An optimal control problem by a first-order hyperbolic system with delay and integral control constraints

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Abstract. A non-classic optimality condition and numerical algorithm for smooth boundary control in semi-linear first-order hyperbolic system with delay are presented. An additional integral control constraint is considered. The suggested approach is based on special variations of admissible continuously differentiable controls. These variations lead to a necessary optimality condition which is a base for a numerical method. An illustrative example demonstrates the effectiveness of the proposed approach.

1. Introduction
Investigation of optimal control problems with delay began from studying of systems of ordinary differential equations. A lot of results have been transferred from systems without delay. For instance, existence of optimal controls was studied in [1]; necessary optimality conditions were suggested in [2, 3].

Existence of optimal control for distributed parameter problems with delay were studied in [4, 5]. In [6] necessary optimality conditions were proposed. Computational methods based on ideas of reductions to simpler problems, dynamic programming, etc. were suggested in [7, 8].

In recent years, several results for optimal control problems with delay have also been obtained. For example, in [9] bilinear system of the differential equations with delay is considered. As a result, sufficient conditions for asymptotic stability were obtained.

Quite recently other types of optimal control problems with continuous controls have been investigated with deriving necessary optimality conditions and practical applications [10, 11, 12]. These developments address new classes of optimal control problems for the so-called sweeping processes governed by discontinuous differential inclusions with controlled moving sets. Controls actions of moving sets are continuous. Necessary optimality conditions for such problems were derived by applying discrete approximations and advanced tools of variational analysis and generalized differentiation. These techniques may be useful for the problems with delay.

In this paper we consider an optimal control problem by boundary differential conditions with delay for semilinear first-order hyperbolic system. Such problems arise in modeling of age-structured biological populations. Boundary delay is connected with environmental factors, reproduction peculiarities, etc. we study the problem in the class of smooth controls that satisfy
the integral constraint. We apply the idea of the general approach [13] based on using a special variation that provides smoothness of controls and satisfaction the constraint.

2. Problem statement
Consider in a rectangle \( P = S \times T, S = [s_0, s_1], T = [t_0, t_1] \) a semilinear first-order hyperbolic system

\[
z_t + D(s,t)z_s = f(z,s,t), \quad (s, t) \in P. \tag{1}
\]

Here independent variables \((s, t) \in P, z = z(s, t)\) is a \(n\)-dimensional state vector-function, \(D(s,t)\) is the given diagonal \((n \times n)\) - matrix function with sign-constant, continuously differentiable diagonal elements \(d_i = d_i(s, t), \quad i = 1, 2, \ldots, n\), the vector-function \(f\) is continuous with respect to its arguments together with partial derivatives \(f_z\).

Define controlled initial-boundary conditions for the system (1) as

\[
\begin{align*}
\frac{dz^+(s_0, t)}{dt} &= p(z^+(s_0, t), z^+(s_0, t - h), u(t), t), \quad t \in T, \\
z(s, t_0) &= z^0(s), \quad s \in S, \quad z^-(s_1, t) = \nu(t), t \in T, \\
z^+(s_0, t) &= q(t), \quad t \in [-h; t_0]; \quad h > 0,
\end{align*}
\]

where \(h\) is a constant delay, the signs (\(+\)) and (\(-\)) denote state subvectors \(z(s, t)\) corresponding to the positive and negative diagonal elements \(d_i(s, t)\).

The admissible control \(u = u(t)\) is a smooth \(r\)-dimensional vector-function on segment \(T\). It satisfies the integral constraint

\[
\int_T \Phi(u(t)) \, dt = L, \tag{3}
\]

where \(L\) is a constant and function \(\Phi\) satisfies the following condition

\[
\Phi(\lambda u) = \lambda^\alpha \Phi(u), \quad \alpha \geq 1.
\]

The objective of the control for the process described by (1)-(3) is to minimize the functional

\[
J(u) = \int_S \varphi(z(s, t_1), s) \, ds, \tag{4}
\]

Denote \(y(t) = z^+(s_0, t - h)\).

