Poisson bracket, deformed bracket
and gauge group actions
in Kontsevich deformation quantization

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Abstract : We express the difference between Poisson bracket and deformed bracket for Kontsevich deformation quantization on any Poisson manifold by means of second derivative of the formality quasi-isomorphism. The counterpart on star products of the action of formal diffeomorphisms on Poisson formal bivector fields is also investigated.

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Introduction

The existence of a star product on any Poisson manifold $(M, \gamma)$ is derived from the more general formality theorem of M. Kontsevich [K], which stipulates the existence of a $L_\infty$-quasi-isomorphism $\mathcal{U}$ (cf. § II) from the differential graded Lie algebra of polyvector fields on any manifold $M$ (with vanishing differential an Schouten bracket) into the differential graded Lie algebra of polydifferential operators on $M$ (with Hochschild differential and Gerstenhaber bracket).

Given such a $L_\infty$-quasi-isomorphism there is a canonical and explicit way to produce a star product $* = *_{\gamma}$ from Poisson bivector field, and more generally from any formal Poisson bivector field $\gamma$. We briefly recall this construction on § II. We call formality star product any star product on $M$ obtained that way. Due to the fact that $\mathcal{U}$ is a $L_\infty$-quasi-isomorphism, any star product is gauge-equivalent to a formality star product [K § 4.4], [AMM § A.2].

Let $f, g \in C^\infty(M)[[\hbar]]$, and let $H_f = [\gamma, f], \quad H_g = [\gamma, g]$ the associated hamiltonian (formal) vector fields. We compute here the second derivative at $h\gamma$ of quasi-isomorphism

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$\mathcal{U}$ evaluated at $(H_f, H_g)$, and more generally at $(Y, H_g)$ where $Y$ stands for any formal vector field on $M$. The main result is theorem III.3, consisting of three equations, which in turn imply the following formula, relating Poisson bracket, deformed bracket, tangent map $\Phi$ at $\hbar\gamma$ of $\mathcal{U}$ and second derivative $\Psi$ at $\hbar\gamma$ of $\mathcal{U}$:

$$\Psi(H_f, H_g) = \frac{1}{\hbar} \left( \Phi([f, g]) - \frac{\Phi(f) \ast \Phi(g) - \Phi(g) \ast \Phi(f)}{\hbar} \right).$$

There is another consequence of theorem III.3 in terms of gauge group action: namely we try to understand the star product $\ast_{g, \gamma}$ obtained from formal Poisson bivector field $g.\gamma$ where $g$ is a formal diffeomorphism of the manifold $M$. Formal diffeomorphisms also act naturally on star-products via action on $C^\infty(M)$, but it is quite obvious that $\ast_{g, \gamma}$ is not the image of $\ast_{\gamma}$ by the action of $g$ in that sense.

Gauge group $G_1$ of formal diffeomorphisms however acts on formality star-products in a more subtle way: considering the set of all formality star-products as a formal pointed manifold ($FSP$), the action we seek amounts to a nontrivial embedding of $G_1$ into the group $\mathcal{G}$ of formal diffeomorphisms of ($FSP$).

I. Super-grouplike elements in cofree cocommutative graded coalgebras

Let $V = \bigoplus V^{(n)}$ be a $\mathbb{Z}$-graded vector space over a field $k$ with zero characteristic, and let $\mathcal{C} = S(V)$ its symmetric algebra in the graded sense. We will throughout the paper denote by $\pi$ the projection of $S(V)$ onto $V$. Defining a coproduct on elements of $V$ by:

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

and extending it to an algebra morphism from $S(V)$ to the tensor product $S(V) \otimes S(V)$ (in the graded sense) we endow $S(V)$ with a structure of graded bialgebra. The set of primitive elements is precisely $V$, and the co-unity is given by the projection on constants.

Let $\mathfrak{m}$ be a (projective limit of) commutative finite dimensional nilpotent algebra(s). We will consider the (completed) tensor product $V \hat{\otimes} \mathfrak{m}$ as a topologically free $\mathfrak{m}$-module and we will see the topologically free $\mathfrak{m}$-module $\mathcal{C}_\mathfrak{m} = S(V) \hat{\otimes} \mathfrak{m} \oplus k.1$ as a topological bialgebra over $\mathfrak{m}$.

