Lessons from supersymmetry: 
“Instead-of-Confinement” Mechanism

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Abstract

We review physical scenarios in different vacua of $\mathcal{N} = 2$ supersymmetric QCD deformed by the mass term $\mu$ for the adjoint matter. This deformation breaks supersymmetry down to $\mathcal{N} = 1$ and, at large $\mu$, the theory flows to $\mathcal{N} = 1$ QCD. We focus on dynamical scenarios which can serve as prototypes of what we observe in real-world QCD. The so-called $r = N$ vacuum is especially promising in this perspective. In this vacuum an “instead-of-confinement” phase was identified previously, which is qualitatively close to the conventional QCD confinement: the quarks and gauge bosons screened at weak coupling, at strong coupling evolve into monopole-antimonopole pairs confined by non-Abelian strings. We review genesis of this picture.

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1 Introduction

Fifty years ago quarks introduced by Gell-Mann and Zweig \cite{1}, which shortly after became three-colored \cite{2,3} opened the gate to modern high-energy physics. Discovery of asymptotic freedom in Yang-Mills theories \cite{4} and creation of quantum chromodynamics (QCD) \cite{3,4} completed this process. Gradually it became clear that a peculiar phenomenon in QCD – quark (or, more generally, color) confinement presented a novel dynamical phase in field theory with which one had never encountered before. This phase is inherent to Yang-Mills theories at strong coupling. A possible physical explanation of the confinement phase – the so called dual Meissner effect – was suggested in the mid-1970s \cite{5} but nobody managed to demonstrate it analytically under controllable conditions until much later.

The advent of supersymmetry changed that. In the mid-1990s Seiberg and Witten considered $\mathcal{N} = 2$ super-Yang-Mills theory with the $SU(2)$ gauge group. In this model a vacuum manifold exists, i.e. a (complex) flat direction. Seiberg and Witten identified \cite{6} two points on this manifold in which a monopole or a dyon become massless. A crucial fact was that supersymmetry allowed them to use analytical extrapolations to give a precise meaning to monopoles and dyons in non-Abelian field theory. Then they showed that a weak deformation of the above model forces the monopoles/dyons to condense, thus triggering the dual Meissner effect in super-Yang-Mills and ensuing color confinement. The Seiberg-Witten vacua \cite{6} are referred to as the monopole vacua.

The Seiberg-Witten solution is based on a cascade gauge symmetry breaking. At a high energy scale the non-Abelian gauge group is broken down to $U(1)$ (or a generic Abelian subgroup in more general models) by condensation of the adjoint scalar field $A$. An effective low-energy theory near the monopole vacuum includes the Abelian gauge fields and magnetically charged matter represented by light monopoles or dyons.

The latter condense upon a small $\mathcal{N} = 2$ breaking deformation $\mu A^2$ at a much lower scale $\mu$. Their vacuum expectation values (VEVs) are of the order of $\Lambda_2$. Simultaneously, formation of confining color-electric flux tubes (strings) occurs. Their tension is minute being proportional to $\mu \Lambda_2$.

The Seiberg-Witten solution gave a strong impetus for studies of Yang-

\footnote{We introduce a shorthand notation for the dynamical scale parameters $\Lambda_2$ and $\Lambda_1$, for the scale parameters $\Lambda_{\mathcal{N}=2}$ and $\Lambda_{\mathcal{N}=1}$ appearing in $\mathcal{N} = 2$ and $\mathcal{N} = 1$ theories, respectively.}
Mills dynamics at strong coupling. Another inspiration came from the so-called Seiberg duality \cite{7,8}. Pairs of completely different $\mathcal{N} = 1$ super-Yang-Mills theories (supersymmetric QCD, or SQCD for short) were identified which were related as follows: one theory from the pair in the ultraviolet domain (UV), flows to the second theory from the pair in the infrared (IR) limit. Of course, this can happen only if the second theory is at strong coupling. Formally the Seiberg duality is an extension of the notion of the ’t Hooft anomaly matching \cite{9}. Its roots are deeper, however, and shortly after its discovery were found to ascend to string theory.

Before the Seiberg-Witten solution theorists were aware of three basic phases in gauge theories: the Higgs, Coulomb and confining regimes. It turned out that the phase structure of super-Yang-Mills theory is much richer. For instance, contrived matter sectors were shown \cite{10} to lead to a number of exotic phases unheard of previously.

The Seiberg duality refers to $\mathcal{N} = 1$ massless SQCD. Several years ago we started a project of detailing Seiberg’s duality in various vacua by using the exact Seiberg-Witten solution as an intermediate step in our derivations, see e.g. \cite{11,12,13,14}. Detailing means that we isolated discrete vacua with specific properties (to be discussed below) by adjusting quark mass parameters which are present in our $\mathcal{N} = 2$ deformed theory. To pass from $\mathcal{N} = 2$ to $\mathcal{N} = 1$ we had to consider large rather than small values of the parameter $\mu$ in front of the deformation term. In these studies we discovered yet a novel phase of super-Yang-Mills which we called “instead of confinement” \cite{15}. A review of this phase is one of the main goals of this article.

