Full characterisation of attractors of two tangentially intersected asynchronous Boolean automata cycles

Tarek Melliti¹, Mathilde Noual², Damien Regnault¹, Sylvain Sené³,⁴, and Jérémy Sobieraj¹

¹ IBISC, EA4526, Université d’Évry Val-d’Essonne, 91000 Évry, France (tarek.melliti, damien.regnault}@ibisc.univ-evry.fr, jeremy.sobieraj@gmail.com)
² I3S, UMR7271 CNRS et Université de Nice Sophia Antipolis, 06900 Sophia Antipolis, France (mathilde.noual@unice.fr)
³ Aix-Marseille Université, CNRS, LIF UMR 7279, 13000 Marseille, France (sylvain.sene@univ-amu.fr)
⁴ IXXI, Institut rhône-alpin des systèmes complexes, 69000 Lyon, France

Abstract. The understanding of Boolean automata networks dynamics takes an important place in various domains of computer science such as computability, complexity and discrete dynamical systems. In this paper, we make a step further in this understanding by focusing on their cycles, whose necessity in networks is known as the brick of their complexity. We present new results that provide a characterisation of the transient and asymptotic dynamics, i.e. of the computational abilities, of asynchronous Boolean automata networks composed of two cycles that intersect at one automaton, the so-called double-cycles. To do so, we introduce an efficient formalism inspired by algorithms to define long sequences of updates, that allows a better description of their dynamics than previous works in this area.

Keywords: Interaction networks, Boolean automata networks, double-cycles, asynchronous dynamics.

1 Introduction

Interaction networks occupying a ceaselessly increasing space in the knowledge of the objects that surround or even constitute us (e.g. genetic regulation networks) as well as in our daily life (e.g. social networks), it is now necessary to understand more their intrinsic properties. This paper follows this statement by using automata networks (ANs) as models of interaction networks. ANs have been chosen for two major reasons. First, although this computational model is among the firsts developed [10,12], lots of their intrinsic computational properties are not known nowadays. Second, their simplicity, and the concepts and
parameters needed to define them, make them particularly adapted to capture
the essence of, and model, real interaction systems at a high abstraction level,
such as physical, biological and social systems [5,8,20]. The present work pre-
cisely takes place at the frontier of theoretical computer science and fundamental
bio-informatics, that aims at analysing and explaining formally the dynamics of
biological regulations, that have constituted the core of molecular biology [6,7].

Fundamental bio-informatics gives rise to many theoretical and applied ques-
tions. In this context, Boolean automata networks (BANs) play a leading role.
Indeed, since the seminal works of Kauffman [8,9] and Thomas [21,22] in the-
oretical biology, computer scientists have not stopped trying to answer their
questions/conjectures. Among the latter, those that are central in this work are
Thomas’ ones, for which solutions have been proven in the discrete framework
at the end of 2000’s [16,17,18]. These results, together with those of Robert [19],
highlighted that the ability of ANs to admit complex asymptotic behaviours only
comes from the presence of cycles in their architecture. However, although the
fact that interacting cycles are the engines of dynamical complexity is known,
we don’t really/perfectly know how yet. That explains why many recent studies
focused on these specific patterns. Among them, [1] gave the characterisation
in parallel of the dynamical behaviours of Boolean automata cycles (BACs).
Then, time was attached to analyse the relations between the dynamical prop-
erties of cycles subjected to distinct updating modes, with a special attention
paid to the asynchronous and the parallel ones [14]. Once the cycle dynamics
finely understood, the natural idea was to study more complex networks. But
to obtain general results for any kind of network remains an open problem that
seems intractable at present. So, following a constructive approach and as a first
step, studies have been led on specific patterns combining cycles, such as the
double-cycles in parallel [1] and the flower-graphs [2] for instance. In addition,
other studies have dealt with the convergence time of specific classes of BANs,
like circular XOR networks [15] and networks without negative cycles [11].

This paper follows the same lines and solves a question that remained open
until now: how do Boolean automata double-cycles that evolve asynchronously
over time behave? The answer is given by emphasising original methods for the
domain in the sense that they are very algorithmic. In particular, they allow
to show that recurrent configurations are not all similar (some have peculiar
features). Some of them can be reached by following paths of linear size accord-
ing to the network sizes whereas other need quadratic sequences of updates to
be reached. In fact, the results presented give a deeper characterisation of the
attractors.

The paper is organised as follows: Section 2 gives the main definitions
and notations used in the paper, in particular those related to the double-cycles and
the asynchronous updating mode; Section 3 gives the definition of the tools and
methods developed here; finally, Section 4 is dedicated to the main contributions
of this paper.
2Definitions and notations

BANs. Consider \( \mathbb{B} = \{0, 1\} \) and \( V = \{0, \ldots, n-1\} \) a set of \( n \) Boolean automata so that \( \forall i \in V, x_i \in \mathbb{B} \) denotes the state of \( i \). A configuration of a BAN \( \mathcal{N} \) of size \( n \) instantiates the state of any \( i \) of \( V \) and is classically denoted as a vector, such that \( x \in \mathbb{B}^n \), or as a binary word. Formally, a BAN \( \mathcal{N} \), whose automata set is \( V \), is a set of \( n \) Boolean functions, which means that \( \mathcal{N} = \{f_i : \mathbb{B}^n \to \mathbb{B} \mid i \in V\} \). Given \( i \in V \), \( f_i \) is the local transition function of \( i \) that predetermines its evolution for any configuration \( x \). Actually, that means that if \( i \) is updated in \( x \), its state switches from \( x_i \) to \( f_i(x) \). Let us define now the sign of an interaction from \( j \) to \( i \) (\( i, j \in V \)) in configuration \( x \in \mathbb{B}^n \) with \( \text{sign}_{x}(j,i) = s(x_j)(f_i(x) - f_i(\overline{x_j})) \), where \( s : \mathbb{B} \to \mathbb{I} \), with \( \mathbb{I} = \{-1, 1\} \), is defined as \( s(b) = b - \overline{b} \) and \( \forall i \in V \), \( \overline{x_i} = (x_0, \ldots, x_{i-1}, \overline{x_i}, x_{i+1}, \ldots, x_{n-1}) \). Interactions that are effective in \( x \) belongs to the set \( \mathcal{A}(x) = \{(j, i) \in V^2 \mid \text{sign}_{x}(j,i) \neq 0\} \). From this is derived the interaction graph of \( \mathcal{N} \) that is the digraph \( G = (V, \mathcal{A}) \), where \( \mathcal{A} = \bigcup_{x \in \mathbb{B}^n} \mathcal{A}(x) \) is the set of interactions.

In this paper, the focus is put on BANs associated with simple interaction graphs: if there exists \( (j, i) \in \mathcal{A} \), it is unique and such that \( \forall x \in \mathbb{B}^n, \text{sign}_{x}(j,i) \neq 0 \) and is constant. As a consequence, \( \text{sign}(j, i) \in \mathbb{I} \). If \( \text{sign}(j, i) = +1 \) (resp. \( -1 \)), \( (j, i) \) is an activating (resp. inhibiting) interaction so that the state of \( i \) tends to mimic (resp. negate) that of \( j \). We call the signed interaction graph of \( \mathcal{N} \) the digraph obtained by labelling each arc \( (i, j) \in \mathcal{A} \) with \( \text{sign}(i, j) \).

In order not to burden the reading, we also denote it by \( G \). Abusing notations, a cycle \( C \) of \( G \) is said to be positive (resp. negative) if the product of the signs of the interactions that compose it equals \( +1 \) (resp. \(-1\)).

Asynchronous transition graphs. In a BAN \( \mathcal{N} \), a couple of configurations \((x, y) \in \mathbb{B}^n \times \mathbb{B}^n \) such that \( y \) is obtained by updating the state of a unique automaton of \( x \) is an asynchronous transition, and is denoted by \( x \xrightarrow{\text{Hamming distance d}} y \) (the Hamming distance \( d(x, y) \leq 1 \)). If \( x \neq y \), \( x \xrightarrow{\text{effective}} y \) is said to be effective. Let \( T = \{x \rightarrow y \mid x, y \in \mathbb{B}^n\} \) be the set of asynchronous transitions of \( \mathcal{N} \). Digraph \( \mathcal{G} = (\mathbb{B}^n, T) \) is then the asynchronous transition graph (abbreviated simply by transition graph) of \( \mathcal{N} \), which actually represents the non-deterministic "perfectly" asynchronous discrete dynamical system related to \( \mathcal{N} \).

