Perturbation Expansion in Phase Ordering Kinetics

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Abstract

A consistent perturbation theory expansion is presented for phase ordering kinetics in the case of a nonconserved scalar order parameter. At lowest order in this formal expansion one obtains the theory due to Ohta, Jasnow and Kawasaki (OJK). At next order, worked out explicitly in d dimensions, one has small corrections to the OJK result for the nonequilibrium exponent $\lambda$ and the introduction of a new exponent $\nu$ governing the algebraic component of the decay of the order parameter scaling function at large scaled distances.
Significant progress has been made on the theory of phase ordering kinetics using methods that introduce auxiliary fields that are taken to have gaussian statistics. The methods developed in OJK and TUG each have appealing aspects and separately give good descriptions of different aspects of the ordering problem. A major lingering question is why these methods work as well as they do and how they can be reconciled and improved. Thus there has been a search for the field theory description where OJK or TUG is the zeroth order approximation in some systematic expansion. Such an expansion is presented in this paper. It has the OJK result as its zeroth order approximation.

The system studied here is the domain wall dynamics generated by the time-dependent Ginzburg-Landau model satisfied by a nonconserved scalar order parameter $\psi(\vec{r},t)$:

$$\Lambda(1)\psi(1) = -V'[\psi(1)]$$

where, in dimensionless units, $\Lambda(1) = \frac{\partial}{\partial t_1} - \nabla_1^2$ is the diffusion operator and 1 denotes $(r_1, t_1)$. We expect our main results to hold for any symmetric degenerate double-well potential $V[\psi]$ and to be independent of the exact nature of the initial and final states, provided they are disordered and ordered states respectively. Therefore it is convenient to quench to zero temperature and set any noise term to zero.

It is well established that for late times following a quench from the disordered to the ordered phase the order-parameter correlation function has a scaling form $C(12) \equiv \langle \psi(1)\psi(2) \rangle = \psi_0^2 F(x,t_1/t_2)$ where $\psi_0$ is the magnitude of the order-parameter in the ordered phase and the scaled length $x$ is defined as $\vec{x} = (\vec{r}_1 - \vec{r}_2)/L$ where $L = \sqrt{4T} \equiv \sqrt{2(t_1 + t_2)}$ is a growing characteristic length. In the perturbation theory developed here we find that

$$F(x,t_1/t_2) = 2 \pi \sin^{-1} [ \Phi(t_1/t_2)f(x,t_1/t_2)]$$

with the normalizations $f(0,t_1/t_2) = 1$ and $\Phi(1) = 1$. The quantity $\Phi(t_1/t_2)$, related to the on-site order parameter autocorrelation function, has the form

$$\Phi(t_1/t_2) = \left( \frac{\sqrt{t_1t_2}}{T} \right)^{\lambda} (1 + \Delta \Phi(t_1/t_2))$$

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where \( \Delta \Phi(t_1/t_2) \) is a regular function of \( t_1/t_2 \) which is of second order in the expansion. The nonequilibrium exponent \( \lambda \), which gives the decay for \( t_1 \gg t_2 \) and vice versa, is given to second order by

\[
\lambda = \frac{d}{2} + \omega^2 \frac{2^d M_d}{3^{d/2+1}}.
\]

The dimensionality dependent quantities, \( \omega \), \( K_d \) and \( M_d \), are given by

\[
2\omega + \omega^2 2^d \left( K_d + \frac{M_d}{3^{d/2+1}} \right) = 1 + \frac{d}{2}
\]

(5)

\[
K_d = \int_0^1 dz \frac{z^{d/2-1}}{[(1+z)(3-z)]^{d/2}}
\]

(6)

and

\[
M_d = \int_0^1 dz \frac{z^{d/2-1}}{[1+z]^d} = \frac{1}{2} \frac{\Gamma^2(d/2)}{\Gamma(d)}.
\]

(7)

\( f(x,t_1/t_2) \) can be written in the form

\[
f(x,t_1/t_2) = \frac{f_0(x)}{(1+x^2)^{\nu/2}} [1 + \Delta f(x,t_1/t_2)]
\]

