Feynman rules for string field theories with discrete target space

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We derive a minimal set of Feynman rules for the loop amplitudes in unitary models of closed strings, whose target space is a simply laced (extended) Dynkin diagram. The string field Feynman graphs are composed of propagators, vertices (including tadpoles) of all topologies, and leg factors for the macroscopic loops. A vertex of given topology factorizes into a fusion coefficient for the matter fields and an intersection number associated with the corresponding punctured surface. As illustration we obtain explicit expressions for the genus-one tadpole and the genus-zero four-loop amplitude.

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1. Introduction

One of the most important problems in a theory of strings is the construction of the corresponding second quantized theory, i.e., a field theory in the space of loops [1]. A minimal requirement for a string field theory is to give simple rules for the perturbative expansion, i.e., a prescription how to decompose the integral over world surfaces with different topologies into a sum of Feynman diagrams built from string propagators and vertices.

In the last several years the simplest noncritical string theories were solved using large-$N$ techniques in matrix models (see, for example, [2]). A matrix model is essentially a system of free fermions and the closed strings are represented there as collective excitations of fermions. A possible way to derive the genus expansion in the string theory is to reformulate the matrix model in terms of these collective fields. Suitable for this purpose are the ADE and ÂDÊ matrix models proposed in [3], describing string theories in which the matter degrees of freedom are labeled by the nodes of a Dynkin diagram $X$ [4]. These models were generalized to describe both closed and open strings and reformulated in terms of the collective loop fields in ref. [5]. The world sheet of the string represents a triangulated surface immersed in the graph $X$. The diagrammatic rules for the interactions of the string fields share some common features with the explicit construction of the interaction in the critical closed string theory [8]. The interaction is described by a nonpolynomial action, with an elementary vertex for every higher genus amplitude. The vertices are essentially the correlation functions for topological gravity and have the geometrical interpretation of punctured surfaces with various topologies localized at a single point $x$ of the target space $X$.

The aim of the present letter is to complete the results of [5] where only the general form of the closed string vertices was obtained. We find here the explicit Feynman rules for a string field theory whose possible backgrounds are classified by the ADE and ÂDÊ Dynkin diagrams. The basic idea of our approach is that the higher genus interactions are concentrated in the vicinity of the edge of the eigenvalue distribution where the singularity is always of square root one. Near this point the collective fields can be expanded in half-integer powers which leads naturally to the KdV picture of topological gravity.

1. Description of the target space

The target spaces of these models have very similar geometrical properties, which allows to consider them simultaneously. The graph representing the target space $X$ consists of a set of nodes $x$, and a number of bonds $\langle xx' \rangle$ between nodes. Two nodes are called adjacent ($\sim$) on $X$ if they are connected by a single bond. (We do not allow more than one bond between two nodes.) The graph $X$ is completely determined by its adjacency matrix

$$A_{xx'} = \begin{cases} 1, & \text{if } x \sim x'; \\ 0, & \text{if not.} \end{cases} \quad (1.1)$$

The corresponding string theory has a stable vacuum only if all the eigenvalues of $A$ are smaller or equal to 2. This condition restricts the choice of possible graphs to Dynkin

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1 This is not always the best way to attack the problem. In the standard matrix-quantum-mechanics formulation of the $C = 1$ string the fermionic formalism seems to be more efficient than the Das-Jevicki collective theory, which is well defined only on tree level.
diagrams of A,D,E and \( \hat{A},\hat{D},\hat{E} \) types. In this case the adjacency matrix is diagonalized as

\[
A_{xx'} = \sum_{p \in P} S_p^x 2 \cos(\pi p) S_{p'}^x,
\]

(1.2)

where \( P \) is the “momentum space” dual to \( X \). The spectrum of momenta is of the form \( p = \frac{m}{h} \) where \( h \) is the Coxeter number and the integer \( m \) are the Coxeter exponents of the corresponding Lie algebra. The Fourier image of a function \( f_x \) will be denoted by \( f_p \)

\[
f_x = \sum_{p \in P} S_p^x f_p, \quad f_p = \sum_{x \in X} S_p^x f_x.
\]

(1.3)

The eigenvectors \( S_p \) can be interpreted as the eigenstates of a quantum particle living on the Dynkin diagram. The string background (i.e., the disc amplitude) is proportional to the Perron-Frobenius eigenvector \( S_x \) labeled by the minimal momentum \( p_0 \)

\[
S_x = S_p^x, \quad p_0 = \min\{p \mid p \in P\}.
\]

(1.4)

For the ADE backgrounds \( p_0 = \frac{1}{h} \) and for the \( \hat{A}\hat{D}\hat{E} \) backgrounds \( p_0 = 0 \). The ratios

\[
\chi_x^{(p)} = \frac{S_p^x}{S_x}
\]

