Abstract

We introduce a new basis of the Temperley-Lieb algebra. It is defined using a bijection between noncrossing partitions and fully commutative elements and a basis introduced by Zinno, which is obtained by mapping the simple elements of the Birman-Ko-Lee braid monoid to the Temperley-Lieb algebra. The combinatorics of the new basis involve the non-natural Bruhat order on noncrossing partitions. As an application we can derive properties of the coefficients of the change of basis matrix between Zinno’s basis and the well-known diagram or Kazhdan-Lusztig basis of the Temperley-Lieb algebra, in particular we give closed formulas for some of the coefficients of the expansion of an element of the diagram basis in the Zinno basis.

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1 Introduction

The Temperley-Lieb algebra $\mathcal{T}_n$ (of type $A_n$) is an associative, unital $\mathbb{Z}[v, v^{-1}]$-algebra of dimension equal to the $(n+1)^{\text{th}}$ Catalan number $C_{n+1} = \frac{1}{n+2} \binom{2n+1}{n+1}$. It is generated by $b_1, \ldots, b_n$, with relations

\begin{align*}
    b_j b_i b_j &= b_j \quad \text{if } |i - j| = 1, \\
    b_i b_j &= b_j b_i \quad \text{if } |i - j| > 1, \\
    b_i^2 &= (v + v^{-1}) b_i.
\end{align*}

Alternatively, it can be viewed as a quotient algebra of the Iwahori-Hecke algebra $\mathcal{H}$ of type $A_n$. There is well-known diagrammatic version of $\mathcal{T}_n$, which is due to Kauffman (see [14]). The corresponding diagram basis is indexed by fully commutative elements of the symmetric group. It is a monomial basis in the generators $b_1, \ldots, b_n$, which is also the projection of the canonical basis of Kazhdan and Lusztig of $\mathcal{H}$ (see [15]). The Kazhdan-Lusztig theory in $\mathcal{T}_n$ is very simple, as reflected by the diagrammatic properties: for example, any product of the generators is proportional to an element of the basis.

Other bases of $\mathcal{T}_n$ are known. A particularly mysterious one is a basis introduced by Zinno in [18]. There is a multiplicative homomorphism from the braid group on $n + 1$ strands to the Temperley-Lieb algebra. The Zinno basis is obtained by mapping the so-called canonical factors of the braid group to $\mathcal{T}_n$ (via $\mathcal{H}$). The canonical factors are a set of distinguished elements of the Birman-Ko-Lee braid monoid (see [3]), later generalized in dual braid monoid by Bessis (see [1]). The Birman-Ko-Lee or dual braid monoid embeds in the braid group, but is generated by a copy of the set of all the transpositions or reflections. The dual braid monoids are examples of Garside monoids (see [6]) and the more standard name for the canonical factors in that setting is the simple elements. They can be seen as lifts of noncrossing partitions (viewed as elements of the symmetric group) in the braid group. The basis defined by Zinno is therefore naturally indexed.
by noncrossing partitions, which is another set enumerated by the Catalan number.

Zinno shows that the images of the simple elements in $\text{TL}_n$ form a $\mathbb{Z}[v, v^{-1}]$-linear basis of it by defining a bijection between noncrossing partitions and fully commutative elements as well as an order on the set of simple elements. He then shows that there exists a matrix with respect to this defined order which is upper triangular with invertible coefficient on the diagonal, allowing one to pass from the diagram basis to the set of images of simple elements. The bijection is then read on the diagonal of the matrix. The bijection is given by an algorithm allowing one to extract a subword of a specific braid word chosen to represent the simple element. The obtained subword is then (after surjection into the symmetric group) shown to be fully commutative. The approach is indirect and there is no description of the inverse bijection.

In this paper, we reformulate Zinno’s bijection in a simple way, allowing one to explicitly compute the inverse bijection. We then use this bijection to introduce a new basis of the Temperley-Lieb algebra, which we call the simplex-basis. This basis will allow us to control a part of the change of basis matrix between the Zinno and diagram bases and find closed formulas for some of the coefficients of the matrix. Surprisingly, the new basis involves considering the non-natural Bruhat order on noncrossing partitions. Such an order is in fact stronger than the order defined by Zinno on noncrossing partitions to have triangularity. As a consequence, the simplex-basis is an intermediate basis between the diagram and Zinno basis, and both change of basis matrices between them and the simplex-basis are upper triangular with invertible coefficients on the diagonals if one orders the set of noncrossing partitions by any linear extension of the Bruhat order.

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2 Bijections between fully commutative elements and noncrossing partitions

2.1 Fully commutative elements

Let $(\mathcal{W}, S)$ be a Coxeter system.

**Definition 2.1.** An element $w \in \mathcal{W}$ is fully commutative if one can pass from any reduced $S$-decomposition of $w$ to any other by applying a sequence of relations of the form $st = ts$, where $s, t \in S$. 
We denote by $\mathcal{W}_f$ the set of fully commutative elements. In this paper, the Coxeter system considered are of type $A_n$. In that case there are many well-known equivalent characterizations of fully commutative elements. We list those we shall need in this paper below:

**Proposition 2.2.** Let $(\mathcal{W}, \mathcal{S})$ be of type $A_n$, with $\mathcal{W} \cong S_{n+1}$ and $\mathcal{S} := \{s_i = (i, i+1)\}_{i=1}^n$. Let $w \in \mathcal{W}$. Then the following are equivalent:

1. The element $w$ is fully commutative,

2. If $s_{i_1} \cdots s_{i_k}$ is a reduced $\mathcal{S}$-decomposition of $w$, then for all $i = 1, \ldots, n$, the integer $n_i(w) := |\{j \mid i_j = i\}|$ is independent of the chosen reduced $\mathcal{S}$-decomposition,

3. The element $w$ has a reduced $\mathcal{S}$-decomposition of the form

   $$(s_{i_\ell}s_{i_{\ell-1}}\cdots s_{j_\ell})(s_{i_{\ell-1}}s_{i_{\ell-2}}\cdots s_{j_{\ell-1}})\cdots (s_{i_1}s_{i_2}\cdots s_{j_1})$$

   where all the indices lie in $\{1, \ldots, n\}$, $i_\ell < i_{\ell-1} < \cdots < i_1$, $j_\ell < j_{\ell-1} < \cdots < j_1$ and $j_m \leq i_m$ for all $m = 1, \ldots, \ell$,

4. If $s_{i_1} \cdots s_{i_k}$ is a reduced $\mathcal{S}$-decomposition of $w$ with $s_{i_j} = s_i = s_{i_d}$, $j > d$ and $s_{i_k} \neq s_i$ for all $j < k < d$, then $(s_{i_{j+1}}, \ldots, s_{i_{d-1}})$ has exactly one entry equal to $s_{i+1}$ and exactly one entry equal to $s_{i-1}$.

**Proof.** The equivalence $1 \iff 2$ is clear and true for any Coxeter system such that the only even entry of the Coxeter matrix is 2. The last two conditions are often considered in the case of the so-called reduced words of the Temperley-Lieb algebra but the results still hold in the symmetric group and the proofs can easily be adapted: for the equivalence $1 \iff 3$, see for example ([9], §2.8); the existence of canonical reduced $\mathcal{S}$-decompositions as in 3 have been noticed by Jones in the Temperley-Lieb case in ([11], §3.5). For the equivalence $1 \iff 4$, see for example ([13], Theorem 1).

**Notation.** For $w \in \mathcal{W}_f$, the set $\{j_1, \ldots, j_\ell\}$ from point 3 of Proposition 2.2 is written $J_w$ and the set $\{i_1, \ldots, i_k\}$ is written $I_w$.

**Remark 2.3.** Notice that $i \in I_w$ if and only if in any reduced $\mathcal{S}$-decomposition of $w$, there is no occurrence of $s_{i+1}$ before the first occurrence of $s_i$. Similarly, one has that $i \in J_w$ if and only if in any reduced $\mathcal{S}$-decomposition of $w$, there is no occurrence of $s_{i-1}$ after the last occurrence of $s_i$. 


2.2 Noncrossing partitions and dual braid monoid

From now and unless otherwise specified, \((W, S)\) will be of type \(A_n\), with the notations introduced in Proposition 2.2. Let \(T\) be the set of transpositions of \(W\) and \(\ell_T : W \to \mathbb{Z}_{\geq 0}\) be the reflection length. There is a partial order \(\prec_T\) on \(W\) defined by \(u \prec_T v\) if the equality
\[
\ell_T(u) + \ell_T(u^{-1}v) = \ell_T(v)
\]
is satisfied. Let \(c\) be any Coxeter element, that is, any product of all the \(s_i\) in some order. The restriction of \(\prec_T\) to \(P_c := \{x \in W \mid x \prec_T c\}\) endows \(P_c\) with a lattice structure. The obtained lattice is isomorphic to the lattice of noncrossing partitions (for the "is finer than" order) as shown in [2]. In case \(c = s_1 \cdots s_n\), one obtains the noncrossing partition corresponding to \(x \in P_c\) by looking at the decomposition of \(x\) into a product of disjoint cycles.

