Gravity Modification with Yukawa-type Potential: Dark Matter and Mirror Gravity

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Abstract

The nature of the gravitational interaction between ordinary and dark matter is still open, and deviations from universality or Newtonian law may also modify the standard assumption of collisionless dark matter. On the other hand, obtaining a Yukawa-like large-distance modification of the gravitational potential is a nontrivial problem, that has so far eluded a consistent realization even at linearized level. We propose here a theory providing an Yukawa-like potential, by coupling non-derivatively the two metric fields related respectively to the visible and dark matter sectors, in the context of massive gravity theories where the local Lorentz invariance is broken by the different coexisting backgrounds. This gives rise to the appropriate mass pattern in the gravitational sector, producing a healthy theory with the Yukawa potential. Our results are of a special relevance in the scenario of dark matter originated from the mirror world, an exact duplicate of the ordinary particle sector.
1 Introduction

The problem of obtaining a Yukawa-like potential in a consistent theory of gravity is a nontrivial task, unlike in (spontaneously broken) gauge theories, and attempts in this direction date back to 1939 when Fierz and Pauli (FP) added a mass term $m$ to the free spin-2 action of the graviton \[1\]. However, the Lorentz-invariant massive FP theory is unfit to be a consistent modification of GR because of the van Dam-Veltman-Zakharov (vDVZ) discontinuity \[2\]: even in the limit $m \to 0$ the light bending is 25% off than that is predicted from the GR and confirmed experimentally with an extremely high precision. Another theoretical problem is the fine-tuning needed to single out a ghost-free action at linearized level which however is probably spoiled by interactions and a sixth ghost-like mode starts to propagate making the whole theory unstable \[3\] and in any case unreliable below some (unacceptably large) distance scale. The problem was reexamined in the framework of effective field theory realising that the reason behind the misbehavior of FP massive gravity is strong coupling of the scalar sector \[4\].

It has been shown that the sickness of the FP theory has its roots in the Lorenz invariance \[5\]. Indeed, retaining only rotational invariance, one can avoid the vDVZ discontinuity and the propagation of ghost-like states \[6, 7\] (for a different approach, see \[8\]). In these models gauge invariance is broken by Lorenz-breaking mass terms, so that the gauge modes that should start propagating, acquire a well behaved kinetic term, or do not propagate at all. What happens is that via Lorenz-breaking one can cure the ‘spatial’ problem of the discontinuity, while avoiding ghost-like propagating states.

In the context of bigravity theories \[9\], a suitable realization of the Lorenz-breaking (LB) massive phase of gravity can be obtained \[10\]. In this approach, in addition to our metric field $g_{1\mu\nu}$ coupled to the Lagrangian $\mathcal{L}_1$ of normal matter (sector 1 in the following), one introduces another metric tensor $g_{2\mu\nu}$ related to a hidden sector 2 (dark matter) with a Lagrangian $\mathcal{L}_2$. Therefore, the visible and dark components can be associated to separate gravities. The action of this theory consists of two Einstein-Hilbert terms and a mixed term $V$:

$$S = \int d^4x \left[ \sqrt{g_1} \left( M_1^2 R_1 + \mathcal{L}_1 \right) + \sqrt{g_2} \left( M_2^2 R_2 + \mathcal{L}_2 \right) + \epsilon^4 \left( g_1 g_2 \right)^{1/4} V(X) \right],$$

where $M_{1,2}$ are the “Planck” masses of the two sectors and $\epsilon$ is some small mass scale which essentially will define the graviton mass trough see-saw type relation $m_g \sim \epsilon^2/M_P$, $M_P$ being the Planck mass. The interaction potential $V$ between the two metrics is assumed to be non-derivative and it can be always taken as a scalar function of $X^{\mu}_\nu = g_{1\mu\alpha} g_{2\nu\sigma};$ the metric determinants are denoted $g_1$ and $g_2$. The invariance under diffeomorphisms is not broken. Local Lorentz invariance, on the other hand, is spontaneously broken because in general there is no local Lorentz frame in which two metric tensors $g_{1\mu\nu}$ and $g_{2\mu\nu}$ are proportional. Nonetheless, because each
matter sector is minimally coupled to its own metric, the weak equivalence principle is respected and the breaking of local Lorentz invariance is transmitted only through the gravitational interactions. Once a flat rotationally invariant (double) background is found, a Lorentz-non invariant mass term for the gravitational perturbations arises in a natural way, by expanding the total action \( I \) in the weak field limit \[10\].

We point out that when the interacting potential \( V \) is absent the (gauge) symmetry is enlarged: one can transform \( g_{1 \mu \nu} \) and \( g_{2 \mu \nu} \) by using two independent diffeomorphisms. The interaction potential \( V \) leaves unbroken the common diffeomorphisms group, corresponding to general covariance, and hence the gravitational fields always include a massless sector including normal graviton. In the massive sector on the other hand one finds a massive graviton, and when Lorentz-invariance is broken there are no additional propagating modes: in particular vector and scalar degrees of freedom do not propagate \[10\]. The Newtonian potential is modified in the infrared, but it is not Yukawa-like. In fact, at linearized level, the deviation from a \( 1/r \) potential is a linearly growing term \[10, 7\].

In order to have a massive phase with a Yukawa-like potential, bigravity must then be enlarged. In this paper we generalize the above construction and show that one can use a further rank-2 field \( g_{3 \mu \nu} \) as a Higgs field to achieve the Yukawa potential. The size of the fluctuations of \( g_{3 \mu \nu} \) is controlled by the 3rd "Planck" mass \( M_3 \) entering in its EH action. We will show that, in the limit \( M_3 \gg M_{1, 2} \), \( g_{3 \mu \nu} \) can be consistently decoupled and one is left with an effective bigravity theory with a Yukawa-like component of the gravitational potential. The tensor \( g_{3 \mu \nu} \) plays the role of a symmetry-breaking field, communicating the breaking of Lorentz invariance to \( g_{1 \mu \nu} \) and \( g_{2 \mu \nu} \) and thus introducing Lorentz-breaking mass terms to their fluctuations. Even in the limit of the decoupling of \( g_3 \) the resulting phase of gravity features a Yukawa-modified static potential while still avoiding any propagation of ghosts and the vDVZ discontinuity.

This situation can open new possibilities for the nature of dark matter. In the present paradigm the visible matter amounts only to about 4% of the present energy density of the Universe while the fraction of dark matter is about 5 times bigger. Cosmological observations are consistent with the hypothesis of cold dark matter. On the other hand, the flattening of galactic rotational curves can be also explained by the presence of cold (collisionless) dark matter distributed, differently from the visible matter, along the galactic halos. The implicit assumption behind this scenario is that gravitational interaction between the two kinds of matter is universal, and that it is Newtonian. Relaxing these hypotheses may radically modify our view and phenomenological modelling.

