The Ground, a String, Two Elastic Springs.

Simple Exactly Solvable Models of One-Dimensional Scalar Fields with Concentrated Factors.

S.A. Chorošavin

Abstract

This paper is a basis of a part of my set of lectures, subject: ‘Formal methods in solving differential equations and constructing models of physical phenomena’. Addressed, mainly: postgraduates and related readers. Content: a discussion of the simple models ( I would rather say, toy models ) of the interaction based on equation arrays of the kind:

\[
\frac{\partial^2 u(t,x)}{\partial t^2} = c^2 \frac{\partial^2 u(t,x)}{\partial x^2} - 4\gamma_a c \delta(x - x_a)Q_a(t) - 4\gamma_b c \delta(x - x_b)Q_b(t),
\]

\[
Q_a(t) = u(t,x_a) \\
Q_b(t) = u(t,x_b)
\]

Central mathematical points: d’Alembert-Kirchhoff-like Formulae, Finite Rank Perturbations.
Introduction

As long as the researcher takes an interest in resolvent formulae of finite rank perturbed operators, in other words, as long as he or she, the researcher, prefers to remain in the space-FREQUENCE framework, so long he has an infinite series of good papers, articles, books, manuals...

But as soon as the researcher is turning his attention to the problems of the "space-TIME", in other words, as soon as he has need to solve the associated wave equation, as soon as he has need of a suitable d’Alembert-like formula, or so, - in that moment the situation is changing dramatically.

I cannot say "there is no paper on the subject at all", I cannot say "I have not seen it", but if anyone asks me "where have you seen it?, where is this ‘there’?", I will become very "pensativo", and I am in doubt that I will be "solitario" in this state.

So, I tried, try and (I hope) will try to collect the suitable examples of simple exactly solvable models of wave’n’particle.

Whether my collection is worth to discuss it, you solve.

1 Models of Two-Point Interaction with an only one-dimensional Scalar Field

1.1 Preliminaries

In this paper we fix measure units and let $x$ be dimensionless position parameter, i.e.,

$$\text{physical position coordinate} = [\text{length unit}] \times x + \text{const}.$$  

Otherwise a confusion can occur, in relating to the definition

$$\int_{-\infty}^{\infty} \delta(x - x_0)f(x)dx = f(x_0).$$

We assume the standard formalism, where

$$\delta(x - x_0) = \frac{\partial 1_+(x - x_0)}{\partial x}$$

and where $1_+$ stands for a unit step function (Heaviside function):

$$1_+(\xi) := 1(\xi \geq 0) := \begin{cases} 1, & \text{if } \xi \geq 0, \\ 0, & \text{if } \xi < 0. \end{cases}$$

Another feature of the notations is this. We will handle the functions which have a special variable, $t$, that means no doubt "time", and we will be interested in the case where $t \geq 0$. So, we could consider the restrictions of these functions, onto positive $t$-half-line. But it will be technically more convenient to redefine the functions, putting them zero on negative $t$-half-line. For more details of this feature of the notations see The $\cdot 1_+()$ Convention, in the next subsection.

A few words about the models: Recently I have already presented some models of ONE particle of FINITE mass, interacting with scalar field, and the interaction has been concentrated at one, only ONE, point.

Now I discuss models of TWO-point concentrated interaction, but the mass of the particle (or particles) I assume to be INFINITE, or, more precisely, I assume the particle(s) to be of infinite mass and immobile: motionless, fixed.
A naive formulation of the situation is: let us look at the Ground (the infinite immobile mass), the Earth or the Moon say, connected with a String (very-very long thin tensed cord) by means of two Ideal (we never approve Imperfection, don’t we?) Springs. What will we tell then, ideally?
1.2 D’Alembert-Kirchhoff-like formulae

Recall that a standard D’Alembert-Kirchhoff-like formula reads: if
\[ \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + f, \quad u = u(t, x), \quad f = f(t, x), \quad (t \geq 0) \]  

\[ f = -4\gamma_c \delta(x - x_a) \left( F_{src a}(t) \right) - 4\gamma_b \delta(x - x_b) \left( F_{src b}(t) \right) \quad x_a \leq x_b \]

and given initial data, \( u(0, \cdot) \) and \( \frac{\partial u(t, \xi)}{\partial t} \bigg|_{t=0} \), then, for \( t \geq 0 \),

\[ u(t, x) = -2\gamma_a \int_0^t \frac{t - |x - x_a|}{c} \left( F_{src a}(\tau) \right) d\tau \cdot 1_+(t - |x - x_a|/c) \]
\[ -2\gamma_b \int_0^t \frac{t - |x - x_b|}{c} \left( F_{src b}(\tau) \right) d\tau \cdot 1_+(t - |x - x_b|/c) \]
\[ + u_0(t, x) \]

where

\[ u_0(t, x) := c_+(x + ct) + c_-(x - ct) \]
\[ = \frac{1}{2} \left( u(0, x + nt) + u(0, x - ct) \right) \]
\[ + \frac{1}{2c} \left( \ddot{u}(0, x + ct) - \ddot{u}(0, x - ct) \right) \]

and where \( \ddot{u} \) stands for any function defined by

\[ \frac{\partial \ddot{u}(0, \xi)}{\partial \xi} = \left( \frac{\partial u(t, \xi)}{\partial t} \right) \bigg|_{t=0}. \]

1

If we put

\[ F_a(t) := -2\gamma_a \int_0^t \left( F_{src a}(\tau) \right) d\tau \cdot 1_+(t), \]
\[ F_b(t) := -2\gamma_b \int_0^t \left( F_{src b}(\tau) \right) d\tau \cdot 1_+(t), \]

for short, then

---

1Note that
\[ \ddot{u}(0, x + ct) - \ddot{u}(0, x - ct) \]
does not depend on what the primitive is which one has chosen!!! Moreover, we need only \( \ddot{u}|_{t=0} \) and not \( u|_{t=0} \) itself!
\[ u(t, x) = F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t, x) \quad (t \geq 0) \]

Another and a little more correct form of this expression is:

\[ u(t, x) \cdot 1_+(t) = F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t, x) \cdot 1_+(t) \]

Now we set

\[ T := |x_a - x_b|/c \]

and concentrate on

\[ x = x_a, x_b \]

by writing

\[ u(t, x_a) = F_a(t) + F_b(t - T) + u_0(t, x_a) \quad (t \geq 0) \]

\[ u(t, x_b) = F_a(t - T) + F_b(t) + u_0(t, x_b) \quad (t \geq 0) \]

