Abstraction-Based Output-Feedback Control with State-Based Specifications

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Abstract—We consider abstraction-based design of output-feedback controllers for non-linear dynamical systems against specifications over state-based predicates in linear-time temporal logic (LTL). In this context, our contribution is two-fold: (I) we generalize feedback-refinement relations for abstraction-based output-feedback control to systems with arbitrary predicate and observation maps, and (II) we introduce a new algorithm for the synthesis of abstract output-feedback controllers w.r.t. LTL specifications over unobservable state-based predicates.

Our abstraction-based output-feedback controller synthesis algorithm consists of two steps. First, we compute a finite state abstraction of the original system using existing techniques. This process typically leads to an abstract system with non-deterministic predicate and observation maps which are not necessarily related to each other. Second, we introduce an algorithm to compute an output-feedback controller for such abstract systems. Our algorithm is inspired by reactive synthesis under partial observation and utilizes bounded synthesis.

I. INTRODUCTION

Abstraction-based control design (ABCD) is a well known technique to synthesize correct-by-design control software for cyber-physical systems. In particular, ABCD allows to consider continuous-state dynamical systems in combination with discrete, temporal control objectives and computes controllers almost fully automatically.

ABCD comes in various flavors implemented in different tools and applicable to different types of dynamical systems and classes of temporal specifications, e.g. [16], [6], [7], [14], [10]. However, almost all works on ABCD rely on the fact that the state of the system is observable. This can be a very restrictive assumption in practice. Recently, this requirement was relaxed and abstraction-based output-feedback control design (ABoCD) was considered and follows mostly two different approaches.

In the first approach, classical observers for the original dynamical system are computed and incorporated into the abstraction process [12], [5], [15], [1]. This handles the complexity of output-feedback control in the pre-abstraction phase and thereby requires particular properties of the underlying dynamics and observation maps to allow for observer design. The second approach does not assume any “niceties” of the underlying dynamics or observation maps and moves the complexity of output-feedback control to the abstract layer by considering a partial-observation game for discrete output-feedback control [13], [11], [8].

Within this paper, we follow the second approach. Here, existing works are either limited to control problems where specifications are defined over observables [11], only safety specifications over abstract states are considered [13], or it is required that the computed abstraction is detectable, i.e., after a finite number of steps the true abstract state can be determined [8].

Within this paper, we relax these assumptions on ABoCD as exemplified in the robotic motion planning example depicted in Fig. 1. Here, a mobile robot (with non-linear disturbed dynamics) can only sense its location by detecting the color of the tile it is currently moving over. This can for example be realized by a downward pointing sensor that detects different colored lines on the floor which indicate tile boundaries. Using this restricted (but very cheap) position measurement, the robot should be controlled such that it alternates between the “pick up” and “drop off” location, while avoiding the black
obstacles. We see that specification predicates (i.e., the black and gray regions) are not expressible in terms of the (restricted) observations and hence are not observable.

In order to synthesize an output-feedback controller for such a system, a straightforward approach is to first employ a uniform grid-based abstraction technique, as e.g. in SCOTS [17], to generate a finite abstraction. For the example in Fig. 1, the abstract system would have one abstract state per grid cell. When computing this abstraction, we additionally need to transform the predicate- and observation maps. As we see from Fig. 1, typically multiple predicates or tiles intersect with a single boxed grid cell. This leads to non-deterministic predicate and observation maps on the abstraction.

This example demonstrates that ABoCD with non-observable predicates requires to

(I) extend the notion of feedback-refinement relations (FRR) [16] to systems with non-deterministic predicate and observation maps, and

(II) to develop an algorithm that synthesizes output-feedback controllers for such systems.

Within this paper, we tackle challenge (I) in Sec. III where we define extended feedback-refinement relations (eFRR) and sound abstract specifications. As our first contribution, this provides a new framework for sound ABoCD in the presence of unobservable predicates.

Afterwards, we address challenge (II) in two steps. We first show in Sec. IV that employing a standard grid-based abstraction technique, as in SCOTS [17] with the obvious transformation of predicate- and observation maps, yields a sound finite abstraction which allows for an eFRR to the original system. As our second contribution we then show in Sec. V how an abstract output-feedback controller can be synthesized for this finite abstraction which has possibly non-deterministic predicate and observation maps. We emphasize that this algorithm does not require any pre-processing of the predicate map. i.e., we do not need to smartly “expand” or “shrink” predicates to render predicate maps deterministic, as required for state-based ABCD in SCOTS.

Our resulting ability to synthesize output-feedback controllers for finite systems with non-deterministic output maps, also allows us to tackle a slightly different abstraction-based control problem exemplified in Fig. 2, which is more closely related to the problems studied in the first approach to ABoCD (e.g., in [1]). Here, the position of the robot can be measured with some error ε. When abstracting this system using a grid-based approach, the resulting abstract observation map also becomes non-deterministic, as shown in the right side of Fig. 2 even if we choose the abstract output space identical to the abstract state space.

![Fig. 2. Another application scenario for our ABoCD technique. Here the position of the robot is measured with some error bounded by ε. After grid-based abstraction this results in a non-deterministic output-map (indicated by the blue dashed line on the right) even if the abstract state and output spaces coincide.](image)

II. Preliminaries

Notation. We use the symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{R}_{\geq 0}$ to denote the sets of natural numbers, integers, reals, and positive reals, respectively. Given $a, b \in \mathbb{R}$ s.t. $a \leq b$, we denote by $[a,b] \in \mathbb{Z}$ a closed interval and define $[a;b] := \{a, b\} \cap \mathbb{Z}$ as its integer counterpart. For a set $W$, we write $W^*$ and $W^\omega$ for the sets of finite and infinite sequences over $W$, respectively, and define $W^\infty := W^* \cup W^\omega$.

For $w \in W^*$, we write $|w|$ for the length of $w$ and $\varepsilon$ for the empty string with $|\varepsilon| = 0$; the length of $w \in W^\omega$ is $\infty$. We define $\text{dom}(w) = \{0, \ldots, |w| - 1\}$ if $w \in W^*$, and $\text{dom}(w) = \mathbb{N}$ if $w \in W^\omega$. For $k \in \text{dom}(w)$ we write $w(k)$ for the $k$-th symbol of $w$ and $w|[0;k]$ for the restriction of $w$ to the domain $[0;k]$. Given two sets $A$ and $B$, $f : A \supseteq B$ and $f : A \rightarrow B$ denote a set-valued and ordinary map, respectively. The inverse mapping $f^{-1} : B \supseteq A$ is defined via its respective binary relation: $f^{-1}(b) = \{a \in A : b \in f(a)\}$. By slightly abusing notation, we lift maps to subsets of their domain in the usual way, i.e., for a set-valued map $f : A \supseteq B$ and $\alpha \subseteq A$ we have $f(\alpha) = \{b : \exists a \in \alpha. b \in f(a)\}$, and similarly for ordinary maps. For any set $A$ we denote the identity function over $A$ by $\iota$ i.e., $\iota(a) = a$ for all $a \in A$.

Systems. A system $S = (X, X_0, U, F, Y, H)$ consists of a state space $X$, a set of initial states $X_0 \subseteq X$, an input space $U$, a transition function $F : X \times U \rightarrow X$, an output space $Y$, and an output function $H : X \rightarrow Y$. The only restriction we impose on such systems is $H(x) \neq \emptyset$ for all $x \in X$, i.e., we require that $Y$ is a cover of $X$. The system $S$ is called finite if $X$, $U$ and $Y$ are finite sets.

We lift the functions $F$ and $H$ to sets of states $A \subseteq X$ and $B \subseteq U$ s.t. $F(A,B) := \bigcup_{a \in A} \bigcup_{b \in B} F(a, b)$, and $H(A) = \bigcup_{a \in A} H(a)$. If not explicitly defined otherwise, we apply this “lifting” of maps from single elements to sets via their union to all maps defined in this paper.
Given a state $x \in X$ we define the set of enabled inputs in $x$ as $\text{Enab}(x) := \{ u \in U \mid F(x, u) \neq \emptyset \}$. We lift this map to sets of states in a slightly unusual fashion by taking intersection rather than union. That is, given a set $A \subseteq X$ we define $\text{Enab}(A) := \bigcap_{a \in A} \text{Enab}(a)$.

**Trace Semantics.** A (maximal) path of $S$ is a sequence $\pi = x_0u_0x_1u_1\ldots$ such that $x_0 \in X_0$, for all $k \in \text{dom}(\pi)$ we have $x_k \in F(x_{k-1}, u_{k-1})$, and if $\text{dom}(\pi) = k < \infty$ we have $F(x_k, u) = \emptyset$ for all $u \in U$. The set of all paths over $S$ is denoted by $\text{Paths}(S)$. The prefix up to $x_n$ of a path $\pi$ over $S$ is denoted by $\pi|_{[0;n]} = x_0u_0x_1u_1\ldots x_n$ with length $|\pi|_{[0;n]}| = n + 1$ and last element $\text{Last}(\pi|_{[0;n]}) = x_n$. The set of all such prefixes is denoted by $\text{Pref}(S)$.

