A BERNSTEIN THEOREM FOR TWO-VALUED MINIMAL GRAPHS IN DIMENSION FOUR

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Abstract. We prove a Bernstein-type theorem for two-valued minimal graphs in the four-dimensional Euclidean space $\mathbb{R}^4$. This states that two-valued functions defined on the entire $\mathbb{R}^3$, and whose graph is a minimal surface, must necessarily be linear. This is a two-valued analogue of the classical Bernstein theorem, which asserts that in dimensions up to $n + 1 \leq 8$, an entire single-valued minimal graph is linear. The main contrast with the single-valued theory is the presence of a large set of singularities in the graphs of two-valued functions. Indeed two-valued minimal graphs are neither area-minimising, nor is the regularity theory of elliptic PDE directly available in this setting. We obtain structure results for the blowdown cones of two-valued minimal graphs, valid in dimension $n + 1 \leq 7$, proving in particular that they are smoothly immersed away from an $(n-2)$-rectifiable set that includes its branch points. In dimension four we go further, and completely classify the possible blowdown cones using a combinatorial argument. We show that they must be a union of two three-dimensional planes: this is the key to the proof of the Bernstein theorem.

Introduction

The Bernstein theorem is one of the most emblematic results in the theory of minimal surfaces. It answers the following question. Let $u : \mathbb{R}^n \to \mathbb{R}$ be a globally defined function whose graph $G$ is minimal. Must $u$ be affine linear? This question was first studied by Bernstein [Ber27], who answered the question affirmatively when $n = 2$, using methods from complex analysis. However his argument proved hard to generalise, and Fleming [Fle62] opened the way to higher-dimensional extensions by framing the problem in measure-theoretic terms. We summarise the novelty of Fleming’s approach, which culminated in a proof of Bernstein’s theorem in all dimensions up to $n + 1 = 8$. Fleming [Fle62] showed that $G$ is weakly asymptotic at large scales to an $n$-dimensional area-minimising cone $C$. Moreover if $u$ is non-linear then $C$ must be singular. De Giorgi [DG61] next observed that additionally the blowdown cone $C$ of a non-linear $G$ must be cylindrical: there is an $(n-1)$-dimensional area-minimising cone $C_0$ in $\mathbb{R}^n$ so that $C = C_0 \times \mathbb{R}e_{n+1}$. Consequently if none of the $(n-1)$-dimensional area-minimising cones in $\mathbb{R}^n$ are singular then there cannot be any non-linear minimal graphs in $\mathbb{R}^{n+1}$ either. Fleming [Fle62] showed that this is true when $n = 3$; next this was extended to $n = 4$ by Almgren [Alm66], and finally by Simons [Sim68] to
\[ n \leq 7. \] This is sharp, as Bombieri–de Giorgi–Giusti [BDGG69] showed that the cones \( C_p = \{(X, Y) \in \mathbb{R}^p \times \mathbb{R}^p : |X| = |Y|\} \) are area-minimising for all \( p \geq 4 \). They also constructed, for every one of these cones, a minimal graph in \( \mathbb{R}^{2p+1} \) which at infinity is asymptotic to \( C_p \times \mathbb{R}^{2p+1} \), thus proving that the Bernstein theorem could not hold in dimensions larger than \( n + 1 \geq 9 \).

Here we propose to study the analogous question for two-valued functions: must entire two-valued minimal graphs be affine linear? Historically multi-valued functions were introduced to geometric measure theory by Almgren [Alm00] in his monumental regularity theory for area-minimising currents, valid in any codimension \( k \). To this end Almgren used functions taking values in the set of unordered \( Q \)-tuples \( \mathcal{A}_Q(\mathbb{R}^k) \). (Almgren’s regularity theory has been streamlined by De Lellis–Spadaro in a series of papers, including an initial paper revisiting the theory of \( Q \)-valued functions [DLS11, DLS14, DLS16a, DLS16b].)

Here we work exclusively in codimension one, with two-valued functions defined to be those taking values in \( \mathcal{A}_2(\mathbb{R}) \), the set of unordered pairs of real numbers \( \{a_1, a_2\} = \{a_2, a_1\} \) with \( a_1, a_2 \in \mathbb{R} \). The graph \( G \) of a two-valued function is called minimal if it is a critical point for the \( n \)-dimensional area functional. Such two-valued minimal graphs provide the local picture of stable minimal hypersurfaces near multiplicity-two branch points. Indeed, a result of Wickramasekera [Wic20] proves the following: near a point where it has a multiplicity two tangent hyperplane, a stationary codimension one varifold with stable regular part is equal to the graph of a two-valued function. Their inherent interest aside, this provides an additional motivation for the study of entire two-valued minimal graphs \( G \), defined in terms of some \( u : \mathbb{R}^n \to \mathcal{A}_2(\mathbb{R}) \), because they arise as blow-up models along degenerating sequences.

Unlike their single-valued counterparts, two-valued minimal graphs are not area-minimising. Indeed, they contain branched and immersed singularities, neither of which exist in area-minimising hypersurfaces. Although a two-valued minimal graph \( G \) is still asymptotic at infinity to a stationary cone \( C \), this is not area-minimising either. In short, the regularity theory of area-minimising cones in \( \mathbb{R}^{n+1} \) is of no use here, even when \( n + 1 \leq 8 \). Nonetheless, two-valued minimal graphs can be shown to be stable (see Lemma 2.9): the second variation of the area functional is non-negative. The presence of a large singular set (that includes branch points) may therefore come as a surprise at a first sight. Indeed, perhaps the most celebrated regularity theorems for stable stationary hypersurfaces—respectively obtained by Schoen–Simon [SS81] and Wickramasekera [Wic14]—give that their singular set is small, and has codimension at least seven. However neither result applies here, because both impose a priori hypotheses on the singular set that two-valued minimal graphs fail to satisfy: Schoen–Simon ask that it be \( \mathcal{H}^{n-2} \)-negligible, and Wickramasekera imposes a (significantly weaker)
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structural hypothesis, forbidding so-called classical singularities. In particular, all the results we obtain are completely independent of the regularity theory developed by Wickramasekera [Wic14].

There are additional difficulties associated to working with two-valued minimal graphs. A real-valued function $u$ whose graph $G$ is minimal satisfies an elliptic quasilinear PDE: the minimal surface equation

$$(1 + |Du|^2)\Delta u - D_iuD_juD_{ij}u = 0.$$ 

This makes the elliptic regularity theory available. For example, assuming only that $u$ is $C^{1,\alpha}$-regular for some $\alpha \in (0, 1)$ one finds that in fact $u$ must be smooth. (In fact via de Giorgi–Nash–Moser theory it suffices to consider Lipschitz-regular $u$ to obtain the same conclusion.) This is not applicable in the two-valued setting. Simon–Wickramasekera [SW07] constructed two-valued minimal graphs defined on bounded subsets of $\mathbb{R}^n$ by solving an adapted Dirichlet problem. Their examples have an $(n - 2)$-dimensional branch set, along which they look approximately like functions of the form $(x_1, \ldots, x_n) \mapsto \text{Re}(x_1 + ix_2)^{k/2}$, where $k \geq 3$ is an odd integer. (See also the subsequent work of Krummel [Kru19] for an extension of these constructions to higher codimensions.) Simon–Wickramasekera [SW16] later showed that a two-valued minimal graph of class $C^{1,\alpha}$ for some $\alpha \in (0, 1)$ is necessarily $C^{1,1/2}$-regular. The examples above with $k = 3$ show that this is optimal.

We overcome these difficulties and prove the following theorem, which is our main result; see Theorem 12.1.

**Theorem 1.** Let $\alpha \in (0, 1)$ and $u \in C^{1,\alpha}(\mathbb{R}^3; A_2(\mathbb{R}))$ be a two-valued function whose graph $G$ is minimal. Then $u$ is linear, and its graph is a union of two three-dimensional planes.

Rosales [Ros16] has proved a similar result in dimension $n + 1 = 3$. However the proofs bear few similarities, because Rosales obtains curvature estimates via a logarithmic cut-off trick, an argument unavailable in larger dimensions. It is crucial here that no bound is assumed for the growth of $u$ or its derivatives. Notice in particular the strong contrast with two-valued harmonic functions, of which an abundance of non-linear examples exists without such a hypothesis: take for example the functions $(x_1, \ldots, x_n) \mapsto \text{Re}(x_1 + ix_2)^{k/2}$ for odd $k \geq 3$ we already mentioned above. (This mirrors a phenomenon already present in the single-valued setting, where the Liouville theorems for harmonic functions require a bound on $u$ or its derivatives.)

When one does in fact assume a bound for the growth of $u$, then the conclusion of Theorem 1 extends to all dimensions; see Theorem 11.1.

**Theorem 2.** Let $\alpha \in (0, 1)$ and $u \in C^{1,\alpha}(\mathbb{R}^n; A_2(\mathbb{R}))$ be a two-valued function whose graph $G$ is minimal, and for which there exists a constant $C > 0$ so that $\sup_{D_R} \{|u_1| + |u_2|\} \leq CR$ for all $R \geq 1$. Then $u$ is linear, and its graph is a union of two $n$-dimensional planes.
The method of proof of Theorem 2 diverges from the classical, single-valued theory. There the corresponding result—that a single-valued minimal graph $u: \mathbb{R}^n \to \mathbb{R}$ with bounded growth is necessarily linear—can essentially be proved using standard Harnack inequalities. We cannot follow this line of argument here, and instead develop an alternative proof. The key here is a regularity lemma, where we prove that Lipschitz two-valued minimal graphs must in fact be $C^{1,\alpha}$-regular for some $\alpha \in (0,1)$. This result is not completely new, as Becker-Kahn and Wickramasekera have obtained the same conclusion via a frequency function argument [BKW20]. Our approach here is different and original: it combines a geometric, inductive argument with anterior results of Becker-Kahn [BK17]; see Section 5.

Of the two results, Theorem 1 is by far the more difficult to prove. However both results require a priori estimates, among which we cite the stability inequality (see Lemma 2.9), an area estimate (see Proposition 3.1) and an interior gradient estimate (see Lemmas 4.1 and 4.2). It is imperative that all three hold in the presence of branch point singularities. This means that, although we can adapt the proofs used in the single-valued theory, these modifications introduce significant technical intricacies.

Broadly speaking, the proof of Theorem 1 follows a similar argument as the proof of the single-valued Bernstein theorem. Let $G = \text{graph } u$ be a minimal two-valued graph, corresponding to some function $u \in C^{1,\alpha}(\mathbb{R}^n; A_2(\mathbb{R}))$. Next let $G_j = \text{graph } u_j$ be a sequence of two-valued minimal graphs obtained by homothetically rescaling $G$ along some sequence of factors $\lambda_j \to +\infty$: 

$$u_j(x) = \lambda_j^{-1}u(\lambda_j x)$$

for all $x \in \mathbb{R}^n$. The area bounds imply that a subsequence of these is convergent, with weak limit a so-called blowdown cone $C$ of $G$. Using the well-known monotonicity formula for area, it suffices to prove that $\|C\|(B_1) = 2\omega_n$—the same area as the sum of two $n$-dimensional planes—to deduce that $u$ is linear. Because $\|C\|(B_1) = \lim_{j \to \infty} \mathcal{H}^n(G_j \cap B_1)$, one might harbour hope that the initial area estimates can be sharpened to obtain this. Although improvements on the initial bounds—see Section 3.2—indeed play an important role in our argument, these do not directly lead to a proof. Instead we classify the cones that can arise from such a blowdown construction, and ultimately prove that in dimension $n + 1 = 4$ the cone $C$ must be a sum of two three-dimensional planes $\Pi_1, \Pi_2 \in Gr(3,4)$, with multiplicity one: $C = |\Pi_1| + |\Pi_2|$. In this sense our argument is similar to the single-valued theory, where one instead shows that, provided $n + 1 \leq 8$, the only possible blowdown cones are $n$-dimensional planes. This is a direct consequence of the regularity of area-minimising currents. As this is unavailable here, even in low dimensions the blowdown cone $C$ could in principle contain a large set of singularities $\text{sing } C$, which can be immersed, branched or more complicated yet. The singular set $\text{sing } C$ can be divided into strata $S^0(C), \ldots, S^n(C)$, where $S^k(C)$ is the set of points whose tangent cones have a spine of dimension at most $k$, along which it is translationally invariant. The Almgren–Federer stratification theorem [Alm00] gives that $\dim_{\mathcal{H}} S^k(C) \leq k$; this was later improved
by Naber–Valtorta [NV15], who proved that in fact $\mathcal{S}^k(\mathcal{C})$ is countably $k$-rectifiable using quantitative stratification methods. The lower strata can thus be gathered into a set with codimension at least two, which can be excised using capacity arguments.

The top stratum $(\mathcal{S}^n \setminus \mathcal{S}^{n-1})(\mathcal{C})$ is the set of branch points $\mathcal{B}(\mathcal{C})$, where at least one tangent cone is of the form $Q|\Pi|$, where $Q \geq 2$ and $\Pi \in \text{Gr}(n,n+1)$. It turns out to be relatively easy to prove that the multiplicity at these points is $Q = 2$. This allows the application of the results of Wickramasekera [Wic20] and Krummel–Wickramasekera [KW20], which together imply that $\mathcal{B}(\mathcal{C})$ too is $(n-2)$-rectifiable and can be excised. A large portion of the text is therefore dedicated to $(\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2})(\mathcal{C})$, the only stratum for which these excision arguments are impossible on account of its size. This is composed of those points $X \in \text{sing} \mathcal{C}$ that have at least one so-called classical tangent cone: this is a cone $\mathcal{P} \in \text{VarTan}(\mathcal{C},X)$ of the form $\mathcal{P} = \sum_i Q_i|\pi_i|$, where $Q_i \geq 1$ and $\pi_i$ and $n$-dimensional half-planes which meet along an $(n-1)$-dimensional axis $L$ say. Via a diagonal extraction argument, one obtains a new sequence $(G_j \mid j \in \mathbb{N})$ of two-valued minimal graphs so that $|G_j| \to \mathcal{P}$ as $j \to \infty$. (We use the same notation as for our original sequence for sake of simplicity.) The aim is therefore to classify the classical cones that can arise as weak limits of two-valued minimal graphs. We prove that, provided $n+1 \leq 7$, there exist two $n$-dimensional planes $\Pi_1, \Pi_2 \in \text{Gr}(n,n+1)$ so that $\mathcal{P} = |\Pi_1| + |\Pi_2|$: Sections 8 to 10 are devoted to this classification.

(The dimension restriction is related to our improved area estimates, see Corollary 3.4. When $n+1 \leq 7$, these exclude a priori the possibility that $\mathcal{P} = 2(|\Pi_1| + |\Pi_2| + |\Pi_3|)$ for example. When $n+1 \geq 8$, and this initial reduction is unavailable, our classification arguments come up short.) This being established, the results of Wickramasekera [Wic20] then demonstrate that the support of $\mathcal{C}$ is smoothly immersed away from an $(n-2)$-dimensional rectifiable set that includes its branch points. In fact this is true for any varifold that arises as a weak limit from a sequence of two-valued minimal graphs. For blowdown cones—those that arise from a sequence of homothetic rescalings of $G = \text{graph} \ u$ for some $u \in C^{1,\alpha}(\mathbb{R}^n; A_2(\mathbb{R}))$—we obtain the following stronger conclusion; see Corollary 10.6 and Theorem 11.3.

**Theorem 3.** Let $\alpha \in (0,1)$ and $2 \leq n \leq 6$. Let $u \in C^{1,\alpha}(\mathbb{R}^n; A_2)$ be an entire two-valued minimal graph and $\mathcal{C}$ be a blowdown cone of $|G|$ at infinity. Then $\mathcal{C}$ is stationary, stable, smoothly immersed away from an $(n-2)$-rectifiable set, and in fact

(i) either $\mathcal{C}$ is cylindrical of the form $\mathcal{C} = C_0 \times \mathbb{R}e_{n+1}$,

(ii) or $\mathcal{C} = |\Pi| + C_0 \times \mathbb{R}e_{n+1}$ where $\Pi \in \text{Gr}(n,n+1)$,

(iii) or $\mathcal{C}$ is the sum of two planes $\Pi_1, \Pi_2 \in \text{Gr}(n,n+1)$, $\mathcal{C} = |\Pi_1| + |\Pi_2|$. 

Finally, in dimension $n+1 = 4$ we complete the classification of blowdown cones: we first show that actually the blowdown cone $\mathcal{C}$ is equal to a sum of at most three planes, each of which has multiplicity one, and then apply a combinatorial argument to a sort of dual cellular decomposition to conclude
that there are two three-dimensional planes $\Pi_1, \Pi_2 \in Gr(3, 4)$ so that $C = |\Pi_1| + |\Pi_2|$. By the monotonicity formula for the area, this will conclude the proof of Theorem 1.

**Overview.** In Section 1 we set notation for two-valued functions. In Section 2 we first define two-valued minimal graphs, next quote results of Simon–Wickramasekera [SW16] and Krummel–Wickramasekera [KW20], and finally establish some of their basic properties, including stability. In the next two sections we prove estimates for two-valued minimal graphs: bounds for their area in Section 3 and an interior gradient estimate in Section 4. In Section 5 we prove a regularity lemma, which establishes that Lipschitz two-valued minimal graphs are in fact $C^{1,\alpha}$-regular. In Section 6 we prove a lemma for single-valued minimal graphs in the style of Jenkins–Serrin: this is used in our classification of classical limit cones; see Section 10. In Section 7 we show that the branch points of limits of two-valued minimal graphs have multiplicity two, and establish some preliminary lemmas in anticipation of Section 10. The following three sections are devoted to the classification of classical limit cones. In Section 8 we adapt an argument of Schoen–Simon for a preliminary analysis. Section 9 treats limit cones that are not vertical, and Section 10 concludes the analysis by considering vertical cones. In Section 11 we prove Theorems 2 and 3. Finally we give a proof of the main result, Theorem 1 in Section 12. In Appendix A we quote some results from geometric measure theory, notably the statement of the branched sheeting theorem of Wickramasekera [Wic20] in Section A.3.

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### 1. Two-valued functions

1.1. **Unordered pairs.** Let $A_2(\mathbb{R})$ be the set of unordered pairs of real numbers, abbreviated $A_2$ when no confusion is possible. An element of $A_2(\mathbb{R})$ is written $\{x, y\} = \{y, x\}$. One can define $A_2(\mathbb{R})$ as the set obtained by taking the quotient of $\mathbb{R}^2$ under the action by the group $\mathbb{Z}_2$ which transposes the two elements. The quotient map is then $\mathbb{R}^2 \to A_2(\mathbb{R}) : (x, y) \mapsto \{x, y\}$. Alternatively one could define $A_2(\mathbb{R})$ as the non-empty subsets of $\mathbb{R}$ counting at most two elements. More generally we can also take unordered pairs of elements of any set $X$, thus forming $A_2(X)$. Here $X$ is usually either a finite-dimensional vector space $E$ say or a subset thereof. Apart from $E = \mathbb{R}$ we also use $E = \mathbb{R}^{n+1}$ and $E = L(\mathbb{R}^n; \mathbb{R})$, the space of linear functions $\mathbb{R}^n \to \mathbb{R}$. Consider the example where $X = E$ is a real vector space. Scalar multiplication is unambiguously defined, but $A_2(E)$ is not naturally
a vector space, because the sum (or difference) of two elements \( u, v \in A_2(\mathbb{E}) \) cannot be made sense of as an element in \( A_2(\mathbb{E}) \).

Any map \( \Phi : X \to Y \) between two sets can be extended to a two-valued analogue by setting \( \Phi : A_2(X) \to A_2(Y) : \{ x, y \} \mapsto \{ \Phi(x), \Phi(y) \} \). For example we define \( \{ \{ u_1, u_2 \} \} = \{ |u_1|, |u_2| \} \in A_2(\mathbb{R}_{\geq 0}) \). Given a pair \( \{ x_1, x_2 \} \in A_2(\mathbb{R}) \) of real numbers, we define its average and symmetric difference by \( x_a = \frac{1}{2}(x_1 + x_2) \) and \( x_s = \{ \pm(x_1 - x_2) \} \) respectively. Moreover write \( x_+ = \max\{x_1, x_2\} = x_1 \lor x_2 \) and \( x_- = \min\{x_1, x_2\} = x_1 \land x_2 \).

Let \( E \) be a vector space with norm \( \| \cdot \| \). For a pair \( \{ x, y \} \in A_2(E) \) we write \( \| \{ x, y \} \| = \| x \| + \| y \| \). We define a metric on \( A_2(E) \) by \( \mathcal{G} : A_2(E) \times A_2(E) \to \mathbb{R}_{\geq 0} \) with \( \mathcal{G}(u, v) = \min(\| u_1 - v_1 \| + \| u_2 - v_2 \|, \| u_1 - v_1 \| + \| u_2 - v_2 \|) \). The analogous construction works for arbitrary metric spaces.

### 1.2. Definition and function spaces.

Let \( A \subset \mathbb{R}^n \) be an arbitrary set. A two-valued function on \( A \) is a function \( A \to A_2(\mathbb{R}) \). To \( u \) we can associate its average \( u_a = \frac{1}{2}(u_1 + u_2) \) and symmetric difference \( u_s = \{ \pm(u_1 - u_2) \} \). Similarly we write \( u_+ = \max\{u_1, u_2\} = u_1 \lor u_2 \) and \( u_- = \min\{u_1, u_2\} = u_1 \land u_2 \).

More generally we consider two-valued maps \( A \to A_2(X) \), where \( X \) is an arbitrary set. For use most often \( X \) will either be \( \mathbb{R} \) or a real vector space, say \( \mathbb{R}^n \) or \( L(\mathbb{R}^n; \mathbb{R}) \). Those taking values in \( A_2(\mathbb{R}^n) \) we shall call two-valued vector fields. Consider a map \( \Phi : X \to \mathbb{R} \). Given \( u : A \to A_2(X) \) we can compose to \( \Phi \circ u : A \to A_2(\mathbb{R}) \). If we are given two different functions \( \Phi, \Psi : X \to \mathbb{R} \) then we may take the sum \( \Phi \circ u + (\Psi \circ u) := (\Phi + \Psi) \circ u \), although one the sum of two-valued functions cannot be defined as a two-valued function. The same of course holds for other binary operations.

As \( A_2 = A_2(\mathbb{R}) \) is a metric space, endowing it with the corresponding topology and Borel \( \sigma \)-algebra allows us to define measurable and continuous two-valued functions. Given \( \alpha \in (0, 1) \), a two-valued function on \( A \) is called \( \alpha \)-Hölder continuous if \( \lim \sup_{\| x - y \| \to 0} \| x - y \|^{-\alpha} \mathcal{G}(u(x), u(y)) < \infty \), and that \( u \) is Lipschitz continuous if this holds with \( \alpha = 1 \). Let \( \Omega \subset \mathbb{R}^n \) be open. We write \( C^{0,\alpha}(\Omega; A_2) \) and \( \text{Lip}(\Omega; A_2) \) for the functions that are locally \( \alpha \)-Hölder continuous or locally Lipschitz respectively. Both notions are again defined for functions taking values in \( A_2(X) \) for any metric space \( (X, d) \), for example \( X = L(\mathbb{R}^n; \mathbb{R}) \).

We say that a function \( l : \mathbb{R}^n \to A_2 \) is linear if there exist two single-valued linear functions \( l_i \in L(\mathbb{R}^n; \mathbb{R}) \) so that \( l = \{ l_1, l_2 \} \). A function \( u : \Omega \to A_2 \) is called differentiable at \( x \) if there is a two-valued linear function \( l = \{ l_1, l_2 \} \) so that for all \( v \in \mathbb{R}^n \), \( t^{-1} \mathcal{G}(u(x + tv), \{ u_1(x) + l_1(tv) \}) \to 0 \) as \( t \to 0 \). If this exists, we write \( Du(x) = l \) and call this the derivative of \( u \) at \( x \). This defines a two-valued function \( Du : \Omega \to A_2(L(\mathbb{R}^n; \mathbb{R})) \). Moreover we write \( u \in C^1(\Omega; A_2) \) if it is differentiable at all points \( x \in \Omega \) and the function \( Du \) is continuous, and for \( \alpha \in (0, 1) \) we write \( u \in C^{1,\alpha}(\Omega; A_2) \) if \( Du \in C^{0,\alpha}(\Omega; A_2(L(\mathbb{R}^n; \mathbb{R}))) \).
The corresponding Hölder norms are defined in the usual way; given \( \alpha \in (0, 1) \) we write \( C^{0}(\Omega; A_{2}), C^{0,\alpha}(\Omega; A_{2}), C^{1}(\Omega; A_{2}), C^{1,\alpha}(\Omega; A_{2}) \) for the spaces formed by those functions on \( \Omega \) for which the respective norms are finite.

### 1.3. Selections and singularities

Consider a two-valued function \( u : \Omega \to A_{2} \), and let \( \Omega' \subset \Omega \) be an open subset. We say that two functions \( u_{1}, u_{2} : \Omega \to \mathbb{R} \) define a selection for \( u \) on \( \Omega' \) if \( u = \{ u_{1}, u_{2} \} \) on \( \Omega' \). Most often one seeks selections with favourable properties: for example a selection is called continuous if \( u_{1}, u_{2} \in C^{0}(\Omega') \), of class \( C^{1} \) if \( u_{1}, u_{2} \in C^{1}(\Omega') \), and smooth if \( u_{1}, u_{2} \in C^{\infty}(\Omega') \).

The touching set of a function \( u \in C^{1}(\Omega; A_{2}) \) is \( Z_{u} = \{ x \in \Omega \mid u_{1}(x) = u_{2}(x) \} \), and the critical set of \( u \) is \( K_{u} = \{ x \in \Omega \mid u_{1}(x) = u_{2}(x), Du_{1}(x) = Du_{2}(x) \} \). We also set \( C_{u} = Z_{u} \cap K_{u} = \{ x \in \Omega \mid u_{1}(x) = u_{2}(x), Du_{1}(x) \neq Du_{2}(x) \} \). This is a relatively open subset of \( Z_{u} \), and \( B_{u}, K_{u} \subset Z_{u} \) are relatively closed. For every point \( y \in \Omega \setminus K_{u} \) there is \( \sigma > 0 \) so that on \( D_{\sigma}(y) \) there are two functions \( u_{1,y}, u_{2,y} \in C^{1}(D_{\sigma}(y)) \) so that \( u = \{ u_{1,y}, u_{2,y} \} \). A point \( x \in \Omega \) is called a branch point if there is no radius \( \sigma > 0 \) for which \( u \) admits a \( C^{1} \) selection on \( D_{\sigma}(x) \). They form the set \( B_{u} \subset K_{u} \). Points in \( K_{u} \setminus B_{u} \) are sometimes called false branch points.

Let \( u : \Omega \to A_{2} \) be a two-valued function. Its graph \( G \subset \Omega \times \mathbb{R} \) is the set
\[
\text{graph } u = G = \{(x, X^{n+1}) \in \Omega \times \mathbb{R} : X^{n+1} = u_{1}(x) \mbox{ or } u_{2}(x)\}.
\]
This may be considered as a varifold inside \( \Omega \times \mathbb{R} \), in which case we write \( |G| = |\text{graph } u| \in IV_{n}(\Omega \times \mathbb{R}) \) as customary. (We emphasise that throughout we do not consider the graph as a subset of \( \Omega \times A_{2}(\mathbb{R}) \), as one might expect by interpreting the term more literally.) In general of course \( G = \text{graph } u \) if \( u \in C^{1}(\Omega; A_{2}) \) is not regular and instead has an immersed set of singularities \( C(G) \), where its tangent cone is a union of two distinct \( n \)-dimensional planes, and the branch set \( B(G) \), where its tangent cone is a hyperplane with multiplicity two. They are related to the singularities of \( u \) via the orthogonal projection \( P_{0} \) onto \( \mathbb{R}^{n} \times \{0\} \), \( P_{0}(\text{sing } G \cup \{ X \in \text{reg } G \mid \Theta(\|G\|, X) = 2 \}) = Z_{u} \), \( P_{0}(B(G) \cup \{ X \in \text{reg } G \mid \Theta(\|G\|, X) = 2 \}) = K_{u} \), and \( P_{0}(C(G)) = C_{u} \).

We briefly remark on the case where \( n = 1 \) and \( \Omega = I \subset \mathbb{R} \) is an open interval in the real line. Let \( u \in C^{1}(I; A_{2}) \). This automatically has \( B_{u} = \emptyset \), even while \( K_{u} \) may be non-empty. In other words, we can always find \( u_{1}, u_{2} \in C^{1}(I) \) so that \( u = \{ u_{1}, u_{2} \} \) on \( I \), although some arbitrary choices have to be made if \( K_{u} \neq \emptyset \). We will later use this elementary observation in the following context. Let \( \Omega \subset \mathbb{R}^{n} \) be open, and \( u \in C^{1}(\Omega; A_{2}) \). Let \( y \in \Omega \) and \( v \in \mathbb{R}^{n} \) be arbitrary. Write \( l_{y} \subset \{ y + tv : t \in \mathbb{R} \} \cap \Omega \) for the connected component containing \( y \). This corresponds to an interval \( I \subset \mathbb{R} \). Via this identification, the restriction of \( u \) to \( l_{y} \) defines a two-valued function in \( C^{1}(I; A_{2}) \). Hence we can find two functions \( u_{1,y}, u_{2,y} \in C^{1}(I) \) so that \( u(y + tv) = \{ u_{1,y}(t), u_{2,y}(t) \} \) even though possibly \( l_{y} \cap B_{u} \neq \emptyset \). Moreover if
u \in C^{1,\alpha}(\Omega; A_2) \) for some \( \alpha \in (0, 1) \) then we can impose \( u_{1,y}, u_{2,y} \in C^{1,\alpha}(I) \) as well.

1.4. Integrals of two-valued functions. Let \( p \in [1, +\infty) \). We write \( L^p(\Omega; A_2) \) for the space of two-valued functions \( u : \Omega \to A_2 \) with \( \int_\Omega ||u||^p < \infty \), and \( L^\infty(\Omega; A_2) \) for those functions for which \( ||u|| \) is essentially bounded. We define the integral of a two-valued function \( u \in L^1(\Omega; A_2) \) by \( \int_\Omega u = 2 \int_\Omega u_a = \int_\Omega u_1 + u_2 \). Sometimes we also write \( \int_\Omega u(x) \, dx = \int_\Omega u \).

Let \( u \in C^1(\overline{\Omega}; A_2) \) and \( \Phi : \mathbb{R} \to \mathbb{R} \) be so that \( \Phi \circ u \in L^1(\Omega; A_2) \). Then we have that \( \int_\Omega (\Phi \circ u)(1 + |Du|^2) = \int_G \Phi \, d\mathcal{H}^n \), where on the right-hand side the integral is over \( G = \text{graph} \, u \subset \Omega \times \mathbb{R} \).

Consider \( u \in L^1(\Omega; A_2), A \subset \mathbb{R} \) be a Borel subset, and \( 1_A \) be its indicator function. As \( u \) takes values in \( A_2 \), the pre-image of \( A \) under \( u \) is not defined. However, we can define the integral \( \int_{u \in A} u := \int u(1_A \circ u) = \int u_1(1_A \circ u_1) + u_2(1_A \circ u_2) \). This can be generalised further if we consider \( \Phi : \mathbb{R} \to \mathbb{R} \) so that \( \Phi \circ u \in L^1(\Omega; A_2) \), by setting \( \int_{u \in A} \Phi \circ u = \int (\Phi \circ v)(1_A \circ v) \), where recall the product of the two functions is a well-defined two-valued function in this specific context. We will often use a variant of this, where in fact \( \Phi : \Omega \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) is so that the two-valued function \( x \in \Omega \mapsto \Phi(x, Du(x), u(x)) \) is integrable, and we consider integrals of the form \( \int_{u \in A} \Phi(x, Du(x), u(x)) \, dx \), or say \( \int_{u \in A, Du \in B} \Phi(x, Du(x), u(x)) \, dx \) where \( B \subset \mathbb{R}^n \) is another Borel subset.

1.5. Some results for two-valued functions. From the usual Rademacher theorem, see for example [Sim84, Thm. 5.2], one easily derives a two-valued analogue.

**Lemma 1.1.** Let \( u \in \text{Lip}(D_1; A_2) \) be a two-valued Lipschitz function. Then \( u \) is differentiable \( \mathcal{H}^n \)-a.e. in \( D_1 \).

Using a general form of the Arzelà–Ascoli theorem, for example [Mun00, Thm. 47.1], we obtain the following lemma specialised to sequences of two-valued Lipschitz functions.

**Lemma 1.2.** Let \( (u_j \mid j \in \mathbb{N}) \) be a sequence of two-valued Lipschitz functions on \( D_1 \). If there is \( C > 0 \) so that \( \sup_{D_1} ||u_j|| + ||Du_j|| \leq C \) for all \( j \in \mathbb{N} \) then there is a subsequence which converges locally uniformly to a Lipschitz function \( u \in \text{Lip}(D_1; A_2) \).

2. Two-valued minimal graphs

2.1. Definition and basic properties. Let \( \Omega \subset \mathbb{R}^n \) be an open set, and let \( \alpha \in (0, 1) \). We say that \( u \in C^{1,\alpha}(\Omega; A_2(\mathbb{R})) \) defines a two-valued minimal graph if its graph

\[ |G| = |\text{graph} \, u| \in \text{IV}_n(\Omega \times \mathbb{R}) \]

is stationary as a varifold in the open cylinder \( \Omega \times \mathbb{R} \). We will often abbreviate this by saying that \( u \in C^{1,\alpha}(\Omega; A_2) \) is a two-valued minimal graph.
When an open subdomain \( \Omega' \subset \Omega \setminus B_u \) is simply connected then there is a selection \( u_1, u_2 \in C^2(\Omega') \) so that \( u = \{u_1, u_2\} \) in \( \Omega' \). Inside \( \Omega' \times \mathbb{R} \) the graph can be decomposed like \( |G| = |\text{graph } u_1| + |\text{graph } u_2| \), which we frequently abbreviate by writing \( G_i = \text{graph } u_i \).