Another goal – discussion of confining strings which are as close as possible to the conjectured QCD strings – also requires the large-$\mu$ limit. Indeed, the Seiberg-Witten confinement at small $\mu$ is essentially Abelian \cite{16,17,18,19}, with strings of the Abrikosov type. We wanted to construct non-Abelian strings, i.e. those with non-Abelian moduli on their world sheet. At large $\mu$ adjoint scalars of the Seiberg-Witten model decouple and we then approach $\mathcal{N} = 1$ limit in which the confinement mechanism is expected to be non-Abelian. Moreover, it is believed that confinement in real-world QCD is also non-Abelian.

Thus, there were two reasons why we wanted, starting from the small-$\mu$ limit to pass to large values of $\mu$ (or, equivalently, the $\mathcal{N} = 1$ limit). We hoped that in passing from $\mathcal{N} = 2$ SQCD via the decoupling of the adjoint scalars, we would be able to see how the Abelian Seiberg-Witten confinement transforms itself into a regime with non-Abelian confinement.
This program turned out to be challenging. The effective (dual) theory which describes low-energy physics of \( \mu \)-deformed \( \mathcal{N} = 2 \) SQCD below the scale of the adjoint VEVs is in fact the Abelian gauge theory with light monopoles. It is infrared-free and stays at weak coupling as long as VEVs of the monopoles (not to be confused with the adjoint VEVs) are small enough. As was mentioned above, these VEVs are proportional to \( \mu A_2 \); thus, at small \( \mu \) the condition for week coupling is met. However, moving toward the desired large-\( \mu \) limit breaks this condition. The effective theory goes into a strong coupling regime. The analytic control is lost. This was a genuine obstacle.

Recent breakthrough developments allowed us to resolve this problem. First, we developed a benchmark model which was slightly different from that of Seiberg and Witten. In our benchmark model the gauge group is \( U(N) \) (rather than \( SU(N) \)) and the number of quark flavors is \( N_f \) where \( (N+1) < N_f < 3/2N \). In this version of \( \mathcal{N} = 2 \) SQCD with the \( \mu \) deformation switched on one can identify a number of the so-called called \( r \) vacua which are characterized by a parameter \( r \), the number of the condensed (s)quarks in the classical domain of large and generic quark mass parameters \( m_A \) (here \( A = 1, ..., N_f \)). Clearly, \( r \) cannot exceed \( N \), the rank of the gauge group. The monopole vacua (there are \( N \) of them) corresponds to \( r = 0 \). Quarks do not condense in the monopole vacua.

The key observation is as follows \[14, 13\]. In this model there is a subset of \( r \) vacua (to be referred to as zero vacua) which can be found at \( r < (N_f - N) \). In these vacua the gaugino condensate is parametrically small provided the quark mass parameters are small too. This subset includes a part of the monopole vacua. The smallness of the gaugino condensate ensures that quantum effects are small in the zero vacua, hence physics can be described in terms of weakly coupled dual theory.

In \[14, 13\] we demonstrated that the Seiberg-Witten confinement, present in the zero vacua at small \( \mu \), disappears at large \( \mu \). This happens because the scale of the gaugino condensate controls the confinement radius (tensions of the confining strings) in a certain sector of the dual theory. As we increase \( \mu \), confinement becomes weaker and weaker in this sector, and eventually quarks are “liberated”. Effective dual theory has \( U(N_f - N) \) gauge group. This theory is infrared-free and stays at weak coupling at low energies.

It appears to be in the mixed Coulomb-Higgs phase for quarks. Namely, \( r \) quarks condense, while the \( U(N_f - N - r) \) subgroup is in the Coulomb phase, see \[14, 13\] for a more rigorous and detailed discussion. Therefore, the zero vacua, (in particular, the monopole zero vacua of strongly coupled
μ-deformed SQCD) are not good prototypes for physics in real-world QCD.

Nevertheless, μ-deformed SQCD is a rich theory with a variety of r vacua with different infrared behaviors. We discovered that certain other vacua in this theory are more promising in providing us with a prototype for real-world QCD dynamics. This will be described below.

As was mentioned above, the zero vacua support a weakly coupled dual description at large μ due to the smallness of the gaugino condensate. Another exceptional vacuum is the r = N vacuum in which the maximal number of quarks condense (in the weakly coupled domain of the large quark masses). In this vacuum the gaugino condensate is identically zero. This vacuum also has a weakly coupled – the so-called r-dual – description, in the large-μ limit [12, 15, 13].

In this paper we review infrared dynamics in μ-deformed SQCD focusing mostly on the r = N vacuum. It is just this vacuum which supports a new phase, namely, the “instead-of-confinement” phase [11, 12, 15, 13]. In this phase quarks and gauge bosons screened at weak coupling evolve at strong coupling into monopole-antimonopole pairs confined by non-Abelian strings.

These monopole-antimonopole stringy mesons have “correct” (adjoint or singlet) quantum numbers with respect to the global group, in much the same way as mesons in real-world QCD. Moreover, they lie on the Regge trajectories. Thus, this phase is qualitatively rather similar to what we observe in real-world QCD. The role of QCD constituent quarks is played by monopoles.

