Spanning tests for assets with option-like payoffs:  
the case of hedge funds

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Abstract

We draw on the skewness literature to propose regression-based performance evaluation tests designed for investments with option-like returns. These tests deliver conclusions valid for all risk-averse mean-variance-skewness investors and can better account for non-linearities in returns than option-based factor models. Applied to mutual funds and hedge funds, our tests usually suggest selecting different funds than standard tests, and find that a significant fraction, 11%, of hedge funds add value to investors, whereas this is an insignificant 4% for mutual funds. We also analyze the economic significance of these option-like returns, and their out-of-sample persistence.

JEL Classification: G10, G11.

Keywords: hedge funds; mutual funds; writing options; performance evaluation; mean-variance-skewness spanning; prudence; portfolio choice.

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1 Introduction

Performance evaluation of mutual funds and hedge funds is commonly based on linear factor models such as the CAPM, the Fama-French three-factor model, or the Fama-French-Carhart four-factor model (see also Harvey et al., 2015 for a comprehensive list of additional factors). This approach is fully justified for mean-variance investors, possibly in a multi-factor economy (Fama, 1996), because a positive alpha is equivalent to an improvement in their investment opportunity set. Acknowledging that trading strategies, particularly for hedge funds, involve dynamic trading, derivative usage, and/or leverage, non-linearities and skewness in returns arise which are not captured by these linear factor models and make the mean-variance assumption unappealing.

This paper argues that a remedy for the shortcoming of linear factor models is to account for investor’s skewness preference in performance evaluation and develops a general framework to assess whether assets with option-like returns improve an investment opportunity set. Our approach generalizes the mean-variance spanning and intersection tests of Huberman and Kandel (1987) with a risk-free asset. We use our framework to better understand the costs and benefits of skewness in hedge fund returns compared with mutual fund returns and to shed new light on the usefulness of hedge funds for investors.

To see why non-linearities and skewness are important empirically and not captured by linear factor models, Figure 1(a) plots the monthly excess returns on an S&P 500 out of the money put writing strategy (henceforth, put1) versus the excess returns on the total return index. The figure shows that Jensen’s alpha of put1 is economically large with 0.59% per month (t-statistic of 5.60). In addition, the relation between put1 and index returns is not entirely captured by the covariance—a measure only of linear dependence. The relation between put1 and index returns is not only increasing; it is also concave. Thus, put1 has low returns (relative to the linear relationship) when squared market returns are large—commonly considered as bad times. More formally, there is a significant negative relation between the residual of the CAPM regression and the squared demeaned index returns, or residual co-skewness, as evidenced in Figure 1(c). Therefore, the alpha

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1 Skewness preference is easily justified theoretically (see, e.g., Arditti, 1967; Kimball, 1990), an important feature of positive theories of investor choice (Jouini et al., 2014), status concerns (Roussanov, 2010), and supported by experimental evidence (Ebert, 2015) and empirical asset pricing research (Kraus and Litzenberger, 1976; Harvey and Siddique, 2000). Non-linear stochastic discount factors, which are necessary to ensure positivity on the whole domain to guarantee the absence of arbitrage opportunities, also imply skewness preference (Scott and Horvath, 1980).

2 The mean-variance spanning approach has been widely used in the literature to assess international diversification benefits (e.g., Bekaert and Urias, 1996; Eruzione et al., 1999; DeRoon et al., 2001) and the regression-based tests of mean-variance spanning and intersection are related to tests of mean-variance efficiency, performance measurement, and portfolio choice (DeRoon and Nijman, 2001).
Panel (a) plots the excess returns of put writing strategy $r_{p1}$ versus the S&P500 TR index $r_s$, and Panel (b) plots put writing strategy $2 r_{p2}$ versus $r_{p1}$. These panels also show the coefficients and $t$-statistics with White (1980) standard errors of a regression of $r_{p1}$ on $r_s$ and $r_{p2}$ on $r_{p1}$. Panel (c) and (d) plot the residuals of these regressions, $\epsilon_{p1}$ and $\epsilon_{p2}$, versus the respective squared demeaned independent variable, $r_s^2$ and $r_{p1}^2$, and show the coefficients and $t$-statistics of the corresponding regressions. Their slopes are proportional to residual co-skewness, i.e., $\text{Cov}(\epsilon, r^2)$.

The put writing strategies are constructed following Jurek and Stafford (2015), and use Option metrics data on S&P500 index options from 1996 to 2014. $r_{p1}$ ($r_{p2}$) uses a leverage ratio of 2 (4) and shorts options with strikes approximately 7% (14%) below prevailing index levels.

of put1 comes at the cost of co-skewness, which deteriorates portfolio skewness, and many investors may prefer not to invest in put1.

A common approach to capture non-linearities is to include factors with non-linear returns, such as option payoffs, next to the standard factors (e.g., Fung and Hsieh, 2001; Agarwal and Naik, 2004). However, this approach does not suffice in general because option-like payoffs can still generate a positive alpha at the expense of negative skewness.\footnote{Notice that adding the squared benchmark return directly to the linear factor model, as in Kraus and Litzenberger (1976)’s quadratic market model, does not help for performance evaluation. Indeed, the slope on the squared market return is negative for strategies with negative residual co-skewness, and estimated alphas are then higher because average squared returns are positive.} To show this, it suffices to construct an alternative put writing strategy (henceforth, put2), which shorts further out of the money puts,
to again obtain an alpha with respect to that benchmark. This is conveyed in Figure 1(b), which plots put2 against put1. The alpha with put1 as a factor is 0.23% per month (t-statistic of 3.42).

Consider an investment opportunity set of benchmark assets versus a larger set of benchmark plus additional assets, we propose two tests for whether risk-averse mean-variance-skewness investors (henceforth, skewness investors) benefit from additional assets. Throughout the paper, we refer to “one investor” as a particular combination of preferences over mean, variance, and skewness, and “all investors” as any combination of these preference intensities. First, a spanning test considers the hypothesis that no investor benefits from the additional assets versus the alternative where at least one investor benefits. It is the equivalent of the mean-variance spanning test to the mean-variance-skewness case. Second, an overlap test considers the hypothesis that at least one investor does not benefit versus the alternative: all investors benefit. This overlap concept is new and recognizes that for a group of investors the positive alpha of assets such as those depicted in Figure 1 is offset by the negative co-skewness with the benchmark. In a sense the overlap test generalizes the mean-variance intersection test. Where the mean-variance intersection test is for the hypothesis that there is one mean-variance investor that does not benefit versus the alternative that all investors do benefit, our overlap test tests for the hypothesis that there is a group of mean-variance-skewness investors that does not benefit versus the same alternative.

We study the costs and benefits of option-like characteristics in returns with a large sample of live and dead hedge funds and mutual funds from Morningstar in the period from 1994 to 2014. Considering investors who currently invest in stocks and bonds, we obtain different results with our method compared with standard spanning tests: among the funds which benefit all mean-variance investors, 73% still do so for the Fama-French-Carhart four-factor model and the Fung-Hsieh eight-factor model, but only 15% improve the investment opportunity set of all skewness investors. Thus, the majority of the hedge funds that are a attractive from a mean-variance point of view, have a trade-off with negative co-skewness with stocks and bonds that do not make them attractive for groups of mean-variance-skewness investors. More generally, all skewness investors improve their investment opportunity set with around 11% of the hedge funds but with less than 4% of the mutual funds.

To gain more economic understanding about our results, we analyze the relation between co-skewness and hedge funds’ characteristics and strategies. Although hedge funds characterized by longer lock-up and advance notice periods have more negative co-skewness, we find that the type of

\footnote{All rejection rates throughout the paper are at the 5% significance level.}
strategy better explains the cross-sectional variation in co-skewness. Event and relative value funds mostly follow arbitrage strategies, and we find that these funds have more negative co-skewness. Global derivatives funds instead have desirable positive co-skewness and are also those most likely to be beneficial to all skewness investors. A detailed subsample analysis shows that rejection rates are highest among funds following systematic futures strategies. These strategies prosper when markets demonstrate sustained bullish or bearish trends (Morningstar, 2011)—i.e., when market returns are large in absolute value, or equivalently squared market returns are large. This generates positive co-skewness and explains their attractiveness for all skewness investors.

We evaluate the economic significance of our results by calculating the utility gains of taking higher moments into account for the funds which significantly improve the investment opportunity set of all skewness investors. More specifically, we compare Jensen’s alphas to alphas adjusted for skewness preference. This skewness adjustment is 0.35% per year on average for hedge funds and attains 1% on average in the highest co-skewness quintile. For mutual funds, it is 0.21% on average and attains 0.67% in the highest co-skewness quintile. Going beyond skewness, we find that ignoring higher moments has little impact. The differences between alphas adjusted for skewness and all moments are mostly below 0.10% per year suggesting that our framework provides a good summary of the option-like characteristics in hedge and mutual fund returns.

Finally, because performance analysis always uses observed returns, most of our results are obtained from an in-sample analysis. In robustness checks, we study whether our test statistics are persistent. We find that hedge funds desirable for all skewness or all mean-variance investors also tend to be desirable out-of-sample. By contrast, mutual funds which have been desirable for both types of investors do not tend to be so out-of-sample. Thus, our results suggest that a significant subset of hedge funds add value to all investors, in and out-of-sample and even to those concerned about skewness.

2 Related literature

Our research is related to the literature on hedge fund performance and risk evaluation. This literature finds that hedge funds are different from mutual funds because additional risk factors are needed to explain their returns, and their risk exposures show strong time variation. Even after controlling for additional risk factors, previous research finds that a subset of hedge funds delivers alpha (Kosowski et al., 2007; Buraschi et al., 2014a). This research builds on the evidence

\footnote{See Fung and Hsieh (1997, 2001); Agarwal and Naik (2004); Buraschi et al. (2014b) for risk factors in hedge funds returns, and Patton and Ramadorai (2013) for time variation of risk exposures.}
that higher-order risk and tail risk help explain the cross-section of hedge fund returns \cite{Agarwal2009, Kelly2012, Hübner2015} which suggests that traditional performance metrics may not accurately measure the “alpha” of hedge funds.

The most widely used method to assess hedge fund performance is a linear factor model with the factors of Fung and Hsieh (2001) or Agarwal and Naik (2004). Motivated by Glosten and Jagannathan (1994), these factors include returns on option-based strategies to capture the option-like characteristics of hedge fund returns. We show that because these models effectively restrict additional assets to be a fixed linear combination of non-linear returns, they are unable to account for general forms of non-linearities. Our method instead jointly considers alpha and co-skewness and outperforms these models due to its increased flexibility.

Our methodology differs from approaches that allow for time-varying betas or, equivalently, managed portfolios to capture the non-linearities in hedge fund returns. For example, Ferson and Schadt (1996) and Patton and Ramadorai (2013) augment the set of benchmark assets with managed portfolios based on conditioning information to account for dynamic trading. Patton and Ramadorai (2013) find that alphas tend to be larger, on average, with time-varying exposure models. Time-varying exposure models and our approach are both motivated by non-linearities in returns. But our approach explicitly accounts for investor’s preference for skewness by looking at both alpha and co-skewness. In fact, in our methodology the set of benchmark assets can be augmented with the managed portfolios suggested by Patton and Ramadorai (2013) to analyze whether skewness investors benefit from an investment.

Existing research on skewness has studied the portfolio choice problem with skewness. In addition, Bali and Murray (2013) have introduced the concept of “skewness asset” which is a portfolio of options and the underlying stock designed to take a position in the risk-neutral skewness of an underlying. Bali and Murray document a strong negative relation between risk-neutral skewness and subsequent skewness asset returns. Bali and Murray (2013) and our paper share a similar motivation—skewness preference—but our focus is to propose a performance evaluation approach which allows for investors with skewness preference. In such a framework, a positive alpha and/or a positive co-skewness are valid reasons to add an asset to a portfolio. In the context of hedge funds, Bali et al. (2012) analyze the predictive power of variance, skewness, and kurtosis for returns. They find that variance is positively related to subsequent returns, and that skewness or kurtosis do not

\footnote{An incomplete reference list is de Athayde and Flores (2004); Jondeau and Rockinger (2006); Mitton and Vorkink (2007); Guidolin and Timmermann (2008); Martellini and Ziemann (2010).}

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predict returns. We do not aspire to introduce a new predictor of hedge fund returns and conduct an out-of-sample analysis only in a robustness check to verify that investor’s can identify ex-ante the funds which are desirable ex-post from a skewness perspective.

We provide simple tests for whether all skewness investors benefit from additional assets. The most closely related research in this direction is Mencia and Sentana (2009) who propose a parametric test for mean-variance-skewness spanning. Under their distributional assumptions, the efficient frontier can be generated by only three funds, which is not the case in general, and hence their spanning conditions are different from the conditions derived in this paper and do not nest the Huberman and Kandel (1987) tests as a special case. In addition, we contribute to this literature by introducing the overlap concept and test.

Existing research on portfolio allocation with hedge funds finds that hedge funds improve the mean-variance trade-off in a portfolio of stocks and bonds at the expense of lower skewness (see Amin and Kat, 2003; Davies et al., 2009). Our paper provides formal tests to analyze the cost of skewness for investors and finds that about 11% of the hedge funds provide both mean-variance and skewness benefits for stock and bond investors. In a related study, Almeida et al., (2018) propose a performance evaluation method based on families of non-linear discount factors and also find a lower performance of hedge funds. The difference to our study is that we focus on the first three moments, a different set of benchmark assets, and use a spanning methodology which is closely linked to portfolio choice by construction. Finally, Back et al., (2018) analyze residual co-skewness in mutual fund returns, and find a trade-off between alpha and residual co-skewness across investment styles and for active funds. Their analysis motivates us to include mutual funds in our comparison sample.

3 Evaluating investments with option-like returns

To facilitate the exposition, we limit the discussion here and in the empirical section to the case of investing with a risk-free asset. The general case without a risk-free asset can be found in Chapter 3 of Karehknke (2014).

We start by stating the popular mean-variance spanning restrictions and the additional skewness restriction, and provide the formal derivation in the next subsection. Let the vectors of excess returns on the k-benchmark and n-test assets be denoted by $r_{x,t}$ and $r_{y,t}$, respectively. Mean-variance spanning tests use the intercepts of the multivariate regression

$$r_{y,t} = \alpha + Br_{x,t} + \epsilon_t,$$

(1)
where $\epsilon_t$ is the $n$-dimensional vector of residuals, $\alpha$ is the $n$-dimensional vector of intercepts, and $B$ is the $n \times k$-dimensional matrix of slope coefficients. In the standard setting of a multiple regression with simple returns and without risk-free asset, mean-variance spanning implies that the intercepts are zero and the slope coefficients for each regression sum to one. With a risk-free asset, there are no constraints on portfolio weights and spanning only requires that the intercepts in (1) are zero. This result provides the rationale to advise investors to buy funds or stocks which have a positive alpha in a linear factor model. For benchmark and test assets with non-normally distributed returns and investors with preferences over higher moments, this result no longer applies. The next subsection shows that in a mean-variance-skewness framework additional conditions are imposed on the $n \times k^2$-dimensional co-skewness matrix of the residual $\epsilon_t$ with the benchmark assets

$$S_{\epsilon xx} = \begin{bmatrix} S_{\epsilon x_1x_1} & \cdots & S_{\epsilon x_1x_k} \\ \vdots & \ddots & \vdots \\ S_{\epsilon x_kx_1} & \cdots & S_{\epsilon x_kx_k} \end{bmatrix},$$

(2)

where $S_{\epsilon x_ix_j} = E(\epsilon_i \tilde{r}_{x_i} \tilde{r}_{x_j})$, $\tilde{r}$ are demeaned returns, $i = 1, \ldots, n$ and $j = 1, \ldots, k$. This result highlights that factor models augmented with a skewness factor representing the return on a low minus high co-skewness portfolio cannot account for an investor’s skewness preference. In addition, the result that it is the residual co-skewness that matters can be traced back to the asset pricing study (although not explicitly named) of [Ingersoll (1975)]. This residual co-skewness is identical to the numerator of [Harvey and Siddique (2000)]’s empirically motivated co-skewness measure when the aggregate stock market is the only benchmark asset. More recent asset pricing derivations of residual co-skewness are in [Back (2014)]. The theoretical contribution of the next subsection is to derive these results in a spanning and intersection setting with multiple benchmark and additional assets, and to introduce the new “overlap” case.

### 3.1 The theory

Consider the portfolio choice problem of skewness investors who can either invest in the risk-free security and the benchmark assets $r_x$, or in a larger universe, which additionally consists of the test assets $r_y$. If the optimal portfolio of *only one* investor is the same with the benchmark assets only as with the benchmark and test assets, the mean-variance-skewness frontiers of $r_x$ and $(r_x, r_y)$ intersect ([Huberman and Kandel, 1987]). If the optimal portfolio of *at least one* investor is the same with the benchmark assets only as with the benchmark and test assets, the mean-variance-skewness frontiers of $r_x$ and $(r_x, r_y)$ overlap. If the optimal portfolio of $r_x$ and $(r_x, r_y)$ is the same for all skewness investors, the benchmark assets are said to span the test assets. In the following, we...
develop these concepts formally.