The stationarity of \( |G| \) is inherited by \( |G_1| \) and \( |G_2| \). This in turn means that \( u_1, u_2 \) are both smooth, that is \( u_1, u_2 \in C^\infty(\Omega') \) and they separately solve the minimal surface equation

\[
\text{div } T(Du_i) = 0 \quad \text{in } \Omega \quad \text{for } i = 1, 2,
\]

where here and throughout we write, for all \( p \in \mathbb{R}^n \)

\[
T_k(p) = \frac{p_k}{\sqrt{1 + |p|^2}} \quad \text{for } k = 1, \ldots, n.
\]

The vector \( T(Du) \in \mathbb{R}^n \) is the horizontal part of \(-\nu\), the downward-pointing unit normal to the graph \( G \). We habitually write \( v = \sqrt{1 + |Du|^2} \), so that also \( T(Du) = Du/v \).

If we interpret the equation (2.1) in a weak sense we can show that for all test functions \( \phi \in C^1_c(\Omega) \)

\[
\int_\Omega \langle T(Du), D\phi \rangle = 0,
\]

using a partition of unity argument to reduce to the case where \( \phi \) is supported in a simply connected domain \( \Omega' \subset \Omega \setminus B_u \). Note that as \( u \) is a two-valued function, the integral above is understood to be \( \int_\Omega \langle T(Du_1), D\phi \rangle + \langle T(Du_2), D\phi \rangle \), as explained in Section 1.4.

This can be generalised to arbitrary test functions \( \phi \in C^1_c(\Omega) \) by reasoning as follows. By assumption, there is a compact subset \( K \subset \Omega \) so that \( \phi \) vanishes identically outside \( K \). Take a sequence \( (\eta_j \mid j \in \mathbb{N}) \) of functions in \( C^1_c(\Omega) \) with

\[
\begin{align*}
(1) & \quad 0 \leq \eta_j \leq 1 \quad \text{in } \Omega \quad \text{for all } j, \\
(2) & \quad \eta_j \equiv 0 \quad \text{on } B_u \cap K \quad \text{for all } j, \\
(3) & \quad \eta_j \to 1 \quad \mathcal{H}^{n-1}\text{-a.e. in } \Omega, \\
(4) & \quad \int_\Omega |D\eta_j| \to 0 \quad \text{as } j \to \infty.
\end{align*}
\]

Such a sequence exists because \( \mathcal{H}^{n-1}(B_u) = 0 \), see Section 2.3 and [EG15, Ch. 5.6] for example. Actually by [KW20] the branch set is countably \((n-2)\)-rectifiable, see again Section 2.3. (There we also explain that analogous sequences can be constructed to cut off the critical set \( K_u \), provided only that the set \( G \) is not in fact equal to a single-valued minimal graph, with multiplicity two.)

Following the steps in the proof of the following proposition one gets \( \int_\Omega \langle T(Du), D\phi \rangle = 0 \) for all \( \phi \in C^1_c(\Omega) \). We skip over the details of this, and move on to the proof of the more general identity expressed in the proposition, which allows the test function to depend on \( u \).
Proposition 2.1. Let \( u \in C^{1,\alpha}(\Omega; A_2) \cap C^0(\overline{\Omega}; A_2) \) define a two-valued minimal graph and let \( \Phi \in \text{Lip}(\Omega \times \mathbb{R} \times \mathbb{R}^n) \) have \( \text{spt } \Phi \subset \Omega' \times \mathbb{R} \times \mathbb{R}^n \) for some \( \Omega' \subset \subset \Omega \). If

\[
(2.3) \quad \int_{\Omega \setminus B_u} |D(\Phi(x, u, Du))| < +\infty
\]

then

\[
(2.4) \quad \int_{\Omega} (T(Du), D(\Phi(x, u, Du))) = 0.
\]

Proof. Define a function \( \phi \in C^1_c(\Omega) \) by setting \( \phi(x) = \Phi(x, u, Du) \) for all \( x \in \Omega \). This is smooth away from \( B_u \) and has support contained inside \( \Omega' \). Let \( \eta_j \subset \mathbb{N} \) be a sequence with the same properties as above. As \( (1 - \eta_j) \phi \) vanishes near \( B_u \) it is a valid test function in the integral identity (2.1), yielding

\[
\int_{\Omega} (T(Du), D\phi)(1 - \eta_j) - \int_{\Omega} (T(Du), D\eta_j) \phi = 0.
\]

That the second integral goes to zero is a direct application of Hölder’s inequality as above. For the first integral, we can bound the integrand like \( |(T(Du), D\phi)(1 - \eta_j)| \leq |D\phi| \) almost everywhere in \( \Omega \) on \( \Omega \setminus B_u \) to be precise. The bounding function is integrable by assumption, so that again we can use dominated convergence to let \( j \to \infty \) and deduce that

\[
\int_{\Omega} (T(Du), D(\Phi(x, u, Du))) = 0.
\]

2.2. Orientation and the current structure. Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( \alpha \in (0, 1) \) and let \( u \in C^{1,\alpha}(\Omega; A_2) \) define a two-valued minimal graph in \( \Omega \times \mathbb{R} \). At all regular points \( X = (x, X^{n+1}) \in \text{reg } G \cap \Omega \times \mathbb{R} \) we write \( \nu(X) \) for the upward-pointing unit normal. In terms of a smooth selection \( u = \{u_1, u_2\} \) for \( u \) on a small disc \( D_\rho(x) \) we have \( \nu(x, u_i(x)) = (1 + |Du_i(x)|^2)^{-1/2}(-Du_i(x), 1) \). This is also defined at the branch points, and hence \( \nu \) defines a continuous vector field on \( (\text{reg } G \cup \overline{\mathcal{B}(G)}) \cap \Omega \times \mathbb{R} \), which is moreover smooth on \( \text{reg } G \). However \( \nu \) cannot be continuously extended to the set of classical singularities \( \mathcal{C}(G) \cap \Omega \times \mathbb{R} \). As this set has \( \mathcal{H}^n(\mathcal{C}(G) \cap \Omega \times \mathbb{R}) = 0 \) we can however still define an integer multiplicity rectifiable current \( [G] \) by integrating over \( \text{reg } G \cap \Omega \times \mathbb{R} \). Moreover \( \partial[G] = 0 \) in \( \Omega \times \mathbb{R} \) and \( [G] \in \mathcal{I}_n(\Omega \times \mathbb{R}) \).

2.3. Properties of the branch set. Let \( \alpha \in (0, 1) \), and let \( u \in C^{1,\alpha}(\Omega; A_2) \) be an arbitrary two-valued minimal graph. Using an approach based on a so-called frequency function, Simon–Wickramasekera [SW16] proved the following.

Theorem 2.2 ([SW16]). Let \( \alpha \in (0, 1) \) and \( u \in C^{1,\alpha}(\Omega; A_2) \) be a two-valued minimal graph. Then the branch set of \( u \) is either empty or \( \dim_H \mathcal{B}_u = n - 2 \) and \( \mathcal{H}^{n-2}(\mathcal{B}_u) \neq 0 \).

In particular \( \mathcal{H}^s(\mathcal{B}_u) = 0 \) for all \( s > n - 2 \). We use the fact that \( \mathcal{H}^{n-1}(\mathcal{B}_u) = 0 \) to derive our area estimates.
Corollary 2.3 ([SW16]). Let \( \alpha \in (0,1) \) and \( u \in C^{1,\alpha}(\Omega;A_2) \) be a two-valued minimal graph. If \( G \) is not equal to a single-valued minimal graph with multiplicity two then \( \dim \mathcal{H} K_u = n - 2 \) and \( \mathcal{H}^{n-2}(K_u) \neq 0 \).

Starting also from a frequency function, this was taken further by Krummel–Wickramasekera [KW20], who proved the following.

Theorem 2.4 ([KW20]). Let \( \alpha \in (0,1) \) and \( u \in C^{1,\alpha}(\Omega;A_2) \) be a two-valued minimal graph defined on \( \Omega \subset \mathbb{R}^n \). Then \( B_u \) is either empty or is countably \( n-2 \)-rectifiable. Moreover if \( G \) is not equal to a single-valued minimal graph with multiplicity two then \( K_u \) is countably \( n-2 \)-rectifiable.

This represents a significant improvement over [SW16] because it allows the excision of the branch set via capacity arguments which we now detail.

Following the presentation given by Evans–Gariepy in [EG15, Ch. 4.7] we let \( K^p \) be the space of functions \( f : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) with \( f \in L^p \) and \( |Df| \in L^p \), where \( p^* = np/(n-p) \) is the Sobolev conjugate of \( p \). The \( p \)-capacity of a set \( A \subset \mathbb{R}^n \) is then defined to be \( \text{cap}_p A = \inf \{ \int |Df|^p : f \in K^p, A \subset \text{int}\{ f \geq 1 \} \} \). When \( A \) is compact then equivalently \( \text{cap}_p A = \inf \{ \int |Df|^p : f \in C_c^\infty(\mathbb{R}^n), f \geq 1_A \} \). We do not list the basic properties of capacity listed in [EG15], and only record here the following two results. For \( p \in (1,n) \), if a set \( A \subset \mathbb{R}^n \) has \( \mathcal{H}^{n-p}(A) < \infty \) then \( \text{cap}_p A = 0 \), and if \( \text{cap}_p A = 0 \) then \( \dim \mathcal{H} A \leq n - p \). From this it follows the countably \((n-p)\)-rectifiable sets have \( p \)-capacity zero. (When \( p = 1 \) then more is true, as a subset \( A \subset \mathbb{R}^n \) has \( \text{cap}_1 A = 0 \) if and only if \( \mathcal{H}^{n-1}(A) = 0 \); see [EG15, Ch. 5.6].)

Now let \( p \in [1,n) \) and \( A \subset \mathbb{R}^n \) be a compact set with \( \text{cap}_p A = 0 \). With only little effort one establishes the existence of a sequence of cutoff function \( \{ \eta_j \mid j \in \mathbb{N} \} \) with the following properties for all \( j \):

(i) \( \eta_j \in C^1_c(\mathbb{R}^n) \),
(ii) \( 0 \leq \eta_j \leq 1 \),
(iii) \( \eta_j \equiv 1 \text{ on } (A)_{r_j} \text{ for some } r_j \to 0 \),
(iv) \( \eta_j \to 0 \text{ } \mathcal{H}^n\text{-a.e.} \),
(v) \( \int_{\mathbb{R}^n} |D\eta_j|^p \to 0 \).

Moreover if \( U \) is an open set containing \( A \) then we can additionally impose that \( \text{spt} \eta_j \subset U \). For the branch set of two-valued minimal graphs this yields the following.

Lemma 2.5. Let \( \alpha \in (0,1) \), and \( u \in C^{1,\alpha}(D_2;A_2) \) be a two-valued minimal graph. Then there is a sequence of functions \( \{ \eta_j \mid j \in \mathbb{N} \} \) with for all \( j \),

(i) \( \eta_j \in C^1_c(D_2) \),
(ii) \( 0 \leq \eta_j \leq 1 \text{ on } D_2 \),
(iii) \( \eta_j \equiv 1 \text{ on } (B_u)_{r_j} \cap D_1 \text{ for some } r_j \to 0 \),
(iv) \( \eta_j \to 0 \text{ } \mathcal{H}^n\text{-a.e. as } j \to \infty \),
(v) \( \int_{D_2} |D\eta_j|^2 \to 0 \text{ as } j \to \infty \).

In the same way as for the results cited above, one obtains a version of this valid for the set \( K_u \) provided the set \( G \) is not equal to a single-valued
minimal graph, with multiplicity two. In various places it is useful to modify
the sequence from Lemma 2.5 and construct it on the graph itself.

Corollary 2.6. Let $\alpha \in (0, 1)$, and $u \in C^{1,\alpha}(D_2; A_2)$ be a two-valued minimal graph. If $\mathcal{B}(G) \neq \emptyset$ then there is a sequence of functions $(\eta_j \mid j \in \mathbb{N})$ with for all $j$,
(i) $\eta_j \in C^1_0(D_2 \times \mathbb{R})$,
(ii) $0 \leq \eta_j \leq 1$ on $D_2 \times \mathbb{R}$,
(iii) $\eta_j \equiv 1$ on $(\mathcal{B}(G))_{r_j} \cap D_1 \times \mathbb{R}$ for some $r_j \to 0$,
(iv) $\eta_j \to 0$ $\mathcal{H}^n$-a.e. on $\text{reg} G \cap D_2 \times \mathbb{R}$ as $j \to \infty$,
(v) $\int_{\text{reg} G \cap D_2 \times \mathbb{R}} |\nabla \eta_j|^2 \, d\mathcal{H}^n \to 0$ as $j \to \infty$.

Proof. Let $(\eta_j^0 \mid j \in \mathbb{N})$ be a sequence with the properties listed in Lemma 2.5, but adapted to the set $K_u$ instead, and we additionally impose that $\text{spt} \eta_j^0 \subset D_{s_j/2}$ for all $j$. Inside this disc there is $A > 0$ so that $-A < \min_{D_{s_j/2}} u_- \leq \max_{D_{s_j/2}} u_+ < A$, where $u_- = \min \{u_1, u_2\}$ and $u_+ = \{u_1, u_2\}$. Let $\tau \in C^1_0(\mathbb{R})$ be a classical cutoff function with $\tau \equiv 1$ on $[-1, 1]$, $\text{spt} \tau \subset [-2, 2]$ and $|\tau'| \leq 2$. For all $j \in \mathbb{N}$, extend $\eta_j^0$ to $D_2 \times \mathbb{R}$ by setting $\eta_j(x, X^{n+1}) = \eta_j^0(x) \tau(X^{n+1}/A)$ at all $X = (x, X^{n+1}) \in D_2 \times \mathbb{R}$.

To obtain the last two properties, let $\delta > 0$ be given, and $d_G$ be the unsigned distance function to $G \cap D_2 \times \mathbb{R}$. For any $j$ we may replace $\eta_j$ with $\bar{\eta}_j$, defined by $\bar{\eta}_{j, \delta}(X) = \eta_j(X) \tau(d_G(X)/\delta)$ at all $X \in D_2 \times \mathbb{R}$. This additionally has $\text{spt} \bar{\eta}_{j, \delta} \subset (G)_{2\delta} \cap D_2 \times \mathbb{R}$. This function inherits most properties from $\eta_j$, but it is only Lipschitz regular because of $d_G$. This however can be easily remedied by a standard mollification argument, taking care to choose the mollification parameter small enough in terms of $r_j, \delta > 0$.

Finally, we may pick any sequence $s_j \to 0$ and apply the construction above with $\delta = s_j/2$, letting $\bar{\eta}_j = \bar{\eta}_{j, s_j/2}$ to conclude. \hfill $\square$

Let us conclude by pointing out the following consequence of [KW20], obtained by combining it with the results of [Wic20], see Theorem A.4.

Corollary 2.7. Let $V \in \mathbf{IV}_n(B_1)$ be a stationary varifold with stable regular part. Then $\mathcal{B}(V) \cap \{\Theta(||V||, \cdot) \leq 2\}$ is countably $n-2$-rectifiable.

2.4 Immersion away from the branch set. Let $\Omega \subset \mathbb{R}^n$ be a connected open set, and let $u \in C^{1,\alpha}(\Omega; A_2)$ be so that $|G| = \text{graph } u \in \mathbf{IV}_n(\Omega \times \mathbb{R})$ is a stationary varifold. In this section we construct a smooth $n$-dimensional manifold $\Gamma$ and a minimal immersion $\iota: \Gamma \to \mathbb{R}^{n+1}$ with image $\iota(\Gamma) = \text{reg } G \cup \mathcal{C}(G) = G \setminus \mathcal{B}(G)$ using a standard gluing construction.

Let $(U_\alpha \mid \alpha \in A)$ be an open cover of $\Omega \setminus \mathcal{B}_u$, chosen so that every $U_\alpha$ is simply connected. For every open set $U_\alpha$ in this collection we can make a smooth selection $u_{\alpha,1}, u_{\alpha,2} \in C^1(U_\alpha)$ so that $u = \{u_{\alpha,1}, u_{\alpha,2}\}$ in $U_\alpha$. Write $G_{\alpha,1} = \text{graph } u_{\alpha,1}$ and $G_{\alpha,2} = \text{graph } u_{\alpha,2}$, so that accordingly $G \cap U_\alpha \times \mathbb{R} = G_{\alpha,1} \cup G_{\alpha,2}$. We then consider the disjoint union of these sets $(G_{\alpha,i} \mid \alpha \in A, i = 1, 2)$, each of which is endowed with an obvious map $\iota_{\alpha,i}: G_{\alpha,i} \to \mathbb{R}^{n+1}$, which may also be composed with the projection
\( P_0 : \mathbb{R}^{n+1} \to \mathbb{R}^n \times \{0\} \) to obtain bijections \( P_0 \circ \iota_{\alpha,i} : G_{\alpha,i} \to U_{\alpha,i} \). We glue these together using the equivalence relation \( \sim \) defined as follows.

Two points \( X \in G_{\alpha,i} \) and \( Y \in G_{\beta,j} \) are equivalent if \( P_0 \circ \iota_{\alpha,i}(X) = P_0 \circ \iota_{\beta,j}(Y) \) and there is a neighbourhood of this point where \( u_{\alpha,i} \) and \( u_{\beta,j} \) coincide.

Given this we simply set

\[
\Gamma = \bigsqcup_{\alpha \in A} G_{\alpha,i} / \sim.
\]

Write \( p \) for the projection \( \sqcup G_{\alpha,i} \to \Gamma \). By construction the map \( \sqcup \iota_{\alpha,i} : \sqcup U_{\alpha,i} \to \mathbb{R}^{n+1} \) passes to the quotient by \( \sim \), thus defining a map \( \iota : \Gamma \to \mathbb{R}^{n+1} \). This map has the property that for every set \( U_{\alpha,i} \) and all \( x \in U_{\alpha,i} \), \( \iota \circ p(x) = \iota_{\alpha,i}(x) \). Then it is not hard to see that \( \Gamma \) is a smooth \( n \)-dimensional manifold, with charts \( \{(p(U_{\alpha,i}), P_0 \circ \iota) \mid \alpha \in A, i = 1, 2\} \).

**Lemma 2.8.**

i. The map \( \iota \) is a smooth immersion, injective away from \( \iota^{-1}(\mathcal{C}(G)) \).

ii. The immersion can be oriented by the upward unit normal \( \nu \).

iii. The map \( \iota \) is proper into \( \mathbb{R}^{n+1} \setminus \mathcal{B}(G) \), but not into \( \mathbb{R}^{n+1} \) unless \( \mathcal{B}(G) = \emptyset \).

iv. The manifold \( \Gamma \) is connected unless \( G \) is the union of two single-valued graphs.

**Proof.** The first two properties follow by construction.

ii. Let \( K \subset \mathbb{R}^{n+1} \setminus \mathcal{B}(G) \) be a compact set, and consider a sequence of points \((X_j \mid j \in \mathbb{N})\) in \( \iota^{-1}(K \cap G) \). Write \( Y_j = \iota(X_j) \) for all \( j \), and extract a convergent subsequence from this, with limit say \( Y'_j \to Y \in \text{reg} G \cup \mathcal{C}(G) \). If \( Y \in \text{reg} G \) then there is \( \rho > 0 \) so that the restriction of \( \iota \) to \( \iota^{-1}(B_\rho(Y) \cap G) \) is a homeomorphism onto \( B_\rho(Y) \cap G \), from whence the property follows. If instead \( Y \in \mathcal{C}(G) \) then we can decompose \( \iota^{-1}(B_\rho(Y) \cap G) = W_1 \cup W_2 \) into two disjoint open sets, so that the restriction of \( \iota \) to either of them is again a homeomorphism. As one of \( W_1, W_2 \) contains infinitely many terms in the sequence, the conclusion follows.

iv. The map \( P_0 \circ \iota : \Gamma \to G \) is a double cover, so \( \Gamma \) has at most two connected components. When \( \Gamma \) is disconnected, then \( \Gamma = \Gamma_1 \cup \Gamma_2 \) and the restriction of \( P_0 \circ \iota \) to either of them is a homeomorphism, and their images are two single-valued graphs.

2.5. **Stability of two-valued minimal graphs.** Let \( \iota : \Gamma \to \mathbb{R}^{n+1} \) be the immersion constructed in the previous section, which maps onto \( \text{reg} G \cup \mathcal{C}(G) \). This admits the positive Jacobi field \( \langle \nu, e_{n+1} \rangle \), which implies the stability of the immersion via the following standard argument. Pick any non-negative test function \( \varphi \neq 0 \in C^2_0(\Gamma) \). On its support \( \langle \nu, e_{n+1} \rangle \) is bounded below away from zero, and thus \( T = \max\{t \in \mathbb{R} \mid t\varphi \leq \langle \nu, e_{n+1} \rangle\} \) is positive. By construction \( \langle \nu, e_{n+1} \rangle - T\varphi \geq 0 \), and it has a zero but does
not vanish identically. If there were a less regular test function \( \varphi \in C^1_c(\Gamma) \) with \( \int_\Gamma |A|_\Gamma^2 \varphi^2 > \int_\Gamma |\nabla \Gamma \varphi|^2 \) then we could mollify this and follow the same argument as above; hence the stability of the regular part of \( G \) is established.

If \( \varphi \) is any function in \( C^1_c(\mathbb{R}^{n+1} \setminus \mathcal{B}(G)) \), then its pullback \( \phi = \varphi \circ \iota \) by the immersion is compactly supported on \( \Gamma \) (because \( \iota \) is proper away from the branch set). If we equate the integrals on \( \Gamma \) with ones on \( G \) we obtain

\[
\int_{\text{reg } G \cap \Gamma} |A_G|^2 \varphi^2 \leq \int_{\text{reg } G \cap \Gamma} |\nabla_G \varphi|^2
\]

for all \( \varphi \in C^1_c(\mathbb{R}^{n+1} \setminus \mathcal{B}(G)) \). Extending this to arbitrary test functions requires a capacity argument.

**Lemma 2.9.** Let \( U \subset \mathbb{R}^{n+1} \) be an open set, and \( G \subset U \) be a two-valued minimal graph. Then \( G \) is ambient stable: for all \( \varphi \in C^1_c(U) \),

\[
(S_G) \quad \int_{\text{reg } G \cap U} |A_G|^2 \varphi^2 \leq \int_{\text{reg } G \cap U} |\nabla_G \varphi|^2.
\]

**Proof.** Let \( \varphi \in C^1_c(U) \) be a test function with \( \text{spt } \varphi \cap \mathcal{B}(G) \neq \emptyset \), and \( (\eta_j | j \in \mathbb{N}) \) be a sequence of functions in \( C^1_c(U) \) with properties analogous to those listed in Corollary 2.6. Then \( (1 - \eta_j) \varphi \in C^1_c(U \setminus \mathcal{B}(G)) \) and thus \( (S_G) \) holds with this test function,

\[
(2.5) \quad \int_{\text{reg } G \cap U} |A_G|^2 (1 - \eta_j)^2 \varphi^2 \leq \int_{\text{reg } G \cap U} |\nabla_G \{(1 - \eta_j) \varphi\}|^2.
\]

The right-hand side can be bounded uniformly in \( j \) because

\[
\int_{\text{reg } G \cap U} |\nabla_G \{(1 - \eta_j) \varphi\}|^2 \leq 2 \int_{\text{reg } G \cap U} |\nabla_G \eta_j|^2 \varphi^2 + (1 - \eta_j)^2 |\nabla_G \varphi|^2
\]

and as \( j \to \infty \) we can separately estimate

\[
(2.6) \quad \int_{\text{reg } G \cap U} |\nabla_G \eta_j|^2 \varphi^2 \to 0
\]

and by dominated convergence

\[
(2.7) \quad \int_{\text{reg } G \cap U} (1 - \eta_j)^2 |\nabla_G \varphi|^2 \to \int_{\text{reg } G \cap U} |\nabla_G \varphi|^2.
\]

In fact we can compute the bounding integral in (2.5) more precisely and show that the cross-term also has

\[
(2.8) \quad 2 \int_{\text{reg } G \cap U} (1 - \eta_j) \varphi \langle \nabla_G (1 - \eta_j), \nabla_G \varphi \rangle \to 0 \text{ as } j \to \infty.
\]

On the left-hand side of (2.5) we may pass to the limit by Fatou’s lemma, so that letting \( j \to \infty \) we obtain the desired inequality

\[
\int_{\text{reg } G \cap U} |A_G|^2 \varphi^2 \leq \int_{\text{reg } G \cap U} |\nabla_G \varphi|^2. \quad \Box
\]

Using the results of [Hut86]—see Proposition A.3—we have the following corollary for sequences of two-valued minimal graphs.
Corollary 2.10. Let \((u_j \mid j \in \mathbb{N})\) be a sequence of two-valued minimal graphs \(u_j \in C^{1,\alpha}(D_2; \mathcal{A}_2)\), and suppose that their graphs converge weakly in the varifold topology to a limit varifold \(V \in IV_n(D_2 \times \mathbb{R})\),

\[|G_j| = |\text{graph } u_j| \to V \text{ as } j \to \infty.\]

Then \(V\) is stationary and ambient stable.

3. Area estimates for two-valued minimal graphs

3.1. Area bounds for two-valued minimal graphs. Here we extend the classical area estimates, which are well-known for single-valued minimal graphs, to two-valued minimal graphs by adapting the arguments presented in [GT98, Ch. 16].

Proposition 3.1. Let \(\alpha \in (0, 1/2)\), and let \(u \in C^{1,\alpha}(D_{2r}; \mathcal{A}_2)\) be a two-valued minimal graph. Then

\[\mathcal{H}^n(G \cap B_r) \leq 2\omega_n(1 + n)r^n.\]

Proof. Let \(\eta \in C^1_c(D_{2r})\) be a test function with \(\eta \equiv 1\) on \(D_r\) and \(|D\eta| \leq 2/r\). Next define \(\Phi(x, z) = \eta(x)z\), where

\[z_r = \begin{cases} 
  r & \text{if } z > r \\
  z & \text{if } -r \leq z \leq r \\
  -r & \text{if } z < -r.
\end{cases}\]

Then by the (two-valued) chain rule we get

\[D(\Phi(x, u)) = D\eta(x)u_r + \{D_2\Phi(x, u_i(x))Du_i(x)\}.\]

This is clearly well-defined at any \(x \in D_{2r} \setminus B_u\); in fact the same is true at branch points \(x \in B_u\) because the two components of \(Du(x) \in \mathcal{A}_2(L(\mathbb{R}^n; \mathbb{R}))\) agree there. We evaluate this expression to be

\[D(\Phi(x, u)) = D\eta(x)u_r(x) + \eta(x)1_{|u| < r}Du(x).\]

As \(\eta\) is compactly supported inside \(D_{2r}\) we get \(\int_{D_{2r}\setminus B_u} |D(\Phi(x, u))| < +\infty\). Then Proposition 2.1 justifies

\[\int_{D_{2r}} u_r(T(Du), D\eta) + \eta(T(Du), Du)1_{|u| < r} = 0,
\]

so that

\[\int_{D_r} 1_{|u| < r} \frac{|Du|^2}{v} \leq 2r \int_{D_{2r}} |D\eta|.
\]

The area of the graph is bounded by the integral \(\mathcal{H}^n(G \cap B_r) \leq \int v1_{|u| < r}\), which we split as

\[\int_{D_r} \frac{1}{v}1_{|u| < r} + \int_{D_r} \frac{|Du|^2}{v}1_{|u| < r} \leq 2(\mathcal{H}^n(D_r) + r\mathcal{H}^{n-1}(\partial D_r)),
\]

whence we conclude by noting \(\mathcal{H}^{n-1}(\partial D_r) = n\omega_n r^{n-1}\). \(\square\)
Table 1. Improved area estimates obtained in Proposition 3.3, for dimensions up to seven.

| n   | 2   | 3   | 4   | 5   | 6   | 7   |
|-----|-----|-----|-----|-----|-----|-----|
| $n\omega_n/\omega_{n-1}$ | $\pi$ | 4   | $3\pi/2$ | 16/3 | 15$\pi/8$ | 32/5 |
| $|n\omega_n/\omega_{n-1}|$ | 3   | 4   | 4   | 5   | 5   | 6   |

A similar argument yields area bounds in the cylinder above the disc $D_r$. A detailed proof in the single-valued case is given in [GT98, Ch.16], to adapt it to two-valued graphs one makes the same modifications as above. These will be used in the proof of the gradient estimates (see the proof of Lemma 4.2).

**Lemma 3.2.** Let $\alpha \in (0,1)$ and $u \in C^{1,\alpha}(D_{2r}; A_2)$ be a two-valued minimal graph. Then

$$\mathcal{H}^n(G \cap D_r \times \mathbb{R}) \leq 2\omega_n r^n (1 + nr^{-1} \sup_{D_{2r}}|u|).$$

### 3.2. Improved estimates for convergent sequences.

These estimates can be significantly improved when one considers a sequence of two-valued minimal graphs that converge weakly to a vertical varifold, that is of the form $V = V_0 \times \mathbb{R}e_{n+1} \in IV_n(D_2 \times \mathbb{R})$. Here we concentrate on the case where additionally $V$ is known to be supported in a union of planes, that is $V = P = \sum_j m_j \Pi_j^0 \times \mathbb{R}e_{n+1}$, where $\Pi_j = \Pi_j^0 \times \mathbb{R}e_{n+1} \in Gr(n, n+1)$.

(However we point out that virtually identical bounds are possible without this hypothesis, at the price of slightly more involved computations.)

**Proposition 3.3.** Let $u_j \in C^{1,\alpha}(D_2; A_2)$ be a sequence of two-valued minimal graphs with $|G_j| \to P = \sum_j m_j \Pi_j^0 \times \mathbb{R}e_{n+1}$. Then $\sum_j m_j \leq [n\omega_n/\omega_{n-1}]$.

**Proof.** For all $j$, $\mathcal{H}^n(G_j \cap B_1) \leq \int_{D_1} \frac{1}{v_j} \mathbb{1}_{|u_j|<1} + \int_{D_1} [Du_j]^2 \mathbb{1}_{|u_j|<1}$. (In fact this bounds the area of the graph in $D_1 \times (-1,1)$.) A quick computation reveals that the varifold convergence $|G_j| \to P$ is strong enough to ensure that $\int_{D_1} \frac{1}{v_j} \mathbb{1}_{|u_j|<1} \to 0$ as $j \to \infty$, so that we may focus on the second term. Referring back to (3.1) and (3.1) we find that this is bounded like $\int_{D_1} [Du_j]^2 \mathbb{1}_{|u_j|<1} \leq 2\mathcal{H}^{n-1}(\partial D_1) = 2n\omega_n$. Hence given any $\epsilon > 0$ there is $J(\epsilon) \in \mathbb{N}$ so that $\mathcal{H}^n(G_j \cap B_1) \leq 2n\omega_n + \epsilon$ for all $j \geq J(\epsilon)$, and in fact by the remark above $\mathcal{H}^n(G_j \cap D_1 \times (-1,1)) \leq 2n\omega_n + \epsilon$. Letting $j \to \infty$ we find that $\|\mathbb{P}(D_1 \times (-1,1)) \leq 2n\omega_n$, whence after replacing the right-hand side by $\sum_j 2m_j \omega_{n-1}$ we find $\sum_j m_j \leq n\omega_n/\omega_{n-1}$. The conclusion follows after taking integer values.

We obtain a small additional improvement of these estimates, which will eventually turn out to be essential. Assume for now the validity of its conclusion. In the special case $n = 3$, using the values from Table 1 one finds
that \( \sum_j m_j \leq (1 - \delta)3\omega_3 / \omega_2 = 4(1 - \delta) \), whence after taking integer values \( \sum_j m_j \leq 3. \)

**Corollary 3.4.** Let \( \alpha \in (0, 1) \), and \( u_j \in C^{1, \alpha}(D_2; A_2) \) be a sequence of two-valued minimal graphs with \( |G_j| \to P = \sum_j m_j|\Pi_0^j| \times \text{Re}_n^{n+1} \) as \( j \to \infty. \) Then there is \( \delta = \delta(P) \in (0, 1) \) so that \( \sum_j m_j \leq [(1 - \delta)\omega_0 / \omega_{n-1}]. \)

**Proof.** As the boundary of the disc is mean-convex, one may use an elementary construction to find an open set \( U \subset D_1 \) with \( \text{spt} V_0 | \cap D_1 \subset U \) and \( \text{Per}(U) < \text{Per}(D_1). \) Given arbitrarily small \( \delta > 0 \), there is \( \eta \in C^1(D_2) \) with \( \eta \equiv 1 \) on \( U \) and \( \int_{D_2} |D\eta| \leq (1 + \delta) \text{Per}(U). \) Now first take \( \tau > 0 \) small enough that \( (U)_\tau \subset \{ \eta = 1 \} \), and next take the index \( j \geq J(\tau) \) large enough that \( G_j \cap D_1 \times (-1, 1) \subset (U)_\tau \times (-1, 1) \). This being fulfilled one proceeds as in the proof of the area bounds, namely by estimating \( \int_{D_1} 1_{|u_j|<1} |Du_j|^2 / v_j^2 \leq 2 \int_{D_2} |D\eta| \leq 2(1 + \delta) \text{Per}(U). \) Now choose \( \delta = \delta(P) \) so that \( \sum_j m_j \leq [(1 - \delta)\omega_0 / \omega_{n-1}]. \)

4. **Gradient estimates for two-valued minimal graphs**

Let \( \alpha \in (0, 1) \) and the dimension \( n \geq 1 \) be arbitrary. Let \( u \in C^{1, \alpha}(D_2; A_2) \) be a two-valued minimal graph. In this section we derive an interior gradient estimate analogous to the classical estimates for smooth, single-valued graphs. These can be found for example in Section 16.2 of [GT98], which we also follow for the structure of the argument. These gradient bounds stem from integral estimates for the function \( w \), defined on \( \text{reg} G \) by the expression (4.1). To ensure the validity of these in the presence of branch points (which are absent in the single-valued case), we rely on the fine properties of the branch set proved by [KW20], specifically that it has zero 2-capacity. The main result in this section is the following.

**Lemma 4.1.** Let \( \alpha \in (0, 1) \) and let \( u \in C^{1, \alpha}(D_2; A_2) \) be a two-valued minimal graph. There is a constant \( C = C(n) > 0 \) so that

\[
\max_{D_1} \|Du\| \leq C \exp(C \sup_{D_2} \|u\|).
\]

We prove the equivalent version below, for discs of arbitrary radius \( r > 0. \)

**Lemma 4.2.** Let \( u \in C^{1, \alpha}(D_{3r}; A_2) \) be a two-valued minimal graph. Then there is a constant \( C = C(n) > 0 \) so that

\[
\max\{|Du_1(0)|, |Du_2(0)|\} \leq C \exp(C \max_{D_{2r}} \|u\|/r).
\]

4.1. **Integral estimates and a mean-value inequality for \( w. \)** Define a function \( w \) at all points \( X \in \text{reg} G \cap D_2 \times \mathbb{R} \) by

\[
(4.1) \quad w(X) = \log v(X) = -\log \langle \nu(X), c_{n+1} \rangle,
\]
where \( \nu(X) \) is the upward-pointing unit normal to \( \text{reg} \ G \) at \( X \) and \( \nu(X) = \langle \nu(X), e_{n+1} \rangle^{-1} \).