We transform these expressions into

\[ F_a(t) = \left( u(t, x_a) - u_0(t, x_a) - F_b(t - T) \right) \cdot 1_+(t) \]

\[ F_b(t) = \left( u(t, x_b) - u_0(t, x_b) - F_a(t - T) \right) \cdot 1_+(t) \]

and then

\[ F_a(t) = \left( u(t, x_a) - u_0(t, x_a) \right) \cdot 1_+(t) - \left( u(t - T, x_b) - u_0(t - T, x_b) \right) \cdot 1_+(t - T) + F_a(t - 2T) \cdot 1_+(t - T) \]

\[ F_b(t) = \left( u(t, x_b) - u_0(t, x_b) \right) \cdot 1_+(t) - \left( u(t - T, x_a) - u_0(t - T, x_a) \right) \cdot 1_+(t - T) + F_b(t - 2T) \cdot 1_+(t - T) \]
\[ F_a(t) = \left( u(t, x_a) - u_0(t, x_a) \right) \cdot 1_+(t) \]
\[ - \left( u(t - T, x_b) - u_0(t - T, x_b) \right) \cdot 1_+(t - T) \]
\[ + \left( u(t - 2T, x_a) - u_0(t - 2T, x_a) \right) \cdot 1_+(t - 2T) \]
\[ - \left( u(t - 3T, x_b) - u_0(t - 3T, x_b) \right) \cdot 1_+(t - 3T) \]
\[ + F_a(t - 4T) \cdot 1_+(t - 3T) \]

\[ F_b(t) = \left( u(t, x_b) - u_0(t, x_b) \right) \cdot 1_+(t) \]
\[ - \left( u(t - T, x_a) - u_0(t - T, x_a) \right) \cdot 1_+(t - T) \]
\[ + \left( u(t - 2T, x_b) - u_0(t - 2T, x_b) \right) \cdot 1_+(t - 2T) \]
\[ - \left( u(t - 3T, x_a) - u_0(t - 3T, x_a) \right) \cdot 1_+(t - 3T) \]
\[ + F_b(t - 4T) \cdot 1_+(t - 3T) \]

and so on.

Take here in account that

\[ F_a(t) = 0, \quad F_b(t) = 0, \quad 1_+(t) = 0 \text{ falls } t < 0; \]

in any case we see that

\[ 1_+(t - NT) = 0, \]
\[ \text{if } t < NT. \]

It follows that

\[ \text{if } T \neq 0, \quad \text{then} \]

\[ F_a(t) = \left( u(t, x_a) - u_0(t, x_a) \right) \cdot 1_+(t) \]
\[ - \left( u(t - T, x_b) - u_0(t - T, x_b) \right) \cdot 1_+(t - T) \]
\[ + \left( u(t - 2T, x_a) - u_0(t - 2T, x_a) \right) \cdot 1_+(t - 2T) \]
\[ - \left( u(t - 3T, x_b) - u_0(t - 3T, x_b) \right) \cdot 1_+(t - 3T) \]
\[ + \cdots \]

\[ F_b(t) = \left( u(t, x_b) - u_0(t, x_b) \right) \cdot 1_+(t) \]
\[ - \left( u(t - T, x_a) - u_0(t - T, x_a) \right) \cdot 1_+(t - T) \]
\[ + \left( u(t - 2T, x_b) - u_0(t - 2T, x_b) \right) \cdot 1_+(t - 2T) \]
\[ - \left( u(t - 3T, x_a) - u_0(t - 3T, x_a) \right) \cdot 1_+(t - 3T) \]
\[ + \cdots \]
The \(1_+()\) Convention.

In the following text, we will often redefine functions of \(t\) by multiplying them by \(1_+(t)\), i.e., by resetting, e.g.,

\[
\begin{align*}
u_0(t, \cdots) &:= u_0(t, \cdots) \cdot 1_+(t) \\
u(t, \cdots) &:= u(t, \cdots) \cdot 1_+(t) \\
u_0(t - T, \cdots) &:= u_0(t - T, \cdots) \cdot 1_+(t - T)
\end{align*}
\]

and so on.

Nevertheless we will sometimes write the multiplier \(1_+()\), although we will mostly do it for emphasis, or there, where the formula does not become too long.

To take an example of this convention implementation, we say that the recent relations concerning \(F_a, F_b\), we may write them as

\[
\begin{align*}
F_a(t) &= u(t, x_a) - u_0(t, x_a) - F_b(t - T) \\
F_b(t) &= u(t, x_b) - u_0(t, x_b) - F_a(t - T)
\end{align*}
\]

In addition to The \(1_+()\) Convention, we will

\textbf{assume that} \(T \neq 0\).
2 Particular Cases. Recurrence Relations

2.1 The Case of $\gamma_b = 0$

We begin the analysis of this case turning to the following relations, which we have seen in the previous section:

\[
\begin{align*}
  u(t, x) &= F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t, x) \quad (t \geq 0) \\
  F_a(t) &= \left( u(t, x_a) - u_0(t, x_a) - F_b(t - T) \right) \cdot 1_+(t) \\
  F_b(t) &= \left( u(t, x_b) - u_0(t, x_b) - F_a(t - T) \right) \cdot 1_+(t)
\end{align*}
\]

Thus, we see, that in the case of $\gamma_b = 0$, these relations become

\[
\begin{align*}
  u(t, x) &= F_a(t - |x - x_a|/c) + u_0(t, x) \quad (t \geq 0) \\
  u(t, x) &= \left( u(t - |x - x_a|/c, x_a) - u_0(t - |x - x_a|/c, x_a) \right) \cdot 1_+(t - |x - x_a|/c) \\
  &+ u_0(t, x) \quad (t \geq 0)
\end{align*}
\]

Take here into account that

\[
\begin{align*}
  u(t, x_a) &= F_a(t) + u_0(t, x_a) \quad (t \geq 0)
\end{align*}
\]

In particular, if

\[
u(t, x_a) \equiv 0 \quad (t \geq 0),\]

then

\[
F_a(t) = -u_0(t, x_a)
\]

and, hence,

\[
\begin{align*}
  \text{if} & \quad u(t, x_a) \equiv 0 \quad (t \geq 0), \\
  \text{then} & \quad u(t, x) = -u_0(t - |x - x_a|/c, x_a) \cdot 1_+(t - |x - x_a|/c) + u_0(t, x) \quad (t \geq 0)
\end{align*}
\]
The restriction \( u(t, x_a) \equiv 0 \quad (t \geq 0) \) states that the string is absolutely motionless, fixed at the point \( x = x_a \). If we replace this restriction by

\[
F_a(t) := -2\gamma_a \int_0^t \left( F_{src, a}(\tau) \right) d\tau \cdot 1_+ (t),
\]

\[
:= -2\gamma_a \int_0^t u(\tau, x_a) d\tau \cdot 1_+ (t),
\]

then we obtain:

\[
\frac{\partial F_a(t)}{\partial t} = -2\gamma_a u(t, x_a) \quad (t \geq 0)
\]

\[
= -2\gamma_a (F_a(t) + u_0(t, x_a)) \quad (t \geq 0)
\]

and, finally, because of \( F_a(0) = 0 \),

\[
F_a(t) = -2\gamma_a \int_0^t e^{-2\gamma_a (t-\tau)} u_0(\tau, x_a) d\tau \cdot 1_+ (t),
\]

\[
u(t, x) = F_a(t - |x - x_a|/c, x_a) + u_0(t, x) \quad (t \geq 0)
\]
2.2 The Case of \( u(t, x_a) = 0, \ u(t, x_b) = 0 \)