Let the $k + n$ vector of excess returns be denoted by $\mathbf{r}^T \equiv [\mathbf{r}_x^T \mathbf{r}_y^T]$, where $^T$ stands for transpose, and let the vector of expected excess returns and the matrix of covariances be denoted by $\mathbf{\mu}$ and $\mathbf{\Sigma}$, respectively. Bold letters denote vectors or matrices throughout the paper and, if it is not specified otherwise, vectors and matrices have the dimension $(k + n) \times 1$ and $(k + n) \times (k + n)$, respectively. In addition, we sometimes use the subscripts $x$ and $y$ to refer to the respective moments of benchmark and test assets (e.g., $\mathbf{\mu}_y$ is the vector of the expected returns of the $n$ test assets and $\mathbf{\Sigma}_{yx}$ is the $n \times k$-matrix of the covariances between test and benchmark assets). The $(k + n) \times (k + n)$ matrix of co-skewness is given by

$$S = E(\tilde{\mathbf{r}} \tilde{\mathbf{r}}^T \otimes \tilde{\mathbf{r}}^T) = \begin{bmatrix} S_{rr_1r} & \cdots & S_{rr_{k+n}r} \end{bmatrix},$$

where $\otimes$ is the Kronecker product, and $S_{rr_i r_j} = E(\tilde{r}_i \tilde{r}_1 \tilde{r}_j)$ for $i, j = 1, \ldots, k + n$.

A skewness investor likes the mean and skewness of his portfolio returns and dislikes the variance. He or she chooses a portfolio $\mathbf{w}$ in the $k + n$ assets to maximize his or her mean-variance-skewness utility

$$\max_{\mathbf{w}} \quad \mathbf{w}^T \mathbf{\mu} - \frac{1}{2} \gamma_1 \mathbf{w}^T \mathbf{\Sigma} \mathbf{w} + \frac{1}{3} \gamma_2 \mathbf{w}^T S (\mathbf{w} \otimes \mathbf{w}),$$

where $\gamma_1$ and $\gamma_2$ are two positive scalars which measure the aversion to variance and preference for skewness (relative to the preference for the mean). In an expected utility framework, $\gamma_1$ can be interpreted as the coefficient of risk aversion and $\gamma_2$ as the coefficient of downside risk aversion of Crainich and Eeckhoudt (2008), or equivalently as the product of risk aversion and prudence. Thus, in our framework, as in expected utility theory (see, e.g., Menezes et al., 1980; Crainich and Eeckhoudt, 2008), aversion to downside risk and preference for skewness are equivalent. Section 1 of the Technical Appendix derives these interpretations for $\gamma_1$ and $\gamma_2$ and discusses possible parameter values for popular utility functions.

Throughout the paper, we assume that the first-order conditions of the investors are necessary and sufficient, which is tantamount to assuming that the mean-variance-skewness investors are risk-averse. Mathematically, this is the case either if the second-order condition of (3) holds, or if we consider the first-order conditions of (3) to be a second-order approximation of the first-order

\footnote{In statistics, skewness usually refers to the third standardized moment (i.e., the third moment divided by the cube of the standard deviation). Here, skewness refers to the third unstandardized moment in line with the portfolio choice literature.}
condition of an investor with a concave utility function. By solving the portfolio choice problem for a specific investor, i.e., a given pair of \((\gamma_1, \gamma_2)\), and imposing the condition that he optimally invests only in the set of benchmark assets, we get the condition for mean-variance-skewness intersection for a specific investor in the next proposition. The proof of this proposition and all other proofs are in Section 2 of the Technical Appendix.

**Proposition 1** The mean-variance-skewness frontiers of \(r_x\) and \((r_x, r_y)\) intersect, if

\[
\mu_y - \Sigma_{yx} \Sigma_{xx}^{-1} \mu_x + \gamma_2 \left\{ \Sigma_{yxx} - \Sigma_{yx} \Sigma_{xx}^{-1} \Sigma_{xxx} \right\} \left( w_x^* \otimes w_x^* \right) = 0,
\]

(4)

holds for one particular pair of preference parameters \((\gamma_1, \gamma_2)\) and corresponding \(w_x^*\).

Using (4) and noting that spanning means that any skewness investor only holds the benchmark assets, we obtain the conditions for spanning.

**Proposition 2** The mean-variance-skewness frontier of \(r_x\) spans the frontier of \((r_x, r_y)\), if

\[
\begin{aligned}
\mu_y - \Sigma_{yx} \Sigma_{xx}^{-1} \mu_x &= 0_n, \\
\Sigma_{yxx} - \Sigma_{yx} \Sigma_{xx}^{-1} \Sigma_{xxx} &= 0_{n \times k^2}.
\end{aligned}
\]

(5)

(6)

Notice that our conditions for spanning and intersection nest the conditions for mean-variance spanning as a special case. We get the conditions for mean-variance spanning—i.e., (5), if we set \(\gamma_2\) in (4) to zero. In addition, to see that \(S_{\epsilon xx}\) in (2) contains the restriction in (6) observe that \(S_{\epsilon xx}\) can be rewritten to \(E\left( e_{r_x}^\top r_x^\top \right) = \Sigma_{yxx} - \Sigma_{yx} \Sigma_{xx}^{-1} \Sigma_{xxx}^\top.

Extensions of mean-variance intersection and spanning tests to take into account short-sale constraints and transaction costs developed by DeRoon et al. (2001) can be adapted to the mean-variance-skewness case. In the following, we present the extension to short-sale constraints. The spanning restrictions (5) and (6) then have to hold for each subset of the benchmark assets on which the short-sale constraints are simultaneously not binding. Let these subsets be denoted by \(x_j\), for \(j = 1, 2, \ldots, M\). Using this notation, we can state the following proposition.

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8This is illustrated in Section 1 of the Technical Appendix, and the global second-order conditions require that \(-\gamma_1 \Sigma + 2 \gamma_2 S (w \otimes I)\), where \(I\) is a \(k + n \times k + n\) identity matrix, is negative semidefinite for all relevant \(w\). This assumption is a necessary working assumption implicitly made also by papers studying the pricing of skewness. It rules out the extreme case in which investors wish to invest an infinite amount in the risky asset(s) due to convex (risk-seeking) utility. For our overlap test and spanning with short-sales constraints, this condition is always satisfied when the elements of \(S\) are non-positive.
Proposition 3 The mean-variance-skewness frontier of \( r_x \) spans the frontier of \( (r_x, r_y) \) without short sales on benchmark and test assets, if

\[
\begin{align*}
\mu_y - \Sigma_{yx} \Sigma_{xx}^{-1} \mu_x & \leq 0_n, \\
S_{yx} & - \Sigma_{yx} \Sigma_{xx}^{-1} S_{xx} & \leq 0_{n \times (L')^2},
\end{align*}
\]

for \( j = 1, \ldots, M \) and where the inequalities apply element-wise.

Because skewness spanning contains the conditions for mean-variance spanning as a special case, skewness spanning may not be satisfied although test assets deteriorate portfolio skewness and some investors prefer to hold only the benchmark assets. To detect this situation, we introduce a new concept, specific to our framework, and label it “overlap.” In the context of no short sales, it asks whether mean-variance-skewness frontiers overlap, and no overlap means that test assets provide diversification benefits for all skewness investors.

Corollary 1 The mean-variance-skewness frontiers of \( r_x \) and \( (r_x, r_y) \) overlap without short sales on benchmark and test assets, if at least one element in \( \mu_y - \Sigma_{yx} \Sigma_{xx}^{-1} \mu_x \) or \( S_{yx} - \Sigma_{yx} \Sigma_{xx}^{-1} S_{xx} \) for \( j = 1, \ldots, M \), is non-positive.

Our framework is closely linked to portfolio choice, and portfolio weights can be expressed as a function of alpha and residual co-skewness. This is shown in the next corollary for \( n = 1 \) and \( k = 1 \). The general case and the portfolio weights in the benchmark asset are in the proof of the corollary in the Technical Appendix.

Corollary 2 Suppose that there is one test asset and one benchmark asset. The portfolio weight in the test asset is implied by

\[
w_y^* = \frac{\alpha}{\gamma_1 \text{Var}(\epsilon)} + \frac{\gamma_2}{\gamma_1 \text{Var}(\epsilon)} \left[ \text{Cov}(\epsilon, \tilde{r}_x^2) w_x^* w_y^* + 2 \text{Cov}(\epsilon, \tilde{r}_x \tilde{r}_y) w_x^* w_y^* + \text{Cov}(\epsilon, \tilde{r}_y^2) w_y^* \right],
\]

where \( \alpha \) and \( \epsilon \) are the intercept and the residual in (1), \( \text{Var}(\epsilon) = \Sigma_{\epsilon \epsilon} \), and \( \text{Cov}(\epsilon, \tilde{r}_x \tilde{r}_y) = S_{\epsilon xy} \).

It is easy to check from (9) that we obtain the familiar mean-variance solution \( w_y^* = \alpha / \gamma_1 \text{Var}(\epsilon) \) when \( \gamma_2 = 0 \). In this case, \( w_y^* \) has the same sign as \( \alpha \), which justifies using \( \alpha \) as a metric for performance measurement. In a skewness framework, \( w_y^* \) may instead equal 0 although \( \alpha \neq 0 \). Indeed, if \( w_y^* = 0 \), (9) can be rewritten to \( \alpha = -\gamma_2 \text{Cov}(\epsilon, \tilde{r}_x^2) w_x^* \), which implies that \( \alpha \) has the opposite
sign of \( \text{Cov}(\epsilon, \tilde{r}_2^2) \). Hence, no skewness investor buys (or shorts) asset \( y \) if \( \alpha = 0 \) and \( \text{Cov}(\epsilon, \tilde{r}_2^2) = 0 \) (spanning). Conversely, all skewness investors buy the asset, if \( \alpha > 0 \) and \( \text{Cov}(\epsilon, \tilde{r}_2^2) > 0 \) (no overlap).

3.2 Tests

We first outline the construction of Wald tests for spanning\(^9\) and then introduce a bootstrap test for overlap. The Technical Appendix contains further details on the implementation of the tests, an analysis which shows that the tests have good size and power properties, and a performance analysis of simulated option strategies.

3.2.1 Spanning

The spanning test with short sales is based on Wald tests with equality constraints. Let \( h \) denote a column vector which contains the \( n + nk(k+1)/2 \)-restrictions in (5)-(6), and \( \text{Var} \left[ h \right] \) its variance. As explained in detail in Section 3.1 of the Technical Appendix, we obtain the sample equivalent of \( h \) denoted by \( \hat{h} \) and its estimated covariance matrix \( \text{Var} \left[ \hat{h} \right] \) from multivariate regressions. The Wald statistic for the null hypothesis \( h = 0 \) is

\[
\text{Span}_{x}^{MVS} = \hat{h}^\top \left( \text{Var} \left[ \hat{h} \right] \right)^{-1} \hat{h}.
\]

Under the null hypothesis and standard regularity assumptions, i.e., that the returns on test and benchmark assets are stationary and ergodic, the Wald statistic has a \( \chi^2 \) limiting distribution with \( n + nk(k+1)/2 \) degrees of freedom (dimension of the column vector \( h \)).

To test for spanning without short sales, we use Wald tests with inequality constraints. Let \( \hat{h}^s \) denote the vector which contains the inequality conditions in (7)-(8), and \( \text{Var} \left[ \hat{h}^s \right] \) its variance. The Wald statistic for the null hypothesis \( h^s \leq 0 \) is

\[
\text{Span}_{x}^{MVS} = \min_{h^s \leq 0} \left( \hat{h}^s - \hat{h}^s \right)^\top \left( \text{Var} \left[ \hat{h}^s \right] \right)^{-1} \left( \hat{h}^s - \hat{h}^s \right).
\]

Under the null hypothesis and standard regularity conditions, the probability of \( \text{Span}_{x}^{MVS} \) exceeding a certain value is (see \cite{Kodde and Palm 1986})

\[
Pr \left( \text{Span}_{x}^{MVS} \geq c \right) = \sum_{i=0}^{d} Pr \left( \chi^2_{d-i} \geq c \right) \omega \left( d, i, \text{Var} \left[ \hat{h}^s \right] \right),
\]

\(^9\)Intersection tests based on Proposition 1 can also be constructed with Wald tests.
where $\chi^2_d$ has unit mass, $d$ is the number of elements in the vector $h^s$ and $\omega\left(d, i, Var\left[h^s\right]\right)$ is the probability that $i$ of the $d$ elements of a vector with a $N\left(0_d, Var\left[h^s\right]\right)$ distribution are strictly negative. Following Gourieroux, Holly, and Monfort (1982) and DeRoon, Nijman, and Werker (2001), we determine $\omega$ with simulations. In particular, we take 100,000 draws for each Wald statistic from a normal distribution with expectation $0_d$ and variance $Var\left[h^s\right]$, and $\omega\left(d, i, Var\left[h^s\right]\right)$ is then the average number of draws in which $i$ realizations are below zero.\footnote{We have verified that 100,000 draws are sufficient to obtain accurate weights.}

### 3.2.2 Overlap

The Wald test with inequality constraints (and spanning without short sales as a null hypothesis) requires that only one of the inequalities is strict under the alternative. Therefore, the null hypothesis can be rejected although the benefits of the additional assets are ambiguous because assets may improve the mean-variance trade-off at the cost of a lower skewness. Our overlap test based on Corollary \footnote{The vector $h^s$ can be augmented with elements of the skewness matrix (with opposite sign) to test whether the second-order conditions are globally satisfied.} fixes this problem, and is adapted from Patton and Timmermann (2010)'s bootstrap test for monotonic relationships. Under the null, at least one investor does not benefit from the additional assets and, under the alternative, all investors benefit. As emphasized by Patton and Timmermann (2010), the researcher seeks to prove or disprove the alternative hypothesis. Our null hypothesis is that at least one element of $h^s$ is non-positive, and the alternative is $h^s > 0$ which can be rewritten to $\min_{i=1,\ldots,d} h^s_i > 0$, where as before, $d$ is the number of elements in $h^s$.\footnote{Notice that the alternative in the spanning test with short-sales constraints would correspond to $\max_{i=1,\ldots,d} h^s_i > 0$ in this framework.}

Following Patton and Timmermann (2010), the test statistic is

$$\text{Over}^\text{MVS} = \min_{i=1,\ldots,d} \hat{h}^s_i.$$  

Critical values for this test statistic are not known, and we follow Patton and Timmermann (2010) to obtain the critical values with a bootstrap procedure. We obtain $B$ bootstrapped samples from the original sample using randomized block bootstrap with replacement, and calculate $\hat{h}^b$ with the bootstrapped return series. The test statistic for the bootstrapped series is

$$\overline{\text{Over}}^\text{MVS}_{x,b} = \min_{i=1,\ldots,d} \left( h^b_i - \hat{h}^s_i \right), \quad b = 1, 2, \ldots, B,$$

and the p-value of the test is given by $\frac{1}{B} \sum_{b=1}^B 1\{\overline{\text{Over}}^\text{MVS}_{x,b} > \text{Over}^\text{MVS}_x\}$, where $1\{\cdot\}$ is an indicator function equal to 1 if $\{\cdot\}$ is true. To eliminate the impact of cross-sectional heteroscedasticity in test

Electronic copy available at: https://ssrn.com/abstract=2667473
asset returns and different standard errors in alphas and residual co-skewnesses, we follow Patton and Timmermann (2010) and use the studentized version of this bootstrap.

4 Empirical results

4.1 Data

Our analysis takes the perspective of investors who can initially invest in bonds, stocks, and a risk-free asset. The proxy for stocks and bonds are the S&P 500 total return index from Morningstar (henceforth, stocks) and the 10-year US treasury bond index from CRSP (henceforth, bonds). The 30-day t-bill index from CRSP is used as a proxy for the risk-free rate.

The additional assets are net of fee returns on hedge and mutual funds from the Morningstar database. The mutual funds are live and dead open-end funds, which are classified as US equity by the US Category Group and have the USD as a base currency and at least 36 months of return history available. For each fund identifier, we only keep the share class with the longest return history.

The hedge funds are live and dead funds with USD as base currency. We start with all funds available in the database and then apply the following standard filters: First, we delete the first 12 months of return data for each fund to mitigate the instant history or back-fill bias (Fung and Hsieh, 2001; Bali et al., 2013). Second, each fund is required to have at least 36 months of return data available to have enough data at the individual fund level for our analysis (Bali et al., 2013; Patton and Ramadorai, 2013). Finally and to avoid having very similar investments multiple times, we keep the fund with the longest return history for each strategy identifier. Applying these filters reduces our sample from 9,717 to 4,753 funds (3,021 liquidated, 75 merged, and 1,657 live). The data ranges from January 1993 to December 2014 but effectively starts in January 1994 after the first twelve months of returns for each fund have been deleted. We use unsmoothed hedge fund returns, i.e., returns adjusted such that their first-order autocorrelation is zero (Getmansky et al., 2004). Adjusting for smoothing hardly changes the mean and unstandardized skewness of returns but increases the average volatility from 3.61% to 4.28% per month in our dataset. Hence, using unsmoothed returns makes our results more conservative, and we verify in the robustness section that our results are robust to using raw returns.

Morningstar classifies hedge funds in six broad categories (see Morningstar, 2011): directional debt, directional equity, event, global derivatives, multistrategy, and relative value. Throughout

13 See Ait-Sahalia and Brandt (2001) and Kan and Zhou (2012) for studies that use the same set of benchmark assets.
the analysis, we essentially keep the Morningstar classification except that we assign fund of funds, which are included in the multistrategy category by Morningstar, to a distinct fund of funds category. Funds without any Morningstar classification are assigned to an additional category labeled “other.” To get an idea of the performance of the different categories, we analyze net-asset-value-weighted portfolios of hedge funds in each category in addition to individual funds.

Finally, we also consider the two strategies writing S&P 500 Index put options proposed by Jurek and Stafford (2015) and mentioned in the introduction. These option strategies are designed to satisfy exchange margin requirements and bid-ask spreads, and have been shown to successfully replicate aggregate hedge fund performance. We construct the returns on these strategies with OptionMetrics data from January 1996 to December 2014. At the end of each month, these strategies buy puts with an expiration as close as possible but longer than one month, and sell the previously bought puts. Put1 thereby uses a leverage ratio of 2 and shorts an option with strikes on average approximately 7% below prevailing index levels. Put2 uses a leverage ratio of 4 and uses strikes approximately 14% below prevailing index levels.\footnote{These strikes are selected based on Z-scores at each rebalancing date to avoid time-varying systematic risk exposures. We refer to Jurek and Stafford (2015) for more details on the construction and advantage of these strategies.}

The descriptive statistics of the returns on benchmark assets, option strategies, and portfolios of hedge funds are reported in Panel A of Table 1. Panels B to D contain the percentiles of the cross-sectional distribution of the summary statistics of returns and characteristics of individual funds.

