Analysis of A New Time Filter Algorithm for The Unsteady Stokes/Darcy Model

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Abstract

This paper first presents a first order \( \theta \)-scheme which is one parameter family of Linear Multistep Methods for the unsteady Stokes/Darcy model. Then, on the basis of this scheme, we use a time filter algorithm to increase the convergence order to the second order with almost no increasing the amount of computation, from which we get a new efficient algorithm. Finally, we theoretically analyze the stabilities and error estimations of coupled and decoupled schemes respectively, and we also do some numerical experiments to verify the effectiveness, convergence and efficiency.

Keywords: coupled Stokes/Darcy model, Linear Multistep method, time filter, increase accuracy

AMS Subject Classification: 76D05, 76S05, 76D03, 35D05

1 Introduction

In recent years, the coupling problem between free flow and porous media flow has attracted more and more attention. This coupling flow appears in many fields, such as environmental problems of groundwater pollution, transport problems between surface water and groundwater, protection of karst aquifers, exploitation of fracture-vuggy reservoirs, and some technologies related to fluid filtration in industry. These problems are also quite closely related to our life. The Navier-Stokes/Darcy and Stokes/Darcy model are very important for these problems in the research field, and many researchers have carried out in-depth studies on these two models. In this paper, we will focus on the Stokes/Darcy model.

A lot of work has been done on the Stokes/Darcy model. There are numerical methods for the steady Stokes/Darcy model including finite element methods [1, 2, 3, 4, 5], discontinuous Galerkin methods [6, 7, 8, 9], interface relaxation methods [10, 11], Lagrange multiplier methods [12, 13, 14, 15], domain decomposition methods [16, 17, 18, 19], two-grid or multi-grid methods [20, 21, 22, 23], and so on. At present, researchers pay more and more attention to the unsteady model and adopt different methods to obtain time semi-discrete results. Such as the partitioned methods [24, 25, 26, 27], the monolithic discretization methods [28, 29, 30]. All the methods mentioned in the paper are related to theoretical analysis and numerical experiments. However, there are still some shortcomings in these methods, for example, some numerical methods will consume a lot to produce relatively high accuracy, and the results of some algorithms must be limited to the time step under certain conditions, and the stability of some algorithms can only be obtained by adding a stabilization term. Therefore, we need to give more effective algorithms for existing problems and strive to improve the accuracy of algorithms.

In this paper, we first give a first order \( \theta \)-scheme \( (0 < \theta < 1/2) \), which is a parameter family of Linear Multistep methods for the unsteady Stokes/Darcy model. Now time filters are widely used and can be easily modified and implemented in programs, and the accuracy of algorithms can be improved. So based on the first order Linear Multistep method, we think about the effect of adding a simple time filter (see[31]) for the unsteady Stokes/Darcy model. The method is modular and need to add only one additional line of code, which increases the accuracy of the Linear Multistep Method from first to second order. Then stability analysis and
error estimations are performed for both coupled and decoupled schemes respectively. Finally, we do numerical experiments, in the first test, we verify the stabilities of the coupled and decoupled schemes. In the second test, we show that the convergence orders of the coupled and decoupled schemes are increased from the first order to the second order, and by comparing the coupled and decoupled schemes, we get that the decoupled scheme is more computationally efficient.

The rest of the paper is organized as follows. In Section 2, we introduce the Stokes/Darcy model with the Beavers-Joseph Saffman (BJS) interface condition and give the weak formulation. The coupled and decoupled Linear Multistep method plus time filter and their stabilities are given in Section 3. Section 4 is devoted to the error estimations of the time semi-discrete scheme for the coupled and decoupled algorithms. Finally, we present numerical tests to illustrate the theoretical analysis in Section 5.

2 The Stokes-Darcy model and weak formulation

The mixed model for coupling a free fluid flow and a porous media flow is discussed in a bounded smooth domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Here $\Omega = \Omega_f \cup \Gamma \cup \Omega_p$, where $\Omega_f$ and $\Omega_p$ are connected, disjoint and bounded domains of the fluid flow and the porous media flow motion respectively. $\Gamma = \partial \Omega_f \cap \partial \Omega_p$ is the interface. We denote $\Gamma_f = \partial \Omega_f \cap \partial \Omega$ and $\Gamma_p = \partial \Omega_p \cap \partial \Omega$ and we also denote by $n_f$ and $n_p$ the unit outward normal vectors on $\partial \Omega_f$ and $\partial \Omega_p$, respectively. See Figure 1 for a sketch.

![Figure 1: A global domain $\Omega$ consisting of a free fluid flow region $\Omega_f$ and a porous media flow region $\Omega_p$ separated by an interface $\Gamma$.](image)

We consider the Stokes/Darcy model consisting of Stokes equations and Darcy equations. Let $T > 0$ be a finite time. The motion of flow in the free fluid region $\Omega_f$ is given by the Stokes equations: find the fluid flow velocity $u_f : \Omega_f \times [0, T] \to \mathbb{R}^d$, the pressure $p_f : \Omega_f \times [0, T] \to \mathbb{R}^d$ satisfying

$$\frac{\partial u_f}{\partial t} = -\nabla p_f + \nu D(u_f), \quad \text{in } \Omega_f \times (0, T], \quad (2.1)$$

$$\nabla \cdot u_f = 0, \quad \text{in } \Omega_f \times (0, T], \quad (2.2)$$

where

$$T(\nu) = -\nu I + 2\nu D(u_f), \quad D(u_f) = \frac{1}{2}(\nabla u_f + \nabla^T u_f),$$

are the stress tensor and the deformation rate tensor, $\nu > 0$ is the kinetic viscosity and $g_f(x, t)$ is the external force.

The motion of flow in the porous media region $\Omega_p$, is given by the following equations:

$$\frac{S \partial \phi_p}{\partial t} + \nabla \cdot u_p = g_p, \quad \text{in } \Omega_p \times (0, T], \quad (2.3)$$

$$u_p = -K \nabla \phi_p, \quad \text{in } \Omega_p \times (0, T], \quad (2.4)$$

where $S$ is the specific mass storativity coefficient and $\phi_p = z + \frac{p_p}{\rho g}$ is the piezometric (hydraulic) head, where $p_p$ denotes the dynamic pressure, $z$ the height from a reference level, $\rho$ the density and $g$ the gravitational constant. $u_p$ is the flow velocity in the porous medium which is proportional to the gradient of $\phi_p$, namely, the Darcy’s law. $K$ represents the hydraulic conductivity in $\Omega_p$, which is a positive symmetric tensor and is allowed to vary in space, and $g_p(x, t)$ is a source term.
Substituting the second equation (2.4) into the first equation (2.3), we get the Darcy equations: find hydraulic head \( \phi_p : \Omega_p \times [0, T] \to \mathbb{R}^d \) satisfying
\[
\frac{S \partial \phi_p}{\partial t} - \nabla \cdot (K \nabla \phi_p) = g_p, \quad \text{in} \ \Omega_p \times (0, T). 
\] (2.5)

Then we give the following initial conditions:
\[
\begin{align*}
u_f(x, 0) &= u_0^f(x), & \text{in } \Omega_f, \\
\phi_p(x, 0) &= \phi_0^p(x), & \text{in } \Omega_p.
\end{align*}
\] (2.6)

And the following homogeneous Dirichlet boundary conditions:
\[
\begin{align*}
u_f &= 0, & \text{on } \Gamma_f, \\
\phi_p &= 0, & \text{on } \Gamma_p.
\end{align*}
\] (2.7)

The interface conditions are the conservation of mass, the balance of normal forces and the Beavers-Joseph-Saffman (BJS) condition on \( \Gamma \):
\[
\begin{align*}
u_f \cdot \mathbf{n}_f - K \nabla \phi_p \cdot \mathbf{n}_p &= 0, \\
- [\mathbf{T}_f](\nu_f, p_f) \cdot \mathbf{n}_f &= g \phi_p, \\
- [\mathbf{T}_f](\nu_f, p_f) \cdot \mathbf{n}_f \cdot \tau_i &= \frac{K \sigma}{\sqrt{\text{trace}(\Pi)}} u_f \cdot \tau_i.
\end{align*}
\] (2.8)

Here \( \tau_i, i = 1, 2, \ldots, d - 1 \), are the orthonormal tangential unit vectors along \( \Gamma \), \( \alpha \) is an experimentally determined parameter and \( \Pi \) represents the permeability, which has the following relation with the hydraulic conductivity, \( K = \frac{\Pi}{\alpha} \).

Now we give the following Hilbert spaces:
\[
\begin{align*}
\mathbf{H}_f &= \{ \mathbf{v}_f \in (H^1(\Omega_f))^d : \mathbf{v}_f |_{\Gamma_f} = 0 \}, \\
\mathbf{H}_p &= \{ \psi_p \in (H^1(\Omega_p))^d : \psi_p |_{\Gamma_p} = 0 \}, \\
Q_f &= L^2(\Omega_f), \\
U &= \mathbf{H}_f \times \mathbf{H}_p.
\end{align*}
\]

For the domain \( D \), \(( \cdot, \cdot)_D\) refers to the scalar inner product in \( D = \Omega_f \) or \( \Omega_p \). We equip the space \( U \) with the following norms: \( \forall \mathbf{u} = (\mathbf{u}_f, \phi_p) \in U \)
\[
\begin{align*}
\| \mathbf{u} \|_0 &= \sqrt{(u_0^f, u_0^f)_{\Omega_f} + g^2 (\phi_0^p, \phi_0^p)_{\Omega_p}}, \\
\| \mathbf{u} \|_U &= \sqrt{\nu (\nabla u_f, \nabla u_f)_{\Omega_f} + g (K(\nabla \phi_p, \nabla \phi_p)_{\Omega_p}}.
\end{align*}
\]

We denote the \( H^1(\Omega_f; p) \) norm by \( \| \cdot \|_{\mathbf{H}_f; H_p} \), the \( L^2(\Omega_f; p) \) norm by \( \| \cdot \|_{J_f} \) and the \( L^2(\Gamma) \) norm by \( \| \cdot \|_{\Gamma} \), and define the corresponding norms:
\[
\begin{align*}
\| u_f \|_f &= \| u_f \|_{L^2(\Omega_f)}, \quad \| u_f \|_{\mathbf{H}_f} = \| \nu^{\frac{1}{2}} \nabla u_f \|_{L^2(\Omega_f)}, \\
\| \phi_p \|_p &= \| \phi_p \|_{L^2(\Omega_p)}, \quad \| \phi_p \|_{\mathbf{H}_p} = \| \nu^{\frac{1}{2}} \nabla \phi_p \|_{L^2(\Omega_p)}.
\end{align*}
\]

For functions \( v(x, t) \), we define the norms:
\[
\begin{align*}
\| v \|_{L^2(0, T; L^2)} &= ( \int_0^T \| v(\cdot, t) \|_{L^2}^2 dt )^{\frac{1}{2}}, \\
\| v \|_{L^2(0, T; L^\infty)} &= \text{ess sup}_{[0, T]} \| v(\cdot, t) \|_{L^\infty}.
\end{align*}
\]

Now the weak formulation of the mixed Stokes/Darcy model (2.1), (2.2), (2.3), (2.5), (2.6), (2.7) and (2.8) reads as follows: \( g_f \in L^2(0, T; L^2(\Omega_f)) \) and \( g_p \in L^2(0, T; L^2(\Omega_p)) \), find \( \mathbf{u} = (u_f, \phi_p) \in \text{L}^2(0, T; H_f) \cap \text{L}^\infty(0, T; L^2(\Omega_f)) \times L^2(0, T; H_p) \cap \text{L}^\infty(0, T; L^2(\Omega_p)) \) and \( p_f \in L^2(0, T; Q_f) \) such that \( \forall (\mathbf{v}, q_f) \in U \times Q_f \)
\[
\begin{align*}
\left( \frac{\partial u_f}{\partial t}, \mathbf{v} \right) + a(u_f, \mathbf{v}) + b(\mathbf{v}, p_f) &= \mathbf{F}, \quad \mathbf{v} \in U, \\
b(u_f, q_f) &= 0, \\
u(x, 0) &= u_0^f, \\
(\partial u_f / \partial t, \mathbf{v}) + a(u_f, \mathbf{v}) + b(\mathbf{v}, p_f) &= \mathbf{F}, \quad \mathbf{v} \in U,
\end{align*}
\] (2.9)
where
\[
\frac{\partial \mathbf{u}}{\partial t} - \nabla p = (\nabla \phi, \mathbf{v}) + g(\nabla \phi, \nabla \mathbf{v}) + g(\phi, \mathbf{v}), \\
a_r(\mathbf{u}, \mathbf{v}) = a_\Omega(\mathbf{u}, \mathbf{v}) + a_T(\mathbf{u}, \mathbf{v}), \\
a_\Omega(\mathbf{u}, \mathbf{v}) = a_\Omega_r(\mathbf{u}, \mathbf{v}) + a_\Omega_p(\phi, \psi),
\]
\[
a_\Omega_r(\mathbf{u}, \mathbf{v}) = \nu(D(\mathbf{u}), D(\mathbf{v}))_\Omega + \left(\frac{\alpha_\Omega \sqrt{\delta}}{\text{trace}(\mathbf{I})}\right) P_T(\mathbf{u}, \mathbf{v})_\Gamma,
\]
\[
a_\Omega_p(\phi, \psi) = g(\nabla \phi, \nabla \psi)_\Omega,
\]
\[
b(\mathbf{v}, p) = -(p f, \nabla \cdot \mathbf{v})_\Omega,
\]
and \( \mathbf{U} \) is the dual space of \( U \), \( P_T(\cdot) \) is the projection onto the local tangential plane that can be explicitly expressed as \( P_T(\mathbf{v}) = \mathbf{v} - (\mathbf{v} \cdot \mathbf{n}) \mathbf{n} \).

