MONADS FOR FRAMED SHEAVES
ON HIRZEBRUCH SURFACES

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Abstract. We define monads for framed torsion-free sheaves on Hirzebruch surfaces and use them to construct moduli spaces for these objects. These moduli spaces are smooth algebraic varieties, and we show that they are fine by constructing a universal monad.

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1. Introduction

Moduli spaces of framed sheaves over projective surfaces have been the object of some interest over the last few years. In the case of the complex projective plane — generalizing the classical result in [7] — these moduli spaces are resolutions of singularities of the moduli space of ideal instantons on the four-sphere $S^4$ [23], and as such, they have been used to compute Nekrasov’s partition function, i.e., the partition function of a (suitably twisted) $N = 2$ topological super Yang-Mills theory (see [25, 2] and also [10] for the more general case of toric surfaces).

More generally, they are at the basis of the so-called instanton counting [24]. Fine moduli spaces of framed sheaves were constructed by Huybrechts and Lehn [17, 18] by introducing a stability condition. Bruzzo and Markushevich [3] showed that framed sheaves on projective surfaces, even without a stability condition, give rise to fine moduli spaces.

Moduli spaces of framed bundles on the projective plane were considered by Donaldson [7, 8], who established their isomorphism with the moduli of framed instantons on $S^4$. This was extended by King to the case of the blowup of the projective plane at a point [20] and by Buchdahl [6] to the case of multiple blowups. The degenerate case (including torsion-free sheaves) was first considered, as already cited, by Nakajima [23] for the projective plane, and then by Nakajima and Yoshioka [24] for the blown-up plane. The case of multiple blowups was studied by Henni [14].

In this paper we consider torsion-free sheaves on a Hirzebruch surface $\Sigma_n$, for $n > 0$, that are framed to the trivial bundle on a generic line $\ell_\infty$ in $\Sigma_n$. We construct a moduli space for such sheaves by using monads. This allows us to obtain a moduli space which is a smooth quasi-projective variety. These moduli spaces are also shown to be fine, and this implies that they are isomorphic to the moduli spaces constructed in [3] (and therefore embed as open subschemes into Huybrechts-Lehn’s moduli spaces).

The monads we use generalize to torsion-free sheaves those introduced by Buchdahl [5] for the locally-free case (he was actually interested in $\mu$-stable vector bundles on Hirzebruch surfaces with $c_1 = 0$, $c_2 = 2$). Indeed, we show in Corollary 4.6 that for any framed torsion-free sheaf $(\mathcal{E}, \theta)$ on $\Sigma_n$, the underlying sheaf $\mathcal{E}$ is isomorphic to the cohomology of a monad

$$U_k \xrightarrow{\alpha} V_k \xrightarrow{\beta} W_k,$$

whose terms depend only on the Chern character $\text{ch}(\mathcal{E}) = (r, aE, -c - \frac{1}{2}na^2)$ (here $E$ is the exceptional curve in $\Sigma_n$, i.e., the unique irreducible curve in $\Sigma_n$ squaring to $-n$, and we put $k = (n, r, a, c)$). This provides a map

$$(\mathcal{E}, \theta) \mapsto (\alpha, \beta) \in \text{Hom}(U_k, V_k) \oplus \text{Hom}(V_k, W_k).$$
We call $L_{\vec{k}}$ the image of this map and prove in Proposition 4.8 and in Lemma 4.9 that $L_{\vec{k}}$ is a smooth variety. We construct a principal $\text{GL}(r, \mathbb{C})$-bundle $P_{\vec{k}}$ over $L_{\vec{k}}$ whose fibre over a point $(\alpha, \beta)$ is naturally identified with the space of framings for the cohomology of the complex (1.1). This implies that the map (1.2) lifts to a map $(E, \theta) \mapsto -\theta \in P_{\vec{k}}$.

The group $G_{\vec{k}}$ of isomorphisms of monads of the form (1.1) acts on $P_{\vec{k}}$, and the moduli space $\mathcal{M}^\alpha(r, a, c)$ of framed sheaves on $\Sigma_n$ with given Chern character is set-theoretically defined as the quotient $P_{\vec{k}}/G_{\vec{k}}$. Theorem 5.1 proves that $\mathcal{M}^\alpha(r, a, c)$ inherits from $P_{\vec{k}}$ a structure of smooth algebraic variety. As a corollary, $\mathcal{M}^\alpha(r, a, c)$ turns out to be irreducible. Moreover, Lemma 4.7 states that two monads of the form (1.1) are isomorphic if and only if their cohomologies are isomorphic, and this ensures that there is a bijection between $\mathcal{M}^\alpha(r, a, c)$ and set of isomorphism classes of framed sheaves on $\Sigma_n$ (isomorphisms of framed sheaves are introduced in Definition 3.3). This enables us to show that the set of these classes is nonempty if and only if $2c \geq na(1 - a)$ (hence, at least when $a \neq 0$, it turns out that the moduli space can be empty even if it has positive expected dimension).

We prove the fineness of the moduli space by constructing a universal family $(E_{\vec{k}}, \Theta_{\vec{k}})$ of framed sheaves on $\Sigma_n$ parametrized by $\mathcal{M}^\alpha(r, a, c)$ (for the precise notion of family see Definition 4.2).

These results are the basis for further work where we give a detailed description of the moduli spaces when the topological invariants satisfy the lower bound $2c = na(1 - a)$. Moreover, considering the rank one case, they allow us to obtain a rather explicit construction of the Hilbert schemes of points of the total spaces of the line bundles $\mathcal{O}_{\mathbb{P}^1}(-n)$. In both cases the results are achieved by using explicit ADHM descriptions [1].

If not otherwise specified, by “scheme” we mean a noetherian reduced scheme of finite type over $\mathbb{C}$. If $X$ and $S$ are schemes, $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X \times S}$-modules, and $F$ is a morphism between two such sheaves, we shall denote by $\mathcal{F}_s$ (resp. $F_s$) the restriction of $\mathcal{F}$ (resp. $F$) to the fibre of $X \times S \to S$ over the point $s \in S$.

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2. Monads

A monad $M$ on a scheme $T$ is a three-term complex of locally-free $\mathcal{O}_T$-modules, having nontrivial cohomology only in the middle term:

\begin{equation}
M : \quad 0 \to U \overset{a}{\to} V \overset{b}{\to} W \to 0. \tag{2.1}
\end{equation}
The cohomology of the monad will be denoted by $E(M)$. It is a coherent $O_T$-module. A morphism (isomorphism) of monads is a morphism (isomorphism) of complexes.

The display of the monad (2.1) is the commutative diagram (with exact rows and columns)

\[
\begin{array}{ccccccc}
0 & 0 & \downarrow & & \downarrow & & \\
0 & U & \xrightarrow{a} & B & & \xrightarrow{e} & 0 \\
0 & U & \xrightarrow{a} & V & \xrightarrow{b} & A & 0 \\
W & \downarrow & & \downarrow & & \\
0 & 0 & & 0 & & \\
\end{array}
\]

(2.2)

where $A := \text{coker } a$, $B := \ker b$, $E = E(M)$, and all morphisms are naturally induced.

Let us suppose that $T$ is a product $T = X \times S$, where $X$ is a smooth connected projective variety and $S$ a scheme; we denote by $t_i, i = 1, 2$ the canonical projections onto the first and second factor, respectively.

**Lemma 2.1.** Let $\varphi : F \to G$ be a morphism between two coherent sheaves on $T$. If for all closed points $s \in S$ the restricted morphism $\varphi_s$ is surjective (resp. injective), then $\varphi$ is surjective (resp. injective).

**Proof.** For all closed points $s \in S$ and for all closed points $x \in T_s \simeq X$ the germs $(\varphi_s)_x$ are surjective (resp. injective) as a consequence of [9, Corollary 2.9]. At the same time for all $O_T$-modules $N$ and all closed points $(x, s) \in T$ there are isomorphisms

\[ N_{(x,s)} \simeq (N_s)_x \otimes O_{(T_s)_x} O_{T,(x,s)} \]  

(2.3)

and this is a flat base change. This implies that the germs $\varphi_{(x,s)}$ are surjective (resp. injective) for all closed points $(x, s) \in T$. By applying [9, Corollary 2.9] again the thesis follows. \qed

**Lemma 2.2.** Let $M$ be a monad over the product scheme $T = X \times S$ (see eq. (2.1)). The cohomology $E(M)$ is flat over $S$ if and only if the restricted morphisms $a_s$ are injective for all closed points $s \in S$.

**Proof.** By Lemma 2.1.4 in [19], what we need to prove is that if $a_s$ is injective for all closed then it is so for all $s$. Note that $B$ is flat over $S$ as it is locally free and $S$ is a factor of
$T = X \times S$. Since flatness over $S$ is local over $S$, we can assume $S = \text{Spec} R$. If $s$ is not closed, it is the generic point of a closed irreducible subscheme $Z \subseteq S$. We know that $a_{|Z}$ is injective for all closed points $s \in Z$. By Lemma 2.1, $a_{|Z}$ is injective. Since $\{s\} \hookrightarrow Z$ is a flat morphism, we conclude that $a_s$ is injective. $\square$

Suppose now that $\dim X = 2$.

**Proposition 2.3.** If $\mathcal{E}$ and $\mathcal{E}'$ are coherent sheaves on $T$, flat on $S$, and for all closed points $s$ in $S$ the restrictions $\mathcal{E}_s$ and $\mathcal{E}'_s$ are torsion-free, the sheaf $\mathcal{H}om(\mathcal{E}, \mathcal{E}')$ is flat on $S$.

The proof of this Proposition relies on a few intermediate results. Let $A$ be a commutative ring, $B$ and $R$ (not necessarily flat) $A$-algebras, with $R$ noetherian; let $R' := R \otimes_A B$.

**Lemma 2.4.** Let $M$, $N$ be $R$-modules, and suppose that $M$ is finitely generated as an $R$-module, and flat as an $A$-module. Then

$$\operatorname{Ext}_R^i(M, N) \otimes_R R' \simeq \operatorname{Ext}_{R'}^i(M \otimes_R R', N \otimes_R R') \quad \text{for } i \geq 0.$$  

*Proof.* This is a modification of the proof of [21, Prop (3.E)]. The module $M$ admits a resolution by free finitely generated $R$-modules:

$$\cdots \longrightarrow F_2 \xrightarrow{f_1} F_1 \xrightarrow{f_0} F_0 \longrightarrow M \longrightarrow 0.$$  

(2.4)

By applying [13, Prop. III.9.1A (e)] one can prove by induction that the modules $\ker f_i$ are $A$-flat for all $i \geq 0$. Hence, by applying the functor $\otimes_R R' \simeq \otimes_A B$ to eq. (2.4) one gets a resolution of $M \otimes_R R'$ by free finitely generated $R'$-modules:

$$F_\bullet \otimes_R R' \longrightarrow M \otimes_R R' \longrightarrow 0.$$  

By applying the functor $\operatorname{Hom}_{R'}(\cdot, N \otimes_R R')$ the thesis follows as

$$\operatorname{Hom}_{R'}(F_\bullet \otimes_R R', N \otimes_R R') \simeq \operatorname{Hom}_R(F_\bullet, N) \otimes_R R'.$$

$\square$

We note that, if $\mathcal{E}$ is flat on $S$, for every point $s \in S$ there are isomorphisms

$$[\mathcal{E}xt^i_{\mathcal{O}_T}(\mathcal{E}, \mathcal{O}_T)]_s \simeq \mathcal{E}xt^i_{\mathcal{O}_{T_s}}(\mathcal{E}_s, \mathcal{O}_{T_s}) \quad i \geq 0.$$

In particular, for all $s \in S$ one has $(\mathcal{E}^*)_s \simeq (\mathcal{E}_s)^*$. Since $\dim X = 2$ and $X$ is smooth it follows that $(\mathcal{E}^*)_s$ is locally free on $T_s$.

**Lemma 2.5.** The dual $\mathcal{E}^*$ of a coherent $\mathcal{O}_T$-module $\mathcal{E}$ flat on $S$ is locally free.
Proof. We claim that, for any point (closed or not) $s \in S$ one has $(\mathcal{E}^*)_s \simeq (\mathcal{E}_s)^*$. Indeed one has

$$(\mathcal{E}^*)_s = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{E}, \mathcal{O}_T) \otimes_{\mathcal{O}_T} \mathcal{O}_T$$

$$(\mathcal{E}_s)^* = \mathcal{H}om_{\mathcal{O}_T}(\mathcal{E} \otimes_{\mathcal{O}_T} \mathcal{O}_T, \mathcal{O}_T)$$

Since $\mathcal{O}_T = \mathcal{O}_T \otimes_{\mathcal{O}_S} k(s)$ and $\mathcal{E}$ is flat on $S$, the claim follows from Lemma 2.4.

