EXTENSIONS OF THE RATIONAL CHEREDNIK ALGEBRA AND GENERALIZED KZ FUNCTORS.

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ABSTRACT. Ginzburg, Guay, Opdam and Rouquier established an equivalence of categories between a quotient category of the category $\mathcal{O}$ for the rational Cherednik algebra and the category of finite dimension modules of the Hecke algebra of a complex reflection group $W$. We establish two generalizations of this result. On the one hand to the extension of the Hecke algebra associated to the normaliser of a reflection subgroup and on the other hand to the extension of the Hecke algebra by a lattice.

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0. INTRODUCTION

In 1967, Yokonuma [Yok67] introduce a generalization of the Hecke algebra in the context of finite reductive group. Let $G$ be a finite Chevalley group. Let $B$ a Borel subgroup of $G$. Let $U$ be the radical unipotent of $B$ and $T$ the maximal torus of $G$. Let $(W,S)$ be the Weyl group of $G$. Let $k$ be a commutative ring. The Yokonuma-Hecke algebra is the algebra of functions from $G$ to $k$ bi-invariant by the action of $U$ endow with a convolution product. A subalgebra of the Yokonuma-Hecke algebra of type $A$ was introduced by Aicardi and Juyumaya in 2014 [AJ14] for the definition of invariants of tied knots, and was therefore named "algebra of braids and ties".

In 2016, I.Marin [Mar18a] observed that $BT_n$ is an extension of the Hecke algebra of type $A$ by the lattice of reflexion subgroups of the symmetric group. Furthermore, he defined for any Coxeter system $(W,S)$ an extension $C_W$ of the Hecke algebra $H(W)$ by the lattice of full reflection subgroups. By full reflection subgroup we mean that $W_0$ is a reflection subgroup of $W$ and for any reflection in $W_0$, all the reflection with the same reflecting hyperplane belong to $W_0$.

Moreover, I.Marin generalized this construction to any finite complex reflection group. Let us first recall the construction of the Hecke algebra associated to a finite complex reflection group.

In 1998, Broue, Malle and Rouquier introduce in their seminal paper [BMR98], the Hecke algebra of a finite complex reflection group. Let $V$ be a complex $\mathbb{C}$-vector space of finite
dimension. Let $W$ be a finite complex reflection group. Let $\mathcal{R}$ be the set of all reflections of $W$ and let $\mathcal{A}$ be the arrangement of reflecting hyperplanes of $W$. We denote by $X$ the complement of $\mathcal{A}$ in $V$. Let $(k_{H,j})_{j \in \{0, \ldots, m_H-1\}}$ be a set of complex numbers indexed by a reflecting hyperplane of $W$ and integers $j$, where $m_H$ is the order of the pointwise stabilizer of $H$ in $W$, such that for all $w \in W$, $k_{w(H),j} = k_{H,j}$ for all integers $j$. Let $B(W) := \pi_1(X/W)$ be the braid group of $W$, which is generated by braided reflections $\sigma_j$.

The Hecke algebra of $W$ is $H(W) := \mathbb{C}B(W)/I$ where $I$ is a two-sided ideal of $\mathbb{C}B(W)$ generated by the relation $\sigma_j^{m_H} = \sum_{k=0}^{m_H-1} k_{H,j} \sigma_j^k$.

Let us now recall the construction made by I. Marin for $W$ a complex reflection group.

**The Hecke algebra of the pair $(W_0, W)$.** Let $W_0$ be a reflection subgroup of $W$. We denote by $N_{W_0}$ the normalizer of $W_0$ in $W$. We have a bijection between the orbit $[W_0]$ of $W_0$ under $W$ and the quotient $W/N_{W_0}(W_0)$.

Since $\mathbb{C}L \cong \mathbb{C} \cong \mathbb{C}L \times W \cong \bigoplus_{[W_0] \in \mathcal{L}/W} \text{Mat}_{[W_0]}(\mathbb{C}N_W(W_0))$ (Mar18b proposition 2.1), we can expect that $\mathcal{C}(\mathcal{L}, W)$ is Morita equivalent to an algebra of matrices involving the algebra $\mathcal{C}(\mathcal{N}_W(W_0))$.

Indeed, the algebra $\mathcal{C}(\mathcal{L}, W)$ is Morita equivalent to a direct sum of matrix algebras with coefficient in a new Hecke algebra. The Hecke algebra of $\mathcal{N}_W(W_0)$.

The Knizhnik–Zamolodchikov functor. In 2003, Ginzburg, Guay, Opdam and Rouquier [GGOR03] defined a category $\mathcal{O}$ associated to a Cherednik algebra $A(W)$ of a finite complex reflection group $W$ [LeG02]. This category of modules over $A(W)$ is endowed with finiteness conditions, based on the Bernstein-Geleshand-Gelfand category $\mathcal{O}$ for a semisimple Lie algebra. They constructed a functor, called Knizhnik–Zamolodchikov functor $KZ$, from this category $\mathcal{O}$ to the category of finite dimension modules over the Hecke algebra of $W$. The construction of this functor made a critical use of the Dunkl-Opdam differential operators. This functor induces an equivalence of categories between the category $\mathcal{O}/\mathcal{O}_{tor}$ of object in $\mathcal{O}$ supported outside of $X$ and the category of all finite dimensional representations of the Hecke algebra of a complex reflection group.

The aim of this article is to establish two results of the same kind as in [GGOR03] for the case of the algebras $H(W_0, W)$ and $\mathcal{C}(\mathcal{L}, W)$. We start by constructing a Cherednik algebra
for $CN_W(W_0)$ as a symplectic reflection algebra for $N_W(W_0) [EG02]$ depending on a parameter $t$. In this article, only the $t \neq 0$ case will be investigated and up to renormalization, we can consider $t = 1$. Then we introduce a family of commutative differential operators, $N_W(W_0)$-equivariant by considering the covariant derivative of a flat connection involving a 1-differential form defined in [GM21] (proposition 2.6). In order to prove the commutativity we use arguments from [LM10] theorem 2.15. This allows us to define a Dunkl embedding of $A(W_0, W)$ inside an algebra of differential operators on $X$, $N_W(W_0)$-equivariant $\mathcal{D}(X) \rtimes N_W(W_0)$.

Then we define an Euler element $\tilde{e}u_0$, standard objects, a highest-weight category $\mathcal{O}(W_0, W)$ and a functor $KZ_0$ from $\mathcal{O}(W_0, W)$ to the category of finite dimensional representation of $H(W_0, W)$. From an object $\mathcal{O}(W_0, W)$, we get a $\mathcal{D}(X)^{N_W(W_0)}$-modules by applying a localization functor, the Dunkl embedding and some classical Morita’s equivalence. We use a general result of algebraic geometry [CH94] theorem 3.7.1 to prove $\mathcal{D}(X)^{N_W(W_0)} \simeq \mathcal{D}(X/N_W(W_0))$.

We define a flat connection on standard objects of $\mathcal{O}(W_0, W)$ and prove that this connection has regular singularities on $V$. Afterwards, we prove that we can apply the Riemann-Hilbert-Deligne equivalence for any object of $\mathcal{O}(W_0, W)$. We get a $C\pi_1(X/N_W(W_0))$-modules. Furthermore, the [GM21] proposition 2.6 implies that the monodromy action of $C\pi_1(X/W)$ factors through an action of $H(W_0, W)$. As a result, we had built a functor

$$KZ_0 : \mathcal{O}((W_0, W) \to H(W_0, W)\text{-mod}_{f.d}$$

Since this functor is exact we can prove that this functor is representable by a projective object of $\mathcal{O}(W_0, W)$, this allows us to prove that the category image of $KZ_0$ is actually the category $H(W_0, W)\text{-mod}_{f.d}$. By killing all the objects of $\mathcal{O}(W_0, W)$ with support outside of $X$ we get

Theorem 0.2. The functor

$$KZ_0 : \mathcal{O}(W_0, W) / \mathcal{O}_{tor}(W_0, W) \to H(W_0, W)\text{-mod}_{f.d}$$

is an equivalence of categories.

