1. Introduction

Flops are some of the most elementary birational transformations. A deep relationship between birational geometry and derived categories originated in the work of Bondal and Orlov [BO95]. They conjectured that flops have equivalent derived categories; Kawamata refined and generalized this further in his famous K-equivalence implies D-equivalence conjecture [Kaw02].

There are various solutions to this problem for explicit types of flops; for example, Bondal and Orlov [BO95] for the standard/Atiyah flop, Bridgeland [Bri02] for flops in dimension 3, and...
Namikawa and Kawamata for Mukai flops \cite{Nam03,Kaw06}. However there is no agreed upon solution to tackle this problem in full generality.

One of the various successful techniques in addressing this problem, known as “grade-restriction windows”, was introduced by Segal \cite{Seg11} based on ideas from string theory revealed by physicists Herbst, Hori, and Page \cite{HHP09}. Segal’s work and the introduction of this technique quickly lead to significant generalizations by Ballard, Favero and Katzarkov \cite{BFK14} and Halpern-Leistner \cite{HL15} i.e. grade-restriction windows can be used to prove Kawamata’s D-equivalence implies K-equivalence conjecture for certain Variation of Geometric Invariant Theory (VGIT) problems, even in non-abelian cases.

Due to a result of Reid (see \cite[Proposition 1.7]{Tha96}), one can reduce the problem of flops to a VGIT problem. In more detail, given a flop

\[ Y_1 \longrightarrow Y_2, \]

one can find a scheme \( Y \) with a \( \mathbb{G}_m \)-action, such that \( Y_1 \) and \( Y_2 \) are realized as two different GIT quotients. Hence, we may try to prove Bondal-Orlov–Kawamata’s conjecture in general using VGIT. In the known cases, the idea is as follows.

Consider a scheme \( Y \) equipped with an action of a linearly reductive group \( G \). We denote the GIT quotient with respect to an \( G \)-equivariant ample line bundle as \( Y / G \). The \textit{window functor} is a fully faithful functor

\[ \Phi : D(Y / G) \rightarrow D(\text{QCoh}^G Y). \]

The essential image of this functor is referred to as the \textit{window subcategory} or as the window in short. We can also consider a different GIT quotient and construct a different window functor. The proof of the equivalence between the derived categories of the different GIT quotients works by comparing the two window subcategories in \( D(\text{QCoh}^G Y) \).

Though it is not constructed this way in \cite{BFK14,HL15}, the window functor \( \Phi \) is expected to be of geometric origin. Namely, it should be defined by a Fourier-Mukai kernel \( P \) which is an object in \( D(Y / G_m \times [Y / G_m]) \), such that

\[ \Phi(F) = \pi_2^*(P \otimes \pi_1^*F), \]

where \( \pi_1 \) and \( \pi_2 \) are projections from \( Y / G_m \times [Y / G_m] \) to the first and second factor respectively. Recently, Ballard, Diemer, and Favero \cite{BDF17} proposed a consistent way to produce homologically well behaved Fourier-Mukai kernels for these window functors in the case of \( G_m \)-actions, called the \textit{Q-construction}. This was extended to \( GL_n \)-actions for Grassmannian flops in \cite{BCF19}

One of the notable properties is that this \( Q \) behaves well with respect to base change. We note here that even in examples of flops where the derived equivalences are known to hold, there is not always a construction of the Fourier-Mukai kernel that induces this equivalence.

In \cite{BDF17}, the authors recovered grade-restriction windows for smooth affine schemes \( X \) with a \( \mathbb{G}_m \)-action via an explicit construction of a kernel. However, birational geometry demands we study more than smooth schemes. Indeed, even in the setting of smooth flops, the scheme \( X \) is often singular (for example, the Mukai flop). In the case of singular affine schemes Ballard, Diemer, and Favero proposed a \textit{derived Q-construction}, which uses techniques from derived algebraic geometry. Furthermore, they conjecture that this derived Q-construction provides a Fourier-Mukai equivalence for flops (i.e. they propose an explicit method to prove Bondal-Orlov–Kawamata's conjecture).

In order to tackle this problem of VGIT for singular schemes, the idea is to resolve the singular scheme, by a smooth dg-scheme and then apply the Q-construction. Instead of working in the simplicial setting (as \cite{BDF17} does), we use the philosophy of the monoidal Dold-Kan correspondence to work in the dg setting instead.

1.1. \textbf{Windows}. In this paper, we develop a theory of variation of GIT quotients arising from semi-free commutative differential graded algebras (cdga). This uses the language of dg-schemes
as developed in [CFK01, Ric10, BR12, MR15]. We prefer to use this over the more involved machinery of derived algebraic geometry (DAG) as it is explicit and better suited (at least for us) for computations.

Let us start with a \( \mathbb{Z} \)-graded semi-free (i.e., free upon forgetting the differential) cdga \( R \) over a smooth finitely-generated \( k \)-algebra \( T \), where \( k \) is an arbitrary field. We will also assume that the homological degree zero part is \( R^0 = T \). This data gives a dg-scheme with a \( \mathbb{G}_m \)-action (for more details about dg-schemes, see Section 2.2),

\[
\mathbf{X} := (\mathbf{X}, \mathcal{R}).
\]

Here, \( \mathbf{X} = \text{Spec}\ T \), and \( \mathcal{R} \) is the cdga \( R \) considered as a dg-\( \mathcal{O}_\mathbf{X} \)-module on \( \mathbf{X} \). Let us denote the ideal generated by all the positively graded elements in \( T \) by \( J^+ \), and the associated semi-stable locus as

\[
\mathbf{X}^+ := \mathbf{X} - V(J^+).
\]

We can also consider the restriction of the sheaf of cdgas \( \mathcal{R} \) to the semi-stable locus, and denote it by \( \mathcal{R}^+ \). Then, we define the positive semi-stable locus of the dg-scheme \( \mathbf{X} \) as

\[
\mathbf{X}^+ := (\mathbf{X}^+, \mathcal{R}^+).
\]

In Section 3.2 we define a Fourier-Mukai kernel \( Q^+ \) and following [BDF17], we get a fully-faithful functor \( \Phi_{Q^+} \).

**Proposition 1.1** (Lemmas 3.14 and 4.8). There is a fully-faithful (window) functor

\[
\Phi_{Q^+} : \text{Perf}^\mathbb{G}_m(\mathbf{X}^+) \longrightarrow \text{Perf}^\mathbb{G}_m(\mathbf{X}).
\]

In order to identify the essential image of the window functor (known as a window) explicitly, we need to impose a condition on the internal degree of the homological generators of \( R \) over \( T \).

**Theorem 1.2** (Theorem 4.13). Assume that \( R \) is generated as a \( T \)-algebra by non-positive elements. The functor

\[
\Phi_{Q^+} : \text{Perf}^\mathbb{G}_m(\mathbf{X}^+) \longrightarrow \mathbb{W}_X
\]

is an equivalence of categories where \( \mathbb{W}_X \) is étale locally the full subcategory generated by \( \mathcal{R}(i) \) for \( i \) in a prescribed range (see Definition 4.9).

1.1.1. **Comparison with the literature.** Let us now restrict to the setting of singular VGIT problems (which we describe in more detail in Section 1.3). Then, the cdga \( R \) is chosen to be a semi-free dg-resolution of a singular ring. Under certain assumptions called Properties \( L(+) \) and \( A \), [HL15] also defines a window subcategory and proves that the derived category of the GIT quotient is equivalent to this window (via different methods).

We note that our conditions on the degree in Theorem 1.2 need not satisfy Property \( A \) or Property \( L^+ \) of [HL15] (see Example 4.5). In fact, these conditions are roughly complementary to the ones of [HL15]. Moreover, in Example 4.6 where \( L^+ \) and \( A \) are satisfied (but our Theorem 1.2 does not hold), we check that the essential image of our window functor \( \Phi_{Q^+} \) is the same as the window described in [HL15]. This suggests that the Fourier-Mukai kernel that we study in this paper is indeed the right one (for example, to solve the Bondal-Orlov-Kawamata conjectures). We hope that a more refined definition of the window will allow us to remove the restrictions in Theorem 1.2.

1.2. **Wall-crossing for cdgas.** Using the window functor and the explicit description of the window from the previous section, we can compare the derived categories of the positive and negative GIT quotients. This allows us to obtain the following results about wall-crossings.

By considering the ideal \( J^- \) generated by the negatively graded elements of \( T \), we may define the negative semi-stable locus of the dg-scheme \( \mathbf{X} \) analogously to the previous section, i.e.,

\[
\mathbf{X}^- := (\mathbf{X}^-, \mathcal{R}^-).
\]
We can also prove the analogue of the window statement (Theorem 1.2) for $X^+$ if we assume that the cdga $R$ is generated by non-negatively graded elements. We can combine these results to understand the wall-crossings between $X^+$ and $X^-$.

Let $\mu_\pm$ be the sum of the weights of the conormal bundle of $\text{Spec}T/J^\pm$ in $X$. Let $j_\pm : X^\pm \to X$ be the inclusion of dg-schemes.

**Theorem 1.3** (Theorem 4.15). Assume that the dg-algebra generators of $R$ over $T$ have internal degree zero. When, $\mu_+ + \mu_- = 0$, the wall crossing functor

$$\Phi^\text{wc}_+ := j^+_\ast \circ (- \otimes (\mathcal{O}(-\mu_+ - 1))) \circ \Phi_{Q_+} : \text{Perf}_{\mathbb{G}_m}(X^+) \to \text{Perf}_{\mathbb{G}_m}(X^-)$$

is an equivalence of categories. The inverse functor is

$$\Phi^\text{wc}_- := j^-_\ast \circ (- \otimes (\mathcal{O}(-\mu_- + 1))) \circ \Phi_{Q_-}$$

We also study the case where $\mu_+ + \mu_- \neq 0$ and $X$ is isomorphic to an affine space over an arbitrary Noetherian ring $k$. Consider the ring

$$T = k[x, y],$$

where we use the shorthand notation $x$ to mean $x_1, x_2, \ldots, x_l$ and $y$ to mean $y_1, y_2, \ldots, y_m$ with internal $\mathbb{Z}$-grading $\text{deg} x_i > 0$ and $\text{deg} y_i < 0$.

In this setting, we have the following result.

**Theorem 1.4** (Theorem 4.17). Let $k$ be an arbitrary Noetherian ring. Assume that the algebra generators of $R$ over $T = k[x, y]$ have internal degree zero. Consider the (derived) fixed locus $R^\mathbb{G}_m := R/(x, y)$.

(1) When, $\mu_+ + \mu_- > 0$, we have the following semi-orthogonal decomposition

$$\text{Perf}_{\mathbb{G}_m} X^+ \cong (\text{Perf}(R^\mathbb{G}_m)_{\mu_+ + \mu_-}, \ldots, \text{Perf}(R^\mathbb{G}_m)_2, \text{Perf}(R^\mathbb{G}_m)_1, \Phi^\text{wc}_+(\text{Perf}_{\mathbb{G}_m} X^-)).$$

(2) When, $\mu_+ + \mu_- < 0$, we have the following semi-orthogonal decomposition

$$\text{Perf}_{\mathbb{G}_m} X^- \cong (\text{Perf}(R^\mathbb{G}_m)_{\mu_+ + \mu_-}, \ldots, \text{Perf}(R^\mathbb{G}_m)_2, \text{Perf}(R^\mathbb{G}_m)_1, \Phi^\text{wc}_+(\text{Perf}_{\mathbb{G}_m} X^+)).$$

The main application of the above results that we are interested in is the setting of VGIT problems arising from singular schemes (in particular, those obtained as a VGIT presentation of a flop).

