QUANTUM GROUPS

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ABSTRACT

These notes correspond rather accurately to the translation of the lectures given at the Fifth Mexican School of Particles and Fields, held in Guanajuato, Gto., in December 1992. They constitute a brief and elementary introduction to quantum symmetries from a physical point of view, along the lines of the forthcoming book by C. Gómez, G. Sierra and myself.

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1 Introduction

Since the advent of physics as a pleasure for the human mind, those in our trade have played with idealizations of reality which preserve the essence of the phenomenon and yet are simple enough to be modelled by the mathematical tools available. These “toy models”, from the point–like friction-less particle in newtonian mechanics to scalar quantum electrodynamics, constitute most of the syllabus of a physicist’s education. At the very worst, the theories and models we shall discuss in these lectures can be taken as paradigmatic toy models; indeed most of two–dimensional field theory was invented as a theoretical laboratory for confinement, dimensional transmutation, instantons, and other such niceties of the real world. Yet something deep remains hidden in the guts of two–dimensional physics. The historic success of string theory in unifying gauge symmetries and general relativity has given the two-dimensional world a new and fruitful life of its own: from a stringy point of view, enough remains to be learned about the fundamental world–sheet that we need not bother for a while about other dimensions.
Quite surprisingly, condensed matter physicists have also come to appreciate the interest of low-dimensional field theories, motivated by the planar character of the quantum hall effect and high-temperature superconductivity or the technological interest in thin plastic and silicon chips, among other noteworthy phenomena.

These lectures are meant to illustrate some of the beautiful tricks that can be applied to understanding two-dimensional physics. Were we to think of these lectures as a meal, a menu would read more or less as follows. For appetizers, some relativistic dynamics in one spatial dimension. The reader should then sit down to a light salad of Bethe ansatz, followed by a hot soup of integrable vertex models on the plane. The main course consists of Yang–Baxter algebras, with a variety of sauces of various mathematical origins. A few words about the possible generalization of these treats to higher dimensions are left for desert. All the wines and liquors come from the quantum group vintage, distilled at Kyoto and Kharkov from the well-known and now extinct Leningrad stock. To avoid indigestion, the beautiful example of two-dimensional integrability provided by conformal field theories is not presented; Professor Weyers’s lectures in this same volume cover that.

Quantum groups have been discovered relatively recently by physicists and mathematicians concerned with integrable two-dimensional systems. An integrable system has as many integrals of motion (constants) as co-ordinates or, equivalently, momenta; accordingly, in an integrable theory we know that the phase space is spanned by the so-called action–angle variables, essentially a bunch of conserved quantities (hamiltonians) and their conjugate variables (times). Classical two-dimensional statistical physics is equivalent to quantum field theory in one (spatial) dimension. In these lectures we shall investigate two-dimensional systems with an infinite number of degrees of freedom: we shall be concerned with the thermodynamic limit of classical two-dimensional statistical models.

It is perhaps more intuitive to start with quantum field theories in one space and one time dimension, that is quantum field theories on the real line. Integrable field theories in such a small dimension is essentially equivalent to the description of solitons. Indeed, a soliton is a non-dispersive classical solution to the classical equations of motion which survives quantization and acquires a particle–like interpretation. Feynman rules are, in general, rather useless in the description of solitons. Collective phenomena of this sort are not perturbative at all, and in low dimensions we might as well attack the problem directly to find the $n$–point solitonic Green’s functions. In this endeavor, integrability comes in quite handy: we shall see shortly that the existence of an infinity of conserved charges is equivalent to a deceivingly simple cubic equation in the $2 \to 2$ scattering amplitudes, the celebrated Yang–Baxter equation. In order to get a real theory, we shall in addition impose unitarity and crossing symmetry. Unitarity just means that the probability is conserved, so that nothing comes of nothing and something comes of just as much. Crossing symmetry is a more subtle requirement, familiar from string theories, which can be viewed somehow as a strong relativistic invariance, whereby particles moving forward or backward in time can be traded off (suitably) by other particles moving forward or backward in time.

2 Factorizable S–matrices

We shall be interested in integrable field theoretical systems, i.e. systems with an infinite number of mutually commuting conserved charges. One of these charges will be called the hamiltonian, an operator which defines the time evolution of the system.
To each conserved charge one can associate a different time evolution: what we call the hamiltonian is a matter of interpretation.

Consider the scattering of relativistic massive particles in a (1+1)–dimensional space-time. There is only one spatial dimension and therefore the ordering of the particles is well–defined, from left to right, say. In more spatial dimensions we should not expect that the interesting features depending strongly on the ordering of the particles remain valid.

Introduce the rapidity \( \theta \):

\[
p^0 = m \cosh \theta, \quad p^1 = m \sinh \theta
\]

This parametrization ensures the on–shell condition \( \vec{p}^2 = (p^0)^2 - (p^1)^2 = m^2 \).

Alternatively, we could use the light-cone momenta \( p \) and \( \bar{p} \),

\[
p = p^0 + p^1 = m e^{\theta}, \quad \bar{p} = p^0 - p^1 = m e^{-\theta}
\]

which transform under a Lorentz boost \( L_\alpha : \theta \to \theta + \alpha \) as

\[
p \to p e^{\alpha}, \quad \bar{p} \to \bar{p} e^{-\alpha}
\]

Quite generally, an irreducible tensor \( Q_s \) of the Lorentz group in 1 + 1 dimensions is labelled by its spin \( s \) according to the rule \( L_\alpha : Q_s \to e^{s\alpha} Q_s \), so that \( p \) is of spin 1 and its parity conjugate \( \bar{p} \) is of spin \(-1\).

If \( Q_s \) is a local conserved quantity of integer spin \( s > 0 \), then in a scattering process involving \( n \) particles

\[
\sum_{i \in \{\text{in}\}} p_i^s = \sum_{f \in \{\text{out}\}} p_f^s
\]

Similarly, if \( Q_{-s} \) is conserved, then

\[
\sum_{i \in \{\text{in}\}} \bar{p}_i^s = \sum_{f \in \{\text{out}\}} \bar{p}_f^s
\]

Setting \( s = 1 \) in (4) and (5), we recover the usual energy and momentum conservation laws of a relativistic theory.

The physical behavior of integrable systems is quite remarkable. For instance, if (4) and (5) hold for an infinity of different spins \( s \), it follows immediately that the incoming and outgoing momenta must be the same. This means that no particle production or annihilation may ever occur. Also, particles with equal mass may reshuffle their momenta among themselves in the scattering, but particles with different masses may not. Equivalently, we may say that the momenta are conserved individually and that particles of equal mass may interchange additional internal quantum numbers. If all the incoming particles have different masses, then the only effect of the scattering is a time delay (a phase shift) in the outgoing state with respect to the incoming one.

All scattering processes can be understood and pictured as a sequence of two–particle scatterings. This property is called factorizability.

By relativistic invariance, the scattering amplitude between two particles \( A_i \) and \( A_j \) may only depend on the scalar

\[
p_i^\mu p_j^\nu \eta_{\mu\nu} = m_i m_j \cosh (\theta_i - \theta_j)
\]
so that, in fact, it may depend only on the rapidity difference $\theta_{ij} = \theta_i - \theta_j$. The general form of the basic two–particle $S$–matrix is

$$|A_1(\theta_1), A_2(\theta_2)\rangle_{\text{in}} \longrightarrow \sum_{k,\ell} S_{ij}^{k\ell}(\theta_{12}) |A_k(\theta_2), A_\ell(\theta_1)\rangle_{\text{out}} \quad (7)$$

In this notation, $|A_1(\theta_1), A_2(\theta_2)\rangle_{\text{in(out)}}$ stands for the initial (respectively, final) state of two incoming (respectively, outgoing) particles of kinds $A_i$ and $A_j$ and rapidities $\theta_1$ and $\theta_2$.

The second crucial feature of a factorizable $S$–matrix theory, from which such models get their name, is the property of factorizability: the $N$–particle $S$–matrix can always be written as the product of $\binom{N}{2}$ two–particle $S$–matrices.

We choose an initial state of $N$ particles with rapidities $\theta_1 > \theta_2 > \cdots > \theta_N$ arranged in the infinite past in the opposite order, i.e. $x_1 < x_2 < \cdots < x_N$. This presumes simply that no scatterings may have occurred before we study the process, i.e. that we have been looking long before any particles meet. After the $N(N-1)/2$ pair collisions, the particles reach the infinite future ordered along the spatial direction in increasing rapidity. Thus we write

$$S |A_{i_1}(\theta_1), \ldots, A_{i_N}(\theta_N)\rangle_{\text{in}} = \sum_{j_1,\ldots,j_N} S_{i_1 \ldots i_N}^{j_1 \ldots j_N} |A_{j_1}(\theta_N), \ldots, A_{j_N}(\theta_1)\rangle_{\text{out}} \quad (8)$$

Factorization means that this process can be interpreted as a set of independent and consecutive two–particle scattering processes.

The spacetime picture of this multi-particle factorized scattering is obtained by associating with each particle a line whose slope is the particle’s rapidity. The scattering process is thus represented by a planar diagram with $N$ straight world–lines, such that no three ever coincide at the same point. Any world–line will therefore intersect, in general, all the other ones. The complete scattering amplitude associated to any such diagram is given by the (matrix) product of two–particle $S$–matrices. For instance, in a four–particle scattering we could get

$$S_{i_1i_2i_3i_4}^{j_1j_2j_3j_4}(\theta_1, \theta_2, \theta_3, \theta_4) = \sum_{k,\ell,m,n, p,q,r,u} S_{i_1i_2}^{k\ell}(\theta_{12}) S_{i_3i_4}^{mn}(\theta_{13}) \times$$

$$\times S_{k\ell}^{pq}(\theta_{23}) S_{i_4j_4}^{qr}(\theta_{14}) S_{q\ell}^{uj}(\theta_{24}) S_{j_4j_3}^{pu}(\theta_{34}) \quad (9)$$

The kinematical data (the rapidities of all the particles) does not fix a diagram uniquely. In fact, for the same rapidities we have a whole family of diagrams, differing from each other by the parallel shift of some of the straight world–lines. The parallel shift of any one line can (and should) be interpreted as a symmetry transformation. It corresponds to the translation of the (asymptotic in– and out–) $x$ co-ordinates of the particle associated to the line. Requiring the factorizability condition is equivalent to imposing that the scattering amplitudes of diagrams differing by such parallel shifts should be the same.

For the simple case of three particles, the condition that the factorization be independent of parallel shifts of the world–lines amounts to the following noteworthy factorization equation, which is the necessary and sufficient condition for any two diagrams differing by parallel shifts to have equal associated amplitudes:

$$\sum_{p_1,p_2,p_3} S_{i_1i_2}^{p_1p_2}(\theta_{12}) S_{p_2i_3}^{p_3j_3}(\theta_{13}) S_{p_3j_3}^{j_1j_2}(\theta_{23}) =$$
This is the famous Yang–Baxter equation.

To formalize this a bit more, consider a set of operators \( \{ A_i(\theta) \} \) (\( i = 1, \ldots, n \)) associated to each particle \( i \) with rapidity \( \theta \), obeying the following commutation relations:

\[
A_i(\theta_1)A_j(\theta_2) = \sum_{k,\ell} S^{k\ell}_{ij}(\theta_{12})A_k(\theta_2)A_{\ell}(\theta_1)
\]  

(11)

This equation encodes the two–particle scattering process (7), where “collision” has been replaced by “commutation”. Furthermore, the relation between (7) and (11) becomes evident if we interpret \( A_i(\theta) \) as an operator (Zamolodchikov operator) which creates the particle \( |A_i(\theta)\rangle \) when it acts on the Hilbert space vacuum \( |0\rangle \):

\[
A_i(\theta) |0\rangle = |A_i(\theta)\rangle
\]  

(12)

The factorization equation (10) emerges in this context as a “generalized Jacobi identity” of the algebra (11), assumed associative.

The following conditions are needed to guarantee the physical consistency of the Zamolodchikov algebra (11):

i) Normalization:

\[
\lim_{\theta \to 0} S^{k\ell}_{ij}(\theta) = \delta^k_i \delta^\ell_j \quad \Longrightarrow \quad \lim_{\theta \to 0} S(\theta) = 1
\]  

(13)

This condition is obtained by setting \( \theta_1 = \theta_2 \) in (11). In physical terms, it means that no scattering takes place if the relative velocity of the two particles vanishes, i.e. if the two world–lines are parallel.

ii) Unitarity:

\[
\sum_{j_1,j_2} S^{i_1i_2}_{j_1j_2}(\theta)S^{j_1j_2}_{k_1k_2}(-\theta) = \delta^{i_1}_{k_1} \delta^{j_2}_{k_2} \quad \Longleftrightarrow \quad S(\theta)S(-\theta) = 1
\]  

(14)

This follows from applying (11) twice.

iii) Real analyticity:

\[
S^{\dagger}(\theta) = S(-\theta^*)
\]  

(15)

which together with (14) implies the physical unitarity condition \( S^{\dagger}S = 1 \).

iv) Crossing symmetry:

\[
S^{k\ell}_{ij}(\theta) = S^{\overline{j}\overline{k}}_{\overline{i}\overline{l}}(i\pi - \theta)
\]  

(16)

where \( \overline{j} \) and \( \overline{k} \) denote the antiparticles of \( j \) and \( k \), respectively.

As an example, let us consider a theory with only one kind of particle \( A \) and its antiparticle \( \overline{A} \). Due to CPT invariance, there exist only three different amplitudes (we also assume conservation of particle number, i.e. \( \mathbb{Z}_2 \) invariance). The scattering amplitude between identical particles (or antiparticles) is denoted \( S_I \), whereas \( S_T \) and \( S_R \) denote the transmission and reflection amplitudes, respectively:

\[
\begin{align*}
A(\theta_1)A(\theta_2) &= S_I(\theta_{12})A(\theta_2)A(\theta_1) \\
A(\theta_1)\overline{A}(\theta_2) &= S_T(\theta_{12})\overline{A}(\theta_2)A(\theta_1) + S_R(\theta_{12})A(\theta_2)\overline{A}(\theta_1) \\
\overline{A}(\theta_1)A(\theta_2) &= S_T(\theta_{12})A(\theta_2)\overline{A}(\theta_1) + S_R(\theta_{12})\overline{A}(\theta_2)A(\theta_1) \\
\overline{A}(\theta_1)\overline{A}(\theta_2) &= S_I(\theta_{12})\overline{A}(\theta_2)\overline{A}(\theta_1)
\end{align*}
\]  

(17)
It is not hard to check that the factorization equations for this algebra read as

\[
\begin{align*}
S_I S_R' S''_I &= S_T S'_R T' + S_R S'_I S''_I \\
S_I S'_T S''_R &= S_T S'_I S''_R + S_R S'_T S''_I \\
S_R S'_T S''_I &= S_R S'_I S''_T + S_T S'_R S''_I
\end{align*}
\] (18)

where we have set \(S_a = S_a(\theta_{12}), S'_a = S_a(\theta_{13}), S''_a = S_a(\theta_{23})\) for \(a \in \{I, T, R\}\) to lighten the notation.

