COMPACTIFICATION OF BOUNDED SEMIGROUP REPRESENTATIONS

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Abstract. The given study uses the methods to identify compactifications of semigroups $S \subseteq L(X)$, which reside in the space $L(X)$. This method generalizes in some sense the deLeeuw-Glicksberg-Theory to a greater class of functions. The approach provides an abstract approach to several notions of almost periodicity, which mainly involving right semitopological semigroups [31], and the adjoint theory. Moreover, the given setting is refined to the case of bounded $C_0$–semigroups.

1. Introduction

The main idea is the result that $L(Y, X^*)$ is a dual space, and therefore carries a $w^*$–topology. Consequently, for subspaces $Y \subseteq X^*$, the operator space $L(Y) \subseteq L(Y, X^*)$ carries a $w^*$–topology. This case applies to several cases, like sun-dual semigroups

$$X^\odot := \left\{ x^* \in X^* : \| x^* \| - \lim_{t \to 0} T^*(t)x^* = x^* \right\} \subseteq X^*$$

, or bounded uniformly continuous functions. Cases which support results on several notions of almost periodicity, and therefore ergodic results. The advantage that the compactification resides in the same operator space, is that the idempotent becomes a projection on the given space, which serves for a splitting of the space, in so called reversible and flight vectors. In this scope, we consider the weak-star topology instead of the weak topology and address by necessity the well-known fact that bounded linear operators between dual spaces are weak – weak but not necessarily weak* – weak* continuous. This approach gives answers based on locally convex theory, to enlarge the boundaries to which results on almost periodicity are still true. They give an slight overview, why some results apply.

2. Preliminaries and Notation

Throughout this study, we assume $S$ to be a complete regular Abelian semigroup. The main literature on functional analysis is [26] on $C_0$–semigroups, please refer to [11], [25], and [22], and for special results on sun-dual semigroups, to [11] and [24]. To obtain the main definitions and results of deLeeuw–Glicksberg theory [3] and [4], the book of [19] pp. 103ff, 2.4] is sufficient. The results of the harmonic analysis are taken from [10], the general topology results on nets from [12], and the locally convex topological space results from [14] and [13]. For the special results on right semitopological semigroups it is referred to [31]. For the locally convex topology notions recall, $\sigma(X,Y)$ the weakest topology such that the space of continuous linear functionals $(X,\sigma(X,Y))^* = Y$. (i.e., $w$-topology =$\sigma(X,X^*)$, $w^*$-topology = $\sigma(X^*,X)$)

Key words and phrases. right semitopological semigroups, compactification, $C_0$–semigroups, almost periodicity.

The author thanks Professor Ruess for his suggestions and advice.

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3. Dual Representations and Left-Semitopological Semigroups

We begin this section with a remark, which points to the first idea of this study.

**Remark 3.1.** [14, Prop. 3, p.330] Let $X, Y$ be Banach spaces, and let $L(X, Y)$ be the Banach space of bounded and linear operators. As $L(X, Y^*) = (X \otimes \pi Y)^*$, a bounded net $\{T_\lambda\}_{\lambda \in \Lambda} \subset L(X, Y^*)$ is convergent to $T \in L(X, Y^*)$ in the $w^*$-topology if

$$\lim_{\lambda \in \Lambda} < T_\lambda x, y > = < Tx, y > \text{ pointwise on } x \in X, \ y \in Y.$$

For this study it is only of interest, that $B_{L(X,Y^*)}$ is compact with respect to the predefined topology, which surely can be verified by an application of Tychonov’s theorem. Surely, this implies that $L(X,Y^*)$ is a dual space, and on $B_{L(X,Y^*)}$ the $w^*$-topology and the predefined coincide, compare [17].

As we discuss semigroups with a certain continuity property, we recall

**Definition 3.2.** [31, p. 8, Definition 1.2, 1.3]

1. Let $A, B$ and $C$ be topological Hausdorff spaces. A map 
   \[ \pi : A \times B \to C \] 
   \[ (x, y) \mapsto \pi(x, y) \]
   is called right (left) continuous if it is continuous in the right (left) variable; that is, for every fixed $x_0 \in X$ ($y_0 \in Y$), the map \( y \mapsto \pi(x_0, y) \) ($\{x \mapsto \pi(x, y_0)\}$) is continuous.
2. Let $S$ be a nonvoid topological space that is provided with an associative multiplication 
   \[ \mu : S \times S \to S \] 
   \[ (x, y) \mapsto \mu(x, y) = xy. \]
   Then, the pair $(S, \mu)$ is called a right (left) semitopological semigroup if $\mu$ is right (left) continuous.

The next definition can be derived from the idea, to find the semigroup as a restriction of a dual operator. In the definition everything is subtracted, what is unnecessary for the proofs given in this study.

**Definition 3.3.** For an Abelian topological semigroup $S$ and a Banach space $X$, and $Y \subset X^*$ a set $\{T(t)\}_{t \in S} \subset L(Y)$ is called a dual semigroup representation, if

1. $T(s + t) = T(s)T(t)$,
2. for given $x \in X$ and $y \in Y$
   \[ T_y : S \to Y \] 
   \[ s \mapsto < x, T(s)y >, \]
   is continuous.
3. $\{T(t)x : t \in S\}_{w^*} \subset Y$,
4. For all $s \in S$, and $y \in Y$,
   \[ T(s) : (ac \{T(t)y : t \in S\}_{w^*}, w^*) \to (ac \{T(t)y : t \in S\}_{w^*}, w^*) \]
   \[ x \mapsto T(s)x \]
   is continuous.
Definition 3.4.  
(1) Let \( \{T(t)\}_{t \in S} \subset L(Y) \) a dual representation of a semigroup \( S \), then 
\( y \in Y \) is called reversible, if for every net \( \{s_\alpha\}_{\alpha \in A} \subset S \) there exist a net \( \{t_\gamma\}_{\gamma \in \Gamma} \subset S \), 
such that \( w^* - \lim_{t \in \Gamma} T(t_\gamma)T(s_\alpha)y = y \). Let \( Y_{\text{rev}} \) the set of reversible vectors. 
(2) \( y \in Y \) is called a flight vector, if for a net \( \{s_\alpha\}_{\alpha \in A} \subset S \), we have 
\( w^* - \lim_{t \in A} T(s_\alpha)y = 0 \). Let \( Y_0 \) the set of flight vectors.

After the above definition it becomes straightforward.

Proposition 3.5. If \( x \in Y \) is a flight vector and reversible, then \( x = 0 \).

Proposition 3.6. Let \( \{T(t)\}_{t \in S} \) a dual semigroup representation. 
(1) \( \{T(t)\}_{t \in S}, w^* \) is a left semitopological semigroup. 
(2) \( \{T(t)\}_{t \in S} \) has a compactification \( T = \overline{\{T(t)\}_{t \in S}} \subset L(Y) \) which is left semitopological 
semigroup. 
(3) \( \overline{w^*}\{T(t)\}_{t \in S} \subset L(Y) \) is a compact left semitopological semigroup. 
(4) \( w^* \{T(t)\}_{t \in S} \subset L(Y) \) is a compact left semitopological semigroup, and \( T(s) \) commutes 
with every operator \( U \in \overline{w^*}\{T(t)\}_{t \in S} \). 
(5) \( Y = Y_a \oplus Y_0 \), with \( Y_a \subset Y_{\text{rev}} \), and \( Y_0 \subset Y_{fl} \).

Proof. Let \( \{s_\alpha\}_{\alpha \in A} \) a \( w^* \)-convergent net with limit \( s, y \in Y \). Then for given \( V \in L(Y) \) we have 
\( \lim_{\alpha \in A} < x, T(s_\alpha)Vy > = < x, T(s)Vy > . \)

Letting for nets \( \{t_\gamma\}_{\gamma \in \Gamma}, \{s_\alpha\}_{\alpha \in A} \subset S, R = w^* - \lim_{t \in \Gamma} T(t_\gamma), \) and \( S = w^* - \lim_{t \in A} T(s_\alpha) \). Due to 
Def. 3.3, we have \( R, S \in L(Y), \) and due to Def. 3.3(4) 
\( w^* - \lim_{t \in A} T(s_\alpha)T(t_\gamma) = w^* - \lim_{t \in \Gamma} T(t_\gamma)S = RS \)

which proves \( RS \in \overline{T(t)}_{t \geq 0} \). The left continuity is straightforward. For the convex or the 
absolute convex hull the proof is quite similar.

Let \( s \in S, \) and \( R = w^* - \lim_{t \in \Gamma} T(t_\gamma) \). Then due to 3.3(4) 
\( T(s)R = w^* - \lim_{t \in \Gamma} T(s)T(t_\gamma) = w^* - \lim_{t \in \Gamma} T(t_\gamma)T(s) = RT(s) \).

Applying Thm. 5.6 we obtain a minimal idempotent \( P \), hence \( Y = PY \oplus (I - P)Y \). Let 
\( \{s_\alpha\}_{\alpha \in A} \subset \) with \( U = w^* - \lim_{t \in A} T(s_\alpha) \). Applying Thm. 5.6(4), we find \( PV_P U \in \mathcal{T} \), such that 
\( PV_P U P y = P y, \) hence \( P y \in Y_{\text{rev}} \). The identity \( P(I - P)y = 0 \) proves \( (I - P)Y \subset Y_{fl} \).

