Continuous rational functions on real and \( p \)-adic varieties

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A rational function on \( \mathbb{R}^n \) is a quotient of two polynomials

\[
f(x_1, \ldots, x_n) := \frac{p(x_1, \ldots, x_n)}{q(x_1, \ldots, x_n)}.
\]

Strictly speaking, a rational function \( f \) is not really a function on \( \mathbb{R}^n \) in general since it is defined only on the dense open set where \( q \neq 0 \). Nonetheless, even if \( q \) vanishes at some points of \( \mathbb{R}^n \), it can happen that \( f \) extends to an everywhere defined continuous function \( f^c \); such an \( f^c \) is unique. It is customary to identify \( f \) with \( f^c \) and call \( f \) itself a continuous rational function on \( \mathbb{R}^n \). For instance,

\[
\frac{p(x_1, \ldots, x_n)}{x_1^{2m} + \cdots + x_n^{2m}}
\]

is a continuous rational function on \( \mathbb{R}^n \) if every monomial in \( p \) has degree \( > 2m \).

The above definition of continuous rational functions makes sense on any real algebraic set \( X \subset \mathbb{R}^n \), as long as the open set where \( f \) is defined is dense in \( X \) in the Euclidean topology. This condition always holds on smooth varieties; see Definition 7 for real algebraic sets and their basic properties.

The aim of this note is to consider four basic problems on continuous rational functions.

**Question 1** Let \( X \) be a real algebraic set and \( Z \) a closed algebraic subset.

1. Let \( f \) be a continuous rational function defined on \( X \). Is the restriction \( f|_Z \) a rational function on \( Z \)?
2. Let \( g \) be a continuous rational function defined on \( Z \). Can one extend it to a continuous rational function on \( X \)?
(3) Assume that $X \setminus Z$ is Zariski dense in $X$. Is $f$ uniquely determined by the restriction $f|_{X \setminus Z}$?

(4) Which systems of linear equations

$$\sum_j f_{ij} \cdot y_j = g_i \quad i = 1, \ldots, m$$

have continuous rational solutions where the $g_i, f_{ij}$ are polynomials (or rational functions) on $X$.

Section 1 contains a series of examples related to Questions 1.1–4. They show that continuous rational functions behave rather unusually. Even more, they suggest to us that there is no good general definition of continuous rational functions on arbitrary real algebraic varieties.

In Sect. 2 we introduce hereditarily rational functions. These have the good properties that one would expect based on studying rational functions on $\mathbb{R}^n$. Proposition 8 implies that a continuous rational function on a smooth real variety is hereditarily rational. In Theorem 10 this concept is used to give a complete answer to Question 1.2.

The key result says that a continuous hereditarily rational function defined on a closed subvariety extends to a continuous hereditarily rational function on the ambient variety. We give two proofs of this result.

The first version, Proposition 11, is quite explicit and gives optimal control over the regularity of the extension. It uses basic properties of semialgebraic sets, thus it works directly only over $\mathbb{R}$. The second variant, discussed in Sect. 3, works for all locally compact fields. However, it relies on a rather strong version of resolution of singularities and it does not control the regularity of the extension.

Subsequent papers [17, 18] consider the strong extension theorem for Henselian real valued fields using the quantifier elimination theory in the language $\mathcal{L}$ of Denef and Pas [5, 19].

Topological aspects of continuous rational maps between smooth algebraic varieties were investigated in [13]. Hereditarily rational functions are also studied in [7]. Applications of hereditarily rational functions are given in the paper [14] discussing stratified-algebraic vector bundles on real algebraic varieties.

1 Examples

The following example shows that the answer to Questions 1.1–2 is not always positive.

Example 2 Consider the surface $S$ and the rational function $f$ given by

$$S := (x^3 - (1 + z^2)y^3 = 0) \subset \mathbb{R}^3 \quad \text{and} \quad f(x, y, z) := \frac{x}{y}.$$ 

We claim that

(1) $S$ is a real analytic submanifold of $\mathbb{R}^3$,
(2) $f$ is defined away from the $z$-axis,
(3) $f$ extends to a real analytic function $f^c$ on $S$, yet
(4) the restriction of $f^c$ to the $z$-axis is not rational and
(5) $f$ can not be extended to a continuous rational function on $\mathbb{R}^3$.

Proof Note that

$$x^3 - (1 + z^2)y^3 = (x - (1 + z^2)^{1/3}y)(x^2 + (1 + z^2)^{1/3}xy + (1 + z^2)^{2/3}y^2).$$
The first factor defines $S$ as a real analytic submanifold of $\mathbb{R}^3$. The second factor only vanishes along the $z$-axis, which is contained in $S$. Therefore $x - y\sqrt[3]{1 + z^2}$ vanishes on $S$, hence

$$f^c|_S = \frac{x}{y}|_S = \sqrt[3]{1 + z^2}|_S$$

and so $f(0, 0, z) = \sqrt[3]{1 + z^2}$.

Assume finally that $F$ is a continuous rational function on $\mathbb{R}^3$ whose restriction to $S$ is $f$. Then $F$ and $f$ have the same restrictions to the $z$-axis. We show below that the restriction of any continuous rational function $F$ defined on $\mathbb{R}^3$ to the $z$-axis is a rational function. Thus $F|_{z-axis}$ does not equal $\sqrt[3]{1 + z^2}$, a contradiction.

To see the claim, write $F = p(x, y, z)/q(x, y, z)$ where $p, q$ are polynomials. We may assume that they are relatively prime. Since $F$ is continuous everywhere, $x$ can not divide $q$. Hence $F|_{(x=0)} = p(0, y, z)/q(0, y, z)$. By canceling common factors, we can write this as $F|_{(x=0)} = p_1(y, z)/q_1(y, z)$ where $p_1, q_1$ are relatively prime polynomials. As before, $y$ can not divide $q_1$, hence $F|_{z-axis} = p_1(0, z)/q_1(0, z)$ is a rational function.