**Lemma 4.3.** For all compact \( K \subset D_2 \times \mathbb{R} \),

\[
\sup_{K \cap \text{reg} \ G} w + \int_{K \cap \text{reg} \ G} |\nabla_G w|^2 < +\infty
\]

and \( w \) satisfies \( \Delta_G w = |\nabla_G w|^2 + |A_G|^2 \) weakly in the sense that for all \( \varphi \in C^1_c(D_2 \times \mathbb{R}) \),

\[
-\int_{\text{reg} \ G \cap D_2 \times \mathbb{R}} \langle \nabla_G w, \nabla_G \varphi \rangle = \int_{\text{reg} \ G \cap D_2 \times \mathbb{R}} (|\nabla_G w|^2 + |A_G|^2) \varphi.
\]

**Proof.** Let \( K \subset D_2 \times \mathbb{R} \) be an arbitrary compact subset. As the gradient of \( u \) is locally bounded, we also find that \( \sup_{\text{reg} \ G \cap K} w < +\infty \). To prove \( \int_{\text{reg} \ G \cap K} |\nabla_G w|^2 < +\infty \) we use the minimal immersion \( \iota : \Gamma \to \mathbb{R}^{n+1} \) with image \( \iota(\Gamma) = G \setminus \mathcal{B}(G) = \text{reg} \ G \cup \mathcal{C}(G) \) that we constructed in Section 2.4.

Pull back \( w \) to \( w \circ \iota \in C^\infty(\iota^{-1}(\text{reg} \ G)) \), which we subsequently extend across \( \iota^{-1}(\mathcal{C}(G)) \) to yield a function in \( C^\infty(\Gamma) \), still denoted by \( w \circ \iota \). This function satisfies the PDE

\[
\Delta_{\Gamma} (w \circ \iota) - |\nabla_{\Gamma} (w \circ \iota)|^2 - |A_{\Gamma}|^2 = 0
\]

pointwise (and thus also weakly) on \( \Gamma \). From this we may deduce the bound

\[
\int_{\Gamma} |\nabla_{\Gamma} (w \circ \iota)|^2 \varphi^2 \leq 4 \int_{\Gamma} |\nabla_{\Gamma} \varphi|^2,
\]

valid for all \( \varphi \in C^1_c(\Gamma) \).

Indeed if we ignore the curvature term in (4.1)—as we may because it has a favourable sign—and integrate against an arbitrary \( \phi \in C^1_c(\Gamma) \) we see that

\[
\int_{\Gamma} |\nabla_{\Gamma} (w \circ \iota)|^2 \phi \leq -\int_{\Gamma} \langle \nabla_{\Gamma} (w \circ \iota), \nabla_{\Gamma} \phi \rangle.
\]

If instead we use \( \phi^2 \) as a test function, then we find

\[
\int_{\Gamma} |\nabla_{\Gamma} (w \circ \iota)|^2 \phi^2 \\
\leq -2 \int_{\Gamma} \phi \langle \nabla_{\Gamma} (w \circ \iota), \nabla_{\Gamma} \phi \rangle \leq 2 \left( \int_{\Gamma} \phi^2 |\nabla_{\Gamma} (w \circ \iota)|^2 \right)^{1/2} \left( \int_{\Gamma} |\nabla_{\Gamma} \phi|^2 \right)^{1/2}.
\]

Unless \( \int_{\Gamma} |\nabla_{\Gamma} (w \circ \iota)|^2 \phi^2 = 0 \) we may divide both sides by its square root, yielding (4.1). (If the integral vanishes then the inequality is trivially satisfied.)

In particular, if we take \( \varphi \in C^1_c(D_2 \times \mathbb{R} \setminus \mathcal{B}(G)) \) and let \( \phi = \varphi \circ \iota \) and translate (4.1) to the graph, we obtain

\[
\int_{\text{reg} \ G \cap D_2 \times \mathbb{R}} |\nabla_G w|^2 \varphi^2 \leq 4 \int_{\text{reg} \ G \cap D_2 \times \mathbb{R}} |\nabla_G \varphi|^2.
\]

To extend this through the branch point singularities of \( G \), we once again employ the sequence \( (\eta_j \mid j \in \mathbb{N}) \) with properties as described in Lemma 2.5.
Then proceed as in the proof of Lemma 2.9, namely take \( \varphi \in C^1_c(D_2 \times \mathbb{R}) \) and substitute the test function \( \varphi(1 - \eta_j) \) into (4.1), yielding
\[
\int_{\text{reg } G \cap D_2 \times \mathbb{R}} |\nabla_G w|^2 |\nabla_G w|^2 (1 - \eta_j)^2 \leq 4 \int_{\text{reg } G \cap D_2 \times \mathbb{R}} |\nabla_G \varphi|^2 ;
\]
justifying passing to the limit on the left-hand side by an application of Fatou’s lemma, again as in the proof of Lemma 2.9. This justifies (4.3).

To show that \( w \) is a weak solution of the PDE \( \Delta_G w = |\nabla_G w|^2 + |A_G|^2 \) we proceed in much the same way. The integral identity (4.3) is obtained for test functions \( \varphi \in C^1_c(D_2 \times \mathbb{R} \setminus B(G)) \) by working with the immersed \( \Gamma \) instead, as above. Using the same sequence \( (\eta_j | j \in \mathbb{N}) \) of functions, we then obtain
\[
- \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \langle \nabla_G w, \nabla \{ (1 - \eta_j) \varphi \} \rangle = \int_{\text{reg } G \cap D_2 \times \mathbb{R}} (|\nabla_G w|^2 + |A_G|^2) \varphi(1 - \eta_j).
\]
For the right-hand side of the identity, we may let \( j \to \infty \) by dominated convergence, which we can justify using our previously established local \( L^2 \)-bounds for \( A_G \) and \( |\nabla_G w| \) from \((S_G)\) and (4.3) respectively.

For the left-hand side of the identity, we expand
\[
- \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \langle \nabla_G w, \nabla \{ (1 - \eta_j) \varphi \} \rangle = \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \langle \nabla_G w, \nabla_G \eta_j \rangle \varphi - \langle \nabla_G w, \nabla_G \varphi \rangle (1 - \eta_j),
\]
and repeat calculations akin to those in the first part of the proof, noting that the bounds of (4.3) justify both the limit
\[
\int_{\text{reg } G \cap D_2 \times \mathbb{R}} \langle \nabla_G w, \nabla_G \eta_j \rangle \varphi \leq \sup_{D_2 \times \mathbb{R}} |\varphi| \left( \int_{\text{reg } G \cap D_2 \times \mathbb{R}} |\nabla_G w|^2 \right)^{1/2} \left( \int_{\text{reg } G \cap D_2 \times \mathbb{R}} |\nabla_G \eta_j|^2 \right)^{1/2} \to 0 \text{ as } j \to \infty
\]
and the application of dominated convergence to deduce that
\[
\int_{\text{reg } G \cap D_2 \times \mathbb{R}} \langle \nabla_G w, \nabla_G \varphi \rangle (1 - \eta_j) \to \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \langle \nabla_G w, \nabla_G \varphi \rangle \text{ as } j \to \infty.
\]
Thus we have derived the identity (4.3), which concludes the proof of the lemma. □

Remark 4.4. We could have derived the integral estimate for $|\nabla Gw|$ in (4.3) differently, using the local curvature bounds that follow from the stability inequality $(S_G)$. Indeed at all regular points $\langle \nu, e_{n+1} \rangle > 0$, hence

$$|\nabla Gw|^2 \leq |A_G|^2 (\langle \nu, e_{n+1} \rangle^{-2} - 1) \text{ on } \text{reg } G,$$

where we used the fact that $|\nabla G\langle \nu, e_{n+1} \rangle|^2 \leq |A_G|^2 (1 - \langle \nu, e_{n+1} \rangle^2)$. Given any compact subset $K \subset D_2 \times \mathbb{R}$, the term $\langle \nu, e_{n+1} \rangle$ is bounded below, say $\langle \nu, e_{n+1} \rangle \geq \delta_K$ on $\text{reg } G \cap K$. Then integrating (4.4) we obtain

$$\int_{\text{reg } G \cap K} |\nabla Gw|^2 \leq (\delta_K^{-2} - 1) \int_{\text{reg } G \cap K} |A_G|^2,$$

whence we get $\int_{\text{reg } G \cap K} |\nabla Gw|^2 \leq C_K$ for some $C_K > 0$ using $(S_G)$.

We chose to include the longer derivation in our proof, as it yields the more precise $\int_{\text{reg } G \cap D_2 \times \mathbb{R}} |\nabla Gw|^2 \varphi \leq 4 \int_{\text{reg } G \cap D_2 \times \mathbb{R}} |\nabla G\varphi|^2$, valid for all $\varphi \in C_c^1(D_2 \times \mathbb{R})$. We will also use this in the derivation of the interior gradient estimates (see the proof of Lemma 4.2 below). Compare this with the less useful inequality derived by arguing as above, essentially combining (4.4) with the stability inequality $(S_G)$ to yield $\int_{\text{reg } G \cap D_2 \times \mathbb{R}} |\nabla Gw|^2 \varphi^2 \leq (\delta_K^2 - 1) \int_{\text{reg } G \cap D_2 \times \mathbb{R}} |\nabla G\varphi|^2$, where $\varphi \in C_c^1(D_2 \times \mathbb{R})$ and $\text{spt } \varphi \subset K$.

Let us quickly comment on a slightly subtle point. The function $w = -\log \langle \nu, e_{n+1} \rangle$ is only defined on the regular part $\text{reg } G \cap D_2 \times \mathbb{R}$, and cannot be extended continuously across $\mathcal{C}(G) \cap D_2 \times \mathbb{R}$. However after pulling $w$ back via the immersion $\iota : \Gamma \to G \setminus \mathcal{B}(G)$ we obtain a function $w \circ \iota$ which we can extend smoothly through $\iota^{-1}(\mathcal{C}(G))$. This in turn allowed us to integrate by parts, yielding formulas which translate to $G$.

This way one obtains the following identity, valid for all $\varphi \in C_c^2(D_2 \times \mathbb{R} \setminus \mathcal{B}(G))$:

$$\int_{\text{reg } G \cap D_2 \times \mathbb{R}} (\Delta_G \varphi)w = -\int_{\text{reg } G \cap D_2 \times \mathbb{R}} \langle \nabla G\varphi, \nabla Gw \rangle = \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \varphi \Delta_Gw.$$  

From this, we may verify using a capacity argument that for $\varphi \in C_c^2(D_2 \times \mathbb{R})$,

$$-\int_{\text{reg } G \cap D_2 \times \mathbb{R}} \langle \nabla G\varphi, \nabla Gw \rangle = \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \varphi \Delta_Gw.$$  

Let again $(\eta_j \mid j \in \mathbb{N})$ be a sequence of functions with properties as described in Lemma 2.5. By a mollification argument for example, we may additionally impose that $\eta_j \in C_c^\infty(D_2 \times \mathbb{R})$ for all $j$. If $\varphi \in C_c^2(D_2 \times \mathbb{R})$ then $(1 - \eta_j)\varphi \in C_c^2(D_2 \times \mathbb{R} \setminus \mathcal{B}(G))$ is a valid test function in (4.1).

Focus on the two integrals in (4.1). For the first, we may justify taking the limit

$$\int_{\text{reg } G \cap U} \langle \nabla G\{(1 - \eta_j)\varphi\}, \nabla Gw \rangle \to \int_{\text{reg } G \cap U} \langle \nabla G\varphi, \nabla Gw \rangle \text{ as } j \to \infty.$$
as usual, whereas for the second note that on $\text{reg} \ G$

$$|\varphi(1 - \eta_j)\Delta_G w| \leq |\varphi((\nabla G w)^2 + |A G|^2)$$

for all $j$.

As $K = \text{spt} \ \varphi$ is compact, by stability and (4.3) there is a $C_K > 0$ so that

$$\int_{\text{reg} \ G \cap \ K} |\nabla G w|^2 + |A G|^2 \leq C_K.$$ 

Finally, dominated convergence allows taking the limit

$$\int_{\text{reg} \ G \cap \ U} \varphi(1 - \eta_j)\Delta_G w \to \int_{\text{reg} \ G \cap \ U} \varphi\Delta_G w \text{ as } j \to \infty,$$

which confirms the identity (4.1) claimed above.

By dropping the curvature term from the (weakly satisfied) PDE $\Delta_G w = |\nabla G w|^2 + |A G|^2$ one sees that $w$ is weakly subharmonic on $G$ through the branch set. This implies that it satisfies a mean-value inequality, see Corollary 4.5 below. Here the difference between the identities expressed in (4.1) and (4.1) becomes significant. The identity involving the left-most term of (4.1) does not naturally extend across the branch set of $G$ via the capacity argument we just invoked. This is unfortunate, because it prima facie prevents an appeal to the mean-value inequalities that appear in the literature: see for example those derived by Michael–Simon [MS73] or Simon [Sim84, Ch. 18]. For this reason, we give a detailed derivation of the mean-value inequality (4.1) expressed in Corollary 4.5; for this we essentially use a modification of arguments of Simon [Sim84, Ch. 17].

**Corollary 4.5.** Let $X = (x, X^{n+1}) \in D_2 \times \mathbb{R}$. Then for all $0 < \sigma < \rho < 2 - |x|$

$$\rho^{-n} \int_{\text{reg} \ G \cap B_\rho(X)} w - \sigma^{-n} \int_{\text{reg} \ G \cap B_\sigma(X)} w

\geq \int_{\text{reg} \ G \cap B_\rho(X) \setminus B_\sigma(X)} w |D^{1+} r|^2 r^{-n} \geq 0.$$ 

**Proof.** To simplify notation, we may assume without loss of generality that the point $X$ lies at the origin and $\rho < 2$. We use a two-parameter family of Lipschitz cutoff functions $(\gamma_{\delta,s} | \ \delta \in (0,1), s \in (0,\rho))$ constructed by first setting

$$\gamma_{\delta}(t) = \begin{cases} 
1 & \text{if } t \leq 1 - \delta, \\
(1 - t)/\delta & \text{if } 1 - \delta < t < 1, \\
0 & \text{if } t \geq 1
\end{cases}$$

and then rescaling $\gamma_{\delta,s}(t) = \gamma_{\delta}(t/s)$ for all $t \in \mathbb{R}$. Moreover we write $r = |X|$ and define the radial functions $\gamma_{\delta,s}(X) = \gamma_{\delta,s}(r)$ for all $X \in D_2 \times \mathbb{R}$, which all have $\text{spt} \ \gamma_{\delta,s} \subset \subset B_\rho \subset D_2 \times \mathbb{R}$. Fix $\delta \in (0,1)$ and $s \in (0,\rho)$, with the eventual aim of letting $\delta$ tend to zero.

Although the vector field $\gamma_{\delta,s}(r)Xw$ is not Lipschitz we can justify its use in the first variation formula using a quick capacity argument. Let
(η_j \mid j \in \mathbb{N}) be a sequence of cutoff sequences with properties essentially as described in Corollary 2.6, namely η_j \in C^1_c(D_2 \times \mathbb{R} \cap \text{reg} G) with 0 \leq η_j \leq 1 and η_j \equiv 1 on (\mathcal{B}(G))_{r_j} \cap B_{r_j} for some r_j \to 0. Moreover as \( j \to \infty \), η_j \to 0 \text{\mathcal{H}^n}\text{-a.e. on } \text{reg} G \cap B_{r_j} and \int_{\text{reg} G \cap B_{r_j}} |\nabla G η_j|^2 \to 0. As these cut out the branch set, the vector field \((1 - η_j)γ_{δ,s}Xw\) is a valid choice in the first variation formula, and yields \int_{\text{reg} G \cap B_{r_j}} \text{div} G ((1 - η_j)γ_{δ,s}Xw) = 0. We can expand this expression to get \(|\int (1 - η_j) \text{div} G (γ_{δ,s}Xw)| \leq \int |γ_{δ,s}Xw| |\nabla G η_j|. The right-hand side tends to zero by the Cauchy–Schwarz inequality. On the left-hand side we justify the convergence \( \int (1 - η_j) \text{div} G (γ_{δ,s}Xw) \to \int \text{div} G (γ_{δ,s}Xw) \) by dominated convergence, after noticing that \( \int |\text{div} G (γ_{δ,s}Xw)| < \infty \). Hence we have

\begin{equation}
(4.13) \quad \int_{\text{reg} G \cap D_2 \times \mathbb{R}} \text{div} G (γ_{δ,s}Xw) = 0.
\end{equation}

Following the computations in [Sim84, p. 83] we find that \text{div} G (γ_{δ,s}X) = nγ_{δ,s} + rγ'_{δ,s}(1 - |D + r|^2), and (4.1) leads to

\begin{equation}
(4.14) \quad n \int_{\text{reg} G \cap D_2 \times \mathbb{R}} γ_{δ,s}w + \int_{\text{reg} G \cap D_2 \times \mathbb{R}} rγ'_{δ,s}w = \int_{\text{reg} G \cap D_2 \times \mathbb{R}} rγ'_{δ,s}w|D + r|^2 - \int_{\text{reg} G \cap D_2 \times \mathbb{R}} γ_{δ,s}⟨\nabla G w, X⟩.
\end{equation}

To somewhat abbreviate this integral identity we define the two functions \( I_δ, J_δ : (0, ρ) \to \mathbb{R} \) by setting, for all \( s \in (0, ρ) \),

\( I_δ(s) = \int_{\text{reg} G \cap D_2 \times \mathbb{R}} γ_{δ,s}w \),

\( J_δ(s) = \int_{\text{reg} G \cap D_2 \times \mathbb{R}} γ_{δ,s}w|D + r|^2 \),

Notice that these are both differentiable in \( s \) with respective derivatives \( I'_δ(s) = \int \frac{∂γ_{δ,s}}{∂s}w \) and \( J'_δ(s) = \int \frac{∂γ_{δ,s}}{∂s}w|D + r|^2 \). Note \( γ'_{δ,s}(r) = 1/sγ'_δ(r/s) = 1/(rs)γ'_δ(r/s) \), so that \( rγ'_{δ,s}(r) = -s \frac{∂}{∂s}γ_{δ,s}(r) \). Thus we can rewrite (4.1) as

\( nI_δ(s) - sI'_δ(s) = -sJ'_δ(s) - \int_{\text{reg} G \cap D_2 \times \mathbb{R}} γ_{δ,s}⟨\nabla G w, X⟩ \) for all \( s \in (0, ρ) \).

Multiply this equation by \( s^{-n-1} \) and notice that this is

\( \frac{d}{ds}(s^{-n}I_δ(s)) = s^{-n}J'_δ(s) + s^{-n-1} \int_{\text{reg} G \cap D_2 \times \mathbb{R}} γ_{δ,s}⟨\nabla G w, X⟩ \).

Integrate this identity for \( s \in (σ, ρ) \) to get

\begin{equation}
(4.15) \quad ρ^{-n}I_δ(ρ) - σ^{-n}I_δ(σ) = \int_{σ}^{ρ} s^{-n}J'_δ(s) + \int_{σ}^{ρ} s^{-n-1} \int_{\text{reg} G \cap D_2 \times \mathbb{R}} γ_{δ,s}⟨\nabla G w, X⟩. \]

\]
The eventual aim is to let \( \delta \to 0 \) in this identity; this will yield (4.5) and conclude the proof. Before we do this, we separately integrate both integrals on the right-hand side by parts. For the first, we obtain

\[
(4.16) \quad \int_\sigma s^{-n} J_\delta'(s) = \rho^{-n} J_\delta(\rho) - \sigma^{-n} J_\delta(\sigma) + n \int_\sigma s^{-n-1} J_\delta(s) = \int_{\text{reg } G \cap D_2 \times \mathbb{R}} w|D^\perp r|^2 \left\{ \rho^{-n} \gamma_{\delta,\rho} - \sigma^{-n} \gamma_{\delta,\sigma} + n \int_\sigma s^{-n-1} \gamma_{\delta,s} \right\} d\mathcal{H}^n.
\]

For the second integral, first fix \( s \in (0, \rho) \) and notice that \( \nabla_G (r^2 - s^2) = 2X^T \). The identity \( \int_{D_2 \times \mathbb{R}^n \setminus \text{reg } G} \nabla_G (r^2 - s^2) \nabla_G w = 0 \), is equivalent to

\[
2 \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \gamma_{\delta,s} \langle \nabla_G w, X \rangle = - \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \nabla_G (r^2 - s^2) \nabla_G w.
\]

The first integral on the right-hand side is equal to \(- \int_{\text{reg } G \cap D_2 \times \mathbb{R}} (|A_G|^2 + |\nabla_G w|^2) \gamma_{\delta,s} (r^2 - s^2) \), and is non-negative. From the identity we only retain the inequality

\[
2 \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \gamma_{\delta,s} \langle \nabla_G w, X \rangle \geq - \int_{\text{reg } G \cap D_2 \times \mathbb{R}} (r^2 - s^2) \langle \nabla_G \gamma_{\delta,s}, \nabla_G w \rangle.
\]

Notice that

\[
\nabla_G \gamma_{\delta,s} = -(s \delta)^{-1} \frac{X^T}{r} 1_{B_\delta \setminus B_{(1-\delta)s}},
\]

so using the co-area formula we can estimate the integral on the right-hand side like

\[
\int_{\text{reg } G \cap D_2 \times \mathbb{R}} (r^2 - s^2) \langle \nabla_G \gamma_{\delta,s}, \nabla_G w \rangle
\]

\[
= \int_{(1-\delta)s}^s (s \delta)^{-1} (s^2 - \theta^2) \left\{ \int_{\partial B_\theta \cap \text{reg } G} \langle \nabla_G w, \frac{X}{\theta} \rangle \right\} d\theta.
\]

As \( \theta \in (s(1-\delta), s) \) we get \( s^2 - \theta^2 \leq s^2 \delta(2 - \delta) \) and

\[
\left| \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \gamma_{\delta,s} \langle \nabla_G w, X \rangle \right| \leq \left| \int_{\text{reg } G \cap D_2 \times \mathbb{R}} (s^2 - r^2) \langle \nabla_G \gamma_{\delta,s}, \nabla_G w \rangle \right|
\]

\[
\leq \int_{(1-\delta)s}^s \int_{\partial B_\theta \cap \text{reg } G} (2 - \delta)s|\nabla_G w| = (2 - \delta)s \int_{\text{reg } G \cap B_\delta \setminus B_{(1-\delta)s}} |\nabla_G w|.
\]

Then integrating this over \( s \in (\sigma, \rho) \) we obtain a bound for the second integral in (4.1),

\[
(4.17) \quad \left| \int_{\text{reg } G \cap D_2 \times \mathbb{R}} \gamma_{\delta,s} \langle \nabla_G w, X \rangle \right| \leq 2 \int_{\sigma}^{\rho} s^{-n-1} \int_{\text{reg } G \cap B_\delta \setminus B_{(1-\delta)s}} |\nabla_G w|.
\]
As we announced above, we may now let \( \delta \to 0 \) at the same time on the left-hand side of (4.1), in (4.1) and (4.1), justifying the convergence each time by dominated convergence. Thus
\[
\rho^{-n}I(\rho) - \sigma^{-n}I(\sigma) \geq \lim_{\delta \to 0} \left\{ \int_{\delta} s^{-n} J_\delta'(s) \right\},
\]
where we write \( I(s) = \int_{\text{reg} G \cap B_s} w \) for all \( s \in (0, \rho) \). We conclude by evaluating the limit on the left-hand side, which is
\[
\int_{\text{reg} G \cap B_\rho \wedge \mathcal{B}_\sigma} w |D^{1/2}r|^{2r-n}.
\]
Therefore we obtain
\[
\rho^{-n}I(\rho) - \sigma^{-n}I(\sigma) \geq \int_{\text{reg} G \cap B_\rho \wedge \mathcal{B}_\sigma} w |D^{1/2}r|^{2r-n},
\]
which establishes (4.5) and concludes the proof.

4.2. Proof of the gradient bounds. We place ourselves in the situation described in Lemma 4.2. Let \( u \in C^{1,\alpha}(D_{3r}; A_2) \) be a two-valued minimal graph. Write \( u_1(0), u_2(0) \) for the two values of \( u \) at \( 0 \in \mathbb{R}^n \), with corresponding \( Du_1(0), Du_2(0) \in \mathbb{R}^n \). Note that we may consider \( w \) as a two-valued function defined on \( D_{3r} \), including at the singular points of \( u \).

Applying the mean-value inequality at either point \( X_i = (0, u_i(0)) \) we obtain
\[
\rho^{-n} \int_{G \cap B_{\rho}(X_i)} w - \sigma^{-n} \int_{G \cap B_{\rho}(X_i)} w \geq 0 \text{ for all } 0 < \sigma < \rho < 3r.
\]
Fixing \( \rho = r \) and letting \( \sigma \to 0 \) we find
\[
\lim_{\sigma \to 0} \left\{ (\omega_n r^n)^{-1} \int_{\text{reg} G \cap B_{\rho}(X_i)} w \right\} \leq (\omega_n r^n)^{-1} \int_{\text{reg} G \cap B_{r}(X_i)} w.
\]
If \( X_i \) is regular, then the limit on the right-hand side is \( w_1(0) \) and if \( X_i \) is a singularity then it is equal \( w_1(0) + w_2(0) \). Translate the graph so that \( 0 \in G \) and \( w \) the larger of its two values there. In both cases
\[
2 \max\{w_1(0), w_2(0)\} \leq (\omega_n r^n)^{-1} \int_{\text{reg} G \cap B_r} w.
\]
This allows us to reduce the proof of Lemma 4.2 to an estimation of the integral on the right-hand side of (4.2).

Claim 1. There is a constant \( C = C(n) \) so that if \( u \in C^{1,\alpha}(D_{3r}; A_2) \) is a two-valued minimal graph with \( 0 \in G \) then
\[
r^{-n} \int_{\text{reg} G \cap B_r} w \leq C(1 + r^{-1} \sup_{D_{2r}} \|u\|).
\]
Proof. We split the integral \( \int_{\text{reg} G \cap \mathcal{B}} \omega = \int_{|x|^2 + u^2 < r^2} wv \) into the sum of \( \int_{|x|^2 + u^2 < r^2} w \) and \( \int_{|x|^2 + u^2 < r^2} \frac{|Du|^2}{v} w \). The former is easier to estimate, as \( w \leq v \). Therefore \( \int_{|x|^2 + u^2 < r^2} w \leq \mathcal{H}^n(G \cap B_r) \leq C(n)r^n \), using the area bounds of Proposition 3.1.

We estimate the latter under the weaker restriction that \( |x| < r, |u| < r \), for notational convenience. Let \( \eta \in C^1_c(D_{2r}) \) be a standard cutoff function with \( \eta \equiv 1 \) on \( D_r \) and \( |D\eta| \leq 2r^{-1} \). We write

\[
\Phi(x, z, p) = \frac{1}{2} \log(1 + |p|^2)(z_r + r)\eta(x),
\]

where \( z_r \) is defined as in (3.1), meaning that here

\[
z_r + r = \begin{cases} 2r & \text{if } z > r, \\ z + r & \text{if } -r \leq z \leq r, \\ 0 & \text{if } z < -r. \end{cases}
\]

One may then check that \( \Phi \) satisfies the required hypotheses laid out in Proposition 2.1 to justify its use as a test function in the equation (2.1). The only hypothesis we check here is (2.1),

\[
\int_{D_{3r} \setminus \mathcal{B}_u} |D(\Phi(x, u, Du))| < +\infty.
\]

As \( \Phi(x, u, Du) = (u_r + r)\eta w \) we calculate its derivative as

\[
D\Phi(x, u, Du) = Du1_{|x| < r, |u| < r} \eta w + D\eta(u_r + r)w + Dw(u_r + r)\eta.
\]

Only the last term is not locally bounded. Instead we find the following integral bound. (The main observation underlying this bound, as proved in Claim 2, is the pointwise inequality (4.2). The unwieldiness of the proof is caused by the possibility of branch points in \( D_{3r} \)).

Claim 2. For all compact \( K \subset D_{3r} \) there is a constant \( C_K > 0 \) so that

\[
\int_{K \setminus \mathcal{B}_u} |Dw| \leq \int_{\text{reg} G \cap K \times \mathbb{R}} |\nabla Gw| \leq C_K.
\]

Proof. This essentially follows from the integral estimates for \( |\nabla Gw| \) in (4.3). To justify this rigorously we proceed as follows. First we extend the two-valued function

\[
w = \{w_1, w_2\} = \frac{1}{2}(\log(1 + |Du_1|^2), \log(1 + |Du_2|^2))
\]

to a two-valued function defined on the cylinder \( D_{3r} \times \mathbb{R} \) by setting it constant in the vertical variable,

\[
w(x, X^+ n) = w(x) = \{w_1(x), w_2(x)\}
\]

for all \( (x, X^+ n) \in D_{3r} \times \mathbb{R} \).

Near points \( x \in D_{3r} \setminus \mathcal{B}_u \) we can make a local selection for \( u \) by say \( u_1^x, u_2^x \in C^\infty(D_\sigma(x)) \) so that \( u = \{u_1^x, u_2^x\} \) in \( D_\sigma(x) \). This selection is also
valid for $w$, meaning that also $w = \{w_1^\tau, w_2^\tau\}$. Of course this relation also holds for the extension of $w$ to $D_\sigma(x) \times \mathbf{R}$.

In the cylinder $D_\sigma(x) \times \mathbf{R}$ we split the graph into the transverse union

$$G \cap D_\sigma(x) \times \mathbf{R} = \text{graph } u_1^\tau \cup \text{graph } u_2^\tau.$$  

Abbreviating $G^\tau_i = \text{graph } u_i^\tau$ for $i = 1, 2$ we can separately calculate

$$\|\nabla G^\tau_i w_i^\tau(y, Y^{n+1})\|^2 = |D w_i^\tau(y, Y^{n+1})|^2 - |\langle \nu_{G^\tau_i}, D w_i^\tau(y, Y^{n+1}) \rangle|^2$$

at all $Y = (y, Y^{n+1}) \in D_\sigma(x) \times \mathbf{R}$. As the $w_i^\tau$ are both independent of the vertical variable we obtain that for $i = 1, 2$,

$$\|\nu_{G^\tau_i}, e_{n+1}\| |D w_i^\tau| \leq |\nabla G^\tau_i w_i^\tau| \text{ on } D_\sigma(x) \times \mathbf{R}. \quad (4.23)$$

Next we split the integral $\int_{D_\sigma(x)} |D w| = \int_{D_\sigma(x)} |D w_1^\tau| + \int_{D_\sigma(x)} |D w_2^\tau|$, and separately bound the two terms using (4.2), that is for $i = 1, 2$,

$$\int_{D_\sigma(x)} |D w_i^\tau| = \int_{G^\tau_i \cap D_\sigma(x) \times \mathbf{R}} \frac{|D w_i^\tau|}{v_i^\tau} \leq \int_{G^\tau_i \cap D_\sigma(x) \times \mathbf{R}} |\nabla G^\tau_i w_i^\tau|,$$

where recall $v_i^\tau = (1 + |D u_i^\tau|^2)^{1/2} = \langle \nu_{G^\tau_i}, e_{n+1} \rangle^{-1}$. Thus we obtain

$$\int_{D_\sigma(x)} |D w| \leq \int_{\text{reg } G \cap D_\sigma(x)} |\nabla G w|,$$

which is finite by (4.3) for example.

We may now return to the original problem of estimating the integral $\int_{K \setminus B_u} |D w|$. We may take a countable cover of the set $K \setminus B_u$ by some collection of discs $(D_{\sigma_j}(x_j) \mid j \in \mathbb{N})$ centered at points $x_j \in K \setminus B_u$ with

$$D_{\sigma_j}(x_j) \subset D_{3r} \setminus B_u,$$

and let $(\rho_j \mid j \in \mathbb{N})$ be a partition of unity subordinate to the cover. Justifying the permutation of the sum and the integral by monotone convergence for example, we can decompose $\int_{K \cap D_{3r} \setminus B_{u}} |D w|$ like

$$\int_{K \cap D_{3r} \setminus B_u} |D w| = \sum_{j \in \mathbb{N}} \int_{K \cap D_{3r} \setminus B_u} |D w| \rho_j. \quad (4.24)$$

Arguing as in our derivation of the inequality (4.2) we can bound each of the integrals on the right-hand side by

$$\int_{K \setminus B_u} |D w| \rho_j \leq \int_{\text{reg } G \cap D_{\sigma_j}(x_j) \times \mathbf{R}} |\nabla G w| \rho_j,$$

where we extend $\rho_j$ to a function defined on $D_{\sigma_j}(x_j) \times \mathbf{R}$ by setting it constant in the vertical variable, as we did for $w$ in (4.2). Finally we may use monotone convergence once again to permute sums with integrals, yielding the desired

$$\int_{K \setminus B_u} |D w| \leq \int_{\text{reg } G \cap K \times \mathbf{R}} |\nabla G w| \leq C_K,$$

where the last inequality comes from (4.3). \qed
As we imposed that \( \text{spt } \eta \subset D_{2r} \), the inequality (2) from the claim applied with \( K = \overline{D}_{2r} \) confirms the boundedness required by (4.2). We may thus substitute the expression (4.2) we calculated for \( D\Phi(x, u, Du) \) into (2.1) to get

\[
\int_{D_{3r}} \left( \frac{Du}{v}, Du1_{|x|<r,|u|<r}\eta w + D\eta(u_r + r)w + Dw(u_r + r)\eta \right) = 0,
\]

so that

\[
\int_{|x|<r,|u|<r} \frac{|Du|^2}{v} w \leq 2r \int_{|x|<2r,|u|<r} |D\eta| w + |Dw| \eta.
\]

To justify the expression on the left-hand side, simply note that for \( i = 1, 2, \) wherever \( u_i > -r \) we have \( (u_i)_r + r \leq 2r \), and if \( u_i \leq -r \) then \( (u_i)_r + r = 0 \). The proof then boils down to separately estimating the two integrals

\[
\int_{|x|<2r,|u|<r} |D\eta| w \quad \text{and} \quad \int_{|x|<2r,|u|<-r} |Dw| \eta.
\]

The first integral is easier. Indeed using the fact that \( w \leq v \) and the area bounds of Lemma 3.2 we get

\[
\int_{|x|<2r,|u|<r} |D\eta| w \leq 2r^{-1} \mathcal{H}^n(G \cap D_{2r} \times R) \leq C(n)r^{n-1}(1 + Mr^{-1}),
\]

where we set \( M = \sup_{D_{2r}} \|u\| \).

