Dirac operators on lightlike hypersurfaces

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Abstract
In this study, we obtain a spinorial Gauss formula for a lightlike hypersurface in Lorentzian manifold with 4-dimension. Then, we take into account the changes caused by degenerate metric on hypersurface and investigate Dirac operator for lightlike hypersurface. Later, we establish the relation between Dirac operators and Riemannian curvatures of manifold and hypersurface.

Key words: Spin geometry, degenerate spin manifold, lightlike vector.
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1 Introduction
Dirac operator has revealed due to the square root problem of Laplacian operator in the Klein-Gordon equation. The Dirac operator, which was emerged from the studies of Paul Dirac during his investigations on the spin-1/2 particles like fermions and electrons, tries to find out an answer to a question that whether the first order differential equation with \( D = \sqrt{\Delta} \) exists or not. As a result of the growing attention to this equation, many researchers from different branches such as geometricians, researchers from both mathematical physics and analysis, have started to work on this topic. Especially, a great amount of mathematicians also interested in this operator after the relation between the properties of Clifford algebra and its the coefficients of the Dirac operator. The calculation of this operator in vector spaces is relatively easy when it compared to the calculation in the manifolds. So, this operator was studied on vector spaces before considering the manifolds. Then, to eliminate the possible problems and to ensure that the operator is well-defined in the manifold was needed some changes since vector bundles are insufficient to obtain Dirac operator on manifolds. The lack of vector bundles
was eliminated by the associated vector bundle and so, the spin geometry has been revealed. After that, Dirac operator was started to work on manifolds.

While many researchers have been investigating the Dirac operators and their features on the Riemannian and Lorentzian manifolds, recently the Dirac operators on the surfaces have been attracted attention. The investigation of the relations between the Dirac equation solutions and immersions of surfaces in \[9\] could be given as an example. The results that exist for the Laplacian have been investigated by Hijazi and Montiel \[11\] for the Dirac operator. In addition to the given examples, it is known that the existence of the bounds for eigenvalues of Dirac operator has a vital importance and as a result of this knowledge there exists studies \[18\] \[14\] in the literature that cover the discussions about the hypersurfaces. Moreover, Nakad and Roth \[20\] aimed to develop upper bounds for the eigenvalues of Dirac operator, which is defined on the hypersurfaces of the spin manifolds. While a new upper bound for the first eigenvalue of the Dirac operator on hypersurface was examined in \[10\], the Dirac operator for the hypersurfaces has been discussed in all manners in many other studies \[13\] \[19\] \[17\]. In \[12\], some results have been shown by taking into account of the scalar curvature. The obtained results for hypersurfaces have been extended to the submanifold in \[15\]. Also, Dereli et.al. worked on degenerate spin group and Levy-Leblond equation as given in \[4\] \[5\].

The main aim of this work is to investigate the hypersurfaces of Lorentzian spin manifolds with 4-dimension. The existing studies in the literature about this problem have been mainly focused on the timelike and spacelike hypersurfaces of Lorentzian manifolds where the lightlike hypersurfaces have left as an open problem. Those hypersurfaces has an important place in the researches due to their contributions to the applicability of the theory of relativity. Even though lightlike geometry studies may provide many beneficial outcomes, there exist some difficulties on working with them since they are different from many geometries. Considering these difference, we describe the spinorial Gauss formula for the lightlike hypersurfaces and show that it is possible to reduce a spin structure to the lightlike hypersurface \(M\) from the Lorentzian spin manifold \(\tilde{M}\). Then, we build up the relationship between the spinor covariant derivatives for \(M\) and \(\tilde{M}\). Also, we define the Dirac operator for lightlike hypersurface by using Dirac operator of Lorentzian spin manifold. Doing those, we aim to investigate the Dirac operator and to establish a relationship between the Dirac operators of lightlike hypersurfaces and Lorentzian spin manifolds. In addition, we study special lightlike hypersurface like minimal, totally umbilical and etc. in this represented work.

2 Preliminaries

In this section, definitions that will be used later have been given.

**Definition 1** Let \(V\) be a vector space over a commutative field \(k\) and \(Q\) be a quadratic form on \(V\). Let \(T(V) = \sum_{i=0}^{\infty} \otimes^i V\) denotes the tensor algebra of \(V\).
where $\otimes$ is tensor product and $I_Q(V)$ be the ideal in $T(V)$, which is generated by all elements of the form $v \otimes v + Q(v)$ for $v \in V$. Then, the quotient
\[
C\ell(V, Q) \equiv T(V) / I_Q(V)
\]
is called Clifford algebra [10].

Let us choose index $q$ with $0 < q < n$ and $p = n - q$ where $n$ is dimension of vector space. In this situation, the set of all linear isometries $\psi : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^{p,q}$ is the same as the set of all matrices $\Psi \in GL(n, \mathbb{R})$ which preserves scalar product on $\mathbb{R}^{p,q}$ where $GL(n, \mathbb{R})$ is general linear group. Then, it generates a group and is called semi-orthogonal group. Also, it is denoted by $O(p, q)$ for $p = n - q$. Also, the set
\[
SO(p, q) = \{ \Psi \in O(p, q) : \det \Psi = 1 \}
\]
is called special semi-orthogonal group [21].

Definition 2 Let $\mathbb{R}^n$ be a $n$-dimensional real vector space, $g$ be a symmetric bilinear form on $\mathbb{R}^n$ and $e_1, e_2, ..., e_n$ be standard basis vectors on $\mathbb{R}^n$. If the symmetric bilinear form $g$ satisfies the condition
\[
g(e_i, e_j) = \varepsilon_i \delta_{ij}, \varepsilon_i = \begin{cases} -1, & 1 \leq i \leq q \\ 1, & q < i \leq n \end{cases}, \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}
\]
then $g$ is called semi-Euclidean bilinear form on $\mathbb{R}^n$ [2].

