On linear periods

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Abstract

Let $\pi'$ be a cuspidal automorphic representation of $GL_{2n}(\mathbb{A})$ with trivial central character, which is assumed to be the Jacquet-Langlands transfer from a cuspidal automorphic representation $\pi$ of $GL_{2m}(D)(\mathbb{A})$, where $D$ is a division algebra so that $GL_{2m}(D)$ is an inner form of $GL_{2n}$. In this paper, we consider the relation between linear periods on $\pi$ and $\pi'$. We conjecture that the non-vanishing of the linear period on $\pi$ would imply the non-vanishing of that on $\pi'$. We illustrate an approach of relative trace formula towards this conjecture, and prove a partial result which relies on the existence of the smooth transfer over non-archimedean fields.

1 Introduction

Goal of this article Let $k$ be a number field, $A$ be its adele ring, and $D$ be a central division algebra over $k$ of index $d$, that is, $\dim_k D = d^2$. Let $G = GL_{2m}(D)$, viewed as an algebraic group over $k$, which is an inner form of $G' = GL_{2n}$ with $n = md$. Let $\pi$ be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, and $\pi'$ be the irreducible automorphic representation of $G'(\mathbb{A})$ associated to $\pi$ by Jacquet-Langlands correspondence, which is assumed to be cuspidal. We also assume that the central characters of $\pi$ and $\pi'$ are trivial. For Jacquet-Langlands correspondence on general linear group and its inner forms, we refer to [dkv], [ba] and [br] for more details. The main purpose of this paper is to investigate the relation between certain automorphic periods under the Jacquet-Langlands correspondence.

To be more precise, let $Z$ be the center of $G$, which is identified with the center of $G'$, isomorphic to $\mathbb{G}_m$ over $k$. Let $H = GL_m(D) \times GL_m(D)$ (resp. $H' = GL_n \times GL_n$) which is embedded into $G$ (resp. $G'$) diagonally. The periods considered in this paper is given by

$$\ell(\phi) := \int_{H(\mathbb{A})} \phi(h) \, dh, \quad \phi \in \pi,$$

and

$$\ell'(\varphi) := \int_{H'(\mathbb{A})} \varphi(h) \, dh, \quad \varphi \in \pi'.$$

We call them linear periods. This notion was introduced by [6] in the case $(G', H')$. We say that $\pi$ is $H$-distinguished or has linear period if $\ell|_{\pi} \neq 0$, of course, including the case of $(G', H')$. Such period has a close relation with $L$-value. It was shown in [6] that $\pi'$ is $H'$-distinguished if and only if the $L$-value $L^S(s, \pi') \res_{s=1} L^S(s, \pi, \lambda^2)$ is nonzero, using an integral representation of the $L$-functions. What happens if $\pi$ is $H$-distinguished? The partial $L$-functions
attached to $\pi$ and $\pi'$ should be the same, while there is no integral representation for the ones associated to $\pi$. However, since $H$ is an inner form of $H'$, it is natural to make the following conjecture.

**Conjecture 1.1.** If $\pi$ is $H$-distinguished, then $\pi'$ is $H'$-distinguished.

**Remark 1.2.** The converse of the above conjecture, that is if $\pi'$ is $H'$-distinguished then $\pi$ is $H$-distinguished, as pointed out by Prasad, should also hold. Moreover, the Conjecture 2 in [pt] that is in the local setting, can be viewed as a generalization of our global conjecture in both directions, which was also pointed out by Prasad.

In this paper, we illustrate a relative trace formula approach towards this conjecture, and focus on the geometric side of the trace formula. In Theorem 5.9, we establish the smooth transfer over non-archimedean places, which is the initial step in this approach. Note that there is no need to prove the fundamental lemma, since $\mathcal{G}(\mathbb{A}_v), H(\mathbb{A}_v)) \simeq (\mathcal{G}'(\mathbb{A}_v), H'(\mathbb{A}_v))$ for almost all places. Roughly speaking, let $v$ be a finite place of $k$, then the smooth transfer is a “map” $\lambda_v$ from $C_c^\infty(\mathcal{G}(k_v))$ to $C_c^\infty(\mathcal{G}'(k_v))$ such that the orbital integrals of $f \in C_c^\infty(\mathcal{G}(k_v))$ and $\lambda_v(f)$ for any regular semisimple orbit are same, where the orbits are with respect to the left and right translation of $H \times H$ on $G$, resp. $H' \times H'$ on $G'$.

Due to the lack of the information on the spectral side of the trace formula, we can only obtain the following conditional result, using the simple relative trace formula.

**Theorem 1.3.** Suppose that $\pi$ is an irreducible $H$-distinguished cuspidal automorphic representation of $G(\mathbb{A})$, which is supercuspidal at two finite places $v_1$ and $v_2$ such that $D$ is split at these two places, and also $D$ is split at all archimedean places. Let $\pi'$ be the Jacquet-Langlands transfer of $\pi$ to $G'(\mathbb{A})$. Then $\pi'$ is $H'$-distinguished.

**Remark 1.4.** The condition on archimedean places is due to the lack of smooth transfer at archimedean places in the non trivial case. The assumption of supercuspidal at one finite place can make the spectral side of the trace formula convergent, while the assumption of supercuspidal at another finite place can ensure the “density” of the support of the local spherical character and thus can make the geometric side of the trace formula convergent.

Our proof of the smooth transfer is mainly inspired by Wei Zhang’s work [zhw] on the smooth transfer for Jacquet-Rallis relative trace formula towards global Gan-Gross-Prasad conjecture for unitary groups, and Waldspurger’s work [wa1] [wa2] on endoscopic transfer. The first step is to reduce the smooth transfer from groups to Lie algebras, that is, to linearize the question. The second step is to show that, roughly speaking, Fourier transform commutes with the smooth transfer. We will use a global method to show such a property.

**Some related works.** The Conjecture 1.1 is motivated by Jacquet-Martin’s conjecture [jm] on Shalika periods. We briefly recall it. Now let $d = n$ and $G = GL_d(D)$. There is a Shalika subgroup $S$ inside $G$. More precisely, $G$ has a parabolic subgroup $P = MN$ with Levi factor and unipotent radical given by $M \simeq D^{\times} \times D^{\times}$, $N \simeq D$. Let $\psi$ be a non trivial character of $k^\times/k$, which defines a non degenerate character (still denoted by $\psi$) of $N(k) \backslash N(\mathbb{A})$ given
by $\psi(x) := \psi(\text{tr}_D(x))$ for $x \in N(\mathfrak{A}) \simeq D(\mathfrak{A})$, where $\text{tr}_D$ is the reduced trace map on $D$. Then its stabilizer in $P$ is the Shalika subgroup $S = \text{LN}$, where $L$ is $\Delta D^\times$. We can extend $\psi$ to a character of $S(k) \backslash S(\mathfrak{A})$ by $\psi(l \cdot n) = \psi(n)$ for $l \in L(\mathfrak{A})$ and $n \in N(\mathfrak{A})$. One can define the Shalika subgroup $S'$ of $G'$ similarly, where the corresponding parabolic subgroup is $P' = M'N'$ with Levi factor $M' \simeq \text{GL}_m \times \text{GL}_n$. Then the Shalika period $S$ is a linear form on $\pi$ given by

$$S(\phi) = \int_{S(k) \backslash S(\mathfrak{A})} \phi(u)\psi^{-1}(u) \, du,$$

and the Shalika period $S'$ on $\pi'$ is defined similarly. In [jm], they conjectured that if $\pi$ is $S$-distinguished then $\pi'$ is also $S'$-distinguished, both with respect to the Shalika periods. Under some hypothesis, using relative trace formulae, Jacquet and Martin showed that this is true in the case of $n = 2$. However, they did not prove the smooth transfer for the full space $\mathcal{C}_c^\infty(G(k_v))$ of Bratelli-Schwartz functions. Of course, if one aims to completely prove this conjecture by the method of relative trace formula, one has to show the smooth transfer for the full space $\mathcal{C}_c^\infty(G(k_v))$. In the case $n = 2$, this conjecture (including the converse direction) was completely solved by Gan and Takeda [gt] by using theta correspondence, while this method can’t be generalized to the higher rank case. Separately, Jiang, Nien and Qin [jnq] proved this conjecture, under some conditions, for general $n$ using the method of automorphic descent.

There is a relation between linear period and Shalika period on $\pi'$. In fact, $H'$-distinction implies $S'$-distinction, since $\pi'$ is $S'$-distinguished if and only if the exterior $L$-function $L(s, \wedge^2, \pi')$ has a simple pole at $s = 1$. Locally, it was shown by [j] and [ja] that if $\pi_v'$ is $S'(k_v)$-distinguished then it is $H'(k_v)$-distinguished, and it is conjectured that if $\pi_v'$ is generic then $S'(k_v)$-distinction is equivalent to $H'(k_v)$-distinction. Recently, Gan [ga] proved this local conjecture by using local theta correspondence for dual pairs of type II. Therefore one can ask whether there are such relations between linear and Shalika periods on $\pi$, both globally and locally.

As we have said before, our approach to prove the smooth transfer is inspired by [zhw] and [wa2], while there are still some significant differences between our method and theirs and it is fair to say that ours is a combination of theirs. The reduction steps here from showing smooth transfer on groups to showing the property of Fourier transform commuting with smooth transfer (Theorem 5.11) are almost as same as [zhw], while the remains are different since there are no partial Fourier transform here and we could not apply the induction arguments in [zhw, §4] any more. We will follow the idea of [wa2] using global method to show Theorem 5.11. To make such a method valid, we have to study the harmonic analysis on the corresponding $p$-adic symmetric spaces further, and prove several results which are analogues of those appearing in [wa1] and [wa2] as well as some more classical results in [hc1] and [hc2]. In this paper, we just state these results and explain them brieﬂy, since they are direct generalizations of those that have been proved in the case of $(G', H')$ in [zhe]. In [zhc], we considered the relation between the similar periods (twisting a character) for the symmetric pairs $(G', H')$ as before and $(G, H)$ where $G = \text{GL}_n(D), H = \text{GL}_n(k')$ with $D$ being a quaternion algebra over $k$ and $k'$ being a quadratic field extension of $k$ included in $D$. However, in [zhc], we can only prove “half” of the property that Fourier transform commutes with smooth transfer, due to
the fact that the regular semisimple orbits of \((G, H)\) are “fewer” than those of \((G', H')\). We encounter the similar problem in this paper, while, fortunately, we can use a technique to overcome this.

