HOLOMORPHIC FUNCTIONS ON CERTAIN KÄHLER MANIFOLDS

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ABSTRACT. We investigate Liouville theorems and dimension estimates for the space of exponentially growing holomorphic functions on complete Kähler manifolds. While our work is motivated by the study of gradient Ricci solitons in the theory of Ricci flow, the most general results we prove here do not require any knowledge of curvature.

1. Introduction

On the complex Euclidean space $\mathbb{C}^n$, the classical Liouville theorem says that any bounded holomorphic function must be a constant. More generally, any holomorphic function of polynomial growth is necessarily a polynomial. In particular, this implies that the space of holomorphic functions with any fixed polynomial growth order is of finite dimension. It also shows that the ring consisting of all polynomial growth holomorphic functions is finitely generated by the coordinate functions.

On general complete Kähler manifolds, it is obviously of great interest to address these issues. Under suitable curvature assumptions, there are satisfactory results concerning finite dimensionality of the space of polynomial growth holomorphic functions. Indeed, on a complete Kähler manifold with nonnegative Ricci curvature, by the well known result of Yau [14] and the fact that holomorphic functions are harmonic, any bounded holomorphic function must be a constant. More generally, as a consequence of the results of Colding and Minicozzi [7] and P. Li [8] on polynomial growth harmonic functions, the space of polynomial growth holomorphic functions of any fixed order is necessarily of finite dimension. In fact, in a more recent work of L. Ni [13], a sharp dimension upper bound together with a rigidity result was established for such spaces by assuming instead that the bisectional curvature is nonnegative. However, the more significant question of whether the ring of polynomial growth holomorphic functions is finitely generated

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on a complete Kähler manifold with nonnegative bisectional curvature still remains unresolved. This question has been raised by Yau [15] and seems to be motivated by the uniformization conjecture [15] that such Kähler manifold is biholomorphic to \( \mathbb{C}^n \). We refer to the interesting work of Mok [10] for progress and further information.

Our purpose here is to establish some Liouville type results on Kähler manifolds without involving curvature conditions. This is largely motivated by the consideration of the so-called gradient Kähler Ricci solitons. Recall that Riemannian manifold \((M, g)\) is a gradient Ricci soliton if there exists a smooth function \( f \in C^\infty (M) \), called the potential function of the soliton, such that the following equation holds true for some constant \( \lambda \).

\[
\text{Ric} + \text{Hess} (f) = \lambda g.
\]

Here, Ric denotes the Ricci curvature of \( M \) and Hess\((f)\) the hessian of function \( f \). The soliton is called shrinking, steady or expanding, respectively, if \( \lambda > 0, \lambda = 0 \) or \( \lambda < 0 \). Ricci solitons are simply the self-similar solutions to the Ricci flows (see [6]). They may also be viewed as natural generalization of Einstein manifolds. In the case \( M \) is a Kähler manifold, Ricci solitons are called Kähler Ricci solitons. With respect to unitary frames, the defining equation for gradient Kähler Ricci soliton can then be written into

\[
R_{\alpha\overline{\beta}} + f_{\alpha\overline{\beta}} = \lambda \delta_{\alpha\overline{\beta}}
\]

\[
f_{\alpha\beta} = 0.
\]

The important fact to our consideration here is that the potential function \( f \) satisfies \( f_{\alpha\beta} = 0 \) on a gradient Kähler Ricci soliton. This condition may be rephrased (see [2]) as \( \nabla f \) being the real part of a holomorphic vector field, or as \( J(\nabla f) \) being a Killing vector field on \( M \). As we shall see later, the existence of such a function \( f \) leads to Liouville type results without involving any curvature conditions.

In the following and throughout the paper, we denote by \( r(x) \) the distance function from \( x \) to a fixed point \( p \) on manifold \( M \). If the volume \( V_p(R) \) of the geodesic ball \( B_p(R) \) in \( M \) satisfies

\[
V_p(R) \leq C e^{aR}
\]

for all \( R > 0 \), where \( C \) and \( a \) are constants, then \( M \) is said to have exponential volume growth with rate \( a \). If instead one has

\[
V_p(R) \leq C (R + 1)^m
\]

for all \( R > 0 \), where \( C \) and \( m \) are constants, then \( M \) is said to have polynomial volume growth of order \( m \).
Also, we denote by $E(d)$ the space of holomorphic functions $u$ on $M$ of exponential growth rate at most $d$, that is,

$$|u|(x) \leq c e^{dr(x)}$$

for $x \in M$.

Similarly, $P(d)$ is the space of holomorphic functions $u$ on $M$ of polynomial growth order at most $d$, i.e., for some constant $c$,

$$|u|(x) \leq c (r(x) + 1)^d$$

for $x \in M$.

**Theorem 1.** Let $M$ be a complete Kähler manifold. Assume that there exists a proper function $f$ on $M$ such that $f_{\alpha\beta} = 0$ with respect to unitary frames. Then for all $d > 0$,

(a) $\dim E(d) < \infty$ if $|\nabla f|$ is bounded and $M$ has exponential volume growth.

(b) $\dim P(d) < \infty$ if $|\nabla f|$ grows at most linearly and $M$ has polynomial volume growth.

