Transfer Matrix Formalism for Two-Dimensional Quantum Gravity and Fractal Structures of Space-time

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ABSTRACT

We develop a transfer matrix formalism for two-dimensional pure gravity. By taking the continuum limit, we obtain a “Hamiltonian formalism” in which the geodesic distance plays the role of time. Applying this formalism, we obtain a universal function which describes the fractal structures of two dimensional quantum gravity in the continuum limit.
Recent developments of two-dimensional gravity have provided us with an unambiguous definition of quantum gravity. This is based on the equivalence of the continuum formulation \[1\] and the dynamical triangulation.\[2\] In two dimensions, we thus have a regularized quantum gravity which has a definite continuum limit. The remarkable success of the matrix models \[3\] further elucidated topological aspects of the theory. However we still lack a general formulation for describing quantum fluctuations of space-time and for evaluating physical observables such as fractal dimensions. In this paper we propose a new formulation which is a kind of Hamiltonian formalism for quantum gravity. We show that a geodesic distance defined on a dynamically triangulated surface can be regarded as the “time” variable for defining the transfer matrix. We then obtain a “Hamiltonian” in the continuum limit, and analyze the fractal structures of the space-time.

Let us consider a cylinder with an entrance loop($c$) and an exit loop($c'$). (See Fig. 1.) We introduce the following quantity which is formally defined in the continuum framework:

\[
N(L, L'; D; A) = \int \frac{Dg}{Vol(Diff)} \delta(\int d^2 x \sqrt{g} - A) \delta(\int_c \sqrt{g_{\mu\nu} dx^\mu dx^\nu} - L) \delta(\int_{c'} \sqrt{g_{\mu\nu} dx'^\mu dx'^\nu} - L') \prod_{P \in c'} \delta(d(P, c) - D).
\]

(1)

Here in the path integral the total area of the surface and the lengths of $c$ and $c'$ are constrained to $A, L$ and $L'$, respectively. We further impose a constraint that any point on $c'$ has a geodesic distance $D$ from the entrance loop $c$. To be precise, the geodesic distance $d(P, c)$ is defined as the minimum distance between the point $P$ and a point on the loop $c$. Note that the exit loop $c'$ is one of the loops which are composed of points having geodesic distance $D$ from the entrance loop $c$. For later convenience we introduce a making point on the exit loop as is shown in Fig.1.
An important property of $N(L, L'; D; A)$ is the following composition law:

$$N(L, L'; D; A) = \int_0^\infty dA' \int_0^\infty dL'' N(L, L''; D'; A') N(L'', L'; D - D'; A - A').$$

(2)

In other words, if we define $N(L, L'; D)$ by

$$N(L, L'; D) = \int_0^\infty dAN(L, L'; D) e^{-tA},$$

where $t$ is the cosmological constant, it can be regarded as the matrix element $< L | e^{-D\hat{H}} | L' >$ for a “Hamiltonian” $\hat{H}$. In this sense we call $N(L, L'; D)$ the proper time evolution kernel. Roughly speaking, eq.(2) asserts that a cylinder with height $D$ can be decomposed into two cylinders with height $D'$ and $D - D'$. More strictly, there could be several loops which have geodesic distance $D'$ from the entrance loop $c$. From only one of them, however, the exit loop $c'$ has geodesic distance $D - D'$. (See Fig.2)

We now give a constructive definition of $N(L, L'; D; A)$ in terms of dynamical triangulation. The strategy to define its lattice counterpart $\tilde{N}(l, l'; d; n)$ is the following. We first introduce the notion of deforming a loop one step forward, whose precise definition is given in the next paragraph. As we will see there, a loop can split into several loops after a one-step deformation. In that case each loop is independently deformed in the next step. By repeating this procedure $d$ times, we have a $d$-step deformation of a loop. Then $\tilde{N}(l, l'; d; n)$ is defined as the number of possible triangulations of a cylinder satisfying the following conditions which correspond to the four $\delta$-functions in eq.(1): (i) The number of triangles is $n$. (ii) The entrance loop ($c$) consists of $l$ links. (iii) The exit loop ($c'$) consists of $l'$ links. (iv) The exit loop is one of the connected components obtained after a $d$ step deformation of the entrance loop.