(Nota that we seemingly have not used the continuity of $F$: for any rational function $f(x, y, z)$ the above procedure defines a rational function on the $z$-axis. However, if we use $x, y$ in reverse order, we could get a different rational function. This happens, for instance, for $f = x^2z/(x^2 + y^2)$. Here $(f|_{(x=0)})|_{z-axis} = 0$ and $(f|_{(y=0)})|_{z-axis} = z$).

In the above example, the problems arise since $S$ is not normal as a complex algebraic surface. However, the key properties (2.4–5) can also be realized on real points of a normal hypersurface.

**Example 3** Consider

$$W := ((x^3 - (1 + t^2)y^3 + z^6 + y^7 = 0) \subset \mathbb{R}^4$$

and $f(x, y, z, t) := \frac{x}{y}$.

We easily see that the singular locus of the corresponding complex algebraic hypersurface $W(\mathbb{C})$ is the $t$-axis. $W(\mathbb{C})$ is normal since a hypersurface in a smooth complex variety is normal iff its singular set has codimension $\geq 2$; see for instance [20, Chap.III.C,Prop.9] or [6, Thm.11.2].

Let us blow up the $t$-axis. There is one relevant chart, where $x_1 = x/y, y_1 = y, z_1 = z/y$.

We get the smooth threefold

$$W' := ((x_1^3 - (1 + t^2)y_1^3 + z_1^6 + y_1^7 = 0) \subset \mathbb{R}^4$$

Each point $(0, 0, 0, t)$ has only 1 preimage in $W'$, given by $(\sqrt[3]{1 + t^2}, 0, 0, t)$ and the projection $\pi : W' \to W$ is a homeomorphism. Thus $W$ is a topological manifold, but it is not a differentiable submanifold of $\mathbb{R}^4$.

Since $f \circ \pi = x_1$ is a regular function, we conclude that $f \circ \pi$ extends to a continuous (even regular) function $(f \circ \pi)^c$ on $W'$. Since $\pi$ is a homeomorphism, we get the continuous function $f^c := (f \circ \pi)^c \circ \pi^{-1}$ on $X$ extending $f$. By construction, $f(0, 0, 0, t) = \sqrt[3]{1 + t^2}$, thus, as before, $f$ can not be extended to a continuous rational function on $\mathbb{R}^4$.

For any $m \geq 1$ we get similar examples of normal hypersurfaces and rational functions

$$W_m := ((x^3 - (1 + t^2)y^3 + z_1^6 + \cdots + z_m^6 + y^7 = 0) \subset \mathbb{R}^{3+m}$$

and $f := \frac{x}{y}$.

Note that for all $m$, the singular set is still the $t$-axis and, for $m \gg 1$, the $W_m$ have rational, even terminal singularities (see [9] for the definitions of these singularities). In fact, we do not know any natural class of singularities (other than smooth points) where Questions 1.1–2 have a positive answer.
In order to elucidate Question 1.3, next we give an example of a continuous rational function $f$ on $\mathbb{R}^3$ and of an irreducible algebraic surface $S \subset \mathbb{R}^3$ such that $f|_S$ is zero on a Zariski dense open subset of $S$ yet $f^c|_S$ is not identically zero.

**Example 4** On $\mathbb{R}^3$ consider the rational function
\[
f(x, y, z) := z^2 \cdot \frac{x^2 + y^2 z^2 - y^3}{x^2 + y^2 z^2 + y^4}.
\]
Its only possible discontinuities are along the $z$-axis. To analyze its behavior there, rewrite it as
\[
f = z^2 - y(1 + y) \cdot \frac{y^2 z^2}{x^2 + y^2 z^2 + y^4}.
\]
The fraction is bounded by 1, hence $f$ extends to a continuous function $f^c$ on $\mathbb{R}^3$ and $f^c(0, 0, z) = z^2$.

Our example is the restriction of $f$ to the surface $S := \{x^2 + y^2 z^2 - y^3 = 0\} \subset \mathbb{R}^3$. Topologically, $S$ has two parts. One is the $z$-axis, which is also the singular locus of $S$. The other part $S^*$ is the image of the map $\mathbb{R}^2_{st} \to \mathbb{R}^3$ given by
\[(s, t) \mapsto (s(s^2 + t^2), (s^2 + t^2), t).
\]
The two parts intersect only at the origin. Thus $S^*$ is Zariski dense but not Euclidean dense in $S$. We see that $f^c$ vanishes on $S^*$ but not on the $z$-axis.

More generally, let $g(z)$ be any rational function without real poles. Then $g(z)f(x, y, z)$ vanishes on $S^*$ and its restriction to the $z$-axis is $z^2g(z)$.

(The best known example of a surface with a Zariski dense open set that is not Euclidean dense is the Whitney umbrella $W := \{(x^2 = y^2z) \subset \mathbb{R}^3$. The Euclidean closure of $W \setminus \{(x = y = 0)\}$ does not contain the “handle” $(x = y = 0, z < 0)$. In this case, a continuous rational function is determined by its restriction to $W \setminus \{(x = y = 0)\}$. The Euclidean closure of $W \setminus \{(x = y = 0)\}$ contains the half line $(x = y = 0, z \geq 0)$, and a rational function on a line is determined by its restriction to any interval.)

The next example shows that the two natural ways of pulling back continuous rational functions by a morphism can be different.

**Example 5 (Two pull-backs)** Start with $S = \mathbb{R}^2$ and $f = x^3/(x^2 + y^2)$. Note that $f$ extends to a continuous function $f^c$ and $f^c(0, 0) = 0$.

Blow up $(x^3, x^2 + y^2)$ to obtain the surface $S' \subset \mathbb{R}^2_{st} \times \mathbb{R}^1_{st}$ defined by the equation $sx^3 = t(x^2 + y^2)$. The first projection $\pi : S' \to S$ is an isomorphism away from the origin and $\pi^{-1}(0, 0) \cong \mathbb{R}^1$.