Given a path $\pi$ an external sequence $\sigma = y_0u_0y_1u_1\ldots$ is generated by $\pi$ if $y_k \in H(x_k)$ for all $k \in \text{dom}(k)$, denoted by $\sigma \in \text{Ext}(\pi)$. The set of all external sequences of a system $S$ is defined by $\text{EPaths}(S) := \text{Ext}(\text{Paths}(S))$ with its prefix set $\text{EPref}(S) := \text{Ext}(\text{Pref}(S))$.

We lift the map $\text{Last}$ to external sequences and write $x \in \text{LastX}_S(\sigma)$ if there exists $\pi \in \text{Pref}(S)$ s.t. $\sigma \in \text{Ext}(\pi)$ and $x = \text{Last}(\pi)$. For a state $x \in X$ we define all prefixes of $S$ that reach $x$ as $\text{Hist}_S(x) = \{ \pi \in \text{Pref}(S) \mid \text{Last}(\pi) = x \}$ and all external sequences generated by such prefixes as $\text{EHist}_S(x) = \{ \sigma \in \text{EPref}(S) \mid x \in \text{LastX}_S(\sigma) \}$.

**Control Strategies.** We define output-feedback control strategies as functions $C : \text{EPref}(S) \to U$. We say that $C$ is feedback-composable with $S$ if we can iteratively construct their external closed-loop behavior as follows. First, we define $\text{EPref}_{0}(S, C) := H(X_0)$. Then, for all $k \in \mathbb{N}$ we require that $\nu \in \text{EPref}_{k}(S, C)$ implies that $C(\nu)$ is defined and $C(\nu) \in \text{Enab}(\text{LastX}_S(\nu))$. Further, we define $\text{EPref}_{k+1}(S, C) := \{ \nu y \in \text{EPref}(S) \mid \nu \in \text{EPref}_{k}(S, C), u = C(\nu) \}$. We have $\text{EPref}(S, C) := \bigcup_{k \in \mathbb{N}} \text{EPref}_{k}(S, C)$ and define the set of infinite external closed-loop sequences of $S$ under $C$ as the set $\text{EPaths}(S, C) \subseteq Y(UY)^\omega$ s.t. $\sigma \in \text{EPaths}(S, C)$ if $\sigma|_{[0:k]} \in \text{EPref}_{k}(S, C)$ for all $k \in \mathbb{N}$. We further define $\text{CPref}(S, C) := \text{Ext}^{-1}(\text{EPref}(S, C))$ and $\text{CPaths}(S, C) := \text{Ext}^{-1}(\text{EPaths}(S, C))$.

**Specifications.** We consider $\omega$-regular specifications over a finite set of atomic (boolean) propositions $AP$ which are given by a formula $\psi$ in linear temporal logic (LTL). We omit the standard definitions of $\omega$-regular languages and LTL (see, e.g., [20], [21]). We assume that the set of atomic propositions $AP$ can be divided into input propositions $AP_I$ and output propositions $AP_O$ defining the finite sets $T := 2^{AP_I}$ and $A := 2^{AP_O}$ of predicates, which collect all possible sets of currently true propositions.

We interpret a specification $\psi$ on a system $S$ with the help of two predicate maps $P_T : U \to Y$ and $P_A : X \to A$. Given a predicate sequence $\nu = \lambda_0\mu_0\lambda_1\mu_1\ldots$ and a path $\pi = x_0u_0x_1u_1\ldots$ of $S$, we say that $\nu$ is generated by $\pi$, written $\nu \in P(\pi)$, if for all $k \in \text{dom}(\pi)$ holds that $\mu_k \in P_T(u_k)$ and $\lambda_k \in P_A(x_k)$.

**Control Problem.** Given a system $S$, a specification $\psi$ and an interpretation of $\psi$ on $S$ via $P_T$ and $P_A$ defining the map $P$, the output-feedback control problem, written $(S, P, \psi, P)$, asks to find an output-feedback control strategy $C$ which is feedback-composable with $S$ and all closed-loop paths of $S$ under $C$ fulfill the specification.

To formalize this further, we adopt the usual convention to interpret a specification $\psi$ as a $\omega$-regular language $\langle \psi \rangle \subseteq \Lambda(\omega)$ of desired infinite predicate sequences. With this, we can define set $W(S, \psi, P)$ of sound output-feedback control strategies s.t. $C \in W(S, \psi, P)$ iff $C$ is feedback-composable with $S$ and $P(\text{CPaths}(S, C)) \subseteq \{ \psi \}$.

### III. Sound Abstraction-Based Output-Feedback Control

Within this section we extend the notion of feedback-refinement relations (FRR) and sound abstract specifications from [16] to systems with possibly infinite state, input and output spaces and non-deterministic observation and predicate maps. The intuition behind this extension is depicted in Fig. 3. Instead of a single relation between $X$ and $\hat{X}$ as in FRR, our extended feedback refinement relation (eFRR) is a tuple $Q = \langle \alpha, \beta, \gamma \rangle$ of relations between the tuples $(X, U, Y)$ and $(\hat{X}, \hat{U}, \hat{Y})$ (see the purple and pink arrows in Fig. 3). In addition, the notion of sound abstract specifications ensures that the specification is correctly abstracted and interpreted over $\hat{S}$ via $\hat{P}$ (indicated in red in Fig. 3) for systems related via $Q$. As the main result of this section, we show that this extended notion of FRR together with sound abstract specifications allows for sound abstraction-based...
output-feedback control design (ABoCD) in the presents of non-deterministic observation and predicate maps.

**Sound Abstract Specifications.** In order to ensure that the specification is correctly interpreted over \( \hat{S} \) via \( P \) we next introduce the concept of sound abstract specifications in analogy to [16, Def. V2].

**Definition 2:** Let \( S \) and \( \hat{S} \) be systems s.t. \( S \preceq_\mathcal{Q} \hat{S} \). Further, let \( (\psi, P) \) and \( (\hat{\psi}, \hat{P}) \) be specifications interpreted over \( S \) and \( \hat{S} \) respectively. Then we say that \( (\hat{\psi}, \hat{P}) \) is a sound abstract specification associated with \( S, \hat{S}, Q \) and \( (\psi, P) \), written \( (S, (\psi, P)) \preceq_\mathcal{Q} \langle \hat{\psi}, \hat{P} \rangle \) if the following holds. For all \( \pi = x_0u_0x_1 \ldots \in \text{CPaths}(S) \) for which there exists an input sequence \( \bar{u}_0 \bar{u}_1 \ldots \) with \( u_i \in \beta(\bar{u}_i) \) (for all \( i \in \mathbb{N} \)) s.t. for all \( \hat{\pi} = \hat{x}_0\hat{u}_0\hat{x}_1 \ldots \) with \( \hat{x}_i \in \alpha(x_i) \) (for all \( i \in \mathbb{N} \)) holds \( P(\hat{\pi}) \subseteq \{ \hat{\psi} \} \) also holds that \( P(\pi) \subseteq \{ \psi \} \).

Intuitively, Def. 2 ensures that for every “good” path \( \hat{\pi} \) over \( \hat{S} \), i.e., a path that only generates predicate sequences in \( \{ \hat{\psi} \} \), all paths of \( S \) related to \( \hat{\pi} \) via the eFRR \( Q \) are also “good”, i.e., only generate predicate sequences in \( \{ \psi \} \). If this holds, we can use \( (\hat{\psi}, \hat{P}) \) to synthesize a “good” abstract controller which can then be refined into a “good” controller \( C \) for \( S \). This is formalized next.

**Sound Controller Refinement.** As the main result of this section, we now show how output-feedback controllers for sound abstractions under sound abstract specifications can be refined to output-feedback controllers for the original system w.r.t. the original specification.

**Theorem 1 (Sound ABoCD):** Let \( \langle S, (\psi, P) \rangle \preceq_\mathcal{Q} \langle \hat{\psi}, \hat{P} \rangle \) with \( Q = (\alpha, \beta, \gamma) \). Further, let \( \hat{C} \in \mathcal{W}(\hat{S}, \hat{\psi}, \hat{P}) \) and define \( C \) s.t.

\[
\forall \sigma \in \text{EPrefs}(S) . C(\sigma) = \beta(\hat{C}(\Omega_\beta, \gamma(\sigma))),
\]

where \( \gamma(y_0y_1 \ldots) \in \Omega_{\beta, \gamma}(y_0y_1 \ldots) \) iff \( \gamma k \in \gamma(y_k) \) and \( u_k \in \beta(\hat{u}_k) \) for all \( k \in \mathbb{N} \). Then \( C \in \mathcal{W}(S, \psi, P) \).

