Frames of reference in spaces with affine connections and metrics

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Abstract

A generalized definition of a frame of reference in spaces with affine connections and metrics is proposed based on the set of the following differential-geometric objects: (a) a non-null (non-isotropic) vector field, (b) the orthogonal to the vector field subspace, (c) an affine connection and the related to it covariant differential operator determining a transport along the given non-null vector field. On the grounds of this definition other definitions related to the notions of accelerated, inertial, proper accelerated and proper inertial frames of reference are introduced and applied to some mathematical models for the space-time. The autoparallel equation is obtained as an Euler-Lagrange’s equation. Einstein’s theory of gravitation appears as a theory for determination of a special frame of reference (with the gravitational force as inertial force) by means of the metrics and the characteristics of a material distribution.

PACS numbers: 0490, 0450, 1210G, 0240V

1 Introduction.

1.1 Definitions of a frame of reference in classical and relativistic physics

The notion of frame of reference is one of the important notions in physics. It is related, from the one hand, to the mathematical models for description of physical systems and the space-time, and, on the other hand, to the experimental check-up of these models. From philosophical and physical point of view there are many problems in finding out an appropriate definition for inertial and non-inertial (accelerated) frames of reference in classical mechanics [19] and in relativistic physics [29, 30, 26, 39, 16, 17].

In classical mechanics limited notions of absolute rigid body and of material point are used for solving theoretical and experimental problems in many cases without any difficulties [19].
From mathematical point of view the application of limited motions is related to mathematical objects (such as points, curves, co-ordinates etc.) used in definitions of a frame of reference (FR). Unfortunately, until now there is no general acceptable definition of a FR as well as of transitions from one to another FR.

There are at least three types of methods for defining a FR. They are based on three different basic assumptions:

(a) *Co-ordinate's methods.* A frame of reference is identified with a local (or global) chart (co-ordinates) in the differentiable manifold \( M \) (\( \dim M = n, n \geq 2 \)) considered as a model of space-time \( [28] \). Additional conditions for the transformations of the co-ordinates are also proposed to ensure the transition from one to another FR [see for instance, \([30]\) and the references there]. After Roditchev \([30]\) these groups of transformations of the co-ordinates are called transformations of type \( A \).

(b) *Tetrad's methods.* A frame of reference is identified with a set of basic contravariant vector fields \( \{ e_\alpha \} \in T_x(M), \alpha = 1, ..., n \), at every given point \( x \) of the manifold \( M \) considered as a model of space-time. For \( n = 4 \), four linear independent contravariant vectors (or their components in a given basis) are called a tetrad. Additional conditions are imposed for determining a special type of a \( n \)-Bein used as a FR \([26], [31], [32], [14], [17], [27]\). The transition from one to another FR is related to the transformation properties of the \( n \)-Beins. The groups of transformations of the \( n \)-Beins is called group of transformations of type \( B \).

(c) *Monad's methods.* A frame of reference is identified with a non-null (non-isotropic) (time-like) contravariant vector field interpreted as the velocity of an observer (material point). A transition from one to another FR is related to the transformations of the contravariant vector field to another vector field of the same type. Some additional conditions are required for finding out the explicit form of these transformations related to the notions of inertial and non-inertial (accelerated) FR. These groups of transformations of the type \( u' = \Omega(u), \Omega \in \otimes^1_1(M), u', u \in T(M) \), are considered by Roditchev \([28], [30]\), where \( \Omega \) is called affinor. Transformations of this type could be called transformations of type \( C \).

If a contravariant vector field at a point of the manifold \( M \) is defined as a vector belonging to an introduced at this point \( n \)-Bein and further the vectors of the \( n \)-Bein are considered as tangential vectors to the co-ordinates given in a neighbourhood of this point, then all methods for determining a FR could have physical interpretation. Usually, the main requirements to all physical laws are of two types:

(a) All physical laws should be expressed analytically in a general covariant form with respect to transformations of the types \( A \) and \( B \), i.e. they should be represented by means of tensors general covariant with respect to \( A \) and \( B \) \([29]\).

(b) All physical laws should be general covariant with respect to transformations of types \( A \) and \( B \).

The cited above methods are used for describing physical interactions \([5]\) and especially the gravitational interaction \([10], [17], [8]\).

In this paper a generalization of the notion of FR is proposed and considered on the basis of an "extended" monad formalism. In Section 2 the notions of different types of frames of reference (inertial, accelerated, proper inertial, proper accelerated etc.) are introduced. In Section 3 transitions from one to
another FR are determined on the basis of well established external covariant differential operators. In Section 4 some invariant properties of parallel and auto-parallel equations are recalled for the case of spaces of affine connections and metrics \([\mathcal{L}_n, g], (L_n, g)\)-spaces, and their special cases. The auto-parallel equation is obtained as Euler-Lagrange’s equation.

**Remark 1** The reader is kindly asked to refer to [22], [23], or [24], where all basic symbols and definitions used without explanations in this paper are introduced.

### 1.2 What are \((\mathcal{L}_n, g)\)-spaces?

The main characteristics of a \((\mathcal{L}_n, g)\)-space which differ from these of a \((L_n, g)\)-space are based on a weaker definition of the notion of dual bases in finite dual vector spaces over a differentiable manifold \(M\) [24], [25]. Instead of the common in multilinear algebra (canonical) contraction operator \(C\) acting on basic vectors \(\{e^\alpha(x)\} \in N_1|_{x \in M}\) and on basic vectors \(\{e_\beta(x)\} \in N_2|_{x \in M}\) of the vector spaces \(N_1\) and \(N_2\) with equal (finite) dimensions \((\dim N_1 = \dim N_2 = n)\) at a point \(x \in M\):

\[
C : (e^\alpha(x), e_\beta(x)) \rightarrow C(e^\alpha(x), e_\beta(x)) \equiv e^\alpha(e_\beta)|_{x \in M} = \delta^\alpha_\beta \equiv g^\alpha_\beta
\]

with \(\delta^\alpha_\beta \equiv g^\alpha_\beta = 1\) for \(\alpha = \beta\) and \(\delta^\alpha_\beta \equiv g^\alpha_\beta = 0\) for \(\alpha \neq \beta\),

a contraction operator \(S\) has been used:

\[
S : (e^\alpha(x), e_\beta(x)) \rightarrow S(e^\alpha(x), e_\beta(x)) \equiv e^\alpha(e_\beta)|_{x \in M} = f^\alpha_\beta(x)
\]

with \(\det(f^\alpha_\beta) \neq 0\)

