Gell-Mann–Okubo Mass Formula Revisited

L. Burakovsky* and T. Goldman†

Theoretical Division, MS B285
Los Alamos National Laboratory
Los Alamos, NM 87545, USA

Abstract

We show that the application of Regge phenomenology to SU(4) meson multiplets leads to a new Gell-Mann–Okubo mass-mixing angle formula in the SU(3) sector, $3m_1^2 + \cos^2 \theta m_0^2 + \sin^2 \theta m_0'^2 + \sqrt{2} \sin 2\theta (m_0^2 - m_0'^2) = 4m_{1/2}^2$, where $m_1, m_{1/2}, m_0, m_0'$ are the masses of the isovector, isodoublet, isoscalar mostly octet and isoscalar mostly singlet states, respectively, and $\theta$ is the nonet mixing angle. For an ideally mixed nonet, $\theta = \arctan 1/\sqrt{2}$, this formula reduces to $2m_0^2 + 3m_1^2 = 4m_{1/2}^2 + m_0'^2$ which holds with an accuracy of $\sim 1\%$ for vector and tensor mesons. For pseudoscalar mesons, with the $\eta-\eta'$ mixing angle $-\arctan 1/(2\sqrt{2}) \simeq -19.5^\circ$, in agreement with experiment, it leads to the relation $4m_K^2 = 3m_\pi^2 + m_{\eta'}^2$ which holds to an accuracy of better than 1% for the measured pseudoscalar meson masses.

Key words: mesons, Gell-Mann–Okubo, mass relations, Regge phenomenology
PACS: 12.40.Nn, 12.40.Yx, 12.90.+b, 14.40.Aq, 14.40.Cs

The original Gell-Mann–Okubo (GMO) mass formula

$$m_1^2 + 3m_8^2 = 4m_{1/2}^2$$ (1)

relates the masses of the isovector ($m_1$), isodoublet ($m_{1/2}$) and isoscalar octet ($m_8$) states of a meson octet. It is usually cast into a form which relates $m_1$ and $m_{1/2}$ with the masses of the two physical isoscalar states, $m_0'$ and $m_0$, which have $n\bar{n}$ and $s\bar{s}$
quark content, respectively (\(n\) stands for non-strange \(u\)- and \(d\)-quark), assuming the ideal structure of a nonet:

\[
\frac{m_1^2 + m_0^2}{2} + m_0^2 = 2m_{1/2}^2. \tag{2}
\]

Indeed, in general the isoscalar octet (\(\omega_8\)) and singlet (\(\omega_9\)) states get mixed, because of SU(3) breaking, which results in the physical \(\omega_0\) and \(\omega_0'\) states (the \(\omega_0\) is a mostly octet isoscalar):

\[
\omega_0 = \omega_8 \cos \theta - \omega_9 \sin \theta, \\
\omega_0' = \omega_8 \sin \theta + \omega_9 \cos \theta,
\]

where \(\theta\) is the mixing angle. Assuming, as usual, that the relevant matrix elements are equal to the squared masses of the corresponding states, one obtains from the above relations \[2\]

\[
m_0^2 = m_8^2 \cos^2 \theta + m_9^2 \sin^2 \theta - 2m_8^2 \sin \theta \cos \theta, \tag{3}
\]

\[
m_0'^2 = m_8^2 \sin^2 \theta + m_9^2 \cos^2 \theta + 2m_8^2 \sin \theta \cos \theta. \tag{4}
\]

Since \(\omega_0\) and \(\omega_0'\) as physical states are orthogonal, one has further

\[
m_{00'} = 0 = (m_8^2 - m_9^2) \sin \theta \cos \theta + m_{89}(\cos^2 \theta - \sin^2 \theta). \tag{5}
\]

Eliminating \(m_9\) and \(m_{89}\) from (3)-(5) yields

\[
\tan^2 \theta = \frac{m_8^2 - m_0^2}{m_0'^2 - m_8^2}. \tag{6}
\]

It also follows from (3)-(5) that \(m_8^2 = m_0^2 \cos^2 \theta + m_0'^2 \sin^2 \theta\), and therefore, Eq. (1) may be rewritten as

\[
4m_{1/2}^2 - m_1^2 - 3m_0^2 = 3 \left( m_0'^2 - m_0^2 \right) \sin^2 \theta, \tag{7}
\]

which is the Sakurai mass formula \[3\]. For the ideal octet-singlet mixing, \(\tan \theta_{id} = 1/\sqrt{2}\), for which \(\omega_0 = s\bar{s}, \omega_0' = (u\bar{u} + d\bar{d})/\sqrt{2}\), Eq. (7) reduces to (2), the formula for the ideal structure of a nonet.

