Holographic Normal Ordering
and
Multi-particle States in the AdS/CFT Correspondence

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Abstract

The general correlator of composite operators of N=4 supersymmetric gauge field theory is divergent. We introduce a means for renormalizing these correlators by adding a boundary theory on the AdS space correcting for the divergences. Such renormalizations are not equivalent to the standard normal ordering of current algebras in two dimensions. The correlators contain contact terms that contribute to the OPE; we relate them diagrammatically to correlation functions of compound composite operators dual to multi-particle states.

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1 Introduction

Recently much progress has been made in the understanding of superconformal $N = 4$ super Yang-Mills theories through a holographic description in terms of a IIB string (M-) theory on an anti-de Sitter background \[1, 2, 3\]. The low energy effective supergravity of the string theory probes the strong coupling limit of the CFT at large $N_c$; stringy/string-loop corrections correspond to the expansion in inverse powers of the 't Hooft effective coupling $g_{YM}^2 N_c$ and $N_c$. An explicit example is the correspondence between type IIB supergravity on $\text{AdS}_5 \times S_5$ and $N = 4$ super-Yang-Mills. Correlation functions in the latter at large $N_c$ may be determined from the supergravity theory through the relation:

$$\prod_{j=1}^{k} \left( \frac{\delta}{\delta \phi_{0,j}(\vec{z}_j)} \right) e^{iS_{\text{sugra}}[\phi(\phi_0)]} \bigg|_{\phi_{0,j}=0} = \prod_{j=1}^{k} \langle O_j(\vec{z}_j) \rangle_{\text{CFT}} .$$

(1.1)

Here, $S_{\text{sugra}}$ is the bulk action of the supergravity theory considered as a functional of the boundary values of the fields, $\phi_{0,j}$, and $O$ are composite (gauge invariant) operators of the conformal Yang-Mills theory. These operators are dual to the boundary values of the supergravity fields in the sense that the latter act as sources for the former. In recent months CFT correlation functions have been analyzed using the holographic prescription in \[4\]-\[21\].

In calculations exploring the correspondence, one item which has been little addressed is the fact that the correlations of composite operators are in general divergent. For example

$$\langle O_{\Delta}(\vec{x}_1) O_{\Delta}(\vec{x}_2) \rangle = \frac{1}{|\vec{x}_1 - \vec{x}_2|^{2\Delta}} \rightarrow \langle O_{\Delta}(\vec{k}) O_{\Delta}(-\vec{k}) \rangle = \frac{2^{d/2} \Gamma(d/2 - \Delta)}{\Gamma(\Delta)} k^{2(d/2-\Delta)} ,$$

(1.2)

is ill-defined whenever the dimension $\Delta$ of the operators is greater than or equal to $d/2$. For a consistent description of the correspondence one needs to provide a regularization of these short-distance singularities to make the theory finite. This issue has been discussed from the CFT point of view in \[22\] and has been briefly mentioned in \[3\] in the context of the duality with AdS theories. We introduce here a modification of the AdS/CFT correspondence through the addition of a boundary action, as a consequence of which the correlators are made finite. Different sets of boundary terms have been considered in \[3, 4, 7, 10, 13\].

A regulatory scheme is to compute the supergravity Green’s functions at points infinitesimally away from the AdS boundary. This IR cut-off for the gravity theory acts as an UV regulator for the CFT \[23, 24, 25\]. By introducing counterterms with the associated scale we are thus in effect renormalizing the CFT through the AdS boundary theory. Of course this violates conformal invariance. However, the prescription of \[4, 8\] considers not the CFT
as such but its perturbation by conformal operators,

\[ S_{N=4\, SYM} \rightarrow S_{N=4\, SYM} + \int d^4 \vec{x} \phi_{0,j}(\vec{x}) \mathcal{O}^j(\vec{x}), \]

where the source of the operator is the boundary value of a supergravity field. On the other hand conformal symmetry protects the dimensions of chiral primary operators and their descendants. Thus for those operators, whose sources are elementary supergravity fields, the introduction of a regulator in intermediate steps of the calculation will not affect the final answer provided one keeps the operator insertions at distinct points \[3, 5, 6\]. However, correlations of multiple operators may still diverge at short distances and these require the introduction of counterterms \[26\]. This is in contrast to the finiteness of the unperturbed CFT.

Insertions of operators in the SYM Green’s functions would yield counterterms that are related to products of conformal operators at the same point. These compound composite operators, e.g. \( \text{Tr} F^2(\vec{x}) \text{Tr} F^2(\vec{x}) \), are neither primary nor descendants \[27\]. As we will see such product operators are dual to multi-particle supergravity states. Specific multi-particle states were found to be necessary in the AdS/CFT correspondence in \[28, 29\]. Here we propose that the coupling of such operators to the boundary values of supergravity fields is dictated by the above procedure. In \[20\] and \[21\] there has been some speculation on where such states might appear in exchange diagrams between elementary supergravity fields.

In addition to the divergent nature of the correlation functions we find that explicit contact contributions appear in the evaluation of three- and four-point functions. Their Fourier transformed \( k\)-space expressions are divergent but they also produce logarithms in the kinematic invariants; they contain cuts. For example, the four-point function contains a contact term of the following form

\[
\langle \mathcal{O}(\vec{x}_1)\mathcal{O}(\vec{x}_2)\mathcal{O}(\vec{x}_3)\mathcal{O}(\vec{x}_4) \rangle = \Lambda^2 \delta^{(d)}(\vec{x}_1 - \vec{x}_2) \frac{1}{|\vec{x}_1 - \vec{x}_3|^p} \delta^{(d)}(\vec{x}_3 - \vec{x}_4) + \text{permutations},
\]

together with products of multiple delta functions. Through the delta functions this correlator resembles the two-point function of the conformal operator \( \mathcal{O}(\vec{x})\mathcal{O}(\vec{x}) \); dual to a multi-particle supergravity state. The purpose of this work is to focus on coincident points, both how they relate to the regulating of divergences and the appearance of contact terms in the calculation of correlation functions with physical implications. The results suggest a way to compute correlators of CFT compound composite operators from AdS supergravity.

