Toeplitz Quantization and Convexity

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Abstract
Let $T^m_f$ be the Toeplitz quantization of a real $C^\infty$ function defined on the sphere $\mathbb{C}P(1)$. $T^m_f$ is therefore a Hermitian matrix with spectrum $\lambda^m = (\lambda^m_0, \ldots, \lambda^m_m)$. Schur’s theorem says that the diagonal of a Hermitian matrix $A$ that has the same spectrum of $T^m_f$ lies inside a finite dimensional convex set whose extreme points are $\{(\lambda_{\sigma(0)}^m, \ldots, \lambda_{\sigma(m)}^m)\}$, where $\sigma$ is any permutation of $(m+1)$ elements. In this paper, we prove that these convex sets "converge" to a huge convex set in $L^2([0,1])$ whose extreme points are $f^* \circ \phi$, where $f^*$ is the decreasing rearrangement of $f$ and $\phi$ ranges over the set of measure preserving transformations of the unit interval $[0,1]$.

1 Introduction and background

In their papers [3, 4], the authors have described similarities between the infinite and finite dimensional Lie groups. They have strengthened the idea that the set $SDiff(\mathbb{C}P(1))$ of area preserving diffeomorphisms of the Riemann sphere is an infinite dimensional analog of $SU(n)$ by proving an infinite version of the $SU(n)$ Schur-Horn convexity theorem.

In the present paper, we want to show that the convex sets in the two versions (finite and infinite) of Schur-Horn convexity theorem are related. In order to do that, we will show first that each permutation of $(m+1)$ letters determine a measure preserving transformation of the interval $[0,1]$. Secondly we use the fact that the eigenvalues of the Toeplitz quantization of $f$ determine the decreasing rearrangement of $f$.

But since the two convex sets are defined by inequalities arising from the theory of majorization in $\mathbb{R}^{m+1}$ developed in [11, 12] and its generalization to $L^1([0,1])$ by J.Ryff [13, 14, 15], we start by summarizing briefly here the main points. Majorization is a partial ordering in $\mathbb{R}^{m+1}$ defined as it follows:

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For $x \in \mathbb{R}^{m+1}$, let $x^*$ denote the vector obtained by rearranging the components of $x$ in non-increasing order. We say that $x$ majorizes $y$, written $y \prec x$, if

$$y_0^* + y_1^* + \cdots + y_k^* \leq x_0^* + x_1^* + \cdots + x_k^*, \quad 0 \leq k \leq m - 1$$

$$\sum_{k=0}^{k=m} y_k^* = \sum_{k=0}^{k=m} x_k^*$$

Now we can state Schur’ theorem for hermitian matrices.

**Theorem 1.1** Let $\lambda^m = (\lambda_0, \cdots, \lambda_m) \in \mathbb{R}^{m+1}$ be the eigenvalues of a hermitian matrix $A$. Let $\text{diag}(A) = (a_{00}, \cdots, a_{mm})$ be the diagonal of $A$ then $\lambda^m$ majorizes $\text{diag}(A)$.

i.e

$$\text{diag}(A) \prec \lambda^m \quad (1.1)$$

Before going on, let us first fix some notations:

1. Let $x = (x_0, \cdots, x_m) \in \mathbb{R}^{m+1}$. $\sum_m x$ is the orbit of $x$ under the symmetric group of $(m + 1)$ letters, i.e the collection of points $(x_{\sigma(0)}, \cdots, x_{\sigma(m)})$, where $\sigma$ ranges over all $(m+1)!$ permutations.

2. For $C \subset E$ where $E$ is a vector space over $\mathbb{R}$, $\text{co}(C)$ denote the convex hull of $C$.

Majorization and convexity are closely related as it is shown by the following theorem.

**Theorem 1.2** Let $x \in \mathbb{R}^{m+1}$

- (Rado’s theorem)

$$\{y \in \mathbb{R}^{m+1}, y \prec x\} = \text{co}(\sum_m x)$$

- $y \prec x$ if and only if $\sum_{i=0}^{i=m} f(y_i) \leq \sum_{i=0}^{i=m} f(x_i)$ for any convex function whose domain contains all the numbers $x_i, y_i, 0 \leq i \leq m$.

Schur-Horn’ theorem can be therefore restated in the following terms :

**Theorem 1.3 (Schur-Horn’s theorem)** Let $\lambda^m = (\lambda_0, \cdots, \lambda_m) \in \mathbb{R}^{m+1}$. Let $\mathcal{H}_{\lambda^m}$ be the set of all hermitian matrices whose spectrum is $\lambda^m$.