In order to give a precise definition of the deformation, let us consider a loop $c$ on a triangulated surface. To deform $c$ one step forward means to remove triangles attached to $c$ in the forward direction and two-fold links on $c$. A typical example is illustrated in Fig.3, where $c$ is represented by a solid line, and the forward
direction is assumed to be inward. In this example, \( c \) has three two-fold links on it as indicated by \( \alpha, \beta \) and \( \gamma \), but we still regard it as a single loop. After a one-step deformation, this \( c \) will split into three loops which are indicated by the dotted lines. The notion of two-fold links might seem an idle complexity. As we will see, however, it plays an important role in the explicit evaluation of the transfer matrix. Note that the deformation of a loop considered here is closely related to the geodesic distance on the dual lattice. Actually, if a loop \( c' \) is obtained from loop \( c \) after a \( d \)-step deformation, any point on \( c' \) has geodesic distance \( d \) from \( c \). This is why we impose the condition (iv) in order to express the last \( \delta \)-function in eq.(1).

Since \( \tilde{N}(l, l'; d; n) \) is the lattice counterpart of \( N(L, L'; D; A) \), it satisfies the same composition law as eq.(2). Therefore, if we define the \( d \)-step evolution kernel \( \tilde{N}(l, l'; d) \) by \( \tilde{N}(l, l'; d) = \sum_{n=0}^{\infty} \tilde{N}(l, l'; d; n)K^n \) and regard it as the \( l, l' \) element of a matrix \( \tilde{N}(d) \), it can be decomposed into a product of single step evolution kernels:

\[
\tilde{N}(l, l'; d) \equiv (\tilde{N}(d))_{l,l'} = (\tilde{N}(1)^d)_{l,l'},
\]

which means that \( \tilde{N}(1) \) plays the role of a transfer matrix.

Next we show that the transfer matrix \( \hat{N}(1) \) can be evaluated by a combinatorial analysis. To proceed, we first need to evaluate the disk amplitude

\[
\tilde{F}(y; K) = \sum_{l,n=0}^{\infty} y^l K^n F(l; n),
\]

where \( F(l; n) \) is the number of possible triangulations of a disk which has a boundary of length \( l \) and consists of \( n \) triangles. The disk amplitude \( \tilde{F}(y; K) \) can be evaluated by the large-\( N \) \( \phi^3 \) matrix model [4] as

\[
\tilde{F}(y; K) = \frac{1}{y} \left\{ \frac{1}{2} \left( \frac{1}{y} - \frac{K}{y^2} \right) + \frac{K}{2} \left( \frac{1}{y} - c \right) \sqrt{\left( \frac{1}{y} - a \right) \left( \frac{1}{y} - b \right)} \right\},
\]
where \( a, b \) and \( c \) are functions of \( K \) determined by

\[
\begin{align*}
a &= \frac{1}{K} - c + \sqrt{2c(\frac{1}{K} - c)}, \\
b &= \frac{1}{K} - c - \sqrt{2c(\frac{1}{K} - c)}, \\
\frac{1}{K} &= c(\frac{1}{K} - c)(c - \frac{1}{2K}).
\end{align*}
\]

(6)

As is easily seen from (5) and (6), \( \tilde{F}(y; K) \) is analytic in both \( K \) and \( y \) with finite convergence radii. At the critical point where \( \tilde{F}(y; K) \) becomes singular as a function of \( K \), the values of \( a \) and \( c \) coincide and the convergence radius with respect to \( y \) is equal to \( 1/a \). These critical values can be easily obtained from eq.(6):

\[
K_c^2 = \frac{1}{12\sqrt{3}}, \quad y_c = (3^{1/4} - 3^{-1/4})/2, \quad a_c = c_c = \frac{1}{y_c}, \quad b_c = (1-\sqrt{3})/2K_c.
\]

