Information-Sharing and Privacy in Social Networks

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Abstract

We present a new model for reasoning about the way information is shared among friends in a social network, and the resulting ways in which it spreads. Our model formalizes the intuition that revealing personal information in social settings involves a trade-off between the benefits of sharing information with friends, and the risks that additional gossiping will propagate it to people with whom one is not on friendly terms. We study the behavior of rational agents in such a situation, and we characterize the existence and computability of stable information-sharing networks, in which agents do not have an incentive to change the partners with whom they share information. We analyze the implications of these stable networks for social welfare, and the resulting fragmentation of the social network.

1 Introduction

A growing line of work on privacy has investigated ways for people to engage in transactions — purchases, queries, participation in activities, and related types of behavior — while revealing very little or no private information about themselves. This research has implicitly construed the problem of privacy as one of a trade-off between the concrete tasks that a person wants or needs to accomplish, and the “leakage” of personal information that might result from the interactions required to perform the task. From such a framing of the problem, it follows that people should want to perform these tasks while exposing as little information as possible.

If one takes this view of privacy, however, it becomes very hard to reason about the kinds of simple, privacy-revealing activities that are ubiquitous in real social networks, both off-line and on-line. As the most basic example, consider two friends engaged in conversation, each sharing personal — though not necessarily particularly sensitive or important — information about themselves with the other: a child is out sick from school; a scheduled trip was canceled; some needed repairs on the house have just been finished. Here, there is no transaction taking place other than the sharing of the information itself, and it is easy to create scenarios in which any of these seemingly mundane pieces of information could ultimately be used to the detriment of the person revealing it. Yet in everyday life people clearly feel a fundamental incentive to engage in this kind of information-sharing; if we are to understand the full scope of privacy as an issue, we need to be able to model and reason about this kind of activity with the same level of concreteness that we use for on-line purchases, database and search-engine queries, and the other more formal, structured types of transactions that have been the traditional focus of privacy research.

Information-Sharing in Social Networks    As the first step toward developing a model for this kind of activity, it is useful to try articulating some aspects of the unstated social conventions that govern the informal sharing of personal information between two friends. This question touches on complex issues from several research literatures, including sociology, psychology, and legal philosophy, and the work on this topic has elucidated both positive and negative

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aspects of information-sharing practices [1][2][5][19]. Given this complexity, we will try to abstract some of the most basic aspects of information-sharing in social networks into a mathematically tractable model. In particular, if we want to explore the potential for rational agents to engage in information-sharing with friends, we need to formalize sources of positive utility that derive from this activity, to trade off against the the sources of negative utility that have been the dominant focus in the computer science literature on privacy.

With these issues in mind, and drawing on the literature above, we argue that personal conversations between you and a friend are governed by social conventions that, at a general level, contain the following general ingredients.

(i) You derive benefit from learning information about your friend, in part because such exchanges serve to strengthen the social tie between the two of you. Moreover, there is a corresponding benefit in having your friend learn information about you; this too strengthens the social tie.

(ii) It is unrealistic to have all such conversations governed by strict promises of secrecy; all parties involved can expect that some information will spread through the social network to a limited extent via gossip.

(iii) The fact that information spreads through the social network contains sources of both positive and negative utility for you. You may receive positive utility from learning information about friends and having friends learn information about you, even by this form of indirect transmission through gossip. However, there are other people in the network whom you do not want your personal information to reach; you receive negative utility when personal information about you indirectly reaches them via gossip.

(iv) In evaluating whether to share personal information with a friend, you therefore take into account who else you believe this friend engages in information-sharing with — and more generally, what you believe the information-sharing pathways in the network look like. You will avoid sharing information with a friend if you believe that their indirect transmission of your information will yield a net negative utility. Correspondingly, you may avoid sharing information with a friend if by cutting this link, it will encourage others to feel safe in sharing their information with you — provided that this trade-off yields a net utility benefit for you.

These general considerations form the basis for the model we develop next. There are many further, and important, issues that could be incorporated into a model: for example, information comes in many categories, and you may well be happy if person $X$ learns about your personal information related to topic $Y$ but not to topic $Z$; similar contrasts may exist when you consider your personal information classified not by topic but by its level of sensitivity. However, we will see that building a model even from the most basic considerations above already leads to complex questions, with results that provide insight — and appear to accord with natural intuitions — about some of the ways in which personal information moves through social networks.

**Formalizing a Model of Information-Sharing** We now describe a model that takes into account issues (i)-(iv) from the preceding discussion. We begin by describing a model without any strategic component on the part of the people involved, and then we add a strategic aspect to it.

We have a set $V$ of $n$ people; some pairs of these people share personal information with each other (including any indirect information that they’ve learned about others), and some pairs of these people do not share personal information. Sharing of information is symmetric, and so if we let $E$ denote the set of pairs who share information, then we obtain an information-sharing network $G = (V, E)$. If $i, j \in V$ are in the same connected component of $G$, then each will learn personal information about the other, either by direct communication (if there is an $i$-$j$ edge) or indirectly via gossip (if there is only an $i$-$j$ path of length two or more).

Now, for any two people $i, j \in V$, person $i$ receives a utility $u_{ij}$ from being in the same component as $j$, and person $j$ receives a utility $u_{ji}$ from being in the same component as $i$. These utilities can be either positive or negative, corresponding to the dichotomy in point (iii) above between the benefits of indirectly learning about and being known to your friends, and the harms from having personal information reach people you are not friendly with. If $C_G(i)$ denotes the component of $G$ containing $i$, then the total utility of $i$ is equal to $\sum_{j \in C_G(i)} u_{ij}$.

**Strategic Behavior and Information-Sharing** In our model, two people must mutually agree to share information, and we presume that they will do so strategically, to maximize their utilities, based on their expectations about what
others will do. This is the crux of point (iv) in the preceding discussion. (We think of the set \( V \) as being a relatively small community, such that everyone has beliefs about who is talking to whom.)

We are thus faced with a kind of network formation game, in which each player must decide which links to maintain so as to maximize her utility, given the links everyone else has formed. We seek information-sharing networks that satisfy a type of stability; from a stable network \( G \) there will be no incentive for parties to add or drop links. In other words, people will be sharing information with the optimal set of contacts, given the pathways for gossip formed by the behavior of everyone else, and they can trust that the information-sharing structure that has developed is thus in a sense “self-enforcing.” Note also that in contrast to many standard network formation games, there is no explicit “cost” to maintain a link; the costs are implicit, based on the fact that a link exposes you to the risk that your information will reach other nodes with whom you are not friendly.

Our stability notion is a strengthening of pairwise Nash stability \(^{15}\) (see the next section for a review of alternate stability notions). Specifically, we define a defection from the current network \( G \) to consist either of (a) a single node \( i \) deleting a subset of its incident edges, or (b) a pair of nodes \( i, j \) agreeing to form the edge \((i, j)\) and simultaneously to each delete subsets of their (other) incident edges. We then say that a graph \( G \) is stable if there are no defections from \( G \) in which all the participating nodes (\( i \) in case (a), and both \( i \) and \( j \) in case (b)), strictly improve their utilities. From the results that follow, it will become clear that defining a network to be “stable” without allowing two-node coordination of the type in (b) provides a stability concept that is too weak to reflect the strategic information-sharing behavior we are trying to capture.\(^3\)

The ability of two people to coordinate is natural in our model, since pairwise interaction is the fundamental level at which information-sharing is taking place. But it is also interesting to consider the possibility of defections in which larger subsets of people coordinate their actions. Thus, we define a \( k \)-defection to consist of a set \( S \) of up to \( k \) nodes agreeing to form all pairwise edges within \( S \), and simultaneously to each delete subsets of their (other) incident edges. We say that \( G \) is \( k \)-stable if no \( k \)-defections are possible from \( G \)\(^{2} \). In view of this general definition, we will sometimes refer to the defections and stability notion in the previous paragraph as \( 2 \)-defections and \( 2 \)-stability.