(1.5)

are in correspondence with the order parameters of the matter field and satisfy a closed algebra

\[
\chi_x^{(p')} \chi_x^{(p'')} = \sum_p C_0^{p'p'p} \chi_x^{(p)}.
\]

(1.6)

The symmetric structure constants \( C_0^{p'p'p} \) are the fusion coefficients for the order parameters and satisfy the usual relations

\[
C_0^{p_0p_0p'} = \delta_{p,p'}, \quad \sum_p C_0^{p_1p_2p} C_0^{pp_3p_4} = \sum_p C_0^{p_1p_3p} C_0^{pp_2p_4}.
\]

(1.7)

More generally, we define the genus \( g \) fusion coefficients

\[
C_{p_1\cdots p_n}^g = \sum_x S_x^{2-2g} \chi_x^{(p_1)} \cdots \chi_x^{(p_n)},
\]

(1.8)

which represent the target-space component of the the string interaction vertices.

2. A large \( N \) matrix model for the loop fields

The string theory with target space \( X \) will be constructed as the \( 1/N \) expansion of the \( N \times N \) matrix model defined in ref. [3] whose entities are the hermitian matrices \( H_x \)
associated with the sites of the lattice, and the complex matrices $C_{xx'} = C_{x'x}^\dagger$ associated with the links $<xx'>$. The partition function of the model reads

$$Z[V] = \int \prod_x DH_x \prod_x e^{-tr V_x(H_x)} \prod_{<xx'>} DC_{xx'} e^{tr H_x C_{xx'} C_{x'x}}. \quad (2.1)$$

The operator creating a macroscopic loop at the point $x$ is the resolvent of the matrix field $H_x$

$$W_x(z) = \text{tr} \frac{1}{z - H_x}. \quad (2.2)$$

The $n$-loop amplitude is obtained by differentiating with respect to the source

$$\langle \prod_{s=1}^n W_x(z_s) \rangle = \left( \prod_{s=1}^n \int \frac{d\lambda_s}{\lambda_s - z_s} \frac{\delta}{\delta V_x(\lambda_s)} \ln Z[V] \right) \bigg|_{V=V_0}. \quad (2.3)$$

where $V_0$ is a polynomial potential that determines the vacuum state of the theory. Since we restrict ourselves to the unitary models, a polynomial of third degree $V_0(z) = g_1 z + g_2 z^2 + g_3 z^3$ is sufficient.

After integrating with respect to the the $C$-fields, the partition function (2.4) can be defined in terms of the (shifted) eigenvalues $\lambda_{ix}, i = 1, 2, \ldots, N$, of the field $H_x$ as

$$Z[V] = \int \prod_{i,x} d\lambda_{ix} e^{-V_x(\lambda_{ix})} \prod_{x,x'} \prod_{i \neq j} (\lambda_{ix} - \lambda_{jx'}) \delta_{x,x'} \prod_{x,x'} \prod_{i \neq j} |\lambda_{ix} + \lambda_{jx'}|^2 A_{xx'}. \quad (2.4)$$

(The potential will change as well, but we will keep the same letter for it.)

Using the Cauchy identity, we formulate this model as a system of free fermions. This is the way to look for a nonperturbative solution. Our aim is to extract the perturbative piece of the partition function. For this purpose it is most convenient to introduce collective field $\psi$ and the corresponding lagrange multiplier field $v$ describing the fluctuations of the system above the large $N$ saddle point. This is done by inserting the identity

$$1 = \int D\psi Dv \exp \left( \sum_x \int d\lambda \ v_x(\lambda) \left[ \frac{d\psi_x(\lambda)}{d\lambda} - \sum_{i=1}^N \delta(\lambda - \lambda_{ix}) \right] \right) \quad (2.5)$$

in the r.h.s. of (2.4), and integrating over the $\lambda$’s. For each $x$ the integration with respect to the $\lambda_{ix}, i = 1, \ldots, N$, yields the “pure gravity” partition function

$$e^{\mathcal{F}[v]} = \int \prod_{i=1}^N d\lambda_i e^{-v(\lambda_i)} \prod_{i<j} (\lambda_i - \lambda_j)^2 \quad (2.6)$$

with potential $v = v_x$. After that the original partition function (2.4) can be written as a functional integral

$$Z[V] = \int D\psi Dv \ e^{-S[\psi,v,V]}, \quad (2.7)$$
\[
S[\psi, v, V] = \frac{1}{2} \sum_{x, x'} \int d\lambda d\lambda' A_{xx'} \frac{d\psi_x(\lambda)}{d\lambda} \ln |\lambda + \lambda'| \frac{d\psi_x'(\lambda')}{d\lambda'} 
+ \sum_x \int d\lambda \frac{d\psi_x(\lambda)}{d\lambda} \left( \psi_x(\lambda) - v_x(\lambda) \right) - \sum_x \mathcal{F}[v_x].
\] (2.8)

The zero mode in the \(\psi\)-integration is eliminated by imposing a Dirichlet boundary condition \(\psi_x(\infty) = 0\).