This approach provides a generalization of noncrossing partitions to arbitrary finite Coxeter groups (see [3] and [1]). Recall that there is a geometric representation of any noncrossing partition by disjoint unions of polygons having vertices in a set of \(n + 1\) points on a circle labelled with \(1, 2, \ldots, n + 1\) in clockwise order, as in Figure 1. From now, we assume that \(c = s_1 s_2 \cdots s_n\). We will identify a noncrossing partition with the corresponding permutation of \(P_c\).

One can associate to \((W, T, c)\) a dual braid monoid \(B_c^*\) (see [1]; it is a generalization of the Birman-Ko-Lee monoid from [3]). It is generated by a copy \(\{i_c(t) \mid t \in T\}\) of \(T\) and relations
\[
i_c(t) i_c(t') = i_c(t') i_c(t t') \text{ if } tt' \in P_c.
\]
A relation as above is called a dual braid relation. The terminology comes from the fact that there is an embedding of \(i_c : B_c^* \hookrightarrow \text{Frac}(B_c^*) \cong B_{n+1}\), where \(B_{n+1}\) is the braid group on \(n + 1\) strands. Recall that \(B_{n+1}\) is generated
by a copy \{s_i \mid s_i \in S\} of the elements of \( \mathcal{S} \) together with the relations
\[
\begin{align*}
  s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & \forall i \in \{1, \ldots, n-1\} \\
  s_i s_j &= s_j s_i, & \text{if } |i - j| > 1.
\end{align*}
\]

In case \( c = s_1 \cdots s_n \), the image \( \iota_c(i_c(t)) \) of \( i_c(t) \) where \( t = (i, k+1), k \geq i \) in \( B_{n+1} \) is represented by the braid word
\[
  s_{i,k+1} := s_k^{-1} s_{k-1}^{-1} \cdots s_{i+1}^{-1} s_i s_{i+1} \cdots s_k.
\]

Moreover, the monoid \( B^*_c \) shares many properties with the positive braid monoid \( B^+ \); both turn out to be so-called Garside monoids (see [6]). In particular, \( B^*_c \) has a set of distinguished elements called simple, which are lifts of elements of \( \mathcal{P}_c \). They have a simple combinatorial definition as follows: given \( x \in \mathcal{P}_c \) with a reduced \( \mathcal{T} \)-decomposition \( t_1 \cdots t_k \), we define the simple \( i_c(x) \) corresponding to \( x \) as the product
\[
  i_c(t_1)i_c(t_2) \cdots i_c(t_k).
\]

Notice that such a definition makes sense only if such a product is independent of the chosen reduced expression, which holds for any \( x \in \mathcal{P}_c \) (it is a consequence of the dual braid relations).

### 2.3 Bijects

Let \( k \in \mathbb{Z}_{\geq 0} \) with \( k \leq n+1 \). We denote by \( \mathcal{I}_k \) the set of pairs \((X, Y)\) where
\[
X = \{d_1, d_2, \ldots, d_k\}, \quad Y = \{e_1, e_2, \ldots, e_k\}, \quad e_i, d_i \in \{1, \ldots, n+1\}, \quad d_i < d_{i+1},
\]
\[
e_i < e_{i+1} \text{ for each } 1 \leq i < k, \quad d_i < e_i \text{ for each } 1 \leq i \leq k.
\]

Set \( \mathcal{I} := \bigsqcup_{k=1}^n \mathcal{I}_k \).

**Remark 2.4.** There is a bijection \( \mathcal{W}_f \rightarrow \mathcal{I} \) given by \( w \mapsto (J_w, I_w + 1) \). This is just a reformulation of the point 3 of Proposition 2.2.

Any polygon \( P \) from the geometric representation of \( x \in \mathcal{P}_c \) as given in 2.2 is given by an ordered sequence of indices: \( P = [i_1 i_2 \cdots i_k] \) where \( i_1 < i_2 < \cdots < i_k \) and \( i_j \) are the indices indexing the vertices of \( P \). In the example of figure 1 one has two polygons \( P_1 = [235] \) and \( P_2 = [16] \).

**Definition 2.5.** We say that \( \min P := i_1 \) is an initial index and \( \max P := i_k \) a terminal index of \( P \) or \( x \). The longest edge of \( P \) is the edge joining the initial index to the terminal index of \( P \).

**Remark 2.6.** Any polygon \( P = [i_1 i_2 \cdots i_k] \in \text{Pol}(x) \) represents an element \( y_i \in \mathcal{P}_c \). As element of the symmetric group \( y_i \) is the cycle \((i_1, i_2, \ldots, i_k) \).
In the framework of Coxeter theory, a reduced expression for a transposition \((j, k)\) is given by the word \([j, k] := s_{k-1}s_{k-2}\cdots s_{j+1}s_js_{j+1}\cdots s_{k-2}s_{k-1}\).

The word \(m_i\) obtained by the concatenation of such words
\[
m_i := [i_1, i_2] \star [i_2, i_3] \star \cdots \star [i_{k-1}, i_k]
\]
yields a reduced expression for \(y_i\). The concatenation \(m_1 \ast m_2 \ast \cdots \ast m_r\) of the words yields a Coxeter word which is a reduced \(S\)-decomposition of \(x\) (this is easy to see by induction on the number of polygons of \(x\) if one keeps in mind that the Coxeter length \(\ell_s(\sigma)\) of a permutation \(\sigma \in S_{n+1}\) is equal to the number of \(i < j\) such that \(\sigma(i) > \sigma(j)\)). In particular, one has that \(\sum_{i=1}^n \ell_s(y_i) = \ell_s(x)\).

We denote by \(\text{Pol}(x)\) the set of polygons of \(x \in \mathcal{P}_c\). We associate to \(x\) a two sets \(D_x \subset \{1, \ldots, n\}\) and \(U_x \subset \{2, \ldots, n + 1\}\) as follows: \(D_x\) (resp. \(U_x\)) contains exactly the integers \(m\) such that there exists a polygon \(P = [i_1i_2\cdots i_k]\) in the geometric representation of \(x\) as well as \(j \neq k\) (resp. \(j \neq 1\)) such that \(m = i_j\). In other words, \(D_x\) (resp. \(U_x\)) contains all the indices indexing a vertex of a polygon of \(x\) except any terminal (resp. any initial) index. We will write \(m \in P\) to mean that there exists \(1 \leq j \leq k\) such that \(m = i_j\). Notice that \(|D_x| = |U_x|\) and \((D_x, U_x) \in \mathcal{I}\). In the example of figure 1 we have \(D_x = \{1, 2, 3\}, U_x = \{3, 5, 6\}\).

**Definition 2.7.** An index \(m \in \{2, \ldots, n\}\) is nested in a polygon \(P = [i_1i_2\cdots i_k]\) if there exists \(1 < j \leq k\) such that \(i_{j-1} < m < i_j\).

**Example 2.8** In the example of figure 1, the integer 4 is nested in the two polygons \(P_1\) and \(P_2\). The integer 5 is nested in \(P_2\) but not in \(P_1\). The integer 6 is not nested in any polygon of \(x\).

**Lemma 2.9.** Let \((D, U) \in \mathcal{I}\) with \(D \cap U = \emptyset\). There is a unique \(x \in \mathcal{P}_c\) such that \((D_x, U_x) = (D, U)\).

**Proof.** Since \(D \cap U = \emptyset\), the \(x\) we have to find must be represented by a disjoint union of edges. Its set of initial indices must be equal to \(D\) while its set of terminal indices must be equal to \(U\). The proof is by induction on \(|D| = |U|\).

If \(D = U = \emptyset\), then the noncrossing partition \(e\) is the unique one such that \((D_e, U_e) = (\emptyset, \emptyset)\). Now if \(|D| = |U| > 0\), consider the biggest index \(d_j\) in \(D\). It has to be joined to a unique \(u_m \in U\) with \(u_m > d_j\). To respect the noncrossing property \(d_j\) must be joined to the first index \(u_m\) in \(U\) appearing after \(d_j\) when going along the circle in clockwise order. Indeed, assume that it is not joined to the first one. Then there exists \(a \in U\) which is nested in
the edge \((d_k, u_m)\). The line containing the points with index equal to \(d_j, u_m\) defines two half-planes \(H_1, H_2\) and the point with index \(u\) lines in one of them, say \(H_1\). But \(u\) must be joined to an index \(d \in D, d \neq d_k\); but these ones are before \(d_j\) in clockwise order since \(d_j\) is the biggest index in \(D\), hence their points lie in \(H_2\). As a consequence the two segments \((d_j, u_m)\) and \((d, u)\) cross and the noncrossing property fails, a contradiction.

Now if we consider \(D' = D\setminus d_k, U' = U\setminus u_m\) and order them as \(D' = \{d'_1, \ldots, d'_{j-1}\}, U' = \{e'_1, \ldots, e'_{j-1}\}, d'_i < d'_{i+1}, e'_i < e'_{i+1}\), we still have that \(d'_i < e'_i\) since we removed the biggest index \(d_j\) from \(D\), hence \((D', U') \in \mathcal{I}\). Hence by induction, there exists a unique noncrossing partition \(x'\) such that \((D_{x'}, U_{x'}) = (D', U')\). Our noncrossing partition \(x\) is obtained by adding the edge \((d_k, u_m)\) in the representation of \(x'\). It does not cross the segments coming from \(x'\) since they all lie in \(H_2\).