Here, $|\nabla f|$ is said to grow at most linearly if $|\nabla f|(x) \leq b (r(x) + 1)$ on $M$ for some constant $b$. Note that part (a) of the theorem immediately implies that the space $E(d_0)$ only consists of the constant functions for some $d_0 > 0$. Indeed, one may take $d_0 = 1/k$, where $k = \dim E(1)$. To see that $\dim E(d_0) = 1$, note that $1, u, u^2, \ldots, u^k$ are linearly independent and belong to $E(1)$ for a nonconstant function $u \in E(d_0)$. This shows that $\dim E(1) \geq k + 1$, an obvious contradiction. Similarly, part (b) of the theorem implies that any bounded holomorphic function must be constant.

Let us also remark that it is possible to make the dimension estimates explicit in Theorem 1 as to establish these results we have used a direct argument. However, in both cases, the estimates depend on the local geometry of $M$ and behavior of $f$.

As mentioned above, an important class of examples to which the theorem applies is the gradient Kähler Ricci solitons. There, the potential function $f$ automatically satisfies $f_{\alpha\beta} = 0$ with respect to unitary frames.

In the case of the steady solitons, according to [11], the volume growth is subexponential, that is, it is of exponential growth with arbitrarily small rate. It is also well known that $|\nabla f|$ is bounded as $S + |\nabla f|^2$ is a constant by a result of Hamilton and the scalar curvature $S$ is nonnegative by Chen [5].

Therefore, the following corollary follows from part (a) of Theorem 1.
Corollary 2. Let \((M, g, f)\) be a gradient Kähler Ricci steady soliton with proper potential function \(f\). Then the space of exponential growth holomorphic functions satisfies \(\dim E(d) < \infty\) for all \(d > 0\). In particular, any subexponentially growing holomorphic function on \(M\) must be a constant.

The assumption that \(f\) is proper is indeed necessary in the corollary. This is because \(\mathbb{C}^n\) with \(f(x) = \langle a, x \rangle + b\) is a gradient Kähler Ricci steady soliton. Obviously, there are nontrivial polynomial growth holomorphic functions. On the other hand, the well known example of the Kähler Ricci steady soliton, the cigar soliton, given by
\[
\left(\mathbb{R}^2, \frac{dzd\bar{z}}{1 + |z|^2}, -\ln \left(1 + |z|^2\right)\right),
\]
admits nonconstant holomorphic functions \(u(z) = cz\). It can be easily checked that \(u\) grows exponentially. In this case, the potential function \(f\) is proper. The two examples show the sharpness of the corollary. In passing, we note that by \([1]\), there is no nonconstant holomorphic function with finite Dirichlet energy on any gradient steady Ricci soliton.

We would like to mention that under more stringent assumptions that the Ricci curvature is positive and the scalar curvature achieves its maximum, a gradient Kähler steady Ricci soliton is biholomorphic to \(\mathbb{C}^n\). This result was proved by Chau and Tam \([4]\) and Bryant \([1]\). The proof in \([1]\) is by constructing a global holomorphic coordinate system \(z_1, \ldots, z_n\) on \(M\) directly. These coordinate functions are shown to be of exponential growth on \(M\). It is not difficult to see from there that any holomorphic function \(u\) of exponential growth on \(M\) is necessarily a polynomial of \(z_1, \ldots, z_n\). In particular, this implies the corollary and also that the ring of exponential growth holomorphic functions on \(M\) is finitely generated.

For both shrinking and expanding gradient Ricci solitons, the gradient of the potential function \(f\) grows at most linearly. So part (b) of Theorem 1 leads directly to the following conclusion for gradient Kähler Ricci expanding solitons.

Corollary 3. Let \((M, g, f)\) be a gradient Kähler Ricci expanding soliton with proper potential function \(f\) and polynomial volume growth. Then the space of polynomial growth holomorphic functions satisfies \(\dim P(d) < \infty\) for all \(d > 0\).

In the case of shrinking gradient Ricci solitons, stronger result is available. Note that by scaling the metric, we may assume \(\lambda = \frac{1}{2}\). Now the potential function \(f\), after adding a suitable constant, satisfies
In addition, by Chen [5],

\[ S \geq 0. \]

Furthermore, Cao and Zhou [3] have shown that

\[ \frac{1}{4} (d(p, x) - c(n))^2 \leq f(x) \leq \frac{1}{4} (d(p, x) + c(n))^2 \]

for any \( x \in M \) and

\[ V_p(R) \leq c(n) R^n \]

for \( R \geq 1 \). Here, \( p \in M \) is a minimum point for \( f \) and the constant \( c(n) \) depends only on the dimension \( n \) of \( M \).

In particular, one sees that \( f \) is proper, \( |\nabla f| \) grows at most linearly and \( M \) has polynomial volume growth of order \( n \). Applying part (b) of Theorem 1, one concludes that \( P(d) \) is finite dimensional for all \( d > 0 \).

It turns out the dimension of the spaces \( P(d) \) can be estimated by a universal constant only depending on the growth order \( d \) and the dimension \( n \) of the underlying manifold.

**Theorem 4.** Let \((M^n, g, f)\) be a gradient Kähler Ricci shrinking soliton of complex dimension \( n \). Then \( \dim P(d) \leq C(n, d) \), a constant depending only on \( n \) and \( d \), for all \( d > 0 \).

