STOCHASTIC FLOWS OF SDES DRIVEN BY LÉVY PROCESSES WITH IRREGULAR DRIFTS.

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Abstract. In this paper, we investigate the strong solutions to SDE driven by a stable-like Lévy process with Sobolev drift. We show that the singular SDE has a unique strong solution for every starting point and these strong solutions form a $C^1$-stochastic flow. Moreover, we also obtain the Malliavin differentiability of the strong solutions, which extends the main result in [10]. As an application, we show a Davie’s type uniqueness result for the related random ODE.

Keywords: Stochastic flow, Lévy process, Zvonkin’s transform, Malliavin differentiable

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1. Introduction

Consider the following SDE in $\mathbb{R}^d$:

$$dX_t(x) = b(X_t(x))dt + \sigma(X_t(x))dZ_t, \quad X_0(x) = x \in \mathbb{R}^d,$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ are two Borel measurable functions, and $Z$ is a Lévy process with Lévy measure $\nu$. The characteristic exponent $\psi(\xi)$ of $Z_t$ is given by

$$\psi(\xi) := -\log(\mathbb{E}e^{i\xi \cdot Z_1}) = -\int_{\mathbb{R}^d} (e^{i\xi \cdot z} - 1 - i\xi \cdot z 1_{B_1})\nu(dz).$$

In this paper, we always assume that $\nu$ is symmetric in order to make the paper self-contained and the statement simple. By symmetry, $\psi$ can be rewritten as

$$\psi(\xi) = \int_{\mathbb{R}^d} (1 - \cos(z \cdot \xi))\nu(dz).$$

To state our condition on Lévy measure $\nu$, for $\alpha \in (0, 2)$, denote by $\mathbb{L}^{(\alpha)}_{\text{non}}$ the space of all non-degenerate $\alpha$-stable measures $\nu^{(\alpha)}$, that is,

$$\nu^{(\alpha)}(A) = \int_0^\infty \left( \int_{S^{d-1}} \frac{1_A(r\theta)\Sigma(d\theta)}{r^{1+\alpha}} \right) dr, \quad A \in \mathcal{B}(\mathbb{R}^d),$$

where $\Sigma$ is a finite measure over the unit sphere $S^{d-1}$ in $\mathbb{R}^d$ with

$$\inf_{\theta_0 \in S^{d-1}} \int_{S^{d-1}} |\theta_0 \cdot \theta| \Sigma(d\theta) > 0.$$

For $R > 0$, let $B_R := \{ x \in \mathbb{R}^d : |x| < R \}$ be the ball in $\mathbb{R}^d$. We assume

(H) There are $\nu_1, \nu_2 \in \mathbb{L}^{(\alpha)}_{\text{non}}$ and $\rho \in (0, 1)$ such that

$$\nu_1(A) \leq \nu(A) \leq \nu_2(A) \quad \text{for } A \subseteq B_\rho,$$

Research of Guohuan is supported by the German Research Foundation (DFG) through the Collaborative Research Centre(CRC) 1283 Taming uncertainty and profiting from randomness and low regularity in analysis, stochastics and their applications.
\((H_2)\) There are positive constants \(\beta, p, \Lambda\) such that
\[
\beta \in (1 - \frac{q}{2}, 1), \quad p \in \left(\frac{d}{\alpha/2 + \beta - 1}, \infty\right) \quad \text{and} \quad b \in W^\beta_p(\mathbb{R}^d).
\]  
(1.4)
\[
\sigma \in C^1_b, \quad \Lambda^{-1}|\xi| \leq |\sigma(x)\xi| \leq \Lambda|\xi|.
\]  
(1.5)
\((H_3)\) \(\sigma \in C^{1+\delta}_b\) for some \(\delta \in (0, 1)\). \(\nu\) has a compact support and
\[
\text{supp } \nu \subseteq B_{r_0}, \quad r_0 < \|\nabla \sigma\|^{-1}_\infty.
\]  
(1.6)

Our main results are following

**Theorem 1.1.**  
(1) Under assumptions \((H_1)\) and \((H_2)\), there is a unique strong solution to equation \((1.1)\). Moreover, if the jumping size of \(Z_t\) is bounded, then for each \(t \geq 0\), the strong solution \(X_t(x, \omega)\) to \((1.1)\) is Malliavin differentiable.

(2) Under assumptions \((H_1), (H_2)\) and \((H_3)\), for almost surely \(\omega \in \Omega\), the random map \((t, x) \mapsto X_t(x, \omega)\) is a \(C^1\)-stochastic flow.

**Remark 1.2.** Before going on, let us point out that without condition \((1.6)\), the stochastic flow may not exist even if the coefficients are smooth. Just consider the following simple example: \(d = 1\), \(Z_t\) is a standard Poisson process, \(b = 0\), \(\sigma(x) = -x\), \(T\) is the first jumping time of \(Z_t\). It’s easy to see, \(X_T(0) = X_T(1) = 0\).

We will also show the following corollary in Section 5:

**Corollary 1.3.** Suppose \(\nu\) satisfies \((H_1)\), \(\sigma = \mathbb{I}\), \(b \in W^\beta_p\) with \(\beta \in (1 - \frac{q}{2}, 1)\) and \(p \in \left(\frac{d}{\alpha/2 + \beta - 1}, \infty\right)\), for any \(x \in \mathbb{R}^d\) and almost every \(\omega\), the following ODE:
\[
\frac{d\theta_t(\omega)}{dt} = b^\nu_t(\theta_t(\omega)), \quad \theta_0 = x,
\]  
(1.7)
has a unique solution, where \(b^\nu_t(x) = b(x + Z_t(\omega))\).

The stochastic differential equations driven by Brownian motion with singular coefficients are important and naturally arise in physics(see [12] and the references therein). It was proved by Zvonkin [31] in 1970’s that if \(d = 1, \sigma = 1\) and \(b\) is bounded, then \((1.1)\) has a unique strong solution. And later, Veretennikov [26] extended the similar result for \(d \geq 1\). Using Girsanovs transformation and results from PDEs, Krylov and Röckner [12] obtained the existence and uniqueness of strong solutions to \((1.1)\) when \(\sigma\) is the identity matrix and \(b\) satisfies \(\|b\|_{L^q_t L^p_x} < \infty\) with \(\frac{2}{p} + \frac{2}{q} < 1\). One can see also [29] for more delicate results. It should be mentioned that in [15, 14, 17], the authors gave another approach based on Malliavin calculus to study the strong existence. Their method does not rely on a pathwise uniqueness argument and can be used to get the Malliavin differentiability of obtained solutions. And we also need to mention that in [8], Davie proved a remarkable result, it says that if \(b\) is only bounded and measurable, \(W_t\) is a Brownian motion, \(b^\omega_t(x) := b(x + W_t(\omega))\), then the random ODE \(d\theta_t(\omega)/dt = b^\omega_t(\theta_t)\) has a unique solution for almost all \(\omega \in \Omega\). His proof was simplified by Shaposhnikov in [21] by using the flow property of strong solutions of SDE driven by the Brownian motion.

However, things become quite different when \(Z\) is a pure jump Lévy process. For one-dimensional case, Tanaka, Tsuchiya and Watanabe [24] proved that if \(Z\) is a symmetric \(\alpha\)-stable process with \(\alpha \in [1, 2)\), \(\sigma(x) \equiv 1\) and \(b\) is bounded measurable, then pathwise uniqueness holds for SDE \((1.1)\). They further show that if \(\alpha \in (0, 1)\), and even if \(b\) is Hölder continuous, the pathwise uniqueness may fail. For multidimensional case, Priola [19] proved pathwise uniqueness for \((1.1)\) when \(\sigma(x) = \mathbb{I}\), \(Z\) is a non-degenerate symmetric but possibly non-isotropic \(\alpha\)-stable process with \(\alpha \in [1, 2)\) and \(b \in C^\beta(\mathbb{R}^d)\) with \(\beta \in \)
(1 − α/2, 1). This result was extended to drift $b$ in some fractional Sobolev spaces in the subcritical case in Zhang [30] and to more general Lévy processes in the subcritical and critical cases in Priola [20]. In [5], the authors established strong existence and pathwise uniqueness for SDE (1.1) when $σ(x) = I$, $b$ is Hölder continuous and the semigroup of $Z_t$ satisfies some regularity assumptions. It partially answers an open question posted in [20] on the pathwise well-posedness of SDE (1.1) in the supercritical case. However, when $Z$ is a cylindrical $α$-stable process, the result of [5] requires $α > 2/3$. Later, Chen, Zhang and Zhao [7] drop the constraint $α > 2/3$, moreover it is done for the multiplicative noise setting and for a large class of Lévy processes. In [23], Song and Xie extend this method to study singular SDEs driven by Poisson measures. Let us also mention that Haadem and Proske in [10] studied the existence and Malliavin differentiability by the similar approach used in [15, 14]. However, they needed to assume that $Z_t$ is a truncated rotational symmetric $α$-stable process with $α > 1$, $σ = I$ and $b ∈ C^β$ with $β > 2 − α$, which are much stronger than the assumptions here.

