On the Kolkata index as a measure of income inequality

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Abstract

We study the mathematical and economic structure of the Kolkata (k) index of income inequality. We show that the k-index always exists and is a unique fixed point of the complementary Lorenz function, where the Lorenz function itself gives the fraction of cumulative income possessed by the cumulative fraction of population (when arranged from poorer to richer). We show that the k-index generalizes Pareto’s 80/20 rule. Although the k and Pietra indices both split the society into two groups, we show that k-index is a more intensive measure for the poor-rich split. We compare the normalized k-index with the Gini coefficient and the Pietra index and discuss when they coincide. We establish that for any income distribution the value of Gini coefficient is no less than that of the Pietra index and the value of the Pietra index is no less than that of the normalized k-index. While the Gini coefficient and the Pietra index are affected by transfers exclusively among the rich or among the poor, the k-index is only affected by transfers across the two groups.

Keywords: Lorenz function, Gini coefficient, Pietra index, k-index

1. Introduction

In the Lorenz curve (see [7] for more details) one plots the proportion of the total income of the population that is earned by the bottom $p$ proportion of the population. See Figure [1] where we plot accumulated proportions of the population from poorest to richest along the horizontal axis and total income
held by these proportions of the population along the vertical axis. The 45° line represents a situation of perfect equality.

Often we are interested in a summary statistic of the Lorenz function. This is because the Lorenz curves can intersect each other meaning that we cannot order the curves. One way of dealing with this is to rely on a summary statistic (see [1] for more details). The most popular summary statistic is the Gini coefficient (see [6]) which is the ratio of the area between the 45° line and the Lorenz curve to the total area under the 45° line. The Pietra index (see [8]) is the maximum value of the gap between the 45° line and the Lorenz curve (see also [3]).

In this paper, we are specifically interested in a particular summary index called the Kolkata index or the k-index (see [5] for more details) which is that proportion $k_F$ such that $k_F + L_F(k_F) = 1$ where $L_F(p)$ is the Lorenz function (also see [2]). We can understand $k_F$ better as follows. Suppose we split society into two groups: the “poor” who constitute a fraction $p$ of the population and the remaining “rich.” Note that $L_F(p) \leq p$; hence, $p$ is an upper bound on the income share of the poor. The actual share of the rich, on the other hand, is $1 - L_F(p)$. The k-index splits society into two groups in a way that the egalitarian income share of the poor equals the actual income share of the rich.

The k-index takes values in the range $[1/2, 1]$ which makes it different from the Gini coefficient and the Pietra index, both of which take values in the range $[0, 1]$. However, a simple normalization of the k-index, namely $K_F = 2k_F - 1$, achieves this. Like the other two indices, the extreme values of the normalized k-index correspond to complete equality ($K_F = 0$) and complete inequality ($K_F = 1$) respectively.

We show that the k-index is a fixed-point of the function $\hat{L}_F(p) = 1 - L_F(p)$ which we call the complimentary Lorenz function. In particular, we show that the fixed-point exists and is unique for all Lorenz functions. We also show that the k-index generalizes Pareto’s 80/20 rule: “20% of the people own 80% of the income.” The k-index has the property that $[100(1 - k_F)]\%$ of the people own $[100k_F]\%$ of the income. Or, equivalently, $[100k_F]\%$ of the people only have $[100(1 - k_F)]\%$ of the income. We show that both the $k$ and the Pietra indices split society into two groups and we discuss the differences between the two indices in this regard. We compare the normalized k-index with the Gini coefficient and the Pietra index and obtain certain important conclusions in terms of coincidence possibilities between all or any two of these three measures. We show that for any given income distribution the value of Gini coefficient is no less than that of the Pietra index and the value of the Pietra index is no less than that of the normalized k-index. We also demonstrate that while the Gini coefficient and the Pietra index are affected by transfers exclusively among the rich or among the poor, the k-index ranks is only affected by transfers across the two groups.

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1 We will use the terms Lorenz function and Lorenz curve interchangeably in this paper.

2 [9] uses the k-index to define a generalized Gini coefficient.
Figure 1: The red curve is the Lorenz function and the blue curve is the complementary Lorenz function. The normalized $k$-index is $K_F = k_F - L_F(k_F) = 2k_F - 1$ and $p^*_F = L_F^{-1}(\frac{1}{2})$ is the population proportion associated with the point of intersection of the Lorenz and the reverse order Lorenz functions (colour online).

2. The framework

Let $F$ be the distribution function of a non-negative random variable $X$ which represents the income distribution in a society. The left-inverse of $F$ is defined as $F^{-1}(t) = \inf \{ x \mid F(x) \leq t \}$. Assume that the mean income $\mu = \int_0^\infty xdF(x)$ is finite. In this case, we obtain an alternative representation of the mean: $\mu = \int_0^1 F^{-1}(t)dt$.