Consider an arbitrary BAN \( \mathcal{N} \), its transition graph \( \mathcal{G} = (\mathbb{B}^n, T) \) and \( x \in \mathbb{B}^n \) any of its possible configurations. A trajectory of \( x \) is any path in \( \mathcal{G} \) that starts in \( x \). A strongly connected component of \( \mathcal{G} \) that admits no outgoing asynchronous transitions is a terminal strongly connected component (TSCC). A TSCC of \( \mathcal{G} \) represents an asymptotic behaviour of \( \mathcal{N} \), i.e. one of its attractors. A configuration that belongs to an attractor is a recurrent configuration and, for a given attractor, the number of its configurations is said to be its size. An attractor of size \( 1 \) (resp. of size greater than \( 1 \)) is a stable configuration (resp. a stable oscillation). We close this paragraph by defining the convergence time of a configuration \( x \) as the length of the shortest trajectory that leads it to an attractor and the convergence time of a BAN as the highest convergence time of all configurations in \( \mathbb{B}^n \).
Fig. 1. Interaction graphs of the three kinds of canonical BADCs: (a) a canonical positive BADC, (b) a canonical mixed BADC, and (c) a canonical negative BADC.

Boolean automata double-cycles. The literature has put the emphasis on BACs. The reason comes from the three following theorems that show that cycles are necessary for BANs to admit complex asymptotic dynamics. Now, consider $G$ as the asynchronous transition graph of a BAN $N$.

**Theorem 1.** [19] Whatever the updating mode is, if $N$ does not contain any cycle, then it admits a unique attractor, that is a stable configuration.

**Theorem 2.** [16,18,22] If $G$ admits two stable configurations then the interaction graph of $N$ contains a positive cycle.

**Theorem 3.** [16,17,22] If $G$ admits a stable oscillation then the interaction graph of $N$ contains a negative cycle.

On the basis of the theorems above, and in the same lines as [1,13] that characterises the dynamical behaviour in parallel of Boolean automata double-cycles (BADCs), we propose in this paper to study BADCs when updated asynchronously. Informally, a BADC $D$ of size $n+m-1$ is composed of two BACs $C^+$ (of size $n$) and $C^-$ (of size $m$) that intersect tangentially at one automaton that will be denoted specifically, for the sake of clarity in proofs, by $c$ (resp. $c^+_0$, $c^-_0$) when considering $D$ (resp. $C^+$, $C^-$). Notice that in $D$, every automaton admits a unary function as its local transition function that is either $id$ or $neg$, except automaton $c$ that admits a binary function. In this paper, we focus on monotone functions and enforce $f_c$ to be the AND-function without loss of generality for our concern. Also, remark that there exist three different kinds of BADCs: positive BADCs made of two positive BACs, negative BADCs made of two negative BACs, and mixed BADCs made of one positive and one negative cycles. An interesting point is that the study of BADCs of size $n + m - 1$ in general can be reduced to that of three canonical BADCs of size $n + m - 1$ [13,14], presented in Figure 1 because of the isomorphism between their transition graphs. A canonical positive BADC $D^+$ is composed only of positive interactions. A canonical negative BADC $D^-$ is composed only of positive interactions, except the two that have $c$ as their destination. A canonical mixed BADC $D^\pm$ is composed only of positive interactions, except one of those that have $c$ as their destination (we suppose that this interaction belongs to $C^-$). To finish, for easing the proofs, we denote a BADC configuration $x$ by a vector of two binary words, in which the first symbol represents $x_c$. For instance, the null configuration in which all
automata are at state 0 is denoted by \((0^n, 0^m)\). Also, we denote by \(x^L\) (resp. \(x^R\)) the projection of \(x\) on cycle \(C^L\) (resp. \(C^R\)). Thus, \(x = (x^L, x^R)\) and the state of automaton \(c_i\) in configuration \(x\) is \(x_i^L\). Note that \(x_0^L = x_0^R = x_0^R\) since these three notations stand for the state of automaton \(c\) in configuration \(x\).

3 Algorithmic tools

In this section, we introduce the tools that will be used further to study the dynamics of BADCs. We introduce first the expressiveness of a configuration, which counts the number of its 01 patterns. This notion is inspired by works on asynchronous cellular automata that have shown that the occurrence number of this pattern is crucial to understand their behaviour [3]. Then are introduced instructions to represent sequences of updates as classical algorithms. Instructions are used to express long sequences of updates with few lines of code.

3.1 Expressiveness

**Definition 1.** Let \(x\) be a configuration of a BAC \(\mathcal{C}\) of size \(n\). The expressiveness of \(x\) is the number of 01 patterns in \(x\), i.e. \(|\{i \mid 0 \leq i \leq n-1, x_i = 0 \text{ and } x_{i+1} \mod n = 1\}|\).

From Definition 1, we derive easily the expressiveness of a configuration \(x\) of a BADC \(\mathcal{D}\) as the sum of the expressivenesses of \(x^L\) and \(x^R\). Expressiveness is very useful to understand the structure of attractors. The least expressive configurations are \((0^n, 0^m)\) and \((1^n, 1^m)\) and the most expressive ones are \(((10)^\#, (10)^\#)\) and \(((01)^\#, (01)^\#)\) (if \(n\) and \(m\) are even). In the sequel, we will see that: (i) the lowly expressive configurations generally are recurrent and can be reached in linear time by most configurations; (ii) the highly expressive configurations either are not recurrent or can only be reached through very specific update sequences, and they can quickly reach any other configuration. So, for a BADC \(\mathcal{D}\) that admits an attractor of exponential size made of lowly expressive and highly expressive configurations, we conjecture that: (1) the shortest path from a highly expressive configuration to any other configuration is linear in \(n\) and \(m\); (2) the shortest path from a lowly expressive configuration to a highly expressive one is quadratic in \(n\) and \(m\). In other terms, to decrease expressiveness is easy whereas to increase expressiveness is hard.

3.2 Elementary instructions

In this article, lots of proofs rely on exhibiting update sequences between two configurations. However, the length of such sequences is problematic and a human reader would not manage to extract directly from these sequences the proof general ideas. Thus, we propose to view update sequences as instructions that allow to define them and understand their effect on configurations easily.

Let \(\mathcal{D}\) be a BADC, \(\mathcal{C}\) be one of the BACs of \(\mathcal{D}\), \(x\) the current configuration of \(\mathcal{C}\), and \(c_i\) and \(c_j\) be two automata of \(\mathcal{C}\) distinct of \(c\) and such that \(i < j\). In the sequel, we will make particular use of the following elementary instructions:
• **sync**: \( x_c \leftarrow f_c(x) \)  
  \# update of \( c \)
  
  `sync` is the only instruction that updates automaton \( c \) and where both BACs interact with each other. This (key)-instruction will always be called when \( c \) can change its state. `sync` can be used either to set \( c \) at a desired state or to increase the expressiveness from a configuration. Notice that `sync` is the only way to switch a \( 111 \) (resp. \( 000 \)) pattern into a \( 101 \) (resp. \( 010 \)) pattern and, thus, to increase the expressiveness. Remark that the BAC sub-configurations have to be specific for \( c \) to switch its state.

• **update\((c_i)\)**: \( x_{c_i} \leftarrow f_{c_i}(x) \)  
  \# update of \( c_i \)
  
  `update` updates an automaton distinct to \( c \).

• **incUp\((\mathcal{E}, i, j)\)**: for \( k = i \) to \( j \) do `update\((c_k)\)`  
  \# incremental updates
  
  `incUp` updates consecutive automata by increasing order. In fact, `incUp` aims at propagating the state of \( c_{i-1} \) along \( \mathcal{E} \). Notice that if \( j < i \) then no automata are updated. Moreover, since \( i \neq 0 \), \( c \) cannot be updated with `incUp`.

**Property 1.** Let \( x' \) be the result of applying `incUp\((\mathcal{E}, i, j)\)` on configuration \( x \). Then we have: \( \forall k \in \{i, \ldots, j\}, \ x'_k = x_{i-1} \) and \( \forall k \notin \{i, \ldots, j\}, \ x'_k = x_k \).

• **erase\((\mathcal{E})\)**: `incUp\((\mathcal{E}, 1, \text{size}(\mathcal{E}) - 1)\)`
  
  `erase` is a particular case of `incUp` that aims at propagating the state of \( c_0 \) along \( \mathcal{E} \). As a consequence, using this instruction on \( \mathcal{E} \) makes it to be of expressiveness 0, and thus, is really efficient to converge quickly to a stable configuration of least expressiveness (if should be the case).

