A HIGH FIBERED POWER OF A FAMILY OF VARIETIES OF GENERAL TYPE
DOMINATES A VARIETY OF GENERAL TYPE

with a few diagrams and one illustration
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March 19, 2022

0. INTRODUCTION

In Which
We Are Introduced to Our Main Theorem, and the Story Begins.

We work over $\mathbb{C}$.

0.1. Statement. We prove the following theorem:

Theorem 0.1 (Fibered power theorem). Let $X \to B$ be a smooth family of positive dimensional varieties of general type, with $B$ irreducible. Then there exists an integer $n > 0$, a positive dimensional variety of general type $W_n$, and a dominant rational map $X^n_B \dashrightarrow W_n$.

Specifically, let $m_n : X^n_B \dashrightarrow W_n$ be the $n$-pointed birational-moduli map. Then for sufficiently large $n$, $W_n$ is a variety of general type.

The latter statement will be explained in section 3. This solves “Conjecture H” of [CHM], §6.1 as well as the question at the end of remark 1.3 in [R-V].

Following Viehweg’s suggestions in [V3], the fibered power theorem is proved by way of the following theorem:

Theorem 0.2. Let $X \to B$ be a smooth family of positive dimensional varieties of general type, with $B$ irreducible, and $\text{Var}(X/B) = \dim B$. Then there exists an integer $n > 0$ such that the fibered power $X^n_B$ is of general type.

0.2. Main ingredients. The starting point is a theorem of Kollár, which roughly speaking says that given $f : X \to B$ a morphism of smooth irreducible projective varieties whose generic fiber is a variety of general type, and $\text{Var}(X/B) = \dim B$, then for large $m$ the saturation of $f_*(\omega^m_f)$ has many sections. A very useful trick of Viehweg shows that this implies that for large $m$ the sheaf $\omega^m_f$ itself has many sections, that is, $\omega_f$ is big.

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1Partially supported by NSF grant DMS-9503276.
Following [CHM], one would like to use these sections pulled back to the fibered powers $f_n : X^n_B \to B$ of $X$ over $B$ to overcome any possible negativity in $\omega_B$. Unfortunately, the fibered powers may become increasingly singular, and it is not easy to tell who wins in the race between the positivity of $\omega_{f_n}$ and the so called adjoint conditions imposed by the singularities of $X^n_B$. The fact that $\omega_f$ may have positivity “by accident”, as shown by the example in [CHM], §6.1 of plane quartics, shows that something more is needed - the fibration $X \to B$ should be “straightened out” before we can use sections coming from Kollár’s theorem.

Semistable reduction would be sufficient for this purpose, but unfortunately semistable reduction for families of fiber dimension > 2 over a base of dimension > 1 is yet unknown. It is not known whether unipotent monodromies would suffice. The case of curves was done in [CHM] using the moduli space of stable curves, and the case of surfaces was done in [Has] using the moduli space of stable surfaces.

Lacking such constructions in higher dimensions, we will use a variant of semistable reduction, introduced by de Jong [dJ]. This variant involves, after a suitable base change and birational modification, a Galois cover $Y \to X$, such that $Y \to B$ is a composition of families of curves with at most nodes. The fibers now are much better controlled, but we are left with descending differential forms from $Y^n_B$ to $X^n_B$. This is done by studying the behavior of the ramification ideals in the fibered powers.

0.3. Arithmetic background and Applications. Results of this type are motivated by Lang’s conjecture. See, e.g., [CHM], [Has], [N], [N-V], [Pac].

Let $K$ be a number field (or any field finitely generated over $\mathbb{Q}$), and let $X$ be a variety of general type over $K$. According to a well known conjecture of S. Lang (see [L], conjecture 5.7), the set of $K$-rational points $X(K)$ is not Zariski dense in $K$. In [CHM], it is shown that this conjecture of Lang implies the existence of a uniform bound $B(K, g)$ on the number of $K$ rational points on curves of genus $g$ (a stronger implication arises if one assumes a stronger version of Lang’s conjecture). This result was later refined in [N], and the ultimate result of this type was recently obtained by P. Pacelli in [Pac], to wit:

**Theorem 0.3** (Pacelli [Pac], Theorem 1.1). Assume that Lang’s conjecture is true. Let $g \geq 2$ and $d \geq 1$ be integers, and let $K$ be a field as above. Then there exists an integer $P_K(d, g)$, such that for any extension $L$ of $K$ of degree $d$, and any curve $C$ of genus $g$ defined over $L$, one has

$$\#C(L) \leq P_K(d, g).$$
The main geometric ingredient in the above mentioned results is the “Correlation Theorem” of [CHM], which is Theorem 0.1 for curves. In [CHM], §6, a version of Theorem 0.1 was conjectured (“Conjecture H”), with the suggestion that strong uniformity results would follow from such a theorem. This was further investigated in [N-V], where it is shown that Theorem 0.1 gives an alternative proof for Pacelli’s theorem, as well as other strong implication results for curves and higher dimensional varieties. As an example of a result which does not follow from Pacelli’s theorem, we have (see [N-V], Corollary 3.4 and theorem 1.7):

**Corollary 0.4.** Assume that the weak Lang conjecture holds. Fix a number field $K$ and an integer $d$. Then there is a uniform bound $N$ for the number of points of degree $d$ over $K$ on any curve $C$ of genus $g$ and gonality $> 2d$ over $K$. In fact, $N$ depends only on $g, d$ and the degree $[K : \mathbb{Q}]$.

