Effect of feedback on the control of a two-level dissipative quantum system

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We show that it is possible to modify the stationary state by a feedback control in a two-level dissipative quantum system. Based on the geometric control theory, we also analyze the effect of the feedback on the time-optimal control in the dissipative system governed by the Lindblad master equation. These effects are reflected in the function $\Delta_A(\bar{x})$ and $\Delta_B(\bar{x})$ that characterize the optimal trajectories, as well as the switching function $\Phi(t)$ and $\theta(t)$, which characterize the switching point in time for the time-optimal trajectory.

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I. INTRODUCTION

The problem of controlling quantum systems has been an important scientific and technological challenge since the discovery of quantum mechanics more than a century ago. Numerous approaches to the control of a quantum system have been proposed in the past decades, which depending on how the controls enter the system, can be divided into two categories: the open loop scheme (coherent control) and the closed loop scheme. In the open loop scheme the control functions are fixed, namely they can not be modified according to the state of the quantum system. Whereas in the closed loop scheme, the control functions are updated in real time by feeding back some information about the actual state of the system. This scheme is called quantum feedback control. Quantum feedback control may emerge as a natural possible route to develop strategies to prepare entangled states and prevent their deterioration in controlled open systems.

For a real-world quantum system, the coupling of the system to its environment is unavoidable. Because of this unavoidable coupling, the system dynamics is subject to irreversibility, dissipation, and dephasing. As a consequence, some appealing properties of quantum systems, for example entanglement, are usually lost during the time evolution, leading to many typical manifestations of this irreversibility, for example the relaxation of the system to a stationary state. This fact naturally gives rise to the following question, to what extent it is possible to modify the stationary state by the controls? For open loop control, a previous study shows that the stationary state can be modified by indirect control through coupling the open system to an auxiliary two-level system. What about the closed loop control? can a feedback affects the stationary state of the system?

On the other hand, active research has been performed for manipulating an open system with more realistic situations taken into account. Although the controls can not fully compensate the effect of decoherence for an open system governed by the Lindblad master equation, as shown in Ref., an efficient control can still be achieved. These analyses are based on the numerical optimization techniques, and it seems that only controls can be achieved by these numerical methods. By the geometric control theory, quantum system with few levels (e.g., two- or three-level systems) can be formulated analytically. This analysis is for the time-optimal control of a dissipative two-level quantum system without feedback. In this paper we put forward the study of the control by taking the feedback into account, a two-level system governed by the Lindblad master equation will be chosen to detail the analysis.

The paper is organized as follows. In Sec.II, we specify the dynamical settings considered in this paper, and study the effect of feedback control on the stationary state of the system. By using the Pontryagin maximum principle, the time-optimal control of a two-level dissipative system with a feedback control will be studied in Sec.III. Conclusion and discussion are presented in the Appendix.

II. RELAXATION TO STATIONARY STATES OF A TWO-LEVEL DISSIPATIVE SYSTEM WITH FEEDBACK

In this section, we shall show that a Markovian feedback scheme based on the continuous monitoring of quantum jumps, can lead to an improvement of control of the stationary states. Before investigating the influence of feedback on the stationary states, let us first briefly analyze the case without feedback. To start with, we consider a two-level system with the free Hamiltonian $H_0$ and the control Hamiltonian $H_1$ (with a control field $u$). Within the Markov approximation for the system-environment interaction, the time evolution of the two-level system is described by the Lindblad master equation, $\frac{d}{dt}\rho = -i[H, \rho] + \Gamma (\sigma^- \rho \sigma^+ - \frac{1}{2} \sigma^+ \sigma^- \rho - \frac{1}{2} \rho \sigma^+ \sigma^-) + \gamma (\sigma^+ \rho \sigma^- - \frac{1}{2} \sigma^- \sigma^+ \rho - \frac{1}{2} \rho \sigma^- \sigma^+).$ Here $H = H_0 + u H_1$, $\sigma_+ = |e\rangle \langle e| - |g\rangle \langle g|,$ $\sigma_- = |g\rangle \langle e|$, and $\sigma^+ = |e\rangle \langle g|$ are the Pauli matrices. $\Gamma = (\bar{n} + 1)\kappa$ and $\gamma = \kappa \hbar$. We denote by $\bar{n}$ the noise intensity of the environment, and $\kappa$ the spontaneous emission rate of the two-level system. $|e\rangle$ and $|g\rangle$ stand for the excited and ground
states of the two-level system, respectively. Choosing $uH \propto \alpha |g\rangle + \beta |e\rangle$, we obtain the stationary state of the dissipative two-level system (setting $u = u_1 + i u_2$) 

