THE HÖLDER CONTINUITY OF LYAPUNOV EXPONENTS FOR A CLASS OF COS-TYPE QUASIPERIODIC SCHRÖDINGER COCYCLES

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Abstract. In this paper we obtain exact $\frac{1}{2}$-Hölder continuity of the Lyapunov exponents for quasi-periodic Schrödinger cocycles with $C^2$ cos-type potentials, large coupling constants, and fixed Diophantine frequency. Moreover, we prove the locally Lipschitz continuity of the Lyapunov exponent for a full measure spectral set. Furthermore, for any given $r$ between $\frac{1}{2}$ to 1, we can find some energy on the spectrum and on which Lyapunov exponent is exactly $r$-Hölder continuous.

1. Introduction

In this paper we consider the Schrödinger operators on $l^2(\mathbb{Z}) \ni u = (u_n)_{n \in \mathbb{Z}}$:

$$ (H_{\alpha, v, x}u)_n = u_{n+1} + u_{n-1} + \lambda v(x + n\alpha)u_n. $$

Here $v \in C^r(\mathbb{R}/\mathbb{Z}, \mathbb{R})$, $r \in \mathbb{N} \cup \{\infty, \omega\}$ is the potential, $\lambda \in \mathbb{R}$ coupling constant, $x \in \mathbb{R}/\mathbb{Z}$ phase, and $\alpha \in \mathbb{R}/\mathbb{Z}$ frequency. Sometimes we may leave $\alpha, \lambda v$ in $H_{\alpha, v, x}$ implicit. Let $\Sigma(H_x)$ be the spectrum of the operator. As is well known that

$$ \Sigma(H_x) \subset [-2 + |\lambda| \inf v, 2 + |\lambda| \sup v] := I. $$

Moreover, for irrational $\alpha$, $\Sigma(H_x)$ is phase-independent. Let $\Sigma_{\alpha, \lambda v}$ denote the common spectrum in this case.

Consider the eigenvalue equation

$$ H_x u = Eu. $$

Then so-called Schrödinger cocycle map $A^{(E - \lambda v)} : \mathbb{R}/\mathbb{Z} \to SL(2, \mathbb{R})$ is given by

$$ A^{(E - \lambda v)}(x) = \begin{pmatrix} E - \lambda v(x) & -1 \\ 1 & 0 \end{pmatrix}, $$

Then $(\alpha, A^{(E - \lambda v)})$ defines a family of dynamical systems on $\mathbb{R}/\mathbb{Z} \times \mathbb{R}^2$, which is given by

$$ (x, \omega) \to (x + \alpha, A^{(E - \lambda v)}(x)\omega), $$

and is called Schrödinger cocycle. The $n$th iteration of dynamics is denoted by

$$ (\alpha, A^{(E - \lambda v)})^n = (n\alpha, A^{(E - \lambda v)}). $$

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Thus
\[ A_n(x)^{(E-\lambda x)} = A^{(E-\lambda x)}(x + (n-1)\alpha) \cdots A^{(E-\lambda x)}(x), n \geq 1, A_0^{(E-\lambda x)} = I. \]

For \( n \geq 1 \),
\[ A_n^{(E-\lambda x)}(x) = A_n^{(E-\lambda x)}(x - n\alpha)^{-1}. \]

Let \( u \in \mathbb{C}^\mathbb{Z} \) be a solution of equation \( H_{\lambda,x}u = Eu \) (note \( u \) is not necessary in \( \ell^2(\mathbb{Z}) \)); then the relation between the cocycle and the operator is given by
\[ A_n^{(E-\lambda x)}(x)\begin{pmatrix} u_0 \\ u_{-1} \end{pmatrix} = \begin{pmatrix} u_n \\ u_{n-1} \end{pmatrix}. \]

The Lyapunov Exponent \( L(E, \lambda) \) of this cocycle is defined as
\[ \lim_{n \to \infty} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \| A_n^{(E-\lambda x)}(x) \| \, dx = \inf_{n} \frac{1}{n} \int_{\mathbb{R}/\mathbb{Z}} \ln \| A_n^{(E-\lambda x)}(x) \| \, dx \geq 0. \]

The limit exists and is equal to the infimum since \( \{ \int_{\mathbb{R}/\mathbb{Z}} \ln \| A_n^{(E-\lambda x)}(x) \| \, dx \}_{n \geq 1} \) is a subadditive sequence. If in addition \( T \) is \( \mu \)-ergodic, then by Kingman’s subadditive ergodic theorem we also have
\[ L(E, \lambda) = \lim_{n \to \infty} \frac{1}{n} \ln \| A_n^{(E-\lambda x)}(x) \| \]
for almost every \( x \).

In the last twenty years, a large amount of paper was dedicated to study the regularity of Lyapunov exponent for quasi-periodic Schrödinger operators. Eventually, they found that the regularity of LE depends sensitively on the arithmetic property of frequency and the regularity of the potential.

For rational frequency or generic irrational frequency (extremely Liouvillean), LE is not Hölder continuous \([AJ2]\).

The study of the regularity of LE starts from G-S [00-Ann]. They proved that in the positive Lyapunov regime, if the potential is analytic and the frequency is a strong Diophantine number, Lyapunov exponent is Hölder continuous. Later, Bourgain [Bel1] proved that for AMO with the potential \( 2\lambda \cos x \), \( |\lambda| \gg 1 \) and Diophantine frequency, LE is \( \frac{1}{2} - \epsilon \) Hölder continuous. It is generalized by G-S [08-GAFAN] to the result that for arbitrary analytic potential near a trigonometric polynomial of degree \( k \), it holds that LE is \( \frac{1}{k} - \epsilon \) Hölder continuous provided the frequency is Diophantine and the LE is positive. In the zero Lyapunov exponent regime, Puig [P] proved that for AMO with the potential \( 2\lambda \cos x, 0 < |\lambda| \ll 1 \) and Diophantine frequency, LE is locally \( \frac{1}{2} \) Hölder continuous at end points of spectral gaps and cannot be better. Later, it was proved by Amor [Amor1] that in the perturbative regime, if the frequency is Diophantine, then LE is \( \frac{1}{2} \) Hölder continuous. Amor’s result was extended by Avila and Jitomirskaya [AJ2] to the non-perturbative regime and they also proved \( \frac{1}{2} \) Hölder continuous for \( \lambda \neq 0, 1 \) and all Diophantine frequencies. Recently A-J’s result was further extended by Leguil-You-Zhao-Zhou [LYZZ] to general subcritical potential and Diophantine frequency. In contrast, there is no \( \frac{1}{2} \) Hölder continuous result on a general analytic potential in the positive LE regime.

If the potential is not analytic, generally LE is not continuous with respect to the potential in \( C^k \) \( (k = 0, 1, \cdots, \infty) \) topology, see [WY]. Then Wang and Zhang [WZ] provided the first regularity result for a finitely differential potential. They proved that for \( C^2 \) cos-like (Morse) potential with a large coupling, LE is weak-Hölder
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continuous. Subsequently it was improved to be Hölder continuous with a Hölder continuous exponent strictly less than $\frac{1}{2}$ by Liang, Wang and You [LWY].

It can be seen that in [WZ] and [LWY], although the potential is only $C^2$, the cos-like condition is strong enough to make up for the deficiency of the lower regularity. Hence it is reasonable to expect that LE is $\frac{1}{2}$ Hölder continuous for all $C^2$ cos-like potential with a large coupling. This is the aim of this paper.

2. Main results

In this paper, from now on, for fixed $h \geq 2$ we say $v$ is $C^h - \cos - \mathrm{type}$ if $v$ satisfy the following conditions, which was first considered by Sinai [Sin]: $v \in C^h(\mathbb{R}/\mathbb{Z}, \mathbb{R})$; $\frac{dv}{dx} = 0$ at exactly two points, one is minimal and the other maximal, which are denoted by $z_1$ and $z_2$. Moreover, these two extremals are non-degenerate, that is, $\frac{d^2v}{dx^2}(z_j) \neq 0$ for $j = 1, 2$.

Fix two positive constants $\tau, \gamma$. We say $\alpha$ satisfies a Diophantine condition $DC_{\tau, \gamma}$ if

$$|\alpha - \frac{p}{q}| \geq \frac{\gamma}{|q|^{\tau}}$$

for all $p, q \in \mathbb{Z}$ with $q \neq 0$.

It is a standard result that for any $\tau > 2$,

$$DC_{\tau} := \bigcup_{\gamma > 0} DC_{\tau, \gamma}$$

is of full Lebesgue measure. We fix $\tau > 2$ and $\alpha \in DC_{\tau}$.

For any $t_0 \in \mathbb{R}$, we say a function $f$ is Hölder continuous at $t_0$ with a Hölder exponent $0 < r \leq 1$, if there exists some constant $C > 0$ and some nontrivial interval $I$ centered by $t_0$ such that

$$|f(t_0) - f(t)| < C|t - t_0|^r, \text{ for any } t \in I.$$ 

If in addition there also exists $c > 0$ such that

$$|f(t_0) - f(t)| > c|t - t_0|^r, \text{ for any } t \in I,$$

we say $f$ is exactly Hölder continuous at $t_0$ with a Hölder exponent $0 < r \leq 1$.

**Theorem 1.** Let $\alpha$ be as above and $v$ be $C^{2+r} - \cos - \mathrm{type}$ with some fixed $r > 0$. Consider the Schrödinger cocycle with potential $v$ and coupling constant $\lambda$. Let $L(E, \lambda)$ be the associated Lyapunov exponents. Then there exists a $\lambda_1 = \lambda_1(\alpha, v) > 0$ such that for any fixed $\lambda > \lambda_1$, the following hold true:

1. $L(\cdot, \lambda)$ is $\frac{1}{2}$-Hölder continuous on any compact interval $I$ of $E$. That is,

   $$|L(E, \lambda) - L(E', \lambda)| < C|E - E'|^{\frac{1}{2}}, \text{ for any } E, E' \in I,$$

   where $C > 0$ depends on $\alpha, v, \lambda, I$.

2. All ends points of spectrum gaps consists of a dense set in the spectrum and for each such point $E$, $L(\cdot)$ is exactly $\frac{1}{2}$ Hölder continuous at $E$.

3. There exists a subset $F$ of $\Sigma_{\alpha, \lambda v}$ with full measure such that for any $E' \in F$, $L(\cdot)$ is Lipschitz at $E'$.

4. For any fixed $\frac{1}{2} \leq \beta < 1$, there exists some point $E'' \in \Sigma_{\alpha, \lambda v}$ such that $L(\cdot)$ is exactly $\beta$-Hölder continuous at $E''$. 

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Remark(a) on Theorem 1: Due to some technical problems, we consider the condition \( v \in C^{2+r} \) instead of \( v \in C^{2} \). However, we think this condition is necessary, this is, we can construct some \( v \in C^{2} \) such that the Hölder constant of \( LE \) is strictly less than \( \frac{1}{2} \). Besides, otherwise the first and the second conclusion, the remaining results all hold true in the condition \( v \in C^{2} \).

Remark(b) on Theorem 1: Conclusion (1) in Theorem 1 implies that \( \frac{1}{2} \) Hölder continuity cannot be improved. Moreover, the set of \( E \) where the Lyapunov exponent possesses exactly \( \frac{1}{2} \) Hölder continuity is small in the sense of measure but is large in the sense of topology.

Remark(c) on Theorem 1: The Hölder continuity of \( LE \) for Schrödinger cocycles is also expected to play important roles in studying Cantor spectrum, typical localization length, phase transition, etc., for quasi-periodic Schrödinger operators.

Remark(d) on Theorem 1: For Theorem 1, a related result on analytic cases was recently obtained by [KXZ].

The idea for the proof is as follows. Previous results on \( \frac{1}{2} \) Hölder continuous regularity for \( LE \) with small potentials [Puig, AJ] showed that the lowest regularity for \( LE \) should be achieved exactly at the end points of spectral gaps. Thus we first study exact local \( \frac{1}{2} \)– Hölder continuity of \( LE \) at the end points of spectral gaps. It is based on a sharp estimate on the derivative of finite \( LE \), which is not available in previous related works. Then we study local regularity of \( LE \) at any spectral point by estimating the approximation speed for it by end points of spectral gaps, which is based on the fact that the set of EP is dense in the spectrum by [WZ2]. We will show that for almost every spectral point, the approximation speed is slow enough such that \( LE \) is Lipschitz there. For those spectral points which is approximated by end points of spectral gaps with a relatively fast speed, \( LE \) possesses a regularity between \( \frac{1}{2} \) Hölder and Lipschitz. Finally we will show that \( LE \) is globally \( \frac{1}{2} \)– Hölder continuous and cannot be better.

2.1. Remarks on the regularity of Lyapunov exponents. Much work has been devoted to the regularity properties of Lyapunov exponents (LE) and integrated density of states (IDS) as well. Here we focus on the regularity of \( LE \).

Bourgain and Goldstein [BoG] established the first LDT for real analytic potentials with a Diophantine frequency to obtain Anderson Localization. Then Goldstein-Schlag [GS] obtained some sharp version of large deviation theorem (LDT) with a strong Diophantine frequency and developed a powerful tool, the Avalanche Principle, by which they proved Hölder continuity of \( L(E) \) in the regime of positive \( LE \). Then similar results were obtained for all Diophantine and some Liouvillean frequencies by [YZ] and [HZ]. Other type of base dynamics on which regularity of \( LE \) of analytic Schrödinger operators holds true include a shift or skew-shift of a higher dimensional torus by Bourgain-Goldstein-Schlag [BGW], doubling map and Anosov diffeomorphism by Bourgain-Schlag [BoS].

Jitomirskaya-Koslover-Schulteis [JKS] get the continuity of \( LE \) with respect to potentials for a class of analytic quasiperiodic \( M(2, C) \) cocycles which is applicable to general quasi-periodic Jacobi matrices or orthogonal polynomials on the unit circle in various parameters. Jitomirskaya-Marx [JMar1] later extended it to all (including singular) \( M(2, C) \) cocycles.

An arithmetic version of large deviations and inductive scheme were developed by Bourgain and Jitomirskaya in [BoJ] allowing to obtain joint continuity of \( LE \) for \( SL(2, C) \) cocycles, in frequency and cocycle map, at any irrational frequencies.
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This result has been crucial in many important developments, such as the proof of the Ten Martini problem [AJ1], Avila’s global theory of one-frequency cocycles [A1]. It was extended to multi-frequency case by Bourgain [Bo2] and to general $M(2, C)$ case by Jitomirskaya and Marx [JMar2].

Hölder continuity for $GL(d, C)$ cocycles, $d \geq 2$, was recently obtained in Schlag [S] and Duarte-Klein [DK].

All these results on regularity of LE have been obtained by LDT and AP. Without use of LDT or Avalanche principle, Hölder continuity for $M(d, C)$ cocycles was given by Avila-Jitomirskaya-Sadel [AJS].

For lower regularity case, [K1] proved the Hölder continuity for a class of Gevrey potentials. More recently, [WZ1] provides the first positive result on the continuity of LE and weak Hölder continuity of IDS on $E$ for $C^2$ cos-type potentials. Later this weak Hölder continuity result for Diophantine frequencies was improved to be $\gamma$-Hölder continuity of $L(E)$ for Liouville frequencies by Liang-Wang-You [LWY], where $\gamma > 0$ is a small constant. For other related results, one can refer to [AK, JMavi1, JMavi2].

There are many negative results on the positivity and continuity of LE for non-analytic cases. It is well known that in $C^0$-topology, discontinuity of LE holds true at every non-uniformly hyperbolic cocycle, see [Fm], [Fs]. Moreover, motivated by Mañe [M1] and [M2], Bochi [Bje] and [Boc] proved that with an ergodic base system, any non-uniformly hyperbolic $SL(2, R)$-cocycle can be approximated by cocycles with zero LE in the $C^0$ topology. Wang-You [WY1] constructed examples to show that LE can be discontinuous even in the space of $C^\infty$ Schrödinger cocycles. Recently, Wang-You [WY2] improved the result in [WY1] by showing that in $C^r$ topology, $1 \leq r \leq +\infty$, there exists Schrödinger cocycles with a positive LE that can be approximated by ones with zero LE. Jitomirskaya-Marx [JMar2] constructed examples showing that LE of $M(2, C)$ cocycles is discontinuous in $C^\infty$ topology.

The remaining part of the paper is as follows. In Section 3, we state some basic preparation which include some technical lemmas from [WZ1] without proof. In section 4, we present the structure of the spectrum and prove part of Theorem 1. In the last several sections, we focus on the remaining part of Theorem 1. The letters $C$ and $c$ will denote absolute constants.

3. Preparation

In this section, we present some technical lemmas. The proof of them can be found in [WZ1]. In the following, each $SL(2, R)$-matrix is assumed to possess a norm strictly larger than 1. For $\theta \in \mathbb{R}/\mathbb{Z}$, let

$$R_\theta = \begin{pmatrix} \cos 2\pi \theta & -\sin 2\pi \theta \\ \sin 2\pi \theta & \cos 2\pi \theta \end{pmatrix} \in SO(2, R)$$

Define the map

$$s : SL(2, R) \to \mathbb{RP}^1 = \mathbb{R}/(\pi \mathbb{Z})$$

so that $s(A)$ is the most contraction direction of $A \in SL(2, R)$. That is, for a unit vector $\hat{s}(A) \in s(A)$, it holds that $\|A \cdot \hat{s}(A)\| = \|A\|^{-1}$. Abusing the notation a little, let

$$u : SL(2, R) \to \mathbb{RP}^1 = \mathbb{R}/(\pi \mathbb{Z})$$
be determined by \( u(A) = s(A^{-1}) \) and \( \dot{u}(A) \in u(A) \). Then for \( A \in SL(2, R) \), it is clear that
\[
A = R_u \cdot \begin{pmatrix} ||A|| & 0 \\ 0 & ||A||^{-1} \end{pmatrix} \cdot R_{s^{-1}},
\]
where \( s, u \in [0, \pi) \) are angles correspond to the directions \( s(A), u(A) \in R/(\pi Z) \).

The following technical lemma provide an equivalent form of the cocycle map (3), which turns to be more convenient.

**Lemma 2.** Let \( J \) be any compact interval. Let \( \lambda \geq \lambda_0 = \lambda_0(v) \gg 1 \). For \( x \in R/Z \) and \( t \in J \) here \( t \) is \( \frac{E}{X} \), define the following cocycles map:
\[
A(x, t) = \Lambda(x, t) \cdot R_{\phi(x, t)} = \begin{pmatrix} \lambda(x, t) & 0 \\ 0 & \lambda^{-1}(x, t) \end{pmatrix} \cdot \begin{pmatrix} \sqrt{(t - v(x))^2 + 1} & 1 \\ 1 & t - v(x) \end{pmatrix}
\]
where \( \cot \phi(x, t) = t - v(x) \). Assume
\[
\lambda(x, t) > \lambda > \min \left\{ \frac{1}{\min_x \lambda_0(x)} \right\} \quad \text{where } C \text{ depends only on } v.
\]

Consider the product of two \( SL(2, R) \) matrices \( E_1 \) and \( E_2 \). We say they are in a nonresonance case if
\[
|s(E_2) - u(E_1)|^{-1} \ll \min\{\|E_1\|, \|E_2\|\}
\]
Otherwise, we say they are in a resonance case.

The following lemma gives the estimates of nonresonance cases.

**Lemma 3.** Let
\[
E(x) = E_1(x)E_2(x).
\]
Define
\[
e_1 = \|E_1\|, e_2 = \|E_2\|, e_3 = \|E\|, e_0 = \min\{e_1, e_2\}.
\]
Assume
\[
0 < e_0^{-1} \ll \eta \ll 1.
\]
Suppose that
\[
e_1 \ll e_2^C \text{ or } e_2 \ll e_1^C,
\]
and for
\[
\theta(x) = s(E_2(x)) - u(E_1(x))
\]
and each \( x \in J \), it holds that
\[
\left| \frac{d^m e_j(x)}{dx^m} \right| < C e_j^{1+m\eta}, \quad \left| \frac{d^m \theta(x)}{dx^m} \right| < C e_0^\eta, \quad j, m = 1, 2,
\]
where \( C \) depends only on \( e_0 \). Then for \( m = 1, 2 \), we have
\[
\left| \frac{d^m e_3(x)}{dx^m} \right| < e_3^{1+m\eta+2Cm\eta},
\]
and
\[
\left| \frac{d^m s(E(x))}{dx^m} \right| < e_1^{2+2\eta}, \quad \left| \frac{d^m u(E(x) - E_1(x))}{dx^m} \right| < e_2^{-\frac{3}{2}} \quad \text{for } e_1 \ll e_2^C,
\]
or
\[
\left| \frac{d^m u(E(x))}{dx^m} \right| < e_2^{2+2\eta}, \quad \left| \frac{d^m s(E(x) - E_2(x))}{dx^m} \right| < e_1^{-\frac{3}{2}} \quad \text{for } e_2 \ll e_1^C.
\]
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We can generalize the concept of nonresonance to the product of many matrices. More precisely, we consider a sequence of map $E^{(l)} \in C^2(J, SL(2, \mathbb{R}))$ for $0 \leq l < n$. Let

$$s^{(l)} = s(E^{(l)}), \ u^{(l)} = u(E^{(l)}), \ \lambda_l = \|E^{(l)}\|, \ \Lambda^{(l)} = \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l^{-1} \end{pmatrix}.$$  

By the polar decomposition theorem, we have

$$E^{(l)} = R_u^{(l)} \Lambda^{(l)} R_{\frac{\pi}{2} - s^{(l)}}.$$  

Set

$$E_k(x) = E^{(0)}(x)E^{(1)}(x) \cdots E^{(k)}(x), \ 0 \leq k \leq n.$$  

Let

$$s_k = s(E_k), \ u_k = u(E_k), \ l_k = \|E_k\|, \ L_k = \begin{pmatrix} l_k & 0 \\ 0 & l_k^{-1} \end{pmatrix}.$$  

Again by the polar decomposition theorem, we have

$$E_k = R_u L_k R_{\frac{\pi}{2} - s_k}.$$  

Let $I \subseteq \mathbb{R}/\mathbb{Z}$ be any compact interval.