\( \square \)

Example 3.7. Let \( \{T^\alpha(t)\}_{t \in \mathbb{R}^+} \subset L(X) \) a \( C_\alpha \)-semigroup, the \( \{T^\alpha(t)\}_{t \in \mathbb{R}^+} \) is a dual representation 
on \( X^\alpha \).

Proof. Sure \( \{T^\alpha(t)\}_{t \in \mathbb{R}^+} \) are \( w^* - w^* \)-continuous operators. Let \( S \in \mathcal{T} \), then for \( x \in X^\alpha \), 
we have \( T^\alpha(t)Sx = ST^\alpha(t)x \rightarrow Sx, \) when \( t \rightarrow 0 \). Hence 
\( \overline{O(x)}^{w^*} = \{Sx : S \in \mathcal{T} \} \subset X^\alpha, \)
what concludes the proof. \( \square \)

Theorem 3.8. Let \( P \) a minimal idempotent.

(1) If for all \( y \in Y \),
\[
P : \ \{(T(t)y : t \in S)^{w^*}, w^*\} \rightarrow \{(T(t)y : t \in S)^{w^*}, w^*\}
\]
\[
x \mapsto Px
\]
is continuous, then \( PTP \) is a compact group.
(2) If \( P = V^*_Y \) with \( V \in L(X) \), then \( PTP \) is a compact group.

(3) If a minimal idempotent \( P \) commutes with every operator of the compactification, then \( PTP \) is a compact group.

Proof. To verify the compactness of the group \( PTP \), let \( \{ T_\alpha \}_{\alpha \in \mathcal{A}} \) a net. Due to the compactness of \( \mathcal{T} \) has the cluster point \( T \). Without loss of generality \( T \) is the limit. Then \( \{ T_\alpha P x \} \subset \alpha c \{ T(t)P x : t \in S \}^w \), \( w^* - \lim_{\alpha \in \mathcal{A}} T_\alpha P x = TP \). Consequently,

\[
\lim_{\alpha \in \mathcal{A}} < x, P T_\alpha P x > = < x, P T P x > .
\]

which proves the claim. Dual operators are \( w^* - w^* \)-continuous. The case \( P \) commutes is straightforward. \( \square \)

**Corollary 3.9.** If \( \mathcal{T} \) is Abelian, then \( PTP \) is a compact Abelian group.

Proof. Use that the minimal idempotent \( P \in \mathcal{T} \). \( \square \)

Next, we recall some definitions of certain classes of vectors and functions.

**Definition 3.10.** Let \( \{ T(t) \}_{t \in S} \) be a dual representation on \( Y \subset X^* \).

1. The orbit of a vector \( y \in Y \) is given by

\[ O(x) := \{ Tst x : t \in S \} \]

2. A vector \( x \in Y \) is called Eberlein-weak almost periodic (E.-wap) if \( O(x) \) is weakly relatively compact in \( Y \).

3. A vector \( x \in Y \) is called almost periodic if \( S = \mathbb{R} \) and \( O(x) \) is relatively compact in \( Y \).

4. We define for a Banach space \( X \). A function \( f \in C_b(S, X) \) is called E.-wap if the orbit with respect to the translation semigroup is rel. \( \sigma(C_b(\mathbb{R}, X), C_b(\mathbb{R}, X)^* \) \) compact.

\[
W(S, X) := \{ f \in C_b(S, X) : f \text{ is E.-wap} \} ,
\]

\[
W_0(S, X) := \{ f \in W(S, X) : f|_{\mathbb{R}} \rightarrow 0 \text{ weakly for some } \{ t_n \}_{n \in \mathbb{N}} \subset \mathbb{R} \} ,
\]

\[
AP(\mathbb{R}, X) := \{ f \in C_b(\mathbb{R}, X) : f \text{ is almost periodic} \} .
\]

In the scalar valued case the \( X \) is left.

5. Let \( M \subset Y^* \) a closed and separating space, and \( S = \mathbb{R} \), a vector \( y \in Y \) is called \( M \)-weakly almost periodic if \( \{ t \mapsto x^* T(t) x \} \in AP(\mathbb{R}) \), for all \( x^* \in M \).

**Theorem 3.11.** Let \( S \) is Abelian and \( \{ T(t) \}_{t \in S} \) a dual representation on \( Y \). If \( y \in Y \), we have \( UVy^* = VUy \), for all \( U, V \in \mathcal{T} \) if and only if,

\[
i_x : S \rightarrow W(S)
\]

\[
s \mapsto < x, T(s) y >
\]

for a set of separating vectors \( x \in X \).

Proof. Letting for nets \( \{ t_\gamma \}_{\gamma \in \Gamma} \subset S \), \( R = w^* - \lim_\gamma T(t_\gamma) \), and \( S = w^* - \lim_\alpha T(s_\alpha) \). The following identity

\[
< x, RSy > = \lim_{\alpha \in A} \lim_{\alpha \in A} < x, T(t_\gamma + s_\alpha) y > = \lim_{\alpha \in A} \lim_{\alpha \in A} < x, T(t_\gamma + s_\alpha) y > = < x, SRY > ,
\]

proves the claim in both directions. If \( < x, T(s) y > \in W(S) \), the double limit criteria hold, and the operators commute, and if they commute the double limit criteria \( \square \) implies \( < x, T(s) y > \in W(S) \).
Now, let \( U, V \in \mathcal{L}(\mathcal{E}, \mathcal{E}^*) \). Then, we find nets \( \{t_i^\gamma\}_{i \in \mathbb{N}, \gamma \in \Gamma} \), \( \{s_i^\lambda\}_{i \in \mathbb{N}, \lambda \in \Lambda} \subset \mathbb{R}^+ \) and \( \{a_i^\gamma\}_{i \in \mathbb{N}, \gamma \in \Gamma}, \{b_i^\lambda\}_{i \in \mathbb{N}, \lambda \in \Lambda} \subset \mathbb{R} \), with \( \sum_{i=1}^{\infty} |a_i^\gamma| \leq 1 \) and \( \sum_{i=1}^{\infty} |b_i^\lambda| \leq 1 \), such that

\[
U_{\gamma} := \sum_{i=1}^{n_\gamma} a_i^\gamma T(t_i^\gamma) \text{ with } \lim_{\gamma \in \Gamma} U_{\gamma} = U \text{ and } V_{\lambda} := \sum_{i=1}^{m_\lambda} b_i^\lambda T(s_i^\lambda), \text{ with } \lim_{\lambda \in \Lambda} V_{\lambda} = V.
\]

Next, define \( g(s) := \langle x, T(s)x^* \rangle \), which is assumed to be \( E_{-wap} \), and the bounded linear functional \( \delta_t(g) := g(t) \); then, the duality reads as

\[
\langle x, VUx^* \rangle = \lim_{\lambda \in \Lambda} \langle x, V_{\lambda}U^*x^* \rangle
\]

\[
= \lim_{\lambda \in \Lambda} \lim_{\gamma \in \Gamma} \sum_{i=1}^{n_\gamma} \sum_{j=1}^{m_\lambda} a_i^\gamma b_j^\lambda < x, T(t_i^\gamma + s_j^\lambda) x^* >,
\]

\[
= \lim_{\lambda \in \Lambda} \lim_{\gamma \in \Gamma} \sum_{i=1}^{n_\gamma} \sum_{j=1}^{m_\lambda} a_i^\gamma b_j^\lambda g(t_i^\gamma + s_j^\lambda),
\]

\[
= \lim_{\lambda \in \Lambda} \lim_{\gamma \in \Gamma} \langle \sum_{i=1}^{n_\gamma} a_i^\gamma g(\cdot + t_i^\gamma), \sum_{j=1}^{m_\lambda} b_j^\lambda \delta_{s_j^\lambda} > (C_b(S), C_b(S)^*) \rangle
\]

\[
= \lim_{\gamma \in \Gamma} \lim_{\lambda \in \Lambda} \langle \sum_{i=1}^{n_\gamma} a_i^\gamma g(\cdot + t_i^\gamma), \sum_{j=1}^{m_\lambda} b_j^\lambda \delta_{s_j^\lambda} > (C_b(S), C_b(S)^*) \rangle.
\]

As \( O(g) \) is weakly relatively compact in \( C_b(S) \), its closed absolutely convex hull is weakly compact. Furthermore, because \( \|\delta_t\| \leq 1 \), the absolute convex combination is bounded. Hence, we have separated the limits and determined that the interchanged limits coincide.

Next we show how some results of \([3]\) embeds into the given context.

**Proposition 3.12.** Let \( j_X : X \to X^{**} \), denote the canonical embedding, \( S \) an Abelian semigroup, \( X \) a Banach space \( \{T(t)\}_{t \in S} \subset \mathcal{L}(X) \) such that,

1. \( T(s + t) = T(s)T(t) \),
2. for given \( x \in X, x^* \in X^* \)

\[
T_x : S \to \mathbb{C}
\]

\[
s \mapsto < x^*, T(s)x >
\]

is continuous.

3. \( \{T(t)x : t \in S\} \text{ weakly relatively compact.} \)

Then \( \{T_{j_XX}^{**}(t)\}_{t \in S} \) is a dual representation on \( j_XX \), and for all \( x^* \in X^*, x \in X \), we have

\[
\{s \mapsto < x^*, T_{j_XX}^{**}(s)j_Xx > \} \in W(S).
\]

**Proof.** Due to \( T_{j_XX}^{**}(t)j_Xx = j_XT(t)x \), and the weak compactness of the orbit, \( \overline{O(j_Xx)^{w*}} = \overline{O(j_Xx)^{w}} \subset j_XX \). The rest is straightforward.