Panel A shows that the average monthly excess returns on stocks and bonds are 0.63% and 0.28% and their standard deviations are 4.32% and 2.03%, respectively, over the sample period. The option strategies seem to almost dominate bonds and stocks based on the first two moments but have more negative skewness and higher excess kurtosis. Portfolios invested in directional debt, event, and multistrategy have a similar average excess return and a lower standard deviation than stocks and are therefore likely to provide a better mean-variance trade-off than stocks. Portfolios invested in multistrategy and relative value have a higher average excess return and a lower standard deviation than bonds. Panel A also shows that all hedge fund strategies have significantly skewed returns. The critical value at a 5% significance level for the null hypothesis that skewness is zero for a sample size of 252 observations is approximately 0.30 under the normal distribution, and the skewness of every strategy exceeds that threshold in absolute value. The strategies also exhibit significant kurtosis because the critical value for the null of kurtosis equal to three is approximately
Table 1: Summary statistics

|                | T | mean | std | skew | kurt | JB | min | max | #funds |
|----------------|---|------|-----|------|------|----|-----|-----|--------|
|                | (in %) | (in %) |     |       |      |    |     |     |        |
| Panel A. Benchmark assets and portfolios of hedge funds |          |      |     |       |      |    |     |     |        |
| stocks         | 252 | 0.63 | 4.32| −0.70| 4.15 | 0.00| −16.87| 10.93 |
| bonds          | 252 | 0.28 | 2.03| 0.06 | 4.12 | 0.01| −6.75 | 8.51  |
| put1           | 227 | 0.84 | 2.27| −3.38| 20.93| 0.01| −15.31| 5.67  |
| put2           | 227 | 0.89 | 1.86| −4.52| 32.46| 0.00| −13.49| 4.32  |
| directional debt       | 191 | 0.58 | 2.76| −2.26| 21.34| 0.00| −19.91| 9.88  |
| directional equity     | 252 | 0.51 | 3.14| −0.55| 4.74 | 0.00| −12.11| 10.03 |
| event           | 239 | 0.56 | 2.29| −1.45| 8.64 | 0.00| −11.44| 6.80  |
| fund of funds    | 252 | 0.30 | 2.15| −0.69| 5.90 | 0.00| −9.17 | 6.41  |
| global derivatives | 252 | 0.49 | 3.08| 0.34 | 4.52 | 0.00| −9.01 | 11.74 |
| multistrategy    | 222 | 0.60 | 1.84| −1.50| 9.80 | 0.00| −10.72| 5.87  |
| relative value   | 236 | 0.36 | 1.58| −3.13| 21.90| 0.00| −11.52| 4.71  |
| other           | 210 | 0.38 | 2.88| −0.87| 9.62 | 0.00| −16.16| 11.24 |
| all strategies   | 252 | 0.38 | 1.97| −0.51| 4.78 | 0.00| −8.45 | 5.80  |

| T | mean | std | skew | kurt | JB | min | max | fund size | #funds | ($ MM) |
|---|------|-----|------|------|----|-----|-----|----------|--------|-------|
| (in %) | (in %) | (in %) | (in %) | (in %) | (in %) | (in %) | (in %) | (in %) | (in %) |
| Panel B. Hedge funds |          |      |     |       |      |    |     |     |        |
| 25th perc | 57  | 0.12 | 2.19| −0.82| 3.62 | 0.00| −17.64| 5.63  |
| median | 87  | 0.37 | 3.26| −0.28| 4.83 | 0.00| −10.55| 9.29  |
| 75th perc | 133 | 0.69 | 5.27| 0.25 | 7.27 | 0.09| −6.36 | 16.99 |
| mean | 102 | 0.43 | 4.27| −0.32| 7.18 | 0.10| −14.01| 14.02 |
| Panel C. Mutual funds |          |      |     |       |      |    |     |     |        |
| 25th perc | 76  | 0.28 | 4.37| −0.72| 3.76 | 0.00| −20.87| 10.78 |
| median | 133 | 0.56 | 4.96| −0.57| 4.30 | 0.00| −18.02| 12.86 |
| 75th perc | 203 | 0.75 | 5.72| −0.34| 4.98 | 0.04| −15.35| 16.22 |
| mean | 141 | 0.49 | 5.15| −0.53| 4.51 | 0.07| −17.87| 14.38 |
| Panel D. Additional hedge fund characteristics |          |      |     |       |      |    |     |     |        |
| 25th perc | 1.00 | 10.00| 0.00| 30.00| 0.10| 0.00| 2.27 |
| median | 1.50 | 20.00| 0.00| 31.00| 0.50| 0.00| 9.99 |
| 75th perc | 2.00 | 20.00| 12.00| 60.00| 1.00| 0.00| 32.40 |
| mean | 1.44 | 15.63| 5.21| 43.76| 1.15| 0.00| 67.11 |
| reported (in %) | 91.29 | 92.32| 64.97| 73.93| 91.02| 90.15|       |

Panel A contains the number of observations (T), average (mean), standard deviation (std), skewness (skew), kurtosis (kurt), p-value of the Jarque-Bera test in the interval [0.001, 0.5], minimum (min) and maximum (max) of the monthly returns in excess of the risk-free rate on benchmark assets (stocks and bonds), option strategies (put1 and put2), and portfolios invested in all hedge funds and within a Morningstar category. These portfolios are net asset value weighted and at least 10 funds are required to report returns and net asset values in a given month to calculate a portfolio return. The column #funds shows the number of distinct funds in each category. The proxy for the stock and bond markets are the S&P 500 TR index and 10 year US T-Bonds, respectively, and the 30 day T-Bill proxies for the risk-free rate. Panel B and C show the percentiles of the summary statistics for individual hedge and mutual funds. The fund size is reported for 1,619 out of 4,753 hedge funds and 2,165 out of 3,905 mutual funds. Panel D shows additional hedge fund characteristics, i.e., management fee (mfee), performance fee (pfee), lockup period (lockup), advance notice period (adv notice), minimum investment amount (minv), and first available net asset value (first NAV). Hedge fund returns are adjusted for smoothing. The sample period is, after deleting the first 12 months of returns for every hedge fund, from January 1994 to December 2014.
3.60 under the normal distribution. Although kurtosis is statistically significant, we will show in Section [4.4.3] that the effect of kurtosis on utility is economically small. In terms of number of funds in each category, the directional equity category contains the largest number of funds followed by the fund of funds category. All categories contain more than 200 funds.

Portfolios of funds understate the standard deviation of hedge fund returns since the average standard deviation is 4.27% in Panel B, which is more than twice as large as the standard deviation of the portfolios of hedge funds. Individual hedge funds have slightly lower average excess returns than mutual funds but also lower standard deviations. The dispersion in the cross-sectional distribution of skewness of individual hedge fund returns is larger than the dispersion in skewness of mutual funds, and the magnitude of mutual fund skewness is very similar to that of the market, suggesting that skewness is essentially spanned by the market. Finally, our hedge funds have typical hedge fund features such as few assets under management, substantial management and performance fees, withdrawal restrictions, and high minimum investment requirements.

4.2 Portfolios of hedge funds and option returns

The summary statistics suggest that some portfolios of hedge funds may provide substantial diversification benefits to bond and stock investors. Formal tests of this hypothesis are reported in Table 2. We consider in columns 1-4 mean-variance and skewness spanning denoted by the superscripts $^{MV}$ and $^{MVS}$, respectively. The mean-variance tests are included because these are the standard tests used in the literature in the form of hypothesis tests on alphas in performance regressions and they can be compared to our tests to determine whether alpha or co-skewness is driving a rejection. The last two columns of the table contain the results of the overlap tests. To understand which benchmark asset drives the rejection, we also test for overlap with only stocks as the benchmark denoted by the subscript $s$.

Our tests suggest that there is strong evidence against spanning with short sales, and this evidence is slightly weaker without short-sales. Spanning is then rejected for the option strategies, event, global derivatives, multistrategy, and relative value. Notice that the benefits of the portfolio of directional debt in terms of skewness can only be achieved with short-positions, and it is important to take short sales into account when testing for skewness spanning.

For most hedge fund strategies, skewness spanning with short-sale constraints, Span$_{s,b}^{MVS}$, is rejected despite a negative residual co-skewness. As shown by Span$_{s,b}^{MV}$, this happens because a subset of the skewness investors wish to invest in the hedge funds. More precisely, once alpha

15 Throughout the paper “significant” means significant at the 5% level.
| Test                        | Spans_{M,s,b} | Span_{M,s,b} | Spans_{M,s,b} | Span_{M,s,b} | Over_{M,s,b} | Over_{M,V,N} |
|-----------------------------|---------------|--------------|---------------|--------------|--------------|--------------|
| directional debt           | 3.89          | 3.89         | 21.83         | 3.89         | -1.31        | -1.26        |
|                            | (0.05)        | (0.02)       | (0.00)        | (0.18)       | (0.79)       | (0.82)       |
| directional equity          | 2.42          | 2.42         | 5.69          | 2.72         | -0.99        | -0.99        |
|                            | (0.12)        | (0.06)       | (0.22)        | (0.27)       | (0.46)       | (0.68)       |
| event                      | 8.23          | 8.23         | 27.96         | 8.30         | -1.83        | -1.83        |
|                            | (0.00)        | (0.00)       | (0.00)        | (0.02)       | (0.90)       | (0.90)       |
| fund of funds              | 0.59          | 0.59         | 2.68          | 0.71         | -0.90        | -0.92        |
|                            | (0.44)        | (0.22)       | (0.61)        | (0.62)       | (0.53)       | (0.70)       |
| global derivatives         | 3.81          | 3.81         | 7.70          | 7.44         | 0.31         | 1.32         |
|                            | (0.05)        | (0.03)       | (0.10)        | (0.03)       | (0.01)       | (0.01)       |
| multistrategy              | 19.74         | 19.74        | 48.38         | 19.82        | -1.84        | -1.86        |
|                            | (0.00)        | (0.00)       | (0.00)        | (0.00)       | (0.88)       | (0.90)       |
| relative value             | 9.18          | 9.18         | 34.23         | 9.18         | -1.40        | -1.48        |
|                            | (0.00)        | (0.00)       | (0.00)        | (0.02)       | (0.80)       | (0.86)       |
| other                      | 0.40          | 0.40         | 6.82          | 0.40         | -1.61        | -1.55        |
|                            | (0.53)        | (0.26)       | (0.15)        | (0.68)       | (0.87)       | (0.90)       |
| put1                       | 28.84         | 28.84        | 195.08        | 29.00        | -5.06        | -5.11        |
|                            | (0.00)        | (0.00)       | (0.00)        | (0.00)       | (1.00)       | (1.00)       |
| put2                       | 35.34         | 35.34        | 263.64        | 35.34        | -3.47        | -3.44        |
|                            | (0.00)        | (0.00)       | (0.00)        | (0.00)       | (1.00)       | (0.99)       |

The test assets are net asset value weighted portfolios of hedge funds within one Morningstar category and option strategies (put1 and put2). The benchmark assets are stocks (denoted by the subscript s), i.e., the S&P 500 Total Return Index, and bonds (denoted by the subscript b), i.e., 10 year US government bonds. Spans and Span refer to the spanning test with short-sales and without short-sales, respectively, and the superscripts MV refers to the mean-variance, and MVS to the mean-variance-skewness test. Over refers to the overlap test. P-values based on [White 1980] standard errors (columns 1-4) and block bootstrapped standard errors (columns 5-6) are reported in parentheses. Bold font indicates significance at the 5% significance level.
is positive, there will be an investor whose skewness preference is sufficiently small such that the skewness deterioration does not matter to him/her, and spanning can be rejected albeit due to an extreme case. The overlap tests, \( \text{Over}^{MVS}_{s,b} \) and \( \text{Over}^{MVS}_{s} \), reported in the last two columns of Table 2 address this issue. They test whether at least one investor does not benefit from the test assets versus the alternative where all skewness investors benefit. These tests are demanding. Nevertheless, the null is rejected for global derivatives funds. In contrast, for option strategies the null cannot be rejected and the p-values are close to one. Finally, we obtain almost identical results with \( \text{Over}^{MVS}_{s} \) and \( \text{Over}^{MVS}_{s,b} \), suggesting that the tail risk of the portfolio of hedge funds is coming from their exposure to stocks. As we show in the next section, this is not always the case for individual funds; it is likely a by-product of diversification across several hedge funds.

The bottom line is only portfolios of global derivatives seem to offer benefits for all skewness investors. As shown in the descriptive statistics, however, portfolios have a lower standard deviation than individual returns and may therefore overstate the diversification benefits. In addition, most hedge funds have very high minimum investment requirements (see Panel D in Table 1), making it likely that some investors only choose to hold a single hedge fund within their portfolio of bonds and stocks. These issues motivate the following analysis of the diversification benefits of individual hedge funds.

### 4.3 Individual funds

Investors initially invest in stocks, or in stocks and bonds, and face short-sale constraints. We compare the results of our tests to the mean-variance spanning case and to three factor models: the Fama-French-Carhart four-factor model,\(^\text{16}\) the Fung and Hsieh (2001) eight-factor model,\(^\text{17}\) and a two-factor model with put1 and the return on stocks. Given that the analysis relies on rejection rates, we also carry out the analysis on the mutual fund sample to have a basis for comparison. All results are reported in Table 3.

The table confirms that portfolios overstate the diversification benefits of hedge fund investments. While mean-variance spanning without short sales, \( \text{Span}^{MV}_{sb} \), was rejected for more than half of the portfolios (categories) of hedge funds, it is rejected only for 26% of the individual funds. Similarly, \( \text{Span}^{MVS}_{sb} \) is rejected for 31% of the individual funds whereas it was rejected for half of the portfolios of hedge funds. At the same time, portfolios of hedge funds tend to overstate to some

\(^{16}\) See Fama and French (1993). These factors are obtained from: [http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/index.html](http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/index.html)

\(^{17}\) The factors are the market, size, bond, credit spread, emerging market, look-back straddles tracking bond, currency, and commodity trend-following returns; see [https://faculty.fuqua.duke.edu/~dah7/HFRFData.htm](https://faculty.fuqua.duke.edu/~dah7/HFRFData.htm) for further information, and [http://faculty.fuqua.duke.edu/~dah7/DataLibrary/TF-FAC.xls](http://faculty.fuqua.duke.edu/~dah7/DataLibrary/TF-FAC.xls) for the data.
Table 3: Tests with individual funds

|                | Unconditional | Most similar | Least similar | (1) | (2) | (3) | (4) | (5) | (6) | (7) | (8) | (9) |
|----------------|---------------|--------------|--------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| **Hedge Funds**|               |              |              |     |     |     |     |     |     |     |     |     |
| (1) SpanMV     | 26.2          | (7), (4), (9)| (6), (3), (5)| -   | 78  | 51  | 90  | 55  | 33  | 95  | 78  | 79  |
| (2) SpanMV     | 25.2          | (7), (9), (1)| (6), (3), (5)| 75  | -   | 60  | 73  | 64  | 43  | 81  | 66  | 77  |
| (3) OverMV     | 18.3          | (6), (9), (2)| (8), (5), (4)| 36  | 43  | -   | 36  | 35  | 53  | 38  | 38  | 51  |
| (4) SpanMV     | 26.1          | (7), (1), (2)| (6), (3), (5)| 90  | 76  | 51  | -   | 57  | 35  | 91  | 75  | 76  |
| (5) SpanMV     | 31.0          | (2), (7), (9)| (6), (3), (8)| 66  | 79  | 59  | 68  | -   | 49  | 72  | 62  | 69  |
| (6) OverMV     | 11.4          | (3), (9), (2)| (1), (4), (8)| 15  | 19  | 33  | 15  | 18  | -   | 16  | 16  | 20  |
| (7) FF4        | 21.6          | (1), (4), (8)| (6), (3), (5)| 79  | 70  | 45  | 75  | 50  | 31  | -   | 70  | 69  |
| (8) FH         | 26.0          | (7), (1), (4)| (6), (3), (5)| 77  | 68  | 49  | 75  | 52  | 35  | 85  | -   | 69  |
| (9) sput1      | 22.3          | (7), (2), (1)| (6), (5), (8)| 67  | 68  | 62  | 64  | 50  | 39  | 71  | 59  | -   |
| **Mutual Funds**|               |              |              |     |     |     |     |     |     |     |     |     |
| (1) SpanMV     | 6.7           | (9), (4), (7)| (6), (3), (5)| -   | 69  | 17  | 72  | 29  | 11  | 71  | 54  | 76  |
| (2) SpanMV     | 4.8           | (9), (7), (1)| (6), (3), (8)| 49  | -   | 24  | 40  | 37  | 13  | 51  | 35  | 66  |
| (3) OverMV     | 2.6           | (6), (9), (2)| (1), (8), (5)| 6   | 13  | -   | 10  | 21  | 12  | 8   | 21  |
| (4) SpanMV     | 8.7           | (1), (9), (7)| (6), (5), (3)| 94  | 72  | 32  | -   | 32  | 15  | 77  | 67  | 83  |
| (5) SpanMV     | 8.8           | (2), (9), (7)| (6), (8), (4)| 37  | 68  | 32  | 32  | -   | 24  | 41  | 28  | 62  |
| (6) OverMV     | 3.4           | (3), (9), (5)| (1), (8), (4)| 5   | 9   | 27  | 6   | 9   | -   | 8   | 5   | 13  |
| (7) FF4        | 4.2           | (9), (2), (1)| (6), (3), (5)| 45  | 45  | 19  | 37  | 20  | 10  | -   | 38  | 49  |
| (8) FH         | 8.9           | (7), (1), (4)| (6), (3), (5)| 73  | 65  | 26  | 69  | 28  | 15  | 81  | -   | 66  |
| (9) sput1      | 3.2           | (2), (7), (1)| (6), (5), (8)| 36  | 44  | 25  | 30  | 23  | 12  | 37  | 23  | -   |