In the second part of this article, we develop a similar construction in the context of the lattice extension. We start by defining a Cherednik-algebra for the pair $(L, W)$ as an algebra with a triangular decomposition [BT18] (definition 3.1) depending on a parameter $t$. We consider the case $t \neq 0$ and up to a renormalization $t = 1$. Then we introduce a family of commutative differential operators, $W$-equivariant by considering the covariant derivative of a flat connection involving a 1-differential form defined in [Mar20] proposition 2.5. In order to prove the commutativity we use arguments from [LM10] theorem 2.15. Whereas in the previous case the action of $(\mathcal{D}(X) \otimes C\mathcal{L}) \rtimes W$ on $C(V)$ is not obviously faithful. In order to prove the faithfulness, we prove a more general result:

Proposition 0.3. Let $R$ be a simple ring with unity. Let $G$ be a group of outer automorphisms of $R$. Let $X$ be a finite set. If $G$ acts transitively on $X$, then $(R \otimes C^X) \rtimes G$ is simple.

This allows us to define a Dunkl embedding of $A(W_0, W)$ inside the algebra $(\mathcal{D}(X) \otimes C\mathcal{L}) \rtimes W$. Then we define an Euler element $\tilde{e}u$, standard objects with $\text{Irr}(C\mathcal{L} \rtimes W)$, and a highest weight category $\mathcal{O}(L, W)$. We prove that we can endow an object of $\mathcal{O}(L, W)$ with flat connection over a trivial vector bundle on $V$ with fiber $E \in \text{Irr}(C\mathcal{L} \rtimes W)$ and we prove that this connection has regular singularities on $V$. This allows us to apply the Riemann-Hilbert-Deligne correspondence, we then get a finitely generated $(CB(W) \otimes C\mathcal{L}) \rtimes W$-modules. Furthermore, [Mar18], the monodromy action factorizes through an action of $\mathcal{C}(L, W)$. Consequently, we have defined a functor: $KZ : \mathcal{O}(L, W) \to \mathcal{C}(L, W)\text{-mod}_{f.d}$.
Since this functor is exact we can prove that this functor is representable by a projective object of \( \mathcal{O}(\mathcal{L}, W) \), this allows us to prove that the category image of \( KZ \) is the category \( \mathcal{C}(\mathcal{L}, W)\text{-mod}_{f.d} \). Then we kill all the object of \( \mathcal{O}(\mathcal{L}, W) \) with support outside of \( X \), and we get

**Theorem 0.4.** The functor \( \overline{KZ} : \mathcal{O}(\mathcal{L}, W) / \mathcal{O}_{tor}(\mathcal{L}, W) \to \mathcal{C}(\mathcal{L}, W)\text{-mod}_{f.d} \) is an equivalence of categories.

1. The rational Cherednik algebra of the normalizer \( N_W(W_0) \) and the functor \( KZ_0 \).

1.1. The rational Cherednik algebra of the normalizer. Let \( \delta = \prod_{H \in A} \alpha_H \) where \( \alpha_H \in V^* \) such that \( \text{Ker}(\alpha_H) = H \). This is an element of \( \mathbb{C}[V] \) and \( \delta \) vanishes on \( \bigcup A \).

We define the rational Cherednik algebra of \( \mathbb{C}N_W(W_0) \) as the symplectic reflection algebra of \( N_W(W_0) \) \([EG02]\), it depends on a parameter \( t \),

\[
A_t(W_0, W) := \frac{T(V \oplus V^*) \rtimes N_W(W_0)}{J}
\]

where \( J \) is an ideal of \( \mathbb{C}N_W(W_0) \) generated by the relations \([x, x'] = 0\) for all \((x, x') \in V^* \times V^*, [y, y'] = 0\) for all \((y, y') \in V \times V\) and \([y, x] = x(y) + \sum_{H \in A_0} \frac{\alpha_H(y)x(v_H)}{\alpha_H(v_H)} \gamma_H\) for all \((x, y) \in V^* \times V\) and \( v_H \in V \) such that \( \mathbb{C}.v_H \oplus H = V \) and \( \mathbb{C}.v_H \) is \( \text{Fix}_W(H) \) stable. The operator

\[
\gamma_H := \sum_{j=0}^{m_H-1} m_H(k_{H,j+1} - k_{H,j})\epsilon_{H,j}
\]

is a linear combination of primitive orthogonal idempotent of \( \mathbb{C}N_W(W_0) \) where \( m_H \) is the order of \( \text{Fix}_W(H) \) and \((\epsilon_{H,j})_{j=0,\ldots,m_H-1}\) is a system of primitives and orthogonal idempotents of \( \mathbb{C}N_W(W_0) \) defined by

\[
\epsilon_{H,j} := \frac{1}{|N_W(W_0)|} \sum_{w \in N_W(W_0) \setminus \text{Id}} \det(w)^j w
\]

In this article, we focus on the case of \( t \neq 0 \). Up to renormalization, we can reduce to \( t = 1 \). Then the Cherednik algebra of the normalizer is denoted \( A(W_0, W) \).

According to the theorem 1.3 \([EG02]\), we have an isomorphism of vector spaces

\[
A(W_0, W) \simeq \mathbb{C}[V] \otimes \mathbb{C}N_W(W_0) \otimes \mathbb{C}[V^*]
\]

We denote by \( A(W_0, W)_{reg} \) the localization by the multiplicative set defined by \( \delta \),

\[
A(W_0, W)_{reg} := \mathbb{C}[X] \otimes_{\mathbb{C}[V]} A(W_0, W)
\]

1.1.1. Dunkl operator.

**Proposition 1.1.** \([GM21\text{ prop. 2.6}]\) The following differential 1-form is \( W \)-equivariant and integrable on \( X \),

\[
\omega_0 = \sum_{H \in A_0} a_H \frac{d\alpha_H}{\alpha_H} \in \Omega^1(X) \otimes \mathcal{C}W_0
\]

where \( a_H = \sum_{H \in A_0} m_H k_{H,j} \epsilon_{H,j} \).

Let us define a family of Dunkl-Opdam operators by considering the covariant derivative of the connection \( \nabla := \partial + \omega_0 \) on a trivial vector bundle over \( X \).

**Definition 1.2.** (Dunkl-Operator) The covariant derivative of \( \nabla \) along \( y \in V \) is

\[
T_y := \partial_y + \sum_{H \in A_0} \frac{\alpha_H(y)}{\alpha_H} a_H \in \mathcal{D}(X) \times N_W(W_0)
\]

where \( \mathcal{D}(X) \) stands for the algebra of differential operators of \( X \) \([Gin98]\).
Proposition 1.3. (1) For all $(y, y') \in V \times V$, $[T_y, T_{y'}] = 0$.
(2) For all $g \in \mathcal{N}_W(W_0)$ and for all $y \in V$, $gT_yg^{-1} = T_{g(y)}$.

Proof. For the commutativity, the proof is similar to that in [EG02]. □

Then we define a faithful representation of $A(W_0, W)$.

Theorem 1.4. (The Dunkl embedding) The morphism
\[
\Phi : A(W_0, W) \rightarrow \mathcal{D}(X) \rtimes \mathcal{N}_W(W_0)
\]
\[
x \in V^* \mapsto x
\]
\[
y \in V \mapsto T_y
\]
\[
w \in \mathcal{N}_W(W_0) \mapsto w
\]
is an injective morphism of algebras.

Proof. We define a filtration on $A(W_0, W)$ by putting in degree 0, $V^*$ and $\mathcal{N}_W(W_0)$, and $V$ in degree 1. We consider filtration by the order of differential operators on $\mathcal{D}(X) \rtimes \mathcal{N}_W(W_0)$.

