Gradient Tracking: A Unified Approach to Smooth Distributed Optimization

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Abstract—In this work, we study the classical distributed optimization problem over digraphs, where the objective function is a sum of smooth local functions. Inspired by the implicit tracking mechanism proposed in our earlier work, we develop a unified algorithmic framework from a pure primal perspective, i.e., UGT, which is essentially a generalized gradient tracking method and can unify most existing distributed optimization algorithms with constant step-sizes. It is proved that two variants of UGT can both achieve linear convergence if the global objective function is strongly convex. Finally, the performance of UGT is evaluated by numerical experiments.

Index Terms—Gradient tracking, distributed optimization, unified framework.

I. INTRODUCTION

We consider the classical distributed optimization problem

\[
\min_{x \in \mathbb{R}^m} F(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)
\]

over a network consisting of \( n \) agents, where \( f_i(x) : \mathbb{R}^m \rightarrow \mathbb{R} \) is differentiable and only known by agent \( i \). It is assumed that \((P1)\) has at least an optimal solution. The communication topology among agents is modeled by a digraph \( G = (\mathcal{V}, \mathcal{E}) \), where \( \mathcal{V} = \{1, 2, \cdots, n\} \) is the vertex set, \( \mathcal{E} \subset \mathcal{V} \times \mathcal{V} \) is the edge set, and \( (i,j) \in \mathcal{E} \) if agents \( i \) can receive information from agent \( j \). Due to its promising applications on many areas, such as machine learning, distributed control and sensor networks [1], \((P1)\) has drawn lots of research interests and varieties of distributed algorithms have been proposed in the past decades, to name a few, distributed subgradient method [1], EXTRA [3], DLM [4], NIDS [5], Exact diffusion [6], and numerous gradient tracking methods [7]–[12].

In this work, we aim to develop a general algorithmic framework which can unify existing algorithms. Some efforts have been made towards that goal, for example, [13] unifies DIGing and EXTRA, [14] unifies Exact diffusion and NIDS, and the two primal-dual frameworks proposed in [15] and [16] can unify most existing algorithms. However, the two primal-dual frameworks, which are the most general frameworks as far as we know, both rely on the symmetry of weight matrices, which is not necessary for many algorithms [6]–[12].

The above observation inspires us to ask that if there exists a more general framework, which does not require the symmetry of weight matrices. In [15] and [16], gradient tracking algorithms [7]–[12] are seen as primal-dual methods, where the symmetry of weight matrices is further assumed. However, the convergence of all these gradient tracking methods do not rely on the symmetry of weight matrices. When the symmetry of weight matrices is absent, in fact, we can not treat gradient tracking algorithms as primal-dual methods, which is also pointed out in [7]. Different from [15] and [16], our framework is built from a pure primal perspective and does not rely on the symmetry of weight matrices. The key in developing the proposed framework is that we regard most existing distributed algorithms with constant step-sizes as gradient tracking methods but with different kinds of tracking mechanisms: explicit tracking and implicit tracking. To be more specific, the tracking of the global gradient exists generally in most distributed algorithms with constant step-sizes. Though some algorithms do not have the explicit gradient tracking structure [3]–[6], [17], they still possess certain implicit structure that can track the global gradient, which we call the implicit tracking mechanism. The idea originates from our earlier work [18], where the implicit tracking mechanism is proposed and further used to design efficient algorithms for distributed constraint-coupled optimization problems. Therefore, our framework can be seen as a generalized gradient tracking method, which is the reason we call it unified gradient tracking (UGT).

Our major contributions are summarized as follows:

1) We offer a new and unified perspective for understanding existing distributed optimization algorithms with constant step-sizes, i.e., most distributed optimization algorithms with constant step-sizes can be seen as gradient tracking methods in essence, which leads to the birth of UGT.