In particular, whenever $s$ is closed, $\mathcal{E}_s^*$ is locally free since $\dim X = 2$. This implies that, for all closed points $x \in T_s$, the stalk $(\mathcal{E}_s)_x$ is a free $\mathcal{O}_{T_s,x}$-module. By equation (2.3) it follows that for all closed points $(x,s) \in T$ the stalk $\mathcal{E}_{(x,s)}$ is a free $\mathcal{O}_{T(s,s)}$-module. By Exercise 6.2 in [9], $\mathcal{E}_s^*$ is locally free. □

Lemma 2.6. Let $\mathcal{E}$ be a coherent $\mathcal{O}_T$-module such that the restrictions $\mathcal{E}_s$ are torsion-free for all closed $s \in S$. Then $\mathcal{E}$ is torsion-free.

Proof. By hypothesis, for all $x \in T_s$, the stalk $(\mathcal{E}_s)_x$ is torsion-free. By equation (2.3), for all closed $(x,s) \in T$, the stalk $\mathcal{E}_{(x,s)}$ is torsion-free. Let $\mathcal{T}$ be torsion submodule of $\mathcal{E}$, so that one has an exact sequence $0 \to \mathcal{T} \to \mathcal{E} \to \mathcal{E}^{**}$. By localizing at a closed point $(x,s) \in T$, we obtain $\mathcal{T}_{(x,s)} = 0$. By Lemma 2.8 in [9], one has $\mathcal{T} = 0$. □

Proof of Proposition 2.3. The morphism of sheaves

$$\mathcal{H}om(\mathcal{E}, \mathcal{E}') \xrightarrow{f} \mathcal{H}om(\mathcal{E}'^{**}, (\mathcal{E}')^{**})$$

obtained by taking double duals is injective as $\mathcal{E}'$ is torsion-free by Lemma 2.6. Since for all closed $s \in S$ the sheaves $\mathcal{E}_s$ and $\mathcal{E}_s'$ are torsion-free, by applying Lemma 2.4 one deduces that also the restricted morphisms $f_s$ are injective for all closed $s \in S$. At the same time the sheaf $\mathcal{H}om(\mathcal{E}'^{**}, (\mathcal{E}')^{**})$ is flat on $S$ by its local freeness and the flatness of the projection $t_2$. From this we can prove that $\text{coker } f$ is flat on $S$ by a reasoning similar to the proof of Lemma 2.2. The thesis follows from [13, Prop. III. 9.1A.(e)]. □

3. Statement of the main result

Let $\Sigma_n$ be the $n$-th Hirzebruch surface, i.e., the projective closure of the total space of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$. We denote by $F$ the class in $\text{Pic}(\Sigma_n)$ of the natural ruling $\Sigma_n \to \mathbb{P}^1$, by $H$ the class of the section of the ruling squaring to $n$, and by $E$ the class of the section squaring to $-n$. One has $E = H - nF$ and $F^2 = 0$, $H \cdot F = 1$. We fix a curve $\ell_\infty \simeq \mathbb{P}^1$ in $\Sigma_n$ linearly equivalent to $H$ and call it the “line at infinity”.

One has $\text{Pic}(\Sigma_n) = \mathbb{Z}H \oplus \mathbb{Z}F$. For any sheaf $\mathcal{E}$ of $\mathcal{O}_{\Sigma_n}$-modules we shall write

$$\mathcal{E}(p,q) := \mathcal{E} \otimes \mathcal{O}_{\Sigma_n}(pH + qF) \quad p, q \in \mathbb{Z}.$$
Lemma 3.1.

\[ H^0(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \] if and only if \( \begin{cases} p \geq 0 \\ np + q \geq 0 \end{cases} \);

\[ H^1(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \] if and only if \( \begin{cases} p \geq 0 \\ q \leq -2 \end{cases} \) or \( \begin{cases} p \leq -2 \\ q \geq n \end{cases} \);

\[ H^2(\mathcal{O}_{\Sigma_n}(p,q)) \neq 0 \] if and only if \( \begin{cases} p \leq -2 \\ np + q \leq -(n + 2) \end{cases} \).

\[ \text{Proof.} \] Similar to King’s proof for the case \( n = 1 \) \cite[pp. 22-23]{20}.

Among the several definitions of framed sheaves available in the literature, we shall adopt the following.

**Definition 3.2.** A framed sheaf is a pair \((\mathcal{E}, \theta)\), where

1. \( \mathcal{E} \) is a torsion-free sheaf on \( \Sigma_n \) such that
   \[ \mathcal{E}|_{\ell_\infty} \simeq \mathcal{O}_{\ell_\infty}^{\oplus r} \] \quad (3.1)
   
   with \( r = \text{rk}(\mathcal{E}) \).

2. \( \theta \) is a fixed isomorphism \( \theta : \mathcal{E}|_{\ell_\infty} \sim \rightarrow \mathcal{O}_{\ell_\infty}^{\oplus r} \).

Condition (3.1) implies \( c_1(\mathcal{E}) \propto E \). The isomorphism \( \theta \) is the so-called framing at infinity. By “sheaf trivial at infinity” we shall mean a sheaf satisfying condition (3.1) (without any assigned framing).

**Definition 3.3.** An isomorphism \( \Lambda \) between two framed sheaves \((\mathcal{E}, \theta)\) and \((\mathcal{E}', \theta')\) is an isomorphism \( \Lambda : \mathcal{E} \sim \rightarrow \mathcal{E}' \) such that the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{E}|_{\ell_\infty} & \xrightarrow{\theta} & \mathcal{O}_{\ell_\infty}^{\oplus r} \\
\Lambda|_{\ell_\infty} \downarrow & & \downarrow \theta' \\
\mathcal{E}'|_{\ell_\infty} & \xrightarrow{\theta'} & \mathcal{O}_{\ell_\infty}^{\oplus r}
\end{array}
\]

where \( \Lambda|_{\ell_\infty} := \Lambda|_{\ell_\infty} \).

Let \( \mathcal{M}^n(r, a, c) \) be the set of isomorphism classes of framed sheaves on \( \Sigma_n \) having rank \( r \), first Chern class \( aE \), and second Chern class \( c \): we shall prove that this set can be endowed with a structure of a smooth algebraic variety. We restrict ourselves to the case \( n \geq 1 \) and assume that the framed sheaves are normalized in such a way that \( 0 \leq a \leq r - 1 \).

In order to simplify the statement of Theorem we introduce some notation.
We denote by $\vec{k}$ a quadruple $(n, r, a, c)$, and define $k_i \in \mathbb{Z}$, $i = 1, \ldots, 4$ as follows:

$$
\begin{align*}
  k_1 &= c + 1 - na(a-1) \\
  k_2 &= k_1 + na \\
  k_3 &= k_1 + (n-1)a \\
  k_4 &= k_1 + r - a.
\end{align*}
$$

(3.2)

The other definitions needed to state Theorem 3.4 makes sense only under the assumption $k_1 \geq 0$. The Theorem itself will give a deeper meaning to this inequality.

We introduce the locally-free sheaves

$$
\begin{align*}
  U_{\vec{k}} &:= \mathcal{O}_{\Sigma_n}(0, -1)^{\oplus k_1} \\
  V_{\vec{k}} &:= \mathcal{O}_{\Sigma_n}(1, -1)^{\oplus k_2} \oplus \mathcal{O}_{\Sigma_n}^{\oplus k_4} \\
  W_{\vec{k}} &:= \mathcal{O}_{\Sigma_n}(1, 0)^{\oplus k_3}.
\end{align*}
$$

(3.3)

We shall write $U_{\vec{k}, \infty}$ in place of $U_{\vec{k}}|_{\ell_{\infty}}$, etc.

We introduce the vector space

$$
V_{\vec{k}} := \text{Hom} \left( U_{\vec{k}}, V_{\vec{k}} \right) \oplus \text{Hom} \left( V_{\vec{k}}, W_{\vec{k}} \right),
$$

whose elements will be denoted by $(\alpha, \beta)$.

Let $L_{\vec{k}}$ be the affine subvariety of $V_{\vec{k}}$ cut by the equation $\beta \circ \alpha = 0$. One has the associated complex

$$
M(\alpha, \beta) : U_{\vec{k}} \xrightarrow{\alpha} V_{\vec{k}} \xrightarrow{\beta} W_{\vec{k}}.
$$

We define the quasi-affine variety $L_{\vec{k}}$ as the open subset of $L_{\vec{k}}$ characterized by the following four conditions:

(c1) the sheaf morphism $\alpha$ is a monomorphism;
(c2) the sheaf morphism $\beta$ is an epimorphism;
(c3) the vector space morphisms $\alpha \otimes k(y)$ have maximal rank for all closed points $y \in \ell_{\infty}$;
(c4) if we consider the display associated with the monad $M(\alpha, \beta)$ as in eq. (2.2), and we restrict it to $\ell_{\infty}$, after twisting by $\mathcal{O}_{\ell_{\infty}}(-1)$ and taking cohomology, we get a vector space morphism $\Phi := H^0(\nu|_{\ell_{\infty}}(-1)) : H^0(\mathcal{A}|_{\ell_{\infty}}(-1)) \rightarrow H^0(\mathbb{W}_{\ell_{\infty}}(-1))$; note that $h^0(\mathcal{A}|_{\ell_{\infty}}(-1)) = h^0(\mathbb{W}_{\ell_{\infty}}(-1)) = nk_3$. We require that

$$
\det \Phi \neq 0.
$$

(c5) the cohomology $\mathcal{E}_{\alpha, \beta}$ of the monad $M(\alpha, \beta)$ is torsion-free.

We shall prove in Lemma 4.9 that $L_{\vec{k}}$ is a smooth variety. We let $\mathcal{E}_{\alpha, \beta, \infty} = \mathcal{E}_{\alpha, \beta}|_{\ell_{\infty}}$. 
We consider the algebraic group
\[ G \overset{\vec{k}}{=} \text{Aut}(U_{\vec{k}}) \times \text{Aut}(V_{\vec{k}}) \times \text{Aut}(W_{\vec{k}}). \]
This group acts naturally on \( L_{\vec{k}} \) according to the following formulas:
\[
\begin{align*}
\alpha & \mapsto \alpha' = \psi \alpha \phi^{-1} \\
\beta & \mapsto \beta' = \chi \beta \psi^{-1}
\end{align*}
\]
\( \bar{\psi} = (\phi, \psi, \chi) \in G_{\vec{k}} \).
\[(3.4)\]
This action will be called \( \rho_0 : L_{\vec{k}} \times G_{\vec{k}} \to L_{\vec{k}} \).

We introduce (see subsection 4.2) a principal \( \text{GL}(r, \mathbb{C}) \) bundle \( \overset{\tau}{P}_{\vec{k}} \to L_{\vec{k}} \) whose fibre over \((\alpha, \beta)\) is naturally identified with the space of framings at infinity for \( E_{\alpha, \beta} \), namely, a point \( \theta \in P_{\vec{k}} \) is an isomorphism \( \theta : E_{\alpha, \beta, \infty} \overset{\sim}{\to} \mathcal{O}_{\infty}^{\oplus r} \), where \((\alpha, \beta) = \tau(\theta)\).

One can lift \( \rho_0 \) to an action \( \rho : P_{\vec{k}} \times G_{\vec{k}} \to P_{\vec{k}} \) by letting
\[
\rho(\theta, \bar{\psi}) = \theta \circ \Lambda_{\infty}(\alpha, \beta; \bar{\psi})^{-1},
\]
where \((\alpha, \beta) = \tau(\theta)\), and after letting \((\alpha', \beta') = \bar{\psi} \cdot (\alpha, \beta)\), the isomorphism
\[
\Lambda_{\infty}(\alpha, \beta; \bar{\psi}) : E_{\alpha, \beta, \infty} \to E_{\alpha', \beta', \infty}
\]
is induced by \( \bar{\psi} : M(\alpha, \beta) \to M(\alpha', \beta') \). The identity
\[
\Lambda_{\infty}(\alpha, \beta; \bar{\psi} \cdot \bar{\psi}') = \Lambda_{\infty}(\alpha', \beta', \bar{\psi}') \circ \Lambda_{\infty}(\alpha, \beta, \bar{\psi})
\]
ensures that \( \rho \) is indeed an action. We note that the projection \( \tau : P_{\vec{k}} \to L_{\vec{k}} \) becomes a \( G_{\vec{k}} \)-equivariant morphism.

We have now all ingredients needed to state our main result.