We assume that $Q$ is a quadratic form for a semi-Euclidean bilinear form $g$ on $(\mathbb{R}^n, g)$. Then, Clifford algebra $C\ell_{p,q} := C\ell(\mathbb{R}^n, Q)$ is called semi-orthogonal Clifford algebra. For this Clifford algebra, there are
\[
e_i^2 = -\varepsilon_i, \quad i = 1, ..., n
\]
\[
e_i e_j + e_j e_i = 0, \quad i \neq j, i, j = 1, ..., n
\]
and $(1, e_1, ..., e_s, 1 \leq i_1 < ... < i_s \leq n, 1 \leq s \leq n)$ is the basis of $C\ell_{p,q} [2]$.

A semi-orthogonal pin group is a subgroup such that
\[
Pin(p, q) := \{ a_1 \cdot ... \cdot a_l : a_i \in S_q^{n-1} \cup H_q^{n-1} \}
\]
consists of the inverse elements of Clifford algebra $C\ell_{p,q}$ where $S_q^{n-1} = \{ v \in \mathbb{R}^n : g(v, v) = 1 \}$ and $H_q^{n-1} = \{ v \in \mathbb{R}^n : g(v, v) = -1 \}$. If $l$ is even, semi-orthogonal pin group is semi-orthogonal spin group and is denoted by $Spin(p, q) [2]$.

Definition 3 Let $V$ be a real 4-dimensional vector space with a symmetric bilinear form $g$. Then, a subspace $\text{Rad } V$ of $V$ expressed by
\[
\text{Rad } V = \{ \eta \in V : g(\eta, v) = 0, v \in V \}
\]
is called radical space [7].
Also, we assume that $V$ is a vector space with quadratic form $Q$ and $(V, Q)$ has rank $n$. If $(V, Q)$ has a radical subspace, we have $V = V_1 \oplus \text{Rad } V$ where $\dim V_1 = n_1$, $\text{Rad } V$ is the radical of $(V, Q)$ and $Q$ induces a quadratic form $Q_1$ of rank $n_1$ on $V_1$. So, Clifford algebra, which is formed by vector space $V$ with radical space, is called degenerate Clifford algebra and this degenerate Clifford algebra is isomorphic to the graded tensor product of $C\ell (V_1, Q_1)$ and $\Lambda \text{Rad } V$ where $\Lambda$ is an exterior product. If we take $\mathbb{R}^{r,p,q}$ instead of $V$, degenerate Clifford algebra is written as $C\ell (r,p,q)$ where $r$ is the dimension of radical space in $\mathbb{R}^{r,p,q}$.

**Proposition 4** There is a homomorphism $\sigma$ that it is defined onto the group $T$ of isometries of $(V, Q)$ from Clifford group $\Gamma$ where the restriction of $V$ to $\text{rad } V$ is the identity.

Degenerate pin group of $(V, Q)$ is denoted by $\text{Pin}(Q)$ and degenerate spin group of $(V, Q)$ is denoted by $\text{Spin}(Q)$. Every element of $\text{Pin}(Q)$ is a product

$$a_1 \cdot \ldots \cdot a_k \cdot \exp\left(\sum_{i,k} c^{ik} e_k f_i \right)$$

where $V = V_1 \oplus \text{Rad } V$, $Q_1$ is a quadratic form on $V_1$, $e_k$ is an orthogonal basis vector of $(V_1, Q_1)$, $f_i$ is an arbitrary basis vector of $\text{Rad } V$ and $Q(a_i) = \pm 1$ for $a_i \in V_1$. If $k$ is even, it is an element of $\text{Spin}(Q)$. If we take $\mathbb{R}^{r,p,q}$ instead of $V$, the degenerate spin group is written as $\text{Spin}(r,p,q)$.

## 3 Spinor Bundles on Lightlike Hypersurfaces

In this section, we will obtain the necessary relationships to define the spinorial Gauss formula for the lightlike hypersurface of the Lorentzian spin manifold with 4-dimension.

Let $(\widetilde{M}, \widetilde{g})$ be a Lorentzian spin manifold with 4-dimension. Moreover, let $\widetilde{g}$ is given by $\widetilde{g} = (+, +, +, -)$ and $\widetilde{\nabla}$ denotes the Levi Civita connection on tangent bundle $T\widetilde{M}$. We consider an 3-dimensional lightlike hypersurface $(M, \tilde{g})$ of the manifold $(\widetilde{M}, \widetilde{g})$. If there exists a vector field $\eta \neq 0$ on $M$ such that $g(\eta, X) = 0$ for $X \in \Gamma (TM)$, then $g$ is degenerate. A subspace, which consist of tangent vector $\eta_x$ at each point $x \in M$, is called a radical or null space and it is denoted by $\text{Rad } T_x \tilde{M}$. Also, $\text{Rad } T\tilde{M}$ is called a radical distribution of $M$ and if $M$ has the radical distribution, then it is called a lightlike hypersurface of $\tilde{M}$. Here, induced metric $g$ by $\tilde{g}$ is degenerate and $\nabla$ is linear connection on $M$, but it is not Levi Civita connection.

We will show that the spin structure on manifold $\tilde{M}$ could be reduced to lightlike hypersurface $M$. For this, we need to the degenerate special orthogonal group to establish a relationship with the degenerate spin group.

The basis vector of Lie algebra $so(3,1)$ is $E_{ij} = -\varepsilon_j D_{ij} + \varepsilon_i D_{ji}$ where $D_{ij}$ denotes matrices of type $4 \times 4$ whose the components of $(ij)$ are one and the other
components are zero. Also, $\varepsilon_i$ is the signature of vector, that is, $\varepsilon_i = g(e_i, e_i)$ where $e_i$ is the basis vector on $\mathbb{R}^{3,1}$. 

