Leading soft gluon production
in high energy nuclear collisions

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Abstract

The leading soft gluon $p_T$ distribution in heavy ion collisions was obtained by Kovner, McLerran, and Weigert after solving classical Yang-Mills equations. I show explicitly this result can be understood in terms of conventional QCD perturbation theory. I also demonstrate that the key logarithm in their result represents the logarithm in DGLAP evolution equations.

1 Introduction

In ultra-relativistic heavy ion collisions, after initial collisions, the parton system will be dominated by gluons [1]. Understanding the distribution of soft gluons formed in the initial stage of the collision is particularly interesting and important for studying the formation of quark-gluon plasma. In terms of conventional QCD perturbation theory, a calculable cross section in high energy hadronic collisions is factorized into a single collision between two partons multiplied by a probability to find these two partons of momentum fractions $x_1$ and $x_2$, respectively, from two incoming hadrons. The probability is then factorized into a product of two parton distributions $\phi(x_1)$ and $\phi(x_2)$, which are probabilities to find these two partons from the respective hadrons [2]. Because of extremely large number of soft gluons in heavy ion beams, it is natural to go beyond the factorized single-scattering formalism to include

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any possible multiple scattering, and long range correlations between soft gluons from two incoming ions.

McLerran and Venugopalan (MV) developed a new formalism for calculation of the soft gluon distribution for very large nuclei \([3, 4]\). In this approach, the valence quarks in the nuclei are treated as the classical source of the color charges. They argued that the valence quark recoil can be ignored in the limit when the gluons emitted are soft. In addition, because of the Lorentz contraction, the color charge of the valence quarks is treated approximately as an infinitely thin sheet of color charge along the light cone. With these assumptions, the gluon distribution function for very large nuclei may be obtained by solving the classical Yang-Mills Equation \([4, 5]\). Using the classical glue field generated by a single nucleus obtained in the MV formalism as the basic input, Kovner, McLerran, and Weigert (KMW) computed the soft gluon production in a collision of two ultra-relativistic heavy nuclei by solving the classical Yang-Mills equations with the iteration to the second order \([6]\). The two nuclei are treated as the infinitely thin sheets of the classical color charges moving at the speed of light in the positive and the negative \(z\) directions, respectively. Following this approach, the distribution of soft gluons at the rapidity \(y\) and the transverse momentum \(p_T\) in nuclear collisions can be expressed as \([3, 4]\)

\[
\frac{d\sigma}{dyd^2p_T} = \frac{2g^6}{(2\pi)^4} \left( \frac{N_q}{2N_c} \right)^2 N_c(N_c^2 - 1) \frac{1}{p_T^4} \ln \left( \frac{p_T^2}{\Lambda_{\text{cutoff}}^2} \right). \tag{1}
\]

where \(g\) is the strong coupling constant, \(N_c = 3\) is the number of the color, and \(\Lambda_{\text{cutoff}}\) is a cutoff mass scale \([3]\). In Eq. (1), \(N_q/(2N_c) = S_T\mu^2\), where \(\mu^2\) is the averaged color charge squared per unit area of the valence quark, and \(S_T\) is the transverse area of the nuclei. \(N_q\) is the total number of the quarks in the color charge source. Eq. (1) is potentially very useful in estimating the production of mini-jet rates, and the formation of the possible quark-gluon plasma at RHIC \([8]\).

In the following, I show that this result can be understood in terms of QCD perturbation theory. I will also explore under what kind of approximation this result matches the conventional perturbative calculation \([8]\).
2 Partonic process $qq \rightarrow qqg$

KMW’s derivation for Eq. (1) is based on the following physical picture: in ultra-relativistic heavy ion collisions, gluons are produced by the fields of two strongly Lorentz contracted color charge sources, which are effectively equal to the valence quarks of two incoming ions. In order to understand KMW’s result in the language of perturbative QCD, let us consider a specific partonic process: $qq \rightarrow qqg$, as sketched in Fig. 1. If we assume that the incoming quarks $qq$ are the valence quarks in the initial color charge sources, the partonic subprocess in Fig. 1 mimics the physical picture adopted in KMW’s derivation. In this section, I show how to extract the leading contribution to the soft gluon production from this diagram. However, as a Feynman diagram in QCD perturbation theory, the single diagram shown in Fig. 1 is not gauge invariant. In next section, I show that with a proper choice of gauge and certain approximation, other diagrams are suppressed in soft gluon limit.