**Proposition 3.13.** Let \( \mathbb{J} \in \{\mathbb{R}, \mathbb{R}^+\} \), and \( x^* \in X^{**} \).

1. \( \text{Let } x, T^0(\cdot)x^* \in W(\mathbb{J}) \), then for the splitting \( x^* = x^*_a + x^*_b, \) we have \( x, T^0(\cdot)x^*_a \in AP(\mathbb{R})_{\mathbb{J}}, \) and \( x, T^0(\cdot)x^*_b \in W_0(\mathbb{J}). \)

2. \( \text{If } x, T^0(\cdot)x^* \in AP(\mathbb{R})_{\mathbb{J}}, \) for a separating set of vectors \( x \in X, \text{ then } x^* \in X^*_a. \)

3. \( \text{If } x, T^0(\cdot)x^* \in W_0(\mathbb{J}), \text{ for a separating set of vectors } x \in X, \text{ then } x^* \in X^*_b. \)
Proof. As $<x, T_\cdot(t)x_\cdot^\omega> = <x, T_\cdot(t)x_\cdot^\omega_a > + <x, T_\cdot(t)x_\cdot^\omega >$, we learn from the definition of the vector valued splitting, that $<x, T_\cdot(t)x_\cdot^\omega_a >$ is an eigenvector, and $<x, T_\cdot(t)x_\cdot^\omega >$ is reversible, with respect to the scalar translation group on $W(J)$. The splitting of $\mathbb{Z}$ gives $W(J)_{rev} = AP(\mathbb{R})_{J}$, and $W(J)_{fl} = W_0(J)$.

Using almost periodic functions are reversible, we have 

$$<x, T_\cdot(t)x_\cdot^\omega > - <x, T_\cdot(t)x_\cdot^\omega_a > = <x, T_\cdot(t)x_\cdot^\omega >,$$ 

for all $x \in X, t \in \mathbb{R}$.

By Pro 8.17 we find $<x, T_\cdot(t)x_\cdot^\omega > - <x, T_\cdot(t)x_\cdot^\omega_a >= 0$, for a separating set of vectors $x \in X$, and all $t \in \mathbb{R}$, and therefore $x_\cdot^\omega = x_\cdot^\omega_a$. For the last claim use $<x, T_\cdot(t)x_\cdot^\omega > - <x, T_\cdot(t)x_\cdot^\omega_a > > <x, T_\cdot(t)x_\cdot^\omega >$, and similar arguments.

As our main interest reduces to bounded sets in operator spaces such as $L(X, Y^*)$, for a given locally convex topology on a Banach space $X$, we define the convergence of bounded nets.

Definition 3.14. Let $\tau$ be a locally convex topology on $X^*$.

1. $\tau$ is call representation compatible, if for all $y \in Y$ the identity map 

   $$\text{id} : (ac\{T(t)y : t \in S\}^w, w^*) \rightarrow (ac\{T(t)y : t \in S\}^\tau, \tau)$$

   is a homeomorphism.

2. Let $Y \subset X^*$, and for the net 

   $$\{T_\alpha\}_{\alpha \in A} : Y \rightarrow Y,$$

   we call the net $\{T_\alpha\}_{\alpha \in A}$ $\tau - OT$ convergent if there exists a $T \in L(Y)$ with 

   $$\tau - \lim_{\alpha \in A} T_\alpha x^* = Tx^*$$

   for all $x^* \in Y$.

Proposition 3.15. Let $\{T(t)\}_{t \in S}$ a dual representation, and $\tau$ a representation compatible topology. Then 

$$\iota : (\overline{ac}^w\{\{T(t)\}_{t \in S}\}^\tau, w^*) \rightarrow (\overline{ac}_\tau^{\tau - OT}\{\{T(t)\}_{t \in S}\}, \tau_Y - OT)$$

is a homeomorphism, and $(\overline{ac}_\tau^{\tau - OT}\{\{T(t)\}_{t \in S}\}, \tau - OT)$ is a compact right semitopological semigroup.

Proof. By definition, we have, if $T \in \overline{ac}^w\{\{T(t)\}_{t \in S}\}$, then $Tx \in \overline{ac}\{T(t)x : t \in S\}^w$, which concludes the proof by an application of Tychonov.

4. Harmonic analysis of dual representations

The next results provide an abstract approach to these several types of almost periodicity. Using the previous condition, which proves the Abelian structure, we are going apply the existing Haar measure, Pettis measureability criteria, and the Mackey topology, to obtain results in the style of [3]. Additionally, two theorems which give an answer, in some sense, on the distance between the reversible part and almost periodicity.

To identify almost periodicity in [3], the following class of vectors is defined.

Definition 4.1. For $\{T(t)\}_{t \in S}$ a dual representation a vector $y \in Y$ is an eigenvector with a unimodular eigenvalue, if for a map $\lambda : \mathcal{T} \rightarrow \mathbb{C}$ with $|\lambda(T)| = 1$, we have $Ty = \lambda(T)y$ for all $T \in \mathcal{T}$. We define 

$$Y_{\text{ads}} := \overline{\text{span}}\{y \in Y : y \text{ is an eigenvector with a unimodular eigenvalue}\}.$$
Proposition 4.2. Let \( Y \subset X^* \), \( \{ T(t) \}_{t \in S} \) a dual representation, \( P \) a minimal idempotent \( \tau \) be representation compatible topology, and
\[
\{ S \ni t \mapsto \langle T(t)y, x \rangle \} \in W(S) \text{ for all } y \in Y, x \in X.
\]
Then the minimal idempotent, \( P \) is unique, \( G := PTP \) is a compact abelian group. Furthermore, let \( \Gamma \) be the character group of \( G \), \( \gamma \in \Gamma \), and let \( \rho \) denote the normalized Haar measure on \( G \). Then,
\[
S_{\gamma} := \int_G \overline{\gamma}(S)Sd\rho(S)
\]
exists in the sense of [26, Def 3.26, p. 74] in \( (L(Y), \tau - OT) \), and \( S_{\gamma} \in \overline{ac}^{\tau - OT} \left( \{ T(t) \}_{t \in S} \right) \).

Proof. Note that for two minimal idempotents by its definition, \( P_1 = P_1P_2 = P_2P_1 = P_2 \).

From Thm 3.8 we find that \( G \) is semitopological, and from an abstract harmonic analysis [7], we recall that any compact semitopological group is a topological group. Hence, we find the normalized Haar measure \( \rho \) on \( G \), [26, Thm 5.14, p. 123].

To prove the existence of the integral, we apply Theorem [26, Thm. 3.27, pp. 74-75], with respect to topology \( \tau - OT \).

\[
f : (G, \tau - OT) \longrightarrow (L(Y), \tau - OT)
\]
\[
S \quad \longmapsto \quad \overline{\gamma}(S)S
\]
is continuous, and \( f(G) \subset \overline{ac}^{\tau - OT} \left( \{ T(t) \}_{t \in S} \right) \), which is compact by Prop. 3.15.

Consequently, the integral exists and is an element of \( \overline{ac}^{\tau - OT} \left( \{ T(t) \}_{t \in S} \right) \). For the additional proof, note that, for \( x^\circ \in Y \),
\[
\delta_x : (L(Y), \tau - OT) \longrightarrow (Y, \tau)
\]
\[
S \quad \longmapsto \quad Sx
\]
is continuous, and for \( V \in L(Y) \),
\[
S_xV = \delta_{Vx}(S_{\gamma}) = \delta_{Vx} \left( \int_G \overline{\gamma}(S)Sd\rho(S) \right) = \int_G \overline{\gamma}(S)SVd\rho(S),
\]
and the claim becomes a consequence of [26, p.85, Exercise 24]. \( \square \)

Theorem 4.3. Let \( \{ T(t) \}_{t \in S} \) a dual representation, \( \tau \) be representation compatible topology, and
\[
\{ S \ni t \mapsto \langle T(t)y, x \rangle \} \in W(S) \text{ for all } y \in Y, x \in X.
\]
Then,
\[
Y_a^\tau \subset \overline{Y}^{\alpha^\tau}.
\]

Proof of Theorem 4.3. Part 1: In this part, we prove that unimodular eigenvectors are found as an image of the generalized Fourier transforms for a given \( \gamma \in \Gamma \). Similar to Prop. 4.2 for the minimal idempotents \( P \), we find that \( \rho \) is the normalized Haar measure on the Abelian compact topological group \( G = PTP \).
Furthermore, if $\Gamma$ denotes the character group, then for $\gamma \in \Gamma$, we can define

$$S_\gamma := \int_G \tau(S)d\rho(S) \in \overline{ae}^{-OT} \left(\{T(t)\}_{t \in S}\right) \subset L(Y).$$

Consequently, for $y \in Y$, we have $S_\gamma y \in Y$, and because $S_\gamma \in \overline{ae}^{-OT} \left(\{T(t)\}_{t \in S}\right)$ and $\overline{ae}^{-OT} \left(\{T(t)\}_{t \in S}\right)$ is Abelian by Theorem 3.11, we find that $S_\gamma$ commutes with the operators in $G \subset \overline{ae}^{-OT} \left(\{T(t)\}_{t \in S}\right)$. Using Prop. 4.2 for $R \in G, y \in Y$, we find that

$$RS_\gamma y = S_\gamma Ry = \delta_{Rz}(S_\gamma) = \delta_{z}(\int_G \tau(S)d\rho(S)) = \int_G \tau(S)SRyd\rho(S)$$

$$= \int_G \tau(S)RSyd\rho(S) = \gamma(R) \int_G \tau(RS)RSyd\rho(S)$$

$$= \gamma(R) \int_G \tau(S)Syd\rho(S) \text{ apply } [26 \text { Thm } 5.14 (1), (2), p. 123]$$

$$= \gamma(R)S_\gamma y.$$

Similarly, using the fact that $G$ is Abelian, we obtain

$$RS_\gamma = \gamma(R)S_\gamma = S_\gamma R.$$  

Because $P$ is the unit in $G$, we have $\gamma(P) = 1$, and by the previous observation, $PS_\gamma = S_\gamma$. Hence, for $T \in T$, we find $PT \in G$ and

$$TS_\gamma = TPS_\gamma = \gamma(TP)S_\gamma = \gamma(T)S_\gamma.$$  

This means that $S_\gamma Y$ consists of eigenvectors with unimodular eigenvalues $\lambda(T) = \gamma(T)$.

