ABELIAN RIGHT PERPENDICULAR SUBCATEGORIES
IN MODULE CATEGORIES

LEONID POSITSELSKI

Abstract. We show that an abelian category can be exactly, fully faithfully embedded into a module category as the right perpendicular subcategory to a set of modules or module morphisms if and only if it is a locally presentable abelian category with a projective generator, or in other words, the category of models of an additive algebraic theory of possibly infinite bounded arity. This includes the categories of contramodules over topological rings and other examples. Various versions of the definition of the right perpendicular subcategory are considered, all leading to the same class of abelian categories. We also discuss sufficient conditions under which the natural forgetful functors from the categories of contramodules to the related categories of modules are fully faithful.

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Introduction

0.1. The word contramodule roughly means “an object of a locally presentable abelian category with a projective generator”. In fact, cocomplete abelian categories with a projective generator can be described as the categories of algebras/modules over additive monads on the category of sets (see the discussion in the introduction to the paper [32] or in [35] Section 6]). Such a category is locally presentable if and only if the monad \( T: \text{Sets} \to \text{Sets} \) is accessible, i.e., the functor \( T \) commutes with \( \kappa \)-filtered colimits in \( \text{Sets} \) for a large enough cardinal \( \kappa \).

Algebras over a monad \( T \) are sets with (possibly infinitary) operations: for any set \( X \), elements of the set \( T(X) \) can be interpreted as \( X \)-ary operations in \( T \)-algebras. That is why \textit{categories of models of } \( \kappa \)-ary algebraic theories \textit{is another name for the}
categories of algebras over $\kappa$-accessible monads on $\text{Sets}$. Such a category is abelian if and only if it is additive; for this, the monad $T$ needs to be additive, that is it has to contain the operations of addition, zero element, and inverse element, and all the other operations in $T$ have to be additive with respect to the abelian group structure defined by these. In this case, it might be preferable to modify the terminology and speak about “$T$-modules” rather than “$T$-algebras”.

So objects of a cocomplete abelian category with a projective generator can be viewed as module-like structures with infinitary additive operations. Contramodules over topological associative rings provide typical examples. In fact, to any complete, separated topological ring $R$ with a base of neighborhoods of zero formed by open right ideals one can assign an additive, accessible monad $T_R : \text{Sets} \to \text{Sets}$. Then one defines left $R$-contramodules as $T_R$-modules [23, 24, 26, 32, 35, 31].

0.2. Thus, contramodules are sets with infinitary additive operations of the arity bounded by some cardinal. Hence it comes as a bit of a surprise when the natural forgetful functors from the categories of contramodules to the related categories of modules turn out to be fully faithful. An infinitary additive operation is uniquely determined by its finite aspects. Why should it be? Nevertheless, such results are known to hold in a number of situations.

The observation that weakly $p$-complete/Ext-$p$-complete abelian groups carry a uniquely defined natural structure of $\mathbb{Z}_p$-modules that is preserved by any group homomorphisms between them goes back to [17, Lemma 4.13]. The claim that the forgetful functor provides an isomorphism between the category of $\mathbb{Z}_p$-contramodules $\mathbb{Z}_p$- contra and the full subcategory of weakly $p$-complete abelian groups in $\mathbb{Z}$-mod was announced in [23, Remark A.3], where the definition of a contramodule over a topological associative ring also first appeared. Similarly, it was mentioned in [23, Remark A.1.1] that contramodules over the coalgebra $C$ dual to the algebra of formal power series $C' = k[[x]]$ in one variable $x$ over a field $k$ form a full subcategory in the category of $k[x]$-modules; and a generalization to the coalgebras dual to power series in several commutative variables $C' = k[[x_1, \ldots, x_m]]$ was formulated.

A more elaborated discussion of these results for the rings $k[[x]]$ and $\mathbb{Z}_p$ can be found in [26, Section 1.6]; and a generalization to the adic completions of Noetherian rings with respect to arbitrary ideals was proved in [24 Theorem B.1.1]. A version for adic completions of right Noetherian associative rings at their centrally generated ideals was obtained in [25, Theorem C.5.1].

0.3. In the most straightforward terms, the situation can be explained as follows. One says that an $x$-power infinite summation operation is defined on an abelian group $B$ if for every sequence of elements $b_0, b_1, b_2, \ldots \in B$ an element denoted formally by $\sum_{n=0}^{\infty} x^n b_n \in B$ is defined in such a way that the equations of additivity
\[
\sum_{n=0}^{\infty} x^n (b_n + c_n) = \sum_{n=0}^{\infty} x^n b_n + \sum_{n=0}^{\infty} x^n c_n \quad \forall b_n, c_n \in P,
\]
contraunitality
\[
\sum_{n=0}^{\infty} x^n b_n = b_0 \quad \text{if} \quad b_1 = b_2 = b_3 = \cdots = 0,
\]
and contraassociativity
\[ \sum_{i=0}^{\infty} x^i \sum_{j=0}^{\infty} x^j b_{ij} = \sum_{n=0}^{\infty} x^n \sum_{i+j=n} b_{ij} \quad \forall b_{ij} \in B, \quad i, j = 0, 1, 2, \ldots \]
are satisfied. Then it is an elementary linear algebra exercise (see [26, Section 1.6] or [28, Section 3]) to show that an \( x \)-power infinite summation operation in an abelian group \( B \) is uniquely determined by the additive operator \( x: B \rightarrow B \),

\[ xb = \sum_{n=0}^{\infty} x^n b_n, \quad \text{where } b_1 = b \text{ and } b_0 = b_2 = b_3 = b_4 = \cdots = 0. \]

Furthermore, a \( \mathbb{Z}[x] \)-module structure on \( B \) comes from an \( x \)-power infinite summation operation if and only if

(1) \[ \text{Hom}_{\mathbb{Z}[x]}(\mathbb{Z}[x, x^{-1}], B) = 0 = \text{Ext}_{\mathbb{Z}[x]}^1(\mathbb{Z}[x, x^{-1}], B). \]

“An abelian group with an \( x \)-power infinite summation operation” is another name for a contramodule over the topological ring of formal power series \( \mathbb{Z}[[x]] \); so the forgetful functor \( \mathbb{Z}[[x]] \)-contra \( \longrightarrow \mathbb{Z}[x] \)-mod is fully faithful, and the conditions (1) describe its image. Notice that the dual condition

\[ \mathbb{Z}[x, x^{-1}] \otimes_{\mathbb{Z}[x]} M = 0 \]
describes the category of discrete/torsion \( \mathbb{Z}[x] \)-modules as a full subcategory in the category of \( \mathbb{Z}[x] \)-modules. In the case of several commutative variables \( x_1, \ldots, x_m \), rewriting an \( [x_1, \ldots, x_m] \)-power infinite summation operation as the composition of \( x_j \)-power infinite summations over the indices \( j = 1, \ldots, m \) allows to obtain a similar description of the abelian category of \( \mathbb{Z}[[x_1, \ldots, x_m]] \)-contramodules as a full subcategory in the abelian category of \( \mathbb{Z}[x_1, \ldots, x_m] \)-modules [28, Section 4].

0.4. The above-mentioned results on the possibility of recovering infinite summation operations from their finite aspects apply to finite collections of commutative/central variables only. So it was a kind of breakthrough when it was shown in [30, Theorem 2.1] that the forgetful functor from the category of left contramodules over the coalgebra \( \mathcal{C} \) dual to the algebra of formal power series \( \mathcal{C}^\vee = k\{x_1, \ldots, x_m\} \) in non-commutative variables \( x_1, \ldots, x_m \) into the category of left modules over the free associative algebra \( k\{x_1, \ldots, x_m\} \) is fully faithful. In other words, an \( \{x_1, \ldots, x_m\} \)-power infinite summation operation in a \( k \)-vector space/abelian group \( B \) is determined by the collection of linear/additive operators \( x_j: B \rightarrow B \), \( 1 \leq j \leq m \).

The following general result, going back, in some form, to Isbell [16], is discussed in the paper [35]: for any locally presentable abelian category \( \mathcal{B} \) with a projective generator, one can choose a projective generator \( Q \in \mathcal{B} \) such that \( \text{Hom}_\mathcal{B}(Q, -) \) is a fully faithful functor from \( \mathcal{B} \) into the category of modules over the endomorphism ring \( \text{Hom}_{\mathcal{B}}(Q, Q) \). Moreover, locally presentable abelian categories with a projective generator can be characterized as exactly and accessibly embedded, reflective full abelian subcategories in the categories of modules over associative rings. Under the Vopěnka principle, the “accessibly embedded” condition can be dropped.
0.5. Two classes of examples of locally presentable abelian categories with a projective generator are mentioned in the paper [32]: in addition to the categories of contramodules over topological rings [32, Sections 5–7], there are also right perpendicular subcategories to sets of modules of projective dimension at most 1 in the module categories [32, Examples 4.1]. The fact that right perpendicular subcategories to sets/classes of objects of projective dimension \( \leq 1 \) in abelian categories are abelian was discovered by Geigle and Lenzing in [13, Proposition 1.1] and, much later, discussed by the present author in [28, Section 1]. The results of [24, Theorem B.1.1] and [25, Theorem C.5.1] identify the categories of contramodules over certain topological rings with the right perpendicular subcategories to certain (sets of) modules of projective dimension not exceeding 1.

The latter results depend on Noetherianity assumptions. These can be weakened to weak proregularity assumptions (in the sense of [37, 22, 27]), and somewhat further, as we show in this paper. On the other hand, we demonstrate an example of the perpendicular category to a module of projective dimension 1 which is not equivalent to the category of contramodules over any topological ring (see Example 1.3(6)). Perhaps the conclusion should be that the good class of categories of module-like structures with infinitary operations to work with is that of the categories of modules over additive, accessible monads on \( \text{Sets} \), and the categories of contramodules over topological rings are only a certain subclass. (The discussion of flat contramodules and contramodule approximation sequences in [32, Sections 6–7], [33, Section 5], [34, Section 7] shows this subclass to be better behaved in some ways.)

0.6. As it was discussed above, the class of all categories of modules over additive, accessible monads on \( \text{Sets} \) coincides with that of locally presentable abelian categories with a projective generator. The main result of this paper can be described as stating that the class of all right perpendicular subcategories to sets of objects in the categories of modules over associative rings also coincides with the class of locally presentable abelian categories with a projective generator. The caveat is that one has to specify what one means by “right perpendicular subcategories”.

Right perpendicular subcategories to (sets of) modules of projective dimension at most 1 are not enough, and right perpendicular subcategories to arbitrary sets of modules, as defined in [13], do not even need to be abelian. However, one can consider the maximal reasonable class of right perpendicular subcategories in the categories of modules over associative rings, namely, those right perpendicular subcategories to sets of modules that happen to be abelian and exactly embedded.

One can also consider the minimal reasonable class of right perpendicular subcategories in modules, namely, those right \( \text{Ext}^{0,1} \)-perpendicular subcategories which happen to coincide with the right \( \text{Ext}^{0,\infty} \)-perpendicular subcategories to the same set of modules. These are automatically abelian and exactly embedded (in fact, it suffices that the \( \text{Ext}^{0,1} \)-perpendicular and the \( \text{Ext}^{0,2} \)-perpendicular subcategories coincide, for this conclusion to hold). Then we show that both the “maximal” and the “minimal reasonable” classes of right perpendicular subcategories in the categories of
modules over associative rings embody the same classes of abstract abelian categories, viz., the locally presentable abelian categories with a projective generator.

0.7. Actually, given that an exact embedding of a certain abelian category $B$ into the category of modules over some associative $R$ ring is shown to exist, one naturally wants more than just an arbitrary such embedding. One wants the image of this fully faithful functor to be closed under extensions in $R$–mod. One wants the functor $B \to R$–mod to induce isomorphisms on all the Ext groups. In the final analysis, one wants the induced triangulated functor between the unbounded derived categories $D(B) \to D(R$–mod) to be fully faithful.

In this paper we explain that, for a locally presentable abelian category $B$ with a projective generator, the task of finding a fully faithful embedding $B \to R$–mod inducing an isomorphism on the groups $\text{Ext}^i$ for all $i \leq n$ is related to the task of representing $B$ as the right $\text{Ext}^{0,1}$-perpendicular subcategory to a set of objects in $R$–mod coinciding with the right $\text{Ext}^{0,n}$-perpendicular subcategory to the same set of modules. Then we proceed to construct, for an arbitrary locally presentable abelian category with a projective generator, a fully faithful embedding $B \to R$–mod satisfying these conditions for $n = \infty$. In fact, we prove that the induced triangulated functor $D^-(B) \to D^-(R$–mod) is fully faithful.

0.8. Now let us describe the content of this paper in more detail. In Section 1 we discuss the interpretation of an arbitrary cocomplete abelian category with a projective generator as the category of modules over an additive monad on the category of sets. Locally presentable abelian categories with a projective generator correspond to accessible additive monads on Sets. Various examples of such abelian categories and such monads are provided, including contramodules over topological rings and right perpendicular subcategories to morphisms of free modules.

In Section 2 we discuss the situation when the forgetful functor from the category of modules over a monad on Sets to the category of modules over the ring of (some) unary operations in this monad is fully faithful. This section consists mostly of examples, which include contramodules over finitely centrally generated ideals $I$ in associative rings $R$, contramodules over the adic completions $\mathcal{R}$ of such rings $R$ with respect to such ideals $I$, contramodules over central multiplicative subsets $S$ in associative rings $R$, and contramodules over the $S$-completions $\mathcal{R}$ of such rings $R$ with respect to such multiplicative subsets $S$. The proofs of some of the assertions related to these examples are postponed to Section 3, where we prove a rather general sufficient condition for full-and-faithfulness of the forgetful functor from the category of contramodules over a topological ring to the category of modules.

In Section 4 we introduce the notion of a right $n$-perpendicular subcategory in an abelian category, where $n \geq 0$ is an integer or $n = \infty$. We also define the notion of an $n$-good projective generator of a locally presentable abelian category, and establish a connection between the two. Every $n$-good projective generator of a locally presentable abelian category $B$ embeds it as a right $n$-perpendicular subcategory into a module category, but the converse is not true. A counterexample
to this effect is given in Section 5, which also contains the definition of a good module over an associative ring \( R \) and the claim that all good \( n \)-tilting \( R \)-modules are good \( R \)-modules, with the references to [4] and [12, 3, 35].

Section 6 contains the proof of the main result of this paper, namely, that every locally presentable abelian category with a projective generator can be embedded as a right \( \infty \)-perpendicular category into the category of modules over an associative ring, because a big enough copower of its given projective generator is good. The argument is based on a result of [16] and [35], claiming, in our present terminology, that a big enough copower of a given projective generator is 0-good, and the theory of \( \kappa \)-flat modules over associative rings, which is developed in the beginning of this section for this purpose.

The final Section 7 contains counterexamples of various badly behaved, from our point of view, full subcategories in the categories of modules over associative rings which one can obtain by relaxing the conditions imposed in our definition of a right \( n \)-perpendicular subcategory. Moreover, we show that many right \( \text{Ext}^{1-n} \)-orthogonal classes in the categories of modules, \( 1 \leq n \leq \infty \), viewed as abstract categories, can be realized as right \( \text{Ext}^{0-n+1} \)-perpendicular subcategories in the categories of modules over appropriately modified rings.

0.9. One terminological remark is in order. Following [1], we understand the terms such as “\( \kappa \)-presentable”, “\( \kappa \)-accessible”, “\( \kappa \)-generated”, etc., to mean “\(< \kappa \)-presentable”, “\(< \kappa \)-accessible”, “\(< \kappa \)-generated”, etc. Here \( \kappa \) is a regular cardinal. So a module is called \( \kappa \)-generated if it generated by \( \text{less than} \ \kappa \) elements, and it is called \( \kappa \)-presentable if it is isomorphic to the cokernel of a morphism of free modules with \( \text{less than} \ \kappa \) generators. The cokernel of a morphism of free modules with at most \( \lambda \) generators is called \( \lambda^+ \)-presentable, where \( \lambda^+ \) is the successor cardinal of a cardinal \( \lambda \).

Acknowledgement. I am grateful to Jan Šťovíček, Sefi Ladkani, Luisa Fiorot, Silvana Bazzoni, Alexander Slávik, Jan Trlifaj, and Jiří Rosický for helpful conversations and comments. The author’s research is supported by the Israel Science Foundation grant # 446/15 and by the Grant Agency of the Czech Republic under the grant P201/12/G028.

1. Additive Monads on the Category of Sets

A monad on the category of sets is a covariant functor \( T : \text{Sets} \rightarrow \text{Sets} \) endowed with the natural transformations of multiplication \( \phi_T : T \circ T \rightarrow T \) and unit \( \epsilon_T : \text{Id}_{\text{Sets}} \rightarrow T \) satisfying the equations of associativity

\[
T \circ T \circ T \Rightarrow T \circ T \rightarrow T
\]

\( \phi_T(\phi_T \circ T) = \phi_T(T \circ \phi_T) \) and unitality

\[
T \Rightarrow T \circ T \rightarrow T
\]
∅ ⊃ φ = Id_{Sets}. An algebra over \( T \) is a set \( B \) endowed with a map of sets \( \pi_B: \mathbb{T}(B) \to B \) satisfying the equations of associativity

\[
\mathbb{T}(\mathbb{T}(B)) \Rightarrow \mathbb{T}(B) \to B
\]

\[
\pi_B \circ \phi_T(B) = \pi_B \circ (\pi_B) \text{ and unitality}
\]

\[
B \to \mathbb{T}(B) \to B
\]

\( \pi_B \circ \epsilon_T(B) = \text{id}_B \). We denote category of all algebras over a monad \( T \) by \( \mathbb{T}_{-\text{alg}} \).

For any set \( X \), elements of the set \( \mathbb{T}(X) \) are interpreted as \( X \)-ary operations on \( \mathbb{T} \)-algebras in the following way. Let \( B \) be a \( \mathbb{T} \)-algebra, \( b: X \to B \) be a map of sets, and \( t \in \mathbb{T}(X) \) be an element. Then the element \( t_B(b) \in B \) is defined as \( t_B(b) = \pi_B(\mathbb{T}(b)(t)) \), where \( \mathbb{T}(b): \mathbb{T}(X) \to \mathbb{T}(B) \) is the map obtained by applying the functor \( \mathbb{T} \) to the map \( b \), and \( \mathbb{T}(b)(t) \in \mathbb{T}(B) \) is the element obtained by applying the map \( \mathbb{T}(b) \) to the element \( t \). So \( t_B = t_B(B) \) is a map \( t_B: B^X \to B \), that is an \( X \)-ary operation in (the underlying set of) an arbitrary \( \mathbb{T} \)-algebra \( B \).

The following lemma describes the class of monads on the category of sets that we are interested in. A monad satisfying these conditions is called additive.

**Lemma 1.1.** The following conditions on a monad \( T: \text{Sets} \to \text{Sets} \) are equivalent:

(a) the category \( T_{-\text{alg}} \) is additive;

(b) the category \( T_{-\text{alg}} \) is abelian;

(c) there exist elements \( "x + y" \in \mathbb{T}(\{x, y\}) \), \( "-x" \in \mathbb{T}(\{x\}) \), and \( 0 \in \mathbb{T}(\emptyset) \), where \( \{x, y\} \) is a two-element set, \( \{x\} \) is a one-element set, and \( \emptyset \) is the empty set, such that the corresponding operations define an abelian group structure on every \( \mathbb{T} \)-algebra \( B \), and all the other operations \( t_B: B^X \to B \), \( t \in \mathbb{T}(X) \), \( X \in \text{Sets} \) are abelian group homomorphisms with respect to these abelian group structures.

The elements \( "x + y" \in \mathbb{T}(\{x, y\}) \), \( "-x" \in \mathbb{T}(\{x\}) \), and \( 0 \in \mathbb{T}(\emptyset) \) in the condition (c) are unique if they exist.

**Proof.** (a) \(\Rightarrow\) (c): For any monad \( T: \text{Sets} \to \text{Sets} \) and every set \( X \), the map \( \pi_T(X) = \phi_T(X): \mathbb{T}(\mathbb{T}(X)) \to \mathbb{T}(X) \) defines a \( \mathbb{T} \)-algebra structure on the set \( \mathbb{T}(X) \). This \( \mathbb{T} \)-algebra is called the free \( \mathbb{T} \)-algebra generated by the set \( X \). For any \( \mathbb{T} \)-algebra \( B \), there is a natural bijection \( \text{Hom}_\mathbb{T}(\mathbb{T}(X), B) \simeq B^X \), where \( \text{Hom}_\mathbb{T}(C, B) \) denotes the set of all morphisms \( C \to B \) in the category \( \mathbb{T}_{-\text{alg}} \).

In particular, one has \( \text{Hom}_\mathbb{T}(\mathbb{T}(\ast), B) = B \), where \( \ast \) denotes a one-element set. So the forgetful functor \( \mathbb{T}_{-\text{alg}} \to \text{Sets} \) assigning to every \( \mathbb{T} \)-algebra \( B \) its underlying set \( B \) is corepresented by the free \( \mathbb{T} \)-algebra \( \mathbb{T}(\ast) \); and likewise, the functor assigning to a \( \mathbb{T} \)-algebra \( B \) the set \( B^X \) is corepresented by the free \( \mathbb{T} \)-algebra \( \mathbb{T}(X) \). Therefore, natural transformations \( B^X \to B \) correspond bijectively to \( \mathbb{T} \)-algebra morphisms \( \mathbb{T}(\ast) \to \mathbb{T}(X) \), i.e., to elements of the set \( \mathbb{T}(X) \).

Now if the category \( \mathbb{T}_{-\text{alg}} \) is additive, then there must be a naturally defined abelian group structure on the set of morphisms \( B = \text{Hom}_\mathbb{T}(\mathbb{T}(\ast), B) \) for every \( \mathbb{T} \)-algebra \( B \). This means the natural transformations of addition \( B \times B \to B \), inverse element \( B \to B \), and zero element \( 0 \in B \). According to the above, these must be come from some elements \( "x + y" \in \mathbb{T}(\{x, y\}) \), \( "-x" \in \mathbb{T}(\{x\}) \), and \( 0 \in \mathbb{T}(\emptyset) \).
Biadditivity of compositions of morphisms in an additive category implies that all other operations on the underlying sets of $\mathbb{T}$-algebras must be abelian group homomorphisms with respect to this abelian group structure, and the uniqueness of an additive category structure implies uniqueness of the elements “$x + y$”, “$-x$”, and $0$. Alternatively, one can say that any two abelian group structures on a given set that are additive with respect to one another always coincide.

$(c) \implies (b)$: One observes that, for any monad $\mathbb{T} : \text{Sets} \to \text{Sets}$, every map between the underlying sets of two $\mathbb{T}$-algebras $C \to B$ forming commutative squares with the $X$-ary operations $t_\mathbb{T}(C)$ and $t_\mathbb{T}(B)$ for all the sets $X$ and elements $t \in \mathbb{T}(X)$ is a $\mathbb{T}$-algebra morphism (it suffices to take $X = C$). Furthermore, a collection of $X$-ary operations $t_A : A^X \to A$ defined on a given set $A$ for all sets $X$ and elements $t \in \mathbb{T}(X)$ corresponds to a $\mathbb{T}$-algebra structure on $A$ if and only if the maps $t_A$ satisfy the composition and unit relations encoded in the structural natural transformations $\phi_\mathbb{T} : \mathbb{T} \circ \mathbb{T} \to \mathbb{T}$ and $\epsilon_\mathbb{T} : \text{Id} \to \mathbb{T}$ of the monad $\mathbb{T}$.

Then, in the assumption of the condition $(c)$, one repeats the proof of the assertion that the category of modules over an associative ring is abelian, replacing unary finitary additive operations with infinitary ones. The point is that the forgetful functor assigning to every set $\mathbb{T}$-algebra $A$ its underlying abelian group $B$ is exact; so the kernels, images, and cokernels of morphisms in $\mathbb{T}_{\text{alg}}$ can be constructed as the kernels, images, and cokernels of the morphisms of the underlying abelian groups, endowed with the induced operations $t_A$. □

In the case of an additive monad $\mathbb{T}$, we modify our terminology and say “$\mathbb{T}$-modules” instead of “$\mathbb{T}$-algebras”. The abelian category of $\mathbb{T}$-algebras/modules is denoted by $\mathbb{T}_{\text{mod}}$ instead of $\mathbb{T}_{\text{alg}}$ in this case.

**Examples 1.2.** (1) Let $\mathcal{C}$ be a category with set-indexed coproducts, and let $M \in \mathcal{C}$ be an object. Then the functor $\mathbb{T}_M : \text{Sets} \to \text{Sets}$ assigning to every set $X$ the set $\mathbb{T}_M(X) = \text{Hom}_\mathcal{C}(M, M^{(X)})$ of all morphisms in $\mathcal{C}$ from $M$ into the coproduct $M^{(X)}$ of $X$ copies of $M$ in $\mathcal{C}$ a monad on the category of sets. The monad $\mathbb{T}_M$ can be viewed as a combinatorial datum encoding the category structure of the full subcategory formed by all the objects $M^{(X)} \in \mathcal{C}$.

Indeed, given two sets $Y$ and $X$, the set of all morphisms $M^{(Y)} \to M^{(X)}$ in $\mathcal{C}$ can be computed as the set $\mathbb{T}_M(X)^Y$. Suppose that we are given a map $f : Y \to \mathbb{T}_M(X)$; then the composition with the related morphism $M^{(Y)} \to M^{(X)}$ defines a map $\mathbb{T}_M(Y) \to \mathbb{T}_M(X)$. In particular, one can set $Y = \mathbb{T}_M(X)$ and take $f$ to be the identity map; then we obtain a natural map $\mathbb{T}_M(\mathbb{T}_M(X)) \to \mathbb{T}_M(X)$. This is our monad multiplication map $\phi_{\mathbb{T}_M}(X)$.

In a more fancy language, one explains the construction of the monad $\mathbb{T}_M$ as follows. The functor $F : \text{Sets} \to \mathcal{C}$ taking a set $X$ to the object $F(X) = M^{(X)}$ is left adjoint to the functor $G : \mathcal{C} \to \text{Sets}$ taking an object $N \in \mathcal{C}$ to the set $G(N) = \text{Hom}_\mathcal{C}(M, N)$. The functor $\mathbb{T}_M$ is the composition of these two adjoint functors $\mathbb{T}_M = GF : \text{Sets} \to \text{Sets}$; hence the monad structure. The full subcategory formed by all the objects $M^{(X)}$ in $\mathcal{C}$ can be recovered as the *Kleisli category* of the
monad \( T_M \), that is the full subcategory of all free \( T_M \)-algebras in \( T_M \)-\text{alg}. The free \( T_M \)-algebra \( T_M(X) \in T_M \)-\text{alg} corresponds to the object \( M^{(X)} \in C \). 

(2) Now let \( C \) be an additive category with set-indexed coproducts containing the images of idempotent endomorphisms of its objects, and let \( M \in C \) be an object. Then \( T_M \) is an additive monad on \( \text{Sets} \): the element \( "x+y" \in T_M(\{x,y\}) \) corresponds to the diagonal morphism \( M \rightarrow M \oplus M \), the element \( "-x" \in T_M(\{x\}) \) corresponds to the morphism \(-\id_M\): \( M \rightarrow M \), and the element \( 0 \in T_M(\emptyset) \) corresponds to the zero morphism \( M \rightarrow 0 \).

Denote by \( \text{Add}_C(M) \subset C \) the full subcategory of all direct summands of coproducts of copies of \( M \) in \( C \). Then \( \mathcal{B} = T_M \)-mod is the unique abelian category with enough projective objects such that the full subcategory \( \mathcal{B}_{\text{proj}} \subset \mathcal{B} \) of projective objects in \( \mathcal{B} \) is equivalent to the full subcategory \( \text{Add}_C(M) \subset C \) \cite[Theorem 1.1(a)]{36}.