In general, the invariant cross section of the gluon production, shown in Fig. 1, can be written as

$$d\sigma_{qq\rightarrow g}/dyd^2p_T = \frac{1}{2s} |\mathcal{M}|^2 dps ,$$

(2)

where $s = (l_1 + l_2)^2$, and $|\mathcal{M}|^2$ is matrix element square with the initial-spin averaged and the final-spin summed. $l_1$ and $l_2$ as the momenta of the two incoming quarks respectively. In Eq. (2), the phase space

$$dps \propto \frac{d^4k_1}{(2\pi)^4} (2\pi)\delta((l_1 - k_1)^2) \times \frac{d^4k_2}{(2\pi)^4} (2\pi)\delta((l_2 - k_2)^2)$$

$$\times (2\pi)^4 \delta^4(k_1 + k_2 - p) ,$$

(3)
where \( k_1 \) and \( k_2 \) are the momenta for the two gluons emitted from the initial quarks. Because of the gluon propagators, as shown in Fig. 1, the matrix element square \( |M|^2 \) has the following pole structure:

\[
\text{poles} = \frac{1}{k_1^2 + i\epsilon} \frac{1}{k_2^2 - i\epsilon} \frac{1}{k_2^2 + i\epsilon} \frac{1}{k_2^2 - i\epsilon}.
\]  (4)

When integrating over the phase space, we see that the leading contribution comes from the terms with \( k_1^2 \to 0 \) or \( k_2^2 \to 0 \) limit. Note that \( k_1^2 \) and \( k_2^2 \) cannot be zero at the same time, because \( k_1 \) and \( k_2 \) come from different directions, and we have the on-shell condition of \( p^2 = (k_1 + k_2)^2 = 0 \). Therefore, to calculate the leading contribution, we can first calculate the diagram in \( k_1^2 \to 0 \) limit. The total leading contribution is just twice of it, because the diagram is symmetric for \( k_1 \) and \( k_2 \).

To derive the leading contribution at \( k_1^2 \to 0 \) limit, we perform the collinear approximation \( k_1 \approx x l_1 + O(k_1 T) \), with \( k_1 T \sim \Lambda_{\text{cutoff}} << p_T \), where \( \Lambda_{\text{cutoff}} \) is a collinear cutoff scale \([10]\). This approximation means that the leading contribution is from the phase space where almost all transverse momentum of the final-state gluon comes from the gluon of \( k_2 \), and \( k_1 \) is almost collinear to \( l_1 \). After such collinear approximation, the cross section in Eq. (3) can be approximately written in a factorized form \([4]\):

\[
\frac{d\sigma_{gg \to g}}{dy dp_T^2} \approx 2 \left( \frac{1}{2(2\pi)^3} \right) \frac{1}{2s} \int \frac{dx}{x} \mathcal{P}_{l_1 \to k_1} (x, k_1 T < p_T) H(x l_1, l_2, p) + O\left( \frac{\Lambda_{\text{cutoff}}^2}{p_T^2} \right),
\]  (5)
where the overall factor of 2 is due to the fact that the leading contribution come from two regions corresponding to \( k_1^2 \to 0 \) and \( k_2^2 \to 0 \), respectively. In Eq. (5), \( P_{l_1 \to k_1}(x, k_{1T} < p_T) \) represents the probability of finding an almost collinear gluon with the momentum fraction \( x \) from an incoming quark of \( l_1 \), and is represented by Fig. 2. \( H(xl_1, l_2, p) \) in Eq. (5) is effectively the hard scattering between the gluon of \( k_1 = xl_1 \) and the incoming quark of \( l_2 \), and it is represented by the diagram shown in Fig. 3.

Under the collinear expansion \( k_1 \approx xl_1 \), the gluon line which connects the partonic parts \( P_{l_1 \to k_1} \) and \( H(xl_1, l_2, p) \) is effectively on the mass-shell, and therefore, the partonic parts, \( P \) and \( H \) in Eq. (5), are separately gauge invariant. The quark splitting function \( P_{l_1 \to k_1}(x, k_{1T} < p_T) \) can be calculated in \( \vec{n} \cdot A = 0 \) gauge. We have

\[
\frac{P_{l_1 \to k_1}(x, k_{1T} < p_T)}{2N_c} = \left( \frac{g^2}{8\pi^2} \right) \left( 1 + \frac{(1-x)^2}{x} \right) \ln \left( \frac{p_T^2}{\Lambda_{\text{cutoff}}^2} \right).
\]

(6)

In \( \vec{n} \cdot A = 0 \) gauge, we have the hard scattering function

\[
H(xl_1, l_2, p) = (2\pi)g^4 \left( \frac{1}{2} \right) \left( \frac{p_+}{xl_+} \right)^2 \frac{1}{p_T^2} \frac{1}{s-2l_+p_-} \delta(x - \frac{2p_+l_-}{s-2l_+p_-}) \\
\times \left[ (xs-2xl_+p_-)^2 + (2xl_+p_-)(2p_+p_-) \right].
\]

(7)

For soft gluon production, we define the soft gluon limit as

\[
\frac{p_-}{l_-} \ll 1 \quad \text{and} \quad \frac{p_+}{l_+} \ll 1.
\]