Near this critical point we take the continuum limit by parametrizing \( K \) and \( y \) as

\[
K = K_c e^{-\epsilon^2 t}, \quad y = y_c e^{-\epsilon \zeta}.
\]

(7)

Then \( a, b \) and \( c \) are calculated from eq.(6) up to order \( \epsilon \) as

\[
a = a_c - \frac{4}{3^{1/4}\epsilon\sqrt{t}}, \quad b = b_c, \quad c = c_c + \frac{2}{3^{1/4}\epsilon\sqrt{t}}.
\]

(8)

We point out that the parametrization of eq.(7) is a natural one in the following sense. If we replace \( K^n \) and \( y^l \) in eq.(4) with \( K^n = K^n_c e^{-n\epsilon^2 t} \) and \( y^l = y^l_c e^{-l\epsilon \zeta} \), respectively, it is clear that the continuum limit \( \epsilon \rightarrow 0 \) corresponds to taking the \( n \rightarrow \infty \) and \( l \rightarrow \infty \) limit with the physical area \( A = n\epsilon^2 \) and the physical length \( L = l\epsilon \) kept finite. Thus \( t \) and \( \zeta \) are the conjugate variables to the area and the boundary length, respectively.
Substituting eqs.(7) into eq.(5), we obtain the following disk amplitude near the continuum limit:

\[ \tilde{F}(\zeta; t) = \frac{1 + \sqrt{3}}{2} (1 - \sqrt{3} \epsilon \zeta) + \frac{(1 + \sqrt{3})^{5/2}}{4} f(\zeta, \tau) \epsilon^{\frac{3}{2}} + O(\epsilon^2), \]

where

\[ f(\zeta, \tau) = (2 \zeta - \sqrt{\tau}) \sqrt{\zeta + \sqrt{\tau}}, \quad \sqrt{\tau} = \frac{4}{3 + \sqrt{3}} \sqrt{t}. \]

Here \( f(\zeta, \tau) \) is a universal function in the sense that it does not depend on the details of the triangulation prescription.

We now use combinatorics to evaluate the generating function of the matrix element of the transfer matrix \( \tilde{N}(1) \):

\[ \tilde{N}(y, y'; K) \equiv \sum_{l, l'} y^l y'^{l'} (\tilde{N}(1))_{l, l'} \equiv \sum_{l, l', n=0}^{\infty} y^l y'^{l'} K^n \tilde{N}(l, l'; 1; n). \]

What we are going to do is to sum up all possible triangulations of a cylinder such that the exit loop \( c' \) is one of the connected components of the one-step deformation of the entrance loop \( c \). In order to define a natural matrix multiplication of the transfer matrix, we introduce a marking point on \( c' \) but not on \( c \). (See Fig.4.) Let us first consider the triangle which is attached to the marked link. There are only three possible types for this marked triangle; 1), 2), and 3) in Fig.5a, where \( F \) in 3) denotes an arbitrary triangulation of a disk which is connected to the marked triangle through a vertex. After a one-step deformation, \( c \) proceeds to the marked link on \( c' \) and the disk part will be disconnected. As we can see in eq.(11), the power of \( y \), \( y' \), and \( K \) counts the number of links on \( c \), links on \( c' \), and triangles respectively. Then the contribution to \( \tilde{N}(y, y'; K) \) from the marked triangles is easily evaluated as follows: 1) \( yy'^2K \), 2) \( yy'^2K \), 3) \( y^2y'K \tilde{F}(y; K) \).
We then examine the structure of the rest of the triangulation by going along the exit loop $c'$. There are four different types of basic structures 1), 2), 3) and 4) as depicted in Fig.5b. Their weights are easily evaluated as 1) $yy^2K$, 2) $y^2y'K\tilde{F}(y;K)$, 3) $y^2\tilde{F}(y;K)$, and 4) $yK(\tilde{F}(y;K) - 1)$. Starting with the marked triangle, we can attach any of these four structures one by one repeatedly, and then come back to the original marked triangle. Therefore the contribution of these four structures to $\tilde{N}(y, y'; K)$ can be expressed by a geometric series. We thus obtain the following form for the generating function of the transfer matrix elements:

$$
\tilde{N}(y, y'; K) = \{2yy^2K + y^2y'K\tilde{F}(y;K)\}
\times \sum_{n=0}^{\infty} \left\{2yy^2K + y^2y'K\tilde{F}(y;K) + yK(\tilde{F}(y;K) - 1)\right\}^n
= \frac{2yy^2K + y^2y'K\tilde{F}(y;K)}{1 - 2yy^2K - y^2y'K\tilde{F}(y;K) - y^2\tilde{F}(y;K) - yK(\tilde{F}(y;K) - 1)}.
$$

(12)

We next consider the continuum limit of this transfer matrix. In order to see how the continuum limit should be taken, let us express the composition law in terms of the generating functions:

$$
\frac{1}{2\pi i} \int \frac{dz}{z} \tilde{N}(y, z; d_1; K)\tilde{N}(\frac{1}{z}, y'; d_2; K) = \tilde{N}(y, y'; d_1 + d_2; K),
$$

(13)

where $\tilde{N}(y, y'; d; K) = \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{n=0}^{\infty} y^l y'^{l'} K^n \tilde{N}(l, l'; d; n)$. From eq.(13) it is clear that the continuum limit for $y$ and $y'$ should be taken around different values which are inverse to each other. Furthermore, since the structure of the entrance loop is similar to the boundary loop of the disk amplitude, it is natural to expect that the continuum limit can be taken by setting

$$
y = ye^{-\epsilon\zeta}, \quad y' = y^{-1}e^{-\epsilon\zeta'}, \quad K = Ke^{-\epsilon^2t}.
$$

(14)
If this is true, the composition law (13) can be re-expressed as

$$\frac{\epsilon}{2\pi i} \int_{-i\infty}^{i\infty} d\xi \tilde{N}(\zeta, \zeta'; d_1; t) \tilde{N}(-\xi, \zeta'; d_2; t) = \tilde{N}(\zeta, \zeta'; d_1 + d_2; t),$$  \hspace{1cm} (15)

where $\tilde{N}(\zeta, \zeta'; d; t)$ stands for $\tilde{N}(y, y'; d; K)$ for the values of $y$, $y'$ and $K$ given by (14).

The validity of this continuum limit is explicitly checked by substituting (14) into eq.(12). We obtain

$$\tilde{N}(y, y'; K) \equiv \tilde{N}(\zeta, \zeta'; t) = \frac{1}{\epsilon \zeta' + \zeta - \alpha \epsilon^{\frac{3}{2}} f(\zeta, \tau)} + O(\epsilon^0),$$ \hspace{1cm} (16)

where $\alpha = \sqrt{2/(9\sqrt{3} - 1)}$ and $f(\zeta, \tau)$ is given by eq.(10). In this equation we have the following miraculous cancellations, which convince us that the continuum limit we consider here is in fact the right one. First of all $O(\epsilon^0)$ terms in the denominator of $\tilde{N}(y, y'; K)$ cancel out. Secondly, the coefficients of $\zeta$ and $\zeta'$ are 1. Thirdly, the residue of $\tilde{N}(y, y'; K)$ with respect to the $\zeta + \zeta'$ term is $1/\epsilon$, which, as we see below, exactly cancels the factor $\epsilon$ in (15). These are highly nontrivial results.