**Theorem 3.4.** The set \( \mathcal{M}^n(r, a, c) \) is nonempty if and only if \( c + \frac{1}{2} na(a - 1) \geq 0 \). If this is the case, it can be given a structure of smooth algebraic variety of dimension \( 2rc + (r - 1)na^2 \) by representing it as the quotient \( \mathcal{M}^n(r, a, c) = P_{\vec{k}}/G_{\vec{k}} \). Moreover, \( \mathcal{M}^n(r, a, c) \) turns out to be a fine moduli space of framed sheaves on \( \Sigma_n \).

Note that
\[
\dim \mathcal{M}^n(r, a, c) = 2rc + (r - 1)na^2 = 2r \Delta,
\]
where \( \Delta = c_2 - \frac{c_1}{2r}c_1^2 \) is the discriminant of the sheaves parametrized by \( \mathcal{M}^n(r, a, c) \).

**Corollary 3.5.** The moduli space \( \mathcal{M}^n(r, a, c) \) is irreducible.

**Proof.** One knows by instanton counting that \( \mathcal{M}^n(r, a, c) \) is connected \([4]\), hence, being smooth, it is irreducible as well. \( \square \)
4. Families of framed sheaves

In this section we explain how the varieties $L_{\vec{k}}$ and $P_{\vec{k}}$ arise and construct a canonical family $\left(\tilde{\mathcal{E}}_{\vec{k}}, \tilde{\Theta}_{\vec{k}}\right)$ on the product $\Sigma_n \times P_{\vec{k}}$.

For any scheme $S$, let $T = \Sigma_n \times S$ and let $t_i, i = 1, 2$ be the projections onto the first and the second factor, respectively. Analogously, we introduce the product scheme $T_\infty = \ell_\infty \times S$, together with the projections $u_i, i = 1, 2$.

**Definition 4.1.** Let $\vec{k} = (n, r, a, c) \in \mathbb{Z}^4$ with $n \geq 1$, $r \geq 1$ and $0 \leq a \leq r - 1$. A coherent sheaf $F$ on $T$ fulfills condition $\vec{k}$ if and only if it is flat on $S$ and for all closed points $s \in S$

- the restricted sheaf $F_s$ is torsion-free and trivial at infinity on $T_s \simeq \Sigma_n$;
- the Chern character of $F_s$ is $(r, aE, -c - \frac{1}{2}na^2)$.

Note that by Lemma 2.6 $F$ is torsion-free.

**Definition 4.2.** Given a vector $\vec{k}$ and a product scheme $T$ as above, a family of framed sheaves on $\Sigma_n$ is a pair $(F, \Theta)$, where:

1. $F$ is a sheaf on $T$ fulfilling condition $\vec{k}$;
2. $\Theta$ is an isomorphism $\tilde{\mathcal{F}}|_{T_\infty} \rightarrow \mathcal{O}_{T_\infty}^{\oplus r}$.

For any sheaf $G$ of $\mathcal{O}_T$-modules we let $G(p, q) = G \otimes t^*_1 \mathcal{O}_{\Sigma_n}(p, q)$ for all $(p, q) \in \mathbb{Z}$.

**Proposition 4.3.** A sheaf $\tilde{\mathcal{F}}$ on $T$ that satisfies condition $\vec{k}$ is isomorphic to the cohomology of a canonically associated monad $M(\tilde{\mathcal{F}})$ on $T$

$$M(\tilde{\mathcal{F}}) : \quad 0 \longrightarrow \mathcal{U} \xrightarrow{A} \mathcal{V} \xrightarrow{B} \mathcal{W} \longrightarrow 0,$$

where the locally-free sheaves $\mathcal{U}$ and $\mathcal{W}$ are

$$\begin{align*}
\mathcal{U} &= \mathcal{O}_T(0, -1) \otimes t^*_1 R^1 t_2, [\tilde{\mathcal{F}}(-2, n - 1)] \\
\mathcal{W} &= \mathcal{O}_T(1, 0) \otimes t^*_2 R^1 t_2, [\tilde{\mathcal{F}}(-1, 0)].
\end{align*}$$

The locally-free sheaf $\mathcal{V}$ is defined as an extension

$$\begin{align*}
0 &\longrightarrow \mathcal{V}_- \xrightarrow{c} \mathcal{V} \xrightarrow{d} \mathcal{V}_+ \longrightarrow 0, \tag{4.1}
\end{align*}$$

where

$$\begin{align*}
\mathcal{V}_+ := & t^*_2 R^1 t_2, [\tilde{\mathcal{F}}(-2, n)] \\
\mathcal{V}_- := & \mathcal{O}_T(1, -1) \otimes t^*_2 R^1 t_2, [\tilde{\mathcal{F}}(-1, -1)]
\end{align*}$$

and the morphisms $c$ and $d$ are canonically determined by $\tilde{\mathcal{F}}$.

Moreover the association $\tilde{\mathcal{F}} \mapsto M(\tilde{\mathcal{F}})$ is covariantly functorial.
To prove this Proposition, one needs the following result.

**Lemma 4.4.** Let $E$ be a torsion-free sheaf on $\Sigma_n$ trivial at infinity. One has

\[
H^0(E(p, q)) = 0 \quad \text{for} \quad np + q \leq -1 ,
\]

\[
H^2(E(p, q)) = 0 \quad \text{for} \quad np + q \geq -(n + 1).
\]

**Proof.** When $E$ is locally free, the proof is essentially the same as in [20, p. 24]. Otherwise we get the thesis by using the injection $E \rightarrow E^{**}$. □

**Proof of Proposition 4.3.** Let us consider the product scheme $\Sigma_n \times \Sigma_n \times S$, together with the canonical projections $p_{12}, p_{13}$ and $p_{23}$. Buchdahl [5] proved the existence of a three-term locally-free resolution $G^\bullet \rightarrow O_\Delta$ of the structure sheaf $O_\Delta$ of the diagonal $\Delta \subseteq \Sigma_n \times \Sigma_n$.

This is given by

\[
\begin{cases}
G^0 = O_{\Sigma_n \times \Sigma_n} \\
G^{-1} = R^* \\
G^{-2} = O_{\Sigma_n}(-1,-1) \boxtimes O_{\Sigma_n}(-1,n-1)
\end{cases} ;
\]

where $R$ is the extension

\[
0 \rightarrow O_{\Sigma_n}(1,0) \boxtimes O_{\Sigma_n}(1,-n) \rightarrow R \rightarrow O_{\Sigma_n}(0,1) \boxtimes O_{\Sigma_n}(0,1) \rightarrow 0
\]

corresponding to the image in $H^1(\Sigma_n \times \Sigma_n, O_{\Sigma_n}(1,-1) \boxtimes O_{\Sigma_n}(1,-n-1))$ of $t$ under the connecting homomorphism of the exact sequence

\[
0 \rightarrow O_{\Sigma_n}(1,-1) \boxtimes O_{\Sigma_n}(1,-n-1)) \rightarrow O_{\Sigma_n}(1,0) \boxtimes O_{\Sigma_n}(1,-n) \\
\quad \rightarrow O_{\Sigma_n}(1,0) \boxtimes O_{\Sigma_n}(1,-n) \otimes O_Y \rightarrow 0 .
\]

Here $Y$ is a suitable divisor in $\Sigma_n \times \Sigma_n$, while $t$ is a section of $O_{\Sigma_n}(0,1) \boxtimes O_{\Sigma_n}(0,-1) \otimes O_Y$ whose zero locus is $\Delta$.

Given any sheaf $\mathfrak{F}$ on $T$ fullfilling condition $\vec{k}$, we introduce the complex

\[
C^\bullet = C^{-2} \rightarrow C^{-1} \rightarrow C^0 = (p_{12}^* G^\bullet) \otimes [p_{23}^*(\mathfrak{F}(1,0))]
\]

There are two spectral sequences, both abutting to the hyperdirect image $R^i p_{13*}(C^\bullet)$. From the first spectral sequence one gets

\[
R^i p_{13*}(C^\bullet) = \begin{cases} 
\mathfrak{F}(-1,0) & \text{if } i = 0 \\
0 & \text{otherwise} 
\end{cases}
\]

By using Lemma 4.4, one can obtain from the second exact sequence a complex that, when twisted by $O_T(1,0)$, yields $M(\mathfrak{F})$.

The functoriality follows from the construction. □
This proof implies that the sheaves \( R^1t_{2*}(\mathcal{F}(p,q)) \) are locally free for 
\[(p,q) \in \mathcal{I} = \{(-2,n-1),(-1,0),(-2,n),(-1,-1)\}.
\]

4.1. The variety \( L_k \). As a straightforward consequence of Proposition 4.3 we get the following result.

Corollary 4.5. Let the sheaf \( \mathcal{F} \) and the monad \( M(\mathcal{F}) \) be as in Proposition 4.3. Assume that \( S \) is affine, and that the sheaves \( R^1t_{2*}(\mathcal{F}(p,q)) \) are trivial for \((p,q) \in \mathcal{I}\). There are isomorphisms
\[
\mathcal{U} \cong t^*_1\mathcal{U}_k; \quad \mathcal{V} \cong t^*_1\mathcal{V}_k; \quad \mathcal{W} \cong t^*_1\mathcal{W}_k,
\]
the sheaves \( \mathcal{U}_k, \mathcal{V}_k \) and \( \mathcal{W}_k \) being defined as in eq. (3.3).

Proof. A trivialization for \( R^1t_{2*}(\mathcal{F}(p,q)) \) amounts to choosing a closed point \( s_0 \in S \) and an isomorphism
\[
R^1t_{2*}(\mathcal{F}(p,q)) \cong O_S \otimes [R^1t_{2*}(\mathcal{F}(p,q)) \otimes k(s_0)].
\]

Since \( \mathcal{F}_{s_0} \) is torsion-free and trivial at infinity, from Lemma 4.4 and from the Semicontinuity Theorem one obtains the isomorphism
\[
R^1t_{2*}(\mathcal{F}(p,q)) \otimes k(s_0) \cong H^1(\mathcal{F}_{s_0}(p,q)).
\]

The dimensions of the vector spaces \( H^1(\mathcal{F}_{s_0}(p,q)) \) can be computed by means of Riemann-Roch Theorem and Lemma 4.3:

\[
h^1(\mathcal{F}_{s_0}(p,q)) = \begin{cases} k_1 & \text{for } (p,q) = (-2,n-1) \\ k_2 & \text{for } (p,q) = (-1,-1) \\ k_3 & \text{for } (p,q) = (-1,0) \\ k_4 & \text{for } (p,q) = (-2,n) \end{cases}
\]

where \( k_i, i = 1, \ldots, 4 \) are as in eq. (3.2). As a consequence the sheaves \( R^1t_{2*}(\mathcal{F}(-2,n-1)) \), \( R^1t_{2*}(\mathcal{F}(-1,-1)) \), \( R^1t_{2*}(\mathcal{F}(-1,0)) \) and \( R^1t_{2*}(\mathcal{F}(-2,n)) \) are free of ranks \( k_1, \ldots, k_4 \) respectively, so that
\[
\mathcal{U} \cong O_T(0,-1)^{\oplus k_1}; \quad \mathcal{W} \cong O_T(1,0)^{\oplus k_3}; \quad \mathcal{W}_- \cong O_T^{\oplus k_4}; \quad \mathcal{W}_+ \cong O_T(1,-1)^{\oplus k_2}.
\]
The thesis follows for $\mathfrak{U}$ and $\mathfrak{W}$. By plugging $\mathfrak{U}_{-}$ and $\mathfrak{U}_{+}$ into the sequence eq. (4.1), the latter splits, since
\begin{align*}
\text{Ext}^{1}(\mathcal{O}_{T}(1,-1),\mathcal{O}_{T}) & \simeq H^{1}(T,\mathcal{O}_{T}(-1,1)) \\
& \simeq H^{0}(\mathcal{O}_{S},\mathcal{O}_{T}(-1,1)) \\
& \simeq H^{0}(\mathcal{O}_{S},\mathcal{O}_{T}[-1,1]) \\
& \simeq H^{0}(S,\mathcal{O}_{S},\mathcal{O}_{S_{\S}}(-1,1)) = 0
\end{align*}
(the isomorphism $(\ast)$ holds true as $S$ is affine and the vanishing is a consequence of Lemma 4.4). This ends the proof. $\square$

The following result is the absolute case of Proposition 4.3, obtained by letting $S = \text{Spec} \mathbb{C}$, and follows easily from Corollary 4.5.