One of the intriguing possibilities is to consider dark matter as a matter of a hidden gauge sector which is an exact copy of the ordinary particle sector, so that along with

\[1\] This term breaks perturbativity at some large distance \( r > r_{IR} \), but remarkably this behavior is cured by the non-perturbative treatment \[11\].
the ordinary matter: electrons, nucleons, etc. the Universe contains also the mirror matter as mirror electrons, mirror nucleons, etc. with exactly the same mass spectrum and interaction properties. Such a parallel sector, dubbed as mirror world [12], can have many interesting phenomenological and cosmological implications (for reviews, see [13]). In particular, the baryon asymmetry in both sectors can be generated via the out-of-equilibrium, $B-L$ and $CP$ violating processes between the ordinary and mirror particles [13] which mechanism could naturally explain the proportion between the visible and dark matter fractions in the Universe. Such processes can be induced by the some very weak interactions between the ordinary and mirror fields that on the other hand can induce the mixing terms between the neutral particles of two sectors, as e.g. kinetic mixing for photons [16] or mass mixing in the case of the neutrinos and neutrons [17].

Mirror matter, dark in terms of ordinary photons and coupled with ordinary matter via common gravity, can be a viable candidate for dark matter. As it was shown in [15], the cosmological observations on the large scale structure and CMB are consistent with the mirror dark matter picture. However, the essential problem emerges at the galaxy scales. It is difficult to understand how the mirror matter, being as collisional and dissipative as normal matter, could produce extended galactic halos and thus explain the galactic rotational curves.

In this paper we show that the new possibilities can emerge if the mirror symmetry is extended also to the gravitational sector in the form of the action (1), the normal and mirror matters having separate gravities related respectively to the metric fields $g_{1\mu\nu}$ and $g_{2\mu\nu}$ while the Lorentz breaking is induced by the third dynamical metric $g_{3\mu\nu}$ with its ”Planck" mass $M_3$ much larger than the ordinary Planck mass $M_P$. Two gravities - one massless and another with a nonzero mass $m$, lead to the Yukawa-modified gravitational potential along with the normal Newtonian term. The potential felt by the probe particle of the type 1 (normal matter) at the distance $r$ from the source is

$$\phi(r) = \frac{G}{2r} \left[ (m_1 + m_2) + (m_1 - m_2) e^{-r/r_m} \right], \quad (2)$$

where $G$ is the Newton constant, $m_1$ and $m_2$ are respectively the masses of the visible (type 1) and mirror (type 2) matter sources, and $r_m = m^{-1}$ is a Yukawa length scale. Hence, at small distances, $r \ll r_m$, the gravitational forces are not universal between two sectors: the normal and mirror matter do not see each other. At distance $r \ll r_m$ a normal test mass interacts only with $m_1$ through the ordinary Newton potential. But at large distances $r \gg r_m$ the gravity becomes universal, and test particle feels both ordinary and dark matter sources $(m_1 + m_2)$ with an effectively halved Newton constant $G/2$. The main result of this work is to reproduce the potential (2) in a consistent model of gravity.

This scenario can have interesting astrophysical implications. One can show [23] that it allows to reproduce the galactic rotational curves even if dark mirror matter
has the similar ”clumped” distribution as the normal matter, as it is expected from its dissipative character.

The paper is organized as follows: in section 2 we review the linearized analysis of bigravity theories, to be used as building blocks for the model, and describe their Lorentz-breaking and Lorentz-invariant phases. In section 3 we describe the model and show how a Yukawa potential arises in the limit when the additional metric is decoupled. In section 4 we discuss the findings. Finally, appendices A and B contain the detailed expressions for the graviton mass matrices, the details for a specific interaction potential and the general expression of the Yukawa-like potential.

2 Bigravity: A review of the Linearized Analysis

In for bigravity generically one can find bi-flat $SO(3)$ preserving vacuum solutions [10]:

\[ \bar{g}_{1\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1,1,1,1) \]
\[ \bar{g}_{2\mu\nu} = \hat{\eta}_{\mu\nu} = \omega^2 \text{diag}(-c^2,1,1,1); \]

we have set the speed of light in our world (sector 1) to be one in natural units, whereas $c$ is the speed of light in the hidden sector 2 and $\omega$ is a relative constant conformal factor. Once $V$ is given, $c$ and $\omega$ can be computed by solving the equations of motion following from (1), and if $c \neq 1$, Lorentz symmetry is broken. Consider the linearized theory obtained by expanding the total action (1) at quadratic level in the metric perturbations around the bi-flat background (3):

\[ g_{1\mu\nu} = \eta_{\mu\nu} + h_{1\mu\nu}, \quad g_{2\mu\nu} = \hat{\eta}_{\mu\nu} + \omega^2 h_{2\mu\nu}. \]

The gravitational perturbations $h_{1\mu\nu}$ and $h_{2\mu\nu}$ interact with matter 1 and 2 through their conserved EMTs, respectively $T^{\mu\nu}_1$ and $T^{\mu\nu}_2$. Since the background preserves rotations, it is convenient to decompose the perturbations $h_{a\mu\nu}$ ($a = 1, 2$) according to irreducible $SO(3)$ representations

\[ h_{a00} = \psi_a, \quad h_{a0i} = u_{ai} + \partial_i v_a, \]
\[ h_{aij} = \chi_{aij} + \partial_i S_{aj} + \partial_j S_{ai} + \partial_i \partial_j \sigma_a + \delta_{ij} \tau_a. \]

For each perturbation one has a gauge invariant transverse traceless tensor $\chi_{aij}$, two vectors and four scalars. The quadratic Lagrangian $\mathcal{L}$ reads

\[ \mathcal{L} = \mathcal{L}_{\text{kin}} + \mathcal{L}_{\text{mass}} + \mathcal{L}_{\text{src}}, \]
\[ \mathcal{L}_{\text{kin}} = \frac{1}{4} \chi_{ij} \mathcal{K} (C^2 \Delta - \partial_i^2) \chi_{ij} - \frac{1}{2} w_i^j \mathcal{K} \Delta w_i + \phi^i \mathcal{K} \Delta \tau - \frac{1}{2} \tau^i \mathcal{K} (C^2 \Delta - 3 \partial_i^2) \tau. \]
We have introduced a vector notation for the fluctuations: $h_{\mu\nu} = (h_{1\mu\nu}, h_{2\mu\nu})^t$, $\chi_{ij} = (\chi_{1ij}, \chi_{2ij})^t$ and the following $2 \times 2$ matrices: $C = \text{diag}(1, c)$, $K = M_2^2\text{diag}(1, \kappa)$ and $\kappa = M_2^2 / M_1^2\omega c$. Also, in the kinetic term, coming from the expansion of EH terms, the fluctuations enter only through the gauge invariant combinations $w_i = u_i - \partial_t S_i$, $\phi = \psi - 2\partial_t v + \partial_t^2 \sigma$. The mass term $L_{\text{mass}}$ is produced by the expansion of the interaction potential $V$. Finally, $L_{\text{src}}$ describes the gravitational coupling to conserved sources associated to matter fields:

$$L_{\text{src}} = T_0^t C^{-1} W_i - T_0^t C^{-3} \phi - T_0^t C^{-2} \tau - T_0^t C^{-2} \chi_{ij}. \quad (9)$$

