A bound on the exponent of the cohomology of $BC$-bundles
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We give a lower bound for the exponent of certain elements in the integral cohomology of the total spaces of principal $BC$-bundles for $C$ a finite cyclic group. We are mainly interested in the case when the total space is $BG$ for some discrete group $G$ having a central subgroup isomorphic to $C$. As applications we give a proof of the theorem of A. Adem and H.-W. Henn that a $p$-group is elementary abelian if and only if its integral cohomology has exponent $p$, and we exhibit some infinite groups of finite virtual cohomological dimension whose Tate-Farrell cohomology contains torsion of order greater than the l.c.m. of the orders of their finite subgroups. Our examples include a class of groups having similar properties discovered by Adem and J. Carlson. As a third application, we examine the integral cohomology of a class of $p$-groups expressible as central extensions with cyclic kernel and quotient abelian of $p$-rank two. For each such $G$ we determine the minimal $n$ such that almost all (i.e. all but possibly finitely many) of the groups $H^j(BG)$ have exponent dividing $p^n$. The lemma we use to give an upper bound for the exponents of almost all of the groups $H^j(BG)$ applies to any $p$-group and may be of independent interest. Here, and throughout the paper, the coefficients for cohomology are to be the integers when not otherwise stated, and we write $\mathbb{Z}_n$ for the integers modulo $n$. The author gratefully acknowledges that this work was funded by the DGICYT.

**Proposition 1.** Let $C$ be a cyclic group of order $n$, and let $E$ be a principal $BC$-bundle over a connected space $X$, classified by $\xi \in H^2(X; C)$ of order $m$. Then for any $i \geq 0$, any element of $H^i(E)$ restricting to the fibre as a generator for $H^i(BC)$ has order divisible by $mn$.

**Remark.** Note that we do not claim that such elements always exist, nor do we rule out the possibility that they have infinite order.

**Proof.** In [4] Cartan and Eilenberg computed the ring $H^*(BC; R)$ for any coefficient ring $R$. Recall that we have the following ring isomorphisms:

$$H^*(BC) \cong \mathbb{Z}[z]/(nz), \quad H^*(BC; \mathbb{Z}_n) \cong \mathbb{Z}_n[x, y]/(ny, nx, y^2 - ex),$$

where $e = 0$ if $n$ is odd and $e = n/2$ if $n$ is even, and $y$ has degree 1 while $x$ and $z$ have degree two. The natural map from integral to mod-$n$ cohomology sends $z$ to $x$, and if we let $\beta$ stand for the Bockstein for the coefficient sequence

$$0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_n \rightarrow 0,$$

then it is easy to see that $\beta(y) = z$, and that therefore $\beta(yx^i) = z^{i+1}$.

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Now consider the spectral sequence for the given fibration with coefficients in $\mathbb{Z}_n$. By assumption the fundamental group of $X$ acts trivially on the cohomology of $BC$, and so

$$E_2^{i,j} \cong H^i(X; \mathbb{Z}_n) \otimes H^j(BC; \mathbb{Z}_n).$$

Now $1 \otimes yx^j$ represents a generator for $E_2^{0,2j+1}$ and $1 \otimes x^j$ represents a generator for $E_2^{0,2j}$. Comparing this spectral sequence with the spectral sequence for the path-loop fibration over an Eilenberg-MacLane space $K(C, 2)$ it is easy to see that $d_2(1 \otimes y) = \xi$ and $d_2(1 \otimes x) = 0$. (In fact, $d_2(1 \otimes x) = \xi' \otimes 1$, where $\xi'$ is the image of $\beta(\xi)$ under the map from $H^3(X)$ to $H^3(X; \mathbb{Z}_n)$, and $d_2$ may be described using the argument given in [8], but we do not need this here.) Now $d_2(1 \otimes x^j y) = \xi \otimes x^j$ and $d_2(1 \otimes x^j) = 0$, from which it follows that $E_3^{0,2j}$ is generated by $1 \otimes x^j$ and $E_3^{0,2j+1}$ by $m(1 \otimes yx^j)$. The map from $H^*(E; \mathbb{Z}_n)$ to $H^*(BC; \mathbb{Z}_n)$ factors through $E_0^*$, which is a subgroup of $E_3^{0,*}$, and so we see that the image of $H^{2j+1}(E; \mathbb{Z}_n)$ in $H^{2j+1}(BC; \mathbb{Z}_n)$ must be contained in the subgroup generated by $myx^j$.