### 1.3. Applications to singular VGIT problems.

Consider a (possibly singular) subscheme $Y = \text{Spec} S$ of a smooth affine scheme $X = \text{Spec} T$ of finite type over a field $k$, which is equipped with a $\mathbb{G}_m$-action,

$$Y = \text{Spec} S \hookrightarrow X = \text{Spec} T.$$

Then we resolve the $\mathbb{Z}$-graded singular ring $S$ by a $\mathbb{Z}$-graded semi-free cdga $R$, using the Koszul-Tate resolution (which is a generalization of the Koszul resolution for complete intersections) [Tat57].

$$R \simeq S.$$ 

In particular, $R$ has only finitely many generators in each homological degree, but could have infinitely many generators in total. Note that the degree zero part $R^0 = T$. We introduce the following notation for the GIT quotients of $Y$.

$$Y \sslash_\mu := [Y - V(J^\ast)/\mathbb{G}_m].$$

Now we may use Theorem 1.3 to prove the following derived equivalences.
Corollary 1.5 (Corollary 4.18). Assume that the dg-algebra generators of $R$ over $T$ have internal degree zero. When $\mu_+ + \mu_- = 0$, the wall crossing functor

$$\Phi^{\text{wc}} : \text{Perf}(Y_+) \rightarrow \text{Perf}(Y_-)$$

is an equivalence of categories. In particular, if the semi-stable loci are smooth (for example, the setting of smooth flops),

$$\Phi^{\text{wc}} : D^b(Y_+) \rightarrow D^b(Y_-)$$

is an equivalence of categories.

As a special case we recover the local case of the equivalence in [Nam03, Kaw06, Mor18, ADM19], using VGIT and window techniques.

Theorem 1.6 (Corollary 4.19). For the local model of the Mukai flop over a fixed commutative Noetherian ring $k$, the wall crossing functor

$$\Phi^{\text{wc}} : D^b(Y_+) \rightarrow D^b(Y_-)$$

is an equivalence of categories.

While this is a well-known result over $\mathbb{C}$, our proof holds over an arbitrary Noetherian ring $k$.

1.4. Outline of the paper. In Section 2, we first set up notation and introduce the language of dg-schemes. We discuss the derived category of equivariant dg-schemes and the existence of derived functors in Section 2.2. We define the category of perfect complexes in Section 2.3, which is the category we are interested in, and in Section 2.4, we introduce the Variation of GIT quotients problem that we study in this paper.

In Section 3, we define the object $Q$ which is the main object of this paper, and study various properties of it. Section 3.1 is a brief reminder of the construction of $Q$ in the setting of (non-dg) rings, and then we define it for semi-free cdgas in Section 3.2. We study various properties of $Q$ in Section 3.3, which will be important to understand the Fourier-Mukai transforms induced by it. In Section 3.4, we find conditions for fully-faithfulness of the window functor defined using $Q$.

In Section 4, we study the window functors and wall-crossing functors defined using $Q$ as the Fourier-Mukai kernel. We focus on the case where the underlying scheme is affine space in Section 4.1 and the general case of a smooth affine scheme in Section 4.2. Then, we study the induced wall-crossing functors in Section 4.3 and finally discuss applications to VGIT problems arising from flops in Section 4.4.

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2. Setup

Let us set notation and recall relevant definitions and results about dg-schemes following [CFK01, Gaic10, MR15, BR12].

2.1. Notation. Throughout, $k$ will denote a fixed commutative Noetherian ring over which all objects are defined. In this paper, we will only work with commutative differential graded algebras concentrated in non-positive degrees, and henceforth we will simply refer to one as a cdga. Bold letters, e.g. $\mathbf{X}$, will denote a dg-scheme, and the underlying ordinary scheme will be denote by unbolded letters, e.g. $X$. We will assume that all our schemes are separated and Noetherian of finite dimension.

Often, our objects (cdgas, for example) will have a $\mathbb{Z}$-grading coming from a $\mathbb{G}_m$-action (on dg-schemes, for example) which we will refer to as the internal grading, as opposed to the homological
grading coming from the cdga structure. In order to denote the homological grading, we will use upper indices, whereas we use lower indices for the internal grading.

We say that the graded vector space \( k(i) \) has weight \( i \). For a graded module \( M \), the degree 0 piece of the shifted module \( M(i) \) is the degree \( i \) piece of \( M \), notated \( M_i \) so that \( (M(i))_j = M_{i+j} \).

2.2. **dg-schemes.** The data of a dg-scheme \( X \) is the pair

\[ X = (X, \mathcal{A}), \]

where \( X \) is a scheme and \( \mathcal{A} \) is a non-positively graded, commutative dg \( \mathcal{O}_X \)-algebra, such that \( \mathcal{A}^i \) is a quasi-coherent \( \mathcal{O}_X \)-module for any \( i \in \mathbb{Z}_{\geq 0} \). We denote the homological graded pieces of \( \mathcal{A} \) by \( \mathcal{A}^i \).

**Definition 2.1.** Let \( X = (X, \mathcal{A}) \) be a dg-scheme. A quasi-coherent dg-sheaf \( \mathcal{F} \) on \( X \) is a \( \mathcal{A} \)-dg-module such that \( \mathcal{F}^i \) is a quasi-coherent \( \mathcal{O}_X \)-module for any \( i \in \mathbb{Z} \).

We define the derived category of quasi-coherent dg-sheaves by \( D(\text{QCoh} X) \). The derived category is defined in the usual way as the localization of the homotopy category with respect to the class of quasi-isomorphisms. This has the structure of a triangulated category. Note that in the case of ordinary schemes, i.e., if \( \mathcal{A} = \mathcal{O}_X \), the category \( D(\text{QCoh} X) \simeq D(\text{QCoh} X) \).

In [BR12], it is shown that there are enough K-flat and K-injective objects in \( \text{QCoh}(X) \), and hence one can use these to define all right derived functors, and left derived tensor products and pullbacks. They also prove the adjunction between derived pushforwards and pullbacks, the projection formula and a base change formula. It is worth noting that in the context of dg-schemes, flatness is not required for the base-change formula; instead we use the derived fiber product.

In the \( \mathbb{G}_m \)-equivariant setting, i.e., when we have a scheme \( X \) equipped with a \( \mathbb{G}_m \)-action and \( \mathcal{A} \) is equivariant with respect to this \( \mathbb{G}_m \)-action, some of the appropriate generalizations are defined by [MR15] under the following technical assumptions:

1. For any \( \mathcal{F} \) in \( \text{QCoh}^{\mathbb{G}_m}(X) \), there exists a \( P \) in \( \text{QCoh}^{\mathbb{G}_m}(X) \) which is flat over \( \mathcal{O}_X \) and a surjection \( P \to \mathcal{F} \) in \( \text{QCoh}^{\mathbb{G}_m}(X) \).

2. Assume that \( \mathcal{A} \) is locally free over \( \mathcal{A}^0 \), \( \mathcal{A}^0 \) is locally finitely generated as an \( \mathcal{O}_X \)-algebra, and finally that \( \mathcal{A} \) is K-flat as a \( \mathbb{G}_m \)-equivariant \( \mathcal{A}^0 \)-dg-module.

In this paper, we will always work over \( X \) quasi-affine or affine, and hence the first condition is automatically satisfied. We will only consider dg-schemes such that \( \mathcal{A}^0 = \mathcal{O}_X \) and that \( \mathcal{A} \) is semi-free over \( \mathcal{O}_X \), and hence the second condition is always satisfied as well.

We note that [MR15] imposes a slightly stronger technical assumption which is that \( \mathcal{A} \) is locally free of finite rank over \( \mathcal{A}^0 \). However, this assumption is unnecessary in the proof of [MR15] Proposition 2.8. Moreover, the proof of [Ric10] Theorem 1.3.6] carries over to this setting as well. Hence we have the following statement.

**Lemma 2.2.** Consider a \( \mathbb{G}_m \)-equivariant dg-scheme \( X \) satisfying the assumptions above. For any object \( \mathcal{F} \) in \( \text{QCoh}^{\mathbb{G}_m}(X) \), there exists a K-injective equivariant resolution \( \mathcal{F} \to \mathcal{I} \).

**Proof.** First we use [MR15] Proposition 2.8 to find a resolution for bounded below \( \mathcal{A} \)-dg-modules; this resolution has flabby graded components. Then, the proof in [Ric10] Theorem 1.3.6] carries over to this setting. As a brief reminder, we truncate the dg-sheaf \( \mathcal{F} \), resolve the truncations by K-injectives and then take the inverse limit over the truncations. In the proof of [Ric10] Theorem 1.3.6], we take \( \mathcal{B} = \text{QCoh}^{\mathbb{G}_m}(X) \) and consider equivariant open covers of \( X \). \( \square \)

Using the above lemma, we can define all right derived functors. In order to define left derived inverse images and tensor products, we use [MR15] Lemma 2.7] which proves the existence of K-flat resolutions. Using these K-injective and K-flat resolutions it is easy to prove the projection formula and the adjunction between pushforwards and pullbacks by adapting the arguments of [BR12] to our setting, and we will use these in this paper.
2.3. **Perfect objects.** Let us give an ad-hoc definition of the category \( \text{Perf}^\text{Gm} X \) where \( X = (X, \mathcal{A}) \) with the assumptions as before, and \( X \) is quasi-projective. Consider an ample line bundle \( \mathcal{L} \) on \( X \).

**Definition 2.3.** The thick triangulated subcategory of \( D(\text{QCoh}^\text{Gm} X) \) generated by \( \mathcal{L} \otimes \mathcal{O}_X A(i) \) for all \( i, j \in \mathbb{Z} \) is called the category of perfect dg-sheaves, denoted \( \text{Perf}^\text{Gm} X \).

**Remark 2.4.** We expect that the ‘right’ definition of the category of perfect dg-sheaves on \( X = (X, \mathcal{A}) \), i.e. the triangulated category generated by dg-sheaves that are quasi-isomorphic to a finite locally semi-free \( \mathcal{A} \)-module (i.e., finite locally free after forgetting the differential) is equivalent to our ad-hoc definition in this context. In particular, it is easy to show that if we consider an ordinary scheme (if \( A = \mathcal{O} \)) equipped with an ample line bundle then the definitions coincide.