Another conjecture of Lang states that if $X$ is a complex variety of general type, the union of rational curves on $X$ is not Zariski dense. In [N-V] a few implications of this conjecture were investigated, see [N-V] §3. Using [Pac], proposition 2.8, P. Pacelli is able to obtain the following remarkable result ([Pac], corollary 5.4): Lang’s conjecture about rational curves implies that there is a bound $P_{geom}(d, g)$ such that for any complex curve $C$ of genus $g$, and curve $D$ of gonality $d$, the number of nonconstant morphisms $D \to C$ is bounded by $P_{geom}(d, g)$. This result can again be deduced from theorem 0.1, but in an unsatisfactory way – one has to reprove proposition 2.8 of [Pac], repeating some of the steps, and therefore Pacelli’s direct method is more appropriate.

Another direction where our theorem falls short of obtaining a definite result is the logarithmic case. Here again Pacelli’s methods should directly imply results regarding uniform boundedness of stably integral points on elliptic curves (see [Ri] for the definition). One suspects that in the future a fibered power theorem will be available for log-varieties. At the moment, the main difficulties seem to lie in obtaining an appropriate generalization of the theorems of Kollár and Viehweg.

0.4. **Acknowledgements.** I would like to thank F. Hajir and R. Gross, whose question kept me thinking about the problem through a period when no significant result seemed to be possible. Thanks to B. Hassett, J. de Jong, J. Kollár, P. Pacelli, R. Pandharipande, E. Viehweg and F. Voloch, for helpful discussions. The realization that pluri-nodal fibration have mild singularities was inspired by Hassett’s results in [Has], §4. The understanding of the utility of such fibrations was reinforced by Pacelli’s results.
1. PRELIMINARIES

In Which
We Set Up Some Terminology about
Families of Varieties and State a
Lemma about Groups.

1.1. Definitions. A variety is called an **r-G variety** if it has only rational
Gorenstein (and hence canonical) singularities. For a Gorenstein variety \(X\) to be r-G, it is necessary and sufficient that for any resolution of singularities
\(r : Y \to X\) one has \(r_* \omega_Y = \omega_X\) (see [E], II).

We say that a flat morphism of irreducible varieties \(Y \to B\) is **mild**, if for
any dominant \(B_1 \to B\) where \(B_1\) is r-G, we have that \(Y_1 = Y \times_B B_1\) is r-G.
Note that, by induction, if \(Y \to B\) is mild then the fibered powers \(Y^n_B \to B\) are
mild as well.

An **alteration** is a projective, surjective, generically finite morphism of irre-
ducible varieties. This is slightly different from the definition in [dJ], where
propernes is assumed rather than projectivity. An alteration \(B_1 \to B\) is **Galois**
if there exists a finite group \(G \subset \text{Aut}_B B_1\) such that \(B_1/G \to B\) is birational.

A **fibration** is a projective morphism of irreducible normal varieties whose
general fibers are irreducible and normal.

Given a fibration \(X \to B\) and an alteration \(B_1 \to B\) we denote by \(X_B \times B B_1\) the **main component** of \(X \times_B B_1\). Namely, if \(\eta_{B_1}\) is the generic point of \(B_1\),
then \(X_B \times B B_1 = X \times_B \eta_{B_1}\).

A **family** is a flat fibration.

A family \(Y \to Y_1\) is called a **nodal fibration** if every fiber is a curve with
at most ordinary nodes. A family \(Y \to B\) is called a **pluri-nodal fibration**
if it is a composition of nodal fibrations \(Y \to Y_1 \to \cdots \to B\). Note that while
nodal fibrations are generically smooth, this is not the case with pluri-nodal
fibrations.

Given a line bundle \(L\) and an ideal sheaf \(I\) on a variety \(X\), we say that \(L \otimes I\)
is **big** if for some \(k > 0\) the rational map associated to \(H^0(X, L^k \otimes I^k)\) is
birational to the image. It readily follows that, if \(L \otimes I\) is big, and \(J\) is an
ideal sheaf, then for sufficiently large \(k\) we have that \(L^k \otimes I^k J\) is big. The
definition differs somewhat from Kollár’s definition in [Ko]. In section 3 we will
refer to sheaves which are “big” in Kollár’s sense as **weakly big**: we say that
a sheaf \(F\) is **weakly big**, if for any ample \(L\) there is an integer \(a\) such that
\(\text{Sym}^a(F) \otimes L^{(-1)}\) is weakly positive (see [Ko], p.367, (vii)).
1.2. Group theory. For a finite group \( G \) let \( e(G) = \text{lcm}\{\text{ord}(g) | g \in G\} \). We will use the following obvious lemma:

**Lemma 1.1.** Let \( G \) be a finite group. Then for any \( n \), we have \( e(G^n) = e(G) \).

2. Ramification

In Which

We Encounter Ramification Ideals
Measuring Differences Between Pluralcanonical Sheaves in a Quotient Situation, and Show that These Ideals Can Be Controlled in Various Cases.

Let \( V \) be a quasi projective \( r \)-G variety, \( G \subset \text{Aut}(V) \) a finite group. Let \( W = V/G \), and \( q : V \to W \) the quotient map. Let \( r : W_1 \to W \) be a resolution of singularities. Note that \( W \) is normal, therefore it is regular in codimension 1. We can pull back sections of pluricanonical sheaves on the nonsingular locus \( W_{\text{ns}} \) and extend them into the pluricanonical sheaf of \( V \). Thus, for an integer \( n > 0 \) we have an injective morphism \( \phi_n : q^*r^*\omega^n_{W_1} \to \omega^n_V \).