It is possible to obtain the constant \( C(n, d) \) explicitly, as a polynomial of \( d \). Again, it has been shown in [14] that a holomorphic function with finite Dirichlet energy on a gradient Kähler shrinking Ricci soliton is necessarily a constant. For Theorem 4, an important example to keep in mind is the Gaussian shrinking soliton given by \( M = \mathbb{C}^n \) endowed with the Euclidean metric and \( f(z) = \frac{1}{4} |z|^2 \). Clearly, there exist holomorphic functions of polynomial growth. It will be interesting to see if the dimension of the space \( P(d) \) is actually maximized over the Gaussian shrinking soliton among all the gradient Kähler Ricci shrinking solitons. This will be an analogue to the aforementioned result of Ni [13] concerning Kähler manifolds with nonnegative bisectional curvature.

Our technique here does not seem to allow us to address the issue of whether the ring of all polynomial (or exponential) growth holomorphic functions is finitely generated. In view of Theorem 11 one can speculate that this is the case for the ring of all exponential growth holomorphic functions and for the ring of all polynomial growth holomorphic functions under the assumptions of part (a) and (b), respectively.
Finally, we also have a Liouville type result for holomorphic forms in the similar spirit of Theorem 1.

**Theorem 5.** Let $M^n$ be a complete Kähler manifold. Assume that there exists a proper function $f$ on $M$ such that $f_{\alpha\beta} = 0$ with respect to unitary frames and $|\nabla f|$ is bounded. Then for all $0 \leq p \leq n$, $\dim F(p) < \infty$, where $F(p)$ denotes the space of holomorphic $(p,0)$ forms $\omega$ on $M$ with $\int_M |\omega|^2 < \infty$.

2. PROOF OF THEOREM 1

In this section, we give proof to Theorem 1. Throughout, we assume $M$ is a complete Kähler manifold and $f$ a proper function on $M$ such that $f_{\alpha\beta} = 0$ with respect to unitary frames on $M$. Without loss of generality, we may assume $f$ is positive with minimum value $c_0$. Let us denote $D(t) := \{x \in M : f(x) \leq t\}$.

We assume everywhere that $c_0 < t < \sup_M f$ so that $D(t)$ is nonempty. Notice that $D(t)$ is compact for any such $t$ as $f$ is proper.

Let $u$ be a holomorphic function which is not identically zero on $M$. We define a sequence of functions $(u_k)_{k \geq 0}$ as follows. We set $u_0 := u$ and define inductively

$$u_{k+1} := (\nabla u_k, \nabla f) = (u_k)_{\alpha} f_{\alpha}, \quad \text{for } k \geq 0.$$  

**Lemma 6.** $u_k$ is holomorphic for any $k \geq 0$.

**Proof of Lemma 6.** We show this by induction on $k$. For $k = 0$ this is obviously true. Assuming it is true for $k \geq 0$, we prove that $u_{k+1}$ is holomorphic. Indeed, with respect to unitary frames, one has

$$(u_{k+1})_\beta = ((u_k)_\alpha f_\alpha)_\beta = (u_k)_{\alpha\beta} f_\alpha + (u_k)_\alpha f_{\alpha\beta} = 0,$$

where the last equality holds true because $u_k$ is holomorphic by the induction hypothesis and function $f$ satisfies $f_{\alpha\beta} = 0$. 

**Lemma 7.** Let $u$ be a nonzero holomorphic function on $M$. If

$$\int_{D(t)} |u_k|^2 \leq c \mu^k$$

for all $k \geq 0$, where $c$ and $\mu$ are constants independent of $k$, then for any regular value $r$ of $f$ with $r \leq t$ we have

$$\int_{\partial D(r)} |u_1|^2 |\nabla f|^{-1} \leq \mu \int_{\partial D(r)} |u_0|^2 |\nabla f|^{-1}. $$
Proof of Lemma 7. For a regular value $s$ of $f$, let us denote

$$
\rho(s) := \frac{\int_{\partial D(s)} |u_1|^2 |\nabla f|^{-1}}{\int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1}} \geq 0.
$$

Notice that $\rho(s) < \infty$. Otherwise, by the unique continuation property of holomorphic functions, it implies $u_0 = 0$ on $M$.

We now apply the following well known formula

$$
\int_{D(s)} (w \Delta v - v \Delta w) = \int_{\partial D(s)} \left( w \frac{\partial v}{\partial \nu} - v \frac{\partial w}{\partial \nu} \right)
$$

to

$$
w := u_k \quad \text{and} \quad v := u_{k+1},
$$

where $\frac{\partial}{\partial \nu} = \frac{\nabla f}{|\nabla f|}$ is the unit normal vector to $\partial D(s)$.