As noted above, there are many possible generalizations of this model. For example, there could be different categories and different sensitivities of information; information could “attenuate” as it travels over multi-step paths, perhaps being forgotten with some probability at each step; and our model does not include the notion of globally “publishing” personal information through a mechanism like a personal Facebook page, but instead focuses on person-to-person communication. Thus, we can think of the model as capturing the information-sharing relationships for a single kind of information, by direct interaction, and in a coherent enough community that people have expectations about the behavior of others. Extending these assumptions in any of the above directions would be an interesting focus for future work.

Our Results: Existence and Social Welfare Our central goal is to study the most basic version of the model that is still rich enough to yield non-trivial and meaningful outcomes. Thus, for much of the first part of the paper, we focus on the case in which utilities are symmetric \( (u_{ij} = u_{ji}) \) and take values from the set \( \{-\infty, 1\} \). This corresponds to a natural version of the problem in which all pairs of people are either friends or enemies; there is a positive utility in sharing information with friends, but a much stronger negative utility in having enemies find out information about you.

Our first main result is that for any set of symmetric utilities from \( \{-\infty, 1\} \), and every \( k \geq 2 \), a \( k \)-stable network always exists. For \( k = 2 \) — the basic definition of stability — we can find a stable network in polynomial time. For general \( k \), it is NP-hard to construct a \( k \)-stable network.\(^{3}\) The intermediate case of fixed, constant \( k > 2 \) is interesting;

\[^{1}\text{Note that for defections of type (b), we require both } i \text{ and } j \text{ to strictly improve their utilities. This is in keeping with an assumption that utilities are not transferable (so that e.g. } i \text{ cannot pay } j \text{ to join her in a defection), and we will see that it creates a theoretical framework that more naturally connects to related lines of work in strategic network formation.}\

\[^{2}\text{One could also consider a notion of } k \text{-defection in which the } k \text{ nodes in } S \text{ only form a subset of the edges within } S \text{. Since we want to capture the idea that the whole set is mutually coordinating, rather than consisting of two disjoint sets that act simultaneously, we adopt the definition in which all edges are formed. We also note that a variant of the definition in which a connected (but not necessarily complete) subgraph on } S \text{ is formed yields very similar lines of analysis, since nodes’ utilities are derived from the components they belong to, rather than just who they are directly connected to.}\)

\[^{3}\text{In other words, although the decision problem, “Does there exist a } k \text{-stable network?” has the trivial answer “yes,” an algorithm that produces a witness could be used to solve NP-complete problems.}\)
we show how to construct $k$-stable networks in polynomial time for $k = 3$ and $k = 4$, with larger constants $k$ left as open questions.

There are also natural questions related to the notions of social welfare, defined as the sum of utilities of all nodes, and socially optimal networks, defined as those that maximize social welfare. Since a socially optimal network may not be stable, we can ask about the price of stability — the maximum welfare of any stable network relative to the optimum. We find that the price of stability is equal to $1$ for 2-stable and 3-stable networks — in other words, there always exist such networks achieving the social optimum — but it exceeds $1$ for $k > 3$. It is an open question to find a tight bound on the price of stability for $k > 3$.

**Our Results: Connections to Graph Coloring** There is a natural connection between the case of symmetric utilities from $\{-\infty, 1\}$ and the problem of graph coloring. Indeed, if we let $F$ denote the pairs of nodes $(i, j)$ with utility $u_{ij} = -\infty$, and define the conflict graph for the instance of the problem to be $H = (V, F)$, then the components in any stable network $G$ will have to be independent sets of $H$, and hence correspond to a coloring of $H$. The requirements of stability, of course, demand more, and so we in fact get an interesting and novel variant of the graph coloring problem in which we must find a coloring in which nodes in different color classes are all “blocked,” in a certain sense, from wanting to form direct connections with each other.

Using the connection to graph coloring, we can consider the following alternate definition of welfare for an information-sharing network $G$: the number of components it has. This essentially captures the extent to which nodes’ collective avoidance of information leakage has caused the group to “fragment” into non-interacting components. It is natural to want this number of components to be as small as possible, relative to the minimum achievable if we did not require stability; this minimum is $\chi(H)$, the chromatic number of the conflict graph $H$. We show that there is always a 2-stable network with a number of components equal to $\chi(H)$, and hence the analogue of the price of stability is equal to $1$ when the number of components is used to measure welfare. On the other hand, when we consider $n$-stable networks — the extreme case in which we allow defections of arbitrary size — it can be the case that the only $n$-stable networks have a number of components equal to $\Omega(\log n) \cdot \chi(H)$; and we show that this bound is tight, by proving that there is always an $n$-stable network with at most $O(\log n) \cdot \chi(H)$ components.

**Our Results: General Forms of the Model** Let’s now return to the general formulation of the problem, in which for each pair of nodes $i$ and $j$, node $i$ receives a utility $u_{ij}$ from being in the same component as $j$, and we may have $u_{ij} \neq u_{ji}$.

It turns out that many problems involving notions of stability or self-enforcing relations are contained in this general version. For example, the Gale-Shapley Stable Marriage Problem with $n$ men and $n$ women [13] arises as a simple special case of the model, by defining $u_{ij} = -\infty$ for each pair of men and each pair of women, and when a person $i$ has a person $j$ of the opposite gender in position $p$ on his or her preference list, defining $u_{ij} = 1 + n - p$. Related problems such as Becker’s Marriage Game [6, 7] can be similarly reduced to simple forms of the present model.

One downside of this generality is that once we move even a little beyond the case of symmetric utilities in $\{-\infty, 1\}$, the problem quickly becomes intractable. In particular, consider a case that is just slightly more general: symmetric utilities from $\{-\infty, 1, n\}$. In other words, the friendly relations now consist of “weak ties” of weight $1$ and “strong ties” of weight $n$ [14], with the relative values chosen so that the benefit of a single strong tie outweighs the total benefit of any number of weak ties incident to a single node. We show by a simple example that stable networks need not always exist with these kinds of weights. More strongly, we show in fact that for any $k$, deciding whether a given instance contains a $k$-stable network is NP-complete; the proof of this is based on developing the connection with graph coloring more extensively.

Despite this hardness result, the presence of elegant special cases like the Stable Marriage Problem suggests that there is considerable promise in developing a deeper understanding of the structural conditions on utilities that lead to settings in which stable networks always exist, and in which they can be efficiently identified.

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4The simplest such example has four people: Anna has a strong tie to Bob; Claire has a strong tie to Daniel; Bob and Daniel are enemies; and all other relations are weak ties. In any stable network, Anna and Bob would need to belong to the same component; Claire and Daniel would need to belong together in a different component. But then Anna and Claire would have an incentive to form an edge, violating stability.
Organization of the Paper The remainder of the paper is organized as follows. In Section 2 we discuss the connections between our model and related work in economic theory and computer science. In Sections 3, 4, and 5 we discuss our results on the existence, efficient construction, and welfare properties of stable networks for symmetric utilities in $\{\pm 1\}$. Finally, in Section 6 we discuss our results on more general forms of the model.

2 Related work

Closely related to this work is the substantial literature in economics on coalition formation. Coalitional games model the partitioning of a society into collaborative groups that jointly create worth; the value of each group is then shared among its participants. We restrict our discussion here to models involving non-transferable utility, which excludes, for example, the work of Deng and Papadimitriou [10]. It also distinguishes our work from that of Muto [17] and Nakayama and Quintas [18], which further differs ours in that their model does not incorporate network structure and uses a different definition of stability. Existing work on coalition formation differs from our work in two substantial ways: First, the solution concepts and defection models used in the coalition formation literature are fundamentally not suitable for modeling gossip and information leakage in a social network. Second, much of the foundational work in coalition formation focuses on conditions for the existence of these orthogonal solution concepts, whereas we study not only existence but computational issues and consequences for social utility.