The normalization conditions read

\[
S_I(0) = 1 \quad , \quad S_T(0) = 0 \quad , \quad S_R(0) = 1
\] (19)

whereas unitarity requires

\[
\begin{align*}
S_T(\theta) S_T(-\theta) + S_R(\theta) S_R(-\theta) &= 1 \\
S_T(\theta) S_R(-\theta) + S_R(\theta) S_T(-\theta) &= 0
\end{align*}
\] (20)

and the crossing symmetry implies

\[
S_I(\theta) = S_T(i\pi - \theta) \quad , \quad S_R(\theta) = S_R(i\pi - \theta)
\] (21)

The equations (18) imply that the quantity

\[
\Delta = \frac{S_I(\theta)^2 + S_T(\theta)^2 - S_R(\theta)^2}{2S_I(\theta) S_T(\theta)}
\] (22)

is independent of the rapidity \(\theta\). An interesting factorized \(S\)-matrix is provided by the sine–Gordon theory, where the states \(A\) and \(\bar{A}\) of (17) are identified with the soliton and antisoliton.

### 3 Bethe’s diagonalization of spin chain hamiltonians

Consider now a periodic one–dimensional regular lattice (a periodic chain) with \(L\) sites. At each site, the spin variable may be either up or down, so that the Hilbert space of the spin chain is simply \(\mathcal{H}^{(L)} = \bigotimes^L V^\frac{1}{2}\) where \(V^\frac{1}{2}\) is the spin-\(\frac{1}{2}\) irreducible representation of \(SU(2)\) with basis \(\{|\uparrow\rangle, |\downarrow\rangle\}\). By simple combinatorics, the dimension of the Hilbert space is \(\dim \mathcal{H}^{(L)} = 2^L\). On \(\mathcal{H}^{(L)}\), we consider a very general hamiltonian \(H\), subject to three constraints.

First, we assume that the interaction is of short range, for example only among nearest neighbors. Next, we impose that the hamiltonian \(H\) be translationally invariant. Letting \(e^{iP}\) denote the operator which shifts the states of the chain by one lattice unit to the right, then this requirement reads as \([e^{iP}, H] = 0\). From periodicity of the closed chain, we must have \(e^{iPL} = 1\) Finally, we demand that the hamiltonian preserve the third component of the spin:

\[
[H, S^z_{\text{total}}] = \left[ H, \sum_{i=1}^L S^z_i \right] = 0
\] (23)

This requirement allows us to divide the Hilbert space of states into different sectors, each labelled by the third component of the spin or, equivalently, by the total number of spins down. We shall denote by \(\mathcal{H}^{(L)}_M\) the subspace of \(\mathcal{H}^{(L)}\) with \(M\) spins down. Obviously, \(\dim \mathcal{H}^{(L)}_M = \binom{L}{M}\), so that \(\dim \mathcal{H}^{(L)} = \sum_{M=0}^L \dim \mathcal{H}^{(L)}_M\).
We wish to study the eigenstates and spectrum of $H$. The zero-th sector $\mathcal{H}_0^{(L)}$ contains only one state, the “Bethe reference state” with all spins up. The most natural ansatz for the eigenvectors of $H$ in the other sectors is some superposition of “spin waves” with different velocities. For the first sector, i.e. the subspace of states with all spins up except one down, the ansatz for the eigenvector is thus of the form $\ket{\Psi} = \sum_{x=1}^{L} f(x) \ket{x}$ where $\ket{x}$ represents the state with all spins up but for the one at lattice site $x$ ($1 \leq x \leq L$). The unknown wavefunction $f(x)$ determines the probability that the single spin down is precisely at site $x$.

From the complete translational invariance due to periodic boundary conditions, it is reasonable to assume that $f(x)$ is just the wavefunction for a plane wave

$$f(x) = e^{ikx}$$

with some particular momentum $k$ to be fixed by the boundary condition $f(x + L) = f(x)$. Thus $k = 2\pi I/L$, with $I = 0, 1, \ldots, L - 1$. Hence the eigenvectors of $H$ with one spin down span indeed a basis of the Hilbert space $\mathcal{H}_1^{(L)}$, by dimensionality counting.

The wavefunction solving the eigenvalue problem for the sector with two spins down, $H \ket{\Psi} = E_2 \ket{\Psi}$, is of the form $\ket{\Psi} = \sum_{x_1, x_2} f(x_1, x_2) \ket{x_1, x_2}$, where $\ket{x_1, x_2}$ stands for the state with ll spins up except two spins down at positions $x_1$ and $x_2$.

The periodicity condition reads now $f(x_1, x_2) = f(x_2, x_1 + L)$. The most naive ansatz for $f(x_1, x_2)$ generalizing the plane wave is $f(x_1, x_2) = A_{12} \exp(i(k_1 x_1 + k_2 x_2))$. This ansatz is inappropriate, however, because it violates the periodicity condition. Physically, we have forgotten to include the scattering of the two “spin waves” with “quasi-momenta” $k_1$ and $k_2$. The solution to this problem was found by Bethe, who wrote the useful ansatz

$$f(x_1, x_2) = A_{12} e^{i(k_1 x_1 + k_2 x_2)} + A_{21} e^{i(k_1 x_2 + k_2 x_1)}$$

which does satisfy the periodicity condition provided

$$A_{12} = A_{21} e^{ik_1 L}, \quad A_{21} = A_{12} e^{ik_2 L}$$

Note that these two conditions imply, in particular, that $\exp(i(k_1 + k_2)L) = 1$, which reflects the invariance of the wavefunction under a full turn around the chain, i.e. under the shift of $L$ units of lattice space:

$$f(x_1 + L, x_2 + L) = f(x_1, x_2) \iff e^{i(k_1 + k_2)L} = 1$$

This equation must hold if the wavefunction is to be single valued.

The ansatz (25) already assumes that the $S$-matrix for two spin waves is purely elastic. In fact, the only dynamics allowed is the permutation of the quasi-momenta.

To capture the physical meaning behind equations (26), let us introduce the “scattering amplitudes for spin waves”

$$\hat{S}_{12} = \frac{A_{21}}{A_{12}}, \quad \hat{S}_{21} = \frac{A_{12}}{A_{21}}$$

in terms of which (26) read as

$$e^{ik_1 L} \hat{S}_{12}(k_1, k_2) = 1, \quad e^{ik_2 L} \hat{S}_{21}(k_2, k_1) = 1$$

These equations tell us that the total phase shift undergone by a spin wave after travelling all the way around the closed chain is one. This phase shift receives two contributions; one is purely kinematic ($e^{ik_1 L}$ or $e^{ik_2 L}$) and depends only on the quasi-
momentum of the spin waves, while the other reflects the phase shift produced by the interchange of the two spin waves.

Summarizing the previous discussion, we have found that the Bethe ansatz for the eigenvector of the hamiltonian $H$ in the sector $M = 2$ is

$$f(x_1, x_2) = A_{12} \left( e^{i(k_1 x_1 + k_2 x_2)} + \hat{S}_{12}(k_1, k_2) e^{i(k_1 x_2 + k_2 x_1)} \right)$$  \hspace{1cm} \text{(30)}$$

The generic form of a state $|\Psi_M\rangle \in \mathcal{H}_M^{(L)}$ in the sector with $M > 2$ spins down is

$$|\Psi_M\rangle = \sum_{1 \leq x_1 < x_2 < \ldots < x_M \leq L} f(x_1, \ldots, x_M) |x_1, \ldots, x_M\rangle$$  \hspace{1cm} \text{(31)}$$

The Bethe ansatz is now

$$f(x_1, \ldots, x_M) = \sum_{p \in S_M} A_p e^{i(k_{p(1)} x_1 + \ldots + k_{p(M)} x_M)}$$  \hspace{1cm} \text{(32)}$$

where the sum runs over the $M!$ permutations $p$ of the labels of the quasi-momenta $k_i$. The periodicity condition is now

$$f(x_1, x_2, \ldots, x_M) = f(x_2, \ldots, x_M, x_1 + L)$$  \hspace{1cm} \text{(33)}$$

When $M = 3$, we get the following six equations:

$$e^{ik_1 L} = \frac{A_{123}}{A_{231}} = \frac{A_{132}}{A_{321}}, \quad e^{ik_2 L} = \frac{A_{231}}{A_{312}} = \frac{A_{213}}{A_{132}}, \quad e^{ik_3 L} = \frac{A_{312}}{A_{123}} = \frac{A_{321}}{A_{213}}$$  \hspace{1cm} \text{(34)}$$

Thus, in addition to the relations among the quasi-momenta $k_i$ and the amplitudes $A_p$, there exist additional constraints among the amplitudes of three quasi-particles, which were absent in the simpler case with $M = 2$. These relations tell us that the interchange of two particles is independent of the position of the third particle. Locality of the interactions is thus equivalent to the factorization property of the $S$-matrix, according to which the scattering amplitude of $M$ quasi-particles factorizes into a product of $\binom{M}{2}$ two–point $S$-matrices.

The Yang–Baxter content of the Bethe ansatz for $M = 3$ is illustrated with the following equalities:

$$A_{321} = \left\{ \begin{array}{l} \hat{S}_{12} A_{312} = \hat{S}_{12}\hat{S}_{13} A_{132} = \hat{S}_{12}\hat{S}_{13}\hat{S}_{23} A_{123} \\ \hat{S}_{23} A_{231} = \hat{S}_{23}\hat{S}_{13} A_{213} = \hat{S}_{23}\hat{S}_{13}\hat{S}_{12} A_{123} \end{array} \right.$$  \hspace{1cm} \text{(35)}$$

We thus arrive to the all–important “Bethe ansatz equations”

$$e^{ik_i L} = \prod_{j \neq i}^{M} \hat{S}_{ji}(k_j, k_i) \quad \text{for} \quad i = 1, \ldots, M$$  \hspace{1cm} \text{(36)}$$

written in general for a sector with arbitrary $M$. The actual solution to these equations far transcends the framework of these lectures. Suffice it to say that a variety of methods have been devised to attack them.

The spin wave scattering amplitude $\hat{S}_{12}$ depends of course on the detailed form of the hamiltonian, and it can be computed by solving the $M = 2$ eigenvalue equation, which reads more explicitly as

$$E_2 f(x_1, x_2) = \sum_{1 \leq y_1 < y_2 \leq L} \langle x_1, x_2 | H | y_1, y_2 \rangle f(y_1, y_2)$$  \hspace{1cm} \text{(37)}$$
Using (30) in (37), we would find \( \hat{S}_{12} \) as a function of \( k_1, k_2 \) and the matrix elements of \( H \).

Unfortunately, there does not exist a simple criterion to decide when a spin chain hamiltonian is integrable, i.e., when it allows the Bethe construction. As we have shown, however, the Bethe ansatz will work whenever the spin wave \( S \)-matrix satisfies the integrability condition and factorization. Let us stress that the diagonalization of a hamiltonian with the help of the Bethe ansatz does not even work for any translationally invariant and short range hamiltonian preserving the total spin. Only a very special class of such hamiltonians can be diagonalized via the Bethe procedure, namely those which describe integrable models. An important spin chain model, to which the Bethe ansatz technique is applicable, is the \( XXZ \) model

\[
H_{XXZ} = J \sum_{i=1}^{L} \left( \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right)
\]  

(38)

4 Integrable vertex models: the six–vertex model

Let us now turn to classical statistical systems in two spatial dimensions (in equilibrium, so no time dimension) on a lattice. Whereas in the previous section the main problem consisted in diagonalizing a one–dimensional hamiltonian, in this section we address the computation of the partition function of the lattice system.

A vertex model is a statistical model defined on a lattice \( \mathcal{L} \), taken regular and rectangular for simplicity. We shall thus consider an \( L \times L' \) lattice with \( L \) vertical lines (columns) and \( L' \) horizontal lines (rows).

A physical state on this lattice is defined by the assignment to each lattice edge of a state variable, characterized by some labels; allowing for two possible states on each link suffices for our purposes. These two possibilities may be interpreted as spins up or down. Alternatively, if we imagine the lattice links as electric wires with a current of constant intensity running through them, then the two states are associated with the direction of the current.

The dynamics of the model is characterized by the interactions among the lattice variables, which take place at the vertices, whence the name vertex model. The energy \( \varepsilon_V \) associated with a vertex \( V \) depends only on the four states on the edges meeting at that vertex (locality). This is also true for the Boltzmann weights \( W_V = \exp \left( -\varepsilon_V / k_B T \right) \) which measure the probability of each local configuration. Quite generally, it is convenient to represent Boltzmann weights as \( W \left( \begin{array}{c} \beta \\ \mu \\ \nu \\ \alpha \end{array} \right) \), where \( \mu \) and \( \nu \) are the horizontal edge state labels, and \( \alpha \) and \( \beta \) the vertical ones:

\[
W \left( \begin{array}{c} \beta \\ \mu \\ \nu \\ \alpha \end{array} \right) = \begin{array}{c} \beta \\ \mu \\ \nu \\ \alpha \end{array}
\]  

(39)

If we impose that the interaction conserve the total spin or the local current, then all but six Boltzmann weights must vanish. In addition to the spin, particle number or current conservation, we may also impose the \( \mathbb{Z}_2 \) reversal symmetry under \( |\uparrow\rangle \leftrightarrow |\downarrow\rangle \). Under this condition, the independent Boltzmann weights are reduced to three, which
we shall call just $a$, $b$ and $c$. These weights define the symmetric or zero–field six–vertex model, which we shall call the six–vertex model for short.