2.4. **GIT problem.** We start with a \( \mathbb{Z} \)-graded semi-free\(^1\) commutative differential graded algebra (cdga), say \( R \). We denote the homological degree zero piece of \( R \) by \( T := R^0 \), and we require that \( T \) is a finitely generated smooth ring over \( k \). We do not require any finiteness conditions on \( R \). To this data, we associate the affine dg-scheme \( X := \text{Spec} T \), \( X := (X, \mathcal{R}) \), where \( \mathcal{R} \) is the dg-\( \mathcal{O}_X \)-module associated to the cdga \( R \). The \( \mathbb{Z} \)-grading on \( T \to R \) is equivalent to a \( \mathbb{G}_m \)-action on \( X \).

We denote the semi-stable loci (which are dg-schemes) as \( X^\pm := (X^\pm, \mathcal{R}^\pm) \), where we define

\[
X^\pm := X - V(J^\pm), \quad \mathcal{R}^\pm = \mathcal{R}|_{X^\pm}.
\]

Here \( J^\pm \) denotes the ideal generated by all the strictly positive/negative-ly graded elements. Then we define two Geometric Invariant Theory (GIT) quotients \( X_{/\pm} \) of \( X \) with respect to the \( \mathbb{G}_m \)-action. The stacks

\[
X_{/\pm} := [X^\pm/\mathbb{G}_m],
\]

with the sheaf of cdgas obtained by the descent of \( \mathcal{R}^\pm \) to \( X_{/\pm} \) are the GIT quotient “dg-stacks” \( X_{/\pm} = (X_{/\pm}, \mathcal{R}^\pm) \).

**Remark 2.5.** We do not define or introduce dg-stacks in general as the ones we consider are global quotients of dg-schemes, and we are only concerned about the corresponding derived categories. We define the derived category of the global quotient dg-stack merely as the corresponding equivariant derived category of the dg-scheme.

3. **Defining \( Q \)**

We will first recall the \( Q \)-construction briefly in the setting of \( \mathbb{Z} \)-graded (not dg-) rings, and then extend this definition to cdgas. In this section, we continue to use the notations of the previous section. In particular, we remind the reader that \( T \) is a \( \mathbb{Z} \)-graded smooth ring finitely generated over \( k \), \( R \) is a \( \mathbb{Z} \)-graded semi-free cdga over \( T \) and \( X \) is an affine dg-scheme equipped with a \( \mathbb{G}_m \)-action.

\(^1\)Recall that a semi-free cdga \( R \) is one that is free when considered as a commutative algebra by forgetting the differential.
3.1. $Q$ for smooth rings. This section is a brief reminder of the construction of [BDF17]. The $\mathbb{Z}$-grading on the ring $T$ is equivalent to a $\mathbb{G}_m$-action on $X = \text{Spec} T$. Viewed in this manner, we say that $T$ is equipped with the co-projection and co-action morphisms, which we denote by $\pi$ and $\sigma$ respectively. The morphisms act as

\begin{align}
\pi : T &\longrightarrow T[u, u^{-1}] \\
\sigma : T &\longrightarrow T[u, u^{-1}]
\end{align}

for any homogeneous element $t$ in $T$. We assign a $\mathbb{Z} \times \mathbb{Z}$-grading to the equivariant diagonal of $\text{Spec} T$, which is

$$k[\mathbb{G}_m \times X] = T[u, u^{-1}],$$

such that the morphisms $\pi$ and $\sigma$ equivariant. The grading for homogeneous $t$ in $T$ is defined as

$$\deg \pi(t) = (\deg t, 0), \quad \deg \sigma(t) = (0, \deg t) \quad \text{and} \quad \deg u = (-1, 1).$$

Given the smooth ring $T$, we define

$$Q(T) := \langle \pi(T), \sigma(T), u \rangle \subseteq T[u, u^{-1}],$$

as the $k$-subalgebra of $T[u, u^{-1}]$ generated by the image of the co-action and co-projection maps, and $u$. It suffices to keep the image of the negative elements under the co-action map along with $T[u]$, so

$$Q(T) = \langle \bigoplus_{i \leq 0} T^i u^i, T[u] \rangle$$

where $T^i$ denotes the $i$-th graded piece of $T$. The co-projection and co-action maps factor through $Q(T)$, giving the maps $p$ and $s$,

\begin{align}
T &\xrightarrow{p} Q(T) \quad \xleftarrow{s} \Delta(T).
\end{align}

The $T \otimes T$-module $Q(T)$ also inherits the $\mathbb{Z} \times \mathbb{Z}$ grading from $T[u, u^{-1}]$. Hence, we can consider $Q(T)$ as an element of $D(\text{mod}^{\mathbb{G}_m \times \mathbb{G}_m}(T \otimes T))$ using the $p \otimes s$ module structure.

In order to clarify the construction above, let us consider the example where $X$ is an affine space. This example will play an important role in the paper.

**Example 3.1.** Let us consider the ring $T = k[x, y]$, where we are using the shorthand notation $x$ to mean $x_1, \cdots, x_l$, and $y$ to mean $y_1, \cdots, y_m$. We assign the $\mathbb{Z}$-grading

$$\deg x_i = a_i > 0 \quad \deg y_i = b_i < 0$$

to $T$. Then,

$$Q(T) = k[x, z, u],$$

with the $p, s : T \rightarrow Q(T)$ maps given by

$$p(x_i) = x_i \quad s(x_i) = u^{a_i} x_i$$

$$p(y_i) = u^{-b_i} z_i \quad s(y_i) = z_i$$

**Remark 3.2.** For the sake of notational simplicity, we have not added any $x_i$ or $y_j$ of internal degree zero. The $Q$ construction does not affect such degree zero generators and hence we may view it as a part of the Noetherian ring $k$.

There is a geometric motivation for this definition of $Q$, and we refer the reader to [BDF17] for more details. The idea is to define $Q$ as a partial compactification of the $\mathbb{G}_m$-action on $X$, generalizing a construction of [Dri13]. In *loc. cit.*, the authors prove various properties of this object $Q$; one that is worth mentioning here is that the assignment $Q : \text{CR}_{k[\mathbb{G}_m]} \rightarrow \text{CR}_{k[u]^{\mathbb{G}_m}}$ is functorial. We also note that the equivariant injective map $Q(T) \rightarrow \Delta(T)$ provides a natural transformation between the Fourier-Mukai functors $\Phi_{Q(T)} \rightarrow \Phi_{\Delta(T)} = \text{Id}_X$. 
3.2. $Q$ for semi-free cdgas. Now we define $Q$ for semi-free cdgas, which is the case of interest in this paper. Recall that the dg-scheme $X = (\text{Spec} T, R)$ is equipped with a $\mathbb{G}_m$-action. This means that we have the projection and action maps, which we denote by $\pi$ and $\sigma$ respectively,

$$\mathbb{G}_m \times X \xrightarrow{\sigma} X.$$ 

We want to view the cdga associated to $\mathbb{G}_m \times X$ as a dg-sheaf on $X \times X$ using the morphism $\pi \times \sigma$. More precisely, we define a semi-free cdga $\Delta(R)$ (which should be thought of as the equivariant diagonal in the dg-setting)

$$\Delta(R) := R[u, u^{-1}],$$

which is equipped with a $\mathbb{Z} \times \mathbb{Z}$-grading (which, as before, is chosen in order to make the action and projection maps equivariant). The grading for homogeneous $r$ in $R$ is defined as

$$\deg \pi(r) = (\deg r, 0), \quad \deg \sigma(r) = (0, \deg r) \quad \text{and} \quad \deg u = (-1, 1).$$

In order to get the dg-structure, we note that the elements, $u$ and $u^{-1}$ are in homological degree zero, and hence are killed by the differential.

We also have the co-action and co-projection dg-morphisms (which we denote by $\pi$ and $\sigma$ by abuse of notation)

$$(3.3) \quad \pi : R \rightarrow R[u, u^{-1}] \quad \text{and} \quad \sigma : R \rightarrow R[u, u^{-1}]$$

for homogeneous $r$ in $R$.

Consider $\Delta(R)$ as a $R \otimes R$-module with the module structure $\pi \otimes \sigma$. Then, we have the associated quasi-coherent dg-sheaf, also denoted $\Delta(R)$, on $X \times X$. Taking into account the $\mathbb{Z} \times \mathbb{Z}$-grading we view it as an element of $D(\text{QCoh}^{\mathbb{G}_m \times \mathbb{G}_m} \mathbb{X} \times \mathbb{X})$.

**Lemma 3.3.** The object $\Delta(R) \in D(\text{QCoh}^{\mathbb{G}_m \times \mathbb{G}_m} \mathbb{X} \times \mathbb{X})$ is a Fourier-Mukai kernel of the identity functor on $D(\text{QCoh}^{\mathbb{G}_m \mathbb{X}})$.

**Proof.** As all the objects are affine, we may work on the level of cdgas. Then, keeping track of the equivariant structures carefully, we have the following sequence of isomorphisms:

$$\Phi_{\Delta(R)}(A) \cong (\sigma_* \pi^* A)^{\mathbb{G}_m}$$

$$\cong (A[u, u^{-1}])_{(0, \ast)}$$

$$\cong \oplus A^i u^i,$$

where the $R$-module structure is

$$r.(au^i) = (r.a)u^{i+\deg r}.$$ 

Similarly, for the inverse functor,

$$\Phi_{\Delta(R)}(A) \cong (\pi_* \sigma^* A)^{\mathbb{G}_m}$$

$$\cong (A[u, u^{-1}])_{(\ast, 0)}$$

$$\cong \oplus A^i u^{-i},$$

where the $R$-module structure is

$$r.(au^i) = (r.a)u^{-i-\deg r}.$$ 

It is clear that $\oplus A^i u^i$ and $\oplus A^i u^{-i}$ are isomorphic to $A$ with the original $R$-module structure, and hence we are done. 

Now, we define an analogue of the $Q$-construction of $[\text{BDF17}]$ for the semi-free cdga $R$. 

\[ \Box \]
Definition 3.4. The object $Q(R)$ is defined as the following $\mathbb{Z} \times \mathbb{Z}$-graded cdga

$$Q(R) = \langle \pi(R), \sigma(R), u \rangle \subseteq R[u, u^{-1}]$$

generated as a $k$-subalgebra of $R[u, u^{-1}]$. The dg-structure and the $\mathbb{Z} \times \mathbb{Z}$-grading are inherited from the object $\Delta(R) = R[u, u^{-1}]$.

The co-action and co-projection maps to $\Delta(R)$ factor through $Q(R)$, giving the maps $p$ and $s$,

$$R \xrightarrow{p} Q(R) \xrightarrow{s} \Delta(R).$$

The maps $p, s$ extend to $\Delta(R)$ to give the co-projection $\pi$ and co-action $\sigma$ maps on $R$ respectively.

Let us clarify these notions by looking at an extension of Example 3.1.