The Lorenz function, defined as $L_F(p) = (1/\mu) \int_0^p F^{-1}(t)dt$, gives the proportion of total income earned by the bottom 100$p\%$ of the population. The following properties of the Lorenz function are well-known: (i) $L_F(0) = 0$, $L_F(1) = 1$ and $L_F(p) \leq p$ for $p \in (0, 1)$, and (ii) the Lorenz function is continuous (see [2]), non-decreasing and convex.

The complementary Lorenz function is defined as $\hat{L}_F(p) = 1 - L_F(p)$ . It measures the proportion of the total income that is earned by the top 100$(1-p)\%$ of the population:

$$\hat{L}_F(p) := 1 - L_F(p) = 1 - \frac{\int_0^p F^{-1}(t)dt}{\mu} = \frac{\int_0^1 F^{-1}(t)dt}{\mu}. \quad (1)$$

It follows straightforwardly that $\hat{L}_F(0) = 1$, $\hat{L}_F(1) = 0$, and $0 \leq \hat{L}_F(p) \leq 1$ for $p \in (0, 1)$. Furthermore, $\hat{L}_F$ is continuous, non-increasing and concave on $(0, 1)$.

The Gini coefficient is given by $G_F = 2 \int_0^1 (p - L_F(p))dp$. The Pietra index
maximizes \( p - L_F(p) \). It is easy to show that this function is maximized at \( p = F(\mu) \); hence, we can define the Pietra index alternatively as \( \hat{p}_F = F(\mu) - L_F(F(\mu)) \).

3. Structure of the \( k \)-index

3.1. \( k \)-index as a fixed point of the complementary Lorenz function

As mentioned, the \( k \)-index is defined by the solution to the equation \( k_F + L_F(k_F) = 1 \). It has been proposed as a measure of income inequality (see [2], [5] for more details). We can rewrite \( k_F + L_F(k_F) = 1 \) as \( k_F = 1 - L_F(k_F) = L_F(k_F) \). Hence, the \( k \)-index is a fixed point of the complementary Lorenz function. Since the complementary Lorenz function maps \([0, 1]\) to \([0, 1]\) and is continuous, it has a fixed point by Brouwer’s fixed point theorem. Furthermore, since \( L_F(p) \) is non-increasing, the fixed point has to be unique.

We can say a little more about the location of the fixed point. Let \( p^*_F = L_F^{-1}(1/2) \). Observe that \( p^*_F \geq 1/2 \) with the equality holding only if we have an egalitarian income distribution. We claim that the unique fixed point of \( L_F \) lies in the interval \([1/2, p^*_F]\). Let \( Z_F(p) = L_F(p) - p, p \in [0, 1] \). Note that \( Z_F \) is continuous. Since \( L_F(p) \leq p \), we have \( Z_F(1/2) \geq 0 \). Also, \( Z_F(p^*_F) = L_F(p^*_F) - p^*_F \leq L_F(p^*_F) - L_F(p^*_F) = 0 \). It follows from the Intermediate Value Theorem that there exists \( k_F \in [1/2, p^*_F] \) such that \( Z_F(k_F) = 0 \). Therefore, we have established the following:

\[(FP) \quad \text{There exists a } k_F \in [1/2, p^*_F] \text{ such that } L_F(k_F) = k_F \Leftrightarrow L_F(k_F) + k_F = 1 \text{ and this } k_F \text{ is unique.} \]

Observe that if \( L_F(p) = p \) (egalitarian income distribution), then \( k_F = 1/2 \). For any other income distribution, \( 1/2 < k_F < 1 \). It is interesting to note that while the Lorenz curve typically has only two trivial fixed points (the two end points), the complementary Lorenz function has a unique non-trivial fixed point \( k_F \). This fixed point \( k_F \) lies between 50% population proportion and the population proportion \( p^*_F = L_F^{-1}(1/2) \) that we associate with 50% income given the income distribution \( F \).

Case 1. Let \( F \) be the uniform distribution on \([a, b]\) where \( 0 \leq a < b < \infty \). Then, \( L_F(p) = p [1 - \{((1 - p)(b - a))/(b + a)\}] \) and

\[
k_F = \frac{-(3a + b) + \sqrt{5a^2 + 6ab + 5b^2}}{2(b - a)}.
\]

It is interesting to note that if \( a = 0 \), then \( L_F(p) = p^2 \) and \( k_F \) is the reciprocal of the Golden ratio, that is, \( k_F = (\sqrt{5} - 1)/2 = 1/\phi \) where \( \phi = (\sqrt{5} + 1)/2 \) is the Golden ratio.

Case 2. Let \( F \) be the exponential distribution function given by \( F(x) = 1 - e^{-\lambda x} \) where \( x \geq 0 \) and \( \lambda > 0 \). Then, \( L_F(p) = p - (1 - p) \ln(1/(1 - p)), L_F(p) = (1 - p) [1 + \ln \{1/(1 - p)\}] \) and \( k_F \approx 0.6822 \).

