Frame Dragging Effect on Moment of Inertia and Radius of Gyration of Neutron Star

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Accurate to the first order in the uniform angular velocity, the general relativity frame dragging effect of the moments of inertia and radii of gyration of two kinds of neutron stars are calculated in a relativistic \(\sigma - \omega\) model. The calculation shows that the dragging effect will diminish the moments of inertia and radii of gyration.

Keywords: neutron star; frame dragging; moment of inertia; radius of gyration.

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1. Introduction
As we know, Neutron stars may be created in the aftermath of the gravitational collapse of the core of a massive star at the end of its life, or may be created when the mass of an accreting white dwarf exceed the Chandrasekhar limit. The typical character of neutron stars may be described as following: spinning rapidly, often making several hundred rotations per second; having a small stature with a radius \(R\) of \(\sim 12\) km and a heavy avoirdupois with a mass \(M\) on the order of \(1.5\) solar masses \((M_\odot)\); and having a very compact central density as high as several times the nuclear saturation density. Neutron stars are one of the densest forms of matters in the observable universe.

One of the important global property of neutron star is the moment of inertia...
Early estimates of the moment of inertia by the energy-loss rate from pulsars spanned a wide range of $I$, and several researchers have given a lower bound on the moment of inertia of the pulsar as $I \geq 4 \times 10^{37} \text{g cm}^2$. Recently, Kalogera and others also studied the bound of the moment of inertia. In order to investigate the general relativistic frame dragging effect on the moments of inertia of neutron stars, the moments of inertia of the non-rotating and rotating at the Kepler frequency neutron stars will be studied in this work.

Here we adopt the metric signature $- + + +$, $G = c = 1$.

2. Moment of inertia and radius of gyration of neutron star

In relativity, the space-time geometry of a static spherical symmetric star can be described as

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(1)

and the space-time geometry of a rotating star in equilibrium is described by a stationary and axisymmetric metric of the form

$$ds^2 = -e^{2\nu}dt^2 + e^{2\lambda}dr^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}d\theta^2,$$

(2)

where $\omega(r)$ is the angular velocity of the local inertial frame and is proportional to the star’s rotational frequency $\Omega$, which is the uniform angular velocity of the star relative to an observer at infinity.

From the $(t, \phi)$ component of Einstein field equations, accurate to the first order in $\Omega$, one gets

$$\frac{1}{r^3} \frac{d}{dr} \left( r^4 j \frac{d\bar{\omega}}{dr} \right) + \frac{4}{r} \frac{dj}{dr} \bar{\omega} = 0,$$

(3)

where $j(r) = e^{-\gamma} \left[ 1 - 2 M_0(r)/r \right]^{\frac{1}{2}}$, $\bar{\omega} = \Omega - \omega$, which denotes the angular velocity of the fluid relative to the local inertial frame. The boundary conditions are imposed as $\bar{\omega} = \bar{\omega}_c$ at the center, $\frac{d\bar{\omega}}{dr}|_{r=0} = 0$, where $\bar{\omega}_c$ is chosen arbitrarily. Integrating eq.(3) outward from the center of the star, one would get the function $\bar{\omega}(r)$. Outside the star, from eq.(3) one has $\bar{\omega}(r) = \Omega - \frac{2J}{r^2}$, where $J$ is the total angular momentum of the star, which takes the form

$$J = \frac{1}{A} R^4 \frac{d\alpha}{dr}|_{r=R},$$

where $R$ is the surface radius of the star. According to the traditional definition, we have

$$J = I \Omega,$$

(4)

where $I$ is the moment of inertia. At the neutron star’s surface, in term of eqs.(3) and (4), one can get

$$I = -\frac{2}{3} \int_0^R r^3 \frac{dj}{dr} \frac{\bar{\omega}}{\Omega} dr.$$

(5)

From the $(0,0)$ and $(1,1)$ components of Einstein field equations to the Static Spherically Symmetric star, one gets

$$\frac{d\alpha}{dr} = \frac{1}{2r} \left( 1 - e^{2\alpha} + 8\pi r^2 p e^{2\alpha} \right),$$

(6)
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\[
\frac{d\varphi}{dr} = \frac{1}{2r}(e^{2\alpha} - 1 + 8\pi r^2 p e^{-2\alpha}).
\]  