The string coupling constant \(\kappa \sim 1/N\) is contained in the genus expansion of the effective potential \(\mathcal{F}[v_x] = \mathcal{F}_0[v_x] + \mathcal{F}_1[v_x] + \cdots\).

3. Saddle point

The functional measure in (2.7) is a nonrestricted homogeneous measure and the string propagator and vertices are obtained by expanding the effective action (2.8) around the mean field \(\psi^c_x, v^c_x\) determined by the large \(N\) saddle point equations. The genus-zero expectation value of the resolvent (2.2) is related to the classical spectral density \(\rho^c_x(\lambda) = \partial^c_x / \delta v_x(\lambda) [v^c_x]\) by

\[
\langle W_x(z) \rangle_0 = \int d\lambda \frac{\partial^c_x(\lambda)}{z - \lambda}.
\] (3.1)

Along the real axis

\[
\langle W_x(\lambda) \rangle_0 = \frac{1}{2} \partial^c_x(\lambda) - i\pi \partial^c_x(\lambda)
\] (3.2)

and the saddle point equations can be written, in the momentum space, as

\[
2 \text{Re}\langle W_p(\lambda) \rangle_0 + 2 \pi p \langle W_p(\lambda) \rangle_0 = \delta_{p,p_0} V_0(\lambda).
\] (3.3)

The density \(\rho^c = \rho^c_{p_0}\) has a compact support \([a, b]\) with \(b < 0\). We are interested in the scaling regime, which is achieved in the limit \(a/b \to \infty\). To render the equations simpler, we will rescale \(\lambda \to |b|^{-1}\lambda\); then the support of \(\rho^c\) becomes the semi-infinite interval \([-\infty, -1]\). The solutions of the saddle point equation (3.3) behave at infinity as \(z^{\beta}\), where \(\cos \pi \beta = -\cos \pi p_0\). The two branches

\[
\beta = 1 \pm p_0
\] (3.4)

correspond to the dense (−) and dilute (+) critical regimes of the model. Up to an arbitrary normalization the solution in the scaling limit reads

\[
\langle W_x(z) \rangle_0 = -\frac{S_x}{\kappa} \frac{(z + \sqrt{z^2 - 1})^\beta + (z - \sqrt{z^2 - 1})^\beta}{2\pi |\sin \pi \beta|}
\] (3.5)

where \(\kappa \sim L^{1+\beta}/N\) is the renormalized string coupling constant. A string theory with this background can be viewed as a theory of 2d quantum gravity with the central charge of the matter field

\[
C = 1 - \frac{6(\beta - 1)^2}{\beta}.
\] (3.6)

For details see [6] and [10].
4. Gaussian fluctuations and analytic fields

In order to derive a set of Feynman rules one has to define the Hilbert space of one-string states, chose a complete orthonormalized set of eigenstates of the quadratic action, and finally express the interactions in terms of the mode expansion of the string field.

It is consistent with the perturbative expansion to assume that the fluctuating fields are again supported by a semi-infinite interval, but its right end can be displaced with respect to its saddle point value $-1$ due to the fluctuations. Therefore it is possible to represent the collective fields in terms of the analytic functions $\Psi_x(z)$ and $\Phi_x(z)$ defined on the $z$-plane cut along the negative real axis and such that

$$\psi_x(\lambda) = \frac{1}{\pi} \text{Im}\Psi_x(\lambda), \quad v_x(\lambda) = 2\text{Re}\Phi_x(\lambda), \quad \lambda < 0$$

(4.1)
determined up to an entire function of $z$. We will consider the discontinuity

$$\phi_x(\lambda) = \frac{1}{\pi} \text{Im}\Phi_x(\lambda)$$

(4.2)
of the analytic function $\Phi_x(z)$ as independent field variable, instead of $v_x$. The quantum field $\Psi$ is related to the loop operator (2.2) by

$$W_x(z) = \frac{\partial}{\partial z} \Psi_x(z)$$

(4.3)
and can be interpreted geometrically as the operator creating a loop without marked point on the world sheet. The macroscopic loop correlator (2.3) is equal to the derivative $\partial^n/\partial z_1 \cdots \partial z_n$ of the connected correlator of $\langle \Psi(z_1) \cdots \Psi(z_n) \rangle$. In the following by loop correlators we shall understand the correlators of the $\Psi$-field. Since we are interested only in the scaling limit, it is consistent to identify all analytic fields that differ by an entire function.