$$\rho_{\infty} = \frac{1}{2} \begin{pmatrix} 1 - x_3 & x_1 + i x_2 \\ x_1 - i x_2 & 1 + x_3 \end{pmatrix},$$

where

$$x_1 = \frac{4 u_2 x_3}{\Gamma + \gamma},$$

$$x_2 = -\frac{4 u_1 x_3}{\Gamma + \gamma},$$

$$x_3 = \frac{(\Gamma^2 - \gamma^2)}{8|u|^2 + (\gamma + \Gamma)^2},$$

and $x = (x_1, x_2, x_3)$ was defined by

$$x_1 = 2 \text{Re} \rho_{eg},$$

$$x_2 = 2 \text{Im} \rho_{eg},$$

$$x_3 = \rho_{gg} - \rho_{ee}.$$

Two observations can be made from Eq. (1). (1) The control field $u$ drastically changes the stationary state, and $x_1 = x_2 = 0$ when $u = 0$; (2) $x_2$ and $x_1$ are proportional to $x_3$, considering $x_1^2 + x_2^2 + x_3^2 \leq 1$, we thus have $x_3^2 \leq \frac{(\Gamma + \gamma)^2}{(\Gamma + \gamma^2 + 16|u|^2)}$. Therefore it is possible to manipulate the stationary state by using the open loop control $uH_1$, but it can not compensate fully the decoherence effect, for instance with specific $\Gamma$ and $\gamma$, we can not reach all states through relaxing the system to its equilibrium.

Taking the cavity QED system as an example, we now introduce the description of the measurement and feedback scheme, this is schematically shown in figure 1. The cavity output is monitored by a photon detector $D$ whose signal provides the input to the application of the closed-loop control $U_F$. Note that we have two controls in this scheme, one is the open-loop scheme corresponding to the control Hamiltonian $u H_1$, and another is the closed-loop control denoted by $U_F$. The feedback control $U_F$ is triggered immediately only after a detection click, namely a quantum jump occurs. This scheme was used to generate and protect entangled steady state in cavity QED system [14]. The master equation for our system reads [10],

$$\frac{\partial}{\partial t} \rho = -i[H, \rho] + \Gamma(U_F \sigma^- \rho \sigma^+ U_F^\dagger - \frac{1}{2} \sigma^+ \sigma^- \rho - \frac{1}{2} \rho \sigma^+ \sigma^-) + \gamma(\sigma^+ \rho \sigma^- - \frac{1}{2} \sigma^- \sigma^+ \rho - \frac{1}{2} \rho \sigma^- \sigma^+).$$

The jump feedback $U_F \sigma^- \rho \sigma^+ U_F^\dagger$ can be understood as follows. The unitary operator $U_F$ is applied only immediately after a detection event, which is described by the term $\sigma^- \rho \sigma^+$. Intuitively the stationary states depend on the feedback operator $U_F$. So, once the measurement prescription has been chosen, the freedom to design a feedback to produce a stationary state lies in the different choices for the feedback operator $U_F$. Although an enormous range of possibilities for $U_F$ is allowed, even considering the limitations imposed by experimental constraints, we here choose (with the constraint $U_F U_F^\dagger = 1$)

$$U_F = a \sigma_x + b \sigma_y + c \sigma_z.$$

In fact the feedback $U_F$ written in this form covers all allowed possibilities. By Setting $\dot{\rho} = 0$, we arrive at the stationary state,

