An Asymptotically Optimal Two-Part Coding Scheme for Networked Control under Fixed-Rate Constraints

Jonathan Keeler, Tamás Linder and Serdar Yüksel

Abstract—It is known that fixed rate adaptive quantizers can be used to stabilize an open-loop-unstable linear system driven by unbounded noise. These quantizers can be designed so that they have near-optimal rate, and the resulting system will be stable in the sense of having an invariant probability measure, or ergodicity, as well as the boundedness of the state second moment. However, results on the minimization of the state second moment for such quantizers, an important goal in practice, do not seem to be available. In this paper, we construct a two-part adaptive coding scheme that is asymptotically optimal in terms of the second moments. The first part, as in prior work, leads to ergodicity (via positive Harris recurrence) and the second part attains order optimality of the invariant second moment, resulting in near optimal performance at high rates.

I. INTRODUCTION

We consider the linear system

\[ x_{t+1} = ax_t + bu_t + d_t, \]

where \(|a| \geq 1, b \neq 0\), and \(d_t\) is a sequence of i.i.d. Gaussian random variables \(d_t \sim \mathcal{N}(0, \sigma^2)\). The state variable \(x_t\) is driven by the noise \(d_t\) and the aim is to control the state through the action \(u_t\). The system is open-loop-unstable but is stabilizable.

First suppose that the system is fully observed. If one minimizes the average quadratic cost of the state, \(E[ x_t^2 ]\), over any time horizon (with no penalty on control) then the optimal control is \(u_t = -\frac{1}{b} x_t\) and the optimal cost is \(\sigma^2\) (via a Riccati equation optimality argument with no cost on control [1]). In contrast, here we assume that the controller only has access to \(x_t\) through a discrete noiseless channel of capacity \(C\) bits.

Thus, the data rate is fixed, and we assume zero coding delay. In this setup, it becomes necessary to describe not just a control policy, but also a coding scheme with which to communicate information about the current state variable.

For systems of this nature, various authors have obtained the smallest channel capacity above which stabilization is possible, under various assumptions on the system and the admissible coders and controllers. This result is usually referred to as a data-rate theorem, and in the scalar case [1] we consider, it reduces to simply \(C > \log |a|\). Some of the earliest works in this context are [2] and [3]. More general versions of the data-rate theorem have been proven in [4] and [5]. For noisy systems and mean-square stabilization, or more generally, moment-stabilization, analogous data-rate theorems have been proven in [6] and [7], see also [8], [9].

In [10], [11], a joint fixed-rate coding and control scheme is given which stabilizes the system (1) while being nearly rate-optimal, in that the rate used satisfies only \(C > \log |a| + 1\). This is achieved using an adaptive uniform quantization scheme, where the quantizer bin sizes "zoom" in and out exponentially to track the state \(x_t\). Here, the notion of stability is ergodicity and finiteness of all limiting system moments. By increasing a sampling period \(T\) the achievable rate \(\frac{1}{T} \log(|a|^T + 1)\) gets arbitrarily close to \(C > \log |a|\) [12, Theorem 2.3]. The same result also applies to multi-dimensional systems [12]. Furthermore, this scheme leads to a closed loop system which is positive Harris recurrent, and hence, ergodic. For a related recent construction, we refer the reader to [13].

However, despite being near rate-optimal for achieving stability (i.e., finite system moments), the scheme in [10], [11] has not been shown to be asymptotically second-moment optimal as the data rate grows large.

In this paper, we present a variation on the coding and control scheme used in [10], [11] by including an additional quantization stage, which is fixed in time, unlike the first stage of the coding scheme. Crucially utilizing the ergodicity results of the first coding stage, we show that this two-part coding scheme attains second-moment optimality with near optimal rate of convergence. While multi-stage quantization schemes have been studied before in the source coding literature [14], our implementation is novel in that one stage of the code is time-adaptive and stabilizing. An inspiration for this approach also comes to us from Berger [15] and Sahai [16].

In this paper, we show that this new scheme retains key stability and ergodicity properties ensured by the original one-stage scheme, closely appealing to the existing arguments made in [10], [11]. In our main results, Theorems 3.3 and 3.7 we show that the new scheme is asymptotically second-moment optimal, i.e., as the data rate goes to infinity, the system second moment converges to the optimum \(\sigma^2\) at a rate which is arbitrarily close to order-optimal in a polynomial sense. We also illustrate this convergence by numerical results.
II. PRELIMINARIES

A. Quantization

To communicate over a finite capacity channel, it is necessary to employ quantization schemes. We will work solely with the following class of quantizers, as used in [10], [11]. For a given bin size $\Delta > 0$ and even number of bins $M \geq 2$, we define the modified uniform quantizer $Q^\Delta_M$ by:

$$Q^\Delta_M(x) = \begin{cases} \Delta \frac{|x|}{2} + \frac{\Delta}{2}, & \text{if } x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right] \\ \frac{M}{2}\Delta - \frac{\Delta}{2}, & \text{if } x = \frac{M}{2}\Delta \\ 0, & \text{if } |x| > \frac{M}{2}\Delta. \end{cases} \tag{2}$$

This quantizer uniformly quantizes $x \in \left[-\frac{M}{2}\Delta, \frac{M}{2}\Delta\right]$ into $M$ bins of size $\Delta$ and maps all larger $x$ to zero. This requires $M + 1$ output levels.

This quantizer is almost as simple as possible (aside from the overload symbol, this is typical uniform quantization), and its use leads to stability of the closed-loop process with near-rate-optimality, as discussed in Section II-A.

We will consider two primary applications of this class of quantizers. First, we consider adaptive quantizers, whose bin sizes vary with time. Fix an even number of bins $K \geq 2$ and let $\{\Delta_t\}_{t=0}^{\infty}$ be a sequence of strictly positive bin sizes. We will then make use of the adaptive modified uniform quantizer $Q^{\Delta_t}_K$.

Secondly, we consider quantizers fixed in time, whose bin size is a function of the number of bins. For a given (even) number of bins $N \geq 2$, we choose a bin size $\Delta_t(N)$ and consider the quantizer $Q^{\Delta_t}_N$. To distinguish this from the adaptive case, we denote $U_N := Q^{\Delta_t}_N$. We also allow for $N = 0$, for which we write $U_0 \equiv 0$.

B. System Description

Recall the linear system (1) with the information constraints of Figure 1. Since the system is open-loop-unstable, any fixed quantization policy will make the system transient. We will consider a two-part coding scheme, where the first part is adaptive and the second is fixed. The adaptive part will yield stability, and the fixed part will yield an optimal rate of convergence.

Suppose $\{\Delta_t\}_{t=0}^{\infty}$ is a sequence such that $\Delta_{t+1}$ is a function of only $\Delta_t$ and the indicator random variable $\mathbb{1}_{\{x_t \leq \frac{4}{3}\Delta_t\}}$. Also assume that both the encoder and the decoder (controller) know $\Delta_0$. Then so long as $Q^\Delta_K(x_t)$ is sent over the channel, it is possible to synchronize knowledge of $\Delta_t$ between the quantizer and the controller, since $|x_t| \leq \frac{4}{3}\Delta_t$ if and only if $Q^\Delta_K(x_t) \neq 0$.

We now describe the proposed communication and control scheme. For $\{\Delta_t\}_{t=0}^{\infty}$ as above, we calculate the adaptive quantizer output $Q^\Delta_K(x_t)$ and the adaptive system error $e_t := x_t - Q^\Delta_K(x_t)$. Then, using a fixed (i.e., non-adaptive) quantizer $U_N$ with bin size $\Delta_t(N)$ as in Section II-A, we calculate the fixed quantizer output $U_N(e_t)$. We then send $Q^\Delta_K(x_t)$ and $U_N(e_t)$ across the noiseless channel using a total of $C = \log_2(K + 1) + \log_2(N + 1)$ bits.