The six–vertex model is characterized by link variables $\in \mathbb{Z}_2 = \{0, 1\}$ with Boltzmann weights subject to current conservation

$$W\left(\begin{array}{c}
\beta \\
\mu \\
\alpha
\end{array} \right) = 0 \quad \text{unless} \quad \mu + \alpha = \nu + \beta$$

(40)

and reflection symmetry ($\pi = 1 - x$)

$$W\left(\begin{array}{c}
\beta \\
\mu \\
\nu
\end{array} \right) = W\left(\begin{array}{c}
\beta \\
\mu \\
\nu
\end{array} \right)$$

(41)

A compact way to write the weights is

$$W\left(\begin{array}{c}
\beta \\
\mu \\
\alpha
\end{array} \right) = b \delta_{\mu \nu} \delta_{\alpha \beta} + c \delta_{\mu \beta} \delta_{\nu \alpha} + (a - b - c) \delta_{\mu \alpha} \delta_{\nu \beta}$$

(42)

The partition function is

$$Z_{L \times L'}(a, b, c) = \sum_C e^{-E(C)/k_B T} = \sum_C \prod_{V \in \text{row}} W_V$$

(43)

where the sum runs over all possible configurations $C$, of which there are $2^{LL'}$. In the thermodynamic limit, when $L$ and $L'$ tend to infinity, the computation of the sum (43) becomes a rather formidable and apparently insurmountable problem. Lieb’s breakthrough to compute the partition function (43) of the six–vertex model relies basically on rephrasing the problem as the diagonalization of the anisotropic spin–$1/2$ chain, which had been solved already by the Bethe ansatz. First, let us perform the sum over the horizontal variables, which involves only the Boltzmann weights on the same row of the lattice, and then carry out the sum over the vertical variables. The double sum (43) can thus be rearranged as follows:

$$Z_{L \times L'}(a, b, c) = \sum_C \prod_{\text{vertical rows}} \left( \sum_{\text{horizontal states}} \prod_{V \in \text{row}} W_V \right)$$

(44)

The quantity in parenthesis depends on the two sets of vertical states above and below the row of horizontal variables: it is the (row to row) transfer matrix of the model. For conceptual clarity, it is convenient to introduce the “fixed time states” as the set of vertical link variables on the same row:

$$|\alpha\rangle = |\alpha_1 \rangle |\alpha_2 \rangle |\alpha_3 \rangle \cdots |\alpha_{L-1} \rangle |\alpha_L \rangle$$

(45)

The transfer matrix element $\langle \beta|t|\alpha\rangle$ can then be understood as the transition probability for the state $|\alpha\rangle$ to project on the state $|\beta\rangle$ after a unit of time. We are thinking now of the horizontal direction as space, and the vertical one as time:

$$\langle \beta|t(a, b, c)|\alpha\rangle = \sum_{\mu_i} W\left(\begin{array}{c}
\beta_L \\
\mu_L \\
\alpha_L
\end{array} \right) W\left(\begin{array}{c}
\beta_{L-1} \\
\mu_{L-1} \\
\alpha_{L-1}
\end{array} \right) \cdots$$

$$\cdots W\left(\begin{array}{c}
\beta_1 \\
\mu_1 \\
\alpha_1
\end{array} \right) W\left(\begin{array}{c}
\beta_2 \\
\mu_2 \\
\alpha_2
\end{array} \right) \cdots$$

(46)
We agree with the Chinese who think that a picture is better than a formula:

\[
\langle \beta | t | \alpha \rangle = \sum_{\mu_i} \begin{pmatrix} \beta_1 & \beta_2 & \ldots & \beta_{L-1} & \beta_L \\ \mu_1 & \mu_2 & \mu_3 & \mu_{L-1} & \mu_L \\ \alpha_1 & \alpha_2 & \ldots & \alpha_{L-1} & \alpha_L \end{pmatrix} \tag{47}
\]

The transfer matrix \( t(a, b, c) \) plays the role of a discrete evolution operator acting on the Hilbert space \( \mathcal{H}^{(L)} \) spanned by the row states \(|\alpha\rangle\) (dim \( \mathcal{H}^{(L)} = 2^L \)), isomorphic to the one considered above in the diagonalization of the spin-\(\frac{1}{2}\) hamiltonian. The full partition function reads thus \( Z_{L \times L'}(a, b, c) = \text{tr}_{\mathcal{H}^{(L)}}(t(a, b, c))^{L'} \). The trace on \( \mathcal{H}^{(L)} \) implements periodic boundary conditions in the “time” direction. This expression is just the hamiltonian formulation of the partition function (43). Thus evaluating the partition function is in fact equivalent to finding the eigenvalues of the transfer matrix. We are led, therefore, to essentially the same problem considered in the previous section, namely the diagonalization of an operator on \( \mathcal{H}^{(L)} \).

First of all, the local conservation law (40) translates into \( \langle \beta | t | \alpha \rangle = 0 \) unless the total spin is equal for both \(|\alpha\rangle\) and \(|\beta\rangle\), \( \sum_{i=1}^{L} \alpha_i = \sum_{i=1}^{L} \beta_i \). More technically, the number operator \( M = \sum_{i=1}^{L} \alpha_i \) commutes with the transfer matrix:

\[
[t(a, b, c), M] = 0 \tag{48}
\]

This is the analog of equation (23) and the relation between the total spin \( S^z \) and \( M \) is simply \( S^z = \frac{L}{2} - M \). Once again, the Hilbert space \( \mathcal{H}^{(L)} \) can be broken down into sectors \( \mathcal{H}_M^{(L)} \) labelled by \( M \in \{0, 1, \ldots, L\} \). In each of these sectors, the transfer matrix can be diagonalized independently, \( t(a, b, c) |\Psi_M\rangle = \Lambda_M(a, b, c) |\Psi_M\rangle \). The states \(|x_1, \ldots, x_M\rangle\) with \( 1 \)'s at the positions \( x_1, \ldots, x_M \) and \( 0 \)'s elsewhere form a basis of \( \mathcal{H}_M^{(L)} \). Expanding \(|\Psi_M\rangle\) in this basis,

\[
|\Psi_M\rangle = \sum_{1 \leq x_1 < x_2 < \ldots < x_M \leq L} f(x_1, \ldots, x_M) |x_1, \ldots, x_M\rangle \tag{49}
\]

we find the equation for the eigenfunctions \( f(x_1, \ldots, x_M) \):

\[
\sum_{1 \leq y_1 < y_2 < \ldots < y_M \leq L} \langle x_1, \ldots, x_M | t(a, b, c) | y_1, \ldots, y_M \rangle f(y_1, \ldots, y_M) = \Lambda_M(a, b, c) f(x_1, \ldots, x_M) \tag{50}
\]

The transfer matrix connects states with the same number \( M \) of down spins, whose locations may change. The eigenvalue problem (50) can be solved with the help of the Bethe ansatz technique.

The sector \( M = 0 \) contains only one state \(|\Omega\rangle = |00 \ldots 0\rangle\) which is the Bethe reference state. This state plays the role of a vacuum in the construction of the other states, but it need not coincide with the ground state of the model: the physical vacuum minimizes the free energy and may have nothing to do with \(|\Omega\rangle\). From (46) and (50), we obtain

\[
\Lambda_0 = \langle \Omega | t | \Omega \rangle = \sum_{\mu=0,1} \left[ W \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix} \right]^L = a^L + b^L \tag{51}
\]
In the sector \( M = 1 \), we choose \( f(x) = e^{ikx} \) and some elementary algebra yields (with the assumption of periodic boundary conditions)

\[
\Lambda_1(k) = aL P(k) + bL Q(k)
\]  
(52)

with

\[
P(k) = \frac{ab + (c^2 - b^2) e^{-ik}}{a^2 - ab e^{-ik}}, \quad Q(k) = \frac{a^2 - c^2 - ab e^{-ik}}{ab - b^2 e^{-ik}}
\]  
(53)

The sector with \( M = 2 \) excitations contains more structure. The Bethe ansatz reads in this case

\[
f(x_1, x_2) = A_{12} e^{i(k_1 x_1 + k_2 x_2)} + A_{21} e^{i(k_2 x_1 + k_1 x_2)}
\]  
(54)

subject to the periodic boundary conditions, which yields the equations (28) and (29). The eigenvalue of (54) is given by

\[
\Lambda_2 = aL P_1 P_2 + bL Q_1 Q_2
\]  
(55)

where \( P_i = P(k_i) \) and \( Q_i = Q(k_i) \). The ratio of the amplitudes \( A_{12} \) and \( A_{21} \), which again may be interpreted as a spin wave scattering matrix, is thus

\[
\hat{S}_{12} = \frac{A_{21}}{A_{12}} = \frac{1 - 2a^2 + b^2 - c^2}{1 - 2ab} \frac{e^{ik_2} + e^{i(k_1+k_2)}}{e^{ik_1} + e^{i(k_1+k_2)}}
\]  
(56)

For later convenience, we define the anisotropy parameter \( \Delta \) as

\[
\Delta = \frac{a^2 + b^2 - c^2}{2ab}
\]  
(57)

Notice the strong similarity between (57) and (22).

The generalization of the above results to sectors with more than two excitations proceeds through the factorization properties of the higher order Bethe amplitudes \( A_1 \ldots M \) [see equations (35)]. The general formula for the eigenvalue \( \Lambda_M \) of a vector of the form (32) is

\[
\Lambda_M = aL \prod_{i=1}^{M} P(k_i) + bL \prod_{i=1}^{M} Q(k_i)
\]  
(58)

and the quasi-momenta \( k_i \) \((i = 1, \ldots , M)\) must satisfy the Bethe equations (36) which follow from the periodicity (33) of the wave functions and the factorization properties of the Bethe amplitudes. In this case, they read explicitly as

\[
ee^{i k_i L} = (-1)^{M-1} \prod_{j=1}^{M} \frac{1 - 2\Delta e^{ik_i} + e^{i(k_i+k_j)}}{1 - 2\Delta e^{ik} + e^{i(k_1+k_j)}}
\]  
(59)

The final step in the computation of the eigenvalues of the transfer matrix and, ultimately, of the partition function, hinges upon the solution of the Bethe equations (59). It is very important that the Bethe equations associated with the six–vertex model depend on the Boltzmann weights \( a, b \) and \( c \) only through the combination yielding the anisotropy \( \Delta \) in (57). This is the key for understanding the integrability of the six–vertex model. The first immediate consequence from this observation is that two different transfer matrices \( t(a, b, c) \) and \( t(a', b', c') \) sharing the same value for \( \Delta \) have the same eigenvectors and thus they commute:

\[
[ t(a, b, c) , t(a', b', c') ] = 0 \iff \Delta(a, b, c) = \Delta(a', b', c')
\]  
(60)
Therefore, given a value of \( \Delta \), through the Bethe procedure we diagonalize not just a transfer matrix but a whole continuous family of mutually commuting transfer matrices. Since each transfer matrix defines a different time evolution, to each transfer matrix with the same parameter \( \Delta \) is associated a conserved quantity. So the six–vertex model has a large number of conserved quantities, in fact an infinity of them in the thermodynamic limit.

Let us sketch now the relation between the one–dimensional hamiltonian \( H_{XXZ}(\Delta) = H_\Delta \) of the anisotropic Heisenberg chain (38) and the two–dimensional six–vertex model. From the identification of the Bethe ansatz eigenvectors under (57), we see that

\[
\begin{bmatrix}
 H_{\Delta(a,b,c)}, t(a,b,c)
\end{bmatrix} = 0
\]  

Comparison of this expression with (60) leads us to suspect that the hamiltonian \( H_\Delta \) must already be contained somehow in the transfer matrix \( t(a,b,c) \), i.e. it should be one of the conserved quantities in the system. The same argument applies also to the translation operator \( e^{-iP} \) which commutes both with the hamiltonian and with the transfer matrix.

To make these suggestive connections explicit, let us start with the momentum operator \( e^{-iP} \). Suppose we make the following choice of Boltzmann weights:

\[
a = c = c_0, \quad b = 0
\]  

which is consistent with any value of \( \Delta \). Then from (42) we get

\[
W\left(\begin{array}{cc}
\beta & \nu \\
\mu & \alpha
\end{array}\right)\bigg|_{a=c=c_0 \atop b=0} = c_0 \delta_{\mu\beta} \delta_{\nu\alpha}
\]  

which can be imagined as an operator which multiplies by \( c_0 \) the incoming state \( \{\mu, \alpha\} \) but otherwise leaves it untouched: the horizontal state on the left becomes the vertical state on top, and the vertical state below becomes the horizontal state to the right. Thus the transfer matrix from these weights behaves as the shift operator \( e^{-iP} \):

\[
t_0 |\alpha\rangle = t(c_0,0,c_0) |\alpha_1, \alpha_2, \ldots, \alpha_L\rangle = c_0^L |\alpha_L, \alpha_1, \ldots, \alpha_{L-1}\rangle
\]  

and the momentum operator \( P \) is identified with

\[
P = i \log \left(\frac{t_0}{c_0^L}\right)
\]  

This identification is easily checked on one–particle states. From (64), we see that \( t_0 |x\rangle = |x+1\rangle \), and thus in the Fourier transformed states \( |k\rangle \) we find

\[
t_0 |k\rangle = t_0 \sum_{x=1}^{L} e^{ikx} |x\rangle = \sum_{x=1}^{L} e^{ikx} |x+1\rangle
\]  

\[
= \sum_{x=1}^{L} e^{ik(x+1)} e^{-ikx} |x+1\rangle = e^{-ik} |k\rangle
\]  

Similarly, the hamiltonian \( H_\Delta \) can be obtained by expanding the transfer matrix in the vicinity of the parameter point (62), keeping the value of \( \Delta \) constant. Note that this
amounts to expanding the transfer matrix about $t_0$, i.e. about a matrix proportional to the shift operator. So we fix

$$\Delta = \frac{\delta a - \delta c}{\delta b} \tag{67}$$

and obtain

$$t_0^{-1} \delta t = \frac{\delta b}{2c_0} \sum_{i=1}^{L} \left\{ \frac{\delta a + \delta c}{\delta b} 1 + \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \Delta \sigma_i^z \sigma_{i+1}^z \right\} \tag{68}$$

Thus the hamiltonian $H_{\Delta}$ [see (38)] appears in the expansion of the logarithm of the transfer matrix about the shift operator,

$$H_{\Delta} = i \frac{\partial}{\partial u} \log \frac{t(u)}{c_0} \bigg|_{u=0} \tag{69}$$

We urge the reader to carry out the above simple and most instructive calculation explicitly.

Expanding $\log t(u)$ in powers of $u$ we get a whole set of local conserved quantities involving in general interactions over a finite range, not just among nearest neighbors. In this sense, the transfer matrix is the generating functional for a large class of commuting conserved quantities; this follows from the integrability equation (60). Recall that we arrived at that equation after a long analysis involving the Bethe ansatz. Instead, we could have taken equation (60) as the starting point to define a vertex model, and asked ourselves under what conditions the Boltzmann weights of such vertex model lead to integrability. The answer to this question is that, in order that the vertex model be integrable, the Boltzmann weights must satisfy the justly celebrated Yang–Baxter equation:

$$\sum_{\mu', \nu', \gamma} W\left( \gamma \mu' \nu' \alpha \mu \right) W'\left( \beta \nu' \gamma \nu \right) W''\left( \nu'' \mu'' \mu' \gamma \right) =$$

$$= \sum_{\mu'', \mu', \gamma} W''\left( \nu'' \mu'' \mu \nu \right) W'\left( \gamma \mu'' \nu' \mu' \alpha \right) W\left( \beta \nu'' \gamma \right) \tag{70}$$

where $W$, $W'$ and $W''$ are three different sets of Boltzmann weights.

Equation (70) is the Yang–Baxter equation for vertex models. Note that in the two problems worked out so far, the diagonalization of spin chain hamiltonians and the diagonalization of the transfer matrix for vertex models, integrability is encoded in the same mathematical structure, namely the factorization of the spin wave $S$–matrix (36) and the vertex Yang–Baxter equation (70). There exist several other formulations of the Yang–Baxter equation, which is always a cubic equality; it first appeared under the name of star–triangle relation in Onsager’s solution to the two–dimensional Ising model.