$$x_3 = \frac{4 \Gamma (-u_1 f_2 + u_2 f_1) + (\gamma + \Gamma) (\gamma - \Gamma |U_F^{gg}|^2)}{4 u_1 (-2 u_1 - \Gamma f_2) - 4 u_2 (2 u_2 - \Gamma f_1) - (\gamma + \Gamma) (\gamma + \Gamma |U_F^{gg}|^2)},$$

$$x_1 = \frac{(4 u_2 - 2 \Gamma f_1) x_3 + 2 \Gamma f_1}{\Gamma + \gamma},$$

$$x_2 = \frac{(-4 u_1 - 2 \Gamma f_2) x_3 + 2 \Gamma f_2}{\Gamma + \gamma},$$

where $f = U_F^{eg} U_F^{gg} = f_1 + i f_2$ was defined with $U_F^{ij} = \langle i | U_F | j \rangle$, $i, j = e, g$. This stationary state differs from that in the case without feedback at (1) $x_1$ and $x_2$ are not zero even if $u = 0$; (2) $x_1$ and $x_2$ are not proportional to $x_3$; (3) $x_1^2 + x_2^2 + x_3^2$, i.e., the trace of $\rho_{\infty}$ ($\rho_{\infty}$ is the stationary state) does not only depend on $|u|^2$. This indicates
that we may change the reachable set of stationary states by the feedback control. Mathematically, we have only two independent parameters in Eq. (1), when \( \Gamma \) and \( \gamma \) are fixed, whereas there are (at least) three degrees of freedom in Eq. (3), matching (beyond) the number of independent parameter in the two-level system.

### III. Time Optimal Control of the System with Markovian Feedback

Optimal control theory has a long-standing tradition in various fields of physics \[23\]. To our knowledge, one of the first application to a quantum system has been in the field of quantum chemistry \[3\]. Recently, optimal control theory has been extended to dissipative systems \[17, 21, 24\]. In this section, we consider the time-optimal control of a dissipative two-level system described by the master equation \[3\]. This problem was studied in Ref. \[21\] without feedback, here we will focus on the effect of the feedback on the time optimal control of this system. With the notations in Eq. (2), the master equation can be written as,

\[
\dot{x}_1 = 2u_2x_3 - \frac{\Gamma + \gamma}{2}x_1 + \Gamma f_1(1 - x_3),
\]

\[
\dot{x}_2 = -2u_1x_3 - \frac{\Gamma + \gamma}{2}x_2 + \Gamma f_2(1 - x_3),
\]

\[
\dot{x}_3 = 2u_1x_2 - 2u_2x_1 - x_3(\gamma + \Gamma|x_F^g|^2) - (\gamma - \Gamma|x_F^g|^2).
\]

(6)

As defined in Eq. (2), \( x_1, x_2, x_3 \) are three real parameters, while \( u \) is a complex function \( u = u(t) = u_1(t) + iu_2(t) \), and \( f = U_F^gU_F^{gg} = f_1 + if_2 \) is a complex number. As the most important tool for the study of optimal control, the Pontryagin maximum principle (PMP) provides a first order necessary condition for optimality. In the following we shall analyze the optimal control of this two-level system by applying the PMP. The analysis of the optimal control on \( \mathbb{R}^3 \) manifold is considerably complicated. To simplify the study, we restrict the dynamics to a sub-manifold \( \mathbb{R}^2 \) by assuming the control field \( u \) is real and \( f_1 = 0 \) \[25\]. With these assumptions, \( x_2 \) and \( x_3 \) are decoupled from \( x_1 \), leading to

\[
\dot{x}_2 = -2u_1x_3 - \frac{\Gamma + \gamma}{2}x_2 + \Gamma f_2(1 - x_3),
\]

\[
\dot{x}_3 = 2u_1x_2 - (\gamma + \Gamma|x_F^g|^2)x_3 - (\gamma - \Gamma|x_F^g|^2). \tag{7}
\]