Part 2: Let $\Gamma$ be the character group of $G$. We prove that $Y_a$ cannot be separated from

$$M = \overline{\text{span}} \{S_\gamma x : y \in Y, \gamma \in \Gamma\},$$

with a $\tau$–continuous functional $\phi$ and apply Proposition 6.1

Because $M \subset X_{ads} \subset X_a$, we assume that there is a $y \in Y_a \setminus M$. By the assumption, we find a $\tau$ continuous $\phi$ such that for $\phi(Py) = \phi(y) \neq 0$ and $\phi|_M = 0$. Because for $x \in Y, \Lambda(T) := \phi(Tx)$ is $\tau – OT$ continuous, we obtain

$$0 = \phi(S_\gamma z) = \int_G \tau(S)\phi(Sz)d\rho(S)$$

for all $\gamma \in \Gamma$ and $z \in Y$. Because the characters form an orthonormal basis in $L_2(G, \rho)$—see [6, p. 944]—we have

$$\{G \ni S \mapsto \phi(Sy)\} = 0 \text{ a.e.}$$

Because $G$ carries the topology $\tau – OT$, for $\phi \tau$ continuous and $z \in Y$, the functions

$$g : (G, \tau – OT) \rightarrow \mathbb{C}$$

$$S \mapsto \phi(Sz)$$

are continuous. Consequently, $\{G \ni S \mapsto \phi(Sy)\}$ is also zero, and we find a contradiction to $\phi(Py) \neq 0$, which completes the proof.

The above result suggests the almost periodicity of the reversible part of an E.-wap solution. Letting $\mu(\tau)$ be the Mackey topology coming with $\tau$, we have by Mazur’s Theorem the following.
Corollary 4.4. Let \( \{T(t)\}_{t \in S} \) a dual representation, \( \tau \) be representation compatible topology, and
\[
\{ S \ni t \mapsto < T(t)y, x > \} \in W(S), \text{ for all } y \in Y, x \in X.
\]
Then,
\[
Y_{a}^{\mu(\tau)} \subset Y_{uds}^{\mu(\tau)}.
\]
If \( \{T(t)\}_{t \in S} \) is equicontinuous with respect to \( \mu(\tau) \), then \( \overline{O(y)^{\mu(\tau)}} \) is \( \mu(\tau) \)– precompact for all \( y \in Y_{a} \).

Proof of Cor. 4.4. We find a net \( \{y_{\alpha}\}_{\alpha \in A} \subset Y_{uds} \), with \( \mu(\tau) \)-limit \( y \in \overline{Y_{a}^{\mu(\tau)}} \). For every \( W, V, U \in U(0) \), with \( V + V \subset U \), we find a net of finite sets, \( \{F_{\alpha}\}_{\alpha \in A} \), such that \( O(y_{\alpha}) \subset F_{\alpha} + V \). From the convergence and equicontinuity, we find \( W, V, U \) such that \( x - x_{\gamma} \in W \), and \( T(t)(y - y_{\gamma}) \subset V \), for all \( \gamma \geq \alpha_{0}, t \in S \). Hence,
\[
O(y) \subset O(y_{\gamma}) + V \subset F_{\gamma} + U.
\]
\( \square \)

The result of Fréchet [8] for asymptotically almost periodic functions becomes a direct consequence.

Corollary 4.5. Let \( Z \) be a Banach space, let \( \{T(t)\}_{t \geq 0} \) be a bounded semigroup on \( Z \), and let \( O(x) \) be relatively compact for all \( x \in Z \); then, \( Z_{uds} = Z_{a} \).

Proof. Choose \( \tau \) equal to the norm topology and apply Cor 4.4. \( \square \)

Moreover, we obtain the following:

Corollary 4.6. If \( \overline{X^{\sigma(X^{*}, X)}} \) is Abelian, then
\[
\overline{X^{\sigma(X^{*}, X)}}_{uds} = \overline{X^{\sigma(X^{*}, X)}}_{a}.
\]

Separability is a further concept that applies. With the pointwise verification of the Abelian structure, we can give the following criterion for a vector to be a member of \( X_{uds}^{*} \), which is the second main result of this section, based mainly on a harmonic analysis, the Pettis-measurability criteria and Them 3.11.

Theorem 4.7. Let \( \{T(t)\}_{t \in S} \) a dual representation, \( \tau \) be representation compatible topology, with \( \sigma(X^{*}, X) \subset \tau \), and
\[
\{ S \ni t \mapsto < T(t)y, x > \} \in W(S), \text{ for all } y \in Y, x \in X.
\]
Then, the following for \( y \in Y_{a} \) are equivalent:
(1) \( \overline{O(y)^{\tau}} \) is norm separable.
(2) \( y \in Y_{uds}^{*} \).
(3) \( O(y) \) is relatively norm compact.

Proof of Theorem 4.7. As \( \overline{ac_{\tau}^{\sigma}} \) \( \{T(t)\}_{t \in S} \) is Abelian, \( G = PT \) is a compact Abelian topological group [7]. The splitting is a consequence of Prop. From Prop. 4.2 we have for \( y \in Y \),
\[
S_{\gamma}y = \int_{G} \overline{\gamma(S)}Syd\rho(S) \in Y.
\]
By [5] Cor. 4 pp. 42-43], we find that this is a Bochner integral, which is an element of \( X^{*} \). Moreover, due to \( \sigma(X^{*}, X) \subset \tau \), this expression coincides on \( X \) with the integral defined in the proof of Theorem 4.3, hence, it becomes an element of \( Y \).

For \( R \in T \), we have
\begin{align*}
RS_\gamma y &= S_\gamma Ry = \delta_{Ry}(S_\gamma) = \int_G \overline{\pi}(S)SRyd\rho(S) = \int_G \overline{\pi}(S)RSyd\rho(S) = \gamma(R)S_\gamma y.
\end{align*}

Defining

\[M = \overline{\text{span}} \{S_\gamma y : y \in Y, \gamma \in \Gamma\},\]

we have \(M \subset Y_\alpha\).

For \(y \in Y_\alpha\) and \(q : Y_\alpha \to Y_\alpha/M\) as the quotient map, if \(Z = \overline{\text{span}} \{qGy\}\), then by assumption, \((Z, \|\cdot\|)\) is separable.

Consequently, \((B_{Z^*}, w^*)\) is separable (compact metrizable). We choose \(\{z_n^*\}_{n \in \mathbb{N}}\) dense in \((B_{Z^*}, w^*)\).

By definition, \(S_\gamma y \in M\). Consequently, for the sequence of bounded linear functionals

\[U_n : Y \to \mathbb{C},\]

\[u \mapsto \langle qu, z_n^* \rangle,\]

due to the Bochner integrability, we obtain

\[0 = \langle qS_\gamma y, z_n^* \rangle = \int_G \overline{\pi}(S) < qSy, z_n^* > d\rho(S)\]

for all \(\gamma \in \Gamma\) and \(n \in \mathbb{N}\). Using \(\{\gamma\}_{\gamma \in \Gamma}\) as an orthonormal basis in \(L^2(G, \rho)\),

\[\langle qSy, z_n^* \rangle = 0 \text{ a.e. for all } n \in \mathbb{N}.\]

Thus, for sets \(A_n \subset K\), with \(\rho(A_n) = 0\), we have

\[\langle qSy, z_n^* \rangle = 0 \text{ for all } S \in G \setminus A_n, n \in \mathbb{N}.\]

Let \(A = \bigcup_{n \in \mathbb{N}} A_n\); then, \(\rho(A) = 0\), and

\[\langle qSy, z_n^* \rangle = 0 \text{ for all } S \in G \setminus A, n \in \mathbb{N}.\]

Using \(\{z_n^*\}_{n \in \mathbb{N}}\) totally on \(Z\), we find an \(S \in G\) with \(qSy = 0\). Consequently, \(Sy \in M\). Because of Part 1 of Theorem 4.3, equation (2), the space \(M\) is translation invariant, and for \(y \in X_\alpha\), we find that using \(G\) as a group on \(Y_\alpha\), a \(T \in G\) such that \(TSy = y\) and, therefore, \(y \in M \subset X_{uds}^\circ\).