Clearly $L_{\text{kin}}$ and $L_{\text{src}}$ are gauge invariant. For the bi flat background (3) the mass term $L_{m}$ has the following form

$$L_{\text{mass}} = \frac{c^4}{4} \left( h_{00}^t M_0 h_{00} + 2 h_{0i}^t M_1 h_{0i} - h_{ij}^t M_2 h_{ij} + h_{ii}^t M_3 h_{ii} - 2 h_{ii}^t M_4 h_{00} \right) \quad (10)$$

and the explicit value of the mass matrices can be easily computed for any given $V$.

It is however crucial to realize that due to linearized gauge invariance the mass matrices have the following property [10]

$$M_{0,1,4} \left( \frac{1}{c^2} \right) = 0, \quad M_{1,2,3} \left( \frac{1}{1} \right) = 0, \quad M_4' \left( \frac{1}{1} \right) = 0. \quad (11)$$

Thus general covariance forces the mass matrices to be at most of rank one.

**Lorenz-Invariant (LI) phase.** In this case an FP graviton mediates Yukawa-like potential. Indeed, when $c = 1$, two conditions in (11) coincides, allowing a non-zero $M_1$ and all mass matrices are rank one and proportional:

$$M_0 = \lambda_0 \mathcal{P}, \quad M_1 = M_2 = \lambda_2 \mathcal{P}, \quad M_3 = M_4 = (\lambda_2 + \lambda_0) \mathcal{P}, \quad \mathcal{P} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \quad (12)$$

After introducing a canonically normalized graviton field $h^{(c)} = K^{1/2} h$, the mass matrices can be diagonalized by a rotation of an angle $\vartheta$ with $\tan \vartheta = \kappa^{1/2} / M_2 / \omega M_1$, leading to a massless and massive graviton eigenstates. This latter has a standard Lorentz-Invariant mass term and to avoid ghosts one has to choose $\lambda_0 = 0$, leading to a Pauli-Fierz mass term. Then, the massless graviton interacts with both matter sectors and effectively acts like standard Newtonian gravity in the weak field limit. The massive graviton on the other hand is Yukawa-like and thus modifies the static potential at scales larger than $m^{-1}$, where $m = \epsilon^2 \lambda_2^{1/2} |\sin \vartheta| / M_1$ is the graviton mass.

In the most interesting (mirror) case, when $M_1 = M_2 = M$, $\omega = 1$ and so $\tan \vartheta = 1$, we obtain that a static potential for a test particle of type 1, generated
by the point-like sources of mass $m_1$ (type-1) and $m_2$ (type-2) at the same point, is given by

$$\phi_{\text{matter}}(r) = \frac{1}{32\pi M^2} \left( (m_1 + m_2) + \frac{4}{3} e^{-m r} (m_1 - m_2) \right), \quad (13)$$

Therefore, the presence of the massive graviton state mediating Yukawa-like terms makes the effective Newton constant distance dependent: the Newton constant measured experimentally via the gravitational interaction between the type-1 test bodies at small distances $r \ll m^{-1}$ should be identified as $G = G_{UV} = 7/96\pi M^2$ while at large distances $r \gg m^{-1}$ it effectively becomes $G_{IR} = 1/32\pi M^2 = 3G/7$ and is universal between the type-1 and type-2 matters. On the other hand, at $r \ll m^{-1}$ the gravitational forces between the type-1 and type-2 bodies are repulsive, with $G_{12} = -G/7$, which in itself is indication of the instability of the theory. However, more serious problem is related to the vDVZ discontinuity. The static potential felt by the photon is

$$\phi_{\text{light}}(r) = \frac{1}{32\pi M^2} \left[ (m_1 + m_2) + e^{-mr} (m_1 - m_2) \right]. \quad (14)$$

Therefore, for the light bending at distances $r \ll m^{-1}$ we have $G_{\text{light}} = 1/16\pi M^2$, and thus $G/G_{\text{light}} = 7/6$. This discrepancy is somewhat milder than in the FP theory where we have $G/G_{\text{light}} = 4/3$; anyway the deviation from the GR prediction $G/G_{\text{light}} = 1$ is unacceptably large and it is clearly excluded by the post-Newtonian gravity tests [18].

The problems can be softened if the two sectors are not symmetric, $M_1 \neq M_2$ and the mixing angle $\vartheta$ between two gravities is enough small. In this case the static potentials respectively for the test body and test photon of the type 1 read:

$$\phi_{1\xi}(r) = \frac{\cos^2 \vartheta}{16\pi M_1^2} \left[ (m_1 + m_2) + \xi e^{-mr} (m_1 \tan^2 \vartheta - m_2) \right], \quad (15)$$

where $\xi = 4/3$ for a test body and $\xi = 1$ for light. Therefore, at small distance ($r \ll m^{-1}$) the Newton “constant” is $G_{UV} = G(1 + 4/3 \tan^2 \vartheta)$, at large distance ($r \gg m^{-1}$) it tends to $G$.

The ratio of (15) at small distances defines the post-Newtonian parameter $\delta$:

$$\delta = \lim_{m \to 0} \left[ \frac{\phi_{1\xi}(\xi = 4/3)}{\phi_{1\xi}(\xi = 1)} \right]_{m_2=0} = 1 + \frac{1}{3} \sin^2 \vartheta. \quad (16)$$

The current light bending experiments put a constraint $\delta = 1.0000 \pm 0.0001$ [18], and for GR $\delta = 1$, so that the limit of vanishing graviton mass reveals the well known vDVZ discontinuity [2] of Pauli-Fierz massive gravity. In our case the mixing angle $\vartheta$ controls the size of the discontinuity.