2) UGT can unify most existing distributed algorithms with constant step-sizes, without assuming the symmetry of weight matrices. In this sense, UGT is more general than the primal-dual frameworks proposed in [15] and [16]. Meanwhile, the permission of asymmetric weight matrices implies that UGT can be applied to digraphs, which is remarkably different from [15] and [16]. Furthermore, the linear convergence of UGT can be guaranteed when \( f_i(x) \) is smooth and \( F(x) \) is strongly convex, without the need of the strong convexity even convexity of \( f_i(x) \). However, the linear convergences of the primal-dual frameworks proposed in [15] and [16] both rely on the strong convexity of \( f_i(x) \).

3) As a generalized gradient tracking method, we can easily obtain lots of different versions of UGT, benefiting from its special structure. In numerical experiments, it is found that some versions of UGT have far better performance...
than classical gradient tracking algorithms.

Notations: Let \(1_n\) and \(I_n\) be the vector of \(n\) ones and the identity matrix with dimension \(n\) respectively. Note that we might not give the dimension explicitly if it could be inferred from the context. For \(x \in \mathbb{R}^m\), \(||x||\) denotes its Euclidean norm and \(1_n x = I_n \otimes x\). For \(A \in \mathbb{R}^{m \times n}\), \(||A||\) and \(\text{det}[A]\) denote its spectral norm and determinant respectively.

II. Unified Gradient Tracking

Let us first consider the centralized gradient descent method:

\[ x^{k+1} = x^k - \alpha \nabla f(x^k), \]

where \(\alpha > 0\) is a constant step-size. As is well known, when \(f(x)\) is convex, (1) can converge to the optimal solution of (P1) if \(\alpha\) is appropriate. Under the distributed scenario, however, the global gradient \(\nabla f(x^k)\) is not accessible for agents, hence (1) cannot be applied. Note that (P1) is equivalent to

\[
\min_{x \in \mathbb{R}^m} f(x) = \sum_{i=1}^n f_i(x_i), \quad \text{s.t.} \quad x_1 = x_2 = \cdots = x_n,
\]

when \(\mathcal{G}\) is strongly connected, where \(x = [x_1^T, \ldots, x_n^T]^T\). Let \(x^*\) be an optimal solution of (P1), then \(x^* = 1_n \otimes x^*\) is an optimal solution of (P2), vice versa. Consequently, we can solve (P1) by equivalently solving (P2) via the well-known distributed gradient method [2]:

\[ x^{k+1} = W x^k - \alpha(k) \nabla f(x^k), \]

where \(\alpha(k) > 0\) is the time-variant step-size, \(W = W \otimes I_n\), and \(W \in \mathbb{R}^{n \times n}\) is a weight matrix associated with \(\mathcal{G}\). The convergence of (2) can be guaranteed with some assumptions about \(\alpha(k)\) and \(W\), one of whom is \(\lim_{k \to \infty} \alpha(k) = 0\). In fact, (2) can not converge exactly to \(x^*\) without the above condition (for example, let \(\alpha(k)\) be a constant), since \(\nabla f_i(x^*) \neq 0\) in general. Obviously the diminishing step-size \(\alpha(k)\) will slow down the convergence rate greatly, hence there is a dilemma: using a constant step-size, the convergence rate is fast but the exact convergence can not be guaranteed; using a diminish step-size, the exact convergence can be guaranteed but the convergence rate is slow. To get rid of the dilemma, we can resort to the discrete-time dynamic average consensus (DAC) algorithm [19]

\[ g^{k+1} = W g^k + \nabla f(x^{k+1}) - \nabla f(x^k) \]

to track the global gradient. Along with [3], theoretically, \(g^k\) will converge to the global gradient \(\frac{1}{n} \sum_{i=1}^n \nabla f_i(x_i)\), hence \(g^k\) can be seen as the inexact global gradient. Based on the above gradient tracking approach, many algorithms with constant step-sizes have been proposed [7]–[12], one of whom is DIgING [7], [8], a Combine-then-Adapt (CTA) algorithm written as

\[
\begin{align*}
    x^{k+1} &= W x^k - g^k, \\
    g^{k+1} &= W g^k + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)),
\end{align*}
\]

whose Adapt-then-Combine (ATC) counterpart, i.e., ATC tracking [9], [10], is given as

\[
\begin{align*}
    x^{k+1} &= W(x^k - g^k), \\
    g^{k+1} &= W g^k + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)).
\end{align*}
\]

Benefiting from the gradient tracking approach, DIgING and ATC tracking can converge exactly to \(x^*\) even using a constant step-size, since the inexact global gradient \(g^k\) will converge to 0 as \(x^k\) approaches \(x^*\).