The formula (2) is known to hold with a high accuracy for all well established meson multiplets except for the pseudoscalar one. It is widely believed that the reason for the invalidity of Eq. (2) for pseudoscalar mesons is a large dynamical mass of the isoscalar singlet state developed (before its mixing with the isoscalar octet which results in the physical \(\eta\) and \(\eta'\) states) due to axial \(U(1)\) symmetry breakdown. In fact, the observed mass splitting among the pseudoscalar nonet may be induced by the following symmetry breaking terms \[2\],

\[
L_m^{(0)} = \frac{f^2}{4} \left( r \text{Tr} M (U + U^+) + \frac{\alpha}{4N} \left[ \text{Tr} \left( \ln U - \ln U^+ \right) \right]^2 \right), \quad r = \text{const}, \tag{8}
\]
with $M$ being the quark mass matrix,

$$M = \text{diag} \left( m_u, m_d, m_s \right) = m_s \text{diag} \left( x, y, 1 \right), \quad x \equiv \frac{m_u}{m_s}, \quad y \equiv \frac{m_d}{m_s},$$  \hspace{1cm} (9)

in addition to the $\text{U}(3)_L \times \text{U}(3)_R$ invariant non-linear Lagrangian

$$L^{(0)} = \frac{f^2}{4} \text{Tr} \left( \partial_\mu U \partial^\mu U \right),$$  \hspace{1cm} (10)

with

$$U = \exp \left( i \pi / f \right), \quad \pi \equiv \lambda_a \pi^a, \quad a = 0, 1, \ldots, 8,$$

which incorporates the constraints of current algebra for the light pseudoscalars $\pi^a$ \cite{4}. As pointed out in ref. \cite{5}, chiral corrections can be important, the kaon mass being half the typical 1 GeV chiral symmetry breaking scale. Such large corrections are precisely required from the study of the octet-singlet mass squared matrix $M^2$.

In the isospin limit $x = y$ one has

$$M^2 = \frac{1}{3} \begin{pmatrix} 4m_K^2 - m_\pi^2 & -2\sqrt{2}(m_K^2 - m_\pi^2) \\ -2\sqrt{2}(m_K^2 - m_\pi^2) & 2m_K^2 + m_\pi^2 + 3\alpha \end{pmatrix}.$$  \hspace{1cm} (11)

The $\eta-\eta'$ mixing angle $\theta_{\eta\eta'}$ then reads

$$\tan 2\theta_{\eta\eta'} = \frac{2m_{89}^2}{m_9^2 - m_8^2} = 2\sqrt{2} \left( 1 - \frac{3\alpha}{2(m_K^2 - m_\pi^2)} \right)^{-1}.$$  \hspace{1cm} (12)

For $\alpha = 0$, one obtains the ideal mixing and the mass relation $m_{\eta'} = m_\pi$ as the source of the $\text{U}(1)$ problem \cite{7}. The parameter $\alpha$ is assumed to be induced by instantons \cite{6,8} and is determined by the trace condition $\alpha = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 \simeq 0.725 \text{ GeV}^2$; one then obtains from (12) the mixing angle

$$\theta_{\eta\eta'} \simeq -18.5^o,$$  \hspace{1cm} (13)

in agreement with most of experimental data \cite{9}. A more popular way to extract the $\eta-\eta'$ mixing angle, through the relation (6) based on GMO (1) \cite{10}, leads to

$$\theta_{\eta\eta'} \simeq -10.5^o,$$  \hspace{1cm}

in disagreement with experiment \cite{9}. On the other hand, in the octet approximation $m_8 \approx m_\eta$, GMO (1) is thought to be quite successful since it predicts

$$m_8 \approx 567 \text{ MeV},$$

which is within the physical $\eta$ mass of 547.5 MeV with an accuracy of $\sim 3.5\%$. Thus, GMO for pseudoscalar mesons is believed to be

$$4m_K^2 = 3m_\eta^2 + m_\pi^2,$$  \hspace{1cm} (14)
and although it does not reproduce the $\eta$-$\eta'$ mixing angle correctly, it gives the mass of the $\eta$ with a rather high accuracy.

