \in B_{d_X}(y, R')$ for which $x \sim_R y'$.

It is easy to see that given a pattern $\mathcal{P}$, the collection $\hat{\mathcal{P}}$ of equivalence relations as constructed above forms a collage, which we shall call the **induced collage** of $\mathcal{P}$. We shall associate with a pattern $\mathcal{P}$ any of the constructions applicable to its induced collage $\hat{\mathcal{P}}$ (e.g., the pattern metric $d_\mathcal{P}$ shall be defined to be $d_\hat{\mathcal{P}}$).

The properties of FLC and repetitivity are named from their usage in the context of tilings [29]. It will be useful to view $x \sim_R y$ as meaning $x$ and $y$ are equivalent to radius $R$. Thought of in this way, the properties defining a collage are geometrically intuitive. The first property states that a prerequisite for points to be equivalent to radius $R$ is that they are also equivalent to any smaller radius $r < R$. The second property establishes a form of coherence between how the equivalence relations are patched together with respect to the metric. We will show that these properties are enough to generalise the tiling metric and also the description of the completion of the metric space $(X, d_X)$ as an inverse limit of BDHS-complexes (see [3]).

Here, and throughout unless otherwise stated, $(X, d_X)$ will denote a metric space on which some pattern or collage is defined for which we impose the following extra assumption: the intersection of open balls $B_{d_X}(x, \epsilon_1) \cap B_{d_X}(y, \epsilon_2) \neq \emptyset$ for all $x, y$ with $d_X(x, y) < \epsilon_1 + \epsilon_2$ and
\( \varepsilon_1, \varepsilon_2 > 0 \). This property will ensure that the function \( d_{\tilde{P}} \), defined below, satisfies the triangle inequality for a collage \( \tilde{P} \) on \((X, d_X)\); the property could be weakened for a suitable weakening of the triangle inequality. For the main theorem of this section, we shall assume that \((X, d_X)\) is complete; the condition above implies that \((X, d_X)\) has approximate midpoints, which along with completeness in fact implies that \((X, d_X)\) is a length space. We will not be interested in the case where \((X, d_X)\) is not locally compact; complete, locally compact length spaces are always geodesic spaces, that is, each pair of points may be connected by a path of length equal to the distance of the points \([27]\). In summation, we shall assume throughout that \((X, d_X)\) is a locally compact geodesic space.

**Definition 1.3.** Given a collage \( \tilde{P} \) of \((X, d_X)\), define the (pseudo)metric \( d_{\tilde{P}} : X \times X \to \mathbb{R} \) by

\[
d_{\tilde{P}}(x, y) := \inf \left\{ \frac{1}{\sqrt{2}} \right\} \bigcup \{ \varepsilon > 0 \mid x' \sim_{\varepsilon-1} y' \text{ for some } x', y' \in B_{d_X}(x, \varepsilon), y' \in B_{d_X}(y, \varepsilon) \}.
\]

**Lemma 1.4.** The function \( d_{\tilde{P}} \) is a pseudometric on \( X \).

**Proof.** That \( d_{\tilde{P}}(x, y) = 0 \) for all \( x \in X \) follows from the fact that each \( \sim_R \) is reflexive and that \( d_{\tilde{P}} \) is symmetric, follows from the fact that each \( \sim_R \) is symmetric.

Let \( x, y, z \in X \). We intend to show that \( d_{\tilde{P}}(x, y) + d_{\tilde{P}}(y, z) \geq d_{\tilde{P}}(x, z) \). If one of \( d_{\tilde{P}}(x, y) \) or \( d_{\tilde{P}}(y, z) \) are \( \frac{1}{\sqrt{2}} \) then we are done. So suppose not. Let \( x_1 \sim_{\varepsilon_1} y_1 \) and \( y_2 \sim_{\varepsilon_2} z_2 \) for some \( \varepsilon_1, \varepsilon_2 < \frac{1}{\sqrt{2}} \) and \( x_1 \in B_{d_X}(x, \varepsilon_1), y_1 \in B_{d_X}(y, \varepsilon_1), y_2 \in B_{d_X}(y, \varepsilon_2), z_2 \in B_{d_X}(z, \varepsilon_2) \). Then, by our assumption on \((X, d_X)\), there exists some \( y' \) for which \( d_X(y_1, y') < \varepsilon_2 \) and \( d_X(y_2, y') < \varepsilon_1 \). Hence, there exist \( x', z' \) for which \( x' \sim_{\varepsilon_1^{-1} - \varepsilon_2} y' \sim_{\varepsilon_2^{-1} - \varepsilon_1} z' \) for \( x' \in B_{d_X}(x, \varepsilon_1^{-1}), z' \in B_{d_X}(z, \varepsilon_2^{-1}) \) since \( \tilde{P} \) is a collage.

We have \( x' \sim_{\varepsilon_1^{-1}} z' \) where

\[
\varepsilon = \min(\varepsilon_1^{-1} - \varepsilon_2, \varepsilon_2^{-1} - \varepsilon_1).
\]

For any numbers \( 0 < a, b \leq \frac{1}{\sqrt{2}} \), we have that \( a^{-1} - b \geq (a+b)^{-1} \) so, in particular, \( \varepsilon^{-1} \geq (\varepsilon_1 + \varepsilon_2)^{-1} \) and hence \( x' \sim_{(\varepsilon_1 + \varepsilon_2)^{-1}} z' \), proving the triangle inequality. \( \square \)

**Definition 1.5.** Given a collage \( \tilde{P} \) on \((X, d_X)\), one has the quotient maps \( \pi_R : (X, d_X) \to (X, d_X)/\sim_R \); we name these quotient spaces the approximants and denote them by \( K_R \). For \( R \geq r \), we denote the canonical quotient maps between approximants, called the connecting maps, by \( \pi_{R,r} : K_R \to K_r \). This defines an inverse system and we define the pattern space as the inverse limit \( \Omega_{\tilde{P}} = \varprojlim K_R \). One may take the inverse limit over a countable unbounded sequence \( R_1 \leq R_2 \leq R_3 \leq \ldots \) and we will often assume that a collage comes with such a sequence; any such inverse limit will of course be homeomorphic to \( \Omega_{\tilde{P}} \).

Given some topological space \( X \), we say that two points \( x, y \in X \) are topologically indistinguishable if each has the same set of neighbourhoods. Two points which are not topologically indistinguishable are called topologically distinguishable and a space for which distinct points are always topologically distinguishable is called \( T_0 \) or Kolmogorov. Topological indistinguishability clearly defines an equivalence relation on \( X \), which is trivial \((x \sim y \text{ if and only if } x = y)\) if and only if \( X \) is \( T_0 \). Taking the quotient of \( X \) by this equivalence relation produces a \( T_0 \) space, known as the Kolmogorov quotient, which we shall denote here by \( X^{KQ} \). Note that for a pseudometric space \((X, d)\) two points \( x, y \) are topologically indistinguishable if and only if \( d(x, y) = 0 \), so \((X, d)^{KQ}\) is the “usual” metric space associated to \((X, d)\).

A pseudometric space for which every Cauchy sequence converges is called complete. For a pseudometric space \((X, d)\), one can define a pseudometric on the set of Cauchy sequences on \( X \)
by defining \(d((x_n)_n, (y_n)_n) := \lim_{n \to \infty} d(x_n, y_n)\). As a topological space \((X, d)\), the completion of \((X, d)\), is the Kolmogorov quotient of this space i.e., the space given by identifying any two Cauchy sequences for which \(\lim_{n \to \infty} d(x_n, y_n) = 0\). The completion of a pseudometric space is a complete metric space.

We may now state the main theorem of this section:

**Theorem 1.6.** Let \(\hat{P}\) be a collage on a complete metric space \((X, d_X)\). Then we have a homeomorphism:

\[
(\Omega_{\hat{P}})^{KQ} \cong (X, d_{\hat{P}}).
\]

Hence, if \(\Omega_{\hat{P}}\) is Hausdorff (as is usual) then \(\Omega_{\hat{P}} \cong (X, d_{\hat{P}})\)\(^1\). Setting \(P\) to be the pattern on \((\mathbb{R}^d, d_{eucl})\) defined by some tiling (see the next section), this theorem generalises the statement that the continuous hull of a tiling is homeomorphic to the inverse limit of its BDHS-complexes [3]. Although \((X, d_P)\) may be a rather complicated space (e.g., for a repetitive aperiodic tiling it is connected but not locally connected), the above theorem often expresses it as an inverse limit of more well-behaved spaces. In the case of FLC tilings of \(\mathbb{R}\), this theorem often expresses it as an inverse limit of \(\Omega\)-invariant of the approximants, which may themselves be more amenable to computation or even interpretation (c.f., pattern-equivariant functions or cochains [20, 28], shape deformations [29]).

Taking the Kolmogorov quotient is necessary. The pattern \(P\) associated to the point pattern of \(P = \mathbb{Q} \subset \mathbb{R}\) (see the next section) has \(\Omega_{\hat{P}} \cong \mathbb{R}/\mathbb{Q}\) whilst \((\mathbb{R}, d_{\hat{P}})\) is its Kolmogorov quotient, the one point space. Although such examples rarely appear to be interesting, it seems easier to prove the above general theorem rather than introducing conditions for which \(\Omega_{\hat{P}}\) is Hausdorff.

1.3. **Proof of Theorem 1.6.** This subsection will be devoted to proving theorem 1.6 and may be skipped on a first (or any) reading at no loss to the results of later sections.

**Definition 1.7.** Let \((X, d)\) be any metric space and \(\sim_R\) be an equivalence relation on \(X\). We define the quotient (pseudo)metric \(d^R\) on \(X/\sim_R\) as follows:

\[
d^R([x]_R, [y]_R) = \inf \left\{ \sum_{i=1}^{k} d(a_i, b_i) \mid k \in \mathbb{N}, a_0 = x, b_k = y, b_i \sim_R a_{i+1} \right\}
\]

Intuitively, one may pass from \([x]_R\) to \([y]_R\) in the quotient pseudometric by hopping between points in the same equivalence class “at no cost”.

We now have various candidates for approximants whose inverse limits we could have defined to be \(\Omega_{\hat{P}}\), namely \(K_R := (X, d_X)/\sim_R, (X, d_{\hat{P}})/\sim_R\) and \((X/\sim_R, d^R_{\hat{P}})\). The next set of lemmas will in fact show that the each of the associated inverse limits are homeomorphic.

**Lemma 1.8.** The spaces \((X, d_X)/\sim_R\) and \((X, d_{\hat{P}})/\sim_R\) are canonically homeomorphic.

**Proof.** Let \(U\) be an open set of \((X, d_{\hat{P}})/\sim_R\), which is open if and only if its preimage \(\pi^{-1}_R(U) \subset (X, d_{\hat{P}})\) is open. Such a set will clearly be open in \((X, d_X)\) since, given any \(x \in \pi^{-1}_R(U)\), \(B_{d_X}(x, \delta) \subset B_{d_{\hat{P}}}(x, \delta)\), so \(U\) is open in \((X, d_X)/\sim_R\) also.

\(^1\)Conversely, we should not consider a general space \(X\) as being all that much different from its Kolmogorov quotient. In fact, it is not too hard to see that every topological space is homotopy equivalent to its Kolmogorov quotient if and only if the axiom of choice holds.
Conversely, suppose that $U$ is an open set of $(X, d_X)/\sim_R$, which is open if and only if its preimage $\pi_R^{-1}(U) \subset (X, d_X)$ is open. Let $x \in \pi_R^{-1}(U)$ and $\delta > 0$ be such that $B_{d_X}(x, \delta) \subset \pi_R^{-1}(U)$. Pick $C$ such that $C^{-1} + C > R$ and $2C < \delta$. Then for any $y \in B_{d_p}(x, C)$ there exists some $x' \in B_{d_X}(x, C)$ and $y' \in B_{d_X}(y, C)$ with $x' \sim_{C^{-1}} y'$ so that $x'' \sim_{C - 1 - C} y$ and hence $x'' \sim_R y$ for $x'' \in B_{d_X}(x, 2C)$. Hence $x' \in B_{d_X}(\delta) \subset \pi_R^{-1}(U)$ and since $x'' \sim_R y$ we have that $y \in \pi_R^{-1}(U)$.

The following lemma shows that the metrics $d_{\overline{p}}$ and $d_{\overline{R}}$ do not differ by too much for large enough $R$.

**Lemma 1.9.** Let $0 < \delta < 1/2$. Then there exists some $R$ such that if $d_{\overline{R}}([x]_R, [y]_R) < \delta$ then $d_{\overline{R}}(x, y) < 2\delta$.

**Proof.** Set $R$ so that $R - 2\delta > (2\delta)^{-1}$. If $d_{\overline{R}}([x]_R, [y]_R) < \delta$, then there exists the following chain

$$a'_1 \sim_{C^{-1}_1} b'_1$$

$$a'_2 \sim_{C^{-1}_2} b'_2$$

$$\vdots$$

$$a'_k \sim_{C^{-1}_k} b'_k$$

where $a'_1 \in B_{d_X}(a_i, C_i)$, $b'_1 \in B_{d_X}(b_i, C_i)$, $b_i \sim_R a_{i+1}$, $a_1 = x$, $b_k = y$ and $C = \sum C_i < \delta$ (using the definition of the pattern metric, quotient metric and the fact that $d_{\overline{R}}([x]_R, [y]_R) < \delta$).

We show by induction that there exists some $b''_n \in B_{d_X}(b_n, 2\sum_{i=1}^n C_i)$ with $a'_1 \sim_L b''_n$ and $L > (2\delta)^{-1}$.

For $k = 1$, set $b''_1 = b'_1$. Clearly $C^{-1}_1 \geq C^{-1} > (2C)^{-1} > (2\delta)^{-1}$ and $b'_1$ was defined so that $b'_1 \in B_{d_X}(b_1, C_1)$. So suppose the result is true for $n \leq k - 1$. That is, there exists some $b''_n$ such that $a'_1 \sim_L b''_n$ with $L > (2\delta)^{-1}$ and $b''_n \in B_{d_X}(b_n, 2\sum_{i=1}^n C_i)$. This implies that $b''_n \sim_{-2C} a''_{n+1}$ for some $a''_{n+1} \in B_{d_X}(a_{n+1}, 2\sum_{i=1}^n C_i)$. Hence, $a''_{n+1} \in B_{d_X}(a'_1, 2(\sum_{i=1}^n C_i + C_{n+1})$ and $a''_{n+1} \sim_{-2C} b''_{n+1} + b''_{n+1} \in B_{d_X}(b'_1, 2(\sum_{i=1}^n C_i + C_{n+1})$ and hence $b''_{n+1} \in B_{d_X}(b_{n+1}, 2\sum_{i=1}^{n+1} C_i)$. Since $R - 2C, C - 1 - 2C > (2\delta)^{-1}$ (as $C^{-1} - 2C > (2\delta)^{-1}$ for any $0 < C < 1/2$), the induction step follows.

Hence, we have that $d_{\overline{R}}(x, y) \leq 2\delta < 2\delta$, as required.

**Lemma 1.10.** The canonical map $f : \varprojlim(X, d_{\overline{p}}) / \sim_R \rightarrow \varprojlim(X/\sim_R, d_{\overline{R}})$ is a homeomorphism.

**Proof.** That $f$ is continuous follows from the universal property of the quotient of a topological space. So we need to show that $f^{-1}$ is continuous.

The infinite product of quotient spaces $(X, d_{\overline{p}})/\sim_R$ has the usual sub-base for its topology, each such open set given as some open set of a fixed $(X, d_{\overline{p}})/\sim_R$ (with the whole quotient in other coordinates). It is sufficient to check that the restriction of such sets $U$ to the inverse limit are open in $\varprojlim(X/\sim_R, d_{\overline{R}})$.

Let $U \subset K_R$ be open and $x \in \pi_R^{-1}(U)$, so that there exists some $0 < \delta < 1$ for which $B_{d_{\overline{p}}}(x, \delta) \subset \pi_R^{-1}(U)$. Let $\delta' \leq \delta/2$ and be such that $(\delta')^{-1} - \delta > R$. We claim that $B_{d_{\overline{p}}}(y, \delta') \subset \pi_R^{-1}(U)$ for each $y \sim_{R+1} x$.

Indeed, suppose that $z \in B_{d_{\overline{p}}}(y, \delta')$, so that $y' \sim_{(\delta')^{-1}} z'$ with $y' \in B_{d_X}(y, \delta')$ and $z' \in B_{d_X}(z, \delta')$. Then $y'' \sim_{(\delta')^{-1} - \delta} z$ with $y'' \in B_{d_X}(y, \delta')$. We have that $y'' \sim_{R} x''$ with $x'' \in B_{d_X}(x, \delta) \subset B_{d_{\overline{p}}}(x, \delta)$. Hence, $y'' \in \pi_R^{-1}(U)$ also so $z \in \pi_R^{-1}(U)$, as required.

Now, let $\underline{z} \in U \subset K$. We want to show that there exists some $\delta' > 0$ and $R' > 0$ such that if $\underline{z} \in B_{d_{\overline{p}}}(\underline{x}, \delta')$ (considered as the obvious subset of the inverse limit) then $\underline{z} \in U$.
Let $x \in \pi^{-1}_R(x) \subset X$. By the above, there exists some $\delta'$ for which $B_{d_p}(x, \delta') \subset \pi^{-1}_R(U)$ for any such choice of $x$. By lemma 1.9 there exists some $R' > R$ such that $d_p(x, y) < \delta'$ if $d_p^R([x]_{R'}, [y]_{R'}) < \delta'/2$.

Let $\tilde{z} \in B_{d_p^R}(x, \delta'/2)$. Then for all $z' \in \pi^{-1}_R(\tilde{z})$ we have that $d_p(x, z') < \delta'$ for $x \in \pi^{-1}_R(x)$ and so $z' \in \pi^{-1}_R(U)$. This implies that $z' \in U$ as an element of the inverse limit. Since $\tilde{z}$ and $\tilde{z}'$ agree on $K_R$, $\tilde{z}, \tilde{z}' \in U$ also.

Hence, $U$ is open in the quotient metric topology, so the result follows.

**Corollary 1.11.** There is a homeomorphism

$$\Omega^{KQ} \cong \lim_{\leftarrow} \left( (X/\sim_R, d^R_p)^{KQ} \right).$$

**Proof.** By the above, we have homeomorphisms $\Omega_p \cong \lim_{\leftarrow} (X, d_p)/\sim_R \cong \lim_{\leftarrow} (X/\sim_R, d^R_p)$ and hence $\Omega^{KQ} \cong \left( \lim_{\leftarrow} (X/\sim_R, d^R_p) \right)^{KQ} \cong \lim_{\leftarrow} \left( (X/\sim_R, d^R_p) ^{KQ} \right)$. The last homeomorphism follows, for example, by considering the spaces as metric spaces and that there is a natural bijection between them as sets which clearly preserves the metric.

To prove the main theorem, by the above we just need to show that there is a homeomorphism $h: (X, d_p) \to \lim_{\leftarrow} (X/\sim_R, d^R_p)^{KQ}$. Note that each space is a metric space; we define the metric on $\lim_{\leftarrow} (X/\sim_R, d^R_p)^{KQ}$ by restriction of the metric on the countable product of the approximants, defined by $d([a], [b]) := \sum_{i=1}^{\infty} 1/2^i d^R_p([a]_{R_i}, [b]_{R_i})$, which is well defined, since each of the metrics is bounded by $1/\sqrt{2}$.

We shall call a map between metric spaces such that it and its inverse are uniformly continuous a uniform homeomorphism. Such homeomorphisms induce homeomorphisms between their respective completions; in fact, it is sufficient for the map and its inverse to be Cauchy continuous.

There is a canonical map from $(X, d_p)$ into the inverse limit by projection. In fact, this gives a well defined map $h: (X, d_p)^{KQ} \to \lim_{\leftarrow} (X/\sim_R, d^R_p)^{KQ}$, since pairs of points of distance zero are mapped to the same sequence in the inverse limit. We claim that:

1. The inverse limit $\lim_{\leftarrow} (X/\sim_R, d^R_p)^{KQ}$ is complete.
2. The image of $h$ is dense.
3. The map $h$ is a uniform homeomorphism onto its image.