DIgING and ATC tracking, as well as another gradient tracking algorithm Aug-DGM, have an explicit state variable \(g^k\) to track the global gradient, which is the reason we call them explicit gradient tracking algorithms. Besides these explicit gradient tracking algorithms, there also exist lots of other distributed algorithms, like EXTRA, NIDS, Exact diffusion, and so forth. Though they do not possess the explicit gradient tracking structure, we found that they do have an implicit one, hence they can also be seen as gradient tracking algorithms, but with the implicit tracking approach. The implicit tracking mechanism (approach) is first proposed in [18], which is employed to develop efficient algorithms for distributed constraint-coupled optimization problems. Specifically speaking, [18] offers a new perspective to understand a classical continuous-time distributed optimization algorithm [20], which is given as

\[
\begin{align*}
    \dot{x} &= -\alpha \nabla f(x) - z - \beta L x, \\
    \dot{z} &= \alpha \beta L x.
\end{align*}
\]

In [20], it is observed that

\[ x = -\alpha \nabla f(x) - \beta L x \]

cannot converge to the optimal solution of (P2) since local gradients are generally different. Based on this observation, the integral feedback term \(z\) is designed to correct the error among agents caused by local gradients.

Different from [20], (4) is derived from a brand-new way in [18]. Concretely speaking, it is easy to verify that

\[
\frac{1}{n} \sum_{i=1}^n \nabla f_i(x_i) = 0,
\]

\[ x_i = x_j, \quad i, j \in \mathcal{V} \]

is a sufficient and necessary condition for \(x_i = x^*, i \in \mathcal{V}\). Furthermore, (5) is equivalent to

\[ x_i = \frac{1}{n} \sum_{j=1}^n (x_j - \nabla f_j(x_j)), \quad i \in \mathcal{V}. \]

Therefore, if each agent \(i\) takes the following dynamics:

\[
\begin{align*}
    \dot{x}_i(t) &= \frac{1}{n} \sum_{j=1}^n (x_j - \nabla f_j(x_j)) - x_i(t) \\
    &= -\frac{1}{n} \sum_{j=1}^n \nabla f_j(x_j(t)) - \left( x_i(t) - \frac{1}{n} \sum_{j=1}^n x_j(t) \right)
\end{align*}
\]

obviously \(x_i(t)\) will converge to \(x^*\). Though (6) is not distributed, it is natural to apply the continuous-time DAC
Algorithm proposed in [21] to track \( \frac{1}{n} \sum_{j=1}^{n} (x_j - \nabla f_j(x_j)) \) distributedly:

\[
\begin{align*}
\dot{x} &= -\alpha(x - (x - \nabla f(x))) - z - \beta L x, \\
\dot{z} &= \alpha \beta L x,
\end{align*}
\]

which is exactly [4]. As a distributed version of (6), \( \alpha \nabla f_i(x_i) + z_i \) can be naturally seen as agent i’s estimation of the global gradient \( \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x_j(t)) \). In other words, [4] can still be seen as a kind of gradient tracking algorithm, but with the implicit tracking approach, where \( \alpha \nabla f(x) + z \) plays the role in tracking the global gradient. Compared with explicit gradient tracking algorithms, such as DIGing and ATC tracking, there is no explicit state variable to track the global gradient, it is feasible to transform them to explicit tracking mechanisms offers.

