A Short Review of Ergodicity and Convergence of Markov chain Monte Carlo Estimators

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Abstract. This short note reviews the basic theory for quantifying both the asymptotic and preasymptotic convergence of Markov chain Monte Carlo estimators.

In this note we’ll review useful notions of distance between probability distributions and use them to specify the conditions under which a Markov transition will converge to a unique invariant distribution, both asymptotically and preasymptotically. The material will assume familiarity with the basic concepts of measure theory.

1. MARKOV, MARKOV, MARKOV!

Consider an ambient topological space $X$ equipped with the $\sigma$-algebra $\mathcal{X}$. A Markov transition

$$\tau: \mathcal{X} \times X \to \mathbb{R}$$

$$A, x \mapsto \tau(A \mid x)$$

specifies a probability distribution at each point $x \in X$, and sampling from this distribution realizes a transition from that initial point $x_0$ to a new point $x_1$. Applying the Markov transition repeatedly yields a sequence of points $(x_0, x_1, \ldots, x_N)$ called a Markov chain that meanders through the ambient space.

More formally a Markov transition lifts any distribution of initial points $\rho$ into a joint distribution over the product space of initial points and transitions $(x_0, x_1) \in X \times X$, $\tau \times \rho$. Applying the Markov transition again yields a joint distribution over the product space of three states $(x_0, x_1, x_2) \in X \times X \times X$,

$$\tau^2 \times \rho = \tau \times (\tau \times \rho) = \tau \times \tau \times \rho,$$

and applying it $N$ times yields a joint distribution over possible Markov chain configurations $(x_0, \ldots, x_N) \in X^{N+1}$,

$$\tau^N \times \rho = \tau \times \ldots \times \tau \rho.$$

Lifting $\rho$ to the joint distribution $\tau \times \rho$ and then marginalizing to the space of transitions convolves the Markov transition over all possible initial states, yielding the 1-step distribution of possible transitions,

$$\tau \circ \rho \equiv (\varpi_1)_*(\tau \times \rho),$$

where $\varpi_n$ denotes the product space projection operator,

$$\varpi_n: X^{N+1} \to X$$

$$(x_0, \ldots, x_n, \ldots, x_N) \mapsto x_n.$$
Repeating this convolution to the 1-step distribution yields the marginal distribution of second transitions, or the 2-step distribution,
\[
\tau^2 \circ \rho = \tau \circ (\tau \circ \rho),
\]
and repeating the convolution \(N\) times yields the marginal distribution for the \(N + 1\) state in the Markov chain or the \(N\)-step distribution,
\[
\tau^N \circ \rho = \tau \circ \ldots \circ \tau \circ \rho.
\]

Equivalently we can define the \(N\)-step distribution as the marginal distribution of the full joint distribution,
\[
\tau^N \circ \rho = (\tau^N)_\times (\tau^N \times \rho).
\]

A probability distribution \(\pi\) that is invariant to the Markov convolution,
\[
\tau \circ \pi = \pi,
\]
is said to be preserved by the Markov transition in which case \(\pi\) is denoted a stationary or invariant distribution. When a Markov transition is engineered to preserve a given distribution that distribution is also denoted the target distribution.

We often take for granted that when a Markov transition \(\tau\) preserves a given target distribution \(\pi\) the Markov chains generated from \(\tau\) will explore \(\pi\). Unfortunately invariance alone is not guaranteed to fully quantify \(\pi\), even asymptotically as the chains grow to infinite lengths. In order for Markov chains to adequately characterize \(\pi\) the \(N\)-step distribution needs to converge towards \(\pi\) sufficiently quickly. To formally define the desired convergence let alone speed of that convergence, however, we first need to define some notion of distance between probability distributions.

2. QUANTIFYING CONVERGENCE

There are many notions of distance between measures, and hence probability distributions, but a few are particularly useful for the theoretical analysis of Markov chain Monte Carlo. In this section we’ll discuss integral probability metrics, Wasserstein metrics, their duality, and important special cases.

2.1 Integral Probability Metrics

An integral probability metric Müller (1997) defines a notion of distance between measures by differences in the expectation values of certain functions.

Consider two measures \(\mu\) and \(\nu\) over \(X\) and a space of real-valued, measurable functions, \(f : X \to \mathbb{R} \in \mathcal{F}\). If both \(\mathbb{E}_\mu[f]\) and \(\mathbb{E}_\nu[f]\) are finite for all \(f \in \mathcal{F}\) then we can define the integral probability metric for \(\mathcal{F}\) as the largest difference between expectation values,
\[
||\mu - \nu||_{\mathcal{F}} = \sup_{f \in \mathcal{F}} |\mathbb{E}_\mu[f] - \mathbb{E}_\nu[f]|.
\]

One immediate consequence of this definition is that an integral probability metric bounds how well expectations with respect to \(\mu\) approximate expectations with respect to \(\nu\), and vice versa, within the space of test functions \(\mathcal{F}\).

When \(\mathcal{F}\) spans functions of practical interest any constraints on the corresponding integral probability metric are directly applicable in applied settings. More restrictive spaces, however, can still prove useful more indirectly. In particular integral probability metric bounds for restricted spaces of functions can sometimes demonstrate the absence of pathological behavior that would otherwise effect the expectation values of functions outside of that space.
2.2 Wasserstein Metric

Wasserstein metrics Sriperumbudur et al. (2009) define a notion of distance between probability distributions through a distance function defined on the ambient space. In particular let $g : X \times X \to \mathbb{R}^+$ be a positive, symmetric function, $g(x_1, x_2) = g(x_2, x_1) \geq 0$, that vanishes if and only if the two arguments are identical, $g(x_1, x_2) = 0$ if and only if $x_1 = x_2$.

We first assume that the distance function is compatible with the two measures being compared so that the expectation values of both arguments are well-defined,

$\mathbb{E}_\mu[g(-, x)] = \mathbb{E}_\mu[g(x, -)] < \infty, \forall x \in X$

$\mathbb{E}_\nu[g(-, x)] = \mathbb{E}_\nu[g(x, -)] < \infty, \forall x \in X$.

In order to take an expectation value of both arguments of the distance function we need to lift $\mu$ and $\nu$ together to a joint measure on the product space $X \times X$. A coupling of $\mu$ and $\nu$ is any joint distribution $\gamma$ over $X \times X$ that marginalizes to $\mu$ and $\nu$ in each component,

$\gamma(A \times X) = \mu(A), \forall A \in X$

$\gamma(X \times A) = \nu(A), \forall A \in X$.