(8)

where \( \Delta f(x,t_1/t_2) \) is a regular function of its arguments and is second order in the expansion and \( f_0(x) = e^{-\frac{1}{4} x^2} \) is just the OJK result for the scaling function. The exponent \( \nu \) governing the algebraic component of the large \( x \) behavior, shared by \( f \) and \( F \), is given by

\[
\nu = \omega^2 2^{d+1} \left( K_d + \frac{M_d}{3^{d/2+1}} \right).
\]

(9)

At lowest order one has the OJK results \( \lambda = \frac{d}{2} \) and \( \nu = 0 \). While \( K_d \) and \( \omega \) can be worked out analytically for specific values of \( d \), the expressions are not very illuminating. Numerically we have for \( d = 1 \), \( \omega = 0.4601.. \), \( \lambda = 0.6268.. \), \( \nu = 1.1596.. \); \( d = 2 \), \( \omega = 0.6877.. \), \( \lambda = 1.1051.. \), \( \nu = 1.2492.. \); \( d = 3 \), \( \omega = 0.9067.. \), \( \lambda = 1.5828.. \), \( \nu = 1.3732.. \). For large \( d \), \( \omega \rightarrow \frac{d}{2}(\sqrt{2} - 1) \), \( \lambda \rightarrow \frac{d}{2} \), and \( \nu \rightarrow d(3 - 2\sqrt{2}) \). The best estimates are \( d = 1 \), \( \lambda = 1 \), \( d = 2 \), \( \lambda = 1.246 \pm 0.02 \), and \( d = 3 \), \( \lambda = 1.838 \pm 0.2 \). For equal times \( t_1 = t_2 \), \( F(x,1) = F(x) = \frac{2}{\pi} \sin^{-1} f(x,1) \).

For small \( x \), where \( f \) is analytic in an expansion in even powers of \( x \), \( F(x) \) satisfies Porod’s law and the Tomita sum rule.
The perturbative treatment developed here is a two step process. The first step is to introduce an auxiliary field $m(1)$ which satisfies an equation of motion

$$\Lambda(1)\nabla_im(1) = 2\xi(t_1)\rho_i(1) = 2\xi(t_1)\delta(m(1))\nabla_im(1) \quad .$$

(10)

The quantity $\xi(t_1)$ must be chosen to go as $L^{-1}(t_1)$ for large times with a coefficient which must be adjusted if the system is to order, much like the temperature must be set to $T_c$ in critical phenomena. This equation of motion has the appealing feature that $\nabla_im(1)$ is a diffusion field driven by a source $\rho_i(1)$ which is the invariant density of domain walls for the system. We show below that we can organize a perturbation theory expansion for this field theory about a gaussian zeroth order theory.

The second step in the analysis is to show that one can map the phase ordering kinetics problem onto this new field theory. This mapping is via the transformation $\psi = \sigma[m] + u[m]$ where $\sigma[m]$ satisfies the Euler-Lagrange equation for the associated stationary interface problem with $m$ the coordinate: $\sigma_2 = V'[\sigma[m]]$ where we introduce the notation $\sigma_\ell = d^\ell\sigma/dm^\ell$. The idea is that $m$ grows large away from an interface and $\sigma \to \psi_0 sgn(m)$ in the bulk. Inserting this mapping into the equation of motion for $\psi$ we find:

$$\Lambda(1)u(1) + \sigma_1(1)\Lambda(1)m(1) = -V'\sigma(1) + u(1) + \sigma_2(1)(\nabla m(1))^2 \quad .$$

(11)

The most important property of the solution of this equation for $u[m]$ is that $\lim_{|m|\to\infty}u[m] = 0$ and $\psi \to \psi_0 sgn m$. In the regime where $|m|$ is large, $\sigma^2 \approx \psi_0^2$, the derivatives of $\sigma$ go exponentially to zero, the linearized equation for $u$ shows that it has acquired a mass and, in the long-time long-distance limit, $u$ is seen to go exponentially to zero. The universal properties of interest are associated with bulk quantities like $C(12...n) = \langle \psi(1)\psi(2)\cdots\psi(n) \rangle$ where the points $\vec{r}_1, \vec{r}_2, ..., \vec{r}_n$ are not constrained to be close together. Because the field $u(1)$ is nonzero only near interfaces the average $\langle u(1)\sigma(2)\cdots \rangle$ is down by a factor of $1/L^2(t_1)$ relative to $\langle \sigma(1)\sigma(2)\cdots \rangle$. Thus if we focus on these properties we do not need to know $u$ in detail.