Let $\tau : \mathcal{C} \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$ be the signed flip defined by:

$$\tau(v \otimes w) = (-1)^{|v||w|} w \otimes v.$$

A nonzero element $v \in \mathcal{C}_\mathfrak{m}$ will be called super-grouplike if we have:

$$\Delta v = \frac{I + \tau}{2} (v \otimes v).$$

As an example, any even grouplike element is super-grouplike, as well as any $1 + x$ where $x \in V \hat{\otimes} \mathfrak{m}$ and $x$ odd.
Proposition I.1.
Any super-grouplike element in the coalgebra $C^m$ is of the form:

$$g = e^{-v} = 1 + v + \cdots + \frac{1}{n!}(v \cdots v)_{n \text{ times}} + \cdots$$

with $v \in V \hat{\otimes} m$, and conversely any such exponential is super-grouplike.

Proof. Consider the decomposition $g = g_+ + g_-$ of our supergrouplike element into its even and odd components. We have then:

$$\frac{1+\tau}{2}\Delta(g) = \frac{1+\tau}{2}\Delta(g_+ + g_-) = g_+ \otimes g_+ + g_- \otimes g_+ + g_+ \otimes g_-,$$

hence:

$$\Delta(g_+) = g_+ \otimes g_+$$

$$\Delta(g_-) = g_- \otimes g_+ + g_+ \otimes g_-.$$ 

So $g_+$ is nonzero, grouplike in the ordinary sense, and $g_- g_+^{-1}$ is an odd primitive element $v_-$. So $g_+$ writes:

$$g_+ = e^{v_+}$$

with $v_+ \in V$ even. To see this one can write $g_+ = 1 + \varepsilon$ with $\varepsilon \in S(V) \hat{\otimes} m$ and directly check that its logarithm is primitive. We have then:

$$g = (1 + v_) e^{v_+} = e^{-v_-} e^{v_+} = e^{v_- + v_+}.$$

The converse is straightforward.

Let $\tilde{C} = S^+(V) = \bigoplus_{n \geq 1} S^n(V)$ be the cofree cocommutative graded coalgebra without co-unity cogenerated by $V$. The coproduct is given by:

$$\tilde{\Delta}(v) = \Delta(v) - 1 \otimes v - v \otimes 1.$$ 

Let $\tilde{C}_m = \tilde{C} \hat{\otimes} m$. It is easy (and left to the reader) to derive a version of the result above in that setting:

Proposition I.2.
Any super-grouplike element in the coalgebra $\tilde{C}_m$ is of the form:

$$g = e^{-v} - 1 = v + \cdots + \frac{1}{n!}(v \cdots v)_{n \text{ times}} + \cdots$$

with $v \in V \hat{\otimes} m$, and conversely any such element is super-grouplike in $\tilde{C}_m$. 

Let us now compute the image of a super-grouplike element by a certain coderivation in the co-unityless setting:

**Lemma I.3.**
Let $Q$ be a coderivation of coalgebra $\mathcal{C}$ with vanishing Taylor coefficients (cf. § II below) except $Q_2$, and extend it naturally by $m$-linearity to a coderivation of coalgebra $\tilde{\mathcal{C}}_m$. Let $X, Y \in V \otimes m$ with $X$ even and $Y$ odd. Then we have:

$$Q(e^{(X+Y)} - 1) = \frac{1}{2}Q_2(X.X)e^{(X+Y)} + Q_2(Y.X)e^X.$$

*Proof.* We have the following explicit formula for a coderivation in terms of its Taylor coefficients [AMM § III.2]: namely, for any $n$-uple of homogeneous elements in $V$,

$$Q(x_1 \cdots x_n) = \sum_{I \prod J = \{1, \ldots, n\}} \varepsilon_x(I,J)(Q_I(x_I))x_J,$$

where $x_I$ stands for $x_{i_1} \cdots x_{i_k}$ when $I = i_1, \ldots, i_k$, and $\varepsilon_x(I,J)$ is the Quillen sign associated with partition $(I,J)$, i.e. the signature of the trace on odd $x_i$’s of the shuffle associated with partition $(I,J)$. We have then:

$$Q(e^{-X} - 1) = \frac{1}{2}Q_2(X.X)e^{-X}$$

$$Q(Y.e^{-X}) = \frac{1}{2}Q_2(X.X)Y.e^{-X} + Q_2(Y.X)e^{-X}.$$

Summing up the two equalities above we get the result.