In this paper, by using the similar method in [7], as mentioned before, we will show that all the strong solutions from single points are Malliavin differentiable and they form a $C^1$-stochastic flow. Let us now briefly introduce our main approach. As usual, to study the strong well-posedness of SDE (1.1), we shall use Zvonkin’s transform, which requires a deep understanding for the following nonlocal PDE (Resolvent equation):

$$\lambda u - \mathcal{L}u - b \cdot \nabla u = f, \quad (1.8)$$

where

$$\mathcal{L}u(x) := \text{p.v.} \int_{\mathbb{R}^d} (u(x + σ(x)z) - u(x))ν(dz).$$

We mention that when $\mathcal{L}$ is the usual fractional Laplacian $Δ^{α/2} := (-Δ)^{α/2}$ with $α ∈ (0, 2)$, that is, $ν(dz) = |z|^{-d-α}dz$ and $σ = I$ in the above definition, and $b ∈ L^∞([0, T]; C^β)$ with $β ∈ ((1 − α) ∨ 0, 1)$, Silvestre [22] obtained the following a priori interior estimate:

$$\|u\|_{L^∞([0,1];C^{α/2}(B_1))} \leq C \left( \|u\|_{L^∞([0,2]×B_2)} + \|f\|_{L^∞([0,2]×C^β(B_2))} \right),$$

where $B_r := \{x ∈ \mathbb{R}^d : |x| < r\}$. Our approach of studying (1.8) is based on the Littlewood-Paley decomposition and some Bernstein’s type inequalities. As showed in [7], this approach allows us to handle a large class of Lévy’s type operator in a uniform way, in particular, for Lévy’s type operators with singular Lévy measures. However, in [7], the authors worked in the space $B^s_{p,∞}$, this space does not enjoy the localization principle(see Lemma 3.4 below), this makes the usual freezing coefficients method does not work for general nondegenerate $σ ∈ C^1$, so one can not get a global diffeomorphism $Φ$ by using Zvonkin’s transform(see Theorem 3.3 of [7] and the proof of Theorem 1.1 therein). In order to overcome this difficulty, in this paper, we replace the working space $B^s_{p,∞}$ by $B^s_{p,p^*}$, which is coincide with the classic Sobolev-Slobodeckij space $W^s_p$ when $s ∉ \mathbb{N}$. Since localization principle holds for $W^s_p$, due to the classic freezing coefficient method and Zvonkin’s transform, we can get a global $C^1$-diffeomorphism $Φ$ for any nondegenerate $σ ∈ C^1_b$, provided that $b$ satisfies (H₂). This helps us to prove that $X_t(x)$ form a stochastic flow.

This paper is organized as follows: In Section 2, we recall some well-known facts from Littlewood-Paley theory. In Section 3, we study the nonlocal advection equation (1.8) with fraction Sobolev drift $b$, and obtain some apriori estimates in Sobolev spaces. In Section 4, we prove our main theorem by Zvonkin’s transform. In Section 5, we will apply our main result to get a Davies type uniqueness theorem for the related random ODEs.
Finally, we introduce some conventions used throughout this paper: The letter $c$ or $C$ with or without subscripts stands for an unimportant constant, whose value may change in different places. We use $A \asymp B$ to denote that $A$ and $B$ are comparable up to a constant, and use $A \lesssim B$ to denote $A \leq C \cdot B$ for some constant $C$.

2. Preliminary

2.1. Besov space. We first give some definitions about fractional Sobolev space.

**Definition 2.1.** Let $H^s_p := (\mathbb{I} - \Delta)^{-s/2}(L^p)$ be the usual Bessel potential space with norm

$$
\|f\|_{H^s_p} := \|(\mathbb{I} - \Delta)^{s/2}f\|_p \asymp \|f\|_p + \|(-\Delta)^{s/2}f\|_p.
$$

The Sobolev-Slobodeckij semi-norm is defined by

$$
[f]_{\theta,p} := \left(\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^\theta p + d} \, dx \, dy\right)^{\frac{1}{p}}.
$$

Let $s > 0$ be not an integer and set $\theta = s - \lfloor s \rfloor \in (0,1)$. Sobolev-Slobodeckij space $W^s_p$ is defined as

$$
W^s_p := \left\{ f \in W_p^{\lfloor s \rfloor} : \sup_{|\alpha| = \lfloor s \rfloor} [\partial^\alpha f]_{\theta,p} < \infty \right\}, \quad \|f\|_{W^s_p} := \|f\|_{W_p^{\lfloor s \rfloor}} + \sup_{|\alpha| = \lfloor s \rfloor} [\partial^\alpha f]_{\theta,p}.
$$

Suppose $s > 0, \varepsilon > 0, p \geq 1, 0 < s - \frac{d}{p} \notin \mathbb{N}$, then

$$
H_{p}^{s+\varepsilon} \hookrightarrow W^s_p \hookrightarrow H_{p}^{s-\varepsilon}; \quad H_{p}^{s} \hookrightarrow C^{s-\frac{d}{p}}; \quad W^s_p \hookrightarrow C^{s-\frac{d}{p}}. \quad (2.1)
$$

Next we recall some basic facts from the Littlewood-Paley theory. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of all rapidly decreasing functions, and $\mathcal{S}'(\mathbb{R}^d)$ the dual space of $\mathcal{S}(\mathbb{R}^d)$ called Schwartz generalized function (or tempered distribution) space. Given $f \in \mathcal{S}(\mathbb{R}^d)$, let $\mathcal{F} f = \hat{f}$ be the Fourier transform of $f$ defined by

$$
\hat{f}(\xi) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, dx.
$$

For $R, R_1, R_2 \geq 0$ with $R_1 < R_2$, we shall denote

$$
B_R := \{ x \in \mathbb{R}^d : |x| \leq R \}, \quad D_{R_1,R_2} := \{ x \in \mathbb{R}^d : R_1 \leq x \leq R_2 \}.
$$

The following simple fact will be used frequently: Let $f, g \in \mathcal{S}'(\mathbb{R}^d)$ be two tempered distributions with supports in $B_{R_0}$ and $D_{R_1,R_2}$ respectively. Then

$$
\text{supp} f * g \subset D_{(R_1-R_0)\cap,0,R_2+R_0}. \quad (2.2)
$$

Let $\chi : \mathbb{R}^d \to [0,1]$ be a smooth radial function with

$$
\chi(\xi) = 1, \ |\xi| \leq 1, \ \chi(\xi) = 0, \ |\xi| \geq 3/2.
$$

Define

$$
\varphi(\xi) := \chi(\xi) - \chi(2\xi).
$$

It is easy to see that $\varphi \geq 0$ and supp $\varphi \subset B_{3/2} \setminus B_{1/2}$ and

$$
\chi(2\xi) + \sum_{j=0}^k \varphi(2^{-j}\xi) = \chi(2^{-k}\xi) \xrightarrow{k \to \infty} 1. \quad (2.3)
$$

In particular, if $|j - j'| \geq 2$, then

$$
\text{supp} \varphi(2^{-j'}) \cap \text{supp} \varphi(2^{-j'}) = \emptyset.
$$
From now on we shall fix such $\chi$ and $\varphi$, and introduce the following definitions.

**Definition 2.2.** The dyadic block operator $\Delta_j$ is defined by

$$\Delta_j f := \begin{cases} \mathcal{F}^{-1}(\chi(2^j \cdot) \mathcal{F} f), & j = -1, \\ \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} f), & j \geq 0. \end{cases}$$

For $s \in \mathbb{R}$ and $p,q \in [1, \infty]$, the Besov space $B^s_{p,q}$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with

$$\|f\|_{B^s_{p,q}} := \left( \sum_{j \geq -1} 2^{jsq} \|\Delta_j f\|^q_p \right)^{1/q} < \infty;$$

The Triebel-Lizorkin space $F^s_{p,q}$ is defined as the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ with

$$\|f\|_{F^s_{p,q}} := \left( \sum_{j \geq -1} 2^{jsq|\Delta_j f|^q} \right)^{1/q} < \infty.$$

The following two Lemmas can be found in [25].

**Lemma 2.3.** \(\forall s \in \mathbb{R}, p,q \geq 1,\)

$$F^s_{p,2} \subsetneq H^s_p, \quad B^s_{p,p} = F^s_{p,p}.$$ And if $s > 0, s \notin \mathbb{N}$

$$B^s_{\infty,\infty} \subsetneq C^s, \quad B^s_{p,p} \subsetneq W^s_p,$$

where $C^s$ is the usual Hölder space.

Let $h = \mathcal{F}^{-1} \chi$ be the inverse Fourier transform of $\chi$. Define

$$h_{-1}(x) := \mathcal{F}^{-1} \chi(2^d \cdot)(x) = 2^{-d}h(2^{-1}x) \in \mathcal{S}(\mathbb{R}^d),$$

and for $j \geq 0$,

$$h_j(x) := \mathcal{F}^{-1} \varphi(2^{-j} \cdot)(x) = 2^{jd}h(2^j x) - 2^{(j-1)d}h(2^{j-1}x) \in \mathcal{S}(\mathbb{R}^d). \quad (2.4)$$

By definition it is easy to see that

$$\Delta_j f(x) = (h_j * f)(x) = \int_{\mathbb{R}^d} h_j(x-y)f(y)dy, \quad j \geq -1. \quad (2.5)$$

**Lemma 2.4.** (Bernstein’s inequality) For any $1 \leq p \leq q \leq \infty$ and $j \geq 0$, we have

$$\|\nabla^k \Delta_j f\|_q \leq C_p 2^{(k+d(\frac{1}{p} - \frac{1}{q}))j} \|\Delta_j f\|_p, \quad k = 0, 1, \ldots, \quad (2.6)$$

and

$$\|(-\Delta)^{s/2} \Delta_j f\|_q \leq C_p 2^{(s+d(\frac{1}{p} - \frac{1}{q}))j} \|\Delta_j f\|_p. \quad (2.7)$$

Next is an easy commutator estimate.