(2) \(\Rightarrow\) (3): Let \(y \in Y_\alpha\). Then, \(y\) is the limit of linear combinations of the unimodular vectors \(\{x_i^n\}_{i=1}^m, n \in \mathbb{N}\) \(\subset Y_\alpha\), i.e., satisfying \(Tx_i^n = \lambda_i^n(T)x_i^n\). Consequently, \(O(x_i^n)\) is norm compact and therefore the orbit of the linear combination.

It follows that if the vectors \(\{x_n\}_{n \in \mathbb{N}}\) have relatively norm-compact orbits and \(x_n \to x\), then \(O(x)\) is relatively norm compact. Note that for some constant \(C > 0\),

\[\|Tx_n - Tx\| \leq C \|x - x_n\|,\]

which concludes the proof. \(\square\)

As proposed, we show that the separability of the orbit indicates almost periodicity. Note that if \(\{T^\circ(t)\}_{t \in \mathbb{R}^+}\) is a sun-dual semigroup, and \(x^\circ \in X_{uds}^\circ\), then the mapping \(\{t \mapsto T^\circ(t)x^\circ\}\) is almost periodic. We give a criterion when an element in \(X_\alpha^\circ\) is in \(X_{uds}^\circ\).

**Theorem 4.8.** If \(x^\circ \in X_\alpha^\circ\) and \(\{T^\circ(t)\}_{t \in \mathbb{R}^+}\) is Abelian, then the following are equivalent.

1. \(\overline{O(x^\circ)}^{(X^\circ, X)}\) is norm separable.
2. \(x^\circ \in X_{uds}^\circ\).
3. \(O(x^\circ)\) is relatively norm compact.

**Corollary 4.9.** If \(\overline{O(x^\circ)}^{(X^\circ, X)}\) is norm separable for all \(x^\circ \in X_\alpha^\circ\) and \(T\) is Abelian, then \(X_\alpha^* = X_{uds}^*\).
Corollary 4.10. If $X^*$ is norm separable and $\mathcal{T}$ is Abelian, then

$$X^*_a = X^*_{uds}.$$ 

5. Ideal Theory

Definition 5.1. A subset $J$ of a semigroup $S$ is called a right (left) ideal of $S$ if

$$JS := \{xs : x \in J, s \in S\} \subset J \quad (SJ := \{sx : x \in J, s \in S\} \subset J)$$

As subset is called an ideal if it is both a right and a left ideal.

Theorem 5.2 (\[7\]). Every compact right [left] topological semigroup has an idempotent.

Definition 5.3 (\[31, \text{p. 12}\]). The set of idempotents in a semigroup $S$ is denoted $E(S)$. We define relations $\leq_L$ and $\leq_R$ on $E(S)$ by

$$e \leq_L f \quad \text{if} \quad ef = e,$$

$$e \leq_R f \quad \text{if} \quad fe = e.$$ 

If $e$ and $f$ commute, then we omit the indices $L$ and $R$.

Definition 5.4. Let $(A, \leq)$ be a set with a transitive relation. Then, an element $a$ is called $\leq$−maximal $[\leq$−minimal] in $A$ if for every $a' \in A, a \leq a'$ implies $a' \leq a [a' \leq a$ implies $a \leq a'].$

Recalling \[31, \text{p. 14}\], we have the following.

Theorem 5.5. Every compact right topological semigroup contains $\leq_L$−maximal and $\leq_R$−minimal idempotents.

Theorem 5.6 (\[31, \text{p. 21}\]). For an idempotent $e$ in a compact right topological semigroup $S$, the following statements are equivalent:

1. $e$ is $\leq_R$−minimal in $E(S)$.
2. $e$ is $\leq_L$−minimal in $E(S)$.
3. $eS$ is a minimal right ideal of $S$.
4. $e$ is a group, and $e$ is an identity in $eS$.
5. $Se$ is a minimal left ideal of $S$.
6. $SeS$ is the minimal ideal of $S$.
7. $S$ has a minimal ideal $M(S)$ and $e \in M(S)$.

Proposition 5.7. Let $S$ be abelian and $f : S \to \mathcal{C}$−wap and $\{t_\lambda\}_{\lambda \in \Lambda}, \{s_\gamma\}_{\gamma \in \Gamma} \subset S$. Then, we may pass to subnets $\{s_\gamma\}_{\alpha \in A}$ and $\{t_\lambda\}_{\beta \in B}$ such that the iterated limits

$$\nu = \lim_{\alpha \in A} \lim_{\beta \in B} f(t_\lambda + s_\gamma)$$

and

$$\mu = \lim_{\beta \in B} \lim_{\alpha \in A} f(t_\lambda + s_\gamma)$$

exist, and we have $\nu = \mu$.

Proof. Because $f$ is $\mathcal{C}$−wap, $\{f_\lambda\}_{\lambda \in \Lambda}$ is rel. $\sigma(C_b(S), C_b(S)^*)$−compact, $\{\delta_{s_\gamma}\}_{\gamma \in \Gamma}$ is relatively $w^*$ compact, and we may pass to convergent subnets. Using $f(t_\lambda + s_\gamma) = \delta_{s_\gamma} f_\lambda$, we find that the iterated limits exist and that they are equal. \qed

6. Several Lemmas

Next, we recall the consequence of \[14, \text{Cor. 2 (a), p. 127}\], and we have the following.

Proposition 6.1. Let $E \subset F \subset X^*$ and $\tau$ be a locally convex topology on $X^*$, with $\sigma(X^*, X) \subset \tau \subset \|\cdot\|$. If no vector of $E$ can be separated from $F$ by a $\tau$−continuous functional, then $E^* = F^*$. 

Proof. If \( F' \neq F'^{\circ} \), then there exists an \( x \in F' \setminus F'^{\circ} \) and a \( \tau \)–continuous functional \( \phi \) such that \( \phi|_{F'^{\circ}} = 0 \) and \( \phi(x) = 1 \). By definition, we have for net \( \{ x_\lambda \}_{\lambda \in \Lambda} \subset F \) the \( \tau \) convergence \( x_\lambda \to x \). Moreover, we find a subnet that has no intersection with \( F'^{\circ} \). The continuity \( \phi \) leads to an element \( x_{\lambda_0} \), with \( \phi(x_{\lambda_0}) > 1/2 \), which illustrates the contradiction. \( \square \)

We start with the main lemma, which is applied in several ongoing circumstances.

**Lemma 6.2.** Let \( K \subset X^* \) a \( w^* \)–compact and norm-separable set. Then \( (K, w^*) \) is compact metrizable, and the metric is given by

\[
d(x, y) = \sum_{n=1}^{\infty} \frac{|\langle z_n, x - y \rangle|}{1 + |\langle z_n, x - y \rangle|},
\]

for a sequence \( \{z_n\}_{n \in \mathbb{N}} \subset X \).

**Proof.** By the assumption there is a sequence \( \{y_n^*\}_{n \in \mathbb{N}} \) such that

\[
K \subset \overline{\text{span}} \{y_n^*\}_{n \in \mathbb{N}} ||x||.
\]

If \( Z = \text{span} \{y_n^*\} = \{x_n^*\}_{n \in \mathbb{N}} \subset X^* \), then \( B_Z \) is compact metrizable, therefore separable, \( K \subset Z^{w^*} \).

Hence let \( \{z_n^*\}_{n \in \mathbb{N}} \) \( w^* \)–dense in \( B_Z \). Let \( x_n^* \in X^{**} \) the sequence of extensions of the \( z_n^* : Z \to \mathbb{C} \). Note that for the natural embedding \( j_X : X \to X^{**} \), we have \( j_X B_X^{w(X^{**}, X^*)} = B_X^{w**} \), compare [6] p. 424. We define the sequence of open sets,

\[
U_{k, l} := \left\{ x^{**} \in X^{**} : |x^{**}(x_m)| < \frac{1}{k}, 1 \leq m \leq l \right\}.
\]

Then \( U_{k, l} \) is \( w^* \)–open zero-neighbourhood. Consequently, for all \( k, l \in \mathbb{N} \), we find \( x_n^{k, l} \in B_X \), such that \( j_X x_n^{k, l} - x_n^* \in U_{k, l} \). Letting \( H = \text{span} \left\{ x_n^{k, l} : k, l, n \in \mathbb{N} \right\} \), we claim \( (Z, \sigma(Z, H)) \) is Hausdorff. If \( z \in Z \), and \( z(x_m^{k, l}) = 0 \), for all \( m, k, l \in \mathbb{N} \). By definition, for all \( \varepsilon > 0 \) we find an \( x_n^* \), such that \( \|x_n^* - z\| \leq \varepsilon \).

\[
|z_m^*(x_n^*)| = |x_m^{**}(x_n^*)| \leq |(x_m^{**} - k_l) (x_n^*)| + |k_l x_m^{**}(x_n^*)| \\
\leq \frac{1}{k} + \left| x_m^{k_l}(x_n^*) \right|,
\]

for all \( 0 \leq l \leq n \). Further,

\[
\left| x_n^{k_l}(x_m^*) \right| \leq \left| x_m^{k_l}(x_n^* - z) \right| + 0 \\
\leq \|x_n^* - z\| \leq \varepsilon.
\]

Hence \( x_n^* = 0 \), therefore \( z = 0 \). With \( \left\{ |\langle x_n^{k_l}, \cdot \rangle| : k, l \in \mathbb{N} \right\} \) we found a countable set of seminorms, which induce Hausdorff metric on \( K \), which is weaker than the \( w^* \)–topology. \( \square \)

Next we recall a lemma given [23] Lemma 4. Due to its central role, and the given rudimentary arguments in the original study, one more detailed is given below.