We have yet observed that in any case

\[
F_a(t) = \left( u(t, x_a) - u_0(t, x_a) \right) \cdot 1_+(t) \\
- \left( u(t - T, x_b) - u_0(t - T, x_b) \right) \cdot 1_+(t - T) \\
+ \left( u(t - 2T, x_a) - u_0(t - 2T, x_a) \right) \cdot 1_+(t - 2T) \\
- \left( u(t - 3T, x_b) - u_0(t - 3T, x_b) \right) \cdot 1_+(t - 3T) \\
+ \cdots
\]

\[
F_b(t) = \left( u(t, x_b) - u_0(t, x_b) \right) \cdot 1_+(t) \\
- \left( u(t - T, x_a) - u_0(t - T, x_a) \right) \cdot 1_+(t - T) \\
+ \left( u(t - 2T, x_b) - u_0(t - 2T, x_b) \right) \cdot 1_+(t - 2T) \\
- \left( u(t - 3T, x_a) - u_0(t - 3T, x_a) \right) \cdot 1_+(t - 3T) \\
+ \cdots
\]

Thus we infer that, if \( u(t, x_a) = 0, \ u(t, x_b) = 0 \),

then exactly \( u(t, x) \)

\[
u(t, x) = u_0(t, x) \\
- u_0(t - |x - x_a|/c, x_a) - u_0(t - |x - x_b|/c, x_b) \\
+ u_0(t - |x - x_a|/c - T, x_b) + u_0(t - |x - x_b|/c - T, x_a) \\
- u_0(t - |x - x_a|/c - 2T, x_a) - u_0(t - |x - x_b|/c - 2T, x_b) \\
+ u_0(t - |x - x_a|/c - 3T, x_b) + u_0(t - |x - x_b|/c - 3T, x_a) \\
\cdots \ (t \geq 0)
\]

Needs there a comment?
2.3 The Case of \( u(t, x_a) = 0, \quad \gamma_b \neq \infty \)

In this case, we observe that

\[
F_a(t) = 0 - u_0(t, x_a) - F_b(t - T)
\]

\[
F_b(t) = u(t, x_b) - u_0(t, x_b) - F_a(t - T)
\]

\[
F_a(t) = 0 - u_0(t, x_a) - u(t - T, x_b) + u_0(t - T, x_b) + F_a(t - 2T)
\]

\[
F_b(t) = u(t, x_b) - u_0(t, x_b) - 0 + u_0(t - T, x_a) + F_b(t - 2T)
\]

Thus we infer that, if

\[
F_b(t) := -2\gamma_b \int_0^t (F_{src b}(\tau)) d\tau \cdot 1_+(t),
\]

\[
:= -2\gamma_b \int_0^t u(\tau, x_b) d\tau \cdot 1_+(t),
\]

then

\[
F_a(t) = -u_0(t, x_a) - u(t - T, x_b) + u_0(t - T, x_b) + F_a(t - 2T)
\]

\[
-2\gamma_b \int_0^t u(\tau, x_b) d\tau \cdot 1_+(t) = u(t, x_b) - u_0(t, x_b) + u_0(t - T, x_a)
\]

\[-2\gamma_b \int_0^{t-T} u(\tau, x_b) d\tau \cdot 1_+(t - 2T)
\]

We now focus on the latter relation, which we write in the following terms:

\[
u(t, x_b) = -2\gamma_b \int_0^t u(\tau, x_b) d\tau + 2\gamma_b \int_0^{t-2T} u(\tau, x_b) d\tau
\]

\[+ u_0(t, x_b) - u_0(t - T, x_a)
\]

---

3 exploiting The \(-1_+()\) Convention.
Next, let us define

\[ Q_b(t) := u(t, x_b) ; \quad Q_{0b}(t) := u_0(t, x_b) ; \quad Q_{0a}(t) := u_0(t, x_a) ; \]

so that the former relation becomes

\[
\begin{align*}
Q_b(t) &= -2\gamma_b \int_0^t Q_b(\tau) d\tau + 2\gamma_b \int_0^{t-2T} Q_b(\tau) d\tau \\
&\quad + Q_{0b}(t) - Q_{0a}(t - T)
\end{align*}
\]

\[
\begin{align*}
Q_b(t) &= -2\gamma_b \int_{t-2T}^t Q_b(\tau) d\tau \\
&\quad + Q_{0b}(t) - Q_{0a}(t - T)
\end{align*}
\]

\[
\frac{\partial Q_b(t)}{\partial t} = -2\gamma_b Q_b(t) + 2\gamma_b Q_b(t - 2T) \\
&\quad + \frac{\partial (Q_{0b}(t) - Q_{0a}(t - T))}{\partial t}
\]

Let \( 4 \)

\[ Q_b(t) = e^{-2\gamma_b t} C(t) . \]

Then we obtain:

\[
\begin{align*}
Q_b(t - 2T) &= e^{-2\gamma_b t} e^{2\gamma_b T} C(t - 2T) , \\
\frac{\partial Q_b(t)}{\partial t} &= -2\gamma_b e^{-2\gamma_b t} C(t) + e^{-2\gamma_b t} \frac{\partial C(t)}{\partial t} \\
&= -2\gamma_b Q_b(t) + e^{-2\gamma_b t} \frac{\partial C(t)}{\partial t} \\
e^{-2\gamma_b t} \frac{\partial C(t)}{\partial t} &= 2\gamma_b Q_b(t - 2T) \\
&\quad + \frac{\partial (Q_{0b}(t) - Q_{0a}(t - T))}{\partial t}
\end{align*}
\]

\(^4\text{Method of Variation of Constants}\)
\begin{align*}
\frac{e^{-2\gamma_b t} \partial C(t)}{\partial t} &= 2\gamma_b e^{-2\gamma_b t} e^{2\gamma_b T} C(t - 2T) \\
&\quad + \frac{\partial (Q_{0b}(t) - Q_{0a}(t - T))}{\partial t} \\
\frac{\partial C(t)}{\partial t} &= 2\gamma_b e^{2\gamma_b T} C(t - 2T) \\
&\quad + e^{2\gamma_b t} \frac{\partial (Q_{0b}(t) - Q_{0a}(t - T))}{\partial t} \\
\frac{\partial C(t)}{\partial t} &= 2\gamma_b e^{2\gamma_b t} Q_b(t - 2T) \\
&\quad + e^{2\gamma_b t} \frac{\partial (Q_{0b}(t) - Q_{0a}(t - T))}{\partial t}
\end{align*}

\begin{align*}
C(t) &= 2\gamma_b \int_0^t e^{2\gamma_b \tau} Q_b(\tau - 2T) d\tau \\
&\quad + \int_0^t e^{2\gamma_b \tau} \frac{\partial (Q_{0b}(\tau) - Q_{0a}(\tau - T))}{\partial \tau} d\tau + C(0)
\end{align*}