**Lemma 2.5.** For any $j \geq -1$,

$$\|\{\Delta_j, b \cdot \nabla\} u\|_p \lesssim 2^{-3j} \|b\|_{W^3_p} \|\nabla u\|_{L^\infty}.$$  

**Proof.** By (2.5) we have

$$[\Delta_j, b \cdot \nabla] u(x) = \int_{\mathbb{R}^d} h_j(y)(b(x-y) - b(x)) \cdot \nabla u(x-y)dy.$$  

Observing that for any $p \in [1, \infty]$ and $s \in (0,1)$,

$$\|b(\cdot - y) - b(\cdot)\|_p \lesssim C |y|^s \|b\|_{W^s_p}, \quad (2.8)$$
by Hölder’s inequality and (2.4), we have
\[
\| [\Delta_j, b \cdot \nabla] u \|_p \leq \int_{\mathbb{R}^d} h_j(y) \| b(\cdot - y) - b(\cdot) \|_p \| \nabla u \|_{L^\infty} dy
\]
\[
\lesssim \| b \|_{W^\beta_p} \| \nabla u \|_{L^\infty} \int_{\mathbb{R}^d} |h_j(y)| |y|^\beta dy
\]
\[
= \| b \|_{W^\beta_p} \| \nabla u \|_{L^\infty} \int_{\mathbb{R}^d} |2h(2y) - h(y)| |y|^\beta dy
\]
\[
\lesssim 2^{-j \beta} \| b \|_{W^\beta_p} \| \nabla u \|_{L^\infty}.
\]
(2.9)

2.2. Mallivian Derivate for Lévy processes. In this subsection, we introduce some basic conceptions of Mallivian calculus for Lévy processes. One can find more details in [18]. Suppose \(N(dt, dz)\) is a Poisson point process with intensity measure \(\nu(dz)\). Let \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) be the filtration generated by \(N\) and \(\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz) dt\).

**Definition 2.6.** The stochastic Sobolev space \(D_{1/2}^1\) consists of all \(\mathcal{F}_T\) measurable random variables \(F \in L^2(\mathbb{P})\) with chaos expansion
\[
F = \sum_{n=0}^\infty I_n(f_n), \quad f_n \in L^2((\lambda \times \nu)^n)
\]
satisfying
\[
\| F \|_{D_{1/2}^1}^2 = \sum_{n=0}^\infty nn! \| f_n \|_{L^2((\lambda \times \nu)^n)}^2 < \infty,
\]
where \(f_n\) are symmetric functions and
\[
I_n(f_n) := \int_{(0,T] \times \mathbb{R}^d)^n} f_n(t_1, z_1; \cdots; t_n, z_n) \tilde{N}^{\otimes n}(dt, dz).
\]

Define
\[
D_{t,z} F := \sum_{n=1}^\infty n I_{n-1}(f_n(\cdot; t, z)),
\]
then
\[
\| F \|_{D_{1/2}^1}^2 = \| DF \|_{L^2((\lambda \times \nu \times \mathbb{P}))}^2.
\]
The next lemma can be found in [18].

**Lemma 2.7** (Closability of Mallivian derivate). If \(F_n \in D_{1/2}^1\), \(F_n \to F\) in \(L^2(\mathbb{P})\) and
\[
\sup_n \| DF_n \|_{L^2(\lambda \times \nu \times \mathbb{P})} < \infty.
\]
Then, \(F \in D_{1/2}^1\) and
\[
\| F \|_{D_{1/2}^1} = \| DF \|_{L^2((\lambda \times \nu \times \mathbb{P}))} \leq \lim_{n \to \infty} \| DF_n \|_{L^2(\lambda \times \nu \times \mathbb{P})}
\]
3. A STUDY OF NONLOCAL PARABOLIC EQUATIONS

In this section we study the solvability and regularity of nonlocal elliptic equations with Sobolev drift term. First of all, we introduce the nonlocal operator studied in this work. Let $\sigma$ be a $d \times d$-matrix and $\nu$ a symmetric Lévy measure, that is,

$$\int_{\mathbb{R}^d \setminus \{0\}} (|z|^2 \wedge 1) \nu(dz) < \infty.$$  

We define a Lévy-type operator by

$$\mathcal{L}_\sigma f(x) := \text{p.v.} \int_{\mathbb{R}^d} \left( f(x + \sigma z) - f(x) \right) \nu(dz), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

By Fourier’s transform, we have

$$\hat{\mathcal{L}_\sigma f}(\xi) = \psi_\sigma(\xi) \hat{f}(\xi),$$

where the symbol $\psi_\sigma(\xi)$ takes the form

$$\psi_\sigma(\xi) = -\int_{\mathbb{R}^d} (e^{i \xi \sigma z} - 1 - i 1_{B_1} \sigma z \cdot \xi) \nu(dz) = \int_{\mathbb{R}^d} (1 - \cos(\sigma z \cdot \xi)) \nu(dz).$$

Now, let $\sigma(x) : \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ be a Borel measurable function. Define

$$\mathcal{L} f(x) := \mathcal{L}_{\sigma(x)} f(x).$$

In this section we want to study the solvability of the following resolvent equation with Sobolev drift $b(x) : \mathbb{R}^d \to \mathbb{R}^d$,

$$\lambda u - \mathcal{L} u - b \cdot \nabla u = f; \quad \lambda > 0. \quad (3.1)$$

3.1. **Constant coefficient case:** $\sigma(x) = \sigma$. In this subsection we consider equation (3.1) with constant coefficient $\sigma(x) = \sigma \in \mathbb{R}^{d \times d}$. First of all, we establish the following Bernstein’s type inequality for nonlocal operator $\mathcal{L}_\sigma$, which plays a crucial role in the sequel.

**Lemma 3.1.** Suppose $\nu$ satisfies $(\text{H}_1)$, $\Lambda^{-1} \leq \|\sigma\| \leq \Lambda$, then for any $p \geq 2$, there is a constant $c = c(\nu, \Lambda, p) > 0$ such that for $j = 0, 1, \ldots,$

$$\int_{\mathbb{R}^d} |\Delta_j f|^{p-2} \Delta_j f \mathcal{L}_\sigma \Delta_j f dx \leq -c p 2^{\alpha j} \|\Delta_j f\|_p^p, \quad (3.2)$$

and for $j = -1$,

$$\int_{\mathbb{R}^d} |\Delta_{-1} f|^{p-2} \Delta_{-1} f \mathcal{L}_\sigma \Delta_{-1} f dx \leq 0.$$

**Proof.** We only prove (3.2) here. By the following elementary inequality:

$$p(a - b)(|a|^{p-2} a - |b|^{p-2} b) \geq (a|a|^{\frac{p}{2}-1} - b|b|^{\frac{p}{2}-1})^2, \quad \forall p \geq 2, \ a, b \in \mathbb{R}.$$  

We have,

$$\int_{\mathbb{R}^d} f |f|^{p-2} (-\mathcal{L}_\sigma f) dx$$

$$= \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x + \sigma z) - f(x))(|f|^{p-2} f(x + \sigma z) - |f|^{p-2} f(x)) \nu(dz) dx$$

$$\geq \frac{1}{2p} \int_{\mathbb{R}^d \times \mathbb{R}^d} (|f|^{\frac{p}{2}-1} f(x + \sigma z) - f|f|^{\frac{p}{2}-1}(x))^2 \nu(dz) dx$$

$$= \frac{1}{p} \int_{\mathbb{R}^d} \left| (-\mathcal{L}_\sigma)^{\frac{1}{2}} (f|f|^{\frac{p}{2}-1}) \right|^2 dx. \quad (3.3)$$
By Plancherel formula, we obtain
\[
\psi_\sigma(\xi) = \int_{\mathbb{R}^d} (1 - \cos(\sigma z \cdot \xi)) \nu(dz) \geq c \int_{|z| \leq (\Lambda \rho)^{-1}} |z| \cdot |\sigma^T \xi|^2 \nu(dz)
\]
\[
\geq |\sigma^T \xi|^2 \int_{|z| \leq (\Lambda \rho)^{-1}} |z| \cdot |\frac{\sigma^T \xi}{|\sigma^T \xi|}|^2 \nu(dz) \geq \inf_{\theta \in \mathbb{S}^{d-1}} \int_{|z| \leq (\Lambda \rho)^{-1}} |z| \cdot \theta^2 \nu_1(dz)
\]
\[
\geq |\xi|^2 \int_0^{(\Lambda \rho)^{-1}} r^{1-\alpha} dr \gtrsim |\xi|^\alpha,
\]
and it is easy to see that
\[
\psi_\sigma(\xi) \gtrsim 1, \quad \forall |\xi| \leq (\Lambda \rho)^{-1}.
\]

By Plancherel formula,
\[
\int_{\mathbb{R}^d} \left| (-\mathcal{L}_\sigma)^{\frac{1}{2}} (|f|^{\frac{\rho}{\rho-1}}) \right|^{2} dx = \int_{\mathbb{R}^d} \psi_\sigma(\xi) |\mathcal{F}(f^{\frac{\rho}{\rho-1}})(\xi)|^2 dx
\]
\[
\gtrsim \int_{\mathbb{R}^d} |\xi|^\alpha |\mathcal{F}(f^{\frac{\rho}{\rho-1}})(\xi)|^2 dx \tag{3.4}
\]
\[
\gtrsim \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{\rho}{\rho-1}} (|f|^{\frac{\rho}{\rho-1}}) \right|^2 dx.
\]