Let $p^m : \mathcal{H}_{\lambda^m} \to \mathbb{R}^{m+1}$ be the map that picks the diagonal of a matrix.

Then the image of the map $p^m$ is the convex set $\text{co}(\sum_m \lambda^m)$.

The convex set $\text{co}(\sum_m \lambda^m)$ plays a very important role in symplectic geometry: It is the image of a moment map [18].

The concepts of majorization is also extended to integrable functions in the
following sense. Let \((X, \mu)\) be a finite measurable space. For \(f\) measurable function on \(X\), the distribution function of \(f\) is the function \(F_f\) defined by

\[ F_f(t) = \mu(\{\omega; f(\omega) < t\}). \]

Let \(d_f(t) = \mu(X) - F_f(t)\).

**Definition 1.1** The decreasing rearrangement of \(f\) is the function \(f^*\) defined by:

\[ f^*(s) = \inf\{t > 0, d_f(t) < s\}. \]

For \(f, g \in L^1((X, \mu))\), let \(f^*\) and \(g^*\) be their decreasing rearrangement. We say that \(f\) majorizes \(g\) (written \(g \prec f\)) if

\[
\int_0^s g^*(z) \, dz \leq \int_0^s f^*(z) \, dz, \quad 0 \leq s < 1
\]

\[
\int_0^1 g^*(z) \, dz = \int_0^1 f^*(z) \, dz
\]

To stay conform with the notation of [4], let \(X = \mathbb{CP}(1)\) be the Riemann sphere, \(\mu\) the measure defined by the the Fubini-Study symplectic form which in the local coordinate \([1, w]\) is given by

\[
\Omega = \frac{i}{(1 + w\overline{w})^2} \, dw \wedge d\overline{w}.
\]

Set \(w = r \exp(i\theta)\) and introduce the real variable \(z \in [0, 1]\) by \(z = r^2 \backslash (1 + r^2)\). The symplectic form \(\Omega\) becomes \(\Omega = 2 \, dz \wedge d\theta\). The infinite version of Schur theorem is

**Theorem 1.4 (Schur-type Theorems, [3])** Let \(L^2(\mathbb{CP}(1))\) be the set of square integrable functions on the sphere. Let \(P : L^2(\mathbb{CP}(1)) \rightarrow L^2[0, 1]\) be the projection \(P(f) = \frac{1}{2\pi} \int_0^{2\pi} f(z, \theta) \, d\theta\).

Then \(f^*\) majorizes \(P(f)\). i.e

\[ P(f) \prec f^* \]

We have also the equivalent of Rado’s theorem in \(L^2([0, 1])\).

**Theorem 1.5** Let \(f \in L^2([0, 1])\). The set \(\Omega(f) = \{g \in L^2([0, 1]), g \prec f\}\) is weakly compact and convex. Its set of extreme points is \(\{f^* \circ \phi \mid \phi\ \text{is a measure preserving transformation of } [0, 1]\}\).

The paper is organized in three sections: In §2, we have reviewed the topology of the set of measure preserving transformations of the unit interval\([0, 1]\) and showed that the groups, \(\sum_m\) of permutations of \((m+1)\) letters, can be identified...
with a dense subset of the set of all invertible measure preserving transformations of $[0,1]$ endowed with strong operator topology.

In §3 we use Toeplitz quantization to show that the weak closure of the topological lim sup of the sets $\co(\sum_m \lambda^m)$ is the set $\Omega(f^*) = \co(\{f^* \circ \phi, \phi \text{ measure preserving transformation of } [0,1]\})$.

## 2 measure preserving transformations of $[0,1]$)

The Lebesgue measure on the unit interval $I = [0,1]$ will be always denoted by $|\cdot|$. A map $\phi$ from $[0,1]$ to itself is a measure preserving transformation if $|\phi^{-1}(A)| = |A|$, for Borel set $A$.

The set of all (non necessary invertible ) measure preserving transformation of the unit interval will be denoted $\text{smeas}(I)$. The invertible ones will be denoted by $\text{imeas}(I)$.

Each $S \in \text{smeas}(I)$ determine a bounded linear operator $P_S$ on $L^2([0,1])$ by $P_S(f) = f \circ S$. In this way, $\text{smeas}(I)$ can be identified to a subset of the set of bounded linear operators of $L^2[0,1]$ and the strong operator topology induces a topology on $\text{smeas}(I)$.