Observe that

$$
\frac{\partial u_k}{\partial \nu} = \frac{1}{|\nabla f|} \langle \nabla u_k, \nabla f \rangle = \frac{1}{|\nabla f|} u_{k+1}, \quad \text{and} \quad \frac{\partial u_{k+1}}{\partial \nu} = \frac{1}{|\nabla f|} \langle \nabla u_{k+1}, \nabla f \rangle = \frac{1}{|\nabla f|} u_{k+2}.
$$

Also, notice that both $u_k$ and $\overline{u_{k+1}}$ are harmonic as both are holomorphic. We deduce from (2.4) that

$$
\int_{\partial D(s)} (u_k \overline{u_{k+2}} - |u_{k+1}|^2) |\nabla f|^{-1} = 0.
$$

By the Cauchy-Schwarz inequality, we obtain

$$
\left( \int_{\partial D(s)} |u_{k+1}|^2 |\nabla f|^{-1} \right)^2 \leq \left( \int_{\partial D(s)} |u_k| |u_{k+2}| |\nabla f|^{-1} \right)^2
$$

$$
\leq \left( \int_{\partial D(s)} |u_k|^2 |\nabla f|^{-1} \right) \left( \int_{\partial D(s)} |u_{k+2}|^2 |\nabla f|^{-1} \right).
$$

We have thus proved that

$$
\left( \int_{\partial D(s)} |u_{k+1}|^2 |\nabla f|^{-1} \right)^2 \leq \left( \int_{\partial D(s)} |u_k|^2 |\nabla f|^{-1} \right) \times \left( \int_{\partial D(s)} |u_{k+2}|^2 |\nabla f|^{-1} \right)
$$

for any $k \geq 0$.

Multiplying the inequality (2.6) from $k = 0$ to $l$, we conclude
\[
\left( \int_{\partial D(s)} |u_1|^2 |\nabla f|^{-1} \right) \left( \int_{\partial D(s)} |u_1|^2 |\nabla f|^{-1} \right) \\
\leq \left( \int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1} \right) \left( \int_{\partial D(s)} |u_{l+1}|^2 |\nabla f|^{-1} \right).
\]

In view of the definition of \( \rho(s) \) in \([2,3]\), this means that
\[
\int_{\partial D(s)} |u_{l+1}|^2 |\nabla f|^{-1} \geq \rho(s) \int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1}.
\]

Iterating \([2,7]\) from \( l = 0 \) to \( k - 1 \), we arrive at
\[
\int_{\partial D(s)} |u_k|^2 |\nabla f|^{-1} \geq \rho^k(s) \int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1} \quad \text{for all } k \geq 0.
\]

We now integrate the preceding inequality from \( r - \delta \) to \( r \), where \( \delta > 0 \) is small so that any \( s \in [r - \delta, r] \) is a regular value for \( f \). The co-area formula implies that
\[
\int_{D(r) \setminus D(r - \delta)} |u_k|^2 \\
\geq \int_{r - \delta}^r \rho^k(s) \left( \int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1} \right) ds \\
\geq \left( \inf_{r - \delta \leq s \leq r} \int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1} \right) \int_{r - \delta}^r \rho^k(s) ds.
\]

Clearly,
\[
\inf_{r - \delta \leq s \leq r} \int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1} > 0
\]
as otherwise, it would mean \( u_0 = 0 \) on a level set of \( f \), which in turn implies \( u_0 = 0 \) on \( M \).

Thus, we have proved that
\[
\int_{r - \delta}^r \rho^k(s) ds \leq C_1 \int_{D(r) \setminus D(r - \delta)} |u_k|^2 \\
\leq C_1 \int_{D(t)} |u_k|^2 \\
\leq C_1 c \mu^k,
\]
where \( C_1 \) is independent of \( k \) and \( c \) is the constant in the hypothesis.
Rewriting the inequality into
\[ \int_{r-\delta}^{r} \left( \frac{\rho(s)}{\mu} \right)^k ds \leq C_2 \]
and letting \( k \to \infty \), one concludes that
\[ \rho(r) \leq \mu. \]

Hence,
\[ (2.8) \quad \int_{\partial D(r)} |u_1|^2 |\nabla f|^{-1} \leq \mu \int_{\partial D(r)} |u_0|^2 |\nabla f|^{-1}. \]
This proves the lemma. \( \square \)

In view of Lemma 7, it is important to control the growth rate of \( |u_k| \). This is done in the following lemma.

**Lemma 8.** (a) If \( |\nabla f| \leq b \) and \( V_p(R) \leq C e^{a R} \) for all \( R \geq 1 \), where \( V_p(R) \) denotes the volume of the geodesic ball \( B_p(R) \) in \( M \), then for any holomorphic function \( u \) satisfying
\[ |u|(x) \leq c e^{d r(x)}, \]
there exists a constant \( C_0 \) independent of \( k \) such that
\[ \int_{B_p(r)} |u_k|^2 \leq C_0 \left( e^{(2d+a)r} \right)^k \]
for any \( r \geq 2b \).

(b) If \( |\nabla f|(x) \leq b (r(x) + 1) \) and \( V_p(R) \leq C R^m \) for all \( R \geq 1 \), then for any holomorphic function \( u \) satisfying
\[ |u|(x) \leq c (r(x) + 1)^d, \]
we have
\[ \int_{B_p(r)} |u_k|^2 \leq C_0 \left( (b + 1)^2 2^{m+2d+2} \right)^k \]
for a constant \( C_0 > 0 \) independent of \( k \) and any \( r \geq b \).

