Self-interacting Brownian motion*

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Abstract
We prove a property of Brownian bridges whose certain time-equidistant sequences of points are pairwise coupled by an interaction. Roughly saying, if the total time span $t$ of the bridge tends to infinity while the distance of its end points is fixed or increases slower than $\sqrt{t}$, the process asymptotically forgets this distance, just as in the absence of interaction. The conclusion remains valid if the bridge interacts in a similar way also with another set of trajectories. The main example for the interaction is the Coulomb potential.

1 Introduction
The problem we consider in this paper arises in Quantum Statistical Physics when a system of interacting particles is investigated by utilizing the Feynman-Kac formula [F1-2, G1-4]. It is in relation with Bose-Einstein condensation that we do not discuss here. Let $P^\beta_{0x}(d\omega)$ denote the Wiener measure for Brownian paths $\omega : [0, \beta] \to \mathbb{R}^\nu$ with $\omega(0) = 0$, $\omega(\beta) = x$. It is generated by the Gaussian functions
$$\psi_t(x) = \lambda_t^{-\nu} e^{-\pi x^2 / \lambda_t^2}, \quad 0 \leq t \leq \beta. \quad (1.1)$$
In physics $\lambda_t$ is the thermal de Broglie wave length at inverse temperature $t$, $\lambda_t = \sqrt{2\pi \hbar^2 t / m}$, where $m$ is the mass of the particle. In particular,
$$\int P^\beta_{0x}(d\omega) = \lambda_\beta^{-\nu} e^{-\pi x^2 / \lambda_\beta^2}. \quad (1.2)$$
Our starting observation is that
$$\lim_{x^2/\lambda_\beta^2 \to 0} \frac{\int P^\beta_{0x}(d\omega)}{\int P^\beta_{00}(d\omega)} = 1, \quad (1.3)$$
and we wish to prove a similar result in the case of interaction. For this we take a real function $u(x)$ on $\mathbb{R}^\nu$ that depends only on $|x|$; the interaction between points $x$ and $y$ is $u(x - y)$. The time span of the bridge is $n\beta$, where $\beta > 0$ and $n$ is a positive integer. Given a trajectory $\omega : [0, n\beta] \to \mathbb{R}^\nu$, its self-interaction is defined as
$$U(\omega) = \frac{1}{\beta} \sum_{0 \leq k < t < n-1} \int_0^\beta u(\omega(t\beta + t) - \omega(k\beta + t)) \, dt. \quad (1.4)$$

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and our interest is to give a lower bound on
\[ \int P_{0x}^\beta (d\omega) e^{-\beta U(\omega)} \] (1.5)
As to the physical background, let
\[ H_n = -\frac{\hbar^2}{2m} \sum_{i=1}^n \Delta x_i + \sum_{1 \leq i < j \leq n} u(x_j - x_i), \] (1.6)
the energy operator of \( n \) interacting particles of mass \( m \). For suitable choices of \( u \), \( e^{-\beta H_n} \) has an integral kernel, and by the Feynman-Kac formula we have
\[ \int e^{-\beta H_n} (0, x_2, \ldots, x_n; x_1, \ldots, x_n, x) \, dx_2 \ldots dx_n = \int P_{0x}^\beta (d\omega) e^{-\beta U(\omega)}. \] (1.7)
In addition, we shall consider also the problem when, apart from a Brownian bridge \( \omega_0 \) of time span \( n\beta \) and self-interaction \( U_1 \) generated by a pair potential \( u_1 \), there is also a set of \( M \) other trajectories \( \omega_i : [0, \beta] \to \mathbb{R}^\nu \)
\[ \omega^M = (\omega_1, \ldots, \omega_M), \quad \omega_i(0) = x_i, \quad \omega(\beta) = y_i. \] (1.8)
These may interact among themselves; their potential energy is then
\[ U_2(\omega^M) = \frac{1}{\beta} \sum_{1 \leq i < j \leq M} \int_0^\beta u_2(\omega_i(t) - \omega_j(t)) \, dt, \] (1.9)
with a pair potential \( u_2 \). The inclusion of this interaction is optional; what is important is an interaction between \( \omega_0 \) and \( \omega^M \),
\[ V(\omega_0, \omega^M) = \frac{1}{\beta} \sum_{k=0}^{n-1} \sum_{i=1}^M \int_0^\beta u_3(\omega_0(k\beta + t) - \omega_i(t)) \, dt, \] (1.10)
acting as an external field for \( \omega_0 \). Both \( u_2(x) \) and \( u_3(x) \) depend only on \( |x| \). The underlying energy operator in this case is that of \( n \) particles of mass \( m_1 \) with pair interaction \( u_1 \) and \( M \) particles of mass \( m_2 \) with pair interaction \( u_2 \), the two sets pairwise interacting via \( u_3 \):
\[ H_{n,M} = -\frac{\hbar^2}{2m_1} \sum_{i=1}^n \Delta x_i + \sum_{1 \leq i < j \leq n} u_1(x_j - x_i) + \sum_{i=1}^n \sum_{j=1}^M u_3(x_i - y_j) - \frac{\hbar^2}{2m_2} \sum_{i=1}^M \Delta y_i + \sum_{1 \leq i < j \leq M} u_2(y_j - y_i). \] (1.11)
Some special cases of the external field will be presented in Section 3.