The morphism $\Phi$ is a morphism of graded algebras. So it induces a morphism on associated graded algebras
\[
gr(\Phi) : gr(A(W_0, W)) \rightarrow gr(\mathcal{D}(X) \rtimes \mathcal{N}_W(W_0))
\]
We can prove that the composition
\[
\mathbb{C}[V \oplus V^*] \rtimes \mathcal{N}_W(W_0) \xrightarrow{\sim} gr(A(W_0, W)) \xrightarrow{\gr(\Phi)} gr(\mathcal{D}(X) \rtimes \mathcal{N}_W(W_0)) \xrightarrow{\sim} \mathbb{C}[X \oplus V^*] \rtimes \mathcal{N}_W(W_0)
\]
is the identity on the homogenous components. Then $gr(\Phi)$ is injective so is $\Phi$. □

Theorem 1.5. By localizing the previous morphism $\Phi$ becomes an isomorphism of algebras
\[
\Phi_{reg} : A(W_0, W)_{reg} \simeq \mathcal{D}(X) \rtimes \mathcal{N}_W(W_0)
\]

Proof. We just have to prove the surjectivity. The operator
\[
\sum_{H \in \mathcal{A}_0} \frac{\partial a_H(y)}{a_H} a_H
\]
has no pole over $X$. Thus $y - \sum_{H \in \mathcal{A}_0} \frac{\partial a_H(y)}{a_H} a_H$ is an element of $A(W_0, W)_{reg}$. So the image of $\Phi_{reg}$ contains a system of generators of $\mathcal{D}(X) \rtimes \mathcal{N}_W(W_0)$. □

1.1.2. Category $\mathcal{O}(W_0, W)$. We define a category $\mathcal{O}(W_0, W)$ similar to the category $\mathcal{O}$ in [GGOR93]. We denoted by $eu_0 := \sum_{y \in \mathcal{B}} y^* y - \sum_{H \in \mathcal{A}} a_H$ the Euler element, where $\mathcal{B}$ is a basis of $V$.

The operator $\sum_{H \in \mathcal{A}_0} a_H$ lies in the center of $\mathbb{C}\mathcal{N}_W(W_0)$, so it acts by multiplication by a scalar $c_E$ on irreducible representations $E$ of $\mathcal{N}_W(W_0)$. We define a partial order on simple $\mathbb{C}\mathcal{N}_W(W_0)$-modules by $F < E$ if $c_E - c_F \in \mathbb{N}$.

Proposition 1.6. (1) $[eu_0, x] = x$ for all $x \in V^*$.
(2) $[eu_0, y] = y$ for all $y \in V$.
(3) $[eu, w] = 0$ for all $w \in \mathcal{N}_W(W_0)$.

The operator $eu_0$ induces an inner graduation on $A(W_0, W)$ defined by $A(W_0, W)^i = \{a \in A(W_0, W) | [eu_0, a] = ia\}$ for all $i \in \mathbb{Z}$. The standard objects are $\Delta(E) := Ind_{\mathbb{C}[V^*] \otimes \mathcal{N}_W(W_0)}^{A(W_0, W)} E$ where $E$ is a simple $\mathcal{N}_W(W_0)$-module. The standard objects satisfy the same properties as standard objects in the rational case, $\Delta(E)$ admits a simple head $L(E)$ and each simple head admits a projective cover $P(E)$ and each projective cover admit a filtration by $A(W_0, W)$-submodules $P_i$ of $P(E)$ with successive quotient $P_i/P_{i-1}$ isomorphic to a standard object $\Delta(E_i)$ such that $E_i < E$. We denote by $\Lambda$ the set of all simple objects. The category $\mathcal{O}(W_0, W)$ is the full subcategory of finitely generated $A(W_0, W)$-module satisfying the following conditions:
The action of $\mathbb{C}[V^+]$ on $M \in \mathcal{O}(W_0, W)$ is locally nilpotent.

2. $M$ is isomorphic to the direct sum of its generalized $e\mu_0$-eigen space,

$$M \simeq \bigoplus_{\alpha \in \mathcal{C}} W_\alpha(M), \text{ with } W_\alpha(M) := \{m \in M | \exists N > 0, (e\mu_0 - \alpha)^N.m = 0\}$$

The standard objects $\Delta(E)$ are elements of $\mathcal{O}(W_0, W)$. The set $\Lambda$ of all simple objects is a set of complete simple objects $\mathcal{O}(W_0, W)$. According to [GGOR03] theorem 2.19, we can assert that the triple $(\mathcal{O}(W_0, W), \Lambda, <)$ is a highest weight category in the sense of [CPSS88]. Therefore, it exists a quasi-hereditary cover, so there exists a finite dimensional $\mathbb{C}$-algebra $R$ such that $\mathcal{O}(W_0, W)$ is equivalent to the category of finitely generated modules over the algebra $R$.

1.2. The functor $KZ_0$. We have a localization functor $Loc$ from the category of $A(W_0, W)$-modules to the category of $A(W_0, W)_{\text{reg}}$-modules which sends $M \in \mathcal{O}(W_0, W)$-mod to $M_{\text{reg}} := A(W_0, W)_{\text{reg}} \otimes_{A(W_0, W)} M$.

Since $\mathbb{C}[X]$ is flat over $\mathbb{C}[V]$, this functor is exact. Let $M$ be a $A(W_0, W)$-module, we defined $M_{\text{tor}}$ as $\{m \in M | \exists N > 0, \delta^N.m = 0\}$. The full subcategory of $A(W_0, W)$-modules such that $M = M_{\text{tor}}$ is called the category of $A(W_0, W)$-modules with torsion with respect of $\delta$ and it is denoted $(A(W_0, W)-\text{mod})_{\text{tor}}$. Thus, for all $M \in (A(W_0, W)-\text{mod})_{\text{tor}}, Loc(M) = 0$.

Then the localization functor factorized through the quotient category $(A(W_0, W)-\text{mod})_{\text{tor}}$. It induces a faithful functor

$$\frac{A(W_0, W)-\text{mod}}{(A(W_0, W)-\text{mod})_{\text{tor}}} \rightarrow A(W_0, W)_{\text{reg}}-\text{mod}$$

We have a restriction functor $Res : A(W_0, W)_{\text{reg}}-\text{mod} \rightarrow A(W_0, W)-\text{mod}$ sending $M \rightarrow \mathbb{C}[X] \otimes_{\mathbb{C}[V]} M$ which is right adjoint to the localization functor. The functor Loc induces an equivalence of categories $\frac{A(W_0, W)_{\text{mod}}}{(A(W_0, W)-\text{mod})_{\text{tor}}} \rightarrow A(W_0, W)_{\text{reg}}-\text{mod}$ [Gab62].

Let $\mathcal{O}(W_0, W)_{\text{tor}} := \mathcal{O}(W_0, W) \cap (A(W_0, W)-\text{mod})_{\text{tor}}$. For all objects with $\delta$-torsion i.e $M \in \mathcal{O}(W_0, W)_{\text{tor}}, Loc(M) = 0$. The functor Loc induces a fully faithful functor

$$\frac{\mathcal{O}(W_0, W)}{\mathcal{O}(W_0, W)_{\text{tor}}} \rightarrow A(W_0, W)_{\text{reg}}-\text{mod}$$

which is the composition of the functors

$$\frac{\mathcal{O}(W_0, W)}{\mathcal{O}(W_0, W)_{\text{tor}}} \rightarrow \frac{A(W_0, W)-\text{mod}}{(A(W_0, W)-\text{mod})_{\text{tor}}}$$

and

$$\frac{A(W_0, W)-\text{mod}}{(A(W_0, W)-\text{mod})_{\text{tor}}} \rightarrow A(W_0, W)_{\text{reg}}-\text{mod}$$

The first is fully faithful (lemma 3.3 [Rou10]) and the second is the previous equivalence of categories.

The Dunkl embedding induces an equivalence of categories between the category of $A(W_0, W)$-modules and the category of $\mathcal{D}(X) \rtimes N_W(W_0)$-modules.

Let $e := \frac{1}{[N_W(W_0)]} \sum_{g \in N_W(W_0)} g$, this is a central idempotent of $\mathbb{C}N_W(W_0)$. The category of $\mathcal{D}(X) \rtimes \mathbb{C}N_W(W_0)$-modules is equivalent to the category $e.(\mathcal{D}(X) \rtimes \mathbb{C}N_W(W_0)).e$-modules. Since we have $e.(\mathcal{D}(X) \rtimes \mathbb{C}N_W(W_0)).e \simeq \mathcal{D}(X)_{N_W(W_0)}$, the category of $\mathcal{D}(X) \rtimes N_W(W_0)$-modules is equivalent to the category of $\mathcal{D}(X)_{N_W(W_0)}$-modules.