We will develop the field theory generated by the equation of motion, Eq.(10), for $m(1)$ using the standard Martin-Siggia-Rose method in its functional integral form as formulated.
by DeDominicis and Peliti. In the MSR method the field theoretical development requires a doubling of fields to include a field $\hat{m}_i$ conjugate to $\nabla_i m$. We also organize things so that the initial field $m_0(\vec{r})$ is also treated as independent. Following standard procedures, averages of interest are given as functional integrals over the fields $m$, $\hat{m}_i$ and $m_0$ weighted by an action $A$:

$$A(m, \hat{m}, m_0) = i \int d1 \sum_{i=1}^{d} \hat{m}_i(1) [\Lambda(1)\nabla_i m(1) - 2\xi(1)\rho_i(1)]$$

$$- i \int d1 \sum_{i=1}^{d} \hat{m}_i(1) \delta(t_1 - t_0)\nabla_i m_0(1) - \frac{1}{2} \int d^d r_1 \int d^d r_2 m_0(\vec{r}_1)g^{-1}(\vec{r}_1 - \vec{r}_2)m_0(\vec{r}_2)$$

where we use the notation $d1 = dt_1d^d r_1$ and where we assume, as is appropriate in this case, that the initial field is gaussian and has a variance given by $<m_0(\vec{r}_1)m_0(\vec{r}_2)> = g(\vec{r}_1 - \vec{r}_2)$. We will not have to be very specific about the form of the initial correlation function $g$.

It is convenient to generate our correlation functions as functional derivatives in terms of sources, $h(1)$ and $\hat{h}(1)$, which couple to the conjugate fields. Thus we introduce the total action $A_T$:

$$A_T = A + \int d1 \left[ h(1)m(1) + \hat{h}(1) \sum_{i=1}^{d} \nabla_i \hat{m}_i(1) \right] .$$

The fundamental equations of motion are given then by

$$< \frac{\delta}{\delta \Psi(1)} A_T(m, \hat{m}, m_0) >_h = 0$$

where $\Psi(1) = \{m(1), \hat{m}_i(1), m_0(1)\}$ and the averages are over the total action $A_T$. Taking the functional derivatives of $A_T$ and and, using Eq.(14) with $\Psi(1) = m_0(1)$ to eliminate the initial field in terms of the hatted field, we obtain the two basic equations

$$-i [\tilde{\Lambda}(1) < \hat{M}(1) >_h + 2\xi(1) < \hat{\rho}(1) >_h] = h(1)$$

$$i [\Lambda(1) \nabla_i < m(1) >_h - 2\xi(1) < \rho_i(1) >_h] = -\nabla_i^{(1)} \int d2 \tilde{\Pi}_0(12) < \hat{M}(2) >_h + \nabla_i \hat{h}(1) .$$

Here we have introduced the quantities $\Pi_0(12) \equiv \delta(t_1-t_0)\delta(t_1-t_2)g(\vec{r}_1 - \vec{r}_2)$, $\tilde{\Lambda}(1) = \frac{\partial}{\partial \vec{r}_1} + \nabla_1^2$, $\hat{\rho}(1) = \delta(m(1))\hat{M}(1)$, and $\hat{M}(1) = \sum_{i=1}^{d} \nabla_i \hat{m}_i(1)$. Notice that the fundamental equations
Eqs. (15) and (16) depend on $\hat{M}$ and not $\hat{m}_i$ and we have constructed $\hat{h}$ such that it couples to $\hat{M}$. All correlation functions of interest can be generated as functional derivatives of $<m(1)>_h$ or $<\hat{M}(1)>_h$ with respect to $h$ and $\hat{h}$.

Taking the functional derivative of Eqs. (15) and (16) with respect to $h(2)$ gives the equations for the two point response and correlation functions:

$$-i\left[\tilde{\Lambda}(1)G_{\hat{M}m}(12) + \hat{Q}_2(12)\right] = \delta(12)$$ (17)

$$i\left[\Lambda(1)\nabla_i G_{mm}(12) - Q_2^{(i)}(12)\right] = -\nabla_i^{(1)} \int d3 \Pi_0(13)G_{\hat{M}m}(32)$$ . (18)

where we have used

$$\hat{Q}_n(12...n) = \frac{\delta^{n-1}}{\delta h(n)\delta h(n-1)\cdots\delta h(2)} \left[2\xi(1)\hat{\rho}(1)\right]$$