**Property 2.** Let \( x' \) be the result of applying `erase\((\mathcal{E})\)` on configuration \( x \). Then we have: \( \forall k \in \{0, \ldots, \text{size}(\mathcal{E}) - 1\}, \ x'_k = x_0 \).

• **expand\((\mathcal{E})\)**: `incUp\((\mathcal{E}, 1, \kappa - 1 \in \mathbb{N})\)` with
  
  \[
  \kappa = \min_{1 \leq k \leq \text{size}(\mathcal{E}) - 1} \begin{cases} 
  k & \text{if } x_0 = 1 \\
  (x_k = 0 \text{ and } (x_{k+1} \mod \text{size}(\mathcal{E}) = 1)) \text{ if } x_0 = 1 \\
  (x_k = 1 \text{ and } (x_{k+1} \mod \text{size}(\mathcal{E}) = 0)) \text{ if } x_0 = 0
  \end{cases}
  \]
  
  `expand` is another particular case of `incUp` that aims at propagating the state of \( c_0 \) along \( \mathcal{E} \) while neither \( 01 \) nor \( 10 \) patterns are destroyed, which avoids decreasing the expressiveness of \( \mathcal{E} \).

• **decUp\((\mathcal{E}, i, j)\)**: for \( k = j \) down to \( i \) do `update\((c_k)\)`  
  \# decremental updates
  
  `decUp` updates consecutive automata by decreasing order. Once `decUp\((\mathcal{E}, i, j)\)` executed, the information of \( c_j \) is lost and that of \( c_{i-1} \) is possessed by both \( c_{i-1} \) and \( c_i \). In fact, `decUp` aims at shifting partially a BAC section. As for `incUp`, if \( j < i \) then no automata are updated and \( c \) cannot be updated with `decUp`.

**Property 3.** Let \( x' \) be the result of applying `decUp\((\mathcal{E}, i, j)\)` on configuration \( x \). Then we have: \( \forall k \in \{i, \ldots, j\}, \ x'_k = x_{k-1} \) and \( \forall k \notin \{i, \ldots, j\}, \ x'_k = x_k \).

• **shift\((\mathcal{E})\)**: `decUp\((\mathcal{E}, 1, \text{size}(\mathcal{E}) - 1)\)`
  
  `shift` is a particular case of `decUp`. Once executed, every automaton of \( \mathcal{E} \) takes the state of its predecessor, except \( c \) whose state does not change. Automaton \( c_{\text{size}(\mathcal{E}) - 1} \) excluded, all the information contained along \( \mathcal{E} \) is kept safe. To use `shift` is useful to propagate information along a BAC without loosing too much expressiveness (at most one \( 01 \) pattern is destroyed).

6
Remark that
Proof.

4 Results

4.1 More complex instructions

Now, consider a configuration $x$ of BADC $\mathcal{D}$ and an algorithm made of instructions that defines a sequence of updates (abbreviated simply by “sequence” from now) from $x$, denoted by $\text{sequence}(x)$. Abusing language, in the sequel, $\text{sequence}(x)$ represents both the underlying sequence and its result, namely the configuration resulting from the execution of $\text{sequence}(x)$. To end this section, we introduce three other sequences in Table 1 more complex, that will be important later. In particular, Lemma 1 states that $\text{copy}$ allows to transform $x$ into $x'$ if $x$ is expressive enough (highly expressive actually).

**Lemma 1.** Let $\mathcal{D}$ be a BADC and $x$ and $x'$ two of its configurations such that $x_0 = x'_0$. If, for any $s \in \{l, r\}$, one of the following properties holds for $x$:

1. $\forall i \in \{1, \ldots, \text{size}(\mathcal{E}^s) - 1\}$, $x_i^s \neq x_{i-1}^s$,
2. $\forall i \in \{1, \ldots, \text{size}(\mathcal{E}^s) - 2\}$, $x_i^s \neq x_{i-1}^s$ and $x_{\text{size}(\mathcal{E}^s)-1}^s = x_{\text{size}(\mathcal{E}^s)-2}^s$,
3. $\forall i \in \{1, \ldots, \text{size}(\mathcal{E}^s) - 2\}$, $x_i^s \neq x_{i-1}^s$ and $\exists p \in \{1, \ldots, \text{size}(\mathcal{E}^s) - 2\}$, $x_p^s \neq x_{p}^s$,

then $\text{copy}(x, x') = x'$ and this sequence consists in at most $2(n + m - 6)$ updates.

**Proof.** Remark that $\text{sync}$ is never called in $\text{copy}$. Thus, the state of $c$ never switches and $x_0 = x'_0$. Since $\text{copy}$ calls twice $\text{copy}_c$, once on $\mathcal{E}^l$ and then on $\mathcal{E}^r$, let us focus without loss of generality on $\text{copy}_c(x, x', \mathcal{E}^l)$ and prove that this sequence transforms $x^l$ into $x'^l$ (the same kind of reasoning adapts directly to $\text{copy}_c(x, x', \mathcal{E}^r)$).

First, it is important to notice that, if $x^l$ follows either Property 1 or Property 2, which both induce that the value of $j$ is initialised to $n$, the only for-loop that can be executed is that of line 10. Now, the assumption stating that

| $\text{copy}_c(x, x', \mathcal{E}^s)$ | $\text{copy}(x, x')$ |
|---------------------------------|---------------------|
| 01. $n \leftarrow \text{size}(\mathcal{E}^s)$; | 01. $\text{copy}_c(x, x', \mathcal{E}^s)$; |
| 02. if $(x'_{n-1} = x_{n-2}^s$ and $x'_{n-1} \neq x_{n-1}$) then | 02. $\text{copy}_c(x, x', \mathcal{E}^s)$; |
| 03. $j \leftarrow \max\{k \mid k < n - 1$ and $x_k^s \neq x'_k\}$; | 03. $\text{shift}(\mathcal{E}^s)$; |
| 04. else $j \leftarrow n$; | 04. $\text{sync}$; |
| 05. end if | 05. end if |
| 06. for $(k = n - 1)$ down to $(j + 1)$ do | 06. $\text{copy}(x, x')$; |
| 07. update($c^s_{k-1}$); | |
| 08. update($c^s_k$); | |
| 09. done | |
| 10. for $(k = j - 1)$ down to $(1)$ do | |
| 11. if $(x_k^s \neq x'_k)$ then update($c^s_k$); | |
| 12. end if | |
| 13. done | |
\( \forall i \in \{1, \ldots, \text{size}(c^\ell) - 1\}, \ x_i^j \neq x_{i-1}^j \) together with lines 11-13 make \( x^\ell \) to become \( x^\ell \).

Second, let us focus on a configuration \( x^\ell \) for which Property 3 holds but not Properties 1 and 2. Such an \( x^\ell \) necessarily verifies conditions given in line 2, which leads \( j \) to be well defined since, by hypothesis, \( \exists p \in \{1, \ldots, \text{size}(c^\ell) - 2\}, x_p^\ell \neq x_p^\ell \) (notice that \( j \) is set to the greatest \( p \) satisfying this relation). As a consequence, the content of the \texttt{for}-loop of line 6 is executed. Let us now prove that, at the end of the execution of this loop, \( \forall j < k < n - 1, \ x_k^\ell = x_k^\ell \). From this, consider the following loop invariant \( \text{inv}(k) \): “at the beginning of the \( k \)-th iteration, \( x_{k-1}^\ell = x_k^\ell \) and \( x_{k-2}^\ell \neq x_{k-2}^\ell \).”

For the \((n - 1)\)-th iteration, from above, the invariant holds.

Assume that the invariant still holds at the \( k \)-th iteration. Given that \( x_{k-1}^\ell \neq x_{k-2}^\ell \), line 7 makes \( c^\ell_{k-1} \) switch its state that consequently becomes (i) different from that of \( c_k^\ell \) and (ii) equal to that of \( c_{k-2}^\ell \). Then, because of (i), line 8 makes \( c_k^\ell \) switch. Notice that, at this point, the states of \( c_{k-2}^\ell \) and \( c_{k-3}^\ell \) have not been changed and \( x_{k-2}^\ell \neq x_{k-3}^\ell \). Thus, with (ii), the invariant still holds for the \((k - 1)\)-th iteration.