Solution Concepts Requiring Large-Scale Consensus Almost all solution concepts for coalitional stability require that outcomes be individually rational, meaning no player would get higher utility by being alone in a singleton coalition. Beyond this, though, much of the work on solution concepts for coalitional stability pertains to definitions of deviations that require the consensus of a large number of players. We contend that information spreads in a social network not only by centralized dissemination strictly within globally negotiated coalitions, but through relationships negotiated at a local scale by a small number of individuals without the permission of the group as a whole.

The best established solution concept in the coalitional literature is the core, which consists of player partitions and payoff distributions so that there is no subset of the players that all are willing to simultaneously abandon their current coalitions and form a new one (where “willing” means at least one of the deviating players must strictly prefer the deviation). There is a substantial literature on necessary and sufficient conditions for the non-emptiness of the core, including work by Banerjee et al. [3] and Bogomolnaia and Jackson [8].

Bogomolnaia and Jackson [8] also study conditions on player preferences that imply existence of individually stable coalition partitions, which model individual player defections by requiring that every player in the coalition a defecting player wishes to join must agree to the defection. Again, this type of global coordination is not a good model for gossip: inherent in the concept of gossip is that it spreads without the permission or knowledge of the individuals to whom it pertains.

Like us, Dimitrov et al. [11] consider games where players have friend or enemy relationships; they characterize the internally stable coalitions, where no subgroup of any coalition wishes to break off and form a new coalition. While internal stability under small group defections might be a reasonable criterion for stability of a gossip network, this solution concept doesn’t allow for the possibility that two players from different coalitions might benefit from pooling their information. Dimitrov et al. [11] and Elkind and Wooldridge [12] both also study the computational tractability of computing the core.

Barbera and Gerber [4] observe that no solution concept can simultaneously provide a number of desirable properties. Among other things this argument ignores the difficulty of coordinating a defection by a large number of players.

Nash Stability The concept of Nash stability comes closest to our defection model; it describes situations where no player wishes to unilaterally defect to join a different coalition (regardless of whether they would agree to receive her). For our purposes, however, this is too individualistic: the spread of information should require the participation of at least two players. Milchtaich and Winter [16] use the Nash stability concept, and study a model where players prefer to associate with other players who are similar to them, but there is some upper bound on the total number of groups allowed. Here, Nash-stable partitions might not exist. In addition to studying existence, they also are interested
in distributed equilibrium computation: they show that asynchronous myopic randomized better response converges almost surely to a stable partition (under a somewhat limited definition of better response, where defecting players do not account for the impact they would have on the coalition they are joining).

Social Welfare Branzei and Larson [9] consider a model that is similar to ours, where each agent has a value for being in the same coalition as each other agent and utility is non-transferable. They also consider issues of social welfare, but for stability concepts (the core, internal stability) unsuitable for the study of information-sharing.

3 Existence and computation of stable outcomes

3.1 2-stability

We begin with the most basic model described in the introduction; we consider 2-stable networks for the case of symmetric utilities from \( \{-\infty, 1\} \).

We first show that the following (inefficient) algorithm always produces a 2-stable network \( G = (V, E) \). Recall that the conflict graph \( H = (V, F) \) is a graph on the same node set as \( G \), defined by setting \( F = \{(i, j) : u_{ij} = -\infty\} \).

**Algorithm 3.1.**

- Find a maximum-size independent set \( S \) in \( H \).
- Add all pairwise edges on \( S \) to the graph \( G \). This clique on \( S \) will be one of the components of \( G \).
- Iterate on \( V - S \).

**Theorem 3.2.** Algorithm 3.1 produces a 2-stable network.

**Proof.** First, because all components of \( G \) are built from independent sets of \( H \), all nodes have non-negative utility in \( G \). Thus, no node wants to defect by unilaterally deleting incident edges.

Now suppose that there were a defection in which two nodes \( i \) and \( j \) wanted to form the edge \((i, j)\), potentially deleting some of their incident edges. Let \( I \) denote the component of \( G \) containing \( i \), and let \( J \) denote the component of \( G \) containing \( j \). Suppose (by symmetry) that \( I \) was formed by the algorithm before \( J \). Then there is some \( i' \in I \) for which \((i', j) \in F\), since otherwise \( j \) could have been included in \( I \) when \( I \) was formed.

Since \( i \) has edges to all nodes in \( I \), a defection by \( i \) and \( j \) will only be utility-increasing for \( j \) if \( i \) deletes all its incident edges. Given this, a defection by \( i \) can only be utility-increasing if (a) \( j \) retains edges to nodes in \( J \), (b) \( i \) has no \(-\infty\)-edge to any \( j' \in J \), and (c) \(|J| + 1 > |I|\). But in this case, \( J \cup \{i\} \) is an independent set in \( F \) of cardinality strictly greater than \( I \), which means that in the iteration when the algorithm constructed \( I \), it should have constructed \( J \cup \{i\} \) instead. ■

We note that the use of maximum-cardinality independent sets is crucial for this algorithm; the variant that repeatedly identifies and deletes inclusionwise maximal independent sets in \( H \) need not create a 2-stable network.

Given that this algorithm contains the NP-hard maximum independent set problem as a subroutine, we next consider the question of finding a 2-stable network efficiently for any set of symmetric utilities in \( \{-\infty, 1\} \). One approach is to consider iterating the analogue of best-response dynamics for 2-defection: we repeatedly search for a 2-defection from the current graph, and if we find one we have the node or nodes perform the defection that maximizes their improvement in total utility.

Unfortunately, best-response dynamics can cycle indefinitely, as we now show.

**Theorem 3.3.** Best-response dynamics can cycle, with symmetric utilities in \( \{-\infty, 1\} \).

**Proof.** As a starting graph \( G \), we take a large clique \( s \) with identical utilities, plus five additional nodes \( a, b, c, d, e \), depicted in Figure [1].

Throughout the following best-response trajectory, the nodes of \( s \cup \{e\} \) will remain in a clique. The starting network \( G \) will also contain the two additional edges \((a, e), (c, d)\), and we make best-response moves as follows:
1. Change to \((c, e), (c, d)\) because \(e\) and \(c\) make a move: they form an edge and \(e\) drops \(a\). This is a best response for \(e\), who could have connected to \(b\) (equally good) or \(d\) while dropping \(a\) (also equally good). This is a best response for \(c\), who had no other options.

2. Change to \((c, e)\) because \(d\) drops its connection to \(c\). This is a best response for \(d\).

3. Change to \((a, b), (c, e)\) because \(a\) and \(b\) form a link. This is a best response for each of them.

4. Change to \((a, e), (a, b)\) because \(a\) and \(e\) make a move: they form an edge and \(e\) drops \(c\). The argument parallels that in step 1.

5. Change to \((a, e)\) because \(b\) drops its connection to \(a\). The argument parallels that in step 2.

6. Change to \((a, e), (c, d)\) because \(c\) and \(d\) form a link. The argument parallels that in step 3.

We’ve now returned to the initial network \(G\), completing the proof. 

Despite this cycling behavior, we now show how to perform a natural alternate dynamic process that reaches a stable network in polynomial time.

**Theorem 3.4.** We can find a 2-stable network in polynomial time.

**Proof.** We build up a polynomial-length sequence of networks iteratively, ending at a 2-stable network. We start from the network in which each node forms its own component, and we inductively maintain the property that all intermediate networks in the sequence will have connected components consisting of cliques.