The paper is organized as follows. In Sec. 2 we review the r = N vacuum at weak coupling, i.e. at large ξ (the parameter ξ is defined in Eq. (2.4)). In Sec. 3 we turn to r-duality in this vacuum and the “instead-of-confinement” mechanism in the small-μ limit. In Sec. 4 we discuss the large μ-limit. Section 5 briefly summarizes our results and conclusions.

2 The r = N vacuum at large ξ

Our benchmark model reduces to N = 2 SQCD with the U(N) gauge group in the absence of μ-deformation. The matter sector consists of N_f massive quark hypermultiplets. We assume that N_f > N + 1 but N_f < 3/2 N. The latter inequality ensures that the dual theory is infrared free.

This theory is described in detail in [20, 21], see also the reviews in [22]. The field content is as follows. The N = 2 vector multiplet consists of the
U(1) gauge field $A_\mu$ and the SU($N$) gauge field $A^a_\mu$, where $a = 1, \ldots, N^2 - 1$, as well as their Weyl fermion superpartners plus complex scalar fields $a$, and $a^a$ and their Weyl superpartners, respectively. These complex scalar fields present the bosonic sector of the adjoint scalars.

The matter sector of the U($N$) theory contains $N_f$ quark multiplets which consist of the complex scalar fields $q^{kA}$ and $\tilde{q}_{Ak}$ (squarks) and their fermion superpartners — all in the fundamental representation of the SU($N$) gauge group. Here $k = 1, \ldots, N$ is the color index while $A$ is the flavor index, $A = 1, \ldots, N_f$.

In addition, as was mentioned, we add the mass term $\mu$ for the adjoint scalar superfield breaking $N = 2$ supersymmetry down to $N = 1$. This deformation term

$$W_{\text{det}} = \mu \Tr \Phi^2, \quad \Phi \equiv \frac{1}{2} A + T^a A^a$$

(2.1)
does not break $N = 2$ supersymmetry in the small-$\mu$ limit, see [17, 19, 20], however, at large $\mu$ this theory flows to $N = 1$ SQCD. The fields $A$ and $A^a$ in Eq. (2.1) are chiral superfields, the $N = 2$ superpartners of the U(1) and SU($N$) gauge bosons.

In this theory we can find a set of $r$ vacua, where $r$ is the number of condensed (s)quarks in the classical domain of large generic quark masses $m_A$ ($A = 1, \ldots, N_f$, and $r \leq N$). In this review we will focus on the $r = N$ vacua. Dynamical scenarios in the $r < N$ vacua are considered in [14, 13].

These vacua have the maximal possible number of condensed quarks, namely, $r = N$. Moreover, the gauge group U($N$) is completely Higgsed in these vacua, and, as a result, they support non-Abelian strings [23, 24, 20, 25]. These strings result in confinement of monopoles.

First, we will assume that $\mu$ is small, much smaller than the quark masses

$$|\mu| \ll |m_A|, \quad A = 1, \ldots, N_f.$$  

(2.2)

In the quasiclassical domain of large quark masses the squark fields develop VEVs triggered by the deformation parameter $\mu$,

$$\langle q^{kA} \rangle = \langle \tilde{q}^{kA} \rangle = \frac{1}{\sqrt{2}} \left( \begin{array}{cccccc} \sqrt{\xi_1} & \ldots & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & \sqrt{\xi_N} & 0 & \ldots & 0 \end{array} \right),$$

$$k = 1, \ldots, N, \quad A = 1, \ldots, N_f,$$

(2.3)

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where we present the squark fields as matrices in the color \((k)\) and flavor \((A)\) indices, while new parameters \(\xi\) are given (in the quasiclassical approximation) by

\[
\xi_P \approx 2 \mu m_P, \quad P = 1, ..., N. \tag{2.4}
\]

The quark condensate \((2.3)\) implies the spontaneous breaking of both gauge and flavor symmetries. A diagonal global \(SU(N)\) combining the gauge \(SU(N)\) and an \(SU(N)\) subgroup of the flavor \(SU(N_f)\) survives in the limit of equal (or almost equal) quark masses. This is the color-flavor locking.

Thus, the unbroken global symmetry we are left with is

\[
SU(N)_{C+F} \times SU(\tilde{N}) \times U(1), \quad \tilde{N} \equiv N_f - N. \tag{2.5}
\]

Here \(SU(N)_{C+F}\) is an unbroken global color-flavor rotation, which involves only the first \(N\) flavors, while the \(SU(\tilde{N})\) factor refers to the flavor rotation of the remaining \(\tilde{N}\) quarks.

The presence of the global \(SU(N)_{C+F}\) symmetry is the reason for formation of the non-Abelian strings \([23, 24, 20, 25, 21]\). At small \(\mu\) these strings are BPS-saturated \([17, 19]\), and their tensions are determined \([21]\) by the parameters \(\xi_P\) introduced in \((2.4)\),

\[
T_P = 2\pi|\xi_P|, \quad P = 1, ..., N. \tag{2.6}
\]

The above non-Abelian strings, with non-Abelian moduli in the coset

\[
SU(N)_{C+F}/SU(N - 1) \times U(1)
\]
on their world sheet, confine monopoles. In fact, in the \(U(N)\) theories confined elementary monopoles are junctions of two “neighboring” strings with labels \(P\) and \((P + 1)\), see \([22]\) for a more detailed review.

