Difference Equations in Spin Chains with a Boundary

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Abstract

Correlation functions and form factors in vertex models or spin chains are known to satisfy certain difference equations called the quantum Knizhnik-Zamolodchikov equations. We find similar difference equations for the case of semi-infinite spin chain systems with integrable boundary conditions. We derive these equations using the properties of the vertex operators and the boundary vacuum state, or alternatively through corner transfer matrix arguments for the 8-vertex model with a boundary. The spontaneous boundary magnetization is found by solving such difference equations. The boundary S-matrix is also proposed and compared, in the sine-Gordon limit, with Ghoshal–Zamolodchikov’s result. The axioms satisfied by the form factors in the boundary theory are formulated.

1 Introduction

Following the success of conformal field theory in two dimensions, much attention has recently been focused on trying to understand the algebraic structure of solvable lattice models and massive integrable quantum field theories [1,2,3,4]. Certain quantum deformations of affine Lie algebras play a central role in the description of these systems. One of the remarkable results of these studies is the discovery of the difference analogues of the Knizhnik–Zamolodchikov (KZ) equation [5]. These ‘quantum KZ equations’ are satisfied by both the correlation functions and the form factors of integrable models [1,2,3,4]. They allow us to study the off-shell properties of the models.

In this paper we establish similar results for the XYZ Hamiltonian on a semi-infinite chain, with an interaction at the boundary corresponding to a magnetic field in the z-direction:

\[ H_B = -\frac{1}{2} \sum_{k=1}^{\infty} \left( (1 + \Gamma) \sigma_{k+1}^{x} \sigma_{k}^{x} + (1 - \Gamma) \sigma_{k+1}^{y} \sigma_{k}^{y} + \Delta \sigma_{k+1}^{z} \sigma_{k}^{z} \right) + h \sigma_{1}^{z}. \]  \hspace{1cm} (1.1)
Integrable models with special boundary conditions which preserve integrability, have been studied both in lattice and continuum theories, e.g., [7,8]. In general frameworks, the boundary interaction is specified by a reflection matrix $K$ for lattice systems [9], or by a boundary $S$-matrix for quantum field theories [10]. Integrability is guaranteed by the fact that they satisfy the boundary Yang-Baxter equation. The Hamiltonian (1.1) is obtained from a commuting transfer matrix constructed by such a $K$-matrix [11,12]. This model is particularly interesting because the sine-Gordon model [10] is its continuum limit.

In a recent paper [13], it is shown that the space of states of the XXZ spin chain with a boundary can be described in terms of the vertex operators $\Phi_\varepsilon(\zeta)$, $\Psi_\varepsilon^*(\zeta)$ associated with the “bulk” (infinite chain) XXZ model [2]. Explicit expressions for the two vacuum states $|i\rangle_B$, $(i = 0, 1)$ of the boundary XXZ Hamiltonian are obtained by using the bosonization formula for the vertex operators, and then they are used to obtain the boundary excited energy and the boundary $S$-matrix.

It is an interesting problem to extend the analysis of the off-shell quantities such as the correlation functions and the form factors in the bulk theories to the boundary cases. In this paper, we show that the boundary analogues [14] of the quantum KZ equation are satisfied by these quantities in the XXZ and XYZ models. For the XXZ model, our results follow from the results in [13]. Assuming similar properties of the vertex operators [13] and the vacuum states, we can extend the results to the XYZ model. Alternatively, for the correlation functions we can use corner transfer matrix arguments [16,17] to derive the same difference equation.

In Section 2, we will derive the $q$-difference equations satisfied by the correlation functions of the XXZ and XYZ spin chains with a boundary. In Section 3, we derive the same equations for the XYZ chain using a graphical corner transfer matrix argument similar to that used in [17]. In Section 4 we discuss the boundary magnetization (the vacuum expectation value of the boundary spin operator $\sigma_1^z$) of the XYZ model. In the case where $h = 0$, one can solve the corresponding difference equation, and the result is simply the square of the bulk magnetization of the eight-vertex model. A similar phenomenon was observed in the case of the XXZ model with a boundary [13]. We conjecture the form of the boundary $S$-matrix in Section 5. Lacking bosonization formulas for the vertex operators of the XYZ model, we cannot derive it explicitly. However, our result reduces to the appropriate sine-Gordon boundary $S$-matrix of [10] in the scaling limit. Finally, in Section 6, we present the analog of Smirnov’s axioms [6] for the form factors.

2 The boundary $q$-difference equation

In this section, we derive the $q$-difference equation satisfied by the boundary $n$-point function, i.e. the expectation value of the product of type I vertex operators between the vacuum states of the boundary Hamiltonian. This equation was first discovered by Cherednik [14]. We will derive this equation first for the boundary XXZ model in the formulation of [13], and then extend it to the boundary XYZ model.
2.1 The XXZ model

The boundary XXZ model was formulated in [13] in terms of the vertex operators of $U_q(sl_2)$. Let us recall the basic relations for type I vertex operators $\Phi_\varepsilon(\zeta)$ and $\Phi_\varepsilon^*(\zeta) = \Phi_{-\varepsilon}(-q^{-1}\zeta)$, $\varepsilon = \pm 1$, and the boundary ground state $|0\rangle_B$ of the XXZ Hamiltonian defined in [13]. For the definition of the $R$-matrix $R(\zeta)$ and the boundary reflection matrix $K(\zeta)$ we refer the reader to [13] (see (A.1) and (A.2), and (2.2) and (2.3) therein, and see also (A.11) for the scalar $g$ which appears below). The type I vertex operators commute as

$$\sum_{\varepsilon_1,\varepsilon_2} R^{\varepsilon_1,\varepsilon_2}_{\varepsilon_1,\varepsilon_2}(\zeta_1/\zeta_2) \Phi_{\varepsilon_1}(\zeta_1) \Phi_{\varepsilon_2}(\zeta_2) = \Phi_{\varepsilon_1}(\zeta_2) \Phi_{\varepsilon_1}(\zeta_1),$$

(2.1)

and have the inversion property

$$g \Phi_{\varepsilon_1}(\zeta) \Phi_{\varepsilon_2}^*(\zeta) = \delta_{\varepsilon_1 \varepsilon_2} \text{id}.$$  

(2.2)

The boundary ground state and its dual have the following reflection properties with respect to type I vertex operators:

$$\sum_{\varepsilon'} K^{\varepsilon}_{\varepsilon'}(\zeta) \Phi_{\varepsilon'}(\zeta) |0\rangle_B = \Phi_{\varepsilon}(\zeta^{-1})|0\rangle_B,$$

(2.3)

$$\sum_{\varepsilon'} B(0) \Phi_{\varepsilon'}^*(\zeta^{-1}) K^{\varepsilon}_{\varepsilon'}(\zeta) = B(0) \Phi_{\varepsilon}^*(\zeta).$$

(2.4)

The relation between these and the spin chain Hamiltonian will be discussed in the context of the XYZ model in Section 2.3. Here we note only that the (renormalized) XXZ Hamiltonian is given by $H = \text{const.} d T_B(\zeta)/d \zeta \big|_{\zeta=1}$, where the transfer matrix is defined by

$$T_B(\zeta) = g \sum_{\varepsilon,\varepsilon'} \Phi_{\varepsilon}^*(\zeta^{-1}) K^{\varepsilon}_{\varepsilon'}(\zeta) \Phi_{\varepsilon'}(\zeta).$$

(2.5)

The ground state $|0\rangle_B$ satisfies $T_B(\zeta)|0\rangle_B = |0\rangle_B$.

Consider the boundary $n$-point function,

$$G(\zeta_1, \cdots, \zeta_n) = \sum_{\varepsilon_1, \cdots, \varepsilon_n} B(0) \Phi_{\varepsilon_1}(\zeta_1) \cdots \Phi_{\varepsilon_n}(\zeta_n) |0\rangle_B v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_n},$$

(2.6)

where $v^\pm$ are the weight vectors in the 2-dimensional vector space $V = \mathbb{C} v^+ \oplus \mathbb{C} v^-$. Note that $R(\zeta) \in \text{End}_C(V \otimes V)$ and $K(\zeta) \in \text{End}_C(V)$. Equations (2.1), (2.3) and (2.4) imply

$$R_{\zeta_{j+1}/\zeta_j}(\zeta_1, \cdots, \zeta_j, \zeta_{j+1}, \cdots, \zeta_n) = P_{\zeta_{j+1}/\zeta_j} G(\zeta_1, \cdots, \zeta_j, \zeta_{j+1}, \cdots, \zeta_n),$$

$$K_n(\zeta_n) G(\zeta_1, \cdots, \zeta_n) = G(\zeta_1, \cdots, \zeta_n^{-1}),$$

$$\overline{K}_1(\zeta_1) G(\zeta_1^{-1}, \cdots, \zeta_n) = G(q^{-2}\zeta_1, \zeta_2, \cdots, \zeta_n).$$

Here,

$$P_{\zeta_{j+1}/\zeta_j}(v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_j+1} \otimes v^{\varepsilon_j} \otimes \cdots \otimes v^{\varepsilon_n}) = v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_j} \otimes v^{\varepsilon_j+1} \otimes \cdots \otimes v^{\varepsilon_n},$$

$$v^{\varepsilon_j+1} \otimes v^{\varepsilon_j}.$$
\[ R_{j+1}(\zeta)(v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_n}) = \sum_{\varepsilon_j, \varepsilon_{j+1}} R_{\varepsilon_j, \varepsilon_{j+1}}(\zeta)(v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_j} \otimes v^{\varepsilon_{j+1}} \cdots \otimes v^{\varepsilon_n}), \]

\[ K_j(\zeta)(v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_n}) = \sum_{\varepsilon_j} K_{\varepsilon_j}^{\varepsilon_j}(\zeta)(v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_j} \otimes \cdots \otimes v^{\varepsilon_n}), \]

\[ K_j(\zeta)(v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_n}) = \sum_{\varepsilon_j} K_{-\varepsilon_j}^{\varepsilon_j}(-q^{-1}\zeta)(v^{\varepsilon_1} \otimes \cdots \otimes v^{\varepsilon_j} \otimes \cdots \otimes v^{\varepsilon_n}). \]

From these equations we obtain

\[ G(\zeta_1, \cdots, q^{-2}\zeta_j, \cdots, \zeta_n) = R_{j-1}(\zeta_j/q^2\zeta_{j-1}) \cdots R_{j_1}(\zeta_j/q^2\zeta_1)K_j(\zeta_j) \times \]

\[ \times R_{1j}(\zeta_1\zeta_j) \cdots R_{j-1j}(\zeta_{j-1}\zeta_j)R_{j+1j}(\zeta_{j+1}\zeta_j) \cdots R_{n_j}(\zeta_n\zeta_j) \times \]

\[ \times K_j(\zeta_j)R_{jn}(\zeta_j/\zeta_n) \cdots R_{j+1}(\zeta_j/\zeta_{j+1})G(\zeta_1, \cdots, \zeta_n). \]

(2.7)

This is a version of Cherednik’s equation [14].