We’ll denote the space of all couplings of $\mu$ and $\nu$ by $\Gamma(\mu, \nu)$. Because we assumed that the marginal expectation values are well-defined the joint expectation value with respect to any coupling $\mathbb{E}_\gamma[g] \in \mathbb{R}$ will also be well-defined.

The $1$-Wasserstein metric, also known as the Kantorovich-Rubinstein distance, is then defined as the smallest of these joint expectation values over all possible couplings,

$W_{1,g}(\mu, \nu) = \inf_{\gamma \in \Gamma(\mu, \nu)} \mathbb{E}_\gamma[g]$.

2.3 Kantorovich-Rubinstein Theorem

Although they might appear different superficially, integral probability metrics and $1$-Wasserstein metrics actually provide equivalent notions of distance between two measures. This duality is formalized in the Kantorovich-Rubinstein Theorem Kantorović and RubinŠtejn (1958); Dudley (2002).

A distance function $g$, defines a Lipschitz semi-norm on functions by

$\|f\|_g = \sup_{x_1 \neq x_2} \frac{|f(x_1) - f(x_2)|}{g(x_1, x_2)}$.

The Lipschitz semi-norm in turn defines a space of all continuous functions whose semi-norms are less than or equal to $1$,

$\mathcal{F}_g = \{ f \in C^0(X) \mid \|f\|_g \leq 1 \}$.

While the positional metric immediately defines a $1$-Wasserstein metric $W_{1,g}(\mu, \nu)$, this space of Lipschitz functions defines a corresponding integral probability metric, $\|\mu - \nu\|_{\mathcal{F}_g}$. The Kantorovich-Rubinstein Theorem shows that these two metrics are in fact equal provided that all of the relevant expectation values are well-defined.

The ability to jump back and forth between integral probability and $1$-Wasserstein metrics is particularly useful in practice. For example one might be able to theoretically bound convergence in one metric but then use its dual representation to understand the consequences of that bound in practical settings.

2.4 The Total Variation Distance

The total variation distance is a particularly useful metric for the analysis of Markov chain Monte Carlo that can be derived from either of these dual perspectives.

From the $1$-Wasserstein perspective the total variation distance is given by using a degenerate distance function specified by an indicator function,

$g = \mathbb{1}_{D^c}$.
where
\[ D = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\}. \]
and
\[ D_c = \{(x_1, x_2) \in X \times X \mid x_1 \neq x_2\}. \]

In other words the distance between any two distinct points is always 1 while the distance between any point and itself is 0. Working through the coupling definition of the 1-Wasserstein metric one can show that the total variation distance is equal to the largest difference of allocated measure over elements of the defining \( \sigma \)-algebra,
\[ \text{TV}(\mu, \nu) = ||\mu - \nu||_{TV} = \sup_{A \in \mathcal{X}} |\mu(A) - \nu(A)|. \]

The corresponding integral probability metric definition is given by
\[ ||\mu - \nu||_{TV} = \sup_{0 \leq f \leq 1} |E_\mu[f] - E_\nu[f]|, \]
where \(0 \leq f \leq 1\) denotes the space of continuous, real-valued functions whose outputs are bounded between 0 and 1.

This bound on function behavior, and hence the class of test functions being considered, can also be scaled so long as the normalization is modified. More formally
\[ ||\mu - \nu||_{TV} = \frac{1}{f_{\max} - f_{\min}} \sup_{f_{\min} \leq f \leq f_{\max}} |E_\mu[f] - E_\nu[f]|. \]

Taking advantage of this scaling freedom the total variation distance is often defined as
\[ ||\mu - \nu||_{TV} = \frac{1}{2} \sup_{-1 \leq f \leq 1} |E_\mu[f] - E_\nu[f]| \]
\[ = \frac{1}{2} \sup_{|f| \leq 1} |E_\mu[f] - E_\nu[f]|. \]

2.5 Application to Markov chain Monte Carlo

Any of the probability metrics introduced above can provide some quantification for how similar the \(N\)-step distribution of a Markov transition \(\tau^N \circ \rho\) is to the target distribution \(\pi\). If the two distributions become more similar as \(N\) increases then any Markov chain generated by the Markov transition will provide a better approximations to \(\pi\) as it becomes longer.

The challenge at hand is then to show not only that the distance \(||\tau^N \circ \rho_1 - \pi||\) converges towards 0 as \(N\) increases but also provide an explicit bound on how quickly the distance decreases with increasing \(N\).

3. ASYMPTOTIC CONVERGENCE

The invariance of the target distribution provides some constraints on how the \(N\)-step distribution of a Markov chain converges towards the target distribution Roberts and Rosenthal (2004). For example the total variation distance between \(N\)-step distributions from different initializations will be bounded by the distance between those initializations,
\[ ||\tau^N \circ \rho - \tau^N \circ \rho'||_{TV} \leq ||\rho - \rho'||_{TV}. \]

If we initialize the second Markov chain from stationarity, \(\rho' = \pi\), then \(\tau^N \circ \pi = \pi\) and this bound implies
\[ ||\tau^N \circ \rho - \pi||_{TV} \leq ||\rho - \pi||_{TV}. \]

In words the \(N\)-step distributions cannot move any further from the target distribution in total variation distance than the initial distribution.
Moreover if we take $\rho = \tau^M \circ \rho'$ for some initialization $\rho'$ then
\[
||\tau^{N+M} \circ \rho' - \pi||_{TV} \leq ||\tau^M \circ \rho' - \pi||_{TV}.
\]
Consequently once the $N$-step distribution achieves a certain distance it can never move further than that distance in future iterations. In other words the $N$-step distributions are non-expanding in the total variation distance.

This non-expansive behavior of the total variational distance can also be interpreted as a data processing inequality Cover and Thomas (2006). Unfortunately non-expansive behavior in other probability metrics is not guaranteed by the invariance of the target distribution alone.

That said non-expansion in the total variation distance is still not enough to ensure that the total variation distance vanishes asymptotically. In particular if the Markov transition features multiple invariant distributions then we cannot guarantee that Markov chains will focus on the desired target distribution, $\pi$. The $N$-step distribution is guaranteed to converge to $\pi$ asymptotically only when certain undesired behavior can be avoided Roberts and Rosenthal (2004).