To prove some partial results towards Conjecture 1.1, sometimes it suffices to show the existence of the smooth transfer only for certain special Bruhat-Schwartz functions, as Jacquet-Martin did in [jm] and recent work of Feigon-Martin-Whitehouse [fmw], by showing the density of the support of the local spherical characters.

Structure of this article In §3, we introduce the relative trace formulae considered in this paper, which are natural for the conjecture concerned. Using a simple form of it and the results that will be proved in the later sections, we can obtain Theorem 1.3, which is a partial result towards Conjecture 1.1.

To factor the global linear periods into local ones, we need to study the property of multiplicity one for the symmetric pair \((G(k_v), H(k_v))\) at each place \(v\) of \(k\), in other words, to study the space \(\text{Hom}_{H(k_v)}(\pi_v, \mathbb{C})\) for any irreducible admissible representation \(\pi_v\) of \(G(k_v)\). If \(\dim \text{Hom}_{H(k_v)}(\pi_v, \mathbb{C}) \leq 1\) for each irreducible admissible representation \(\pi_v\), we call \((G(k_v), H(k_v))\) a Gelfand-pair. We could not show \((G(k_v), H(k_v))\) is a Gelfand-pair, but we can show a weaker form of it, which is enough for our purpose on factoring the global period. In §4, we follow the a systematically treatment of [ag1] for such a kind of question, i.e. using generalized Harish-Chandra descent to study \(H(k_v) \times H(k_v)\)-invariant distributions on \(G(k_v)\), which is also important for the later study on smooth transfer.

In §5, we introduce the notion of the smooth transfer explicitly, both for groups and Lie algebras. By several reductions steps, we show the existence of smooth transfer (Theorem 5.9) relies on Theorem 5.11 which will be proved in §6.

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2 Notations and conventions

Actions of algebraic groups Let \(k\) be a number field or a \(p\)-adic field. Let \(\pi\) be an action of a reductive group \(M\) on a smooth affine variety \(X\), all defined over \(k\). Denote \(M = M(k)\) and \(X = X(k)\). Recall that for \(x \in X\), we say \(x\) is \(M\)-semisimple if \(Mx\) is Zariski closed in \(X\) (or equivalently, if \(k\) is \(p\)-adic, \(Mx\) is closed in \(X\) for the analytic topology); say \(x\) is \(M\)-regular if the stabilizer \(M_x\) of \(x\) has minimal dimension. We denote by \(X_{\text{rs}}(k)\) or \(X_{\text{rs}}\) for the set of \(M\)-regular semisimple elements in \(X\). If \(k\) is \(p\)-adic, we call an algebraic automorphism \(\tau\) of \(X\) is \(M\)-admissible if (i) \(\tau\) normalizes \(\pi(M)\) and \(\tau^2 \in \pi(M)\), (ii) for any closed \(M\)-orbit \(O \subset X\), \(\tau(O) = O\).
Analysis on ℓ-space

Now let $k$ be a $p$-adic field. For a totally disconnected topological space $X$, we denote by $C^\infty_c(X)$ the space of locally constant and compactly supported $\mathbb{C}$-valued functions on $X$, and $\mathcal{D}(X)$ the space of distributions on $X$ which is the dual of $C^\infty_c(X)$. If there is an action of an $\ell$-group $M$ on $X$, we denote by $\mathcal{D}(X)^M$ the subspace of $\mathcal{D}(X)$ consisting of $M$-invariant distributions.

Now suppose $X$ is a finite dimensional vector space over $k$ with the natural topology induced from that of $k$, $\psi$ is a non-trivial continuous additive character of $k$, and $q$ is a non-degenerate quadratic form on $X$. We equip $X$ the self-dual Haar measure with respect to the bi-character $\psi(q(\cdot,\cdot))$. Define the Fourier transform $f \mapsto \hat{f}$ of $C^\infty_c(X)$ by

$$\hat{f}(x) = \int_X f(y)\psi(q(x,y)) \, dy.$$

Then $\hat{f}(x) = f(-x)$. We write $\gamma_\psi(q)$ for the Weil index associated to $q$ and $\psi$.

3 Relative trace formulae

Let $(G,H)$ and $(G',H')$ be the ones defined in §1. For $f \in C^\infty_c(G(\mathbb{A}))$, define the kernel function

$$K_f(x,y) = \int_{Z(k)\backslash G(k)} \sum_{\gamma \in G(k)} f(zx^{-1}\gamma y) \, dz.$$

We consider the distribution on $G(\mathbb{A})$

$$I(f) = \int_{H(k)Z(\mathbb{A})\backslash H(\mathbb{A})} \int_{H(k)Z(\mathbb{A})\backslash H(\mathbb{A})} K_f(h_1,h_2) \, dh_1 \, dh_2$$

if it is absolutely convergent. We call a function $f \in C^\infty_c(G(\mathbb{A}))$ nice if it is decomposable as $f = \bigotimes \psi f_\psi$ and satisfies:

- at some finite place $v_1$, $f_{v_1}$ is supported on the locus of $H(k_{v_1}) \times H(k_{v_1})$-elliptic elements;
- at some finite place $v_2$, $f_{v_2}$ is an essential matrix coefficient of a supercuspidal representation, which means that $\tilde{f}_{v_2}(g) = \int_{Z(k_{v_2})} f_{v_2}(gz) \, dz$ is a matrix coefficient of a supercuspidal representation;
- $f_\infty$ is $K_\infty$-finite where $K_\infty$ is the maximal compact subgroup of $G(k_\infty)$.

Here we recall that an element in $G(k_v)$ is called $H(k_v) \times H(k_v)$-elliptic if it is $H(k_v) \times H(k_v)$-regular semisimple and its centralizer in $H(k_v) \times H(k_v)$ is an anisotropic torus. Later we will use $G(k_v)_{\text{reg}}$ to denote the locus of $H(k_v) \times H(k_v)$-elliptic elements. It is true that, for $f \in C^\infty_c(G(\mathbb{A}))$ being nice, $I(f)$ is absolutely convergent, and in such a case, we have two ways to decompose $I(f)$: geometric expansion and spectral expansion.

For the geometric side, let’s fix a Haar measure on $H(\mathbb{A})$. If $\gamma \in G(k)$ is $H(k) \times H(k)$-regular semisimple, fix a Haar measure on $H, (\mathbb{A})$, where

$$H_\gamma = \{(h_1,h_2) \in H \times H; h_1 \gamma h_2^{-1} = \gamma\}$$
is the stabilizer of $\gamma$ under the action of $H \times H$. For $f \in C_c^\infty(G(\mathbb{A}))$, denote the orbital integral

$$I_\gamma(f) = \int_{H_1(\mathbb{A}) \backslash (H(\mathbb{A}) \times H(\mathbb{A}))} f(h_1^{-1}\gamma h_2) \, dh_1 \, dh_2,$$

which is absolutely convergent.

For the spectral side, if $\sigma$ is an irreducible cuspidal automorphic representation of $G(\mathbb{A})$ with trivial central character, denote

$$I_\sigma(f) = \int_{H(\mathbb{A})} \int_{K(\mathbb{A}) / H(\mathbb{A})} K_{\sigma,f}(h_1,h_2) \, dh_1 \, dh_2,$$

which is also absolutely convergent, where

$$K_{\sigma,f}(x,y) = \sum_{\varphi} \sigma(f(x)) \overline{\varphi(y)}$$

and $\varphi$ runs over an orthonormal basis for the space of $\sigma$. It is obvious that $\sigma$ is $H$-distinguished if and only if $I_\sigma$ is non zero as a distribution on $G(\mathbb{A})$. The following result establishes a simple trace identity relating $I_\gamma$ and $I_\sigma$, whose proof is standard (cf. [zhw, Theorem 2.3]).

**Proposition 3.1.** If $f \in C_c^\infty(G(\mathbb{A}))$ is nice, then $I(f)$ is equal to

$$\sum_{\gamma \in [G(k)_{\text{cusp}}]} \tau(\gamma) I_\gamma(f) = \sum_{\sigma \text{ cuspidal}} I_\sigma(f). \tag{1}$$

Here $[G(k)_{\text{cusp}}]$ denotes the $H(k) \times H(k)$-elliptic orbits of $G(k)$, $\tau(\gamma)$ denotes the volume $\text{vol}(H_1(k)\mathbb{Z}(\mathbb{A}) \backslash H_1(\mathbb{A}))$, and $\sigma$ runs over cuspidal representations of $G(\mathbb{A})$ with trivial central character.

All the discussions above contain the case of $(G', H')$. We will use the same notations for $(G', H')$ if there is no confusion. If $\gamma \in G(k)$ is $H(k) \times H(k)$-regular semisimple, there is $\delta \in G'(k)$ being $H'(k) \times H'(k)$-regular semisimple such that $\delta$ matches $\gamma$ (see Proposition 5.3 for more details). Then it is true that $H_1$ is isomorphic to $H'_1$. For any regular semisimple $\delta \in G'(k)$ such that there is regular semisimple $\gamma \in G(k)$ matching $\delta$, we fix the Haar measure on $H'_1(\mathbb{A})$ so that it is compatible with that on $H_1(\mathbb{A})$.

Now we can prove the Theorem 1.3 with the aid of Proposition 4.3 and Theorem 5.9 in later sections. The Theorem 5.9 is about the smooth transfer in the $p$-adic case, whose global version is the following lemma, which is obvious.