At each intermediate state, we look for a node \(j\) in a clique \(J\), such that there is some clique with \(|I| \geq |J|\), and no edge \((i, j) \in F\) for any \(i \in I\). If we cannot find such a node, then the network is 2-stable, by an analogue of the argument in the proof of Theorem 3.2. Otherwise, we delete all of \(j\)’s edges to \(J\), and create edges from \(j\) to all nodes in \(I\). (Note that this is not a 2-defection, but we are not producing a run of best-response dynamics, simply a sequence of networks.)

Note that if there is an improving defection, there must be some node \(j\) in a clique \(J\), such that there is some clique with \(|I| \geq |J|\), and no edge \((i, j) \in F\) for any \(i \in I\). By construction, no node wishes to unilaterally drop all her edges, and no two non-singleton nodes improve their utility by forming an edge between them while both dropping all of their edges. If two nodes wish to form an edge while one of them drops all of her edges, this node is such a \(j\). If two nodes wish to form an edge while neither drops all her edges, the node from the smaller (or either, if the cliques are of equal size) clique provides such a \(j\).

Thus, our sequence of networks proceeds by repeatedly moving a node \(j\) from one clique \(J\) into another \(I\), such that \(|I \cup \{j\}| > |J|\). We now show that this process must terminate after passing through at most a polynomial number of networks. For this, we let \(x_0, x_1, x_2, \ldots\) denote the sizes of the cliques in our current graph, and we consider the
potential function $\sum_i x_i^2$. If a player moves from a clique of size $b$ to a clique of size $a \geq b$, then in the potential function we replace the terms $a^2 + b^2$ by $(a + 1)^2 + \left((b - 1)^2 = a^2 + b^2 + 2(a - b) + 2 \geq a^2 + b^2$. Thus, the potential function increases by at least 2 with each move, and since it can’t grow larger than $n^2$, this proves that the construction terminates after passing through at most $O(n^2)$ graphs.

The running time of the full algorithm is also polynomial, since we can easily check for the existence of the required node $j$ in each iteration in polynomial time. ■

3.2 \textit{k-stability}

We now consider the generalization to $k$-defections and the corresponding notion of $k$-stability. We begin by showing that $k$-stable networks exist, for all $k$.

\textbf{Theorem 3.5.} For every $k \geq 2$, every instance admits a $k$-stable network.

\textbf{Proof.} In fact, we show that Algorithm 3.1 finds a network that is $k$-stable for all $k$.

Suppose by way of contradiction that in the network $G$ produced by this algorithm (consisting of disjoint cliques), there were a set $S$ of nodes that wanted to defect. Consider the first clique $I$ in order of formation that contains a node $i \in S$. All nodes in $S - I$ must have $-\infty$-edges to nodes in $I$, so in any defection involving $S$, the node $i$ must drop all its edges into $I$.

Now, let $I'$ be the component that $i$ belongs to after the defection. In order for this to be a defection in which $i$ participates, it must be that $|I'| > |I|$; but then in the iteration when $I$ was produced, the algorithm should have produced $I'$ instead, a contradiction. ■

However, although $k$-stable networks must exist, actually constructing one is NP-hard.

\textbf{Theorem 3.6.} Constructing a $k$-stable network is NP-hard when $k$ is part of the input.

\textbf{Proof.} If $k$ is at least the size of the maximum independent set in the graph $H$, any $k$-stable network contains a maximum independent set of $H$ as one of its connected components. ■

In fact, even deciding whether a given network is $k$-stable is computationally intractable.

\textbf{Theorem 3.7.} Testing stability under $k$-defections is NP-hard.

\textbf{Proof.} The proof is by reduction from finding a $k$-node independent set. Given an $n$-node graph $L$ that is an instance of independent set, assume that each edge in the graph represents a $-\infty$ relationship and that all absent edges are $+1$ relationships. We will add $k - 2$ additional nodes for each node $x_i$. The $k - 2$ nodes for $x_i$ all have $+1$ relationships with each other and with $x_i$ and have $-\infty$ relationships with all other nodes in the graph. The arrangement whose stability we will test consists of $n$ many $(k - 1)$-node cliques, each consisting of a node in the original graph and its $k - 2$ additional nodes. There is a group of $\leq k$ players who wish to defect from this arrangement if and only if there was an independent set of size $k$ in $G$. ■

Now, a natural question is whether it is computationally feasible to construct $k$-stable networks for constant $k$. One approach to this is to follow the style of analysis in the proof of Theorem 3.4 and to use a potential function on the vector of component sizes that always increases, and is bounded by a function of the form $n^{f(k)}$. Here something interesting happens: this approach provides a polynomial bound when $k \in \{3, 4\}$, but we show that such a cardinality-based potential function provably cannot provide a polynomial bound when $k \geq 5$.

To give some first intuition for what goes wrong, suppose we were to try using the function $\sum_i x_i^{f(k)}$, where the $x_i$ are the component sizes. Now, suppose $k = 6$; we consider 5 groups of 5 nodes, and one group with 1 node; and we allow six nodes to defect. Suppose further that we have have one player from each large group all join the group of 1. The initial potential was $5^7 + 1 = 78126$ and the new potential is $5 \cdot 4^6 + 6^6 = 67136$.

We now provide proofs for the cases of $k \in \{3, 4\}$ and $k \geq 5$.

\textbf{Theorem 3.8.} We can find a 3-stable network in polynomial time, using the potential function $\sum_i (x_i + 4)(x_i - 1)/2$. 

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Proof. We define a recurrence relation for a potential function for \( k = 3 \):

\[
F_3(1) = 1 \\
F_3(2) = 3 \\
F_3(3) = 7 \\
F_3(i) = 2F_3(i - 1) - F_3(i - 2) + 1
\]

gives \( F_3(n) = (n + 4)(n - 1)/2 \).

In general, we require that \( F_3(i) < F_3(i + 1) \).

The recurrence for 2 nodes each leaving groups to form a new group requires \( 2F_3(1) < F_3(2) \). The recurrence for 1 node leaving its group to join another is

\[
F_3(i) = 2F_3(i - 1) - F_3(i - 2),
\]

which is strictly less than that given above.

This recurrence covers the worst case for 2 nodes each leaving a separate group and joining a third node. The recurrence for 2 nodes both leaving the same group and joining a third node is

\[
F_3(i) = F_3(i - 1) + F_3(i - 2) - F_3(i - 3) - 1,
\]

which is strictly less. The recurrence for 3 nodes each leaving groups to form a new group requires that \( 3F_3(2) < 3F_3(1) + F_3(3) \). ■

**Theorem 3.9.** We can compute a 4-stable network in polynomial time, using a potential function that is \( O(n^3) \).

Proof. We will solve a recurrence relation to derive a potential function for \( k = 4 \):

\[
F_4(1) = 1 \\
F_4(2) = 3 \\
F_4(3) = 7 \\
F_4(4) = 17 \\
F_4(i) = 3F_4(i - 1) - 3F_4(i - 2) + F_4(i - 3) + 1
\]

which solves to \( F_4(n) = 17(n-3) + (n-5)(n-4)(n-3)/6 + 7(n-5)(n-4)/2 \) for \( n \geq 6 \).

In general, we require that \( F_4(i) < F_4(i + 1) \).

Note that \( F_4(i) \geq F_3(i), \forall i \), and thus we need only address defections by 4 nodes. The given recurrence covers the worst case for 3 nodes each leaving groups and joining a fourth node: If 3 nodes left the same group to join a fourth, the recurrence is \( F_4(i) > F_4(i - 1) + F_4(i - 2) - F_4(i - 3) \). If 2 nodes leave one group and one node leaves another, the recurrence is \( F_4(i) > 2F_4(i - 1) - F_4(i - 4) \). The recurrence for 4 nodes each leaving groups to form a new group requires that \( 3F_4(i) < 3F_4(1) + F_4(4) \). ■

Starting with \( k = 5 \), however, polynomially bounded additive potential functions no longer exist.