The outline of this work is as follows. In section 2 we recall how conformal transformations constrain the form of correlators at distinct points. In section 3 we review the Green’s functions used in the computations and examine the asymptotic forms necessary for the analysis of the contact-terms. In Section 4 we examine the divergences within the correlators by starting with a simple analysis of the two-point functions. Our testing ground will be the
dilaton-axion sector of IIB supergravity on AdS$_5 \times S_5$. In Sections 5 and 6 we do the same for three- and four-point functions. We discuss the multi-particle states in section 7. Contact term contributions are diagrammatically related to the multiparticle state correlators coming from bulk AdS multi-loop supergravity. Lastly, in section 8 we discuss implications and extensions related to this work.

2 Conformal Invariance Constraints

In this section we briefly review the constraints imposed by conformal transformations on correlation functions of conformal operators.

Conformal transformations preserve the line element up to a scale factor:

$$x_\mu \rightarrow x'_\mu(x) \quad \eta_{\mu\nu} dx^\mu dx^\nu = \Omega^{-2}(x) \eta_{\mu\nu} dx'^\mu dx'^\nu. \quad (2.1)$$

In $d$ dimensions they make up the conformal group $SO(d, 2)$ generated by rotations,

$$x'_\mu = R_{\mu\nu} x_\nu \quad R_{\mu_1 \nu} R_{\mu_2 \nu} = \delta_{\mu_1 \mu_2}, \quad \Omega = 1 \quad (2.2)$$

display scale transformations,

$$x'_\mu = \lambda x_\mu \quad \Omega^{-2} = \lambda^2, \quad (2.3)$$

and special conformal transformations,

$$x'_\mu = \Omega^{-1}(x) \left[ x_\mu + v_\mu x^2 \right] \quad \Omega(x) = 1 + 2v \cdot x + v^2 x^2. \quad (2.4)$$

Alternatively we may use inversion,

$$x'_\mu = \frac{x_\mu}{x^2} \quad \Omega(x) = x^2, \quad (2.5)$$

instead of the special conformal transformations to build up the generators of the group.

Conformal operators $O^i$ of scale dimension $\Delta$ transform as

$$T \cdot O^i = \Omega^\Delta D^i_j [R] O^j. \quad (2.6)$$

This means that the correlator of two such operators must behave under an inversion as

$$\left< O_\Delta(z_1) O_\Delta(z_2) \right> \sim \frac{1}{|z_1 - z_2|^{2\Delta}} \rightarrow \frac{1}{|z_1^2 - z_2^2|^{2\Delta}} = \Omega^\Delta(z_1) \Omega^\Delta(z_2) \frac{1}{|z_1^2 - z_2^2|^{2\Delta}}. \quad (2.7)$$

At coincident points, however, these transformations are singular. Conformal invariance constrains the correlators only when the points of the operators are at distinct separated values. Potential contact terms in the correlator are allowed and probe the short-distance structure (UV region) of the conformal field theory. As an example consider free-field QCD.
Green’s functions which are conformally invariant at non-coincident points \([30]\). Only when we investigate the short-distance behaviour and compensate for infinities by the introduction of counterterms do we find scale dependence. These short-distance singularities have been investigated in the context of conformal field theories in \([22]\). Note that there are no UV non-renormalization theorems for correlations of composite operators in \(N = 4\) super Yang-Mills theory, the reason being that they are not finite. Non-renormalization theorems as proposed in \([6, 12]\) refer only to the independence of correlators on the microscopic coupling \(g_{YM}^2\).

3 Asymptotic form of Greens functions

In this section we examine the relevant limits of the bulk-bulk and bulk-boundary kernels necessary for an exact evaluation of the contact contributions to the multi-point correlation functions examined in later sections. The theory we consider is the dilaton-axion sector of IIB supergravity on \(AdS_5 \times S_5\) with action

\[
S = \frac{1}{2\kappa^2} \int d^5x \sqrt{g(x)} (-R + 12/A^2) + g^{\mu\nu} \left[ \partial_\mu \phi \partial_\nu \phi + e^{2\phi} \partial_\mu C \partial_\nu C \right]
\]

Note that the interaction between the dilaton and axion contains derivatives. For the background metric on \(AdS_5\) we will use the half-space Poincaré metric

\[
ds^2 = \frac{A^2}{x_0^2} \left( dx_0^2 + dx^i dx^j \eta_{ij} \right), \quad i = 1, \ldots, d
\]

where \(x_0 \geq 0\). We set the AdS radius \(A^2\) to unity in the remainder and we will keep the dimension \(d = 4\) abstract whenever possible. The metric \(\eta_{ij}\) is Minkowski with mostly plus signature, and \(\eta_{dd} = -1\).

The bulk-bulk correlator for massive scalars is given by

\[
\left\langle \phi(x) \phi(y) \right\rangle \equiv G(x, y)
\]

\[
= (x_0 y_0)^{d/2} \int_0^\infty d\lambda \lambda^\nu \int \frac{d^d k}{(2\pi)^d} e^{i \vec{k} \cdot (\vec{x} - \vec{y})} \frac{J_\nu(\lambda x_0) J_\nu(\lambda y_0)}{\lambda^2 + \vec{k}^2 - i\epsilon},
\]

where \(\nu = \sqrt{m^2 + d^2/4} > 0\) and \(\vec{k} \cdot \vec{x} \equiv \sum_{i=1}^d k_i x_i\). We denote with \(\vec{x}\) a (Minkowski) four-vector on the boundary of AdS. The \(i\epsilon\) prescription was provided in \([19]\). As the dilaton and axion are massless we shall consider this case in the remainder of this work.

The bulk-boundary kernel \(\Delta(\vec{x}, y)\) is found by taking the small \(y_0\) limit,

\[
\triangle(\vec{y}, z) = \lim_{y_0 \to 0} \frac{1}{y_0^{d/2 + \nu - 1}} \partial_{y_0} G(y, z)
\]
The extra factor of $y_0^{-\nu}$ corrects for the asymptotic behaviour of the Green’s function. To evaluate this limit we need the asymptotic behaviour of the Bessel functions. At $z \to 0$ they behave as

$$J_\nu(z) = \frac{1}{\Gamma(1 + \nu)} \left(\frac{z}{2}\right)^\nu + \ldots \quad K_\nu(z) = \frac{\Gamma(\nu)}{2} \left(\frac{2}{z}\right)^\nu + \ldots , \quad (3.6)$$

and at $z \to \infty$ we have,

$$J_\nu(z) = \left(\frac{1}{2\pi z}\right)^{1/2} \cos(z + \pi/2) + \ldots \quad K_\nu(z) = \pi^{-1/2} e^{-z} + \ldots . \quad (3.7)$$