In order to prove Thm. 1 we need to show that the constructed controller \( C \) is feedback-composiable with \( S \) and only generates paths that are compatable with the specfication \( \psi \). Intuitively, this requires to show that at every time step \( k \) every input choice \( u_k \) made by \( C \) via (1) based on the already observed external sequence \( \sigma_k = y_0u_0 \ldots y_k \) ensures that \( u_k \) is available in all possible states reached under this observation and that all possible paths of \( S \) that conform with these observation trances are compliant with \( \{ \psi \} \) via \( P \). In order to prove this claim, we first formalize some observations about all possibly generated \( \sigma_k \) and all compliant paths \( \tau_k \in \text{Ext}(\sigma_k) \) that result from the fact that \( S \preceq_\mathcal{Q} \hat{S} \) in the following lemma, which is proven in App. A.

**Lemma 1:** Given the premises of Thm. 1 the following holds for all \( k \in \mathbb{N} \). For all \( \sigma_{k-1} = y_0u_0 \ldots y_{k-1} \in \text{EPrefs}_{k-1}(S, C) \), \( u_{k-1} \in C(\sigma_{k-1}) \) (if \( k > 0 \)) and \( \sigma_k = y_0u_0 \ldots y_{k-1}u_{k-1}y_k \in \text{EPrefs}(S) \) holds that

(a) \( \sigma_k \in \text{EPrefs}_k(S, C) \),
(b) for all \( \hat{\sigma}_k = \hat{y}_0\hat{u}_0 \ldots \hat{y}_k \in \Omega_{\beta, \gamma} \) s.t. \( \hat{u}_i \in \beta(\hat{x}_i) \) (for all \( i \in [0; k-1] \) holds that \( \hat{\sigma}_k \in \text{EPrefs}_k(\hat{S}, \hat{C}) \),
(c) for all \( \tau_k = x_0u_0 \ldots x_{k-1}u_{k-1}x_k \in \text{Ext}_S^{-1}(\sigma_k) \), \( \hat{\tau}_k \) as in (c), and \( \hat{\tau}_k = \hat{x}_0\hat{u}_0 \ldots \hat{x}_{k-1}\hat{u}_{k-1} = \hat{x}_k \in \Omega_{\alpha, \beta}(\sigma_k) \) s.t. the input sequence \( \hat{u}_0 \ldots \hat{u}_{k-1} \) matches the inputs of \( \hat{\tau}_k \) holds that \( \hat{\tau}_k \in \text{Ext}_\hat{S}^{-1}(\hat{\sigma}_k) \).
With this, it follows from Def. 2 that a BoSy ABoCD framework for an output feedback control step (II) by utilizing the second main contribution of this paper, which is solving the SCOTS class of systems $S$.

Recall from the definition of $P$ it remains to show that (d) that is not surprising that Lem. 1 allows us to prove Thm. 1 resulting traces always stay related. With this intuition it is not surprising that Lem. 1 allows us to prove Thm. 1 under the assumption that $(\hat{\psi}, \hat{P})$ is a sound abstract specification as in Def. 2.

Proof: It immediately follows from Lem. 1 (a) and (d) that $\hat{C}$ is feedback-composeable with $S$ and it remains to show that $P(CPaths(S, C)) \subseteq \{\psi\}$. We recall from the definition of $CPaths(S, C)$ that $\pi = u_0 x_0 x_1 \ldots \in CPaths(S, C)$ iff $|\pi| = \infty$ and $\pi[0:k] \in Ext^{-1}(EPrefs_k(S, C))$ for all $k \in \mathbb{N}$. Now it follows from Lem. 1 (b/c) that there exists an input sequence $\hat{u}_0 \hat{u}_1 \ldots$ s.t. for all $\hat{\pi} = \hat{x}_0 \hat{x}_1 \ldots$ with $\hat{x}_k \in \alpha(x_k)$ holds that $\hat{\pi}[0:k] \in Ext^{-1}(EPrefs_k(\hat{S}, \hat{C}))$ and therefore $\hat{\pi} \in CPaths(\hat{S}, \hat{C})$. As $\hat{C} \in W(\hat{S}, \hat{\psi}, \hat{P})$ we further have $\hat{P}(CPaths(\hat{S}, \hat{C})) \subseteq \{\hat{\psi}\}$ and therefore $\hat{P}(\hat{\pi}_k) \subseteq \{\hat{\psi}\}$. With this, it follows from Def. 2 that $P(\pi) \subseteq \{\psi\}$, what proves the claim.

Algorithmic ABoCD. So far, we have defined a sound ABoCD framework for an output feedback control problem $(S, \psi, P)$. In the remainder of this paper we will target the problem of algorithmically computing

(I) a sound abstraction $\hat{S}$ and a sound abstract specification $(\hat{\psi}, \hat{P})$ s.t. $(S, (\psi, P)) \not\approx (\hat{S}, (\hat{\psi}, \hat{P}))$,

(II) an output feedback controller $\hat{C} \in W(\hat{S}, \hat{\psi}, \hat{P})$.

If we solve these two algorithmic challenges, we can apply Thm. 1 to obtain a sound controller $\hat{C} \in W(S, \psi, P)$ for the original ABoCD problem via (I).

We first discuss step (I) in Sec. IV for a particular class of systems $S$, which is a straightforward extension of the constructions in [16] which are implemented in the tool SCOTS [17]. Then, in Sec. V, we provide the second main contribution of this paper, which is solving step (II) by utilizing bounded synthesis [19] which is implemented in the tool BoSy [4].

IV. CONSTRUCTING SOUND FINITE ABSTRACTIONS

Within this section we follow the grid-based abstraction technique developed in [16] for non-linear systems with disturbances. This abstraction process starts with a continuous non-linear control system $\Sigma$ which is first time-discretized into a system $S$ of the form introduced in Sec. II. This system $S$ has infinite input, state and output spaces. It is therefore further abstracted into a finite system $\hat{S}$ which can be used for symbolic controller synthesis.

In the following we recall this abstraction process from [16] and discuss the special treatment of actuation and observation constrains and the abstraction of the specification. We show that our definition of sound abstractions (Def. 1) and sound abstract specifications (Def. 2) is readily fulfilled by this abstraction procedure.

Control System. A control system $\Sigma = (X, X_0, U, W, F, Y, h)$ consists of a continuous state space $X = \mathbb{R}^n$, a set of initial states $X_0 \subseteq X$, a non-empty compact set of inputs $U \subseteq \mathbb{R}^m$, a continuous output space $Y \subseteq \mathbb{R}^r$, a compact cell $W \subseteq X$, and nonlinear (differential) inclusions

$$\dot{\xi} \in f(\xi(t), u(t)) + W$$

and

$$\nu(t) \in h(\xi(t)),$$

where $f(\cdot, u)$ is locally Lipschitz for all $u \in U$.

Continuous Transition System. A control system can be time-discretized to obtain a system $S$ as defined in Sec. II. I.e., given a time sampling parameter $\tau > 0$, we can define the system $S = (X, X_0, U, F, Y, h)$ associated with $\Sigma$ as follows. First, given an initial state $\xi(0) \in X$, and a constant input trajectory $\mu_u : [0, \tau] \to U$ which maps every $t \in [0, \tau]$ to the same $u \in U$, a solution of the inclusion in (2a) on $[0, \tau]$ is an absolutely continuous function $\xi : [0, \tau] \to X$ that fulfills (2a) for almost every $t \in [0, \tau]$. We collect all such solutions in the set $Sol_f(\xi(0), \tau, u)$. Then the transition and output functions of $S$ are defined s.t. for all $x \in X$ and for all $u \in U$ it holds that $x' \in F(x, u)$ and $y' \in H(x')$ iff there exists a solution $\xi \in Sol_f(x, \tau, u)$ s.t. $\xi(\tau) = x'$ and $y' \in h(x')$.

Finite Abstract System. Following [16] one can now apply a grid-based discretization of the state space of $S$ to obtain a system $\hat{S}$ with finitely many states. For this, one usually fixes a grid parameter $\eta \in \mathbb{R}^m \setminus \{0\}$ and a region of interest defined as a hyper-rectangle $\Theta = [\alpha, \beta]$, s.t. $\beta - \alpha = \eta$ is an integer multiple of $\eta$. Then one defines the finite abstract state space $\hat{X}$ as a set of hyper cells which cover $\Theta$ with grid-aligned cells $[a, b]$ s.t. $b - a = \eta$, while covering the rest of the state space with “overflow-cells” of the form $[\{0\}^n, \alpha]$ and $[\beta, \{\infty\}]^n$. These
“overflow-cells” are then treated as obstacles and added to the specification.