Combing (3.3), (3.4) and using the elementary inequality:
\[
|a| |a|^{\frac{\rho}{\rho-1}} - b |b|^{\frac{\rho}{\rho-1}} \geq c_p |a - b|^p, \quad \forall a, b \in \mathbb{R}, \quad p \geq 2,
\]
we obtain
\[
\int_{\mathbb{R}^d} f |f|^{p-2} (-\mathcal{L}_\sigma f) dx \gtrsim \int_{\mathbb{R}^d} \left| (-\Delta)^{\frac{\rho}{\rho-1}} (|f|^{\frac{\rho}{\rho-1}}) \right|^2 dx
\]
\[
\gtrsim \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x)|^{\frac{\rho}{\rho-1}} - f(y)|^{\frac{\rho}{\rho-1}}}{|x - y|^{d+\alpha}} dx dy
\]
\[
\gtrsim \int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|f(x) - f(y)|^p}{|x - y|^{d+\frac{\alpha}{p}}} dx dy = c_p |f|^p_{p,p}.
\]

Now using Theorem 2.36 of [2], we see
\[
[\Delta_j f]^p_{\frac{p}{p-1},p} = \|f\|_{B_{p,p}}^p \sum_{k=-\infty}^{\infty} 2^{\alpha k} \|\Delta_k \Delta_j f\|_{p}^p \simeq 2^{\alpha j} \|\Delta_j f\|_{p}^p.
\]

Combining the above two inequalities, we complete the proof. \(\square\)

**Remark 3.2.** The above result is proved in [4] for \(\mathcal{L}_\sigma = \Delta^{\alpha/2}\), our proof here is much simpler.

Now we can state our main result of this subsection.

**Theorem 3.3.** Let \(\alpha \in (0, 2)\), \(\beta \in (0 \vee (1 - \alpha), 1)\) and \(p \in (\frac{d}{\alpha + \beta - 1}, \infty)\). Suppose \(b \in W^\beta_p\), \(\nu\) satisfy \((H_1)\), then for any \(\gamma \in (1 - \alpha + \frac{d}{\rho} \vee 0, \beta)\) and \(f \in W^\gamma_p\), there exists a unique solution \(u \in W^{\alpha + \gamma}_p\) to equation (3.1). Moreover, there is a constant \(\lambda_0 > 0\), such that, for all \(\lambda \geq \lambda_0 > 0\),
\[
\lambda \|u\|_{W^\gamma_p} + \|u\|_{W^{\alpha + \gamma}_p} \leq C \|f\|_{W^\gamma_p},
\]
(3.5)

\(C\) depends only on \(d, p, \alpha, \beta, \gamma\) and \(\|b\|_{W^\beta_p}\).
Proof. We first assume

\[ b, f \in \cap_{s \geq 0} W^{s}_p. \]

Under this assumption, it is well-known that PDE (3.1) has a unique smooth solution \( u \).
Our main task is to show the apriori estimates (3.5). Using operator \( \Delta_j \) act on both sides of (3.1), we have

\[ \lambda \Delta_j u = \mathcal{L} \Delta_j u + \Delta_j (b \cdot \nabla u) + \Delta_j f. \]

For \( p \geq 2 \), by the chain rule or multiplying both sides by \( |\Delta_j u|^{p-2} \Delta_j u \) and then integrating in \( x \), we obtain

\[
\lambda \int_{\mathbb{R}^d} |\Delta_j u|^p = \int_{\mathbb{R}^d} |\Delta_j u|^{p-2} \Delta_j u \left[ \mathcal{L} \Delta_j u + \Delta_j (b \cdot \nabla u) + \Delta_j f \right] dx
\]

\[
= \int_{\mathbb{R}^d} |\Delta_j u|^{p-2} \Delta_j u \mathcal{L} \Delta_j u dx + \int_{\mathbb{R}^d} |\Delta_j u|^{p-2} \Delta_j u [\Delta_j, b \cdot \nabla] u dx
\]

\[
+ \int_{\mathbb{R}^d} |\Delta_j u|^{p-2} \Delta_j u (b \cdot \nabla) \Delta_j u dx + \int_{\mathbb{R}^d} |\Delta_j u|^{p-2} \Delta_j u \Delta_j f dx
\]

\[
= : I_j^{(1)} + I_j^{(2)} + I_j^{(3)} + I_j^{(4)}. \]

For \( I_j^{(1)} \), recalling \( \mathcal{L} = \mathcal{L}_\sigma \) and by Lemma 2.4, there is a \( c > 0 \) such that

\[ I_j^{(1)} = c 2^{\beta_j} \|\Delta_j u\|^p_p, \quad j = 0, 1, 2, \ldots. \]

For \( I_j^{(2)} \), using Lemma 2.5 and Hölder's inequality, we have for all \( j = -1, 0, 1, \ldots, \)

\[ I_j^{(2)} \lesssim \|\Delta_j, b \cdot \nabla\|_p \|\Delta_j u\|_p^{-1} \lesssim 2^{-\beta_j} \|b\|_{W^{\beta}_p} \|\nabla u\|_{L^\infty} \|\Delta_j u\|_p^{-1}. \]

For \( I_j^{(3)} \), let us write

\[ I_j^{(3)} = \int_{\mathbb{R}^d} ((b - S_j b) \cdot \nabla) \Delta_j u |\Delta_j u|^{p-2} \Delta_j u dx \]

\[ + \int_{\mathbb{R}^d} (S_j b \cdot \nabla) \Delta_j u |\Delta_j u|^{p-2} \Delta_j u dx =: I_j^{(31)} + I_j^{(32)}. \]

For \( I_j^{(31)} \), by Bernstein's inequality (2.6), we have

\[
I_j^{(31)} \lesssim \sum_{k \geq j} \|\Delta_k b \cdot \nabla\|_p \|\Delta_j u\|_p \|\Delta_j u\|_p^{-1}
\]

\[
\lesssim \sum_{k \geq j} \|\Delta_k b\|_p \|\nabla \Delta_j u\|_{L^\infty} \|\Delta_j u\|_p^{-1}
\]

\[
\lesssim 2^{(1 + d/p)j} \|\Delta_j u\|_p \sum_{k \geq j} \|\Delta_k b\|_p
\]

\[
\lesssim 2^{(1 + d/p - \beta_j)j} \|b\|_{W^{\beta}_p} \|\Delta_j u\|_p. \]

For \( I_j^{(32)} \), by integration by parts formula and (2.6) again, we have

\[
I_j^{(32)} = \frac{1}{p} \int_{\mathbb{R}^d} (S_j b \cdot \nabla) |\Delta_j u|^p dx = - \frac{1}{p} \int_{\mathbb{R}^d} S_j \text{div} b |\Delta_j u|^p dx
\]

\[
\leq \frac{1}{p} \|S_j \text{div} b\|_\infty \|\Delta_j u\|_p \leq \frac{1}{p} \sum_{k \leq j} \|\Delta_k \text{div} b\|_\infty \|\Delta_j u\|_p. \]
\[
\leq \sum_{k \leq j} 2^{k(1+d/p)} \| \Delta_k b \|_p \| \Delta_j u \|_p^p
\]
\[
\leq 2^{(1-\beta + d/p)j} \| b \|_{W_p^\beta} \| \Delta_j u \|_p^p.
\]

Combining the above calculations, we obtain
\[
\lambda \| \Delta_j u \|_p^p + c 2^{\alpha j} \| \Delta_j u \|_p^p \leq C 2^{-\beta j} \| b \|_{W_p^\beta} \| \nabla u \|_{L^\infty} \| \Delta_j u \|_p^{p-1}
\]
\[
+ C 2^{(1-\beta + d/p)j} \| \Delta_j u \|_p^p + C \| \Delta_j u \|_p^{p-1} \| \Delta_j f \|_p
\]

By dividing both sides by \( \| \Delta_j u \|_p^{p-1} \), we get
\[
\lambda \| \Delta_j u \|_p + c 2^{\alpha j} \| \Delta_j u \|_p - C 2^{(1-\beta + d/p)j} \| \Delta_j u \|_p \leq C 2^{-\beta j} \| b \|_{W_p^\beta} \| \nabla u \|_{L^\infty} + C \| \Delta_j f \|_p
\]

Since \( 1 - \beta + d/p < \alpha \), we get, for some \( \lambda \) sufficient large and all \( j \geq 1 \),
\[
\lambda \| \Delta_j u \|_p + c 2^{\alpha j} \| \Delta_j u \|_p \leq C 2^{-\beta j} \| b \|_{W_p^\beta} \| \nabla u \|_{L^\infty} + C \| \Delta_j f \|_p
\]  (3.6)

Multiplying both sides of (3.6) by \( 2^{\gamma j} (\gamma < \beta) \) and then taking \( \ell^p \) norm over \( j \), we obtain
\[
\lambda \| u \|_{W_p^\gamma} + \| u \|_{W_p^\alpha + \gamma} \leq C_1 \left( \| \nabla u \|_{L^\infty} + \| f \|_{W_p^\gamma} \right)
\]
where \( C_1 \) only depends on \( d, p, \alpha, \beta, \gamma, \| b \|_{W_p^\beta} \). Using Sobolev embedding and interpolation theorem, we obtain, \( \| \nabla u \|_{L^\infty} \leq \frac{1}{\lambda} \| u \|_{W_p^\alpha + \gamma} + C' \| u \|_{W_p^\gamma} \). Choosing \( \lambda_0 > 2C_1C'/\lambda \), we complete the proof for (3.5). \[\square\]

3.2. Varying coefficient case. In this subsection we consider the varying coefficient case. We drop the large jump part below, and consider the following operator
\[
\bar{\mathcal{L}} f(x) := \bar{\mathcal{L}}_{\sigma(x)} f(x) = \text{p.v.} \int_{B_R} \left( f(x + \sigma(x)z) - f(x) \right) \nu(dz).
\]  (3.7)

The following lemma(see [25, Theorem 2.4.7]) in order to localize the resolvent equation.