Using (3.6) we find

$$\Delta(\vec{y}, z) = (z_0)^{d/2} \int_0^\infty \lambda d\lambda \int \frac{d^dk}{(2\pi)^d} e^{ik(\vec{z} - \vec{y})} \frac{2}{\Gamma(\nu)} \left(\frac{\lambda}{2}\right)^\nu \frac{J_\nu(\lambda z_0)}{\lambda^2 + k^2 - i\epsilon} \quad (3.8)$$

$$= \frac{2}{\Gamma(d/2)} \int \frac{d^dk}{(2\pi)^d} \left(\frac{|k|z_0}{2}\right)^{d/2} K_\nu(|k|z_0) e^{ik(\vec{z} - \vec{y})} . \quad (3.9)$$

This kernel satisfies the appropriate Dirichlet boundary conditions, as may be verified. For the massless fields $\nu = d/2$ and in this case explicit integration over the Fourier modes $k$ gives the position-space form,

$$\Delta(\vec{y}, z) = \frac{\Gamma(d)}{\pi^{d/2} \Gamma(d/2)} \left(\frac{z_0}{z_0^2 + (\vec{y} - \vec{z})^2}\right)^d . \quad (3.10)$$

The Dirichlet conditions on the bulk-boundary kernel may also be verified using the distributional form above,

$$\lim_{z_0 \to 0} \Delta(\vec{y}, z) = \delta^d(\vec{y} - \vec{z}) , \quad (3.11)$$

and a related identity is

$$\lim_{z_0 \to 0} z_0 \partial_{z_0} \Delta(\vec{y}, z) = 0 . \quad (3.12)$$

In the following we will also need the small $z_0$ limit of the derivative of the bulk-boundary kernel,

$$\mathcal{F}(\vec{y}, \vec{z}) = \lim_{z_0 \to 0} \frac{1}{z_0^{d-1}} \partial_{z_0} \Delta(\vec{y}, z) , \quad (3.13)$$

Straightforward differentiation of (3.9) gives,

$$\mathcal{F}(\vec{y}, \vec{z}) = \lim_{z_0 \to 0} \frac{\Gamma(d + 1)}{\pi^{d/2} \Gamma(d/2)} \left\{ -\frac{z_0^2}{[z_0^2 + (\vec{y} - \vec{z})^2]^{d+1}} + \frac{(\vec{y} - \vec{z})^2}{[z_0^2 + (\vec{y} - \vec{z})^2]^{d+1}} \right\} . \quad (3.14)$$

The functional form of the limit does not permit a naive interpretation as a distribution, and we need to include a divergent coefficient. The limits of (3.14) are:

$$\vec{y} - \vec{z} \neq 0 : \quad \mathcal{F}(\vec{y}, \vec{z}) = \frac{\Gamma(d + 1)}{\pi^{d/2} \Gamma(d/2)} \frac{1}{(\vec{y} - \vec{z})^{2d}} , \quad (3.15)$$
and
\[ \vec{y} - \vec{z} = 0 : \ F(\vec{y}, \vec{z}) = -\frac{\Gamma(d + 1)}{\pi^{d/2} \Gamma(d/2)} \frac{1}{y_0^d}. \] (3.16)

We define the \( z_0 \to 0 \) form of (3.14) to be
\[ \lim_{z_0 \to 0} \frac{1}{z_0^{d-1}} \partial_{z_0} \Delta(\vec{y}, \vec{z}) = \frac{\Gamma(d + 1)}{\pi^{d/2} \Gamma(d/2)} \left\{ \frac{1}{(\vec{y} - \vec{z})^{2d}} - \frac{1}{\mu^d} \delta(\vec{y} - \vec{z}) \right\}. \] (3.17)

The coefficient \( \mu \) may be regarded as an infinitesimal distance from the boundary at \( z_0 = 0 \); this can be explicitly verified by evaluating (3.13) using the momentum space formulation of the bulk-boundary kernel.

4 Two-point Functions

Conformal invariance fixes the form of the two-point function of chiral primary operators with dimension \( \Delta \), the bosonic form of which is,
\[ \langle O_{\Delta_1}(\vec{z}_1) O_{\Delta_2}(\vec{z}_2) \rangle = \frac{\delta_{\Delta_1 \Delta_2}}{|\vec{z}_1 - \vec{z}_2|^{\Delta_1 + \Delta_2}}, \] (4.1)

provided the points are kept distinct. Correlators of descendents easily follow. This two-point function is computed through the holographic correspondence by solving for the boundary-boundary kernel between two different points on the boundary of the anti-de Sitter compactification and integrating over a two-point insertion in the bulk.

The Fourier transform of the general two-point correlator (4.1)
\[ \langle O_{\Delta}(\vec{k}) O_{\Delta}(\vec{-k}) \rangle = \frac{2^d \pi^{d/2} \Gamma(d/2 - \Delta) \bar{k}^{2(d/2 - \Delta)}}{\Gamma(\Delta)}, \] (4.2)
is divergent and must be regularized. This is not an ad hoc requirement, but follows directly from the free-field evaluation of the two-point function of the operator \( \text{Tr} \phi^i \phi^j \) in \( N = 4 \) super Yang-Mills theory. In this example the contribution is a one-loop self-energy graph whose divergence equals that of (4.2).

The necessity of introducing a regulating scale may be considered as a conformal anomaly \[ \text{in the } N = 4 \text{ super Yang-Mills theory. This resembles the naive scale dependence in the tree-level bosonic propagator } \langle X(z)X(v) \rangle \sim \ln(z - v) \text{ in open (or closed) string theory. The value of the composite two-point correlator at coincident points is not determined through conformal invariance but suffers from operator product ambiguities and associated divergences. In } N = 4 \text{ super Yang-Mills theory we find that there are further modifications of higher-point functions.} \]

We will add to the two-point correlator a counterterm that eliminates the pathology at short distance. In \( x \)-space counterterms are known to be provided in the form of distributions
with support at coincident points. Such regularizations and renormalizations have been extensively studied in the differential regularization approach \[30\]. We shall modify the two-point function to the following form,

\[
\langle O_\Delta(\vec{z}_1)O_\Delta(\vec{z}_2) \rangle = \frac{1}{|\vec{z}_1 - \vec{z}_2|^{2\Delta}} + \alpha \mu^{2\Delta - d - 2n} \square^n \delta^{(d)}(\vec{z}_1 - \vec{z}_2) ,
\]

(4.3)

where \(n = [\Delta - d/2]\) is the integer part of \(\Delta - d/2\). \(\mu\) is a dimensionful regulating scale that permits the counterterm to be built out of operators with well-defined classical scaling dimensions: \(\square^n\), where \(n\) is an integer.