### 2.2 Consistency conditions

We have shown that the boundary \(n\)-point function \(G(\zeta_1, \cdots, \zeta_n)\) satisfies (2.7) if relations (2.1), (2.3) and (2.4) are satisfied, without using any further details of the XXZ model. Therefore, if we have more general settings for these relations, we get more general solutions to equation (2.7). We remark that these three relations along with (2.2) imply the following consistency conditions for \(R(\zeta)\) and \(K(\zeta)\):

(i) Yang–Baxter equation

\[ R_{12}(\zeta_1/\zeta_2)R_{13}(\zeta_1/\zeta_3)R_{23}(\zeta_2/\zeta_3) = R_{23}(\zeta_2/\zeta_3)R_{13}(\zeta_1/\zeta_3)R_{12}(\zeta_1/\zeta_2). \]

(2.8)

(ii) Unitarity relation

\[ R_{12}(\zeta_1/\zeta_2)R_{21}(\zeta_2/\zeta_1) = \text{id}. \]

(2.9)

(iii) Crossing relation

\[ R_{\varepsilon_1\varepsilon_2}^{\varepsilon_2 \varepsilon_1}(-q^{-1}\zeta^{-1}) = R_{\varepsilon_2 \varepsilon_1}^{\varepsilon_1 \varepsilon_2}(\zeta). \]

(2.10)

(iv) Boundary Yang–Baxter equation

\[ K_2(\zeta_2)R_{21}(\zeta_1\zeta_2)K_1(\zeta_1)R_{12}(\zeta_1/\zeta_2) = R_{21}(\zeta_1/\zeta_2)K_1(\zeta_1)R_{12}(\zeta_1\zeta_2)K_2(\zeta_2). \]

(2.11)

(v) Boundary unitarity relation

\[ K(\zeta)K(\zeta^{-1}) = \text{id}. \]

(2.12)

(vi) Boundary crossing relation

\[ K_0(\zeta) = \sum_{a, b} R_{b-a}^{b}(-q^{-1}\zeta^{-1})K_0^{b}(\zeta). \]

(2.13)

For completeness, we add a few remarks following the results in [13]. The properties of the vertex operators entails the following relations for (2.5): \([T_B(\zeta_1), T_B(\zeta_2)] = 0, T_B(1) = 1, T_B(\zeta)T_B(\zeta^{-1}) = 0\).
1. $T_B(-q^{-1} \zeta^{-1}) = T_B(\zeta)$. Therefore, if $t(\zeta)$ is an eigenvalue of $T_B(\zeta)$, we have $t(1) = 1$, $t(\zeta)t(\zeta^{-1}) = 1$, and $t(-q^{-1} \zeta^{-1}) = t(\zeta)$. A vector $|v\rangle$ (or $\langle v|$) satisfies $\sum_{\epsilon'} K^{\epsilon'}(\zeta) \Phi^{\epsilon'}(\zeta) |v\rangle = t(\zeta) \Phi_{\epsilon}(\zeta^{-1}) |v\rangle$ (or $\sum_{\epsilon'} \langle v| \Phi^{\epsilon'}_{\epsilon'}(\zeta^{-1}) K^{\epsilon'}(\zeta) = t(\zeta) \langle v| \Phi^{\epsilon'}_{\epsilon}(\zeta)$) if and only if it is an eigenvector of $T_B(\zeta)$ with the eigenvalue $t(\zeta)$. In this situation, the n-point function

$$G(\zeta_1, \ldots, \zeta_n) = \sum_{\epsilon_1, \ldots, \epsilon_n} \langle v| \Phi_{\epsilon_1}(\zeta_1) \cdots \Phi_{\epsilon_n}(\zeta_n) |v\rangle v^{\epsilon_1} \otimes \cdots \otimes v^{\epsilon_n}$$

(2.14) satisfies the same equation (2.7) because the effects of $t(\zeta)$ in $K(\zeta)$ and $K(\zeta)$ cancel. In particular, $B \langle 1| \Phi_{\epsilon_1}(\zeta_1) \cdots \Phi_{\epsilon_n}(\zeta_n) |1\rangle_B$ satisfies (2.7) (see 3.1 in [13]).

2.3 The XYZ model

In this section we consider (2.7) in the context of the XYZ model, extending the setting of Section 2.1. Unlike the XXZ case, we do not have a complete solution for the XYZ model, because we do not have the corresponding mathematical machinery, in particular, a bosonization scheme. Certain results can, however, be generalized. Following [15] we assume the existence of vertex operators. We further assume the existence of the boundary vacuum states, and derive (2.7) for the boundary n-point function. See, however, Section 3, where we give a physical argument that supports the existence of such a setting.

To define the model, we use the variables

$$p = e^{-\frac{K'}{2K}}, \quad -q = e^{-\frac{K}{2K}}, \quad \zeta = e^{\frac{Kn}{2K}}, \quad z = \zeta^2,$$  

(2.15)

where $K$, $K'$ are $I$, $I'$ in Baxter’s notation [16], and $\lambda$ and $u$ are as in (10.4.21) of [10]. The variable $p$ is the elliptic nome, and $p = 0$ corresponds to the XXZ case. We restrict our discussion to the principal regime, $0 < p^{1/2} < -q < \zeta^{-1} < 1$. Let

$$\text{snh}(u) = -\text{sn}(iu), \quad \text{chn}(u) = \text{cn}(iu), \quad \text{dnh}(u) = \text{dn}(iu),$$

$$\left(z; q_1, \ldots, q_m\right) = \prod_{n_1, \ldots, n_m = 0}^{\infty} (1 - q_1^{n_1} \cdots q_m^{n_m} z), \quad \Theta_q(z) = (z; q)\infty (qz^{-1}; q)\infty (q; q)\infty.$$

The elliptic functions $\text{sn}(u)$ etc., are found in the appendix of [16]. Unless otherwise stated, the elliptic nome for these functions is $p$, and the corresponding modulus and conjugate modulus are denoted by $k$ and $k'$. In (2.28) etc., we use elliptic functions with elliptic nome

$$-q = e^{-\frac{K'}{2K}},$$

(2.16)

modulus $k_1$ and conjugate modulus $k'_1$.

We first give solutions $R$ and $K$ to the consistency conditions (i)–(vi). The solution $R(\zeta)$ to equations (i)–(iii) for the XYZ model is

$$R(\zeta) = \begin{pmatrix}
a(\zeta) & d(\zeta) \\
b(\zeta) & c(\zeta) \\
c(\zeta) & b(\zeta) \\
d(\zeta) & a(\zeta)
\end{pmatrix},$$

(2.17)
where the Boltzmann weights are

\[
a(\zeta) = \frac{1}{\mu(\zeta)} \frac{\text{snh}(\lambda - u)}{\text{snh}(\lambda)}, \quad b(\zeta) = \frac{1}{\mu(\zeta)} \frac{\text{snh}(u)}{\text{snh}(\lambda)}, \quad c(\zeta) = \frac{1}{\mu(\zeta)} \frac{1}{\text{snh}(\lambda)}, \quad d(\zeta) = \frac{k}{\mu(\zeta)} \frac{\text{snh}(\lambda - u)\text{snh}(u)}{\text{snh}(\lambda)},
\]

with

\[
\frac{1}{\mu(\zeta)} = \frac{1}{\pi(\zeta)} \frac{(p^2; p^2)^2_\infty}{\Theta_{p^2}(q^2_{\infty}) \Theta_{p^2}(p z)}.
\]

(2.18)

Usually, will not denote the \( p \) and \( q \) dependence explicitly in cases when no confusion should arise as a result of their absence.

The general 3-parameter family of solutions \( K(\zeta) \) to (2.11) was given in [11] and [12]. In the following, we restrict to the diagonal solution \( K(\zeta) \) with a single parameter

\[
r = e^{\frac{2\eta}{\pi}}.
\]

(2.19)

Let

\[
K(\zeta) = \frac{1}{f(\zeta; r)} \hat{K}(\zeta; r), \quad \hat{K}(\zeta; r) = \left( \frac{\text{snh}(\eta + u)}{\text{snh}(\eta - u)} \right),
\]

(2.20)

where

\[
\frac{1}{f(\zeta; r)} = \frac{\varphi(z; r)}{\varphi(z^{-1}; r)}.
\]

\[
\varphi(z; r) = \frac{(pr z^2; q^4_\infty)(pq^2 r^{-1} z; p^2, q^4_\infty)(rq^4 z; p^2, q^4_\infty)(p r^{-1} q^4 z; p^2, q^4_\infty)}{(pr^{-1} q^2 z; p^2, q^4_\infty)(pq^{-1} q^2 z; p^2, q^4_\infty)(rq^{-1} q^2 z; p^2, q^4_\infty)(p r^{-1} q^4 z; p^2, q^4_\infty)}
\]

\[
\times \frac{(q^8 z^2; p^2, q^8_\infty)(pq^8 z^2; p^2, q^8_\infty)(pq^2 q^2 z; p^2, q^8_\infty)(p^2 q^8 z^2; p^2, q^8_\infty)}{(q^8 z^2; q^8_\infty)(pq^8 z^2; q^8_\infty)(pq^2 q^2 z; q^8_\infty)(p^2 q^8 z^2; q^8_\infty)}.
\]

The scalar factor \( f(\zeta; r) \) is chosen so that (2.12), (2.13) hold, and \( K^- (\zeta) \) is analytic in the region \(-q < |r|^{1/2} < \zeta^{-1}\). In terms of the additive parameters defined in (2.13) and (2.19), the region we consider is given by \( 0 < u < -\eta < \lambda < K' \).