**Corollary 4.6.** Any sheaf $\mathcal{E}$ on $\Sigma_{n}$ that is torsion-free and trivial at infinity is isomorphic to the cohomology of a monad $M_{k}(\mathcal{E})$, which is of the form $M(\alpha, \beta)$ for a suitable $\alpha, \beta \in \mathcal{L}_{k}$. Note that we do not require $k_{1} \geq 0$ a priori. As a consequence however, if $k_{1} = c + \frac{1}{2}na(a-1) < 0$, the set $M^{\alpha}(r,a,c)$ is empty.

We fix a $k$ such that $k_{1} \geq 0$. The functoriality of $M_{k}(-)$ implies that $M_{k}(\mathcal{E}) \simeq M_{k}(\mathcal{E}')$ whenever $\mathcal{E} \simeq \mathcal{E}'$. In particular, $M_{k}(-)$ provides a map between the set of isomorphism classes of torsion-free sheaves on $\Sigma_{n}$ that are trivial at infinity and the set of isomorphism classes of monads of the form $M(\alpha, \beta)$. The following two results establish the injectivity of this map, and enable us to characterize its image.

**Lemma 4.7.** Let $(\alpha, \beta), (\alpha', \beta')$ be any two points in $\mathcal{L}_{k}$ satisfying the conditions (c1) and (c2) introduced in section 3 (therefore $M = M(\alpha, \beta)$ and $M' = M(\alpha', \beta')$ are monads). Then
\begin{equation*}
M \simeq M' \quad \text{if and only if} \quad \mathcal{E}(M) \simeq \mathcal{E}(M').
\end{equation*}

**Proof.** The proof of [26, Lemma 4.1.3] holds true for non-locally free sheaves as well. $\square$

**Proposition 4.8.** For any point $(\alpha, \beta) \in \mathcal{L}_{k}$ satisfying conditions (c1) and (c2), the cohomology $\mathcal{E}$ of the monad $M(\alpha, \beta)$ is trivial at infinity if and only if the morphisms $(\alpha, \beta)$ satisfy conditions (c3) and (c4).

**Proof.** Condition (c3) is equivalent to the local freeness of $\mathcal{E}|_{t_{\infty}}$. As for condition (c4), the display of $M(\alpha, \beta)$ produces the exact sequence
\begin{equation*}
H^{0}(\mathcal{E}|_{t_{\infty}}(-1)) \longrightarrow H^{0}(\mathcal{A}|_{t_{\infty}}(-1)) \xrightarrow{\Phi} H^{0}(\mathcal{W}_{e_{\S}}(-1)) \longrightarrow H^{1}(\mathcal{E}|_{t_{\infty}}(-1)).
\end{equation*}
Condition (c4) is equivalent to the vanishing of $H^i(\mathcal{E}|_{\ell_\infty}(-1))$, $i = 0, 1$. The thesis follows easily. □

This result enables us to identify $L_{\vec{k}}$ with the subset of $V_{\vec{k}}$ whose points correspond to cohomology sheaves $E_{\alpha,\beta}$ that are torsion-free and trivial at infinity.

**Lemma 4.9.** The variety $L_{\vec{k}}$ is smooth of dimension $\dim L_{\vec{k}} = \dim V_{\vec{k}} - \dim W_{\vec{k}}$, where $W_{\vec{k}} = \text{Hom}(U_{\vec{k}}, V_{\vec{k}})$.

**Proof.** We define the map

$$\zeta: \ V_{\vec{k}} \longrightarrow \ W_{\vec{k}} \quad (\alpha, \beta) \longmapsto \beta \alpha.$$ 

So $T_{\vec{k}}$ is the set $\{\zeta = 0\}$. The differential $d\zeta$ at the point $(\alpha_0, \beta_0) \in L_{\vec{k}}$ is the linear map

$$(d\zeta)|_{(\alpha_0, \beta_0)}: \ V_{\vec{k}} \longrightarrow \ W_{\vec{k}} \quad (\alpha, \beta) \longmapsto \beta_0 \alpha + \beta \alpha_0.$$ 

The rank of this map is equal to $d = \dim V_{\vec{k}} - \dim L_{\vec{k}}$ on the (non empty) nonsingular locus of $L_{\vec{k}}$, and outside of this set is bounded above by $d$ (see [13, pp. 31-33]).

One can prove as in [26, Lemma II.4.1.7] that for any point $(\alpha, \beta) \in L_{\vec{k}}$, there is an isomorphism

$$\text{coker}(d\zeta)|_{(\alpha, \beta)} \simeq H^2(E_{\alpha,\beta}^* \otimes E_{\alpha,\beta}).$$

Now, $E_{\alpha,\beta}^* \otimes E_{\alpha,\beta}$ is torsion-free and trivial at infinity, so that $H^2(E_{\alpha,\beta}^* \otimes E_{\alpha,\beta}) = 0$ by Lemma [4.4] and $d\zeta$ has maximal rank everywhere on $L_{\vec{k}}$. Since the singular locus of $L_{\vec{k}}$ coincides with the set of points in which $d\zeta$ fails to have maximal rank, $L_{\vec{k}}$ is smooth. □

### 4.2. The variety $P_{\vec{k}}$

Let us introduce the varieties $Y = \Sigma \times L_{\vec{k}}$ and $Y_\infty = \ell_\infty \times L_{\vec{k}}$, together with the canonical projections shown in the following diagram:

\[
\begin{array}{c}
\ell_\infty \longrightarrow \Sigma \\
\downarrow \quad \quad \quad \downarrow \quad \quad \quad \downarrow \\
Y_\infty \quad \longrightarrow \quad Y \\
\quad \downarrow \quad \quad \quad \downarrow \\
Y \quad \longrightarrow \quad L_{\vec{k}} \\
\end{array}
\]

On $Y$ we define the complex

$$\mathbb{M}_{\vec{k}}: \quad \tilde{\iota}_1 U_{\vec{k}} \xrightarrow{f_A} \tilde{\iota}_1 V_{\vec{k}} \xrightarrow{f_B} \tilde{\iota}_1 W_{\vec{k}}$$
where \( f_A \) and \( f_B \) are the defined by the formulas
\[
\begin{align*}
f_A : \quad (\tilde{t}_1^* U_{k}(x,\alpha,\beta)) & \quad \mapsto \quad (\tilde{t}_1^* U_{k}(x,\alpha,\beta)), \\
u & \quad \mapsto \quad \alpha(u) \\
f_B : \quad (\tilde{t}_1^* V_{k}(x,\alpha,\beta)) & \quad \mapsto \quad (\tilde{t}_1^* V_{k}(x,\alpha,\beta)), \\
v & \quad \mapsto \quad \beta(v)
\end{align*}
\]
here \( u \) and \( v \) are points in the fibres over \((x,\alpha,\beta) \in \Sigma_n \times L_k\). This complex is actually a monad by Lemma 2.1: let \( \tilde{E}_k \) be its cohomology. This sheaf satisfies condition \( \tilde{k} \), and more precisely one has the following isomorphism for all points \((\alpha,\beta) \in L_k\):
\[
\tilde{E}_k(\alpha,\beta) \simeq E_{\alpha,\beta,\infty}.
\]
The restriction \( \tilde{M}_{k,\infty} \) of \( \tilde{M}_k \) to \( \Upsilon_\infty \) is isomorphic to the monad
\[
0 \quad \longrightarrow \quad \tilde{u}_2^* U_{k,\infty} \quad \longrightarrow \quad \tilde{u}_2^* V_{k,\infty} \quad \longrightarrow \quad \tilde{u}_2^* W_{k,\infty} \quad \longrightarrow \quad 0.
\]
where \( g_A \) and \( g_B \) are the defined by the formulas
\[
\begin{align*}
g_A : \quad (\tilde{u}_2^* U_{k,\infty}(x,\alpha,\beta)) & \quad \mapsto \quad (\tilde{u}_2^* U_{k,\infty}(x,\alpha,\beta)), \\
u & \quad \mapsto \quad \alpha|_{\ell_\infty}(u) \\
g_B : \quad (\tilde{u}_2^* V_{k,\infty}(x,\alpha,\beta)) & \quad \mapsto \quad (\tilde{u}_2^* V_{k,\infty}(x,\alpha,\beta)), \\
v & \quad \mapsto \quad \beta|_{\ell_\infty}(v)
\end{align*}
\]
The cohomology of \( \tilde{M}_{k,\infty} \) will be denoted by \( \tilde{E}_{k,\infty} \). It turns out that
\[
\tilde{E}_{k,\infty}(\alpha,\beta) \simeq E_{\alpha,\beta,\infty}
\]
for all points \((\alpha,\beta) \in L_k\).

Let \( \mathfrak{N}_k \) denote the direct image \( \tilde{u}_2^* \tilde{E}_{k,\infty} \). Since \( \tilde{E}_{k,\infty} \) is a trivial vector bundle on each fibre of \( \tilde{u}_2 \), the sheaf \( \mathfrak{N}_k \) is locally free of rank \( r \). Let \( P_k \) be its bundle of linear frames, which is a principal \( GL(r) \) bundle on \( L_k \), whose fibre at \((\alpha,\beta) \in L_k\) may be identified with the bundle of linear frames of the vector bundle \( E_{\alpha,\beta,\infty} \). Moreover, if
\[
\tau : P_k \rightarrow L_k
\]
is the projection, the vector bundle \( \tilde{N}_k = \tau^* \mathfrak{N}_k \) is trivial. Note that a framed sheaf \( (E,\theta) \) determines a point of \( P_k \). Indeed the monad whose cohomology is \( E \) gives a point \((\alpha,\beta) \in L_k\); and the framing \( \theta \) gives a point of \( P_k \) in the fibre over \((\alpha,\beta)\).
4.3. The family \((\tilde{E}_k, \tilde{\Theta}_k)\). The geometrical environment of this subsection is provided by the varieties \(\Sigma_n \times P_k^*\) and \(\ell_\infty \times P_k^*\), together with the canonical projections shown in the following diagram:

\[
\begin{array}{ccc}
\ell_\infty & \rightarrow & \Sigma_n \\
\uparrow \tilde{u}_1 & & \downarrow i_1 \\
\ell_\infty \times P_k^* & \rightarrow & \Sigma_n \times P_k^* \rightarrow P_k^* \\
\uparrow \tilde{u}_2 & & \downarrow i_2 \\
\end{array}
\]

**Proposition 4.10.** Let \(\tilde{E}_k := (\text{id}_{\Sigma_n} \times \tau)^* \tilde{E}_k\). This sheaf satisfies condition \(\tilde{k}\), and in particular for any point \(\theta \in P_k^*\) one has the natural isomorphism

\[\left(\tilde{E}_k\right)_\theta \simeq E_{\tau(\theta)} . \quad (4.3)\]

We shall call \(\tilde{E}_{k,\infty}\) the restriction of \(\tilde{E}_k\) to \(\ell_\infty \times P_k^*\), so that \(\tilde{E}_{k,\infty} \simeq (\text{id}_{P^1} \times \tau)^* \tilde{E}_{k,\infty} \simeq \tilde{u}_2^* \tilde{\mathcal{N}}_k\).

**Proof.** The flatness of \(\tilde{E}_k\) on \(P_k^*\) follows from the identification \(\Sigma_n \times P_k^* \simeq P_k^* \times L_k \Upsilon\), while the isomorphism (4.3) comes from [13, Prop 9.3], as \(\ell_\infty \times P_k^* \simeq P_k^* \times L_k \Upsilon_\infty\). The last statement is trivial. \(\square\)

One can extend the action \(\rho\) of \(G_k^*\) on \(P_k^*\) to actions \(\tilde{\rho}\) on \(\Sigma_n \times P_k^*\) and \(\rho_\infty\) on \(\ell_\infty \times P_k^*\) by setting

\[
\left\{ \begin{array}{l}
\tilde{\rho} := \text{id}_{\Sigma_n} \times \rho \\
\rho_\infty := \text{id}_{\ell_\infty} \times \rho
\end{array} \right.
\]

**Proposition 4.11.** The sheaves \(\tilde{E}_k\) and \(\tilde{E}_{k,\infty}\) are, respectively, isomorphic to the cohomologies of the monads

\[
\tilde{M}_k := (\text{id}_{\Sigma_n} \times \tau)^* \tilde{M}_k; \\
\tilde{M}_{k,\infty} := (\text{id}_{\ell_\infty} \times \tau)^* \tilde{M}_{k,\infty}.
\]

Both monads \(\tilde{M}_k\) and \(\tilde{M}_{k,\infty}\) are \(G_k\)-equivariant.