Then we have

**Lemma 4.** Let

$$\lambda' = \min_{0 \leq l < n} \{ \lambda_l \} \gg \eta^{-1} \gg 1.$$  

We further assume that for $x \in I$, $m = 1, 2$ and $0 \leq l < n$,

$$|\frac{d^m \lambda_l}{dx^m}| < C \lambda_l^{1+m\eta}, \ |\frac{d^m s^{(l)}}{dx^m}| < C \lambda^\eta, \ |s^{(l)} - u^{(l-1)}| > C \lambda^{-\eta},$$  

where $C$ depends only on $\lambda'$. Then we have

$$\|u^{(n-1)} - u_n\|_{C^2} < C \lambda_{n-1}^{-2+5\eta}, \ |s^{(0)} - s_n|_{C^2} < \lambda_0^{-2+5\eta},$$  

$$|\frac{d^m l_n}{dx^m}| < C \lambda_l^{1+m\eta}, \ m = 1, 2,$$

$$l_n \geq C \prod_{0 \leq k < n} \lambda_k \cdot \prod_{0 \leq k < n} |s^{(l)} - u^{(l-1)}|.$$  

**Remark 5.** It is not difficult to find that if we consider the functions $\lambda, \theta, e_1, e_2$ and $e_3$ with respect to $t$ (i.e. $\frac{d}{dt}$), we still obtain similar conclusions to the above two lemmas. Actually it is clear that $x$ and $t$ play similar roles in Lemma 2.

The following key lemma is essentially from [WZ1].

**Lemma 6.** Let $\lambda_1, \lambda_2 > 1$ and $\theta \in C^2(\mathbb{R}/(2\pi\mathbb{Z}), J)$. Matrix $A$ is defined as following:

$$A := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}.$$  

Denote $W(\lambda_1, \lambda_2) := \frac{|\cot \theta|}{\cot^2 \theta + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}}$. Then, we have

$$\frac{\partial \|A\|}{\partial \theta} \|A\| = \frac{\text{sgn}(\theta)(1 - \lambda_2^{-4})(1 - \lambda_1^{-4})}{\sqrt{(1 - \frac{1}{\lambda_1 \lambda_2})^2 \cot^2 \theta + \left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)^2 \tan^2 \theta + 2(1 + \frac{1}{\lambda_1 \lambda_2})(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}) - \frac{8}{\lambda_1 \lambda_2}}}.$$
Thus, which implies which implies

$$(1 - \frac{6}{\lambda_1^4} - \frac{6}{\lambda_2^4}) W(\lambda_1, \lambda_2 \theta) \leq \left| \frac{\partial \|A\|}{\partial \theta} \cdot \frac{1}{\|A\|} \right| \leq (1 + \frac{6}{\lambda_1^4} + \frac{6}{\lambda_2^4}) \left( \frac{\tan^2 \theta \sqrt{\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}}}{2} + W(\lambda_1, \lambda_2 \theta) \right).$$

$$\frac{\partial \|A\|}{\partial \lambda_1} \cdot \frac{1}{\|A\|} = \frac{\sgn(\theta) \cdot \lambda_1^{-1} \left( (1 - \frac{1}{\lambda_1^2 \lambda_2^2}) \cot^2 \theta + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan^2 \theta + 2(1 + \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}) - \frac{8}{\lambda_1^2 \lambda_2^2} \right)}{\sqrt{(1 - \frac{1}{\lambda_1^2 \lambda_2^2}) \cot^2 \theta + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan^2 \theta + 2(1 + \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}) - \frac{8}{\lambda_1^2 \lambda_2^2}}}.$$  

which implies $\left| \frac{\partial \|A\|}{\partial \lambda_1} \cdot \frac{1}{\|A\|} \right| \leq \frac{1}{\lambda_1}$;

$$\frac{\partial \|A\|}{\partial \lambda_2} \cdot \frac{1}{\|A\|} = \frac{\sgn(\theta) \cdot \lambda_2^{-1} \left( (1 - \frac{1}{\lambda_1^2 \lambda_2^2}) \cot^2 \theta + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan^2 \theta + 2(1 + \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}) - \frac{8}{\lambda_1^2 \lambda_2^2} \right)}{\sqrt{(1 - \frac{1}{\lambda_1^2 \lambda_2^2}) \cot^2 \theta + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan^2 \theta + 2(1 + \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}) - \frac{8}{\lambda_1^2 \lambda_2^2}}}.$$  

which implies $\left| \frac{\partial \|A\|}{\partial \lambda_2} \cdot \frac{1}{\|A\|} \right| \leq \frac{1}{\lambda_2}$.

**Proof.** It follows from a straightforward calculation that

$$\|A\|^2 + \|A\|^{-2} = (\lambda_1^2 \lambda_2^2 + \lambda_1^{-2} \lambda_2^{-2}) \cos^2 \theta + (\lambda_1^2 \lambda_2^{-2} + \lambda_1^{-2} \lambda_2^2) \sin^2 \theta.$$  

Taking derivatives with respect to $\theta$ for both sides of the above equation, we have

$$2\|A\| \frac{\partial \|A\|}{\partial \theta} - 2\|A\| \frac{\partial \|A\|}{\partial \theta} \|A\|^{-3} = (\lambda_1^2 \lambda_2^{-2} + \lambda_1^{-2} \lambda_2^2 - (\lambda_1^2 \lambda_2^{-2} + \lambda_1^{-2} \lambda_2^2)) \sin 2\theta.$$  

Thus,

$$\frac{\partial \|A\|}{\partial \theta} \cdot \|A\|^{-1} = \frac{(\lambda_1^2 \lambda_2^{-2} + \lambda_1^{-2} \lambda_2^2 - (\lambda_1^2 \lambda_2^{-2} + \lambda_1^{-2} \lambda_2^2)) \sin 2\theta}{2\sqrt{\|A\|^2 + \|A\|^{-2}) - 4}}.$$  

Equivalently, we have

$$\frac{\partial \|A\|}{\partial \theta} \cdot \|A\|^{-1} = \frac{\sgn(\theta) (1 - \lambda_2^{-4})(1 - \lambda_1^{-4})}{2\sqrt{(1 - \frac{1}{\lambda_1^2 \lambda_2^2}) \cot^2 \theta + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan^2 \theta + 2(1 + \frac{1}{\lambda_1^2 \lambda_2^2})(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}) - \frac{8}{\lambda_1^2 \lambda_2^2}}}.$$  

On one hand, it implies

$$\left| \frac{\partial \|A\|}{\partial \theta} \cdot \frac{1}{\|A\|} \right| \geq \frac{(1 - \frac{6}{\lambda_1^4} - \frac{6}{\lambda_2^4})}{\sqrt{1 + \frac{1}{\lambda_1^2 \lambda_2^2}}} \cdot \frac{1}{\sqrt{\cot^2 \theta + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan^2 \theta + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}}} \geq (1 - \frac{6}{\lambda_2^4} - \frac{6}{\lambda_1^4}) \left( \frac{1}{\sqrt{\cot^2 \theta + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan^2 \theta + \frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}}} \right) \geq (1 - \frac{6}{\lambda_2^4} - \frac{6}{\lambda_1^4}) \left( \frac{\left| \cot \theta \right|}{\cot^2 \theta + (\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2})} \right).$$  

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On the other hand,

$$\frac{\partial \|A\|}{\partial \theta} \cdot \frac{1}{\|A\|} \leq \frac{1}{\sqrt{(1 - \frac{1}{\lambda_1 \lambda_2})^2 \cot^2 \theta + (\frac{1}{\lambda_1} - \frac{1}{\lambda_2})^2 \tan^2 \theta + 2(1 + \frac{1}{\lambda_1 \lambda_2})(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}) - \frac{8}{\lambda_1 \lambda_2}}$$

$$\leq \frac{1}{1 - \frac{1}{\lambda_1} - \frac{1}{\lambda_2}} \sqrt{\cot^2 \theta + (\frac{1}{\lambda_1} - \frac{1}{\lambda_2})^2 \tan^2 \theta + \frac{2}{\lambda_1} + \frac{2}{\lambda_2}}$$

$$\leq (1 + \frac{6}{\lambda_2} + \frac{6}{\lambda_1}) \sqrt{\cot^2 \theta + (\frac{1}{\lambda_1} - \frac{1}{\lambda_2})^2 \tan^2 \theta + \frac{2}{\lambda_1} + \frac{2}{\lambda_2}}$$

$$\leq (1 + \frac{6}{\lambda_2} + \frac{6}{\lambda_1}) \left( \frac{1}{\sqrt{\cot^2 \theta + (\frac{1}{\lambda_1} - \frac{1}{\lambda_2})^2 \tan^2 \theta + \frac{2}{\lambda_1} + \frac{2}{\lambda_2}} - \frac{1}{\sqrt{\cot^2 \theta + (\frac{1}{\lambda_1} + \frac{1}{\lambda_2})^2 \tan^2 \theta + \frac{2}{\lambda_1} + \frac{2}{\lambda_2}}} \right)$$

The next two inequalities are clearly followed from a direct calculation. \( \square \)

In the resonance case, \( s(E_1 E_2) \) (or \( u(E_1 E_2) \)) may differ greatly from \( s(E_2) \) (or \( u(E_1) \)). To describe the angle functions, we give the following definitions.

**Definition.** Let \( B(x, r) \subset \mathbb{R}/\mathbb{Z} \) be the ball centered at \( x \in \mathbb{R}/\mathbb{Z} \) with a radius of \( r \). For a connected interval \( J \subset \mathbb{R}/\mathbb{Z} \) and a constant \( a \in (0, 1] \), let \( aJ \) be the subinterval of \( J \) with the same center and whose length is \( a|J| \). Let \( I = B(0, r) \), \( 1 \ll r^{-1} \ll l_0 \). Let \( f \in C^2(I, \mathbb{R}P^1) \).

1. **f is of type I**, if we have the following: \( \|f\|_{C^2} < C \) and \( f(x) = 0 \) has only one solution, say \( x_0 \), which is contained in \( \frac{I}{4} \); \( \frac{df}{dx}(x_0) \neq 0 \) has at most one solution on \( I \) while \( |\frac{df}{dx}| > r^2 \) for all \( x \in B(x_0, \frac{r}{2}) \); let \( J \subset I \) be the subinterval such that \( \frac{df}{dx}(J, \frac{df}{dx}(x_0) \leq 0 \), then \( |f(x)| > cr^3 \) for all \( x \in J \). Let \( J_+ \) denotes the case \( \frac{df}{dx}(x_0) > 0 \) and \( J_- \) for \( \frac{df}{dx}(x_0) < 0 \).

2. **f is of type II**, if we have the following: \( \|f\|_{C^2} < C \) and \( f(x) = 0 \) has at most two solutions which are in \( \frac{I}{4} \); \( \frac{df}{dx} = 0 \) has one solution which is contained in \( \frac{I}{4} \); \( f(x) = 0 \) has one solution if and only if it is the \( x \) such that \( \frac{df}{dx} = 0 \); \( |\frac{df}{dx}| > c \) whenever \( |\frac{df}{dx}| < r^2 \).

3. **f is of type III**, if for \( l : I \to \mathbb{R}^+ \) such that \( l(x) > l_0 \gg 1 \), \( \frac{d^m l(x)}{dx^m} < l(x)^{1+\beta}, \ x \in I, m = 1, 2 \),

\[ f(x) = \arctan(\ell^2 \tan f_1(x)) - \frac{\pi}{2} + f_2(x). \]
Here, either $f_1$ is of type $I_+$ and $f_2$ is of type $I_-$, or $f_1$ is of type $I_-$ and $f_2$ is of type $I_+$. 

**Remark 7.** The graphs of type I and II are easy to be understood. The function of type III, as one can see, can divided to $\arctan(t^2 \tan f_1(x))$, which is similar to a pulse function, and $-f_2(x)$, which is exactly of type I. One can directly calculate that if $x$ locates far from the zero of $f_1$, the first part is small enough to be neglected, and therefore $f \approx -f_2$. If $x$ is near the zero of $f_1$, the first part has a drastic change from $-\pi$ to 0, which leads to a bifurcation of no zero, one zero, and two zeros of $f$. For more details, see [WZ1].

**Lemma 8.** Let $f : I \rightarrow \mathbb{R}^{n^1}$ be of type III and be defined as above. Let $r^2 \leq \eta_j \leq r^{-2}$, $0 \leq j \leq 4$, be some constant. Then 

$$|x_1| < Cl_0^{-\frac{4}{3}}, |x_2 - d| < Cl_0^{-\frac{4}{3}}.$$ 

In particular, if $f(x_1) = f(x_2) = 0$, then 

$$0 < x_1 \leq x_2 < d.$$ 

If $f(x_1) = f(x_2) \neq 0$, then 

$$x_1 = x_2.$$ 

Then we consider the following two different cases:

1. $d \leq \frac{r}{3}$: then there exist two distinct points $x_3, x_4 \in B(x_1, \eta_0 l_0^{-1})$ such that 

$$\frac{df}{ds}(s_j) = 0 \text{ for } j = 3, 4.$$ 

Here we set $x_4$ such that $x_1 \leq x_4 \leq x_2$. Then $x_3$ is a local minimum with

$$f(x_3) > \eta_1 l_1^{-1} - \pi.$$ 

See Fig2 for positions of $x_j, j = 1, 2, 3, 4$. Moreover, it holds that 

\begin{equation}
\left\{ \begin{array}{l}
|d^2 f dx^2(x)| > c \text{ whenever } |df dx(x)| \leq r^2 \text{ for } x \in B(X, \frac{r}{6}); \\
|f(x)| > cr^3, \text{ for all } x \notin B(X, \frac{r}{6}).
\end{array} \right.
\end{equation}

2. $d \geq \frac{r}{3}$: then $\frac{df}{ds} = 0$ may have one or two solutions, among which the one between $x_1$ and $x_2$ always exists. In other words, there might exist $x_3$ or not while $x_4$ always exists. If $x_3$ exists, then (6) still holds. In any case, it always holds that 

\begin{equation}
\left\{ \begin{array}{l}
|f(x)| > cr^3, x \notin B(x_1, Cl_0^{-\frac{4}{3}}) \cup B(x_2, \frac{r}{4}); \\
\|f - f_2\|_{C^2} < Cl_0^{-\frac{4}{3}}, x \in B(x_2, \frac{r}{4}).
\end{array} \right.
\end{equation}

**Finally, we have the following bifurcation as $d$ varies.** There is a $d'$ with $l_1^{-1} < d', l_2 < l_2^{-1}$ such that:

1. if $d > d'$, then $f(x) = 0$ has two solutions;
2. if $d = d'$, then $f(x) = 0$ has exactly one tangential solution. In other words, $x_1 = x_2 = x_4$ and $f(x_4) = 0$;
3. if $0 \leq d < d'$, then $f(x) \neq 0$ for all $x \in I$. Moreover, combining (6), we have 

$$\min_{x \in I} |f(x)| > -\eta_3 l_1^{-1} + \eta_4 d.$$
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No matter which type of function is, it satisfies the following non-degenerate property.

**Lemma 9.** Let $f : I \to \mathbb{R}/\mathbb{Z}$ be of type I, II, or III. Define

$$X = \{ x \in I : \{ f(x) \} = \min_{y \in I} \{ f(y) \} \} = \begin{cases} \{ x_0 \}, & f \text{ is of I}, \\ \{ x_1, x_2 \}, & f \text{ is of type II or III}. \end{cases}$$

In case $f$ is of type III, we further assume

$$d := |x_1 - x_2| < \frac{r}{3}.$$

Then for any $r' \in (0, r)$, we have

$$|f(x)| > cr^3, \ x \notin B(X, r').$$

For the case that $f$ is of type III, we have the same estimate for $C \frac{1}{4} < r' < r$ if $d \geq \frac{r}{3}$.

**3.0.1. Induction theorem.** From now on, let $A = A(x, t)$ be as in [5]. Abusing the notation a little bit, for $n \geq 1$, we define

$$s_n(x, t) = s[A_n(x, t)], \ u_n(x, t) = s[A_{-n}(x, t)].$$

We call $s_n$ (respectively, $u_n$) the $n$–step stable (respectively, unstable) direction.

Obviously, we have that $u_1(x, t) = 0$ and

$$s_1(x, t) = \frac{\pi}{2} - \phi(x, t) = \frac{\pi}{2} - \cot^{-1}[t - v(x)] = \tan^{-1}[t - v(x)].$$

We define $g_1(x, t) = s_1(x, t) - u_1(x, t)$. Thus, it clearly holds that

$$g_1(x, t) = \tan^{-1}[t - v(x)].$$

To consider the Hölder regularity of $L(t)$, for $\lambda > \lambda_0 \gg 1$, we may restrict $t$ to the following interval:

$$t \in J := [\inf v - \frac{2}{\lambda}, \sup v + \frac{2}{\lambda}].$$

It is due to the fact that if $t_0 \notin J$, then $(\alpha, A(\cdot, t_0)) \in UH$, which implies that $L(t)$ possesses the same regularity as that of $v$. See [21] for details.

Here $UH$ is defined as the following:

**Definition.** We say a cocycle $(\beta, B)$ is uniformly hyperbolic ($UH$) if there exist two functions $\bar{s}, \bar{u} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}^1$ such that $\bar{s}(x) \oplus \bar{u}(x) = \mathbb{R}^2$ (here we also consider $\bar{s}(x)$ and $\bar{u}(x)$ as one dimensional subspace of $\mathbb{R}^2$) and the following hold:

1. $B(x) \cdot \bar{s}(x) = \bar{s}(x + \beta)$, $B(x) \cdot \bar{u}(x) = \bar{u}(x + \beta)$. In other words, $\bar{s}$ and $\bar{u}$ are $B$–invariant.

2. There exist $c > 0$ and $\rho > 1$ such that for any unit vectors $\bar{w}^s \in \bar{s}(x)$ and $\bar{w}^u \in \bar{u}(x)$, it holds that

$$\|B_n(x)\bar{w}^s\|, \|B_{-n}(x)\bar{w}^u\| < c\rho^{-n}$$

for all $n \geq 1$ and for all $x \in \mathbb{R}/\mathbb{Z}$. Here we also consider $\bar{s}(x)$ and $\bar{u}(x)$ as one dimensional subspace of $\mathbb{R}^2$.

Set $I_0 = \mathbb{R}/\mathbb{Z}$ for all $t \in J$. Recall $\{\frac{q_n}{p_n}\}_{n \geq 1}$ are the continued fraction approximants of $\alpha$. Fix a large $N = N(v)$. Then at step 1, we have the following.
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(1) First step critical points:
\[ C_1(t) = \{c_{1,1}(t), c_{1,2}(t)\} \]
with \(c_{1,1}(t), c_{1,2}(t) \in I_0\) minimizing \(\{g_1(x,t), \ x \in I_0\}\) for each \(t \in J\).

(2) First step critical interval:
\[ I_{1,j}(t) = \{x : |x - c_{1,j}(t)| \leq \frac{1}{2q_N}\} \text{ and } I_1(t) = I_{1,1}(t) \cup I_{1,2}(t). \]

(3) First step return times:
\[ q_N \leq r_1^+(x,t) : I_1(t) \to \mathbb{Z}^+ \]
are the first return times (back to \(I_1(t)\)) after time \(q_N - 1\). Here \(r_1^+(x,t)\) is the forward return time and \(r_1^-(x,t)\) the backward return time. Let \(r_1(t) = \min r_1^+(t), r_1^-(t)\) with \(r_1^+(t) = \min_{x \in I_1(t)} r_1^+(x,t)\).

(4) The second step angle \(g_2\):
\[ g_2(x,t) = s_{r_1(t)}(x,t) - u_{r_1(t)}(x,t) : D_1 \to \mathbb{R}^1, \]
where we define
\[ D_1 := \{(x,t) : x \in I_1(t), t \in J\}. \]

It is easy to see that for each \(t \in J\), \(g_1(., t) = \tan^{-1}(1 - v(\cdot))\) is either of type I or II. And for the second step, there may exist some \(t \in J\), \(g_2(\cdot, t)\) is of type III. Moreover, for any \(n \geq 2\) and any \(t \in J\), \([WZ1]\) tells us that \(g_n(., t)\) belongs to one of these three types.

More precisely, \([WZ1]\) proved the following conclusion by induction. Assume that for \(i \geq 1\), the following objects are well defined:

(1) \(i\)th step critical points:
\[ C_i(t) = \{c_{i,1}(t), c_{i,2}(t)\} \]
with \(c_{i,j}(t) \in I_{i-1,j}(t)\) minimizing \(\{g_i(x,t), \ x \in I_{i-1,j}(t)\}\). More precise description of \(C_i(t)\) will be given in the following theorem.