Alternatively, the category \( \mathcal{B} \) can be constructed as the category of coherent functors on \( \text{Add}_C(M) \). One observes that the category \( \text{Add}_C(M) \) has weak kernels, hence the category of coherent functors on it is abelian \cite[Lemma 1]{19}. We refer to the introduction to \cite{32} for further references.

(3) In particular, let \( \mathcal{B} \) be an abelian category with set-indexed coproducts and a projective generator \( P \). Then \( \mathcal{B} \) is equivalent to the category of modules over the additive monad \( T_P: X \mapsto \Hom_B(P, P^{(X)}) \) on the category of sets. The equivalence is provided by the functor assigning to every object \( N \in \mathcal{B} \) the set of morphisms \( \Hom_B(P, N) \) with its natural \( T_P \)-module structure (see the discussion in the introduction to \cite{32} and a further discussion in \cite[Section 6.3]{35}).

Conversely, for any additive monad \( \mathcal{T}: \text{Sets} \rightarrow \text{Sets} \), the abelian category \( \mathcal{T} \)-mod has set-indexed coproducts, and the free \( \mathcal{T} \)-module with one generator \( P = \mathcal{T}(\ast) \) is a natural projective generator of \( \mathcal{T} \)-mod.

(4) Let \( R \) be an associative ring and \( \mathcal{B} \subset R \)-mod be a reflective full subcategory in the category of left \( R \)-modules. Denote the reflector (i. e., the functor left adjoint to the embedding \( \mathcal{B} \rightarrow R \)-mod) by \( \Delta: R \)-mod \rightarrow \mathcal{B} \). Then the category \( \mathcal{B} \) has small limits (which coincide with the limits in \( R \)-mod, as \( \mathcal{B} \subset R \)-mod is closed under limits) and colimits (which can be computed by applying the functor \( \Delta \) to the coproducts of object of \( \mathcal{B} \) taken in \( R \)-mod).

Suppose further that the full subcategory \( \mathcal{B} \) is closed under cokernels in \( R \)-mod. Then \( \mathcal{B} \) is an abelian category with set-indexed products and coproducts, and its embedding \( \mathcal{B} \rightarrow R \)-mod is an exact functor preserving products. Applying the functor \( \Delta \) to the free \( R \)-module with one generator \( R \), we obtain a natural projective generator \( P = \Delta(R) \) of the abelian category \( \mathcal{B} \). Thus the category \( \mathcal{B} \) is equivalent to the category of modules over the additive monad \( T_P: \text{Sets} \rightarrow \text{Sets} \).

Denoting by \( R[X] \) the left \( R \)-module freely generated by a set \( X \), one can compute the coproduct of \( X \) copies of \( P \) in \( \mathcal{B} \) as \( P^{(X)} = \Delta(R[X]) \). Hence the monad \( T_P \) assigns to a set \( X \) the underlying set of the abelian group/left \( R \)-module \( \Hom_B(P, P^{(X)}) = \Hom_R(\Delta(R), \Delta(R[X])) = \Hom_R(R, \Delta(R[X])) = \Delta(R[X]) \).
Let \( \mathcal{B} \) be an abelian category with set-indexed coproducts and a projective generator \( P \), and let \( \kappa \) be a regular cardinal. Then the category \( \mathcal{B} \) is locally \( \kappa \)-presentable and the object \( P \in \mathcal{B} \) is \( \kappa \)-presentable if and only if the monad \( T_P \) in Example \([\mathbb{1}2]3\) is \( \kappa \)-accessible, i.e., the functor \( T_P \) preserves \( \kappa \)-filtered colimits.

If the additive category \( \mathcal{C} \) in Example \([\mathbb{1}2]2\) is accessible and the object \( M \in \mathcal{C} \) is \( \kappa \)-presentable (or, more generally, \( \kappa \)-generated), then the monad \( T_M \) is \( \kappa \)-accessible. Assuming Vopěnka’s principle, any full subcategory closed under limits in \( R\text{-mod} \) is (reflective and) locally presentable \([\mathbb{1} \text{ Corollary 6.24}]\), so the monad \( T_P \) in Example \([\mathbb{1}2]4\) is always accessible.

Here are some examples of accessible additive monads on the category of sets.

**Examples 1.3.** (1) Let \( R \) be an associative ring. Then the functor \( T_R \) assigning to a set \( X \) the set \( R[X] \) of all finite formal linear combinations of elements of the set \( X \) with coefficients in \( R \) has a natural structure of monad on the category of sets.

The monad multiplication map \( \phi_{T_R}(X) : R[R[X]] \rightarrow R[X] \) opens the parentheses in a formal linear combination of formal linear combinations of elements of \( X \), assigning to it a formal linear combination of elements of \( X \) with the coefficients computed using the multiplication and addition operations in the ring \( R \). The monad unit map \( \epsilon_{T_R(X)} : X \rightarrow R[X] \) assigns to an element \( x \in X \) the formal linear combination into which the element \( x \) enters with the coefficient 1 and all the other elements of \( X \) enter with the coefficient 0.

The abelian category \( T_R\text{-mod} \) is the category of left \( R \)-modules \( R\text{-mod} \). The natural projective generator \( P \in T_R\text{-mod} \) is the free left \( R \)-module with one generator, \( P = R \).

(2) Let \( \mathcal{R} \) be a separated and complete topological associative ring with a base of neighborhoods of zero formed by open right ideals. For any set \( X \), denote by \( \mathcal{R}[[X]] \) the set of all infinite formal linear combinations of elements of \( X \) with the coefficients converging to 0 in the topology of \( \mathcal{R} \). This means that an expression \( \sum_{x \in X} r_x x \), \( r_x \in \mathcal{R} \), belongs to \( \mathcal{R}[[X]] \) if and only for any open subset \( U \subset \mathcal{R} \), \( 0 \not\in U \), the set of all \( x \in X \) for which \( r_x \) does not belong to \( U \) is finite. In other words, we have \( \mathcal{R}[[X]] = \varprojlim \mathcal{R}/J[X] \), where the projective limit is taken over all the open right ideals \( J \subset \mathcal{R} \) and, for any abelian group \( A \) and a set \( X \), the notation \( A[X] \) stands for the direct sum of \( X \) copies of \( A \).

Then, for any map of sets \( f : X \rightarrow Y \), the induced map \( \mathcal{R}[[f]] : \mathcal{R}[[X]] \rightarrow \mathcal{R}[[Y]] \) is constructed by computing the infinite sums \( \sum_{x, f(x) = y} r_x \) for all \( y \in Y \) as the limits of finite partial sums in the topology of \( \mathcal{R} \). Furthermore, the functor \( T_{\mathcal{R}} : X \mapsto \mathcal{R}[[X]] \) has a natural structure of monad on the category of sets. The monad multiplication map \( \phi_{T_{\mathcal{R}}}(X) : \mathcal{R}[[\mathcal{R}[[X]]]] \rightarrow \mathcal{R}[[X]] \) opens the parentheses, computing the coefficients using the multiplication operation in the ring \( \mathcal{R} \) and the infinite summation in the sense of the passage to the limit of finite partial sums. The above condition on the topology of \( \mathcal{R} \) ensures the convergence \([\mathbb{1}2 \text{ Section 5}]\).

The monad \( T_{\mathcal{R}} \) is additive and \( \lambda^{+} \)-accessible, where \( \lambda^{+} \) is the successor cardinal of the cardinality of a base of neighborhoods of zero in \( \mathcal{R} \). Modules over the monad \( T_{\mathcal{R}} \) are called **left contramodules** over the topological ring \( \mathcal{R} \), and the abelian category
of left $\mathcal{R}$-contramodules is denoted by $T_{\mathcal{R}} \text{- mod} = \mathcal{R} \text{- contra}$. So the abelian category $\mathcal{R} \text{- contra}$ is locally $\lambda^+$-presentable with a natural $\lambda^+$-presentable projective generator, which is the free left $\mathcal{R}$-contramodule with one generator $P = \mathcal{R} = \mathcal{R}[[x]]$.

The category $\mathcal{R} \text{- contra}$ has the additional property that, for every family of projective objects $P_\alpha \in \mathcal{R} \text{- contra}$, the natural morphism $\prod_\alpha P_\alpha \rightarrow \prod_\alpha P_\alpha$ is a monomorphism [32, Section 1.2].

(3) Let $R$ be an associative ring and $M$ be a left $R$-module. Denote by $\mathcal{S} = \text{Hom}_R(M, M)^{\text{op}}$ opposite ring to the ring of endomorphisms of the left $R$-module $M$ (so that the ring $\mathcal{S}$ acts in $M$ on the right). The ring $\mathcal{S}$ is naturally endowed with a complete, separated topological ring structure in which the base of neighborhoods of zero is formed by the annihilators of finitely-generated submodules in $M$. Such annihilators are left ideals in $\text{Hom}_R(M, M)$ and right ideals in $\mathcal{S}$; so open right ideals form a base of neighborhoods of zero in $\mathcal{S}$.

One readily checks that the monad $T_M$ from Examples 1.2(1-2) is naturally isomorphic to the monad $T_{\mathcal{S}}$ from (2) in this case [35, proof of Theorem 7.1]. Thus the abelian category $\mathcal{B}$ from Example 1.2(2) corresponding to an object $M$ of the category of modules over an associative ring $R$ is nothing but the category of left contramodules $\mathcal{S} \text{- contra}$ over the topological associative ring $\mathcal{S} = \text{Hom}_R(M, M)^{\text{op}}$.

Similar results can be obtained for many abelian or additive categories $\mathcal{C}$ other than the category of modules over an associative ring $R$ (including, in particular, locally finitely presentable Grothendieck abelian categories, the categories of comodules and semimodules, etc.) [35, Sections 9–10].

(4) Let $R$ be an associative ring and $f : U^{-1} \rightarrow U^0$ be a morphism of free left $R$-modules. Denote by $\mathcal{B} = f^\perp$ the full subcategory in $R \text{- mod}$ formed by all the left $R$-modules $B$ for which $\text{Hom}_R(f, B) : \text{Hom}_R(U^0, B) \rightarrow \text{Hom}_R(U^{-1}, B)$ is an isomorphism. Then the full subcategory $\mathcal{B}$ is closed under the kernels, cokernels, extensions, and infinite products in $R \text{- mod}$ [28, Remark 1.3]. So $\mathcal{B}$ is an abelian category and the embedding functor $\mathcal{B} \rightarrow R \text{- mod}$ is exact. Furthermore, if $\lambda^+$ is the successor cardinal of the supremum of the cardinalities of the sets of free generators of $U^{-1}$ and $U^0$, then the full subcategory $\mathcal{B} \subset R \text{- mod}$ is closed under $\lambda^+$-filtered colimits.

According to [11, Theorem and Corollary 2.48], it follows that $\mathcal{B}$ is a locally $\lambda^+$-presentable category and a reflective subcategory in $R \text{- mod}$. Denoting the reflector by $\Delta : R \text{- mod} \rightarrow \mathcal{B}$, the object $P = \Delta(R)$ is a $\lambda^+$-presentable projective generator of the abelian category $\mathcal{B}$ (see Example 1.2(4) above, cf. [32, Examples 4.1]). Thus $\mathcal{B}$ is equivalent to the category of modules over a certain $\lambda^+$-accessible additive monad $T_f : \text{Sets} \rightarrow \text{Sets}$.

The monad $T_f$ can be described more explicitly in terms of generators and relations in the following way. Let $V$ and $W$ denote the sets of free generators of the $R$-modules $U^0$ and $U^{-1}$, respectively. For every element $v \in V$, denote also by $v$ the related $R$-module morphism $R \rightarrow U^0$. For any object $B \in \mathcal{B}$, consider the map of abelian groups $\text{Hom}_R(U^{-1}, B) \rightarrow \text{Hom}_R(U^0, B)$ inverse to the isomorphism $\text{Hom}_R(f, B)$, and compose it with the coordinate projection map $\text{Hom}_R(v, B) : \text{Hom}_R(U^0, B) \rightarrow$
B. The resulting composition $t_v : B^W = \text{Hom}_R(U^{-1}, B) \rightarrow B$ is a $W$-ary operation in the set $B$, so $t_v \in T_f(W)$. Together with the operations of addition and multiplication by the elements of $R$, the operations $t_v$, $v \in V$ generate the monad $T_f$. The defining relations between these generating operations are the defining relations for the monad $T_f$ (describing the notion of an $R$-module) together with the relations guaranteeing that the map of sets $(t_v)_{v \in V} : \text{Hom}_R(U^{-1}, B) \rightarrow \text{Hom}_R(U^0, B)$ is inverse to the map $\text{Hom}_R(f, B)$ for any $T_f$-module $B$.

(5) Let $R$ be a commutative ring and $s \in R$ be an element. Denote by $E$ the ring $R[s^{-1}]$, and let $f : U^{-1} \rightarrow U^0$ be a free $R$-module resolution of the $R$-module $E$ [28, proof of Lemma 2.1]. The objects of the full subcategory $B = f^\perp \subset R\text{-mod}$ are called $s$-contramodule $R$-modules [27, 28]. The reflector $\Delta = \Delta_s : R\text{-mod} \rightarrow B$ is described in [27, Proposition 2.1] and [28, Theorem 6.4].

In particular, $P = \Delta_s(R)$ is a projective generator of the abelian category $B$. For any set $X$, the coproduct of $X$ copies of the object $P$ in $B$ can be computed as $P^{(X)} = \Delta_s(R[X])$ (while the infinite products of any objects of $B$ coincide with those computed in $R\text{-mod}$).

Furthermore, for any left $R$-module $M$, there is a natural short exact sequence of left $R$-modules [28, Lemma 6.7]

$$0 \rightarrow \lim_{\leftarrow n} s^n M \rightarrow \Delta_s(M) \rightarrow \lim_{\rightarrow n} M/s^n M \rightarrow 0,$$

where $s^n M \subset M$ denotes the submodule of all elements in $M$ annihilated by $s^n$, the maps in the projective system $sM \leftarrow s^2 M \leftarrow s^3 M \leftarrow \cdots$ are the operators of multiplication with $s$, and the maps in the projective system $M/sM \leftarrow M/s^2 M \leftarrow M/s^3 M \leftarrow \cdots$ are the usual projections.

(6) The following example shows that some of the abelian categories $B = f^\perp$ as in (4) differ from all the abelian categories of contramodules over topological rings $\mathcal{R}$-contra as in (2).

Let $k$ be a field and $R$ be a commutative $k$-algebra generated by an infinite sequence of elements $x_1, x_2, x_3, \ldots$ and an additional generator $s$ with the relations $x_i x_j = 0$ for all $i, j \geq 1$ and $s^i x_i = 0$ for all $i \geq 1$ [27, Example 2.6]. Let $f : U^{-1} \rightarrow U^0$ be a free $R$-module resolution of the ring $R[s^{-1}] \simeq k[s, s^{-1}]$ viewed as an $R$-module, as in (4). Set $B = f^\perp$ to be the abelian category of $s$-contramodule $R$-modules and $P = \Delta_s(R)$ to be its natural projective generator.

Then, for any set $Y$, one computes, in the spirit of the computation in [27, Example 2.6], that the $R$-module $\lim_{\rightarrow n} s^n R[Y]$ is isomorphic to $W_Y[[s]]$, where $W_Y = \prod_{i=1}^\infty k[Y]/\bigoplus_{i=1}^\infty k[Y]$. In particular, the natural map $\lim_{\rightarrow n} s^n R[Y] \rightarrow (\lim_{\rightarrow n} s^n R)^Y$ is not injective when the set $Y$ is infinite, and it follows that the natural map $P^{(Y)} \rightarrow P^Y$ from the coproduct of $Y$ copies of $P$ in $B$ to the product of the same copies is not injective, either (hence not a monomorphism).

(7) On the other hand, in the context of (5), assume that the $s$-torsion in $R$ is bounded (that is, there exists $m \geq 1$ such that $s^n r = 0$ for $r \in R$ and $n \geq 1$ implies $s^m r = 0$). Denote by $\mathcal{R} = \lim_{\leftarrow n} R/s^n R$ the $s$-adic completion of the ring $R$, endowed
with the \( s \)-adic ( = projective limit) topology. Then the abelian category \( B = f^\perp \) is equivalent to the abelian category \( \mathcal{R} \text{-} \text{contra} \), as we will see below in Example 2.2(5).

2. EXAMPLES OF FULLY FAITHFUL CONTRAMODULE FORGETFUL FUNCTORS

The aim of this section is to generalize the results of [24, Theorem B.1.1] and [25, Theorem C.5.1], claiming that the natural forgetful functors from the categories of contramodules to related categories of modules are fully faithful under certain assumptions. We leave aside the result of [30, Theorem 2.1], postponing the job of its generalization to the next Section 3.

Let \( \Delta: \text{Sets} \rightarrow \text{Sets} \) be an additive monad. Then the underlying set of the free \( T \)-module with one generator \( T(*) \) has a natural associative ring structure. This is the ring of unary operations in the monad \( T \), which can be also defined as the opposite ring \( T(*) = \text{Hom}_{T}(T(*), T(*))^\text{op} \) to the ring of endomorphisms of the free \( T \)-module with one generator \( T(*) \in T\text{-mod} \). Let \( R \) be another associative ring and \( \theta: R \rightarrow T(*) \) an associative ring homomorphism. Then the natural bijection \( C \cong \text{Hom}_{T}(T(*), C) \) endows the underlying set of every \( T \)-module \( C \) with a left \( R \)-module structure, providing a forgetful functor \( T\text{-mod} \rightarrow R\text{-mod} \).

Let \( B \subset R\text{-mod} \) be a reflective full subcategory closed under cokernels in \( R\text{-mod} \). Let \( \Delta: R\text{-mod} \rightarrow B \) denote the reflector, and set \( P = \Delta(R) \). Then \( P \) is a projective generator of the abelian category \( B \), and the coproduct of \( X \) copies of \( P \) in \( B \) can be computed as \( P^{(X)} = \Delta(R[X]) \) (see Example 1.2(4)).

The following proposition provides a sufficient condition (in fact, a criterion) for existence of an isomorphism of monads \( T \cong T_{P} \).

**Proposition 2.1.** Suppose that for every set \( X \) there is an isomorphism of left \( R \)-modules \( \Delta(R[X]) \cong T(X) \) forming a commutative diagram with the adjunction morphism \( R[X] \rightarrow \Delta(R[X]) \) and the natural \( R \)-module morphism

\[
R[X] \rightarrow \text{Hom}_{T}(T(*), T(*))[X] \rightarrow \text{Hom}_{T}(T(*), T(X)) = T(X).
\]

Then the forgetful functor \( T\text{-mod} \rightarrow R\text{-mod} \) is fully faithful and its essential image coincides with the full subcategory \( B \subset R\text{-mod} \). So there is an equivalence of abelian categories \( T\text{-mod} \cong B \) identifying the free \( T \)-module \( T(X) \) with the projective object \( P^{(X)} \in B \) for every set \( X \).

Conversely, if the forgetful functor \( T\text{-mod} \rightarrow R\text{-mod} \) is fully faithful and its essential image coincides with the full subcategory \( B \subset R\text{-mod} \), then for every set \( X \) there is a natural isomorphism of left \( R \)-modules \( \Delta(R[X]) \cong T(X) \) forming a commutative diagram with the adjunction morphism \( R[X] \rightarrow \Delta(R[X]) \) and the natural morphism \( R[X] \rightarrow T(X) \).

**Proof.** The forgetful functor \( T\text{-mod} \rightarrow R\text{-mod} \) is exact by construction, and in particular, it preserves cokernels. As every \( T \)-module is the cokernel of a morphism of free \( T \)-modules \( T(X) \rightarrow T(Y) \) for some sets \( X \) and \( Y \), and the cokernel of any
$R$-module morphism $P^{(X)} \to P^{(Y)}$ belongs to $B$, it follows that the image of the forgetful functor is contained in the full subcategory $B \subset R\text{-mod}$.

Furthermore, for any sets $X$ and $Y$ one has

$$\text{Hom}_T(T(X), T(Y)) \simeq \text{Hom}_R(R[X], T(Y)) \simeq \text{Hom}_R(R[X], P^{(Y)}) \simeq \text{Hom}_R(P^{(X)}, P^{(Y)}),$$

so the forgetful functor is fully faithful in restriction to the full subcategory of free $T$-modules in $T\text{-mod}$. Commutativity of the diagram involved in this computation can be checked using the assumption of commutativity of the diagram in the formulation of the proposition. It follows easily that the forgetful functor is fully faithful on the whole abelian category of $T$-modules.

Finally, it remains to notice that every object of $B$ is the cokernel of a morphism $P^{(X)} \to P^{(Y)}$ for some sets $X$ and $Y$, since $P$ is a projective generator on $B$. Hence every object of $B$ belongs to the essential image of the forgetful functor.

Alternatively, one can argue in the following way. Suppose that we already know (e.g., from the first paragraph of this proof) that the image of the forgetful functor $T\text{-mod} \to R\text{-mod}$ is contained in the full subcategory $B \subset R\text{-mod}$. Then, by the adjunction property of the functor $\Delta$, for every set $X$ there exists a unique left $R$-module morphism $\Delta(R[X]) \to T(X)$ forming a commutative triangle diagram with the morphisms $R[X] \to \Delta(R[X])$ and $R[X] \to T(X)$.

According to Example 1.2(4), the category $B$ is equivalent to the category of modules over the monad $T_B = T_P : X \mapsto \Delta(R[X])$. The functor $T\text{-mod} \to B \simeq T_B\text{-mod}$ is a functor between two categories of modules over monads on the category of sets, and it forms a commutative triangle diagram of functors with the forgetful functors $T\text{-mod} \to \text{Sets}$ and $T_B\text{-mod} \to \text{Sets}$. Hence our functor is induced by a morphism of monads $T_B \to T$, which is provided by the above construction. Thus the functor $T\text{-mod} \to B$ is an equivalence of categories whenever the map $\Delta(R[X]) \to T(X)$ is bijective for every set $X$.

Conversely, if the forgetful functor is an equivalence of categories $T\text{-mod} \simeq B$, then this equivalence of categories also identifies the forgetful functors to the category of sets, $T\text{-mod} \to \text{Sets}$ and $B \to R\text{-mod} \to \text{Sets}$. Thus it also forms a commutative diagram with the functors adjoint to the forgetful functors to sets, $X \mapsto T(X) : \text{Sets} \to T\text{-mod}$ and $X \mapsto \Delta(R[X]) : \text{Sets} \to B$. Hence a natural isomorphism of left $R$-modules $\Delta(R[X]) \simeq T(X)$ forming a commutative diagram with the adjunction maps of sets $X \to \Delta(R[X])$ and $X \to T(X)$, and consequently also with the morphisms of left $R$-modules $R[X] \to \Delta(R[X])$ and $R[X] \to T(X)$. $\square$

The observation that (what was then called) an $\text{Ext}_p\text{-complete}$ or $\text{weakly }p\text{-complete}$ abelian group carries a uniquely defined structure of a module over the ring of $p$-adic integers $\mathbb{Z}_p$, and these $\mathbb{Z}_p$-module structures are preserved by any abelian group homomorphisms between such groups, goes back, at least, to $[17, \text{Lemma 4.13}]$. The following series of examples shows how far it can be generalized using the contemporary techniques.
Examples 2.2. (1) Let $R$ be a commutative ring and $I \subset R$ be the ideal generated by a finite set of elements $s_1, \ldots, s_m \in R$. An $R$-module $C$ is said to be an $I$-contramodule if $\text{Hom}_R(R[s_j^{-1}], C) \cong 0 = \text{Ext}_R^1(R[s_j^{-1}], C)$ for all $1 \leq j \leq m$. This property does not depend on a chosen set of generators $s_1, \ldots, s_m$ of an ideal $I \subset R$, but only on the ideal $I$ itself \[28\] Proposition 5.1].

Let $E$ denote the $R$-module $\bigoplus_{j=1}^m R[s_j^{-1}]$, and let $f: U^{-1} \to U^0$ be a two-term free resolution of the $R$-module $E$. Then the full subcategory of $I$-contramodule $R$-modules $R$-mod_{I,ctra} \subset R$-mod coincides with the full subcategory $B = f_{\perp} \subset R$-mod discussed in Example 1.3(4). In particular, the embedding functor $R$-mod_{I,ctra} \to R$-mod has a left adjoint functor $\Delta_I: R$-mod \to $R$-mod_{I,ctra} described in \[27\] Proposition 2.1 and \[28\] Theorem 7.2]. The abelian category $R$-mod_{I,ctra} is equivalent to the category of modules over the additive, $N_j$-accessible monad $\mathcal{T}_{R,I} = \mathcal{T}_f$ on the category of sets, assigning to every set $X$ the underlying set of the $R$-module $\mathcal{T}_{R,I}(X) = \Delta_I(R[X])$.

(2) Let $R$ be a commutative ring and $I \subset R$ be a finitely generated ideal. Denote by $\mathfrak{R} = \varprojlim R/I^n$ the $I$-adic completion of the ring $R$ endowed with the $I$-adic (projective limit) topology. Consider the abelian category of $\mathfrak{R}$-contramodules $\mathfrak{R}$-contra defined in Example 1.3(2). Then the image of the forgetful functor $\mathfrak{R}$-contra \to $R$-mod is contained in the full subcategory of $I$-contramodule $R$-modules $R$-mod_{I,ctra} \subset R$-mod discussed in (1).

Indeed, it suffices to check that the free $\mathfrak{R}$-contramodules are $I$-contramodule $R$-modules, as every $\mathfrak{R}$-contramodule is the cokernel of a morphism of free $\mathfrak{R}$-contramodules. Now, for any set $X$, the free $\mathfrak{R}$-contramodule $\mathfrak{R}[X] = \varprojlim R/I^n[X]$ is an $I$-adically separated and complete $R$-module, and all such $R$-modules are $I$-contramodules \[28\] Lemma 5.7].

Furthermore, it follows from an appropriate generalization of \[30\] Theorem 2.1] that the forgetful functor $\mathfrak{R}$-contra \to $R$-mod is fully faithful for any finitely-generated ideal $I$ in a commutative ring $R$ (see Examples 3.6(2-3) below). Thus the abelian category $\mathfrak{R}$-contra is a full subcategory in the abelian category $R$-mod_{I,ctra}.

(3) In the same context, denote by $\Lambda_I$ the $I$-adic completion functor assigning to an $R$-module $M$ the $R$-module $\Lambda_I(M) = \varprojlim M/I^n M$. Then for any $R$-module $M$ there exists a unique $R$-module morphism $\Delta_I(M) \to \Lambda_I(M)$ forming a commutative diagram with the natural morphisms $M \to \Delta_I(M)$ and $M \to \Lambda_I(M)$. According to Proposition 2.1] we can conclude that the forgetful functor $\mathfrak{R}$-contra \to $R$-mod_{I,ctra} is an equivalence of abelian categories if and only if the natural morphism $\Delta_I(R[X]) \to \Lambda_I(R[X])$ is an isomorphism for every set $X$.

By \[27\] Lemma 2.5], the latter condition holds when $I \subset R$ is a weakly proregular ideal in the sense of \[37\] 22]. Hence the forgetful functor $\mathfrak{R}$-contra \to $R$-mod_{I,ctra} is an equivalence of categories for any weakly proregular finitely generated ideal $I$ in a commutative ring $R$. Moreover, it suffices to require the weak proregularity condition (the homology of the Koszul/telescope complexes forming a pro-zero projective system) to hold in the homological degree 1, i.e., for the modules $H_1(\text{Hom}_R(T^n_s(R; s_1, \ldots, s_m, R))$. Indeed, the
natural morphism $\Delta_f(R[X]) \to \Lambda_f(R[X])$ is always surjective with the kernel $$\lim_{n \to \infty} H_1(\text{Hom}_R(T^n(\mathbb{Z}/p^n(R; s_1, \ldots, s_m), R[X])))$$ \cite[Lemma 7.5]{28}. (See Example 5.2(6) below for further details.)