Evidently, a sequence $S_n$ converges to $S$ in the strong operator topology if for every function $f$, $f \circ S_n$ converges to $f \circ S$ in $L^2([0,1])$.

To state our first main result, we need to define dyadic permutations.

Let $I^m_k = [k \cdot (m + 1), (k + 1) \cdot (m + 1)]$, $k = 0, 1, \ldots, m$; $m = 0, 1, \ldots$.

Let $\sum_m$ be the group of permutations of $(m+1)$ letters. For $\sigma \in \sum_m$, $\hat{\sigma}$ is the invertible measure preserving transformation that sends the interval $I^m_k$ to the interval $I^m_{\sigma(k)}$ by ordinary translation.

We call $\hat{\sigma}$ a permutation of rank $m$. In this way, we can identified the group of permutations $\sum_m$ with a subgroup of $\text{imeas}(I)$.

Halmos ([10]) shows that the set of all permutations of different rank is dense in $\text{imeas}(I)$ for the strong operator topology.

Also Brown, in [6] has proved that $\text{smeas}(I)$ is the closure of $\text{imeas}(I)$ for the strong operator topology.

We can summarize these results in the following theorem.

**Theorem 2.1** Let $\sum_m$ be the symmetric group of $(m+1)$ letters. There exists a one to one group homomorphism $\Psi^m : \sum_m \rightarrow \text{imeas}(I)$ that sends $\sigma$ to $\hat{\sigma}$ and if we identify $\sum_m$ with $\Psi^m(\sum_m)$, then $\bigcup_m \sum_m$ is a dense set in $\text{smeas}(I)$ for the strong operator topology.
3 Toeplitz Quantization and Convexity

3.1 Toeplitz Quantization

Consider the Riemann sphere $\mathbb{CP}(1)$ with the Fubini-Study symplectic form in the local coordinate $[1, w]$

$$\Omega = \frac{i}{(1 + w\overline{w})^2} dw \wedge d\overline{w}$$

and the standard hyperplane bundle $L$. The tensor power $L^{} \otimes^m$ has $(m+1)$ linearly independent sections which in the local coordinate $w$ are just $1, w, \ldots, w^m$. The bundle $L^{} \otimes^m$ comes equipped with the Hermitian metric

$$\langle s_1, s_2 \rangle(w) = \frac{1}{(1 + \left|w\right|^2)^m} s_1(w)\overline{s_2(w)}.$$ 

Now let $\Gamma^{m}_{2}$ be the space of square-integrable sections of $L^{} \otimes^m$ and $\Gamma^{m}_{hol}$ the space of holomorphic sections (the span of $1, w, \ldots, w^m$). The orthogonal projection $\Gamma^{m}_{2} \rightarrow \Gamma^{m}_{hol}$ is denoted by $P^m$. The Toeplitz quantization of $f$ is the map $T^m f : \Gamma^{m}_{hol} \rightarrow \Gamma^{m}_{hol}$ defined by

$$T^m f = P^m \circ M^{} f \circ P^m$$

where $M^{} f$ is multiplication by $f$. We refer the interested reader to [5] for a detailed exposition on Toeplitz quantization.

The crucial result is the following theorem

**Theorem 3.1 (Distribution of the Eigenvalues of Toeplitz Quantization)**

Let $\lambda^m = (\lambda^m_0, \lambda^m_1, \ldots, \lambda^m_m)$ be the eigenvalues of $T^m f$ arranged in non-increasing order and let $\Lambda^m(s)$ be the real step function defined on the interval $[0, 1]$ by

$$\Lambda^m \left( \frac{k}{m+1}, \frac{k+1}{m+1} \right) = \lambda^m_k, \quad 0 \leq k \leq m.$$ (3.1)

Then the sequence $(\Lambda^m(s))_m$ converges point-wise almost everywhere to the decreasing rearrangement $f^*(s)$ of the function $f$.