The gaussian fluctuations of the collective fields are those that do not shift the edge of the eigenvalue interval; the fluctuations displacing of the edge are described by nongaussian terms in the effective action that represent the $n$-string interactions. Therefore the leading (genus-zero) term of the effective action (2.8) can be split into gaussian and interacting parts

$$S_0 = S_{\text{free}} + \sum_{n \geq 3} S_{0,n}$$

(4.4)
where $S_{0,n}$ is the genus-zero $n$-string interaction.

Taking the genus-zero contribution to the mean field free energy (2.6)

$$F_0[v_x] = \frac{1}{\pi} \int d\lambda (\text{Re}\Phi_x - v_x) \text{Im} \frac{d\Phi_x}{d\lambda} = -\frac{1}{\pi} \int d\lambda \text{Re}\Phi_x \text{Im} \frac{d\Phi_x}{d\lambda}$$

(4.5)
we find for the gaussian action

$$S_{\text{free}} = \frac{1}{\pi} \sum_x \int_{-\infty}^{-1} d\lambda [\text{Re}\Phi_x(\lambda) \frac{d}{d\lambda} \text{Im}\Phi_x(\lambda) - 2\text{Re}\Phi_x(\lambda) \frac{d}{d\lambda} \text{Im}\Psi_x(\lambda)$$

$$+ \sum_{x'} A_{xx'} \psi_x(-\lambda) \frac{d}{d\lambda} \text{Im}\Psi_x(\lambda) + V_x(\lambda) \frac{d}{d\lambda} \text{Im}\Psi_x(\lambda)]$$

(4.6)
In writing (4.6) we used that $\text{Im} \Psi_x(-\lambda) = 0$, $\lambda < 0$.

We define the Hilbert space $\mathcal{H}$ of one-string states as the space of real functions $\phi_x(\lambda)$ defined on the interval $-\infty < \lambda < -1$, with scalar product

$$\langle \phi | \psi \rangle = \sum_{x \in X} \int_{-\infty}^{-1} \frac{d\lambda}{\sqrt{\lambda^2 - 1}} \psi_x(\lambda) \phi_x(\lambda). \quad (4.7)$$

For the purpose of diagonalizing the quadratic action it is very useful to introduce the map

$$z(\tau) = z(-\tau) = \cosh \tau. \quad (4.8)$$

transforming the space of meromorphic functions in the $z$-plane cut along the interval $[-\infty, -1]$ into the space of entire even analytic functions of $\tau$. The cut $z$-plane is parametrized by the semi-infinite strip $\{ \text{Re} \tau \geq 0, -\pi \leq \text{Im} \tau \leq \pi \}$ so that the two sides of the cut are parameterized by the boundaries $\{ \tau \pm i\pi, \tau > 0 \}$ of the strip

$$\lambda = \cosh (\tau \pm i\pi) = -\cosh \tau, \quad \tau \geq 0. \quad (4.9)$$

Due to the symmetry of the map (4.8) the contour integrals around the cut transforms into integrals along the shifted real axis $\tau - i\pi$, $-\infty < \tau < \infty$. In the following we will keep the same letters for the fields considered as functions of $\tau$ and denote $\Phi(\tau) \equiv \Phi(z(\tau))$.

The disc amplitude (3.5) as a function of $\tau$-variable is

$$\langle W_x(\tau) \rangle_0 = S_x \frac{\cosh \beta \tau}{2\pi |\sin \pi \beta|} \frac{\partial \Phi^c_x(\tau)}{\partial \cos \tau}. \quad (4.10)$$

It is quite evident that the plane waves

$$\langle x, \tau | p, E \rangle = S_p^x e^{iE\tau} \quad (4.11)$$

form a complete set of (delta-function) normalized wave functions diagonalizing the quadratic action (4.6). The Fourier components of the fields $\phi$ and $\psi$ are related to these of $\Phi$ and $\Psi$ by $\phi(p, E) = \frac{1}{\pi} \sinh(\pi E) \Phi(p, E), \psi(p, E) = \frac{1}{\pi} \sinh(\pi E) \Psi(p, E)$. The action (4.6) reads, in terms of these fields,

$$S^\text{free}[\psi, \phi] = \sum_{p \in P} \int_0^\infty dE \left( \left( \frac{\phi(p, E) - 2\psi(p, E)}{2\pi} \right) \frac{\pi E \cosh \pi E}{\sinh \pi E} \phi(p, E) + \psi(p, E) \frac{\pi E \cos p}{\sinh \pi E} \psi(p, E) - EV(p, E) \psi(p, E) \right) \quad (4.12)$$

By inverting the quadratic form in (4.12) we find the propagators in the $(E, p)$ space

$$G_{\psi\psi}(E, p) = G_{\phi\psi}(E, p) = G(E, p),$$

$$G_{\phi\phi}(E, p) = G(E, p) \frac{\cos p}{\cosh \pi E} = (G(E, p) - G(E, \frac{1}{2})). \quad (4.13)$$