Also, the basis of Lie algebra $so(1,2,0)$ for the hypersurface $M$ is

$$E_{01} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, E_{02} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{12} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{8}$$

When we examine Lie algebras $so(1,2,0)$ and $so(3,1)$, we see that there exists an immersion between the Lie algebras. So, it is obvious that $SO(1,2,0)$ is a subgroup of $SO(3,1)$. So, we establish the relationship between special orthogonal groups. Similar to this relationship, a connection between spin groups is also needed. Since spin groups consist of inverse elements of Clifford algebra, we use the features of Clifford algebra to establish this connection. Then, there exists the following homomorphism because Clifford algebra $Cl_{1,2,0}$ is immersed in Clifford algebra $Cl_{3,1}$.

**Lemma 5** There is an algebra homomorphism between $Cl_{1,2,0}$ and $Cl_{3,1}$.

**Proof.** According to the universal property of the Clifford algebra, an algebra homomorphism on the Clifford algebra is found by using a linear map, which is defined between vector space and algebra. Then, the existence of such an algebra homomorphism is easily demonstrated by regarding this property.

Let us consider the orthonormal basis $\{e_1, e_2, e_3, e_4\}$ of $\mathbb{R}^{3,1}$ such that $\tilde{Q}(e_4) = -1$ and $\tilde{Q}(e_i) = 1, i = 1, 2, 3$ where $\tilde{Q}$ is a quadratic form for $\mathbb{R}^{3,1}$. Also, let the basis of $\mathbb{R}^{1,2,0}$ is given by $\{e_0, e_1, e_2\}$ such that $Q(e_i) = 1, i = 1, 2$ and $Q(e_0) = 0$. We assume that a map $f$ is defined by $f : \mathbb{R}^{1,2,0} \rightarrow Cl_{3,1}$, $f(v) = v$. This map is linear and it is necessary to provide the condition $f(v)^2 = -Q(v) \cdot 1$. To show that, if we write $v = v_0e_0 + v_1e_1 + v_2e_2$ for $v \in \mathbb{R}^{1,2,0}$, then we have

$$f(v)^2 = \left\{ \frac{1}{\sqrt{2}}v_0(e_3 + e_4) + v_1e_1 + v_2e_2 \right\}\left\{ \frac{1}{\sqrt{2}}v_0(e_3 + e_4) + v_1e_1 + v_2e_2 \right\} = -Q(v)$$

In this situation, the map $f$ expands to $\tilde{f} : Cl_{1,2,0} \rightarrow Cl_{3,1}$ and so, $\tilde{f}$ is the algebra homomorphism. ■

It is defined by

$$\tilde{i} : Cl_{1,2,0} \rightarrow Cl_{3,1} \tag{9}$$

$$e_1 \mapsto e_1$$

$$e_2 \mapsto e_2$$

$$e_0 \mapsto \frac{1}{\sqrt{2}}(e_3 + e_4)$$

where $e_1, e_2$ and $e_0$ are the spacelike and lightlike basis vectors on $\mathbb{R}^{1,2,0}$, respectively. Also, $e_1, e_2, e_3$ and $e_4$ are the spacelike and timelike basis vectors on $\mathbb{R}^{3,1}$, respectively.
Normally, when we pass from the degenerate Clifford algebra to the nondegenerate Clifford algebra, the degenerate vectors is written as nondegenerate, such as \( e_0 \rightarrow \frac{1}{\sqrt{2}} (e_3 + e_4) \). But, we write \( e_0 \) for shortness.

This map, which is defined between the Clifford algebras, could be restricted to the spin groups. Then, we obtain the following diagram

\[
\begin{array}{ccc}
Spin (1, 2, 0) & \xrightarrow{i} & Spin (3, 1) \\
\downarrow \varphi & \downarrow \tilde{\varphi} & \\
SO (1, 2, 0) & \xrightarrow{i} & SO (3, 1)
\end{array}
\]

which is commutative for adjoint maps \( \varphi \) and \( \tilde{\varphi} \). Also, using the relationship between \( SO (1, 2, 0) \) and \( SO (3, 1) \), the principal bundle for \( \tilde{M} \) is constituted by the principal bundle \( Prin_{SO (3, 1)} \tilde{M} \). From the principal bundle for the group \( SO (3, 1) \) on manifold \( \tilde{M} \), we write a map \( \pi : Prin_{SO (3, 1)} \tilde{M} \rightarrow \tilde{M} \) and so, there exists a diffeomorphism \( \vartheta : \pi^{-1} (\tilde{U}) \rightarrow U \times SO (3, 1) \) for an open set \( \tilde{U} \subseteq \tilde{M} \). Then, when this principal bundle is restricted to hypersurface \( M \), we have \( \pi : Prin_{SO (3, 1)} \tilde{M} \big|_M \rightarrow M \) and this map is subjective. Also, we get

\[
\vartheta : \pi^{-1} (\tilde{U}) \big|_M \rightarrow U \times SO (3, 1) \big|_M = \left( \tilde{U} \cap M \right) \times SO (1, 2, 0)
\]

Using the map \( \vartheta \), we need to show that the principal bundle with spin group on \( M \) occurs if we restrict the principal bundle with spin group on \( \tilde{M} \) to \( M \). For this, according to the definition of pullback of principal bundle in [16], we write the following commutative diagram since there exist the continuous map \( \vartheta \) restricted to \( M \) is a diffeomorphism. If \( Prin_{SO (1, 2, 0)} M \) is the principal bundle for \( M \), then the relationship between the principal bundle \( Prin_{SO (3, 1)} \tilde{M} \big|_M \) and the principal bundle \( Prin_{SO (1, 2, 0)} M \) could be established. For this, let us define a continuous map

\[
\xi : Prin_{SO (1, 2, 0)} M \rightarrow Prin_{SO (3, 1)} \tilde{M} \big|_M.
\]