Now, let us briefly discuss the elementary excitation spectrum in the bulk. Since both \(U(1)\) and \(SU(N)\) gauge groups are broken by the squark condensation, all gauge bosons become massive. To the leading order in \(\mu\), \(\mathcal{N} = 2\) supersymmetry is not broken. In fact, with nonvanishing \(\xi_P\)’s (see Eq. \((2.4)\)), both the quarks and adjoint scalars combine with the gauge bosons to form long \(\mathcal{N} = 2\) supermultiplets \([19]\). In the equal quark mass limit \(\xi_P \equiv \xi\), and all states come in representations of the unbroken global group \((2.5)\), namely, in the singlet and adjoint representations of \(SU(N)_{C+F}\),

\[(1, 1), \quad (N^2 - 1, 1), \tag{2.7}\]
and in the bifundamental representations

\[(\bar{N}, \tilde{N}), \quad (N, \bar{N})]. \quad (2.8)\]

The representations in (2.7) and (2.8) are marked with respect to two non-Abelian factors in (2.5). The singlet and adjoint fields are (i) the gauge bosons, and (ii) the first \(N\) flavors of squarks \(q^{P} (P = 1, \ldots, N)\), together with their fermion superpartners. The bifundamental fields are the quarks \(q^{KK}\) with \(K = N + 1, \ldots, N_f\). Quarks transform in the two-index representations of the global group (2.5) due to the color-flavor locking.

The above quasiclassical analysis is valid if the theory is at weak coupling. From (2.3) we see that the weak coupling condition is

\[\sqrt{\xi} \sim \sqrt{\mu m} \gg \Lambda^2, \quad (2.9)\]

where we assume all quark masses to be of the same order \(m_A \sim m\). This condition means that the quark masses are large enough to compensate for the smallness of \(\mu\).

3 \(r\)-Dual theory

In this section we review non-Abelian \(r\) duality in the \(r = N\) vacua first established in [11, 26] at small \(\mu\). This is an important part of our consideration on which we base further analysis, in particular the conclusion of the instead-of-confinement phase.

Let us relax the condition (2.9) and pass to the strong coupling domain at

\[|\sqrt{\xi_P}| \ll \Lambda_2, \quad |m_A| \ll \Lambda_2, \quad (3.1)\]

while keeping \(\mu\) small.

In non-supersymmetric theories such as QCD this step cannot be carried out analytically. This is the point where supersymmetry becomes important. More exactly, we exploit the exact Seiberg-Witten solution on the Coulomb branch [6] in our theory. We start at large \(\xi \sim \mu m\) (in the equal quark mass limit) and then go to the equal mass small-\(\xi\) limit via the domain of large \(\Delta m \sim \Delta m_{AB} \equiv (m_A - m_B)\).

At \(\Delta m \sim \Lambda_2\) the theory enters a strong coupling regime and undergoes a crossover. We use the Seiberg-Witten curve to find the dual gauge group.
The domain (3.1) can be described in terms of weakly coupled (infrared free) $r$-dual theory with the gauge group

$$U(\tilde{N}) \times U(1)^{N-\tilde{N}},$$

and $N_f$ flavors of light quark-like dyons. Note, that we refer to our dual theory as the “$r$ dual” because $N = 2$ duality described here can be generalized to other $r$ vacua with $r > N_f/2$.

This leads to a theory with the dual gauge group $U(N_f - r) \times U(1)^{N-N_f+r}$ [15]. However, the $N = 1$ deformation of these $r$ dual theories at larger $\mu$ can be performed at weak coupling only in the $r = N$ vacuum [14], which we will discuss below.

The light dyons $D^{lA}$ ($l = 1, ..., \tilde{N}$ and $A = 1, ..., N_f$) are in the fundamental representation of the gauge group $SU(\tilde{N})$ and are charged under the Abelian factors indicated in Eq. (3.2). In addition, there are $(N - \tilde{N})$ light dyons $D^J$ ($J = \tilde{N} + 1, ..., N$), neutral under the $SU(\tilde{N})$ group, but charged under the $U(1)$ factors.

The color charges of all these dyons are identical to those of quarks. This is the reason why we call them quark-like dyons. However, these dyons are not quarks. As we will show below they belong to a different representation of the global color-flavor locked group. Most importantly, condensation of these dyons still leads to confinement of monopoles.

The dyon condensates have the form [21, 12]:

$$\langle D^{lA} \rangle = \langle \tilde{D}^{lA} \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \cdots & 0 & \sqrt{\xi_1} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & 0 & \cdots & \sqrt{\xi_{\tilde{N}}} \end{pmatrix},$$

$$\langle D^J \rangle = \langle \tilde{D}^J \rangle = \sqrt{\frac{\xi_J}{2}}, \quad J = (\tilde{N} + 1), ..., N.$$  

The important feature apparent in (3.3), as compared to the squark VEVs in the original theory (2.3), is a “vacuum leap” [11]. Namely, if we pick

2The $SU(\tilde{N})$ gauge group was first identified at the root of the baryonic Higgs branch in $\mathcal{N} = 2$ $SU(N)$ SQCD with massless quarks and vanishing $\xi$ parameters using the Seiberg-Witten curve in [27].