For the second integral \( \int_{u>,-r} |Dw| \eta \) we may start by arguing as in the proof of Claim 2 to justify that

\[
(4.25) \quad \int_{u>,-r} |Dw| \eta \leq \int_{\overline{G \cap D_{2r} \times (-r,\infty)}} |\nabla_G w| \eta.
\]

To estimate this, recall the helpful integral inequality (4.1) we used in the proof of Lemma 4.3. To use it in the present context, we extend \( \eta \) to a test function compactly supported in the cylinder \( D_{2r} \times R \) by multiplying by a cutoff function \( \tau \in C^0_c(R) \) in the vertical direction. We further impose that \( \tau \equiv 1 \) on \( (-r, M) \) and spt \( \tau \subset (-2r, M + r) \) with \( |\tau'| \leq 2r^{-1} \). Then from (4.1) we obtain

\[
(4.26) \quad \int_{\overline{G \cap D_{2r} \times (-r,\infty)}} |\nabla_G w|^2 \tau^2 \eta^2 \leq 8 \int_{\overline{G \cap D_{2r} \times (-r,\infty)}} |\nabla_G \tau|^2 \eta^2 + \tau^2 |\nabla_G \eta|^2 \leq 64r^{-2} \mathcal{H}^n(G \cap \text{spt} \phi).
\]

Estimate the right-hand side of (4.2) with Hölder’s inequality,

\[
\int_{\overline{G \cap D_{2r} \times (-r,\infty)}} |\nabla_G w| \eta
\]

\[
\leq (\mathcal{H}^n(G \cap D_{2r} \times (-2r, M + r)))^{1/2} \left( \int_{G \cap D_{2r} \times (-r,M)} |\nabla_G w|^2 \eta^2 \right)^{1/2}.
\]
and combining this with (4.2),
\[ \int_{\text{reg } G \cap D_{2r} \times (-r, \infty)} |\nabla_G w| \eta \leq 8r^{-1} \mathcal{H}^n(G \cap D_{2r} \times \mathbb{R}). \]
This in turn can be bounded using the area estimates from Lemma 3.2: \( \mathcal{H}^n(G \cap D_{2r} \times \mathbb{R}) \leq C(n) r^n (1 + Mr^{-1}) \). This yields the desired bound for \( \int_{u>-r} |Dw| \eta \) via (4.2), and concludes the proof of the claim. □

5. A REGULARITY LEMMA FOR TWO-VALUED MINIMAL GRAPHS

5.1. A maximum principle near branch point singularities.

Lemma 5.1. Let \( \alpha \in (0,1) \) and \( u \in C^{1,\alpha}(D_2; \mathbb{A}_2) \) be a two-valued minimal graph. Suppose that at the origin \( u(0) = \{0,0\} \) and \( Du(0) = \{0,0\} \).

Let \( e \in \mathbb{R}^n \times \{0\} \) be a fixed unit vector. If \( \langle Du(X), e \rangle \leq 0 \) for all \( X \in \text{reg } G \cap D_2 \times \mathbb{R} \) then it vanishes identically.

Proof. Let \( X \in \text{reg } G \cap D_2 \times \mathbb{R} \), and note that \( \langle Du(X), e \rangle \leq 0 \) if and only the upward-pointing unit normal to graph \( u \) has \( \langle \nu(X), e \rangle \geq 0 \). We argue by contradiction, assuming that \( \langle \nu, e \rangle \) is non-negative but does not vanish identically. It is well-known that the function \( \langle \nu, e \rangle \) is a Jacobi field for \( G \), that is it satisfies the equation \( \Delta \langle \nu, e \rangle + |A_G|^2 \langle \nu, e \rangle = 0 \) both pointwise on \( \text{reg } G \) and weakly through singularities of \( G \). (The justification for this can be made in essentially the same way as when working with \( \langle \nu, e_{n+1} \rangle \) in the above.) Moreover, by the standard strong maximum principle, we know that \( \langle \nu, e \rangle > 0 \) on \( \text{reg } G \), meaning that we can define a smooth function \( w_e \in C^2(\text{reg } G) \) by
\[ w_e(X) = -\log \langle \nu(X), e \rangle \] at all \( X \in \text{reg } G \).

The hypotheses of the claim ensure that \( \langle \nu(0), e \rangle = 0 \) at the origin, which also means that \( w \) diverges there. To avoid technical difficulties related to this, we perturb the vector \( e \) slightly. Let \( \theta > 0 \) be a small angle, through which we rotate \( e \) in the two-dimensional plane span\( \{e, e_{n+1}\} \), yielding the vector
\[ e_\theta = (\cos \theta)e + (\sin \theta)e_{n+1}. \]

Unless we are in the pathological case where \( u \) diverges near the boundary of \( D_2 \), the function \( \langle \nu, e_{n+1} \rangle = \frac{1}{\sqrt{1+|Du|^2}} \) is positive and bounded below, say \( \langle \nu, e_{n+1} \rangle \geq \alpha > 0 \) on \( \text{reg } G \cap B_1 \).

Should \( u \) in fact diverge near the boundary, we can rescale it around the origin by a factor \( \lambda > 1 \) close to one, and restrict the resulting function to \( D_2 \) to reduce to the case where \( u \) is bounded near the origin.

As a consequence the rotated vector \( e_\theta \) also has
\[ \langle \nu, e_\theta \rangle \geq \alpha \sin \theta > 0 \] on \( \text{reg } G \cap B_1 \).
We can then define the function
\[ w_\theta(X) = -\log(\langle \nu(X), e_\theta \rangle) \]
without running the risk of it diverging anywhere. This function has, locally for all compact \( K \subset D_2 \times \mathbb{R} \) that
\[
\int_{K \cap \text{reg } G} w_\theta^2 + |\nabla w_\theta|^2 < +\infty,
\]
and it satisfies the equation
\[
\Delta w_\theta - |\nabla w_\theta|^2 - |A_G|^2 = 0
\]
weakly on \( G \cap D_2 \times \mathbb{R} \). (These facts can be checked in much the same way as we did for the function \( -\log(\langle \nu, e_{n+1} \rangle) \) in Section 4, see Lemma 4.3.)

In particular the function \( w_\theta \) is weakly subharmonic on \( G \), and thus by the mean-value inequality, we get that for all \( 0 < r < s < 1 \) and all points \( X \in B_1 \),
\[
\frac{1}{\omega_n r^n} \int_{\text{reg } G \cap B_r(X)} w_\theta \leq \frac{1}{\omega_n s^n} \int_{\text{reg } G \cap B_s(X)} w_\theta.
\]
Applying this at the origin and letting \( r \to 0 \) (as is justified by Fatou’s lemma), we obtain that for all \( 0 < s < 1 \),
\[
2w_\theta(0) \leq \frac{1}{\omega_n s^n} \int_{\text{reg } G \cap B_s} w_\theta. \tag{5.1}
\]

For all \( \theta > 0 \) we write
\[
M_\theta = \sup_{\text{reg } G \cap B_1} w_\theta \text{ and } m_\theta = \inf_{\text{reg } G \cap B_1} \langle \nu, e_\theta \rangle,
\]
which are related by \( M_\theta = -\log m_\theta \). Then
\[
w_\theta(0) \geq M_\theta + \log \alpha,
\]
where recall \( \alpha = \inf \langle \nu, e_{n+1} \rangle \). Indeed at the origin \( \langle \nu(0), e_\theta \rangle = \sin \theta \), whereas \( m_\theta \geq \alpha \sin \theta \). Therefore \( \langle \nu(0), e_\theta \rangle \leq m_\theta / \alpha \). Translating this to \( w_\theta \) we obtain \( w_\theta(0) \geq -\log(m_\theta / \alpha) = -\log m_\theta + \log \alpha \). Let \( \delta > 0 \) be given. As \( M_\theta \to \infty \) as \( \theta \to 0 \), we may choose a small value \( \theta_0 > 0 \) in terms of \( \alpha \) so that for all \( \theta \in (0, \theta_0) \) we also have
\[
w_\theta(0) \geq (1 - \delta/2)M_\theta.
\]
Substitute this into the mean-value inequality (5.1) with radius \( s = 1 \), obtaining that
\[
(2 - \delta)M_\theta \leq \frac{1}{\omega_n} \int_{\text{reg } G \cap B_1} w_\theta.
\]
Let \( \lambda > 0 \) be a parameter whose value we will fix later. Then we may split the integral on the right-hand side by conditioning on the event that \( \{ w_\theta \geq \lambda \} \), obtaining two integrals which we can separately bound by
\[
\int_{\text{reg } G \cap B_1 \cap \{ w_\theta \geq \lambda \}} w_\theta \leq M_\theta \mathcal{H}^n(\text{reg } G \cap B_1 \cap \{ w_\theta \geq \lambda \})
\]
and
\[
\int_{\text{reg} G \cap B_1 \cap \{ w_\theta \leq \lambda \}} w_\theta \leq \lambda \mathcal{H}^n(\text{reg} G \cap B_1 \cap \{ w_\theta \leq \lambda \}).
\]

Recall that \( w \) is large only near \( \{ \langle \nu, e \rangle = 0 \} \subset \text{sing} G \cap B_1 \), and hence is bounded away from the singular set. Since the functions \( w_\theta \) converge to \( w \) pointwise on \( \text{reg} G \cap B_1 \), and uniformly in compact subsets \( K \subset \text{reg} G \cap B_1 \), we obtain uniform bounds for the sequence too: for all compact \( K \subset \text{reg} G \cap B_1 \) there exist \( \theta_K > 0 \) and \( D_K > 0 \) so that for all \( \theta \in (0, \theta_K) \),

\[(5.2) \quad w_\theta \leq D_K \text{ on } K.\]

Working in the larger ball \( B_{3/2} \), we see that \( \mathcal{H}^n(\text{sing} G \cap B_{3/2}) = 0 \), which also means that small neighbourhoods of this set have arbitrarily small Hausdorff measure. That is, given any \( \epsilon > 0 \) we may find a finite open cover of \( \text{sing} G \cap B_{3/2} \) by balls \( B_{r_1}(X_1), \ldots, B_{r_N}(X_N) \) with \( \sum_{k=1}^N r_k^n \leq \epsilon \). Perhaps after slightly increasing the radii of the balls in the cover, we may arrange for \( \text{reg} G \cap B_1 \setminus \bigcup_{k=1}^N B_{r_k}(X_k) \) to lie a positive distance away from the singular set. Using the bound above in (5.1), we see that there must be a constant \( D > 0 \) so that for all \( \theta \in (0, \theta_0) \)

\[ w_\theta(X) \leq D \text{ at all } X \in \text{reg} G \cap B_1 \setminus \bigcup_{k=1}^N B_{r_k}(X_k), \]

after adjusting \( \theta_0 \) to a smaller value if necessary. We may then set \( \lambda = D \), and see that for all \( \theta \in (0, \theta_0) \),

\[(5.3) \quad \mathcal{H}^n(\text{reg} G \cap B_1 \cap \{ w_\theta > D \}) \leq \sum_{k=1}^N \omega_n r_k^n \leq \omega_n \epsilon.\]

The divergence of \( M_\theta \) as \( \theta \to 0 \) additionally lets us impose that \( \theta \) be small enough that \( M_\theta \geq C(n)D \), where \( C(n) \) is a constant so that \( \mathcal{H}^n(\text{reg} G \cap B_1) \leq \omega_n C(n) \), available via the area bounds of Proposition 3.1. With \( \theta \) as small as this, we get

\[(5.4) \quad D/\omega_n \mathcal{H}^n(\text{reg} G \cap B_1 \cap \{ w_\theta \leq D \}) \leq M_\theta.\]

Substituting (5.1) and (5.4) into our decomposition for \( 1/\omega_n \int_{\text{reg} G \cap B_1} w_\theta \) we obtain the inequality

\[ (2 - \delta)M_\theta \leq \frac{1}{\omega_n} \int_{\text{reg} G \cap B_1} w_\theta \leq (\epsilon + 1)M_\theta, \]

which is absurd provided \( \delta, \epsilon \) are small enough. \( \square \)

The analogous statement is a lot easier to prove for single-valued, smooth minimal graphs. In fact, this is an immediate application of the classical, strong maximum principle, and indeed we used this in the proof above to deduce that \( \langle \nu(X), e \rangle > 0 \) for all regular points \( X \in \text{reg} G \cap D_2 \times \mathbb{R} \).
5.2. Regularity by a geometric argument. Here we show that Lipschitz two-valued minimal graphs are automatically regular. This is somewhat well-known among experts in the field, although an explicit proof is absent from the literature. Here we follow a strategy suggested to us by S. Becker-Kahn, using the results developed in his thesis [BK17]. Our aim is to prove the following two results simultaneously, using an inductive argument on the dimension $n$.

**Theorem 5.2.** Let $u \in \text{Lip}(D_2; A_2)$ be a two-valued minimal graph with Lipschitz constant $L$. Then there is $\alpha = \alpha(L, n) \in (0, 1)$ so that

$$u \in C^{1, \alpha}(D_2; A_2).$$

This turns out to be equivalent to the following, seemingly weaker lemma.

**Lemma 5.3.** Let $u \in \text{Lip}(\mathbb{R}^n; A_2)$ be a two-valued minimal graph with Lipschitz constant $L$. If additionally $u$ is homogeneous,

$$u(\lambda x) = \lambda u(x) \text{ for all } \lambda > 0, x \in \mathbb{R}^n,$$

then $u$ is linear.

**Proof.** Before moving on to the inductive argument, let us explain how Lemma 5.3 implies Theorem 5.2. Let $u \in \text{Lip}(D_2; A_2)$ be a two-valued minimal graph with Lipschitz constant $L$. This is smooth away from away from $\mathcal{B}_u$, so consider an arbitrary $x \in \mathcal{B}_u \cap D_2$. Write $X = (x, X^{n+1}) \in \mathcal{B}(G)$ be the corresponding point in the graph, at height $X^{n+1} = u_1(x) = u_2(x)$. For any tangent cone $C_X \in \text{VarTan}(|G|, X)$—a priori these are not unique—, there is a two-valued Lipschitz function $U_X \in \text{Lip}(\mathbb{R}^n; A_2)$ with the same Lipschitz constant, so that $C_X = |\text{graph } U_X|$. This function $U_X$ is homogeneous as in (5.3), and thus by Lemma 5.3 this must be linear. In other words there is $\Pi_X \in Gr(n, n + 1)$ so that $C_X = 2|\Pi_X|$. Then by [BK17] there is $0 < \gamma = \gamma(n, L) < 1$ so that for some $\rho > 0$, $u \in C^{1, \gamma}(D_\rho(x); A_2)$. As the branch point $x$ was chosen arbitrarily and $\alpha(n, L) := \gamma$ does not depend on it, we get $u \in C^{1, \gamma}(D_2; A_2)$.

The base of the inductive argument is simple, as when $n = 1$ then two-valued minimal graphs are automatically linear. For the induction step, assume that Theorem 5.2 holds in dimension $n-1 \geq 1$. We prove Lemma 5.3 in dimension $n$, and for that purpose consider an arbitrary minimal graph $u \in \text{Lip}(\mathbb{R}^n; A_2)$, homogeneous as in (5.3). We claim that there is $\gamma = \gamma(n, L) > 0$ so that $u \in C^{1, \gamma}(\mathbb{R}^n \setminus \{0\}; A_2)$. To see this, let $X = (x, X^{n+1}) \neq 0 \in \text{sing } G \cap D_2 \times \mathbb{R}$. Every tangent cone $C_X \in \text{VarTan}(G, X)$ is the graph of a two-valued function $U_X \in \text{Lip}(\mathbb{R}^n; A_2)$ with the same Lipschitz constant. By a standard dimension reduction argument, $C_X$ is invariant under translation by $tX$ for all $t \in \mathbb{R}$. By the induction hypothesis $U_X$ is linear, and $C_X$ is

(1) either a sum of two multiplicity one planes, $C_X = |\Pi_1^X| + |\Pi_2^X|$, 
(2) or a single multiplicity two plane, $C_X = 2|\Pi_1^X|$. 

Then there is $n$.
By [BK17] there exists $0 < \gamma = \gamma(n, L) < 1$ so that for some $0 < \rho < |x|$, $u \in C^{1,\gamma}(D_{\rho}(x); A_2)$ regardless of whether $X = (x, X^{n+1})$ is a classical singularity or a branch point. As $x$ is arbitrary and $\gamma$ can be chosen independently of it, we get $u \in C^{1,\gamma}(\mathbb{R}^n \setminus \{0\}; A_2)$.

To extend this across the origin, define a function $w$ on the regular set,

$$w(X) = -\log(\langle \nu(X), e_{n+1} \rangle) \text{ for all } X \in \text{reg } G.$$  

This is non-negative and bounded, and we may let

$$M = \sup_{\text{reg } G \cap \partial B_1} w > 0,$$

and consider a sequence of points $X_i \in \text{reg } G \cap \partial B_1$ with $X_i \to Z \in G \cap \partial B_1$ and $w(X_i) \to M$ as $i \to \infty$.

**Claim 3.** If $Z = (z, Z^{n+1}) \notin B(G) \cap \partial B_1$ then $u$ is locally linear near $Z$ in the sense that there is $\rho > 0$ and a smooth selection $u_1, u_2 \in C^{\infty}(D_{\rho}(z))$ with $u_1$ linear and $Z \in \text{graph } u_1$.

**Proof.** As the graph $|G|$ is invariant under homotheties, so is $w$, whence if $X \in \text{reg } G$ then $\lambda X \in \text{reg } G$ and $w(\lambda X) = w(X)$ for all $\lambda > 0$. Therefore (5.2) also means $M = \sup_{\text{reg } G \cap B_1} w$.

When $Z \in \text{reg } G \cap \partial B_1$ then by the classical strong maximum principle $w$ is locally constant near $Z$, and thus so is so $|Du|$. Pick a small radius $\rho > 0$ so that a smooth selection $\{u_1, u_2\}$ can be made for $u$ on $D_{\rho}(z)$. We arrange for $Z \in \text{graph } u_1$. As $|Du_1|$ is constant in $D_{\rho}(z)$, it is harmonic by inspection. By the Bochner formula, $|D^2u_1|^2 = \Delta |Du_1|^2 \equiv 0$ on $D_{\rho}(z)$. Thus $u_1$ is affine linear, and the homothety-invariance of $G$ means that it must in fact be linear.

The argument is similar when $Z \in C(G) \cap \partial B_1$. Make a smooth selection $\{u_1, u_2\}$ for $u$ on $D_{\rho}(z)$ and write $G_i = \text{graph } u_i$. Define the two functions $w_1, w_2$ on $G_1, G_2$ respectively, using the analogue of (5.2). Without loss of generality assume that $G_1$ contains infinitely many points of $\{X_i \mid i \in \mathbb{N}\}$. As $w_1$ is continuous at the point $Z$, we get $w_1(Z) = M$. From then on, one can argue in the same way as when $Z$ is regular. \qed

Now suppose $Z \in B(G) \cap \partial B_1$. This argument is a bit more involved, but revolves around the same idea. Because $Du_1(z) = Du_2(z)$, the function $w$ can be continuously extended to $Z$. (The same is true for all branch points.) Write $\nu(Z) \in \mathbb{R}^{n+1}$ for the unit normal, and $2|\Pi_Z| \in \text{VarTan}(G, Z)$ for the tangent plane to $G$ at $Z$. As $w(Z) = M > 0$, this plane is not horizontal, and $\nu(Z) \neq e_{n+1}$. Let $P = \text{span}\{e_{n+1}, \nu(Z)\} \subset \mathbb{R}^{n+1}$. This intersects $\Pi_Z$ in a one-dimensional line, from which we pick a vector $e \in \Pi_Z \cap P$ with $\langle e, e_{n+1} \rangle > 0$. Write $e = ae_{n+1} + b\nu(Z)$ for $a, b \in \mathbb{R}$, which are constrained by $0 = \langle e, \nu(Z) \rangle = a\langle e_{n+1}, \nu(Z) \rangle + b$, or equivalently $b = -a(\langle e_{n+1}, \nu(Z) \rangle)$. Let $X \in \text{reg } G$ be an arbitrary regular point near $Z$. Then

$$\langle \nu(X), e \rangle = \langle \nu(X), ae_{n+1} - a\langle e_{n+1}, \nu(Z) \rangle \nu(Z) \rangle$$

$$= a\langle \nu(X), e_{n+1} \rangle - a\langle e_{n+1}, \nu(Z) \rangle \nu(X) \nu(Z).$$
Take $X$ close enough to $Z$ that $0 < \langle \nu(Z), \nu(X) \rangle \leq 1$, say this holds for $X \in \text{reg} G \cap B_\rho(Z)$ for example. Moreover by construction $\langle e_{n+1}, \nu(Z) \rangle \leq \langle e_n, \nu(X) \rangle$, and hence

\begin{equation}
\langle \nu(X), e \rangle \geq 0 \text{ for all } X \in \text{reg} G \cap B_\rho(Z).
\end{equation}

Upon decreasing $\rho > 0$ there is $U_Z \subset C^{1,\gamma}(B_\rho(Z) \cap (Z + \Pi_Z); \Pi_Z^\perp)$ so that $G \cap B_\rho(Z) \subset \text{graph } U_Z$, by Wickramasekera’s Theorem A.4. By construction $e \in \Pi_Z$, and we can apply Lemma 5.1 to $U_Z$ to deduce from (5.2) that $\langle \nu(X), e \rangle = 0$ for all $X \in \text{reg} G \cap B_\rho(Z)$. Returning to (5.2) we see that this is only possible if for these points we have both $\langle \nu(X), e_{n+1} \rangle = \langle \nu(Z), e_{n+1} \rangle$ and $\langle \nu(Z), \nu(X) \rangle = 1$. Either would suffice to conclude that $w(X) = w(Z)$ for all $X \in \text{reg} G \cap B_\rho(Z)$. Once we have derived this, we may reason as in the proof of Claim 3 to draw the analogous conclusion.

We use this to show that $G$ must be a union of planes, using an argument similar to that used to prove Lemma 11.4. Let $\mathcal{R}$ be the set of connected components of $\text{reg } G$, of which there are at most countably many. Among them we write $\mathcal{R}_f \subset \mathcal{R}$ for those $\Sigma \in \mathcal{R}$ that are flat in the sense that $|A_\Sigma| \equiv 0$. The rest is denoted $\mathcal{R}_c = \mathcal{R} \setminus \mathcal{R}_f$. We decompose $\{G\} = C_f + C_c \in IV_n(R^{n+1})$, respectively defined by $C_f = \sum_{\Sigma \in \mathcal{R}_f} \Theta_\Sigma|\Sigma|$ and $C_c = \sum_{\Sigma \in \mathcal{R}_c} \Theta_\Gamma|\Gamma$. Here given $\Sigma \in \mathcal{R}$ we write $\Theta_\Sigma \in Z_{>0}$ for its multiplicity, which is constant by [Sim84, Thm. 41.1]. Both $C_f, C_c$ are invariant under homotheties, and stationary. To justify the latter, it suffices to prove that $C_f, C_c$ are stationary near points in $\mathcal{C}(G)$, as the other singularities do not contribute to the first variation. Pick some point $X \in \text{spt} \|C_f\| \cap \|C_c\| \cap \mathcal{C}(G)$, and let $\rho > 0$ be so that we can decompose $G \cap B_\rho(X) = \Sigma_1 \cup \Sigma_2$ into a union of two surfaces embedded in $B_\rho(X)$, which meet transversely along $\text{sing} G \cap B_\rho(X)$. By a unique continuation argument we may arrange for $\Sigma_1 \subset \text{spt} \|C_f\|$ and $\Sigma_2 \subset \text{spt} \|C_c\|$. Both $\Sigma_i$ are stationary in $B_\rho(X)$, whence $C_f, C_c$ are stationary inside $B_\rho(X)$ too. As $X$ was arbitrary, they are stationary in $R^{n+1}$. The argument above shows that $\mathcal{R}_f \neq \emptyset$ and $C_f \neq 0$, and by Lemma 12.2 it is supported in a union of planes, say $\text{spt} \|C_f\| = \Pi_1 \cup \cdots \cup \Pi_D$ with $D \leq 2$. If $C_c = 0$ then we are done, otherwise $D = 1$ and $C_f = |\Pi_1|$. In this case too one ultimately finds that $\|G\| = |\Pi_1| + |\Pi_2|$, for instance using [Sim77].}

\section*{6. A Jenkins–Serrin type lemma for single-valued minimal graphs}

We make a brief excursion to single-valued minimal graphs, with the aim of proving two basic technical results that will turn out essential in the classification of vertical limit cones; see Section 10 and Lemma 10.2 in particular.

Throughout this section, $\Omega \subset R^n$ be a bounded, convex domain with Lipschitz-regular boundary. Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a single-valued function, whose graph $G \in \mathcal{L}_n(\Omega \times R)$ is minimal. Using the convexity of $\Omega$, it is well-known that the current $\|G\|$ is area-minimising: any current...
$T \in \mathbf{I}_n(\mathbb{R}^{n+1})$ with $\partial T = \partial [G]$ has larger area: $\mathcal{H}^n(G) \leq \|T\|_n(\mathbb{R}^{n+1})$. Next, given $A < B \in \mathbb{R}$ let $G_{A,B} = G \cap \Omega \times (A, B)$. As the set $\Omega \times (A, B) \subset \mathbb{R}^{n+1}$ is convex, this retains the area-minimising property: any current $T \in \mathbf{I}_n(\mathbb{R}^{n+1})$ with $\partial T = \partial [G_{A,B}]$ has $\mathcal{H}^n(G \cap \Omega \times (A, B)) \leq \|T\|_n(\mathbb{R}^{n+1})$.

For the remainder we work in a more specialised, polyhedral setting. Let $\Pi_1, \ldots, \Pi_N \in Gr(n, n+1)$ be vertical planes, that is $\Pi_i = \Pi_i^0 \times \mathbb{R}^n_{n+1}$, consider points $p_1, \ldots, p_N \in \mathbb{R}^n$ and the affine planes $p_1 + \Pi_i, \ldots, p_N + \Pi_N$. In terms of some choice of unit normals $n_1, \ldots, n_N \in \mathbb{R}^n$ these can be written

$p_1 + \Pi_i^0 = \{ x \in \mathbb{R}^n \mid \langle x, n_i \rangle = a_i \}$,

where we set $a_i = \langle p_i, n_i \rangle$. Let $\Omega \subset \mathbb{R}^n$ be a convex polyhedral domain bounded by these planes, and assume that the normals point into $\Omega$: $\Omega = \cap_{i=1}^N \{ x \in \mathbb{R}^n \mid \langle x, n_i \rangle > a_i \}$. The boundary of $\Omega$ contains what we call the faces $F_j = \{ x \in \partial \Omega \mid \langle x, n_i \rangle = a_i \}$ and the edges $E_{ij} = F_j \cap F_j$, which have $\mathcal{H}^{n-1}(E_{ij}) = 0$.

Let $T = T_0 \times \mathbb{R}^n_{n+1} \in \mathbf{I}_n(\mathbb{R}^{n+1})$ be a vertical current so that the support of $T_0$ is equal to a union of some subcollection of the faces in $\partial \Omega$. Write $F_T$ for the faces and, likewise $E_T$ for the edges contained in $\text{spt}\|T_0\|$. Let $(\Omega_j | j \in \mathbb{N})$ be a sequence of bounded domains with $\text{dist}_H(\Omega_j, \Omega) \to 0$ as $j \to \infty$; these need neither be convex, nor polyhedral or have piecewise smooth boundary. For each $j \in \mathbb{N}$ there is a regular, single-valued $u_j \subset C^2(\Omega_j)$ whose graph $G_j$ is minimal. We prove the following lemma for their limit, inspired by the work of Jenkins–Serrin in dimension three [JS66a, JS66b].

**Theorem 6.1.** Let $\Omega, \Omega_j \subset \mathbb{R}^n$ and $u_j \subset C^2(\Omega_j)$ be as above. Suppose that $\text{dist}_H(\Omega_j, \Omega) \to 0$, $\|G_j\| \to T$ and that for all $0 < \sigma < 1 < A$ there is $J = J(\sigma, A) \in \mathbb{N}$ so that $\text{spt} \partial [G_j \cap \Omega_j \times (-A, A)] \subset (E_T)_\sigma$ for all $j \geq J$. Then $\|T_0\| \leq \mathcal{H}^{n-1}(\partial \Omega)/2$.

**Proof.** We obtain the conclusion by constructing a comparison surface. We begin the construction by making the following two assumptions:

1. $\Omega \subset \Omega_j$ for all $j$,
2. and for all $0 < \tau < 1 < A$ there is $J = J(\tau, A) \in \mathbb{N}$ so that $|u_j| > 2A$ on $\Omega \setminus (F_T)_\tau$ for all $j \geq J$.

We explain at the end of the proof why this can be done without restricting the generality of the argument.

Let $0 < \tau < \sigma < 1 < A$ be given, with the eventual aim of letting $\sigma, \tau \to 0$ and $A \to \infty$. We may perturb these by a small amount to guarantee that the level sets $\{ u_j = \pm A \}$ are regular inside $\Omega_j$, justifying this by Sard’s lemma; whenever we adjust the values of $\tau, \sigma$ or $A$ we do so in a way which preserves this property.

First we adjust $\tau$ in terms of $\sigma$ so that $(F_T)_\tau \cap \partial \Omega \subset F_T \cup (E_T)_\sigma$ and the $\tau$-tubular neighbourhoods of any two distinct faces are either disjoint or meet inside $(E_T)_\sigma$. Hence we can decompose $(F_T)_\tau \setminus (E_T)_\sigma$ into a disjoint union, with one connected component for each face. We define the open subset $\Omega_{j, \tau, \sigma} \subset \Omega_j$ by $\Omega_{j, \tau, \sigma} = \Omega_j \cap \{ \Omega \cup (F_T)_\tau \setminus (E_T)_\sigma \}$ and take $j \geq J(\tau, \sigma, A)$ large enough that $|u_j| > 2A$ on $\Omega_{j, \tau, \sigma}$. 


Consider the open set $W_{j, \tau, \sigma, A} \subset \Omega_{j, \tau, \sigma} \times (-A, A)$ defined by
\[ W_{j, \tau, \sigma, A} = \{(x, X^{n+1}) \in \Omega_{j, \tau, \sigma} \times (-A, A) \mid u_j(x) < X^{n+1}\}. \]

This is a Caccioppoli set, whose current boundary we decompose into
\[ -\partial[W_{j, \tau, \sigma, A}] = \left[\left(\{u_j < A\} \cap \Omega_{j, \tau, \sigma}\right) \times \{A\}\right] \]
\[ + \left[\left(\{u_j < -A\} \cap \Omega_{j, \tau, \sigma}\right) \times \{-A\}\right] \]
\[ + \left[\partial \Omega \setminus ([E_T]_{\sigma} \cup F_T) \times (-A, A)\right] \]
\[ + \left[\partial W_{j, \tau, \sigma, A} \cap \partial (E_T)_{\sigma} \times (-A, A)\right] \]
\[ + \left[G_j \cap \Omega_{j, \tau, \sigma} \times (-A, A)\right]. \]

We estimate the areas of all the summands separately. Let a small $\delta > 0$ be given. For the first two we respectively find
\[ \mathcal{H}^n((\{u_j < \pm A\} \cap \Omega_{j, \tau, \sigma}) \times \{\pm A\}) \leq \mathcal{H}^n(\Omega_{j, \tau, \sigma}) \leq \mathcal{H}^n(\Omega_j). \]

Using the convergence $\Omega_j \to \Omega$ in the Hausdorff distance, we find that for large $j \geq J(\tau, \sigma, A, \delta)$ so that $\mathcal{H}^n(\Omega_j) \leq \mathcal{H}^n(\Omega) + \delta$. The third term, $[\partial \Omega \setminus ([E_T]_{\sigma} \cup F_T) \times (-A, A)]$ does not depend on $j$, and taking $\sigma, \tau$ small enough in terms of $\delta$ we get
\[ \mathcal{H}^n(\partial \Omega \setminus ([E_T]_{\sigma} \cup F_T) \times (-A, A)) \]
\[ \leq 2A(H^{n-1}(\partial \Omega) - H^{n-1}(\text{spt} T_0)) + 2A\delta. \]

For the fourth, we bound
\[ \mathcal{H}^n(\partial W_{j, \tau, \sigma, A} \cap \partial (E_T)_{\sigma} \times (-A, A)) \leq 2A\mathcal{H}^{n-1}(\partial (E_T)_{\sigma}) \]
\[ \leq 4\pi A\sigma \mathcal{H}^{n-2}(E_T) + 2A\delta. \]

Next, the convergence assumed in the statement allows the lower bound
\[ \mathcal{H}^n(G_j \cap \Omega_{j, \tau, \sigma} \times [-A, A]) \]
\[ \geq 2A(\mathcal{H}^{n-1}(F_T) - \mathcal{H}^{n-1}((E_T)_{\sigma})) - 2A\delta \]
\[ \geq 2A(\mathcal{H}^{n-1}(F_T) - 2\pi \sigma \mathcal{H}^{n-2}(E_T) - \delta) - 2A\delta, \]

provided we change $\sigma$ to a suitably small value in terms of $\delta$.