In fact, the octet approximation and the corresponding relation (14) mean

$$\eta \approx \eta_8 = \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}},$$

consistent with the $(1/3 \ n, \ 2/3 \ s)$ quark content of the $\eta$, in agreement with the Gell-Mann–Oakes-Renner relations (to first order in chiral symmetry breaking) \[11\]

$$m^2_\pi = 2B m_n,$$
$$m^2_K = B (m_s + m_n),$$
$$m^2_\eta = \frac{2}{3} B (2m_s + m_n),$$

(15)

$$m_n \equiv \frac{m_u + m_d}{2}; \ B = \text{const};$$

however, the actual quark content of the $\eta$, due to the $\eta_8$-$\eta_9$ mixing angle $\simeq -19^\circ$, is \[12\]

$$\eta \simeq 0.58 (u\bar{u} + d\bar{d}) - 0.57 s\bar{s} \approx \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{3}},$$

(16)

i.e., $(2/3 \ n, \ 1/3 \ s)$, quite different from that provided by (14). Thus, a natural suspicion is that $m_8 \approx m_\eta$ is purely numerical coincidence, and the actual Gell-Mann–Okubo relation should be different from Eq. (14). Moreover, since the $(1/3 \ n, \ 2/3 \ s)$ quark content corresponds to the $\eta'$ meson, in view of \[12\]

$$\eta' \simeq 0.40 (u\bar{u} + d\bar{d}) + 0.82 s\bar{s} \approx \frac{u\bar{u} + d\bar{d} + 2s\bar{s}}{\sqrt{6}},$$

(17)

one may also suspect that the true mass formula should relate the masses of the $\pi$, $K$ and $\eta'$.

In this paper we derive such a formula which we call the Gell-Mann–Okubo mass formula revisited (GMO$_r$). We shall use Regge phenomenology which proved to be quite successful in producing hadronic (both mesonic \[13, 14\] and baryonic \[15\]) mass relations in both the light and heavy quark sectors.

As discussed in detail in our previous papers \[13, 14, 15\], Regge phenomenology for mesons is based on the following two relations among the masses and Regge slopes of the states which belong to a given meson multiplet:

$$\alpha'_{ii} m^2_{ii} + \alpha'_{jj} m^2_{jj} = 2\alpha'_{ji} m^2_{ji},$$

(18)

$$\frac{1}{\alpha'_{ii}} + \frac{1}{\alpha'_{jj}} = \frac{2}{\alpha'_{ji}},$$

(19)

In the light quark sector, $i = n$ ($= u, d$), $j = s$, one has $\alpha'_{ss} \approx \alpha'_{sn} \approx \alpha'_{nn}$; with the definition

$$m^2_{nn} \equiv m^2_{nn}(I = 1) + m^2_{nn}(I = 0) \frac{2}{2},$$

(20)
(I stands for isospin), Eq. (18) then reduces to the formula (2).

We shall also use the following mass relations among vector and pseudoscalar meson mass squared:

\[
m_\rho^2 - m_\pi^2 \approx m_{K^*}^2 - m_K^2
\]

\[
\approx m_{D^*}^2 - m_D^2 \approx m_{D^*}^2 - m_{D^*}^2 \approx 0.57 \text{ GeV}^2.
\]

(21)

This relation is easily obtained in the constituent quark model [16] and an algebraic approach to QCD [17]. They lead, e.g., to

\[
m_\phi^2 - \frac{m_\rho^2 + m_\omega^2}{2} = 2 \left( m_{K^*}^2 - \frac{m_\rho^2 + m_\omega^2}{2} \right) \approx 2 \left( m_K^2 - m_\pi^2 \right),
\]

(22)

in view of (2) and \( m_\omega \approx m_\rho \).