(2) \(i\)th step critical interval:
\[ I_{i,j}(t) = \{x : |x - c_{i,j}(t)| \leq \frac{1}{2q_{N+i-1}}\} \text{ and } I_i(t) = I_{i,1}(t) \cup I_{i,2}(t). \]

(3) \(i\)th step return times:
\[ q_{N+i-1} \leq r_i^+(x,t) : I_i(t) \to \mathbb{Z}^+ \]
are the first return times (back to \(I_i(t)\)) after time \(q_{N+i-1} - 1\). Here \(r_i^+(x,t)\) is the forward return time and \(r_i^-(x,t)\) backward. Let \(r_i(t) = \min r_i^+(t), r_i^-(t)\) with \(r_i^+(t) = \min_{x \in I_i(t)} r_i^+(x,t)\).

(4) \((i+1)\)th step angle \(g_{i+1}\):
\[ g_{i+1}(x,t) = s_{r_i(t)}(x,t) - u_{r_i(t)}(x,t) : D_i \to \mathbb{R}^1, \]
where we define
\[ D_i := \{(x,t) : x \in I_i(t), t \in J\}. \]

The next theorem, which is from \([WZ1]\)’s induction theorem, shows the precise description of the several important quantities mentioned above.

**Theorem 10.** For any \(\epsilon > 0\), there exists a \(\lambda_0 = \lambda_0(\nu, \alpha, \epsilon) > 0\) such that for all \(\lambda > \lambda_0\), the following holds for each \(i \geq 2\).
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(1) For each $t \in J$, $g_i(\cdot, t) : I_{i-1}(t) \to \mathbb{R}P^1$ is of type I, II or III, which are denoted by case (1)$_1$, (i)$_{12}$ and (i)$_{11}$, respectively. In case (1)$_1$ and (1)$_{11}$, as functions on $I_{i-1}(t)$, it holds that

$$
\|g_i - g_{i-1}\|_{C^2} \leq C\lambda^{-\frac{1}{2}r_{i-1}},
$$

where $C$ depends only on $\lambda$ and $v$. Moreover, we have the following.

(a) In case (i)$_1$, $I_{i-1}(t) \cap I_{i-2}(t) = \emptyset$. Moreover, if $g_i(\cdot, t)$ is of type $I_-$ on $I_{i-1}(t)$, then it is of type $I_-$ on $I_{i-2}(t)$, and vice versa. In addition, $c_{i,j}(t)$ is the only point such that $g_i(x, t) = 0$ on $I_{i-1}(t)$.

(b) In case (i)$_{12}$, $I_{i-1}(t) \cap I_{i-2}(t) \neq \emptyset$ and $g_i$ is of type II as a function on the connected interval $I_{i-1}(t)$. Moreover, $c_{i,j}(t)$ is the only point minimizing $|g_i(x, t)|$ on $I_{i-1}(t)$ (note that it is possible that $c_{i-1}(t) = c_{i,2}(t)$).

(c) In case (i)$_{11}$, $I_{i-1}(t) \cap I_{i-2}(t) = \emptyset$. We have multiple points minimizing $|g_i(x, t)|$ on each $I_{i-1}(t)$ for $j = 1, 2$. Furthermore, $c_{i,j}(t)$ can be defined as the minimal point of $g_i : I_{i-1}(t) \to \mathbb{R}P^1$ that corresponds to $x_2$ of in Lemma 10 for case $d > d'$.

(2) For each $i \geq 1$ and $t \in J$, it holds that

$$
|c_{i-1,j}(t) - c_{i,j}(t)| < C\lambda^{-\frac{1}{2}r_{i-2}}, j = 1, 2;
$$

(3) For all $x \in I_{i-1}(t)$ and $m = 1, 2$, it holds that

$$
\|A_{\pm r_{i-1}^v}(x, t)\| > \lambda^{(1 - \epsilon)r_{i-1}^v(x, t)} \geq \lambda^{(1 - \epsilon)q_{N+i-2}}
$$

and

$$
\frac{\partial^m(\|A_{\pm r_{i-1}^v}(x, t)\|)}{\partial v^m} < \|A_{\pm r_{i-1}^v}(x, t)\|^{1 + \epsilon}, v = x \text{ or } t.
$$

(4) In case (i)$_{11}$, there exists a unique $k$ such that $1 \leq |k| < q_{N+i-2}$ and

$$
I_{i-2}(t) \cap (I_{i-1}(t) + k\alpha) \neq \emptyset.
$$

Moreover, there exist points $d_{i,j}(t) \in I_{i-1}(t)$ such that

$$
g_i(d_{i,j}(t), t) = g_i(c_{i,j}(t), t), \quad j = 1, 2,
$$

and the following hold.

(a) If $|g_i(c_{i,j}(t), t)| > C\lambda^{-\frac{1}{2}r_{i-1}^v}$, $j = 1$ or 2, then so are $|g_i(c_{i,j'}(t), t)|$ for $j' \neq j$ and $|g_{i+1}(c_{i,j}(t), t)|$ for $j = 1$ and 2;

(b) If $|g_i(c_{i,j}(t), t)| < C\lambda^{-\frac{1}{2}r_{i-1}^v}$, $j = 1$ or $j = 2$, then

$$
\|c_{i,1}(t) + k\alpha - d_{i,2}(t)\|_{\mathbb{R}/\mathbb{Z}}, \quad \|c_{i,2}(t) + k\alpha - d_{i,1}(t)\|_{\mathbb{R}/\mathbb{Z}} < C\lambda^{-\frac{1}{2}r_{i-1}^v},
$$

where $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ denotes the distance to the nearest integer.

Remark 11. (12) implies that as $i$-th step critical points, $d_{i,1}(t)$ essentially is $-k$-iteration of $c_{i,2}(t)$ while $d_{i,2}(t)$ essentially is the $k$-iteration of $c_{i,1}(t)$ under $x \mapsto x + \alpha$ on $\mathbb{R}/\mathbb{Z}$. Thus, we can change the notation from

$$
d_{i,1}(t) \to c_{i,2}^{-k}(t), \quad d_{i,2}(t) \to c_{i,1}^k(t).
$$

Remark 12. In Theorem 10, the size of the 'critical interval' $I_{i,j}$, is chosen to be $2^{-i}q_{N+i-1}^{-2\epsilon}$. However, there is no any difference if we set $|I_{i,j}| = q_{N+i-1}^{-C}$ for any large $C > 0$ depending on the choice of $\lambda$. 

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In Theorem [11], all the derivatives of $g_i(x, t)$ and $\|A_r(x, t)\|$ are on the variable $x$. However, all the necessary technical lemmas in [WZ1] (lemma 2-5) can be directly applied to the derivative estimates of $g_i$ as a two-variable function in $(x, t)$.

4. The resonance and the classification of the spectrum

In this section, we consider the classification of the spectrum according to the occurrence of resonance in the iteration process.

From [WZ2], we know the spectrum is a cantor set. In particular, the endpoints of spectral gaps are dense in the spectrum.

For any fixed $t \in \frac{1}{\lambda} \Sigma$, we write $c_{n,1}(t) - c_{n,2}(t)$ as $d_n(t)$ in brief.

First we give the important definitions.

**Definition.** Denote $F_{C,n,k} \triangleq \{ t \in \frac{1}{\lambda} \Sigma \mid d_n(t) - k\alpha > \frac{1}{q_{N+n-1}} \}$ and $E_{C,n,k} \triangleq \{ t \in \frac{1}{\lambda} \Sigma \mid d_n(t) - k\alpha \leq C\lambda^{-\frac{q_{N+n-1}}{q_{N+n+k}}} \}$.

Then we define

1. $\Sigma_1 := \bigcap_{l=1}^{\infty} \bigcap_{n \geq 1} \bigcap_{k \geq 1} F_{C,n,k}$, which is finitely-resonant condition, FR in brief.
2. $\Sigma_2 := \frac{1}{\lambda} \Sigma - \Sigma_1$, which is infinitely-resonant condition, IR in brief.
3. $\Sigma_3 := \bigcup_{k \in \mathbb{Z}} \bigcap_{n \geq N(|k|)} E_{C,n,k}$

where $N(|k|)$ satisfies $q_{N+N(|k|)-2} < |k| \leq q_{N+N(|k|)-1}$.

**Remark 13.** Here the choice of constant $C$ is flexible (it is noteworthy that the constant $C$ only determines the choice of $\lambda$ in the iteration process), for instance, we can take $C = 100\tau^{100}$. Anyway we can select constants according to specific needs in the later proof process. Similarly the choice of $-\frac{1}{10}$ before $r_n-1$ is also nonessential. This point is very clear in the next lemma. In fact, $\Sigma_3$ is the endpoints of gaps, thus this will be proved soon.

It holds directly by definition that $\Sigma_1 \bigcup \Sigma_2 = \Sigma, \Sigma_1 \bigcap \Sigma_2 = \emptyset$.

We claim that Now we prove

$\Sigma_3 \subset \Sigma_2$.

In fact, for any fixed $k_0 \in \mathbb{Z}$ and $t \in \bigcap_{n \geq N(|k_0|)} E_{C,n,k_0}$, by the help of $E_{C,n,k_0} \subset F_{C,n,k_0} (\triangleq \frac{1}{\lambda} \Sigma - F_{C,n,k_0})$, we have the following containment.

$\bigcap_{n \geq N(|k_0|)} E_{C,n,k_0} \subset \bigcap_{n \geq N(|k_0|)} F_{C,n,k_0} \subset \bigcup_{n \geq N(|k_0|)} F_{C,n,k_0} \subset \bigcup_{n \geq N(|k_0|)} F_{C,n,k_0} \subset \Sigma_2$.

It hold from the $E_{C,n,k} \subset \frac{1}{\lambda} \Sigma - F_{C,n,k}$, which implies $\Sigma_3 \subset \Sigma_2$.

It’s clear that the definition of $\Sigma_3$ directly implies $\lim_{n \to \infty} d_n(t) = k\alpha$.

In fact, the converse is also right.

**Lemma 14.** For any fixed $t \in \frac{1}{\lambda} \Sigma$,

*If there exists some $k \in \mathbb{Z}$ such that $\lim_{n \to \infty} d_n(t) = k\alpha$, then $t \in \Sigma_3$.***
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**Proof.** We will directly prove that

$$t \in \bigcap_{n \geq N(|k|)} E_{C,n,k}.$$ 

If it is not true, then we have there exists some $n' \geq N(|k|)$ such that $t \notin \frac{1}{\lambda} \Sigma - E_{C,n',k}$. Then we have

$$|d_{n'}(t) - k\alpha| > \lambda^{-\frac{N+n'+2}{16}}. \tag{13}$$

By the help of Induction Theorem of [WZ1],

$$|d_n(t) - d_{n-1}(t)| \leq \lambda^{-\frac{N+n}{2}}$$

for $n \geq 2$ and $t \in \frac{1}{\lambda} \Sigma$. Combine it with (13), we have

$$\inf_{m \geq n'} |d_m(t) - k\alpha| \geq \lambda^{-\frac{N+n'+2}{16}}.$$

Then a conflict is formed with the fact $\lim_{n \to \infty} d_n(t) = k\alpha$.

\[\square\]

So, we essentially have

$$\Sigma_3 = \left\{ t \in \frac{1}{\lambda} \Sigma \mid \exists k \in \mathbb{Z}, \text{s.t.} \lim_{n \to \infty} (d_n(t)) = k\alpha \right\}.$$

The following several lemmas are useful for our further proof and they will show the important fact: When we move the parameter $t$ in a small neighborhood, the image of the angle function $g_n(t)$ also move the same (a constant multiple) distance.

**Lemma 15.** Fixed $t \in \mathbb{I}$, for any $g_n$ being **type II** or **III**, there exist some constant $1 > c > 0$ dependent on $v$ and $\alpha$ such that

$$c^{-1} > \left| \frac{\partial g_n(x,t)}{\partial t} \right| > c, \tag{14}$$

for any $x \in \mathcal{J}_n \triangleq [\tilde{c}_n(t), \check{c}_n(t) + \frac{1}{q_{N+n-1}}] (or x \in [\tilde{c}_n(t) - \frac{1}{q_{N+n-1}}, \check{c}_n(t)])$ with $c_n(t) \in \mathcal{J}_n$, where $\tilde{c}_n(t) := \{ x \in I_n | \frac{\partial g_n(x,t)}{\partial t} = 0 \}$ and $c_n$ is the essential zero point of $g_n(t)$.

Meanwhile, if $g_n$ is **type I**, (14) holds for any $x \in I_n(t)$.

**Remark 16.** In fact, take $x = \check{c}_n$, it’s easy to see that the first part of the lemma directly implies the estimate for the length of gaps. From the proof, it’s easy to check that

$$\lambda^{\kappa(t)} > \left| \frac{\partial g_n(x,t)}{\partial t} \right|$$

for $g_n$ being any type and $x \in I_n(t)$, where $\kappa(t) = \max \{ |g(t)| \text{ is Type I and } g_{t+1} \text{ is Type III} \}$.

**Proof.** We process the proof by induction.

**Case 0 → 1:**

For $n = 0$ it is clear that

$$1 \geq \left| \frac{\partial g_1(t,x)}{\partial t} \right| = \left| \frac{-1}{(t-v)^2 + 1} \right| > \frac{1}{(\sup |v| + 3)^2 + 1} := c_1, \forall x \in I_1$$

while $\text{sgn}(\frac{\partial g_1(t,x)}{\partial t}) = \text{sgn}(\frac{\partial g_1(t,x)}{\partial t})$ on $I_1$.

Denote $\delta_n = c_1^{-1} \lambda^{-q_{N+n-2}}$, $n \in \mathbb{N}$, and $q_{N-1} = 1$. For $n = 1$, three cases may occur.
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(1) $g_2$ is of type I.
Due to the $|\frac{\partial g_2(t,x)}{\partial t} - \frac{\partial g_1(t,x)}{\partial t}|c'(t_n) \leq \lambda^{-\frac{3}{2}}$, we have
$$1 + \lambda^{-1} \geq |\frac{\partial g_2(t,x)}{\partial t}| \geq c_1 - \lambda^{-\frac{3}{2}} \geq c_1(1 - \delta_1).$$
(2) $g_2$ is of type II.
It's similar to the first case.
(3) $g_2$ is of type III.
There exits some $1 \leq k(1) \leq q_N$ such that $|c_{1,1} - c_{2,1} - k(1)\alpha| \leq C\lambda^{-k(1)}$,
thus $g_2 = \arctan \lambda^2 \tan \tilde{g}_{1,1} - \frac{\pi}{2} + \tilde{g}_{1,2}$, where $\lambda_{k(1)} > \frac{\lambda}{\lambda^k(1)}$, $\tilde{g}_{1,1}, \tilde{g}_{1,2}$
are all Type I satisfying
$$1 + \lambda^{-1} \geq |\frac{\partial \tilde{g}_{1,1}}{\partial t}| > c_1(1 - \delta_1)$$
and
$$1 + \lambda^{-1} \geq |\frac{\partial \tilde{g}_{1,2}}{\partial t}| > c_1(1 - \delta_1)$$
with $\text{sgn}(\frac{\partial \tilde{g}_{1,1}}{\partial t}) = \text{sgn}(\frac{\partial \tilde{g}_{1,2}}{\partial t})$.

Therefore (for convenience, we write $\frac{\partial u}{\partial t}$ as $\cdot^t$), we have the lower bound at follow.
$$\geq \frac{\lambda^2 \frac{\tilde{g}_{1,1}}{\lambda^k(1) \cos^2 \tilde{g}_{1,1}} + \tilde{g}_{1,2} + \frac{2\lambda \lambda_{k(1)} \lambda_{k(1)} \tan \tilde{g}_{1,1}}{1 + \lambda^4 \lambda_{k(1)} \tan^2 \tilde{g}_{1,1}}}{1 + \lambda^4 \lambda_{k(1)} \tan^2 \tilde{g}_{1,1}}$$
$$\geq \frac{\frac{\partial g_2(t,x)}{\partial t}}{1 + \lambda^4 \lambda_{k(1)} \tan^2 \tilde{g}_{1,1}}$$
$$= \frac{2\lambda \lambda_{k(1)} \lambda_{k(1)} \tan \tilde{g}_{1,1} + \lambda^2 \frac{\tilde{g}_{1,1}}{\lambda^k(1) \cos^2 \tilde{g}_{1,1}} + \tilde{g}_{1,2}}{1 + \lambda^4 \lambda_{k(1)} \tan^2 \tilde{g}_{1,1}}$$
$$\geq |\frac{\lambda^2 \frac{\tilde{g}_{1,1}}{\lambda^k(1) \cos^2 \tilde{g}_{1,1}} + \tilde{g}_{1,2}}{1 + \lambda^4 \lambda_{k(1)} \tan^2 \tilde{g}_{1,1}}|$$
$$\geq c_1(1 - \delta_1)(1 + 0.5\lambda_{k(1)}^{-2}) - 2\lambda_{k(1)}^{-\frac{3}{2}}$$
$$\geq c_1(1 - \delta_1) - \lambda_{k(1)}^{-\frac{3}{2}} \geq c_1(1 - 2\delta_1).$$

For the upper bound, note that if $|\tilde{g}_{1,1}| > \lambda_{k(1)}^{-\frac{3}{2}}$, then we can rewrite $g_2$ as
$$-\hat{C}\lambda_{k(1)}^{-2}(\frac{\partial \tilde{g}_{1,1}}{\partial x}(c_{1,1}^1 + o(\frac{1}{q_N^2}))\tilde{g}_{1,1})^{-1} + d_1 - (\frac{\partial \tilde{g}_{1,2}}{\partial x}(c_{1,2}^1 + o(\frac{1}{q_N^2}))\tilde{g}_{1,1}),$$
where $\hat{C}$ is an absolute constant. Therefore, it follows from the estimate in [WZ1] that
$$g_2 = -C\lambda_{k(1)}^{-2}(c_{1,1}^1(x - c_{1,1}^1))^{-1} + d_1 - c_{1,2}^1(x + k(1)\alpha - c_{1,2}^1),$$
where
$$c_{1,i}^1 = \frac{\partial \tilde{g}_1}{\partial x}(c_{1,i}), \ i = 1, 2$$
with $\|\delta_{i}^t(x,t)\|_{C^2} \leq \frac{1}{q_N}.$

By a direct calculation, we have $\tilde{c}_{2,i} = c_{1,i} + (\sqrt{c_{1,i}^1 \cdot c_{1,i}^2})^{-1}\lambda_{k(1)}^{-1}, i = 1, 2.$
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Hence, for any $x \in [\hat{c}_2(t), \hat{c}_2(t) + \frac{1}{\varphi_{N_n+1}}]$, it holds that

$$\left| \frac{\partial q_2(x, t)}{\partial t} \right| = \left| \frac{2\lambda_{k(1)} \lambda_{k(1)}^* \tan \gamma_{1,1} + \lambda_{k(1)}^* \gamma_{1,1} + \gamma_{1,2}^\prime}{(1+\lambda_{k(1)}^* \tan^2 \gamma_{1,1})} + \gamma_{1,2} \right| \leq \left| \frac{\lambda_{k(1)}^2 \tan^{2} \gamma_{1,1}}{1+\lambda_{k(1)}^* \tan^2 \gamma_{1,1}} + \gamma_{1,2}^\prime \right| + \left| \frac{2\lambda_{k(1)} \lambda_{k(1)}^* \tan \gamma_{1,1}}{1+\lambda_{k(1)}^* \tan^2 \gamma_{1,1}} \right| + \left| \gamma_{1,2} \right|$$

(15)

Therefore, we obtain

$$\frac{1}{q_N} + 1 \geq \left| \frac{\partial q_2(x, t)}{\partial t} \right| > c_1 - 2\delta_1$$

for any $y \in [\hat{c}_2(t), \hat{c}_2(t) + \frac{1}{\varphi_{N_n+1}}]$.

**Case** $n \to n+1$:

Suppose for all $2 \leq l \leq n$, when $g_l$ is type I, we have obtained

$$1 + \sum_{i=1}^{l-1} \frac{1}{q_{N_{n+1}}-1} \geq \left| \frac{\partial q_l(x, t)}{\partial t} \right| > (c_1 - 2) \sum_{i=1}^{l-1} \delta_i$$

for any $x \in I_l(t)$; when $g_l$ is type II or III, we have

$$1 + \sum_{i=1}^{l-1} q_{N_{n+1}-1} \geq \left| \frac{\partial q_l(x, t)}{\partial t} \right| > c_1 - 2 \sum_{i=1}^{l-1} \delta_i,$$

for $x \in [\hat{c}_l(t), \hat{c}_l(t) + \frac{1}{q_{N_{n+1}-1}}]$.

For the case $n+1$, similarly we consider three cases as follows.

1. **$g_{n+1}$ is of type I.**
   
   Due to the $\left| \frac{\partial q_{n+1}(x, t)}{\partial t} \right|_{C^1(I_{n+1})} \leq \lambda^{-\frac{4}{2}} q_{N_{n+1}-2}$, we have
   
   $$1 + \sum_{i=1}^{n-1} q_{N_{n+1}-1} \geq$$

   $$1 + \sum_{i=1}^{n-1} q_{N_{n+1}-1} + \lambda^{-\frac{4}{2}} q_{N_{n+1}-2} \geq \left| \frac{\partial g_{n+1}(x, t)}{\partial t} \right| \geq (c_1 - 2) \sum_{i=1}^{n-1} \delta_i - \lambda^{-\frac{4}{2}} q_{N_{n+1}-2} \geq c_1 (1 - 2) \sum_{i=1}^{n-1} \delta_i$$

   for any $x \in I_{n+1}(t)$.