$$Q_n^{(i)}(12...n) = \frac{\delta^{n-1}}{\delta h(n)\delta h(n-1)\cdots\delta h(2)} \left[2\xi(1)\rho_i(1)\right]$$ ,

evaluated for $n = 2$ and introduced the notation

$$G_{\hat{M}mm...m}(123...n) = \frac{\delta^{n-1}}{\delta h(2)\delta h(3)\cdots\delta h(n)} <\hat{M}(1)>_h$$

$$G_{n}(12\cdots n) = \frac{\delta^{n-1}}{\delta h(n)\delta h(n-1)\cdots\delta h(2)} <m(1)>_h$$ .

The functional derivative of Eq. (16) with respect to $\hat{h}(2)$ is redundant compared with Eq. (17) because of the relation $G_{m\hat{M}}(12) = G_{\hat{M}m}(21)$ . The equations governing the $n^{th}$ order cumulants are given by

$$-i\left[\tilde{\Lambda}(1)G_{\hat{M}mm...m}(12...n) + \hat{Q}_n(12...n)\right] = 0$$

$$i\left[\Lambda(1)\nabla_i G_n(12...n) - Q_n^{(i)}(12...n)\right] = -\nabla_i^{(1)} \int d\bar{I} \Pi_0(1\bar{I})G_{\hat{M}mm...m}(12...n)$$ .

The point now is to show that there is a consistent perturbation expansion for this theory. After introducing the one-point probability distribution $P_h(x_1,1) = <\delta(x_1-m(1))>_h$, it is easy to show
\[
Q_1^{(i)}(1) = 2\xi(1) < \rho_i(1) >_h = \nabla_1^{(i)} \int dx_1 \xi(1) sgn(x_1) P_h(x_1, 1) 
\]

\[
\hat{Q}_1(1) = 2\xi(1) < \hat{\rho}(1) >_h = 2\xi(1) \left[ < \hat{M}(1) >_h + \frac{\delta}{\delta h(1)} \right] P_h(0, 1) . 
\]

Then any perturbation theory expansion for \( P_h(x_1, 1) \) will lead immediately to an expansion for \( Q_n^{(i)} \) and \( \hat{Q}_n \) by functional differentiation. The expansion for \( P_h(x_1, 1) \) is straightforward. Using the integral representation for the \( \delta \)-function we have

\[
P_h(x_1, 1) = \int \frac{dk_1}{2\pi} e^{-ik_1 x_1} < e^{H(1)} >_h = \int \frac{dk_1}{2\pi} e^{-ik_1 x_1} exp \left[ \sum_{n=1}^{\infty} \frac{1}{n!} G_H^{(n)}(1) \right] 
\]

where \( H(1) \equiv ik_1 m(1) \) and \( G_H^{(n)}(1) \) is the \( n \)th order cumulant for the field \( H(1) \). Since \( H(1) \) is proportional to \( m(1) \) these are, up to factors of \( ik_1 \) to the \( n \)th power, just the cumulants for the \( m \) field. It turns out that in zero external field we can take \( G_n \) to be of \( \mathcal{O}(\frac{\pi}{2} - 1) \).

Keeping terms up to the 4-point cumulant it is easy to see that

\[
P_h(x_1, 1) = \left[ 1 - \frac{1}{3!} G_3(111) \frac{d^3}{dx_1^3} + \frac{1}{4!} G_4(1111) \frac{d^4}{dx_1^4} + \cdots \right] P_h^{(0)}(x_1, 1) 
\]

where

\[
P_h^{(0)}(x_1, 1) = \int \frac{dk_1}{2\pi} \Phi_0(k_1, 1) e^{-ik_1 x_1} 
\]

and

\[
\Phi_0(k_1, 1) = e^{-ik_1(x_1-G_1(1))} e^{-\frac{1}{2}k_1^2 G_2(11)} . 
\]