It is clear that the bilinear form \( a(\cdot, \cdot) \) is continuous and coercive:
\[
a(\mathbf{u}, \mathbf{v}) \leq C_{\text{con}} \| \mathbf{u} \|_U \| \mathbf{v} \|_U, \quad \forall \mathbf{u}, \mathbf{v} \in U,
\]
\[
a(\mathbf{u}, \mathbf{u}) \geq C_{\text{co}} \| \mathbf{u} \|_U^2, \quad \forall \mathbf{u} \in U,
\]
\[
a_\Omega_r(\mathbf{u}, \mathbf{u}) \geq \tilde{C}_{\text{co}} \| \mathbf{u} \|_H^2, \quad \forall \mathbf{u} \in H,
\]
\[
a_\Omega_p(\phi, \phi) \geq \tilde{C}_{\text{co}} \| \phi \|_H^2, \quad \forall \phi \in H.
\]

Furthermore,
\[
a_T(\mathbf{u}, \mathbf{v}) = -a_T(\mathbf{v}, \mathbf{u}) \quad \text{and} \quad a_T(\mathbf{u}, \mathbf{u}) = 0, \quad \forall \mathbf{u}, \mathbf{v} \in U. \quad (2.10)
\]

The well-posedness of the mixed Stokes-Darcy model can be found in papers for the stationary case and is assumed to hold similarly for the non-stationary case. In this paper, we focus on its numerical solution.

For the purpose of later analysis, we need recall the Poincaré, trace and Sobolev inequalities: there exist constants \( C_p, C_t \) and \( \tilde{C} \) which only depend on the domain \( \Omega_f \) and \( \tilde{C}_p, C_t \) and \( \tilde{C} \) which only depend on the domain \( \Omega_p \) such that for all \( \mathbf{v}_f \in H_f \) and \( \psi_p \in H_p \),
\[
\| \mathbf{v}_f \|_f \leq C_p \| \nabla \mathbf{v}_f \|_f, \quad \| \psi_p \|_f \leq \tilde{C}_p \| \nabla \psi_p \|_p,
\]
\[
\| \mathbf{v}_f \|_f \leq C_t \| \nabla \mathbf{v}_f \|_f, \quad \| \psi_p \|_f \leq \tilde{C}_t \| \nabla \psi_p \|_p.
\]

\[
(2.11)
\]

\section{Numerical algorithms and stabilities}

For any given small parameter \( h > 0 \), we construct the regular triangulations \( T_h, T_{fh} \) and \( T_{ph} \) of \( \Omega_f \) and \( \Omega_p \). For simplicity, we assume that \( \Omega_f \) and \( \Omega_p \) are smooth domains. Let MINI elements(P1b-P1) \( H_{fh} \subset H_f \), \( Q_{fh} \subset Q_f \) and the linear Lagrangian elements(P1) \( H_{ph} \subset H_p \) are finite element spaces such that the space pair \( (H_{fh}, Q_{fh}) \) satisfies the discrete LBB condition:
\[
\inf_{q_{fh} \in Q_{fh}} \sup_{\mathbf{v}_{fh} \in H_{fh}} \frac{(q_{fh}, \nabla \cdot \mathbf{v}_{fh})_\Gamma}{\| q_{fh} \|_{Q_f} \| \mathbf{v}_{fh} \|_{H_f}} \geq \beta, \quad (3.1)
\]
and define \( U_h = H_{fh} \times H_{ph} \).

We define the linear projection operator (see \( \tilde{\text{33}} \)): for all \( \mathbf{v}_h \in U_h \), \( q_{fh} \in Q_{fh} \) and \( t \in (0, T] \), \( P_h : (\mathbf{u}(t), p_f(t)) \in U \times Q_f \rightarrow (P_h^u \mathbf{u}(t), P_h^p p_f(t)) \in U_h \times Q_{fh} \) satisfies
\[
a(P_h^u \mathbf{u}(t), \mathbf{v}_h) + b(\mathbf{v}_h, P_h^p p_f(t)) = a(\mathbf{u}(t), \mathbf{v}_h) + b(\mathbf{v}_h, p_f(t)),
\]
\[
b(P_h^p p_f(t), q_{fh}) = 0. \quad (3.2)
\]

Besides, we assume that \( (\mathbf{u}(t), p_f(t)) \) is smooth enough and the projection \( (P_h^u \mathbf{u}(t), P_h^p p_f(t)) \) of \( (\mathbf{u}(t), p_f(t)) \) satisfies following approximation properties:
\[
\| P_h^u \mathbf{u}(t) - \mathbf{u}(t) \|_0 \leq C h^2 \| \mathbf{u}(t) \|_{H^2},
\]
\[
\| P_h^u \mathbf{u}(t) - \mathbf{u}(t) \|_U \leq C h \| \mathbf{u}(t) \|_{H^2},
\]
\[
\| P_h^p p_f(t) - p_f(t) \|_{L^2} \leq Ch^2 \| p_f(t) \|_{H^1}. \quad (3.3)
\]
Furthermore, we will use the inverse inequalities: there exist constants $C_1$ and $\tilde{C}_1$, which depend on the angles in the finite element mesh, such that for all $v_{fh} \in H_{fh}$ and $\psi_{ph} \in H_{ph}$,
\[
\|v_{fh}\|_{H_1} \leq C_1 h^{-1} \|v_{fh}\|_f, \quad \|\psi_{ph}\|_{H_p} \leq \tilde{C}_1 h^{-1} \|\psi_{ph}\|_p.
\] (3.4)

Now we give two Lemmas that will be used in later analyses.  

**Lemma 3.1.** Let $(\rho, \sigma)$ be the two-step A-stable scheme, and let $\delta = \beta_2 - \frac{3\alpha}{2} > 0$. Then the coefficients $\alpha_i$ and $\beta_i$ satisfy the following relation:
\[
2 \left( \sum_{i=0}^{2} \alpha_i \zeta_i \right) \left( \sum_{i=0}^{2} \beta_i \zeta_i \right) \geq (\alpha_2 + \delta) \zeta_2^2 - (2\alpha_2 - 1) \zeta_1^2 - ((\alpha_2 - 1)^2 + \delta) \zeta_0^2 - 2(\alpha_2 - 1 + \delta) (\zeta_2 \zeta_1 - \zeta_1 \zeta_0), \quad \forall \zeta_0, \zeta_1, \zeta_2 \in \mathbb{R}.
\]

**Lemma 3.2.** for all $v_{fh} \in H_{fh}, \phi_{ph} \in H_{ph}$, there exists $C_k > 0$ such that $\forall \epsilon > 0$,
\[
|\epsilon_1(v_{fh}, \phi_{ph})| \leq \frac{1}{4\epsilon} \|v_{fh}\|_{H_1}^2 + C \|\phi_{ph}\|_{H_p}^2.
\] (3.5)

In addition, for all $v_{fh} \in H_{fh}, \phi_{ph} \in H_{ph}$, there exists $\tilde{C}_k > 0$ such that $\forall \bar{\epsilon} > 0$,
\[
|\epsilon_1(v_{fh}, \phi_{ph})| \leq \frac{1}{4\bar{\epsilon}} \|v_{fh}\|_{H_1}^2 + C \bar{\epsilon} h^{-1} \|\phi_{ph}\|_{H_p}^2.
\] (3.6)

In this section, we propose the coupled and decoupled schemes for the unsteady Stokes-Darcy model. We choose a uniform distribution of discrete time levels with $t_m = m \Delta t, m = 0, 1, \ldots, N$, where $\Delta t = \frac{T}{N}$ is the time step, here $(u_{fh}^{m+1}, p_{fh}^{m+1}) = (u_{fh}^{m+1}, p_{fh}^{m+1}, \phi_{ph}^{m+1})$ denotes the full discrete approximation by the scheme to $(u_h(t_{m+1}), p_{fh}(t_{m+1})) = (u_{fh}(t_{m+1}), p_{fh}(t_{m+1}), \phi_{ph}(t_{m+1}))$.

### 3.1 The coupled scheme

The coupled Linear Multistep method plus time filter is described below.

**The Linear Multistep method (First Order):**

Given $(\bar{u}_{fh}^0, \bar{p}_{fh}^0)$ and $(\bar{u}_{fh}^1, \bar{p}_{fh}^1)$, find $\bar{u}_{fh}^m = (\bar{u}_{fh}^{m+1}, \bar{p}_{fh}^{m+1}) \in U_f, (\bar{p}_{fh}^{m+1}, \phi_{ph}^{m+1}) \in Q_{fh}$ with $m = 1, 2, \ldots, N - 1$, such that $\forall \bar{u}_h \in U_h$ and $\forall q_{fh} \in Q_{fh}$,
\[
\begin{align*}
\left( \bar{u}_{fh}^{m+1} - \bar{u}_{fh}^m, \bar{u}_h \right) + a_1 (1 - \theta) \bar{u}_{fh}^m + \theta \bar{u}_{fh}^{m+1} + \bar{u}_h, b_1 (1 - \theta) \bar{p}_{fh}^{m+1} + \theta \bar{p}_{fh}^m, \bar{u}_{fh}^m, \bar{u}_h \right) &= < (1 - \theta) F^{m+1} + \theta F^m, \bar{u}_h >, \\
b_1 (1 - \theta) \bar{u}_{fh}^m + \bar{u}_{fh}^{m+1} + \bar{p}_{fh}^m &= 0.
\end{align*}
\] (3.7)

**The Time Filter (Second Order):**

Update the previous solutions $(\bar{u}_{fh}^{m+1}, \bar{p}_{fh}^{m+1})$ by time filter,
\[
\begin{align*}
\bar{u}_{fh}^{m+1} &= \bar{u}_{fh}^m - \frac{1 - 2\theta}{3 - 2\theta} \bar{u}_{fh}^{m+1} - 2\bar{u}_{fh}^m + \bar{u}_{fh}^{m-1}, \\
\bar{p}_{fh}^{m+1} &= \bar{p}_{fh}^m - \frac{1 - 2\theta}{3 - 2\theta} \bar{p}_{fh}^{m+1} - 2\bar{p}_{fh}^m + \bar{p}_{fh}^{m-1}.
\end{align*}
\] (3.8)

**Remark 1:** Whether the pressure $\bar{p}_{fh}^{m+1}$ is filtered or not has little effect for the results.

The stability theorem for the coupled Linear Multistep method plus time filter is introduced below.

**Theorem 3.1.** (coupled stability) Let $u_{fh}^{m+1}$ be the solution of the Linear Multistep method plus time filter. For $N \geq 2$, we have
\[
\|u_{fh}^N\|_U^2 + \frac{4(1 - \theta) C_2 \Delta t}{3 - 2\theta} \sum_{m=1}^{N-1} \|u_{fh}^m\|_U^2 \leq C \left( \|g_f\|^2_{L^2(0,T;L^2(\Omega_f))} + \|g_p\|^2_{L^2(0,T;L^2(\Omega_p))} + \|u_{fh}^1\|^2_0 + \|u_{fh}^0\|^2_0 + \|u_{fh}^0\|_0 \|u_{fh}^0\|_0 \right),
\]
where $C$ is a positive constant, which is independent of $h, \Delta t$ or other parameters, and it may have different values at different occasions.
Proof. From (3.8), we get
\[
\dot{u}^{m+1}_h = \frac{3 - 2\theta}{2} u^{m+1}_h - \frac{1}{2} \frac{u^m_h}{2} - (1 - 2\theta) u^m_h + \frac{1 - 2\theta}{2} u^{m-1}_h,
\]
\[
\dot{p}^{m+1}_f = \frac{3 - 2\theta}{2} p^{m+1}_f - (1 - 2\theta) p^m_f + \frac{1 - 2\theta}{2} p^{m-1}_f,
\]
then take them into (3.7)
\[
\left(\frac{3 - 2\theta}{2} u^{m+1}_h - 4(1 - \theta) u^m_h + (1 - 2\theta) u^{m-1}_h, \psi_h\right)_{2\Delta t}
+ a \left(\frac{1 - \theta}{2}(3 - 2\theta) u^{m+1}_h + (4\theta - 1 - 2\theta^2) u^m_h + \frac{1 - \theta}{2} (1 - 2\theta) u^{m-1}_h, \psi_h\right)
+ b \left(\frac{1 - \theta}{2}(3 - 2\theta) p^{m+1}_f + (4\theta - 1 - 2\theta^2) p^m_f + \frac{1 - \theta}{2} (1 - 2\theta) p^{m-1}_f, q^m_f\right) = 0,
\forall \psi_h \in U_h,
\forall q^m_f \in Q_f.
\]
Setting \(\psi_h = 2\Delta t \left\{ \frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_h + (4\theta - 1 - 2\theta^2) u^m_h + \frac{(1 - \theta)(1 - 2\theta)}{2} u^{m-1}_h \right\} \) and observing the first equation in (3.9), we use Lemma 3.1 because of \(\frac{(1 - \theta)(3 - 2\theta)}{2} - \frac{3 - 2\theta}{2} > 0,\) the first term on the left-hand side becomes
\[
2 \left(\frac{3 - 2\theta}{2} u^{m+1}_h - 2(1 - \theta) u^m_h + \frac{1 - 2\theta}{2} u^{m-1}_h, (1 - \theta)(3 - 2\theta) u^{m+1}_h + (4\theta - 1 - 2\theta^2) u^m_h + \frac{1 - \theta}{2} (1 - 2\theta) u^{m-1}_h \right)
\geq (2\theta^2 - 5\theta + 3) \|u^{m+1}_h\|^2 - 2(1 - \theta) \|u^m_h\|^2 - (1 - 2\theta)(1 - \theta) \|u^{m-1}_h\|^2
- (1 - 2\theta)(3 - 2\theta) \|u^{m+1}_h\|_U \|u^m_h\|_U - \|u^m_h\|_U \|u^{m-1}_h\|_U.
\]
Then using the coercivity of the bilinear form \(a(\cdot, \cdot),\) we get
\[
2\Delta t a \left(\frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_h + (4\theta - 1 - 2\theta^2) u^m_h + \frac{(1 - \theta)(1 - 2\theta)}{2} u^{m-1}_h, \right)
\geq 2C_{\text{coec}} \Delta t \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_h + (4\theta - 1 - 2\theta^2) u^m_h + \frac{(1 - \theta)(1 - 2\theta)}{2} u^{m-1}_h \right\|_{U}^2.
\]
At last, for the forcing term on the right-hand side, we get
\[
2 \Delta t \left\{ \frac{(1 - \theta)}{2} F^{m+1}_h + \frac{(1 - \theta)}{2} F^m_h + \frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_h + (4\theta - 1 - 2\theta^2) u^m_h + \frac{(1 - \theta)(1 - 2\theta)}{2} u^{m-1}_h \right\}
\leq 2\theta^2 \Delta t \frac{C_{\text{coec}}}{C_{\text{coec}}} \frac{\|F^m\|^2}{\|F^m\|^2_U} + \frac{2(1 - \theta)^2 \Delta t}{C_{\text{coec}}} \frac{\|F^{m+1}\|^2}{\|F^{m+1}\|^2_U}
+ C_{\text{coec}} \Delta t \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_h + (4\theta - 1 - 2\theta^2) u^m_h + \frac{(1 - \theta)(1 - 2\theta)}{2} u^{m-1}_h \right\|_{U}^2.
\]
Combining the (3.10) and (3.12) and sum them over \(m = 1, 2, \ldots N - 1,\) we get
\[
(2\theta^2 - 5\theta + 3) \sum_{m=1}^{N-1} \|u^{N-1}_h\|^2 + (1 - 2\theta)(1 - \theta) \|u^{N-1}_h\|^2 - (1 - 2\theta)(3 - 2\theta) \|u^{N-1}_h\|_U \|u^{N-1}_h\|_0
+ C_{\text{coec}} \Delta t \sum_{m=1}^{N-1} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_h + (4\theta - 1 - 2\theta^2) u^m_h + \frac{(1 - \theta)(1 - 2\theta)}{2} u^{m-1}_h \right\|_{U}^2
\leq 2\theta^2 \Delta t \frac{C_{\text{coec}}}{C_{\text{coec}}} \sum_{m=1}^{N-1} \|F^m\|^2 + \frac{2(1 - \theta)^2 \Delta t}{C_{\text{coec}}} \sum_{m=1}^{N-1} \|F^{m+1}\|^2
+ (2\theta^2 - 5\theta + 3) \sum_{m=1}^{N-1} \|u^m\|^2 + (1 - 2\theta)(1 - \theta) \|u^m\|^2 - (1 - 2\theta)(3 - 2\theta) \|u^m\|_U \|u^m\|_0
- \|u^m\|_U \|u^{m-1}\|_U - \|u^{m-1}\|_U \|u^{m-2}\|_U.
\]
Then note that