**Proposition 2.10.** The map \(\varepsilon : \mathcal{P}_c \to \mathcal{I}, x \mapsto (D_x, U_x)\) is a bijection.

**Proof.** Given \((D, U) \in \mathcal{I}\), we need to show that there exists a unique \(x \in \mathcal{P}_c\) such that \((D, U) = (D_x, U_x)\). Set \(I := D \setminus (D \cap U)\) and \(T := U \setminus (D \cap U)\). Write \(I = \{d_1, d_2, \ldots, d_j\}, T = \{e_1, e_2, \ldots, e_j\}\) where \(d_i < d_{i+1}, e_i < e_{i+1}\). By definition of \(\mathcal{I}\), one has \(d_i < e_i\) if \(1 \leq i \leq j\). If there exists \(x \in \mathcal{P}_c\) with \((D, U) = (D_x, U_x)\), then \(I\) must be the set of initial indices and \(T\) the set of terminal indices of \(x\). In particular, \(|I| = |T|\) is equal to the number of polygons of \(x\) and any of the \(e_i\) is joined to a unique \(d_k\) such that \((e_i, d_k)\) is the longest edge of a polygon of \(x\).

We now show by induction on \(D \cap U\) that we can always find such an \(x\) and that it is unique. The case where \(D \cap U = \emptyset\) is given by lemma 2.9. Assume that \(D \cap U \neq \emptyset\). Let \(r \in D \cap U\) and consider the pair \((D \setminus r, U \setminus r)\). It lies in \(\mathcal{I}\) again and one has \((D \setminus r) \cap (U \setminus r) = (D \cap U) \setminus r\). Hence by induction, there exists a unique noncrossing partition \(x'\) such that \((D_{x'}, U_{x'}) = (D \setminus r, U \setminus r)\). It now suffices to show that \(r\) is nested in at least one polygon of \(x'\): the only possibility in then to enlarge the polygon which is the closest to \(r\) among the ones in which it is nested to obtain a noncrossing partition \(x\) with \((D_x, U_x) = (D, U)\); moreover it will be uniquely determined since \(x'\) is. But if \(r\) is nested in no polygon, it means that \(|\{a \in D \mid a < r\}| = |\{a \in U \mid a < r\}|\), hence that there exists \(1 \leq i \leq j\) such that \(d_i = e_i = r\), which is a contradiction with the fact that \((D, U)\) lies in \(\mathcal{I}\).

Thanks to Proposition 2.10 together with Remark 2.3 we have the following:

**Theorem 2.11.** There exists a bijection \(\varphi : \mathcal{P}_c \to \mathcal{W}_f\) characterized by the equality \((J_{\varphi(x)}, I_{\varphi(x)}) = (D_x, U_x - 1)\), for all \(x \in \mathcal{P}_c\).
We therefore have a characterization of the inverse bijection \( \psi : \mathcal{W}_f \to \mathcal{P}_c \) by the equality

\[
(D_{\psi(w)}, U_{\psi(w)}) = (J_w, I_w + 1), \text{ for all } w \in \mathcal{W}_f.
\]

Example 2.12 For \( x \) as in the example of Figure 1, we have

\[
(D_x, U_x) = ([1, 2, 3], [3, 5, 6]).
\]

Hence \( (J_{\varphi(x)}, I_{\varphi(x)}) = ([1, 2, 3], [2, 4, 5]) \). Hence if we write \( \varphi(x) \) as in point 3 of Proposition 2.2, we have

\[
\varphi(x) = (s_2s_1)(s_4s_3s_2)(s_5s_4s_3).
\]

3 Zinno basis of the Temperley-Lieb algebra

The aim of this section is to introduce results by Zinno (see [18]) on the Temperley-Lieb algebra and explain the relation with the previously introduced bijections. We first introduce the Temperley-Lieb algebra and explain the link with the braid group.

3.1 Temperley-Lieb algebra and braid group

We denote by \( \text{TL}_n \) the Temperley-Lieb algebra. It is the associative, unital \( \mathbb{Z}[v, v^{-1}] \)-algebra having as generators \( b_1, \ldots, b_n \) and relations

\[
\begin{align*}
  b_j b_i b_j &= b_j \text{ if } |i - j| = 1, \\
  b_i b_j &= b_j b_i \text{ if } |i - j| > 1, \\
  b_i^2 &= (v + v^{-1})b_i.
\end{align*}
\]

It has a basis indexed by fully commutative elements:

**Proposition 3.1** (Jones, [11]). Let \( w \in \mathcal{W}_f \). One associates to any reduced \( S \)-decomposition \( s_{i_1}s_{i_2} \cdots s_{i_k} \) of \( w \) the element \( b_{i_1}b_{i_2} \cdots b_{i_k} \) of \( \text{TL}_n \).

1. The element \( b_{i_1}b_{i_2} \cdots b_{i_k} \) is independent of the choice of the reduced expression for \( w \). We will therefore denote it by \( b_w \).

2. The set \( \{b_w\}_{w \in \mathcal{W}_f} \) is a \( \mathbb{Z}[v, v^{-1}] \)-basis of \( \text{TL}_n \).

3. Given any sequence \( j_1j_2 \cdots j_m \) of integers in \( \{1, \ldots, n\} \), there exists a unique pair \( (x, k) \in \mathcal{W}_f \times \mathbb{Z}_{\geq 0} \) such that

\[
b_{j_1}b_{j_2} \cdots b_{j_m} = (v + v^{-1})^kb_x.
\]
The basis \( \{ b_w \}_{w \in W_f} \) has a well-known interpretation by planar diagrams. There are two quotient maps

\[
\omega : \mathbb{Z}[v, v^{-1}]B_{n+1} \to \text{TL}_n
\]

\[
s_i \mapsto v^{-1} - b_i,
\]

where \( \mathbb{Z}[v, v^{-1}]B_{n+1} \) is the group algebra of \( B_{n+1} \) over \( \mathbb{Z}[v, v^{-1}] \), and

\[
\omega' : \mathbb{Z}[v, v^{-1}]B_{n+1} \to \text{TL}_n
\]

\[
s_i \mapsto b_i - v.
\]

**Remark 3.2.** In fact, both \( \omega \) and \( \omega' \) factor through the Iwahori-Hecke algebra \( \mathcal{H} \) via the natural quotient map \( \pi : \mathbb{Z}[v, v^{-1}]B_{n+1} \to \mathcal{H} \), \( s_i \mapsto vT_s \), where \( T_s \) is the standard generator of \( \mathcal{H} \) (see for example [15] or [12], §4.2.1). There are then two ways to realize \( \text{TL}_n \) as a quotient of \( \mathcal{H} \) by maps \( \theta, \theta' \) defined by

\[
\theta(C_w) = (-1)^{\ell_S(w)}b_w \text{ if } w \in W_f, \text{ while } \theta(C'_w) = 0 \text{ if } w \notin W_f.
\]

Similarly, one has \( \theta'(C'_w) = b_w \) if \( w \in W_f \), while \( \theta'(C''_w) = 0 \) if \( w \notin W_f \) (see [7]).

**Definition 3.3.** The basis \( \{ b_w \}_{w \in W_f} \) is the diagram or Kazhdan-Lusztig basis of \( \text{TL}_n \).

In this paper, we will work with the quotient map \( \omega \). That is, given a braid word, we will obtain its image in \( \text{TL}_n \) by replacing \( s_i \) by \( v^{-1} - b_i \) and \( s_i^{-1} \) by \( v - b_i \) (recall that \( (v^{-1} - b_i)(v - b_i) = 1 \)). Of course, our results can be adapted if one prefers to use the quotient map \( \omega' \).

### 3.2 Zinno basis

The images \( \iota_c(i_c(x)) \), \( x \in \mathcal{P}_c \) of the simple elements of \( B_c^* \) in \( B_{n+1} \), which can be considered as lifts of noncrossing partitions in the braid group, are called canonical factors (shortly canfac) by Zinno; we will call them simple even if they are viewed in \( B_{n+1} \) since this is a more standard name for them. In fact, the way Zinno writes the canfaces corresponds to the choice of the Coxeter element \( c' = s_n \cdots s_2 s_1 \). Since we are working with \( c = s_1 s_2 \cdots s_n \),
we need to reverse the order of the braid words considered in [18]. Recall that for $t = (i, k+1)$, $k \geq i$, the image of $i_c(t)$ in $B_{n+1}$ is given by the braid word

$$s_{i,k+1} := s_k^{-1}s_{k-1}^{-1} \cdots s_{i+1}^{-1}s_is_{i+1} \cdots s_k.$$  

We will abuse notation and also write $i_c(x)$ for the image of it in the braid group which we previously denoted by $i_c(i_c(x))$ since it will make no possible confusion. A braid word such as $s_{i,k+1}$ is called a syllable by Zinno. The braid group generator $s_i$ is the center of the syllable, splitting the syllable into a left part $s_k^{-1}s_{k-1}^{-1} \cdots s_{i+1}^{-1}$ and a right part $s_{i+1} \cdots s_k$. The letters $s_k^{\pm 1}$ are at the top of the syllable. A noncrossing partition $x \in \mathcal{P}_c$ which is a cycle, that is, which is represented by a single polygon is still called a cycle in [18] after lifting in the braid group. Zinno uses the following braid word to represent $i_c(x)$: firstly write $x = (i_1, i_2, \ldots, i_k)$, where $i_1 < i_2 < \cdots < i_k$. We have

$$x = (i_1, i_2)(i_2, i_3) \cdots (i_{k-1}, i_k)$$

and $\ell_T(x) = k$. Then $i_c(x)$ is represented by the braid word

$$s_{i_1,i_2}s_{i_2,i_3} \cdots s_{i_{k-1},i_k}.$$ 

Now if $x$ has more than one cycle, we will represent $i_c(x)$ by the braid word obtained by concatenating the cycles, ordered by the maximal index in each cycle (that is, the terminal index of the associated polygon), in ascending order, and refer to such a word as to the standard form of a simple element of the dual braid monoid. We denote the obtained braid word by $m_x$.