However, $N_W(W_0)$ is not necessarily a complex reflection group, so we do not have an a priori an isomorphism between the algebra of differential operators invariants by the action of $N_W(W_0)$ and the algebra of differential operators on $X/N_W(W_0)$. We use a general result from algebraic geometry proved by Halland and Cannings in [CH93] theorem 3.7. Let us recall this result. Let $k$ be an algebraically closed field. Let $A$ be a reduced, finitely generated $k$-algebra. Let $X := \text{Spec}(A)$. Let $G$ be a finite group acting on $A$ by automorphisms,
then it acts on \( X \). The algebra \( A^G \) is also reduced and finitely generated. We denote by \( X/G := \text{Spec}(A^G) \). We denote by \( \Phi : A^G \to A \) and \( \Phi' : \text{Spec}(A) \to \text{Spec}(A^G) \), we get a morphism of sheaves \( \Phi^* : \mathcal{O}_{\text{Spec}(A^G)} \to \Phi_*\mathcal{O}_{\text{Spec}(A)} \). Let \( p \) be a point in \( X \) and \( I_p^1(p) := \{ g \in G | (f - g) \in p, \forall f \in A \} \) the inertia group of \( p \).

Let \( \tilde{V} := \{ p \in X | I(p) = 1 \} \). We said \( G \) act generically without inertia if \( G \) acts trivially and if for all generic points of \( X \), \( I(p) = 1 \). We can assume \( G \) act generically without inertia. If \( \tilde{V} = X \), we stated that \( G \) acts without fixed points.

**Theorem 1.7.** ([CH94] theorem 3.7)

1. If \( G \) acts without fixed points then \( \mathcal{D}(A^G) \simeq \mathcal{D}(A)^G \)

Let us apply this theorem to the case of \( G = N_W(W_0) \) and \( A = \mathbb{C}[X] \). Since \( N_W(W_0) \) acts on \( X \) without fixed points, we get the desired isomorphism \( \mathcal{D}(X)^{N_W(W_0)} \simeq \mathcal{D}(X/N_W(W_0)) \).

Let us investigate the structure of \( \mathcal{D}(X/N_W(W_0)) \)-module. Let \( E \) be a simple \( \mathbb{C}N_W(W_0) \)-modules. We have \( \Delta(E)_{\text{reg}} \simeq \mathbb{C}[X] \otimes E \), it is a free \( \mathbb{C}[X] \)-module of rank \( \text{dim}(E) \). Thus, it corresponds to a trivial vector bundle over \( X \) of fiber \( E \). The structure of \( \mathcal{D}(X) \times N_W(W_0) \)-module is given by the action of \( x \in V^* \), \( T_y \) with \( y \in V \) and \( w \in N_W(W_0) \) on \( \Delta(E)_{\text{reg}} \). The element \( w \) acts diagonally on \( \Delta(E)_{\text{reg}} \), \( x \) acts by multiplication on the left and \( T_y(P \otimes v) = \partial_y(P \otimes v) + \sum_{H \in \mathcal{A}_0} \alpha_H(y) a_H(P \otimes v) \) where \( P \in \mathbb{C}[X] \) and \( v \in E \). Since \( y.v = 0 \) for all \( y \in V \) and \( v \in E \) then \( T_y(v) = 0 \) so \( \partial_y(v) = -\sum_{H \in \mathcal{A}_0} \alpha_H(y) a_H v \). Therefore, we get

\[
\partial_y(P \otimes v) = \partial_y P \otimes v + \sum_{H \in \mathcal{A}_0} \frac{\alpha_H(y)}{\alpha_H} \sum_{j=0}^{m_H-1} m_H k_{H,j} P \otimes \epsilon_{H,j} v.
\]

This formula defines a covariant derivative \( \nabla^0 := \partial_y \). The associated connection is \( \nabla^0 := d \otimes \text{Id} + \sum_{H \in \mathcal{A}_0} \frac{d_H}{\alpha_H}(\sum_{j=0}^{m_H-1} m_H k_{H,j}(\text{Id} \otimes \epsilon_{H,j})) \).

**Proposition 1.8.** The algebraic connection \( \nabla^0 \) is flat, \( N_W(W_0) \)-equivariant with regular singularities on \( V \).

**Proof.** According to the proposition 2.1 we obtain the flatness of the connection.

Since for all \( \alpha \in V^* \) and for all \( g \in N_W(W_0), g.d\alpha = d(g.\alpha) \) and the operator \( \sum_{H \in \mathcal{A}_0} \frac{d_H}{\alpha_H} a_H^* \) is \( N_W(W_0) \)-equivariant if and only if for all \( H \in \mathcal{A}_0 \) and for all \( w \in N_W(W_0), a_H^*(H) = w a_H^*(H) \), the connection \( \nabla^0 \) is \( N_W(W_0) \)-equivariant.

Let us prove the claim about the singularities. Let us follow [Del06] and try to apply the Deligne’s regularity criterion for an integrable connection \( \nabla \) on a smooth complex algebraic variety \( X \). It states that \( \nabla \) is regular along the irreducible divisor at infinity in some fixed normal compactification of \( X \) if and only if the restriction of \( \nabla \) to every smooth curve on \( X \) is Fuchsian.

Let \( j : V \to \mathbb{P}(V \otimes \mathbb{C}) =: Y \) be a compactification \( N_W(W_0) \)-equivariant of \( V \). We consider \( M_Y := \mathcal{O}_Y \otimes \Delta(E) \) a sheaf of \( \mathcal{O}_Y \)-module.

Let \( H \in \mathcal{A}_0 \) and \( x_H \in H \). We know that there is a vector \( v_H \in V \) such that the line generated by \( v_H \) is supplementary to \( H \) and \( W_H \)-stable. Let us consider the affine line directed by \( L_H \) passing through \( x_H \) denoted \( L_H := x_H + \mathbb{C}.v_H \).

Let \( D_H^* = \{ x_H + z.v_H | 0 < |z| < 2 \} \) such that \( D_H^* \subset X \). Then \( \alpha_H(x_H + z.v_H) = \alpha_H(x_H) + z.\alpha_H(v_H) = 0 + z.\alpha_H(v_H) \). For \( H \neq H', \alpha_H(x_H + z.v_H) \neq 0 \) on \( D_H^* \). We restricted the connection to the affine line, and we got

\[
\nabla_{L_H^*} = d \otimes \text{Id} - dz \sum_{j=0}^{m_H-1} m_H k_{H,j} \text{Id} \otimes \epsilon_{H,j} - \sum_{H' \neq H} \frac{\alpha_{H'}(v_H)dz}{\alpha_{H'}(x_H + z.v_H)} \sum_{j=0}^{m_H-1} m_H k_{H,j} \text{Id} \otimes \epsilon_{H,j}
\]

It is deduced that on each \( H \) the singularities are regulars.
Now let us prove the regularity at infinity. We start by defining a change of chart. Let \( \phi : V \rightarrow \mathbb{C} \) be a linear form of \( V \). Let \( \phi \) such that \( \phi(\hat{\phi}) = 1 \). We define a chart of \( Y \), noted \( V_\phi \) by \( V_\phi = \{ v \in V | \phi(v) = 1 \} \oplus \mathbb{C} \subset V \oplus \mathbb{C} \). Let \( (e_1, \ldots, e_n) \) be the canonical basis of \( V \) and \( (e_1^*, \ldots, e_n^*) \) its dual basis. The elements \( V_{e_i^*} \) are the classical affine carts \([x_1: \ldots: 1: \ldots: x_{n+1}]\). We consider the pullback on \( V_\phi \) of the 1-form of connection. It is equivalent to a replacement of variables \( (v, t) \rightarrow (\frac{v}{t}, 1) \). We get the formula:

\[
d \otimes \text{Id} + \sum_{H \in A} a_H \frac{-1}{t^2} \alpha_H(v) dt = B \sum_{H \in A} \frac{1}{t} \alpha_H(v) dt + \frac{1}{t^2} \alpha_H(v) dt = A
\]

Part \( A \) is regular by the previous step and does not depend on \( t \). Part \( B \) is regular in \( t = \infty \) after the change of variable \( (t' = 1/t) \). Therefore, \( \nabla^0 \) is regular. \( \square \)

Every standard object is endowed with a flat, \( N_{X}^{\mathcal{O}(W_0, W)} \)-equivariant, connection with regular singularities on \( V \). So every object of the category \( \mathcal{O}(W_0, W) \) is endowed with a connection with regular singularities over \( V \). We can apply the Riemann-Hilbert-Deligne equivalence, we get a \( \mathbb{C} \pi_1(X/N_{X}^{\mathcal{O}(W_0, W)})\)-module \( f_{d,d} \).

