A simple tensorial proof for the completely symmetric property of the Bel-Robinson tensor

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Abstract

The Bel-Robinson tensor $T_{\alpha\beta\mu\nu}$ was proposed in 1958. The main application of this tensor is for describing gravitational energy. It is known that $T_{\alpha\beta\mu\nu}$ has many nice properties such as being completely symmetric. It is easy to prove this property using spinors as shown in Penrose’s book. The main purpose of the present paper is to verify that the Bel-Robinson tensor is indeed completely symmetric using a basic tensorial method. After we have this result we learned that Senovilla in 2000 has already used the similar idea to obtain the same result. However, keep using the tensorial method, we propose another easier proof that $T_{\alpha\beta\mu\nu}$ is indeed totally symmetric. Moreover, we also found that the well known equation in vacuum, $R_{\alpha\lambda\sigma\tau}R_{\beta\lambda\sigma\tau} \equiv \frac{1}{4} g_{\alpha\beta} R_{\rho\lambda\sigma\tau} R^{\rho\lambda\sigma\tau}$, which can be proven by the same tensorial method.

1 Introduction

The famous Bel-Robinson tensor $T_{\alpha\beta\mu\nu}$ [1] was proposed in 1958. Nowadays it is called a superenergy tensor [2]. It is believed that the gravitational field energy is related to $T_{\alpha\beta\mu\nu}$ [3] as it gives a positivity energy density quasilocally. The quasilocal idea (i.e., within a closed 2-surface) is physical, which means that the gravitational energy density is well defined at the quasilocal level theoretically [4, 5]. There were many papers in the literature using $T_{\alpha\beta\mu\nu}$ to describe the gravitational energy [6, 7, 8, 9]. The Bel-Robinson tensor has many nice properties such as being completely symmetric, totally trace free and divergence free. It also fulfills the dominant energy condition. There are different ways to define $T_{\alpha\beta\mu\nu}$, one of the common expressions is

$$T_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R_{\beta\lambda\nu\sigma} * R_{\alpha\lambda\mu\sigma} * R_{\beta\lambda\nu\sigma}$$

$$= R_{\alpha\lambda\mu\sigma} R_{\beta\lambda\nu\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta\lambda\mu\sigma} - \frac{1}{2} g_{\alpha\beta} R_{\mu\lambda\sigma\tau} R_{\nu\lambda\sigma\tau}$$

$$= R_{\alpha\lambda\mu\sigma} R_{\beta\lambda\nu\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta\lambda\mu\sigma} - \frac{1}{8} g_{\alpha\beta} g_{\mu\nu} R_{\rho\lambda\sigma\tau} R^{\rho\lambda\sigma\tau},$$

(1)

where $*R_{\alpha\lambda\mu\sigma} = \frac{1}{2} \epsilon^{\alpha\lambda\xi\epsilon} R_{\xi\mu\sigma}$ is the dual of the Riemann curvature tensor and we have made use the vacuum relation $R_{\alpha\lambda\sigma\tau} R_{\beta\lambda\sigma\tau} \equiv \frac{1}{4} g_{\alpha\beta} R_{\rho\lambda\sigma\tau} R^{\rho\lambda\sigma\tau}$. It is transparent to see some of the symmetry properties

$$T_{\alpha\beta\mu\nu} = T_{(\alpha\beta)(\mu\nu)} = T_{(\mu\nu)(\alpha\beta)},$$

(2)

but the totally symmetric property is not obvious. Thus Penrose and Rindler [10] say “The symmetry properties of $T_{\alpha\beta\mu\nu}$ are by no means apparent from the tensor
formula, but they follow directly from the spinor expression ...”, p241. We already know that this tensor should be completely symmetric because it is an analog of the symmetric trace-free divergence-free tensor, the energy-momentum tensor for the electromagnetic field,

$$T_{\alpha\beta} := \frac{1}{2}(F_{\alpha\lambda}F_{\beta}{}^{\lambda} + *F_{\alpha\lambda} *F_{\beta}{}^{\lambda}) = F_{\alpha\lambda} F_{\beta}{}^{\lambda} - \frac{1}{4} g_{\alpha\beta} F_{\rho\lambda} F_{\rho}{}^{\lambda},$$  \hspace{1cm} (3)

where $*F_{\alpha\beta}$ is the dual 2-form of electromagnetic field strength tensor $F_{\alpha\beta}$. As $T_{\alpha\beta}$ possesses the dominant energy condition, then $T_{\alpha\beta\mu\nu}$ should also.