**Proof of Lemma 8.** We prove part (a) first. Consider the cut-off function
\[ \phi(x) := \begin{cases} 
1 & \text{on } B_p(t) \\
\frac{1}{r} (t + r - d(p, x)) & \text{on } B_p(t + r) \setminus B_p(t) \\
0 & \text{on } M \setminus B_p(t + r). 
\end{cases} \]
Integrating by parts, we get
\[
\int_M |\nabla u_k|^2 \phi^2 = - \int_M (\Delta u_k) \phi^2 - 2 \int_M \langle \nabla u_k, \nabla \phi \rangle \phi u_k
\]
\[
\leq 2 \int_M |\nabla u_k| |\nabla \phi| |\phi| u_k
\]
\[
\leq \frac{1}{2} \int_M |\nabla u_k|^2 \phi^2 + 2 \int_M |u_k|^2 |\nabla \phi|^2.
\]
Consequently, this proves that
\[
\int_{B_p(t)} |\nabla u_k|^2 \leq \int_M |\nabla u_k|^2 \phi^2
\]
\[
\leq 4 \int_M |u_k|^2 |\nabla \phi|^2
\]
\[
= \frac{4}{r^2} \int_{B_p(t+r)} |u_k|^2.
\]
However,
\[
|u_{k+1}| = |\langle \nabla u_k, \nabla f \rangle|
\]
\[
\leq |\nabla u_k| |\nabla f|
\]
\[
\leq b |\nabla u_k|.
\]
Combining this with the above estimate, we conclude
\[
(2.9) \quad \int_{B_p(t)} |u_{k+1}|^2 \leq \frac{4 b^2}{r^2} \int_{B_p(t+r)} |u_k|^2.
\]
This is true for any \( t > 0 \) and for any \( k \geq 0 \). Iterating (2.9) leads to
\[
\int_{B_p(t)} |u_{k+1}|^2 \leq \frac{4 b^2}{r^2} \int_{B_p(t+r)} |u_k|^2
\]
\[
\leq \left( \frac{4 b^2}{r^2} \right)^2 \int_{B_p(t+2r)} |u_{k-1}|^2
\]
\[
\leq \left( \frac{4 b^2}{r^2} \right)^{k+1} \int_{B_p(t+(k+1)r)} |u_0|^2.
\]
Choosing now \( t = r \) in the above inequality implies
\[
(2.10) \quad \int_{B_p(r)} |u_k|^2 \leq \left( \frac{4 b^2}{r^2} \right)^k \int_{B_p((k+1)r)} |u_0|^2.
\]
By the volume growth assumption, we know that
\[
V_p((k+1)r)) \leq C e^{a(k+1)r}.
\]
In view of the growth assumption on $u$, we conclude
\[
\int_{B_p(r)} |u_k|^2 \leq \left( \frac{4b^2}{r^2} \right)^k \left( \sup_{B_p((k+1)r)} |u_0|^2 \right) V_p((k+1)r)) \\
\leq c^2 C \left( \frac{4b^2}{r^2} \right)^k e^{2d(2b+1) r}.
\]

So the conclusion of part (a) follows by noticing that $r \geq 2b$.

To prove part (b), we start from the inequality
\[
\int_{B_p(r)} |
abla u_k|^2 \leq 4 \int_{B_p(2r)} |u_k|^2.
\]

Since $|\nabla f|(x) \leq b (r(x) + 1)$ for all $x \in M$, it follows that
\[
|u_{k+1}|(x) = |\langle \nabla u_k, \nabla f \rangle|(x) \leq (b+1) r(x) |\nabla u_k|(x)
\]
for $r(x) \geq b$.

Combining this with the above estimate, for $r \geq b$, we have
\[
(2.11) \quad \int_{B_p(r)} |u_{k+1}|^2 \leq 4(b+1)^2 \int_{B_p(2r)} |u_k|^2.
\]

This is true for any $k \geq 0$. By iterating (2.11) it follows that
\[
(2.12) \quad \int_{B_p(r)} |u_k|^2 \leq (4(b+1)^2)^k \int_{B_p(2^kr)} |u_0|^2.
\]

Since
\[
V_p(R) \leq C R^m \quad \text{for all } R \geq 1
\]
and
\[
|u|(x) \leq c (r(x) + 1)^d,
\]
it follows that
\[
\int_{B_p(r)} |u_k|^2 \leq (4(b+1)^2)^k \left( \sup_{B_p(2^kr)} |u_0|^2 \right) V_p(2^kr) \\
\leq c^2 C \left( 4(b+1)^2 2^{m+2d} \right)^k r^{m+2d}.
\]

We have thus proved that
\[
(2.13) \quad \int_{B_p(r)} |u_k|^2 \leq C_0 \mu^k
\]
for all $k \geq 0$ and $r \geq b$, where $\mu := (b+1)^2 2^{m+2d+2}$. This proves the lemma.
We are now in position to prove Theorem 1. For the sake of convenience, we state it again here.

**Theorem 9.** Let $M$ be a complete Kähler manifold. Assume that there exists a proper function $f$ on $M$ such that $f_{\alpha\beta} = 0$ with respect to unitary frames. Then for all $d > 0$,

(a) $\dim E(d) < \infty$ if $|\nabla f|$ is bounded and $M$ has exponential volume growth.

(b) $\dim P(d) < \infty$ if $|\nabla f|$ grows at most linearly and $M$ has polynomial volume growth.