Case 3. The Pareto distribution is given by \( F(x) = 1 - (x_m/x)^\alpha \) on the support \([x_m, \infty]\) where \( \alpha > 1 \) and the minimum income is \( x_m > 0 \). Then, \( L_F(p) = 1 - (1 - p)^{1-\alpha} \) and \( L_F(p) = (1 - p)^{1-\alpha} = k_F \). The \( k \)-index is a solution to \((1 - k_F)^{1-\alpha} = k_F \). If \( \alpha = \ln 5/\ln 4 \approx 1.16 \), then \( k_F = 0.8 \) and we get what is known as the Pareto principle or the 80/20 rule.
3.2. \textit{k-index as a generalization of the Pareto principle}

The Pareto principle is based on Pareto’s observation (in the year 1906) that approximately 80\% of the land in Italy was owned by 20\% of the population. The evidence, though, suggests that the income distribution of many countries fails to satisfy the 80/20 rule (see \cite{5}). The \textit{k-index} can be thought of as a generalization of the Pareto principle. Note that \( L_F(k_F) = 1 - k_F \); hence, the top 100\%(1 - k_F)% of the population has 100\%(1 - (1 - k_F)) = 100k_F\% of the income. Hence, the “Pareto ratio” for the \textit{k-index} is \( k_F/(1 - k_F) \). Observe, however, that this ratio is obtained endogenously from the income distribution and in general, there is no reason to expect that this ratio will coincide with the Pareto principle.

Given any income distribution \( F \), for any \( p \in [0, p_F^*] \) with \( p_F^* = L_F^{-1}(1/2) \), let \( r_F(p) = L_F^{-1}(1-p) \). Consider the interval \( C(p) = [\min\{p, r_F(p)\}, \max\{p, r_F(p)\}] \). To understand what \( C(p) \) signifies, let \( p = 0.2 \). That is, we consider the poorest 20\% (or 100p\%) as undeniably poor. To identify the dividing line between the poor and the rich, one strategy is to eliminate those who are undeniably rich. We do this by considering the fraction of the rich whose income share is exactly 100\%(1-p)\%. That is, we identify \( p' \) such that \( L_F(p') = 0.8 = 1 - p \). Eliminating the poor and the undeniably rich, we find that the dividing line between poor and rich must lie in the interval \([\min\{p, r_F(p)\}, \max\{p, r_F(p)\}]\). We now ask the question: What proportions are in the set \([\min\{p, r_F(p)\}, \max\{p, r_F(p)\}]\) for all \( p \in [0, p_F^*] \)? The answer is that only \( k_F \) meets this criterion. Specifically, for any \( p \in [0, p_F^*] \), define the potential income disparity division set as \( C(p) = \{t | \min\{p, r_F(p)\} \leq t \leq \max\{p, r_F(p)\}\} \). We show in the appendix that

\[
k_F = \cap_{p \in [0, p_F^*]} C(p). \tag{2}
\]

3.3. \textit{Interpreting the k-index in terms of rich-poor disparity}

The \textit{Gini coefficient}, as is well-known, measures inequality by the area between the Lorenz curve and the 45-degree line. For any \( p \in [0, 1] \), we can decompose this coefficient into three parts: two representing the \textit{within-group inequality} and one representing the \textit{across-group inequality}. In Figure 2 below, the unshaded area bounded by the Lorenz curve and the line from (0, 0) to \((p, L_F(p))\) is the within-group inequality of the poor. It represents the extent to which inequality can be reduced by redistributing incomes among the poor. Similarly, the area bounded by the Lorenz curve and the line segment from \((p, L_F(p))\) to \((1, 1)\) represents the within-group inequality of the rich. The shaded area represents the across-group inequality.

An easy computation shows that the extent of across-group inequality between the bottom \( p \times 100\% \) and top \((1 - p) \times 100\% \) is the (across-group) disparity function \( D_F(p) = (1/2)[p - L_F(p)] \). One can ask for what value of \( p \) is the across-group inequality maximized? The answer is that this is maximized at the proportion associated with the \textit{Pietra index}. It is well-known that the Pietra index (see \cite{8}) is given by

\[
\mathcal{P}_F := \max_{p \in [0, 1]} 2D_F(p) = \max_{p \in [0, 1]} [p - L_F(p)] = F(\mu) - L_F(F(\mu)).
\]

Hence, \( F(\mu) \) is the proportion where the disparity is maximized. Therefore, one way of understanding the Pietra index is that it splits society into two
groups in a way such that inter-group inequality is maximized. This provides a different perspective on the Pietra index.

What about the $k$-index? Let us divide society into two groups, the “poorest” who constitute a fraction $p$ of the population and the “rich” who constitute a fraction $1 - p$ of the population. Given the Lorenz curve $L_F(p)$, we look at the distance of the “boundary person” from the poorest person on the one hand and the distance of this person from the richest person on the other hand. These distances are given by $\sqrt{p^2 + L_F(p)^2}$ and $\sqrt{(1 - p)^2 + (1 - L_F(p))^2}$ respectively. Then, the $k$-index divides society into two groups in a manner such that the Euclidean distance of the boundary person from the poorest person is equal to the distance from the richest person.