A side comment: in statistical mechanics, the local energies must be real and thus the Boltzmann weights must be real and positive. It is nevertheless useful to allow the Boltzmann weights to be complex in general. This freedom is useful from the technical viewpoint, but it is also physically meaningful. In particular, the region of parameter space in which the one–dimensional spin chain hamiltonian is hermitian need not coincide with that in which the two–dimensional Boltzmann weights are real and positive. Thus the physical spin chain hamiltonian is indeed an analytic extension of the hamiltonian derived from realistic Boltzmann weights.
5 The Yang–Baxter algebra

Let us analyze in more detail the structure of the Yang–Baxter equation (70). Our goal is to capture in a general algebraic framework the integrability properties of the models studied above. The transfer matrix is an endomorphism

\[ t(a, b, c) : V_1 \otimes \cdots \otimes V_L \to V_1 \otimes \cdots \otimes V_L \]  

where \( V_i \) stands for the spin-1/2 representation space at the \( i \)-th position of the lattice. This operator is built up by multiplying Boltzmann weights on the same row and summing over the horizontal states connecting them, while keeping the vertical states above and below them fixed [see eq. (46)]. To make this distinction even clearer, we shall refer to the space of horizontal states as auxiliary, and denote it by \( V_a \). The space of vertical states on which the transfer matrix acts we shall call quantum and denote it as \( H(L) = V_1 \otimes \cdots \otimes V_L \), as in (71). For the six–vertex model, \( V_a \) is also a spin–1/2 representation space, like \( V_i \) (\( i = 1, \ldots, L \)).

According to these definitions, it is natural to interpret the Boltzmann weights associated to the \( i \)-th vertex as an operator \( R_{ai} : \) 

\[ R_{ai} : V_a \otimes V_i \to V_a \otimes V_i \]  

where the subindices in \( R \) label the vector spaces it acts upon. The operator \( R_{ai} \) is defined by its matrix elements

\[
\begin{align*}
\beta_i & \quad \mu_{i+1} = W \left( \begin{array}{c}
\beta_i \\
\mu_i \\
\alpha_i
\end{array} \right) \equiv R^\mu_{i+1} \beta_i \\
= a \langle \mu_{i+1} | \otimes i \langle \beta_i | R_{ai} | \mu_i \rangle_a \otimes | \alpha_i \rangle_i
\end{align*}
\]

Note that if \( R \) appears with two subindices, they label the spaces \( R \) acts upon, whereas if \( R \) appears with two subindices and two superindices, they label the basis vectors of the spaces \( R \) is acting between. Using these notations, the transfer matrix (46) can be written as

\[
(\beta | t | \alpha) = \sum_{\mu,s} R^\mu_{\mu_L} \beta_L R^\mu_{\mu_L-1} \beta_{L-1} \cdots R^\mu_{\mu_2} \beta_2 R^\mu_{\mu_1 \alpha_1}
\]

We have reversed the order of multiplication of the Boltzmann weights to agree with the conventions for multiplying matrices in the auxiliary space, namely \( (XY)^\mu_{\nu} = X^\mu_{\lambda} Y^\lambda_{\nu} \). We thus arrive finally to a label–independent expression for the transfer matrix:

\[ t = \text{tr}_a \left( R_{aL} R_{aL-1} \cdots R_{a2} R_{a1} \right) \]  

Here, \( \text{tr}_a \) denotes the trace over the auxiliary space \( V_a \).

After these preliminaries, we may ask whether two transfer matrices \( t \) and \( t' \), derived from two sets of Boltzman weights \( R \) and \( R' \), do commute. Of course, \( t \) and \( t' \) must act on the same quantum space \( V_1 \otimes \cdots \otimes V_L \), but the auxiliary spaces for each of them may be different. We multiply \( t \) and \( t' \) and for clarity we label their respective auxiliary spaces as \( V_a \) and \( V_b \), even in the case when these spaces are isomorphic:

\[ t t' = \text{tr}_{a \times b} \left( R_{aL} R'_{bL} \cdots R_{a1} R'_{b1} \right) \]  

where \( \text{tr}_{a \times b} \) denotes the trace on \( V_a \otimes V_b \). Similarly, multiplying \( t' \) and \( t \) we get

\[ t' t = \text{tr}_{a \times b} \left( R'_{bL} R_{aL} \cdots R'_{b1} R_{a1} \right) \]  

We thus arrive finally to a label–independent expression for the transfer matrix:

\[ t = \text{tr}_a \left( R_{aL} R_{aL-1} \cdots R_{a2} R_{a1} \right) \]  

Here, \( \text{tr}_a \) denotes the trace over the auxiliary space \( V_a \).
Hence $t$ commutes with $t'$ if and only if there exists an invertible matrix $X_{ab}$ such that
\[ R_{bi} R_{ai} = X_{ab} R_{ai} R'_{bi} X^{-1}_{ab} \quad \forall i = 1, \ldots, L \] (78)
Indeed, using the cyclicity of the trace we find
\[ t' t = \text{tr}_{ab} \left( X_{ab} R_{aL} R'_{bL} X^{-1}_{ab} X_{ab} R_{aL-1} R'_{bL-1} X^{-1}_{ab} \cdots X_{ab} R_{a1} R'_{b1} X^{-1}_{ab} \right) = t t' \] (79)
Moreover, the matrix $X_{ab}$ may be interpreted as arising from Boltzmann weights on the space $V_a \otimes V_b$: we shall call them $R_{ab}$. The integrability condition (78) is the Yang–Baxter equation in operator formalism:
\[ R''_{ab} R_{ai} R'_{bi} = R'_{bi} R_{ai} R''_{ab} \] (80)
With some minor changes in notation, equation (80) can be written as
\[ R_{12} R'_{13} R''_{23} = R''_{23} R'_{13} R_{12} \] (81)
where $R_{12}$, $R'_{13}$ and $R''_{23}$ are Yang–Baxter matrices acting on the spaces $V_1 \otimes V_2$, $V_1 \otimes V_3$ and $V_2 \otimes V_3$, respectively. In components, the operator Yang–Baxter equation (81) reads as follows:
\[ \sum_{j_1, j_2, j_3} R^{k_1 k_2}_{j_1 j_2} R'^{l_{j_1 k_3}}_{j_1 j_3} R''^{m_{j_2 j_3}}_{j_2 j_3} = \sum_{j_1, j_2, j_3} R'^{n_{k_1 k_2}}_{j_1 j_3} R'^{m_{j_2 j_3}}_{j_1 j_3} R''^{l_{j_2 j_3}}_{j_2 j_3} \] (82)
Equation (82) is the most general form of the Yang–Baxter equation for vertex models, in the sense that the spaces $V_1$, $V_2$ and $V_3$ need not be isomorphic. We shall not consider this possibility, but keep all these vector spaces two–dimensional, so that the $R$ operator is a $4 \times 4$ matrix which, in the case of the six–vertex model, is
\[ R^{(6v)}(a, b, c) = \begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix} \] (83)
If $R^{(6v)}$ is to be invertible, then we must require $a \neq 0$ and $b \neq \pm c$.

Taking now three six–vertex $R$–matrices $R = R^{(6v)}(a, b, c)$, $R' = R^{(6v)}(a', b', c')$ and $R'' = R^{(6v)}(a'', b'', c'')$, then the Yang–Baxter equation holds provided
\[ \Delta(a, b, c) = \Delta(a', b', c') = \Delta(a'', b'', c'') \] (84)
in full agreement with equation (60). The Yang–Baxter equation captures completely the integrability of the six–vertex model, encoded in (84).

Expressing the weights $a$, $b$ and $c$ in terms of $u$, we find that the Yang–Baxter matrix $R(u) = R^{(6v)}(a(u), b(u), c(u))$ satisfies the Yang–Baxter equation (81) in the form
\[ R_{12}(u) R_{13}(u') R_{23}(u'') = R_{23}(u'') R_{13}(u') R_{12}(u) \] (85)
with $u''$ fixed in terms of $u$ and $u'$. Now, on a sphere all points are equivalent in the sense that any point can be mapped to any other one by a conformal transformation. We may therefore choose the functions $a(u)$, $b(u)$ and $c(u)$ in such a way that $u''$ is just $u' - u$. Then (85) adopts the usual additive form
\[ R_{12}(u) R_{13}(u + v) R_{23}(v) = R_{23}(v) R_{13}(u + v) R_{12}(u) \] (86)
valid for any complex $u$ and $v$. 

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The monodromy matrix $T(u)$ is defined in the same manner as the transfer matrix, except that we do not trace over the first (or last, due to periodic boundary conditions) horizontal states in (74), that is to say

$$T(u) = R_{aL} R_{aL-1} \cdots R_{a2} R_{a1}$$  \hspace{1cm} (87)

The trace of the monodromy matrix on the auxiliary space is just the transfer matrix

$$t(u) = \text{tr}_a T(u)$$  \hspace{1cm} (88)

Using $i, j, \ldots$ as labels in the auxiliary space $V_a$, we see that $T(u)$ is in fact a matrix $T^j_i(u)$ of operator valued functions which act, in this case, on the Hilbert space $\mathcal{H}^{(L)} = V_1 \otimes \cdots \otimes V_L$. These operators will be represented graphically as

$$T^j_i(u) = \begin{array}{c|c}
  i & j \end{array}$$  \hspace{1cm} (89)

with the double line standing for the Hilbert space $\mathcal{H}^{(L)}$. The characteristic feature of these operators is that they satisfy an important set of quadratic relations reflecting their behavior under monodromy:

$$\mathcal{R}_{ab}(u-v) (T_a(u) \otimes T_b(v)) = (T_b(v) \otimes T_a(u)) \mathcal{R}_{ab}(u-v)$$  \hspace{1cm} (90)

This equation constitutes the cornerstone of the quantum inverse scattering method; it is also at the origin of the quantum group. The subindices $a$ and $b$ are short-hand for the two auxiliary spaces $V_a$ and $V_b$ on which the operators $T$ and $R$ act. For extra clarity, we have indulged in a notational redundancy indicating the tensor product in (90) which is taken over these auxiliary spaces, while the quantum indices (not shown) are multiplied as ordinary matrix indices. The proof of (90) uses the Yang–Baxter equation (86) repeatedly and elucidates the index interplay:

$$\mathcal{R}_{ab}(u-v) (T_a(u) \otimes T_b(v)) =$$

$$= \mathcal{R}_{ab}(u-v) R_{aL}(u) R_{bL}(v) \cdots R_{a1}(u) R_{b1}(v)$$

$$= R_{bL}(v) R_{aL}(u) \cdots R_{b1}(v) R_{a1}(u) \mathcal{R}_{ab}(u-v)$$  \hspace{1cm} (91)

$$= (T_b(v) \otimes T_a(u)) \mathcal{R}_{ab}(u-v)$$

For practical purposes, it is often convenient to write equation (90) in components:

$$\sum_{j_1, j_2} \mathcal{R}^{k_1 k_2}_{j_1 j_2} (u-v) T(v)^{j_1}_{i_1} T(v)^{j_2}_{i_2} = \sum_{j_1, j_2} T(v)^{k_1}_{j_1} T(u)^{k_2}_{j_2} \mathcal{R}^{j_1 j_2}_{i_1 i_2} (u-v)$$  \hspace{1cm} (92)

Given (90), it is an easy task to prove the commutativity of the transfer matrices, i.e. $[\text{tr} T_a(u), \text{tr} T_b(u)] = 0$.

Equation (92) has been derived for the six–vertex model, but it can be taken as the starting point for the construction of integrable vertex models, at least for those with the difference property. To this end, we shall introduce the formal notion of a Yang–Baxter algebra.

A Yang–Baxter algebra $\mathcal{A}$ consists of a couple $(\mathcal{R}, T)$, where $\mathcal{R}$ is an $n^2 \times n^2$ invertible matrix and $T^j_i(u)$ ($i, j \in \{1, \ldots, n\}; u \in \mathbb{C}$) are the generators of $\mathcal{A}$. They must satisfy the quadratic relations (92), whose consistency implies the Yang–Baxter equation (86) for $\mathcal{R}(u)$. The entries of the matrix $\mathcal{R}(u)$ play the role of structure constants of the algebra $\mathcal{A}$. This is quite analogous to a Lie algebra, or better yet to its universal enveloping algebra, which is also defined in terms of a set of generators and structure
constants. Following this analogy, the Yang–Baxter relation plays the role of the Jacobi identity: they both reflect the associativity of the corresponding algebras.

An important property of Yang–Baxter algebras is their “addition law”, called co-product or co-multiplication $\Delta$, which maps the algebra $A$ into the tensor product $A \otimes A$ while preserving the algebraic relations of $A$:

\[
\Delta : A \rightarrow A \otimes A
\]

\[T_i^j(u) \mapsto \sum_k T_i^k(u) \otimes T_k^j(u) \quad (93)\]

The diagrammatic representation of the co-product follows from (89):

\[
\Delta \left( \begin{array}{c|c} i & j \end{array} \right) = \sum_k \left( \begin{array}{c|c} i & k \end{array} \right) \otimes \left( \begin{array}{c|c} k & j \end{array} \right) \quad (94)\]

It is left as an exercise for the reader to check that $\Delta T_i^j$ satisfy the same relations as $T_i^j$ in (92). The algebra $A$ has thus both a multiplication and a co-multiplication; $A$ is called a bi-algebra.

The definition of a Yang–Baxter algebra just provided is general, and it can be applied to any $R$–matrix satisfying the Yang–Baxter relation. We confine ourselves once more to the Yang–Baxter algebra constructed from the $R$–matrix of the six–vertex model.