The proof of this Theorem is based on the following theorem

**Theorem 3.2 (Szegő-type Theorem, [9] p: 248)** Given a smooth real-valued function $f$ on $\mathbb{CP}(1)$, let $T^m f$ be Toeplitz quantization of $f$ and let $\mu^m$ be its spectral measure. Then $\frac{\mu^m_0}{m+1}$ tends weakly to a limiting measure as $m$ tends to infinity, this limiting measure being

$$\mu(\phi) = \frac{1}{2\pi} \int_{\mathbb{CP}(1)} \phi(f(x)) d\Omega, \quad \text{for } \phi \in \mathcal{C}(\mathbb{R}).$$

i.e if $(\lambda^m_0, \lambda^m_1, \ldots, \lambda^m_m)$ are the eigenvalues of $T^m f$ ordered in non-increasing order, then

$$\lim_{m \rightarrow +\infty} \sum_{k=0}^{k=m} \frac{\phi(\lambda^m_k)}{m+1} = \frac{1}{2\pi} \int_{\mathbb{CP}(1)} \phi(f(x)) d\Omega.$$ (3.2)
If we use the step function $\Lambda^m$ defined by (3.1) then (3.2) can be written

$$\lim_{m \to +\infty} \int_0^1 \phi(\Lambda^m) = \frac{1}{2\pi} \int_{CP(1)} \phi(f(x))d\Omega. \quad (3.3)$$

But since $f$ and $f^*$ are equi-measurable, we have

$$\frac{1}{2\pi} \int_{CP(1)} (\phi \circ f)d\Omega = \int_0^1 \phi \circ f^*(t)dt$$

and (3.3) becomes

$$\lim_{m \to +\infty} \int_0^1 \phi(\Lambda^m)(t) dt = \int_0^1 \phi(f^*)(t) dt. \quad (3.4)$$

where $\Lambda^m$ is defined by (3.1).

Relation (3.4) is equivalent to: the sequence of step functions $\Lambda^m$ converges in distribution to the real function $f^*$. (One can see [7] for more details on convergence in distribution.)

In general convergence in distribution does not imply convergence point-wise. Nevertheless, there exists another sequence $g_n$ with the same distribution as $f_n$ that converges point-wise to a function $g$, that has the same distribution of $f$. That is the content of Skorokhod’s Theorem.
Theorem 3.3 (Skorokhod) Let $(X, \Sigma, \mu)$ be a finite measure space, and $f, f_n : X \to \mathbb{R}$ be a sequence of measurable functions such that $f_n$ converge in distribution to $f$. Then on the Lebesgue measure space $(I, B, |\cdot|)$, where $I = (0, \mu(X))$, $B$ is the Borel $\sigma$-algebra of $I$, and $|\cdot|$ is the Lebesgue measure, there exists measurable functions $g_n, g : I \to \mathbb{R}$ such that $g_n(t) \to g(t)$ a.e.$|\cdot|$, and
\[
\mu(\{\omega : f_n(\omega) < x\}) = |\{t : g_n(t) < x\}|
\]
\[
\mu(\{\omega : f(\omega) < x\}) = |\{t : g(t) < x\}|
\]
x \in \mathbb{R}, n \geq 1.

Let $F_n$ and $F$ be the distribution functions of $f_n$ and $f$. We can take $g_n$ and $g$ to be just the generalized inverse of $F_n$ and $F$:
\[
g_n(t) = \inf\{x : F_n(x) > t\}, \quad g(t) = \inf\{x : F(x) > t\}, \quad 0 < t < 1.
\]
It easily seen then that
\[
g_n(1 - t) = g_n^*(t) = f_n^*(t), \quad g(1 - t) = g^*(t) = f^*(t).
\]

(11) \cite[p141-144].

Applying Skorokhod’s Theorem to the the sequence $\Lambda^m$, we deduce that the generalized inverses of the functions $\Lambda^m$ converges point-wise almost everywhere to the generalized inverse of the function $f$.

Consequently, the sequence of the decreasing rearrangements of $\Lambda^m$ converges to the decreasing rearrangement $f^*$ of $f$.
\[
\lim_{m \to +\infty} (\Lambda^m)^*(s) = f^*(s).
\]

But since $\Lambda^m(s)$ is a decreasing function, $(\Lambda^m)^*(s) = \Lambda^m(s)$, and therefore we have
\[
\lim_{m \to +\infty} \Lambda^m(s) = f^*(s)
\]
and that ends the proof of the theorem.

3.2 Convexity

Let $p^m : \mathbb{R}_{m+1} \to L^2([0, 1])$ be the map that associates to the point $(a_0, a_1, \ldots, a_m)$ the step function $p^m(a_0, a_1, \ldots, a_m) = \sum_{k=0}^m a_k \chi_{I_k^m}$, where $I_k^m$ is the interval $\left[\frac{k}{m+1}, \frac{k+1}{m+1}\right]$ and $\chi_{I_k^m}$ is the characteristic function of $I_k^m$. Set $E_m = p^m(\text{co}(\sum_m \lambda^m))$.

Now we are ready to state the main result about convexity.