**Lemma 3.2.** For each $f$ in $C_c^\infty(G(\mathbb{A}))$ there exists $f'$ in $C_c^\infty(G'(\mathbb{A}))$ such that for each $\delta \in G'(k)_{\text{rs}}$

$$I_\delta(f') = \begin{cases} I_\gamma(f), & \text{if there is } \gamma \in G(k)_{\text{rs}} \text{ such that } \gamma \leftrightarrow \delta, \\ 0, & \text{otherwise.} \end{cases}$$

**Proof of Theorem 1.3.** Since $\pi$ is $H$-distinguished, $I_\pi \neq 0$ as a distribution on $G(\mathbb{A})$. First we will choose a nice function $f = \bigotimes_v f_v \in C_c^\infty(G(\mathbb{A}))$ such that $I_\pi(f) \neq 0$. 

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At all places \(v\), \(\pi_v\) is \(H(k_v)\)-distinguished. Since \(\pi_v\) is a local component of a cuspidal representation, it is unitary. Therefore, by Corollary 4.4, at each place \(v\) of \(k\), we have \(\dim \text{Hom}_{H(k_v)}(\pi_v, \mathbb{C}) = 1\). Thus we can factor the global linear period \(\ell\) as the local ones \(\ell = \prod_v \ell_v\), where \(\ell_v \in \text{Hom}_{H(k_v)}(\pi_v, \mathbb{C})\) at each place \(v\). We will specify the choice of \(\ell_v\). For each \(\ell_v\), there is a spherical character \(I_{\pi_v}\) associated to it, which is a distribution on \(G(k_v)\), defined by

\[
I_{\pi_v}(f_v) = \sum_{\varphi} \ell_v(\pi_v(f_v)\varphi)\overline{\ell_v(\varphi)}, \quad \text{for } f_v \in C_c^{\infty}(G(k_v)),
\]

where \(\varphi\) runs over an orthonormal basis for the space of \(\pi_v\). Fix a finite set \(S\) of places of \(k\), including all archimedean places, such that for \(v \notin S\), \(D\) is split at \(v\), \(\pi_v\) is unramified, and \(\ell_v(1_{K_v}) \neq 0\), where \(1_{K_v}\) is the characteristic function of \(K_v = \text{GL}_{2n}(O_{k_v})\). We choose \(\ell_v\) at \(v \notin S\) such that \(\ell_v(1_{K_v}) = 1\), and choose \(\ell_v\) at \(v \in S\) arbitrarily such that \(\ell = \prod \ell_v\).

To choose \(f_v \in C_c^{\infty}(G(k_v))\), we first require that, at each place \(v\), \(\ell_v(f_v)\) is non zero. The other additional conditions are as follows. At \(v_1\), by [imw, Proposition 4.2.(2)], we can choose \(f_{v_1}\) to be supported on \(G(k_{v_1})\). At \(v_2\), we can choose \(f_{v_2}\) to be an essential matrix coefficient of a supercuspidal representation. At each archimedean place \(v\), choose \(f_v\) to be \(K_v\)-finite. At \(v \notin S\), let \(f_v = 1_{K_v}\). At all other places, we choose arbitrary \(f_v\), but of course, require \(\ell_v(f_v) \neq 0\). Thus we obtain a nice function \(f = \bigotimes_v f_v\).

At each place \(v\) where \(D\) is split, we fix an isomorphism \(\iota_v : G(k_v) \rightarrow \widetilde{G}(k_v)\) whose restriction on \(H(k_v)\) is onto \(H'(k_v)\), and an isomorphism \(G_c^{\infty}(G(k_v)) \rightarrow C_c^{\infty}(G'(k_v))\) induced by \(\iota_v\) which is still denoted by \(\iota_v\). At these places \(v\) which split \(D\), put \(f'_v \in C_c^{\infty}(G'(k_v))\) to be \(\iota_v(f_v)\). At all other places \(v\), by Theorem 5.9, there is a smooth transfer \(f'_v \in C_c^{\infty}(G'(k_v))\) of \(f_v\). Set \(f' = \bigotimes f'_v\), then \(f'\) is also nice. Such a choice ensures that

\[
\sum_{\gamma \in [G(k_v):\text{un}]} \tau(\gamma)I_{\pi}(f) = \sum_{\delta \in [G'(k_v):\text{un}]} \tau(\delta)I_{\pi'}(f').
\]

Hence, by Proposition 3.1, we have

\[
\sum_{\sigma \text{ cuspidal}} I_{\sigma}(f) = \sum_{\sigma' \text{ cuspidal}} I_{\sigma'}(f').
\]

By the strong multiplicity one (cf. [ba] and [br]) and the principle of infinite linear independence of characters (cf. [al]), we have \(I_{\pi}(f) = I_{\pi'}(f')\), which implies that \(I_{\pi'}(f')\) is non zero, that is, \(\pi'\) is \(H'\)-distinguished.

4 Multiplicity one

The global linear period \(\ell\) belongs to the space \(\text{Hom}_{H(k)}(\pi, \mathbb{C})\). To factor it into local ones, we need to study the space \(\text{Hom}_{H(k_v)}(\pi_v, \mathbb{C})\) for each place \(v\) of \(k\). We hope the so-called multiplicity one holds at each place \(v\), that is, if \(\pi_v\) is an irreducible admissible representation of \(G(k_v)\), then \(\dim \text{Hom}_{H(k_v)}(\pi_v, \mathbb{C}) \leq 1\). If \((G(k_v), H(k_v))\) satisfies the property of multiplicity one, we call them a Gelfand pair. It was proved by [ir] in the non-archimedean case and by [agl] in
the archimedean case that \((G', k_e), H'(k_v))\) is a Gelfand pair. When \(m = 1, v\) is non-archimedean and \(D\) is a general division algebra, \((G(k_v), H(k_v))\) is also a Gelfand pair, which was proved by Prasad [pr]. We could not show that, for general \(m\) and \(D\), \((G(k_v), H(k_v))\) is a Gelfand pair, while we can prove a weaker result, that is, Proposition 4.3 and Corollary 4.4, which is enough for our purpose.

In this section, we focus on the study of non-archimedean case, while the archimedean case is similar and the details are left to the reader. From now on, and until the end of the paper, we fix a \(p\)-adic field \(F\). We will follow the same line as that of [ag1] where an effective way to prove results like multiplicity one for symmetric pairs is systematically studied, and we refer the readers to [ag1] for more details.

**Symmetric pairs** Let \(G\) and \(H\) be the ones introduced in §1, both defined over \(F\) now. Write \(G = G(F)\) and \(H = H(F)\). Let \(\epsilon = \begin{pmatrix} 1_m & 0 \\ 0 & -1_m \end{pmatrix}\) and define an involution \(\theta\) on \(G\) by \(\theta(g) = \epsilon g \epsilon\). Then \(H = G^\theta\), that is, \((G, H, \theta)\) is a symmetric pair. When there is no confusion, we write \((G, H)\) instead of \((G, H, \theta)\) for simplicity. Let \(\iota\) be the anti-involution on \(G\) defined by \(\iota(g) = \theta(g^{-1})\). Denote \(G' = \{g \in G; \iota(g) = g\}\) and define a symmetrization map

\[
s : G \to G', \quad s(g) = g\iota(g).
\]

Via this map we can view the \(p\)-adic symmetric space \(S = G/H\) as an open and closed subset of \(G'(F)\). Since \(H^1(F, H)\) is trivial, we also have \(S = (G/H)(F)\). We will always identify \(S\) with its image in \(G'(F)\).

Let \(\theta\) act by its differential on \(g = \text{Lie}(G)\). Denote \(\mathfrak{h} = \text{Lie}(H)\), then \(\mathfrak{h} = \{X \in g; \theta(X) = X\}\). Denote \(\mathfrak{s} = \{X \in g; \theta(X) = -X\}\), which can be viewed as the “Lie algebra” of \(G/H\), and on which \(H\) acts by adjoint action. It’s easy to see that \(\mathfrak{s} \simeq \mathfrak{gl}_m(D) \oplus \mathfrak{gl}_m(D)\), and the action of \(H\) on \(\mathfrak{s}\) is \((h_1, h_2) \cdot (X_1, X_2) = (h_1 X_1 h_2^{-1}, h_2 X_2 h_1^{-1})\) for \((h_1, h_2) \in H\) and \((X_1, X_2) \in \mathfrak{s}\).

We fix the non degenerate symmetric bilinear form \(\langle , \rangle\) on \(g(F)\) defined by

\[
\langle X, Y \rangle = \text{tr}(XY), \quad \text{for } X, Y \in g(F),
\]

where \(\text{tr}\) is the reduced trace map on \(\mathfrak{gl}_m(D)\). Notice that the form \(\langle , \rangle\) is both \(G\)-invariant and \(\theta\)-invariant. When we want to emphasize on the index \(m\), we write \(G_m, H_m, \theta_m, \mathfrak{s}_m\). Notice that the case of \(m = n\) and \(D = F\) is just the case denoted by \((G', H')\) in §1. Therefore, we can treat them systematically.

We will consider the action of \(H \times H\) on \(G\) by left and right translation, and the adjoint action of \(H\) on \(S\) or \(\mathfrak{s}(F)\).

Now we recall some notions of certain properties for a general symmetric pair \((G, H, \theta)\), we refer the readers to [ag1] and [ag2] for more details. Define \(Q(\mathfrak{s}) = \mathfrak{s}/\mathfrak{s}^H\). There is a canonical embedding \(Q(\mathfrak{s}) \to \mathfrak{s}\). Denote \(\phi : \mathfrak{s} \to \mathfrak{s}/H\) the standard projection, \(\Gamma(\mathfrak{s}) = \phi^{-1}(\phi(0) \cap Q(\mathfrak{s}))\), and \(R(\mathfrak{s}) = Q(\mathfrak{s}) - \Gamma(\mathfrak{s})\). We call an element \(g \in G\) admissible if (i) \(\text{Ad}(g)\) commute with \(\theta\) and (ii) \(\text{Ad}(g)|_{\mathfrak{s}}\) is \(H\)-admissible. Notice that, in our case, \(Q(\mathfrak{s}) = \mathfrak{s}\), \(\Gamma(\mathfrak{s}) = \mathfrak{N}\) is the nilpotent cone of \(\mathfrak{s}\), and \(R(\mathfrak{s}) = \mathfrak{s} - \mathfrak{N}\).

