Disorder effects in the thermodynamic properties of a ideal Bose gas confined in a semi-infinite multi-layer structure within a box of thickness $L$ and infinite lateral extent, are analyzed. The layers are first modeled by a periodic array of $M$ Dirac delta-functions of equal intensity. Then, we introduce structural and compositional disorder, as well as a random set of layer vacancies in the system to calculate the internal energy, chemical potential and the specific heat for different configurations. Whereas structural and compositional disorder does not reveal a significant change, a dramatic increase in the maximum of the specific heat is observed when the system is depleted a fraction of the order of 0.1 to 0.2 of random layers compared to the original, fully periodic array. Furthermore, this maximum, which is reminiscent of a Bose-Einstein condensation for an infinite array, occurs at higher temperatures.

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which in 3D become surfaces of strength \( v_j \) located at \( z_j \). For the ordered or periodic, reference system, \( \Delta_j = a \) and we assume box boundary conditions

\[
\psi_j(0) = \psi_{M+1}(L) = 0.
\]  

For a system of \( M \) potentials we have

\[
\begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = \left( \prod_{j=1}^{M} T_j \right) \begin{pmatrix} A_{M+1} \\ B_{M+1} \end{pmatrix} = \tau \begin{pmatrix} A_{M+1} \\ B_{M+1} \end{pmatrix}
\]

where \( \tau = \prod_{j=1}^{M} T_j \) is a \( 2 \times 2 \) matrix and the dispersion relation that defines the \( z \)-direction energy levels is given in terms of the elements of the product matrix \( \tau \) by

\[
\text{Im} \left[ \left( \tau_{11} + \tau_{12} \right) e^{-i\kappa L} \right] = 0.
\]

For a periodic system, Eqs. (11) and (11) are equivalent. In each band, there are \( M \) discrete levels due to the barriers and an additional level at the top of the \( n \)-band with \( \kappa_{M+1} a = n\pi \). When a finite number of layers is suppressed an equal number of levels in each band is moved down to the forbidden region. For example, in Fig. 2 we show the first two energy bands with \( M + 1 = 11 \) levels when one layer (b) or different sets of two layers are removed (c–e). Level degeneracy is observed when the removed layers are at symmetrical positions as shown in the diagram of Fig. 2(e).

### III. THERMODYNAMIC PROPERTIES

When there is no interaction between the particles, the partition function for this system is

\[
Z = \prod_{n=1}^{M+1} \prod_{i=1}^{2} \left( 1 - z e^{-\beta \varepsilon_{i,n}} \right)^{-1},
\]
where \( z = e^{\beta \mu} \) is the fugacity and the single-particle energies are
\[
\varepsilon_{n,i} = \frac{\hbar^2 (k_x^2 + k_y^2)}{2m} + \varepsilon_{\kappa_{i,n}}
\]
The thermodynamic properties of the system are obtained from the grand potential
\[
\Omega(T, V, \mu) = -k_B T \ln Z = k_B T \sum_{n,i} \ln \left[ 1 - e^{-\beta (\varepsilon_{n,i} - \mu)} \right].
\]

When the system size in the \( x- \) and \( y- \)directions is infinite, sums become integrals and the number equation \( N = \partial \Omega / \partial \mu \) becomes
\[
N = \sum_{n,i} \left( \frac{L}{2\pi} \right)^2 \int dk_x dk_y \left\{ \alpha_{\kappa_{i,n}}^{-1} \exp \left[ \frac{\lambda^2}{2\pi} (k_x^2 + k_y^2) \right] - 1 \right\}^{-1}.
\]

where
\[
\alpha_{\kappa_{i,n}} = e^{i (\mu - \varepsilon_{\kappa_{i,n}})},
\]
\[
\lambda^2 = \frac{\hbar^2}{2\pi m k_B T}.
\]
Integration over \( k_x, k_y \) yields
\[
N = - \left( \frac{L}{\pi} \right)^2 \sum_{n=1}^{M+1} \sum_{i=1}^{\infty} \ln(1 - \alpha_{\kappa_{i,n}}),
\]
which may be expressed in terms of the ideal boson gas (IBG) condensation temperature \( T_0 \) and the corresponding thermal wavelength \( \lambda_0 = \hbar / \sqrt{2\pi m k_B T_0} \) as
\[
1 = - \frac{\lambda_0 T}{\zeta(3/2)(M+1) \alpha T_0} \sum_{n=1}^{M+1} \sum_{i=1}^{\infty} \ln \left[ 1 - e^{-\beta (\mu - \varepsilon_{\kappa_{i,n}})} \right].
\]