Besides the continuous-time algorithm [4], the implicit tracking mechanism exists generally in many discrete-time algorithms, such as EXTRA, NIDS, and Exact diffusion. Though these algorithms do not have an explicit state variable to track the global gradient, it is feasible to transform them to explicit tracking forms. Let us take EXTRA and Exact diffusion as CTA and ATC examples respectively. EXTRA updates as

\[
x^{k+1} = \tilde{W} x^{k+1} - \tilde{W} x^k - \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)),
\]

where \( \tilde{W} = \frac{I + W}{2} \). Define

\[
g^{k+1} = -\tilde{W} (x^{k+1} - x^k) + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)),
\]

then we can obtain the explicit tracking form of EXTRA:

\[
x^{k+1} = \tilde{W} x^k - g^k,
\]

\[
g^{k+1} = \tilde{W} g^k + (I - \tilde{W}) \tilde{W} x^k + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)).
\]

Exact diffusion updates as

\[
x^{k+1} = \tilde{W} (2x^{k+1} - x^k - \alpha(\nabla f(x^{k+1}) - \nabla f(x^k))).
\]

Define

\[
g^{k+1} = x^k - x^{k+1} + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k))
\]

we have

\[
x^{k+1} = \tilde{W} (x^k - g^k)
\]

and

\[
g^{k+1} = x^k - \tilde{W} (x^k - g^k) + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)) = \tilde{W} g^k + (I - \tilde{W}) x^k + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)),
\]

then the explicit tracking form of Exact diffusion is given as

\[
x^{k+1} = \tilde{W} (x^k - g^k),
\]

\[
g^{k+1} = \tilde{W} g^k + (I - \tilde{W}) x^k + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)).
\]

Apart from EXTRA and Exact diffusion, many other distributed optimization algorithms also have the implicit gradient tracking structure and we can transform them to explicit tracking forms, which inspires the design of the unified framework. Our unified framework, i.e., UGT, has two variants, one is CTA-UGT:

\[
x^{k+1} = W_1 x^k - g^k,
\]

\[
g^{k+1} = W_2 g^k + \beta (I - W_2) W_1 x^k + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)),
\]

and another one is ATC-UGT:

\[
x^{k+1} = W_1 (x^k - g^k),
\]

\[
g^{k+1} = W_2 g^k + \beta (I - W_2) x^k + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)),
\]

where \( \beta \geq 0 \) is a tunable parameter, \( W_1 = W_1 \otimes I_m \), \( W_2 = W_2 \otimes I_m \), \( W_1 \) and \( W_2 \) are weight matrices, and \( g^0 = \alpha \nabla f(x^0) \). By choosing different \( W_1 \), \( W_2 \), and \( \beta \), UGT can recover most existing algorithms, as shown in Table II.

Remark 1: As a matter of fact, UGT is a generalized gradient tracking algorithm. Compared with the two classical

| Algorithms | Tracking type | \(W_1\) | \(W_2\) | \(\beta\) |
|------------|--------------|--------|--------|--------|
| DIGing [7,8] | E | \(W\) | \(W\) | 0 |
| EXTRA [3] | I | \(\frac{1+W}{2}\) | \(\frac{1+W}{2}\) | 1 |
| DLM [4] | I | \(I - cL\) | \(I - cL\) | 1 |
| ATC tracking [9,10] | E | \(W\) | \(W\) | 0 |
| Aug-DGM [11,12] | E | \(W^2\) | \((I - (I + W)^2)\) | 1 |
| Exact diffusion [6] | I | \(\frac{1+W}{2}\) | \(\frac{1+W}{2}\) | 1 |
| NIDS [5] | I | \(I - c(I - W)\) | \(I - c(I - W)\) | 1 |
gradient tracking algorithms (DIGing and ATC tracking), UGT has an extra term; for CTA-UGT, it is $\beta(I - W_2)x^k$; for ATC-UGT, it is $\beta(I - W_2)x^k$, which we call the modified term. It is worth noting that the introduce of the modified term does not increase the number of communication rounds in each iteration, which is obvious for ATC-UGT, while for CTA-UGT, we only need to notice that the updating of $g^k$ can be rewritten as

$$g^{k+1} = (1 - \beta)W_2g^k + \beta\left[g^k + (I - W_2)x^{k+1}\right] + \alpha(\nabla f(x^{k+1}) - \nabla f(x^k)),$$

hence the number of communication rounds of CTA-UGT in each iteration is still 2.

