SUBCLASSES OF ANALYTIC FUNCTIONS
WITH RESPECT TO SYMMETRIC AND
CONJUGATE POINTS CONNECTED WITH
Q-ANALOGUE OF THE BESSEL FUNCTION

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Abstract. By using $q$-analogue of the Bessel function, we introduce new subclasses of starlike functions with respect to symmetric and conjugate points and obtain some useful properties of these subclasses.

1. Introduction

Srivastava [23] presented and motivated about brief expository overview of the classical $q$-analysis versus the so-called $(p,q)$-analysis with an obviously redundant additional parameter $p$. We also briefly consider several other families of such extensively and widely-investigated linear convolution operators as (for example) the Dziok–Srivastava, Srivastava–Wright and Srivastava–Atiya linear convolution operators (see also [21, 22]), together with their extended and generalized versions. The theory of $(p,q)$-analysis has important role in many areas of mathematics and physics. Our usages here of the $q$-calculus and the fractional $q$-calculus in geometric function theory of complex analysis are believed to encourage and motivate significant further developments on these and other related topics (see also Srivastava and Karlsson [24, pp. 350–351]). Our main objective in this survey-cum-expository article is based chiefly upon the fact that the recent and future usages of the classical $q$-calculus and the fractional $q$-calculus in geometric function theory of complex analysis have the potential to encourage and motivate significant future researches on many of these and other related subjects. Jackson [13, 14] was the first that gave some application of $q$-calculus and introduced the $q$-analogue of derivative and integral operator (see also [1]).

Let $A$ denote the class of all analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U := \{z \in \mathbb{C} : |z| < 1\},$$

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If $k \in \mathcal{A}$ is given by
\[ k(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad z \in \mathcal{U}, \]
then, the Hadamard (or convolution) product of $f$ and $k$ is defined by
\[ (f * k)(z) := z + \sum_{k=2}^{\infty} a_k b_k z^k, \quad z \in \mathcal{U}. \]

If $f$ and $F$ are analytic functions in $\mathcal{U}$, we say that $f$ is subordinate to $F$, written as $f \prec F$ or $f(z) \prec F(z)$ if there exists a Schwarz function $w$, which is analytic in $\mathcal{U}$, with $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathcal{U}$, such that $f(z) = F(w(z))$, $z \in \mathcal{U}$. Furthermore, if the function $F$ is univalent in $\mathcal{U}$, then we have the following equivalence (see [5] and [16]):
\[ f(z) \prec F(z) \Leftrightarrow f(0) = F(0) \text{ and } f(\mathcal{U}) \subset F(\mathcal{U}). \]

Sakaguchi [18] introduced a class $S^*_s$ of functions starlike with respect to symmetric points, which consists of functions $f \in \mathcal{A}$ satisfying the inequality
\[ \Re \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0, \quad z \in \mathcal{U}, \]
Obviously the class of univalent functions and starlike with respect to symmetric points include the classes of convex functions and odd functions starlike with respect to the origin (see [18]).

Also, Aouf et al. [2] introduced and studied the class $S^*_s, T(1, 1)$ of functions $n$-starlike with respect to symmetric points, which consists of functions $f \in \mathcal{A}$ with $a_k \leq 0$ for $k \geq 2$, and satisfying the inequality
\[ \Re \left( \frac{D^{n+1} f(z)}{D^n f(z) - D^n f(-z)} \right) > 0, \quad z \in \mathcal{U}, \]
where $D^n$ is the Sălăgean operator [19].

El-Ashwah and Thomas [6] introduced and studied the class namely $S^*_c$ consisting of functions starlike with respect to conjugate points if it satisfies the following condition:
\[ \Re \left( \frac{zf'(z)}{f(z) + f(\overline{z})} \right) > 0, \quad z \in \mathcal{U}, \]
and, Aouf et al. [2] introduced and studied the class $S^*_c, T(1, 1)$ of functions $n$-starlike with respect to conjugate points, which consists of functions $f \in \mathcal{A}$ with $a_k \leq 0$ for $k \geq 2$, and satisfying the inequality
\[ \Re \left( \frac{D^{n+1} f(z)}{D^n f(z) + D^n f(\overline{z})} \right) > 0, \quad z \in \mathcal{U}. \]