We compare the graph $[G_j \cap \Omega_{j, \tau, \sigma} \times (-A, A)]$ to the integral current $T_{j, A} = -\partial[W_{j, \tau, \sigma, A}] + [G_j \cap \Omega_{j, \tau, \sigma} \times (-A, A)]$. These have the same boundary, whence $\mathcal{H}^n(G_j \cap \Omega_{j, \tau, \sigma} \times (-A, A)) \leq \|T_{j, A}\|(\mathbb{R}^{n+1})$. Substituting our term-by-term calculations into this inequality we find
\[ 2A(\mathcal{H}^{n-1}(F_T) - 2\pi \sigma \mathcal{H}^{n-2}(E_T) - \delta) - 2A\delta \]
\[ \leq 2\mathcal{H}^n(\Omega) + 2\delta + 2A(\mathcal{H}^{n-1}(\partial \Omega) - \mathcal{H}^{n-1}(F_T)) + 2A\delta + 4\pi A\sigma \mathcal{H}^{n-2}(E_T) + 2A\delta, \]

whence after dividing by $2A,$
\[ \mathcal{H}^{n-1}(F_T) - 2\sigma \pi \mathcal{H}^{n-2}(E_T) - 2\delta \]
\[ \leq \mathcal{H}^n(\Omega)/A + \mathcal{H}^{n-1}(\partial \Omega) - \mathcal{H}^{n-1}(F_T) + 2\sigma \pi \mathcal{H}^{n-2}(E_T) + \delta(2 + 1/A). \]
This simplifies to
\[ 2\mathcal{H}^{n-1}(F_T) \leq \mathcal{H}^n(\Omega)/A + \mathcal{H}^{n-1}(\partial \Omega) + 4\sigma \pi \mathcal{H}^{n-2}(E_T) + \delta(4 + 1/A). \]

The desired inequality follows after letting \( A \to \infty, \delta, \sigma, \tau \to 0 \) and \( j \geq J(\sigma, \tau, A, \delta) \to \infty \). To conclude it only remains to justify the two assumptions (1) and (2).

(1) After translating \( \Omega \) we may assume that it contains the origin. The convexity of \( \Omega \) and the piecewise regularity of its boundary that for all \( \tau > 0 \) there is \( \delta > 0 \) so that \( \eta_{0,(1+\delta)\bar{\Omega}} \subset \Omega \setminus (\partial \Omega)_\tau \) and \( (\Omega)_\tau \subset \eta_{0,(1+\delta)^{-1}\bar{\Omega}}. \) Moreover as \( \tau \to 0 \) we may impose that \( \delta \to 0 \) also. Given \( \tau > 0 \) take \( J(\tau) \in \mathbb{N} \) so that \( \Omega \setminus (\partial \Omega)_\tau \subset \Omega_j \subset (\Omega)_\tau. \) Taking \( \delta > 0 \) as above we get that \( \eta_{0,1+\delta}\bar{\Omega} \subset \Omega_j \subset \eta_{0,(1+\delta)^{-1}\bar{\Omega}}. \) After rescaling we find a sequence \( (\eta_{0,1+\delta}[G_j] \mid j \geq J(\tau)) \) of single-valued minimal graphs respectively defined over \( \eta_{0,1+\delta}\bar{\Omega} \). Moreover \( \eta_{0,1+\delta}[G_j] \to \eta_{0,1+\delta\Omega} \) as \( j \to \infty \) and \( \eta_{0,1+\delta}T \to T \) as \( \delta \to 0 \). We may then diagonally extract a subsequence of graphs \( [G_j] \) and find a sequence of positive scalars with \( \delta_j \to 0 \) as \( j \to \infty \), so that \( \Omega \subset \eta_{0,1+\delta}\Omega_j \) for all \( j \) and \( \eta_{0,1+\delta\Omega_j} \to T \) as \( j \to \infty \). Upon replacing our original sequence by this rescaled subsequence, we may assume throughout that \( \Omega \subset \Omega_j \) without restriction of generality.

(2) Let any \( 0 < \tau < \delta < 1 < A \) be given, and consider the open set \( \Omega' = \Omega \setminus [\partial \Omega \setminus \text{spt} T_0]. \) By construction \( \Omega' \setminus \text{spt} T_0 \) lies a distance at least \( \tau > 0 \) away from \( \partial \Omega \), and thus also \( \partial \Omega_j \) for all \( j \). Were it not guaranteed that \( |u_j| > 2A \) in \( \Omega' \setminus \text{spt} T_0 \) for large enough \( j \), then there would exist a sequence of points \( X_j \in \Omega' \setminus \Omega \times [-2A, 2A] \setminus \text{spt} T \) belonging to a subsequence of the two-valued graphs. From this we can extract yet another subsequence so that \( X_j \to X \in \Omega' \setminus [-2A, 2A] \setminus \text{spt} T \) as \( j \to \infty \). By the monotonicity formula and upper semicontinuity of density \( X \in \text{spt} T \), but that is manifestly absurd. Replacing the original domain by \( \Omega' \) and following the reasoning in the proof yields \( \mathcal{H}^n(\text{spt}(T_0 \mathbb{L} \Omega')) \leq 1/2 \mathcal{H}^n(\partial \Omega'). \) Letting \( \delta \to 0 \) one finds the desired conclusion in terms of \( \Omega \).

\[ \square \]

The following special case is of particular importance in what follows. Let \( \pi = \pi_0 \times \mathbb{R} e_{n+1}, \pi' = \pi_0' \times \mathbb{R} e_{n+1} \) be two \( n \)-dimensional half-planes meeting along an axis \( L = L_0 \times \mathbb{R} e_{n+1} \) at which they form a positive angle \( 0 < \theta < \pi \), taken in the counterclockwise direction. Let \( N, N' \) be their respective unit normals, which we both take pointing in the counterclockwise direction. Further let \( p, p' \in L^\perp \) be the two unit vectors so that \( \pi = \{Y + tp \mid Y \in L, t \geq 0\} \) and \( \pi' = \{Y + tp' \mid Y \in L, t \geq 0\}. \) Any point in \( \mathbb{R}^n \) can be written \( x = y + z = y + tp + t'p \) with \( y \in L_0, z \in L_0^\perp. \) Define \( Q = \{y + z \in \mathbb{R}^n \mid |y| < 1, |z| < 1\} \) and the wedge-shaped region \( V = \{x \in Q \mid \langle x, N \rangle > 0, \langle x, N' \rangle < 0\} \).

\[ \text{Lemma 6.2.} \] Let \( \pi, \pi' \) and \( V \subset \mathbb{R}^n \) be as above, and let the current \( T = [(\pi \cup \pi') \cap \partial V \times \mathbb{R}] \) be oriented inward.

Then there does not exist a sequence of minimal graphs \( G_j = \text{graph } u_j \) defined over domains \( \Omega_j \) with \( \text{dist}_{\mathcal{H}}(\Omega_j, V) \to 0 \), \( [G_j] \to 0 \) and so that for all
$0 < \sigma < 1 < A$ there is $J = J(\sigma, A) \in \mathbb{N}$ so that $\text{spt} \partial [G_j \mathbf{1}_{\Omega_j \times (-A, A)}] \subset ((\pi \cup \pi') \cap \partial V)_\sigma$ for all $j \geq J$.

**Proof.** This is essentially a direct consequence of Theorem 6.1, although we need to construct a subdomain $\Delta_a \subset V$ to make the area comparison work, where $0 < a < 1/2$ is a small parameter whose value we leave undetermined for now. Let $\Delta_a = \{ y + tp + \tau'p' \in V \mid |y| < 1/2, t, t' > 0, t + t' < a \}$, which is convex and has piecewise smooth boundary. When we fix $y_0 \in L_0$ with $|y_0| < 1/2$ then $\Delta_a \cap \{ x = y_0 + z \}$ is an isosceles triangle with two sides of length $a$. Moreover, apart from $\pi \cap \partial \Delta_a$ and $\pi' \cap \partial \Delta_a$, the boundary of $\Delta_a$ contains only two subsets $\Gamma_1, \Gamma_2$ with positive area, namely $\Gamma_1 = \{|y| = 1/2\}$ and $\Gamma_2 = \{|y| < 1/2, t + t' = a\}$. On the one hand

$$\mathcal{H}^{n-1}(\Gamma_1) = a^2(n - 2)\omega_{n-2} \sin(\theta) 2^{-n+2},$$

$$\mathcal{H}^{n-1}(\Gamma_2) = a\omega_{n-2} 2^{-n+3} \sin(\theta/2),$$

and on the other hand

$$\mathcal{H}^{n-1}(\pi \cap \partial \Delta_a) = a\omega_{n-2} 2^{-n+2} = \mathcal{H}^{n-1}(\pi' \cap \partial \Delta_a).$$

Comparing the two we have

$$\mathcal{H}^{n-1}(\Gamma_1 \cup \Gamma_2) = a\omega_{n-2} 2^{-n+3} ((n - 2) a \sin(\theta)/2 + \sin(\theta/2))$$

$$< a\omega_{n-2} 2^{-n+3} = \mathcal{H}^{n-1}(\pi \cup \pi' \cap \partial \Delta_a)$$

provided $a$ is small enough. If there were a sequence of minimal graphs of $u_j \in C^2(\Omega_j)$ as in the statement, then by restricting them to $\Omega_j \cap \Delta_a$ and letting $j \to \infty$ we would obtain a contradiction to Theorem 6.1. \qed

### 7. Multiplicity and branch points of limit cones

Let $\alpha \in (0, 1)$ and $(u_j \mid j \in \mathbb{N})$ be a sequence of two-valued minimal graphs with $u_j \in C^{1,\alpha}(D_2; A_2)$. Here we examine the situation in which these graphs converge to a plane weakly in the varifold topology, $[G_j] \to m|\Pi$ as $j \to \infty$, where $\Pi \in Gr(n, n + 1)$ and $m \in \mathbb{Z}_{>0}$.

#### 7.1. An a priori multiplicity bound.

**Lemma 7.1.** If $|G_j| = |\text{graph } u_j| \to m|\Pi$ then $m \leq 2$.

**Proof.** The proof is slightly easier when the plane $\Pi \in Gr(n, n + 1)$ is vertical, that is of the form $\Pi = \Pi_0 \times \mathbb{R}e_{n+1}$. Let $\epsilon > 0$ be a given, arbitrarily small constant. Arguing as in the proof of Proposition 3.1 we may take $J(\epsilon) \in \mathbb{N}$ so that $\int_{D_1} \frac{1}{|_{|u_j|<1}} \leq \epsilon$ when $j \geq J(\epsilon)$. Thus for large $j$, $\mathcal{H}^n(G_j \cap D_1 \times (-1, 1)) \leq \epsilon + \int_{D_1} |Du_j|^2 1_{|u_j|<1}$. Let $U \subset D_1$ be a tubular neighbourhood of $\Pi_0$ inside the disc, narrow enough that $\text{Per}(U) \leq 2\mathcal{H}^{n-1}(\Pi_0 \cap D_1) + \delta$. We may find a function $\eta \in C^1_0(D_2)$ with $\eta \equiv 1$ on $U$ and $\int_{D_2} \eta \leq \text{Per}(U) + \delta \leq 2\mathcal{H}^{n-1}(\Pi_0 \cap D_1) + 2\delta$. Updating $j \geq J(\epsilon, \delta)$ so that $G_j \cap D_1 \times (-1, 1) \subset U \times (-1, 1)$ we get $\mathcal{H}^n(G_j \cap D_1 \times (-1, 1)) \leq \epsilon + 4(\mathcal{H}^{n-1}(\Pi_0 \cap D_1) + \delta)$. Letting
$\epsilon, \delta \to 0$ and $j \geq J(\tau, \epsilon) \to \infty$ we find $mH^n(\Pi \cap D_1 \times (-1, 1)) \leq 4H^{n-1}(\Pi_0)$, which gives the desired conclusion.

Now for the case where the plane $\Pi$ is not vertical. There nothing to prove if $m = 1$, so we may assume that $m \geq 2$. Let a small constant $0 < \tau < 1$ be given, and take $j \geq J(\tau)$ large enough that inside the cylinder $G_j \cap D_2 \times \mathbb{R} \cap (\Pi)_1 \subset (\Pi)_\tau$. In fact we have the same control over $G_j$ in the whole cylinder, that is $G_j \cap D_2 \times \mathbb{R} \subset (\Pi)_\tau$. Indeed, if this were to fail then $G_j \cap D_2 \times \mathbb{R}$ would be disconnected, and we could write $G_j \cap D_2 \times \mathbb{R} = \Gamma_{j,1} \cup \Gamma_{j,2}$ where $\Gamma_{j,1} \cap (\Pi)_\tau \neq \emptyset$ and $\Gamma_{j,2} \cap (\Pi)_1 = \emptyset$. But then $H^n(\Gamma_{j,1}) \leq H^n(\Gamma_{j,1})$, and taking limits as $j \to \infty$ would yield $m = 1$. This is absurd as we initially assumed that $m$ is at least two, and hence we have confirmed that $G_j \cap D_2 \times \mathbb{R} \subset (\Pi)_\tau$. Let $L = \max\{X^{n+1} \mid X = (x, X^{n+1}) \in \Pi \cap D_2 \times \mathbb{R}\}$, then $\|u_j(x)\| \leq 2(\tau + L)$ for all $x \in D_2$. Using the interior gradient estimates, we find that there is a constant $C = C(n, L)$ so that eventually $\|u_j\|_{1; D_1} \leq C$. Up to extracting a subsequence we find that the $u_j$ converge to a two-valued Lipschitz graph defined on $D_1$. As by assumption $|G_j| \to m|\Pi|$ we can conclude that $m = 2$. \qed

7.2. Multiplicity in limit varifolds. Combining the previous lemma with a diagonal extraction argument, we obtain the following result.

**Corollary 7.2.** Let $\alpha \in (0, 1)$, and let $(u_j \mid j \in \mathbb{N})$ be a sequence of two-valued minimal graphs in $C^{1,\alpha}(D_2; A_2)$. Suppose that there are half-planes $\pi_i$ and $m_i \in \mathbb{Z}_{>0}$ so that $|G_j| \to \sum_i m_i|\pi_i|$. Then $m_i \leq 2$.

Similarly, though in a more general context, we can combine the estimate from Lemma 7.1 with the work of Krummel–Wickramasekera [KW20].

**Corollary 7.3.** Let $G_j = \text{graph} u_j$ be a sequence of two-valued minimal graphs, where $u_j \in C^{1,\alpha}(D_2; A_2)$ for all $j$ for some $\alpha \in (0, 1)$. Suppose that $|G_j| \to V \in IV_n(D_2 \times \mathbb{R})$ weakly in the topology of varifolds. Then for all $Z \in \text{reg } V$, $\Theta(|V|, Z) \leq 2$. If $Z \in B(V)$ then $\Theta(|V|, Z) = 2$, and the branch set is countably $n-2$-rectifiable.

7.3. Local description near vertical planes. We return to the situation where $|G_j| \to 2|\Pi|$ to some vertical plane $\Pi = \Pi_0 \times \mathbb{R}e_{n+1} \in \text{Gr}(n, n+1)$. The limit in the current topology is supported in the same plane, $[G_j] \to l[\Pi]$ for some non-negative $l \in \mathbb{Z}$ with $l \leq 2$. By Allard’s regularity theorem the multiplicity is either $l = 0$ or $2$. The following result considers the case where the mass of the currents vanishes in the limit.

**Lemma 7.4.** Let $\alpha \in (0, 1)$, $u_j \in C^{1,\alpha}(D_2; A_2)$ be a sequence of two-valued minimal graphs, and let $\Pi = \Pi_0 \times \mathbb{R}e_{n+1}$ be a vertical plane. Suppose $|G_j| \to 2|\Pi|$ and $[G_j] \to 0$ as $j \to \infty$. Then $B(G_j) \cap B_1 = \emptyset$ for large $j$.

**Proof.** Let $N$ be either unit normal to $\Pi$, and define $f_j = \langle u_j, N \rangle$ on $\text{reg } G_j$. Given any $\delta > 0$, $f_j$ restricted to $\text{reg } G_j \cap B_{3/2}$ takes values in $[-1, -1 + \delta) \cup (1 - \delta, 1]$ by [Wic20], at least for large enough $j \geq J(\delta)$. Write $\mathcal{R}_j$ for
the connected components of \( \text{reg} \ G_j \cap B_{3/2} \), which we further divide into \( \mathcal{R}_j^\pm \) according to the sign of \( f_j \). Accordingly we may decompose \([G_j]\) = \( T_j^+ + T_j^-\) into the sum of the two currents obtained by integrating over \( \mathcal{R}_j^\pm \) respectively, in a way that \([G_j]\) = \( T_j^\pm \) and \(|G_j| = |T_j^\pm| + |T_j^-|\). The two currents only meet along classical, immersed singularities of \( G_j \), where they moreover intersect transversely. Therefore they are both separately stationary with \( \partial T_j^\pm = 0 \) in \( B_{3/2} \). By assumption \( T_j^+ + T_j^- \to 0 \) and \(|T_j^\pm| + |T_j^-| \to 2|\Pi|\mathbb{L}B_{3/2}\) as \( j \to \infty \) in the current and varifold topologies respectively. Moreover by Federer–Fleming compactness \( T_j^\pm \to T^\pm \) separately as \( j \to \infty \). The limit currents satisfy \( T^+ + T^- = 0 \), and thus they are equal to the plane \( \Pi \) with multiplicity, but with opposite orientations. By Allard regularity both \( T_j^+ \llcorner B_1 \) and \( T_j^- \llcorner B_1 \) can be written as smooth graphs defined on \( \Pi \), and thus do not support any branch points. \( \square \)

Return to the general case, where \(|G_j| \to m|\Pi|\) and \([G_j]\) \( \to l|\Pi|\) for some vertical plane \( \Pi = \Pi_0 \times \mathbb{R}^{n+1} \). If \( l \neq 0 \) then we let \( N \) be the unit normal to \( \Pi \) corresponding to the orientation induced on the plane by \([G_i]\), and if \( l = 0 \) then we pick our orientation arbitrarily. Thus we can divide \( D_1 \setminus \Pi_0 \subset \mathbb{R}^n \) into two connected components \( D_1^\pm = \{ x \in D_1 \mid \pm \langle x, N \rangle > 0 \} \).

For each \( j \) define a function \( F_j : D_1 \to \{0, 1, 2\} \) by

\[
F_j(x) = \sum_{X \in P_0^{n+1}(\{x\}) \atop X^{n+1} < -1} \Theta(|G_j|, X).
\]

(7.1)

This returns the number of points in \( G_j \) which lie below \( x \in D_1 \), counted with multiplicity. (We could equally well have worked with a function counting the points lying above \( x \in D_1 \), although formulas such as (7.5) would have the opposite sign.) These functions are eventually locally constant away from the plane \( \Pi_0 \), in the following sense.

**Claim 4.** Let \( \tau > 0 \) be arbitrary. Then there is \( j \geq J(\tau) \) so that \( F_j \) is constant on the two components \( D_1^\pm \setminus (\Pi_0)_\tau \).

*Proof.* The proof is identical for both components, so we just work with \( D_1^+ \). By the convergence of the graphs \( G_j \) in the Hausdorff distance, we may take \( j \geq J(\tau) \) large enough that \( G_j \cap D_1 \times (-1, 1) \subset (\Pi)_\tau \). Hence eventually \( G_j \cap D_1^+ \times \mathbb{R} \setminus (\Pi)_\tau \subset \{X^{n+1} > 1\} \).

There are three possibilities:

1. either \( G_j \cap D_1^+ \times \mathbb{R} \setminus (\Pi)_\tau \cap \{X^{n+1} < -1\} = \emptyset \),
2. or \( G_j \cap D_1^+ \times \mathbb{R} \setminus (\Pi)_\tau \subset \{X^{n+1} < -1\} \),
3. or \( G_j \cap D_1^+ \times \mathbb{R} \setminus (\Pi)_\tau \) contains points with positive and negative values for \( X^{n+1} \).

Going through these cases in the same order we find that at all points \( x \in D_1^+ \setminus (\Pi)_\tau \) the function \( F_j(x) \) takes the values 0, 2 or 1. \( \square \)
Lemma 7.5. Let $\alpha \in (0,1)$, $u_j \in C^{1,\alpha}(D_2;A_2)$ be a sequence of two-valued minimal graphs, and $\Pi = \Pi_0 \times \mathbb{R}_{e_{n+1}}$ be a vertical plane. Suppose that $|G_j| \to m|\Pi|$ and $\|G_j\| \to l|\Pi|$ as $j \to \infty$. Then the $F_j$ are eventually constant away from $\Pi_0$, taking the values $F^\pm$ on $D_1^\pm$ respectively, and

\begin{equation}
F^+ - F^- = l.
\end{equation}

Proof. There are three possible cases:

(1) either $m = 2$ and $l = 2$,
(2) or $m = 2$ and $l = 0$,
(3) or $m = 1$ and $l = 1$.

The proof is basically the same in all three cases, so we only consider the first. Consider the line $l \subset \mathbb{R}^n \times \{0\}$ directed by $N$ and passing through the origin. Identify $l$ with $\mathbb{R}$ via a unit-speed parametrisation. Then there exist two functions $u_{j,t}^1, u_{j,t}^2 \in C^1(\mathbb{R})$ so that $u_j(tN) = \{u_{j,t}^1(tN), u_{j,t}^2(tN)\}$ for all $t \in \mathbb{R}$. Moreover, as $l = 2$ we get that $u_{j,t}^1(t) \wedge u_{j,t}^2(t) < -1$ on $(1/2,1)$ and $u_{j,t}^1(t) \vee u_{j,t}^2(t) > 1$ for $t \in (-1,-1/2)$, provided $j$ is large enough.

Now let $0 < \tau < 1$ be an arbitrary small constant. By Claim 4, the function $F_j$ is constant on the two components of $D_1 \setminus (\Pi)_\tau$, at least provided $j \geq J(\tau)$ is chosen large enough. Combining this with our calculations above, we find that for points $x \in D_1 \setminus (\Pi)_\tau$, $F_j(x) = 2$ if $x \in D_1^+$ and $F_j(x) = 0$ if $x \in D_1^-$. These values do not change with large values of $j$, and we may set $F^+ = 2, F^- = 0$, which confirms that indeed $F^+ - F^- = 2 = l$. As explained above, the other cases can be argued similarly. $\square$

8. Classical limit cones: initial analysis

Let $\alpha \in (0,1)$, and $(u_j \mid j \in \mathbb{N})$ be a sequence of two-valued minimal graphs, with $u_j \in C^{1,\alpha}(D_2;A_2)$ for all $j$. We assume that they converge to a classical cone in the varifold topology, say $|G_j| \to P$. By this we mean that there are $n$-dimensional half-planes $\pi_1, \ldots, \pi_N$ meeting along a common $n-1$-dimensional axis $L \in Gr(n-1,n+1)$ and integers $m_1, \ldots, m_N \in \mathbb{Z}_{>0}$ so that $P = \sum_i m_i|\pi_i|$. By the graphs are endowed with the orientation corresponding to their upward-pointing unit normal we obtain a sequence of currents which we may also assume convergent, say $\|G_j\| \to T \in \mathcal{I}_n(D_2 \times \mathbb{R})$, extracting a subsequence if necessary. This limit too has a similar form to the above, namely $T = \sum_i l_i \|\pi_i\|$ where $0 \leq l_i \leq m_i$. Here the half-planes are given the orientations induced by $T$, where we pick an arbitrary orientation for those $\pi_i$ which have $l_i = 0$. Our main theorem in this section is the following.

Theorem 8.1. Let $\alpha \in (0,1)$, and let $(u_j \mid j \in \mathbb{N})$ be a sequence of two-valued minimal graphs with $u_j \in C^{1,\alpha}(D_2;A_2)$. Suppose that $|G_j| \to P$ and $\|G_j\| \to T$ as $j \to \infty$, where $P$ and $T$ are classical cones. Then there exist
planes $\Pi_1, \ldots, \Pi_D \in \text{Gr}(n, n + 1)$ and integers $0 \leq l_i \leq m_i \leq 2$ so that

$$\mathbf{P} = \sum_{i=1}^{D} m_i |\Pi_i| \quad \text{and} \quad T = \sum_{i=1}^{D} l_i |\Pi_i|.$$ 

Our proof follows an approach adapted from the work of Schoen–Simon in [SS81]. In preparation for this, we introduce an orthogonal decomposition of $\mathbb{R}^{n+1}$ adapted to $L$, writing $\mathbb{R}^{n+1} = \{Y + Z \mid Y \in L, Z \in L^\perp\}$. In keeping with this, we call the two-dimensional affine space $\{Y + Z \mid Z \in L^\perp\}$ obtained by fixing a point $Y \in L$ a slice through $Y$. In this notation the cone $\mathbf{P}$ is supported in $\{Y + \sum_{i=1}^{N} t_i p_i \mid Y \in L, t_i \geq 0, i = 1, \ldots, N\}$, where the $p_i \in L^\perp$ are unit vectors so that $\pi_i = \{Y + tp_i \mid Y \in L, t \geq 0\}$. Letting $R_i$ be the ray generated by $p_i$, we have $\mathbf{P} = L \times \sum_{i=1}^{N} m_i |R_i|$.

Let a small $\sigma > 0$ be given. For sufficiently large $j \geq J(\sigma)$,

$$G_j \cap \{Y + Z \mid Z \in L^\perp, \sigma/2 < |Z| < \sigma\} = \cup_{k=1}^{M} \gamma_{j,Y}^{k}$$

for each point $Y \in L$ with $|Y| \leq 1$, where $M = \sum_{i} m_i$. The $\gamma_{j,Y}^{k}$ are $C^{1,\alpha}$ embedded Jordan arcs with endpoints in $\{Y + Z \mid Z \in L^\perp, |Z| = \sigma/2$ or $\sigma\}$. Moreover as $j \to \infty$ we have the uniform limits

$$\text{dist}_H(\cup_{k=1}^{M} \gamma_{j,Y}^{k}, \cup_{i=1}^{N} \{Y + tp_i \mid \sigma/2 < t < \sigma\}) \to 0$$

and for every $k$,

$$\min_{i \in \{1,\ldots, N\}} \sup_{X \in \gamma_{j,Y}^{k}} \langle \nu_j(X), N_i \rangle \to 0,$$ 

for all $Y \in L$ with $|Y| \leq 1$, where the $N_i$ are the normal vectors to the $\pi_i$. The latter is justified using either Allard regularity or the branched sheeting theorem of Wickramasekara, quoted in Theorem A.4, depending on the multiplicity of the ray.

Let $0 < \tau < 1$ be chosen small enough in terms of $\sigma$ that the tubular neighbourhoods $\{\{tp_i \mid \sigma/2 < t < \sigma\}\}$ are two-by-two disjoint in $L^\perp$. Next taking $j \geq J(\tau, \sigma)$ large enough that $\text{dist}_H(\cup_{k=1}^{M} \gamma_{j,Y}^{k}, \cup_{i=1}^{N} \{Y + ti p_i \mid \sigma/2 < t_i < \sigma\}) < \tau$ we ensure that there lie precisely $m_i$ Jordan arcs near every line segment $\{tp_i \mid \sigma/2 < t < \sigma\}$.

By [KW20]—whose results we quote in Theorem 2.4—the branch set of $G_j$ is countably $(n - 2)$-rectifiable for all $j$. Let $P_L$ be the orthogonal projection onto $L$. Then the projection of $B_{G_j} \cap \{|Y| \leq 1, |Z| \leq \sigma\}$ is a compact subset of $L$ with

(8.1) \hspace{1cm} \mathcal{H}^{n-1}(P_L(B_{G_j} \cap \{|Y| \leq 1, |Z| \leq \sigma\})) = 0.

Together with Sard’s theorem, we find that there is an open subset $U_{j,\sigma} \subset L \cap \{|Y| \leq 1\}$ with full $\mathcal{H}^{n-1}$-measure of points $Y \in U_{j,\sigma}$ we call unbranched for which

$$G_j \cap \{Y + Z \mid |Z| < \sigma\} = \cup_{k=1}^{P} \gamma_{j,Y}^{k} \cup \cup_{i=1}^{Q} \Delta_{j,Y}^{i},$$
where the \( Y^k_{j,Y} \) are smooth properly embedded Jordan arcs with endpoints in \( \{Y + Z \mid |Z| = \sigma\} \) and the \( \Delta^k_{j,Y} \) are smooth properly embedded Jordan curves. Given a small \( \kappa > 0 \), we also single out those \( Y \in U_{j,\sigma} \) for which
\[ \int_{G_j \cap \{Z + Y \mid |Z| < \sigma\}} |A_{G_j}| \, d\mathcal{H}^1 < \kappa \]; these form a set denoted \( U_{j,\sigma}(\kappa) \). The stability of the \( G_j \) forbids curvature concentration near the axis \( L \), from whence one obtains the following; this is a point made in [SS81]. We use a similar result at a later stage, Lemma 9.1, valid under slightly different hypotheses.

There we also give a detailed argument, whose steps are easily transcribed to the present context.

**Lemma 8.2.** For all \( \kappa > 0 \), \( \mathcal{H}^{n-1}(U_{j,\sigma} \setminus U_{j,\sigma}(\kappa)) \to 0 \) as \( \sigma \to 0 \) and \( j \geq J(\sigma) \to \infty \).

This is the key ingredient in the proof of Theorem 8.1, which we give now.

**Proof.** We list four ways in which the cones \( P \) and \( T \) can fail to conform to the conclusion of the theorem, each assuming the negation of the preceding:

1. \( P \) is not supported in a union of planes,
2. \( P \) is not a sum of planes,
3. \( T \) is not supported in a union of planes,
4. \( T \) is not a sum of planes.

In each of the four cases we prove the existence of constants \( \varphi_1, \varphi_2, \varphi_3 > 0 \) so that \( U_{j,\sigma}(\varphi_i) \) is empty, independently of \( \sigma \) and \( j \geq J(\sigma) \). Specifically we show that for all \( Y \in U_{j,\sigma} \) there is a curve \( \Upsilon \in \{Y^1_{j,Y}, \ldots, Y^P_{j,Y}\} \) along which the unit normal \( \nu_j \) varies by a positive amount, say \( 2\varphi_1 \leq \sup_{X,Y \in \Upsilon} |\nu_j(X) - \nu_j(Y)| \). Next let \( \varphi \) be the minimum of \( \varphi_1, \varphi_2, \varphi_3 \). Identifying \( \Upsilon \) with a smooth parametrisation on \( [0, 1] \), there are \( 0 < s_j < t_j < 1 \) so that \( \varphi \leq |\nu_j(\Upsilon(s_j)) - \nu_j(\Upsilon(t_j))| \).

Then
\[
\varphi \leq \int_{s_j}^{t_j} |(\nu_j \circ \Upsilon)'(t)| \, dt = \int_{s_j}^{t_j} |(\nabla G_j \nu_j(\Upsilon(t)), (\Upsilon'(t))| \, dt
\leq \int_{s_j}^{t_j} |A_{G_j}(\Upsilon(t))| \, dt \leq \int_{\text{reg} G_j \cap \{Y + Z \mid Z \in L, |Z| < \sigma\}} |A_{G_j}| \, d\mathcal{H}^1.
\]

Hence \( Y \in U_{j,\sigma} \setminus U_{j,\sigma}(\varphi) \), and as this was chosen arbitrarily we find that \( U_{j,\sigma}(\varphi) \) is empty. We may then let \( \sigma \to 0 \) and \( j \geq J(\sigma) \to \infty \) to obtain a contradiction with Lemma 8.2.

1. If the cone \( P \) is not supported in a union of planes, then we can relabel the rays to arrange \( -p_1 \in \{p_1, \ldots, p_M\} \). Let \( \Upsilon \in \{\Upsilon^1_{j,Y}, \ldots, \Upsilon^P_{j,Y}\} \) be an arc with at least one endpoint near the ray \( R_1 \). If its other endpoint lies near another ray \( R_i \), then picking a point \( X \) near one endpoint and \( Y \) near another we find \( 2\varphi_1 \leq \sup_{X,Y \in \Upsilon} |\nu_j(X) - \nu_j(Y)| \), where \( \varphi_1 \) depends only on the angle between the two rays. Similarly if both endpoints lie near \( R_0 \) then \( 1 \leq \sup_{X,Y \in \Upsilon} |\nu_j(X) - \nu_j(Y)| \).

2. Thus we may assume that \( \text{spt} \|P\| = \bigcup_{i=1}^P \Pi_i \), and we can relabel the half-planes so that \( \Pi_i = \pi_i \cup \pi_{i+D} \) for all \( i \), taking the indices module \( 2D = \)
The only way for $P$ not to be a sum of these planes (with some multiplicities) is if $m_i \neq m_{i+D}$ for some $i$, say $m_1 \neq m_{D+1}$ after relabelling. A pigeonhole argument demonstrates the existence of an arc $\Upsilon \in \{\Upsilon^1_{1,Y}, \ldots, \Upsilon^P_{1,Y}\}$ satisfying one of the following. Either $\Upsilon$ has both endpoints near $R_1$ or $R_{D+1}$, or it has one endpoint one of $R_1, R_{D+1}$ and the other near a third ray $R_i$. Combining the two cases we find $\min\{1, 2, 3\} \leq \sup_{X,Y} |\nu_j(X) - \nu_j(Y)|$, where $\varphi_2$ depends only on the respective angles that $R_i$ forms with $R_1, R_{D+1}$.

(3) Hence we may take $P = \sum_{i=1}^{D} m_i[\Pi_i]$, and if $|T| = \sum_{i=1}^{D} (l_i[\pi_i] + l_{i+D}[\pi_{i+D}])$ is not a sum of planes then $l_i \neq l_{i+D}$. Relabelling the half-planes once again if necessary, we may assume that $l_1 = 0$ and $l_{D+1} = 2$, inequivalent cases being excluded by Allard regularity. Then $m_1 = 2 = m_{D+1}$, and for large enough $j$ we can argue as above to prove the existence of two arcs $\Upsilon_-, \Upsilon_+ \in \{\Upsilon^1_{j,Y}, \ldots, \Upsilon^P_{j,Y}\}$ lying near the rays $R_1, R_{D+1}$, lest $\nu_j$ vary by a positive amount along the arcs containing the portions of $G_j$ near them. We label these so that $\pm(\nu_j, N_1)$ is positive on $\Upsilon_+ \sim R_1$. This therefore changes sign along $\Upsilon_-$, and $1 \leq \sup_{X,Y} |\nu_j(X) - \nu_j(Y)|$.