According to what has just been explained, at the end of the loop, every automaton \( c_k^\ell, \ j < k < n - 1 \) has switched twice (and thus has recovered its initial state) whereas automata \( c_j^\ell \) and \( c_{n-1}^\ell \) have switched once (and thus do have changed their state). As a consequence, we now have that \( x_n^\ell = x_{n-1}^\ell \) and \( x_j^\ell = x_{j+1}^\ell \). All this ensures that at line 9, \( \forall j \leq k \leq n - 1, \ x_k^\ell = x_k^\ell \).

For ending the proof, with \( 0 \leq k \leq j - 1 \), it suffices to follow the \texttt{for}-loop of line 10 whose effect has been explained in the previous paragraph. Also, we have just seen that in \( c^\ell \), \texttt{copy c} can lead \((n - 2)\) automata (except \( c_j \) and \( c_{n-1} \) as said before) to switch twice in the worst case, \textit{i.e.} when \( j \) is initialised to 1.

As a consequence, the execution of \texttt{copy} takes at most \( 2(n - 2) - 2 + 2(m - 2) - 2 = 2(n + m - 6) \) updates. \( \square \)

From this first result that gives strong insights about the power of instructions and sequences to reveal possible trajectories between configurations, let us now focus on the dynamical behaviours of double-cycles.

### 4.2 Positive BADCs

Since results of [13][14] have shown that positive BADCs behave as positive BACs, and because stable configurations are conserved between distinct updating modes [3], it is easy to show that the asymptotic dynamics of positive BADCs consists in two stable configurations \( x \) and \( \overline{x} \) (where \( \overline{x} \) denotes the negation of \( x \)). In the case of canonical BADCs, these stable configurations are \((0^n, 0^m)\) and \((1^n, 1^m)\). Here, let us focus on an arbitrary positive BADC \( \mathcal{G}^+ \). We show that two new sequences \texttt{fix0} and \texttt{fix1} (cf. Table 2) can respectively transform any configuration with at least one automaton at state \( 0 \) into \((0^n, 0^m)\), and any configuration with at least one automaton at state \( 1 \) in both cycles into \((1^n, 1^m)\).
Table 2. The sequences fix0 and fix1.

| fix0(x) | fix1(x) |
|---------|---------|
| 01. if \((x_0 = 1)\) then \(i \leftarrow \min\{k \mid x_k = 0\}\); incUp(\(C^\ell, i + 1, n - 1\)); sync; erase(\(C^\ell\)); erase(\(C^r\)); | 01. if \((x_0 = 0)\) then \(i \leftarrow \min\{k \mid x_k = 1\}\); \(j \leftarrow \min\{k \mid x_k = 0\}\); incUp(\(C^r, j + 1, m - 1\)); sync; erase(\(C^l\)); erase(\(C^r\)); |

Theorem 4. Let \(D^+\) be a canonical positive BADC and \(x\) one of its unstable configuration. If \(x\) admits one automaton at state 0, then \(\text{fix}_0(x) = (0^n, 0^m)\). Also, if \(x\) admits one automaton at state 1 in both its cycles, then \(\text{fix}_1(x) = (1^n, 1^m)\). The convergence time of \(D^+\) is at most \(2(n + m) - 5\).

Proof. Let us focus on the case of \(\text{fix}_1\) and consider an unstable configuration \(x\) of \(D^+\) with at least one \(1\) in both cycles. First, if \(c\) is at state 1, erase of lines 5 and 6 make every automaton of \(C^\ell\) and \(C^r\) to take state 0 and the obtained configuration is then \((1^n, 1^m)\) which is stable.

Second, consider that \(c\) is at state 0. So, instructions of lines 2-6 are executed. Since, by hypothesis, there is at least a \(1\) in \(C^\ell\) and \(C^r\), after the execution of incUp at line 3, \(\forall k \in \{1, \ldots, n - 1\}, x_k = 1\), and, after the execution of incUp at line 5, \(\forall k \in \{j, \ldots, m - 1\}, x_k = 1\). As a consequence, the effect of sync at line 6 is to fix \(c\) at state 1 and we get back to the case above.

Now, notice that the case of \(\text{fix}_0\) is very similar, by considering with no loss of generality that at least one automaton is at state 0 in \(C^\ell\) and that we need to set \(x_{n-1}^\ell\) to 0 before the execution of sync at line 4.

Finally, notice that the number of effective updates made by \(\text{fix}_0\) (resp. \(\text{fix}_1\)) is at most \(2n + m - 3\) (resp. \(= 2(n + m) - 5\)). \(\square\)

4.3 Mixed BADCs

Now, we pay attention to mixed BADCs. From the same works that showed also that asynchronism keeps only recurrent configurations of least global instability, we know that their asymptotic dynamics consists only in a stable configuration. In particular, the attractor of canonical mixed BADCs is \((0^n, 0^m)\). Let us focus on their convergence time. To do so, we will make particular use of new sequence simp (cf. Table 3) that gives a way of converging to this stable configuration from any initial configuration \(x\), by reducing progressively its expressiveness.

Theorem 5. Let \(D^\pm\) be a canonical mixed BADC. For any configuration \(x\) of \(D^\pm\), \(\text{simp}(x) = (0^n, 0^m)\) holds. The convergence time of \(D^\pm\) is at most \(2n + m - 2\).

Proof. First, if \(c\) is at state 0, erase of lines 5 and 6 make every automaton of \(C^\ell\) and \(C^r\) to take state 0 and the stable configuration \((0^n, 0^m)\) is obtained.
Table 3. The sequences simp, comp1 and comp2.

\[
\text{ simp}(x) \\
01. \text{ if } (x_0 = 1) \text{ then} \\
02. \text{ erase}(C^\ell); \\
03. \text{ sync}; \\
04. \text{ end if} \\
05. \text{ erase}(C^r);
\]

\[
\text{ comp1}(x) \\
01. \text{ for } (i = 1) \text{ to } (n - 1) \text{ do} \\
02. \text{ sync}; \\
03. \text{ expand}(C^\ell); \\
04. \text{ erase}(C^r); \\
05. \text{ done}
\]

\[
\text{ comp2}(x) \\
01. \text{ if } (x^r_0 = 1^m) \text{ then} \\
02. \text{ sync}; \\
03. \text{ erase}(C^r); \\
04. \text{ end if} \\
05. \text{ sync}; \\
06. \text{ expand}(C^r); \\
07. \text{ for } (i = 1) \text{ to } (m - 2) \text{ do} \\
08. \text{ shift}(C^r); \\
09. \text{ sync}; \\
10. \text{ expand}(C^r); \\
11. \text{ done}
\]

Second, consider that \(c\) is at state 1. Instructions of lines 2 and 3 are thus executed. So, \text{ erase} makes every automaton of \(C^\ell\) take state 1, and \text{ sync} makes \(c\) take state 0. And we get back to the case above.

Finally, notice that the number of effective updates made by \text{ simp} is at most \(2n + m - 2\). \(\square\)

4.4 Negative BADCs

In this section, we interest in negative BADCs. Contrary to BADCs of other sorts, the previous results of [13,14] obtained under the parallel updating mode are not helpful for dealing with the asynchronous updating mode. Indeed, in parallel, negative BADCs admit an exponential number of attractors. In our asynchronous framework, we will show that they admit a unique stable oscillation of exponential size that depends on the parity of underlying cycles. In particular, the study that follows is divided in two axes: the first one deals with BADCs made of two negative cycles of even sizes (abbreviated by \(D^-e\)), the second one with the others where at least one cycle of odd size (abbreviated by \(D^-o\)).

Both cycles are even Here, we show that any BADC \(D^-e\) admits only one stable oscillation of size \(2^n + m - 1\). In other terms, all configurations are recurrent and the convergence time is null. However, although all configurations are accessible from each other, those of high expressiveness are hard to reach. The proof of this result follows three points (they will be referred to Points 1, 2 and 3 later) in which it is respectively shown that:

1. any configuration can reach the least expressive one \((0^n, 0^m)\) in linear time;
2. configuration \((0^n, 0^m)\) can reach the highest expressive one \(((10)^\frac{n}{2}, (10)^\frac{m}{2})\) in quadratic time;
3. any configuration can be reached from \(((10)^\frac{n}{2}, (10)^\frac{m}{2})\) in linear time.

Notice that Point 2 above is the hardest part. Indeed, to reach \(((10)^\frac{n}{2}, (10)^\frac{m}{2})\) from \((0^n, 0^m)\) needs \(O(n^2 + m^2)\) updates. We will see that this upper bound is
tight and that to increase a configuration expressiveness by $\delta$ requires at least $\delta^2$ updates (cf. Theorem 7).

