Noncommutativity Approach to Supersymmetry on the Lattice:
SUSY Quantum Mechanics and an Inconsistency

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It is argued that the noncommutativity approach to fully supersymmetric field theories on the lattice suffers from an inconsistency. Supersymmetric quantum mechanics is worked out in this formalism and the inconsistency is shown both in general and explicitly for that system, as well as for the Abelian super BF model.

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I. INTRODUCTION

In supersymmetric theories on the lattice it is very useful to maintain exact SUSY invariance, at least for some of the supersymmetry transformations (for a review see e.g. 1,2), in order to reduce fine-tuning in the continuum limit. A well-known obstacle against this comes from the failure of the Leibniz rule (the product rule of differentiation) for the difference operator on the lattice. The latter also spoils the Leibniz rule for the supersymmetry transformations is

\[ \{Q_1, Q_2\} = \{Q_2, Q_1\} = 2H, \]

\[ \{Q_1, Q_2\} = 0, \quad [H, Q_i] = 0 \quad \text{for } i = 1, 2, \]

where \( Q_1 \) and \( Q_2 \) are two real supercharges and \( H \) is the Hamiltonian. A superspace representation of this algebra is given by

\[ H = \partial_t, \quad Q_1 = \partial_{\theta^1} + \theta^1 \partial_t, \quad Q_2 = \partial_{\theta^2} + \theta^2 \partial_t, \]

with Majorana Grassmann coordinates \( \theta^i \) fulfilling

\[ \{\theta^i, \theta^j\} = 0, \quad \{\partial_{\theta^i}, \theta^j\} = \delta_{ij}, \quad [t, \theta^i] = 0. \]

Having numerical simulations on the lattice in mind, we have moved to a Euclidean description. A Hermitian superfield

\[ \Phi(t, \theta^1, \theta^2) = \phi(t) + i \theta^1 \psi_1(t) + i \theta^2 \psi_2(t) + i \theta^1 \theta^2 D(t) \]

contains two real bosonic fields, \( \phi \) and \( D \), and two Majorana fermions \( \psi_{1,2} \). The action of supersymmetric quantum mechanics is given by

\[ S = \int dt d\theta^1 d\theta^2 \frac{1}{2} D^2 \Phi D \Phi + i F(\Phi), \]

where \( F(\Phi) \) is the superpotential. The superderivatives \( D_{1,2} \) are defined as

\[ D_1 = \partial_{\theta^1} - \theta^1 \partial_t, \quad D_2 = \partial_{\theta^2} - \theta^2 \partial_t. \]

They satisfy the following algebra

\[ \{D_1, D_1\} = \{D_2, D_2\} = -2 \partial_t, \quad \{D_1, D_2\} = 0, \]

\[ 1 \text{ Just like gauge invariance is kept in lattice gauge theories.} \]

\[ 2 \text{ A modification of the supersymmetry transformations yielding an exact lattice supersymmetry has been written down in 1,2, but to be practical needs to be expanded in powers of the coupling constant.} \]
and anticommute with the supercharges.

We take the superpotential to be \( F(\Phi) = \frac{1}{2}m\Phi^2 + \frac{1}{3}g\Phi^3 \), the choice \( \frac{1}{3}g\Phi^3 \) for the interaction term works fully analogously. After integrating out the Grassmann variables the action reads

\[
S = \int dt \left[ \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}D^2 - \frac{1}{2}(\psi_1 \partial_t \psi_1 + \psi_2 \partial_t \psi_2) - i(m + 3g\phi^2)\psi_1 \psi_2 - D(m\phi + g\phi^3) \right].
\]

The field \( D \) is nondynamical and using its equation of motion \( D = -m\phi - g\phi^3 \) one arrives at the on-shell action

\[
S = \int dt \left[ \frac{1}{2}(\partial_t \phi)^2 + \frac{m^2}{2} \phi^2 - \frac{1}{2}(\psi_1 \partial_t \psi_1 + \psi_2 \partial_t \psi_2) - i(m + 3g\phi^2)\psi_1 \psi_2 + mg\phi^4 + \frac{g^2}{2} \phi^6 \right].
\]

By letting the supercharges act on the superfield \( \Phi \) it is a straightforward exercise to find the supersymmetry variations \( \delta_1 = \epsilon^i Q_1 \) and \( \delta_2 = \epsilon^2 Q_2 \) of the component fields, see Table I. Omitting the \( \epsilon \) parameters, the supersymmetry transformations of the component fields are denoted by \( s_i \), e.g. \( s_1 \phi = i\psi_1 \). Since the supercharges obey the Leibniz rule, it follows that the latter also holds for the supersymmetry transformations \( s_i \):

\[
s_i \left[ f_1 f_2 \right] = \left[ s_i f_1 \right] f_2 + (-1)^{f_1} f_1 \left[ s_i f_2 \right],
\]

where \( |f| \) is 0 for bosonic \( f \) and 1 for fermionic \( f \).