**II. Kontsevich’s formality theorem**

Let $M$ be any real $C^\infty$ manifold, let $\mathfrak{g}_1$ the differential graded Lie algebra of polyvector fields on $M$ with zero differential and Schouten-Nijenhuis bracket, and let $\mathfrak{g}_2$ the differential graded Lie algebra of polydifferential operators on $M$ with Gerstenhaber bracket and Hochschild differential. The gradings are such that a degree $n$ homogeneous element in $\mathfrak{g}_1$ (resp. $\mathfrak{g}_2$) is a $n+1$-vector field (resp. a $n+1$-differential operator).

We can consider the shifted spaces $\mathfrak{g}_1[1]$ and $\mathfrak{g}_2[1]$ as formal graded pointed manifolds: it means that for $i = 1, 2$ we have a coderivation $Q^i$ of degree 1 on coalgebra without co-unity $S^+(\mathfrak{g}_i[1])$ satisfying the *master equation*:

$$[Q^i, Q^j] = 0,$$

where $\mathfrak{g}_i[1]$ is meant for space $\mathfrak{g}_i$ with grading shifted by 1: a degree $n$ homogeneous element in $\mathfrak{g}_1[1]$ (resp. $\mathfrak{g}_2[1]$) is now a $n+2$-vector field (resp. a $n+2$-differential operator).
Theorem II.1 (M. Kontsevich).
There exists a $L_\infty$-quasi-isomorphism from formal graded pointed manifold $g_1[1]$ to formal graded pointed manifold $g_2[1]$: namely, there exists a coalgebra morphism:

$$U : S^+(g_1[1]) \longrightarrow S^+(g_2[1])$$

such that:

$$U \circ Q^i = Q^2 \circ U,$$

and such that the restriction $U_1$ of $U$ to $g_1[1]$ is a quasi-isomorphism of cochain complexes*.

Let us briefly recall how formality theorem is related to deformation quantization: due to the universal property of cofree cocommutative coalgebras, coderivations $Q^i$ and $L_\infty$-quasi-isomorphism $U$ are uniquely determined by their Taylor coefficients:

$$Q^i_k : S^k(g_1[1]) \longrightarrow g_i[2]$$

$$U_k : S^k(g_1[1]) \longrightarrow g_2[1],$$

$k \geq 1, i = 1, 2$, obtained by composing $Q^i$ and $U$ on the right by the canonical projection: $S^+(g_i) \rightarrow g_i$ (resp. $S^+(g_2) \rightarrow g_2$). Let $m = h\mathbb{R}[[h]]$ the projective limit of finite-dimensional nilpotent commutative algebras $m_r = h\mathbb{R}[[h]]/h^r\mathbb{R}[[h]]$. Let $h\gamma = h(\gamma_0 + h\gamma_1 + h^2\gamma_2 + \cdots)$ be an infinitesimal formal Poisson bivector field, i.e. a solution in $g_1(1) \otimes m$ of Maurer-Cartan equation:

$$(hd\gamma) - \frac{1}{2}[h\gamma, h\gamma] = 0,$$

which amounts exactly to the more geometrical equation:

$$Q^1(e^{h\gamma} - 1) = 0,$$

where $e^{h\gamma} - 1$ is grouplike (in the usual sense) in coalgebra $S^+(g_1[1]) \otimes m$. Then $U(e^{h\gamma} - 1)$ is grouplike in coalgebra $S^+(g_2[1]) \otimes m$. So we have:

$$U(e^{h\gamma} - 1) = e^{h\tilde{\gamma}} - 1$$

with:

$$\tilde{\gamma} = \sum_{k \geq 1} \frac{h^k}{k!} U_k(\gamma^{(k)}).$$

Due to the fact that $Q$ vanishes at $e^{h\gamma} - 1$ the element $h\tilde{\gamma}$ verifies Maurer-Cartan equation in $g_2 \otimes m$:

$$hd\tilde{\gamma} - \frac{1}{2}[h\tilde{\gamma}, h\tilde{\gamma}] = 0.$$

* According to [AMM] one should replace Schouten bracket with minus the bracket taken in the reverse order. This bracket coincides with Schouten bracket modulo a minus sign when the odd elements are involved so it does not matter in what follows.
We denote by $m$ the particular bidifferential operator $f \otimes g \mapsto fg$, and we set $\ast = m + \hbar \tilde{\gamma}$.  

Maurer-Cartan equation for $\hbar \tilde{\gamma}$ is equivalent to:

\[[\ast, \ast] = 0,\]

i.e. $\ast$ is an associative product on $C^\infty(M)[[\hbar]]$. Starting from a Poisson bivector field $\gamma = P$ on $M$ we construct then explicitly a star product from $P$ and $L_\infty$-morphism $U$.

**Remark**: the expression $e^{\hbar \gamma} - 1$ is nothing but an algebraic way to express “the point $\hbar \gamma$ in the formal graded pointed manifold”. One can be convinced by looking at the delta distribution at $\hbar \gamma$ and expressing it at 0 by means of Taylor expansion. The expression $e^{\hbar \gamma} - 1$ is then just the difference between the delta distribution at $\hbar \gamma$ and the delta distribution at 0. I would like to thank Siddhartha Sahi for having brought this nice geometrical interpretation to my attention.

**III. On particular super-grouplike elements**

Let $\gamma$ be a formal Poisson 2-tensor on manifold $M$, and let $\ast = m + \hbar \tilde{\gamma}$ the star-product constructed from these data with Kontsevich’s $L_\infty$-quasi-isomorphism $U$ following the formula recalled in previous paragraph. Let us consider for any $g \in C^\infty(M)[[\hbar]]$ and for any formal vector field $Y$ the super-grouplike element $e^{\hbar \gamma} - 1$. We will denote by $H_g$ the hamiltonian formal vector field $[\gamma, g]$. As a straightforward application of lemma I.3 we get the following result:

**Lemma III.1.**

With the same notations as in § II we have:

\[Q^1(e^{\hbar (\gamma + Y + g)} - 1) = \hbar^2(H_g e^{\hbar (\gamma + Y + g)} + [Y, \gamma + g] e^{\hbar (\gamma + g)}).\]

Any morphism of graded coalgebras, in particular $L_\infty$-quasi-isomorphism $U$, preserves super-grouplike elements. Due to this fact and to proposition I.1 we have then:

**Proposition III.2.**

There exists a formal differential operator $\Phi(Y)$ and a formal series $\Phi(g) \in C^\infty(M)[[\hbar]]$ such that:

\[U(e^{\hbar (\gamma + Y + g)} - 1) = e^{\ast - m + \hbar \Phi(Y) + \hbar \Phi(g)} - 1,\]

with:

\[\Phi(Y) = U_1(Y) + \hbar U_2(Y, \gamma) + \frac{\hbar^2}{2} U_3(Y, \gamma, \gamma) + \cdots\]

\[\Phi(g) = U_1(g) + \hbar U_2(g, \gamma) + \frac{\hbar^2}{2} U_3(g, \gamma, \gamma) + \cdots\]
Correspondence $\Phi$ is precisely the tangent map at $h\gamma$ of quasi-isomorphism $U$ \cite{K8 8.1}.