The coefficient \(\alpha\) is determined by enforcing finiteness on the Fourier transformed two-point correlator, and is divergent. With the addition of the counterterm and after dimensional continuation the Fourier transform of (4.3) yields

\[
\langle O_\Delta(\vec{k}_1)O_\Delta(\vec{k}_2) \rangle = (2\pi)^d \delta^{(d)}(\vec{k}_1 + \vec{k}_2) \frac{2^{d} \pi^{d/2}}{\Gamma(\Delta)} \frac{1}{k^{\Delta - 2n}} \left( \gamma_1 + \gamma_2 \ln(\vec{k}^2/\mu^2) \right) .
\]

(4.4)

and is finite. The above follows from minimal subtraction with \(\alpha \sim 1/(\Delta - d/2 - n)\). All of the two-point functions may be regularised in this manner. The renormalization scale \(\mu\) may be thought of as an infinitesimal distance from the boundary of \(\text{AdS}\).

Rather than modifying the correlation functions by adding contact terms by hand, we add a boundary action with these counterterms to the bulk anti-de Sitter supergravity. Boundary term additions have been considered before within the purely gravitational anti-de Sitter action, with the addition of the “Gibbons-Hawking” term \[3, 5, 7\] and in the work of \[4, 13\]. In our case we include the boundary term,

\[
S_{b.t.} = \frac{\alpha}{2} \mu^{2\Delta - d - 2n} \int d^d x \phi_0(x) \square^n \phi_0(x) ,
\]

(4.5)

where the boundary values of the supergravity fields, \(\phi_0(\vec{x})\) are dual to the composite fields \(O(\vec{x})\). The functional variation of (4.5),

\[
\frac{\delta}{\delta \phi_0(\vec{z}_1)} \frac{\delta}{\delta \phi_0(\vec{z}_2)} S_{b.t.} = \alpha \mu^{2\Delta - d - 2n} \int d^d \vec{x} \delta^{(d)}(\vec{z}_1 - \vec{x}) \square^n \delta^{(d)}(\vec{z}_2 - \vec{x}) ,
\]

(4.6)

reproduces the counterterm in (4.3). This is exactly the procedure of composite operator renormalization in the CFT. In this case, since the two-point function is given to all orders by its free-field value, this can be explicitly verified. The perturbed CFT,

\[
S_{N=4 \text{ SYM}} \to S_{N=4 \text{ SYM}} + \int d^4 \vec{x} J_j(\vec{x}) O^j(\vec{x}) ,
\]

(4.7)

requires counterterms consisting of all operators \(O\) of similar dimension or less. In particular graphs with no external fundamental fields will require counterterms formed by products.
\( J^n(\vec{x}) \cdot \mathbb{I} \). In the AdS/CFT correspondence the sources \( J(\vec{x}) \) are the boundary values of supergravity fields.

For two-point functions of correlators of operators corresponding to the dilaton and the axion we have \( \Delta = d \) and \( n = [d/2] \). In this case the constants \( \alpha \) and \( \gamma_1 \) and \( \gamma_2 \) are explicitly

\[
\alpha = \frac{(-1)^n}{n!} \frac{1}{d/2 - n}, \quad \gamma_1 = \frac{3(-1)^n}{2n!}, \quad \gamma_2 = \frac{(-1)^n}{n!}.
\]

(4.8)

Similar terms occur for the boundary values of all the other supergravity fields.

5 Three-point Functions

We next analyze how similar divergent behaviour of the three-point functions can be compensated by adding additional interaction terms on the boundary of the anti-de Sitter space. We will examine the correlator

\[ \langle \text{Tr} F \tilde{F}(\vec{x}_1) \text{Tr} F \tilde{F}(\vec{x}_2) \text{Tr} F^{2}(\vec{x}_3) \rangle \]

whose supergravity dual is the unamputated \( \langle C(\vec{x}_1)C(\vec{x}_2)\phi(\vec{x}_3) \rangle \) amplitude. It arises within the AdS/CFT correspondence by considering the contributions from the \( CC\phi \) vertex,

\[
S_{CC\phi} = \frac{1}{2\kappa^2} \int d^{d+1}x \sqrt{g(x)} \partial \mu g^{\mu\nu} \partial_\nu C \partial_\sigma C,
\]

(5.1)

where \( g_{\mu\nu}(x) \) is the background anti-de Sitter metric, eq. (3.2). The value of the correlator is computed through the use of the bulk-boundary kernel to be,

\[
A_{CC\phi}(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \frac{1}{2} \int d^d y \frac{1}{y_{0}^{d-1}} \left[ \Delta_1 \partial_{y_0} \Delta_2 \Delta_3 - \Delta_1 \Delta_2 \partial_{y_0} \Delta_3 \right]_{y_0=0}^{y_0=\infty}.
\]

(5.2)

Here the parenthesis, (12), denote symmetrization, \( A(1 B_2) = A_1 B_2 + A_2 B_1 \). The bulk term vanishes due to the fact that the bulk-boundary kernel in (3.9) solves Laplace’s equation. We have used the shorthand \( \Delta_i = \Delta(\vec{x}_i, y) \).