(4) For the last step we may assume that $T = \sum_{i=1}^{D} l_i([\pi_i] + [\pi_{i+D}])$ and it remains to show that the pairs $\pi_i, \pi_{i+D}$ are oriented in a consistent fashion. Were this to fail for $\pi_1, \pi_{D+1}$ for example then we could once again find an arc $\Upsilon \in \{\Upsilon^1_{j,Y}, \ldots, \Upsilon^P_{j,Y}\}$ with either one endpoint near one of $R_1, R_{D+1}$ and the other near a third ray, or both endpoints near $R_1, R_{D+1}$ but along which $\langle \nu_j, N_1 \rangle$ changes sign. This concludes the proof. \hfill $\Box$

9. Classical limit cones: non-vertical cones

Let $\alpha \in (0, 1)$ and $(u_j \mid j \in \mathbb{N})$ be a sequence of two-valued graphs with $u_j \in \text{C}^1(\mathbb{R}^2; \mathcal{A}_2)$. Let $D \in \mathbb{Z}_{>0}$ and for $i = 1, \ldots, D$ let $\Pi_i \in \text{Gr}(n, n+1)$ be $n$-dimensional planes which meet along a common axis $L \in \text{Gr}(n-1, n+1)$. Recall that we call a plane $\Pi$ vertical if it is of the form $\Pi = \Pi_0 \times \mathbb{R}e_{n+1}$. Here we consider the situation where they are not all vertical, and we may relabel them to arrange for $\Pi_1$ to be non-vertical. Assume that $|G_j| \to \sum_{i=1}^{D} m_i[\Pi_i] = P$ and $[G_j] \to \sum_{i=1}^{D} l_i[\Pi_i] = T$ as $j \to \infty$, where $0 \leq l_i \leq m_i$.

9.1. Slicing at an acute angle. Broadly speaking we again use an approach based on [SS81]. However from a technical standpoint the arguments from the previous section are maladapted to the current situation, as we want to exploit the two-valued graphicality of the $G_j$. Instead of taking slices orthogonal to the axis $L$ of the cone, we proceed as follows. Let $v \in \mathbb{R}^{n+1}$ be a unit vector with $\langle v, e_{n+1} \rangle = 0$, so that $v, e_{n+1}, L$ span $\mathbb{R}^{n+1}$. Let $V = \text{span}\{v, e_{n+1}\}$, and write $Z = tv + ze_{n+1} \in V$. Then $\mathbb{R}^{n+1} = L + V$, and every point $X \in \mathbb{R}^{n+1}$ can uniquely be written $X = Y + Z = y + Y^{n+1}e_{n+1} + tv + ze_{n+1}$. We emphasise that in general $L$ and $V$ do not meet at a right angle, and this decomposition is not orthogonal. (The exception being the case where $L = \mathbb{R}^n \times \{0\}$.) The slices we take are adapted
Analogous to (8) we have
\[ H^0(Y + Z, |Z| < \sigma) = \{ |y| < 1, |Z| < \sigma \} = \{ Y + Z | Y \in L, Z \in V, |y| < 1, |Z| < \sigma \} \]

The corresponding projection map is written \( Q : X = Y + Z \in \mathbb{R}^{n+1} \mapsto Y \).

To this decomposition, using sets of the form \( \{ |y| < 1, |Z| < \sigma \} \), and in the same vein write for example \( \{ |y| < 1, |Z| < \sigma \} = \{ Y + Z | Y \in L, Z \in V, |y| < 1, |Z| < \sigma \} \).

The corresponding projection map is written \( Q : X = Y + Z \in \mathbb{R}^{n+1} \mapsto Y \).

Analogous to (8) we have \( H^{n-1}(Q(B(G_j) \cap \{|y| \leq 1, |Z| \leq 1\})) = 0 \). Let \( \sigma > 0 \) be given. Together with Sard’s theorem we find that for all \( j \) there is an open subset \( V_{j,\sigma} \subset L \cap \{|y| \leq 1\} \setminus Q(B(G_j) \cap \{|y| \leq 1, |Z| \leq 1\}) \) of full measure, so that for all \( Y \in V_{j,\sigma}, G_j \cap \{ Z + Y | |Z| < \sigma \} = \cup_k \mathcal{Y}_{j,Y}^k \) where the \( \mathcal{Y}_{j,Y}^k \) are smooth properly embedded Jordan arcs with endpoints in the set \( \{ Y + Z | |Z| = \sigma \} \). (This cannot contain any closed curves because of the graphicality of \( G_j \).

In the same vein, given \( \kappa > 0 \) we write \( V_{j,\sigma}(\kappa) \subset V_{j,\sigma} \) for the measurable subset of \( V_{j,\sigma} \) formed by those points \( Y \in V_{j,\sigma} \) with \( \int_{\text{reg} G_j \cap \{ Z + Y | |Z| < \sigma \}} |A_{G_j}| \, dH^1 < \kappa \).

**Lemma 9.1.** For all \( \kappa > 0 \), \( H^{n-1}(V_{j,\sigma} \setminus V_{j,\sigma}(\kappa)) \to 0 \) as \( \sigma \to 0 \) and \( j \geq J(\sigma) \to \infty \).

**Proof.** By definition of \( V_{j,\sigma}(\kappa) \), we can integrate over points \( Z \in V_{j,\sigma} \setminus V_{j,\sigma}(\kappa) \) to obtain the inequality
\[
H^{n-1}(V_{j,\sigma} \setminus V_{j,\sigma}(\kappa)) \leq \int_{V_{j,\sigma}(\kappa)} \left\{ \int_{\text{reg} G_j \cap \{ Z + Y | |Z| < \sigma \}} |A_{G_j}| \, dH^1 \right\} \, dH^{n-1}(Y).
\]

On the right-hand side we may increase the domain of integration to now be over the set \( \{ |Y| < 1, |Z| < \sigma \} \). By the Cauchy–Schwarz inequality, this larger integral is bounded like
\[
\int_{\text{reg} G_j \cap \{ |Y| < 1, |Z| < \sigma \}} |A_{G_j}| \, dH^n \leq H^n(\text{reg} G_j \cap \{ |Y| < 1, |Z| < \sigma \})^{1/2} \left( \int_{\text{reg} G_j \cap \{ |Y| < 1, |Z| < \sigma \}} |A_{G_j}|^2 \, dH^n \right)^{1/2}
\]

The integral on the right-hand side can be bounded by the stability inequality. Let \( \phi \in C^1_c(\mathbb{R}^{n+1}) \) be a test function with \( 0 \leq \phi \leq 1, \phi \equiv 1 \) on \( \{ |Y| < \sqrt{7}/2, |Z| < 1/4 \} \), \( \phi \equiv 0 \) outside \( \{ |Y| < \sqrt{7}/2, |Z| < 1/2 \} \) and \( |D\phi| \leq 8 \). Taking \( \sigma \in (0, 1/4) \) and \( j \geq J(\sigma) \) we find that
\[
\int_{\text{reg} G_j \cap \{ |Y| < 1, |Z| < \sigma \}} |A_{G_j}|^2 \, dH^n \leq \int_{\text{reg} G_j \cap \{ |Y| < (\sqrt{7} - 1)/2, |Z| < 1/4 \}} |A_{G_j}|^2 \phi^2 \, dH^n
\]
\[
\leq \int_{\text{reg} G_j \cap \{ |Y| < \sqrt{7}/2, |Z| < 1/2 \}} |\nabla G_j \phi|^2 \, dH^n
\]
\[
\leq 64H^n(\text{reg} G_j \cap \{ |Y| < \sqrt{7}/2, |Z| < 1/2 \})
\]
\[
\leq 128\|P\|\{ |Y| < \sqrt{7}/2, |Z| < 1/2 \}.
\]
the last inequality being guaranteed to hold for large \( j \). From this we retain only that there is a constant \( B > 0 \) independent of \( \sigma \) so that

\[
\int_{\text{reg} G_j \cap \{ |Y| < 1, |Z| < \sigma \}} |A_{G_j}|^2 \, d\mathcal{H}^n \leq B
\]

for large enough \( j \geq J(\sigma) \). Increasing the value of this constant if necessary, we additionally find \( \mathcal{H}^n(\text{reg} G_j \cap \{ |Y| < 1, |Z| < \sigma \}) \leq B\sigma \), and combining the two with (9.1) and (9.1) yields

\[
\mathcal{H}^{n-1}(V_{j,\sigma} \setminus V_{j,\sigma}(\kappa)) \kappa \leq \int_{\text{reg} G_j \cap \{ |Y| < 1, |Z| < \sigma \}} |A_{G_j}| \, d\mathcal{H}^n \leq B\sigma^{1/2},
\]

at least for \( j \geq J(\sigma) \) large enough. To force \( \mathcal{H}^{n-1}(V_{j,\sigma} \setminus V_{j,\sigma}(\kappa)) \to 0 \), it suffices to let \( \sigma \to 0 \) and \( j \geq J(\sigma) \to \infty \). \( \square \)

9.2. Initial reduction. If at least one of the planes in \( \text{spt}||P|| \) is non-vertical, then the axis \( L \) along which the planes meet cannot be vertical either. However, any vertical \( \Pi \in \{ \Pi_1, \ldots, \Pi_D \} \) must contain \( L \), and thus be of the form \( \Pi = \text{span}\{ L, e_{n+1} \} \). This uniquely determines the plane, and thus at most one plane in the support of \( P \) is vertical.

**Lemma 9.2.** Suppose \( G_j \to \sum_i m_i |\Pi_i| \), and that \( \Pi_1 \) is not vertical.

(i) If \( \Pi_2 \) is vertical then \( P = |\Pi_1| + m_2 |\Pi_2| \).

(ii) If \( \Pi_2 \) is not vertical then \( P = |\Pi_1| + |\Pi_2| \).

**Proof.** We make a preliminary observation: arguing as in the proof of Theorem 8.1 one proves the existence of a constant \( \varphi > 0 \) so that for all \( 0 < \tau < \sigma \) and \( j \geq J(\tau, \sigma) \) the following is true. If \( Y \in V_{j,\sigma}(\varphi) \) then for every \( \Upsilon \in \{ \Upsilon^1_{j,Y}, \ldots, \Upsilon^P_{j,Y} \} \) there is a plane \( \Pi \in \{ \Pi_1, \ldots, \Pi_D \} \) so that \( \Upsilon \subset (\Pi)_\tau \).

Using this, we first show that

\[
(9.3) \quad P = m_1 |\Pi_1| + m_2 |\Pi_2|
\]

regardless of whether or not \( \Pi_2 \) is vertical, arguing by contradiction. Suppose that there are at least three distinct planes \( \Pi_1, \Pi_2, \Pi_3 \subset \text{spt}||P|| \). Let the constant \( \varphi > 0 \) be as above, let \( \tau > 0 \) be small and take \( j \geq J(\tau, \sigma) \) large enough that \( \mathcal{H}^{n-1}(V_{j,\sigma}(\varphi)) > 0 \) as per Lemma 9.1. We may thus take any point \( Y \in V_{j,\sigma}(\varphi) \) and decompose \( G_j \cap \{ Z + Y | |Z| < \sigma \} \) as above. Next let \( \Upsilon_1, \Upsilon_2, \Upsilon_3 \in \{ \Upsilon^1_{j,Y}, \ldots, \Upsilon^P_{j,Y} \} \) be three of the curves, lying respectively near \( \Pi_1, \Pi_2, \Pi_3 \), in the sense that \( \Upsilon_i \subset (\Pi_i)_\tau \). Possibly after extending \( \Upsilon_1 \) slightly beyond its endpoints, we have that \( \{ tv | |t| < \sigma \} \subset P_0(\Upsilon_1) \). Now if \( \tau > 0 \) is small enough in terms of \( \sigma > 0 \) then \( \Upsilon_2 \) and \( \Upsilon_3 \) intersect in at least one point, say \( Y + Z_0 = Y + t_0 v + z_0 e_{n+1} \) with \( |Z_0| < \sigma \). This is absurd because the density of \( ||G_j|| \) at such a point is at least two, and hence writing
This is impossible for a two-valued graph, which proves (9.2).

Next we prove (i). First pick two arcs \( \Upsilon_1, \Upsilon_2 \in \{ \Upsilon^1_{j,Y}, \ldots, \Upsilon^P_{j,Y} \} \) so that \( \Upsilon_1, \Upsilon_2 \subset (\Pi_1)_r \). Arguing as above, we may extend the two arcs slightly beyond their respective endpoints and get that \( \{ y + tv \mid |t| < \sigma \} \subset P_0(\Upsilon_1) \cap P_0(\Upsilon_2) \), where \( Y = (y, Y^{n+1}) \). If there were another curve \( \Upsilon_3 \) say in that slice, then at any point \( Z_0 + Y = tv_0 + z_0 \epsilon_{n+1} + Y \in \Upsilon_3 \setminus (\Upsilon_1 \cup \Upsilon_2) \) we would obtain a contradiction, as

\[
\sum_{X \in P_0^{-1}(\{y+tv_0\})} \Theta(||G_j||, X) \\
\quad \quad \quad \geq \sum_{X \in P_0^{-1}(\{y+tv_0\}) \cap (\Upsilon_1 \cup \Upsilon_2)} \Theta(||G_j||, X) + \Theta(||G_j||, Z_0 + Y) \geq 3.
\]

It remains to prove (ii), where we do not need the decomposition of \( G_j \) in the slice. Pick any point \( x \in D_1 \setminus P_0(L) \). Then there exist two distinct points \( X_1, X_2 \in \Pi_1, \Pi_2 \) respectively so that \( P_0(X_1) = x = P_0(X_2) \). As both planes have multiplicity two at most, we can apply the regularity theory of Wickramasekera if either of the two planes has multiplicity two, and Allard regularity otherwise, to guarantee that when \( j \) is large enough, then

\[
\sum_{X \in P_0^{-1}(\{x\})} \Theta(||G_j||, X) \geq m_1 + m_2 \geq 2
\]

where \( m_1, m_2 \) are the respective multiplicities of \( \Pi_1, \Pi_2 \). Anything but equality in (9.2) would be absurd, whence \( m_1 = 1 = m_2 \). \( \square \)

### 9.3. Horizontal multiplicity one.

We introduce some useful additional notation. As the arguments only simplify when \( L \subset \mathbb{R}^n \times \{0\} \), we assume throughout that this is not the case. For an arbitrary point \( X \in \mathbb{R}^{n+1} \) we write \( X = Y + sf + ze_{n+1} \) where \( Y \in L \) and \( s, z \in \mathbb{R} \). Sometimes it is also convenient to write \( Y = (y, Y^{n+1}) \) where \( y = P_0(Y) \).

We define open domains \( Q, Q_\tau \subset \mathbb{R}^n \times \{0\} \) by \( Q = \{ y + sf \mid |y| < 1, s^2 < 1 \} \) and \( Q_\tau = Q \cap (\Pi_2)_r = \{ y + sf \mid |y| < 1, s^2 < \tau^2 \} \). A point \( X = Y + sf + ze_{n+1} \in Q \times \mathbb{R} \) is said to lie north of \( \Pi_1 \) if \( z > 1 \) and south of \( \Pi_1 \) if \( z < -1 \). In the same vein we say that a set \( E \subset Q \times \mathbb{R} \) lies north (resp. south) of \( \Pi_1 \).

**Lemma 9.3.** Let \( \Pi_1, \Pi_2 \in Gr(n, n+1) \) be so that \( \Pi_1 \) is not vertical, but \( \Pi_2 \) is. Suppose \( |G_j| \to ||\Pi_1|| + 2||\Pi_2|| \). For all \( \tau > 0 \) there is \( J(\tau) \in \mathbb{N} \) so that for all \( j \geq J(\tau) \), sing \( G_j \cap Q \times \mathbb{R} \subset Q_\tau \times \mathbb{R} \) and we can decompose
Corollary 9.4. Let the two planes $\Pi_1, \Pi_2 \in Gr(n, n+1)$ be so that $\Pi_1$ is not vertical, but $\Pi_2$ is. Then $|G_j| \to |\Pi_1| + 2|\Pi_2|$ is impossible.

Proof. Fix a small value for $\tau > 0$, depending only on $\Pi_1, \Pi_2$ and a corresponding $J(\tau) \in \mathbb{N}$ so that without loss of generality, $G_j \cap Q \times R \cap \{z < -1\}$ is a non-empty subset of $Q_\tau \times R \cap \{z < -1\}$ and $\text{sing } G_j \cap Q_\tau \times R \cap \{z < -1\} = \emptyset$.

Then $G_j \backslash Q \times R \cap \{z < -1\}$ is equal to the graph of a single-valued, smooth function $u_{j, s}$ defined on some subset $\Omega_{j, s} \subset Q_\tau$. From this we only retain that the current $\|G_j\| \backslash Q \times R \cap \{z < -1\}$ is area-minimising. As $j \to \infty$ we get that $\|G_j\| \backslash Q \times R \cap \{z < -1\} \to 2|\Pi_2| \backslash Q \times R \cap \{z < -1\}$ in the varifold topology. At the same time, by inspection one finds that in the current topology $\|G_j\| \backslash Q \times R \cap \{z < -1\} \to 0$ as $j \to \infty$. This mass cancellation is absurd in light of the well-known compactness for area-minimising currents.

Before we give a proof of the lemma, we use its conclusions to derive the following corollary.

Corollary 9.4. Let the two planes $\Pi_1, \Pi_2 \in Gr(n, n+1)$ be so that $\Pi_1$ is not vertical, but $\Pi_2$ is. Then $|G_j| \to |\Pi_1| + 2|\Pi_2|$ is impossible.

Proof. Fix a small value for $\tau > 0$, depending only on $\Pi_1, \Pi_2$ and a corresponding $J(\tau) \in \mathbb{N}$ so that without loss of generality, $G_j \cap Q \times R \cap \{z < -1\}$ is a non-empty subset of $Q_\tau \times R \cap \{z < -1\}$ and $\text{sing } G_j \cap Q_\tau \times R \cap \{z < -1\} = \emptyset$.

Then $G_j \backslash Q \times R \cap \{z < -1\}$ is equal to the graph of a single-valued, smooth function $u_{j, s}$ defined on some subset $\Omega_{j, s} \subset Q_\tau$. From this we only retain that the current $\|G_j\| \backslash Q \times R \cap \{z < -1\}$ is area-minimising. As $j \to \infty$ we get that $\|G_j\| \backslash Q \times R \cap \{z < -1\} \to 2|\Pi_2| \backslash Q \times R \cap \{z < -1\}$ in the varifold topology. At the same time, by inspection one finds that in the current topology $\|G_j\| \backslash Q \times R \cap \{z < -1\} \to 0$ as $j \to \infty$. This mass cancellation is absurd in light of the well-known compactness for area-minimising currents. 

Now for the proof of Lemma 9.3.

Proof. Note first that every singular point $X = Y + sf + ze_{n+1}$ in $\text{sing } G_j \cap Q \times R$ automatically belongs to $Q_\tau \times R$, that is has $s^2 < \tau^2$. Indeed the Allard regularity theorem can be applied near $\Pi_1$ because it has multiplicity one in the limit, which guarantees that away from $L$ the $G_j$ converge to the plane like smooth single-valued graphs. Counting the pre-images of points $x \in Q \backslash \overline{Q}_\tau$, we find that $\text{sing } u_j \cap Q \backslash \overline{Q}_\tau = \emptyset$, or equivalently $\text{sing } G_j \cap Q \times R \subset \overline{Q}_\tau \times R$.

Although $Q \backslash \overline{Q}_\tau$ is not simply connected, its two connected components, which lie on either side of $\Pi_2$, both are. We may thus make a smooth selection $u_{j, \tau}, u_{j, \tau} \in C^\infty(Q \backslash \overline{Q}_\tau)$ for $u_j$, arranging for the graph of $u_{j, \tau}$ to lie near $\Pi_1$. Both graphs are disconnected, and we write graph $u_{j, \tau} = \Sigma_{j, -} \cup \Sigma_{j, +}$ and graph $u_{j, \tau} = \Sigma_{j, -} \cup \Sigma_{j, +}$. As the graphs $G_j$ locally converge to $\Pi_1 \cup \Pi_2$ in the Hausdorff distance, we may take an even larger $j \geq J(\tau)$ to get

$$G_j \cap Q \times R \cap \{z^2 \leq 1\} \subset (\Pi_1 \cup \Pi_2)_\tau.$$

Thus $\Sigma_{j, \pm} \subset \{z^2 < (1 - (e, e_{n+1})^2)^{-1}\tau^2\}$ and

$$\Sigma_{j, \pm} \subset \{z^2 > 1\} = \{z > 1\} \cup \{z < -1\}.$$
We show that in fact either $\Sigma_{j,\pm}^2 \subset \{z > 1\}$ or $\Sigma_{j,\pm}^2 \subset \{z < -1\}$. Recall from our initial analysis that using Sard’s theorem one finds an open subset of ‘unbranched’ points $U_j \subset L \cap \{|y| < 1\}$ with $\mathcal{H}^{n-1}(L \cap \{|y| < 1\} \setminus U_j) = 0$ so that for all $Y \in U_j$,

$$G_j \cap \{Y + sf + ze_{n+1} | s^2 < 1, z^2 < 1\} \cap B_{G_j} = \emptyset,$$

and in fact can be decomposed into a union of three, smooth properly embedded Jordan arcs $\gamma_{j,Y}^1, \gamma_{j,Y}^2, \gamma_{j,Y}^3$ with endpoints in $\{Y + sf + ze_{n+1} | s^2 = 1\}$ or $z^2 = 1$).

Given $\kappa > 0$ we define the subset $U_j(\kappa) \subset U_j$ by

$$U_j(\kappa) = \{Y \in U_j \mid \int_{\text{reg} G_j \cap \{Y + sf + ze_{n+1} | s^2, z^2 < \tau^2\}} |A_{G_j}| \, d\mathcal{H}^n < \kappa\}.$$

Arguing as in Lemma 9.1 we can show that here we can take $\tau > 0$ small enough and $j \geq J(\tau)$ to make $\mathcal{H}^{n-1}(U_j \setminus U_j(\kappa))$ as small as we like. For our purposes we may take for example $\kappa = 1/2$, $\tau > 0$ small and $j \geq J(\tau)$ large enough that $\mathcal{H}^{n-1}(U_j \setminus U_j(\kappa)) < \omega_{n-1}$, as then automatically $U_j(\kappa) \neq \emptyset$. If we then take a point $Y \in U_j(\kappa)$, then we may relabel the curves so that $\gamma_{j,Y}^1 \subset (\Pi_1)_\tau$, and $\gamma_{j,Y}^2 \cup \gamma_{j,Y}^3 \subset (\Pi_2)_\tau$.

Consider $Y = (y, Y^{n+1}) \in U_j$ and let $l_y = \{y + sf | s^2 < 1\} \subset \mathbb{R}^n \times \{0\}$. On this line segment we can make a smooth selection $u_{j,2}^1, u_{j,2}^2 \in \mathcal{C}^\infty(-1,1)$, where we identify $l_y$ with $(-1,1) \subset \mathbb{R}$. Then $G_j \cap \{Y + sf + ze_{n+1} | s^2 < 1, z^2 < 1\} \subset \text{graph } u_{j,2}^1 \cup \text{graph } u_{j,2}^2$. We may furthermore make our selection in such a way that $\gamma_{j,Y}^1 \subset \text{graph } u_{j,2}^1 \subset (\Pi_1)_\tau$ and $\gamma_{j,Y}^2 \cup \gamma_{j,Y}^3 \subset \text{graph } u_{j,2}^2$. In what follows we also write $l_y(a,b) = \{y + sf | a < s < b\}$, where $-1 < a < b < 1$. The graph of $u_{j,2}^2$ restricted to the short segment $l_y(-2\tau, 2\tau)$ is a single smooth curve, which by inspection has both its endpoints lying on the same side of $\Pi_1$ that is either both lie north or both lie south. But graph $u_{j,2}^2 \cap l_y(-2\tau, -\tau) \times \mathbb{R} \subset \Sigma_{j,-}^2$ and likewise graph $u_{j,2}^2 \cap l_y(\tau, 2\tau) \times \mathbb{R} \subset \Sigma_{j,+}^2$ whence we find that $\Sigma_{j,-}^2$ and $\Sigma_{j,+}^2$ must lie on the same side of $\Pi_1$—that is either both lie north or both lie south of $\Pi_1$. This concludes the proof of (9.3).

To prove (9.3) we start by making a few general observations. First, by (9.3) we may assume without loss of generality that $\Sigma_{j,\pm} \subset \{z > 1\}$. Let $Y = (y, Y^{n+1}) \in L$ be an arbitrary point with $|y| < 1$, not necessarily in $U_j$. By the two-valued sheeting theorem of [Wic20] applied in the region $Q \times \mathbb{R} \cap \{\tau^2 < s^2 \lor z^2 < 1\}$, we find that in this slice the graph can be decomposed into six differentiable curves,

$$G_j \cap \{Y + sf + ze_{n+1} | \tau^2 < s^2 \lor z^2 < 1\} = \bigcup_{k=1}^2 \gamma_{j,Y,k}^1 \cup \bigcup_{l=1}^4 \gamma_{j,Y,l}^2,$$

where $\gamma_{j,Y,k}^1 \subset (\Pi_1)_\tau$ and $\gamma_{j,Y,l}^2 \subset (\Pi_2)_\tau$. (Were the slice $\{Y + sf + ze_{n+1} | \tau^2 < s^2 \lor z^2 < 1\}$ to contain a branch point of $\mathcal{B}(G_j)$, then some arbitrary choices would have to be made in this decomposition, with no impact on the argument.) These curves taken together have two endpoints $\{z = -1, s^2 <
\(\tau^2\}, counted with multiplicity. Hence \(\#G_j \cap \{Y + sf - e_{n+1} \mid s^2 < \tau^2\} \leq 2\) and
\[
\sum_{s^2 < \tau^2} \Theta(\|G_j\|, Y + sf - e_{n+1}) = 2.
\]

There exist two functions \(u_{j,y}^1, u_{j,y}^2 \in C^1(-1,1)\) so that
\[
G_j \cap \{Y + sf + z e_{n+1} \mid s^2 < 1, z \in \mathbb{R}\} = \text{graph } u_{j,y}^1 \cup \text{graph } u_{j,y}^2.
\]
Moreover their graphs \(\text{graph } u_{j,y}^i\) are two differentiable curves which do not meet the region south of \(\Pi_1\), except in the thin strip near \(\Pi_2\) where \(s^2 < \tau^2\).

In other words, \((\text{graph } u_{j,y}^1 \cup \text{graph } u_{j,y}^2) \cap \{s^2 \geq \tau^2, z < -1\} = \emptyset\).

Notice that \(\text{sing } G_j \cap \{Y + sf + z e_{n+1} \mid s^2 < 1, z \in \mathbb{R}\} = \text{graph } u_{j,y}^1 \cap \text{graph } u_{j,y}^2\). Hence, if \(\{Y + sf + z e_{n+1} \mid s^2 < 1, z < -1\}\) contained a singular point of \(G_j\), then \(\text{graph } u_{j,y}^1\) and \(\text{graph } u_{j,y}^2\) would both contain portions lying in that region. This is impossible, because if both curves intersected that region they would both need to pass through the set \(\{Y + s^2 - e_{n+1} \mid s^2 < \tau^2\}\) twice each, because neither meets the set \(\{s^2 \geq \tau^2, z < -1\}\). This in turn would be a contradiction to the decomposition of the graph as we obtained in (9.3) above, and (9.3) in particular. As the point \(Y\) was chosen arbitrarily, we have \(\text{sing } G_j \cap \{s^2 < 1, z < -1\} = \emptyset\), which concludes the proof. \(\square\)

10. CLASSICAL LIMIT CONES: VERTICAL CONES

Let \(\alpha \in (0,1)\) and \(\{u_j \mid j \in \mathbb{N}\}\) be a sequence of two-valued minimal graphs \(u_j \in C^{1,\alpha}(D_2;A_2)\). Let \(D \in \mathbb{Z}_{>0}\) and for \(i = 1, \ldots, D\), let \(\Pi_i = \Pi_i^0 \times \mathbb{R} e_{n+1}\) be vertical planes which meet along a common \((n-1)\)-dimensional axis \(L = L_0 \times \mathbb{R} e_{n+1}\), and suppose that as \(j \rightarrow \infty\), \(|G_j| \rightarrow \sum_i m_i |\Pi_i|\) and \(|\tau_j| \rightarrow \sum_i l_i |\Pi_i|\).

Here \(0 \leq l_i \leq m_i \leq 2\) are integers and we pick arbitrary orientations for those planes with \(l_i = 0\). When this notation is convenient we write \(P = \sum_i m_i |\Pi_i|\) and \(T = \sum_i l_i |\Pi_i|\).

To make statements less awkward, we allow the possibility that \(P\) and \(T\) are supported in a single plane, although technically these would not be called classical cones.

Label the planes \(\Pi_1, \ldots, \Pi_D\) so that they lie in counterclockwise order around \(L\). From now on we consider their indices modulo \(D\), and write \(\Pi_i = \pi_i \cup \pi_{i+D}\), where \(\pi_i, \pi_{i+D}\) are two half-planes which meet along \(L\). The indices of the half-planes are considered modulo \(2D\). For every \(\pi_i\) let \(N_i\) be its unit normal pointing in the counterclockwise direction; note that \(N_i = -N_{i+D}\). Write \(n_i\) for the unit normal induced on \(\Pi_i\) as limits of the \([G_j]\), and let \(s_i = \langle n_i, N_i\rangle\) equal to \(\pm 1\) depending on whether or not \(n_i\) agrees with the counterclockwise orientation. With this notation, \(s_i = -s_{i+D}\).

We say that two half-planes \(\pi_i, \pi_j\) are oriented in the same direction if they are both oriented in the clockwise or counterclockwise direction, or equivalently if \(s_i = s_j\).
Let \( Q = \{ x \in \mathbb{R}^n \mid \text{dist}(x, L) < 1, \text{dist}(x, L^\perp) < 1 \} \). Extend the functions \( F^j \) from (7.3) to \( Q \) using the same formula, counting the number of points in \( G_j \) lying below \( x \) with multiplicity. These functions are eventually constant away from \( \bigcup_i \Pi_i \), that is given \( \tau > 0 \) we can take \( j \geq J(\tau) \) large enough that the \( F^j \) are constant on every connected component of \( Q \setminus (\bigcup_i \Pi_i)_\tau \). Write \( Q \setminus \bigcup_i \Pi_i \) as a disjoint union of wedge-shaped connected components \( V_1, \ldots, V_{2D} \). Each \( V_i \) lies between \( \pi_i, \pi_{i+1} \),

\[
V_i = \{ x \in Q \mid \langle x, N_i \rangle > 0, \langle x, N_{i+1} \rangle < 0 \}.
\]

By the above there is \( F_i = F(V_i) \in \{ 0, 1, 2 \} \) so that \( F^j(x) = F_i \) at all \( x \in V_i \setminus (\pi_i \cup \pi_{i+1})_\tau \) provided \( j \geq J(\tau) \). Although the notation is slightly ambiguous, no confusion should arise between the value \( F_i = F(V_i) \) and the functions \( F^j \).

### 10.1. Results in arbitrary dimensions

Applying Lemma 7.5 in the present context, we can relate consecutive values of \( F_i \).

**Lemma 10.1.** For all \( i \), \( F_i - F_{i-1} = s_i l_i \).

Similarly using Lemma 6.2, we obtain the following result.

**Lemma 10.2.** If \( F_i = 1 \) and \( \pi_i \neq \pi_{i+1} \) then both half-planes have multiplicity one and are oriented in the same direction.

**Proof.** Start with the observation that \( 1 = F_i = F_{i+1} - s_{i+1}l_{i+1} + F_{i-1} + s_i l_i \), so \( l_i, l_{i+1} \in \{ 0, 1 \} \). The possibilities are as follows:

1. \( m_i = 1 = m_{i+1} \) and \( l_i = 1 = l_{i+1} \),
2. \( m_i = 2, m_{i+1} = 1 \) and \( l_i = 0, l_{i+1} = 1 \),
3. \( m_i = 1, m_{i+1} = 2 \) and \( l_i = 1, l_{i+1} = 0 \),
4. \( m_i = 2 = m_{i+1} \) and \( l_i = 0 = l_{i+1} \).

The proof is similar in all four cases. Every case is argued by contradiction, eventually reaching a conclusion that Lemma 6.2 forbids. We give a detailed proof for \( m_i = 1 = m_{i+1} \) and \( l_i = 1 = l_{i+1} \) and explain the necessary modifications for the remaining cases.

Assume without loss of generality that \( \pi_i \) and \( \pi_{i+1} \) both point into \( V_i \), that is \( n_i = N_i \) and \( n_{i+1} = -N_{i+1} \). Let two small \( 0 < \tau < \sigma < 1 \) and a large \( A > 1 \) be given, and define the open subset \( U_{j,i} \subset Q \setminus [L]_\sigma \) by

\[
U_{j,i} = U_{j,i}(\tau, \sigma, A) = (V_i)_\tau \setminus [L]_\sigma \cap \{ u_j^- < 2A \} = \{ x \in Q \setminus [L]_\sigma \mid \langle N_i, x \rangle > -\tau, \langle N_{i+1}, x \rangle < \tau, u_j^-(x) < 2A \},
\]

where recall \( u_j^- = u_j^1 \wedge u_j^2 \).

As \( F_i = 1 \) we know that for large enough \( j \geq J(\tau, \sigma, A) \),

\[
V_i \setminus ([\pi_i]_\tau \cup [\pi_{i+1}]_\tau \cup [L]_\sigma) \subset U_{j,i}
\]
and \( \text{sing} \, G_j \cap U_{j,i} \cap (-\infty, -2A) = \emptyset \). Hence any singular points of \( u_j \) would have to lie in \( U_{j,i} \cap [\pi_i \cup \pi_{i+1}]_\tau \). As \( m_i = 1 = m_{i+1} \) we may use Allard regularity inside \( Q \times (-9/4A, 9/4A) \cap (\pi_i \cup \pi_{i+1})_{2\tau} \) and find that in fact

\[
\text{sing} \, u_j \cap U_{j,i} = \emptyset,
\]

at least provided \( j \geq J(\tau, \sigma, A) \) is large enough.