For the discretization of the output space $Y$ one can impose a very similar grid-based discretization with a possibly different grid parameter $\eta'$. This would allow us to capture the example in Fig. 2. On the other hand, we can also consider the case where given observation constraints impose a finite set of observations $\hat{Y}$ (as in Fig. 1). In both cases, $\hat{Y}$ is a finite cover of $Y$.

To discretize the input space, one usually restricts attention to a finite subset of “representative” inputs $\hat{U} \subseteq U$. This conforms, on one hand, with a grid-based discretization of inputs (similar to $X$ and $Y$) and picking one representative per grid cell.

Given these finite state, input and output sets one can define the finite abstract system $\hat{S} = (\hat{X}, \hat{Y}, \hat{F}, \hat{Y}, \hat{H})$ of $S$ s.t. $\hat{x} \in F(\hat{x}, \hat{u})$ iff $\{x \in \hat{x} : \text{Sol}(x, \tau, \hat{u})\} \cap \hat{x'} \neq \emptyset$, and $\hat{y} \in \hat{H}(\hat{x})$ iff $H(\hat{x}) \cap \hat{y} \neq \emptyset$.

In addition to the abstract system, we define the maps $\hat{P}_T$ and $\hat{P}_A$ to interpret $\psi$ over $\hat{S}$ s.t., $\mu \in \hat{P}_T(\hat{u})$ iff $\mu \in P_T(\hat{u})$ and $\lambda \in \hat{P}_A(\hat{x})$ iff $\lambda \in P_A(\hat{x})$. In this case we have $\hat{\psi} := \psi$.

**Soundness.** We have the following expected result on the soundness of the outlined abstraction procedure.

**Theorem 2:** Let $S$ be the time-discretized system associated with the control system $\Sigma$ and $\hat{S}$ its grid-based abstraction. Further, let $\psi$ be an LTL specification interpreted over $S$ and $\hat{S}$ via the maps $P_T$ and $P_A$, and $\hat{P}_T$ and $\hat{P}_A$, respectively. Then

$$\langle S, (\psi, P) \rangle \preceq \langle \hat{S}, (\psi, \hat{P}) \rangle.$$  

**Proof:** We define $\alpha$, $\beta$ and $\gamma$ s.t. $\hat{x} \in \alpha(x)$ iff $x \in \hat{x}$, $\beta(\hat{u}) = \hat{u}$ and $\hat{y} \in \gamma(y)$ iff $y \notin \hat{y}$. We first prove that $S \models S$ by showing that (A1)-(A3) in Def. 1 holds.

- **(A1)** Follows from the fact that the initial states are not restricted (i.e., $X_0 = X$ and $\hat{X}_0 = \hat{X}$) and $\hat{X}$ is a cover of $X$.
- **(A2.i)** Follows from the fact that $S$ is input enabled by definition, i.e., $\text{Enab}(S(x) = U$ for all $x \in X$. With this, it follows from the definition of $F$ that $\hat{S}$ is also fully input enabled, i.e., $\text{Enab}(\hat{S}(\hat{x}) = \hat{U}$ for all $\hat{x} \in \hat{X}$. With this, the claim directly follows.
- **(A2.ii)** We pick $\hat{u} \in \hat{U}$ and observe that $\beta(\hat{u}) = \hat{u}$ by construction. Now it follows from the construction of $\hat{F}$ that for every $\hat{x} \in \alpha(x)$ we have $\hat{x}' \in F(\hat{x}, \hat{u})$ if there exists some $x' \in F(x, \hat{u})$ s.t. $x' \in \hat{x}'$. This immediately implies the claim.

- **(A3)** Follows directly from the definition of $\hat{H}$.

It remains to show that $(\psi, \hat{P})$ is a sound abstract specification. As $\hat{\psi} = \psi$ and $\beta(\hat{u}) = \hat{u}$ this, however, follows immediately from the definition $\hat{P}$.

V. OUTPUT-FEEDBACK CONTROLLER SYNTHESIS FOR STATE-BASED SPECIFICATIONS

Within this section we consider a system $\hat{S} = (\hat{X}, \hat{X}_0, \hat{U}, \hat{F}, \hat{Y}, \hat{H})$ with finite sets $\hat{X}$, $\hat{U}$ and $\hat{Y}$ which allows to interpret the original LTL specification $\psi$ over $\hat{S}$ via maps $\hat{P}_T : \hat{U} \Rightarrow \hat{Y}$ and $\hat{P}_A : \hat{X} \Rightarrow \Lambda$, defining the path map $\hat{P}$. Our goal is to construct an output-feedback controller $\hat{C} \in \mathcal{W}(\hat{S}, \psi, \hat{P})$.

In order to formalize this construction we need to introduce some additional notation.

A. Preliminaries

**Transition Systems.** We define finite transition systems as tuples $T = (Q, Q_0, \Sigma, \delta)$ where $Q$ is a finite set of states, $Q_0 \subseteq Q$ is a set of initial states, $\Sigma$ is a finite alphabet and $\delta : Q \times \Sigma \Rightarrow Q$ is a set valued transition function. We call $T$ deterministic if $|\delta(q, \sigma)| \leq 1$ for all $q \in Q$ and $\sigma \in \Sigma$.

A sequence $\pi = q_0 \sigma_0 q_1 \sigma_1 \ldots$ is a (maximal) path over $T$ if $q_0 \in Q_0$ and $q_k+1 \in \delta(q_k, \sigma_k)$ for all $k \in \text{dom}(\pi)$ and either $|\pi| = \infty$ or $\delta(q_k, \sigma_k) = \emptyset$ for all $\sigma \in \Sigma$.

Similar to systems we collect all maximal paths of $T$ in the set $\text{Paths}(T)$ and their corresponding external sequences $\sigma_0 \sigma_1 \ldots$ in the set $\text{EPaths}(T)$.

Given an infinite string $\sigma = \sigma_0 \sigma_1 \ldots \in \Sigma^\omega$ we say that a path $\pi$ over $T$ is compliant with $\sigma$ if $q_k+1 \in \delta(q_k, \sigma_k)$ for all $k \in \text{dom}(\pi)$ and either (i) $|\pi| = \infty$ or (ii) $|\pi| = k < \infty$ and $\delta(q_k, \sigma_k) = \emptyset$. We say that $T$ is complete if for all $\sigma \in \Sigma^\omega$ there exists at least one infinite path over $T$ compliant with $\sigma$.

**Universal Co-Büchi Automata.** We define automata as tuples $A = (T, F)$ where $T$ is a finite transition system and $F \subseteq Q$ is a set of rejecting states. A universal Co-Büchi automata (UCA) is an automaton where $F$ is interpreted as a Co-Büchi condition and external sequences $\sigma$ are accepted based on universal branching. I.e., a path $\pi$ of $T$ is accepted by a UCA $A$ if $\pi$ visits the set $F$ only finitely often. Further, an external sequence $\sigma$ of $T$ is accepted by a UCA $A$ iff all runs of $\pi$ compliant with $\sigma$ are accepted by $A$.

1This is in contrast to non-deterministic acceptance, where only one compliant run needs to be accepting.
we have $M = 2$ and define over a finite set of atomic propositions $\Sigma$. Bounded Synthesis are summarizing the necessary features and constructions systems $M$ byproduct, bounded synthesis generates small transition Safra’s construction [18], and utilizes a direct synthesis procedure avoids the computationally expensive step of Bounded Synthesis an overview). Within this paper, we use a rather modern computed in many different ways (see e.g. [3, Ch.1] for an unrestricted environment, i.e., assuming $EPaths(\langle \Sigma I \rangle) = \Sigma I = \Lambda$ and $\Sigma O = \Upsilon$ for $M$ we have $\Sigma I = \Lambda$ and $\Sigma O = \Upsilon$. While this requires a determinization step which is exponential in the state-size of the NBA). In the third and last step, the deterministic observation system determined from $\hat{S}$ is combined with this game. Due to the deterministic relationship of all involved variables the resulting game is still deterministic and can be solved with common techniques from reactive synthesis.

In the presence of non-deterministic relationships between the alphabets of the observer automaton and the game, the combination of both yields a non-deterministic game that must again be determinized before it can be solved. This causes another exponential blow-up (see e.g. [2] for a discussion of this aspect).