**Remark 2:** The parameter $\beta$ in the modified term plays a critical role in the performance of UGT. In existing algorithms, there are only two possible values of $\beta$: 0 and 1, as shown in Table I. Nevertheless, $\beta$ can be set to any values, as long as the convergence can be guaranteed (the upper bound of $\beta$ is derived in Section III). In our opinion, UGT with different $\beta$ are essentially different algorithms, and whose convergence rates may have huge differences in numerical experiments. Therefore, it is convenient to develop efficient versions of UGT by choosing an appropriate value for $\beta$.

## III. Convergence Analysis

We first give the following assumption about weight matrices $W_1$ and $W_2$.

**Assumption 1:** $W_1$ and $W_2$ are primitive and doubly stochastic.

$W_1$ and $W_2$ are associated with $G$. $W_1(W_2) = [w_{ij}]$ can be constructed as: $w_{ii} > 0$; $w_{ij} > 0$ if $(i, j) \in \mathcal{E}$, otherwise $w_{ij} = 0$. Based on this constructed rule, $W_1$ and $W_2$ are primitive if $G$ is strongly connected. Given Assumption 1, $W_1$ and $W_2$ have the following property [8]:

$$\sigma_1 = \left\| W_1 - \frac{1}{n}11^\top \right\| \in (0, 1),$$

$$\sigma_2 = \left\| W_2 - \frac{1}{n}11^\top \right\| \in (0, 1).$$

Though the above property is derived under the assumption that $G$ is undirected and connected in [8], it is trivial to prove it for our case by feat of the Perron–Frobenius theory.

**Remark 3:** Different from [15] and [16], we do not assume the symmetry of $W$. Consequently, Assumption 1 can not only be satisfied by undigraphs, but also digraphs that permit doubly stochastic weight matrices, which is the reason we use the digraph to model the communication topology. On the converse, the communication topology can only be modeled by an undigraph in [15] and [16], due to the symmetry of $W$.

**Assumption 2:** $F$ is $\mu$-strongly convex and $f_i$ is $l_i$-smooth, $i \in \mathcal{V}$.

Let $l = \min_{i \in \mathcal{V}} l_i$, $\bar{l} = \frac{1}{n} \sum_{i=1}^{n} l_i$, $\eta = \min\{|1 - \alpha\mu|, |1 - \alpha\bar{l}|\}$, $\sigma_1 \in (0, 1)$ and $\sigma_2 \in (0, 1)$ be the second largest singular values of $W_1$ and $W_2$ respectively. Define two non-negative matrices as

$$G_1(\alpha, \beta) = \begin{bmatrix} \sigma_1 & 1 & 0 \\ \beta\sigma_1(1 + \sigma_2) + \alpha l(1 + \sigma_1 + \alpha l) & \sigma_2 + \alpha l(\alpha l)^2 \\ \alpha l & 0 & \eta \end{bmatrix}$$

and

$$G_2(\alpha, \beta) = \begin{bmatrix} \sigma_1 & \sigma_1 & 0 \\ \beta(1 + \sigma_2) + \alpha l(1 + \sigma_1 + \alpha l) & \sigma_2 + \alpha l(\alpha l)^2 \\ \alpha l & 0 & \eta \end{bmatrix}.$$