The value of the disparity function at the $k$-index is given by $D_F(k_F) = k_F - 1/2$. The interpretation of this is quite transparent since it measures the gap between the proportion $k_F$ of the poor from the 50–50 population split. As long as we do not have a completely egalitarian society, $k_F > 1/2$ and hence it is one way of highlighting the rich-poor disparity with $k_F$ defining the income proportion of the top $(1 - k_F)$ proportion of the rich population. The other measures do not have as nice an interpretation. For instance, the value of the disparity function at the proportion corresponding to the Pietra index is $D_F(F(\mu)) = |F(\mu) - L_F(F(\mu))|/2$. This number has no obvious interpretation.

### 3.4. The $k$-index as a solution to optimization problems

The $k$-index is the unique solution to the following surplus maximization problem:

$$k_F = \arg \max_{P \in [0,1]} \int_0^P (L_F(t) - t)dt.$$  \hspace{1cm} (3)
Therefore, $k_F$ is that fraction of the lower income population for which the area between the complementary Lorenz function and the income distribution line associated with the egalitarian distribution is maximized. Condition (3) follows since $L_F(p) \geq p$ for all $p \in [0, k_F]$ and $L_F(p) < p$ for all $p \in (k_F, 1]$. For the same reason the $k$-index is also the unique solution to the following surplus minimization problem (which is the dual of the problem in (3)):

$$k_F = \arg\min_{F \in [0,1]} \int_0^1 ((1 - t) - L_F(t))\,dt.$$  \hspace{1cm} (4)

Therefore, $(1 - k_F)$ is that fraction of the higher income population for which the area between the income distribution line associated with the egalitarian distribution and the Lorenz function is minimized.

4. Comparing the normalized $k$-index with the Pietra index and the Gini coefficient

We start our comparison by specifying a family of Lorenz functions for each of which the Gini coefficient coincides with the Pietra index and yet the normalized $k$-index is different.

Case 4. Consider the $p$-oligarchy Lorenz function discussed in [3] that has the following functional form: For any fraction $a \in (0, 1)$,

$$L_F(p) = \begin{cases} 0 & \text{if } p \in [0, a], \\ \frac{(p-a)}{(1-a)} & \text{if } p \in (a, 1]. \end{cases}$$  \hspace{1cm} (5)

See Figure 3 where the Lorenz function given by (5) is represented by the piece-wise linear red line OBA (colour online). It is easy to verify that the proportion

\hspace{1cm}

associated with the Pietra index is $F(\mu) = a$ and $P_F = F(\mu) - L_F(F(\mu)) = ...
\[ a - 0 = a. \] One can also verify that the Gini coefficient coincides with the Pietra index, that is, \( G_F = P_F = a. \) However, the \( k \)-index fraction \( k_F \) is a solution to the equation \( (k_F - a)/(1 - a) + k_F = 1 \) and it gives \( k_F = 1/(2 - a) \). Moreover, the normalized \( k \)-index yields \( K_F = 2k_F - 1 = a/(2 - a) \). Therefore, for any given \( a \in (0, 1) \) and any associated Lorenz function given by (5), we have

\[ G_F = P_F = a > a^2 - a = K_F. \] (6)

Therefore, Case 4 suggests that the \( k \)-index in itself has properties that are different from the other two measures and hence deserves a special theoretical analysis.

### 4.1. Coincidence of the \( k \)-index and the Pietra index

The Lorenz function \( L_F(p) \) is symmetric if for all \( p \in [0, 1] \),

\[ L_F(\hat{L}_F(p)) = 1 - p \text{ or equivalently } L_F(p) + r_F(p) = 1, \] (7)

where \( r_F(p) = L_F^{-1}(1 - p) \). The idea of symmetry is explained in Figure 4.

![Figure 4: Symmetry condition (7) requires that for any proportion \( p \), the distance \( L_F(p) \) between the points \( A = (p, L_F(p)) \) and \( B = (p, 0) \) must be the same as the distance \( 1 - L_F^{-1}(1 - p) \) between the points \( C = (L_F^{-1}(1 - p), (1 - p)) \) and \( D(1, 1 - p) \).](image)

**Case 5.** Suppose that the Lorenz function is given by \( L_F(p) = 1 - \sqrt{1 - p^2} \) (see Figure 5). Observe that \( L_F(\hat{L}_F(p)) = L_F(\sqrt{1 - p^2}) = 1 - p \) and hence the Lorenz function \( L_F(p) = 1 - \sqrt{1 - p^2} \) is symmetric.