Example 3.5. Let $T = k[x, y]$ as in Example 3.1 Consider a homogeneous regular sequence of length $n$, say $(h_1, h_2, \ldots, h_{n_1+n_2})$ in $T$. Assume further, without loss of generality, that the internal degree of $h_i$ is $d_i$, and that

$$d_i \geq 0 \text{ for } i \in [0, n_1], \quad d_i < 0 \text{ for } i \in [n_1 + 1, n_1 + n_2].$$

Then, we define the dg-algebra $R$ to be the Koszul resolution on this regular sequence.

$$R = k[x, y, e_1, \ldots, e_{n_1}, f_1, \ldots, f_{n_2}], \quad de_i = h_i, \quad df_j = h_{n_1+j}.$$

Then we can compute $Q(R)$ to be

$$Q(R) = k[x, y, e_1, \ldots, e_{n_1}, g_1, \ldots, g_{n_2}],$$

where the maps $p, s : R \to Q(R)$ are given by

$$p(x_i) = x_i \quad \quad s(x_i) = u^{d_i} x_i$$
$$p(y_i) = u^{-b_i} z_i \quad \quad s(y_i) = z_i$$
$$p(e_i) = e_i \quad \quad s(e_i) = u^{d_i} e_i$$
$$p(f_i) = u^{-d_{n_1+i}} g_i \quad \quad s(f_i) = g_i$$

Let us return to the general setting now. The maps $p, s$ give $Q(R)$ the structure of a $\mathbb{Z} \times \mathbb{Z}$-graded $R \otimes R$ dg-module with the $p \otimes s$-module structure. We will view the associated dg-sheaf, also denoted $Q(R)$, as an element of $D(Q\text{Coh}^{G_n \times G_m} X \times X)$.

3.2.1. Explicit description of $Q(R)$. Let us introduce some more notation and describe $Q(R)$ explicitly. As $R$ is a semi-free cdga over $T$, let us choose a set of (possibly infinite) homogeneous (in the internal grading) algebra generators. We denote the positive generators by $e_i$ and the negative generators by $f_i$, where $i$ takes values in a possibly countably infinite set. We will use the short hand notation $e$ ($f$) to refer to the set of all $e_i$ ($f_i$). Using this notation,

$$R = T[e, f].$$

Defining $g_i := u^{d_{n_1+i}} f_i$, we can express $Q(R)$ explicitly as

$$Q(R) = Q(T)[e, g],$$

with the $p$ and $s$ dg-module structures given by

$$p : R \to Q(R) \quad \quad s : R \to Q(R)$$

$$t \mapsto p_T(t) \quad \quad t \mapsto s_T(t)$$
$$e_i \mapsto e_i \quad \quad e_i \mapsto u^{d_{n_1+i}} e_i$$
$$f_i \mapsto u^{-d_{n_1+i}} g_i \quad \quad f_i \mapsto g_i,$$

where the maps $p_T$ and $s_T$ are the maps defined in equation 3.2 but we have added the subscript $T$ for clarity.
We also note the bi-degrees of the elements in $Q(R)$ for convenience:

$$\text{bi-deg } e_i = (\deg e_i, 0) \quad \text{bi-deg } g_i = (0, \deg f_i),$$

and the degrees of the elements in $Q(T)$ are as defined in Section 3.1.

### 3.3. Some properties of $Q(R)$.

Let us denote the $G_m \times G_m$-equivariant inclusion of $Q(R)$ into $\Delta(R)$ by $\eta$,

$$\eta : Q(R) \to \Delta(R) = R[u, u^{-1}].$$

Let us study the relation of $\Delta(R)$ to $\Delta(R)$ further. In particular, we show that they become isomorphic if we localize by an element in $T$ of non-zero internal degree.

**Lemma 3.6.** Consider $Q(R)$ as a $R \otimes R$ dg-module with the $p \otimes s$ structure. Let $t$ in $T$ be a homogenous element, with the corresponding localization map $R_t = T_i[e, f] \to R = T[e, f]$. If $\deg t > 0$,

$$1 \otimes_s \eta : R_t \otimes_s Q(R) \to R_t \otimes_s \Delta(R)$$

is an isomorphism. If $\deg t < 0$,

$$1 \otimes_p \eta : R_t \otimes_p Q(R) \to R_t \otimes_p \Delta(R)$$

is an isomorphism.

**Proof.** The morphisms are injective as $R_t$ is flat over $R$.

To check that the morphisms are surjective, we just need to check that we get $u^{-1}$ in the image. In the first case, $t^{-1} \otimes u^{\deg t - 1} t$ maps to $u^{-1}$; in the second case, $t^{-1} \otimes s(t) u^{-\deg t - 1}$ maps to $u^{-1}$. □

We need to study some properties of the object $Q(R) \otimes_p Q(R)$ here as it will play a role when we discuss fully faithfulness of the window functor in the next subsection. The object $Q(R) \otimes_p Q(R)$ inherits a $G_m^3$-action, where the $\mathbb{Z}^3$ grading is as follows:

$$\deg q \otimes 1 = (a, b, 0) \quad \deg 1 \otimes q = (0, a, b)$$

if $q \in Q(R)$ is a homogeneous element of degree $(a, b)$.

We would like to understand what happens when we take middle degree invariants; we denote this by $(M)_0$ where $M$ is $\mathbb{Z}^3$-graded.

**Lemma 3.7.** The following diagram commutes

\[
\begin{array}{ccc}
(Q(R) \otimes \Delta)_0 & \cong & (Q(R) \otimes_p Q(R))_0 \\
\downarrow & & \downarrow \\
(Q(R) \otimes_p Q(R))_0 & \cong & Q(R)
\end{array}
\]

**Proof.** Consider the upper right arrow

\[
(Q(R) \otimes \Delta)_0 = Q(R)[v, v^{-1}]_0 \\
\cong \left( \bigoplus_{i \leq 0} R_i, R[u, v][v, v^{-1}] \right)_0 \\
\cong \left( \bigoplus_{i \leq 0} R_i v^i, R[uv] \right) \\
\cong Q(R).
\]

In order to take the middle degree zero invariants, we used that $\deg u = (-1, 1, 0)$ and $\deg v = (0, -1, 1)$. Similarly, one can prove the same for the bottom row, and the commutativity is clear. □
We consider the following morphism,

\[(3.6) \quad \rho : (Q(R) s \otimes_p Q(R))_0 \rightarrow (Q(R) s \otimes_p Q(R))_0 \rightarrow Q(R)\]

where the first morphism is the map from the left derived functor to the (underived) functor, and the second morphism is the one constructed in Lemma 3.6. Fully faithfulness of the window functor is related to properties of the morphism \(\rho\) as we will see in Section 3.4.

Our goal is to construct a Fourier-Mukai kernel for the window functors

\[D(Q\text{Coh}^{G_m} X) \rightarrow D(Q\text{Coh}^{G_m} X),\]

Using the \(G_m\)-equivariant morphism of dg-schemes

\[j : X^+ \rightarrow X,\]

we define

\[Q_+ := (j \times \text{Id})^* Q(R),\]

and consider it as an object of \(D(Q\text{coh}^{G_m \times G_m} X^+ \times X)\). We have dropped the \(R\) in \(Q_+\) in the interest of notational convenience. We will focus on the positive GIT quotient, but the arguments are analogous for the negative GIT quotient.

3.4. Fully faithfulness. In this section, we find sufficient conditions for the Fourier-Mukai functor \(\Phi_{Q_+}\) to be fully faithful. Checking faithfulness is easy and follows the arguments of [BDF17].

**Lemma 3.8.** The composition \(j^* \circ \Phi_{Q_+}\) is naturally isomorphic to the identity. In particular, the functor

\[\Phi_{Q_+} : D(Q\text{coh}^{G_m} X^+) \rightarrow D(Q\text{Coh}^{G_m} X),\]

is faithful.

**Proof.** We want to show that \(j^* \circ \Phi_{Q_+}\) is the identity functor. On the level of the kernels, it suffices to show that the morphism

\[(j \times j)^* Q \rightarrow (j \times j)^* \Delta(R)\]

is an isomorphism. We can do this locally on the \(G_m\)-invariant affine cover of \(X^+ \times X^+\), obtained by inverting the positive elements \(t \in J^+\) in \(T = k[X]\). This is precisely the content of the first part of Lemma 3.6. \(\square\)

Fullness of the functor is more involved and is best phrased in the language of Bousfield localizations. Let us recall some of the definitions and results that we need about Bousfield (co)-localizations. The existence of a Bousfield triangle produces a semi-orthogonal decomposition, and we show that the essential image of our functor is equivalent to one part of the semi orthogonal decomposition.

**Definition 3.9.** Let \(\mathcal{T}\) be a triangulated category. A Bousfield localization is an exact endofunctor \(L : \mathcal{T} \rightarrow \mathcal{T}\) equipped with a natural transformation \(\delta : 1_\mathcal{T} \rightarrow L\) such that:

- a) \(L\delta = \delta L\) and
- b) \(L\delta : L \rightarrow L^2\) is invertible.

A Bousfield co-localization is given by an endofunctor \(C : \mathcal{T} \rightarrow \mathcal{T}\) equipped with a natural transformation \(\epsilon : C \rightarrow 1_\mathcal{T}\) such that:

- a) \(Ce = \epsilon C\) and
- b) \(Ce : C^2 \rightarrow C\) is invertible.

**Definition 3.10.** Assume there are natural transformations of endofunctors

\[C \xleftarrow{\epsilon} 1_\mathcal{T} \xrightarrow{\delta} L\]
of a triangulated category $\mathcal{T}$ such that

$$C_x \xleftarrow{\epsilon_C} x \xrightarrow{\delta} L_x$$

is an exact triangle for any object $x$ of $\mathcal{T}$. Then we refer to $C \rightarrow 1_\mathcal{T} \rightarrow L$ as a Bousfield triangle for $\mathcal{T}$ when any of the following equivalent conditions are satisfied:

1) $L$ is a Bousfield localization and $C(\epsilon_x) = \epsilon_{C_x}$
2) $C$ is a Bousfield co-localization and $L(\delta_x) = \delta_{L_x}$
3) $L$ is a Bousfield localization and $C$ is a Bousfield co-localization.

For a proof that the above properties are indeed equivalent, we refer the reader to [BDF17, Definition 3.33].

**Remark 3.11.** Any Fourier-Mukai functor $\Phi_P$ with a morphism $\Delta \rightarrow P$ satisfies the condition a) to be a Bousfield localization. It is easy to see that $\Phi_P(\delta_A) = \delta_{\Phi_P(A)}$. Analogously, any Fourier-Mukai functor $\Phi_{P'}$ with a morphism $P' \rightarrow \Delta$ satisfies the condition a) to be a Bousfield co-localization.

**Lemma 3.12.** (Property $P$) The triangle of functors

$$\Phi_Q(R) \longrightarrow \text{Id} \longrightarrow \Phi_{\text{cone}(\eta)}$$

is a Bousfield triangle if the morphism

$$\rho : (Q(R) \otimes^L_{\mathcal{P}} Q(R))_0 \longrightarrow Q(R),$$

is an isomorphism.

**Proof.** The proof follows the arguments of the proof of [BDF17, Lemma 3.3.6].

We have a morphism $Q(R) \rightarrow \Delta(R)$, and a morphism $\Delta \rightarrow \text{cone}(\eta)(R)$. Hence (using Remark 3.11), we only need to check the second condition for $\Phi_Q(R)$ to be Bousfield co-localization. This condition translates to

$$(Q(R) \otimes^L_{\mathcal{P}} Q(R))_0 \cong Q(R).$$

If $Q(R)$ satisfies the condition of Lemma 3.12, we will say that $Q(R)$ satisfies Property $P$.

