Gravity in the brane-world for two-branes model with stabilized modulus

Takahiro Tanaka¹,² and Xavier Montes³

¹IAFE, Departament de Fisica, Universitat Autonoma de Barcelona, 08193 Bellaterra (Barcelona), Spain;
²Department of Earth and Space Science, Graduate School of Science Osaka University, Toyonaka 560-0043, Japan.

We present a complete scheme to discuss linear perturbations in the two-branes model of the Randall and Sundrum scenario with the stabilization mechanism proposed by Goldberger and Wise. We confirm that under the approximation of zero mode truncation the induced metric on the branes reproduces that of the usual 4-dimensional Einstein gravity. We also present formulas to evaluate the mass spectrum and the contribution to the metric perturbations from all the Kaluza-Klein modes. We also conjecture that the model has tachyonic modes unless the background configuration for the bulk scalar field introduced to stabilize the distance between the two branes is monotonic in the fifth dimension.
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I. INTRODUCTION

Recently, there has been a growing interest in considering extra-dimensions in non-trivial form. One important suggestion was done by Randall and Sundrum. They realized that if one considers 5-dimensional gravity with a negative cosmological constant bounded by two positive and negative tension branes, the induced gravity on the negative tension brane can possess a very small gravitational constant as compared with the energy scale introduced in the original Lagrangian. This fact potentially gives a solution to the hierarchy problem of explaining the extraordinary weakness of the gravitational coupling.

Later, the same authors pointed out that the model with the positive tension brane alone is also possible. In this case, the extension of the extra dimension is infinite. Nevertheless, the induced gravity on the brane can be expected to mimic Einstein gravity.

There are many discussions about the cosmology based on these scenarios. The behavior of gravity on these models has been also investigated by many authors. In this direction, an explicit method to deal with the linear order metric perturbations induced by the matter fields confined on the branes was developed in the paper by Garriga and Tanaka. In that paper, considering the zero mode truncation approximation, it was shown that the induced gravity on the branes becomes of the Brans-Dicke type in the two-branes model. On the other hand, Einstein gravity is recovered in the single-brane model.

However, the analysis of the two-branes model in Paper I was not complete. The stabilization mechanism of the distance between the two branes was not taken into account. Moreover, the Kaluza-Klein contribution was not estimated for the two-branes model. In this paper we consider a model with stabilization mechanism, and we confirm that the 4-dimensional Einstein gravity is recovered on both branes under the approximation of zero mode truncation. An approximate formula for the mass of the lowest massive mode in the scalar-type perturbations is also obtained. Furthermore, we present a method to evaluate the metric perturbations taking into account all the KK contributions under a contact interaction approximation. This approximation is valid if the source which excites the KK modes is smoothly distributed compared with the scale corresponding to the mass of the lowest massive mode.

This paper is organized as follows. In Sec. II we describe the background model that we consider, and derive the basic equations for the linear perturbations on it. In Sec. III we discuss the mass spectrum of the perturbations. We identify the mode functions for the massless degrees of freedom, and also derive the formula for the mass and the mode function of the lowest massive mode. In Sec. IV the mechanism to recover Einstein gravity is explained. This result is expected from the notion of the mass spectrum but its derivation is not so trivial when we consider the explicit construction of metric perturbations. We shall find that the contribution from the massive modes of the scalar-type perturbations must be also taken into account in some sense. For this purpose, we introduce a contact interaction approximation. In Sec. V we apply the same technique to the KK modes of the tensor-type perturbations to complete the description of the linear perturbations of this system. Section VI is devoted to summary.

II. MODEL AND BASIC EQUATIONS

We consider linear perturbations of the model proposed by Randall and Sundrum with the stabilization mechanism by Goldberger and Wise. The fields existing on the 5-dimensional spacetime (bulk) are the usual 5-dimensional gravity with a negative cosmological constant Λ, and a 5-dimensional scalar field ϕ introduced to stabilize the distance between the two branes. We de-
note the 5-dimensional gravitational constant by \( G_5 \). The unperturbed metric is supposed to take the form,

\[
d s^2 = a^2(y) \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,
\]

where \( \eta_{\mu\nu} \) is the 4-dimensional Minkowski metric with \((-+++)\) signature. We use \( \gamma_{\mu\nu} = a^2(y) \eta_{\mu\nu} \) to raise 4-dimensional tensor indices. The \( y \)-direction is bounded by two branes located at \( y = y^{(\pm)} \). On these two branes, \( \mathbb{Z}_2 \)-symmetry is imposed. The Lagrangian for the bulk scalar field is

\[
\mathcal{L} = -\frac{1}{2} g^{ab} \phi_{,a} \phi_{,b} - V_B(\phi) - \sum_{\sigma = \pm} V^{(\sigma)}(\phi) \delta(y - y^{(\sigma)}).
\]

(2.2)

For most of the present analysis, we do not need to specify the explicit form of the potentials \( V_B(\phi) \) and \( V^{(\pm)}(\phi) \). In the bulk, the background scale factor and the scalar field, \((a, \phi_0)\), must satisfy

\[
\bar{H}(y) = \frac{\kappa}{3} \dot{\phi}_0^2(y),
\]

\[
H^2(y) = \frac{\kappa}{6} \left( \frac{1}{2} \dot{\phi}_0^2(y) - V_B(\phi_0(y)) - \kappa^{-1} \Lambda \right),
\]

\[
\phi_0(y) + 4H(y) \dot{\phi}_0(y) - V_B'(\phi_0(y)) = 0,
\]

(2.3)

where \( H(y) := \dot{a}(y)/a(y) \approx -\sqrt{-\Lambda} / 6 \) and \( \kappa = 8\pi G_5 \). Ordinary matter fields reside on both branes, and the values of the 4-dimensional vacuum energy on both branes are adjusted to realize a static background configuration. This way of model construction is different from the standard approach which assumes the model potential and the values of the vacuum energy on the branes from the very beginning. In the present approach, we obtain a constraint on the model potential by fixing the distance between the two branes. Conversely, in the standard approach, the stabilized distance is determined for each given model (although some tuning of one of the model parameters is necessary to realize a static configuration). We call the brane at \( y = y^{(+)}(y^{(-)}) \) the positive (negative) tension brane, and adopt the convention \( y^{(+)} < y^{(-)} \).

We first consider the metric perturbation \( \delta(ds^2) = h_{ab} dx^a dx^b \) and the scalar field perturbation \( \delta \phi \) in the bulk. We are using the convention that Latin indices run through 0, 1, 2, 3, 5, while Greek indices run through 0, 1, 2, 3. In the bulk, we can always impose the “Newton gauge” condition,

\[
h_{55} = 2\phi^N, \quad h_{5\mu} = 0, \quad h_{\mu\nu} = h^{(TT)}_{\mu\nu} - \phi^N \gamma_{\mu\nu},
\]

\[
\delta \phi = \frac{3}{2\kappa \phi_0} [\partial_y + 2H] \phi^N,
\]

(2.4)

where \( h^{(TT)}_{\mu\nu} \) satisfies the transverse-traceless condition, and \( \phi^N \) is the 5-dimensional “Newtonian potential”.