One can think of the pull-back of $f^c$ to $S'$ in two different ways.

First, $f^c \circ \pi$ is the composite of two continuous maps, hence it is a continuous function. This interpretation gives a continuous function $f^c \circ \pi$ which vanishes along $\pi^{-1}(0, 0)$.

Second, one can view $f \circ \pi$ as a rational function on $S'$. This interpretation views $f \circ \pi$ as a rational map $f \circ \pi : S' \setminus \pi^{-1}(0, 0) \dashrightarrow \mathbb{R}^1$. Note that $f \circ \pi$ agrees with the second projection $S' \to \mathbb{R}^1_{st}$ hence it is regular on $S'$. Thus we get a continuous (even regular) map $(f \circ \pi)^c : S' \to \mathbb{R}^1$ whose restriction to $\pi^{-1}(0, 0)$ is an isomorphism $\pi^{-1}(0, 0) \cong \mathbb{R}^1$.

This confusion is possible only because $\pi^{-1}(\mathbb{R}^2 \setminus (0, 0))$ is not Euclidean dense in $S'$. Its Euclidean closure contains only one point of $\pi^{-1}(0, 0) \cong \mathbb{R}^1$. The two versions of $f \circ \pi$ agree on $\pi^{-1}(\mathbb{R}^2 \setminus (0, 0))$, hence also on its Euclidean closure, but not everywhere.
In general, let $\pi : X' \to X$ be a morphism of real varieties and $f$ a rational function on $X$ that is regular on an open set $U \subset X$. Let $U' \subset X'$ be the Euclidean closure of $\pi^{-1}(U)$; it is a semialgebraic subset of $X'$. All possible definitions of a continuous pull-back of $f$ agree on $U'$ but, as the above example shows, they may be different on $X' \setminus U'$.

Finally we turn to Question 1.4 for a single equation

$$\sum_i f_i(x) \cdot y_i = g(x),$$

where $g$ and the $f_i$ are polynomials in the variables $x = (x_1, \ldots, x_n)$. Such equations have a solution where the $y_i$ are rational functions provided not all of the $f_i$ are identically zero. The existence of a solution where the $y_i$ are continuous functions is studied in [8] and [11].

One could then hope to prove that if there is a continuous solution then there is also a continuous rational solution. [8, Sec.2] proves that if there is a continuous solution then there is also a continuous semialgebraic solution. The next example shows, however, that in general there is no continuous rational solution.

**Example 6** We claim that the linear equation

$$x_1^2 x_2 \cdot y_1 + (x_1^3 - (1 + x_2^2) x_2^3) \cdot y_2 = x_1^4$$

has a continuous semialgebraic solution but no continuous rational solution.

A continuous semialgebraic solution is given by

$$y_1 = (1 + x_2^3)^{1/3} \quad \text{and} \quad y_2 = \frac{x_1^3}{x_1 + (1 + x_2^3)^{1/3} x_1 x_2 + (1 + x_2^3)^{2/3} x_2^2}. \quad (2)$$

(Note that $x_1^2 + 1 + x_2^3)^{1/3} x_1 x_2 + (1 + x_2^3)^{2/3} x_2^2 \geq \frac{1}{2}(x_1^2 + x_2^2)$, so $|y_2| \leq 2x_1$ and it is indeed continuous. A solution by rational functions is $y_1 = x_1/2$ and $y_2 = 0$.

To see that (1) has no continuous rational solution, restrict any solution $(y_1, y_2)$ to the semialgebraic surface $S := (x_1 - (1 + x_2^3)^{1/3} x_2 = 0)$. Since $x_1^3 - (1 + x_2^3)x_2^3$ is identically zero on $S$, we conclude that $y_1|_S = x_1^4/(x_1^3 x_2)|_S = (x_1/x_2)|_S$. The latter is equal to $\sqrt[3]{1 + x_2^2}$, thus

$$y_1|_{x_3 \text{-axis}} = \frac{3}{\sqrt[3]{1 + x_2^2}},$$

which is not a rational function. As we saw in Example 2, this implies that $y_1$ is not a rational function.

## 2 Hereditarily rational functions

In order to move from examples to proofs, we need some definitions.

**Definition 7** In this note, a real algebraic variety is a quasi-projective variety $X$ defined over $\mathbb{R}$ as in [21]. We always assume that $X$ is reduced but we allow it to be reducible. The set of real points is denoted by $X(\mathbb{R})$. We do not assume that $X(\mathbb{R})$ is Zariski dense in $X$. For the final applications this generality is not relevant, but it is necessary for some inductive arguments.

It is easy to see that there is always an open affine subvariety $X^0 \subset X$ that contains $X(\mathbb{R})$. Thus, as in [3], one can always view $X(\mathbb{R})$ as a closed subset of $\mathbb{R}^n$ defined by polynomial equations.
We are interested in continuous functions \( \phi: X(\mathbb{R}) \to \mathbb{R} \) that “come from” algebraic geometry. The simplest examples are regular functions; these are the restrictions of regular functions (in the sense of [21, p.24]) to \( X(\mathbb{R}) \). That is, \( \phi \) is regular if every point \( x \in X(\mathbb{R}) \) has an open affine neighborhood \( x \in U_\chi \subset X \) and an embedding \( U_\chi \hookrightarrow \mathbb{A}^n \) such that one can write \( \phi = (p/q)|_{U_\chi(\mathbb{R})} \) where \( p, q \in \mathbb{R}[x_1, \ldots , x_N] \) are polynomials and \( q(x) \neq 0 \).

Let \( X \subset \mathbb{A}^n \) be a closed real algebraic variety. By [3, 3.2.3] \( \phi: X(\mathbb{R}) \to \mathbb{R} \) is regular iff one can write \( \phi = (p/q)|_{X(\mathbb{R})} \) where \( p, q \in \mathbb{R}[x_1, \ldots , x_n] \) are polynomials and \( q \) is everywhere positive on \( \mathbb{R}^n \).