The result follows, since (1) and (2) show that the completion of the image of $h$ is the inverse limit whilst (3) shows that $h$ induces a homeomorphism between the completions of $(X, d_p)$ and the image of $h$.

**Lemma 1.12.** For complete $(X, d_X)$ the metric space $\lim_{\leftarrow} (X/\sim_R, d^R_p)^{KQ}$ is also complete.

**Proof.** Let $x_n$ be a Cauchy sequence in $\lim_{\leftarrow}$. This implies that $x_n$ is a Cauchy sequence in each approximant. We shall show that it converges to a point in each approximant, which is sufficient to prove the result.

We set $R$ arbitrary, $\epsilon_i = 1/2^{i+n}$ (with $n$ set so that $\epsilon_i^{-1} \geq R + 2$) and $R_i \geq R + 2$ so that if $d^R_i < \epsilon_i/2$ then $d_p < \epsilon_i$, as in lemma 1.9. Since the sequence is Cauchy in each approximant, we can pick pairs of point $a_i, b_{i+1} \in X$ so that $\pi_{R_i}(a_i) = x_i, \pi_{R_i}(b_{i+1}) = x_{i+1}$ and $d_p(a_i, b_{i+1}) < \epsilon_i$, (upon passing to a subsequence of $x_i$, for which we omit the extra notation).

Define $z_1 := a_1$. Suppose that $z_1, \ldots, z_k$ have been defined so that $\pi_{R+2-2\sum_{j=1}^{i-1} \epsilon_j}(x_i) = z_i$ and $d_X(z_{i-1}, z_i) < 2 \epsilon_i$ for $i > 1$. 
Note that $d_P(a_k, b_{k+1}) < \epsilon_k$, so that $a'_k \sim_{\epsilon_k} b_{k+1}$ with $a'_k \in B_{d_X}(a_k, 2\epsilon_k)$. Now, $a_k \sim_{R+2-2\sum_{i=1}^{k-1} \epsilon_i} z_k$ so we have that $b_{k+1} \sim_{\epsilon_k} b'_k \sim_{R+2-2\sum_{i=1}^{k} \epsilon_i} z_{k+1}$ for $z_{k+1}$ with $d_X(z_k, z_{k+1}) < 2\epsilon_k$.

The sequence $z_i$ agrees with $x_i$ on the $R$ approximant and converges in $X$ since it is Cauchy by our choice of $\epsilon_i$. Hence, $x_i$ converges in the $R$ approximant to the equivalence class of the limit of the $z_i$.

\[\square\]

Lemma 1.13. The map $h: (X, d_P)^{KQ} \rightarrow \lim_{\leftarrow}((X \sim_R, d_P^R)^{KQ})$ as defined above is a uniform homeomorphism onto its image which is dense in $\lim_{\leftarrow}((X \sim_R, d_P^R)^{KQ})$.

Proof. It is clear that the image of $h$ has dense image since it is surjective onto each approximant.

The map $h$ is also injective; both spaces are metric spaces, so we just need to show that pairs of points of non-zero distance in $(X, d_P)^{KQ}$ are mapped to pairs of points with non-zero distance in $\lim_{\leftarrow}$. This follows from lemma 1.9.

Finally we must show that $h$ is a uniform homeomorphism. That $h$ is uniformly continuous is clear. Indeed, $d(h(x), h(y)) \leq d_P(x, y)$. For $h^{-1}$, we use again lemma 1.9. Let $\epsilon > 0$ be given and set $R$ so that $d_P(x, y) < \epsilon$ whenever $d_P^R([x]_R, [y]_R) < \epsilon/2$. We set $\delta := (\epsilon/2)(1/2^R)$. Then if $d(h(x), h(y)) < \delta$, we must have that $d_P^R([x]_R, [y]_R) < \epsilon/2$ and $d_P(x, y) < \epsilon$, as required. \[\square\]

1.4. Patterns as Semigroups. In the previous subsection it was observed that a pattern $P$ naturally defines an inverse semigroup $S(P)$. One may ask whether the pattern space $\Omega_P$ can be recovered from $S(P)$. Unfortunately, this is not the case, for example simply consider “capping” the Euclidean metric $d_{eu}$ to $d_1 = \min\{d_{eu}, 1\}$. A partial isometry of $(\mathbb{R}^d, d_{eu})$ is a partial isometry of $(\mathbb{R}^d, d_{1})$ but the equivalence relations $\sim_R$ and hence pattern spaces $\Omega_P$ will generally depend vitally on the choice of metric. We shall present here a way of enriching the structure on $S(P)$ so that the original pattern may be recovered.

For $R \in \mathbb{R}_{\geq 0}$ let $\Phi_1 \leq_R \Phi_2$ if $\Phi_1$ is a restriction of $\Phi_2$ for which $\text{dom}(\Phi_2)$ contains all points of distance less than or equal to $R$ of a point of $\text{dom}(\Phi_1)$ and $\text{ran}(\Phi_2)$ contains all points of distance less than or equal to $R$ of a point of $\text{ran}(\Phi_1)$. One may easily check the following:

Proposition 1.14. The inverse semigroup $S(P)$ equipped with $\leq_R$ for $R \in \mathbb{R}_{\geq 0}$ as above satisfies:

1. $\Phi_1 \leq_0 \Phi_2$ if and only if $\Phi_1$ is a restriction of $\Phi_2$, that is, $\leq_0$ corresponds to the natural partial order on $S(P)$.
2. The idempotents of $S(P)$ are precisely the morphisms $\text{Id}_U$ for open $U \subset X$ and $\text{Id}_U \leq_R \text{Id}_V$ if and only if every point of distance less than or equal to $R$ of a point of $U$ is contained in $V$.
3. $\Phi_1 \leq_R \Phi_2$ if and only if $\Phi_1 = \text{Id}_{\text{dom}(\Phi_1)} \cdot \Phi_2 = \Phi_2 \cdot \text{Id}_{\text{ran}(\Phi_1)}$ with $\text{Id}_{\text{dom}(\Phi_1)} \leq_R \text{Id}_{\text{dom}(\Phi_2)}$ and $\text{Id}_{\text{ran}(\Phi_1)} \leq_R \text{Id}_{\text{ran}(\Phi_2)}$.
4. If $\Phi_1 \leq_R \Phi_2$ then $\Phi_1 \leq_r \Phi_2$ for all $r \leq R$.
5. If $\Phi_1 \leq R \Psi_1$ and $\Phi_2 \leq R \Psi_2$ then $\Phi_1 \cdot \Phi_2 \leq R \Psi_1 \cdot \Psi_2$.
6. If $\Phi_1 \leq R \Phi_2$ then $\Phi_1^{-1} \leq R \Phi_2^{-1}$.
7. If $\Phi_1 \leq R_1 \Phi_2 \leq R_2 \Phi_3$ then $\Phi_1 \leq R_1 + R_2 \Phi_3$.

Item (7) may be proved under the assumption that $(X, d_X)$ is a geodesic space, which we assume throughout (see the discussion following definition 1.3). If the pattern $P$ is defined on a general metric space then $\leq_R \leq R_1 + R_2$ may be replaced with $\leq_{max\{R_1, R_2\}}$.

The above extra structure on $S(P)$ is not only enough to recover the space $\Omega_P$ up to homeomorphism, but in fact is enough to recover the pattern $P$, up to a relabelling of points of
(X, d_X). We shall say that the patterns P_X, defined on (X, d_X), and P_Y, defined on (Y, d_Y), are isomorphic if and only if there exists some isometry f: (X, d_X) → (Y, d_Y) such that f and its inverse f preserve the semigroup structures of P_X and P_Y. That is, we have that for each Φ ∈ P_X the partial isometry f(Φ): f(dom(Φ)) → f(ran(Φ)) defined by f(Φ)(y) = (f ◦ Φ ◦ f^{-1})(y) is an element of P_Y (and similarly for the inverse).

Proposition 1.15. Let P be a pattern. Then the pair (S(P), ≤_R) as constructed above determines the pattern P up to isomorphism.

Proof. The map sending an open set U to the morphism Id_U is an isomorphism between the locale associated to (X, d_X) and the lattice of idempotents of S(P) with partial order ≤_0. Given Φ ∈ S(P), we have that dom(Φ) corresponds to the map Id_{dom(Φ)} = Φ · Φ^{-1}, ran(Φ) corresponds to the map Id_{ran(Φ)} = Φ^{-1} · Φ and the preimage of an open set V in the range of Φ is the open set U satisfying Id_U = Φ · Id_V · Φ^{-1}. Since all of the spaces involved are Hausdorff and hence sober, there is a natural homeomorphism from (X, d_X) to its space of points, so we may recover the topological space (X, d_X) and the partial isometries of P defined on it up to homeomorphism preserving this structure of partial isometries.

Points of (X, d_X) may be identified as the completely prime filters of the locale associated to (X, d_X) (see, for example, [20]). These correspond precisely to filters F_x of the form U ∈ F_x if and only if x ∈ U for some choice of x ∈ X. For r ∈ R_{>0} consider the open set U^r_x which is defined to be the interior of \bigcap U where the intersection is taken over all open sets U with Id_{V_x} ≤_r Id_U for some V_x ∈ F_x. That is, U^r_x is the meet of these open sets in the locale of (X, d_X). It is easy to see that B_{d_X}(x, r) ⊂ U^r_x and if y ∈ U^r_x then d_X(x, y) ≤ r. It follows that ∪_{r<R} U^r_x = B_{d_X}(x, R). Since d_X(x, y) = inf\{ R | y ∈ B_{d_X}(x, R) \}, we may recover the metric on (X, d_X).

The order structure ≤_R on S(P) puts a “grading” on the elements of S(P). Given Φ ∈ S(P), one may write Φ ∈ S^R(P) if and only if there exists some element \Phi^R ∈ P for which Φ ≤_R \Phi^R. By proposition [1,14] each S^R(P) is a pattern on (X, d_X) and if Φ ∈ S^R(P) then Φ ∈ S^R(P) for all r ≤ R.

This graded structure of patterns may be used to recover the pattern space more directly. Notice that if there exists some \Phi ∈ S^R(P) with \Phi(x) = y then x ∼_R y. Conversely, for every \epsilon > 0, if x ∼_{R+\epsilon} y then there exists some \Phi ∈ S^R(P) with \Phi(x) = y. It follows that \Omega_\epsilon \cong \lim_{\downarrow R} (X, d_X) ∼_R where x ∼_R y if there exists some \Phi ∈ S^R(P) with \Phi(x) = y. Hence, as in the above proof, since all of the spaces involved are sober, the pattern space may be recovered up to homeomorphism from the pattern S(P) along with this directed system of sub-inverse semigroups.

2. Tilings

The main motivation for the patterns defined in the last section comes from the theory of aperiodic tilings. We provide here a definition of a tiling general enough to cover the examples of interest to us in this paper.

Let (X, d_X) be a metric space. A protatile is a subspace p ⊂ (X, d_X). A protatile set is some (usually finite) set of prototiles. In case we wish to distinguish congruent prototiles, we may also allow each prototile to be coupled with some element of a label set L. Given a prototile set P, a tile is some subspace congruent to one of the prototiles, that is, a subspace t ⊂ (X, d_X) which is the image of a prototile p under some isometry \Phi: p → t (and t should be coupled with the same label l ∈ L as p, in case we demand that our prototiles be labelled). We shall always assume, unless otherwise stated, that tiles are the closures of their interiors.
A patch is some union of tiles for which distinct tiles intersect on at most their boundaries. A tiling is a patch for which the tiles cover all of \( X \). Given a patch \( T \) and some subset \( U \subset X \), we define \([U]_T\) to be the patch of tiles of \( T \) which have non-trivial intersection with \( U \). For a (partial) isometry \( \Phi \) (whose domain contains \( T \)), one can define the patch \( \Phi(T) \) by setting \( \Phi(T) \) to be the set of images of tiles \( t \in T \) under \( \Phi \).

Many of the tilings we encounter will be cellular tilings. By this we mean that the prototiles of \( T \) have a CW-decomposition (which induces CW-decomposition on the tiles) so that the union of the subcells of the tiles produces a CW-structure on \((X,d_X)\). That is, for any two subcells of tiles, their intersection is a union of subcells (this condition is sometimes summarised as saying that the tiles meet full-face to full-face). We should consider a decomposition of a tile into cells (along with its possible label) as part of the decoration of the tile. So we should consider two patches \( P \) and \( Q \) as being equal only if they have the same set of tiles when taken with their decoration of label and CW-decomposition.

Remark 2.1. For what is to follow, there seems no reason, a priori, to force tiles to be the closures of their interiors, for distinct tiles of patches to intersect only on their boundaries, or that the tiles of a tiling cover \((X,d_X)\) (other than to conform to intuition of what a tiling should be). There are cases where dropping these assumptions will be useful. For example, one may consider a (coloured) point pattern as a tiling where each prototile is a (labelled) point; the notation as set up works conveniently for point patterns as well as tilings, and there is little reason to add the notational baggage required to separate these cases. We do not disallow infinite prototile sets, but in such a case one should usually like extra structure than just a label set; see [25].

### 2.1. Patterns of Tilings

Let \((X,d_X)\) be some metric space on which a tiling \( T \) is defined. We shall set \( S \) to be some pattern of allowed partial isometries of \((X,d_X)\). In most cases of interest to us here, the allowed partial isometries will be restrictions (to the open sets of \((X,d_X)\)) of a group of global isometries of \((X,d_X)\) e.g., restrictions of translations, or orientation-preserving isometries (which we name rigid motions) of \((\mathbb{R}^d,d_{euc})\) or, for a Lie group \( L \) with left invariant metric \( d_L \), restrictions of those isometries on \((L,d_L)\) given by left multiplication \( L_g(x) = gx \) for \( g \in L \). However, see example [5.1.6] for a situation where it is necessary to consider partial isometries which aren’t restrictions of global isometries of the space.

Definition 2.2. Let \( T \) be a tiling of \((X,d_X)\) with allowed partial isometries \( S \). Let \( \Phi \in S \) be an element of \( T_S \) if and only if \( \Phi \) is the restriction of some isometry \( \Psi \) with \( \text{dom}(\Psi) = [\text{dom}(\Phi)]_T \), \( \text{ran}(\Psi) = [\text{ran}(\Phi)]_T \) and \( \Psi([\text{dom}(\Phi)]_T) = [\text{ran}(\Phi)]_T \).

We will frequently drop the \( S \) from \( T_S \), where the allowed partial isometries are understood.

Proposition 2.3. The collection of partial isometries defining \( T_S \) forms a pattern.

Proof. Since \( \text{Id}_U \in S \) and \( \text{Id}|_{[U]_T}([U]_T) = [U]_T \), we have that \( \text{Id}_U \in T_S \) for all open \( U \subset X \).

Suppose that \( \Phi \in T_S \). Then \( \Phi^{-1} \in S \) since \( S \) is closed under inverses. Since \( \Phi \in T_S \) we have that \( \Phi \) is the restriction of some isometry \( \Psi: [\text{dom}(\Phi)]_T \rightarrow [\text{ran}(\Phi)]_T \) with \( \Psi([\text{dom}(\Phi)]_T) = [\text{ran}(\Phi)]_T \). Then \( \Phi^{-1} \in T_S \) since it is the restriction of \( \Psi^{-1}: [\text{ran}(\Phi)]_T \rightarrow [\text{dom}(\Phi)]_T \), for which one has that \( \Psi^{-1}([\text{ran}(\Phi)]_T) = \Psi^{-1}(\Psi([\text{dom}(\Phi)]_T)) = [\text{dom}(\Phi)]_T \).

Finally, suppose that \( \Phi_1, \Phi_2 \in T_S \). Then \( \Phi_2 \circ \Phi_1 \in S \) since \( S \) is closed under composition. For \( i = 1, 2 \) we have that \( \Phi_i \) is a restriction of some isometry \( \Psi_i: [\text{dom}(\Phi_i)]_T \rightarrow [\text{ran}(\Phi_i)]_T \) with \( \Psi_i([\text{dom}(\Phi_i)]_T) = [\text{ran}(\Phi_i)]_T \). Then \( \Phi_2 \circ \Phi_1 \in T_S \) since it is the restriction of

\[
\Psi_2 \circ \Psi_1\mid_{\Phi_1^{-1}(\text{ran}(\Phi_1) \cap \text{dom}(\Phi_2))}_T : [\Phi_1^{-1}(\text{ran}(\Phi_1) \cap \text{dom}(\Phi_2))]_T \rightarrow [\Phi_2(\text{ran}(\Phi_1) \cap \text{dom}(\Phi_2))]_T
\]
for which we have that \( \Psi_2 \circ \Psi_1 ([\Phi_1^{-1}(\text{ran}(\Phi_1) \cap \text{dom}(\Phi_2))]_T) = \Psi_2([\text{ran}(\Phi_1)]_T \cap [\text{dom}(\Phi_2)]_T) = [\Phi_2(\text{ran}(\Phi_1) \cap \text{dom}(\Phi_2))]_T = [\text{ran}(\Phi_2 \circ \Phi_1)]_T. \)

### 2.2. Topology of the Pattern Space

Let \( T \) be a tiling of \((X, d_X)\), with pattern \( \mathcal{T} \) associated to some collection of allowed partial isometries. Two points \( x, y \) of the tiling are close in the \( d_T \) metric if they have nearby points \( x', y' \) for which there is an allowed partial isometry mapping \( x' \) to \( y' \) preserving a large radius of tiles about \( x' \) and \( y' \).

For certain non-FLC tilings (e.g., for tilings with “fault lines”), we may wish to consider other ways for which patches of tiles should be considered as close; we may allow the tiles of two patches to approximate each other pairwise in shape or label (see [25]). The motivations here are not to provide a good metric for non-FLC tilings. For a given pattern \( \mathcal{P} \), the metric space \((X, d_P)\) can be expressed as an inverse limit of approximants which are quotients of the ambient space \((X, d_X)\). Hence it is to be expected that such patterns will not be able to accommodate for non-FLC tilings for which the approximants to the tiling space need to have extra dimensions to the ambient space.

#### 2.3. Patterns \( \mathcal{T}_1, \mathcal{T}_0 \) and \( \mathcal{T}_{\text{rot}} \)

Let \( T \) be a tiling of \((\mathbb{R}^d, d_{\text{euc}})\). We present here three different patterns \( \mathcal{T}_1, \mathcal{T}_0 \) and \( \mathcal{T}_{\text{rot}} \). The first pattern is only suitable for tilings which have FLC with respect to translations, while the latter two are suitable for tilings which have FLC with respect to rigid motions. The pattern spaces \( \Omega_{\mathcal{T}_1}, \Omega_{\mathcal{T}_0} \) and \( \Omega_{\mathcal{T}_{\text{rot}}} \) correspond to the tiling spaces commonly seen in the literature; they are named \( \Omega^1, \Omega^0 \) and \( \Omega_{\text{rot}} \), resp., in [3], for example.

**Definition 2.4.** Given a tiling \( T \) of \((\mathbb{R}^d, d_{\text{euc}})\), define \( \mathcal{T}_1 \) (resp.) to be the pattern \( \mathcal{T}_S \) for \( T \), where the allowed partial isometries \( \mathcal{S} \) are taken to be restrictions of translations (rigid motions, resp.) to the open sets of \((\mathbb{R}^d, d_{\text{euc}})\).

So \( x \sim_R y \) in \( \mathcal{T}_1 \) (resp.) if and only if the tilings \( T - x \) and \( T - y \) have the same patch of tiles within radius \( R \) of the origin (after a rotation at the origin, resp.). Note that the properties of a tiling being FLC or repetitive, as in the usual sense, with respect to translations (rigid motions, resp.) correspond precisely to the pattern \( \mathcal{T}_1 \) (resp.) having these properties. The approximants \( K_R \) are exactly those of [3], and the description of the pattern space \( \Omega_{\mathcal{T}_1} \) in theorem [1] as the inverse limit \( \lim_{\leftarrow R} K_R \) is simply a restatement of the fact that the tiling space is the inverse limit of BDHS-complexes of increasing patch radii. The space \( \Omega_{\mathcal{T}_1} \) has a natural \( \mathbb{R}^d \)-action given by translation making \((\Omega_{\mathcal{T}_1}, \mathbb{R}^d)\) a dynamical system.