(4) In particular, all ideals in a Noetherian commutative ring are weakly proregular. Thus we have obtained a new proof, based on Proposition \cite[Theorem 3.1.1]{27}, of the result of \cite[Theorem 3.3(c)]{28}, according to which the forgetful functor $\mathcal{R}\text{-contra} \to R\text{-mod}$ is fully faithful and its image coincides with the full subcategory $R\text{-mod}_{I,\text{contra}} \subset R\text{-mod}$ for any ideal $I$ in a Noetherian commutative ring $R$.

(5) In the special case of a principal ideal $I = (s)$ in a commutative ring $R$, the weak proregularity condition mentioned in (3) means that the $s$-torsion in $R$ is bounded. Thus the forgetful functor $\mathcal{R}\text{-contra} \to R\text{-mod}_{I,\text{contra}}$ is an equivalence of abelian categories in this case (cf. Example 1.3(7) above and Example 5.2(7) below).

Examples 2.3. (1) Let $R$ be an associative ring and $s \in R$ be a central element. A left $R$-module $C$ is said to be an $s$-\textit{contramodule} if $\text{Hom}_R(R[s^{-1}], C) = 0 = \text{Ext}^1_R(R[s^{-1}], C)$ (notice that $R[s^{-1}]$ is a left $R$-module of projective dimension $\leq 1$, so all the higher Ext groups vanish automatically). More generally, for an arbitrary element $s \in R$, one can say that a left $R$-module $C$ is an $s$-contramodule if, viewed as a module over the polynomial ring $\mathbb{Z}[s]$, it is an $s$-contramodule in the sense of Example 2.2(1) and \cite[Section 2]{28}, that is, $\text{Hom}_{\mathbb{Z}[s]}(\mathbb{Z}[s, s^{-1}], C) = 0 = \text{Ext}^1_{\mathbb{Z}[s]}(\mathbb{Z}[s, s^{-1}], C)$.

Let $s_1, \ldots, s_m \in R$ be a finite set of central elements in $R$, and let $s = \sum_{i=1}^m s_m x_m$ be an element of the ideal $I \subset R$ generated by $s_1, \ldots, s_m$. Then any left $\mathcal{R}$-module $C$ that is an $s_j$-contramodule for every $j = 1, \ldots, m$, is also an $s$-contramodule. One can prove this assertion by constructing the $s$-power infinite summation operation in $C$ \cite[Section 3]{28} by the rule

$$\sum_{n=0}^{\infty} s^n c_n = \sum_{n_1=0}^{\infty} s_1^{n_1} \cdots \sum_{n_m=0}^{\infty} s_m^{n_m} (p_{n_1, \ldots, n_m}(x_1, \ldots, x_m)c_{n_1+\cdots+n_m}),$$

for any elements $c_0, c_1, c_2, \ldots \in C$, where $p_{n_1, \ldots, n_m}$ is the appropriate noncommutative polynomial of polydegree $(n_1, \ldots, n_m)$ in the variables $x_1, \ldots, x_m$. Then it remains to apply \cite[Theorem 3.3(c)]{28} (cf. \cite[first proof of Theorem 5.1]{28}).

It follows that the property of a left $R$-module $C$ to be an $s_j$-contramodule for every element $s_j$ of a given set of central generators of an ideal $I \subset R$ depends only on the centrally generated ideal $I$ and not on the chosen set of its central generators. A left $R$-module $C$ with this property is called an $I$-\textit{contramodule}. The full subcategory of $I$-contramodule left $R$-modules $R\text{-mod}_{I,\text{contra}} \subset R\text{-mod}$ is closed under the kernels, cokernels, extensions, and infinite products in $R\text{-mod}$. So, in particular $R\text{-mod}_{I,\text{contra}}$ is an abelian category and its embedding $R\text{-mod}_{I,\text{contra}} \to R\text{-mod}$ is an exact functor.

(2) Let $R$ be an associative ring and $I \subset R$ be the ideal generated by a finite set of central elements $s_1, \ldots, s_m \in R$. As in Example 2.2(1), denote by $E$ the left $R$-module $\bigoplus_{j=1}^m R[s_j^{-1}]$, and let $f: U^{-1} \to U^0$ be a two-term free resolution of the left $R$-module $E$. Then the full subcategory of $I$-contramodule $R$-modules $R\text{-mod}_{I,\text{contra}} \subset R\text{-mod}$ coincides with the full subcategory $B = f^\perp \subset R\text{-mod}$ from Example 1.3(4). In particular, the embedding functor $R\text{-mod}_{I,\text{contra}} \to R\text{-mod}$ has a
left adjoint functor $\Delta_I: R\text{-}\text{mod} \to R\text{-}\text{mod}_{I\text{-ctra}}$. The abelian category $R\text{-}\text{mod}_{I\text{-ctra}}$ is equivalent to the category of modules over the additive, $\aleph_1$-accessible monad $\mathbb{T}_{R,I} = \mathbb{T}_I: \text{Sets} \to \text{Sets}$, assigning to every set $X$ the underlying set of the left $R$-module $\mathbb{T}_{R,I}(X) = \Delta_I(R[X])$.

(3) Let $R$ be an associative ring and $I \subset R$ be an ideal generated by a finite set of central elements. As in Example 2.2(2), denote by $\mathfrak{R} = \varprojlim_n R/I^n$ the $I$-adic completion of the ring $R$ endowed with the $I$-adic (= projective limit) topology. Consider the abelian category of left $\mathfrak{R}$-contramodules $\mathfrak{R}\text{-}\text{contra}$ defined in Example 1.3(2). Then the forgetful functor $\mathfrak{R}\text{-}\text{contra} \to R\text{-}\text{mod}$ is fully faithful (see Examples 3.6(2-3) below), and its image is contained in the full subcategory of $I$-contramodule left $R$-modules $R\text{-}\text{mod}_{I\text{-contra}} \subset R\text{-}\text{mod}$. Thus the abelian category $\mathfrak{R}\text{-}\text{contra}$ is a full subcategory in the abelian category $R\text{-}\text{mod}_{I\text{-contra}}$.

As in Example 2.2(3), denote by $\Lambda_I$ the $I$-adic completion functor $M \mapsto \Lambda_I(M) = \varprojlim_n M/I^nM$. Then for any left $R$-module $M$ the left $R$-module $\Lambda_I(M)$ is an $I$-contramodule, hence there exists a unique left $R$-module morphism $\Delta_I(M) \to \Lambda_I(M)$ forming a commutative diagram with the natural morphisms $M \to \Delta_I(M)$ and $M \to \Lambda_I(M)$. By Proposition 2.1, it follows that the forgetful functor $\mathfrak{R}\text{-}\text{contra} \to R\text{-}\text{mod}_{I\text{-contra}}$ is an equivalence of categories if and only if the natural morphism $\Delta_I(R[X]) \to \Lambda_I(R[X])$ is an isomorphism for every set $X$.

By [28, Lemma 7.5], the morphism $\Delta_I(R[X]) \to \Lambda_I(R[X])$ is surjective with the kernel $\varprojlim_n H_1(\text{Hom}_R(T_n^*(R; s_1, \ldots, s_m), R[X]))$, where $s_1, \ldots, s_m$ is some finite set of central generators of the ideal $I \subset R$. Hence the morphism in question is an isomorphism whenever the $R$-$R$-bimodules $H_1(\text{Hom}_R(T_n(R; s_1, \ldots, s_m), R))$, $n \geq 1$, form a pro-zero projective system. In particular, for a principal centrally generated ideal $I = (s)$, where $s$ is a central element in $R$, the latter condition means that the $s$-torsion in $R$ should be bounded. (See Example 5.3(4) below for further details.)

(4) Arguing as in [27, Lemma 2.4] or [22, Theorem 4.24] (see also [27, Lemma 2.3]), one shows that the projective system $H_1(\text{Hom}_R(T_n(R; s_1, \ldots, s_m), R))$ is pro-zero if and only if for every injective left $R$-module $J$ the augmented Čech complex $\check{C}_s^*(J)^\sim$ has no cohomology in cohomological degree 1, or equivalently, for every injective right $R$-module $J$ the complex $\check{C}_s^*(J)^\sim$ has no cohomology in degree 1 (where $s$ is a shorthand notation for the sequence $s_1, \ldots, s_m$).

Set $X = \text{Spec} S$, where $S$ is the subring in $R$ generated by $s_1, \ldots, s_m$ over $\mathbb{Z}$. Suppose that the ring $R$ is left Noetherian. Then $X$ is a Noetherian affine scheme and the quasi-coherent algebra $\mathcal{R}$ over $X$ corresponding to the $S$-algebra $R$ is (left) Noetherian in the sense of [6, Section 1.2]. Let $\mathcal{J}$ be the quasi-coherent left $\mathcal{R}$-module associated with an injective left $R$-module $J$. Denote by $Z$ the closed subscheme in $X$ defined by the equations $\{s_1 = 0, \ldots, s_m = 0\}$, and set $U = X \setminus Z$. According to [6, Theorem A.3], the quasi-coherent left $\mathcal{R}|_U$-module $\mathcal{J}|_U$ is injective and $\mathcal{J}$ is flasque sheaf of abelian groups on $X$. Arguing as in [27, Section 1], one can conclude that $H^i(\check{C}_s^*(J)^\sim) = 0$ for all $i > 0$. 

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Thus the forgetful functor $\mathcal{R} \text{-contra} \rightarrow R \text{-mod}_{S\text{-contra}}$ is an equivalence of abelian categories whenever the ring $R$ is either left or right Noetherian. We have obtained a new proof, based on Proposition 2.1, of the result of [25, Theorem C.5.1].

Given an object $N$ in an abelian category $A$, we denote by $\text{pd}_A N \in \{-\infty\} \cup \mathbb{Z}_{\geq 0} \cup \{+\infty\}$ the projective dimension of the object $N \in A$. The projective dimension of a left module $N$ over an associative ring $R$ is denoted by $\text{pd}_R N$.

**Examples 2.4.** (1) Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset. Assume that the projective dimension of the $R$-module $S^{-1}R$ does not exceed 1. An $R$-module $C$ is said to be an $S$-contramodule if $\text{Hom}_R(S^{-1}R, C) = 0 = \text{Ext}^1_R(S^{-1}R, C)$ [29, Section 1].

Let $f: U^{-1} \rightarrow U^0$ be a two-term free resolution of the $R$-module $S^{-1}R$. Then the full subcategory of $S$-contramodule $R$-modules $R\text{-mod}_{S\text{-contra}} \subset R\text{-mod}$ coincides with the full subcategory $B = f^+ \subset R\text{-mod}$ discussed in Example [14,4]. In particular, $R\text{-mod}_{S\text{-contra}}$ is an abelian category with an exact embedding functor $R\text{-mod}_{S\text{-contra}} \rightarrow R\text{-mod}$, which has a left adjoint functor $\Delta_S: R\text{-mod} \rightarrow R\text{-mod}_{S\text{-contra}}$.

The latter functor can be computed as $\Delta_S(M) = \text{Ext}^1_R(K^\bullet, M) = \text{Hom}_{D\text{-mod}}(K^\bullet, M[1])$ for every $R$-module $M$, where $K^\bullet$ denotes the two-term complex $R \rightarrow S^{-1}R$ with the term $R$ placed in the cohomological degree 0 and the term $S^{-1}R$ placed in the cohomological degree 1 [29, Theorem 3.4]. The abelian category $R\text{-mod}_{S\text{-contra}}$ is equivalent to the category of modules over the additive, accessible monad $T_{R,S} = T_f$ on the category of sets, assigning to every set $X$ the underlying set of the $R$-module $T_{R,S}(X) = \Delta_S(R[X])$.

(2) Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset. Denote by $\mathcal{R} = \lim_{S \in S} R/sR$ the $S$-completion of the ring $R$, endowed with the projective limit topology [29, Section 2]. Then $\mathcal{R}$ is a complete, separated topological commutative ring with open ideals forming a base of neighborhoods of zero. So we can consider the abelian category of $\mathcal{R}$-contramodules $\mathcal{R} \text{-contra}$ defined in Example [14,3](2). The conditions under which the forgetful functor $\mathcal{R} \text{-contra} \rightarrow R\text{-mod}$ is fully faithful are discussed in Examples [13,7] below.

Assume that $\text{pd}_R S^{-1}R \leq 1$. Then the image of the forgetful functor $\mathcal{R} \text{-contra} \rightarrow R\text{-mod}$ is contained in the full subcategory of $S$-contramodule $R$-modules $R\text{-mod}_{S\text{-contra}} \subset R\text{-mod}$ discussed in (1). Indeed, it suffices to check that the free $\mathcal{R}$-contramodules are $S$-contramodule $R$-modules, as every $\mathcal{R}$-contramodule is the cokernel of a morphism of free $\mathcal{R}$-contramodules. Now, for any set $X$, the free $\mathcal{R}$-contramodule $\mathcal{R}[[X]] = \lim_{S \in S} R/sR[X]$ is an $S$-contramodule $R$-module [29, Lemma 2.1(a)].

(3) Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset such that $\text{pd}_R S^{-1}R \leq 1$. Denote by $\Lambda_S$ the $S$-completion functor assigning to every $R$-module $M$ the $R$-module $\Lambda_S(M) = \lim_{s \in S} M/sM$. Then for any $R$-module $M$ there exists a unique $R$-module morphism $\Delta_S(M) \rightarrow \Lambda_S(M)$ forming a commutative diagram with the natural morphisms $M \rightarrow \Delta_S(M)$ and $M \rightarrow \Lambda_S(M)$ [29, Lemma 2.1(b)]. Applying Proposition 2.1 we can conclude that the forgetful functor $\mathcal{R} \text{-contra} \rightarrow$
$R\text{-mod}_{S,\text{contra}}$ is an equivalence of abelian categories if and only if the natural morphism $\Delta_S(R[X]) \to \Lambda_S(R[X])$ is an isomorphism for every set $X$.

According to [29] Theorem 2.5(c) or Corollary 2.7, the latter condition holds when the $S$-torsion in the ring $R$ is bounded (i.e., there exists an element $t \in S$ such that $sr = 0$, $s \in S$, $r \in R$ implies $tr = 0$). Hence the forgetful functor $\mathcal{R}\text{-contra} \to R\text{-mod}_{S,\text{contra}}$ is an equivalence of categories for any commutative ring $S$ with a multiplicative subset $S$ such that $\text{pd}_R S^{-1}R \leq 1$ and the $S$-torsion in $R$ is bounded. Furthermore, all the $S$-contramodule $R$-modules are $\Delta_S(R)$-modules, so it suffices to require the $S$-torsion in $\Delta_S(R)$, rather than in $R$, to be bounded (see Examples 5.4 for further details.)

**Examples 2.5.** (1) Let $R$ be an associative ring and $S \subset R$ be a multiplicative subset consisting of central elements. Assume that the projective dimension of the left $R$-module $S^{-1}R$ does not exceed 1. A left $R$-module $C$ is said to be an $S$-contramodule if $\text{Hom}_R(S^{-1}R, C) = 0 = \text{Ext}_R(S^{-1}R, C)$.

Let $f: U^{-1} \to U^0$ be a two-term free resolution of the left $R$-module $S^{-1}R$. Then the full subcategory of $S$-contramodule left $R$-modules $R\text{-mod}_{S,\text{contra}} \subset R\text{-mod}$ coincides with the full subcategory $\mathcal{B} = f^\perp \subset R\text{-mod}$ from Example 1.3(4). In particular, $R\text{-mod}_{S,\text{contra}}$ is an abelian category with an exact embedding functor $R\text{-mod}_{S,\text{contra}} \to R\text{-mod}$, which has a left adjoint functor $\Delta_S: R\text{-mod} \to R\text{-mod}_{S,\text{contra}}$.

In the same way as in the commutative case of [29] Theorem 3.4] (as mentioned in Example 2.4(1)), one shows that the latter functor can be computed as $\Delta_S(M) = \text{Ext}^1_R(K^\bullet, M) = \text{Hom}_{R\text{-mod}}(K^\bullet, M[1])$ for every left $R$-module $M$, where $K^\bullet$ denotes the two-term complex $R \to S^{-1}R$ (see [3] Proposition 3.2(b)] for a more general result). The abelian category $R\text{-mod}_{S,\text{contra}}$ is equivalent to the category of modules over the additive, accessible monad $T_{R,S} = T_f$ on the category of sets, assigning to every set $X$ the underlying set of the left $R$-module $T_{R,S}(X) = \Delta_S(R[X])$.

(2) Let $R$ be an associative ring and $S \subset R$ be a multiplicative subset consisting of central elements. Denote by $\mathcal{R}$ the ring $\varprojlim_{s \in S} R/sR$, endowed with the projective limit topology. Then $\mathcal{R}$ is a complete, separated topological associative ring with open two-sided ideals forming a base of neighborhoods of zero. So we can consider the abelian category of left $\mathcal{R}$-contramodules $\mathcal{R}\text{-contra}$ defined in Example 1.3(2).

(3) Let $R$ be an associative ring and $S \subset R$ be a multiplicative subset of central elements such that $\text{pd}_R S^{-1}R \leq 1$. Denote by $\Lambda_S$ the $S$-completion functor assigning
to every left $R$-module $M$ the left $R$-module $\Lambda_S(M) = \lim_{s \in S} M/sM$. Then for any left $R$-module $M$ the left $R$-module $\Lambda_S(M)$ is an $S$-contramodule, hence there exists a unique left $R$-module morphism $\Delta_S(M) \to \Lambda_S(M)$ forming a commutative diagram with the natural morphisms $M \to \Delta_S(M)$ and $M \to \Lambda_S(M)$. Applying Proposition 2.1, we once again conclude that the forgetful functor $\mathcal{R} \text{-contra} \to R\text{-mod}_{S,\text{contra}}$ is an equivalence of abelian categories if and only if the natural morphism $\Delta_S(R[X]) \to \Lambda_S(R[X])$ is an isomorphism for every set $X$.

Let $Z \subset R$ be a central subring containing $S$. Then both the functors $\Delta_S$ and $\Lambda_S$ computed in the categories of $Z$-modules and left $R$-modules agree (since the two-term complex $(R \to S^{-1}R)$ is isomorphic to the tensor product $R \otimes_Z (Z \to S^{-1}Z)$, which coincides with the derived tensor product $R \otimes^Z_2 (Z \to S^{-1}Z)$). Notice that we do not know what the projective dimension of the $Z$-module $S^{-1}Z$ might be. But nevertheless it follows from [29, Theorem 2.5(c)] applied to the ring $Z$ with the multiplicative subset $S \subset Z$ that the morphism $\Delta_S(R[X]) \to \Lambda_S(R[X])$ is an isomorphism for every set $X$ provided that the $S$-torsion in $R$ is bounded.

Thus the forgetful functor $\mathcal{R} \text{-contra} \to R\text{-mod}_{S,\text{contra}}$ is an equivalence of abelian categories for any associative ring $R$ with a multiplicative subset of central elements $S$ such that $\text{pd}_{S^{-1}R} \leq 1$ and the $S$-torsion in $R$ is bounded. Furthermore, all the $S$-contramodule $R$-modules are modules over the ring $\Delta_S(R)$, so it suffices to require the $S$-torsion in $\Delta_S(R)$, rather than in $R$, to be bounded (see Examples 5.5 for further details.)

3. Full-and-Faithfulness for Contramodules over Topological Rings

In this section, we are interested in modules over the monads $T = T_{\mathcal{R}}$ associated with topological associative rings $\mathcal{R}$, as defined in Example 1.3(2). We refer to [23, Remark A.3], [24, Section 1.2], [26, Section 2.1], [32, Section 5], [35, Section 6], or [31, Section 2.7] for further and more detailed discussions of the definition of a contramodule over a topological ring (see also [23, Section D.5.2], [24, Section 1.10] or [26, Section 2.3] for comparison with the definition of a contramodule over a topological associative algebra over a field).

Let $\mathcal{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals. As in Section 2 we observe that the datum of an associative ring $R$ together with an associative ring homomorphism $\theta: R \to \mathcal{R}$ defines an exact forgetful functor $\mathcal{R} \text{-contra} \to R\text{-mod}$.

Let $\mathfrak{J} \subset \mathcal{R}$ be a right ideal. We will say that a finite set of elements $s_1, \ldots, s_m \in \mathfrak{J}$ strongly generates the right ideal $\mathfrak{J}$ if for every family of elements $r_x \in \mathfrak{J}$, indexed by some set $X$ and converging to zero in the topology of $\mathcal{R}$, there exist families of elements $r_{j,x} \in \mathcal{R}$, $j = 1, \ldots, m$, each of them indexed by the set $X$ and converging to zero in the topology of $\mathcal{R}$, such that $r_x = \sum_{j=1}^m s_j r_{j,x}$ for all $x \in X$. Since any finite family of elements in $\mathcal{R}$ converges to zero in the topology of $\mathcal{R}$, any finite set of elements of a right ideal $\mathfrak{J} \subset \mathcal{R}$ strongly generating the ideal $\mathfrak{J}$ also generates the right ideal $\mathfrak{J}$ in the conventional sense (cf. [24, Section B.4]).
The following theorem is a generalization of [30] Theorem 2.1, and its proof is similar to that in [30] (see Examples [3.4] below for a discussion).

**Theorem 3.1.** Let \( \mathcal{R} \) be a complete, separated topological associative ring, \( R \) be an associative ring, and \( \theta : R \rightarrow \mathcal{R} \) be a ring homomorphism with a dense image. Assume that \( \mathcal{R} \) has a countable base of neighborhoods of zero consisting of open two-sided ideals, each of which, viewed as a right ideal, is strongly generated by a finite set of elements lying in the image of the map \( \theta \). Then the forgetful functor \( \mathcal{R} \text{-contra} \rightarrow R \text{-mod} \) is fully faithful.

**Proof.** Given a set \( X \) and a complete, separated topological abelian group \( \mathfrak{A} \) with a base of neighborhoods of zero formed by open subgroups \( \mathcal{U} \subset \mathfrak{A} \), denote by \( \mathfrak{A}[[X]] \) the abelian group \( \varprojlim_{U \subset \mathcal{U}} \mathfrak{A}/U[X] \) of all infinite formal linear combinations of elements of \( X \) with the coefficients converging to zero in the topology of \( \mathfrak{A} \). Following the notation in [25, Section D.1], [32, Sections 5–6], [35, Section 7.3], and [31, Section 2.8], for any closed subgroup \( \mathfrak{A} \subset \mathcal{R} \) and any left \( \mathcal{R} \)-contra-module \( B \), we denote by \( \mathfrak{A} \times B \subset B \) the image of the composition \( \mathfrak{A}[[B]] \rightarrow B \) of the natural embedding \( \mathfrak{A}[[B]] \rightarrow \mathfrak{R}[[B]] \) and the contraaction map \( \pi_B : \mathfrak{R}[[B]] \rightarrow B \). The map \( \mathfrak{A}[[B]] \rightarrow B \) is an abelian group homomorphism, so \( \mathfrak{A} \times \mathfrak{B} \) is a subgroup in \( B \).

As usually, for any left \( \mathfrak{R} \)-module \( M \) and any subgroup \( A \subset \mathfrak{R} \), we denote by \( AM \subset M \) the subgroup generated by the products \( am \), where \( a \in A \) and \( m \in M \). So we have \( \mathfrak{A} B \subset \mathfrak{A} \times B \) for any closed subgroup \( \mathfrak{A} \subset \mathcal{R} \) and any left \( \mathcal{R} \)-contra-module \( B \). For any left ideal \( I \subset \mathcal{R} \) and any left \( \mathcal{R} \)-module \( M \), the subgroup \( IM \subset M \) is a left \( \mathcal{R} \)-submodule in \( M \). For any closed left ideal \( J \subset \mathcal{R} \) and any left \( \mathcal{R} \)-contra-module \( B \), the subgroup \( J \times B \subset B \) is a left \( \mathcal{R} \)-subcontra-module in \( B \). On the other hand, when a closed right ideal \( J \subset \mathfrak{A} \) is strongly generated by a finite set of elements \( s_1, \ldots, s_m \in J \), one has \( J \times B = J B = s_1 B + \cdots + s_m B \) for any left \( \mathfrak{R} \)-contra-module \( B \).

Let \( B \) and \( C \) be two left \( \mathfrak{R} \)-contra-modules, and let \( f : B \rightarrow C \) be a left \( R \)-module morphism. The contraaction map \( \pi_B : \mathfrak{R}[[B]] \rightarrow B \) is a surjective morphism of left \( \mathfrak{R} \)-contra-modules. In order to show that \( f \) is an \( \mathfrak{R} \)-contra-module morphism, it suffices to check that the composition \( \mathfrak{R}[[B]] \rightarrow B \rightarrow C \) is an \( \mathfrak{R} \)-contra-module morphism. Hence we can replace \( B \) with \( \mathfrak{R}[[B]] \) and suppose that \( B = \mathfrak{R}[[X]] \) is a free left \( \mathfrak{R} \)-contra-module generated by a set \( X \).

Then the composition \( X \rightarrow C \) of the natural embedding \( X \rightarrow \mathfrak{R}[[X]] \) with a left \( R \)-module morphism \( f : \mathfrak{R}[[X]] \rightarrow C \) can be extended uniquely to a left \( \mathfrak{R} \)-contra-module morphism \( f' : \mathfrak{R}[[X]] \rightarrow C \). Setting \( g = f - f' \), we have a left \( \mathfrak{R} \)-module morphism \( g : \mathfrak{R}[[X]] \rightarrow C \) taking the elements of the set \( X \) to zero elements in \( C \). We have to show that \( g = 0 \). Without loss of generality, we can assume that the left \( \mathfrak{R} \)-contra-module \( C \) is generated by its subset \( g(\mathfrak{R}[[X]]) \subset C \) (otherwise, replace \( C \) with its subcontra-module generated by \( g(\mathfrak{R}[[X]]) \)). Then we have to show that \( C = 0 \).

Let \( I \subset \mathfrak{R} \) be an open two-sided ideal strongly generated, as a right ideal, by a finite set of elements \( s_1, \ldots, s_m \in I \) belonging to the image of \( \theta \). Denote by \( I \subset R \) the full preimage of the ideal \( I \) with respect to ring homomorphism \( \theta \); so \( I \) is a two-sided ideal in \( R \) and \( R/I \cong \mathfrak{R}/\mathfrak{I} \) (since the image of \( \theta \) is dense in \( \mathfrak{R} \)).
Then we have \( \mathfrak{J}[[X]] = \mathfrak{J} \otimes \mathbb{R}[[X]] = s_1 \mathbb{R}[[X]] + \cdots + s_m \mathbb{R}[[X]] = I \mathbb{R}[[X]] \) and \( \mathfrak{J} \otimes \mathfrak{C} = s_1 \mathfrak{C} + \cdots + s_m \mathfrak{C} = IC \).

Now the induced left \((R/I)\)-module morphism \( g/I : \mathbb{R}[[X]]/I \mathbb{R}[[X]] \rightarrow C/IC \) vanishes, because the left \((R/I)\)-module \( \mathbb{R}[[X]]/I \mathbb{R}[[X]] = \mathbb{R}[[X]]/\mathfrak{J}[[X]] = \mathbb{R}/\mathfrak{J}[X] = R/I[X] \) is generated by elements from \( X \). So the image of the morphism \( g \) is contained in \( IC \). Since the left \( \mathbb{R}\)-contramodule \( C \) is generated by \( g(\mathbb{R}[[X]]) \) and \( IC = \mathfrak{J} \otimes \mathfrak{C} \) is a left \( \mathbb{R}\)-subcontramodule in \( C \), it follows that \( C = IC \).

We have shown that \( C = \mathfrak{J} \otimes \mathfrak{C} \) for a countable set of open two-sided ideals \( \mathfrak{J} \subset \mathbb{R} \) forming a base of neighborhoods of zero in \( \mathbb{R} \). According to the contramodule Nakayama lemma ([25, Lemma D.1.2] or [32, Lemma 6.14]), it follows that \( C = 0 \). \( \square \)

A generalization of Theorem 3.1 to complete, separated topological associative rings \( \mathbb{R} \) with a countable base of neighborhoods of zero consisting of open right ideals can be found in [31, Theorem 6.2].