We will now compute both terms $\pi U Q^1(e^{h(\gamma+Y+g)} - 1)$ and $\pi Q^2U(e^{h(\gamma+Y+g)} - 1)$, and try to get some information from the fact that they coincide, by the very definition of a $L_{\infty}$-morphism. Let us introduce for any pair $(Y, Z)$ of polyvector fields the second derivative term:

$$\Psi(Y, Z) = \sum_{k \geq 0} \frac{h^k}{k!} U_{k+2}(Y, Z, \gamma, k).$$

This expression is symmetric (in the graded sense) in $Y, Z \in g_1[1]$ and is of degree $|Y| + |Z| - 2$ in $g_2[1]$, so it belongs to $O^{\infty}(M)[[h]]$ when $|Y| + |Z| = 0$ in $g_1$. The expression is skew-symmetric in $(Y, Z)$ when $Y$ and $Z$ are both vector fields, and symmetric when $Y$ is a function and $Z$ is a bivector field. We easily compute:

$UQ^1(e^{h(\gamma+Y+g)} - 1) = h^2 U(H_g e^{h(\gamma+Y+g)} + [Y, \gamma + g] e^{h(\gamma+g)})$

$$= h^2 U((H_g + h H_g Y + [Y, g] + [Y, \gamma]) e^{h(\gamma+g)}).$$

From degree considerations we easily derive:

$$\pi U Q^1(e^{h(\gamma+Y+g)} - 1) = h^2 \pi U((H_g + h H_g Y + [Y, g] + [Y, \gamma] + h[Y, \gamma, g]) e^{h(\gamma)}),$$

so that we finally get:

$$\pi U Q^1(e^{h(\gamma+Y+g)} - 1) = h^2 (\Phi(H_g) + \Phi([Y, g]) + h\Psi(H_g, Y) + \Phi([Y, \gamma]) + h\Psi([Y, \gamma, g])).$$

On the other hand we have to compute:

$$\pi Q^2U(e^{h(\gamma+Y+g)} - 1) = \pi Q^2(e^\delta - 1)$$

$$= [\delta, m] + \frac{1}{2} Q^2_2(\delta, \delta),$$

with $\delta = * - m + h(\Phi(Y) + \Phi(g))$, according to proposition III.2. We have then:

$$\pi Q^2U(e^{h(\gamma+Y+g)} - 1) = [* - m + h\Phi(Y) + h\Phi(g), m]$$

$$+ \frac{1}{2} Q^2_2([* - m + h\Phi(Y) + h\Phi(g), (*) - m + h\Phi(Y) + h\Phi(g)]).$$

$$= [* , m] + h[\Phi(Y), m] + h[\Phi(g), m]$$

$$- \frac{1}{2} [* , m] - \frac{h}{2} [* , \Phi(Y)] + \frac{h}{2} [* , \Phi(g)]$$

$$- \frac{1}{2} [m, *] + \frac{h}{2} [m, \Phi(Y)] - \frac{h}{2} [m, \Phi(g)]$$

$$+ \frac{h}{2} [\Phi(Y), *] - \frac{h}{2} [\Phi(Y), m] + \frac{h^2}{2} [\Phi(Y), \Phi(g)]$$

$$+ \frac{h}{2} [\Phi(g), *] - \frac{h}{2} [\Phi(g), m] - \frac{h^2}{2} [\Phi(Y), \Phi(g)]$$

$$= - h[* , \Phi(Y)] + h[* , \Phi(g)] + h^2 [\Phi(Y), \Phi(g)].$$
In the computation above the relation between the second Taylor coefficient $Q_2^2$ and Gerstenhaber bracket is the following:

$$Q_2^2(x,y) = (-1)^{|x||y|-1}|x,y|.$$

This extra sign one must take care of comes from the identification of $S^k(g_2[1])$ with $\Lambda^k(g_2)[k]$ which goes as follows:

$$x_1 \cdots x_k \mapsto \varepsilon.x_1 \wedge \cdots \wedge x_k,$$

where $\varepsilon$ is the signature of the unshuffle storing even elements on the left and odd elements on the right [AMM § II.4], [K § 4.2].