It may need to be emphasized that the completely symmetric property of $T_{\alpha\beta\mu\nu}$ is important [2, 11]. Penrose [10] used spinors to verify that $T_{\alpha\beta\mu\nu}$ is really totally symmetric long ago. However, could it be possible to use the traditional tensorial method to prove this nice property? This may help someone who is not familiar with spinor techniques especially for the beginner to study the general relativity. The answer is yes. The main purpose of the present paper is to verify that the Bel-Robinson tensor is indeed completely symmetric using a basic tensorial method. After we have this result we learned that Senovilla [2] (see Proposition 6.3) in 2000 has already used the similar idea to obtain the same result. However, insist using the tensorial method and making use the formal dual of left and right, we propose another easier proof that $T_{\alpha\beta\mu\nu}$ is indeed completely symmetric. Moreover, using this symmetric property, we also found that the vacuum relation $R_{\alpha\lambda\sigma\tau} R_{\beta}{}^{\lambda\sigma\tau} \equiv \frac{1}{4} g_{\alpha\beta} R_{\rho\lambda\sigma\tau} R_{\rho}{}^{\lambda\sigma\tau}$ can be obtained using the same method.

## 2 Technical background

In order to prove the completely symmetric property of $T_{\alpha\beta\mu\nu}$, we need the following relation which is only valid in vacuum,

$$\epsilon_{\rho\lambda\xi\kappa} R_{\xi\kappa}{}^{\sigma\tau} + \epsilon_{\rho\sigma\xi\kappa} R_{\xi\kappa}{}^{\tau\lambda} + \epsilon_{\rho\tau\xi\kappa} R_{\xi\kappa}{}^{\lambda\sigma} \equiv 0,$$  \hspace{1cm} (4)

where $\epsilon_{\rho\lambda\xi\kappa}$ is the totally skew-symmetric Levi-Civita tensor. This equation looks like making a dual on the first Bianchi identity

$$* R_{\rho\lambda\sigma\tau} + * R_{\rho\sigma\tau\lambda} + * R_{\rho\tau\lambda\sigma} \equiv 0.$$  \hspace{1cm} (5)

However, it is not true in general but only true in vacuum. The detailed verification will be demonstrated in the next paragraph. After some simple algebra using (4), the anti-symmetric property of $\epsilon_{\alpha\beta\mu\nu}$ and $R_{\alpha\beta\mu\nu}$, we can obtain one more relation

$$\epsilon_{\rho\lambda\xi\kappa} R_{\xi\kappa}{}^{\sigma\tau} \equiv \epsilon_{\sigma\tau\xi\kappa} R_{\xi\kappa}{}^{\rho\lambda} \iff * R_{\rho\lambda\sigma\tau} \equiv R_{\sigma\tau\rho\lambda}.$$  \hspace{1cm} (6)

Once again, this looks like a property of the Riemann curvature tensor, but it is only valid in vacuum. In fact, this result is well known, the left dual and right dual and they are equal in vacuum. Explicitly

$$\epsilon_{\rho\lambda\xi\kappa} R_{\xi\kappa}{}^{\sigma\tau} \equiv R_{\rho\lambda}{}^{\xi\kappa} \epsilon_{\xi\kappa\sigma\tau} \iff * R_{\rho\lambda\sigma\tau} \equiv R_{\sigma\tau\rho\lambda}.$$  \hspace{1cm} (7)
Here we verify the relation (4) using differential forms. Define the dual of the curvature 2-form $R$ as follows
\[(∗R)_{µν} := \frac{1}{2} ǫ_{µνκξ} R^κ_ξ θ^κ ∧ θ^ξ. \tag{8}\]
Consider the wedge of a frame $θ^ν$
\[(∗R)_{µν} ∧ θ^ν = \frac{1}{4} ǫ_{µνκξ} R^κ_ξ θ^κ ∧ θ^ν ∧ θ^ν = -G^ρ_{µ} η_ρ, \tag{9}\]
where $G^ρ_{µ}$ is the Einstein tensor and $η_ρ$ is the 3-form (i.e., $η_ρ = ∗θ_ρ$). Taking the triple anti-symmetrization
\[G^ρ_{µ} η_ρ = -\frac{1}{12} \left( ǫ_{µνκξ} R^κ_ξ θ^κ ∧ θ^ν + ǫ_{µκξν} R^κ_ξ θ^ν ∧ θ^λ + ǫ_{µσκξ} R^κ_ξ θ^ν ∧ θ^λ ∧ θ^σ \right) \]
\[= -\frac{1}{12} \left( ǫ_{µνκξ} R^κ_ξ θ^κ + ǫ_{µκξν} R^κ_ξ θ^ν + ǫ_{µσκξ} R^κ_ξ θ^ν \right) θ^λ ∧ θ^σ ∧ θ^ν. \tag{10}\]
As the Einstein tensor vanishes in vacuum, the result follows.