Thus, the definition of dual vector bases has been weaken for vector spaces (as fibres of vector bundles) over a differentiable manifold \(M\). [The weaken definition of dual vector bases is meaningless in the multilinear algebra but it is very interesting in the case of vector spaces over a differentiable manifold.] It leads to the possibility of introducing two different affine connections (whose components differ not only by sign) for the tangent and cotangent vector spaces and, respectively, to two different affine connections for contravariant and covariant tensor fields over \(M\). All formulas written in index-free form are identical and valid in their form (but not in their contents) for \((L_n, g)\)- and \((\mathcal{L}_n, g)\)-spaces. The difference between them appear only if they are written in a given (co-ordinate or non-co-ordinate) basis. For instance,

\[
g(u, u) = g_{ij} \cdot u^i \cdot u^j
\]

in a \((L_n, g)\)-space, but

\[
g(u, u) = g_{ij} \cdot f^{i^*_j} \cdot f^{j^*_l} \cdot u^{k^*} \cdot u^l = g_{ij} \cdot u^7 \cdot u^7 = g_{kl} \cdot u^k \cdot u^l
\]

with \(f^{i^*_j}(x^l) = S(dx^l, \partial_j) = S(\partial_j, dx^l) = dx^l(\partial_j)\)

in a \((\mathcal{L}_n, g)\)-space;

\[
g(u) = g_{kl} \cdot u^l \cdot dx^k
\]
in a \((L_n, g)\)-space, but
\[
g(u) = g_{kl} \cdot f^l \cdot u^m \cdot dx^k = g_{kl} \cdot u^T \cdot dx^k = g_{\alpha\beta} \cdot u_{\alpha} \cdot dx_{\beta}
\]
in a \((L_n, g)\)-space. In a \((L_n, g)\)-space
\[
S(dx^i, \partial_j) = C(dx^i, \partial_j) = \delta^i_j \equiv g^i_j.
\]

All formulas with indices can be very easily specialized for \((L_n, g)\)-spaces by omitting the bars of all indices and taking into account that \(\delta^i_j; k \equiv g^i_j; k = 0\). \(\delta^i_j\) are the Kronecker symbols identical with the components \(g^i_j\) of the Kronecker tensor \(K_r = g^i_j \cdot \partial_i \otimes dx^j = g^\alpha_\beta \cdot e_\alpha \otimes e_\beta\).

2 Generalized definition of a frame of reference

Let us take a closer look at the third type of methods for introduction of a frame of reference. The method is called by different names [method of \(\tau\)-field [26], monad formalism, \([(n - 1) + 1]\)-representation, conception of the solitary (lonely) observer etc.]. It is proposed by Eckard (1941) [2] and Lief (1951) [see [26]], and later on refunded and applied by many authors [38], [37]. The consideration of a vector field \(u\) as the velocity of an observer induces (at least locally) a tangential sub space \(T^u(M)\), orthogonal to \(u\) over \(M\). All observed physical events and systems are projected to the direction of the vector field \(u\) and its sub space by the use of the corresponding to \(u\) contravariant and covariant metrics \(h^u = \mathcal{F} - \frac{1}{2} \cdot u \otimes u\) and \(h_u = g - \frac{1}{2} \cdot g(u) \otimes g(u)\). The invariant projections of the different tensor characteristics of an observed physical system were then given the corresponding physical interpretation [37], [20]. The world line of the observer is determined by its tangential vector \(u\) at every of its points. A tangential vector \(u\) at a point \(x \in M\), considered as an initial point, is transported to all other points of the world line by means of a transport [22], [23], determined by a preliminary given contravariant affine (linear) connection. With other words, the world line of the observer is determined by the transport of its tangential vector and, therefore, by the affine connection respectively. On this basis, we can conclude that it is not enough for a definition of a frame of reference a non-null (time-like) vector field \(u\) [with its corresponding orthogonal to it sub space \(T^u(M)\)] to be given. A \(FR\) should be determined in the third type of methods (c) by the set of four geometric objects:

(a) A non-null (time-like (if \(\dim M = 4\)) vector field \(u \in T(M)\).

(b) Orthogonal to \(u\) tangential sub space \(T^u_x(M)\) at every point \(x \in M\), where \(u\) is defined.

(c) (Contravariant) affine connection \(\nabla = \Gamma\). It determines the type of transport along the trajectory to which \(u\) is a tangent vector field. \(\Gamma\) is related to the covariant differential operator \(\nabla_u\) along \(u\) [23], [24].

(d) Metric tensor field \(g\) at every point \(x \in M\), where \(u\) is defined. It enables one to measure lengths and angles.

Remark 2 In the further considerations the last point (d) will not be considered explicitly. It is only assumed that a covariant metric tensor field \(g\) and its corresponding contravariant vector field \(\mathcal{F}\) exist among the other structures of a frame of reference. The existence of the metric tensor field \(g\) is implicitly given in the definition of \(T^u(M)\).
Now we can define the notion of frame of reference.

**Definition 3** The set $FR \sim [u, T^\perp u(M), \nabla = \Gamma, \nabla_u]$ is called frame of reference in a differentiable manifold $M$ considered as a model of the space or of the space-time.

Every $FR$ determines a congruence of curves (world lines) (a set of non-intersecting curves) in the range of the definition of the vector field $u$ and the action of the covariant differential operator $\nabla_u$.

A differentiable manifold $M$, provided with affine connection $\nabla$, i.e. the pair $(M, \nabla)$, is called space with affine connection [22] - [25]. A space with affine connection and metrics determines the existence of a set of frames of reference $FR$ which is invariant with respect to transformations of the types $A$ and $B$. The different types of frames of reference [inertial, non-inertial (accelerated) frame of reference] could be defined on the grounds of the above basic definition. For every special type of a $FR$ additional conditions should be imposed characterizing its properties related to the action of $\nabla_u$ on the vector field $u$ and on its corresponding orthogonal vector fields $\xi_\perp \in T^\perp u(M)$.