**Lemma 3.4** (localization principle). Let \( c > 0 \), \( \zeta_k \in \mathcal{C}^\infty_c, k = 1, 2, \cdots \). Assume for any multi-index \( \alpha \) and \( x \in \mathbb{R}^d \),
\[
\sup_{x \in \mathbb{R}^d} \sum_k | \partial^\alpha \zeta_k(x) | \leq C_\alpha < \infty.
\]

Then, there is a constant \( C \) such that
\[
\sum_k \| u \zeta_k \|_{W_p^\gamma} \leq C \| u \|_{W_p^\gamma}.
\]

Moreover, if
\[
\sum_k | \zeta_k(x) |^p \geq c > 0,
\]
then we have
\[
\| u \|_{W_p^\gamma} \geq \sum_k \| u \zeta_k \|_{W_p^\gamma}.
\]  (3.8)

The following lemma is taken from [16, Lemma 5].

**Lemma 3.5.** Suppose \( s \in (0, 1) \), \( p > \frac{4}{s} + 1 \), then
\[
\left\| \sup_{z \neq 0} \left| f(\cdot + z) - f(\cdot) \right| \right\|_p \leq C \| f \|_{H_p^s}.
\]  (3.9)
The main result of this subsection is

**Theorem 3.6.** Under assumptions \((H_1), (H_2)\), for any \( R > 0 \), \( \mathcal{L} \) is defined as (3.7), then for any \( \gamma \in (1 - \alpha + \frac{d}{p} \vee 0, \beta) \) and \( f \in W^\gamma \), the following equation:

\[
\lambda u - \nabla u - b \cdot \nabla u = f
\]

has a unique solution in \( W^{\alpha + \gamma} \). Moreover, we have

\[
\lambda \|u\|_{W^\gamma} + \|u\|_{W^{\alpha + \gamma}} \leq C\|f\|_{W^\gamma}.
\]  

(3.10)

In order to prove the above theorem, we need a commutator estimate under the following assumption.

\((H^\theta)\) There are \( \theta \in (0, 1), \varepsilon \in (0, r) \) and \( \Lambda \geq 1 \) such that

\[
\|\sigma(x) - \sigma(y)\| \leq \Lambda|x - y|^{\varepsilon}, \quad |x| \geq \varepsilon, \quad \Lambda^{-1}|\xi|^{2} \leq |\sigma(0)\xi|^{2} \leq \Lambda|\xi|^{2}, \quad \xi \in \mathbb{R}^{d}.
\]  

(3.11)

(3.12)

**Lemma 3.7.** Under \((H^\theta)\), for any \( p > 1 \), we have

\[
\left\|\left[\Delta^{s/2}, \mathcal{L}\right]u\right\|_{p} \leq C \left\{ \begin{array}{ll}
\varepsilon^{\beta - s/d + \varepsilon p} \|u\|_{C^{\beta}}, & \alpha \in (0, 1), \delta \in (\alpha, 1), s \in (0, \theta); \\
\varepsilon^{\theta - s/d + \varepsilon p} \|\nabla u\|_{C^{\beta - 1}}, & \alpha \in [1, 2), \delta \in (\alpha, 2), s \in (0, \theta),
\end{array} \right.
\]

where \( [\Delta^{s/2}, \mathcal{L}]u := \Delta^{s/2}u - \mathcal{L}\Delta^{s/2}u \), and the constant \( C > 0 \) is independent of \( \varepsilon \).

The above lemma was proved in [7], we give its proof here for reader’s convenience.

**Proof.** We only prove it for \( \alpha \in [1, 2) \) since the case \( \alpha \in (0, 1) \) is similar. We write

\[
\Gamma^{\sigma}_{u}(x, y, z) := u(x + y + \sigma(x + y)z) - u(x + y + \sigma(x)z) - (\sigma(x + y) - \sigma(x))z \cdot \nabla u(x + y),
\]

and

\[
\left\|\left[\Delta^{s/2}, \mathcal{L}\right]u\right\|_{p} = \left( \int_{|x| \leq 2\varepsilon} + \int_{|x| > 2\varepsilon} \right) \left\|\left[\Delta^{s/2}, \mathcal{L}\right]u(x)\right\|_{p} dx =: J_{1} + J_{2}.
\]

Let \( \delta \in (\alpha, 2) \). By \((H^\theta)\), we have

\[
|\Gamma^{\sigma}_{u}(x, y, z)| \lesssim (|y| \wedge \varepsilon)^{\theta} \int_{0}^{1} |\nabla u(x + y + (1 - r)\sigma(x + y)z + r\sigma(x)z) - \nabla u(x + y)| dr
\]

\[
\lesssim (|y| \wedge \varepsilon)^{\theta} \|\nabla u\|_{C^{\beta - 1}},
\]

and by definition,

\[
[\Delta^{s/2}, \mathcal{L}]u(x) = \int_{|z| < r_{0}} \nu(dz) \int_{\mathbb{R}^{d}} \frac{\Gamma^{\sigma}_{u}(x, y, z)}{|y|^{d+s}} dy.
\]

Thus, for \( J_{1} \) we have

\[
J_{1} \lesssim \|\nabla u\|_{C^{\beta - 1}}^{p} \int_{|z| \leq 2\varepsilon} \int_{|z| < r_{0}} \nu(dz) \int_{\mathbb{R}^{d}} \frac{|z|^{\delta} (|y| \wedge \varepsilon)^{\theta}}{|y|^{d+s}} dy dx
\]

\[
\lesssim \varepsilon^{\delta} \|\nabla u\|_{C^{\beta - 1}}^{p} \int_{|y| \leq \varepsilon} \frac{|y|^{\theta} dy}{|y|^{d+s}} + \int_{|y| > \varepsilon} \frac{\varepsilon^{\theta} dy}{|y|^{d+s}} \lesssim \|\nabla u\|_{C^{\beta - 1}}^{p} \varepsilon^{\theta-s \varepsilon^{p+d}}.
\]
For $J_2$, since $\Gamma_{\alpha}^\sigma(x, y, z) = 0$ for $|x|, |x + y| > \varepsilon$ by $(H_\varepsilon^\theta)$, we have
\[
J_2 = \int_{|x| > 2\varepsilon} \left| \int_{|z| < r_0} \nu(dz) \int_{|x+y| \leq \varepsilon} \frac{\Gamma_{\alpha}^\sigma(x, y, z)}{|y|^{d+s}} dy \right|^p dx \\
\lesssim \|\nabla u\|_{C^{\beta-1}}^p \int_{|x| > 2\varepsilon} \left| \int_{|z| < r_0} \nu(dz) \int_{|x+y| \leq \varepsilon} \frac{|z|^d (|y| \wedge \varepsilon) \theta}{|y|^{d+s}} dy \right|^p dx \\
\lesssim \|\nabla u\|_{C^{\beta-1}}^p \varepsilon^{\theta p} \int_{|x| > 2\varepsilon} \left| \int_{|x+y| \leq \varepsilon} \frac{1}{|y|^{d+s}} dy \right|^p dx \\
\lesssim \|\nabla u\|_{C^{\beta-1}}^p \varepsilon^{\theta p + dp} \int_{|x| > 2\varepsilon} \frac{1}{(|x| - \varepsilon)^{p(d+s)}} dx \lesssim \|\nabla u\|_{C^{\beta-1}}^p \varepsilon^{(\theta - s)p + d}.
\]
Combining the above calculations, we obtain the desired estimate.

**Lemma 3.8.** Under $(H_\varepsilon^\theta)$, for any $p \in (\frac{d^2}{\alpha \Lambda_\theta}, \infty)$, we have
\[
\|(\tilde{L}_{\sigma(-)} - \tilde{L}_{\sigma(0)}) f\|_{W^p_\gamma} \leq c_\varepsilon \left\{ \begin{array}{ll}
\|f\|_{W^{\alpha + \gamma}_p}, & \alpha \in (0, 1), \theta \in (1 - \alpha + \frac{d}{p}, 1), \gamma \in (0, \theta); \\
\|f\|_{W^{\alpha + \gamma}_p}, & \alpha \in [1, 2), \theta \in (\frac{d}{\alpha_1}, 1), \gamma \in (0, \theta),
\end{array} \right.
\]
where $c_\varepsilon \to 0$ as $\varepsilon \to 0$.

**Proof.** For simplicity of notation, we drop the time variable $t$ and write
\[
T_\sigma f := \tilde{L}_{\sigma(-)} f - \tilde{L}_{\sigma(0)} f.
\]
We prove the estimate for $\alpha \in (0, 1)$. The case $\alpha \in [1, 2)$ is similar.