We now identify the homogeneous components of degrees 1, 0 and $-1$ in the two expressions, so we get the following three equations:

**Theorem III.3.**

1) $[*, \Phi(Y)] = h\Phi([\gamma, Y])$
2) $[*, \Phi(g)] = h\Phi(H_g)$
3) $[\Phi(Y), \Phi(g)] = \Phi([Y, g]) + h(\Psi(H_g, Y) - \Psi([\gamma, Y], g)).$

Equation 1) implies the following: tangent map $\Phi$ sends derivations of $C^\infty(M)[[h]]$ with commutative product leaving $\gamma$ invariant to derivations of $C^\infty(M)[[h]]$ with deformed product. From equation 2) we see that hamiltonian formal vector fields are sent to inner derivations of the deformed algebra: these two facts proceed more directly from the fact that the tangent map is a morphism of cochain complexes [K § 8.1]. Equation 3) rewrites as follows:

$$R(Y, g) = \frac{1}{h}(\Phi([Y, g]) - [\Phi(Y), \Phi(g)]),$$

where skew-symmetric bilinear term:

$$R(Y, g) = \Psi([\gamma, g], Y) - \Psi([\gamma, Y], g)$$

can be seen as a kind of curvature. As a particular case we can take for $Y$ the hamiltonian vector field $H_f$ for any $f \in C^\infty(M)[[h]]$. From equations 2) and 3) we immediately get:

**Theorem III.4.**

For any $f, g$ in $C^\infty(M)$ the following formula holds true:

$$\Psi(H_f, H_g) = \frac{1}{h}\left(\Phi(\{f, g\}) - \frac{\Phi(f) \ast \Phi(g) - \Phi(g) \ast \Phi(f)}{h}\right).$$
Remark 1: Introducing the new star-product:

\[ f \# g = \Phi^{-1}(\Phi(f) \star \Phi(g)) \]

the formula of theorem III.4 can be rewritten as follows:

\[ \Phi^{-1} \circ \Psi(H_f, H_g) = \frac{1}{\hbar} (\{f, g\} - \frac{f \# g - g \# f}{\hbar}) \]

Remark 2: It is immediate to see from symmetry properties of star products that the right-hand side is \( O(\hbar) \). In the flat case \( M = \mathbb{R}^d \) one can use quasi-isomorphism of \([K \ § \ 6]\), and recover this fact from the left-hand side by computing the constant term: it involves only the following graph:

\[ \text{the weight of which is zero} \ [K \ § \ 7.3.1]. \]

IV. Gauge transformations

Recall (from \([K \ § \ 3.2]\)) that the gauge group associated with any differential graded Lie algebra \( g \) is by definition the pro-nilpotent group \( G \) associated with pro-nilpotent Lie algebra \( g^{(0)} \hat{\otimes} m \). The gauge group acts on \( g^{(1)} \hat{\otimes} m \) by affine transformations, and the action of \( G \) is defined by exponentiation of the infinitesimal action of \( g^{(0)} \hat{\otimes} m \):

\[ \alpha \otimes \gamma \in g^{(0)} \otimes g^{(1)} \mapsto \alpha \cdot \gamma := d\alpha + [\alpha, \gamma]. \]

It is easy to check that gauge group \( G \) acts on the set of solutions of Maurer-Cartan equation in \( g^{(1)} \hat{\otimes} m \).

We try in this last paragraph to give a geometrical meaning to equation 1) of theorem III.3. We let \( \hbar \gamma \) run inside the set \((MC)_1\) of infinitesimal formal Poisson 2-tensors on manifold \( M \), i.e. the subset of solutions of Maurer-Cartan equation in \( g^{(1)}_1 \hat{\otimes} m \). We denote by \( \ast_\gamma \) the star-product constructed from \( \gamma \) along the lines of \( \S \) II. We will use the notation \( U'_{\hbar \gamma} \) instead of \( \Phi \) to emphasize the dependance on \( \gamma \). Equation 1) of theorem III.3 is then rewritten as follows:

\[ [U'_{\hbar \gamma}(Y), \ast_\gamma] = \hbar U'_{\hbar \gamma}([Y, \gamma]) \]

for any vector field \( Y \) on manifold \( M \). Let \( V^1_Y \) be the vector field (of degree 0) on formal pointed manifold \( g^{(1)}_1 \hat{\otimes} m \) equal to \([Y, \hbar \gamma]\) at point \( \hbar \gamma \). It is the coderivation of coalgebra \( S^+(g^{(1)}_1) \hat{\otimes} m \) such that:

\[ V^1_Y(e^{\hbar \gamma} - 1) = [Y, \hbar \gamma]e^{\hbar \gamma}. \]