Using the map \( \xi \), we need to show that the principal bundle with spin group on \( M \) occurs if we restrict the principal bundle with spin group on \( \tilde{M} \) to \( M \). For this, according to the definition of pullback of principal bundle in [16], we write the following commutative diagram since there exist the continuous map \( \xi \) and a principal bundle. So, we have

\[
\xi^* \left( Prin_{Spin (3, 1)} \tilde{M} \big|_M \right) = P_{Spin (1, 2, 0)} M \xrightarrow{\pi^*} Prin_{Spin (3, 1)} \tilde{M} \big|_M
\]

where \( \pi^\prime \) defines principal bundle \( Prin_{Spin (1, 2, 0)} M \), \( \pi \) defines principal bundle \( Prin_{Spin (3, 1)} \tilde{M} \big|_M \), and \( \xi^* : Prin_{Spin (3, 1)} \tilde{M} \big|_M \rightarrow Prin_{Spin (1, 2, 0)} M \). So, there
exists the principal bundle $Prin_{Spin(3,1)}\tilde{M}|_M$ such that
\[
\xi^* \left( Prin_{Spin(3,1)}\tilde{M}|_M \right) = \{(x,y) : \xi(x) = \pi(y), x \in Prin_{SO(1,2,0)}M, \quad y \in Prin_{Spin(3,1)}\tilde{M}|_M \}. 
\] (14)

It is seen that the bundle formed by the restriction of $\tilde{M}$ to $M$ is a principal bundle with the spin group. Thus, we show that the restrictions of the principal bundles with special or spin groups of $\tilde{M}$ to $M$ have similar properties with $\tilde{M}$. Then, using these results, we could define a spin structure on the restriction of $\tilde{M}$ to $M$. Accordingly, using the spinor bundle on $\tilde{M}$, we could define a spin structure on the restriction of $\tilde{M}$ to $M$.

So, we obtain the homomorphism for the group $\Aut (\mathbb{R}^4)$ by
\[
\phi_r = [\bar{s}, \alpha_r] \quad \text{where} \quad \bar{s} \in Prin_{Spin(3,1)}\tilde{M} \quad \text{and} \quad \alpha_r \in \mathbb{R}^4. 
\] (15)

when we restrict the spinor field $\phi_r$ to $M$. Since $\sim$ could not be an equivalence relation for $M$, the equivalence relation $\sim$ should be revised. So, it will be $\bar{s} \in Prin_{Spin(3,1)}\tilde{M}|_M$ when $\bar{s}$ is restricted to $\tilde{U} \cap M$. Also, $u \in Spin(3,1)$ should be the element of $Spin(1,2,0)$. In that case, if we use the map $i$, then we restate the homomorphism $\bar{\rho}_{3,1}$ for the spin group on $M$. It is given by
\[
Spin(1,2,0) \xrightarrow{i} Spin(3,1) \\
\bar{\rho}_{3,1} \circ i \quad \Downarrow \quad \bar{\rho}_{3,1} \\
\Aut (\mathbb{R}^4) 
\] (17)

So, we obtain the homomorphism for the group $Spin(1,2,0)$. Then, we write equivalence relation, which gives the spinor bundle on $M$. Thus, we have
\[
[\bar{s}|_{\tilde{U}\cap M}, \alpha_r |_{\tilde{U}\cap M}] \sim [\bar{s}|_{\tilde{U}\cap M}, (\bar{\rho}_{3,1} \circ i)(u^{-1}) \alpha_r |_{\tilde{U}\cap M}] 
\] (18)

for $u \in Spin(1,2,0)$. From there, we find
\[
\tilde{S}M|_M = Prin_{Spin^{+}(1,2,0)}M \times_{\bar{\rho}_{3,1} \circ i} \Delta M 
\] (19)
where $Spin^+ (1, 2, 0)$ is connected component of $Spin (1, 2, 0)$ and $\Delta_M$ is a module of representation $\tilde{\rho}_{3,1} \circ i$.

Now, let us express the Clifford multiplication for the hypersurface. So, reduced Clifford multiplication from $\tilde{M}$ to $M$ is obtained as following since $\rho_{3,1} \circ i$ provides the Clifford multiplication for $M$. It is defined by

$$\rho_{3,1} : Cl_{3,1} \rightarrow Hom (\mathbb{R}^4, \mathbb{R}^4)$$

$$\phi \mapsto \tilde{\rho}_{3,1} (\phi) (v) \equiv \phi \cdot v$$

for $\phi \in Cl_{3,1}$ and $v \in \mathbb{R}^4$. So, we obtain that

$$Cl_{1,2,0} \overset{i}{\rightarrow} Cl_{3,1} \overset{\tilde{\rho}_{3,1}}{\rightarrow} Hom (\mathbb{R}^4, \mathbb{R}^4)$$

$$\phi \mapsto \phi \mapsto \phi \cdot v$$

### 4 Spinorial Gauss formula for lightlike hypersurfaces

Let $\tilde{M}$ be a 4-dimensional Lorentzian spin manifold and $M$ be a lightlike hypersurface in $\tilde{M}$. Then, a complementary vector bundle $S (TM)$ of $Rad TM$ in $TM$ is called a screen distribution on $M$ and there exists $TM = Rad TM \perp S (TM)$. Moreover, we have the following decompositions.

$$T \tilde{M} = S (TM) \perp (Rad TM \oplus ltr (TM)) = TM \oplus ltr (TM)$$

where $ltr (TM)$ is a complementary vector bundle to $TM$ in $T \tilde{M}$ and it is called lightlike transversal bundle of $M$.