Inserting the kernels and the limits from section 3 we obtain two types of contributions to the three-point function, \( A = A^{(2)} + A^{(3)} \), distinguished by the number of delta functions present. We set the gravitational coupling \( \kappa \) to unity from here on. The form \( A^{(2)} \) contributes at doubly coincident points \( \vec{x}_i \),

\[
A^{(2)}_{CC\phi} = \frac{\Gamma(d+1)}{\pi^{d/2} \Gamma(d)} \int d^d y \frac{1}{(\vec{y} - \vec{z}_2)^{2d}} \delta^{(d)}(\vec{y} - \vec{z}_1) \delta^{(d)}(\vec{y} - \vec{z}_3)
+ \frac{1}{(\vec{y} - \vec{z}_1)^{2d}} \delta^{(d)}(\vec{y} - \vec{z}_2) \delta^{(d)}(\vec{y} - \vec{z}_3)
- \frac{1}{(\vec{y} - \vec{z}_3)^{2d}} \delta^{(d)}(\vec{y} - \vec{z}_1) \delta^{(d)}(\vec{y} - \vec{z}_2).
\]

(5.3)
The second one, \( A^{(3)} \), contributes at triply-coincident points,

\[
A^{(3)}_{CC\phi} = -\frac{1}{\mu^d \pi^{d/2} \Gamma(d)} \int d^d \vec{y} \ \delta^{(d)}(\vec{y} - \vec{z}_1) \delta^{(d)}(\vec{y} - \vec{z}_2) \delta^{(d)}(\vec{y} - \vec{z}_3) \ .
\] (5.5)

The fact that this three-point function only contains contact terms is in accordance with the constraints of conformal invariance \([3, 12]\).

To have a well-defined Fourier transform for the expressions in \( A^{(2)} \) we add the boundary action

\[
S^{(2)}_{bdy, CC} = \alpha \int d^d \vec{x} \ C(\vec{x}) C(\vec{x}) \Box^{k+2} \phi(\vec{x}) + C(\vec{x}) \Box^{k+2} C(\vec{x}) \phi(\vec{x}) + \Box^{k+2} C(\vec{x}) C(\vec{x}) \phi(\vec{x}) \ .
\] (5.6)

Setting \( \alpha \) to the same value as in eq.\((4.8)\) the divergence in the three-point correlator \( A^{(2)}_{CC\phi} \) is nullified. We may also include a counterterm of the form

\[
S^{(3)}_{bdy, CC} = \beta \int d^d \vec{x} \ C(\vec{x}) C(\vec{x}) \phi(\vec{x}) \ ,
\] (5.7)

to eliminate the contribution in \( A^{(3)}_{CC\phi} \), with \( \beta \) determined from \((5.5)\).

## 6 Four-point Functions

Four-point correlation functions have the new feature that there are two types of holographic Feynman diagrams to analyze: the one built from two three-point bulk vertices exchanging an intermediate supergravity field and the contribution from a bulk four-point vertex. Scalar exchange contributions to the first diagram have been analyzed several times \([14, 15, 19, 21]\), and are known to be reducible to an effective four-point vertex plus total derivatives in the bulk coordinate. As we found in the previous section these total derivative terms contribute as contact terms to the correlator. Some of these have physical significance and contain cuts in the kinematic invariants.

It will suffice to consider the \( s \)-channel scalar exchange contribution to the correlator of four axions, \( \langle CCCC \rangle \),

\[
A^{s}_{CCCC}(\vec{x}_i) = \int d^{d+1} y \sqrt{g(y)} \int d^{d+1} z \sqrt{g(z)} \left[ g^{\mu\nu}(y) \partial_\mu \Delta_1 \partial_\nu \Delta_2 \right] \times G(y, z) \left[ g^{\alpha\beta}(z) \partial_\alpha \Delta_3 \partial_\beta \Delta_2 \right] ,
\] (6.1)

Following the steps in \([14, 15]\) we partially integrate \( (6.1) \) with respect to both the \( y_0 \) and \( z_0 \) coordinates. After partially integrating the \( y_0 \) coordinate symmetrically we obtain a bulk four-point vertex contribution

\[
A^{s, \ bulk}_{CCCC} = \frac{1}{2} \int d^{d+1} y \sqrt{g(y)} \int d^{d+1} z \sqrt{g(z)} \left[ \Delta_1 \Delta_2 \left( \frac{1}{\sqrt{g}} \partial_\mu \sqrt{gg^{\mu\nu}\partial_\nu G(y, z)} \right) \right. \\
- \Delta_1 \left( \frac{1}{\sqrt{g}} \partial_\mu \sqrt{gg^{\mu\nu}\partial_\nu G(y, z)} \right) G(y, z) \left[ g^{\alpha\beta}(z) \partial_\alpha \Delta_3 \partial_\beta \Delta_4 \right] .
\] (6.2)
plus boundary terms. The second bulk term vanishes through the field equation for the massless scalar and the first reduces to an effective four-point vertex. This contribution will be cancelled by those from the t- and u-channel exchange diagrams \[15\]. Partially integrating the \( z_0 \)-coordinate there remain three types of boundary terms

\[
A^{\text{bdy}}_{CCCC} = M_1 + M_2 + M_3
\]

\[
M_1 = \frac{1}{4} \int d^d \bar{z} d^d \bar{y} \frac{1}{(y_0 z_0)^{d-1}} \partial_{y_0} \triangle_1 \partial_{z_0} \triangle_2 G(y, z) \partial_{z_0} \triangle_3 \triangle_4 
\]

\[
M_2 = -\frac{1}{4} \int d^d \bar{z} d^d \bar{y} \frac{1}{(y_0 z_0)^{d-1}} \triangle_1 \partial_{y_0} \partial_{z_0} G(y, z) \partial_{z_0} \triangle_3 \triangle_4 
\]

\[
M_3 = \frac{1}{4} \int d^d \bar{z} d^d \bar{y} \frac{1}{(y_0 z_0)^{d-1}} \triangle_1 \partial_{y_0} \partial_{z_0} G(y, z) \triangle_3 \triangle_4 
\]

in addition to similar contributions from the t- and u-channel.

It is straightforward to see that the \( y_0, z_0 \) contributions at \( \infty \) all vanish and the only surviving ones are at the \( y_0 = z_0 = 0 \) boundary. Evaluating these limits with the aid of section 3 we find that for \( M_1 \) the lower limit also vanishes. For \( M_2 \) we have two different contributions with triply and quadruply coincident points respectively,

\[
M^{(a)}_2 = -\frac{\Gamma(d+1)}{4 \pi^d \Gamma(d/2)} \int d^d \bar{y} d^d \bar{z} \delta^{(d)}(\bar{z}_1 - \bar{y}) \delta^{(d)}(\bar{z}_2 - \bar{y}) \delta^{(d)}(\bar{y} - \bar{z}) \times \delta^{(d)}(\bar{z}_4 - \bar{z}) \frac{1}{(\bar{z}_3 - \bar{z})^{2d}} + (3 \leftrightarrow 4) ,
\]

and

\[
M^{(b)}_2 = \frac{\Gamma(d+1)}{4 \mu^d \pi^d \Gamma(d/2)} \int d^d \bar{y} d^d \bar{z} \delta^{(d)}(\bar{z}_1 - \bar{y}) \delta^{(d)}(\bar{z}_2 - \bar{y}) \delta^{(d)}(\bar{y} - \bar{z}) \times \delta^{(d)}(\bar{z}_4 - \bar{z}) \delta^{(d)}(\bar{z}_3 - \bar{z}) + (3 \leftrightarrow 4) .
\]

plus contributions from \( \bar{z}_1, \bar{z}_2 \leftrightarrow \bar{z}_3, \bar{z}_4 \).