**Example 3.4** Let $x = (1,6)(2,3,5)$ as in figure 1. We have two polygons $P_1 = [235]$ and $P_2 = [16]$. They have as corresponding standard form $s_{1,6}$ and $s_{2,3}s_{3,5}$. We have $P_2 < P_1$ since the maximal index of $P_2$ is 5 and that of $P_1$ is 6. Hence

$$m_x = s_2s_4^{-1}s_3s_4s_5^{-1}s_4^{-1}s_3^{-1}s_2^{-1}s_1s_2s_3s_4s_5.$$ 

**Remark 3.5.** Notice that a braid group generator can be the center of at most one syllable, hence it occurs twice in any other syllable in which it occurs, once in the left part with negative exponent and once in the right part with positive exponent. The way the polygons (equivalently the cycles) are ordered implies that if $s_i$ is the center of a syllable (which is equivalent as saying that $i$ is a non terminal index of a polygon of $x$), then the first occurrence of $s_i^{\pm 1}$ in $m_x$ when reading the word from the left to the right is at the center of that syllable and hence with positive exponent.

If we replace each $s_i^{\pm 1}$ by $s_i$ in $m_x$, we obtain a reduced $\mathcal{S}$-decomposition $m_x$.
of \( x \in \mathcal{P}_c \) which we also call the standard form of \( x \) (and we will also call \( m_t \) for \( t \in \mathcal{T} \) a syllable with a center, left part, etc.); a more general definition of this Coxeter word is given in [8] where we work with arbitrary Coxeter elements. It turns out that the Coxeter word \( m_t \) plays an important role in the study of a basis of the Temperley-Lieb algebra discovered by Zinno (which we will introduce in a few lines) and its generalizations to arbitrary Coxeter elements.

Let \( x \in \mathcal{P}_c \). Set \( Z_x := \omega(i_c(x)) \). Notice that \( \mathcal{S} \subset \mathcal{P}_c \), whence \( Z_s = \omega(s) = v^{-1} - b_i \).

**Theorem 3.6 (Zinno, [18]).** The set \( \{Z_x\}_{x \in \mathcal{P}_c} \) is a \( \mathbb{Z}[v,v^{-1}] \)-linear basis of TL\(_n\).

Before explaining Zinno’s strategy to prove Theorem 3.6 let us point out that there is an alternative proof by Lee and Lee (see [16]) of Zinno’s theorem which is simpler than the original proof, where it is shown that the set above generates the whole algebra, implying for cardinality reasons that it gives a basis of it. However, Zinno proves something stronger which gives some information on the coefficients of the change of basis matrix between \( \{Z_x\}_{x \in \mathcal{P}_c} \) and \( \{b_w\}_{w \in \mathcal{W}_f} \). Hence its approach turns out to be more convenient here since we are interested in a better understanding of the change of basis matrix.

Zinno proves that there are total orders on \( \{Z_x\}_{x \in \mathcal{P}_c} \) and \( \{b_w\}_{w \in \mathcal{W}_f} \) such that there exists an upper triangular matrix with an explicitly computed invertible coefficient on the diagonal allowing one to pass from \( \{b_w\} \) to \( \{Z_x\} \). Since \( \{b_w\} \) is a basis it proves that \( \{Z_x\} \) is also a basis. To this end, he processes as follows: given \( x \in \mathcal{P}_c \), Zinno considers the braid word \( m_x \) representing \( i_c(x) \). He then extracts a subword \( w_x \) of \( m_x \) as follows: firstly, if a syllable has at least one letter indexed by \( i \) (the letters indexed by \( i \) are \( s_i \) and \( s_i^{-1} \)), then that syllable must contribute to the subword exactly one of its letters indexed by \( i \). In particular each center contributes since it is the only letter with its index in a syllable. Secondly, the contributions are as follows: if \( s_i \) is the center of a syllable and occurs in another syllable, then such a syllable contributes the \( s_i^{\pm 1} \) which has positive exponent. If \( s_i \) is not the center of a syllable but there are syllables containing letters indexed by \( i \), then these syllables must contribute their \( s_i^{-1} \) to the subword. In this way we extract a subword \( w_x \). By replacing in that word the \( s_i^{\pm 1} \) by \( s_i \) we get an element \( w_x \) of the Coxeter group.

Thanks to remark 3.5 the rules given above are equivalent to the rules given by the following algorithm: read the word \( m_x \) from left to right. If the first letter \( s_i^{\pm 1} \) occurring in \( m_x \) has positive (resp. negative) exponent, then all the occurrences of \( s_i \) (resp. of \( s_i^{-1} \)) in \( m_x \) and only those must
contribute to the subword \( w_x \). Apply the same process to the next generator \( s_j^{\pm 1}, j \neq i \) occurring right to the first \( s_i^{\pm 1} \) in \( m_x \), until you have considered all the indices \( k \) such that \( s_k^{\pm 1} \) occurs in \( m_x \).

Zinno then shows that \( w_x \) is a reduced expression of a fully commutative element (we will therefore often abuse notation and identify \( w_x \) with the fully commutative element it represents) and that the map \( a : P_c \to \mathcal{W}_f \) defined by \( x \mapsto w_x \) is surjective. Since \(|P_c| = |\mathcal{W}_f|\) the map is bijective. An example of Zinno’s algorithm to extract the fully commutative element \( w_x \) as a subword of a standard form \( m_x \) of a \( i.e(x) \) for \( x \) as in figure 1 is given in example 3.7 below.

Example 3.7 (Zinno’s algorithm to extract \( w_x = a(x) \) as a subword of \( m_x \))

Let \( x = (2, 3, 5)(1, 6) \in P_c \)

\[
\begin{align*}
m_x &= s_2(s_4s_3s_4)(s_5s_4s_3s_4s_5) \\
m_x &= s_2(s_4s_3s_4)(s_5s_4s_3s_4s_5) \\
m_x &= s_2(s_4s_3s_4)(s_5s_4s_3s_4s_5) \\
m_x &= s_2(s_4s_3s_4)(s_5s_4s_3s_4s_5) \\
m_x &= s_2(s_4s_3s_4)(s_5s_4s_3s_4s_5) \\
m_x &= s_2(s_4s_3s_4)(s_5s_4s_3s_4s_5)
\end{align*}
\]

\[ \leadsto w_x = s_2s_4s_3s_5s_4s_3 = (s_2s_1)(s_4s_3s_2)(s_5s_4s_3) \in \mathcal{W}_f. \]

\[ \leadsto w_x = s_2s_4s_5s_3s_4s_5s_3 = (s_2s_4s_5s_3s_4s_5) \in \mathcal{W}_f. \]

Notice that in that special case, we have thanks to Examples 3.7 and 2.12 that \( a(x) = w_x = \varphi(x) \). This is a general fact:

**Proposition 3.8.** The bijection \( \varphi \) and the bijection described in ([18], Theorems 3 and 6) which we denoted by \( a \) are the same.

**Proof.** For \( w \in \mathcal{W}_f \), we will use the characterization of the sets \( I_w \) and \( J_w \) given in remark 2.3.

Let \( x \in P_c \) and write \( w_x = a(x) \in \mathcal{W}_f \) for the element as described above and given by ([18], Theorem 6). By the characterization of \( \varphi \) from Theorem 2.11 we must show that \( i \in I_{w_x} \) if and only if \( i + 1 \in U_w \) and \( i \in J_{w_x} \) if and only if \( i \in D_w \). Let \( i \in I_{w_x} \). It implies that the first occurrence of \( s_i \) in \( w_x \) must come from a syllable \( w \) of \( w_x \) whose first letter is \( s_i^{\pm 1} \): otherwise \( s_{i+1}^{-1} \) would occur in \( w \) on the left of the \( s_i^{\pm 1} \) contributed and that \( s_{i+1}^{-1} \) would be contributed to \( w_x \) in case \( s_{i+1} \) is not a center; in case \( s_{i+1} \) is a center, the occurrence of \( s_{i+1} \) at the center must be the first in the word (by Remark 3.5), hence before \( w \), must be contributed. Hence the first occurrence of \( s_i \)
in \( w_x \) must come from a \( s_i^{\pm 1} \) which is the first letter of its syllable \( w \). But \( s_i^{\pm 1} \) is the first letter of a syllable if and only if \( s_i^{\pm 1} \) is at the top of a syllable if and only if \( i + 1 \in U_x \). Hence \( i + 1 \) lies in \( U_x \).