\[-(1 - 2\theta)(3 - 2\theta)\|\mathbf{u}^N_n\|_0\|\mathbf{u}^{N-1}_n\|_0 \geq -\frac{(1 - 2\theta)(3 - 2\theta)^2}{4(1 - \theta)} \|\mathbf{u}^N_n\|_0^2 - (1 - 2\theta)(1 - \theta)\|\mathbf{u}^{N-1}_n\|_0^2,\]

so we can get

\[
\frac{3 - 2\theta}{4(1 - \theta)} \|\mathbf{u}^N_n\|_0^2 + C_{\text{coe}} \Delta t \sum_{m=1}^{N-1} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \mathbf{u}^{m+1}_n + (4\theta - 1 - 2\theta^2)\mathbf{u}^m_n + \frac{(1 - \theta)(1 - 2\theta)}{2} \mathbf{u}^{m-1}_n \right)^2_{\mathcal{U}} \\
\leq \frac{2\theta \Delta t}{C_{\text{coe}}} \sum_{m=1}^{N-1} \|\mathbf{F}^m\|_{\mathcal{U}}^2 + \frac{2(1 - \theta)^2 \Delta t}{C_{\text{coe}}} \sum_{m=1}^{N-1} \|\mathbf{F}^{m+1}\|_{\mathcal{U}}^2,
\]

+ \left(2\theta^2 - 5\theta + 3\right)\|\mathbf{u}_0\|_0^2 + (1 - 2\theta)(1 - \theta)\|\mathbf{u}^0\|_0^2 - (1 - 2\theta)(3 - 2\theta)\|\mathbf{u}^1\|_0\|\mathbf{u}^0\|_0,

then we end the proof. \(\Box\)

### 3.2 The decoupled scheme

The decoupled Linear Multistep method plus time filter is described below.

\(\blacktriangledown\) The Linear Multistep method (First Order):

Give \((\mathbf{u}^0_{fh}, \mathbf{p}^0_{fh})\) and \((\mathbf{u}^m_{fh}, \mathbf{p}^m_{fh}) \in (\mathbf{H}_{fh}, \mathbf{Q}_{fh})\) with \(m = 1, 2, \ldots, N - 1\), such that for \(\forall \mathbf{v}_{fh} \in \mathbf{H}_{fh}\) and \(\forall q_{fh} \in \mathbf{Q}_{fh}\),

\[
\left(\mathbf{u}^m_{fh} - \mathbf{u}^m_{fh}, \mathbf{v}_{fh}\right)_{\Omega_f} + a_{\Omega_f} \left(1 - \theta\right)\mathbf{u}^{m+1}_{fh} + \theta \mathbf{u}^m_{fh}, \mathbf{v}_{fh}\right) + b \left(\mathbf{v}_{fh}, (1 - \theta)\mathbf{p}^m_{fh} + \theta \mathbf{p}^m_{fh}\right)
\]

\[
= \left(1 - \theta\right)\mathbf{g}^m_{fh} + \theta \mathbf{g}^m_{fh}, \mathbf{v}_{fh}\right)_{\Omega_f} - c_{\Omega_f} \left(\mathbf{v}_{fh}, (2 - \theta)\phi^m_{ph} - (1 - \theta)\phi^m_{ph}\right),
\]

\[
\left(b \left(1 - \theta\right)\mathbf{u}^m_{fh} + \theta \mathbf{u}^m_{fh}, q_{fh}\right) = 0.
\]

Give \(\phi^0_{ph}\) and \(\phi^1_{ph}\), find \(\phi^{m+1}_{ph} \in \mathbf{H}_{ph}\) with \(m = 1, 2, \ldots, N - 1\), such that for \(\forall \psi_{ph} \in \mathbf{H}_{ph}\),

\[
g \left(\frac{\phi^{m+1}_{ph} - \phi^m_{ph}}{\Delta t}, \psi_{ph}\right)_{\Omega_p} + a_{\Omega_p} \left(1 - \theta\right)\phi^{m+1}_{ph} + \theta \phi^m_{ph}, \psi_{ph}\right)
\]

\[
= g \left((1 - \theta)\mathbf{g}^m_{ph} + \theta \mathbf{g}^m_{ph}, \psi_{ph}\right)_{\Omega_p} + c_{\Omega_p} \left((2 - \theta)\mathbf{u}^m_{ph} - (1 - \theta)\mathbf{u}^m_{ph}, \psi_{ph}\right).
\]

\(\blacktriangledown\) The Time Filter (Second Order):

Update the previous solutions \((\mathbf{u}^m_{fh}, \mathbf{p}^m_{fh}, \phi^m_{ph})\) by time filter,

\[
\mathbf{u}^{m+1}_{fh} = \mathbf{u}^m_{fh} - \frac{1 - 2\theta}{3 - 2\theta} (\mathbf{u}^{m+1}_{fh} - 2\mathbf{u}^m_{fh} + \mathbf{u}^{m-1}_{fh}),
\]

\[
\mathbf{p}^{m+1}_{fh} = \mathbf{p}^m_{fh} - \frac{1 - 2\theta}{3 - 2\theta} (\mathbf{p}^{m+1}_{fh} - 2\mathbf{p}^m_{fh} + \mathbf{p}^{m-1}_{fh}),
\]

\[
\phi^{m+1}_{ph} = \phi^m_{ph} - \frac{1 - 2\theta}{3 - 2\theta} (\phi^{m+1}_{ph} - 2\phi^m_{ph} + \phi^{m-1}_{ph}).
\]

**Remark 2:** Here, we use the second-order extrapolation method to approximate \(\mathbf{u}_f^{m+1}\) with \(2\mathbf{u}_f^m - \mathbf{u}_f^{m-1}\) and \(\phi_{ph}^{m+1}\) with \(2\phi_{ph}^m - \phi_{ph}^{m-1}\) in the interface coupled term. At the same time, similar to the coupled scheme, whether the pressure \(\mathbf{p}_f^{m+1}\) is filtered or not has little influence on the result.

The stability theorem for the decoupled Linear Multistep method plus time filter is introduced below.

**Theorem 3.2.** (Decoupled stability) Let \(\mathbf{u}^{m+1}_n, \phi^{m+1}_n\) be the solution of the Linear Multistep method plus time
Proof. From (3.15), we get
\[
\frac{3 - 2\theta}{4(1 - \theta)} \| u_{n+1}^N \|^2 + C_{\text{cor}} \Delta t \sum_{m=1}^{N-1} \left\| (1 - \theta)(3 - 2\theta) \frac{u_{n+1}^m}{2} + (4\theta - 1 - 2\theta^2) u_{f_{n+1}}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} u_{f_{n+1}}^{m-1} \right\|_{H_f}^2 \\
+ \frac{g(3 - 2\theta)}{4(1 - \theta)} \| \phi_{ph}^N \|^2 + g C_{\text{cor}} \Delta t \sum_{m=1}^{N-1} \left\| (1 - \theta)(3 - 2\theta) \frac{\phi_{ph}^m}{2} + (4\theta - 1 - 2\theta^2) \phi_{ph}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \phi_{ph}^{m-1} \right\|_{H_p}^2 \\
\leq C(T) \left( \| g \|_{L^2(0,T;L^2(\Omega_p))} + \| g \|_{L^2(0,T;L^2(\Omega_p))} \right) + (2\theta^2 - 5\theta + 3) \| u_{f_{n+1}} \|^2 + g(2\theta^2 - 5\theta + 3) \| \phi_{ph} \|^2 \\
+ (1 - 2\theta)(1 - \theta) \| u_{f_{n+1}} \|^2 + g(1 - 2\theta)(1 - \theta) \| \phi_{ph} \|^2 \\
- (1 - 2\theta)(3 - 2\theta) \| u_{f_{n+1}} \|_f \| \phi_{n+1} \|_f - g(1 - 2\theta)(3 - 2\theta) \| \phi_{ph} \|^2 \| \phi_{ph} \|^2,
\]
where \( C(T) = \exp \left( \sum_{m=1}^{N-1} \max \left( \frac{8(1 - \theta)\Delta t}{2(3 - 2\theta)C_{\text{cor}}}, \frac{8(1 - \theta)\Delta t}{2(3 - 2\theta)C_{\text{cor}}} \right) \right) \).

From (3.15), we get
\[
\begin{align*}
\phi_{nf+1} &= \frac{3 - 2\theta}{2} \phi_{nf+1} - (1 - 2\theta) \phi_{nf}, \\
\phi_{nf+1} &= \frac{3 - 2\theta}{2} \phi_{nf+1} - (1 - 2\theta) \phi_{nf},
\end{align*}
\]
and observing the first equation in (3.16), we use Lemma 3.1 because of \( \frac{(1 - \theta)(3 - 2\theta)}{2} - \frac{3 - 2\theta}{2} > 0 \), the first two
term on the left-hand side becomes
\[
2 \left( \frac{3-2\theta}{2} \right) u_{fh}^{m+1} - 2(1-\theta)u_{fh}^{m} + \frac{1-2\theta}{2} u_{fh}^{m-1},
\]
\[
\frac{(1-\theta)(3-2\theta)}{2} u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^{m} + \frac{(1-\theta)(3-2\theta)}{2} u_{fh}^{m-1}
\]
\[
\geq (2\theta^2 - 5\theta + 3) \| u_{fh}^{m+1} \|^2 - 2(1-\theta)\| u_{fh}^{m} \|^2 - (1-\theta)(1-\theta^2) \| u_{fh}^{m-1} \|^2
\]
\[
- (1-\theta)(3-2\theta) \left( \| u_{fh}^{m+1} \|_f u_{fh}^{m} - \| u_{fh}^{m} \|_f u_{fh}^{m-1} \|_f \right),
\] (3.17)
and
\[
2g \left( \frac{3-2\theta}{2} \right) \phi_{ph}^{m+1} - 2(1-\theta)\phi_{ph}^{m} + \frac{1-2\theta}{2} \phi_{ph}^{m-1},
\]
\[
\frac{(1-\theta)(3-2\theta)}{2} \phi_{ph}^{m+1} + (4\theta - 1 - 2\theta^2) \phi_{ph}^{m} + \frac{(1-\theta)(1-2\theta)}{2} \phi_{ph}^{m-1}
\]
\[
\geq g \left( 2\theta^2 - 5\theta + 3 \right) \| \phi_{ph}^{m+1} \|^2_p - 2(1-\theta)\| \phi_{ph}^{m} \|^2_p - (1-\theta)(1-\theta^2) \| \phi_{ph}^{m-1} \|^2_p
\]
\[
- (1-\theta)(3-2\theta) \left( \| \phi_{ph}^{m+1} \|_p \| \phi_{ph}^{m} \|_p - \| \phi_{ph}^{m} \|_p \| \phi_{ph}^{m-1} \|_p \right). \] (3.18)

Then using the coercivity of the bilinear form \( a(\cdot, \cdot) \), we get
\[
2\Delta t a_{\Omega_f} \left( \frac{(1-\theta)(3-2\theta)}{2} u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^{m} + \frac{(1-\theta)(1-2\theta)}{2} u_{fh}^{m-1},
\]
\[
\frac{(1-\theta)(3-2\theta)}{2} u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^{m} + \frac{(1-\theta)(1-2\theta)}{2} u_{fh}^{m-1}
\]
\[
\geq 2C_{coec} \Delta t \left\| \frac{(1-\theta)(3-2\theta)}{2} u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^{m} + \frac{(1-\theta)(1-2\theta)}{2} u_{fh}^{m-1} \right\|_{H_f}^2,
\] (3.19)
and
\[
2\Delta t a_{\Omega_p} \left( \frac{(1-\theta)(3-2\theta)}{2} \phi_{ph}^{m+1} + (4\theta - 1 - 2\theta^2) \phi_{ph}^{m} + \frac{(1-\theta)(1-2\theta)}{2} \phi_{ph}^{m-1},
\]
\[
\frac{(1-\theta)(3-2\theta)}{2} \phi_{ph}^{m+1} + (4\theta - 1 - 2\theta^2) \phi_{ph}^{m} + \frac{(1-\theta)(1-2\theta)}{2} \phi_{ph}^{m-1}
\]
\[
\geq 2gC_{coec} \Delta t \left\| \frac{(1-\theta)(3-2\theta)}{2} \phi_{ph}^{m+1} + (4\theta - 1 - 2\theta^2) \phi_{ph}^{m} + \frac{(1-\theta)(1-2\theta)}{2} \phi_{ph}^{m-1} \right\|_{H_p}^2. \] (3.20)