Define the \( n \)-th ramification ideal \( J_n = J_n(G, V) = \text{Ann Coker} \phi_n \).

**Lemma 2.1.**

1. We have \( J_n \otimes \omega^n_V \cong q^*r^*\omega^n_{W_1} \).
2. For any integer \( k > 0 \) we have \( J_n^k \subset J_{nk} \).
3. The ideals \( J_n \) are locally defined: if \( V' \subset V \) is a \( G \)-invariant open subset, then \( J_n(G, V') = J_n(G, V)|_{V'} \).
4. The ideals \( J_n \) are independent of the choice of resolution \( r : W_1 \to W \).
5. The ideals \( J_n \) can be also obtained if we use a partial resolution \( r' : W_1 \to W \) where \( W_1 \) is \( r \)-G.

**Proof.** Since \( \omega_V \) is by assumption invertible, we have (1). For the same reason (2) follows: if \( \omega = \prod_{i=1}^k \omega_i \) where \( \omega_i \) are local sections of \( \omega^n_V \) and if \( f = \prod_{i=1}^k f_i \) where \( f_i \in \mathcal{J}_n \), then we can write \( f_i \cdot \omega_i = \sum g_{i,j} \cdot (q^*r^*\eta_{i,j}) \) and expanding we get that \( f\omega \) is a local section of \( q^*r^*\omega^{nk}_{W_1} \). It would be nice to have an actual equality for high \( n \). Part (3) follows by definition. Parts (4) and (5) follow by noticing that for a birational morphism \( r' : W_2 \to W_1 \) with \( W_2 \) nonsingular, we have \( r'_*\omega^{nk}_{W_2} = \omega^{nk}_{W_1} \) in both cases. \( \square \)

Traditionally, one studies ramification by reducing to the case where both \( V \) and \( W \) are regular. Most of the results below follow that line, with the exception of Proposition 2.7, where the author finds it liberating, if not essential, to avoid unnecessary blowups.

The ideals \( J_n \) give conditions for invariant differential forms to descend to regular forms on the quotient:
Proposition 2.2. Given an integer \( n > 0 \) we have
\[
(q_*(\omega^n_V \otimes J_n))^G = r_*\omega^n_{W_1}.
\]

Proof: A local section of \((q_*(\omega^n_V \otimes J_n))^G\) can be written as \(\sum q_*(f_i)r_*(s_i)\), where \(f_i\) are \(G\) invariant, therefore \(f_i = q^*g_i\).

The ideals \(J_n\) are bounded below in terms of multiplicities (here we first use the assumption on rational singularities):

Proposition 2.3 ([CHM] §4.2, lemma 4.1). Let \(\Sigma_{G,V} = \Sigma \subset V\) be the locus of fixed points:
\[
\Sigma = \{x \in V \mid \exists g \in G, g(x) = x\},
\]
viewed as a closed reduced subscheme, with ideal \(I_\Sigma\). Then \(I^{-1}_{\Sigma} \subset J_n\).

Proof. Let \(V_1\) be the normalization of \(W_1\) in \(\mathbb{C}(V)\). Let \(W'_1 \subset W_1\) be the open subset over which both \(V_1\) and the branch locus \(B_{W_1/W_1}\) are nonsingular. The codimension of \(W_1 \setminus W'_1\) is at least 2. Let \(V'_1\) be the inverse image of \(W'_1\). We have a diagram
\[
\begin{array}{ccc}
V'_1 & \xrightarrow{s} & V \\
\downarrow q_1 & & \downarrow q \\
W'_1 & \xrightarrow{r'_1} & W
\end{array}
\]
Let \(\omega\) be a \(G\)-invariant \(n\)-canonical form on \(V\), vanishing to order \(n \cdot (e(G) - 1)\) on \(\Sigma\). To show that \(\omega\) descends to \(W_1\) it suffices to descend it to \(W'_1\), since the codimension of the complement is at least 2. Since \(V\) is \(r\)-\(G\), \(\omega' = s^*\omega\) is a regular \(n\)-canonical form on \(V'_1\), vanishing to order \(n \cdot (e(G) - 1)\) on \(B_{V_1/W_1}\). The subgroup fixing a general point of a component of \(B_{V_1/W_1}\) is cyclic, and the action is given formally by \(u_1 \mapsto \zeta_k u_1, u_i \mapsto u_i\) for some root of unity \(\zeta_k, k \leq e(G)\), where \(u_i\) are local parameters, \(u_1\) a uniformizer for \(B_{V_1/W_1}\). Formally at such a point, the quotient map is given by \(w_1 = u_1^k, w_i = u_i, i > 1\). By assumption, \(\omega'\) can be written in terms of local parameters as \(\omega' = f(u)(u^{k-1}dw_1 \wedge \cdots \wedge dw_m)^n = f(u)q_1^*(dw_1 \wedge \cdots \wedge dw_m)^n\). The invariance implies that \(f(u) = q_1^*g(w)\) and therefore \(\omega' = q_1^*g(w)(dw_1 \wedge \cdots \wedge dw_m)^n\). \(\square\)

Remark. It is not difficult to obtain the following refinement of this proposition (see analogous case in [Ko], lemma 3.2): let \(B = q(\Sigma)_{\text{red}}\), and let \(I_B\) be the defining ideal. Then \(q^{-1}I_B^{\frac{1}{n(1 - \alpha_G)}} \subset J_n\).