We represent the four generators $T_i^j$ of $A^{(6v)}$ as operators acting on a Hilbert space $H$. We shall name them as follows, for convenience:

\[
T_0^0(u) = A(u) \quad , \quad T_0^0(u) = B(u) \]

\[
T_1^0(u) = C(u) \quad , \quad T_1^1(u) = D(u) \quad (95)\]

Just as the structure constants of Lie algebras provide a representation of the algebra (the adjoint), the $R^{(6v)}$ matrix provides a representation of $A^{(6v)}$ of dimension two under the identification

\[
\left( T_i^j(u) \right)_{k \ell}^k = R_{i \ell}^{j k}(u) \quad (96)\]

or explicitly

\[
A(u) = \begin{pmatrix} a(u) & 0 \\ 0 & b(u) \end{pmatrix} = \frac{a + b}{2} 1 + \frac{a - b}{2} \sigma_3
\]

\[
B(u) = \begin{pmatrix} 0 & 0 \\ c(u) & 0 \end{pmatrix} = c \sigma^- \]

\[
C(u) = \begin{pmatrix} 0 & c(u) \\ 0 & 0 \end{pmatrix} = c \sigma^+ \quad (97)\]

\[
D(u) = \begin{pmatrix} b(u) & 0 \\ 0 & a(u) \end{pmatrix} = \frac{a + b}{2} 1 - \frac{a - b}{2} \sigma_3
\]

Equations (97) yield what we might call the spin $\frac{1}{2}$ representation of the algebra (92). This nomenclature is appropriate since $C(u)$ and $B(u)$ act as raising and lowering operators, respectively, while $A(u)$ and $D(u)$ span the Cartan subalgebra of $SU(2)$. Using now the bi-algebra structure of $A^{(6v)}$ defined by the co-multiplication (93), we may obtain a representation of $A^{(6v)}$ on the space $H^{(L)} = \bigotimes^L V_{\frac{1}{2}}$. In particular, for
\[ L = 2 \text{ we get} \]
\[
\Delta(A(u)) = A(u) \otimes A(u) + C(u) \otimes B(u) \\
\Delta(B(u)) = B(u) \otimes A(u) + D(u) \otimes B(u) \\
\Delta(C(u)) = C(u) \otimes D(u) + A(u) \otimes C(u) \\
\Delta(D(u)) = D(u) \otimes D(u) + B(u) \otimes C(u)
\]  
(98)

It is an easy exercise to check that \( \Delta C \) annihilates the reference state \( |\Omega\rangle \equiv |00\rangle = |\uparrow \uparrow\rangle \):
\[
\Delta C(u) |00\rangle = 0 \quad (99)
\]

Under this interpretation of \( B(u) \) and \( C(u) \) as creation and annihilation operators, it follows that \( \Delta B(u) \) acting on the reference state \( |\Omega\rangle \) yields a state in the sector with the number of spins down equal to one. We can rewrite this state as
\[
\Delta B(u) |\Omega\rangle = |\Psi_1\rangle = \sum_x f(x) |x\rangle = f(1) |10\rangle + f(2) |01\rangle \quad (100)
\]

with
\[
f(1) = c(u)a(u) \quad , \quad f(2) = b(u)c(u) \quad (101)
\]

Comparing (101) with (24), we deduce the relation between Boltzmann weights and quasi-momenta:
\[
\frac{b(u)}{a(u)} = e^{ik} \quad (102)
\]

This method for lowering spins (i.e. creating 1’s) from a reference state by means of the \( B \) operators can be extended to a lattice with \( L > 2 \) sites. To do so we recall the definition (95) of the operator \( B \) as the entry \( T^0_{01} \) of the monodromy matrix; thanks to the co-product (93), it can be made to act on the space \( \mathcal{H} = \otimes^L V_\frac{1}{2} \):
\[
B(u) = \Delta^{L-1} \left( T^0_{01}(u) \right) \quad (103)
\]

where
\[
\Delta^{L-1} : \mathcal{A} \rightarrow \underbrace{\mathcal{A} \otimes \cdots \otimes \mathcal{A}}_{L \text{ times}} \quad (104)
\]
is the associative generalization of (93), \( \Delta^{L-1} = (1 \otimes \Delta)\Delta^{L-2} \) with \( L \geq 2 \). Hence a state with \( M \) spins down can be built as follows:
\[
|\Psi_M\rangle = \prod_{i=1}^M B(u_i) |00 \cdots 0\rangle = \prod_{i=1}^M B(u_i) |\Omega\rangle \quad (105)
\]

The states (105) are called algebraic Bethe ansatz states [“algebraic” in contrast with the “co-ordinate” description (31)], and constitute a very good starting point for solving the eigenvalue problem of the transfer matrix. In order to show this, let us work out more explicitly the relations satisfied by the generators of the six–vertex Yang–Baxter algebra.

From (83) and (92) we obtain, for arbitrary \( u \) and \( v \),
\[
B(u)B(v) = B(v)B(u) \quad (106a)
\]
\[
A(u)B(v) = \frac{a(v-u)}{b(v-u)} B(v)A(u) - \frac{c(v-u)}{b(v-u)} B(u)A(v) \quad (106b)
\]
\[ D(u)B(v) = \frac{a(u - v)}{b(u - v)} B(v)D(u) - \frac{c(u - v)}{b(u - v)} B(u)D(v) \quad (106c) \]
\[ C(u)B(v) - B(v)C(u) = \frac{c(u - v)}{b(u - v)} (A(v)D(u) - A(u)D(v)) \quad (106d) \]

Equation (106a) implies that the algebraic Bethe ansatz state (105) is independent of the ordering in which the \( B \) operators are multiplied.

The transfer matrix of the six–vertex model can be written from equations (88) and (95) as
\[ t^{(6\nu)}(u) = \text{tr}_a T^{(6\nu)}(u) = A(u) + D(u) \quad (107) \]

Therefore, the problem of diagonalizing the transfer matrix (107) in the algebraic Bethe ansatz basis (105) amounts to finding a choice of the parameters \( \{u_i, i = 1, \ldots, M\} \) such that
\[ t^{(6\nu)}(u) |\Psi_M\rangle = [A(u) + B(u)] \prod_{i=1}^{M} B(u_i) |\Omega\rangle = \Lambda_M(u; \{u_i\}) \prod_{i=1}^{M} B(u_i) |\Omega\rangle \quad (108) \]

The advantage of using the algebraic Bethe ansatz states is that the whole computation involved in (108) reduces to a systematic use of the commutation relations (106), in addition to the obvious relations
\[ A(u) |\Omega\rangle = a(u) L |\Omega\rangle \quad , \quad D(u) |\Omega\rangle = b(u) L |\Omega\rangle \quad (109) \]

Indeed, using (106b), (106c) and (109), we find
\[ (A(u) + D(u)) \prod_{i=1}^{M} B(u_i) |\Omega\rangle = \text{unwanted terms} + \\
+ \left[ a^L(u) \prod_{i=1}^{M} \frac{a(u_i - u)}{b(u_i - u)} + b^L(u) \prod_{i=1}^{M} \frac{a(u - u_i)}{b(u - u_i)} \right] \prod_{i=1}^{M} B(u_i) |\Omega\rangle \quad (110) \]

The first term of the right–hand side of this equation gives us the eigenvalue of the transfer matrix:
\[ \Lambda_M(u; \{u_i\}) = a^L(u) \prod_{i=1}^{M} \frac{a(u_i - u)}{b(u_i - u)} + b^L(u) \prod_{i=1}^{M} \frac{a(u - u_i)}{b(u - u_i)} \quad (111) \]

From the second summands in (106b) and (106c), however, we also obtain terms which are not of the desirable form \( \prod_{i=1}^{M} B(u_i) |\Omega\rangle \). If these terms are present, the algebraic Bethe ansatz does not work. The unwanted terms actually cancel, for the six–vertex model, under a judicious choice of the \( u_i \) parameters. The condition that the parameters \( u_i \) should satisfy to guarantee the cancellation of the unwanted terms is precisely the Bethe equations, written in the form
\[ \left( \frac{a(u_i)}{b(u_i)} \right)^L = \prod_{j \neq i}^{M} \frac{a(u_j - u_i) b(u_j - u_i)}{a(u_j - u_i) b(u_i - u_j)} \quad (112) \]
Equations (111) and (112) are the final outcome of the diagonalization of the transfer matrix through the algebraic Bethe ansatz, which we now can compare with equations (58) and (59). Matching the eigenvalues (58) and (111) yields

\[
\frac{a(u_i - u)}{b(u_i - u)} = P(k_i) = \frac{a(u)b(u) + (c^2(u) - b^2(u)) e^{-ik_i(u_i)}}{a^2(u) - a(u)b(u) e^{-ik_i(u_i)}} \quad (113a)
\]

\[
\frac{a(u - u_i)}{b(u - u_i)} = Q(k_i) = \frac{a^2(u) - c^2(u) - a(u)b(u) e^{-ik_i(u_i)}}{a(u)b(u) - b^2(u) e^{-ik_i(u_i)}} \quad (113b)
\]

Choosing \( u = 0 \) in (113a) and using the fact that \( a(0) = c(0) \neq 0, b(0) = 0 \) (see the explicit parametrization below), we get

\[
\frac{b(u_i)}{a(u_i)} = e^{ik_i(u_i)} \quad (114)
\]

Hence the comparison between (112) and (59) ends up producing

\[
\hat{S}_{ji} = \frac{a(u_j - u_i)b(u_i - u_j)}{a(u_i - u_j)b(u_j - u_i)} = -\frac{1 - 2\Delta e^{ik_i} + e^{(k_i + k_j)}}{1 - 2\Delta e^{ik_j} + e^{(k_i + k_j)}} \quad (115)
\]

which confirms the result (102). Equations (113), (114) and (115) provide the map between the quasi-momenta \( k_i \) and the uniformization variables \( u_i \) used in the algebraic Bethe ansatz construction.

Let us use the following uniformization of the Boltzmann weights of the six–vertex model:

\[
a(u) = R_{00}^{11}(u) = R_{11}^{11}(u) = \sinh(u + i\gamma)
\]

\[
b(u) = R_{10}^{10}(u) = R_{01}^{01}(u) = \sinh u
\]

\[
c(u) = R_{10}^{10}(u) = R_{01}^{01}(u) = i \sin \gamma
\]

where the parameter \( \gamma \) is related to the anisotropy \( \Delta \) by the relation \( \Delta = \cos \gamma \). Using now the map

\[
e^{ik_j} = \frac{\sinh u_j}{\sinh(u_j + i\gamma)} \quad (117)
\]

it is easy to check that equations (113) and (115) are satisfied with the six–vertex \( R \)–matrix (116), and that the Bethe equations can be written as

\[
\left( \frac{\sinh(u_j + i\gamma)}{\sinh u_j} \right)^L = \prod_{k=1}^{M} \frac{\sinh(u_j - u_k + i\gamma)}{\sinh(u_j - u_k - i\gamma)} \quad (118)
\]

We may calculate the energy of the Bethe ansatz state (105) from the eigenvalue of the transfer matrix (111) if we recall that the hamiltonian is defined as

\[
H = i \frac{\partial}{\partial u} \log \left( \frac{t(u)}{a(u)L} \right) \bigg|_{u=0}
\]

\[
= \frac{1}{2 \sin \gamma} \sum_{j=1}^{L} \left[ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cos \gamma \left( \sigma_j^z \sigma_{j+1}^z - 1 \right) \right]
\]

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\]

\[
= \frac{1}{2 \sin \gamma} \sum_{j=1}^{L} \left[ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cos \gamma \left( \sigma_j^z \sigma_{j+1}^z - 1 \right) \right]
\]
Indeed, carrying out the computation explicitly we find

\[ E_M (\{u_j\}) = i \left. \frac{\partial}{\partial u} \log \left( \frac{\Lambda_M (u, \{u_j\})}{a(u)^L} \right) \right|_{u=0} \]

\[ = - \sum_{j=1}^{M} \frac{\sin \gamma}{\sinh u_j \sinh (u_j + i\gamma)} \]  

(120)

Similarly, the total momentum of the same Bethe state is

\[ P_M (\{u_j\}) = i \left. \log \left( \frac{t(u)}{a(u)^L} \right) \right|_{u=0} = i \sum_{j=1}^{M} \log \frac{\sinh (u_j + i\gamma)}{\sinh u_j} \]

(121)

Equations (113)–(115) explain the integrability content of the Bethe equations of the six–vertex model, which is codified in the Yang–Baxter equation. The factorization properties for \( S \)–matrices simply reflect the consistency of the Yang–Baxter algebra. We have thus reinterpreted the factorization properties as integrability of the six–vertex model.

6 Physical spectrum of the Heisenberg spin chain

Let us abandon formalism for a while and come back to the simple system of the one–dimensional antiferromagnetic Heisenberg model (the XXX model)

\[ H = J \sum_{i=1}^{L} (\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} - 1) \quad J > 0 \]  

(122)

Since \( J \) is positive, neighboring spins tend to align antiparallel. If \( J \) was negative, then it would be favored for all spins to align in the same direction, and we would be in the ferromagnetic phase.

The first issue we should address about a one–dimensional spin system like (122) is whether a particle interpretation exists. By this, we mean whether it is possible to define a Fock representation of the Hilbert space of the model, such that the vacuum \( |0\rangle \) of the Fock space \( \mathcal{F} \) corresponds to the ground state and the many–particle states to the low lying excitations. A particle interpretation is readily available if we establish the correspondence

\[ \mathcal{H}^{f.t.}_\infty \rightarrow |0\rangle \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \cdots = \mathcal{F} \]

(123)

where \( \mathcal{H}^{f.t.}_n \) represents the Hilbert space of the low lying excitations of the hamiltonian (122) in the limit \( L \rightarrow \infty \), and \( \mathcal{H}_n \) is the Hilbert space of \( n \) elementary excitations. The Fock vacuum \( |0\rangle \) represents the antiferromagnetic ground state.

The two crucial questions about the elementary excitations are

i) The dispersion relation \( \epsilon (k) \), from which we get information about the existence of a mass gap. If the mass gap is zero, then the theory may correspond in the continuum limit to a massless or critical field theory, i.e. to a conformal field theory.

ii) The internal quantum numbers (spin) of the elementary excitations.

To get some flavor for why the answer to these two questions is so difficult, it is necessary to reflect for a moment on the richness of the antiferromagnetic vacuum. Recall that to solve the model (122) we must diagonalize the hamiltonian in the basis of spin waves. A state with \( M \) spin waves is gotten by flipping \( M \) spins down from the
reference state with all spins up. For the trivial case of only one spin wave in a periodic chain, the dispersion relation for the spin wave of the \( XXX \) model is given by

\[
E(k) = 4J(\cos k - 1)
\]  

(124)

and thus the different physical behavior of the ferromagnetic phase and the antiferromagnetic one can be easily distinguished.

In the ferromagnetic regime, the coupling constant \( J \) is negative and the energy of the spin wave is positive, so the Bethe reference state coincides with the ground state of minimal energy; this is the ordered phase, and spins tend to align. The solution to the physical problem is relatively straightforward.

The antiferromagnetic regime, with a positive coupling \( J > 0 \), is trickier. The energy of the spin wave is negative, and flipping one spin down is energetically favored over keeping all spins up. The Bethe reference state has nothing to do with the ground state, which we expect to be a singlet of the global \( SU(2) \) symmetry, in fact a state with \( S_{\text{total}} = 0 \). To get such a state in our picture, we need a “condensate” of spin waves. This physical intuition, together with the fact that the energy of the spin wave for \( J > 0 \) is negative, can be combined thanks to the concept of a Dirac sea. Identifying the vacuum as the Dirac sea filled up to the Fermi surface, the elementary excitations will be thought of as holes in the Dirac sea. The integrability of the model, which amounts to the factorizability of the scattering matrix for spin waves, allows us to construct the sea starting from the Bethe equations.