### 3.1 Reducibility

One important undesired behavior is that Markov chain initialized within some neighborhood might not be able to explore every other relevant neighborhood in $X$. To formalize relevance here we define the $\pi$-null sets as those measurable neighborhoods with vanishing target probability,
\[
\mathcal{X}_{\pi\text{-null}} = \{ A \in \mathcal{X} | \pi(A) = 0 \},
\]
and the complementary $\pi$-non-null sets as those measurable neighborhoods with non-zero target probability
\[
\mathcal{X}_{\pi\text{-non-null}} = \{ A \in \mathcal{X} | \pi(A) > 0 \}.
\]
In order to completely explore the target distribution every Markov chain needs to be able to eventually explore every $\pi$-non-null set.

Markov transitions that generate Markov chains that cannot reach every relevant set, more formally when there is at least one $\pi$-non-null set $A \in \mathcal{X}_{\pi\text{-non-null}}$ with $(\tau^N \circ \rho)(A) = 0$ for all $N$, are denoted $\pi$-reducible (Figure 1). Reducible Markov transitions often partition the ambient space into non-overlapping sets in which realized Markov chains are confined (Figure 2). In other words exploration of the ambient space is “reduced” to the local exploration of these divided neighborhoods.

If the $N$-step distribution from every point initialization $x_0 \in X$,
\[
\tau^N \circ \delta_{x_0},
\]
allocates non-zero probability to every $\pi$-non-null set $A$,
\[
\tau^N \circ \delta_{x_0}(A) > 0,
\]
for some $0 < N(x) < \infty$, then at least some realized Markov chains from every initialization will reach every $\pi$-non-null set. In this case we say that the Markov transition is $\pi$-irreducible.

One convenient circumstance where $\pi$-irreducibility is guaranteed is when the density of an $N$-step distribution with respect to the target distribution is everywhere positive,
\[
\frac{d(\tau^N \circ \delta_{x_0})}{d\pi}(x) > 0
\]
for all $x_0, x \in X$.

Critically irreducibility is determined not by the precise probability $\pi(A)$ allocated to measurable sets but rather the classification of measurable sets into null and non-null sets. If $\pi(A) = 0$ whenever $\phi(A) = 0$ for some $\sigma$-finite measure $\phi$ then $\pi$ and $\phi$ will share the same null sets, and $\phi$-irreducibility will immediately imply $\pi$-irreducibility. In this case we say that $\phi$ dominates $\pi$ or, equivalently, that $\pi$ is absolutely continuous with respect to $\phi$. 
\[ \pi(\mathcal{A}) > 0 \quad (\tau^N \circ \rho)(\mathcal{A}) = 0 \]

Fig 1. A $\pi$-reducible Markov transition $\tau$ initialized from $\rho$ features at least one set $\mathcal{A}$ with non-zero invariant probability $\pi(\mathcal{A}) > 0$ that cannot be reached by all Markov chains initialized from $\rho$ no matter how large $N$ is.

Figs 2. A $\pi$-reducible Markov transition $\tau$ often induces a partition of the ambient space. Here Markov chains initialized at positive values will never reach negative values while Markov chains initialized at negative values will never reach positive values.

Consequently $\pi$-irreducibility is also guaranteed whenever the density of an $N$-step distribution with respect to a dominating base measure is everywhere positive,

\[
\frac{d(\tau^N \circ \delta_{x_0})}{d\phi}(x) > 0
\]

for all $x_0, x \in X$.

The same result also holds if only a component of the $N$-step distribution is absolutely continuous with respect to $\phi$. More formally if we can write $N$-step distribution as a mixture

\[
\tau^N \circ \delta_{x_0} = \lambda \rho_1 + (1 - \lambda) \rho_2
\]
A Markov transition is periodic whenever there is a sequence of disjoint, $\pi$-non-null sets that trap Markov chains into cyclic transitions. Here all Markov transitions that start in $A_1$ are confined to $A_2$, those that start in $A_2$ are confined to $A_3$, and those that start in $A_3$ are confined to $A_1$. Once a Markov chain wanders into any of these sets it will be forever doomed to cycle between the three sets and unable to explore the rest of the ambient space.

with $0 < \lambda(x, x_0) \leq 1$ then

$$\frac{d\rho_1}{d\phi}(x) > 0 \text{ for all } x_0, x \in X,$$

implies that the Markov transition $\tau$ is $\pi$-irreducible even if $\rho_2$ is not dominated by the base measure $\phi$. This form is especially convenient when working with Metropolis-Hastings methods that mix a continuous proposal transition with a singular rejection transition.

### 3.2 Periodicity

Irreducibility ensures that at least some realized Markov chains from any initialization will reach every $\pi$-non-null set at least once, but it doesn’t guarantee that those sets will be visited repeatedly. Even if a Markov transition is $\pi$-irreducible the exploration of some Markov chains can be obstructed by cyclic behavior that forever traps them within some subset of the ambient space.

Formally consider a collection of at least two $\pi$-non-null sets,

$$A_1, \ldots, A_j, \ldots, A_J \in \mathcal{X}$$

that are disjoint,

$$A_j \cap A_{j'} = \emptyset \text{ for } j \neq j'.$$

If $\tau(A_{j+1} \mid x) = 1$ for all $x \in A_j$ then all transitions from points in $A_j$ will be confined to $A_{j+1}$. Moreover if $\tau(A_1 \mid x) = 1$ for all $x \in A_J$ then those transitions will eventually return to $A_1$ where the cycle begins anew (Figure 3). In this case the Markov chain is said to be $\pi$-periodic with period $J$ and periodic decomposition $A_1, \ldots, A_J$.

A period decomposition serves as a sink that absorbs any Markov chains that venture into any of the component sets and preventing them from exploring anywhere else. In order to ensure that Markov chains have the opportunity to explore the entire target distribution over and over again we have to avoid this absorbing behavior by ensuring that our Markov transition is $\pi$-aperiodic.