For the first two terms on the right-hand side, we get
\[
2\Delta t \left( \frac{(1-\theta)}{2} g_{fh}^{m+1} + \frac{1-\theta}{2} g_{fh}^{m} + (4\theta - 1 - 2\theta^2) g_{fh}^{m} + \frac{(1-\theta)(1-2\theta)}{2} g_{fh}^{m-1} \right)
\]
\[
\leq \frac{2\theta^2 \Delta t}{C_{coec}} \| g_{fh}^m \|_{H_f}^2 + \frac{2(1-\theta)^2 \Delta t}{C_{coec}} \| g_{fh}^{m+1} \|_{H_f}^2
\]
\[+ \frac{C_{coec} \Delta t}{2} \left\| \frac{(1-\theta)(3-2\theta)}{2} u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^{m} + \frac{(1-\theta)(1-2\theta)}{2} u_{fh}^{m-1} \right\|_{H_f}^2,
\] (3.21)
and
\[
2g \Delta t \left( \frac{(1-\theta)}{2} g_{ph}^{m+1} + \frac{1-\theta}{2} g_{ph}^{m} + (4\theta - 1 - 2\theta^2) g_{ph}^{m} + \frac{(1-\theta)(1-2\theta)}{2} g_{ph}^{m-1} \right)
\]
\[
\leq \frac{2\theta^2 \Delta t}{C_{coec}} \| g_{ph}^m \|_{H_p}^2 + \frac{2g(1-\theta)^2 \Delta t}{C_{coec}} \| g_{ph}^{m+1} \|_{H_p}^2
\]
\[+ \frac{gC_{coec} \Delta t}{2} \left\| \frac{(1-\theta)(3-2\theta)}{2} \phi_{ph}^{m+1} + (4\theta - 1 - 2\theta^2) \phi_{ph}^{m} + \frac{(1-\theta)(1-2\theta)}{2} \phi_{ph}^{m-1} \right\|_{H_p}^2.
\] (3.22)
Then by using the Lemma 3.2 and take $\varepsilon_1 = \frac{1}{c_{\text{coe}}}$, $\varepsilon_2 = \frac{\theta}{c_{\text{coe}}}$ into it, the interface terms on the right-hand side can be rewritten as

$$-2\Delta t C_{\text{coe}} \left( \frac{1 - \theta}{2} (3 - 2\theta) u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} u_{fh}^{m-1}, \xi_{\text{rh}}(1 - \theta) \phi_{ph}^m - (1 - \theta) \phi_{ph}^{m-1} \right)$$

$$+ 2\Delta t C_{\text{coe}} \left( (2 - \theta) u_{fh}^m - (1 - \theta) u_{fh}^{m-1}, \xi_{\text{rh}}(1 - \theta) (3 - 2\theta) \phi_{ph}^m + (4\theta - 1 - 2\theta^2) \phi_{ph}^{m-1} + \frac{(1 - \theta)(1 - 2\theta)}{2} \phi_{ph}^{m-1} \right)$$

$$\leq \frac{C_{\text{coe}} \Delta t}{2} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} u_{fh}^{m-1} \right\|_{\mathbb{H}_f}^2$$

$$+ \frac{g C_{\text{coe}} \Delta t}{2} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \phi_{ph}^m + (4\theta - 1 - 2\theta^2) \phi_{ph}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \phi_{ph}^{m-1} \right\|_{\mathbb{H}_p}^2$$

$$+ \frac{C_{\text{coh}} \Delta t}{2} \left\| (2 - \theta) \phi_{ph}^m - (1 - \theta) \phi_{ph}^{m-1} \right\|_{p}^2 + \frac{2g C_{\text{ coh}} \Delta t}{C_{\text{coe}}} \left\| (2 - \theta) u_{fh}^m - (1 - \theta) u_{fh}^{m-1} \right\|_{f}^2 \right.$$ 

where the last inequality follows from properly chosen constant $C_1$ and $C_2$. Combining the (3.17) and (3.23), and summing over $m = 1, 2, \ldots, N - 1$, we get

$$(2\theta^2 - 5\theta + 3) \left\| u_N^N \right\|_{f}^2 + (1 - 2\theta)(1 - \theta) \left\| u_{N-1}^{N-1} \right\|_{f}^2 + g(2\theta^2 - 5\theta + 3) \left\| \phi_{ph}^N \right\|_{p}^2 + g(1 - 2\theta)(1 - \theta) \left\| \phi_{ph}^{N-1} \right\|_{p}^2$$

$$- (1 - 2\theta)(3 - 2\theta) \left\| u_{N-1}^N \right\|_{f}^2 - g(1 - 2\theta)(3 - 2\theta) \left\| \phi_{ph}^N \right\|_{p}^2 + g(1 - 2\theta)(1 - \theta) \left\| \phi_{ph}^{N-1} \right\|_{p}^2$$

$$+ C_{\text{coh}} \Delta t \sum_{m=1}^{N-1} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} u_{fh}^{m-1} \right\|_{\mathbb{H}_f}^2$$

$$+ g C_{\text{coh}} \Delta t \sum_{m=1}^{N-1} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \phi_{ph}^m + (4\theta - 1 - 2\theta^2) \phi_{ph}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \phi_{ph}^{m-1} \right\|_{\mathbb{H}_p}^2 \right.$$ 

$$- (1 - 2\theta)(3 - 2\theta) \left\| u_{fh}^m \right\|_{f} \left\| u_{fh}^{m-1} \right\|_{f} + (1 - 2\theta)(1 - \theta) \left\| \phi_{ph}^m \right\|_{p} \left\| \phi_{ph}^{m-1} \right\|_{p}$$

Then note that

$$- (1 - 2\theta)(3 - 2\theta) \left\| u_{fh}^m \right\|_{f} \left\| u_{fh}^{m-1} \right\|_{f} \geq - \frac{(1 - 2\theta)(3 - 2\theta)}{4(1 - \theta)} \left\| u_{fh}^m \right\|_{f} \left\| u_{fh}^{m-1} \right\|_{f}$$

$$- (1 - 2\theta)(3 - 2\theta) \left\| \phi_{ph}^m \right\|_{p} \left\| \phi_{ph}^{m-1} \right\|_{p} \geq - \frac{(1 - 2\theta)(3 - 2\theta)}{4(1 - \theta)} \left\| \phi_{ph}^m \right\|_{p} \left\| \phi_{ph}^{m-1} \right\|_{p},$$

so we can get

$$\frac{3 - \theta}{4(1 - \theta)} \left\| u_{fh}^m \right\|_{f}^2 + \frac{g(3 - \theta)}{4(1 - \theta)} \left\| \phi_{ph}^m \right\|_{p}^2 + g C_{\text{coh}} \Delta t \sum_{m=1}^{N-1} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} u_{fh}^{m+1} + (4\theta - 1 - 2\theta^2) u_{fh}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} u_{fh}^{m-1} \right\|_{\mathbb{H}_f}^2$$

$$+ \frac{g}{4(1 - \theta)} \left\| \phi_{ph}^m \right\|_{p}^2 + g C_{\text{coh}} \Delta t \sum_{m=1}^{N-1} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \phi_{ph}^m + (4\theta - 1 - 2\theta^2) \phi_{ph}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \phi_{ph}^{m-1} \right\|_{\mathbb{H}_p}^2 \right.$$ 

$$\leq \sum_{m=1}^{N-1} \left\| \phi_{fh}^m \right\|_{\mathbb{H}_f}^2 + \frac{2g (1 - \theta)^2 \Delta t}{C_{\text{coh}}} \sum_{m=1}^{N-1} \left\| g_m^m \right\|_{\mathbb{H}_f}^2 + \frac{2g (1 - \theta)^2 \Delta t}{C_{\text{coh}}} \sum_{m=1}^{N-1} \left\| g_m^{m-1} \right\|_{\mathbb{H}_p}^2 \right.$$
4 Error analysis

In this section, we will analyze the errors of coupled and decoupled schemes. Let us denote \((u^{m+1}, p^{m+1}_f) = (u(t_{m+1}), p_f(t_{m+1}))\) where \(u^{m+1} = (u^{m+1}_f, p^{m+1}_f)\) and \(u(t_{m+1}) = (u_f(t_{m+1}), p_f(t_{m+1}))\), at the same time to define error functions:

- \(e_u = u^{m+1}_u - u^{m} + p^{m}_h u^{m} - p^{m}_h u^{m}_u - u^{m} = \xi_u^{m}\),
- \(e_p = p^{m+1}_p - p^{m}_f + p^{m}_h p^{m}_p - p^{m}_h p^{m}_f - p^{m}_f = \xi_p^{m}\),
- \(e_{u_f} = u^{m+1}_f - u^{m}_f = u^{m}_f - p^{m}_h u^{m}_f - p^{m}_h u^{m}_f - u^{m}_f = \xi_{u_f}^{m}\),
- \(e_{\phi_p} = \phi^{m+1}_p - \phi^{m}_p + p^{m}_h \phi^{m}_p + p^{m}_h \phi^{m}_p - \phi^{m}_p = \xi_{\phi_p}^{m}\).

Obviously, we have

\[
\begin{align*}
\|\xi_u^{m}\|_0 &\leq Ch^2\|u(t)\|_{H^2}, & \|\xi_u^{m}\|_U &\leq Ch\|u(t)\|_{H^2}, & \|\xi_u^{m}\|_{L^2} &\leq Ch\|p_f(t)\|_{H^2}, \\
\|\xi_{u_f}^{m}\|_f &\leq Ch^2\|u_f(t)\|_{H^2}, & \|\xi_{u_f}^{m}\|_{H_f} &\leq Ch\|u_f(t)\|_{H^2}, & \|\xi_{\phi_p}^{m}\|_{L^2} &\leq Ch\|\phi_f(t)\|_{H^2}.
\end{align*}
\]

(4.1)

Note that \(\eta_u^0 = 0, \eta_p^0 = 0, \eta_{u_f}^0 = 0\) and \(\eta_{\phi_p}^0 = 0\).

Assume the solution satisfies the following regularity conditions:

\[
\begin{align*}
u_f &\in L^\infty(0, T; H^2), u_{f,t} &\in L^2(0, T; H^1) \cap L^\infty(0, T; L^2), \quad u_{f,ttt} &\in L^2(0, T; H^2), \quad u_{f,tt} &\in L^2(0, T; H^2), \quad u_{f,ttt} &\in L^2(0, T; H^2), \\
\phi_p &\in L^\infty(0, T; H^2), \phi_{p,t} &\in L^2(0, T; H^1) \cap L^\infty(0, T; L^2), \quad \phi_{p,tt} &\in L^2(0, T; L^2), \quad \phi_{p,ttt} &\in L^2(0, T; H^1).
\end{align*}
\]

(4.2)

And the external force \(g_f\) and \(g_p\) also need to be satisfied

\[
\begin{align*}
g_{f,t} &\in L^2(0, T; L^2), \quad g_{f,tt} &\in L^2(0, T; H^1), \\
g_{p,tt} &\in L^2(0, T; L^2), \quad g_{p,ttt} &\in L^2(0, T; H^1).
\end{align*}
\]

(4.3)

4.1 The error estimate of the coupled scheme

**Theorem 4.1.** (second-order convergence for the coupled scheme) Under the assumption of (4.2) and (4.3), for \(N \geq 2\) we have the estimates

\[
\frac{3-2\theta}{4(1-\theta)} \left\| \frac{u}{\bar{u}} \right\|_0^2 + C_{\text{coer}} \Delta t \sum_{m=1}^{N-1} \left\| \left( \frac{1-\theta}{2} + \frac{3-2\theta}{4(1-\theta)} \right) e^{m+1}_u + \left( \frac{1-\theta}{2} + \frac{3-2\theta}{4(1-\theta)} \right) e^{m-1}_u \right\|_U^2 \leq C(\Delta t^4 + h^4).
\]

Proof. First of all, let us multiply (2.9) by \(\frac{1-\theta}{2} + \frac{3-2\theta}{4(1-\theta)} \) and \(\frac{1-\theta}{2} + \frac{3-2\theta}{4(1-\theta)} \) at \(t_{m+1}, t_m\) and \(t_{m-1}\), respectively. Then, subtracting the summation of three equations from (3.9), by the definition of the error
functions and the properties of the project operator \[3.2\], we get

\[
\left(\frac{3 - 2\theta}{2}\eta_{u}^{m+1} - 4(1 - \theta)\eta_{u}^{m} + (1 - 2\theta)\eta_{u}^{m-1}\right) \frac{2\Delta t}{\Delta t}
+ a\left(\frac{(1 - \theta)(3 - 2\theta)}{2}\eta_{u}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{u}^{m} + \frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{u}^{m-1}, \eta_{h}\right)
+ b\left(\frac{(1 - \theta)(3 - 2\theta)}{2}\eta_{p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{p}^{m} + \frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{p}^{m-1}\right)
\]

\[
= \left(\frac{3 - 2\theta}{2}\eta_{u}^{m+1} - 4(1 - \theta)\eta_{u}^{m} + (1 - 2\theta)\eta_{u}^{m-1}\right) \frac{2\Delta t}{\Delta t}
- \left(\frac{(1 - \theta)(3 - 2\theta)}{2}\eta_{u}^{m+1} - 4(1 - \theta)\eta_{u}^{m} + \frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{u}^{m-1}, \eta_{h}\right)
- \left(\frac{(3 - 2\theta)\eta_{u}^{m+1} - 4(1 - \theta)\eta_{u}^{m} + (1 - 2\theta)\eta_{u}^{m-1}}{2\Delta t}, \eta_{h}\right)
\]

\[
\left(\frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{u}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{u}^{m} + (1 - \theta)(1 - 2\theta)\eta_{u}^{m-1}, q_{fh}\right) = 0, \quad \forall q_{fh} \in Q_{fh},
\]

\[
\eta_{h} = 2\Delta t \left(\frac{(1 - \theta)(3 - 2\theta)}{2}\eta_{u}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{u}^{m} + \frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{u}^{m-1}\right),
\]

similar to Theorem \[3.1\], we use Lemma \[3.1\] the first term on the left hand side can be handled as

\[
2 \left(\frac{3 - 2\theta}{2}\eta_{u}^{m+1} - 2(1 - \theta)\eta_{u}^{m} + \frac{1 - 2\theta}{2}\eta_{u}^{m-1},
\right.
\]

\[
\left.\frac{(1 - \theta)(3 - 2\theta)}{2}\eta_{u}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{u}^{m} + \frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{u}^{m-1}\right)
\geq (2\theta^2 - 5\theta + 3)\|\eta_{u}^{m+1}\|_u^2 - 2(1 - \theta)\|\eta_{u}^{m}\|_u^2 - (1 - 2\theta)(1 - \theta)\|\eta_{u}^{m-1}\|_u^2
- (1 - 2\theta)(3 - 2\theta)(\|\eta_{u}^{m+1}\|_u\|\eta_{u}^{m}\|_u - \|\eta_{u}^{m}\|_b\|\eta_{u}^{m-1}\|_b).
\]