Instead of taking this three-step approach, we first combine $\hat{S}$ and the UCA $A_\psi$ induced by $\psi$ into a new UCA $\hat{A}_\psi$ over their product state space and then use bounded synthesis to extract a model for this product UCA. While this requires a determinization step which is exponential in the size $\hat{A}_\psi$, only one such step is required. The main contribution of this section is to show that this procedure leads to a model $M$ over $\hat{Y} \times \hat{U}$ that allows to extract a sound controller $\hat{C} \in \langle \hat{S}, \psi, \hat{P} \rangle$.

Combining $\hat{S}$ and $A_\psi$. We first describe the formal combination of $\hat{S}$ and $A_\psi$ and then discuss the properties of the resulting UCA.

Proposition 1: [9, Thm.5.1] Let $\psi$ be an LTL formula over a finite set of atomic propositions $AP = AP_I \cup AP_O$ and define $\Upsilon := 2^{AP_I}$ and $\Lambda := 2^{AP_O}$. Then one can construct a complete UCA $A_\psi = (Q, Q_0, \Upsilon \times \Lambda, \delta, F)$ with $2O(|\psi|)$ number of states and with language $L(A_\psi) = \langle \psi \rangle$. We call $A_\psi$ the UCA induced by $\psi$.

Proposition 2 (Bounded Synthesis [19]): Let $A$ be a UCA. Then there exists an algorithm to compute a maximal model $M$ of $A$ if it exists.

C. Output-Feedback Controller Synthesis

The main obstacle in utilizing reactive synthesis to solve the output-feedback control problem $\langle \hat{S}, \psi, \hat{P} \rangle$, as indicated by the dotted green box in Fig. 4, lies in the fact that for $\hat{S}$ we have $\Sigma I = \hat{U}$ and $\Sigma O = \hat{Y}$ and for $M$ we have $\Sigma I = \Lambda$ and $\Sigma O = \Upsilon$ (as $\langle \psi \rangle \subseteq (\Upsilon \Lambda)^{\omega}$) while their relationship is in general non-deterministic.

In the usual setting of ABCD with state-feedback and deterministic predicate maps (as in [16]) or ABoCD with output-feedback and specifications defined over the observables (as, e.g. in [11], [8]), the relationship of the inputs and outputs of $\hat{S}$ and $\hat{M}$ is deterministic. In this case, utilizing reactive synthesis for control design is computationally simpler and done as follows.

First, one constructs a deterministic observer of $\hat{S}$ over $\hat{U}$ and $\hat{Y}$ (which is at most exponential in the state-size of $\hat{S}$), by using for example subset construction. Second, one uses the usual machinery in reactive synthesis to translate the specification $\psi$ first into a non-deterministic Büchi automaton (NBA) which is then determinized into a deterministic two-player game (again at most exponential in the state-size of the NBA). In the third and last step, the deterministic observation system determined from $\hat{S}$ is combined with this game. Due to the deterministic relationship of all involved variables the resulting game is still deterministic and can be solved with common techniques from reactive synthesis.

In the presence of non-deterministic relationships between the alphabets of the observer automaton and the game, the combination of both yields a non-deterministic game that must again be determinized before it can be solved. This causes another exponential blow-up (see e.g. [2] for a discussion of this aspect).

Instead of taking this three-step approach, we first combine $\hat{S}$ and the UCA $A_\psi$ induced by $\psi$ into a new UCA $\hat{A}_\psi$ over their product state space and then use bounded synthesis to extract a model for this product UCA. While this requires a determinization step which is exponential in the size $\hat{A}_\psi$, only one such step is required. The main contribution of this section is to show that this procedure leads to a model $M$ over $\hat{Y} \times \hat{U}$ that allows to extract a sound controller $\hat{C} \in \langle \hat{S}, \psi, \hat{P} \rangle$.
Definition 3: Let $\hat{S} = (X, \hat{X}_0, \hat{U}, \hat{F}, \hat{Y}, \hat{H})$ be a finite system with predicate maps $\hat{P}_Y : \hat{U} \Rightarrow \hat{Y}$ and $\hat{P}_X : X \Rightarrow \Lambda$. Further, let $\psi$ be an LTL formula with induced complete UCA $\mathcal{A}_\psi = (Q, Q_0, X \Rightarrow \Lambda, \delta, \mathcal{F})$. Then we define the product of $\hat{S}$ and $\mathcal{A}_\psi$ as the UCA $\mathcal{A}_x = (Q, Q_0, \hat{X} \times \hat{U}, \hat{\delta}, \hat{\mathcal{F}})$ s.t.

- $Q = (\hat{X} \times Q) \cup \{\bot\}$
- $Q_0 = \hat{X}_0 \times Q_0$
- $\hat{F} = (\hat{X} \times \hat{F}) \cup \{\bot\}$
- $\{\bot\} = \hat{\delta}_x(\hat{q}, (\hat{y}, \hat{u}))$ iff either $\hat{q} = \bot$ or $\hat{q} = (\hat{x}, q)$, $\hat{y} \in \hat{H}(\hat{x})$ and $\hat{F}(\hat{x}, \hat{u}) = \emptyset$, and
- $\bot \neq (x', q') \in \hat{\delta}_x(\hat{x}, q), (\hat{y}, \hat{u})$ iff $\hat{y} \in \hat{H}(\hat{x})$, $\hat{x}' \in \hat{F}(\hat{x}, \hat{u})$ and there exists $\mu \in \hat{P}_T(\hat{u})$ and $\lambda \in \hat{P}_Y(\hat{y})$ s.t. $q' \in \delta(q, (\mu, \lambda))$.

We call $\mathcal{A}_x$ the UCA induced by $\hat{S}$, $\psi$ and $\hat{P}$.

Intuitively, the UCA $\mathcal{A}_x$ has the following properties.
(i) Every path $\pi$ of $\hat{S}$ corresponds to a set of paths in $\mathcal{A}_x$, each describing exactly one predicate sequence induced by $\pi$. This implies that an external sequence $\sigma \in \mathcal{E} \mathcal{P}_\mathcal{X}$ is only accepted by $\mathcal{A}_x$ if all predicate sequences generated by any compliant path of $\sigma$ over $\hat{S}$ are allowed by the specification.
(ii) Every blocking path of $\hat{S}$ is extended to an infinite path in $\mathcal{A}_x$. In order to formalize this observation we define the set of blocking external sequences $\mathcal{B} \mathcal{X}(\hat{S})$ as follows.

Definition 4: Given a system $\hat{S}$ we define its set of blocking sequences $\mathcal{B} \mathcal{X}(\hat{S}) \subseteq \mathcal{Y}(\hat{U})^\omega$ as follows. An external sequence $\hat{\sigma} = \hat{y}_0 u_0 \hat{y}_1 u_1 \ldots \in \mathcal{Y}(\hat{U})^\omega$ is said to be blocking on $\hat{S}$, i.e., $\hat{\sigma} \in \mathcal{B} \mathcal{X}(\hat{S})$, if there exists an index $k \in \mathbb{N}$ s.t. $\nu = \hat{\sigma}|_{[0,k]} \in \mathcal{E} \mathcal{P}_\mathcal{X}(\hat{S})$ but $\hat{u}_k \notin \text{Enab}($LastX$\mathcal{S}(\nu))$.

We emphasize that the map Enab is lifted to sets of states using intersection. Hence, a string $\sigma$ is called blocking on $\hat{S}$ as defined in Def. 4, if there exists some state $q \in \text{LastX}\mathcal{S}(\nu)$ in which $u_k$ is not enabled, even if there might exist other states $q' \in \text{LastX}\mathcal{S}(\nu)$ which allow progress on $u_k$.

We are now ready to formalize the above observations in the following proposition, which is proven in App. B.

Proposition 3: Given the premises of Def. 3, let $\hat{\sigma} = \hat{y}_0 u_0 \hat{y}_1 u_1 \ldots \in \mathcal{Y}(\hat{U})^\omega$ be an external sequence. Then (i) $\hat{\sigma} \in \mathcal{E} \mathcal{P}_\mathcal{X}(\hat{S}) \cap \mathcal{L}(\hat{A}_x)$ iff $\hat{P}(\text{Ext}-1(\hat{\sigma})) \subseteq \{\psi\}$, (ii) $\hat{\sigma} \in \mathcal{L}(\hat{A}_x)$ implies $\hat{\sigma} \notin \mathcal{B} \mathcal{X}(\hat{S})$.

Extracting a control strategy. As $\hat{A}_x$ is a UCA over $\hat{Y} \times \hat{U}$ we can use Prop. 2 to extract a maximal model $M$ of $\hat{A}_x$, if it exists. We now show how we can define an output-feedback control strategy $\hat{C}$ from $M$.