Let us rename \( u_{j,i,S} = u_{j,i}^\tau \) and \( u_{j,i,N} = u_{j,i}^\sigma \). These two functions give a smooth selection for \( u_j \) on \( \hat{U}_{j,i} \). By construction

\[
G_j \cap (V_i)_\tau \setminus [L]_\sigma \times (-\infty, A) \subset \text{graph} \, u_{j,i,S}.
\]

From this we obtain a contradiction with Lemma 6.2. Indeed as we let \( \sigma, \tau \to 0 \), \( A \to \infty \) and \( j \geq J(\tau, \sigma, A) \to \infty \), we have \( \text{dist}_H(U_{j,i}, V_i) \to 0 \) and

\[
\begin{align*}
\text{[graph} \, u_{j,i,S} \text{]} \to [\pi_i] + [\pi_{i+1}]L_Q \times \mathbb{R}.
\end{align*}
\]

This concludes the proof in the first case. In the next case \( m_i = 2 \), \( m_{i+1} = 1 \) and \( l_i = 0 \), \( l_{i+1} = 1 \). As \( l_i = 0 \) the half-plane \( \pi_i \) has no well-defined orientation induced by \( T \). The half-plane \( \pi_{i+1} \) may be assumed oriented in the clockwise direction without loss of generality, that is \( n_{i+1} = -n_{i+1} \). By Lemmas 7.5 and 7.4 we may take \( j \geq J(\tau, \sigma) \) large enough that

\[
\text{sing} \, u_j \cap (V_i \cup V_{i-1}) \setminus ([\pi_{i-1} \cup \pi_i \cup \pi_{i+1}]_\tau \cup [L]_\sigma) = \emptyset,
\]

\[
B_{u_j} \cap (V_{i-1} \cup V_i \cup \pi_i) \setminus ([\pi_{i-1} \cup \pi_i]_\tau \cup [L]_\sigma) = \emptyset.
\]

Additionally the set \( (V_{i-1} \cup V_i \cup \pi_i) \setminus ([\pi_{i-1} \cup \pi_i]_\tau \cup [L]_\sigma) \) is simply connected, so we can make a smooth selection \( \{u_{j,i,S}, u_{j,i,N}\} \) for \( u_j \) on it, arranging the indices in a way that graph \( u_{j,i,S} \) lies south of \( V_i \). (Here \( u_{j,i,S} \) is not equal \( u_{j,i}^- \) anymore.) Define the region \( U_{j,i} = (V_i)_{\tau} \setminus [L]_\sigma \cap \{u_{j,i,S} < 2A\} \subset (V_i)_{\tau} \setminus [L]_\sigma \).

As above (10.1) holds, and near \( \pi_i \) we have that given any \( \delta > 0 \),

\[
\langle u_j(X), N_i \rangle > 1 - \delta
\]

for all \( X \in \text{graph} \, u_{j,i,S} \cap \text{reg} \, G_j \cap (\pi_i)_{\tau} \setminus \{[X^{n+1}] < A\} \), at least after updating \( j \geq J(\tau, \sigma, A, \delta) \). As we let \( \tau, \sigma, \delta \to 0 \) and \( A \to \infty \) both \( \text{dist}_H(U_{j,i}, V_i) \to 0 \) and (10.1) hold as \( j \geq J(\tau, \sigma, A, \delta) \to \infty \). The last inequality (10.1) guarantees that the limit current has the right orientation to apply Lemma 6.2, which immediately yields a contradiction. The third case, when \( m_i = 1, m_{i+1} = 2 \) and \( l_i = 1, l_{i+1} = 0 \) can be argued in precisely the same way, with reversed roles of \( \pi_i \) and \( \pi_{i+1} \).

The last remaining case is \( m_i = 2, m_{i+1} = 1 \) and \( l_i = 0, l_{i+1} = 1 \). Arguing as above we find

\[
\text{sing} \, u_j \cap (V_{i-1} \cup V_i \cup V_{i+1}) \setminus ([\pi_{i-1} \cup \pi_i \cup \pi_{i+1} \cup \pi_{i+2}]_\tau \cup [L]_\sigma) = \emptyset,
\]

\[
B_{u_j} \cap (V_{i-1} \cup V_i \cup V_{i+1} \cup \pi_i \cup \pi_{i+1}) \setminus ([\pi_{i-1} \cup \pi_{i+2}]_\tau \cup [L]_\sigma) = \emptyset.
\]

We may make a smooth selection \( \{u_{j,i,S}, u_{j,i,N}\} \) for \( u_j \) on the latter set \( (V_{i-1} \cup V_i \cup V_{i+1} \cup \pi_i \cup \pi_{i+1}) \setminus ([\pi_{i-1} \cup \pi_{i+2}]_\tau \cup [L]_\sigma) \). We arrange for \( u_{j,i,S} \) to lie south of \( V_i \), and moreover

\[
\langle u_j(X), N_i \rangle > 1 - \delta \quad \text{and} \quad \langle u_j(X), -(N_i+1) \rangle > 1 - \delta
\]
at all points \( X \in \text{graph} \ u_{j,i} \cap \text{reg} \ G_j \cap (\pi_i)_\tau \cap \{ |X^{n+1}| < A \} \) and graph \( u_{j,i} \cap \text{reg} \ G_j \cap (\pi_i+1)_\tau \cap \{ |X^{n+1}| < A \} \) respectively. From that point on we can argue in the same way as above, ultimately leading to a contradiction with Lemma 6.2. This exhausts the list of possible cases, and concludes the proof. \( \square \)

Using this, we can immediately conclude mass cancellation in all but one case. Indeed \(|T| \neq P\) if and only if \( t_i = 0 \) for at least one plane. But then \( F_{i-1} = 1 = F_i \) by Lemma 7.5, which contradicts Lemma 10.2 unless \( \pi_i = \pi_i+1 \) and \( D = 1 \). For the remainder of this section, we may assume that \( t_i = m_i \) for all \( i \). As a consequence also \( F_i-F_{i-1} = s_im_i \) for all \( i \).

**Lemma 10.3.** Let \(|G_j| \to \sum_{i=1}^{D} m_i |\Pi_i|\). Then

(i) any consecutive \( \pi_i, \pi_{i+1}, \ldots, \pi_{i+J} \) have \( \sum_{j=1}^{J} s_{i+j} m_{i+j} \leq 2 \).
(ii) if \( \pi_i \) has multiplicity two then \( \sum_{j=1}^{D-1} s_{i+j} m_{i+j} = 0 \).

**Proof.** (i) Iterating Lemma 10.1 we find that \( F_1 = \sum_{j=1}^{J} s_{i+j} m_{i+j} \).

As \( F_1 \leq 2 \) we get \( \sum_{j=1}^{J} s_{i+j} m_{i+j} \leq 2 \) as desired.

(ii) Here we consider the \( D-1 \) consecutive planes \( \pi_i, \ldots, \pi_{i+D-1} \), which stop just shy of \( \pi_i \) and \( \pi_{i+D} \). Because \( \pi_i \cup \pi_{i+D} = \Pi \) is a plane, they must have \( s_i = -s_{i+D} \); without loss of generality \( s_i = 1 \). By Lemma 10.1, \( F_i = F_{i-1} + 1 \) and \( F_{i+D} = F_{i+D-1} + 2 \), so \( F_i = 2 = F_{i+D-1} \). Iterating the same lemma over the half-planes \( \pi_{i+1}, \ldots, \pi_{i+D-1} \) we find \( F_i = \sum_{j=1}^{D-1} s_{i+j} m_{i+j} = 0 \), as desired. \( \square \)

**Corollary 10.4.** Let \(|G_j| \to \sum_{i=1}^{D} m_i |\Pi_i|\). Then

(i) if \( m_1 = \cdots = m_D = 1 \) then \( D \equiv 2 \) (mod 4),
(ii) if \( m_1 = \cdots = m_D = 2 \) then \( D \) is odd,
(iii) if \( m_1 = 2, m_2 = \cdots = m_D = 1 \) then \( D \equiv 1 \) (mod 4).

**Proof.** The result is obtained by listing the orientations of the half-planes \( \pi_1, \ldots, \pi_{2D} \) weighted by their respective multiplicities, and excluding certain subsequences from this. We start by considering three consecutive half-planes \( \pi_{i-1}, \pi_i, \pi_{i+1} \) with multiplicities \( m_{i-1}, m_i, m_{i+1} = 1 \), and show that then \( (s_{i-1}, s_i, s_{i+1}) \in \{ \pm(1, -1, -1), \pm(1, 1, -1) \} \). First note that the two sequences \( (s_{i-1}, s_i, s_{i+1}) = \pm(1, 1, 1) \) are excluded by Lemma 10.1. For the remaining cases we argue by contradiction, and assume that \( (s_{i-1}, s_i, s_{i+1}) = (1, -1, -1) \). Then on the one hand by Lemma 10.2, applied between \( \pi_{i-1}, \pi_i \) and \( \pi_i, \pi_{i+1} \) respectively we find that \( F_{i-1}, F_i \neq 1 \). This is absurd, as on the other hand \( F_i = F_{i+1} + 1 \). One reasons similarly when \( (s_{i-1}, s_i, s_{i+1}) = (-1, 1, 1) \).

(i) When the multiplicities are all equal \( m_1, \ldots, m_D = 1 \) then the only possibility is that \( \sum_{j=1}^{D} s_{i+j} m_{i+j} = (1, 1, -1, -1, \ldots) \) or a cyclic permutation thereof. Hence \( D \) must be even. As \( s_1, s_2 = 1 \) we get \( s_{D+1}, s_{D+2} = -1 \), whence \( s_{D-1}, s_D = 1 \) and \( D \equiv 0 \) (mod 4).
Corollary 10.6. Suppose $10.3$. Multiplicity and mass cancellation. Then either a stable minimal surface $\Sigma$ is forbidden by Corollary 10.4 is $\Theta(\|V\|, X) = 2$ for $\mathcal{H}^n$-a.e. $X \in \text{reg} V$ and there is a smooth, stable minimal surface $\Sigma$ so that $V = 2|\Sigma|$, or $\Theta(\|V\|, X) = 1$ for $\mathcal{H}^n$-a.e. $X \in \text{reg} V$ and

(i) $\text{spt} \|V\|$ is immersed near points of $S^{n-2}(V)$, 
(ii) the set $S^{n-2}(V) \cup \mathcal{B}(V)$ is countably $(n-2)$-rectifiable.

(ii) From Lemma 10.3 (i) either $(s_1, \ldots, s_{2D}) = (1, 1, -1, -1, \ldots)$ or a cyclic permutation thereof, so $D$ must be odd.

(iii) Without loss of generality $s_1 = 1$, and by Lemma 10.1 the sequence $(m_is_i)$ starts $(2s_1, s_2, s_3) = (2, -1, -1)$. The orientations of half-planes with multiplicity one alternate in pairs, so this continues $(s_4, s_5, s_6, s_7, \ldots) = (1, 1, -1, -1, \ldots)$. As $2s_{D+1} = -2$ we get $(s_{D-1}, s_D, 2s_{D+1}) = (1, 1, -2)$. Combining the two observations, $D - 1 \equiv 0 \pmod{4}$. 

$\square$

10.2. Classification in dimensions up to seven. Here too, as in the previous section, we consider a sequence of two-valued minimal graphs $u_j \in C^{1,\alpha}(D_2; A_2)$ which converge in the varifold topology, $|G_j| = |\text{graph } u_j| \to \sum_i m_i|\Pi_i|$ as $j \to \infty$. These planes are assumed to meet along a single $n - 1$-dimensional vertical axis $L = L_0 \times \mathbb{R}e_{n+1} \in Gr(n-1, n+1)$.

**Corollary 10.5.** Let $|G_j| \to \sum_{i=1}^{D} m_i|\Pi_i|$. If $2 \leq n \leq 6$ then this is either $2|\Pi_1|$ or $|\Pi_1| + |\Pi_2|$.

**Proof.** As the graphs have dimension up to six, the area estimates of Proposition 3.3 give $\sum_j m_j \leq [n\omega_n/\omega_{n-1}] \leq 5$. Hence $D \leq 5$, and the possibilities for $(m_1, \ldots, m_D)$ are listed in Table 2 up to cyclic permutation. Of these, the only not forbidden by Corollary 10.4 is $(m_1, m_2, m_3) = (2, 2, 1)$. However $2s_2 + s_3 \neq 0$ is forbidden by Lemma 10.1. 

$\square$

10.3. Multiplicity and mass cancellation.

**Corollary 10.6.** Suppose $2 \leq n \leq 6$. Let

$|G_j| \to V \neq 0 \in \mathcal{IV}_n(D_1 \times \mathbb{R})$ as $j \to \infty$.

Then either $\Theta(\|V\|, X) = 2$ for $\mathcal{H}^n$-a.e. $X \in \text{reg} V$ and there is a smooth, stable minimal surface $\Sigma$ so that $V = 2|\Sigma|$, or $\Theta(\|V\|, X) = 1$ for $\mathcal{H}^n$-a.e. $X \in \text{reg} V$ and

(i) $\text{spt} \|V\|$ is immersed near points of $S^{n-1}(V) \setminus S^{n-2}(V)$,
(ii) the set $S^{n-2}(V) \cup \mathcal{B}(V)$ is countably $(n-2)$-rectifiable.
Proof. We only need to show that the multiplicity of regular points of \( V \) is either one or two; the conclusion follows by combining our limit cone classification with the results of [SS81, KW20, Wic20]. We may assume without loss of generality that \( G_j \cap D_1 \times \mathbb{R} \) is connected for all \( j \). As the graphs converge locally in \( D_1 \times \mathbb{R} \) with respect to Hausdorff distance, \( \text{spt} \| V \| \cap D_1 \times \mathbb{R} \) is also connected. Let \( \mathcal{R} \) be the set of connected components of \( \text{reg} V \), which we group into the two sets \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) according to their respective multiplicities. (The multiplicities are constant on every component by [Sim84, Thm. 41.1].)

We use a contradiction argument to show that one of the two is empty. Let \( V_1 = \sum_{\Sigma \in \mathcal{R}_1} |\Sigma| \) and \( V_2 = \sum_{\Gamma \in \mathcal{R}_2} 2|\Gamma| \). These are both stationary in \( D_1 \times \mathbb{R} \) away from \( \text{spt} \| V_1 \| \cap \text{spt} \| V_2 \| \cap D_1 \times \mathbb{R} \). By our classification of limit cones, \( \mathcal{C}(V) \cap \text{spt} \| V_2 \| = \emptyset \). As \( \mathcal{B}(V) \cup S^{n-2}(V) \) is countably \((n-2)\)-rectifiable, this means \( \mathcal{H}^{n-1}(\text{spt} \| V_1 \| \cap \text{spt} \| V_2 \| \cap D_1 \times \mathbb{R}) = 0 \), whence \( V_1, V_2 \) are in fact stationary in \( D_1 \times \mathbb{R} \) without restrictions. One argues in the same way to justify their stability in \( D_1 \times \mathbb{R} \). As the support of \( V_2 \) contains neither genuine branch points nor classical singularities, [SS81] implies that there is a smooth embedded minimal surface \( \Sigma_2 \subset D_1 \times \mathbb{R} \) so that \( V_2 = 2|\Sigma_2| \). Take \( X \in \text{spt} \| V_1 \| \cap \Sigma_2 \cap D_1 \times \mathbb{R} \), and \( \rho > 0 \) small enough enough that \( B_\rho(X) \setminus \Sigma_2 \) has two connected components, say \( U_\pm \). Let \( V_1^\pm \) be the two varifolds made up of the portions of \( V_1 \) lying in \( U_\pm \) respectively. Arguing as above we find that \( V_1^\pm \) are both stationary in \( B_\rho(X) \). Without loss of generality \( X \in \text{spt} \| V_1^+ \| \). As \( V_1^+ \) lies above \( \Sigma_2 \), we have reached a contradiction with the maximum principle of Solomon–White [SW89].

Corollary 10.7 (No mass cancellation). Let \( 2 \leq n \leq 6 \). Suppose that as \( j \to \infty \), \( |G_j| \to V \in \text{IV}_n(D_1 \times \mathbb{R}) \) and \( \|G_j\| \to T \in \text{I}_n(D_1 \times \mathbb{R}) \). If \( T \neq 0 \) then \( |T| = V \).

Proof. The limit current \( T \) necessarily has \( |T| \ll V \), with equality if and only if it is no mass cancellation. This cannot occur at points of multiplicity one, so by Corollary 10.6 we may assume that \( \Theta(\| V \|, X) = 2 \) for \( \mathcal{H}^n\text{-a.e.} \) \( X \in \text{reg} V \) and there is a smooth embedded minimal surface \( \Sigma \) with \( V = 2|\Sigma| \). Thus also \( \text{spt} \| T \| \subset \Sigma \), and by the Constancy Theorem [Sim84, Thm. 41.1] there is \( m \in \mathbb{Z}_{\geq 0} \) so that \( |T| = m|\Sigma| \). As \( m \leq 2 \), and \( m \neq 1 \) as otherwise we could use Allard’s regularity theory, we have that either \( m = 2 \) and \( |T| = V \) or \( m = 0 \) and \( T = 0 \).

11. Blowdown cones and asymptotic analysis

11.1. Entire graphs with bounded growth

Theorem 11.1. Let \( \alpha \in (0, 1) \) and \( n \geq 2 \) be arbitrary. Let \( u \in C^{1, \alpha}(\mathbb{R}^n; A_2) \) be an entire two-valued minimal graph with \( u(0) = 0 \). If

\[
\limsup_{r \to \infty} \left( \| u \|_{0, D_r} / r \right) < +\infty
\]
then \( u \) is linear. Otherwise the support of every blowdown cone at infinity contains the half-line \( L_+ = \{ t e_{n+1} \mid t \geq 0 \} \) or its reflection \(-L_+\).

**Proof.** Suppose first that \( u \) has bounded growth, say \( \sup_r r^{-1} \| u \|_{0,D_r} \leq C \) for some \( C > 0 \). Let \( (\lambda_j \mid j \in \mathbb{N}) \) be a sequence of positive scalars with \( \lambda_j \to \infty \), along which we blow down \( u \). For all \( j \in \mathbb{N} \), define \( u_j \in C^{1,\alpha}(\mathbb{R}^n;A_2) \) by setting \( u_j(x) = \lambda_j^{-1} u(\lambda_j x) \) for all \( x \in \mathbb{R}^n \). Using the interior gradient estimates, we find that for all \( r > 0 \) there is a constant \( C(r) \) so that \( \sup_j \| u_j \|_{1,D_r} \leq C(r) \). By the two-valued Lipschitz theorem we can extract a subsequence so that that there is a two-valued Lipschitz function \( U \in \text{Lip}(\mathbb{R}^n;A_2) \) so that simultaneously \( u_j \to U \) locally uniformly and \( |G_j| \to |\text{graph} U| \). By Lemma 5.3, \( U \) must be linear and its graph is a union of two possibly equal planes. By the monotonicity formula the same holds for \( u \). This concludes the proof of the first half of the lemma.

Now assume instead that \( \sup_{r>0} r^{-1} \max_{D_r} \{ u_1, u_2 \} = +\infty \), without loss of generality. We show that \( \{ t e_{n+1} \mid t \geq 0 \} \) is contained in the support of any blowdown cone of \( |G| \). As above, we blow down \( |G| \) along the sequence \( \lambda_j \to \infty \). Let \( \delta > 0 \) be a small parameter, whose value we eventually let tend to zero. Inside the disc \( D_\delta \) the functions \( u_j \) have \( \max_{D_\delta} \{ u_1^j, u_2^j \} \to +\infty \) as \( j \to \infty \), so that \( \max_{D_\delta} \{ u_1^j, u_2^j \} \geq 1 \) for large enough \( j \geq J(\delta) \). Hence there exists a sequence of points \( X_j = (x_j, X_j^{n+1}) \in G_j \cap D_\delta \times \mathbb{R} \) with \( X_j^{n+1} > 1 \) for all \( j \). As \( G_j \cap D_\delta \times \mathbb{R} \) is connected there is a continuous path \( \gamma_j : [0,1] \to \mathbb{R}^{n+1} \) with image \( \gamma_j([0,1]) \subset G_j \cap D_\delta \times \mathbb{R} \) and endpoints \( \gamma_j(0) = 0 \) and \( \gamma_j(1) = X_j \). This path must cross the solid disc \( D_\delta \times \{1\} \) at height one, so that by picking a point in this intersection we can construct a sequence of points \( (Y_j \mid j \geq J(\delta)) \) that respectively belong to \( \gamma_j([0,1]) \cap D_\delta \times \{1\} \).

Given any positive sequence \( \{ \delta_m \mid m \in \mathbb{N} \} \) with \( \delta_m \to 0 \) repeat the argument with \( \delta = \delta_m \). Via a diagonal extraction argument we obtain a subsequence of indices \( (j_m \mid m \in \mathbb{N}) \) and \( (Y_m \mid m \in \mathbb{N}) \) with \( Y_m = Y_{j_m}, \delta_m \in G_{j_m} \cap D_{\delta_m} \times \{1\} \). They converge to the point \( (0,1) \in \mathbb{R}^{n+1} \) at height one, which thus is in \( \text{spt} \| C \| \). As \( C \) is a cone, this confirms that \( \{ t e_{n+1} \mid t \geq 0 \} \subset \text{spt} \| C \| \). \( \square \)

11.2. **General results in low dimensions.** Combining the results from Section 5.1 with the work of \[Wic20\] we obtain the following technical lemma, which will be useful in what follows.

**Corollary 11.2.** Let \( V \in IV_n(D_2 \times \mathbb{R}) \) be the limit of a sequence of two-valued graphs \( G_j = \text{graph} u_j \), where there is \( \alpha \in (0,1) \) so that \( u_j \in C^{1,\alpha}(D_2;A_2) \) for all \( j \). Suppose that at the point \( Z \in \text{spt} \| V \| \), there is a tangent cone of the form

\[
\{ \Pi_1^0 \times \mathbb{R} e_{n+1}, 2|\Pi_1^0| \times \mathbb{R} e_{n+1}, \Pi_1^0 \times \mathbb{R} e_{n+1} \}, \quad (\Pi_0^0 + |\Pi_2^0|) \times \mathbb{R} e_{n+1} \in \text{VarTan}(V, Z),
\]

where \( \Pi_1^0, \Pi_0^0 \) are two distinct \( n - 1 \)-dimensional planes in \( \mathbb{R}^n \). Then there is \( \rho > 0 \) so that

\[
(\nu(X), e_{n+1}) = 0 \quad \text{for all } X \in \text{reg} V \cap B_\rho(Z).
\]
This holds in arbitrary dimensions, but for the remainder we restrict to the range $2 \leq n \leq 6$.

**Theorem 11.3.** Let $\alpha \in (0, 1)$ and $2 \leq n \leq 6$. Let $u \in C^{1,\alpha}(\mathbb{R}^n; \mathcal{A}_2)$ be an entire two-valued minimal graph and $C$ be a blowdown cone of $|G|$ at infinity. Then

(i) either $C$ is cylindrical of the form $C = C^0 \times \mathbb{R}e_{n+1}$,
(ii) or $C = |\Pi| + C^0 \times \mathbb{R}e_{n+1}$ where $\Pi \in Gr(n, n+1)$,
(iii) or $C$ is the sum of two possibly equal planes $\Pi_1, \Pi_2 \in Gr(n, n+1)$, $C = |\Pi_1| + |\Pi_2|$.

The remainder is dedicated to proving this theorem, starting by decomposing the blowdown cone $C$ into a vertical and a horizontal part. We construct this decomposition as follows. We consider the set $\mathcal{R}$ of connected components of $\text{reg} \ C$. (This set has at most countably many elements by a classical separability argument.) By [Sim84, Thm. 41.1] every $\Sigma \in \mathcal{R}$ has has constant multiplicity $\Theta_\Sigma \in \mathbb{Z}_{>0}$. We say that $\Sigma$ is vertical if $\langle \nu, e_{n+1} \rangle \equiv 0$ on $\Sigma$, and horizontal if instead $\langle \nu, e_{n+1} \rangle > 0$. Thus $\mathcal{R} = \mathcal{R}_v \cup \mathcal{R}_h$, both of which are allowed to be empty.

**Lemma 11.4.** Let $\alpha \in (0, 1)$ and $2 \leq n \leq 6$. Let $u \in C^{1,\alpha}(\mathbb{R}^n; \mathcal{A}_2)$ be an entire two-valued minimal graph and $C \in IV_n(\mathbb{R}^{n+1})$ be a blowdown cone of $|G|$ at infinity. Then $C_v = C^0_v \times \mathbb{R}e_{n+1} = \sum_{\Sigma \in \mathcal{R}_v} \Theta_\Sigma |\Sigma|$ and $C_h = \sum_{\Gamma \in \mathcal{R}_h} \Theta_\Gamma |\Gamma|$ are stationary integral varifolds, and

\[
(11.1) \quad C = C_v + C_h \in IV_n(\mathbb{R}^{n+1}).
\]

**Proof.** First, the convergence of the two sums can be justified because their weight measures are bounded by $\|C\|$. Let us write the argument out explicitly for $C_v$. Assume that $\mathcal{R}_v$ is countably infinite, enumerated by $\mathcal{R}_v = \{ \Sigma_i | i \in \mathbb{N} \}$ say. For every compact subset $K \subset \mathbb{R}^{n+1}$, \( \sum_{\Sigma \in \mathcal{R}_v} \Theta_\Sigma \|\Sigma\|(|K|) \leq \|C\|(|K|) \leq C_K \), so that the partial sums $\sum_{i=1}^{k} \Theta_{\Sigma_i} |\Sigma_i| \rightarrow 0$ when $k \rightarrow \infty$ as varifolds. Thus the sum $\sum_{i=1}^{\infty} \Theta_{\Sigma_i} |\Sigma_i|$ is convergent, with limit $C_v \in IV_n(\mathbb{R}^{n+1})$. As every $|\Sigma_i| \in IV_n(\mathbb{R}^{n+1})$ is invariant under homotheties, the same holds for their limit, which we are thus justified in denoting by $C_v$.

Similarly one may check that indeed $C_v$ is vertical, meaning it is of the form $C_v = C^0_v \times \mathbb{R}e_{n+1}$ for some $C^0_v \in IV_{n-1}(\mathbb{R}^n)$.

Proceeding similarly one can justify the construction of $C_h$, and confirm that $C = C_v + C_h$ as in (11.4). Moreover, the stationarity of $C$ means that $C_v$ is stationary in the open set $\mathbb{R}^{n+1} \setminus \text{spt} \|C_h\|$ and vice-versa for $C_h$. The only way either of the two cones could fail to be stationary in $\mathbb{R}^{n+1}$ is if

\[
\mathcal{H}^{n-1}(\text{spt} \|C_v\| \cap \text{spt} \|C_h\| \cap B_1) > 0.
\]

By construction $\text{spt} \|C_v\| \cap \text{spt} \|C_h\| \subset \text{sing} \ C$, which is stratified like

\[
S^0 \subset \cdots \subset S^{n-2} \subset S^{n-1} \subset S^n,
\]
Lemma 11.5.\: Intersecting. Both

Proof.\: Define a function \( \Pi \) two-valued minimal graph and \( C \) \( B \) complicated case of the three remains, in which both \( C \) mean curvature, and in particular they are both stationary near the number of points in \( \text{spt} \) was chosen arbitrarily, this proves that both \( C \) gradient bounds for \( \sum y \). Let \( \varifolds \) in \( IV \) \( C \) immediate, while in the last \( C \) \( h \) have separately pointwise vanishing mean curvature, and in particular they are both stationary near \( X_0 \). As \( X_0 \) was chosen arbitrarily, this proves that both \( C \) and \( h \) are stationary as varifolds in \( IV_n(\mathbb{R}^{n+1}) \).

The classification of classical tangent cones established in the previous section (e.g. see Corollary 10.6 again), and valid for the range of dimensions \( 2 \leq n \leq 6 \) prescribed in the hypotheses, implies that \( \text{spt}(C) \) must be immersed near \( X_0 \). Therefore both \( C_v \) and \( C_h \) must be embedded near \( X_0 \), say \( B(X_0, \rho_0) \cap \text{spt}(C) \subset \text{reg} C_h \cap \text{reg} C_v \) for some \( \rho_0 > 0 \), which are transversely intersecting. Both reg \( C_h \) and reg \( C_v \) have separately pointwise vanishing mean curvature, and in particular they are both stationary near \( X_0 \). As \( X_0 \) was chosen arbitrarily, this proves that both \( C_v \) and \( C_h \) are stationary as varifolds in \( IV_n(\mathbb{R}^{n+1}) \).

\( \square \)

The three cases (i), (ii) and (iii) listed in Theorem 11.3 correspond to the following situations. In the first case \( C_h = 0 \), and the conclusion is immediate, while in the last \( C_v = 0 \) and we conclude using the uniform gradient bounds for \( u \), as demonstrated in Theorem 11.1. Probably the most complicated case of the three remains, in which both \( C_v \neq 0 \) and \( C_h \neq 0 \).

Lemma 11.5. Let \( \alpha \in (0, 1) \) and \( 2 \leq n \leq 6 \). Let \( u \in C^{1,\alpha}(\mathbb{R}^n; A_2) \) be a two-valued minimal graph and \( C = C_v + C_h \in IV_n(\mathbb{R}^{n+1}) \) be a blowdown cone of \( |G| \) at infinity. If \( C_v \neq 0 \) and \( C_h \neq 0 \) then \( C_h = |\Pi| \) for some plane \( \Pi \in Gr(n, n+1) \).

Proof. Define a function \( Q: \mathbb{R}^n \rightarrow [0, \infty] \) which for all \( y \in \mathbb{R}^n \) counts the number of points in \( \text{spt}(C_h) \cap P_{0}^{-1}(\{y\}) \) with multiplicity: \( Q(y) = \sum_{y \in P_{0}^{-1}(\{y\})} \Theta(\|C_h\|, Y) \). As we are working with cones, this function \( Q \) is constant along open rays through the origin: for all \( y \in \mathbb{R}^n \) and \( \lambda > 0 \), \( Q(\lambda y) = Q(y) \). Let \( K = P_{0}(S^{n-2}(C) \cup B(C) \cup \text{sing} C_v) \). The convergence \( |G_j| \rightarrow C \) means that at all \( y \in \mathbb{R}^n \setminus K \), \( Q(y) \in \{0, 1, 2\} \). The function \( Q \) is locally constant on the set \( U = \mathbb{R}^n \setminus K \cap \{Q > 0\} \). We show that \( U \) is connected, and that in fact it is equal to \( \mathbb{R}^n \setminus K \).

First off, the set \( U \) is open. Let \( y \in U \) and \( Y = (y, Y^{n+1}) \in \text{spt}(\|C_h\|) \). Depending on whether \( \Theta(\|C_h\|, Y) = 1 \) or \( 2 \), the tangent cone to \( C_h \) at \( Y \) is either a single plane \( |\Pi_Y| \) with \( \Pi_Y \in Gr(n, n+1) \) or \( |\Pi_{Y,1}| + |\Pi_{Y,2}| \) with \( \Pi_{Y,1}, \Pi_{Y,2} \in Gr(n, n+1) \). Moreover by Corollary 11.2 these planes cannot be vertical. Applying either Allard’s regularity theorem if \( \Theta(\|C_h\|, Y) = 1 \) or the two-valued branched sheeting theorem of Wickramasekera [Wis20], Theorem A.4 if \( \Theta(\|C_h\|, Y) = 2 \) one finds that there is a radius \( \rho > 0 \) and either a single smooth function \( u_Y \in C^{\infty}(D_{\rho}(y)) \) so that \( \text{spt}(\|C_h\| \cap B_{\rho}(Y) \subset \text{graph} u_Y \), or else two such functions \( u_{Y,1}, u_{Y,2} \in C^{\infty}(D_{\rho}(y)) \) so that \( \text{spt}(\|C_h\| \cap B_{\rho}(Y) \subset \text{graph} u_{Y,1} \cup \text{graph} u_{Y,2} \). In particular \( Q > 0 \) in a neighbourhood of \( y \).
Next we show that \( U \subset \mathbb{R}^n \setminus K \) is relatively closed. Let \( \{y_k | k \in \mathbb{N}\} \) be a sequence of points in \( U \), with \( y_k \to y \) as \( k \to \infty \) for some \( y \in \mathbb{R}^n \setminus K \). For all \( k \in \mathbb{N} \) write \( Y_k = (y_k, Y_k^{n+1}) \in \text{spt}\|C_h\| \), and let \( Z_k = Y_k/|Y_k| \). After extracting a subsequence, which we do without relabelling, we have \( Z_k \to Z \) for some \( Z \in \partial B_1 \). By upper semicontinuity of density, \( Q(y) = Q(z) \geq \Theta(\|C_h\|, Z) \geq 1 \). Finally, arguing as in the proof of Lemma A.1 of [SW16] one finds that \( \mathbb{R}^n \setminus K \) is connected which means that \( U \) is connected as well. In fact \( U = \mathbb{R}^n \setminus K \), and \( Q \) is constant on it, taking the value \( q \in \{1, 2\} \).

We next show that \( q = 1 \), by assuming that \( q = 2 \) instead, and deriving a contradiction with our tangent cone classification, which we obtain near points where the supports of \( C_v, C_h \) intersect. There are two cases to distinguish:

\[
(11.2) \quad \text{either } \text{reg} \ C^0_v \setminus P_0(\text{sing} \ C_h) \neq \emptyset \text{ or } \text{reg} \ C^0_v \subset P_0(\text{sing} \ C_h).
\]

We start with the first, and pick an arbitrary point \( y \in \text{reg} \ C^0_v \setminus P_0(\text{sing} \ C_h) \). This has two pre-images \( Y = (y, Y^{n+1}) \) under \( P_0 \), which by assumption are also in \( \text{reg} \ C_v \). By Corollary 11.2 the tangent cones to \( C \) at the two points are \([\Pi^i_{1}], [\Pi^h_{1}] \in \text{VarTan}(C, Y_i)\) where \( \Pi^i, \Pi^h \in Gr(n, n+1) \) and \( \Pi^i \) is vertical but \( \Pi^h \) is not. Because the two planes \( \Pi^i, \Pi^h \) are not vertical, there exists a radius \( \rho > 0 \) and \( u_i \in C^2(D_\rho(y)) \) so that \( \text{spt}\|C_h\| \cap B_\rho(Y_i) \subset \text{graph} \ u_i \); moreover there is \( \theta \in (0, 1) \) depending only on the inclination of the two planes so that \( \text{graph} \ u_i \cap D_{\rho\theta}(y) \times \mathbb{R} \subset \text{spt}\|C_h\| \cap B_\rho(Y_i) \). From the convergence \( G_j \to C \) and invoking again Theorem A.4 of Wickrascenko, we find that, after perhaps slightly adjusting the value of \( \rho \), and taking large enough \( j \), there exist functions \( u_{ji} \in C^2(D_{\rho\theta}(y)) \) so that \( \text{graph} \ u_{ji} \cap D_{\rho\theta}(y) \times \mathbb{R} \subset G_j \cap B_\rho(Y_i) \). Now notice that in fact the two functions define a selection for the two-valued \( u_{j} \) on the disc \( D_{\rho\theta}(y) \), and in fact \( G_j \cap D_{\rho\theta}(y) \times \mathbb{R} = \langle \text{graph} \ u_{ji} \cup \text{graph} \ u_{j2} \rangle \cap D_{\rho\theta}(y) \times \mathbb{R} \). As \( \text{graph} \ u_{ji} \cap L \in \text{reg} \ C_h \cap \text{reg} \ C_h \) as \( j \to \infty \) we find that \( C \) cannot have a vertical component, in contradiction to our original assumption that \( C_v \neq 0 \).