Now let us recall the vertex operators defined for the elliptic algebra corresponding to the \( R \)-matrix (2.17) in [13]. The type I vertex operator

\[
\Phi_\varepsilon (\zeta) : \mathcal{H}^{(i)} \to \mathcal{H}^{(1-i)}
\]

acts on graded vector spaces

\[
\mathcal{H}^{(i)} = \bigoplus_{d=0}^\infty \mathcal{H}^{(i)}_d \quad (i = 0, 1)
\]

with character

\[
\sum_{d=0}^\infty \dim \mathcal{H}^{(i)}_d t^d = \frac{1}{\prod_{n=0}^\infty (1 - t^{2n+1})}.
\]

\[1\text{In [17], } \Phi_\varepsilon (\zeta) \text{ was introduced satisfying (2.1), and (2.2) with a different normalization, i.e., } g = 1. \text{ In this paper we follow the normalization given in [13].}\]
The vertex operator is expanded in the form

$$\Phi_\varepsilon(\zeta) = \sum_{n \in \mathbb{Z}} \Phi_{\varepsilon,n} \zeta^{-n} = (-1)^{i+(1+i)/2} \Phi_{\varepsilon}(-\zeta),$$

where $\Phi_{\varepsilon,n} : \mathcal{H}_d^{(i)} \rightarrow \mathcal{H}_d^{(i+1)}$. The normalizations are such that $\Phi_{-,0}|0\rangle = |1\rangle$, and $\Phi_{+,0}|1\rangle = |0\rangle$, where $\mathcal{H}_d^{(i)} = \mathbb{C}|i\rangle$. In this normalization, we find that Eq. (2.17) holds with the elliptic $R$-matrix (2.17), and Eq. (2.2) holds with

$$g = \frac{(pq^2;p)_\infty(pq^6;p,q^4)_\infty(pq^2;p,q^4)_\infty(q^2;q^4)_\infty}{(p;p)_\infty(pq^4;p,q^4)^2(q^4;q^4)_\infty}. \quad (2.21)$$

We now turn to the boundary XYZ hamiltonian. In analogy with the XXZ case, we now assume that there exists a vacuum vector $|0\rangle_B$ (resp. $B|0\rangle$) in an appropriate completion of $\oplus_{d,\text{even}} \mathcal{H}_d^{(0)}$, (resp. $\oplus_{d,\text{even}} \mathcal{H}_d^{(0)*}$), which satisfies (2.3) (resp. (2.4)). In Section 3, we give a graphical argument for the construction of these states. However, since we do not have a bosonization formula for the vertex operators, we cannot give an explicit bosonic construction of this state, as we did in [13] for the XXZ Hamiltonian.

In the XXZ model, we found that there exists another vacuum state, the boundary bound state $|1\rangle_B$, with an excitation energy

$$\Lambda(\zeta;r) = \frac{K^e_0(\zeta;r)}{K^{-e}_0(\zeta;r)} = \frac{1}{\zeta^2} \frac{\Theta_{q^4}(rq^2)\Theta_{q^4}(q^2r\zeta^{-2})}{\Theta_{q^4}(r\zeta^{-2})\Theta_{q^4}(q^2r\zeta^{-2})}. \quad (2.22)$$

It turns out that this formula remains valid without any change for the elliptic $K(\zeta)$ given by (2.20). Therefore, for $|r| \geq |q|$, we assume that there exists a vector $|1\rangle_B$ (resp. $B|1\rangle$) in an appropriate completion of $\oplus_{d,\text{even}} \mathcal{H}_d^{(1)}$, (resp. $\oplus_{d,\text{even}} \mathcal{H}_d^{(1)*}$), which satisfies

$$\sum_{e'} K^e_{e'}(\zeta) \Phi_{e'}(\zeta)|1\rangle_B = \Lambda(\zeta;r) \Phi_{e}(\zeta^{-1})|1\rangle_B, \quad (2.23)$$

$$\sum_{e'} B\langle 1| \Phi^*_e(\zeta^{-1})K^e_{e'}(\zeta) = \Lambda(\zeta;r)B\langle 1| \Phi^*_e(\zeta). \quad (2.24)$$

Under these assumptions, we can formulate the boundary XYZ model in an analogous way to the boundary XXZ model. In particular, the correlation function

$$\sum_{e_1,\ldots, e_n} B\langle i| \Phi_{e_1}(\zeta_1) \cdots \Phi_{e_n}(\zeta_n)|i\rangle_B \nu^{e_1} \otimes \cdots \otimes \nu^{e_n}$$

with elliptic vertex operators satisfies the $q$-difference equation (2.7).

We define the transfer matrix of the boundary theory as in (2.7) with the appropriate elliptic generalizations of $g$, $K$ and $\Phi_{e}(\zeta)$. The renormalized form of the Hamiltonian (1.1) is defined from this transfer matrix by

$$H^\text{renor}_B = -\frac{\pi \snh(\lambda, k)}{4K} \frac{d}{d\zeta} T_B(\zeta) \bigg|_{\zeta=1}. \quad (2.25)$$

We find that the parameters in equation (1.1) are related to the elliptic parameters above by

$$\Gamma = k \snh^2(\lambda, k), \quad \Delta = -\cnh(\lambda, k) \dnh(\lambda, k), \quad h = -\snh(\lambda, k) \frac{\cnh(\eta, k) \dnh(\eta, k)}{2\snh(\eta, k)}. \quad (2.26)$$
This Hamiltonian is normalized so that its lowest eigenvalue is 0. From (2.22) and (2.23), the eigenvalues \( e^{(i)}(r) \quad (i = 0, 1) \) of \( H_B \) on \( |i\>_B \) are

\[
e^{(0)}(r) = 0, \tag{2.27}
\]

\[
e^{(1)}(r) = \begin{cases} 
\epsilon(1)\text{sn}(-2I'\eta + iK)/\lambda, k'_f & \text{if } -1 < r < -|q|; \\
\epsilon(1)k'_f\text{sn}(-2I'\eta/\lambda, k'_f) & \text{if } |q| < r < 1.
\end{cases} \tag{2.28}
\]

where we have defined the following function

\[
\epsilon(\xi) = \frac{I}{K}\text{snh}(\lambda, k)\text{dn}(2I\pi \theta, k_I), \quad \xi = -ie^{i\theta},
\]

\[
\epsilon(1) = \frac{Ik'_f}{K}\text{snh}(\lambda, k).
\]

The reader should be careful about the name of the elliptic functions. (See (2.16).) The function \( \epsilon(\xi) \) differs from the one in the XXZ chain [13] due to the difference of the normalization factor in front of the derivative of \( T_B(\zeta) \) in (2.25). It reduces to the XXZ excitation energy when \( p = 0 \).

The excited states are created by the action of type II vertex operators \( \Psi^*_\mu(\xi)(\mu = \pm) \) [15] on the vacuum vectors \( |i\>_B \):

\[
\Psi_{\mu_m}(\xi_m) \cdots \Psi_{\mu_1}(\xi_1)|i\>_B. \tag{2.29}
\]

The energy spectrum of the one particle excitation is evaluated as \( e^{(i)}(r) + \epsilon(\xi) \). The excitation is therefore massive with mass \( \epsilon(1) \) [18].

3 The Corner Transfer Matrix

In this section, we show how within the context of an inhomogeneous boundary vertex model it is possible to construct lattice realizations of \( \Phi_\epsilon(\zeta), \Phi^*_\epsilon(\zeta), |0\>_B \) and \( B|0\> \) that obey relations (2.1), (2.3) and (2.4). We then present a rather physical picture of the origin of the difference equations for correlation functions of local operators in this vertex model.

3.1 The Partition Function

Following Sklyanin [3], we build a lattice with \( 2M \) vertical and \( 2N \) horizontal lines as in Figure 1. Here, in the vertical direction we impose the cyclic boundary condition. Let us call this lattice \( L_{MN} \). Let the horizontal lines carry alternating spectral parameters \( \zeta \) and \( 1/\zeta \), and the vertical lines \( \xi \). We consider the region \( 0 < p^{1/2} < -q < |r|^{1/2} < |\zeta/\xi|^{-1}, |\zeta|^{-1} < 1 \). We use the word ground state to refer to the lowest energy states in the \( h = 0, \Gamma = 0, \Delta \rightarrow -\infty \) limit. There are two such ground states, one of which is shown in Figure 2. Note that if \( h \neq 0 \) then the degeneracy of the ground state energies is resolved, and only one of the ground states corresponds to the lowest energy. Nevertheless, we call both of them ground states for all values of \( h \). Corresponding to these two ground states, we have two antiferromagnetic sectors. Let us label these sectors by \( i = 0, 1 \). When we consider the partition function and correlation functions, we choose one of the ground
Figure 1: The inhomogeneous $2M \times 2N$ lattice $\mathcal{L}_{MN}$. The circles on the right hand side denote reflection matrices $K(\zeta)$, and those on the left hand side denote $K(-q^{-1}\zeta^{-1})^t$.

Figure 2: The $i = 0$ ground state
state sectors, and take configuration sums over such states that are different from the ground state at finitely many edges.

Let us restrict to the region $h \geq 0$. We conjecture that in the limit $M,N \to \infty$ the partition function $Z_{MN}^{(i)}$ of this lattice behaves as

$$\log Z_{MN}^{(i)} \sim MN \left( \log \mu^{(i)}(\zeta/\xi) + \log \mu^{(i)}(\xi/\zeta) \right) + N \left( \log \nu^{(i)}(\zeta) + \log \nu^{(i)}(-q^{-1}\zeta^{-1}) \right).$$

Here $\mu^{(i)}$ is the partition function per site in the bulk theory, and $\nu^{(i)}$ is the partition function per boundary site, which in the present normalization are given by

$$\mu^{(i)}(\zeta) = 1 \quad \text{for } i = 0, 1,$$

$$\nu^{(0)}(\zeta) = 1, \quad \nu^{(1)}(\zeta) = \Lambda(\zeta; r).$$

These conjectures have been suggested by an argument similar to the inversion trick in the bulk theory, on the basis of the properties $2.8$–$2.13$ and $2.22$ for $R$ and $K$ matrices. Here we will not discuss the details.