Definition 5: Let $\hat{\mathcal{A}}_x$ be the UCA induced by $\hat{S}$, $\psi$ and $\hat{P}$ and $M = (Z, \{a_0\}, Y \times \hat{U}, \alpha)$ a maximal model of $\hat{A}_x$. Then we say that $\hat{C} : \mathcal{E} \mathcal{P}_\mathcal{X}(\hat{S}) \rightarrow \hat{U}$ is an output-feedback control strategy induced by $M$ if for all $\nu = \hat{y}_0 u_0 \hat{y}_1 \ldots \hat{y}_k \in \mathcal{E} \mathcal{P}_\mathcal{X}(\hat{S})$ we have that (i) $\hat{C}(\nu) = \hat{u}_k$ implies the existence of $z :\text{LastX}M(\hat{y}_0 u_0 \hat{y}_1 \ldots \hat{y}_{k-1} \hat{u}_{k-1})$ s.t. $|\alpha(z, (\hat{y}_k, \hat{u}_k))| = 1$ and (ii) $\hat{C}(\nu) = \emptyset$ only if $\nu \notin \mathcal{E} \mathcal{P}_\mathcal{X}(M)$.

The next theorem, which is the main result of this section, shows that the construction of an output-feedback controller $\hat{C}$ for a finite system $\hat{S}$ with specification $(\psi, \hat{P})$ via Prop. 1, Def. 3, Prop. 2 and Def. 5, is sound and complete. I.e., if the controller $\hat{C}$ obtained from $M$ is non-empty, it is sound (i.e., $\hat{C} \in \mathcal{W}(\hat{S}, \psi, \hat{P})$), and if no non-empty controller can be derived from $M$ no solution to the given control problem exists (i.e., $\mathcal{W}(\hat{S}, \psi, \hat{P}) = \emptyset$).

Theorem 3: Let $\hat{A}_x$ be the UCA induced by $\hat{S}$, $\psi$ and $\hat{P}$, and $M$ and $\hat{M}$ be maximal models of $\hat{A}_x$. Then (i) we have $\hat{C} \in \mathcal{W}(\hat{S}, \psi, \hat{P})$ for every non-empty controller $\hat{C}$ induced by $M$, and (ii) if no non-empty controller induced by $M$ exists, we have $\mathcal{W}(\hat{S}, \psi, \hat{P}) = \emptyset$.

The proof of Thm. 3 is given in App. C. For soundness (Thm. 3 (i)) the claim intuitively follows from the observation that Prop. 3 (ii) ensures that $\hat{C}$ is feedback-composable with $\hat{S}$ and (i) ensures that all paths of $(\hat{S}, \hat{C})$ are compatible with $\psi$.

Conversely, completeness (Thm. 3 (ii)) is established by showing that whenever $\mathcal{W}(\hat{S}, \psi, \hat{P}) \neq \emptyset$ there exists a non-empty controller $\hat{C}$ induced by $M$. Intuitively, this follows from the observations that (a) $\mathcal{W}(\hat{S}, \psi, \hat{P}) \neq \emptyset$ implies the existence of some infinite path $\sigma \in \mathcal{E} \mathcal{P}_\mathcal{X}(\hat{S})$ s.t. $\mathcal{P}(\text{Ext}-1(\sigma)) \subseteq \{\psi\}$, which in turn implies (from Prop. 3 (i)) that (b) $\sigma \in \mathcal{L}(\hat{A}_x)$ and therefore (c) $M$ is non-empty (as $M$ is defined to be maximal), implying the existence of a non-empty $\hat{C}$ by definition.

VI. RELATED WORK

Our paper is closest related to [16], [11], [8], [13]. Our definition of sound abstractions and sound abstract specification is based on the definition of the same notions for state-feedback control in [16]. Indeed, targeting all maps used in this paper to the special case discussed in [16] we observe that both definitions coincide. In particular, we recover the situation discussed in [16] if $Y = U$, $\Lambda = X$, $\hat{T} = \hat{U}$ and $\hat{A} = \hat{X}$, and therefore $P_T = P_\Lambda = \hat{P}_T = \hat{P}_\Lambda = \iota$. Further, we choose $\beta$ to be an identity map that is strictly defined on $\hat{U}$ and have $Y = X$ and $H = \iota$. Therefore $\gamma$ coincides with $\alpha$. With this, Def. 1 coincides with the definition of FRR in [16, Def. V.2] and Def. 2 coincides with [16, Def. VI.2].
In [11] the authors consider output-feedback control design with specifications defined over a finite set of observables. This corresponds to the setting in this paper when defining \( \Lambda = Y = \hat{A} = \hat{Y} \) and \( Y = U = \hat{Y} = \hat{U} \) implying that \( P = H, \hat{P} = \hat{H} \) and \( P = \hat{P} \) are identity maps. Further, as \( Y = \hat{Y} \) and \( U = \hat{U} \) the maps \( \beta \) and \( \gamma \) are also the identity maps. With this Def. 1 coincides with [11, Def.3.1]. While the abstraction algorithm in [11] does not require a grid-based discretization of the state space, it does not allow to handle state-based specifications.

Recently, Khaled et.al. proposed a similar notion of sound abstractions for symbolic output-feedback control in [8]. Unfortunately, it seems to the authors that their definition of sound abstractions does not allow to prove a soundness result similar to Thm. 1. Further, the authors only consider the abstract output-feedback controller synthesis problem either for specifications over the observables, similar to [11], or for abstractions that are detectable, i.e., where the state of the system becomes observable after a finite number of observations. Our controller synthesis procedure does not require these assumptions.

Finally, [13] solves the outlined ABoCD problem for safety specifications only, while we can handle arbitrary LTL properties.

References

[1] W. Apaza-Perez, A. Girard, C. Combastel, and A. Zolghadri. Symbolic observer-based controller for uncertain nonlinear systems. *IEEE Control Systems Letters*, 5(4):1297–1302, 2020.
[2] K. Chatterjee and L. Doyen. The complexity of partial-observation parity games. In *International Conference on Logic for Programming Artificial Intelligence and Reasoning*, pages 1–14, Springer, 2010.
[3] R. Ehlers. Symmetric and efficient synthesis. PhD thesis, Saarland University, 2003. https://www.ruediger-ehlers.de/papers/DissertationEhlers.pdf.
[4] P. Faymonville, B. Finkbeiner, and L. Tentrup. Bosy: An experimentation framework for bounded synthesis. In *International Conference on Computer Aided Verification*, pages 325–332, Springer, 2017.
[5] S. Haesaert, A. Abate, and P. M. Van den Hof. Correct-by-design output feedback of lti systems. In *CDC’15*, pages 6159–6164. IEEE, 2015.
[6] K. Hsu, R. Majumdar, K. Mallik, and A.-K. Schmuck. Multi-layered abstraction-based controller synthesis for continuous-time systems. In *HSCC’18*, pages 120–129. ACM, 2018.
[7] M. Khaled and M. Zamani. pfaces: an acceleration ecosystem for symbolic control. In *HSCC’19*, pages 252–257. ACM, 2019.
[8] M. Khaled, K. Zhang, and M. Zamani. Output-feedback symbolic control, 2020.
[9] O. Kapferman and M. Y. Vardi. Safraless decision procedures. In *46th Annual IEEE Symposium on Foundations of Computer Science (FOCS’05)*, pages 531–540, 2005.

4We remark that there was a typo in [11, Def. 3.1]. The set \( \hat{Y} \) in the definition of \( S \) must be \( Y \).

[10] Y. Li and J. Liu. Rocs: A robustly complete control synthesis tool for nonlinear dynamical systems. In *HSCC’18*, pages 130–135. ACM, 2018.
[11] R. Majumdar, N. Ozay, and A.-K. Schmuck. On abstraction-based controller design with output feedback. *arXiv preprint arXiv:2002.02687*, 2020.
[12] O. Mickelin, N. Ozay, and R. M. Murray. Synthesis of correctness-by-construction control protocols for hybrid systems using partial state information. In *ACC’14*, pages 2305–2311. IEEE, 2014.
[13] M. Mizoguchi and T. Ushio. Deadlock-free output feedback controller design based on approximately abstracted observers. *Nonlinear Analysis: Hybrid Systems*, 30:58–71, 2018.
APPENDIX

A. Proofs of Lem. 1

We prove all claims by induction over k.

- Base case (k = 0):
  - (a) We have \( \sigma_0 = y_0 \in \text{EPrefs}(S) \) iff \( y_0 \in H(x_0) \). As \( \text{EPrefs}_k(S, \tilde{C}) = H(x_0) \) by definition, the claim follows.
  