The optimal control problem (1)-(4) is considered under the following assumptions:

1) functions \(z^0(s), \nu(t), q(t)\) are continuous with respect to their arguments on the sets \(S\) and \(T\) respectively and satisfy the conditions

\[
\nu(t_0) = (z^0(s_1))^-, \quad q(t_0) = (z^0(s_0))^+;
\]

2) functions \(f(z,s,t)\) and \(\varphi(z,s)\) are continuous with respect to their arguments, and they have continuous and bounded partial derivatives with respect to the state function \(z\):

3) function \(p(z^+, y, u, t)\) is continuous and continuously differentiable, and it has bounded partial derivatives with respect to \(z^+, y\) and \(u\).

The solution of the initial-boundary problem for the hyperbolic system is understood in a generalized sense. Each component of this solution is continuously differentiable along the corresponding characteristic family [14]. So, instead of the left side of system (1) we consider the differential operator

\[
\left( \frac{dz}{dt} \right)_D = \begin{pmatrix} \frac{dz_1}{dt} \\frac{dz_2}{dt} \cdots \frac{dz_n}{dt} \end{pmatrix}_D,
\]

where \(\frac{dz_i}{dt}_D\) is the derivative of \(i\)-th component of the state vector along the corresponding family of characteristic curves.
3. Increment formula for the functional
Consider the problem for two admissible processes \( \{u, z\} \) and \( \{\tilde{u} = u + \Delta u, \tilde{z} = z + \Delta z\} \):
\[
\left( \frac{d\Delta z}{dt} \right)_D = \Delta f(z, s, t),
\]
\[
\Delta z^+(s_0, t) = 0, \quad t \in [-h; t_0]; \quad \Delta z(s, t_0) = 0, \quad s \in S; \quad \Delta z^-(s_1, t) = 0, \quad t \in T;
\]
\[
\Delta z^+_t(s_0, t) = \Delta p(z^+(s_0, t), y(t), u(t), t),
\]
(5)
where
\[
\Delta p(z^+(s_0, t), y(t), u(t), t) = p(\tilde{z}^+, \tilde{y}, \tilde{u}, t) - p(z^+, y, u, t) =
\]
\[
\Delta \gamma^p(z^+, y, u, t) + \Delta \gamma^p(z^+, y, \tilde{u}, t) + \Delta \gamma^p(\tilde{z}^+, \tilde{y}, \tilde{u}, t),
\]
Consider the following expression
\[
\Delta J(u) = \int_S \Delta \varphi(z(s, t_1), s) ds + \int_P \langle \psi(s, t), (\frac{d\Delta z}{dt})_D - \Delta f(z, s, t) \rangle ds dt +
\]
\[
\int_T \langle g(t), \Delta z^+_t(s_0, t) - \Delta p(z^+(s_0, t), y(t), u(t), t) \rangle dt,
\]
where \( \psi(s, t) \) and \( g(t) \) are still undefined vector-functions having the same dimensions and smoothness properties as \( z(s, t) \) and \( z^+(t) \) respectively. Here \( \langle \ldots \rangle \) is a designation of a scalar product in Euclidean space of a corresponding dimension.

Introduce the following auxiliary functions
\[
H(\psi(s, t), z(s, t), s, t) = \langle \psi(s, t), f(z, s, t) \rangle,
\]
\[
h(g(t), z^+(s_0, t), y(t), u(t), t) = \langle g(t), p(z^+(s_0, t), y(t), u(t), t) \rangle.
\]
Let functions \( \psi(s, t), g(t) \) be the solutions of the following adjoint problem
\[
\left( \frac{d\psi}{dt} \right)_D + D_s \psi = -H_\psi(\psi, z, s, t), \quad \psi(s, t_1) = -\varphi_\psi(z(s, t_1), s),
\]
(6)
\[
\psi^+(s_1, t) = 0; \quad \psi^-(s_0, t) = 0, \quad t \in T;
\]
\[
g_L = -h_z[t] - h_y[t + h] - D^+(s_0, t)\psi^+(s_0, t), \quad t \in [t_0; t_1 - h],
\]
\[
g_L = -h_z[t] - D^+(s_0, t)\psi^+(s_0, t), \quad t \in [t_1 - h; t_1],
\]
\[
g(t_1) = 0; \quad g(t) \equiv 0, \quad t > t_1,
\]
(7)
where \( D^+ \) is a submatrix of the matrix \( D \) corresponding to the positive diagonal elements \( d_i(s, t) \).
Here
\[
h_z[t] = h_z(g(t), z^+(s_0, t), y(t), u(t), t),
\]
\[
h_y[t + h] = h_y(g(t + h), z^+(s_0, t + h), z^+(s_0, t), u(t + h), t + h).
\]
Then, the increment formula for the functional takes the following form:
\[
\Delta J(u) = - \int_T \Delta \gamma^p(g(t), z^+(s_0, t), y(t), u(t), t) dt + \eta,
\]
(8)
where