The equation for \( h^{(TT)}_{\mu\nu} \) in the bulk is given by

\[
\left[ \frac{1}{a^2} \Box^{(4)} + \hat{L}^{(TT)} \right] h^{(TT)}_{\mu\nu} = 0,
\]

(2.5)

where \( \Box^{(4)} \) is the 4-dimensional d’Alembertian operator with respect to \( \eta_{\mu\nu} \), and we have introduced the operator

\[
\hat{L}^{(TT)} := \frac{1}{a^2} \partial_y a^4 \partial_y \frac{1}{a^2}.
\]

(2.6)

For the scalar-type perturbations, we obtain

\[
\left[ \Box^{(4)} + \hat{L}^{(N)} \right] \phi^N = 0,
\]

(2.7)

where \( \hat{L}^{(N)} \) is the operator defined by

\[
\left[ \Box^{(4)} + \hat{L}^{(N)} \right] := a^2 \partial_y^2 \partial_y a^2 - \frac{2\kappa}{3} a^2 \partial_y^2.
\]

(2.8)

As discussed in Paper I, the \( y \) = constant hypersurface does not correspond to the location of the branes in this gauge. Following it, we introduce two Gaussian normal coordinate systems near both branes, respectively. We denote quantities in these coordinate systems by associating a bar, such as \( \bar{y} \).

Then, the junction condition for the bulk scalar field is given by \( \pm 2 \delta \phi = V''(\bar{y})(\bar{y}) \) at \( \bar{y} = \bar{y}^{(\pm)} \). Hence, its perturbation become

\[
\pm 2\delta \phi = V''(\bar{y})(\bar{y}) \delta \bar{y}, \quad (\bar{y} = \bar{y}^{(\pm)}). \quad (2.9)
\]

The junction condition for the metric perturbation is

\[
\pm (\partial_y - 2H) \bar{h}_{\mu\nu} = -\kappa \left[ T_{\mu\nu} - \frac{1}{3} \gamma_{\mu\nu} T \right]^{(\pm)}
\]

\[
\mp 2\kappa \gamma_{\mu\nu} \phi_0 \partial_y \delta \bar{y}, \quad (\bar{y} = \bar{y}^{(\pm)}). \quad (2.10)
\]

Here, we have introduced the energy momentum tensor of the matter fields confined on the branes, \( T^{(\pm)}_{\mu\nu} \), and \( T^{(\pm)} := \gamma^{\mu\nu} T_{\mu\nu}^{(\pm)} \). \( T^{(\pm)}_{\mu\nu} \) satisfies the 4-dimensional conservation law \( T^{(\pm)}_{\mu\nu} = 0 \). Notice that there appears a contribution from the perturbation of the scalar field \( \phi \) which was not present in the models discussed in Paper I.

Whereas the junction conditions have been easily derived using Gaussian normal coordinates, the equations of motion for the perturbations are simpler in the “Newton gauge”. So we need to consider the gauge transformation which relates normal coordinates and the “Newton gauge”. This is given by \( h_{ab} = \bar{h}_{ab} + \xi_{ab} + \xi_{b/a} \) with

\[
\xi^{(\pm)}_a = \int_{y^{(\pm)}}^y \phi^N(y') dy' = \int_{y^{(\pm)}}^y \phi^N(y') dy' + \xi^{(5)}_a,
\]

\[
\xi^{(\pm)} = -\int_{y^{(\pm)}}^y \gamma^{\mu\nu}(y')dy' \int_{y^{(\pm)}}^y \phi^N(y'') dy''
\]

\[
- \int_{y^{(\pm)}}^y \gamma^{\mu\nu}(y')dy' \int_{y^{(\pm)}}^y \phi^N(y'') dy'' + \xi^{(5)}, \quad (2.11)
\]

\[
= \int_{y^{(\pm)}}^y \gamma^{\mu\nu}(y')dy' \int_{y^{(\pm)}}^y \phi^N(y'') dy'' + \xi^{(5)}_a, \quad (2.12)
\]
where $\hat{\xi}^{5}_{(\pm)}$ and $\hat{\xi}^5_{(-)}$ are independent of $y$. Then, we have
\begin{equation}
\begin{aligned}
\delta \bar{\varphi}(y) &= \delta \varphi(y) - \bar{\varphi}_0(y) \left[ \int_{y'_{(\pm)}} y_0(y')dy' + \hat{\xi}^{5}_{(\pm)} \right], \\
\bar{h}_{\mu\nu}(y) &= h_{\mu\nu}(y) + 2a^2(y) \int_{y'_{(\pm)}} y_0(y')dy' + \phi^N_{\mu\nu}(y'_{(\pm)})dy' \\
&= -2H\gamma_{\mu\nu}(y) \int_{y'} \phi^N(y')dy'.
\end{aligned}
\end{equation}

Here we should stress that the arguments in the l.h.s. are not $\bar{y}$ but $y$.

Combining Eqs. (2.10) and (2.13), the junction condition in the "Newton gauge" for the $TT$ part is obtained as
\begin{equation}
\pm(\theta - 2H)h_{\mu\nu}^{(TT)} = -\kappa \Sigma_{\mu\nu}^{(\pm)}, \quad (y = y^{(\pm)}),
\end{equation}
where we have defined
\begin{equation}
\Sigma_{\mu\nu}^{(\pm)} := T_{\mu\nu} - \frac{1}{4} \gamma_{\mu\nu} T^\pm + \frac{2}{\kappa} \left[ \hat{\xi}^{5}_{(\pm),\mu\nu} - \frac{1}{4} \gamma_{\mu\nu} \hat{\xi}^{5}_{(\pm)} \right].
\end{equation}

Combining Eq. (2.14) with Eq. (2.5), we obtain the equation for $h_{\mu\nu}^{(TT)}$,
\begin{equation}
\left[ \frac{\Box^{(4)}}{a^2} + \hat{L}^{(TT)} \right] h_{\mu\nu}^{(TT)} = -2\kappa \sum_{\sigma = \pm} \Sigma^{(\sigma)}_{\mu\nu} \delta(y - y^{(\sigma)}).
\end{equation}

The resulting $h_{\mu\nu}^{(TT)}$ is automatically consistent with the traceless condition, but the transverse condition gives us the equation which determines $\hat{\xi}^{5}_{(\pm)}$,
\begin{equation}
\frac{1}{a^2_{(\pm)}} \Box^{(4)} \hat{\xi}^{5}_{(\pm)} = \pm \frac{\kappa}{6} T_{\pm},
\end{equation}
where we have defined $a_{(\pm)} := a(y^{(\pm)})$. The set of Eqs. (2.16) and (2.17) is exactly the same that was obtained in Paper I once we specialize the background bulk geometry to pure anti de Sitter.

Using Eq. (2.17), it can be easily seen that the trace part of the metric junction condition is trivially satisfied.