**Corollary 4.12.** The sheaf \(\tilde{E}_k\) admits a \(G_k\)-linearization \(\Psi\) satisfying the isomorphism

\[+ + + \left(\Psi_{\ell_\infty \times P_k^* \times G_k}\right)_{(\theta, \bar{\psi})} \simeq \Lambda_\infty (\alpha, \beta; \bar{\psi})\]

for any point \((\theta, \bar{\psi}) \in P_k^* \times G_k\).
Proof. One defines a morphism of monads 
\( m_{12}^* \tilde{M}_k \to (\text{id}_{\Sigma_n} \times \rho_0)^* \tilde{M}_k \) by letting 
\[
(m_{12}^* \tilde{M}_k)(x,\alpha,\beta,\tilde{\psi}) \mapsto ((\text{id}_{\Sigma_n} \times \rho_0)^* \tilde{M}_k)(x,\alpha,\beta,\tilde{\psi}).
\]

where \( m_{12} : \Upsilon \times G_k \to \Upsilon \) is the canonical projection and \( \rho_0 \) was defined in eq. (3.4). This induces by pullback a morphism \( \Psi : m_{12}^* \tilde{E}_k \to (\text{id}_{\Sigma_n} \times \rho_0)^* \tilde{E}_k \) which satisfies the property in the statement of this Corollary. \( \square \)

Since \( P_k \) is the bundle of linear frames of \( \tilde{N}_k \), there exists a canonical isomorphism \( \tilde{N}_k \to O_{P_k}^{\oplus r} \), which can be regarded as a framing \( \tilde{\Theta}_k \) for the sheaf \( \tilde{E}_k \). As a consequence of eq. (3.5), the morphism \( \tilde{\Theta}_k \) is \( G_k \)-equivariant, namely, the following diagram commutes

\[
\begin{array}{ccc}
I_{12}^* \tilde{E}_{k,\infty} & \xrightarrow{l_{12}^* \tilde{\Theta}_k} & O_{\ell_{\infty} \times P_k \times G_k}^{\oplus r} \\
\Psi_\infty \downarrow & & \downarrow \rho_\infty^* \tilde{\Theta}_k \\
\rho_\infty^* \tilde{E}_{k,\infty} & \xrightarrow{} & \end{array}
\]

where \( \Psi_\infty = \Psi|_{\ell_{\infty} \times P_k \times G_k} \) and \( l_{12} : \ell_{\infty} \times P_k \times G_k \to \ell_{\infty} \times P_k \) is the projection.

5. The moduli space \( \mathcal{M}^n(r, a, c) \)

In this section we give the moduli space \( \mathcal{M}^n(r, a, c) \) a scheme structure, and prove the first part of the Main Theorem. The space \( \mathcal{M}^n(r, a, c) \) can be set-theoretically identified with the quotient \( P_k / G_k \). We denote by \( \pi : P_k \to \mathcal{M}^n(r, a, c) \) the natural projection.

Theorem 5.1. The orbit space \( \mathcal{M}^n(r, a, c) \) is a smooth algebraic variety, and \( P_k \) is a locally trivial principal \( G_k \)-bundle over it.

In order to prove this Theorem, we need to investigate some properties of the \( G_k \)-action on \( P_k \).

Lemma 5.2. Let \( \mathcal{E} \) and \( \mathcal{E}' \) be sheaves on \( \Sigma_n \) that are torsion-free and trivial at infinity. There is an injection

\[
0 \to \text{Hom}(\mathcal{E}, \mathcal{E}') \xrightarrow{R} \text{Hom}(\mathcal{E}|_{\ell_{\infty}}, \mathcal{E}'|_{\ell_{\infty}}) \cong \text{End}(\mathcal{C}')
\]

where \( R \) is the restriction morphism.

Proof. If \( \mathcal{E} \) and \( \mathcal{E}' \) are locally free, one has \( \text{Hom}(\mathcal{E}, \mathcal{E}') \cong H^0(\mathcal{E}^* \otimes \mathcal{E}') \). The sheaf \( \mathcal{E}^* \otimes \mathcal{E}' \) is locally free and trivial at infinity, so that the result follows by twisting the structure.
sequence of $\ell_\infty$ by it and taking cohomology (see Lemma 4.4). In general, one can conclude because the morphism (2.3) is injective, and $E^{**}$ and $E^{***}$ are locally free. □

This result generalizes to the relative situation. Let $S$ be a scheme, and let $\mathcal{F}$ and $\mathcal{F}'$ be two sheaves on $T = \Sigma_n \times S$ satisfying condition $\vec{k}$.

**Corollary 5.3.** The restriction morphism

$$\text{Hom} (\mathcal{F}, \mathcal{F}') \to \text{Hom} (\mathcal{F}|_{T_\infty}, \mathcal{F}'|_{T_\infty})$$

is injective.

**Proof.** Let $\mathcal{H} = \text{Hom} (\mathcal{F}, \mathcal{F}')$. Since both $\mathcal{F}$ and $\mathcal{F}'$ are locally free along $T_\infty$ one gets $\text{Tor}_i (\mathcal{H}, \mathcal{O}_{T_\infty}) = 0$ for $i > 0$ (see for example [11, p. 700]). Thus, if we twist the structure sequence of the divisor $T_\infty$ by $\mathcal{H}$, we get

$$0 \longrightarrow \mathcal{H}(-T_\infty) \longrightarrow \mathcal{H} \longrightarrow \mathcal{H}|_{T_\infty} \longrightarrow 0,$$

where $\mathcal{H}|_{T_\infty} \simeq \text{Hom} (\mathcal{F}|_{T_\infty}, \mathcal{F}'|_{T_\infty})$. It follows that

$$\ker R = H^0 (\mathcal{H}(-T_\infty)) = H^0 (t_{2*} (\mathcal{H}(-T_\infty))).$$

By Propositions 4.3 and 2.3, the sheaf $\mathcal{H}(-T_\infty)$ is flat on $S$. Moreover Lemmas 2.4 and 5.2 yield the following vanishing result for all closed points $s \in S$:

$$H^0 (\mathcal{H}(-T_\infty)_s) = H^0 (\text{Hom} (\mathcal{F}_s, \mathcal{F}'_s) (-\ell_\infty)) = 0.$$

The Semicontinuity Theorem entails the vanishing of the sheaf $t_{2*} (\mathcal{H}(-T_\infty))$. This ends the proof. □

**Corollary 5.4.** The action of $G_{\vec{k}}$ on $P_{\vec{k}}$ is free.

**Proof.** Let $(\alpha, \beta; \vec{\psi}) \in L_{\vec{k}} \times G_{\vec{k}}$, and put $(\alpha', \beta') = \vec{\psi} \cdot (\alpha, \beta)$. It follows from Lemma 5.2 that a morphism $\Lambda \in \text{Hom} (\mathcal{E}_{\alpha, \beta}, \mathcal{E}_{\alpha', \beta'})$ is fully determined by its restriction $\Lambda_\infty$ to $\ell_\infty$. It is not difficult to see [26, Lemma 4.1.3] that $\Lambda$ is induced by a unique isomorphism $\vec{\psi} : M(\alpha, \beta) \to M(\alpha', \beta')$ between the corresponding monads.

Whenever $\vec{\psi}$ lies in the stabilizer of a point $\theta \in P_{\vec{k}}$, one has $\Lambda_\infty (\alpha, \beta; \vec{\psi}) = \text{id}_{\mathcal{E}_{\alpha, \beta, \infty}}$, where $(\alpha, \beta) = \tau(\theta)$. Since $\vec{\psi}$ is uniquely determined, this implies $\vec{\psi} = \text{id}_{G_{\vec{k}}}$.

□

**Proposition 5.5.** The graph $\Gamma$ of the action $\rho$ is closed in $P_{\vec{k}} \times P_{\vec{k}}$.

**Proof.** Let $x = (\theta_{\alpha, \beta}, \theta'_{\alpha', \beta'})$ be a point in $\Gamma$; by $\theta_{\alpha, \beta}$ we mean that $\theta$ belongs to the fibre over $(\alpha, \beta)$. One has

$$\Lambda_\infty := (\theta'_{\alpha', \beta'})^{-1} \circ \theta_{\alpha, \beta} \in \text{Iso} (\mathcal{E}_{\alpha, \beta, \infty}, \mathcal{E}_{\alpha', \beta', \infty}).$$
We define the vector space $\text{Hom}_\Lambda (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'})$ as the fibre product
\[
\begin{array}{ccc}
\text{Hom}_\Lambda (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}) & \xrightarrow{i} & \mathbb{C} \\
\downarrow j & & \downarrow \Lambda_\infty \\
\text{Hom} (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}) & \xrightarrow{R} & \text{Hom} (\mathcal{E}_{\alpha,\beta,\infty}, \mathcal{E}_{\alpha',\beta',\infty})
\end{array}
\] (5.1)

where $R$ is the restriction morphism to $\ell_\infty$, and $\cdot \Lambda_\infty$ is the multiplication by $\Lambda_\infty$. Both morphisms $i$ and $j$ are injective, since $R$ is injective by Lemma 5.2, while $\cdot \Lambda_\infty$ is injective by the invertibility of $\Lambda_\infty$ [15, Lemma 1.2].

Thus, $\text{Hom}_\Lambda (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'})$ is the subspace of homomorphisms between $\mathcal{E}_{\alpha,\beta}$ and $\mathcal{E}_{\alpha',\beta'}$ that at infinity reduce to multiples of $\Lambda_\infty$. By Lemma [15, Lemma 1.1] one has the short exact sequence
\[
0 \rightarrow \text{Hom}_\Lambda (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}) \rightarrow \mathbb{C} \oplus \text{Hom} (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}) \xrightarrow{(\Lambda_\infty, -(-)_{\ell_\infty})} \text{Hom} (\mathcal{E}_{\alpha,\beta,\infty}, \mathcal{E}_{\alpha',\beta',\infty}) \rightarrow 0. \quad (5.2)
\]

Since the morphism $i$ in the diagram (5.1) is injective, one has $\dim \text{Hom}_\Lambda (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}) \leq 1$, so that $\text{Hom}_\Lambda (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'})$ is either zero or is generated by $\Lambda_\infty$. Hence,
\[
\Gamma = \{ (\theta_{\alpha,\beta}, \theta'_{\alpha',\beta'}) \in P_k \times P_k \mid \dim \text{Hom}_\Lambda (\mathcal{E}_{\alpha,\beta}, \mathcal{E}_{\alpha',\beta'}) = 1 \}. \quad (5.3)
\]

Let us consider the following product varieties, along with the associated canonical projections:

\[
\mathcal{X} := \Sigma_n \times P_k \times P_k \xrightarrow{q_{12}} \Sigma_n \times P_k \xrightarrow{q_{13}} P_k \times P_k,
\]
\[
\mathcal{Y} := \ell_\infty \times P_k \times P_k \xrightarrow{p_{12}} \ell_\infty \times P_k.
\]

One can pull-back the family $\left( \overline{\mathcal{E}}_k, \overline{\Theta}_k \right)$ to $\mathcal{X}$ in two different ways, getting $\left( q_i^* \overline{\mathcal{E}}_k, p_i^* \overline{\Theta}_k \right)$ with $i = 2, 3$. Out of these two pairs one defines

\[
\mathcal{K} = \text{Hom} (q_{12}^* \overline{\mathcal{E}}_k, q_{13}^* \overline{\mathcal{E}}_k)
\]
\[
\Omega = \left( p_{13}^* \overline{\Theta}_k \right)^{-1} \circ \left( p_{12}^* \overline{\Theta}_k \right) \in \text{Iso} \left( p_{12}^* \overline{\mathcal{E}}_k, \ell_\infty, p_{13}^* \overline{\mathcal{E}}_k, \ell_\infty \right).
\]
Since \( q_{1i}^* \mathcal{E}_k \) for \( i = 2, 3 \) are locally free along \( \mathfrak{Y} \), there is an isomorphism

\[
\mathcal{K}_\infty := \text{Hom} \left( p_{12}^* \mathcal{E}_k, \mathcal{K} \right) \cong \mathcal{K} | \mathfrak{Y}.
\]

We introduce the sheaf \( \mathcal{K}_\Omega \) by means of the exact sequence

\[
0 \to \mathcal{K}_\Omega \to \mathcal{O}_\mathfrak{Y} \oplus \mathcal{K} (\cdot \Omega, \cdot | \mathfrak{Y}) \to \mathcal{K}_\infty \to 0.
\]

By Lemma 2.4 for any point \( x = (\theta_{\alpha, \beta}, \theta'_{\alpha', \beta'}) \in P^k \times P^k \) the restriction of this sequence to the fibre of \( q_{23} \) over \( x \) is isomorphic to the sequence (5.2). In particular one gets the isomorphism

\[
H^0 (\mathcal{K}_\Omega, x) \cong \text{Hom} (\mathcal{E}_{\alpha, \beta}, \mathcal{E}_{\alpha', \beta'}). \quad (5.4)
\]

Since, by Proposition 2.3, the sheaf \( \mathcal{K} \) is flat on \( P^k \times P^k \), the sheaf \( \mathcal{K}_\Omega \) is flat on \( P^k \times P^k \) as well [13, Prop. III. 9.1A.(e)]. Equation (5.4) and the Semicontinuity Theorem ensure that \( \Gamma \), as characterized in (5.3), is closed.