Theorem 3.4 Let $f \in C^\infty(\mathbb{CP}(1))$. Let $f^*$ be the decreasing rearrangement of $f$. Let $T_{f^*}^\mu$ be the Toeplitz quantization of $f$ and let $\lambda^m = (\lambda^m_0, \ldots, \lambda^m_m)$ be the eigenvalues of $T_{f^*}^\mu$. Let $\Omega(f^*) = \{g \in L^2([0, 1], g \prec f^*)$. Then the closed convex hull of the set $\text{Smeas}(I) \cdot f^* = \{f^* \circ \phi \mid \phi \in \text{Smeas}(I)\}$ is the weak closure of the topological lim sup of the convex sets $E_m = p^m(\text{co}(\sum_m \lambda^m))$.
We recall the definition of the closed limit.

**Definition 3.1 ([2], p. 114)** Let \( \{ E_m \} \) be a sequence of subsets of a topological space \( X \). Then, a point \( x \) in \( X \) belongs to the **topological lim sup** of \( E_m \), denoted \( LsE_m \), if for every neighborhood \( V \) of \( x \) there are infinitely many \( m \) with \( V \cap E_m \neq \emptyset \).

Clearly, \( LsE_m \) is a closed set and moreover

\[
LsE_m = \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} E_k.
\]

In our case, \( X = L^2([0,1]) \) and \( E_m = p^m(co(\sum_m \cdot \lambda_m)) \).

Since \( L^2([0,1], \| \cdot \|_2) \) is a normed vector space, every point of \( L^2([0,1]) \) has a countable basis of neighborhoods and \( LsE_m \) can be characterized in terms of sequences:

\( g \in LsE_m \) if and only if there exists a subsequence \( (g_{m_k})_k \) such that \( g_{m_k} \in E_{m_k} \) and \( g_{m_k} \) converges to \( g \in L^2([0,1]) \).

Now we claim

(A) \( Smeas(I) \cdot f^* \subset LsE_m \).

(B) \( \Omega(f^*) \subset \overline{LsE_m}^{\text{weak}} \subset \Omega(f^*) \).

**Proof of (A):**

Let \( f^* \circ \phi \in Smeas(I) \cdot f^* \).

The set of dyadic permutations is a countable set and from theorem (2.1) is dense in \( Smeas(I) \) for the strong operator topology; therefore there exists a sequence \( (\sigma_n)_n \) of permutations that converges to \( \phi \), i.e.

\[
\forall g \in L^2([0,1]), \forall \epsilon > 0, \exists N_0, \forall n \geq N_0, \| g \circ \sigma_n - g \circ \phi \|_2 < \frac{\epsilon}{2}.
\]

In particular for \( g = \Lambda^m \), we have,

\[
\forall m, \forall \epsilon > 0, \exists N_0, \forall n \geq N_0, \| \Lambda^m \circ \sigma_n - \Lambda^m \circ \phi \|_2 < \frac{\epsilon}{2}.
\] (3.5)

Now since the sequence \( \Lambda^m \) converges almost everywhere to \( f^* \) and \( \forall m, \forall x \in [0,1], |\Lambda^m(x)| \leq ||f^*||_\infty \), it converges also to \( f^* \) in \( L^2([0,1]) \), i.e.

\[
\forall \epsilon > 0, \exists M_0, \forall m \geq M_0, ||\Lambda^m \circ \phi - f^* \circ \phi||_2 < \frac{\epsilon}{2}.
\] (3.6)

We conclude then from (3.5) and (3.6) that \( \forall \epsilon > 0, \exists M_0, \exists N_0, \forall n \geq N_0, \forall m \geq M_0 \)

\[
||\Lambda^m \circ \sigma_n - f^* \circ \phi||_2 \leq ||\Lambda^m \circ \sigma_n - \Lambda^m \circ \phi||_2 + ||\Lambda^m \circ \phi - f^* \circ \phi||_2 \leq \epsilon.
\] (3.7)

Now if \( \sigma_n \) is of order \( k_n \), and if we let \( m = k_n \) in (3.7) we get
∀ε > 0, ∃M₀, ∃N₀ for every n ≥ N₀ such that kₙ ≥ M₀, we have

\[ \| \Lambda^{kₙ} \circ σₙ - f^* \circ φ \|_2 \leq \epsilon. \]

Since \( \Lambda^{kₙ} \circ σₙ \in E_{kₙ} \), we conclude therefore that \( f^* \circ φ \in LsE_m \).