Recall that if a group \(G\) acts on a variety \(V\), a line bundle \(L\) and an ideal \(I\) then the ring of invariant sections \(\oplus_{k \geq 0} H^0(Y, L^k \otimes I^k)^G\) has the same dimension as the ring of sections \(\oplus_{k \geq 0} H^0(Y, L^k \otimes \mathcal{I}^k)\). This allows us to have:
Corollary 2.4 (See more general statement in [Pac], Lemma 4.2). Let $X$ be a variety of general type and let $G = \text{Aut}_C(\mathcal{C}(X))$ be its birational automorphism group. Then for some $n > 0$ the quotient variety $X^n/G$, where $G$ acts diagonally, is of general type.

Proof. Applying Hironaka’s equivariant resolution of singularities, we may assume that $X$ is regular and $G = \text{Aut}_X$. Let $p_i : X^n \to X$ be the projection onto the $i$-th factor. Choose $n$ large enough so that $\omega_X^n \otimes \mathcal{I}_{\Sigma_{G,X}}$ is big. Therefore $\omega_X^n \otimes (\sum p_i^{-1}\mathcal{I}_{\Sigma_{G,X}})^n$ is big. But

$$(\sum p_i^{-1}\mathcal{I}_{\Sigma_{G,X}})^n \subset \mathcal{I}_{\Sigma_{G,X^n}} \subset J_n(G, X^n),$$

giving the result.

Let $\Sigma \subset V$ be the locus of fixed points, and let $\Sigma = \Sigma_1 \cup \Sigma_2$ be a closed decomposition. Then $J_n$ is supported along $\Sigma$, and can be written as $J_n = J_{n,\Sigma_1} \cap J_{n,\Sigma_2}$. Applying 2.3 we obtain:

Corollary 2.5. We have $(\mathcal{I}_{\Sigma_2}^{(G)})^n \cdot J_{n,\Sigma_1} \subset J_n$.

Our goal is to apply our propositions to powers of mild families. First, let $f : V \to B$ be mild. Assume that $B$ is r-G. As before, let $G \subset \text{Aut}_B(V)$, $W = V/G$, and $q : V \to W$ the quotient map.

Let $p_i : V^m_B \to V$ be the $i$-th projection. We naturally have $G^m \subset \text{Aut}_B(V^m_B)$ acting on all components. We denote by $q_m : V^m_B \to W^m_B$ the associated map. Let $r : W_1 \to W$ be a resolution of singularities.

Define $J_{m,n} = \prod p_i^{-1} J_n$.

Lemma 2.6. Assume that $W_1 \to B$ is mild. Then $J_{m,n} \subset J_n(G^m, V^m_B)$.

Proof. Denote $r_m : W_m = (W_1)^m_B \to W^m_B$ and $p_{i,W} : W_m \to W_1$ the $i$-th projection. Since $V \to B$ and $W_1 \to B$ are mild, we have that

$$\omega^n_{V^m_B/B} = \otimes_i p_i^*(\omega^n_{V/B}) \quad \text{and} \quad \omega^n_{W_m/B} = \otimes_i p_{i,W}^*(\omega^n_{W_1/B}).$$

Suppose a local section $w$ of $\omega^n_{V^m_B/B}$ is a monomial written as $w = \Pi p_i^*w_i$, and suppose $f \in J_{m,n}$ is a monomial written as $f = \Pi p_i^*f_i$. Then $fw = \Pi p_i^*(fw_i)$ is a local section of $q_m^*r_m^*\omega^n_{W_m/B}$.

Proposition 2.7. There exists a closed subset $F \subset B$ such that

$$(\mathcal{I}_F^{(G)})^n \cdot J_{m,n} \subset J_n(G^m, V^m_B).$$

Proof. Let $F \subset B$ be the discriminant locus of $W_1 \to B$, and $U = B \setminus F$. Now apply 2.6 and 2.5.
Remark. It follows from the remark after 2.3 that already
\[(\mathcal{I}_F^{[n(1-\frac{1}{m})]} \cdot \mathcal{J}_{m,n}) \subset \mathcal{J}_n(G^m, V_B^m).\]

We will often need to perform base changes for fibrations. We need to find a condition on the base changed fibration which guarantees that the original variety is of general type. This is provided by the following proposition (which is probably well known):

**Proposition 2.8.** Given an alteration \(\rho : B_1 \to B\) between smooth projective varieties, there exists an ideal sheaf \(\mathcal{I} \subset \mathcal{O}_{B_1}\) with the following property: given a fibration \(f : Y \to B\), with \(Y_1 \to Y \tilde{\times}_B B_1\) a resolution of singularities, \(f_1 : Y_1 \to B_1\) the induced projection, such that \(\omega_1 \otimes f_1^{-1} \mathcal{I}\) is big, then \(Y\) is of general type.

First a lemma:

**Lemma 2.9.** 1. Let \(g : Y_1 \to Y\) be a generically finite morphism of smooth projective varieties. Let \(B \subset Y\) be the branch locus. Then there exists an effective \(g\)-exceptional divisor \(E\) on \(Y_1\) and an injection \(\omega_{Y_1}(-g^*B) \to g^*\omega_Y \otimes \mathcal{O}_{Y_1}(E)\).

2. If \(\omega_{Y_1}(-g^*B)\) is big, then \(\omega_Y\) is big as well.

**Proof.** The pull-back morphism \(g^*\omega_Y \to \omega_{Y_1}\) gives \(g^*\omega_Y = \omega_{Y_1}(-R - E)\) where \(E\) is an effective exceptional divisor and \(R\) is the ramification divisor. Clearly \(R < g^*B\).