Now recall that the image of the Bockstein $\beta$ defined above is exactly the elements of integral cohomology of order dividing $n$. Let $f : BC \to E$ be the inclusion of the fibre of the above fibration. Now let $\chi$ be an element of $H^*(E)$ such that $f^*(\chi) = z^{j+1}$ for some $j$. If $\chi$ has infinite order then there is nothing to prove. Otherwise, the order of $\chi$ must be a multiple of $n$ (the order of $z^{j+1}$), say $m' n$, and it remains to show that $m$ divides $m'$. Now $m' \chi$ has order $n$, so there exists $\chi' \in H^{2j+1}(E; \mathbb{Z}_n)$ such that $\beta(\chi') = m' \chi$. However, the spectral sequence argument shows that $f^*(\chi')$ is in the subgroup of $H^{2j+1}(BC; \mathbb{Z}_n)$ generated by $myx^j$ and hence $\beta f^*(\chi')$ is in the subgroup of $H^{2j+2}(BC)$ generated by $mx^{j+1}$, but $\beta f^*(\chi') = f^*(\beta(\chi')) = f^*(\chi') = mx^j z^{j+1}$.

\[\square\]

**Corollary 1.** Let $C$ be a cyclic subgroup of order $n$ of a group $G$. If there exists an element of $H^*(BG)$ of order $n$ whose image in $H^*(BC)$ is a generator for $H^{2i}(BC)$ for some $i$, then $C$ is a direct factor of its centraliser in $G$.

**Proof.** This is just Proposition 1 applied to the principal $BC$-bundle with total space the classifying space of the centraliser of $C$.

**Corollary 2.** Let $G$ be a discrete group expressible as a central extension with kernel $C$ cyclic of order $n$. Let $Q$ be the quotient $G/C$, and let the extension class of $G$ in $H^2(BQ; C)$ have order $m$. If $G$ has a normal subgroup $N$ of finite index whose intersection with $C$ is trivial (for example, if $G$ is finite or residually finite), then for infinitely many $i$, $H^{2i}(BG)$ contains elements of order $mn$.

**Remark.** The condition that the extension class of $G$ has order $m$ may be rephrased as follows: If $D$ is the smallest subgroup of $C$ such that $G/D$ is isomorphic to $(C/D) \times Q$, then $D$ has order $m$.  

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Proof. Let $G'$ be the quotient $G/N$, and let $C'$ be the image of $C$ in $G'$. Then $C'$ is isomorphic to $C$ and $G'$ is finite. By either Evens’ argument using the Norm map from $H^*(BC)$ to $H^*(BG)$ [5,6] or Venkov’s argument using Chern classes of a representation of $G'$ restricting faithfully to $C'$ [10], we see that for infinitely many $i$ there exists $\chi' \in H^{2i}(BG')$ whose image in $H^{2i}(BC')$ is a generator. If $\chi$ is the image of $\chi'$ in $H^*(BG)$, then $\chi$ has finite order (dividing the order of $G'$) and its image in $H^{2i}(BC)$ is a generator. Hence by Proposition 1, some multiple of $\chi$ has order exactly $mn$.

The first example of a group whose Tate-Farrell cohomology contains elements of order greater than the l.c.m. of the orders of its finite subgroups is due to Adem [2]. The following application of Corollary 2 is more closely related to some other examples due to Adem and Carlson [3]. In particular, Corollary 3 may be compared with Theorem 3.1 of [3], which gives stronger cohomological information about a smaller class of groups.

Corollary 3. With notation and hypotheses as in Corollary 2, assume also that $Q$ has finite cohomological dimension (or equivalently, assume that there is a finite-dimensional CW-complex $BQ$). Then

a) $G$ has finite virtual cohomological dimension and hence the Tate-Farrell cohomology groups $\hat{H}^i(G)$ are defined,

b) $C$ consists of all the elements of $G$ of finite order, and

c) $\hat{H}^i(G)$ contains elements of order $mn$ for infinitely many $i$.

Proof. The subgroup $N$ of $G$ has finite index and is isomorphic to a subgroup of $Q$, so has cohomological dimension less than or equal to that of $Q$. Hence $G$ has finite vcd. The group $Q$ is torsion-free, and so any element of $G - C$ has infinite order because its image in $Q$ does. If $i$ is greater than $\text{vcd}G$ then $\hat{H}^i(G)$ is isomorphic to $H^i(BG)$, and so the third claim follows from Corollary 2.

The following Corollary is due to Adem [1] and Henn [7].

Corollary 4. Let $G$ be a finite $p$-group. Then $G$ is not elementary abelian if and only if $H^i(BG)$ contains elements of order $p^2$ for some $i$ if and only if $\hat{H}^i(G)$ contains elements of order $p^2$ for infinitely many $i$.

Proof. If $G$ is elementary abelian (i.e. is isomorphic to a product of cyclic groups of order $p$) then $H^i(G)$ has exponent $p$ for $i > 0$ by the Künneth theorem. Conversely, if $G$ is not elementary abelian then $G$ contains a central subgroup of order $p$ which is not a direct factor, or equivalently, $C$ of order $p$ such that the extension class of $G$ in $H^2(BG/C; C)$ has order $p$. The result now follows by applying Corollary 2.

The following application of Proposition 1 is new.

Proposition 2. For positive integers $\alpha$, $\beta$, $\gamma$, $\delta$ satisfying the inequalities $0 \leq \gamma - \delta \leq \min\{\alpha, \beta\}$, let $G = G(\alpha, \beta, \gamma, \delta)$ be a $p$-group with the following
Now let \( \epsilon \) be \( \max\{\alpha, \beta, 2\gamma - \delta\} \). Then for infinitely many \( i \), \( H^i(BG) \) has exponent \( p^\epsilon \), and at most finitely many of the groups \( H^i(BG) \) have higher exponent.