Conversely, consider an index \( i + 1 \) which labels a vertex of a polygon \( P \) of \( x \) and which is not initial (equivalently, consider \( i + 1 \in U_x \)). Write \( (i_1, \ldots, i_m), \ i_1 < i_2 < \cdots < i_m \) for the cycle corresponding to \( P \). Let \( i + 1 = i_{\ell}, \ \ell \neq 1 \). Then \( s_i^{\pm 1} \) is the first letter of the syllable \( w = s_{i_{m-1},i_m} \) of the cycle corresponding to \( P \). We will show that this letter contributes to \( w_x \), that it is the first occurrence of \( s_i^{\pm 1} \) in \( m_x \) and that there is no occurrence of \( s_i^{\pm 1} \) in \( m_x \) at its left. These properties together imply that \( i \in I_{w_x} \). If there is another letter \( s_i^{\pm 1} \) before \( w \), then it must be in a cycle corresponding to a polygon \( Q \neq P \). Suppose that it occurs as a center of a syllable of the cycle corresponding to \( Q \). It means that \( x \) has a polygon \( Q \) with a non terminal vertex indexed by \( i \) and another polygon \( P \) with a non initial vertex indexed by \( i + 1 \), contradicting the noncrossing property. If it is a center, it cannot be at a top since \( s_i^{\pm 1} \) is already at the top of \( w \) and there can be at most one syllable having it at its top. It implies that it has to be a letter of a syllable \( s_{k,k'} \) where \( k < \min P, \ k' > \max P \) since the polygons are disjoint. If \( Q \) is the polygon whose cycle has as syllable \( s_{k,k'} \), we would have \( \max Q > \max P \), hence \( s_{k,k'} \) would occur after \( s_{i_{m-1},i_m} \) in \( m_x \). Hence our \( s_i^{\pm 1} \) from \( w \) is the first occurrence of \( s_i^{\pm 1} \) in \( m_x \). Now if \( s_{i+1} \) occurs in a syllable, if it is the center then it is in \( P \) and the corresponding syllable appears just after \( w \). If it is not the center, then to respect the noncrossing property one must again have that the syllable containing it appears after \( w \). Therefore we have \( i \in I_{w_x} \).

One shows with a similar argument that \( i \in J_{w_x} \) if and only if \( i \in D_x \). As a consequence one gets that \( a(x) = w_x = \varphi(x) \). Since this holds for any \( x \in P_c \), we have \( \varphi = a \).

4 A new basis of the Temperley-Lieb algebra

Let us first recall some basic facts about the Bruhat order on a Coxeter group.

4.1 Bruhat order

We recall the definition and various characterizations of the Bruhat order on a Coxeter system \((\mathcal{W}, \mathcal{S})\). For \( w, w' \in \mathcal{W} \), we define a relation by \( w \rightarrow w' \) if there exists \( t \in \mathcal{T} \) such that \( w' = tw \) and \( \ell_\mathcal{S}(w') > \ell_\mathcal{S}(w) \). We then extend this relation to a partial order \( <_\mathcal{S} \) by setting \( w <_\mathcal{S} w' \) if there exists \( w_1, \ldots, w_k \in \mathcal{W} \) such that \( w \rightarrow w_1 \rightarrow w_2 \rightarrow \cdots \rightarrow w_k \rightarrow w' \). It is the
Bruhat order of the Coxeter system \((W, S)\). The following characterization is classical (see for example [4], Corollary 2.2.3):

**Proposition 4.1.** For \(w, w' \in W\), the following are equivalent:

1. One has \(w <_S w'\),
2. Any \(S\)-reduced expression for \(w'\) has a subword that is an \(S\)-reduced expression for \(w\),
3. There exists an \(S\)-reduced expression for \(w'\) which has a subword that is an \(S\)-reduced expression for \(w\).

### 4.2 A new basis

In this section, we are working exclusively with the Coxeter element \(c = s_1s_2 \cdots s_n\). We will denote by \(<_S\) the Bruhat order on \(S_{n+1}\). For \(w \in W_f\) we set

\[
L(w) = \{ s \in S \mid sw <_S w \}, \\
R(w) = \{ s \in S \mid ws <_S w \}.
\]

**Remark 4.2.** Notice that if \(s, t \in L(w)\), then \(ts = st\). The same holds if both \(s, t\) lie in \(R(w)\). Moreover, if \(s \in L(w)\) (resp. \(R(w)\)), then \(sw\) (resp. \(ws\)) lies again in \(W_f\).

Given a transposition \((i, j) \in T\), assume that a polygon \(P\) occurring in the geometrical representation of \(x \in P_c\) has an edge joining the point with index \(i\) to the point with index \(j\). We will also denote this edge by \((i, j)\) (notice that if \((i, j)\) is an edge or a diagonal of a polygon of \(x\), it implies that \((i, j) <_T x\)).

**Proposition 4.3.** Let \(w \in W_f\), \(s_i \in S\). Then

1. \(s_i \in L(w)\) if and only if \(\{(i, i+1)\) is an edge of a polygon \(P \in \text{Pol}(\psi(w))\) with \(i\) initial\} or \(\{\text{the point with index } i \text{ is not a vertex of a polygon of } \psi(w) \text{ but there exists a polygon } P \in \text{Pol}(\psi(w)) \text{ having an edge } (k, i+1) \text{ for some } k < i\}\).

2. \(s_i \in R(w)\) if and only if \(\{(i, i+1)\) is an edge of a polygon \(P \in \text{Pol}(\psi(w))\) with \(i + 1\) terminal\} or \(\{\text{the point with index } i + 1 \text{ is not a vertex of a polygon of } \psi(w) \text{ but there exists a polygon } P \in \text{Pol}(\psi(w)) \text{ having an edge } (i, i+k) \text{ with } k > 1\}\).
Proof. One has that \( s \in L(w) \) if and only if \( i \in I_w, i - 1 \notin I_w \) if and only if \( i + 1 \notin U_{\psi(w)} \) or \( \{i, i + 1\} \) is an edge of a polygon of \( \psi(w) \) with \( i \) initial. Hence we again have \( \ell_S(y') = \ell_S(y) - 1 \). But if a noncrossing partition \( x \in \mathcal{P}_c \) has decomposition into disjoint cycles \( y_1y_2 \cdots y_k \), one has (see Remark 2.6) that

\[
\ell_S(y) = \sum_{j=1}^{k} \ell_S(y_j),
\]

which concludes. In case \( i \) is not an index of a vertex of a polygon of \( \psi(w) \), but there is a polygon \( P' \) obtained from \( P \) by removing the vertex labelled by \( i \). If the set of indices of vertices of \( P \) is given by \( d_1, \ldots, d_k \), \( d_j < d_{j+1} \) with \( d_m = k, d_{m+1} = i + 1 \), an \( S \)-reduced expression of \( y \) is given by the concatenation

\[
[d_1, d_2][d_2, d_3] \cdots [d_{k-1}, d_k],
\]

where \( [j, \ell] = s_{\ell-1}s_{\ell-2} \cdots s_{j+1}s_js_{j+1} \cdots s_{\ell-2}s_{\ell-1} \) (see Remark 2.6). Adding the vertex \( i \) replaces in the product above the subword \([d_m, d_{m+1}]\) by \([d_m, i][i, d_{m+1}]\) and this just removes one occurrence of \( s_i \). Hence we again have \( \ell_S(s_iy) = \ell_S(y) - 1 \) and the same argument as for the first case gives the conclusion. The proof of the case where \( s \in R(w) \) is similar.

**Corollary 4.5.** Let \( w \in \mathcal{W}_f \). Then

\[
s_i \in L(w) \cap R(w) \iff s_i <_T \psi(w) \text{ and } s_i\psi(w) = \psi(w)s_i
\]
\( \iff \) There exists a polygon of \( \psi(w) \) which is reduced to the edge \((i, i+1)\).

**Proof.** It is an immediate consequence of proposition 4.3 (\(i, i+1\)) must be a vertex of a polygon with both \(i\) initial and \(i+1\) terminal (the two other conditions together give a contradiction to the noncrossing property).

**Corollary 4.6.** Let \( w \in W_f \). Let \( s \in L(w), t \in R(w), \) with \( s \neq t \). Then

\[
s\psi(w) = \psi(w)t \iff s = s_j, \ t = s_{j-1} \text{ for some index } j.
\]

**Proof.** Let \( s = s_j, t = s_k \) and suppose \( s\psi(w) = \psi(w)t \). Thanks to proposition 4.3 applying \( s_j \) on the left of \( \psi(w) \) either adds or removes the vertex with index \( j \) (and possibly the vertex with index \( j + 1 \) but in that case, one would have \( s \in R(w) \); since \( t \neq s \) the reflection \( t \) would then remove a vertex with index \( k \) distant from \( j \) since any two reflections in \( R(w) \) commute with each other, a contradiction to \( s\psi(w) = \psi(w)t \) since the operation of \( s \) in the left hand side does not change the vertex with index \( k \)). So we can suppose that \( s \) removes or adds the vertex with index \( j \), leaving all other vertices of the polygons unchanged. This means that \( t \) also has to remove or add the vertex with index \( j \). This is possible only if \( t = s_{j-1} \) or \( t = s_j \) but the last case is excluded. Conversely, the assumption implies by the above proposition that \( \psi(w) \) has a polygon \( P \) having an edge \((j-1, j+1)\). We then have that \( s_j\psi(w) = \psi(w)s_{j-1} \) and in the geometrical representation, it corresponds to adding the vertex with index \( j \) to the polygon \( P \). \( \square \)

**Notation.** Let \( w \in W_f, L \subset L(w) \) and \( R \subset R(w) \). We build new sets \( L', R' \) from \( L \) and \( R \) by doing the following: if \( s \in L \cap R \), we either remove \( s \) from \( L \) or remove it from \( R \). If \( s_j \in L \) and \( s_{j-1} \in R \), then we either remove \( s_j \) from \( L \) or remove \( s_{j-1} \) from \( R \). At the end of the process we get two (non canonically defined) sets \( L' \subset L \) and \( R' \subset R \). It is clear that if \((L', R')\) and \((\bar{L}', \bar{R}')\) are two distinct sets with these properties, one has \(|L' \cup R'| = |\bar{L}' \cup \bar{R}'|\).