Assume that \(\omega_{Y_1}(-g^*B)\) is big. Then \(g^*\omega_Y \otimes \mathcal{O}_{Y_1}(E)\) is big. Let \(Y_1 \xrightarrow{g_1} Y' \xrightarrow{s} Y\) be the Stein factorization. Since \(Y'\) is normal and \(E\) is \(g_1\)-exceptional we have that \(s^*\omega_Y \otimes g_1^*\mathcal{O}_{Y_1}(E) = s^*\omega_Y\) therefore \(s^*\omega_Y\) is big. Since \(s\) is finite we have that \(\omega_Y\) is big.

**Proof of 2.8.** Choose a nonzero ideal \(\mathcal{I}_0 \subset \mathcal{O}_{B_1}\) with an injection \(\mathcal{I}_0 \subset \omega_{B_1}\), and an ideal \(\mathcal{I}_1 \subset \mathcal{O}_B\) such that \(\omega_B \otimes \rho^{-1}\mathcal{I}_1 \subset \rho^*\omega_B\). Given a fibration \(f : Y \to B\), with \(g : Y_1 \to Y\) as above, we have that the ideal \(\mathcal{I}_1\) vanishes on the branch locus of \(g\). Set \(\mathcal{I} = \mathcal{I}_1 \rho^{-1}\mathcal{I}_2\). Assume that \(\omega_{Y_1/B_1} \otimes g^{-1}\mathcal{I}\) is big, then \(\omega_{Y_1} \otimes (\rho \circ g)^{-1}\mathcal{I}_2\) is big, therefore \(\omega_{Y_1}(-g^*B)\) is big, and by the lemma we have that \(\omega_Y\) is big. \(\square\)

3. MAXIMAL VARIATION AND KOLLÁR’S THEOREMS

**In Which**

We Reduce Our Theorem to the Maximal Variation Case, and Quote a Big Theorem Of Kollár Producing Sections.
3.1. **Pointed birational moduli.** The following is an immediate generalization of Kollár’s generic moduli theorem ([Ko], 2.4):

**Theorem 3.1 (Pointed birational moduli theorem).** Let \( f : X \to B \) be a family of varieties of general type. There exist open sets \( U \subset B \) and \( V \subset f^{-1}U \), and projective varieties \( Z \) and \( W_n \), \( n \geq 1 \), with a diagram:

\[
\begin{array}{cccc}
V^n_B & \to & V^{n-1}_B & \to & \cdots & \to & U \\
\downarrow m_n & & \downarrow m_{n-1} & & & & \downarrow m_0 \\
W_n & \to & W_{n-1} & \to & \cdots & \to & Z
\end{array}
\]

satisfying the following requirements:

1. The morphisms \( m_n \) are dominant.
2. If \( P = (P_1, \ldots, P_n), P' = (P'_1, \ldots, P'_n) \in V^n_B, f_n(P) = b, f_n(P') = b' \in U \), then \( m_n(P) = m_n(P') \) if and only if there exists a birational map \( g : V_b \to V_{b'} \) which is defined and invertible at \( P_i \), such that \( g(P_i) = P'_i \).
3. For general \( b \in U \), let \( G \) be the birational automorphism group of \( X_b \), then the fiber of \( W_n \) over \( m_0(b) \) is birational to \( X^n_b / G \), where \( G \) acts diagonally.
4. There are canonical generically finite rational maps \( W_n^k \to (W_n)^k_Z \).

**Sketch of proof:** Parts (3) and (4) follow from (2). The proof of (1) and (2) is a simple modification of ([Ko], 2.4), where we let \( PGL \) act on the universal family over the Hilbert scheme and its fibered powers. \( \square \)

3.2. **Reduction of theorem 0.1 to theorem 0.2.** Recall by corollary 2.4 that for sufficiently large \( n \) the general fiber of \( W_n \to Z \) is of general type. Also, a simple lemma below shows that for large \( n \) the family \( W_n \to Z \) is of maximal variation. Assuming that theorem 0.2 holds true, we have that for large \( k \) the variety \( (W_n)^k_Z \) is of general type, therefore \( W_{nk} \) is of general type. For any \( n' > nk \), applying the additivity theorem (Satz III of [V1]) to \( W_{nk} \to W_{nk} \) we have that the variety \( W_{nk} \) is of general type. Therefore \( X^n_B \) dominates a variety of general type.

**Lemma 3.2.** Suppose \( X \to B \) is a one dimensional family of varieties of general type, \( \text{Var}(X/B) = 1 \), and \( G \subset \text{Aut}_B X \). Then for sufficiently large \( n \), the quotient by the diagonal action \( W_n = X^n_B / G \to B \) has \( \text{Var}(W_n/B) = 1 \).

**Proof.** This is immediate from the theorems of Kobayashi-Ochiai (see [D-M]) and Maehara (see [Mor]). Using Proposition 2.4, choose \( n \) so that the general fiber of \( W_{n_0} \) over \( B \) is of general type. We show that \( \text{Var}(W_{n_0+1}/B) = 1 \), and by induction this follows for any higher \( n \). Assume the opposite. We have the projection map \( W_{n_0+1} \to W_{n_0} \). The theorem of Maehara implies that \( \text{Var}(W_{n_0}/B) = 0 \): a family of varieties of general type dominated by a fixed
variety is isotrivial. The theorem of Kobayashi-Ochiai implies that the map \( W_{n_0+1} \to W_{n_0} \) is isotrivial: a family of rational maps from a fixed variety to a fixed variety of general type is isotrivial. But the general fiber of \((W_{n_0+1})_b \to (W_{n_0})_b\) is isomorphic to \(X_b\) (one only needs to avoid the fixed point set of the action) - implying that \(X \to B\) is isotrivial.