When $M_2 \gg M_1$, we have $\vartheta \to \pi/2$ and $\delta = 4/3$, unacceptably large. In this limit sector 2 is very weakly coupled, and the discontinuity is mainly shifted to sector 1, that approaches a normal Fierz-Pauli massive gravity.
Conversely when $M_2 \ll M_1$ we have $\vartheta \to 0$ and the discontinuity is shifted to sector 2; $h_+$ and $h_-$ almost coincide with $h_1$ and $h_2$ and gravity is stronger in sector 2. The experimental bound on $\delta$ translates into $\vartheta \simeq 0.02$, that amounts to roughly $M_2 \simeq \vartheta M_1$. In this case, if $m_2$ is interpreted as dark matter, it gives a sizable contribution, increasing the gravitational force in the region $r \gtrsim m^{-1}$. Notice incidentally that for small $r$, the potential is repulsive. This result contradicts observations in the gravitationally bounded systems as cluster and galaxies, for this reason is ruled out.

**Lorenz-Breaking (LB) phase.** In this phase, $c \neq 1$, conditions (11) imply that $M_1=0$ and for other masses one has

$$M_0 = \lambda_0 C^{-2} \mathcal{P} C^{-2}, \quad M_{2,3} = \lambda_{2,3} \mathcal{P}, \quad M_4 = \lambda_4 \mathcal{P} C^{-2}. \quad (17)$$

In this situation all the scalar and vector perturbations become non-dynamical [10]. The vanishing of $M_1$ in the LB phase is the reason why no ghosts or tachyons appear in the theory and only gravitational waves propagate. However, this is also the reason behind the absence of Yukawa-like gravitational potential. The resulting modification was studied in detail in [11] both at linear and non-linear level.

### 3 Three metrics: Effective Higgs Phase

In order to find a phenomenologically healthy Yukawa phase, we introduce one more rank-2 field $g_3$ which couples with both $g_1$ and $g_2$:

$$S = \int d^4x \left[ \sqrt{g_1} \left( M_2^2 R_1 + \mathcal{L}_1 \right) + \sqrt{g_2} \left( M_2^2 R_2 + \mathcal{L}_2 \right) + M_3^2 \sqrt{g_3} R_3 + \epsilon^4 \left( g_1 g_2 g_3 \right)^{1/6} V(g_1, g_2, g_3) \right]. \quad (18)$$

The only non trivial tensors that can be formed are: $X_{12} = g_1^{-1} g_2$, $X_{13} = g_1^{-1} g_3$, $X_{23} = g_2^{-1} g_3$, that satisfy the identity $X_{13} = X_{12} X_{23}$. Therefore $V$ can be taken as a scalar function of two of them.

We also introduce in [18] a discrete symmetry under the exchange $1\leftrightarrow 2$, so that the potential $V$ is symmetric and the two sectors 1, 2 have equal Planck masses $M$. The third Planck mass on the other hand will be eventually taken much larger, $M_3 \gg M$, and the fluctuations of the third field will be effectively decoupled.

The first step is to find a suitable background. As for bigravity, we look for flat solutions, for which a consistent LB ansatz is the following

$$\bar{g}_{1\mu\nu} = \bar{g}_{2\mu\nu} = \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1) \quad \bar{g}_{3\mu\nu} = \bar{\eta}_{\mu\nu} = \omega^2 \text{diag}(-c^2, 1, 1, 1), \quad (19)$$

2For the simplicity, we assume that the third auxiliary sector is purely gravitational and does not contain the respective matter. In principle, any tensor condensate e.g. emerging via a strongly coupled hidden gauge sector can be also used for inducing the Lorentz-breaking background [19].
so that $\bar{X}_{12} = \mathbb{I}$ and $\bar{X}_{13} = \eta^{-1}\hat{\eta}$. The background (19) is a solution of the equation of motion if the following equations are satisfied:

\[
\begin{align*}
V I + 6 \frac{\partial V}{\partial X_{21}} X_{21} + 6 \frac{\partial V}{\partial X_{31}} X_{31} &= 0 \\
V I + 6 \frac{\partial V}{\partial X_{12}} X_{12} + 6 \frac{\partial V}{\partial X_{32}} X_{32} &= 0 \\
V I + 6 \frac{\partial V}{\partial X_{13}} X_{13} + 6 \frac{\partial V}{\partial X_{23}} X_{23} &= 0,
\end{align*}
\]

(20)

where $X_{ba} = X_{ab}^{-1}$. Then, due to the identity

\[
\frac{\partial V}{\partial X_{ab}} X_{ab} = - \frac{\partial V}{\partial X_{ba}} X_{ba},
\]

(21)

and due to the $1 \leftrightarrow 2$ exchange symmetry of the EOM and of the background, we have $\partial V/\partial X_{12} = 0$ and the EOM reduce to

\[
V = 0, \quad \frac{\partial V}{\partial X_{13}} = 0.
\]

(22)

These are three independent equations for the two parameters $\omega$ and $c$: thus one fine-tuning is needed to have the present flat solution. This fine tuning is analogous to the cosmological constant in standard GR, and can be easily realized for instance by introducing a cosmological constant in sector 3.

Once a background solution is found, one can study small fluctuations defined by

\[
g_{1\mu\nu} = \eta_{\mu\nu} + h_{1\mu\nu}, \quad g_{2\mu\nu} = \eta_{\mu\nu} + h_{2\mu\nu}, \quad g_{3\mu\nu} = \hat{\eta}_{\mu\nu} + \omega^2 h_{3\mu\nu}.
\]

(23)

The structure of the quadratic Lagrangian for the fluctuations is the same as in (6)-(9) except that now the tensor, vector, scalar and source fields all have 3 components, $h_{\mu\nu} = (h_{1\mu\nu}, h_{2\mu\nu}, h_{3\mu\nu})$. Also,

\[
\mathcal{K} = \text{diag}(M^2, M^2, M^2_3/\omega^2 c), \quad \mathcal{C} = \text{diag}(1, 1, c)
\]

(24)

and the masses $\mathcal{M}_i$ are $3 \times 3$ matrices, entering the usual mass Lagrangian:

\[
\mathcal{L}_{\text{mass}} = h_{00}^i \mathcal{M}_0 h_{00} + 2 h_{0i}^i \mathcal{M}_1 h_{0i} - h_{ij}^i \mathcal{M}_2 h_{ij} + h_{ii}^i \mathcal{M}_3 h_{ii} - 2 h_{ii}^i \mathcal{M}_4 h_{00}.
\]

(25)

Diagonal diffeomorphisms invariance constrains the form of these matrices:

\[
\mathcal{M}_{1,2,3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathcal{M}_4^T \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathcal{M}_{0,1,4} \begin{pmatrix} 1 \\ 1 \\ c^2 \end{pmatrix} = 0.
\]