It is easily seen that the relations (18),(19) may be applied only to pure \( q\bar{q} \) states, and neither \( \eta \) nor \( \eta' \) is such a state. Therefore, in order to apply Eqs. (18),(19) to pseudoscalar mesons, one has first to construct the proper states \( \eta_n \) and \( \eta_s \) (as linear combinations of the physical \( \eta \) and \( \eta' \)),

\[
\eta_n = \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}, \quad \eta_s = s\bar{s},
\]

(23)

which have the masses \( m_{\eta_n} \) and \( m_{\eta_s} \), respectively, which we determine later on. For these states, we apply Eqs. (18),(19) with \( (i, j) = (n, c) \) and \( (i, j) = (s, c) \) using the experimental fact that the slope of the \( c\bar{c} \)-trajectory is less than that of the \( n\bar{n} \)- and \( s\bar{s} \)-trajectories. One has, in agreement with (19),

\[
\alpha'_{c\bar{s}} \approx \alpha'_{c\bar{n}} = \frac{\alpha'_{c\bar{n}}}{1 + x} \approx \frac{\alpha'_{s\bar{s}}}{1 + x}, \quad \alpha'_{c\bar{c}} = \frac{\alpha'_{c\bar{n}}}{1 + 2x} \approx \frac{\alpha'_{s\bar{s}}}{1 + 2x}, \quad x > 0.
\]

(24)

Therefore,

\[
m_{\eta_n}^2 + \frac{m_{\phi}^2 + m_{\omega}^2}{2} = 2 \frac{m_{c\bar{n}}^2}{1 + x},
\]

(25)

\[
m_{\eta_s}^2 + \frac{m_{\phi}^2 + m_{\omega}^2}{2} = 2 \frac{m_{c\bar{s}}^2}{1 + x},
\]

(26)

with \( m_{n\bar{n}}^2 \) defined in (20), in agreement with Eq. (2) in the light quark sector.

For pseudoscalar and vector mesons, Eqs. (25),(26) may be rewritten as

\[
m_{\eta_n}^2 + \frac{m_{\phi}^2 + m_{\omega}^2}{2} = 2 \frac{m_{D}^2}{1 + x},
\]

\[
m_{\eta_s}^2 + \frac{m_{\phi}^2 + m_{\omega}^2}{2} = 2 \frac{m_{D^*}^2}{1 + x},
\]

\[
m_{\rho}^2 + \frac{m_{J/\psi}^2}{1 + 2x} = 2 \frac{m_{D^*}^2}{1 + x},
\]

\[
m_{\phi}^2 + \frac{m_{J/\psi}^2}{1 + 2x} = 2 \frac{m_{D^*}^2}{1 + x},
\]

(27)
where
\[ m_{\eta_n}^2 \equiv \frac{m_\pi^2 + m_{\eta_n}^2}{2}, \quad m_\rho^2 \equiv \frac{m_\pi^2 + m_\omega^2}{2}. \] (28)

It follows from (21),(22),(27) that
\[ m_{\eta_n}^2 - m_{\eta_n}^2 \approx \frac{2}{1 + x} \left( m_{D_s}^2 - m_D^2 \right) \approx \frac{2}{1 + x} \left( m_K^2 - m_\pi^2 \right) \]
\[ = m_\phi^2 - m_\rho^2 \approx 2 \left( m_K^2 - m_\pi^2 \right). \] (29)

One then obtains, from (28),(29),
\[ 2m_{\eta_n}^2 - m_{\eta_n}^2 \approx 4m_K^2 - 3m_\pi^2, \] (30)
which is a new Gell-Mann–Okubo (GMO) mass formula for pseudoscalar mesons, which however cannot be applied to them directly since \( m_{\eta_n} \) and \( m_{\eta_n} \) are not known.

Therefore, the last step in derivation of the analog of Eq. (30) applicable to pseudoscalar states is to determine the values of \( m_{\eta_n} \) and \( m_{\eta_n} \), in terms of the physical \( m_{\eta} \) and \( m_{\eta'} \), in order to use them in Eq. (30).