\begin{align*}
Q_b(t) &= e^{-2\gamma_b t} 2\gamma_b \int_0^t e^{2\gamma_b \tau} Q_b(\tau - 2T) d\tau \\
&\quad + e^{-2\gamma_b t} \int_0^t e^{2\gamma_b \tau} \frac{\partial (Q_{0b}(\tau) - Q_{0a}(\tau - T))}{\partial \tau} d\tau + e^{2\gamma_b t} Q_b(0)
\end{align*}

\begin{align*}
Q_b(t) &= 2\gamma_b \int_0^t e^{-2\gamma_b (t - \tau)} Q_b(\tau - 2T) d\tau \\
&\quad + \int_0^t e^{-2\gamma_b (t - \tau)} d\tau \frac{\partial (Q_{0b}(\tau) - Q_{0a}(\tau - T))}{\partial \tau} + e^{-2\gamma_b t} Q_b(0)
\end{align*}

\begin{align*}
Q_b(t) &= 2\gamma_b \int_0^t e^{-2\gamma_b (t - \tau)} Q_b(\tau - 2T) d\tau \\
&\quad + I_{0b}(t),
\end{align*}

where

\begin{align*}
I_{0b}(t) &= \int_0^t e^{-2\gamma_b (t - \tau)} d\tau (Q_{0b}(\tau) - Q_{0a}(\tau - T)) + e^{-2\gamma_b t} Q_b(0)
\end{align*}
and then, after returning to the terms of \(u, u_0\), we infer that

\[
u(t, x_b) = 2\gamma_b \int_{2T}^{t} e^{-2\gamma_b(t-\tau)} u(\tau - 2T, x_b) d\tau + \mathcal{I}_{0b}(t),
\]

where

\[
\mathcal{I}_{0b}(t) := \int_{0+0}^{t} e^{-2\gamma_b(t-\tau)} d\tau (u_0(\tau, x_b) - u_0(\tau - T, x_a)1_+(\tau - T)) + e^{-2\gamma_b t} u(0, x_b)
\]

Finally, recall that

\[
u(t, x) = F_a(t - |x - x_a|/c) - F_b(t - |x - x_b|/c) + u_0(t, x) \quad (t \geq 0)
\]

where

\[
F_a(t) = \left(0 - u_0(t, x_a)\right) \cdot 1_+(t)
- \left(u(t - T, x_b) - u_0(t - T, x_b)\right) \cdot 1_+(t - T)
+ \left(0 - u_0(t - 2T, x_a)\right) \cdot 1_+(t - 2T)
- \left(u(t - 3T, x_b) - u_0(t - 3T, x_b)\right) \cdot 1_+(t - 3T)
+ \cdots
\]

\[
F_b(t) = \left(u(t, x_b) - u_0(t, x_b)\right) \cdot 1_+(t)
- \left(0 - u_0(t - T, x_a)\right) \cdot 1_+(t - T)
+ \left(u(t - 2T, x_b) - u_0(t - 2T, x_b)\right) \cdot 1_+(t - 2T)
- \left(0 - u_0(t - 3T, x_a)\right) \cdot 1_+(t - 3T)
+ \cdots
\]
2.4 The Case of $\gamma_a \neq \infty$, $\gamma_b \neq \infty$

As before, we begin to analyse the situation looking at the relations

$$u(t, x) \cdot 1(t) = F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t, x) \cdot 1(t)$$

$$u(t, x_a) = F_a(t) + F_b(t - T) + u_0(t, x_a) \quad (t \geq 0)$$

$$u(t, x_b) = F_a(t - T) + F_b(t) + u_0(t, x_b) \quad (t \geq 0)$$

We focus on the two latter. Putting, for short,

$$Q_a(t) := u(t, x_a); Q_{0a}(t) := u_0(t, x_a); Q_b(t) := u(t, x_b); Q_{0b}(t) := u_0(t, x_b);$$

$$(V_T f)(t) := f(t - T);$$

we transform them into

$$\begin{pmatrix} Q_a \\ Q_b \end{pmatrix} = \begin{pmatrix} I & V_T \\ V_T & I \end{pmatrix} \begin{pmatrix} F_a \\ F_b \end{pmatrix} + \begin{pmatrix} Q_{0a} \\ Q_{0b} \end{pmatrix}$$

Now take into account the fact that the current situation is such that

$$F_a(t) = -2\gamma_a \int_0^t u(\tau, x_a)d\tau = -2\gamma_a \int_0^t Q_a(\tau)d\tau$$

$$F_b(t) = -2\gamma_b \int_0^t u(\tau, x_b)d\tau = -2\gamma_b \int_0^t Q_b(\tau)d\tau$$

As a result,

$$\frac{\partial}{\partial t} \begin{pmatrix} F_a \\ F_b \end{pmatrix} = \begin{pmatrix} -2\gamma_a Q_a \\ -2\gamma_b Q_b \end{pmatrix}$$

$$= \begin{pmatrix} -2\gamma_a & 0 \\ 0 & -2\gamma_b \end{pmatrix} \begin{pmatrix} Q_a \\ Q_b \end{pmatrix}$$

$$= \begin{pmatrix} -2\gamma_a & 0 \\ 0 & -2\gamma_b \end{pmatrix} \begin{pmatrix} I & V_T \\ V_T & I \end{pmatrix} \begin{pmatrix} F_a \\ F_b \end{pmatrix} + \begin{pmatrix} -2\gamma_a & 0 \\ 0 & -2\gamma_b \end{pmatrix} \begin{pmatrix} Q_{0a} \\ Q_{0b} \end{pmatrix}$$

$$= \begin{pmatrix} -2\gamma_a & 0 \\ 0 & -2\gamma_b \end{pmatrix} \begin{pmatrix} I & V_T \\ V_T & I \end{pmatrix} \begin{pmatrix} F_a \\ F_b \end{pmatrix} - \begin{pmatrix} 2\gamma_a Q_{0a} \\ 2\gamma_b Q_{0b} \end{pmatrix}$$

$$= \begin{pmatrix} -2\gamma_a & -2\gamma_a V_T \\ -2\gamma_b V_T & -2\gamma_b \end{pmatrix} \begin{pmatrix} F_a \\ F_b \end{pmatrix} - \begin{pmatrix} 2\gamma_a Q_{0a} \\ 2\gamma_b Q_{0b} \end{pmatrix}$$
To continue the analysis, make some preparations. First, for any real $\gamma$, introduce an operation, $E\gamma$, defining it by

$$(E\gamma f)(t) := e^{-2\gamma t}f(t).$$

In addition, let $I_0$ stands for the operation defined by

$$(I_0 f)(t) := \int_0^t f(\tau)d\tau.$$ 

Now let $C_a, C_b$ be defined by

$$\left( \begin{array}{c} F_a \\ F_b \end{array} \right) = \left( \begin{array}{c} E\gamma_a C_a \\ E\gamma_b C_b \end{array} \right)$$