If we compose this functor with the previous construction we get the following functor \( \mathcal{O}(W_0, W) \rightarrow \mathbb{C} \pi_1(X/N_{X}^{\mathcal{O}(W_0, W)})\)-module \( f_{d,d} \), \( M \rightarrow (((M_{\text{reg}})^{\mathcal{O}(W_0, W)})^{\mathcal{O}(W_0, W)})^{\mathcal{O}(W_0, W)} \) where \((-)^{\mathcal{O}(W_0, W)} \) is the analytisation functor and \((-)^{\mathcal{O}(W_0, W)} \) the horizontal section functor of \( \nabla^0 \).

According to [GM21] proposition 2.6 the monodromy representation factorizes through \( H(W_0, W) \). Finally, we get a functor

\[
KZ_0 : \mathcal{O}(W_0, W) \rightarrow H(W_0, W)^{-}\text{mod}_{f,d}
\]

such that \( M \rightarrow (((M_{\text{reg}})^{\mathcal{O}(W_0, W)})^{\mathcal{O}(W_0, W)})^{\mathcal{O}(W_0, W)} \) which induces a faithful functor \( KZ_0 : \mathcal{O}(W_0, W)^{-}\text{mod}_{f,d} \rightarrow H(W_0, W)^{-}\text{mod}_{f,d} \).

The quasi-hereditary cover of \( \mathcal{O}(W_0, W) \) and the exactness of \( KZ_0 \) allow us to use a kind of Watt’s theorem. The claim of the following general proposition has been communicated to us by R.Rouquier.

**Proposition 1.9.** Let \( A \) and \( B \) two \( k \)-algebras of finite dimensions. Let \( F \) be an exact functor from the category of finitely generated \( A \)-module to the category of finitely generated \( B \)-modules. Then \( F \) is isomorphic to the functor \( \text{Hom}_A(\text{Hom}_{A^{op}}(F(A), A), -) \).

**Proof.** What follows is a mere outline of the proof: firstly, based on the proposition 4.4.b [ARS97] the functor \( \text{Hom}_A(F(A), A) \otimes_A - \) is isomorphic to the functor \( \text{Hom}_A(F(A), -) \).

Secondly, based on the corollary 5.47 [Rot08] the functor \( F(A) \otimes_A - \) is isomorphic to the functor \( F(-) \). Thirdly, based on proposition 4.3.b [Ari95], \( \text{Hom}_A(\text{Hom}_{A^{op}}(F(A), A), A) \) is isomorphic to \( F(A) \). \( \square \)

Let us apply this theorem to our example, the functor \( KZ_0 \) is representable by a projective object of \( \mathcal{O}(W_0, W) \) called \( P_{KZ_0} \).

**Theorem 1.10.** The morphism of algebra \( \Phi : H(W_0, W) \rightarrow \text{End}_\mathcal{O}(P_{KZ_0})^{\text{op}} \) is actually an isomorphism.

**Proof.** Let us follow the arguments proposed in [Bel12] part 4.6.

**Lemma 1.11.** ([Bel12] lemma 4.6.4) Let \( A \) be an abelian category. We assume \( A \) is Artinian. Let \( B \) be an abelian full subcategory of \( A \), closed under sub-object and quotient. Then the functor \( F : B \rightarrow A \) has a left adjoint \( G : A \rightarrow B \) sending an object \( M \in \text{Obj}(A) \) on its largest quotient in \( B \) and the co-unit \( \eta : \text{Id}_A \rightarrow F \circ G \) induces a family of surjective morphisms for all objects in \( A \).
Let us apply this lemma to $\mathcal{A} = H(W_0, W)$-mod, $f, d$ and $\mathcal{B} = Im(KZ_0)$. This proves the surjectivity of $\Phi$.

Since

$$P_{KZ_0} = \bigoplus_{E \in Irr(N_W(W_0))} \dim(KZ_0(L(E)))P(E)$$

we can calculate explicitly $\dim(\text{End}_{\mathcal{O}_W(W_0)}(P_{KZ_0})^{op})$.

We have $\dim(\text{End}_{\mathcal{O}_W(W_0)}(P_{KZ_0})^{op}) = \dim(H(W_0, W))$. This implies that $\Phi$ is an isomorphism. □

**Theorem 1.12.** The functor $KZ_0$ is an equivalence of categories.

**Proof.** The only thing which remains to prove is the essential surjectivity. The category $Im(KZ_0)$ is a full subcategory of $H(W_0, W)$-mod, closed under quotients, sub-object and direct sum. Since $H(W_0, W)$ is isomorphic to $\text{End}_{\mathcal{O}_W(W_0)}(P_{KZ_0})^{op}$, $H(W_0, W)$ is an object of $Im(KZ_0)$, so $H(W_0, W)^{ns}$ is an object of $Im(KZ_0)$. Then every $H(W_0, W)$-modules finitely generated lies into $Im(KZ_0)$. Therefore, $KZ_0$ is essentially surjective. □

1.2.1. Forgetting the ambient group $W$. We can provide a related result involving only $W_0$, and not the ambient group $W$. This is done in our article [Fal22]. The general setting is as follows. Let $G$ be a finite subgroup of $GL(V)$. Let $G_0$ be a normal subgroup of $G$ generated by reflexions. Let $R_0$ be the set of reflexions of $G_0$ and $\mathcal{A}_0$ the arrangement of reflecting hyperplanes of $G_0$. The first goal is to build up a Hecke algebra for $G$ from the Hecke algebra of $G_0$ generalizing $H(W_0, W)$ for $G = N_W(W_0)$.

Let $X^+$ be the subspace of $V$ on which $G$ acts freely and let $X_0$ be the subspace of $V$ on which $G_0$ acts freely. The manifold $X_0 \setminus X^+$ is of codimension $> 2$ then $\pi_1(X^+) \simeq \pi_1(X_0)$ [God71] theorem 2.3. We get two short exact sequences.

$$
\begin{array}{cccccc}
1 & \rightarrow & \pi_1(X^+) & \rightarrow & \pi_1(X^+/G_0) & \rightarrow & G_0 & \rightarrow & 1 \\
\downarrow & & \uparrow \simeq & & \uparrow & & \uparrow & & \uparrow \\
1 & \rightarrow & \pi_1(X_0) & \rightarrow & \pi_1(X_0/G_0) & \rightarrow & G_0 & \rightarrow & 1
\end{array}
$$

The exactness and the commutativity of the diagram together imply

$$\pi_1(X^+/G_0) \simeq \pi_1(X_0/G_0)$$

The braid group $B_0$ of $G_0$ is a normal subgroup of $B := \pi_1(X^+/G)$, we get a short exact sequence

$$
\begin{array}{cccccc}
1 & \rightarrow & B_0 := \pi_1(X_0/G_0) & \rightarrow & \pi_1(X^+/G) & \rightarrow & G/G_0 & \rightarrow & 1
\end{array}
$$

Let $I$ be the ideal of $CB_0$ generated by the relations $\sigma^m_H = \sum_{k=0}^{m-1} a_{H,k} \sigma^k_H$ for $\sigma_H$ a braided reflection associated to $H \in \mathcal{A}_0$. Then the Hecke algebra of $G_0$ is the quotient $H_0 := \frac{CB_0}{I}$. According to the now proven BMR freeness conjecture (see the references of [GM21] or its weaker version in Characteristic 0 [Eti17]) it is an algebra finitely generated of dimension $|G_0|$. Let $I^+ = CB \otimes_{CB_0} I$ be the ideal which define the Hecke algebra of $G$, $H(G) := \frac{CB_0}{I^+} \simeq CB \otimes_{CB_0} H_0$ is of dimension $|G|$.