A symmetric pair \((G, H, \theta)\) is called good, if for any closed \(H \times H\) orbit \(O\) in \(G\), \(\iota(O) = O\).
A symmetric pair \((G, H, \theta)\) is called regular if for any admissible \(g \in G\) such that \(\mathcal{D}(R(s))(F)^H \subset \mathcal{D}(R(s))(F)^{\text{Ad}(g)}\) we have
\[\mathcal{D}(Q(s))(F)^H \subset \mathcal{D}(Q(s))(F)^{\text{Ad}(g)}\].

For each nilpotent element \(X \in \mathfrak{s}(F)\), there exists an \(\mathfrak{s}_2\)-triple \((X, \mathfrak{d}(X), Y)\) with \(Y \in \mathfrak{s}(F)\) being nilpotent and \(\mathfrak{d}(X) \in \mathfrak{h}(F)\) being semisimple. We say a symmetric pair \((G, H, \theta)\) is of negative defect if for any nilpotent \(X \in \mathfrak{s}(F)\) we have
\[\text{Tr}(\text{ad}(\mathfrak{d}(X))|_{\mathfrak{h}_X}) < \dim Q(s),\]
where \(\mathfrak{h}_X\) is the centralizer of \(X\) in \(\mathfrak{h}\). By [ag1, Proposition 7.3.7], one can see that if a symmetric pair is of negative defect it is regular.

Recall that for \(g \in G\) being \(H \times H\)-semisimple and \(x = s(g)\), the triple \((G_x, \mathbf{H}_x, \theta|_{G_x})\) is still a symmetric pair, which is called a descendant of \((G, H, \theta)\). Notice that the Lie algebra of \(G_x/\mathbf{H}_x\) can be identified with \(\mathfrak{s}_x\), where \(\mathfrak{s}_x\) is the centralizer of \(x\) in \(\mathfrak{s}\). It was shown in [ag1, Theorem 7.4.5] that if \((G, H, \theta)\) is a good symmetric pair such that all its descendants are regular then it is a GK-pair. The GK-pair means that:
\[\mathcal{D}(G)^{H \times H} \subset \mathcal{D}(G)^+\].

**Descendants**  
Now we return to the specific symmetric pair concerned in the paper. To study the property of multiplicity one, as we have explained, it is important to classify all descendants of \((G, H, \theta)\). We follow the computation of [jr] to get the following list of all the descendants, and we omit the details of the proof.

**Proposition 4.1.** All descendants of \((G, H, \theta)\) are products of the following types

1. \((R_{l/F}(\text{GL}_r(D')) \times \text{GL}_r(D')), \Delta(R_{l/F}(\text{GL}_r(D')), \delta)\) for some field extension \(L/F\) and some central division algebra \(D'\) over \(L\),

2. \((R_{l/F}(\text{GL}_r(D') \otimes L L')_1, R_{l/F}(\text{GL}_r(D'), \gamma)\) for some field extension \(L/F\), its quadratic extension \(L'/L\) and some central division algebra \(D'\) over \(L\),

3. \((\text{GL}_{2r}(D), \text{GL}_{r}(D) \times \text{GL}_r(D), \theta)\).

**Remark 4.2.** Here we use \(\Delta\) to denote the diagonal embedding and use \(R_{l/F}\) to denote the Weil restriction with respect to the field extension \(L/F\). The involution \(\delta\) in (1) of the above proposition is \((x, y) \mapsto (y, x), \gamma\) in (2) is \(x \mapsto \bar{x}\) which is induced from \(\text{Gal}(L'/L)\), and \(\theta\) in (3) is the one introduced before.

**Multiplicity one**  
From the above classification of the descendants, we can see that, for any \(H \times H\)-semisimple \(g \in G\), \(H^1(F, H_{s(g)})\) is trivial. By [ag1, Corollary 7.1.5], this implies that the symmetric pair \((G, H)\) is good.

Such a classification also implies that all descendants of \((G, H)\) are regular. The reason is that, after base change to some extension field \(F'\), they are of negative defect (proved in [jr]) over \(F'\), thus they are of negative defect over \(F\) by [ag2, Lemma 4.2.8]. Therefore \((G, H)\) is a GK-pair. In particular, by [ag1, Corollary 8.1.6], it satisfies the following property, which is a weaker form of Gelfand pairs.

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Proposition 4.3. For any irreducible admissible representation \( \pi \) of \( G \) we have

\[
\dim \Hom_H(\pi, \mathbb{C}) \cdot \dim \Hom_H(\tilde{\pi}, \mathbb{C}) \leq 1,
\]

where \( \tilde{\pi} \) is the contragredient of \( \pi \).

Corollary 4.4. If \( \pi \) is an irreducible unitary admissible representation of \( G \), we have

\[
\dim \Hom_H(\pi, \mathbb{C}) \leq 1.
\]

Proof. The following "well-known" argument was pointed out by Prasad to the author. If \( \pi \) is unitary, then \( \bar{\pi} = \tilde{\pi} \), where \( \bar{\pi} \) denotes the complex conjugate of \( \pi \). Observe that if \( \pi \) has an \( H \)-invariant form \( \ell \), taking the complex conjugate of the form, one can obtain an \( H \)-invariant form \( \bar{\ell} \) on \( \bar{\pi} \simeq \tilde{\pi} \).

Remark 4.5. In general, we expect that \((G, H)\) is a Gelfand pair. When \( D = F \), this is proved by [jr]; for \( m = 1 \) and general \( D \), it was proved by Prasad [pr]. For general \( m \) and \( D \), we do not know how to prove it, because we do not know how to realize the contragredient representation of an irreducible admissible one. However, if \( D \) is a quaternion algebra, there is an \( \Ad(G) \)-admissible anti-automorphism \( \tau \) of \( G \) such that \( \tau(H) = H \) (cf. [ra, Theorem 3.1]), which implies the following result.

Corollary 4.6. If \( D \) is a quaternion algebra, for any irreducible admissible representation \( \pi \) of \( G \) we have

\[
\dim \Hom_H(\pi, \mathbb{C}) \leq 1.
\]

5 Smooth transfer

We keep the notations as before, and still let \( F \) be a \( p \)-adic field. In this section, we will show the existence of the smooth transfer with respect to the relative trace formulae concerned in this paper. Our strategy here follows the somewhat standard procedure, that was used to study smooth transfers in other cases (cf. [zhw] or [zhc] for more details), to reduce the question to prove the property that "Fourier transform commutes with smooth transfer", which arises from the work of Waldspurger on endoscopic transfer (cf. [wa2]). This property is exactly what Theorem 5.11 states, which will be proved in section 6.

Motivated by the global orbital integrals, the local ones that we consider are

\[
O(g, f) = \int_{H_1 \backslash (H \times H)} f(h^{-1}1g) \, dh_1 \, dh_2,
\]

where \( g \in G \) is \( H \times H \)-regular semisimple, and \( f \in C_c^\infty(G) \). However, the quotient map \( q : G \to G/H = S \) gives rise a surjection \( \tilde{q} : C_c^\infty(G) \to C_c^\infty(S) \). Denote \( x = s(g) \) which is \( H \)-regular semisimple with respect to the adjoint action of \( H \) on \( S \), and denote \( \tilde{f} = \tilde{q}(f) \). Then the original orbital integral becomes

\[
O(g, f) = O(x, \tilde{f}) = \int_{H_1 \backslash H} \tilde{f}(h^{-1}xh) \, dh.
\]

Therefore it suffices to consider the orbital integrals for \( C_c^\infty(S) \) with respect to the \( H \)-action. Eventually, we also have to consider the orbital integrals for \( C_c^\infty(s(F)) \) with respect to the \( H \)-action.
Orbits  First, we classify all $H$-semisimple orbits of $S$ and $\mathfrak{s}(F)$. We remark that the results here on the orbits still hold when we replace $F$ by a number field $k$. For each semisimple $x$ in $S$ or $\mathfrak{s}(F)$, it is also important to determine the couple $(H_x, \mathfrak{s}_x)$ which are also called the descendant of $x$. The reason to consider semisimple element $x$ and its descendant is that we can reduce the orbital integrals associated to regular semisimple orbits around $x$ to orbital integrals for $C_x^\times(\mathfrak{s}_x(F))$ with respect to $H_x$-action. Therefore, if $x$ is not the identity, we can argue by induction, according to the form of $(H_x, \mathfrak{s}_x)$.

Proposition 5.1.  1. Each semisimple element $x$ of $S$ is $H$-conjugate to an element of the form

$$x(A,m_1,m_2) := \begin{pmatrix} A & 0 & 0 & A - 1_r & 0 & 0 \\ 0 & 1_{m_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & -1_{m_2} & 0 & 0 & 0 \\ A + 1_r & 0 & 0 & A & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{m_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & -1_{m_2} \end{pmatrix},$$

with $m = r + m_1 + m_2$, $A \in \mathfrak{gl}_r(D)$ being semisimple without eigenvalues $\pm 1$ and unique up to conjugation. Moreover, $x(A,m_1,m_2)$ is regular if and only if $m_1 = m_2 = 0$ and $A$ is regular in $\mathfrak{gl}_m(D)$ without eigenvalue $\pm 1$.