Then

$$\frac{\partial}{\partial t} \left( \begin{array}{c} F_a \\ F_b \end{array} \right) = \left( \begin{array}{cc} -2\gamma_a E\gamma_a C_a \\ -2\gamma_b E\gamma_b C_b \end{array} \right) + \left( \begin{array}{cc} E\gamma_a \frac{\partial}{\partial t} C_a \\ E\gamma_b \frac{\partial}{\partial t} C_b \end{array} \right)$$

$$= \left( \begin{array}{cc} -2\gamma_a & 0 \\ 0 & -\gamma_b \end{array} \right) \left( \begin{array}{c} F_a \\ F_b \end{array} \right) + \left( \begin{array}{cc} E\gamma_a & 0 \\ 0 & E\gamma_b \end{array} \right) \frac{\partial}{\partial t} \left( \begin{array}{c} C_a \\ C_b \end{array} \right)$$

and therefore

$$\left( \begin{array}{cc} E\gamma_a & 0 \\ 0 & E\gamma_b \end{array} \right) \frac{\partial}{\partial t} \left( \begin{array}{c} C_a \\ C_b \end{array} \right) = \left( \begin{array}{cc} 0 & -2\gamma_a V_T \\ -2\gamma_b V_T & 0 \end{array} \right) \left( \begin{array}{c} F_a \\ F_b \end{array} \right) - \left( \begin{array}{c} 2\gamma_a Q_{0a} \\ 2\gamma_b Q_{0b} \end{array} \right)$$

Note that $F_a(0) = 0$, $F_b(0) = 0$, $C_a(0) = 0$, $C_b(0) = 0$. Hence

$$\left( \begin{array}{c} C_a \\ C_b \end{array} \right) = \left( \begin{array}{cc} 0 & -2\gamma_a I_0 E\gamma_a^{-1} V_T \\ -2\gamma_b I_0 E\gamma_b^{-1} V_T & 0 \end{array} \right) \left( \begin{array}{c} F_a \\ F_b \end{array} \right) - \left( \begin{array}{c} 2\gamma_a I_0 E\gamma_a^{-1} Q_{0a} \\ 2\gamma_b I_0 E\gamma_b^{-1} Q_{0b} \end{array} \right)$$

and then

$$\left( \begin{array}{c} F_a \\ F_b \end{array} \right) = \left( \begin{array}{cc} 0 & -2\gamma_a E\gamma_a I_0 E\gamma_a^{-1} V_T \\ -2\gamma_b E\gamma_b I_0 E\gamma_b^{-1} V_T & 0 \end{array} \right) \left( \begin{array}{c} F_a \\ F_b \end{array} \right) - \left( \begin{array}{c} 2\gamma_a E\gamma_a I_0 E\gamma_a^{-1} \gamma_b Q_{0a} \\ 2\gamma_b E\gamma_b I_0 E\gamma_b^{-1} Q_{0b} \end{array} \right)$$

i.e.,

$$F_a(t) = -2\gamma_a \int_0^t e^{-2\gamma_a(t-\tau)} F_b(\tau - T) d\tau - 2\gamma_a \int_0^t e^{-2\gamma_a(t-\tau)} Q_{0a}(\tau) d\tau$$

$$F_b(t) = -2\gamma_b \int_0^t e^{-2\gamma_b(t-\tau)} F_a(\tau - T) d\tau - 2\gamma_b \int_0^t e^{-2\gamma_b(t-\tau)} Q_{0b}(\tau) d\tau$$

5Method of Variation of Constants
equivalently,

\[
F_a(t) = -2\gamma_a \int_0^t e^{-2\gamma_a(t-\tau)} F_b(\tau - T) d\tau - 2\gamma_a \int_0^t e^{-2\gamma_a(t-\tau)} u_0(\tau, x_a) d\tau \\
F_b(t) = -2\gamma_b \int_0^t e^{-2\gamma_b(t-\tau)} F_a(\tau - T) d\tau - 2\gamma_b \int_0^t e^{-2\gamma_b(t-\tau)} u_0(\tau, x_b) d\tau
\]

Finally, recall that

\[
\begin{align*}
  u(t, x) &= F_a(t - |x - x_a|/c) + F_b(t - |x - x_b|/c) + u_0(t, x) & (t \geq 0)
\end{align*}
\]

\[\text{Note that } F_a(0) = 0, F_b(0) = 0, C_a(0) = 0, C_b(0) = 0.\]
3 Explicit Relations

3.1 The Algebra of $I_{\gamma,T_0}$

In the previous section we have seen the recurrence relations, which normally contain "retarded integral operations". When solving such relations, a usual machinery involves a treatment of the various compositions of the operations that build the relations. So, we try to find an effective representation of the referred compositions.

Now, let $I_{\gamma,T_0}$ denote the operation defined by

$$\left(I_{\gamma,T_0}f\right)(t) := \int_{T_0}^{t} e^{-2\gamma(t-\tau)} f(\tau - T_0) d\tau$$

$$= \int_{T_0}^{t} e^{-2\gamma(t-\tau)} f(\tau - T_0) \cdot 1_+(t - T_0)$$

$$= \int_{0}^{t-T_0} e^{-2\gamma(t-T_0-\tau)} f(\tau) d\tau \cdot 1_+(t - T_0)$$

$$= \int_{0}^{t-T_0} e^{-2\gamma(t-T_0-\tau)} f(\tau) d\tau.$$  

Firstly, notice that

$$\left(I_{\gamma,T_0}f\right)(t) = \int_{0}^{t} e^{-2\gamma(t-T_0-\tau)} f(\tau) \cdot 1_+(t - T_0 - \tau) d\tau \cdot 1_+(t)$$

In particular, the operations under consideration are usual convolution operations, and as such they are commutative:

$$I_{\gamma_1,T_1}I_{\gamma_2,T_2} = I_{\gamma_2,T_2}I_{\gamma_1,T_1}.$$  

In addition notice that

$$\left(I_{\gamma,T_0}f\right)(t + T_0) = \int_{0}^{t} e^{-2\gamma(t-\tau)} f(\tau) d\tau \cdot 1_+(t)$$

$$= \left(I_{\gamma,T_0}f\right)(t)$$

Next, let us calculate

$I_{\gamma,T_1}I_{\gamma,T_2}$

The usual way is:

$$\left(I_{\gamma,T_1}I_{\gamma,T_2}f\right)(t)$$

$$= \int_{0}^{t-T_1} e^{-2\gamma(t-T_1-\tau)} \left(I_{\gamma,T_2}f\right)(\tau) d\tau \cdot 1_+(t - T_1)$$

$$= \int_{0}^{t-T_1} e^{-2\gamma(t-T_1-\tau)} \left(I_{\gamma,T_2}f\right)(\tau) \cdot 1_+(\tau - T_2) d\tau \cdot 1_+(t - T_1)$$

$$= \int_{0}^{t-T_1} e^{-2\gamma(t-T_1-\tau)} \left(I_{\gamma,T_2}f\right)(\tau) \cdot 1_+(\tau - T_2) d\tau \cdot 1_+(t - T_1 - T_2)$$