The controller then applies the control given by

$$u_t = -\frac{a}{b} \left( Q^\Delta_K(x_t) + U_N(e_t) \right).$$

This coding and control scheme is illustrated below.

We note that the term $U_N(e_t)$ distinguishes our control scheme from [10], [11]. This control is chosen to mirror the optimal fully observed control, where $Q^\Delta_K(x_t) + U_N(e_t)$ is a good estimate of the true state $x_t$. By (1), this results in the state dynamics

$$x_{t+1} = a(x_t - Q^\Delta_K(x_t) - U_N(e_t)) + d_t$$

$$= a(e_t - U_N(e_t)) + d_t. \tag{3}$$

In the case $N = 0$, this reduces to [10], [11] with

$$x_{t+1} = a(x_t - Q^\Delta_K(x_t)) + d_t. \tag{4}$$

We briefly motivate the given scheme. For the bin update rules we will describe below, the pair $(x_t, \Delta_t)$ forms a Markov chain. The update dynamics (4) with no fixed quantization ($N = 0$) ensure that the system is stochastically stable and ergodic, due to results of [11].

Our main contribution is analyzing the performance of the proposed two-stage scheme. Through an intricate stochastic stability analysis, we ultimately prove that the addition of a fixed quantization stage leads to optimal convergence of the second moment with near-optimality in the rate of convergence.

Finally, we describe the bin update dynamics. As in [10], [11], a simple zooming scheme is employed. Assuming that $K > a$ (which ensures stability), choose $\frac{|a|}{K} < \alpha < 1$, $\delta > 0$ and $L > 0$. The update is:

$$\Delta_{t+1} = \begin{cases} |a| + \delta \Delta_t, & \text{if } |x_t| > \frac{K}{2}\Delta_t \\ \alpha \Delta_t, & \text{if } |x_t| \leq \frac{K}{2}\Delta_t, \Delta_t \geq L \\ \Delta_t, & \text{if } |x_t| \leq \frac{K}{2}\Delta_t, \Delta_t < L. \end{cases} \tag{5}$$

The above rules imply that $\Delta_t \geq \alpha L$. We choose an arbitrary initial $\Delta_0 > 0$.

Then we consider the process $\{\Delta_t\}_{t=0}^{\infty} = \{(x_t, \Delta_t)\}_{t=0}^{\infty}$ with the dynamics described above. This process is a Markov chain. The state space of this process highly depends on the following "countability condition" utilized in [10], [11].
Condition A. There exist relatively prime integers $j,k \geq 1$ such that $\alpha^j(|a| + \delta)^k = 1$. Equivalently, $\log_\alpha(|a| + \delta)$ is rational.

If this condition holds, then starting from an arbitrary $\Delta_0 > 0$ there exists $\kappa, b \in \mathbb{R}$ such that $\log_\alpha \Delta_i$ always belongs to a subset of $\mathbb{Z}\kappa + b = \{n\kappa + b : n \in \mathbb{Z}\}$. If the condition fails, then starting from any fixed $\Delta_0$, the set of reachable bin sizes is a dense but countable subset of $\mathbb{R}_{>0}$.

We restrict our analysis to the case where Condition A holds. We let the state space for $\Delta_i$ be

$$\Omega_\Delta := \{\alpha^j(|a| + \delta)^k \Delta_0 : j,k \in \mathbb{Z}_{\geq 0}\}.$$ 

The state space for the Markov chain $\{(x_i, \Delta_i)\}_{i=0}^{\infty}$ is then $\mathbb{R} \times \Omega_\Delta$.

C. Stochastic Stability

Suppose $\{(\phi_t)\}_{t=0}^\infty$ is a Markov chain with state space $\mathbb{X}$, where $\mathbb{X}$ is a complete separable metric space that is locally compact; its Borel sigma algebra is denoted $\mathcal{B}(\mathbb{X})$. The transition probability is denoted by $P$, so that for any $\phi \in \mathbb{X}$ and $A \in \mathcal{B}(\mathbb{X})$, the probability of moving in one step from state $\phi$ to the set $A$ is given by $P(\phi_{t+1} \in A|\phi_t = \phi) = P(x,A)$. The $n$-step transitions are obtained in the usual way, $P(\phi_{t+n} \in A|\phi_t = \phi) = P^n(x,A)$ for any $n \geq 1$.

The transition law acts on measurable functions $f : \mathbb{X} \to \mathbb{R}$ and measures $\mu$ on $\mathcal{B}(\mathbb{X})$ via

$$Pf(\phi) = \int_\mathbb{X} P(\phi,dy)f(y), \quad \text{for all } \phi \in \mathbb{X}$$

and

$$\mu P(A) = \int_\mathbb{X} \mu(d\phi) P(\phi, A), \quad \text{for all } A \in \mathcal{B}(\mathbb{X}).$$

A probability measure $\pi$ on $\mathcal{B}(\mathbb{X})$ is called invariant if $\pi P = \pi$, that is:

$$\int_\mathbb{X} \pi(d\phi) P(\phi, A) = \pi(A), \quad \text{for all } A \in \mathcal{B}(\mathbb{X}).$$

For any initial probability measure $\nu$ on $\mathcal{B}(\mathbb{X})$ we can construct a stochastic process with transition law $P$ and $\phi_0 \sim \nu$. We let $P_\nu$ denote the resulting probability measure on the sample space, with the usual notation $\nu = \delta_{\phi_0}$ (i.e., $\nu(\{\phi_0\}) = 1$) when the initial state is $\phi \in \mathbb{X}$. When $\nu = \pi$ the resulting process is stationary.

There is at most one stationary solution under the following irreducibility assumption. For a set $A \in \mathcal{B}(\mathbb{X})$ we denote,

$$\tau_A := \min\{t \geq 1 : \phi_t \in A\}.$$ 

Definition 2.1. Let $\varphi$ denote a $\sigma$-finite measure on $\mathcal{B}(\mathbb{X})$.

(i) The Markov chain is called $\varphi$-irreducible if for any $\phi \in \mathbb{X}$ and $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$ we have

$$P_\phi(\tau_B < \infty) > 0.$$ 

(ii) A $\varphi$-irreducible Markov chain is Harris recurrent if $P_\phi(\tau_B < \infty) = 1$ for any $\phi \in \mathbb{X}$ and any $B \in \mathcal{B}(\mathbb{X})$ satisfying $\varphi(B) > 0$. It is positive Harris recurrent if in addition there is an invariant probability measure $\pi$.

Notably, the positive Harris recurrence property leads to ergodicity of the closed-loop process: for every initial state and every $g \in L_1(\pi)$, the following holds almost surely:

$$\lim_{N \to \infty} \frac{1}{N} \sum_{k=0}^{N-1} g(\delta_k) = \int \pi(d\phi) g(\phi).$$

This almost sure sample path convergence also holds in expectation under mild Lyapunov conditions [17] (which will be the case in our analysis).

III. ANALYSIS OF SCHEME

A. Supporting Lemma on Optimality at High–Rates

For the class of fixed quantizers $U_N$ introduced in Section II-A we present the following result bounding the expected MSE of a family of well-behaved random variables.