The \( k \)-index associated with the Lorenz function \( L_F(p) = 1 - \sqrt{1 - p^2} \) is \( k_F = 1/\sqrt{2} \). Moreover, since \( L_F'(p) = p/\sqrt{1 - p^2} \), at the proportion \( F(\mu) \) associated with the Pietra index \( P_F \), we have \( L_F'(F(\mu)) = F'(\mu)/\sqrt{1 - (F(\mu))^2} = 1 \) implying \( F(\mu) = k_F \). Therefore, for the Lorenz function given by \( L_F(p) = 1 - \sqrt{1 - p^2} \), the normalized \( k \)-index \( K_F = 2k_F - 1 = \sqrt{2} - 1 \) coincides with the Pietra index \( P_F = F(\mu) - L_F(F(\mu)) = \sqrt{2} - 1 \). Moreover, one can verify that the Gini coefficient is different and is given by \( G_F = \pi/2 - 1 > P_F = K_F. \)
Case 5 provides an example of a symmetric and differentiable Lorenz function for which \( k_F = F(\mu) \) and hence \( K_F = P_F \). This result is true in general and in the appendix we prove the following general result.

\[(KP)\] If the Lorenz function is symmetric and differentiable, then the proportion \( F(\mu) \) associated with the Pietra index coincides with the proportion \( k_F \) of the \( k \)-index. Hence, we also have \( K_F = P_F \).

Observe that \((KP)\) provides a sufficient condition for the coincidence. It is not necessary as the following example shows that we can find a Lorenz function which is not symmetric and yet we have the coincidence of the normalized \( k \) and the Pietra indices.

\[
L_F(p) = \begin{cases} 
1 - \sqrt{1 - p^2} & \text{if } p \in [0, 1/\sqrt{2}], \\
1 - \frac{\sqrt{3}}{2 - \sqrt{2}}(1 - p) & \text{otherwise.}
\end{cases}
\] (8)

Note that in (8) we have simply replaced the curve DB in Figure 5 by a straight line between the two points. Even though this Lorenz curve is not symmetric, we can “convert” it into a symmetric one by replacing the segment OD by a corresponding straight line. This change leaves \( K_F \) and \( P_F \) unchanged. It is clear that this can be done in general: given any non-symmetric Lorenz curve where \( K_F \) and \( P_F \) coincide, we can derive a symmetric Lorenz curve such that the two indices coincide by replacing the Lorenz curves for the poor and the rich by straight lines. This suggests that the symmetry condition is almost necessary.

4.2. Coincidence of the normalized \( k \)-index and the Gini coefficient

As an instance of coincidence between \( G_F \) and \( K_F \) we consider the following family of Lorenz functions.

\[ \begin{align*}
C &= (0,1) \\
A &= (0,0) \\
B &= (0,1) \\
D &= (0,0)
\end{align*} \]

Figure 5: \( K_F = P_F = \sqrt{2} - 1 < G_F = \pi/2 - 1. \)
Case 6. For any fraction $K \in [1/2, 1)$, consider the associated Lorenz function $L_F(p)$ given by

$$L_F(p) = \begin{cases} \frac{1-K}{K} p & \text{if } p \in [0, K], \\ \frac{K}{1-K} (p-K) & \text{if } p \in (K, 1]. \end{cases} \quad (9)$$

In Figure 6, the Lorenz function given by (9) is depicted by the piecewise linear red lines OQ and QB (colour online).

It is immediate that $L_F(K) + K = (1-K) + K = 1$ implying that $k_F = K$. Moreover, $\int_{t=0}^{1} L_F(t) dt = 1 - K$ and hence $G_F = 1 - 2(1 - K) = 2K - 1 = 2k_F - 1 = k_F$. Also note that the difference $p - L_F(p)$ is maximized at $p = K$ and hence $F(\mu) = k_F = K$. Therefore, we have

$$G_F = K_F = P_F = 2K - 1. \quad (10)$$

Therefore, Case 6 shows that for the family of Lorenz functions given by (9), $G_F$ coincidences with $K_F$ and as a result $P_F$ also coincides. We claim that this is no exception. More generally, in the appendix we show the following:

\[ \text{(GK-P)} \quad \text{For any income distribution } F, \ G_F \geq P_F \geq K_F. \ \text{Moreover, if } G_F = P_F = K_F, \text{ then the Lorenz function is given by (9).} \]

5. Ranking the Lorenz functions using the normalized $k$-index, the Pietra index and the Gini coefficient

One important aspect of summary statistics is to rank different Lorenz curves. Here we demonstrate that the three indices can provide very different rankings.

\[ \text{Observe that if } K = 1/2, \text{ then from (10) we have the Lorenz function for egalitarian income, that is, } L_F(p) = p \text{ for all } p \in [0, 1] \text{ and in that case } G_F = K_F = P_F = 0. \]
Case 7. Consider the following Lorenz functions:

\[ L_{F_1}(p) = \begin{cases} \frac{3p}{2} & \text{if } p \in [0, 1/3], \\ \frac{9p^2 - 1}{8} & \text{if } p \in (1/3, 1]. \end{cases}\]  (11)

\[ L_{F_2}(p) = \begin{cases} \frac{3p}{2} & \text{if } p \in [0, 7/8], \\ \frac{9p^2 - 7}{9} & \text{if } p \in (7/8, 1]. \end{cases}\]  (12)