**Definition 1.1.** [15] Let $\Omega$ be the family of functions $w(z)$ which are analytic in $\mathcal{U}$ and satisfy the conditions $w(0) = 0$ and $|w(z)| < 1$ for $z \in \mathcal{U}$. Next, for arbitrary fixed numbers $A$ and $B$, such that $-1 \leq B < A \leq 1$, denote by $\mathcal{P}[A, B]$ the family of functions $p(z) = 1 + b_1 z + b_2 z^2 + \cdots$, is analytic in $\mathcal{U}$ and such that
By using the $\lambda$-calculus and recalling the definition and notations. The $q$-shifted factorial is defined

$$\Gamma_q(z) = \prod_{k=0}^{\infty} (1 - \lambda q^{k+1}), \quad (|q| < 1),$$

and, Arif et al. [4] introduced the subclass of $S(b)$ is said to be in the class of starlike functions of complex order $b$, if

$$\{ f(z) \in A : \Re \left\{ 1 + \frac{1}{b} \left( \frac{z f'(z)}{f(z)} - 1 \right) \right\} > 0 \ (b \in \mathbb{C} \setminus \{0\}, \ z \in \mathbb{U}) \}. $$

God and Mehrok [11] introduced a subclass of $S^*_b$ denoted by $S^*_b(A, B)$, defined as follows

$$\{ f(z) \in A : \frac{2zf'(z)}{f(z) - f(-z)} < \frac{1 + Az}{1 + Bz} \quad (-1 \leq B < A \leq 1, \ z \in \mathbb{U}) \}. $$

For $-1 \leq B < A \leq 1$, $b \in \mathbb{C}^*$, $z \in \mathbb{U}$. Aouf et al. [3] with $\Phi(z) = \frac{1 + Az}{1 + Bz}$, $n = 0$ and Arif et al. [4] introduced a subclass of $S^*_b$ denoted by $S^*_b(b, A, B)$ defined as follows

$$(1.2) \quad \{ f(z) \in A : 1 + \frac{1}{b} \left( \frac{2zf'(z)}{f(z) - f(-z)} - 1 \right) < \frac{1 + Az}{1 + Bz} \},$$

and, Arif et al. [4] introduced another subclass of $S^*_b$ denoted by $C^*_b(b, A, B)$ defined as follows

$$\{ f(z) \in A : 1 + \frac{1}{b} \left( \frac{2(zf'(z))'}{(f(z) - f(-z))'} - 1 \right) < \frac{1 + Az}{1 + Bz} \}.$$

Srivastava. 23, 25 made use of various operators of q-calculus and fractional q-calculus and recalling the definition and notations. The q-shifted factorial is defined for $\lambda, q \in \mathbb{C}$ and $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ as follows

$$(\lambda; q)_k = \begin{cases} 1 & k = 0, \\ (1 - \lambda)(1 - \lambda q) \ldots (1 - \lambda q^{k-1}) & k \in \mathbb{N}. \end{cases}$$

By using the $q$-gamma function $\Gamma_q(z)$, we get

$$(q^\lambda; q)_k = \frac{(1 - q)^{\lambda+k}}{\Gamma_q(\lambda)}, \quad (k \in \mathbb{N}_0),$$

where (see [10])

$$\Gamma_q(z) = (1 - q)^{-z} \frac{(q^z; q)_\infty}{(q^z; q)_\infty}, \quad (|q| < 1).$$

Also, we note that $$(\lambda; q)_\infty = \prod_{k=0}^{\infty} (1 - \lambda q^k), \quad (|q| < 1),$$

and, the $q$-gamma function $\Gamma_q(z)$ is known $\Gamma_q(z+1) = [z]_q \Gamma_q(z)$, where $[k]_q$ denotes the basic $q$-number defined
by

\[(1.3) \quad [k]_q := \begin{cases} \frac{1-q^k}{1-q}, & k \in \mathbb{C}, \\ \prod_{i=1}^{k-1} q^i, & k \in \mathbb{N}. \end{cases} \]

Using definition formula (1.3) we have the next two products:

(i) For any non negative integer \( k \), the \( q \)-shifted factorial is given by

\[
[k]_q! := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{n=1}^{k-1} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}
\]

(ii) For any positive number \( r \), the \( q \)-generalized Pochhammer symbol is defined by

\[
[r]_{q,k} := \begin{cases} 1, & \text{if } k = 0, \\ \prod_{n=r}^{r+k-1} [n]_q, & \text{if } k \in \mathbb{N}. \end{cases}
\]

It is known in terms of the classical (Euler's) gamma function \( \Gamma(z) \), that

\[
\Gamma_q(z) \to \Gamma(z) \text{ as } q \to 1^-.
\]

Also, we observe that

\[
\lim_{q \to 1^-} \left\{ \frac{(q^\lambda; q)_k}{(1-q)^k} \right\} = (\lambda)_k,
\]

where \((\lambda)_k\) is the familiar Pochhammer symbol defined by

\[
(\lambda)_k = \begin{cases} 1, & \text{if } k = 0, \\ \lambda\lambda+1\cdots(\lambda + k - 1), & \text{if } k \in \mathbb{N}. \end{cases}
\]