Lemma 3.1. Suppose $\{X_N\}_{N=2}^\infty$ are random variables satisfying the following uniform moment bound:

$$E|X_N|^m \leq B_m \quad \text{for all } m = 1,2,...$$

where $\{B_m\}_{m=1}^\infty$ is independent of $N$. Let $\varepsilon > 0$ and set the bin size for $U_N$ as $\Delta_{(N)} = AN^{-\frac{1}{2}+\frac{1}{2}\varepsilon}$, for fixed $A > 0$. Then we have

$$E[(X_N - U_N(X_N))^2] = O\left(\frac{1}{N^{2-\varepsilon}}\right),$$

i.e., there exists a constant $C > 0$ which depends on $\varepsilon$ with

$$E[(X_N - U_N(X_N))^2] \leq CN^{-2+\varepsilon}, \quad \text{for all } N \geq 2.$$

Proof: For brevity, denote $Y_N = X_N - U_N(X_N)$. First note that when $|X_N| \leq \frac{1}{2}N\Delta_{(N)}$, we have $|Y_N| \leq \frac{1}{2}\Delta_{(N)}$.

Therefore,

$$E\left[Y_N^2 1\{|X_N| \leq \frac{1}{2}N\Delta_{(N)}]\right] \leq \frac{1}{4}\Delta_{(N)}^2 = \frac{1}{4}A^2N^{-2+\varepsilon}. \quad (8)$$

Next we will consider the region $|X_N| > \frac{1}{2}N\Delta_{(N)}$. Here, note that $Y_N = X_N$ since $U_N(X_N) = 0$. Therefore,

$$E\left[Y_N^2 1\{|X_N| > \frac{1}{2}N\Delta_{(N)}\}\right] = E\left[X_N^2 1\{|X_N| > \frac{1}{2}N\Delta_{(N)}\}\right].$$

We then apply H"older’s inequality with conjugates $\frac{2+\varepsilon}{\varepsilon}$ and $\frac{2}{2+\varepsilon}$:

$$E\left[X_N^2 1\{|X_N| > \frac{1}{2}N\Delta_{(N)}\}\right] \leq E\left[|X_N|^{2+\frac{2}{2+\varepsilon}}\right]^\frac{2+\varepsilon}{2} E\left[1\{|X_N| > \frac{1}{2}N\Delta_{(N)}\}\right]^\frac{2}{2+\varepsilon} \leq B_m \frac{2+\varepsilon}{2+\varepsilon} \leq (B_m)^\frac{2+\varepsilon}{2}.$$

(9)

Now, choose an integer $m \geq 2 + \frac{1}{2}$ such that $-\frac{m}{2+\varepsilon} \leq -2$. First, note that by Jensen’s inequality and the uniform moment bound (7),

$$E\left[|X_N|^{2+\frac{2}{2+\varepsilon}}\right]^\frac{2+\varepsilon}{2} \leq E\left[|X_N|^m\right]^\frac{2+\varepsilon}{2} (2+\frac{2}{2+\varepsilon}) \leq (B_m)^\frac{2+\varepsilon}{2}.$$ 

Secondly, note that by Markov’s inequality and the uniform moment bound, for any $u > 0$ we have

$$P(|X_N| > u) \leq \frac{E[|X_N|^m]}{u^m} \leq B_m u^{-m}.$$
Therefore,
\[ P\left(|X_N| > \frac{1}{2} N \Delta(N) \right) \leq (B_m) \left( \frac{2}{N \Delta(N)} \right)^{\frac{m}{2+m}} \cdot \left( \frac{1}{2} N \Delta(N) \right)^{-\frac{m}{2+m}}. \]
We conclude that the product in (9) is upper bounded by
\[ (B_m) \left( \frac{2}{N \Delta(N)} \right)^{\frac{m}{2+m}} \left( \frac{1}{2} N \Delta(N) \right)^{-\frac{m}{2+m}}. \]
Combining (8) and (10) gives the upper bound
\[ O \left( N^{-2+\epsilon} \right), \]
completing the proof.

The above lemma will allow us, via the second part of our code, to bound the rate at which the MSE for a random variable decreases to zero as we increase the number of quantization bins \( N \).

**B. System Stability and Rate Optimality**

First, we will establish key stability results. As stated before, we will assume that Condition A holds. Consider the case where \( N = 0 \). By [11, Theorem 3.1], if \( K > |a| \), then \( \{ (x_t, \Delta_t) \}_{t=0}^{\infty} \) is a positive Harris recurrent Markov chain with unique invariant measure \( \pi \). We offer the following extension:

**Theorem 3.2.** For all \( N \geq 0 \), \( \{ (x_t, \Delta_t) \}_{t=0}^{\infty} \) is positive Harris recurrent.

*Sketch of Proof:* The case \( N \geq 2 \) is different from the case of no fixed quantization only in that the state update dynamics are decreased deterministically. This allows us to extend the proof program due to [11] for positive Harris recurrence for the case \( N = 0 \) with only minor modifications to some of the intermediate expressions. \( \square \)

Now suppose that the system is positive Harris recurrent with fixed parameters \((a, \sigma^2, K, \alpha, \beta)\) and given \( N \geq 2 \) (this is possible so long as \( K > |a| \)). Let \( \pi_N \) be the resulting invariant measure depending on \( N \), and let \((x_N^* , \Delta_N) \sim \pi_N \). Since the adaptive quantization error \( e_t \) is a deterministic function of \((x_t, \Delta_t)\), this invariant measure will also induce a stationary distribution for \( e_t \), which we denote by \( e_N \sim \pi_N^e \).

**Theorem 3.3.** For all \( N \geq 2 \), for every initial condition,
\[ \lim_{t \to \infty} E[x_t^2] = E[(x_N^*)^2] = \int \pi_N(dx, d\Delta) x^2. \]

*Proof:* This result follows from [11, Theorem 2.1(iii)] and the drift criteria we establish in Section II-D. \( \square \)

It is obvious from the state update equation (3) that \( E[(x_N^*)^2] \geq \sigma^2 \), so we wish to derive bounds for the "rate" at which \( E[(x_N^*)^2] - \sigma^2 \to 0 \) as \( N \to \infty \). To make this precise, we will say that \( E[(x_N^*)^2] \) converges to \( \sigma^2 \) at a rate \( r > 0 \) if
\[ \limsup_{N \to \infty} N^r (E[(x_N^*)^2] - \sigma^2) < \infty. \]

The key contributions of this section are as follows. First, we show in Lemma 3.4 that regardless of our choice of \( \Delta(N) \), any such \( r \) satisfies \( r \leq 2 \). Secondly, in Theorem 3.7 we show that for any \( \varepsilon > 0 \), an appropriate choice of \( \Delta(N) \) ensures that the second moment converges at a rate \( r \geq 2 - \varepsilon \). This provides a control scheme leading to a convergence rate which is arbitrarily close to optimal in a polynomial sense.

The following result implies that any rate for the second moment satisfies \( r \leq 2 \).

**Lemma 3.4.** The following lower bound on \( E[(x_N^*)^2] \) holds:
\[ E[(x_N^*)^2] - \sigma^2 \geq \frac{a^2 \sigma^2}{(K + 1)^2(N + 1)^2 - a^2}. \]

To prove this result, we will need the following intermediate result due to [18].

**Proposition 3.5** ([18, Theorem 11.3.2]). Consider the linear Gaussian system \( x_{t+1} = ax_t + bu_t + d_t \) with \( b \neq 0 \) and \( \{ d_t \} \) a zero-mean Gaussian i.i.d. sequence with variance \( \sigma^2 \). To satisfy
\[ E[(x_t^*)^2] - \sigma^2 \geq \frac{a^2 \sigma^2}{(K + 1)^2(N + 1)^2 - a^2}. \]

Remark. It is clear that any such \( d \) must satisfy \( d > \sigma^2 \). More subtly, it must be the case that \( 2C > |a| \). Assuming the converse allows one to show that the above inequality holds in the reverse direction, for all \( d > \sigma^2 \).