(26)
From these conditions and from the $1 \leftrightarrow 2$ symmetry it follows that the matrices can be written as the following combinations of projectors

$$
\begin{align*}
\mathcal{M}_0 &= a_0 \mathcal{P}_{12} + b_0 C^{-2} (\mathcal{P}_{13} + \mathcal{P}_{23}) C^{-2} \\
\mathcal{M}_1 &= a_1 \mathcal{P}_{12} \\
\mathcal{M}_2 &= a_2 \mathcal{P}_{12} + b_2 (\mathcal{P}_{13} + \mathcal{P}_{23}) \\
\mathcal{M}_3 &= a_3 \mathcal{P}_{12} + b_3 (\mathcal{P}_{13} + \mathcal{P}_{23}) \\
\mathcal{M}_4 &= a_4 \mathcal{P}_{12} + b_4 (\mathcal{P}_{13} + \mathcal{P}_{23}) C^{-2} ,
\end{align*}
$$

where

$$
\begin{align*}
\mathcal{P}_{12} &= \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\mathcal{P}_{13} &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \\
\mathcal{P}_{23} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}
\end{align*}
$$

and $a_i$, $b_i$ are constants that depend on the quadratic expansion of the interaction term $V$.

Since we are interested in the gravitational potential we will first focus on the scalar sector. The full Lagrangian is:

$$
\mathcal{L}_{\text{scalars}} = \phi^i \mathcal{K}^2 \Delta \tau - \tau^i \mathcal{K}_i^2 \left( \mathcal{O}^2 \Delta - 3 \partial_\tau^2 \right) + \frac{1}{4} \left[ \psi \mathcal{M}_0 \psi - 2 \Delta \psi \mathcal{M}_1 \psi - (\tau + \Delta \sigma) \mathcal{M}_2 (\tau + \Delta \sigma) - 2 \tau \mathcal{M}_2 \tau \\
+ (3 \tau + \Delta \sigma) \mathcal{M}_3 (3 \tau + \Delta \sigma) - 2 (3 \tau + \Delta \sigma) \mathcal{M}_4 \psi \right]
$$

$$
- \phi \frac{C^{-3}}{2} T_{00} - \tau^i \frac{C}{2} T_{ii}.
$$

In order to disentangle the different fluctuations we decompose the system by defining a ‘tilded’ basis where the 1,2 fluctuations are rotated:

$$
[\psi, v, \sigma, \tau] = S \left[ \tilde{\psi}, \tilde{v}, \tilde{\sigma}, \tilde{\tau} \right], \quad \tilde{\mathcal{M}}_i = S^t \mathcal{M}_i S \quad S = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}.
$$

In this basis the mass matrices take the block-diagonal form

$$
\begin{align*}
\tilde{\mathcal{M}}_0 &= \begin{pmatrix} 2a_0 + b_0 & 0 & 0 \\ 0 & b_0 & -b_0/c^2 \\ 0 & -b_0/c^2 & b_0/c^4 \end{pmatrix}, \\
\tilde{\mathcal{M}}_1 &= \begin{pmatrix} 4a_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \\
\tilde{\mathcal{M}}_{2,3} &= \begin{pmatrix} 2a_{2,3} + b_{2,3} & 0 & 0 \\ 0 & b_{2,3} & -b_{2,3} \\ 0 & -b_{2,3} & b_{2,3} \end{pmatrix}, \\
\tilde{\mathcal{M}}_4 &= \begin{pmatrix} 2a_4 + b_4 & 0 & 0 \\ 0 & b_4 & -b_4/c^2 \\ 0 & -b_4 & b_4/c^2 \end{pmatrix}
\end{align*}
$$
Because the kinetic structure commutes with $S$, we see that in the new basis the system splits into two sectors: a single massive gravity plus a bigravity sector, associated with the $2\times2$ sub-matrices in (31). Due to the third background $\bar{g}_3$, in both sectors the mass pattern is Lorentz-breaking. The first sector can be analysed as in [6], while for the second the analysis of [10] applies. This allows for a consistent theory, free of ghosts and of instabilities at linearized level.

Indeed, in the single massive gravity sector ghosts can be avoided, if the relevant entry 1-1 in $\tilde{\mathcal{M}}_0$ vanishes. We have thus the condition:

$$a_0 = -b_0/2.$$  \hfill (32)

The bigravity sector on the other hand is automatically free of ghosts as shown in [10] thanks to the vanishing of $\tilde{\mathcal{M}}_1$ in the relevant block.

At this point we can study in the new basis the static gravitational potential associated in each sector with the gauge invariant field $\tilde{\phi}_a = \tilde{\psi}_a - 2\partial_t \tilde{v}_a + \partial^2 \tilde{\sigma}_a$ ($a = 1, 2, 3$). It is convenient to define also the rotated and $M^2$-normalized sources $\tilde{t}_{\mu\nu} = S t_{\mu\nu} = S (T_{\mu\nu}/M^2)$.

The field $\tilde{\phi}_1$ is separated from the bigravity sector and gives the Yukawa-like static potential. It turns out that in general $\tilde{\phi}_1$ is a combination of two Yukawa potentials, with two parametrically different mass scales (see appendix A for the details). For simplicity, by tuning the parameters one can also have a single mass scale:

$$\tilde{\phi}_1 = \frac{\tilde{t}_{1}}{2\Delta - m^2}, \quad \text{with} \quad m^2 = 3(2a_4 + b_4) \frac{\epsilon^4}{M^2}. \hfill (33)$$

In this sector, in addition to the propagating massive graviton (two polarizations) also a vector and a scalar field propagate (respectively two and one degrees of freedom). All these fields are massive with mass given by the relative 1-1 entry of $\mathcal{M}_2$. The vector and the scalar can have well behaved properties, i.e. no ghosts when condition (32) is enforced. In [6] it was also argued that the scale of strong coupling is high enough, coinciding with $\Lambda_2 \simeq \sqrt{Mm}$, with $m$ the characteristic mass scale in this sector.