2 Isolated Brownian motion

Here we treat a Brownian bridge of time span \( n\beta \) under self-interaction alone.

**Theorem 2.1** Let \( u \) be twice differentiable outside \( 0 \). If \( \Delta u \leq 0 \), then
\[ \lim_{n \to \infty, x^2/n \to 0} \frac{1}{\sqrt{n}} \int P_{0x}^\beta (d\omega) e^{-\beta U(\omega)} \geq 1. \] (2.1)

**Proof.** We start with the identity
\[ \int P_{0x}^\beta (d\omega) \phi(\omega) = \exp \left\{ -\frac{\pi x^2}{n\lambda^2} \right\} \int P_{00}^\beta (d\omega) \phi(\omega) \] (2.2)
where
\[ \tilde{\omega}(t) = \omega(t) + \frac{t}{n\beta} x, \] (2.3)
see [G1]. Applying this to \( \phi(\omega) = \exp(-\beta U(\omega)) \) and using the notation
\[ E_\omega(x) = \sum_{0 \leq k < l \leq n-1} \int_0^{\beta} u \left( \omega(l\beta + t) - \omega(k\beta + t) + \frac{l-k}{n} x \right) dt \] (2.4)
we obtain
\[ \int P_{00}^{n\beta}(d\omega)e^{-\beta U(\omega)} = \exp\left\{ \frac{-\pi x^2}{n\lambda_\beta^2} \right\} \int P_{00}^{n\beta}(d\omega)e^{-E_\omega(x)}. \] (2.5)