Now suppose that for some choice of control, the second moment \( \lim_{t \to \infty} E[x_t^2] \) exists and is finite. Setting \( d = \lim_{t \to \infty} E[x_t^2] \) and rearranging with the above remark in mind yields the following:
\[ \lim_{t \to \infty} E[x_t^2] \geq \frac{(2C \sigma)^2}{(2C)^2 - a^2}. \]

We proceed with the proof that any rate \( r \) must satisfy \( r \leq 2 \).

*Proof of Lemma 3.4* The system (11) is of the form needed for the previous proposition. Furthermore, as we will show in Section II-D, the limiting second moment exists and is finite. Therefore with \( C = \log_2(K + 1) + \log_2(N + 1) \), (11) yields
\[ E[(x_N^*)^2] - \sigma^2 \geq \frac{(K + 1)^2(N + 1)^2 - a^2}{(K + 1)^2(N + 1)^2 - a^2} \]
which completes the proof. \( \square \)

To show that we can achieve \( r \) arbitrarily close to 2, we will apply Lemma 3.4 to the stationary adaptive error \( e_N \), so it will be necessary to show that \( e_N \) admits uniformly bounded moments:
Lemma 3.6. The stationary adaptive error \( \{e_N\}_{N=2}^{\infty} \) has uniformly bounded moments, i.e.,
\[
E[|e_N|^m] \leq B_m \quad \text{for all } m = 1, 2, \ldots
\]
for some \( \{B_m\}_{m=1}^{\infty} \) independent of \( N \).

Remark. While this result is not surprising, the proof program is quite involved. We discuss this in further detail in Section III-D.

Finally, it follows from Lemma 3.1 that we can achieve a rate \( r \geq 2 - \epsilon \), for any \( \epsilon > 0 \).

Theorem 3.7. Choose any \( \epsilon > 0 \) and set the bin size for \( U_N \) as \( \Delta(N) = AN^{-1+\frac{1}{2} \epsilon} \), for fixed \( A > 0 \). Then we have
\[
E[\{x_N^*\}^2] - \sigma^2 = O\left(\frac{1}{N^{2-\epsilon}}\right),
\]
i.e., there exists \( C > 0 \) which depends on \( \epsilon \), such that
\[
E[\{x_N^*\}^2] - \sigma^2 \leq CN^{-2+\epsilon}, \quad \text{for all } N \geq 2.
\]

Proof. Let \( (x_N^*, \Delta(N)) \sim \pi_N \) and let \( e_N = x_N^* - Q_N^\Delta(x_N^*) \) so that \( e_N \sim \pi_N^t \). Now let \( x' \) be the one-step pushforward of \( x_N^* \) using \( e_N \), that is
\[
x' = a(e_N - U_N(e_N)) + Z,
\]
where \( Z \sim N(0, \sigma^2) \) is independent of \( e_N \). Since we started under the invariant measure, \( x' \) is distributed identically to \( x_N^* \). Furthermore, since \( E[e_N]\) is finite by Lemma 3.6, it follows from (13) that \( E[\{x_N^*\}^2] = E[\{x'\}^2] \) is finite. For brevity, let \( q_N = e_N - U_N(e_N) \), then by total expectation:
\[
E[\{x_N^*\}^2] = E[\{x'\}^2] = E[\{x'\}^2 | e_N] = E\left[ a^2 q_N^2 + 2aq_N Z + Z^2 | e_N \right] = a^2 E[q_N^2] + 2a E[q_N] E[Z] + E[Z^2]
\]
\[
= a^2 \left( e_N - U_N(e_N) \right)^2 + \sigma^2.
\]
Therefore, \( E[\{x_N^*\}^2] - \sigma^2 = a^2 E[(e_N - U_N(e_N))^2] \).

Since \( e_N \) has uniformly bounded moments, Lemma 3.1 implies that our choice of \( \Delta(N) \) causes this MSE to be \( O\left(\frac{1}{N^{2-\epsilon}}\right) \). The result follows since the optimality gap is linearly related to this MSE.

C. Drift Criteria for Moment Stability

In this section, we provide a result which will be critical to the proof program for Lemma 3.6. To this end, we introduce a set of random-time Lyapunov drift criteria and show that if these criteria are satisfied, we can upper bound the expectation of functions under invariant measure. As in Section II-C, we consider a quite general Markov chain \( \{\phi_t\}_{t=0}^{\infty} \) with state space \( \mathbb{X} \).

Suppose that \( \{T_k : k \in \mathbb{N}\} \) is a sequence of strictly increasing stopping times with \( T_0 = 0 \). Then for a measurable function \( V : \mathbb{X} \to (0, \infty) \), measurable functions \( f, d : \mathbb{X} \to [0, \infty) \), a constant \( b \) and a set \( C \subseteq B(\mathbb{X}) \), we say that \( \{\phi_t\}_{t=0}^{\infty} \) satisfies the random-time Lyapunov drift criteria if for all \( z = 0, 1, 2, \ldots \)
\[
E\left[ V(\phi_{T_{z+1}}) | F_T \right] \leq V(T_z) - d(\phi_{T_z}) + bI_{\{\phi_{T_z} \in C\}},
\]
and
\[
E \left[ \sum_{t=T_z}^{T_{z+1}-1} f(\phi_t) \mid F_T \right] \leq d(\phi_{T_z}). \quad (14)
\]

These drift criteria (as well as variations) can be used to establish important stability properties, in combination with irreducibility and other conditions (see [11, Theorem 2.1]). We present the following application for moment stability.

Lemma 3.8. Suppose \( \{\phi_t\}_{t=0}^{\infty} \) is positive Harris recurrent with unique invariant measure \( \pi \) and satisfies the random-time Lyapunov drift criteria (14). Then, \( E_{\pi}[f(\phi)] \leq b \).

Proof. We will utilize supermartingale arguments for this proof. Define the sequence of random variables \( \{M_z\}_{z \geq 0} \) by \( M_0 = V(\phi_0) \), and for any \( z \geq 0 \)
\[
M_{z+1} = V(\phi_{T_{z+1}}) + \sum_{t=T_z}^{T_{z+1}-1} f(\phi_t) - b \sum_{k=0}^{z} I_{\{\phi_{T_k} \in C\}}.
\]

Then the supermartingale property for \( M_z \) follows from the assumed drift criteria (14):
\[
E[\left. M_{z+1} \mid F_{T_z} \right] - M_z = E[\left. V(\phi_{T_{z+1}}) \mid F_{T_z} \right] - V(\phi_{T_z}) + \sum_{t=T_z}^{T_{z+1}-1} f(\phi_t) - b \sum_{k=0}^{z} I_{\{\phi_{T_k} \in C\}}
\leq -d(\phi_{T_z}) + bI_{\{\phi_{T_z} \in C\}} + d(\phi_{T_z}) - bI_{\{\phi_{T_z} \in C\}} = 0.
\]

Therefore, \( \{M_z\}_{z \geq 0} \) is a supermartingale sequence. It follows that for any integer \( n \geq 1 \) we have \( E[M_n \mid F_0] \leq M_0 \). This expands as
\[
E\left[ V(\phi_{T_n}) + \sum_{t=0}^{T_n-1} f(\phi_t) - b \sum_{k=0}^{n-1} I_{\{\phi_{T_k} \in C\}} \mid F_0 \right] \leq V(\phi_0).
\]

Since \( V \) is strictly positive and \( \sum_{k=0}^{n-1} I_{\{\phi_{T_k} \in C\}} \geq M \), we can relax the above to
\[
E\left[ \sum_{t=0}^{T_n-1} f(\phi_t) \mid F_0 \right] \leq V(\phi_0) + bn
\]
\[
\iff \frac{1}{n} E\left[ \sum_{t=0}^{T_n-1} f(\phi_t) \mid F_0 \right] \leq \frac{1}{n} V(\phi_0) + b. \quad (15)
\]