By [6, (2.19)], we have
\[
\|\tilde{L}_{\sigma_1} f - \tilde{L}_{\sigma_2} f\|_p \lesssim (\|\sigma_1 - \sigma_2\|^\alpha + 1) \|f\|_{H^\alpha_p}.
\]
By (3.11),
\[
|T_\sigma f(x)| \leq |\tilde{L}_{\sigma(x)} f(x) - \tilde{L}_{\sigma(0)} f(x)| \leq \sup_{|\sigma - \sigma(0)| \leq \Lambda \varepsilon^\theta} |\tilde{L}_{\sigma} f(x) - \tilde{L}_{\sigma(0)} f(x)|,
\]
since $p > d^2/\alpha$, by [6, Lemma 2.2] and (3.13), we have
\[
\|T_\sigma f\|_p \leq \sup_{|\sigma - \sigma(0)| \leq \Lambda \varepsilon^\theta} \|\tilde{L}_{\sigma} f(\cdot) - \tilde{L}_{\sigma(0)} f(\cdot)\|_p \lesssim \varepsilon^{\alpha \theta} \|f\|_{H^\alpha_p}.
\]
By Minkowski’s inequality,
\[
\|\Delta_i T_\sigma f\|_p \leq \sum_{j > i} \|\Delta_i T_\sigma \Delta_j f\|_p + \sum_{j \leq i} \|\Delta_i T_\sigma \Delta_j f\|_p =: J_i^{(1)} + J_i^{(2)}.
\]
For $J_i^{(1)}$, by (3.14),
\[
\|\Delta_i T_\sigma \Delta_j f\|_p = \|\Delta_i (\mathbb{I} - \Delta)^{1/2} (\mathbb{I} - \Delta)^{-1/2} T_\sigma \Delta_j f\|_p \\
\lesssim 2^i \|\mathbb{I} - \Delta\|^{-1/2} T_\sigma \Delta_j f\|_p \\
\lesssim \varepsilon^{\alpha \theta} 2^i \|\Delta_j f\|_{H^\alpha_p} \\
\lesssim \varepsilon^{\alpha \theta} 2^i \|\Delta_j f\|_{H^\alpha_p}.
\]
Choosing $l = \alpha + 1$, we obtain
\[
J_i^{(1)} \leq \sum_{j > i} 2^{\alpha i} 2^{-(j-i)} \|\Delta_j f\|_p.
\]
For $J_i^{(2)}$, choose $\delta = 1$ in Lemma 3.7, $s \in (\gamma, \theta)$. By Bernstein’s inequality and Lemma 3.7, we have
\[
J_i^{(2)} = \sum_{j \leq i} \| \Delta_i \Delta^{-s/2} \Delta^{s/2} T_\sigma \Delta_j f \|_p \lesssim 2^{-si} \sum_{j \leq i} \| \Delta^{s/2} T_\sigma \Delta_j f \|_p \\
\leq 2^{-si} \sum_{j \leq i} \left( \| [\Delta^{s/2}, T_\sigma] \Delta_j f \|_p + \| T_\sigma \Delta^{s/2} \Delta_j f \|_p \right) \\
= 2^{-si} \sum_{j \leq i} \left( \| [\Delta^{s/2}, \widetilde{T}] \Delta_j f \|_p + \| T_\sigma \Delta^{s/2} \Delta_j f \|_p \right) \\
\lesssim 2^{-si} \sum_{j \leq i} \left( \varepsilon^{\theta - s + d/p} \| \Delta_j f \|_{C^1} + \varepsilon^{\theta} \| \Delta^{s/2} \Delta_j f \|_{H^p_0} \right) \\
\lesssim 2^{-si} \sum_{j \leq i} \left( \varepsilon^{\theta - s + d/p} 2(1+s/d/p) \| \Delta_j f \|_p + \varepsilon^{\theta} 2(s+\alpha) \| \Delta_j f \|_p \right) \\
\lesssim \varepsilon^{\theta s - d + d/p} 2^{-si} \sum_{j \leq i} 2^{(\alpha+s)j} \| \Delta_j f \|_p.
\]

Denoting $a_j = 2^{(\alpha+\gamma)j} \| \Delta_j f \|_p$, $b_j = 2^{-(s-\gamma)|j|}$ and combining the above calculations, we obtain,
\[
2^{\gamma_i} \| \Delta_i T_\sigma f \|_p \leq 2^{\gamma_i} J_i^{(1)} + 2^{\gamma_i} J_i^{(2)} \\
\leq c_\varepsilon \left[ \sum_{j > i} a_j 2^{(\alpha+\gamma)(i-j)} 2^{-(j-i)} + \sum_{j \leq i} a_j 2^{-(\gamma-s)(i-j)} \right] \\
\leq c_\varepsilon \sum_j a_j 2^{-(s-\gamma)|i-j|} = c_\varepsilon (a \ast b)_i.
\]

By Young’s inequality,
\[
\| T_\sigma f \|_{W^{\alpha}_\nu_p} = 2^{\gamma_i} \| \Delta_i T_\sigma f \|_p \mathcal{O}_p \\
\leq c_\varepsilon \| (a \ast b) \|_{\mathcal{O}_p} \leq c_\varepsilon \| a \|_{\mathcal{O}_p} \| b \|_{\mathcal{O}_1} \\
\leq c_\varepsilon \| f \|_{W^{\alpha+\gamma}_p}.
\]

Now we are on the position of proving Theorem 3.6.

Proof of Theorem 3.6. Like before, we only give the aprior estimate here. Let $\{\zeta_k\}_{k \in \mathbb{N}}$ be a standard partition of unity, such that, for any $k$, the support of $\zeta_k$ lies in a ball $B_k$ of radius $\varepsilon/8$, where $\varepsilon$ will be determined later. Denote by $y_k$ the center of $B_k$. Also for any $k$, we take functions $\eta_k, \xi_k \in C^\infty$, such that, $\eta_k = 1$ on $B_{\varepsilon/4}(y_k)$, $\eta_k = 0$ outside $B_{\varepsilon/2}(y_k)$, and $0 \leq \eta_k \leq 1; \xi_k = 1$ on $B_{\varepsilon/2}(y_k)$, $\xi_k = 0$ outside $B_{\varepsilon}(y_k)$, and $0 \leq \xi_k \leq 1$. Define $\sigma_k(x) = \xi_k(x) \sigma(x) + (1 - \xi_k(x)) \sigma(y_k)$,
\[
\mathcal{L}_k f(x) := \text{p.v.} \int_{|z| < R} (f(x + \sigma_k(x)z) - f(x)) \nu(dz).
\]
\[
\mathcal{L}_k f(x) := \text{p.v.} \int_{|z| < R} (f(x + \sigma(y_k)z) - f(x)) \nu(dz).
\]

Multiplying $\zeta_k$ on both side of (1.8), we get
\[
\lambda(u \zeta_k) - \mathcal{L}_k(u \zeta_k) - b \cdot \nabla(u \zeta_k) = f \zeta_k + \zeta_k(b \cdot \nabla u) - b \cdot \nabla(u \zeta_k) + \zeta_k(\mathcal{L}u) - \mathcal{L}_k(\zeta_k u) \quad (3.17)
\]
So Theorem 3.3 implies
\[ \lambda \| u \zeta \|_{W_p^c} + \| u \zeta \|_{W_p^{c+\gamma}} \lesssim (\| f \zeta \|_{W_p^c} + \| u b \cdot \nabla \zeta \|_{W_p^c} + \| \zeta (\widetilde{D} u) - \widetilde{L}_k (\zeta_k u) \|_{W_p^c}) . \]

Hence, using Lemma 3.4 we have,
\[ \lambda^p \| u \|_{W_p^c}^p + \| u \|_{W_p^{c+\gamma}}^p \lesssim \sum_k \lambda^p \| u \zeta \|_{W_p^c}^p + \| u \zeta \|_{W_p^{c+\gamma}}^p \lesssim \sum_k \left( \| f \zeta \|_{W_p^c}^p + \| u b \cdot \nabla \zeta \|_{W_p^c}^p + \| \zeta (\widetilde{D} u) - \widetilde{L}_k (\zeta_k u) \|_{W_p^c}^p \right) . \tag{3.18} \]

Again by Lemma 3.4,
\[ \sum_k \| f \zeta \|_{W_p^c}^p \approx \| f \|_{W_p^c}^p, \quad \sum_k \| u b \cdot \nabla \zeta \|_{W_p^c}^p \approx \| u b \|_{W_p^c}^p \lesssim (\| u \|_{L}\infty^p + \| u \|_{W_p^c}^p) . \tag{3.19} \]

Next we estimate the third term in the last line of (3.18),
\[ \zeta_k (x)(\widetilde{D} u)(x) - \widetilde{L}_k (\zeta_k u)(x) = I_k^{(1)} (x) + I_k^{(2)} (x) + I_k^{(3)} (x) \]
\[ = \left[ \widetilde{D} (u \zeta_k)(x) - \widetilde{L}_k (u \zeta_k)(x) \right] \eta_k (x) + \left[ \widetilde{D} (u \zeta_k)(x) - \widetilde{L}_k (u \zeta_k)(x) \right] (1 - \eta_k (x)) \]
\[ + \left\{ u (x) \widetilde{D} \zeta_k (x) + \int_{|z|<\varepsilon R} \left[ u (x + \sigma (x) z) - u (x) \right] \cdot \left[ \zeta_k (x + \sigma (x) z) - \zeta_k (x) \right] \nu (dz) \right\} . \]