Letting the supersymmetry transformations \( s_1 \) and \( s_2 \) act on the action \( S \), the Leibniz rule is used to show that the Lagrangian is supersymmetry invariant (up to a total derivative).

### B. Definition on the Lattice

In this section the noncommutativity approach to SUSY on the lattice \( \mathbb{R}^4 \mathbb{R}^2 \mathbb{R}^2 \mathbb{R}^2 \) is introduced by means of SUSYQM. Periodic boundary conditions will be assumed for all fields.

Naturally, the derivative \( \partial_t \) is replaced on the lattice by the forward/backward difference operator \( \Delta_\pm \)

\[
\Delta_\pm f(t) = \pm \frac{1}{2n} (f(t \pm 2n) - f(t)),
\]

where \( n \) corresponds to the shift of one lattice spacing. Why the difference is taken over two lattice spacings will become clear later. This difference operator does not obey the conventional Leibniz rule, but rather a ‘modified Leibniz rule’:

\[
\Delta_\pm \left[ f_1(t) f_2(t) \right] = [\Delta_\pm f_1(t)] f_2(t) + f_1(t \pm 2n) [\Delta_\pm f_2(t)].
\]

Taking the lattice supercharges to be \( Q_1 = \partial_{\theta^i} + \theta^i \Delta_\pm \) it is obvious that they do not obey the Leibniz rule either and a naive approach would run into problems defining supersymmetric actions on the lattice.

The main idea of the noncommutativity approach is to introduce a noncommutativity between the bosonic and Grassmann coordinates of superspace\(^3\) in such a way that both supersymmetry transformations \( \epsilon^i Q_1 \) and \( \epsilon^2 Q_2 \) obey the normal Leibniz rule and then to proceed along the lines of continuum SUSY.

When the lattice supercharges act on products of functions similar to Eq. (12), \( \theta^i \) has to be ‘pulled through’ the first function \( f_1 \) and this can be used to restore the Leibniz rule for \( \theta^i \Delta_\pm \) if one demands

\[
\theta^i f(t) = f(t - a_i) \theta^i \leftrightarrow [t, \theta^i] = a_i \theta^i \quad \text{(no sum)},
\]

with \( a_i = \pm 2n \). However, this new relation implies another noncommutativity between \( \partial_{\theta^i} \) and \( t \), with the opposite sign

\[
[t, \partial_{\theta^i}] = -a_i \partial_{\theta^i} \quad \text{(no sum)},
\]

which makes it impossible to obtain the Leibniz rule for both terms in the supercharges at the same time.

In contrast, the supersymmetry transformations \( \epsilon^i Q_1 \) (no sum) can obey the Leibniz rule, provided the same noncommutativity is introduced for the variation parameters \( \epsilon^i \)

\[
[t, \epsilon^i] = a_i \epsilon^i \quad \text{(no sum)}.
\]

Now the Leibniz rule is obvious for the terms \( \epsilon^i \partial_{\theta^i} \), while for the terms \( \epsilon^i \theta^i \Delta_\pm \) it leads to \( 2a_i = \pm 2n \). The corresponding Jacobi identities have been checked to give no further constraints.

Altogether, the supercharges

\[
Q_1 = \partial_{\theta^i} + \theta^i \Delta_\pm, \quad Q_2 = \partial_{\theta^2} + \theta^2 \Delta_\pm,
\]

(with independent signs) fulfill the algebra

\[
\{Q_1, Q_1\} = 2\Delta_\pm, \{Q_2, Q_2\} = 2\Delta_\pm, \{Q_1, Q_2\} = 0
\]

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\(^3\) Interestingly, a calculus with \( [x_\mu, dx_\nu] = dx_\mu \) has been worked out in \[12,13\] keeping the Leibniz rule for the discrete derivative. It seems attractive to identify the supercoordinates \( \theta_{1,2} \) with differentials \( dx_{1,2} \) (which are Grassmannians, too), but this introduces new coordinates \( x_{1,2} \).
with the same signs as in Eq. (16). Their supersymmetry transformations $\epsilon^1 Q_1$ and $\epsilon^2 Q_2$ obey the Leibniz rule, provided the noncommutativity relations (13), (14) and (15) hold with

$$a_1 = \pm n, \quad a_2 = \pm n,$$

again with the signs as in Eq. (16). The Grassmannians $T^\theta$ and $\epsilon^i$ still anticommute.