One common way to avoid periodic behavior is to ensure that the Markov transition always admits a nonzero probability of staying at the initial point. If $\tau(x \mid x) > 0$ then $\tau(A_{j+1} \mid x) = 1$ only if $x \in A_{j+1}$. Consequently transitions initialized in a set $A_j$ cannot be confined to a disjoint set $A_{j+1}$, obstructing a periodic decomposition.
3.3 Recurrence

If a Markov transition with an invariant distribution $\pi$ is both $\pi$-irreducible and $\pi$-aperiodic then $\pi$ is the unique invariant distribution. In this case the Markov transition is said to be $\pi$-recurrent. The Markov chains generated from $\pi$-recurrent Markov transitions will explore every $\pi$-non-null set well enough to ensuring asymptotic convergence in the total variation distance,

$$\lim_{N \to \infty} |\tau^N \circ \delta_x - \pi| = 0$$

for $\pi$-almost-all point initializations $x \in X$. This also implies that the empirical average of any measurable function $f : X \to \mathbb{R}$ with a well-defined expectation value $\mathbb{E}_{\pi}[f] < \infty$,

$$\hat{f}_N(x_0, \ldots, x_N) = \frac{1}{N+1} \sum_{n=0}^{N} f \circ \omega_n(x_0, \ldots, x_N) = \frac{1}{N+1} \sum_{n=0}^{N} f(x_n)$$

converges to that exact expectation value,

$$\lim_{N \to \infty} \hat{f}_N = \mathbb{E}_{\pi}[f]$$

for almost all Markov chain realizations $\{x_1, \ldots, x_n, \ldots\}$.

Extending this result to all point initializations requires a stronger condition than $\pi$-recurrence alone. If for all $\pi$-non-null sets $A \in \mathcal{X}_{\text{non-null}}$ and point initializations $x \in X$ we have

$$\sum_{n=1}^{N(x)} (\tau^N \circ \delta_x)(A) = 1$$

for some finite $N(x) < \infty$ then the Markov chain is said to be Harris recurrent (Harris 1956; Tierney 1994; Chan and Geyer 1994). Harris recurrence ensures that all Markov chain realizations from all initializations will visit every $\pi$-non-null set infinitely often. This then implies that

$$\lim_{N \to \infty} |\tau^N \circ \delta_x - \pi| = 0$$

for all $x \in X$.

While a non-zero density of the $N$-step distribution relative to the target distribution, is a useful sufficient condition for $\pi$-irreducibility, a non-zero density of the 1-step distribution,

$$\frac{d(\tau \circ \delta_{x_0})}{d\pi}(x) > 0$$

for all $x_0, x \in X$, is sufficient for Harris recurrence. For any dominating measure $\phi$ Harris recurrence is also implied by

$$\frac{d(\tau \circ \delta_{x_0})}{d\phi}(x) > 0$$

for all $x_0, x \in X$. As with $\pi$-irreducibility we can also verify Harris recurrence using only mixture components of 1-step distribution to avoid any singular behavior.

4. PREASYMPTOTIC CONVERGENCE

Given the inherent difficulty of probabilistic computation we can’t take asymptotic convergence of Markov chain Monte Carlo estimators for granted. To avoid problematic behavior we have to carefully verify the recurrence properties of a given Markov transition, or rely on Markov transitions that have been vetted by experts. Unfortunately asymptotic convergence alone does not guarantee that Markov chain Monte Carlo will behave well in practice, where we can only ever realize finite Markov chains.

Practical Markov chain Monte Carlo performance is instead determined by the preasymptotic convergence where $N$ is large but finite, which is a much more delicate property. In this section we will review common strategies for quantifying the preasymptotic convergence of Markov chain Monte Carlo estimators.
4.1 Bounding Convergence

While asymptotic convergence determines to where a Markov $N$-step distribution converges, preasymptotic convergence determines how it gets there. Typically we quantify preasymptotic convergence by bounding the distance between the $N$-step distribution and the target distribution as a function of the number of iterations,

$$||\tau^N \circ \rho - \pi||_F \leq b(\rho, N)$$

for some monotonically decreasing function $b(\rho, N)$ that converges to zero,

$$\lim_{N \to \infty} b(\rho, N) = 0,$$

If we can bound the distance from every point initialization,

$$||\tau^N \circ \delta_x - \pi||_F \leq b(x, N),$$

then we can bound the distance from any distributional initialization $\rho$ with an expectation value,

$$b(\rho, N) = \mathbb{E}_\rho [b(\cdot, N)] \leq \sup_{x \in X} b(x, N).$$

Consequently we will focus on bounding the distance between the $N$-step distribution and the target distribution from point initializations.

These $N$-step distribution bounds allow us to immediately quantify the preasymptotic behavior of Markov chain Monte Carlo estimators. For example we can work out the preasymptotic bias of the Markov chain Monte Carlo estimator for any test function $f \in \mathcal{F}$,

$$\left| \mathbb{E}_{\tau^N \circ \delta_x} [\hat{f}_N] - \mathbb{E}_\pi [f] \right| = \left| \left( \frac{1}{N+1} \sum_{n=0}^{N} \mathbb{E}_{\tau^N \circ \delta_x} [f \circ \omega_n] \right) - \mathbb{E}_\pi [f] \right|$$

$$= \left| \frac{1}{N+1} \sum_{n=0}^{N} \mathbb{E}_{\tau^N \circ \delta_x} [f \circ \omega_n] - \frac{1}{N+1} \sum_{n=0}^{N} \mathbb{E}_\pi [f] \right|$$

$$= \left| \frac{1}{N+1} \sum_{n=0}^{N} \left( \mathbb{E}_{\tau^N \circ \delta_x} [f] - \mathbb{E}_\pi [f] \right) \right|$$

$$\leq \frac{1}{N+1} \sum_{n=0}^{N} \left| \mathbb{E}_{\tau^N \circ \delta_x} [f] - \mathbb{E}_\pi [f] \right|$$

$$\leq \frac{1}{N+1} \sum_{n=0}^{N} ||\tau^N \circ \delta_x - \pi||_F$$

$$\leq \frac{1}{N+1} \sum_{n=0}^{N} b(x, N).$$

In other words the bias of Markov chain Monte Carlo estimators decays linearly with $N$ provided that $\sum_{n=0}^{N} b(x, N) < N+1$. Bounds on the estimator variance requires understanding the correlations between Markov chain states and a more delicate calculation; see for example Joulin and Ollivier (2010).