Now
\[ \lim_{n \to \infty, x^2/n \to 0} \exp\left\{ \frac{-\pi x^2}{n\lambda_\beta^2} \right\} = 1, \] (2.6)
so for
\[ I^{n\beta}(x) := \int P_{00}^{n\beta}(d\omega)e^{-E_\omega(x)} \] (2.7)
we must prove that
\[ \lim_{n \to \infty, x^2/n \to 0} \frac{I^{n\beta}(x)}{I^{n\beta}(0)} \geq 1. \] (2.8)
Actually, the stronger result
\[ \frac{I^{n\beta}(x)}{I^{n\beta}(0)} \geq 1, \] (2.9)
without taking the limit, also holds. Note that \( I^{n\beta}(x) \) is a spherical function, thus it has an extremum at \( 0 \). We show that \( I^{n\beta}(0) \) is a global minimum by proving that \( \Delta I^{n\beta}(x) \geq 0 \) everywhere. Indeed, this implies that \( I^{n\beta} \) is a convex even function of \( x \) along every axis passing through \( 0 \). Now
\[ \nabla I^{n\beta}(x) = -\int P_{00}^{n\beta}(d\omega)e^{-E_\omega(x)} \nabla E_\omega(x), \] (2.10)
\[ \Delta I^{n\beta}(x) = \int P_{00}^{n\beta}(d\omega)e^{-E_\omega(x)} \left( |\nabla E_\omega(x)|^2 - \Delta E_\omega(x) \right). \] (2.11)
So \( \Delta I^{n\beta}(x) \geq 0 \) if \( \Delta E_\omega(x) \leq 0 \), and the result follows from
\[ \Delta E_\omega(x) = \sum_{k \leq l} \int_0^\beta \Delta_x u \left( \omega(l\beta + t) - \omega(k\beta + t) + \frac{l-k}{n} x \right) dt \]
\[ = \sum_{k \leq l} \left( \frac{l-k}{n} \right)^2 \int_0^\beta \Delta u \left( \omega(l\beta + t) - \omega(k\beta + t) + \frac{l-k}{n} x \right) dt. \] (2.12)
The general solution of \( \Delta u \leq 0 \) is easily obtained. Define \( f : [0, \infty) \to \mathbb{R} \cup \{ \infty \} \) by \( u(y) = f(y^2) \). Because
\[ \Delta u(y) = 2 \left[ \nu f'(y^2) + 2y^2 f''(y^2) \right], \] (2.13)
one must find those \( f \) satisfying
\[ \nu f'(s) + 2sf''(s) \leq 0 \quad \text{for any } s > 0. \] (2.14)
Let \( g(s) \) be a nonnegative function, and consider the differential equation
\[ \nu f'(s) + 2sf''(s) = -g(s). \] (2.15)
Choose some \(a, b > 0\), then for the boundary conditions
\[ f(a) = c_1, \quad f'(b) = c_2b^{-\nu/2} \] the solution of Eq. (2.15) is
\[ f(s) = c_1 - \frac{1}{2} \int_a^s dx x^{-\nu/2} \int_b^x dt g(t)t^{\nu/2-1} + c_2 \begin{cases} s^{1-\nu/2}g^{1-\nu/2} - 1, & \nu \neq 2, \\ \ln s - \ln a, & \nu = 2. \end{cases} \]

(2.17)

**Examples.** For \(g = 0\) we can obtain the \(\nu\)-dimensional Coulomb potential of both signs. In general, \(f(s) = s^{-\alpha}\) is a solution of Eq. (2.15) for \(g(s) = \alpha(\nu - 2 - 2\alpha)s^{-\alpha-1}\) which is nonnegative if \(\alpha \leq \nu/2 - 1\). Another interesting solution is obtained in dimensions \(\nu \geq 3\) if
\[ a = b = +\infty, \quad c_1 = c_2 = 0, \quad g(s) = s^{-\alpha}, \quad \alpha > \nu/2 > 1. \]

It reads
\[ f(s) = \frac{-s^{-\alpha+1}}{(\alpha - 1)(2\alpha - \nu)} \quad \text{or} \quad u(x) = \frac{-|x|^{-2\alpha+2}}{(\alpha - 1)(2\alpha - \nu)}. \]

(2.18)

In three dimensions with \(\alpha = 4\) this is the form of the induced dipole-dipole interaction which appears in the Lennard-Jones potential.

### 3 Self-interacting Brownian bridge in an external field

The external field is created by a set of other trajectories, and is given by Eq. (1.10). Let
\[ E_{\omega_0}(x) = \sum_{0 \leq k < l \leq n-1} \int_0^\beta u_1 \left( \omega_0(l\beta + t) - \omega_0(k\beta + t) + \frac{t}{n}x \right) dt, \]
\[ E_{\omega_0, \omega_0^M}(x) = \sum_{k=0}^{n-1} \sum_{i=1}^M \int_0^\beta u_3 \left( \omega_0(k\beta + t) - \omega_i(t) + \frac{k\beta + t}{n\beta}x \right) dt, \]

(3.1)

and
\[ I^{n\beta}_{\omega_0}(x) = \int P_{00}^{n\beta}(d\omega_0)e^{-E_{\omega_0}(x) - E_{\omega_0, \omega_0^M}(x) - \beta U_2(\omega^M)}. \]

(3.2)

As before,
\[ \int P_{0\lambda}^{n\beta}(d\omega_0)e^{-\beta[U_1(\omega_0) + V(\omega_0, \omega^M) + U_2(\omega^M)]} = e^{-\pi x^2/\lambda_{n\beta}^2} I^{n\beta}_{\omega_0^M}(x), \]

(3.3)

where \(\lambda_{n\beta} = \sqrt{2\pi n^2\beta/m_1}\).

