Minimality of Linear Switched Systems with known switching signal

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1 Introduction

We consider linear switched system of the form

\[
\Sigma_\sigma : \begin{cases} 
\dot{x}_q(t) = A_{\sigma(t)}x_q(t) + B_{\sigma(t)}u(t), & t \in (t_q, t_{q+1}), \\
x_q(t_q^+) = J_{\sigma(t_q^+),\sigma(t_q)}x_q(t_q^-), \\
y(t) = C_{\sigma(t)}x_q(t), & t \in \mathbb{R},
\end{cases}
\]  

where \(x_q : (t_q, t_{q+1}) \rightarrow \mathbb{R}^{n_q}\) is the absolutely continuous \(q\)-th piece of the state, \(u : \mathbb{R} \rightarrow \mathbb{R}^m\) is the input and \(y\) is the measured output. The switching signal \(\sigma : \mathbb{R} \rightarrow \mathcal{Q} = \{1, 2, \ldots, f\} \subset \mathbb{N}\) is a given piecewise constant function with finitely many switching times: \(\{t_q \mid q \in \mathcal{Q}, t_1 < t_2 < \cdots < t_f\}\) in the bounded interval \((t_1, t_{f+1})\) of interest. For each \(q \in \mathcal{Q}\), the matrices \(A_q, B_q, C_q\), are of appropriate \(q\)-dependent size. We need a jump map \(J_{q^-} : \mathbb{R}^{n_{q^-}} \rightarrow \mathbb{R}^{n_{q^+}}\) to relate different state-space dimensions and simplify the notation \(J_{\sigma(t_q^+),\sigma(t_q)} = J_{q^-} = J_q\).

The general idea of minimal realization is to construct a state-space model from a given input-output behavior of the system. In particular, finding a minimal realization could be seen as the first step towards model reduction. In [1], we have presented a time-varying model reduction approach for linear switched system which was not a switched system anymore. Therefore, our aim is to gain insight into a more suitable model-reduction approach by studying the minimal realization problem for switched systems of the form (1) within this system class.

Several approaches have been discussed in the cases of arbitrary and constrained switching e.g. in [1–5] where switching signal are viewed as input to the switched systems. It can be seen that (minimal) realization in general depends on the specifically given switching signal, so in contrast to the existing literature, we view the switched system (1) as a piecewise-constant time-varying linear system. We begin with the formal definition of minimality.

Definition 1.1 For \(\Sigma_\sigma\) as in (1), the total dimension is defined by \(\dim \Sigma_\sigma := \sum_{q \in \mathcal{Q}} n_q\). Furthermore, we define its input-output behaviour as follows

\[
\mathcal{B}_\sigma^{io} := \{ (u, y) \mid \forall q \in \mathcal{Q} \exists x_q : (t_q, t_{q+1}) \rightarrow \mathbb{R}^{n_q} \text{ satisfying (1)} \text{ and } x_1(t_1^+) = 0 \}.
\]

A linear switched system \(\hat{\Sigma}_\sigma\) with corresponding input-output behavior \(\hat{\mathcal{B}}_\sigma^{io}\) is said to be a minimal realization of switched system \(\Sigma_\sigma\) if 1) \(\mathcal{B}_\sigma^{io} = \hat{\mathcal{B}}_\sigma^{io}\) and 2) for any \(\Sigma_\sigma\) with \(\mathcal{B}_\sigma^{io} = \mathcal{B}_\sigma^{io}\) satisfies \(\dim \hat{\Sigma}_\sigma \leq \dim \Sigma_\sigma\).

Remark 1.2 The above definition of minimality is not specifying any method to obtain a minimal realization from a given switched system as in (1). In general, a minimal realization can only be obtained by considering each mode individually (and by properly taking the effect on the other modes into account).

2 Minimal realization of single switch switched system

We propose a method to find a minimal realization of linear switched system of the form

\[
\Sigma_\sigma : \begin{cases} 
\dot{x}_1 = A_1 x_1 + B_1 u, & \text{on } (t_1, t_2), \quad x_1(t_1^+) = 0, \\
\dot{x}_2 = A_2 x_2 + B_2 u, & \text{on } (t_2, t_3), \quad x_2(t_2^+) = J_2 x_1(t_1^++).
\end{cases}
\]
The Kalman decomposition (KD), [6], is a well known method to find a minimal realization of a system, however, this method is based on the assumption that the initial value is zero. In system (2), we have seen that second mode starts with nonzero initial values which are not completely arbitrary, but are constraint to the reachable space of the first mode. By taking into account the reachable subspace of the first mode, we construct an input-extended system which is input-output equivalent to the second mode (cf. [7] in the context of model reduction). Then we extend the first mode by taking into account the observable states of the second mode. Finally we define the jump map from mode 1 to mode 2.

Overall, the algorithm of the proposed method is summarized as follows.