(8)

Combining Eq. (6), Eq. (7) and Eq. (5), and taking the soft gluon limit, we have

\[
\frac{d\sigma_{qq \to g}}{dyd^2p_T} = \frac{2g^6}{(2\pi)^4} \left( \frac{1}{2N_c} \right)^2 N_c(N_c^2 - 1) \left( \frac{1}{p_T^4} \right) \ln \left( \frac{p_T^2}{\Lambda_{\text{cutoff}}^2} \right).
\]

(9)

Eq. (9) reproduced Eq. (1), which is obtained by solving classical Yang-Mills equations, at \( N_q = 1 \). If we consider the total number of the quarks in the charge sources of both sides, we need to multiply \( N_q^2 \) to Eq. (2).
Figure 4: The rest of Feynman diagrams contributing to the process: $qq \rightarrow qg$, in addition to the diagram in Fig. 1.

3 Gauge Invariance

For the gluon production in the process $qq \rightarrow qg$, the single diagram shown in Fig. 1 is not gauge invariant. In general, we also need to consider the radiation diagrams shown in Fig. 4. Similar to the diagram in Fig. 1, the contribution of Fig. 4a and Fig. 4b can also be written in the same factorized form:

$$
E \frac{d\sigma^{\text{rad}}_{qq \rightarrow g}}{d^3p} \approx \frac{1}{2(2\pi)^3} \frac{1}{2s} \int \frac{dx}{x} P_{l_{1}\rightarrow k_{1}}(x, k_{1T} < p_{T}) H_{i}(x l_{1}, l_{2}, p)
$$

$$
+ O(\frac{\Lambda^2_{\text{cutoff}}}{p_{T}^2}),
$$

(10)

with $i = a, b$. $P_{l_{1}\rightarrow k_{1}}(x, k_{1T} < p_{T})$ is the quark splitting function given by Eq. (4). $H_{a}(x l_{1}, l_{2}, p)$ and $H_{b}(x l_{1}, l_{2}, p)$ are the hard scattering parts from the diagrams in Fig. 4a and Fig. 4b, and they are represented by Fig. 5a and Fig. 5b, respectively. With the contribution from diagrams in Fig. 4, Eq. (3) changes to

$$
E \frac{d\sigma_{qq \rightarrow g}}{d^3p} \approx 2 \left( \frac{1}{2(2\pi)^3} \frac{1}{2s} \right) \int \frac{dx}{x} P_{l_{1}\rightarrow k_{1}}(x, k_{1T} < p_{T})
$$

$$
\times \left[ H(x l_{1}, l_{2}, p) + H_{a}(x l_{1}, l_{2}, p) + H_{b}(x l_{1}, l_{2}, p) \right]
$$
Figure 5: Feynman diagrams contributing to \( H_a(x_{l_1}, l_2, p) \) (a), and \( H_b(x_{l_1}, l_2, p) \) (b).

\[ \text{with the approximation } k_1 = x_{l_1} + O(k_{1T}) \text{.} \]

Feynman diagrams shown in Fig. 3 and Fig. 5 form a gauge invariant subset for calculating the hard scattering parts, \( H(x_{l_1}, l_2, p) \)'s in Eq. (10). With our choice of gauge, \( H_i/H \sim p_-/l_- << 1 \) in the soft gluon limit. Therefore, the contribution from diagrams in Fig. 4a and Fig. 4b can be neglected in comparison with the contribution from the diagram in Fig. 1 at \( k_{1T} \sim 0 \).

Similarly, the contributions from diagrams shown in Fig. 4c and Fig. 4d can be neglected in the soft gluon limit, when compared with the contribution from the diagram in Fig. 1 at \( k_{2T} \sim 0 \). Therefore, with the approximation of \( k_{1T} \sim 0 \) (or \( k_{2T} \sim 0 \)) and the soft gluon limit, and a proper choice of the gauge, the contributions extracted from the diagram in Fig. 1 to the leading soft gluon production in Eq. (1) is gauge invariant.

4 Discussion

When we consider the collision between two nuclei, we can treat the two incoming quarks in Fig. 1 as coming from two nuclei respectively. In this picture, the number of the valence quark \( N_q \) is replaced by the quark distribution in the nuclei. In terms of the parton model, the cross section between the two nuclei \( A \) and \( B \) can be expressed in the following form:

\[ \frac{d\sigma_{AB \to g}}{dy dp_T^2} = \int dz_1 dz_2 f_{q/A}(z_1) f_{q/B}(z_2) E \frac{d\sigma_{qq \to g}}{d^3p}. \]  