Let us consider Point 1. It is easy to see that sequence $\text{simp}$ is still efficient to reach $(0^n, 0^m)$ and thus, that the following Lemma holds.

**Lemma 2.** For any configuration $x$ of $\mathcal{G}_e^-$, $\text{simp}(x) = (0^n, 0^m)$ holds and takes at most $2n + m - 2$ updates.

**Proof.** This proof is identical to that of Theorem 5, except the fact that $(0^n, 0^m)$ is not a stable configuration anymore.

Now, let us pay attention to Point 2 that asks for increasing the expressiveness of $C$. To do so, let us proceed in two steps. The first one aims at increasing the expressiveness of $C$ by means of sequence $\text{comp}1$ (cf. Lemma 3), while the second one aims at increasing that of $C'$ while ensuring not to decrease that of $C$ by means of $\text{comp}2$ (cf. Lemma 4). Then, we get directly Lemma 5 with the composition $\text{comp} = \text{comp}2 \circ \text{comp}1$.

**Lemma 3.** In a BADC $\mathcal{G}_e^-$, $\text{comp}1((0^n, 0^m)) = ((10)^2, 1^m)$ holds and takes at most $(n - 1)(n + m - 2)$ updates.

**Proof.** In this proof, we show that invariant $\text{inv}(i)$ defined as “at the end of the $i$th iteration of the loop, the configuration is $\left\{ (1^{n-i-1}(10)^{i/2}, 1^m) \text{ if } i \text{ is odd, } (0^{n-i-1}(01)^{i/2}, 0, 0^m) \text{ otherwise} \right\}$ holds for all $i \in \{1, \ldots, n - 1\}$. Notice that we denote by arrow $x \xrightarrow{k} x'$ the transformation of $x$ into $x'$ by the execution of line $k$ of the sequence considered (i.e. $\text{comp}1$ here).

At the initialisation step ($i = 1$), we have:

\[
(0^n, 0^m) \xrightarrow{02} (10^{n-1}, 10^{m-1}) \xrightarrow{03} (1^{n-1}0, 10^{m-1}) \xrightarrow{04} (1^{n-1}0, 1^m) = (1^{n-2}(10), 1^m)
\]

and $\text{inv}(1)$ is true.

At the maintenance steps, we have:

- if $i \equiv 0 \mod 2$, at the beginning of the iteration, the configuration comes from iteration $i-1$ (that is odd) and is consequently $(1^{n-(i-1)-1}(10)^{(i-1)+1}, 1^m)$. Thus we have:

\[
(1^{n-(i-1)}(10)^{i/2}, 1^m) \xrightarrow{02} (01^{n-i-1}(10)^{i/2}, 01^{m-1}) \xrightarrow{03} (0^{n-i}(10)^{i/2}, 01^{m-1}) = (0^{n-i-1}(01)^{i/2}, 0, 01^{m-1}) \xrightarrow{04} (0^{n-i-1}(01)^{i/2}, 0, 0^m)
\]

- if $i \equiv 1 \mod 2$, at the beginning of the iteration, the configuration comes from iteration $i-1$ (that is even) and is consequently $(0^{n-(i-1)-1}(01)^{i/2}, 0, 0^m)$. Thus we have:

\[
(0^{n-(i-1)}(01)^{i/2}, 0, 0^m) \xrightarrow{02} (0^{n-i}(01)^{i/2}, 0, 0^m) = (0^{n-i-1}(01)^{i/2}, 0, 0^m)
\]
\[
\begin{align*}
(0^{n-i}(01)^{i+1}0, 0^m) \overset{0_2}{\rightarrow} & (10^{n-i-1}(01)^{i+1}0, 0^{10^m-1}) \\
\overset{0_3}{\rightarrow} & (1^{n-i}(01)^{i+1}0, 10^{10^m-1}) \\
& = (1^{n-i-1}(10)^{i+1}, 10^{10^m-1}) \\
\overset{0_4}{\rightarrow} & (1^{n-i-1}(10)^{i+1}, 1^m)
\end{align*}
\]

and \( \text{inv}(i), \) \( 2 \leq i \leq n - 1 \), still holds.

At the termination step, since \( \mathcal{E}^t \) size is even by hypothesis, \( n - 1 \) is odd and \( \text{inv}(n - 1) \) holds.

Thus \( (0^n, 0^m) \) is transformed into \( (1^{n-(n-1)-1}(10)^{2(n-m-2)}, 1^m) = ((10)^{\frac{n}{2}}, 1^m) \), which is the expected result. Moreover, remark that the number of effective updates made by \( \text{comp}1 \) is at most \((n - 1)(n + m - 2)\).

**Lemma 4.** In a BADC \( \mathcal{D}_\mathcal{E}^- \), \( \text{comp}2(((10)^{\frac{n}{2}}, 1^m)) = ((10)^{\frac{n}{2}}, (10)^{\frac{n}{2}}) \) holds and takes at most \((m - 2)(n + m - 2) + (2m - 1)\) updates.

**Proof.** This proof is similar to that of Lemma 3. Indeed, we show that invariant \( \text{inv}(i) \) defined as “at the end of the \( i \)th iteration of the loop, the configuration is \( (((01)^{\frac{n}{2}}, 0^{m-1}(10)^{\frac{i+1}{2}}) \) if \( i \) is odd, holds for all \( i \in \{1, \ldots, m - 2\} \).

Let us first consider lines 1 to 6 of \( \text{comp}2 \), before we enter the loop. Since the configuration is \( (((10)^{\frac{n}{2}}, 1^m)) \) initially, these lines transform it into \( (((10)^{\frac{n}{2}}, 1^{m-2}(10)) \) with respect to the following changes:

\[
(10)^{\frac{n}{2}}, 1^m \overset{01}{\rightarrow} (00)(10)^{\frac{n}{2}-1, 01^{m-1}) \\
\overset{02}{\rightarrow} (00)(10)^{\frac{n}{2}-1, 0^m} \\
\overset{03}{\rightarrow} (10)^{\frac{n}{2}, 10^{m-1}} \\
\overset{04}{\rightarrow} (10)^{\frac{n}{2}, 1^{m-2}} = (10)^{\frac{n}{2}, 1^{m-2}(10)}
\]

Now, at the initialisation step \((i = 1)\) of the loop, we have:

\[
(10)^{\frac{n}{2}, 1^{m-2}(10)) \overset{06}{\rightarrow} (10)^{\frac{n}{2}, 1^{m-2}(10)} \\
\overset{07}{\rightarrow} (0)(10)^{\frac{n}{2}-1, 10^{m-3}(10)) = ((01)^{\frac{n}{2}}, 01^{m-2}0) \\
\overset{08}{\rightarrow} (01)^{\frac{n}{2}, 0^{m-2}(01))
\]

and \( \text{inv}(1) \) is true.

At the maintenance steps, we have:

- if \( i \equiv 0 \mod{2} \):
  \[
  (01)^{\frac{n}{2}, 0^{m-i}(10)^{\frac{i+1}{2}}}) \overset{06}{\rightarrow} (00)(01)^{\frac{n}{2}-1, 0^{m-i}(10)^{\frac{i+1}{2}}) \\
  \overset{07}{\rightarrow} (10)^{\frac{n}{2}, 0^{m-i-1}(10)^{\frac{i+1}{2}}} = ((01)^{\frac{n}{2}, 1^{m-i-2}(10)^{\frac{i+1}{2}}}) \\
  \overset{08}{\rightarrow} (01)^{\frac{n}{2}, 0^{m-2}(01))
  \]

- if \( i \equiv 1 \mod{2} \):
  \[
  (10)^{\frac{n}{2}, 1^{m-i-1}(10)^{\frac{i+1}{2}}}) \overset{06}{\rightarrow} (10)^{\frac{n}{2}-1, 1^{m-i-1}(10)^{\frac{i+1}{2}}} \\
  \overset{07}{\rightarrow} (00)(10)^{\frac{n}{2}-1, 10^{m-i-2}(10)^{\frac{i+1}{2}}} \\
  \overset{08}{\rightarrow} (01)^{\frac{n}{2}, 0^{m-i-2}(10)^{\frac{i+1}{2}}})
  \]

\]
The proof that \( \text{inv}(i), 2 \leq i \leq m - 2 \), still holds.

At the termination step, since \( \mathcal{C}^r \) size is even by hypothesis, \( m - 2 \) is even and \( \text{inv}(m - 2) \) holds.