By a rational function on a variety \( X \) we mean a function that is regular on some Zariski open dense subvariety \( X^0 \subset X \). Thus if \( f \) is rational, it defines a continuous (even regular) function \( f|_{X^0(\mathbb{R})} : X^0(\mathbb{R}) \to \mathbb{R} \). It can happen that \( f|_{X^0(\mathbb{R})} \) can be extended to a continuous function \( f^c : X(\mathbb{R}) \to \mathbb{R} \).

A continuous function \( \phi: X(\mathbb{R}) \to \mathbb{R} \) is called rational if there is a rational function \( f \) that is regular on a Zariski open dense subvariety \( X^0 \subset X \) such that \( X^0(\mathbb{R}) \) is Euclidean dense in \( X(\mathbb{R}) \) and \( f|_{X^0(\mathbb{R})} = \phi|_{X^0(\mathbb{R})} \). If \( X(\mathbb{R}) \) is Zariski dense in \( X \) then such an \( f \) is unique. In this case we also call \( f \) a continuous rational function and write \( f^c := \phi \). (We do not define what a continuous rational function should be if \( X^0(\mathbb{R}) \) is not Euclidean dense in \( X(\mathbb{R}) \). The examples of Sect. 1 show several unexpected properties of this case.)

First we show that Question 1.1 has a positive answer on smooth varieties.

**Proposition 8** Let \( X \) be a real algebraic variety and \( f \) a rational function on \( X \) that is regular on \( X^0 \subset X \). Assume that \( f|_{X^0(\mathbb{R})} \) has a continuous extension \( f^c : X(\mathbb{R}) \to \mathbb{R} \). Let \( Z \subset X \) be an irreducible subvariety that is not contained in the singular locus of \( X \).

Then there is a Zariski dense open subset \( Z^0 \subset Z \) such that \( f^c|_{Z^0(\mathbb{R})} \) is a regular function.

**Proof** By replacing \( X \) with a suitable open subvariety, we may assume that \( X \) and \( Z \) are both irreducible and smooth. The claim is vacuous if \( Z(\mathbb{R}) = \emptyset \).

Let \( X^0 \subset X \) be a Zariski open set such that \( f \) is regular on \( X^0 \). Since \( X \) is smooth, \( X(\mathbb{R}) \) has pure topological dimension \( = \dim X \) and \( X(\mathbb{R}) \setminus X^0(\mathbb{R}) \) has topological dimension \( < \dim X \). Thus \( X^0(\mathbb{R}) \) is Euclidean dense in \( X(\mathbb{R}) \). In particular, \( f^c \) is uniquely determined by \( f \) and \( f^c(x) = f(x) \) whenever \( f \) is regular at \( x \).

Assume first that \( Z \) has codimension 1. Then the local ring \( \mathcal{O}_{X,Z} \) is a principal ideal domain [21, Sec.II.3.1]; let \( t \in \mathcal{O}_{X,Z} \) be a defining equation of \( Z \). We can write \( f = t^mu \) where \( m \in \mathbb{Z} \) and \( u \in \mathcal{O}_{X,Z} \) is a unit. Here \( m \geq 0 \) since \( f \) does not have a pole along \( Z \), hence \( f \) is regular along a Zariski dense open subset \( Z^0 \subset Z \). Thus \( f|_{Z^0} \) is a regular function and

\[
(f|_{Z^0})|_{Z^0(\mathbb{R})}.
\]

If \( Z \) has higher codimension, note that \( Z \) is a local complete intersection at its smooth points [21, Sec.II.3.2]. That is, there is a sequence of subvarieties \( Z_0 \subset Z_1 \subset \cdots \subset Z_m = X_0 \subset X \) where each \( Z_i \) is a smooth hypersurface in \( Z_{i+1} \) for \( i = 0, \ldots , m - 1 \) and \( Z_0 \) (resp. \( X_0 \)) is open and dense in \( Z \) (resp. \( X \)). We can thus restrict \( f^c \) to \( Z_{m-1}(\mathbb{R}) \), then to \( Z_{m-2}(\mathbb{R}) \) and so on, until we get that \( f^c|_{Z_0(\mathbb{R})} \) is regular.

This suggests that we should focus on those functions for which Question 1.1 has a positive answer. Then we show that for such functions Question 1.2 also has a positive answer.

**Definition 9** Let \( X \) be a real algebraic variety and \( \phi \) a continuous function on \( X(\mathbb{R}) \). We say that \( \phi \) is hereditarily rational if every irreducible, real subvariety \( Z \subset X \) has a Zariski dense open subvariety \( Z^0 \subset Z \) such that \( \phi|_{Z^0(\mathbb{R})} \) is regular.
Examples 2 and 3 show that not every continuous rational function is hereditarily rational.  
If $\phi$ is hereditarily rational then there is a Zariski dense open set $X^0 \subset X$ such that $\phi|_{X^0(\mathbb{R})}$ is regular. We can repeat this process with the restriction of $\phi$ to $X \setminus X^0$, and so on. Thus we conclude that a continuous function $\phi$ is hereditarily rational iff there is a sequence of closed subvarieties $\emptyset = X_{-1} \subset X_0 \subset \cdots \subset X_m = X$ such that for $i = 0, \ldots, m$ the restriction of $\phi$ to $X_i(\mathbb{R}) \setminus X_{i-1}(\mathbb{R})$ is regular.

If it is convenient, we can also assume that each $X_i \setminus X_{i-1}$ is smooth of pure dimension $i$.

Proposition 8 says that on a smooth variety every continuous rational function is hereditarily rational.

The pull-back of a hereditarily rational function by any morphism is again a (continuous and) hereditarily rational function.

The following result shows that hereditarily rational functions constitute the right class for Question 1.2.

Theorem 10 Let $Z$ be a real algebraic variety and $\phi : Z(\mathbb{R}) \to \mathbb{R}$ a continuous function. The following are equivalent.