The investment opportunity set in rows (1)-(3) consists of only stocks indicated by $s$, and stocks and bonds indicated by $s_b$, in rows (4)-(6). SpanMV (SpanMV) refers to the corresponding mean-variance-skewness (mean-variance) spanning tests without short-sales, and OverMV is the mean-variance-skewness overlap test. We compare these spanning tests to the rejection of the null of a non-positive alpha in the Fama-French-Carhart model in row (7) FF4, and the Fung-Hsieh factor model in row (8) FH, and a two factor models with put1 and market return in row (9) sput1. The column unconditional reports the average rejection of the null hypothesis at a 5% significance level, and the other columns report rejection frequencies conditional on rejecting the test indicated with the column number. For example, for hedge funds and in row (8) and column (6), the null of a non-positive Fung and Hsieh factor alpha is rejected 35% of the time when the null of the OverMV test is rejected. The columns ‘most similar’ and ‘least similar’ report in descending order the three tests which would be considered as such based on the conditional rejection frequencies. All tests use White (1980) standard errors, except OverMV and OverMV which use block bootstrapped errors.
extend the co-skewness of hedge funds. The overlap test with only stocks as benchmark, $\text{Overlap}^{\text{MVS}}_s$, is rejected 18% of the time, while it was rejected only once for portfolios of hedge funds. The lower rejection rate of 11% for $\text{Overlap}^{\text{MVS}}_{s,b}$ suggests that some hedge funds have a negative co-skewness with bonds.\footnote{One may ask whether a rejection rate of 11% is significantly different from 5% (the level of the test). In an analysis available upon request, we calculated that the bootstrapped 95% confidence interval of this rejection frequency ranges from 10% to 12% supporting the claim that 11% is a significant fraction.}

The table also answers the question of how similar the conclusions of the proposed spanning tests are, especially when compared with mean-variance spanning and the factor models. While overall rejection rates for $\text{Span}^{\text{MV}}$ and $\text{Span}^{\text{MVS}}$ are similar, in the cross-section the two tests are often rejected for different funds, particularly with bonds and stocks as benchmark assets. The explanation is that $\text{Span}^{\text{MVS}}$ accounts also for co-skewness(es) and thus even if alpha is positive, the joint test of alphas and co-skewnesses may not have enough power to reject the null. In addition, negative alpha funds may have benefits in terms of co-skewness and thus $\text{Span}^{\text{MVS}}$ can be rejected. Among all tests, the overlap tests stand out as being the most different. $\text{Overlap}^{\text{MVS}}_s$ rejects the null ranging from 34% to 54% of the time conditional on the null being rejected with one of the other tests, and this number ranges from 15% to 33% for $\text{Overlap}^{\text{MVS}}_{s,b}$. This is perhaps unsurprising given that these tests take into account an additional performance attribute, co-skewness, in addition to alpha, and require funds to be attractive in terms of both alpha and co-skewness. When compared to each other, these tests can still lead to different conclusions because funds’ co-skewnesses with bonds and the product of stocks and bonds, which enter the calculation of $\text{Overlap}^{\text{MVS}}_{s,b}$, can differ from a funds’ stock co-skewness. $\text{Span}^{\text{MV}}_s$ and $\text{Span}^{\text{MV}}_{s,b}$ are instead more similar suggesting that hedge fund returns have low betas with bonds.

Next, we compare our hedge fund results to those obtained with mutual funds as reported in the bottom part of Table 3. We find that considerably fewer mutual funds significantly improve the portfolios for stock and bond investors. For example, $\text{Overlap}^{\text{MVS}}_{s,b}$ suggests that less than 4% of the mutual funds reliably improve the investment opportunity set of all skewness investors. Given our size analysis in Section 4 of the Technical Appendix, by “luck” and if all mutual fund returns are generated by the same underlying no-skill distribution (after fees), this number equals 5%. The lower number then suggests a data-generating process in which funds have negative skill (measured by negative co-skewness and/or alpha). More generally, the rejection rates for all tests are low, and in the range of 2.5% to 9.2%. Perhaps these low rejection rates are unsurprising given that due to “equilibrium accounting” the alphas before fees (and the residual co-skewness) of all investment
strategies need to sum to zero, and thus the net-of-fee alphas of most mutual funds should and are generally found to be negative \cite{Fama2010}. Taking skewness into account also yields different conclusions for mutual funds, and the overlap tests again generate results that are most different. For example, \text{Over}_s^{MVS} is only rejected 7\% of the time conditional on \text{Span}_s^{MV} being rejected, which suggests a trade-off between alpha and co-skewness for mutual funds as documented by \cite{Back2018}. In sum, the results suggest that (co-)skewness matters for mutual funds as well.

By explicitly comparing different tests including the Fung-Hsieh eight-factor model, an option-based factor model, and simple mean-variance spanning tests, Table 3 shows that the overlap tests win the horse race of taking into account non-linearities. In particular, the two competing models tend to be most similar to mean-variance spanning tests. The Fung-Hsieh model is most similar to the Fama-French-Carhart model, \text{Span}_s^{MV} and \text{Span}_{s,b}^{MV}, suggesting that the additional factors included in the Fung-Hsieh and Fama-French-Carhart model are unable to explain the alpha of hedge funds. While the option-based factor model does slightly better by including \text{Span}_{s}^{MVS} in the top three comparison models, it otherwise includes the Fama-French-Carhart model and \text{Span}_{s}^{MV}. This confirms that hedge fund’s beta with respect to put1 does not subsume the co-skewness with respect to stocks.

Notice that \text{Span}_{s,b}^{MV} tends to yield similar conclusions as \text{Span}_{s}^{MV} and the three factor models: the rejection frequencies of \text{Span}_{s,b}^{MV} conditional on rejecting with these models range from 76\% to 91\%. The corresponding opposite rejection rates—rejecting with one of the factor models conditional on rejecting with a mean-variance spanning test—are also high and range from 64\% to 90\%. Thus, we only analyze the rejections of \text{Over}_s^{MVS}, \text{Over}_{s,b}^{MVS}, \text{Span}_{s,b}^{MV}, and \text{Span}_{s,b}^{MVS} in the remainder of the paper.

4.4 Economic interpretation

4.4.1 Relation with skewness and timing ability

Skewness investors may wish to narrow down the search for the appropriate fund by focusing on hedge funds with a high skewness. To analyze this scenario, we assign funds to quintiles based on their sample skewness and calculate rejection rates of spanning and overlap in each group. The results in Table 4 show that the rejection rates of spanning and overlap increase with skewness, and the hedge funds in the top skewness group are most likely to improve the investment opportunity set of all skewness investors. For mutual funds, the pattern is less clear-cut and the rejection rates of spanning and overlap for the funds in the highest quintile are lower than in the fourth quintile.
Table 4: Tests with individual funds sorted by skewness and timing ability

| skewness      | Q1   | Q2   | Q3   | Q4   | Q5   | timing  | market  | macro  |
|---------------|------|------|------|------|------|---------|---------|--------|
| Q1            |      |      |      |      |      | neg     | no      | pos    |
| Q2            |      |      |      |      |      | 19.2    | 25.4    | 30.8   |
| Q3            |      |      |      |      |      | 29.5    | 30.7    | 32.9   |
| Q4            |      |      |      |      |      | 3.8     | 9.7     | 20.3   |
| Q5            |      |      |      |      |      | 10.3    | 16.5    | 27.7   |
| low skew      |      |      |      |      |      | 156     | 3741    | 856    |
| high skew     |      |      |      |      |      | 775     | 3799    | 179    |
| hedge funds   |      |      |      |      |      | 951     | 950     | 951    |
| Span_{MV}     | 16.2 | 20.9 | 22.8 | 28.3 | 42.7 | 19.2    | 25.4    | 30.8   |
| Span_{MVS}    | 24.4 | 25.5 | 30.3 | 33.7 | 41.3 | 29.5    | 30.7    | 32.9   |
| Over_{MVS}    | 6.9  | 10.2 | 11.7 | 14.0 | 14.4 | 3.8     | 9.7     | 20.3   |
| Over_{MVS}    | 9.0  | 9.6  | 17.4 | 24.8 | 30.6 | 10.3    | 16.5    | 27.7   |
| Over_{MVS}    | 6.9  | 10.2 | 11.7 | 14.0 | 14.4 | 3.8     | 9.7     | 20.3   |
| Over_{MVS}    | 9.0  | 9.6  | 17.4 | 24.8 | 30.6 | 10.3    | 16.5    | 27.7   |
| Over_{MVS}    | 6.9  | 10.2 | 11.7 | 14.0 | 14.4 | 3.8     | 9.7     | 20.3   |
| Over_{MVS}    | 9.0  | 9.6  | 17.4 | 24.8 | 30.6 | 10.3    | 16.5    | 27.7   |
| #funds        | 951  | 950  | 951  | 950  | 951  | 156     | 3741    | 856    |
| mutual funds  |      |      |      |      |      | 775     | 3799    | 179    |
| Span_{MV}     | 5.9  | 8.6  | 11.0 | 10.8 | 7.6  | 3.3     | 8.9     | 8.9    |
| Span_{MVS}    | 5.6  | 6.1  | 9.0  | 11.7 | 11.3 | 11.5    | 8.9     | 6.9    |
| Over_{MVS}    | 0.4  | 2.0  | 3.7  | 5.8  | 4.9  | 3.3     | 3.0     | 6.4    |
| Over_{MVS}    | 0.5  | 1.3  | 2.8  | 6.1  | 2.3  | 1.6     | 2.4     | 4.9    |
| Over_{MVS}    | 6.9  | 10.2 | 11.7 | 14.0 | 14.4 | 3.8     | 9.7     | 20.3   |
| Over_{MVS}    | 9.0  | 9.6  | 17.4 | 24.8 | 30.6 | 10.3    | 16.5    | 27.7   |
| Over_{MVS}    | 6.9  | 10.2 | 11.7 | 14.0 | 14.4 | 3.8     | 9.7     | 20.3   |
| Over_{MVS}    | 9.0  | 9.6  | 17.4 | 24.8 | 30.6 | 10.3    | 16.5    | 27.7   |
| Over_{MVS}    | 6.9  | 10.2 | 11.7 | 14.0 | 14.4 | 3.8     | 9.7     | 20.3   |
| Over_{MVS}    | 9.0  | 9.6  | 17.4 | 24.8 | 30.6 | 10.3    | 16.5    | 27.7   |
| #funds        | 781  | 781  | 781  | 781  | 781  | 122     | 3377    | 406    |
| mutual funds  |      |      |      |      |      | 788     | 3096    | 21     |

Funds are assigned to bins either based on their skewness (quintiles Q1 to Q5, header: skewness) and timing ability (header: timing). Following Henriksson and Merton (1981), market-timing ability is estimated as the coefficient on a variable equal to the maximum of zero and excess market returns in a multiple regression with the fund excess returns as the dependent variable and the excess market return as the control variable. Following Bali et al. (2014) and using their uncertainty index, we estimate a similar regression in which the excess market return is replaced by the uncertainty index, and macro-timing ability is measured by the coefficient on a variable equal to the uncertainty index when the index is above its time-series median and zero otherwise. We assign funds into bins depending on whether the slope coefficient of the timing variable is significantly negative (neg), positive (pos), or insignificant (no). The table reports the number of funds in each bin #funds, and the rejection rates for the tests outlined in the caption of Table 3.

A possible explanation is that funds in the high skewness quintile have an inferior performance because they have a lower skill in risk management and invest in lottery-type stocks that help attract investor flows, especially from retail investors.\(^6\)

The table also analyses spanning and overlap for funds classified by their market and macro-timing ability. For market timing ability, we follow Henriksson and Merton (1981) and run a regression of fund returns on an intercept, excess market returns, and the market-timing variable equal to the excess market return when it is positive and zero otherwise.\(^20\) Successful market timing involves an increase in exposure prior to a market rise, and translates into a positive and significant coefficient on the timing variable. Macro-timing ability is estimated following Bali et al. (2014) and using their macroeconomic risk index.\(^21\) Specifically, we run a regression of fund returns on an intercept, the macroeconomic risk index, and the macro-timing variable equal to the macroeconomic

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\(^6\) We thank the associate editor for suggesting this interpretation.

\(^20\) An alternative way to assess market timing is to follow Treynor and Mazuy (1966) and use the quadratic market model. Here, the market timing variable is the coefficient on the squared market return, which is almost perfectly correlated with co-skewness and would obfuscate the distinction between co-skewness and market timing.

\(^21\) The macroeconomic risk index is available on Turan Bali’s website: [http://faculty.msb.edu/tgb27/workingpapers.html](http://faculty.msb.edu/tgb27/workingpapers.html). We thank him for making his data available.
risk index when it is above its median and zero otherwise. A positive and significant coefficient on the timing variable indicates macro-timing ability. The right-hand side of Table 4 reports the rejection rates separately for funds with negative, positive, and insignificant market and macro-timing ability. The results suggest a positive relationship between spanning/overlap and timing ability. All skewness investors are most likely to benefit from hedge funds with positive market timing ability and positive macro-timing ability. The rejection rates for mutual funds also suggest that all skewness investors are most likely to benefit from mutual funds with positive market timing ability, although the total fraction of funds with positive market timing is only half as large. In addition, only very few mutual funds—21 out of 3905—have macro-timing ability. As a consequence, these rejection rates should be interpreted with caution.

Taken together, the results show that hedge funds are better able to actively vary their exposure to market and macroeconomic risk in an advantageous way than mutual funds. As a result they generate both positive alpha and desirable co-skewness, and are more likely to be attractive to all skewness investors.

4.4.2 Relation to fund characteristics and investment strategies

To get more insight about how our tests are related to hedge fund characteristics and investment strategies, we run cross-sectional regressions of the intercept of the mean-variance regression in $\text{Span}^{\text{MV}}_{k,b}$ on fund characteristics and dummies for the investment strategies.\footnote{The dependent variable in these regressions is estimated, which can introduce a bias if the measurement error is correlated with the regressors. In our setting as in, e.g., \cite{Aragon2007} or \cite{Teo2011}, this is unlikely to be the case because the regressors are all predetermined.} We also run these regressions for each component of residual co-skewness: the co-skewness with stocks, with bonds, and the interaction of stocks and bonds. Table 5 reports the results.

Simple alphas are positively related to management and performance fees. This upholds the intuition that better funds are able to charge higher management fees and that performance fees better align the incentives of managers and investors. In addition, simple alphas are positively related to minimum investment requirements. To the extent that high minimum investment requirements make an investment more illiquid, this positive relation may reflect that investor capital is competitively supplied to the hedge fund industry \cite{Aragon2007}. For residual co-skewness with stocks, there is no robust relationship with compensation, but a significant negative relation with the lockup and advance notice period—a proxy for managerial discretion. This can be explained if funds choose their capital structure and investment strategies jointly in the vein of \cite{HombertThesmar2014}: funds focusing on investments with lower residual co-skewness may optimally
### Table 5: Hedge fund characteristics, alpha, and residual co-skewness

| dep var | alpha in % | s² | s × b | b² |
|---------|------------|----|-------|----|
| model (1) (2) (3) (4) (5) (6) (7) (8) |
| mfee | 0.069 (0.00) | 0.003 | -0.073 | -0.043 | -0.067 | 0.024 | 0.037 |
| pfee | 0.012 (0.00) | 0.017 | 0.003 | -0.002 | -0.001 | 0.002 | 0.001 |
| dhwater | -0.028 (0.44) | 0.049 | 0.087 | 0.041 | 0.034 | 0.013 | 0.022 |
| minv | 0.028 (0.00) | 0.032 | 0.032 | 0.020 | 0.017 | -0.007 | -0.002 |
| loglocknot | 0.009 (0.42) | 0.170 | -0.095 | 0.011 | 0.026 | 0.003 | -0.004 |
| logsize | -0.023 (0.00) | 0.016 | 0.004 | -0.001 | 0.001 | -0.001 | -0.001 |
| ddebt | 0.049 (0.51) | -0.479 | 0.042 | -0.047 | 0.032 | 0.151 |
| dequity | -0.029 (0.60) | 0.090 | 0.053 | 0.067 | 0.046 | 0.000 |
| event | 0.064 (0.35) | -0.431 | 0.084 | -0.030 | 0.033 | 0.061 |
| fof | -0.107 (0.07) | 0.097 | 0.248 | 0.097 | 0.248 | 0.097 |
| gderivatives | 0.052 (0.41) | 0.697 | 0.248 | 0.097 | 0.248 | 0.097 |
| mstrategy | 0.101 (0.16) | -0.211 | 0.114 | -0.003 | 0.000 | 0.104 |
| rvalue | -0.120 (0.06) | -0.601 | -0.003 | -0.003 | -0.003 | 0.095 |
| AdjR2 | 4.4% | 5.3% | 9.9% | 3.1% | 0.2% | 0.7% | 0.0% | 0.9% |
| #obs | 2451 | 2451 | 2451 | 2451 | 2451 | 2451 | 2451 | 2451 |

Each column reports a regression of the indicated dependent variable depvar on hedge fund characteristics. Residual co-skewness with stocks (bonds) is denoted by $s^2 (b^2)$, and the cross residual co-skewness is denoted by $s \times b$. The independent variables are (intercepts not tabulated) the management fee in percent (mfee), the performance fee in percent (pfee), a dummy equal to 1 if the fund has a high watermark provision (dhwater), the minimum investment amount in million of USD (minv), the logarithm of 1 plus the sum of the number of lockup months and the advanced notice days divided by 30 (loglocknot), the logarithm of the net asset value of the fund at inception (logsize), and dummies for all Morningstar categories except other, i.e., directional debt (ddebt), directional equity (dequity), event (event), fund of funds (fof), global derivatives (gderivatives), multistrategy (mstrategy), and relative value (rvalue). The dependent and independent variables (except dummies) are winsorized at the 1% percent level. P-values are reported under the coefficients in parentheses. The last two rows of the table report the adjusted $r^2$ and the number of observations in the regression. Coefficients significant at the 5% percent level are highlighted with bold font.
choose more managerial discretion to avoid liquidating their positions after large temporary losses. There is no robust relationship between fund characteristics and the co-skewness with the bonds and the product of stocks and bonds.