**Theorem 1:** Suppose Assumptions 1 and 2 hold and choose $\alpha > 0$ and $\beta \geq 0$ such that $\rho(G_1(\alpha, \beta)) < 1$, then $x^k$ generated by CTA-UGT converges to $x^\ast$ with the R-linear rate $O(\rho(G_1(\alpha, \beta)^k))$. Furthermore, $\rho(G_1(\alpha, \beta)) < 1$ if

$$0 \leq \beta < \frac{(1 - \sigma_1)(1 - \sigma_2)}{\sigma_1(1 + \sigma_2)}$$

and

$$0 < \alpha < \min \left\{ \frac{1 - \sigma_2}{\bar{l}}, -\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right\},$$

where $a = l^3 + \mu l^2$, $b = 2\mu l$, and $c = \mu[\beta\sigma_1(1 + \sigma_2) - (1 - \sigma_1)(1 - \sigma_2)]$.

**Theorem 2:** Suppose Assumptions 1 and 2 hold and choose $\alpha > 0$ and $\beta \geq 0$ such that $\rho(G_2(\alpha, \beta)) < 1$, then $x^k$ generated by ATC-UGT converges to $x^\ast$ with the R-linear rate $O(\rho(G_2(\alpha, \beta)^k))$. Furthermore, $\rho(G_2(\alpha, \beta)) < 1$ if

$$0 \leq \beta < \frac{(1 - \sigma_1)(1 - \sigma_2)}{\sigma_1(1 + \sigma_2)}$$

and

$$0 < \alpha < \min \left\{ \frac{1 - \sigma_2}{\bar{l}}, -\frac{-b + \sqrt{b^2 - 4ac}}{2a} \right\},$$

where $a = \sigma_1(l^3 + \mu l^2)$, $b = (1 + \sigma_2 l)\mu l$, and $c = \mu[\beta\sigma_1(1 + \sigma_2) - (1 - \sigma_1)(1 - \sigma_2)]$.

For the sake of readability, the proofs of Theorems 1 and 2 are placed in the appendix.

## IV. Numerical Experiments

In this section, we evaluate the performance of UGT by solving the following quadratic programming problem:

$$\min_{x \in \mathbb{R}^m} F(x) = \frac{1}{n} \sum_{i=1}^{n} x_i^\top Q_i x_i + e_i^\top x_i,$$

where $Q_i \in \mathbb{R}^{m \times m}$ is a positive definite matrix. In particular, different kinds of graphs are considered, included four digraphs: directed cycle graph, directed exponential graphs with $e = 2, 4, 6$, and four undigraphs: cycle graph, Erdos–Renyi random graphs with the connectivity probability $p = 0.05, 0.1, 0.3$. The directed exponential graph with $n$ nodes is generated by the rule that each node $i$ can send
In this work, the classical distributed optimization problem is studied. Inspired by the implicit tracking mechanism, UGT, a unified algorithmic framework, is developed from a pure primal perspective, which can unify most existing distributed optimization algorithms with constant step-sizes. When the global objective function is strongly convex, we prove that two variants of UGT can both achieve linear convergence. Finally, numerical experiments are taken to evaluate the performance of UGT.

APPENDIX

Since the proofs of Theorems 1 and 2 are quite similar, we only provide the proof of the former for the sake of conciseness. We first give some lemmas which are necessary for the subsequent convergence analysis. For CTA-UGT, define

\[
\overline{\alpha} = \text{argmax}_{\alpha} \sum_{i=1}^{n} \nabla f_i(x_i^{(k)}). 
\]

Lemma 1: Suppose Assumption II holds, then

\[
\begin{align*}
\bar{x}^{k+1} &= \bar{x}^k - \bar{g}^k, \\
\bar{g}^{k+1} &= \alpha \nabla f(\bar{x}^{k+1}).
\end{align*}
\]

Proof. The first equality is obvious. Note that \(1^T(I - W) = 0\), we have

\[
\bar{g}^{k+1} = \bar{g}^k + \alpha(\nabla f(\bar{x}^{k+1}) - \nabla f(\bar{x}^k)),
\]
then we can obtain the second equality since $g^0 = \alpha \nabla f(x^0)$.