If the distance between the $N$-step distribution and target distribution is bounded by a geometrically decreasing function of $N$ that is independent of the initialization,

$$||\tau^N \circ \delta_x - \pi||_F \leq b \cdot r^N$$

where $r$ is the geometric rate of convergence, then we can work out the preasymptotic bias for a test function $f \in \mathcal{F}$.
for $0 \leq r < 1$, then we say that the Markov transition is *uniformly geometrically ergodic* in the associated metric. In this case

$$\left| \mathbb{E}[\hat{f}_N] - \mathbb{E}_\pi[f] \right|$$

$$\leq \frac{1}{N+1} \sum_{n=0}^N \left| \mathbb{E}_{\tau^n \circ \rho}[f] - \mathbb{E}_\pi[f] \right|$$

$$\leq \frac{1}{N+1} \sum_{n=0}^N b r^n$$

$$\leq \frac{b}{N+1} \sum_{n=0}^N r^n$$

$$\leq \frac{b}{N+1} \frac{1 - r^{N+1}}{1 - r}.$$  

Because $r < 1$, the bias will monotonically decay at a geometric rate.

Similarly if the metric is bounded by a geometrically decreasing function of $N$ that depends on the initialization,

$$||\tau^N \circ \rho - \pi||_F \leq b(\rho) r^N$$

then we say that the Markov transition is *geometrically ergodic* in the associated metric. Here the bias will still decay monotonically, but the distance will be affected by the initial condition,

$$\left| \mathbb{E}[\hat{f}_N] - \mathbb{E}_\pi[f] \right| \leq \frac{b(\rho)}{N+1} \frac{1 - r^{N+1}}{1 - r}.$$  

Uniform geometric ergodicity is rare and for general problems geometric ergodicity is usually the best we can hope to establish. That said establishing geometric ergodicity is no easy task itself, with many bounds readily obstructed by not-uncommon interactions between a Markov transition and its stationary distribution.

### 4.2 Coupling Methods

In Section 2.2 we used couplings between measures to construct the 1-Wasserstein metric. Couplings between two Markov chains are also useful for studying the ergodicity properties of a given Markov transition.

Consider a Markov transition $\tau$ and two different initializations, $\rho$ and $\omega$, from which we can construct two $N$-step distributions, $\tau^N \circ \rho$ and $\tau^N \circ \omega$. A *Markov coupling* of $\tau$ is any Markov transition $\tau_\gamma$ defined on the product space $X \times X$ and initialized from the product distribution $\rho \times \omega$ such that every $N$-step distribution $\tau^N_\gamma \circ (\rho \times \omega)$ is a coupling between $\tau^N \circ \rho$ and $\tau^N \circ \omega$.

A Markov coupling is said to be *contractive* if the distance between the two marginal distributions is bounded by a monotonically decreasing function of $N$,

$$||\tau^N \circ \rho - \tau^N \circ \omega||_F \leq b(\rho, \omega, N).$$

While the total variation distance between $N$-step distributions from different initializations is always non-expanding the distance from other metrics will be non-expanding, let alone explicitly contractive, only in special cases.

With clever choices of $\rho$ and $\omega$ a contractive Markov coupling will bound the convergence of the Markov transition $\tau$. For example if we take $\omega = \pi$ then

$$\tau^N \circ \omega = \tau^N \circ \pi = \pi$$

and

$$||\tau^N \circ \rho - \pi||_F = ||\tau^N \circ \rho - \tau^N \circ \omega||_F \leq b(\rho, \pi, N).$$
In other words a coupling between Markov chain initialized from an arbitrary initialization \( \rho \) and a Markov chain given a \textit{warm} initialization from the stationary distribution can be used to bound the convergence of the \( N \)-step distribution \( \tau^N \circ \rho \).

4.2.1 Wasserstein Bounds The most common approach to constructing general 1-Wasserstein bounds is to consider the 1-step distributions from two point initializations, \( \tau \circ \delta_{x_1} \) and \( \tau \circ \delta_{x_2} \). Scaling the 1-Wasserstein distance between these two distributions by the defining distance function gives the \textit{coarse Ricci curvature} Ollivier (2009); Joulin and Ollivier (2010)

\[
\kappa(x, x') = 1 - \frac{W_{1,g}(\tau \circ \delta_x, \tau \circ \delta_{x'})}{g(x, x')}
\]

If the course Ricci curvature is uniformly lower bounded by some constant,

\[
0 < \kappa \leq \kappa(x, x'), \quad \forall x, x' \in X
\]

then the 1-Wasserstein distance between the \( N \)-step distribution from any point initialization and the target distribution is bounded by

\[
W_{1,g}(\tau^N \circ \delta_x, \pi) \leq (1 - \kappa)^N \cdot W_{1,g}(\tau \circ \delta_x, \pi) \leq (1 - \kappa)^N \cdot \mathbb{E}_\pi[g(x, -)].
\]

In order to bound the Ricci curvature we have to bound the 1-Wasserstein distance between the 1-step distributions,

\[
W_{1,g}(\tau \circ \delta_x, \tau \circ \delta_{x'}) = \inf_{\gamma \in \Gamma(\tau \circ \delta_x, \tau \circ \delta_{x'})} \mathbb{E}_\gamma[g],
\]

This, in turn, is bounded by the expectation of the distance function with respect to \textit{any} coupling between the 1-step distributions from two different point initializations. Engineering any mathematically-convenient coupling \( \gamma \in \Gamma(\tau \circ \delta_x, \tau \circ \delta_{x'}) \) that satisfies

\[
\mathbb{E}_\gamma[g] \leq g(x, x')
\]

for all \( x, x' \in X \) immediately establishes an upper bound on the 1-Wasserstein distance,

\[
W_{1,g}(\tau \circ \delta_x, \tau \circ \delta_{x'}) \leq \mathbb{E}_\gamma[g] \leq g(x, x'),
\]

which then establishes a uniform lower bound on the coarse Ricci curvature,

\[
1 - \kappa = \frac{W_{1,g}(\tau \circ \delta_x, \tau \circ \delta_{x'})}{g(x, x')} \leq 1.
\]

The more strongly Markov chains initialized from two points contract \textit{towards} each other in expectation the larger the lower bound we can establish on the Ricci curvature and the smaller the upper bound we can establish on the convergence of the \( N \)-step distribution in the 1-Wasserstein metric.

This expected point-wise contraction is particularly convenient to study when a Markov transition can be written as a distribution over deterministic transformations Diaconis and Freedman (1999), especially deterministic trajectories. In this case we can sometimes demonstrate expected point-wise contraction by coupling the trajectories in a way that they tend to evolve towards each other.