3Because of monodromies [6, 28] the quarks pick up at strong coupling root-like color-magnetic charges in addition to their weight-like color-electric charges [11].
up the vacuum with nonvanishing VEVs of the first $N$ quark flavors in the original theory at large $\xi$, and then reduce $\xi$ below $\Lambda_2$, the system undergoes a crossover transition and ends up in the vacuum of the $r$-dual theory with the dual gauge group (3.2) and nonvanishing VEVs of $\tilde{N}$ last dyons (plus VEVs of $(N - \tilde{N})$ dyons that are SU($\tilde{N}$) singlets).

The parameters $\xi_P$ in (3.3) and (3.4) are determined [21] by the quantum version of the classical expressions (2.4). They can be written in terms of roots of the Seiberg–Witten curve.

The first $\tilde{N}$ parameters $\xi_P$ which determine VEVs of the non-Abelian dyons in (3.3) are small,

$$\xi_P = 2\mu \, m_{P+N} \sim \xi^{\text{small}} \sim \mu m, \quad P = 1, \ldots, \tilde{N}. \quad (3.5)$$

This is a reflection of the fact that the non-Abelian sector of the dual theory is infrared free and is at weak coupling in the domain (3.1). Other $\xi$’s which determine VEVs of the Abelian dyons in (3.4) are large,

$$\xi_P \sim \xi^{\text{large}} \sim \mu \Lambda_{N=2}, \quad P = \tilde{N} + 1, \ldots, N. \quad (3.6)$$

As long as we keep $\xi_P$ and masses small enough (i.e. in the domain (3.1)) the coupling constants of the infrared-free $r$-dual theory (frozen at the scale of the dyon VEVs) are small: the $r$-dual theory is at weak coupling.

### 3.1 “Instead-of-confinement” mechanism

Now, we are ready to explain the regime which we called “instead-of-confinement.” Let us consider the limit of almost equal quark masses. Both, the gauge group and the global flavor SU($N_f$) group, are broken in the vacuum. The form of the dyon VEVs in (3.3) shows that the $r$-dual theory is also in the color-flavor locked phase. Namely, the unbroken global group of the dual theory is

$$\text{SU}(N) \times \text{SU}(\tilde{N})_{C+F} \times \text{U}(1), \quad (3.7)$$

where this time the SU($\tilde{N}$) global group arises from color-flavor locking.

In much the same way as in the original theory, the presence of the global SU($\tilde{N}$)$_{C+F}$ symmetry is the reason behind formation of the non-Abelian strings. Their tensions are still given by Eq. (2.6), where the parameters...
\(\xi_P\) are determined by (3.5) [21, 12]. These strings still confine monopoles [11].

In the equal-mass limit the global unbroken symmetry (3.7) of the dual theory at small \(\xi\) coincides with the global group (2.5) of the original theory in the \(r = N\) vacuum at large \(\xi\). However, this global symmetry is realized in two very distinct ways in the dual pair at hand. As was already mentioned, the quarks and U(\(N\)) gauge bosons of the original theory at large \(\xi\) come in the following representations of the global group (2.5):

\[(1, 1), (N^2 - 1, 1), (\bar{N}, \bar{N}), \text{ and } (N, \bar{N})\,.
\]

At the same time, the dyons and U(\(\bar{N}\)) gauge bosons of the \(r\)-dual theory form

\[(1, 1), (1, \bar{N^2} - 1), (N, \bar{N}), \text{ and } (\bar{N}, N)\]

(3.8) representations of (3.7). We see that the adjoint representations of the color-flavor locked subgroup are different in two theories.

The quarks and gauge bosons which form the adjoint \((N^2 - 1)\) representation of SU(\(N\)) at large \(\xi\) and the quark-like dyons and dual gauge bosons which form the adjoint \((\bar{N^2} - 1)\) representation of SU(\(\bar{N}\)) at small \(\xi\) are, in fact, distinct states [11].

Thus, the quark-like dyons are not quarks. At large \(\xi\) they are heavy solitonic states. However below the crossover at small \(\xi\) they become light and form fundamental “elementary” states \(D^\text{IA}\) of the \(r\)-dual theory. And vice versa, quarks are light at large \(\xi\) but become heavy below the crossover.

This raises the question: what exactly happens to quarks when we reduce \(\xi\)?

They are in the “instead-of-confinement” phase. The Higgs-screened quarks and gauge bosons at small \(\xi\) decay into the monopole-antimonopole pairs on the curves of marginal stability (the so-called wall crossing) [11, 26].

The general rule is that the only states that exist at strong coupling inside the curves of marginal stability are those which can become massless on the

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4 An explanatory remark regarding our terminology is in order. Strictly speaking, the dyons carrying root-like electric charges are confined as well. We refer to all such states collectively as to “monopoles.” This is to avoid confusion with the quark-like dyons which appear in Eqs. (3.3) and (3.4). The latter dyons carry weight-like electric charges. As was already mentioned, their color charges are identical to those of quarks, see [11] for further details.
Figure 1: Meson formed by a monopole-antimonopole pair connected by two strings. Open and closed circles denote the monopole and antimonopole, respectively.