For the remaining bigravity sector the gravitational potential can be computed by solving the equations of motion as in [10]. The result is

$$\tilde{\phi}_2 = \frac{\tilde{t}_{200} + \tilde{t}_{2iii}}{2\Delta} + \mu^2 \frac{\tilde{t}_{200}}{\Delta^2} \hfill (34)$$

$$\tilde{\phi}_3 = -\mu^2 \left( \frac{M}{M_3} \right)^2 \frac{2\epsilon^2 \tilde{t}_{200}}{\Delta^2} \hfill (35)$$

where

$$\mu^2 = \frac{\epsilon^4}{M^2} \left[ b_2 \frac{3b_1^2}{2} + b_0 (b_2 - 3b_3) \right]. \hfill (36)$$

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When $M_3 \gg M$, the third sector has a sub-leading impact on the other gravitational potentials. In the limit $M_3 \to \infty$, the third sector decouples and $g_3$ just produces a LB fixed background $\hat{\eta}$. Going back to the original basis, the potentials are:

\[
\phi_1 = \frac{t_{100} + t_{1ii} + t_{200} + t_{2ii}}{4\Delta} + \frac{t_{100} + t_{1ii} - t_{200} - t_{2ii}}{4\Delta - 2m^2} + \mu^2 \frac{t_{100} + t_{200}}{2\Delta^2},
\]

\[
\phi_2 = \frac{t_{100} + t_{1ii} + t_{200} + t_{2ii}}{4\Delta} - \frac{t_{100} + t_{1ii} - t_{200} - t_{2ii}}{4\Delta - 2m^2} + \mu^2 \frac{t_{100} + t_{200}}{2\Delta^2},
\]

\[
\phi_3 = -\mu^2 \left(\frac{M}{M_3}\right)^2 \frac{c\omega^2(t_{100} + t_{200})}{\Delta^2}.
\]

The final potentials $\phi_{1,2}$ contain a Newtonian term, a Yukawa-like term, and a linearly growing term, originating from $\mu^2/\Delta^2$.

This latter linear term is the same appearing in the bigravity case, as found in [10] [20]. It would invalidate perturbation theory at distances larger than $r_{IR}^{-1} \sim G\mu^2 M_3$ from a source $M_3$ [10], but remarkably the full nonlinear solutions found in [11] shows that its linear growth is replaced by a non-analytic power $\sim r^\gamma$, where $\gamma$ depends on the coupling constants in the potential. Moreover, in the full solution for a realistic star, also the magnitude of this new term is proportional to $\mu^2$, therefore the effect can be eliminated by setting $\mu^2 = 0$. This can be achieved by simple fine-tuning, or by adopting a particular scaling symmetry of the potential, as discussed in [10]. We can thus obtain a pure Yukawa modification of the gravitational potential, by setting $\mu^2 = 0$, that here amounts to the condition $b_0 = -3b_3/(b_2 - 3b_1)$.

The analysis of vector modes is identical to that carried out in [1] for the single gravity sector and to the one of [10] for the bigravity one. In the single-gravity sector there is a vector state propagating with nonlinear dispersion relation: at high energy its speed is $(2a_2 + b_2)/(2a_1 + b_1)$ and at low momentum it has a mass gap given by $b_2/M^2$. In the bigravity sector on the other hand vector states do not propagate.

The analysis of tensor modes is similar and is best carried out in the original basis. In the limit of $M_3 \to \infty$, the equation of motion for the canonically normalized fields becomes:

\[
\left[\begin{array}{c}
\square \\
\hat{\square}
\end{array}\right] + \frac{1}{M^2} \left[\begin{array}{ccc}
b_2 + a_2 & b_2 - a_2 & 0 \\
b_2 - a_2 & b_2 + a_2 & 0 \\
0 & 0 & 0
\end{array}\right] \left[\begin{array}{c}
\chi^c_{1ij} \\
\chi^c_{2ij} \\
\chi^c_{3ij}
\end{array}\right] = \left[\begin{array}{c}
t_{1ij} \\
t_{2ij} \\
0
\end{array}\right]
\]

where $\square = \gamma^{\mu\nu} \partial_\mu \partial_\nu$, $\hat{\square} = \hat{\gamma}^{\mu\nu} \partial_\mu \partial_\nu$ and we used the form of the projectors (27). We see that the massless spin two state decouples (it is a superimposition of mostly $\chi_3$) and we are left with two massive gravitons, with two polarizations each, travelling at the normal speed of light. Their mass matrix can be diagonalized, and the resulting graviton masses are $m^{2}_{g_1} = (2a_2 + b_2)c^4/M^2$, $m^{2}_{g_2} = b_2c^4/M^2$. 

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As an explicit example, consider the simplest case of a potential quadratic in $X_{12}, X_{13}, X_{23}$ plus two cosmological terms, satisfying the $1 \leftrightarrow 2$ exchange symmetry (taking, for simplicity $\omega = 1$):

$$V(g_1, g_2, g_3) = \xi_0 + \xi_1 (\text{Tr}[X_{13}^2] + \text{Tr}[X_{23}^2]) + \xi_2 \text{Tr}[X_{13}X_{23}] + \xi_3 (\text{Tr}[X_{13}]^2 + \text{Tr}[X_{23}]^2) + \xi_4 \text{Tr}[X_{13}]\text{Tr}[X_{23}] + \xi_5 (\text{Tr}[X_{12}]^2 + \text{Tr}[X_{12}^{-1}]^2) + \xi_6 (\text{Tr}[X_{12}^2] + \text{Tr}[(X_{12}^{-1})^2]) + \xi_7 ((\text{det}X_{12})^{-1/6}(\text{det}X_{13})^{-1/6} + (\text{det}X_{12})^{1/6}(\text{det}X_{23})^{-1/6}) + \xi_8 (\text{det}X_{13})^{1/6}(\text{det}X_{23})^{1/6}$$

(39)

The EOM for flat backgrounds require to solve for three constants (e.g. $\xi_3, \xi_7, \xi_8$) and then the coefficients of the projectors in the mass matrices $a_i$’s and $b_i$ are a function of the remaining coupling constants (see appendix B).

The no-ghost condition $b_0 = -2a_0$, the condition for the absence of the linear term $\mu^2 = 0$ and the condition for having a single Yukawa scale (see Appendix A), can be solved for $\xi_{1,2,4}$ and reduce finally the dependence to only four couplings. Then, the Yukawa scale $m$ and the graviton masses $m^2_{g_1}$ and $m^2_{g_2}$, only depend on $\xi_5$ and $\xi_6$:

$$m^2 = [p_0(c)\xi_5 + q_0(5)\xi_6] \frac{\epsilon^4}{M^2}$$

$$m^2_{g_1} = [p_1(c)\xi_5 + q_1(c)\xi_6] \frac{\epsilon^4}{M^2}$$

$$m^2_{g_2} = [p_2(c)\xi_5 + q_2(c)\xi_6] \frac{\epsilon^4}{M^2}$$

(40)

where $p_i(c)$’s and $q_i(c)$’s are given in appendix B.\footnote{When $\xi_6 = 0$ and all the masses above depend only on $\xi_5$, one can check that they are positive, for $1.41 \leq c \leq 2.05$.}

To summarize, in the limit where the third metric is decoupled the theory has two massive gravitons and the potential felt by a test particle of type 1 is:

$$\phi_1(r) = \frac{G m_1}{r} \left( \frac{1 + e^{-mr}}{2} \right) + \frac{G m_2}{r} \left( \frac{1 - e^{-mr}}{2} \right),$$

(41)

where $G = 1/16\pi M^2$. This shows that the vDVZ discontinuity is absent, and we have obtained the potential [2] while avoiding the troubles of the Lorentz-invariant FP theory.