We assume the \( \eta - \eta' \) mixing angle to take the “ideal” value
\[ \theta_{\eta\eta'} = \arctan \frac{1}{2\sqrt{2}} \approx -19.5^\circ, \] (31)
in agreement with experimental data \[9\]. This value was first predicted by Bramon \[18\] from a simple quark model and duality constraints for the set of scattering processes \( \pi\eta \rightarrow \pi\eta, \pi\eta \rightarrow \pi\eta', \pi\eta' \rightarrow \pi\eta', \eta K \rightarrow (\pi, \eta, \eta') K, \) and \( \eta\eta \rightarrow \eta\eta, \eta\eta \rightarrow \eta\eta' \). Since the ideal mixing of a nonet corresponds to \( \theta_{id} = \arctan 1/\sqrt{2} \approx 35.3^\circ \), one has from (31)
\[ 2\theta_{id} - \theta_{\eta\eta'} = \frac{\pi}{2}. \] (32)

In view of (32) and
\[ \left( \begin{array}{c} \eta \\ \eta' \end{array} \right) = \left( \begin{array}{cc} \cos \theta_{\eta\eta'} & -\sin \theta_{\eta\eta'} \\ \sin \theta_{\eta\eta'} & \cos \theta_{\eta\eta'} \end{array} \right) \left( \begin{array}{c} \eta_8 \\ \eta_9 \end{array} \right), \]
\[ \left( \begin{array}{c} \eta_s \\ \eta_n \end{array} \right) = \left( \begin{array}{cc} \cos \theta_{id} & -\sin \theta_{id} \\ \sin \theta_{id} & \cos \theta_{id} \end{array} \right) \left( \begin{array}{c} \eta_8 \\ \eta_9 \end{array} \right), \]
one has
\[ \left( \begin{array}{c} \eta_s \\ \eta_n \end{array} \right) = \left( \begin{array}{cc} \cos(\theta_{id} - \theta_{\eta\eta'}) & -\sin(\theta_{id} - \theta_{\eta\eta'}) \\ \sin(\theta_{id} - \theta_{\eta\eta'}) & \cos(\theta_{id} - \theta_{\eta\eta'}) \end{array} \right) \left( \begin{array}{c} \eta \\ \eta' \end{array} \right) \]
\[ = \left( \begin{array}{cc} \sin \theta_{id} & -\cos \theta_{id} \\ \cos \theta_{id} & \sin \theta_{id} \end{array} \right) \left( \begin{array}{c} \eta \\ \eta' \end{array} \right). \] (33)

Assuming, as previously, that the relevant matrix elements are equal to the squared masses of the corresponding states, and using the orthogonality of the \( \eta \) and \( \eta' \) as physical states, we obtain
\[ m_{\eta_n}^2 = \frac{2}{3} m_\eta^2 + \frac{1}{3} m_{\eta'}^2, \] (34)
\[ m_{\eta_n}^2 = \frac{1}{3} m_\eta^2 + \frac{2}{3} m_{\eta'}^2. \] (35)

\textsuperscript{1}The mixing angle (31) predicts the suppression of the \( K_2^* \rightarrow K\eta \) decay \[10\], in excellent agreement with experiment \[10\].
in agreement with naive expectations from the quark content of these states. Thus, \(2m_{\eta s}^2 - m_{\eta n}^2 = m_{\eta f}^2\); the use of this result in Eq. (30) leads finally to

\[
4m_K^2 \simeq 3m_\pi^2 + m_{\eta f}^2, \tag{36}
\]

which is our final form of GMO\(_r\) for pseudoscalar mesons.

With the measured masses of the states entering Eq. (36) \([10]\),

\[
m_\pi = 137.3 \pm 2.3 \text{ MeV}, \quad m_K = 495.7 \pm 2.0 \text{ MeV}, \quad m_{\eta f} = 957.8 \text{ MeV},
\]

it gives (in GeV\(^2\)) \(0.983 \pm 0.008\) on the l.h.s. vs. \(0.974 \pm 0.002\) on the r.h.s.; its accuracy is therefore \(\simeq 0.9\%\).