Then using the coercivity of the bilinear form \(a(\cdot, \cdot)\), the second term on the left hand side can be handled as

\[
2\Delta t a \left(\frac{(1 - \theta)(3 - 2\theta)}{2}\eta_{u}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{u}^{m} + \frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{u}^{m-1},
\right.
\]

\[
\left.\frac{(1 - \theta)(3 - 2\theta)}{2}\eta_{u}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{u}^{m} + \frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{u}^{m-1}\right)
\geq 2C_{\text{coerc}}\Delta t \left(\frac{(1 - \theta)(3 - 2\theta)}{2}\eta_{u}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{u}^{m} + \frac{(1 - \theta)(1 - 2\theta)}{2}\eta_{u}^{m-1}\right)^2_u.
\]

We consider the first term on the right side, by Taylor expansion with the integral remainder,

\[
\eta_{u}^{m} = \eta_{u}^{m+1} - \Delta t \eta_{u}^{m+1} + \frac{\Delta t^2}{2} \eta_{u}^{m+1} + \frac{1}{2} \int_{t^{m+1}}^{t^{m}} (t^{m} - t)^2 \eta_{ttt} dt,
\]

\[
\eta_{u}^{m-1} = \eta_{u}^{m+1} - 2\Delta t \eta_{u}^{m+1} + 2\Delta t^2 \eta_{ttt}^{m+1} + \frac{1}{2} \int_{t^{m+1}}^{t^{m}} (t^{m} - t)^2 \eta_{ttt} dt,
\]

\[
\eta_{u}^{m} = \eta_{u}^{m+1} - \Delta t \eta_{u}^{m+1} + \frac{1}{2} \int_{t^{m+1}}^{t^{m}} (t^{m} - t) \eta_{ttt} dt,
\]

\[
\eta_{u}^{m-1} = \eta_{u}^{m+1} - 2\Delta t \eta_{u}^{m+1} - \frac{1}{2} \int_{t^{m+1}}^{t^{m}} (t^{m} - t) \eta_{ttt} dt.
\]
So we get

\[
\begin{align*}
(3-2\theta)\mathbf{u}^{m+1} - 4(1-\theta)\mathbf{u}^m + (1-2\theta)\mathbf{u}^{m-1} \\
= \frac{2\Delta t}{2} (1-\theta)(3-2\theta)\mathbf{u}^{m+1} - (4\theta - 1 - 2\theta^2)\mathbf{u}^m - \frac{1-\theta)(1-2\theta)\mathbf{u}^{m-1}
\end{align*}
\]

Similarly, for the second term on the right side,

\[
\begin{align*}
3\Delta \int_{t_m}^{t_{m+1}} (t-m)^2 \mathbf{u}_{ttt} dt - \frac{2\Theta}{4} \int_{t_{m-1}}^{t_{m+1}} (t_{m-1}-t)^2 \mathbf{u}_{ttt} dt \\
- 4\Theta - 1 - 2\Theta^2 \int_{t_{m-1}}^{t_{m+1}} (t_{m-1} - t) \mathbf{u}_{ttt} dt = \frac{(1-\Theta)(1-2\Theta)}{2} \int_{t_{m-1}}^{t_{m+1}} (t_{m-1} - t) \mathbf{u}_{ttt} dt.
\end{align*}
\]

By using Cauchy-Schwarz inequality,

\[
\begin{align*}
\int_{t_m}^{t_{m+1}} (t-m)^2 \mathbf{u}_{ttt} dt &\leq \frac{\Delta t^5}{5} \int_{t_m}^{t_{m+1}} \mathbf{u}_{ttt}^2 dt, \\
\int_{t_{m-1}}^{t_{m+1}} (t_{m-1} - t)^2 \mathbf{u}_{ttt} dt &\leq \frac{32\Delta t^5}{5} \int_{t_{m-1}}^{t_{m+1}} \mathbf{u}_{ttt}^2 dt, \\
\int_{t_m}^{t_{m+1}} (t-m) \mathbf{u}_{ttt} dt &\leq \frac{\Delta t^3}{3} \int_{t_m}^{t_{m+1}} \mathbf{u}_{ttt}^2 dt, \\
\int_{t_{m-1}}^{t_{m+1}} (t_{m-1} - t) \mathbf{u}_{ttt} dt &\leq \frac{8\Delta t^3}{3} \int_{t_{m-1}}^{t_{m+1}} \mathbf{u}_{ttt}^2 dt.
\end{align*}
\]

Then

\[
\begin{align*}
2\Delta t \left( (3-2\theta)\mathbf{u}^{m+1} - 4(1-\theta)\mathbf{u}^m + (1-2\theta)\mathbf{u}^{m-1} \right) \\
- \frac{1-\theta)(3-2\theta)}{2} \mathbf{u}^{m+1} - (4\theta - 1 - 2\theta^2)\mathbf{u}^m - \frac{(1-\theta)(1-2\theta)}{2} \mathbf{u}^{m-1}, \\
(1-\theta)(3-2\theta)\eta_{m+1}^m + (4\theta - 1 - 2\theta^2)\eta_{m+1}^m - \frac{(1-\theta)(1-2\theta)}{2} \eta_{m-1}^m
\end{align*}
\]

\[
\leq \frac{3\Delta t}{C_{coe}^2} \left(\frac{(3-2\theta)}{2} \mathbf{u}^{m+1} - 4(1-\theta)\mathbf{u}^m + (1-2\theta)\mathbf{u}^{m-1} \right) \\
- \frac{(1-\theta)(3-2\theta)}{2} \mathbf{u}^{m+1} - (4\theta - 1 - 2\theta^2)\mathbf{u}^m - \frac{(1-\theta)(1-2\theta)}{2} \mathbf{u}^{m-1} \right) \left(\frac{C_{coe}}{3} \right) \left(\frac{(1-\theta)(3-2\theta)}{2} \eta_{m+1}^m + (4\theta - 1 - 2\theta^2)\eta_{m+1}^m - \frac{(1-\theta)(1-2\theta)}{2} \eta_{m-1}^m \right) \left(\frac{\Delta t}{U} \right)
\]

\[
\leq \frac{(6\theta^4 - 20\theta^3 + 25\theta^2 - 130\theta + 24)\Delta t^4}{5C_{coe}} \int_{t_{m-1}}^{t_{m+1}} \mathbf{u}_{ttt}^2 dt \\
+ \frac{C_{coe}}{3} \left(\frac{(1-\theta)(3-2\theta)}{2} \eta_{m+1}^m + (4\theta - 1 - 2\theta^2)\eta_{m+1}^m - \frac{(1-\theta)(1-2\theta)}{2} \eta_{m-1}^m \right) \int_{t_{m-1}}^{t_{m+1}} \mathbf{u}_{ttt}^2 dt.
\]

Similarly, for the second term on the right side,

\[
\mathbf{u}^{m} = \mathbf{u}^{m+1} + \int_{t_m}^{t_{m+1}} \mathbf{u}_t dt, \\
\mathbf{u}^{m-1} = \mathbf{u}^{m+1} + \int_{t_{m-1}}^{t_{m+1}} \mathbf{u}_t dt.
\]

Then

\[
\begin{align*}
(3-2\theta)\mathbf{c}_u^{m+1} - 4(1-\theta)\mathbf{c}_u^m + (1-2\theta)\mathbf{c}_u^{m-1} \\
= \frac{2\Delta t}{2} (1-\theta)(3-2\theta)\mathbf{c}_u^{m+1} - (4\theta - 1 - 2\theta^2)\mathbf{c}_u^m - \frac{(1-\theta)(1-2\theta)}{2} \mathbf{c}_u^{m-1}
\end{align*}
\]

\[
\begin{align*}
= \frac{1}{\Delta t} \left[ (P_{h}^n - I) (3-2\theta)\mathbf{c}_u^{m+1} - 2(1-\theta)\mathbf{c}_u^m + \frac{(1-2\theta)}{2} \mathbf{c}_u^{m-1} \right] \\
= \frac{1}{\Delta t} \left[ 2(1-\theta) \int_{t_m}^{t_{m+1}} (P_{h}^n - I) \mathbf{u}_t dt - \frac{1-2\theta}{2} \int_{t_{m-1}}^{t_{m+1}} (P_{h}^n - I) \mathbf{u}_t dt \right].
\]

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So we get

\[
2 \Delta t \left( \frac{(3 - 2\theta)\xi^m + 4(1 - \theta)\xi^{m-1}}{2 \Delta t} \right),
\]

\[
\frac{(1 - \theta)(3 - 2\theta)\eta^{m+1}_u + (4\theta - 1 - 2\theta^2)\eta^{m}_u + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m-1}_u}{2 \Delta t},
\]

\[
\leq 3 \Delta t \frac{C_{\text{coe}}}{2 \Delta t} \left( \frac{(3 - 2\theta)\xi^m + 4(1 - \theta)\xi^{m-1}}{2 \Delta t} \right) \sum_{m=1}^{N-1} \left\| \frac{(1 - \theta)(3 - 2\theta)\eta^{m+1}_u + (4\theta - 1 - 2\theta^2)\eta^{m}_u + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m-1}_u}{2 \Delta t} \right\|_U^2.
\]

At last, being aware of

\[
F^{m+1} = F^m + \Delta t F^m - \int_{t_m}^{t_{m+1}} (F^m - t) dt,
\]

\[
F^{m-1} = F^m - \Delta t F^m - \int_{t_m}^{t_{m-1}} (F^m - t) dt,
\]

then

\[
- \frac{(1 - \theta)(1 - 2\theta)}{2} F^{m+1} + (1 - \theta)(1 - 2\theta) F^m - \frac{(1 - \theta)(1 - 2\theta)}{2} F^{m-1},
\]

\[
= \frac{(1 - \theta)(1 - 2\theta)}{2} \int_{t_m}^{t_{m+1}} (\xi^{m+1} - t) dt - \frac{(1 - \theta)(1 - 2\theta)}{2} \int_{t_{m-1}}^{t_{m-1}} (\xi^{m-1} - t) dt.
\]

So we get

\[
2 \Delta t \left( - \frac{(1 - \theta)(1 - 2\theta)}{2} F^{m+1} + (1 - \theta)(1 - 2\theta) F^m - \frac{(1 - \theta)(1 - 2\theta)}{2} F^{m-1},
\]

\[
\frac{(1 - \theta)(3 - 2\theta)\eta^{m+1}_u + (4\theta - 1 - 2\theta^2)\eta^{m}_u + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m-1}_u}{2 \Delta t},
\]

\[
\leq \frac{(1 - \theta)^2(1 - 2\theta)^2 \Delta t^4}{4 C_{\text{coe}}} \int_{t_m}^{t_{m+1}} \| F^m \|_U^2 dt
\]

\[
+ \frac{C_{\text{coe}} \Delta t}{3} \left\| \frac{(1 - \theta)(3 - 2\theta)\eta^{m+1}_u + (4\theta - 1 - 2\theta^2)\eta^{m}_u + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m-1}_u}{2 \Delta t} \right\|_U^2.
\]

Combining the (4.5) - (4.9) and sum the (4.4) over \( m = 1, 2, \ldots, N - 1 \), and using the same method as Theorem 3.1 we get

\[
\frac{3 - 2\theta}{4(1 - \theta)} \| u_N \|_U^2 + C_{\text{coe}} \Delta t \sum_{m=1}^{N-1} \left\| \frac{(1 - \theta)(3 - 2\theta)\eta^{m+1}_u + (4\theta - 1 - 2\theta^2)\eta^{m}_u + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m-1}_u}{2 \Delta t} \right\|_U^2,
\]

\[
\leq \sum_{m=1}^{N-1} \left( \frac{60\theta^4 - 200\theta^3 + 257\theta^2 - 130\theta + 24}{5 C_{\text{coe}}} \int_{t_{m-1}}^{t_m} \| u_{ttt} \|_U^2 dt + \frac{3(20\theta^2 - 36\theta + 17)}{4 C_{\text{coe}}} \int_{t_{m-1}}^{t_m} \| (P^m_h - I) u \|_U^2 dt
\]

\[
+ \frac{(1 - \theta)^2(1 - 2\theta)^2 \Delta t^4}{4 C_{\text{coe}}} \int_{t_{m-1}}^{t_{m+1}} \| F^m \|_U^2 dt.
\]

Then using the triangle inequality, we end the proof. \( \square \)
4.2 The error estimate of the decoupled scheme

**Theorem 4.2.** (second-order convergence for the decoupled scheme) Under the assumption of (4.2) and (4.3), for $N \geq 2$ we have the estimates

\[
\frac{3 - 2\theta}{4(1 - \theta)} \|e_f\|^2 + C_{\text{coerc}} \Delta t \sum_{m=1}^{N-1} \left| \frac{(1 - \theta)(3 - 2\theta)}{2} e_{u_f}^{m+1} + (4\theta - 1 - 2\theta^2) e_{u_f}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} e_{u_f}^{m-1} \right|^2_{H_f} \\
+ \frac{g(3 - 2\theta)}{4(1 - \theta)} \|e_p\|^2 + gC_{\text{coerc}} \Delta t \sum_{m=1}^{N-1} \left| \frac{(1 - \theta)(3 - 2\theta)}{2} e_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2) e_{\phi_p}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} e_{\phi_p}^{m-1} \right|^2_{H_p} \\
\leq C(\Delta t^4 + h^4).
\]