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2 The choice of the scalar product is a matter of convenience and does not affect the final results. It is not conserved by the linear transformations relating different functional realizations of the one-string configuration space [11], [12].
where

\[ G(E, p) = \frac{\sinh \pi E}{\cosh \pi E - \cos \pi p} = \frac{2}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{E^2 + (p + 2n)^2}. \]  

(4.14)

The three propagators have the following diagrammatic meaning: \( G_{\phi\psi} \) (\( G_{\phi\phi} \)) is associated with the external (internal) lines of a Feynman diagram, and \( G_{\psi\psi} \) is the genus-zero loop-loop correlator \[4]\n
\[ \langle \Psi_p(z) \Psi_p(z') \rangle_0 = \int_0^\infty dE \frac{E}{\sinh \pi E} \frac{\sin E \tau \cdot \sin E \tau'}{\cosh \pi E - \cos \pi p} \]  

(4.15)

5. Interactions

The genus \( g \), \( n \)-string interactions, \( 2g + n - 2 > 0 \), are determined by the genus expansion of the one-matrix free energy \( \mathcal{F}[v] \) defined by (2.6). In the scaling limit \( \mathcal{F}[v] \) is the generating functional for the correlation functions

\[ \{k_1 \cdots k_n \}_g = \langle \sigma_{k_1} \cdots \sigma_{k_n} \rangle_g \]  

(5.1)

of the scaling operators \( \sigma_k, k = 1, 2, \ldots, \) in topological gravity \[13\]. The correlation functions (5.1) are the intersection numbers on the moduli space \( \mathcal{M}_{g,n} \) of algebraic curves of genus \( g \) with \( n \) marked points \[14\]. Each operator \( \sigma_k \) represents a \( 2k \)-form in the moduli space \( \mathcal{M}_{g,n} \) and the correlation function (5.1) is nonzero only if the product is a volume form, i.e., if \( 2(k_1 + \cdots + k_n) = \dim \mathcal{M}_{g,n} = 2(3g - 3 + n) \). The intersection numbers can be obtained from a system of recurrence relations equivalent to the loop equations \[15\]. In particular, the genus-zero the intersection numbers coincide with the multinomial coefficients

\[ \{k_1 \cdots k_n \}_0 = \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!}, \quad k_1 + \cdots + k_n = n - 3. \]  

(5.2)

For generic potential \( v = 2\text{Re}\Phi \), the deformation parameters \( t_0, t_1, \ldots \), are proportional to the coefficients in the expansion of the analytic function \( \Phi(z) \) in the half-integer powers of \( z - z_0 \) for some \( z_0 \). Sometimes these parameters are named KdV coordinates of the potential \( v \). The genus \( g \) term in the perturbative expansion of \( \mathcal{F} \) reads

\[ \mathcal{F}_g[\Phi] = \kappa_0^{2g-2} \sum_{n \geq 0} \frac{1}{n!} \sum_{k_1, \ldots, k_n \geq 0} \{k_1 \cdots k_n \}_g t_{k_1} \cdots t_{k_n}, \]  

(5.3)

where \( \kappa_0 \) is the string coupling and the sum is restricted to \( k_1 + \cdots + k_n = 3g - 3 + n \).

There is certain freedom in the choice of the coupling constants and we will use it in order to get simpler formulas. We choose \( z_0 = -1 \) and define the deformation parameters \( t_{kx} = t_k(\phi_x) \) by\[3\]

\[ \delta_{k,1} - t_{kx} = \frac{\langle k|\phi_x \rangle}{\langle 1|\phi_x \rangle} \]  

(5.4)

\[3\] As mentioned above, we are considering \( \phi = \frac{1}{\pi} \text{Im}\Phi \) as independent variable.
where the linear functionals \( \langle k | \phi \rangle \), \( k = 0, 1, \ldots \), are determined by the generating function

\[
\sum_{k=0}^{\infty} \frac{u^k}{k!} \langle k | \phi \rangle = \sqrt{2} \int_{0}^{\infty} \frac{dz}{2\pi i} \frac{d\Phi/dz}{\sqrt{z} + 1 - 2u} \quad (5.5)
\]

and \( \phi_x^c \) is the mean field corresponding to \( v_x^c \) and \( \psi_x^c \). With this definition the first two KdV coordinates of the string background vanish, \( t_0(\phi_x^c) = t_1(\phi_x^c) = 0 \), and the sum in (5.3) contains only finite number of terms. The topological-gravity coupling constant corresponding to the normalization (5.4) is \( \kappa_x = -\langle 1 | \phi_x^c \rangle^{-1} \). Note that its value generically depends on the point \( x \) in the target space. This is natural, since the mean field is \( x \)-dependent. The genus \( g \), \( n \)-string interaction is a linear functional in the tensor product \( \mathcal{H}^\otimes n \)

\[
\mathcal{S}_{g,n}(\phi) = \langle \mathcal{V}_{g,n} | \phi \rangle \cdots | \phi \rangle.
\]