### 4 Conclusions

Motivated by the interesting possibility to relax the assumption that dark and visible matter feel the same gravitational interaction, in this work we have addressed the
possibility to obtain a Yukawa-like large-distance modification of the gravitational potential, while avoiding ghosts or classical instabilities.

The request to generate an Yukawa potential from a consistent theory led us to consider Lorenz-Breaking backgrounds in enlarged models of bigravity. For example, bigravity theories while giving rise to a healthy Lorentz-Breaking massive phase, do not produce a Yukawa potential. Here we have generalized this picture and have shown that if an additional field $g_3$ is introduced, a Yukawa modification is allowed. The extra field $g_3$ can be harmlessly decoupled by freezing it to a Lorentz violating background configuration. The two remaining sectors represent two interacting massive gravities, of which one features a Yukawa potential. This pattern then leads to the desired modified gravity, because standard matter (type 1) couples to all the mass eigenstates.

On the technical side, the price to be paid to solve the problem is that two fine-tunings are needed, one to have a ghost-free spectrum at linear level, the other to avoid the linearly growing potential. The first one has been shown to follow (in single massive gravity theories) from extra unbroken partial diffeomorphisms invariance \[21\] and it would be interesting to extend that symmetry arguments also to the present model. The other can also be understood as the consequence of a scaling symmetry of the potential \[10\].

Let us note also that while the theory presented of three rank-two fields only propagates 9 well behaved modes at quadratic level (three spin-2 with 2 polarizations each, one spin-1 with 2 polarizations and one scalar) one may expect that the total number of propagating modes would be 18, i.e. $3 \times 10$ minus 8 by unbroken gauge conditions minus 4+4 for the broken relative diffeomorphisms (à la Proca). The missing 9 modes could then propagate at non-linear level. The real non-perturbative question, to be addressed in future work, is then at which scale these non-linearities would show up.

The resulting setup, featuring two separate metric fields responsible for gravity for the visible and dark matter, allows to consider also collisional and dissipative dark matter, as mirror matter, if the potential felt by ordinary matter, and generated by the ordinary and dark matter sources of mass $m_1$, $m_2$, is distance dependent as in \[41\]\(\footnote{\text{4 Let us remark also that the weak equivalence principle does not exclude the possibility of direct (non-gravitational) interactions between the normal (type 1) and dark (type 2) matter components. To the action \[11\], besides the mixed gravitational term $V$, the mixed matter term \[ \int d^4x (g_1 g_2)^{1/4} L_{\text{mix}} \] can be added with the Lagrangian $L_{\text{mix}}$ including for example, the photon kinetic mixing term $\varepsilon F_{\mu \nu} F^*_{\mu \nu}$ \[16\] or the neutrino interaction terms \[17\]. This also makes possible the direct detection of dark matter via such interactions \[22\], with interesting implications.}}\)

The result is very different from a standard picture when normal and dark matters both have an universal Newtonian gravity; in fact the potential \[41\] can be used to fit galactic rotational curves using similar density profiles for the visible and dark sectors, alleviating the problems of profile formation \[23\]. It interesting to note
that the effective Newton constant relative to the type 1 - type 1 and type 2 - type 2 interactions is distance dependent: \( G_N(r \ll m^{-1}) = G \) and \( G_N(r \gg m^{-1}) = G/2 \). Let us remark also, that as far as at large cosmological distances only the massless gravity is effective with a halved Newton constant \( G/2 \), the observed Hubble constant would imply for the total energy density of the universe twice as bigger than in the standard cosmology when the Newton constant at the cosmological distances remains the canonical \( G \), i.e. now we must have \( \rho_{cr} = 3H_0^2/4\pi G \) instead of \( \rho_{cr} = 3H_0^2/8\pi G \) implied by the standard cosmology.

We then conclude that at linear level a Yukawa modification of the Newtonian gravitational potential is possible and it also opens up to the possibility to have collisional dark matter, coupled to ordinary matter via a modified gravitational interaction in a physically nontrivial way.

**Acknowledgments**

We thank D. Comelli for useful discussions. The work is supported in part by the MIUR grant for the Projects of National Interest PRIN 2006 "Astroparticle Physics", and in part by the European FP6 Network "UniverseNet" MRTN-CT-2006-035863.

### A General Yukawa-like potential

The degrees of freedom in the single gravity sector consists of a metric fluctuation with mass term that we can write as

\[
\mathcal{L}_{mass} = \frac{M^2}{2} \left( m_0^2 h_{00} h_{00} + 2 m_1^2 h_{0i} h_{0i} - m_2^2 h_{ij} h_{ij} + m_3^2 h_{ii} h_{jj} - 2 m_4^2 h_{00} h_{ii} \right). \tag{42}
\]

In our case effectively \( m_0 = 0 \). In this case, if \( m_1 \neq 0 \), there is a (healthy) propagating scalar degree of freedom \( \tau \) as well as a healthy propagating vector [6]. The scalar perturbations obey the equations:

\[
2\Delta \tau - m_4^2 (\Delta \sigma + 3 \tau) = t_{00} \tag{43}
\]

\[
2\partial_0^2 \tau - m_1^2 \nu = \frac{1}{\Delta} \partial_0 t_{00} \tag{44}
\]

\[
2\partial_0^2 \sigma - m_2^2 \Delta \sigma - m_1^2 m_3 \Delta \tau + 3 m_3^2 \tau - m_4^2 (\phi + 2 \partial_0 \nu - \partial_0^2 \sigma) = \frac{1}{\Delta} \partial_0^2 t_{00} \tag{45}
\]

\[
2\Delta \phi - 2\Delta \tau + 2 m_2^2 \Delta \sigma = t_{ii} - \frac{3}{\Delta} \partial_0^2 t_{00} \tag{46}
\]

where \( t_{\mu\nu} = T_{\mu\nu}/M^2 \). These can be solved with respect to \( \phi \) to get the static Newtonian potential. One finds