2. Let $x = x(A,m_1,m_2)$ in $S$ be semisimple. Then its descendant $(H_x, \mathfrak{s}_x)$ is isomorphic to the product

$$(\mathfrak{gl}_r(D)_A, \mathfrak{gl}_r(D)_A) \times (H_{m_1}, \mathfrak{s}_{m_1}) \times (H_{m_2}, \mathfrak{s}_{m_2}).$$

Here $\mathfrak{gl}_r(D)_A$ and $\mathfrak{gl}_r(D)_A$ are the centralizer of $A$ in $\mathfrak{gl}_r(D)$ and $\mathfrak{gl}_r(D)$ respectively, and $\mathfrak{gl}_r(D)_A$ acts on $\mathfrak{gl}_r(D)_A$ by adjoint.

Proposition 5.2.  1. Each semisimple element $X$ of $\mathfrak{s}(F)$ is $H$-conjugate to an element of the form

$$X(A) = \begin{pmatrix} 0 & 0 & 1_r & 0 \\ 0 & 0 & 0 & 0 \\ A & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with $A \in \mathfrak{gl}_r(D)$ being semisimple and unique up to conjugation. Moreover, $X(A)$ is regular if and only if $r = m$ and $A \in \mathfrak{gl}_m(D)$ is regular.

2. Let $X = X(A)$ in $\mathfrak{s}(F)$ be semisimple. Then its descendant $(H_X, \mathfrak{s}_X)$ is isomorphic to the product

$$(\mathfrak{gl}_r(D)_A, \mathfrak{gl}_r(D)_A) \times (H_{m-r}, \mathfrak{s}_{m-r}).$$

Matching between the orbits  Now we give a description for the matching between $H$-semisimple orbits in $S$ or $\mathfrak{s}(F)$ and $H'$-semisimple orbits in $S'$ or $\mathfrak{s}'(F)$. Let $L$ be a field extension of $F$ with degree $d$ contained in $D$, and fix an embedding $D \hookrightarrow \mathfrak{gl}_d(L)$. This induces embeddings $(G,H) \hookrightarrow (G'(L), \mathbf{H}'(L))$, $S \hookrightarrow G'(L)/\mathbf{H}'(L)$ and $\mathfrak{s}(F) \hookrightarrow \mathfrak{s}'(L)$. We identify $G'(L)/\mathbf{H}'(L)$ with its image in $G'(L)$. The following proposition is obvious.
Proposition 5.3. For each semisimple element $x$ of $S$, resp. $\mathfrak{s}(F)$, there exists $h \in H'(L)$ such that $hxh^{-1}$ belongs to $S'$, resp. $\mathfrak{s}'(F)$. This establishes an injection of the $H$-semisimple orbits in $S$, resp. $\mathfrak{s}(F)$, into the $H'$-semisimple orbits in $S'$, resp. $\mathfrak{s}'(F)$, which carries the orbit of $x(A, m_1, m_2)$ in $S$ to the orbit of $x(B, m_1d, m_2d)$ such that $A \in \mathfrak{gl}_{m_1-m_2}(D)$ and $B \in \mathfrak{gl}_{(m_1-m_2)d}(F)$ have the same characteristic polynomial, and carries the orbit of $X(A)$ in $\mathfrak{s}(F)$ to the orbit of $X(B)$ such that $A \in \text{GL}_r(D)$ and $B \in \text{GL}_{rd}(F)$ have the same characteristic polynomial.

Definition 5.4. We say $x \in S$, resp. $X \in \mathfrak{s}(F)$, matches $y \in S'$, resp. $Y \in \mathfrak{s}'(F)$, and write $x \leftrightarrow y$, resp. $X \leftrightarrow Y$, if the above map sends the orbit of $x$, resp. $X$, to the orbit of $y$, resp. $Y$.

Smooth transfer Now we can introduce the notion of the smooth transfer and state the main theorem of this section. We first fix Haar measures on $H$ and $H'$, and fix a Haar measure on $H'_y$ for each $y$ in $S'_rs$ or $\mathfrak{s}'_rs(F)$. For any $x$ in $S_\text{rs}$ or $\mathfrak{s}_\text{rs}(F)$, choose any $y$ in $S'_rs$ or $\mathfrak{s}'_rs(F)$ respectively so that $x \leftrightarrow y$. Then $H_x \simeq H'_y$. We choose the Haar measure on $H_x$ compatible with that on $H'_y$.

Definition 5.5. For $x \in S_\text{rs}$, resp. $x \in \mathfrak{s}_\text{rs}(F)$, and $f \in C_c^\infty(S)$, resp. $f \in C_c^\infty(\mathfrak{s}(F))$, we define the orbital integral to be

$$O(x, f) = \int_{H_x \setminus H} f(xh) \, dh.$$ 

Definition 5.6. For $f \in C_c^\infty(S)$, resp. $C_c^\infty(\mathfrak{s}(F))$, we say $f' \in C_c^\infty(S')$, resp. $C_c^\infty(\mathfrak{s}'(F))$, is a smooth transfer of $f$ if for each $y \in S'_rs$, resp. $y \in \mathfrak{s}'_rs(F)$,

$$O(y, f') = \begin{cases} O(x, f), & \text{if there is } x \in S_\text{rs}, \text{ resp. } x \in \mathfrak{s}_\text{rs}(F), \text{ such that } x \leftrightarrow y, \\ 0, & \text{otherwise}. \end{cases}$$

(3)

Sometimes we will write transfer instead of smooth transfer for short. If $f$ and $f'$ are transfers of each other, we write $f \leftrightarrow f'$ for simplicity.

Remark 5.7. Conversely, for $f' \in C_c^\infty(S')$, resp. $f' \in C_c^\infty(\mathfrak{s}'(F))$, satisfying the above condition (3), we say $f \in C_c^\infty(S)$, resp. $f \in C_c^\infty(\mathfrak{s}(F))$, is a smooth transfer of $f'$ if for each $x \in S_\text{rs}$, resp. $x \in \mathfrak{s}_\text{rs}(F)$,

$$O(x, f) = O(y, f') \quad \text{if } x \leftrightarrow y.$$ 

Remark 5.8. For semisimple $x \in S$, resp. $x \in \mathfrak{s}(F)$, and semisimple $y \in S'$, resp. $y \in \mathfrak{s}'(F)$, such that $x \leftrightarrow y$, we can also define smooth transfers for $f \in C_c^\infty(\mathfrak{s}_x(F))$ and $f' \in C_c^\infty(\mathfrak{s}'_y(F))$ determined by the orbital integrals with respect to the action of $H_x$ on $\mathfrak{s}_x(F)$ and the action of $H'_y$ on $\mathfrak{s}'_y(F)$.

Our main theorem on the smooth transfer is the following.

Theorem 5.9. For each $f \in C_c^\infty(S)$, its smooth transfer in $C_c^\infty(S')$ exists.

Showing the existence of the smooth transfer essentially is the local issue, and via Luna Slice Theorem and descent of orbital integrals, one can reduce to prove the smooth transfers on the descendants for each semisimple $x \in S$. 

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and $y \in S'$ such that $x \leftrightarrow y$. By the description of the descendants and the inductive arguments, one reduces to prove the following Lie algebra’s version of the smooth transfer. We refer the readers to [zhw, §3] or [zhe, §5.3] for more details of such reduction steps.

**Theorem 5.10.** For each $f \in C^\infty_c(s(F))$, its smooth transfer in $C^\infty_c(s'(F))$ exists.

To prove Theorem 5.10, the following theorem, which roughly says that Fourier transform commutes with smooth transfer, is the key input.

**Theorem 5.11.** There exists a non zero constant $c \in \mathbb{C}$ satisfying that: if $f' \in C^\infty_c(s'(F))$ is a transfer of $f \in C^\infty_c(s(F))$, then $f'$ is a transfer of $cf$.

**Remark 5.12.** We now briefly recall why Theorem 5.11 implies Theorem 5.10. Repeating the above arguments again and by the induction hypothesis, one can assume that we have showed the existence of the smooth transfer for $f \in C^\infty_c(s(F) - \mathcal{N})$, where $\mathcal{N}$ is the nilpotent cone of $s(F)$. We have explained that the symmetric pair $(G, \mathcal{H})$ is of negative normalized defect, which implies the following fact: let $T$ be an $H$-invariant distribution on $s(F)$ such that $\text{Supp}(T) \subset \mathcal{N}$ and $\text{Supp}(\hat{T}) \subset \mathcal{N}$, then $T$ must be zero. This fact has the following direct consequence: let $c_0 = \bigcap_T \ker(T)$ where $T$ runs over all $H$-invariant distributions on $s(F)$, then each $f \in C^\infty_c(s(F))$ can be written as $f = f_0 + f_1 + \hat{f}_2$ with $f_0 \in c_0$ and $f_i \in C^\infty_c(s(F) - \mathcal{N})$ for $i = 1, 2$. Therefore, it remains to prove the existence of the smooth transfer for $\hat{f}$ with $f$ belonging to the space $C^\infty_c(s(F) - \mathcal{N})$, which is exactly what Theorem 5.11 shows.

**Remark 5.13.** To prove the smooth transfer in the converse direction, as Remark 5.7 stated, it suffices to prove that each $f \in C^\infty_c(s'(F))$ satisfying condition (3) can be written as $f = f_0 + f_1 + \hat{f}_2$ with $f_0 \in c_0$ and $f_i \in C^\infty_c(s'(F) - \mathcal{N})$ satisfying (3) for both $i = 1, 2$. However we do not know how to prove such a decomposition now.

6 **Local orbital integrals**

Let $F$ be a $p$-adic field as before. This section is devoted to prove Theorem 5.11. We employ the ideas and techniques used by Harish-Chandra on harmonic analysis for $p$-adic groups (cf. [hc1] and [hc2] or [ko]), and the ones used by Waldspurger on endoscopic smooth transfer (cf. [wa1] and [wa2]), to establish various analogues results for the $p$-adic symmetric spaces considered here. A large part of this section can be viewed as a generalization of the results in [zhe].