**Proposition 3.1** If \(u_1\) and \(u_3\) are twice differentiable outside the origin and \(\Delta u_1 \leq 0, \Delta u_3 \leq 0\), then \(\Delta I^{n\beta}_{\omega_0^M} \geq 0\).

**Proof.** Again, a direct computation gives
\[ \Delta I^{n\beta}_{\omega_0^M}(x) = \int P_{0\lambda}^{n\beta}(d\omega_0)e^{-E_{\omega_0}(x) - E_{\omega_0, \omega_0^M}(x) - \beta U_2(\omega^M)} \times \left[ \nabla E_{\omega_0}(x) + \nabla E_{\omega_0, \omega_0^M}(x) \right]^2 - \Delta E_{\omega_0}(x) - \Delta E_{\omega_0, \omega_0^M}(x). \]

(3.4)

Recalling the identity (2.12) and
\[ \Delta E_{\omega_0, \omega_0^M}(x) = \sum_{k=0}^{n-1} \sum_{i=1}^M \int_0^\beta \left( \frac{k\beta + t}{n\beta} \right)^2 \Delta u_3 \left( \omega_0(k\beta + t) - \omega_i(t) + \frac{k\beta + t}{n\beta}x \right) dt \]

(3.5)
One can differentiate inside the integral over \( \omega \) for Theorem 3.1. \( \omega \) preserves some dependence on \( \omega \) and therefore possibly

\[
E_{\omega_0, \omega M}(x) \gg E_{\omega_0, \omega M}(0) \tag{3.7}
\]

and therefore \( I^{n\beta}_{\omega M}(x) \leq I^{n\beta}_{\omega M}(0) \).

Let \( S \left( I^{n\beta}_{\omega M} \right)(x) \) denote a spherical symmetrization of \( I^{n\beta}_{\omega M}(x) \), obtained by integration over a set of \( \omega M \). Examples will be given below. As a function of \( x \), \( S \left( I^{n\beta}_{\omega M} \right)(x) \) depends only on \(|x|\), and may preserve some dependence on \( \omega M \). From Eq. (3.3),

\[
S \left( \int P^\beta_{0x}(d\omega_0)e^{-\beta(U_1(\omega_0)+V(\omega_0,\omega M)+U_2(\omega M))} \right)(x) = e^{-\pi x^2/\lambda^2} S \left( I^{n\beta}_{\omega M} \right)(x). \tag{3.8}
\]

One can differentiate inside the integral over \( \omega M \), so

\[
\Delta S \left( I^{n\beta}_{\omega M} \right)(x) = S \left( \Delta I^{n\beta}_{\omega M} \right)(x). \tag{3.9}
\]

**Theorem 3.1** For \( u_1 \) and \( u_3 \) as in Proposition 3.1,

\[
\lim_{n \to \infty, x^2/n \to 0} \frac{S \left( \int P^\beta_{0x}(d\omega_0)e^{-\beta(U_1(\omega_0)+V(\omega_0,\omega M)+U_2(\omega M))} \right)(x)}{S \left( \int P^\beta_{0x}(d\omega_0)e^{-\beta(U_1(\omega_0)+V(\omega_0,\omega M)+U_2(\omega M))} \right)(0)} \geq 1. \tag{3.10}
\]

**Proof.** \( \Delta I^{n\beta}_{\omega M} \geq 0 \) implies \( \Delta S \left( I^{n\beta}_{\omega M} \right) \geq 0 \) and, because \( S \left( I^{n\beta}_{\omega M} \right)(x) \) is spherical,

\[
S \left( I^{n\beta}_{\omega M} \right)(x) \geq S \left( I^{n\beta}_{\omega M} \right)(0).
\]