We have another Bousfield triangle given as follows.

**Lemma 3.13.** The following triangle is a Bousfield triangle

$$\Phi_{\text{cone}(\gamma)[-1]} \longrightarrow \text{Id} \longrightarrow j_* \circ j^*,$$

where $\gamma$ is the morphism $\Delta \rightarrow (\text{Id} \times j)_* (\text{Id} \times j)^* \Delta$.

**Proof.** First, we check that $j_* j^*$ is a Bousfield localization. We have the pair of adjoint functors $j^* \dashv j_*$. We apply base change to the following Cartesian square

$$\begin{array}{ccc}
X^+ & \xrightarrow{\text{Id}} & X^+ \\
\downarrow\text{Id} & & \downarrow j \\
X^+ & \xrightarrow{j} & X,
\end{array}$$

to see that the co-unit map

$$j^* j_* \xrightarrow{\sim} \text{Id},$$

is an isomorphism. Now, we can show that the functor $j_* j^*$ equipped with the unit $\text{Id} \rightarrow j_* j^*$ is a Bousfield localization (using base change for example). This is similar to the arguments of [HR17, Example 1.2].

Finally, we take the cone of the unit natural transformation to get the desired Bousfield triangle

$$\Phi_{\text{cone}(\gamma)[-1]} \longrightarrow \text{Id} \longrightarrow j_* \circ j^*.$$
Notice that we may do this on the level of the Fourier-Mukai kernels as we have explicit kernels.

For convenience, we define $C := \Phi_{\text{cone}(\gamma)}[-1]$.

**Lemma 3.14.** If $Q(R)$ satisfies property $P$, there is a weak semi-orthogonal decomposition\[ D(Q\text{Coh}^G_{X}) = \{ \text{Im} \Phi_{\text{cone}(\eta)}, \text{Im} \Phi_{Q(R)} \circ C, \text{Im} \Phi_{Q+]}, \} \]
where $\text{Im}$ denotes the essential image. Furthermore, the functor\[ \Phi_{Q+] : D(Q\text{Coh}^G_{X^+}) \to D(Q\text{Coh}^G_{X}) \]
is fully faithful.

**Proof.** We want to use [BDF17, Lemma 3.3.5] to prove this statement. This says that if $C_1 \to \text{Id} \to L_1$ and $C_2 \to \text{Id} \to L_2$ are Bousfield triangles in a triangulated category $\mathcal{T}$ such that $L_1 C_2 \to L_1$ is an isomorphism, there is a weak semi-orthogonal decomposition\[ \mathcal{T} = \{ \text{Im} C_2 \circ L_1, \text{Im} C_2 \circ C_1, \text{Im} L_2 \}. \]

This induces a fully-faithful functor\[ F : \mathcal{T}/\text{Im} C_1 \to \mathcal{T}. \]

We shall take the two Bousfield triangles from Lemma 3.13 and Lemmas 3.12 to be the triangles 1 and 2 above respectively. Hence we need to show that\[ j_* \circ j^* \circ \Phi_{Q(R)} = j_* \circ j^*. \]
The Fourier-Mukai transform $j^* \circ \Phi_{Q(R)}$ is the one induced by the kernel $(\text{Id} \times j)^* Q(R)$. Note that the $R \otimes R$ module structure on $Q(R)$ is $p \otimes s$. By Lemma 3.6 applied to an open cover of $X \times X^+$, $(\text{Id} \times j)^* Q(R)$ is isomorphic to $(\text{Id} \times j)^* \Delta(R)$, as we are inverting a positive element with the $s$-module structure. Noting that $\Phi_{(\text{Id} \times j)^* \Delta(R)} = j^*$ proves the claim.

The functor $F$ mentioned above is the functor $\Phi_{Q+]$. □

**Remark 3.15.** We mention here that [BDF17] proves a semi-orthogonal decomposition result for the homotopy category of the category of spectra in simplicial graded modules over $R$ (viewed as a simplicial graded commutative ring), which is morally equivalent to a part of the semi-orthogonal decomposition (Lemma 3.14) that we prove. The relevant result is [BDF17, Proposition 5.4.7] for the interested reader.

### 4. Windows and Wall-crossings

In this section, we study the object $Q_+$ defined earlier as a Fourier-Mukai kernel for the window functor,\[ \Phi_{Q_+} : \text{Perf}^G_{X^+} \to \text{Perf}^G_{X}. \]

We use the various properties of $Q$ proved in the previous section to prove that the functor is always fully faithful. Then, we identify the essential image of this functor explicitly under certain assumptions on our dg-scheme $X$, which proves our first main theorem, Theorem 1.3. Then, we look at wall-crossing functors between the GIT quotients and prove that it is an equivalence in some cases, and this gives our second main result, Theorem 1.4. We also study the cases where the wall-crossing functor is not an equivalence, but only fully-faithful. Then, under certain conditions on $X$, we find a semi-orthogonal decomposition for a GIT quotient, where one of the pieces of the SOD is the derived category of the other GIT quotient.

For ease of exposition, we first tackle the case that $X$ is affine space (i.e., Example 3.1) in Section 4.1 as the calculations are more explicit. The case of general smooth affine schemes $X$ will reduce to this using the Luna slice theorem in Section 4.2. Wall-crossings are studied in Section 4.3.

Finally, we also look at applications to flops and the Bondal-Orlov conjecture, and prove the derived equivalence result for the Mukai flop in Section 4.4.
4.1. **Windows over affine space.** Here, we consider the case where the underlying scheme of the dg-scheme $X$ is affine space. On the level of the rings, this means that we will choose (as in Example 3.1 and 3.3)

$$T = k[x, y],$$

where we use the shorthand notation $x$ to mean $x_1, x_2, \ldots, x_l$ and $y$ to mean $y_1, y_2, \ldots, y_m$ with internal $\mathbb{Z}$-grading $\deg x_i = a_i > 0$ and $\deg y_i = b_i < 0$. We will repeatedly use the above convention for bold letters for notational convenience. We also define the following two integers:

$$\mu_+ := \sum a_i, \quad \mu_- := \sum b_i.$$

**Lemma 4.1.** When $T = k[x, y]$ with the notations above, the object $Q(R)$ satisfies property $P$, i.e., the morphism

$$\rho : (Q(R) \otimes_\mathbb{C} Q(R))_0 \longrightarrow Q(R),$$

from equation (3.1) is an isomorphism.

**Proof.** In order to compute the derived tensor product, we need to find a semi-free resolution of $Q(R)$. We claim that the following cdga $K$ is a semi-free resolution of $Q(R)$

$$K := (R \otimes R[u, \kappa, \lambda, \mu, \nu], d),$$

such that

$$d\kappa = (x^2 - u^{-deg} x^1), \quad d\lambda = (e^2 - u^{-deg} e^1), \quad d\mu = (y^1 - u^{-deg} y^2), \quad d\nu = (f^1 - u^{-deg} f^2).$$

In order to make the notation compact, we are using $\kappa$ to denote a set of elements (as many as the number of $x_i$s); and similarly for $\lambda, \mu$ and $\nu$. The superscripts (1 or 2) refer to which copy of $R$ the variable belongs to; for example $x^1$ denotes $x^1 \in R \otimes R$ (we apologize for this unfortunate choice of notation).

It is easy to see that the morphism

$$K \longrightarrow Q$$

sending $\kappa, \lambda, \mu$ and $\nu$ to zero is a quasi-isomorphism.

Now, we explicitly calculate $(Q(R) \otimes_\mathbb{C} Q(R))$

$$(Q(R) \otimes_\mathbb{C} Q(R)) = T \otimes T[\kappa, \lambda, \mu, \nu, u] \otimes_{\mathbb{C}} k[x, e, g, v]$$

$$(\kappa, \nu) = k[x, x^1, e^1, z, y^1, g, f^1, \kappa, \lambda, \mu, \nu]$$

where the dg structure remains unchanged on all the elements except $\kappa, \lambda, \mu, \nu$ on which it is given by

$$d\kappa = (x - u^{-deg} x^1), \quad d\lambda = (e - u^{-deg} e^1), \quad d\mu = (y - u^{-deg} y^2), \quad d\nu = (f - u^{-deg} f^2).$$

Now, we see that this cdga is quasi-isomorphic (by “solving out” the above relations) to

$$k[x^1, e^1, z, g, u, v],$$

where the tri-degree of the generators are as follows

$$\text{tri} - \deg x^1 = (\deg x_i, 0, 0) \quad \text{tri} - \deg e^1 = (\deg e_i, 0, 0) \quad \text{tri} - \deg z = (0, 0, \deg y_i),$$

$$\text{tri} - \deg g = (0, 0, \deg f_i) \quad \text{tri} - \deg u = (-1, 1, 0) \quad \text{tri} - \deg v = (0, -1, 1).$$

Taking middle degree 0 invariants we get

$$(Q(R) \otimes_\mathbb{C} Q(R))_0 \simeq k[x^1, e^1, z, g, u]$$

$$\simeq Q(R).$$

If we carefully trace through the series of isomorphisms in the proof, we see that it is induced by the morphism $\rho$.

**Lemma 4.2.** The category of perfect objects $\text{Perf}^G_m(X^+) \text{ is generated by } j^* \mathcal{R}(i)$ for $i \in (-\mu_+, 0]$. 
Proof. The structure sheaf $\mathcal{O}_{X^*}$ is an ample line bundle as $X^*$ is quasi-affine. By Definition 2.3, the category is generated by $j^* \mathcal{R}(i)$ where $i \in \mathbb{Z}$. In fact, we can restrict to a smaller set of generators, $j^* \mathcal{R}(i)$ where $i \in (-\mu_+, 0]$ due to the following argument.

Consider the Koszul complex $K_\bullet(x_1, x_2, \ldots, x_n)$ on $X$

$$
\begin{array}{cccccccc}
0 & \longrightarrow & \mathcal{O}_X(-\mu_+) & \longrightarrow & \cdots & \longrightarrow & \oplus_i \mathcal{O}_X(-a_i) & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}
$$

This is acyclic when restricted to $X^*$ and the twists in all intermediate terms lie in $(-\mu_+, 0]$. Now, we may apply $\otimes_{\mathcal{O}_X^\text{flat}} j^* \mathcal{R}$ to see that $j^* \mathcal{R}(-\mu_+)$ is generated as claimed. We can twist the Koszul complex by $\mathcal{R}(p)$ where $p < 0$ to show that $j^* \mathcal{R}(i)$ for $i \leq -\mu_+$ is generated as claimed. Similarly, we twist by $\mathcal{R}(q)$ where $q > 0$, to show that $j^* \mathcal{R}(i)$ for $i > 0$ is generated as claimed.

Now, we would like to find a set of generators for the essential image of $\Phi_{Q_s}$. For this result we need to make the assumption that all the positive generators $e_i$ vanish, i.e., $R$ is semi-free over $T$, where all the (homological) generators $f_i$ of $R$ over $T$ are of non-positive internal degree.