Having a normal Leibniz rule for the supersymmetry transformations, we proceed to build a supersymmetric lattice action. We take the Hermitian superfund $\Phi$ like in the continuum with $t$ now labeling the lattice points. Acting with the supercharges on it, one immediately sees that Table I is also valid on the lattice upon replacing $\partial_1$ by $\Delta_\pm$. But the noncommutativity shows up when moving the $\theta$’s through the fields, e.g.

$$\Phi = \phi(t) - i\psi_1(t - a_1)\theta^1 - i\psi_2(t - a_2)\theta^2 + iD(t - a_1 - a_2)\theta^2\theta^1,$$

and especially when bringing the $\theta$’s to the left in the computation of the action.

From (19) it can be seen that $t - a_1 = t \pm n$ needs to be a lattice point, which is why we took the difference over two lattice spacings in Eq. (11).

In order to construct the kinetic term for this multiplet, we use the lattice superderivatives

$$D_1 = \partial_1 - \theta^1\Delta_\pm, \quad D_2 = \partial_2 - \theta^2\Delta_\pm$$

with the signs as in Eq. (16) and allow for a relative shift $\alpha$ in $D_2\Phi D_1\Phi$.

$$n \sum_t \int d\theta^2 d\theta^1 \frac{1}{2} D_2\Phi(t - \alpha)D_1\Phi(t) =$$

$$\frac{n}{2} \sum_t [\Delta_\pm \phi(t + a_1 - \alpha)\Delta_\pm \phi(t) - D(t + a_2 - \alpha)D(t) - (\psi_1(t)\Delta_\pm \psi_1(t - \alpha) + \psi_2(t + a_1 + a_2 - \alpha)\Delta_\pm \psi_2(t))]].$$

This action is supersymmetry invariant for any choice of the shift $\alpha$.

At this point we invoke two physical arguments. When integrating out $D$ later, it is desirable to have an equation of motion like $D(t) = f[\phi]$ which comes out only if the corresponding term in the action reads $D^2(t)/2$, i.e. if the shift $\alpha$ is fixed to $a_2$ (all other choices would lead to a nonlocal equation $D(t + T) + D(t - T) = f[\phi]$ with some $T$).

Moreover, we wish to obtain a strictly positive kinetic term $(\Delta^2 \phi(t))/2$ for the bosonic component. This gives more constraints depending on the choice of the difference operators in (16). By virtue of $\Delta_\pm f(t) = \Delta_\pm f(t + 2n)$ and changes in the summation variable $t$ all solutions to these constraints are equivalent to taking just the plus signs in Eqs. (16), (17), (18) and (20). This leads to the following kinetic part of the action

$$\frac{n}{2} \sum_t [(\Delta^2 \phi(t))^2 - D^2(t) - (\psi_1(t + n)\Delta_\pm \psi_1(t) + \psi_2(t + n)\Delta_\pm \psi_2(t))].$$

Observe that the argument $t + n$ of the spinors $\psi_{1,2}$ is right in the middle of the two lattice points used in the difference operator.

For the mass and interaction terms there seems to be no particular reason to introduce relative shifts and we find the lattice analogue of the continuum action (3) to be

$$S = n \sum_t \frac{1}{2} (\Delta^2 \phi)^2 - \frac{1}{2} D^2$$

$$- \frac{1}{2} (\psi_1(t + n)\Delta_\pm \psi_1(t + n) \Delta_\pm \psi_2(t + n)) / 2$$

$$- (D^2(t + n) + D^2(t)) / 6,$$

$$- D^2(m\phi(1) + g\phi(3))]$$

(23)

where unshifted arguments ($t$) are not displayed for readability. We used $\phi^{(a)}$ to abbreviate the lattice terms that converge to the corresponding powers $\phi^a$ in the continuum limit

$$\phi^{(1)}(t) = [\phi(t + 2n) + \phi(t)] / 2,$$

$$\phi^{(2)}(t) = [\phi(t + 2n) + \phi(t + n) + \phi(t)] + \phi^2(t + 2n) + \phi^2(t) / 6,$$

$$\phi^{(3)}(t) = [\phi^2(t + 2n) + \phi^2(t)] \phi(t + 2n) + \phi(t) / 4.$$ (24) (25) (26)

While the kinetic part of the lattice action (26) only differs by one shift from the naive translation of the continuum, the complicated structure of the $\phi^{(a)}$’s is the nontrivial result of the noncommutativity approach and is needed for supersymmetry invariance to be discussed in Section II. However, the arguments of the fields in the $\phi^{(a)}$’s are never shifted over more than two lattice spacings.