The Bessel function of the first kind of order \( \nu \) is defined by the infinite series

\[
J_{\nu}(z) := \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{z}{2})^{2k+\nu}}{k! \Gamma(k+\nu+1)}, \quad (z \in \mathbb{C}, \nu \in \mathbb{R}),
\]

where \( \Gamma \) stands for the Gamma function. Recently, Szász and Kupán \cite{szasz2016} investigated the univalence of the normalized Bessel function of the first kind \( g_{\nu} : U \to \mathbb{C} \) defined by (see also \cite{hamza2012, hamza2014})

\[
g_{\nu}(z) := 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_{\nu}(z^{\frac{1}{2}}) = z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(\nu + 1)}{4^{k-1}(k-1)! \Gamma(k+\nu)} z^k, \quad (z \in U, \nu \in \mathbb{R}).
\]

For \( 0 < q < 1 \), the \( q \)-derivative operator for \( g_{\nu} \) is defined by

\[
D_q g_{\nu}(z) = D_q \left[ z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(\nu + 1)}{4^{k-1}(k-1)! \Gamma(k+\nu)} z^k \right] := \frac{g_{\nu}(qz) - g_{\nu}(z)}{z(q-1)} = 1 + \sum_{k=2}^{\infty} \frac{(-1)^{k-1} \Gamma(\nu + 1)}{4^{k-1}(k-1)! \Gamma(k+\nu)} [k, q] z^{k-1}, \quad z \in U,
\]
For $\nu > 0$, $\lambda > -1$, and $0 < q < 1$, El-Deeb and Bulboaca [8] (see also [7, 9, 26]) define the function $T_{\nu,q}^\lambda : U \to \mathbb{C}$ by $T_{\nu,q}^\lambda(z) = zD_qg_{\nu}(z)$, $z \in U$, where the function $M_{q,\lambda+1}$ is given by

$$
M_{q,\lambda+1}(z) := z + \sum_{k=2}^{\infty} \frac{[\lambda + 1]_q}{[k - 1]_q} z^k, \ z \in U
$$

Then

$$
T_{\nu,q}^\lambda(z) := z + \sum_{k=2}^{\infty} \frac{(-1)^{k-1}\Gamma(\nu + 1)}{4\Gamma(k+\nu)} \frac{\Gamma(k + \nu)}{\Gamma(k+\nu)} \frac{[k]_q!}{[\lambda + 1]_q} z^k, \ z \in U,
$$

$$(\nu > 0, \lambda > -1, 0 < q < 1).$$

El-Deeb and Bulboaca [8] used the definition of $q$-derivative along with the idea of convolutions to introduce the linear operator $N_{\nu,q}^\lambda : A \to A$ defined by

$$
N_{\nu,q}^\lambda f(z) := T_{\nu,q}^\lambda(z) * f(z) = z + \sum_{k=2}^{\infty} \psi_k a_k z^k, \ z \in U,
$$

(1.4)

where

$$
\psi_k := \frac{(-1)^{k-1}\Gamma(\nu + 1)}{4\Gamma(k+\nu)} \frac{\Gamma(k + \nu)}{\Gamma(k+\nu)} \frac{[k]_q!}{[\lambda + 1]_q},
$$

(1.5)

From definition relation (1.4), we can easily verify that the next relations hold for all $f \in A$:

(i) $[\lambda + 1, q] N_{\nu,q}^\lambda f(z) = [\lambda, q] N_{\nu,q}^{\lambda+1} f(z) + q^\gamma z \partial_q (N_{\nu,q}^{\lambda+1} f(z))$, $z \in U$;

(ii) $\lim_{q \to 1^{-}} N_{\nu,q}^\lambda f(z) = T_{\nu,1}^\lambda * f(z) =: T_{\lambda}^\nu f(z) = z + \sum_{k=2}^{\infty} \phi_k a_k z^k, \ z \in U$,

(1.6)

where

$$
\phi_k := \frac{k!}{(\lambda + 1)_k} \frac{(-1)^{k-1}\Gamma(\nu + 1)}{4\Gamma(k+\nu)}, \ z \in U.
$$

The class defined in (1.2) could be generalized by introducing the next class of functions, defined with the aid of the $N_{\nu,q}^\lambda$ operator.

**Definition 1.2.** Let the function $f \in A$ is said to be in the class $S^\lambda_{\nu,q}^\gamma(\gamma, A, B)$ if and only if

$$
1 + \frac{1}{\gamma} \left[ \frac{2z(N_{\nu,q}^\lambda f(z))'}{N_{\nu,q}^\lambda f(z) - N_{\nu,q}^\lambda f(-z)} - 1 \right] < \frac{1 + Az}{1 + Bz}
$$

$$
(-1 \leq A \leq B \leq 1, \nu > 0, \lambda > -1, 0 < q < 1, \gamma \in \mathbb{C}^+).
$$

Putting $q \to 1^{-}$ in the class $S^\lambda_{\nu,q}^\gamma(\gamma, A, B)$, we obtain that

$$
\lim_{q \to 1^{-}} S^\lambda_{\nu,q}^\gamma(\gamma, A, B) := G^\lambda_{\nu} \gamma(\gamma, A, B),
$$

where

$$
G^\lambda_{\nu} \gamma(\gamma, A, B) := \left\{ 1 + \frac{1}{\gamma} \left[ \frac{2z(T_{\lambda}^\nu f(z))'}{T_{\lambda}^\nu f(z) - T_{\lambda}^\nu f(-z)} - 1 \right] < \frac{1 + Az}{1 + Bz} \right\},
$$
(-1 ≤ A ≤ B ≤ 1, ν > 0, λ ≥ -1, γ ∈ C*).