A different way of taking the full orientation-preserving isometry group into account is to define a pattern on the group \( E^+(d) \) of rigid motions of \( \mathbb{R}^d \). Elements of \( E^+(d) \cong \mathbb{R}^d \rtimes SO(d) \) can be uniquely described as a translation \( b \in \mathbb{R}^d \) followed by a rotation \( A \in SO(d) \). One can define a metric on \( E^+(d) \) by setting \( d_E(f, g) := \max\{d_{\text{euc}}(f(x), g(x)) \mid \|x\| \leq 1\} \). Then \( E^+(d) \) acts on itself as a group of isometries by post-composition, so we may take the restrictions of such isometries to the open sets of \( E^+(d) \) to be a set of allowed partial isometries. We temporarily allow ourselves to use tiles with empty interior for the following definition:

**Definition 2.5.** Let \( T \) be a tiling of \((\mathbb{R}^d, d_{\text{euc}})\). We define a tiling \( T_{\text{rot}} \) on \((E^+(d), d_E)\) in the following way. The prototiles of \( T_{\text{rot}} \) are the prototiles of \( T \) as subsets of \( \mathbb{R}^d \), where we consider \( \mathbb{R}^d \subset E^+(d) \) as the subset of translations. A tile \( t_{\text{rot}} \in T_{\text{rot}} \) if and only if \( t_{\text{rot}} = g(t) \) for some \( t \in T \) and \( g \in SO(d) \). Define \( T_{\text{rot}} \) to be the pattern associated to this tiling (with set of allowed partial isometries restrictions of those given by the action of \( E^+(d) \) on itself, as above).

Note that the tiles of \( T_{\text{rot}} \) here are contained in the cosets of \( \mathbb{R}^d \leq E^+(d) \); one may like to think of the tiling as a union of copies of the tiling \( T \), one copy on each fibre \( \mathbb{R}^d \hookrightarrow E^+(d) \to SO(d) \).
for each rotation $g \in SO(d)$. We have that $f \sim_R g$ in $T_{rot}$ if and only if the patches of tiles within distance $R$ of the origin of $f^{-1}(T_{rot})$ and $g^{-1}(T_{rot})$ are the same. This implies that the patch of tiles within distance $R$ of the origin of $f^{-1}(T)$ and $g^{-1}(T)$ are the same, which is also sufficient (for then this is also true of their rotates). That is, $f \sim_R g$ in $T_{rot}$ if and only if $f^{-1}(T)$ and $g^{-1}(T)$ have the same patch of tiles within distance $R$ of the origin. Similarly to the case for $T_1$, the space $\Omega_{T_{rot}}$ has a natural $E^+(d)$-action by rigid motions making $(\Omega_{T_{rot}}, E^+(d))$ a dynamical system.

Examples 2.6. Let $T$ be the periodic tiling of unit squares of $(\mathbb{R}^2, d_{euc})$ with the vertices of the squares lying on the integer lattice. Then for $i = 1$ ($i = 0$, resp.) $(T_i)_{x,y}$ consists of (restrictions to $B_{d_{x}}(x, R)$ of) translations (rigid motions, resp.) taking $x$ to $y$ preserving the integer lattice. That is, $|(T_i)_{x,y}| \neq \emptyset$ if and only if $y - x \in \mathbb{Z}^2$; for $y - x \in \mathbb{Z}^2$ we have that $(T_i)_{x,y} = \{t_{y-x}|B_{d_{x}}(x,R)\}$. The set of morphisms $(T_0)_{x,y}$ may have more than one element; when $x,y$ both lie in the centre of an edge then $|(T_0)_{x,y}| = 2$ and if $x,y$ both lie in the centre of a square or both lie on a vertex then $|(T_0)_{x,y}| = 4$.

For $T_{rot}$, we have that $|(T_{rot})_{f,g}| \neq \emptyset$ if and only if $f(T) = g(T)$, in which case $(T_{rot})_{f,g} = g \circ f^{-1}|B_{d_{f}}(f,R)$.

2.4. Hierarchical Tilings. A hierarchical tiling here will be a set of tilings $T_{\omega} = \{T_0,T_1,\ldots\}$ of $(X,d_{y})$ along with allowed partial isometries $S$ with the following property: the local configurations of tiles in $T_{n-1}$ are determined by the local configurations of tiles in $T_n$. More concretely, we impose that if $\Phi \in T_n$ then $\Phi \in T_{n-1}$ also.

Definition 2.7. For an isometry $\Phi$, temporarily write $rad(\Phi) \geq R$ to mean that there exists some $x$ for which $B_{d_{x}}(x, R) \subset \text{dom}(\Phi)$ and $B_{d_{x}}(\Phi(x), R) \subset \text{ran}(\Phi)$. Let $k: \mathbb{R}_{>0} \to \mathbb{N}_0$ be a non-decreasing unbounded function. Then define the pattern $T_{\omega}$ for $T_{\omega}$ by setting $\Phi \in T_{\omega}$ if and only if $\Phi \in T_{k(R)}$ for any $R$ with $rad(\Phi) \geq R$.

It follows rather immediately from the definitions that $\omega$ is a pattern (note that $rad(\Phi_2 \circ \Phi_1) \geq R$ implies that $rad(\Phi_1), rad(\Phi_2) \geq R$). For a hierarchical tiling of $(\mathbb{R}^d,d_{euc})$ we have that $(T_{\omega})_{x,y} = (T_{k(R)})_{x,y}$. This doesn’t hold for general metric spaces since one may have $B_{d_{x}}(x, r) = B_{d_{x}}(y, R)$ for $r < R$ e.g., for the half real line $\mathbb{R}_{\geq 0}$. For this reason, for simplicity, we shall only consider hierarchical tilings of metric spaces for which open balls determine their centre and radii.

The idea here is that one removes those supposedly “large” isometries which don’t preserve patches of the tilings high up the hierarchy. Two points are close in the pattern metric for a hierarchical tiling if they have nearby points which agree on a large patch of a tiling $T_k$ for some large $k$. An interesting feature of these patterns is that they don’t satisfy the “glueing” axiom of classical pseudogroups; one cannot determine $T_{\omega}$ from those morphisms of small domain alone.

The most important examples of hierarchical tilings are given by substitution tilings of $(\mathbb{R}^d, d_{euc})$. For such tilings one has a set of prototiles, an inflation constant $\lambda > 1$ and a substitution rule which replaces a prototile with a patch of the same support with prototiles scaled down by a factor of $\lambda^{-1}$, see [2] for details. This defines an inflation map on patches of tiles by first subdividing each tile, and then inflating the patch by $\lambda$. A tiling of $\mathbb{R}^d$ is admissible under the substitution rule if the tiles cover $\mathbb{R}^d$ and are such that every finite patch is a sub-patch of some iteratively substituted tile. Under suitable conditions, one can show that this set of tilings is non-empty and that the induced substitution $\omega$ on this set of tilings is surjective. That is, for every such tiling there is a supertiling of supertiles (which are the original tiles
scaled by a factor of $\lambda$) which decomposes under the substitution rule into this tiling, and is itself an (inflated) admissible tiling under the substitution rule.

Given such a substitution rule $\omega$, we can define a hierarchical tiling as some $T_\omega = \{T_0, T_1, \ldots\}$ where each tiling $T_i$ is a tiling of the original prototiles inflated by a factor of $\lambda^i$ and $T_i = \omega(T_{i+1})$. That is, we take a tiling $T_0$ admissible under the substitution rule and set $T_i$ to be some tiling of super-$i$-tiles, which are the original prototiles of $T_0$ scaled by a factor of $\lambda^i$, such that the substitution rule subdivides $T_{i+1}$ into $T_i$. More generally, one could consider several substitutions on the prototiles; in symbolic dynamics, one speaks of “s-adic systems” (see, for example, [22]) and multi/mixed-substitution systems in arbitrary dimensions have been considered too, (see [15]). A more general framework to describe hierarchical tilings is presented in [24].

The topological space $\Omega_{T_\omega} \cong ([R^d, d_{T_\omega}])$ was independent of the choice of function $k : R_{>0} \rightarrow N_0$ (given two different such functions, one can easily construct invertible continuous maps between the associated inverse limits). Let $T_\omega$ be some hierarchical tiling coming from a substitution with inflation factor $\lambda$. Then taking some $\epsilon > 0$ (typically small with relative to the size of the tiles) it is convenient to pick $k(t) := \max\{[\log(\lambda/t)]_0, 0\}$; that is $k(t)$ is the smallest $n \in N_0$ for which $t/\lambda^n \leq \epsilon$. We have that $x \sim_{\lambda^n} y$ if and only if the patches of tiles within distance $\lambda^n \epsilon$ of $x$ and $y$ agree up to an isometry taking $x$ to $y$ in the $n^{th}$ supertile composition $T_n$. Then (when the allowed partial isometries are given as translations or rigid motions) each of the approximants $K_{\lambda^n}$ for $n \in N_0$ are homeomorphic and in a way which makes all of the connecting maps between them the same – these are just the BDHS-complexes for the tiling with a fixed parameter and the connecting maps are those induced by substitution.

We have not yet talked about invertibility of the substitution map. Suppose that we allowed, and then defined $k$ to be the constant function $t \mapsto 0$. In this case $\Omega_{T_\omega} \cong \Omega_T$, the tiling space of $T_0$. For any function $k$ there will always exist a (surjective) continuous map $f : \Omega_{T_\omega} \rightarrow \Omega_T$. Under certain circumstances one can also find an inverse. One can define a continuous map in the other direction if one can determine the local configuration of supertiles given enough information about the local configuration of tiles; the notion to consider here is that of “recognisability”. In such a situation, the hierarchical tiling is determined by $T_0$. But where the substitution is not invertible, the pattern spaces will differ. One may like to view the pattern $T_\omega$ in such a case as instead a tiling which “knows” where its tiles are as well as how to combine them into supertiles, how to combine those into super-$i$-tiles and so on. We illustrate this in the following example.

**Examples 2.8.** The $d$-dimensional dyadic solenoid $D^d_2$ is the inverse limit $\lim_{\rightarrow} \frac{R^d}{2^\times \times \times 2^d}$ of $d$-dimensional tori under the times two map induced from the map $x \mapsto 2x$ in $R^d$. Although the dyadic solenoid is not the translational hull for any tiling of $R^d$ because of its equicontinuous $R^d$-action, it can be realised as the translational hull of a hierarchical tiling. That is, we can define a hierarchical tiling $T_\omega$ such that, with the collection of allowed partial isometries given as translations, the pattern space $\Omega_{T_\omega}$ is the dyadic solenoid. It comes from the substitution of a single $d$-cube prototile, which subdivides to $2^d$ $d$-cubes in the obvious way. Such a hierarchical tiling consists of a sequence $T_0, T_1, \ldots$ of periodic tilings $T_i$ of $d$-cubes with sides length $2^i$ such that $T_i$ subdivides to $T_{i-1}$ (for example, pick each as the periodic tiling with vertex of a cube at the origin).
3. Pattern-Equivariant Homology

3.1. Preliminaries. We denote by $\Delta^n$ the standard $n$-simplex. A singular $n$-simplex of a topological space $X$ is a continuous map $\sigma: \Delta^n \to X$. We denote by $C_n(X)$, the group of singular $n$-chains, the free abelian group generated by the singular $n$-simplexes of $X$. The usual boundary map $\partial_n: C_n(X) \to C_{n-1}(X)$ makes $C_*(X)$ a chain complex.

Given an arbitrary sum $\sigma = \Sigma_i c_i \sigma_i$, where each $c_i \in \mathbb{Z}$ and $\sigma_i$ is a singular $n$-simplex, for a set $S \subset X$ define $\sigma^S = \Sigma c_i \sigma_i$, where $c_i^S = c_i$ if $S \cap \sigma_i(\Delta^n) \neq \emptyset$ and $c_i^S = 0$ otherwise. We set $\sigma^x = \sigma(x)$ for elements $x \in X$. We say that a sum $\sigma = \Sigma_i c_i \sigma_i$ is a Borel-Moore $n$-chain if the number of non-zero coefficients $c_i^K$ is finite for each compact $K \subset X$. The groups $C_n^{BM}(X)$ of Borel-Moore $n$-chains fit into a chain complex by extending the usual boundary map of singular chains (note that this boundary map is well defined since the image of a singular simplex is compact). The Borel-Moore homology $H_{BM}^n(X)$ is defined to be the homology of this chain complex.\footnote{This definition of course differs greatly from that of \cite{2}; the resulting homology groups will coincide only on a class of “reasonable” spaces.}

A continuous map $f: X \to Y$ is called proper if the preimages of compact subsets are compact. The homology theory $H_{BM}^\bullet$ is a covariant functor over the category of topological spaces and proper maps. One can generalise to coefficients in an arbitrary abelian group here and throughout, so we will often omit the coefficient group from our notation.

3.2. PE Homology. Given a pattern $P$ a Borel-Moore chain $\sigma = \Sigma_i c_i \sigma_i \in C_n^{BM}(X)$ is said to be $P$-equivariant (PE for short or pattern-equivariant when the pattern is understood) to radius $R$ if each $\sigma_i$ has radius of support bounded by $\frac{3}{4} R/2$ and, for each pair of elements $x, y \in X$ and each element $\Phi \in \mathcal{P}_{x,y}^R$, we have that $\Phi_* (\sigma^x) = \sigma^y$. One may like to say that $\sigma$ looks the same locally at $x$ and $y$ via the isometries $\Phi \in \mathcal{P}_{x,y}^R$.

We denote by $C_n^{P,R}(X)$ the group of all $n$-chains with $\text{PE}$ radius $R$. A Borel-Moore chain is said to be $P$-equivariant if it is $P$-equivariant to some radius $R$ and we denote the set of all $\text{PE}$ $n$-chains by $C_n^{P}(X)$. The lemma below will establish that

$$C_n^P(X) = (0 \leftarrow C_0^n(X) \leftarrow C_1^n(X) \leftarrow C_2^n(X) \leftarrow \cdots)$$

forms a subchain complex of $C_*^{BM}(X)$, the homology of which, called the $P$-equivariant homology, we shall denote by $H^n_P(X)$ (and similarly for $H_*^{P,R}(X)$, the $\text{PE}$ homology to radius $R$).

It will be useful in our consideration of a cellular $\text{PE}$ homology to restrict attention to $\text{PE}$ chains contained in some subset. Given a pattern $P$ and any subset $A \subset X$, one can form the chain complex

$$C_\bullet^P(A) = (0 \leftarrow C_0^P(A) \leftarrow C_1^P(A) \leftarrow C_2^P(A) \leftarrow \cdots)$$

of the chain groups $C_i^P(A)$ whose elements are those $\text{PE}$ chains of $X$ contained in the subset $A$ (and similarly for $C_i^{P,R}(A)$). The boundary maps are the restrictions of the usual boundary maps to these chain groups. The homology of this chain complex is denoted by $H_\bullet^P(A)$.

For $B \subset A \subset X$, one can form the quotient complex $C_\bullet^P(A, B) := C_\bullet^P(A)/C_\bullet^P(B)$, for which we denote the homology $H_\bullet^P(A, B)$. Of course, we have the usual exact sequence of chain complexes and long exact sequence in homology

$$0 \leftarrow H_0^P(A, B) \leftarrow H_0^P(A) \leftarrow H_0^P(B) \leftarrow H_1^P(A, B) \leftarrow \cdots.$$\footnote{We say that $A \subset (X, d_X)$ has radius bounded by $\alpha$ if there exists some $x$ with $A \subset B_{d_X}(x, \alpha)$. A function $f: Y \to (X, d_X)$ has radius of support bounded by radius $\alpha$ if its image has radius bounded by $\alpha$.}
We emphasise that $C^P_*(A)$ is considered as some subgroup of $C^P_*(X)$. When $A$ is not closed in $X$, it is not necessarily true that Borel-Moore chains of $A$ are Borel-Moore chains of $X$; in our notation, elements of $C^P_*(A)$ are both.

**Lemma 3.1.** Given any $\sigma \in C^P_n(A)$, the following holds:

- $\sigma \in C^P_R(A)$ for any $R > r$,
- $\partial_n(\sigma) \in C^P_R(A)$.

**Proof.** The first item follows from the fact that restrictions of elements of $\mathcal{P}_R$ to $B_d(x,r)$ are elements of $\mathcal{P}_R$, so the requirements for $\sigma$ to be $\mathcal{P}E$ to radius $R$ are weaker than for it to be $\mathcal{P}E$ to radius $r$. For the second note that, since the transformations $\Phi \in \mathcal{P}_R$ are injective, $\Phi_* (\sigma^x) = (\Phi_* (\sigma))^y$. We also have that $\partial_n (\sigma^x) = \partial_n (\sigma^y)$ so $\Phi_* (\partial_n (\sigma^x)) = \Phi_* (\partial_n (\sigma^y)) = \Phi_* (\partial_n (\sigma^y)) = \partial_n (\sigma^y)$. \hfill $\square$

By a $\mathcal{P}$-set, we shall mean a set $U = \pi^{-1}_R(S)$ for some $R > 0$, $S \subset K_R$. An indexed collection of $\mathcal{P}$-sets $U = \{ U_i | i \in I \}$ of $X$ will be assumed to consist of pullbacks of sets from $K_R$ for some fixed $R$, in case the indexing set is infinite (we shall say that the $\mathcal{P}E$ radius of $U$ is $R$, in such a case). Note that finite unions, finite intersections and complements of $\mathcal{P}$-sets are also $\mathcal{P}$-sets.

For $D > 0$, temporarily denote by $C^D_\mathcal{P}$ the chain complex of $\mathcal{P}E$ chains for which the singular simplexes have radius of support bounded by $D$ and, for an open cover $U$ of $X$, denote by $C^D_\mathcal{P}$ the chain complex of $\mathcal{P}E$ chains for which the singular simplexes are contained in the cover $U$.

**Lemma 3.2.** For any pattern $\mathcal{P}$ on $X$ and $Y \subset X$ we have:

1. $C_\mathcal{P}^P(Y)$ is a subchain complex of $C^BM_\mathcal{P}(Y)$.
2. $C^P_\mathcal{P}(Y)$ is a subchain complex of $C^P_\mathcal{P}(Y) \leftarrow \lim_{\to R} C^P(R)$. 
3. The inclusion $\iota : C^D_\mathcal{P}(Y) \rightarrow C^P_\mathcal{P}(Y)$ is a quasi-isomorphism.
4. For a collection of $\mathcal{P}$-sets $U$ whose interiors cover $Y$ (with the subspace topology, relative to $Y$), the inclusion $\iota : C^{\mathcal{P}}_\mathcal{P}(Y) \rightarrow C^P_\mathcal{P}(Y)$ is a quasi-isomorphism.
5. Under the same conditions above, with $U = \{ A, B \}$, the obvious inclusion of chain complexes $\iota : C^P_\mathcal{P}(B, A \cap B) \rightarrow C^P_\mathcal{P}(Y, A)$ is a quasi-isomorphism. Hence, $H^P_n(Y - Z, A - Z) \cong H^P_n(Y, A)$ for $\mathcal{P}$-sets $Z \subset A \subset Y$ with (in the subspace topology relative to $X$) the closure of $Z$ contained in the interior of $Y$.
6. Under the same conditions as above, we have the following long exact sequence:

$$0 \leftarrow H^P_0(Y) \leftarrow \cdots \leftarrow H^P_{n-1}(A \cap B) \leftarrow H^P_n(Y) \leftarrow H^P_n(A) \oplus H^P_n(B) \leftarrow H^P_n(A \cap B) \leftarrow \cdots$$

as well as the usual relative version.