Let the locally orthonormal frame of the tangent bundle $T \tilde{M}$ be $\{ s_1, s_2, s_3, s_4 \}$ such that $\{ s_1, s_2, s_3 \}$ and $\{ s_4 \}$ are spacelike and timelike vectors according to $\tilde{g}$, respectively. Considering these vectors, it is possible to construct lightlike vectors. We write that $s_0 = \frac{1}{\sqrt{2}} (s_3 + s_4)$ and $N = \frac{1}{\sqrt{2}} (s_3 - s_4)$ where these vectors satisfy the conditions

$$g (s_0, N) = 1, g (s_0, s_i) = g (N, s_i) = 0, i = 1, 2.$$  \hspace{1cm} (23)

Thus, the quasi orthonormal basis of $\tilde{M}$ is given by $\{ N, s_0, s_1, s_2 \}$ and the basis of 3-dimensional lightlike subbundle $TM$ of $T \tilde{M}$ is $\{ s_0, s_1, s_2 \}$ and $N$ is a normal vector field for the hypersurface $M$.

We write Gauss-Weingarten formula for lightlike case to obtain the induced geometric objects. Let $\tilde{\nabla}$ be Levi Civita connection on $\tilde{M}$ and $\nabla$ be a linear connection on $M$. So, we have

$$\tilde{\nabla}_X Y = \nabla_X Y + h (X, Y) N,$$

$$\tilde{\nabla}_X N = - A_N (X) + \nabla_X N$$

8
for $X, Y \in \Gamma(TM)$ where $\nabla_X Y, A_N (X) \in \Gamma(TM)$ and $N, \nabla^i X N \in ltr(TM)$. Moreover, $h$ is a symmetric bilinear form on $\Gamma(TM)$, $A_N$ is a shape operator of $M$ in $\tilde{M}$ and $\nabla^i$ is a linear connection on $ltr(TM)$ [3].

Also, if $f'$ is defined by

$$f': TM \to T^*M$$

$$X \mapsto f(X)(Y) = g(PX, Y) + \eta(X)\eta(Y)$$

then it is an isomorphism where $P$ is projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ and $\eta$ is 1-form defined by $\eta(X) = \tilde{g}(N, X)$ [1].

Now, we get the spinorial Gauss formula for the lightlike hypersurface with these informations. Let $SM, SM$ be spinor bundles of $\tilde{M}, M$, and spinorial connections on the spinor bundles $S\tilde{M}, SM$ are denoted by $\tilde{\nabla}^s, \nabla^s$, respectively. The connection on the spinor bundle for causal structure $(3, 1)$ is given by

$$\tilde{\nabla}^s_X \Phi = X(\Phi) + \frac{1}{2} \sum_{i<j=1}^4 \varepsilon_{ij} \tilde{g} \left( \tilde{\nabla}^s_X s_i, s_j \right) s_i \cdot s_j \cdot \Phi$$

for $X \in \Gamma(TM), \Phi \in \Gamma(S\tilde{M})$ [2].

**Theorem 6** Let $\tilde{M}$ be 4-dimensional Lorentzian spin manifold with a metric tensor $\tilde{g} = (+, +, +, -)$ and connection on the spinor bundle $SM$ be $\tilde{\nabla}^s$. We assume that $(M, g)$ is a 3-dimensional lightlike hypersurface of $(\tilde{M}, \tilde{g})$ and $\nabla^s$ is the connection on spinor bundle $SM$. The relation between these connections is given by

$$\tilde{\nabla}^s_X \varphi = \nabla^s_X \varphi + \frac{1}{2} \sum_{i=2}^4 h(X, s_i) s_i \cdot N \cdot \varphi$$

for $X \in \Gamma(TM), \varphi \in \Gamma(SM)$. Here, $N$ is a normal vector field on $M$, $s_i$ is a locally orthonormal basis vector field on $TM$ and $h$ is a symmetric bilinear form.

**Proof.** If we write more clearly [26], we have

$$\tilde{\nabla}^s_X \Phi = X(\Phi) + \frac{1}{2} \left( \tilde{g} \left( \tilde{\nabla}^s_X s_1, s_2 \right) s_1 \cdot s_2 \cdot \Phi + \tilde{g} \left( \tilde{\nabla}^s_X s_1, s_3 \right) s_1 \cdot s_3 \cdot \Phi - \tilde{g} \left( \tilde{\nabla}^s_X s_1, s_4 \right) s_1 \cdot s_4 \cdot \Phi \right)$$

$$- \tilde{g} \left( \tilde{\nabla}^s_X s_2, s_3 \right) s_2 \cdot s_3 \cdot \Phi + \tilde{g} \left( \tilde{\nabla}^s_X s_2, s_4 \right) s_2 \cdot s_4 \cdot \Phi - \tilde{g} \left( \tilde{\nabla}^s_X s_3, s_4 \right) s_3 \cdot s_4 \cdot \Phi$$