The value of \(m_{\eta s}^2\) may be obtained from (6),(31):

\[
\frac{m_{\eta s}^2 - m_{\eta f}^2}{m_{\eta f}^2 - m_{\eta s}^2} = \frac{1}{8},
\]

therefore

\[9m_{\eta s}^2 = 8m_{\eta}^2 + m_{\eta f}^2, \quad \text{and} \quad m_{\eta s} \approx 607 \text{ MeV}, \tag{37}\]

in excellent agreement with

\[
m_{\eta s}^2 = \frac{4}{3}m_K^2 - \frac{1}{3}m_\pi^2 - \frac{2}{3}(4\pi f_\pi)^2 \ln \frac{m_K^2}{\mu^2} \approx 610 \text{ MeV}, \quad \mu \approx 1 \text{ GeV} \tag{38}
\]

obtained from chiral perturbation theory \([19]\).

We now wish to extend the application of this new Gell-Mann–Okubo mass formula (30), which we rewrite here as

\[
2m_{ss}^2 + 3m_{nn}(I = 1) = 4m_{ss}^2 + m_{nn}(I = 0).
\]

In the case of the ideal mixing of a nonet, it differs from (2) only by a term proportional to explicit isospin variation: \(m_{nn}^2(I = 1) - m_{nn}^2(I = 0)\). However, in contrast to (2), it has correct (3 and 1, respectively) isospin degeneracies for the isovector and non-strange isoscalar states. Moreover, as we have shown in the example of pseudoscalar mesons, this formula may work when its counterpart (2) does not; namely, in case of a non-ideal nonet mixing. As clear from its derivation, the formula (2) will hold only for an ideally mixed nonet; in contrast, Eq. (30) is obtained from Regge phenomenology based on Eqs. (18),(19) which relate the masses of pure \(q\bar{q}\) states but not the nonet mixing angle, and will therefore hold even if a nonet mixing differs from the ideal one (e.g., if the isoscalar octet mass is shifted from its “GMO value” (1), Eq. (6) will not be compatible with \(\theta = \theta_{id}\), even if \(\omega_0 = s\bar{s}, \omega_{0'} = n\bar{n}\)). The generality of Eqs. (18),(19) which are the basis of the formula (30), and of the arguments used for its derivation suggests that this formula should be applicable to any meson multiplet, not only to the pseudoscalar one. Also, as discussed above, in cases when the nonet mixing is not ideal but two isoscalars are almost pure \(n\bar{n}\) and \(s\bar{s}\) states (which are realized in the real world in some cases), we expect this formula to hold with better accuracy than Eq. (2).
In order to test this, we shall apply the formula (30) to two well-established meson multiplets, vector and tensor mesons.

For vector mesons, we write down this formula (assuming $\omega \approx n\bar{n}$, $\phi \approx s\bar{s}$) as

$$2m_{\rho}^2 = 3m_{\phi}^2 + m_{\omega}^2.$$  \hspace{1cm} (39)

For the measured meson masses entering Eq. (39), it gives (in GeV$^2$) 3.85 on the l.h.s. vs 3.81 $\pm$ 0.02 on the r.h.s.; the accuracy is therefore 1.1%. For comparison, Eq. (2) for vector mesons gives (in GeV$^2$) 1.64 vs. 1.60 $\pm$ 0.02, with the accuracy of 2.5%, a factor of two worse than that of (39).

For tensor mesons, Eq. (30) should be written down (with $f_2 \approx n\bar{n}$, $f_2' \approx s\bar{s}$) as

$$2m_{f_2}^2 + 3m_{a_2}^2 = 4m_{K^*}^2 + m_{f_2}^2.$$  \hspace{1cm} (40)

which for the measured meson masses gives (in GeV$^2$) 9.86 $\pm$ 0.03 on the l.h.s. vs. 9.79 $\pm$ 0.07 on the r.h.s., with the accuracy of 0.7%. Eq. (2) in this case gives (in GeV$^2$) 4.01 $\pm$ 0.02 vs. 4.08 $\pm$ 0.03, with the accuracy of 1.7%, again, a factor of two worse than that of (40).

Finally, we cast Eq. (30) into a form which involves the physical meson masses and nonet mixing angle, and therefore is applicable to every meson nonet.