$$= \int_{T_2}^{t-T_1} e^{-2\gamma(t-T_1-\tau)} \left(I_{\gamma,T_2}f\right)(\tau) d\tau \cdot 1_+(t - T_1 - T_2)$$

$$= \int_{0}^{t-T_1-T_2} e^{-2\gamma(t-T_1-T_2-\tau)} \left(I_{\gamma,T_2}f\right)(\tau + T_2) d\tau \cdot 1_+(t - T_1 - T_2)$$

\[\text{recall that, in the context, } f_a^b \text{ means } f_a^b \cdot 1_+(b - a)\]
Thus, we have seen that
\[
I_{\gamma, T_1} I_{\gamma, T_2} = I_{\gamma, T_1 + T_2} I_{\gamma, 0}.
\]
Consequences which we need:
\[
I_{\gamma_0, 2T}^N = I_{\gamma_0, 2NT} I_{\gamma_0, 0}^{N-1},
\]
\[
(I_{\gamma_0, T})^{2N} = I_{\gamma_0, 2NT} I_{\gamma_0, 0}^{2N-1},
\]
\[
(I_{\gamma_0, T})^{2N+1} = I_{\gamma_0, (2N+1)T} I_{\gamma_0, 0}^{2N},
\]
\[
(I_{\gamma_0, T} I_{\gamma_0, T})^N = (I_{\gamma_0, T} I_{\gamma_0, T})^N = I_{\gamma_0, N T} I_{\gamma_0, N T} I_{\gamma_0, 0}^{N-1} I_{\gamma_0, 0}^{N-1}.
\]
In addition, let \( E_{\gamma} \) denote the operation defined by
\[
(E_{\gamma} f)(t) := e^{\gamma t} f(t).
\]
Then we can write
\[
I_{\gamma, 0} = E_{\gamma}^{-1} I_{0, 0} E_{\gamma},
\]
and then
\[
I_{\gamma, 0}^2 = E_{\gamma}^{-1} I_{0, 0} E_{\gamma} E_{\gamma}^{-1} I_{0, 0} E_{\gamma} = E_{\gamma}^{-1} I_{0, 0}^2 E_{\gamma}.
\]
Iterating this kind of arguments we infer that
\[
I_{\gamma, T_0}^N = I_{\gamma, NT_0} E_{\gamma}^{-1} I_{0, 0}^{N-1} E_{\gamma},
\]
\[
I_{\gamma, T_0}^N I_{\gamma, 0} = I_{\gamma, NT_0} E_{\gamma}^{-1} I_{0, 0}^N E_{\gamma}
\]
and then
\[
(I_{\gamma, T_0}^N f)(t) = \int_0^{t-NT_0} \int_{\gamma_0}^{\gamma} \cdots \int_{\gamma_0}^{\gamma_0} e^{-2\gamma_0 (t-NT_0-\tau_{N-2})} f(\tau_{N-2}) d\tau_{N-2} \cdots d\tau_1 \ d\tau
\]
\[
= \int_0^{t-NT_0} \frac{(t-NT_0-\tau)^{N-1}}{(N-1)!} e^{-2\gamma (t-NT_0-\tau)} f(\tau) \ d\tau
\]
\[
= \int_0^{t} \frac{(t-NT_0-\tau)^{N-1}}{(N-1)!} e^{-2\gamma (t-NT_0-\tau)} f(\tau) \cdot 1_+(t-NT_0-\tau) \ d\tau,
\]
Finally,
\[
(I_{\gamma, T_0}^N I_{\gamma, 0} f)(t) = \int_0^{t-NT_0} \frac{(t-NT_0-\tau)^N}{N!} e^{-2\gamma (t-NT_0-\tau)} f(\tau) \ d\tau
\]
\[
= \int_0^{t} \frac{(t-NT_0-\tau)^N}{N!} e^{-2\gamma (t-NT_0-\tau)} f(\tau) \cdot 1_+(t-NT_0-\tau) \ d\tau.
\]
Now, we go on to see consequences of these consequences.
3.2 Explicit Relations: The Case of \( u(t, x_a) = 0, \quad \gamma_b \neq \infty \)

We have seen that

\[
\begin{align*}
 u(t, x_b) &= 2\gamma_b \int_0^t e^{-2\gamma_b(t-\tau)} u(\tau - 2T, x_b) d\tau \\
&\quad + I_{0b}(t), \\
\end{align*}
\]

where

\[
I_{0b}(t) := \int_{0+0}^t e^{-2\gamma_b(t-\tau)} d\tau (u_0(\tau, x_b) - u_0(\tau - T, x_a) 1_+(\tau - T)) + e^{-2\gamma_b t} u(0, x_b).
\]

Recall, \( I_{\gamma, T_0} \) denotes the operation defined by

\[
\left( I_{\gamma, T_0} f \right)(t) := \int_{T_0}^t e^{-2\gamma(t-\tau)} f(\tau - T_0) d\tau
\]

\[
= e^{2\gamma T_0} \int_0^{t-T_0} e^{-2\gamma(t-\tau)} f(\tau) d\tau
\]

\[
= e^{2\gamma T_0} \int_{-\infty}^{\infty} e^{-2\gamma(t-\tau)} f(\tau) \cdot 1_+(t - T_0 - \tau) \cdot 1_+(\tau) d\tau \cdot 1_+(t - T_0).
\]

Thus, we can write

\[
\begin{align*}
 u(t, x_b) &= I_{0b}(t), \quad \text{(if } 0 \leq t < 2T) \\
 u(t, x_b) &= \left( (1 + (2\gamma_b) I_{\gamma, 2T}) I_{0b} \right)(t), \quad \text{(if } 0 \leq t < 4T) \\
 u(t, x_b) &= \left( (1 + (2\gamma_b) I_{\gamma, 2T} + (2\gamma_b)^2 I_{\gamma, 2T}^2 + \cdots + (2\gamma_b)^N I_{\gamma, 2T}^N) I_{0b} \right)(t), \quad \text{(if } 0 \leq t < (N+1)2T) \\
\end{align*}
\]

So, iterating,

\[
\begin{align*}
 u(t, x_b) &= \left( (1 + (2\gamma_b) I_{\gamma, 2T} + (2\gamma_b)^2 I_{\gamma, 2T}^2 + \cdots + (2\gamma_b)^N I_{\gamma, 2T}^N) I_{0b} \right)(t)
\end{align*}
\]