2. **$g_{n+1}$ is of type II.**
   
   It’s similar to the first case.

3. **$g_{n+1}$ is of type III.**
   
   There exits some $k(n) \ll \lambda^{c(k(n))} \leq k(n+1) \leq q_{N_{n+1}}$, such that $|c_{n,1} - c_{n,2} - k(n+1)| \leq \lambda^2 \tan \gamma_{n,1} - \frac{q}{2} + g_{n,2}$, where $\lambda_{n} > \lambda^{\lambda^{c(k(n))}}$, $g_{n,1}, g_{n,2}$ are all Type I satisfying

   $$1 + \sum_{i=1}^{n-1} q_{N_{n+1}-1} \geq \left| \frac{\partial g_{n+1}(x, t)}{\partial t} \right| > c_1 (1 - 2) \sum_{i=1}^{n-1} \delta_i$$
and

\[
1 + \sum_{i=1}^{n-1} q_{i+N-1}^{-1} \geq \frac{\partial \hat{g}_{n,2}}{\partial t} > c_1(1 - \sum_{i=1}^{n-1} \delta_i)
\]

with \(\text{sgn}(\hat{g}_{n,1}) = \text{sgn}(\hat{g}_{n,2})\).

Therefore (for convenience, we write \(\frac{\partial}{\partial t}\) as \(\dot{}\)), we have the lower bound as follow

\[
|\frac{\partial g_n(1, x)}{\partial t}| = \left| \frac{2\lambda_k(n+1) \tan \hat{g}_{n,1} + \lambda_k' \hat{g}_{n,2}}{1 + \lambda_k(n+1) \tan \hat{g}_{n,1}} - \frac{\hat{g}_{n,1}' + \hat{g}_{n,2}'}{\lambda_k(n+1) \tan \hat{g}_{n,1}} \right| \\
\geq c_1(1 - 2\sum_{i=1}^{n-1} \delta_i)(1 + 0.5\lambda_k^{-2} k(n+1)) - 2\lambda_k^{-2} k(n+1) \\
\geq c_1(1 - 2\sum_{i=1}^{n-1} \delta_i) - \lambda_k^{-2} k(n+1) \geq c_1(1 - 2\sum_{i=1}^{n-1} \delta_i).
\]

For the upper bound, note that if \(|\hat{g}_{n,1}| > \lambda_k^{-2} k(n+1)\), then we can rewrite \(g_{n+1}\) as

\[
\dot{C}\lambda_k^{-2} k(n+1) (\frac{\partial g_{n,1}}{\partial x}(c_{n,1}) + o(\frac{1}{q_N}))\hat{g}_{n,1})^{-1} + d_1 - (\frac{\partial g_{n,2}}{\partial x}(c_{n,2}) + o(\frac{1}{q_N}))\hat{g}_{n,2},
\]

where \(\dot{C}\) is an absolute constant. Therefore, it follows from the estimate in [WZ1] that

\[
g_{n+1} = C_1\lambda_k^{-2} k(n+1) (c_{n,1}'(x - c_{n,1}'))^{-1} + d_1 - c_{2}^n(x + k(n+1)\alpha - c_{n,2}),
\]

where

\[
c^n_i = \frac{\partial g_n}{\partial x}(c_{n,i}) + \sum_{i=1}^{n} \delta_i(x, t) \leq \frac{1}{q_N+n-1}, i = 1, 2
\]

with \(||\delta^n_i(x, t)||_{C^2} \leq \frac{1}{q_N+n-1}\).

By a direct calculation, we have \(\dot{c}_{n+1,i} = c_{n,i}' + \dot{C}\dot{\lambda}_k^{2}((\sqrt{c_{n,1}^2 + c_{n,2}^2})^{-1})\lambda_k^{-1}, i = 1, 2\).
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Hence, for any $x \in [\bar{c}_{n+1}(t), \hat{c}_{n+1}(t) + \frac{1}{q_{N+n-1}}]$, we have

$$\left| \frac{\partial g_{n+1}(x, t)}{\partial t} \right|$$

$$= \frac{2\lambda_{k(n+1)} \lambda_{k(n+1)}'}{1 + \lambda_{k(n+1)}^2} \tan \hat{g}_{n,1} + \frac{\lambda_{k(n+1)}^2}{1 + \lambda_{k(n+1)}^2} \tan^2 \hat{g}_{n,1} + \frac{\lambda_{k(n+1)}^2}{1 + \lambda_{k(n+1)}^2} \tan^2 \hat{g}_{n,1} + \frac{\lambda_{k(n+1)}^2}{1 + \lambda_{k(n+1)}^2} \tan^2 \hat{g}_{n,1} + o(\lambda_{k(n+1)}^2)$$

$$\leq \frac{\lambda_{k(n+1)}^2}{1 + \lambda_{k(n+1)}^2} \tan^2 \hat{g}_{n,1} + \frac{\lambda_{k(n+1)}^2}{1 + \lambda_{k(n+1)}^2} \tan^2 \hat{g}_{n,1} + \frac{\lambda_{k(n+1)}^2}{1 + \lambda_{k(n+1)}^2} \tan^2 \hat{g}_{n,1} + \frac{\lambda_{k(n+1)}^2}{1 + \lambda_{k(n+1)}^2} \tan^2 \hat{g}_{n,1} + o(\lambda_{k(n+1)}^2)$$

$$\leq 1 + \sum_{i=1}^{n} q_{N+i-1}^{-1},$$

where the last inequality follows from (17). Therefore, we obtain

$$1 + \sum_{i=1}^{n} q_{N+i-1}^{-1}$$

$$\geq \left| \frac{\partial g_{n+1}(\bar{c}_{n+1}, t, t)}{\partial t} \right|$$

$$> c_1(1 - 2 \sum_{i=1}^{n} \delta_i), i = 1, 2.$$
Proof. For the first case.

On one hand, according to Lemma 15, we obtain
\[
c(\alpha, v) \leq \left| \frac{\partial g_i(x, t)}{\partial t} \right| \leq c^{-1}(\alpha, v)
\]
for any \(x \in I_i(t)\).

On the other hand, since \(g_i\) is of type I, we clearly have (for sufficiently large \(\lambda\))
\[
1 \geq \left| \frac{\partial g_i}{\partial x} \right| \geq q_{N+i-1}^C(\alpha, v, \lambda), \forall x \in I_i.
\]

Then it follows from the Implicit Function Theorem that
\[
c(\alpha, v) \leq \left| \frac{dc_{1,1}(t)}{dt} \right| = \left| \frac{\partial g_i(c_{1,1}(t), t)}{\partial t} \frac{\partial g_i(c_{1,1}(t), t)}{\partial x} \right| \leq c^{-1}(\alpha, v)q_{N+i-1}^C(\alpha, v, \lambda),
\]
where \(0 < c(\alpha, v) < 1\).

Moreover, we have
\[
\left| \frac{dc_{1,1}(t)}{dt} \right| - \left| \frac{dc_{1,2}(t)}{dt} \right| = \left( \frac{\partial g_i(c_{1,1}(t), t)}{\partial t} \frac{\partial g_i(c_{1,1}(t), t)}{\partial x} \right) \cdot \left( \frac{\partial g_i(c_{1,1}(t), t)}{\partial t} \frac{\partial g_i(c_{1,1}(t), t)}{\partial x} \right)
\]
< 0
\]
Thus, it holds that
\[
2C(\alpha, v, \lambda)q_{N+i-1}^C(\alpha, v, \lambda) \geq \left| \frac{dd_i(t)}{dt} \right| = \left| \frac{dc_{1,1}(t)}{dt} - \frac{dc_{1,2}(t)}{dt} \right| > 2c(\alpha, v).
\]

The second case can be similarly handled with the help of Lemma 15, the left inequality of (18) and the Implicit Function Theorem.

\(\square\)

Now, we are ready to show the ‘leaness’ of \(IR\) in the sense of measure.

Lemma 18. \(\text{Leb}(IR) = 0\).

Proof. Denote
\[
B(n, k) := \{t|d_n(t) - k\alpha| \geq \frac{1}{q_{N+n-1}^{1000C}}\};
\]
\[
P_1(n) = \{t|\min |g_n(t)| = 0\} \text{ and } P_2(n) = \{t|\min |g_n(t)| > 0\};
\]
\[
\bigcup_{k \leq q_{N+n-1}} B^c(n, k) := C(n).
\]

Clearly,
\[
\Sigma_1 = \bigcap_{l \geq 1} \bigcup_{n \geq l} \bigcup_{k \leq q_{N+n-1}} B(n, k);
\]
\[
\Sigma_2 = \bigcap_{l \geq 1} \bigcup_{n \geq l} \bigcup_{k \leq q_{N+n-1}} B^c(n, k) = \bigcup_{l \geq 1} \bigcup_{n \geq l} C(n);
\]

By Borel Contelli Lemma, it is suffice to prove \(\sum_{n=1}^{+\infty} \text{Leb}(C(n)) < +\infty\).

Clearly, \(g_n\) is of type III, by the help of Lemma 15 and Lemma 17 it’s not difficult to see:

\(a\): If \(\min |g_n(t)| = 0\), when we continuously move \(t\) to \(t'\) with \(\min |g_n(y)| = 0\) for any \(y\) between \(t\) and \(t'\), the difference between the critical points \(c_n,1, 2\) is at least \(c(\alpha, v)|t - t'|\), that is
\[
|d_n(t) - d_n(t')| \geq c(\alpha, v) \cdot |t - t'|.
\]
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Therefore, for any $|t' - t| \geq \frac{1}{q_{N+n-1}}$ we obtain

$$|d_n(t') - k\alpha| \geq c(\alpha, v) \frac{1}{q_{N+n-1}^{1000\epsilon}} - \frac{1}{q_{N+n-1}^{500\epsilon}} \geq \frac{1}{q_{N+n-1}^{1000\epsilon}},$$

which implies

$$\text{Leb} \left\{ C(n) \cap P_2(n) \right\} = \text{Leb} \left\{ \{t| |d_n(t) - k\alpha| < \frac{1}{q_{N+n-1}^{1000\epsilon}} \} \cap P_2(n) \right\} \leq \frac{1}{q_{N+n-1}^{500\epsilon}}.$$

\textbf{b:} If $\min |g_n(t)| > 0$, then according to Lemma 15 we are allowed to continuously move $t$ until $t''$ such that $\min |g_n(t'')| = 0$ with $|t - t''| \leq c(\alpha, v)\lambda^{-\frac{1}{2}r_n} \ll \frac{1}{q_{N+n-1}^{500\epsilon}}$.

Thus, combining a and b we have

$$\text{Leb}(B^c(n, k)) = \text{Leb}(B^c(n, k) \cap P_1(n)) + \text{Leb}(B^c(n, k) \cap P_2(n)) \leq \frac{1}{q_{N+n-1}^{500\epsilon}} + \frac{1}{q_{N+n-1}^{500\epsilon}}.$$

(20)

Therefore

$$\sum_{n=1}^{\infty} \text{Leb}(C(n)) \leq \sum_{n=1}^{\infty} \sum_{k \leq q_{N+n-1}} \text{Leb}(B^c(n, k)) \leq \sum_{n=1}^{\infty} \frac{1}{q_{N+n-1}^{1000\epsilon}} < \infty.$$

Let $EP := \{\text{all the endpoints of spectrum gaps}\}$ and $LP := \Sigma - EP$. Clearly, we obtain that $LP$ is a full measure set in $\Sigma$.

when $g_n$ is of type II or III, let

$$\tilde{c}_{n,1}(t) := \{x \in I_n \mid \frac{d}{dx}g_n(x) = 0 \text{ and } x \text{ is between } c_{n,2} \text{ and } c_{n,1}\}.$$

Similarly, let

$$\tilde{c}_{n,2}(t) := \{x \in I_n \mid \frac{d}{dx}g_n(x) = 0 \text{ and } x \text{ is between } c_{n,1} \text{ and } c_{n,2}\}.$$

The following lemma states the important property of $t \in EP$, which results in $\frac{1}{2}$-Hölder continuity.

**Lemma 19.** The following are equivalent.

1. $t \in EP$;
2. $t \in \frac{1}{\lambda} \Sigma$, there exists some $N(t)$ such that $g_n(t)$ always is of type II or III for $n \geq N(t)$;
3. $t \in \frac{1}{\lambda} \Sigma$, there exists some $N(t)$ such that $|g_n(t, \tilde{c}_{n,1}(t))| < \lambda^{-\frac{1}{2}r_N+n-2}$ for $n \geq N(t)$.

First, we prove the following lemma.

**Lemma 20.** For any fixed $t \in J$ if $g_n(t)$ is of type I for some $n \in \mathbb{N}$, it holds that $[t - \lambda^{-0.5q_N+n(t)-1}, t + \lambda^{-0.5q_N+n(t)-1}] := I(n(t)) \cap \Sigma \neq \emptyset$.

**Proof:** Clearly, it follows from \cite{Yi} that $\lim \min_{n \to \infty} g_n(t) = 0$ implies $t \in \Sigma$. Thus, without loss of generality, we assume that there exists some $N(t) > n(t)$ such that $g_N$ is of type III and $g_i(t), n(t) \leq i \leq N - 1$, are of type I. We denote $t_0 = t$ and consider the following two conditions.
Therefore, we can find some $t_1 \in [t - \lambda^{-\frac{3}{4}r_{N-1}}, t + \lambda^{-\frac{3}{4}r_{N-1}}]$ such that $|g_N(t_1)(\tilde{c}_N)| \geq \lambda^{-\frac{3}{2}r_{N-1}}$ and $|g_N| = 0$. Since

\[ \sum_{0 \leq i \leq +\infty} \|g_{N+i} - g_{N+i+1}\| < C\lambda^{-\frac{3}{4}r_{N-1}} \ll \lambda^{-\frac{3}{2}r_{N-1}}, \]

there exists some $N' > N$ such that $g_{N'}$ is of type I and $g_i, N \leq i \leq N' - 1$, are of type III. Clearly, we have $q_{N+N'-1}^C \leq |I_{N'}| \leq \lambda^{-\frac{3}{4}r_{N-1}}$, thus

(21) \[ \lambda^{-\frac{3}{4}r_{N'}} \leq \lambda^{-\frac{3}{4}r_{N+N'-1}} \leq \lambda^{-\frac{3}{4}r_{N-1}} \leq \lambda^{-\frac{3}{2}r_{N-1}}. \]

Therefore, it’s not difficult to see that if we start from $g_{N'}$, by the same process as above we can find some $t_2 \in [t - \lambda^{-\frac{3}{4}r_{N-1}}, t + \lambda^{-\frac{3}{4}r_{N-1}}]$ such that $|g_N(t_2)(\tilde{c}_{N'})| \geq \lambda^{-\frac{3}{2}r_{N(t_2) - 1}}$ and $|g_{N'}| = 0$. Meanwhile $|g_{N'}(t_2)(\tilde{c}_{N'})| \geq (1 - e_1)\lambda^{-\frac{3}{2}r_{N-1}}$ and $|g_{N'}| = 0$ with $e_1 = \lambda^{-\frac{3}{4}r_{N-1}}$.

Repeat this process. We can find the sequences $t_i$ and $N(t_i)$ such that $g_{N(t_j)}$ is of type I and $|g_{N(t_j)}(t_j)(\tilde{c}_{N(t_j)})| \geq \lambda^{-\frac{3}{2}r_{N(t_j) - 1}}$, $|g_{N(t_j)}| = 0$. Meanwhile for each $j < i$, we have $|g_{N(t_j)}(t_i)(\tilde{c}_{N(t_j)})| \geq (1 - \sum_{i=1}^{j-1} e_i)\lambda^{-\frac{3}{2}r_{N(t_j) - 1}}$ and $|g_{N(t_j)}(t_i)| = 0$ with $e_i = \lambda^{-\frac{3}{4}r_{N-1}}$.

Denote $t_0 = \lim_{i \to \infty} t_i$. Clearly, $t_0 \in I(n(t))$. Finally, we find a sequence $g_{N_i}(t_0)$ such that $\lim_{i \to \infty} |g_{N_i}(t_0)| = 0$, thus $t_0 \in \Sigma$ as require.

\[ \square \]

**The proof of lemma 15.** 1 → 2 :Note that if there exists some subsequence $g_n(t)$ being of type I. By lemma 15 it holds that for any $t_i = t - \frac{1}{q_{N+n(i)-1}^C}, g_n(t_i)$ is of type I. Similarly, for any $t^+_i = t + \frac{1}{q_{N+n(i)-1}^C}, g_n(t^+_i)$ is of type I. Therefore, it follows from lemma 20 that there exists some spectral point $t^+_i \in B(t_i^{\pm}, \lambda^{-\frac{3}{4}r_{n-1}})$. Hence at each side of $t$, there exists two sequence $t_i^+$ and $t_i^-$, which come from the spectrum, such that $t_i^+ \to t$. This is a contradiction since for any endpoint, there must be no spectral point on one side.

Therefore, there exists some $N$ such that $g_n(t)$ are of type II or III $\forall n \geq N$.

2 → 3 : note that if $|g_{m}(\tilde{c}_{m,1})| \geq C\lambda^{-\frac{3}{4}r_{m-1}}$ for some $m > N$, only the following two cases might occur.

(1) $|g_{m}(\tilde{c}_{m,1})| = 0$.

Since (21) holds, there must exist some $N' > m$ such that $g_{N'}$ is of type I, this contradicts the fact $g_n(t)$ are of type II or III $\forall m \geq N$.

\[ \square \]
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\[(2) \min |g_m(\tilde{c}_{m,1})| > 0.\]

Again by [21], it follows that $t \in \Sigma^c$, which also contradicts the fact $t \in \Sigma$.

$3 \rightarrow 1$: For any fixed $t$ satisfies $3$, by the help of Lemma [15] we have a uniformly (respect with $n$ and $t$) lower bound

$$\left| \frac{\partial g_n(x,t)}{\partial t} \right| > c$$

for any $x \in J_n \equiv [\tilde{c}_n(t), \tilde{c}_n(t) + \frac{1}{q_{N+n-1}}]$ or $x \in [\tilde{c}_n(t) - \frac{1}{q_{N+n-1}}, \tilde{c}_n(t)] \in I_n(t)$. Thus for fixed large $n_0 > \lambda^{N(t)}$ and $t'$ satisfied $q_{N+n_0}^{-1000} < |t' - t| < q_{N+n_0-2}^{-1000}$, we have

$$\left| \frac{\partial g_{n_0}(c_{n_0}(t'), t')} {\partial t} - \frac{\partial g_{n_0}(c_{n_0}(t), t)} {\partial t} \right| \geq c|t - t'| \geq q_{N+n_0-1}^{-1000}.$$ Combining this with the help of [22], we have that for $n \geq n_0$, \[(22) \min \{ g_n(t') \} \geq c|t' - t| - \sum_{i=n_0}^{+\infty} \lambda^{-\frac{1}{2}r_i} \geq q_{N+n_0}^{-3000} \equiv C(n_0) > 0,\]

which implies $\mathcal{UH}$. And according the definition of type III, it’s clear that (22) holds for only one of the following two cases. \[(1)[t - q_{N+n_0-2}^{-1000}, t - q_{N+n_0-1}^{-1000}] \quad \text{WLOG, we assume (1) holds true, which implies } [t - q_{N+n_0-2}^{-1000}, t - q_{N+n_0-1}^{-1000}] \subset \frac{1}{\lambda} \Sigma^c. \text{ Hence, } [t - q_{N+n_0}^{-1000}, t) = \bigcup_{t \geq n_0} [t - q_{N+n_0-2}^{-1000}, t - q_{N+n_0-1}^{-1000}] \subset \frac{1}{\lambda} \Sigma^c. \text{ Therefore, } t \in EP. \]

At the end of the section, we prove one of the important property of $EP$.

**Lemma 21.** $\Sigma_3 = EP$.

**Proof.** Combine Lemma [19] with the definition of $\Sigma_3$, we obtain that for $t \in \Sigma_3$ there exists $N(t)$ such that $g_n$ is of type III for each $n \geq N(t)$, which implies $EP \supseteq \Sigma_3$. For another direction, we only need to prove $\Sigma_3 \subseteq EP^c$. Lemma [19] shows that $\Sigma_1 \cap EP = \emptyset$. Since $\Sigma_3 \subseteq \Sigma_2$, it’s enough to show $\Sigma_2 = \Sigma_3 \cap EP = \emptyset$. Again by the definition of $\Sigma_3$, we can find a sequence $0 < k_1 < k_2 < \cdots < k_n < \cdots \rightarrow \infty$ such that

$$|d_{N(k_i)} - k_i \alpha| \geq \lambda^{-\frac{1}{2}r_{k_i}^{-1}}.$$ 

Thus there must exists some type I cases between each $k_i$ and $k_{i+1}$. Again by Lemma [19] we finish the proof. □

### 5. Indicator function for measuring the intensity of resonance

In this section, we introduce a function to measure the intensity of resonance. Let

\[(23) \beta(t) \equiv \min \{|\liminf_{n \rightarrow +\infty} \frac{1}{2} + \log \|A_{k_n}\|} {2 \log |c_{n,1}(t) - c_{n,2}(t) - k_n(t)\alpha|^{-1}} \}, \]

Here $k_n$ denote the **resonance-distance**, which is defined as follows.

\[k_n := \min \{ k |I_{n,1} + k\alpha \cap I_{n,2} \neq \emptyset \}, \]

$n \geq 1$.
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It’s easy to find that either there exists some \( k \geq 0 \) and \( N > 0 \) such that \( k_n = k \) for any \( n \geq N \), or \( k_n \to +\infty \). In fact, for any \( t \in EP \), \( t \) satisfies the first case. With slight offense to the definition, we denote \( \|A_{k_n}\| := \|A_{k_n}(x)\| \) for a fixed \( x \in I_n \). It’s essentially well defined since lemma \([27]\) implies

\[
\left| \frac{d \|A_{k_n}\|}{dx} \right| \leq \|A_{k_n}(x)\| q_{N+n-1}^C
\]

for \( x \in I_n \) and

\[
q_{N+n-1} \ll \lambda^k_n \leq \|A_{k_n}\| \leq \lambda^k_n,
\]

which means \( \|A_{k_n}(x)\| \) is almost a constant on \( I_n \). It’s easy to check that

\[
\frac{1}{2} \leq \beta(t) \leq 1.
\]

Besides, it’s worth noting that \( \beta(t) \) represents the intensity of resonance. One can check that the definition is equivalent to the following one.