\[
\eta = \int_S \alpha \sigma (|\Delta z (s, t_1)|) \, ds + \int_\mathcal{T} \int_P (o_H (|\Delta z (s, t)|) \, ds \, dt +
\]

\[
\int_\mathcal{T} [o_H (|\Delta z^+ (s_0, t)|) + \langle \Delta \tilde{x}^+ h_y (g (t), z^+ (s_0, t), y (t), u (t), \Delta z^+ (s_0, t)) \rangle \, dt +
\]

\[
\int_\mathcal{T} [o_H (|\Delta y (t)|) + \langle \Delta \tilde{y}^+ h_y (g (t), z^+ (s_0, t), y (t), u (t), \Delta y (t)) \rangle \, dt +
\]

\[
\int_\mathcal{T} \langle \Delta \tilde{y}^+ h_y (g (t), z^+ (s_0, t), y (t), u (t), \Delta y (t)) \rangle \, dt.
\]

Under the condition (5) the estimation of a state increment (analogously to [14]) takes the form

\[
\gamma (t) = \max_{(\xi, \tau) \in \mathcal{P} (t)} |\Delta z (\xi, \tau)| \leq M_3 \int_{t_0}^t |\Delta u (\tau)| \, d\tau,
\]

\[
P (t) = \{ (\xi, \tau) \in \mathcal{P} : \tau \leq t \}.
\]

Here \( M_3 = \sqrt{\pi} M_2 \cdot e^{\sqrt{\pi} M_2 (t_1 - t_0)} \), where \( M_2 = M_1 \cdot e^{2 M_1 (t_1 - t_0)} \), \( M \) is a Lipschitz constant for the function \( f \), \( M_1 \) is a Lipschitz constant for the function \( p \).

4. An optimality condition and iterative method

Consider non-classic variation of control. Let \( u = u (t) \) be an admissible control. Give the varied control by the formula

\[
u_{\varepsilon, \delta} (t) = \mu (t) u (t + \varepsilon \delta (t)) \quad \mu (t) = (1 + \varepsilon \delta (t))^{\frac{1}{\alpha}}, \quad t \in T,
\]

where \( \varepsilon \in [0, 1] \) is a parameter of variation.

Here \( \delta (t) \) is a twice continuously differentiable function, \( t_0 \leq t + \delta (t) \leq t_1, \quad t \in T \). It is easy to verify that the function (10) is an admissible control too, if \( \delta (t_0) = \delta (t_1) = 0, \quad |\delta (t)| \leq 1, \quad t \in T \) [13]. Originally, the inner variation has been used for proving a necessary optimality condition in optimal control problems by ordinary differential equations with delay [15]. However, this variation turned out to be very efficient for the problems with integral constraints on control function.

Then, it follows from (8)

\[
\Delta J (u) = -\varepsilon \int_\mathcal{T} \langle \{ h_u, u (t) \} - \frac{1}{\alpha} \langle h_u, u \rangle \delta (t) \rangle \, dt + o (\varepsilon).
\]

We have the following theorem from the increment formula (11) and (9).

**Theorem.** Consider the problem (1)-(4). For an optimal control \( u^* \) the following condition holds

\[
\langle h_u, u^* \rangle - \frac{1}{\alpha} \langle h_u, u^* \rangle_t = 0, \quad t \in T.
\]

**Remark.** If integral constraint is linear with respect to control \( (\alpha = 1) \), then (12) can be written in a simpler form

\[
\langle h_u, u^* \rangle = 0, \quad t \in T.
\]

This optimality condition allows to construct an iterative procedure.