for $X \in \Gamma(TM), \varphi \in \Gamma(SM)$ where $s_i$ is a locally orthonormal basis vector field on $T\tilde{M}$ and $\tilde{\nabla}^s$ is the connection on the spinor bundle $S\tilde{M}$. If we use Gauss
So, we get
\[ \nabla X \Phi \big|_M = X(\Phi) \big|_M + \frac{1}{2} g(\nabla X s_1, s_2) s_1 \cdot s_2 \cdot \Phi \big|_M \]
also, we find that we have
\[ \tilde{\nabla}^s_X \Phi \big|_M = X(\Phi) \big|_M + \frac{1}{2} g(\nabla X s_1, s_2) s_1 \cdot s_2 \cdot \Phi \big|_M \]
\[ + \frac{1}{4} \left[ \{ g(\nabla X s_1, N) + h(X, s_1) \} s_1 \cdot (s_0 + N) \cdot \Phi \big|_M \right] \]
\[ - \{ [h(X, s_1) - g(\nabla X s_1, N)] s_1 \cdot (s_0 - N) \cdot \Phi \big|_M \} \]
\[ + \{ [g(\nabla X s_2, N) + h(X, s_2)] s_2 \cdot (s_0 + N) \cdot \Phi \big|_M \} \]
\[ - \{ [-g(\nabla X s_2, N) + h(X, s_2)] s_2 \cdot (s_0 - N) \cdot \Phi \big|_M \} \]
Also, we find that
\[ \tilde{\nabla}^s_X (\Phi \big|_M) = X(\Phi) \big|_M + \frac{1}{2} \left[ g(\nabla X s_1, s_2) s_1 \cdot s_2 \cdot \Phi \big|_M + g(\nabla X s_1, N) s_1 \cdot s_0 \cdot \Phi \big|_M \right] \]
\[ + h(X, s_1) s_1 \cdot N \cdot \Phi \big|_M + h(X, s_2) s_2 \cdot N \cdot \Phi \big|_M \]
\[ + g(\nabla X s_2, N) s_2 \cdot s_0 \cdot \Phi \big|_M \]
from \[ X(\Phi) \big|_M = X(\Phi) \big|_M \] and \[ (\tilde{\nabla}^s_X \Phi) \big|_M = \tilde{\nabla}^s_X (\Phi \big|_M) \]. If we show as \[ \Phi \big|_M = \varphi \], we obtain
\[ \tilde{\nabla}^s_X \varphi = \nabla_X \varphi + \frac{1}{2} \left[ g(\nabla X s_1, s_2) s_1 \cdot s_2 \cdot \varphi + g(\nabla X s_1, N) s_1 \cdot s_0 \cdot \varphi \right] \]
\[ + h(X, s_1) s_1 \cdot N \cdot \varphi + h(X, s_2) s_2 \cdot N \cdot \varphi \]
\[ + g(\nabla X s_2, N) s_2 \cdot s_0 \cdot \varphi \].

So, we get
\[ \tilde{\nabla}^s_X \varphi = \nabla_X \varphi + \frac{1}{2} \sum_{i=1}^{2} h(X, s_i) s_i \cdot N \cdot \varphi \]
since the covariant derivative on spinor bundle \( SM \) is
\[ \nabla_X \varphi = X(\varphi) + \frac{1}{2} \left[ -g(\nabla X s_1, N) s_0 \cdot s_1 \cdot \varphi - g(\nabla X s_2, N) s_0 \cdot s_2 \cdot \varphi \right] \]
\[ + g(\nabla X s_1, s_2) s_1 \cdot s_2 \cdot \varphi \].

\[ \square \] So, the obtained this formula is called spinorial Gauss formula for lightlike hypersurfaces.

**Theorem 7** Let \( \tilde{M} \) be a 4-dimensional Lorentzian spin manifold whose Riemannian curvature is denoted by \( R \) and \( M \) be a hypersurface of \( M \) whose
Riemannian curvature associated with spinor bundle is denoted by \( R \). The relationship between their Riemannian curvatures is as the following.

\[
\tilde{R}(X,Y)\varphi = R(X,Y)\varphi - g(R(X,Y)s_0,N)s_0 \cdot N \cdot \varphi
\]

\[
+ [g(\nabla_Xs_0,A_N(Y)) - g(\nabla_Ys_0,A_N(X))]s_0 \cdot N \cdot \varphi
\]

\[
+ [g(\nabla_Xs_0,N)\nabla_Ys_0 - g(\nabla_Ys_0,N)\nabla_Xs_0]N \cdot \varphi
\]

\[
+ \left[g(\nabla_Xs_0,N)\nabla_YN - g(\nabla_Ys_0,N)\nabla_XN\right]s_0 \cdot \varphi.
\]

**Proof.** We assume that \( s_i \) is locally frame field for \( U \subset M \), \( N \) is a normal vector field on \( M \) and \( h \) is symmetric bilinear form, which is coefficient of the second fundamental form. So, for \( X, Y \in \Gamma(TM) \) and \( \varphi \in \Gamma(SM) \), we have

\[
\tilde{R}(X,Y)\varphi = \nabla_X^* (\nabla_Y^* \varphi) + \frac{1}{2} \sum_{i=1}^{2} h(X,s_i) s_i \cdot N \cdot \nabla_Y^* \varphi
\]

\[
+ \frac{1}{2} \sum_{i=1}^{2} g(\nabla_Xh)(Y,s_i) s_i \cdot N \cdot \varphi + h(\nabla_XY,s_i) s_i \cdot N \cdot \varphi
\]

\[
+ h(Y,\nabla_Xs_i) s_i \cdot N \cdot \varphi + h(Y,s_i) \nabla_Xs_i \cdot N \cdot \varphi
\]

\[
+ h(Y,s_i) s_i \cdot \nabla_XN \cdot \varphi
\]

If we use Gauss-Weingarten equations, then \( \nabla_XY - \nabla_YX = \nabla_XY - \nabla_YX \) and \( N \cdot N = 0 \). Thus, we obtain

\[
\tilde{R}(X,Y)\varphi = R(X,Y)\varphi + \frac{1}{2} \sum_{i=1}^{2} \left[X(h(Y,s_i)) - Y(h(X,s_i))\right] s_i \cdot N \cdot \varphi
\]

\[
+ \frac{1}{2} \sum_{i=1}^{2} \left[-h([X,Y],s_i) s_i \cdot N \cdot \varphi + h(Y,s_i) \nabla_Xs_i \cdot N \cdot \varphi
\]

\[
- h(X,s_i) \nabla_Ys_i \cdot N \cdot \varphi
\]

\[
- h(Y,s_i) s_i \cdot \nabla_YN \cdot \varphi.
\]

\[\blacksquare\]
Theorem 8 Let \( \tilde{M} \) be 4-dimensional Lorentzian spin manifold and \( M \) be a lightlike hypersurface. If \( M \) is a totally geodesic, the spinor covariant derivative of hypersurface \( M \) and manifold \( \tilde{M} \) are the same.