When comparing the explanatory power of characteristics and investment strategies, characteristics better explain cross-sectional variations in alpha but investment strategies have more explanatory power for the co-skewnesses. As suggested by the portfolio analysis in Section 4.2 and compared to other funds, global derivatives funds have positive co-skewnesses with the benchmark assets, which are significant for the co-skewnesses with stocks and the product of stocks and bonds. Mitchell and Pulvino (2001) show that risk arbitrage hedge funds have returns that are similar to those obtained from short-selling index put options. In the Morningstar classification, event and relative value funds mostly follow arbitrage strategies, and the table suggests that these funds have more negative co-skewness (although only at the 10% significance level for event funds). Finally, directional equity funds have a larger co-skewness with bonds.

In Table TA.3 of the Technical Appendix, we analyze the relation of investment strategies with residual co-skewness further with rejection rates in Morningstar subcategories (see Morningstar 2011 for a description of these categories). This analysis shows that in the four global derivatives subcategories—currency, global macro, systematic futures, and volatility—the desirable positive co-skewness is most pronounced in the systematic futures category. The rejection rates within the fund of funds subcategories confirm this result because the rejection rates are the highest for the macro/systematic category and of similar magnitude than those in the systematic futures category. Taken together, this evidence suggests that the systematic futures strategy, which involves trend-following strategies in liquid global futures, options, and foreign-exchange contracts, is most appropriate for all skewness investors. This makes intuitive sense because these strategies prosper when markets demonstrate sustained bullish or bearish trends (Morningstar 2011) or, in other words, when monthly market returns are large in absolute value. This generates a positive relation between the excess returns on the strategy and squared market returns.

4.4.3 Magnitude of utility gains

Having shown in Section 4.3 that 11% of the hedge funds significantly improve the investment opportunity set of all skewness investors, we analyze below the magnitude of the utility gains of including these funds into a portfolio. We measure utility gains with alphas, and distinguish between

23The analysis of Morningstar subcategories discussed below also points to the merger arbitrage strategies, which lose most of their appeal once co-skewness is accounted for: the rejection rate of 81% for mean-variance spanning is reduced to 9.5% when all skewness investors are considered.
the gains due to the first two return moments, and the gains due to skewness. To obtain an alpha-like measure for skewness, it is necessary to further specify the risk preferences of the investor. This corresponds to Proposition 1, and we can define the alpha adjusted for skewness (henceforth also referred to as prudent alpha) as

$$\alpha_{Pr}^i = \alpha_i + \gamma_2 E[\epsilon_i \tilde{r}_p^2],$$

(10)

where \(\alpha_i\) is the intercept of the linear regression of asset \(i\)'s return on the benchmark asset returns (i.e., the simple or Jensen’s alpha), \(\epsilon_i\) is the residual of that regression, and \(\tilde{r}_p\) is the demeaned return on the optimal benchmark portfolio of the investor. Notice that if the individual with preferences \((\gamma_1, \gamma_2)\) optimally holds portfolio \(r_p\), then \(\alpha_{Pr}^i = 0\). If there is no overlap, \(\alpha_{Pr}^i\) measures the additional return the asset delivers in excess of the return required by the investor.

The comparison of simple and prudent alphas measures the magnitude of the utility gains associated to skewness. In addition, prudent alpha can be compared with the generalized alpha (i.e., the alpha adjusted for all higher moments), and the kurtosis adjusted alpha to assess how important the skewness adjustment is. The generalized alpha for a CRRA utility function \([\text{Leland}, 1999]\) is

$$\alpha_i^{CRRA} = E[r_i] - \beta_i^{CRRA}E[r_p],$$

(11)

where \(\beta_i^{CRRA} = \text{Cov}(r_i, (R_f + r_p)^{-\gamma}) / \text{Cov}(r_p, (R_f + r_p)^{-\gamma})\), \(\gamma\) is the coefficient of relative risk aversion of the investor and \(R_f\) is the gross risk-free rate. Below we also calculate the impact of the next moment, kurtosis. The alpha adjusted for moments up to kurtosis will be referred to as temperant alpha, and it can be obtained in the same way as the prudent alpha. It equals

$$\alpha_{Te}^i = \alpha_{Pr}^i - \gamma_3 E[\epsilon_i \tilde{r}_p^3],$$

(12)

where \(\gamma_3\) is a positive scalar which measures the aversion to kurtosis (relative to the preference for the mean).

Next, we calculate these adjusted alphas for the individual funds for which Over\(_{b,b}\) is rejected to

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\(^{24}\)The term prudence was coined by \([\text{Kimball}, 1990]\), and refers to a positive third derivative of a utility function.

\(^{25}\)To obtain the expression for prudent alpha, assume that \(\alpha_i\) is not equal to zero but equal to \(\alpha_{Pr}^i\) for test asset \(i\). Then notice that \(\alpha_i = \mu_i - \Sigma_{y,x}^{-1} \mu_x\) and \(E[\epsilon_i \tilde{r}_p^2] = \{ S_{y,x} - \Sigma_{y,x} \Sigma_{x,x}^{-1} S_{x,x} \} (w_x \otimes w_p).\)

\(^{26}\)Temperance is a term coined by \([\text{Kimball}, 1992]\) to refer to a negative fourth derivative of the utility function which measures the aversion to the fourth moment.

\(^{27}\)Following the approach of Appendix 1 \(\gamma_3\) equals \(-\frac{1}{6} u''''(R_f + \mu_p)\) in an expected utility framework.
assess the magnitude of the utility gains when overlap is rejected. The adjusted alphas are calculated for a CRRA utility function with relative risk aversion of 10 which implies $\gamma_1 = \gamma/(R_f + \mu_p)$ and $\gamma_2 = \frac{1}{2}(\gamma + 1)/(R_f + \mu_p)^2 = 55/(R_f + \mu_p)^2$. We also report the alphas for $\gamma = 4$ to show that the choice of $\gamma$ hardly matters for the alpha adjustments. For each fund, the adjustment is calculated for a portfolio of stocks and bonds, which is optimally chosen to maximize the average CRRA utility over the sample period. The results are reported in Table 6. To maximize the cross-sectional dispersion, we sort the funds in residual co-skewness quintiles. The table reports the annualized average of $\alpha_i$, $\alpha^{pr}_i - \alpha_i$, $\alpha^{te}_i - \alpha^{pr}_i$, and $\alpha^{CRR\mathrm{A}}_i - \alpha^{pr}_i$.

The average annualized simple alpha of hedge funds is 6%, and the average difference between prudent and simple alphas is 0.35% (for $\gamma = 10$). In the cross-section, this difference attains 1% per year on average in the highest quintile. Accounting for moments beyond skewness only leads to modest additional utility gains: The differences between prudent and generalized alphas, which take into account all higher moments, are 0.05% on average (0.14% in the high residual co-skewness quintile). The kurtosis adjustments are thereby mostly below 0.05% on average, suggesting that the next moment beyond skewness has less economic significance. For mutual funds, the average simple alpha is 2.3%, and the skewness adjustments average 0.21% (0.68% in the high residual co-skewness quintile). Adjusting further for kurtosis and all higher moments changes alphas in absolute value by up to 0.04% and 0.1%, respectively. Finally, skewness adjustments are similar for $\gamma = 4$ and $\gamma = 10$. Indeed, a lower relative risk aversion decreases prudence but increases optimal portfolio allocations in the risky assets. These two opposing effects approximately cancel out each other in the skewness adjustments.

Overall, these results suggest that taking into account skewness in performance evaluation has significant economic value for investors. In addition, the skewness framework seems to capture the most meaningful difference between the mean-variance framework and a more specific framework with all higher moments.

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28 A relative risk aversion of 10 is considered to be in the range of reasonable values of risk aversion and yields realistic portfolio allocations.
29 In results available upon request, we have constructed a similar table by sorting on residual co-kurtosis. The high-minus-low quintile difference between prudent and temperate alphas increases then to up to 0.23%.
30 This conclusion is also supported by experimental and empirical asset pricing evidence (Dittmar, 2002; Ebert and Wiesen, 2014; Trautmann and van de Kuilen, 2018).
Table 6: Utility gains

|               | Q1  | Q2  | Q3  | Q4  | Q5  | all  |
|---------------|-----|-----|-----|-----|-----|------|
|               | low coskew | high coskew |
| hedge funds   |     |     |     |     |     |      |
| $\alpha$      | 5.08 | 4.83 | 4.49 | 6.08 | 9.76 | 6.05 |
| $\alpha^{pr} - \alpha$ | 0.03 | 0.12 | 0.22 | 0.36 | 1.04 | 0.35 |
| $\alpha^{te} - \alpha^{pr}$ | 0.00 | -0.01 | 0.01 | 0.02 | 0.05 | 0.01 |
| $\alpha^{CRRA} - \alpha^{pr}$ | -0.01 | 0.00 | 0.05 | 0.06 | 0.14 | 0.05 |
| mutual funds  |     |     |     |     |     |      |
| $\alpha$      | 2.21 | 1.73 | 2.35 | 2.40 | 3.00 | 2.34 |
| $\alpha^{pr} - \alpha$ | 0.02 | 0.07 | 0.12 | 0.19 | 0.67 | 0.21 |
| $\alpha^{te} - \alpha^{pr}$ | -0.01 | -0.01 | 0.01 | 0.03 | 0.04 | 0.01 |
| $\alpha^{CRRA} - \alpha^{pr}$ | -0.01 | -0.02 | 0.01 | 0.04 | 0.13 | 0.03 |

Panel A. $\gamma = 10$

Panel B. $\gamma = 4$

This table evaluates the utility gains for investors when overlap with stocks and bonds as benchmark assets is rejected. It reports the alpha of individual funds with respect to the benchmark assets ($\alpha$), the alpha adjusted for higher order co-moments up to order three (prudent alpha; $\alpha^{pr}$), up to order four (temperant alpha; $\alpha^{te}$), and all higher co-moments ($\alpha^{CRRA}$). To adjust alpha for higher order co-moments, we consider a portfolio of stocks and bonds which is chosen optimally on the respective sample for each hedge fund for a CRRA investor with a coefficient of relative risk aversion of $\gamma = 10$ in Panel A and $\gamma = 4$ in Panel B. The resulting average fractions invested in stocks and bonds for the hedge fund (mutual fund) sample are 0.29 and 0.41 (0.95 and 0.97) for $\gamma = 10$, and 0.71 and 2.35 (0.99 and 2.39) for $\gamma = 4$. Funds are sorted in quintiles according to the residual co-skewness of the fund with the portfolio of benchmark assets. All figures are annualized and in percent, and significance at the 5% level is highlighted with bold font.
4.5 Robustness

4.5.1 Different fund types

Having shown that rejection rates are considerably higher for hedge funds than for mutual funds, in Table 7 we investigate the robustness of this result using subsets of hedge funds and alternative modeling assumptions. The insights can be summarized as follows. First, rejection rates are very similar for small and large funds, as measured by the funds’ first net asset value. Second, older funds—funds with longer return histories—tend to perform better by mean-variance metrics whereas younger funds are more interesting from a skewness perspective. Third, over time the fraction of funds that provides mean-variance spanning benefits is cut in half while the fraction of funds that provides skewness benefits decreases less and even increases in the case of Over$^{MV5}$. This suggests that skewness is especially important in the recent subsample. It may also mean that hedge funds are becoming increasingly sophisticated in managing skewness.

Fourth, rejection rates within Morningstar categories are similar to the results with portfolios in Table 2: global derivatives stand out as being most attractive for all skewness investors. In addition, directional debt, event, and multistrategy funds are much less attractive from a skewness perspective than they are from a mean-variance perspective. Fifth, using simple returns (instead of smoothing adjusted returns) or deleting the first 24 returns for each fund (instead of deleting only the first 12 returns) to adjust for back-fill bias yields similar results. Sixth, including de-listing returns for liquidated funds of $-25\%$ and $-50\%$ as suggested by Aiken et al. (2013) reduces the rejection rates considerably. But because all funds that stop reporting returns to the database are classified as liquidated by Morningstar, these results are likely to be lower bounds on the rejection rates.

Finally, using additional benchmark assets to reflect returns in other asset classes, in particular commodities and currencies in the form of the GSCI total return index and Lustig et al. (2011)’s currency factor, hardly changes the conclusions and rejection rates. While overlap is harder to reject with currencies than stocks as benchmarks, the overall rejection rates with currencies or commodities combined with stocks and bonds are similar. This last panel also separately reports rejection rates for global derivatives funds which include currency funds. Even here, rejection rates of overlap are about the same as previously. Hence, it is unlikely that global derivatives achieve their

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31 We use this measure because it is identifiable at launch by investors and not mechanically correlated with a fund’s lifetime performance.

32 The data on the currency factor is available at https://people.stanford.edu/hlustig/data-and-code, and we use the net-of-fee factor with all countries, which is the factor with the highest Sharpe ratio and skewness.
superior performance mechanically by investing in assets not spanned by the benchmarks. Overall, the rejection rates in Table 7 are very similar to the rejection rates in the main analysis.

4.5.2 Out-of-sample analysis

While performance can only be analyzed ex-post, some readers may be interested to see whether investors can select hedge funds and mutual funds ex-ante which provide desirable skewness properties ex-post. The rejection rates presented so far suggest that some hedge funds provide desirable higher moment exposure to stocks or bonds. This could indicate skill and hence persistence. Mutual funds, on the other hand, do not seem to exhibit desirable higher moment and alpha properties, suggesting no skill and no performance persistence.

To investigate whether there is further evidence for this conjecture, we form equally weighted decile portfolios at the end of every year based on $\text{Over}_{\text{MVS}}$ and $\text{Over}_{\text{MVS,b}}$ calculated on rolling windows with the previous 24 months of data. For comparison, we also report portfolios based on the past t-statistics of the intercept in the Span$_{\text{MV}}$ regression. In the latter sort, using t-statistics makes the results more comparable with $\text{Over}_{\text{MVS}}$, which is studentized, and also produces better out-of-sample results by taking into account the noise in estimated alphas. Table 8 reports the results. We focus on the most important deciles, the lowest and the highest, and their difference. Results in the intermediate deciles show a similar pattern.

The results suggest that alphas and co-skewnesses with stocks are very persistent for hedge funds but not for mutual funds. Portfolios formed based on hedge funds with high past values of Span$_{\text{MV,b}}$ and $\text{Over}_{\text{MVS}}$ have returns for which Span$_{\text{MV,b}}$ and $\text{Over}_{\text{MVS}}$ is rejected. The respective hypothesis tests on the low minus high decile portfolios are rejected with a p-value of 0.01 in both sorts. For mutual funds, these p-values are above 0.12 and 0.39 suggesting no persistence. Sorting on $\text{Over}_{\text{MVS,b}}$ does not produce clear-cut results for hedge funds and mutual funds, suggesting that the co-skewness with bonds and/or with the product of stocks and bonds is less persistent. At the same time, a portfolio for which $\text{Over}_{\text{MVS,b}}$ is rejected can be achieved by sorting on past $\text{Over}_{\text{MVS,b}}$, confirming the evidence from Table 5 that $\text{Over}_{\text{MVS}}$ and $\text{Over}_{\text{MVS,b}}$ is rejected for similar hedge funds. Overall, the results highlight that non-linearities are important even out-of-sample for hedge funds, but less so for mutual funds.

Motivated by extensive evidence that hedge fund returns exhibit option-like characteristics and the general inability of linear factor models to account for these non-linearities, this research extends the mean-variance spanning and intersection approach of [Huberman and Kandel](1987) to skewness.