**Proof.** The first and second items follow trivially from the above lemma.

To prove the third, we can follow the classical approach (see [17]). One defines an operator $\rho$ on the singular chain complex, defined in terms of the barycentric subdivision operator and the number $m(\sigma_i)$ which, for each singular simplex $\sigma_i$, is the smallest natural number such that subdividing the simplex $m(\sigma_i)$ times yields a chain whose singular simplexes have radius of support bounded by $D$. Going through all of the details again would be tedious, but note that $m(\sigma_i)$ is invariant under the isometries $\mathcal{P}_R$ which are applied to $\sigma_i$. So $\rho$ “commutes” with the maps $\Phi_*$ and, since $\rho(\sigma^x) = \rho(\sigma)^y$, $\rho$ will send a $\mathcal{P}E$ chain of radius $R$ to a $\mathcal{P}E$ chain of radius $R$, mimicking the proof of the lemma 3.1.

For the fourth item, one may show that given a $\mathcal{P}E$ chain $\sigma$ of $\mathcal{P}E$ radius $R$ (pick $R \geq R_U$, the $\mathcal{P}E$ radius of the cover $U$), it has a subdivision $\rho(\sigma)$ contained in the cover with $\mathcal{P}E$ radius
quasi-isomorphism as desired. The proofs is similar to the above. If the cover has a Lebesgue number, one can of course subdivide the chains as above so that their radius of supports are smaller than the Lebesgue number of the cover by (3). If it doesn’t, pick \( \epsilon > 0 \) and first subdivide to a \( \mathcal{PE} \) chain with simplexes of support bounded by radius \( \epsilon/2 \) as described above. Let \( m(\sigma_i) \) be the smallest number such that subdividing \( \sigma_i \) \( m(\sigma_i) \) times produces a singular chain contained in the cover. Then \( m(\sigma_i) = m(\Phi(\sigma_i)) \) for every \( \Phi \in \mathcal{P}^{R+\epsilon} \) since \( \pi_R = \pi_R \circ \Phi \) on \( B_{dx}(x, \epsilon) \), which contains the support of \( \sigma_i \). It follows that the chain given by subdividing each singular simplex \( \sigma_i \) \( m(\sigma_i) \) times is \( \mathcal{PE} \) to radius \( R + \epsilon \).

The proof of excision, again, follows from a simple adaptation of the classical case. We note that the constructions above allow us to canonically define a quasi-isomorphism \( C^P(Y)/C^P(A) \to C^P(X)/C^P(A) \). There is an obvious isomorphism \( C^P(B)/C^P(A \cap B) \to C^P(Y)/C^P(A) \) (elements of both can be identified with \( \mathcal{PE} \) chains contained in \( B \) but not \( A \)), so we obtain the quasi-isomorphism as desired.

Given \( \mathcal{P} \)-sets \( Z \subset A \subset Y \), we have that \( Y - Z \) is also a \( \mathcal{P} \)-set, so we obtain the result that \( H_n^P(Y - Z, A - Z) \cong H_n^P(Y, A) \) by setting \( B = Y - Z \) and applying the above.

Finally, the Mayer-Vietoris sequence follows, since the above gives us that the inclusion \( C^P(Y) \hookrightarrow C^P(X) \) is a quasi-isomorphism, which implies the result from long exact sequence associated to the short exact sequence of chain complexes

\[
0 \to C^P(X) \otimes C^P(\mathcal{A}) \xrightarrow{\phi} C^P(Y) \oplus C^P(B) \xrightarrow{\psi} C^P(X) \to 0
\]

where \( \phi(\sigma) = (\sigma, -\sigma) \) and \( \psi(\sigma, \tau) = \sigma + \tau \).

It follows that the pattern-equivariant homology groups \( H_n^P(X) \) are well defined and are isomorphic to the direct limits \( \lim_{\rightarrow R} H_n^{\mathcal{P},R}(X) \). The Mayer-Vietoris exact sequence of the above lemma will be useful when showing the equivalence of the singular and cellular versions of \( \mathcal{PE} \) homology, as well as in section 3.3 for Poincaré duality.

### 3.3. Cellular Pattern-Equivariant Homology

To make explicit computations, we require a cellular version of \( \mathcal{PE} \) homology. For this, we need the notion of a cellular pattern, which may represent a given pattern in a certain sense. Much like for PE cohomology, one must be careful when dealing with rotational symmetry. We show that, with certain conditions on either the divisibility of the coefficient group or the CW-decomposition with respect to the symmetry of the pattern, singular and cellular \( \mathcal{PE} \) homology coincide. We shall often identify a cell of \( (X, dx) \) with the image of its characteristic map from the closed \( d \)-disc \( c : D^d \to X \).

**Definition 3.3.** A cellular pattern \( \mathcal{C} \) (on \( (X, dx) \)) consists of the following data:

- A finite dimensional (and locally finite) CW-decomposition of \( (X, dx) \) for which the cells are bounded in radius.
- For each \( R > 0 \) a groupoid \( \mathcal{C}^R \).

The groupoids \( \mathcal{C}^R \) are required to satisfy the following:

1. The set of objects of each \( \mathcal{C}^R \) is the set of cells of the CW-decomposition. The set of morphisms between \( k \)-cells \( c \) and \( d \), denoted \( \mathcal{C}^R_{c,d} \), is a finite set of cellular isometries of \( c \) onto \( d \), where composition is given by the usual composition of isometries. Further, these sets of isometries should be tame: for each \( c \) there exists some neighbourhood in \( c \) of \( \partial c \), the boundary of \( c \), which deformation retracts to \( \partial c \) equivariantly. That is, for a \( k \)-cell \( c \), \( k > 0 \), there exists a deformation retraction \( F : [0, 1] \times U \to U \) of some closed neighbourhood \( U \subset c \) of \( \partial c \) with \( F(t, x) = x \) for all \( x \in \partial c \), \( F(0, -) = Id_U \), \( F(1, x) \subset \partial c \) and \( F \circ \Phi = \Phi \circ F \) for all \( \Phi \in \mathcal{C}^R_{c,c} \).
(2) For \( r \leq R \), if \( \Phi \in C^R_{c,d} \) then \( \Phi \in C^r_{c,d} \). One may like to say that \( C^R \subset C^r \).

(3) For every \( R > 0 \) there exists some \( R' > 0 \) satisfying the following: for every \( \Phi \in C^R_{c,d} \) between \( k \)-cells \( c \) and \( d \) there exists a bijection \( f \) between the \( (k+1) \)-cells containing \( c \) and the \( (k+1) \)-cells containing \( d \) for which, for each \( (k+1) \)-cell bounding \( c \), there exists some \( \Phi' \in C^R_{c',f(c')} \) which restricts to \( \Phi \).

For the “tameness” of the isometries it is sufficient, for example, that there exists some such neighbourhood \( U \) of \( \partial c \) which is fibred as \( U \cong \partial c \times [0,\epsilon] \) in a way such that \( \Phi(x,t) = (\Phi(x),t) \) for all \( \Phi \in C^R_{c,c} \). Whilst this condition seems reasonable, it is not obvious to the author that this condition must always hold. Group actions of closed disks, even of finite groups, can have properties which depart from intuition sufficiently to cause some doubt; for example, there exist finite group actions of disks which have no fixed points \[11\]. In most situations of interest here, the tameness of the isometries won’t be an issue. For example, a sufficient condition is that the open cells are isometric to star domains of Euclidean space. It isn’t too hard to show that the tameness of the isometries won’t be an issue.

Choosing an orientation for each \( k \)-cell \( c \) with boundary \( \partial c \), we may identify the singular homology \( H_i(c,\partial c;G) \) as \( G \) for \( i = k \) and as being trivial otherwise. A map \( \Phi \in C^R_{c,d} \) induces an isomorphism between the coefficient groups of the cells \( c \) and \( d \), the isomorphism being determined by the choice of orientation of the cells and whether or not \( \Phi \) preserves or reverses orientations. We may then define the cellular pattern-equivariant homology chains as those chains which are “equivariant” with respect to these maps:

**Definition 3.4.** Let \( C \) be a cellular pattern on \((X,d_X)\). A cellular Borel-Moore \( k \)-chain \( \sigma \) on \((X,d_X)\) is called pattern-equivariant (with PE radius \( R \), with respect to \( C \)) if there exists some \( R > 0 \) such that for any \( k \)-cell \( c \) and \( \Phi \in C^R_{c,d} \) then \( \Phi_*(\sigma)(c) = \sigma(d) \). The homology of this chain complex, the cellular pattern-equivariant homology of \( C \), will be denoted by \( H^C_*(X;G) \).

It is easy to see that condition (3) in the definition of a cellular pattern ensures that the usual cellular boundary map of cellular Borel-Moore chains restricts to a well-defined boundary map on cellular PE chains.

**Definition 3.5.** Let \( \mathcal{P} \) be a pattern and \( C \) be a cellular pattern of \((X,d_X)\). We shall say that \( C \) is a cellular representation of \( \mathcal{P} \) if:

- For all \( R > 0 \) there exists some \( R' > 0 \) such that if \( \Phi \in \mathcal{P}^{R'}_{x,y} \) then \( \Phi|_{c_x} \in C^R_{c_x,\Phi(c_x)} \) for each cell \( c_x \) containing \( x \).
- For all \( R > 0 \) (with \( R/2 \) greater than the radii of the cells) there exists some \( R' > 0 \) such that if \( \Phi \in C^R_{c,d} \) then there exists some \( \Phi' \in \mathcal{P}^R_{x,y} \) with \( x \) contained in \( c \) and \( \Phi'|_c = \Phi \).

We may loosely summarise the above by saying that for arbitrarily large \( R \) there exists some \( R' \) with \( \mathcal{P}^{R'} \subset C^R \) and \( C^R \subset \mathcal{P}^{R'} \).

**Examples 3.6.**

(1) Let \( G \) be a (discrete) group which acts on \((X,d_X)\) by isometries. That is, there exist isometries \( \Phi_g : (X,d_X) \to (X,d_X) \) for each \( g \in G \) satisfying \( \Phi_g \circ \Phi_h = \Phi_{gh} \) and \( \Phi_e = I_{d_X} \) for \( e \in G \) the identity of \( G \). Then we can define the pattern \( \mathcal{G} \) on \((X,d_X)\) by setting \( \Phi \in \mathcal{G} \) if and only if \( \Phi = \Phi_g|_U : U \to \Phi_g(U) \) for some \( \Phi_g \in G \) and open set \( U \subset X \). The \( G \)-action on \( X \) induces a \( G \)-action on the chain complex of (bounded) singular Borel-Moore chains on \( X \). It isn’t too hard to see that the singular \( \mathcal{G} \cdot E \) chain complex will correspond precisely to the complex of bounded \( G \)-invariant singular Borel-Moore chains on \( X \).
Suppose that $X$ is equipped with a CW-decomposition of cells of bounded radius which is invariant under the action of $G$ (that is, if $c$ is a cell of $X$ then so is $\Phi_g(c)$). Let $C_{c,d}^R$ to be the set of isometries between $c$ and $d$ which are restrictions of the isometries $\Phi_g$. Then (given that the isometries are tame, and each $C_{c,d}^R$ is finite) $C$ is a cellular representation of $G$. Of course, the cellular $G$E chains are precisely cellular Borel-Moore chains which are invariant under the action of $G$. This example indicates that “pattern-invariant” would be a justifiable adjective for the homology groups defined here. The name “pattern-equivariant” was chosen due to the pre-established term as used in the context of tilings, see below.

(2) Let $T$ be a cellular tiling of $\mathbb{R}^d$. The CW-decomposition of $\mathbb{R}^d$ given by the tiling can easily be made into cellular patterns that represent the patterns $T_i$ or $T_0$ (see the following subsection). With a little more effort, one can also find a CW-decomposition of $E_+^i(d)$ and make it a cellular pattern representing $T_{\text{rot}}$ (this CW-decomposition is described in [29]). The $T_i$E cellular homology groups (for $i = 0, 1$) are discussed in detail below in subsection 3.3.

(3) Let $P \subset \mathbb{R}^d$ be a Delone set. One may consider a pattern $\mathcal{P}$ associated to $P$ (see remark 2.1). There is a CW-decomposition associated to $P$, the Voronoi tiling of $P$; naturally it can be made into a cellular pattern which represents $\mathcal{P}$.

Let $G$ be a commutative ring with identity $1 \in G$. The group structure on $G$ is a $\mathbb{Z}$-module in the usual way; we shall say that $G$ has division by $n$ if $n.1$ is invertible in $G$. This defines, for each $g \in G$, a unique element $g/n := n^{-1}g$ for which $n(g/n) = g$.\footnote{More generally, one may define the notion of an abelian group $G$ having division by $n$ whenever $nG = G$. Of course, elements $g/n$ which satisfy $n(g/n) = g$ are not necessarily unique, but with care the arguments here could be adapted to this setting.} We may now state the main theorem of this subsection:

**Theorem 3.7.** Let $\mathcal{P}$ be a pattern and $\mathcal{C}$ be a cellular pattern that represents it. Suppose further that $G$ has division by $|\mathcal{C}_{c,d}^R|$ (for $|\mathcal{C}_{c,d}^R| \neq 0$) for each $c,d$ and sufficiently large $R$. Then $H^P_{\mathcal{C}}(X,G) \cong H^C_{\mathcal{C}}(X,G)$.

One should draw attention to the requirement in the above of having an appropriate coefficient group or CW-decomposition if the pattern has local symmetries. To see the need for these restrictions in a simple example, consider the unit disc $\mathbb{D}^2$ together with its boundary $S^1$ and rotation $\tau$ about its centre by $\pi$. This defines a group action of $\mathbb{Z}_2$ on the relative singular chain group of $(\mathbb{D}^2, S^1)$ which commutes with the boundary map and so one may construct a chain complex consisting of those chains which are invariant under the induced action of $\tau$.

Over $\mathbb{Z}$-coefficients, there is two-torsion in the degree zero homology group – one may only “remove” singular 0-simplexes at the origin in multiples of two. The rotation invariant homology calculations will be incorrect unless the coefficient groups have division by two or the CW-decomposition has the origin as a zero-cell.

The remainder of this subsection will be devoted to proving the above theorem. The following technical lemma is used to show that the pairs $(X^k, X^{k-1})$ of the $k$ and $(k-1)$-skeleta are “good” in a $\mathcal{P}$-equivariant sense:

**Lemma 3.8.** Let $\mathcal{P}$ be a pattern. Suppose that $F\colon [0,1] \times A \to A$ is a proper homotopy with $A \subset X$ for which:

- $F_0 = Id_X$.
- $\sup\{d_X(F_t(x), F_{t'}(x)) \mid t, t' \in [0,1]\} < D$ for some $D_F > 0$.


\[ \Phi \circ F_\epsilon = F_\epsilon \circ \Phi \text{ on } B_{d_\mathcal{X}}(x, D + \epsilon) \cap A \text{ for all } x \in A, \Phi \in \mathcal{P}_x \mathcal{R} \text{ for some } \epsilon, R > 0 \text{ (in particular, } R_F \text{ should be chosen so that these maps are well defined on } B_{d_\mathcal{X}}(x, D_F + \epsilon) \cap A). \]

Then, for all \( n \)-chains \( \sigma \) with \( \mathcal{P}E \) radius \( R \geq R_F \) with radius of support bounded by \( \epsilon/2 \) and contained in \( A \) we have that \( F_{1*}(\sigma) \) is \( \mathcal{P}E \) to radius \( R + D_F + \epsilon \) and \( F_{1*}(\sigma) \sim \sigma \) in \( C^n_\mathcal{P}(X) \).

**Proof.** We have that \((Id_A)_* \simeq F_{1*}\) on Borel-Moore chains. We need to show that the singular chain resulting from the prism operator given by the homotopy is \( \mathcal{P}E \).

The homotopy gives, for each singular simplex \( \omega: \Delta^n \to A \), a singular prism \( F(\omega): [0,1] \times \Delta^n \to A \). We could consider, given some chain \( \sigma \), the “singular prism chain” \( F(\sigma) \), where each singular prism inherits a coefficient from \( \sigma \) and the homotopy \( F \) in the obvious way. Using analogous notation as for singular chains, if \( F_*(F(\sigma)^y) = F(\sigma)^y \) for a \( \mathcal{P}E \) chain \( \sigma \) the result follows since the same will hold for the corresponding singular chain after we apply the prism operator.

Let \( \sigma \) have \( \mathcal{P}E \) radius \( R \geq R_F \) and singular simplexes with radius of support bounded by \( \epsilon/2 \). Set \( S = B_{d_\mathcal{X}}(x, D_F) \). Then \( F_*(\sigma^S) = (\Phi(S))^* \) for all \( \Phi \in \mathcal{P}_x \mathcal{R} \). To see this, note that \( F_B(x', R) \in \mathcal{P}_x \mathcal{R} \) with \( x' \in B_{d_\mathcal{X}}(x, D_F + \epsilon) \). If \( \Phi_*(\sigma^S) \neq \sigma^\Phi(S) \) then there would exist some singular simplex \( \tau \) with non-zero coefficient in \( \Phi_*(\sigma^S) - \sigma^\Phi(S) \) with support contained in \( B_{d_\mathcal{X}}(y, D_F + \epsilon) \). But then \( \Phi_*(\sigma^S(x')) \neq \sigma^\Phi(x') \) for \( x' \) in the support of \( \Phi^{-1}_*(\tau) \).

The second assumption in the above lemma implies that \( (F(\sigma'))^y = (F(\sigma))^y \). The third implies that \( \Phi_*(F(\sigma^S)) = F(\Phi_*(\sigma^S)) \) since the singular simplexes of \( \sigma^S \) with non-zero coefficients are contained in \( B_{d_\mathcal{X}}(x, D_F + \epsilon) \) on which \( F \) and \( \Phi \) “commute”.

Hence, we have that \( \Phi_*(F(\sigma^y)) = \Phi_*(F(\sigma^S))^y = (\Phi_*(F(\sigma^S)))^y = (F(\Phi_*(\sigma^S)))^y = (F(\sigma))^y \), so we get the required result after application of the prism operator. \( \square \)

For a CW-complex, the pair \((X^k, X^{k-1})\) is good, that is, there exists a deformation retraction of some closed neighbourhood \( U \subset X^k \) of \( X^{k-1} \) onto \( X^{k-1} \). It follows that \( H_*(X^k, X^{k-1}) \simeq H_*(X^k - X^{k-1}, U - X^{k-1}) \) by excision. Cellular patterns will have the same property with respect to the pattern:

**Lemma 3.9.** Let \( \mathcal{P} \) be a pattern on \( X \) which has some cellular representation \( \mathcal{C} \). Then \( H_\mathcal{P}(X^k, X^{k-1}) \simeq H_\mathcal{P}(X^k - X^{k-1}, U^{k-1} - X^{k-1}) \) for \( U^{k-1} \subset X^k \) some neighbourhood of \( X^{k-1} \).

**Proof.** Elements \( \Phi_{x,y} \in \mathcal{P}_x \mathcal{R} \) induce isometries between the cells intersecting \( B_{d_\mathcal{X}}(x, D + \epsilon) \) and \( B_{d_\mathcal{X}}(y, D + \epsilon) \) for sufficiently large \( R \). Consider forming a deformation retraction of a closed neighbourhood \( U^{k-1} \) of the \((k-1)\)-skeleton on each isometry class of \( k \)-cell in a way such that points are moved by distance less than some \( D \) and which is equivariant with respect to the maps \( \mathcal{C}_c.d^R \) (such deformation retractions exist by our assumption of “tameness” of the isometries of the cellular pattern). Then the induced deformation retraction on the whole complex satisfies the above lemma 3.8. Since every \( \mathcal{P}E \) chain is homologous to one with radius of support of arbitrarily small size by barycentric subdivision (see lemma 3.2), it follows that every \( \mathcal{P}E \) chain of \( U^{k-1} \) is homologous to one contained in \( X^{k-1} \), so that the inclusion \( X^{k-1} \hookrightarrow U^{k-1} \) induces an isomorphism \( H_\mathcal{P}(X^k, X^{k-1}) \simeq H_\mathcal{P}(X^k, U^{k-1}) \). By excision (lemma 3.2), we have that \( H_\mathcal{P}(X^k, X^{k-1}) \simeq H_\mathcal{P}(X^k - X^{k-1}, U^{k-1} - X^{k-1}) \). \( \square \)

**Lemma 3.10.** Let \( \mathcal{P} \) be a pattern with cellular representation \( \mathcal{C} \). Suppose that \( G \) has division by \( |\mathcal{C}_c.d^R| \) (for \( \mathcal{C}_c.d^R \neq 0 \)) for sufficiently large \( R \). Then \( H_i^\mathcal{P}(X^k - X^{k-1}, U^{k-1} - X^{k-1}; G) \) is trivial for \( i \neq k \) and for \( i = k \) is canonically isomorphic to the group of cellular \( \mathcal{P}E \) \( k \)-chains of \( \mathcal{C} \), that is, the group given by assigning coefficients to each oriented \( k \)-cell in a way which is equivariant with respect to the isometries \( \Phi \in \mathcal{C}^R \) for some sufficiently large \( R \).
Proof. We will show that the inclusion of chain complexes \( i: C^P_\bullet(X - X^{k-1}, U^{k-1} - X^{k-1}; G) \to C^{BM}_\bullet(X - X^{k-1}, U^{k-1} - X^{k-1}; G) \) induces inclusions on homology. That is, if \( \sigma \) is a \( \mathcal{P}E \) relative chain and \( \sigma = \partial(\tau) \) for some relative Borel-Moore chain \( \tau \), then \( \tau \) can in fact be chosen to be \( \mathcal{P}E \).