Also, by using the \( N_{\nu,q}^\lambda \) operator, we define another class as follows.

**Definition 1.3.** Let the function \( f \in A \) is said to be in the class \( S_{\lambda,\nu,q}^\gamma(c)(1, A, B, \nu) \) if and only if

\[
1 + \frac{1}{\gamma} \left( z N_{\nu,q}^\lambda f(z) - 1 \right) < \frac{1 + A}{1 + B} z,
\]

\((-1 ≤ A ≤ B ≤ 1, ν > 0, λ ≥ -1, 0 < q < 1, \gamma ∈ C^*)

Putting \( q \to 1^- \) in the class \( S_{\lambda,\nu,q}^\gamma(c)(1, A, B, \nu) \), we obtain that

\[
\lim_{q \to 1^-} S_{\lambda,\nu,q}^\gamma(c)(1, A, B, \nu) := G_{\lambda,\nu}^\gamma(c)(1, A, B, \nu),
\]

where

\[
G_{\lambda,\nu}^\gamma(c)(1, A, B, \nu) := \left\{ 1 + \frac{1}{\gamma} \left( z N_{\nu,q}^\lambda f(z) - 1 \right) < \frac{1 + A}{1 + B} z \right\},
\]

\((-1 ≤ A ≤ B ≤ 1, ν > 0, λ ≥ -1, γ ∈ C^*)

The following lemmas will be needed to prove our results.

**Lemma 1.1.** [11, Lemma 2] If \( p(z) = 1 + p_1 z + p_2 z^2 + \cdots ∈ P[A,B] \), then

\[
|p_n| ≤ A - B.
\]

**Lemma 1.2.** [11, Lemma 3] If \( N \) be analytic and \( M \) starlike functions in \( U \) with \( N(0) = M(0) = 0 \), then

\[
\frac{|N'(z)/M'(z) - 1|}{|A - B(N'(z)/M'(z))|} < 1, -1 ≤ A ≤ B ≤ 1
\]

implies

\[
\frac{|N(z)/M(z) - 1|}{|A - B(N(z)/M(z))|} < 1, (z ∈ U).
\]

2. **Properties of the subclass \( S_{\lambda,\nu,q}^\gamma(c)(1, A, B) \)**

Unless otherwise mentioned, we shall assume in the reminder of this paper that \(-1 ≤ B ≤ A ≤ 1, ν > 0, λ ≥ -1, 0 < q < 1, γ ∈ C^*, \) and the powers are understood as principle values. Throughout this work, we use the following notation

\[
\prod_{i=k}^{k-1} A(i) = 1.
\]

**Theorem 2.1.** Let \( f(z) ∈ S_{\lambda,\nu,q}^\gamma(c)(1, A, B) \), then the following condition

\[
1 + \frac{1}{\gamma} \left( z N_{\nu,q}^\lambda \psi(z) - 1 \right) < \frac{1 + A}{1 + B} z,
\]

is satisfied for the odd function \( \psi \), where

\[
\psi(z) := \frac{f(z) - f(-z)}{2}.
\]
Proof. If \( f \in \mathcal{S}_{\lambda, q}^{\gamma, A, B} \), then there exists \( h \in \mathcal{P}[A, B] \), such that
\[
(2.3) \quad h(z) = 1 + \frac{1}{\gamma} \left[ \frac{2z(N_{\nu, q}^\lambda f(z))'}{N_{\nu, q}^\lambda f(z) - N_{\nu, q}^\lambda f(-z)} - 1 \right].
\]
It follows that
\[
\gamma(h(z) - 1) = \frac{2z(N_{\nu, q}^\lambda f(z))'}{N_{\nu, q}^\lambda f(z) - N_{\nu, q}^\lambda f(-z)} - 1,
\]
\[
\gamma(h(-z) - 1) = \frac{-2z(N_{\nu, q}^\lambda f(-z))'}{N_{\nu, q}^\lambda f(z) - N_{\nu, q}^\lambda f(-z)} - 1,
\]
which implies that
\[
(2.4) \quad \frac{h(z) + h(-z)}{2} = 1 + \frac{1}{\gamma} \left[ \frac{z(N_{\nu, q}^\lambda \psi(z))'}{N_{\nu, q}^\lambda \psi(z)} - 1 \right].
\]
On the other hand, \( h(z) \prec 1 + \frac{A z}{1 + B z} \), and \( \frac{A z}{1 + B z} \) is univalent, so by (1.1), we have
\[
\frac{h(z) + h(-z)}{2} \prec \frac{1 + A z}{1 + B z}.
\]
It follows (2.1). \( \square \)