We will explain below in Theorem 4.6(a) and Example 5.1(b) how to distinguish the objects of the full subcategory \( \mathbb{R}\text{-contra} \) among the objects of the ambient category \( R\text{-mod} \) in the context of Theorem 3.1.

In the rest of this section we list some examples of topological rings \( \mathbb{R} \) together with ring homomorphisms \( R \rightarrow \mathbb{R} \) into \( \mathbb{R} \) from a discrete ring \( R \) for which one can show that the forgetful functor \( \mathbb{R}\text{-contra} \rightarrow R\text{-mod} \) is fully faithful using Theorem 3.1 (or parts of the argument from its proof). We also provide a certain nonexample and a certain counterexample.

**Remark 3.2.** Let \( \mathbb{R} \) be a complete, separated topological associative ring with a countable base of neighborhoods of zero consisting of open right ideals. Let \( \mathfrak{K} \subset \mathbb{R} \) be a closed two-sided ideal and \( \mathcal{G} = \mathbb{R}/\mathfrak{K} \) be the quotient ring endowed with the quotient topology. Then \( \mathcal{G} \) is also a complete, separated topological associative ring with a countable base of neighborhoods of zero consisting of open right ideals, and the functor of restriction of scalars \( \mathcal{G}\text{-contra} \rightarrow \mathbb{R}\text{-contra} \) is fully faithful. Indeed, any family of elements of \( \mathcal{G} \) converging to zero in the topology of \( \mathcal{G} \) can be lifted to a family of elements of \( \mathbb{R} \) converging to zero in the topology of \( \mathbb{R} \).

Moreover, if a homomorphism of associative rings \( \theta : R \rightarrow \mathbb{R} \) satisfies the assumptions of Theorem 3.1 then so does the composition \( \overline{\theta} : R \rightarrow \mathcal{G} \) of the homomorphism \( \theta \) with the natural surjective homomorphism \( \mathbb{R} \rightarrow \mathcal{G} \). Indeed, for any open right/two-sided ideal \( \mathfrak{J} \subset \mathbb{R} \), one can consider the open right/two-sided ideal \( \overline{\mathfrak{J}} = (\mathfrak{J} + \mathfrak{K})/\mathfrak{K} \subset \mathcal{G} \). Then any family of elements of \( \overline{\mathfrak{J}} \) converging to zero in the topology of \( \mathcal{G} \) can be lifted to a family of elements of \( \mathfrak{J} \) converging to zero in the topology of \( \mathbb{R} \), hence the image of any finite set of elements strongly generating the right ideal \( \mathfrak{J} \subset \mathbb{R} \) strongly generated the right ideal \( \overline{\mathfrak{J}} \subset \mathcal{G} \).

**Examples 3.3.** (1) Let \( k \) be a commutative ring and \( \mathbb{R} = k\{\{x_1, \ldots, x_m\}\} \) be the \( k\)-algebra of noncommutative formal Taylor power series in the variables \( x_1, \ldots, x_m \) with the coefficients in \( k \), endowed with the formal power series topology (or, in other words, the \( \mathfrak{J}\)-adic topology for the ideal \( \mathfrak{J} = (x_1, \ldots, x_m) \subset \mathbb{R} \)). We observe that the ideal \( \mathfrak{J}^n \subset \mathbb{R} \) of all the formal power series with vanishing coefficients at
all the noncommutative monomials of the total degree less than \( n \) in \( x_1, \ldots, x_m \) is strongly generated, as a left ideal in \( \mathcal{R} \), by the finite set of all the noncommutative monomials of the total degree \( n \) in \( x_1, \ldots, x_m \). Denoting by \( R = k\{x_1, \ldots, x_m\} \) the \( k \)-algebra of noncommutative polynomials in \( x_1, \ldots, x_m \) and by \( \theta: R \to \mathcal{R} \) the natural embedding, we conclude, by applying Theorem 3.1, that the forgetful functor \( \mathcal{R} \text{-contra} \to R \text{-mod} \) is fully faithful.

(2) More generally, let \( R \) be a quotient algebra of the algebra \( k\{x_1, \ldots, x_m\} \) of noncommutative polynomials in the variables \( x_1, \ldots, x_m \) over a commutative ring \( k \) by a two-sided ideal \( K \subset k\{x_1, \ldots, x_m\} \). Let \( \mathcal{R} = \lim_{\leftarrow n} R/I^n \) be the adic completion of the algebra \( R \) with respect to the two-sided ideal \( I = (x_1, \ldots, x_m) \subset \mathcal{R} \), endowed with the projective limit topology (\( = \) \( I \)-adic topology of the left or right \( R \)-module \( \mathcal{R} \)). Then \( \mathcal{R} \) is the topological quotient ring of the algebra of noncommutative formal power series \( k\{x_1, \ldots, x_m\} \) by the closure of the image of the ideal \( K \subset k\{x_1, \ldots, x_m\} \) in \( k\{x_1, \ldots, x_m\} \). In view of Remark 3.2 the forgetful functor \( \mathcal{R} \text{-contra} \to R \text{-mod} \) is fully faithful.

(3) Even more generally, let \( \mathcal{R} \) be the quotient ring of the topological algebra of noncommutative formal power series \( k\{x_1, \ldots, x_m\} \) by a closed two-sided ideal \( \mathcal{R} \subset k\{x_1, \ldots, x_m\} \), endowed with the quotient topology. Then the forgetful functor \( \mathcal{R} \text{-contra} \to R \text{-mod} \) is fully faithful.

(4) One can also drop the commutativity assumption on the ring \( k \), presuming only that the elements of \( k \) commute with the variables \( x_1, \ldots, x_m \) (while the variables do not commute with each other and the elements of \( k \) do not necessarily commute with each other). All the assertions of (1–3) remain valid in this setting.

Examples 3.4. (1) Let \( k \) be a field and \( C \) be a coassociative, counital \( k \)-coalgebra. Then the dual vector space \( C^\vee \) to the \( k \)-vector space \( C \) has a natural topological \( k \)-algebra structure. The category \( C \text{-contra} \) of left contramodules over the coalgebra \( C \) is isomorphic to the category \( C^\vee \text{-contra} \) of left contramodules over the topological algebra \( C^\vee \) [23, Section 1.10], [26, Section 2.3].

(2) In particular, when \( C \) is a conilpotent coalgebra over \( k \) with a finite dimensional cohomology space \( H^1(C) \), the topological algebra \( C^\vee \) is a topological quotient algebra of the algebra of noncommutative formal power series \( k\{x_1, \ldots, x_m\} \), where \( m = \dim_k H^1(C) \), by a closed two-sided ideal, as in Example 3.3(3). This allows to recover the result of [26, Theorem 2.1] as a particular case of our Theorem 3.1 (A change of variables may be needed in order to ensure that an arbitrary dense subalgebra \( R \subset C^\vee \) contains the images of the generators \( x_j \in k\{x_1, \ldots, x_m\} \)).

(3) Let \( C = k \oplus V \oplus k \) be the coalgebra from [23, Section A.1.2] for which the category of left \( C \)-contramodules is equivalent to the category of pairs of \( k \)-vector spaces \( P_1 \) and \( P_2 \) endowed with a \( k \)-linear map \( \text{Hom}_k(V, P_1) \to P_2 \). The category of left \( C^\vee \)-modules is equivalent to the category of pairs of \( k \)-vector spaces \( M_1 \) and \( M_2 \) endowed with a \( k \)-linear map \( V^\vee \otimes_k M_1 \to M_2 \). Then the forgetful functor \( C \text{-contra} \to C^\vee \text{-mod} \) is clearly not fully faithful when \( V \) is infinite-dimensional.
This example shows that the finite generatedness condition on the ideals in the topological ring \( R \) in Theorem 3.1 cannot be readily replaced with a countable generatedness condition.

**Example 3.5.** To give another example in which Theorem 3.1 is *not* applicable, let us consider contramodules over the Virasoro Lie algebra \( \mathbb{Vir} \). The Virasoro Lie algebra is the topological vector space \( \mathbb{Vir} = k((z))d/dz \oplus kC \) of vector fields with coefficients in the field of Laurent power series in one variable \( z \) over a field \( k \) of characteristic 0, extended by adding a one-dimensional vector space \( kC \). The vectors are fully faithful for the positively-graded subalgebras \( \mathbb{Vir} \subset \mathbb{Vir} \). We do not know whether the above-described infinite summation operation with the coefficients \( L_i \) is fully faithful. Perhaps a more natural question would be to show that the forgetful functor \( \mathbb{Vir}_{-\text{contra}} \to \mathbb{Vir}_{-\text{mod}} \) is fully faithful for the positively-graded subalgebras \( \mathbb{Vir}_{+} = z^2k[[z]]d/dz \subset \mathbb{Vir} \) and

\[
\[L_i, L_j\] = (j - i)L_{i+j} + \delta_{i,j}\frac{i^3 - i}{12}C, \quad [L_i, C] = 0, \quad i, j \in \mathbb{Z},
\]

where \( \delta \) is the Kronecker symbol.

A \( \mathbb{Vir} \)-contramodule \( P \) is a \( k \)-vector space endowed with a linear operator \( C : P \to P \) and an infinite summation operation assigning to every sequence \( p_i, q_i \in P \) a vector denoted formally by \( \sum_{i=-\infty}^{\infty} L_i p_i \). The equations of agreement

\[
\sum_{i=-N}^{\infty} L_i p_i = \sum_{i=-M}^{\infty} L_i p_i \quad \text{when} \quad -N < -M \quad \text{and} \quad p_{-N} = \cdots = p_{-M+1} = 0,
\]

linearity

\[
\sum_{i=-N}^{\infty} L_i (a p_i + b q_i) = a \sum_{i=-N}^{\infty} L_i p_i + b \sum_{i=-N}^{\infty} L_i q_i \quad \forall a, b \in k, \quad p_i, q_i \in P,
\]

centrality of \( C \)

\[
C \sum_{i=-N}^{\infty} L_i p_i = \sum_{i=-N}^{\infty} L_i (C p_i),
\]

and contra-Jacobi identity

\[
\sum_{i=-N}^{\infty} L_i \left( \sum_{j=-M}^{\infty} L_j p_{ij} \right) - \sum_{j=-M}^{\infty} L_j \left( \sum_{i=-N}^{\infty} L_i p_{ij} \right) = \sum_{n=-N-M}^{\infty} L_n \left( \sum_{i+j=n}^{i \geq -N, j \geq -M} (j - i)p_{ij} \right) + C \sum_{i+j=0}^{i \geq -N, j \geq -M} \left( \frac{i^3 - i}{12} p_{ij} \right)
\]

have to be satisfied [23, Section D.2.7], [26, Section 1.7].

We do *not* know whether the forgetful functor \( \mathbb{Vir}_{-\text{contra}} \to \mathbb{Vir}_{-\text{mod}} \) from the abelian category of \( \mathbb{Vir} \)-contramodules to the abelian category of modules over the Lie algebra \( \mathbb{Vir} \) is fully faithful. Perhaps a more natural question would be about full-and-faithfulness of the forgetful functor \( \mathbb{Vir}_{-\text{contra}} \to \mathbb{Vir}_{-\text{mod}} \) from the category of \( \mathbb{Vir} \)-contramodules to the category of modules over a discrete version \( \mathbb{Vir} = k[z, z^{-1}]d/dz \oplus kC \subset \mathbb{Vir} \) of Virasoro Lie algebra. We do *not* know whether this functor is fully faithful. In other words, we do not know whether the above-described infinite summation operation with the coefficients \( L_i \) in a vector space \( P \) can be uniquely recovered from the action of the linear operators \( C \) and \( L_i \) in \( P \).

It would be sufficient to show that the forgetful functor \( \mathbb{Vir}_{+} \to \mathbb{Vir}_{+} \) is fully faithful for the positively-graded subalgebras \( \mathbb{Vir}_{+} = z^2k[[z]]d/dz \subset \mathbb{Vir} \) and
Vir$_+$ = $z^2k[z]d/dz \subset$ Vir of (the topological and discrete versions of) the Virasoro Lie algebra. This would mean recovering an infinite summation operation \((p_i)_{i>0} \longrightarrow \sum_{i>0} L_ip_i\) with the coefficients \(L_i, i > 0\), satisfying the equations similar to the above, from the linear operators \(L_i: P \longrightarrow P\). But even though the Lie algebra Vir$_+$ is generated by two elements \(L_1\) and \(L_2\), the relevant completion of the enveloping algebra \(U(\text{Vir}_+)\) [23, Sections D.5.1–3] is not a topological quotient algebra of the algebra of noncommutative formal power series \(k\{\{L_1, L_2\}\}\), so this example does not reduce to Example 3.3(3). The assumptions of Theorem 3.1 are not satisfied for the topological enveloping algebra of Vir$_+$ (still less of \(\text{Vir}\)). The topological enveloping algebra \(U^-(\text{Vir})\) does not even have a base of neighborhoods of zero consisting of two-sided open ideals, and the two-sided open ideals in the topological enveloping algebra \(U^-(\text{Vir}_+)\) are not strongly finitely generated as right ideals.

Examples 3.6. (1) Let \(R\) be an associative ring with a two-sided ideal \(I \subset R\). Consider the associated graded ring \(\text{gr}_I R = \bigoplus_{n=0}^{\infty} I^n/I^{n+1}\) and the \(I\)-adic completion \(\mathcal{R} = \lim \leftarrow n R/I^n\) (viewed as a topological ring in the projective limit topology). Assume that the ideal \(\text{gr}_I I = \bigoplus_{n=1}^{\infty} I^n/I^{n+1}\) in the graded ring \(\text{gr}_I R\) is generated by a finite set of central elements \(\bar{s}_1, \ldots, \bar{s}_m\) of grading 1.

Choose some liftings \(s_1, \ldots, s_m \in I\) of the elements \(\bar{s}_1, \ldots, \bar{s}_m \in I/I^2\). Then the open two-sided ideal \(\mathcal{I}_n = \lim \leftarrow n I^n/I^{n+1}\) in the topological ring \(\mathcal{R}\), viewed as a right ideal, is strongly generated by the images of the monomials of degree \(n\) in the elements \(s_1, \ldots, s_m\). According to Theorem 3.1, it follows that the forgetful functor \(\mathcal{R} \rightarrow \mathcal{R} \rightarrow \mathcal{R}\) is fully faithful.

(2) In particular, let \(R\) be an associative ring and \(I \subset R\) be the ideal generated by a finite set of central elements \(s_1, \ldots, s_m \in R\). Let \(\mathcal{R} = \lim \leftarrow n R/I^n\) be the \(I\)-adic completion of the ring \(R\), endowed with the projective limit (= \(I\)-adic) topology. Then the forgetful functor \(\mathcal{R} \rightarrow \mathcal{R} \rightarrow \mathcal{R}\) is fully faithful. This assertion was mentioned in Examples 2.2(2) and 2.3(3) above.

(3) Here is an alternative argument, not based on Theorem 3.1, proving the assertion in (2). Let us show that the exact forgetful functor \(\mathcal{R} \rightarrow \mathcal{R} \rightarrow \mathcal{R}\) is fully faithful. The abelian category \(\mathcal{R} \rightarrow \mathcal{R}\) is the category of modules over the monad on \(\text{Sets}\) assigning to a set \(X\) the set \(\mathcal{R}[X] = \text{L}_f(R[X])\), while the abelian category \(\mathcal{R} \rightarrow \mathcal{R}\) is the category of modules over the monad assigning to a set \(X\) the set \(\Delta_f(R[X])\). The functor \(\mathcal{R} \rightarrow \mathcal{R} \rightarrow \mathcal{R}\) is induced by the morphism of monads \(\Delta_f(R[X]) \rightarrow \text{L}_f(R[X])\), and surjectivity of this map for every set \(X\) immediately implies that this functor is fully faithful.

(4) It may be worth noting that, while \(\Delta_f: \mathcal{R} \rightarrow \mathcal{R} \rightarrow \mathcal{R}\) is the left adjoint functor to the exact embedding \(\mathcal{R} \rightarrow \mathcal{R}\), the left adjoint functor to the exact embedding \(\mathcal{R} \rightarrow \mathcal{R}\) can be computed as the 0-th left derived functor \(\text{L}_0\Lambda_f\) of the \(I\)-adic completion functor \(\Lambda_f: \mathcal{R} \rightarrow \mathcal{R}\). The functor \(\Lambda_f\) itself is neither left, nor right exact. Its derived functor was considered in [22, Section 3]. (Indeed, both the reflector onto \(\mathcal{R} \rightarrow \mathcal{R}\) in \(\mathcal{R} \rightarrow \mathcal{R}\) and \(\text{L}_0\Lambda_f\) are right exact and take the left \(\text{R}\)-module \(R[X]\) to the left \(\mathcal{R}\)-contramodule \(\mathcal{R}[X]\).)
For any left $R$-module $M$, there are natural surjective $R$-module morphisms $\Delta_I(M) \to \mathbb{L}_0\Lambda_I(M) \to \Lambda_I(M)$. We refer to Examples 5.2(6) and 5.3(4) below for further discussion.

**Examples 3.7.** (1) Let $R$ be an associative ring and $S \subset R$ be a multiplicative subset consisting of some central elements in $R$. Let $\mathcal{R} = \lim_{s \in S} R/sR$ denote the $S$-completion of the ring $R$, viewed as a topological ring in the projective limit topology. Assume that the projective limit topology of $\mathcal{R}$ coincides with the $S$-topology of $R$-module $R$, and moreover, assume that for every set $X$ the projective limit topology of the free $\mathcal{R}$-contramodule $\mathcal{R}[[X]] = \lim_{s \in S} R/sR[X]$ coincides with the $S$-topology of the $R$-module $\mathcal{R}[[X]]$ (cf. [29, Theorem 2.3]).

The latter condition can be expressed by saying that for any $X$-indexed family of elements $r_x \in \mathcal{R}$, converging to zero in the topology of $\mathcal{R}$ and belonging to the kernel ideal of the natural ring homomorphism $R \to R/sR$, there exists an $X$-indexed family of elements $t_x \in \mathcal{R}$, converging to zero in the topology of $\mathcal{R}$, such that $r_x = st_x$ for all $x \in X$. In other words, it means that the kernel ideal of the ring homomorphism $\mathcal{R} \to R/sR$ is strongly generated by (the image in $\mathcal{R}$ of) an element $s$, for every $s \in S$.

Following the proof of Theorem 3.1, we would be able to conclude that the forgetful functor $\mathcal{R} \to R$-mod is fully faithful if we knew that, for every left $\mathcal{R}$-contramodule $C$, the equations $C = sC$ for all $s \in S$ imply $C = 0$. This condition holds whenever every $S$-divisible left $R$-module is $S$-h-divisible (see the discussion in [29, Section 1] and the references therein).

To sum up, the forgetful functor $\mathcal{R} \to R$-mod is fully faithful whenever the $S$-completion of the free left $R$-module $R[X]$ is $S$-complete for every set $X$ and all the $S$-divisible left $R$-modules are $S$-h-divisible. Notice that, unlike in Examples 2.4 and 2.5 above, no condition on the projective dimension of the $R$-module $S^{-1}R$ was needed for our present discussion.

(2) In particular, for any countable multiplicative subset $S$ consisting of some central elements in an associative ring $R$, the forgetful functor $\mathcal{R} \to R$-mod is fully faithful (see [29, Proposition 2.2]). In this case, the category $\mathcal{R} \to R$-mod is a full subcategory in $R$-mod (see Examples 2.4(2) and 2.5(2), and [29, Lemma 1.9]).

While $\Delta_S$ is the left adjoint functor to the exact embedding $R$-mod \to $R$-mod, the left adjoint functor to the exact embedding $\mathcal{R} \to R$-mod can be computed as the 0-th left derived functor $\mathbb{L}_0\Lambda_S$: $R$-mod \to $\mathcal{R}$-mod of the $S$-completion functor $\Lambda_S$. For any left $R$-module $M$, there are natural surjective left $R$-module morphisms $\Delta_S(M) \to \mathbb{L}_0\Lambda_S(M) \to \Lambda_S(M)$. We refer to Examples 5.4(2) and 5.5(2) below for further discussion.
The definition of the perpendicular subcategory in [13, Section 1] has many obvious versions and generalizations. In the next series of definitions, we list the most important ones in our context.

Let $A$ be an abelian category and $B \subset A$ be a full subcategory. We will say that $B$ is a \textit{right 0-perpendicular subcategory} in $A$ if there exists a class of morphisms $F$ in the category $A$ such that

- an object $B \in A$ belongs to $B$ if and only if for every morphism $f \in F$, $f : U' \to U''$, the morphism of abelian groups $\text{Hom}_A(f, B) : \text{Hom}_A(U'', B) \to \text{Hom}_A(U', B)$ is an isomorphism.

In this case, we will say that $B$ is the right 0-perpendicular subcategory to the class of morphisms $F$ in $A$ and write $B = F \perp_0 \subset A$.

Furthermore, we will say that $B$ is a \textit{right 1-perpendicular subcategory} in $A$ if there exists a class of objects $E$ in the category $A$ such that

- an object $B \in A$ belongs to $B$ if and only if for every object $E \in E$ one has $\text{Hom}_A(E, B) = 0 = \text{Ext}^1_A(E, B)$.

In this case, we will say that $B$ is the right 1-perpendicular subcategory to the class of objects $E$ in $A$ and write $B = E \perp_0, 1 \subset A$.

Let $n \geq 1$ be an integer. We will say that a full subcategory $B \subset A$ is a \textit{right $n$-perpendicular subcategory} in $A$ if there exists a class of objects $E$ in the category $A$ such that

- an object $B \in A$ belongs to $B$ if and only if for every object $E \in E$ one has $\text{Hom}_A(E, B) = 0 = \text{Ext}^i_A(E, B)$, and
- at the same time, for any objects $E \in E$ and $B \in B$ one has $\text{Ext}^i_A(E, B) = 0$ for all $0 \leq i \leq n$.

So an $n$-perpendicular subcategory is a 1-perpendicular subcategory satisfying an additional condition. We will say that $B$ is the right $n$-perpendicular subcategory to the class of objects $E$ in $A$ and write $B = E \perp_0, 1 = E \perp_0, n$.

Finally, we will say that a full subcategory $B \subset A$ is a \textit{right $\infty$-perpendicular subcategory} in $A$ if there exists a class of objects $E \subset A$ such that

- an object $B \in A$ belongs to $B$ if and only if for every object $E \in E$ one has $\text{Hom}_A(E, B) = 0 = \text{Ext}^i_A(E, B)$, and
- for any objects $E \in E$ and $B \in B$, one has $\text{Ext}^i_A(E, B) = 0$ for all $i \geq 0$.

In this case, we will say that $B$ is the right $\infty$-perpendicular subcategory to the class of objects $E$ in $A$ and write $B = E \perp_0, 1 \subset E \perp_0, \infty$.

\textbf{Lemma 4.1.} (a) For any abelian category $A$ and any integer $n \geq 0$, every right $(n + 1)$-perpendicular subcategory in $A$ is at the same time a right $n$-perpendicular subcategory in $A$.

(b) For any locally presentable abelian category $A$ and any integer $n \geq 0$, every right $(n + 1)$-perpendicular subcategory to a set of objects in $A$ is at same time a right $n$-perpendicular subcategory to a set of objects/morphisms in $A$. 


Proof. Both assertions are obvious for $n \geq 1$, as any right $(n + 1)$-perpendicular subcategory to a class of objects $E \subset A$ is, by the definition, at the same time a right $n$-perpendicular subcategory to the same class of objects $E \subset A$ for $n \geq 1$.

The nontrivial case is $n = 0$. Part (a): given a class of objects $E \subset A$, denote by $F$ the class of all $E$-monomorphisms in $A$, i.e., all the monomorphisms in $A$ with the cokernels belonging to $E$. Then $F_{\perp 0} = E_{\perp 0}$. Part (b): given a set of $\kappa$-presentable objects $E$ in a locally $\kappa$-presentable abelian category $A$ (where $\kappa$ is a regular cardinal), denote by $F$ the class of all $E$-monomorphisms with $\kappa$-presentable codomains in $A$. Then $F_{\perp 0} = E_{\perp 0, 1}$ [32, Lemma 3.4].

In particular, in the simplest case when there are enough projective objects in $A$, it suffices to pick an epimorphism from a projective object $F_E \to E$ onto every object $E \in E$. Let $G_E \to F_E$ be the kernel of the morphism $F_E \to E$; then the class/set $F$ of all the morphisms $G_E \to F_E$ has the property that $F_{\perp 0} = E_{\perp 0, 1}$. □

Our definition of a right 0-perpendicular subcategory is (the abelian categories-related particular case of) what appears under the name of an “orthogonality class” in the book [1] (where nonabelian, nonadditive categories are generally considered). What we would call “the right 0-perpendicular subcategory to a set of morphisms” is called a “small-orthogonality class” in [1].

The right 0-perpendicular subcategory to a single morphism between free left modules over an associative ring $R$ in the abelian category $A = R$–mod was discussed in Example [13, 4]. Part (d) of the next lemma generalizes some of the properties mentioned there to the case of the right 0-perpendicular subcategory to an arbitrary class of morphisms between projective objects in an abelian category.

What we call a right 1-perpendicular subcategory was called simply a “right perpendicular subcategory” in [13]. The case of the right perpendicular subcategory to a class of objects of projective dimension $\leq 1$ played a special role in [13] (and also in [28, Section 1]), where it was noticed that such a subcategory $B \subset A$ is always abelian with an exact embedding functor $B \to A$.

In fact, the right perpendicular subcategory to a class/set of objects of projective dimension $\leq 1$ (in the sense of [13]) is a right $n$-perpendicular subcategory to the same class/set of objects for every $n \geq 1$ (in the sense of our definitions). Part (c) of the next lemma generalizes the related results of [13, Proposition 1.1] and [28, Theorem 1.2] to the case of right 2-perpendicular subcategories to arbitrary classes of objects.

**Lemma 4.2.** Let $A$ be an abelian category. Then

(a) any right 0-perpendicular subcategory $B \subset A$ is closed under arbitrary limits, and in particular, under infinite products and kernels in $A$;

(b) any right 1-perpendicular subcategory $B \subset A$ is closed under infinite products, kernels, and extensions in $A$;

(c) any right 2-perpendicular subcategory $B \subset A$ is closed under infinite products, kernels, extensions, and cokernels in $A$. Hence any right 2-perpendicular subcategory $B \subset A$ is abelian and its embedding functor $B \to A$ is exact;

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(d) the right 0-perpendicular subcategory $B \subset A$ to any class of morphisms between projective objects in $A$ is closed under infinite products, kernels, extensions, and cokernels in $A$. Hence any such full subcategory $B \subset A$ is abelian and its embedding functor $B \rightarrow A$ is exact.

*Proof.* Part (a) is [11 Observation 1.34]. Part (b) is [13 Proposition 1.1]. In part (c), one notices that any full subcategory in $A$ closed under the kernels of all morphisms and the cokernels of monomorphisms is also closed under the cokernels of all morphisms. Checking that $B = E^{1,0} = E^{1,0,2}$ is closed under the cokernels of monomorphisms is easy. Part (d) holds, because the full subcategory of isomorphisms in the category of morphisms in the category of abelian groups (or in any other abelian category generally) is closed under infinite products, kernels, extensions, and cokernels (see [28] proof of Theorem 1.2 and Remark 1.3) for an alternative argument. □

**Example 4.3.** The following notable examples of left and right perpendicular subcategories serve to illustrate the above definitions rather well. Let $X$ be a scheme with the structure sheaf $O_X$. Consider the following preadditive category (or, which is the same, a ring with several objects [21]) $A_X$. The objects of $A_X$ are affine open subschemes $U \subset X$. For any two affine open subschemes $U$ and $V \subset X$, the group of morphisms $V \rightarrow U$ in $A_X$ is the group $O_X(V)$ if $V \subset U$ and the zero group otherwise. The compositions of morphisms in $A_X$ are defined in the obvious way.