Like in the continuum, the field $D$ can be replaced by $-m\phi^{(1)} - g\phi^{(3)}$. We will consider this action in slightly more detail now, in order to compare to existing SUSYQM lattice actions. With Dirac spinors $\psi = (\psi_1 + i\psi_2) / \sqrt{2}$ it reads

$$S_{\text{spin}} = n \sum_{t,t'} \frac{1}{2} \phi(t)B_{tt'} \phi(t') + \tilde{\psi}(t)F_{tt'} \psi(t'),$$

where the fermion enters the mass and interaction term in the form $\psi^{(1)}$ of (24), too. Without interactions we write the action in the bilinear form

$$S_{\text{spin}}^{\text{gen}} = n \sum_{t,t'} \frac{1}{2} \phi(t)B_{tt'} \phi(t') + \tilde{\psi}(t)F_{tt'} \psi(t'),$$

(27)

(28)

with the following bosonic and fermionic matrix:

$$B_{tt'} = \frac{1}{2} \left( \frac{1}{n^2} + m^2 \right) \delta_{tt'}$$

$$+ \frac{1}{4} \left( \frac{1}{n^2} + m^2 \right) (\delta_{t+1,t'} + \delta_{t-1,t'}),$$

(29)

$$F_{tt'} = \frac{1}{2} \left( \frac{1}{n} - m \right) \delta_{t-1,t'} + \frac{1}{2} \left( \frac{1}{n} - m \right) \delta_{t+1,t'}.$$ (30)
The supersymmetry of the action expresses itself by the fact that $B = F TF$ such that the corresponding determinants in the path integral cancel each other. It can straightforwardly be seen that in the noninteracting case our action amounts to the one given in [6] by Eq. (2.18) putting $r/a = -m$, up to a relative minus sign for the fermion terms. However, for the case of a $\Phi^4$-interaction that action gives a diagonal term coupling the fermions to $\phi^2$, while in our case there will always be terms like $\phi(t) \phi(t) \psi(t)$.

### III. THE INCONSISTENCY

#### A. The General Noncommutativity Approach and its Inconsistency

In this section we will show that the noncommutativity approach to lattice SUSY suffers from an inconsistency, independent of the dimensionality of space-time and the number of supersymmetries. Before doing so, we first review the general formalism briefly.

In higher dimensional spaces, like in the case of SUSYQM, the derivative operator $\partial_\mu$ is replaced on the lattice by the difference operator $\Delta_{\pm \mu}$ defined by

$$\Delta_{\pm \mu} f(x) = \pm \frac{1}{2} n_\mu (f(x \pm 2n_\mu) - f(x)),$$

where $n_\mu$ corresponds to the shift of one lattice spacing in the $\mu$-direction. Each $\Delta_{\pm \mu}$ obeys the modified Leibniz rule [12].

In a superspace representation with coordinates $(x^\mu, \theta^A)$ the supercharges can generically be written as

$$Q_A = \partial_\theta A + \frac{1}{2} f^\mu_{AB} \theta^B \Delta_{\pm \mu},$$

where we have immediately moved to the lattice using $\Delta_{\pm \mu}$. The general SUSY algebra reads

$$\{Q_A, Q_B\} = f^\mu_{AB} \Delta_{\pm \mu}.$$  \hspace{1cm} (33)

In complete analogy to the case of SUSYQM the supercharges will not obey the Leibniz rule, but the variations

$$[x, \theta^A] = a_A \theta^A, \quad [x, \epsilon^A] = a_A \epsilon^A \quad \text{(no sum)}$$

hold. The shift parameters $a_A$ (now vectors) ensure the Leibniz rule for the first term in $Q_A$, while the second one imposes the following conditions

$$a_A + a_B = \pm 2n_\mu \quad \text{for} \quad f^\mu_{AB} \neq 0.$$  \hspace{1cm} (35)

As before, the Jacobi identities are seen to give no further constraints.

4 We follow the conventions of [11].

The authors of [3] have found a solution to these relations for the twisted $N = 2$ supersymmetry algebra in two dimensions as well as for the twisted $N = 4$ supersymmetry algebra in four dimensions. As explained above, the relations can also be satisfied for the $N = 2$ supersymmetry algebra in one dimension.

At this point one can proceed as in the continuum with building a supersymmetric action out of (chiral) superfields. The superderivatives

$$D_A = \partial_\theta A - \frac{1}{2} f^\mu_{AB} \theta^B \Delta_{\pm \mu}$$

obey the algebra

$$\{D_A, D_B\} = -f^\mu_{AB} \Delta_{\pm \mu}, \quad \{D_A, Q_B\} = 0.$$  \hspace{1cm} (37)

The transformation rules $\epsilon_{AS} A$ for the component fields (like in Table I) can be derived by comparing the $\theta$-expansions of the superfields $\Phi_i$ of the theory with their supersymmetry variations $\epsilon_{AS} Q_A \Phi_i$ or, equivalently, by doing the same with the superfields $D_A \Phi_i, D_A D_B \Phi_i, \ldots$ (because the superderivatives and supercharges anticommute).