Finally, recall that the sequence of stopping times \( T_k \) is strictly increasing with \( T_0 = 0 \). It follows that \( T_k \geq k \) for all \( k \geq 0 \), so that \( n \leq T_n \). In light of (15) this gives us:
\[
\frac{1}{n} E\left[ \sum_{t=0}^{n-1} f(\phi_t) \mid F_0 \right] \leq \frac{1}{n} E\left[ \sum_{t=0}^{T_n-1} f(\phi_t) \mid F_0 \right] \leq \frac{1}{n} V(\phi_0) + b. \quad (16)
\]

For integer \( k \geq 1 \), let \( f_k(x) := \min\{f(x), k\} \). Due to positive Harris recurrence of \( \{\phi_t\}_{t=0}^{\infty} \), ergodicity holds for
bounded functions. Since $f_k$ is bounded and $f_k \leq f$, we have
\[
E_\pi[f_k(\phi)] = \lim_{n \to \infty} \frac{1}{n} E \left[ \sum_{t=0}^{n-1} f_k(\phi_t) \mid \mathcal{F}_0 \right] \\
\leq \lim_{n \to \infty} \frac{1}{n} E \left[ \sum_{t=0}^{n-1} f(\phi_t) \mid \mathcal{F}_0 \right] \\
\leq b
\]
where the last inequality holds by (16). Finally, since $f_k$ converges monotonically to $f$ from below, it follows from the monotone convergence theorem that
\[
E_\pi[f(\phi)] = E_\pi \left[ \lim_{k \to \infty} f_k(\phi) \right] = \lim_{k \to \infty} E_\pi[f_k(\phi)] \leq b
\]
which completes the proof.

Remark. If the stopping times $\{T_k : k \in \mathbb{N}\}$ are the sequential return times to a set $B \in \mathcal{B}(X)$, that is for $k \geq 0$,
\[
T_{k+1} = \min \{ t > T_k : \phi_t \in B \}
\]
then Lemma 3.8 holds if the drift criteria (14) is established only for $z = 0$, where we assume that $\phi_0 \in B$. Once we show this, the full drift criteria follows for all $z \geq 1$ by induction. What is missing is only the case $z = 0$ where $\phi_0 \in B^c$, but the proof of Lemma 3.8 can be made to work with the sequence of stopping times $\{T_k : k \in \mathbb{N}\} \setminus \{T_0\}$.

D. Proof Program for Uniformly Bounded Moments

The proof program for Lemma 3.6 is quite long. Therefore, we motivate this section by stating its key result first. As in Section III-B, we let $(x_N^*, \Delta_N^*) \sim \pi_N$ where $\pi_N$ is the invariant measure depending on $N$, and we denote the stationary adaptive error as $e_N$. First, note that if the state $x_N^*$ admits uniformly bounded moments, then so too does $e_N$. This can be argued by setting $e_N = x_N^* - \bar{Q}K_N N(x_N^*)$ and recognizing that $|e_N| \leq |x_N^*|$. Therefore, it suffices to show that the state $x_N^*$ admits uniformly bounded moments. To do so, we will use Lemma 3.8. We define the following sequence of "in-view" stopping times,
\[
T_{k+1} = \min \{ t > T_k : |x_t| \leq \frac{K}{2} \Delta_t \}
\]
where $T_0 = 0$. These are the times for which the state $x_t$ is properly quantized by the adaptive quantizer $Q_kN$. Let $m \geq 1$ be arbitrary and choose arbitrary $\gamma \in (0, 1 - \alpha m)$. Let $\beta > 0$ and $D > 0$, then set
\[
V(x, \Delta) = \Delta^m, \\
d(x, \Delta) = \gamma \Delta^m, \\
f(x, \Delta) = \gamma \beta \left( \frac{2}{\bar{K}} \right)^m |x|^m, \\
C = \{(x, \Delta) : \Delta \leq D \}.
\]

Proposition 3.9. There exists choice of $\beta$ sufficiently small, $D$ sufficiently large, and $b$ sufficiently large, all independent of $N$, so that the system $\{(x_t, \Delta_t)\}_{t=0}^\infty$ satisfies the random-time Lyapunov drift criteria (14). It follows directly from Lemma 3.8 that
\[
E[|x_N^*|^m] \leq \frac{b}{\gamma^m} \left( \frac{K}{2} \right)^m =: B_m
\]
and so the state $x_N^*$ admits uniformly bounded moments. By our initial discussion, so too does the stationary adaptive error $e_N$, completing the proof of Lemma 3.6.

We note that the stopping times $\{T_k : k \in \mathbb{N}\}$ are the sequential stopping times of the system to the in-view set,
\[
\Lambda := \left\{ (x, \Delta) \in \mathbb{R} \times \Omega_\Delta : |x| \leq \frac{K}{2} \Delta \right\}.
\]
Therefore, as remarked in Section III-C it suffices to assume that $(x_0, \Delta_0) \in \Lambda$ and that the drift criteria (14) holds for only $z = 0$. The remainder of this section will be dedicated to proving this. The following concentration bound for Gaussian random variables will be useful.

Proposition 3.10. Suppose $Z \sim N(\mu, \sigma^2)$. Then for $t > |\mu|$,\[
P(|Z| > t) \leq 2 \exp \left( -\frac{1}{2\sigma^2} (t - |\mu|)^2 \right).
\]

Proof. We use the following more well-known concentration bound,\[
P(|Z - \mu| > t) \leq 2 \exp \left( -\frac{1}{4\sigma^2} x^2 \right), \quad x \geq 0.
\]
Now suppose $b > 0$. Since $x < |x|$ for all $x \in \mathbb{R}$ and by the reverse triangle inequality we have
\[
P(|Z| > |\mu| + b) = P(|Z| - |\mu| > b) \\
\leq P(|Z| - |\mu| > b) \\
\leq P(|Z - \mu| > b) \\
\leq 2 \exp \left( -\frac{1}{2\sigma^2} b^2 \right).
\]
The proof concludes by letting $t = |\mu| + b$.

Before proving Proposition 3.9 we give the following tail bound for $T_1$.

Lemma 3.11. Define the constants\[
\xi := \frac{|\alpha| + \delta}{|\alpha|}, \quad W := \frac{K \alpha}{|\alpha| + \delta}.
\]
For $(x_0, \Delta_0) \in \Lambda$, we have for any $k \geq 1$ that
\[
P_{x_0, \Delta_0}(T_1 \geq k + 1) \leq 2 \exp \left( -\frac{\Delta_0^2}{8\sigma^2} \left( \frac{W \xi^k - 1}{\sqrt{k}} \right)^2 \right).
\]
Furthermore, for any $\omega > 0$,
\[
P_{x_0, \Delta_0}(T_1 \geq k + 1) \leq B_\omega \Delta_0^{-2\omega} \cdot \frac{k^\omega}{(W \xi^k - 1)^{2\omega}},
\]
where
\[
B_\omega := 2 \left( \frac{8\sigma^2 \omega}{e} \right)^\omega.
\]
Proof. Starting from \((x_0, \Delta_0) \in \Lambda\), let \(e_0 := x_0 - Q^K_{\Delta_0}(x_0)\). Then the following holds for all \(1 \leq t \leq T_1\):

\[
x_t = a^t \left( e_0 - U_N(e_0) + \sum_{m=0}^{t-1} a^{-m} d_m \right)
\]

\[
\Delta_t = \alpha \Delta_0 (|a| + \delta)^{t-1}.
\]

(17)

Let \(\mu := \frac{2}{\Delta_0} (e_0 - U_N(e_0))\) and \(\sigma^2 := \frac{4}{\Delta_0^2} \frac{a^2(1-a^{-2})}{a^{-1} - \frac{1}{\frac{1}{2}a^2}} \sigma^2\), then we can simplify the state equation as \(x_t = \frac{\Delta_0}{\Delta_t} a^t Z_t\) where \(Z_t \sim N(\mu, \sigma^2)\). Since \((x_0, \Delta_0) \in \Lambda\), it follows that \(|\mu| \leq 1\). Also note that \(\sigma^2 \leq \frac{4}{\Delta_0^2} a^2 \sigma^2\). If \(|a| = 1\), this is exact.