**Lemma 6.3** ([23]). Let \( K \) compact Hausdorff and \( M \subset C(K) \) closed bounded and separable with respect to the pointwise topology. Then \( M \) is norm separable in \( C(K) \).
Proof. We start defining an equivalence on $K$.
\[ t \sim s :\Leftrightarrow \sup_{g \in M} \|g(t) - g(s)\| = 0. \]
The rules for the relations simply are given, and let $\bar{K} := K/\sim$, the quotient with the quotient topology, and $q : K \to \bar{K}$ the quotient map. Considering for $M \subset B_{C(K)}(0, L)$, and $D \subset M$ the countable dense set, define
\[ U : K \to \prod_{g \in D} [-L, L] \]
\[ t \mapsto g(t), \]
then the product is compact metric, and $\varphi(K)$ is compact subset. Moreover, $U = \bar{U} \circ q$, with
\[ \bar{U} : \bar{K} \to \prod_{g \in D} [-L, L] \]
\[ t \mapsto g(t). \]
Similar, we have for $f \in M$, that $f = \hat{f} \circ q$. Due to the continuity of $f$, we have $\hat{f}$ is continuous [12, Thm. 9, p. 95]. The continuity of $q$, and the compactness of $K$ gives $\bar{K}$ compact. Its simply to verify that $\bar{U}$ is injective. Hence $\bar{K}$ and $U(K)$ are homeomorphic, and
\[ \hat{f} := \hat{f} \circ \bar{U}^{-1}, \text{ or } f(t) = \hat{f} \circ U(t). \]
The first identity proves that $\hat{f}$ is continuous. Additionally, $\|g(t)\| = \|\hat{g}(U(t))\|$ the pointwise denseness gives $\|f(t)\| = \|\hat{f}(U(t))\|$. Therefore $\|f\|_\infty = \|\hat{f}\|_\infty$. Consequently,
\[ V : M \to C(U(K)) \]
\[ f \mapsto \hat{f} \]
is an isometry, and the norm-separability of $C(U(K))$ implies $T(M)$ is norm separable, and therefore $M$. \qed

With the aim to obtain the translation operator on $BUC(S; X)$ as a dual operator, the following proposition points on a sufficient condition.

Proposition 6.4. Let $S$ a Borel measure space, and the measure $\mu$ such that for all $f \in BUC(S)$,
\[ \sup_{g \in B_{L^1(S)}} \int_S |fg| \, d\mu = \sup_{t \in S} |f(t)|. \]
Then, using $L^1(S, X^*) = (L^1(S) \otimes_\pi X)^* = L(L^1(S), X^*)$, for
\[ t : BUC(S, X^*) \to L(L^1(S), X^*) \]
\[ f \mapsto \left\{ g \mapsto \left( x \mapsto \int_S <g(r) \otimes x, f(r) > \, d\mu(r) \right) \right\} \]
we have $\|\nu(f)\|_\infty = \|f\|_\infty$. The duality for $(L^1(S, X), BUC(S, X^*))$ is given by,
\[ <g, f> := \int_S <g(r), f(r) > \, d\mu(r). \]
If in addition $S$ is a subsemigroup of a group, for $t \in S$ the translation operator
\[ T(t) : BUC(S, X^*) \to BUC(S, X^*) \]
\[ f \mapsto \{ s \mapsto f(t + s) \}, \]
Lemma 7.5. \( Y \) is norm-separable in \( \tau \) together with the topology \( \tau \).

Proof. Due to the \( \tau \), the orbit \( \{t \mapsto x(t)\}_{t \in \mathbb{R}^+} \) is compact metrizable.

Corollary 7.1. Let \( \{T(t)\}_{t \in \mathbb{R}^+} \) a bounded \( C_0 \)-semigroup, \( x^\circ \in X^\circ \), and \( \{T^\circ(t)x^\circ\}_{t \in \mathbb{R}^+} \) norm-separable in \( X^\circ \). Then \( \{T^\circ(t)x^\circ\}_{t \in \mathbb{R}^+} \) is compact metrizable.

Proof. Apply Lemma 6.2 to \( \{T^\circ(t)x^\circ\}_{t \in \mathbb{R}^+} \subset X^\circ \) compact and separable.

Corollary 7.2. Let \( \{T(t)\}_{t \in \mathbb{R}^+} \) a bounded \( C_0 \)-semigroup, \( Y \subset X^\circ \), \( y \in Y \), and \( \{T^\circ(t)y\}_{t \in \mathbb{R}^+} \) together with the topology \( \tau \) a dual semigroup representation. If \( \{T^\circ(t)y\}_{t \in \mathbb{R}^+} \) \( \tau \)-compact and norm-separable in \( Y \). Then \( \{T^\circ(t)y\}_{t \in \mathbb{R}^+} \) is compact metrizable.

Proof. Due to the \( \tau \)-compactness of \( \{T^\circ(t)y\}_{t \in \mathbb{R}^+} \), recalling the definition of the dual semigroup representation, the \( w^* \)-topology coincides with the stronger topology \( \tau \).

Next we apply the previous result to obtain a general metrizability criteria for bounded \( C_0 \)-semigroups on a Banach space \( X \). Let \( j_X : X \to X^\circ \) the canonical embedding.

Corollary 7.3. Let \( \{T(t)\}_{t \in \mathbb{R}^+} \) a bounded \( C_0 \)-semigroup, \( x \in X \), and \( \{T^\circ(t)x\}_{t \in \mathbb{R}^+} \) separable in \( X^\circ \). Then \( \{T^\circ(t)x\}_{t \in \mathbb{R}^+} \) is metrizable.

Proof. Use that \( \{T^\circ(t)x\}_{t \in \mathbb{R}^+} \subset \{T^\circ(t)x\}_{t \in \mathbb{R}^+} \) and apply Lemma 7.1.

In the view of asymptotics we define the following classes of vectors.

Definition 7.4. Let \( M \subset X^* \) a closed and separating space.

1. A vector \( x \in X \) is called \( M \)-weakly almost periodic, if for every \( x \in M \), there is a \( g \in AP(\mathbb{R}) \), such that \( \{t \mapsto x^*T(t)x\} = g_{\mathbb{R}^+}(t) \). A \( C_0 \)-semigroup \( \{T(t)\}_{t \in \mathbb{R}} \) is called \( M \)-weakly almost periodic on \( X \), if every vector in \( X \) is an \( M \)-weakly almost periodic vector.

2. A vector \( x \in X \) is called \( M \)-weakly Ewap (M-w-Ewap), if for every \( x^* \in M \), the mapping \( \{t \mapsto x^*T(t)x\} \in W(\mathbb{R}^+) \). A \( C_0 \)-semigroup \( \{T(t)\}_{t \in \mathbb{R}} \) is called \( M \)-weakly Ewap (M-w-Ewap) on \( X \), if every vector in \( x \) is M-w-Ewap.

Lemma 7.5. If \( \{T(t)\}_{t \geq 0} \) a bounded \( C_0 \)-semigroup and \( x \in X \) a \( X^* \)-w-Ewap vector, then \( \{T(t)x\}_{t \geq 0} \) is metrizable.

Proof. Apply, that for \( K \) weakly compact \( j_XK \subset j_XX \), and Cor. 7.3.
Recalling Lybich, Yu.I. and Kadets [16] Thm.2 they ask for reasons, which condition Banach spaces have to fulfill, that a weakly almost periodic semigroup \( t \mapsto x^*(T(t)x) \in AP(\mathbb{R}) \), for all \( x^* \in X^* \) implies, that it is a almost periodic.

**Theorem 7.6.** [16, Thm. 2] Let a Banach space \( X \) have the following property: the weak* sequential closure of each of its separable subspaces \( Y \) in the second conjugate space \( Y^{**} \) is separable. Then each \( X^* \)-weakly almost periodic group acting on \( X \) is almost periodic.

**Proof.** Considering the dual-semigroup \( \{T^\ominus(t)\}_{t \geq 0} \) with \( jT(t)x = T^\ominus(t)jx \), which is an extension to a dual semigroup. By Prop. 3.13 \( jx \in X_a^\ominus \), and we are in the situation of Theorem 3.12. It leaves to verify that the \( w^* \)-closure of the orbit is sequentially separable in \( X^\ominus \). Lemma 7.1 implies the closure and the sequentially closure of the orbit coincide. The prove concludes using the assumption \( O(x)^{w^*} = O(x)^{seq-w^*} \subset \overline{JZ}^{seq-w^*} \), is separable when \( Z \) is separable. Choose \( Z = \text{span} \{ T(\mathbb{R})x \} \).

**Corollary 7.7.** Let \( X \) a Banach space, and \( \{T(t)\}_{t \in \mathbb{R}^+} \) a bounded \( C_0 \)-semigroup, If \( X^\ominus \) is separable, then for every \( X \)-weakly almost periodic vector \( x^\gamma \), the function \( \{t \mapsto T^\ominus(t)x^\gamma \} = g|_{\mathbb{R}^+}(t) \), with \( g \in AP(\mathbb{R}, X^\gamma) \).