The internal energy \( U \) is obtained from
\[
U = \sum_{n,i} \left( \frac{L}{2\pi} \right)^2 \int dk_x dk_y \frac{\varepsilon_{k_x} + \varepsilon_{k_y} + \varepsilon_{\kappa_{i,n}}}{e^{\beta (\varepsilon_{k_x} + \varepsilon_{k_y} + \varepsilon_{\kappa_{i,n}} - \mu)} - 1}.
\]
Again, integration over \( k_x, k_y \) and using the IBG condensation temperature yields
\[
\frac{U}{Nk_B T} = \frac{\lambda_0}{\zeta(3/2)(M+1) \alpha} \sum_{n,i} \left[ \frac{T}{T_0} g_2(\alpha_{\kappa_{i,n}}) + \frac{\varepsilon_{\kappa_{i,n}}}{k_B T_0} g_1(\alpha_{\kappa_{i,n}}) \right],
\]
where \( g_\sigma(z) \) is the \( \sigma- \)th order Bose function \[17\].

The specific heat at constant volume is \( C_V = (\partial U / \partial T)_V \) yielding
\[
\frac{C_V}{Nk_B} = \frac{\sqrt{4\pi \gamma}}{\zeta(3/2)(M+1)} \sum_{n,i} \left[ \frac{2T}{T_0} g_2(\alpha_{\kappa_{i,n}}) + 2\gamma \pi g_1(\alpha_{\kappa_{i,n}}) + (\gamma \pi_{i,n})^2 \frac{T_0}{T} g_0(\alpha_{\kappa_{i,n}}) + f(\bar{\mu}) \right],
\]
where \( \gamma = \lambda_0^2 / 4\pi a^2 \), \( \zeta \) is the Riemann Zeta-function and
\[
f(\bar{\mu}) = \left( \frac{T}{\pi T_0} \frac{\partial^2 C_V}{\partial T^2} - \bar{\mu} \right) \left[ \sqrt{\frac{\gamma T_0}{4\pi T}} + \gamma \pi \varepsilon_{\kappa_{i,n}} g_0(\alpha_{\kappa_{i,n}}) \right].
\]

where we have used dimensionless units, namely \( \bar{\mu} = \mu / (\hbar^2 / 2ma^2) \) and \( \pi_{i,n} = \varepsilon_{i,n} / (\hbar^2 / 2ma^2) \).

The effects of disorder in the specific heat of the system may be analyzed by introducing random variations in the positions of the barriers (structural disorder) through \( z_j \rightarrow (j + \delta_j) a \) with \( |\delta_j| < 1 \), a random number. Alternately, a random variation on the strengths of the barriers can mimic compositional disorder by setting \( A_j \rightarrow (1 + \delta_j) A \). In the former case, we probe a sample of \( M = 10 \) barriers at varying positions. The specific heat

for several trials in the choice of \( \delta_j \) for the positions of the layers is shown in Fig. 3 where a 10 percent random variation is adopted. The case of compositional disorder is shown in Fig. 4 for a 90 percent random variation of \( \delta_j \) in the strengths of the delta potentials. In both cases, the maximum of the specific heat increases and its temperature is always higher than the corresponding values of the reference, ordered system, as well as the average curve from several random trials labeled 1,2,3,4 in the figures. However, when vacancies are introduced a dramatic effect in the specific heat is observed as shown in the next Section.