The smooth algebraic varieties \( P^k \) and \( G^k \) have unique compatible structures of complex manifolds \( P^k \) and \( G^k \). Note that \( \Gamma \) is closed in \( P^k \times P^k \) as well.

**Corollary 5.6.** The action of \( G^k \) on \( P^k \) is locally proper.

**Proof.** Let \( K_{\theta_0} \) be a compact neighbourhood of a point \( \theta_0 \in P^k \). We consider the morphism

\[
\gamma_{\theta_0} : \quad K_{\theta_0} \times G^k \to P^k \times P^k \quad \left( \theta, \psi \right) \mapsto \left( \theta, \psi \cdot \theta \right).
\]

Since the action of \( G^k \) is free, \( \gamma_{\theta_0} \) is injective, so that its image is

\[
\text{im} \gamma_{\theta_0} = \Gamma \cap (K_{\theta_0} \times P^k).
\]

We have to prove that the counterimage \( \left( \rho |_{K_{\theta_0} \times G^k} \right)^{-1} (K) \) of any compact subset \( K \subset P^k \) is compact. But it is easy to see that

\[
\left( \rho |_{K_{\theta_0} \times G^k} \right)^{-1} (K) = \gamma_{\theta_0}^{-1} (\Gamma \cap (K_{\theta_0} \times K)).
\]

As \( \Gamma \) is closed by Proposition 5.5 the thesis follows.

We recall that an algebraic group \( G \) is said to be special if every locally isotrivial principal \( G \)-bundle is locally trivial [27] (a fibration is said to be isotrivial if it is trivial in the étale topology).

**Lemma 5.7.** The group \( G^k \) is special.
Proof. For any two positive integers \(p, q\), let \(H_{p,q}\) be the subgroup of \(GL(p + q, \mathbb{C})\) whose elements are the matrices
\[
\begin{pmatrix}
1_q & A \\
0 & 1_p
\end{pmatrix},
\]
where \(A \in \text{Hom}(\mathbb{C}^p, \mathbb{C}^q)\).

This group is isomorphic to the direct product of copies of the additive group \(\mathbb{C}\), and therefore it is special [12, Prop. 1]. We have
\[
G^+_k \simeq GL(k_1, \mathbb{C}) \times \text{Aut}(V_k^+) \times GL(k_3, \mathbb{C}),
\]
where \(\text{Aut}(V_k^+)\) can be embedded as a closed subgroup in \(GL(nk_2 + k_4, \mathbb{C})\). Moreover \(H_{k_4, nk_2}^+\) is a normal subgroup of \(\text{Aut}(V_k^+)\), and we get the short exact sequence of groups:
\[
1 \to H_{k_4, nk_2}^+ \to G^+_k \to GL(k_1, \mathbb{C}) \times GL(k_2, \mathbb{C}) \times GL(k_4, \mathbb{C}) \times GL(k_3, \mathbb{C}) \to 1.
\]
Since the group \(GL(p, \mathbb{C})\) is special for any \(p\) [27, Thm. 2], it turns out that \(G^+_k\) is special as well [27, Lemma 6].

We have now all ingredients to prove Theorem 5.1.

Proof of Theorem 5.1. Since \(\mathcal{M}^n(r, a, c)\) is defined as a quotient set, the canonical projection \(\pi\) induces both the quotient topology, which makes \(\mathcal{M}^n(r, a, c)\) into a noetherian topological space, and a canonical structure of locally ringed space (see for example [28]).

Let \(\mathcal{M}^n(r, a, c)_{an} := P^\text{an}_k / G^\text{an}_k\) and let \(\pi^\text{an} : P^\text{an}_k \to \mathcal{M}^n(r, a, c)_{an}\) be the projection. Since the action of \(G^\text{an}_k\) on \(P^\text{an}_k\) is free and locally proper, [16, Satz 24] implies that \(\mathcal{M}^n(r, a, c)_{an}\) with its natural structure of locally ringed space is a complex manifold.

We have a commutative diagram of locally ringed spaces:
\[
\begin{array}{ccc}
P^\text{an}_k & \longrightarrow & P^+_k \\
\downarrow \pi^\text{an} & & \downarrow \pi \\
\mathcal{M}^n(r, a, c)_{an} & \longrightarrow & \mathcal{M}^n(r, a, c).
\end{array}
\]
It follows plainly that \(\mathcal{M}^n(r, a, c)\) is an algebraic variety, and is smooth since \(\mathcal{M}^n(r, a, c)_{an}\) is (see [29, p. 109]).

Moreover, the morphism \(P^+_k \times G^+_k \to P^+_k \times P^+_k\) defined by \(\rho\) is a closed immersion, and \(\mathcal{M}^n(r, a, c)\) is a geometric quotient of \(P^+_k\) modulo \(G^+_k\). It follows from [22, Prop 0.9] that \(P^+_k\) is a principal \(G^+_k\)-bundle over \(\mathcal{M}^n(r, a, c)\), in particular it is locally isotrivial. Lemma 5.7 says that \(P^+_k\) is actually locally trivial, and this completes the proof.

We can now prove the Main Theorem.
Proof of the Main Theorem, first part. From Corollary 4.6 if \( k_1 < 0 \) the set \( \mathcal{M}^n(r, a, c) \) is empty. Vice versa, let \( \vec{k} = (n, r, a, c) \) be such that \( k_1 \geq 0 \), and define the sheaf \( \mathcal{E}_{\vec{k}} \) as follows:

\[
\mathcal{E}_{\vec{k}} = \begin{cases} 
\mathcal{I}_{c \cdot x} & \text{if } r = 1 \ (\Rightarrow a = 0) \\
\mathcal{I}_{c \cdot x} \oplus \mathcal{O}_{\Sigma_n}(aE) & \text{if } r = 2 \\
\mathcal{I}_{c \cdot x} \oplus \mathcal{O}_{\Sigma_n}(aE) \oplus \mathcal{O}_{\Sigma_n}^{\oplus (r-2)} & \text{if } r > 2
\end{cases}
\]

where \( \mathcal{I}_{c \cdot x} \) is the ideal sheaf of a 0-dimensional scheme of length \( c \) concentrated at a point \( x \notin \ell_\infty \). The sheaf \( \mathcal{E}_{\vec{k}} \) has a natural framing at infinity. Its Chern character is \((r, aE, -c - \frac{1}{2}na^2)\). It follows that \( \mathcal{M}^n(r, a, c) \) is empty if and only if \( k_1 < 0 \).

By Theorem 5.1 \( \mathcal{M}^n(r, a, c) \) is a smooth algebraic variety, and its dimension can be computed from the dimensions of \( L_{\vec{k}} \), \( \text{GL}(r, \mathbb{C}) \) and \( G_{\vec{k}} \).

\[ \Box \]

6. THE UNIVERSAL FAMILY

In this section we prove that the moduli space \( \mathcal{M}^n(r, a, c) \) is fine by constructing a universal family of framed sheaves on \( \Sigma_n \). Let us define the varieties \( \Xi = \Sigma_n \times \mathcal{M}^n(r, a, c) \) and \( \Xi_\infty = \ell_\infty \times \mathcal{M}^n(r, a, c) \), together with the canonical projections shown in the following diagram:

\[
\begin{array}{ccc}
\ell_\infty & \to & \Sigma_n \\
\downarrow & & \downarrow_{t_1} \\
\Xi_\infty & \to & \Xi \\
\downarrow & & \downarrow_{t_2} \\
\mathcal{M}^n(r, a, c) & .
\end{array}
\]

Let \( q : t_2^*P_{\vec{k}} \to \Xi \) and \( p : u_2^*P_{\vec{k}} \to \Xi_\infty \) be the natural projections; note that \( t_2^*P_{\vec{k}} \simeq \Sigma_n \times P_{\vec{k}} \), \( u_2^*P_{\vec{k}} \simeq \ell_\infty \times P_{\vec{k}} \), \( q = \text{id}_{\Sigma_n} \times \pi \), \( p = \text{id}_{\ell_\infty} \times \pi \), where \( \pi : P_{\vec{k}} \to \mathcal{M}^n(r, a, c) \) is the quotient morphism. We define the sheaf

\[
\mathfrak{c}_{\vec{k}} = (q_\ast \mathcal{E}_{\vec{k}})^G
\]

on \( \Xi \), where \((\cdot)^G\) denotes taking invariants with respect to the action of \( G_{\vec{k}} \) on \( P_{\vec{k}} \).

**Proposition 6.1.** \( \mathfrak{c}_{\vec{k}} \) is a rank \( r \) coherent sheaf, satisfying condition \( \vec{k} \). Actually, for any point \([\theta] \in \mathcal{M}^n(r, a, c)\) with \( \tau(\theta) = (\alpha, \beta) \) one has the isomorphism

\[
(\mathfrak{c}_{\vec{k}})_{[\theta]} \simeq \mathcal{E}_{\alpha, \beta}.
\]
Furthermore, by considering the restriction at infinity $E_{\vec{k},\infty} := E_{\vec{k}|_{\Sigma}}$ we get
\[ E_{\vec{k},\infty} \simeq \left( p_* E_{\vec{k},\infty} \right)^G. \] (6.2)

We need a few preliminary results. First, we take the monad $\tilde{M}_{\vec{k}} = (id_{\Sigma_n} \times \tau)^* \tilde{M}_{\vec{k}}$ as in Proposition 4.11, and we let
\[ \tilde{M}_{\vec{k}} = 0 \rightarrow \tilde{U}_{\vec{k}} \rightarrow \tilde{V}_{\vec{k}} \rightarrow \tilde{W}_{\vec{k}} \rightarrow 0. \]
(See Equation (4.2)).

We introduce the subsheaves
\[ U_{\vec{k}} = (q^* \tilde{U}_{\vec{k}})^G; \quad V_{\vec{k}} = (q^* \tilde{V}_{\vec{k}})^G; \quad W_{\vec{k}} = (q^* \tilde{W}_{\vec{k}})^G; \]
\[ U_{\vec{k},\infty} = (p^* \tilde{U}_{\vec{k},\infty})^G; \quad V_{\vec{k},\infty} = (p^* \tilde{V}_{\vec{k},\infty})^G; \quad W_{\vec{k},\infty} = (p^* \tilde{W}_{\vec{k},\infty})^G. \]

**Lemma 6.2.** The sheaves $U_{\vec{k}}$, $V_{\vec{k}}$, and $W_{\vec{k}}$ are locally free of rank $k_1$, $k_2 + k_4$, and $k_3$, respectively. Furthermore, there are isomorphisms
\[ q^* U_{\vec{k}} \simeq \tilde{U}_{\vec{k}}; \quad q^* V_{\vec{k}} \simeq \tilde{V}_{\vec{k}}; \quad q^* W_{\vec{k}} \simeq \tilde{W}_{\vec{k}}; \]
\[ p^* U_{\vec{k},\infty} \simeq \tilde{U}_{\vec{k},\infty}; \quad p^* V_{\vec{k},\infty} \simeq \tilde{V}_{\vec{k},\infty}; \quad p^* W_{\vec{k},\infty} \simeq \tilde{W}_{\vec{k},\infty}. \]

**Proof.** We prove the thesis for the sheaf $\tilde{U}_{\vec{k}}$, since the other cases are analogous.

We claim that there exists an open cover of $t^*_2 P_{\vec{k}}$ over which $\tilde{U}_{\vec{k}}$ can be $G_{\vec{k}}$-equivariantly trivialized. Let $V \subseteq M^a(r, a, c)$ be an open subset over which $P_{\vec{k}}$ trivializes. This implies the existence of a morphism
\[ q^{-1}(\Sigma_n \times V) \ni (x, \theta) \mapsto (\phi(x, \theta), \psi(x, \theta) x(x, \theta)) \in G_{\vec{k}}. \]

This enables us to define an automorphism $\eta$ of $\tilde{U}_{\vec{k}}$ by the equation
\[ \eta: \begin{array}{c} (\tilde{U}_{\vec{k}})_{(x, \theta)} \rightarrow (\tilde{U}_{\vec{k}})_{(x, \theta)} \\ u \mapsto \phi(x, \theta) u \end{array}. \]

At the same time, let $g: U_{\vec{k}}|_U \rightarrow \mathcal{O}_U^{d_{k_1}}$ be a trivialization of $U_{\vec{k}}$ over a suitable open subset $U \subseteq \Sigma_n$. On the intersection $W = q^{-1}(\Sigma_n \times V) \cap \tilde{t}^{-1}(U) = U \times \pi^{-1}(V)$ it makes sense to pursue the composition $(\tilde{t}^*_2 g) \circ \eta$, and it is easy to prove that this isomorphism is a $G_{\vec{k}}$-equivariant trivialization for $\tilde{U}_{\vec{k}}$ over $W$. Since we can cover $t^*_2 P_{\vec{k}}$ with such open subsets the claim follows.
The equivariant trivializations induce isomorphisms such
\[ \mathcal{U}_k|_{U \times V} \rightarrow (q_*\mathcal{O}_W^\oplus k_1)^G = \mathcal{O}_{U \times V}^\oplus k_1, \]
and this proves the first statement.