3 Tensorial proof for the completely symmetric of the Bel-Robinson tensor

Here we present a detailed proof of the complete symmetry of the Bel-Robinson tensor using the basic tensorial method. Consider
\[T_{αβµν} := R_{αλµσ} R^λ_β ν σ + R_{αλµσ} R^λ_β ν σ \]
\[= R_{αλµσ} R^λ_β ν σ + ½ ǫ_{ακξκ} R^κ_σ µσ ½ g_τ β γ_ν ǫ^τ_ρ π R^γ_ρ π σ \]
\[= R_{αλµσ} R^λ_β ν σ + ½ ǫ_{ακξκ} R^κ_σ µσ ½ g_τ β γ_ν \left( -ε^γ_ρ π R^σ_ρ π σ λ - ε^σ_ρ π R^γ_ρ π λ \right) \]
\[= R_{αλµσ} R^λ_β ν σ + ½ ǫ_{ακξκ} R^κ_σ µσ ½ g_τ β γ_ν \left( -ε^σ_ρ π R^γ_ρ π τ γ + ε^σ_ρ π R^γ_ρ π λ \right) \]
\[= R_{αλµσ} R^λ_β ν σ + ¼  δ^σ_κ γ_ν R^κ_σ µσ R^γ_ρ π τ γ + R_{αλµσ} R^λ_β σ \]
\[= \left( \frac{1}{2} R_{αμλσ} R^λ_β ν σ + R_{αλµσ} R^λ_β ν σ \right) - \frac{1}{2} R_{αμλσ} R^λ_β σ + R_{αλµσ} R^λ_β σ \]
\[= T_{ανµβ}. \tag{11}\]

From (2) and (11), the completely symmetric property easily follows. After we found this method we learned that which is similar to the ones used in Senovilla paper in 2000 [2] (see Proposition 6.3).

However, can now we propose another easier proof the completely symmetric of the Bel-Robinson tensor? It is possible. The idea is simply making use of the property indicated in (5), the formal dual of the first Bianchi identity (which is only valid in
vacuum. The basic idea is that we can treat the formal dual of the Riemann curvature tensor as the usual tensor manipulation. For example, we know \( R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta} \) in general and this can immediately to treat the relation in vacuum \( *R_{\alpha\beta\mu\nu} = *R_{\mu\nu\alpha\beta} \), this is another representation of the standard left dual and right dual denoted in (7).

Although the idea and the associated result is simple, it turns out that it is very useful and practical. Here we give the detail derivation of the completely symmetric of the Bel-Robinson tensor as follows

\[
T_{\alpha\mu\beta\nu} - T_{\alpha\nu\beta\mu} = R_{\alpha\lambda\beta\sigma} R_{\mu\nu}^{\lambda\sigma} - R_{\alpha\lambda\beta\sigma} R_{\mu\nu}^{\lambda\sigma} + *R_{\alpha\lambda\beta\sigma} * R_{\mu\nu}^{\lambda\sigma} - *R_{\alpha\lambda\beta\sigma} * R_{\mu\nu}^{\lambda\sigma}
\]

\[
= \frac{1}{2} R_{\alpha\beta\lambda\sigma} R_{\mu\nu}^{\lambda\sigma} + \frac{1}{2} * R_{\alpha\beta\lambda\sigma} * R_{\mu\nu}^{\lambda\sigma}
\]

\[
= 0,
\]

(12)

using (6). Consider the first two terms, employing the Bianchi identity \( B_{\alpha[\beta\mu\nu]} = 0 \)

\[
R_{\alpha\lambda\beta\sigma} (R_{\mu\nu}^{\lambda\sigma} - R_{\nu\mu}^{\lambda\sigma}) = R_{\alpha\beta\lambda\sigma} R_{\mu\nu}^{\lambda\sigma} = \frac{1}{2} R_{\alpha\beta\lambda\sigma} R_{\mu\nu}^{\lambda\sigma}.
\]

(13)