Since $\nabla = \Gamma$ could fulfill the condition $\nabla = \Gamma = 0$ under a special choice of the basic contravariant vector fields $\{e_{\alpha}\}$ at least at a point or on a trajectory to which $u$ is a tangent vector field, it is always possible a FRIF to be transform to an IFR (frame of reference without inertial forces).

The existence of a $FR$ with inertial forces is due to the fact that in some cases inappropriate basic vector fields $\{e_{\alpha}\}$ are chosen. In the table below $F \nabla_u$ denotes a Fermi-Walker transport [22], [23].

**Table 1. Different types of frames of reference**

| Type of a frame of reference | Symbol | Conditions, determining the type of a frame of reference |
|-----------------------------|--------|----------------------------------------------------------|
| General frame of reference  | $FR$   | $[u, T^\perp u(M), \nabla = \Gamma, \nabla_u]$           |
| Accelerated frame of reference | $AFR$ | $FR + [\nabla_u u = a \neq 0]$                         |
| Inertial frame of reference  | $IFR$ | $FR + [\nabla = \Gamma = 0, \nabla_u u = a = 0]$         |
| Proper accelerated frame of reference | $PAFR$ | $FR + [\nabla_u u = a \neq 0, F \nabla_u = \nabla_u - A_u$, $F \nabla_u \xi_\perp = 0, \xi_\perp \in T^\perp u(M)$] |
| Proper inertial frame of reference | $PIFR$ | $FR + [\nabla = \Gamma = 0, \nabla_u u = a = 0$, $F \nabla_u = \nabla_u - A_u$, $F \nabla_u \xi_\perp = 0, \xi_\perp \in T^\perp u(M)$] |
| Frame of reference without inertial forces | $FRWIF$ | $FR + [\nabla = \Gamma = 0, \nabla_u u = a \neq 0]$ |
| Frame of reference with inertial forces | $FRIF$ | $FR + [\nabla = \Gamma \neq 0, \nabla_u u = a = 0]$ |
3 Transition from one to another frame of reference

A FR \( \sim [u, T^{\perp u}(M)] \), \( \nabla = \Gamma \), \( \nabla_u \) will determine by means of its affine connection the model of a space or of a space-time, where the physical systems and events (at least locally) could be considered. Therefore, a transition from one to another affine connection could be related to the transition from one to another FR. At that, there are two possibilities for a transition from one to another affine connection.

(a) Transition from \( \nabla \) to \( \nabla' \) on the grounds of the transformation properties of \( \nabla \) under changing the basic vector fields (or co-ordinates) used in the corresponding frame of reference. Let us recall some well known facts.

If the basic vectors \( \{\partial_i\} \) or \( \{e_i\} \) are transforming as
\[
e\alpha = A_\alpha^i \cdot \partial_i = A^k_\alpha \cdot e_k ,
\]
and if \( \nabla_{e\beta} e\alpha = \Gamma^\gamma_{\alpha\beta} \cdot e\gamma \), \( \nabla_{\partial_i} \partial_i = \Gamma^k_i \cdot \partial_k \),
then \( \nabla = \Gamma \) will transform in \( \nabla' = \Gamma' \) as
\[
\Gamma'^i_{\alpha\beta} = A_\alpha^i \cdot A^j_\beta \cdot A^k_\gamma \cdot \Gamma^j_{k\gamma} + A^j_\beta \cdot A^k_\gamma \cdot A^i_\alpha \cdot \Gamma^j_{k\gamma} .
\]

A transformation of the above type (as a transformation of type A or B) does not change the form of the transports of the vector field \( u \), i.e. \( \nabla_u u = a \) is invariant to the transition of \( \nabla = \Gamma \) to \( \nabla' = \Gamma' \). To every affine connection \( \nabla = \Gamma \) and a vector field \( u \) corresponds a covariant differential operator \( \nabla_u \) invariant under the above transformation of the basic vector fields (co-ordinates).

(b) Transition from \( \nabla = \Gamma \) to \( \epsilon \nabla = \Gamma \) on the grounds of a change (deformation) of \( \Gamma \) by the use of a tensor field of third rank \( \overline{A} := \overline{A}^i_{jk} \cdot \partial_i \otimes dx^j \otimes dx^k \in \otimes^3 \mathbb{R}(M) \) in the form \( [22], [23] \)
\[
\Gamma^i_{jk} = \Gamma'^i_{jk} - \overline{A}^i_{jk} .
\]

The corresponding to \( \overline{\Gamma} \) covariant differential operator \( \epsilon \nabla_u \) (called extended covariant differential operator) could be written in the form
\[
\epsilon \nabla_u = \nabla_u - \overline{A}_u ,
\]
where \( \overline{A}_u := \overline{A}^i_{jk} \cdot u^k \cdot \partial_i \otimes dx^j \).

This type of transformation is invariant under the change of the basic vector fields (and co-ordinates). It induces a new frame of reference \( [u, T^{\perp u}(M)] \), \( \epsilon \nabla, \epsilon \nabla_u u \) which being induced by the frame of reference \( [u, T^{\perp u}(M)] \), \( \nabla, \nabla_u u \) differs from it. Therefore, we can relate a transition from one to another FR to a transition from one \( \nabla_u \) to another \( \epsilon \nabla_u \) covariant differential operator acting on tensor fields over a differentiable manifold \( M \).