Taking \( q \to 1^- \) in Theorem 2.1, we obtain the following corollary:

**Corollary 2.1.** Let \( f(z) \in \mathcal{G}_{\lambda, q}^{\gamma, A, B} \), then the following condition
\[
1 + \frac{1}{\gamma} \left[ \frac{z(N_{\nu, q}^\lambda \psi(z))'}{N_{\nu, q}^\lambda \psi(z)} - 1 \right] \prec \frac{1 + A z}{1 + B z},
\]
is satisfied for the odd function \( \psi \) given by (2.2).

**Theorem 2.2.** A function \( f \in \mathcal{S}_{\lambda, q}^{\gamma, A, B} \), if and only if there exists \( p \in \mathcal{P}[A, B] \) such that
\[
(2.5) \quad (N_{\nu, q}^\lambda f(z))' = (\gamma(h(z) - 1) + 1) \exp \left( \frac{\gamma}{2} \int_0^z \frac{h(t) + h(-t) - 2}{t} \, dt \right).
\]

**Proof.** From Theorem 2.1, we have (2.4), it implies
\[
\frac{(N_{\nu, q}^\lambda \psi(z))'}{N_{\nu, q}^\lambda \psi(z)} = \frac{1}{z} + \frac{\gamma}{2} \left( \frac{h(z) + h(-z) - 2}{z} \right).
\]
Integrating the above equation,
\[
(2.6) \quad N_{\nu, q}^\lambda \psi(z) = z \exp \left( \frac{\gamma}{2} \int_0^z \frac{h(t) + h(-t) - 2}{t} \, dt \right).
\]
Since \( f \in \mathcal{S}_{\lambda, q}^{\gamma, A, B} \), then from (2.3), we obtain
\[
z(N_{\nu, q}^\lambda f(z))' = (\gamma(h(z) - 1) + 1)N_{\nu, q}^\lambda \psi(z)
\]
Using (2.4) and above equation, we get (2.5). \( \square \)

Taking \( q \to 1^- \) in Theorem 2.2 we obtain the following corollary:
Corollary 2.2. A function \( f \in \mathcal{G}_b^{\lambda, \nu}(\gamma, A, B) \), if and only if there exists \( p \in \mathcal{P}[A, B] \) such that

\[
(\mathcal{I}_b^\lambda f(z))' = (\gamma(h(z) - 1) + 1) \exp \left( \frac{\gamma}{2} \int_0^z \frac{h(t) + h(-t) - 2 \omega(t)}{t} \, dt \right).
\]

Theorem 2.3. If \( f(z) \in \mathcal{S}_b^{\lambda, \nu, \vartheta}(\gamma, A, B) \), then for all \( n \geq 1 \),

\[
|a_{2n}| \leq \frac{\gamma|(A - B)|}{2^n n! \vartheta^n} \prod_{k=1}^{n-1} (\gamma|(A - B)| + 2k),
\]

\[
|a_{2n+1}| \leq \frac{\gamma|(A - B)|}{2^n n! |\vartheta|^{n+1}} \prod_{k=1}^{n} (\gamma|(A - B)| + 2k).
\]

where \( \vartheta_k \), for all \( k \geq 2 \) are given by \( |\vartheta| \).

Proof. Since \( f \in \mathcal{S}_b^{\lambda, \nu, \vartheta}(\gamma, A, B) \), Definition 1.2 yields

\[
1 + \frac{1}{\gamma} \left[ \frac{2z(N_{\vartheta, \vartheta}^\lambda f(z))'}{N_{\vartheta, \vartheta}^\lambda f(z) - N_{\vartheta, \vartheta}^\lambda f(-z)} - 1 \right] = 1 + \frac{A_\vartheta(z)}{1 + B\vartheta(z)}.
\]

Assuming that

\[
h(z) = 1 + \sum_{k=1}^{\infty} c_k z^k = \frac{1 + A\vartheta(z)}{1 + B\vartheta(z)}
\]

In view of (2.9) and (2.10), we get

\[
2z(N_{\vartheta, \vartheta}^\lambda f(z))' = (N_{\vartheta, \vartheta}^\lambda f(z) - N_{\vartheta, \vartheta}^\lambda f(-z)) \left( 1 + \gamma \sum_{k=1}^{\infty} c_k z^k \right).
\]