Finally, by recalling that

\[
\left( I_{\gamma, T_0, f} \right)(t) = \int_0^t \frac{(t - NT_0 - \tau)^{N-1}}{(N-1)!} e^{-2\gamma(t-NT_0-\tau)} f(\tau) \cdot 1_+(t - NT_0 - \tau) d\tau
\]

and after introducing

\[
Exp(\lambda, T_0, t)
\]

\[
:= 1 \cdot 1_+(t) + \lambda(t - T_0) \cdot 1_+(t - T_0) + \lambda^2 \frac{(t - 2T_0)^2}{2!} \cdot 1_+(t - 2T_0) \\
+ \cdots + \lambda^N \frac{(t - NT_0)^N}{N!} \cdot 1_+(t - NT_0) \cdots
\]

we conclude that

\[
\begin{align*}
 u(t, x_b) &= I_{0b}(t) \\
&\quad + (2\gamma_b) \int_0^t e^{-2\gamma_b(t-\tau)} I_{0b} \left( \tau - 2T \right) d\tau
\end{align*}
\]
where

\[ I_{0b}(t) := \int_{0+0}^{t} e^{-2\gamma_b(t-\tau)} d\tau (u_0(\tau, x_b) - u_0(\tau - T, x_a)1_{+}(\tau - T)) + e^{-2\gamma_b t} u(0, x_b). \]
3.3 Explicit Relations: The Case of $\gamma_a \neq \infty$, $\gamma_b \neq \infty$

Imitating the arguments of the previous subsections, we observe that

$$\begin{pmatrix} F_a \\ F_b \end{pmatrix} = \begin{pmatrix} 0 & -2\gamma_a I_{\gamma_a,T} \\ -2\gamma_b I_{\gamma_b,T} & 0 \end{pmatrix} \begin{pmatrix} F_a \\ F_b \end{pmatrix} + \begin{pmatrix} 2\gamma_a I_{\gamma_a,0}Q_{0a} \\ 2\gamma_b I_{\gamma_b,0}Q_{0b} \end{pmatrix}$$

So, we infer that

$$\begin{pmatrix} F_a \\ F_b \end{pmatrix} = -\sum_{n=0}^{\infty} (4\gamma_a \gamma_b)^n (I_{\gamma_a,T}I_{\gamma_b,T})^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -2\gamma_a I_{\gamma_a,T} \\ -2\gamma_b I_{\gamma_b,T} & 0 \end{pmatrix}^{2n+1} = -(4\gamma_a \gamma_b)^n (I_{\gamma_a,T}I_{\gamma_b,T})^n \begin{pmatrix} 0 & 2\gamma_a I_{\gamma_a,T} \\ 2\gamma_b I_{\gamma_b,T} & 0 \end{pmatrix},$$

we infer that

$$\begin{pmatrix} F_a \\ F_b \end{pmatrix} = -\sum_{n=0}^{\infty} (4\gamma_a \gamma_b)^n (I_{\gamma_a,T}I_{\gamma_b,T})^n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2\gamma_a I_{\gamma_a,0}Q_{0a} \\ 2\gamma_b I_{\gamma_b,0}Q_{0b} \end{pmatrix}$$

$$+ \sum_{n=0}^{\infty} (4\gamma_a \gamma_b)^n (I_{\gamma_a,T}I_{\gamma_b,T})^n \begin{pmatrix} 0 & 2\gamma_a I_{\gamma_a,T} \\ 2\gamma_b I_{\gamma_b,T} & 0 \end{pmatrix} \begin{pmatrix} 2\gamma_a I_{\gamma_a,0}Q_{0a} \\ 2\gamma_b I_{\gamma_b,0}Q_{0b} \end{pmatrix}$$

Unfortunately, I do not know a good representation of this formula, in the general case. Nevertheless, if

$$\gamma_a = \gamma_b =: \gamma_0,$$

then the situation becomes very similar to that of the previous subsection: we infer that

$$\begin{pmatrix} F_a \\ F_b \end{pmatrix} = -\sum_{n=0}^{\infty} (2\gamma_0)^{2n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} I_{\gamma_0,T}^{2n} \begin{pmatrix} 2\gamma_0 I_{\gamma_0,0}Q_{0a} \\ 2\gamma_0 I_{\gamma_0,0}Q_{0b} \end{pmatrix}$$

$$+ \sum_{n=0}^{\infty} (2\gamma_0)^{2n+1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} I_{\gamma_0,T}^{2n+1} \begin{pmatrix} 2\gamma_0 I_{\gamma_0,0}Q_{0a} \\ 2\gamma_0 I_{\gamma_0,0}Q_{0b} \end{pmatrix}$$

$$= -\sum_{n=0}^{\infty} (2\gamma_0)^{2n} \begin{pmatrix} 2\gamma_0 I_{\gamma_0,0}Q_{0a} \\ 2\gamma_0 I_{\gamma_0,0}Q_{0b} \end{pmatrix}$$

$$+ \sum_{n=0}^{\infty} (2\gamma_0)^{2n+1} \begin{pmatrix} 2\gamma_0 I_{\gamma_0,0}Q_{0a} \\ 2\gamma_0 I_{\gamma_0,0}Q_{0b} \end{pmatrix}$$
Finally, as in the previous subsection, by recalling that
\[ (I_{\gamma,T}^{2n} I_{\gamma,0} f)(t) = \int_0^t \frac{(t - 2nT_0 - \tau)^{2n}}{(2n)!} e^{-2\gamma(t-2nT_0-\tau)} f(\tau) \cdot 1_+(t - 2nT_0 - \tau) d\tau \]
\[ (I_{\gamma,T}^{2n+1} I_{\gamma,0} f)(t) = \int_0^t \frac{(t - (2n + 1)T_0 - \tau)^{2n+1}}{(2n + 1)!} e^{-2\gamma(t-(2n+1)T_0-\tau)} f(\tau) \cdot 1_+(t - (2n + 1)T_0 - \tau) d\tau \]
and after introducing

\[
\begin{align*}
\text{Sinh}(\lambda, T_0, t) &= \lambda(t - T_0) \cdot 1_+(t - T_0) + \lambda^3 \frac{(t - 3T_0)^3}{3!} \cdot 1_+(t - 3T_0) \\
&\quad + \ldots + \lambda^{2n+1} \frac{(t - (2n + 1)T_0)^{2n+1}}{(2n + 1)!} \cdot 1_+(t - (2n + 1)T_0) \\
\text{Cosh}(\lambda, T_0, t) &= 1 \cdot 1_+(t) + \lambda^2 \frac{(t - 2T_0)^2}{2!} \cdot 1_+(t - 2T_0) \\
&\quad + \ldots + \lambda^{2n} \frac{(t - 2nT_0)^{2n}}{(2n)!} \cdot 1_+(t - 2nT_0) 
\end{align*}
\]
we observe that
\[ \left( \sum_{n=0}^{\infty} (2\gamma_0)^{2n} I_{\gamma_0,T}^{2n} I_{\gamma_0,0} f \right)(t) = \int_0^t \text{Cosh}(2\gamma_0 e^{2\gamma_0 T}, t - \tau) e^{-2\gamma_0(t-\tau)} f(\tau) d\tau \]
\[ \left( \sum_{n=0}^{\infty} (2\gamma_0)^{2n+1} I_{\gamma_0,T}^{2n+1} I_{\gamma_0,0} f \right)(t) = \int_0^t \text{Sinh}(2\gamma_0 e^{2\gamma_0 T}, t - \tau) e^{-2\gamma_0(t-\tau)} f(\tau) d\tau \]
The conclusion is evident.