**Proof of Theorem 9.** Fix a regular value $t_0$ of $f$ and choose $r_0 > 0$ so that $D(t_0) \subset B_p(r_0)$. By Lemma 8, for $u \in E(d)$ under the assumptions of (a) or for $u \in P(d)$ under the assumptions of (b), we have

$$\int_{D(t_0)} |u_k|^2 \leq \int_{B_p(r_0)} |u_k|^2 \leq C_0 \mu^k$$

for some constant $\mu$ independent of $k$. So by Lemma 7

$$\int_{\partial D(t)} |u_1|^2 |\nabla f|^{-1} \leq \mu \int_{\partial D(t)} |u_0|^2 |\nabla f|^{-1}$$

for any regular value $t$ of $f$ such that $c_0 \leq t \leq t_0$. Since $u_1$ is zero whenever $\nabla f = 0$, it is easy to see, by the co-area formula, that

$$\int_{D(t_0)} |u_1|^2 \leq \mu \int_{D(t_0)} |u_0|^2.$$

Since

$$\int_{D(t_0)} \langle \nabla |u_0|^2, \nabla f \rangle = \int_{D(t_0)} u_0 \bar{u}_1 + \bar{u}_0 u_1$$

$$\leq 2 \int_{D(t_0)} |u_0||u_1|$$

$$\leq \int_{D(t_0)} |u_0|^2 + \int_{D(t_0)} |u_1|^2,$$

we conclude

$$\int_{D(t_0)} \langle \nabla |u_0|^2, \nabla f \rangle \leq (1 + \mu) \int_{D(t_0)} |u_0|^2.$$  

Plugging this into the following integration by parts formula

$$\int_{D(t_0)} \langle \nabla |u_0|^2, \nabla f \rangle = -\int_{D(t_0)} |u_0|^2 \Delta f + \int_{\partial D(t_0)} |u_0|^2 \frac{\partial f}{\partial \nu}$$
and noting that $\frac{\partial f}{\partial \nu} = |\nabla f|$, we arrive at
\begin{equation}
(2.16) \quad \int_{\partial D(t_0)} |u_0|^2 |\nabla f| \leq \left( \sup_{D(t_0)} |\Delta f| + 1 + \mu \right) \int_{D(t_0)} |u_0|^2.
\end{equation}
Since $t_0$ is a regular value of $f$,
\begin{equation*}
C_1(t_0) := \inf_{\partial D(t_0)} |\nabla f| > 0.
\end{equation*}
Therefore, we may rewrite (2.16) into
\begin{equation}
\int_{\partial D(t_0)} |u_0|^2 |\nabla f|^{-1} \leq C_2(t_0) \int_{D(t_0)} |u_0|^2,
\end{equation}
where $C_2(t_0)$ is a constant depending on $\mu$, $\sup_{D(t_0)} |\Delta f|$ and $C_1(t_0)$.
For the regular value $t_0$ of $f$, we let $\varepsilon > 0$ be sufficiently small so that any $t$ with $t_0 - \varepsilon \leq t \leq t_0$ is a regular value of $f$ as well. Such $\varepsilon$ depends on $\sup_{D(t_0)} |\operatorname{Hess}(f)|$. The preceding argument also implies
\begin{equation}
(2.17) \quad \int_{\partial D(t)} |u_0|^2 |\nabla f|^{-1} \leq C(t_0) \int_{D(t)} |u_0|^2
\end{equation}
with the constant $C(t_0)$ now depending on $\mu$, $\sup_{D(t_0)} |\Delta f|$ and $C_3(t_0)$, where
\begin{equation*}
C_3(t_0) := \inf_{D(t_0) \setminus D(t_0 - \varepsilon)} |\nabla f| > 0.
\end{equation*}
Integrating (2.17) from $t := t_0 - \varepsilon$ to $t := t_0$ implies
\begin{equation}
(2.18) \quad \int_{D(t_0)} |u_0|^2 \leq e^{\varepsilon C(t_0)} \int_{D(t_0 - \varepsilon)} |u_0|^2.
\end{equation}
The inequality (2.18) is true for any $u = u_0 \in E(d)$ in case of (a) and $u = u_0 \in P(d)$ in case of (b). It is well known that this implies a dimension estimate as claimed in the theorem. We will follow [9] to supply some details here. Denote by $H$ to be $E(d)$ in case of (a) and $P(d)$ in case of (b). By a result of P. Li (see [9]), there exists $u_0 \in H$ so that
\begin{equation}
(2.19) \quad \int_{D(t_0 - \varepsilon)} |u_0|^2 \leq \frac{n V(D(t_0 - \varepsilon))}{\dim H} \sup_{D(t_0 - \varepsilon)} |u_0|^2.
\end{equation}
On the other hand, applying the Moser iteration argument to the subharmonic function $|u_0|$, one obtains that
\begin{equation}
(2.20) \quad \sup_{D(t_0 - \varepsilon)} |u_0|^2 \leq \frac{C}{V(D(t_0))} \int_{D(t_0)} |u_0|^2.
\end{equation}
for some constant $C$ depending on $D(t_0)$. By combining these two inequalities, it follows that
\[
\int_{D(t_0-\epsilon)} |u_0|^2 \leq \frac{C}{\dim H} \int_{D(t_0)} |u_0|^2.
\]
In view of (2.18), one concludes $\dim H \leq C$. The theorem is proved. □

3. Shrinking solitons

In this section we prove Theorem 4 which is restated below.