Observe that the population fraction associated with the Pietra index is \( F_1(\mu_1) = 1/3 \) for the Lorenz function \( L_{F_1}(p) \) and is \( F_2(\mu_2) = 7/8 \) for the Lorenz function \( L_{F_2}(p) \). Since \( \mathcal{P}_{F_i} = F_i(\mu_i) - L_{F_i}(F_i(\mu_i)) \) for \( i = 1, 2 \), we get

\[ \mathcal{P}_{F_1} = \frac{1}{12} < \mathcal{P}_{F_2} = \frac{7}{72}. \]  (13)

The \( k \)-index fraction \( k_{F_1} \) associated with the Lorenz function \( L_{F_1}(p) \) is a solution to the equation \( (9k_{F_1} - 1)/8 + k_{F_1} = 1 \) and it gives \( k_{F_1} = 9/17 \). The \( k \)-index fraction \( k_{F_2} \) associated with the Lorenz function \( L_{F_2}(p) \) is a solution to the equation \( 8k_{F_2}/9 + k_{F_2} = 1 \) and it also gives \( k_{F_2} = 9/17 \). Therefore, \( k_{F_1} = k_{F_2} = 9/17 \) and hence the normalized \( k \)-indices are also identical, and, in particular, we have

\[ K_{F_1} = K_{F_2} = \frac{1}{17} < \mathcal{P}_{F_1} = \frac{1}{12} < \mathcal{P}_{F_2} = \frac{7}{72}. \]  (14)

Case 8. Now consider the following two Lorenz functions:

\[ L_{F_3}(p) = p^2, \quad \forall \ p \in [0, 1]. \]  (15)

\[ L_{F_4}(p) = \begin{cases} p^2 & \text{if } p \in [0, 3/4], \\ 1 - \frac{7(1-p)}{4} & \text{if } p \in (3/4, 1]. \end{cases}\]  (16)

The \( k \)-index associated with both Lorenz functions \( L_{F_3}(p) \) and \( L_{F_4}(p) \) is a solution to the equation \( K^2 + K = 1 \) and it gives \( k_{F_3} = k_{F_4} = K = 1/\phi \) where \( \phi = (\sqrt{5} + 1)/2 \) is the Golden ratio. Therefore, \( K_{F_3} = K_{F_4} = 2/\phi - 1 \simeq 0.23607 \). However, Gini coefficient associated with the two Lorenz functions \( L_{F_3}(p) \) and \( L_{F_4}(p) \) are different. In particular, one can show that \( G_{F_3} = 2 \int_0^{3/4} [t - t^2]dt = 1/3 \) and \( G_{F_4} = 2 \int_0^{3/4} [t - t^2]dt + \int_{3/4}^1 [(3/4) - t]dt = 21/64 \).

\[ K_{F_3} = K_{F_4} = 2/\phi - 1 < G_{F_4} = 21/64 < G_{F_3} = 1/3. \]  (17)

Case 8 demonstrates an important difference between \( K \) and \( G \). The Gini is affected by transfers within a group. In particular, the poor are unaffected but the rich have become more egalitarian while moving from \( L_{F_3} \) to \( L_{F_4} \). The normalized \( k \)-index on the other hand is unaffected with such intra-group transfers. This suggests that if we are interested in reducing inequality between groups, then the normalized \( k \)-index is a better indicator.

6. Summary

We summarize the main results of this paper:
1. The \( k \)-index always exists and is a unique fixed point of the complementary Lorenz function. While the Lorenz function has two trivial fixed points, the complementary Lorenz function has one non-trivial fixed point \( k_F \) and it gives the value of the Kolkata index or the \( k \)-index (see Section 3.1).

2. The \( k \)-index generalizes Pareto’s 80/20 rule. The \( k \)-index has the property that \( [100(1-k_F)] \% \) of the people own \( [100k_F] \% \) of the income. We also provide an argument as to why \( k_F \) is a correct and endogenously obtained dividing population proportion between the rich and the poor in a society with income distribution \( F \) (see condition (2) on Section 3.2).

3. Although the \( k \) and Pietra indices both split the society into two groups, the \( k \)-index is more transparent measure for the poor-rich split.

4. The \( k \)-index also has interpretations as a solution to optimization problems. The \( k \)-index maximizes the area between the complementary Lorenz function and the income distribution line associated with the egalitarian distribution. Hence, \( (1-k_F) \) minimizes the area between the income distribution line associated with the egalitarian distribution and the Lorenz function.