(5.6)

Expanding \( \mathcal{F}_g[\phi_x] \) around the saddle point \( \phi_x^c \) we find the vertex \( \mathcal{V}_{g,n} \) in an operator form

\[
\langle \mathcal{V}_{g,n} \rangle = \frac{1}{n!} \sum \langle 1 | \phi_x^c \rangle^{2-2g-n} \langle k_1 \cdots k_{n+m} \rangle \frac{t_{k_{n+1}} \cdots t_{k_{n+m}}}{m!} \langle k_1, x | \cdots | k_n, x \rangle, \quad (5.7)
\]

where the sum goes over \( x \in X, m = 0, 1, \ldots ; k_1, \ldots, k_{n+m} \geq 0; k_1 + \cdots + k_{n+m} = 3g-3+n \), and \( \langle k, x | \phi \rangle \equiv \langle k | \phi_x \rangle \). In order to find the functional representation of the vertex \( \mathcal{V}_{g,n} \) we have to calculate the KdV coordinates of the plane waves (4.11). In the \( \tau \)-parametrization, the generating function (5.3) reads

\[
\sum_{n \geq 0} \frac{u^n}{n!} \langle n | \phi \rangle = \sqrt{2} \int_{0}^{\infty} d\tau \frac{\partial \tau \cos \pi \partial \tau \phi(\tau)}{\cosh \tau + 1 - 2u}. \quad (5.8)
\]

For \( \phi(\tau) = \sin E\tau \) the r.h.s. of (5.8) is a double series in \( u \) and \( E \)

\[
\frac{\sqrt{2}}{\pi} \int_{0}^{\infty} d\tau \frac{\cosh \pi E \cos E\tau}{\cosh \tau + 1 - 2u} = P_\frac{1}{2+iE} (1 - 2u) = \sum_{k=0}^{\infty} \frac{u^k}{k!} \Pi_k(iE), \quad (5.9)
\]

where

\[
\Pi_0(a) = 1; \quad \Pi_k(a) = \frac{1}{k!} \prod_{j=0}^{k-1} \left( (j + \frac{1}{2})^2 - a^2 \right), \quad k = 1, 2, \ldots \quad (5.10)
\]

Therefore the KdV coordinates of a plane wave are

\[
\langle k | E \rangle = \pi E \Pi_k(iE), \quad k = 0, 1, 2, \ldots \quad (5.11)
\]

The KdV coordinates \( t_k^c \) of the string background can be readily found using the fact that the derivative \( d\Phi^c(z)/dz \), eq. (4.10), is a plane wave with \( E = i\beta \), and the identity \( \langle k | d\phi/dz \rangle = \frac{1}{2} \langle k + 1 | \phi \rangle \). We get

\[
\langle k | \phi_x^c \rangle = -\frac{S_x}{\kappa} (\delta_{k,1} - t_k^c) \quad (5.12)
\]
with
\[ t_0^c = t_1^c = 0, \quad t_k^c = -\Pi_{k-1}(\beta), \quad k = 2, 3, \ldots \] (5.13)

Combining (5.7), (5.11) and (5.13) we find the explicit form of the vertex \( V_{g,n} \) in the \((p, E)\) space

\[
\langle V_{g,n}|p_1, E_1 \rangle \cdots |p_n, E_n \rangle = \frac{k^{2g-2+n}}{n!} \sum_{p_1, \ldots, p_n} C_{p_1 \cdots p_n}^g \sum_{m \geq 0} \left\{ k_1 \cdots k_{n+m} \right\}_{g} \frac{t_{k_{n+1}}^c \cdots t_{k_{n+m}}^c}{m!} \langle k_1|E_1 \rangle \cdots \langle k_n|E_n \rangle, \tag{5.14}
\]

where the sum goes over \( m = 0, 1, \ldots; k_1, \ldots, k_{n+m} \geq 0; k_1 + \cdots + k_{n+m} = 3g - 3 + n \).

The vertex (5.14) is represented in the \( \tau \)-space by a distribution supported by the point \( \tau = 0 \). Indeed, eq. (5.11) means that the linear functional \( \langle k | \) acts in the space of odd functions \( \phi(\tau) = -\phi(-\tau) \) smooth at \( \tau = 0 \), as

\[
\langle k | \phi \rangle = \pi (\Pi_k(\partial_\tau) \partial_\tau \phi(\tau))_{\tau=0}. \tag{5.15}
\]

Therefore the scattering of string states occurs only along the edge of the half-space \((x, \tau)\). This renders the discrete-space formulation of the string theory simpler than the continuous one where the interaction only falls exponentially at \( \infty \).