(1) $\phi$ is hereditarily rational.

(2) For every real algebraic variety $X$ that contains $Z$ as a closed subvariety, $\phi$ extends to a hereditarily rational function $\Phi$ on $X(\mathbb{R})$.

(3) For every smooth real algebraic variety $X$ that contains $Z$ as a closed subvariety, $\phi$ extends to a continuous rational function $\Phi$ on $X(\mathbb{R})$.

(4) Let $X_0$ be a smooth real algebraic variety that contains $Z$ as a closed subvariety. Then $\phi$ extends to a continuous rational function $\Phi_0$ on $X_0(\mathbb{R})$.

Proof It is clear that (2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (4) $\Rightarrow$ (1) holds by Proposition 8. Thus we need to show that (1) $\Rightarrow$ (2).

We can embed $X$ into a smooth real algebraic variety $X'$. If we can extend $f$ to $X'$ then its restriction to $X$ gives the required extension. Thus we may assume to start with that $X$ is smooth (or even that $X = \mathbb{A}^N$ for some $N$).

We prove the following more precise version.

Proposition 11 Let $X$ be a smooth real algebraic variety and $W \subset Z \subset X$ closed sub-varieties. Let $\phi$ be a continuous, hereditarily rational function on $Z(\mathbb{R})$ that is regular on $Z(\mathbb{R}) \setminus W(\mathbb{R})$. Then $\phi$ extends to a continuous, hereditarily rational function $\Phi$ on $X(\mathbb{R})$ that is regular on $X(\mathbb{R}) \setminus W(\mathbb{R})$.

Proof As we noted in Definition 7, we may assume that $X$ is affine. We may also assume that $Z(\mathbb{R})$ (resp. $W(\mathbb{R})$) is Zariski dense in $Z$ (resp. $W$). Thus there is a unique rational function $f$ on $Z$ such that $f|_{Z(\mathbb{R})} = \phi$.

We use induction on dim $Z$. The case dim $Z = 0$ is obvious.

If $W$ is replaced by a smaller set, the assertion gets stronger. Hence we may assume that $W \subset Z$ is the smallest set such that $\phi$ is regular on $Z(\mathbb{R}) \setminus W(\mathbb{R})$. Since every rational function is regular on a Zariski dense open set, $W$ is nowhere dense in $Z$. In particular, dim $W \prec$ dim $Z$.

Since $\phi$ is hereditarily rational, $\phi|_W$ is also hereditarily rational. Thus, by induction, there is a continuous, hereditarily rational function $\Phi_1$ on $X(\mathbb{R})$ that is regular on $X(\mathbb{R}) \setminus W(\mathbb{R})$ and such that $\Phi_1|_W(\mathbb{R}) = \phi|_W(\mathbb{R})$. Set $\phi_2 := \phi - \Phi_1|_{Z(\mathbb{R})}$. Then $\phi_2$ vanishes on $W(\mathbb{R})$ and it is enough to show that $\phi_2$ extends to a continuous, hereditarily rational function $\Phi_2$ on $X(\mathbb{R})$ that is regular on $X(\mathbb{R}) \setminus W(\mathbb{R})$. 

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There is a unique rational function \( f_2 \) on \( Z \) such that \( f_2|_{Z(\mathbb{R})} = \phi_2 \). Let \( I_2 \) be the ideal of those regular functions \( g \) on \( Z \) such that \( g f_2 \) is regular. Let \( g_1, \ldots, g_r \) be a set of generators of \( I_2 \) and \( q := g_2^2 + \cdots + g_r^2 \). Then \( p := q f_2 \) is regular, \( q \) vanishes precisely at the points of \( Z(\mathbb{R}) \) where \( f \) is not regular and \( f_2 = p/q \). Note further that \( W(\mathbb{R}) \) is the zero set of \( q|_{X(\mathbb{R})} \) and the zero set of \( p|_{X(\mathbb{R})} \) contains \( W(\mathbb{R}) \).

Since \( X \) is affine, \( p \) (resp. \( q \)) extend to regular functions \( P \) (resp. \( Q \)) on \( X \). Usually \( P/Q \) is not continuous on \( X(\mathbb{R}) \) but we will improve it in two steps.

Let \( H \) be a regular function on \( X \) whose zero set on \( X(\mathbb{R}) \) equals \( Z(\mathbb{R}) \). Then \( (Q^2 + H^2)|_{X(\mathbb{R})} \) vanishes only on \( W(\mathbb{R}) \) and its restriction to \( Z \) equals \( q^2 \). Thus

\[
G := \frac{PQ}{Q^2 + H^2}
\]

is a rational function on \( X \) that is regular on \( X(\mathbb{R}) \setminus W(\mathbb{R}) \) and whose restriction to \( Z \setminus W \) equals \( f_2 \). However, usually \( G \) is not continuous on \( X(\mathbb{R}) \) near \( W(\mathbb{R}) \); see Example 13. Thus we need one more correction.

**Claim** For \( n \gg 1 \) the rational function

\[
F_{2n} := G \cdot \frac{Q^{2n}}{Q^{2n} + H^2} = \frac{PQ}{Q^2 + H^2} \cdot \frac{Q^{2n}}{Q^{2n} + H^2}
\]

restricts to a continuous function \( \Phi_{2n} \) on \( X(\mathbb{R}) \).

It is clear that the restriction of \( F_{2n} \) to \( Z \setminus W \) equals \( f_2 \), thus, once the Claim is proved, \( \Phi := \Phi_1 + \Phi_{2n} \) satisfies all the requirements of the Theorem.

In order to prove the claim, we work on the variety \( \pi : X_1 \to X \) obtained by blowing up the ideal \((PQ, Q^2 + H^2)\). Equivalently, \( X_1 \) is the Zariski closure of the graph of \( G \) in \( X \times \mathbb{P}^1 \).