The remaining junction condition is the one for the scalar field, Eq. (2.5). After some computations, it reduces in the "Newton gauge" to
\begin{equation}
\pm 2 \frac{2\kappa}{3} (\delta \varphi - \bar{\varphi}_0 \hat{\xi}^{5}_{(\pm)}) = \frac{c_{(\pm)}}{a^2 \bar{\varphi}_0} \Box^{(4)} \phi^N, \quad (y = y^{(\pm)}),
\end{equation}
where we have defined
\begin{equation}
c_{(\pm)} := \frac{2}{V^{(\pm)}(\pm)} \left[ \frac{\hat{\xi}^{5}_{(\pm),\mu\nu}}{\mp \varphi_0} \right].
\end{equation}

Combining Eq. (2.18) with Eq. (2.7), we obtain the equation for $\phi^N$,
\begin{equation}
\hat{L}(\phi^N) \phi^N - \sum_{\sigma = \pm} \frac{4\kappa a^2}{3} \hat{\xi}^{5}_{(\sigma)} \delta(y - y^{(\sigma)}) = -\Box^{(4)} \phi^N \left( 1 + \sum_{\sigma = \pm} 2c_{(\sigma)} \delta(y - y^{(\sigma)}) \right).
\end{equation}

### III. MASS SPECTRUM

#### A. Zero modes

Let us first consider the solution of the source free equations by setting $T_{\mu\nu}^{(\pm)} = 0$. We will first consider the zero eigenmodes of $\Box^{(4)}$, which are the most important because they correspond to the 4-dimensional massless field responsible for the propagation of a long range force.

If we set $\Box^{(4)} = 0$ in Eq. (2.16), the solution for $h_{\mu\nu}^{(TT)}$ is
\begin{equation}
h_{\mu\nu}^{(TT)} = \hat{h}_{\mu\nu}^{(1)}(x^\rho) a^2(y) + \hat{h}_{\mu\nu}^{(2)}(x^\rho) a^2(y) \int_{y'} \frac{dy'}{a^4(y')}.
\end{equation}

where $\hat{h}_{\mu\nu}^{(1)}$ and $\hat{h}_{\mu\nu}^{(2)}$ are 4-dimensional $TT$ tensors independent of $y$ satisfying $\Box^{(4)} \hat{h}_{\mu\nu}^{(i)} = 0$. Furthermore, the junction condition (2.14) gives
\begin{equation}
\hat{h}_{\mu\nu}^{(2)} = -2a^2 \hat{\xi}^{5}_{(\pm),\mu\nu}.
\end{equation}

Hence, there is no extra constraint on $\hat{h}_{\mu\nu}^{(1)}$, but $\hat{h}_{\mu\nu}^{(2)}$ must be written as the second derivative of a scalar function. The former mode corresponds to 4-dimensional gravitational wave perturbations. Although there seem to exist 5 independent degrees of freedom in this mode, three of them are pure gauge. On the other hand, the latter mode should be classified as scalar-type perturbations. If we forget about the stabilization mechanism, this mode corresponds to the mode called radion in Ref. [2].

Notice that Eq. (3.2) gives a relation between $\hat{\xi}^{5}_{(\pm)}$ and $\hat{\xi}^{5}_{(-)}$,
\begin{equation}
a^2_{(\pm)} \hat{\xi}^{5}_{(\pm)} = a^2_{(-)} \hat{\xi}^{5}_{(-)}.
\end{equation}

In the present case with stabilization, $\hat{\xi}^{5}_{(\pm)}$ also appears in the junction condition for the scalar-type perturbations. Hence, the scalar field perturbation must also be chosen to be compatible with this condition. Here we should note that this relation holds only when we restrict our considerations to zero modes. In the general case, which will be considered later, there is no reason for such a relation to be satisfied.

Let us now consider the solution of the zero eigenvalue scalar-type perturbation. One solution is
The boundary condition on the negative tension brane is given by \[ \partial_z u(y) = 0 \] for simplicity, here we set \[ u(y) = e^{-y/\ell} \]. However, this solution can be transformed to nothing by a gauge transformation with parameters \( \xi^0 = a^{-2} f^{(1)}/2, \xi^i = -f^{(1)} \int^y a^{-2}(y') \gamma^{0i}(y')dy' \).

The other solution is

\[
\phi^N = \left(1 - \frac{2H}{a^2} \int^y a^2(y')dy' \right) f^{(2)}(x^\nu), \\
\delta \varphi = -f^{(2)}(x^\nu) \frac{\dot{\varphi}_0}{a^2} \int^y a^2(y')dy'.
\]

However, this mode is not compatible with condition \(2.3\). Hence, if we include the stabilization mechanism, no physical massless mode is present in the scalar-type perturbation spectrum, in contrast with the case discussed in Paper I.

B. Massive modes

Let us now consider the lowest massive mode. They are also important because they give the leading order correction to the zero mode truncation approximation. Furthermore, if tachyonic modes are present, such a model must be rejected.

To estimate the lowest mass eigenvalue in the general case, one needs numerical calculations. In order to obtain analytical approximations, here we assume that the effect of the bulk scalar field back reaction to the background geometry is small. Namely, we assume

\[
\frac{|\dot{H}|}{H^2} = \frac{\kappa \varphi_0^2}{3H^2} \ll 1.
\]

For the metric, we can use the pure anti-de Sitter form

\[ a(y) = e^{-y/\ell}. \]

For simplicity, here we set

\[ y^{(+)} = 0, \quad y^{(-)} = d. \]

First we consider the tensor-type perturbations. The expression for the mode functions \( u_i(y) \) for the operator \( L^{(TT)} \) satisfying the junction condition \[ \partial_y - 2H u_i(y) = 0 \] on the positive tension brane is found in Refs. \(2.18\). Denoting the eigenvalue corresponding to \( u_i(y) \) by \( m_i^2 \), the mode function is given in terms of Bessel functions, \( u_i(y) \propto \{ J_1(m_i\ell e^{d/\ell}) - Y_1(m_i\ell e^{d/\ell}) \}/J_0(m_i\ell e^{d/\ell}) \). The boundary condition on the negative tension brane reduces to

\[ \{ J_1(m_i\ell e^{d/\ell}) Y_1(m_i\ell e^{d/\ell}) - Y_1(m_i\ell) J_1(m_i\ell e^{d/\ell}) \} = 0, \]

and determines the discrete eigenvalues \( m_i \). For small \( m_i \), this condition becomes \( J_1(m_i\ell e^{d/\ell}) \approx 0 \), and hence the eigenvalues are given by \( m_i \approx e^{-d/\ell} j_i \ell^{-1} \), where \( j_i \) is the \( i \)-th zero-point of \( J_1 \). Hence the physical mass of the lowest massive KK mode on the positive tension brane is given by \( \approx 3.8 \ell^{-1} \) while that on the negative tension brane by \( \approx 3.8 \ell^{-1} \).

Next, let us consider the scalar-type perturbations. When we discuss the scalar-type perturbations, we should not assume an scale factor of the form \(3.7\) from the beginning, because the non-vanishing second or higher derivative of \( a(y) \) can be important. Hence we will derive formula \(3.19\) for the lowest mass eigenvalue without assuming \(3.7\). Only to obtain an estimate of this expression, we will use the scale factor \(3.7\).