We turn to the second case, where it is assumed that \( \text{reg} \ C^0_v \subset P_0(\text{sing} \ C_h) \). As \( \mathcal{H}^{n-1}(P_0(\text{sing} \ C_h \setminus \text{C}(C_h))) = 0 \) but \( \text{reg} \ C^0_v \) has positive \( \mathcal{H}^{n-1} \)-measure, we can pick a point \( y \in \text{reg} \ C^0_v \setminus P_0(\text{sing} \ C_h \setminus \text{C}(C_h)) \) which further has

\[
P_0^{-1}(\{y\}) \cap \text{spt}\|C_h\| \subset \text{C}(C_h).
\]

In fact more is true, because

\[
(11.3) \quad P_0(\text{C}(C_h)) \cap P_0(\text{sing} \ C_h \setminus \text{C}(C_h)) = 0.
\]

To see this, take \( z \in D_1 \cap P_0(\text{C}(C_h)) \) and argue as above to show that \( P_0^{-1}(\{z\}) \cap \text{spt}\|C_h\| \subset \text{sing} \ C_h \). Let \( Z = (z, Z^{n+1}) \in \text{C}(C_h) \). Write \( |\Pi_{Z,1}| + |\Pi_{Z,2}| \in \text{VarTan}(C_h, Z) \), neither of which is vertical by Corollary 11.2. By Theorem A.4, for small \( \rho > 0 \) and large enough \( j \), there are two smooth functions \( U_{ji} \in C^\infty(\Pi_{Z,1} \cap B_\rho(Z); \Pi_{Z,2}) \) so that \( \text{VarTan}(U_{ji}) \) have a smooth selection for \( u_j \) on \( D_{\rho\theta}(z) \); there are
$u_{j,1}, u_{j,2} \in C^\infty(D_{\theta\rho}(z))$ with graph $u_{ji} \subset \text{graph } U_{ji}$. Write $Z_1 = Z$. If there were a point $Z_2 \in P_0^{-1}(\{z\}) \cap \text{sing } C_h \setminus \mathcal{C}(C_h)$ then no matter how small $\sigma > 0$ is chosen we could pick $j$ large enough that $G_j \cap B_\sigma(Z_2) \neq \emptyset$. This, however, is absurd if the radius is so small that $\sigma < \theta\rho$ and $B_\rho(Z_1) \cap B_\sigma(Z_2) = \emptyset$. In short $P_0^{-1}(\{z\}) \cap \text{sing } C_h \subset \mathcal{C}(C_h)$, which demonstrates the validity of (11.2).

Therefore, having chosen the point $y \in \text{reg } C_v \setminus P_0(\text{sing } C_h \setminus \mathcal{C}(C_h))$ we can pick a radius $\rho > 0$ small enough that

$$\text{reg } C_v \cap D_\rho(y) \subset P_0(\mathcal{C}(C_h)).$$

Next let $Y \in P_0^{-1}(\{y\}) \cap \mathcal{C}(C_h)$ be the unique singular point lying above $y$, and denote its tangent cone

$$[\Pi_{Y,1}] + [\Pi_{Y,2}] \in \text{VarTan}(C_h, Y),$$

with neither plane $\Pi_{Y,1}, \Pi_{Y,2}$ vertical. By assumption the point $Y$ also belongs to $\text{reg } C_v$, with respect to which it has the tangent cone

$$[\Pi_{Y}] = [\Pi_{y,0}] \times \mathbb{R} e_{n+1} \in \text{VarTan}(C_v, Y).$$

Blowing up $C$ at $Y$ one finds $[\Pi_{Y,1}] + [\Pi_{Y,2}] + [\Pi_{y}] \in \text{VarTan}(C, Y)$. Let $L_Y = \Pi_{Y,1} \cap \Pi_{Y,2}$ be the $(n-1)$-dimensional axis along which the two non-vertical planes. When blowing up the assumed inclusion $\text{reg } C_v \subset P_0(\text{sing } C_h)$, one finds $\Pi_{y,0} \subset P_0(L_Y)$. Therefore the three planes meet along this very same axis $L_Y$, and the cone $[\Pi_{Y,1}] + [\Pi_{Y,2}] + [\Pi_{y}] \in \text{VarTan}(C, Y)$ is classical. This is absurd, because it contradicts our classification of classical limit cones.

We have thus shown, in both cases described in (11.2), that $Q \equiv 1$ on $D_1 \setminus K$. Using Allard regularity for example there is a smooth, single-valued function $U_h \in C^\infty(D_1 \setminus K)$ so that $C_h \{ D_1 \setminus K \} \times \mathbb{R} = |\text{graph } U_h|$. As $\mathcal{H}^{n-1}(K) = 0$ and $K$ is a locally compact subset of $D_1$, we are precisely in a setting investigated by Simon [Sim77], whose results prove that in fact $U_h$ can be extended smoothly across $K$. It is then a standard fact that $U_h$ is linear, that is $|\text{graph } U_h| = |\Pi_h| \{ D_1 \setminus K \} \times \mathbb{R}$ for some horizontal plane $\Pi_h \in Gr(n, n+1)$. Then also $|C_h| = |\Pi_h|$, which is precisely what was to prove. 

\section{12. The Bernstein theorem in four dimensions}

In this section we complete the proof of the main result: the Bernstein theorem in dimension four.

\textbf{Theorem 12.1.} Let $n + 1 = 4$, $\alpha \in (0, 1)$, and $u \in C^{1,\alpha}(\mathbb{R}^n; A_2)$ be a two-valued function whose graph $G$ is minimal. Then $u$ is linear, and there are two three-dimensional planes $\Pi_1, \Pi_2 \in Gr(3, 4)$ so that $|G| = |\Pi_1| + |\Pi_2|$.

\subsection{12.1. Stability and the logarithmic cutoff trick.} By Theorem 11.3 the cone $C$ must take one of the following three forms:

\begin{equation}
C^0 \times \mathbb{R} e_4, |\Pi_1| + C^0 \times \mathbb{R} e_4 \text{ or } |\Pi_1| + |\Pi_2|,
\end{equation}
where \( \Pi_1, \Pi_2 \) are two non-vertical, possibly equal planes, and \( \mathbf{C}^0 \in IV_2(\mathbb{R}^3) \) is a stationary integral cone. The remainder of this section is dedicated to excluding the first two cases, that is necessarily \( \mathbf{C} = \|\Pi_1\| + \|\Pi_2\| \). The proof starts with the observation that \( \mathbf{C}^0 \) inherits the ambient stability from \( \mathbf{C} \). Using the so-called logarithmic cutoff trick one finds that \( \mathbf{C}^0 \) has \( A_{\mathbf{C}^0} \equiv 0 \) on \( \text{reg} \mathbf{C}^0 \).

**Lemma 12.2.** Suppose \( n \geq 2 \). Let \( \mathbf{C} \in IV_n(\mathbb{R}^{n+1}) \) be a stationary integral cone, with support immersed outside of the origin and \( |A_{\mathbf{C}}| \equiv 0 \) on \( \text{reg} \mathbf{C} \). Then \( \mathbf{C} \) is supported in a finite union of \( n \)-dimensional planes.

**Proof.** Let \( \Pi \in Gr(n, n+1) \) be any \( n \)-dimensional linear plane with \( \text{reg} \mathbf{C} \cap \Pi \neq \emptyset \). The set \( \Pi \setminus \{0\} \) is connected because \( n \geq 2 \). As \( \text{spt} \|\mathbf{C}\| \cap \Pi \setminus \{0\} \) is a relatively closed subset of \( \Pi \setminus \{0\} \), we only need to show that it is open to obtain \( \Pi \subseteq \text{spt} \|\mathbf{C}\| \). A point \( X \in \text{spt} \|\mathbf{C}\| \cap \Pi \setminus \{0\} \) is either regular or an immersed classical singularity. In both cases the fact that \( |A_{\mathbf{C}}| \equiv 0 \) on \( \text{reg} \mathbf{C} \) means that near \( X \) the support of \( \mathbf{C} \) is either a plane or a union of planes. By assumption \( X \in \Pi \), so one of these planes must be \( \Pi \) itself. This shows that \( \text{spt} \|\mathbf{C}\| \cap \Pi \setminus \{0\} \) is open inside \( \Pi \setminus \{0\} \), and thus \( \Pi \subseteq \text{spt} \|\mathbf{C}\| \). Repeating this, we find a finite collection of planes \( \Pi_1, \ldots, \Pi_D \) so that \( \mathcal{H}^n(\text{spt} \|\mathbf{C}\| \setminus \cup_i \Pi_i) = 0 \), and hence \( \text{spt} \|\mathbf{C}\| \subseteq \cup_i \Pi_i \) using the monotonicity formula. \( \square \)

To apply Lemma 12.2 to \( \mathbf{C}^0 \) we first need to show that it is immersed outside the origin. This is essentially a consequence of Corollary 10.6, which offers the following two possibilities.

1. Either \( \Theta(\|\mathbf{C}\|, X) = 2 \) for \( \mathcal{H}^3 \)-a.e. \( X \in \text{reg} \mathbf{C} \), and then \( \mathbf{C} \) is smooth embedded. This makes it impossible that \( \mathbf{C} = \mathbf{C}^0 \times \mathbb{R} e_4 + \Pi_1 \), and in the case where \( \mathbf{C} = \mathbf{C}^0 \times \mathbb{R} e_4 \) we find that \( \mathbf{C}^0 = 2|\Pi^0| \) for some two-dimensional plane \( \Pi^0 \subset \mathbb{R}^3 \). Thus \( \mathbf{C} = 2|\Pi^0| \times \mathbb{R} e_4 \), and we can conclude by the monotonicity formula.

2. The second possibility is that the density of \( \mathbf{C} \) is \( \mathcal{H}^3 \)-a.e. equal one, and thus automatically also \( \Theta(\|\mathbf{C}^0\|, X) = 1 \) for \( \mathcal{H}^2 \)-a.e. \( X \in \text{reg} \mathbf{C}^0 \). Recall that the singular set of \( \mathbf{C}^0 \) is stratified like \( \mathcal{S}^0(\mathbf{C}^0) \subset \mathcal{S}^1(\mathbf{C}^0) \subset \mathcal{S}^2(\mathbf{C}^0) \). Invoking Corollary 10.6 again we find that that \( \text{spt} \|\mathbf{C}^0\| \) is immersed near points of \( \mathcal{S}^1(\mathbf{C}^0) \setminus \mathcal{S}^0(\mathbf{C}^0) \), and the remaining singularities necessarily have \( \mathcal{S}^0(\mathbf{C}^0) \cup B(\mathbf{C}^0) \subset \{0\} \).

In both cases \( \mathbf{C}^0 \) is immersed outside the origin, and by Lemma 12.2 we find that \( \mathbf{C}^0 \) is supported in a union of planes. In the remainder we need only consider the second possibility, where the density of \( \mathbf{C}^0 \) at all regular points is one, and the cone is equal to a sum of planes, all of which are vertical and have multiplicity one. (Note Lemma 12.2 only gives that \( \mathbf{C}^0 \) is supported in a union of planes.) In the remainder we write \( \mathbf{C}^0 = \mathbf{P}^0 = \sum_{j=2}^D |\Pi_j^0| \) to reflect this. At this stage of the proof we have reduced the possible forms of the blowdown cone given in (12.1) to

\[
\mathbf{P}^0 \times \mathbb{R} e_4 \text{ or } |\Pi_1| + \mathbf{P}^0 \times \mathbb{R} e_4.
\]
The two require different approaches, and we treat the latter first.

12.2. **Non-vertical blowdown cones.** Write \( L^0 \subset \mathbb{R}^3 \) for the union of one-dimensional lines along which the planes in the support of \( P^0 \) meet. The singularities of the blowdown cone are \( \text{sing } C = (\Pi_1 \cap \text{spt } |P^0 \times R e_4|) \cup L^0 \times R e_4 \) and those lying in \( \Pi_1 \cap \text{spt } |C| \setminus (L^0 \times R e_4) \) are all immersed. Let \( \tau > 0 \) be given. Using Allard’s regularity theorem near the points of \( \Pi_1 \cap \text{reg } C \) and Wickramasekera’s stable sheeting theorem near those in \( \Pi_1 \cap \text{sing } C \setminus (L^0 \times R e_4) \) we find the existence of a smooth function \( u_{j,1} \in C^\infty(D_1 \setminus [L^0]_{\tau}) \) with \( G_{j,1} = \text{graph } u_{j,1} \subset G_j \cap (\Pi_1)_{\tau} \), at least provided \( j \geq J(\tau) \) is large enough. We obtain a smooth selection \( u_{j,1}, u_{j,2} \in C^\infty(D_1 \setminus [L^0]_{\tau}) \) by picking the remaining value of \( u_j \) for \( u_{j,2} \) above every point. (It is not enough to observe that eventually \( B_{u_j} \cap D_1 \subset [L^0]_{\tau} \), as the set \( D_1 \setminus [L^0]_{\tau} \) is not simply connected regardless of how small \( \tau > 0 \) is.) As \( \tau \to 0 \) and \( j \geq J(\tau) \to \infty \) we find that by construction \( |G_j^1| \to |\Pi_1| \) and \( |G_j^2| \to P^0 \times R e_4 \). As the \( G_j^2 \) are all single-valued graphs we find that \( P^0 \) is necessarily supported in a single plane, say \( P^0 = |\Pi_2| \). (There are various ways of confirming this in more detail, all boiling down to the fact that \( P^0 \) cannot be the limit of a sequence of area-minimising currents if it is supported in more than one plane. To give but one example, revisiting the arguments used to prove the improved area estimates we obtain that there is \( \delta > 0 \) so that \( \|P^0\|(D_1) \leq (2 - \delta)\omega_2 \). Therefore \( |G_j| \to |\Pi_1| + |\Pi_2| \), as desired.

12.3. **Vertical blowdown cones: the adjacency graph.** Here the blowdown sequences converges to a vertical cone, \( |G_j| \to P = P^0 \times R e_4 = \sum_{j=1}^D |\Pi^0_j| \times R e_4 \). In the current topology \( |G_j| \to \sum_{j=1}^D |\Pi^0_j| \times R e_4 \) where the planes are respectively oriented by unit normals \( n_1, \ldots, n_D \). The improved area estimates give \( D \leq 3 \), see Corollary 3.4.

The only problematic value is \( D = 3 \). We exclude this by a combinatorial argument, constructing what we call the **adjacency graph** by a kind of dual cellular decomposition. The planes \( \Pi_1, \Pi_2, \Pi_3 \) divide \( \mathbb{R}^3 \) into a finite number of connected components \( \Omega_1, \ldots, \Omega_N \subset \mathbb{R}^3 \). These are all polyhedral, with respective boundaries \( \partial \Omega_1, \ldots, \partial \Omega_N \subset \Pi_1 \cup \Pi_2 \cup \Pi_3 \). We say that two regions \( \Omega \neq \Omega' \) are adjacent if they meet along a face. To every component \( \Omega \) we associate a vertex \( v \), forming a set \( V \). Connect two distinct vertices \( v, v' \in V \) by an edge \( e \) if the corresponding regions \( \Omega, \Omega' \) are adjacent. If \( \Omega, \Omega' \) are adjacent then they meet along a single plane \( \Pi_i \). We orient \( e \) so that it agrees with the orientation of this plane, meaning if \( n_i \) points away from \( \Omega \) and into \( \Omega' \) then \( e \) is directed from \( v \) to \( v' \) and vice-versa. Thus we obtain a set of directed edges denoted \( E \). We call the finite, directed graph \( H = (V, E) \) the **adjacency graph** of \( P \). Label the vertices of the graph by a function \( F : V \to \{0, 1, 2\} \) which returns the number \( F(v) \) of sheets of \( G_j \) eventually lying over the corresponding region \( \Omega \cap D_1 \). This is well-defined by Lemma 7.5 for example. Let \( v, v' \in V \) be two adjacent vertices, and suppose that \( e \) points from \( v \) to \( v' \). By Lemma 10.1, \( F(v') = F(v) + 1 \). As
an immediate consequence we find that $H$ cannot contain directed paths of length more than two. Indeed if $H$ contained three edges $e_1, e_2, e_3$ so that $e_i$ points from $v_i$ to $v_{i+1}$ then $F(v_i) = F(v_1) + 3$, which is absurd.

There are essentially only two ways in which the planes $\Pi_1, \Pi_2, \Pi_3$ can be arranged. Let $\Pi_3 = \{ x \in \mathbb{R}^3 \mid \langle x, n_3 \rangle \equiv 1 \} \subset \mathbb{R}^3$ be the affine plane parallel to $\Pi_3$ at height one. The two planes $\Pi_1$ and $\Pi_2$ intersect this transversely in a pair of affine lines $l_1, l_2$. If these lines were parallel, then the planes $\Pi_1, \Pi_2, \Pi_3$ would meet along a common axis, making $P = P^0 \times \mathbb{R}e_4$ a classical cone. As we have already dealt with these, we may assume this is not the case.

Hence we may assume the two lines $l_1, l_2$ intersecting, and compute the adjacency graph. The set $\Pi_3 \setminus (l_1 \cup l_2)$ has four connected components. Each of these leads to a pair of adjacent vertices in $V$, which correspond to regions meeting along a face in $\Pi_3$. Thus $H$ contains eight vertices, arranged as four pairs of vertices lying on either side of $\Pi_3$. Additionally the four vertices corresponding to the regions contained inside $\{ x \in \mathbb{R}^3 \mid \langle x, n_3 \rangle > 0 \}$ are arranged in a square in $H$, with parallel edges oriented in the same direction. The same holds for the regions lying in the half-space $\{ x \in \mathbb{R}^3 \mid \langle x, n_3 \rangle < 0 \}$. In short, $H$ is a cube with eight vertices and twelve edges, with parallel edges pointing in the same direction. As this graph contains a directed path of length three, we have reached a contradiction.

**Appendix A. Singularities and Regularity of Minimal Surfaces**

Here we record some results from geometric measure theory, and define some notation that we rely on in our arguments. We mainly work with integral varifolds, but we sometimes also require some results for integer-density rectifiable currents. For their basic theory one may consult the book of Simon [Sim84], or indeed any of the other standard references. Here we concentrate on the regularity of stationary integral varifolds, and quote some results concerning their singularities. We state all results for codimension one surfaces, although some of them—notably those regarding the stratification of their singular sets—remain valid in higher dimensions.

**A.1. Stratification of the singular set.** Let $U \subset \mathbb{R}^{n+1}$ be an open set. We write $IV_n(U)$ for the space of $n$-dimensional integer-density rectifiable varifolds in $U$, and $I_n(U)$ for the space of integral currents in $U$, namely those integer-density currents in $U$ whose boundary also defines an integer-density rectifiable current. (This includes the $n$-dimensional cycles, which have zero boundary by definition.)

Let $V \in IV_n(U)$ be a stationary varifold. A point $X \in U \cap \text{spt} \|V\|$ is called regular if there is a radius $\rho > 0$ so that $B_\rho(X) \cap \text{spt} \|V\|$ is an embedded surface. A point which is not regular is called singular. We denote the regular set $\text{reg} V$ and the singular set $\text{sing} V$.

Consider a point $X \in U \cap \text{sing} V$ and a tangent cone $C \in \text{VarTan}(V, X)$. Let $V \in \mathbb{R}^{n+1}$ be a vector, and $\tau_V : X \in \mathbb{R}^{n+1} \mapsto X - V \in \mathbb{R}^{n+1}$ be the corresponding translation by $-V$. The set $S(C) = \{ V \in \mathbb{R}^{n+1} \mid \tau_V \# C =$
$C$} of vectors that leave $C$ invariant forms a vector space called the spine of $C$. For $0 \leq m \leq n$ let $S^m(V) = \{X \in U \cap \text{sing } V | \dim S(C) \leq m \text{ for all } C \in \text{VarTan}(V,X)\}$. If $\dim S(C) = m$ then there is a stationary cone $C' \in IV_{n-m}(R^{n-m+1})$ so that $C = C' \times S(C)$. Let $C \in IV_n(R^{n+1})$ be a stationary integral cone. Then for $X \in \text{spt}||C|| \setminus \{0\}$ every tangent cone $C_X \in \text{VarTan}(C,X)$ has $S(C_X) \neq \{0\}$ because $R \cdot X \subset S(C_X)$. This means that there exists $C' \in IV_{n-1}(R^n)$ so that $C_X = C'_X \times R \cdot X$. Then we have the so-called Almgren–Federer stratification theorem.

**Theorem A.1** ([Alm00]). Let $U \subset R^{n+1}$ be open, and $V \in IV_n(U)$ be a stationary integral varifold. Then $\dim_H S^m(V) \leq m$ for all $0 \leq m \leq n$.

Under the same hypotheses, Naber–Valtorta [NV15] improve this to the following.

**Theorem A.2** ([NV15]). Let $U \subset R^{n+1}$ be open, and $V \in IV_n(U)$ be a stationary integral varifold. Then $S^m(V)$ is countably $k$-rectifiable for all $0 \leq m \leq n$.

In the literature both $S^k(V)$ and $(S^k \setminus S^{k-1})(V)$ are sometimes called strata of the singular set of $V$. The singular set of a stationary $V \in IV_n(U)$ can be divided into $\text{sing } V = S^{n-2}(V) \cup (S^n \setminus S^{n-2})(V)$. By Naber–Valtorta’s result, the lower strata gathered into $S^{n-2}(V)$ can be excised by suitable sequences of test functions, using a capacity argument.

The top stratum $(S^n \setminus S^{n-1})(V)$ is also called the branch set of $V$, and denoted $\mathcal{B}(V)$. A singular point $X$ belongs to $\mathcal{B}(V)$, and is called a branch point if at least one cone $C \in \text{VarTan}(V,X)$ is of the form $C = Q[\Pi]$ for some $n$-dimensional plane $\Pi \in Gr(n,n+1)$ with multiplicity $Q \in Z_{>0}$. (By Allard regularity the multiplicity of such a branch point must be at least two.) Note that a point $X \in \text{reg } V$ near which $V$ coincides with an embedded minimal surface with multiplicity two would not be considered a branch point according to this convention. (These points are sometimes called false branch points.)

The next stratum $(S^{n-1} \setminus S^{n-2})(V)$ is formed by those points $X \in U \cap \text{sing } V$ near which at least one tangent cone $C \in \text{VarTan}(V,X)$ is of the form $C = \sum_{i=1}^D m_i |\pi_i|$ where $m_i \in Z_{\geq 1}$ and the $\pi_i$ are $n$-dimensional half-planes meeting along a common $(n-1)$-dimensional axis $L$. We call these classical cones. This is motivated by the term classical singularities, which refers to the points in $\text{sing } V$ near which the support of $V$ is a union of minimal hypersurfaces meeting along a common boundary. We write $\mathcal{C}(V)$ for the set of classical singularities in $V$. This is the case for example for singularities that are smoothly immersed. A classical singularity in $V$ necessary belongs to $(S^{n-1} \setminus S^{n-2})(V)$, but in principle not all points in $(S^{n-1} \setminus S^{n-2})(V)$ are classical singularities.

An example of particular relevance to us is where $D = 4, m_1, \ldots, m_4 = 1$ and the half-planes $\pi_1, \ldots, \pi_4$ form a union of two $n$-dimensional planes $\Pi_1, \Pi_2 \in Gr(n,n+1)$, say $\Pi_1 = \pi_1 \cup \pi_3$ and $\Pi_2 = \pi_2 \cup \pi_4$. In other words $C =$
Such tangent cones arise for example at immersed singularities, where there are two sheets $\Sigma_1, \Sigma_2$ which are separately embedded and meet transversely along the $(n-1)$-dimensional singular axis $\Sigma_1 \cap \Sigma_2$. For stable minimal hypersurfaces the converse holds as well, see Section A.3 below.

Let $\Omega \subset \mathbb{R}^n$ be an open domain, and $u \in \text{Lip}(\Omega; \mathcal{A}_2)$ be a two-valued, locally Lipschitz function. Its graph defines an integral varifold $[G] \in \mathcal{IV}_n(\Omega \times \mathbb{R})$, taken with multiplicity one. (When endowed with an orientation it yields an integral current $[G] \in \mathcal{I}_n(\Omega \times \mathbb{R})$.) When in fact $u \in C^1(\Omega; \mathcal{A}_2)$ then $[G]$ can only have two kinds of singularities, namely branch points and classical, immersed singularities, respectively forming the sets $\mathcal{B}(G)$ and $\mathcal{C}(G)$.

A.2. The second variation formula and stability. Here to start off we assume that $\text{reg} \, V$ is orientable, and let $\xi \in C^1_c(U \setminus \text{sing} \, V; \mathbb{R}^{n+1})$ be an arbitrary vector field. Its restriction to the support of $V$ can be any compact set containing $\text{spt} \, \varphi$. Writing $(\Phi_t)$ for the flow generated by $\xi$, the second variation of the area of $V$ can be expressed as

$$
(A.1) \quad \frac{d^2}{dt^2} \|\Phi_t \# V\|(K)|_{t=0} = \int_{K \cap \text{reg} \, V} |\nabla_V \varphi|^2 - |A_V|^2 \varphi^2 \, d\|V\|,
$$

where $\nabla_V$ and $A_V$ are the gradient operator and second fundamental form on $\text{reg} \, V$ respectively.

A varifold $V \in \mathcal{IV}_n(U)$ is said to have stable regular part if this is non-negative for all perturbations $\varphi \in C^1_c(\text{reg} \, V)$, that is $\int |A_V|^2 \varphi^2 \, d\|V\| \leq \int |\nabla_V \varphi|^2 \, d\|V\|$; this is the stability inequality. This automatically gives integral curvature bounds away from the singular set. Indeed, let $X \in U \cap \text{reg} \, V$ be a point with $\text{dist}(X, \text{sing} \, V) > 2R$. If we let $\varphi \in C^1_c(B_2R(X))$ be a standard cutoff function, with $\varphi = 1$ on $B_R(X)$ and $|D\varphi| \leq 2/R$ then (A.2) yields

$$
\int_{\text{reg} \, V \cap B_2R(X)} |A_V|^2 \varphi^2 \, d\|V\| \leq 4R^{-2} \|\varphi\|(B_2R(X)).
$$

On $\text{reg} \, V$ we can define the linear, elliptic Jacobi operator $L_V = \Delta_V + |A_V|^2$. Let $W \subseteq U \setminus \text{sing} \, V$ be another open set, and write (without $V$) for simplicity the spectrum of $L_V$ with zero Dirichlet eigenvalues on $\partial W \cap \text{reg} \, V$. Although this is not necessary, assume for simplicity that the regular part is connected, so that by the Constancy Theorem [Sim84, Thm 41.1] the density of $V$ is constant on $\text{reg} \, V$. If $\text{reg} \, V$ is stable in $U$, then the stability inequality becomes

$$
\int_{U \cap \text{reg} \, V} |A_V|^2 \varphi^2 \, d\mathcal{H}^n \leq \int_{U \cap \text{reg} \, V} |\nabla_V \varphi|^2 \, d\mathcal{H}^n.
$$

After integrating by parts one gets that $\lambda_p(W) \geq 0$ for all $W \subseteq U \cap \text{reg} \, V$ and all $p \in \mathbb{N}$. Additionally we point out that the symmetries of $V$ lead to Jacobi fields, namely functions $f \in C^2(\text{reg} \, V)$ which solve $\Delta_V f + |A_V|^2 f = 0$ in the classical, pointwise sense. In particular by translating the varifold in the direction of a vector $v \in \mathbb{R}^{n+1}$ we find $\Delta_V (\langle N, v \rangle) + |A_V|^2 \langle N, v \rangle = 0$.

Although we imposed orientability of $\text{reg} \, V$ in our derivation, stability is a posteriori well-defined regardless of this. This remains true for the second notion of stability we introduce, which we call ambient stability. Suppose that for all compact $K \subset U$, $\int_{K \cap \text{reg} \, V} |A_V|^2 \, d\|V\| < \infty$. Then we say that $V$
is ambient stable if the inequality

\[(S_V) \quad \int_{U \cap \text{reg} V} |A_V|^2 \varphi^2 \, d\|V\| \leq \int_{U \cap \text{reg} V} |\nabla_V \varphi|^2 \, d\|V\|\]

holds for all \( \varphi \in C_c^2(U) \). We emphasise here that both the local bounds for the curvature and the stability inequality hold for any compact subsets of \( U \), not just those that avoid the singularities of \( V \).

This in turn allows the application of the theory developed by Hutchinson in [Hut86], where it was proved that ambient stability is preserved under weak convergence of varifolds.

**Proposition A.3** ([Hut86]). Let \( U \subset \mathbb{R}^{n+1} \) be open, and let \( (V_j) \) be a sequence of stationary varifolds in \( U \) satisfying \((S_V)\). Suppose that \( V_j \rightharpoonup V \in \mathcal{IV}_n(U) \) weakly in the varifold topology. Then \( V \) is stationary and ambient stable.

A.3. Wickramasekera’s branched sheeting theorem.

**Theorem A.4** ([Wic20]). There is \( \epsilon = \epsilon(n) > 0 \) so that if \( V \in \mathcal{IV}_n(B_2) \) is stationary with stable regular part, \( \|V\|(B_2)/(\omega_n 2^n) < 2 + \epsilon \), \( \Theta(\|V\|, Y) \neq 3/2 \) for all \( Y \in B_2 \) and

(i) either \( \int_{B_2} \text{dist}(X, \Pi_2) \, d\|V\|(X) < \epsilon \) for some \( \Pi \in \text{Gr}(n, n+1) \),

(ii) or \( \int_{B_2} \text{dist}(X, \Pi_1)^2 \wedge \text{dist}(X, \Pi_2)^2 \, d\|V\|(X) + \int_{B_2} \text{dist}(X, \text{spt}\|V\|)^2 \, d(\|\Pi_1\| + \|\Pi_2\|)(X) < \epsilon \) for some \( \Pi_1 \neq \Pi_2 \in \text{Gr}(n, n+1) \).

Then

(i) either \( B_1 \cap \text{spt}\|V\| \subset \text{reg} V \cup \mathcal{C}(V) \cup \mathcal{B}(V) \) and there is \( \gamma = \gamma(n, \epsilon) \in (0,1) \) and a two-valued function \( u \in C^{1,\gamma}(B_1 \cap \Pi; A_2(\Pi^\perp)) \) so that \( B_1 \cap \text{spt}\|V\| \subset \text{graph} u \subset \text{reg} V \cup \mathcal{C}(V) \cup \mathcal{B}(V) \), and there is \( C = C(n, \epsilon) > 0 \) so that \( |u|_{1,\gamma; B_1 \cap \Pi} \leq C \left( \int_{B_2} \text{dist}(X, \Pi_2) \, d\|V\|(X) \right)^{1/2} \).

(ii) or \( B_1 \cap \text{spt}\|V\| \subset \text{reg} V \cup \mathcal{C}(V) \) and there is \( \gamma = \gamma(n, \epsilon) \in (0,1) \) and two single-valued functions \( u_i \in C^{1,\gamma}(B_1 \cap \Pi_i; \Pi_i^\perp) \) so that \( B_1 \cap \text{spt}\|V\| \subset \bigcup_i \text{graph} u_i \subset \text{reg} V \cup \mathcal{C}(V) \), and there is \( C = C(n, \epsilon) > 0 \) so that \( |u_i|_{1,\gamma; B_1 \cap \Pi_i} \leq C \left( \int_{B_2} \text{dist}(X, \Pi_1)^2 \wedge \text{dist}(X, \Pi_2)^2 \, d\|V\|(X) + \int_{B_2} \text{dist}(X, \text{spt}\|V\|)^{1/2} \, d(\|\Pi_1\| + \|\Pi_2\|)(X) \right)^{1/2} \).

Elliptic regularity theory is not available for two-valued minimal graphs. However, Simon–Wickramasekera [SW16] have shown that necessarily \( u \in C^{1,1/2}(B_1 \cap \Pi; A_2(\Pi^\perp)) \). Of course, in the second case described above, where \( V \) is close to \( \Pi_1 + |\Pi_2| \) and \( V \cap \text{spt}\|V\| \subset \text{reg} V \cup \mathcal{C}(V) \) the two functions \( u_i \) are regular. The usual, single-valued elliptic regularity then gives that for all \( l \in \mathbb{Z}_{>0} \) there is \( C(l) = C(n, l) \) so that \( |u_i|_{l; B_1 \cap \Pi_i} \leq C(l) \left( \int_{B_2} \text{dist}(X, \Pi_1)^2 \wedge \text{dist}(X, \Pi_2)^2 \, d\|V\|(X) + \int_{B_2} \text{dist}(X, \text{spt}\|V\|)^{1/2} \, d(\|\Pi_1\| + \|\Pi_2\|)(X) \right)^{1/2} \).

**Theorem A.5** ([Wic20]). Let \( U \subset \mathbb{R}^{n+1} \) be open and \( V \in \mathcal{IV}_n(U) \) be stationary with stable regular part, and \( \Theta(\|V\|, Y) \neq 3/2 \) for all \( Y \in U \).
Then there is $\epsilon = \epsilon(n) \in (0, 1)$ so that if $Y \in U \cap \text{spt} \|V\|$ has $\Theta(\|V\|, Y) \leq 2 + \epsilon$ then $Y \in \text{reg } V \cup \mathcal{B}(V) \cup \mathcal{C}(V)$. Let us quickly comment on these three possibilities. Let $V \in \text{IV}_n(U)$ be as above, and let $Y \in U \cap \text{spt} \|V\|$ have $\Theta(\|V\|, Y) = 2$. Then either $Y \in \mathcal{B}(V)$ or else there is $0 < \rho < \text{dist}(Y, \partial U)$ so that

1. either $B_\rho(X) \cap \text{spt} \|V\| \subset \text{reg } V$ and there is a smooth embedded $\Sigma$ so that $V \mathcal{L} B_\rho(X) = 2|\Sigma|$, 
2. or $B_\rho(X) \cap \text{spt} \|V\| \subset \text{reg } V \cup \mathcal{C}(V)$ is immersed and there are two smooth embedded surfaces $\Sigma_1, \Sigma_2$ which meet transversely along an axis of immersed, classical singularities, so that $V \mathcal{L} B_\rho(Y) = |\Sigma_1| + |\Sigma_2|$ and $\text{sing } V \cap B_\rho(Y) = \Sigma_1 \cap \Sigma_2$.

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