5. We compare the normalized \( k \)-index \( (K_F := 2k_F - 1) \) with the Gini coefficient \( G_F \) and the Pietra index \( P_F \). If the Lorenz function is symmetric, then the normalized \( k \)-index coincides with the Pietra index (see Section 4.1). We show for any given income distribution, \( G_F \geq P_F \geq K_F \). We have also identified the complete set of Lorenz functions for which the coincidence between the normalized \( k \)-index with the Gini coefficient and the Pietra index takes place (see Section 4.2).

6. Finally, we show that the ranking of Lorenz functions from the \( k \)-index is different from that of the Pietra index as well as from the Gini coefficient. The Gini coefficient and the Pietra index are affected by transfers exclusively among the rich or among the poor, the \( k \)-index ranks is only affected by transfers across the two groups (see Section 5).

7. Appendix

Proof of (2): If \( p = k_F \), then \( \min\{k_F, r_F(k_F)\} = \max\{k_F, r_F(k_F)\} = k_F \) implying \( C(k_F) = \{k_F\} \). If \( p \in [0, k_F) \), then \( L_F'(k_F) = 1 - k_F < 1 - p \) and from non-decreasingness of \( L_F(.) \) we get \( k_F \leq r(p) \). Therefore, if \( p \in [0, k_F) \), then \( p < k_F \leq r_F(p) \) and \( k_F \in C(p) \). Similarly, if \( p \in (k_F, p^*_F] \), then \( L_F'(k_F) = 1 - k_F > 1 - p \Rightarrow k_F \geq r_F(p) \). Therefore, if \( p \in (k_F, p^*_F] \), then \( p > k_F \geq r_F(p) \) and \( k_F \in C(p) \).

Proof of (KP): Specifically, using the symmetry and differentiability of the Lorenz function it follows that \( - (1/L_F'(r_F(p))) + L_F'(p) = 0 \) and, given \( L_F'(F(\mu)) = 1 \) at the population fraction \( F(\mu) \) associated with Pietra index, it follows that \( L_F'(r_F(F(\mu))) = 1 \Rightarrow F^{-1}(r_F(F(\mu))) = \mu \Rightarrow L_F^{-1}(1 - F(\mu)) = F(\mu) \Rightarrow F(\mu) + L_F(F(\mu)) = 1 \) implying \( F(\mu) = k_F \).

Proof of (GK-P): Consider any Lorenz function \( L_F(p) \) and for any \( q \in (0, 1) \) define the induced Lorenz function

\[
L_F(p) = \begin{cases} 
\frac{L_F(q)}{(1-q)} p - \frac{L_F(q)}{(1-q)} & \text{if } p \in [0, q], \\
\frac{L_F(q)}{(1-q)} p - \frac{L_F(q)}{(1-q)} & \text{if } p \in (q, 1]. 
\end{cases}
\]
In Figure 7, we depict how given any $q \in (0, 1)$ we get the induced Lorenz function $\bar{L}_F(p)$ from any given Lorenz function $L_F(p)$.

![Figure 7](image-url)  

Figure 7: The red curve OAB depicts any Lorenz function $L_F(p)$ and, for any $q \in (0, 1)$, the dotted piecewise linear blue line OAB is the induced Lorenz function $\bar{L}_F(p)$ (colour online).

From Figure 7 it is clear that $\bar{L}_F(p) \geq L_F(p)$ for all $p \in [0, 1]$. Hence,

$$
\int_0^1 L_F(t) dt \leq \int_0^q L_F(t) dt + \int_q^1 \bar{L}_F(t) dt \\
= \int_0^q \frac{L_F(q)}{q} dt + \int_q^1 \left\{ \frac{(1 - L_F(q))}{(1 - q)} - \frac{(q - L_F(q))}{(1 - q)} \right\} dt \\
= \frac{q L_F(q)}{2} + L_F(q) - q + \frac{(1 - L_F(q))(1 + q)}{2} \\
= \frac{1 + L_F(q) - q}{2}.
$$

Since $\int_{t=0}^1 L_F(t) dt = (1 - G_F)/2$, it follows that $G_F \geq q - L_F(q)$ for any $q \in [0, 1]$. Since Pietra index maximizes the function $q - L_F(q)$ over all $q \in [0, 1]$, it follows that $G_F \geq F(\mu) - L_F(F(\mu)) \geq k_F - L_F(k_F)$ implying $G_F \geq P_F \geq K_F$.