The terms of \( M_3 \) differs from \( M_2 \) in the arguments of the delta functions. It also produces functions contributing at triply and quadruply coincident points but at different pairs,

\[
M^{(a)}_3 = \frac{\Gamma(d+1)}{4 \pi^d \Gamma(d/2)} \int d^d \bar{y} d^d \bar{z} \delta^{(d)}(\bar{z}_1 - \bar{y}) \delta^{(d)}(\bar{z}_2 - \bar{y}) \delta^{(d)}(\bar{z}_3 - \bar{z}) \times \delta^{(d)}(\bar{z}_4 - \bar{z}) \frac{1}{(\bar{y} - \bar{z})^{2d}} ,
\]

and

\[
M^{(b)}_3 = -\frac{\Gamma(d+1)}{4 \mu^d \pi^d \Gamma(d/2)} \int d^d \bar{y} d^d \bar{z} \delta^{(d)}(\bar{z}_1 - \bar{y}) \delta^{(d)}(\bar{z}_2 - \bar{y}) \delta^{(d)}(\bar{z}_3 - \bar{z}) \times \delta^{(d)}(\bar{z}_4 - \bar{z}) \delta^{(d)}(\bar{y} - \bar{z}) .
\]
Using the shorthand
\[ \delta_{ij} \equiv \delta^{(d)}(\vec{z}_i - \vec{z}_j) \, , \] (6.10)
the final expression for the correlator yields,
\[ M_2^{(a)} = -\frac{\Gamma(d+1)}{4\pi d/2 \Gamma(d/2)} \left\{ 2\delta_{12}(\delta_{13} + \delta_{14}) \frac{1}{(\vec{z}_4 - \vec{z}_3)^2d} \right\} \] (6.11)
\[ M_2^{(b)} = \frac{1}{4\mu d} \frac{\Gamma(d+1)}{\pi^{d/2} \Gamma(d/2)} \{ 4\delta_{12}\delta_{14}\delta_{13} \} \, , \] (6.12)
The expression for \( M_3 \) is contained in,
\[ M_3^{(a)} = \frac{\Gamma(d+1)}{4\pi d/2 \Gamma(d/2)} \left\{ \delta_{12}\delta_{34} \frac{1}{(\vec{z}_3 - \vec{z}_1)^2d} \right\} \, , \] (6.13)
and
\[ M_3^{(b)} = -\frac{1}{4\mu d} \frac{\Gamma(d+1)}{\pi^{d/2} \Gamma(d/2)} \{ \delta_{12}\delta_{14}\delta_{13} \} \, , \] (6.14)
Using \( \delta_{12}f(x_1) = \delta_{12}f(x_2) \) the result is symmetric under \( \vec{z}_1 \leftrightarrow \vec{z}_2 \) and \( \vec{z}_3 \leftrightarrow \vec{z}_4 \), as is manifest in the original graph. Finally we need to also include the \( t \)- and \( u \)-channel diagrams to find full Bose symmetry. The sum of terms does not cancel.

Proceeding as before we remove the infinities by the introduction of a four-point contribution to the boundary AdS theory. The pure (divergent) contact contributions of \( M_2^{(b)} \) and \( M_3^{(b)} \) are completely removed similar to the \( A^{(3)} \) contribution in the previous section. Further modification of the boundary theory by finite terms may also modify their correlations in the AdS picture. As for the other terms, their Fourier transforms contain imaginary parts indicating that they contribute physically. In particular, the Fourier transform of the scalar exchange to the \( \langle CCCC \rangle \) correlator \cite{19} contains the \( s \)-channel cut,
\[ \text{Im} A_{CCCC}^{\phi,s} = -\frac{\pi}{\kappa^2} \delta^{(4)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \frac{1}{8}(\vec{k}_1 + \vec{k}_2)^2 \, , \] (6.15)
and arises solely from the contact terms in (6.13) above. This is equivalent to the result in \cite{19} after simplification.

Though we have not computed beyond four-point functions it is clear from the previous results that the pattern of contact terms persists in higher order.

7 Multi-particle states

Composite operator insertions in field theory lead to additional UV ambiguities in correlation functions; their renormalizations require the addition of counterterms of products of
composite operators to the correlation-functions. Recent studies have focused on position space, where for non-coincident points conformal symmetry imposes tight restrictions on the correlator expressions. Operators at coincident points and counterterms have not been considered in detail. In this section we discuss these singularities based on the renormalization of composite operators.

The AdS/CFT prescription comes with a natural regulator: the infinitesimal distance from the boundary of AdS. We have found that the holographic supergravity description of the CFT suffers from similar divergences when one tries to take this distance to zero. Since one expects the full string theory embedding to be finite one might ask how string theory will account for these in the low-energy limit. This will not come from including stringy effects. The free-field result for the two- and three-point functions are conjectured to be exact to all orders in \( g_{YM}^2 N_c \) (stringy) and \( N_c \) (string loop) corrections, yet the answer is still divergent. It could be that these divergences are an additional feature of string theory in a D-brane (Ramond-Ramond)-background, that is not yet understood.