- (b/c) Observe that \( \hat{y}_0 \in \Omega_{\beta, \gamma}(y_0) \) implies \( \hat{y}_0 \in \gamma(y_0) \). Further, \( x_0 \in \text{Ext}^{-1}(y_0) \) implies \( x_0 \in X_0 \) and \( y_0 \in H(x_0) \). It then follows from (A1) that \( \emptyset \neq \alpha(x_0) \subseteq \hat{X}_0 \). Now observe that all \( \alpha(x_0) = \Omega_{\alpha, \beta}(x_0) \) and it follows from (A3) that for all \( x_0 \in \alpha(x_0) \) holds that \( \hat{y}_0 \in \hat{H}(x_0) \) implying \( x_0 \in \text{Ext}^{-1}(y_0) \), proving (c). As \( \text{EPrefs}_0(S, \tilde{C}) := \hat{H}(X_0) \) by definition, we further conclude \( \hat{y}_0 \in \text{EPrefs}_0(S, \tilde{C}) \), proving (b).

- (d) First, recall that \( \tilde{C} \) is feedback compatible with \( S \). Therefore, \( \hat{y}_0 \in \text{EPrefs}_0(S, \tilde{C}) \) implies that \( \hat{C}(\hat{y}_0) \in \text{Enab}(\text{LastX}_{S}(\hat{y}_0)) \neq \emptyset \). Further, it follows from (b) that for all \( x_0 \in \text{Ext}^{-1}(y_0) \) holds that \( \alpha(x_0) \subset \text{LastX}_{S}(\hat{y}_0) \). This implies \( \hat{u}_0 := \hat{C}(\hat{y}_0) \in \text{Enab}(\alpha(x_k)) \). Then it follows from (A2,i) that for any \( u_0 \in \beta(\hat{u}_0) \) holds that \( u_0 \in \text{Enab}(x_0) \) and hence \( u_0 \in \text{Enab}_S(\text{LastX}_{S}(y_0)) \). With this, the claim follows from the construction of \( C \).

- Induction step from \( k \) to \( k + 1 \): It follows from induction hypothesis (d) that \( \emptyset \neq C(\sigma_{k+1}) \subseteq \text{Enab}_S(\text{LastX}_{S}(\sigma_{k-1})) \) for any choice of \( \sigma_{k-1} \). We consider any coice \( u_{k+1} \in C(\sigma_{k-1}) \).

  - (a) follows from the definition of \( \text{EPrefs}_k(S, \tilde{C}) \).

  - (b/c) We fix \( \pi_k = x_0u_0 \ldots x_k \) and \( \sigma_k = y_0u_0 \ldots y_k \) as required. Then it follows from induction hypothesis (c) that \( \pi_{k-1} \in \text{Ext}^{-1}(\sigma_{k-1}) \) for any \( \sigma_{k-1} \) compatible with \( \pi_{k-1} = \pi_k|_{[0:k-1]} \) and \( \sigma_{k-1} = \sigma_k|_{[0:k-1]} \). This implies \( \alpha(x_{k-1}) \subseteq \text{LastX}_{S}(\sigma_{k-1}) \) by construction.

  As \( \sigma_{k-1} \in \text{EPrefs}_{k-1}(S, \tilde{C}) \) from induction hypothesis (b) and \( \tilde{C} \) is a sound controller for \( S \) we know that \( \hat{u}_{k-1} \in \text{Enab}(\text{LastX}_{S}(\sigma_{k-1})) \neq \emptyset \). This implies \( \hat{u}_{k-1} \in \text{Enab}(\alpha(x_{k-1})) \). Then it follows from (A2,i) that \( \hat{u}_{k-1} \in \text{Enab}(x_{k-1}) \) and from (A2,ii) that \( \alpha(F(x_{k-1}, u_{k-1})) \subseteq F(\hat{u}_{k-1}, \hat{u}_{k-1}) \) for all \( \hat{u}_{k-1} \in \alpha(x_{k-1}) \). Now recall that \( \hat{x}_k \in F(x_{k-1}, u_{k-1}) \) which implies \( \hat{x}_k \in F(\hat{x}_{k-1}, \hat{u}_{k-1}) \) for any \( \hat{x}_{k-1} \in \alpha(x_{k-1}) \) and any \( \hat{u}_k \in \alpha(x_k) \). This implies by construction that \( \hat{x}_k \in \text{Ext}^{-1}(\sigma_k) \) for any considered \( \hat{x}_k \), which proves claim (b).

  Now, recall from the construction of \( \sigma_k \) in (c) that \( \hat{y}_k \in \gamma(H(\hat{x}_k)) \). Then it follows from (A3) that \( \hat{y}_k \in H(\hat{x}_k) \) for any \( \hat{x}_k \in \alpha(x_k) \). With this, claim (c) follows from the definition of \( \text{EPrefs}_k(S, \tilde{C}) \).

- (d) It follows from (b) that there exists a \( \sigma_k \in \Omega_{\beta, \gamma}(\sigma_k) \) s.t. \( \hat{C}_k \in \text{EPrefs}_k(S, \tilde{C}) \). As \( \tilde{C} \) is a sound controller for \( S \) it follows that for all such \( \sigma_k \) we have \( \hat{C}(\sigma_k) \neq \emptyset \) and \( \hat{C}(\sigma_k) \in \text{Enab}(\text{LastX}_{S}(\sigma_k)) \). Now it follows form (c) that for any \( \pi_k = x_0u_0 \ldots x_k \) \( x_0 \in \text{Ext}^{-1}(\sigma_k) \) holds that \( \alpha(x_k) \subseteq \text{LastX}_{S}(\sigma_k) \). This implies \( \hat{u}_k := \hat{C}(\sigma_k) \in \text{Enab}(\alpha(x_k)) \). Then it follows from (A2,i) that for any \( u_k \in \beta(\hat{u}_k) \) holds that \( u_k \in \text{Enab}(x_k) \) and hence \( u_k \in \text{Enab}_S(\text{LastX}_{S}(\sigma_k)) \). With this, the claim follows from the conctuction of \( C \) in (1).

B. Additional Proofs for Prop. 3

In this section we formalize the intuition behind Prop. 3. In particular, we prove a stronger claim, formalized in the following Prop. 4, from which Prop. 3 follows as a corollary.

**Proposition 4:** Given the premises of Def. 3, let \( \sigma = y_0u_0y_1u_1 \ldots \in Y(UY)^\omega \) be an external sequence. Then \( \sigma \in \mathcal{L}(\bar{A}_x) \) iff \( \sigma \notin \text{IBlock}(\tilde{S}) \) and either

(i) \( \sigma \notin \text{EPaths}(\tilde{S}) \), or

(ii) \( \sigma \in \text{EPaths}(\tilde{S}) \) and \( \hat{P}(\text{Ext}^{-1}(\sigma)) \subseteq \psi \).

We now prove Prop. 4 in multiple steps. We first show that an external sequence \( \sigma \) has a compliant infinite path over \( \bar{A}_x \) iff either \( \sigma \in \text{EPaths}(\tilde{S}) \) or \( \sigma \in \text{IBlock}(\tilde{S}) \). We further show that these two cases are disjointed and that we know that \( \sigma \in \text{IBlock}(\tilde{S}) \) iff there exists a compliant infinite path over \( \bar{A}_x \) that visits \( \bot \) infinitely often. This is formalized in the following lemma.

**Lemma 2:** Given the premises of Def. 3, let \( \bar{\sigma} \in \bar{Y}(UY)^\omega \). Then there exists an infinite path \( \hat{\pi} \) compliant with \( \bar{\sigma} \) iff \( \bar{\sigma} \in \text{EPaths}(\tilde{S}) \cup \text{IBlock}(\tilde{S}) \). Further, there exists a path \( \hat{\pi} \) compliant with \( \bar{\sigma} \) that visits \( \bot \) infinitely often if \( \bar{\sigma} \in \text{IBlock}(\tilde{S}) \).

**Proof:** “\( \Rightarrow \):” First recall that \( A_\psi \) is complete in \( \Lambda \times Y \). With this, it follows from the definition of \( \bar{A}_x \) that for any path of \( \tilde{S} \) there exists a path over \( \bar{A}_x \) with the same length. With this, it immediately follows that whenever \( \bar{\sigma} \in \text{EPaths}(\tilde{S}) \), there is by definition an infinite path over \( \bar{S} \) compliant with \( \bar{\sigma} \) and therefore this implies that there is an infinite compliant path over \( \bar{A}_x \) as well.