Now consider the lattice shown in Figure 3. We divide this lattice into the four sections indicated by the dotted lines. We denote the NW and SW corner transfer matrices (CTM’s), by $A_{NW}^{(i)}(\zeta, \xi)$ and $A_{SW}^{(i)}(\zeta, \xi)$, and the upper and lower lines of $K(\zeta)$ matrices by $|B\rangle^{(i)}$ and $\langle B|$ as maps between, or elements of, the vectors spaces $|\bar{H}\rangle^{(i)}$ and $|\bar{H}\rangle^{(i)}$ (and their duals $|H\rangle^{(i)}$ and $|H\rangle^{(i)}$). These vector spaces are...
defined as the spans of the half-infinite pure tensor vectors

\[ \cdots \otimes v_p(3) \otimes v_p(2) \otimes v_p(1), \quad \text{with} \quad p(j) = (-)^{j+i}, \quad j \gg 1, \quad \text{for} \quad H(i), \]

\[ p(j) = (-)^{1+i}, \quad j \gg 1, \quad \text{for} \quad \tilde{H}(i). \]

The identification is that

\[ A_{NW}^{(i)}(\zeta, \xi) : \tilde{H}(i) \rightarrow H(i), \]
\[ A_{SW}^{(i)}(\zeta, \xi) : H(i) \rightarrow \tilde{H}(i), \]
\[ (i|B) \in \tilde{H}^{(i)}, \]
\[ (i)\langle B | \in H^{(i)}. \]

The partition function of the lattice is then given by

\[ Z^{(i)}(\zeta, \xi) = (i|B|A_{SW}^{(i)}(\zeta, \xi)A_{NW}^{(i)}(\zeta, \xi)|B) \]

### 3.2 Vertex Operators

We define ‘vertex operators’ \( \phi_{\epsilon}^{(1-i,i)}(\xi), \phi_{\epsilon}^{(1-i,i)}(\xi), \phi_{\epsilon}^{U,(1-i,i)}(\xi, \xi), \) and \( \phi_{\epsilon}^{D,(1-i,i)}(\zeta, \xi) \) by the half-infinite lattice insertions shown in Figure 4. The superscripts \((1-i,i)\) indicate how the vertex operators act on \( H(i) \) and \( \tilde{H}(i) \). Namely,

\[ \phi_{\epsilon}^{(1-i,i)}(\xi) : H(i) \rightarrow H(1-i) \]
\[ \phi_{\epsilon}^{U,(1-i,i)}(\zeta, \xi) : \tilde{H}(i) \rightarrow \tilde{H}(1-i) \]
\[ \phi_{\epsilon}^{D,(1-i,i)}(\zeta, \xi) : \tilde{H}(i) \rightarrow \tilde{H}(1-i). \]

Henceforth these superscripts will be suppressed.

We now argue that the vertex operators (3.3) and CTM’s obey a set of relations that will allow us to use them to construct lattice realizations of \( \Phi_{\epsilon}(\zeta), \Phi^{*}_{\epsilon}(\zeta), |0\rangle_{B} \) and \( B|0\rangle \) obeying relations (2.1), (2.3) and (2.4). These relations are

\[ \phi_{\epsilon}(\xi) = \phi_{\epsilon}^{*}(-q\xi), \]
\[ \sum_{\xi_1, \xi_2} R_{\xi_1, \xi_2}^{\epsilon, \epsilon}(\xi_1/\xi_2)\phi_{\epsilon_1}(\xi_1)\phi_{\epsilon_2}(\xi_2) = \phi_{\epsilon_2}(\xi_2)\phi_{\epsilon_1}(\xi_1), \]
\[ \sum_{\epsilon'} K_{\epsilon'}^{\epsilon}(\xi')\phi_{\epsilon'}^{U}(\zeta, \xi')|B) \]
\[ = \nu^{(i)}(\epsilon')\phi_{\epsilon}^{U}(\zeta, 1/\xi')|B), \]
\[ \sum_{\epsilon'} (i)|B)\phi_{\epsilon'}^{D}(\zeta, 1/\epsilon')K_{\epsilon'}^{\epsilon}(\epsilon') = \nu^{(i)}(\epsilon')|B)\phi_{\epsilon}^{D}(\zeta, \epsilon'), \]
\[ A_{NW}^{(i)}(\zeta, \xi)\phi_{\epsilon}^{U}(\zeta, \xi') = \phi_{\epsilon}(\xi')A_{NW}^{(1-i)}(\zeta, \xi), \]
\[ A_{SW}^{(i)}(\zeta, \xi)\phi_{\epsilon}^{U}(\xi'/\xi) = \phi_{\epsilon}^{D}(\zeta, \epsilon')A_{SW}^{(1-i)}(\zeta, \xi), \]

where \( \nu^{(i)}(\epsilon') \) are given in (3.2). Properties (3.4) and (3.5) are consequences of the crossing symmetry of the matrix and the Yang-Baxter equation (in the infinite lattice limit) respectively. They are discussed in reference [17]. The point in the argument for (3.3) is as follows. If we compare the graphical expressions for both sides of (3.3), the left hand side contains one more site (or vertex),
Figure 4: The graphical definition of vertex operators

Figure 5: The boundary reflection property
Figure 6: The homogeneous CTM $A_{NW}^{(i)}(\zeta_1/\zeta_2)$

i.e., $R_{\zeta_1:\zeta_2}(\xi_1/\xi_2)$, in the formula, than the right hand site. The effect of the additional site is to multiply by the partition function per site $\mu^{(i)}$. Since we normalized $R$ so that $\mu^{(i)} = 1$, we have (3.5). Equation (3.6) follows from a similar graphical argument. See Figure 5. We use the boundary Yang Baxter equation in order to move the $K(\xi')$ matrix up the boundary to infinity. This time we get the factor $\nu^{(i)}$. Equation (3.7) follows by a similar argument applied to the lower boundary. The argument leading to (3.8) requires the introduction of the CTM of the homogeneous lattice [17] denoted by $A_{NW}^{(i)}(\xi)$ with one less argument. This is the CTM of the same vertex model defined on the homogeneous lattice of Figure 6. In [17] a simple graphical argument is given that leads to the equality

$$A_{NW}^{(i)}(\xi)\phi_{\xi}(\xi) = \phi_{\xi}(1)A_{NW}^{(1-i)}(\xi).$$

(3.10)

This argument extends without change to the inhomogeneous lattice discussed here, giving

$$A_{NW}^{(i)}(\zeta,\xi)\phi_{\xi}(\zeta,\xi) = \phi_{\xi}(1)A_{NW}^{(1-i)}(\zeta,\xi).$$

(3.11)

Baxter has shown that the asymptotic (i.e. infinite lattice) limit of the CTM of the homogeneous 8-vertex model is given by $A_{NW}^{(i)}(\xi) = f(\xi)D^{(i)}$, where $D^{(i)}$ is the CTM Hamiltonian (which is independent of $\xi$) and $f(\xi)$ is a scalar function. It is then a consequence of (3.10) that

$$A_{NW}^{(i)}(\xi)\phi_{\xi}(\xi') = \phi_{\xi}(\xi'/\xi)A_{NW}^{(1-i)}(\xi).$$

(3.12)

Finally, given (3.12) and (3.11), property (3.8) follows if

$$A_{NW}^{(i)}(\xi)A_{NW}^{(i)}(\zeta,\xi') = g(\zeta,\xi,\xi')A_{NW}^{(i)}(\zeta,\xi\xi'),$$

(3.13)

where $g(\zeta,\xi,\xi')$ is some scalar function. In Appendix A we present a derivation of equation (3.13) using a generalization of Baxter’s argument [16] for the asymptotic behavior of the CTM of the
homogeneous 8-vertex model. If we apply $A_{NW}^{(i)}(\zeta, \xi)$ to both side of (3.2), and then use (3.8), we obtain
\[
\sum_{\epsilon'}^{} K_{\epsilon'}^{(i)}(\xi') \phi_{\epsilon'}(\xi') A_{NW}^{(i)}(\zeta, \xi)|B\rangle^{(i)} = \nu^{(i)}(\xi') \phi_{\epsilon}(\frac{1}{\xi'} \zeta) A_{NW}^{(i)}(\zeta, \xi)|B\rangle^{(i)}.
\]
Similarly,
\[
\sum_{\epsilon'}^{(i)} \langle B|A_{SW}^{(i)}(\zeta, \xi) \phi_{\epsilon'}^*(\xi') \frac{1}{\xi'} K_{\epsilon'}^{(i)}(\xi') = \nu^{(i)}(\xi') \langle B|A_{SW}^{(i)}(\zeta, \xi) \phi_{\epsilon'}^*(\xi') / \xi' \rangle.
\]
Specializing to the case $\xi = 1$, and making the identifications
\[
\Phi_{\epsilon}(\zeta) \sim \phi_{\epsilon}(\zeta), \quad \Phi_{\epsilon}^*(\zeta) \sim \phi_{\epsilon}^*(\zeta),
\]
we obtain equations (2.3) and (2.4) of Section 2. The remaining property (2.1) comes from (3.5).

Thus, the vertex operators $\phi_{\epsilon}(\zeta)$ and $\phi_{\epsilon}^*(\zeta)$, and the boundary states $A_{NW}^{(0)}(\zeta, 1)|B\rangle^{(0)}$ and $(0)\langle B|A_{SW}^{(0)}(\zeta, 1)$ are lattice realizations of the corresponding objects discussed in section 2. We speculate that the states $|B\rangle^{(0)}$ and $(0)\langle B|$ will correspond to initial and final times states in the sense of [10] but now in an axial quantization scheme.

3.3 Correlation functions and difference equations

As in [14], correlation functions of local operators of the inhomogeneous vertex model are defined in terms of
\[
\langle B|\phi_{\epsilon_1}^D(\zeta, \xi) \cdots \phi_{\epsilon_N}^D(\zeta, \xi) A_{SW}^{(i+1)}(\zeta, \xi) A_{NW}^{(i+1)}(\zeta, \xi) \phi_{\epsilon_N}^U(\zeta, \xi) \cdots \phi_{\epsilon_1}^U(\zeta, \xi)|B\rangle^{(i)},
\]
where the superscripts on the CTM’s are understood as modulo 2. From properties (3.8) and (3.9) we see that it is possible to move all the vertex operators $\phi_{\epsilon_i}^U(\zeta, \xi)$ and $\phi_{\epsilon_i}^D(\zeta, \xi)$ inside of the $A_{SW}^{(i)}(\zeta, \xi) A_{NW}^{(i)}(\zeta, \xi)$ product. We can thus consider the more general spectral parameter dependent expression
\[
G^{(i)}(\zeta, \xi_1, \cdots, \xi_N)_{\epsilon_1, \cdots, \epsilon_N} = \langle B|A_{SW}^{(i)}(\zeta, \xi) \phi_{\epsilon_1}(\xi_1) \cdots \phi_{\epsilon_N}(\xi_N) A_{NW}^{(i)}(\zeta, \xi)|B\rangle^{(i)}, \text{ with } N \text{ even.}
\]
From (3.14) $G^{(0)}(\zeta, 1 \xi_1, \cdots, \xi_N)_{\epsilon_1, \cdots, \epsilon_N}$ is the lattice realization of the corresponding correlation function $G(\xi_1, \cdots, \xi_N)_{\epsilon_1, \cdots, \epsilon_N}$ of equation (2.6). $G^{(0)}(\zeta, 1 \xi_1, \cdots, \xi_N)_{\epsilon_1, \cdots, \epsilon_N}$ thus obeys the difference equation (2.7) from the arguments already given in section 2. Here we shall attempt to give some insight into the physical origin of this difference equation by pointing out the sequence of steps necessary in order to derive it within the context of the lattice theory. For simplicity, we drop the $\epsilon$ subscripts. At each step we refer to the vertex operator that arises as a result of the previous step.