Considering \( N = 4 \) super Yang-Mills theory as a consistent theory by itself, one would like to correct for the divergences by the introduction of counterterms. We have corrected for the divergences in the CFT correlation functions by the introduction of a boundary action to the AdS bulk theory consisting of a polynomial in the sources for the conformal operators. Complete composite operator renormalization in field theory also yields counterterms consisting of compound composite operators which also come in a power series in the source for the “simple” composite operator \[26\]. In double insertions of \( \text{Tr} F \tilde{F} \) in Green’s functions, for instance, one would correct for the UV infinities by adding to the \( N = 4 \) SYM action a term

\[
S = S_{N=4 \text{ SYM}} + \int d^4 \vec{x} \ J(\vec{x}) \text{Tr} F \tilde{F}(\vec{x}) + S^{(\text{counter})} \tag{7.1}
\]

\[
S^{(\text{counter})} = \ldots + \frac{c_i}{\mu^{p-d}} \int d^d \vec{x} \ J(\vec{x}) J(\vec{x}) \mathcal{O}_p(\vec{x}) + \ldots \tag{7.2}
\]

where \( \mathcal{O}_p \) are operators having dimension \( p \ (p \leq 8) \) consistent with the symmetries. In particular the compound composite : \( \text{Tr} F \tilde{F}(\vec{x}) \text{Tr} F \tilde{F}(\vec{x}) \) : is one of them, though \( N \)-counting arguments show that this term is suppressed as \( 1/N \). This term reflects on the supergravity side how the compound composite operator is dual to a multi-particle state.

We therefore conjecture that such additional couplings should be included in the AdS/CFT correspondence from the beginning (the renormalization breaks scale invariance). This means we should consider \( N = 4 \) SYM theory plus the space of all deformations

\[
S_{\text{CFT}}^{N=4} = \ldots + \sum \mu_{j_1\ldots j_k} \int d^d \vec{x} \phi_{0,j_1} \ldots \phi_{0,j_k} \mathcal{O}_{j_1} \ldots \mathcal{O}_{j_k} , \tag{7.3}
\]

with the sources corresponding to boundary values of the AdS fields as dictated by the renormalization of Green’s functions. The \( \mu_{j_1\ldots j_k} \) are dimensionful coupling constants. Their
scale depends on the renormalization scale $\mu$, the distance to the boundary, naturally. Functionally differentiation of these operators in the theory gives rise to correlators of compound composites after one interprets the delta functions $\delta^{(d)}(0) = \mu^{-d}$. Following the prescription in \[2, 3\] the correlation functions of products of composite operators would then be given by loop-like calculations in supergravity (Point-splitting on the boundary indicates that these couplings would arise from coincident point limits of single-particle correlation functions and that the legs connecting to the boundary would be given by the bulk-boundary kernel). Diagrammatically this is understood from pinching boundary points of the external fields in the original holographic Feynman diagram. This loop-like picture describes how the multi-particle states interact with others, though such loops contribute at the same order as the regular tree-level diagrams after appropriate normalization \[10\].

Composite operator renormalization suggests these multiparticle couplings as in (7.2) and (7.3). At the same time we have physical contact term contributions to the correlators of “simple” operators. They resemble diagrammatically the multiparticle correlators; for instance, the two separate delta function contributions $A_{CC\phi}^{(2)}$ and $A_{CC\phi}^{(3)}$ in (5.4) and (5.5) are pictorially identical to such one- and two-loop supergravity diagrams on AdS, respectively. Our calculations in earlier sections indicate the presence in the OPE of contact terms which modify the expansion via

$$O_n(\vec{x})O_n(\vec{y}) = \sum_j \frac{O_{\Delta_j}}{(\vec{x} - \vec{y})^{2n - \Delta_j}} + \delta^{(d)}(\vec{x} - \vec{y})O_{2n-d} + \ldots .$$  

Such contact contributions are usually required for consistency with the Ward-identities of the theory \[22, 31, 32, 33\] and are therefore not subject to renormalization ambiguities.

Finally it is worth noting that the AdS/CFT correspondence requires the existence of multi-particle states which occur in specific long-multiplets of the AdS supergroup \[28, 29\]. The above proposal provides a way of computing correlators involving such states.

8 Conclusion

We have provided a scheme for the renormalization of composite operator correlation functions within the AdS/CFT correspondence in the large $N$ limit involving the addition of a boundary supergravity theory to the bulk gauged supergravity theory. The general correlation function is made finite by modifying the correspondence as,

$$\prod_{j=1}^{k} \left( \frac{\delta}{\delta \phi_{0,j}(\vec{z}_j)} \right) e^{iS_{\text{sugra}}[\phi(\phi_0)] + iS_{\text{bt}}} \bigg|_{\phi_{0,j}=0} = \prod_{j=1}^{k} \langle O^j(\vec{z}_j) \rangle_{\text{CFT}}^{\text{ren}} ,$$  

where order by order the boundary theory $S_{\text{bt}}$ is chosen to contain counterterms renormalizing the correlators, as calculated in this work. This scheme does not maintain conformal
invariance. Multi-trace states, whose dimensions are not protected, appear at coincident points where ultraviolet singularities occur. Although the microscopic $N = 4$ super Yang-Mills theory is UV finite [34] there are no non-renormalization theorems for correlators of composite operators. Straightforward calculations of the two-point and three-point functions, for example, show the presence of divergences and the need for a renormalization scheme. It would be interesting to find the renormalization in the string context.

The correlators we have examined possess both contact terms in their explicit expressions and singularities which render their Fourier transform divergent. The contact terms in the form of delta function distributions contribute at coincident points - they appear discontinuously in the OPE. Diagrammatically they resemble correlators involving multi-particle states which are dual in the AdS/CFT correspondence to products of single-trace operators $O(\vec{x})$ at the same point. The dimensions of the latter are in general not protected which is related to the divergences in the theory which occur at short distances (when points collide). The renormalization and effective theory description of the correspondence allows one to determine the couplings of such compound composite operators to supergravity fields.

Given these results it appears necessary to examine in greater detail the conformal structure of the correspondence in the field theory and possible deformations of the conformal theory with the operators discussed here.

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References

[1] J. Maldacena, The large $N$ limit of superconformal field theories and supergravity, Adv. Theor. Math. Phys. 2 231 (1998), hep-th/9711200.

[2] S. Gubser, I. Klebanov, A. Polyakov, Gauge theory correlators from noncritical String theory, Phys. Lett. 428B (1998) 105, hep-th/9802109.

[3] E. Witten, Anti-de-Sitter space and holography, Adv. Theor. Math. Phys. 2 253 (1998), hep-th/9802150.
[4] M. Henningson, K. Sfetsos, Spinors and the AdS/CFT correspondence, Phys.Lett. B431 63 (1998), hep-th/9803251.

[5] W. Muck, K.S. Viswanathan, Conformal field theory correlators from classical scalar field theory on AdS_{d+1}, Phys. Rev. D 58 (1998) 041901, hep-th/9804033.