**Theorem 3.10.** Any additive potential function for \( k \)-defections on \( n \) nodes for \( k \geq 5 \) is in \( \Omega(2^n) \).

Proof. We will lower bound the value of any potential function \( F \) for 5-defections. First, let \( F(1) = 1 \). Note that \( 2F(1) < F(2) \) in order to increase the potential when 2 singleton nodes defect to form a group. In order to increase the potential when 3 nodes defect from groups of size 2 to form a new group, \( 3F(2) < 3F(1) + F(3) \). Similarly, we get \( 4F(3) < 4F(2) + F(4) \) and \( 5F(4) < 5F(3) + F(5) \). Solving, this gives \( F(2) \geq 3, F(3) \geq 7, F(4) \geq 17, F(5) \geq 51 \), and thus \( F(i) \geq 2^{i-1} \) for \( i \leq 5 \).

We now consider defections where \( k - 1 = 4 \) nodes each defect from groups of size \( i - 1 \) to join a group of size \( i-k+1 = i-4 \), resulting in a group of size \( i \) and 4 groups of size \( i-2 \). Thus, \( F(i) > 4(F(i-1) - F(i-2)) + F(i-4) \) for \( i > 5 \). So certainly \( F(i) \geq 4(F(i-1) - F(i-2)) \) for \( i > 5 \), which solves to \( F(i) \geq 2^{i-1} \). ■
Figure 2: Depicted edges \((i, j)\) represent \(u_{ij} = 1\); absent edges have \(u_{ij} = -\infty\).

4 Social Welfare: Total Utility

As noted in the introduction, there are two natural measures of welfare for an information-sharing network \(G\): the sum of node utilities, and the number of components of \(G\). We first observe that optimizing each of these (over all networks, not just stable ones) is NP-hard.

**Observation 4.1.** Maximizing the total utility on a 3-partite graph is equivalent to partitioning the graph into induced triangles, which is NP-hard.

**Observation 4.2.** Minimizing the number of groups in a partition is equivalent to determining the chromatic number of the graph, and thus cannot be approximated to within \(n^{1-\epsilon}\) for any \(\epsilon > 0\).

The two notions of welfare are also quite distinct: networks that are ideal for one may not be optimal for the other.

**Theorem 4.3.** There exist instances where no network minimizing the number of connected components also maximizes the total utility.

**Proof.** Figure 2 has only one network that minimizes the number of connected components while not placing any \(-\infty\)-edges within a connected component: the four pairs \(\{a, a_1\}, \{b, b_1\}, \{c, c_1\}, \{d, d_1\}\). Any network with fewer than four connected components would necessarily place at least two of \(\{a_1, b_1, c_1, d_1\}\) in the same component.

This conflict graph also has only one network that maximizes its total utility: the 4-clique plus 4 isolated vertices. Both of these networks are stable: the \(x1\) nodes cannot form any additional edges, and no other node would wish to join a pair containing a player she dislikes.

Despite these negative results, one can still study the quality of \(k\)-stable networks relative to these optima as baselines. We consider the sum of utilities in this section, and the number of components in the next section.

### 4.1 2-defections

For the total utility metric, we can make the following strong statement: *every network that maximizes the total utility is 2-stable.*

**Theorem 4.4.** In every instance, every network that maximizes the total utility is stable. Thus, the price of stability for total utility under 2-deviations is 1.

**Proof.** Consider a network \(G\) that maximizes the total utility. We may assume that each component of \(G\) is a clique. Suppose that \(G\) is not 2-stable. Clearly, no player wishes to defect by simply dropping edges, no two players can form an edge without dropping any edges (if they could, the network wasn’t optimal), and no two players wish to both drop
Figure 3: Depicted edges \((i, j)\) represent \(u_{ij} = 1\); absent edges have \(u_{ij} = -\infty\).

edges to form a pair. So we must consider two players \(u\) and \(v\) in cliques of size \(n_1 \geq n_2\), respectively, who wish to defect by forming an edge between them while \(v\) drops all of her other edges. But the resulting total utility will increase by

\[
\frac{1}{2} \left((n_1 + 1)^2 + (n_2 - 1)^2 - n_1 - n_2 - (n_1^2 + n_2^2 - n_1 - n_2)\right) = n_1 - n_2 + 1,
\]

which is strictly greater than 0 for any \(n_1 \geq n_2\), so we have arrived at a contradiction, and no player wishes to defect.

We now show that the price of anarchy is strictly greater than 1 for both the sum of utilities and the number of components.

**Theorem 4.5.** There exist stable networks that neither maximize total utility nor minimize the number of connected components. Hence the price of anarchy for both measures is \(> 1\).

**Proof.** In Figure 3, the minimum number of connected components (of four) is achieved by the partition \(\{x_1, y_1, z_1\}; \{x_2, y_2, z_2\}; \{x_3, y_3, z_3\}; \{x_4, y_4, z_4\}\). The maximum total utility (value twelve) is achieved by the partition \(\{x_1, x_2, x_3, x_4\}; \{y_1, y_2, y_3, y_4\}; \{z_1\}; \{z_2\}; \{z_3\}; \{z_4\}\). But there is another stable network, with value ten and with five connected components: \(\{x_1, x_2, x_3, x_4\}; \{y_1, z_1\}; \{y_2, z_2\}; \{y_3, z_3\}; \{y_4, z_4\}\). This network is stable: no player in a pair can entice a player in the group of four to drop all her links, none wishes to join the group of four otherwise. Similarly, no two players in different pairs wish to defect.

In fact, the price of anarchy for total welfare is much worse.

**Theorem 4.6.** The price of anarchy for total welfare for \(k = 2\) is at least \(n/2\).

**Proof.** Consider a conflict graph with two cliques of size \(n/2\) with a matching between them except on one pair of unmatched vertices. The matching is 2-stable, but the two cliques optimize total welfare.

### 4.2 \(k\)-deviations

We now turn to the problem of characterizing the welfare properties of stable networks under the more general \(k\)-defection model.
Price of Stability for total welfare  Unlike for \( k = 2 \), for larger \( k \), not every network maximizing total welfare is stable.

Observation 4.7. For \( k = 3 \), Figure 5 demonstrates a network that maximizes total welfare but is not stable.

Despite this, for \( k = 3 \), there is always a \( k \)-stable network maximizing total welfare.

Theorem 4.8. The price of stability for total welfare is 1 for \( k = 3 \).

Proof. Consider some network maximizing total welfare, and suppose it is not 3-stable. We know from the proof of Theorem 4.4 that there is no defection where fewer than 3 players participate. So we will consider every type of 3-player defection, and perform defections until no more exist. This will not cycle, because, as we show below, each possible defection strictly increases the potential function that is the sum of the cubes of the connected component sizes.

- If the three players defect to form a clique and all drop all existing edges, the new total utility is \( 9 + a^2 + b^2 + c^2 \), and the previous utility was \( (a + 1)^2 + (b + 1)^2 + (c + 1)^2 \), for \( a, b, c, \in \{0, 1\} \). Thus the new utility is at least the old utility, so this defection will result in a new network maximizing total welfare. The potential function strictly increases.

- Suppose two players drop all existing edges to join a third. Then if the original total utility was \( a^2 + b^2 + c^2 \), the new value is \( (a - 1)^2 + (b - 1)^2 + (c + 2)^2 \). Since \( a + 1 \geq a \) and \( c + 1 \geq b \), this defection results in a strict increase in total utility (of at least two), which is a contradiction.