Eqs. (21), (22) and (23) inserted back into Eqs. (19) and (20) gives a relation between the \( m \)-field cumulants and the quantities \( Q_n^{(i)} \) and \( \hat{Q}_n \). These quantities are ultimately related to derivatives of \( P_h^{(0)}[0, 1] \). Because of this we see that each term in the perturbation theory expansion in zero external fields is weighted by factors of

\[
\phi_n(1) = \int \frac{dk_1}{2\pi} k_1^{2n} e^{-\frac{1}{2}k_1^2 S(1)} = \left( -2 \frac{\partial}{\partial S(1)} \right)^n \phi_0(1) 
\]

where \( \phi_0(1) = \frac{1}{\sqrt{2\pi S(1)}} \) and \( S(1) = < m^2(1) > \). The structure of the perturbation theory becomes clear if one looks at two classes of contributions to \( Q_{2n}^{(i)} \). The first class corresponds
to the $2n - 1$ derivatives which all act on $\Phi_0(k_1, 1)$ to give contributions to $Q^{(i)}_{2n}(12 \cdots, 2n)$ of the form

$$\nabla_i^{(1)} \xi(1) \phi_{n-1}(1) G_2(12) G_2(13) \cdots G_2(1, 2n)$$

There is another set of contributions where all of the derivatives except the first acts on the factor multiplying $\Phi_0(k_1, 1)$ and gives a contribution to $Q^{(i)}_{2n}(12 \cdots, 2n)$ of the form

$$\nabla_i^{(1)} \xi(1) \phi_0(1) G_{2n}(12 \cdots, 2n)$$

Inserting these results into the expression for the $2n$ point cumulant we see term by term that the leading order behavior goes as $G_{2n} \approx \phi_{n-1}$. Orders in the expansion correspond to the sum of the labels $n$ on $\phi_n$ in products of $\phi$’s. Thus a term with factors $\phi_1 \phi_2 \phi_1$, typically associated with different times, is of $O(4)$. It should be emphasized that at this stage that this is a formal expansion. At order $n$ it is true that $\phi_n \approx L^{-(2n+1)}$ is small, however it will be multiplied, depending on the quantity expanded, by positive factors of $L(t)$ such that each term in the expansion in $\phi_n$ has the same overall leading power with respect to $L(t)$.

It is then straightforward to work out the expansion for the two-point cumulant and obtain at zeroth order:

$$\hat{Q}_2(12) = \omega_0(1) G_{\hat{M}m}(12)$$

$$Q^{(i)}_2(12) = \nabla_i^{(1)} \omega_0(1) G_2(12)$$

where we have defined $\omega_n(1) = 2 \xi(1) \phi_n(1)$. The equations of motion at zeroth order for the two point response and correlation functions is given by Eqs. (17) and (18) with $\tilde{\Lambda}$ and $\tilde{\Lambda}^{(i)}$ given by Eqs. (23) and (26):

$$- i \left[ \tilde{\Lambda}(1) + \omega_0(1) \right] G^{(0)}_{\tilde{M}m}(12) = \delta(12)$$

and

$$i \left[ \Lambda(1) - \omega_0(1) \right] G^{(0)}_2(12) = - \int d\bar{1} \Pi_0(1\bar{1}) G^{(0)}_{\tilde{M}m}(\bar{1}2)$$
The equation for the response function has the solution
\[ G_{Mm}^{(0)}(r, t_1 t_2) = -i\theta(t_2 - t_1)R(t_2 t_1) \frac{e^{-\frac{r^2}{4(t_2 - t_1)}}}{4\pi(t_2 - t_1)^{d/2}} = G_{mM}^{(0)}(r, t_2 t_1) \] (29)
where we have introduced
\[ R(t_1 t_2) = e^{\int_{t_2}^{t_1} d\tau \omega_0(\tau)} . \] (30)

The equation for the correlation function, Eq.(28), can be integrated up into the symmetric form, suppressing the step functions in time,
\[ G_{2}^{(0)}(r, t_1 t_2) = R(t_1 t_0)R(t_2 t_0)\int \frac{d^d q}{(2\pi)^d} e^{i\vec{q}\cdot\vec{r}} \tilde{g}(q)e^{-2q^2T} \] (31)
where \( T = \frac{t_1 + t_2}{2} \) and \( \tilde{g}(q) \) is the Fourier transform of \( g(r) \). We focus on the long-time limit where in the integral over wavenumber we can replace \( \tilde{g}(q) \to \tilde{g}(0) \) to leading order in powers of \( 1/T \) and obtain for large \( T \):
\[ G_{2}^{(0)}(r, t_1 t_2) = R(t_1 t_0)R(t_2 t_0)\tilde{g}(0)\frac{e^{-r^2/8T}}{(8\pi T)^{d/2}} . \] (32)