By substituting (16) into (15) for $d_1 = 1$ and $d_2 = d$, we obtain

$$\tilde{N}(\zeta, \zeta'; d + 1; t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\xi \frac{1}{\zeta + \xi - \alpha \epsilon^{\frac{3}{2}} f(\zeta, \tau)} \tilde{N}(-\xi, \zeta'; d; t),$$ \hspace{1cm} (17)

which leads to

$$\tilde{N}(\zeta, \zeta'; d + 1; t) - \tilde{N}(\zeta, \zeta'; d; t) = -\alpha \sqrt{\epsilon} f(\zeta, \tau) \frac{\partial}{\partial \zeta} \tilde{N}(\zeta, \zeta'; d; t).$$ \hspace{1cm} (18)

We then take the continuum limit of eq.(18) by introducing $D \equiv \alpha \sqrt{\epsilon} d$ and taking $\epsilon \to 0$. We thus obtain the following continuum differential equation for $\psi(\zeta; D) =$
\[ \tilde{N}(\zeta, \zeta'; D; t): \]

\[ \frac{\partial}{\partial D} \psi(\zeta, D) = -f(\zeta, \tau) \frac{\partial}{\partial \zeta} \psi(\zeta, D). \] (19)

In other words, the proper time evolution kernel can be identified with the following matrix element:

\[ \tilde{N}(\zeta, \zeta'; D; \tau) = \langle \zeta | e^{-DH} | \zeta' \rangle, \]

where \( H \) is a “Hamiltonian” given by

\[ H = f(\zeta, \tau) \frac{\partial}{\partial \zeta}. \] (20)

It should be noted that the miraculous cancelations that occurred in eq.(16) are crucial in the derivation of the continuum differential equation (19).

The initial value problem for the differential equation (19) can be easily solved by the use of the characteristic curve method as \( \psi(\zeta; D) = \psi(\zeta'; 0) \), where \( \zeta' \) is defined by \( \int_{\zeta}^{\zeta'} d\zeta''/f(\zeta'', \tau) = D \) as a function of \( \zeta \) and \( D \). We can now evaluate the proper time evolution kernel \( N(L, L'; D) \) which is given by the inverse Laplace transformation of \( \tilde{N}(\zeta, \zeta'; D; \tau) \) for \( \zeta \) and \( \zeta' \). By carrying out an inverse Laplace transformation for the solution of (19) with the initial condition \( \psi(\zeta; 0) = e^{-L'\zeta} \), we obtain

\[ N(L, L'; D) = \frac{1}{2\pi i} \int_{-\infty}^{i\infty} d\zeta e^{L\zeta} e^{-L'\zeta'}. \] (21)

Note that eq.(21) indeed reproduces the corresponding initial condition \( N(L, L'; 0) = \delta(L - L') \), because we have \( \zeta' = \zeta \) for \( D = 0 \).

For later use we give \( N(L, L'; D) \) for small values of \( L \) and \( \tau \):

\[ N(L, L'; D) = \frac{L'}{\sqrt{\pi L}} \left\{ \frac{2}{D^3} - \frac{\tau}{10} (D + \frac{L'}{D}) + \frac{\tau^2}{7} \frac{1}{2} (D^3 + \frac{1}{2}L'D) + O(\tau^2) \right\} e^{-L'/D^2} \]

\[ + O(L^0). \] (22)

\[ \star \quad N(L, L'; D) \] is related to its discrete version \( \tilde{N}(l, l'; d) \) by \( N(L, L'; D) = e^{-1}y e^{L'-l'} \tilde{N}(l, l'; d) \).
We also calculate the inverse Laplace transformation of the disk amplitude as a double series expansion with respect to $\epsilon$ and $t$:

\[
F(L) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\zeta \ e^{L\zeta} \tilde{F}(\zeta; t) e^{-3/2}
\]

\[
= \frac{1 + \sqrt{3}}{2} \delta(L - \sqrt{3}\epsilon) e^{-3/2} + \frac{3(1 + \sqrt{3})^{5/2}}{8\sqrt{\pi}} \left( L^{-5/2} - \frac{\tau}{2} L^{-1/2} + \frac{\tau^{3/2}}{3} L^{1/2} \right) + O(\tau^2),
\]

where $\tilde{F}(\zeta; t)$ is given by eq.(9).