[6] D.Z. Freedman, S. Mathur, A. Matusis, L. Rastelli, Correlation functions in the CFT_d/AdS_{d+1} correspondence, hep-th/9804058.

[7] H. Liu, A. Tseytlin, D=4 Super Yang-Mills, D=5 gauged supergravity and D=4 conformal supergravity, Nucl. Phys. B533 88 (1998), hep-th/9804076.

[8] T. Banks, M. Green, Nonperturbative effects in AdS_{5}∗S^{5} string theory and d = 4 SUSY Yang-Mills, JHEP 9805 002 (1998), hep-th/9804170.

[9] G. Chalmers, H. Nastase, K. Schalm, R. Siebelink, R-current correlators in N = 4 super-Yang-Mills theory from Anti-de-Sitter supergravity, hep-th/9805105, to appear in Nucl. Phys. B.

[10] W. Muck and K.S. Viswanathan, Conformal field theory correlators from classical field theory on anti-de Sitter space, 2. Vector and Spinor Fields, Phys. Rev. D58 106006,1998, hep-th/9805143; W. Muck, K.S. Viswanathan, The graviton in the AdS-CFT correspondence: solution via the Dirichelet boundary value problem, hep-th/9810151.

[11] V. Balasubramanian, P. Kraus, A. Lawrence, Bulk vs. Boundary dynamics in anti-de Sitter spacetime, hep-th/9805174; V. Balasubramanian, P. Kraus, A. Lawrence, S. Trivedi, Holographic probes of anti-de Sitter spacetimes, hep-th/9808017.

[12] S. Lee, S. Minwalla, M. Rangami, N. Seiberg, Three-point functions of chiral operators in D=4 N = 4 SYM at large N, hep-th/9806194.

[13] G. Arutyunov, S. Frolov, On the origin of supergravity boundary terms in the AdS/CFT correspondence, hep-th/9806216; Anti-symmetric tensor field on AdS_5, hep-th/9807046; Quadratic action for type IIB supergravity on AdS_5 × S^5, hep-th/9811106; Three-point Green function of the stress-energy tensor in the AdS/CFT correspondence, hep-th/9901121.

[14] H. Liu, A. Tseytlin, On four point functions in the CFT/AdS correspondence, hep-th/9807097.

[15] D.Z. Freedman, S. Mathur, A. Matusis, L. Rastelli, Comments on 4-point functions in the CFT/AdS correspondence, hep-th/9808006.
[16] T. Banks, M.R. Douglas, G.T. Horowitz, E. Martinec, *AdS dynamics from conformal field theory*, hep-th/9808016.

[17] J.H. Brodie, M. Gutperle, *String corrections to four-point functions in the AdS/CFT correspondence*, hep-th/9809067.

[18] E. D’Hoker, D. Z. Freedman, *Gauge Boson Exchange in AdS_{d+1}*, hep-th/9809179.

[19] G. Chalmers, K. Schalm, *The large N_c limit of four-point functions in N = 4 super Yang-Mills theory from anti-de Sitter supergravity*, hep-th/9810051.

[20] H. Liu, *Scattering in Anti-de Sitter Space and Operator Product Expansion*, hep-th/9811152.

[21] E. D’Hoker, D.Z. Freedman, *General scalar exchange in AdS_{d+1}*, hep-th/9811257.

[22] H. Osborn, A. Petkos, *Implications of conformal invariance in field theories for general dimensions*, Ann. Phys. 231 311 (1994), hep-th/9307010; A.Petkou, *Conserved currents, consistency relations and operator product expansions in the conformally invariant O(N) vector model*, Ann. Phys. 249 180 (1996), hep-th/9410093.

[23] L. Susskind, E. Witten, *The holographic bound in anti-de Sitter space*, hep-th/9805114.

[24] A. Peet, J. Polchinski *UV/IR relations in AdS dynamics*, hep-th/9809022.

[25] I. Klebanov, *From Three-branes to large N gauge theories*, hep-th/9901018.

[26] S. Joglekar and B. Lee, Ann. Phys. 97, 160 (1976); S. Joglekar, Ann. Phys. 108, 233 (1977); Ann. Phys. 109, 210 (1977); J. Zinn-Justin, *Quantum field theory and critical phenomena*, Oxford University Press (1989).

[27] L. Adrianopoli, S. Ferrara, *On short and long SU(2, 2|4) multiplets in the AdS/CFT correspondence*, hep-th/9812067.

[28] J. de Boer, *Six-Dimensional Supergravity on S^3 X AdS_3 and 2d Conformal Field Theory*, hep-th/9806104.

[29] F. Larsen, *The Perturbation Spectrum of Black Holes in N=8 Supergravity*, Nucl. Phys. B536 (1998) 258, hep-th/9805208.

[30] D.Z. Freedman, K. Johnson, R. Munoz-Tapia, X. Vilasis-Cardona, *A cutoff procedure and counterterms for differential renormalization*, Nucl. Phys. B395 454 (1993), hep-th/9206028. D.Z. Freedman, G. Grignani, K. Johnson, N. Rius, *Conformal symmetry and differential regularization of the three-gluon vertex*, Annals Phys. 218 75 (1992),
D.Z. Freedman, K. Johnson, J.I. Latorre, *Differential regularization and renormalization: a new method of calculations in quantum field theory*, Nucl. Phys. B371 353 (1992).

[31] N. Seiberg, *Observations On The Moduli Space Of Superconformal Field Theories*, Nucl. Phys. B303 286 (1988).

[32] M. Green, N. Seiberg, *Contact Interactions in superstring theory*, Nucl. Phys. B299 559 (1988).

[33] D. Kutasov, *Geometry On The Space Of Conformal Field Theories And Contact Terms*, Phys. Lett. B220 153 (1989).

[34] M. Grisaru, M. Rocek and W. Siegel, *Superloops 3, beta 0: a calculation in N = 4 super Yang-Mills theory*, Nucl. Phys. B183 141 (1981); M. Grisaru, M. Rocek and W. Siegel, *Zero three loop beta function in N = 4 super Yang-Mills theory*, Phys. Rev. Lett. 45 1063 (1980); S. Mandelstam *Light cone superspace and the ultraviolet finiteness of the N = 4 model*, Nucl. Phys. B213 149 (1983); L. Brink, O. Lindgren, B.E.W. Nilsson, *N=4 Yang-Mills theory on the light cone*, Nucl. Phys. B212 401 (1983).