For \( I_k^{(1)} \), notice \( \sigma_k (x) = \sigma (x) \) when \( x \) belongs to the support of \( \eta_k \), so we have
\[ I_k^{(1)} (x) = \left[ \widetilde{D} (u \zeta_k)(x) - \widetilde{L}_k (u \zeta_k)(x) \right] \eta_k (x) . \]
\( \widetilde{L}_k \) satisfies assumption \( (H_{\tilde{\xi}}) \), by Lemma 3.8, we have
\[ \| I_k^{(1)} \|_{W_p^c} \leq c_e \| u \zeta_k \|_{W_p^{c+\gamma}} \quad (c_e \to 0 \text{ as } e \to 0) . \tag{3.20} \]

For \( I_k^{(2)} \), since \( 1 - \eta_k (x) = 0 \) if \( |x - y_k| \leq \frac{\varepsilon}{4} \) and \( u \zeta_k (x) = 0 \) if \( |x - y_k| > \frac{\varepsilon}{4} \), we have
\[ I_k^{(2)} (x) = \int_{\frac{\varepsilon}{8}\gamma \leq |z|<\varepsilon R} \left[ u \zeta_k (x + \sigma (x) z) - u \zeta_k (x + \sigma (y_k) z) \right] (1 - \eta_k (x)) \nu (dz) . \]

By Minkowski inequality, Lemma 3.5 and interpolation theorem, for any \( s \in (0, 1 \wedge (\alpha + \gamma - 1)) \),
\[ \| I_k^{(2)} \|_p \leq \int_{\frac{\varepsilon}{8}\gamma \leq |z|<\varepsilon R} \| u \zeta_k (\cdot + \sigma (\cdot) z) - u \zeta_k (\cdot + \sigma (y_k) z) \|_p \nu (dz) \]
\[ \lesssim \| u \zeta_k \|_{H_p}^s \int_{\frac{\varepsilon}{8}\gamma \leq |z|<\varepsilon R} |z|^s \nu (dz) \lesssim \varepsilon^{\alpha - \alpha} \| u \zeta_k \|_{H_p} \]
\[ \lesssim \varepsilon \| u \zeta_k \|_{W_p^{c+\gamma}} + C_e \| u \zeta_k \|_p . \]

And
\[ \nabla I_k^{(2)} (x) = \int_{\frac{\varepsilon}{8}\gamma \leq |z|<\varepsilon R} \left[ (\nabla (u \zeta_k)(x + \sigma (x) z))(1 + \nabla \sigma (x)) - \nabla (u \zeta_k)(x + \sigma (y_k) z) \right] (1 - \eta_k (x)) \nu (dz) \]
\[ + u \zeta_k (x + \sigma (x) z) - u \zeta_k (x + \sigma (y_k) z) \right] \nabla \eta_k (x) \nu (dz) \]
\[ = \int_{\frac{\varepsilon}{8}\gamma \leq |z|<\varepsilon R} \left[ (\nabla (u \zeta_k)(x + \sigma (x) z) - \nabla (u \zeta_k)(x)) \cdot \nabla \sigma (x) z \right. (1 - \eta_k (x)) \]
similarly, we have,
\[ \|\nabla I_k^{(2)}\|_p \lesssim \int_{\mathbb{R}^d} \sum_{x \in \mathbb{R}^d} \left( \|\nabla(u_\zeta_k)(\cdot + \sigma(\cdot)z) - \nabla(u_\zeta_k)(\cdot + \sigma(y_k)z)\|_p \cdot \|\nabla\sigma\|_\infty |z| + \|\nabla\eta_k\|_\infty \cdot \|u_\zeta_k(\cdot + \sigma(\cdot)z) - u_\zeta_k(\cdot + \sigma(y_k)z)\|_p \right) \nu(dz) \]

\[ \leq \varepsilon \|u_\zeta_k\|_{W^{\alpha+\gamma}} + C_\varepsilon \|u_\zeta_k\|_p. \]  

(3.21)

For \( I_k^{(3)}(x) \), for any \(|z| < R\), it’s not hard to see,
\[ \sup_{x \in \mathbb{R}^d} \sum_k |\zeta_k(x + \sigma(x)z) - \zeta_k(x)|^p \lesssim |z|^p, \quad \sup_{x \in \mathbb{R}^d} \sum_k |(\widetilde{\zeta}_k)(x)|^p \lesssim 1. \]

Hence, for any \( s \in ((\alpha - 1) \lor 0, \alpha \land 1) \),
\[ \left( \sum_k \|I_k^{(3)}\|_p^p \right)^{1/p} \leq \left( \sum_k \int_{\mathbb{R}^d} |u(x)|^p \widetilde{\zeta}(x)^p dx \right)^{1/p} + \int_{|z| < R} \nu(dz) \]
\[ \left\{ \int_{\mathbb{R}^d} |u(x + \sigma(x)z) - u(x)|^p \sum_k |\zeta_k(x + \sigma(x)z) - \zeta_k(x)|^p dx \right\}^{1/p} \]
\[ \lesssim \|u\|_p + \int_{|z| < R} \|u\|_{H^s_p} |z|^{s+1} \nu(dz) \]
\[ \leq \varepsilon \|u\|_{W^{\alpha+\gamma}} + C_\varepsilon \|u\|_p. \]

Similarly, we can prove
\[ \left( \sum_k \|\nabla I_k^{(3)}\|_p^p \right)^{1/p} \leq \varepsilon \|u\|_{W^{\alpha+\gamma}} + C_\varepsilon \|u\|_p. \]

(3.22)

Now using Lemma 3.4, combining (3.18), (3.19), (3.20), (3.21), (3.22) and choosing \( \varepsilon \) sufficient small, \( \lambda_0 \) sufficient large, we complete our proof. \( \square \)

4. Proof of Theorem 1.1

Let \( N(dt, dz) \) be the Poisson random measure associated with \( Z \), that is,
\[ N((0, t] \times E) = \sum_{t \leq s} 1_E(\Delta Z_s), \quad E \in \mathcal{B}(\mathbb{R}^d), \quad \Delta Z_s := Z_s - Z_{s-}, \]
whose intensity measure is given by $dt\nu(dz)$. Let $\tilde{N}(dt, dz) = N(dt, dz) - dt\nu(dz)$ be the compensated Poisson random martingale measure. By Lévy-Itô’s decomposition, we have

$$Z_t = \int_0^t \int_{|z|<R} z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq R} zN(ds, dz).$$

Thus, SDE (1.1) can be written as

$$X_t = X_0 + \int_0^t b(X_s)ds + \int_0^t \int_{|z|<R} \sigma(X_s-) z\tilde{N}(ds, dz) + \int_0^t \int_{|z|\geq R} \sigma(X_s-) zN(ds, dz). \quad (4.1)$$

Below we shall fix a complete and right continuous filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}; (\mathcal{F}_t)_{t \geq 0})$ so that all the processes are defined on it.

**Proof of Theorem 1.1.** (1). For the well-posedness, one can assume $\nu$ compactly supported on $B_R$ i.e. $\sup_{t \geq 0} |\Delta Z_t| < R$, otherwise, we can take $\tau_0 := 0, \tau_k := \inf\{t > \tau_{k-1} : \Delta Z_t \geq R\}$ for any $k \geq 1$, and solve the SDE step by step.

Let $u$ be the solution of equation:

$$\lambda u - \tilde{\mathcal{H}} u - b \cdot \nabla u = b$$

By Theorem 3.6, for any $\mu \in (\alpha/2, \alpha - 1 - d/p)$, we have $u \in C^{1+\mu}$ with $\|u\|_{C^{1+\mu}} = c(\lambda, \mu)$ and $c(\lambda, \mu) \to 0$ as $\lambda \to \infty$. So by Itô’s formula (c.f. [19]),

$$u(X_t) = u(X_0) + \int_0^t [\tilde{\mathcal{H}} u + b \cdot \nabla u](X_s)ds + \int_0^t \int_{B_R} [u(X_{s-} + \sigma(X_{s-})z) - u(X_{s-})] \tilde{N}(ds, dz).$$

Let $\Phi(x) = x + u(x)$, choose $\lambda$ sufficient large, $x \mapsto \Phi(x)$ is a $C^1$-diffeomorphism and

$$Y_t := \Phi(X_t) = \Phi(X_0) + \int_0^t \lambda u(X_s)ds + \int_0^t \int_{B_R} [\Phi(X_{s-} + \sigma(X_{s-})z) - \Phi(X_{s-})] \tilde{N}(ds, dz)$$

$$= \Phi(X_0) + \int_0^t a(Y_s)ds + \int_0^t \int_{B_R} g(Y_{s-}, z) \tilde{N}(ds, dz),$$

where

$$a(y) := \lambda u(\Phi^{-1}(y));$$

$$g(y, z) := \Phi(\Phi^{-1}(y) + \sigma(\Phi^{-1}(y))z) - y$$

$$= u(\Phi^{-1}(y) + \sigma(\Phi^{-1}(y))z) - u(\Phi^{-1}(y)) + \sigma(\Phi^{-1}(y))z.$$ 