Then the category $\text{mod} - A_X$ of right $A_X$-modules (or, which is the same, contravariant functors from $A_X$ to the category of abelian groups $\mathbb{Z} \text{-mod}$) is the category of presheaves of $O_X$-modules on the affine open subschemes in $X$. The category $A_X \text{-mod}$ of left $A_X$-modules ( = covariant functors from $A_X$ to $\mathbb{Z} \text{-mod}$) is the category of copresheaves of $O_X$-modules on the affine open subschemes in $X$.

Given an affine open subscheme $W \subset X$, denote by $P_W$ the left $A_X$-module assigning to an affine open subscheme $V \subset X$ the group $P_W(V) = O_X(W)$ if $W \subset V$ and 0 otherwise. So $P_W$ is the covariant functor $A_X \rightarrow \mathbb{Z} \text{-mod}$ corepresented by the object $W$, or, in another language, the free left $A_X$-module with one generator at the object $W$. Furthermore, given two embedded affine open subschemes $W \subset U$ in $X$, denote by $E_{W,U}$ the left $A_X$-module assigning to an affine open subscheme $V \subset X$ the group $E_{W,U}(V) = O_X(W)$ if $W \subset V \subset U$ and 0 otherwise. Then there is a short exact sequence $0 \rightarrow O_X(W) \otimes O_X(U) P_U \rightarrow P_W \rightarrow E_{W,U} \rightarrow 0$ in the abelian category $A_X \text{-mod}$ (where the tensor product $O_X(W) \otimes O_X(U) P_U$ assigns to an affine open subscheme $V \subset X$ the group $O_X(W) \otimes O_X(U) P_U(V)$).

Furthermore, denote by $J_{W,U}$ the right $A_X$-module assigning to an affine open subscheme $V \subset X$ the abelian group $J_{W,U}(V) = \text{Hom}_Z(E_{W,U}(V), \mathbb{Q} / \mathbb{Z})$. Then the left $A_X$-module $E_{W,U}$ has projective dimension at most 2 (as an object of the abelian category $A_X \text{-mod}$) and flat dimension at most 1 (in the sense of the theory of the tensor product functor $\otimes_{A_X}$ and the derived tensor product functor $\text{Tor}^1_{A_X}$ over a preadditive category $A_X$, as developed, e. g., in [21] Section 6)), while the right $A_X$-module $J_{W,U}$ has injective dimension at most 1.

Now the description of the category of quasi-coherent sheaves $X \text{-qcoh}$ on a scheme $X$ suggested by Enochs and Estrada in [10] Section 2] can be reformulated by saying...
that \( X-\text{qcoh} \) is equivalent to the full subcategory in \( \text{mod-}A_X \) consisting of all the presheaves of \( \mathcal{O}_X \)-modules \( F \) on affine open subschemes in \( X \) such that \( F \otimes_{A_X} \mathcal{E}_{W,U} = 0 = \text{Tor}^1_{A_X}(F, \mathcal{E}_{W,U}) \) for all affine open subschemes \( W \subset U \) in \( X \), or equivalently, \( \text{Hom}_{A_X}^i(F, \mathcal{J}_{W,U}) = 0 \) for all \( W \subset U \subset X \). Indeed, for any presheaf \( F \) the groups \( F \otimes_{A_X} \mathcal{E}_{W,U} \) and \( \text{Tor}^1_{A_X}(F, \mathcal{E}_{W,U}) \) are, respectively, the cokernel and the kernel of the natural map \( \mathcal{O}_X(W) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U) \to \mathcal{F}(W) \). So, in other words, one can say that \( X-\text{qcoh} \) is the left 1-perpendicular (and \( \infty \)-perpendicular) subcategory to a set of objects of injective dimension \( \leq 1 \) in \( \text{mod-}A_X \).

Similarly, the description of the category of contraherent cosheaves \( X-\text{ctrh} \) on a scheme \( X \) in [25, Section 2.2] can be reformulated by saying that \( X-\text{ctrh} \) is equivalent to the full subcategory in \( A_X-\text{mod} \) consisting of all the copresheaves of \( \mathcal{O}_X \)-modules \( Q \) on affine open subschemes in \( X \) such that \( \text{Ext}^i_{A_X}(\mathcal{E}_{W,U}, Q) = 0 \) for all pairs of embedded affine open subschemes \( W \subset U \) in \( X \) and all \( 0 \leq i \leq 2 \) (or, equivalently, for all \( i \geq 0 \)). In fact, the contraherence condition on \( Q \) means the vanishing of \( \text{Ext}^i_{A_X}(\mathcal{E}_{W,U}, Q) \) for \( i = 0 \) and \( 1 \), while the contraadjustness condition is equivalent to this vanishing for \( i = 2 \). More precisely, for any copresheaf \( Q \) the groups \( \text{Hom}_{A_X}(\mathcal{E}_{W,U}, Q) \) and \( \text{Ext}^1_{A_X}(\mathcal{E}_{W,U}, Q) \) are, respectively, the kernel and the cokernel of the natural map \( \mathcal{Q}(W) \to \text{Hom}_{\mathcal{O}_X(U)}(\mathcal{O}_X(W), \mathcal{Q}(U)) \), while the group \( \text{Ext}^2_{A_X}(\mathcal{E}_{W,U}, Q) \) is isomorphic to \( \text{Ext}^1_{\mathcal{O}_X(U)}(\mathcal{O}_X(W), \mathcal{Q}(U)) \).

So, denoting by \( A \) the abelian category \( A_X-\text{mod} \) and by \( E \) the set of objects \( \{ \mathcal{E}_{W,U} \} \), one can write that \( X-\text{ctrh} = E^{1,0,2} \subseteq E^{1,0,3} \) in general. Hence it is clear that the full subcategory \( X-\text{ctrh} \subset A_X-\text{mod} \) is closed under extensions, direct summands, and infinite products. But it is not closed under kernels or cokernels, in general (as the example of \( X = \text{Spec} \mathbb{Z} \) already demonstrates). Moreover, one can see that there are morphisms in the category of contraherent cosheaves over \( X = \text{Spec} \mathbb{Z} \) that do not have kernels in \( X-\text{ctrh} \) at all. Thus, for an arbitrary scheme \( X \), the category \( X-\text{ctrh} \) has a natural exact category structure, but it is not an abelian category, in general (while the category \( X-\text{qcoh} \) is abelian).

Now we return to the discussion of right \( n \)-perpendicular subcategories \( B \) in abelian categories \( A \). In the rest of this paper (with the exception of the last Section 7 where counterexamples are discussed), we are interested in the right \( n \)-perpendicular subcategories \( B \subset A \) that are closed under cokernels. According to Lemmas [4.1(a)] and [4.2(a)], for any \( n \geq 0 \), such a full subcategory \( B \) is closed under infinite products, kernels, and cokernels in \( A \); so \( B \) is an abelian category and its embedding functor \( B \to A \) is exact. In other words, we are interested in the abelian, exactly embedded right \( n \)-perpendicular subcategories in abelian categories. According to Lemma [4.2(c)], for \( n \geq 2 \) this additional condition holds automatically.

**Lemma 4.4.** Let \( A \) be a locally presentable abelian category and \( B \subset A \) be an abelian, exactly embedded right \( n \)-perpendicular subcategory to a set of objects or morphisms in \( A \). Then \( B \) is a locally presentable abelian category, accessibly embedded into \( A \) and reflective in \( A \).
Proof. In view of Lemma 4.1(b), it suffices to consider the case of the right 0-perpendicular subcategory $B = F_{\perp 0}$ to a set of morphisms $F$ in $A$. Suppose that $A$ is locally $\kappa$-presentable and $F$ consists of morphisms between $\kappa$-presentable objects (where $\kappa$ is a regular cardinal). Then the full subcategory $B$ is $\kappa$-accessibly embedded into $A$ (which means that it is closed under $\kappa$-filtered colimits in $A$). As the full subcategory $B \subset A$ is also closed under arbitrary limits, [1] Theorem and Corollary 2.48 apply, proving that $B$ is locally $\kappa$-presentable and reflective in $A$.

Alternatively, one can apply directly [1] Theorem 1.39. (Notice that the proof of the implication (ii) $\Rightarrow$ (i) in [1] Theorem 1.39 is erroneous, and the implication itself is only true for uncountable cardinals, see [14, 15] and the references therein; but we are not using this implication here.) \[\square\]

Assuming Vopěnka’s principle, any right 0-perpendicular subcategory in a locally presentable abelian category is the right 0-perpendicular subcategory to a set of morphisms [1, Corollary 6.24], so this condition can be dropped.

The following lemma is a direct generalization of Example 1.2(4).

Lemma 4.5. Let $A$ be an abelian category with a projective generator, and let $B$ be an abelian, exactly embedded, reflective full subcategory in $A$. Then $B$ is also an abelian category with a projective generator.

Proof. Denote the reflector by $\Delta: A \to B$, and let $Q$ be a projective generator of $A$. Then $P = \Delta(Q)$ is a projective generator of $B$. \[\square\]

From this point on, we are interested in the abelian, exactly embedded right $n$-perpendicular subcategories $B$ to sets of objects or morphisms in the categories of modules over associative rings, $A = R\text{-mod}$. According to Lemmas 4.4 and 4.5, any such abelian category $B$ is locally presentable and has a projective generator. Our aim is to prove the converse assertion, for every $n \geq 0$ and $n = \infty$.

Let $B$ be a locally $\kappa$-presentable abelian category, $R$ be an associative ring, and $\Theta: B \to R\text{-mod}$ be an exact functor preserving infinite products and $\kappa$-filtered colimits. Then the functor $\Theta$ is a right adjoint [1] Theorem 1.66, so it has a left adjoint functor $\Delta: R\text{-mod} \to B$. Set $P = \Delta(R)$, where $R$ is the free left $R$-module with one generator; then $P$ is a projective object in $B$ endowed with a right action of the ring $R$, that is, with a homomorphism of associative rings $R \to \text{Hom}_B(P, P)^{\text{op}}$. The functor $\Theta$ is corepresented by the object $P \in B$: one has $\Theta(B) = \text{Hom}_B(P, B)$ for all $B \in B$, with the left $R$-module structure on the abelian group $\text{Hom}_B(P, B)$ coming from the right action of $R$ in $P$.

The projective object $P$ is a projective generator of $B$ if and only if the exact functor $\Theta$ is conservative, i.e., it takes nonisomorphisms to nonisomorphisms, or equivalently, takes nonzero objects to nonzero objects. An exact functor between abelian categories is conservative if and only if it is faithful. Thus conservative exact functors $\Theta: B \to R\text{-mod}$ preserving infinite products and $\kappa$-filtered colimits are indexed by projective generators $P \in B$ endowed with an associative ring homomorphism $R \to \text{Hom}_B(P, P)^{\text{op}}$. We are interested in those of such functors $\Theta$ that are not only conservative, but, actually, fully faithful.
The following theorem is the main result of this section.

**Theorem 4.6.** Let $\mathcal{B}$ be a locally presentable abelian category with a projective generator $P$, let $R$ be an associative ring, and let $\theta: R \to \text{Hom}_\mathcal{B}(P, P)$ be an associative ring homomorphism. Assume that the related functor $\Theta = \text{Hom}_\mathcal{B}(P, -): \mathcal{B} \to R\text{-mod}$ is fully faithful. Then

(a) the full subcategory $\Theta(\mathcal{B}) \subset R\text{-mod}$ is the right $0$-perpendicular subcategory to a set of morphisms in $R\text{-mod}$;

(b) assuming additionally that $\theta$ is injective, the full subcategory $\Theta(\mathcal{B}) \subset R\text{-mod}$ is a right $1$-perpendicular subcategory (to a set of objects) in $R\text{-mod}$ if and only if it is closed under extensions;

(c) assuming that $\theta$ is injective and given an integer $n \geq 1$, the full subcategory $\Theta(\mathcal{B}) \subset R\text{-mod}$ is the right $n$-perpendicular subcategory to a set of objects in $R\text{-mod}$ whenever the induced maps between the Ext groups $\Theta: \text{Ext}^i_R(\Theta(\mathcal{B}), \Theta(C)) \to \text{Ext}^i_B(B, C)$ are isomorphisms for all $B, C \in \mathcal{B}$ and $i \leq n$;

(d) assuming that $\theta$ is injective, the full subcategory $\Theta(\mathcal{B}) \subset R\text{-mod}$ is the right $\infty$-perpendicular subcategory to a set of objects in $R\text{-mod}$ whenever the induced maps $\Theta: \text{Ext}^i_B(B, C) \to \text{Ext}^i_R(\Theta(\mathcal{B}), \Theta(C))$ are isomorphisms for all $B, C \in \mathcal{B}$ and all $0 \leq i < \infty$.

**Proof.** According to Example 1.23, the functor $\text{Hom}_\mathcal{B}(P, -)$ identifies the category $\mathcal{B}$ with the category of modules over the additive monad $T: X \mapsto \text{Hom}_\mathcal{B}(P, P(X))$. Assume that the category $\mathcal{B}$ is locally $\kappa$-presentable and the object $P$ is $\kappa$-presentable; then the monad $T$ is $\kappa$-accessible. Identifying $\mathcal{B}$ with $\mathcal{T}\text{-mod}$, the functor $\Theta$ gets identified with the forgetful functor $\text{Hom}_\mathcal{T}(T(\ast), -)$ discussed in the beginning of Section 2. In particular, for every set $X$ we have the free $T$-module $T(X) \in \mathcal{T}\text{-mod}$ corresponding to the object $P(X) \in \mathcal{B}$, and there is a natural left $R$-module morphism $R[X] \to T(X)$, as in Proposition 2.1. Denote this morphism by $\theta_X$.

Part (a): we claim that a left $R$-module $C$ belongs to the full subcategory $\Theta(\mathcal{B}) \subset R\text{-mod}$ if and only if the morphism of abelian groups

$$\text{Hom}_R(\theta_Z, C): \text{Hom}_R(T(Z), C) \to \text{Hom}_R(R[Z], C)$$

is an isomorphism for all sets $Z$ of cardinality less than $\kappa$. Indeed, if $C \in \Theta(\mathcal{B})$, then

$$\text{Hom}_R(T(X), C) \simeq C^X \simeq \text{Hom}_R(R[X], C)$$

for all sets $X$, since the functor $\Theta$ is fully faithful by assumption.

Conversely, assume that $\text{Hom}_R(\theta_Z, C)$ is an isomorphism for all sets $Z$ of cardinality less than $\kappa$. Then $\text{Hom}_R(\theta_X, C)$ is an isomorphism for all sets $X$, because $R[X] = \lim_{Z \subset X} R[Z]$ and $T(X) = \lim_{Z \subset X} T(Z)$ in the category of left $R$-modules, where the $\kappa$-filtered colimit is taken over all the subsets $Z \subset X$ of cardinality less than $\kappa$.

In particular, we have $\text{Hom}_R(T(C), C) = \text{Hom}_R(R[C], C) = C^C$, so there is a natural surjective morphism of left $R$-modules $T(C) \to C$ corresponding to the identity map $C \to C$. Let $K$ denote the kernel of this morphism; then the morphism $\text{Hom}_R(\theta_X, K)$ is an isomorphism for all sets $X$, since the morphisms $\text{Hom}_R(\theta_X, T(C))$ and $\text{Hom}_R(\theta_X, C)$ are. Hence we also have a natural surjective morphism of left
$R$-modules $\mathbb{T}(K) \rightarrow K$. Now $C$ is the cokernel of the composition $\mathbb{T}(K) \rightarrow K \rightarrow \mathbb{T}(C)$, and it follows that $C \in \Theta(B)$, since $\mathbb{T}(K), \mathbb{T}(C) \in \Theta(B)$ and the full subcategory $\Theta(B) \subset R\text{-}mod$ is closed under cokernels.

Part (b): obviously, $\Theta(B)$ is closed under extensions in $R\text{-}mod$ if and only if $\Theta$ induces isomorphisms on the groups $\text{Ext}^1$. If $\Theta(B)$ is a right 1-perpendicular subcategory in $R\text{-}mod$, then $\Theta(B) \subset R\text{-}mod$ is closed under extensions by Lemma 4.2(b).

Conversely, if $\Theta$ induces isomorphisms on $\text{Ext}^1$, then $\text{Ext}^1_R(\mathbb{T}(X), C) = 0$ for all $C \in \Theta(B)$, as the free $R$-modules $\mathbb{T}(X)$ are projective objects in $\mathbb{T}\text{-}mod$. We claim that a left $R$-module belongs to the full subcategory $\Theta(B) \subset R\text{-}mod$ if and only if

$$\text{Hom}_R(\mathbb{T}(Z)/R[Z], C) = 0 = \text{Ext}^1_R(\mathbb{T}(Z)/R[Z], C)$$

for all sets $Z$ of cardinality less than $\kappa$, where $\mathbb{T}(Z)/R[Z]$ is the cokernel of the morphism $\theta_Z$. Notice that the morphism $\theta_Z$ is injective in our present assumptions (since the homomorphism $\theta$ is injective).

Indeed, it suffices to consider the long exact sequence

$$0 \rightarrow \text{Hom}_R(\mathbb{T}(Z)/R[Z], C) \rightarrow \text{Hom}_R(\mathbb{T}(Z), C) \rightarrow \text{Ext}^1_R(\mathbb{T}(Z)/R[Z], C) \rightarrow \text{Ext}^1_R(\mathbb{T}(Z), C)$$

and recall that a left $R$-module $C$ belongs to $\Theta(B)$ if and only if $\text{Hom}_R(\theta_Z, C)$ is an isomorphism for all sets $Z$ of the cardinality less than $\kappa$, and that $\text{Ext}^1_R(\mathbb{T}(X), C) = 0$ for all sets $X$ and all $C \in \Theta(B)$.

Part (c): for every set $X$, we have $\text{Ext}^i_R(\mathbb{T}(X)/R[X], C) = \text{Ext}^i_R(\mathbb{T}(X), C)$ for all left $R$-modules $C$ and all $i \geq 2$. If the functor $\Theta$ induces isomorphisms of the groups $\text{Ext}^i$ for all $i \leq n$, then one has

$$\text{Ext}^i_R(\mathbb{T}(X)/R[X], C) = \text{Ext}^i_R(\mathbb{T}(X), C) = \text{Ext}^i_B(P(X), \Theta^{-1}(C)) = 0$$

for all $C \in \Theta(B)$ and $2 \leq i \leq n$, where $\Theta^{-1}(C) \in B$ is an object such that $\Theta(\Theta^{-1}(C)) = C$. Hence $\Theta(B) = E_{\neq 0.1} = E_{\neq 0..n}$, where $E \subset R\text{-}mod$ is the set of all the left $R$-modules $\mathbb{T}(Z_\mu)/R[Z_\mu]$, where $\mu$ runs over all the cardinals smaller than $\kappa$ and $Z_\mu$ is a set of cardinality $\mu$.

Part (d) is provable in the same way as part (c). \[\square\]

**Remark 4.7.** It is clear from the proof of Theorem 4.6(c) that, denoting by $E \subset R\text{-}mod$ the set of all left $R$-modules $\mathbb{T}(Z_\mu)/R[Z_\mu]$ and assuming that $\Theta(B) = E_{\neq 0.1}$, one has $E_{\neq 0.1} = E_{\neq 0..n}$ if and only if $\text{Ext}^i_R(\mathbb{T}(Z_\mu), C) = 0$ for all the cardinals $\mu$ involved, all $C \in \Theta(B)$, and all $1 \leq i \leq n$. If we knew that, moreover, $\text{Ext}^i_R(\mathbb{T}(X), C) = 0$ for all sets $X$, all objects $C \in \Theta(B)$, and all $1 \leq i \leq n$, it would follow that the functor $\Theta: B \rightarrow R\text{-}mod$ induces isomorphisms of the groups $\text{Ext}^i$ for $i \leq n$.

Nevertheless, there are examples of associative (and even commutative) rings $R$ with a set $E \subset R\text{-}mod$ of left $R$-modules of projective dimension 1 for which the embedding functor $\Theta: B \rightarrow R\text{-}mod$ of the right $\infty$-perpendicular subcategory $B = E_{\neq 0.1} = E_{\neq 0..\infty}$ does not induce an isomorphism of the groups $\text{Ext}^i$ for some $i \geq 2$. Such examples can even be found among the embedding functors $\Theta: B \rightarrow \text{mod}$.  

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Let \( R \) be a locally presentable abelian category with a projective generator \( P \), let \( R \) be an associative ring, let \( \theta: R \to \Hom_B(P, P)^{\text{op}} \) be a ring homomorphism, and let \( \Theta = \Hom_B(P, -): B \to R-\text{mod} \) be the related exact functor. Then the induced maps \( \Theta: \Ext^i_B(B, C) \to \Ext^i_R(\Theta(B), \Theta(C)) \) are isomorphisms for all \( B, C \in B \) and all \( 0 \leq i < \infty \) if and only if the induced triangulated functor between the bounded above derived categories \( \Theta: D^-(B) \to D^-(R-\text{mod}) \) is fully faithful.

**Proof.** Clearly, an exact functor between abelian categories \( B \to A \) induces isomorphisms on all the Ext groups if and only if the induced triangulated functor between the bounded derived categories \( D^-(B) \to D^-(A) \) is fully faithful if and only if the functor \( \Theta: D^-(B) \to D^-(A) \) is, provided that there are enough projective objects in the category \( A \) and the exact functor \( B \to A \) has a left adjoint \( \Delta: A \to B \).

In the situation at hand, obviously there are enough projectives in \( R-\text{mod} \), so it remains to recall from the discussion before Theorem 4.6 that the functor \( \Theta \) has a left adjoint. Alternatively, one can construct the functor \( \Delta \) explicitly, as the unique right exact functor taking the free left \( R \)-module \( R[X] \) to the object \( P(X) \in B \) for all sets \( X \) and acting on morphisms between free left \( R \)-modules in the natural way (determined by the ring homomorphism \( \theta \)). \( \square \)

Now let us restrict to the particular case when the ring \( R \) coincides with the endomorphism ring \( \Hom_B(P, P)^{\text{op}} \) and \( \theta \) is the identity isomorphism. Let \( B \) be a locally presentable abelian category with a projective generator \( P \). We will say that the projective generator \( P \in B \) is 0-good if the conservative exact functor \( \Theta_P = \Hom_B(P, -): B \to R-\text{mod} \), where \( R = \Hom_B(P, P)^{\text{op}} \), is fully faithful.

A projective generator \( P \in B \) is 0-good if and only if the full subcategory on one object \( \{P\} \subset B \) is dense in the category \( B \) [1 Proposition 1.26(i)] (cf. [1] Remark 1.23 and Example 1.24(4)]). Such full subcategories were called “left adequate” in the terminology of the paper [16].

Furthermore, we will say that a projective generator \( P \in B \) is 1-good if the functor \( \Theta_P \) is fully faithful and the full subcategory \( \Theta_P(B) \subset R-\text{mod} \) is closed under extensions. More generally, for any \( n \geq 0 \), we will say that a projective generator \( P \in B \) is \( n \)-good if the functor \( \Theta_P \) induces isomorphisms on the groups \( \Ext^i \) for all \( 0 \leq i \leq n \).

Finally, we will say that a projective generator \( P \in B \) is good (or \( \infty \)-good) if it is \( n \)-good for all \( n \geq 0 \). According to Proposition 1.8, a projective generator \( P \in B \) is good if and only if the functor \( \Theta_P \) induces a fully faithful functor between the bounded above derived categories

\[
D^-(B) \to D^-(R-\text{mod}).
\]

Clearly, any \((n + 1)\)-good projective generator, \( n \geq 0 \), is at the same time \( n \)-good.

The following corollary summarizes the assertions of Theorem 1.6 in the case \( \theta = \text{id} \).
Corollary 4.9. Let $\mathcal{B}$ be a locally presentable abelian category and $P \in \mathcal{B}$ be a projective generator. Then

(a) if $P$ is 0-good, then the full subcategory $\Theta_P(\mathcal{B}) \subset R\text{-}\text{mod}$ is an abelian, exactly embedded right 0-perpendicular subcategory to a set of morphisms in $R\text{-}\text{mod}$;

(b) assume that $P$ is 0-good; then $P$ is 1-good if and only if $\Theta_P(\mathcal{B}) \subset R\text{-}\text{mod}$ is a right 1-perpendicular subcategory (to a set of objects) in $R\text{-}\text{mod}$;

(c) if $P$ is $n$-good for some $n \geq 1$, then $\Theta_P(\mathcal{B}) \subset R\text{-}\text{mod}$ is a right $n$-perpendicular subcategory to a set of objects in $R\text{-}\text{mod}$;

(d) if $P$ is good, then $\Theta_P(\mathcal{B}) \subset R\text{-}\text{mod}$ is a right $\infty$-perpendicular subcategory to a set of objects in $R\text{-}\text{mod}$. □

For any cardinal $\lambda$, we denote its successor cardinal by $\lambda^+$. The proof of the following theorem will be given in Section 6.

Theorem 4.10. Let $\mathcal{B}$ be an abelian category with a projective generator $P$ and $\lambda$ be a cardinal such that the category $\mathcal{B}$ is locally $\lambda^+$-presentable and the object $P \in \mathcal{B}$ is $\lambda^+$-presentable. Then $Q = P^{(\lambda)}$ is a good projective generator of $\mathcal{B}$.

The following corollary shows that the classes of abelian categories embodied by the (abelian and exactly embedded) right $n$-perpendicular subcategories to sets of objects/morphisms in the categories of modules over associative rings are the same for all integers $n \geq 0$ and for $n = \infty$.

Corollary 4.11. The following classes of abelian categories (viewed as abstract categories irrespectively of a particular embedding into a category of modules) coincide with each other, and with the class of all locally presentable abelian categories with a projective generator:

- abelian, exactly and accessibly embedded full subcategories in the categories of modules over associative rings, closed under infinite products;
- abelian, exactly embedded right 0-perpendicular subcategories to sets of morphisms in the categories of modules over associative rings;
- abelian, exactly embedded right 1-perpendicular subcategories to sets of objects in the categories of modules over associative rings;
- right $n$-perpendicular subcategories to sets of objects in the categories of modules over associative rings (where $n \geq 2$ is some fixed integer);
- right $\infty$-perpendicular subcategories to sets of objects in the categories of modules over associative rings.

Proof. Any right $(n + 1)$-perpendicular subcategory to a set of objects in $R\text{-}\text{mod}$ is at the same time a right $n$-perpendicular subcategory to a set of objects/morphisms in $R\text{-}\text{mod}$ by Lemma 4.1. Any right 2-perpendicular subcategory in an abelian category is abelian and exactly embedded by Lemma 4.2.

Any right 0-perpendicular subcategory to a set of morphisms in $R\text{-}\text{mod}$ is accessibly embedded and closed under infinite products (in fact, arbitrary limits) by [1, Observation 1.34 and Proposition 1.35]. Any abelian, exactly and accessibly embedded full subcategory in $R\text{-}\text{mod}$ is locally presentable and reflective by [1, Theorem
and Corollary 2.48] (see also [1, Theorem 1.39] and Lemma 4.4 above). Any abelian, exactly embedded, reflective full subcategory in $R$–mod has a projective generator by Lemma 4.5. This proves that all the itemized classes of categories consist of locally presentable abelian categories with a projective generator.

Conversely, if $\mathcal{B}$ is a locally $\kappa$-presentable abelian category with a $\kappa$-presentable projective generator $P$, and $\lambda$ is a cardinal such that $\lambda^+ \geq \kappa$, then $Q = P^{(\lambda)}$ is a good projective generator of $\mathcal{B}$ by Theorem 4.10. Set $S = \text{Hom}_{\mathcal{B}}(Q, Q)^{op}$; then the functor $\Theta_Q: \mathcal{B} \rightarrow S$–mod is fully faithful and its image is the right $\infty$-perpendicular subcategory to a set of objects in $S$–mod by Corollary 4.9(d).