1. Move $\phi(\xi_j)$ to the right of $\phi(\xi_{j+1}) \cdots \phi(\xi_N)$ using the exchange relation (3.3).
2. Move $\phi(\xi_j)$ to the right of the CTM $A_{NW}(\zeta, 1)$ using the commutation property (3.8).
3 ‘Reflect’ \( \phi^U(\zeta, \xi_j) \) off the upper boundary using (3.6).

4 Move \( \phi^U(\zeta, 1/\xi_j) \) back to the left of the CTM \( A_{NW}(\zeta, 1) \) using (3.8).

5 Move \( \phi(1/\xi_j) \) to the left of \( \phi(\xi_1) \cdots \phi(\xi_{j-1}) \phi(\xi_{j+1}) \cdots \phi(\xi_N) \) using (3.5).

6 ‘Cross’ \( \phi(1/\xi_j) \) using the crossing relation (3.4).

7 Move \( \phi^*(-q/\xi_j) \) to the left of \( A_{SW}(\zeta, 1) \) using the commutation property (3.9).

8 Reflect \( \phi^D(\zeta, -q/\xi_j) \) off the lower boundary using (3.7).

9 Move \( \phi^D(\zeta, -\xi_j/q) \) to the right of \( A_{SW}(\zeta, 1) \) using (3.9).

10 Cross \( \phi^*(-\xi_j/q) \) using (3.4).

11 Move \( \phi(\xi_j/q^2) \) to the right of \( \phi(\xi_1) \cdots \phi(\xi_{j-1}) \) using (3.5).

This sequence of steps leads to the difference equation (2.7). One can of course derive the same equation by moving \( \phi(q^2-2\xi_j) \) to the left of \( \phi(\xi_1) \cdots \phi(\xi_{j-1}) \), through \( A_{SW}(\zeta, 1) \), reflecting off the lower boundary, etc.

4 Boundary spontaneous magnetization

In this section, we investigate the difference equation obeyed by the two point function of the XYZ spin chain. The solution of this equation, with a certain specialization of the spectral parameters, gives the boundary spontaneous magnetization. For general values of the boundary magnetic field, the solution does not factorize, but in the case of free boundary conditions, we find a factorized solution. As we show below, the value of the boundary spontaneous magnetization is (minus) the square of the spontaneous staggered polarization of the usual eight-vertex model without boundary.

4.1 Two point functions

Setting \( n = 2 \) in the elliptic version of (2.14), we have

\[
G(q^{-2}\zeta_1, \zeta_2) = K_1(\zeta_1)R_{21}(\zeta_1, \zeta_2)K_1(\zeta_1)R_{12}(\zeta_1, \zeta_2)G(\zeta_1, \zeta_2), \quad (4.1)
\]

\[
G(\zeta_1, q^{-2}\zeta_2) = R_{21}(q^{-2}\zeta_2/\zeta_1)K_2(\zeta_2)R_{12}(\zeta_1, \zeta_2)K_2(\zeta_2)G(\zeta_1, \zeta_2), \quad (4.2)
\]

where

\[
G(\zeta_1, \zeta_2) = \sum_{\varepsilon_1, \varepsilon_2} G_{\varepsilon_1, \varepsilon_2}(\zeta_1, \zeta_2)v^{\varepsilon_1} \otimes v^{\varepsilon_2},
\]

\[
G_{\varepsilon_1, \varepsilon_2}(\zeta_1, \zeta_2) = B\langle i|\Phi_{\varepsilon_1}(\zeta_1)\Phi_{\varepsilon_2}(\zeta_2)|i\rangle_B.
\]

In the following paragraph, we restrict our attention to the free boundary condition, \( r = -1 \), or \( h = 0 \). The reflection matrices \( K \) and \( \overline{K} \) in this limit are

\[
K(\zeta) = \frac{1}{f(\zeta)}I, \quad \overline{K}(\zeta) = \frac{1}{f(-q^{-1}\zeta)}I.
\]

Here \( I \) denotes the 2\times2 unit matrix and \( f(\zeta) = f(\zeta; -1) \).
Let
\[ G_{\pm}(\zeta_1, \zeta_2) = G_+(\zeta_1, \zeta_2) = G_-(\zeta_1, \zeta_2). \] (4.3)

One can reduce equations (4.1), (4.2) to the following scalar difference equations:
\[ G_\varepsilon(q^{-2}\zeta_1, \zeta_2) = \frac{\nu(\varepsilon\zeta_1/\zeta_2)}{f(\zeta_1)} G_\varepsilon(\zeta_1, \zeta_2), \]
\[ G_\varepsilon(\zeta_1, q^{-2}\zeta_2) = \frac{\nu(\varepsilon q^{-2}\zeta_2/\zeta_1)}{f(\zeta_2)} G_\varepsilon(\zeta_1, \zeta_2), \]

where
\[ \nu(\zeta) = \frac{1}{\pi(z)} \frac{(-q^{-1}\zeta; p)_{\infty} (-pq^{-1}q^{-1}; p)_{\infty}}{(-q^{-1}\zeta; p)_{\infty} (-pq^{-1}q^{-1}; p)_{\infty}}, \]
and \[ \pi(z) \] is defined in (2.18). We solve these equations by setting
\[ G_\varepsilon(\zeta_1, \zeta_2) = A(\zeta_1) A(\zeta_2) B_\varepsilon(\zeta_1 \zeta_2) B_\varepsilon(\zeta_1/\zeta_2). \]

The problem is now reduced to the following equations:
\[ \frac{A(q^{-2}\zeta)}{A(\zeta)} = \frac{1}{f(\zeta)f(-q^{-1}\zeta)}, \]
\[ \frac{B_\varepsilon(q^{-2}\zeta)}{B_\varepsilon(\zeta)} = \varepsilon \nu(\varepsilon\zeta). \]

The solutions which are analytic in the region \(-q < |\zeta| < q^{-2}\) are given by
\[ A(\zeta) = \frac{(q^2 z^{-1}; p, q^4)_{\infty} (q^6 z^{-2}; p, q^4, q^8)_{\infty} (pq^2 z^{-2}; p, q^4, q^8)_{\infty}}{(pz^{-1}; p, q^4)_{\infty} (q^4 z^{-2}; p, q^4, q^8)_{\infty} (pq^4 z^{-2}; p, q^4, q^8)_{\infty}} \]
\[ \times \frac{(q^4 z; p, q^4)_{\infty} (q^{10} z^2; p, q^4, q^8)_{\infty} (pq^6 z^2; p, q^4, q^8)_{\infty}}{(pq^2 z; p, q^4)_{\infty} (q^8 z^2; p, q^4, q^8)_{\infty} (pq^8 z^2; p, q^4, q^8)_{\infty}}, \]
\[ B(\zeta) = \frac{(q^{-1} \zeta^{-1}; p, q^2)_{\infty} (q^2 z^{-1}; p, q^4, q^4)_{\infty} (pq^2 z^{-1}; p, q^4, q^4)_{\infty}}{(q^{-1} \zeta^{-1}; p, q^2)_{\infty} (q^4 z^{-1}; p, q^4, q^4)_{\infty} (pq^4 z^{-1}; p, q^4, q^4)_{\infty}} \]
\[ \times \frac{(q^4 z; p, q^4)_{\infty} (q^8 z^2; p, q^4, q^4)_{\infty} (pq^8 z^2; p, q^4, q^4)_{\infty}}{(q^2 z; p, q^4)_{\infty} (q^4 z^2; p, q^4, q^4)_{\infty} (pq^4 z^2; p, q^4, q^4)_{\infty}}, \]
\[ B(\zeta) = B(\zeta^{-1}), \]
where \( z = \zeta^2. \)

### 4.2 Spontaneous magnetization

The boundary magnetization is the vacuum expectation value of the boundary spin operator \( \sigma_i^z \).

As in the XXZ chain, it is given by the value at \( \zeta = 1 \) of the following function:
\[ \mathcal{M}^{(0)}(\zeta; r) = \frac{\langle B| \hat{E}_z^{(0)}(\zeta, \zeta) - \hat{E}_z^{(0)}(-\zeta, -\zeta) \rangle_0 |B \rangle}{\langle B|0 \rangle_B}, \]

where
\[ \hat{E}_z^{(i)}(\zeta_1, \zeta_2) = \Phi^{(i,1-i)}(\zeta_1) \Phi^{(1-i, i)}(\zeta_2). \]
In terms of the functions $G_\varepsilon(\zeta_1, \zeta_2)$ of \(4.3\),
\[
\mathcal{M}^{(0)}(\zeta; r) = -\frac{G_\varepsilon(-q^{-1}\zeta, \zeta)}{G_\varepsilon(-q^{-1}\zeta, \zeta)}.
\tag{4.6}
\]
Using the result in the last section for the case $r = -1$, we obtain
\[
\mathcal{M}^{(0)}(\zeta; -1) = -\frac{(-pz; p, q^2)_{\infty}(-pz^{-1}; p, q^2)_{\infty}(-p; p, q^2)_{\infty}^2}{(pz; p, q^2)_{\infty}(pz^{-1}; p, q^2)_{\infty}(p; p, q^2)_{\infty}^2}
\times \frac{(q^2 z; p, q^2)_{\infty}(q^2 z^{-1}; p, q^2)_{\infty}(q^2; p, q^2)_{\infty}^2}{(-q^2 z; p, q^2)_{\infty}(-q^2 z^{-1}; p, q^2)_{\infty}(-q^2; p, q^2)_{\infty}^2}.
\]
The value at $\zeta = 1$ gives the spontaneous magnetization of the boundary spin operator:
\[
\mathcal{M}^{(0)}(1; -1) = -\frac{(q^2; q^2)_{\infty}^4(-p; p)_{\infty}^4}{(-q^2; q^2)_{\infty}^2(p; p)_{\infty}^2}.
\tag{4.7}
\]
The XXZ limit $p \to 0$ coincides with the previous result of \[13\]. We again note the remarkable fact that the spontaneous magnetization $-\mathcal{M}^{(0)}(1; -1)$ is exactly the square of the corresponding bulk quantity, i.e. the spontaneous staggered polarization in the eight-vertex model \cite{19,17}. The same phenomenon was observed in the case of the XXZ chain in \[13\].