**Example 4.7** Let \( w = s_2s_1s_3 \). Then \( L(w) = \{s_2\}, R(w) = \{s_1, s_3\} \). Let \( L = L(w), R = R(w) \). One can choose \( L' = \{s_2\}, R' = \{s_3\} \). Another possible choice is \( L' = \emptyset, R' = \{s_1, s_3\} \).

The following proposition is a generalization of corollary 4.4

**Proposition 4.8.** Let \( w \in W_f, L \subset L(w), R \subset R(w) \). Then

\[
x_{L', R'} := (\prod_{s \in L'} s)\psi(w)(\prod_{s \in R'} s)
\]

is independent of the choice of \( L' \) and \( R' \) and will therefore be denoted by \( x_{L, R} \). Moreover, \( x_{L, R} \) lies in \( P_c \), \( x_{L, R} \leq_S \psi(w) \) and \( \ell_S(x_{L, R}) = \ell_S(\psi(w)) - |L' \cup R'| \).
Proof. One can argue by induction on \(|L' \cup R'\)|. If it is equal to zero, it means that \(L = \emptyset = R\), in which case the claim is trivially true. If \(L' \cup R'\) is a singleton, the claim is true by corollaries 4.4, 4.5, and 4.6. Now suppose that \(|L' \cup R'| > 1\) and remove an arbitrary reflection \(s_j\) from \(L' \cup R'\), say from \(L'\), the other case being similar. Write \(L'' = L' \setminus \{s_j\}\). One can choose \((L'')' = L'', (R')' = R'\). Since \(s \in L(w)\), it means by proposition 4.3 that in the representation of \(\psi(w)\) by disjoint unions of polygons, we has one of the two following configurations: either \((j, j + 1)\) is an edge of a polygon of \(\psi(w)\) with \(j\) initial, or \(j\) does not index any vertex of a polygon of \(\psi(w)\) but one has a polygon of \(\psi(w)\) with an edge \((k, j + 1)\) where \(k < j\). Now any reflection in \(L''\) is distant from \(s_j\) and \(R'\) contains neither \(s_j\) nor \(s_{j-1}\). Using proposition 4.3 again this implies that any of the two possible configurations are preserved when reducing from \(\psi(w)\) to \(y := (\prod_{s \in L''} s) \psi(w) (\prod_{s \in R'} s)\) (the configuration with an edge \((j, j + 1)\) is preserved and since \(s_{j-1} \notin R'\) the only thing that can change the edge \((k, j + 1)\) of the second configuration is in case we have an edge \((k, j + 1)\) with \(k < j - 1\) and \(s_k \in R'\); in that case the edge \((k, j + 1)\) is replaced by an edge \((k + 1, j + 1)\) in \(y\) and \(y\) still has the second configuration since \(k + 1 < j\)). In particular, using the same proposition, we get \(s_j \in L(\varphi(y))\). Induction together with corollary 4.4 conclude. 

Notation. For \(w \in \mathcal{W}_f\), \(x \in \mathcal{P}_c\), we set

\[\alpha_w(x) := \ell_x(w) + \ell_x(\psi(w)) - \ell_y(x) \in \mathbb{Z}.\]

Definition 4.9. To each fully commutative element \(w \in \mathcal{W}_f\), we will associate an element \(X_w\) of the Temperley-Lieb algebra called the simplex of \(w\).

Set

\[Q_w := \{ x_{L,R} \mid L \subseteq L(w), R \subseteq R(w) \}.\]

We then define \(X_w\) by its coefficients when expressed in Zinno’s basis:

\[X_w := \sum_{x \in \mathcal{P}_c} p_x^w Z_x,\]

where \(p_x^w = 0\) unless \(x \in Q_w\). If \(x \in Q_w\) then set

\[p_x^w := (-1)^{\alpha_w(x)} v^{(\ell_x(w) - \ell_y(\psi(w))}.\]

Remark 4.10. As a consequence of proposition 4.8, one has \(sQ_w = Q_w\) for any \(s \in L(w)\) and \(Q_w s = Q_w\) for any \(s \in R(w)\). In particular, \(|Q_w|\) is always a power of two and is at least two if \(w \neq e\) since for any \(w \in \mathcal{W}_f\), \(L(w) \cup R(w) \neq \emptyset\).

Example 4.11 \(w = s_1 s_4 s_3 s_2\), \(\psi(w) = s_1 s_4 s_3 s_2 s_3 s_4\)
Remark 4.12. Notice that for $s_i \in S$,

$$X_{s_i} = p_{s_i}^{e_i} Z_{s_i} + p_{s_i}^{e_i} = -Z_{s_i} + v^{-1} = b_i = b_{s_i}.$$  

In general $X_w \neq b_w$.

**Proposition 4.13.** The set $\{X_w\}_{w \in W_f}$ is a basis of the Temperley-Lieb algebra.

**Proof.** It suffices to order Zinno basis by the order on $P_c$ given by any linear extension of the order induced by the length function $\ell_S$ restricted to $P_c$. One then orders the set $\{X_w\}_{W_f}$ by the order on $W_f$ obtained as the image of the order we put on $P_c$ under the bijection $\varphi$. Thanks to proposition 4.8, one then gets an upper triangular matrix with the invertible coefficients $\{p_{\varphi(w)}\}_{w \in W_f}$ on the diagonal, passing from the basis $\{Z_x\}_{x \in P_c}$ to the set $\{X_w\}_{w \in W_f}$. Theorem 3.6 then concludes. \qed

Remark 4.14. To have triangularity (and invertibility of the diagonal coefficients) of the change of basis matrix between $b_w$ and $Z_x$, Zinno order $P_c$ by any linear extension of the order induced by the lengths of the lifts $\{m_x\}_{x \in P_c}$. Since the braid words $\{m_x\}_{x \in P_c}$ are reduced, it is the same function as the Coxeter length function $\ell_S$ on the indexing set $P_c$, hence the same order as the one we consider in the proof above. As a consequence, these orders also give triangularity of the change of basis matrix between $X_w$ and $b_w$, with invertible coefficient on the diagonal.

**Lemma 4.15.** Let $w \in W_f$, $s \in L(w)$. Then

$$b_sX_w = (v + v^{-1})X_w.$$  

**Proof.** Let $x \in Q_w$ such that $sx <_S x$. Since $s \in L(w)$ one has that $sx \in Q_w \subset P_c$ thanks to remark 4.10. One has either $sx <_T x$, in which case
In this section, we use the newly introduced basis \(\{Z_i\}_{i=1}^\ell\) to compute some coefficients of the change of basis matrix between the diagram and Zinno bases and give a necessary condition for the coefficients to be nonzero.

One has \(\ell(x) = \ell(x) - 1\) hence \(p_x^w = -vp_x^w\) so we get

\[
\begin{align*}
b_s(p_{sx}^w Z_{sx} + p_x^w Z_x) &= (v - Z_s)(p_{sx}^w Z_{sx} + p_x^w Z_x) \\
&= v^{-1}p_{sx}^w Z_{sx} - p_{sx}^w Z_s Z_{sx} + v^{-1}p_x^w Z_x - p_x^w Z_s Z_{sx} \\
&= v^{-1}p_{sx}^w Z_{sx} + 2v^{-1}p_x^w Z_x - p_x^w (Z_s(v^{-1} - v) + 1) Z_{sx} \\
&= (v + v^{-1})(p_{sx}^w Z_{sx} + p_x^w Z_x).
\end{align*}
\]

Now assume that \(x < T x\). One has \(\ell_T(x) = \ell_T(x) + 1\) hence \(p_x^w = -v^{-1}p_{sx}^w\) so we get

\[
\begin{align*}
b_s(p_{sx}^w Z_{sx} + p_x^w Z_x) &= (v - Z_s^{-1})(p_{sx}^w Z_{sx} + p_x^w Z_x) \\
&= vp_{sx}^w Z_{sx} - p_{sx}^w Z_x + vp_x^w Z_x - p_x^w Z_s Z_x \\
&= vp_{sx}^w Z_{sx} + 2vp_x^w Z_x - p_x^w ((v - v^{-1}) + Z_s) Z_x \\
&= (v + v^{-1})(p_{sx}^w Z_{sx} + p_x^w Z_x).
\end{align*}
\]

Summing these equalities on all the couples \((sx, x)\) one gets the result. \(\Box\)

Remark 4.16. One has of course a similar statement for \(s \in R(w)\).

5 Application: coefficients of the change of basis matrix between the diagram and Zinno bases

In this section, we use the newly introduced basis \(\{X_w\}_{w \in W}\) to explicitly compute some coefficients of the change of basis matrix between the diagram and Zinno bases and give a necessary condition for the coefficients to be nonzero.