**Lemma 22.**

\[
\beta(t) \triangleq \min \{ \liminf_{k \to +\infty} \left( \frac{1}{2} + \frac{k \log \lambda}{2 \log |c_{\infty,1} - c_{\infty,2} - k\alpha|} \right), 1 \},
\]

**Proof.**

1: If there exist a subsequence \( n_i \) such that \( |c_{n_i,1}(t) - c_{n_i,2}(t) - k_{n_i}(t)\alpha| \leq \lambda^{-\frac{\delta}{2}r_{n_i}} \), then by the help of the Induction Theorem of \([WZ1]\), it’s clear that \( |c_{\infty,1}(t) - c_{\infty,2}(t) - k_{n_i}(t)\alpha| \leq \lambda^{-\frac{\delta}{2}r_{n_i}} \). Thus, it follows from the fact \( \|A_{k_{n_i}}\| \leq \lambda^{k_{n_i}} \leq \lambda^{\frac{\delta}{2}r_{n_i}} \) and \( \lambda^{1 - \sum_{i=1}^{\infty} \frac{\log q_{N+n-1}}{n_i}} \leq \lambda_{\infty} \leq \lambda \) that

\[
\lim_{j \to +\infty} \log \|A_{k_{n_j}}\| = \lim_{j \to +\infty} \frac{k_{n_j} \log \lambda_{\infty}}{\log |c_{\infty,1} - c_{\infty,2} - k_{n_j}\alpha|} = 0.
\]

Note that the definition shows \( \beta(t) \geq \frac{1}{2} \), then it’s not difficult to see that

\[
\beta(t) = \min \{ \liminf_{k \to +\infty} \left( \frac{1}{2} + \frac{k \log \lambda_{\infty}}{2 \log |c_{\infty,1} - c_{\infty,2} - k\alpha|} \right), 1 \}
\]

\[
= \min \{ \liminf_{n \to +\infty} \left( \frac{1}{2} + \frac{\log \|A_{k_n}\|}{2 \log |c_{n,1} - c_{n,2} - k_n\alpha|} \right), 1 \} = \frac{1}{2}.
\]

2: If for all \( n \in \mathbb{N}, |c_{n,1}(t) - c_{n,2}(t) - k_n(t)\alpha| > \lambda^{-\frac{\delta}{2}r_n} \), then it again follows from the Induction Theorem of \([WZ1]\) that

\[
|c_{n,1}(t) - c_{n,2}(t) - k_n\alpha| - \lambda^{-\frac{\delta}{2}r_n} \leq |c_{\infty,1}(t) - c_{\infty,2}(t) - k_{n}\alpha| \leq |c_{n,1}(t) - c_{n,2}(t) - k_{n}\alpha| + \lambda^{-\frac{\delta}{2}r_n},
\]

which implies

\[
1 - \lambda^{-r_n} \leq \frac{|c_{\infty,1}(t) - c_{\infty,2}(t) - k_{n}\alpha|}{|c_{n,1}(t) - c_{n,2}(t) - k_{n}\alpha|} \leq 1 + \lambda^{-r_n}.
\]

Combine this with \( |\log \lambda_n - \log \lambda_{\infty}| \leq \sum_{i=n}^{\infty} \frac{\log q_{N+n-1}}{q_{N+n-1}} \triangleq \epsilon_n \), it holds that

\[
(1-\epsilon_n)(1-\lambda^{-r_n}) \leq \frac{\log \|A_{k_n}\| \log |c_{\infty,1}(t) - c_{\infty,2}(t) - k_{n}\alpha|}{k_n \log \lambda_{\infty} \log |c_{n,1}(t) - c_{n,2}(t) - k_{n}\alpha|} \leq (1+\epsilon_n)(1+\lambda^{-r_n}).
\]
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Therefore, on one hand, for any given $t \in \Sigma$, if we assume there exists a sequence $k_n$ such that (23) convergence to the limit inferior($\beta(t)$), then with the same sequence we have

$$\min\{\liminf_{n \to +\infty} \frac{1}{2} + \frac{k_n \log \lambda_\infty}{2 \log |c_{\infty,1} - c_{\infty,2} - k_n \alpha|}, 1\} = \beta(t),$$

which implies

$$\min\{\liminf_{n \to +\infty} \frac{1}{2} + \frac{k \log \lambda_\infty}{2 \log |c_{\infty,1} - c_{\infty,2} - k \alpha|}, 1\} \leq \beta(t).$$

On the other hand, the definition of $k_n$ shows that $|c_{\infty,1} - c_{\infty,2} - j \alpha| \geq \frac{1}{10} |J_j| - \|A_{k_n-1}\|^{-1} - \lambda^{-\frac{2}{20} j} \geq \lambda^{-\frac{1}{20} j}$, for $k_n - 1 \leq j \leq k_n$. Thus $|c_{\infty,1} - c_{\infty,2} - k \alpha| \geq \lambda^{-\frac{1}{20} k}$, for each $k \in \mathbb{N}$, which directly implies

$$\min\{\liminf_{n \to +\infty} \frac{1}{2} + \frac{k \log \lambda_\infty}{2 \log |c_{\infty,1} - c_{\infty,2} - k \alpha|}, 1\} \geq \beta(t).$$

In conclusion we obtain (22). \qed

It is easy to see that If $t \in EP$, then $\beta(t) = \frac{1}{2}$, if $t \in FR$, then $\beta(t) = 1$.

6. Regularity of LE at the end points of spectral gaps

In this section, we prove the exact $\frac{1}{2}$-Hölder continuity of Lyapunov exponent $L(E)$ for $E \in \Sigma$. That is, for each end point $E_0$ of spectral gaps, there exists a $\delta = \delta(E_0) > 0$ and two positive constants $\tilde{c} = \tilde{c}(E_0), \tilde{c}' = \tilde{c}'(E_0)$ such that for each $E \in (E_0 - \delta, E_0 + \delta)$, it holds that $\tilde{c}|E - E_0|^\frac{1}{2} \leq |L(E) - L(E_0)| \leq \tilde{c}'|E - E_0|^\frac{1}{2}$.

The $\frac{1}{2}$-Hölder continuity is proved by improving the estimates in a recent work \cite{LWY}. Based on a Large Deviation Theorem (LDT) of the form

$$\text{Leb}\{x \in \mathbb{R}/\mathbb{Z}: \frac{1}{i} \log \|A_i(x)\| \geq \frac{9}{10} \log \lambda\} < \lambda^{-c_i}, \quad 0 < c < 1/2,$$

the $c$–Hölder continuity of Lyapunov exponent was obtained in \cite{LWY}. The idea for the proof of \cite{LWY} is as follows.

Since $L_{N_0}(E) = \frac{1}{N_0} \int \log \|A_{N_0}(x, E)\| dx$, from the facts that $\|A_{N_0}(x, E)\| \geq 1$ and $|\partial E| |A_{N_0}(x, E)| \leq C N_0$ with $C > 1$ depending only on $A(x)$, we have

$$|L'_{N_0}(E)| = \left| \frac{1}{N_0} \int \frac{\partial E}{|A_{N_0}(x, E)|} \|A_{N_0}(x, E)\| dx \right| \leq C N_0.$$

With the help of LDT and the Avalanche Principle, for each pair $(E_0, E)$, there exist $c > 0, C' > C$ and (large) $N_0$ satisfying $|E - E_0| \approx C'^{-N_0}$ such that

$$|L(E) - L(E_0)| \leq |L_{N_0}(E) - L_{N_0}(E_0)| + |L_{2N_0}(E) - L_{2N_0}(E_0)| + 2 e^{-c N_0}.$$

It follows that

$$|L(E) - L(E_0)| \leq 2 C N_0 |E - E_0| + 2 e^{-c N_0} \leq |E - E_0|^{\frac{c'}{2}}$$

with $c' \approx \min\{\log C', \frac{c'}{2}, c \cdot \log C, \epsilon\} > 0$.

Note that the estimate on the upper bound for $L'_{N_0}(E)$ in the above argument is too bad. In fact, in \cite{LWY} it is estimated by $\|\frac{\partial E}{|A_{N_0}(x, E)|} \|_{L^2(\mathbb{Z})} \|_{C^0(\mathbb{Z})} \approx C N_0$. However, the upper bound for $L'_{N_0}(E)$ only depends on $\|\frac{\partial E}{|A_{N_0}(x, E)|} \|_{L^2(\mathbb{Z})}$, which should
be much less than \( \| \cdot \|_{C^0(S^1)} \). The reason lies in the observation that the set of bad \( x \) satisfying
\[
\| \frac{\partial x}{\partial N_0(x,E)} \|_{A_0(x,E)} \| \approx C N_0
\]
is of a small measure depending on \( N_0 \). By a complicated computation, we will be able to obtain a sharp upper bound estimate on \( L^2_{N_0}(E) \) which is sufficient to prove \( \frac{1}{2} \)-Hölder continuity.

On the other hand, for different \( E \), the measure of bad \( x \) is of different order. More precisely, the measure of the bad set for end points of spectral gaps is larger than the one for each point in some set \( FR \) of full measure. Thus for an end point of spectral gaps we can obtain a lower bound on \( L^2_{N_0}(E) \) which implies the regularity cannot be better than \( \frac{1}{4} \)-Hölder continuity, while for each point in the set \( FR \) we can obtain the Lipschitz continuity.

From now on, the scale of the relevant quantities in \([WZ1]\) will be slightly modified as follows. For any \( n \geq 0 \) Let \( I_n := B(c_n, 2^n q_{N+n+1}) \bigcup B(c_n, 2^n q_{N+n+1}) \). Define \( q_{N+n+1}^2 < r_n^j : I_n \to \mathbb{Z}^+ \) to be the smallest positive number \( j \) such that \( j > q_{N+n+1}^2 \) and \( T^{j}x \in I_n \) for \( x \in I_n \). Note that the choice of “8\( n \)” is nonessential, and any large constant \( C \) is adequate. It’s not difficult to see the different choices only affect on the error function between the two iterations steps in \([WZ1]\).

For our purpose, LDT in \([LWY]\) is needed. For the convenience of readers, we provide a simple proof of it in Appendix A.4.

**Theorem 23.** Let \( v \) and \( \alpha \) be as in Theorem 1. Then there exists \( \lambda_1 = \lambda(v, \alpha), i_0 = i_0(\alpha) \in \mathbb{Z}^+ \) and \( 0 < c < 1 \), such that for each \( \lambda > \lambda_1 \) and each \( i \geq i_0 \), it holds that
\[
\text{Leb}\{x \in \mathbb{R}/\mathbb{Z}| \frac{1}{i} \log \| A_i(x) \| \geq \frac{9}{10} \log \lambda \} < \lambda^{-ci}.
\]

**Remark 24.** The version of LDT is not sharp but it is enough for our purpose.

The following lemma is a corollary of Avalanche Principle of \([GS]\), see \([BoJ]\).

**Lemma 25.** Let \( E^{(1)}, \cdots, E^{(n)} \) be a finite sequence in \( SL(2, \mathbb{R}) \) satisfying the following conditions
\[
\min_{1 \leq j \leq n} \left\| E^{(j)} \right\| \geq \mu \geq n.
\]

and
\[
\min_{1 \leq j \leq n} \left| \log \left\| E^{(j+1)} \right\| + \log \left\| E^{(j)} \right\| - \log \left\| E^{(j+1)} E^{(j)} \right\| \right| < \frac{1}{2} \log \mu,
\]

then
\[
\left| \log \left\| E^{(n)} \cdots E^{(1)} \right\| + \sum_{j=1}^{n-1} \log \left\| E^{(j)} \right\| - \sum_{j=1}^{n-1} \log \left\| E^{(j+1)} E^{(j)} \right\| \right| \leq C \frac{n}{\mu}.
\]

With the help of LDT and Avalanche Principle, we have

**Lemma 26.** Let \( v, \alpha \) and \( \lambda \) be as in Theorem 23. Then for all large \( n \in \mathbb{Z}^+ \) and for all \( E \in [\lambda \inf v - 2, \lambda \sup v + 2] \), it holds that
\[
\left| L_n(E) + L(E) - 2 L_2n(E) \right| < \lambda^{-2n}.
\]
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Proof. For any $l \in \mathbb{N}$, Let

$$ n = [\lambda^{2l}], E^{(j)} = A_l(x + (j - 1)l\alpha). $$

From lemma 23, there is an exceptional set $\Omega \subset \mathbb{R}/\mathbb{Z}$ with $Leb(\Omega) < \lambda^{-\frac{5}{2}}$ such that if $x \notin \Omega$, then

$$ \frac{9}{10} \log \lambda < \frac{1}{l} \log \|A_l(x + j\alpha)\|, \frac{1}{2l} \log \|A_{2l}(x + j\alpha)\| < \log \lambda. $$

Hence for $x \notin \Omega$,

$$ |\log \|E^{(j+1)}\| + \log \|E^{(j)}\| - \log \|E^{(j+1)}E^{(j)}\|| < \frac{1}{5} \log \lambda < \frac{4}{2} \log \lambda. $$

Thus taking $\mu = \lambda^{\frac{5}{2}}$, conditions (25) and (26) of lemma 25 are clearly fulfilled. For $x \notin \Omega$ the conclusion (27) is that

$$ |\log \|A_{ln}(x)\| + \sum_{2 \leq j \leq n-1} \log \|A_l(x + (j-1)l\alpha)\| - \sum_{1 \leq j \leq n-1} \log \|A_{2l}(x + (j-1)l\alpha)\|| \leq \frac{Cn}{\mu}. $$

Divide the above inequality by $ln$ and integrate it in $x \in \mathbb{R}/\mathbb{Z}$. Splitting the integration as $(\mathbb{R}/\mathbb{Z}) - \Omega$ and $\Omega$, we get

$$ |L_{ln}(E) + \frac{n-2}{n}L_l(E) - \frac{2(n-1)}{n}L_{2l}(E)| < C(\mu^{-1}l^{-1} + Leb(\Omega)) < C\lambda^{\frac{5}{2}}. $$

Note here $C$ depends on $\lambda$. Hence we obtain

(28) $$ |L_{ln}(E) + L_l(E) - 2L_{2l}(E)| < C\lambda^{-\frac{5}{2}}. $$

Applying (28) to $n = n_1 = l$ large enough and $\log n_2 \sim n_1$ yields

(29) $$ |L_{n_1}(E) + L_{n_1}(E) - 2L_{2n_1}(E)| < C\lambda^{-\frac{5}{2}n_1}. $$

Applying (28) to $n = n_1 = l$ and $2n_2$ yields

$$ |L_{2n_2}(E) + L_{n_1}(E) - 2L_{2n_1}(E)| < C\lambda^{-\frac{5}{2}n_1}. $$

Therefore, we get

(30) $$ |L_{2n_2}(E) - L_{n_2}(E)| < C\lambda^{-\frac{5}{2}n_1}. $$

Clearly, in (29) and (30), we may replace $n_1$ and $n_2$ by any $n_s$ and $\log n_{s+1} \sim n_s$. Thus we obtain

$$ |L(E) - L_{n_2}(E)| \leq \sum_{s \geq 2} |L_{n_{s+1}}(E) - L_{n_s}(E)| $$

$$ \leq 2 \sum_{s \geq 2} (|L_{n_s}(E) - L_{n_s}(E)| + C\lambda^{\frac{5}{2}n_s}) $$

$$ \leq 4 \sum_{s \geq 1} C\lambda^{\frac{5}{2}n_s} $$

$$ < C\lambda^{-\frac{5}{2}n}. $$

Thus, replacing $L_{n_2}(E)$ by $L(E)$ in (29) yields lemma 26.

\[ \square \]
For any fixed $n$ and $x$, we can always assume that $r_i(t, x) \equiv r_i(t', x)$ for any $i \leq n$ when $|t - t'|$ small enough, which means $r_i(x, t)$ is locally independent on $t$ for $i \leq n$. (This holds from continuity of $c_i(t)$ on some small neighborhood and the fact: $|x + r_i(x, t)\alpha - c_i| \leq 0.5|I_i|$ and $|x + s\alpha - c_i(t)| > 0.5|I_i|$). We first deal with the case $s = 1$. Let $\hat{r}_i(x, t) = r_i(x)$ for $i \leq n$ in the next proofs since the derivative represent the property on any small neighborhood. In the same manner, for any fixed $\hat{r}_i(x, t)$ and $n$, we assume $r_i(x, \hat{t}) \equiv r_i(x_0, \hat{t})$, when we estimate 
$$
\frac{1}{\|A_n^{-1}(x, t)\|} \|\frac{d}{dx}A_n^{-1}(x, \hat{t})\|(x_0).
$$

Now we are at a position to estimate 
$$
\frac{1}{\|A_n(x, E)\|} \|\frac{\partial \|A_n(x, E)\|}{\partial E}\| \leq \max(r_i^n(x), q_{N+n-1})^{C(t, \alpha, v)}.
$$

Moreover, for any $t \in EP$, which is labeled by $k \in N$, let $\tilde{N}(k)$ satisfy $q_{N+\tilde{N}(k)-1} \leq k \leq q_{N+\tilde{N}(k)-2}$. There exist some constant $C'(\alpha, v)$ (independent on $t$) such that

$$
\int_{I_{\tilde{N}(k), t}} \frac{1}{\|A_{r_{\tilde{N}(k)}(x, t)}\||\frac{\partial \|A_{r_{\tilde{N}(k)}(x, t)}\|}{\partial t}| dx | \leq r_{\tilde{N}(k)}^{C'(\alpha, v)}.
$$

Proof: It follows from the definition of $FR$ that there exists some integer $L$ such that the series $\{g_s\}_{s \geq L}$ are of type $I$, which implies

$$
c(L)^{-1} \geq \left| \frac{dg_s(x)}{dx} \right| \bigg|_{I_{s-1}} > c(L),
$$

where $c(L)$ can be chosen as $|I_L|^{100}$.

Next, we’ll prove the lemma by induction.

For $n = L$, let $C_0 = \lambda^{2R_L(t)}$, where $R_L(t) := \max_{x \in \mathbb{R}/\mathbb{Z}} \{r_i^+(x), r_i^-(x)\}$. Clearly, for any $x \in \mathbb{R}/\mathbb{Z}$ we have 
$$
\frac{\|A_n(x, t)\|}{\|A_n^{-1}(x, t)\|} \leq \|A_n^{-1}(x, t)\|^{1+\epsilon} \leq \lambda^{2r_n^+(x)} \leq \lambda^{2R_L(t)} \leq C_0(t).
$$

Let $s_n := \left\lceil \frac{50n^3 \log q_{N+n-1}}{\log \lambda} \right\rceil$.
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Assume that for $n = k$, we have

$$\frac{1}{\|A_{x_k}^+(x,t)\|} \left| \frac{\partial \|A_{x_k}^+(x,t)\|}{\partial t} \right| \leq r_k^+ (C_0 + 2^{2k+1} q_{N+k-1}^{50\tau^2})$$

for any $x \in \mathbb{R}/\mathbb{Z}$.

Consider the condition $n = k + 1$. If $r_{k+1}^+ (x) = r_k^+ (x)$, then we need do nothing. Otherwise, let $0 \leq j_m \leq r_{k+1}^+$, $0 \leq m \leq p$ be the return times such that

$$j_{m+1} - j_m \geq q_{N+k-1}^2, \quad 1 \leq m \leq p - 1 \quad \text{and} \quad x + j_m \alpha \in I_k \setminus I_{k+1}, \quad 1 \leq m \leq p - 1,$$

where $j_0 = 0$, $j_p = r_{k+1}^+$ and $p > 0$. Now consider the sequence

$$A_{j_1} (x), A_{j_2 - j_1} (x + j_1 \alpha), \ldots, A_{j_p - j_{p-1}} (x + j_{p-1} \alpha).$$

From the definition of return time, we have

$$|g_{k+1} (x + j_m \alpha)| \geq 2^{-2k-2} q_{N+k}^{-16\tau} \leq 2^{-2k-2} q_{N+k}^{-200\tau}$$

for $1 \leq m \leq p - 1$.

Let $g_{k+1}^+ := \frac{x}{2} - s_{r_k} + u (A_{j_1} (x))$. Then we need to consider the following two cases.

(1) $j_1 > s_k$.