To discuss scalar-type perturbations, it is convenient to introduce a new variable defined by

\[ q := \frac{3a^2}{2\kappa A} \phi^N, \]

where a prime ′ denotes derivative with respect to the conformal coordinate \( z \) defined by \( dz = a(y)^{-1}dy \), and we have introduced \( A = \pm a^{1/2} \varphi_0^2 \). The signature in the definition of \( A \) is chosen so that \( A \) is continuous on the branes. The junction condition for \( q \) is obtained from \(2.18\),

\[ 2[\partial_z + (A'/A)]q = \pm 2A \delta^5_{(+)}(y = y^{(\pm)}). \]

The perturbation equation in terms of \( q \) becomes

\[
\left[ \nabla^{(4)} + \partial_z + A \left( \frac{1}{A} \right)' - \frac{2\kappa}{3} \varphi_0^2 \right] q = \sum \pm 2A \delta^5_{(+)} \delta(y - y^{(\pm)}).
\]

Integrating the above equation once at the vicinity of the branes, we correctly reproduce the junction condition \(3.11\).

To obtain the solution of Eq. \(3.12\), we will construct the Green function for the differential operator appearing in it. It is given by

\[ G_q = -\int \frac{dk}{(2\pi)^4} e^{ik_z\Delta x^0} \sum_i \frac{q_i(z)q_i(z')}{m_i^2 + k^2}, \]

where \( m_i \) is the mass eigenvalue. The mode functions satisfy

\[ \left[ \partial_z^2 - A \left( \frac{1}{A} \right)' - \frac{2\kappa}{3} \varphi_0^2 \right] q_i = -m_i^2 q_i, \]

with the boundary condition \[ \partial_z + (A'/A)]q_i = 0, \] at \( y = y^{(\pm)} \).

In the weak back reaction case, the term \( \delta U := 2\kappa \varphi_0^2/3 \) in Eq. \(3.14\) is assumed to be small. If we
perturbatively expand the mode function and the mass eigenvalue with respect to powers of $dU$ like $q_i = q_0^{(i)} + \cdots$ and $m_i = m_0^{(i)} + m_1^{(i)} + \cdots$, the lowest eigenmode for the unperturbed system is trivially given by

$$q_0^{(0)} = \frac{N}{A}$$

(3.15)

with $(m_0^{(0)})^2 = 0$, where $N$ is a normalization constant.

It is straightforward to obtain the next order correction. We find

$$q_0^{(1)} = \frac{N}{A} \int \frac{d^2 z'}{A^2} \int d^\prime \frac{d^\prime z''}{A^2} \left( \frac{2 \kappa}{3} \varphi_0'' - (m_0^{(1)})^2 \right).$$

(3.16)

In terms of $\delta \varphi$, the above first order correction is given by

$$\delta \varphi_0^{(1)} = N \varphi_0(y) \int_{y^{(+)}(\pm)}^y dy' \left( \frac{2 \kappa}{3\alpha^2} \frac{(m_0^{(1)})^2}{a^4(y')^2 \varphi_0(y')} \right).$$

(3.17)

The junction condition on the positive tension brane is already imposed due to the choice of the integration constant. Here, for simplicity, we have taken the $\epsilon(\pm) \to 0$ limit.

This expression is seen to be regular even when $\varphi_0$ vanishes at some points. Denoting the point of vanishing $\varphi_0$ by $y_0$, we can show that $a^4 \varphi_0^2$ can be expanded around this point as

$$a^4 \varphi_0^2 = [a^4(V_B')^2]_{y=y_0} (y-y_0)^2 + \alpha^2 (y-y_0)^4 + \cdots,$$

(3.18)

where $\alpha^2 := \frac{1}{3} (2H^2 + V''_B)|_{y=y_0}$. From this, it is easy to see that the expression for $\delta \varphi_0^{(1)}$ is regular at $y = y_0$.

The junction condition (3.11) on the negative tension brane gives the formula for the lowest mass eigenvalue,

$$m_2 \approx \frac{2 \kappa}{3} \left[ \int_{y^{(+)}(\pm)}^y \frac{dy}{a^2} \right] \left[ \int_{y^{(+)}(\pm)}^y \frac{dy}{a^4 \varphi_0^2} \right].$$

(3.19)

This expression is positive definite if there is no point at which $\varphi_0$ vanishes. However, when $\varphi_0$ vanishes, the integral in the denominator changes its signature at that point. Hence, $m_2^2$ can be negative. As seen from Eq. (3.18), the integrand does not have residue at this point. Hence the integral is independent of the way of modification of the integration path. It is expected that the dominant contribution to this integral comes from the region near the point $y_0$, where $\varphi_0$ vanishes. By substituting equation (3.18), we can approximately evaluate the denominator in Eq. (3.19) as

$$\approx \int dy (y-y_0)^2 + \alpha^2 (y-y_0)^4 = \left[ \frac{\pi \alpha}{a^4(V_B')^2} \right]_{y=y_0},$$

which becomes negative. Although we cannot give a proof here, this seems to be the case in general. Here we simply conjecture it. If tachyonic modes appear, they mean the breakdown of the model. Hence, we do not consider the case in which $\varphi_0$ vanishes at some point, and hereafter we assume that $\varphi_0$ has a definite sign.

Without explicitly specifying any model, we can make a crude estimate of the lowest mass eigenvalue. To evaluate the numerator in the r.h.s of Eq. (3.11), we can use the form (3.7) for the scale factor $a$. To evaluate the denominator, we use the following fact. Besides some exceptional cases, $a^4 \varphi_0^2$ will have a minimum (cases with no minima will be discussed later). Then $\varphi_0 + 2H \varphi_0 = V_B - 2H \varphi_0 = 0$, and accordingly the integrand has a rather sharp peak. We denote this point by $y_c$. At this point, the second derivative of $a^4 \varphi_0^2$ is evaluated as $(8H^2 + 2V''_B - 4H)a^4 \varphi_0^2$. Hence, we can approximate the integral like $\int (1/\alpha^4 \varphi_0^2) dy \approx (4H^2/a^4(V_B')^2)|_{y=y_c} \int \exp[-(4H^2 + V''_B)y] dy = \frac{\varphi_0^2}{4} dy$. Under these approximations, the formula for the mass of the lowest eigenmode is obtained as

$$m_2^2 \approx \frac{\kappa e^{2d/\ell^2}}{6\sqrt{\pi}} \left[ e^{-4y_B/Y''} \left( \frac{V''_B}{4} \right)^{1/2} \right]_{y=y_c}. \quad (3.20)$$

To proceed further, we consider the specific model for the bulk potential $V_B(\varphi)$ discussed in Ref. [18],

$$V_B(\varphi) = \frac{M^2 \varphi^2}{2}. \quad (3.21)$$

For this model, the background solution for the bulk scalar field in the weak back reaction case is already given by

$$\varphi_0 = B_1 e^{\nu_1 y} + B_2 e^{\nu_2 y}, \quad (3.22)$$

where $\nu_1 = 2e^{-1} - \sqrt{4\ell^{-2} - M^2}$, $\nu_2 = 2e^{-1} - \sqrt{4\ell^{-2} + M^2}$ and

$$B_1 \approx e^{-\nu_1 d} (\varphi(-) - \varphi(+)) e^{\nu_1 d}, \quad B_2 \approx \varphi(+) - \varphi(-) e^{-\nu_1 d}. \quad (3.23)$$

$\varphi(+)$ and $\varphi(-)$ are the values of $\varphi_0$ on the positive and the negative tension branes, respectively.