**Proof of (B):**

It is shown in [3, p 523] that \( Ω(f^*) = \{ g \in L^2([0,1]), g \prec f^* \} \) is weakly compact and convex. Its extreme points are precisely the elements \( f^* \circ φ, φ \in Smeas(I) \).

It is also indicated in [15, p1030] that the set of extreme points is dense in \( Ω(f^*) \) for the weak topology. But we have just seen in part (A) of our claim that \( Smeas(I) \cdot f^* \subset LsE_m \). Taking the closure in the weak topology we get

\[ Ω(f^*) = Smeas(I) \cdot f^*_{weak} \subset LsE_m_{weak}. \]

It remains to show that \( LsE_m_{weak} \subset Ω(f^*) \).

Let \( g \in LsE_m \). Then by definition, there exists a subsequence \((g_{m_k})_k \) such that \( g_{m_k} \in E_{m_k} \), and \((g_{m_k})_k \) converges in \( L^2([0,1]) \) to \( g \).

Now by Rado’s theorem (1.2), we have

\[ g_{m_k} \in E_{m_k} \iff g_{m_k} \prec Λ_{m_k} \]

But the sequence \((g_{m_k})_k \) being convergent in \( L^2([0,1]) \), we can extract a subsequence \((g_{m_{ki}})_l \) that converges pointwise to \( g \) a.e. We have then

\[ g_{m_{ki}} \prec Λ_{m_{ki}}. \]

And taking limit (simple convergence) of both sides, we get \( g \prec f^* \) i.e \( LsE_m \subset Ω(f^*) \). Taking the weak closure of both sets we conclude therefore that

\[ Ω(f^*) = LsE_m_{weak}. \]

Our next goal is to compare the co-adjoint orbits of \( SU(m+1) \) and the co-adjoint orbits of \( SDiff(\mathbb{C}P(1)) \), the group of area preserving diffeomorphisms of the sphere.

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**References**

[1] M.F. Atiyah, *Convexity and commuting Hamiltonians*, Bull. London Math. Soc **14**, 1-15, (1982)

[2] C. D. Aliprantis, K. C. Border, *Infinite Dimensional Analysis*, a Hitchhiker’s Guide, 3rd edition, Springer (2005).
[3] A. Bloch, H. Flaschka, T. Ratiu, A Shur-Horn-Kostant Convexity Theorem for the Diffeomorphism Group of the Annulus, Inv.Math. 113, 511-529, (1993).

[4] A. Bloch, M. El Hadrami, H. Flaschka, T. Ratiu, Maximal Tori Of Some Symplectomorphism Groups And Applications To Convexity, Proceedings of Ascona Meeting, June 1996. (D. Sternheimer, J. Rawnsley, S. Gutt, eds.), Mathematical Physics Studies 20, Kluwer Academic Publishers, 201-222, (1997).

[5] M. Bordemann, E. Meinrenken and M. Schlichenmaier, Toeplitz Quantization Of Kähler Manifolds And gl(N), N → +∞ Limits., Commun. Math. Phys. 165, 281-296, (1994).

[6] J. R. Brown, Approximation theorems for Markov Operators, Pac.J.Maths. 16, 13-23, (1966).

[7] Richard Durrett Probability: Theory And Examples, Wadsworth & brooks/cole Advanced Books & Software, 1991.

[8] V. Guillemin and S. Sternberg, Convexity properties of the momentum mapping I, Invent. Math, 67, 491-513, (1982).

[9] Victor Guillemin, Some Classical Theorems In Spectral Theory Revisited, Seminar on Singularities of Solutions of Linear Partial Differential Equations (L. Hörmander ed) Annals of Mathematics Studies, 91, Princeton University Press and University of Tokyo Press, [1979], 219–259.

[10] P. Halmos Approximation Theorems for Measure Preserving Transformations, Trans.AMS 55, 1-18, (1944).

[11] M. M. Rao Measure Theory And Integration, 2nd Edition, Marcel Dekker Inc, (2004).

[12] Jean-Michel Rakotoson Réarrangement Relatif, Un Instrument d’Estimation dans les Problèmes aux Limites, Mathématiques & Applications, 64, Springer, (2008).

[13] J. V. Ryff, On the Representation of Doubly Stochastic Operators, Pac.J.Maths. 13, 1379-1386, (1963).

[14] J. V. Ryff, Orbits of L^1 Functions Under Doubly Stochastic Transformations, Trans.AMS 117, 92-100, (1965).

[15] J. V. Ryff, Extreme points of some convex subsets of L^1(0,1), Proc. AMS 18, 1026-1034, (1967).

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