It follows from 1.3 that

\[
z + 2 \vartheta_2 a_2 z^2 + 3 \vartheta_3 a_3 z^3 + 4 \vartheta_4 a_4 z^4 + \cdots + 2n \vartheta_{2n} a_{2n} z^{2n} + (2n + 1) \vartheta_{2n+1} a_{2n+1} z^{2n+1} + \cdots
\]

\[
= \left( z + \vartheta_2 a_2 z^2 + \vartheta_3 a_3 z^3 + \cdots + \vartheta_{2n-1} a_{2n-1} z^{2n-1} + \vartheta_{2n} a_{2n} z^{2n} + \vartheta_{2n+1} a_{2n+1} z^{2n+1} + \cdots \right)
\]

\[
\cdot \left( 1 + \gamma c_1 z + \gamma c_2 z^2 + \cdots \right)
\]

Equating the coefficients of the like powers of \( z \), we obtain

\[
2 \vartheta_2 a_2 = \gamma c_1, \quad 2 \vartheta_3 a_3 = \gamma c_2,
\]

\[
4 \vartheta_4 a_4 = \gamma c_3 + \gamma c_1 \vartheta_3 a_3, \quad 4 \vartheta_5 a_5 = \gamma c_4 + \gamma c_2 \vartheta_3 a_3,
\]

\[
2n \vartheta_{2n} a_{2n} = \gamma c_{2n-1} + \gamma c_{2n-3} \vartheta_3 a_3 + \gamma c_{2n-5} \vartheta_5 a_5 + \cdots + \gamma c_1 \vartheta_{2n-1} a_{2n-1},
\]

\[
2n \vartheta_{2n+1} a_{2n+1} = \gamma c_{2n} + \gamma c_{2n-2} \vartheta_3 a_3 + \gamma c_{2n-4} \vartheta_5 a_5 + \cdots + \gamma c_2 \vartheta_{2n-1} a_{2n-1}.
\]

We prove (2.11) and (2.13) using mathematical induction. Using Lemma 2.1, 2.14 and (2.12) respectively, we get

\[
|a_2| \leq \frac{\gamma}{2|\vartheta_2|(A - B)}, \quad |a_3| \leq \frac{\gamma}{2|\vartheta_3|(A - B)},
\]
\[ |a_4| \leq \frac{|\gamma|(A - B)}{8|\psi|} (2 + |\gamma|(A - B)), \quad |a_5| \leq \frac{|\gamma|(A - B)}{8|\psi|} (2 + |\gamma|(A - B)). \]

It follows that (2.7) and (2.8) hold for \( n = 1, 2 \). Equation (2.13) in conjunction with Lemma (4) yields

\[ |a_{2n}| \leq \frac{|\gamma|(A - B)}{2n|\psi|^{2n}} \left( 1 + \sum_{r=1}^{n-1} |\psi|^{2r+1}|a_{2r+1}| \right) \]

Next, we assume that (2.7) and (2.8) hold for \( 3, 4, \ldots, n - 1 \). Thus the above inequality yields

\[ (2.15) \quad |a_{2n}| \leq \frac{|\gamma|(A - B)}{2n|\psi|^{2n}} \left( 1 + \sum_{r=1}^{n-1} \frac{|\gamma|(A - B)}{2^{2r} r!} \prod_{k=1}^{r-1} (|\gamma|(A - B) + 2k) \right). \]

To complete the proof it is sufficient to show that

\[ (2.16) \quad \frac{|\gamma|(A - B)}{2m|\psi|^{2m}} \left( 1 + \sum_{r=1}^{m-1} \frac{|\gamma|(A - B)}{2^{2r} r!} \prod_{k=1}^{r-1} (|\gamma|(A - B) + 2k) \right) \]

\[ = \frac{|\gamma|(A - B)}{2m|\psi|^{2m}} \left( 1 + \frac{m-1}{2^{2m-1}(m-1)!} \prod_{k=1}^{m-2} (|\gamma|(A - B) + 2k) \right) \]

It is easy to see that (2.10) is valid for \( m = 3 \). Now, suppose that (2.16) is true for \( 4, \ldots, m - 1 \). Then (2.15) follows that

\[ \frac{|\gamma|(A - B)}{2m|\psi|^{2m}} \left( 1 + \sum_{r=1}^{m-1} \frac{|\gamma|(A - B)}{2^{2r} r!} \prod_{k=1}^{r-1} (|\gamma|(A - B) + 2k) \right) \]

\[ = \frac{m-1}{m|\psi|^{2m-2}} \left( \frac{|\gamma|(A - B)}{2^{2m-1}(m-1)!|\psi|^{2m-2}} \prod_{k=1}^{m-2} (|\gamma|(A - B) + 2k) \right) \]

\[ \quad + \frac{|\gamma|(A - B)}{2m|\psi|^{2m}} \frac{|\gamma|(A - B)}{2^{2m-1}(m-1)!} \prod_{k=1}^{m-2} (|\gamma|(A - B) + 2k) \]