Then,

\[
\lim_{n \to \infty, x^2/n \to 0} \frac{S \left( \int P^\beta_{0x}(d\omega_0)e^{-\beta(U_1(\omega_0)+V(\omega_0,\omega M)+U_2(\omega M))} \right)(x)}{S \left( \int P^\beta_{0x}(d\omega_0)e^{-\beta(U_1(\omega_0)+V(\omega_0,\omega M)+U_2(\omega M))} \right)(0)} = \lim_{n \to \infty, x^2/n \to 0} \frac{S \left( I^{n\beta}_{\omega M} \right)(x)}{S \left( I^{n\beta}_{\omega M} \right)(0)} \geq 1. \tag{3.11}
\]

\( \square \)

**Examples.**

(i) Fix some \( L > 0 \) and define

\[
S \left( I^{n\beta} \right)(x) = \sum_{\pi \in S_M} \prod_{i=1}^M \int_{|x_i| < L} dx_i \int P^\beta_{x_i, x_{\pi(i)}}(d\omega_i) I^{n\beta}_{\omega M}(x) \tag{3.12}
\]

where \( S_M \) is the set of permutations of \( \{1, \ldots, M\} \). This is the typical choice when the particles associated with \( \omega M \) are bosons. The physically interesting case is \( n < M \ll L' \) and \( n \to \infty \) replaced by \( L \to \infty \). If \( n \ll L' \), for \( x^2/n \to 0 \) to hold one must have \(|x| = o(L'/2)\). In \( \nu \geq 3 \) dimensions all \( x \) of length \(|x| = O(L)\) satisfy this condition. When \( n = M \), \( u_1 = u_2 = -u_3 = u \) with \( \Delta u = 0 \), the system is a neutral two-component plasma.
(ii) The second set of particles can be treated classically. This is the $m_2 = \infty$ limit in which both the kinetic and the potential energy of these particles (the last two terms of the Hamiltonian (1.11)) can be dropped. Now $\omega_i(t) \equiv x_i$, $\omega^M = x^M = (x_1, \ldots, x_M)$, and the interaction between the two sets becomes

$$V(\omega_0, x^M) = \frac{1}{\beta} \sum_{k=0}^{n-1} \sum_{i=1}^{M} \int_0^{\beta} u_3(\omega_0(k\beta + t) - x_i) \, dt. \quad (3.13)$$

Spherical symmetrization can be done by keeping all $|x_i - x_j|$ unchanged. Let $g$ denote a general element of the rotation group $SO(\nu)$, and let $\mu$ be the Haar measure on $SO(\nu)$. Define

$$S\left(I^{n\beta}_{(x_1, \ldots, x_M)}(x)\right) = \int I^{n\beta}_{(gx_1, \ldots, gx_M)}(x) \mu(\,dg). \quad (3.14)$$

Because $u_1$ and $u_3$ are spherical functions and the measure $P_{\nu \nu}$ is rotation invariant, one actually has

$$S\left(I^{n\beta}_{(x_1, \ldots, x_M)}(x)\right) = \int I^{n\beta}_{(x_1, \ldots, x_M)}(gx) \mu(\,dg). \quad (3.15)$$

When $n = M$, $u_1 = -u_3 = u$ with $\Delta u = 0$, we have a neutral system of light charged particles moving in the field of immobile heavy ions.

References

[Fe1] Feynman R. P.: Space-time approach to non-relativistic quantum mechanics. Rev. Mod. Phys. 20, 367-387 (1948).

[Fe2] Feynman R. P.: Atomic theory of the $\lambda$ transition in helium. Phys. Rev. 91, 1291-1301 (1953).

[G1] Ginibre J.: Some applications of functional integration in Statistical Mechanics. In: Statistical Mechanics and Quantum Field Theory, eds. C. De Witt and R. Stora, Gordon and Breach (New York 1971).

[G2] Ginibre J.: Reduced density matrices of quantum gases. I. Limit of infinite volume. J. Math. Phys. 6, 238-251 (1965).

[G3] Ginibre J.: Reduced density matrices of quantum gases. II. Cluster property. J. Math. Phys. 6, 252-262 (1965).

[G4] Ginibre J.: Reduced density matrices of quantum gases. III. Hard-core potentials. J. Math. Phys. 6, 1432-1446 (1965).