A common example of this approach considers the circumstance where the ambient space \( X \) is a smooth Riemannian manifold and the Markov transition is a distribution over deterministic, geodesic flows. If the Riemannian sectional curvatures are strictly positive in some neighborhood then nearby geodesics will converge towards each other, at least for sufficiently short integration times Lee (2018) (Figure 4). Consequently when the transitions utilize sufficiently short geodesics with high enough probability then the local contraction of the distance function can imply contraction in expectation, and hence bound the 1-Wasserstein distance of the 1-step distribution.

Relying on the contraction of deterministic trajectories, however, can be severely limiting in practice. For example in many systems trajectories contract only in small neighborhoods, and we will be able to establish the desired bounds only for Markov transitions that strongly concentrate around the initial point and hence Markov chains that explore the target distribution only very slowly. Alternatively we might be able to establish the desired contraction but only for relatively simple target distributions, which limits the relevance of the resulting conclusions.
4.2.2 Total Variation Bounds The probability definition of the total variation metric admits a particularly useful coupling method for bounding distances between $N$-step distributions and stationary distributions Roberts and Rosenthal (2004).

A $\pi$-non-null set $A \in \mathcal{X}$ with $\pi(A) > 0$ naturally defines two lifted sets, $A \times X \in \mathcal{X} \times \mathcal{X}$ and $X \times A \in \mathcal{X} \times \mathcal{X}$. By definition the probability any coupling between any two probability distributions $\mu$ and $\nu$ allocates to these sets is just the corresponding marginal probabilities,

$$
\gamma(A \times X) = \mu(A),
$$

$$
\gamma(X \times A) = \nu(A).
$$

Another natural set to consider when coupling probability distributions together is the equality or diagonal set,

$$
D = \{(x_1, x_2) \in X \times X \mid x_1 = x_2\},
$$

and its complement, the inequality set $D^c$. In particular we can always decompose any set on the product space into disjoint intersections with the diagonal set and its complement (Figures 5, 6),

$$
A_1 \times A_2 = (A_1 \times A_2) \cap D \cup (A_1 \times A_2) \cap D^c.
$$

This decomposition allows us write the total variation distance between two probability distributions $\mu$ and $\nu$ as

$$
\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{X}} |\mu(A) - \nu(A)|
$$

$$
= \sup_{A \in \mathcal{X}} |\gamma(A \times X) - \nu(X \times A)|
$$

$$
= \sup_{A \in \mathcal{X}} \left| \gamma\left( ((A \times X) \cap D) \cup ((A \times X) \cap D^c) \right) - \gamma\left( ((X \times A) \cap D) \cup ((X \times A) \cap D^c) \right) \right|
$$

Because each intersection is disjoint the allocated probabilities decompose into independent contributions,

$$
\|\mu - \nu\|_{TV} = \sup_{A \in \mathcal{X}} \left| \gamma\left( (A \times X) \cap D \right) + \gamma\left( (A \times X) \cap D^c \right) - \gamma\left( (X \times A) \cap D \right) - \gamma\left( (X \times A) \cap D^c \right) \right|
$$

$$
+ \gamma\left( (X \times A) \cap D^c \right) - \gamma\left( (X \times A) \cap D \right).
$$
FIG 5. Any measurable set $A \in \mathcal{X}$ lifts into two measurable sets on the product space $X \times X$. These two lifted sets have exactly the same intersection with the diagonal set, $D$.

FIG 6. The intersections of the lifted sets $A \times X$ and $X \times A$ with the inequality set $D^c$ are not the same, but they both define subsets of the inequality set.

The sets $(A \times X) \cap D$ and $(X \times A) \cap D$, however, are exactly the same and so too must be the probabilities allocated to them,

$$||\mu - \nu||_{TV} = \sup_{A \in \mathcal{X}} \left| \gamma((A \times X) \cap D) - \gamma((X \times A) \cap D) \right|$$
\begin{align*}
&\sup_{A \in \mathcal{X}} \left| \gamma((A \times X) \cap D^c) - \gamma((X \times A) \cap D^c) \right| \\
&= \sup_{A \in \mathcal{X}} 0 \\
&+ \gamma((A \times X) \cap D^c) - \gamma((X \times A) \cap D^c) \\
&= \sup_{A \in \mathcal{X}} \left| \gamma((A \times X) \cap D^c) - \gamma((X \times A) \cap D^c) \right|.
\end{align*}

We can now employ the triangle inequality to separate the absolute value into the sum of two probabilities,

\[ ||\mu - \nu||_{TV} = \sup_{A \in \mathcal{X}} \left| \gamma((A \times X) \cap D^c) - \gamma((X \times A) \cap D^c) \right| \leq \sup_{A \in \mathcal{X}} \left( |\gamma((A \times X) \cap D^c)| + |\gamma((X \times A) \cap D^c)| \right). \]

Because the remaining intersection sets are both subsets of $D^c$,

\[
(A \times X) \cap D^c \subseteq D^c \\
(X \times A) \cap D^c \subseteq D^c,
\]

we have

\[
\gamma((A \times X) \cap D^c) \leq \gamma(D^c) \\
\gamma((X \times A) \cap D^c) \leq \gamma(D^c),
\]

and

\[
||\mu - \nu||_{TV} \leq \sup_{A \in \mathcal{X}} \left| \gamma((A \times X) \cap D^c) \right| + \left| \gamma((X \times A) \cap D^c) \right| \\
\leq \sup_{A \in \mathcal{X}} 2 |\gamma(D^c)| \\
\leq \sup_{A \in \mathcal{X}} 2 \gamma(D^c)
\]

Finally because this bound is independent of $A$ we can drop the supremum to give

\[
||\mu - \nu||_{TV} \leq \sup_{A \in \mathcal{X}} \gamma(D^c) \leq \gamma(D^c).
\]

Consequently if we can bound the probability allocated to the inequality set for any coupling between $\mu$ and $\nu$ then we can bound the total variation distance between the two probability distributions.

Under certain conditions we can construct a Markov coupling between $\tau^N \circ \rho$ and $\tau^N \circ \pi$ that concentrates enough probability on $D^c$ to ensure a finite bound. Conceptually this splitting coupling evolves the marginal Markov chains independently until they both fall into a small set $C \in X$ at the same iteration. At this point the Markov chains merge with a certain probability; if they don’t merge then they continue to transition independently until the next meeting, but if they do merge then they transition to the same point and share the exact same transition for all future iterations (Fig 7).

The probability $\gamma(D^c)$ allocated by this coupling is determined by probability that the marginal Markov transitions $\tau^N \circ \rho$ and $\tau^N \circ \pi$ merge once in the small set. In order to ensure that this probability is non-zero the Markov transition being coupled needs to admit minorization and drift conditions.