However, this noncommutativity approach is spoiled by an inconsistency. At the heart of it lies the fact that the supersymmetry transformations $s_A$ obey a modified Leibniz rule when acting on a product of (component) fields. This can be seen by evaluating the transformations of a product of fields

$$f_1 f_2 \rightarrow (f_1 + \epsilon_{AS} A_1)(f_2 + \epsilon_{AS} A_2) = f_1 f_2 + [\epsilon_{AS} A_1] f_2 + f_1 [\epsilon_{AS} A_2] + O(\epsilon^2) \equiv f_1 f_2 + \epsilon_{AS} A_1 f_2 + O(\epsilon^2)$$

This is the normal Leibniz rule for the supersymmetry transformations $\epsilon_{AS} A$. For the transformations $s_A$ we bring $\epsilon_A$ to the left using [24], omit it, and obtain

$$s_A [f_1(x) f_2(x)] = [s_A f_1(x)] f_2(x) + (-1)^{|f_1|} f_1(x + a_A) [s_A f_2(x)].$$  \hspace{1cm} (39)

Because the product $f_1 f_2$ of component fields is equal to $(-1)^{|f_1| |f_2|} f_2 f_1$, one would expect the corresponding supersymmetry transformations to agree, too. However, because of the different shifts induced by $\epsilon^A$, one gets

$$s_A [(-1)^{|f_1| |f_2|} f_2(x) f_1(x)] = [s_A f_1(x)] f_2(x + a_A) + (-1)^{|f_1|} f_1(x + a_A) [s_A f_2(x)].$$  \hspace{1cm} (40)

This means that the supersymmetry transformations do not evaluate to the same expression when acting on the product $f_1 f_2$ and the equivalent expression
\((-1)^{|f_1||f_2|} f_2 f_1,\) respectively. Instead, the difference of the two can be written as

\[
[s_A f_1(x)](f_2(x + a_A) - f_2(x))
\]

\[+ (-1)^{|f_1|}(f_1(x) - f_1(x + a_A))[s_A f_2(x)],
\]

which is not even a total difference, i.e. does not lead to a boundary term (in which case the difference would be irrelevant).

Because all shifts are proportional to the lattice spacing, the ambiguity disappears in the continuum limit, but to keep exact supersymmetry at finite spacing remains a problem\(^6\).

In the derivation we have taken the natural point of view that the supersymmetry transformations of all combinations of component fields follow – via the Leibniz rule – from the transformations of the individual fields, Table I. Having obtained an action and the supersymmetry transformations of component fields (plus the noncommutativity of \(t\) and \(\epsilon\)) one should be allowed to neglect the auxiliary superspace formalism. The lattice action resulting from this formalism is a particular discretization of the continuum action, but the supersymmetry transformations have been shown to be inconsistent.

Supersymmetry transformations of products of component fields derived from products of superfields, e.g. by \(\theta\)-expanding \(\Phi^2\) and \(\Delta \Phi^2\), should agree with the ones used so far for consistency. If other supersymmetry rules for the products of component fields are obtained in this way, then an unacceptable memory to superspace is introduced and, what is more, products of fields are treated like new independent fields.

Also in this approach we are able to derive the inconsistency. Any of the superfields \(\Phi_1, D_A \Phi_1, D_A D_B \Phi_1, \ldots\) can be written as

\[
F = f + \sum_B \theta^B s_B f + O(\theta^2),
\]

because then \(\epsilon^A Q_A F = \epsilon^A s_A f + O(\theta)\) gives the correct supersymmetry transformation for \(f\). This first component \(f\) runs over all component fields in the theory when taking all such superfields \(F\). Now we consider the product

\[
F_1(x) F_2(x) = f_1(x) f_2(x) + \sum_B \theta^B \{[s_B f_1(x)] f_2(x) + (-1)^{|f_1|} f_1(x + a_B)[s_B f_2(x)]\} + O(\theta^2).\]

the variation of which is given by

\[
\epsilon^A Q_A [F_1(x) F_2(x)] = \epsilon^A \{[s_A f_1(x)] f_2(x) + (-1)^{|f_1|} f_1(x + a_A)[s_A f_2(x)]\} + O(\theta).
\]

From the first components of (13) and (14) we read off Eq. (39) for the supersymmetry transformation of the product of any two component fields.

In general, the product of two superfields depends on their order (because of the noncommutativity \(\Delta\)), but the first component is always the same (up to a sign). Therefore, looking in the same manner at the variation of

\[
F_2(x) F_1(x) = f_2(x) f_1(x) + \sum_B \theta^B \{[s_B f_2(x)] f_1(x) + (-1)^{|f_2|} f_2(x + a_B)[s_B f_1(x)]\} + O(\theta^2)
\]

and multiplying with the factor \((-1)^{|f_1||f_2|}\) one arrives at Eq. (40) for the supersymmetry transformation of the product of two component fields with the opposite order. Hence, the inconsistency is redetermined, because it is the disagreement between (39) and (40).