We now apply the formalism developed here to the analysis of the fractal structures of the space-time.[5, 6, 7] We consider a large enough space-time with spherical topology for two-dimensional pure gravity. We take an arbitrary point $P$ and consider the set of points $S(P; D)$ whose geodesic distances from $P$ are less than or equal to $D$. The boundary of $S(P; D)$ usually consists of many loops with various lengths. Let $\rho(L; D) dL$ be the number of loops belonging to the boundary of $S(P; D)$ whose lengths lie between $L$ and $L + dL$. As we explain below, the quantity $\rho(L; D)$ can be evaluated as

\[
\rho(L; D) = \lim_{L_0 \to 0, \tau \to 0} \frac{\partial^2}{\partial \tau^2} N(L_0, L; D) \frac{1}{L} F(L),
\]

\[
= \frac{1}{D^2} G(x) + \frac{4 \epsilon^{-3/2}}{7(1 + \sqrt{3})^{3/2}} D^3 \left( 1 - \frac{x}{2} \right) \delta(L - \sqrt{3}\epsilon),
\]

where

\[
G(x) \equiv \frac{3}{7\sqrt{\pi}} \left( x^{-5/2} + \frac{1}{2} x^{-3/2} + \frac{14}{3} x^{1/2} \right) e^{-x},
\]

with $x = L/D^2$ as a scaling parameter. In eq.(24), the limit $L_0 \to 0$ corresponds to shrinking the entrance loop to a point $P$, and the limit $\tau \to 0$ is taken to have
the thermodynamic limit. The second derivatives with respect to \( \tau \) in eq.(24) are introduced to avoid small area dominance in the \( \tau \to 0 \) limit. Any higher derivative works for this purpose without changing the final answer. The numerator on the RHS corresponds to an operation of gluing a disk with a boundary of length \( L \) to the exit loop of the cylinder. The factors \( 1/L_0 \) and \( 1/L \) in front of \( F \)'s are introduced to convert marked boundaries to unmarked ones. As we have already explained, there are several loops which have geodesic distance \( D \) from the entrance loop. Recognizing that the exit loop is one of these loops, we easily realize that the ratio in eq.(24) counts the number of loops which have the boundary length \( L \).

One of the surprising properties of the function \( \rho(L; D) \) is that it is essentially a universal scaling function of the scaling parameter \( x = L/D^2 \). For small values of \( L \), however, \( \rho(L; D) \) includes non-universal parts which have negative power dependence on the lattice constant \( \epsilon \). In order to examine the scaling property, it is convenient to introduce the following quantities:

\[
<L^n> = \int_0^{\infty} dL \ L^n \rho(L; D).
\]

(26)

From eq.(24) it is easy to show that

\[
<L^0> \propto const \times D^3 \epsilon^{-3/2} + const \times D \epsilon^{-1/2} + const \times D^0,
\]

\[
<L^1> \propto const \times D^3 \epsilon^{-1/2} + const \times D^2,
\]

\[
<L^n> \propto const \times D^{2n} \quad (n \geq 2).
\]

(27)

Since \( D^2 \rho(L; D) \) is essentially the universal function \( G(x) \) of the dimensionless parameter \( x \), it is an ideal quantity to measure in computer simulations. In fact a numerical study for pure gravity shows a clear universal scaling behavior according to \( G(x) \).[8] As we can see in eq.(27), \( <L^0> \) and \( <L^1> \) include \( \epsilon \) dependent non-universal part as a dominant contribution while \( <L^n> \) for \( n \geq 2 \) includes
only universal part and thus is expected to show clear fractal behaviors. The contro-
troversial numerical results for the fractal dimension by Agishtein and Migdal [6] 
can be understood by the non-universal behavior of the corresponding quantities. 