A Markov transition admits a minorization condition if there exists a positive real number $\epsilon > 0$, a positive integer $M$, and measure $\nu$ that dominates $\pi$ such that

\[
(\tau^M \circ \delta_x)(A) \geq \epsilon \nu(A)
\]
The splitting Markov coupling evolves two Markov chains independently until they reach a small set, $C \in X$. Once in the small set the splitting Markov coupling mixes (left) a transition that merges the two marginal Markov transitions together for all future iterations and (right) a transition that continues to evolve them independently. The probability of merging bounds the probability allocated to the inequality set, $\gamma(D^c)$.

A minorization condition ensures that the Markov transitions from any two point initializations in the small set $C \in X$ have a non-zero overlap on any $\pi$-non-null set $A$ after a finite number of iterations. In other words once in the small set all Markov transitions look somewhat similar, which allows for two independent Markov chains in a splitting coupling to merge with non-zero probability.

for all $A \in X$ and points $x$ in the set $C \subset X$ (Figure 8). Conceptually the minorization condition ensures that the $M$-step distributions from all point initializations in the small set, $\tau^M \circ \delta_x$, share a finite overlap. This overlap then allows two coupled Markov transitions to merge with finite probability once in the small set.

If a minorization condition with overlap $\epsilon$ holds after $M$ iterations then the probability that the splitting coupling does not merge two Markov chains initialized within the small set after $M$ transitions is no greater than $(1 - \epsilon)^M$. This then ensures the bound

$$||\tau^M \circ \delta_{x_1} - \tau^M \circ \delta_{x_2}||_{TV} \leq \gamma(D^c) \leq (1 - \epsilon)^M$$

for $x_1, x_2 \in C$.

This minorization bound is only relevant, however, if two Markov chains are able to meet in the small set in the first place. In particular the splitting Markov coupling needs to assign sufficient probability to the lifted set $C \times C$. One way to guarantee a finite meeting probability is a drift condition which requires the existence of a drift function $V : X \rightarrow [1, \infty) \subset \mathbb{R}$
A drift condition ensures that Markov chains steadily converge towards smaller values of the associated drift function. When the drift function is minimized within a small set \( C \) then Markov chains will steadily converge towards \( C \), and multiple Markov chains have a finite probability of meeting in \( C \) at the same time.

Satisfying

\[
\mathbb{E}_{\tau \delta_x}[V] \leq \lambda V(x) + b \mathbb{1}_C(x)
\]

for constants \( 0 < \lambda < 1 \) and \( b < \infty \) and all \( x \in X \). If the minimum of this drift function is within the small set \( C \) then a drift condition ensures that Markov chains initialized outside of the small set will transition towards the small set with high probability (Figure 9). This then ensures that the probability of two independent Markov chains not meeting in the small set after \( N \) iterations is bounded by a geometric function of \( N \).

In other words the drift condition ensures a geometric bound on two Markov chains not meeting in the small set, while the minorization condition ensures a bound on \( \gamma(D^c) \) once the Markov chains have met. Together with \( \phi \)-irreducibility these conditions guarantee a geometric bound on the total variation distance between the two \( N \)-step distributions,

\[
||\tau^N \circ \delta_x - \tau^N \circ \omega||_{TV} \leq C V(x) r^N,
\]

and hence the convergence of any \( N \)-step distribution towards the target distribution,

\[
||\tau^N \circ \delta_x - \pi||_{TV} = ||\tau^N \circ \delta_x - \tau^N \circ \pi||_{TV} \leq C V(x) r^N.
\]

One interesting consequence of this construction is that the drift condition ensures not only a geometric convergence bound in the total variation distance but also the \( V \)-norm,

\[
||\tau^N \circ \delta_x - \pi||_{\mathcal{F}_V}
\]

where

\[
\mathcal{F}_V = \{ f \in C^0(X) \mid |f(x)| < V(x) \}.
\]

The faster the drift function grows the larger this space of test functions will be and the more applicable it might be in practice.

Unfortunately even when the space of test functions is large the explicit convergence bound guaranteed by this splitting coupling construction is often extremely loose. The configuration of a Markov chain Monte Carlo algorithm motivated by an explicit bound, such as for how many iterations we need to run each Markov chain to achieve a given estimator error, is often drastically conservative.
4.3 The Markov Chain Monte Carlo Central Limit Theorem

Although geometric total variation bounds might not be directly useful for quantifying the preasymptotic behavior of Markov transitions, they are still extremely useful in practice. In particular geometric ergodicity in the total variation metric guarantees the existence of a Markov chain Monte Carlo central limit theorem that allows us to empirically quantify preasymptotic convergence of Markov chain Monte Carlo estimators for any sufficiently integrable function Roberts and Rosenthal (2004).

Let \( f : X \to \mathbb{R} \) be a real-valued, measurable function with \( \mathbb{E}_\pi[f^\delta] \) finite for any \( \delta > 0 \). For any realized Markov chain \( (x_0, \ldots, x_N) \), we can construct a corresponding Markov chain Monte Carlo estimator

\[
\hat{f}_N(x_0, \ldots, x_N) = \frac{1}{N+1} \sum_{n=0}^{N} f(\omega_n(x_0, \ldots, x_N)) = \frac{1}{N+1} \sum_{n=0}^{N} f(x_n),
\]

If the Markov transition is Harris recurrent then the distribution of estimator values derived from the possible Markov chain realizations initialized from \( \rho \), \( (\hat{f}_N)_*(\tau^N \times \rho) \), asymptotically converges to a Dirac distribution at the exact expectation value,

\[
\lim_{N \to \infty} (\hat{f}_N)_*(\tau^N \times \rho) = \delta_{\mathbb{E}_\pi[f]}.
\]

Moreover if \( \tau \) is geometrically ergodic in the total variation distance then the probability density function of \( (\hat{f}_N)_*(\tau^N \times \rho) \) satisfies a central limit theorem,

\[
\lim_{N \to \infty} \frac{(\hat{f}_N)_*(\tau^N \times \rho)(y) - \mathbb{E}_\pi[f]}{\sigma_N} = \text{normal}(y \mid 0, 1),
\]

where \( \sigma_N^2 \) is the asymptotic variance.