\[ = \frac{m-1}{m|\psi|^{2m-2}} \left( \frac{|\gamma|(A - B)}{2^{2m-1}(m-1)!|\psi|^{2m-2}} \prod_{k=1}^{m-2} (|\gamma|(A - B) + 2k) \right) \]

\[ \quad + \frac{|\gamma|(A - B)}{2m|\psi|^{2m}} \frac{|\gamma|(A - B)}{2^{2m-1}(m-1)!} \prod_{k=1}^{m-2} (|\gamma|(A - B) + 2k) \]
where $\phi_m$. That is, (2.16) holds for $\forall m$. From (2.15) and (2.16) we obtain (2.7). Similary we can prove (2.8). This completes the proof of Theorem 2.3.

Taking $q \to 1^-$ in Theorem 2.3 we obtain the following corollary:

**Corollary 2.3.** Let $f(z) \in S_\gamma^\lambda,\nu(q, A, B)$, then for all $n \geq 1$,

\[ |a_{2n}| \leq \frac{|\gamma|(A - B)}{2^n m!|\phi_{2n}|} \prod_{k=1}^{n-1} (|\gamma|(A - B) + 2k), \]

\[ |a_{2n+1}| \leq \frac{|\gamma|(A - B)}{2^n m!|\phi_{2n+1}|} \prod_{k=1}^{n-1} (|\gamma|(A - B) + 2k), \]

where $\phi_k$, for all $k \geq 2$ are given by (1.6).

**Theorem 2.4.** If the function $f \in S_\gamma^\lambda,\nu(q, A, B)$, then $F \in S_\gamma^\lambda,\nu(q, A, B)$, where

\[ F(z) = \frac{2}{z} \int_0^z f(t) \, dt \]

**Proof.** From (2.17) it is easy to see that

\[ 1 + \frac{1}{\gamma} \left[ \frac{2z(N_\nu^\lambda_f(z))^\prime}{N_\nu^\lambda(f(z) - N_\nu^\lambda_f(-z))} - 1 \right] \]

\[ = \frac{2zN_\nu^\lambda_f(z) + (\gamma - 3) \int_0^z N_\nu^\lambda_f(t) \, dt + (\gamma - 1) \int_0^z N_\nu^\lambda_f(-t) \, dt}{\gamma(\int_0^z N_\nu^\lambda_f(t) \, dt + \int_0^z N_\nu^\lambda_f(-t) \, dt)}. \]

Define $N$ and $M$ be the numerator and denominator functions respectively. Therefore,

\[ \frac{zM'(z)}{M(z)} = \frac{zN_\nu^\lambda_f(z) - zN_\nu^\lambda_f(-z)}{\int_0^z N_\nu^\lambda_f(t) \, dt + \int_0^z N_\nu^\lambda_f(-t) \, dt} \]

\[ = \frac{1}{2} \left( \frac{2zG'(z)}{G(z) - G(-z)} + \frac{2(-z)G'(-z)}{G(z) - G(-z)} \right). \]
where \( G(z) = \int_0^z N_{\nu,q}^f(t) \, dt \). Since \( f \in S_{\lambda}^{\gamma,\nu,q}(\gamma, A, B) \), it follows that
\[
1 + \frac{1}{\gamma} \left[ \frac{2zG''(z)}{G(z) - G'(-z)} - 1 \right] < \frac{1 + Az}{1 + Bz},
\]
and \( G(z) \in C_{\ast}(b, A, B) \subset S_{\ast}(b, A, B) \subset S_{\ast}^{\gamma} \). From \( (3.4) \), it follows that \( M(z) \) is starlike functions. In addition to

From Lemma 1.2, we have
\[
(3.4)
\]

Assuming that \( \gamma \in S \), for all \( \nu, q \) are given by \( (2.17) \), we obtain the following corollary:

**Corollary 2.4.** If the function \( f \in G_{\ast}^{\gamma,\nu}(\gamma, A, B) \), then \( F \) given by \( (2.17) \) belongs to the class \( G_{\ast}^{\gamma,\nu}(\gamma, A, B) \).

**The subclass \( S_{\ast}^{\lambda,\nu,q}(\gamma, A, B) \)**

**Theorem 3.1.** Let \( f \in S_{\ast}^{\lambda,\nu,q}(\gamma, A, B) \), then for all \( n \geq 1 \),
\[
(3.1)
\]

\[
(3.2)
\]

where \( \psi_k \) for all \( k \geq 2 \) are given by \( (1.8) \).