Then it holds that

$$\|g_{k+1} - g_{k+1}\| \leq \|u (A_{j_1} (x) - u_{r_k})\|$$

$$\leq C \|A_{j_1} (x)\|^{\frac{\tau}{2}}$$

$$\leq C \lambda^{-\frac{\tau}{2} (1 - \epsilon) j_1}$$

$$\leq C \lambda^{-\frac{\tau}{2} (1 - \epsilon) s_k}$$

$$\leq C \lambda^{-s_k}$$

$$\leq C q_{N+k-1}^{-50\tau^2},$$

which implies

$$|g_{k+1} (x + j_1 \alpha)| \geq 2^{-2k-2} q_{N+k}^{-16\tau} - C q_{N+k-1}^{-50\tau^2} \geq 2^{-2k-3} q_{N+k}^{-16\tau} \geq 2^{-2k-2} q_{N+k}^{-200\tau}.$$

Thus by lemma 6, it follows from induction hypothesis that

$$\frac{1}{\|A_{x_{k+1}}^+(x,t)\|} \left| \frac{\partial \|A_{x_{k+1}}^+(x,t)\|}{\partial t} \right| \leq \sum_{0 \leq m \leq p-1} \frac{1}{\|A_{j_{m+1} - j_m}^+(x,t)\|} \left| \frac{\partial \|A_{j_{m+1} - j_m}^+(x,t)\|}{\partial t} \right| +$$

$$\left( \sum_{2 \leq m \leq p-1} 2 \tan (g_{k+1} (x + j_m \alpha)) + 2 \tan (g_{k+1} (x + j_1 \alpha)) \right)$$

$$\leq \sum_{0 \leq m \leq p-1} (C_0 (j_{m+1} - j_m) + 2^{2k+1} q_{N+k-1}^{200\tau^2}) \leq C (p - 2) 2^{2k+2} q_{N+k}^{16\tau} + 2^{2k+2} q_{N+k}^{200\tau}$$

$$\leq r_{k+1}^+ (C_0 + 2^{2k+1} q_{N+k-1}^{200\tau^2} + 2^{2k+2} q_{N+k}^{200\tau^2})$$

$$\leq r_{k+1}^+ (C_0 + 2^{2k+3} q_{N+k}^{200\tau^2}).$$
According to Lemma 16, we have

\[ \| A_{r_{k+1}^+}(x, t) \| \leq \sum_{1 \leq m \leq p-1} \frac{1}{\| A_{j_{m+1}-j_m}(x, t) \|} \left( \frac{\| \partial \| A_{r_{k+1}^+}(x, t) \|}{\partial t} \right) + \frac{1}{\| A_{j_1}(x, t) \|} \left( \frac{\| \partial \| A_{j_1}(x, t) \|}{\partial t} \right) + \sum_{2 \leq m \leq p-1} (2 \tan(g_{j_1+1}(x+j_1\alpha)) + 2 \tan(g_{j_1+1}(x+j_1\alpha))) \]

Then also by Lemma 6 and induction hypothesis, it holds that

\[ \| A_{r_{k+1}^+}(x, t) \| \leq \sum_{1 \leq m \leq p-1} (C_0(j_{m+1} - j_m) + 2^{2k+1} q_{N+k-1}^{200r^2}) + \| A_{j_1} \| + (p-2)2^{2k+2} q_{N+k}^{200r^2} + \| A_{j_1} \| ^e \]

Thus, for any \( n > L \) and \( x \in \mathbb{R}/\mathbb{Z} \) we obtain

\[ \| A_{r_{n}^+}(x, t) \| \leq r_n^+(C_0 + 2^{2n+1} q_{N+n-1}^{200r^2}) \]

\[ \leq C(t, \alpha, v) \max \{ r_n^+(x), q_{N+n-1} \}^{200r^2} \]

\[ \leq \max \{ r_n^+(x), q_{N+n-1} \} C(t, \alpha, v) \]

for proper constant \( C \).

For the case \( t \in EP \), it’s easy to see that the proof is quite similar as the the condition \( t \in FR \). In fact, for any fixed \( t \in EP \), it follows from lemma 19 that for some sufficient large \( N_0 \), we have \( g_t \) is always of type III(I) or II) for \( t \geq N_0 \) and \( |g_t(\hat{e}_n, 1)| < C\lambda^\frac{2}{\mu-1} \) for any \( t \geq N_0 \). Thus for \( n \) sufficiently large, one can check that \( g_t \) is locally similar as type II. So the remain part of the proof is obvious if we note that \( \theta \) between two matrices we’ve divided still has the property as we want since the second derivative of \( g_t(x) \) have an uniform lower bound.

For the remaining part, note that \( g_\hat{S} \) is of type I and \( I_{\hat{S}, 1} + k\alpha = I_{\hat{S}, 2} \). We rewrite \( A_{r_{\hat{S}}} \) as \( A_{r_{\hat{S}}-k} A_k = R_{u(A_{r_{\hat{S}}-k})} \text{diag} \{ \| A_{r_{\hat{S}}-k} \|, \| A_{r_{\hat{S}}-k} \|^{-1} \} R_{\hat{S}} \text{diag} \{ \| A_{r_k} \|, \| A_{r_k} \|^{-1} \} R_{\hat{S}-s(A_k)} \).

According to Lemma 16, we have

\[ \frac{\| \partial \| A_{r_{\hat{S}}-k} \|}{\partial t} \leq c^{-1} \]

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with some absolute constant $c$. Therefore, by the help of Lemma 6 we have

$$\left| \frac{1}{\| A_{r_N} \|} \frac{\partial}{\partial t} \| A_{r_N} \| \right| \leq \left| \frac{1}{\| A_{r_N} \|} \frac{\partial}{\partial t} \| A_{r_{N-k}} \| \right| + \left| \frac{1}{\| A_k \|} \frac{\partial}{\partial t} \| A_k \| \right| + \left| \frac{2}{\| A_{r_N} \|} \frac{\partial}{\partial t} \| A_{r_N} \| \right|

\leq (r_N - k)^C + k^C + c^{-1} \left| \frac{2}{\cot^2 \theta_k + \| A_{r_{N-k}} \|^2 + \| A_k \|^2} \right|

\leq r_N^C \sqrt{(x - c_{N,2})^2 + \frac{2}{\| A_{r_{N-k}} \|^2 + \| A_k \|^2}}

\leq r_N^C \sqrt{(x - c_{N,2})^2 + \frac{1}{\lambda^2}}

Therefore,

$$\left| \int_{I_{r_N(1)}} \frac{1}{\| A_{r_N(x,t)} \|} \frac{\partial}{\partial t} \| A_{r_N(x,t)} \| \right| \leq \int_{I_{r_N(1)}} r_N^C \sqrt{(x - c_{N,2})^2 + \frac{1}{\lambda^2}} \, dx

\leq \int_{0}^{2^{N+\lambda} - 1} r_N^C \sqrt{(x - c_{N,2})^2 + \frac{1}{\lambda^2}} \, dx

\leq C r_N^C 1 \log \lambda

\leq r_N^C. $$

The next lemma is absolutely similar to the first part of the lemma above. So we omit the proof of it.

**Lemma 28.** For any fixed $i \in FR$ or $\hat{i} \in EP$, $x_0 \in \mathbb{R}/\mathbb{Z}$, and sufficiently large $n$, there exists some constant $C$ depending on $v$, $\alpha$, and $t$, such that

$$\left| \frac{1}{\| A_{r_n} \|} \frac{\partial}{\partial t} \| A_{r_n} \| \right| (x_0) \leq \max \{ -r_n^C (x_0), q_{N+n-1} \}^{C(v,\alpha)}.

Now we deal with the case $x \in S_1$. In fact, the following Lemma immediately implies what we desire.

**Corollary 29.** For any $t \in FR$ or $\hat{t} \in EP$, sufficiently large $n$, $q_{N+n-1} \leq m \leq q_{N+n}$ and $x \in \{ x \in \mathbb{R}/\mathbb{Z} | T^l(x) \notin I_{n+1} \}$ for any $0 \leq l \leq m$. There exists some constant $C$ depending on $v$, $\alpha$, and $t$, such that

$$\left| \frac{1}{\| A_m \|} \frac{\partial}{\partial t} \| A_m \| \right| \leq \max \{ m, q_{N+n-1} \}^{C(v,\alpha)}.

**Proof:** Consider a fixed $x$ satisfying the requirement as above and sufficiently large $n$. Without loss of generality, we assume that there exists some $0 \leq l \leq m$ such that $T^l(x) \in I_n$. Consider matrices

$$A_{j_1}(x), A_{j_2-j_1}(x + j_1 \alpha), \cdots, A_{j_{p-l-1}}(x + j_{p-l-1} \alpha), A_{m-j_p}(x)(x + j_p \alpha)$$
where $0 \leq j_k \leq m$, $0 \leq k \leq p + 1$ denote the times such that

$$j_{k+1} - j_k \geq q_{n+N+n-1}, \quad 1 \leq k \leq p - 1 \text{ and } x + j_k \alpha \in I_n \setminus I_{n+1}, \quad 1 \leq k \leq p - 1,$$

with $j_0 = 0$ and $j_p + 1 = m$.

Let $g_{k+1}^\prime := \frac{3}{2} - s_{r_n} + u(A_{j_1}(x))$ and $g_{k+1}'' := \frac{3}{2} - s(A_{m-j_p}) + u_{r_n}$.

(1) If $j_1, m - j_p > s(n)$, it is clear that

$$\|g_{k+1}^\prime - g_{k+1}\| \leq \|u(A_{j_1}(x)) - u_{r_n}\|$$

$$\leq C\|A_{j_1}(x)\|^{-\frac{3}{4}}$$

$$\leq CA^{-\frac{5}{4}(1-\varepsilon)j_1}$$

$$\leq CA^{-\frac{5}{4}(1-\varepsilon)s_n}$$

$$\leq CA^{-s_k}$$

$$\leq Cq_n^{-50\tau^2},$$

which implies

$$|g_{k+1}^\prime(x + j_1\alpha)| \geq 2^{-2n-2}q_n^{-16\tau} - Cq_n^{-50\tau^2} \geq 2^{-2n-3}q_n^{-16\tau} \geq 2^{-2n-2}q_n^{-200\tau^2}.$$

Similarly, we have

$$\|g_{k+1}'' - g_{n+1}\| \leq \|s(A_{m-j_p}(x) - s_{r_n})\|$$

$$\leq C\|A_{m-j_p}(x)\|^{-\frac{3}{4}}$$

$$\leq CA^{-\frac{5}{4}(1-\varepsilon)(m-j_p)}$$

$$\leq CA^{-\frac{5}{4}(1-\varepsilon)s_n}$$

$$\leq CA^{-s_n}$$

$$\leq Cq_n^{-50\tau^2},$$

which implies

$$|g_{k+1}''(x + j_1\alpha)| \geq 2^{-2n-2}q_n^{-16\tau} - Cq_n^{-50\tau^2} \geq 2^{-2n-3}q_n^{-16\tau} \geq 2^{-2n-2}q_n^{-200\tau^2}.$$

Thus by lemma it follows from induction hypothesis that

$$\frac{1}{\|A_m(x,t)\|} \left\| \frac{\partial}{\partial t} A_m(x,t) \right\| \leq \sum_{0 \leq k \leq p} \frac{1}{\|A_{j_{k+1} - j_k}(x,t)\|} \left\| \frac{\partial}{\partial t} A_{j_{k+1} - j_k}(x,t) \right\| +$$

$$\left( \sum_{2 \leq k \leq p} 2 \tan(g_{n+1}(x + j_m \alpha)) + 2 \tan(g_{n+1}'(x + j_1 \alpha)) + 2 \tan(g_{n+1}''(x + j_p \alpha)) \right)$$

$$\leq \sum_{0 \leq k \leq p-1} ((j_{k+1} - j_k)^C) + Cp2^{2n+2}q_n^{16\tau}$$

$$\leq m^C + mq_n^C$$

$$\leq \max\{m, q_{n+n-1}\}^C,$$

for proper constant C.
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(2) If $j_1 \leq s(n)$ and $i - j_p > s(n)$. Straightly calculation shows that
\[
\frac{1}{\|A_m(x,t)\|} \left| \frac{\partial}{\partial t} \|A_{m}(x,t)\| \right| \leq \sum_{1 \leq k \leq p} \frac{1}{\|A_{j_{k+1}-j_k}(x,t)\|} \left| \frac{\partial}{\partial t} \|A_{j_{k+1}-j_k}(x,t)\| \right| + \frac{1}{\|A_j(x,t)\|} \left| \frac{\partial}{\partial t} \|A_j(x,t)\| \right|
\]
\[
+ \left( \sum_{2 \leq m \leq p} 2 \tan(g_{n+1}(x + j_m \alpha)) + 2 \tan(g'_{n+1}(x + j_1 \alpha)) + 2 \tan(g''_{n+1}(x + j_p \alpha)) \right)
\]
\[
\leq \sum_{1 \leq k \leq p} (j_{m+1} - j_m)^C + \|A_{j_{k}}\|^3 + (p - 1)2^{2k+2}q_{N+n}^2 + \|A_j\|^\epsilon
\]
\[
\leq m^C + m q_{N+n}^C
\]
\[
\leq \max\{m, q_{N+n-1}\} C,
\]
for some proper constant $C$.

The cases $j_1 \leq s(n)$ and $i - j_p \leq s(n)$ and $j_1 > s(n)$ and $i - j_p \leq s(n)$ is quite similar as above. Thus we finish the proof.

When we use Induction Theorem of [WZ1], it’s noteworthy that we do not require the orbit of $x$ starting from and ending at critical interval $I_n$, the following lemma shows the fact.

**Lemma 30.** For any fixed $t$, $n \in \mathbb{N}$, $q_{N+n-2}^2 < l_1, l_2 \leq q_{N+n-1}^2$, $x \in I_n$, denote $\min\{l_1, r_n^+(x)\} = m_1^+, \min\{l_2, r_n^-(x)\} = m_2^-$ and $s(A_m^+) - u(A_m^-) \triangleq g_{m_1^+, m_2^-}^n$, where $r_n^+$ is the first returning time to $I_n$. The following holds true,
\[
\|A_{m_1^+}^\| \geq \lambda(1-\epsilon)m_1^+, \quad \epsilon = \sum_{i=1}^{\infty} \frac{\log q_{N+i}}{q_{N+i-1}}, \quad i = 1, 2;
\]
\[
\|g_{m_1^+, m_2^-}^n - (s(A_{r_n^+}(x)) - u(A_{r_n^-}(x)))\|_{C^2} \leq \lambda^{-\frac{1}{2}r_n^--1}.
\]

The proof of Lemma 30 is located in Appendix A.5.

The following lemma is in preparation for later proofs. It is a stronger version of the estimate for the angle of lemma 3 in [WZ1].

**Lemma 31.** For fixed $t \in FR$, $n \in \mathbb{N}$ and $x \in I_n$ satisfying $g_n$ and $g_{n-1}$ are same type (Type I, II, or III with the same resonance), let $s_i^n$ be all the returning times (second returning time for type III) for $x$ to $I_{n-1}$, and we denote $\|A_{s_i^n-s_{i-1}^n}\| \triangleq \lambda_i$ and $\theta_j = s(A_{s_i^n-s_{i-1}^n}) - u(A_{s_i^n-s_{i-1}^n})$ for $i = 1, 2, \cdots, t_{n-1}$, $j = 1, 2, \cdots, t_{n-2}$. Then the following holds true.
\[
\frac{d}{dx} \frac{1}{\|A_{r_n}\|} \leq \sum_{j=1}^{t_{n-1}} \frac{\chi_j^n}{\lambda_j} + \sum_{j=1}^{t_{n-2}} |\tan \theta_j \theta_j'|;
\]
\[
\frac{d^2}{dx^2} \frac{1}{\|A_{r_n}\|} \leq C \left( \sum_{1 \leq k \neq j \leq t_{n-1}} \frac{\chi_j^n \chi_k^n}{\lambda_j \lambda_k} + \sum_{j=1}^{t_{n-1}} \left( \frac{\chi_j^n}{\lambda_j} + |\theta_j'| \tan \theta_j \right) + \sum_{j,k=1}^{t_{n-1}} |\theta_j | \right)
\]

**Proof.** It follows from lemma 38 that
\[
\|A_{r_n}\| \leq 2 \left( \prod_{i=1}^{t_{n-1} - 1} \lambda_i \right) \left( \prod_{i=1}^{t_{n-2}} |\cos \theta_i| \right)
\]
and

\[(32) \quad \|A_{r_n}\| \geq \left( \prod_{i=1}^{t_n-1} \lambda_i \right) \left( \prod_{i=1}^{t_n-2} |\cos \theta_i| \right).\]

Then it holds from (31) and the direct calculation, we have

\[
\frac{d\|A_{r_n}\|}{dx} \leq \sum_{j=1}^{t_n-1} \left( \lambda_j^t \prod_{i \neq j}^{t_n-2} \lambda_i \prod_{i=1}^{t_n-2} |\cos \theta_j| \right) + \sum_{j=1}^{t_n-2} \left[ \left( \prod_{i \neq j}^{t_n-2} |\cos \theta_j| \right) \prod_{i=1}^{t_n-1} \lambda_i \prod_{i \neq j}^{t_n-2} |\cos \theta_j| \right] ;
\]

\[
\frac{d^2\|A_{r_n}\|}{dx^2} \leq \sum_{j=1}^{t_n-1} \left( \lambda_j^t \prod_{i \neq j}^{t_n-2} \lambda_i \prod_{i=1}^{t_n-2} |\cos \theta_j| \right) + \sum_{1 \leq j \neq k \leq t_n-1} \left( \lambda_j \lambda_k \prod_{i \neq j,k}^{t_n-2} \lambda_i \prod_{i=1}^{t_n-1} \lambda_i \prod_{i \neq j,k}^{t_n-2} |\cos \theta_j| \right) + \sum_{j=1}^{t_n-2} \left[ \left( \prod_{i \neq j}^{t_n-2} |\cos \theta_j| \right) \prod_{i=1}^{t_n-1} \lambda_i \prod_{i \neq j}^{t_n-2} |\cos \theta_j| \right] \]

\[
+ \sum_{j=1}^{t_n-1} \sum_{k=1}^{t_n-2} \left( \lambda_j^t \prod_{i \neq j}^{t_n-2} \lambda_i \prod_{i=1}^{t_n-2} |\cos \theta_j| \right) + \sum_{j=1}^{t_n-2} \left[ \left( \prod_{i \neq j}^{t_n-2} |\cos \theta_j| \right) \prod_{i=1}^{t_n-1} \lambda_i \prod_{i \neq j}^{t_n-2} |\cos \theta_j| \right] .
\]

Combine this with (32), we obtain the conclusion. \(\square\)

**Remark 32.** It’s not difficult to see that lemma (31) holds for each \(|A_{r_n^i}|, i = 1, 2, \cdots, t_n-1\).

**Lemma 33.** For any fixed \(t \in EP\) or \(FR\), \(m \leq n \in \mathbb{N}\) and \(x_m \in I_m\) satisfying \(r_m^+(x_m) \geq \left[ \frac{C \log q_n}{\log \lambda + n-1} \right]\), and \(r_n^+(x_m) \geq q_{N+n-1}\), where \(r_n^+(t)\) denote the first returning time to \(I, i \in \mathbb{N}^+\) and \(C\) can be chosen as \(1000r^{1000} \cdot \kappa(\alpha)\) with \(\kappa(\alpha) > 1\) satisfying \(\sum_{i=1}^{[\frac{1}{\alpha}]} \text{Leb} \{I \cap (I + i \cdot \alpha)\} > 0, \forall \text{ open interval } I \subset \mathbb{R}\). Then we have

\[(33) \quad \|A_{r_n^+(x_m)}\| \geq \lambda^{1-\varepsilon} r_n^+(x_m) ;\]

\[(34) \quad \|s(A_{r_n^+(x_m)}) - s(A_{r_n^+(x_m)})\|_{C^2} \leq \lambda^{-1.5} r_n^+(x_m) .\]

**Remark 34.** The existence of \(\kappa\) is a standard result for Diophantine translation on torus, one can see the Lemma 6 of ADZ for a simple proof. The lower bound of \(r_n^+(x_m)\) implies that there are no other strong resonance between \(m\)-th step and \(n\)-th step.

We put the proof at Appendix A.6.

**Lemma 35.** For any fixed \(t \in EP\) or \(FR\), \(l \in \mathbb{N}\) large and \(x \in I_{L(l)}\), the following holds true.

\[
\frac{d\|A_l\|}{dx} + \frac{d^2\|A_l\|}{dx^2} \leq \|A_l\| \|l(C,l)\| ,
\]

for some proper constant \(C\) depending on \(\alpha\) and \(t\), where \(L(l) = \{m|q_{N+m-2}^2 \leq l \leq q_{N+m-1}^2\}\).

**Proof.** We process it by induction. Note that for large \(N_0\), it’s enough to consider \(t \in FR\) since there essentially exist only one critical interval in the case \(t \in EP\). \(N_0\) can be chosen as \(N_0 \geq \min \{m|g_n \text{ is Type I for } n \geq m\}\) (or \(N_0 \geq \min \{m|g_n \text{ is Type III for } n \geq m\}\) in the case \(t \in EP\).
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Starting Step $N_0$: Denote $\lambda_0' \triangleq \max\{\frac{d|A_{s_i}(x)|}{dx}|k \leq r_{N_0}(x), x \in I_{N_0}\}$ and $\lambda_0'' \triangleq \max\{\frac{d^2|A_{s_i}(x)|}{dx^2}|k \leq r_{N_0}(x), x \in I_{N_0}\}$.