Let us make a link between this new algebra and the algebra $H(W_0, W)$. We defined $H(W_0, W)$ as a quotient of the algebra $CB_0$. We defined $\tilde{B}_0$ as the quotient of $\pi_1(X/N_W(W))$ by $K := \text{Ker}(\pi_1(X) \rightarrow \pi_1(X_0))$. Since $X_0 \setminus X^+$ has codimension $> 2$

$$K = \text{Ker}(\pi_1(X) \rightarrow \pi_1(X_0)) \simeq \text{Ker}(\pi_1(X/N_W(W)) \rightarrow \pi_1(X^+/N_W(W)))$$

And $\tilde{B}_0 \simeq \pi_1(X^+/N_W(W))$ is our group $\pi_1(X^+/G) =: B$. As a result, the algebra $H(W_0, W)$ is the same as $H(G)$.

Let us consider the category $\mathcal{O}_{tor}$, the full subcategory of $\mathcal{O}$ of module annihilated by a power of $\delta_0 := \prod_{H \in \mathcal{A}_0} \alpha_H$. We have
Theorem 1.13. ([Fal22] theorem 5 ) \( KZ_0 \) is fully faithful and essentially surjective from the category \( \mathcal{O}^{\tor} \) to the category of finite dimension \( H(G) \)-modules.

A priori \( \mathcal{O}^{\tor} \) and \( \mathcal{O}_{\tor}^0 \) are different. Actually, we can prove that these two categories are the same [Fal22].

2. THE RATIONAL CHEREDNIK ALGEBRA OF THE PAIR \((\mathcal{L}, \mathcal{W})\) AND THE FUNCTOR \(\tilde{KZ}\).

2.1. The rational Cherednik algebra of the pair \((\mathcal{L}, \mathcal{W})\). Let us denote the Cherednik algebra of the pair \((\mathcal{L}, \mathcal{W})\) by \(\mathcal{A}(\mathcal{L}, \mathcal{W})\). As a vector space \(\mathcal{A}(\mathcal{L}, \mathcal{W})\) is \(\mathbb{C}[V] \otimes \mathcal{L} \otimes \mathcal{C}W \otimes \mathbb{C}[V^*]\).

Let us define a product of algebra by adding relations between generators of the sub-algebras \(\mathbb{C}[V], \mathbb{C}[V^*], \mathcal{C}L \rtimes \mathcal{W} : [x,x'] = 0\) for all \((x,x') \in V^* \times V^*\), \([y,y'] = 0\) for all \((y,y') \in V \times V\), \([\epsilon_H, \epsilon_{H'}] = 0\) for all \((H,H') \in \mathcal{L} \times \mathcal{L}\) and

\[
[y,x] = t.x(y) + \sum_{H \in A} \frac{\alpha_H(y)x(v_H)}{\alpha_H(v_H)} \gamma_H e_H
\]

where \(t \in \mathbb{C}\). Likewise to the case of \(\mathcal{A}(W_0, \mathcal{W})\), we just consider the case \(t \neq 0\). Up to renormalization, we can consider \(t = 1\).

2.1.1. Dunk operator. In [Mar18b] is introduced a differential 1-form, \(W\)-equivariant and integrable

\[
\tilde{\omega} := \sum_{H \in A} dH a_H e_H \in \Omega^1(X) \otimes \mathcal{C}L \otimes \mathcal{C}W
\]

We define the Dunkl operator as the covariant derivative associated to the connection \(\nabla := d + \tilde{\omega}\), in the direction of \(y \in V\).

Definition 2.1. (Dunkl operator of the pair \((\mathcal{L}, \mathcal{W}))\) For all \(y \in V\),

\[
\tilde{T}_y := \partial_y + \sum_{H \in A} \frac{\alpha_H(y)}{\alpha_H} a_H e_H \in (\mathcal{D}(X) \otimes \mathcal{C}L) \rtimes \mathcal{W}
\]

This family of differential operators satisfies the following properties:

Proposition 2.2. (1) For all \(y,y' \in V \times V\), \([\tilde{T}_y, \tilde{T}_{y'}] = 0\).

(2) For all \(y \in V\) and \(g \in W\), \(g.\tilde{T}_y.g^{-1} = \tilde{T}_{g(y)}\).

Proof. (1) We can follow the method of [EM10] owing to the fact that the action of \((\mathcal{D}(X) \otimes \mathcal{C}^1) \rtimes \mathcal{W}\) on \(\mathcal{C}(V)\) is faithful, with \(c \in \mathcal{L}/\mathcal{W}\). Let us prove a more general result. An outer automorphism of rings \(R\) is an automorphism of \(R\) which is not inner. If each non identity element of a group \(G\) induces an outer automorphism of \(R\), then \(G\) is called a group of outer automorphisms.

Proposition 2.3. Let \(R\) be a simple ring with unity. Let \(G\) be a group of outer automorphisms. Let \(X\) be a finite set. If \(G\) acts transitively on \(X\), then \((R \otimes \mathbb{C}^X) \rtimes G\) is simple.

Proof. Let us start by proving the reducibility of the support of an element of \((R \otimes \mathbb{C}^X) \rtimes G\). Let \(J\) be a two-sided ideal of \((R \otimes \mathbb{C}^X) \rtimes G\) not reduced to \(\{0\}\). There exists \(x \in J, x \neq 0\), of minimal support, \(x = \sum_{g \in G} r_g \cdot g = \sum_{g \in G \times X} r_{g \lambda} \cdot g \epsilon_{\lambda g}\). Since \(x\) is different from \(0\), \((r_g)_g\) are not all \(0\). There exists \(\lambda_0 \in X\) such that \(r_{\lambda_0, g \cdot \epsilon_{\lambda_0}} = 0\).

\[
g_0^{-1} x = \sum_{g \in G} g_0^{-1} r_g g = g_0^{-1} r_{g_0} g_0 + \sum_{g \in G, g \neq g_0} g_0^{-1} r_g g
\]

\[
= r_{1,1} + \sum_{g \in G, g \neq g_0} g_0^{-1} r_g g
\]
Up to multiply $x$ by $g_0^{-1}$, we can assume the existence of $\lambda_0 \in X$ such that $r_{1, \lambda_0, \epsilon_{\lambda_0}} \neq 0$. The ideal $Rr_{1, \lambda_0}R$ is a two-sided ideal of $R$ not reduced to $\{0\}$. Since $R$ is simple $Rr_{1, \lambda_0}R = R$, so there exists $(x_j, y_j) \in R \times R$, such that $\sum_j x_j r_{1, \lambda_0} y_j = 1$. All depends on $\lambda_0$. Let us cancel this dependency.