![FIG. 3: (Color online) Specific heat for random 10 percent variations in the position of the barriers for a system of \( M = 10 \) planes of equal strength, \( \Lambda = 10 \). Each curve denotes a different set of random trials](image)

**IV. EFFECT OF VACANCIES**

Using a small sample of \( M = 10 \) planes, we calculate \( C_V \) when a number of vacancies is produced by setting \( \Lambda_j = 0 \) for some \( j \)-s. We start by removing walls
in progressive positions. When one wall is removed, a level in each energy package of the ordered case jumps down thereby modifying the energy gap between the lowest level and the following one. This phenomenon produces an effective energy gap in the first set of energy levels whose magnitude depends on the position of the removed wall.

\[
\begin{array}{|c|c|c|c|}
\hline
T_{\text{max}} & C_V(T_{\text{max}}) & \text{gap } [\hbar^2/2ma^2] & \text{site} \\
1.39100100 & 2.590968641 & 4.617840120 & 1 \\
1.40013969 & 2.610722976 & 4.741202730 & 2 \\
1.40019796 & 2.610862299 & 4.791629539 & 3 \\
1.40019848 & 2.610863260 & 4.791629539 & 4 \\
1.40019849 & 2.610863267 & 4.835378530 & 5 \\
\hline
\end{array}
\]

TABLE I: Specific heat height \(C_V(T_{\text{max}})\) at \(T_{\text{max}}\) for systems with one site vacancy indicated in column 4.

The largest gap appears when one layer near the center of the sample is removed. In Table 1 we show the specific heat maxima \(C_V(T_{\text{max}})\) as well as the temperature \(T_{\text{max}}\) where the specific heat attains its maximum, and the magnitude of the energy gap when the layer in a specific site is removed. The gap, \(C_V(T_{\text{max}})\) and \(T_{\text{max}}\) increases as the position of the removed layer varies from site 1 (near the edge) to site 5. We claim that the increase in \(C_V(T_{\text{max}})\) and \(T_{\text{max}}\) is caused by the appearance of this gap since for a 3D infinite ideal Bose gas with a quadratic dispersion relation plus a gap, both the BEC critical temperature and specific heat height at its critical temperature increase as a function of the gap magnitude. Since our system is semi-infinite the specific heat is unable to develop a sharp peak which is a signature of a phase transition. Instead a pronounced maximum is observed which suggests a precursor of a BEC transition since it becomes sharper as the size of the system grows as shown in Figs. 6 to 8. The removal of additional walls has a comparable effect in the energy levels. Each suppression moves down one level and if the removed walls were in symmetrical sites, the lowest levels are degenerate.

A similar behavior is observed in a larger sample with \(M = 100\) planes as shown in Figs. 9 and 10. In Fig. 9 we show the effects when there are two vacancies compared to the ordered case and the free boson gas. The removal
at the edge of one or two planes shows a dramatic increase in the first maximum of the specific heat at somewhat larger temperatures. This maximum is even more pronounced when there are two consecutive vacancies in the middle of the sample. Figure 6 shows the effects of additional vacancies in the system. The specific heat in the ordered system has two maxima as a function of temperature. The maximum at the lower temperature is a signature of a BEC transition for a Bose gas confined in an infinite layered structure that would be present in the infinite system which appears at a critical temperature lower than $T_0$ as discussed in Ref. [1]. The second maximum signals the onset of the BEC of an ideal Bose gas between two consecutive walls. When vacancies are introduced, either at symmetric or at random sites as shown in Figs. 7 and 8 respectively, the second maximum is suppressed and the first peak has a sharp increase both, in its magnitude and in the temperature.

![Figure 7](image1.png)

**FIG. 7:** (Color online) $M = 100$, $P_0 = 10$, $a/\lambda_0 = 1$. Specific heat for a system with 100 delta-function planes (ordered system) and with a proportion of removed planes from symmetric sites.

![Figure 8](image2.png)

**FIG. 8:** (Color online) $M = 100$, $P_0 = 10$, $a/\lambda_0 = 1$. Same as the preceding figure but with a random proportion of removed planes.

random number of planes in the $z$–direction a dramatic effect is observed. In the case of a small sample with $M = 10$ walls, the lower temperature maximum of the specific heat increases by a large amount and it occurs at larger temperatures compared to the ordered and the free Bose gas cases. In addition, the maximum at higher temperatures disappears. As more walls are removed, the magnitude of the first maximum decreases.

V. CONCLUSIONS

Lattice disorder is known to produce localization in Fermi and, more recently, in Bose systems in the thermodynamic limit. In this work, we have studied the effects of lattice disorder in the specific heat of finite, Bose systems by introducing random variations in the positions and/or strengths of the planes. The overall effect of these variations is to modify the position, as a function of the temperature, of the maximum of the specific heat to higher temperatures. Its magnitude also increases when a large number of random samples is considered. However, when disorder is caused by the suppression of a

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