6. Feynman rules in the KdV representation

The most efficient way of calculating Feynman diagrams is to represent all entities by their KdV coordinates. The KdV representation of the internal propagator is given by the symmetric matrix

\[
G_{kk'}(p) = \langle k | (k') G_{\phi \phi}(p) \rangle = \pi^2 \left( \Pi_k(\partial_\tau) \Pi_{k'}(\partial_{\tau'}) \partial_\tau \partial_{\tau'} \left( G_{\phi \phi}(\tau, \tau'; p) \right) \right)_{\tau=\tau'=0}. \tag{6.1}
\]

The coordinate representation of the propagator (4.14) is

\[
G(\tau, \tau'; p) = \int_0^\infty \frac{4 \sin E \tau}{\pi} \sin E\tau' \ G(E, p) dE = G(\tau - \tau'; p) - G(\tau + \tau'; p) \tag{6.2}
\]

where

\[
\frac{\partial G(\tau; p)}{\partial \tau} = -\frac{1}{\pi^2} \sum_{n \in \mathbb{Z}} e^{-|p+2n|\tau} = -\frac{1}{\pi^2} \frac{\cosh(1-p)\tau}{\sinh \tau}, \quad 0 \leq p < 2. \tag{6.3}
\]

The internal propagator \( G_{\phi \phi}(\tau, \tau'; p) \) is regular at the origin

\[
G_{\phi \phi}(\tau, \tau'; p) = \frac{1}{\pi^2} \left( p - \frac{1}{2} \right) \left( p - \frac{3}{2} \right) \tau \tau' \left[ 1 + \frac{1}{12} \left( p^2 - 2p - \frac{3}{4} \right) (\tau^2 + \tau'^2) + \cdots \right] \tag{6.4}
\]
and from its Taylor expansion one extracts the matrix elements (6.1)

\[
G_{00}(p) = -\Pi_1(1-p), \quad G_{01}(p) = G_{10}(p) = -\Pi_2(1-p), \\
G_{11}(p) = -2\Pi_2(1-p) - 2\Pi_3(1-p), \quad \ldots
\] (6.5)

The external line factors in the corresponding Feynman diagrams are given by

\[
\langle k,p | G_{\phi\psi} | \tau,p \rangle = \frac{\pi}{\sin \pi \partial_\tau} \int_0^\infty \frac{dE}{\pi} \sin E\tau \ E \Pi_k(iE)G(E,p)
\]

\[
= \Pi_k(\partial_\tau) \frac{\sinh(1-p)\tau}{\sin \pi p \sinh \tau}.
\] (6.6)

The Feynman rules are summarized in Fig. 1. It is understood that the dressed vertices are composed by adding tadpoles to the bare vertices.

As an illustration we present some simple examples.

(i) Let us first check that the Feynman rules reproduce the loop correlation functions in the topological gravity when \( X = A_1, t_k^c = 0, k \geq 0 \). The momentum space contains a single momentum \( p_0 = 1/2 \), hence the propagator \( G_{\phi\phi} \) is identically zero and the leg factors are

\[
\Pi_k(\partial_\tau) \frac{\sinh(\tau/2)}{\sinh \tau} = \left(-2 \frac{d}{dz}\right)^k \frac{1}{\sqrt{2(z+1)}}.
\] (6.7)

In the spherical limit we get, using the explicit form of the intersection coefficients (5.2), the compact formula found originally in [16]

\[
\langle \prod_{a=1}^n \Psi(z_a) \rangle_0 = \sum_{k_1+\ldots+k_n=n-3} \frac{(k_1 + \ldots + k_n)!}{k_1! \ldots k_n!} \prod_{a=1}^n \left(-2 \frac{d}{dz_a}\right)^{k_a} \frac{1}{\sqrt{2(z_a + 1)}}
\]

\[
= \left(\frac{d^{n-3}}{du^{n-3}} \prod_{a=1}^n \frac{1}{\sqrt{2(z_a + 1 - 2u)}}\right)_{u=0}.
\] (6.8)
(ii) The three-loop genus-zero amplitude is just a product of a vertex \( \{000\}_0 C_{p_1,p_2,p_3}^0 \) and of three leg factors (Fig. 2)

\[
\langle \prod_{s=1}^3 \Psi_{p_s}(z_s) \rangle_0 = \kappa C_{p_1,p_2,p_3}^0 \prod_{s=1}^3 \frac{\sinh(1 - p_s)\tau_s}{\sin \pi p_s \sinh \tau_s}.
\]

One can check [17] that for \( \beta = 1 \) this amplitude coincides, after being transformed to the \( x \)-space, with the three-loop amplitude in the string theory with continuous target space, calculated in [12].

\[
\sum_{p' \in \mathcal{P}} \frac{1}{2} \left[ \frac{1}{24} \left( \beta^2 - \frac{\partial^2}{\partial \tau^2} \right) + \frac{1}{2} \left( p' - \frac{1}{2} \right) \left( p' - \frac{3}{2} \right) \right] \sinh(1 - p_0)\tau 
\]

This expression is in accord with the continuum limit of the genus-one loop amplitude for the \( O(n) \) model on a random lattice, \( n = 2 \cos \pi p_0 \), obtained recently by B. Eynard and C. Kristjansen [18] (in this case the sum contains only one term, \( p' = p_0 \)).