Let us briefly elaborate on the stabilization distance. For simplicity, we will assume that the ratio $\varphi(+)/\varphi(-)$ is not extremely large, and also that all the input scales are similar, i.e. $\kappa \approx \ell^3$. The condition for weak back reaction is

$$\varphi_0^2 \ll \frac{1}{\kappa \ell^2}. \quad (3.24)$$

Since $\varphi^2$ does not have its maximum in the bulk, it is sufficient to consider the conditions for weak back reaction on the boundaries. Then, we can evaluate the values of $\varphi_0$ on boundaries,

$$\varphi_0(\gamma^{(+)}) \approx \nu_2 \varphi(+),$$
\[ \dot{\phi}_0(y^-) \approx \nu_1 \phi_0 - 4 \ell^{-1} \sqrt{1 + (M^2 \ell^2/4)} e^{-d/\ell} \phi_0. \]  

(3.25)

The conditions for weak back reaction on both branes become

\[ \left( \frac{\nu_2 \dot{\rho}(\pm)}{4 \ell^{-5/2}} \right)^2 \ll \frac{\kappa^3}{\nu} \]  

and

\[ \left( \frac{\nu_1 \dot{\rho}(\pm)}{4 \ell^{-5/2}} \right)^2 \left( 1 - \frac{4 e^{-d/\ell} \nu_1}{\nu_2} \phi_0 \sqrt{1 + \frac{M^2 \ell^2}{4}} \right) \ll \frac{\kappa^3}{\nu}. \]  

(3.26)

The condition on the positive tension brane is satisfied by choosing (case A) \( \phi(\pm) \ll \ell^{-3/2} \) with \( M^2 \lesssim \ell^{-2} \) or (case B) \( M^2 \ll \ell^{-2} \) with \( \phi(\pm) \gtrsim \ell^{-3/2} \).

In case A, the condition on the negative tension brane is automatically satisfied. In this case, the stabilization distance is sensitive to the change of the values of the vacuum energy on the branes. Hence, without specifying the complete model, we cannot estimate the stabilization distance. If we consider the case in which \( M^2 \ell^2 \) is not small, we find that it is natural to assume that the signature of \( \dot{\phi}_0 \) is different from that of \( \dot{\phi}_0 \). If \( \phi(+) \) and \( \phi(-) \) have the same signature, the ratio between them must be chosen to be extremely large to find a node-less solution of \( \dot{\phi}_0 \) with sufficiently large brane separation \( d \).

In the case that \( \phi(+) \) and \( \phi(-) \) have different signature, the solution of \( \dot{\phi}_0 \) becomes node-less for any choice of parameters.

Case B is the case discussed in Ref. [18]. In this case, to satisfy the condition for weak back reaction on the negative tension brane,

\[ \frac{d}{\ell} \approx \frac{1}{-\nu_2 \ell} \ln \left( \frac{4 \ell^{-1} \phi(+) \sqrt{1 + \frac{M^2 \ell^2}{4}}}{\nu_1 \phi(-)} \right) \]  

(3.27)

is required. Hence, \( \phi(+) \) and \( \phi(-) \) must have the same signature. To realize a sufficiently large value of \( d/\ell \), it is necessary that \( |\phi(+)\phi(-)| \) is slightly larger than \( |\phi(-)| \).

We take the small \( M^2 \) limit, the above expression for the stabilization distance coincides with that obtained in Ref. [18].

Let us now return to the computation of \( m_0^2 \). First we consider the case in which \( a^4 \dot{\phi}_0^2 \) has a minimum and the estimate (3.20) is valid. As we shall see below, case B is exceptional in this sense. From the condition that \( V'_0 - 2H \dot{\phi}_0 = 0 \) at \( y = y_c \), we have \( e^{(\nu_2 - \nu_1)} y_c = \nu_1 B_1 / \nu_2 B_2 \). Substituting this estimate of \( y_c \) into (3.20), we obtain

\[ m_0^2 \approx 8 \kappa M^2 e^{-d/\ell} \sqrt{1 + (M^2 \ell^2/4)} / 3 \sqrt{\pi} (-B_1 B_2). \]  

(3.28)

To solve the hierarchy problem on the negative tension brane, we need to set \( e^{d/\ell} \sim 10^{19} \text{(GeV)}/10^{16} \text{(GeV)} = 10^{16} \). Hence, \( d/\ell \approx 37 \). Therefore, it is rather natural to suppose that \( M^2 \ell d \) is larger than unity. If \( |\phi(+)\phi(-)| \) is not much larger than \( |\phi(-)| \), as assumed, we can approximate \( -B_1 B_2 \approx |\phi(+)\phi(-)\phi(-)| e^{-d/\ell} \).

In this case, the mass eigenvalue \( m_0^2 \) becomes

\[ O(\kappa|\phi(+)\phi(-)| M^2 e^{-2d/\ell} \sqrt{1 + (M^2 \ell^2/4)}) \]  

and the physical squared mass on the negative tension brane becomes

\[ O(\kappa|\phi(+)\phi(-)| M^2 e^{-2d/\ell} \sqrt{1 + (M^2 \ell^2/4)}) \]  

while that on the positive tension brane is the same as \( m_0^2 \). As for \( |\phi(+)\phi(-)| \), they must be smaller than \( \ell^{-3/2} \) for the approximation of weak back reaction to be valid. Also, there is a factor \( M^2 e^{-2d/\ell} \sqrt{1 + (M^2 \ell^2/4)} \) which takes its maximum value \( (0.2/\ell)^2 \) at \( M \approx 0.33 \ell^{-1} \). Hence, the mass scale on the negative tension brane tends to be smaller than the typical background energy scale \( \ell^{-1} \).

Finally, we consider the special case in which \( a^4 \dot{\phi}_0^2 \) in Eq. (3.19) has no minima. This happens when one of the terms in \( \dot{\phi}_0 = \nu_1 B_1 e^{\nu_1 y} + \nu_2 B_2 e^{\nu_2 y} \) can be totally neglected. This condition becomes

\[ |\phi(-) - \phi(+) e^{d/\ell}| \ll \frac{\nu_2}{\nu_1} |\phi(-)|, \]  

(3.29)

or

\[ |\phi(+) - \phi(-) e^{-d/\ell}| \ll \frac{\nu_1}{\nu_2} |\phi(+)|. \]  

(3.30)

In the former case we can set \( B_1 \approx 0 \), and in the latter case we can set \( B_2 \approx 0 \). Then, for the former case, formula (3.19) gives

\[ m_0^2 = \frac{4 \kappa}{3} \nu_2^2 \phi(+) e^{-2d/\ell} \left[ \frac{\sqrt{1 + (M^2 \ell^2/4)}}{1 - e^{-4 \sqrt{1 + (M^2 \ell^2/4)} d/\ell}} \right]. \]  

(3.31)

It is easy to see that case B corresponds to this case. The model discussed in Sec. 4.1 of Ref. [19] with negative small value of \( b \) also corresponds to this case (for the definition of \( b \), see Ref. [19]). The factor in the square brackets is almost unity. Substituting the approximation \( \nu_2 \approx -M^2 \ell^2/4 \), we recover the result obtained in [18]. Note that their \( M^3 \) and \( k \) are our \( 1/(4 \kappa) \) and \( \ell^{-1} \), respectively.