**Proof.** Let us multiply (2.9) by $\frac{(1 - \theta)(3 - 2\theta)}{2}$, $(4\theta - 1 - 2\theta^2)$ and $\frac{(1 - \theta)(1 - 2\theta)}{2}$ at $t_{m+1}$, $t_m$ and $t_{m-1}$, respectively. Then, subtracting the summation of three equations from (3.10), by the definition of the error functions and the properties of the project operator (3.2), we obtain

\[
\begin{align*}
\left(3 - 2\theta\right)\eta_{u_f}^{m+1} - 4(1 - \theta)h_{u_f}^m + (1 - 2\theta)h_{u_f}^{m-1} \\
\frac{2\Delta t}{2} \\
+ a_{\Omega_f} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{u_f}^{m+1} + (4\theta - 1 - 2\theta^2) \eta_{u_f}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{u_f}^{m-1}, v_f \right) \\
+ a_{\Omega_p} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2) \eta_{\phi_p}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{\phi_p}^{m-1}, \psi_p \right) \\
+ b \left( v_f, \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2) \eta_{\phi_p}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right) \\
\leq \left( \frac{3 - 2\theta}{2} \right) u_f^{m+1} - 4(1 - \theta) u_f^m + (1 - 2\theta) u_f^{m-1}
\end{align*}
\]
Review the proof of the Theorem 4.1, there holds
\[
2 \left( 3 - \frac{2\theta}{2} \eta_{a_f}^{m+1} - 2(1 - \theta)\eta_{a_f}^m + \frac{1 - 2\theta}{2} \eta_{a_f}^{m-1} \right),
\]
\[
(1 - \theta)(3 - 2\theta) \eta_{a_f}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{a_f}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{a_f}^{m-1} \right) \geq \left(2\theta^2 - 5\theta + 3\right)\|\eta_{a_f}^{m+1}\|_f^2 - (2(1 - \theta))\|\eta_{a_f}^m\|_f^2 - (1 - 2\theta)(1 - \theta)\|\eta_{a_f}^{m-1}\|_f^2
\]
\[-(1 - 2\theta)(3 - 2\theta)(\|\eta_{a_f}^{m+1}\|_f\|\eta_{a_f}^m\|_f - \|\eta_{a_f}^m\|_f\|\eta_{a_f}^{m-1}\|_f) ,
\]
and
\[
2g \left( 3 - \frac{2\theta}{2} \eta_{a_f}^{m+1} - 2(1 - \theta)\eta_{a_f}^m + \frac{1 - 2\theta}{2} \eta_{a_f}^{m-1} \right),
\]
\[
(1 - \theta)(3 - 2\theta) \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right) \geq g \left[ (2\theta^2 - 5\theta + 3)\|\eta_{\phi_p}^{m+1}\|_{\phi_p}^2 - (2(1 - \theta))\|\eta_{\phi_p}^m\|_{\phi_p}^2 - (1 - 2\theta)(1 - \theta)\|\eta_{\phi_p}^{m-1}\|_{\phi_p}^2
\]
\[-(1 - 2\theta)(3 - 2\theta)(\|\eta_{\phi_p}^{m+1}\|_{\phi_p}\|\eta_{\phi_p}^m\|_{\phi_p} - \|\eta_{\phi_p}^m\|_{\phi_p}\|\eta_{\phi_p}^{m-1}\|_{\phi_p}) ,
\]
and
\[
2\Delta t u_{\Omega_f} \left( 2 \left( 3 - \frac{2\theta}{2} \eta_{a_f}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{a_f}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{a_f}^{m-1} \right),
\]
\[
(1 - \theta)(3 - 2\theta) \eta_{a_f}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{a_f}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{a_f}^{m-1} \right) \geq 2\tilde{C}_{coe} \Delta t \left( \|\eta_{a_f}^{m+1}\|_f^2 - (2(1 - \theta))\|\eta_{a_f}^m\|_f^2 - (1 - 2\theta)(1 - \theta)\|\eta_{a_f}^{m-1}\|_f^2
\]
\[-(1 - 2\theta)(3 - 2\theta)(\|\eta_{a_f}^m\|_f\|\eta_{a_f}^{m-1}\|_f) ,
\]
and
\[
2\Delta t u_{\Omega_p} \left( 2 \left( 3 - \frac{2\theta}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right),
\]
\[
(1 - \theta)(3 - 2\theta) \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right) \geq 2g\tilde{C}_{coe} \Delta t \left( \|\eta_{\phi_p}^{m+1}\|_{\phi_p}^2 - (2(1 - \theta))\|\eta_{\phi_p}^m\|_{\phi_p}^2 - (1 - 2\theta)(1 - \theta)\|\eta_{\phi_p}^{m-1}\|_{\phi_p}^2
\]
\[-(1 - 2\theta)(3 - 2\theta)(\|\eta_{\phi_p}^m\|_{\phi_p}\|\eta_{\phi_p}^{m-1}\|_{\phi_p}) ,
\]
and
\[
2\Delta t \left( 3 - \frac{2\theta}{2} \eta_{f}^{m+1} - 4(1 - \theta)\eta_{f}^m + (1 - 2\theta)\eta_{f}^{m-1} \right)
\]
\[
2\Delta t \left( \|\eta_{\phi_p}^m\|_{\phi_p}^2 - (2(1 - \theta))\|\eta_{\phi_p}^m\|_{\phi_p}^2 - (1 - 2\theta)(1 - \theta)\|\eta_{\phi_p}^{m-1}\|_{\phi_p}^2
\]
\[-(1 - 2\theta)(3 - 2\theta)(\|\eta_{\phi_p}^m\|_{\phi_p}\|\eta_{\phi_p}^{m-1}\|_{\phi_p}) ,
\]
\[
\left\| \frac{(3 - 2\theta)\eta_{f}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{f}^m + (1 - \theta)(1 - 2\theta)\eta_{f}^{m-1}}{2\Delta t} \right\|_f^2
\]
\[
\left\| \frac{(3 - 2\theta)\eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + (1 - \theta)(1 - 2\theta)\eta_{\phi_p}^{m-1}}{2\Delta t} \right\|_{\phi_p}^2
\]
\[
\leq \frac{2(6\theta^4 - 200\theta^3 + 257\theta^2 - 130\theta + 24)\Delta t}{5\tilde{C}_{coe}} \int_{t_{m-1}}^{t_m} \|\eta_{TT}^m\|^2 dt
\]
\[
+ \frac{\tilde{C}_{coe} \Delta t}{6} \left( \|\eta_{a_f}^{m+1}\|_f^2 - (2(1 - \theta))\|\eta_{a_f}^m\|_f^2 - (1 - 2\theta)(1 - \theta)\|\eta_{a_f}^{m-1}\|_f^2
\]
\[-(1 - 2\theta)(3 - 2\theta)(\|\eta_{a_f}^m\|_f\|\eta_{a_f}^{m-1}\|_f) ,
\]
\[ 2g \Delta t \left( \frac{(3 - 2\theta)\phi^{m+1}_p - 4(1 - \theta)\phi^m_p + (1 - 2\theta)\phi^{m-1}_p}{6\Delta t} \right) - \frac{(1 - \theta)(3 - 2\theta)}{2} \phi^{m+1}_t - (4\theta - 1 - 2\theta^2)\phi^m_t - \frac{(1 - \theta)(1 - 2\theta)}{2} \phi^{m-1}_t, \]
\[ (1 - \theta)(3 - 2\theta)\eta^{m+1}_{\phi_p} + (4\theta - 1 - 2\theta^2)\eta^m_{\phi_p} + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m-1}_{\phi_p} \right) \]
\[ \leq 2g(600\Delta t - 200\theta^2 - 257\theta^2 - 130\theta + 24)\Delta t^4 \int_{t_{m-1}}^{t_m} \|\phi_{tt}\|^2_{H_p} dt \]
\[ + \frac{gC_{coe}}{6} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \eta^{m+1}_{\phi_p} + (4\theta - 1 - 2\theta^2)\eta^m_{\phi_p} + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m-1}_{\phi_p} \right\|_{H_p}^2, \]

and

\[ 2\Delta t \left( \frac{(3 - 2\theta)\xi^{m+1}_{u_f} - 4(1 - \theta)\xi^m_{u_f} + (1 - 2\theta)\xi^{m-1}_{u_f}}{6\Delta t} - \frac{(1 - \theta)(3 - 2\theta)}{2} \xi^{m+1}_{u_f} + (4\theta - 1 - 2\theta^2)\xi^m_{u_f} + \frac{(1 - \theta)(1 - 2\theta)}{2} \xi^{m-1}_{u_f} \right) \]
\[ \leq \frac{3(20\theta^2 - 36\theta + 17)}{2C_{coe}} \int_{t_{m-1}}^{t_m} \|\xi_{u_f} - I\|_{H_f}^2 dt \]
\[ + \frac{C_{coe} \Delta t}{6} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \xi^{m+1}_{u_f} + (4\theta - 1 - 2\theta^2)\xi^m_{u_f} + \frac{(1 - \theta)(1 - 2\theta)}{2} \xi^{m-1}_{u_f} \right\|_{H_f}^2, \]

and

\[ 2g \Delta t \left( \frac{(3 - 2\theta)\xi^{m+1}_{\phi_p} - 4(1 - \theta)\xi^m_{\phi_p} + (1 - 2\theta)\xi^{m-1}_{\phi_p}}{2\Delta t} - \frac{(1 - \theta)(3 - 2\theta)}{2} \xi^{m+1}_{\phi_p} + (4\theta - 1 - 2\theta^2)\xi^m_{\phi_p} + \frac{(1 - \theta)(1 - 2\theta)}{2} \xi^{m-1}_{\phi_p} \right) \]
\[ \leq \frac{3g(20\theta^2 - 36\theta + 17)}{2C_{coe}} \int_{t_{m-1}}^{t_m} \|\xi_{\phi_p} - I\|_{H_p}^2 dt \]
\[ + \frac{gC_{coe} \Delta t}{6} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \xi^{m+1}_{\phi_p} + (4\theta - 1 - 2\theta^2)\xi^m_{\phi_p} + \frac{(1 - \theta)(1 - 2\theta)}{2} \xi^{m-1}_{\phi_p} \right\|_{H_p}^2, \]

and

\[ 2\Delta t \left( \frac{(1 - \theta)(1 - 2\theta)}{2} \xi^{m+1}_{f} + (1 - \theta)(1 - 2\theta)\xi^m_{f} - \frac{(1 - \theta)(1 - 2\theta)}{2} \xi^{m-1}_{f}, \right) \]
\[ (1 - \theta)(3 - 2\theta)\eta^{m+1}_{\phi_f} + (4\theta - 1 - 2\theta^2)\eta^m_{\phi_f} + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m-1}_{\phi_f} \right) \]
\[ \leq \frac{(1 - \theta)^2(1 - 2\theta)^2\Delta t^4}{2C_{coe}} \int_{t_{m-1}}^{t_m} \|\xi_{f} - I\|_{H_f}^2 dt \]
\[ + \frac{C_{coe} \Delta t}{6} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \xi^{m+1}_{u_f} + (4\theta - 1 - 2\theta^2)\xi^m_{u_f} + \frac{(1 - \theta)(1 - 2\theta)}{2} \xi^{m-1}_{u_f} \right\|_{H_f}^2, \]

and

\[ 2g \Delta t \left( \frac{(1 - \theta)(1 - 2\theta)}{2} g^{m+1}_p + (1 - \theta)(1 - 2\theta)g^m_p - \frac{(1 - \theta)(1 - 2\theta)}{2} g^{m-1}_p, \right) \]
\[ (1 - \theta)(3 - 2\theta)\eta^{m+1}_{\phi_p} + (4\theta - 1 - 2\theta^2)\eta^m_{\phi_p} + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta^{m-1}_{\phi_p} \right) \]
\[ \leq \frac{g(1 - \theta)^2(1 - 2\theta)^2\Delta t^4}{2C_{coe}} \int_{t_{m-1}}^{t_m} \|g_{tt}\|^2_{H_p} dt. \]
Here, we focus on the interface terms on the right-hand side, they can be rewritten as

\[
- 2\Delta t \mathcal{L}_p \left\{ \left( \frac{1 - \theta}{2} \right) \left[ (3 - \theta) \eta_{\alpha p}^{m+1} + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi p}^{m+1} \right] + (4\theta - 1 - 2\theta^2) \eta_{\phi p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi p}^{m-1} \right\}.
\]

(4.21)
\[
\frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1}.
\]

Using linear properties, the sum of the first four terms equals zero. For the following four terms, we use (2.10) and lemma 3.2 to plug in \(\varepsilon_3 = \varepsilon_5 = \frac{1}{C_{\text{coc}}}\) and \(\varepsilon_4 = \varepsilon_6 = \frac{1}{C_{\text{coc}}}\), we get

\[
-2\Delta t C_{\text{coc}} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1}, \right.
\]

\[
(1 - \theta)(3 - 2\theta) \xi_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\xi_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m-1}
\]

\[
+ 2\Delta t C_{\text{coc}} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\xi_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m-1}, \right.
\]

\[
(1 - \theta)(3 - 2\theta) \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1},
\]

\[
-2\Delta t C_{\text{coc}} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1}, \right.
\]

\[
(1 - \theta)(3 - 2\theta) \xi_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\xi_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m-1}
\]

\[
\leq \frac{\tilde{C}_{\text{coc}} \Delta t}{6} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right)^2
\]

\[
+ \frac{6C_3 h^{-1} \Delta t}{C_{\text{coc}}} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\xi_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m-1} \right)^2
\]

\[
+ \frac{g^2 C_3 h^{-1} \Delta t}{6} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right)^2
\]

\[
+ \frac{6g C_3 h^{-1} \Delta t}{C_{\text{coc}}} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\xi_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m-1} \right)^2
\]

\[
+ \frac{\tilde{C}_{\text{coc}} \Delta t}{3} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right)^2
\]

\[
+ \frac{g^2 C_3 h^{-1} \Delta t}{6} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\xi_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m-1} \right)^2
\]

\[
\leq \frac{\tilde{C}_{\text{coc}} \Delta t}{6} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right)^2
\]

\[
+ \frac{6g C_3 h^{-1} \Delta t}{C_{\text{coc}}} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\xi_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m-1} \right)^2
\]

\[
+ \frac{\tilde{C}_{\text{coc}} \Delta t}{3} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\eta_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi_p}^{m-1} \right)^2
\]

\[
+ \frac{g^2 C_3 h^{-1} \Delta t}{6} \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m+1} + (4\theta - 1 - 2\theta^2)\xi_{\phi_p}^m + \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{\phi_p}^{m-1} \right)^2
\]
\[ + \frac{6\Delta t}{C_{\text{cor}}} \left\| \phi^m_p + \Delta t \phi^m_t - \int_{t_m}^{t_{m+1}} (t^m - t) \phi_t \, dt \right\|_f^2, \]

where the last inequality follows from properly chosen constant \( C_3, C_4, C_5 \) and \( C_6 \).