For the isotropic Heisenberg model (122) in the antiferromagnetic phase \( J > 0 \), the two questions above were pretty much solved by Faddeev and Takhtadjan in the early eighties. The dispersion relation for the low lying excitations turns out to be of the form

\[
e(k) = 2\pi J \sin k \quad 0 \leq k \leq \pi
\]  

(125)

which means that the system has no mass gap. This is consistent with a continuum limit described by a free massless scalar field. The surprising result has to do with the spin of the low lying excitations. Since an excitation corresponds to flipping one local spin up into a spin down, with a net change of one unit of angular momentum, you might have guessed from the one–particle hamiltonian that the elementary excitations would have spin one. Instead, it turns out that the particle–like excitations over the antiferromagnetic Dirac sea have spin \( \frac{1}{2} \). The Fock space of the model is thus, for a chain with an even number of sites,

\[
\mathcal{F} = \bigoplus_{n=0}^{\infty} \int_{0}^{\pi} \cdots \int_{0}^{\pi} dk_{1} \cdots dk_{2n} \otimes C_{2}^{n}
\]  

(126)

where the integrations run over the possible values of the momenta and \( C_{2}^{n} \) represents the internal spin \( \frac{1}{2} \) space of dimension two. Proper symmetrization of the states in (126) must be taken into account as well. The excitations come in pairs [whence the \( 2n \) in (126)], for otherwise the total spin of a chain with an even number of sites would not be an integer. Let us stress that the internal quantum numbers of the elementary excitations are a completely unexpected collective result, impossible to predict \textit{a priori}. This is the motivation for invoking particles with strange statistics (anyons) in attempts to understand high temperature superconductors.
7 Yang–Baxter algebras and braid groups

The basic relation studied two section ago is the Yang–Baxter equation (86). In components, it reads as

$$\sum_{j_1,j_2,j_3} R_{j_1j_2}^{k_1k_2}(u) R_{j_1j_3}^{k_1k_3}(u + v) R_{j_2j_3}^{k_2k_3}(v) = \sum_{j_1,j_2,j_3} R_{j_2j_3}^{k_2k_3}(v) R_{j_1j_3}^{k_1j_3}(u + v) R_{i_1i_2}^{j_1j_2}(u)$$

(127)

where all the indices run from 1 to $n = \dim V$ ($n = 2$ for the six–vertex model). An interesting way to write this equation calls for the permuted $R$ matrix,

$$R = P \mathcal{R} : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$$

(128)

where $P$ is the permutation map

$$P : V_1 \otimes V_2 \rightarrow V_2 \otimes V_1$$

(129)

with $\{ e^{(i)}_r, r = 1, \ldots, n \}$ a basis of $V_i$. The relation between the entries of $R$ and $\mathcal{R}$ is straightforward:

$$R = P \mathcal{R} \quad \iff \quad R_{ij}^{kl} = \mathcal{R}_{ij}^{kl}$$

(130)

With the help of the permuted $R$–matrix, the Yang–Baxter equation (127) can be written as

$$(1 \otimes R(u)) (R(u + v) \otimes 1) (1 \otimes R(v)) = (R(v) \otimes 1) (1 \otimes R(u + v)) (R(u) \otimes 1)$$

(131)

Every operator in parentheses acts on the space $V \otimes V \otimes V$:

$$(R(u) \otimes 1) e_{i_1} \otimes e_{i_2} \otimes e_{i_3} = R_{i_1i_2}^{j_1j_2}(u) e_{j_2} \otimes e_{j_1} \otimes e_{i_3}$$

$$\quad (1 \otimes R(u)) e_{i_1} \otimes e_{i_2} \otimes e_{i_3} = R_{i_2i_3}^{j_2j_3}(u) e_{i_1} \otimes e_{j_3} \otimes e_{j_2}$$

(132)

Note that $R$ is very close to a factorizable $S$–matrix.

The reason for writing the Yang–Baxter equation in the form (131) comes from its relation to the braid group $B_L$ on $L$ strands, which is generated by $L - 1$ elements $\sigma_i$ ($i = 1, \ldots, L - 1$) subject to the relations

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

(133a)

$$\sigma_i \sigma_j = \sigma_j \sigma_i \quad |i - j| \geq 2$$

(133b)

$$\sigma_i^{-1} \sigma_{i-1} = \sigma_i^{-1} \sigma_i = 1$$

(133c)

The generator $\sigma_i$ braids the $i$–th strand under the $(i + 1)$–th strand, whereas $\sigma_i^{-1}$ effects the inverse braiding, i.e. it takes the $i$–th strand over the $(i + 1)$–th strand.

Trying to make (131) look more like (133), we define the operators $R_i(u)$ (for $i = 1, \ldots, L - 1$) on $\bigotimes_{i=1}^L V_i$, which act on the spaces $V_i \otimes V_{i+1}$ as $R(u)$ and as the identity elsewhere:

$$R_i(u) = 1 \otimes \cdots \otimes R(u) \otimes 1 \otimes \cdots \otimes 1$$

(134)
Then equation (131) becomes

\[ R_{i+1}(u)R_i(u+v)R_{i+1}(v) = R_i(v)R_{i+1}(u+v)R_i(u) \]  

(135)

while obviously

\[ R_i(u)R_j(v) = R_j(v)R_i(u) \quad |i - j| > 1 \]  

(136)

The identification of the Yang–Baxter equation in the form (135) with the braid group relation (133a) cannot be realized yet due to the presence of the rapidity variable \( u \). This should be no problem, however, in a situation where \( u \) depends on the difference of \( u \) and \( v \), which has two solutions:

i) \( u = v = 0 \),

ii) \( u = v, |u| = \infty \).

The first solution is trivial, since from (116) and (130) we get merely

\[ R(u = 0) = i \sin \gamma 1 \]  

(137)

And thus \( \mathcal{R} \) is just proportional to a permutation \( P \). Solution (ii) is known as the braid limit. Up to constant factors,

\[ \lim_{|u| \to \pm \infty} e^{-|u|} R(u) \sim P \exp [(\pm i \gamma / 2) \sigma^z \otimes \sigma^z] \]  

(138)

where we have assumed \( u \) real. This limit provides us with a representation of the braid group in terms of, essentially, permutations. The permutation group of \( L \) elements \( S_L \) satisfies the same defining relations as the braid group \( B_L \), except for the crucial difference that the square of a transposition is the identity, and therefore one cannot distinguish overcrossings from undercrossings. A good representation of the braid group should be able to distinguish between \( \sigma \) and \( \sigma^{-1} \). In the limit \( u \to +\infty \), the Boltzmann weights behave as

\[ a(u) \to \frac{1}{2} e^{u} e^{i \gamma}, \quad b(u) \to \frac{1}{2} e^{u}, \quad c(u) = i \sin \gamma \]  

(139)

Hence the information contained in the weight \( c(u) \) is washed out in the limit, which accounts for the “triviality” of the result (138). In order not to lose information in the limits \( u \to \pm \infty \), we perform a \( u \)-dependent rescaling of the basis elements, i.e. a \( u \)-dependent “diagonal” change of basis \( \tilde{e}_r(u) = f_r(u)e_r(u) \) \((r = 1, \ldots, n)\). Recalling that the definition of \( R \) is

\[ R(e_{r_1}(u_1) \otimes e_{r_2}(u_2)) = R_{r_1 r_2}^{r_1' r_2'}(u_1 - u_2)e_{r_1'}(u_2) \otimes e_{r_2'}(u_1) \]  

(140)

we deduce that the \( R \)-matrix in the new basis \( \tilde{e}_r \) is given by

\[ \tilde{R}_{r_1 r_2}^{r_1' r_2'}(u_1, u_2) = \frac{f_{r_1}(u_1)f_{r_2}(u_2)}{f_{r_1'}(u_1)f_{r_2'}(u_2)} R_{r_1 r_2}^{r_1' r_2'}(u_1 - u_2) \]  

(141)

The trick is to preserve the difference property of the \( R \)-matrix under this change of basis, and thereby fix the scaling functions \( f_r(u) \). Indeed, if \( \tilde{R}(u_1, u_2) \) is to still depend only on the difference \( u_1 - u_2 \), the functions in the change of basis must be \( f_r(u) = e^{au} \). Since \( \tilde{R} \) conserves the total quantum number, that is \( \tilde{R}_{r_1 r_2}^{r_1' r_2'}(u_1, u_2) = 0 \) unless \( r_1 + r_2 = r_1' + r_2' \), we may write the rescaled \( \tilde{R} \) matrix explicitly as

\[ \tilde{R}_{r_1 r_2}^{r_1' r_2'}(u_1 - u_2, \alpha) = e^{a(u_1 - u_2)(r_1 - r_1')} \tilde{R}_{r_1 r_2}^{r_1' r_2'}(u_1 - u_2) \]  

(142)
We take the braid limit of (142) at a special value of $\alpha$:

$$R \equiv 2 e^{-i\gamma/2} \lim_{u \to \pm\infty} e^{-u} \tilde{R}(u, \alpha = 1)$$  \hspace{1cm} (143)$$

obtaining

$$R^{00}_{00} = R^{11}_{11} = e^{i\gamma/2}$$

$$R^{01}_{10} = R^{10}_{01} = e^{-i\gamma/2}$$

$$R^{10}_{10} = e^{-i\gamma/2} \left( e^{i\gamma} - e^{-i\gamma} \right)$$

$$R^{01}_{01} = 0$$  \hspace{1cm} (144)$$

or, in matrix form,

$$R = \begin{pmatrix}
0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & q^{-\frac{1}{2}} & q^{-\frac{1}{2}} (q - q^{-1}) \\
0 & 0 & 0 & q^{\frac{1}{2}}
\end{pmatrix}$$  \hspace{1cm} (145)$$

with

$$q = e^{i\gamma}$$  \hspace{1cm} (146)$$

These $R$–matrices, first derived by Jimbo, satisfy the Yang–Baxter relation (131) without spectral parameter, and appear often in the literature. It can be checked explicitly that $(R)^2 \neq 1$ for $\gamma \neq 0$, so that we have indeed obtained a genuine representation of the braid group. In the isotropic case ($\gamma = 0$) we fall back to the previous result (138). The inverse matrix $R^{-1}$ can be obtained as the other real infinite limit of the same rescaled $R$:

$$R^{-1} = -2 e^{i\gamma/2} \lim_{u \to -\infty} e^{u} R(u, \alpha = 1)$$  \hspace{1cm} (147)$$

8 Yang–Baxter algebras and quantum groups

In the new basis $\{\tilde{e}_r\}$, the $R$–matrix (without the permutation in $R$) and the monodromy matrix $T$ are related to those in the basis $\{e_r\}$ by the equations

$$\tilde{R}_{i_1i_2}^{j_1j_2}(u) = e^{u(i_1 - j_1)} R_{i_1i_2}^{j_1j_2}(u)$$  \hspace{1cm} (148a)$$

$$\tilde{T}(u)_i^j = e^{u(i - j)} T(u)_i^j$$  \hspace{1cm} (148b)$$

The reader may check that $\tilde{R}(u)$ and $\tilde{T}(u)$ do satisfy indeed equation (92).

Now just like we took the limit $u \to \pm\infty$ of the matrix $\tilde{R}(u)$, we may take the limit of the monodromy matrix $\tilde{T}(u)$. To get a feeling for what this limit may yield, let us consider the spin $\frac{1}{2}$ representation given by equations (97):

$$\lim_{u \to \pm\infty} \tilde{A}(u) = \lim_{u \to \pm\infty} \frac{1}{2} e^{u} e^{i\gamma/2} \left( \begin{array}{cc}
e^{-i\gamma/2} & e^{i\gamma/2} \\
e^{i\gamma/2} & e^{-i\gamma/2} \end{array} \right) = \frac{1}{2} e^{u} q^{1/2} q^{S_z}$$

$$\lim_{u \to -\infty} \tilde{A}(u) = \lim_{u \to -\infty} -\frac{1}{2} e^{-u} e^{-i\gamma/2} \left( \begin{array}{cc}
e^{i\gamma/2} & e^{-i\gamma/2} \\
e^{-i\gamma/2} & e^{i\gamma/2} \end{array} \right) = -\frac{1}{2} e^{-u} q^{-1/2} q^{-S_z}$$  \hspace{1cm} (149)$$
where \( S^z = \frac{1}{2} \sigma^z \) is the Cartan generator in the spin-\( \frac{1}{2} \) representation of \( SU(2) \) and we recall that \( q = e^{i\gamma} \). The limits \( u \to \pm \infty \) of \( \tilde{B}(u), \tilde{C}(u) \) and \( \tilde{D}(u) \) can be evaluated similarly, whereby the braid limits of the monodromy matrix \( \tilde{T}(u) \) are

\[
T_+ \equiv 2q^{-1/2} \lim_{u \to +\infty} e^{-u} \begin{pmatrix} T_0^0 & T_0^1 \\ T_1^0 & T_1^1 \end{pmatrix} = \begin{pmatrix} q^{S^z} & 0 \\ q^{-1/2}(q - q^{-1})S^- & q^{-S^z} \end{pmatrix} \tag{150a}
\]

\[
T_- \equiv -2q^{1/2} \lim_{u \to -\infty} e^u \begin{pmatrix} T_0^0 & T_0^1 \\ T_1^0 & T_1^1 \end{pmatrix} = \begin{pmatrix} q^{-S^z} & -q^{1/2}(q - q^{-1})S^+ \\ 0 & q^{S^z} \end{pmatrix} \tag{150b}
\]

where \( S^\pm = (\sigma^x \pm i\sigma^y)/2 \) are the off–diagonal generators of \( SU(2) \) in the spin–\( \frac{1}{2} \) irrep.

The fun starts when we take the various limits \( u \to \pm \infty, v \to \pm \infty \) in the \( \mathcal{R}TT = TTR \) equation (90). All the extra factors work out nicely so that the result is

\[
\mathcal{R}_{12} (T_+)_{1} (T_+)_{2} = (T_+)_{2} (T_+)_{1} \mathcal{R}_{12} \\
\mathcal{R}_{12} (T_+)_{1} (T-)_{2} = (T-)_{2} (T_+)_{1} \mathcal{R}_{12} \\
\mathcal{R}_{-1} (T-)_{1} (T-)_{2} = (T-)_{2} (T-)_{1} \mathcal{R}_{-1} \tag{151}
\]

with \( \mathcal{R} = PR \ [R \text{ given by } (144)] \) and \( T_\pm \) from (150).

This system of equations appears rather complicated at first sight. They are, in fact, equivalent to the following algebraic relations between \( S^z, S^+ \) and \( S^- \):

\[
\begin{pmatrix} S^z, S^\pm \end{pmatrix} = \pm S^\pm \tag{152a}
\]

\[
\begin{pmatrix} S^+, S^- \end{pmatrix} = \frac{q^{2S^z} - q^{-2S^z}}{q - q^{-1}} \tag{152b}
\]

These are the defining relations for the quantum group \( U_q(sl(2)) \), which is some kind of deformation of the Lie algebra \( sl(2) \) with \( q = e^{i\gamma} \) acting as deformation parameter.