A general transition from one to another FR could have the form
\[
\text{Transition} : [u, T^{\perp u}(M)], \nabla = \Gamma, \nabla_u \] \[\longrightarrow [\overline{\pi}, T^{\perp \overline{\pi}}(M)], \epsilon \nabla = \overline{\Gamma}, \epsilon \nabla_u \] \[\quad (4) \]
where
\[
\overline{\pi} := \Omega(u) = \pi^i \cdot \partial_i = \Omega^i_j \cdot u^j \cdot \partial_i \quad , \quad \Omega \in \otimes^3 \mathbb{R}(M) ,
\]
\[
T^{\perp \overline{\pi}} = \{ \xi \in T(M) : g(\overline{\pi}, \xi) = 0 \} , \quad \epsilon \nabla = \nabla - \overline{A} .
\]

The last type of transformations of \( \Gamma \), \( D : \Gamma \rightarrow \overline{\Gamma} \), could be called transformations of type D (because they are related to deformations of an affine connection \( \Gamma \)).
3.1 Transition from an accelerated frame of reference to an inertial frame of reference

A transition from an AFR \( \sim [u, T^{-1}u(M), \nabla, \nabla_u u = a \neq 0] \) to an IFR \( \sim [u, T^{-1}u(M), \nabla, \nabla_u u = 0] \) could be interpreted as a transition from an accelerated FR to an inertial FR under conserving the vector field (velocity) of the observer. This means that the vector field in both frames of reference is one and the same but the transport of this vector field is different in the two frames of reference. This type of transition could be done by two steps.

(a) Transition from \( [u, T^{-1}u(M), \nabla = \Gamma, \nabla_u u = a \neq 0] \) to \( [u, T^{-1}u(M), \nabla = \Gamma \neq 0, \nabla_u u = 0] \), i.e. \( AFR \rightarrow FRIF \).

(b) Transition from \( [u, T^{-1}u(M), \nabla = \Gamma \neq 0, \nabla_u u = 0] \) to \( [u, T^{-1}u(M), \nabla = \Gamma \neq 0, \nabla_u u = 0] \), i.e. \( FRIF \rightarrow IFR \).

Let us consider every step separately from the other.

3.1.1 Transition from an accelerated frame of reference to a frame of reference with inertial forces

The finding out of a transition from \( [u, T^{-1}u(M), \nabla = \Gamma, \nabla_u u = a \neq 0] \) to \( [u, T^{-1}u(M), \nabla = \Gamma \neq 0, \nabla_u u = 0] \) is related to the problem of finding out an extended covariant differential operator \( \nabla_u = \nabla_u - A_u \) such that \( \nabla_u u = 0 \).

The equation \( \nabla_u u = 0 \) could be written in a co-ordinate basis in a \((\mathbf{T}_m, g)\)-space in the form

\[
\sum_{i,j} u^i_{,j} \cdot u^j = \left( \Gamma_{jk}^{i} + \Gamma_{jk}^{i} \cdot u^{k} \cdot u^{j} = a^{i} \right) ,
\]

or in the form

\[
\sum_{i,j} u^i_{,j} \cdot u^j + (\Gamma^{i}_{jk} - \frac{1}{e} \cdot a^{i} \cdot g_{jk}) \cdot u^{k} \cdot u^{j} = 0 , \quad e = g(u, u) \neq 0.
\]

On the other side, \( \nabla_u u = 0 \) could be written in the form

\[
\left( \frac{\partial}{\partial x^{i}} - (\Gamma^{i}_{jk} - \frac{1}{e} \cdot a^{i} \cdot g_{jk}) \right) \cdot u^{k} = 0 .
\]

If we chose the form of \( A^{i}_{jk} \) as \( A^{i}_{jk} = \frac{1}{e} \cdot a^{i} \cdot g_{jk} \), then we find \( \Gamma^{i}_{jk} \) in the form

\[
\Gamma^{i}_{jk} = \Gamma^{i}_{jk} - \frac{1}{e} \cdot a^{i} \cdot g_{jk} .
\]

The tensor \( A_u \) has the form

\[
A_u = \frac{1}{e} \cdot a \otimes g(u) = \frac{1}{e} \cdot a^{i} \cdot g_{jk} \cdot u^{k} \cdot \partial_{i} \otimes dx^{j} .
\]

Therefore, \( \nabla_u u = \nabla_u u - a = 0 \).

In the new frame of reference \( [u, T^{-1}u(M), \nabla = \Gamma, \nabla_u u = 0] \) the vector field appears as an auto-parallel vector field with respect to the new affine connection \( \nabla = \Gamma \). On this basis we can make the conclusion that every "real" force inducing an acceleration \( a \) with respect to an affine connection \( \nabla = \Gamma \) could
be considered as an inertial force with respect to the corresponding to \( \nabla = \Gamma \) affine connection \( ^e \nabla = ^\Gamma \) fulfilling the condition (8).

The action of the extended contravariant differential operator \( ^e \nabla_u \) on a contravariant vector field \( \xi \in T(M) \) can be found as

\[
^e \nabla_u \xi = \nabla_u \xi - \frac{1}{c} \cdot a \cdot g(u, \xi) = \nabla_u \xi - \frac{l}{c} \cdot a , \quad l = g(u, \xi) .
\]  

(10)

If \( \xi \) is orthogonal to \( u \), i.e. if \( l = 0 \), then \( ^e \nabla_u \xi \perp = ^\nabla_u \xi \perp = \frac{g}{[h_u(\xi)]} \). Therefore, the new extended covariant differential operator \( ^e \nabla_u \) does not change the type of the transport (induced by \( ^\nabla_u \)) of the vectors of the sub space \( T^\perp u(M) \).

### 3.1.2 Transition from a frame of reference with inertial forces to an inertial frame of reference

Let us now consider the transition from \([u, T^\perp u(M)], \nabla = \Gamma \neq 0, \nabla_u, \nabla_u u = 0\] to \([u, T^\perp u(M)], \nabla = \Gamma = 0, ^\nabla_u, \nabla_u u = 0\]. From

\[
\nabla_{\varepsilon_i} \varepsilon_i = \nabla_{\varepsilon_i} \varepsilon_i , \quad e_\alpha = A_\alpha^i \cdot e_i , \quad e_i = A_i^\alpha \cdot e_\alpha , \quad A_\alpha^i \neq 0 , \quad A_i^\alpha \cdot A_\beta^j = g^j_\alpha , \quad A_i^\alpha \cdot A_j^\beta = g^\beta_\alpha ,
\]

and from the condition

\[
\nabla_{e_\beta} e_\gamma = \nabla_{e_\beta} e_\gamma = 0 ,
\]

we get

\[
^\Gamma_{\alpha\beta} = 0 .
\]

(11)