By Taylor expansion with the integral remainder,

\[ \phi^{m+1}_p = \phi^m_p + \Delta t \phi^m_t - \int_{t_m}^{t_{m+1}} (t^m - t) \phi_t \, dt, \]

then

\[ (1 - \theta)(3 - 2\theta)\phi^m_p = \frac{(1 - \theta)(3 - 2\theta)}{2} \phi^m_p - \frac{(1 - \theta)(3 - 2\theta)}{2} \phi^{m+1}_p \]

\[ = \frac{(1 - \theta)(3 - 2\theta)}{2} \int_{t_m}^{t_{m+1}} (t^m - t) \phi_t \, dt - \frac{(1 - \theta)(3 - 2\theta)}{2} \int_{t_{m-1}}^{t_m} (t^m - t) \phi_t \, dt, \]

similarly,

\[ (1 - \theta)(3 - 2\theta)u^m_f = \frac{(1 - \theta)(3 - 2\theta)}{2} u^m_f - \frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_f \]

\[ = \frac{(1 - \theta)(3 - 2\theta)}{2} \int_{t_m}^{t_{m+1}} (t^m - t) u_t \, dt - \frac{(1 - \theta)(3 - 2\theta)}{2} \int_{t_{m-1}}^{t_m} (t^m - t) u_t \, dt. \]

So using the lemma \( 3.2 \) and taking \( \varepsilon_7 = \frac{3}{C_{\text{cor}}}, \varepsilon_8 = \frac{3\delta}{C_{\text{cor}}} \), we get

\[ - 2\Delta t \varepsilon_7 \left( 1 - \theta \right) (3 - 2\theta) \eta^{m+1}_{\Omega_f} + (4\theta - 1 - 2\theta^2) \eta^m_{\Omega_f} + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m+1}_{\Omega_f}, \]

\[ (1 - \theta)(3 - 2\theta) \phi^m_p - \frac{(1 - \theta)(3 - 2\theta)}{2} \phi^{m+1}_p, \]

\[ + 2\Delta t \varepsilon_7 \left( 1 - \theta \right) (3 - 2\theta) u^{m+1}_f + (4\theta - 1 - 2\theta^2) u^m_f + \frac{(1 - \theta)(1 - 2\theta)}{2} u^{m+1}_f, \]

\[ \leq \hat{C}_{\text{cor}} \Delta t \left\| \left( 1 - \theta \right) (3 - 2\theta) \eta^{m+1}_{\Omega_f} + (4\theta - 1 - 2\theta^2) \eta^m_{\Omega_f} + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m+1}_{\Omega_f} \right\|_H^2 \]

\[ + \frac{6\hat{C}_h h^{-1} \Delta t}{C_{\text{cor}}} \left\| \left( 1 - \theta \right) (3 - 2\theta) \phi^m_p - \frac{(1 - \theta)(3 - 2\theta)}{2} \phi^{m+1}_p \right\|_p^2 \]

\[ + \frac{g\hat{C}_h h^{-1} \Delta t}{6} \left\| \left( 1 - \theta \right) (3 - 2\theta) u^m_f - \frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_f \right\|_f^2, \]

\[ \leq \hat{C}_{\text{cor}} \Delta t \left\| \left( 1 - \theta \right) (3 - 2\theta) \eta^{m+1}_{\Omega_f} + (4\theta - 1 - 2\theta^2) \eta^m_{\Omega_f} + \frac{(1 - \theta)(1 - 2\theta)}{2} \eta^{m+1}_{\Omega_f} \right\|_H^2 \]

\[ + \frac{6\hat{C}_h \Delta t}{C_{\text{cor}}} \left\| \left( 1 - \theta \right) (3 - 2\theta) \phi^m_p - \frac{(1 - \theta)(3 - 2\theta)}{2} \phi^{m+1}_p \right\|_p^2 \]

\[ + \frac{g\hat{C}_h \Delta t}{6} \left\| \left( 1 - \theta \right) (3 - 2\theta) u^m_f - \frac{(1 - \theta)(3 - 2\theta)}{2} u^{m+1}_f \right\|_f^2. \]

\[ \hat{C}_{\text{cor}} \Delta t \left\| \phi^{m+1}_p - \phi^m_p - \Delta t \phi^m_t + \frac{(1 - \theta)(3 - 2\theta)}{2} \phi^{m+1}_p \right\|_f^2, \]

\[ \leq \hat{C}_{\text{cor}} \Delta t \left\| \phi^m_p - \phi^{m+1}_p \right\|_f^2, \]

\[ \leq \hat{C}_{\text{cor}} \Delta t \left\| \phi^m_p - \phi^{m+1}_p \right\|_f^2. \]
\[
+ \frac{\hat{C}_{\text{co}} \Delta t}{6} \left\| (1 - \theta)(3 - 2\theta) \eta_{\eta, \theta}^{m+1} + (4\theta - 1 - 2\theta^2) \eta_{\eta, \theta}^{m} + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi, \theta}^{m-1} \right\|_{H_p}^2
+ \frac{(1 - \theta)^2(3 - 2\theta)^2 \Delta t}{2C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|u_{\text{tr}}\|^2 dt + \frac{g(1 - \theta)(3 - 2\theta)^2 \Delta t}{2C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|\phi_{\text{tr}}\|^2 dt,
\]

where the inequality follows from properly chosen constant $C_7$ and $C_8$.

Then the interface terms on the right-hand side can be rewritten as

\[
- 2\Delta t C \left( \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\theta, \theta}^{m+1} + (4\theta - 1 - 2\theta^2) \eta_{\theta, \theta}^{m} + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi, \theta}^{m-1}, \quad (2 - \theta) \phi_{\text{ph}, \theta}^{m} - (1 - \theta)(3 - 2\theta) \phi_{\text{ph}, \theta}^{m+1} - (4\theta - 1 - 2\theta^2) \phi_{\text{ph}, \theta}^{m} - \frac{(1 - \theta)(3 - 2\theta)}{2} \phi_{\text{ph}, \theta}^{m-1} \right)
+ 2\Delta t C \left( (1 - \theta) \phi_{\text{h}, \theta}^{m+1} - (1 - \theta) \phi_{\text{h}, \theta}^{m} - \frac{(1 - \theta)(3 - 2\theta)}{2} \phi_{\text{h}, \theta}^{m+1} - (4\theta - 1 - 2\theta^2) \phi_{\text{h}, \theta}^{m} - \frac{(1 - \theta)(3 - 2\theta)}{2} \phi_{\text{h}, \theta}^{m-1} \right)
\]

\[
\leq \frac{\hat{C}_{\text{co}} \Delta t}{2} \left\| (1 - \theta)(3 - 2\theta) \eta_{\theta, \theta}^{m+1} + (4\theta - 1 - 2\theta^2) \eta_{\theta, \theta}^{m} + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi, \theta}^{m-1} \right\|_{H_p}^2
+ \frac{g(1 - \theta)(3 - 2\theta)^2 \Delta t}{2C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|u_{\text{tr}}\|^2 dt + \frac{g(1 - \theta)(3 - 2\theta)^2 \Delta t}{2C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|\phi_{\text{tr}}\|^2 dt.
\]

Combining the (1.11) and (4.24) together and sum the (4.10) over $m = 1, 2, ... N - 1$, and using the same method as Theorem 3.2 we get

\[
\frac{3 - 2\theta}{4(1 - \theta)} \left\| \eta_{\theta}^{m} \right\|^2 + \hat{C}_{\text{co}} \Delta t \sum_{m=1}^{N-1} \left\| (1 - \theta)(3 - 2\theta) \eta_{\theta}^{m+1} + (4\theta - 1 - 2\theta^2) \eta_{\theta}^{m} + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi}^{m-1} \right\|_{H_p}^2
+ \frac{g(3 - 2\theta)}{4(1 - \theta)} \left\| \eta_{\phi}^{m} \right\|^2 + \hat{C}_{\text{co}} \Delta t \sum_{m=1}^{N-1} \left\| (1 - \theta)(3 - 2\theta) \eta_{\phi}^{m+1} + (4\theta - 1 - 2\theta^2) \eta_{\phi}^{m} + \frac{(1 - \theta)(3 - 2\theta)}{2} \eta_{\phi}^{m-1} \right\|_{H_p}^2
\leq \sum_{m=1}^{N-1} \left( (2(6\theta^4 - 200\theta^3 + 257\theta^2 - 130\theta + 24) \Delta t) \int_{t_{m-1}}^{t_{m+1}} \|u_{\text{tr}}\|^2 dt + \frac{2(6\theta^4 - 200\theta^3 + 257\theta^2 - 130\theta + 24) \Delta t}{5C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|\phi_{\text{tr}}\|^2 dt \right.
\]

\[
+ \frac{3(2\theta^2 - 36\theta + 17)}{2C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|g_{\text{tr}}\|^2 dt + \frac{3g(2\theta^2 - 36\theta + 17)}{2C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|g_{\text{tr}}\|^2 dt + \frac{(1 - \theta)^2(1 - 2\theta)^2 \Delta t^4}{2C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|u_{\text{tr}}\|^2 dt + \frac{g(1 - \theta)(3 - 2\theta)^2 \Delta t^4}{2C_{\text{co}}} \int_{t_{m-1}}^{t_{m+1}} \|\phi_{\text{tr}}\|^2 dt
\]
Then using the triangle inequality, we end the proof.

\[ \frac{6g \Delta t}{C_{coe}} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{m+1}^{p \phi} + \frac{(1 - \theta)(1 - 2\theta)}{2} \xi_{m-1}^{p \phi} \right\|_p^2 + \frac{6 \Delta t}{C_{coe}} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{m+1}^{u \phi} + \frac{(1 - \theta)(1 - 2\theta)}{2} \xi_{m-1}^{u \phi} \right\|_p^2 + \frac{6g \Delta t}{C_{coe}} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{m+1}^{m \phi} - \frac{(1 - \theta)(1 - 2\theta)}{2} \xi_{m-1}^{m \phi} \right\|_p^2 + \frac{6 \Delta t}{C_{coe}} \left\| \frac{(1 - \theta)(3 - 2\theta)}{2} \xi_{m+1}^{f \phi} - \frac{(1 - \theta)(1 - 2\theta)}{2} \xi_{m-1}^{f \phi} \right\|_p^2. \]

Figure 2: Speed contours and velocity streamlines with $\theta = 1/6$

5 Numerical experiments

In this section, we do two numerical experiments. In the first test, we verify the effectiveness of the coupled and decoupled schemes. In the second test, we show that the convergence orders of the coupled and decoupled schemes are increased from the first order to the second order, and the decoupled scheme is more efficient than the coupled scheme. The following numerical experiments are implemented using the Software package FreeFEM++, and we set all the physical parameters $n$, $\rho$, $g$, $\nu$, $K$, $S$ and $\alpha$ are equal to 1, and the initial conditions, boundary conditions and the source terms follow from the exact solution.

5.1 Test of the effectiveness

Here we apply the numerical example from [35], let the computational domain $\Omega$ be composed of $\Omega_f = (0, \pi) \times (0, 1)$ and $\Omega_p = (0, \pi) \times (-1, 0)$ with the interface $\Gamma = (0, \pi) \times 0$. The Taylor-Hood element (P2-P1) and the piecewise quadratic polynomials (P2) are used for the free fluid equation and the porous media flow equation. The exact solution is given by

\[
\begin{align*}
    u_f &= \left[ \frac{1}{\pi} \sin(2\pi y) \cos(x)e^t, -2 + \frac{1}{\pi^2} \sin^2(\pi y) \sin(x)e^t \right], \\
    p_f &= 0, \\
    \phi_p &= (e^y - e^{-y}) \sin(x)e^t.
\end{align*}
\]
Figures 2-4 show speed contours and velocity streamlines of coupled Linear Multistep method, decoupled Linear Multistep method, coupled Linear Multistep method plus time filter and decoupled Linear Multistep method plus time filter with $\theta = 1/6, 1/4, 1/3$, respectively. From these figures, we can see that these algorithms can effectively simulate fluid motion.