One can check formula (4.7) by comparing it with the derivative with respect to the external magnetic field $h$ of the energy difference $\Delta e(r) = e^{(1)}(r) - e^{(0)}(r)$. From (2.28), we obtain
\[
\frac{\partial \Delta e(r)}{\partial h} = 2 \frac{(q^2; q^2)_{\infty}^4(r^2 q^2; q^4)_{\infty}(r^{-2} q^2; q^4)_{\infty}^2 (p; p^2)_{\infty}^2 (r p; p^2)_{\infty}^2 (r^{-1} p; p)_{\infty}^2}{(q^2; q^2)_{\infty}^2(q^2 r; q^2)_{\infty}^2(q^2 r^{-1}; q^2)_{\infty}^2(p; p^2)_{\infty}^2 (r^{-2} p; p^2)_{\infty}^2 (r^{-2} p^{-1}; p^2)_{\infty}^2}.
\]
This quantity is equal to the difference of the magnetizations $\mathcal{M}^{(1)}(1; r) - \mathcal{M}^{(0)}(1; r)$, which agrees, at $r = -1$, with the result (4.7).

One can also verify the result for $\mathcal{M}^{(0)}(1; -1)$ directly using perturbation theory in $\varepsilon = -q/(1 + q^2)$. As in \[13\], we solve order by order the equation
\[
\left( \sum_{k \geq 1} \left( \frac{1}{2} \sigma^+_k \sigma^-_k + 1 + c_k(\varepsilon) \right) - 2\varepsilon(Q + \Gamma Q') \right) |0\rangle_B = 0,
\]
where
\[
Q = \sum_{k \geq 1} (\sigma^+_k \sigma^-_k + \sigma^-_k \sigma^+_k), \quad Q' = \sum_{k \geq 1} (\sigma^+_k \sigma^-_k + \sigma^-_k \sigma^+_k).
\]
The c-number normalization term $c_k(\varepsilon) = \sum_{j \geq 1} c_{k,j}\varepsilon^j$ is included in order to ensure that the eigenvalue is zero. Solving for $|0\rangle_B$, we find
\[
\mathcal{M}^{(0)}(1; -1) = -1 + 8\varepsilon^2 + 8(5\varepsilon^2 - 1)\varepsilon^4 + O(\varepsilon^6).
\]
In terms of $q$ and $p$, this expansion becomes
\[
\mathcal{M}^{(0)}(1; -1) = -1 + 8q^2 - 24q^4 - 8p + 0(q^6, p q^2, p^2),
\]
which agrees with the expression (4.7).
5 Boundary $S$ matrix

We now come to a brief discussion about the $S$-matrices which describe scatterings of quasi-particles with themselves or with the boundary. These $S$-matrices obey the same type of Yang-Baxter equations, unitarity and crossing relations, as do the $R$- and the $K$-matrices used to construct the model, but they are not the same. In fact, for the XYZ chain in the bulk, one expects [13] that the two-particle $S$-matrix has the same form as $R$ but with the elliptic parameter $p$ being changed:

$$ S(ξ;p,q) = -R(ξ;p^*,q), \quad p^* = pq^{-2}. \quad (5.1) $$

A similar phenomenon was observed in [13], where the the boundary $S$-matrix $M(ξ)$ for the XXZ chain was shown to be of the same form as the $K$-matrix, up to an overall scalar factor, wherein the parameter $r$ is changed to $r^* = rq^{-1}$.

We are then led to speculate that the $M$-matrix for the XYZ chain with a boundary (see (6.2) for the characterization) is proportional to the $K$-matrix, with both $p$ and $r$ scaled in the same way as above; namely

$$ M(ξ;p,q,r) = \frac{1}{f(ξ;p,q,r)} \hat{K}(ξ;p^*,r^*), \quad p^* = pq^{-2}, \quad r^* = rq^{-1}, \quad (5.2) $$

where by $\hat{K}(ξ;p^*,r^*)$ we mean the matrix $\hat{K}(ξ;p,r) = \hat{K}(ξ;r)$ in (2.20) with the parameters $p, r$ being replaced by $p^*, r^*$ respectively. The scalar factor $f(ξ;p,q,r)$ is fixed by solving the boundary unitarity and crossing relations,

$$ M(ξ)M(ξ^{-1}) = 1, $$
$$ M^b_a(-q^{-1}ξ^{-1}) = \sum_{α,β} S^c_{β,a}(-qξ^2)M^c_β(ξ). $$

with the condition that it reduces to the known result [13] in the XXZ limit $p \to 0$. Explicitly we have

$$ \frac{1}{f(ξ;p,q,r)} = -ξ^{-2} \frac{\varphi(ξ^2;p^*,q,r^*)}{\varphi(ξ^{-2};p^*,q,r^*)}, $$

where

$$ \varphi(z;p^*,q,r^*) = \varphi_0(z;p^*,q)\varphi_1(z;p^*,q,r^*), $$

$$ \varphi_0(z;p^*,q) = \frac{(q^2z^2;p^*,q^6)_{∞}}{(q^8z^2;p^*,q^8)_{∞}} \frac{(p^*q^6z^2;p^*,q^8)_{∞}}{(p^*z^2;p^*,q^8)_{∞}}, $$

$$ \varphi_1(z;p^*,q,r^*) = \frac{(p^*q^8z^2;p^*z^2,q^8)_{∞}}{(p^*z^2;p^*,q^8)_{∞}} \frac{(p^*q^4z^2;p^*,q^8)_{∞}}{(p^*z^2;p^*,q^8)_{∞}} \frac{(q^4r^*z;p^*,q^4)_{∞}}{(q^2r^*z;p^*,q^4)_{∞}} \frac{(p^*r^*-1z;p^*,q^4)_{∞}}{(p^*q^2r^*-1z;p^*,q^4)_{∞}} \frac{(p^*z^2;p^*,q^4)_{∞}}{(p^*r^*-1z;p^*q^2z^2)_{∞}}. $$

In the case of the XXZ chain, it was possible to derive the $M$-matrix since the boundary vacuum states are known explicitly in terms of the bosonic oscillators. Here we do not have such a construction, and [5,2] is no more than a plausible guess. We find it difficult to check it by
perturbative methods. As an alternative argument in its favor, we study below the continuum limit and show that the resulting formulas agree with those for the sine-Gordon model with a boundary \[ R \].

Introduce

\[ \hat{\lambda} = \frac{2\lambda}{\pi - 2\lambda} \]

and write

\[ \xi^2 = (p^*)^{-\hat{\lambda}u}/\pi, \quad q^2 = (p^*)^{\hat{\lambda}}, \quad r^* = (p^*)^{1/2 - \hat{\eta}/\pi}. \] (5.3)

By the continuum limit we mean the limit \( K \to \infty, K' \to \pi/2 \), so that \( p^* \to 1, \hat{\lambda} \to \hat{\lambda} \), while keeping \( u \) and \( \hat{\eta} \) fixed. We find the following:

\[ \lim S(\xi) = \rho(u) \times \left( \sin(\hat{\lambda}\pi) \frac{1}{2} \left( 1 + \sigma^y \otimes \sigma^y \right) + \sin(\hat{\lambda}(\pi - u)) \frac{1}{2} \left( \sigma^x \otimes \sigma^x + \sigma^z \otimes \sigma^z \right) \right. \]
\[ \left. + \sin(\hat{\lambda}u) \frac{1}{2} \left( \sigma^x \otimes \sigma^x - \sigma^z \otimes \sigma^z \right) \right), \]

\[ \lim M(\xi) = -R_0(u)\hat{\sigma}(0, u)\hat{\sigma}(\hat{\eta}, u)\cos(\hat{\lambda}u) \times \left( 1 - \frac{\sin(\hat{\lambda}u)}{\cos \hat{\eta}} \sigma^z \right). \] (5.4)

Here the functions \( \rho(u), R_0(u), \sigma(x, u) \) are given in (5.7), (5.21), (5.23) in \[ 10 \], respectively, and \( \hat{\sigma}(x, u) = \sigma(x, u)/\sigma(x, 0) \), wherein our \( \hat{\lambda} \) and \( \hat{\eta} \) are to be identified with \( \lambda \) and \( \eta \) there. \(^2\)

The formula (5.4) differs from the standard formula (5.6-7) \[ 10 \] for the two-particle \( S \)-matrix of the sine-Gordon theory by a gauge transformation. To see this, let

\[ U = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}, \quad \bar{U} = \sigma^z U \sigma^z, \]

which has the property \( U\sigma^x U^{-1} = \sigma^y, U\sigma^y U^{-1} = \sigma^z, U\sigma^z U^{-1} = \sigma^x \). Let further

\[ \psi^{(j)}_\mu(\xi) = \sum_\nu U^{(j)\nu}_\mu \Psi^{(j)}_\nu(\xi), \]

where \( U^{(j)} = U \) for \( j \) odd and \( = \bar{U} \) for \( j \) even. Define the new basis of eigenstates by

\[ |\xi_n, \cdots, \xi_1\rangle_{\mu_n, \cdots, \mu_1;i}B = \psi^{(n)}_{\mu_n}(\xi_n) \cdots \psi^{(2)}_{\mu_2}(\xi_2) \psi^{(1)}_{\mu_1}(\xi_1)|i\rangle_B. \] (5.6)

In this basis the bulk and the boundary \( S \)-matrices are given by

\[ S'_{\xi}(\xi) = (\bar{U} \otimes U) S(\xi) (U \otimes \bar{U})^{-1}, \quad M'(\xi) = U M(\xi) U^{-1}. \]

Then \( \lim S'_{\xi}(\xi) \) coincides with the formula (5.6-7) of \[ 10 \]. The limit of the \( M' \)-matrix is to be compared with a special case of the boundary \( S \)-matrix in \[ 10 \]

\[ \hat{\xi} = 0, \quad \vartheta = 0, \]

where \( \hat{\xi} \) denotes the parameter \( \xi \) in \[ 10 \]. With this specialization the \( \lim M'(\xi) \) agrees with (5.12), (5.21–25) in \[ 10 \] up to an overall sign. In particular, the free boundary condition (5.29) \[ 10 \] of the sine-Gordon theory is given by \( \hat{\eta} = \pi(\hat{\lambda} + 1)/2 \). It corresponds exactly to the free boundary condition \( r = -1 \) in the XYZ chain. This is an indication that our speculative choice of the parameter \( r \) in the \( M \)-matrix \[ 5.2 \] is exact.