**Lemma 2**: [8] Suppose Assumption 1 holds, then
\[
\|W_1 x - 1\bar{x}\| \leq \sigma_1 \|x - 1\bar{x}\|, \\
\|W_2 x - 1\bar{x}\| \leq \sigma_2 \|x - 1\bar{x}\|.
\]

**Lemma 3**: \[\|x - 1\bar{x}\| \leq \|x\|\]

**Proof**.
\[
\|x - 1\bar{x}\|^2 = \|x\|^2 + \|1\bar{x}\|^2 - 2 \sum_{i=1}^{n} x_i^T \bar{x} \\
= \|x\|^2 + \|1\bar{x}\|^2 - 2n\bar{x}^T \bar{x} \\
\leq \|x\|^2,
\]
which completes the proof.

**Lemma 4**: [8] Suppose Assumption 2 holds, then
\[
\|x - \alpha \nabla F(x) - x^*\| \leq \eta \|x - x^*\|,
\]
where $\eta = \min\{|1 - \alpha \mu|, |1 - \alpha \tilde{\mu}|\}$.

**Lemma 5**: [22] Consider a nonnegative and irreducible matrix $B = [b_{ij}] \in \mathbb{R}^{3 \times 3}$ which satisfies that $b_{ii} < \lambda^*$, where $\lambda^* > 0$ and $i = 1, 2, 3$. Then $\rho(B) < \lambda^*$ iff $det(\lambda^* I - B) > 0$.

**Proof of Theorem 1**. According to CTA-UGT and Lemma 1 we have
\[
\|x^{k+1} - 1\bar{x}^{k+1}\| \\
= \|W_1 x^k - 1\bar{x}^k - (g^k - 1g)\| \\
\leq \sigma_1 \|x - 1\bar{x}\| + \|g^k - 1g^k\|,
\]
where the inequality holds due to Lemma 2.

Notice that
\[
\|(I - W_2)W_1 x^k\| \\
= \|W_2 W_1 x^k - W_1 x^k - (W_1 x^k - 1\bar{x}^k)\| \\
\leq (1 + \sigma_2)\|W_1 x^k - 1\bar{x}^k\| \\
\leq (1 + \sigma_2)\|x^k - 1\bar{x}^k\|,
\]
where the two inequalities hold because of Assumption 1 and Lemma 2 then we have
\[
\|g^{k+1} - 1g^{k+1}\| \\
\leq \|W_2 g^k - 1g^k\| + \beta \|(I - W_2)W_1 x^k\| \\
+ \|\alpha (\nabla f(x^{k+1}) - \nabla f(x^k)) - 1(g^{k+1} - g^k)\| \\
\leq \sigma_2\|g^k - 1g^k\| + (1 + \sigma_2)\|x^{k+1} - 1\bar{x}^k\| \\
+ \|\alpha (\nabla f(x^{k+1}) - \nabla f(x^k))\| \\
\leq \sigma_2\|g^k - 1g^k\| + (1 + \sigma_2)\|x^k - 1\bar{x}^k\| \\
+ \alpha l\|x^{k+1} - x^k\|,
\]
where the second inequality holds due to Lemmas 1 and 3 and the last inequality holds since $f$ is $l$-smooth. In addition, we have
\[
\|x^{k+1} - x^k\| = \|W_1x^k - 1x^k - (x^k - 1x^k) - g^k\| 
\leq (1 + \sigma_1)\|x^k - 1x^k\| + \|g^k\|
\]
and
\[
\|g^k\| \leq \|g^k - 1g^k\| + \|1g^k - 1\alpha\nabla F(x^k)\| + \alpha\|1\nabla F(x^k) - 1\nabla F(x^*)\| 
\leq \|g^k - 1g^k\| + \alpha\|x^k - 1x^k\| + \alpha\sqrt{n}\|x^k - x^*\|,
\]
where the letter holds because
\[
\|g^k - \alpha\nabla F(x^k)\| = \alpha\left\|\frac{1}{n}\sum_{i=1}^{n} \nabla f_i(x_i^k) - \nabla f_i(x^k)\right\| 
\leq \alpha l \sum_{i=1}^{n} \frac{\|x_i^k - \bar{x}^k\|}{n} 
\leq \alpha l \sum_{i=1}^{n} \frac{\|x_i^k - \bar{x}^k\|^2}{n} 
\leq \frac{\alpha l}{\sqrt{n}} \|x^k - 1x^k\|.
\]