Further, if \( \bar{\sigma} \in \text{IBlock}(\tilde{S}) \) instead, we know that there exists a prefix \( \nu = \bar{\sigma}|_{[0:k]} \) s.t. \( \text{LastX}_{\tilde{S}}(\nu) \neq \emptyset \) and \( \hat{u}_k \notin \text{Enab}(\text{LastX}_{\tilde{S}}(\nu)) \). Then it again follows from the completeness of \( A_\psi \) that there exists a path over \( \bar{A}_x \) that is compliant with \( \nu \) and for which \( \hat{u}_k \notin \text{Enab}(\text{LastX}_{\tilde{S}}(\nu)) \). This implies from the construction that \( \bar{q}_k = \bot \). As \( \bot \) is absorbing this implies that there is an infinite compliant path for \( \bar{\sigma} \) that visits \( \bot \) infinitely often.
“⇒”: Let \( \overline{\pi} \) be an infinite path compliant with \( \overline{\sigma} \). This implies that either (i) there exists some \( k \in \mathbb{N} \) s.t. \( \overline{q}_k = (\overline{x}_k, q_k) \neq \perp \) and \( \overline{q}_{k'} = \perp \) for all \( k' > k \) or (ii) \( \overline{x}_{k+1} \in \overline{F}(\overline{x}_k, \overline{u}_k) \) and \( \overline{y}_k \in \overline{H}(\overline{x}_k) \) for all \( k \in \mathbb{N} \). If (i) holds, we see that \( \overline{\pi} \) visits \( \perp \) infinitely often. We also see that, by construction, \( \overline{F}(\overline{x}_k, \overline{u}_k) = \emptyset \) and hence \( \overline{u}_k \notin Enab(\overline{x}_k) \).

With this it follows from the lifting of Enab to sets via intersection that \( \overline{u}_k \notin Enab(LastX_S(\overline{\sigma}_{|0,k})) \) while \( \overline{x}_k \in LastX_S(\overline{\sigma}_{|0,k}) \neq \emptyset \). Therefore, \( \overline{\sigma} \in IBlock(S) \). If (ii) holds it immediately follows from the definition that \( \overline{\sigma} \in Paths(S) \) and hence, \( \overline{\sigma} \in EPaths(S) \).

After establishing Lem. 2, we see that all external sequences that only have finite compliant paths over \( \overline{A}_x \) must not be contained in \( EPaths(S) \) and obviously also not in \( IBlock(S) \). With this, it remains to show that for all sequences \( \overline{\sigma} \in EPaths(S) \) with a compliant infinite play over \( \overline{A}_x \) holds that all compliant sequences over \( \overline{A}_x \) fulfill the specification. This is formalized in the following lemma.

**Lemma 3:** Given the premises of Def. 3, let \( \overline{\sigma} \in EPaths(S) \). Then \( \overline{\sigma} \in L(\overline{A}_x) \) iff \( \overline{P}(\text{Ext}^{-1}_S(\overline{\sigma})) \subseteq \{ \psi \} \).

**Proof:** First observe that the definition of \( \text{Ext}^{-1}_S(\overline{\sigma}) \) only contains infinite paths \( \nu = \overline{x}_0 \overline{u}_1 \overline{x}_1 \overline{u}_2 \overline{x}_2 \ldots \) s.t. for all \( k \in \mathbb{N} \) we have \( \overline{y}_k \in \overline{H}(\overline{x}_k) \) and \( \overline{u}_k \in \overline{F}(\overline{x}_k, \overline{u}_k) \). As \( \overline{A}_x \) is complete on \( \Lambda \times \Upsilon \) we know that for all \( \rho \in \overline{P}(\nu) \) (for any \( \nu \in \text{Ext}^{-1}_S(\overline{\sigma}) \), i.e., for all \( \rho \in \overline{P}(\text{Ext}^{-1}_S(\overline{\sigma})) \)) there exists an infinite path \( \alpha = q_0(\lambda_0, \mu_0)q_1 \ldots \) over \( A_x \). Now it follows from the construction of \( \overline{A}_x \) that for any such \( \nu \) and \( \alpha \) there is an infinite path \( \pi = (\overline{x}_0, q_0)(\overline{u}_0, \overline{y}_0)(\overline{x}_1, q_1) \ldots \) over \( \overline{A}_x \). We see that any such path \( \pi \) is compliant with \( \overline{\sigma} \) and never visits \( \perp \). We therefore know that \( \pi \) visits \( \overline{\pi} \) infinitely often iff \( \alpha \) visits \( F \) infinitely often. With this we have \( \overline{\sigma} \in L(\overline{A}_x) \) iff \( \rho \in L(\overline{A}_x) = \{ \psi \} \). As this reasoning holds for all \( \rho \in \overline{P}(\text{Ext}^{-1}_S(\overline{\sigma})) \) the claim follows.

With this, Prop. 4 becomes a direct consequence of Lem. 2 and Lem. 3 as formalized in the final proof below.

**Proof:** We proof both directions separately.

- “⇒”: Let \( \overline{\sigma} = q_0(\overline{u}_0, \overline{y}_0)(\overline{u}_1, \overline{y}_1) \ldots \in L(\overline{A}_x) \). This implies that all paths \( \overline{\pi} = q_0(\overline{y}_0, \overline{u}_0)(\overline{y}_1, \overline{u}_1, \overline{y}_2) \ldots \) over \( \overline{A}_x \) that are compliant with \( \overline{\sigma} \) are either finite or are infinite and visit \( \overline{F} \) only finitely often.

Now it follows from Lem. 2 that whenever all compliant paths are finite, we know that \( \overline{\sigma} \notin EPaths(S) \) and \( \overline{\sigma} \notin IB(S) \) and \( EPaths(S) \) and \( IB(S) \) are disjoint. We now consider the case that there exists an infinite compliant path. This path only visits \( \overline{F} \) finitely often it follows from Lem. 2 that we again have \( \overline{\sigma} \notin IB(S) \). We therefore know that \( \overline{\sigma} \notin EPaths(S) \). With this, it follows from Lem. 3 and the fact that \( \overline{\sigma} \in L(\overline{A}_x) \) that we also have \( \overline{P}(\text{Ext}^{-1}_S(\overline{\sigma})) \subseteq \{ \psi \} \).

- “⇐”: We have \( \overline{\sigma} \notin IB(S) \) and consider two cases. If \( \overline{\sigma} \notin EPaths(S) \) it follows from Lem. 2 that all paths over \( \overline{A}_x \) that are compliant with \( \overline{\sigma} \) are finite, and hence \( \overline{\sigma} \in L(\overline{A}_x) \) from the definition of acceptance of UCA. Now let \( \overline{\sigma} \in EPaths(S) \) and \( \overline{P}(\text{Ext}^{-1}_S(\overline{\sigma})) \subseteq \{ \psi \} \).

Then it follows from Lem. 3 that \( \overline{\sigma} \in L(\overline{A}_x) \).
consider some $\hat{\sigma} \in \text{Ext}_{\hat{S}}(\pi)$. Then we know from the definition of CPaths($\hat{S}, \hat{C}$) that $\hat{\sigma} \in \text{EPaths}(\hat{S}, \hat{C})$ and hence $\text{Ext}^{-1}_{\hat{S}}(\hat{\sigma}) \subseteq \text{CPaths}(\hat{S}, \hat{C})$. With this, it follows that $\hat{P}(\text{Ext}^{-1}_{\hat{S}}(\hat{\sigma})) \subseteq \langle \psi \rangle$ which proves the claim.

Lemma 7: If $\mathcal{W}(\hat{S}, \hat{\psi}, \hat{P}) \neq \emptyset$ then $\text{EPaths}(\hat{S}) \cap \mathcal{L}(\hat{A}_x) \neq \emptyset$.

Proof: We know from Lem. 6 that there exists at least one $\hat{\sigma} \in \text{EPaths}(\hat{S})$ s.t. $\hat{P}(\text{Ext}^{-1}_{\hat{S}}(\hat{\sigma})) \subseteq \langle \psi \rangle$. Then it follows from Lem. 3 that $\hat{\sigma} \in \mathcal{L}(\hat{A}_x)$. This proves the claim.

Lemma 8: If $\text{EPaths}(\hat{S}) \cap \mathcal{L}(\hat{A}_x) \neq \emptyset$ then there exists a non-empty output-feedback control strategy induced by $\mathcal{M}$.

Proof: As $\text{EPaths}(\hat{S}) \cap \mathcal{L}(\hat{A}_x) \neq \emptyset$ we know that there exists a $\hat{\sigma} \in \text{EPaths}(\hat{S}) \cap \mathcal{L}(\hat{A}_x)$. With this, it follows from Lem. 2 that there exists an infinite path $\pi$ over $\hat{A}_x$ that is compliant with $\hat{\sigma}$. This implies that $\pi$ is also a path of $\mathcal{M}$ and hence, $\hat{\sigma} \in \text{EPaths}(\mathcal{M})$. As we know that $\hat{\sigma} \in \text{EPaths}(\hat{S})$ from above, we see that there exists a non-empty control strategy induced by $\mathcal{M}$ from Def. 5.

With this, Thm. 3, Part (ii) becomes a direct consequence of Lem. 7 and Lem. 8.