To shorten the notation, we omit the index 1 of \( u_1 \) and set \( \vec{x} = (x_2, x_3) \). Equation (7) becomes \( \dot{\vec{x}} = F + uG \), where

\[
F = \begin{pmatrix}
\frac{\Gamma + \gamma}{2} & \Gamma f_2 - \Gamma f_2 x_3 \\
\Gamma f_1 & -\frac{\Gamma + \gamma}{2} x_3 - (\gamma - \Gamma|x_F^g|^2)
\end{pmatrix}
\]

and

\[
G = \begin{pmatrix}
-2x_2 \\
x_3
\end{pmatrix}
\]

(8)

(9)

For each \( \vec{x} \), we have calculated \( \Delta_A(\vec{x}) = Det(F, G) \) and \( \Delta_B(\vec{x}) = Det(G, [F, G]) \) defined in \[26\],

\[
\Delta_A(\vec{x}) = -2(\gamma + \Gamma|x_F^g|^2)x_2^2 - 2(\gamma - \Gamma|x_F^g|^2)x_3^2 - (\gamma + \Gamma)x_2^2 - 2\Gamma f_2 x_3 + 2\Gamma f_2 x_2,
\]

\[
\Delta_B(\vec{x}) = 4(\gamma - \Gamma|x_F^g|^2)x_2 + 4(\gamma - \Gamma + 2\Gamma|x_F^g|^2)x_2 x_3 - 4\Gamma f_2 x_2^2 + 4\Gamma f_2 x_3 + 4\Gamma f_2 x_3.
\]

(10)

where \( \Delta_A(\vec{x}) \) is useful for studying abnormal extremals, and \( \Delta_B(\vec{x}) \) is for detecting singular trajectories. We can find from Eq. (10) that the feedback control \( U_F \) play an important role in \( \Delta_A(\vec{x}) \) and \( \Delta_B(\vec{x}) \), which are crucial in the time-optimal problem. We denote by \( C_A \) and \( C_B \) the two sets of points \( \Delta_A^{-1}(0) \) and \( \Delta_B^{-1}(0) \), respectively. \( C_A \) and \( C_B \) are responsible for quantitative modification of the optimal trajectories. For \( U_F = 1 \) (without feedback), \( \Delta_A(\vec{x}) \) and \( \Delta_B(\vec{x}) \) reduce to

\[
\Delta_A(\vec{x}) = -2(\gamma + \Gamma)x_2^2 - 2(\gamma - \Gamma)x_3^2 - (\gamma + \Gamma)x_2^2,
\]

\[
\Delta_B(\vec{x}) = 4(\gamma - \Gamma)x_2 + 4(\gamma + \Gamma)x_2 x_3.
\]

(11)

In this case the set \( C_B \) consists of the following two lines, \( x_2 = 0 \) and \( x_3 = (\gamma - \Gamma)/(\gamma + \Gamma) \), while the solutions of the polynomial equation \( 2(\gamma + \Gamma)x_2^2 + 2(\gamma - \Gamma)x_3 + (\gamma + \Gamma)x_2^2 = 0 \) belong to the set \( C_A \) (as shown in figure \[2\](b)). The two sets \( C_A \) and \( C_B \) change when the feedback is added to the system. This is illustrated in figure \[2\](a), where we plot \( C_A \) and \( C_B \) with \( \Gamma = 0.6, \gamma = 0.3, f = \cos \frac{\pi}{3} \sin \frac{\pi}{5} \), and \( U_F = \cos^2 \frac{\pi}{3} \). Without feedback, the sub-manifold of \( \mathbb{R}^2 \) is symmetrically divided by the line \( x_2 = 0 \), as figure \[1\](b) shows. This symmetry is broken by the feedback (see figure \[1\](a)), indicating that the symmetry in the optimal trajectory disappears. This is confirmed by figure \[3\] where the two trajectories with \( u = \pm 1 \) are plotted. Assuming the field \( u \) is bounded by \( |u| \leq 1 \), we now analyze the optimal control of the two-level dissipative system with the constraint of minimizing the total time of the control, i.e., time-optimal control. The Pontryagin maximum principle tells us that for the