(7)

Accurate to the first order in \(\Omega\), substituting eqs. (6) and (7) into eq. (5), one has:

\[
I = \frac{8\pi}{3} \int_0^R r^4 e^{-\varphi} \frac{\rho + p}{\sqrt{1 - 2M_0 \over r}} \cdot \overline{\omega} \Omega dr.
\]  

(8)

This is just the approximate expression of the moment of inertia accurate to the first order in \(\Omega\). To the non-rotating neutron stars, \(\overline{\omega} \Omega = 1\). As a first order approximation, it is means that there is only consideration of the frame dragging effect and without consideration of the deformation caused by the rotation. According to eq.(8), we can estimate that, to the same neutron star, the moment of inertia of the non-rotating star will be bigger than that of the rotating star, which will be testified in our later numerical calculation. This is just the frame dragging effect to the moment of inertia. Approximating in week field, according to eqs.(6) and (7), one gets:

\[
\frac{dj}{dr} \approx -4\pi rp.
\]  

(9)

In Newtonian theory, \(\overline{\omega} = \Omega\), so from eq.(5), we have

\[
I = \frac{8\pi}{3} \int_0^R r^4 \rho dr,
\]  

this is just the calculating expression of the moment of inertia of a spherical symmetric star in Newtonian theory.

Similar to the Newtonian theory, in general relativity, we also define the radius of gyration of a spherical symmetric star as

\[
R_g = \left(\frac{I}{M}\right)^{1/2}.
\]  

(11)

3. Numerical results and discussion

There are several models to deal with the superdense matters, such as non-relativistic models, relativistic field theoretical models. To different models, the fractions of particles in the superdense matters are different, and then the bulk properties of superdense matters are different, that is, the EOS of them are different. In this work, two kind of EOS will be employed, one is for the traditional neutron stars (TNS), in which \(n, p, e, \mu\) are the main elements; the other is for the hyperon stars (HS), in which \(n, p, e, \mu, \Lambda, \Sigma, \Xi, \Delta\) are the main elements. The EOS will be considered in the relativistic \(\sigma - \omega\) model. Fig.(1) shows the EOS of the THS and HS.

By using above EOS and eqs.(8) and (11), the moments of inertia and radii of gyration of neutron stars were calculated. Fig.(2) presents the moment of inertia of non-rotating and rotating (at the Kepler frequency) neutron stars as a function of central density. It is clear that, because of the frame dragging effect, as a first
order approximation, the moments of inertia of the non-rotating TNS (or HS) are bigger than that of the corresponding rotating stars at the same central density. From this figure, one can also see that, with the same central density, as the star rotating at its Kepler frequency, the frame dragging effect on the moment of inertia of TNS is bigger than that of HS. The reason of this is that the EOS of TNS is stiffer than the EOS of HS, so there is a bigger Kepler frequency of TNS at the same central density, and then corresponds a bigger frame dragging effect. In order to show the frame dragging effect of the moments of inertia entirely, we present them in a three-dimension figure, fig.3 just presents the moments of inertia of TNS as a function of the central densities and the central angular velocities relative to the local inertial frame.

Fig. 4 gives the radii of gyration of non-rotating and rotating (at the Kepler frequency) neutron stars as a function of central density. From this figure one can see that, to the same central density, there is a big gap between the lines of the non-rotating and rotating stars, that is to say, there is a remarkable frame dragging effect to the radii of gyration, and as the central density increasing, the effect increases. We also present the radii of gyration of TNS in a three-dimension figure, see fig.5.

From our calculation, one can see that to a star at the same central density, the frame dragging effect will diminish its moment of inertia and radius of gyration, and as the star rotates faster, the dragging effect will become more obviously.

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Fig. 2. The moment of inertia of non-rotating and rotating (at the Kepler frequency) neutron stars as a function of central density.

Fig. 3. The moment of inertia of TNS as a function of central density and the angular velocity relative to the local inertial frame at the center.

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Fig. 4. The radius of gyration of non-rotating and rotating (at the Kepler frequency) neutron stars as a function of central density.

Fig. 5. The radius of gyration of TNS as a function of central density and the angular velocity relative to the local inertial frame at the center.

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