Although we have only treated pure gravity \((c = 0)\) in this paper, it is im-
portant to investigate quantum gravity coupled to matter fields. In the limit 
\(c \to -\infty\) it is known that the two-dimensional gravity system approaches to 
the classical limit. We thus expect that the wildly branching space-time sur-
face approaches to a smooth surface. On the smooth surface there is only a 
single boundary at a geodesic distance \(D\) from a given point. We then have 
\[
\rho(L; D) = \delta(L - 2\pi D) = 1/D\delta(L/D - 2\pi),
\]
which again suggests a scaling function with respect to a scaling parameter \(x = L/D\). It is thus very natural to expect that 
the gravity, for example, with \(c = -2\) matter shows the similar scaling behavior.

In fact our recent numerical investigation shows a clear scaling behavior of the 
function \(\rho(L; D)\) which has a similar scaling parameter \(x = L/D^{1.9\pm 0.2}\) but has a 
slightly different power behavior from (25).[9] It would be very interesting if we 
can find an analytic formulation for deriving \(\rho(L; D)\) in a model with \(c \neq 0\).

In this paper we have given an explicit form of “Hamiltonian” by eq.(20) which 
could be viewed as a time evolution generator of closed string, where time is iden-
tified with the geodesic distance. This formulation may provide a new formulation 
of closed string field theory.

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REFERENCES

1. V. G. Knizhnik, A. M. Polyakov and A. A. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819;
   F. David, Mod. Phys. Lett. **A3** (1988) 651;
   J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509.

2. F. David, Nucl. Phys. **B257** (1985) 45;
   V. A. Kazakov, Phys. Lett. **150B** (1985) 282;
   D. V. Boulatov, V. A. Kazakov, I. K. Kostov and A. A. Migdal, Nucl. Phys. **B275** (1986) 641;
   J. Ambjørn, B. Durhuus and J. Fröhlich, Nucl. Phys. **B257** (1985) 433.

3. E. Brézin and V. A. Kazakov, Phys. Lett. **236B** (1990) 144;
   M. Douglas and S. Shenker, Nucl. Phys. **B335** (1990) 635;
   D. J. Gross and A. A. Migdal, Phys. Rev. Lett. **64** (1990) 127.

4. E. Brézin, C. Itzykson, G. Parisi and J. B. Zuber, Commun. Math. Phys. **59** (1978) 35.

5. H. Kawai and M. Ninomiya, Nucl. Phys. **B336** (1990) 115.

6. M. E. Agishtein and A. A. Migdal, Int. J. Mod. Phys. **C1** (1990) 165; Nucl. Phys. **B350** 690 (1991).

7. N. Kawamoto, V. A. Kazakov, Y. Saeki and Y. Watabiki, Phys. Rev. Lett. **68** (1992) 2113; Nucl. Phys. **B(Proc. Suppl.)26** (1992) 584.

8. N. Tsuda and T. Yukawa, private communications.

9. H. Kawai, N. Kawamoto, T. Mogami, Y. Saeki and Y. Watabiki, in preparation.
FIGURE CAPTIONS

Fig. 1 Cylinder with entrance loop $c$ and exit loop $c'$.  

Fig. 2 Decomposition of a cylinder with height $D$ into two cylinders with height $D'$ and $D - D'$.  

Fig. 3 A Typical example of the deformation of a loop. The dotted loops are obtained after a one-step deformation of the solid loop.  

Fig. 4 A triangulation contributing to $\tilde{N}(y; y'; K)$. The dotted loop ($c'$) is one of the components of the one-step deformation of the solid line ($c$).  

Fig. 5 a) Three types for the triangle attached to the marked link. b) Four basic structures which may appear along $c$. The solid and dotted lines stand for a part of the entrance($c$) and the exit($c'$) loops, respectively.
Fig. 1
Fig. 2
Fig5  a)