Using this state relaxation, it follows that for \(k \geq 1\),

\[
\{ T_1 \geq k + 1 \} \iff \bigcap_{t=1}^{k} \{|x_t| > \frac{K}{2} \Delta_t \}
\]

\[
\iff \bigcap_{t=1}^{k} \{|a|^t |Z_t| > K \alpha (|a| + \delta)^{t-1} \}
\]

\[
\iff \bigcap_{t=1}^{k} \{|Z_t| > \frac{K \alpha}{|a| + \delta} \left( \frac{|a| + \delta}{|a|} \right)^t \}
\]

\[
\iff \bigcap_{t=1}^{k} \{|Z_t| > W \xi^t \}.
\]

Therefore, the following simple upper bound holds:

\[
P_{x_0,\Delta_0}(T_1 \geq k + 1) = P_{x_0,\Delta_0}\left( \bigcap_{t=1}^{k} \{|Z_t| > W \xi^t \} \right)
\]

\[
\leq P_{x_0,\Delta_0}(\{|Z_k| > W \xi^k \}).
\]

It follows from their definitions that \(W > \xi^{-1}\) so for all \(k \geq 1\), \(W \xi^k > 1 \geq |\mu|\). Therefore we may apply Proposition 3.10 to find that

\[
P_{x_0,\Delta_0}(T_1 \geq k + 1) \leq 2 \exp \left( -\frac{1}{2 \sigma_k^2} (W \xi^k - |\mu|^2) \right).
\]

This inequality is relaxed further when we replace \(|\mu|\) by its upper bound of 1 and \(\sigma_k^2\) by its upper bound of \(\frac{1}{\Delta_0^2} \sigma^2\). Applying these relaxations gives

\[
P_{x_0,\Delta_0}(T_1 \geq k + 1) \leq 2 \exp \left( -\frac{\Delta_0^2}{8 \sigma^2} \left( \frac{W \xi^k - 1}{\sqrt{k}} \right)^2 \right)
\]

which is exactly the claimed bound. To arrive at the second bound, it suffices to note that for any \(\omega > 0\), the bound \(e^{-x} \leq (\frac{x}{\omega})^\omega x^{-\omega}\) holds for all \(x > 0\). This can be seen by verifying that the maximum of \(f(x) = (\frac{x}{\omega})^\omega x^{-\omega} e^{-x}\) on \(x > 0\) is \(f(\omega) = 1\) (e.g. by derivative test).

This result enables us to complete the proof program. We note that the proof program presented here closely follows the proof of [11, Theorem 3.2], but emphasis is placed on showing independence in \(N\) of the drift criteria.

Proof of Proposition 3.9: Recall that we need only show that the drift criteria (14) holds in the case \(z = 0\), where we assume that \((x_0, \Delta_0) \in \Lambda\). We will start with the first inequality, which simplifies to

\[
E_{x_0,\Delta_0}[\Delta_{T_1}] - (1 - \gamma) \Delta^m_0 \leq b \lambda \Delta_0 \leq D.
\]

(18)

Note that starting from \((x_0, \Delta_0) \in \Lambda\) we have

\[
\Delta_{T_1} = (\alpha \Delta_0 (|a| + \delta)^{T_1-1})^m
\]

\[
= (\alpha \Delta_0)^m (|a| + \delta)^m (T_1-1)
\]

where \(r := (|a| + \delta)^m\) is written for brevity. It then follows that

\[
E_{x_0,\Delta_0}[\Delta_{T_1}] = (\alpha \Delta_0)^m \sum_{k=0}^{\infty} P_{x_0,\Delta_0}(T_1 = k + 1) r^k
\]

\[
\leq (\alpha \Delta_0)^m \sum_{k=0}^{\infty} P_{x_0,\Delta_0}(T_1 \geq k + 1) r^k.
\]

(19)

In view of Lemma 3.11 the above sum can be further bounded as

\[
\sum_{k=0}^{\infty} P_{x_0,\Delta_0}(T_1 \geq k + 1) r^k = 1 + \sum_{k=1}^{\infty} P_{x_0,\Delta_0}(T_1 \geq k + 1) r^k
\]

\[
\leq 1 + B_r \Delta_0 \sum_{k=1}^{\infty} \frac{k \omega^k}{(W \xi^k - 1)^{2\omega}}
\]

where the sum \(M_r := \sum_{k=1}^{\infty} \frac{k \omega^k}{(W \xi^k - 1)^{2\omega}}\) converges so long as \(\omega > \frac{1}{2} \log_r r\). This is because the summand has the asymptotic behaviour \(k \omega^k r^{-2\omega} (\xi^k - 1)^{2\omega}\) which converges if and only if \(r \xi^{-2\omega} < 1\). Note that \(\omega > \frac{1}{2} \log_r r \implies \omega > \frac{m}{T_1}\).

Define \(C_r := a r M_r \Delta_0 \omega^{-2\omega}\), then \(19\) becomes

\[
E_{x_0,\Delta_0}[\Delta_{T_1}] \leq \Delta_0^m \left( a^m + C_r \Delta_0 \omega^{-2\omega} \right).
\]

Since we aim to satisfy \(18\), we write this as

\[
E_{x_0,\Delta_0}[\Delta_{T_1}] - (1 - \gamma) \Delta^m_0 \leq -\Delta_0^m ((1 - \alpha^m) - \gamma) + C_r \Delta_0 \omega^{-2\omega}.
\]

(20)

Note that since \(\gamma < 1 - \alpha^m\), the first term tends to \(-\infty\) as \(\Delta_0 \to \infty\), while since \(\omega > \frac{m}{T_1}\), the second term tends to 0 as \(\Delta_0 \to \infty\). Therefore, for \(D\) sufficiently large, if \(\Delta_0 \geq D\) it follows that \(20\) is upper bounded by 0 so that \(18\) is satisfied for \((x_0, \Delta_0) \notin \Lambda\).

It remains to ensure that \(18\) holds for \((x_0, \Delta_0) \in \Lambda \cap C\), that is with \(\Delta_0 \leq D\). This is straightforward, as it follows from \(20\) that \(E_{x_0,\Delta_0}[\Delta_{T_1}] - (1 - \gamma) \Delta^m_0\) is bounded on \(\Lambda \cap C\), so we may take

\[
b = \sup_{(x_0, \Delta_0) \in \Lambda \cap C} E_{x_0,\Delta_0}[\Delta_{T_1}] - (1 - \gamma) \Delta^m_0 < \infty.
\]

This fully satisfies the first drift inequality.