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Supplementary: The results are similar for rolling windows of 36 months.
Table 7: Spanning and overlap for different fund types

| Panel A. Size | Span_{MV,s,b} | Span_{MVS,s,b} | Over_{MV,s,b} | Over_{MVS,s,b} | Span_{MV,c,b} | Span_{MVS,c,b} | Over_{MV,c,b} | Over_{MVS,c,b} |
|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|
| small         | 27.1          | 31.5          | 11.9          | 19.1          | 25.2          | 30.1          | 11.3          | 17.6          |
| large         |               |               |               |               |               |               |               |               |
| Panel B. Age  | Span_{MV,s,b} | Span_{MVS,s,b} | Over_{MV,s,b} | Over_{MVS,s,b} | Span_{MV,c,b} | Span_{MVS,c,b} | Over_{MV,c,b} | Over_{MVS,c,b} |
| young         | 18.9          | 32.1          | 9.8           | 16.6          | 33.8          | 30.0          | 13.2          | 20.2          |
| old           |               |               |               |               |               |               |               |               |
| Panel C. Subperiod | Span_{MV,s,b} | Span_{MVS,s,b} | Over_{MV,s,b} | Over_{MVS,s,b} | Span_{MV,c,b} | Span_{MVS,c,b} | Over_{MV,c,b} | Over_{MVS,c,b} |
| Jan 1994 to Jun 2004 | 46.0 | 38.6 | 7.0 | 13.6 | 17.6 | 32.5 | 13.0 | 18.1 |
| July 2004 to Dec 2014 |               |               |               |               |               |               |               |               |
| Panel D. Morningstar categories | Span_{MV,s,b} | Span_{MVS,s,b} | Over_{MV,s,b} | Over_{MVS,s,b} | Span_{MV,c,b} | Span_{MVS,c,b} | Over_{MV,c,b} | Over_{MVS,c,b} |
| directional debt | 39.1          | 38.6          | 9.2           | 20.8          | 26.4          | 31.8          | 11.7          | 18.6          |
| directional equity |               |               |               |               |               |               |               |               |
| event fund of funds | 48.2          | 41.3          | 4.9           | 14.2          | 16.0          | 21.9          | 11.3          | 11.5          |
| global derivatives | 25.4          | 36.0          | 18.4          | 35.4          | 38.1          | 40.1          | 11.3          | 19.8          |
| multistrategy |               |               |               |               |               |               |               |               |
| relative value | 31.9          | 37.7          | 8.6           | 16.6          | 30.6          | 31.5          | 4.6           | 15.3          |
| other |               |               |               |               |               |               |               |               |
| Panel E. Simple returns and deleting first 24 returns | Span_{MV,s,b} | Span_{MVS,s,b} | Over_{MV,s,b} | Over_{MVS,s,b} | Span_{MV,c,b} | Span_{MVS,c,b} | Over_{MV,c,b} | Over_{MVS,c,b} |
| simple returns | 35.4          | 33.8          | 9.2           | 15.8          | 23.7          | 29.0          | 11.3          | 17.5          |
| delete first 24 returns |               |               |               |               |               |               |               |               |
| Panel F. Delisting returns for liquidated funds | Span_{MV,s,b} | Span_{MVS,s,b} | Over_{MV,s,b} | Over_{MVS,s,b} | Span_{MV,c,b} | Span_{MVS,c,b} | Over_{MV,c,b} | Over_{MVS,c,b} |
| delisting return of -25% | 14.4          | 23.8          | 8.5           | 12.4          | 11.7          | 22.4          | 7.1           | 10.0          |
| delisting return of -50% |               |               |               |               |               |               |               |               |
| Panel G. Additional benchmark assets: currencies \( fx \) or commodities \( c \) | Span_{MV,s,b} | Span_{MVS,s,b} | Over_{MV,s,b} | Over_{MVS,s,b} | Span_{MV,c,b} | Span_{MVS,c,b} | Over_{MV,c,b} | Over_{MVS,c,b} |
| all | 22.2          | 28.4          | 7.6           | 9.7           | 25.2          | 34.9          | 16.8          | 22.6          |
| global derivatives | 26.1          | 37.0          | 12.8          | 18.4          | 26.7          | 47.5          | 21.1          | 37.5          |

Panel A reports the rejection rates in percent separately for small funds (first net asset value below median) and large funds (otherwise), Panel B compares young funds (36 to 89 return observations) and old funds (at least 90 return observations), Panel C examines rejection rates for the earlier subsample until June 2004 and the more recent subsample, Panel D reports rejection rates for each Morningstar category, Panel E reports rejection rates for simple returns, i.e., not adjusted for return smoothing, and the smoothing adjusted returns when the first 24 observation are deleted for each fund instead of the first 12 observation as in the main line analysis, and Panel F examines the effect of including a delisting return of -25% and -50% for liquidated funds. Panel G adds the currency factor of Lustig et al. [2011], denoted by the subscript \( fx \), and the return on the Goldman Sachs Commodity Index (GSCI) TR Index, denoted by \( c \), to the set of benchmark assets. All rejection rates are at the 5% significance level and the Wald statistics use White [1980] standard errors.
and propose regression-based tests. We use our tests to study option-like characteristics in option strategies, mutual fund returns, and hedge fund returns; and to ask whether hedge funds add value to investors. In summary, our findings highlight the importance of skewness in performance evaluation, and suggest that in the cross-section and even after controlling for co-skewness, a subset of hedge funds improves the investment opportunity set.

This paper focuses on hedge and mutual fund returns. Our method can also be applied to other asset classes. In particular, the returns of currency trading strategies and emerging markets have been in the limelight for their skewness (see, Brunnermeier et al. 2008 and Ghysels et al. 2016, respectively) so that is a possible area for future research.

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|                     | Panel A. Span$_{MV}^{\text{D1, D10, D10-D1}}$ | Panel B. Over$_{\text{MVS}}^{\text{D1, D10, D10-D1}}$ | Panel C. Over$_{\text{MVS}}^{\text{D1, D10, D10-D1}}$ |
|---------------------|---------------------------------------------|---------------------------------------------|---------------------------------------------|
| Hedge funds         |                                             |                                             |                                             |
| Span$_{MV}^{\text{D10}}$ | (0.05) (0.00) (0.01)                         | (0.02) (0.00) (0.08)                         | (0.00) (0.00) (0.17)                         |
| Span$_{MVS}^{\text{D10}}$ | (0.13) (0.00) (0.09)                         | (0.11) (0.00) (0.03)                         | (0.01) (0.01) (0.38)                         |
| Over$_{\text{MVS}}^{\text{D10}}$ | (0.40) (0.74) (0.44)                         | (0.92) (0.02) (0.18)                         | (0.52) (0.23) (0.62)                         |
| Over$_{\text{MVS}}^{\text{D10}}$ | (0.67) (0.82) (0.13)                         | (0.93) (0.03) (0.02)                         | (0.70) (0.16) (0.05)                         |
| Mutual funds        |                                             |                                             |                                             |
| Span$_{MV}^{\text{D10}}$ | (0.50) (0.20) (0.11)                         | (0.50) (0.22) (0.24)                         | (0.50) (0.16) (0.14)                         |
| Span$_{MVS}^{\text{D10}}$ | (0.96) (0.55) (0.42)                         | (0.96) (0.62) (0.30)                         | (0.96) (0.55) (0.37)                         |
| Over$_{\text{MVS}}^{\text{D10}}$ | (0.74) (0.78) (0.54)                         | (0.63) (0.59) (0.79)                         | (0.61) (0.71) (0.97)                         |
| Over$_{\text{MVS}}^{\text{D10}}$ | (0.87) (0.86) (0.74)                         | (0.77) (0.73) (0.41)                         | (0.57) (0.79) (0.75)                         |

At the end of each year, we sort funds into deciles and form equally-weighted portfolios which are held for the following year. The table shows the $p$-values of the spanning and overlap tests for the lowest and highest decile and their difference, separately for hedge funds and mutual funds. In Panel A, we sort based on the t-statistic of the intercept in the mean-variance spanning regression, and Panel B and C sort based on the test statistic of the overlap test. All these statistics are calculated with the past 24 months of returns.
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Technical Appendix to accompany “Spanning Tests for Assets with Option-Like Payoffs: The Case of Hedge Funds” (Not for publication)

This Technical Appendix relates the mean-variance-skewness framework to expected utility, and reports the proofs of the propositions and corollaries. The appendix also contains implementation details of the spanning and overlap tests, the size and power analysis, a simulation experiment with option strategies, and additional subsample results.

1 Relation to expected utility

This appendix explains how $\gamma_1$ and $\gamma_2$ in (3) are related to the coefficients of risk aversion, i.e., $A(x) \equiv -u''(x)/u'(x)$, and prudence, i.e., $P(x) \equiv -u'''(x)/u''(x)$, of a von Neumann-Morgenstern utility function $u$. To start take a Taylor approximation of marginal utility derived from end of period wealth around expected wealth

$$u'(R_f + r_p) = \sum_{i=0}^{\infty} \frac{1}{i!} (r_p - \mu_p)^i u^{i+1} (R_f + \mu_p),$$

where $u^i (\cdot)$ is the $i$th derivative of $u$ with respect to $x$ (i.e., $u^1 (\cdot) = u' (\cdot)$, etc.), $R_f$ is the gross risk-free rate, $r_p$ is the excess portfolio return and $\mu_p$ its expectation.

Truncating the approximation at the second-order and inserting the approximation in the first-order condition for asset $i$ yields

$$E [r_i u' (R_f + r_p)] = E [r_i] u' (\kappa) + u'' (\kappa) E [r_i (r_p - \mu_p)] + \frac{1}{2} u''' (\kappa) E [r_i (r_p - \mu_p)^2],$$

(TA.1)

where $\kappa \equiv R_f + \mu_p$ and we ignore that the equalities in (TA.1) are approximations. If the first-order condition of the investor holds, the right-hand side of (TA.1) can be rewritten to

$$E [r_i] + \frac{u'' (\kappa)}{u' (\kappa)} E [r_i (r_p - \mu_p)] + \frac{1}{2} \frac{u''' (\kappa)}{u' (\kappa)} E [r_i (r_p - \mu_p)^2] = 0,$$

(TA.2)

which is very similar to the first-order condition of the investor with mean-variance-skewness utility and, therefore, $\gamma_1 = -\frac{u'' (\kappa)}{u (\kappa)}$ and $\gamma_2 = \frac{1}{2} \frac{u''' (\kappa)}{u (\kappa)}$. Hence, the aversion to variance relative to the preference for the mean is the absolute risk aversion, i.e., $\gamma_1 = A (\kappa)$, and the preference for skewness relative to the preference for the mean is one half of the product of risk aversion and prudence, i.e.,

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34 As a reference and for the technical details see Jurczenko and Maillet (2006) and the references cited therein.
\[ \gamma_2 = \frac{1}{2} A(\kappa) P(\kappa) \]  

Possible values for \( \gamma_1 \) and \( \gamma_2 \) can now be obtained from standard utility functions like constant absolute risk aversion (CARA) utility functions or constant relative risk aversion (CRRA) utility functions. For CARA, \( u(x) = -Ae^{-Ax} \) where \( A \) is the coefficient of absolute risk aversion, \( \gamma_1 = A \) and \( \gamma_2 = \frac{1}{2} A^2 \). For CRRA, \( u(x) = x^{1-\gamma}/(1-\gamma) \) where \( \gamma \neq 1 \), \( A(x) = \gamma/x \) and \( P(x) = (\gamma + 1)/x \) which in turn implies \( \gamma_1 = \gamma/\kappa \) and \( \gamma_2 = \frac{1}{2} \gamma (\gamma + 1)/\kappa^2 \). A reasonable value for \( \gamma \) is \( \gamma = 4 \) (see Gollier, 2001), which yields, for \( w = 1 \), \( \gamma_1 = 4 \) and \( \gamma_2 = 10 \). While prudence is a function of risk aversion for CRRA utility, hyperbolic absolute risk aversion (HARA) functions, for instance, introduce more flexibility in modeling \( \gamma_1 \) and \( \gamma_2 \) separately. They are given by \( u(x) = \xi(\eta + x/\gamma)^{1-\gamma} \) and defined on a domain of \( x \) such that \( \eta + x/\gamma > 0 \) (Gollier, 2001, p. 26). This implies \( A(x) = \gamma / (\gamma \eta + x) \) and \( P(x) = (\gamma + 1) / (\gamma \eta + x) \). A widely used special case of HARA is additive habit utility. Setting \( \xi = \left( \frac{1}{1-\gamma} \right) \left( \frac{1}{1-\gamma} \right) \) and \( \eta = -k/\gamma \), we obtain \( u(x) = (x - k)^{1-\gamma}/(1-\gamma) \), where \( k \) is the (constant) habit level and \( \gamma \neq 1 \). These values imply \( A(x) = \frac{\gamma}{x - k} \) and \( P(x) = \frac{\gamma + 1}{x - k} \), and \( \gamma_2 = \frac{1}{2} \gamma_1 \left( \frac{9}{1-\gamma} \right) \). Low relative risk aversion and very high relative prudence can now be achieved with \( \gamma \) close to zero and \( k \) very close to \( \kappa \).  

2 Proofs  

**Proof.** Proof of Proposition 1. The optimal portfolio \( \mathbf{w}^* \) satisfies the first-order conditions  

\[
\begin{pmatrix}
\mu_x \\
\mu_y \\
\end{pmatrix} - \gamma_1 \begin{pmatrix}
\Sigma_{xx} & \Sigma_{xy} \\
\Sigma_{yx} & \Sigma_{yy} \\
\end{pmatrix} \begin{pmatrix}
w_x^* \\
w_y^* \\
\end{pmatrix} + \gamma_2 \begin{pmatrix}
S_{xxx} & S_{xxy} & S_{xyx} & S_{yyx} \\
S_{yx} & S_{yy} & S_{yy} & S_{yy} \\
S_{xy} & S_{yx} & S_{yy} & S_{yy} \\
S_{yy} & S_{yy} & S_{yy} & S_{yy} \\
\end{pmatrix} \begin{pmatrix}
w_x^* \otimes w_x^* \\
w_y^* \otimes w_y^* \\
w_x^* \otimes w_y^* \\
w_y^* \otimes w_y^* \\
\end{pmatrix} = 0, \quad (TA.3)
\]

where the subscripts \( x \) and \( y \) refer to the \( k \) benchmark assets and \( n \) test assets, respectively, \( w_x^* \) and \( w_y^* \) are the subvectors of \( w^* \), \( \mu_x \) and \( \mu_y \) are the subvectors of \( \mu \), \( \Sigma_{xx} \), \( \Sigma_{xy} \), \( \Sigma_{yx} \) and \( \Sigma_{yy} \) are the submatrices of \( \Sigma \), \( S_{xxx} \), \( S_{xxy} \), \( S_{xyx} \), \( S_{yyx} \), \( S_{yx} \), \( S_{yy} \), \( S_{yy} \), \( S_{xy} \), \( S_{yx} \), \( S_{yy} \), and \( S_{yy} \) are the submatrices

---

\(^{35}\) Notice that \( E \left[ r_i (r_p - \mu_p)^2 \right] \) is the covariance of asset \( i \) with the optimal portfolio. \( E \left[ r_i (r_p - \mu_p)^2 \right] \) is not exactly the co-skewness of asset \( i \) with the optimal portfolio because \( E \left[ r_i (r_p - \mu_p)^2 \right] - E \left[ (r_i - \mu_i) (r_p - \mu_p)^2 \right] = -\mu_i \sigma_p^2 \). Hence, interpretations of \( \gamma_1 \) and \( \gamma_2 \) in terms of expected utility hold only approximately. But this is reasonable because empirically alphas adjusted for skewness according to the mean-variance-skewness framework with \( \gamma_1 = -\frac{\mu_i' \sigma_p^2}{\mu_i' \sigma_p^2} \) and \( \gamma_2 = \frac{1}{2} \frac{\mu_i' \sigma_p^2}{\mu_i' \sigma_p^2} \) are very close to alphas adjusted using an expected utility framework.

\(^{36}\) E.g., \( \gamma = 1/10000 \), \( \kappa = 1 \) and \( k = 9999/10000 \) imply \( \gamma_1 = 1 \) and \( \gamma_2 = 5000.5 \).
of $S$. If there is intersection (i.e., $w^*_y = 0$), (TA.3) becomes

$$
\begin{pmatrix}
\mu_x \\
\mu_y
\end{pmatrix} - \gamma_1 \begin{pmatrix}
\Sigma_{xx}w^*_x \\
\Sigma_{yx}w^*_x
\end{pmatrix} + \gamma_2 \begin{pmatrix}
S_{xxx}(w^*_x \otimes w^*_x) \\
S_{yxx}(w^*_x \otimes w^*_x)
\end{pmatrix} = 0. 

$$

(TA.4)

The first $k$ rows of (TA.4) can then be written as

$$
w^*_x = \frac{1}{\gamma_1} \Sigma_{xx}^{-1} \mu_x + \frac{\gamma_2}{\gamma_1} \Sigma_{xx}^{-1} S_{xxx}(w^*_x \otimes w^*_x),
$$

and using this to rewrite the last $n$ rows of (TA.4) gives the result of the proposition

$$
\mu_y - \Sigma_{yx} \Sigma_{xx}^{-1} \mu_x + \gamma_2 \left( S_{yxx} - \Sigma_{yx} \Sigma_{xx}^{-1} S_{xxx} \right) (w^*_x \otimes w^*_x) = 0_n.
$$

Proof. Proof of Proposition 2) Spanning requires that (4) is satisfied for all values of $\gamma_2$ and associated $w^*_x$. Hence, sufficient conditions are

$$
\begin{align*}
\mu_y - \Sigma_{yx} \Sigma_{xx}^{-1} \mu_x &= 0_n, \\
S_{yxx} - \Sigma_{yx} \Sigma_{xx}^{-1} S_{xxx} &= 0_{n \times k^2}.
\end{align*}
$$

Proof. Proof of Proposition 3) The portfolio problem with short-sales constraints is

$$
\max_w \quad w^\top \mu - \frac{1}{2} \gamma_1 w^\top \Sigma w + \frac{1}{3} \gamma_2 w^\top S (w \otimes w),
$$

s.t. $w_i \geq 0, \forall i.$

Let the vector $\delta$ contain the Kuhn-Tucker multipliers for the restriction that portfolio weights are

\[\text{Note that the mean-variance-skewness portfolio problem has no closed form solution for portfolio weights. In addition, there is no three fund separation for arbitrary distributions because it is not possible to write the optimal portfolio of any investor as a function of three distinct funds. Three fund separation can be obtained with additional distributional assumptions as for example in Mencia and Sentana (2009).}\]
non-negative. The mean-variance-skewness efficient portfolio \( w^* \) satisfies

\[
\mu + \delta = \gamma_1 \Sigma w^* - \gamma_2 S ( w^* \otimes w^* ), \tag{TA.5}
\]

\[
w^*_i, \delta_i \geq 0, \ \forall i,
\]

\[
w^*_i \delta_i = 0, \ \forall i,
\]

\[
w^{*\top} 1 = 1.
\]