3.3. **Kollár’s big theorem.** Here we introduce the main source for global sections.

**Theorem 3.3** (Kollár’s big theorem, [Ko], I, p. 363). Suppose that \( \pi : X \to B \) is a fibration of positive dimensional varieties of general type, and \( \text{Var}(X/B) = \dim B \). Assume both \( X \) and \( B \) are smooth. There is an integer \( n > 0 \) such that the sheaf \( \pi_\ast \omega^n \) is weakly b´ıg.

Kollár’s use of **weakly b´íg** requires saturations, which means that the sections obtained may have poles over exceptional divisors of the map \( X \to B \). From this one first deduces:

**Corollary 3.4** ([V2], Corollary 7.2). Suppose that \( \pi : X \to B \) is as above. There is a divisor \( D \) on \( X \) such that \( \text{codim}(\pi(\text{supp}B)) > 1 \), and such that \( \omega_\pi(D) \) is big.

We still have the annoying divisor \( D \). Our method below will allow us to ignore it, but actually a trick of Viehweg ([V2], lemma 7.3) makes it easier. Viehweg simply applies the theorem above to \( X' \to B' \) where \( X' \) is a desingularization of a flattening of \( X \), where any exceptional divisor for \( X'/B' \) is exceptional for \( X'/X \). Since \( \omega_{B'/B} \) is effective, one immediately obtains:

**Theorem 3.5** (Kollár-Viehweg). Suppose that \( \pi : X \to B \) is as above. Then \( \omega_\pi \) is big.

4. **PLURI-NODAL REDUCTION**

*In Which*

We Prove a Pluri-Nodal Reduction
Lemma and Show That Pluri-Nodal Families Are Mild

4.1. **Statement.** Let \( X_0 \to B_0 \) be a fibration. We need to dominate it by a pluri-nodal fibration, so that it becomes a quotient by the action of a finite group.

To this end, we prove the following theorem, which is a variant of de Jong’s results in [dJ], sections 6-7. The proof is based on that of de Jong.
Lemma 4.1 (Galois pluri-nodal reduction lemma). There exists a diagram
\[ \begin{array}{c}
Y & \to & X_0 \\
\downarrow & & \downarrow \\
B_1 & \to & B_0
\end{array} \]
and a finite group \( G \subset \text{Aut}_{X_0 \times B_0} B_1 \) such that \( B_1 \to B_0 \) is an alteration, \( Y/G \to X_0 \times B_0 B_1 \) is birational and \( Y \to B_1 \) is a pluri-nodal fibration.

Proof. We proceed by induction. The setup is as follows: suppose we have \( X \to Z \to B \) a pair of fibrations, where \( X \to Z \) is pluri-nodal, and a finite group \( G_0 \subset \text{Aut}_B(X \to Z) \). We also assume that we have a birational morphism \( X/G_0 \to X_0 \times B_0 B_1 \). We will produce a diagram
\[ \begin{array}{c}
X' & \to & Z'' & \to & Z' & \to & B' \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
X & \to & Z & \to & B
\end{array} \]
with the following properties:

1. the vertical arrows are alterations,
2. the horizontal arrows are fibrations,
3. the morphism \( Z'' \to Z' \) is a nodal fibration,
4. \( X' = X \times_Z Z'' \), and therefore \( X' \to Z' \) is pluri-nodal,
5. there is a finite group \( G' = G_0 \times G'' \subset \text{Aut}_B(X' \to Z'' \to Z') \), and
6. the morphism \( X'/G' \to X_0 \times B_0 B' \) is birational, and therefore \( X'/G' \to X_0 \times B_0 B' \) is birational.

The basis of the induction is \( X_0 \to X_0 \to B_0 \) with \( G_0 \) trivial. The induction ends with \( Z' \to B' \) being birational, in which case we set \( Y := X' \), \( B_1 := Z' \), \( G := G' \) and the lemma will be proved.

Let \( G_Z \subset \text{Aut}_B Z \) be the image of \( G_0 \), and denote \( W = Z/G_Z \).

Lemma 4.2. There exists a dominant rational map \( Z/G_Z \dashrightarrow P \to B \), where \( P \to B \) is a projective bundle, such that \( \dim(P) = \dim(Z) - 1 \), and such that the generic fiber of \( Z \) over \( P \) is geometrically irreducible.

Proof. This is obvious in case rel. \( \dim(Z/B) = 1 \), so assume rel. \( \dim(Z/B) > 1 \). Denote this relative dimension by \( r \). Since we are looking for a rational map, we may replace \( B \) by its generic point \( \eta \), and replace \( Z \) by \( Z_\eta \). Let \( W = Z/G \), choose an embedding \( W \subset \mathbb{P}^N \), and let \( f : Z \to \mathbb{P}^N \) be the induced morphism. According to [I], 6.3(4), for general hyperplane \( H \subset \mathbb{P}^N \) we have \( f^{-1}H \) geometrically irreducible. Continuing by induction, there is a linear series \((\mathbb{P}^{r-1})^* \) of dimension \( r - 1 \) of hyperplanes in \( \mathbb{P}^N \) such that the general fiber of \( Z \dashrightarrow \mathbb{P}^{r-1} \) is a geometrically irreducible curve. \( \square \)
The normalization of the closure of the graph of the rational map $Z \to P$ gives a $G_Z$-equivariant resolution of indeterminacies

$$Z_1 \to P$$

Let $X_1 = X \times_Z Z_1$. Then $X_1 \to Z_1$ is pluri-nodal, and the action of $G_0$ on $X$ lifts to $X_1$ (if $x_1 = (x, z_1) \in X_1$ and $g \in G_0$ then $(g(x), g(z_1)) \in X_1$ as well).