For the latter case, we have

\[ m_0^2 = \frac{4 \kappa}{3} \nu_1^2 \phi(-) e^{-2d/\ell} \left[ \frac{\sqrt{1 + (M^2 \ell^2/4)}}{e^{4 \sqrt{1 + (M^2 \ell^2/4)} d/\ell - 1}} \right]. \]  

(3.32)

The model discussed in Sec. 4.1 of Ref. [19] with positive small value of \( b \) corresponds to this case. To keep the mass sufficiently large, we have to choose \( |\phi(+)\phi(-)| \) extremely large, which is not compatible with the weak back reaction condition.

IV. RECOVERY OF EINSTEIN GRAVITY

As we have shown in the preceding section, in two-brane models with stabilization mechanism, a physical massless mode is absent in the scalar-type perturbation spectrum, which is different from what happens in the
case without stabilization mechanism discussed in Paper I. This fact indicates that the resulting 4-dimensional effective gravity can resemble Einstein gravity at linear order. Let us show how it can be recovered.

To compute the induced metric on the branes, it is convenient to transform back to Gaussian coordinates. Thus, applying “minus” the gauge transformation (2.12), the induced metric on eachbrane is given by
\[
\tilde{h}_{\mu\nu}^{(\pm)} = h_{\mu\nu}^{(0)}(y^{(\pm)}) + h_{\mu\nu}^{(KK)}(y^{(\pm)}) - \gamma_{\mu\nu} \left( \phi^N(y^{(\pm)}) + 2H\tilde{\xi}^5(y^{(\pm)}) \right) - \left( \xi^5_{(\mu\nu)} + \xi^5_{\nu\mu} \right). \tag{4.1}
\]
Here we have decomposed \(h_{\mu\nu}^{(TT)}\) into two parts, the zero mode contribution \(h_{\mu\nu}^{(0)}\) and the KK contribution \(h_{\mu\nu}^{(KK)}\). The last term represents the residual 4-dimensional gauge transformation. As we have already noted, Eq. (2.16) and Eq. (2.17) determining \(h_{\mu\nu}^{(TT)}(y^{(\pm)})\) and \(\tilde{\xi}^5(y^{(\pm)})\) are essentially the same as the ones obtained in Paper I for the case without stabilization mechanism. There, it was found that the induced gravity approximated by the zero mode truncation becomes of the Brans-Dicke type [11]. Here we only quote the result for \(h_{\mu\nu}^{(0)}(y^{(\pm)})\) up to 4-dimensional gauge transformation,
\[
h_{\mu\nu}^{(0)}(y^{(\pm)}) = \left[ \frac{\Box^{(4)}}{a^2} \right]^{-1} \sum_{\sigma = \pm} 16\pi G(\sigma) \left[ T_{\mu\nu} - \gamma_{\mu\nu} \frac{T}{3} \right]^{(\sigma)}, \tag{4.2}
\]
where \(8\pi G^{(\pm)} := \kappa N a^2\) and
\[
N := \left[ 2 \int_{y^{(\pm)}} a^2 dy \right]^{-1}. \tag{4.3}
\]
Hence, the only possible source of an additional contribution is that from \(h_{\mu\nu}^{(KK)}\) or \(\phi^N(y^{(\pm)})\). Although we have just shown that there are no massless degrees of freedom in the scalar-type perturbations, we will find that the contribution from \(\phi^N(y^{(\pm)})\) gives the correct long range force required to recover Einstein gravity.

First we consider the weak back reaction case discussed in Sec. III-B. From the normalization condition of \(q^0(0)\) we obtain \(N^2 = \left[ 2 \int_0^d dz (1/A)^2 \right]^{-1}\). Comparing it with the formula for the mass (3.19), we find a simple relation
\[
m_0^2 \approx \frac{4\kappa N^2}{3} \int_0^d \frac{dy}{a^2} \approx \frac{2\kappa L N^2}{3}(e^{2d/\ell} - 1). \tag{4.4}
\]
Then, taking into account the contribution from the modes with the lowest mass eigenvalue alone, the Green function for \(q\), Eq. (3.13), is approximated by
\[
G_q \approx - \int \frac{d^4k}{(2\pi)^4} \frac{3e^{ik_\mu A^\mu} m_0^2 A^{-1}(y) A^{-1}(y')}{2\kappa L (e^{2d/\ell} - 1)} \times \left[ \delta^4(\Delta x^\mu) - \frac{d^4k}{(2\pi)^4} \frac{k^2 e^{ik_\mu A^\mu} m_0^2}{m_0^2 + k^2} \right]. \tag{4.5}
\]
Using this Green function, and with the aid of (5.10) and (5.12), we find that \(\phi^N\) is given by
\[
\phi^N \approx \frac{2\ell^{-1}}{a^2(e^{2d/\ell} - 1)} \left[ (\xi^{(5)}_\mu + \xi^{(5)}_{\mu\nu}) - \frac{k}{6} m_0^2 - \Box^{(4)} \right]^{-1} (a^2_{(+)T} T^{(+)}) + a^2_{(-)T} T^{(-)} \tag{4.6}
\]
Only the first term inside the square brackets gives a long range contribution to the induced metric. The propagation of this force is essentially due to \(\xi^{(5)}_{\pm}\). The source for the second term exactly becomes the force due to the energy momentum tensor of the ordinary matter field. Hence, the gravitational field is propagated through the scalar-type perturbations, we will find that the force required to recover Einstein gravity.

\[
\frac{\Box^{(4)}}{a^2} \tilde{h}_{\mu\nu}^{(\pm)} = \sum_{\sigma = \pm} 16\pi G^{(\sigma)} \left[ T_{\mu\nu} - \gamma_{\mu\nu} \frac{T}{2} \right]^{(\sigma)}. \tag{4.7}
\]
where we have used the fact that \(8\pi G^{(\pm)} = \kappa \ell^{-1} a^2 (1 - e^{-2d/\ell})\) in the weak back reaction case. Substituting this into Eq. (4.1), and using Eq. (4.2) and Eq. (4.17), the zero mode truncation reproduces the formula for the linearized Einstein gravity,
\[
\frac{\Box^{(4)}}{a^2} \tilde{h}_{\mu\nu}^{(\pm)} = \sum_{\sigma = \pm} 16\pi G^{(\sigma)} \left[ T_{\mu\nu} - \gamma_{\mu\nu} \frac{T}{2} \right]^{(\sigma)}. \tag{4.8}
\]
In the above derivation of Eq. (4.7), we have used several approximations. This derivation has the merit of having a result with a rather easy intuitive interpretation. The gravitational field is propagated through the massless field \(\xi^{(5)}_{\pm}\). At a point far from the source \(T_{\mu\nu}\), \(\xi^{(5)}_{\pm}\) generates a cloud of metric perturbations through the interaction with the massive KK modes. However, the above approximate derivation is not completely satisfactory because some aspects of general relativity are already tested with very high precision. Hence, we present an alternative and more complete treatment below.