5.1 Zinno basis and Bruhat order

Remark 5.1. Let \(x \in P_c\). Recall that \(m_x\) is a reduced braid word representing the simple element \(i_c(x)\) in the braid group. As we previously noticed, of one replaces any \(s_i^+\) by \(s_i\) in \(m_x\), one obtains a Coxeter word \(m_x\) that is an \(S\)-reduced expression of \(x\). After being mapped to the Temperley-Lieb algebra, any letter \(s_i\) is replaced by \(v^{-1} - b_i\) while each letter \(s_i^{-1}\) is replaced by \(v - b_i\). As a consequence, if we expand the image of \(m_x\) in \(TL_m\), we obtain a linear combination of elements of the form \(b_{i_1} b_{i_2} \cdots b_{i_k}\) where \(s_{i_1} s_{i_2} \cdots s_{i_k}\) is a subword of \(m_x\). If \(s_{i_1} s_{i_2} \cdots s_{i_k}\) is not an \(S\)-reduced expression of a fully
commutative element, then \( b_i b_{i_2} \cdots b_{i_k} \) is not a reduced word, but it is equal to \((v + v^{-1})^k b_w\) for a unique pair \((k, w) \in \mathbb{Z}_{>0} \times \mathcal{W}_f\) and as a consequence of the Temperley-Lieb relations, \( w \) has an \( S \)-reduced expression which is a subword of \( s_{i_1} s_{i_2} \cdots s_{i_k} \). But \( s_{i_1} s_{i_2} \cdots s_{i_k} \) was itself a subword of \( m_x \). Since \( m_x \) is an \( S \)-reduced expression of \( x \), it follows that \( w <_S x \). Hence in the linear combination of \( Z_x \) in the diagram basis, the \( w \in \mathcal{W}_f \) indexing the \( b_w \) which occur must satisfy \( w <_S x \).

As mentioned in Remark 4.14, Zinno orders \( \mathcal{P}_c \) by any linear extension of the order induced by \( \ell_S \). He then proves the following theorem, which is rewritten here using our notations and Remark 5.1:

**Theorem 5.2** (Zinno, [18], Theorem 5). Let \( x \in \mathcal{P}_c \) and assume \( w <_S x \). If \( w \neq \varphi(x) \), there exists an element \( y \in \mathcal{P}_c \) such that \( w <_S y \) and \( \ell_S(y) < \ell_S(x) \).

It turns out that in Zinno’s proof, various cases are done and in all the cases considered, one sees that the element \( y \) which is built is a subword of \( m_x \), hence we can refine his conclusion by \( y <_S x, y \neq x \), which will be useful for a study of the coefficients of the change of basis matrix between the Zinno and diagram bases. In the following we will use this refinement. Zinno then uses this theorem to prove that with the same assumptions as in the theorem, one has then \( \ell_S(\psi(w)) < \ell_S(x) \). Again, it is not difficult to see from Zinno’s proof that one can refine the conclusion by \( \psi(w) <_S x, \psi(w) \neq x \). One argues as follows: consider the set

\[
Y_w := \{ y \in \mathcal{P}_c \mid w <_S y <_S x, y \neq x \}.
\]

Using the refinement of Theorem 5.2, we know that \( Y_w \neq \emptyset \). Let \( y \in Y_w \) with minimal Coxeter length. If \( y \neq \psi(w) \), then \( \varphi(y) \neq w \). Applying once again the refinement of Theorem 5.2 we get that there exists \( y' \in \mathcal{P}_c \) such that \( w <_S y' <_S y, y \neq y' \), a contradiction to the minimality of the Coxeter length of \( y' \). In other words, \( Y_w \) has a unique element of minimal Coxeter length. Hence \( y = \psi(w) \in Y_w \). So we have the following:

**Proposition 5.3.** Let \( w \in \mathcal{W}_f, x \in \mathcal{P}_c \) and assume that \( w <_S x \). Then

\[
\psi(w) <_S x.
\]

**Remark 5.4.** The fact that the order hidden behind the Zinno basis is the Bruhat order on \( \mathcal{P}_c \) is a surprising fact since it is a highly non-natural order on \( \mathcal{P}_c \) (the natural order being \( <_T \), which in the framework of noncrossing partitions is just the "is finer than" order). In [8], we give a criterion for the
Bruhat order on $\mathcal{P}_c$ and prove that the poset $(\mathcal{P}_c, <_S)$ is a lattice, isomorphic to the lattice of order ideals in the root poset of type $A_n$, which is also isomorphic to the lattice of Dyck paths for inclusion. The criterion given there can also be used to give another proof of the refinement of Theorem 5.2 mentioned above.

5.2 The new basis as an intermediate basis

We now consider the linear expansion of an element $b_w$ in the basis $X_w'$

$$b_w = \sum_{w' \in \mathcal{W}_f} q_{w, w'}^{w'} X_{w'}$$

and we would like to understand for which $w'$ one can have $q_{w, w'}^{w'} \neq 0$. To this end, we write the element $X_w$ in the Kazhdan-Lusztig basis as

$$X_w = \sum_{y \in \mathcal{W}_f} r_{w, y}^{w} b_y.$$

**Notation.** To each fully commutative element $w \in \mathcal{W}_f$ we associate a subset $F_w \subset \mathcal{W}_f$ defined by

$$F_w = \{ y \in \mathcal{W}_f \mid L(y) \supset L(w), R(y) \supset R(w) \text{ and } \psi(y) <_S \psi(w) \}.$$ 

**Remark 5.5.** Obviously one has $w \in F_w$ and if $y \in F_w$, then $F_y \subset F_w$. The inclusion of these sets defines a new partial order on $\mathcal{W}_f$.

**Proposition 5.6.** If $r_{y, y}^{w} \neq 0$, then $y \in F_w$.

**Proof.** Let $s \in L(w)$. Thanks to lemma 4.15 one has that

$$b_s \left( \sum_{y \in \mathcal{W}_f} r_{y, y}^{w} b_y \right) = (v + v^{-1}) \left( \sum_{y \in \mathcal{W}_f} r_{y, y}^{w} b_y \right).$$

Among all the $y$ for which $r_{y, y}^{w}$ is nonzero, choose an element $y$ such that $\ell_S(y)$ is maximal. It follows from this equality and the maximality of $\ell_S(y)$ that in case $\ell_S(sy) > \ell_S(y)$, then $sy$ cannot be a fully commutative element. In other words, when reducing $b_s b_y$, one has to apply the relation $b_s^2 = (v + v^{-1}) b_s$ (in case $sy <_S y$) or the relation $b_s b_{s \pm 1} b_s = b_s$ where $s = s_i$ (in case $sy >_S y$). In the first case $y$ has an $S$-reduced expression beginning with $s$, hence $s \in L(y)$ implying $b_s b_y = (v + v^{-1}) b_y$. In the second case since
$b_y$ also appears in the right hand side of the equality above it means that there exists a fully commutative element $y'$ such that $r^{w}_{y'} \neq 0$ having an $S$-reduced expression beginning with $b_{s_{i \pm 1}}b_s$. But such an element also occurs in the right hand side and cannot obviously come from an element $b_yb_{y''}$ with $y'' \in W_f$, a contradiction. Hence it means that our element $y$ has an $S$-reduced expression beginning with $s$, that it, $s \in L(y)$ and that we can remove $b_yb_y = (v + v^{-1})b_y$ from both sides of the equality above obtaining

$$b_s \left( \sum_{z \in W_f, z \neq y} r^{w}_{z}b_z \right) = (v + v^{-1}) \left( \sum_{z \in W_f, z \neq y} r^{w}_{z}b_z \right).$$

One can then choose another element $z$ with maximal Coxeter length among the remaining ones with nonzero coefficient and give the same argument to obtain that $s \in L(z)$ and so on until we run out of all the elements with nonzero coefficient. This proves that for any $s \in L(w)$, $s \in L(y)$ for any $y$ such that $r^{w}_{y} \neq 0$. Doing the same for any $s \in R(w)$ one gets that for any $y$ such that $r^{w}_{y} \neq 0$, $L(y) \supset L(w)$ and $R(y) \supset R(w)$.

Now if $y$ is such that $r^{w}_{y} \neq 0$, one must have $y <_{S} x$ for at least one $x \in Q_w$ by remark 5.1. Thanks to Proposition 5.3, we have $\psi(y) <_{S} x$ and thanks to proposition 4.8 one also has that $x <_{S} \psi(w)$ giving $\psi(y) <_{S} \psi(w)$. Therefore we have that $y \in F_w$.

**Proposition 5.7.** If $q^{w}_{w'} \neq 0$, then $w' \in F_w$.

**Proof.** One argues by induction of $\ell_{S}(\psi(w))$. If $\ell_{S}(\psi(w)) = 1$ then $w$ is a simple reflection. In that case by remark 4.12 one has $b_w = X_w$ and the claim is trivially true since $F_w = \{ w \}$. Now suppose that $\ell_{S}(\psi(w)) > 1$. Thanks to the previous proposition we have that

$$X_w = \sum_{y \in F_w} r^{w}_{y}b_y,$$

in particular, $\psi(y) <_{S} \psi(w)$, hence $\ell_{S}(\psi(y)) < \ell_{S}(\psi(w))$ in case $w \neq y$. Hence by induction one has that

$$b_y = \sum_{z \in F_y} q^{y}_{z}X_z$$

which we replace in the previous equality:

$$X_w = r^{w}_{w}b_w + \sum_{y \in F_w, y \neq w} r^{w}_{y} \left( \sum_{z \in F_y} q^{y}_{z}X_z \right).$$
But since $y \in F_w$, one has that $F_y \subset F_w$ (see remark 5.5), hence the equality can be rewritten as

$$X_w = r_w^w b_w + \sum_{y \in F_w, y \neq w} \tilde{q}_y^w X_y$$

for suitable polynomials $\tilde{q}_y^w$, which concludes since $r_w^w$ is invertible (see Remark 4.14).