Step $N_0 + 1$: For any $q_{N_0+N_0-1}^2 \leq l \leq q_{N_0+N_0}^2$, we write $A_l$ as $A_{s_{i_1}^j-s_{i_1-1}^j} \cdots A_{s_2^j-s_1^j} A_{s_1^j}$, where $s_i^j$ are all the returning timings for $x$ to $I_{N_0}$. Note that $l \leq q_{N_0+N_0}^2$ implies $x$ doesn’t return back to $I_{N_0+1}$. We denote $M_1 \triangleq [100r \frac{\log q_{N_0+N_0}}{\log \lambda_0}]$ and $\theta_i^j \triangleq s(A_{s_{i+1}^j-s_{i}^j}) - u(A_{s_{i}^j})$, $i = 0, 1, \cdots, t_1 - 1$.

1: $s_{i+1}^j - s_i^j \leq M_1$: On one hand,

$$||A_{s_{i+1}^j-s_i^j}|| \leq CL_{s_{i+1}^j-s_i^j-1} \leq q_{N_0+N_0}^{150r},$$

On the other hand,

$$||A_{s_{i+1}^j-s_i^j-1}|| \geq \lambda^{(1-\epsilon)s_{i+1}^j-1} = \lambda^{(1-\epsilon)(l-(s_{i+1}^j-s_i^j-1))} \geq \lambda^{(1-\epsilon)(q_{N_0+N_0}^{-M_1})} \gg q_{N_0+N_0}^{150r}.$$

We write $A_l = A_{s_{i+1}^j-s_i^j-1} A_{s_i^j}$. It follows from lemma 29 that

$$||A_l|| \lesssim _2 ||A_{s_{i+1}^j-s_i^j-1}|| ||A_{s_i^j-s_i^j-1}||^{-1} |\sin(\frac{\pi}{2} - \theta_i^j)| \triangleq \lambda_1 \lambda_2^2 \sin \theta,$$

Therefore,

$$\frac{d||A_l||}{dx} \leq \lambda_1 \lambda_2^{-1} \sin \theta + \lambda_1 \lambda_2^{-2} \lambda_2^\prime \sin \theta + \lambda_1 \lambda_2^{-1} \cos \theta'',$$

$$\frac{d^2||A_l||}{dx^2} \leq 2(\lambda_1 \lambda_2^{-1} \sin \theta + \lambda_1 \lambda_2^{-2} \lambda_2^\prime \sin \theta + \lambda_1 \lambda_2^{-1} \cos \theta') + \lambda_1 \lambda_2^{-2} \lambda_2^\prime \sin \theta + \lambda_1 \lambda_2^{-1} \sin \theta + \lambda_1 \lambda_2^{-1} \cos \theta'' + \lambda_1 \lambda_2^{-3} \lambda_2^\prime \sin \theta + \lambda_1 \lambda_2^{-3} \lambda_2^\prime \sin \theta + \lambda_1 \lambda_2^{-1} \sin \theta + \lambda_2^\prime \sin \theta |\sin(\theta')^2 + \sin(\theta'')^2|.$$
We use the same notation as the previous case. We focus on \( i \) \( \frac{(38)\lambda_i'}{2} \leq \frac{\lambda_i''}{2} \).

Combine these with \( (36) \) and \( (37) \), we have

\[ \lambda_0' + \frac{\lambda_1'}{2} + \frac{\lambda_2'}{2} + \sum_{i=1}^{t_n-3} |\tan \theta_i \theta_i'| + \sum_{i,j} |\theta_i' \theta_j'| \]

\[ \leq C(t_n - 2)^2 \lambda_0' + C(t_n - 2)^2 + C(t_n - 3)q_{N+N_0}^{24r} \leq C(t_n - 2)^2 (\lambda_0' + q_{N+N_0}^{24r}). \]

It follows from lemma \( 33 \) that \( \frac{1}{\|A_i'\|} \frac{d\|A_i'\|}{dx} \leq C(t_n - 1)(\lambda_0 + q_{N+N_0}^{24r}) \leq C(t_n - 1)(\lambda_0 + q_{N+N_0}^{30r}); \)

\[ \frac{1}{\|A_i\|} \frac{d^2\|A_i\|}{dx^2} \leq C(t_n - 1)^2 (\lambda_0^2 + q_{N+N_0}^{24r}) \leq C(t_n - 1)^2 (\lambda_0^2 + q_{N+N_0}^{30r}). \]

2: \( s_{t_1}^1 - s_{t_1-1}^1 > M_1 \): We use the same notation as the previous case. We focus on \( A_{s_{t_1}^1 - s_{t_1-1}^1} \).

Note that there must exist some \( 0 \leq p \leq N_0 \), such that \( r_p \geq \left[ \frac{C \log q_{N+N_0}}{\log \lambda} \right] C \).

Without loss of generality, we assume that \( r_{N_0 - 1}(x) \geq \left[ \frac{C \log q_{N+N_0}}{\log \lambda} \right] \). We denote \( A_{s_{t_1}^1 - s_{t_1-1}^1} \equiv A_{r_{N_0 - 1}} \).

It follows from lemma \( 33 \) that \( \|A' A_{s_{t_1}^1 - s_{t_1-1}^1}\| \geq \lambda_0^2 - \lambda_0^{1-\epsilon} \lambda_0^{(1-\epsilon)(t_1-1)} \) and \( \|A'\| \geq \lambda(1-\epsilon)(s_{t_1}^1 - s_{t_1-1}^1 - r_{N_0 - 1}) \). Therefore, similar to the previous case, we denote \( \tilde{\lambda} \equiv \|A' A_{s_{t_1}^1 - s_{t_1-1}^1}\|, \tilde{\lambda}_1 \equiv \|A'\|, \tilde{\lambda}_2 \equiv \|A_{s_{t_1}^1 - s_{t_1-1}^1}\| \) and \( \tilde{\theta} \equiv -\frac{\tilde{\lambda}}{\tilde{\lambda}_2} - u(A_{s_{t_1}^1 - s_{t_1-1}^1}) \).

It’s not difficult to see that

\[ \frac{\tilde{\lambda}}{\tilde{\lambda}_2} \leq \frac{\tilde{\lambda}_1 \tilde{\lambda}_2}{\tilde{\lambda}} \cos \tilde{\theta}; \]

\[ \frac{1}{\|A_i\|} \frac{d\|A_i\|}{dx} \leq \frac{\tilde{\lambda}}{\lambda} + C \lambda_0; \]

\[ \frac{1}{\|A_i\|} \frac{d^2\|A_i\|}{dx^2} \leq \frac{\tilde{\lambda}_1''}{\lambda} + C \lambda_0; \]

It follows from lemma \( 33 \) that \( |\tilde{\theta}(x) - g_{N_0}| \leq \|A'\|^{-1.5} \leq \frac{C}{q_{N+N_0}^{30r}} \). Thus

\[ (38) \quad |\cos \tilde{\theta}(x)| \geq \frac{C}{q_{N+N_0}^{30r}} - \frac{C}{q_{N+N_0}^{30r}} \geq \frac{1}{q_{N+N_0}^{30r}}. \]

By lemma \( 31 \) it’s clear that

\[ \frac{\tilde{\lambda}}{\lambda} \leq \frac{C \tilde{\lambda}_1}{\lambda_1} + C \frac{\tilde{\lambda}_2}{\lambda_2} + |\tan \tilde{\theta} \tilde{\theta}'|; \]

\[ \frac{\tilde{\lambda}_1''}{\lambda} \leq C \frac{\tilde{\lambda}_1'}{\lambda_1} + C \frac{\tilde{\lambda}_2}{\lambda_2} + |\tan \tilde{\theta} \tilde{\theta}'| + \sum_{i=1}^{t_n-3} |\tan \tilde{\theta} \tilde{\theta}'| + \sum_{i,j} |\tilde{\theta}_i' \tilde{\theta}_j'| \]

\[ + C \frac{\tilde{\lambda}_1'}{\lambda_1} + C \frac{\tilde{\lambda}_2}{\lambda_2} + \theta^2. \]

Note that the estimate for \( \tilde{\lambda}_2 \) has been done in the previous case.

Combine this with \( (38) \), we have

\[ \frac{\tilde{\lambda}}{\lambda} \leq C(t_n - 1)(\lambda_0 + q_{N+N_0}^{24r}) + C \lambda_0 + q_{N+N_0}^{30r} \leq C(t_n - 1)(\lambda_0 + q_{N+N_0}^{30r}); \]
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\[ \hat{\lambda}' / \lambda \leq C(t_n - 1)^2(\lambda_0^2 + q_{N,N_0}^{2+}) + C\lambda_0 + q_{N,N_0}^{307} \leq C(t_n - 1)^2(\lambda_0^2 + q_{N,N_0}^{307}). \]

Therefore,

\[ \frac{d\|A_t\|}{dx} \leq C(t_n - 1)(\lambda_0 + q_{N,N_0}^{307}); \]
\[ \frac{d^2\|A_t\|}{dx^2} \leq C(t_n - 1)^2(\lambda_0^2 + q_{N,N_0}^{307}). \]

In conclusion, \( \frac{d\|A_t\|}{dx} + |\frac{d^2\|A_t\|}{dx^2}| \leq C(t_n - 1)^2(\lambda_0 + q_{N,N_0}^{307}) \leq C t^2(\lambda_0 + q_{N,N_0}^{307}). \)

Assume that for any \( x \in \mathbb{R}/\mathbb{Z} \) and \( p > N_0 \), the following holds:

\[ \frac{d\|A_{t_N + p}(x)\|}{dx} + \frac{d^2\|A_{t_N + p}(x)\|}{dx^2} \leq C r^2 N_{N+p}(x)(\lambda_0 + \sum_{i=N_0}^{p-1} q_{N+i}). \]

**Step \( N_0 + p+1 \):** For any \( q_{N,N_0+p-1}^2 \leq l \leq q_{N,N_0+p}^2 \), we write \( A_t = A_{s_{p-1}} \cdots A_{s_p} \), where \( s_p \) are all the returning timings for \( x \) to \( I_{N_0+p+1} \). Note that \( l \leq q_{N,N_0+p}^2 \) implies \( x \) doesn’t return back to \( I_{N_0+p+1} \). We denote \( M_p \triangleq \lfloor \frac{100r \log q_{N,N_0+p}}{\log \lambda} \rfloor \) and \( \theta_{s_p} \triangleq s(A_{s_{p+1}} - s_p) - u(A_{s_p}) \), \( i = 0, 1, \ldots, t_p - 1 \).

1: \( s_{p-1} \leq M_p \): On one hand,

\[ \|A_{s_{p-1}} - s_{p-1}^p\| \leq C \lambda^{t_{s_p} - s_{p-1}^p} \leq q_{N,N_0+p}^{150r}. \]

On the other hand,

\[ \|A_{s_{p-1}} - s_{p-1}^p\| \geq \lambda^{1-(1-\epsilon)(s_{p-1}^p - s_{p-1}^p)} \geq \lambda^{1-(1-\epsilon)(q_{N,N_0+p}^2 - M_p)} \gg q_{N,N_0+p}^{150r}. \]

We write \( A_t = A_{s_{p-1}} \cdots A_{s_p} \). It follows from lemma [3] that

\[ \|A_t\| \geq \|A_{s_{p-1}}\| \|A_{s_{p-1}} - s_{p-1}^p\| \|A_{s_{p-1}} - s_{p-1}^p\|^{-1} |\sin(\pi/2 - \theta_{s_{p-1}})| \triangleq \lambda_1 \lambda_2^{-1} |\sin \theta|. \]

Therefore,

\[ \frac{d\|A_t\|}{dx} \leq \lambda_1 \lambda_2^{-1} |\sin \theta| + \lambda_1 \lambda_2^{-2} \lambda_2 |\sin \theta| + \lambda_1 \lambda_2^{-1} |\cos \theta \delta |; \]
\[ \frac{d^2\|A_t\|}{dx^2} \leq 2(\lambda_1 \lambda_2^{-1} |\sin \theta| + \lambda_1 \lambda_2^{-2} |\sin \theta| + \lambda_1 \lambda_2^{-1} |\cos \theta \delta | + \lambda_1 \lambda_2^{-2} \lambda_2 |\sin \theta| + \lambda_1 \lambda_2^{-1} |\sin \theta| + \lambda_1 \lambda_2^{-2} \lambda_2 |\sin \theta| + \lambda_1 \lambda_2^{-1} |\sin \theta| + \lambda_1 \lambda_2^{-2} \lambda_2 |\sin \theta| + \lambda_1 \lambda_2^{-1} |\sin \theta| + \lambda_1 \lambda_2^{-2} \lambda_2 |\sin \theta| + \lambda_1 \lambda_2^{-1} |\sin \theta| + \lambda_1 \lambda_2^{-2} \lambda_2 |\sin \theta| + \lambda_1 \lambda_2^{-1} |\sin \theta|. \]

Note that \( \|A_t\| \approx \sqrt{\lambda_1 \lambda_2^{-2} |\cos \theta | + \lambda_1 \lambda_2^{-2} |\sin \theta|} \geq \lambda_1 \lambda_2^{-1} |\sin \theta| \). Thus, by direct calculation, we have

\[ \frac{1}{\|A_t\|} \frac{d\|A_t\|}{dx} \leq \lambda_1 \lambda_2^{-1} + \lambda_2 \lambda_2^{-1} + |\cos \theta \delta |; \]
\[ \frac{1}{\|A_t\|} \frac{d^2\|A_t\|}{dx^2} \leq \lambda_1 \lambda_1^{-1} + (\lambda_1 \lambda_1^{-1})(\lambda_2 \lambda_2^{-1}) + (\lambda_1 \lambda_1^{-1}) |\cos \theta \delta | + (\lambda_2 \lambda_2^{-1}) |\cos \theta \delta | + (\lambda_2 \lambda_2^{-1})^2 \]
\[ + \lambda_2 \lambda_2^{-1} + (\theta \delta)^2 + |\cos \theta \delta | \]

Note that either \( t \in FR \) or \( t \in EP \), we have

\[ |\cos \theta \delta | \leq 1; |\delta|_{C^2} \leq C(t), \]

(39)
where $C$ only depends on $t$. Therefore, the following hold true.

\begin{equation}
\left| \frac{1}{\|A_t\|} \frac{d\|A_t\|}{dx} \right| \leq C(\bar{\lambda}_1 - \lambda_0 - \bar{\lambda}_2 - 1) \leq C \lambda_0 + \bar{\lambda}_2 - 1
\end{equation}

\begin{equation}
\left| \frac{1}{\|A_t\|} \frac{d^2\|A_t\|}{dx^2} \right| \leq C(\bar{\lambda}_1 - \lambda_0 - \bar{\lambda}_2 - 1) \leq C \lambda_0 + \bar{\lambda}_2 - 1
\end{equation}

We denote $\|A_{s_i} - s_{s_{i-1}}\| \triangleq \bar{\lambda}_{2,i-1}$ and $s(A_{s_i} - s_j) - u(A_{s_{i-1}} - s_{j-1})$ for $i, j = 1, 2, \cdots, t_p - 2$. It follows from remark 32 and Induction hypothesis that

\[ \bar{\lambda}_{2,i} \leq C \sum_{i=1}^{t_p-2} \bar{\lambda}_{2,i} + \sum_{i=1}^{t_p-3} |\tan \theta_i \theta_i'| \leq C \sum_{i=0}^{t_p-1} (s_{p+1}^i - s_p^i)(\lambda_0 + \sum_{i=0}^{t_p-1} q_{N,N+i}^{30}) + (t_{p-2})^2 q_{N,N+p}^{30} \leq C \lambda_0 + \sum_{i=0}^{t_p-1} q_{N,N+i}^{30} \]

\[ \frac{\lambda_{p+1} - \lambda_0}{\lambda_2} \]

\begin{equation}
\leq C \left( \sum_{1 \leq i \neq j \leq t_p-2} \frac{\lambda_{p+1}}{\lambda_2} + \sum_{i=1}^{t_p-3} \frac{\lambda_{p+1}^i}{\lambda_2} \right) + \sum_{i=1}^{t_p-3} |\tan \theta_i \theta_i'| + \sum_{i,j} |\theta_i \theta_j'| \leq C \lambda_0 + \sum_{i=0}^{t_p-1} q_{N,N+i}^{30} + C(t_{p-2})^2 + C(t_{p-3})q_{N,N+p}^{30} \leq C(t_{p-2})^2 \left( \lambda_0 + \sum_{i=0}^{t_p-1} q_{N,N+i}^{30} \right). \]

Combine these with (40) and (41), we have

\begin{equation}
\left| \frac{1}{\|A_t\|} \frac{d\|A_t\|}{dx} \right| \leq C(t_p - 1)(\lambda_0 + \sum_{i=0}^{t_p-1} q_{N,N+i}^{30})
\end{equation}

\begin{equation}
\left| \frac{1}{\|A_t\|} \frac{d^2\|A_t\|}{dx^2} \right| \leq C(t_p - 1)^2(\lambda_0^2 + \sum_{i=0}^{t_p-1} q_{N,N+i}^{30}).
\end{equation}

We use the same notation as the previous case. We focus on $A_{s_{t_p-1}} - s_{s_{t_p-1}}$.

Note that there must exist some $0 \leq u \leq N_0$, such that $r_{u}(x) \geq \frac{C \log q_{N,N+p}}{\log \lambda}(C$ is from the Lemma 33). Without loss of generality, we assume that $r_{N_0+p-1}(x) \geq \frac{C \log q_{N,N+p}}{\log \lambda}$. We denote $A_{s_{t_p-1}} - s_{s_{t_p-1}} \triangleq A_{r_{N_0+p-1}} - A_{s_{t_p-1}}$. It follows from lemma 33 that $\|A_{s_{t_p-1}}\| \geq \lambda^{(1-\epsilon)}(s_{t_p} - s_{t_p-1} - r_{N_0+p-1})$ and $\|A'\| \geq \lambda^{(1-\epsilon)}(s_{t_p} - s_{t_p-1} - r_{N_0+p-1})$. Therefore, similar to the previous case, we denote $\bar{\lambda} \triangleq \|A_{s_{t_p-1}}\|$, $\bar{\lambda}_1 \triangleq \|A'\|$, $\bar{\lambda}_2 \triangleq \|A_{s_{t_p-1}}\|$ and
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$$\tilde{\theta} \triangleq \frac{\theta}{2} - (s(A') - u(A_{r_n-1})).$$ It’s not difficult to see that

$$\tilde{\lambda} \lesssim 2 \tilde{\lambda}_1 \tilde{\lambda}_2 \cos \tilde{\theta};$$

$$\frac{1}{\|A_i\|} \left( \frac{d\|A_i\|}{dx} \right) \leq \frac{\tilde{\lambda}'}{\lambda} + C \lambda_0;$$

$$\frac{1}{\|A_i\|} \left( \frac{d^2\|A_i\|}{dx^2} \right) \leq \frac{\tilde{\lambda}''}{\lambda} + C \lambda_0;$$

It follows from lemma 33 that $|\tilde{\theta}(x) - g_{N_0+p-1}| \leq \|A'\|^2 \leq \frac{C}{q_{N+N_0+p}}.