Elementary calculation yields,

$$\nabla a(y) = \lambda \nabla u(\Phi^{-1}(y)) \nabla \Phi^{-1}(y);$$

$$\nabla_y g(y, z) = [\nabla u(\Phi^{-1}(y) + \sigma(\Phi^{-1}(y))z) - \nabla u(\Phi^{-1}(y))] \nabla \Phi^{-1}(y)$$

$$+ \nabla u(\Phi^{-1}(y) + \sigma(\Phi^{-1}(y))z) \nabla \sigma(\Phi^{-1}(y)) \nabla \Phi^{-1}(y)z$$

$$+ \nabla \sigma(\Phi^{-1}(y)) \nabla \Phi^{-1}(y)z. \quad (4.3)$$

Fix $\mu \in (\alpha/2, \alpha - 1 - d/p)$, noticing that $u \in C^{1+\mu}$ with $\|u\|_{C^{1+\mu}} = c_\lambda$, we have,

$$\|\nabla a\|_{C^\mu} < \infty \quad (4.4)$$
and
\[
\|\nabla g(\cdot, z)\|_\infty \leq \|\nabla u\|_{C^0}\|\nabla \Phi^{-1}\|_\infty (\|\sigma\|_\infty \cdot |z|)^\alpha \\
+ \|\nabla u\|_\infty \|\nabla \Phi^{-1}\|_\infty \|\sigma\|_\infty \|\nabla \Phi^{-1}\|_\infty |z| \|\nabla \Phi^{-1}\|_\infty |z| \] \tag{4.5}
\]

Now let
\[
Y^0_t = Y_0 = \Phi(X_0); \quad Y^{n+1}_t = Y_0 + \int_0^t a(Y^n_s) \, ds + \int_0^t \int_{B_R} g(Y^n_{s-}, z) \tilde{N}(ds, dz),
\]
then by Doob’s inequality and (4.4), (4.5), we get
\[
\mathbb{E} \sup_{0 \leq s \leq t} |Y^{n+1}_s - Y^n_s|^2 \leq C \|\nabla a\|_{\infty}^2 \mathbb{E} \int_0^t |Y^n_s - Y^{n-1}_s|^2 \, ds \\
+ C \mathbb{E} \int_0^t \int_{|z| < R} |g(Y^n_{s-}, z) - g(Y^{n-1}_{s-}, z)|^2 \, \nu(dz) \, ds \\
\leq C \left( \|\nabla a\|_{\infty}^2 + \int_{|z| < R} \|\nabla g(\cdot, z)\|_{\infty}^2 \, \nu(dz) \right) \mathbb{E} \int_0^t |Y^n_s - Y^{n-1}_s|^2 \, ds \\
\leq C \mathbb{E} \int_0^t |Y^n_s - Y^{n-1}_s|^2 \, ds.
\]
Hence,
\[
\lim_{n,m \to \infty} \mathbb{E} \sup_{0 \leq t \leq T} |Y^m_t - Y^n_t|^2 = 0,
\]
and the limit point \(Y\) is the unique strong solution to (4.2), which implies (1.1) has a unique strong solution.

For the Mallivian differentiability of \(X_t\), like above we only need to show that \(Y_t\) is Mallivian differentiable. And by the closability of Mallivian derivates, we only need to show that for each \(n \in \mathbb{N}\), \(t > 0\), \(Y^n_t\) is Mallivian differentiable and
\[
\sup_n \|D_{r,z}Y^n_t\|_{L^2(\lambda \times \mu \times \mathbb{P})} < \infty. \tag{4.6}
\]
Assume \(Y^n_t\) is Mallivian differentiable, then by Theorem 12.8 and Theorem 12.15 of [18],
\[
D_{r,z}a(Y^n_s) = a(Y^n_s + D_{r,z}Y^n_s) - a(Y^n_s),
\]
\[
D_{r,z} \int_0^t \int_{B_R} g(Y^n_{s-}, z) \tilde{N}(ds, dz) = \int_0^t \int_{B_R} \left[ g(Y^n_{s-} + D_{r,z}Y^n_{s-}, \eta) - g(Y^n_{s-}, \eta) \right] \tilde{N}(ds, d\eta) + g(Y^n_{t-}, z).
\]
So we obtain \(Y^{n+1}_t \in \mathbb{D}_2^1\) and
\[
D_{r,z}Y^{n+1}_t = g(Y^{n+1}_{t-}, z) + \int_0^t \left[ a(Y^n_s + D_{r,z}Y^n_s) - a(Y^n_s) \right] \, ds \\
+ \int_0^t \int_{B_R} \left[ g(Y^n_{s-} + D_{r,z}Y^n_{s-}, \eta) - g(Y^n_{s-}, \eta) \right] \tilde{N}(ds, d\eta)
\]
Denote
\[
f^n_r(t) = \mathbb{E} \left[ \int_{B_R} \sup_{r \leq s \leq t} (D_{r,z}Y^n_s)^2 \, \nu(dz) \right]
\]
For any \(t \in [0, T]\), again by Doob’s inequality,
\[
f^{n+1}_r(t) = \mathbb{E} \left[ \int_{B_R} \sup_{r \leq s \leq t} (D_{r,z}Y^{n+1}_s)^2 \, \nu(dz) \right]
\]
Moreover, by (L), the proof of Theorem 1.1 was replaced by Remark 4.1. By Theorem 1.1, we will show that

\[ \phi \in \mathbb{L}^{\infty} \]

\[ \int_{B_{r,z}} |D_{r,z}Y_{r,z}^{n}|^2 \nu(dz) \leq C \left\{ 1 + \int_{R} ds \mathbb{E} \left[ \sup_{B_{r,z}} (D_{r,z}Y_{r,z}^{n})^2 \nu(dz) \right] \right\} \]

\[ = C + C \int_{R} f_{r}(s)ds, \]

here \( C \) is independent with \( n, r \) and \( t \), by this we get

\[ \sup_{n \in \mathbb{N}} \mathbb{E} \left[ \int_{B_{r,z}} (D_{r,z}Y_{r,z}^{n})^2 \nu(dz) \right] < \infty. \]

which implies (4.6). So we complete our proof.

(2). By (H3), (4.5) and choosing \( \lambda \) sufficient large, for any \( z \in B_{r_0} \), we have

\[ |\nabla y g(y, z)| \leq \|\nabla \sigma\|_{\infty} |z| + c_{\lambda}(1 - c_{\lambda})^{-1} (\|\sigma\|_{\infty} |z|^\mu + \|\nabla \sigma\|_{\infty} |z|) \]

\[ \leq r_0 \|\nabla \sigma\|_{\infty}^{-1} + C \cdot c_{\lambda} < 1, \]

which implies that for each \( z \in \text{supp} \nu \subseteq B_{r_0} \) the map \( y \mapsto y + g(y, z) \) is homeomorphic and \( I + \nabla y g(y, z) \) is invertible. Again by (4.5), for any \( z \in B_{r_0} \), \( \|\nabla y g(y, z)\|_{\infty} \leq K(z) \leq |z|^\mu \).

Since \( 2 \mu > \alpha \), by (H1),

\[ \int_{B_{r_0}} K(z)^2 \nu(dz) \leq C \int_{B_{r_0}} |z|^{2\mu} \nu_2(dz) \leq C r_0^{2\mu - \alpha} < \infty. \]

Moreover, by (4.3) and the regularity estimates for \( u \), one can also check that

\[ |\nabla y g(y, z) - \nabla y g(y', z)| \leq L(z) |y - y'|^{\delta \cdot \mu} \]

and \( L(z) \leq 1 \), hence

\[ \int_{B_{r_0}} L(z)^2 \nu(dz) < \infty. \]

Thanks to [13, Theorem 3.11], \( \{Y_t(x)\} \) defines a \( C^1 \)-stochastic flow, so does \( \{X_t(x)\} \). \( \square \)

**Remark 4.1.** By Theorem 3.3 and the proof of Theorem 1.1, one can see that if \( \sigma \) is a constant positive defined matrix, then the conclusions in Theorem 1.1 still hold if (H2) was replaced by \( b \in W_{p}^{\beta} \) with \( \beta \in (1 - \frac{\alpha}{2}, 1) \) and \( p \in \left( \frac{d}{\alpha/2 + \beta - 1}, 1 \right) \).

**5. PROOF OF COROLLARY 1.3**

For the proof of Corollary 1.3, we follow the argument in [21], since we have established flow property of strong solutions.

**Proof.** Suppose \( \theta \) solves (1.7). Denote

\[ \phi(t) := X_{1-t}(y_t). \]

We will show that \( \phi(t, \omega) \) are constant functions almost surely. For any \( 0 \leq s \leq r \leq t \leq 1 \) and \( \delta \in (0, 1) \),

\[ |\phi(t) - \phi(s)| = |X_{1-t}(y_t) - X_{1-t}(X_{t-s}(y_s))| \leq K|y_t - X_{t-s}(y_s)|^\delta, \quad (5.1) \]
here $K$ is an integrable variable depending on $\delta$.

\[
|y_t - X_{t-s}(y_s)| = |(y_t - y_s) - (X_{t-s}(y_s) - y_s)| = \left| \int_s^r b(y_u) du - \int_s^r b(X_{u-s}(y_s)) du \right| \\
\leq 2\|b\|_{\infty} |r - s|.
\]

Hence,

\[
|y_t - X_{t-s}(y_s)| = \left| \int_s^t b(y_r) dr - \int_s^t b(X_{r-s}(y_s)) dr \right| \\
\leq |b|_{\infty} \int_s^t |y_r - X_{r-s}(y_s)|^\beta dr \\
\leq C\|b\|^2_{C^{\beta}} \int_s^t |r - s|^\beta dr \\
\leq C|t - s|^{1+\beta}.
\]

Combining (5.1) and above inequalities, we obtain

\[
|\phi(t) - \phi(s)| \leq C|K| |t - s|^{\beta(1+\beta)},
\]

by choosing $\delta > (1 + \beta)^{-1}$, we obtain for almost surely $\omega \in \Omega$

\[
X_1(x, \omega) = \phi(0, \omega) = \phi(1, \omega) = X_0(y_1(\omega), \omega) = y_1(\omega).
\]

\[\square\]

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