First we consider to apply the same trick used in the weak back reaction case discussed in Paper I for \(\xi^{(5)}_{\pm}\) to the complete Green function containing all massive modes. Then the long range part of the Green function reduces to \(G_q \approx -\delta^4(\Delta x^\mu) \sum q_i(z) q_i(z')/m_i^2\). This replacement is exactly valid when we focus on the long range force. Now one can notice that to use this Green function neglecting the terms corresponding to the short range force is
equivalent to solve the equation for $\phi^N$ \((2.20)\) by setting $\square^{(4)} = 0$ from the beginning.

For the case with $\square^{(4)} = 0$, we have already obtained the general solutions to Eq. \((2.20)\), i.e., Eqs. \((3.4)\) and \((3.7)\). For convenience, here we quote the previous results in a slightly different notation, namely

$$
\phi^N = \sum_{\sigma = \pm} u_{\sigma} f^{(\sigma)}(x^\rho),
$$

with

$$
u := 1 - \frac{2H}{a^2} \int_{y(t)}^{y} a^2(y')dy'.
\hspace{1cm}
(4.10)$$

Then the junction condition \((2.18)\) with $\square^{(4)} = 0$ determines $f^{(\pm)}$ as $f^{(\pm)} = \mp 2Na^2_{(\pm)}N^{(\pm)}$. Substituting these back into the expression for $\phi^N$, we obtain

$$
\phi^N(y^{(\pm)}) = -2H\phi_{(\pm)}^2 - 2N \sum_{\sigma = \pm} \left[ \sigma a^2(\sigma) \phi_{(\sigma)}^{(\pm)} \sigma \right].
\hspace{1cm}
(4.11)$$

Adding this contribution to the contributions coming from the zero mode TT part and $\phi_{(\pm)}^2$ in Eq. \((4.1)\) for the induced metric on the branes, Einstein gravity at the linear order is recovered.

It will be illustrative to give the next order correction. The source term for the next order correction is $-\square^{(4)}\phi^N_N\left(1 + \sum_{\sigma = y} 2e^{(\sigma)}(y - y^{(\sigma)})\right)$, where we have denoted the solution of the lowest order approximation \((4.11)\) by $\phi^N_0$. By using Eq. \((2.17)\), $\square^{(4)}\phi^N_0$ is evaluated as

$$
\square^{(4)}\phi^N_0 = \frac{\kappa N}{3} \sum_{\sigma = \pm} u_{\sigma}(y)a^4_{(\sigma)}T^{(\sigma)}. \hspace{1cm} (4.12)
$$

Then, by using the standard Green function method, we obtain

$$
\phi^N_1(y) = u_+ \left[ \int_{y(t)}^{y} \frac{u_+}{y_t - y} \left( 3N \frac{\sigma}{\kappa} \phi^N_0 \right) dy' + K_+ \right] - u_- \left[ \int_{y(-)}^{y} \frac{u_-}{y_t - y} \left( 3N \frac{\sigma}{\kappa} \phi^N_0 \right) dy' - K_- \right],
\hspace{1cm} (4.13)
$$

where $K_+$ and $K_-$ are integration constants determined from the junction condition as

$$
K_\pm = \left[ \frac{N^2}{a^2} \right] \epsilon^{(\pm)}
\times \left[ \left( 1 \pm \frac{H\kappa a^2}{N^2} \right) a^4 T^{(\pm)} \right].
\hspace{1cm} (4.14)
$$

For simplicity, we take again the $\epsilon^{(\pm)} \to 0$ limit. Then, we have $K_\pm = 0$, and we obtain

$$
\phi^N_1(y) = N^2 \sum_{\sigma = \pm} I_{\pm, \sigma} a^4_{(\sigma)}T^{(\sigma)},
\hspace{1cm} (4.15)
$$

where we have defined

$$
I_{i,j} := \int_{y(t)}^{y} u_{i}(y)u_{j}(y)dy.
\hspace{1cm} (4.16)
$$

In the weak back reaction case, we can approximately evaluate this integral by using the following facts. First, we note that the combination $a^2u_\pm$ can be shown to be a slowly changing function for $y \gtrsim y_c$ in the weak back reaction case. As before, $y_c$ is the peak location of $(a^4\dot{\phi}_0^{(4)})^{-1}$. To show this, we use the fact that $\dot{\phi}_0$ is approximated by $\nu_1B_1e^{\sigma(y)}$ for $y \gtrsim y_c$. This constancy of $a^2u_\pm$ indicates that the integrand of Eq. \((4.16)\) is approximately proportional to $(a^4\dot{\phi}_0^{(4)})^{-1}$. Then by using the background geometry defined by Eqs. \((3.7)\) and \((3.8)\), $I_{+, +}$ is evaluated like

$$
I_{+, +} \approx \left[ a^2u_+ \right]^2 \int_{y(t)}^{y} dy \approx \frac{\kappa \ell}{4m_0^2} e^{-2d/\ell},
$$

where we have used \((8.13)\). The other components are also evaluated in a similar way to obtain the relations $I_{+, +} \approx e^{-2d/\ell}I_{+, -} \approx e^{-2d/\ell}I_{-, -}$. The correction obtained by substituting these estimates for $I_{i,j}$ into \((4.15)\) is consistent with the contribution from the second term in the square brackets in Eq. \((4.1)\). If we set $\square^{(4)} = 0$ there, we recover the same result.

It is easy to check that the treatment of taking $\square^{(4)} = 0$ to be small is consistent if the distribution of the energy momentum tensor is sufficiently smooth. In the treatment presented here, we have taken into account the contribution from all the massive modes simultaneously. The approximation is completely valid as long as we consider smooth matter distribution as compared with the mass scale of the lowest massive mode. However, the lowest mass on the positive tension brane becomes very small if we consider the case in which we are living on the negative tension brane. In this sense, the treatment presented here is rather restrictive when we discuss perturbations caused by the matter fields on the positive tension brane.