Let \( \sigma \) be \( \mathcal{P}E \) to radius \( R_1 \), with \( R_1 \) large enough so that \( G \) has division by \( |C^R_{x,y}| \), and \( \sigma = \partial(\tau) \) with \( \tau \) not necessarily \( \mathcal{P}E \). Let the cells of \( \mathcal{C} \) have radius of support bounded by \( D \). Since \( \mathcal{C} \) is a cellular representation of \( \mathcal{P} \), there exists some \( R_2 \) such that whenever \( \Phi \in \mathcal{C}_{c,d} \) then there exists some \( \Phi' \in \mathcal{P}_{x,y} \) which restricts to \( \Phi \). We replace \( \tau \) by an “averaged” version \( \tau_A \). First, for each orbit of \( k \)-cell in \( \mathcal{C}_{c,d} \) pick some representative cell \( c \). Restrict \( \tau \) to it and replace each coefficient \( g \) of singular chain on it by \( g/|C^R_{c,d}| \). Now transport it to each equivalent \( k \)-cell by each of the maps \( \Phi \in \mathcal{C}_{c,d} \). Repeating for each orbit of \( k \)-cell, we denote the resulting chain by \( \tau_A \). Of course, \( \tau_A \) is defined by its restrictions \( \tau_A^d \) to each \( k \)-cell \( d \); for \( c \) the representative of the orbit of \( d \), we have that \( \tau_A^d = \sum_{\Phi \in \mathcal{C}_{c,d}} \Phi_*^d(\tau^c/|C^R_{c,d}|) \). We must check that \( \tau \) is \( \mathcal{P}E \) and that \( \sigma = \partial(\tau) = \partial(\tau_A) \).

Since \( \mathcal{C} \) is a cellular representation of \( \mathcal{P} \), there exists some \( R_3 \) such that whenever \( \Phi \in \mathcal{P}_{x,y} \), then \( \Phi|_{c_x} \in \mathcal{C}_{c_x,\Phi(x)} \) for \( c_x \ni x \). We claim that \( \tau_A \) is \( \mathcal{P}E \) to radius \( R_3 \). To see this, let \( \Phi \in \mathcal{P}_{x,y} \) for \( x \in X^k \) and \( x \in c_x \), \( y \in c_y \) and \( c \) the representative of \( c_x \). We shall show that \( \Phi_*^\ast(\tau_A^c) = \tau_A^c \), from which it follows that \( \tau_A \) is \( \mathcal{P}E \). Since (the restriction of) \( \Phi \) is in \( \mathcal{C}_{c_x,\Phi(x)} \), we have that the set of isometries \( \Phi \circ \mathcal{C}_{c_x,\Phi(x)} = \mathcal{C}_{c_x,c_y} \). Hence

\[
\Phi_*^\ast(\tau_A^c) = \Phi_* \sum_{\Psi \in \mathcal{C}_{c_x,c_y}} \Psi_*^\ast(\tau^c/|C^R_{c_x,c_y}|) = \sum_{\Psi \in \mathcal{C}_{c_x,c_y}} (\Phi \circ \Psi)_\ast^\ast(\tau^c/|C^R_{c_x,c_y}|) = \sum_{\Psi \in \mathcal{C}_{c_x,c_y}} \Psi_*^\ast(\tau^c/|C^R_{c_x,c_y}|) = \tau_A^c.
\]

To show that \( \partial(\tau_A^c) = \sigma = \partial(\tau_A) \), it is sufficient to prove that \( \partial(\tau_A)^d = \sigma^d \) for each \( k \)-cell \( d \). Let \( \Phi \in \mathcal{C}_{c_d} \). Since \( \Phi = \Phi|_{c} \) for some \( \Phi' \in \mathcal{P}_{x,y} \) with \( x \in c \), it follows that \( \Phi \) is a restriction of some \( \Phi' \in \mathcal{P}_{x,y}^R \) for each \( x \in c \). Hence, \( \Phi_*^\ast(\sigma^c) = \sigma^d \) for each \( \Phi \in \mathcal{C}_{c} \) since \( \sigma \) is \( \mathcal{P}E \) to radius \( R_1 \). It follows that

\[
\partial(\tau_A)^d = \partial \sum_{\Phi \in \mathcal{C}_{c_d}} \Phi_*^\ast(\tau^c/|C^R_{c,d}|) = \sum_{\Phi \in \mathcal{C}_{c_d}} \Phi_*^\ast(\sigma^c/|C^R_{c,d}|) = \sum_{\Phi \in \mathcal{C}_{c_d}} \sigma^d/|C^R_{c,d}| = (|C^R_{c,d}|\sigma^d)/|C^R_{c,d}| = \sigma^d.
\]

Hence, we have an inclusion \( i_*: H^P_\bullet(X - X^{k-1}, U - X^{k-1}; G) \to H^{BM}_\bullet(X - X^{k-1}, U - X^{k-1}; G) \). It follows that \( H^P_\bullet(X - X^{k-1}, U - X^{k-1}; G) \simeq 0 \) for \( i \neq k \). For \( i = k \), it isn’t too hard to see that the image of \( i_* \) consists precisely of those chains of the form described in the lemma (one can construct arbitrary such equivariant relative singular chains similarly to the construction of \( \tau_A \)).

The above lemmas allow one to replicate the usual proof that cellular homology, defined in terms of the relative groups \( H_\ast(X^\bullet, X^\bullet - 1) \), coincides with the singular homology groups; one uses the usual diagram chases, see for example [17] pg. 137. The identification of the relative chain groups \( H^P_\bullet(X^\bullet, X^\bullet - 1) \) with the chain groups of \( H^C_\bullet \) as given in the previous lemma completes the proof that \( H^P_\bullet(X; G) \simeq H^C_\bullet(X; G) \).

3.4. Pattern-Equivariant Homology of Tilings. We postponed giving an explicit description of a cellular representation \( \mathcal{C} \) for the pattern \( T_i \) (\( i = 0, 1 \)) of a cellular tiling of \( \mathbb{R}^d \) in examples [3.6] we shall provide one here. For two \( k \)-cells \( c_1, c_2 \) of the tiling \( T \), let \( \Phi \in (\mathcal{C}_i)_{c_1, c_2} \) if and only if \( \Phi \) is a translation taking \( c_1 \) to \( c_2 \) which also takes all of the tiles within distance \( R \) of every point of \( c_1 \) to the equivalent
patch at \( c_2 \) (of course preserving decorations of the tiles, if necessary). It is easy to see that \( \mathcal{C}_1 \) is a cellular representation of \( \mathcal{T}_1 \). Since \( |(\mathcal{C}_1|_{c_1,c_2}) = 0 \) or 1, it follows from the previous section that \( H^\theta_{\mathcal{T}_1}(\mathbb{R}^d; G) \cong H^\theta_{\mathcal{C}_1}(\mathbb{R}^d; G) \) for any coefficient group \( G \). Let two cells of the tiling \( c_1 \sim_R c_2 \) if \( |(\mathcal{C}_1|_{c_1,c_2}) \neq 0 \). A cellular \( k \)-chain \( \sigma \in \mathcal{C}_1 \) is a cellular Borel-Moore \( k \)-chain of the tiling with the following property: there exists some \( R \) such that if \( c_1 \sim_R c_2 \) then \( \sigma(c_1) = \sigma(c_2) \). We shall say that \( \sigma \) is a chain of \( \mathcal{C}_1 \) radius \( R \). Then the group of \( k \)-chains with \( \mathcal{C}_1 \) radius \( R \), denoted \( \mathcal{C}_1^{c:R}(\mathbb{R}^d; G) \), can be identified with an assignment of coefficient from \( G \) to each equivalence class of \( k \)-cell \([c]_R \).

A useful feature of this choice of \( \mathcal{C}_1 \) is that if \( \sigma \in \mathcal{C}_1^{c:R} \) then \( \partial(\sigma) \in \mathcal{C}_{k-1}^{c:R} \), so we may write \( H^\sigma_{\mathcal{T}_1} \cong \lim_{\rightarrow R} H^\sigma_{\mathcal{C}_1} \). We see from the above that these chain groups have the analogous description to the PE cohomology groups of the tiling [28], except with the cellular coboundary map replaced with the cellular boundary map.

Of course, we may analogously define a pattern \( \mathcal{C}_0 \), replacing translations by rigid motions. It is not necessarily the case that \( |(\mathcal{C}_0|_{c,d}) = 0 \) or 1 since some cells may have rotational symmetries preserving their patches of tiles. Also note that for \( \sigma \in \mathcal{C}_0^{c:R} \) here, we require that \( \sigma(c) = -\sigma(c) \) if there exists some \( \Phi \in \mathcal{C}_0^{c,c} \) reversing the orientation of \( c \). If \( |(\mathcal{C}_0|_{c,d}) \) may have order larger than 1 then it is not necessarily true that \( H^\mathcal{C}_0(\mathbb{R}^d; G) \cong H^\mathcal{C}_1(\mathbb{R}^d; G) \) for a general coefficient group \( G \). To obtain this isomorphism for a general coefficient group, one should use a finer CW-decomposition for the tiling. For example, for a tiling of \( \mathbb{R}^2 \), one should take a CW-decomposition for the tiling so that points of local rotational symmetry are contained in the 0-skeleton.

An alternative way of defining a cellular representation for a cellular tiling is to consider the patches of all tiles within an \( R \)-neighbourhood of a cell. Let \( T \) be a cellular tiling on \((X, d_X)\) with allowed partial isometries \( S \). Then \( \mathcal{T}_S \) has a cellular representation, with CW-decomposition given by the original tiling, where we set \( \Phi \in \mathcal{C}_0^{c,c} \) if and only if \( \Phi \) is a restriction of some \( \Phi^r \in (\mathcal{T}_S)^R_{x,y} \) for every \( x \in c \). Provided that the isometries fixing a cell are tame (and finite), it is easily checked that \( \mathcal{C} \) as so defined is a cellular representation for \( \mathcal{T}_S \).

3.5. Rotationally Invariant PE chains. Let \( T \) be an FLC (with respect to translations) cellular tiling of \( \mathbb{R}^d \). Suppose that a rotation group \( \Theta < SO(d) \) acts on the translation classes of patches of the tiling. That is, for any finite patch of the tiling, the rotate of that patch by \( \theta \in \Theta \) is a (translate of) a patch of the tiling. Then \( \Theta \) canonically acts on the chain groups \( \mathcal{C}_k^{c:R}(\mathbb{R}^d; G) \), which induces a well-defined action on \( \mathcal{C}_1(\mathbb{R}^d; G) \). We assume further that if two sufficiently large patches are equivalent up to a rigid motion, then they are so for some rotation in \( \Theta \) followed by a translation. Then the cellular \( \mathcal{T}_0 \) chain groups can be identified with those \( \mathcal{T}_1 \) chain groups which are invariant with respect to the action of \( \Theta \). Just as for PE cohomology, the \( \mathcal{T}_0 \) homology groups can be identified as the subgroups of the \( \mathcal{T}_1 \) homology groups of elements represented by rotationally invariant elements:

**Lemma 3.11.** Let \( G \) have division by \( |\Theta| \). Then the inclusion of chain complexes \( i: \mathcal{C}_0^{c}(\mathbb{R}^d; G) \to \mathcal{C}_1^{c}(\mathbb{R}^d; G) \) induces an injective map on homology.

**Proof.** Suppose that \( \tau \) is \( \mathcal{C}_1 \) and \( \sigma = \partial(\tau) \) is \( \mathcal{C}_0 \). Similarly as to in the proof of lemma [3.10] by “averaging” \( \tau \) one can show that there is a \( \mathcal{T}_0 \) chain \( \tau_A \) with the same boundary as \( \tau \).

In more detail, since \( \tau \) is \( \mathcal{T}_1 \), there is some \( R \) such that \( \tau \in \mathcal{C}_k^{c:R} \). Replace \( \tau \) by the chain \( \tau_A \), which assigns to the equivalence class \([c]_R \) the average \( \sum_{\theta \in \Theta} \tau(\theta[c]_R)/|\Theta| \). Then \( \tau_A \) is \( \mathcal{T}_0 \) since it assigns the same coefficient to any two cells \( c \sim_R d \). Further, since \( \theta_A \partial(\tau) = \partial(\tau) \) as
\( \partial(\tau) \) is \( T_0 \mathbb{E} \), we have that
\[
\partial(\tau_A) = \partial \sum_{\theta \in \Theta} \theta \cdot \tau / |\Theta| = \sum_{\theta \in \Theta} \theta \cdot \partial(\tau) / |\Theta| = \sum_{\theta \in \Theta} \partial(\tau) / |\Theta| = (|\theta| \sigma) / |\Theta| = \sigma.
\]

It follows that if \( i(\tau) \) is the boundary of a \( T_0 \mathbb{E} \) chain, then it is also the boundary of a \( T_0 \mathbb{E} \) chain, and so the induced map on homology is injective. \( \square \)

**Examples 3.12.** We discuss a simple periodic example, the periodic tiling of equilateral triangles of the plane. For periodic tilings, the (singular and cellular) \( T_1 \) and \( T_0 \)-equivariant homology groups will be the homology of the chain complexes of (singular and cellular) chains invariant under the action of translations and rigid motions, resp., preserving the tiling.

The cellular chain complex associated to the pattern \( T_1 \) for the periodic tiling of equilateral triangles (with the obvious CW-decomposition of \( \mathbb{R}^2 \)) is \( 0 \leftarrow \mathbb{Z} \leftarrow \mathbb{Z}^3 \leftarrow \mathbb{Z}^2 \leftarrow 0 \), as one has one vertex-type, three edge-types and two face-types, up to translation. One may compute that \( H_0^{T_1}(\mathbb{R}^2) \cong \mathbb{Z}, \ H_1^{T_1}(\mathbb{R}^2) \cong \mathbb{Z}^2 \) and \( H_2^{T_1}(\mathbb{R}^2) \cong \mathbb{Z}_2 \). In the next section, we shall see that we have Poincaré between the PE homology and cohomology of this pattern. The PE cohomology agrees with cohomology of the tiling space of \( T_1 \) which in this case is the 2-torus, agreeing with the (regraded) PE homology groups here.

Now consider the pattern \( T_0 \), for which chains should be invariant with respect to rigid motions. Due to complications with rotationally symmetric points, one must either choose to use coefficients with division by 6, or a finer CW-decomposition of \( \mathbb{R}^2 \) for which the rotationally symmetric points are contained in 0-cells. For the former case, using \( \mathbb{R} \)-coefficients, there is only one vertex and face-type. There are no generators associated to 1-cells; for such a cellular 1-chain \( \sigma \), the rotational symmetry of the 1-cells means that \( \sigma = -\sigma \), and hence is zero over \( \mathbb{R} \)-coefficients. It follows that \( H_0^{T_0}(\mathbb{R}^2; \mathbb{R}) \cong \mathbb{R}, \ H_1^{T_0}(\mathbb{R}^2; \mathbb{R}) \cong 0 \) and \( H_2^{T_0}(\mathbb{R}^2; \mathbb{R}) \cong \mathbb{R} \), which are precisely the rotationally invariant parts of \( H_1^{T_1}(\mathbb{R}^2; \mathbb{R}) \).

In the latter case, so that one can calculate the correct homology with \( \mathbb{Z} \)-coefficients, one must choose a finer CW-decomposition. A barycentric subdivision of the original complex is sufficient. One has three vertex and edges types and two face types. We calculate \( H_0^{T_0}(\mathbb{R}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_6, \ H_1^{T_0}(\mathbb{R}^2) \cong 0 \) and \( H_2^{T_0}(\mathbb{R}^2) \cong \mathbb{Z} \). To compute \( H_0 \), consider the map \( H_0 \to \mathbb{Z} \) which takes the value \( S = A + 2B + 3C \) on a class represented by assigning coefficients \( A, B, C \) to the vertices of rotational symmetry order 6, 3 and 2, respectively. It is easy to see that this is a well defined homomorphism (in fact, \( S \) can be seen as the (rescaled) “density” of the cycle, which must be invariant under boundaries). One may always represent such a class by one for which \( 0 \leq B < 3 \) and \( 0 \leq C < 2 \) and, given such a class, since \( A \) is determined by \( S, B \) and \( C \), one sees that the map sending this class to \( (S, [C]_2, [B]_3) \in \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \cong \mathbb{Z} \oplus \mathbb{Z}_6 \) is an isomorphism. The generator of \( \mathbb{Z} \) may be given as (1, 0, 0, 0), of \( \mathbb{Z}_2 \) as \((-3, 0, 1, 0)\) and of \( \mathbb{Z}_3 \) as \((-2, 1, 0, 0)\). Note that \( H_0^{T_0} \) doesn’t correspond to a subgroup of \( H_1^{T_1} \) generated by rotationally invariant elements, the torsion elements are nullhomologous only by non-rotationally-invariant \( T_1 \mathbb{E} \) boundaries.

This confirms our above calculation over \( \mathbb{R} \)-coefficients using the universal coefficient theorem. It seems common that torsion groups are picked up when one has two or more points of rotational symmetry in the pattern. Repeating the above calculations for the square tiling, one obtains \( H_0^{T_0} \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4, \ H_1^{T_0} \cong 0 \) and \( H_2^{T_0} \cong \mathbb{Z} \). Note that the spaces \( \Omega T_0 \cong S^2 \) in both cases, despite the different homologies (over \( \mathbb{Z} \)-coefficients) which seem to retain information about the different symmetries of the two tilings.

**3.6. Uniformly Finite Homology.** Much of the motivation for the definition of the PE chains here is derived from an effort to describe topological invariants of tiling spaces in a highly geometric way. As an aside, we shall briefly mention here a possible alternative route.
In [6], Block and Weinberger construct aperiodic tilings on non-amenable spaces (see also [23]). The idea is to construct so called “unbalanced tile sets”. This is made possible through the vanishing of certain homology groups, the (degree zero) uniformly finite homology groups. These homology groups are coarse invariants: if two metric spaces are coarsely quasi-isometric then these metric spaces have isomorphic uniformly finite homology groups.

It should be possible to construct the PE homology groups here from uniformly finite homology chains, instead of singular Borel-Moore chains. Indeed, given a tiling $T$ and a $\mathcal{T}E$ homology chain on it, under reasonable conditions, this chain may be considered as a uniformly finite homology chain by mapping a simplex to the image of its vertices and extending linearly. This defines a chain map into the uniformly finite homology chain complex, with image those chains with coefficients assigned in a way which “only depends on the local tile arrangement to some radius”. In fact, many of the uniformly finite chains constructed in [6] are of this nature: the “impurity” of a tiling determines a uniformly finite 0-chain which is defined by the local tiles, and this impurity is homologous to zero via the boundary of a uniformly finite 1-chain which is locally determined by the way pairs of tiles meet on their boundaries.

4. Poincaré Duality

For an oriented (not necessarily compact) manifold $M$ of dimension $d$ one has Poincaré duality $H^k(M) \cong H^{d-k}_{BM}(M)$, induced by taking the cap product with the fundamental class of $M$. In many cases of interest here, one may restrict this isomorphism to the pattern-equivariant groups.