We now show that if the Gini coefficient coincides with the normalized $k$ index, then the Lorenz function must be given by (9). Consider any population proportion $K \in [1/2, 1)$ and given such a $K$ consider any income distribution $F$ such that $k_F = K$. Then, the Gini coefficient $G_F = 1 - 2 \int_{t=0}^{t=1} L_F(t) dt$ coincides with the normalized $k$-index $K_F = K - L_F(K)$ if and only if

$$
\int_{t=0}^{t=1} L_F(t) dt = L_F(K) \Leftrightarrow \int_{t=0}^{t=K} \{L_F(K) - L_F(t)\} dt = \int_{t=0}^{t=K} \{L_F(t) - L_F(K)\} dt.
$$

(19)

---

2Possibility of such a selection is guaranteed by the family of Lorenz functions defined by (9).
Consider Figure 8 where the area of integral \( \int_{t=0}^{t=K} \{L_F(K) - L_F(t)\} dt \) is depicted by the region OAB. Given convexity of the Lorenz function, the area OAB is minimized if OAB represents the area of a triangle with base length \( K \) and altitude length \((1-K)\). Therefore, we have

\[
\int_{t=0}^{t=K} \{L_F(K) - L_F(t)\} dt \geq \frac{K(1-K)}{2}.
\]  

(20)

Similarly, in Figure 8 the area of integral \( \int_{t=K}^{t=1} \{L_F(t) - L_F(K)\} dt \) is depicted by the region ACD. Given convexity of the Lorenz function, the area ACD is maximized if ACD represents the area of a triangle with base length \((1-K)\) and altitude length \(K\). Therefore, we also have

\[
\int_{t=K}^{t=1} \{L_F(t) - L_F(K)\} dt \leq \frac{K(1-K)}{2}.
\]  

(21)

From (20) and (21) it follows that

\[
\int_{t=0}^{t=K} \{L_F(K) - L_F(t)\} dt \geq \frac{K(1-K)}{2} \geq \int_{t=K}^{t=1} \{L_F(t) - L_F(K)\} dt.
\]  

(22)

Applying (22) in (19) we get

\[
\int_{t=0}^{t=K} \{L_F(K) - L_F(t)\} dt = \frac{K(1-K)}{2} = \int_{t=K}^{t=1} \{L_F(t) - L_F(K)\} dt.
\]  

(23)
Simplification of the first equality in (23) gives
\[
\int_{t=0}^{t=K} L_F(t) dt = \frac{K(1-K)}{2} = \int_{t=0}^{t=K} H(t) dt, \tag{24}
\]
where \( H(t) := \frac{1-K}{K} t \) for all \( t \in [0, K] \).\(^5\) Observe that \( L_F(0) = H(0) = 0 \) and \( L_F(K) = H(K) = 1 - K \). For any \( t \in [0, K] \), \( H(t) \) is increasing and linear in \( t \) and \( L_F(t) \) is non-decreasing and convex in \( t \) and hence \( L_F(t) \leq H(t) \) for all \( t \in [0, K] \). Therefore, given \( \int_{t=0}^{t=K} L_F(t) dt = \int_{t=0}^{t=K} H(t) dt \) (condition (21)), we have \( L_F(t) = H(t) \) for all \( t \in [0, K] \), that is,
\[
L_F(t) = \frac{(1-K)}{K} t, \quad \forall \ t \in [0, K]. \tag{25}
\]
Similarly, simplification of the second equality in (23) gives
\[
\int_{t=K}^{t=1} L_F(t) dt = \frac{K(1-K)}{2} + (1-K)^2 = \int_{t=0}^{t=K} I(t) dt, \tag{26}
\]
where \( I(t) := (1-K) + \frac{K}{1-K} (t-K) \) for all \( t \in [K, 1] \).\(^6\) Observe that \( L_F(K) = I(K) = 1 - K \) and \( L_F(1) = I(1) = 1 \). For any \( t \in [K, 1] \), \( I(t) \) is increasing and linear in \( t \) and \( L_F(t) \) is non-decreasing and convex in \( t \) and hence \( L_F(t) \leq I(t) \) for all \( t \in [K, 1] \). Therefore, given \( \int_{t=K}^{t=1} L_F(t) dt = \int_{t=K}^{t=1} I(t) dt \) (condition (26)) we get \( L_F(t) = I(t) \) for all \( t \in [K, 1] \), that is,
\[
L_F(t) = (1-K) + \frac{K}{1-K} (t-K), \quad \forall \ t \in [K, 1]. \tag{27}
\]

Therefore, if for any income distribution \( F \), the Gini coefficient \( G_F \) coincides with the normalized \( \bar{k} \)-index \( K_F \), then from (25) and (27) (and due to the fact that while selecting any income distribution \( F \) such that \( k_F = K \), the selection of \( K \in [1/2, 1) \) was arbitrary) it follows that the Lorenz function must be of the form given by (20). □

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\(^5\) Note that \( \int_{t=0}^{t=K} H(t) dt = \int_{t=0}^{t=K} \frac{1-K}{K} dt = \frac{1-K}{2K} \left( t^2 \right)_{t=0}^{t=K} = \frac{K(1-K)}{2} \).

\(^6\) Note that \( \int_{t=K}^{t=1} I(t) dt = \int_{t=K}^{t=1} \left( (1-K) + \frac{K}{1-K} (t-K) \right) dt = (1-K)^2 + \frac{K}{2(1-K)^2} \left( (t-K)^2 \right)_{t=K}^{t=1} = (1-K)^2 + \frac{K(1-K)}{2} \).
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