Theorem 10. Let $(M^n, g, f)$ be a gradient Kähler Ricci shrinking soliton of complex dimension $n$. Then $\dim P(d) \leq C(n, d)$, a constant depending only on $n$ and $d$, for all $d > 0$.

Proof of Theorem 10. Recall that on a shrinking soliton, by normalizing the metric so that $\lambda = \frac{1}{2}$ and adding a suitable constant to the potential function $f$, one has
\[
S + |\nabla f|^2 = f \quad \text{and} \quad S + \Delta f = \frac{n}{2}.
\]
Note that by Chen [5]
\[
S \geq 0.
\]
So we have
\[
|\nabla f|^2 \leq f.
\]
Also, by Cao and Zhou [3],
\[
\frac{1}{4} (d(p, x) - c(n))^2 \leq f(x) \leq \frac{1}{4} (d(p, x) + c(n))^2
\]
and
\[
V_p(R) \leq c(n) R^n
\]
for $R \geq 1$, where $p \in M$ is a minimum point for $f$ and the constant $c(n)$ depends only on the dimension $n$ of $M$.

In particular, one concludes that $f$ is proper on $M$ and
\[
|\nabla f|(x) \leq \frac{1}{2} r(x) + c(n).
\]
So by Lemma 8 for $u \in P(d)$ and $r \geq c(n)$,
\[
\int_{B_p(r)} |u_k|^2 \leq C \mu^k
\]
for all $k \geq 0$, where $\mu = c(n, d)$. 

Now by Lemma 7 we have

\[ \int_{\partial D(s)} |u_1|^2 |\nabla f|^{-1} \leq \mu \int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1}, \]

for any regular value \( s \) of \( f \). Using (3.4) and the co-area formula, we obtain

\[
\int_{D(t)} |u_1|^2 e^{-f} = \int_0^t e^{-s} \left( \int_{\partial D(s)} |u_1|^2 |\nabla f|^{-1} \right) ds
\leq \mu \int_0^t e^{-s} \left( \int_{\partial D(s)} |u_0|^2 |\nabla f|^{-1} \right) ds
= \mu \int_{D(t)} |u_0|^2 e^{-f}.
\]

Hence, we have established that

\[ \int_{D(t)} |u_1|^2 e^{-f} \leq \mu \int_{D(t)} |u_0|^2 e^{-f} \]

for all \( t \). Using (3.1) we have that

\[ \int_{D(t)} |u_0|^2 \left( f - \frac{n}{2} - 1 - \mu \right) e^{-f} \leq 0 \]

for all \( t \). By (3.6), this means that

\[ \int_{D(t)} |u_0|^2 \left( f - \frac{n}{2} - 1 - \mu \right) e^{-f} \leq 0 \] for all \( t \).
Since the constant $\mu = c(n, d)$, by choosing sufficiently large $t = t_0$ depending only on $n$ and $d$ it follows that

$$\int_{D(5t_0)} |u_0|^2 \leq K \int_{D(t_0)} |u_0|^2,$$

where $K$ is a constant depending only on $n$ and $d$. In view of (3.2), $t_0$ can be chosen in such a way that

$$D(t_0) \subset B_p(3r_0)$$

and

$$B_p(4r_0) \subset D(5t_0),$$

where $r_0 = \sqrt{t_0}$. Hence, we conclude

$$\int_{B_p(4r_0)} |u_0|^2 \leq K \int_{B_p(3r_0)} |u_0|^2$$

for some $r_0$ depending only on $n$ and $d$.

Notice that (3.8) is true for any $u_0 \in P(d)$.

This implies that $\dim P(d) \leq C(n, d)$. Indeed, by a result of P. Li (see [9]), there exists nontrivial $u_0 \in P(d)$ so that

$$\int_{B_p(3r_0)} |u_0|^2 \leq \frac{n V_p(3r_0)}{\dim P(d)} \sup_{B_p(3r_0)} |u_0|^2.$$

On the other hand, the Sobolev constant of $B_p(4r_0)$ can be controlled by a constant depending only on $n$ and $r_0$ (see [12]). Therefore, applying the Moser iteration argument to the subharmonic function $|u_0|$, we obtain that

$$\sup_{B_p(3r_0)} |u_0|^2 \leq \frac{C(n, d)}{V_p(4r_0)} \int_{B_p(4r_0)} |u_0|^2.$$

By combining (3.9) and (3.10), it follows that

$$\int_{B_p(3r_0)} |u_0|^2 \leq \frac{C(n, d)}{\dim P(d)} \int_{B_p(4r_0)} |u_0|^2.$$

In view of (3.8), we have $\dim P(d) \leq C(n, d)$. This proves the theorem.  

\[\square\]

4. Holomorphic forms

In this section, we will deal with the space of holomorphic forms and prove Theorem 5 which is restated below.
Theorem 11. Let $(M^n, g)$ be a complete Kähler manifold. Assume that there exists a proper smooth function $f$ on $M$ with bounded gradient and $f_{\alpha \beta} = 0$ in unitary frames. Then the dimension of the space of $L^2$ holomorphic $(p, 0)$ forms is finite for $0 \leq p \leq n$.

Proof of Theorem 11. We use induction on $p$. For $p = 0$, the statement is clear as any $L^2$ holomorphic function must be a constant (see [16]).