Thus \( ((10)^\frac{r}{2}, 1^{m-(m-2)-2}(10)^\frac{m-2}{2}+1) = ((10)^\frac{r}{2}, (10)^\frac{r}{2}) \) is indeed reached by \( ((10)^\frac{r}{2}, 1^m) \), which is the expected result. Moreover, remark that the number of effective updates made by \( \text{comp}2 \) is at most \((m - 2)(n + m - 2) + (2m - 1)\).  

\[\text{Lemma 5. In a BADC } \mathcal{R}^-, \text{ comp}(0^n, 0^m)) = ((10)^\frac{r}{2}, (10)^\frac{r}{2}) \text{ holds and takes at most } (n + m)^2 - 5(n - 1) - 3m \text{ updates.}\]

\[\text{Proof. The proof that } \text{comp}(0^n, 0^m)) = ((10)^\frac{r}{2}, (10)^\frac{r}{2}) \text{ holds directly derives from Lemmas } 3 \text{ and } 4. \text{ Concerning the number of updates that are needed in the worst case, it suffices to add the maximum number of updates done by } \text{comp}1 \text{ and } \text{comp}2. \text{ So, we have:}\]

\[
(n - 1)(n + m - 2) + (m - 2)(n + m - 2) + (2m - 1) \\
= (n^2 + nm - 3n - m + 2) + (m^2 + nm - 2n - 2m + 4) + (2m - 1) \\
= n^2 + m^2 + 2nm - 5n - 3m + 5 \\
= (n + m)^2 - 5(n - 1) - 3m,
\]

which is the expected result. \( \square \)

Point 3 is developed in Lemma 6 in which we make particular use of \( \text{copy}_p \) (cf. Table 1).

\[\text{Lemma 6. In a BADC } \mathcal{R}^-, \text{ for any } x', \text{ copy}_p(((10)^\frac{r}{2}, (10)^\frac{r}{2}), x') \text{ transforms configuration } ((10)^\frac{r}{2}, (10)^\frac{r}{2}) \text{ into } x' \text{ in at most } 3(n + m - 4) - 1 \text{ updates.}\]

\[\text{Proof. Let us consider } \text{copy}_p \text{ where } x = ((10)^\frac{r}{2}, (10)^\frac{r}{2}) \text{ and } x' \text{ is an arbitrary configuration. The proof is done by considering two cases, that of } x_0 \neq x'_0 \text{ and that of } x_0 = x'_0. \text{ The general idea of this proof is to show that } \text{copy} \text{ allows to find a sequence from } x \text{ to } x' \text{ if } c \text{ and } c' \text{ are at the same state. Obviously, in the first case, } x \text{ needs to be transformed for } \text{copy} \text{ to apply correctly. That is what is done in lines 1-5 of } \text{copy}_p \text{ that transform } x \text{ into the other most expressive configuration.}\]

So, let us focus on the case \( x_0 \neq x'_0 \), which means that \( c'_0 = 0 \), and the transformations that are performed on it by lines 1-5. We have:

\[
((10)^\frac{r}{2}, (10)^\frac{r}{2}) \stackrel{0}{\rightarrow} (1(10)^\frac{r}{2} - 1, (10)^\frac{r}{2}) \\
\quad \rightarrow (1(10)^\frac{r}{2} - 1, 1(01)^\frac{r}{2} - 1) \\
\quad \rightarrow (0(10)^\frac{r}{2} - 1, 0(01)^\frac{r}{2} - 1) = ((01)^\frac{r}{2}, (01)^\frac{r}{2})
\]

On this basis, at line 6, just before \( \text{copy} \) is executed, \( c \) and \( c' \) necessarily have the same state and Property 1 of Lemma 7 holds for \( x \) and can thus be applied for ending the proof. Also, notice that, in the worst case, every automaton of \( \mathcal{C}^r \) and \( \mathcal{C}^l \) are updated before the execution of \( \text{copy} \). Thus, this sequence takes at most \( 3(n + m - 4) - 1 \) updates. \( \square \)
By combining Lemmas 2, 5 and 6 for all configurations $x$ and $x'$, the composition $\text{copy}(\text{comp}(\text{simp}(x)), x') = x'$ holds, which shows that there exists a unique attractor of size $2^{n+m-1}$. From this is derived the following theorem.

**Theorem 6.** A BADC $\mathcal{D}_e^-$ admits a unique attractor of size $2^{n+m-1}$. In this stable oscillation, any configuration can be reached by any other one in $O(n^2 + m^2)$. However, some configurations are specific: $(0^n, 0^m)$ and $(1^n, 1^m)$ can be reached from any other one in $O(n + m)$, and configurations $((01)^{\frac{n}{2}}, (01)^{\frac{m}{2}})$ and $((10)^{\frac{n}{2}}, (10)^{\frac{m}{2}})$ can reach any configuration in $O(n + m)$.

Now we show that the bound $O(n^2 + m^2)$ of Theorem 6 above is tight.

**Theorem 7.** Let $x$ be a configuration of a BADC $\mathcal{D}_e^-$. To increase the expressiveness of $x$ by $\delta \in \mathbb{N}$ needs $\Omega(\delta^2)$ updates.

**Proof.** Although this theorem deals with configuration $x$, let us focus with no loss of generality on how the expressiveness of $x^\ell$ can be increased by $\delta$. First, notice that the only way to increase the expressiveness of $\mathcal{E}^\ell$ needs to use sync. However, to execute two sync puts $c_0$ at its initial state. So, to be efficient, the two sync have to be separated by a sequence of updates. Take for instance the following sequence of instructions for $i \in \{1, \ldots, n\}$: sync: $\text{incUp}(\mathcal{E}^\ell, 1, i)$; sync: $\text{incUp}(\mathcal{E}^\ell, 1, i)$. With this sequence, the second call to $\text{incUp}$ leads to replace all the information created by the first one and contained by automata $c_1, \ldots, c'_i$. As a consequence, to create $\delta$ new patterns $01$ in $\mathcal{E}^\ell$ needs the calls of sync to be separated by specific sequences which propagate along the cycle the patterns generated by the previous call to sync. Now, since there is at least $\delta$ calls to sync, just after its $i$th call, the pattern has to be propagated at least until automaton $c_{\delta-(i-1)}$. Thus, the $i$th call to sync has to be followed by at least $\delta - (i - 1)$ updates in order to ensure that the pattern is effectively kept. As a result, at the end, to increase the expressiveness of $\mathcal{E}^\ell$ by $\delta$ patterns needs $\Omega(\delta^2)$ updates. □

Corollary 1 is then directly derived from the two previous theorems, considering that $\delta = \frac{n}{2}$ for $\mathcal{E}^\ell$ and $\delta = \frac{m}{2}$ for $\mathcal{E}^r$.

**Corollary 1.** In a BADC $\mathcal{D}_e^-$, to reach $((10)^{\frac{n}{2}}, (10)^{\frac{m}{2}})$ from $(0^n, 0^m)$ requires $\Theta(n^2 + m^2)$ steps.

**At least one cycle is odd** Like BADCs $\mathcal{D}_e^-$, BADCs $\mathcal{D}_o^-$ admit only one attractor but contrary to the latter, they also admit a set $I$ of specific non-recurrent configurations, from which updates are “irreversible” (i.e. configurations of $I$ are not accessible). In the sequel, abusing language, these configurations are said to be irreversive. Lemma 7 below shows the irreversibility of some configurations.

**Lemma 7.** Let us consider a BADC $\mathcal{D}_o^-$. The following properties hold:

1. If $\mathcal{E}^s$, $s \in \{\ell, r\}$, is of odd size $k > 1$, then configuration $x$ such that $x^s = ((10)^{\frac{k}{2}} - 1)$ is irreversive.
2. If both $\mathcal{C}$ and $\mathcal{C}'$ are of odd sizes $n > 1$ and $m > 1$, then configuration 
$(01)^{\frac{n-1}{2}}0, (01)^{\frac{m-1}{2}}0)$ is irreversible.

Proof. Let us consider an arbitrary BADC $D_2^-$. The proof is divided into two parts. First, without loss of generality, we show that when $n$ is odd, configurations $((10)^{\frac{n-1}{2}}1.,)$ (where . denotes any configuration $x^r$ of $\mathcal{C}'$) are irreversible. Second, we prove the irreversibility of configuration $((01)^{\frac{n-1}{2}}0, (01)^{\frac{m-1}{2}}0)$ when $n$ and $m$ are odd.