It follows that
\[
\|g^{k+1} - 1g^{k+1}\| 
\leq [(1 + \sigma_2)\sigma_1 + \alpha(1 + \sigma_1 + \alpha l)]\|x^k - 1x^k\| 
+ (\sigma_2 + \alpha l)\|g^k - 1g^k\| + (\alpha l)^2 \sqrt{n}\|x^k - x^*\|.
\]

Based on Lemma 1, we have
\[
\|x^{k+1} - x^*\| 
\leq \|x^k - \alpha\nabla F(x^k) - x^*\| + \|\alpha\nabla F(x^k) - \bar{g}^k\| 
\leq \eta\|x^k - x^*\| + \frac{\alpha l}{\sqrt{n}}\|x^k - 1x^k\|,
\]
where the last inequality holds because of Lemma 4 and 8.

Combining (7), (9), and (10), we can obtain that
\[
e^{k+1} \leq G_1(\alpha, \beta)e^k,
\]
where
\[
e^k = \begin{bmatrix} \|x^k - 1x^k\| \\ \|g^k - 1g^k\| \\ \sqrt{n}\|x^k - x^*\| \end{bmatrix}.
\]

If $G_1$ is nonnegative, i.e., all entities of $G_1$ are nonnegative, we have
\[
e^{k+1} \leq \rho(G_1)^k e^0,
\]

Fig. 3: The convergence rates of different versions of CTA-UGT over undigraphs.
which means that CTA-UGT converges linearly if $\rho(G_1) < 1$.

According to Lemma 5 to guarantee $\rho(G_1) < 1$, we first need to assure $0 \leq g_{ii} < 1$, which holds when $0 < \alpha < \frac{1 - \sigma_2}{\sigma_1} < \frac{1}{7}$, since $\sigma_1 < 1$ and $\eta = 1 - \alpha \mu$ if $\alpha < \frac{1}{7} < \frac{1}{1}$. Furthermore, we must guarantee that $\det[I - G_1] > 0$. Note that

\[
\det[I - G_1] = \det[ - (al)^3 + (1 - \eta)[- (al)^2 - 2al + (1 - \sigma_1)(1 - \sigma_2) - \beta \sigma_1 (1 + \sigma_2)] ,
\]

hence it must hold that

\[
(\alpha l)^3 + \alpha \mu [(\alpha l)^2 + 2al + \beta \sigma_1 (1 + \sigma_2) - (1 - \sigma_1)(1 - \sigma_2)] < 0,
\]

which is equivalent to

\[
\alpha^2 t^3 + \mu [(\alpha l)^2 + 2al + \beta \sigma_1 (1 + \sigma_2) - (1 - \sigma_1)(1 - \sigma_2)] < 0,
\]

rearranging it gives that

\[
\alpha^2 (l^3 + \mu l^2) + 2\mu l \alpha + \mu [\beta \sigma_1 (1 + \sigma_2) - (1 - \sigma_1)(1 - \sigma_2)] < 0.
\] (11)

Recall the expressions of $a$, $b$, and $c$ in Theorem 4 and $\beta < \frac{(1 - \sigma_1)(1 - \sigma_2)}{\sigma_1 (1 + \sigma_2)}$, then (11) can be guaranteed by letting

\[
\alpha < \frac{-b + \sqrt{b^2 - 4ac}}{2a},
\]

which finishes the proof.

Fig. 4: The convergence rates of different versions of ATC-UGT over undigraphs.

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