Coulomb branch \[ \text{Coulomb branch} \ [6, 28] \]. For the \( r \)-dual theory these are light dyons shown in Eq. (3.3), gauge bosons of the dual gauge group and monopoles.

The monopoles and antimonopoles produced at small nonvanishing values of \( \xi \) in the decay process of the adjoint \((N^2 - 1, 1)\) states cannot escape from each other and fly to opposite infinities because they are confined. Therefore, the (screened) quarks and gauge bosons evolve into stringy mesons (in the strong coupling domain of small \( \xi \)) shown in Fig. 1, namely monopole-antimonopole pairs connected by two strings \[11, 12\).

The flavor quantum numbers of stringy monopole-antimonopole mesons were studied in \[26\] in the framework of an appropriate two-dimensional \( CP(N - 1) \) model which describes world-sheet dynamics of the non-Abelian strings \[23, 24, 20, 25\]. In particular, confined monopoles are seen as kinks in this world-sheet theory. If two strings in Fig. 1 are “neighboring” strings \( P \) and \( P + 1 \) \((P = 1, ..., (N - 1))\), each meson is in the two-index representation \( M^B_A(P, P + 1) \) of the flavor group, where the flavor indices are \( A, B = 1, ..., N_f \). It splits into a singlet, adjoint and bifundamental representations of the global unbroken group \(3.7\). In particular, at small \( \xi \) the adjoint representation of \( SU(N) \) contains former (screened) quarks and gauge bosons of the original theory.

The picture of the crossover is schematically shown in Fig. 2. The left and right sides of this figure correspond to large and small values of \( \xi \), respectively. Quarks are light at large \( \xi \). They evolve into monopole-antimonopole stringy mesons at small \( \xi \). Moreover, heavy monopole-antimonopole stringy mesons present at large \( \xi \) become light at small \( \xi \) and form “fundamental” charged matter of the \( r \)-dual theory, namely, quark-like dyons.

We see that the monopole-antimonopole stringy mesons have “correct” (adjoint or singlet) quantum numbers with respect to the global group, in much the same way as mesons in real-world QCD. Let us explain this. For example, in the actual world a \( U(1) \) subgroup of the global flavor group is
Figure 2: Schematic picture of the crossover from large to small $\xi$. Blue circles denote light quarks, while green circles denote light quark-like dyons. Monopole-antimonopole mesons are shown as in Fig. 1.

gauged with respect to electromagnetic interactions. The correct (adjoint or singlet) global quantum numbers of the monopole-antimonopole stringy mesons mean, in particular, that they have integer rather than fractional (like 2/3 or $-1/3$) electric charges.

Moreover, because these mesons are formed by strings, they lie on the Regge trajectories. Thus, the monopole-antimonopole mesons of the instead-of-confinement phase are qualitatively similar to real-world QCD mesons. The role of QCD constituent quarks is played by monopoles.

4 Flowing to $\mathcal{N} = 1$ QCD

In this section we will discuss what happens to the $r$-dual theory in the $r = N$ vacuum once we increase $\mu$, see [12, 15, 13]. We also discuss the relation of our dual theory to the Seiberg’s duality.

4.1 Intermediate $\mu$: emergence of the $U(\tilde{N})_{\text{gauge}}$

Combining Eqs. (3.3), (3.4), (3.5) and (3.6) we see that in the domain (3.1) the VEVs of the non-Abelian dyons $D^{iA}$ are much smaller than those of the Abelian dyons $D^I$. This circumstance is the most crucial. It allows us to increase $\mu$ and decouple the adjoint fields without violating the weak coupling condition in the dual theory [12].

First let us consider intermediate values of $\mu$ which are large enough to
decouple the adjoint matter \[12, 15\]. We uplift \( \mu \) to the intermediate domain

\[ |\mu| \gg |m_A|, \quad A = 1, \ldots, N_f, \quad \mu \ll \Lambda_2. \tag{4.1} \]

The VEVs of the Abelian dyons \[3.4\] are large. This makes the \( U(1) \) gauge fields of the dual group \[3.2\] heavy. Decoupling these gauge factors, together with the adjoint matter and the Abelian dyons themselves, we obtain the low-energy theory with the \( U(\tilde{N}) \) gauge fields and a set of non-Abelian dyons

\[ D^{lA}, \quad l = 1, \ldots, \tilde{N}, \quad A = 1, \ldots, N_f. \tag{4.2} \]

The superpotential for \( D^{lA} \) has the form \[12\]

\[ W = -\frac{1}{2\mu} (\tilde{D}_A D^B)(\tilde{D}_B D^A) + m_A (\tilde{D}_A D^A), \tag{4.3} \]

where the color indices are contracted inside each parentheses. Minimization of this superpotential leads to the VEVs \[3.3\] for the non-Abelian dyons determined by \( \xi^{\text{small}} \), see \[3.5\].