- Suppose two players leave the same group to join the third player’s group. If the total utility of the initial network was \( a^2 + b^2 \), the new total utility is \( (a + 2)^2 + (b - 2)^2 \), for a strict increase of at least 4, which is a contradiction.

Note that we need not consider defections in which more than one player retains links to her current clique, since in that case there would exist a two player defection where these two players form an edge between them.

Theorem 4.9. The price of stability for total welfare, when \( k \geq 4 \), is strictly greater than 1.

Proof. In Figure 4, the only 4-stable network has components corresponding to the \( K_4 \) plus the four pairs, but this has lower total utility than the four triangles.

The ratio of 6/5 implied by the proof of Theorem 4.9 is the strongest lower bound on the price of stability for total welfare that we know of for any \( k \); it is an interesting open question to find the correct asymptotic bound for this objective function.
Price of Anarchy for Total Utility  The worst $k$-stable network can have a factor $\frac{n/k-1}{k-1}$ smaller total utility than is optimal.

**Theorem 4.10.** The price of anarchy for total welfare for $k > 2$ is $\Omega(\frac{n}{k-1})$.

*Proof.* Again, consider the graph with $n/k$ rows of elements, with each row forming a clique and each column forming a clique. As before, the columns are stable under $k$-defections; they give total utility $\frac{1}{2}(nk - n)$, whereas the rows give total utility $\frac{1}{2}(n^2/k - n)$. 

For sufficiently large $k$, however, every stable network gives good total utility.

**Theorem 4.11.** When there exists a socially optimal network where all cliques are of size $\leq k$, the worst $k$-stable network has at most a factor 2 smaller utility than is optimal.

*Proof.* Consider a clique of size $c \leq k$ that is present in the socially optimal network, and then consider the utility of each of those players in some $k$-stable network. At least one of those players must have utility at least $c - 1$; otherwise the $c$ players would all strongly prefer to defect to their original clique. Ignoring this first player, there must be some other player from the clique with utility at least $c - 2$; otherwise the $c - 1$ remaining players would all prefer to defect and form a clique. Similarly, there must be players achieving utilities at least $(c - 3), \ldots, 2, 1, 0$.

The utility that this clique contributes to the social optimum is $\frac{1}{2}c(c - 1)$. The utility those players must contribute to any stable network is at least $\frac{1}{2}\sum_{i=1}^{c}(i - 1) = \frac{1}{2}c(c - 1)$ (note that our definition of total utility counts each edge once, not twice). Thus, the total maximum utility is at most twice that of any stable network.

In general, the worst $k$-stable network has at most a factor $\frac{n(n - 1)}{(k - 1)(n - k/2)}$ smaller utility than is optimal.

**Theorem 4.12.** The price of anarchy for total utility is $O(\frac{n(n - 1)}{(k - 1)(n - k/2)})$.

*Proof.* As in the proof of Theorem 3.11 consider a clique of size $c$ that is present in the socially optimal network. It contributes $\frac{1}{2}c(c - 1)$ to the social optimum, and its constituents must contribute at least $\frac{1}{2}(0 + 1 + \ldots + (k - 2) + (k - 1) + \ldots + (k - 1) = \frac{1}{2}(k - 1)(c - k/2)$. The ratio of the sums of these clique contributions is maximized when we consider a single clique of size $n$.

## 5 Social Welfare: Number of Components

We now consider the quality of stable networks with respect to the number of components they contain; the ideal outcome is to have a number of components close to $\chi(H)$, the chromatic number of the conflict graph $H$.

### 5.1 2-stability

There always exists a 2-stable network minimizing the number of connected components.

**Theorem 5.1.** In any instance, there exists a 2-stable network with a number of components equal to $\chi(H)$. Thus the price of stability for the number of components under 2-deviations is 1.

*Proof.* Given an instance, and a partition $\Pi$ of the nodes into $\chi(H)$ sets, each of which is independent in $H$, we first build a network $G$ by placing a clique on each set in $\Pi$, with no other edges between.

Suppose that $G$ is not 2-stable. It cannot be the case that there exists a single player who wishes to defect by dropping edges, since by definition $G$ does not place any node in a clique with players it dislikes. It also cannot be the case that two players wish to defect by forming an edge between them without dropping any edges, since the resulting network would have a number of color classes strictly less than $\chi(H)$. Nor would both players wish to drop edges, since their resulting clique would contain only two players. So, finally, consider the case where two nodes wish to defect by forming an edge between them while one drops all its edges. Next, form all edges between the defecting node and its new group members. This results in a new network that also minimizes the number of components (or reduces, it, if the smaller clique was of size one), and the defection increases the potential function corresponding to the sum of the squared group sizes. Thus we can allow players to repeatedly defect in this manner until no defections are available, and the procedure (which started with an arbitrary network minimizing the number of components) will yield a stable network minimizing the number of components.
5.2 \(k\)-stability

Price of Stability for Number of Components

Theorem 5.2. For \(k = 3\), the price of stability for the number of components is \(> 1\).

Proof. Figure 5 shows a conflict graph with chromatic number 3, but every \(k\)-stable network for this instance has more than 3 components. ■

Theorem 5.3. The price of stability for the number of components is \(O(\log n)\), for any \(k\).

Proof. Algorithm 3.1 is in fact performing the greedy set-cover algorithm on the set system of independent sets of \(H\). It produces a network \(G\) that is \(k\)-stable for all \(k\), and by the approximation properties of the greedy set-cover algorithm, it produces a number of components that is at most \(\ln s\) times larger than \(\chi(H)\), where \(s \leq n\) is the maximum independent set in \(H\). ■

We now show a matching asymptotic lower bound on the price of stability when \(k = n\).

Theorem 5.4. The price of stability for \(n\)-defections is \(\Omega(\log n)\).

Proof. Define the graph \(B_n\) to have nodes \(x_1, \ldots, x_n, y_1, \ldots, y_n\), with edges \((x_i, y_j)\) and \((x_j, y_i)\) for each pair \((i, j)\) with \(j \leq i/3\). We define an instance with symmetric utilities in \((-\infty, 1)\) by giving the edges of \(B_n\) weight \(-\infty\), and all other pairs of nodes weight 1. Since \(B_n\) is bipartite, we have \(\chi(B_n) = 2\). We claim that \(B_n\) has a unique maximum independent set \(S_1\), equal to \(\{x_i, y_i : i > n/3\}\). To see why \(S_1\) is the unique maximum independent set, consider any other independent set \(R\). Let \(a = \max\{i : x_i \in R\}\) and \(b = \max\{j : y_j \in R\}\). Now, since \(x_a \in R\), we cannot have \(y_j \in R\) for any \(j \leq a/3\); and since \(y_b \in R\), we cannot have \(x_i \in R\) for any \(i \leq b/3\). Thus, \(|R| \leq (a - b/3) + (b - a/3) = 2(a + b)/3\). Now, if \(a = b = n\), then \(R \subseteq S_1\); and if \(\max(a, b) < n\), then \(|R| < 2(n + n)/3 = 4n/3 = |S_1|\). Thus, \(S_1\) is the unique maximum independent set in \(B_n\).