If this system is to order, \( < m^2(1) > \) is to grow large, then we must choose, for large time \( t, \omega_0(t) = 2\xi(t)\phi_0(t) = \frac{\omega}{t} \) where \( \omega \) is a constant we will determine. Then, \( R(t_1 t_2) \) defined by Eq.(30), is given by \( R(t_1 t_2) = \left( \frac{t_1}{t_2} \right)^{\omega} \) and
\[ G_{2}^{(0)}(r, t_1 t_2) = \tilde{g}(0)\left( \frac{t_1}{t_0} \right)^{\omega}\left( \frac{t_2}{t_0} \right)^{\omega} \frac{e^{-r^2/8T}}{(8\pi T)^{d/2}} . \] (33)

If we are to have a self-consistent scaling equation then the autocorrelation function \( (r = 0) \), at large equal times \( t_1 = t_2 = t \) must satisfy
\[ G_{2}^{(0)}(0, tt) = S^{(0)}(t) = A_0 t = t^{2\omega-d/2} \frac{1}{(t_0)^{2\omega}} \frac{\tilde{g}(0)}{(8\pi)^{d/2}} \] (34)
which fixes the exponent \( \omega = \frac{1}{2}(1 + \frac{d}{2}) \) and the amplitude \( A_0 \). This determination of \( \omega \) must be repeated order by order in perturbation theory. Returning to the correlation function, we can eliminate \( \tilde{g}(0) \) using the expression in terms of the amplitude and obtain
\[ G_{2}^{(0)}(r, t_1 t_2) = \sqrt{S(t_1)S(t_2)}\Phi_0(t_1/t_2)f_0(x) \] where \( \Phi_0(t_1/t_2) = \left( \frac{\sqrt{tt_2}}{T} \right)^{d/2} \) and \( f_0(x) = e^{-x^2/2} \).
One can read off from these results the nonequilibrium exponent \( \lambda = \frac{d}{2} \) given by the OJK result.

Going forward, it is easy to find that the next nonzero contribution to \( Q_2 \) and \( \hat{Q}_2 \) is of \( O(2) \) and given by

\[
Q_2^{(i)}(12)^{(2)} = -\nabla_i^{(1)} \left[ \frac{\omega_1(1)}{3!} G_4(1112) \right] \tag{35}
\]

\[
\hat{Q}_2(12)^{(2)} = -\frac{\omega_1(1)}{2} G_{Mmm}(1112) \tag{36}
\]

These require that we evaluate the four point cumulants at the leading first order and obtain

\[
G_{Mmm}(1234) = G_{Mm}^{(0)}(1\bar{I})[-i\omega_1(\bar{I})] \tag{37}
\]

\[
\times \left[ G_{Mm}^{(0)}(\bar{I}2)G_2^{(0)}(\bar{I}3)G_2^{(0)}(\bar{I}4) + G_{Mm}^{(0)}(\bar{I}3)G_2^{(0)}(\bar{I}2)G_2^{(0)}(\bar{I}4) + G_{Mm}^{(0)}(\bar{I}4)G_2^{(0)}(\bar{I}2)G_2^{(0)}(\bar{I}3) \right]
\]

\[
G_4(1234) = G_2^{(0)}(1\bar{I})[-i\omega_1(\bar{I})] \left[ G_{Mm}^{(0)}(\bar{I}2)G_2^{(0)}(\bar{I}3)G_2^{(0)}(\bar{I}4) \right] + \left[ G_{Mm}^{(0)}(\bar{I}3)G_2^{(0)}(\bar{I}2)G_2^{(0)}(\bar{I}4) + G_{Mm}^{(0)}(\bar{I}4)G_2^{(0)}(\bar{I}2)G_2^{(0)}(\bar{I}3) \right]
\]

where we integrate over the repeated barred index. Notice that \( G_4 \) is properly symmetric under interchange of its labels. Inserting these results back into the equations for the two point quantities we have for the response function

\[
G_{Mm}(12) = G_{Mm}^{(0)}(12) + \int d\bar{I}d\bar{2}G_{Mm}^{(0)}(1\bar{I})\Sigma_{Mm}(\bar{I}2)G_{Mm}^{(0)}(22) \tag{39}
\]

and the lowest order self-energy contribution is given by

\[
\Sigma_{Mm}(12) = \frac{1}{2}[ -i\omega_1(1)]G_{Mm}^{(0)}(12)G_2^{(0)}(12)[-i\omega_1(2)] \tag{40}
\]