Under Vopěnka’s principle, the set-theoretical conditions (“accessibly embedded”, “to sets of”) in Corollary 4.11 can be dropped. The related classes of abelian categories (that is, the classes of all abelian, exactly embedded right $n$-perpendicular subcategories to classes of objects or morphisms in the categories of modules over associative rings, for various values of $n$) will still be the same, and coincide with the class of all locally presentable abelian categories with a projective generator. This is provable using the above arguments together with [1, Corollary 6.24].

5. Examples of Good Projective Generators

The aim of this section is to provide some examples to the theory developed in Section 4, mostly based on Examples 1.2–1.3 and examples from Sections 2–3.

Examples 5.1. (1) Let $\mathcal{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals, $R$ be an associative ring, and $\theta: R \rightarrow \mathcal{R}$ be an associative ring homomorphism such that the forgetful functor $\mathcal{R}$–contra $\rightarrow R$–mod is fully faithful. Then the forgetful functor $\mathcal{R}$–contra $\rightarrow \mathcal{R}$–mod is also fully faithful. So, by the definition, the free left $\mathcal{R}$-contramodule with one generator $P = R = \mathcal{R}[[*]]$ is a 0-good projective generator of $\mathcal{R}$–contra. This includes all the topological rings satisfying the assumptions of Theorem 3.1, and in particular, all the topological rings from Examples 3.3 and 3.6 (and those topological rings from Examples 3.7 which satisfy the conditions mentioned there).

See Examples 5.2(6-8), 5.3(4), 5.4(2), and 5.5(2) for further discussion.

(2) Let $\mathcal{R}$ be a complete, separated topological associative ring with a base of neighborhoods of zero formed by open right ideals, $R$ be an associative ring, and $\theta: R \rightarrow \mathcal{R}$ be an associative ring homomorphism such that the related forgetful functor $\Theta: \mathcal{R}$–contra $\rightarrow R$–mod is fully faithful. Then the argument from the proof of Theorem 4.6 shows how to distinguish the objects of the full subcategory $\mathcal{R}$–contra among the objects of the ambient category $R$–mod.

Specifically, let $Y$ be a set of the cardinality greater or equal to the cardinality of a base of neighborhoods of zero in $\mathcal{R}$. Denote by $\theta_Y$ the natural $R$-module morphism $R[Y] \rightarrow \mathcal{R}[[Y]]$. Then a left $R$-module $C$ belongs to the full subcategory $\Theta(\mathcal{R}$–contra) $\subset R$–mod if and only if the morphism of abelian groups
Hom$_R(\theta_Y, C) = \text{Hom}_R(\mathfrak{R}[[Y]], C) \to \text{Hom}_R(R[Y], C) = C^Y$ is an isomorphism. Notice that if $Z$ is a subset in $Y$ and the map Hom$_R(\theta_Y, C)$ is an isomorphism, then the map Hom$_R(\theta_Z, C)$ is also an isomorphism, because the left $R$-module morphism $\theta_Z : R[Z] \to \mathfrak{R}[[Z]]$ is a direct summand of the morphism $\theta_Y$.

In particular, in the context of Theorem 3.11, it suffices to take a countable set $Y$.

Examples 5.2. (1) Let $R$ be a Noetherian commutative ring and $I \subseteq R$ be a finitely generated ideal; denote by $\mathfrak{R} = \varprojlim R/I^n$ the $I$-adic completion of the ring $R$. Let $B$ be the locally $\aleph_1$-presentable abelian category $\mathfrak{R}_{contra} \simeq R{-\text{mod}_{fctra}}$ (see Examples 2.2) and $P = \mathfrak{R} = \Delta_I(R)$ be its natural projective generator. Then the forgetful/natural embedding functor $\Theta : B \to R{-\text{mod}}$ is corepresented by the projective generator $P \in B$ with the natural action of the ring $R$ in it (provided by the natural ring homomorphism $\theta : R \to \mathfrak{R} = \text{Hom}_B(P, P)^{opp}$).

According to [24 Theorem B.8.1], the functor $\Theta$ induces isomorphisms of the groups Ext$^i$ for all $0 \leq i < \infty$. Moreover, one can show using [24 Propositions B.9.1 and B.10.1] that the triangulated functor between the bounded above derived categories $D^-(B) \to D^-(R{-\text{mod}})$ induced by exact functor $\Theta$ is fully faithful.

In particular, $\mathfrak{R}$ is also a Noetherian commutative ring; so one can replace $R$ with $\mathfrak{R}$ and consider the case $R = \mathfrak{R}$. Then we have $\Theta = \Theta_P$, so we can conclude that $P = \mathfrak{R}$ is a good projective generator of $B$.

(2) More generally, let $R$ be a commutative ring and $I \subseteq R$ be a weakly proregular finitely generated ideal. We keep the notation $\mathfrak{R} = \varprojlim R/I^n$, $B = \mathfrak{R}_{contra} = R{-\text{mod}_{fctra}}$, and $P = \mathfrak{R} = \Delta_I(R)$ (see Example 2.2(3)). As in (1), $B$ is a locally $\aleph_1$-presentable abelian category, $P$ is an $\aleph_1$-presentable projective generator of $B$, and the forgetful/natural embedding functor $\Theta : B \to R{-\text{mod}}$ is corepresented by $P$.

According to [27 Theorem 2.9], the triangulated functor between the unbounded derived categories $D(B) \to D(R{-\text{mod}})$ induced by $\Theta$ is fully faithful. Hence the functor $\Theta$ induces isomorphisms of the groups Ext$^i$ for all $0 \leq i < \infty$.

By [27 Lemma 5.3(b) or 5.4(b)], the ideal $\mathfrak{RI} = \varprojlim I/I^n$ in the ring $\mathfrak{R}$ is also weakly proregular; so one can replace $R$ with $\mathfrak{R}$ and consider the case $R = \mathfrak{R}$. Then $\Theta = \Theta_P$, so we can conclude that $P = \mathfrak{R}$ is a good projective generator of $B$.

(3) Even more generally, let $R$ be a commutative ring and $I \subseteq R$ be a finitely generated ideal. Set $B = R{-\text{mod}_{fctra}}$ and $P = \Delta_I(R)$ (see Example 2.2(1)). As in (2), $B$ is a locally $\aleph_1$-presentable abelian category, $P$ is an $\aleph_1$-presentable projective generator of $B$, and the natural embedding functor $\Theta : B \to R{-\text{mod}}$ is corepresented by the projective generator $P \in B$ with the natural action of the ring $R$ in it (provided by the natural ring homomorphism $\theta : R \to \Delta_I(R) = \text{Hom}_B(P, P)^{opp}$).

Following the argument in [27, proof of Theorem 2.9] based on [27 Lemma 2.7], the triangulated functor $D(B) \to D(R{-\text{mod}})$ induced by $\Theta$ is fully faithful provided that $H_i(\text{Hom}_R(T^*(R; s_1, \ldots, s_m), R[X])) = 0$ for all sets $X$ and all $i > 0$. Conversely, arguing as in [29 second paragraph of Remark 6.8] (with torsion modules replaced by contramodules), one can show that $H_i(\text{Hom}_R(T^*(R; s_1, \ldots, s_m), R[X])) = 0$ for all $i > 0$ whenever the functor $D^b(B) \to D^b(R{-\text{mod}})$ is fully faithful. Thus,
given any derived category symbol $\star = b, +, -$, or $\otimes$, the triangulated functor $\mathcal{D}^\star(B) \rightarrow \mathcal{D}^\star(R\text{-mod})$ induced by $\Theta$ is fully faithful if and only if for all sets $X$ one has $H_i(\text{Hom}_R(T^\star(R; s_1, \ldots, s_m), R[X])) = 0$ for $i > 0$, or equivalently, if and only if $\lim_{\leftarrow n} H_i(\text{Hom}_R(T^\star_n(R; s_1, \ldots, s_m), R[X])) = 0$ for $i \geq 1$ and $\lim_{\leftarrow n} H_i(\text{Hom}_R(T^\star_n(R; s_1, \ldots, s_m), R[X])) = 0$ for $i \geq 2$.

Arguing as in [29, Proposition 3.1], one can show that $\Delta_f(R)$ is a commutative ring. All the $I$-contramodule $R$-modules admit a unique natural extension of their $R$-module structure to a $\Delta_f(R)$-module structure. So one can replace $R$ with $\Delta_f(R)$ and $I$ with $\Delta_f(R)I$, and consider the case $R = \Delta_f(R)$. Then $\Theta = \Theta_P$. So $P$ is always a 1-good projective generator of $B$, since $B = R\text{-mod}_I$-contra is closed under extensions in $R\text{-mod}$. As we have seen, $P$ is a good projective generator of $B$ if and only if $H_i(\text{Hom}_R(T^\star_n(R; s_1, \ldots, s_m), \Delta_f(R)[X])) = 0$ for all sets $X$ and all $i > 0$.

(4) Let $E_1 \leftarrow E_2 \leftarrow E_3 \leftarrow \cdots$ be a projective system of abelian groups. Clearly, one has $\lim_{\leftarrow n} E_n[X] = 0$ for all sets $X$ if and only if $\lim_{\leftarrow n} E_n = 0$. Furthermore, one has $\lim_{\leftarrow n} H_n(\text{Hom}_R(T^\star_n(R; s_1, \ldots, s_m), \Delta_f(R)[X])) = 0$ for all sets $X$ if and only if the projective system $E_n$ satisfies the Mittag-Leffler condition [9, Corollary 6]. Finally, one easily checks that a projective system $(E_n)$ satisfies both the condition of the vanishing of the projective limit $\lim_{\leftarrow n} E_n = 0$ and the Mittag-Leffler condition if and only if it is pro-zero.

Thus, in the context of (3), the triangulated functor $\mathcal{D}^\star(R\text{-mod}_I)$-contra $\rightarrow \mathcal{D}^\star(R\text{-mod})$ is fully faithful if and only if the projective system $H_i(\text{Hom}_R(T^\star_n(R; s_1, \ldots, s_m), R))$ is pro-zero for $i \geq 2$ and $\lim_{\leftarrow n} H_1(\text{Hom}_R(T^\star_n(R; s_1, \ldots, s_m), R)) = 0$.

(5) In particular, let $R$ be the commutative algebra over a field $k$ generated by the elements $s, t,$ and $x_i, i \geq 1,$ with the relations $x_ix_j = 0$ and $s^it^jx_i = 0$ for all $i, j \geq 1$ (cf. [27, Example 2.6]). Let $I = (s, t) \subset R$ be the ideal generated by the elements $s$ and $t$. Then $H_1(\text{Hom}_R(T^\star_n(R; s, t), R)) = 0$ for all $n \geq 1$ and $\lim_{\leftarrow n} H_2(\text{Hom}_R(T^\star_n(R; s, t), R[X])) = 0$ for all sets $X$, while $\lim_{\leftarrow n} H_2(\text{Hom}_R(T^\star_n(R; s, t), R)) \neq 0$. So $\Delta_f(R[X]) = \Delta_f(R[X]) = \mathfrak{R}[[X]]$ for all $X$.

Set $B = R\text{-mod}_I$-contra and $P = \Delta_f(R) = \mathfrak{R}$. One has $\bigcap_n I^n = 0$ in $R$, hence the morphism $\theta: R \rightarrow \mathfrak{R}$ is injective. Thus the category $B$ with its projective generator $P$ and the morphism $\theta$ provide a counterexample to the converse assertions to Theorem 4.6(c-d) promised in Remark 13. Denoting by $C$ the $I$-contramodule $R$-module $H_1(\text{Hom}_R(T^\star_n(R; s, t), R)) = \lim_{\leftarrow n} H_2(\text{Hom}_R(T^\star_n(R; s, t), R))$, the complex of $R$-modules $\text{Hom}_R(T^\star(R; s, t), R)$ represents a nontrivial extension class in $\text{Ext}_R^2(P, C)$.

Furthermore, one has $H_1(\text{Hom}_R(T^\star_n(R; s, t), \mathfrak{R})) = 0$ for all $n \geq 1$ and $\lim_{\leftarrow n} H_2(\text{Hom}_R(T^\star_n(R; s, t), \mathfrak{R}[X])) = 0$ for all sets $X$, while $\lim_{\leftarrow n} H_2(\text{Hom}_R(T^\star_n(R; s, t), \mathfrak{R}[Z])) \neq 0$ for a countable set $Z$. Thus the category $B$ with its projective generator $P$ (and the identity isomorphism $\theta = \text{id}_\mathfrak{R}$) provide a counterexample to the converse assertions to Corollary 4.7(c-d). Setting $Q = \Delta_f(R[Z]) = \Delta_f(\mathfrak{R}[Z]) = \mathfrak{R}[[Z]]$ and denoting by $C$ the $(\mathfrak{R})$-contramodule $\mathfrak{R}$-module $H_1(\text{Hom}_R(T^\star(R; s, t), \mathfrak{R}[Z])) = \ldots$
\[ \lim_{n} H_2(\text{Hom}_R(T_n^\bullet (R; s, t), \mathfrak{R}[Z])) \], the complex of \( \mathfrak{R} \)-modules \( \text{Hom}_R(T^\bullet (R; s, t), \mathfrak{R}[Z]) \) represents a nontrivial extension class in \( \text{Ext}^2_{\mathfrak{R}}(Q, C) \).

(6) Let \( R \) be a commutative ring and \( I \subset R \) be a finitely generated ideal. Set \( \mathfrak{R} = \lim_{n} R/I^n, \mathcal{B} = \mathfrak{R} - \text{contra}, \) and \( P = \mathfrak{R} \) (see Examples [27, 2.2(1)], and [27, 3.6(2-4)]). Then the abelian category \( \mathcal{B} \) is exactly embedded full subcategory of the abelian category \( R \text{-mod}_{\mathfrak{R} \text{-contra}} \). We claim that every object of \( R \text{-mod}_{\mathfrak{R} \text{-contra}} \) is an extension of two objects from \( \mathfrak{R} - \text{contra} \).

Indeed, every object of \( R \text{-mod}_{\mathfrak{R} \text{-contra}} \) has the form \( \Delta_I(C) \) for some \( C \in R \text{-mod} \). According to [28, Lemma 7.5], there is a natural short exact sequence of \( R \)-modules \( 0 \rightarrow K \rightarrow \Delta_I(C) \rightarrow \Lambda_I(C) \rightarrow 0 \) with an \( R \)-module \( K \) of the form \( K = \lim_{n} D_n \), where \( D_n \) is a sequence of \( R \)-modules, each of which is annihilated by some power of the ideal \( I \). Now we have \( D_n \in \mathfrak{R} - \text{contra} \), and it follows that \( K \in \mathfrak{R} - \text{contra} \), as the full subcategory \( \mathfrak{R} - \text{contra} \subset R \text{-mod} \) is closed under the cokernels and infinite products. Similarly, we have \( \Lambda_I(C) \in \mathfrak{R} - \text{contra} \).

So the full subcategory \( \mathfrak{R} - \text{contra} \subset R \text{-mod} \) is closed under extensions if and only if it coincides with \( R \text{-mod}_{\mathfrak{R} \text{-contra}} \). According to (4) and Example [27, 8.2(3)], this holds if and only if the projective system \( H_1(\text{Hom}_R(T_n^\bullet (R; s_1, \ldots, s_m), R)) \) satisfies the Mittag-Leffler condition (where \( s_1, \ldots, s_m \) is some set of generators of the ideal \( I \)). Furthermore, it follows that, for any derived category symbol \( * = \emptyset, +, - \), or \( \mathfrak{S} \), the triangulated functor \( D^\bullet (\mathfrak{R} - \text{contra}) \rightarrow D^\bullet (R \text{-mod}) \) is fully faithful if and only if the projective system \( H_1(\text{Hom}_R(T_n^\bullet (R; s_1, \ldots, s_m), R)) \) is pro-zero for all \( i \geq 1 \), or in other words, the ideal \( I \subset R \) is weakly proregular.

It was mentioned in Example [4.1(1)] that the projective generator \( P \in \mathfrak{R} - \text{contra} \) is 0-good. Now we see that \( P \) is 1-good if and only if the full subcategory \( \mathfrak{R} - \text{contra} \subset R \text{-mod} \) coincides with \( R \text{-mod}_{\mathfrak{R} \text{-contra}} \), and this holds if and only if the projective system \( H_1(\text{Hom}_R(T_n^\bullet (R; s_1, \ldots, s_m), \mathfrak{R})) = 0 \) satisfies the Mittag-Leffler condition. The projective generator \( P \) is good if and only if the ideal \( \mathfrak{R} I \subset \mathfrak{R} \) is weakly proregular.

(7) In particular, let \( I = (s) \subset R \) be a principal ideal. Then \( H_1(\text{Hom}_R(T_n^\bullet (R; s), R)) \) is the submodule of elements annihilated by \( s^n \) in \( R \), and the maps in the projective system formed by these modules are the multiplications with (the powers of) \( s \). Hence \( \lim_{n} H_1(\text{Hom}_R(T_n^\bullet (R; s), R)) = 0 \) if and only if there is no \( s \)-divisible \( s \)-torsion in \( R \), and the projective system \( H_1(\text{Hom}_R(T_n^\bullet (R; s), R)) \) is pro-zero if and only if the \( s \)-torsion in \( R \) is bounded.

Furthermore, the rings \( \Delta_s(R) \) and \( \mathfrak{R} = \Lambda_s(R) \) are \( s \)-contra modules, so they cannot contain \( s \)-divisible \( s \)-torsion. Thus \( \mathfrak{R} - \text{contra} = R \text{-mod}_{\mathfrak{R} \text{-contra}} \) if and only if the \( s \)-torsion in \( \Delta_s(R) \) is bounded; and \( \mathfrak{R} - \text{contra} = R \text{-mod}_{\mathfrak{R} \text{-contra}} \) if and only if the \( s \)-torsion in \( \mathfrak{R} \) is bounded. According to [28, Remark 6.9], the latter two conditions are equivalent. The projective generator \( P = \mathfrak{R} \in \mathfrak{R} - \text{contra} \) is 1-good if and only if it good, and if and only if these equivalent conditions hold.

(8) In particular, if \( R \) is the commutative \( k \)-algebra from Example [4.3(6)] and [27, Example 2.6], and \( I \) is the principal ideal \( I = (s) \), then the projective generator
\( P \in \mathfrak{R} \text{-contra} \) is not 1-good. Indeed, the \( s \)-torsion in \( \mathfrak{R} \) is not bounded, so the morphism \( \Delta_I(\mathfrak{R}[Z]) \to \mathfrak{R}[Z] \) has a nontrivial kernel for a countable set \( Z \).

**Examples 5.3.** (1) Let \( R \) be a right Noetherian associative ring and \( I \subseteq R \) be an ideal generated by a finite set of central elements. Denote by \( \mathfrak{R} \) the ideal generated by a finite set of central elements. Denote by \( \mathfrak{R} \) the ideal generated by \( s \) and \( \mathfrak{R} \) is the locally \( \mathfrak{R} \)-presentable abelian category \( \mathfrak{R} \text{-contra} \cong R \text{-mod}_{\mathfrak{R}} \) (see Examples 2.3) and \( P = \Delta_I(R) \) is its natural projective generator. Then the forgetful/natural embedding functor \( \Theta: \mathfrak{R} \to R \text{-mod} \) is corepresented by the projective generator \( P \in \mathfrak{B} \) with the natural right action of the ring \( R \) in it (provided by the natural ring homomorphism \( \theta: R \to \mathfrak{R} = \text{Hom}_\mathfrak{B}(P, P)^{op} \)).

According to [25, Corollary C.5.6(b)], the functor \( \Theta \) induces isomorphisms of the groups \( \text{Ext}^i \) for all \( 0 \leq i < \infty \). Moreover, one can show using [25] Proposition C.5.5 and Corollary C.5.6(a)] that the triangulated functor \( D^-(\mathfrak{B}) \to D^-(R \text{-mod}) \) induced by the exact functor \( \Theta \) is fully faithful.

In particular, \( \mathfrak{R} \) is also a right Noetherian ring, so one can replace \( R \) with \( \mathfrak{R} \) and consider the case \( R = \mathfrak{R} \). Then we have \( \Theta = \Theta_P \), so we can conclude that \( P = \mathfrak{R} \) is a good projective generator of \( \mathfrak{B} \).

(2) More generally, let \( R \) be an associative ring and \( I \subseteq R \) be the ideal generated by a weakly proregular finite sequence of central elements \( s_1, \ldots, s_m \in R \), i.e., a finite sequence of central elements such that the projective system of \( R \text{-}R \)-bimodules \( \text{Hom}_R(T_n(R; s_1, \ldots, s_m, R)) \), \( n \geq 1 \), is pro-zero for all \( i > 0 \) (see [37, 22, 27]; cf. Example 2.3(3)). We keep the notation \( \mathfrak{R} = \varprojlim R/I^n \), \( \mathfrak{B} = \mathfrak{R} \) is the ideal generated by \( s \), \( \mathfrak{R} \text{-contra} = \mathfrak{R} \text{-mod}_{\mathfrak{R}} \), and \( P = \mathfrak{R} = \Delta_I(R) \). As above, \( \mathfrak{B} \) is a locally \( \mathfrak{R} \)-presentable abelian category, \( P \) is an \( \mathfrak{R} \)-presentable projective generator of \( \mathfrak{B} \), and the forgetful/natural embedding functor \( \Theta: \mathfrak{B} \to R \text{-mod} \) is corepresented by \( P \).

Arguing as in [27] Section 2] (see [29] Theorem 6.4 for additional details), one can show that the triangulated functor \( D(\mathfrak{B}) \to D(R \text{-mod}) \) induced by \( \Theta \) is fully faithful. Hence the functor \( \Theta \) induces isomorphisms of the groups \( \text{Ext}^i \) for all \( 0 \leq i < \infty \).

Using the appropriate generalization of [27] Lemma 5.3(b)], one can show that the sequence of central elements \( s_1, \ldots, s_m \) is also weakly proregular in the ring \( \mathfrak{R} \). So one can replace \( R \) with \( \mathfrak{R} \) and consider the case \( R = \mathfrak{R} \). Then \( \Theta = \Theta_P \), and one can conclude that \( P = \mathfrak{R} \) is a good projective generator of \( \mathfrak{B} \).

(3) Even more generally, let \( R \) be an associative ring and \( I \subseteq R \) be the ideal generated by a finite set of central elements \( s_1, \ldots, s_m \in R \). Set \( \mathfrak{B} = R \text{-mod}_{\mathfrak{R}} \) and \( P = \Delta_I(R) \) (see Example 2.3(1)). As above, \( \mathfrak{B} \) is a locally \( \mathfrak{R} \)-presentable abelian category, \( P \) is an \( \mathfrak{R} \)-presentable projective generator of \( \mathfrak{B} \), and the natural embedding functor \( \Theta: \mathfrak{B} \to R \text{-mod} \) is corepresented by the projective generator \( P \in \mathfrak{B} \) with the natural right action of the ring \( R \) in it (provided by the natural ring homomorphism \( \theta: R \to \Delta_I(R) = \text{Hom}_\mathfrak{B}(P, P)^{op} \)).

Arguing as in Example 5.2(3), one shows that the triangulated functor \( D^*(\mathfrak{B}) \to D^*(R \text{-mod}) \) induced by \( \Theta \), where \( * = b, +, -, \) or \( \otimes \), is fully faithful if and only
if $H_i(\text{Hom}_R(T^*(R; s_1, \ldots, s_m), R[X])) = 0$ for all sets $X$ and all $i > 0$, or equivalently, if and only if $\lim_{\leftarrow n} H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), R[X])) = 0$ for $i \geq 1$ and $\lim^1 H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), R)) = 0$ for $i \geq 2$. As explained in Example 5.2(4), this holds if and only if the projective system $H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), R))$ is pro-zero for $i \geq 2$ and $\lim_{\leftarrow n} H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), R)) = 0$.

Furthermore, all the $I$-contramodule left $R$-modules admit a unique natural extension of their left $R$-module structure to a left $\Delta_I(R)$-module structure. Central elements in $R$ remain central in $\Delta_I(R)$. So one can replace $R$ with $\Delta_I(R)$ and $I$ with $\Delta_I(R)I$, and consider the case $R = \Delta_I(R)$, then $\Theta = \Theta_P$. Hence $P$ is always a 1-good projective generator of $\mathcal{B}$, as $\mathcal{B} = \mathcal{R} \text{-mod}_{I, \text{contra}}$ is closed under extensions in $R \text{-mod}$. As we have seen, $P$ is a good projective generator of $\mathcal{B}$ if and only if $H_i(\text{Hom}_R(T^*(R; s_1, \ldots, s_m), \Delta_I(R)[X])) = 0$ for all sets $X$ and all $i > 0$, and if and only if the projective system $H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), \Delta_I(R)))$ is pro-zero for $i \geq 2$ and $\lim_{\leftarrow n} H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), \Delta_I(R))) = 0$.

(4) Let $R$ be an associative ring and $I \subset R$ be an ideal generated by a finite set of central elements. Set $\mathcal{R} = \lim_{\leftarrow n} R/I^n$, $\mathcal{B} = \mathcal{R} \text{-contra}$, and $P = \mathcal{R}$ (see Examples 1.3(2), 2.3(3), and 3.6(2-4)).

Then the abelian category $\mathcal{B}$ is an exactly embedded full subcategory of the abelian category $\mathcal{R} \text{-mod}_{I, \text{contra}}$. In the same way as in Example 5.2(6), one shows that every object of $R \text{-mod}_{I, \text{contra}}$ is an extension of two objects from $\mathcal{R} \text{-contra}$. Hence the full subcategory $\mathcal{R} \text{-contra} \subset R \text{-mod}$ is closed under extensions if and only if it coincides with $R \text{-mod}_{I, \text{contra}}$, which holds if and only if the projective system $H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), R))$ satisfies the Mittag-Leffler condition. Furthermore, for any derived category symbol $* = \text{b}, +, −, \text{or } \emptyset$, the functor $D^*(\mathcal{R} \text{-contra}) \to D^*(R \text{-mod})$ is fully faithful if and only if the projective system $H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), R))$ is pro-zero for all $i \geq 1$.

According to Example 5.4(1), the projective generator $P \in \mathcal{R} \text{-contra}$ is 0-good. Now we see that $P$ is 1-good if and only if the full subcategory $\mathcal{R} \text{-contra} \subset \mathcal{R} \text{-mod}$ coincides with $\mathcal{R} \text{-mod}_{\Delta(I) \text{-contra}}$, and $P$ is good if and only if the projective system $H_i(\text{Hom}_R(T^*_n(R; s_1, \ldots, s_m), \mathcal{R}))$ is pro-zero for all $i \geq 1$.

In particular, when $I = (s)$ is a principal ideal, one shows, as in Example 5.2(7), that $\mathcal{R} \text{-contra} = R \text{-mod}_{s \text{-contra}}$ if and only if the $s$-torsion in $\Delta_s(R)$ is bounded, if and only if the $s$-torsion in $\mathcal{R}$ is bounded, and if and only if $\mathcal{R} \text{-contra} = \mathcal{R} \text{-mod}_{s \text{-contra}}$. These are the necessary and sufficient conditions for the projective generator $P \in \mathcal{R} \text{-contra}$ to be 1-good or, which is equivalent in this case, good.