(iii) The genus-one tadpole (Fig. 3).

\[
\langle \Psi_p(z) \rangle_1 = \kappa \left[ C_p^1 \{1\}_1 \left( \frac{1}{4} - \frac{\partial^2}{\partial \tau^2} \right) + C_p^1 \{02\}_1 t_2^c \right] + \kappa \delta_{p,p_0} \sum_{p' \in \mathcal{P}} \frac{1}{2} \left[ \frac{1}{24} \left( \beta^2 - \frac{\partial^2}{\partial \tau^2} \right) + \frac{1}{2} \left( p' - \frac{1}{2} \right) \left( p' - \frac{3}{2} \right) \right] \sinh(1 - p_0)\tau \]
(iv) The four-loop genus-zero amplitude (Fig. 4).

\[
\left\langle \prod_{s=1}^{4} \Psi_{p_s}(z_s) \right\rangle_0 = \kappa^2 \left[ \left( \beta^2 - \frac{1}{4} + \sum_{s=1}^{4} \left( \frac{1}{4} - \frac{\partial^2}{\partial \tau_s^2} \right) \right) C_{p_1 p_2 p_3 p_4}^0 
+ \sum_{p} \left( C_{p_1 p_2 p}^0 C_{p p_3 p_4}^0 + C_{p_1 p_3 p}^0 C_{p p_2 p_4}^0 + C_{p_1 p_4 p}^0 C_{p p_2 p_3}^0 \right) \left( p - \frac{1}{2} \right) \left( p - \frac{3}{2} \right) \right] \tag{6.11}
\]

\[
\prod_{s=1}^{4} \frac{\sinh(1 - p_s) \tau_s}{\sin \pi p_s \sinh \tau_s}
\]

\[
\langle \prod_{s=1}^{4} \Psi_{p_s}(z_s) \rangle_0 = \delta^{(2)} \sum_{i=1}^{4} p_i \left[ 4 - 2|p_1 + p_2| - 2|p_1 + p_3| - |p_1 + p_4| \right]
+ \sum_{s=1}^{4} \left( p_s^2 - \frac{\partial^2}{\partial \tau_s^2} \right) \prod_{s=1}^{4} \frac{\sinh(1 - p_s) \tau_s}{\sin \pi p_s \sinh \tau_s}. \tag{6.12}
\]

Fig. 4: Four-loop genus-zero amplitude

In the limit \( \beta \to 1 \) of one-dimensional target space the fusion coefficient represents a periodic delta-function, \( C_{p_1 \cdots p_n}^0 = \delta^{(2)}(p_1 + \cdots + p_n) \), and eq. (6.11) reproduces the four-loop amplitude found in \([17]\).
7. Conclusions

In this work, we have constructed a string field diagram technique for $C \leq 1$ backgrounds. An essential feature of this diagram technique is the factorization of the vertices into a matter-dependent (fusion coefficient) and gravity-dependent (intersection number) parts. We have found a minimal representation of the Feynman rules characterized by discrete quantum numbers, which allows to efficiently calculate the loop amplitudes on a surface with arbitrary genus.

The discreteness of the target space was crucial for the derivation of our Feynman rules. However, as it was argued in [17], there might be a one-to-one correspondence between the string theories with discrete and continuous target spaces. Such a correspondence would confirm the idea of Klebanov and Susskind about the appearance of a minimal length in the target space [19]. In ref. [17] the target spaces $A_\infty \sim \mathbb{Z}$ and $\mathbb{R}$ were compared. We expect that, more generally, every string theory with discrete target space can be mapped onto a string theory with one-dimensional space of orbifold type. However, after translating the Feynman rules into the continuum language, the vertices will lose their beautiful factorized form and the so called "special states" with integer momenta will appear as a remnant of the periodicity in the momentum space of the original vertices and propagator.

Another important issue to be addressed is to derive the string field perturbative expansion from more fundamental geometrical principles, without reference to a matrix model. For this one has to find the underlying algebraic structure and the related symmetry. One of the possible approaches relies on the $W$-algebra symmetry [13]. However, it seems that a more natural structure, following directly from the loop equations, is described by a direct sum of Virasoro symmetries, one for each point of the target space [20].

Finally, let us mention that our discrete Feynman rules can be readily extended to the open string sector using the explicit expressions for the interactions of open strings derived in ref. [3].

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