**Lemma 4.3.** Assume that the degrees of the generators $f_i$ of $T \to R$ are non-positive, i.e., $\deg f_i \leq 0$. Then, the image of $\Phi_{Q_s} : \text{Perf}^{G_m}(X^*) \longrightarrow \text{Perf}^{G_m}(X)$ is the full subcategory of $\text{Perf}^{G_m}(X)$ generated by $\mathcal{R}(i)$ where $i \in (-\mu_+, 0]$. Moreover, we have

$$
\Phi_{Q_s} \circ j^*(\mathcal{R}(i)) = \mathcal{R}(i) \quad \text{where} \quad i \in (-\mu_+, 0].
$$

**Proof.** It suffices to find the image of $\Phi_{Q_s}$ on the generators obtained in Lemma 4.2, $j^* \mathcal{R}(i)$ for $i \in (-\mu_+, 0]$. We have the following sequence of isomorphisms.

$$
\Phi_{Q_s}(j^* \mathcal{R}(i)) = (R\pi_{2*}(Q_+ \otimes \pi_1^* j^* \mathcal{R}(i)))^{G_m}
$$

$$
= (R\pi_{2*} Q_+ (i, 0))(0, *)
$$

$$
= (\pi_{2*}(Q_+ \otimes C_{X^* \times X} (i, 0))(0, *)
$$

where $C_{X^* \times X}$ denotes the Čech resolution obtained by inverting the $x_i$. We claim that we can use the Čech resolution to compute the derived pushforward $R\pi_{2*}$ due to the following argument.

The morphism $\pi_2 : X^* \times X \to X$ factors as

$$
X^* \times X \xrightarrow{\text{for}} (X^* \times X, \mathcal{O}_{X^*} \otimes \mathcal{R}) \xrightarrow{\pi_0^0} (X, \mathcal{R}) = X,
$$

where the first morphism is the forgetful morphism. To compute $R\pi_0^0$, we can use the Čech resolution and then to compute $R\pi_{2*}$, we use the composition of derived functors.

Notice that we are tensoring over the $p$ module structure and hence the result of the cohomology computation will have the $s$ module structure.

We claim that the morphism

$$
\pi_{2*}(Q_+^b (i, 0)) \longrightarrow \pi_{2*}(Q_+^b \otimes C_{X^* \times X} (i, 0)).
$$

is an isomorphism for all $b \leq 0$ after taking internal degree $(0, *)$ invariants. Here, the upper index $b$ denotes the homological degree. First, let us compute the right hand side using a standard Čech cohomology computation. Notice that

$$
H^c(\pi_{2*}(Q_+^b \otimes C_{X^* \times X} (i, 0))) = 0 \quad \text{for} \quad c \neq 0, l.
$$

The $c = l$ term is generated by

$$
\prod_{i=1}^l x_i^{-p_i} k[z, u] \prod_{\text{hdeg} f_i = b} g_i,
$$

where $p_i > 0$ for all $0 \leq i \leq l$. Here, we use hdeg to denote the homological degree, and where all the $a_i$ are strictly positive. Upon taking degree $(i, *)$ invariants, this term vanishes as $i \in (-\mu_+, 0]$. The
left hand side of (4.1) is just a module and by taking its cohomology, we just mean taking degree 0 invariants. This can be computed directly as
\[ \pi_{2*}(Q^b)_{(i,*)} \cong u^{-i}k[u^\deg x, x, z] \prod_{\Sigma \deg f_i = b} g_i \cong s(\mathcal{R})^b(i) \]

Finally, we apply \( H^0 \) and \((0,*)\) invariants to equation (4.1) to get
\[ s(\mathcal{R})^b(i) \rightarrow H^0(\pi_{2*}(Q^b \otimes C_{X,x,x}))_{(i,*)} \cong u^{-i}k[u^\deg x, x, z] \prod_{\Sigma \deg f_i = b} g_i, \]
which is clearly an isomorphism and this proves our claim.

Now, this claim implies that
\[ \pi_{2*}(Q_+ \otimes C_{X,x,x}))_{(i,*)} \cong \mathcal{R}(i), \]
with the \( s \)-module structure and hence we are done.

\[ \square \]

We can do the same analysis for \( \Phi_{Q_+} : \text{Perf}_m^G(X^-) \rightarrow \text{Perf}_m^G(X) \) where \( Q_- := Q|_{X-x \times X} \). Again, we need to make the assumption that all the negative generators \( f_i \) vanish, i.e., \( R \) is semi-free over \( T \), where all the generators \( e_i \) are of non-negative degree.

**Lemma 4.4.** Assuming that the degrees of the generators \( e_i \) of \( T \rightarrow R \) are non-positive, i.e., \( \deg e_i \geq 0 \), the image of \( \Phi_{Q_-} : \text{Perf}_m^G(X^-) \rightarrow \text{Perf}_m^G(X) \) is the full subcategory of \( \text{Perf}_m^G(X) \) generated by \( \mathcal{R}(i) \) where \( i \in \{0, -\mu_\_\} \).

**Proof.** This holds by symmetry. \[ \square \]

Now, we look at an example of a hypersurface in affine space, in order to compare our conditions on the degree of \( e_i \) with Property \((L+)\) and Property \((A)\) of [HL15].

**Example 4.5.** We consider the hypersurface case of Example 3.5 This is also studied in [HL15, Example 2.5]. However, as a warning to the reader, the convention on weights in loc. cit. appears to be the opposite of our work. Consider the cdga \( R = k[x, y, f] \), where \( df = h \). The internal \( \mathbb{Z} \)-grading is given by
\[ \deg x_i > 0, \quad \deg y_i < 0, \]
as before. If we have the condition
\[ \deg h \leq 0, \]
Lemma 4.3 identifies the window subcategory explicitly. By comparison, Property \((L+)\) and Property \((A)\) of [HL15] can be satisfied either by imposing \( \deg h \geq 0 \) or by requiring that \( h \mod x \) is non-zero and linear in at least one of the \( y_i \). Hence our condition on the degree appears to be almost complementary to the one imposed in loc. cit.

Now, let us look at a specific case of Example 4.5 with \( \deg h \leq 0 \).

**Example 4.6.** We consider the cdga \( R = k[x_1, x_2, f] \) with \( df = x_1 x_2 \) and internal \( \mathbb{Z} \)-grading \( \deg x_1 = \deg x_2 = 1 \). Note that \( \deg f = 2 \) in this example, and hence Lemma 4.3 does not apply. However, we wish to understand the essential image of the window functor \( \Phi_{Q_+} \) in this example. It is easy to do so by direct computation. As we wish to compare our window to [HL15], we also use the isomorphism
\[ \text{Perf}_m^G(R) \cong \text{Perf}_m^G(k[x_1, x_2]/(x_1 x_2)). \]

It suffices to find the image of \( j^* \mathcal{R} \) as \( j^* \mathcal{R}(-1) \cong j^* \mathcal{R} \). So, we compute \( \Phi_{Q_+}(j^* \mathcal{R}) \) as in Lemma 4.3 to get
\[ \Phi_{Q_+}(j^* \mathcal{R}) = (Q_+ \otimes C_{X,x,x})_{(0,*)} = (k[x_1, x_2, x_1^{-1}, e, u] \oplus k[x_1, x_2, x_2^{-1}, e, u])_{(0,*)}. \]
To get this, observe that the other term of the Čech complex is
\[ k[x_1, x_2, x_1^{-1}, x_2^{-1}, e, u] \cong 0 \]
as \( de = x_1x_2 \) is inverted.

Under the isomorphism [4.2] we can compute the right hand side of equation [4.3] to get
\[ (k[x_1, x_2, x_1^{-1}, u]/(x_1x_2) \oplus k[x_1, x_2, x_2^{-1}, u]/(x_1x_2))(0,*) \cong (k[x_1, x_1^{-1}, u] \oplus k[x_2, x_2^{-1}, u])(0,*) \cong k[u] \oplus k[x_2]. \]

Now, one can check that the window subcategory as defined in [HL15] is generated by \( k[x_1] \) and \( k[x_2] \).

4.2. Windows in general. Now, we consider the general case (for us), where \( T \) is a finitely generated smooth \( \mathbb{Z} \)-graded ring over a field \( k \). We need to restrict to a field \( k \) in order to use the Luna slice theorem. We recall the following result from [BDF17] for (non-dg) \( k \)-algebras.

**Lemma 4.7.** For any smooth finitely generated \( \mathbb{Z} \)-graded \( k \)-algebra \( T \), \( Q(T) \) satisfies Property \( P \) i.e.,
\[ \rho_T : (Q(T) \otimes^L Q(T))_0 \cong Q(T) \]

**Proof.** The proof uses the Luna slice theorem to reduce to the case of affine space. See [BDF17] Lemma 4.2.5, Proposition 4.2.6. \( \square \)

**Lemma 4.8.** When \( T \) is a finitely generated smooth \( \mathbb{Z} \)-graded ring over a field \( k \), the object \( Q(R) \) satisfies property \( P \), i.e., the morphism
\[ \rho : (Q(R) \otimes^L Q(R))_0 \rightarrow Q(R), \]
is an isomorphism.

**Proof.** Consider a semi-free resolution \( K \) of \( Q(T) \) as a \( T \otimes T \)-module. This provides a semi-free resolution \( K[e, g] \) of \( Q(R) \) and we can compute the derived tensor product,
\[ Q(R) \otimes^L Q(R) \cong (K[e, g] \otimes^L Q(T)[e', g']) \]
\[ \cong K \otimes_{T'} Q(T)[e, g'] \]
\[ \cong Q(T) \otimes^L_{T'} Q(T)[e, g']. \]

Hence
\[ (Q(R) \otimes^L Q(R))_0 \cong (Q(T) \otimes^L_{T'} Q(T))_0 [e, g'] \quad \text{since } e, g' \text{ have middle degree 0} \]
\[ \cong Q(T)[e, g'] \quad \text{by Lemma 4.7} \]
\[ \cong Q(R). \]

Tracing through the chain of maps, we see that the map inducing the isomorphism is \( \rho \). \( \square \)

4.2.1. Defining windows. Let us recall the definition of windows in the setting of ordinary schemes as in [BDF17] now. Let \( T \) be a \( \mathbb{Z} \)-graded smooth ring over a field \( k \). Let \( \mu \) be the sum of the weights of the conormal bundle of \( \text{Spec} T/J^+ \) in \( \text{Spec} T = X \), where \( J^+ \) is the ideal generated by all the positive/negative-ly graded elements. For simplicity, we assume that the fixed locus \( V(J^+) \cap V(J^-) \) is connected. If not, we need to define different \( \mu \) for different connected components.

Then we define the grade restriction window \( \mathbb{W}^X_\mu \) to be the full subcategory of \( \text{D}^b(QCoh^{\leq m} X) \) generated by objects \( A \) such that for any fixed point \( y \in X \), and some affine étale neighborhood \( V = \text{Spec} S \) in \( X = \text{Spec} T \), the restriction
\[ A \otimes_T S \]
is generated by \( S(i) \) for \( i \in (-\mu, 0] \). Notice that this definition of the window matches the one from Lemma 4.3 in the case where \( X \) is affine space.