We get $r_1 = \sum_{\lambda \in X} r_{\lambda, 1, \epsilon_{\lambda}} = r_{\lambda_0, 1, \epsilon_{\lambda_0}} + \sum_{\lambda \in X, \lambda \neq \lambda_0} r_{\lambda, 1, \epsilon_{\lambda}}$. Let us consider $x' = x \cdot \epsilon_{\lambda_0} \in J \setminus \{0\}$. Then

$$x' = r_{\lambda_0, 1, \epsilon_{\lambda_0} \cdot 1} + \sum_{g \neq 1} r'_{g, g}$$

where $r'_{g, g} = \sum_{\lambda \in X} r_{\lambda, g, \epsilon_{\lambda}} \epsilon_{\lambda_0} = r_{\lambda_0, g, \epsilon_{\lambda_0}}$.

but $\sum_j x_j x' y_j = \sum_{g \in G, j} x_j r'_{g, g} g y_j = \sum_{g \in G, j} x_j r'_{g, g} g(y_j) g$. For $g = 1$, we have

$$P'_1 := \sum_j x_j r_{1, 1} y_j = \sum_j x_j r_{1, \lambda_0, 1} \epsilon_{\lambda_0} y_j = \epsilon_{\lambda_0}$$

We can reduce $P'_1$ to 1. Let

$$x'' := \sum_{h \in G} hx'' h = \sum_{h \in G} h (r'_1) h g h^{-1} = \sum_{h \in G} h (r'_1) + \sum_{h \in G, g \neq 1} h g h^{-1}$$

but $\sum_{h \in G} h (r'_1) = \sum_{h \in G} h \epsilon_{\lambda_0} = \sum_{h \in G} \epsilon_{h \cdot \lambda_0} = \text{cst.} \cdot 1$. Finally, the support of $\sum_j x_j x'' y_j$ is equal to the support of $x$ and

$$\sum_{j} x_j x'' y_j, x = \sum_{g \in G} P_{g}^" x r - r P_{g}^" g$$

$$= P_{1}^" r - r P_{1}^" + \sum_{g \in G, g \neq 1} (P_{g}^" x r - r P_{g}^" g)$$

At least, one term misses in the support of $[x, r]$, so by the minimality of the support of $x$, $[x, r] = 0$. So we can reduce the support of an element.

Let $\tilde{R} := R \otimes \mathbb{C}^X = \bigoplus_{\lambda \in X} R \cdot \epsilon_{\lambda}$. We have $[x, r] = \sum_{g \in G} (r_{g} g(r) - r, r_{g}) g = 0$, due to the fact that $(R \otimes \mathbb{C}^X) \times G$ is a $G$-module, where the elements of $G$ form a basis, $r_{g} g(r) = r, r_{g}$ for all $g \in G$ and $r \in R \otimes \mathbb{C}^X$. So $\tilde{R} g = \tilde{g} \tilde{R}$.

Let $g \in G$ such that $r_{g} \neq 0$, we can assume the existence of $\mu_0 \in X$ such that $r_{\mu_0, g} \neq 0$. Then $r_{g} \epsilon_{g(\lambda)} = \sum_{\lambda \in X} r_{\lambda, g, \epsilon_{g(\lambda)}} = r_{g(\lambda), g, \epsilon_{g(\lambda)}}$ and

$$\epsilon_{\lambda} r_{g(\lambda)} = \sum_{\lambda \in X} \epsilon_{\lambda} r_{\mu, g, \epsilon_{g(\lambda)}} = \sum_{\mu \in X} r_{\mu, g, \epsilon_{g(\lambda)}} = r_{\lambda, g, \epsilon_{g(\lambda)}}$$

For $\lambda = \mu_0$, we get $r_{g} \epsilon_{g(\mu_0)} = r_{g(\mu_0), g, \epsilon_{g(\lambda)}} = \epsilon_{\mu_0} r_{g}$; so $g(\mu_0) = \mu_0$. As a result, for all $\mu_0$ such that $r_{\mu_0, g} \neq 0$ then $g(\mu_0) = \mu_0$. Let $r$ be an element of $R \epsilon_{\mu_0}$, we can write $r = \rho \epsilon_{\mu_0}$ and $r_{\mu_0, g} \neq 0$,

$$r_{g} g(r) = r_{g} (g, \rho) g \epsilon_{g, \mu_0}$$

$$= \sum_{\mu \in X} r_{\mu, g, \epsilon_{g(\mu_0)}} g \rho \epsilon_{g(\mu_0)} = \sum_{\mu \in X} r_{\mu, g, \rho} g \epsilon_{g(\mu_0)} = g \cdot \rho r_{\mu_0, g} \epsilon_{\mu_0}$$

and

$$r^{2} \mu_{0} g = \rho \epsilon_{\mu_0} \sum_{\mu \in X} r_{\mu, g, \epsilon_{\mu}} = \sum_{\mu \in X} \rho r_{\mu, g} \epsilon_{\mu} = \rho r_{\mu_0, g} \epsilon_{\mu_0} = g \cdot \rho r_{\mu_0, g} \epsilon_{\mu_0}$$

Because of $r_{\mu_0, g} \rho = r_{\mu_0, g} \cdot R = R r_{\mu_0, g}$ then $r_{\mu_0, g}$ is two-sided invertible. The action of $g$ on $\rho$ would be by inner automorphism, which is absurd. □
Let us apply this result to $X = c \in \mathcal{L}/W$, $G = W$ and $R = \mathcal{D}(X)$. The algebra $(\mathcal{D}(X) \otimes \mathbb{C}^n) \rtimes W$ is simple, therefore $\bigoplus_{c \in \mathcal{L}/W} (\mathcal{D}(X) \otimes \mathbb{C}^n) \rtimes W$ is simple. But this algebra is $\mathcal{D}(X) \otimes \mathcal{L} \rtimes W$. So the action of $(\mathcal{D}(X) \otimes \mathcal{L}) \rtimes W$ on $\mathbb{C}(V)$ is fully faithful. The rest of the proof is similar to the proof for the theorem 2.16 in [EM10].

$\Box$

2.1.2. Dunkl embedding. We consider the filtration $(F_i)_{i \in \mathbb{Z}}$ on $(\mathcal{D}(X) \otimes \mathcal{L}) \rtimes W$ by the order of the differential operator. Let us define a graduation on $(\mathcal{L}, \mathcal{L})$ of generators of $(\mathcal{D}(X) \otimes \mathcal{L}) \rtimes W$, by

$$B_i = \sum_{\alpha, \beta, \lambda, g \in \mathbb{N}^n, \beta \in \mathbb{N}^m, \lambda \in \mathcal{L}, g \in W} a_{\alpha, \beta, \lambda, g} x^\alpha y^\beta e_\lambda g$$

for all $i \in \mathbb{Z}$. We get an isomorphism of algebras $\psi : \text{grad}_F(\mathcal{D}(X) \otimes \mathcal{L}) \rtimes W) \simeq (\mathbb{C}[X \oplus V^*] \otimes \mathcal{L}) \rtimes W$. Let us define a filtration $(A_i)_{i \in \mathbb{Z}}$ on $A(\mathcal{L}, W)$ by putting in degree 0, $\mathcal{L}$, $V^*$, $W$ and $V$ in degree 1. Let us define a graduation on $(\mathbb{C}[V \oplus V^*] \otimes \mathcal{L}) \rtimes W$ by

$$A_i := \sum_{\alpha, \beta, \lambda, g \in \mathbb{N}^n, \beta \in \mathbb{N}^m, \lambda \in \mathcal{L}, g \in W, |\beta| = i} a_{\alpha, \beta, \lambda, g} x^\alpha y^\beta e_\lambda g$$

We get an isomorphism of algebras $\Phi : \text{grad}_F(A(\mathcal{L}, W)) \simeq (\mathbb{C}[V \oplus V^*] \otimes \mathcal{L}) \rtimes W$.

**Theorem 2.4.** The application

$$\Phi : A(\mathcal{L}, W) \longrightarrow (\mathcal{D}(X) \otimes \mathcal{L}) \rtimes W$$

$$x \in V^* \longmapsto x$$

$$y \in V \longmapsto T_y$$

$$g \in W \longmapsto g$$

is an injective morphism of algebras. After localization, $\Phi$ becomes an isomorphism of algebras from $A(\mathcal{L}, W)_{\text{reg}}$ to $(\mathcal{D}(X) \otimes \mathcal{L}) \rtimes W$, denoted $\Phi_{\text{reg}}$.

**Proof.** We observe that $\Phi$ is a morphism of filtered algebras because of $\Phi(F_i) \subset F_i$. Therefore, $\Phi$ induces a morphism of algebras between graded algebras denoted by $\text{gr}(\Phi)$. Then we can prove that $\text{gr}(\Phi)$ is injective by considering the following composition

$$\begin{array}{ccc}
(\mathbb{C}[V \oplus V^*] \otimes \mathcal{L}) \rtimes W & \longrightarrow & \text{grad}_F(A(\mathcal{L}, W)) \\
& & \longrightarrow (\mathcal{D}(X) \otimes \mathcal{L}) \rtimes W
\end{array}$$

This is the identity. Therefore, $\text{gr}(\Phi)$ is injective, which implies that the morphism $\Phi$ is injective.