Next, we see that in any stable network \(G_n\) for the instance defined by \(B_n\), we must have \(S_1\) as one of the components; otherwise, the nodes in \(S_1\) could defect and form a clique on themselves. But since \(B_n - S_1 = B_{n/3}\), we can proceed inductively. \(B_n - S_1\) has a unique maximum independent set \(S_2\), equal to \(\{x_i, y_i : n/9 < i \leq n/3\}\). Since \(S_1\) is a component in any stable partition of \(B_n\), it follows that \(S_2\) must also be a component in any stable network for the instance defined by \(B_n\); otherwise, all nodes in \(S_2\) will belong to distinct components in \(B_n - S_1\), with each component an independent set of \(B_n - S_1\), and so they could all improve their utility by defecting to form a clique on themselves. Now define \(S_k = \{x_i, y_i : n/3^k < i \leq n/3^{k-1}\}\). Continuing by induction, the set \(S_k\) is the unique maximum independent set in the graph \(B_n - (\cup_{i=1}^{k-1} S_i) = B_{n/3^{k-1}}\), and must be a component in any stable network for the instance defined by \(B_n\). But this implies that any stable network for the instance defined by \(B_n\) must have \(\Omega(\log n)\) components. ■
Price of Anarchy for Number of Components

The worst $k$-stable network can have a factor $\frac{n}{k^2}$ more connected components than $\chi(H)$.

**Theorem 5.5.** The price of anarchy for number of components is $\Omega(\frac{n}{k^k})$.

**Proof.** Consider a graph with $\frac{n}{k}$ rows of elements, with each row forming a clique and each column forming a clique. Under size-$k$ defections, the columns are a stable network, but the chromatic number $k$ is achieved by the rows. □

6 Generalizations of the model

As noted in the introduction, there are several aspects of our model one might consider varying, including

- Are the values $u_{ij}$ symmetric ($u_{ij} = u_{ji}$ $\forall i, j$) or asymmetric?
- What utility values are allowed?

Unfortunately, simple versions of both generalizations result in instances for which there is no stable network.

6.1 Asymmetric Cost Functions

**Theorem 6.1.** Under asymmetric preferences, 2-stable networks need not exist even when all utilities are in $\{-\infty, 1\}$.

**Proof.** Consider a set of four nodes $x, v_1, v_2, v_3$ and $u_{v_1v_2} = u_{v_2v_3} = u_{v_3v_1} = -\infty$; all other utilities are 1. In any 2-stable network, none of the nodes $v_i$ can be in a component together. Now, who else can be in a component with $x$? If $x$ were in a singleton connected component, this would not be stable, since forming an edge with one of the nodes $v_i$ node would be an improving defection. If $x$ were in the same component as (without loss of generality) $v_1$, then nodes $v_2$ and $x$ would want to gossip, forming a deviation. Thus, such a network would not be 2-stable either. □

6.2 General Symmetric Weights

As we saw in the introduction, stable networks need not exist with symmetric utilities that can take general values. In fact, stable networks may not exist even in a very mild generalization of our basic model with symmetric utilities in $\{-\infty, 1\}$.

**Theorem 6.2.** In the model with all $u_{ij} \in \{-\infty, +1, +c\}$ for $c > n$, stable networks need not exist.

**Proof.** Suppose we have four nodes $w_1, w_2, m_1, m_2$ with utilities $u_{w_1m_1} = c; u_{w_2m_2} = c; u_{m_1m_2} = -\infty$ and all other utilities equal to 1. Any stable network must put the pairs with utility $c$ in two distinct components; but then $w_1$ and $w_2$ will wish to defect by forming the edge between them. □

One might ask whether the simple techniques we used in the case of symmetric utilities in $\{-\infty, +1\}$ can be generalized to find stable networks under generalized weights, when such networks exist. Unfortunately, this is not the case. First, repeated formation of maximum-size cliques does not result in a stable network in the generalized weight setting setting, since it may place a player in an earlier clique that is worse for her (but where she increases its maximum value). Second, note that sequential improving moves by single players from one clique to another always increase the potential function

$$\sum_i \sum_{j \in C_G(i)} u_{ij},$$

where $C_G(i)$ denotes the component of $i$ in the current network $G$. The potential function is bounded by the sum of the positive $u$ values. However, a chain of sequential improving moves need not arrive at a stable network, because players might still wish to make other types of deviations.

Finally, we will show that determining whether an instance has a stable network is intractable, even under a very mild generalization of the possible utilities. We represent the utilities using a weighted complete graph $W$, in which
the weights on the edges of $W$ define the utilities. The nodes of $W$ are $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$; there is an edge of weight $c > n$ between each pair $(x_i, y_j)$; there are edges of weight $-\infty$ between certain pairs $(x_i, x_j)$ and $(y_k, y_l)$; and there edges of weight 1 between all other pairs. We call this a $(-\infty, +1, +c)$-instance with a matching structure.

**Theorem 6.3.** For $c > n$, the problem of determining whether a $(-\infty, +1, +c)$-instance with a matching structure has a stable network is NP-complete.

**Proof.** We define a set of related problems for the purposes of the reduction.

- The first is the **3-coloring problem**: given a graph $H$, determining whether it is 3-colorable.
- The second is a problem we’ll call **3-coloring of a triangle-partitioned graph (3CTPG)**. In this problem, we are given a graph $H$ together with a partition of its nodes into triples, each of which induce a triangle in $H$, and we want to know whether $H$ is 3-colorable.
- The third problem is **Stable Coloring of Bichromatic Graphs (SCBG)**. In this problem, we are given a graph $K$ in which each edge is colored either red or blue. We allow $K$ to have parallel edges of different colors. We want to partition $K$ into independent sets $\{S_i\}$ with the property that if $(v, w)$ is an edge of $K$, with $v \in S_i$ and $w \in S_j$, then at least three of the following four kinds of edges are present:
  1. A red edge from $v$ to a node in $S_j$;
  2. A blue edge from $v$ to a node in $S_j$;
  3. A red edge from $w$ to a node in $S_i$;
  4. A blue edge from $w$ to a node in $S_i$.

Such a partition will be called a **stable coloring** of $K$. The problem is to determine whether $K$ has a stable coloring.

**NP-Completeness of 3CTPG** We claim that 3CTPG is NP-complete, by a reduction from (standard) 3-coloring. Given a graph $H$ for which we want 3-colorability, we construct a graph $H'$ as follows: for each node $v \in V(H)$, we add new nodes $v'$ and $v''$, with new edges $(v, v'), (v, v'')$, and $(v', v'')$. We then present $H'$ together with the sets $\{(v, v', v'') : v \in V(H)\}$ as an instance of 3CTPG. Now, if $H'$ is 3-colorable, then we can use the 3-coloring of $V(H) \subseteq V(H')$ as a 3-coloring of $H$. Conversely, if $H$ is 3-colorable, we can extend this to a 3-coloring of $H'$ by coloring each $v'$ and $v''$ with the two colors not used for $v$. Thus $H$ is 3-colorable if and only if $H'$ is, and hence 3CTPG is NP-complete.

**NP-Completeness of SCBG** We next show that SCBG is NP-complete, by a reduction from 3CTPG. Suppose we are given an instance of 3CTPG, consisting of a graph $H = (V, E)$ and a partition $\Pi$ of the nodes into sets of size three. For a node $v \in V$, we let $\pi(v)$ denote the partition $v$ belongs to. We construct an equivalent instance of SCBG as follows, consisting of a red-blue-colored graph $K$. For each triangle in $\Pi$, we create a triangle of parallel red and blue edges on these nodes in $K$. For all other edges of $H$, we add only a blue edge to $K$. Note, crucailly, that red edges thus only appear in the collection of disjoint triangles defined by $\Pi$.

Now we claim that $K$ has a stable coloring if and only if $H$ has a 3-coloring. First, suppose that $H$ has a three-coloring, with color classes $A$, $B$, and $C$. Then we use this same partition of $K$ into three independent sets. Clearly, for each triangle in $\Pi$, one node goes in each of $A$, $B$, and $C$. As a result, for each pair of nodes $v, w$ in $K$ belonging to different color classes, all four types of edges (i)-(iv) are present, since the other two members of $\pi(v)$ belong to the two color classes $v$ is not in, and $v$ has both red and blue edges to them; and likewise for $w$. Thus, this is a stable coloring of $K$.