6.1 **Preparations**

**Inequalities** Fix a non zero $X_0$ in the nilpotent cone $\mathcal{N}$ of $s(F)$. Let $(X_0, d, Y_0)$ be an $s_t$-triple with $d \in h(F)$ and $Y_0 \in \mathcal{N}$. We denote $r = \dim_F s Y_0$ and $m = \frac{1}{2} \text{tr} (\text{ad}(-d)|_{s Y_0})$. The inequalities below are used to bound the orbital integrals for $C^\infty_c(s(F))$ along a Cartan subspace of $s$. Recall that a Cartan subspace of $s$ is by definition a maximal abelian $F$-subspace of $s$.

**Proposition 6.1.** We have the relation
1. \( r \geq n \),

2. \( r + m > n^2 + \frac{2}{r} \).

Proof. Let \( L \) be an extension field of \( F \) with degree \( d \) contained in \( D \). Then

\[
(G \times_F L, H \times_F L) \simeq (G' \times_F L, H' \times_F L) =: (G'', H'').
\]

Denote by \( s'' \) the Lie algebra associated to \( G'' / H'' \), and we can view \( X_0, Y_0 \in s''(L) \) and \( d \in h''(L) \) \((h'':=\text{Lie}(H''))\) in a canonical way. Let \( r' = \dim_F s''_0 \) and \( m' = \frac{1}{2} \Tr \left( \text{ad}(d) \big|_{s''_0} \right) \). It’s not hard to see that \( r = r' \) and \( m = m' \). Therefore the required inequalities can be deduced from that of \([zhc, \text{Proposition 4.4}]\). \( \square \)

Representability. With the aid of Proposition 6.1, we can generalize all the results in \([zhc, \S5, \S6, \S7]\) when \( d = 1 \) to the more general case at hands. We will only state the results and omit the proofs since they are almost word by word check of those in \([zhc]\).

Let \( X \in s_{rs}(F) \), which lies in a Cartan subspace \( \mathfrak{c} \) of \( \mathfrak{s} \). Let \( T \) be the centralizer of \( \mathfrak{c} \) in \( \mathfrak{H} \), which is a torus, and denote \( \mathfrak{t} = \text{Lie}(T) \). We will consider the normalized orbital integral:

\[
I(X,f) = |D^s(X)||\frac{1}{2}O(X,f)|, \quad \text{for } f \in C^\infty_c(s(F)),
\]

which is a distribution on \( s(F) \), and also consider its Fourier transform:

\[
\hat{I}(X,f) := I(X,\hat{f}), \quad \text{for } f \in C^\infty_c(s(F)).
\]

Here the factor \( |D^s(X)|_F \) is defined to be

\[
|D^s(X)|_F = |\det(\text{ad}(X)); h/\mathfrak{t} \oplus \mathfrak{s}/\mathfrak{c}|_F^\dagger.
\]

Notice that if \( X \leftrightarrow Y \) with \( X \in s_{rs}(F) \) and \( Y \in s'_{rs}(F) \) then

\[
|D^s(X)|_F = |D^s(Y)|_F.
\]

Hence it doesn’t matter if we consider the smooth transfer with respect to the normalized orbital integrals. The following theorem is a generalization of \([zhc, \text{Theorem 6.1}]\). The ingredients of its proof are the analogues of parabolic induction, Howe’s finiteness theorem and bounding the normalized orbital integrals along Cartan subspaces of \( s(F) \), all in the setting of our symmetric spaces.

**Theorem 6.2.** For each \( X \in s_{rs}(F) \), there is a locally constant \( H \)-invariant function \( \hat{i}_X \) defined on \( s_{rs}(F) \) which is locally integrable on \( s(F) \), such that for any \( f \in C^\infty_c(s(F)) \) we have

\[
\hat{I}(X,f) = \int_{s_{rs}(F)} \hat{i}_X(Y)|f(Y)||D^s(Y)|_F^{-1/2} \, dY.
\]

We will need a proposition appearing in the process of the proof. Now suppose that \( X \in s_{rs}(F) \) is of the form \( \left( \begin{array}{cc} 0 & 1_m \\ A & 0 \end{array} \right) \) and \( A \) is not elliptic. Write \( s = s^+ \oplus s^- \), where we identify \( s^\pm = \mathfrak{gl}_m(D) \). Then there is a proper Levi
Here we identify \( \mathfrak{m}_0 \) of \( \text{GL}_m(D) \) such that \( A \in M_0 := \mathfrak{m}_0(F) \). Let \( \tau = \tau^+ \oplus \tau^- \subset \mathfrak{g} \). Here we identify \( \tau^\pm = \mathfrak{m}_0 \), where \( \mathfrak{m}_0 \) is the Lie algebra of \( M_0 \). Then \( X \) lies in \( \tau(F) \) and is regular semisimple with respect to the adjoint action of \( M = M_0 \times M_0 \) on \( \tau(F) \). Choosing a Haar measure on \( M \), we can also consider the orbital integral with respect to the action of \( M \) on \( \tau(F) \):

\[
I^\tau(X, f') = |D^\tau(X)|_F^{\frac{1}{2}} \int_{H_X \setminus M} f'(X^m) \, dm, \quad \text{for } f' \in C_c^\infty(\tau(F)),
\]

which is a distribution on \( \tau(F) \), and consider its Fourier transform

\[
\hat{I}^\tau(X, f') := I^\tau(X, \hat{f'}).
\]

Here \( |D^\tau(X)|_F = |\det(\text{ad}(X); \mathfrak{m}/\tau \oplus \tau/\varepsilon)|_F^\frac{1}{2} \), with \( \tau \) and \( \varepsilon \) introduced before. Then, with suitable choices of Haar measures, there is a relation between orbital integrals \( I(X, \cdot) \) and \( I^\tau(X, \cdot) \), the so-called parabolic descent of orbital integrals,

\[
I(X, f) = I^\tau(X, f(t)), \quad \text{for all } f \in C_c^\infty(\mathfrak{g}^\tau(F)).
\]

Here \( f(t) \in C_c^\infty(\tau(F)) \) is induced from \( f \) (see [zhe, §6.1] for more details). There is a locally constant \( M \)-invariant function \( \hat{i}_X^\tau \) defined on \( \tau_{rs}(F) \) which is locally integrable on \( \tau(F) \), such that for any \( f' \in C_c^\infty(\tau(F)) \) we have

\[
\hat{I}^\tau(X, f') = \int_{\tau_{rs}(F)} \hat{i}_X^\tau(Y)f'(Y)|D^\tau(Y)|_F^{-\frac{1}{2}} \, dY.
\]

The following formula for \( \hat{i}_X^\tau \) will be needed. Not surprisingly, there should be a relation between \( i_X^\tau \) and \( \hat{i}_X^\tau \) for such non-elliptic \( X \).

**Proposition 6.3.** Keep the notations and assumptions as above. We have

\[
\hat{i}_X^\tau(Y) = \sum_{Y'} \hat{i}_X^\tau(Y'), \quad Y \in \tau_{rs}(F),
\]

where \( Y' \) runs over a finite set of representatives for the \( M \)-conjugacy classes of elements in \( \tau(F) \) which are \( H \)-conjugate to \( Y \). In particular, if there is no element in \( \tau(F) \) being \( H \)-conjugate to \( Y \), we have

\[
\hat{i}_X^\tau(Y) = 0.
\]

**Limit formula** We also write \( \hat{i}(X, Y) \) for \( \hat{i}_X^\tau(Y) \). There is a limit formula for \( \hat{i}(X, Y) \) shown in [zhe, Proposition 7.1] when \( d = 1 \) and twisted by a quadratic character. The similar formula still holds for the case at hands, which will be stated below. Notice that \( \hat{i}(X, Y) \) depends on the choices of the Haar measures on \( H \) and \( H_X \) up to a scalar, we will not specify them, but refer the reader to [zhe] for more details, while other results do not depend on the choices of the measures.

Let \( \varepsilon \) be a Cartan subspace of \( \mathfrak{g} \), \( \mathbf{T} \) be the centralizer of \( \varepsilon \) in \( H \), and denote \( t = \text{Lie}(\mathbf{T}) \). For \( X, Y \in \varepsilon_{\text{reg}}(F) \), we define a non degenerate, bilinear and symmetric form \( q_{X,Y} \) on \( \mathfrak{h}(F)/t(F) \) by

\[
q_{X,Y}(Z, Z') = \langle [Z, X], [Y, Z'] \rangle,
\]

where the pairing \( \langle \cdot, \cdot \rangle \) is the one introduced before. One can verify that \( q_{X,Y} = q_{Y,X} \). We will write \( \gamma_\psi(X, Y) = \gamma_\psi(q_{X,Y}) \) for simplicity.
Proposition 6.4. Let \( X \in \mathfrak{s}_n(F) \) and \( Y \in \mathfrak{c}_{reg}(F) \). Then there exists \( N \in \mathbb{N} \) such that if \( \mu \in F^\times \) satisfying \( v_F(\mu) < -N \), we have the equality

\[
\widehat{\iota}(\mu X, Y) = \sum_{h \in T \setminus H, \ h \cdot X \in \mathfrak{c}} \gamma_{\psi}(\mu h \cdot X, Y) \psi((\mu h \cdot X, Y)).
\]

Construction of test functions For \( X, Y \in \mathfrak{c}_{reg}(F) \), there is a formula for \( \gamma_{\psi}(X, Y) \), which is exhibited in [zhc, Proposition 7.3] when \( d = 1 \). The formula for general \( d \) has the same form. We will not state it here, since it involves much more notations. However, we point out the following fact, which is an analogy of [zhc, Lemma 7.5].

Let \( c' \) be a Cartan subspace of \( \mathfrak{c}' \), and \( t' \) be the Lie algebra of \( c' \)'s centralizer in \( \mathfrak{h}' \). Recall that for \( h \in \mathfrak{c}_{reg}(F) \) and \( Y \in \mathfrak{c}_{reg}(F) \) such that \( X \leftrightarrow Y \), there are isomorphisms \( \varphi_1 : t \rightarrow t' \) and \( \varphi_\epsilon : \epsilon \rightarrow c' \) defined over \( F \) such that \( \varphi_\epsilon(X) = Y \). The morphisms \( \varphi_\epsilon \) and \( \varphi_1 \) are also isometries with respect to \( \langle \cdot, \cdot \rangle \) on \( t, t', \epsilon \) and \( c' \). The following lemma is used to construct certain test functions required in Proposition 6.6 below.