### 7.2. Vector-valued functions and their Orbits.

We consider the space of bounded uniformly continuous functions, which is the space for the translation semigroup, or group repsectively. A special type functions are for bounded \( C_0 \)-semigroups, \( f(\cdot) := S(\cdot)x \). This view moves behavior from the translation semigroup to general \( C_0 \)-semigroups.

We start verifying that the translation semigroup on the bounded uniformly continuous functions is in a sense a restriction semigroup, a notion introduced a previous section. In doing this, topologies come into play. We start with the \( w^* \)-compact open topology.

**Lemma 7.8.** On bounded sets \( A \subset BUC(\mathbb{R}, X^*) \) the \( w^* \)-compact-open topology is stronger than \( \sigma(BUC(\mathbb{R}, X^*), L^1(\mathbb{R}, X)) \).

**Proof.** Let \( \tau_{w^*-co}^A \), the bounded topology due to weak* compact-open convergence, coming with \( A_n := 2^n B_{BUC(\mathbb{R}, X^*)} A := \{ A_n \}_{n \in \mathbb{N}} \). Further, let \( \{f_\gamma\}_{\gamma \in \Gamma} \subset BUC(\mathbb{J}, X^*) \) a bounded and \( \tau_{w^*}-co \) convergent net with the limit \( f \). Further, \( g \in L^1(\mathbb{J}, X) \), and \( \varepsilon > 0 \), we find \( \{ K_i \}_{i=1}^n \subset \mathcal{P}(\mathbb{R}) \), and \( \{x_i^\gamma \}_{i=1}^n \subset X \), such that for \( \varphi = xK_i \), we have, \( \| g - \sum_{i=1}^n \varphi_i x_i \|_1 < \varepsilon \). Define \( \varphi := \sum_{i=1}^n \varphi_i x_i \).

Then, \( \{ f_\gamma - f \}_{\gamma \in \Gamma} \) is bounded and

\[
|< f_\gamma - f, g > | \leq |< f_\gamma - f, \varphi > | + |< f_\gamma - f, g - \varphi > |.
\]

Hence, it leaves to prove the convergence on \( L^1(\mathbb{J}) \otimes_x X \) But for \( x \in X \),

\[
T_x : (BUC(\mathbb{J}, X^*), \tau_{w^*}-co^A) \rightarrow (BUC(\mathbb{J}), \tau_{co})
\]

\[
f \mapsto < x, f > .
\]

is continuous,

\[
i_T : (BUC(\mathbb{J}), \tau_{co}) \rightarrow (BUC(\mathbb{J}), \sigma(BUC(\mathbb{J}), L^1(\mathbb{J})))
\]

\[
f \mapsto f.
\]

Hence, the mapping

\[
i_A \circ T_x : (BUC(\mathbb{J}, X^*), \tau_{w^*}-co^A) \rightarrow (BUC(\mathbb{J}), \sigma(BUC(\mathbb{J}), L^1(\mathbb{J})))
\]

and therefore,

\[
id : (BUC(\mathbb{J}, X^*), \tau_{w^*}-co^A) \rightarrow (BUC(\mathbb{J}, X^*), \sigma(BUC(\mathbb{J}, X^*), L^1(\mathbb{J}, X))) .
\]
After this comparison we are ready to verify the first compactness result, which steps into the provided theory.

**Lemma 7.9.** Let $X$ a Banach space and $f \in BUC(\mathbb{R}^+, X^*)$, then $$\overline{ac}^{\sigma(BUC(\mathbb{R}, X^*), L^1(\mathbb{R}, X))} \{f_t : t \in \mathbb{R}^+\} \subset BUC(\mathbb{R}^+, X^*),$$ is compact, and the translations are dual representation on $BUC(\mathbb{R}, X)$. Therefore independent of the a.p. we have (a not necessarily unique splitting) $$BUC(\mathbb{R}^+, X^*) = BUC(\mathbb{R}^+, X^*)_a \oplus BUC(\mathbb{R}^+, X^*)_1,$$ and the splitting is nontrivial because $$AP(\mathbb{R}, X^*) \subset BUC(\mathbb{R}, X^*)_a, \text{ and } W_0(\mathbb{R}, X^*) \subset BUC(\mathbb{R}, X^*)_1.$$  

*Proof.* Let $\{t_\gamma\}_{\gamma \in \Gamma, i \in \mathbb{N}} \subset \mathbb{R}$, $\{\lambda_i^\gamma\}_{\gamma \in \Gamma, i \in \mathbb{N}} \subset \mathbb{C}$, and $\{n_\gamma\}_{\gamma \in \Gamma} \subset \mathbb{N}$, with $\sum_{i=1}^{n_\gamma} |\lambda_i^\gamma| = 1$. Using $$\left\{\left\{\sum_{i=1}^{n_\gamma} \lambda_i^\gamma f(t + t_\gamma^i)\right\}_{t \in \mathbb{R}}\right\}_{\gamma \in \Gamma} \subset \prod_{t \in \mathbb{R}} \overline{ac}^{\sigma^*} \{f(t) : t \in \mathbb{R}\}, w^*$$ by Tychonov’s theorem we find a subnet converging pointwise $w^*$ to a bounded function element $\tilde{g}$, and the lower semi continuity of the the $w^*$-topology of $X^*$ gives $\tilde{g}$ is uniformly continuous with the modulus of $f$. Hence by the equicontinuity this net is $\tau_{w^* - co}$ convergent. The previous lemma [7.8] gives, $$\sigma(BUC(\mathbb{R}, X^*), L^1(\mathbb{R}, X)) - \lim_{\gamma \in \Gamma} \sum_{i=1}^{n_\gamma} \lambda_i^\gamma f(t + t_\gamma^i) = \tilde{g} \in acO(f)^{w^*}.$$ Consequently, $\tilde{g} \in L^1(\mathbb{R}, X)^{\circ} \cap BUC(\mathbb{R}, X^*)$, and Prop. 3.6 concludes the proof.

7.3. The setting for vector-valued functions.

**Lemma 7.10.** If $X$ is separable, then $(BUC(\mathbb{R}, X), \sigma(BUC(\mathbb{R}, X), L^1(\mathbb{R}, X^*)), \text{then every } \sigma(BUC(\mathbb{R}, X), L^1(\mathbb{R}, X^*))$-compact subset is metric.

*Proof.* Due to the compact metrizability of $B_X^*$, let $\{x_m^*\}_{m \in \mathbb{N}}$ a dense sequence. Additionally, chose $\{v_n\}_{n \in \mathbb{N}}$ dense in $L^1(\mathbb{R})$. If $f \in BUC(\mathbb{R}, X)$ such that $\int x_m^*(f(t))v_n(t)dt = 0$, for all $n.m \in \mathbb{N}$. Then by the denseness of $\{v_n\}_{n \in \mathbb{N}}$, we have $x_m(f(t)) = 0$, for $m \in \mathbb{N}$, and therefore $f = 0$. Let $\tau_{pd}$ the induced topology by this countable set of elements, every compact subset becomes metric with respect to this coarser topology.  

**Proposition 7.11.** Let $\{T(t)\}_{t \geq 0}$ the translation semigroup on $L^1(\mathbb{R}, X^*)$, and $L^1(\mathbb{R}, X)$ respectively. Then  

$$i_1 : BUC(\mathbb{R}, X) \longrightarrow L^1(\mathbb{R}, X^*)^\circ \quad f \quad \longmapsto \quad \left\{ g \mapsto \int_{\mathbb{R}} < g(\tau), f(\tau) > d\tau \right\}$$

and

$$i_2 : BUC(\mathbb{R}, X^*) \longrightarrow L^1(\mathbb{R}, X)^\circ \quad f \quad \longmapsto \quad \left\{ g \mapsto \int_{\mathbb{R}} < g(\tau), f(\tau) > d\tau \right\}$$
are isometries. \( \|f\|_\infty = \sup_{g \in B_{L^1(\mathbb{R}, X^*)}} |\int_{\mathbb{R}} g(\tau), f(\tau) > d\tau| \) Consequently, every uniformly closed subspace of \( BUC(\mathbb{R}, X) \) is a closed subspace of \( (L^1(\mathbb{R}, X^*), \|\|) \), and \( BUC(\mathbb{R}, X^*) \subset L^1(\mathbb{R}, X^*) \) respectively.

**Proof.** Let \( \{t_m\}_{m \in \mathbb{N}} \subset \mathbb{R} \), such that \( \lim_{m \to \infty} \|f(t_m)\| = \|f\|_\infty \). By the Hahn-Banach theorem we find \( \{x_m^*\}_{m \in \mathbb{N}} \subset B_{X^*} \) with \( x_m^*(f(t_m)) = \|f(t_m)\| \). (or choose \( \varepsilon_1 \) such that \( x_m(f(t_m)) = \|f(t_m)\| - \varepsilon_1 \). Let \( \varphi \in C_0^\infty(\mathbb{R}) \) with \( \int_{\mathbb{R}} \varphi(t)dt = 1 \). Then define \( \varphi_\varepsilon(t) = \frac{1}{\varepsilon} \varphi(\frac{t}{\varepsilon}) \). If \( g_{\varepsilon,m}(s) := \varphi_\varepsilon(s - t_m)x_m^* \), then all \( g_{\varepsilon,m} \in L^1(\mathbb{R}, X^*) \) have norm one, and the claim is proved using regularization arguments for the bounded and uniformly continuous functions having the same modulus of continuity \( \{t \to x_m^*(f(t))\} \). Hence \( (\int_{\mathbb{R}} g_{\varepsilon,m} f d\mu \to \|f(t_m)\|) \). \( (\int_{\mathbb{R}} g_{\varepsilon,m} f d\mu \to \|f(t_m) - \varepsilon_1\|) \). The second claim uses surely the same methods using that for we find \( \{x_m\}_{m \in \mathbb{N}} \subset B_{X^*} \), such that \( \|f(t_m)\| - 1/m \geq f(t_m)(x_m) \).