We will now perform a canonical nodal reduction for $Z_1 \to P$ using the Kontsevich space of stable maps. The generic fiber of $Z_1 \to P$ is a normal curve, and therefore smooth. Choose a projective embedding $Z_1 \subset \mathbb{P}^N$. Let $d$ be the degree of the generic fiber of $Z_1 \to P$ and let $g$ be its genus. By [dJ-O], theorem 3.14, there exists a proper Deligne-Mumford stack $\overline{M}_{g,0}(Z_1, d)$ parametrizing stable maps $C \to Z_1$ of curves of genus $g$ and degree $d$. By [Pan] this stack admits a projective coarse moduli space. In particular, this implies that there is a finite cover $\rho : M \to \overline{M}_{g,0}(Z_1, d)$ where $M$ is a projective scheme admitting a stable map $(C \to M, f : C \to Z_1)$ whose moduli map is $\rho$.

Let $\eta \in P$ be the generic point. The pair $((Z_1)_\eta \to \eta, (Z_1)_\eta \hookrightarrow Z_1)$ is a stable map of genus $g$ and degree $d$, therefore we have a rational map $P \to \overline{M}_{g,0}(Z_1, d)$. We can choose a normal resolution of indeterminacies

$$P_2 \to M$$

such that there is a finite group $G_1 \subset \text{Aut}_P P_2$ with $P_2/G_1 \to P$ birational. Let $Z_2 = C \times_M P_2$. We have an induced stable map $(Z_2 \to P_2, f_2 : Z_2 \to Z_1)$, in particular $Z_2 \to P_2$ is nodal. Over the generic point of $P_2$ this coincides with $Z_1 \times_P P_2$.

Since stable reduction over a normal base is unique when it exists (see [dJ-O], 2.3), the action of $G_1$ lifts to $Z_2$, and it lifts to $X_2 = X_1 \times_Z Z_2$ as well by pulling back as before. Let $P_2 \to B_2 \to B$ be the Stein factorization. Since the Stein factorization is unique we have canonically an action of $G_1$ on $B_2$. Let $G_2 \subset G_1$ be the subgroup acting trivially on $B_2$. Then $G = G_0 \times G_2 \subset \text{Aut}_{B_2}(X_2 \to P_2)$.

We have $X_2 \to P_2$ pluri-nodal, and $X_2/G_2 \to X_\times B B_2$ birational. If we denote $X' := X_2$, $Z'' := Z_2$, $Z' := P_2$, $B' := B_2$ and $G'' := G_2$ we have obtained the goal of the induction step.

4.2. Mild Singularities. We want to show that pluri-nodal fibrations are mild. This seems to be well known (see [Has], §4), but in our case we can
give a proof which is sufficiently short to include here. The following lemma is well known (see [V2], lemma 3.6):

**Lemma 4.3.** Let $Y \to B$ be a nodal fibration such that $B$ is smooth and the discriminant locus is a divisor of normal crossings. Then $Y$ is $r$-G.

(The proof is by taking formal coordinates near a singular point of the form $xy = t_1^{k_1} \cdots t_r^{k_r}$, and either resolving singularities explicitly or noting that this is a toroidal singularity.)

**Proposition 4.4.** Let $Y \to B$ be a nodal fibration such that $B$ is $r$-G. Then $Y$ is $r$-G.

**Proof.** Let $r : B_1 \to B$ be a resolution of singularities, $Y_1 \to B_1$ the pullback, and assume that the discriminant locus of $Y_1 \to B_1$ is a divisor of normal crossings. Let $f : Y_1 \to Y$ be the induced map. Then $r_* \omega_{B_1} = \omega_B$ and $f^* \omega_{Y/B} = \omega_{Y_1/B_1}$, and by the projection formula we obtain that $f_* \omega_{Y_1} = \omega_Y$. \hfill \square

By induction we obtain:

**Corollary 4.5.** If $\pi : Y \to B$ is a pluri-nodal fibration where $B$ is $r$-G, then $Y$ is $r$-G. In particular the $n$-th fibered power $Y^n_B$ is $r$-G.

Thus pluri-nodal fibrations are mild.

5. PROOF OF THE THEOREM

In Which

Our Main Theorem Arrives at an Enchanted Place, and We Leave It There.

Let $X_0 \to B_0$ be a smooth projective family of varieties of general type of maximal variation. Choose a model $X \to B$ where both $X$ and $B$ are projective nonsingular. By \[\square\] we may assume, after an alteration $B_1 \to B$, that we have a birational morphism $g_0 : Y/G = X_1 \to X_{\tilde{B}B_1}$ where $\pi_Y : Y \to B_1$ is a pluri-nodal fibration and $G \subset \text{Aut}_{B_1}Y_1$ a finite group. Choose a resolution of singularities $r : X_2 \to X_1$ and denote by $\pi_2 : X_2 \to B_1$ the projection. We have a diagram:

\[
\begin{array}{ccc}
Y & \xrightarrow{q} & X \\
\downarrow & & \downarrow \pi_2 \\
X_2 & \xrightarrow{r} & X_1 \\
\downarrow \pi_2 & & \downarrow \\
B_1 & \to & B
\end{array}
\] (1)
According to \([2.7]\) (where we set \(V = Y\) and \(W = X_1\)) there is an ideal \(\mathcal{I}_F \subset \mathcal{O}_{B_1}\) such that \((\mathcal{I}_F m(G) n) \cdot \mathcal{I}_{m,n} \subset \mathcal{J}_{n}(G_{m}, Y_{B_1}^m)\). For arbitrary integer \(m > 0\) let \(X_m \to X_B\) be a resolution of singularities of the main component, and let \(\mathcal{W}_m \to (X_1)_B^m\) be a resolution of singularities, dominating \(X_m\). According to \([2.8]\) (applied to \(B_1 \to B\)) there is an ideal \(\mathcal{I} \subset \mathcal{O}_{B_1}\), such that for any \(m\), if \(\omega_{\mathcal{W}_m/B_1} \otimes \mathcal{I}\) is big then \(X_m\) and (therefore \((X_0)_B^m\)) is of general type.

By the Kollár - Viehweg theorem \([3.3]\), \(\omega_{\pi_2}\) is big. Therefore \(q^* r_\ast \omega_{\pi_2}\) is big. We have by definition that \(\omega_{\pi_X} \otimes \mathcal{I}_1(G, Y)\) is big. Therefore, for sufficiently large \(n\) we have that \(\omega^n_{\pi_Y} \otimes \mathcal{J}_1(G, Y)\) is big. Pulling back along all the projections \(p_i : Y_B^m \to Y\) we have that \(\omega^n_{Y_B^m/B_1} \otimes \mathcal{J}_{m,n} \mathcal{I}_F m(G)\) is big. In particular, if \(m > n\), we have that \(\omega^n_{Y_B^m/B_1} \otimes \mathcal{J}_{m,n} \mathcal{I}_F m(G)\) is big. By \([2.7]\) we have that \(\omega^n_{Y_B^m/B_1} \otimes \mathcal{J}_n(G_{m}, Y_{B_1}^m)\) is big. Taking invariants and using \([2.2]\), \(\omega_{\mathcal{W}_m/B_1} \otimes \mathcal{I}\) is big, and by \([2.8]\) we have that \((X_0)_B^m\) is of general type for large \(m\).

5.1. An alternative approach. The following argument gives a variation on the proof which is more in line with \([K\&Q]\) and \([V2]\). Having chosen the diagram \([\dag]\), we can alter it as follows: using semistable reduction in codimension 1 (see \([KKMS]\) II, and \([K\&a]\), theorem 17), we can find a nonsingular alteration \(B'_1 \to B_1\), a variety \(X'_2 \to B'_1\), and a birational morphism \(X'_2 \to X_2\), satisfying the following conditions

1. The discriminant locus \(\Delta\) of \(X'_2 \to B'_1\) is a divisor of normal crossings. Set \(F = Sing(\Delta)\) and \(U = B'_1 \setminus F \hookrightarrow B'_1\).
2. The restriction \(X'_2|_U \to U\) is semistable, in particular it is mild (see \([V2]\), lemma 3.6).

Let \(X'_1 = X_1 \times_{B_1} B'_1\) and \(Y' = Y \times_{B_1} B'_1\). We can replace \(B_1, X_1, X_2, Y\) by \(B'_1, X'_1, X'_2, Y'\) and assume that conditions (1) and (2) are satisfied.

Let \(\pi_{X(m)} : X(m) \to B_1\) be the main component of \((X_1)_B^m\). Choose a resolution of singularities \(W_m \to X(m)\), and let \(\pi_{W_m} : W_m \to B_1\) be the associated projection. Denote \(\mathcal{F}_{m,n} = \pi_{2\ast} \omega_{\pi_{W_m}}\) and \(\mathcal{G}_{m,n} = (\mathcal{F}_{m,n})^{\ast\ast}\). Since the restriction of \(W_m\) to \(U\) is mild, we have that \(\mathcal{G}_{m,n} = i_{\ast} i_{\ast} \mathcal{F}_{m,n}\). Applying \([2.13]\), we obtain:

1. We have natural morphisms
   \[\mathcal{G}_{m_1,n} \otimes \mathcal{G}_{m_2,n} \to \mathcal{G}_{m_1+m_2,n}\]
   (by pulling back sections to \(W_{m_1+m_2}\) over \(U\), multiplying and extending).
2. We have natural morphisms
   \[\mathcal{G}_{m_1,n_1} \otimes \mathcal{G}_{m_2,n_2} \to \mathcal{G}_{m_1+n_1,n_2}\]
(by multiplying sections).

3. We have

\[ G_{m,n} \otimes \mathcal{I}_F^{n(e(G)-1)} \subset \mathcal{F}_{m,n} \subset G_{m,n} \]

(by 2.7). Notice that the remark after 2.7 shows that \( T_F^a \) suffices).

By Kollar’s theorem \( G_{1,n} \) is weakly big for sufficiently large \( n \). We can choose an ideal \( I \) as in 2.8. By (2) above, for sufficiently large \( n \) we have that \( G_{1,n} \otimes \mathcal{I}_F^{n(G)} \) is big, and using (1) above we have that for sufficiently large \( m \), we have that \( G_{m,n} \otimes \mathcal{I}_F^{m(e(G)-1)} \) is big, therefore by (3) \( \mathcal{F}_{m,n} \otimes \mathcal{I}_F^m \) is big, which is what we need.

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... “Come on!”
“Where?” said Pooh.
“Anywhere,” Said Christopher Robin.

A.A. Milne,
The House at Pooh Corner