Proposition 6.5. Let the notations be as above. Then for any \( X' \in \mathfrak{c}_{reg}(F) \) we have the equality

\[
\gamma_{\psi}(X, X') = \gamma_{\psi}(h(F))\gamma_{\psi}(h'(F))^{-1}\gamma_{\psi}(\varphi_\epsilon(X), \varphi_\epsilon(X')).
\]

The following proposition is an analogy of [zhc, Proposition 7.6], whose proof involves Propositions 6.4 and 6.5. It plays an important role in proving smooth transfer by global method.

Proposition 6.6. Let \( X_0 \in \mathfrak{c}_{reg}(F) \) and \( Y_0 \in \mathfrak{c}_{reg}(F) \) such that \( X_0 \leftrightarrow Y_0 \). Then there exist functions \( f \in C_0^\infty(\mathfrak{g}(F)) \) and \( f' \in C_0^\infty(\mathfrak{g}'(F)) \) satisfying the following conditions.

1. If \( X \in \text{Supp}(f) \), \( X \) is \( H \)-conjugate to an element in \( \mathfrak{c}_{reg}(F) \). If \( Y \in \text{Supp}(f') \), there exists \( X' \in \mathfrak{c}_{reg}(F) \) such that \( X' \leftrightarrow Y \);
2. \( f' \) is a transfer of \( f \);
3. There is an equality

\[
\widehat{I}(X_0, f) = c\widehat{I}(Y_0, f') \neq 0,
\]

where \( c = \gamma_{\psi}(h(F))\gamma_{\psi}(h'(F))^{-1} \).

6.2 Proof of Theorem 5.11

In this subsection, we fix two \( C_0^\infty \)-functions \( f' \in C_0^\infty(\mathfrak{g}'(F)) \) and \( f \in C_0^\infty(\mathfrak{g}(F)) \) such that \( f \leftrightarrow f' \). The proof of Theorem 5.11 can be divided into two parts:

1. the first part is to prove that \( \widehat{I}(Y, f') = 0 \) for any \( Y \in \mathfrak{g}'_n(F) \) such that there is no element in \( \mathfrak{s}_n(F) \) matching with \( Y \);
2. the second part is to search for a non zero constant \( c \in \mathbb{C} \), independent of \( f \) and \( f' \), such that

\[
\widehat{I}(Y, f') = c\widehat{I}(X, f)
\]

for any \( X \in \mathfrak{s}_n(F), Y \in \mathfrak{g}'_n(F) \) such that \( X \leftrightarrow Y \).
First part of the proof  Now we fix a $Y_0 \in \mathfrak{s}'_{\text{reg}}(F)$ such that there is no element in $\mathfrak{s}_{\text{reg}}(F)$ matching with $Y_0$. Suppose that $Y_0$ belongs to a Cartan subspace $\mathfrak{c}'$ of $\mathfrak{s}'$. By Theorem 6.2 (in the case of $d = 1$) and Weyl integration formula, we have

$$
\hat{I}(Y_0, f') = \int_{\mathfrak{s}'_{\text{reg}}(F)} \hat{\iota}_{Y_0}(Z) f'(Z) |D'(Z)|^{-\frac{2}{d}} \, dZ
$$

$$
= \sum_{c'} w_{c'} \int_{\mathfrak{c}'_{\text{reg}}(F)} \hat{\iota}_{Y_0}(Z) I(Z, f') \, dZ,
$$

(4)

where $c'$ runs over a finite set of representatives for the $H'$-conjugacy classes of Cartan subspaces in $\mathfrak{s}'$ and $w_{c'}$ is the cardinality of relative Weyl group associated to $c'$. We denote by $\mathcal{G}^D$ the set of Cartan subspaces $\mathfrak{c}'$ of $\mathfrak{s}'$ such that there is a Cartan subspace $\mathfrak{c}$ of $\mathfrak{s}$ such that $\mathfrak{c} \leftrightarrow \mathfrak{c}'$. Here we say that $\mathfrak{c} \leftrightarrow \mathfrak{c}'$ if there exist $X \in \mathfrak{c}_{\text{reg}}(F)$ and $Y \in \mathfrak{c}'_{\text{reg}}(F)$ such that $X \leftrightarrow Y$. If $\mathfrak{c} \leftrightarrow \mathfrak{c}'$, there is an isomorphism $\varphi_{\mathfrak{c}} : \mathfrak{c} \leftrightarrow \mathfrak{c}'$ defined over $F$ such that $X \leftrightarrow \varphi_{\mathfrak{c}}(X)$ for any $X \in \mathfrak{c}_{\text{reg}}(F)$. By the condition on $Y_0$, we see that $c_0 \notin \mathcal{G}^D$.

For any $\mathfrak{c}' \notin \mathcal{G}^D$, we automatically have $I(Z, f') = 0$ for each $Z \in \mathfrak{c}'_{\text{reg}}(F)$ by the condition on $f'$. If $\mathfrak{c}' \notin \mathcal{G}^D$, we claim that $\hat{\iota}_{Y_0}(Z) = 0$ for any $Z \in \mathfrak{c}'_{\text{reg}}(F)$.

We can assume that $Y_0$ is of the form $\begin{pmatrix} 0 & 1_n \\ A & 0 \end{pmatrix}$ with $A \in \text{GL}_n(F)_{\text{rs}}$. By the condition of $Y_0$, there is an irreducible factor (over $F$) of the characteristic polynomial of $A$ with degree $r$ such that $d \nmid r$. Then there is a subspace $\mathfrak{r}$ of $\mathfrak{s}$ in the form of $\mathfrak{r} = (\mathfrak{gl}_r \oplus \mathfrak{gl}_{n-r}) \oplus (\mathfrak{gl}_r \oplus \mathfrak{gl}_{n-r})$ such that $Y_0 \in \mathfrak{r}(F)$ (see Proposition 6.3 for the notation). Since $\mathfrak{c}' \notin \mathcal{G}^D$, there is no element in $\mathfrak{r}(F)$ being $H'$-conjugate to any $Z \in \mathfrak{c}'_{\text{reg}}(F)$. Thus the claim follows from Proposition 6.3. Therefore, in any case, we showed the terms appearing in the sum of (4) are zero, thus, obtain that $\hat{I}(Y_0, f') = 0$.

Second part of the proof  The arguments in this part are almost as same as those in [zie, §8]. We shall briefly explain it.

Now, we fix $f \in C^\infty_c(\mathfrak{s}(F))$ and $f' \in C^\infty_c(\mathfrak{s}'(F))$ which is a transfer of $f$, and fix $X_0 \in \mathfrak{s}_{\text{reg}}(F), Y_0 \in \mathfrak{s}'_{\text{reg}}(F)$ such that $X_0 \leftrightarrow Y_0$. Next, we list the choices made on the global data.

- **Fields.** We choose a number field $k$ and a central division algebra $\mathbb{D}$ over $k$ so that:

1. $k$ is totally imaginary;
2. there exists a finite place $w$ of $k$ such that $k_w \simeq F$ and $\mathbb{D}(k_w) \simeq D$;
3. there exists another finite place $u$ of $k$ such that $D$ does not split over $k_u$.

By conditions 1 and 2, such finite place $u$ exists.

From now on, we identify $k_w$ with $F$. We denote by $\mathcal{O}_k$ the ring of integers of $k$, and by $\mathbb{A}$ the adele ring of $k$. We fix a maximal order $\mathcal{O}_D$ of $\mathbb{D}$ containing $\mathcal{O}_k$.

We fix a continuous character $\psi$ on $\mathbb{A} / k$ such that the original fixed character $\psi$ of $F$ is the local component of $\psi$ at $w$.
• **Groups.** We define a global symmetric pair \((G, H)\) over \(k\) with respect to \(D\), so that the base change of \((G, H)\) to \(k_v\) is isomorphic to \((G, H)\). Thus if the index of \(\mathcal{D}\) is \(d'\), let \((G, H) = (\text{GL}_{2m}(\mathcal{D}), \text{GL}_{m'}(\mathcal{D}) \times \text{GL}_{m''}(\mathcal{D}))\) where \(m'd' = n\). We still denote the Lie algebra of \(G/H\) by \(\mathfrak{s}\). Define the symmetric pair \((G', H') = (\text{GL}_{2m}, \text{GL}_{m} \times \text{GL}_{m})\) over \(k\) as usual, and still use \(\mathfrak{s}'\) to denote the Lie algebra of \(G'/H'\). Hence \(X_0 \in \mathfrak{s}(k_w)\) and \(Y_0 \in \mathfrak{s}'(k_w)\).

• **Places.** Denote by \(V\) (resp. \(V_{\infty}, V_{\ell}\)) the set of all (resp. archimedean, non-archimedean) places of \(k\). Fix two \(\mathcal{O}_k\)-lattices: \(L = \text{gl}_{m'}(\mathcal{O}_D) \oplus \text{gl}_{m''}(\mathcal{O}_D)\) in \(\mathfrak{s}(k)\) and \(L' = \text{gl}_m(\mathcal{O}_D) \oplus \text{gl}_m(\mathcal{O}_D)\) in \(\mathfrak{s}'(k)\). For each \(v \in V_{\ell}\), denote \(L_v = L \otimes_{\mathcal{O}_k} \mathcal{O}_{k,v}\) and \(L'_v = L' \otimes_{\mathcal{O}_k} \mathcal{O}_{k,v}\). We fix a finite set \(S \subset V\) such that:

1. \(S\) contains \(u, w\) and \(V_{\infty}\);
2. for each \(v \in V - S\), everything is unramified, i.e. \(G\) and \(G'\) are unramified over \(k_v\), \(L\) and \(L'_v\) are self-dual with respect to \(\psi_v\) and \(\langle \cdot, \cdot \rangle\). We denote by \(S'\) the subset \(S - V_{\infty} - \{w\}\) of \(S\).