In our setting of dg-schemes we will view the scheme \( X = \text{Spec} T \) as a dg-scheme \((X, \mathcal{O}_X)\), but by abuse of notation denote it by just \( X \). We use \( j' \) for the following inclusion:

\[ j' : (X^+, \mathcal{O}_X^+) \rightarrow (X, \mathcal{O}_X). \]

For the dg-scheme \( X = (X, R) \) that we are interested in, we will define the window \( W_X^+ \) as the pullback of the window \( W_X^+ \) under the forgetful morphism. Just as in the case of ordinary schemes, we assume that the fixed locus is connected for simplicity. In more detail, we have the following.

**Definition 4.9.** Let

\[ f : X = (X, R) \rightarrow X = (X, \mathcal{O}_X) \]

be the forgetful morphism. The *grade restriction window* \( W_X^+ \) to be the full subcategory of \( \text{Perf}^G_{\text{gm}}(X) \) generated by objects \( f^* A \) with \( A \in \text{Perf}^G_{\text{gm}}(X) \) such that for any fixed point \( y \in X \), and some affine étale neighborhood \( V = \text{Spec} S \) in \( X = \text{Spec} T \), the restriction

\[ A \otimes_T S \]

is generated by \( S(i) \) for \( i \in (-\mu, 0] \) i.e.

\[ W_X^+ := \{ f^* A, A \text{ in } W_X^+ \}. \]

We also introduce the forgetful morphism on the semi-stable locus

\[ f^+ : X^+ \rightarrow X^+, \]

and the forgetful morphism

\[ \alpha := f^+ \times f : X^+ \times X \rightarrow X^+ \times X. \]

Let us reiterate that we will denote the dg-scheme \((X, \mathcal{O}_X)\) by just \( X \), as opposed to the dg-scheme \( X = (X, R) \). The following commutative diagrams summarize the notation:

\[
\begin{array}{cc}
\pi_1 & \alpha \\
\downarrow \pi_1 & \downarrow \pi_2 \\
X^+ \times X & X^+ \times X
\end{array}
\]

and

\[
\begin{array}{cc}
\pi_1 & \alpha \\
\downarrow \pi_1 & \downarrow \pi_2 \\
X^+ & X
\end{array}
\]

The following result about base change for \( Q \) under the forgetful morphism \( f : X \rightarrow X \) will help us compute the window \( W_X^+ \).

**Lemma 4.10.** Assuming that the degrees of the generators \( f_i \) of \( T \rightarrow R \) are non-positive, i.e., \( \deg f_i \leq 0 \), the following is an isomorphism of functors

\[ \Phi_{Q(R)} \circ f^* \cong f^* \circ \Phi_{Q(T)}, \]

where \( Q(R) \) and \( Q(T) \) are considered with the \( p \otimes s \)-module structure.

**Proof.** We claim that the following morphism of \( \mathbb{Z} \times \mathbb{Z} \)-graded \( R \otimes R \)-dg-modules

\[ Q(T)_s \otimes_T R \rightarrow Q(T)[g] \]

\[ q \otimes 1 \mapsto q \]

\[ 1 \otimes f_i \mapsto g_i, \]

\[ 1 \otimes t \mapsto s(t), \]

(4.4)
where \( q \in Q(T), f_i \in R \) and \( t \in T \subset R \), is an isomorphism.

If we forget the dg-structure, it is clear that the map is surjective and injective, and respects the grading when we view \( R \) in \( Q(T)_s \otimes_T R \) with the grading \((0, \ast)\). The morphism also respects the dg-structure; if \( df_i \) is in \( T \),
\[
d(1 \otimes f_i) = 1 \otimes df_i = s(df_i) \otimes 1 \mapsto s(df_i) = dg_i.
\]

This proves the isomorphism
\[
Q(T)_s \otimes_T R \cong Q(R).
\]

Now, let \( M \) be an object of \( D(X) \). Then
\[
\Phi_{Q(R)} \circ f^*(M) = \pi_{2\ast}((M \otimes_T R) \otimes_R pQ(R))^{G_m}
\]
\[
= \pi_{2\ast}((M \otimes_T R) \otimes_R pQ(T) \otimes_T R)^{G_m}
\]
\[
= \pi_{2\ast}((M \otimes_T pQ(T)) \otimes_T R)^{G_m}
\]
\[
= f^* \circ \Phi_{Q(T)}(M)
\]

To get the second to last line of the above chain of isomorphisms, we use that \( R \) has grading \((0, \ast)\) and that the \( G_m \)-invariants is with respect to the first grading. \( \square \)

We recall one of the main results of [BDF17].

**Lemma 4.11 ([BDF17, Theorem 4.2.9]).** When \( X \) is a smooth scheme over a field \( k \), the following
\[
\Phi_{Q(T)_s} : D^b(X_+) \cong W_X^+ : j'^\ast
\]
is an equivalence of categories.

Finally, we show that our window functor \( \Phi_Q \) lands in the window \( W_X^+ \).

**Lemma 4.12.** Assuming that the degrees of the generators \( f_i \) of \( R \) as a \( T \)-algebra are non-positive, i.e., \( \deg f_i \leq 0 \), the image of \( \Phi_Q : \text{Perf}^{G_m}(X^+) \to \text{Perf}^{G_m}(X) \) lies in the window subcategory \( W_X^+ \subset \text{Perf}^{G_m}(X) \).

**Proof.** We have the following isomorphisms:
\[
f^* \circ \Phi_{Q(T)} \circ j' \cong \Phi_{Q(R)} \circ f^* \circ j' 
\]
\[
\cong \Phi_{Q(R)} \circ j \circ f'^\ast
\]
\[
\cong \Phi_{Q(R)} \circ f'^\ast
\]
where we used the isomorphism of Lemma 4.10 in the first line. The second line follows using the projection formula,
\[
j_\ast \circ f'^\ast(F) = j_\ast(j'^\ast R \otimes F)
\]
\[
= R \otimes j'_\ast F
\]
\[
= f^* \circ j'_\ast(F)
\]

This gives us the following isomorphism of functors.
\[
\Phi_{Q(R)} \circ f'^\ast \cong f^* \circ \Phi_{Q(T)_s}.
\]

Clearly, \( \text{Perf}^{G_m}(X^+) \) is generated by the essential image of \( f'^\ast \). Hence we only need to find the image of \( \Phi_Q \) on objects of the form \( f'^\ast j'^\ast A \), where \( A \) is in \( W_X^+ \).
Now, if we consider an object $A$ in $\mathbb{W}_X^+$,
\[
\Phi_{Q(R)}(f^{**} j^* A) \cong f^* \circ \Phi_{Q(T)}(j^* A) \cong f^* A
\]
where we used Lemma 4.11 in the last line.
This shows that the essential image of $\Phi_{Q(R)}$ lands in the window $\mathbb{W}_X^+$. □

The previous statement can now be upgraded to the following theorem.

**Theorem 4.13.** Assume that $R$ is generated as a $T$-algebra by non-positive elements. The functor
\[
\Phi_{Q^+} : \text{Perf}^{G_m}(X^+) \to \mathbb{W}_X^+
\]
is an equivalence of categories with inverse functor $j^*$.

**Proof.** The fullness follows from Lemma 3.14 using Lemma 4.8. The faithfulness was proved in Lemma 3.8. Lemma 4.12 implies that a set of generators of $\mathbb{W}_X^+$ lies in the essential image of $\Phi_{Q^+}$. This shows that $\Phi_{Q^+}$ is an equivalence. The inverse functor is $j^*$ by the proof of Lemma 4.12. □

Similarly, we can define the negative window functor and the negative window for the negative GIT quotient, when the generators $e_i$ of $T \to R$ are non-negative.

**Theorem 4.14.** Assume that $R$ is generated as a $T$-algebra by non-negative elements. The functor
\[
\Phi_{Q^-} : \text{Perf}^{G_m}(X^-) \to \mathbb{W}_X^-
\]
is an equivalence of categories with inverse functor $j^-$, where $j^- : X^- \to X$ is the inclusion of dg-schemes.

**Proof.** This is true by symmetry. □

### 4.3. Wall-crossings

By combining Theorem 4.13 and Theorem 4.14, we can prove results about the wall-crossing functors when all the generators $e_i$ of $T \to R$ have degree 0. Hence, we assume that the generators $e_i$ of the semi-free cdga $R$ over $T$ have internal degree 0 in this section.

We consider the easiest case first. Recall that $T$ is a $\mathbb{Z}$-graded smooth ring over $k$, and that $J^\pm$ are generated by the positively/negatively graded elements of $T$. Let $\mu_\pm$ be the sum of the weights of the conormal bundle of $\text{Spec} T/J^\pm$ in $\text{Spec} T$. Our first result is when $\mu_+ + \mu_- = 0$, often referred to as the Calabi-Yau condition.

**Theorem 4.15.** Let $R$ be a semi-free cdga equipped with a $G_m$-action such that $T = R^0$ is smooth, and the generators $e_i$ of $R$ over $T$ have internal degree 0. Let $\mu_\pm$ be the sum of the weights of the conormal bundle of $T/J^\pm$ in $T$. Let $j^- : X^- \to X$ be the inclusion of dg-schemes. When, $\mu_+ + \mu_- = 0$, the wall crossing functor
\[
\Phi_{wc} := j^- \circ (\otimes \mathcal{R}(\mu_+ - 1)) \circ \Phi_{Q^+} : \text{Perf}^{G_m}(X^+) \to \text{Perf}^{G_m}(X^-)
\]
is an equivalence of categories.

**Proof.** Theorem 4.13 shows that $\Phi_{Q^+}$ gives an equivalence of categories
\[
\text{Perf}^{G_m}(X^+) \cong \mathbb{W}_X^+.
\]
A similar analysis for $\Phi_{Q^-}$ shows that
\[
\text{Perf}^{G_m}(X^-) \cong \mathbb{W}_X^-.
\]
We have the condition $\mu_+ + \mu_- = 0$ which ensures that applying $(\otimes \mathcal{R}(\mu_+ - 1))$ exchanges the positive and negative windows
\[
\mathbb{W}_X^+ \otimes \mathcal{R}(\mu_+ - 1) \cong \mathbb{W}_X^-.
\]
As $j^-$ is the inverse functor to $\Phi_{Q^-}$, we have the result. □
Now, we would like to understand the case when the windows are of different lengths, i.e., when \( \mu_+ + \mu_- \neq 0 \). We only consider the (easier) case when \( X \) is an affine space.

4.3.1. **Affine space.** Recall the setting of Section 4.1 where

\[
T = k[x, y],
\]

with internal \( \mathbb{Z} \)-grading \( \deg x_i = a_i > 0 \) and \( \deg y_i = b_i < 0 \). The assumption that the generators \( e_i \) of \( R = T[e] \) are of internal degree zero still stands.