Conversely, suppose that $K$ has a stable coloring. We first claim that this coloring must consist of at most three non-empty independent sets. Indeed, suppose that this coloring included at least four non-empty independent sets. Consider a node $v$ in one of the independent sets $A$, and let $\pi(v) = \{v', v''\}$ with $v' \in B$ and $v'' \in C$. Now, since we are assuming there are at least four non-empty independent sets, let $D$ be another non-empty independent set containing a node $w \notin \pi(v)$. At least one of $A, B, \text{or } C$ contains no node of $\pi(w)$; suppose (by symmetry) that it is $C$. Then $v''$ and $w$ belong to different independent sets in the coloring, $v''$ has no red edge to any node in $D$, and $w$ has no red edge to any node in $C$; this contradicts the stability of the coloring. It follows that the coloring must consist
of at most three non-empty independent sets. Consequently, the stable coloring of \( K \) is also a 3-coloring of \( H \); since this completes the converse direction, we’ve shown that \( K \) has a stable coloring if and only if \( H \) has a 3-coloring.

**NP-Completeness of determining whether a \((-\infty, +1, +c)\)-instance with a matching structure has a stable network** This is the final step, showing that our original problem is NP-complete. We reduce from SCBG.

Suppose we are given an instance of SCBG, consisting of a graph \( K = (V, E) \) with each edge colored red or blue. We construct a \((-\infty, +1, +c)\)-instance with a matching structure as follows, using a weighted complete graph to encode the utilities. For each \( v \in V \), we create two nodes \( x_v \) and \( y_v \) in \( W \), and we join them by an edge of weight \( c \). Then, for each \( (v, w) \in E \) colored red, we create an edge \((x_v, x_w)\) of weight \(-\infty\); for each \( (v, w) \in E \) colored blue, we create an edge \((y_v, y_w)\) of weight \(-\infty\). We include edge of weight 1 between all other pairs of nodes in \( W \). Now we claim that \( K \) has a stable coloring if and only if \( W \) has a stable network.

We prove the two directions of this as follows. First, if there is a stable network \( G \) for the instance \( W \), then for each \( v \), the nodes \( x_v \) and \( y_v \) must be in the same component \( S_a \) of \( G \), since otherwise (by the fact that \( c > n \)) they would have an incentive to drop all the edges to their current sets and form an edge between each other. We define a subset \( S'_a \subseteq V(K) \) consisting of all \( v \) for which \( \{x_v, y_v\} \subseteq S_a \), and we claim that these sets \( \{S'_a\} \) form a stable coloring of \( K \). Indeed, suppose there existed nodes \( v \in S'_a \) and \( w \in S'_b \) for which two of the four types of edges (i)-(iv) were not present. Then this would imply that one of \( x_v \) or \( y_v \) would be able to gossip with one of \( x_w \) or \( y_w \) without either of them receiving a negative utility from information spreading into the other’s component. This contradicts the stability of the network \( G \) for the instance \( W \).

For the converse direction, we must show that if there is a stable coloring of \( K \), into sets \( \{S'_a\} \), then there is a stable network \( G \) for the instance \( W \). For this, we define a set \( S_a \) containing both \( x_v \) and \( y_v \), for each \( v \in S_a \), and include in \( G \) a clique of edges on \( S_a \). We put no other edges in \( G \). Since \( S'_a \) is an independent set, \( S_a \) has no internal \(-\infty\) edges. Also, suppose a node in \( z_v \in S_a \) were able to gossip with a node in \( z_w \in S_b \), without either of these nodes dropping any edges, where \( z_v \) denotes one of the nodes \( x_v \) or \( y_v \), and \( z_w \) likewise denotes one of the nodes \( x_w \) or \( y_w \). Then the corresponding nodes \( v \) and \( w \) in \( K \) would each lack at least one color of edge into the other’s set, contradicting the stability of the coloring of \( K \). But neither \( z_v \) nor \( z_w \) will be able to increase utility if they drop edges: since the sets \( S_a \) and \( S_b \) are cliques, the only ways that \( v \) and \( w \) can break paths to any other nodes in \( S_a \) or \( S_b \) involve breaking their edges to the nodes to whom they’re connected by edges of weight \( c \), which would result in a net loss of utility. Thus, no nodes have an incentive to change their connections in this partition of \( G \), and so it is a stable network.

This shows that \( K \) has a stable coloring if and only if there is a stable network for the instance \( W \). and hence establishes the NP-completeness of our original problem. ■

**References**

[1] Rebecca G. Adams and Rosemary Blieszner. An integrative conceptual framework for friendship research. _Journal of Social and Personal Relationships_, 11(2):163–184, May 1994.

[2] Elizabeth J. Aries and Fern L. Johnson. Close friendship in adulthood: Conversational content between same-sex friends. _Sex Roles: A Journal of Research_, 9(12):1183–1195, December 1983.

[3] S. Banerjee, H. Konishi, and T. Sönmez. Core in a simple coalition formation game. _Social Choice and Welfare_, 18(1):135–153, 2001.

[4] S. Barberà and A. Gerber. A note on the impossibility of a satisfactory concept of stability for coalition formation games. _Economics Letters_, 95(1):85–90, 2007.

[5] Max Bazerman, Robert Gibbons, Leigh Thompson, and Kathleen Valley. Can negotiators outperform game theory? In Jennifer J. Halpern and Robert N. Stern, editors, _Debating Rationality: Nonrational aspects of organizational decision-making_, pages 78–98. Cornell University Press, 1998.

[6] G.S. Becker. A theory of marriage: Part I. _Journal of Political Economy_, 81(4):813, 1973.

[7] G.S. Becker. A Theory of Marriage: Part II. _Journal of Political Economy_, 82(S2):11, 1974.
[8] A. Bogomolnaia and M.O. Jackson. The stability of hedonic coalition structures. *Games and Economic Behavior*, 38(2):201–230, 2002.

[9] S. Brânzei and K. Larson. Coalitional affinity games and the stability gap. In *Proceedings of The 8th International Conference on Autonomous Agents and Multiagent Systems-Volume 2*, pages 1319–1320. International Foundation for Autonomous Agents and Multiagent Systems, 2009.

[10] X. Deng and C.H. Papadimitriou. On the complexity of cooperative solution concepts. *Mathematics of Operations Research*, 19(2):257–266, 1994.

[11] D. Dimitrov, P. Borm, R. Hendrickx, and S.C. Sung. Simple priorities and core stability in hedonic games. *Social Choice and Welfare*, 26(2):421–433, 2006.

[12] E. Elkind and M. Wooldridge. Hedonic coalition nets. In *Proceedings of The 8th International Conference on Autonomous Agents and Multiagent Systems-Volume 1*, pages 417–424. International Foundation for Autonomous Agents and Multiagent Systems, 2009.

[13] D. Gale and L. S. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[14] Mark Granovetter. The strength of weak ties. *American Journal of Sociology*, 78:1360–1380, 1973.

[15] Matthew O. Jackson. *Social and Economic Networks*. Princeton University Press, 2008.

[16] I. Milchtaich and E. Winter. Stability and segregation in group formation. *Games and Economic Behavior*, 38(2):318–346, 2002.

[17] S. Muto. Resale-Proofness and coalition-proof Nash Equilibria. *Games and Economic Behavior*, 2:337–361, 1990.

[18] M. Nakayama and L. Quintas. Stable payoffs in resale-proof trades of information. *Games and Economic Behavior*, 3:339–349, 1991.

[19] Jeffrey H. Reiman. Privacy, intimacy, and personhood. *Philosophy and Public Affairs*, 6(1):26–44, Autumn 1976.