We now assume that the result is true for all $0 \leq p \leq q - 1$ and prove it for $p = q$. Let $F(p)$ denote the vector space of holomorphic $(p, 0)$ forms in $L^2(M)$. For any $(q, 0)$ form

$$\omega := \frac{1}{q!} \omega_{i_1 \ldots i_q} dz^{i_1} \wedge \ldots \wedge dz^{i_q},$$

we associate a $(q - 1, 0)$ form $\Theta^\omega$ by contracting it with $\nabla f$. So

$$\Theta^\omega := \omega (\ldots, \cdot, \nabla f)$$

$$\Theta^\omega = \frac{1}{(q - 1)!} \left( \omega_{i_1 \ldots i_q} f_{\bar{\alpha}} \right) dz^{i_1} \wedge \ldots \wedge dz^{i_{q-1}}.$$

We notice that $\Theta^\omega$ is a holomorphic form. Indeed,

$$\nabla_\alpha \Theta^\omega_{i_1 \ldots i_{q-1}} = \nabla_\alpha \left( \omega_{i_1 \ldots i_q} f_{\bar{\alpha}} \right) = 0$$

by using that $\omega$ is holomorphic and $f_{\alpha \beta} = 0$. Moreover, $\Theta^\omega$ is in $L^2(M)$ as $|\nabla f|$ is bounded on $M$. By the induction hypothesis, we know that the vector space

$$\{ \Theta^\omega : \omega \text{ is } L^2 \text{ holomorphic } (p, 0) \text{ form } \} \subset F(q - 1)$$

is finite dimensional. Therefore, to finish the proof it suffices to show that the space

$$F := \{ \omega \in F(q) : \omega (\ldots, \cdot, \nabla f) = 0 \} \subset F(q)$$

is finite dimensional as well. For this, we consider $\omega \in F$. First, observe that $\omega$ is closed as it is harmonic and in $L^2$. It follows that

$$\langle \nabla |\omega|^2, \nabla f \rangle = \frac{1}{q!} \langle \nabla |\omega_{i_1 \ldots i_q}|^2, \nabla f \rangle$$

$$= \frac{2}{q!} \text{Re} \left\{ (\nabla_\alpha \omega_{i_1 \ldots i_q}) f^{\bar{\alpha}} \omega_{\bar{i}_1 \ldots \bar{i}_{q-1}} \right\}$$

$$= \frac{2}{q!} \text{Re} \left\{ \left( \sum_k \varepsilon(k) \nabla_i \omega_{i_1 \ldots \alpha \ldots i_q} \right) f^{\bar{\alpha}} \omega_{\bar{i}_1 \ldots \bar{i}_{q-1}} \right\}$$

for some $\varepsilon(k) \in \{-1, 1\}$. On the other hand, note that

$$\left( \nabla_i \omega_{i_1 \ldots \alpha \ldots i_q} \right) f^{\bar{\alpha}} = \nabla_i \left( \omega_{i_1 \ldots \alpha \ldots i_q} f^{\bar{\alpha}} \right) - \omega_{i_1 \ldots \alpha \ldots i_q} f_{i k \bar{\alpha}}$$

$$= -\omega_{i_1 \ldots \alpha \ldots i_q f_{ik \bar{\alpha}}}.$$
Therefore, from (4.1), one concludes
\[
\left| \langle \nabla |\omega|^2, \nabla f \rangle \right| \leq C |\omega|^2
\]
for some constant $C$ depending on the Hessian of $f$ on $D(r)$.

Plugging this into the following equation
\[
(4.2) \quad \int_{D(r)} \langle \nabla |\omega|^2, \nabla f \rangle = -\int_{D(r)} |\omega|^2 \Delta f + \int_{\partial D(r)} |\omega|^2 |\nabla f|,
\]
where $r$ is a regular value of $f$, we have
\[
(4.3) \quad \int_{\partial D(r)} |\omega|^2 |\nabla f|^{-1} \leq C \int_{D(r)} |\omega|^2
\]
for a constant $C$ depending on the Hessian of $f$ on $D(r)$ and on a lower bound of $|\nabla f|$ on the set $\partial D(r) = \{ f = r \}$.

For a fixed regular value $r_0$ of $f$, choose $\varepsilon > 0$ so that for $r$ is also a regular value for $r_0 - \varepsilon \leq r \leq r_0$. Integrating (4.3) from $r_0 - \varepsilon$ to $r_0$, we obtain that
\[
(4.4) \quad \int_{D(r_0)} |\omega|^2 \leq C \int_{D(r_0 - \varepsilon)} |\omega|^2
\]
for any $\omega \in F$.

Notice that by the Bochner formula,
\[
\Delta |\omega| \geq -c |\omega|
\]
on $D(r_0)$, where $c$ depends on the curvature bounds of $D(r_0)$. So a mean value inequality of the following form holds for the function $|\omega|$.
\[
(4.5) \quad \sup_{D(r_0 - \varepsilon)} |\omega|^2 \leq \frac{C}{V(D(r_0))} \int_{D(r_0)} |\omega|^2
\]
Together with (4.4), this implies that $F$ is finite dimensional as indicated in the proof of Theorem 1. The dimension of $F$ depends on $q$, $r_0$, the bounds of $f$ on $D(r_0)$ and also on a curvature bound on $D(r_0)$. This proves the theorem.

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