If there is intersection, (TA.5) can be rewritten to

\[
\begin{pmatrix}
\mu_x \\
\mu_y
\end{pmatrix} - \gamma_1 \begin{pmatrix}
\Sigma_{xx} w^*_x \\
\Sigma_{yx} w^*_x
\end{pmatrix} + \gamma_2 \begin{pmatrix}
S_{xxx} (w^*_x \otimes w^*_x) \\
S_{yyx} (w^*_x \otimes w^*_x)
\end{pmatrix} + \delta = 0. \tag{TA.6}
\]

We proceed in a similar fashion as [DeRoon et al. (2001)] and take the mean-variance-skewness efficient portfolio for a particular value of \((\gamma_1, \gamma_2)\). Let \( r^x \) refer to the \( L \)-dimensional subvector of \( r \) which contains only the returns of the assets for which short-sales constraints are not binding and let superscripts \( \eta \) refer to this subset. (TA.6) becomes then

\[
\mu_x^{\eta} - \gamma_1^{\eta} \Sigma_{x^{\eta} x^{\eta}} w^{\eta} + \gamma_2^{\eta} S_{x^{\eta} x^{\eta}} (w^{\eta} \otimes w^{\eta}) = 0_L, \tag{TA.7}
\]

and

\[
\begin{pmatrix}
\mu_x \\
\mu_y
\end{pmatrix} - \gamma_1^{\eta} \begin{pmatrix}
\Sigma_{x^{\eta} x^{\eta}} w^{\eta} \\
\Sigma_{y^{\eta} x^{\eta}} w^{\eta}
\end{pmatrix} + \gamma_2^{\eta} \begin{pmatrix}
S_{x^{\eta} x^{\eta}} (w^{\eta} \otimes w^{\eta}) \\
S_{y^{\eta} x^{\eta}} (w^{\eta} \otimes w^{\eta})
\end{pmatrix} + \delta = 0.
\]

Using (TA.7) we get the condition on the test assets for intersection

\[
\mu_y - \Sigma y^{\eta} \Sigma_{x^{\eta} x^{\eta}}^{-1} \mu x^{\eta} + \gamma_2^{\eta} \{ S y^{\eta} x^{\eta} - \Sigma y^{\eta} \Sigma_{x^{\eta} x^{\eta}}^{-1} S x^{\eta} x^{\eta} \} (w^{\eta} \otimes w^{\eta}) + \delta = 0_L,
\]

or

\[
\mu_y - \Sigma y^{\eta} \Sigma_{x^{\eta} x^{\eta}}^{-1} \mu x^{\eta} + \gamma_2^{\eta} \{ S y^{\eta} x^{\eta} - \Sigma y^{\eta} \Sigma_{x^{\eta} x^{\eta}}^{-1} S x^{\eta} x^{\eta} \} (w^{\eta} \otimes w^{\eta}) \leq 0_L. \tag{TA.8}
\]

Spanning implies that (TA.8) holds for all relevant values of \( \gamma_2^{\eta} \). Again, we follow the exposition of [DeRoon et al. (2001)] to derive the spanning conditions. Let \( \Gamma^j \) be the set of \( \gamma_2^{\eta} \), for which the subset of assets for which the short-sales constraints in the mean-variance-skewness efficient portfolios are not binding is the same. In addition, let the \( L^j \)-dimensional vector of returns of these assets be denoted by \( r^{x^j} \), i.e., \( r^{x^j} = r^{x^\eta} \) if and only if \( \gamma_2^{\eta} \in \Gamma^j \). As before, each variable which refers to the
set \( r^{x^j} \), \( j = 1, 2, \ldots, M \), is denoted by a superscript \( j \). Hence, we have spanning if and only if the \( M \) conditions,

\[
\mu_y - \Sigma_{yx}^{-1} \mu_x + \gamma_2 \left\{ S_{yx} \Sigma_{yx}^{-1} S_{xx} \right\} (w^y \otimes w^x) \leq 0_n, \quad (TA.9)
\]

\( \forall \gamma_2 \in \Gamma^j \), hold. Note that a sufficient condition for part B of (TA.9) to be non-positive is that all elements of \( S_{yx} \Sigma_{yx}^{-1} S_{xx} \) are non-positive because \( \gamma_2^2 \) is non-negative and all elements of \( w^y \) are positive. Sufficient conditions for mean-variance-skewness spanning without short-sales are then

\[
\begin{align*}
\mu_y - \Sigma_{yx}^{-1} \mu_x & \leq 0_n, \\
\Sigma_{yx}^{-1} S_{xx} - 1_n & \leq 0_n, \\
S_{yx} - \Sigma_{yx}^{-1} S_{xx} & \leq 0_{n \times (L^j)^2},
\end{align*}
\]

for \( j = 1, \ldots, M \). \( \blacksquare \)

**Proof.** Proof of Corollary \( \square \)Immediate. \( \blacksquare \)

**Proof.** Proof of Corollary \( \heartsuit \)To simplify the equations, we use \( S_{xy} (w^x \otimes w^y) = S_{yx} (w^y \otimes w^x) \) and \( S_{xy} (w^x \otimes w^y) = S_{yx} (w^y \otimes w^x) \). The first \( k \) rows of (TA.3) can then be written as

\[
w_x^* = \frac{1}{\gamma_1} \Sigma_{xx}^{-1} \mu_x - \Sigma_{xx}^{-1} \Sigma_{xy} w_y^* \\
+ \frac{\gamma_2}{\gamma_1} \Sigma_{xx}^{-1} \left[ S_{xx} (w_x^* \otimes w_x^*) + 2 \Sigma_{xy} (w_x^* \otimes w_y^*) + \Sigma_{yy} (w_y^* \otimes w_y^*) \right],
\]

and using this to rewrite the last \( n \) rows of (TA.3) gives

\[
\begin{align*}
\mu_y - \Sigma_{yx} \Sigma_{yx}^{-1} \mu_x & + \gamma_1 \left[ \Sigma_{yx} \Sigma_{yx}^{-1} \Sigma_{xy} - \Sigma_{yy} \right] w_y^* \\
+ \gamma_2 \left\{ \left( S_{yx} - \Sigma_{yx} \Sigma_{yx}^{-1} S_{xx} \right) (w_x^* \otimes w_x^*) + 2 \left( S_{yx} - \Sigma_{yx} \Sigma_{yx}^{-1} S_{xx} \right) (w_x^* \otimes w_y^*) \right. \\
& \left. + \left( S_{yy} - \Sigma_{yx} \Sigma_{yx}^{-1} S_{xy} \right) (w_y^* \otimes w_y^*) \right\} & = 0_n.
\end{align*}
\]

(TA.10)

Now consider the regression

\[ r_{y,t} = \alpha + Br_{x,t} + \epsilon_t. \]
It is easy to check that we have the following relations

\[ B = \Sigma_{yx} \Sigma_{xx}^{-1}, \]
\[ \alpha = \mu_y - \Sigma_{yx} \Sigma_{xx}^{-1} \mu_x, \]
\[ \Sigma_{ee} = \Sigma_{yy} - \Sigma_{yx} \Sigma_{xx}^{-1} \Sigma_{xy}, \]
\[ S_{eex} = S_{yxx} - \Sigma_{yx} \Sigma_{xx}^{-1} S_{xxx}, \]
\[ S_{exy} = S_{yx} - \Sigma_{yx} \Sigma_{xx}^{-1} S_{xy}, \]
\[ S_{eey} = S_{yyy} - \Sigma_{yx} \Sigma_{xx}^{-1} S_{eyy}. \]

These relations enable us to rewrite (TA.10) to

\[ \alpha - \gamma 1 \Sigma_{ee} w^*_y + \gamma 2 \left[ S_{eex} (w^*_x \otimes w^*_x) + 2 S_{exy} (w^*_x \otimes w^*_y) + S_{eey} (w^*_y \otimes w^*_y) \right] = 0_n. \]

We get then

\[ w^*_y = \frac{1}{\gamma 1} \Sigma_{ee}^{-1} \alpha + \frac{\gamma 2}{\gamma 1} \Sigma_{ee}^{-1} \left[ S_{eex} (w^*_x \otimes w^*_x) + 2 S_{exy} (w^*_x \otimes w^*_y) + S_{eey} (w^*_y \otimes w^*_y) \right], \]
\[ w^*_x = \frac{1}{\gamma 1} \Sigma_{xx}^{-1} \left[ \mu_x - \Sigma_{xy} \Sigma_{ee}^{-1} \alpha \right] + \frac{\gamma 2}{\gamma 1} \Sigma_{xx}^{-1} \left[ \left( S_{xxx} - \Sigma_{xy} \Sigma_{ee}^{-1} S_{exx} \right) (w^*_x \otimes w^*_x) + 2 \left( S_{xx} - \Sigma_{xy} \Sigma_{ee}^{-1} S_{exy} \right) (w^*_x \otimes w^*_y) + \left( S_{xy} - \Sigma_{xy} \Sigma_{ee}^{-1} S_{eyy} \right) (w^*_y \otimes w^*_y) \right]. \]

It is easy to see that for \( \gamma 2 = 0 \), we recover the familiar mean-variance case:

\[ w^*_y = \frac{1}{\gamma 1} \Sigma_{ee}^{-1} \alpha, \]
\[ w^*_x = \frac{1}{\gamma 1} \Sigma_{xx}^{-1} \left[ \mu_x - \Sigma_{xy} \Sigma_{ee}^{-1} \alpha \right]. \]

Finally, if there is only one benchmark asset \( n = 1 \) and \( x = x \) and one test asset \( k = 1 \) and \( y = y \), we obtain

\[ w^*_y = \frac{\alpha}{\gamma 1 \Sigma_{xx}} + \frac{\gamma 2}{\gamma 1 \Sigma_{ee}} \left[ S_{eex} w^*_x + 2 S_{exy} w^*_x w^*_y + S_{eey} w^*_y \right], \]
\[ w^*_x = \frac{1}{\gamma 1 \Sigma_{xx}} \left[ \mu_x - \Sigma_{xy} \Sigma_{ee}^{-1} \alpha \right] + \frac{\gamma 2}{\gamma 1 \Sigma_{xx}} \left[ \left( S_{xxx} - \Sigma_{xy} \Sigma_{ee}^{-1} S_{exx} \right) w^*_x + 2 \left( S_{xx} - \Sigma_{xy} \Sigma_{ee}^{-1} S_{exy} \right) w^*_x w^*_y + \left( S_{xy} - \Sigma_{xy} \Sigma_{ee}^{-1} S_{eyy} \right) w^*_y \right]. \]
3 Implementation of the test

3.1 Spanning

We obtain consistent estimates \( \hat{h} \) for \( h \) and the asymptotic covariance matrix of \( \hat{h} \), \( \text{Var}(\hat{h}) \), with multivariate regressions. Below we sketch how to obtain \( \hat{h} \) and \( \text{Var}(\hat{h}) \) with simple OLS standard errors for the ease of exposition. The corresponding Matlab code is available for download at \( \text{http://tiny.cc/9fzt4y} \). This code also offers the possibility to obtain estimates of \( \text{Var}(\hat{h}) \) based on \( \text{White} \) \( (1980) \) standard errors (as used in our analysis) and \( \text{Newey and West} \) \( (1987) \) standard errors.

**Step 1:** Run the multivariate regression

\[
\begin{align*}
\mathbf{r}_{y_{i,t}} &= \alpha_i + \mathbf{\beta}_i^\top \mathbf{r}_{x,t} + \epsilon_{i,t}, \quad \text{for} \quad i = 1, \ldots, n, \quad \text{and} \quad t = 1, \ldots, T, \quad (TA.11)
\end{align*}
\]

where \( \mathbf{r}_{y_{i,t}} \) is the excess return on test asset \( i \) in \( t \), \( \mathbf{r}_{x,t} \) is the \( k \)-dimensional vector of excess returns on the \( k \) benchmark assets. Let \( \mathbf{b}_{MV} \equiv [\alpha_1, \mathbf{\beta}_1^\top, \ldots, \alpha_n, \mathbf{\beta}_n^\top] \) be the \((k+1)n\)-dimensional vector of coefficients, and

\[
\mathbf{X} \equiv \mathbf{1}_n \otimes \begin{bmatrix}
\mathbf{r}_{x,1}^\top \\
\vdots \\
\mathbf{r}_{x,T}^\top
\end{bmatrix},
\]

where \( \otimes \) denotes the Kronecker product and \( \mathbf{1}_n \) is an \( n \)-dimensional vector of ones. Let \( \hat{\mathbf{b}}_{MV} \) be the OLS estimate for \( \mathbf{b}_{MV} \). A consistent estimate for the asymptotic covariance matrix of \( \hat{\mathbf{b}}_{MV} \) is

\[
\hat{\mathbf{Q}}_{MV} = \text{Cov}(\epsilon, \epsilon) \otimes \left( \mathbf{X}^\top \mathbf{X} \right)^{-1}, \quad (TA.12)
\]

where \( \text{Cov}(\epsilon, \epsilon) \) is the sample covariance matrix of the estimated residuals in \( (TA.11) \).

**Step 2:** Run the multivariate regression

\[
\begin{align*}
\mathbf{z}_{i,t} &= \alpha_{S,i} + \mathbf{\beta}_{S,i} \mathbf{X}_{S,i,t} + \mathbf{u}_{i,t}, \quad \text{for} \quad i = 1, \ldots, k^2, \quad \text{and} \quad t = 1, \ldots, T, \quad (TA.13)
\end{align*}
\]

where \( \mathbf{z}_{i,t} \) is the \( n \)-dimensional vector \( \mathbf{z}_{i,t} \equiv \sigma_{X_{S,i}}^2 \epsilon_{t+1} \), and \( \mathbf{X}_{S,i,t} \) is the \( i \)th element in the \( k^2 \)-dimensional vector \( \mathbf{X}_{S,t} \equiv \mathbf{\tilde{r}}_{x,t} \otimes \mathbf{\tilde{r}}_{x,t} \) (\( \mathbf{\tilde{r}}_{x,t} \) is the vector of demeaned excess returns in \( t \) on the benchmark assets). Note that the left-hand-side variable in \( (TA.13) \) is scaled by the variance of
the right-hand-side variable to recover the residual co-skewnesses in the slope coefficients. Let

\[ \mathbf{b}_S \equiv \left[ \text{vec} \left( \mathbf{\alpha}_{S,1}^\top, \mathbf{\beta}_{S,1}^\top \right), \ldots, \text{vec} \left( \mathbf{\alpha}_{S,k^2}^\top, \mathbf{\beta}_{S,k^2}^\top \right) \right] \]

be the \( 2nk^2 \)-dimensional vector of coefficients (\text{vec} denotes the vec operator), and

\[ \mathbf{X}_{S_i} \equiv \begin{bmatrix} 1_T & \mathbf{X}_{S_i,1} \\ & \vdots \\ & \mathbf{X}_{S_i,T} \end{bmatrix} . \]  

\[(TA.14)\]

Let \( \hat{\mathbf{b}}_S \) denote the OLS estimate for \( \mathbf{b}_S \). The OLS estimate for the asymptotic covariance matrix of \( \hat{\mathbf{b}}_S \) is

\[ \mathbf{Q}_S = \begin{bmatrix} \text{Cov}(\hat{\mathbf{u}}_1, \hat{\mathbf{u}}_1) \otimes (\mathbf{X}_{S_i}^\top \mathbf{X}_{S_i})^{-1} & \cdots & \text{Cov}(\hat{\mathbf{u}}_n, \hat{\mathbf{u}}_n) \otimes \left( (\mathbf{X}_{S_i}^\top \mathbf{X}_{S_i})^{-1} (\mathbf{X}_{S_i}^\top \mathbf{X}_{S_i}) \left( \mathbf{X}_{S_i}^\top \mathbf{X}_{S_i} \right)^{-1} \right) \\ \vdots & \ddots & \vdots \\ \text{Cov}(\hat{\mathbf{u}}_n, \hat{\mathbf{u}}_1) \otimes \left( (\mathbf{X}_{S_i}^\top \mathbf{X}_{S_i})^{-1} (\mathbf{X}_{S_i}^\top \mathbf{X}_{S_i}) \left( \mathbf{X}_{S_i}^\top \mathbf{X}_{S_i} \right)^{-1} \right) & \cdots & \text{Cov}(\hat{\mathbf{u}}_n, \hat{\mathbf{u}}_n) \otimes (\mathbf{X}_{S_i}^\top \mathbf{X}_{S_i})^{-1} \end{bmatrix} . \]

\[ \text{Step 3: Calculate the asymptotic covariance matrix } \hat{\mathbf{b}}_{\text{MV}} \text{ and } \hat{\mathbf{b}}_S \]

\[ \text{Cov} \left( \hat{\mathbf{b}}_{\text{MV}}, \hat{\mathbf{b}}_S \right) = \begin{bmatrix} \text{Cov}(\hat{\mathbf{e}}, \hat{\mathbf{e}}) \otimes \left( (\mathbf{X}^\top \mathbf{X})^{-1} (\mathbf{X}^\top \mathbf{X}_{S_i}) (\mathbf{X}_{S_i}^\top \mathbf{X}_{S_i})^{-1} \right) & \cdots & \text{Cov}(\hat{\mathbf{e}}, \hat{\mathbf{e}}_k) \otimes \left( (\mathbf{X}^\top \mathbf{X})^{-1} (\mathbf{X}^\top \mathbf{X}_{S_k}) (\mathbf{X}_{S_k}^\top \mathbf{X}_{S_k})^{-1} \right) \end{bmatrix} . \]

To compute the test statistics, it is convenient to combine the OLS estimates in one big vector \( \hat{\mathbf{b}} \equiv [\hat{\mathbf{b}}_{\text{MV}}, \hat{\mathbf{b}}_S] \), which has the asymptotic covariance matrix

\[ \mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{\text{MV}} & \text{Cov} \left( \hat{\mathbf{b}}_{\text{MV}}, \hat{\mathbf{b}}_S \right) \\ \text{Cov} \left( \hat{\mathbf{b}}_{\text{MV}}, \hat{\mathbf{b}}_S \right)^\top & \mathbf{Q}_S \end{bmatrix} . \]  

\[(TA.15)\]

\[ \text{Step 4: To select the alphas and co-skewnesses used for the spanning tests, we define the matrix} \]

\[ \mathbf{H} \equiv \begin{bmatrix} \mathbf{I}_n \otimes \begin{bmatrix} 1 & 0_k^\top \\ 0 & 0_k^\top \end{bmatrix} & \mathbf{0}_{2n \times 2nk^2} \\ \mathbf{0}_{2nk^2 \times 2n} \otimes \mathbf{A} \otimes \begin{bmatrix} 0 & 1 \end{bmatrix} \end{bmatrix} , \]

where \( \mathbf{I}_n \) is a \( n \times n \) identity matrix, and \( \mathbf{A} \) is a diagonal matrix for which the elements on the diagonal equal \( \text{vec} (\mathbf{T}_k) \otimes 1_n \) (\( \mathbf{T}_k \) is a \( k \times k \) upper triangular matrix in which all non-zero entries equal one). The purpose of \( \mathbf{A} \) is to eliminate the repeated rows in \( \hat{\mathbf{b}}_S \) and the corresponding asymptotic covariance matrix \( \mathbf{Q}_S \).
Table TA.1: Simulated rejection rates

|          | Panel A. Span_{MVS}^{s,b} | Panel B. Span_{MVS}^{s,b} | Panel C. Over_{MVS}^{s,b} |
|----------|----------------------------|----------------------------|-----------------------------|
|          | 0.10 0.05 0.01            | 0.10 0.05 0.01            | 0.10 0.05 0.01             |
| 3 years  | 0.114 0.062 0.015         | 0.120 0.062 0.016         | 0.100 0.066 0.030          |
| 5 years  | 0.107 0.052 0.011         | 0.116 0.060 0.013         | 0.104 0.062 0.025          |
| 10 years | 0.092 0.047 0.010         | 0.105 0.052 0.013         | 0.101 0.059 0.018          |
| 25 years | 0.094 0.046 0.009         | 0.107 0.058 0.012         | 0.105 0.059 0.017          |

Column headers refer to the significance levels for which the average rejection rates of the null hypothesis (small sample size) of the tests in 10,000 simulations are reported. Samples range from 3 to 25 years of monthly data. Spanning tests with short-sales and without short-sales are in Panel A and Panel B, respectively, and the overlap tests are in Panel C. The simulated data assumes that the returns on the benchmark assets follow a multivariate skew normal distribution (Azzalini and Valle, 1996) and that the new asset is spanned by the benchmark assets.