Examples 5.4. (1) Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset such that $pd_R S^{-1}R \leq 1$. Set $\mathcal{B} = R \text{-mod}_{S \text{-contra}}$ and $P = \Delta_S(R)$ (see Example 2.3(1)). Then $\mathcal{B}$ is a locally presentable abelian category, $P \in \mathcal{B}$ is a projective generator, and the natural embedding functor $\Theta : \mathcal{B} \to R \text{-mod}$ is corepresented by the projective generator $P$ with the natural action of the ring $R$ in it (provided by the natural ring homomorphism $\theta : R \to \Delta_S(R) = \text{Hom}_R(P, P)^{op}$).
According to [29, first paragraph of Remark 6.8], the triangulated functor $D(B) \rightarrow D(R{-}\text{mod})$ induced by $\Theta$ is fully faithful provided that there is no $S$-h-divisible $S$-torsion in $R$. Conversely, arguing as in [29] second paragraph of Remark 6.8 (with torsion modules replaced by contramodules), one shows that there is no $S$-h-divisible $S$-torsion in $R$ whenever the functor $D^b(B) \rightarrow D^b(R{-}\text{mod})$ is fully faithful. Thus, for any derived category symbol $\ast = b, +, \cdot, \circ, \emptyset$, the triangulated functor $D^\ast(B) \rightarrow D^\ast(R{-}\text{mod})$ induced by $\Theta$ is fully faithful if and only if there is no $S$-h-divisible $S$-torsion in the $R$-module $R$.

According to [29, Proposition 3.1], $\Delta_S(R)$ is a commutative ring. Since $S^{-1}R$ is a flat $R$-module, the condition that $pd_R S^{-1}R \leq 1$ implies that $pd_{\Delta_S(R)} S^{-1}\Delta_S(R) \leq 1$. All the $S$-contramodule $R$-modules admit a unique natural extension of their $R$-module structure to a $\Delta_S(R)$-module structure. So one can replace $R$ with $\Delta_S(R)$ and $S$ with its image in $\Delta_S(R)$, and consider the case $R = \Delta_S(R)$, so that $\Theta = \Theta_P$. Then the $R$-module $R$ is $S$-h-reduced, and therefore has no $S$-h-divisible $S$-torsion. Thus $P$ is always a good projective generator of $B$.

(2) Let $R$ be a commutative ring and $S \subseteq R$ be a countable multiplicative subset. Set $\mathfrak{R} = \lim_{\leftarrow n} R/s^nR$, $B = \mathfrak{R}{\text{-contra}}$, and $P = \mathfrak{R}$ (see Examples 2.4(2) and 3.7(2)). Then the abelian category $B$ is an exactly embedded full subcategory of the abelian category $R{-}\text{mod}_{S{-}\text{stra}}$. Both of these are locally $\aleph_1$-presentable abelian categories.

We claim that every object of $R{-}\text{mod}_{S{-}\text{stra}}$ is an extension of two objects from $\mathfrak{R}{\text{-contra}}$ (cf. Example 5.2(6)). Indeed, every object of $R{-}\text{mod}_{S{-}\text{stra}}$ has the form $\Delta_S(C)$ for some $C \in R{-}\text{mod}$. The functor $\Delta_S$ can be computed as $\Delta_S(C) = \text{Ext}_R(K^*, C)$ (see Example 2.4(1)).

Let $s_1, s_2, s_3, \ldots$ be a sequence of elements of $S$ containing every element of $S$ infinitely many times. Set $t_0 = 1$ and $t_n = s_1 s_2 \cdots s_n$ for all $n \geq 1$. Then the two-term complex $K^*$ is the inductive limit of the sequence of two-term complexes $R \rightarrow R$ (with the morphism of complexes $(R \rightarrow R) \rightarrow (R \rightarrow R)$ acting by $1$ on the leftmost terms of the complexes and by $s_n$ on the rightmost terms). Using this description of $K^*$, one can compute that for any $R$-module $C$ there is a short exact sequence of $R$-modules $0 \rightarrow \lim_{\leftarrow n} t_n C \rightarrow \Delta_S(C) \rightarrow \lim_{\leftarrow n} C/t_n C = \Lambda_S(C) \rightarrow 0$, where $tM \subset M$ denotes the submodule of all elements annihilated by an element $t \in R$ in an $R$-module $M$, and the maps $t_{n+1} C \rightarrow t_n C$ are the multiplications with $s_n$ (see [33, Lemma 3.2]). Since any $R$-module annihilated by an element from $S$ is an $\mathfrak{R}$-contramodule, and the full subcategory $\mathfrak{R}{\text{-contra}} \subset R{-}\text{mod}$ is closed under the kernels, cokernels, and infinite products, it follows that both $\Lambda_S(C)$ and the kernel of the surjective morphism $\Delta_S(C) \rightarrow \Lambda_S(C)$ belong to $\mathfrak{R}{\text{-contra}}$.

So the full subcategory $\mathfrak{R}{\text{-contra}} \subset R{-}\text{mod}$ is closed under extensions if and only if it coincides with $R{-}\text{mod}_{S{-}\text{stra}}$. According to Example 2.4(3) and [9, Corollary 6] (cf. Example 5.2(4)), this holds if and only if the projective system $(\iota_n R)_{n \geq 0}$ satisfies the Mittag-Leffler condition. Furthermore, can easily see that $\lim_{\leftarrow n} t_n R = 0$ if and only if there is no $S$-h-divisible $S$-torsion in $R$; and the projective system $(\iota_n R)_{n \geq 0}$ is pro-zero if and only if the $S$-torsion in $R$ is bounded (which means that there exists $t \in S$ such that $sr = 0$, $s \in S$, $r \in R$, implies $tr = 0$). Comparing these observations
with (1), we conclude that the functor $D^*(\mathcal{R}\text{-contra}) \to D^*(\mathcal{R}\text{-mod})$ is fully faithful if and only if the $S$-torsion in $R$ is bounded.

Arguing as in Example 5.2(7) and using an appropriate version of [28] Remark 6.9, one can show that $\mathcal{R}\text{-contra} = \mathcal{R}\text{-mod}_{\text{contra}}$ if and only if the $S$-torsion in $\mathcal{R}$ is bounded, if and only if the $S$-torsion in $\mathcal{R}$ is bounded, and if and only if $R\text{-mod}_{\text{contra}} = \mathcal{R}\text{-mod}_{\text{contra}}$. According to Example 5.1(1), the projective generator $P \in \mathcal{R}\text{-contra}$ is $0$-good. Now we see that $P \in \mathcal{R}\text{-contra}$ is $1$-good if and only if it is good, and if and only if these equivalent conditions hold.

**Examples 5.5.** (1) Let $R$ be an associative ring and $S \subset R$ be a multiplicative subset of central elements in $R$ such that $\text{pd}_R S^{-1}R \leq 1$. Set $\mathcal{B} = R\text{-mod}_{\text{contra}}$ and $P = \Delta_S(R)$ (see Example 2.5(1)). Then $\mathcal{B}$ is a locally presentable abelian category, $P \in \mathcal{B}$ is a projective generator, and the natural embedding functor $\Theta : \mathcal{B} \to R\text{-mod}$ is corepresented by the projective generator $P$ with the natural right action of the ring $R$ in it (provided by the natural ring homomorphism $\theta : R \to \Delta_S(R) = \text{Hom}_\mathcal{B}(P, P^{\text{op}})$).

Arguing as in Example 5.4(1), one shows that, for any derived category symbol $\ast = b, +, -, \text{ or } \emptyset$, the triangulated functor $D^*(\mathcal{B}) \to D^*(R\text{-mod})$ induced by $\Theta$ is fully faithful if and only if there are is no $S$-h-divisible $S$-torsion in the left $R$-module $R$.

Furthermore, central elements in $R$ remain central in $\Delta_S(R)$. Since $S^{-1}R$ is a flat left $R$-module, the condition that $\text{pd}_R S^{-1}R \leq 1$ implies that $\text{pd}_{\Delta_S(R)} S^{-1}\Delta_S(R) \leq 1$.

All the $S$-contramodule left $R$-modules admit a unique natural extension of their left $R$-module structure to a left $\Delta_S(R)$-module structure. So one can replace $R$ with $\Delta_S(R)$ and $S$ with its image in $\Delta_S(R)$, and consider the case $R = \Delta_S(R)$, so that $\Theta = \Theta_P$. Then the left $R$-module $R$ is $S$-h-reduced (i.e., there are no nonzero morphisms into it from left $S^{-1}R$-modules), and therefore has no $S$-h-divisible $S$-torsion. Thus $P$ is always a good projective generator of $\mathcal{B}$.

(2) Let $R$ be an associative ring and $S \subset R$ be a countable multiplicative subset of central elements. Set $\mathcal{R} = \lim_{\longrightarrow_{s \in S}} R/sR$, $\mathcal{B} = \mathcal{R}\text{-contra}$, and $P = \mathcal{R}$ (see Examples 2.5(2-3) and 3.7(2)). The abelian category $\mathcal{B}$ is an exactly embedded full subcategory of the abelian category $R\text{-mod}_{\text{contra}}$. Both $\mathcal{B}$ and $R\text{-mod}_{\text{contra}}$ are locally $\aleph_1$-presentable categories. Arguing as in Example 5.4(2), one shows that every object of $R\text{-mod}_{\text{contra}}$ is an extension of two objects from $\mathcal{R}\text{-contra}$.

Thus the full subcategory $\mathcal{R}\text{-contra} \subset R\text{-mod}$ is closed under extensions if and only if it coincides with $R\text{-mod}_{\text{contra}}$. The functor $D^*(\mathcal{R}\text{-contra}) \to D^*(R\text{-mod})$ is fully faithful if and only if the $S$-torsion in $R$ is bounded. Furthermore, one has $\mathcal{R}\text{-contra} = R\text{-mod}_{\text{contra}}$ if and only if the $S$-torsion in $\Delta_S(R)$ is bounded, if and only if the $S$-torsion in $\mathcal{R}$ is bounded, and if and only if $\mathcal{R}\text{-contra} = R\text{-mod}_{\text{contra}}$. The projective generator $P \in \mathcal{R}\text{-contra}$ is always $0$-good. The previous several equivalent conditions are necessary and sufficient for $P$ to be $1$-good or, which is equivalent in this situation, good.

The next series of examples explains the reason for the “good projective generator” terminology.
Examples 5.6. (1) Let $C$ be an accessible additive category (notice that any such category contains the images of idempotent endomorphisms of its objects [11 Observation 2.4]). Let $M \in C$ be an object. According to Example [12](2), there exists a unique abelian category $B$ with enough projective objects for which there is an equivalence of additive categories $B_{\text{proj}} \simeq \text{Add}_C(M)$. The category $B$ comes endowed with a natural projective generator $P \in B_{\text{proj}}$ corresponding to the free $T_M$-module with one generator $T_M(*) \in T_M-\text{mod}$ and to the object $M \in \text{Add}_C(M)$. If the object $M \in C$ is $\kappa$-presentable, where $\kappa$ is a regular cardinal, then the abelian category $B$ is locally $\kappa$-presentable and the object $P \in B$ is $\kappa$-presentable.

We will say that an object $M \in C$ is $n$-good, where $n \geq 0$ is an integer, if the projective generator $P \in B$ is $n$-good. In other words, denoting by $\mathcal{G}$ the ring $\text{Hom}_C(M, M)^{op}$, the object $M \in C$ is $n$-good if and only if the functor $\Theta = \text{Hom}_B(P, -) : \mathcal{B} \to \mathcal{G}-\text{mod}$ induces isomorphisms on all the groups $\text{Ext}^i$ with $0 \leq i \leq n$. An object $M \in C$ is good if it is $n$-good for all $n \geq 0$, or in other words, if the projective generator $P \in B$ is good. According to Proposition 4.8, an object $M$ is good if and only if the triangulated functor $D^-(\mathcal{B}) \to D^-(\mathcal{G}-\text{mod})$ induced by $\Theta$ is fully faithful.

According to Theorem 4.10 if the object $M \in C$ is $\kappa$-presentable (or, more generally, $\kappa$-generated), then for any cardinal $\lambda$ such that $\lambda^+ \geq \kappa$ the object $M^{(\lambda)} \in C$ is good.

(2) In particular, let $R$ be an associative ring and $M$ be a left $R$-module. Substituting $C = \mathcal{R}-\text{mod}$ into the definitions in (1), we obtain the definition of what it means for $M$ to be $n$-good or good. Specifically, it means that the forgetful functor $\mathcal{G}-\text{contra} \to \mathcal{G}-\text{mod}$, where $\mathcal{G} = \text{Hom}_R(M, M)^{op}$ is the topological ring from Example [13](3), should induce isomorphisms on all the groups $\text{Ext}^i$ with $0 \leq i \leq n$, or with $i \geq 0$, respectively. An $R$-module $M$ is good if and only if the triangulated functor $D^-(\mathcal{G}-\text{contra}) \to D^-(\mathcal{G}-\text{mod})$ is fully faithful. If a left $R$-module $M$ admits a set of generators of cardinality $\lambda$, then the left $R$-module $M^{(\lambda)}$ is good.

(3) Any finitely generated left $R$-module $M$ is good. Indeed, in this case the monad $T_M$ is isomorphic to the monad $T_S$ associated with the discrete ring $S = \text{Hom}_R(M, M)^{op}$ as in Example [13](1), so the functor $\Theta = \text{Hom}_B(P, -) : \mathcal{B} \to S-\text{mod}$ is an equivalence of abelian categories.

(4) Let $R$ be a commutative ring and $S \subset R$ be a multiplicative subset such that all elements of $S$ are nonzero-divisors in $R$. Assume that $pd_RS^{-1}R \leq 1$ and set $K = S^{-1}R/R \in \mathcal{R}-\text{mod}$. Then $B = \mathcal{R}-\text{mod}_{S\text{-stra}} = \mathcal{R}-\text{contra}$, where $\mathcal{R} = \Delta_S(R) = \Delta_S(R)$ (see Example [24](3)), is the corresponding locally presentable abelian category with enough projective objects such that $B_{\text{proj}} \simeq \text{Add}_R(K)$, and $P = \mathcal{R} \in B$ is its natural projective generator. Indeed, the functors $Q \mapsto K \otimes_R Q$ and $N \mapsto \text{Hom}_R(K, N)$ establish an equivalence between the additive categories $B_{\text{proj}} \equiv Q$ and $\text{Add}_R(K) \equiv N$ assigning the object $P \in B_{\text{proj}}$ to the object $K \in \text{Add}_R(K)$ and the object $P(X) = \Delta_S(R[X]) = \Delta_S(R[X]) \in B_{\text{proj}}$ to the object $K(X) \in \text{Add}_R(K)$ for all sets $X$ (cf., e. g., [29 Corollary 5.2]). According to Example [54](1), the projective generator $P \in B$ is good, so the $R$-module $K = S^{-1}R/R$ is good.
More generally, let $R$ be an associative ring and $S \subset R$ be a multiplicative subset consisting of some central nonzero-divisors in $R$. Assume that $\text{pd}_R S^{-1}R \leq 1$ and set $K = S^{-1}R/R \in R$–mod. Then $B = R$–mod$_{S,\text{contra}} = \mathcal{R}$–contra, where $\mathcal{R} = \Delta_S(R) = \Lambda_S(R)$ (see Example 2.5(3)), is the corresponding locally presentable abelian category with enough projective objects such that $B_{\text{proj}} \cong \text{Add}_R$–mod$(K)$, and $P = \mathcal{R} \in B$ is its natural projective generator. Indeed, just as in (4), the functors $Q \mapsto K \otimes_R Q$ and $N \mapsto \text{Hom}_R(K, N)$ (where $K$ is viewed as an $R$–$R$-bimodule) establish an equivalence between the additive categories $B_{\text{proj}}$ and $\text{Add}_R$–mod$(K)$ assigning $P \in B_{\text{proj}}$ to $K \in \text{Add}_R$–mod$(K)$ for all sets $X$ (cf. [11, Theorem 4.7] or [5, Theorem 1.3]).

According to Example 5.5(1), the projective generator $P \in B$ is good, so the left $R$-module $K = S^{-1}R/R$ is good.

(6) Let $T$ be a good $n$-tilting left $R$-module in the sense of [4] (where $n \geq 0$ is an integer). Then, according to [1] Theorem 2.2(2), the triangulated functor $\mathbb{R}\text{Hom}_R(T, -): D(R$–mod$) \rightarrow D(\mathcal{G}$–mod$)$ is fully faithful.

On the other hand, by [12] Proposition 2.3 or [3] Theorem 4.5 (cf. the discussion in [35] Proposition 8.2), the triangulated functor $\mathbb{R}\text{Hom}_R(T, -): D(R$–mod$) \rightarrow D(\mathcal{G}$–contra$)$ is an equivalence of triangulated categories. (Notice that, by [3] Proposition 4.3 and our Examples 1.2(2) and 1.3(3), the heart $B \subset D^b(R$–mod$)$ of the tilting t-structure associated with $T$ is equivalent to the abelian category $\mathcal{G}$–contra $[35$, Proposition 2.6 and Theorem 7.1,]).

Thus the triangulated functor $D(\mathcal{G}$–contra$) \rightarrow D(\mathcal{G}$–mod$)$ induced by the forgetful functor $\mathcal{G}$–contra $\rightarrow \mathcal{G}$–mod is fully faithful, so $T$ is a good left $R$-module in the sense of our definition.

6. $\kappa$-Flat Modules and the Fully Faithful Triangulated Functor

The aim of this section is to prove Theorem 4.10. For this purpose, we develop the theory of $\kappa$-flat modules over associative rings, generalizing the Govorov–Lazard characterization of flat modules to arbitrary regular cardinals.

Let $R$ be an associative ring and $\kappa$ be a regular cardinal.

**Theorem 6.1.** The following conditions on a left $R$-module $F$ are equivalent:

(a) every morphism into $F$ from a $\kappa$-presentable left $R$-module factorizes through a projective left $R$-module;

(b) every morphism into $F$ from a $\kappa$-presentable left $R$-module factorizes through a free left $R$-module with less than $\kappa$ generators;

(c) $F$ is the colimit of a $\kappa$-filtered diagram of projective left $R$-modules;

(d) the cocone formed by all the morphisms into $F$ from projective left $R$-modules with less than $\kappa$ generators (and all the morphisms between the latter forming commutative triangles with the morphism into $F$) is a $\kappa$-filtered colimit cocone;
(e) the cocone formed by all the morphisms into \( F \) from free left \( R \)-modules with less than \( \kappa \) generators (and all the morphisms between the latter forming commutative triangles with the morphism into \( F \)) is a \( \kappa \)-filtered colimit cocone.

Proof. (a) \( \implies \) (b): given a left \( R \)-module morphism \( E \to F \), where \( E \) is \( \kappa \)-presentable, one first factorizes the morphism \( E \to F \) through a projective left \( R \)-module \( P \), then replaces \( P \) with a free left \( R \)-module \( P' \) in which \( P \) is a direct summand, and finally replaces \( P' \) with its free submodule \( P'' \) with less than \( \kappa \) generators containing the image of the morphism \( E \to P' \).

(c) \( \implies \) (a): let \( F = \lim_{\leftarrow i} P_i \) be a presentation of \( F \) as the colimit of a \( \kappa \)-filtered diagram of projective left \( R \)-modules, and let \( E \) be a \( \kappa \)-presentable left \( R \)-module. Then \( \text{Hom}_R(E, \lim_{\leftarrow i} P_i) = \lim_{\leftarrow i} \text{Hom}_R(E, P_i) \), hence any left \( R \)-module morphism \( E \to F \) factorizes through one of the modules \( P_i \).

(b) \( \implies \) (d): denote our cocone by \( (P_i \to F)_{i \in I} \) (where \( I \) is the set of all pairs \( i = (P_i, f_i) \), with \( P_i \) a projective left \( R \)-module with less than \( \kappa \) generators and \( f_i : P_i \to F \) a left \( R \)-module morphism). Since the free left \( R \)-module with one generator \( R \) is a projective left \( R \)-module with less than \( \kappa \) generators, and every element of \( F \) belongs to the image of some left \( R \)-module morphism \( R \to F \), the natural morphism \( \lim_{\leftarrow i} P_i \to F \) is surjective.

Furthermore, given two morphisms \( f_i : P_i \to F \) and \( f_j : P_j \to F \) in the cocone, and two elements \( p' \in P_i \) and \( p'' \in P_j \) such that \( f_i(p') = f_j(p'') \in F \), there exist left \( R \)-module morphisms \( R \to P_i \) and \( R \to P_j \) taking the unit element \( 1 \in R \) to \( p' \) and \( p'' \), respectively. Setting \( P_k = R \) and denoting by \( f_k : P_k \to F \) the morphism taking \( 1 \) to \( f_i(p') = f_j(p'') \), we have morphisms \( k \to i \) and \( k \to j \) in the diagram \( I \). If \( I \) is filtered, there exists a morphism \( f_l : P_l \to F \) in the cocone and morphisms \( i \to l, j \to l \) in the diagram \( I \) such that the square diagram \( k \to i \to l, k \to j \to l \) is commutative. It follows that images of the elements \( p' \in P_i \) and \( p'' \in P_j \) coincide in \( P_l \), and hence also in \( \lim_{\leftarrow i} P_i \). Thus the morphism \( \lim_{\leftarrow i} P_i \to F \) is an isomorphism whenever \( I \) is filtered.

It remains to show that the diagram \( I \) is \( \kappa \)-filtered whenever (b) holds. Any collection of less than \( \kappa \) morphisms \( L_i \to F \), where \( L_i \) are projective left \( R \)-modules with less than \( \kappa \) generators, factorizes through the morphism \( M = \bigoplus_i L_i \to F \), and \( M \) is also a projective left \( R \)-module with less than \( \kappa \) generators.

Now let \( h_i : L \to M \) be a collection of less than \( \kappa \) morphisms between two projective left \( R \)-modules \( L \) and \( M \) with less than \( \kappa \) generators each. Then the coequalizer \( E \) of the whole system of morphisms \( h_i : L \to M \) is a \( \kappa \)-presentable left \( R \)-module. Given left \( R \)-module morphisms \( f : L \to F \) and \( g : M \to F \) forming commutative triangles with all the morphisms \( h_i \), we have the induced left \( R \)-module morphism \( E \to F \). Assuming (b), the latter morphism factorizes through a projective left \( R \)-module \( P \) with less than \( \kappa \) generators. Hence we have a morphism \( M \to P \) in the diagram \( I \) whose compositions with all the morphisms \( h_i : L \to M \) are equal to one and the same morphism \( L \to P \).

The proof of (b) \( \implies \) (e) is similar to (b) \( \implies \) (d). The implications (d) \( \implies \) (e), (e) \( \implies \) (c), and (b) \( \implies \) (a) are obvious. \( \square \)
A left $R$-module $F$ is said to be $\kappa$-flat if it satisfies one of the equivalent conditions of Theorem [5.1]

**Lemma 6.2.** (a) The class of all $\kappa$-flat left $R$-modules is closed under extensions, kernels of surjective morphisms, and $\kappa$-filtered colimits.

(b) Any short exact sequence of $\kappa$-flat left $S$-modules is a $\kappa$-filtered colimit of split short exact sequences of $\kappa$-flat left $S$-modules.

**Proof.** If $G = \lim_{\rightarrow} F_i$ is the colimit of a $\kappa$-filtered diagram of modules and $E$ is a $\kappa$-presentable module, then any morphism $E \to G$ factorizes through one of the modules $F_i$. If, in addition, any morphism $E \to F_i$ factorizes through a projective module, then any morphism $E \to G$ also factorizes through a projective module.

Let $0 \to F \to G \to H \to 0$ be a short exact sequence of left $R$-modules. Suppose that the left $R$-modules $G$ and $H$ are $\kappa$-flat. Let $E$ be a $\kappa$-presentable left $R$-module and $E \to F$ be a left $R$-module morphism. Then the composition $E \to F \to G$ factorizes through a free left $R$-module $P$ with less than $\kappa$ generators. Now the composition $P \to G \to H$ factorizes through the cokernel $C = P/E$ of the morphism $E \to P$. Since the left $R$-module $C$ is also $\kappa$-presentable, the morphism $C \to H$ factorizes through a projective left $R$-module $Q$.

Pick an arbitrary lifting $Q \to G$ of the morphism $Q \to H$ and subtract the composition $P \to C \to Q \to G$ from the morphism $P \to G$ that we have. The resulting morphism $P \to G$ is annihilated by the composition with the morphism $G \to H$, and therefore factorizes through the monomorphism $F \to G$. We have obtained a morphism $P \to F$ whose composition with the morphism $E \to P$ is equal to our original morphism $E \to F$. Thus the left $R$-module $F$ is $\kappa$-flat.

Now suppose that the left $R$-modules $F$ and $H$ are $\kappa$-flat. Then $H$ is a $\kappa$-filtered colimit of projective left $R$-modules $Q_i$, hence it follows that the short exact sequence $0 \to F \to G \to H \to 0$ is a $\kappa$-filtered colimit of split short exact sequences $0 \to F \to F \oplus Q_i \to Q_i \to 0$ (proving the assertion (b)). In particular, the left $R$-module $G$ is a $\kappa$-filtered colimit of left $R$-modules isomorphic to $F \oplus Q_i$. The latter are obviously $\kappa$-flat; and a $\kappa$-filtered colimit of $\kappa$-flat modules is $\kappa$-flat, as we have already seen. Thus the left $R$-module $G$ is $\kappa$-flat. \qed

**Remark 6.3.** We are not aware of a definition equivalent to our notion of $\kappa$-flatness appearing anywhere in the previously existing literature, but certainly there are all kinds of similar or related definitions known for many years.

In particular, there is the classical notion of a $\kappa$-free module [8]. In the similar spirit (cf. [7]), one can define $\kappa$-projective modules. Specifically, an $R$-module $M$ is said to be $\kappa$-projective if it admits a system of submodules $M_\alpha$ (which is said to witness the $\kappa$-projectivity of $M$) such that (1) every $R$-module $M_\alpha$ is projective with less than $\kappa$ generators; (2) every subset of $M$ of the cardinality less than $\alpha$ is contained in one of the modules $M_\alpha$; and (3) the set of submodules $M_\alpha \subset M$ is closed under unions of well-ordered chains of length smaller than $\kappa$. An $R$-module $M$ is said to be $\kappa$-projective in the weak sense if it has a witnessing system of submodules satisfying the conditions (1) and (2) [8 Section IV.1].
An $R$-module is $\kappa$-projective in the weak sense if and only if it is the colimit of a $\kappa$-filtered diagram of projective $R$-modules with less than $\kappa$ generators and injective morphisms between them. Indeed, given a module that is $\kappa$-projective in the weak sense, all its projective submodules with less than $\kappa$ generators form such a diagram. Thus $\kappa$-projectivity is a stronger condition than $\kappa$-flatness. When $R$ is a left hereditary ring, there is no difference between $\kappa$-projectivity and $\kappa$-projectivity in the weak sense for left $R$-modules. Assuming additionally that every left ideal in $R$ is generated by less than $\kappa$ elements (so any $\kappa$-generated left $R$-module is $\kappa$-presentable), these two conditions are also equivalent to $\kappa$-flatness. Indeed, let $F$ be a $\kappa$-flat left $R$-module. Given a subset of cardinality less than $\kappa$ in $F$, denote by $E$ the $R$-submodule in $F$ generated by this subset. Then $E$ is $\kappa$-presentable, so the injective morphism $E \rightarrow F$ factorizes through a projective left $R$-module $P$. It follows that $E$ is a submodule in $P$, hence a projective module itself.

Another classical concept closely related to $\kappa$-flatness is that of $\kappa$-purity. A left $R$-module morphism $f: P \rightarrow Q$ is said to be a $\kappa$-pure epimorphism if, for any $\kappa$-presentable left $R$-module $E$, any left $R$-module morphism $E \rightarrow Q$ factorizes through $f$. If this is the case, $Q$ is said to be a $\kappa$-pure quotient of $P$ [18, Chapter 7], [2]. One can easily see that an $R$-module is $\kappa$-flat if and only if it is a $\kappa$-pure quotient of a projective $R$-module.

Finally, some of the assertions and proofs of our Theorem 6.1 and Lemma 6.2 resemble those of the theory of modules with support in a subcategory (of finitely presented modules), as developed in [20] (see [20, Propositions 2.1–2]).