To give this asymptotic variance an explicit form let \( \mu_f = \mathbb{E}_\pi[f] \) and \( \sigma_f^2 = \mathbb{E}_\pi[(f - \mu_f)^2] \) and define the lag-\( l \) autocovariance of a Markov transition as

\[
\xi_l[f] = \mathbb{E}_{\tau^N \times \rho}[(f \circ \omega_{l+n} - \mu_f)(f \circ \omega_n - \mu_f)],
\]

with the corresponding lag-\( l \) autocorrelation defined as

\[
\zeta_l[f] = \frac{\xi_l[f]}{\operatorname{Var}_\pi[f]}.
\]

The asymptotic variance is then given by

\[
\lim_{N \to \infty} N \cdot \sigma_N^2 = \sum_{l=-\infty}^{\infty} \xi_l[f] = \operatorname{Var}_\pi[f] \sum_{l=-\infty}^{\infty} \zeta_l[f] = \operatorname{Var}_\pi[f] \cdot \left( 1 + 2 \cdot \sum_{l=1}^{\infty} \zeta_l[f] \right).
\]

Although there is an initial bias in Markov chain Monte Carlo estimators from geometrically ergodic Markov transitions, that bias decays linearly with the number of iterations \( N \). Consequently for large enough \( N \) the bias becomes negligible and we can use the central limit theorem to approximate the probability density function of estimator values as

\[
(\hat{f}_N)_*(\tau^N \times \rho)(y) = \text{normal}(y \mid \mathbb{E}_\pi[f], \sigma_N).
\]

For fixed \( N \) this defines a Markov chain Monte Carlo standard error that quantifies how well a realized value of \( f_N \) estimates the exact expectation value. In practice the error is typically
written as

\[ N \cdot \sigma_N^2 = \text{Var}_\pi[f] \cdot \left(1 + 2 \cdot \sum_{l=1}^\infty \zeta_l[f]\right) \]

\[ \sigma_N = \sqrt{\text{Var}_\pi[f] \cdot \left(1 + 2 \cdot \sum_{l=1}^\infty \zeta_l[f]\right)} \]

\[ \sigma_N = \sqrt{\text{Var}_\pi[f] \cdot \frac{1}{\text{ESS}[f]}} \]

where the effective sample size

\[ \text{ESS}[f] = \frac{N}{\left(1 + 2 \cdot \sum_{l=1}^\infty \zeta_l[f]\right)} \]

moderates the precision of the estimator. The less the values of \( f \) are autocorrelated in the realized Markov chains the larger the effective sample size, and the more precise Markov chain Monte Carlo estimators, will be for any fixed Markov chain length \( N \).

In practice if we don’t know \( E_\pi[f] \) then it’s highly unlikely that we will know the exact variance or effective sample size, either. If \( E_\pi[f^4+\delta] \) is finite, however, then we can use a realized Markov chain to estimate these quantities and hence the Markov chain standard error. The error in this estimation is comparable to the error introduced by assuming that the central limit theorem holds for finite \( N \), and consequently negligible when \( N \) is large enough.

When we know that a Markov transition is geometrically ergodic in the total variation distance then the implementation of Markov chain Monte Carlo is straightforward. Given some initialization distribution \( \rho \) we generate a Markov chain by sampling an initial point from \( \rho \) and then applying the Markov transition \( N \) times. For any function \( f : X \to \mathbb{R} \) we can construct a Markov chain Monte Carlo estimator for the corresponding expectation value, as well as the variance and the autocorrelations needed to estimate the corresponding error estimate.

For small \( N \) all of these estimates suffer from a bias that monotonically decays with \( N^{-1} \). As \( N \) increases the bias becomes negligible and the Markov chain Monte Carlo central limit theorem kicks in, allowing us to empirically quantify the error of the estimator using only the history of the realized Markov chain. Letting \( N \) grow to infinity the normal approximation given by the central limit theorem continues to narrow until it finally converges to a Dirac distribution in the asymptotic limit (Figure 10).

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REFERENCES

CHAN, K. S. and GEYER, C. J. (1994). Discussion: Markov Chains for Exploring Posterior Distributions. The Annals of Statistics 22 1747–1758.

COVER, T. M. and THOMAS, J. A. (2006). Elements of information theory, Second ed. Wiley-Interscience [John Wiley & Sons], Hoboken, NJ.

DIACONIS, P. and FREEDMAN, D. (1999). Iterated Random Functions. SIAM review 41 45–76.

DUDLEY, R. M. (2002). Real analysis and probability. Cambridge Studies in Advanced Mathematics 74. Cambridge University Press, Cambridge.

HARRIS, T. E. (1956). The existence of stationary measures for certain Markov processes. In Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, 1954–1955, vol. II 113–124. University of California Press, Berkeley and Los Angeles.

JOULIN, A. and OLLIVIER, Y. (2010). Curvature, Concentration and Error Estimates for Markov Chain Monte Carlo. The Annals of Probability 38 2418–2442.

KANTOROVIČ, L. V. and RUBINŠTEIN, G. V. (1958). On a space of completely additive functions. Vestnik Leningrad. Univ. 13 52–59.
Markov chain Monte Carlo estimators from geometrically ergodic Markov transitions converge in three distinct phases. In the initial phase where the number of iterations small, \( N \approx 0 \), the estimators are biased towards the initialization. As the number of iterations grows this bias decays, and for large enough iterations \( N \gg 0 \) the distribution of estimator values is well-approximated by a normal density function centered on the exact expectation value. After an infinite number of iterations this normal approximation collapses to a Dirac distribution centered on the exact expectation value.

Lee, J. M. (2018). Introduction to Riemannian manifolds. Graduate Texts in Mathematics 176. Springer, Cham.

Müller, A. (1997). Integral probability metrics and their generating classes of functions. Adv. in Appl. Probab. 29 429–443.

Ollivier, Y. (2009). Ricci Curvature of Markov Chains on Metric Spaces. Journal of Functional Analysis 256 810–864.

Roberts, G. O. and Rosenthal, J. S. (2004). General State Space Markov Chains and MCMC Algorithms. Probability Surveys 1 20–71.

Sriperumbudur, B. K., Fukumizu, K., Gretton, A., Schölkopf, B. and Lanckriet, G. R. G. (2009). On integral probability metrics, \( \phi \)-divergences and binary classification. arXiv e-prints 0901.2698.

Tierney, L. (1994). Markov chains for exploring posterior distributions. Ann. Statist. 22 1701–1762.