5.2 Test of the convergence and efficiency

Here we use the example from [33], considering the model problem on $\Omega_f = (0,1) \times (1,2)$ and $\Omega_p = (0,1) \times (0,1)$ with the interface $\Gamma = (0,1) \times 1$. We use the well-known MINI elements(P1b-P1) for the fluid
Table 1: The convergence orders of coupled Linear Multistep method at time $T = 1$, with varying time step $\Delta t$ but fixed mesh size $h = \frac{1}{8}$.

| $\Delta t$ | $\|u_h^{\Delta t} - u_h^\Delta t\|_{L^2}$ | $\rho_{u_h}$ | $\|p_h^{\Delta t} - p_h^\Delta t\|_{L^2}$ | $\rho_{p_f}$ | $\|\phi_h^{\Delta t} - \phi_h^\Delta t\|_{L^2}$ | $\rho_{\phi_p}$ | CPU(s) |
|------------|----------------------------------|----------|----------------------------------|----------|----------------------------------|----------|-------|
| $\frac{1}{32}$ | 3.44699e-05 | 1.96 | 0.004841111 | 1.98 | 4.64995e-05 | 1.97 | 0.924 |
| $\frac{1}{64}$ | 1.75611e-05 | 1.98 | 0.00245076 | 1.99 | 2.36541e-05 | 1.98 | 1.889 |
| $\frac{1}{128}$ | 8.86308e-06 | 1.99 | 0.00123278 | 1.99 | 1.19296e-05 | 1.99 | 3.917 |
| $\frac{1}{256}$ | 4.5231e-06 | 2.00 | 0.000618217 | 2.00 | 5.99062e-06 | 2.00 | 7.704 |
| $\frac{1}{512}$ | 2.23136e-06 | - | 0.00309563 | - | 3.00179e-06 | - | 15.411 |

Table 2: The convergence orders of coupled Linear Multistep method plus time filter at time $T = 1$, with varying time step $\Delta t$ but fixed mesh size $h = \frac{1}{8}$.

| $\Delta t$ | $\|u_h^{\Delta t} - u_h^\Delta t\|_{L^2}$ | $\rho_{u_h}$ | $\|p_h^{\Delta t} - p_h^\Delta t\|_{L^2}$ | $\rho_{p_f}$ | $\|\phi_h^{\Delta t} - \phi_h^\Delta t\|_{L^2}$ | $\rho_{\phi_p}$ | CPU(s) |
|------------|----------------------------------|----------|----------------------------------|----------|----------------------------------|----------|-------|
| $\frac{1}{32}$ | 0.0003555654 | 4.17 | 0.004973097 | 4.19 | 0.000108397 | 4.19 | 0.986 |
| $\frac{1}{64}$ | 8.53369e-05 | 4.09 | 0.00246429 | 4.10 | 2.58998e-05 | 4.10 | 1.98  |
| $\frac{1}{128}$ | 2.08824e-05 | 4.04 | 0.00123112 | 4.05 | 6.32373e-06 | 4.05 | 4.058 |
| $\frac{1}{256}$ | 5.16381e-06 | 4.04 | 0.000615749 | 4.02 | 1.56196e-06 | 4.02 | 8.229 |
| $\frac{1}{512}$ | 1.28384e-06 | - | 0.00309563 | - | 3.88112e-06 | - | 16.387 |

equations and the linear Lagrangian elements(P1) for the porous media flow equation. The exact solution is:

$$u_f = ((x^2(y-1)^2+y)\cos(t), -\frac{2}{3}x(y-1)^3\cos(t) + (2 - \pi \sin(\pi x))\cos(t)),$$

$$p_f = (2 - \pi \sin(\pi x)) \sin(\frac{1}{2} \pi y) \cos(t),$$

$$\phi_p = (2 - \pi \sin(\pi x))(1 - y - \cos(\pi y)) \cos(t).$$

In order to demonstrate the convergence of coupled and decoupled Linear Multistep methods plus time filter, we consider the Linear Multistep method when $\theta = 1/3$ and list the error, convergence order and CPU time with varying time step $\Delta t$ and mesh size $h$ in Table 3.

We first calculate the convergence orders of the coupled and decoupled Linear Multistep methods and the Linear Multistep methods plus time filter by varying the time step $\Delta t$ with a fixed mesh size $h$, using the same method as in [33]. So we define

$$\rho_{v} = \frac{\|v_h^{\Delta t} - v_h^\Delta t\|_{L^2}}{\|v_h^\Delta t - v_h^\Delta t\|_{L^2}}$$

where $v = u_f, \phi_p, p_f, \nabla u_f$ and $\nabla \phi_p$, then $\rho_{v} \approx \frac{\Delta t - \Delta t}{\Delta t - \Delta t}$. In particular, $\rho \approx 4$ for $\gamma = 2$ and $\rho \approx 8$ for $\gamma = 3$, when the corresponding order of convergence in time is of $O(\Delta t^2)$ and $O(\Delta t^3)$, respectively. So we change the time step $\Delta t$ from $\frac{1}{32}$ to $\frac{1}{512}$, but fix the mesh size $h = \frac{1}{8}$. We get a set of values for $\rho_{v}$ in the Table 3 from Table 2 and Table 4, we can clearly state that the coupled and decoupled Linear Multistep methods plus time filter are convergent in time $\Delta t$ and the orders of convergence both are $O(\Delta t^2)$.

At the same time, we compare Table 3 with Table 2 and Table 3 with Table 4. It can be found that the convergence orders of coupled and decoupled Linear Multistep methods are first order, while the convergence orders of coupled and decoupled Linear Multistep methods plus time filter are second order, that is, the convergence orders of Linear Multistep method can be improved from first order to second order by adding time filter algorithm. And by comparing the CPU time in the tables, we can find that the computational efficiency of the Linear Multistep method plus time filter is slightly lower than the Linear Multistep method, but the difference between the two schemes is not large, so we can know the Linear Multistep method plus time filter is more efficient because it can achieve a higher order of convergence with similar computational efficiency.

Then we calculate the convergence orders of the coupled and decoupled Linear Multistep methods and the Linear Multistep methods plus time filter by varying the mesh size $h$ with a fixed time step $\Delta t$. Thus, the approximation error is mainly determined by the mesh size $h$, so here we estimate the corresponding convergence order by

$$\rho_{h,v} = \frac{\log((h_1)/(h_2))}{\log((h_1)/(h_2))}$$
Table 3: The convergence orders of decoupled Linear Multistep method at time $T = 1$, with varying time step $\Delta t$ but fixed mesh size $h = \frac{1}{8}$.

| $\Delta t$ | $\|u_h^{\Delta t} - u_{1/2}^{\Delta t}\|_{L^2}$ | $\rho_{u_{1/2}}$ | $\|p_h^{\Delta t} - p_{1/2}^{\Delta t}\|_{L^2}$ | $\rho_{p_{1/2}}$ | $\|\phi_h^{\Delta t} - \phi_{1/2}^{\Delta t}\|_{L^2}$ | $\rho_{\phi_{1/2}}$ | CPU (s) |
|------------|----------------------------------|----------------|----------------------------------|----------------|----------------------------------|----------------|--------|
| $\frac{1}{20}$ | 3.26496e-05                      | 1.93           | 0.00485986                       | 1.98           | 8.50835e-05                      | 2.58           | 0.606  |
| $\frac{1}{40}$ | 1.69223e-05                      | 1.95           | 0.00245478                       | 1.99           | 3.2967e-05                       | 2.32           | 1.247  |
| $\frac{1}{80}$ | 8.68426e-05                      | 1.97           | 0.0012337                        | 1.99           | 1.4211e-05                       | 2.17           | 2.482  |
| $\frac{1}{160}$ | 4.04543e-06                      | 1.98           | 0.00061844                       | 1.99           | 6.55462e-06                      | 2.09           | 5.25   |
| $\frac{1}{320}$ | 2.12938e-06                      | -              | 0.000309618                      | -              | 3.14196e-06                      | -              | 10.534 |

Table 4: The convergence orders of decoupled Linear Multistep method plus time filter at time $T = 1$, with varying time step $\Delta t$ but fixed mesh size $h = \frac{1}{8}$.

| $\Delta t$ | $\|u_h^{\Delta t} - u_{1/2}^{\Delta t}\|_{L^2}$ | $\rho_{u_{1/2}}$ | $\|p_h^{\Delta t} - p_{1/2}^{\Delta t}\|_{L^2}$ | $\rho_{p_{1/2}}$ | $\|\phi_h^{\Delta t} - \phi_{1/2}^{\Delta t}\|_{L^2}$ | $\rho_{\phi_{1/2}}$ | CPU (s) |
|------------|----------------------------------|----------------|----------------------------------|----------------|----------------------------------|----------------|--------|
| $\frac{1}{20}$ | 0.000358215                      | 4.17           | 0.00498508                       | 2.02           | 0.000123915                      | 4.19           | 0.654  |
| $\frac{1}{40}$ | 8.59611e-05                      | 4.09           | 0.00247109                       | 2.00           | 2.95761e-05                      | 4.10           | 1.278  |
| $\frac{1}{80}$ | 2.10364e-05                      | 4.04           | 0.0012332                        | 2.00           | 7.21654e-06                      | 4.05           | 2.659  |
| $\frac{1}{160}$ | 5.20270e-06                      | 4.04           | 0.00061632                       | 2.00           | 1.78181e-06                      | 4.02           | 5.346  |
| $\frac{1}{320}$ | 1.29337e-06                      | -              | 0.00030815                       | -              | 4.42655e-07                      | -              | 10.648 |

where $e_v(h)$ is the error computed by the algorithm with time step $\Delta t$. So we change the mesh size $h$ from $\frac{1}{4}$ to $\frac{1}{8}$, but fix the time step $\Delta t = 0.01$. Similarly, we get a set of values for $\rho_{h,v}$ in the Table 5-8, from them we can clearly state that the coupled and decoupled Linear Multistep methods plus time filter are convergent in mesh size $h$ and the orders of convergence both are $O(h^2)$. Finally, compared to the coupled Linear Multistep method plus time filter, the decoupled Linear Multistep method plus time filter takes less computation time, that is, it is more computationally efficient.

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Table 5: The convergence orders of coupled Linear Multistep method at time $T = 1$, with varying mesh size $h$, but fixed time step $\Delta t = 0.01$.

| $h$  | $\|u_f - u_{fh}\|_{L^2}^{\rho_{h,u_f}}$ | $\|u_f - u_{fh}\|_{L^2}^{\rho_{h,u_f}}$ | $\|p_f - p_{fh}\|_{L^2}^{\rho_{h,p_f}}$ | $\|\phi_p - \phi_{ph}\|_{L^2}^{\rho_{h,\phi_p}}$ | CPU (s) |
|------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|---------|
| $1/4$ | 0.0670348                       | 1.99                            | 0.959017                        | 1.71                            | 0.194706 0.288 |
| $1/8$ | 0.0168619                       | 2.00                            | 0.292807                        | 1.61                            | 0.054276 4.606 |
| $1/16$ | 0.00422473                      | 2.00                            | 0.0956361                       | 1.55                            | 0.014021 14.415 |
| $1/32$ | 0.00105852                      | 1.98                            | 0.0325627                       | 1.67                            | 0.003566 54.096 |
| $1/64$ | 0.000269156                     | -                               | 0.0102568                       | -                               | -        213.743 |

Table 6: The convergence orders of coupled Linear Multistep method plus time filter at time $T = 1$, with varying mesh size $h$, but fixed time step $\Delta t = 0.01$.

| $h$  | $\|u_f - u_{fh}\|_{L^2}^{\rho_{h,u_f}}$ | $\|p_f - p_{fh}\|_{L^2}^{\rho_{h,p_f}}$ | $\|\phi_p - \phi_{ph}\|_{L^2}^{\rho_{h,\phi_p}}$ | CPU (s) |
|------|---------------------------------|---------------------------------|---------------------------------|---------|
| $1/4$ | 0.0670315                       | 1.99                            | 0.959261                        | 1.71                            | 0.194682 2.361 |
| $1/8$ | 0.0168558                       | 2.00                            | 0.292947                        | 1.61                            | 0.054239 4.757 |
| $1/16$ | 0.00421773                      | 2.00                            | 0.0957064                       | 1.55                            | 0.013980 14.633 |
| $1/32$ | 0.00195118                      | 2.00                            | 0.0325911                       | 1.68                            | 0.003523 53.459 |
| $1/64$ | 0.000262099                     | -                               | 0.0101905                       | -                               | -        213.075 |

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Table 7: The convergence orders of decoupled Linear Multistep method at time $T = 1$, with varying mesh size $h$, but fixed time step $\Delta t = 0.01$.

| $h$ | $\|u_f - u_{fh}\|_{L^2}$ | $\rho_{h,u_f}$ | $\|p_f - p_{fh}\|_{L^2}$ | $\rho_{h,p_f}$ | $\|\phi_p - \phi_{ph}\|_{L^2}$ | $\rho_{h,\phi_p}$ | CPU ($s$) |
|-----|-----------------|----------------|----------------|----------------|----------------|----------------|---------|
| $\frac{1}{4}$ | 0.0670347 | 1.99 | 0.959015 | 1.71 | 0.194708 | 1.84 | 1.935 |
| $\frac{1}{8}$ | 0.0168618 | 2.00 | 0.292805 | 1.61 | 0.0542803 | 1.95 | 3.13 |
| $\frac{1}{16}$ | 0.00422463 | 2.00 | 0.0956351 | 1.55 | 0.0140256 | 1.97 | 7.715 |
| $\frac{1}{32}$ | 0.00105841 | 1.98 | 0.0325623 | 1.67 | 0.0035706 | 1.93 | 32.77 |
| $\frac{1}{64}$ | 0.000269047 | - | 0.0102574 | - | 0.000934248 | - | 126.761 |

Table 8: The convergence orders of decoupled Linear Multistep method plus time filter at time $T = 1$, with varying mesh size $h$ but fixed time step $\Delta t = 0.01$.

| $h$ | $\|u_f - u_{fh}\|_{L^2}$ | $\rho_{h,u_f}$ | $\|p_f - p_{fh}\|_{L^2}$ | $\rho_{h,p_f}$ | $\|\phi_p - \phi_{ph}\|_{L^2}$ | $\rho_{h,\phi_p}$ | CPU ($s$) |
|-----|-----------------|----------------|----------------|----------------|----------------|----------------|---------|
| $\frac{1}{4}$ | 0.0670315 | 1.99 | 0.959258 | 1.71 | 0.194686 | 1.84 | 2.034 |
| $\frac{1}{8}$ | 0.0168557 | 2.00 | 1.292945 | 1.61 | 0.0544244 | 1.96 | 3.244 |
| $\frac{1}{16}$ | 0.00421761 | 2.00 | 0.0957054 | 1.55 | 0.0139853 | 1.99 | 8.096 |
| $\frac{1}{32}$ | 0.00105106 | 2.02 | 0.0325911 | 1.68 | 0.00352877 | 1.98 | 27.95 |
| $\frac{1}{64}$ | 0.000261987 | - | 0.0101928 | - | 0.000891959 | - | 114.218 |

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