Theorem 9 Let \( \tilde{M} \) be 4-dimensional Lorentzian spin manifold and \( M \) be a lightlike hypersurface. If \( M \) is a totally umbilical, there exists the relation

\[
\begin{align*}
    s_k &= s_0 \text{ ise, } \bar{\nabla}_{s_k}^s \varphi = \nabla_{s_k}^s \varphi \\
    s_k &\neq s_0 \text{ ise, } \bar{\nabla}_{s_k}^s \varphi = \nabla_{s_k}^s \varphi + \frac{1}{2} \varepsilon_{kl} c_k s_k \cdot N \cdot \varphi
\end{align*}
\]

between the spinor covariant derivative of \( M \) and \( \tilde{M} \). Here, \( c_k \) is constant, \( s_i \) is a locally frame field for open set \( U \subset M \) and \( N \) is the normal vector field on \( M \).

5 Dirac Operator for Lightlike Hypersurfaces

Theorem 10 Let \( (\tilde{M}, \tilde{g}) \) be 4-dimensional Lorentzian spin manifold and \( M \) be a lightlike hypersurface of \( \tilde{M} \). Dirac operator reduced by \( \tilde{M} \) on \( M \) is given as

\[
D = \sum_{i=1}^{2} s_i \cdot \nabla_{s_i}^s + s_0 \cdot \nabla_{s_0}^s
\]

where \( s_i \) is locally frame field for \( U \subset M \), \( s_0 \) is lightlike vector field on \( TM|_U \), \( N \) is a normal vector field on \( M \) and \( \nabla^s \) is connection on spinor bundle \( SM \).

Proof. Dirac operator is defined by

\[
D : \Gamma(SM) \xrightarrow{\nabla^s} \Gamma(T^*M \otimes SM) \xrightarrow{\mu} \Gamma(TM \otimes SM) \xrightarrow{\eta} \Gamma(SM)
\]

where \( \nabla^s \) is connection on spinor bundle \( SM \), \( \mu \) is Clifford multiplication and \( \eta \) is a map \( \eta : \Gamma(T^*M \otimes S) \rightarrow \Gamma(TM \otimes S) \). It should be an isomorphism to pass between these maps. In this situation, if \( f' \) is defined by

\[
f' : TM \rightarrow T^*M
\]

\[
X \rightarrow f'(X)(Y) = g(\tilde{g}(X,Y) + \eta(X) \eta(Y)
\]

then it is isomorphism. According to [23], \( P \) is projection morphism of \( \Gamma(TM) \) on \( \Gamma(S(TM)) \) and \( \eta \) is 1-form defined by \( \eta(X) = \tilde{g}(N,X) \). Let \( \{s_0, s_1, s_2\} \) be a locally basis field on \( U \subset M \). Then, \( f' \) is given by

\[
f' : TM \rightarrow T^*M
\]

\[
s_i \mapsto w^i
\]

for the basis vector fields where \( \{w^i\} \) is dual basis of \( \{s_i\} \) for \( i = 0, 1, 2 \). So, we write \( f : T^*M \rightarrow TM \) since \( f' \) is isomorphism. Also, the condition \( w^i(s_j) = \delta_{ij} \) should be satisfied.
• For \(i = 1, 2\), we find

\[
\begin{align*}
    w^i(s_i) &= g(P(f(w^i)), s_i) + \eta(f(w^i)) \eta(s_i) \\
    &= g(f(w^i), s_i) + \bar{g}(N, f(w^i)) \bar{g}(N, s_i) \\
    &= g(f(w^i), s_i)
\end{align*}
\]

So, we have \(f(w^i) = \varepsilon_i s_i\) from \(g(f(w^i), s_i) = 1\).

• For \(i = 0\), we obtain

\[
\begin{align*}
    w^0(s_0) &= g(P(f(w^0)), s_0) + \eta(f(w^0)) \eta(s_0) \\
    &= g(f(w^0), s_0) + \bar{g}(N, f(w^0)) \bar{g}(N, s_0) \\
    &= \bar{g}(N, f(w^0))
\end{align*}
\]

So, we have \(f(w^0) = s_0\).

Then, Dirac operator is given by

\[
D = \mu \circ f \circ \nabla^s_{s_i}
\]

\[
= \mu \circ f(w^i) \otimes \nabla^s_{s_i}
\]

\[
= \mu(f(w^i) \otimes \nabla^s_{s_i})
\]

\[
= \sum_{i=1}^{2} \varepsilon_i s_i \cdot \nabla^s_{s_i} + s_0 \cdot \nabla^s_{s_0}
\]

\[
= \sum_{i=1}^{2} s_i \cdot \nabla^s_{s_i} + s_0 \cdot \nabla^s_{s_0}.
\]

\[\blacksquare\]