![Figure 2: The sub-manifold of \((x_2, x_3)\) is divided by \(C_A = \Delta_A^{-1}(0)\)(solid lines) and \(C_B = \Delta_B^{-1}(0)\)(dashed lines). \(\Gamma = 0.6, \gamma = 0.3\) were chosen for this plot. (a) and (b) are for the case with and without feedback, respectively.](image-url)
model system considered here, the extremal field \( u \) may take either \(-1\) or \(1\) according to \( u = \text{sgn}[\Phi(t)] \), if \( \Phi(t) \neq 0 \), where

\[
\Phi(t) = \vec{p} \cdot \mathbf{G} = -2p_2x_3 + 2p_3x_2 \tag{12}
\]

with \( \vec{p} \) satisfying

\[
p_2 = -\frac{\partial H}{\partial x_2} = \frac{\Gamma + \gamma}{2}p_2 - 2up_3,
\]

\[
p_3 = -\frac{\partial H}{\partial x_3} = 2up_2 + \Gamma f_2p_2 + (\gamma + \Gamma |U|^2)p_3. \tag{13}
\]

If \( \Phi(t) \) vanishes on an interval \([t_0, t_1]\), the corresponding control in this interval is \( u = \phi \), where \( \phi \) can be calculated by

\[
\frac{d}{dt} \Delta_B = \frac{\partial \Delta_B}{\partial x_2} \dot{x}_2 + \frac{\partial \Delta_B}{\partial x_3} \dot{x}_3 = 0, \tag{14}
\]

leading to

\[
u = \frac{K_{33}x_3 + K_{23}x_2x_3 + K_{22}x_2^2 + K_{33}x_3 + K_{22}x_2}{K_{33}x_3 + K_{23}x_2x_3 + K_{22}x_2^2 + K_{33}x_3 + K_{22}x_2} \tag{15}
\]

in our case. Here

\[
K_{33} = -(\Gamma + \gamma)\Gamma \gamma_2, \\
K_{23} = (\gamma - \Gamma + 2\Gamma |U_g|^2)(\frac{\Gamma}{2} + \frac{3\gamma}{2} + \Gamma |U_g|^2) + 2\Gamma^2 f_2^2, \\
K_{22} = (\Gamma + \gamma)\Gamma \gamma_2, \\
K_3 = (\Gamma - \Gamma)\Gamma \gamma_2, \\
K_2 = (\gamma - \Gamma)|U_g|^2(\frac{\Gamma}{2} + \frac{3\gamma}{2} + 2\Gamma |U_g|^2) - 2\Gamma^2 f_2^2, \\
K_{33}' = -2(\gamma - \Gamma + 2\Gamma |U_g|^2), \\
K_{23}' = -8\Gamma \gamma_2, \\
K_{22}' = 2(\gamma - \Gamma + 2|U_g|^2), \\
K_3' = -2(\gamma - \Gamma |U_g|^2), \\
K_2' = 2\gamma \gamma_2. \tag{16}
\]

Notice that if \( \Phi(t) \) has no zeros then \( u \) is almost everywhere constantly equal to \pm 1. Hence we are interested in determining when the control may change sign. This problem can be studied in two different ways: either by means of the switching function \( \Phi(t) \) (as above) or using the function \( \theta(t) \). Here \( \theta(t) \) is defined as the angle of rotation of the adjoint vector \( \vec{v} = (v_2, v_3) = (\dot{x}_2, \dot{x}_3) \) with respect to its initial position. By the definition of \( \vec{v} \), we obtain

\[
\dot{v}_2 = -(2u + \Gamma \gamma_2)v_3 - \frac{\Gamma + \gamma}{2}v_2, \\
\dot{v}_3 = 2uv_2 - (\gamma + \Gamma |U|^2)v_3. \tag{17}
\]

with the initial conditions \( v_2(0) = -2ux_3(0) - 0.5(\Gamma + \gamma)x_2(0) + \Gamma f_2(1 - x_3(0)) \) and \( v_3(0) = 2ux_2(0) - (\gamma + \Gamma |U|^2)x_2(0) - (\gamma - \Gamma |U|^2) \).