4
Let an admissible control $u^0$ be given and $u^k$ be calculated by the method. Calculate

$$\omega_k(t) = \langle h_u(u^k(t), \dot{u}^k(t)) \rangle - \frac{1}{\alpha} \langle h_u(u^k(t), t), u^k(t) \rangle_t.$$ 

Note that we need to integrate the adjoint problem (6)-(7). Construct an admissible function $\delta_k(t)$ which has the coinciding sign with $\omega_k(t)$. The ways of constructing such functions were considered in [13]. Then we determine one-parametric collection of controls $u^k_\varepsilon$ (see (10)) and solve a problem

$$J(u^k_\varepsilon) \to \min, \quad \varepsilon \in [0, 1].$$

The next approximation is given by the formula

$$u^{k+1}(t) = u^k_{\varepsilon_k}(t).$$

The corresponding convergence theorem to the necessary optimality condition (12) is given in [14]. The given iterative process is relaxative and convergent in the sense

$$\mu(u^k) = \int_T \delta_k(t) \omega_k(t) \, dt \to 0, \quad k \to \infty$$

and additional standard conditions (boundedness from below of a functional and Lipschitz inequality for partial derivatives with respect to $z$ from right-hand sides of a system and objective functions).

5. Illustrative example

The results described above were applied to solving a test example. The algorithm was coded in Matlab 7.0. In the square $[0, 1] \times [0, 1]$ we consider the optimal control problem

$$z_{1t} + z_{1s} = z_1 + z_2 + e^s \cos t,$$

$$z_{2t} - z_{2s} = z_2 - \sin t,$$

$$z_{1t}(0, t) = u \cdot z_1(0, t - 0.15), \quad \int_T u(t) \, dt = \frac{1}{3}, \quad z_1(0, t) = 0.1 \cdot t, \quad t \in [-0.15; 0];$$

$$z_1(1, t) = 0.2 \cdot t, \quad z_2(s, 0) = s + 0.3, \quad z_2(s, 0) = s^2.$$

The cost functional

$$J(u) = \frac{1}{2} \int_S (z_1(s, 1) - z_1(s))^2 + (z_2(s, 1) - z_2(s))^2 \, ds \to \min, \quad u \in U,$$

where $z_1(s) = z_1(s, 1)$, $z_2(s) = z_2(s, 1)$ are evaluated for the control $u(t) = \frac{2}{3} (1 - t)$. We solved the problem by the described method under different initial approximations:

1) The initial control is $u^0(t) = \frac{1}{3} (\sin 2\pi t + 1)$. The value of the functional is $J(u^0) = 1.6723$.

We have obtained the following results: the value of the cost functional on the procedure output is $J(u^k) = 0.0002571$, the optimality error is $\max_{t \in T} \left| \omega_k(t) \right| = 0.0006125$, the total number of iteration equals 27, the stop criterion consisted in attaining the given accuracy with respect to the functional value.

2) The initial control is $u^0(t) = t^2$. The functional value is $J(u^0) = 1.9346$. 
We have obtained the following results: \( J(u^k) = 0.003248 \), \( \max_{t \in T} \mid \omega_k(t) \mid = 0.007346 \), the total number of iteration equals 19. The stop criterion is \( \varepsilon_k \leq 10^{-5} \) (there is no improvement of the functional on the method step).

3) The initial control is \( u^0(t) = \frac{1}{3} \). The functional value is \( J(u^0) = 1.4758 \).

We have obtained the following results: \( J(u^k) = 0.04123 \), \( \max_{t \in T} \mid \omega_k(t) \mid = 0.03287 \), the total number of iteration equals 14. The stop criterion is \( \varepsilon_k \leq 10^{-5} \) (there is no improvement of the functional on the method step).

6. Conclusion
In this paper we considered the optimal control problem by a semilinear first-order hyperbolic system with delay. We proposed the optimality condition in a class of smooth controls satisfied the integral constraint. We used a non-classic variation for control that provides smoothness of control function and satisfaction the constraint. The series of numerical experiments is carried out. Numerical experiments showed the efficiency and applicability of the method.

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