To get the second statement, consider the natural morphism
\[ q^*\mathcal{U}_k \hookrightarrow q^*\left(q_*\tilde{\mathcal{U}}_k\right) \rightarrow \tilde{\mathcal{U}}_k. \tag{6.3} \]
By using equivariant trivializations one can show that the restriction of this morphism to all subsets like \( W \) is invertible, and this completes the proof.

\[ \square \]

Remark 6.3. Since the restriction morphism \( \cdot|_{\ell_\infty \times P_k} \) is \( G_k \)-equivariant, the triples of sheaves \((\mathcal{U}_k, \mathcal{V}_k, \mathcal{W}_k)\) on \( \Xi \) and \((\mathcal{U}_{k,\infty}, \mathcal{V}_{k,\infty}, \mathcal{W}_{k,\infty})\) on \( \Xi_{\infty} \) are related by restriction at infinity, that is, \( \mathcal{U}_{k,\infty} \simeq \mathcal{U}_k|_{\Xi_{\infty}}; \mathcal{V}_{k,\infty} \simeq \mathcal{V}_k|_{\Xi_{\infty}} \) and \( \mathcal{W}_{k,\infty} \simeq \mathcal{W}_k|_{\Xi_{\infty}} \).

We introduce now the “universal” monad.

Proposition 6.4. One has the following commutative diagram of monads on \( \Xi \):
\[ \begin{array}{c c c c c c}
M_k &: & 0 & \rightarrow & \mathcal{U}_k & \rightarrow & \mathcal{V}_k & \rightarrow & \mathcal{W}_k & \rightarrow & 0 \\
M_{k,\infty} &: & 0 & \rightarrow & \mathcal{U}_{k,\infty} & \rightarrow & \mathcal{V}_{k,\infty} & \rightarrow & \mathcal{W}_{k,\infty} & \rightarrow & 0
\end{array} \tag{6.4} \]
where
\[ \begin{align*}
A_k &= (q_*f_A)|_{\mathcal{U}_k} \\
B_k &= (q_*f_B)|_{\mathcal{V}_k}
\end{align*} \quad \text{and} \quad \begin{align*}
A_{k,\infty} &= (p_*g_A)|_{\mathcal{U}_{k,\infty}} \\
B_{k,\infty} &= (p_*g_B)|_{\mathcal{V}_{k,\infty}}
\end{align*} \]
The sheaves \( \mathcal{E}_k \) and \( \mathcal{E}_{k,\infty} \) are isomorphic to the cohomologies of the monads \( M_k \) and \( M_{k,\infty} \), respectively.

Proof. The morphisms of \( M_k \) are well defined due to the \( G_k \)-equivariance of \( f_A \) and \( f_B \).
The condition \( B_k \circ A_k = 0 \) follows from the functoriality of \( q_* \). The injectivity of \( A_k \) is apparent since \( A_k \) is the restriction of the injective morphism \( q_*f_A \). Lemma 6.2 implies \( q^*B_k \simeq f_B \) and \( q^*(\operatorname{coker} B_k) \simeq \operatorname{coker} f_B = 0 \); the vanishing of \( \operatorname{coker} B_k \) follows from the faithful flatness of \( q \).
The proof for \( M_{k,\infty} \) is analogous, and the commutativity of the diagram is an easy consequence of Remark 6.3.

We call \( \mathcal{E} \) the cohomology of \( M_k \). By applying Proposition 4.11 to the display of \( M_k \) one obtains a natural injection \( \iota: \mathcal{E} \hookrightarrow \mathcal{E}_k \). We claim that \( \iota \) is invertible. By pulling-back the
display of $M_k$ to $t_2^* P_k$ and by applying Lemma 6.2 one gets an isomorphism $\kappa: q^* \mathcal{E} \to \tilde{E}_k$ which fits into a commutative triangle
\[
q^* \mathcal{E} \xrightarrow{q^* \iota} q^* \mathfrak{e}_k \xrightarrow{\varphi} \tilde{E}_k
\]
where $\varphi$ is defined analogously to (6.3). We deduce that $\varphi$ is surjective. Since $\tilde{\Theta}_k$ is a $G_k$-equivariant trivialization for $\tilde{E}_kB$, it follows that $\varphi|_{u^*_1 P_k}$ is invertible, and the same argument works for a generic element of the linear system $|u^*_1 P_k|$. This implies that there exists a dense $G_k$-invariant open subset $\Omega \subset t_1^* P_k$ such that $\varphi|_{\Omega}$ is invertible, so that the rank of ker $\varphi$ is zero. Since $\tilde{E}_kB$ is torsion-free, $E_k$ and $q^* E_k$ are torsion-free as well, so that ker $\varphi$ must be zero, i.e., $\varphi$ is invertible. We deduce from the diagram (6.5) that $q^* \iota$ is an isomorphism, and since $q$ is faithfully flat, $\iota$ is invertible as well. This completes the proof.

Proof of Proposition 6.1. The coherence of $\mathfrak{e}_k$ follows from $\mathfrak{e}_k \simeq E_k$, and its rank can be computed easily from the ranks of $U_k$, $V_k$ and $W_k$ given by Lemma 6.2. From Lemma 6.2 and Proposition 6.4 we get isomorphisms $q^* M_k \simeq \tilde{M}_k$, $q^* E_k \simeq \tilde{E}_k$ and $q^* A_k \simeq \tilde{A}_k$. Note that $(A_k)|_{[\theta]} = \alpha$ for all points $[\theta] \in \mathcal{M}^n(r,a,c)$, with $\tau(\theta) = (\alpha, \beta)$. Now 2.2 implies the flatness of $\mathfrak{e}_k$, and eq. (6.1) follows from eq. (4.3). This is enough to show that $\mathfrak{e}_k$ satisfies condition $\tilde{k}$.

Finally, eq. (6.2) is a consequence of the commutativity of the diagram (6.4).

Definition 6.5. We define the isomorphism $\Theta_k$ as the restriction of $p_* \tilde{\Theta}_k$ to the $G_k$-invariant subsheaf $\mathfrak{e}_k\infty$:
\[
\Theta_k := (p_* \tilde{\Theta}_k)|_{\mathfrak{e}_k\infty} : \mathfrak{e}_k\infty \xrightarrow{\sim} \mathcal{O}_{\Sigma^n_r}.
\]
We shall call $\Theta_k$ the universal framing.

We show how to canonically associate a scheme morphism $f_{((\mathfrak{F}, \Theta))} : S \to \mathcal{M}^n(r,a,c)$ with an isomorphism class $[(\mathfrak{F}, \Theta)]$ of families of framed sheaves on $T = \Sigma_n \times S$. We begin by describing some properties of the monad $M(\mathfrak{F})$.

Lemma 6.6. Let $\mathfrak{F}$ be a sheaf on $T$ which satisfies condition $\tilde{k}$, and let $M(\mathfrak{F})$ be the canonically associated monad.
For any closed point \( s \in S \) there is an isomorphism of complexes
\[
M(\mathfrak{F})_s \simeq M(\alpha(s), \beta(s)) ,
\]
where \((\alpha(s), \beta(s)) = (A_s, B_s) \in L_k\).

The restriction \( M(\mathfrak{F})_\infty \) of the monad \( M(\mathfrak{F}) \) to \( T_\infty \) is a monad, whose cohomology is isomorphic to \( \mathfrak{F}|_{T_\infty} \). For any closed point \( s \in S \) there is an isomorphism
\[
M(\mathfrak{F})_\infty, s \simeq U_k \xrightarrow{A_s} V_k \xrightarrow{B_s} W_k .
\]

**Proof.** The proof splits in two steps. 1. Let \( s \in S \) be any closed point. By Corollary \[4.5\] there is an isomorphism
\[
M(\mathfrak{F})_s \simeq U_k \xrightarrow{A_s} V_k \xrightarrow{B_s} W_k .
\]

It is enough to show that \( A_s \) is injective. This is a consequence of the application of \[19\] Lemma 2.1.4] to the short exact sequence.

\[
0 \longrightarrow \Omega \xrightarrow{A} \ker B \longrightarrow \mathfrak{F} \longrightarrow 0 .
\]

Hence \( M(\mathfrak{F})_s \) is a monad, whose cohomology is isomorphic to \( \mathfrak{F}_s \); the latter sheaf is torsion free and trivial at infinity. Proposition \[4.8\] implies \((A_s, B_s) \in L_k\).

2. Equation (6.7) follows from eq. (6.6), since the condition \((\alpha(s), \beta(s)) \in L_k\) implies that \( \alpha(s)|_{T_\infty} \) is injective. By Lemma 2.1 this condition ensures that \( A_\infty \) is injective. \( \square \)

**Remark 6.7.** Suppose that \((\mathfrak{F}, \Theta)\) is a family of framed sheaves on \( T \). For any closed point \( s \in S \), let \( \theta(s) = \Theta_s \) be the restricted framing. One has
\[
\theta(s) \in \text{Iso} \left( E_{\alpha(s), \beta(s), \infty}, \mathcal{O}_{\Sigma_n}^{\oplus d} \right) .
\]

\( \triangle \)

We proceed with the construction of the morphism \( f_{(\mathfrak{F}, \Theta)} : S \rightarrow \mathcal{M}^n(r, a, c) \) by defining it first on closed points. We choose an open affine cover \( S = \bigcup_{a \in \mathcal{U}} S_a = \bigcup_{a \in \mathcal{U}} \text{Spec} \mathcal{F}_a \) where each \( S_a \)’s satisfies the conditions required of the scheme \( S \) in Corollary \[4.3\]. Moreover, if \((A, B)\) is the pair of morphisms in the monad \( M(\mathfrak{F}) \), we introduce the following notation:
\[
(A_a, B_a, \Theta_a) := (A|_{\Sigma_n \times S_a}, B|_{\Sigma_n \times S_a}, \Theta|_{T_\infty \times S_a}) .
\]

Recall that \( t_2 : T \rightarrow S \) is the projection. Note that by applying the functor \( t_2^* \) to the monad \( M(\mathfrak{F}) \) restricted to \( \Sigma_n \times S_a \) we obtain a complex of trivial sheaves on \( S_a \), so that
\[
(t_2^* A_a, t_2^* B_a) \in \mathcal{V}_k \otimes \mathcal{F}_a \simeq \mathcal{F}_a^{\oplus d} .
\]
If we define \((\alpha_a(s), \beta_a(s)) = (t_2 A_a, t_2 B_a) \otimes_{\mathcal{F}_a} k(s)\) we obtain the same morphisms as in Lemma \([6.6]\). We define a morphism \(\tilde{f}_a : S_a \to L_k\) by letting \(\tilde{f}_a(s) = (\alpha_a(s), \beta_a(s))\). We complete this to a scheme morphism by defining the ring homomorphism
\[
\tilde{f}_a^\#: \mathbb{C}[z_1, \ldots, z_d] \to \mathcal{S}_a
\]
which maps the polynomial \(g\) to \(\tilde{g}(A_a, B_a)\), where \(\tilde{g}\) is the natural extension of \(g\) to the ring \(\mathcal{S}_a[z_1, \ldots, z_d]\).

In the same way, we define \(\theta_a(s) = u_2 \Theta \otimes_{\mathcal{S}_a} k(s)\). This allows us to lift the morphisms \(\tilde{f}_a\) to morphisms \(\tilde{f}_a: \mathcal{S}_a \to \mathcal{P}_{\vec{k}}\) which we define on closed points as \(\tilde{f}_a(s) = \theta_a(s)\). Again, these extend to scheme morphisms. By composing these morphisms with the projection \(\pi: \mathcal{P}_{\vec{k}} \to M_n(r, a, c)\) we obtain morphisms \(f_a: \mathcal{S}_a \to M_n(r, a, c)\), which glue to a morphism \(f[\vec{F}, \vec{\Theta}]: \mathcal{S} \to M_n(r, a, c)\), since on overlaps the different \(\tilde{f}_a\)'s differ by the action of \(G_{\vec{k}}\).