• **Orbits.** For each \(v \in V_{\ell}\), we choose an open compact subset \(\Omega_v\) of \(\mathfrak{s}(k_v)\) such that:

1. if \(v = w\), we require that: \(X_0 \in \Omega_w\) and \(\Omega_w \subset \mathfrak{s}(k_w)\).
2. \(\Omega_u \subset \mathfrak{s}(k_u)\);
3. if \(v \in S\) but \(v \neq w, u\), choose \(\Omega_v\) to be any open compact subset;
4. if \(v \in V_{\ell} - S\), let \(\Omega_v = L_v\).

Then by the strong approximation theorem, there exists \(X^0 \in \mathfrak{s}(k)\) such that \(X^0 \in \prod_{v \in V_{\ell}} \Omega_v\). Furthermore, by the condition (2), \(X^0 \in \mathfrak{s}(k)\). Take an element \(Y^0 \in \mathfrak{s}(k)\) such that \(X^0 \leftrightarrow Y^0\).

• **Functions.** For each \(v \in V\), choose Bruhat-Schwartz functions \(\phi_v \in \mathcal{S}(\mathfrak{s}(k_v))\) and \(\phi'_v \in \mathcal{S}(\mathfrak{s}'(k_v))\) in the following way:

1. if \(v = w\), let \(\phi_v = f\) and \(\phi'_v = f'\);
2. if \(v \in S'\), by Proposition 6.6, we can require that:
   - if \(X_v \in \text{Supp}(\phi_v)\), there exists \(X'_v \in \mathfrak{c}_{\mathfrak{X}^0}(k_v)\) such that \(X_v\) and \(X'_v\) are \(\mathbb{H}(k_v)\)-conjugate, where \(\mathfrak{c}_{\mathfrak{X}^0}\) is the Cartan subspace of \(\mathfrak{s}\) containing \(\mathfrak{X}^0\); if \(Y_v \in \text{Supp}(\phi'_v)\), there exists \(X_v \in \mathfrak{c}_{\mathfrak{X}^0}(k_v)\) such that \(X_v \leftrightarrow Y_v\);
   - \(\phi'_v\) is a transfer of \(\phi_v\);
   - \(\hat{I}(X^0, \phi_v) = c_v \hat{I}(Y^0, \phi'_v)\), where \(c_v = \gamma_v(h(k_v))\gamma_v(b'(k_v))^{-1}\);
3. for \(v \in V - S\), since we required \(G\) is unramified over \(k_v\), that is to say, \(\mathcal{D}\) is split over \(k_v\), we can identify \(G(k_v) = G'(k_v)\), \(L_v = L'_v\), and set \(\phi_v = \phi'_v = 1_{L_v}\); moreover, since \(L_v\) is self-dual with respect to \(\psi_v\) and \(\langle \cdot, \cdot \rangle\), \(\phi_v = \phi'_v\);
4. for $v_0 \in V$, identifying $(\mathbb{H}(k_{v_0}), s(k_{v_0}))$ with $(\mathbb{H}'(k_{v_0}), s'(k_{v_0}))$, we can choose $\phi_{v_0} = \phi'_{v_0} \in S(\mathfrak{s}(k))$ such that:

- $\tilde{I}(X^0, \phi_{v_0}) = \tilde{I}(Y^0, \phi'_{v_0}) \neq 0$;
- if $X \in \mathfrak{s}(k)$ is $\mathbb{H}(k)$-conjugate to an element in the support of $\phi_v$ at each place $v \in V$, then $X$ is $\mathbb{H}(k)$-conjugate to $X^0$;
- if $Y \in \mathfrak{s}'(k)$ is $\mathbb{H}'(k_e)$-conjugate to an element in the support of $\phi'_v$ at each place $v \in V$, then $Y$ is $\mathbb{H}'(k)$-conjugate to $Y^0$.

Now we set $\phi \in S(\mathfrak{s}(A))$ and $\phi' \in S(\mathfrak{s}'(A))$ to be:

$$\phi = \prod_{v \in V} \phi_v, \quad \phi' = \prod_{v \in V} \phi'_v.$$  

**Final proof.** As showed in [zhc, Theorem 8.2] the following integrals $I(\phi)$ and $I(\phi')$ are absolutely convergent:

$$I(\phi) = \int_{\mathbb{H}(k) \backslash \mathbb{H}(A)^1} \sum_{X \in [s_{\text{cell}}(k)]} \phi(X^h) \, dh, \quad I(\phi') = \int_{\mathbb{H}'(k) \backslash \mathbb{H}'(A)^1} \sum_{Y \in [s'_{\text{cell}}(k)]} \phi'(Y^h) \, dh,$$

where

$$\mathbb{H}(A)^1 = \bigcap_{\chi \in \text{Hom}_k(\mathbb{H}, \mathbb{G}_m)} \ker |\chi|, \quad \mathbb{H}'(A)^1 = \bigcap_{\chi \in \text{Hom}_k(\mathbb{H}', \mathbb{G}_m)} \ker |\chi|.$$  

Actually [zhc, Theorem 8.2] only treat the case for $(\mathbb{G}', \mathbb{H}')$, but the arguments also go for $(\mathbb{G}, \mathbb{H})$. It’s obvious that

$$I(\phi) = \sum_{X \in [s_{\text{cell}}(k)\backslash]} \tau(X) \prod_v I(X, \phi_v),$$

and

$$I(\phi') = \sum_{Y \in [s'_{\text{cell}}(k)\backslash]} \tau(Y) \prod_v I(Y, \phi'_v),$$

where $[s_{\text{cell}}(k)]$ denotes the $\mathbb{H}(k)$-orbits of $s_{\text{cell}}(k)$,

$$\tau(X) = \text{vol}(\mathbb{H}_X(k) \backslash (\mathbb{H}_X(A) \cap \mathbb{H}(A)^1)),$$

and the notions of $[s'_{\text{cell}}(k)]$ and $\tau(Y)$ are similiar. If $X \in s_{\text{cell}}(k), Y \in s'_{\text{cell}}(k)$ such that $X \leftrightarrow Y$, then $\mathbb{H}_X \simeq \mathbb{H}'_Y$, and we choose Haar measures on $\mathbb{H}_X(A)$ and $\mathbb{H}'_Y(A)$ so that they are compatible, in particular,

$$\tau(X) = \tau(Y).$$

According to the conditions on $\phi_u$ (resp. $\phi'_u$), we know that if $X \in s(k)$ (resp. $Y \in s'(k)$) such that $X \in \text{Supp}(\phi)^{\mathbb{H}(A)}$ (resp. $Y \in \text{Supp}(\phi')^{\mathbb{H}'(A)}$), then $X \in s_{\text{cell}}(k)$ (resp. $Y \in s'_{\text{cell}}(k)$), where we use $\text{Supp}(\phi)^{\mathbb{H}(A)}$ to denote the $\mathbb{H}(A)$-orbits intersecting $\text{Supp}(\phi)$. By the conditions on $\phi'_v$ at each place $v$, we know that, if $Y \in s'_{\text{cell}}(k)$ such that $I(Y, \phi'_v) \neq 0$ for each $v \in V$, there exists

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$X_v \in \mathfrak{g}_m(k_v)$ such that $X_v \leftrightarrow Y$ at each place $v \in V$ and hence there exists $X \in \mathfrak{g}_d(k)$ such that $X \leftrightarrow Y$. Therefore we have

$$I(\phi) = I(\phi'),$$

since $\phi_v$ is a transfer of $\phi'_v$ at each place $v \in V$ by the requirement.

On the other hand, according to the conditions on $\hat{\phi}_v$ and $\hat{\phi}'_v$, we know that if $X \in \mathfrak{g}(k)$ (resp. $Y \in \mathfrak{g}'(k)$) such that $X \in \text{Supp}(\hat{\phi})\mathbb{H}(A)$ (resp. $Y \in \text{Supp}(\hat{\phi}')\mathbb{H}'(A)$), $X$ is $\mathbb{H}(k)$-conjugate to $X^0$ (resp. $Y$ is $\mathbb{H}'(k)$-conjugate to $Y^0$).

By Poisson summation formula, we have

$$\sum_{X \in \mathfrak{g}(k)} \phi(X^h) = \sum_{X \in \mathfrak{g}(k)} \hat{\phi}(X^h), \quad \forall \ h \in \mathbb{H}(A),$$

and

$$\sum_{Y \in \mathfrak{g}'(k)} \phi'(Y^h) = \sum_{Y \in \mathfrak{g}'(k)} \hat{\phi}'(Y^h), \quad \forall \ h \in \mathbb{H}'(A).$$

Therefore, by the conditions on $\phi$ and $\phi'$, we have

$$I(\phi) = I(\hat{\phi}), \quad I(\phi') = I(\hat{\phi}').$$

Hence we have

$$I(\hat{\phi}) = I(\hat{\phi'}),$$

which amounts to say,

$$\tau(X^0) \prod_{v \in V} \hat{I}(X^0, \phi_v) = \tau(Y^0) \prod_{v \in V} \hat{I}(Y^0, \phi'_v).$$

For $v \in V - S$ or $v \in V_\infty$, we have

$$\hat{I}(X^0, \phi_v) = \hat{I}(Y^0, \phi'_v) \neq 0.$$

For $v \in S'$ we have

$$\hat{I}(X^0, \phi_v) = c_v \hat{I}(Y^0, \phi'_v) \neq 0.$$

Therefore

$$c \hat{I}(X^0, f) = \hat{I}(Y^0, f'),$$

where

$$c = \prod_{v \in S'} c_v = \prod_{v \in S'} \gamma_{\psi}(b(k_v))\gamma_{\psi}(b'(k_v))^{-1}.$$
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