In figure 4, we illustrate the angle \( \theta(t) \) as a function of time for the system without (figure (a)) and with (figure (b)) feedback. The times at which the control \( u \) can switch are marked by arrows. We found from figure 4(a)
that the two trajectories corresponding to \( u = \pm 1 \) may switch at the same times when there is no feedback, resulting in symmetry in the optimal trajectories. This result is changed by the feedback as figure [4](b) shows, the two trajectories switch almost at different times. In addition, the trajectories can switch to an opposite control when \( \theta > 0 \) and \( \dot{\theta} > 0 \) or \( \theta < 0 \) and \( \dot{\theta} < 0 \), this condition together with the results in figure [4] yield the switching points in time. Define \( sgn_{\theta} = sgn(\theta) - sgn(\dot{\theta}) \), the allowed switching points in terms of \( sgn_{\theta} \) are plotted in figure [4].

IV. CONCLUSION

For a dissipative two-level system, we have shown that the manipulation of stationary state by feedback is possible. The dependence of the stationary state on the feedback has been calculated and discussed. The feedback together with the open loop control can broaden the reachable set of the stationary state, which provides us a new method to prepare quantum states by relaxing the system to its equilibrium states. In addition, we assume that \( u = \pm 1 \), simple algebra gives the exact solutions to Eq. ([18]). We assume that \( u = \pm 1 \), simple algebra gives the exact solutions to Eq. ([18]),

\[
\begin{align*}
\alpha_2, \alpha_3 &= \frac{A_2 + B_3}{2} \pm \frac{1}{2} \sqrt{(A_2 + B_3)^2 - 4(A_2B_3 - A_3B_2)}, \\
L' &= \frac{C_3B_2 - C_2B_3}{A_2B_3 - A_3B_2}, \\
M' &= \frac{L'(A_2 + B_3) + B_2x_3(0) - B_3x_2(0) + C_2 + (x_2(0) - L')\alpha_2}{\alpha_2 - \alpha_3}, \\
N' &= x_2(0) - L' - M', \\
L &= \frac{A_3C_2 - A_2C_3}{A_2B_3 - A_3B_2}, \\
M &= \frac{L(A_2 + B_3) - A_2x_3(0) + A_3x_2(0) + C_3 + (x_3(0) - L)\alpha_2}{\alpha_2 - \alpha_3}, \\
N &= x_3(0) - L - M,
\end{align*}
\]

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APPENDIX: ANALYTICAL DETERMINATION OF THE DYNAMICS

In this section, we solve analytically the dynamics of the system given by Eq. ([18]). To simplify the expression, we rewrite the equations as

\[
\begin{align*}
x_2 &= A_2x_2 + B_2x_3 + C_2, \\
x_3 &= A_3x_2 + B_3x_3 + C_3,
\end{align*}
\]

where \( A_2 = -\frac{\Gamma_2}{2}, B_2 = -2u - \Gamma f_2, C_2 = \Gamma f_2, A_3 = 2u, B_3 = -(\gamma + \Gamma |U_{\parallel}^g|^2), \) and \( C_3 = -(\gamma - \Gamma |U_{\parallel}^g|^2) \), We assume that \( u = \pm 1 \), simple algebra gives the exact solutions to Eq. ([18]),

\[
\begin{align*}
x_2(t) &= L' + M'e^{\alpha_2 t} + N'e^{\alpha_3 t}, \\
x_3(t) &= L + Me^{\alpha_2 t} + Ne^{\alpha_3 t},
\end{align*}
\]

where

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By choosing two degenerate levels and setting the energy to be zero, we may safely drop \( H_0 \) in the discussion. Alternatively, the situation discussed here can be understood as a case where the cavity field resonantly couples to the two-level system, and let \( u = u(t)e^{i\omega t} \) with \( \omega \) being the level difference (also the frequency of the cavity field). In the interaction picture, we can have the results in the context.

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[25] This is the case for \( a = 0, b = \sin \beta \) and \( c = \cos \beta \) in \( U_F \), where \( \beta \in [0, 2\pi) \). Clearly \( U_F^* U_F = 1 \) satisfies the constraint. \( x_1(0) \) may be set to be zero, hence \( x_1(t) = 0 \).

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