The Latin indices \( i, j, k, \ldots \), belong to the indices of the basis \( \{e_k, k = 1, \ldots, n\} \) and the Greek index \( \alpha \) belongs to the indices of the basis \( \{e_\alpha\} \). Therefore, for every given index \( \alpha \) and \( j \), and given components \( ^\Gamma_{ij}^\alpha \) of the affine connection \( \Gamma \) in the basis \( \{e_k\} \) we have a system of \( k \) partial differential equations for \( A_\alpha^i \) in the type (12). Further, if we consider the basis \( \{e_k\} \) as a co-ordinate basis, i.e. \( \{e_k\} := \{\partial_k\} \), then we can write the system of partial differential equations (PDEs) in the same form, but \( A_\alpha^k \) could be considered as components \( ^k A_\alpha \) of \( \alpha \) vector fields \( A_\alpha \) in a co-ordinate basis which are transported parallel to a basic vector field \( \partial_j \) with respect to the affine connection \( \Gamma \) with components \( ^\Gamma_{ij}^k \) in a co-ordinate basis \( \{\partial_k\} \):

\[
\nabla_{\partial_j} A_\alpha = 0 , \quad A_\alpha^k \cdot \partial_j = 0 .
\]

(13)

The last equations mean that we have to find \( \alpha \) covariant constant vector fields \( A_\alpha \) with respect to the affine connection \( \nabla = \Gamma \).

The system of PDEs (13) appears as a necessary and sufficient condition for \( ^\Gamma = 0 \) (with components \( ^\Gamma_{\alpha\beta} = 0 \) in the contravariant vector basis \( \{e_\alpha\} \)). The proof of this statement is trivial if we use the above relations.

A necessary but not sufficient condition following from (12) is the condition for a given vector field \( u \):

\[
\nabla_u A_\alpha = 0 , \quad u = \frac{d}{ds} = \frac{dx^i}{ds} \cdot \partial_i \in T(M)
\]

(14)
The last \( \alpha \) equations are equivalent to the equations for \( \alpha \) vector fields \( A \) transported parallel along an auto-parallel vector field \( u (\nabla_u u = 0) \)

\[
\frac{dA^k_\alpha}{ds} + \Gamma^k_{ij} \cdot A^i_\alpha \frac{dx^j}{ds} = 0 , \quad \alpha = 1, \ldots, n .
\]  

(15)

Since the vector field \( u = u^k \cdot \partial_k \) is transported parallel to itself, we can find \( n - 1 \) additional vector fields \( A_\alpha \) (with \( \alpha = 1, \ldots, n - 1 \)) transported parallel to \( u \) and orthogonal to \( u \). The solutions of the equations (15) determine \( \alpha \) vector fields \( A_\alpha (s) \) transported parallel along a curve with parameter \( s \) and tangential to it vector field \( u \). The components \( A_\alpha^k (s) \) of these \( \alpha \) vector fields \( A_\alpha (s) \) in the basis \( \{ e_k = \partial_k \} \) determine the components \( A_\alpha^k (s) \) of the transformation matrix \( (A_\alpha^k (s)) \) for a transition from the basis \( \{ e_k \} \) to the basis \( \{ e_\alpha \} \) along with the transition from \( \Gamma (s) \) with components in \( \{ e_k = \partial_k \} \) to \( \overline{\Gamma} (s) = 0 \) with vanishing components \( \overline{\Gamma}^\alpha_{\beta \gamma} (s) = 0 \) in the new basis \( \{ e_\alpha \} \).

**Remark 4** The above considerations are a short representation of the results given in [4] - [15] and [6] expressed in a more evident form for our investigations.

### 3.1.3 Relations between the components of the (contravariant) curvature tensor for the affine connection \( \Gamma \) and that for the affine connection \( \overline{\Gamma} \)

The components \( R^i_{jkl} \) of the (contravariant) curvature tensor \( Riem \) induced by the affine connection \( \Gamma \) could be written in a co-ordinate basis \( \{ \partial_k \} \) in the form

\[
R^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl} \cdot \Gamma^i_{mk} - \Gamma^i_{jk} \cdot \Gamma^m_{ml} .
\]

The components \( \overline{R}^i_{jkl} \) of the (contravariant) curvature tensor \( \overline{Riem} \) induced by the affine connection \( \overline{\Gamma} \) written in the same co-ordinate basis will have the form

\[
\overline{R}^i_{jkl} = \Gamma^i_{jl,k} - \Gamma^i_{jk,l} + \Gamma^m_{jl} \cdot \Gamma^i_{mk} - \Gamma^i_{jk} \cdot \Gamma^m_{ml} .
\]

By the use of (3) we can express \( \overline{R}^i_{jkl} \) by means of \( R^i_{jkl} \) in the form

\[
\overline{R}^i_{jkl} = R^i_{jkl} + F^i_{jkl} ,
\]

where

\[
F^i_{jkl} = (\frac{1}{e} \cdot a^i \cdot g^m_{jk} ) l - (\frac{1}{e} \cdot a^i \cdot g^m_{jl} ) k + \frac{1}{e^2} \cdot a^i \cdot a^m \cdot (g^m_{jl} \cdot g^k_{ml} - g^m_{jk} \cdot g^l_{ml}) + \frac{1}{e} \cdot (a^m \cdot g^j_{ml} \cdot \Gamma^i_{mk} + a^i \cdot g^m_{ml} \cdot \Gamma^j_{ik}) + \frac{1}{e} \cdot (a^m \cdot g^j_{mk} \cdot \Gamma^i_{jl} + a^i \cdot g^m_{mk} \cdot \Gamma^j_{il}) .
\]

(16)

**Special case**: Flat space \( M_n; \quad \Gamma^i_{jk} \equiv 0 ; \quad R^i_{jkl} \equiv 0 .
\]

\[
\overline{R}^i_{jkl} = F^i_{jkl} = (\frac{1}{e} \cdot a^i \cdot g^m_{jk} ) l - (\frac{1}{e} \cdot a^i \cdot g^m_{jl} ) k + \frac{1}{e^2} \cdot a^i \cdot a^m \cdot (g^m_{jl} \cdot g^k_{ml} - g^m_{jk} \cdot g^l_{ml}) .
\]