Next, we proceed to showing the second inequality of \(14\). For brevity, let \(\lambda := \gamma \beta \left( \frac{\Delta_0}{\Delta_0^m} \right)\) so that \(f(x, \Delta) = \lambda |x|^m\).
Again, we assume that \((x_0, \Delta_0) \in \Lambda\). First, we have that
\[
E_{x_0, \Delta_0} \left[ \sum_{t=0}^{T_1-1} f(x_t, \Delta_t) \right] = \lambda E_{x_0, \Delta_0} \left[ \sum_{t=0}^{T_1-1} |x_t|^m \right] \\
= \lambda |x_0|^m + \lambda E_{x_0, \Delta_0} \left[ \sum_{t=1}^{T_1-1} |x_t|^m \right].
\]
(21)

First, note that since \((x_0, \Delta_0) \in \Lambda\),
\[
\lambda |x_0|^m = \gamma \beta \left( \frac{2}{K} |x_0| \right)^m \leq \gamma \left( \frac{2}{K} \frac{\Delta_0}{2} \right)^m \\
= \gamma \Delta_0^m \cdot \beta.
\]

We will focus next on the summation in (21) for \(t \geq 1\).
Recall (17) so that for \(1 \leq t \leq T_1\), we may write the state as \(x_t = \frac{\mu}{\Delta_0} a^t(\mu + W_t)\) where \(W_t \sim \mathcal{N}(0, \sigma_t^2)\). Here, \(\mu = \frac{\Delta_0}{\Delta_0} (e_0 - U_N(e_0))\) and \(\sigma_t^2 = \frac{4}{\Delta_0} \frac{\sigma^2(1 - a^{-2t})}{a^{t+1}} \sigma^2\) are as before. Using this, the summation in (21) becomes
\[
\lambda E_{x_0, \Delta_0} \left[ \sum_{t=1}^{T_1-1} |a^t(\mu + W_t)|^m \right] \\
= \gamma \beta \left( \frac{2}{K} \frac{\Delta_0}{2} \right)^m E_{x_0, \Delta_0} \left[ \sum_{t=1}^{T_1-1} |a^t(\mu + W_t)|^m \right] \\
\leq \gamma \Delta_0^m \cdot \beta \frac{2}{2} E_{x_0, \Delta_0} \left[ \sum_{t=1}^{T_1-1} |a^t(\mu + W_t)|^m \right]
\]
since \(K \geq 2\). For brevity, denote
\[
Y := \sum_{t=1}^{T_1-1} |a^t(\mu + W_t)|^m.
\]
It follows that (21) is bounded by
\[
\lambda E_{x_0, \Delta_0} \left[ \sum_{t=0}^{T_1-1} |x_t|^m \right] \leq \lambda \Delta_0^m \cdot \beta \left( 1 + \frac{1}{2} E_{x_0, \Delta_0} [Y] \right).
\]

If we can show that \(\sup_{(x_0, \Delta_0) \in \Lambda} E_{x_0, \Delta_0} [Y] < \infty\) then the proof is essentially complete as we may take \(\beta\) small enough so that the factor multiplying \(d(x_0, \Delta_0) = \gamma \Delta_0^m\) is at most 1 over \(\Lambda\). The remainder of the proof is dedicated to showing this fact.

First, assume that \(m\) is odd. We may do this without loss of generality, as the nesting of \(L^p\) spaces ensures that if all odd moments exist, then all even moments exist also. We apply Hölder’s inequality as follows:
\[
E_{x_0, \Delta_0} [Y] = E_{x_0, \Delta_0} \left[ \sum_{t=1}^{\infty} |a^t(\mu + W_t)|^m \right] \\
= \sum_{t=1}^{\infty} |a|^{tm} E_{x_0, \Delta_0} \left[ |a^t(\mu + W_t)|^m \right] \\
\leq \sum_{t=1}^{\infty} |a|^{tm} E_{x_0, \Delta_0} \left[ \left[ \sum_{t=1}^{\infty} |a|^{tm} \right]^{m+1} \right] \cdot E_{x_0, \Delta_0} \left[ |a+\mu+\Delta_0|^m \right].
\]
(22)

We will handle the two expectations separately. The first is straightforward:
\[
E_{x_0, \Delta_0} \left[ \sum_{t=1}^{\infty} |a^t(\mu + W_t)|^m \right] \leq P_{x_0, \Delta_0} \left[ \sum_{t=1}^{\infty} |a|^m \right] \leq 2 \sum_{t=1}^{\infty} |a|^m \\
= \lambda |x_0|^m + \lambda E_{x_0, \Delta_0} \left[ \sum_{t=1}^{T_1-1} |x_t|^m \right].
\]

which by Lemma 3.11 is bounded by
\[
\lambda |x_0|^m + \lambda E_{x_0, \Delta_0} \left[ \sum_{t=1}^{T_1-1} |x_t|^m \right] \\
\leq \left( \frac{\Delta_0}{8} \lambda \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^2 \right)^{\frac{m}{m+1}} \\
= \left( \frac{\Delta_0}{8} \lambda \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^2 \right)^{\frac{m}{m+1}}.
\]

As in the second bound of Lemma 3.11 for any \(\omega > 0\) we may use the bound \(e^{-x} \leq (\frac{x}{\omega})^\omega e^{-\omega x}\), yielding:
\[
\leq \lambda |x_0|^m + \lambda E_{x_0, \Delta_0} \left[ \sum_{t=1}^{T_1-1} |x_t|^m \right] \\
\leq 2 \frac{\Delta_0}{8} \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^2 \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^{-2} \omega \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^{-2} \\
= \left( \frac{\Delta_0}{8} \lambda \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^2 \right)^{\frac{m}{m+1}} \\
\leq 2 \frac{\Delta_0}{8} \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^2 \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^{-2} \omega \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^{-2} \\
= \frac{c_1 \left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^2}{\left( \frac{W_\xi^t - 1}{\sqrt{t}} \right)^{-2}}.
\]
(23)

where \(c_1\) is constant in terms of \((x_0, \Delta_0)\). Next, we will upper bound the second expectation in (22). First, since \(m\) is odd, note that \(|s|^{m+1} = s^{m+1}\). Therefore,
\[
E_{x_0, \Delta_0} \left[ |a^t(\mu + W_t)|^m \right] = E_{x_0, \Delta_0} \left[ |(\mu + W_t)|^m \right] \\
= \frac{m+1}{k} \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{1}{k} W_t^k
\]
By the binomial theorem, this is
\[
= E_{x_0, \Delta_0} \left[ \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{1}{k} W_t^k \right] \\
= \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{1}{k} E_{x_0, \Delta} \left[ W_t^k \right].
\]
Since \(W_t\), conditioned on \((x_0, \Delta_0)\), is Gaussian with zero mean, it follows that \(E_{x_0, \Delta_0} \left[ W_t^k \right] = 0\) for all odd \(k\). Thus, after removing the odd terms and re-indexing, this sum becomes
\[
= \sum_{k=0}^{m+1} \binom{m+1}{k} \frac{1}{2k} E_{x_0, \Delta} \left[ W_t^{2k} \right].
\]
(24)

For a Gaussian random variable \(Z \sim \mathcal{N}(0, \sigma^2)\) its \((2k)th\) moment is given by \(\gamma^{2k} \Gamma(2k - 1)\) where \(n!!\) is the double factorial which yields the product of \(n\) and all smaller natural numbers with the same parity. For \(n = 2k - 1\) odd, it admits the representation
\[
(2k - 1)!! = \frac{(2k)!}{2k!}.
\]
It therefore holds that
\[
\left(\frac{m+1}{2k}\right)(2k-1)! = \frac{(m+1)!}{2^k k!} (2k)! = \left(\frac{m+1}{k}\right) \frac{(m+1)!}{(m+1-2k)!} \frac{(2k)!}{k!}.
\]

\[
\leq \left(\frac{m+1}{k}\right) \cdot 1 \cdot (m+1)! = (m+1)! \left(\frac{m+1}{k}\right).
\]

Therefore, returning to (24), where \(W_1 \sim \mathcal{N}(0, \sigma_t^2)\) we have again by the binomial theorem that
\[
(24) \leq (m+1)! \sum_{k=0}^{m+1} \left(\frac{2k}{m+1}\right) \left(\mu^2\right)^{\frac{m+1-k}{m+1}} \left(\sigma_t^2\right)^k
\]
\[
= (m+1)! \left(\mu^2 + \sigma_t^2\right)^{\frac{m+1}{m+1}}.
\]

Finally, recall that the expectation in (22) was raised to the power \(\frac{m}{m+1}\). Applying this, the second expectation of (22) is upper bounded by:
\[
\left(\frac{m}{m+1}\right)^{\frac{m}{m+1}} \left(1 + \frac{4}{(aL)^2 \sigma^2}\right)^{\frac{m+1}{m+1}}. \tag{25}
\]

Applying the upper bounds (23) and (25) to the summation (22), the key observation is that none of the terms involve \(x_0\) or \(\Delta_0\), so if the sum converges, it constitutes an upper bound for all \((x_0, \Delta_0) \in \Lambda\), yielding a finite supremum.

