Entanglement and corrections to Bekenstein-Hawking entropy

Saurya Das(1), S. Shankaranarayanan(2,3)∗ and Sourav Sur(4)

(1) University of Lethbridge, 4401 University Drive, Lethbridge, Alberta, Canada T1K 3M4
(2) Institute of Cosmology and Gravitation, University of Portsmouth, Portsmouth, U.K.
(3) School of Physics, Indian Institute of Science Education and Research-Trivandrum, CET campus, Thiruvananthapuram 695 016, India.
(4) Theory Division, Saha Institute of Nuclear Physics Bidhannagar, Kolkata 700064, India

In this talk, we focus on the corrections to Bekenstein-Hawking entropy by associating it with the entanglement between degrees of freedom inside and outside the horizon. Using numerical techniques, we show that the corrections proportional to fractional power of area result when the field is in a superposition of ground and excited states. We explain this result by identifying that the degrees of freedom contributing to such corrections are different from those contributing to Bekenstein-Hawking entropy.

Keywords: black-holes, Bekenstein-Hawking entropy, entanglement

The entropy of a black-hole (BH), given by the well-known Bekenstein-Hawking relation

\[
S_{\text{BH}} = \left( \frac{k_B}{4} \right) \frac{A_H}{\ell_{\text{Pl}}^2}, \quad \left( \ell_{\text{Pl}} \equiv \sqrt{\frac{G\hbar}{c^3}} = \text{Planck length}, \ A_H = \text{Horizon area} \right),
\]

is characteristically distinct from that of other physical systems, e.g. ideal gases, because it is finite only for a quantum description of gravity and(or) matter fields. It is therefore expected that any viable quantum gravity (QG) model should explain the origin of the BH entropy, identifying the microscopic degrees of freedom (DOF) that give rise to it. The area law is shown to hold in several approaches, starting from those that count microstates assuming fundamental structures (string, loop, etc.),3–5 to the entanglement of quantum modes inside and outside of the BH horizon.3–5

However, \( S_{\text{BH}} \) in Eq. (1) being, by origin, a semi-classical result, there is no reason to believe it to be the complete answer conceivable from a correct QG theory, and valid even for small Planck-sized BHs (i.e., \( A_H \sim \ell_{\text{Pl}}^2 \)). Therefore, it is imperative for any approach to QG to go beyond \( S_{\text{BH}} \) and provide generic subleading corrections. Another crucial thing is to identify which of the quantum DOF contribute to \( S_{\text{BH}} \) and which to the corrections, for if these DOF are different then possibly one may isolate their individual contributions for a deeper view on the mechanism of entropy generation. This can be illustrated in the quantum entanglement approach6–8 of BH entropy, which predicts generic power-law corrections to \( S_{\text{BH}} \), as we discuss below.

Entanglement and its connection to black-hole entropy:

On a product of two Hilbert spaces \( \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \), if a wave-function \( \Psi \) cannot be factorized into wave-functions \( \Psi_1 \) and \( \Psi_2 \) on \( \mathcal{H}_1 \) and \( \mathcal{H}_2 \) respectively, i.e., \( \Psi \neq \Psi_1 \otimes \Psi_2 \), then the states described by \( \Psi_1 \) and \( \Psi_2 \) are said to be entangled. The associated entropy, called the entanglement entropy is given by the Von Newmann

∗Speaker, E-mail: shanki@iisertvm.ac.in
relation $S_{\text{ent}} = -\text{Tr}[\rho_\alpha \ln(\rho_\alpha)]$, where $\rho_\alpha$ ($\alpha \in \{1,2\}$) is the reduced density matrix of a system in any one of the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$.

$S_{\text{ent}}$, being a quantum effect without classical analogue and associated with the existence of a horizon, similar to BH entropy, may serve as the source of the latter.\(^8\)

**Entanglement entropy computation — setup and assumption:**

We consider the propagation of a massless scalar field\(^a\) ($\varphi$) in an asymptotically flat BH background, and choose to work in the Lemaître coordinates $(\tau, \xi, \theta, \phi)$ rather than in the Schwarzschild coordinates $(t, r, \theta, \phi)$ because of some advantages: (i) the BH line element in Lemaître coordinates is non-singular at the horizon ($r_H$), unlike that in Schwarzschild coordinates, (ii) $\xi$ (or $\tau$) is space(or, time)-like everywhere, as opposed to $r$ which is space-like only for $r > r_H$. However, the scalar field Hamiltonian in Lemaître coordinates is explicitly time-dependent.