(17)
The curvature components $\mathbf{R}_{ijkl}$ of the corresponding space-time are induced by the acceleration $a = a^i \cdot \partial_i$ of the observer (related to its accelerated frame of reference). On the other hand,

$$a^i = \frac{e}{n} \cdot (\Gamma^i_{jk} - \Gamma^i_{jk}) \cdot g^{jk} , \quad n = 4 , \quad e = g(u, u) \neq 0 . \quad (18)$$

Every acceleration $a$ could be determined by the difference of two affine connections, the existing metrics, and the contravariant vector field $u$. For a flat space ($\Gamma^i_{jk} = 0$), $a^i = -\frac{e}{n} \cdot \Gamma^i_{jk} \cdot g^{jk}$. Therefore, every field theory in a flat space-time (corresponding to an accelerated FR without inertial forces) using the equation $\nabla_u u = a$ as an equation for a particle moving with an acceleration $a$ in an external field could be considered as a field theory in a $(L^n, g)$-space, where the same particle moves without acceleration, i.e. its velocity obeys an auto-parallel equation ($^e\nabla_u u = 0$) in a $(L^n, g)$-space (corresponding to a FR with inertial forces).

### 3.2 Transition preserving the form of an auto-parallel equation

Let us now consider the transition of a FR of the type $[u, T^i u(M), \nabla, ^e\nabla_u u = 0]$ to a $FR \sim [u, T^i u(M), ^e\nabla, ^e\nabla_u u = 0]$. The following problem could be investigated and some solutions of it could be found.

**Problem.** Let an auto-parallel equation $\nabla_u u = 0$ be given with respect to a contravariant affine connection $\nabla = \Gamma$. Find a new contravariant affine connection $^e\nabla = \Gamma$ with a corresponding extended covariant differential operator $^e\nabla_u$ such that $^e\nabla_u u = 0$.

**Solution.** The auto-parallel equation $\nabla_u u = 0$ (with respect to $\nabla$) and the new auto-parallel equation $^e\nabla_u u = 0$ (with respect to $^e\nabla$) can be written in the forms respectively

$$u^i \cdot j \cdot u^j + \Gamma^i_{jk} \cdot u^j \cdot u^k = 0 , \quad u^i \cdot j + (\Gamma^i_{jk} - ^eA^i_{jk}) \cdot u^j \cdot u^k = 0 . \quad (19)$$

After substraction of the second equation from the first one, we obtain a condition for $^eA^i_{jk} \cdot u^j \cdot u^k$ in the form

$$^eA^i_{jk} \cdot u^j \cdot u^k = 0 . \quad (20)$$

Different solutions are possible for $^eA^i_{jk}$ fulfilling the above condition. Let us give some of them.

1. $^eA^i_{jk} := q^i \cdot h_{jk}$. The tensor $\mathbf{A}$ will have the form

$$\mathbf{A} = q \otimes h_u = q^i \cdot h_{jk} \cdot \partial_i \otimes dx^j \cdot dx^k \in \otimes_1^1 2(M) , \quad q \in T(M) , \quad (21)$$

and the condition $\mathbf{A}_u = \mathbf{A}(u) = 0$ is fulfilled leading to the equation $^e\nabla_u u = \nabla_u u = 0$.

2. $^eA^i_{jk} := t^i \cdot \omega_{jk} + q^i \cdot h_{jk}$. The tensor $\mathbf{A}$ will have the form

$$\mathbf{A} = t \otimes \omega + q \otimes h_u , \quad \omega \in \Lambda^2(M) , \quad t, q \in T(M) , \quad (22)$$

and the condition $\mathbf{A}_u(u) = \mathbf{A}(u, u) = 0$ is fulfilled.
3. \( \mathbf{A}_{ij} := t^i \cdot \omega^j + q^i \cdot (h^j \cdot \eta^k \cdot p_k + h^j \cdot \eta^j \cdot p_j) \). The tensor \( \mathbf{A} \) will have the form
\[
\mathbf{A} = t \otimes \omega + q \otimes [h_u(\eta) \otimes p + p \otimes h_u(\eta)] , \quad t, q, \in T(M) , \quad p \in T^*(M) ,
\]
and the condition \( \mathbf{A}_u(u) = \mathbf{A}(u, u) = 0 \) is also fulfilled.

All three possible solutions for \( \mathbf{A}_{ij} \) and the corresponding solutions for \( \mathbf{A} \) preserve the form of \( \nabla_u u = 0 \). On the other hand, solutions 2. and 3. for \( \mathbf{A} \) and \( \mathbf{e}^{\nabla_u} \) induce new type of transports (different from that induced by \( \nabla_u \)) for the vector fields orthogonal to \( u \):

\[
\begin{align*}
(a) \quad \mathbf{e}^{\nabla_u} \xi - \mathbf{A}(u, \xi) &= \nabla_u \xi - \omega(u, \xi) \cdot t , \quad \xi, \; t, \; u \in T(M). \\
(b) \quad \mathbf{e}^{\nabla_u} \xi - \mathbf{A}(u, \xi) &= \nabla_u \xi - \omega(u, \xi) \cdot t - p(u) \cdot h_u(\eta, \xi) \cdot q .
\end{align*}
\]

Therefore, the transitions of the type \( \nabla_u \rightarrow \mathbf{e}^{\nabla_u} \), leading to the form invariance of \( \mathbf{A}_u(u) = 0 \), do not lead in general to the invariance of the transport of the vector fields orthogonal to \( u \).

4 Canonical representation of the parallel and the auto-parallel equations

4.1 Canonical representation of a parallel equation

Let a congruence \( x^i(\tau, \lambda) \) be given described by the two parameters \( \tau \) and \( \lambda \) and by the tangent vector fields
\[
\begin{align*}
u := \frac{\partial}{\partial \tau} &= \frac{\partial x^i}{\partial \tau} \cdot \partial_i , \quad \text{and} \quad \xi := \frac{\partial}{\partial \lambda} = \frac{\partial x^j}{\partial \lambda} \cdot \partial_j
\end{align*}
\]
respectively. Let us consider a parallel transport of the vector field \( \xi \) along the vector field \( u \)
\[
\nabla_u \xi = f \cdot \xi , \quad f \in C^r(M) .
\]

An equation of this type is called recurrent equation (or recurrent relation for the vector field \( \xi \)). Three types of invariance of this equation could be found.