\(^2\)It seems that in (5.22), \[ 10 \], one should use \( \hat{\sigma}(x, u) = \sigma(x, u)/\sigma(x, 0) \) in place of \( \sigma(x, u) \), to ensure \( \hat{R}_1(0) = 1/\cos \xi \).
6 Properties of boundary form factors

We now consider the boundary form factors, defined as

$$F_{n}^{(i)}(\xi_1, \ldots, \xi_n) = \sum_{\xi_1, \ldots, \xi_n} v_{\xi_1}^{*} \otimes \cdots \otimes v_{\xi_n}^{*} B\langle i| \mathcal{O}\Psi_{\xi_1}^{*}(\xi_1) \cdots \Psi_{\xi_n}^{*}(\xi_n)|i\rangle_B ,$$  \hspace{1cm} (6.1)

where \( \mathcal{O} \) is some local operator, which commutes with type II vertex operators. Here, \( v_{\xi}^{*} \) denote basis vectors in the space dual to \( V \). The boundary form factor satisfies properties analogous to Smirnov’s axioms for form factors, but with some differences due to the presence of a boundary. These properties follow from the properties of type II vertex operators and the boundary vacuum states. The commutation relations for type II vertex operators for the XYZ model were presented in [13]. In particular,

$$\Psi_{\xi_1}^{*}(\xi_1)\Psi_{\xi_2}^{*}(\xi_2) = \sum_{\varepsilon_1, \varepsilon_2} M_{\varepsilon_1, \varepsilon_2}(\xi_1) S_{\varepsilon_1, \varepsilon_2}(\xi_1/\xi_2) ,$$

where the \( S \)-matrix is defined in (5.1).

By definition of the boundary \( S \)-matrix, type II vertex operators have the following property with respect to the boundary bound states:

$$\Psi_{\xi}^{*}(\xi)|i\rangle_B = \sum_{\varepsilon} M_{\varepsilon}(\xi)\Psi_{\varepsilon}^{*}(\xi^{-1})|i\rangle_B ,$$

$$B\langle i|\Psi_{\xi}^{*}(\xi^{-1}) = \sum_{\varepsilon} B\langle i|\Psi_{\varepsilon}^{*}(q^{-2}\xi) \overline{M}_{\varepsilon}(\xi) ,$$  \hspace{1cm} (6.2)

where the boundary \( S \)-matrices \( \overline{M}_{\varepsilon}(\xi) = \sigma^x M_{\varepsilon}(\xi) \sigma^{-x} \) and \( M_{\varepsilon}(\xi) = M(\xi) \) of Eq. (5.2), and \( M(\xi; r) = \sigma^x M_{\varepsilon}(\xi; r^{-1}) \sigma^{-x} \).

Let

$$\prod_{\varepsilon_{i+1}} v_{\xi_{i+1}}^{*} \cdots v_{\xi_{i}}^{*} \cdots \cdots) S_{i,j}(\xi) = \sum_{\varepsilon_{i+j}} \prod_{\varepsilon_{i+1}} v_{\xi_{i+1}}^{*} \cdots v_{\xi_{i}}^{*} \cdots \cdots) S_{i,j}(\xi) ,$$

$$\prod_{\varepsilon_{i+1}} v_{\xi_{i+1}}^{*} \cdots v_{\xi_{i}}^{*} \cdots \cdots) P_{i,j+1} = \cdots v_{\xi_{i}}^{*} \cdots v_{\xi_{i+1}}^{*} \cdots \cdots ,$$

$$\prod_{\varepsilon_{j}} v_{\xi_{j}}^{*} \cdots \cdots) M_{j}^{(i)}(\xi) = \sum_{\varepsilon_{j}} \prod_{\varepsilon_{j+1}} v_{\xi_{j+1}}^{*} \cdots \cdots) M_{j}^{(i)}(\xi) ,$$

etc.. Then the boundary form factors defined in (6.1) have the following properties, analogous to Smirnov’s form factor axioms.

**Axiom I:** Due to the exchange relation between type II vertex operators, which remains unchanged in the boundary theory, this property is identical to the usual case in the absence of a boundary:

$$F_{n}^{(i)}(\xi_1, \ldots, \xi_j, \xi_{j+1}, \ldots, \xi_n) = F_{n}^{(i)}(\xi_1, \ldots, \xi_{j+1}, \xi_j, \ldots, \xi_n) P_{j,j+1} S_{j,j+1}(\xi_j/\xi_{j+1}) .$$

**Axiom II:** The analog of the periodicity condition is the difference equation satisfied by the form factors, which is similar to Cherednik’s equation (2.7). It is a consequence of Axiom I and the
The residue of the form factor at the point $\xi$ and boundary vacuum states, we showed that they obey the appropriate commutation relations to Sklyanin’s transfer matrix. By constructing a lattice realization of type I vertex operators

$\text{(B.9)}$ in $[20]$ we cannot obtain an explicit formula for the vacuum states $|\pi_\xi\rangle$ from the poles in the difference equations satisfied by the correlation functions of the $C$ where $Axiom III$: The form factor $g$ has simple annihilation poles due to the relation (see equation $\text{(6.3)}$)

\begin{align}
\Psi_{\xi_1}^{(i,1-i)}(\xi_1)\Psi_{\xi_2}^{(1-i,i)}(\xi_2) &= \frac{g^* \delta_{\xi_1,-\xi_2}}{1-q^{2}\xi_2/\xi_1} \left(-\frac{\xi_2}{q\xi_1}\right)^{i+1} + \cdots, \quad \xi_1 \to \pm q^{-1}\xi_2,
\end{align}

where “…” refers to regular terms and the scalar $g^*$ is as in Eq. $\text{(2.21)}$ with $p$ replaced by $p^* = p/q^2$. The residue of the form factor at the point $\xi_j = -q^{-1}\xi_n$ is

\begin{align}
\text{Res}F^{(i)}_n(\xi_1, \ldots, \xi_j, \ldots, \xi_n) d(\xi_j/\xi_n) &= \frac{g}{2} F^{(i)}_{n-2}(\xi_1, \ldots, \xi_{j-1}, \xi_{j+1}, \ldots, \xi_{n-1}) \otimes C
\end{align}

\begin{align}
&\times P_{n-2,n-1} \cdots P_{j+1} \left( S_{j,n-1}(-\xi_n/q\xi_{n-1}) \cdots S_{j,j+1}(-\xi_n/q\xi_{j+1}) 
- M_{n}(\xi_n) S_{j,n-1}(-1/q\xi_{n-1}\xi_n) \cdots S_{j,j+1}(-1/q\xi_{j+1}\xi_n) S_{j,j-1}(-1/q\xi_{j-1}\xi_n) 
\times \cdots S_{j,1}(-1/q\xi_{1}\xi_n) M_{j}^{(i)}(-q^{-1}\xi_n) S_{1,j}(-q\xi_1/\xi_n) \cdots S_{j-1,j}(-q\xi_{j-1}/\xi_n) \right), \quad (6.3)
\end{align}

where $C = \sum_{\xi_1,\xi_2} v^*_\xi \otimes v^*_{-\xi}$. There are additional poles corresponding to bound states, which come from the poles in the $S$-matrix and boundary $S$-matrix.

7 Discussion

Let us summarize our results. We have shown that correlation functions of the semi-infinite XXZ spin chain obey the quantum Knizhnik–Zamolodchikov equation with reflection. This is a simple consequence of the exchange algebra of the vertex operators and the reflection property $\text{(2.3)},\text{(2.4)}$ of the boundary vacuum state and its dual.

In the case of the XYZ chain, we assumed such a boundary vacuum state exists with a similar reflection property. Again using the exchange algebra between the elliptic vertex operators of $[13]$ and the reflection property with the elliptic $K$-matrix of $[11,12]$ we were able to derive the $q$-difference equations satisfied by the correlation functions of the XYZ spin chain with a boundary. We cannot obtain an explicit formula for the vacuum states $|i\rangle_B$ in this case, however, due to a lack of a bosonization formula for the elliptic vertex operators.

The assumptions made in the latter case are supported by an alternative method of derivation using the corner transfer matrix of the eight vertex model with a boundary corresponding to Sklyanin’s transfer matrix. By constructing a lattice realization of type I vertex operators and boundary vacuum states, we showed that they obey the appropriate commutation relations

\footnote{There is an error in equation (B.9): The first factor is the inverse of the correct expression.}
and reflection condition, and therefore that the correlation functions obey the correct difference equations. Using the difference equations, we computed the boundary spontaneous magnetization \( \mathcal{M}^{(0)}(1; -1) \), i.e., the magnetization in the case where the boundary field vanishes.

We also conjectured a natural form for the boundary \( S \)-matrix for the XYZ model, which is the reflection matrix associated with type II vertex operators. In the continuum limit the boundary \( S \)-matrix becomes (up to a gauge transformation) that of the boundary sine-Gordon model \([11]\).

Finally, using this boundary \( S \)-matrix and the exchange algebra of the elliptic type II vertex operators, we formulated the analogues of Smirnov’s axioms for form factors \([6]\).

For the XYZ chain, neither the bulk nor the boundary \( S \)-matrix have been derived using an alternative method to the vertex operator approach. It should be possible to obtain these using the Bethe ansatz approach. The same can be said for the boundary \( S \)-matrix of the XXZ chain.

In our boundary \( S \)-matrix, one can find a pole \( \xi^2 = rq - 1 \) in the physical strip \( 1 < |\xi| < |q| - 2 \). This pole yields a boundary bound state, and as in the XXZ chain case \([13]\) this state should be identified with the second vacuum state \( |1\rangle_B \). The coincidence of the vanishing point \( r = q, -q \) of \( \partial \Delta e/\partial h \) and the point where the pole leaves the physical strip gives a consistency check for the conjectured form of the boundary \( S \)-matrix.

In section 6, we derived the difference equation for form factors of the semi-infinite spin chain. In the continuum limit, one can derive the difference equations satisfied by form factors in massive integrable quantum field theory on a semi-infinite line. For example, for the sine-Gordon theory, using the setting given in Section 5 and taking the limit \( p^* \to 1 \) of the equation (6.3), one finds

\[
F_n^\prime(\beta_1, \cdots, \beta_n) = F_n(\beta_1, \cdots, \beta_n) \cdot S_{j_1} \cdots S_{j_n} M(\beta_j - 2\pi i),
\]

where \( u = -i\beta \) and

\[
\lim F_n^\prime(\xi_1, \cdots, \xi_n) = F_n(\beta_1, \cdots, \beta_n), \\
\lim S^\prime(\xi) = S(\beta), \quad \lim M^\prime(\xi) = M(\beta).
\]

Here the primed quantities are the gauge transformed ones (see Sec.5). To solve this equation is an open problem. It seems possible by applying the method developed by Smirnov \([9]\) with modification by Sklyanin’s Bethe ansatz scheme \([8]\).

