Evidence for a conjecture of Pandharipande

Jim Bryan

1. Introduction

In [3], Pandharipande studied the relationship between the enumerative geometry of certain 3-folds and the Gromov-Witten invariants. In some good cases, enumerative invariants (which are manifestly integers) can be expressed as a rational combination of Gromov-Witten invariants. Pandharipande speculated that the same combination of invariants should yield integers even when they do not have any enumerative significance on the 3-fold. In the case when the 3-fold is the product of a complex surface and an elliptic curve, Pandharipande has computed this combination of invariants on the 3-fold in terms of the Gromov-Witten invariants of the surface [4]. This computation yields surprising conjectural predictions about the genus 0 and genus 1 Gromov-Witten invariants of complex surfaces. The conjecture states that certain rational combinations of the genus 0 and genus 1 Gromov-Witten invariants are always integers. Since the Gromov-Witten invariants for surfaces are often enumerative (as oppose to 3-folds), this conjecture can often also be interpreted as giving certain congruence relations among the various enumerative invariants of a surface.

In this note, we state Pandharipande’s conjecture and we prove it for an infinite series of classes in the case of $\mathbb{CP}^2$ blown-up at 9 points. In this case, we find generating functions for the numbers appearing in the conjecture in terms of quasi-modular forms (Theorem 3.1). We then prove the integrality of the numbers by proving a certain a congruence property of modular forms that is reminiscent of Ramanujan’s mod 5 congruences of the partition function (Theorem 3.2).

2. The conjecture

Let $X$ be a smooth complex projective surface (or more generally, a symplectic 4-manifold), let $K$ be its canonical class, and let $\chi(X)$ be its Euler characteristic. Let $\beta \in H_2(X, \mathbb{Z})$ and let $g(\beta)$ be defined by $2g(\beta) - 2 = \beta \cdot (K + \beta)$. Define $c(\beta)$ to be $-\beta \cdot K$ and assume that $c(\beta) > 0$. Let $N^r(\beta)$ be the genus $r$ Gromov-Witten invariant of $X$ in the class $\beta$ where we have imposed $c(\beta) + r - 1$ point constraints. By convention we will say $N^r(0) = 0$.

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**Conjecture 2.1** (Pandharipande). Define \( a(\beta) \) by

\[
a(\beta) = -\frac{1}{12} g(\beta) N^0(\beta)
\]

and define \( b(\beta) \) by

\[
b(\beta) = \frac{1}{2880} \left( 12 g(\beta)^2 + g(\beta) c(\beta) - 24 g(\beta) \right) N^0(\beta)
\]

\[
+ \frac{1}{240} \chi(X) N^1(\beta)
\]

\[
+ \frac{1}{240} \sum_{\beta' + \beta'' = \beta} \left( \frac{c(\beta) - 1}{c(\beta')} \right) (\beta' \cdot \beta'')(\beta' \cdot \beta'') N^1(\beta') N^0(\beta'').
\]

Then \( a(\beta) \) and \( b(\beta) \) are integers.

**Remark 2.1.** This conjecture is related to the proposal of Gopakumar and Vafa that relates the Gromov-Witten invariants of Calabi-Yau 3-folds to conjecturally integer valued invariants ("BPS state counts", or "BPS invariants"). Pandharipande has generalized the Gopakumar-Vafa formula to Fano classes in non-Calabi-Yau 3-folds (see [3]). In this formulation, the numbers \( a(\beta) \) and \( b(\beta) \) are respectively genus 1 and genus 2 "BPS invariants" for the surface cross an elliptic curve. The reason that these are expressible in terms of ordinary Gromov-Witten invariants of the surface is that the Hodge class in \( \overline{M}_g \) (which appears in the computation of the virtual class) is readily expressible in terms of boundary classes for \( g = 1 \) and \( g = 2 \). For arbitrary \( g \) there will also be predictions for the invariants of the surface, but they will involve gravitational descendants in general.

3. The case of \( \mathbb{CP}^2 \) blown-up at 9 points

Let \( X \) be \( \mathbb{CP}^2 \) blown up at nine points. Let \( F = -K \) be the anti-canonical class and let \( S \) be the exceptional divisor of one of the blow-ups (so if \( X \) is elliptically fibered, then \( F \) is the fiber and \( S \) is a section). Let \( \beta_n = S + nF \). Then \( N^r(\beta_n) \) was computed in [1]. We will find a nice generating functions for the numbers \( a(\beta_n) \) and \( b(\beta_n) \) and will prove that they are integers thus verifying Pandharipande’s conjecture for \( X \) for this infinite series of classes.