**Theorem 11** Let \(\tilde{M}\) be a 4-dimensional Lorentzian spin manifold whose Dirac operator is denoted by \(\tilde{D}\) and \(M\) be a hypersurface of \(\tilde{M}\) whose Dirac operator is denoted by \(D\). The relationship between their Dirac operators is

\[D\varphi = \tilde{D}\varphi - s_0 \cdot \nabla^s_{N} \varphi + (s_0 - N) \cdot \nabla^s_{s_0} \varphi - \sum_{k=1}^{2} h(s_k, s_0) \cdot s_k \cdot N \cdot \varphi + HN \cdot \varphi (31)\]

for any \(\varphi \in \Gamma(SM)\) where \(s_i\) is locally frame field for \(U \subset M\), \(H\) is mean curvature and \(\nabla^s\) is connection on spinor bundle \(\tilde{S}M\).
Proof. From Dirac operator and spinorial Gauss formula, we obtain

\[
D\varphi = \sum_{i=1}^{2} s_{i} \cdot \nabla_{s_{i}}^{s} \varphi + s_{0} \cdot \nabla_{s_{0}}^{s} \varphi
\]

\[
= \sum_{i=1}^{2} s_{i} \cdot \left[ \nabla_{s_{i}}^{s} \varphi - \frac{1}{2} \sum_{k=1}^{2} \left[ h(s_{k}, s_{i}) s_{k} \cdot N \cdot \varphi \right] \right]
+ s_{0} \cdot \left[ \nabla_{s_{0}}^{s} \varphi - \frac{1}{2} \sum_{k=1}^{2} \left[ h(s_{k}, s_{0}) s_{k} \cdot N \cdot \varphi \right] \right]
\]

\[
= \sum_{i=1}^{2} s_{i} \cdot \nabla_{s_{i}}^{s} \varphi - \frac{1}{2} \sum_{i,k=1}^{2} h(s_{k}, s_{i}) s_{i} \cdot s_{k} \cdot N \cdot \varphi + s_{0} \cdot \nabla_{s_{0}}^{s} \varphi
\]

\[
- \frac{1}{2} \sum_{k=1}^{2} h(s_{k}, s_{0}) s_{0} \cdot s_{k} \cdot N \cdot \varphi
\]

for \( \varphi \in \Gamma(SM) \). If we add and substract \( s_{3} \cdot \nabla_{s_{3}}^{s} \varphi - s_{4} \cdot \nabla_{s_{4}}^{s} \varphi \) to this equation, then we find

\[
D\varphi = \sum_{i=1}^{2} s_{i} \cdot \nabla_{s_{i}}^{s} \varphi + s_{0} \cdot \nabla_{s_{0}}^{s} \varphi + N \cdot \nabla_{s_{0}}^{s} \varphi - s_{0} \cdot \nabla_{s_{0}}^{s} \varphi - N \cdot \nabla_{s_{0}}^{s} \varphi
\]

\[
- \frac{1}{2} \sum_{i,k=1}^{2} h(s_{k}, s_{i}) s_{i} \cdot s_{k} \cdot N \cdot \varphi + s_{0} \cdot \nabla_{s_{0}}^{s} \varphi - \frac{1}{2} \sum_{k=1}^{2} h(s_{k}, s_{0}) s_{0} \cdot s_{k} \cdot N \cdot \varphi.
\]

Using \( H = \frac{1}{n} \sum h(e_{i}, e_{i}) \) and \( s_{i}s_{k} = -s_{k}s_{i} \), we have

\[
D\varphi = \widetilde{D}\varphi - s_{0} \cdot \nabla_{N}^{s} \varphi + (s_{0} - N) \cdot \nabla_{s_{0}}^{s} \varphi - \frac{1}{2} \sum_{k=1}^{2} h(s_{k}, s_{0}) s_{0} \cdot s_{k} \cdot N \cdot \varphi + HN \cdot \varphi.
\]

Corollary 12 Let \( \widetilde{M} \) be a 4-dimensional Lorentzian spin manifold and \( M \) be lightlike hypersurface of \( \widetilde{M} \). If \( M \) is a minimal hypersurface, the relation between Dirac operators of \( M \) and \( \widetilde{M} \) is

\[
D\varphi = \widetilde{D}\varphi - s_{0} \cdot \nabla_{N}^{s} \varphi + (s_{0} - N) \cdot \nabla_{s_{0}}^{s} \varphi - \frac{1}{2} \sum_{k=1}^{2} h(s_{k}, s_{0}) s_{0} \cdot s_{k} \cdot N \cdot \varphi + HN \cdot \varphi.
\]

for \( \varphi \in \Gamma(SM) \).

Example 13 Let \((\mathbb{R}^{3,1}, \widetilde{g})\) be the Minkowski space with signature \((+,+,+,-)\) of the canonical basis \((\partial_{1}, \partial_{2}, \partial_{3}, \partial_{4})\). \((M, g)\) is the lightlike hypersurface given by

\[
M = \left\{ (-x, y - z, -y - z, -x) \in \mathbb{R}^{4}_{1} : x, y, z \in \mathbb{R} \right\}
\]
Then, \( \text{Rad } TM \) and \( \text{ltr } (TM) \) are defined by

\[
\text{Rad } TM = Sp \{ s_0 = -\partial_0 - \partial_3 \}
\]

\[
\text{ltr } (TM) = Sp \{ N = -\partial_0 + \partial_3 \}
\]

So, the screen distribution \( S(TM) \) is spanned by

\[
s_1 = \partial_1 - \partial_2, \quad s_2 = -\partial_1 - \partial_2
\]

In this situation, we obtain the vector fields \( N, s_0, s_1, s_2 \) satisfying the following conditions.

\[
g(s_0, N) = 1, \quad g(s_0, s_i) = g(N, s_i) = 0, \quad i = 1, 2
\]

Then, we obtain that for \( i, j = 0, 1, 2 \),

\[
h(s_i, s_j) = \tilde{g}(\tilde{\nabla}_{s_i} s_j, s_0) = 0
\]

Thus, relation between the spinorial covariant derivatives

\[
\tilde{\nabla}_{s_i}^s \varphi = \nabla_{s_i}^s \varphi + \frac{1}{2} \sum_{j=1}^2 h(s_i, s_j) s_j \cdot N \cdot \varphi \implies \tilde{\nabla}_{s_i}^s \varphi = \nabla_{s_i}^s \varphi
\]

for \( \varphi \in \Gamma (SM) \) and \( s_i, i = 0, 1, 2 \).

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