We assume that the Hamiltonian evolves adiabatically, implying that the evolution of the late-time modes leading to Hawking particles are negligible. In the microcanonical ensemble (fixed total energy), this also means that the wave functional $\Psi[\varphi(\xi), \tau]$ describing the quantum state in the Schrödinger representation has a weak time-dependence.\(^9\) Under canonical transformation and at fixed Lemaître time Hamiltonian reduces to free scalar field propagating in flat space-time.\(^8\)

**Entanglement entropy computation — procedure:**

First, we discretize the Hamiltonian on a spherical lattice with number of points $N$ ($\gg 1$) and of spacing $a$ ($\ll L = (N + 1)a$ — the infrared cutoff), whence the Hamiltonian reduces to that of $N$-coupled harmonic oscillators.

Then we choose, for simplicity, the quantum state described by $\Psi$, to be in a superposition\(^b\) of the vacuum (ground) state and the 1-particle (excited) states described by $\Psi_0$ and $\Psi_1$ respectively, i.e., $\Psi = c_0 \Psi_0 + c_1 \Psi_1$, with $|c_0|^2 + |c_1|^2 = 1$.

Finally, we obtain the reduced density matrix by tracing over $n$ of the $N$ oscillators, and evaluate the entropy $S_{\text{ent}}$ using the above Von Newmann relation. Due to extreme difficulty in analytical computations, we adopt numerical techniques.

**Entanglement entropy computation — results:**

The best fit for the entanglement entropy for the superposition (or mixing) of vacuum and 1-particle states gives a power-law correction to $S_{\text{BH}}$ (see Fig. 1)\(^8\)

$$S_{\text{ent}} = S_{\text{BH}} \left[ 1 + a_1 \left( \frac{A_H}{r_H^2} \right)^{-\beta} \right] \quad a_1, \beta > 0. \quad (2)$$

The parameter $a_1$ increases with $|c_1|$; and when $c_1 = 0$, $a_1 = 0$ and $S_{\text{ent}} = S_{\text{BH}}$. Thus, the entanglement entropy of the ground state obeys the area law. Only the excited state contributes to the correction, and more excitations produce more deviation from the area law. The parameter $\beta$ lies between 1 and 2. So for large BHs

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\(^a\)Motivated from the point of view of the metric perturbations of black-hole space-times,\(^9\) that correspond to test scalar fields in these space-times.\(^8\)

\(^b\)n oscillators are supposed to provide the DOF inside the horizon $r_H$, and the tracing is therefore over the region enclosing the horizon [$\xi \to (r_H, \infty)$] in Lemaître coordinates.\(^8\)
(i.e., large $A_H$) the correction term falls off rapidly and the area law is recovered, whereas for the small BHs the correction is significant. This can be interpreted as follows: for large $A_H$, i.e., at low energies, it is difficult to excite the modes and hence, the ground state modes contribute to most of the $S_{\text{ent}}$. However, for small horizon area, a large number of field modes can be excited and contribute significantly to the correction causing large deviation from the area law. In fact, the increase in the deviation with excitations may be attributed to the fact that the total entropy gets increasing contributions from the quantum field DOF that are far from the horizon, rather than immediately close to it$^c$. See refs 6,7 for details.

In conclusion, let us make the following remarks: (i) As $S_{\text{ent}}$ is obtained for a scalar field in flat space-time, $S_{\text{BH}}$ and its correction can be identified uniquely with the quantum state correlations. (ii) Although we have considered the microcanonical ensemble, the identification of the power-law correction to the kinematical properties of the horizon can be done by obtaining $S_{\text{ent}}$ in the canonical ensemble$^{10,11}$.

SSh is supported by Marie Curie Incoming International Grant IIF-2006-039205.

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$^c$The near-horizon DOF of course contribute to the bulk of the entropy even for $A_H/\ell_{Pl}^2\sim O(10)$, thus keeping the power-law correction sub-leading for fairly small-sized black-holes.