With both upper bounds applied, there are three terms to consider. The first term is \(|a|^{tm}\). The second term is (23), which for the purposes of series convergence acts asymptotically like \(t^2 \xi^{-2t}\). The final term is (25), which again for series convergence acts asymptotically as \(t^{\frac{m}{m+1}}\). In total, the summand acts asymptotically as \(t^{\frac{m}{m+1}} \left(\xi^{-2t} |a|^{m}\right)^{\frac{m+1}{m-1}}\). This will converge if and only if \(\xi^{-2t} |a|^{m} < 1\).

First note that \(|a| = 1\) implies this condition to always be true. Then supposing \(|a| > 1\), since \(\omega > 0\) was chosen otherwise arbitrarily, we are free to let \(\omega = \frac{m}{\log |a| \xi}\), which satisfies the convergence requirement as:
\[
\xi^{-2m |a|^m} \leq \left(\frac{\xi^{-2m} |a|}{|a|} \right) = |a|^{-m} < 1.
\]

Therefore, \(\sup_{(x_0, \Delta_0) \in \Lambda} E_{x_0, \Delta_0} [Y] < \infty\), which completes the proof.

\[\square\]

IV. A SIMULATION

Let us note that a key issue in a numerical simulation here is that for \(\varepsilon\) too close to 0, it is not feasible to effectively demonstrate the rate of convergence. To be precise, suppose \(N \leq 2^{2/\varepsilon}\). Then the quantizer support size satisfies \(\frac{1}{2} N \Delta(N) \leq A\). Therefore, all schemes with less than \(2^{2/\varepsilon}\) bins will fail to quantize the region \((-\infty, A) \cup (A, \infty)\). As a direct result of this, the overload distortion will not go to zero as \(N\) increases up to this point. This is especially problematic for small \(\varepsilon\). As an example, suppose \(\varepsilon = 0.001\). We would then require our fixed quantizer to have \(N \gg 2^{2000}\) bins in order to demonstrate the convergence rate \(O(N^{-1.999})\). It is infeasible on typical computing software (e.g. MATLAB) to simulate a scheme with this many bins. In view of this, we now proceed with an example simulation. To ensure that the convergence is observable, we modestly choose \(\varepsilon = 1\) so that the rate of convergence is at worst \(O(1/N)\). The system parameters are \(a = 2\) and \(\sigma^2 = 1\). \(K = 16\) bins are used to stabilize the system, with zooming parameters \(a = 0.5\), \(\delta = 0.2\), and \(L = 1\). We start the simulation from \(x_0 = 0\), \(\Delta_0 = L\). With the parameters above, the system was run for all \(N \in \{10, 12, \ldots 100\}\) and the average second moment was recorded. The second moment achieved in each trial for \(N\) is shown below. As the convergence is at worst \(O(1/N)\), we include an estimate of this order for the tail of the data.

![Fig. 3. Convergence to the optimum \(\sigma^2 = 1\). The order of convergence is approximately \(O(N^{-1.966})\) for the tail end of the data, overlaid in red.](Image)

V. CONCLUSION

In this paper, we studied the optimal coding and control problem for a linear system driven by unbounded noise over a finite-rate noiseless channel. We presented a two-part coding scheme where the first part, via an adaptive zooming scheme, led to stochastic stability and positive Harris recurrence of a controlled linear system over a channel and the second part was a time-invariant high-rate quantizer. We showed the stochastic stability and order optimality of this coding scheme, where optimality was established via convergence to a lower bound in high-rates. Crucial to our analysis was the positive Harris recurrence of the first part of the coding scheme. In future work, we aim to generalize the analysis to multi-dimensional systems and multi-sensor systems [12] and non-linear systems [19], where related ergodicity and stochastic stability results are available.
REFERENCES

[1] D. Bertsekas, *Dynamic Programming and Optimal Control Vol. 1*. Athena Scientific, 2000.

[2] W. S. Wong and R. W. Brockett, “Systems with finite communication bandwidth constraints - part II: Stabilization with limited information feedback,” *IEEE Transactions on Automatic Control*, vol. 42, pp. 1294–1299, September 1997.

[3] J. Baillieul, “Feedback designs for controlling device arrays with communication channel bandwidth constraints,” in *Proceedings of the 4th ARO Workshop on Smart Structures*, State College, PA, August 1999.

[4] S. Tatikonda and S. Mitter, “Control under communication constraints,” *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1056–1068, 2004.

[5] L. V. J. Hespanha, A. Ortega, “Towards the control of linear systems with minimum bit-rate,” in *Proc. 15th Int. Symp. on Mathematical Theory of Networks and Systems (MTNS)*, Citeseer, 2002.

[6] G. N. Nair and R. J. Evans, “Stabilizability of stochastic linear systems with finite feedback data rates,” *SIAM J. Control and Optimization*, vol. 43, pp. 413–436, July 2004.

[7] A. Sahai and S. Mitter, “The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link part I: Scalar systems,” *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3369–3395, 2006.

[8] N. C. Martins, M. A. Dahleh, and N. Elia, “Feedback stabilization of uncertain systems in the presence of a direct link,” *IEEE Transactions on Automatic Control*, vol. 51, no. 3, pp. 438–447, 2006.

[9] A. S. Matveev and A. Savkin, “An analogue of Shannon information theory for detection and stabilization via noisy discrete communication channels,” *SIAM J. Control Optim*, vol. 46, pp. 1323–1367, 2007.

[10] S. Yüksel, “Stochastic stabilization of noisy linear systems with fixed-rate limited feedback,” *IEEE Transactions on Automatic Control*, vol. 55, pp. 2847–2853, December 2010.

[11] S. Yüksel and S. P. Meyn, “Random-time, state-dependent stochastic drift for Markov chains and application to stochastic stabilization over erasure channels,” *IEEE Transactions on Automatic Control*, vol. 58, pp. 47 – 59, January 2013.

[12] A. Johnston and S. Yüksel, “Stochastic stabilization of partially observed and multi-sensor systems driven by unbounded noise under fixed-rate information constraints,” *IEEE Transactions Automatic Control*, vol. 59, pp. 792–798, March 2014.

[13] V. Kostina, Y. Peres, G. Ranade, and M. Sellke, “Exact minimum number of bits to stabilize a linear system,” *IEEE Transactions on Automatic Control*, 2021.

[14] B.-H. Juang and A. Gray, *Multiple stage vector quantization for speech coding*, vol. 7. 1982.

[15] T. Berger, “Information rates of Wiener processes,” *IEEE Transactions on Information Theory*, vol. 16, pp. 134–139, 1970.

[16] A. Sahai, “Coding unstable scalar Markov processes into two streams,” in *Proceedings of the IEEE International Symposium on Information Theory*, p. 462, 2004.

[17] S. P. Meyn and R. Tweedie, *Markov Chains and Stochastic Stability*. London: Springer-Verlag, 1993.

[18] S. Yüksel and T. Başar, *Stochastic Networked Control Systems*. Springer, 2013.

[19] S. Yüksel, “Stationary and ergodic properties of stochastic non-linear systems controlled over communication channels,” *SIAM J. on Control and Optimization*, vol. 54, pp. 2844–2871, 2016.