Since \( W(\mathbb{R}) \) is the real part of the common zero set \((PQ = Q^2 + H^2 = 0)\), the real exceptional set of \( \pi \) is \( E(\mathbb{R}) \cong W(\mathbb{R}) \times \mathbb{R}\mathbb{P}^1 \) and \( \pi \) is an isomorphism over \( X(\mathbb{R}) \setminus W(\mathbb{R}) \). We prove that

\[
(F_{2n}|_{X \setminus W}) \circ \pi
\]

extends to a continuous function \( (F_{2n} \circ \pi)^c \) on \( X_1(\mathbb{R}) \) that vanishes on the exceptional set \( E(\mathbb{R}) \). Thus \( (F_{2n} \circ \pi)^c \) descends to a continuous function on \( X(\mathbb{R}) \) that vanishes on \( W(\mathbb{R}) \).

The existence of such a continuous extension is a local question on \( X_1(\mathbb{R}) \) and we use different arguments on different charts. A slight complication is that one of our charts is only semialgebraic.

Let \( Z^* \subset X_1(\mathbb{R}) \) be the Euclidean closure of \( \pi^{-1}(Z(\mathbb{R}) \setminus W(\mathbb{R})) \). Note that \( Z^* \) is a closed semialgebraic subset by [3, Prop.2.2.2] but it is usually not real algebraic.

On \( X_1 \) one can identify the rational function \( G \circ \pi \) with the restriction of the second projection \( \pi_2 : X \times \mathbb{P}^1 \to \mathbb{P}^1 \).

On \( Z^* \) we thus have a continuous function \( \phi_2 \circ \pi \) that vanishes on \( Z^* \cap E(\mathbb{R}) \) and a regular (hence continuous) map \( \pi_2|_{Z^*} : Z^* \to \mathbb{R}\mathbb{P}^1 \). Using the natural identification \( \mathbb{R}^1 = \mathbb{R}\mathbb{P}^1 \setminus \{\infty\} \), these two agree on the open set \( \pi^{-1}(Z(\mathbb{R}) \setminus W(\mathbb{R})) \), hence they agree on \( Z^* \). (Note that, as in Example 5, the two functions might not agree on the Zariski closure of \( \pi^{-1}(Z(\mathbb{R}) \setminus W(\mathbb{R})) \); this is why we work with \( Z^* \).) Thus \( \pi_2^{-1}(\infty) \) is disjoint from \( Z^* \), and hence there is a Zariski open neighborhood \( U^* \subset X_1(\mathbb{R}) \) of \( Z^* \) such that \( \pi_2 \) defines a regular function on \( U^* \). This gives the extension of \( (G|_{X \setminus W}) \circ \pi \) to a continuous function on \( U^* \) that vanishes along \( Z^* \cap E(\mathbb{R}) \). (It is not clear that \( (G \circ \pi)|_{U^*} \) vanishes along all of \( U^* \cap E \).)
Note further that
\[
\frac{Q^{2n}}{Q^{2n} + H^2} \circ \pi
\]
is a bounded regular function on \(X_1(\mathbb{R}) \setminus E(\mathbb{R})\). Therefore the restriction of the product
\[
(F_{2n} \circ \pi)\Big|_{U^*} = (G \circ \pi) \cdot \left(\frac{Q^{2n}}{Q^{2n} + H^2} \circ \pi\right)\Big|_{U^*}
\]
extends to a function that vanishes and is continuous at every point of \(Z^* \cap E(\mathbb{R})\). However, we have not proved so far that it is defined along \((U^* \cap E(\mathbb{R})) \setminus Z^*\).

The other chart is \(V^* := X_1(\mathbb{R}) \setminus Z^*\); it is semialgebraic and Euclidean open in \(X_1(\mathbb{R})\). Here we write \(F_{2n} \circ \pi\) in the form
\[
F_{2n} \circ \pi = (P \circ \pi) \cdot \left(\frac{Q^{2n-1}}{H^2} \circ \pi\right) \cdot \left(\frac{Q^2}{Q^2 + H^2} \circ \pi\right) \cdot \left(\frac{H^2}{Q^{2n} + H^2} \circ \pi\right).
\]

The last two factors are bounded regular functions on \(V^* \setminus E(\mathbb{R})\) and \(P\) is a regular function that vanishes along \(E(\mathbb{R}) \cap V^*\).

Note that on \(V^*\) the function \(H \circ \pi\) vanishes only along \(E(\mathbb{R})\) and \(Q \circ \pi\) also vanishes along \(E(\mathbb{R})\). We can thus apply Theorem 12 to conclude that \(\left(\frac{Q^{2n-1}}{H^2}\right) \circ \pi\) extends to a continuous (and semialgebraic) function on \(V^*\) for \(n \gg 1\). Thus \(F_{2n} \circ \pi\) extends to a continuous function on \(V^*\) that vanishes along \(E(\mathbb{R}) \cap V^*\) for \(n \gg 1\).

Putting the two charts together we conclude that \(F_{2n} \circ \pi\) is defined on \(X_1(\mathbb{R})\) that vanishes along \(E(\mathbb{R})\).

\[\Box\]

We have used the following version of the Łojasiewicz inequality given in [3, Thm.2.6.6], see also [15, 16].

**Theorem 12 [Łojasiewicz Inequality]** Let \(V\) be a locally closed, semialgebraic subset of \(\mathbb{R}^N\) and \(\phi, \psi: V \to \mathbb{R}\) continuous semialgebraic functions. Assume that \(\{x \in V : \phi(x) = 0\} \subset \{x \in V : \psi(x) = 0\}\). Then there exist a positive integer \(n\) and a continuous semialgebraic function \(\rho\) such that \(\psi^n = \rho \phi\).

The following example shows that, even in very simple cases, the extension of continuous rational functions is not entirely trivial.

**Example 13** Let \(X = \mathbb{R}^2\) and \(Z \subset \mathbb{R}^2\) the cuspidal cubic with equation \((x^2 - y^3) = 0\). Consider the rational function \(f(x, y) := y^2/x\). \(Z\) can be parametrized as \(x = t^3, y = t^2\) and then \(f(t^3, t^2) = t\) is clearly continuous.