Take for example the pattern $\mathcal{T}_1$ of a simplicial tiling of $\mathbb{R}^d$. There will exist a dual tiling which gives a pairing between the $k$ and dual $(d-k)$-cells which in turn induces an isomorphism between the PE complexes. The dual tiling will share the same PE cohomology to the original (since it is MLD to it), and so Poincaré duality follows.

We will show here the Poincaré duality isomorphism on the singular complexes by assuming the existence of, and then capping with, a PE fundamental class. It follows, for example, that in the cases where we obtain duality, one may view the cup product in cohomology as the intersection product in homology. For the pattern $\mathcal{T}_0$, a singular behaviour is apparent at the points of rotational symmetry, which often breaks the Poincaré duality and leads to extra torsion in the $\mathcal{T}_0E$ homology.

4.1. Singular Pattern-Equivariant Cohomology.

**Definition 4.1.** Let $C^\bullet_p(X)$ be the subcomplex of $C^\bullet(X)$ (of singular cochains on $X$) of pullback cochains i.e., $C^\bullet_p(X) := \varinjlim_{R \to R} \pi^*_R(C^\bullet(K_R))$, taken with the usual coboundary map. The cohomology of this cochain complex, $H^\bullet_p(X)$, is called the pattern-equivariant cohomology of $\mathcal{P}$.

Since $\pi^*_R \circ \delta = \delta \circ \pi^*_R$ we have that $\delta(\pi^*_R(C^\bullet(K_R))) = \pi^*_R(\delta(C^\bullet(K_R)))$ so that $C^\bullet_p(X)$ is a well-defined cochain complex and, defining $C^\bullet_{\mathcal{P};R}(X) = \pi^*_R(C^\bullet(K_R))$, we have that $C^\bullet_p(X) = \varinjlim_{\mathcal{P};R}(X)$. Let $U$ be a $\mathcal{P}$-set, that is $U = \pi^*_R(S)$ for some set $S \subset K_R$. Then we define $H^\bullet_p(U)$ by letting $C^\bullet_p(U) := \varinjlim_{R \to R} \pi^*_R(C^\bullet(\pi^*_1R(S)))$. Of course, given $\mathcal{P}$-sets $V \subset U$ we have the cochain map induced by inclusion $i^\bullet: C^\bullet_p(U) \to C^\bullet_p(V)$, and we may define the relative cochain groups $H^\bullet_p(U, V)$ in the usual way.

\footnote{That the tiling was simplicial was not essential here; if it wasn’t one may convert it to an MLD simplicial tiling using some (subdivision of an) associated Voronoi tiling.}
Lemma 4.2. Let $\mathcal{P}$ be a pattern and suppose that, for some unbounded sequence $R_1 < R_2 < \ldots$, each approximant $K_{R_i}$ is a CW-complex for which singular simplexes on $K_R$ with support contained in some cell lift to $(X, d_X)$. Then there is a canonical isomorphism $\tilde{H}^\bullet(\Omega_\mathcal{P}) \cong H^\bullet_P(X)$.

Proof. Denote by $\overline{C}_\bullet(K_R)$ the chain complex of singular chains on $K_R$ which lift to $X$. Since every singular chain on $K_{R_i}$ is homologous to a cellular chain we have that the inclusion of free abelian chain complexes $\iota^\bullet: \overline{C}_\bullet(K_{R_i}) \to \overline{C}_\bullet(K_R)$ is a quasi-isomorphism. By a result of homological algebra, the dual of this map $\iota^\bullet: C^\bullet(K_R) \to C^\bullet(K_{R_i})$ is also a quasi-isomorphism. Notice that this quasi-isomorphism is natural with respect to the connecting maps, in that $\iota^\bullet \circ \pi^\bullet_{R_{i+1}, R_i} = \pi^\bullet_{R_{i+1}, R_i} \circ \iota^\bullet$.

It is easily inspected that the restriction $\pi^\bullet_{R_i}: \overline{C}_\bullet(K_{R_i}) \to C^\bullet_P(X)$ of the usual pullback is a cochain map. It is also a cochain isomorphism: it is surjective (by definition) and is injective since each chain of $\overline{C}_\bullet(K_{R_i})$ lifts. It follows that $\pi^\bullet_{R_i} = \pi^\bullet_{R_i} \circ \iota^\bullet: C^\bullet(K_{R_i}) \to C^\bullet_P(X)$ is a quasi-isomorphism.

Since each $K_{R_i}$ is a CW-complex, we have that $\tilde{H}^\bullet(K_{R_i})$ is naturally isomorphic to $H^\bullet(K_{R_i})$. Since Čech cohomology is a continuous contravariant functor, we have that $\tilde{H}^\bullet(\lim_{\leftarrow} K_{R_i}) \cong \lim_{\leftarrow} \tilde{H}^\bullet(K_{R_i})$. It follows that

$$\tilde{H}^\bullet(\Omega_\mathcal{P}) \cong \lim_{\leftarrow} H^\bullet(K_{R_i}) \cong \lim_{\leftarrow} H^\bullet_{P; R_i}(X) \cong H^\bullet_P(X).$$

$\square$

Notice that the above isomorphism will extend to the ring structures on $\tilde{H}^\bullet(\Omega_\mathcal{P})$ and $H^\bullet_P(X)$.

Corollary 4.3. We have that $\tilde{H}^\bullet(\Omega_\mathcal{P}) \cong H^\bullet_{\mathcal{P}; X}(X)$ for $i = 0, 1, \text{rot}$.

Proof. Each $K_R$ is a BDHS complex \(^3\) which satisfies the above. Alternatively, one may use the Gähler approximants to describe the inverse limit $\Omega_\mathcal{P}$ (see \(^2\)). Clearly the pullback cochains are unchanged by using these approximants and it is trivial to see that they will satisfy the required properties of the $K_R$ of the above lemma. $\square$

Lemma 4.4. The cap product of a $\mathcal{P}E$ chain (with singular simplexes of support bounded by $\epsilon/2$) and cochain with $\mathcal{P}E$ radius $R$ is a $\mathcal{P}E$ chain (with $\mathcal{P}E$ radius $R + \epsilon$, for any $\epsilon > 0$).

Proof. Since $\mathcal{P}$ is a pattern we have that $\pi_{R, \epsilon} = \pi_{R, \epsilon} \circ \Phi$ on the open ball of radius $\epsilon$ at $x$ for all $\Phi \in \mathcal{P}_{x, y}^{R + \epsilon}$. Hence, if $x \sim_{R + \epsilon} y$ then $\Phi$ sends the front and back faces of $\sigma$ at $x$ to the front and back faces of $\sigma$ at $y$, resp., and since these chains have support bounded by $\epsilon/2$ the cochain will evaluate to the same value on each. Hence $\Phi$ sends the cap product at $x$ to the cap product at $y$. $\square$

The above shows that the cap product defined at the level of chains and cochains $\nearrow: C_p^{BM}(X) \times C^q(X) \to C_p^{BM}(X)$ restricts to a cap product on the pattern-equivariant complexes $\nearrow: C_p^\mathcal{P}(X) \times C_q^\mathcal{P}(X) \to C_p^{BM}(X)$. Of course, the usual formula $\partial(\sigma \nearrow \psi) = (-1)^q(\partial\sigma \nearrow \psi - \phi \nearrow \delta\psi)$ holds, so the cap product defines a homomorphism $\nearrow: H_p^\mathcal{P}(X) \times H_q^\mathcal{P}(X) \to H_p^{BM}(X)$ (and in fact makes $H_p^\mathcal{P}(X)$ a right $H_p^{BM}(X)$-module).

4.2. Poincaré Duality. For $D \subset X$, let $H^\bullet(X|D) = H^\bullet(X, X - D)$. The compactly supported cohomology of a space is defined by the cochain complex with cochain groups $C^\bullet_c(X) = \lim_{\leftarrow} C^\bullet(X|K)$ over all compact $K \subset X$, with inclusions as connecting maps. In other words, one restricts to those cochains which vanish outside of some compact set.
The classical proof of Poincaré duality (see, for example, [17]) is to show that capping with a fundamental class induces an isomorphism between compactly supported cohomology and homology, firstly for convex sets of \( \mathbb{R}^d \). One then uses commutative diagrams of Mayer-Vietoris sequences to “stitch together” these isomorphisms to deduce duality for general manifolds. Analogous arguments will often follow in the pattern-equivariant setting.

Let \( M \) be an oriented manifold. For \( K \subset L \subset M \) we have the following diagram:

\[
\begin{array}{c}
H_d(M|L) \times H^k(M|L) \\
\downarrow i_* \quad \downarrow i^* \\
H_d(M|K) \times H^k(M|K)
\end{array}
\]

\[
\cong \quad H_{d-k}(M)
\]

For each compact \( L \subset M \), one can show that there is a unique element \( \mu_L \in H_d(M|L) \) which restricts to some given orientation of \( M \) so that \( i_*(\mu_L) = \mu_K \). One has, by naturality of the cap product, that \( \mu_K \sim x = \mu_L \sim i^*(x) \). By varying \( K \) over the compact sets of \( M \), one has that the maps \( \sim \) induce in the limit the duality homomorphism \( D_M : H_c^k(M) \to H^{n-k}(M) \).

The main technicality of the proof of classical Poincaré duality is to show that there exists a commutative diagram of the form (see [17], lemma 3.36):

\[
\begin{array}{c}
\cdots \to H_c^k(U \cap V) \to H_c^k(U) \oplus H_c^k(V) \to H_c^k(M) \to H_c^{k+1}(U \cap V) \to \cdots \\
\downarrow D_{U \cap V} \quad \downarrow D_{U \oplus V} \quad \quad \quad \quad \downarrow D_M \quad \downarrow D_{U \cap V} \\
\cdots \to H_{d-k}(U \cap V) \to H_{d-k}(U) \oplus H_{d-k}(V) \to H_{d-k}(M) \to H_{d-k-1}(U \cap V) \to \cdots
\end{array}
\]

where \( M = U \cup V \) are manifolds. In more detail, one in fact has a diagram:

\[
\begin{array}{c}
\cdots \to H^k(M|K \cap L) \to H^k(M|K) \oplus H^k(M|L) \to H^k(M|K \cup L) \to \cdots \\
\downarrow \\
\cdots \to H_{d-k}(U \cap V) \to H_{d-k}(U) \oplus H_{d-k}(V) \to H_{d-k}(M) \to \cdots
\end{array}
\]

for compact sets \( K \subset U \) and \( L \subset V \) which in the limit produces the required diagram.

For the proof that the above exact sequence commutes, one uses excision for \( \hat{H}^* \), that \( K, L \) are closed and that the unique fundamental classes \( \mu_K \in H_d(M|K), \mu_L \in H_d(M|L) \) compatible with the orientation exist. Replacing the manifolds \( M = U \cup V \) with \( \mathcal{P} \)-open sets of \( X \) and subsets \( K \subset U, L \subset V \) with \( \mathcal{P} \)-closed sets of \( X \) such that the required fundamental classes exist, the analogous diagram commutes when restricting to \( \mathcal{P} \) chains and cochains; the proof is just as in the original, with the obvious alterations of terminology. In certain cases, Poincaré duality between \( \mathcal{P} \) homology and \( \mathcal{P} \) cohomology (“vanishing near the boundary” of open subsets) can be seen on a suitable basis of open sets. In such cases (see the next subsection), duality will follow replicating the usual proof of Poincaré duality. In this case, one may define the product of two chains as the Poincaré dual of the cup product of their duals. When the notion of transversality makes sense on \( X \), since the duality seen is simply the restriction of usual Poincaré duality, the product of two transverse chains may be alternatively viewed as the intersection product of those chains.
4.3. Poincaré Duality for Tilings. As discussed in the introduction to this section, there is a rather direct argument for Poincaré duality between $\mathcal{T}_1 E$ homology and cohomology, by considering dual tilings. We present here an alternative argument, by inducing the isomorphism by capping with a $\mathcal{T}_1 E$ fundamental class, which is analogous to the usual general proof of Poincaré duality. This approach makes clear the reason for Poincaré duality failing for the pattern $\mathcal{T}_0$, since it fails locally at points of rotational symmetry. This suggests an alternative complex, for which we do obtain duality.

**Theorem 4.5.** Let $T$ be an FLC tiling of $(\mathbb{R}^d,d_{\text{euc}})$ with respect to translations. Then we have Poincaré duality

$$H^{d-\bullet}_{\mathcal{T}_1 E}(\mathbb{R}^d) \cong H^\bullet_{\mathcal{T}_1 E}(\mathbb{R}^d)$$

induced by taking the cap product with a $\mathcal{T}_1 E$ fundamental class for $\mathbb{R}^d$.

**Proof.** One sees that a $\mathcal{T}_1 E$ fundamental class $\mu$ (that is, a fundamental class for the manifold $\mathbb{R}^d$ which is also $\mathcal{T}_1 E$) exists for $\mathcal{T}_1$ simply by considering the cellular version of $\mathcal{T}_1 E$ homology, for which its existence is obvious. For any $r \in \mathbb{R}$ and $z \in \mathbb{R}^d$, the set $S_{z,r}$ of points $y \sim_r z$ is a Delone set and, in particular, for sufficiently small $\epsilon > 0$, an $\epsilon$-neighbourhood $U$ of it is a $\mathcal{T}_1$-open set of disjoint $\epsilon$-balls around each element of $S_{z,r}$.

The relative cohomology of $(U,U_0)$, with $U_0$ a small $\mathcal{T}_1$-open neighbourhood of the boundary of $U$, will correspond to an assignment of coefficient to each $\epsilon$-ball in degree $d$ (and is trivial in other degrees). The relative $\mathcal{T}_1 E$ cohomology in degree $d$, on the other hand, will be precisely the subgroup consisting of an assignment of coefficient to each $x \in S_{z,r}$ for which there exists some $R$ such that, whenever $x \sim_R y$ in $S_{z,r}$, then $x$ and $y$ are assigned the same coefficient.

There exists a $\mathcal{T}_1 E$ fundamental class for the pair $(U,U_0)$ which agrees with the orientation given by $\mu$ on $\mathbb{R}^d$ (take a barycentric subdivision of $\mu$, if necessary). The groups $H^i_{\mathcal{T}_1 E}(U)$ will be trivial in degrees $i \neq 0$ (this follows easily from the fact that these groups are trivial in usual singular homology). The group $H^i_{\mathcal{T}_1 E}(U)$ will correspond to an assignment of coefficient to each $x \in S_{z,r}$ for which there exists some $R$ such that, whenever $x \sim_R y$, then $x$ and $y$ are assigned the same coefficient. Hence the (regraded) $\mathcal{T}_1 E$ homology groups have precisely the same description as the $\mathcal{T}_1 E$ cohomology groups, and the cap product with the fundamental class induces an isomorphism between them.

By limiting over shrinking neighbourhoods $U_0$ of the boundary, it follows that we have Poincaré between the $\mathcal{T}_1 E$ cohomology vanishing at the boundary of $U$ and the $\mathcal{T}_1 E$ homology of $U$. Taking a covering of such subsets for $\mathbb{R}^d$ such that Poincaré duality can be similarly checked for each, as well as their intersections, Poincaré duality follows using the usual arguments (see [17]), piecing together the duality isomorphisms using the Mayer-Vietoris sequences as seen in the last subsection. \hfill $\square$

An analogous argument to the above can also be used for the pattern $\mathcal{T}_{\text{rot}}$ defined on the Euclidean group $E^+(d)$. Example 3.12 (and aperiodic examples in the next section), however, show that the argument cannot extend to the pattern $\mathcal{T}_0$ (for integral coefficients). Consider an FLC tiling $T$ of $\mathbb{R}^2$ with $n$-fold symmetry at the origin. Then (for sufficiently large $R$) the map $\pi_R: U \to K_R$ from a small open disk $U$ of $x$ into the approximant $K_R$ corresponds to the map $re^{i\theta} \mapsto re^{ni\theta}$ onto its image, with singular point at the origin. Hence, one may only realise $n$ times the usual generator of the compactly supported cohomology at $U$ in $\mathcal{T}_0 E$ cohomology. For the $\mathcal{T}_0 E$ homology, however, we may still realise the usual generator, by marking those points which are equivalent to radius $R$ to $x$. Hence, the duality map here will have cokernel $\mathbb{Z}/n\mathbb{Z}$ over $\mathbb{Z}$-coefficients.
The above discussion shows how we may retain duality by modifying the $T_0E$ homology groups. When the tiling has symmetry to radius $R$ at a point $x$ of order $n$, one should allow only rotationally invariant chains of $T_0E$ radius $R$ there to have coefficient divisible by $n$.

**Definition 4.6.** Let $\sigma$ be an element of the *modified singular pattern-equivariant chain group* $C^P_k(X)$ if and only if $\sigma$ is a $PE$ singular $k$-chain with $PE$ radius $R$ for which, for each $x \in X$, we have that $\sigma^x = \sum_{\Phi \in P_k x} \Phi(\tau)$ for some singular chain $\tau$. With the standard boundary map we thus define the *modified singular pattern-equivariant homology groups* as the homology groups $H^P_k(X)$ of this chain complex.

Equivalently, we allow a $PE$ chain $\sigma$ to be in the modified $PE$ chain group if the coefficient of some singular simplex $\tau$ of $\sigma$ is divisible by the order of the subgroup $F \leq P^E_{x,x}$ fixing that singular simplex. Similarly defining the *modified cellular pattern-equivariant groups*, so that $PE$ chains of radius $R$ are such that the coefficients assigned to cells $c$ are divisible by $|C^R_{c,c}|$, one may verify that the results of section 3 follow in their modified form, with only minor modifications to the proofs.

Returning to the example of a pattern $T_0$ associated to an FLC tiling of $\mathbb{R}^2$, by considering the modified $T_0$ homology groups, we see that one removes the “singular behaviour” of chains at rotationally symmetric points. Following the above proof of Poincaré duality, we now see that we have Poincaré duality $H^{d-k}_0(\mathbb{R}^2) \cong H^k_0(\mathbb{R}^2)$.

Let the tiling $T$ be cellular and in a way such that each point of local rotational symmetry is contained in the 0-skeleton. There is an obvious inclusion of cellular $T_0E$ complexes $C^\bullet_0 \to C^\bullet_0$. The modified cellular chain complex differs from the unmodified one only in degree zero: a cellular 0-chain in the modified complex should be a $T_0E$ cellular 0-chain for which there exists some $R$ such that, if there is some rotation of order $n$ preserving some 0-cell and its patch of tiles to radius $R$, then the coefficient assigned to that cell should be divisible by $n$. Consider the exact sequence of cellular chain complexes

$$0 \to C^\bullet_0 \to C^\bullet_0 \to C^\bullet_0 / C^\bullet_0 \to 0.$$ 

The quotient homology groups have a simple description:

**Proposition 4.7.** The above quotient homology groups are trivial in degrees not equal to zero and the degree zero group is isomorphic to $\bigoplus_p \mathbb{Z}^{T(p)}$ where the direct sum is taken over all primes $p \in \mathbb{N}$ with $T(p)$ equal to the sum $\sum_{T_i} n_i$ over all tilings $T_i \in \Omega_{T_0}$ with rotational symmetry $k.p^n$ ($p \nmid k$) in case this sum is finite and $T(p) = |\mathbb{N}|$ otherwise.

**Proof.** Since the groups $C^T_0$ and $C^T_0$ have the same elements for $k \neq 0$, it is clear that the quotient homology groups are trivial for $k \neq 0$ and that the degree zero homology group is naturally isomorphic to $C^T_0 / C^T_0$.

The tiling space $\Omega_{T_0}$ consists of tilings (modulo rotation) which are *locally isomorphic* to $T$, that is, they are those tilings which have identical finite patches to $T$, up to rigid motion. If there are finitely many such tilings with rotational symmetry, the proposition is clear: each rotationally invariant tiling $T_i$ defines a generator of $C^T_0 / C^T_0$ which is given by assigning coefficient one to each vertex of $T$ whose patch of tiles agrees with the central patch of $T_i$ to radius $R$ (and every other vertex is assigned coefficient zero), where $R$ is chosen so that the patches of tiles of each $T_i$ to radius $R$ are pairwise distinct up to rotation.