Taking the quasi-classical limit of our difference equation, one can derive the boundary analog of the KZ equation. It is however unclear whether this equation governs the boundary conformal field theory in the way that the KZ equation does for the bulk theory. It would be nice if one could relate the equation to the one obtained by Cardy \([8]\).

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approach to boundary problems, and his colleagues at RIMS for their hospitality during the period in which much of this work was carried out. This work is partly supported by Grant-in-Aid for Scientific Research on Priority Areas 231, the Ministry of Education, Science and Culture. H. K. is supported by Soryushi Shyogakukai. R. K. is supported by the Japan Society for the Promotion of Science.
A The Asymptotic Behaviour of the CTM

In this appendix we derive the asymptotic form of the CTM of the inhomogeneous 8-vertex model using a generalisation of Baxter’s argument for the homogeneous model. Consider the transfer matrix $T(\{\zeta_i\}, \{\xi_i\})$ shown in Figure A1.

![Figure A.1: The finite completely inhomogeneous transfer matrix $T(\{\zeta_i\}, \{\xi_i\})$](image)

It is the trace of the product of $N$ R-matrices, each with independent vertical and horizontal rapidities $\zeta_i$ and $\xi_i$, $i = 1, \ldots, N$. Two such transfer matrices $T(\{\zeta_i\}, \{\xi_i\})$ and $T(\{\zeta'_i\}, \{\xi'_i\})$ commute if $(\xi_i/\zeta_i)/(\xi'_j/\zeta'_j) =$const, independent of $i$. This means that the normalised eigenvectors of $T(\{\zeta_i\}, \{\xi_i\})$ depend only on the ratios $(\xi_i/\zeta_i)/(\xi_j/\zeta_j)$ for $i \neq j$.

The first step is to specialise the transfer matrix to the case when all the horizontal rapidities on the left half of the lattice are equal to $\xi'$, the left vertical rapidities are alternately $\zeta'$ and $1/\zeta'$, the right horizontal rapidities are all $\xi$ and the right vertical rapidities are alternately $\zeta$ and $1/\zeta$. Then we identify the infinite product of such transfer matrices (taking $N \to \infty$ as well) with the product $A_{SW}(\zeta', \xi')A_{NW}(\zeta, \xi)$ of suitably normalised CTM’s. Such an identification implies

$$A_{SW}(\zeta', \xi')A_{NW}(\zeta, \xi) = \tau(\zeta', \zeta, \xi')X(\zeta', \zeta, \xi/\xi'). \quad (A.1)$$

Here and elsewhere we adopt the convention that lower case letters represent scalar functions and that upper case letters represent matrix functions. We also suppress the $i$ superscripts on the CTM’s. Now send $\xi \to \chi/\xi'$ and $\xi' \to \chi/\xi$ (where $\chi$ is an arbitrary constant) and eliminate $X$. Then,

$$\tau(\zeta', \chi/\xi, \chi/\xi')A_{SW}(\zeta', \xi')A_{NW}(\zeta, \xi) = \tau(\zeta', \zeta, \xi')A_{SW}(\zeta', \chi/\xi)A_{NW}(\zeta, \chi). \quad (A.2)$$

This equation immediately tells us that the product

$$\tilde{A}_{NW}(\zeta, \xi) = A_{NW}(\zeta, \xi)A_{NW}(\zeta, \mu)^{-1} \quad (A.3)$$

(where $\mu$ is another constant) depends on $\zeta$ only through a scalar function. Setting $\xi = \mu$ in (A.2), solving for $A_{SW}(\zeta', \xi')$ and substituting in (A.1) gives

$$\tilde{A}_{NW}(\zeta, \chi/\xi')\tilde{A}_{NW}(\zeta, \xi) = c(\zeta', \zeta, \xi')Y(\zeta', \zeta, \xi/\xi'), \quad (A.4)$$

24
where
\[ c(\zeta', \zeta, \xi', \xi) = \frac{\tau(\zeta', \zeta, \xi, \chi)}{\tau(\zeta, \zeta', \xi)} \]
\[ Y(\zeta', \zeta, \xi/\xi') = A_{NW}^{-1}(\zeta', \chi/\mu)X(\zeta', \zeta, \xi/\xi')A_{NW}^{-1}(\zeta, \mu). \]  \hspace{1cm} (A.5)

(We shall suppress dependencies on the constants \(\chi\) and \(\mu\).) Setting \(\zeta = \zeta' = 1\), we can rewrite (A.4) as
\[ \bar{A}_{NW}(1, \xi') \bar{A}_{NW}(1, \xi) = d(\xi', \xi)Z(\xi') \],
\hspace{1cm} (A.6)
where \(d(\xi', \xi) = c(1, 1, \chi/\xi', \xi)\) and \(Z(\xi') = Y(1, 1, \xi'\xi/\chi)\). We shall solve this equation for \(\bar{A}_{NW}(1, \xi)\) and then reclaim
\[ \bar{A}_{NW}(\zeta, \xi) = r(\zeta, \xi)\bar{A}_{NW}(1, \xi), \]  \hspace{1cm} (A.7)
where \(r(\zeta, \xi)\) is the scalar function \(r(\zeta, \xi) = \bar{A}_{NW}(\zeta, \xi)A_{NW}^{-1}(1, \xi)\).

Equation (A.6) is the generalisation of (13.5.17) of Baxter’s book [16], and the argument now proceeds as in [16] with some minor modifications. Interchanging \(\xi'\) and \(\xi\) and eliminating \(Z\) gives,
\[ d(\xi, \xi')\bar{A}_{NW}(1, \xi')\bar{A}_{NW}(1, \xi) = d(\xi', \xi)\bar{A}_{NW}(1, \xi)\bar{A}_{NW}(1, \xi'). \]  \hspace{1cm} (A.8)
Now consider a representation in which \(\bar{A}_{NW}(1, \xi')\) is diagonal. Then (A.8) implies
\[ d(\xi, \xi')\bar{A}_{NW}(1, \xi')\bar{A}_{NW}(1, \xi) = d(\xi', \xi)\bar{A}_{NW}(1, \xi)\bar{A}_{NW}(1, \xi'). \]  \hspace{1cm} (A.9)
Thus \(\bar{A}_{NW}(1, \xi')\), \(\bar{A}_{NW}(1, \xi)\) and \(Z(\xi')\) commute and have common eigenvectors, independent of \(\xi'\) and \(\xi\). We can clearly renormalise in the following way:
\[ \bar{A}_{NW}^R(1, \xi) = \bar{A}_{NW}(1, \xi)/\alpha(\xi), \]
\[ Z^R(\xi) = Z(\xi)/\beta(\xi), \]  \hspace{1cm} (A.10)
where \(\alpha(\xi)\) and \(\beta(\xi)\) are the respective eigenvalues of some common eigenvector of \(\bar{A}_{NW}(1, \xi)\) and \(Z(\xi)\). Then equation (A.6) becomes
\[ \bar{A}_{NW}^R(\xi')\bar{A}_{NW}^R(\xi) = Z^R(\xi'). \]  \hspace{1cm} (A.11)
Now we can diagonalise,
\[ \bar{A}_{NW}^d(\xi) = P^{-1}\bar{A}_{NW}^R(\xi)P, \]  \hspace{1cm} (A.12)
where \(P\) is the matrix of eigenvectors of \(\bar{A}_{NW}^R(\xi)\). Define \(Z^d(\xi')\) similarly such that
\[ \bar{A}_{NW}^d(\xi')\bar{A}_{NW}^d(\xi) = Z^d(\xi'). \]  \hspace{1cm} (A.13)
Differentiating with respect to \(\xi\), it is apparent that the general solution is
\[ \bar{A}_{NW}^{dr}(\xi) = m_r\xi^{-\alpha_r} \quad r = 1, 2, \ldots, \]  \hspace{1cm} (A.14)
where \(m_r\) and \(\alpha_r\) are independent of \(\xi\) and \(\zeta\). Using the redefinitions (A.3), (A.7), (A.10) and (A.12), we find,
\[ A_{NW}(\zeta, \xi) = a(\zeta, \xi)P\bar{A}_{NW}^d(\xi)Q^{-1}(\zeta), \]  \hspace{1cm} (A.15)
where $Q^{-1}(\xi) = P^{-1}A_{NW}(\xi, \mu)$ and the scalar function $a(\xi, \xi) = r(\xi, \xi)\alpha(\xi)$. Setting $\xi = 1$ in (A.15), solving for $Q^{-1}(\xi)$ and substituting back into the same equation gives

$$A_{NW}(\xi, \xi) = \frac{a(\xi, \xi)}{a(\xi, 1)} P\tilde{A}^{-d}_{NW}(\xi)\tilde{A}^{-d}_{NW}(1) P^{-1}A_{NW}(\xi, 1).$$

(A.16)

Defining the operator $D^d$ as that with diagonal entries $\alpha_1, \alpha_2, \ldots$, gives

$$A_{NW}(\xi, \xi) = \frac{a(\xi, \xi)}{a(\xi, 1)} \xi^{-D} A_{NW}(\xi, 1),$$

(A.17)

where $\xi^{-D} = P\xi^{-D} P^{-1}$.

We may now specialise this formula to the case when $\xi = 1$ and all the horizontal arrows of $A_{NW}$ point to the left (the argument is independent of the direction of the arrows). The CTM is that of Figure 4, and is denoted by $A_{NW}(\xi)$. Since $A_{NW}(1) = \text{id}$, this gives

$$A_{NW}(\xi) = \frac{\tilde{a}(1, \xi)}{\tilde{a}(1, 1)} \xi^{-D},$$

(A.18)

where $D$ is unchanged and the bar indicates that the scalar factors are those relevant to the homogeneous CTM $A_{NW}(\xi)$. Hence we obtain the desired result (3.13) that

$$A_{NW}(\xi) A_{NW}(\xi, \xi') = g(\xi, \xi') A_{NW}(\xi, \xi' \xi) \text{ where,}$$

$$g(\xi, \xi') = \frac{a(\xi, \xi') \tilde{a}(1, \xi)}{a(\xi, \xi') \tilde{a}(1, 1)}.$$  

(A.19)
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