--- Irreversibility of $((10)^{\frac{n-1}{2}}1.,)$

In order to simplify this part, we consider a BAN composed of a negative cycle $\mathcal{C}$ of odd size $n$ and of an automaton $c^*$. This BAN is defined by the following $n + 1$ local transition functions:

$$f_{c_i}(x) = ¬x_{n-1} \land x_{c^*}, \quad \forall i \in \{1, \ldots, n-1\}, \quad f_{c_i}(x) = x_{i-1}.$$  

The configurations of this BAN will be denoted by $(x_0 \ldots x_{n-1}, x_{c^*})$. Remark that the idea underlying automaton $c^*$ is to represent an atomic element that acts on $\mathcal{C}$, as another cycle should do. However, this interacting element is more expressive than a cycle since its state switches as soon as it is updated (it plays the role of an oscillator). In fact, in the context of BADCs, $c^*_{m-1}$ plays the role of $c^*$. However, the effective updates of $c^*_{m-1}$ are clearly more restricted than that of $c^*$ since they directly depend on the configuration of $\mathcal{C}'$ and indirectly on that of $\mathcal{C}$. 

Now, let us consider configurations $x = ((10)^{\frac{n-1}{2}}1.,)$ and $x'$ obtained by executing update$(c_i)$ on $x$, for any $i \in \{0, \ldots, n-1\}$. Given the nature of $x$, notice that $x' \neq x$. In order to prove the result, we have to show that there are no sequences to reach $x$ from $x'$. To do so, we reason by contradiction.

Let us suppose that there exists a sequence $\sigma$ composed of update and sync instructions that transforms $x'$ into $x$. Since $x'$ is made from $x$ by updating $c_i$, in order to get back to $x$, $c_i$ in $x'$ needs to be switched again in order to have $x'_i = x_i$, which implies that $c_{i-1}$ has to be updated. Thus, sequence $\sigma$ contains at least one update$(c_{i-1})$. Furthermore, if we want $c_{i-1}$ to change its state, $c_{i-2}$ has to be switched, and so on. Thus, by iterating this argument, $c_0$ has to be switched too, which implies that $\sigma$ necessarily contains at least one sync, that leads $c_0$ to take state $0$. From this, while $x'_{n-1} = 1$, if we want $c_0$ to get back to state $1$, $c_{n-1}$ has to switch to state $0$, which imposes that $\sigma$ contains also at least one update$(c_{n-1})$. Furthermore, if $i = 0$ (i.e. $x' = x$) or $i = n - 1$ (i.e. $x' = x^{n-1}$), it is obvious that $\sigma$ leads the process to get back to $x$ to a punctual configuration where the state of $c_{n-1}$ is $0$. As a consequence, any sequence that transforms $x'$ into $x$ needs to temporarily transform $x'$ into a configuration in which $c_{n-1}$ is at state $0$. Let us focus essentially on $c_{n-1}$ in order to determine properties that need to hold in $\sigma$. It is easy to understand that the two following properties are necessary:

(a) $\sigma$ must contain at least one update$(c_{n-1})$.
(b) Any time $\sigma$ passes through a configuration $y$ where $y_0 = 0$ and $y_{n-1} = 1$, then the sequence of transformations made by $\sigma$ on $y$ to get back to $x$ contains at least two occurrences of update$(c_{n-1})$. 

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Let us now consider the suffix of $\sigma$ that begins within the last occurrence of $\text{update}(c_{n-1})$. According to (a), such a suffix exists. Furthermore, according to (b), this suffix does not contain the instruction $\text{sync}$. Indeed, if it was the case, it would mean that the suffix would temporarily lead $y$ to be transformed into $y'$ such that $y'_0 = 0$ since $c_{n-1}$ is at state 1. Now, according to (b), in such a case, the suffix should contain two occurrences of $\text{update}(c_{n-1})$, which contradicts the hypothesis made on the suffix according to which it cannot contain any $\text{update}(c_{n-1})$.

From above, if we denote by $z$ the configuration obtained from $x'$ by applying the instructions of $\sigma$ until the last $\text{update}(c_{n-1})$ (included), because there are no instructions $\text{sync}$ and no more instructions $\text{update}(c_{n-1})$ in the suffix, $z_0$ and $z_{n-1}$ have to equal 1. Now, for $n > 1$ (i.e., $c_0 \neq c_{n-1}$), since the last update of $c_{n-1}$ leads it to state 1, this means that $z_{n-2} = 1$. Now, from that, the only possible way to get back to $x$ from $z$ would be to put $c_{n-1}$ to state 0 if $i$ is even and to state 1 otherwise. But this implies $c_0$ to be at state 0 at a certain step, which would need to call to $\text{sync}$ again, which contradicts the definition of the suffix as stated above. As a result, such sequence $\sigma$ does not exist.

We have just shown that configuration $((10)^{n-1}1., (01)^{n-1}0)$ cannot be reached, even if an over-expressive element $c'$ acting on $c_0$ is considered instead of a second negative cycle. So, $((10)^{n-1}1., (01)^{n-1}0)$ is irreversible, and with trivial extensions, we obtained the expected result given in Point 1 of Lemma 7.

---

Irreversibility of $((01)^{n-1}0, (01)^{n-1}0)$

Consider here a BADC $\mathcal{P}_c$ where $\mathcal{C}_c$ and $\mathcal{C}_{c'}$ are of odd sizes, respectively such that $n > 1$ and $m > 1$. The proof of the irreversibility of $((01)^{n-1}0, (01)^{m-1}0)$ is done by contradiction too.

Let $x'$ be the configuration that results from the update of one automaton in configuration $x = ((01)^{n-1}0, (01)^{m-1}0)$. With the same argument as above, any sequence $\sigma$ that allows to transform $x'$ into $x$ must pass temporarily through a configuration $y$ such that $y_0 = 0$ and $y'_{n-1} = 1$ or $y'_{m-1} = 1$. From $y$, in order that $\sigma$ transforms it into $x$, $\sigma$ needs to contain at least one $\text{update}(c_{n-1})$ or one $\text{update}(c_{m-1})$ depending on what automaton $c_{n-1}$ or $c_{m-1}$ is at state 1 (maybe both). Consider now the last occurrence of $\text{update}(c_{n-1})$ or $\text{update}(c_{m-1})$. After that occurrence, $c_{n-1}$ and $c_{m-1}$ are at state 0. Otherwise, $\sigma$ cannot lead to $x$. From that, we derive that:

(i) the temporary configuration obtained $z \neq x$ is such that $z_0 = 0$, $z'_{n-1} = z''_{n-1} = 0$ and $z''_{m-2} = 0$ or $z''_{m-2} = 0$ (maybe both), and

(ii) no instructions $\text{sync}$ can follow because this would imply the presence of other $\text{update}$ on $c_{n-1}$ or $c_{m-1}$ which contradicts the hypothesis that we focused on the last occurrence of such an $\text{update}$.

Now, consider for instance that $z''_{n-2} = 0$, with no loss of generality. From that, as above, the only possible way to get back to $x$ from $z$ would be to put $c_{n-1}$ to state 0 if $i$ is even and to state 1 otherwise. But this implies $c_0$ to be at state 0 at a certain step, which would need to call to $\text{sync}$ for $c_0$ to switch
to state 1. A contradiction with (ii) appears, which shows that such sequence \( \sigma \) does not exist and, consequently, that \((01)^{1+1}0, (01)^{1+1}0\) is irreversible, as stated in Point 2 of Lemma \(7\).

Let \( I \) be the set of irreversible configurations of a BADC \( \mathcal{D}_o^- \) given by Lemma \(7\). Theorem \(8\) below proves that \( I \) contains in fact all the irreversible configurations and, from this set, generalises Theorem \(6\) for any sort of negative BADCs. Notice that the complexity bounds remain valid. They are consequently not given again.

**Theorem 8.** Let \( \alpha : \mathbb{N} \to \{0, 1\} \) with \( \alpha(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \equiv 1 \mod 2 \\ 1 & \text{otherwise} \end{cases} \)

Any negative BADC \( \mathcal{D}_o^- \) admits one attractor of size \( 2^{n+m-1} - |I| \), where \( |I| = \alpha(n-1) \times 2^{m-1} - \alpha(m-1) \times 2^{n-1} \).