Where there are infinitely many rotationally invariant tilings, such an isomorphism is less obvious. By FLC, there are only finitely many integers $n > 1$ for which there are tilings $T_i \in \Omega_{T_0}$...
with \(n\)-fold symmetry. We see that \(C^{T_0}_0 / C^{T_0}_0\) is a countable abelian group of bounded exponent. It follows from Prüfer’s first theorem \([13]\) that \(C^{T_0}_0 / C^{T_0}_0\) is isomorphic to a countable direct sum of cyclic groups \(\bigoplus_p \mathbb{Z}_p^{r(p)}\). This group is isomorphic to \(\bigoplus_p \mathbb{Z}_p^{T(p)}\) if and only if \(r(p) = T(p)\) for each prime, which follows from direct consideration of the group \(C^{T_0}_0 / C^{T_0}_0\).

Let \(f_1, \ldots, f_k\) be the list of primes appearing as a divisor of the order of rotational symmetry for only finitely many tilings \(T_i \in \Omega_{T_0}\). Let \(R\) be such that the tilings having such symmetries are distinguishable to radius \(R\). Then the subgroup generated by elements which assign coefficient \(n.c\) with \(n \in \mathbb{Z}\) to a patch (of radius larger than \(R\)) with rotational symmetry \(c \cdot f_1^{a_1} \cdots f_k^{a_k}\) (with \(f_i \not| c\)) corresponds precisely to the direct sum of the \(f_i\) parts of the group, which confirms that \(r(f_i) = T(f_i)\). For the other primes, we have that \(r(p) = |N|\) when there are infinitely many tilings with rotational symmetry \(p.k\) since one may easily construct infinitely many distinct elements of order \(p\). Clearly there are no elements of order \(p\) in \(C^{T_0}_0 / C^{T_0}_0\) whenever \(T(p) = 0\), so the result follows.

It follows from the above that \(H^T_0(\mathbb{R}^d) \cong \hat{H}^{2-i}(\Omega_{T_0})\) for \(i = 1, 2\), and in degree zero the \(\mathcal{T}_0\)E homology fits into the short exact sequence:

\[
0 \to \hat{H}^2(\Omega_{T_0}) \to H^0_0(\mathbb{R}^d) \to \bigoplus_p \mathbb{Z}^{T(p)}_p \to 0.
\]

A similar statement will hold for two-dimensional hierarchical tilings. In higher dimensions, one may expect the quotient homology groups to be non-trivial in larger dimensions. For example, a tiling of \(\mathbb{R}^3\) may be invariant under rotations fixing a one-dimensional line, which will lead to non-trivial elements of \(C^{T_0}_1 / C^{T_0}_1\).

5. HOMOLOGY GROUPS OF HIERARCHICAL TILINGS

Given an FLC substitution tiling \(T\) with substitution map \(\omega\), its PE cohomology groups can be calculated using the fact that the tiling space is the inverse limit of the BDHS \([3]\) or (collared) Anderson-Putnam \([2]\) complexes along with their self-maps induced by \(\omega\). We give here an alternative method for carrying out these computations using PE homology.

We shall set a coefficient group \(G\) and suppress its notation in the homology groups in this subsection. We shall assume throughout that \(T_\omega = \{T_0, T_1, \ldots\}\) is an FLC hierarchical tiling of \((X, d_X)\), with respect to the allowed partial isometries \(S\), which has the following properties:

- \((X, d_X)\) is homeomorphic to \(\mathbb{R}^d\) for some \(d \in \mathbb{N}\).
- Each \(T_i\) is a cellular tiling for a common CW-decomposition \(X^*\) of \((X, d_X)\).
- Each tile is homeomorphic to a closed \(d\)-disk.

Each of the tilings \(T_i\) share the same CW-decomposition of \((X, d_X)\). To apply the main theorem of this section, we shall need the cells of \(X^*\) to get relatively small compared to the tiles of \(T_i\) as \(i \to \infty\) (c.f., the tiles of a substitution tiling relative to the supertiles). We assume that for each \(i\) the tiles of \(T_i\) are given an alternative CW-decomposition, making each \(T_i\) a cellular tiling with this new CW-decomposition of the tiles, and in a way such that the \(k\)-skeleton of the induced CW-decomposition \(T_i^k\) is contained in the original \(k\)-skeleton \(X^k\) (for practical computations we shall usually want CW-decompositions which are “combinatorially equivalent” in a certain sense for each \(T_i\)).

**Definition 5.1.** For a cell \(c\) of \(T^k_i\) let \(P(c)\), the *incident patch of \(c)*, be the patch of tiles in \(T_i\) which intersect the open cell of \(c\). Given two \(k\)-cells \(c_1, c_2\) of \(T^k_i\), let them be considered as equivalent (via the equivalence \(\Phi\)) if \(\Phi\) is an isometry from \(P(c_1)\) to \(P(c_2)\) sending \(c_1\) to \(c_2\).
(and preserving decorations of tiles) which when restricted to the interior of the incident patch of \(c_1\) is an element of \(S\). We call a class of \(k\)-cell under these equivalences a \textit{k-cell type}. For \(k = 0, 1, 2\) we call \(k\)-cell types \textit{vertex}, \textit{edge} and \textit{face types}, respectively.

For example, in \((\mathbb{R}^2, d_{euc})\) with \(S\) given by (restrictions of) translations, a vertex type consists of a 0-cell along with all of the tiles which contain it, taken up to translation. An edge type consists of a 1-cell along with those tiles intersecting its interior, taken up to translation. A face type is a 2-cell of the tiling along with the tile it is contained in, taken up to translation. Notice that the composition of an equivalence \(\Phi_1: P(c_1) \to P(c_2)\) from \(c_1\) to \(c_2\) with an equivalence \(\Phi_2: P(c_2) \to P(c_3)\) from \(c_2\) to \(c_3\) is an equivalence from \(c_1\) to \(c_3\). For each \(k\)-cell type \([c]\) there is a well-defined number \(n_{[c]}\), which is defined as the number of distinct equivalences of \(c\) with itself when restricted to \(c\), where \(c\) is any representative of \([c]\). For example, \(n_{[v]} = 1\) for any vertex type \([v]\). We shall usually want \(n_{[c]} = 1\) for each \(k\)-cell type \([c]\); for example, for a tiling of \((\mathbb{R}^2, d_{euc})\) with \(S\) given as (restrictions of) rigid motions (which, here, are orientation-preserving isometries), this is satisfied if all points of local rotational symmetry in the tiling are contained in the 0-skeleton.

The \(k\)-cell types induce a subcomplex of the cellular \(T_\ast\)E chain complex. We denote by \(C^{A_i}_\ast\), the \(i\)th approximant chain complex, the chain complexes of cellular Borel-Moore chains of \(T^\ast_i\) which are invariant under equivalences of \(k\)-cell types. One may naturally identify a cellular \(k\)-chain of \(T^\ast_i\) with the obvious cellular chain of the (usually) finer CW-decomposition \(X^\ast\), the cellular chains here will always be assumed to be of the original CW-decomposition \(X^\ast\). Given \([c] \in H^{A_i}_k\), we would like to say that it only depends on the local \(k\)-cell types of \(T_{i+1}\) also. Unfortunately, the simple inclusion map isn’t necessarily cellular since the CW-decompositions \(T^k_i\) vary as \(i\) increases. However, we can always take a “nice” boundary which takes \(\tau\) to a cellular chain of \(H^{A_{i+1}}_k\):

\begin{proposition}
Suppose that the coefficient group \(G\) has division by \(n_{[c]}\) for every \(k\)-cell type \([c]\) of each \(T_i\). Let \([\sigma] \in H^{A_i}_k\). Then there exists some cellular Borel-Moore \((k + 1)\)-chain \(\tau\) such that \([\sigma + \delta(\tau)] \in H^{A_{i+1}}_k\). The chain \(\tau\) may be chosen so that \(\tau(c_1) = \tau(c_2)\) for any two \((k + 1)\)-cells which are isometric via a restriction of an equivalence of \((k + 1)\)-cells of \(T^{k+1}_{i+1}\).

Further, given any other such choice \(\tau'\) in replacement for \(\tau\) satisfying this property, we have that \([\sigma + \tau] = [\sigma + \tau'] \in H^{A_{i+1}}_k\).
\end{proposition}

\begin{proof}
If \(\sigma \in H^d_k\) then it must be equal to some multiple of the fundamental class of \(\mathbb{R}^d\) and so the claim is immediate. So let \(\sigma \in H^d_k\) with \(k < d\). Away from \(T^{d-1}_{i+1}\), the coefficients assigned to \(k\)-cells by \(\sigma\) are invariant under equivalences of \(d\)-cell types. Hence, we may choose a cellular \((k + 1)\)-chain which takes \(\sigma\) to \(T^{d-1}_i\) by making such a choice for each \(d\)-cell type. If a \(d\)-cell type possesses non-trivial symmetries, we can ensure that the boundary chain chosen is invariant with respect to them by “averaging” the chain over the equivalences (c.f, the proof of lemma \[\ref{3.10}\]). We may continue this process until \(\sigma\) is made homologous to a chain contained in \(T^k_{i+1}\).

By construction, \(\tau\) assigns coefficients in a way which only depends on the local \((k + 1)\)-cell types up to equivalence and \(\sigma + \delta(\tau) \in C^{A_{i+1}}_k\).

Take a possibly different such choice \(\tau'\) of \((k + 1)\)-chain satisfying the above and consider the chain \(\tau - \tau'\). Of course, \(\tau - \tau'\) has trivial boundary and, as above, \(\tau - \tau'\) is homologous to a \((k + 1)\)-chain \(\rho\) contained in \(T^{k+1}_{i+1}\) which is invariant under equivalences of \((k + 1)\)-cell types of \(T^k_{i+1}\). But then \(((\sigma - \partial(\tau')) - (\sigma - \partial(\tau)) = \partial(\tau - \tau') = \partial(\rho)\) and so is homologous to 0 in \(H^{A_{i+1}}_k\).
\end{proof}
The above proposition induces a well-defined homomorphism \( \omega^i_k : H^{A_i}_k \rightarrow H^{A_i}_{k+1} \), which we shall name the substitution map.

It will be useful in the proof of the theorem below to consider singular and cellular PE chains. We define cellular PE chains for \( T_i \) (with respect to the allowed partial isometries \( S \) and CW-decomposition \( X^\bullet \)) analogously as to in the discussion in subsection 3.4. A cellular Borel-Moore \( k \)-chain will be consider \( T_\omega E \) if it is \((T_\omega(R))_SE\) to radius \( R \) where \( k : \mathbb{R}^+ \rightarrow \mathbb{N}_0 \) is a non-decreasing unbounded function (the choice of such a \( k \) is unimportant). By the results of section 3 (given conditions on the divisibility of the coefficient group \( G \) with respect to the local symmetries of cells), there is a canonical isomorphism between the singular and cellular PE homology groups.

Now, note that each element of \( C_{\omega}^{A_i} \) may be considered as a cellular \( T_\omega E \) chain, since it has as elements chains which depend only on their incident patches of tiles in \( T_i \). Since each of the substitution maps is defined by an inclusion followed by a \( T_\omega E \) boundary, and each chain of \( C_{\omega}^{A_i} \) is a \( T_\omega E \) cellular chain, we have a homomorphism \( f : \lim_{\omega \downarrow} H^{A_i}_\bullet \rightarrow H^{T_\omega}_{\bullet}(X) \). Under reasonable conditions, this map is in fact an isomorphism:

**Theorem 5.3.** Let \( T_\omega \) be a hierarchical tiling satisfying the above conditions. Suppose that the coefficient group \( G \) has division by \( n_\omega \) for each \( k \)-cell type \([c] \) of each \( T_i \) and that, for each \( R > 0 \), for sufficiently large \( i \) there exist deformation retractions \( F^p_i : [0,1] \times A^p_i \rightarrow A^p_i \) for \( 0 \leq p \leq d-1 \) of a neighbourhood \( A^p_i \subset T^{p+1}_i \) of the \( p \)-skeleton \( T^p_i \) onto \( T^p_i \) satisfying the following:

1. Each \( F^p_i \) is equivariant with respect to equivalences of \( p \)-cell types. That is, \( A^p_i \) and \( F^p_i \) are defined cell-wise, for each \( p \)-cell type, and are invariant under self equivalences of a \( p \)-cell type onto itself.

2. Points of \( T^{p+1}_i - A^p_i \) have \( R_p \)-neighbourhoods intersecting only those tiles in the incident patch of the cell containing the point. We have that \( R_{d-1} = R \) (for \( R_p \) with \( p \neq d \) see below).

3. For each \( F^p_i \) we have that \( \sup\{d_X(F^p_i(t,x),F^p_i(t',x)) \mid t,t' \in [0,1] \} < D_p \) for some \( D_p > 0 \).

4. \( D_p \geq R_p \) and \( R_{p-1} \geq 3D_p + 2\epsilon \) for some \( \epsilon > 0 \).

Then there exists an isomorphism:

\[
\lim_{\omega \downarrow} H^{A_i}_\bullet \cong H^{T_\omega}_{\bullet}(X).
\]

**Proof.** By our assumptions on the divisibility of the coefficient group \( G \) and the results of section 3, we have canonical isomorphisms between the singular and cellular PE homology groups of \( T_i \) and \( T_\omega \). Let \([\sigma] \in H^{T_\omega}_{k}(X)\) be represented by the singular \( T_\omega E \) chain \( \sigma \). Then there is some \( T_j \) for which \( \sigma \) is \( T_j E \) to radius \( R \). Pick \( i \geq j \) for which \( T_i \) has the above deformation retractions for value \( R \).

By assumption (2), away from \( A^{d-1}_i \) we have that \( \sigma \) is invariant up to equivalences of \( d \)-cell types since the chain is \( T_\omega E \) to radius \( R \). Suppose that \( \sigma \) is a \( k \)-chain with \( k < d \) (and, without loss of generality due to lemma 3.2 has singular simplexes of small radius of support). Then one may choose a boundary \( \rho \) with \( \sigma + \partial(\rho) \subset A^{d-1}_i \) for which \( \rho \) is invariant under equivalences of \( d \)-cell types (if necessary we average the chain over self-equivalences of \( d \)-cell types, c.f., the proof of lemma 3.10). Again, by lemma 3.2 we may assume that the resulting cycle in \( A^{d-1}_i \) has singular simplexes with radius of support bounded by \( \epsilon/2 \). As in lemma 3.8 the deformation retraction induces a boundary taking \( \sigma \) to a chain contained in \( T^{d-1}_i \). We have that \( \Psi \circ F_i^{d-1} = F_i^{d-1} \circ \Psi \) for equivalences \( \Psi \) between \( d \)-cell types, and so for any \( \Phi : B_{d_X}(x, 2D + \epsilon) \rightarrow B_{d_X}(x, 2D + \epsilon) \) which is a restriction of equivalences of \( d \)-cell types we
have that $\Phi \circ F_{i}^{d-1} = F_{i}^{d-1} \circ \Phi$ on $B_{d\chi}(x, D + \epsilon)$. As in lemma 3.8, we see that $\sigma$ is homologous to a chain contained in $T_{i}^{d-1}$ which only depends on the $d$-cell types to radius $3D_{d-1} + 2\epsilon$.

By assumption (4) we may inductively repeat the above process until $[\sigma]$ is represented by a chain contained in $T_{i}^{k}$ which only depends on the local $d$-cell types to radius $3D_{k} + 2\epsilon$. Since, away from $A_{i}^{k-1}$, the chain is locally determined by $k$-cell types, it follows that $[\sigma]$ is represented by a cellular chain of $T_{i}^{k}$ which is invariant under equivalences of $k$-cell types. Hence, the map $f$ is surjective.

Showing injectivity of the map is similar to the above. Let $\sigma$ be a $k$-cycle of the direct limit for which $\sigma = \partial(\tau')$ for some $T_{i}^{k}E$ $(k + 1)$-chain $\tau'$. Then $\tau'$ may be represented as a singular chain which is $T_{j}^{k}E$ to radius $R$. Pick $i \geq j$ for which $T_{i}$ has the above deformation retractions for value $R$.

As in the construction of the substitution map, we may find a boundary $\rho$ which only depends on incident patches of $(k + 1)$-cells and makes $\sigma$ homologous to a chain contained in $T_{i}^{k}$. Similarly to above, one can show that $\tau := \tau' + \rho$ is represented by a chain of the direct limit: one inductively pushes $\tau$ to the $(k + 1)$-skeleton ending with a $T_{i}^{k}E$ chain contained in $T_{i}^{k+1}$ which, away from $A_{i}^{k+2}$, only depends on local $(k + 1)$-cells up to equivalences, and so the result follows.

The assumption of the existence of the deformation retractions in the above theorem is analogous to the assumption of the tameness of the isometries in definition 3.3 for cellular patterns. We require here, though, that the deformation retractions can be taken so as to retract arbitrarily large neighbourhoods of the skeleta. In particular the cells of $T_{i}^{k}$ should grow arbitrarily large in diameter as $i \to \infty$. This is a key assumption for the above theorem: chains whose coefficients may depend on a large diameter of tiles in some $T_{i}$ may be seen as chains whose coefficients only depend on a small diameter of tiles in some $T_{i}$, relative to the size of the tiles.

For a cellular substitution tiling of $(\mathbb{R}^{d}, d_{euc})$, with allowed partial isometries given by translations or rigid motions, the above theorem makes the $T_{i}^{k}E$ homology groups computable since the approximant homology groups are then all isomorphic and in a way such that the induced substitution maps between them are the same; see the examples below. Hence, by Čech duality, this gives an alternative way of computing the Čech cohomology groups $\tilde{H}^{d-\bullet}(\Omega_{T_{i}}) \cong H^{d-\bullet}_{C}(\mathbb{R}^{d}) \cong H^{d}_{*}(\mathbb{R}^{d})$ when we take (restrictions of) translations as the allowed partial isometries. When we take our allowed partial isometries to be given by (restrictions of) rigid motions of $\mathbb{R}^{d}$, we don’t in general obtain duality with the Čech cohomology $\tilde{H}^{*}(\Omega_{T_{i}})$ (over non-divisible coefficients). One may however use the modified homology groups, as in the discussion at the end of section 4, using only multiples of chains at rotationally invariant cells. The above methods may be adjusted in the obvious way so as to compute the modified homology groups, which are Poincaré dual to the PE cohomology groups.

It is interesting to note that the duals of the approximant complexes have been considered in the work of Gonçalves [16]. There they were incorporated into a method for computation of the $K$-theory of the stable equivalence relation associated to a substitution tiling. The calculations indicate that there is a duality present between the $K$-theory of the stable and unstable equivalence relations (which for low dimension tilings correspond simply to direct sums of the Čech cohomology groups of the tiling space). The appearance of these approximant complexes here indicate that this is indeed the case, although extra analysis of the substitution maps is needed to confirm the relationship.

5.1. Examples.
5.1.1. The Fibonacci Tiling. The Fibonacci tiling of $\mathbb{R}^1$ is given as a substitution on two intervals, named here 0 and 1 of lengths the golden ratio $\phi$ and one, resp., and the substitution $0 \mapsto 01$, $1 \mapsto 0$ (see figure 5.1). We have 2 edge types, corresponding to the two types of tiles, and 3 vertex types (denoted by 0.1, 1.0 and 0.0) up to translation. We set an orientation of our tiles all pointing to the right, so that the boundary of the 0 tile is the formal sum $0.1 - 1.0$ (since the 0 vertex can by found to the right of the 0 tile, 1.0 to the left) and similarly the boundary operator maps the 1 tile to the formal sum $1.0 - 0.1$.

Hence, the approximant homology is $H_0^A = \mathbb{Z}^2$, $H_1^A = \mathbb{Z}$, where $H_0^A$ is generated by $a = 0.1$ and $b = 0.0$ (0.1 and 1.0 are identified in homology) and $H_1^A$ (as usual) is generated by the fundamental class of $\mathbb{R}^1$ by assigning coefficient one to each edge type.