The notation \( U_q(sl(2)) \) clarifies that the quantum group consists of all the formal powers and linear combinations of \( S^+, S^- \) and \( S^z \), subject to the relations (152). Traditionally, \( U(sl(2)) \) denotes the universal enveloping algebra of \( sl(2) \), that is all the formal powers and linear combinations of \( S^\pm \) and \( S^z \) modulo the standard Lie algebra relations. In the isotropic limit \( \gamma \to 0 \), i.e., \( q \to 1 \), we recover from (152) the usual \( sl(2) \) algebra. The limit \( \gamma \to 0 \) is called classical, in the sense that the “quantum” group \( U_q(sl(2)) \) becomes the “classical” universal enveloping algebra \( U(sl(2)) \). From the viewpoint of rigid nomenclature, it is perhaps unfortunate that the classical limit of a quantum group is (the universal enveloping algebra of) a Lie algebra; beware of misled distinctions between quantum groups and quantum algebras!

Finally, we may take the braid limits \( u \to \pm \infty \) in the co-multiplication rule (93) to find the co-multiplication for the generators of the quantum group \( U_q(sl(2)) \):

\[
\Delta(q^{S^z}) = q^{S^z} \otimes q^{S^z} \tag{153a}
\]

\[
\Delta(S^\pm) = S^\pm \otimes q^{S^z} + q^{-S^z} \otimes S^\pm \tag{153b}
\]

The co-multiplication preserves the algebraic relations (152), as can be checked by using the fact that \( \Delta \) is a homomorphism, that is to say \( \Delta(ab) = \Delta(a)\Delta(b) \). Note that the
non–trivial addition rule (153b) is consistent with the non–trivial commutator (152b), and vice versa. Compare also with $\Delta B(u)$ in equation (98).

We have derived an interesting algebraic structure, the quantum group $U_q(s\ell(2))$, by letting the rapidities become infinite in the Yang–Baxter elements $T^j_i(u)$. More precisely, the quadratic relations between the monodromy matrices ($RTT$ equations of the Yang–Baxter algebra) give us the defining relations of $U_q(s\ell(2))$, while the co–multiplication of $A^{(6\nu)}$ implies that of $U_q(s\ell(2))$.

Let us return once again to the Yang–Baxter equation satisfied by the $\mathcal{R}$–matrix:

$$
\sum_{j_1,j_2,j_3} \mathcal{R}^{k_1 k_2}_{j_1 j_2} (u-v) \mathcal{R}^{j_1 j_2}_{i_1 i_2} (u) \mathcal{R}^{j_2 j_3}_{i_2 i_3} (v) =
\sum_{j_1,j_2,j_3} \mathcal{R}^{k_1 k_3}_{j_2 j_3} (v) \mathcal{R}^{j_1 j_3}_{i_1 i_3} (u) \mathcal{R}^{j_1 j_2}_{i_1 i_2} (u-v)
$$

The $RTT$ equation is based on the identification of $T^j_i$ in the representation of dimension two with the $\mathcal{R}$–matrix itself:

$$
(T^j_i (u))^{\beta}_\alpha = \mathcal{R}^{j \beta}_{i \alpha} = i \quad \begin{array}{c} \beta \\ \alpha \end{array} \quad j
$$

where $i, j$ are indices of the auxiliary space (that is, labels for the elements of $A^{(6\nu)}$) and $\alpha, \beta$ are indices of the quantum space (indicating the representation of $A^{(6\nu)}$).

We wish to emphasize that in all this construction, the $\mathcal{R}$ matrix has played a rather auxiliary role, as indeed its indices in $RTT = TT\mathcal{R}$ are auxiliary: we would like to see the $\mathcal{R}$–matrix playing a role in quantum space as well.

Let us take advantage of an interesting property satisfied by the $\mathcal{R}$–matrix of the six–vertex model, the parity symmetry:

$$
\mathcal{R}^{j_1 j_2}_{i_1 i_2} (u) = \mathcal{R}^{j_2 j_1}_{i_2 i_1} (u)
$$

or equivalently

$$
P \mathcal{R}(u) P = \mathcal{R}(u)
$$

with $P$ the permutation operator (129). With the help of equation (156), we may rewrite (154) as

$$
\sum_{j_1,j_2,j_3} \mathcal{R}^{k_1 k_2}_{j_1 j_2} (u-v) \mathcal{R}^{k_3 j_1}_{j_2 j_3} (u) \mathcal{R}^{j_2 j_3}_{i_3 i_2} (v) =
\sum_{j_1,j_2,j_3} \mathcal{R}^{k_1 k_3}_{j_2 j_3} (v) \mathcal{R}^{k_3 j_2}_{j_3 j_1} (u) \mathcal{R}^{j_1 j_2}_{i_1 i_2} (u-v)
$$

Note that in (154) the space $V_1 \otimes V_2$ is auxiliary and $V_3$ quantum, whereas now $V_3$ is auxiliary but both $V_1$ and $V_2$ are quantum. We have thus gotten $\mathcal{R}$ into quantum space. Using (155), we rewrite (158) as

$$
\mathcal{R}(u-v) \left( T^j_i (u) \otimes T^k_i (v) \right) = \left( T^j_i (u) \otimes T^k_i (v) \right) \mathcal{R}(u-v)
$$

where the tensor product takes place in $V_1 \otimes V_2$ and $\mathcal{R}(u-v) \in \text{End}(V_1 \otimes V_2)$.

What are the consequences of the $RTT$ equation (159) with $\mathcal{R}$ in quantum space? First of all, for consistency with the parity symmetry, there must exist some function
\( \rho(u) \) such that
\[ \mathcal{R}(u) \mathcal{R}(-u) = \rho(u) \rho(-u) 1 \] (160)

This can be derived by acting on both sides of (159) with the permutation operator \( P \) and then using (157). Equation (160) can also be derived from the unitarity condition (16), namely \( \mathcal{R}(u) P \mathcal{R}(u) P \sim 1 \) for parity invariant \( \mathcal{R} \)-matrices, i.e. satisfying (159). For the \( \mathcal{R} \)-matrix of the six–vertex model, \( \rho(u) = a(u) \) in (160).

Letting \( u = v \) in (159) and knowing that
\[ \mathcal{R}(0) = \rho(0) P \] (161)
we obtain
\[ T_i^j(u) \otimes T_j^k(u) = P \left( T_j^k(u) \otimes T_i^j(u) \right) P \] (162)
which establishes the equivalence between the co-multiplication (93) and its transpose. More generally, when \( u \neq v \), equation (159) establishes an equivalence between the two ways of co-multiplying arbitrary elements of \( \mathcal{A} \). Note that \( u \) and \( v \) are labels of the representation spaces \( V_1 \) and \( V_2 \), respectively. If, recalling (93), we define
\[ \Delta_{u,v} \left( T_i^k \right) = T_i^j(u) \otimes T_j^k(v) \] (163a)
\[ \Delta'_{v,u} \left( T_i^k \right) = T_j^k(v) \otimes T_i^j(u) \] (163b)
we can write (159) as
\[ \Delta_{u,v} \left( T_i^k \right) = \mathcal{R}(u - v) \Delta'_{v,u} \left( T_i^k \right) \mathcal{R}^{-1}(u - v) \] (164)
This equation means that \( \mathcal{R}(u - v) \) intertwines the two possible co-multiplications \( \Delta_{u,v} \) and \( \Delta'_{v,u} \). In Drinfeld’s definition of quantum groups as quasi-triangular Hopf algebras, equation (164) is one of the basic postulates.

To understand how the \( \mathcal{R} \)-matrix intertwines between a co-product and its transpose, let us take a closer look at the braid limit \( u \to \infty \) of equation (159), or rather of its analog for the rescaled \( \tilde{\mathcal{R}}(u) \) and \( \tilde{T}(u) \) introduced in (148), namely
\[ \tilde{\mathcal{R}}(v - u) \left( \tilde{T}_i^j(u) \otimes \tilde{T}_j^k(v) \right) = \left( \tilde{T}_j^k(v) \otimes \tilde{T}_i^j(u) \right) \tilde{\mathcal{R}}(v - u) \] (165)
Using (143) and (150) we find the braid limit of (165), which is the braid limit of (164):
\[ \Delta'(g) = \mathcal{R} \Delta(g) \mathcal{R}^{-1} \quad g \in \left\{ q^{\pm S^z}, S^\pm \right\} \] (166)
Here, the \( \mathcal{R} \)-matrix is \( \mathcal{R} = PR \) with \( R \) the Jimbo matrix (144), the co-product \( \Delta(g) \) is given by (153), and the transposed co-product \( \Delta' \) is, explicitly,
\[ \Delta'(q^{S^z}) = q^{S^z} \otimes q^{S^z} \]
\[ \Delta'(S^\pm) = S^\pm \otimes q^{-S^z} + q^{S^z} \otimes S^\pm \] (167)

Equation (166) should really be written, to avoid confusion, as
\[ \Delta_{1/2,1/2}'(g) = \mathcal{R}_{1/2,1/2} \Delta_{1/2,1/2} \left( \mathcal{R}_{1/2,1/2}^{-1} \right) \] (168)
where \( \Delta_{1/2,1/2} \) denotes the restriction of \( \Delta(g) \) to the irrep \( 1/2 \otimes 1/2 \) of \( U_q(sl(2)) \) and the indices on \( \mathcal{R} \) remind us that we are using the representation of \( \mathcal{R} \) on the vector space \( 1/2 \otimes 1/2 \). It is nevertheless worth stressing that (166) makes sense even at the level of
the quantum group $U_q(s\ell(2))$, prior to the construction of its representations. By this we mean that $R$ in (167) can be viewed as an element of $U_q(s\ell(2)) \otimes U_q(s\ell(2))$, rather than as a numerical matrix as in (168). For this reason, the matrix $R$ is called the “universal $R$–matrix”, and its existence guarantees that the co-multiplications $\Delta$ and $\Delta'$ of $U_q(s\ell(2))$ are equivalent at the purely algebraic level.

9 Affine quantum groups

We have found that the integrability of the six–vertex model in the braid limit (when the dependence of the Boltzmann weights on the rapidity drops out) is encoded in the quantum group $U_q(s\ell(2))$. Motivated by these results, we may ask ourselves whether a mathematical structure similar to the quantum group could be associated with the rapidity–dependent $R$–matrix solutions to the Yang–Baxter equation.

The $s\ell(2)$ Lie algebra ($A_1^1$ in Cartan’s classification) has three Chevalley generators $E, F$ and $H$ with the following non–vanishing commutators

$$ [E, F] = H $$
$$ [H, E] = 2E $$
$$ [H, F] = -2F $$

The usual spin generators are related to these by

$$ E = S^+ , \quad F = S^- , \quad H = 2S^z $$

The affine extension of $A_1$, called $A_1^{(1)}$ in Kac’s classification, has six Chevalley generators $E_i, F_i$ and $H_i$ ($i = 0, 1$). Suppose we have an irreducible representation of $A_1$. It can be affinized, i.e. promoted to an irreducible representation of $A_1^{(1)}$ through the identifications:

$$ E_0 = e^u F \quad E_1 = e^u E $$
$$ F_0 = e^{-u} E \quad F_1 = e^{-u} F $$
$$ H_0 = -H \quad H_1 = H $$

where $x = e^u$ is a complex affinization parameter.

Different irreps of $A_1^{(1)}$ (of zero central extension) may be labelled by the affine parameter $e^u$ and the Casimir of the corresponding representation of $A_1$. For example, the irreducible $(e^u, \frac{1}{2})$ representation of $A_1^{(1)}$ derives from the usual spin $\frac{1}{2}$ irrep of $A_1$:

$$ E_0 = \begin{pmatrix} 0 & e^u & 0 \\ e^u & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad E_1 = \begin{pmatrix} 0 & e^u \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} $$
$$ F_0 = \begin{pmatrix} 0 & 0 & e^{-u} \\ 0 & 0 & 0 \\ e^{-u} & 0 & 0 \end{pmatrix} \quad F_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} $$

$$ H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} $$

Let us turn to the quantum deformations of $A_1$ and $A_1^{(1)}$, which we shall denote by $U_q(A_1)$ and $U_q(A_1^{(1)})$, respectively. If we define the operator $K$ as

$$ K = q^H $$

we may rewrite equations (152) in terms of $E, F$ and $K$ as

$$ KE = q^2 EK $$

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\[KF = q^{-2}F K\]
\[[E, F] = \frac{K - K^{-1}}{q - q^{-1}}\]

We shall denote by \(U_q(A_1) = U_q(sl(2))\) the algebra generated by \(E, F\) and \(H\) subject to (152).

Affinization of the spin– \(\frac{1}{2}\) irrep of \(U_q(A_1)\) yields an irrep \((e, \frac{1}{2})\) of \(U_q(A_1^{(1)})\):

\[
\begin{align*}
E_0 &= \begin{pmatrix} 0 & 0 \\ e^u & 0 \end{pmatrix} & E_1 &= \begin{pmatrix} 0 & e^u \\ 0 & 0 \end{pmatrix} \\
F_0 &= \begin{pmatrix} 0 & e^{-u} \\ 0 & 0 \end{pmatrix} & F_1 &= \begin{pmatrix} 0 & 0 \\ e^{-u} & 0 \end{pmatrix} \\
K_0 &= \begin{pmatrix} q^{-1} & 0 \\ 0 & q \end{pmatrix} & K_1 &= \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}
\end{align*}
\]  

which is the same as representation (172) of the classical group \(A_1^{(1)}\) provided we take into account relation (173). For the fundamental irrep, as the doublet of \(sl(2)\), it is always true that the classical and quantum representations coincide.

We expect that the irrep \((e, \frac{1}{2})\) of \(U_q(A_1^{(1)})\) should be intimately related to the spin– \(\frac{1}{2}\) representation (97) of the generators \(A(u), B(u), C(u)\) and \(D(u)\) of \(A^{(6c)}\). This is so, indeed:

\[
\begin{align*}
A(u) &= \frac{1}{2} \left( e^u q^{1/2} K^{1/2} - e^{-u} q^{-1/2} K^{-1/2} \right) \\
B(u) &= \frac{1}{2} \left( q - q^{-1} \right) q^{-1/2} F K^{1/2} \\
C(u) &= \frac{1}{2} \left( q - q^{-1} \right) q^{-1/2} E K^{1/2} \\
D(u) &= \frac{1}{2} \left( e^u q^{1/2} K^{1/2} - e^{-u} q^{-1/2} K^{-1/2} \right)
\end{align*}
\]

Please check that the algebraic relations (106) follow from those of the quantum group \(U_q(A_1)\), equations (174).