The correlation function is then given to second order by
\[ G_2(12) = G_2^{(0)}(12) + G_2^{(2,1)}(12) + G_2^{(2,1)}(21) + G_2^{(2,2)}(12) \]  

where

\[ G_2^{(2,1)}(12) = \int d\tilde{t} d\tilde{t}' G_2^{(0)}(1\tilde{t}) \Sigma_{\tilde{M}m}(1\tilde{t}) G_2^{(0)}(\tilde{M}m, 212) \]  

and

\[ G_2^{(2,2)}(12) = -\int d\tilde{t} d\tilde{t}' G_2^{(0)}(1\tilde{t}) \Pi^{(2)}(1\tilde{t}) G_2^{(0)}(\tilde{M}m, 212) \]  

where the self-energy is the same as for the response function and

\[ \Pi^{(2)}(12) = -\frac{1}{3!} [-i\omega_1(1)] G_2^{(0)}(12)^3 [-i\omega_1(2)] . \]

The spatial integrals giving \( G_2^{(2,1)} \) and \( G_2^{(2,2)} \) can be evaluated in \( d \)-dimensions since they involve products of displaced gaussians. After rescaling the internal time integrations \( \tilde{t}_1 = Ty_1 \) and \( \tilde{t}_2 = Ty_2 \) we obtain

\[ G_2^{(2,1)}(12) = \sqrt{S(1)S(2)2^{d-1}} \omega^2 \Phi_0(t_1/t_2) J_1(x, t_1/T, T) \]  

\[ J_1(x, t_1/T, T) = \int_{t_0/T}^{t_1/T} dy_1 \int_{t_0/T}^{y_1} dy_2 \frac{y_1^{d/2-1} y_2^{d/2-1}}{[(y_1 + y_2)(3y_1 - y_2 - (y_1 - y_2)^2)]^{d/2}} e^{-\frac{1}{2} \frac{3y_1 - y_2}{3y_1 - y_2 - (y_1 - y_2)^2} x^2} \]  

\[ G_2^{(2,2)}(12) = \sqrt{S(1)S(2)} \frac{3^{d-1}}{3} \omega^2 \Phi_0(t_1/t_2) J_2(x, t_1, t_2) \]  

\[ J_2(x, t_1, t_2) = \int_{t_0/T}^{t_1/T} dy_1 \int_{t_0/T}^{t_2/T} dy_2 \frac{y_1^{d/2-1} y_2^{d/2-1}}{[(y_1 + y_2)(3y_1 - y_2 - (y_1 - y_2)^2)]^{d/2}} e^{-\frac{1}{2} \frac{3y_1 - y_2}{3y_1 - y_2 - (y_1 - y_2)^2} x^2} \] .

The first thing we should do with these results is look at the contribution at this order to the on site equal-time \( t_1 = t_2 = t \) correlation function. The integrals \( J_1(0, 1, t) \) and \( J_2(0, t, t) \) are logarithmically divergent as \( t \to \infty \) and we have

\[ S(1) = S^{(0)}(t) \left[ 1 + \omega^2 2^d \left( K_d + \frac{M_d}{3^{d/2+1}} \right) ln(t/t_0) + \cdots \right] \]  

where \( K_d \) is defined by Eq.(\ref{eq:Kd}) and \( M_d \) is defined by Eq.(\ref{eq:Md}). Remembering that at lowest order \( S^{(0)}(t) = A_0 t^{2\omega-d/2} \) and after exponentiation of the \( ln(t/t_0) \) term in Eq.(\ref{eq:47}) we find that to obtain \( S(t) = At \), we require that \( \omega \) be given by Eq.(\ref{eq:5}).
The $t_0 \to 0$ singularities in $J_1$ and $J_2$ can be regulated by turning our attention from $G_2(12)$ to the scaled quantities $\Phi(t_1/t_2)$ and $f(x,t_1/t_2)$ which appear in Eq.(2):

$$f(12) = \frac{G_2(12)}{\sqrt{S(1)S(2)}} = \Phi(t_1/t_2)f(x,t_1/t_2) \quad (48)$$