(a) Invariance with respect to a change of the co-ordinates (charts) or the basic vector fields in the manifold \( M \). This invariance (of type A or B) is obvious because it follows from the index-free form of the equation.

(b) Form invariance with respect to a change of the vector \( \xi \) with a collinear to it vector \( \eta := \varphi \cdot \xi \).

Remark 5 In \((\mathbf{T}_n, g)\)-spaces the transformation \( \xi \rightarrow \eta \) can not be related to a transformation of type \( C \) \( \Omega : \xi \rightarrow \Omega(\xi) := \eta := \varphi \cdot \xi , \) with \( \Omega = \varphi \cdot Kr \) as it is the case in \((L_n, g)\)-spaces.

(c) Form invariance with respect to a change of the parameter \( \lambda \) determining \( \xi \).

The proofs of all types of invariance are trivial.
4.2 Canonical representation of an auto-parallel equation

Let us consider the auto-parallel equation \( \nabla_u u = f \cdot u \) as a special case of a parallel equation for \( \xi = u, f = f(x^k(\tau)) \). In this case

\[
\lambda = \tau, \quad u = \frac{d}{d\tau}, \quad u^i = \frac{dx^i}{d\tau} \quad \text{and} \quad \sigma = \sigma(\tau), \quad \tau = \tau(\sigma), \quad \frac{d\sigma}{d\tau} \neq 0.
\]

Then

\[
u_i = \frac{dx^i}{d\sigma} \cdot \frac{d\sigma}{d\tau} = \bar{u}^i \cdot \frac{d\sigma}{d\tau}, \quad \bar{u}^i = \frac{dx^i}{d\sigma},
\]

\[
u^i_{;j} \cdot u^j - f \cdot u^i = \left(\frac{d\sigma}{d\tau}\right)^2 \left(\frac{d\sigma}{d\tau} + \Gamma^i_{jk} \cdot u^j\right) + \bar{u}^i \cdot \left(\frac{d^2\sigma}{d\tau^2} - f(\tau) \frac{d\sigma}{d\tau}\right) = 0.
\]

One can choose as condition for determining the function \( \sigma = \sigma(\tau) \) as a function of \( \tau \) the vanishing of the last term of the above equation

\[
\frac{d^2\sigma}{d\tau^2} - f(\tau) \cdot \frac{d\sigma}{d\tau} = 0.
\]

The last equation is of the type

\[
y' - f \cdot y = 0, \quad \text{where} \quad y = \frac{d\sigma}{d\tau}, \quad y' = \frac{d^2\sigma}{d\tau^2}.
\]

Then

\[
\sigma = \sigma_0 + \sigma_1 \cdot \int \exp \left(\int f(\tau) \cdot d\tau\right), \quad \sigma_0 = \text{const.}, \quad \sigma_1 = \text{const.}
\]

After the introduction of the new parameter \( \sigma = \sigma(\tau) \) (called canonical parameter), the auto-parallel equation will have the form

\[
\nabla_{\bar{u}} \bar{u} = 0 \quad \Rightarrow \quad \bar{u}^i_{;j} \cdot u^j = 0, \quad \bar{u}^i = \frac{d}{d\sigma}, \quad \bar{u}^i = \frac{dx^i}{d\sigma}.
\]

The last equation for \( \bar{u} \) is called auto-parallel equation in canonical form.

4.3 Auto-parallel equations as Euler-Lagrange’s equations

In pseudo-Riemannian spaces without torsion \((V_n\text{-spaces})\) the geodesic equation (identical with the auto-parallel equation \( \nabla_u u = 0 \)) can be found on the ground of the variation \( \delta S = 0 \) of an action \( S \) identified with the length of a curve with parameter \( s \)

\[
S = \int ds + s_0 : \delta S = 0 \Rightarrow \nabla_u u = 0, \quad \text{with} \quad \nabla = \Gamma = \{ \}.
\]
is the Levi Civita (symmetric) affine connection. The same method has been used for finding out the geodesic equation in a \((\mathcal{T}_n, g)\)-space \([22]\). Since the geodesic equation (interpreted as an equation for motion of a moving free test particle in an external gravitational field) differs from the auto-parallel equation in a \((\mathcal{T}_n, g)\)- or \((L_n, g)\)-space, the old question arises as what is the right equation for description of a free moving particle in a \((\mathcal{T}_n, g)\)-space: the geodesic equation (G) or the auto-parallel equation (A). The most authors believe that the geodesic equation is the more appropriate equation. One of their major arguments is that the geodesic equation is related to a variational principle (as a basic principle in classical physics) in contrast to the auto-parallel equation. So the so called G-A problem induces investigations of possible ways for finding its solution \([31]-[35],[18],[3],[4]\). Unfortunately, no general solution was found until now. In our opinion, the failure is related to the attempt of using analogous variational expression as in the case of the geodesic equation. In \((\mathcal{T}_n, g)\)- and \((L_n, g)\)-spaces the auto-parallel equation has much more complicated structure (related to torsion and nonmetricity) than the geodesic equation. On the other side, an auto-parallel contravariant vector field induces additional structures such as the orthogonal to it sub space \(T_u\perp M\) which should be taken into account if we wish to find the auto-parallel equation on the basis of a variational principle. If we use the basic arguments for introducing a generalized definition of a FR we can also find a solution of the G-A problem by the use of the method of Lagrangians with covariant derivatives (MLCD) \([24]\).