First we claim that there are no regular functions \(P, Q\) such that \(P|_Z = y^2, Q|_Z = x\) and \(P/Q\) is continuous. In fact, \(P/Q\) can not even be bounded. Indeed, any such extension would be of the form
\[
P = y^2 + P_1(x^2 - y^3) \quad \text{and} \quad Q = x + Q_1(x^2 - y^3)
\]
where \(P_1, Q_1\) are regular. Thus
\[
\left.\frac{P}{Q}\right|_{y \text{-axis}} = \frac{y^2 - y^3 P_1}{-y^3 Q_1} = \frac{1 - y P_1}{-y Q_1}
\]
has a pole at \(y = 0\).
Our first two improvements are
\[
y^2x \quad \frac{x^2}{x^2 + (x^2 - y^3)^2} \quad \frac{y^2x}{x^2 + (x^2 - y^3)^2} \quad \frac{x^2}{x^2 + (x^2 - y^3)^2}.
\]
For both of these, the limit along the curve \((t^2, t)\) is 1, hence they are not continuous at the origin. The next improvement is
\[
F_4 := y^2x \quad \frac{x^2}{x^2 + (x^2 - y^3)^2} \quad \frac{x^4}{x^4 + (x^2 - y^3)^2}.
\]
This turns out to be continuous as can be seen either by two blow-ups or by the following direct computation. Performing the change of variables \(x = u^3, \quad y = u^2v\) we get
\[
\left| F_4(u^3, u^2v) \right| = \left| \frac{uv^2}{(1 + u^6(1 - v^3)^2) \cdot (1 + (1 - v^3)^2)} \right| \leq |u| \cdot \frac{v^2}{1 + (1 - v^3)^2}.
\]
The last factor is uniformly bounded by a constant \(C\) for \(v \in \mathbb{R}\), thus we conclude that
\[
\left| F_4(x, y) \right| \leq C|x|^{1/3}.
\]

3 Varieties over \(p\)-adic fields

Let \(K\) be any topological field. The \(K\)-points \(X(K)\) of any \(K\)-variety \(X\) inherit from \(K\) a topology, called the \(K\)-topology. One can then consider rational functions \(f\) on \(X\) that are continuous on \(X(K)\). This does not seem to be a very interesting notion in general, unless \(K\) satisfies the following.

Definition 14 A topological field \(K\) satisfies the density property if the following equivalent conditions hold.

1. If \(X\) is a smooth, irreducible \(K\)-variety and \(\emptyset \neq U \subset X\) is a Zariski open subset then \(U(K)\) is dense in \(X(K)\) in the \(K\)-topology.
2. If \(C\) is a smooth, irreducible \(K\)-curve and \(\emptyset \neq C^0 \subset C\) is Zariski open then \(C^0(K)\) is dense in \(C(K)\) in the \(K\)-topology.
3. If \(C\) is a smooth, irreducible \(K\)-curve then \(C(K)\) has no isolated points.

Examples of such fields are complete real valued fields with non-discrete topology; this can be deduced by means of Hensel’s lemma for restricted formal power series over real valued fields (see e.g. [4, Chap.III]).

If \(K\) is algebraically closed, for instance \(K = \mathbb{C}\), then every continuous rational function on a normal variety is regular, so we do not get a new notion. In general, the study of continuous rational functions leads to the concept of seminormality and seminormalization; see [1,2] or [12, Sec.10.2] for a recent treatment.

The proofs of Sect. 2, except that of Proposition 11, all work over such general topological fields \(K\) that satisfy the density property. As for Proposition 11, there seem to be 3 issues.

- We need to show that every rational function that is regular at the \(K\)-points of a subvariety extends to a rational function that is regular at the \(K\)-points of the ambient variety. This is proved in Lemma 15 by a slight modification of the usual arguments that apply when the base field is algebraically closed [21, Sec.I.3.2] or real closed [3, 3.2.3].
The proof of Proposition 11 relied on semialgebraic sets. Although it is doubtful that definable sets behave sufficiently nicely for arbitrary topological fields, one could use them for the case of $p$-adic fields and even Henselian real valued fields; see [17,18]. Here we go around this problem by using a strong form of resolution of singularities and transformation to simple normal crossing; see [10, Chap.III] for references and relatively short proofs.

At the end of the proof we need to show that if $\sigma : Y \to X$ is a birational morphism between smooth varieties and $f_Y$ is a rational function that is continuous on $Y(K)$ and constant on the fibers of $Y(K) \to X(K)$ then $f_Y$ descends to a rational function $f_X$ that is continuous on $X(K)$. If $K$ is locally compact, then $Y(K) \to X(K)$ is proper, hence the continuity of $f_X$ is clear. The similar assertion does not hold for arbitrary fields satisfying the density property, not even if $X$ is normal, but we do not know what happens in the smooth case. In the papers [17,18] it is proven that such a descent property holds over Henselian real valued fields.

Next we turn to the details supporting the above discussion.

**Lemma 15** (Extending regular functions) Let $k$ be a field, $X$ an affine $k$-variety and $Z \subset X$ a closed subvariety. Let $f$ be a rational function on $Z$ that is regular at all points of $Z(k)$. Then there is a rational function $F$ on $X$ that is regular at all points of $X(k)$ and such that $F|_Z = f$.

**Proof** If $k$ is algebraically closed, then $f$ is regular on $Z$ hence it extends to a regular function on $X$.

If $k$ is not algebraically closed, then, as an auxiliary step, we claim that there are polynomials $G_r(x_1, \ldots, x_r)$ in any number of variables whose only zero on $k^r$ is $(0, \ldots, 0)$.