**Remark 7.12.** Considering the translation semigroup, and using the topology on \( BUC(\mathbb{R}, X) \) coming with \( L^1(\mathbb{R}, X^*) \), in view of Prop. \[\text{[7.1]}\] it becomes straightforward to consider the embedding

\[ t : BUC(\mathbb{R}, X) \to BUC(\mathbb{R}, X^*) \]

\[ j : BUC(\mathbb{R}, X^*) \to C((B_{X^*}, w^*) \times T) \]

This lead to a dual semigroup representation on \( BUC(\mathbb{R}, X^*) \) coming with a restriction of the sun-dual of the translation semigroup defined on \( L^1(\mathbb{R}, X^*) \).

**8. Separability of Orbits**

Next we present the main Lemma providing norm separability of \( w^* \)-closure of the orbit.

**Lemma 8.1.** Let \( f \in BUC(\mathbb{R}, X^*) \), \( \{T(t)\}_{t \in \mathbb{R}} \) the translation group, \( f(\mathbb{R})^{w^*} \) norm-separable in \( X^* \), and for some \( s \in \mathbb{R} \),

\[ F : ((B_{X^*}, w^*) \times (T, w^*)) \to C \]

\[ (x^*, S) \to (x^*, (Sf)(s)) \]

is continuous. Then \( \bar{O}(f)^{w^*} \) is separable in \( BUC(\mathbb{R}, X^*) \).

**Proof.** By Lemma \[\text{[7.9]}\] the semigroup of translation on \( BUC(\mathbb{R}, X^*) \) is a dual representation, therefore let \( T \) the \( w^* \)-closure of the translations in \( L(BUC(\mathbb{R}, X)) \). For \( f \in BUC(\mathbb{R}, X^*) \)

\[ j : \bar{O}(f)^{w^*} \to C((B_{X^*}, w^*) \times T) \]

\[ f \to \{(x^*, S) \to (x^*, (Sf)(s))\} \]

fulfills \( \|j f\|_\infty = \|f\|_\infty \). Let by assumption \( \bar{O}(x_n)_{n \in \mathbb{N}}^{\cdot} = \bar{f(\mathbb{R})}^{w^*} \), \( x^* \in B_{X^*} \), and \( S \in T \). Then for \( \varepsilon > 0 \)

\( |(x^*, (Sf)(s)) - x_n| \leq \|x^*\| \cdot \|(Sf)(s) - x_n\| \leq \varepsilon \) for a suitable \( x_n \). Hence, \( j(\bar{O}(f))^{w^*} \) is pointwise separable, and application of Lemma \[\text{[5.3]}\] gives,

\[ \bar{O}(f)^{w^*} \subset C((B_{X^*}, w^*) \times T), \|\cdot\|_\infty \) is norm separable.

That is, there is \( \{f_n\}_{n \in \mathbb{N}} \subset \bar{O}(f)^{w^*} \), such that for every \( g \in \bar{O}(f)^{w^*} \),

\[ \sup_{x^* \in B_{X^*}, S \in T} \|(Sf_i)(s) - (Sg)(s)\| \leq \varepsilon, \]

for some \( i \in \mathbb{N} \). Hence

\[ \|f - g\|_\infty = \sup_{t \in \mathbb{R}} \|T(t)f_i(s) - T(t)g(s)\| \leq \sup_{S \in T^{w^*}} \|(Sf_i)(s) - (Sg)(s)\| \leq \varepsilon, \]
Proposition 8.2. Let for $J \in \{\mathbb{R}, \mathbb{R}^+\}$, and a closed subspace $M$ with $X \subset M \subset X^{**}$,

$$WAP_M(\mathbb{R}, X^*) := \{f \in BUC(\mathbb{R}, X^*) : \{t \rightarrow < x^{**}, f(t) > \} \in AP(\mathbb{R}) \text{ for all } x^{**} \in M\},$$

$$WW_M(J, X^*) := \{f \in BUC(J, X^*) : \{t \rightarrow < x^{**}, f(t) > \} \in W(J) \text{ for all } x^{**} \in M\}.$$

Define by the seminorms the topology,

$$\tau_{M-\infty} := \left\{ p_{x_1^{**}, \ldots, x_n^{**}}(f) := \sup_{t \in \mathbb{R}, 1 \leq i \leq n} |< x_i^{**}, f(t) >| : \{x_i^{**}\}_{i=1}^n \subset M \right\}.$$

Then the translation semigroup on $WW_M(J, X^*)$ is a dual semigroup representation, with representation compatible topology $\tau_{M-\infty}$, and $WW(J, X^*)_a = WAP(\mathbb{R}, X^*)_J$.

Proof. Let $\{t_i^\gamma\}_{\gamma \in \Gamma, i \in \mathbb{N}} \subset \mathbb{R}$, $\{\lambda_i^\gamma\}_{\gamma \in \Gamma, i \in \mathbb{N}} \subset \mathbb{C}$, and $\{n_i^\gamma\}_{\gamma \in \Gamma} \subset \mathbb{N}$, with $\sum_{i=1}^{n^\gamma} |\lambda_i^\gamma| = 1$. Using the weak topology on $W(J)$, and that for weakly compact sets the absolute convex hull is weakly compact an application of Tychonov's theorem gives

$$\left\{ \left\{ \sum_{i=1}^{n^\gamma} \lambda_i^\gamma f_{t+i^\gamma} \right\} \right\} \subset \Pi_{x \in M} (aw^w \{< x, f_t > : t \in \mathbb{R}\}, w)$$

which lead to $\tau_{M-\infty}$ convergent subnet. Applying Lemma 7.8 the compatibility of the topologies on bounded sets

$$\sigma(WW(J, X^*), L^1(\mathbb{R}, X)) \subset \tau_{w^{*-\infty}} \subset \tau_{X-\infty} \subset \tau_{M-\infty},$$

concludes the proof.

Proposition 8.3. Let $X$ a Banach space, and $\{T(t)\}_{t \in \mathbb{R}}$ the translation group. Then

$$WAP_X(\mathbb{R}, X^*) \subset BUC(\mathbb{R}, X^*)_a.$$

Proof. From the vector valued splitting we learn, $g(t) = < x, f(t) > = < x, f_a(t) > + < x, f_0(t) >$, with $< x, f_0 >$ a flight vector, and $< x, f_a >$ reversible with respect to the scalar translation semigroup. Using that almost periodic functions are reversible, we find $< x, f > - < x, f_a > = < x, f_0 >$, for all $x \in X$, therefore $f = f_a$, by Prop. 3.5.

Theorem 8.4. Let $f \in WW_X^*(\mathbb{R}, X^{**})$, $\bar{f}(\mathbb{R})^{w^*}$ is norm-separable, and the map and for some $s \in \mathbb{R}$,

$$F : (B_X^{**}) \times (T, w^*) \longrightarrow \mathbb{C}$$

$$(x^*, S) \longmapsto < x^*, (Sf)(s) >$$

is continuous. Then $f \in AP(\mathbb{R}, X^{**})$.

Proof. By Prop. 8.2 we are in the situation of Thm. 4.7. And Lemma 8.1 verifies the separability condition.

The next corollary seems to be in contradiction with the example given in [21] 4. p. 75, but the constructed counterexample fails to be uniformly continuous. It seems they construct on an interval $I$ a sequence of scalar valued norm one functions with disjoint supports, and extend them to all of the reals. Then these function have the common period, which is exactly the length of the interval. Then they define,

$$f : \mathbb{R} \longrightarrow l^2(\mathbb{N})$$

$$t \longmapsto \{\phi_k(t)\}_{k \in \mathbb{N}}.$$
The function is weakly almost periodic in their sense, applying [I, XII, p.51], but it is not an element of $WW^{l^2(N)}(\mathbb{R}, l^2(N))$. Let $\phi_k(t_k) = 1$. Choose $s_k$ in the boundary of $\{t : \phi_k(t) \neq 0\}$, with $[s_k, t_k] \subset \text{supp} \{\phi_k\}$. By the boundedness of $I$, we have $t_k - s_k \to 0$, but
\[
\|f(t_k) - f(s_k)\|_2 = |\phi_k(t_k) - \phi_k(s_k)| = |1 - 0| = 1.
\]
Hence $f$ fails to be uniformly continuous, and therefore, as they claim, it fails to be almost periodic.

**Corollary 8.5.** Let $f \in WW_X^\ast((\mathbb{R}, X)_a$ $f(\mathbb{R})$ is weak relatively compact, and for some $s \in \mathbb{R}$,

\[
F : ((B_X^\ast, w^\ast) \times (T, w^\ast)) \longrightarrow \mathbb{C}
\]

\[
(x^\ast, S) \longmapsto < x^\ast, (Sf)(s) >
\]

is continuous. Then $f \in AP(\mathbb{R}, X)$.

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