For the substitution map $\omega_0: H_0^A \to H_0^A$, note that all 0.1 vertices are found in the interior of 1 tiles (distance $\phi$ from the left endpoint of the interval) and all 0 vertices lie on a 1.0 vertex in the supertile decomposition of the tiling. The PE chain marking the interior point of the 0 tiles is homologous to the formal sum $0.0 + 0.1$ (i.e., the sum of all vertex types with the 0 tile to the left, see figure 5.1) and the 1.0 vertex type is homologous to 0.1 as above. Hence, $\omega_0(a, b) = (a + b, a)$, which is an isomorphism so $H_0^{T_\omega} = \lim_{\omega_0} \mathbb{Z}^2 = \mathbb{Z}^2$ and, as is usual, $H_1^{T_\omega} = \mathbb{Z}$ is generated by the fundamental class for $\mathbb{R}^1$ given by assigning coefficient one to each edge type.

5.1.2. The Thue-Morse Tiling. The Thue-Morse tiling of $\mathbb{R}^1$ is given as a substitution on two intervals 0 and 1 each of unit length, and substitution given by $0 \mapsto 01$, $1 \mapsto 10$. There are 2 edge types, 0 and 1, and 4 vertex types 0.0, 0.1, 1.0 and 1.1 up to translation. The boundary map will again send the 0 tile to the formal sum 0.1-1.0 and the 1 tile to 1.0-0.1, so that $H_0^A = \mathbb{Z}^3$.

In the supertile decomposition, vertices of tiles can be found either on the vertices of the supertile decomposition, or in the centre of the supertiles. Applying the substitution map to the vertex types generating $H_0^A$, we see that $\omega_0(0.1) = 1.1 + \bar{0}$, $\omega_0(0.0) = 1.0$ and $\omega_0(1.1) = 0.1$ where 0 is the PE chain marking the centres of the 0 tiles. Since $\bar{0} \sim 0.1 + 0.0$ and 1.0 $\sim 0.1$, we have the $\omega_0(a, b, c) = (a + b + c, a, a)$. This linear map has eigenvectors with eigenvalues 0,-1 and 2, although they do not span $\mathbb{Z}^3$. With some further careful analysis, the direct limit can be identified as $H_0^{T_\omega} = \lim_{\omega_0} \mathbb{Z}^3 = \mathbb{Z} \oplus \mathbb{Z}[1/2]$.

5.1.3. The Dyadic Solenoid. One can put a hierarchical structure onto the periodic triangle tiling analogously to example 2.8 from the substitution of the triangle into four of half the size. The approximant homology with respect to translations has already been computed and is the homology of the periodic tiling without the added hierarchical structure. It is easy to compute the induced substitutions $\omega_\uparrow$ as the times 4.2 and 1 maps in degree 0.1 and 2, respectively. Hence, we have that $H_0 \cong \mathbb{Z}[1/4]$, $H_1 \cong \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/2]$ and $H_2 \cong \mathbb{Z}$ which, via Poincaré duality, agrees with the observation that $\Omega_{T_\uparrow} \cong D_2^2$, the two-dimensional dyadic solenoid.
One may also compute the homology with respect to rigid motions. Again, the approximant homology will be as for the homology of the triangle tiling, \( H_0 \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \) \( H_1 \cong 0 \) and \( H_2 \cong \mathbb{Z}. \) One finds that, with the generators as described in example 3.12, \( \omega_0(a, b, c) = (4a, a + b, c). \) This map is conjugate via an automorphism of \( \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \) to the map sending \( (a, b, c) \mapsto (4a, b, c) \) and hence the PE homology is \( H_0 \cong \mathbb{Z}[1/4] \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \) \( H_1 \cong 0 \) and \( H_2 \cong \mathbb{Z}. \) One may similarly compute the homology with respect to rigid motions for the square model of the dyadic solenoid; the substitution map \( \omega_0: \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \rightarrow \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4 \) maps \( (a, b, c) \mapsto (4a, c, c) \), so \( H_0 \cong \mathbb{Z}[1/4] \oplus \mathbb{Z}_4. \)

5.1.4. The Penrose Tiling. We shall compute the PE homology of the Penrose Kite and Dart Tilings with respect to rigid motions. There are 7 vertex types (sun, star, ace, deuce, jack, queen and king, in Conway’s notation) and 7 edge types \((E_1 - 7)\) up to rigid motion, see figure 5.2 where each of the vertex and edge types are listed in their respective orders as listed. The two 2 face types correspond to the Kite and Dart tiles on which there is defined the usual kite and dart substitution.

The \( \partial_1 \) boundary map, with ordered bases as listed above, is represented as a matrix by:

\[
\begin{pmatrix}
5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -5 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & -1 & 0 & 0 & 1 \\
1 & 0 & 1 & -1 & -1 & -1 & 0 \\
-1 & 0 & 0 & 0 & 1 & 1 & -2 \\
0 & -2 & 0 & 0 & 1 & 1 & -1
\end{pmatrix}
\]

The first column, for example, is \((5, 0, -1, 0, 1, -1, 0)^T\) since at any sun vertex there are 5 incoming \(E_1\) edges, at an ace there is an outgoing \(E_1\), at a jack there is an incoming \(E_1\) and at a queen there is an outgoing \(E_1\).

Some simple calculations using the Smith normal form show that \( H_0^A \cong \mathbb{Z}^2 \oplus \mathbb{Z}_5, \) with basis of the free abelian part given by \( e_1 = \) sun and \( e_2 = \) star and the torsion part generated by \( t = \) sun + star − queen. The torsion element is illustrated in figure 5.3, we have that \( 5t + \partial_1(-E_1 + E_2 - E_4 - 2E_7) = 0. \)

For the \( \omega_0 \) substitution map, we first note that each sun vertex lies on one of a star, queen or king vertex in the supertiling, each star lies on a sun and each queen lies on a deuce. Hence, up to homology, we compute that \( \omega_0(e_1) \simeq 3e_1 - e_2 + 2t, \omega_0(e_2) \simeq e_1 \) and \( \omega_0(t) = t. \) We see that the \( \omega_0 \) is an isomorphism on \( H_0^A \) and hence \( H_0^{T_0} \cong \mathbb{Z}^2 \oplus \mathbb{Z}_5. \)
Figure 5.3. Torsion Element $t$ with $5t + \partial_1(-E1 + E2 - E4 - 2E7) = 0$.

The degree one group $H^1_A$ is generated by $E3 + E4$, that is, the cycle which trails the bottom of the dart tiles (with appropriate orientation). This is illustrated in figure 5.4 in red. The analogous 1-cycle trailing the bottoms of the superdarts in the supertiling is illustrated in green. It is evident from the figure that the two cycles (with opposite orientations) are homologous via the locally defined 2-chain given as the boundary of the dart tiles. We see that $\omega_1$ is an isomorphism on $H^1_A$ and so $H^1_T \cong \mathbb{Z}$. As usual, $H^2_T \cong \mathbb{Z}$ is generated by a fundamental class for $\mathbb{R}^2$ by assigning coefficient one to each 2-cell.

Every star and sun vertex has five-fold rotational symmetry. One may consider the modified chain complexes, where one allows only multiples of 5 for the coefficients of the star and sun vertices. One replaces the boundary map $\partial_1$ with the modified version which has entries 1 and $-1$ instead of 5 and $-5$ for the edge types terminating and initiating from the star and sun vertex types, respectively. One may repeat the above calculations for these modified groups. The resulting homology groups are Poincaré dual to the Čech cohomology groups $\check{H}^\bullet(\Omega_0)$ and, indeed, we calculate that $H^0_\check{T} \cong \mathbb{Z}^2$, $H^1_\check{T} \cong \mathbb{Z}$ and $H^2_\check{T} \cong \mathbb{Z}$. As in the discussion at the end of section 4, the modified and unmodified groups fit into an exact sequence which at degree zero is

$$0 \to \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z}^2 \oplus \mathbb{Z}_5 \xrightarrow{g} \mathbb{Z}_5^2 \to 0$$

where $f(a, b) = (a + 2b, a - 3b, [4a + 3b]_5)$ and $g(a, b, [c]_5) = ([a + c]_5, [b + c]_5)$. 
5.1.5. **Point Patterns on a Free Group.** Consider the free group on two generators \( F(a, b) \) generated by \( a, b \). We may consider its associated Cayley graph \( C \) (with these generators), which may be given a metric by extending the word metric on the 0-cells to the 1-skeleton (giving each edge unit length). The group \( F(a, b) \) naturally acts on this space as a group of isometries.

One may construct “periodic” point patterns of this space in the following way. Consider a group \( G \) along with a homomorphism \( f: F(a, b) \to G \). Then the kernel of this map determines a point pattern of the Cayley graph (contained in the vertex set). If the group \( G \) is finite then \( \ker(f) \) is of finite index and so the group (which of course is \( \ker(f) \)) of isometries preserving the point pattern acts cocompactly on \( C \).

We shall construct a hierarchical point pattern of a similar nature to the dyadic solenoid construction of example 2.8. For each \( n \in \mathbb{N} \) consider the map \( f_i: F(a, b) \to \mathbb{Z}_{2^n} \) defined by sending \( a, b \) to \([1]_{2^n}\). This defines point patterns \( P_n \) on \( C \) by marking points of the kernel. For each \( n \), define a tile of \( C \) as a “cross” of 1-cells connecting the centre of the cross to \( a^n, b^n, a^{-n} \) and \( b^{-n} \). By placing such a tile with centre at each point of the point pattern, we form a tiling \( T_n \); each 1-cell is covered by precisely one tile, although tiles may overlap on more than just their boundaries. Each of the point patterns locally determines the corresponding tiling and
vice versa. Note that the patch of tiles at a point of $T_n$ is determined by the patch of tiles at that point of $T_{n+1}$, so we may consider the collection $T_\omega = \{T_1, T_2, \ldots\}$ as a hierarchical tiling.

Although this example doesn’t satisfy all of the properties required for the application of the methods of the previous subsection, with adjustments one may still compute the $T_\omega$ homology groups in a similar manner for this simple example. We have that the approximant groups $H_0^{A^i}(C)$ are isomorphic to $\mathbb{Z}$, as represented by assigning coefficient 1 to the vertices of $\ker(f_i)$. Other 0-chains whose coefficients are locally determined by the tiling are homologous to these by moving the non-zero coefficients of the 0-chain locally, for example, to the nearest element of $\ker(f_i)$ to their right-hand side (which, again, will be a 1-chain whose coefficients are locally determined by the tiling). The connecting maps $\omega_0^i: \mathbb{Z} \to \mathbb{Z}$ are the times two maps, since a generator for $H_0^{A_i}$ is mapped to the chain assigning coefficient one to the centres and boundaries of each cross in $T_{i+1}$, hence $H_0^{T_\omega} \cong \mathbb{Z}[1/2]$. The group $H_1^{T_\omega}$ will be free abelian, and it isn’t too hard to see that it has countable many generators. The pattern space here, $(\overline{C}, d_{T_\omega})$ is connected but not locally path-connected, and is such that the Cayley graph $C$ maps bijectively and continuously onto each path component (that is, the space has leaves homeomorphic to $C$). We do not obtain Poincaré duality here, since we don’t have Poincaré duality for the space $C$; Poincaré duality fails locally at the vertices where the compactly supported cohomology is isomorphic to $\mathbb{Z}^3$.

5.1.6. A “Regular” Pentagonal Tiling of the Plane. In \cite{Bowers} Bowers and Stephenson define an interesting tiling coming from a combinatorial substitution of pentagons. Of course, one may not tile ($\mathbb{R}^2$, $d_{eucl}$) by regular pentagons. However, the combinatorial substitution defines a CW-decomposition of $\mathbb{R}^2$ for which each 2-cell has 5 bounding 0-cells and 1-cells. One may assign the unit edge metric on the 1-skeleton and extend this metric to the faces so that each face is isometric to a regular pentagon. Defining the distance between two points as the length of the shortest path connecting them, the resulting metric space (which is homeomorphic to $\mathbb{R}^d$) can be tiled by the regular pentagon. The tiling shares many properties of usual substitution tilings of Euclidean space, and in fact the techniques as described here are directly applicable to this tiling.

An interesting feature of this example is that group of global isometries of the metric space constructed above (which we shall call $P$) is rather small. Of course, the metric space is locally non-Euclidean at the vertices of the tiling, and as one can easily imagine this allows the tiling, that is, the locations of the regular pentagons in $P$, to be determined by the metric. One sees that isometries of $K$ correspond precisely to combinatorial automorphisms preserving the CW-decomposition of $K$. However, the tiling is aperiodic in the following sense: $P/Iso(T) = P/Iso(P)$ is non-compact. Indeed, the substitution is recognisable, one may invert the subdivision process using only local information. Any isometry of $P$ must preserve this hierarchy of tilings and so there can be no cocompact group action preserving the space. Hence, one may not define a sensible tiling metric in terms of preservation of large patches by global isometries of the space $P$. However, this issue doesn’t cause any complications here, we may define the tiling metric in terms of partial isometries.

We shall let $S$ be the collection of partial isometries between open sets of $P$ which preserve the orientation of $P$. We define a hierarchical tiling on $P$ by setting $T_0 = T$ and $T_{i+1}$ to be the unique tiling of $P$ of super$^{i+1}$-tiles (which are formed from glueing together $6^{i+1}$ regular pentagons, as described by the substitution rule) which subdivides to $T_i$. For $\mathbb{R}$-coefficients, since the coefficient group is divisible, one may take the CW-decomposition of $T_0$ to be that of the original tiling and the analogous such CW-decompositions for each subsequent $T_i$. We see that there are two vertex types, those which lie at the meeting of three or of four pentagons, one
edge type and one face type. Since there is an equivalence of the edge type with itself reversing orientations, the approximant chain complex is 0 \rightarrow \mathbb{R}^2 \rightarrow 0 \rightarrow \mathbb{R} \rightarrow 0. The substitution maps induce isomorphisms in each degree and so \( H^T_0 \cong \mathbb{R}^2 \), \( H^T_1 \cong 0 \) and \( H^T_2 \cong \mathbb{R} \).

To compute the groups over \( \mathbb{Z} \)-coefficients, one needs to take a finer CW-decomposition of \( P \) for each tiling. Of course, each such CW-decomposition will be analogous for each tiling. One should include the centres of pentagons and edges in the 0-skeleton of \( T_0 \) since the tiling has local symmetry at these points. We compute that the approximant homologies are \( H^A_0 \cong \mathbb{Z}^2 \), \( H^A_1 \cong 0 \) and \( H^A_2 \cong \mathbb{Z} \). As usual, \( H^T_0 \) is generated by the fundamental class of \( P \), so we just need to compute the substitution map on \( H^T_0 \).

The matrix has eigenvalues 1 and 6 with eigenvectors which span \( \mathbb{Z}^2 \). Hence, we have that \( H^T_0 \cong \mathbb{Z} \oplus \mathbb{Z}[1/6], H^T_1 \cong 0 \) and \( H^T_2 \cong \mathbb{Z} \).

One may repeat these calculations for the modified homology groups. We calculate them as being isomorphic to the unmodified groups. As a result of the substitution being recognisable, we have that the pattern space \( \Omega_{T_0} \) of \( T_0 \) is homeomorphic to the tiling space \( \Omega_T \cong (P, d_T) \) of \( T \). Hence, by Poincaré duality, we have that \( \tilde{H}^0((P, d_T)) \cong \mathbb{Z}, \tilde{H}^1((P, d_T)) \cong 0 \) and \( \tilde{H}^2((P, d_T)) \cong \mathbb{Z} \oplus \mathbb{Z}[1/6] \).

References

[1] A. M. Abd-Allah and R. Brown, A Compact-Open Topology on Partial Maps with Open Domain, J. London Math. Soc. (2) 21 (1980), no. 3, 480-486.
[2] J. E. Anderson and I. F. Putnam, Topological Invariants for Substitution Tilings and their Associated C*-Algebras, Ergodic Theory Dynam. Systems 18 (1998) no. 3, 509-537.
[3] M. Barge, B. Diamond, J. Hunton and L. Sadun, Cohomology of Substitution Tiling Spaces, Ergodic Theory Dynam. Systems 30 (2010), no. 6, 1607-1627.
[4] J. Bellissard, Gap Labelling Theorems for Schrödinger’s Operators, From number theory to physics (Les Houches, 1989), 538-630, Springer, Berlin, 1992.
[5] R. Berger, The Undecidability of the Domino Problem, Mem. Amer. Math. Soc. 66 (1966) 72 pp.
[6] J. Block and S. Weinberger, Aperiodic Tilings, Positive Scalar Curvature, and Amenability of Spaces, J. Amer. Math. Soc. 5 (1992), no. 4.
[7] A. Borel and J. C. Moore, Homology Theory for Locally Compact Spaces, Michigan Math. J. 7 (1960).
[8] P. L. Bowers and K. Stephenson, A “Regular” Pentagonal Tiling of the Plane, Conform. Geom. Dyn. 1 (1997).
[9] A. Clark and L. Sadun, When Shape Matters: Deformations of Tiling Spaces, Ergodic Theory Dynam. Systems 26 (2006), no. 1.
[10] K. Culik II, An Aperiodic Set of 13 Wang Tiles, Discrete Math. 160 (1996), no. 1-3.
[11] E. Floyd and R. Richardson, An Action of a Finite Group on an n-cell Without Fixed Points, Bull. Amer. Math. Soc. 65 (1959).
[12] A. Forrest, J. Hunton and J. Kellendonk, Topological Invariants for Projection Method Patterns, Mem. Amer. Math. Soc. 159 (2002).
[13] L. Fuchs (1970), Infinite Abelian Groups, Vol. I. Pure and Applied Mathematics, Vol. 36. New York-London: Academic Press 1970.
[14] F. Gähler, J. Hunton and J. Kellendonk, Torsion in Tiling Homology and Cohomology, preprint, arXiv:math-ph/0505048.
[15] F. Gähler and G. Maloney, Cohomology of One-Dimensional Mixed Substitution Tiling Spaces, Topology Appl. 160 (2013), no. 5.
[16] D. Gonçalves, On the K-theory of the Stable C*-Algebras from Substitution Tilings, J. Funct. Anal. 260 (2011).
[17] A. Hatcher, Algebraic Topology, Cambridge University Press (2002).
[18] J. Kellendonk, Pattern-Equivariant Functions and Cohomology, J. Phys. A 36 (2003), no. 21.
[19] J. Kellendonk and M. V. Lawson, Tiling Semigroups, J. Algebra 224 (2000), no. 1.
[20] J. Kellendonk and I. F. Putnam, Tilings, C*-algebras, and K-theory, in Directions in Mathematical Quasicrystals M. Baake and R. V. Moody, editors, CRM Monograph Series, Amer. Math. Soc., Providence, R.I., 2000.

[21] M. V. Lawson, Inverse Semigroups: The Theory of Partial Symmetries, World Scientific, Singapore, 1998.

[22] J. Leroy, Some improvements of the S-adic Conjecture, Adv. in Appl. Math. 48 (2012), no. 1.

[23] M. Marcinkowski and P. W. Nowak, Aperiodic Tilings of Manifolds of Intermediate Growth, preprint, arXiv:math-ph/1205.0495.

[24] N. Priebe-Frank and L. Sadun, Fusion: A General Framework for Hierarchical Tilings of R^d, Geom. Dedicata (2013), DOI: 10.1007/s10711-013-9893-7.

[25] N. Priebe-Frank and L. Sadun, Fusion Tilings with Infinite Local Complexity, Topology Proc. 43 (2014).

[26] J. Picado and A. Pultr, Frames and Locales: Topology Without Points, Frontiers in Mathematics (Springer, Basel, 2012).

[27] J. Roe, Lectures on Coarse Geometry, (University Lecture Series, 31), Amer. Math. Soc., Providence, R.I., 2003.

[28] L. Sadun, Pattern-Equivariant Cohomology with Integer Coefficients, Ergodic Theory Dynam. Systems 27 (2007), no. 6, 1991-1998.

[29] L. Sadun, Topology of Tiling Spaces, (University Lecture Series 46), Amer. Math. Soc., 2008.

Department of Mathematics, University of Leicester,
University Road, Leicester LE1 7RH, UK
E-mail address: jjw19@le.ac.uk