Indeed, take a polynomial $g(t) = t^d + a_1 t^{d-1} + \cdots + a_d \in k[t]$, $d > 1$, which has no roots. Then its homogenization $G_2(x_1, x_2) = x_1^d + a_1 x_1^{d-1} x_2 + \cdots + a_d x_2^d$ is a polynomial in two variables we are looking for. Further, we can recursively define polynomials $G_r$ by putting $G_{r+1}(x_1, \ldots, x_r, x_{r+1}) := G_2(G_r(x_1, \ldots, x_r), x_{r+1})$.

Now we construct the extension of $f$ as follows.

For every $z \in Z(k)$ we can write $f = p_z/q_z$ where $q_z(z) \neq 0$. After multiplying both $p_z, q_z$ with a suitable polynomial, we can assume that $p_z, q_z$ are regular on $Z$ and then extend them to regular functions on $X$. By assumption, $\bigcap_{z \in Z(k)} (q_z = 0)$ is disjoint from $Z(k)$. Choose finitely many $z_1, \ldots, z_m \in Z(k)$ such that

$$\bigcap_{i=1}^m (q_{z_i} = 0) = \bigcap_{z \in Z(k)} (q_z = 0).$$

Let $q_{m+1}, \ldots, q_r$ be defining equations of $Z \subset X$. Set $q_i := q_{z_i}$ and $p_i := p_{z_i}$ for $i = 1, \ldots, m$ and $p_i = q_i$ for $i = m+1, \ldots, r$. Write (non-uniquely) $G_r = \sum G_{r,i} x_i$ and finally set

$$F := \sum_{i=1}^r G_{r,i}(q_1, \ldots, q_r) p_i.$$

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Since the $q_i$ have no common zero on $X(k)$, we see that $F$ is regular at all points of $X(k)$.

Along $Z$, $p_i = f q_i$ for $i = 1, \ldots, m$ by construction and for $i = m+1, \ldots, r$ since then both sides are 0. Thus

$$F|_Z := \sum_{i=1}^r G_{ri} (q_1, \ldots, q_r) f q_i / G_r (q_1, \ldots, q_r) |_Z = f .$$

From now on $K$ denotes a locally compact field of characteristic 0 that is not algebraically closed. That is, $K$ is either $\mathbb{R}$ or a finite extension of a $p$-adic field $\mathbb{Q}_p$.

**Proposition 16** Let $X$ be a smooth $K$-variety and $Z \subset X$ a closed subvariety. Let $f$ be a continuous, hereditarily rational function on $Z(K)$. Then $f$ extends to a continuous, hereditarily rational function $F$ on $X(K)$.

**Proof** Let $W \subset Z$ denote the closed subvariety consisting of those points $z \in Z$ such that either $f$ is not regular at $z$ or $Z$ is singular at $z$. As in the proof of Proposition 11, an induction argument reduces the problem to the following special case.

**Claim** The conclusion of Proposition 16 holds if we assume in addition that $f$ vanishes on $W(K)$.

In order to prove the Claim, take an embedded resolution of singularities $\sigma : X' \to X$ with the following properties.

1. $\sigma$ is an isomorphism over $X \setminus W$,
2. the birational transform $Z' := \sigma^{-1}_*(Z)$ is smooth,
3. the rational map $f \circ \sigma$ restricts to a morphism $f' : Z' \to \mathbb{P}^1$,
4. the exceptional set $E := \sigma^{-1}(W)$ is a simple normal crossing divisor and
5. $Z' \cup E$ is a simple normal crossing subvariety.

The last assertion means that for every point $p \in Z' \cup E$ one can choose local coordinates $y_1, \ldots, y_N$ such that, in a neighborhood of $p$, $Z' = (y_1 = \cdots = y_n = 0)$ and the irreducible components of $E$ are given by the equations $\{ y_j = 0 : j \in J \}$ for a suitable subset $J \subset \{ n+1, \ldots, N \}$.

Since $Z'$ is smooth, $f'|Z'(K)$ agrees with the topological pull-back $f^\circ \circ \sigma$. Thus $f'|Z'(K)$ is a regular function on $Z'(K)$ that vanishes on $Z'(K) \cap E(K)$.

Consider next the rational function $g$ on $Z' \cup E$ whose restriction to $Z'$ equals $f'$ and to $E$ equals 0. We claim that $g$ is regular at the $K$-points of $Z' \cup E$. This is a local problem and the only points in question are in $Z' \cap E$.

So pick a $K$-point $p \in Z' \cap E$, an open neighborhood $p \in U \subset X'$ and local coordinates $y_1, \ldots, y_N$ as above. We know that $f'$ is regular on $Z'(K) \cap U(K)$, hence it extends to a regular function $\tilde{f}$ on $U(K)$. Since $f'$ vanishes along $Z'(K) \cap E(K)$, the extension $\tilde{f}$ also vanishes along $Z'(K) \cap E(K) \cap U(K)$. Note that the ideal of $Z' \cap E \cap U$ is generated by the functions $y_1, \ldots, y_n, \prod_{j \in J} y_j$, hence we can write (non-uniquely)

$$\tilde{f} = p_1 y_1 + \cdots + p_n y_n + q \prod_{j \in J} y_j$$

where the $p_i$ and $q$ are regular on $U(K)$. Thus

$$\tilde{f} - p_1 y_1 - \cdots - p_n y_n = q \prod_{j \in J} y_j$$

is a regular function near $p$ which vanishes on $E(K)$ and whose restriction to $Z'(K)$ agrees with $f'$. Thus the restriction of $\tilde{f}$ to $Z'(K) \cup E(K)$ is regular and equals $g$. 
Since $g$ is regular at all $K$-points of $Z' \cup E$, by Lemma 15 it extends to a rational function $G$ on $X'$ that is regular on $X'(K)$. Furthermore $G$ vanishes on the exceptional set $E(K)$. Finally, $\sigma : X'(K) \to X(K)$ is a proper surjection and $G$ is constant on every fiber of $\sigma|_{X'(K)}$. Thus $G$ descends to the required continuous function $F$ on $X(K)$. \hfill \Box

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