From the last step, we obtain \( \hat{h} = H\hat{b} \) and \( \text{Var}\left(\hat{h}\right) = HQH^\top \). Note that the vector \( \hat{h} \) has \( 2n+2nk^2 \) rows but \( nk(3k-1)/2 \) of them are zero. These zero-rows do not affect the asymptotic distribution of the tests and can be removed (in which case \( \text{Var}\left(\hat{h}\right) \) needs to be adjusted accordingly).

3.2 Overlap

Following Patton and Timmermann (2010), the overlap test is implemented with the Politis and Romano (1994) stationary bootstrap. As parameters, we choose 1000 bootstrapped samples and a block length of 6.

4 Small sample size and power

This section analyzes the small sample properties of our tests with simulations. Table IA.1 contains the average number of rejections of overlap and spanning with \( n = 1 \) and \( k = 2 \) in 10,000 simulations for the asymptotic significance levels 0.01, 0.05, and 0.10. The data-generating process of the two benchmark assets is a multivariate skew normal distribution (Azzalini and Valle, 1996) which has the parameters chosen to fit the first three moments of the benchmark assets in the empirical analysis. The data-generating process of the test asset assumes spanning (e.g., \( \Pi \) holds with \( \alpha = 0 \) and each slope coefficient equals 0.5). The regression residual is generated from a skew normal distribution with variance 0.23% and skewness \(-0.04\), which correspond to the average sample values in the empirical section. We simulate 3-25 years of monthly data and report the average number of rejections for the spanning test with short sales in Panel A and without short sales in Panel B. Panel C reports the size of the overlap test.

Table IA.1 shows that the finite sample size is very close to the asymptotic size. For the 5% significance level, differences between the asymptotic size and the small sample size is less than 2% in all cases. The small sample size of the overlap test is also close to its asymptotic size, suggesting that all our tests have good small sample size properties.
Panel (a) shows the power as a function of the intercept of the mean-variance regression (alpha) for the spanning tests. Panel (b) contains the corresponding rejection rates for the overlap test. The power is derived for each value of the intercept from 1,000 simulations with 100 monthly observations each, when the significance level is 5 percent.

Figure TA.1 contains the rejection rates of spanning and overlap as a function of the intercept of the mean-variance regression. The power of the spanning test with short-sales seems to be slightly larger than its theoretical power. In addition, the power of the test with inequality constraints generally exceeds the power of the test with equality constraints, which is in line with the results for the mean-variance case of DeRoon et al. (2001). The figure also contains the rejection rates for the overlap test. While a positive alpha does not invalidate the null hypothesis here, it appears that the null is still rejected more often for positive alphas.\footnote{This is in line with Patton and Timmermann (2010). They show that their test still rejects the hypothesis of a non-monotonic relationship when some excess return differences in portfolio sorts are insignificant.}

Overall, the size and power analysis suggests that any possible small sample bias is likely to be too small to affect any of our conclusions.
5 Simulated option strategies

To provide evidence on the effectiveness of our approach, we conduct a performance analysis of call writing strategies in a simulated economy with lognormally distributed market returns. While such an economy is complete, a positive Jensen’s alpha can be achieved at the expense of negative co-skewness with strategies that sell put or call options—i.e., concave investment strategies. In our simulations, factor models in the vein of Fung and Hsieh (2001) and Agarwal and Naik (2004) (those which include option returns or polynomials of the market return as factors) usually conclude that the alphas of option strategies are significant. This happens because these models restrict strategy returns to be a fixed linear combination of factors and therefore cannot account for concave relations between strategy and benchmark returns as long as the option factors have different strikes than the strategy to be explained. Because our approach jointly considers alphas and co-skewness, it is more flexible and can account for the concave relation between benchmark assets and strategy returns. In the simulations, our test concludes that option strategies do not improve the investment opportunity set of investors concerned about skewness.

To show that our method is able to determine whether benchmark assets span test assets with option-like returns, we assess the performance of option strategies in a simulated economy. As in Leland (1999), the market return follows a geometric Brownian motion with constant drift and volatility, and options are priced with Black-Scholes formulas. This setting is attractive because the economy is complete and the representative investor is known to have CRRA utility (He and Leland, 1993). We consider strategies shorting call options. The proceeds from the sale and the margin requirements are invested at the risk-free rate. The strategies are calculated for different strike levels, and margin requirements are determined as in Santa-Clara and Saretto (2009). We calculate the returns on these strategies for 100 observations and repeat the simulations 10,000 times. Table 1A.2 reports the results.

Call writing strategies have positive Jensen’s alphas on average and also in most simulations, and they are significant around 50% of the time. The generalized Leland (1999) alphas calculated with power utility are also positive but smaller in magnitude and usually insignificant. The residual co-skewness of the option strategies is negative in all simulations and usually significant. Hence, the mean-variance benefits of these strategies come at the cost of more left-tail risk, and the overlap

39In unreported robustness checks, we have also considered other strategies such as shorting puts or straddles, and found very similar results.

40We have verified that the generalized alphas get even closer to zero as we increase the number of observations from 100 to a larger number, in line with the results in Leland (1999) and theory.
Table TA.2: Simulated option strategies

| Strike  | 95   | 105  | 115  | Strike  | 95   | 105  | 115  |
|---------|------|------|------|---------|------|------|------|
| $\alpha$ in % | 2.56 | 6.35 | 7.26 | $\alpha_1$ in % | -1.57 | 2.21 | 5.03 |
|          | [0.90] | [0.93] | [0.96] |          | [0.01] | [0.99] | [0.99] |
|          | (0.54) | (0.51) | (0.47) |          | (0.67) | (0.64) | (0.57) |
| $\alpha_{CRRA}$ in % | 0.36 | 0.73 | 0.70 | $\alpha_2$ in % | -0.55 | 0.46 | 0.33 |
|          | [0.58] | [0.56] | [0.56] |          | [0.12] | [0.87] | [0.86] |
|          | (0.05) | (0.05) | (0.05) |          | (0.40) | (0.31) | (0.21) |
| Coskew $\times 10^3$ | -2.88 | -8.18 | -10.53 | $\alpha_3$ in % | 0.16 | -0.25 | -0.47 |
|          | [0.00] | [0.00] | [0.00] |          | [0.61] | [0.43] | [0.47] |
|          | (0.72) | (0.93) | (0.91) |          | (0.43) | (0.30) | (0.20) |
| Cokurt $\times 10^4$ | 0.18 | -4.97 | -13.26 | $\alpha_4$ in % | 6.72 | 17.93 | 21.72 |
|          | [0.55] | [0.27] | [0.09] |          | [1.00] | [1.00] | [1.00] |
|          | (0.05) | (0.07) | (0.10) |          | (1.00) | (1.00) | (1.00) |
| Over$_{MVS}$ | (0.00) | (0.00) | (0.00) | Span$_{MVS}$ | (0.52) | (0.49) | (0.44) |

Each column analyzes the performance of call writing strategies with the indicated strikes in a simulated economy with a log-normally distributed market return with expected return of 12% and volatility of 15%. The initial value of the market is 100, and the risk-free rate is 5%. These values are as in Leland (1999). The strategies short a call option with a maturity of 1.5 periods, and buy the shorted option one period later. Options are traded at Black-Scholes prices, and there are CBOE type margin requirements as in Santa-Clara and Saretto (2009). The statistics are calculated for 100 realizations of market returns, and the simulations are repeated 10,000 times. $\alpha$ is the simple alpha, and $\alpha_{CRRA}$ is Leland (1999)'s generalized alpha using the relative risk aversion implied by the market return parameters, 3.63. Coskew and cokurt are the residual co-moments, and Span$_{MVS}$ is the rejection rate for the spanning test with short-sale constraints, and Over$_{MVS}$ for the overlap test. The last rows report the intercepts of a one factor model with a short call (strike 100), $\alpha_1$; and two factor models with: two short calls (strikes 100 and 120), $\alpha_2$; market and short call (strike 100), $\alpha_3$; and Kraus and Litzenberger (1976)'s quadratic market model with market and squared market return, $\alpha_4$. The table reports the average value of these coefficients across all simulations, the fraction of times in which the coefficient was positive in brackets, and significantly different from zero at the 5% level in parentheses.
test correspondingly does not reject the null hypothesis in any simulation. Notice that the spanning test with short-sale constraints rejects the null in almost half of the simulations because a subset of the investors—the mean-variance investors—wish to invest in the option strategies. We also report results for residual co-kurtosis. It turns out that residual co-kurtosis is significant in at most 10% of the simulations, in line with the idea that the skewness framework captures the main features of option-like payoffs in this simple setting.

The table also reports the intercepts of popular factor models used to capture non-linear payoffs such as a single-factor model with call returns, and two-factor models with the market and call returns, or two call returns. These specifications are inspired by the popular factor models of Fung and Hsieh (2001) and Agarwal and Naik (2004) who use returns on option strategies to capture hedge funds’ non-linear returns. To be clear, these papers aim to explain the variation in hedge fund returns, whereas we ask whether linear factor models augmented with option returns can evaluate the performance of non-linear investment strategies. It turns out this is not the case, and the option-based factor models tend to find that the call option strategies have significant intercepts in 20% to 67% of the simulations. This is quite striking because the test assets are deliberately constructed to differ only from the factors by their strikes. Unreported alphas and rejection rates are even higher if different strategies such as selling straddles are used as test assets. Finally, Kraus and Litzenberger (1976)’s quadratic market model, with the excess market return and the squared excess market return, yields alphas that are large, positive, and significant in all simulation draws. The quadratic market model is unable to assess the alpha of these strategies, because the slope on the squared market return is negative (in line with negative residual co-skewness) and the estimated intercepts are higher because average squared returns are positive.

Overall, the evidence in Table TA.2 suggests the overlap test is very effective in assessing that these option strategies do not add value to investors concerned about skewness, whereas linear factors are in general unable to get to this conclusion. Our simulation analysis is deliberately kept simple, as we provide further evidence on the effectiveness of our approach with real-world option strategies in the empirical section of the paper.

6 Spanning and overlap in subsamples

This section analyzes the rejection rates within the subcategories explained in Morningstar (2011). Table TA.3 reports the results, and the number of funds in each subcategory. The rejection rates suggest that among the global derivatives funds, the systematic futures funds are most likely to improve the investment opportunity set of all skewness investors. This is expected because
they follow trend-following strategies which prosper when markets demonstrate sustained bearish or bullish trends, thus generating a positive relation between funds excess returns and squared stock market returns. Other strategies, such as event driven funds lose most of their appeal once all skewness investors are considered.

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Table TA.3: Spanning and overlap within Morningstar’s hedge-fund categories

| Directional Equity                          | \( \text{Span}_{s.b}^{M_N} \) | \( \text{Span}_{s.b}^{M_V} \) | \( \text{Over}_{s.b}^{M_NV} \) | \( \text{Over}_{s.b}^{M_VV} \) | #funds |
|---------------------------------------------|-------------------------------|-------------------------------|-------------------------------|-------------------------------|--------|
| Asia/Pacific Long/Short Equity              | 15.7                          | 33.1                          | 12.4                          | 15.7                          | 121    |
| Bear Market Equity                          | 16.7                          | 16.7                          | 0.0                           | 11.1                          | 18     |
| China Long/Short Equity                     | 33.3                          | 66.7                          | 33.3                          | 33.3                          | 3      |
| Emerging Markets Long-Only Equity           | 2.4                           | 14.3                          | 2.4                           | 4.8                           | 42     |
| Emerging Markets Long/Short Equity          | 22.7                          | 27.9                          | 14.2                          | 13.7                          | 233    |
| Europe Long/Short Equity                    | 31.9                          | 36.1                          | 16.8                          | 31.9                          | 119    |
| Global Long/Short Equity                    | 26.2                          | 34.8                          | 9.4                           | 16.7                          | 233    |
| U.S. Long/Short Equity                      | 31.6                          | 33.1                          | 12.1                          | 20.5                          | 580    |
| U.S. Small Cap Long/Short Equity            | 26.7                          | 29.5                          | 9.6                           | 20.5                          | 146    |
| Long-Only Equity                            | 19.6                          | 33.3                          | 9.8                           | 11.8                          | 51     |
| Directional Debt                            |                               |                               |                               |                               |        |
| Long/Short Debt                             | 37.9                          | 37.3                          | 10.2                          | 18.6                          | 177    |
| Long-Only Debt                              | 46.7                          | 46.7                          | 3.3                           | 33.3                          | 30     |
| Event                                       |                               |                               |                               |                               |        |
| Distressed Securities                       | 52.1                          | 41.5                          | 2.1                           | 12.8                          | 94     |
| Event Driven                                | 40.2                          | 34.1                          | 6.1                           | 15.9                          | 132    |
| Merger Arbitrage                            | 81.0                          | 85.7                          | 9.5                           | 9.5                           | 21     |
| Global Derivatives                          |                               |                               |                               |                               |        |
| Currency                                    | 32.4                          | 35.3                          | 17.6                          | 29.4                          | 34     |
| Global Macro                                | 24.5                          | 37.1                          | 14.3                          | 26.2                          | 237    |
| Systematic Futures                          | 24.7                          | 35.4                          | 22.7                          | 45.1                          | 308    |
| Volatility                                  | 34.8                          | 34.8                          | 4.3                           | 8.7                           | 23     |
| Multi-strategy                              |                               |                               |                               |                               |        |
| Multistrategy                               | 38.1                          | 40.1                          | 11.3                          | 19.8                          | 247    |
| Fund of Funds                               |                               |                               |                               |                               |        |
| Fund of Funds - Macro/Systematic            | 22.6                          | 32.1                          | 27.4                          | 34.9                          | 106    |
| Fund of Funds - Debt                        | 10.2                          | 20.4                          | 8.2                           | 4.1                           | 49     |
| Fund of Funds - Equity                      | 11.5                          | 21.6                          | 13.2                          | 8.0                           | 425    |
| Fund of Funds - Event                       | 14.7                          | 14.7                          | 6.7                           | 9.3                           | 75     |
| Fund of Funds - Multi-strategy              | 16.7                          | 20.7                          | 9.8                           | 10.3                          | 551    |
| Fund of Funds - Other                       | 27.3                          | 28.3                          | 6.1                           | 17.2                          | 99     |
| Fund of Funds - Relative Value              | 17.1                          | 17.1                          | 2.9                           | 5.7                           | 70     |
| Relative Value                              |                               |                               |                               |                               |        |
| Convertible Arbitrage                       | 15.2                          | 42.4                          | 3.0                           | 18.2                          | 66     |
| Debt Arbitrage                              | 40.7                          | 44.4                          | 9.9                           | 17.3                          | 81     |
| Diversified Arbitrage                       | 34.5                          | 24.1                          | 13.8                          | 17.2                          | 29     |
| Equity Market Neutral                       | 34.3                          | 34.3                          | 9.5                           | 15.3                          | 137    |
| Other                                       |                               |                               |                               |                               |        |
| Empty                                       | 30.0                          | 31.1                          | 4.2                           | 14.7                          | 100    |
| Long-Only Other                             | 34.6                          | 34.6                          | 7.7                           | 19.2                          | 26     |

Columns 1-4 show the rejection rates of the spanning and overlap tests within the Morningstar hedge-fund categories, and Column 5 reports the number of funds in each subcategory. These categories are described in Morningstar [2011], and the tests are constructed as explained in the main text.
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