**Remark.** It is easy to see that any group having a presentation of the above form for arbitrary \((\alpha, \beta, \gamma, \delta)\) also has a presentation of the above form in which the inequalities are satisfied: If \( \gamma \) is less than \( \delta \), then \( c^{p\delta} = c^{p\gamma} = 1 \), and so in this case \( G(\alpha, \beta, \gamma, \delta) \) is isomorphic to \( G(\alpha, \beta, \gamma, \gamma) \). On the other hand, the order of \([a, b] = c^{p\delta}\) is bounded by the orders of \(a\) and \(b\) given that \(c\) is central, and so the order of \(c\) is bounded by \(p^{\alpha + \delta}\) and \(p^{\beta + \delta}\). Thus given a presentation as above but not satisfying the second inequality we could replace \( \gamma \) by \( \gamma' = \min\{\alpha + \delta, \beta + \delta\} \) and obtain another presentation of the same group.

**Proof.** First we recall that for any \( G \) and any split surjection from \( G \) onto \( Q \), \( H^i(BQ) \) occurs as a direct summand of \( H^i(BG) \). Now the above group \( G \) may be expressed as a split extension with kernel \( \langle a, c \rangle \) and quotient \( \langle b \rangle \cong \mathbb{Z}/p^\beta \), or as a split extension with kernel \( \langle b, c \rangle \) and quotient \( \langle a \rangle \cong \mathbb{Z}/p^\alpha \). Hence we deduce that \( H^{2i}(BG) \) has elements of exponents \( p^\alpha \) and \( p^\beta \) for all \( i > 0 \).

\( G \) may also be viewed as a central extension with kernel \( \langle c \rangle \) which is isomorphic to \( \mathbb{Z}/p^\gamma \), and quotient isomorphic to \( \mathbb{Z}/p^\alpha \oplus \mathbb{Z}/p^\beta \) generated by the images of \(a\) and \(b\). The extension class of this extension is easily seen to have order \( p^{\gamma - \delta} \), and so it follows from Corollary 1 that for infinitely many \( i \), \( H^{2i}(BG) \) contains elements of order \( p^{2\gamma - \delta} \).

For the partial converse, note that \( G \) has subgroups \( \langle a, c \rangle, \langle b, c \rangle \), and \( \langle a, b^{p^{\gamma - \delta}} \rangle \) of index \( p^\alpha, p^\beta \) and \( p^{2\gamma - \delta} \) respectively whose intersection is trivial, and then apply the following Lemma.

**Lemma 1.** Let \( G \) be a (finite) \( p \)-group, let \( H_1, \ldots, H_k \) be a family of subgroups of \( G \) such that the index \( [G : H_j] \) of each \( H_j \) is less than or equal to \( p^n \), and suppose that the intersection

\[
\bigcap_{g \in G, 1 \leq j \leq k} H_j^g
\]

of the conjugates of the subgroups \( H_j \) is trivial. Then \( H^i(BG) \) has exponent dividing \( p^n \) for all but finitely many \( i \).

**Proof.** Let \( \Sigma_m \) be the symmetric group on \( m \) symbols and let \( G_n \) be the Sylow \( p \)-subgroup of \( \Sigma_{p^n} \). Since the index of \((\Sigma_m)^p \) in \( \Sigma_{mp} \) divides exactly once by \( p \) an easy induction argument using the transfer shows that for all \( i > 0 \) and all \( n \), \( H^i(BG_n) \) has exponent dividing \( p^n \). If \( H \) is a subgroup of \( G \), then the kernel of the permutation representation of \( G \) on the cosets of \( H \) is the intersection of the conjugates of \( H \). Hence if \( G \) has subgroups \( H_1, \ldots, H_k \) as in the statement
then $G$ occurs as a subgroup of a product of $k$ symmetric groups on at most $p^n$ symbols, and hence as a subgroup of $(G_n)^k$. The result now follows from the observation due to Adem [1] that for any group $G'$ and any subgroup $G$, the finite generation of $H^*(BG)$ as an $H^*(BG')$-module implies that at most finitely many of the groups $H^i(BG)$ can have higher exponent than the reduced cohomology $\tilde{H}^*(BG')$.

\begin{remark}
The bound given by Lemma 1 for the exponent of almost all of the integral cohomology groups of a $p$-group is attained for many groups. For example, Proposition 2 shows that the bound is attained for the groups $G(\alpha, \beta, \gamma, \delta)$. We were tempted to conjecture that the bound is always attained, but have recently found a group of order 128 whose index four subgroups intersect non-trivially and whose integral cohomology has exponent four [9]. Adem has conjectured that for $G$ a finite group, if $H^i(BG)$ contains elements of order $p^n$ for some $i$, then it does so for infinitely many $i$ [1], and Henn has asked if this is the case [7]. We do not know if this holds for the groups $G(\alpha, \beta, \gamma, \delta)$.
\end{remark}

\begin{references}
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\end{references}