As a consequence, there is an ambiguity in showing supersymmetry invariance of lattice actions. Indeed, suppose starting with a term \(f_1 f_2\) in a lattice action \(S\) one can show supersymmetry invariance of this action, i.e. \(s_A S = 0\). The same action will not be supersymmetry invariant when writing this term as \((-1)^{|f_1||f_2|} f_2 f_1\), because now \(s_A S\) will be equal to a difference term of the form \((41)\), which does not vanish. Therefore, the noncommutativity approach is seen to be inconsistent.

### B. Explicit Calculation

**Supersymmetric Quantum Mechanics**

In this section we would like to demonstrate how the inconsistency emerges when checking the supersymmetry invariance of the proposed SUSYQM action (22). We do this by explicitly calculating the supersymmetry variation of the mass terms (which of course have to be invariant on their own). For the variation under \(\delta_1\) one can group the four terms proportional to \(m\) into two pairs \(M\) and \(N\) (which, however, get mixed under \(\delta_2\))

\[
M = D(t) \phi(t) - i \psi_2(t + n) \psi_1(t),
\]

\[
N = D(t) \phi(t + 2n) + i \psi_1(t + n) \psi_2(t).
\]

Using the Leibniz rule for \(\delta_1\) and the supersymmetry variations of Table I we get

\[
\delta_1 M = -\epsilon^1 \Delta_+ \psi_2(t) \phi(t) + i D(t) \epsilon^1 \psi_1(t)
\]

\[
- i \epsilon^1 D(t + n) \psi_1(t) + \psi_2(t + n) \epsilon^1 \Delta_+ \phi(t).
\]

\(^6\) From the corresponding modified Leibniz rule for the difference operator one might be tempted to conclude that this operator suffers from the same problem, but its definition \(\pm 2n \Delta_{\pm} f f = [f f_2](t + 2n) - [f f_2](t)\) is invariant under interchanging \(f_1\) and \(f_2\).
Moving $e^1$ to the left results in two shifts (and a sign change)

$$\delta_1 M = e^1 [-\Delta_+ \psi_2(t) \phi(t) + iD(t+n) \psi_1(t) - iD(t+n) \psi_1(t) - \psi_2(t+2n) \Delta_+ \phi(t)]$$

Since the total difference vanishes under the sum over $t$, this term is supersymmetry invariant as it should. The various terms cancel as in the continuum.

However, if we interchange $D(t)$ and $\phi(t)$, which does not change $M$,

$$\tilde{M} = \phi(t) D(t) - i \psi_2(t+n) \psi_1(t),$$

then $e^1$ induces different shifts leading to

$$\delta_1 \tilde{M} = e^1 [i \psi_1(t) D(t) - \phi(t+n) \Delta_+ \psi_2(t) - iD(t+n) \psi_1(t) - \psi_2(t+2n) \Delta_+ \phi(t)]$$

Hence, written in this form, the term turns out not to be supersymmetry invariant.

Actually, the other mass term $N$ is not invariant as it stands, but

$$\tilde{N} = \phi(t+2n) D(t) + i \psi_1(t+n) \psi_2(t)$$

is.

Since $M = \tilde{M}$ and $N = \tilde{N}$ (the fields are treated as (anti)commuting functions in a path integral), the definition of the supersymmetry transformations of the mass terms is not unique, nor is the question whether (the sum of) these terms are supersymmetry invariant.

Of course this problem does not occur for squares (like $(\Delta_+ \phi)^2$ in (28)). It can be checked that it neither appears for the kinetic term of the spinors (the second line in (28)). However, the ambiguity problem persists for the on-shell action (which is not just a sum of squares) because of the third line in (28), which as a fermionic term does not change when integrating out $D$.

### The Abelian Super BF Model

As a second concrete example of the inconsistency, the Abelian super BF model in two dimensions is investigated. This model has been formulated using twisted $N = 2$ supersymmetry in Paragraph 5.1 of [3] (the conventions of [3] are followed). In this algebra, there are four different supersymmetry transformations, $\delta, \tilde{\delta}, \delta_1, \tilde{\delta}_2$, each with its own shift parameter, respectively $\alpha, \tilde{\alpha}, \alpha_1, \tilde{\alpha}_2$. The relations (49) can in this case be solved and written as

$$\alpha + \tilde{\alpha} + \alpha_1 + \tilde{\alpha}_2 = 0, \quad \alpha + \alpha_1 = \alpha_2, \quad \alpha + \alpha_2 = \alpha_1. \quad (53)$$