**Step 1a.** Compute the reachable subspace \( R_1 = \text{im} R_1 \) of first subsystem \((A_1, B_1, C_1)\) and extend the input matrix of the second mode to

\[
B_{2,e} := \text{im}[B_2, J_2R_1].
\]

**Step 1b.** Calculate the KD of \((A_2, B_{2,e}, C_2)\) with corresponding transformation matrix \(V_2\) and left- and right-projectors \(W_2, V_2\) (i.e. the corresponding rows and columns of \(V_2^{-1}\) and \(V_2\)) and let

\[
(\hat{A}_2, \hat{B}_2, \hat{C}_2) = (W_2A_2V_2, W_2B_2, C_2V_2).
\]

**Step 2a.** Calculate the space \( L_2 = R_1 \cap K_2 := \text{im} L_2 \) of additional observable states, where \( K_2 = \text{im} K_2 \) for some full column rank matrix \( K_2 \in \mathbb{R}^{n_2 \times n_2} \) such that \( J_2K_2 = V_2^T \) for a full column rank matrix \( V_2^T \in \mathbb{R}^{n_2 \times n_2} \) with \( V_2^T := \text{im} V_2 \). Then extend the output matrix of the first mode as

\[
C_{1,e} := \text{im} \begin{bmatrix} C_1 \\ L_2^T \end{bmatrix}.
\]

**Step 2b.** Calculate the KD of \((A_1, B_1, C_{1,e})\) with corresponding transformation matrix \(V_1\) and left- and right-projectors \(W_1, V_1\) (i.e. the corresponding rows and columns of \(V_1^{-1}\) and \(V_1\)) and let

\[
(\hat{A}_1, \hat{B}_1, \hat{C}_1) = (W_1A_1V_1, W_1B_1, C_1V_1).
\]

**Step 3.** The reduced jump \( \hat{J}_2 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_2} \) is calculated as \( \hat{J}_2 := W_2J_2V_1 \).

The overall reduced switched system is then given by

\[
\hat{\Sigma}_\sigma : \begin{cases} 
\dot{\hat{x}}_1 = \hat{A}_1\hat{x}_1 + \hat{B}_1u, & \text{on } (t_1, t_2), \quad \hat{x}_1(t_1^+) = 0, \\
\dot{\hat{x}}_2 = \hat{A}_2\hat{x}_2 + \hat{B}_2u, & \text{on } (t_2, t_f), \quad \hat{x}_2(t_2^+) = \hat{J}_2\hat{x}_1(t_2^+). 
\end{cases}
\] (3)

The above algorithm ensures following observations. Due to page limitation, we ignore details.

**Theorem 2.1** Consider the switched system \( \Sigma_\sigma \) and the reduced system \( \hat{\Sigma}_\sigma \) obtained via the above algorithm. Then both systems are input-output equivalent in the sense of Definition 1.1. Also, \( \hat{\Sigma}_\sigma \) has minimal total dimension under all possible input-output equivalent system of \( \Sigma_\sigma \).

The proposed approach is illustrated by the following example.

**Example 2.2** Consider a switched system as in (2) with modes

\[
(A_1, B_1, C_1) = \left( \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \right) \quad \text{and} \quad (A_2, B_2, C_2, J_2) = \left( \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \right).
\]

It is easily seen, that each modes is unreachable and unobservable, however, the switched system is reachable and observable. We apply the proposed method. Via the KD of the extended 2nd mode \((A_2, [B_2, J_2R_1, C_2])\) and the extended 1st mode \((A_1, B_1, [C_1^T, L_2^T])\) respectively, we obtain the left- and right-projectors \(W_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\), \(V_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\), and \(W_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\), \(V_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}\) with \(R_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\), \(L_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\). The corresponding input-output equivalent minimal switched system is given by

\[
(\hat{A}_1, \hat{B}_1, \hat{C}_1) = (W_1A_1V_1, W_1B_1, C_1V_1) = \left( \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \right),
\]

\[
(\hat{A}_2, \hat{B}_2, \hat{C}_2) = (W_2A_2V_2, W_2B_2, C_2V_2) = \left( \begin{bmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \end{bmatrix} \right) \quad \text{and} \quad \hat{J}_2 = W_2J_2V_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.
\]

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References

[1] M. S. Hossain and S. Trenn, IFAC-PapersOnLine 53(2), 5629–5634 (2020), 21th IFAC World Congress.
[2] M. Petreczky and J. H. van Schuppen, IEEE Trans. Autom. Control 55(10), 2282–2297 (2010).
[3] M. Baştuğ, M. Petreczky, R. Wisniewski, and J. Leth, Automatica 74, 162–170 (2016).
[4] M. Petreczky, ESAIM Control Optim. Calc. Var. pp. 446–471 (2011).
[5] I. V. Gosea, M. Petreczky, A. C. Antoulas, and C. Fiter, Advances in Computational Mathematics 44(6), 1845–1886 (2018).
[6] R. E. Kalman, SIAM J. Control Optim. 1, 152–192 (1963).
[7] M. Heinkenschloss, T. Reis, and A. C. Antoulas, Automatica 47(3), 559–564 (2011).