**Proof.** In this proof, we focus on BADC \( \mathcal{D}_o^- \) since the case of negative BADCs composed of two cycles of even sizes has been treated previously. Let us begin by showing that \( \mathcal{D}_o^- \) admits only one attractor that contains all the configurations except those belonging to \( I \).

First, we have to prove that any configuration \( x \in \mathcal{D}_o^- \) can be transformed into the lowest expressive configuration \((0^n, 0^m)\). Following the proof of Lemma \(2\), we get \( \text{simp}(x) = (0^n, 0^m) \) and the related complexity still holds.

Second, let us focus on the increase of the expressiveness of configurations. To do so, let us consider two cases: (a) only one cycle is of odd size and we consider that it is \( \mathcal{C}^\ell \) with no loss of generality; (b) both cycles are of odd sizes. According to both these cases, we have:

(a) \( x0 = ((01)^{1+1}0, (01)^{1+1} \mathcal{T}) \) and \( x1 = ((10)^{1+1}0, (10)^{1+1} \mathcal{T}) \) are two of the three most expressive configurations that do not belong to \( I \) (the third one is \( x1 \) that has not to be taken into account because the results for \( x1 \) extend to it directly). Notice that \( x0 \) can be transformed into \( x1 \) by means of sequence \( \sigma_a = \text{shift}(\mathcal{C}^\ell); \text{shift}(\mathcal{C}^\ell); \text{update}(c_{n-1}); \text{sync}. \). Conversely, the sequence \( \sigma'_a = \text{shift}(\mathcal{C}^\ell); \text{shift}(\mathcal{C}^\ell); \text{sync}; \) allows to reach \( x0 \) from \( x1 \).

(b) \( x0 = ((01)^{1+1}1, (01)^{1+1}0) \) and \( x1 = ((10)^{1+1}0, (10)^{1+1}0) \) are two of the three most expressive configurations that do not belong to \( I \) (the third one is \( x1 \) and is not considered for the same reason as above). In this case, \( x0 \) can be transformed into \( x1 \) by means of sequence \( \sigma_b = \text{shift}(\mathcal{C}^\ell); \text{shift}(\mathcal{C}^\ell); \text{update}(c_{n-1}); \text{update}(c_{m-1}); \text{sync}. \). Also, the sequence \( \sigma'_b = \text{shift}(\mathcal{C}^\ell); \text{shift}(\mathcal{C}^\ell); \text{update}(c_{n-1}); \text{sync}; \) allows to reach \( x0 \) from \( x1 \).

From the reasoning given in the proofs of Lemmas \(3\); \(4\); and \(5\) it can be derived that \( \text{comp}(0^n, 0^m) = x0 \), which shows together with \( \sigma_a \) and \( \sigma_b \) the accessibility of the most expressive configurations from the least expressive ones in any case. Notice that the bound \( \Theta(n^2 + m^2) \) remains valid in this case.

Third, consider now a new version of \( \text{comp} \) that takes as parameters a configuration and either \( 0 \) or \( 1 \). More precisely, this new version of \( \text{comp} \) is defined.
such that:
\[
\text{comp}(z, 0) = \text{comp}(z) \quad \text{and} \quad \text{comp}(z, 1) = \begin{cases} 
\text{comp}(z); & \text{for case (a)} \\
\text{comp}(z); & \text{for case (b)}
\end{cases}
\]

Let \( x \) and \( y \) be two configurations that do not belong to \( I \). First, remark that the state of \( c_0 \) in \( \text{comp}((0^n, 0^m), y_0) \) equals \( y_0 \). Consequently, since we have \( \text{comp}((0^n, 0^m), y_0) = y_0 \), thanks to the proofs of Lemmas 7 and 8 and the fact that \( \text{comp}((\text{simp}(x), y_0)) \) corresponds to one of the most expressive configurations in any case, \( y = \text{copy}(\text{comp}((\text{simp}(x), y_0), y) \) holds. As a result, all the configurations that do not belong to \( I \) are recurrent and are reachable from each other, which implies that they compose a unique attractor. Notice also that from this result, we have easily the intermediary result stating that the number of updates to reach any configuration from any of the most expressive configurations is linear. Indeed, since \( x0 \) and \( x1 \) can reach each other through the distinct linear sequences \( \sigma \) and since we have just shown that they can reach any configuration \( y \notin I \) by using \( \text{copy} \), it is direct that the most expressive configurations can transform themselves linearly into any other configuration.

To complete this part, by basing ourselves on what has been done until now, let us focus on the cases where either \( n \) or \( m \) equals 1. Consider without loss of generality that \( m = 1 \). We have to distinguish two cases:

\( n \equiv 0 \mod 2 \): This case is trivial because no irreversible configurations exist.

\( n \equiv 1 \mod 2 \): First, remark that \( \text{sync}((1(10)^{n-1}, 1)) = (01)^{n-1}0, 0) \), which is thus not irreversible. Second, according to Lemma 7, \((10)^{n-1}1, 1) \) is irreversible. Consequently, it is the only one that cannot be reached. As a result, such a BADC admits one attractor of size \( 2^n - 1 \) and \( I = \{(10)^{n-1}1, 1)\} \).

We have proven that a negative BADC \( \mathcal{D}^- \) has only one attractor, whatever the cycle parity. However, the size of this attractor depends on the cardinal of \( I \), on which we focus from now. Several cases have to be taken into account:

\( n = 1 \) or \( m = 1 \): In this case, as stated just above, if the cycle of size 1 intersect with a cycle of even size (resp. odd size), there are no irreversible configurations and \( |I| = 0 \) (resp. there is one irreversible configuration and \( |I| = 1 \).

\( n \) and \( m \) are greater than 1:

- If both cycles are of even size: such BADCs admits a unique attractor of size \( 2^{n+m-1} \) and \( |I| = 0 \) (cf. Theorem 6).
- If only one of the cycles is of odd size: if this cycle is \( C^\ell \) (resp. \( C^r \)) then the configurations of the form \((10)^{n-1}1, 1) \) (resp. \((10)^{n-1}1, 1) \)) are irreversible and \( |I| = 2^{m-1} \) (resp. \( |I| = 2^{n-1} \)).
- If both cycles are of odd sizes: \( I \) is in this case composed of configurations of the forms \((10)^{n/2}0, (10)^{n-1}1) \) and \((10)^{n/2}1, (10)^{n-1}0) \). That means that \(|I| = 2^{m-1} + 2^{n-1}\).
From this, we derive the following generalisation that states that any negative BADC $\mathcal{D}^-$ admits a unique attractor, a stable oscillation of size

$$2^{n+m-1} - \alpha(n-1) \times 2^{m-1} + \alpha(m-1) \times 2^{n-1},$$

where $\alpha : \mathbb{N} \to \{0, 1\}$ with $\alpha(k) = \begin{cases} 0 & \text{if } k = 0 \text{ or } k \equiv 1 \mod 2 \\ 1 & \text{otherwise} \end{cases}$. □

5 Conclusion and perspectives

This paper followed the lines of [11,13] and focused on the dynamical properties of BADCs subjected to the asynchronous updating mode. Again, the focus on BADCs is explained by the fact that although cycles have been known to be the engines of complexity in interaction networks since the 1980’s, their influence on network dynamics is not really known, contrary to the common beliefs. This needs to be changed if we want to understand precisely interaction network complexity. However, because of the intrinsic difficulties to bring such studies in general frameworks (in general BANs for instance), we needed to restrain the spectrum of intersections considered to the “simplest” kinds, the tangential ones. In this setting, our contribution was twofold: (i) we gave a complete characterisation of the dynamical behaviour of asynchronous BADCs by means of (ii) new algorithmic tools that bring a new way to view updates in networks and a nice understanding of how information is relayed. Obviously, these tools have been built for our purpose and their use is consequently limited. Nevertheless, remark that they can be applied almost directly in some more complex networks, in particular those with tangential cycle intersections, such as flower graphs for which they will allow to provide characterisation results regarding their behaviours that will generalise the existence results given in [2]. Furthermore, another perspective would consist in adapting these tools in order them to apply to more complex intersections. Beyond the dynamical aspects, notice that the algorithmic tools owe the benefits to represent concisely long sequences of updates. About this abstraction, we would like to understand to what extent we can characterise network architectures when update sequences (that represent only pieces of dynamics) are given. For instance, the latter could be very useful to find networks of specific dynamics complexity classes (in terms of convergence time for instance, or even in terms of number of attractors). To finish, this work together with that of [13] (and the differences they present) raises once again the matter of the fundamental differences between synchronism and asynchronism whose study deserves to be pursued.

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