Following the noncommutativity approach the lattice action for the abelian super BF model is given by

$$S_{\text{BF}} = \sum_{x} \{ \phi(x-a) \epsilon_{\mu \nu} \Delta_{+ \mu} \omega_{\nu}(x+a) \} + \hat{b}(x-a) \Delta_{- \mu} \omega_{\nu}(x+a) - i \hat{c}(x) \Delta_{+ \mu} \Delta_{- \nu} \omega(x)$$

$$+ i \hat{\rho}(x-a) \Delta_{+ \mu} \Delta_{- \nu} \omega(x), \quad (54)$$

The supersymmetry transformations for $s$ are given in the table below, with $a_A$ taken to be 2a (cmp. Eq. (18) in [3]).

| $\phi$          | $\hat{s}\phi$ |
|-----------------|---------------|
| $\phi(x+a)$     | 0             |
| $\phi(x+a)$     | $i\hat{s}(x+a)$ |
| $\omega_\nu(x+a) $ | $\Delta_{+ \mu} \omega_\nu(x+a) $ |
| $\omega(x) $    | $-ib(x+a)$    |
| $\phi(x+a)$     | $i\hat{s}(x+a)$ |
| $\phi(x+a)$     | 0             |
| $\omega(x) $    | 0             |

**TABLE II:** The $N = 2$ twisted supersymmetry transformation $\hat{s}$ for the abelian super BF model, copied from [3]. The fields above the line are bosonic, the ones below fermionic.

Applying $\hat{s}$ to the action it follows that

$$\hat{s} S_{\text{BF}} = \sum_{x} \{ i \hat{s}(x+a) \epsilon_{\mu \nu} \Delta_{+ \mu} \omega_{\nu}(x+a) \} + \hat{s}(x-a) \epsilon_{\mu \nu} \Delta_{+ \mu} \omega_{\nu}(x-a)$$

$$- i \hat{s}(x+a) \epsilon_{\mu \nu} \Delta_{- \mu} \omega_{\nu}(x+a) \} + \hat{s}(x) \epsilon_{\mu \nu} \Delta_{+ \mu} \Delta_{- \nu} \omega(x)$$

$$+ i \hat{s}(x-a) \epsilon_{\mu \nu} \Delta_{+ \mu} \Delta_{- \nu} \omega(x), \quad (55)$$

where identities for the shift parameters and $\epsilon_{\mu \nu} \Delta_{+ \mu} \Delta_{+ \nu} = 0$ have been used. In other words, the action seems to be invariant under the supersymmetry transformation $\hat{s}$.

Up to this point we agree with the authors of [3]. However, interchanging the first two fields in the action, i.e.

7 In the conventions of [3] the transformation $\hat{s}$ ‘carries a shift’, e.g. $\hat{s}\phi(x) = -ib(x+a)$, whereas the transformations $s_i$ in our conventions do not, e.g. $s_1 \phi(x) = i\psi_1(x)$, Table 1. This difference in convention is due to a different way of defining the superfield, compare Eq. (4.29) or (5.1) in [3] with our Eq. 4.
writing the action as
\[ S_{\text{lat}}^{\text{BF}} = \sum_x \left[ \epsilon_{\mu\nu} \Delta_{\mu\nu} \phi(x + a_\nu) \phi(x - a_\mu) + \text{rest unchanged} \right] \] (56)
and applying the supersymmetry transformation \( \tilde{\phi} \) to this 'new form' of the action leads to
\[ S_{\text{lat}}^{\text{BF}} = \sum_x \left[ \epsilon_{\mu\nu} \Delta_{\mu\nu} + \epsilon \xi(x) \phi(x - \tilde{a}) \right. 
+ \epsilon_{\mu\nu} \Delta_{\mu\nu} \xi(x) (x + a_\nu + 2a) i \rho(x + a - \tilde{a}) 
+ \tilde{b}(x + a) \Delta_{\mu\nu} \xi(x) - \tilde{b}(x + a) \Delta_{\mu\nu} \xi(x) 
- i \rho(x + a - \tilde{a}) \epsilon_{\mu\nu} \Delta_{\mu\nu} \xi(x) (x + a_\nu - a_1 - a_2 - a - \tilde{a}) \right] 
= \sum_x i \rho(x + a - \tilde{a}) \epsilon_{\mu\nu} \Delta_{\mu\nu} \xi(x) (x + a_\nu + 2a) - \xi(x) (x + a_\nu) \right] \neq 0. \] (57)
Hence now the action appears not to be invariant under the supersymmetry transformation.

### IV. DISCUSSION AND CONCLUSIONS

We have presented the noncommutativity formalism to put supersymmetric quantum mechanics on the lattice, including masses and typical field-theoretic interactions. In the corresponding lattice action, Eq. (23), the different powers of the bosonic field \( \phi \) come in very special combinations, see Eqs. (24) and (26).