V. TT PART REVISITED

Applying the same technique that we have used in the preceding section for the scalar-type perturbations, we can also deal with the KK mode contribution for the TT part $h_{\mu\nu}^{(TT)}$. By using the Green function method, we can easily calculate the zero mode contribution like

$$
h_{\mu\nu}^{(0)} = -2N\kappa a^2(y)(\square^{(4)})^{-1} \sum_{\sigma = \pm} a^2(\sigma)\Sigma_{\mu\nu}. \hspace{1cm} (5.1)
$$

After using Eq. \((2.17)\), we can recover Eq. \((4.1)\). Substituting $h_{\mu\nu}^{(TT)} = h_{\mu\nu}^{(0)} + h_{\mu\nu}^{(KK)}$ into \((2.18)\), we obtain the equation for the KK contribution,

$$
a^{-2\square^{(4)}} + \hat{L}_\mu^{(TT)} \hat{h}_{\mu\nu}^{(KK)}.
$$

8
Hence we have already subtracted the zero mode contribution.

\[ h^{(KK)} = 2Nk \sum_{\sigma = \pm} a_\sigma^2 \left[ a_\sigma^2 N - \delta(y - y^{(\sigma)}) \right]. \]  

(5.2)

As before, neglecting the \( \Box^{(4)} \)-term for the massive KK contribution, we can solve this equation like

\[ h^{(KK)}_{\mu\nu} = 2Nk \sum_{\sigma = \pm} a_\sigma^2 \sigma^{(\sigma)} a^2(y) \times \left[ \int_{y^{(-)}}^{y^{(+)}} \frac{dy'}{a^2(y')} \int_{y'^{(-)}}^{y'^{(+)}} dy'' a^2(y'') - C_{\sigma} \right], \]  

(5.3)

where \( C_+ \) and \( C_- \) are constants. We should recall that we have already subtracted the zero mode contribution. Hence \( h^{(KK)}_{\mu\nu} \) must be orthogonal to the zero mode. This requires

\[ \int_{y^{(-)}}^{y^{(+)}} dy h^{(KK)}_{\mu\nu}(y) = 0. \]  

(5.4)

The constants \( C_+ \) and \( C_- \) in the solution (5.3) are determined by imposing this condition.

For simplicity we again adopt (3.7) with (3.8) as the background geometry. Then, from condition (5.4), \( C_\pm \) are explicitly calculated, and the resulting KK contribution becomes

\[ h^{(KK)}_{\mu\nu} = \frac{k\ell}{4(1 - e^{-2d/\ell})} \sum_{\sigma = \pm} \sigma^{(\sigma)} \]  

\[ \times \left[ e^{2y -2d/\ell} - 2a_\sigma^2 + e^{-2y/\ell} \left( 4a_\sigma^2 d\ell^{-1} - 1 \right) \right]. \]  

(5.5)

Hence, on the respective branes, the contributions from the KK modes become

\[ h^{(KK)}_{\mu\nu}(+) \approx -\frac{k\ell}{4} \left( (3 - 4d/\ell) \Sigma^{(+)\mu\nu} + \Sigma^{(-)\mu\nu} \right), \]  

\[ h^{(KK)}_{\mu\nu}(-) \approx -\frac{k\ell}{4} \left( -\Sigma^{(+)\mu\nu} + \Sigma^{(-)\mu\nu} \right), \]  

(5.6)

where we have assumed \( d/\ell \gg 1 \). Here we should recall the remaining degrees of freedom for the 4-dimensional gauge transformation. Using these degrees of freedom, \( \Sigma^{\mu\nu} \) can be replaced with \( T^{\mu\nu} - \gamma_{\mu\nu} T/3 \) with the aid of Eq. (2.17). Hence, the KK modes, of course, do not give any long range force contribution. If one takes the \( d/\ell \to \infty \) limit, \( h^{(KK)}_{\mu\nu}(+) \) seems to diverge. But this is just due to the breakdown of the approximation. In this limit, the mass difference of the KK modes becomes zero.

VI. SUMMARY

In this paper we have developed a systematic procedure to evaluate the perturbations in the 5-dimensional brane world model proposed by Randall and Sundrum supplemented with the moduli stabilization mechanism by Goldberger and Wise [13].

We have first investigated in detail the mass spectrum of this model. In the case without stabilization mechanism, there was a scalar-type massless mode, which was called radion in Ref. [20]. We have clarified how this massless mode disappears once we switch on the stabilization mechanism.

We have also estimated the mass eigenvalue and the mode function corresponding to the lowest mass eigenmodes for both tensor-type and scalar-type perturbations, assuming that the back reaction to the background geometry due to the bulk scalar field introduced for the moduli stabilization is not large. The physical mass of the lowest tensor-type mode becomes \( \approx \ell^{-1} \) on the negative tension brane and \( \approx e^{-d/\ell}\ell^{-1} \) on the positive tension brane. Here \( \ell \) is the curvature scale of the background geometry, and \( d \) is the proper distance between the two branes. In the original model, \( \ell^{-1} \) is supposed to be TeV scale.

For the physical mass of the lowest scalar-type mode, we have obtained rather general formulas, (3.19) and (3.20). To proceed further in estimating the mass, we have specified a model for the bulk potential of the scalar field. We have considered a simple quadratic potential whose mass is given by \( M \), and we have assumed that the 5-dimensional gravitational constant is also \( \approx \ell^2 \). We pointed out that in this model there are two regimes in which the weak back reaction condition holds. The first case is the one in which the vacuum expectation values of the scalar field on the branes, \( \varphi(\pm) \), are sufficiently small compared with the background energy scale, i.e., \( \varphi(\pm) \ll \ell^{-3/2} \). In this case, we found that the mass of the lowest scalar-type mode becomes \( \approx \sqrt{|\varphi(\pm)|}\ell^3 M e^{-\sqrt{1+(M^2\ell^4/4) - 1/d/\ell}} \). (For a more precise formula, see Eq. (3.23).) If we take into account the fact that we need to set \( d/\ell \approx 37 \) to solve the hierarchy problem on the negative tension brane, this factor is at most \( \approx 0.2\ell^{-1}\sqrt{|\varphi(-)|\varphi(+)\ell^3} \) for \( M \approx 0.33 \). The second case is the one in which \( M \) is small compared with \( \ell^{-1} \), which is the situation discussed in Ref. [18]. In this case, we reproduced the same result found there. Namely, the mass is approximately given by \( (|\varphi(\pm)|\ell^3/2) M^2 \ell \). As pointed out in Ref. [18], the mass scale on the negative tension brane tends to be smaller than the typical background energy scale \( \ell^{-1} \). We confirmed that this is a general feature within the context of weak back reaction. However, looking at these formulas, it seems that we can raise the mass by taking a slightly larger vacuum expectation values of the bulk scalar field on both branes, although a large value of \( |\varphi(\pm)| \) is inconsistent with our approximation. Hence, if we remove the technical limitation of weak back reaction, it is not clear whether the physical mass of the lowest scalar-type mode is always smaller than that of the lowest tensor-type mode. We also mention that the mass scale on the positive tension brane is smaller by a factor of \( e^{-d/\ell} \).
Next, we have developed a method to evaluate the explicit form of the perturbations caused by the matter fields confined on the branes. This is a generalization of the results presented in the previous paper [11]. We found that, as is expected, Einstein gravity is exactly recovered for the long range force at the order of linear perturbations when we impose the stabilization mechanism. The formulas for the leading correction from the scalar-type perturbations (4.15) and that from the KK modes (5.6) are also derived without specifying the model for the potential of the bulk scalar field. From these formulas, we can read the coupling of the metric perturbations induced on the branes to the matter fields on both branes.

As we have confirmed in Sec.V, there is no pathological behavior in all perturbation modes at the level of linear perturbations. This is a rather expected result because almost all important information at the level of linear perturbations is contained in the mass spectrum except for the strength of coupling to the matter fields. However, it is still unclear what happens once we take into account the non-linearity of gravity. To investigate this issue, the formulas obtained in this paper will be useful.

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