Let us define a Lagrangian invariant in the form
\[
L = p_0 + h_0 \cdot g[\nabla_u (\rho \cdot u) \cdot \xi] = p_0 + h_0 \cdot g_{ij}^{\cdot} \cdot (\rho u^i)_j \cdot u^k \cdot \xi^j , \\
p_0, h_0 = \text{const.}, \quad \rho \in C^r(M) , \quad u, \xi \in T(M) .
\]
(27)
with the additional condition for the contravariant vector fields \(u\) and \(\xi\): \(g(u, \xi) = l = 0\). The corresponding action \(S\) could be written in the form
\[
S = \int \sqrt{-d_g} \cdot (L + \lambda \cdot l) \cdot d^{(n)} x = \int (L + \lambda \cdot l) \cdot d\omega , \quad d_g = \det(g_{ij}) < 0 .
\]
(28)
where \(\lambda\) is a Lagrangian multiplier. \(L\) is interpreted as the pressure \(p\) of a physical system, \(u\) is the velocity of the particles, \(\rho\) is their proper mass density, and \(\xi\) is a vector, orthogonal to \(u\). By the use of the MLCD we obtain the covariant Euler-Lagrange equations for the vector fields \(u\) and \(\xi\) obeying the condition \(l = 0\)
\[
\frac{\delta L}{\delta \xi^i} = 0 : \quad u^i \cdot \xi^i = \left[\frac{\lambda}{h_0} - u(\log \rho)\right] \cdot u^i ,
\]
(29)
\[
\frac{\delta L}{\delta u^i} = 0 : \quad \xi^i \cdot \xi^j = (q - \delta u + \frac{\lambda}{h_0}) \cdot \xi^i + g_{ii} \cdot u^k \cdot g^{ji} \cdot \xi^l - [g^{ij} \cdot (g_{m}^{\cdot})_{m} \cdot u^m - g_{ik}^{\cdot} \cdot u^j] \cdot \xi^k .
\]
(30)
\[
\frac{\delta L}{\delta \lambda} = 0 : \quad g(u, \xi) = l = 0 .
\]
(31)
In index-free form the equations for \( u \) and \( \xi \) would have the forms:

\[
\nabla_u u = k \cdot u, \quad k = \frac{\lambda}{h_0} - u(\log \rho),
\]

(32)

\[
\nabla_u \xi = m \cdot \xi + K - N, \quad m = q - \delta u + \frac{\lambda}{h_0},
\]

(33)

\[
q = q_j \cdot u^j, \quad q_j = T_{kj}^k - \frac{1}{2} g^k_{ij} g^m_{kl} \cdot g_{ul}^j \\
\delta u = u^k \cdot \partial_k, \quad K = K^i \cdot \partial_i = (g^k_{ij} \cdot u^k \cdot g^j_{ij} \cdot \xi^k) \cdot \partial_i,
\]

(34)

\[
N = N^i \cdot \partial_i, \quad N^i = [g^{ij} \cdot (g^m_{ij} \cdot u^m - g^k_{ij} \cdot u^j) \cdot \xi^k].
\]

(36)

The Euler-Lagrange’s equation (32) is just the auto-parallel equation in a non-canonical form. For \( \rho = \text{const.} \), it will have the form \( \nabla_u u = \frac{\lambda}{h_0} \cdot u \). After changing the parameter of the curve to which \( u \) is a tangential vector field the auto-parallel equation could be found in its canonical form \( \nabla_u u = 0 \).

The Euler-Lagrange’s equation for \( \xi \) (33) has in general a more complicated form than the parallel equation for \( \xi \) along \( u \) (\( \nabla_u \xi = g \cdot \xi \)). For different affine connections (and the corresponding models of space-time) this equation would have different solutions. Therefore, if we consider an auto-parallel equation as a result of a variational principle we should take into account the corresponding orthogonal to \( u \) sub space.

Remark 6 The Lagrangian invariant \( L \) could be defined without the requirement \( \xi \) to be orthogonal to \( u \). The covariant Euler-Lagrange’s equations will be then found for \( u \) and a vector field \( \xi \in T(M) \). For \( \rho = \text{const.} \) the auto-parallel equation will have its canonical form. The orthogonality condition for \( u \) and \( \xi \) could be introduced after solving the Euler-Lagrange equations for \( u \) or \( \xi \).

4.4 Einstein’s theory of gravitation as a theory for finding out an appropriate frame of reference for describing the gravitational interaction

Let us consider the Einstein theory of gravitation (ETG) from the point of view of the definition of a generalized FR. For that we should see how every element in the definition for a \( FR \sim [u, T^u(M), \nabla = \Gamma, \nabla_u] \) is related to the theory (\( \dim M = 4 \)).

The vector field \( u \) in the ETG for a material distribution is usually related to the 4-velocity of the material points. For Einstein’s equations in vacuum \( u \) is not uniquely determined. An assumption is made that a free particle in an external gravitational field is moving on a geodesic world line [i.e. its 4-velocity obeys the auto-parallel equation \( \nabla_u u = 0 \) (identical with the geodesic equation in a \( V^4 \)-space)]. The affine connection \( \nabla = \Gamma = \{ \} \) (Levi-Civita connection) has components \( \Gamma^i_{jk} \) in a basis \( \{ \partial_k \} \) determined by the metric tensor field \( g = g_{ij} \cdot dx^i \cdot dx^j \). These components are found on the basis of the Einstein
equations in vacuum for the metric $g$. Therefore, ETG from a point of view of a FR is a theory for description of the gravitational interaction on the basis of an appropriate FR determined by the use of the Einstein equations, in which FR the gravitational force in vacuum appears as an inertial force.

**Remark 7** Conditions under which a sub space $T^\perp_{x_1}(M)$ at a given point $x_1 \in M$ on a curve with tangent vector $u$ could (or could not) intersect the sub space $T^\perp_{x_2}(M)$ of another point $x_2 \in M$ lying on the same curve as well as Fermi-Walker and conformal transports related to frames of reference in $(\mathbb{L}_n, g)$- and $(\mathbb{L}_n, g)$-spaces and different from these already proposed in the literature \cite{22}, \cite{23} will be considered elsewhere.

## 5 Conclusion

In the present paper a generalized definition of the notion of frame of reference has been introduced and considered. It leads to the hypothesis that every FR determines a model of space-time used for description of physical systems and events. For instance, the Einstein theory of gravitation could be formulated either in a (pseudo) Riemannian space or in a Minkowski space $\mathbb{L}$. On the other hand, every type of FR as a mankind’s construction could cause problems in attempts for describing the physical events in the best possible way $\mathbb{L}$.

**Acknowledgments**

This work is supported by the National Science Foundation of Bulgaria under Grant No. F-642.

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