Note that \( c(\beta_n) = 1 \), \( g(\beta_n) = n \), and \( \chi(X) = 12 \). Since for \( N^0(\beta'') \) to be non-zero, we need \( c(\beta'') = 1 \), the sum must have \( c(\beta'') = 1 \) and \( c(\beta') = 0 \). It follows that \( \beta'' \) and \( \beta' \)
are of the form $S + kF$ and $(n - k)F$ respectively. Thus we have

$$a(\beta_n) = -\frac{1}{12}nN^0(\beta_n)$$

$$b(\beta_n) = \frac{1}{2880}(12n^2 - 23n)N^0(\beta_n)$$

$$+ \frac{1}{20}N^1(\beta_n)$$

$$+ \frac{1}{240} \sum_{k=0}^{n-1} (n - k)(2k - 1)N^1((n - k)F)N^0(\beta_k).$$

Define

$$A(q) = \sum_{n=0}^{\infty} a(\beta_n)q^n,$$

$$B(q) = \sum_{n=0}^{\infty} b(\beta_n)q^n.$$

We will find an expression for $A(q)$ and $B(q)$ in terms of quasi-modular forms. Let $\sigma(k) = \sum_{d|k} d$ and let $p(k)$ be the number of partitions of $k$. Define

$$G(q) = \sum_{k=1}^{\infty} \sigma(k)q^k,$$

$$P(q) = \sum_{k=1}^{\infty} p(k)q^k$$

$$= \prod_{m=1}^{\infty} (1 - q^m)^{-1},$$

$$P_\alpha(q) = (P(q))^\alpha,$$

$$D = \frac{d}{dq}.$$

Note that $G$ and $P$ are closely related to well known (quasi-) modular forms: $G - 1/24$ is the Eisenstein series $G_2$ and $q^{1/24}P_{-1}$ is the Dedekind $\eta$ function.

With this notation, the results of [1] (Theorem 1.2) give

$$\sum_{n=0}^{\infty} N^0(\beta_n)q^n = P_{12}$$

$$\sum_{n=1}^{\infty} N^1(\beta_n)q^n = P_{12}DG.$$
Furthermore, one can show that

\[ N^1(lF) = \frac{1}{l} \sigma(l) \]

(when the blow-up points are generic, this comes from the multiple covers of the unique elliptic curve in the class \( F \)). We thus have

\[
A(q) = -\frac{1}{12} DP_{12} \\
B(q) = \frac{1}{2880} (12D^2 - 23D)P_{12} + \frac{1}{20} P_{12}DG \\
+ \frac{1}{240} \sum_{m \geq 1} \sum_{k \geq 0} (2k - 1)\sigma(m)N^0(\beta_k)q^{m-k}q^k
\]

\[
= \frac{1}{240} D^2P_{12} - \frac{23}{2880} DP_{12} + \frac{1}{20} P_{12}DG \\
+ \frac{1}{240} \sum_{m \geq 1} \sum_{k \geq 0} (2k - 1)\sigma(m)N^0(\beta_k)q^kq^m
\]

\[
= \frac{1}{240} D^2P_{12} - \frac{23}{2880} DP_{12} + \frac{1}{20} P_{12}DG + \frac{1}{240} G(2DP_{12} - P_{12})
\]

Now, by a standard calculation, \( G = P_{-1}DP \) and so \( DP_{12} = 12P_{12}G \). Substituting and simplifying we arrive at:

**Theorem 3.1.** The following equations holds:

\[ A(q) = -P_{12} \cdot G \]

\[ B(q) = \frac{1}{10} P_{12} \left\{ 7G^2 - G + DG \right\}. \]

This theorem immediately shows that the coefficients of \( A \) are integers. On the other hand, the integrality of the coefficients of \( B \) requires the following theorem:

**Theorem 3.2.** The following equation holds:

\[ 7G^2 - G + DG \equiv 0 \pmod{10}. \]

**Proof:** By a simple calculation mod 5, we have:

\[ 7G^2 - G + DG \equiv 3P_{-2}(D^2 - D)P_2 \pmod{5} \]

and so to prove that the above expression is 0 mod 5, it suffices to prove that \((D^2 - D)P_2 \equiv 0 \pmod{5}\). Using the Jacobi triple product formula and the Euler inversion formula, it is easy to show that the \( k \)th coefficient of \( P_2 = P_{-3}P_5 \) is divisible by 5 unless \( k \) is 0 or 1 mod 5 (see [2]). In other words:

\[ P_2(q) \equiv r(q^5) + qs(q^5) \pmod{5}. \]

It follows that \( DP_2 \equiv qs(q^5) \pmod{5} \) and so \( D^2P_2 \equiv DP_2 \pmod{5} \) as desired.
On the other hand, it is easy to compute that
\[ 7G^2 - G + DG \equiv P_5(D^2 + D)P \pmod{2}. \]
This expression is 0 mod 2 since the \( k \)th coefficient of \((D^2 + D)P\) is \( k(k + 1)p(k) \).

Thus we have established that \( 7G^2 - G + DG \) is 0 mod 2 and mod 5 and so the theorem is proved.

References

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Department of Mathematics, Tulane University
E-mail address: jbryan@math.tulane.edu