\[
\phi = \frac{(t_{00} + t_{ii})(\zeta_1 - 1) \zeta_2 \Delta + [t_{ii} + t_{00}(3 \zeta_1 - 1) \zeta_2] \zeta_2 m_4^2}{2(\zeta_1 - 1) \zeta_2 \Delta^2 + (4 \zeta_2 - 1)m_4^2 \Delta - 3 \zeta_2 m_4^2}, \tag{47}
\]
with $\zeta_1 = m_3^2/m_2^2$ and $\zeta_2 = m_2^2/m_4^2$. The potential can be split in two Yukawa-like terms:
\[
\phi = \frac{t_+}{2\Delta - m^2} + \frac{t_-}{2\Delta - m^2},
\]
where
\[
t_\pm = \frac{1}{2}(t_{00} + t_{ii}) \left( 1 \pm \frac{1}{\delta} \right) \pm t_{00} \left( \delta - \frac{1}{\delta} \right),
\]
\[
m^2_\pm = m_4^2 \left( \frac{4\zeta_2 - 1 \pm \delta}{2\zeta_2(1 - \zeta_1)} \right),
\]
\[
\delta = \sqrt{1 + 8\zeta_2(3\zeta_1\zeta_2 - \zeta_2 - 1)}.
\]

Recall [6] that the conditions $\zeta_2 > 1/4$ and $\zeta_1 < 1$ ensure that the theory as no derivative instabilities neither in the UV nor in the IR. Moreover, if $\zeta_1 > (8\zeta_2^2 + 8\zeta_2 - 1)/24\zeta_2^2$, the two masses $m_\pm$ are real and positive, and the theory has no instabilities also at intermediate scales. Accordingly, the potential is the sum of two “genuine” Yukawa-like terms.

Finally, if $\zeta_1 = (1 + \zeta_2)/3\zeta_2$, then $\delta = 1$ and $t_-$ vanishes, so that one is left with a single Yukawa potential:
\[
\phi = \frac{t_{00} + t_{ii}}{2\Delta - 3m_4^2}.
\]

B **Explicit solution for potential (39)**

Mass coefficients as a function of the coupling constants for the potential (39), after solving the EOM (22):
\[
\begin{align*}
a_0 &= -\frac{\xi_4c^4}{2} - \frac{2\xi_1c^2}{9} - \frac{\xi_0}{72} - \frac{c^2}{18} (9c^2 + 2) \xi_2 + \frac{35\xi_5}{9} + \frac{50\xi_6}{9} \\
b_0 &= -\frac{\xi_0}{72} - \frac{c^2 (39 - 23c^2)}{18 (c^2 + 3)} \xi_1 - \frac{c^2 (39 - 23c^2)}{36 (c^2 + 3)} \xi_2 - \frac{\xi_5}{9} - \frac{\xi_6}{9} \\
a_1 &= \frac{c^2 \xi_2}{2} - 4\xi_5 \\
a_2 &= \frac{\xi_3}{2} - \xi_5 \\
b_2 &= (c^2 - 1) \xi_1 + \frac{1}{2} (c^2 - 1) \xi_2 \\
a_3 &= -\frac{2\xi_1c^2}{9} - \frac{\xi_2c^2}{9} - \frac{\xi_0}{72} - \frac{\xi_4}{2} + \frac{26\xi_5}{9} + \frac{50\xi_6}{9} \\
b_3 &= -\frac{\xi_0}{72} + \frac{(5c^2 - 6c)\xi_1}{18} + \frac{(5c^2 - 6c)\xi_2}{36} - \frac{\xi_5}{9} - \frac{4\xi_6}{9} \\
a_4 &= -\frac{2\xi_1c^2}{9} - \frac{\xi_2c^2}{9} - \frac{\xi_4c^2}{2} - \frac{\xi_0}{72} - \frac{\xi_5}{9} + \frac{50\xi_6}{9} \\
b_4 &= -\frac{\xi_0}{72} - \frac{(13c^4 + 3c^2)\xi_1}{18 (c^2 + 3)} - \frac{(13c^4 + 3c^2)\xi_2}{36 (c^2 + 3)} - \frac{\xi_5}{9} - \frac{4\xi_6}{9}
\end{align*}
\]
Functions $p$ appearing in the graviton masses \( (40) \):

\[
p_0 = \frac{1}{C_2} \left[ 6 \left( c^2 \left( 3 (1850c^8 - 7725c^6 - 31099c^4 + 154507c^2 - 92547) c^2 + 7C_1 - 168318 \right) + 5C_1 + 117936 \right) - 18C_1 \right] \tag{53}
\]

\[
q_0 = \frac{1}{C_2} \left[ 12 \left( c^2 \left( (3610c^{10} - 15312c^8 - 58045c^6 + 296415c^4 - 187461c^2 + 14C_1 - 115371) c^2 - 10C_1 + 88452 \right) - 36C_1 \right] \tag{54}
\]

\[
p_1 = -\frac{1}{C_3} \left[ 2 \left( 95c^{18} - 5674c^{16} + 21090c^{14} + 95053c^{12} - 447746c^{10} + 243567c^8 + 194157c^6 - 37C_1 \left( 6c^8 - 125117c^4 + 5c^2 + 42 \right) \right) \right] \tag{55}
\]

\[
q_1 = -\frac{1}{C_3} \left[ 2 \left( 380c^{18} - 7716c^{16} + 22284c^{14} + 123114c^{12} - 529730c^{10} + 334098c^6 - 148 + C_1 \left( 24c^8 - 205631c^4 + 20c^2 + 1682386 \right) \right) \right] \tag{56}
\]

\[
p_2 = -\frac{1}{C_5} \left[ (95c^{10} - 324c^8 - 405c^6 + 378c^4 + 6C_1) C_4 \right] \tag{57}
\]

\[
q_2 = -\frac{4}{C_5} \left[ (95c^{10} - 324c^8 - 405c^6 + 378c^4 + 6C_1) C_4 \right], \tag{58}
\]

where

\[
C_1 = c^4 \left( 5c^4 - 26c^2 + 21 \right) \sqrt{13c^4 + 42c^2 + 9}
\]

\[
C_2 = (1404 - 1773c^2 + 6c^4 + 107c^6)(c^4(-2 + c^2)(-21 + 5c^2)) \tag{60}
\]

\[
C_3 = c^8(58968 - 117990c^2 + 62235c^4 - 4557c^6 - 3287c^8 + 535c^{10}) \tag{61}
\]

\[
C_4 = 2 \left( c^4 + 2c^2 - 3 \right) \tag{62}
\]

\[
C_5 = c^6(-29484 + 44253c^2 - 8991c^4 - 2217c^6 + 535c^8). \tag{63}
\]

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