Since $\sum_{H \in \mathcal{A}} \frac{a_H(y)}{a_H} a_H e_H$ is well defined on $A(\mathcal{L}, W)_{\text{reg}}$, the image of $\Phi_{\text{reg}}$ contains a system of generators of $(\mathcal{D}(X) \otimes \mathcal{L}) \rtimes W$. $\Box$

2.1.3. Category $\mathcal{O}(\mathcal{L}, W)$. Let $e \in B := \sum_{y \in B} y^*y - \sum_{H \in \mathcal{A}} a_H e_H$ be the Euler element, where $\mathcal{B}$ is a basis of $V$.

**Proposition 2.5.** $[e, x] = x$ for all $x \in V^*$, $[e, y] = -y$ for all $y \in V$, $[e, e_H] = 0$ for all $e_H \in \mathcal{L}$, $[e, w] = 0$ for all $w \in W$.

Therefore, $e$ induces an inner graduation on $A(\mathcal{L}, W)$,

$$A(\mathcal{L}, W)^i := \{ a \in A(\mathcal{L}, W) | [e, a] = ia \}$$

for all $i \in \mathbb{Z}$.

The element $\sum_{H \in \mathcal{A}} a_H e_H$ belongs to $Z(\mathcal{L} \rtimes W)$. Thus, it acts by scalar multiplication on simple $(\mathcal{L} \rtimes W)$-modules. We denote by $c_E$ the associated scalar where $E$ is a simple $(\mathcal{L} \rtimes W)$-module.
Let us define a partial order on simple $\mathbb{C}L \times W$-modules by: $E' < E$ if $c_E - c_{E'} \in \mathbb{N}^*$. Let us define the standard objects associated to a simple $\mathbb{C}L \times W$-module, by $\Delta(E) := \text{Ind}_{\mathbb{C}[V^*]}^{\mathbb{C}[L \times W]} E$. This object admits a simple head $L(E)$ and each simple head admits a projective cover $P(E)$. Each projective cover $P(E)$ admits a standard filtration. We denote by $\Lambda$ the set of all simple heads.

The category $\mathcal{O}(L,W)$ is the full subcategory of $A(L,W)$-module finitely generated, locally nilpotent for the action of $\mathbb{C}[V^*]$ and isomorphic to the direct sum of their generalized $\mathfrak{sl}_n$-eigen spaces. The triple $(\mathcal{O}(L,W), \Lambda, <)$ is a highest weight category in the sense of $[\text{CPSS}SS]$. Then $\mathcal{O}(L,W)$ admits a quasi-hereditary cover, this means that there exists a finite dimensional $\mathbb{C}$-algebra $\tilde{R}$ such that $\mathcal{O}(L,W)$ is equivalent to the category of finitely generated $\tilde{R}$-modules.

2.2. The functor $\widehat{KZ}$. Let $M$ be a $A(L,W)$-module. Let $M_{tor} := \{m \in M | \exists n > 0 \delta^n m = 0\}$ and $(A(L,W)$-module)$_{tor} = \{M \in (A(L,W)$-mod)$| M_{tor} = M\}$. This is a Serre subcategory of $A(L,W)$-module.

Let $\text{Loc}: A(L,W)$-modules $\to A(L,W)_{reg}$-modules the localization functor, $\text{Loc}(M) = M_{reg} := \mathbb{C}[X] \otimes_{\mathbb{C}[V]} M$, this is an exact functor, because $\mathbb{C}[X]$ is a flat $\mathbb{C}[V]$-module and $\text{Loc}(M) = 0$ if and only if $M \in (A(L,W)$-mod)$_{tor}$. Therefore, the functor $\text{Loc}$ factorizes through the quotient category $\frac{A(L,W)_{mod}}{(A(L,W)$-mod)$_{tor}}$ and then the induced quotient functor $\frac{A(L,W)_{mod}}{(A(L,W)$-mod)$_{tor}} \to A(L,W)_{reg}$-mod is an equivalence of categories.

We can restrict this functor to the category $\mathcal{O}(L,W)$. Let us introduce the category $\mathcal{O}(L,W)_{tor} := \mathcal{O}(L,W) \cap (A(L,W)$-mod)$_{tor}$. This is a Serre subcategory of $\mathcal{O}(L,W)$.

The lemma 3.3 $[\text{Rou10}]$ implies that $\frac{\mathcal{O}(L,W)}{\mathcal{O}(L,W)_{tor}} \to \frac{A(L,W)_{mod}}{(A(L,W)$-mod)$_{tor}}$ is a fully faithful functor. By composing this with the previous equivalence of categories, we get a fully faithful functor $\mathcal{O}(L,W)_{tor} \to A(L,W)_{reg}$-mod. Then we apply the Dunkl embedding.

Let us figure out the structure of $(D(X) \otimes \mathbb{C}L) \times W$-module, on a standard object $\Delta(E)$, $E \in Irr(\mathbb{C}L \times W)$. The localization $\Delta(E)_{reg}$ of $\Delta(E)$ correspond to a trivial vector bundle over $X$ of dimension $\text{dim}(E)$. We can endow this vector bundle with a connection by considering the action of $\hat{T}_y$ on an element $P \otimes v \in \Delta(E)_{reg}$. We get the formula

$$\nabla_y (P \otimes v) := \partial_y P \otimes v + \sum_{H \in A} \frac{\alpha_H(y)}{\alpha_H} \sum_{j=0}^{m_H-1} m_H k_{H,j} P \otimes \epsilon_{H,j} v$$

Proposition 2.6. $\nabla_y$ is an algebraic, flat and $W$-equivariant connection with regular singularities on $V$.

Proof. The proof is similar to that proposed for the $A(W_0, W)$ case. \hfill $\square$

Since the connection $\nabla_y$ has regular singularities, we can apply the Riemann-Hilbert-Deligne correspondence and we get a $\mathbb{C} \pi_1(X/W) \times \mathbb{C}L$-mod$_{f, d}$. The category of connection with regular singularities is thick and for all simple $\mathbb{C}L \times W$-modules $E$ we get a short exact sequence

$$\Delta(E) \to L(E) \to 0$$

So $L(E)$ is endowed with a connection with regular singularities. Every object of $\mathcal{O}(L,W)$ admits a finite Jordan-Hölder series, so all objects of $\mathcal{O}(L,W)$ can be endowed with a connection with regular singularities.

According to proposition 5.6 and 5.7 $[\text{GM21}]$ this monodromy action factorizes through $\mathcal{C}(L,W)$. We obtain an exact functor

$$\widehat{KZ}: \mathcal{O}(L,W) \to \mathcal{C}(L,W)$-mod$_{f,d}$

$$M \mapsto (((M_{reg})^W)_{an})^\nabla$$
According to [1.9] the functor $\widetilde{KZ}$ is representable by a projective object in $\mathcal{O}(\mathcal{L}, \mathcal{W})$, denoted by $P_{\widetilde{KZ}}$. The image of the functor $\widetilde{KZ}$ is a full abelian subcategory of the category of $\mathcal{C}(\mathcal{L}, \mathcal{W})$-modules finitely generated, closed under quotient, sub-objects and direct sum. We get the result:

**Proposition 2.7.** The morphism $\Phi : \mathcal{C}(\mathcal{L}, \mathcal{W}) \to \text{End}_{\mathcal{O}(\mathcal{L}, \mathcal{W})}(P_{\widetilde{KZ}})$ is an isomorphism of algebras.

**Theoreme 2.8.** The functor $\widetilde{KZ}$ is essentially surjective. Hence, the induced functor $\widetilde{KZ} : \mathcal{O}(\mathcal{L}, \mathcal{W})_{\text{tor}} \to \mathcal{C}(\mathcal{L}, \mathcal{W})$-modules f.d is an equivalence of categories.

The proofs of these results are similar to those in the previous section.

**Acknowledgements:**

These results are part of my PhD-Thesis at University Picardie Jules Verne under the supervision of Prof. Ivan Marin. I would like to thank Cedric Bonnafe, which suggested forgetting the ambient group $W$.

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