Looking at the second order contributions to $\Phi(t_1/t_2)$ we find for $t_1 \gg t_2$ or $t_2 \gg t_1$ there are contributions which are logarithmically divergent in the quantity $\sqrt{t_1t_2}/T$ which can be exponentiated to obtain the result given by Eq.(3). $f(x,t_1/t_2)$ is an analytic function of $x$. One can easily work out, for example, the lowest order term in the power series expansion in $x^2$ for $f(x,t_1/t_2)$ which is regular for all $d$. It is this term which gives the coefficient in Porod’s law. There is one last regularization that must be carried out before the perturbation theory expression for the correlation function can be used for all values of $x$. If we look at large $x$ we find that $f(x,t_1/t_2)$ is logarithmically divergent in $1 + x^2$ which, on exponentiation leads to the expression given by Eq.(8).

We turn next to the connection between the correlation function for the auxiliary field $m$ and the order parameter correlation function. As discussed earlier, since $u(1)$ vanishes exponentially for large $|m(1)|$, then the averages over these fields are down by a factor of $L^{-2}$ relative to the averages over the field $\sigma$ and in the scaling regime $C(12) = \langle \sigma(1)\sigma(2) \rangle$. The perturbation expansion for this quantity follows closely the development for the one-point quantity except we must treat the two-point generalization of $P_h(x_1,1)$. One can then show

$$C(12) = \frac{2}{\pi} \sin^{-1} f(12) + O(3) \quad (49)$$

where $f(12)$ is defined by Eq.(48). Since we have worked out $f(12)$ to second order previously, we have consistently to this order, the result given by Eq.(2).

Clearly one can go to higher order in this perturbation theory and introduce more sophisticated graphical methods. It is interesting to ponder the implications of the ideas developed here for the conserved and vector order parameter cases.

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REFERENCES

1 See A.J. Bray, Adv. Phys. 43, 357 (1994) for a recent review.

2 T. Ohta, D. Jasnow, K. Kawasaki, Phys. Rev. Lett. 49, 1223 (1982).

3 G. F. Mazenko, Phys. Rev. B 42, 4487 (1990).

4 A. J. Bray and K. Humayun, Phys. Rev. E 48, R1609 (1993).

5 G.F. Mazenko, Phys. Rev. E 49, 3717 (1994).

6 The quantities $\Delta \Phi(t_1/t_2)$ and $\Delta f(x, t_1/t_2)$ have been determined but the expressions for
them are long and not very illuminating.

7 The nature of the expansion is discussed below. Convergence properties, if any, remain to
be investigated.

8 D. S. Fisher and D. A. Huse, Phys. Rev. B 38, 373 (1988).

9 The exponent $\nu$ was first introduced in Ref. (3) in section V.B.4.

10 F. Liu and G.F. Mazenko, Phys. B 44, 9185 (1991).

11 G. Porod, in Small Angle X-ray Scattering, ed. by O. Glatter and L. Kratky (Academic,
New York, 1983).

12 H. Tomita, Prog. Theor. Phys. 72, 656 (1984); 75, 482 (1986).

13 The quantity $\rho_i$ is the scalar order parameter defect density. Since we expect that $m$ and
$\psi$ share the same zeros, we have $\rho_i = \delta(\psi) \nabla_i \psi$. As discussed in G. F. Mazenko, Phys.
Rev. Lett. 78, 401 (1997), $\rho_i$ is conserved for $n = d = 1$. It is easy to see that this holds
for all $d$ for the scalar case.

14 This is just the transformation introduced in Ref. (3) and used in Ref. (4).

15 P.C. Martin, E.D. Siggia and H.A. Rose, Phys. Rev. A 8, 423 (1973).
16 C. de Dominicis and L. Peliti, Phys. Rev. B18, 353 (1978). See, relating to the current application, G. F. Mazenko, O. T. Valls, and M. Zannetti, Phys. Rev. B38, 520 (1988).

17 One point to keep in mind is the well known property of the MSR method that the equal-time correlation functions with hatted fields must vanish if one is to satisfy causality: $G_{\hat{M}m}(11) = 0$.

18 One can regulate the theory at short times by introducing the time cutoff $t_c$ via $\omega_0(t) = \omega/(t + t_c)$. 