Below the scale \( \mu \), our theory becomes dual to \( N = 1 \) SQCD. This \( r \)-dual theory has the scale

\[ \tilde{\Lambda}_1^{N-2\tilde{N}} = \frac{\Lambda_2^{N-\tilde{N}}}{\mu^\tilde{N}}. \tag{4.4} \]

In order to keep this infrared-free theory in the weak coupling regime we impose the constraint

\[ |\sqrt{\mu m}| \ll \tilde{\Lambda}_1. \tag{4.5} \]

This means that at large \( \mu \) we must keep the quark masses sufficiently small.

Note that for the intermediate \( \mu \) we assume that \( \mu \ll \Lambda_2 \). This condition guarantees that the heavy Abelian \( U(1)^{N-\tilde{N}} \) sector is at weak coupling too, and is indeed heavy. If we relax the condition \( \mu \ll \Lambda_2 \) this sector enters a strong coupling regime, and certain states could in principle become light and show up in our low-energy \( U(\tilde{N}) \) theory.

### 4.2 Connection to Seiberg’s duality

The gauge group of our \( r \)-dual theory is \( U(\tilde{N}) \), the same as the gauge group of the Seiberg’s dual theory \[7, 8\]. This suggests that there should be a close relation between two duals. For intermediate values of \( \mu \) this relation was found in \[29, 14\].

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Originally Seiberg’s duality was formulated for $\mathcal{N} = 1$ SQCD which in our set-up corresponds to the limit $\mu \to \infty$. Therefore, in the original formulation Seiberg’s duality referred to the monopole vacua with $r = 0$. Other vacua, with $r \neq 0$, have condensates of $r$ quark flavors $\langle \tilde{q}q \rangle_A \sim \mu m_A$ and, therefore, disappear in the limit $\mu \to \infty$: they become runaway vacua.

However, Seiberg’s duality can be (and in fact, was) generalized to the case of $\mu$-deformed $\mathcal{N} = 2$ SQCD [30, 31]. If the mass term $\mu$ is large then $\mu$-deformed $\mathcal{N} = 2$ SQCD flows to $\mathcal{N} = 1$ SQCD with an additional quartic quark superpotential. This theory has all $r$ vacua which were present in the original $\mathcal{N} = 2$ theory in the small-$\mu$ limit.

The generalized Seiberg dual theory in the case of $\mu$-deformed $U(N)$ $\mathcal{N} = 2$ SQCD at large but finite $\mu$ has the $U(\tilde{N})$ gauge group, $N_f$ flavors of Seiberg’s “dual quarks” $\tilde{h}^A_l$ (here $l = 1, ..., \tilde{N}$ and $A = 1, ..., N_f$) and the following superpotential:

$$W_S = -\kappa^2 \frac{2\mu}{\kappa} \text{Tr}(M^2) + \kappa m_A M^A_A + \tilde{h}_A l^B M^A_B, \quad (4.6)$$

where $M^B_A$ is the Seiberg neutral mesonic field defined as

$$\langle \tilde{q}A q^B \rangle = \kappa M^B_A. \quad (4.7)$$

The parameter $\kappa$ above has dimension of mass and is needed to formulate Seiberg’s duality [7, 8]. Two last terms in (4.6) were originally suggested by Seiberg, while the first term is a generalization to finite $\mu$. This generalization originates from the quartic quark potential [30, 31].

Now, let us assume the fields $M^B_A$ to be heavy and integrate them out. This implies that $\kappa$ is large. Integrating out the $M$ fields in (4.6) we arrive at

$$W_{LE}^S = \frac{\mu}{2\kappa} (\tilde{h}_A l^B) (\tilde{h}_B l^A) + \frac{\mu}{\kappa} m_A (\tilde{h}_A l^A). \quad (4.8)$$

The change of variables

$$D^A_l = \sqrt{-\frac{\mu}{\kappa}} h^A_l, \quad l = 1, ..., \tilde{N}, \quad A = 1, ..., N_f \quad (4.9)$$

brings this superpotential to the form

$$W_{LE}^S = \frac{1}{2\mu} (\tilde{D}_A l^B)(\tilde{D}_B l^A) - m_A (\tilde{D}_A l^A). \quad (4.10)$$
We see that the \( r \)-dual and Seiberg’s dual theories match each other. At intermediate \( \mu \) the Seiberg \( M \) meson is heavy and should be integrated out implying the superpotential (4.10) which agrees with the superpotential (4.3) obtained in the \( r \)-dual theory.

This match, together with the identification (4.9), reveals the physical nature of Seiberg’s “dual quarks.” They are not monopoles as one could naively think. Instead, they are quark-like dyons appearing in the \( r \)-dual theory below the crossover. Their condensation leads to confinement of monopoles and the “instead-of-confinement” phase [15] for quarks and gauge bosons of the original theory.

4.3 Large \( \mu \)

Finally, we pass to the large-\( \mu \) domain. Increasing \( \mu \) we simultaneously reduce \( m \) keeping \( \xi \) sufficiently small, see (4.5). Namely, we assume
\[
\xi \sim \mu m \ll \bar{\Lambda}_1, \quad \mu \gg \Lambda_1,
\]
(4.11)
where \( \Lambda_1 \) is the scale of the original \( \mathcal{N} = 1 \) SQCD,
\[
\Lambda_1^{2N-N} = \mu^N \Lambda_2^{N-N}.
\]
(4.12)
This ensures that our low-energy \( U(\bar{N}) \) \( r \)-dual theory is at weak coupling. However, the Abelian \( U(1)^{N-N} \) sector ultimately enters the strong coupling regime. As was already mentioned, we lose analytic control over this sector and, in particular, certain states can become light and show up in our low-energy \( U(\bar{N}) \) theory.