And in the same formal way \( *R_{\alpha\lambda\beta\sigma} (* R_{\mu\nu}^{\lambda\sigma} - * R_{\nu\mu}^{\lambda\sigma} \) becomes \( *R_{\alpha\beta\lambda\sigma} * R_{\mu\nu}^{\lambda\sigma} \) which is \( \frac{1}{2} * R_{\alpha\beta\lambda\sigma} * R_{\mu\nu}^{\lambda\sigma} \)

Moreover, using Lanczos identity, there is a well known equation in empty space [12]

\[
R_{\alpha\lambda\beta\sigma} R_{\beta\lambda\sigma\tau} \equiv \frac{1}{4} g_{\alpha\beta} R_{\rho\lambda\sigma\tau} R_{\beta\rho\lambda\sigma\tau}.
\]

(14)

One can verify this identity in vacuum by employing an orthonormal frame. We first define

\[
E_{ab} := R_{0a0b}, \quad H_{ab} := * R_{0a0b},
\]

(15)

where \( E_{ab} \) and \( H_{ab} \) are the electric and magnetic parts of the Weyl tensor in vacuum. Indeed we have verified this identity recently [13]. It is a simple straightforward but tedious calculation. However, instead of making use the Lanczos identity, we can reproduce this result using a simple tensorial method similar to the above. From the completely symmetric property, simply consider the two cases

\[
T_{\alpha\beta\mu\nu} := R_{\alpha\lambda\mu\sigma} R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta\mu}^{\lambda\sigma} - \frac{1}{2} g_{\alpha\beta} R_{\mu\lambda\sigma\tau} R_{\nu\lambda\sigma\tau},
\]

(16)

\[
T_{\mu\nu\alpha\beta} := R_{\alpha\lambda\mu\sigma} R_{\beta\nu}^{\lambda\sigma} + R_{\alpha\lambda\nu\sigma} R_{\beta\mu}^{\lambda\sigma} - \frac{1}{2} g_{\mu\nu} R_{\alpha\lambda\sigma\tau} R_{\beta\lambda\sigma\tau}.
\]

(17)

We know that these two equations are equivalent in vacuum, consider the last terms of (16) and (17)

\[
g_{\alpha\beta} R_{\mu\lambda\sigma\tau} R_{\nu}^{\lambda\sigma\tau} \equiv g_{\mu\nu} R_{\alpha\lambda\sigma\tau} R_{\beta}^{\lambda\sigma\tau}.
\]

(18)

Either taking the trace on \( g_{\alpha\beta} \) or \( g_{\mu\nu} \), the result appears as shown in (14).
4 Conclusion

The Bel-Robinson tensor possesses many nice properties such as completely symmetric. Penrose used spinors to prove that indeed it is true. We have a proof to show that $T_{\alpha\beta\mu\nu}$ has this symmetric property. Soon after we have this result we learned that Senovilla in 2000 has the result using a similar method as we did. However, keep using the tensorial method and making use the formal dual of left and right, we propose another easier proof that $T_{\alpha\beta\mu\nu}$ is indeed completely symmetric.

Here we provide a basic and straightforward tensorial method to verify that the Bel-Robinson tensor is really completely symmetric. One may ask why we prefer a tensorial method to prove something that it was well known. In particular, why not keep using spinors. The reason is that although spinors are very power and elegant, it may be worthwhile using a basic and simple tensorial way to understand this symmetric property especially for someone who is not familiar with spinors. In fact, people usually learning general relativity starts with the tensorial and then study spinors afterwards. This is the reason we presented this simple verification for the symmetric property of the Bel-Robinson tensor.

Moreover, during the proof of the completely symmetric property of the Bel-Robinson tensor, we have discovered something extra, which is the well known relation in vacuum $R_{\alpha\lambda\sigma\tau}R_{\beta\lambda\sigma\tau} = \frac{1}{4}g_{\alpha\beta}R_{\rho\lambda\sigma\tau}R^{\rho\lambda\sigma\tau}$. It is amazing that we can reproduce this result only using such a simple tensorial method. Of course, one can simply use orthonormal frames to verify this identity, but it is a testing method fundamentally. In other words, it is not a deduction. However, making use of the known symmetric property of the Bel-Robinson tensor, we have recovered the one-quarter identity. This is a simple and nice proof.

After using the tensorial method with some successful results, one may wonder whether the basic tensorial method is very useful so that we do not need any other method. In particular, spinors. The answer seems negative. This is because the tensorial method has its own limitations. There are still some things that using the tensorial method are not easy to proof. Then we have to use another method. This may be the reason why people invented spinors and used then for a long time.

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