Now write the expansion of an element $b_w$ in Zinno basis as

$$b_w = \sum_{x \in \mathcal{P}_x} h_x^w Z_x.$$

As an immediate consequence of the proposition above we get:

**Corollary 5.8.** If $x \notin \bigcup_{y \in F_w} Q_y$, then $h_x^w = 0$.

**Lemma 5.9** (Zinno, [18]). Let $w \in \mathcal{W}_f$, $x = \psi(w)$. The coefficient of $b_w$ in the expansion of $Z_x$ in the diagram basis is equal to

$$(−1)^{\ell_S(w)} v^{-2k_w + \ell_S(w) - \ell_T(x)},$$

where $k_w$ is the number of letters of $m_x$ which have negative exponent and contribute to $w_x$.

**Proof.** The coefficient on the diagonal is explicitly computed by Zinno in [18] at the end of section 6. Since we have different notations and conventions we sketch a proof. Let $x = \psi(w)$. Recall that $Z_x$ is the image of the element of the braid group represented by the word $m_x$ in the Temperley-Lieb algebra. It is obtained by replacing each letter $s_i$ in $m_x$ by $v^{-1} - b_i$ and each letter $s_i^{-1}$ by $v - b_i$. Hence if we expand without reducing, we obtain $2^{\ell_S(x)}$ different terms: for each $s_i^\pm$ occurring in $m_x$ we can either choose the $-b_i$ or the $v^\pm$. Recall that there is a rule to read $w_x$ which is a reduced expression for $w$ as a subword of $m_x$ that we recalled in example 3.7 and in the paragraphs above it. Zinno proves that among the $2^{\ell_S(x)}$ terms which are (possibly non reduced) words in the $b_i$ multiplied by a power of $v$, the term obtained by taking the $b_i$ from any $s_i^\pm$ contributing to $w_x$ and taking the $v^\pm$ from any other $s_i^\pm$ is the only term among the $2^{\ell_S(x)}$ which is proportional to $b_w$ (reference a ajouté). But its coefficient is easily computed: each $b_i$ which is contributed is multiplied by $-1$, and since a $b_i$ is contributed exactly from the $s_i^\pm$ contributing to $w_x$ and since moreover $w_x$ is an $S$-reduced expression of $w$, this gives rise to a sign $(-1)^{\ell_S(w)}$. Now each $s_i^\pm$ not contributing to $w_x$ must contribute its $v^\pm$. For any $s_i^{-1}$ contributing to $w_x$, there is an $s_i \mapsto v^{-1} - b_i$ at its right which does not contribute, giving a coefficient $v^{-k_w}$.
Now if a \( s_i \) contributes to \( w_x \), it means that \( s_i \) is the center of a syllable. As a consequence all the \( s_i^{-1} \) do not contribute to \( w_x \). We need to count them. The number of occurrences of all the various \( s_i^\pm 1 \) with \( s_i \) occurring at a center is given by \( \ell_S(x) - 2k_w \). We then need to subtract the centers and there are \( \ell_T(x) \) many of them. We then need to divide the result by two since we have here all the \( s_i^\pm 1 \) such that the instance with positive exponent contribute with the centers removed, but any instance \( s_i \) of one of these comes with an instance of \( s_i^\pm 1 \) in the same syllable since we removed the centers. Hence the power of \( v \) we obtain from the \( s_i \) not contributing to \( w_x \) is equal to

\[
\frac{\ell_S(x) - 2k_w - \ell_T(x)}{2}
\]

so the power of \( v \) we obtain before our \( b_w \) in the expansion is

\[-k_w + \frac{\ell_S(x) - 2k_w - \ell_T(x)}{2}.
\]

One gets the claim using the equality

\[
\frac{\ell_S(x) - \ell_T(x)}{2} = \ell_S(w) - \ell_T(x)
\]

which holds since all the centers contribute to \( w_x \); hence the left hand side is equal to all the contribution to \( w \) different from the centers (recall that any syllable contributes any of its reflections exactly once to \( w \) and that if \( s_i \) is not at the center, it occurs twice in the syllable).

**Proposition 5.10.** Let \( w \in \mathcal{W}_f \). Let \( x \in Q_w \). Then

\[
h_x^w = (-1)^{\alpha_w(x)} v^{2k_w + \ell_T(x) - \ell_S(w)},
\]

where \( k_w \) is the number of letters of \( m_{\psi(w)} \) which have negative exponent and contribute to \( w_{\psi(w)} \).

**Proof.** This is a consequence of the fact that if \( y \in F_w, y \neq w \), then \( Q_w \cap Q_y = \emptyset \). Indeed, assume that \( x \in Q_w \cap Q_y \). Then there exist two sets \( L' \subset L(w) \), \( R' \subset R(w) \) such that

\[
(\prod_{s \in L'} s)x(\prod_{s \in R'} s) = \psi(w).
\]

Since \( L(y) \supset L(w) \) and \( R(y) \supset R(w) \) and \( x \in Q_y \), one also has using remark 4.10 that

\[
(\prod_{s \in L'} s)x(\prod_{s \in R'} s) \in Q_y.
\]

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Theorem 5.11. Let \( F_w, \psi(y) \prec_S \psi(w) = x, \psi(w) \neq \psi(y) \) and any element \( z \in Q_y \) satisfies \( z \prec_S \psi(y) \). Hence \( x \prec_S \psi(y) \prec_S \psi(w) = x \), a contradiction.

As a consequence of this observation together with corollary 5.8 if one knows the coefficient of \( Z_{\psi(w)} \) in the expansion of \( b_w \), one knows the coefficient of any \( Z_x \) for \( x \in Q_w \) since the only element of the simplex-basis which can contribute elements \( Z_x \) for \( x \in Q_w \) is \( X_w \). Using lemma 5.9 we have that the inverse coefficient of \( b_w \) in the expansion of \( Z_{\psi(w)} \) is equal to \((-1)\ell_S(w)\nu^{2k_w-\ell_S(w)+\ell_T(\psi(w))}\). Therefore since the change of basis matrix is upper triangular with invertible coefficient on the diagonal one has that the coefficient of \( Z_{\psi(w)} \) in the expansion of \( b_w \) is given by

\[
(-1)\ell_S(w)\nu^{2k_w-\ell_S(w)+\ell_T(\psi(w))}.
\]

Using the fact that

\[
b_w = \sum_{w' \in F_w} q_{w'}^w \cdot X_{w'}
\]

and that any element \( Z_x \) with \( x \in Q_w \) is contributed exclusively by \( X_w \), one has that

\[
q_{w'}^w \cdot p_{\psi(w)} = (-1)^{\ell_S(w)}\nu^{2k_w-\ell_S(w)+\ell_T(\psi(w))},
\]

hence \( q_{w'}^w = \nu^{2k_w-\ell_S(w)+\ell_T(\psi(w))} \) since \( p_{\psi(w)} = (-1)^{\ell_S(w)} \). Hence for any \( x \in Q_w \) we obtain

\[
h_x^w = q_{w'}^w \cdot p_x = (-1)^{\alpha_w(x)}\nu^{2k_w+\ell_T(x)-\ell_S(w)},
\]

as claimed. \( \square \)

Putting Corollary 5.8 and Proposition 5.10 together we have:

**Theorem 5.11.** Let \( w \in \mathcal{W}_f, x \in \mathcal{P}_c \).

1. If \( x \notin \bigcup_{y \in F_w} Q_y \), then \( h_x^w = 0 \).

2. If \( x \in Q_w \), then \( h_x^w = (-1)^{\alpha_w(x)}\nu^{2k_w+\ell_T(x)-\ell_S(w)} \).

**Remark 5.12.** Theorem 5.11 explicitly gives the coefficient \( h_x^w \) of the change of basis matrix between the Zinno and diagram bases except in case \( x \in \bigcup_{y \in F_w, y \neq w} Q_y \). There are two main difficulties in that case: firstly, examples show that the condition of 5.8 is not sufficient, and secondly, one may have \( y, y' \in F_w \setminus w \) such that \( Q_y \cap Q_{y'} \neq \emptyset \), hence different sets \( Q_y, Q_{y'} \) may contribute the same element \( Z_x \) of Zinno basis for \( x \in Q_y \cap Q_{y'} \); the sum of the various contributions may be zero, but in some cases they are not, hence can yield a coefficient \( h_x^w \) which is not a monomial. It is the case for example, as computations with GAP show, in type \( A_4 \) for \( x = s_4s_3s_2s_1s_2s_3s_4 = (1, 5) \) and \( w = c^{-1} = s_4s_3s_2s_1 \) where \( h_x^w \) is not a monomial.
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