Here \( z_1 \) and \( z_2 \) are the momentum fractions of the quarks, and \( f_{q/A}(z_1) \) and \( f_{q/B}(z_2) \) are the quark distributions (or quark number densities) of the two
nuclei. If we denote $p_A$ and $p_B$ as the momenta for the two nuclei respectively, then $z_1 = l_1/p_A$ and $z_2 = l_2/p_B$. Substituting Eq. (5) into Eq. (12), we have

$$
\frac{d\sigma_{AB\rightarrow g}}{dydp_T^2} \approx \frac{1}{2(2\pi)^3} \frac{1}{2S} \int \frac{dz_1}{z_1} \frac{dz_2}{z_2}
$$

$$
\times \left[ \int \frac{dx_1}{x_1} f_{q/A}(z_1) f_{q/B}(z_2)
$$

$$
\times P_{l_1 \rightarrow k_1}(x_1, k_{1T} < p_T) H(l_1, l_2, p)
$$

$$
+ \int \frac{dx_2}{x_2} f_{q/A}(z_1) f_{q/B}(z_2)
$$

$$
\times P_{l_2 \rightarrow k_2}(x_2, k_{2T} < p_T) H(l_1, l_2, p) \right] (13)
$$

where the overall factor 2 in Eq. (5) is now represented by the two terms, and $S = (p_A + p_B)^2 \approx 2p_A \cdot p_B$. In Eq. (13), $x_i = k_i/l_i$ with $i = 1, 2$, and $l_1 = z_1 p_A$ and $l_2 = z_2 p_B$. If we denote the momentum fraction of gluon $k_1$ with respect to $p_A$ as $z_1' = k_1/p_A$, and $k_2$ with respect to $p_B$ as $z_2' = k_2/p_B$, we can rewrite Eq. (13) in terms of $z_1'$ and $z_2'$:

$$
\frac{d\sigma_{AB\rightarrow g}}{dydp_T^2} \approx \frac{1}{2(2\pi)^3} \frac{1}{2S}
$$

$$
\times \left\{ \int \frac{dz_1'}{z_1'} \frac{dz_2}{z_2} \left[ \int \frac{dz_1}{z_1} f_{q/A}(z_1) P_{l_1 \rightarrow k_1}(z'/z_1, k_{1T} < p_T) \right]
$$

$$
\times f_{q/B}(z) H(z_1' p_A, z_2 p_B, p)
$$

$$
+ \int \frac{dz_2'}{z_2'} \frac{dz_1}{z_1} \left[ \int \frac{dz_2}{z_2} f_{q/B}(z_2) P_{l_2 \rightarrow k_1}(z'/z_2, k_{2T} < p_T) \right]
$$

$$
\times f_{q/A}(z_1) H(z_1 p_A, z_2' p_B, p) \right\} (14)
$$

According to the QCD factorization theorem [2], we see that the part inside the square brackets is actually the gluon distribution from nuclei $A$ (or $B$) at the factorization scale $\mu_F^2 = p_T^2$, with only the quark splitting function [1],

$$
f_{g/A}(z_1', \mu_F^2 = p_T^2) = \int \frac{dz_1}{z_1} f_{q/A}(z_1) P_{l_1 \rightarrow k_1}(z'/z_1, k_{1T} < p_T)
$$

$$
+ \text{term from gluon splitting.} (15)
$$
Using Eq. (15), we can then reexpress Eq. (14) as:

\[
\frac{d\sigma_{AB\rightarrow g}}{dydp_{T}^{2}} \approx \frac{1}{2(2\pi)^{3}} \frac{1}{2S} \times \int \frac{dz' d\bar{z}}{z} \left[ f_{g/A}(z', \mu_{F}^{2} = p_{T}^{2}) f_{g/B}(z) H(z' p_{A}, \bar{z} p_{B}, \bar{p}) + f_{q/A}(z) f_{g/B}(z', \mu_{F}^{2} = p_{T}^{2}) H(z p_{A}, z' p_{B}, \bar{p}) \right],
\]

which is the factorized formula for two-to-two subprocesses in the conventional perturbative QCD for the nucleus-nucleus collisions. In KMW formalism, only the valence quark color charge was used as the source of the classical charge of colors. As a result, the gluon splitting term in Eq. (15) is neglected for the distribution \( f_{g/A} \).

From the above comparison, we conclude that by solving the classical Yang-Mills Equation to the second order in iteration, KMW’s result is consistent with conventional perturbative QCD at the leading logarithmic approximation. The logarithmic dependence shown in KMW’s result basically describes the logarithmic DGLAP evolution of the quark distributions \([12]\). However, the iteration in this approach is different from the expansion series in conventional perturbative QCD. The McLerran-Venugopalan formalism was later further developed to include the harder gluons into the charge density \( \mu^{2} \) and treat the charge source as an extended distribution which depends on the rapidity \([13]\). It will be potentially very useful if this new approach, after including higher orders of iteration, can include the parton recombination \([14]\) and other non-perturbative effects which are not apparent in the normal perturbative calculation.

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