It follows from (33) that

$$m_n^2 = \sin^2 \xi m_0^2 + \cos^2 \xi m_{0'}^2,$$

$$m_s^2 = \cos^2 \xi m_0^2 + \sin^2 \xi m_{0'}^2,$$  \hspace{1cm} (41)

where $m_0$, $m_{0'}$ are the masses of the isoscalar mostly octet and singlet states, respectively, and $\xi \equiv \theta_{id} - \theta$, $\theta$ being the nonet mixing angle. Using these expressions for $m_n^2$ and $m_s^2$ in Eq. (30), one obtains

$$\left(3\cos^2 \xi - 1\right)m_0^2 + \left(2 - 3\cos^2 \xi\right)m_{0'}^2 = 4m_{1/2}^2 - 3m_1^2,$$  \hspace{1cm} (42)

where $m_1$, $m_{1/2}$ are the masses of the isovector and isodoublet states, respectively, which finally reduces, through

$$\cos \xi = \frac{\sqrt{2}\cos \theta + \sin \theta}{\sqrt{3}},$$  \hspace{1cm} (43)

to

$$3m_1^2 + \cos^2 \theta m_0^2 + \sin^2 \theta m_{0'}^2 + \sqrt{2}\sin 2\theta \left(m_0^2 - m_{0'}^2\right) = 4m_{1/2}^2,$$  \hspace{1cm} (44)

which is a new nonet mass-mixing angle relation. For an ideally mixed nonet, $\tan \theta = 1/\sqrt{2}$, it reduces to

$$2m_0^2 + 3m_1^2 = m_{0'}^2 + 4m_{1/2}^2,$$

which is equivalent to (30); for the pseudoscalar nonet, $\tan \theta = -1/(2\sqrt{2})$, it leads to

$$3m_1^2 = 4m_{1/2}^2 - m_{0'}^2,$$

which is equivalent to (36).

The new mass-mixing angle relation (44) is the main result of this paper.
Concluding remarks

We have derived a new Gell-Mann–Okubo mass formula, Eq. (30), by applying Regge phenomenology to pseudoscalar and vector mesons. For pseudoscalar mesons, using the $\eta$-$\eta'$ mixing angle $\simeq -19.5^\circ$, in agreement with experiment, this formula may be reduced to Eq. (36) which relates the masses of the $\pi$, $K$ and $\eta'$ mesons. This relation predicts the mass of the $\eta'$ meson, $m_{\eta'} = 962.4 \pm 5.1$ MeV, within the physical mass of 957.8 MeV with an accuracy of $\simeq 0.5\%$. Since no additional assumption except the linearity of the corresponding trajectories and the additivity of the inverse slopes, Eq. (19), has been made in deriving Eqs. (30) and (36) (Eq. (36) is based on the $\eta$-$\eta'$ mixing angle $\simeq -19.5^\circ$ which is provided by duality constraints [15]), we conclude that Regge phenomenology suffices to describe the $\eta'$ mass, which has been a mystery for a long time. The question remains, however, about the mass of the isoscalar singlet state (before its mixing with the isoscalar octet which results in the physical $\eta$ and $\eta'$ states): since, independent of the mixing angle, $m_{\eta_8}^2 + m_{\eta_9}^2 = m_\eta^2 + m_{\eta'}^2$ (as seen, e.g., in Eqs. (3),(4)), the value of $m_{\eta_8}$ (37) leads to $m_{\eta_9} \approx 921$ MeV. Thus, compared to a 40 MeV shift of the mass of the $\eta_8$ from its GMO value by chiral corrections, which is only 7\% of its bare (GMO) mass, the mass of the $\eta_9$ is shifted by $\sim 500$ MeV, taking its “GMO” value as $\simeq (2m_K^2 - m_{\eta_8}^2)^{1/2}$. We believe that such a large shift of the mass of the pseudoscalar isoscalar octet state is due to instanton effects discussed in detail in refs. [8, 20].

It is clear from our arguments given above that Eq. (30) or its mass-mixing angle form (44) should also hold for scalar mesons. It would be very interesting to consider the scalar meson case and shed some light on the long-standing problem of the correct $q\bar{q}$ assignment for this nonet. We plan to do this in a forthcoming publication.