The affine quantum group \(U_q(A_1^{(1)})\) enjoys also a bi-algebra structure, determined by the co-product

\[
\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i
\]
\[
\Delta(F_i) = F_i \otimes K_i^{-1} + K_i \otimes F_i
\]
\[
\Delta(K_i) = K_i \otimes K_i
\]  

It is thus natural to look for an intertwiner \(R\)-matrix for the tensor product of two spin– \(\frac{1}{2}\) irreps \((e^{u_1}, \frac{1}{2}) \otimes (e^{u_2}, \frac{1}{2})\) of \(U_q(A_1^{(1)})\):

\[
R(e^{u_1}, e^{u_2}) \Delta e^{u_1}, e^{u_2} (g) = \Delta e^{u_1}, e^{u_2} (R(e^{u_1}, e^{u_2}))(g) \forall g \in U_q(A_1^{(1)})
\]  

This is nothing but the affinized version of the intertwiner condition (168) for \(U_q(A_1)\). At the risk of offending the reader, we show the transposed co-multiplication \(\Delta'\) of the generators of \(U_q(A_1^{(1)})\):

\[
\begin{align*}
\Delta'(E_i) &= E_i \otimes 1 + K_i \otimes E_i \\
\Delta'(F_i) &= F_i \otimes K_i^{-1} + 1 \otimes F_i \\
\Delta'(K_i) &= K_i \otimes K_i
\end{align*}
\]
Comparing (177) with (153), we see that the relation between $E$, $F$ and $S^+$, $S^-$ is

$$ E = K^{\frac{1}{2}} S^+ , \quad F = K^{-\frac{1}{2}} S^- $$  \hbox{(180)}

After some straightforward manipulations, from (178) we get the affine $\mathcal{R}^{\frac{1}{2}\frac{1}{2}}(e^{u_1}, e^{u_2})$ matrix:

$$
\begin{align*}
\mathcal{R}^{01}_{01}(e^{u_1}, e^{u_2}) &= e^{u_1-u_2} - e^{u_2-u_1} \\
\mathcal{R}^{00}_{00}(e^{u_1}, e^{u_2}) &= q e^{u_1-u_2} - q^{-1} e^{u_2-u_1} \\
\mathcal{R}^{10}_{01}(e^{u_1}, e^{u_2}) &= q - q^{-1}
\end{align*}
$$  \hbox{(181)}

with $\mathcal{R}^{00}_{00} = \mathcal{R}^{11}_{11}$, $\mathcal{R}^{01}_{01} = \mathcal{R}^{10}_{10}$, $\mathcal{R}^{10}_{01} = \mathcal{R}^{01}_{10}$, and all other matrix elements of $\mathcal{R}$ equal to zero.

Identifying now $q = e^{i\gamma}$ and $e^u = e^{u_1-u_2}$, we see that $\mathcal{R}(e^{u_1}, e^{u_2})$ is just the six–vertex $\mathcal{R}$–matrix (116). With this happy result we conclude our preview of the relation between the six–vertex model and the affine quantum group $U_q(A_1^{(1)})$.

### 10 Hopf algebras

Let $(\mathcal{A}, m, \iota)$ be an algebra whose multiplication $m : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is associative, that is

$$ [m(m \otimes 1)](a \otimes b \otimes c) = [m(1 \otimes m)](a \otimes b \otimes c) \quad \forall a, b, c \in \mathcal{A} $$  \hbox{(182)}

We write $ab = m(a \otimes b)$, $\forall a, b \in \mathcal{A}$. If $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, to make formal sense of $\lambda a \in \mathcal{A}$ we need the unit map $\iota : \mathbb{C} \rightarrow \mathcal{A}$ of $\mathcal{A}$, which is intimately tied to the identity $1 \in \mathcal{A}$:

$$ \iota : \lambda \in \mathbb{C} \mapsto \lambda 1 \in \mathcal{A} $$  \hbox{(183)}

The unit $\iota$ and the multiplication $m$ are compatible in the sense that

$$ m(a \otimes \iota(\lambda)) = a\lambda = \lambda a = m(\iota(\lambda) \otimes a) \quad \forall a \in \mathcal{A} \quad \forall \lambda \in \mathbb{C} $$  \hbox{(184)}

Let us consider now a co-multiplication or co-product $\Delta : \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{A}$, which should be “co-associative”:

$$ (\Delta \otimes 1)(\Delta(a)) = (1 \otimes \Delta)(\Delta(a)) \quad \forall a \in \mathcal{A} $$  \hbox{(185)}

We also need a co-unit map $\epsilon : \mathcal{A} \rightarrow \mathbb{C}$ to define a co-algebra $(\mathcal{A}, \Delta, \epsilon)$; it must satisfy

$$ (1 \otimes \epsilon)\Delta = (\epsilon \otimes 1)\Delta = 1 $$  \hbox{(186)}

A simultaneous algebra and co-algebra $(\mathcal{A}, m, \iota, \Delta, \epsilon)$ is called a bi-algebra if the co-multiplication $\Delta$ and the co-unit $\epsilon$ are consistent with the multiplication $m$, that is if they are homomorphisms:

$$ \epsilon(ab) = \epsilon(a)\epsilon(b) \quad , \quad \Delta(ab) = \Delta(a)\Delta(b) $$  \hbox{(187)}

Actually, the unit $\iota$ and co-unit $\epsilon$ must also be compatible:

$$ \iota(\epsilon(a)) = \epsilon(a)1 \quad \forall a \in \mathcal{A} $$  \hbox{(188)}

A Hopf algebra is a bi-algebra enjoying an antipode $\gamma : \mathcal{A} \rightarrow \mathcal{A}$, which is an anti-homomorphism

$$ \gamma(ab) = \gamma(b)\gamma(a) $$  \hbox{(189)}

satisfying the following condition:

$$ m(\gamma \otimes 1)\Delta(a) = m(1 \otimes \gamma)\Delta(a) = \epsilon(a)1 \quad \forall a \in \mathcal{A} $$  \hbox{(190)}

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This condition involves all the ingredients of the bi-algebra structure. Hopf algebras are much more interesting than mere bi-algebras.

If the multiplication \( m \) is commutative (respectively, not commutative), we call the algebra commutative (respectively, non–commutative). Just like the multiplication, the co-multiplication may or may not be commutative. The Hopf algebra is accordingly co-commutative or non–co-commutative. Of primary interest are those Hopf algebras which are neither commutative nor co-commutative.

Introduce the permutation map

\[
\sigma : A \otimes A \rightarrow A \otimes A
\]

\[
a \otimes b \mapsto b \otimes a
\]

which merely interchanges the order of the operands. Commutativity means thus

\[
ab \equiv m(a \otimes b) \equiv m(\sigma(a \otimes b)) \equiv m(b \otimes a) \equiv ba \quad \forall a, b \in A
\]

On the other hand, if the algebra is co-commutative, then

\[
\Delta(a) = \sigma \cdot \Delta(a) \equiv \Delta'(a) \quad \forall a \in A
\]

Given a co-multiplication \( \Delta \), it is not hard to check that the operation \( \Delta' = \sigma \circ \Delta \in \text{End}(A \otimes A) \) is also a co-multiplication, with modified antipode \( \gamma'(a) = [\gamma(a)]^{-1} \), (\( \forall a \in A \)). Given a Hopf algebra \( A \), it is called quasi-triangular if there exists a universal \( R \)–matrix \( R \in A \otimes A \) such that

\[
\Delta'(a) = R \Delta(a) R^{-1} \quad \forall a \in A
\]

and

\[
(1 \otimes \Delta)R = R_{13} R_{12} = \sum_{i,j} A_i A_j \otimes B_j \otimes B_i
\]

\[
(\Delta \otimes 1)R = R_{13} R_{23} = \sum_{i,j} A_i \otimes A_j \otimes B_i B_j
\]

and

\[
(\gamma \otimes 1)R = (1 \otimes \gamma^{-1})R = R^{-1}
\]

where \( R \) is called the universal \( R \)–matrix. We write \( R = \sum_i A_i \otimes B_i \) and let

\[
R_{12} = \sum_i A_i \otimes B_i \otimes 1
\]

\[
R_{13} = \sum_i A_i \otimes 1 \otimes B_i
\]

\[
R_{23} = \sum_i 1 \otimes A_i \otimes B_i
\]

Essentially, quasi-triangularity means that the co-multiplication \( \Delta \) and its “transposed” \( \Delta' \) are related linearly. In some sense, it establishes an equivalence between two different ways of “adding things up”.

A co-commutative algebra is trivially quasi-triangular, with \( R = 1 \otimes 1 \). A Hopf algebra is called triangular if \( R_{12} R_{21} = 1 \otimes 1 \), where \( R_{21} = \sum B_i \otimes A_i \).

A non–co-commutative quasi-triangular Hopf algebra is called a quantum group.
The interest in quasi-triangular Hopf algebras is that they produce solutions to the Yang–Baxter equation naturally. Indeed, from (194) and (195) the Yang–Baxter equation in a “universal form” without spectral parameter may be derived:

\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \]  

(198)

The proof goes like this:

\[ [\sigma \circ \Delta \otimes 1] R = \sum_i \Delta'(A_i) \otimes B_i \]

\[ = \sum_i R_{12} \Delta(A_i) R_{12}^{-1} \otimes B_i \]

\[ = R_{12} \left( \sum_i \Delta(A_i) \otimes B_i \right) R_{12}^{-1} \]  

(199)

\[ = R_{12} \left[ (\Delta \otimes 1) R \right] R_{12}^{-1} \]

\[ = R_{12} R_{13} R_{23} R_{12}^{-1} \]

On the other hand,

\[ [\sigma \circ \Delta \otimes 1] R = \sigma_{12} (\Delta \otimes 1) R \]

\[ = \sigma_{12} (R_{13} R_{23}) \]  

(200)

\[ = R_{23} R_{13} \]

and thus (198) follows.

11 The quantum group \( U_q(G) \)

In this section, we present the quantum semi-simple algebras due to Drinfeld and Jimbo, generalizing the construction of \( U_q(sl(2)) \) above.

Let \( G \) be a semi-simple Lie algebra with \( A = (a_{ij}) \) \((i, j = 1, \ldots, n = \text{rank } G)\) the corresponding Cartan matrix and \( D = (D_i) \) the vector or diagonal matrix such that \( D_i a_{ij} = a_{ij} D_i \).

The quantum group \( U_q(G) \) is defined as the algebra of formal power series in \( q \) with generators \( e_i, f_i, k_i \) \((i = 1, \ldots, n = \text{rank } G)\) subject to the following relations:

\[ k_i k_j = k_j k_i \]

\[ k_i e_j = q_i^{a_{ij}} e_j k_i \]

\[ k_i f_j = q_i^{-a_{ij}} f_j k_i \]

\[ e_i f_j - f_j e_i = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}} \]

\[ \sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \left[ \frac{1-a_{ij}}{\ell} \right]_{q_i} e_i^{1-a_{ij}-\ell} e_j^{\ell} e_i^{\ell} = 0 \quad i \neq j \]  

(202a)

\[ \sum_{\ell=0}^{1-a_{ij}} (-1)^\ell \left[ \frac{1-a_{ij}}{\ell} \right]_{q_i} f_i^{1-a_{ij}-\ell} f_j^{\ell} f_i^{\ell} = 0 \quad i \neq j \]  

(202b)

We have used the notations

\[ q_i = q^{D_i}, \quad [x]_{q_i} = \frac{q_i^x - q_i^{-x}}{q_i - q_i^{-1}} \]  

(203)
The equations (202) are called the quantum Serre relations. For the simplest case of $G = A_1$, there are no Serre relations and (201) coincide with (174).

The generators $e_i, f_i$ and $k_i$, the index $i$ ranging over the positive simple roots, constitute the Chevalley basis of the algebra. Supplemented with the Serre relations, it is equivalent to the Cartan basis (one raising operator for each positive root). In the quantum case, the Chevalley basis is much more convenient than the Cartan basis, due to the profusion of $q$–factors.

The co-multiplication of $U_q(G)$ is given by

\[
\Delta(k_i) = k_i \otimes k_i \\
\Delta(e_i) = e_i \otimes k_i + 1 \otimes e_i \\
\Delta(f_i) = f_i \otimes 1 + k_i^{-1} \otimes f_i
\]  
(204)

and the antipode by

\[
\gamma(e_i) = -e_i k_i^{-1} \quad , \quad \gamma(f_i) = -k_i f_i \quad , \quad \gamma(k_i) = k_i^{-1}
\]  
(205)

To write the universal $R$–matrix of $U_q(G)$ we need the “logarithm” of $k_i$,

\[
H_i = \frac{1}{2hD_i} \log \left( k_i^2 \right)
\]  
(206)

where we have set $q = \exp h$. Then

\[
\mathcal{R} = \exp \left[ h \sum_{i,j=1}^{n} \left( B^{-1} \right)_{ij} H_i \otimes H_j \right] \left[ 1 + \sum_{i=1}^{n} \left( 1 - q_i^{-2} \right) e_i \otimes f_i + \cdots \right]
\]  
(207)

where $B_{ij} = D_i a_{ij}$ is the symmetrized Cartan matrix. This $\mathcal{R}$–matrix follows form Drinfeld’s quantum double construction.

In practical applications, we are interested in the $R$–matrix in some representation. For example, the $R$–matrix in the fundamental of $U_q(sl(n))$, $R_n = P \mathcal{R}^{(n,n)}$, is

\[
R_n = q \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ij} \otimes e_{ji} + \left( q - q^{-1} \right) \sum_{i < j} e_{jj} \otimes e_{ii}
\]  
(208)

where $e_{ij}$ is an $n \times n$ matrix whose only non–zero entry is the $(i, j)$-th one. In matrix notation,

\[
(R_n)^{k\ell}_{ij} = \begin{cases}
q & \text{if } i = j = k = \ell \\
1 & \text{if } i = \ell \neq k = j \\
q - q^{-1} & \text{if } i = k < \ell = j \\
0 & \text{otherwise}
\end{cases}
\]  
(209)

It is not hard to verify that

\[
(R_n)^{-1} = q^{-1} \sum_{i=1}^{n} e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ij} \otimes e_{ji} + \left( q^{-1} - q \right) \sum_{i > j} e_{jj} \otimes e_{ii}
\]  
(210)

and thus

\[
R_n - (R_n)^{-1} = \left( q - q^{-1} \right) 1_V \otimes V
\]  
(211)

where $V$ is the $n$–dimensional space on which $U_q(sl(n))$ is represented.
12 Comments

In these lectures, we have tried to give a flavor for some of the main ideas and techniques underlying two–dimensional integrable systems. Much has been left out, notably the thorny eight–vertex model, all the face models (closely linked with conformal field theories), and the funny models arising from quantum groups when \( q \) is a root of unit. This last point alone deserves a full course. Also, we have bypassed all the applications of and to knot theory. In a finite amount of time, however, only so much information can be humanly absorbed. The subject of theories with quantum symmetries is under active research, particularly from the point of view of string theory: the emphasis there is on continuum two–dimensional field theories with \( N = 2 \) supersymmetry.

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13 References

It is perhaps more useful to point out a few good review articles and books; references to the original literature can be found easily starting from the works cited below.

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