**Proof.** Since \( f \in S_{\ast}^{\lambda,\nu,q}(\gamma, A, B) \), Definition 1.3 yields
\[
(3.3)
\]

Assuming that
\[
(3.4)
\]

from \( (3.3) \) and \( (3.4) \), we obtain
\[
2z(N_{\nu,q}^f(z))' = (N_{\nu,q}^f(z) + N_{\nu,q}^f(z)) \left( 1 + \gamma \sum_{k=1}^{\infty} c_k z^k \right).
\]
It follows from (1.1) that
\[ z + 2\psi_2 a^2 z^2 + 3\psi_3 a^3 z^3 + 4\psi_4 a^4 z^4 + \cdots + 2n\psi_{2n} a_{2n} z^{2n} + (2n + 1)\psi_{2n+1} a_{2n+1} z^{2n+1} + \cdots \]
\[ = (z + \psi_2 a^2 z^2 + \psi_3 a^3 z^3 + \psi_4 a^4 z^4 + \cdots + \psi_{2n} a_{2n} z^{2n} + \psi_{2n+1} a_{2n+1} z^{2n+1} + \cdots) \cdot (1 + \gamma_1 z + \gamma_2 z^2 + \cdots) \]

Equating the coefficients of the like powers of \( z \), we obtain
\[ \psi_2 a_2 = \gamma c_2, \quad 2\psi_3 a_3 = \gamma c_2 + \gamma c_1 \psi_2 a_2, \]
\[ 3\psi_4 a_4 = \gamma c_3 + \gamma c_2 \psi_3 a_2 + \gamma c_1 \psi_3 a_3 + 4\psi_5 a_5 = \gamma c_4 + \gamma c_3 \psi_3 a_2 + \gamma c_2 \psi_3 a_3 + \gamma c_1 \psi_4 a_4, \]
\[ (2n + 1)\psi_{2n+1} a_{2n+1} = \gamma c_{2n+1} + \gamma c_{2n+2} \psi_{2n+2} a_2 + \cdots + \gamma c_{2n} \psi_{2n-2} a_{2n-2} + \gamma c_{2n} \psi_{2n-1} a_{2n-1}, \]
\[ 2n\psi_{2n} a_{2n} = \gamma c_{2n} + \gamma c_{2n-1} \psi_{2n-2} a_2 + \cdots + \gamma c_{2n-1} \psi_{2n-2} a_{2n-2} + \gamma c_{2n-1} \psi_{2n-3} a_{2n-3} + \gamma c_{2n} \psi_{2n-3} a_{2n-3}. \]

Using Lemma 1.1 (3.5) and 3.6 respectively, we get
\[ |a_2| \leq \frac{\gamma |(A - B)|}{|\psi_2|} (1 + \gamma |(A - B)|), \quad |a_3| \leq \frac{\gamma |(A - B)|}{2|\psi_3|} (1 + \gamma |(A - B)|), \]
\[ |a_4| \leq \frac{\gamma |(A - B)|}{2.3|\psi_4|} (1 + \gamma |(A - B)|), \]
\[ |a_5| \leq \frac{\gamma |(A - B)|}{2.3.4|\psi_5|} (1 + \gamma |(A - B)|). \]

It follows that (3.1) and (3.2) hold for \( n = 1, 2 \). Equation (3.7) in conjunction with Lemma 1.1 yields
\[ |a_{2n}| \leq \frac{\gamma |(A - B)|}{(2n - 1)|\psi_{2n}|} \left( 1 + \sum_{r=1}^{n-1} |\psi_{2r}| |a_{2r}| + \sum_{r=1}^{n-1} |\psi_{2r+1}| |a_{2r+1}| \right) \]

Next, we assume that (3.4) and (3.5) hold for \( 3, 4, \ldots, n - 1 \). Thus the above inequality yields
\[ |a_{2n}| \leq \frac{\gamma |(A - B)|}{(2n - 1)|\psi_{2n}|} \left( 1 + \sum_{r=1}^{n-1} |\psi_{2r}| \frac{|(A - B)|}{(2r - 1)!} \prod_{i=1}^{2r-2} (i + \gamma |(A - B)|) \right. \]
\[ + \sum_{r=1}^{n-1} |\psi_{2r}| \frac{|(A - B)|}{2r!} \prod_{i=1}^{2r-1} (i + \gamma |(A - B)|) \right). \]

In order to complete the proof it is sufficient to show that
\[ \frac{\gamma |(A - B)|}{(2m - 1)|\psi_{2m}|} \left( 1 + \sum_{r=1}^{m-1} \frac{|\psi_{2r}| |(A - B)|}{(2r - 1)!} \prod_{i=1}^{2r-2} (i + \gamma |(A - B)|) \right. \]
\[ + \sum_{r=1}^{m-1} \frac{|\psi_{2r}| |(A - B)|}{2r!} \prod_{i=1}^{2r-1} (i + \gamma |(A - B)|) \right). \]
That is, (3.10) is holds for \( m = n \). From (3.9) and (3.10), we obtain (3.11). Similarly we can prove (3.2). \( \square \)
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