Also, the generalization of the relations for meson masses and mixing angles discussed in the paper to finite temperature and/or baryon density would be very important for the understanding of the in-medium hadron behavior and its possible consequences for the decay widths and particle spectra, in view of ongoing experimental activity of different groups all around the world in the search for the new phases of matter.

Acknowledgements

Correspondence of one of the authors (L.B.) with L.P. Horwitz during the preparation of the present paper is greatly acknowledged.

References

[1] S. Okubo, Prog. Theor. Phys. 27 (1962) 949, 28 (1962) 24
M. Gell-Mann and Y. Ne’eman, The Eightfold Way, (Benjamin, NY, 1964)
[2] D.H. Perkins, *Introduction to High Energy Physics*, 3rd ed., (Addison Wesley, Reading, MA, 1987), section 5.6

[3] J.J. Sakurai, *Currents and Mesons*, (University of Chicago Press, Chicago, 1969)

[4] H. Georgi, *Weak Interactions and Modern Particle Theory*, (Benjamin, New York, 1984)

[5] D.B. Kaplan and A.V. Manohar, Phys. Rev. Lett. 56 (1986) 2004

[6] C. Rosenzweig, J. Schechter and C.G. Trahern, Phys. Rev. D 21 (1980) 3388
P. Di Vecchia and G. Veneziano, Nucl. Phys. B 171 (1980) 253
E. Witten, Ann. Phys. 128 (1980) 363

[7] S. Weinberg, Phys. Rev. D 11 (1975) 3583

[8] V. Dmitrasinovic, Phys. Rev. C 53 (1996) 1383; Phys. Rev. D 56 (1997) 247

[9] F.J. Gilman and R. Kauffman, Phys. Rev. D 36 (1987) 2761
Crystal Barrel Collaboration (C. Amsler *et al.*), Phys. Lett. B 294 (1992) 451
P. Ball, J.-M. Frere and M. Tytgat, Phys. Lett. B 365 (1996) 367
A. Bramon, R. Escribano and M.D. Scadron, Phys. Lett. B 403 (1997) 339

[10] Particle Data Group, Phys. Rev. D 54 (1996) 1

[11] M. Gell-Mann, R.J. Oakes and B. Renner, Phys. Rev. 175 (1968) 2195

[12] J.F. Donoghue, E. Golowich and B.R. Holstein, *Dynamics of the Standard Model*, (Cambridge University Press, 1996), p. 194

[13] L. Burakovsky, T. Goldman and L.P. Horwitz, New mass relations for heavy quarkonia, Preprint LA-UR-97-1494 [hep-ph/9704440], to be published
L. Burakovsky, T. Goldman and L.P. Horwitz, New mass relation for meson 25-plet, Preprint LA-UR-97-1720 [hep-ph/9704432], to be published

[14] L. Burakovsky, T. Goldman and L.P. Horwitz, New quadratic mass relations for heavy mesons, Preprint FERMILAB-PUB-97/265-T [hep-ph/9708xxx], to be published

[15] L. Burakovsky, T. Goldman and L.P. Horwitz, New quadratic baryon mass relations, Preprint LA-UR-97-2333 [hep-ph/9706464], to be published

[16] W. Lucha, F.F. Schöberl and D. Gromes, Phys. Rep. 200 (1991) 127

[17] S. Oneda and K. Terasaki, Prog. Theor. Phys. Suppl. 82 (1985) 1

[18] A. Bramon, Phys. Lett. B 51 (1974) 87

[19] D.T. Cornwall, Nucl. Phys. B51 (1973) 16
P. Langacker and H. Pagels, Phys. Rev. D 10 (1974) 2904
J.F. Donoghue, B.R. Holstein and Y.-C.R. Lin, Phys. Rev. Lett. 55 (1985) 2766
[20] W.H. Blask, U. Bohn, M.G. Huber, B.C. Metsch and H.R. Petry, Z. Phys. A 337 (1990) 327
E. Klempt, B.C. Metsch, C.R. Münz and H.R. Petry, Phys. Lett. B 361 (1995) 160
C. Ritter, B.C. Metsch, C.R. Münz and H.R. Petry, Phys. Lett. B 380 (1996) 431