Remark.
\[ (I_{\gamma_0,T}^N I_{\gamma_0,0} f)(t) = \int_0^{t-NT} \frac{(t - NT - \tau)^N}{N!} e^{-2\gamma_0(t-NT-\tau)} f(\tau) d\tau \]
\[ = \int_{NT}^t \frac{(t - \tau)^N}{N!} e^{-2\gamma_0(t-\tau)} f(\tau - NT) d\tau \]
Hence
\[ \left( \sum_{n=0}^{\infty} (2\gamma_0)^{2n} I_{\gamma_0,T}^{2n} I_{\gamma_0,0} f \right)(t) = \sum_{n=0}^{\infty} (2\gamma_0)^{2n} \int_{2nT}^t \frac{(t - \tau)^{2n}}{(2n)!} e^{-2\gamma_0(t-\tau)} f(\tau - 2nT) d\tau \]
\[ \left( \sum_{n=0}^{\infty} (2\gamma_0)^{2n+1} I_{\gamma_0,T}^{2n+1} I_{\gamma_0,0} f \right)(t) = \sum_{n=0}^{\infty} (2\gamma_0)^{2n+1} \int_{(2n+1)T}^t \frac{(t - \tau)^{2n+1}}{(2n + 1)!} e^{-2\gamma_0(t-\tau)} f(\tau - (2n + 1)T) d\tau \]
References

[AK] S. Albeverio and P. Kurasov, Singular Perturbations of Differential Operators. Solvable Schrödinger Type Operators, London Mathematical Society: Lecture Note Series. 271, 1999. CAMBRIDGE UNIVERSITY PRESS

[BF] F.A. Berezin, L.D. Faddeev, Remark on the Schrödinger equation with singular potential, Dokl. Akad. Nauk. SSSR, 137 (1961) 1011-1014 (in Russian).

[BST56] Budak, B.M.; Samarskij, A.A.; Tikhonov, A.N. Aufgabensammlung zur mathematischen Physik. (Russian) Moskau: Staatsverlag für technisch-theoretische Literatur 1956. 684 S. (1956).

[BST80] Budak, B.M.; Samarskij, A.A.; Tikhonov, A.N. ( = Tichonov, Tichonow, Tychonoff) A collection of problems of mathematical physics (English) Translated by A.R.M. Robson. Translation edited by D.M. Brink (International Series of Monographs in Pure and Applied Mathematics Vo. 52) Oxford-London-New York-Paris: Pergamon Press 1964, XII, 768 p. 80 s. (1964).

[Do] W. Donoghue, On the perturbation of spectra, Comm. Pure App. Math. 18 (1965) 559-579

[Fog] S.R. Foguel, Finite Dimensional Perturbations in Banach Spaces, American Journal of Mathematics, Volume 82, Issue 2 (Apr., 1960), 260-270

[Jack] J.D. Jackson: Classical electrodynamics. John Wiley & Sons, Inc. New York-London, 1962.

[RT79] Razumov, A.V.; Tarano, A.Ju. Dipole interaction of an oscillator with a scalar field. (Russian) Teoret. Mat. Fiz. 38 (1979), no. 3, 355-363. MR 80b:81034

Electronic Print:
Mathematical Physics Preprint Archive mp_arc
(http://www.ma.utexas.edu/mp_arc/index.html):

[BrDeB] mp_arc 01-275
L. Bruneau, S. DeBiévre A Hamiltonian model for linear friction in a homogeneous medium (548K, postscript) Jul 17, 01
http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=01-275
http://mpej.unige.ch/mp_arc-bin/mpa?yn=01-275
http://www.maia.ub.es/mp_arc-bin/mpa?yn=01-275

[DerFr] mp_arc 02-275
Derezinski J., Fruboes R. Renormalization of the Friedrichs Hamiltonian (16K, LATEX 2e)
http://www.ma.utexas.edu/mp_arc-bin/mpa?yn=02-275
http://mpej.unige.ch/mp_arc-bin/mpa?yn=02-275
http://www.maia.ub.es/mp_arc-bin/mpa?yn=02-275
http://LANL E-Print (http://arXiv.org/):
[AMN] Paper: physics/0001009
From: adolfo@lafexSu1.lafex.cbpf.br (Adolfo Malbouisson)
Date: Wed, 5 Jan 2000 19:20:08 GMT (15kb)
Title: An Exact Approach to the Oscillator Radiation Process in an Arbitrarily Large Cavity
Authors: N.P. Andion, A.P.C. Malbouisson and A. Mattos Neto
Comments: 27 pages
Subj-class: Atomic Physics; Mathematical Physics

[CGS] Paper: quant-ph/0307232
From: Ricardo Moritz Cavalcanti rmoritz@if.ufrj.br
Date: Wed, 30 Jul 2003 18:14:48 GMT (20kb)
Title: Decay in a uniform field: An exactly solvable model
Authors: R. M. Cavalcanti, P. Giacconi and R. Soldati
Comments: 21 pages, 2 figures
Subj-class: Quantum Physics; Atomic Physics; Mathematical Physics
(http://arXiv.org/abs/quant-ph/0307232, 20kb)

[Ch1] Paper: math.DS/0301167
From: Sergej A. Choroszavin sergius@pve.vsu.ru
Date: Thu, 16 Jan 2003 04:34:16 GMT (18kb)
Title: An Interaction of An Oscillator with An One-Dimensional Scalar Field. Simple Exactly Solvable Models based on Finite Rank Perturbations Methods. I: D’Alembert-Kirchhoff-like formulae
Author: Sergej A. Choroszavin
Comments: Latex 2.09
Subj-class: Dynamical Systems; Mathematical Physics
(http://arXiv.org/abs/math/0301167, 18kb)

[Ch2] Paper: math-ph/0302038
From: Sergej A. Choroszavin sergius@pve.vsu.ru
Date: Sat, 15 Feb 2003 02:14:09 GMT (15kb)
Title: An Interaction of An Oscillator with An One-Dimensional Scalar Field. Simple Exactly Solvable Models based on Finite Rank Perturbations Methods. II: Resolvents formulae
Author: Sergej A. Choroszavin
Comments: Latex 2.09
Subj-class: Mathematical Physics; Dynamical Systems
(http://arXiv.org/abs/math-ph/0302038, 15kb)

[Ch3] Paper: math-ph/0307029
From: Sergej Choroszavin sergius@pve.vsu.ru
Date: Sun, 13 Jul 2003 22:32:33 GMT (14kb)
Title: 1D Particle, 1D Field, 1D Interaction. Simple Exactly Solvable Models based on Finite Rank Perturbations Methods. III. Linear Friction as Radiation Reaction
Authors: Sergej A. Choroszavin
Comments: Latex 2.09
Subj-class: Mathematical Physics; Dynamical Systems
(http://arXiv.org/abs/math-ph/0307029, 14kb)
From: physth@ulb.ac.be
Date: Fri, 10 Jul 92 16:52:54 +0200 (20kb)

Title: On the Problem of the Uniformly Accelerated Oscillator
Authors: S. Massar, R. Parentani, R. Brout
Comments: 14 pages (+postscript figures attached), ULB-TH-03/92