The following result, whose nonadditive version goes back to [16, Section 2.2], was rediscovered and discussed in the additive/abelian context in [35, Theorem 6.10].

**Theorem 6.4.** Let $\mathcal{B}$ be a locally $\kappa$-presentable abelian category and $P \in \mathcal{B}$ be a $\kappa$-presentable projective generator of $P$ (where $\kappa$ is a regular cardinal). Let $\lambda$ be a cardinal such that $\lambda^+ \geq \kappa$ and let $Q = P(\lambda) \in \mathcal{B}$ be the coproduct of $\lambda$ copies of $P$ in $\mathcal{B}$. Denote by $S$ the ring $\text{Hom}_\mathcal{B}(Q, Q)^{\text{op}}$. Then the exact functor $\Theta = \text{Hom}_\mathcal{B}(Q, -): \mathcal{B} \rightarrow S\text{-mod}$ corepresented by $Q$ is fully faithful.

**Sketch of proof.** Let us identify $\mathcal{B}$ with the category of modules over the $\kappa$-accessible monad $T_P: X \mapsto \text{Hom}_\mathcal{B}(P, P(\lambda))$ on the category of sets. Let $Y$ be a set of the cardinality $\lambda$; then the projective object $P \in \mathcal{B}$ corresponds to the free $T_P$-module with one generator $T_P(*)$ and the projective object $Q \in \mathcal{B}$ corresponds to the free $T_P$-module with $\lambda$ generators $T_P(Y)$. The functor $\Theta: T_P\text{-mod} \rightarrow S\text{-mod}$ assigns to a $T_P$-module $B$ the $S$-module $\Theta(B) = \text{Hom}_{T_P}(T_P(Y), B) = B^Y$.

Let $\delta_B: B \rightarrow B^Y$ be the diagonal embedding and by $\text{pr}_y: B^Y \rightarrow B$ the projection onto the component indexed by $y$. Then the composition $\delta_B \text{pr}_y: B^Y \rightarrow B^Y$ is equal to the action of the element $s_y \in S$ corresponding to the composition of the natural morphism $P(Y) \rightarrow P$ with the coproduct injection of the $y$-indexed component $P \rightarrow P(Y)$. Thus the morphism $g$
forms a commutative diagram with the maps $\delta_{B \text{pr}, B}$ and $\delta_{C \text{pr}, C}$:

\[
\begin{array}{ccc}
B^Y & \xrightarrow{\text{pr}_Y, B} & B \\
\downarrow g & & \downarrow g \\
C^Y & \xrightarrow{\text{pr}_Y, C} & C \\
\end{array}
\]

As this holds for all elements $y \in Y$, it follows easily that there is a map $f: B \to C$ such that $g = f^Y$.

Now, for any $Y$-ary operation $t \in T_P(Y)$ in the monad $T_P$, the composition of the natural morphism $P(Y) \to P$ with the morphism $t: P \to P(Y)$ defines an element $s_t \in S$. The action of $s_t$ in $B^Y$ is equal to the composition of the $Y$-ary operation $t_{T_P}(B): B^Y \to B$ in $B$ (see Section 1) with the diagonal embedding $\delta_B: B \to B^Y$. Commutativity of the diagram

\[
\begin{array}{ccc}
B^Y & \xrightarrow{t_{T_P}(B)} & B \\
\downarrow f^Y & & \downarrow f^Y \\
C^Y & \xrightarrow{t_{T_P}(C)} & C \\
\end{array}
\]

means that the map $f: B \to C$ preserves the operation $t_{T_P}$ in $B$ and $C$. As this holds for all $t \in T_P(Y)$ and all operations in the monad $T_P$ depend essentially on at most $\lambda$ arguments, it follows that $f$ is a morphism of $T_P$-modules. □

The argument deducing Theorem 4.10 from Theorem 6.4 is based on a technique summarized in the following proposition.

Let $A$ be an abelian category with enough projective objects and $B \subset A$ be a full subcategory closed under the kernels and cokernels in $A$; so $B$ is also an abelian category and the embedding functor $B \to A$ is exact. Assume that there exists a functor $\Delta: A \to B$ left adjoint to the fully faithful embedding functor $B \to A$. Then the functor $\Delta$ takes projective objects in $A$ to projective objects in $B$, and it follows easily that there are enough projectives in $B$ (cf. Lemma 4.5). Let $L_n \Delta: A \to B$, $n \geq 0$, denote the left derived functor of the right exact functor $\Delta$.

**Proposition 6.5.** The following four conditions are equivalent:

(a) $L_n \Delta(P) = 0$ for every object $P \in B_{\text{proj}} \subset A$ and all $n \geq 1$;

(b) $L_n \Delta(B) = 0$ for every object $B \in B \subset A$ and all $n \geq 1$;

(c) the triangulated functor $D^-(B) \to D^-(A)$ induced by the exact embedding functor $B \to A$ is fully faithful;

(d) the triangulated functor $D^b(B) \to D^b(A)$ induced by the exact embedding functor $B \to A$ is fully faithful.

**Proof.** This is an infinite homological dimension version of [29, Theorem 6.4] (see also [27, proofs of Theorems 1.3 and 2.9]).
(a) \iff (b): The composition \( B \to A \to B \) of the embedding \( B \to A \) with its left adjoint functor \( \Delta: A \to B \) is the identity functor \( \text{Id}_B \), since the embedding functor \( B \to A \) is fully faithful. Let \( P_\bullet \to B \) be a left projective resolution of an object \( B \in B \). In the assumption of (a), since \( \mathbb{L}_n \Delta(P_i) = 0 \) for all \( i \geq 0 \) and \( n \geq 1 \), the complex \( \Delta(P_\bullet) \) computes the derived functor \( \mathbb{L}_n \Delta(B) \). However, we have \( \Delta(P_\bullet) = P_\bullet \), hence \( \mathbb{L}_n \Delta(B) = 0 \) for \( n \geq 1 \).

(b) \iff (c): Denote by \( A_{\Delta-\text{adj}} \subset A \) the full subcategory of all objects \( A \in A \) such that \( \mathbb{L}_n \Delta(A) = 0 \) for all \( n \geq 1 \). Then the full subcategory \( A_{\Delta-\text{adj}} \subset A \) is closed under extensions and the kernels of epimorphisms in \( A \); so \( A_{\Delta-\text{adj}} \) inherits an exact category structure from the abelian category \( A \). Furthermore, every object of \( A \) is the image of an epimorphism from an object of \( A_{\Delta-\text{adj}} \). It follows that the triangulated functor \( D^{-}(A_{\Delta-\text{adj}}) \to D^{-}(A) \) induced by the exact embedding \( A_{\Delta-\text{adj}} \to A \) is a triangulated equivalence (e. g., by [25, Proposition A.3.1(a)]; or simply because both the derived categories in question are equivalent to the homotopy category of bounded above complexes of projective objects in \( A \)).

The restriction of the functor \( \Delta \) to the exact subcategory \( A_{\Delta-\text{adj}} \subset A \) is an exact functor \( \Delta: A_{\Delta-\text{adj}} \to B \). Applying the functor \( \Delta \) to bounded above complexes of objects from \( A_{\Delta-\text{adj}} \), one constructs a triangulated functor

\[
\mathbb{L}\Delta: D^{-}(A) \to D^{-}(B),
\]

which is left adjoint to the triangulated functor \( D^{-}(B) \to D^{-}(A) \) induced by the embedding \( B \to A \) [23, Lemma 8.3].

Now the functor \( D^{-}(B) \to D^{-}(A) \) is fully faithful if and only if its composition \( D^{-}(B) \to D^{-}(A) \to D^{-}(B) \) with its left adjoint functor \( \mathbb{L}\Delta: D^{-}(A) \to D^{-}(B) \) is the identity functor \( D^{-}(B) \to D^{-}(B) \). In the situation of (b), we have \( B \subset A_{\Delta-\text{adj}} \). Since the composition \( B \to A \to B \) is the identity functor, it follows that so is the composition \( D^{-}(B) \to D^{-}(A) \to D^{-}(B) \).

Conversely, if \( D^{-}(B) \to D^{-}(A) \to D^{-}(B) \) is the identity functor, we obviously have \( \mathbb{L}_n \Delta(B) = 0 \) for all \( B \in B \) and \( n \geq 1 \).

The implication (c) \implies (d) is obvious.

(d) \implies (b): Fix an integer \( m \geq 0 \), and denote by \( D^{b,\geq-m}(B) \subset D^{b}(B) \) the full subcategory of bounded complexes with the cohomology objects concentrated in the cohomological degrees \( \geq -m \). The embedding functor \( D^{b,\geq-m}(B) \to D^{-}(B) \) has a left adjoint, which is the canonical truncation functor \( \tau^{\leq-m}: D^{-}(B) \to D^{b,\geq-m}(B) \) (taking a complex \( \cdots \to B^{-m-2} \to B^{-m-1} \to B^{-m} \to B^{-m+1} \to \cdots \) to the complex \( \cdots \to 0 \to 0 \to B^{-m}/\text{im} B^{-m-1} \to B^{-m+1} \to \cdots \)). Hence the composition of functors \( D^{b,\geq-m}(B) \to D^{-}(B) \to D^{-}(A) \) also has a left adjoint, which can be computed as the composition of the two left adjoints \( \tau^{\leq-m} \mathbb{L}\Delta \).

On the other hand, the same functor \( D^{b,\geq-m}(B) \to D^{-}(A) \) can be obtained as the composition \( D^{b,\geq-m}(B) \to D^{b}(B) \to D^{b}(A) \to D^{-}(A) \). In the assumption of (d), the functor \( D^{b}(B) \to D^{b}(A) \) induced by \( \Theta \) is fully faithful. Since the embedding functors \( D^{b,\geq-m}(B) \to D^{b}(B) \) and \( D^{b}(A) \to D^{-}(A) \) are fully faithful, too, it follows that so is the composition \( D^{b,\geq-m}(B) \to D^{-}(A) \).
Therefore, the composition of the two adjoint functors \( D^b_{\geq -m}(B) \to D^{-}(A) \to D^b_{\geq -m}(B) \) is isomorphic to the identity functor. In other words, it means that the adjunction morphism \( B \to \tau_{\geq -m} L \Delta(B) \) is an isomorphism for every \( B \in B \). Since \( m \geq 0 \) is an arbitrary integer, it follows that \( \mathbb{L}_n \Delta(B) = 0 \) for all \( n \geq 0 \).

\[ \square \]

**Proof of Theorem 4.10.** Denote by \( S \) the ring \( \text{Hom}_B(Q, Q^{op}) \). First of all, we already know from Theorem 6.4 that the functor \( \Theta = \text{Hom}_B(Q, -) : B \to S \text{-mod} \) is fully faithful. So \( Q \) is a 0-good projective generator of \( B \).

To prove that \( Q \) is a good projective generator, we have to check that the triangulated functor \( D^b(B) \to D^b(S \text{-mod}) \) induced by \( \Theta \) is fully faithful. It was explained in the proof of Proposition 4.8 that the exact embedding functor \( \Theta : B \to S \text{-mod} \) has a left adjoint functor \( \Delta : S \text{-mod} \to B \). According to Proposition 6.5, it remains to show that the projective objects of the abelian full subcategory \( B \subseteq S \text{-mod} \) are adjusted to \( \Delta \), that is \( \mathbb{L}_n \Delta(\Theta(P)) = 0 \) for all \( P \in B_{\text{proj}} \) and all \( n \geq 1 \).

As any projective object in \( B \) is a direct summand of an object of the form \( Q^{(X)} \), where \( X \) is some set, it suffices to check that the left \( S \)-module \( \Theta(Q^{(X)}) \) is adjusted to \( \Delta \) for all sets \( X \). Now, the functor \( \Theta \) preserves \( \lambda^+ \)-filtered colimits (since the object \( Q \in B \) is \( \lambda^+ \)-presentable). Hence we have \( \Theta(Q^{(X)}) = \varinjlim_{Z} \Theta(Q^{(Z)}) \), where the colimit is taken over all the subsets \( Z \subseteq X \) of cardinality not exceeding \( \lambda \).

Furthermore, \( Q^{(Z)} \simeq Q \) in \( B \) for all nonempty sets \( Z \) of cardinality \( \leq \lambda \), so the left \( S \)-module \( \Theta(Q^{(Z)}) \) is isomorphic to \( S \). By Lemma 6.2(a), it follows that the left \( S \)-module \( \Theta(Q^{(X)}) \) is \( \lambda^+ \)-flat for all sets \( X \), as a \( \lambda^+ \)-filtered colimit of free left \( S \)-modules. So the functor \( \Theta \) takes the projective objects of \( B \) to \( \lambda^+ \)-flat left \( S \)-modules (this observation improves upon the result of [35] Lemma 6.13, where it was noticed that \( \Theta \) takes the projective objects of \( B \) to flat left \( S \)-modules).

Let us show that all the \( \lambda^+ \)-flat left \( S \)-modules are adjusted to \( \Delta \), that is \( \mathbb{L}_n \Delta(F) = 0 \) for all \( \lambda^+ \)-flat left \( S \)-modules \( F \) and all \( n \geq 1 \). Since the projective left \( S \)-modules are \( \lambda^+ \)-flat and the class of all \( \lambda^+ \)-flat left \( S \)-modules is closed under the kernels of surjective morphisms, we only need to check that the functor \( \Delta \) preserves exactness of short exact sequences of \( \lambda^+ \)-flat left \( S \)-modules.

By Lemma 6.2(b), every short exact sequence of \( \lambda^+ \)-flat \( S \)-modules is a \( \lambda^+ \)-filtered colimit of split short exact sequences. It remains to point out that the functor \( \Delta \) preserves all colimits, and that \( \lambda^+ \)-filtered colimits are exact in \( B \) (e. g., because they are preserved by the conservative exact functor \( \Theta \) and exact in \( R \text{-mod} \); cf. [1] Proposition 1.59 and [32] Definition 2.1).

\[ \square \]

7. **Nonabelian Subcategories Defined by Perpendicularity Conditions**

The aim of this section is to show what can happen if one relaxes the conditions in the definitions of right \( n \)-perpendicular subcategories in Section 4. We start with one positive assertion before proceeding to present various counterexamples.

**Lemma 7.1.** Let \( A \) be a locally presentable abelian category and \( B \subseteq A \) be a right \( n \)-perpendicular subcategory to a set of objects or morphisms in \( A \), where \( n \geq 0 \). Then
\textbf{B} is an additive category with kernels, cokernels, infinite direct sums and products. The kernels and products in \textbf{B} coincide with those computed in \textbf{A}, while the cokernels and coproducts in \textbf{B} can be obtained by applying the reflector $\Delta: \textbf{A} \rightarrow \textbf{B}$ to the cokernels and coproducts computed in \textbf{A}.

\textit{Proof.} In view of Lemma 4.1(b), it suffices to consider the case $n = 0$. Then, by [1, Theorem 1.39] (cf. the discussion in the proof of Lemma 4.4), the category \textbf{B} is locally presentable and reflective as a full subcategory in \textbf{A}. By the definition and by [1, Corollary 1.28], any locally presentable category is complete and cocomplete. It remains to observe that the full subcategory $\textbf{B} \subset \textbf{A}$ is closed under limits (Lemma 4.2(a)), while the reflector $\Delta: \textbf{A} \rightarrow \textbf{B}$, being a left adjoint functor, preserves colimits. Before we finish, let us recall that, when $n \geq 2$, the cokernels in \textbf{B} also coincide with those computed in \textbf{A} by Lemma 4.2(c). \hfill \Box

Our counterexamples are produced by the following construction inspired by Example 4.3. Let $R \rightarrow S$ be a homomorphism of associative rings. Denote by \textbf{A} the abelian category whose objects are triples $(N, M, f)$, where $N$ is a left $S$-module, $M$ is a left $R$-module, and $f: N \rightarrow M$ is an $R$-module morphism. The morphisms $(N', M', f') \rightarrow (N'', M'', f'')$ in \textbf{A} are pairs consisting of an $S$-module morphism $N' \rightarrow N''$ and an $R$-module morphism $M' \rightarrow M''$ forming a commutative square with $f'$ and $f''$. One can easily interpret \textbf{A} as the category of modules $A = T\text{-mod}$ over an appropriate matrix ring $T$.

Denote by $E$ the object $(S, 0, 0) \in \textbf{A}$, where $S$ is viewed as a free left $S$-module with one generator.

\textbf{Lemma 7.2.} Let $L = (N, M, f)$ be an object of \textbf{A}. Then the following Ext computations hold:

(a) $\text{Hom}_A(E, L) = \ker(N \rightarrow \text{Hom}_R(S, M))$,
(b) $\text{Ext}^1_A(E, L) = \text{coker}(N \rightarrow \text{Hom}_R(S, M))$,
(c) $\text{Ext}^i_A(E, L) = \text{Ext}^{i-1}_R(S, M)$ for $i \geq 2$,

where the $S$-module morphism $N \rightarrow \text{Hom}_R(S, M)$ is induced by the $R$-module morphism $N \rightarrow M$.

\textit{Proof.} There is a short exact sequence

$0 \rightarrow (0, S, 0) \rightarrow (S, S, \text{id}_S) \rightarrow (S, 0, 0) \rightarrow 0$

in the category \textbf{A}, with a projective object $(S, S, \text{id}_S) \in \textbf{A}$. Computing $\text{Hom}_A(E, L)$ and $\text{Ext}^1_A(E, L)$ in terms of this one-step resolution of the object $E$ produces the natural isomorphisms (a-b). Furthermore, let $\cdots \rightarrow P_1 \rightarrow P_0 \rightarrow S \rightarrow 0$ be a projective resolution of the left $R$-module $S$. Then the following is a projective resolution of the object $E \in \textbf{A}$:

$\cdots \rightarrow (0, P_1, 0) \rightarrow (0, P_0, 0) \rightarrow (S, S, \text{id}_S) \rightarrow (S, 0, 0) \rightarrow 0$.

Computing $\text{Ext}^i_A(E, L)$ in terms of this resolution produces the natural isomorphism (c) as well. \hfill \Box
Given a left $R$-module $G$ and an integer $n \geq 0$, we denote by $G^{+,n} \subset R\text{-mod}$ the full subcategory in $R\text{-mod}$ formed by all the $R$-modules $M$ such that $\text{Ext}_R^i(G,M) = 0$ for all $1 \leq i \leq n$. So, in particular, we have $G^{+,0} = R\text{-mod}$. Similarly, we denote by $E^{+,n} \subset A$ the full subcategory formed by all the objects $L \in A$ such that $\text{Ext}_A^i(E,L) = 0$ for all $0 \leq i \leq n$. For $n = \infty$, the notation $G^{+,\infty} \subset R\text{-mod}$ and $E^{+,\infty} \subset A$ also has the obvious meaning.

**Corollary 7.3.** The functor assigning to an $R$-module $M$ the object $(\text{Hom}_R(S,M), M, f_M) \in A$, where $f_M : \text{Hom}_R(S,M) \rightarrow M$ is the $R$-module morphism induced by the ring homomorphism $R \rightarrow S$, provides an equivalence between the full subcategory $E^{+,1} \subset A$ and the abelian category $R\text{-mod}$. For every $n \geq 1$, this equivalence identifies the full subcategories $E^{+,n+1} \subset A$ and $S^{+,n} \subset R\text{-mod}$ (where $S$ is viewed as a left $R$-module), providing a category equivalence between them.

**Proof.** The first assertion follows from Lemma 7.2, and the second one from Lemma 7.2(c). \qed

**Example 7.4.** Let $R$ be an associative ring and $G$ be a left $R$-module admitting a right $R$-module structure which makes it an $R$-$R$-module. Set $S = R \oplus G$ to be the trivial ring extension of $R$ by $G$ (so the product of any two elements of $G$ is zero in $S$, while the products of two elements of $R$ and of one element from $R$ and one element from $G$ in $S$ are defined using the ring structure on $R$ and the $R$-$R$-bimodule structure on $G$). Then the above construction provides an associative ring $T$ with a left $T$-module $E$ such that, by Corollary 7.3, the full subcategory $E^{+,1} \subset T\text{-mod}$ is equivalent to $R\text{-mod}$, and this equivalence restricts to equivalences between the full subcategories $E^{+,n+1} \subset T\text{-mod}$ and $G^{+,n} \subset R\text{-mod}$ for all $n \geq 1$.

Notice that, by Lemma 7.2, the projective dimensions of the left $R$-module $G$ and the left $T$-module $E$ are related by the rule $\text{pd}_T E = \text{pd}_R G + 1$.

**Examples 7.5.** (1) In the context of Example 7.4, let $R$ be a commutative ring and set $G = \bigoplus_{s \in R} R[s^{-1}]$. Then $\text{pd}_R G \leq 1$ and the orthogonal class $G^{+,1} = G^{+,\infty} \subset R\text{-mod}$ is called the class of all contraadjusted $R$-modules [25, Section 1.1], [28, Sections 2 and 8]. The full subcategory $G^{+,1} \subset R\text{-mod}$ is equivalent to the category of all contraherent cosheaves on $\text{Spec } R$ (cf. Example 1.3).

Set $R = \mathbb{Z}$ to be the ring on integers, and consider the abelian group $A = \mathbb{Z}_p(\omega)$, that is, the direct sum of a countable family of copies of the group of $p$-adic integers (where $p$ is a fixed prime number). Set $B = \mathbb{Q} \otimes_{\mathbb{Z}} A$ and $C = \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} A = \text{coker}(A \rightarrow B)$. Being divisible (= injective) abelian groups, $B$ and $C$ are obviously contraadjusted. We claim that the surjective morphism $B \rightarrow C$ has no kernel in the full subcategory of contraadjusted abelian groups $G^{+,1} \subset \mathbb{Z}\text{-mod}$.

Indeed, suppose that $K \rightarrow B$ is such a kernel. Since the class of contraadjusted modules is closed under the passages to arbitrary quotient modules, the image of the morphism $K \rightarrow B$ is also contraadjusted. Hence the morphism $K \rightarrow B$ has to be injective, and we can consider $K$ as a subgroup in $B$. Since the composition $K \rightarrow B \rightarrow C$ has to vanish, we have $K \subset A$. Now, we have $A = \mathbb{Z}_p(\omega)$, and every summand $\mathbb{Z}_p \subset A$ in this direct sum is a contraadjusted abelian group. Hence $K$
contains every one of these direct summands, so it follows that $K = A$. However, $A$ is not a contraadjusted abelian group, as $\text{Ext}^1_R(\mathbb{Z}[p^{-1}], A) \neq 0$ [28, Section 12].

Thus we have constructed an associative ring $T$ and a left $T$-module $E$ of projective dimension 2 such that in the full subcategory $B = E^{1,0,2} = E^{1,0,\infty} \subset T-\text{mod}$ there are morphisms which have no kernel. Let us emphasize what this means: not only the full subcategory $B$ is not closed under kernels in $T-\text{mod}$, but some morphisms in $B$ do not even have any kernel internal to $B$.

(2) Again in the context of Example 7.4, let $R$ be a commutative ring and set $G = \bigoplus_i R/I$, where the direct sum is taken over all the ideals $I \subset R$. Then, by Baer’s criterion, the orthogonal class $G^{1,1,1} = G^{1,1,\infty} \subset R-\text{mod}$ is the class of all injective $R$-modules. Choose $R$ to be a commutative ring of global dimension 3, and let $A$ be a $R$-module of projective dimension 3. Choose exact sequences of $R$-modules $0 \rightarrow A \rightarrow J \rightarrow B \rightarrow 0$ and $0 \rightarrow B \rightarrow H \rightarrow C \rightarrow 0$, where $J$ and $H$ are injective $R$-modules. We claim that the morphism $J \rightarrow H$ has no cokernel in the full subcategory of injective modules $G^{1,1,1} \subset R-\text{mod}$.

Indeed, suppose that $H \rightarrow K$ is such a cokernel. Then the composition $J \rightarrow H \rightarrow K$ vanishes, so we have an $R$-module morphism $C \rightarrow K$. Furthermore, for every injective $R$-module $K'$ any any $R$-module morphism $C \rightarrow K'$ there exists a unique $R$-module morphism $K \rightarrow K'$ making the triangle diagram $C \rightarrow K \rightarrow K'$ commutative. Choosing $K'$ to be an injective $R$-module such that there is an injective $R$-module morphism $C \rightarrow K'$, we see that the morphism $C \rightarrow K$ is injective. Choosing $K'$ to be an injective $R$-module such that there is an injective $R$-module morphism $K/C \rightarrow K'$, we conclude that the morphism $C \rightarrow K$ is an isomorphism. Hence $C$ is an injective $R$-module, which contradicts the assumption that $\text{pd}_R A = 3$.

Thus we have constructed an associative ring $T$ and a left $T$-module $E$ of projective dimension 4 such that $E^{1,0,2} = E^{1,0,4} = E^{1,0,\infty} \subset T-\text{mod}$ and in the full subcategory $B = E^{1,0,2} \subset T-\text{mod}$ there are morphisms which have no cokernel. As in (1), this means that not only the full subcategory $B$ is not closed under kernels in $T-\text{mod}$, but some morphisms in $B$ do not even have any cokernel internal to $B$.

Example 7.6. This example is a naive version of the category of contraherent cosheaves on the projective line $\mathbb{P}^1_k$ (where $k$ is a field). Consider two polynomial rings $R' = k[x]$ and $R'' = k[x^{-1}]$ in the variables $x$ and $x^{-1}$, and the ring of Laurent polynomials $S = k[x, x^{-1}]$ containing $R'$ and $R''$ as subrings. Denote by $C$ the abelian category whose objects are quintuples $(N, M', M'', f', f'')$, where $N$ is an $S$-module, $M'$ is an $R'$-module, $M''$ is an $R''$-module, $f': N \rightarrow M'$ is an $R'$-module morphism, and $f'': N \rightarrow M''$ is an $R''$-module morphism. Morphisms in the category $C$ are defined in the obvious way (similar to the construction of the category $A$ above). It is not difficult to construct a matrix ring $U$ such that $C = U-\text{mod}$.

Consider the two objects $E' = (S, 0, S, 0, \text{id}_S)$ and $E'' = (S, S, 0, \text{id}_S, 0) \in C$, and set $E = E' \oplus E''$. Then $E$ is an object of projective dimension 2 in $C$, and the full subcategory $E^{1,0,1} \subset C$ is equivalent to the category of all pairs of modules $M'$ over $R'$ and $M''$ over $R''$ endowed with an isomorphism of $S$-modules $\text{Hom}_{R'}(S, M') \cong$
Hom_{R'}(S, M''). The full subcategory $E^{1\cdot2} \subset E^{1\cdot1}$ is defined by the additional conditions $\text{Ext}^1_{R'}(S, M') = 0$ and $\text{Ext}^1_{R''}(S, M'') = 0$ (cf. Example 4.3).

By Lemma 7.1, the full subcategory $B = E^{1\cdot1} \subset U\text{-mod}$ is an additive category with kernels and cokernels (and, also, infinite direct sums and products). Let us show that the category $B$ is not abelian. Indeed, consider the two objects $A = (0, R', R'', 0, 0)$ and $C = (k, k, k, \text{id}_k, \text{id}_k)$, where $x$ acts in $k$ by the identity operator. Then there is a morphism $g: A \rightarrow C$ in $B$ whose components $R' \rightarrow k$ and $R'' \rightarrow k$ take $x^n$ to 1 for all $n \in \mathbb{Z}$. Now, the cokernel of $g$ vanishes, so $C$ is the kernel of the cokernel of $g$. On the other hand, the kernel of $g$ is the morphism $h: A \rightarrow A$ with the components $1 - x: R' \rightarrow R'$ and $1 - x^{-1}: R'' \rightarrow R''$. The cokernel of the kernel of $g$ is isomorphic to the direct sum of two copies of $C$.

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Institute of Mathematics, Czech Academy of Sciences, Žitná 25, 115 67 Prague 1, Czech Republic

Email address: positselski@math.cas.cz