6.1. Fineness. In this section we prove that \(M_n(r, a, c)\) represents a moduli functor, i.e., it is a fine moduli space. Let \(\mathcal{Sch}\) the category of noetherian reduced schemes of finite type over \(\mathbb{C}\) and \(\mathcal{Sets}\) the category of sets.

**Definition 6.8.** For any vector \(\vec{k}\) such that \(k_1 \geq 0\) we introduce the contravariant functor \(\mathcal{Mod}_{\vec{k}}: \mathcal{Sch} \to \mathcal{Sets}\) by the following prescriptions:

- for any object \(S \in \text{Ob}(\mathcal{Sch})\) we define the set \(\mathcal{Mod}_{\vec{k}}(S)\) as
  \[\mathcal{Mod}_{\vec{k}}(S) := \{\text{families } (\mathcal{F}, \Theta) \text{ of framed sheaves on } \Sigma_n \times S \} / \{\text{framing-preserving isomorphisms}\};\]

- for any morphism \(\varphi: S' \to S\) we define the set-theoretic map
  \[\mathcal{Mod}_{\vec{k}}(\varphi): \mathcal{Mod}_{\vec{k}}(S) \to \mathcal{Mod}_{\vec{k}}(S')\]
  \[
  \left[(\mathcal{F}, \Theta)\right] \mapsto \left[(\varphi_* \mathcal{F}, \varphi^*_\Theta)\right],
  \]
  where \(\varphi_\Sigma = \text{id}_{\Sigma}\times \varphi\) and \(\varphi_\infty = \text{id}_{\infty} \times \varphi\).

(Notations such as \(\varphi_\Sigma\) and \(\varphi_\infty\) will be used repeatedly in the following.) Observe that \(\mathcal{Mod}_{\vec{k}}(\text{Spec} \mathbb{C})\) is the set underlying \(\mathcal{M}^n(r, a, c)\). The key property of this functor is the following.

**Proposition 6.9.** The functor \(\mathcal{Mod}_{\vec{k}}(-)\) is represented by the scheme \(\mathcal{M}^n(r, a, c)\), that is, there is a natural isomorphism of functors
\[
\mathcal{Mod}_{\vec{k}}(-) \cong \text{Hom}(-, \mathcal{M}^n(r, a, c)).
\]
This implies that \(\mathcal{M}^n(r, a, c)\) is a fine moduli space of framed sheaves on \(\Sigma_n\). The pair \((\mathcal{E}_{\vec{k}}, \Theta_{\vec{k}})\) on \(\Sigma_n \times \mathcal{M}^n(r, a, c)\) is a universal family of framed sheaves on \(\Sigma_n\).
We divide the proof of this Proposition in three Lemmas.

**Lemma 6.10.** Let $S$ be any scheme.

- We define the map $\mu_S$ as
  
  $$
  \mu_S : \text{Mod}_k(S) \rightarrow \text{Hom} (S, \mathcal{M}^n(r, a, c))
  $$
  $$
  \left[\left(\mathfrak{N}, \Theta\right)\right] \mapsto f_{\left[\left(\mathfrak{N}, \Theta\right)\right]}.
  $$

- We define the map $\nu_S$ as
  
  $$
  \nu_S : \text{Hom} (S, \mathcal{M}^n(r, a, c)) \rightarrow \text{Mod}_k(S)
  $$
  $$
  f \mapsto \left[\left(\varphi^* f^* \mathcal{E}_k, \varphi^* \varphi^* \Theta\right)\right].
  $$

In this way we get natural transformations $\mu : \text{Mod}_k(-) \rightarrow \text{Hom} (-, \mathcal{M}^n(r, a, c))$ and $\nu : \text{Hom} (-, \mathcal{M}^n(r, a, c)) \rightarrow \text{Mod}_k(-)$.

**Proof.** The naturality of $\nu$ is straightforward since

$$
\left[\left(\varphi^* f^* \mathcal{E}_k, \varphi^* \varphi^* \Theta\right)\right] = \left[\left((f \circ \varphi)^* \mathcal{E}_k, (f \circ \varphi)^* \varphi^* \Theta\right)\right],
$$

whenever a composition of morphisms $S' \overset{\varphi}{\longrightarrow} S \overset{f}{\longrightarrow} \mathcal{M}^n(r, a, c)$ is given.

To prove the naturality of $\mu$ we need to show that, given any morphism $S' \overset{\varphi}{\longrightarrow} S$ and any family $(\mathfrak{N}, \Theta)$ on $T$, the following equality holds true:

$$
f_{\left[\left(\mathfrak{N}, \Theta\right)\right]} \circ \varphi = f_{\left[\left(\varphi^* \mathfrak{N}, \varphi^* \Theta\right)\right]}.
$$

(6.8)

To simplify the notation we let $f = f_{\left[\left(\mathfrak{N}, \Theta\right)\right]}$ and $f' = f_{\left[\left(\varphi^* \mathfrak{N}, \varphi^* \Theta\right)\right]}$. We can assume $S = \text{Spec} \mathcal{I}$ and $S' = \text{Spec} \mathcal{I}'$, so that $\varphi$ is induced by a ring homomorphism $\mathcal{I} \overset{\varphi}{\longrightarrow} \mathcal{I}'$.

If $(A, B)$ and $(A', B')$ are the morphisms in the monads $M(\mathfrak{N})$ and $M(\varphi^* \mathfrak{N})$, respectively, one has

$$(A', B') = (\varphi^* \otimes_d (A, B))
$$

and $\bar{f} \circ \varphi = \bar{f}' : S' \longrightarrow L_\mathfrak{N}$. We can assume that $\text{im} \bar{f}'$ is contained in an open affine subset $V = \text{Spec} \mathcal{L}$ of $L_\mathfrak{N}$ which trivializes $\mathfrak{N}_\mathfrak{N}$, and we put $N_\mathfrak{N} = \Gamma (V, \mathfrak{N}_\mathfrak{N})$ which turns out to be a free $\mathcal{L}$-module of rank $r$. We put $N = \Gamma (T_\infty, \mathfrak{N}|_{T_\infty})$ which is a free $\mathcal{I}$-module of rank $r$, and similarly we put $N' = \Gamma (T'_\infty, \varphi^* \mathfrak{N}|_{T'_\infty})$ which is a free $\mathcal{I}'$-module of the same rank; By the homomorphisms $f^*$ and $\varphi^*$ one has $N \simeq N_\mathfrak{N} \otimes_{\mathcal{I}} \mathcal{I}$ and $N' \simeq N \otimes_{\mathcal{I}} \mathcal{I}'$. Moreover one has $\varphi^* \Theta \simeq \Theta \otimes_{\mathcal{I}} 1_{\mathcal{I}'}$.

For any polynomial $h \in \text{Sym}_{\mathcal{I}} (H^*)$, where $H \simeq \text{Hom}_{\mathcal{I}} (N_\mathfrak{N}, \mathcal{L}^\oplus)$, we get

$$
ev_{\varphi^* \Theta} (h \otimes_{\mathcal{I}} 1_{\mathcal{I}'}) = ev_{\Theta \otimes_{\mathcal{I}} 1_{\mathcal{I}'}} [(h \otimes_{\mathcal{I}} 1_{\mathcal{I}'}) \otimes_{\mathcal{I}} 1_{\mathcal{I}'}] = [ev_{\Theta} (h \otimes_{\mathcal{I}} 1_{\mathcal{I}'})] \otimes_{\mathcal{I}} 1_{\mathcal{I}'}
$$
where \( \text{ev} \) is the evaluation of polynomials. Hence
\[
\tilde{f} \circ \varphi = \tilde{f}' : S' \to P_k.
\]
By applying the projection \( \pi \) to both sides of this equation, we obtain eq. (6.8). \( \square \)

**Lemma 6.11.** For any scheme \( S \) one has \( \nu_S \circ \mu_S = \text{id}_{\text{Mod}_k(S)} \).

**Proof.** We need to prove that
\[
(f_\Sigma^* \mathcal{E}_k, f_\Sigma^* \Theta_k) \simeq (\tilde{\mathcal{F}}, \Theta),
\]
for any family \((\tilde{\mathcal{F}}, \Theta)\) of framed sheaves on \( \Sigma_n \) parametrized by \( S \), where \( f = f_{((\tilde{\mathcal{F}}, \Theta))} \). It is enough to show that there is an isomorphism
\[
f_\Sigma^* M_k \simeq M(\tilde{\mathcal{F}}) \tag{6.9}
\]
and that this isomorphism is compatible with the framings. By Lemma 6.2 there are isomorphisms
\[
f_\Sigma^* \mathcal{F}_k \simeq \tilde{t}_1 \mathcal{F}_k, \quad f_\Sigma^* \mathcal{V}_k \simeq \tilde{t}_1 \mathcal{V}_k, \quad f_\Sigma^* \mathcal{W}_k \simeq \tilde{t}_1 \mathcal{W}_k,
\]
together with \((q^* A_k, q^* B_k, p^* \Theta_k) \simeq (\tilde{A}_k, \tilde{B}_k, \tilde{\Theta}_k)\). When \( S \) is affine and satisfies the conditions of Corollary 4.5, we have in addition the isomorphisms
\[
\mathcal{U} \simeq \tilde{t}_1 \mathcal{U}_k, \quad \mathcal{V} \simeq \tilde{t}_1 \mathcal{V}_k, \quad \mathcal{W} \simeq \tilde{t}_1 \mathcal{W}_k,
\]
and we have
\[
f_\Sigma^* (A_k, B_k) = \tilde{f}_* q^* (\tilde{A}_k, \tilde{B}_k) = \tilde{f}_* (\tilde{A}_k, \tilde{B}_k) = \tilde{f}_* \tilde{t}_2^* \tau^* (\text{id}_{L_k}) = \tilde{t}_2 \tilde{\tau}^* (\text{id}_{L_k}) = (A, B).
\]
where \( \tilde{f}_* = \text{id}_{\Sigma_n} \times \tilde{f} \). This proves eq. (6.9) locally, and similarly the compatibility of \( \Theta_k \) can be shown. By Corollary 5.3 we get the thesis. \( \square \)

**Lemma 6.12.** For any vector \( \vec{k} \) such that \( k_1 \geq 0 \) and for any scheme \( S \) one has
\[
\mu_S \circ \nu_S = \text{id}_{\text{Hom}(S, \mathcal{M}^n(r, a, c))}.
\]

**Proof.** Let \( g : S \to \mathcal{M}^n(r, a, c) \) be any scheme morphism. We need to show that:
\[
g = f_{[(g_\Sigma^* \mathcal{E}_k, g_\Sigma^* \Theta_k)]};
\]
for simplicity we set \( f = f_{[(g_\Sigma^* \mathcal{E}_k, g_\Sigma^* \Theta_k)]} \). Let \( M (g_\Sigma^* \mathcal{E}_k) \) be the monad
\[
0 \to \mathcal{U} \xrightarrow{A} \mathcal{V} \xrightarrow{B} \mathcal{W} \to 0
\]
canonically associated with \( g_\Sigma^* \mathcal{E}_k \). We can work locally by assuming that \( S = \text{Spec} \mathcal{F} \) satisfies the hypotheses of Corollary 4.5 for the sheaves \( \mathcal{U}, \mathcal{V}, \mathcal{W} \) and that \( \text{im} g \subseteq W \),
where $W \subseteq \mathcal{M}^n(r, a, c)$ is a trivializing open subset for the $G_k$-principal bundle $P_k$. Thus, there exists a local section $\sigma : W \to P_k$ lifting $g$ to $P_k$:

$$
\begin{array}{c}
\sigma \circ g \\
\downarrow \\
S \\
\downarrow \\
\mathcal{M}^n(r, a, c).
\end{array}
$$

Under our assumptions, the complex

$$
g^*_S \mathcal{M}_k : 0 \to g^*_S \mathcal{U}_k \to g^*_S \mathcal{A}_k \to g^*_S \mathcal{W}_k \to 0
$$

is a monad. Indeed the morphism $g^*_S \mathcal{A}_k$ is injective, as it follows from diagram (6.10) and Lemma 2.1. This monad is isomorphic to $M (g^*_S \mathcal{E}_k)$: as a matter of fact, their cohomologies are isomorphic and [26, Lemma 4.1.3] applies. Because of this isomorphism, in view of Proposition 6.1 one has

$$
(g^*_S \mathcal{E}_k)_s \simeq \mathcal{E}_{\alpha(s), \beta(s)},
$$

where $g(s) = [\theta(s)]$ and $(\alpha(s), \beta(s)) = \tau(\theta(s))$ for any closed point $s \in S$. This ends the proof. □

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