At first sight the approach keeps all supersymmetries exact. Indeed, showing the invariance of the lattice action in terms of superfields (the analogue of Eq. 5) works as in the continuum (using the Leibniz rule of \( e^A Q_A \)). However, we have pointed to an inconsistency, namely that the supersymmetry transformation of products of component fields is not uniquely defined. Therefore, the supersymmetry invariance of the lattice action in terms of component fields is ambiguous. We have shown this problem to be present in the noncommutativity approach in general and also demonstrated it extensively for (the mass term of) SUSYQM and the abelian super BF model. We have not considered the recent SUSY lattice gauge theories \([14]\), where there is a chance that the inconsistency is absent, because the link nature of the variables prevents one from naively interchanging fields.

In our view the inconsistency is a remainder of the noncommutative algebra \([t, \epsilon] \neq 0\). This structure is somewhat hidden in the action, which seems to contain only functions of ordinary numbers (integers) \( t \) as is very useful for numerical simulations. However, \( t \) cannot be just a number, as any number commutes with everything in any algebra. It seems to be the too naive treatment of (fields as functions of) the coordinates which spoils the definition of supersymmetry in such systems.

A natural step to establish a more profound formalism would be to promote all objects to \( T \times T \)-matrices \((T \) being the number of lattice sites), to replace the sum over \( t \) by the trace and so on. We have tried to represent the component fields by diagonal matrices and \( \epsilon \) (and \( \theta \)) by an off-diagonal one, in order to realize the noncommutativity. However, the supersymmetry variations turned out to be either trivially vanishing or to suffer from the same inconsistency. The only way out seems to be non-sparse matrices, which increases the number of degrees of freedom considerably (and slows down numerical simulations).

An alternative route to follow could be to define a modified product of fields, as is typical for functions over non-
commutative spaces.

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[1] S. Catterall, “Dirac-Kaehler fermions and exact lattice supersymmetry”, PoS LAT2005 (2005) 006, [arXiv:hep-lat/0509136].
[2] J. Giedt, “Deconstruction and other approaches to supersymmetric lattice field theories”, [arXiv:hep-lat/0602007].
[3] A. d’Adda, I. Kanamori, N. Kawamoto and K. Nagata, “Twisted superspace on a lattice”, Nucl. Phys. B 707 (2005) 100, [arXiv:hep-lat/0406029].
[4] A. d’Adda, I. Kanamori, N. Kawamoto and K. Nagata, “Twisted $N=2$ exact SUSY on the lattice for BF and Wess-Zumino”, Nucl. Phys. Proc. Suppl. 140 (2005) 754, [arXiv:hep-lat/0409092].
[5] A. D’Adda, I. Kanamori, N. Kawamoto and K. Nagata, “$N=D=2$ twisted supersymmetry on a lattice”, Nucl. Phys. Proc. Suppl. 140 (2005) 757.
[6] J. Giedt, R. Koniuk, E. Poppitz and T. Yavin, “Less naive about supersymmetric lattice quantum mechanics”, JHEP 0412 (2004) 033, [arXiv:hep-lat/0410041].
[7] S. Catterall and E. Gregory, “A lattice path integral for supersymmetric quantum mechanics”, Phys. Lett. B 487 (2000) 349, [arXiv:hep-lat/0006013].
[8] S. Catterall, “Notes on (twisted) lattice supersymmetry”, [arXiv:hep-lat/0510054].
[9] M.F.L. Golterman and D.N. Petcher, “A Local Interactive Lattice Model with Supersymmetry”, Nucl. Phys. B 319 (1989) 307.
[10] M. Bonini and A. Feo, “Wess-Zumino Model with Exact Supersymmetry on the Lattice”, JHEP 0409 (2004) 011, [arXiv:hep-lat/0402034].
[11] P. Cooper and B. Freedman, “Aspects Of Supersymmetric Quantum Mechanics”, Annals Phys. 146 (1983) 262.
[12] A. Dimakis, F. Müller-Hoissen and T. Striker, “Noncommutative differential calculus and lattice gauge theory”, J. Phys. A 26 (1993) 1927.
[13] I. Kanamori and N. Kawamoto, “Dirac-Kaehler fermion from Clifford product with noncommutative differential form on a lattice”, Int. J. Mod. Phys. A 19 (2004) 695, [arXiv:hep-th/0305094].
[14] A. d’Adda, I. Kanamori, N. Kawamoto and K. Nagata, “Exact extended supersymmetry on a lattice: Twisted $N=2$ super Yang-Mills in two dimensions”, Phys. Lett. B 633 (2006) 645, [arXiv:hep-lat/0507029].