This is exactly what happens at large values of \( \mu \) and is, in fact, required by the ’t Hooft anomaly matching [9]. Large values of \( \mu \) require a chiral limit of small \( m \) due to the condition (4.11). In this limit we need to match global anomalies in terms of original and dual theories. In fact, without light Seiberg \( M \) meson the anomalies do not match. This was checked initially in [7] for the limit \( \mu \to \infty \) and presented a basis for the discovery of Seiberg’s duality. Moreover, recently it was confirmed [13] for our \( \mu \)-deformed theory at large but finite \( \mu \) and massive quarks in the domain (4.11).

Natural candidates for the Seiberg \( M \) mesons in the \( r \)-dual theory are the stringy mesons \( M_{A}^{B}(P, P + 1) \) (with \( P = \bar{N}, ..., (N - 1) \)) from the Abelian \( U(1)^{N-N} \) sector. This sector is at strong coupling at large \( \mu \); therefore, certain states from this sector can become light. Perturbative states from this sector
(quark-like dyons and Abelian gauge fields) are singlets with respect to the global group (3.7) and cannot play the role of the $M$ mesons. Stringy mesons $M^B_A(P, P + 1)$ (where $P = 1, ..., (\tilde{N} - 1)$) from the $U(\tilde{N})$ low-energy theory also cannot play the role of the $M$ mesons. First, they are represented in the $U(\tilde{N})$ low-energy theory as nonperturbative solitonic states and cannot be added to this theory as new “fundamental” or “elementary” fields. Second, they are too heavy, with masses of the order of $\sqrt{\xi_{\text{small}}}$, determined by the tensions of the non-Abelian strings, which can be calculated at weak coupling.

Thus, we proposed in [13] that the Seiberg $M^B_A$ mesons come from a multitude of the monopole-antimonopole stringy mesons $M^B_A(P, P + 1)$ (where $P = \tilde{N}, ..., (N - 1)$) from the Abelian $U(1)^{N-\tilde{N}}$ sector. At large $\mu$ the $M$ meson should become light, with mass of the order of $m$. It should be incorporated in the $U(\tilde{N})$ low-energy theory as a new “fundamental” or “elementary” field. Note, that other states from the Abelian sector are still heavy and decouple.

Since our $U(\tilde{N})$ r-dual theory is at weak coupling we can write down its effective action. Using the procedure described in Sec. 4.2 in the opposite direction we “integrate the $M$-meson in” the superpotential (4.3). In this way we arrive at

$$W = \frac{\kappa^2}{2\mu} \text{Tr} (M^2) - \kappa m_A M^A_A + \frac{\kappa}{\mu} \tilde{D}_{AB} D^B M^A_A, \quad (4.13)$$

where

$$\kappa \sim \begin{cases} \mu^2 \Lambda_{N=2}^2, & \mu \ll \Lambda_{N=2}, \\ \sqrt{\mu m}, & \mu \gg \Lambda_{N=2}. \end{cases} \quad (4.14)$$

This dependence guarantees that the $M$ meson is heavy, with mass of the order of $\sqrt{\xi_{\text{large}}}$ at intermediate $\mu$, and becomes light, with mass of the order of $m$ at large $\mu$, see [13] for details.

5 Conclusions

Quarks, gluons, and other notions of which M. Gell-Mann was a pioneer got a new life in the era of supersymmetry, when supersymmetry-based methods became powerful – and quite often, unique – tools in the studies of confinement and other nontrivial features of gauge dynamics at strong coupling.
In this brief article we summarized some applications of non-Abelian strings and reviewed phases of $\mathcal{N} = 1$ SQCD obtained from $\mu$-deformed $\mathcal{N} = 2$ SQCD in the limit of large $\mu$. We identified “promising” vacua among all $r$ vacua – promising in the quest of confinement similar to that inherent to QCD. The number of $r$ vacua as a function of $r$ is shown in Fig. 3.

The zero vacua represent a subset of vacua at $r < \tilde{N}$ with parametrically small gaugino condensate in the limit of the small quark masses. In this limit the zero vacua are described in terms of a weakly coupled dual infrared-free $U(\tilde{N})$ gauge theory with $r$ condensed quarks. This theory is in the mixed Coulomb-Higgs phase [14, 13].

The $\Lambda$ vacua (of which we said little, if at all) have no weak coupling description at large $\mu$ and small $m$ [14]. In a certain limit they flow into a conformal Argyres-Douglas-like strongly coupled regime [32].

The regime closest to what we observe in real-world QCD is represented by the instead-of-confinement phase which occurs in the $r = N$ vacuum at strong coupling. The monopole-antimonopole stringy mesons formed in this phase are qualitatively similar to mesons in QCD. They have “correct” quantum numbers with respect to the global group (singlet plus adjoint) and lie on the Regge trajectories.
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