For convenience, we also introduce the notation

\[
\mathbb{W}_{[a, b]} = \langle R(a), R(a + 1), \ldots, R(b) \rangle \subset \text{Perf}^{G_m}(X), \quad a < b \text{ in } \mathbb{Z}
\]

Let us consider the case when \( \mu_+ + \mu_- > 0 \), i.e., the positive window \( \mathbb{W}^+_{X} \) is longer than the negative window \( \mathbb{W}^-_{X} \). Then, we have the following semi-orthogonal decomposition for the window.

**Lemma 4.16.** Assume \( b - a \geq \mu_+ \). The following is a semi-orthogonal decomposition

\[
\mathbb{W}_{[a, b + 1]} = \langle \text{Perf}(R^{G_m}), \mathbb{W}_{[a, b]} \rangle.
\]

where \( R^{G_m} := R/(x, y) \) is the dg fixed locus. Furthermore, \( \text{Perf}(R^{G_m}) \) is the category generated by \( R/x(b + 1) \).

**Proof.** First, we show that the category \( W_{[a, b + 1]} \) is generated by \( R/x(b + 1) \) and \( W_{[a, b]} \). The only missing generator is \( R(b + 1) \), and we can get this by considering the following Koszul resolution

\[
R \otimes K^*(x_1, \ldots, x_l) (b + 1) \cong R/x(b + 1).
\]

The top term of the resolution is \( R(b + 1) \) and the rest of the terms are in \( W_{[a, b]} \). Hence \( R(b + 1) \) is generated by \( \langle R/x(b + 1), W_{[a, b]} \rangle \).

Now, it suffices to verify the semi-orthogonal decomposition condition on the generators. Hence we need to check that

\[
R\text{Hom}(R(i), R/x(b + 1)) = 0 \quad \text{for} \ a \leq i \leq b.
\]

We have

\[
R\text{Hom}(R(i), R/x(b + 1)) = (R/x)_{(b + 1 - i)} = k[y]_{(b + 1 - i)}
\]

as \( b + 1 - i > 0 \) and hence, we are done.

We now compute the dg-endomorphism ring of \( R/x(i) \) as:

\[
R\text{Hom}(R/x, R/x) = R\text{Hom}(R \otimes K(x_1, \ldots, x_l), R/x)
\]

\[
= \left( \bigwedge^i \bigoplus_{i=1}^{a_i} k(a_i) \otimes R/x \right)_0
\]

\[
= R^{G_m}
\]

The derived category of perfect dg-modules of the endomorphism ring \( \text{Perf}(R^{G_m}) \) is then equivalent to the category generated by \( R/x(b + 1) \) \([\text{Kel06}, \text{Theorem 3.8b}]\) (note that \( R/x(b + 1) \) is idempotent-complete as it is generated by a compact object of the category \( D(X) \) which is idempotent-complete as it admits countable co-products). \( \square \)

Using this, we can get a semi orthogonal decomposition for \( \text{Perf}^{G_m}(X^+) \) when \( \mu > 0 \).

**Theorem 4.17.** The following is a semi-orthogonal decomposition

\[
\text{Perf}^{G_m}(X^+) = \langle \text{Perf}(R^{G_m})_{\mu}, \ldots, \text{Perf}(R^{G_m})_{1}, \Phi^{wc}(\text{Perf}^{G_m}(X^+)) \rangle
\]

where \( \text{Perf}(R^{G_m})_{i} \cong \text{Perf}(R^{G_m}) \) denotes the full subcategory generated by \( j^*_i R/x(i) \).
Proof. First, we use Lemma 4.16 inductively to get the semi-orthogonal decomposition

\[ W_{[-\mu_+, 0]} = (R/x(\mu), \ldots, R/x(2), R/x(1), W_{[\mu_-, 1, 0]}), \]

in \( \text{Perf}^{G_m}(X) \). Then, we pull back along \( j_+ : X^+ \to X \) to get the result.

\[
\square
\]

4.4. Applications to flops. As discussed in the introduction, one of the major applications of our results that we have in mind is to the Bondal-Orlov conjecture on derived equivalences of flops [BO95]. By an observation of Reid [Tha96, Proposition 1.7], a flop between smooth projective varieties can be realized as different GIT quotients of \( G_m \) acting on a scheme \( Y \). Let us consider the local setting and assume that \( Y \) is affine. In general, \( Y \) is singular, and the idea is to resolve it as a semi-free cdga in order to construct the wall-crossing functor. We can prove the derived equivalence under certain conditions, using the results of the previous sections.

To be precise, consider a sub-variety \( Y \) of a smooth affine variety \( X = \text{Spec} T \), equipped with a \( G_m \)-action. Now, consider the Koszul-Tate resolution of \( k[Y] = T/\mathfrak{I} \) to get a semifree cdga \( R \) over \( T \); in particular, the cdga \( R \) can be chosen such that there are only finitely many generators in each homological degree [Sta19, Lemma 23.6.9]. In general, we require an infinite number of dg-algebra generators. Then, we have

\[ R = (T[e_1, e_2, \ldots, d], \text{ such that } H^0(R) \simeq k[Y]. \]

If \( Y \) is a complete intersection in \( X \) defined by functions \( (h_1, \ldots, h_n) \), we can take the semi-free cdga \( R \) to be the Koszul resolution; in this case, the number of dg-algebra generators is finite, and the differential acts as

\[ de_i = h_i, \quad i \in [1, n]. \]

In either case, if we assume that the internal degree of all the \( e_i \) is zero, we can use the results of the previous section, in particular Theorem 4.15, to obtain the following result.

**Corollary 4.18.** Assume that the condition \( \mu_+ + \mu_- = 0 \) is satisfied. Also assume that the dg-algebra generators \( e_i \) in \( R \) have internal degree zero. Then the wall crossing functor

\[ \Phi_{wc} : \text{Perf}(Y^+) \to \text{Perf}(Y^-) \]

is an equivalence of categories.

**Proof.** This is a direct application of Theorem 4.15 to the setting of this section. We use the fact that quasi-isomorphisms of dg-schemes induce equivalences of derived categories, which is an immediate extension of [Ric10, Proposition 1.5.6] to the equivariant setting, i.e.,

\[ D^{G_m}(X^+, \mathcal{R}^+ \simeq D^{G_m}(Y^+, \mathcal{O}) \]

as \( \mathcal{R} \) is a dg-resolution of \( k[Y] \). As mentioned in Remark 2.3, it is easy to see that our ad-hoc definition of the category of perfect objects coincides with the standard one in the case of ordinary schemes, and hence we are done.

\[
\square
\]

Moreover, if we do not have the condition \( \mu_+ + \mu_- = 0 \), the results on wall crossings (Theorem 4.17) can be carried over to our setting in this section. These statements are straightforward to write down, but we leave it to the interested reader.

4.4.1. Mukai flop. In particular, this allows us to give a VGIT proof of the derived equivalence for the Mukai flop. Let us briefly remind the reader about the VGIT construction of the Mukai flop. Here, \( k \) will be an arbitrary Noetherian ring, and we recall Example 3.1. Consider the \( \mathbb{Z} \)-graded ring \( T = k[x_1, \ldots, x_l, y_1, \ldots, y_l] \) with internal degree

\[ \deg x_i = 1 \quad \deg y_i = -1. \]
Consider the cdga

\[ R = T[e], \quad de = \sum_{i=1}^{l} x_i y_i, \]

which is quasi-isomorphic to the ring

\[ S := k[x_1, \ldots, x_l, y_1, \ldots, y_l]/(\sum_{i=1}^{l} x_i y_i). \]

The (local model of the) Mukai flop is the birational transformation between the two GIT quotients

\[ Y \dashrightarrow Y, \]

where \( Y = \text{Spec} S \). As the \( e \) has internal degree zero, we can apply Corollary 4.18 to get the derived equivalence.

**Corollary 4.19.** Consider the VGIT presentation of the Mukai flop as above. The wall crossing functor:

\[ \Phi_{wc} : D^b(Y) \rightarrow D^b(Y) \]

is an equivalence.

**Proof.** This is a direct application of Corollary 4.18. We note that as the GIT quotients \( Y \) are smooth, the category of perfect complexes is equivalent to the bounded derived category of coherent sheaves. \( \square \)

**Remark 4.20.** This result is well known, however, all the existing proofs in the literature are over \( \mathbb{C} \) [Nam03, ADM19, Mor18]. Our results hold over an arbitrary Noetherian ring \( k \).

**References**

[ADM19] Nicolas Addington, Will Donovan, and Ciaran Meachan. Mukai flops and \( \mathbb{P} \)-twists. *J. Reine Angew. Math.*, 748:227–240, 2019.

[BCF+19] Matthew R. Ballard, Nitin K. Chidambaram, David Favero, Patrick K. McFaddin, and Robert R. Van der Veen. Kernels for Grassmann Flops. *arXiv e-prints*, page arXiv:1904.12195, Apr 2019.

[BDF17] Matthew Ballard, David Favero, and Ludmil Katzarkov. A category of kernels for equivariant factorizations and its implications for hodge theory. *Publications mathématiques de l’IHÉS*, 120(1):1–111, Nov 2014.

[BO95] A. Bondal and D. Orlov. Semiorthogonal decomposition for algebraic varieties. In *eprint arXiv:alg-geom/9506012*, June 1995.

[BR12] Roman Bezrukavnikov and Simon Riche. Affine braid group actions on derived categories of Springer resolutions. *Ann. Sci. Éc. Norm. Supér. (4)*, 45(4):535–599 (2013), 2012.

[Bri02] Tom Bridgeland. Flops and derived categories. *Invent. Math.*, 147(3):613–632, 2002.

[CFK01] Ionut Ciocan-Fontanine and Mikhail Kapranov. Derived Quot schemes. *Ann. Sci. École Norm. Sup. (4)*, 34(3):403–440, 2001.

[Dri13] V. Drinfeld. On algebraic spaces with an action of \( G_m \). *ArXiv e-prints*, August 2013.

[HHP09] M. Herbst, K. Hori, and D. Page. *B-type D-branes in toric Calabi-Yau varieties*, volume 757 of *Lecture Notes in Phys.* Springer, Berlin, 2009.

[HR17] Jack Hall and David Rydh. The telescope conjecture for algebraic stacks. *J. Topol.*, 10(3):776–794, 2017.

[Kaw02] Yujiro Kawamata. D-Equivalence and K-Equivalence. *J. Differential Geom.*, 61(1):147–171, 05 2002.

[Kaw06] Yujiro Kawamata. Derived equivalence for stratified Mukai flop on \( G(2,4) \). In *Mirror symmetry. V*, volume 38 of *AMS/IP Stud. Adv. Math.*, pages 285–294. Amer. Math. Soc., Providence, RI, 2006.

[Kel06] Bernhard Keller. On differential graded categories. In *International Congress of Mathematicians. Vol. II*, pages 151–190. Eur. Math. Soc., Zürich, 2006.

[Mor18] Hayato Morimura. Derived equivalence for Mukai flop via mutation of semiorthogonal decomposition. *arXiv e-prints*, page arXiv:1812.06413, Dec 2018.

[MR15] Ivan Mirković and Simon Riche. Iwahori-Matsumoto involution and linear Koszul duality. *Int. Math. Res. Not. IMRN*, (1):150–196, 2015.
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