It restricts to the submanifold \((MC)_1\).
The $L_\infty$-quasi-isomorphism $U$ is injective, by injectivity of first Taylor coefficient $U_1$. Let $(MC)_2$ be the set of solutions of Maurer-Cartan equation in $\mathfrak{g}^{(1)}_2 \otimes \mathfrak{m}$. Let $V^2_Y$ be the vector field (of degree 0) on formal pointed manifold $U(\mathfrak{g}^{(1)}_1 \otimes \mathfrak{m})$ equal to $[U'_{h \gamma}(Y), *_{\gamma}]$ at point $h \gamma$. It is the coderivation of the image coalgebra $U(S^+(\mathfrak{g}^{(1)}_1 \otimes \mathfrak{m}))$ such that:

$$V^2_Y(e^{h \gamma} - 1) = [U'_{h \gamma}(Y), *_{\gamma}] e^{h \gamma}.$$ 

Clearly vector field $V^2_Y$ restricts to $U((MC)_1) \subset (MC)_2$. Adding multiplication $m$ we identify $(MC)_2$ with the set of all star products, and $U((MC)_1)$ with the set $(FSP)$ of formality star products.

**Proposition IV.1.**

$$U \circ V^1_Y = V^2_Y \circ U.$$

*Proof.* We have:

$$U \circ V^1_Y(e^{h \gamma} - 1) = U'_Y([Y, h \gamma]) e^{h \gamma}$$

and:

$$V^2_Y \circ U(e^{h \gamma} - 1) = [U'_{h \gamma}(Y), *_{\gamma}] e^{h \gamma}.$$

The result follows then immediately from equation 1) of theorem III.3.

It is clear that we have $V^2_Y(\tau) = O(h)$ for any $\tau \in U((MC)_1)$. We have then:

**Corollary IV.2.**

Let $\gamma_Y = e^{ad h Y} \gamma$ be the transformation of formal Poisson 2-tensor $\gamma$ under the formal diffeomorphism $e^{h Y}$. It clearly belongs to $(MC)_1$ and the star-product $*_{\gamma_Y}$ constructed from $\gamma_Y$ by means of $L_\infty$-morphism $U$ is the transformation of star-product $*_{\gamma}$ under the formal diffeomorphism $e^{V^2_Y}$ of formal pointed manifold $(FSP)$.

Of course differential operator $U'_Y(Y)$ is not a vector field on $M$ in general, so diffeomorphism $e^{V^2_Y}$ of formal pointed manifold $(FSP)$ does not come from a formal diffeomorphism of $M$. Correspondences $Y \mapsto V^1_Y$ and $Y \mapsto V^2_Y$ are injective and respect brackets, i.e.:

$$V^1_{[Y,Z]} = [V^1_Y, V^1_Z],$$

and, as a consequence of proposition IV.1:

$$V^2_{[Y,Z]} = [V^2_Y, V^2_Z].$$

By exponentiation we get then the following result:
Theorem IV.3.
Let $G_1$ and $G_2$ denote the gauge groups of $g_1$ and $g_2$ respectively, and let $\mathcal{G}$ be the group of formal diffeomorphisms of $(FSP)$. The correspondence:

$$\iota : G_1 \rightarrow \mathcal{G}$$

$$e^\gamma \mapsto e^{\iota_2 \gamma}$$

is an embedding of $G_1$ into the group $\mathcal{G}$ of formal diffeomorphisms of $(FSP)$ such that:

$$*_{g, \gamma} = \iota(g) *_{\gamma}.$$ 

Remark: On one hand we have natural embeddings:

$$G_1 \subset G_2, \quad G_1 \subset \mathcal{G},$$

but we must stress on the other hand that vector fields $\mathcal{V}_1^Y$ are linear whereas vector fields $\mathcal{V}_2^Y$ are not, and not even affine (this is due to the non-linearity of $\mathcal{U}$). The embedding of $G_1$ constructed above is then nontrivial, in the sense that the image of $G_1$ is not even contained in the second gauge group $G_2$.

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