Thus

$$|\cos \tilde{\theta}(x)| \geq \frac{C}{q_{N+N_0+p}} \frac{C}{q_{N+N_0+p}} \geq \frac{1}{q_{N+N_0+p}}.$$ By lemma 31, it’s clear that

$$\tilde{\lambda}' \leq C \frac{\tilde{\lambda}'}{\lambda_1} + C \frac{\tilde{\lambda}'}{\lambda_2} + |\tan \tilde{\theta}'|;$$

$$\tilde{\lambda}'' \leq C \frac{\tilde{\lambda}''}{\lambda_1} + C \frac{\tilde{\lambda}''}{\lambda_2} + |\tan \tilde{\theta}''| + \sum \frac{\tilde{\lambda}'}{\lambda_1} \frac{\tilde{\lambda}'}{\lambda_2}

+ C \frac{\tilde{\lambda}''}{\lambda_1} + C \frac{\tilde{\lambda}''}{\lambda_2} + \theta^2.$$ Note that the estimate for $\tilde{\lambda}_2$ has been done in the previous case. Combine this with 32, we have

$$\frac{\tilde{\lambda}'}{\lambda} \leq Cl(\lambda_0 + \sum_{i=0}^{\frac{30r}{N+N_0+i}} q_{N+N_0+i}) + C \lambda_0 + q_{N+N_0+i} \leq Cl(\lambda_0 + \sum_{i=0}^{\frac{30r}{N+N_0+i}} q_{N+N_0+i});$$

$$\frac{\tilde{\lambda}''}{\lambda} \leq Cl^2(\lambda_0^2 + \sum_{i=0}^{\frac{30r}{N+N_0+i}} q_{N+N_0+i}) + C \lambda_0 + q_{N+N_0+i} \leq Cl^2(\lambda_0^2 + \sum_{i=0}^{\frac{30r}{N+N_0+i}} q_{N+N_0+i}).$$

Therefore,

$$\left| \frac{1}{\|A_i\|} \left( \frac{d\|A_i\|}{dx} \right) \right| \leq C l(\lambda_0 + \sum_{i=0}^{\frac{30r}{N+N_0+i}} q_{N+N_0+i});$$

$$\left| \frac{1}{\|A_i\|} \left( \frac{d^2\|A_i\|}{dx^2} \right) \right| \leq C l^2(\lambda_0^2 + \sum_{i=0}^{\frac{30r}{N+N_0+i}} q_{N+N_0+i}).$$

In summary, for any fixed $l$, we have

$$\left| \frac{1}{\|A_i\|} \left( \frac{d\|A_i\|}{dx} + \frac{d^2\|A_i\|}{dx^2} \right) \right| \leq C l^2 \left( \lambda_0^2 + \sum_{i=0}^{\frac{30r}{N+N_0+i}} q_{N+N_0+i} \right) \leq C l^2 L(l) q_{N+N_0+i} \leq C l^{100r^2}.$$

**Lemma 36.** Let $l$ be as above, the following estimate holds true.

$$|s(A_i) - s(A_{r_{L(l)}})|_{C^2} + |u(A_i) - u(A_{r_{L(l)}})|_{C^2} \leq \|A_i\|^2 \|C^2 \lambda_0^2 \| N_0 \|q_{N+N_0+i} \| \leq C l^{100r^2}.$$

for some proper constant $C$ depending on $\alpha$ and $v$. 39
Proof. It’s an easy extension of the lemma 3 of [WZ1]. We denote $M_1 \triangleq 100r^{\frac{\log 2N + L(t)}{\log \lambda}}$. Note that there must exist some $N'$ such that $r_{N'} \leq M(t)$. Without loss of generality, we assume that $r_{N + L(t)} - 1 \leq M(t)$ and rewrite $A_t$ as $A_2 A_1 \triangleq A_t - r_{N + L(t)} - 1 A_{N + L(t)} - 1$, where $\|A_2\| \geq \lambda^{\frac{1}{2}} \gg 2^{100r_{N + L(t)}} \geq \|A_1\|$. It follows from lemma 3 that

$$|u(A_t) - u(A_2)|_{C^2} \leq \|A_1\|^2 \|A_2\|^{-2} = (\|A_1\| \|A_2\|)^{-2} \|A_1\|^4 \leq \|A_1\|^{-2} 2^{100r_{N + L(t)}} \leq \|A_1\|^{-2} 500r^6.$$  

Thus, it’s enough to estimate $|u(A_2) - u_{r_{N + L(t)}}|_{C^2}$. Let $\|A_2\| = e_1$ and the angle between $A_2 = A_{r_{N + L(t)}}$ and $A_{r_{N + L(t)} - r_{N + L(t)}}$ is $\theta_1$. Thus $|\theta_1 - \frac{\pi}{2}| \geq \|I_{N + L(t)}\| \geq q^{N + L(t)} \geq I^C$ for some constant $C$. The direct calculation and lemma 3 show that

$$|s(A_t) - s(A_{r_{L(t)}})| \leq |\arctan(e_1^2 \cot \theta)| \leq e_1^{-2} I^C;$$

$$\left|\frac{d^2 s(A_t)}{dx^2} - \frac{d^2 s(A_{r_{L(t)}})}{dx^2}\right| \leq C \left(\frac{2e''_1}{e_1^2 \cot \theta} + C \frac{\theta'}{e_1^2 \cot^2 \theta}\right) \leq e_1^{-2} I^C,$$

which complete the proof.

In the next two lemmas we deal with the case $x \in S_2$, which separately implies the $\frac{1}{2}$ Hölder continuity on the two subinterval centered by $t$. We denote $k(t)$ the largest resonance-distance of $t \in EP$.

**Lemma 37.** Let $n' A_t, A_{n-1}$ and $\theta_1$ be defined as in the beginning of the section and $c$ be as in lemma 29. For any $t_0 \in EP$ and $\epsilon > 0$, there exists some $N'' = N'(t_0)$, and absolute constant $\lambda^{-\frac{1}{2}(k(t_0))} < e''_0 < \lambda^{\frac{1}{2}(1-\epsilon)k(t_0)}$ such that for $n \geq N''$, $x \in S_2$ and $t_0 - \lambda^{-\frac{1}{2}N + M_n} \leq t_1 \leq t_0 - \lambda^{-\frac{1}{2}N + M_n - 1}$, we have

$$\left|\int_{T^t \times I_{L(t)}} \frac{d\theta_t}{dt} \frac{d|A_n|}{dA_n} \frac{1}{\|A_n\|} dx \right| \leq e''_0 \sqrt{I_1 - t_0^{-\frac{1}{2}}}.$$  

**Proof.** By a straight calculation, it holds that

$$\frac{1}{\|A_n\|} \frac{d|A_n|}{dt} = \frac{1}{\|A_n\|} \frac{d|A_n|}{d|A_{n-1}|} \|A_{n-1}\| + \frac{1}{\|A_n\|} \frac{d|A_n|}{dA_n} \frac{dA_n}{dt} + \frac{d\theta_t}{dt} \frac{1}{\|A_n\|} \frac{d|A_n|}{dA_n} \frac{dA_n}{dt}$$

By the second part of Lemma 9 it holds that $\left|\frac{1}{\|A_n\|} \frac{d|A_n|}{dA_n}\right| \leq \frac{1}{\|A_{n-1}\|}$. Subsequently by Lemma 27 it holds that

$$\left|\int_{T^t \times I_{L(t)}} \frac{1}{\|A_{n-1}\|} \frac{d|A_n|}{d|A_{n-1}|} \frac{d|A_{n-1}|}{dt}\right| \leq \int_{T^t \times I_{L(t)}} \frac{1}{\|A_{n-1}\|} \frac{d|A_{n-1}|}{dt} \leq k^C(t_0) \leq N^C(t_0);$$

Similarly, we have

$$\left|\int_{T^t \times I_{L(t)}} \frac{1}{\|A_n\|} \frac{d|A_n|}{dA_n} \frac{dA_n}{dt}\right| \leq \int_{T^t \times I_{L(t)}} \frac{1}{\|A_n\|} \frac{d|A_n|}{dA_n} \frac{dA_n}{dt} \leq k^C(t_0) \leq N^C(t_0),$$

where $k$ is the resonance distance of $g_{N''}$.

Next we estimate $\frac{d\theta_t}{dt} \frac{1}{\|A_n\|} \frac{d|A_n|}{d\theta_t}$. Fixed any $t_0 \in EP$. By lemma 13 it holds that for some sufficient large $N_0$, we have $g_1$ is always of type III(or II) and $|g_i(\tilde{e}_1, 1)| < 1$.
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$C\lambda^{-\frac{1}{p}}r_{i-1}, \left|\frac{\partial^2 \theta(x,t_0)}{\partial x^2}\right| \geq c > 0$ for any $i \geq N_0$. It implies that there exists some $N$ such that for $n \geq N$ we have $M(n) \geq R_{N_0}^0$. Moreover, due to the nonexistence of resonance for the case $t_0 \in EP$, it holds that

$$\|g_i - g_{i+1}\|_{C^2(I_{i+1})} \leq \lambda^{-\frac{1}{p}}r_{i-1} \ll \frac{1}{q_{N+i-1}}$$

$i \geq N_0$ for $i \geq N_0$.

Without loss of generality, we can always assume that $\frac{\partial^2 \theta(x,t_0)}{\partial x^2} > 0$, for any $x \in I_M$. Besides, we only need to consider one of the critical intervals. For instance, we consider $I_{M,1}$ and we still write it as $I_M$.

The inequality $\lambda^{-\frac{1}{p}}r_{q(N+M(n))} \leq |t_0 - t_1| \leq \lambda^{-\frac{1}{p}}r_{q(N+M(n))-2}$ is equivalent to

$$t_0 + \lambda^{-\frac{1}{p}}r_{q(N+M(n))} < t_1 < t_0 + \lambda^{-\frac{1}{p}}r_{q(N+M(n))}$$

or

$$t_0 - \lambda^{-\frac{1}{p}}r_{q(N+M(n))} < t_1 < t_0 - \lambda^{-\frac{1}{p}}r_{q(N+M(n))} - 1.$$

Let $t_0 - t_1 := \epsilon_1$. It’s obvious that $n' = o(n)$. Firstly, we consider the case $\epsilon_1 < 0$. It follows from lemma [30] that

$$\|\theta_i - g_{M+1}\|_{C^2(I_M)} \leq \|u(A_1) - u_{r,M}\|_{C^2(I_M)} + \|s(A_{n-l} - s_{r,M})\|_{C^2(I_M)} \leq \lambda R_{n,0} \epsilon_1,$$

for any $\frac{1}{4}n' \leq n \leq 1 - \frac{1}{4}n'$.

Let $\bar{x}_i := \{y \in I_{M+1}| \frac{\partial \theta}{\partial x}(y) = 0\}$. It is well defined due to the fact that $g_i, i \gg 1$ is of type III (hence $\{y \in I_{M+1}| \frac{\partial \theta_{M+1}}{\partial x}(y) = 0\}$ is well defined) and [30].

We claim that there exist constants $0 < c < C, 0 < c' < C''$ and $d > 0$ independent of $t, l, n$ such that

$$c(T^l x - \bar{x}_i)^2 - C' \epsilon_1 \leq \theta_i(t_1, x) \leq C(T^l x - \bar{x}_i)^2 - c' \epsilon_1, \quad |T^l x - \bar{x}_i| \leq d.$$  

In fact, if we denote $J_{t_0,n} = \{t_0 - \lambda^{-\frac{1}{p}}r_{q(N+M(n))} - 1, t_0 + \lambda^{-\frac{1}{p}}r_{q(N+M(n))} - 1\}$ and $I_{\bar{x}_i,n} = [\bar{x}_i - \lambda^{-\frac{1}{p}}r_{q(N+M(n))} - 1, \bar{x}_i + \lambda^{-\frac{1}{p}}r_{q(N+M(n))} - 1]$, then by Taylor expand $\theta_i(x,t) \in C^{2+h}(I_{\bar{x}_i,n} \times J_{t_0,n})$ at $(\bar{x}_i,t_0)$, we have

$$\theta_i(x,t) = \theta_i(\bar{x}_i,t_0) + (x - \bar{x}_i)$$(51)

and

$$\frac{\partial \theta_i}{\partial x}(x) = \frac{\partial \theta_i}{\partial x}(\bar{x}_i,t_0) + \frac{\partial \theta_i}{\partial x^2}(x - \bar{x}_i)^2, \quad \frac{\partial^2 \theta_i}{\partial x^2}(x) = \frac{\partial^2 \theta_i}{\partial x^2}(\bar{x}_i,t_0) + \frac{\partial^2 \theta_i}{\partial x^2}(x - \bar{x}_i)^2 + \frac{\partial^2 \theta_i}{\partial x^2}(\bar{x}_i,t_0) + \frac{\partial^2 \theta_i}{\partial t^2}(x - \bar{x}_i)^2 + \frac{\partial^2 \theta_i}{\partial x^2}(\bar{x}_i,t_0) + \frac{\partial^2 \theta_i}{\partial t^2}(x - \bar{x}_i)^2.$$  

Note that $\frac{\partial \theta_i}{\partial x}(\bar{x}_i,t_0) = 0$, which follows from the definition of $\bar{x}_i$.

Thus,

$$\theta_i(x,t) = \theta_i(\bar{x}_i,t_0) + [C(\bar{x}_i,t_0) + o(x - \bar{x}_i)^h](x - \bar{x}_i)^2 + [D(x,t,\bar{x}_i,t_0) + o(t - t_0)^{1+h}](t - t_0),$$

where

$$C(\bar{x}_i,t_0) = \frac{1}{2} \frac{\partial^2 \theta_i}{\partial x^2}(\bar{x}_i,t_0); \quad D(x,t,\bar{x}_i,t_0) = \frac{\partial \theta_i}{\partial t}(\bar{x}_i,t_0) + (x - \bar{x}_i) \frac{\partial \theta_i}{\partial x}(\bar{x}_i,t_0) + \frac{1}{2} (t - t_0) \frac{\partial^2 \theta_i}{\partial t^2}(\bar{x}_i,t_0).$$

Clearly, it holds from Lemma 17 that $\frac{\partial \theta_i}{\partial x}(\bar{x}_i,t_0) > c$ with some absolute constant $c \geq \frac{1}{10}$. And by directly calculation and the choice of $n$, we have

$$\frac{\partial \theta_i}{\partial x}(\bar{x}_i,t_0), \frac{\partial^2 \theta_i}{\partial x^2}(\bar{x}_i,t_0), \frac{\partial^2 \theta_i}{\partial x^2}(\bar{x}_i,t_0) \sim \lambda^k(t_0) \ll q_{N+M(n)-1} \ll |I_{\bar{x}_i}| + |J_{t_0}|.$$

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Therefore,

\begin{equation}
C(\tilde{x}_1, t_0) \sim \frac{1}{2} \lambda^{k(t_0)} \geq 1 \gg |I_{\tilde{x}_1,n}|^h.
\end{equation}

\begin{equation}
D(x, t, \tilde{x}_1, t_0) \sim \frac{\partial \theta(t_0)}{\partial t} > c \geq \frac{1}{10} \gg (|I_{\tilde{x}_1,n}| + |J_{t_0}|) \cdot \lambda^{k(t_0)} + |J_{t_0}|^{1+h}.
\end{equation}

We denote \( c_l \triangleq \frac{2\pi}{\delta(x, t)} (\tilde{x}_1, t_0) \) and \( \delta \triangleq \theta_t(\tilde{x}_1, t_0) \). Hence, combine (52), (53) with (54), we obtain

\begin{equation}
(c_l - |I_{\tilde{x}_1,n}|^{\frac{1}{2}})(x - \tilde{x}_1)^2 - \lambda^{R^{N_0}c_1} + \delta \leq \theta(t, x) \leq (c_l + |I_{\tilde{x}_1,n}|^{\frac{1}{2}})(x - \tilde{x}_1)^2 - \epsilon_1 + \delta.
\end{equation}

Then (50) is followed by \( |\delta| \leq \lambda^{-\frac{1}{2}} + L^{-1} \ll \epsilon_1 \).

Let \( \lambda_1 := \|A_1\|, \lambda_2 := \|A_{n-1}\| \). Then by Lemma (51) and (52), it holds that

\begin{equation}
\left( \begin{array}{c}
dh(t) \\
\frac{dA_1}{dt}
\end{array} \right) \leq \frac{1}{\|A_1\|} \left( \begin{array}{c}
\cot \theta_1 t\|A_1\| \\
\frac{dA_1}{dt}
\end{array} \right) \leq \frac{1}{\|A_1\|} \left( \begin{array}{c}
\cot \theta_1 t\|A_1\| \\
\frac{dA_1}{dt}
\end{array} \right)
\end{equation}

Now we estimate \( \int_{T^1 \in I_{L,+1}} \hat{\theta}_1(T^1x)|dx = \int_{x \in I_{L,+1}} \hat{\theta}_1(x)|dx = \int_{x \in I_{L,+1,1}} + \int_{x \in I_{L,+1,2}} \). We only consider the case \( x \in I_{L,+1,1} \) and the other one is similar. It’s worth noting that, \( I_{L,+1,1} \) consist of two main parts: on the first part, the image of \( g_{L,+1} \) is very steep and rapidly cross \([\pi, \pi]\), and on the second part, the image of \( g_{L,+1} \) is seriously flat and slowly move away from the extremum point. Due to the difference between the two cases, we have to separately consider them.

By (WZ1), we can write \( \theta_1 \) as \( \arctan(\|A_k\| \tan \theta_1) - \frac{\pi}{2} + \theta_2 \) (we omit the dependence on \( t_0 \) for \( k \)), where \( \theta_1 \) and \( \theta_2 \) are of type I before the resonance with two zero points \( c_{N,1} \) and \( c_{N,2} \). Denote \( \{c_{L+1,1}, c_{L+1,1}, c_{L+1,2}, c_{L+1,2}\} = \{x \in I_{L+1,1} \cup I_{L+1,2} | \hat{\theta}_1(x) = 0\} \). Clearly, without loss of generality, we assume that

\begin{align*}
&c_{L+1,2} - k \alpha > c_{L+1,1} > c_{L+1,2} > c_{N,1}; \\
&c_{N,2} > c_{L+1,2} > c_{L+1,1} > c_{L+1,1} + k \alpha,
\end{align*}

where

\begin{align*}
|c_{N,i} - c_{L+1,i}| + |c_{N,i} + k \alpha| - c_{L+1,i} - k \alpha| \leq C_1\|A_k\|^{-1}, \\
\text{for } i = 1, 2; \\
C_2\|A_k\|^{-\frac{1}{2}} > |c_{N,i} + k \alpha| - c_{N,i},
\end{align*}

for \( i = 1, 2 \); and

\begin{align*}
|c_{L+1,i} - c_{L+1,j}| \leq C_3\|A_k\|^{-1} \\
\text{for } i \neq j, i, j = 1, 2. \text{ We denote } C'' = \{C \in \mathbb{R} | g_{L+1}(c_{L+1,2} + C\|A_k\|^{-1}) = \max\{g_{L+1}\} \} \text{ and } I' := \{c_{L+1,2} + C''\|A_k\|^{-1}, c_{L+1,2} + \frac{1}{2} + q_{N,L+1,1}\}. \text{ Next we separately consider the case } x \in I' \text{ and } x \in I_{L+1,1} - I'.
\end{align*}
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**A: The case $x \in I'$**: For $x \in I'$, it follows from lemma 27 and $r_{L+1}^C \geq |k|^{-C} \geq |\frac{\partial \theta_k}{\partial t}|_{I_{L+1},}$, $|k|^{-C} \geq r_{L+1}^{-C}$, where $k' = \max \{l < L+1 | \text{g}_l \text{ is of type I} \}$, that

$$
\frac{\partial \theta_k}{\partial t} \leq \frac{2\|A_k\|\|\theta\|_{L+1}^2 (1 + \tan^2 \theta) \theta'}{1 + \|A_k\|^4 \tan^2 \theta} + |\theta'|^2
\leq \frac{r_{L+1}^C}{\|A_k\|^2 \tan \theta} + \frac{r_{L+1}^C}{\|A_k\|^2 \tan^2 \theta} + r_{L+1}^C
\leq \frac{r_{L+1}^C}{\|A_k\|^2} + r_{L+1}^C + r_{L+1}^C \leq k^C.
$$

Therefore, it’s sufficient to consider the following simplified form, that is

$$
\int_{0}^{\frac{\pi}{2} + \frac{\pi}{N+L}} k^C \text{sgn}(\theta) |\hat{\theta}(T^l x)| dx.
$$

We denote $x^* \triangleq \{ x > \bar{x}|\theta|_l(x) = \frac{\pi}{2}\}$, $c_1 \triangleq \theta''_l(x^*)$, $I \triangleq [\frac{1}{2}, \frac{1}{2} + q_{N+L}]$ and $\hat{\theta}(T^l x)| dx = \int_{I_{L+1}} C \text{sgn}(\theta) |\hat{\theta}(T^l x)| dx
\leq R_1 + R_2 + R_3

For $R_1$, note that $\cot \theta + (\lambda_4^2 - \lambda_2^2)^2 \tan^2 \theta \geq 2|\lambda_4^2 - \lambda_2^2|$. Therefore, it follows from the direct calculation that

$$
R_1 = \int_{I_{m}} C \text{sgn}(\theta) |\hat{\theta}(T^l x)| dx
\leq \int_{I_{m}} \frac{C}{\cot \theta + (\lambda_4^2 - \lambda_2^2)^2 \tan^2 \theta + \lambda_1^2 + \lambda_2^4}
\leq \text{Leb}(I_{m}) \cdot C \min \{\lambda_1^2, \lambda_2^4\}
\leq \frac{C}{\sqrt{\frac{c_m}{c_m}} - \frac{c_m}{c_m}} \cdot C \min \{\lambda_1^2, \lambda_2^4\}
\leq C \frac{\lambda_1^2 + \lambda_2^2}{c_1} \cdot C \min \{\lambda_1^2, \lambda_2^2\}
\leq C \epsilon_1^{-1}.
$$

For $R_2$, note that

$$
|\cot \theta| \geq |\frac{c_m}{c_1} x^2 - \frac{c_m}{c_1} \epsilon_2^2| \geq \epsilon_2 \sqrt{\lambda_4^2 + \lambda_2^2} \geq \sqrt{\lambda_4^2 - \lambda_2^2}|.
$$

(60)
For the function \( C^0(\mathbb{R}) \ni f \triangleq \frac{1}{\sqrt{x^2 + (\lambda_1^{-1} - \lambda_2^{-1})^2 x^{-2} + \lambda_1^{-1} + \lambda_2^{-1}}} \), it is clear that \( f \) is nondecreasing on \([\sqrt{\lambda_1^{-4} - \lambda_2^{-4}}, +\infty)\).

Therefore,

(61)

\[
R_2 = \int_{I_m} C\text{sgn}(\theta_1)\hat{\theta}_1(T^1 x) dx
\]

\[
\leq (1 + O(\lambda_1^{-6} + \lambda_2^{-6})) \int_{I_m} \left( \frac{1}{\sqrt{1 + \epsilon^2 (\tan^2 \theta_1 + \frac{1}{\cot^2 \theta_1 + \epsilon^2})}} \right) dx
\]

\[
- (1 - O(\lambda_1^{-6} + \lambda_2^{-6})) \int_{I_m} \left( \frac{1}{\sqrt{1 + \epsilon^2 (\cot \theta_1) / (\theta_1 + \epsilon^2)}} \right) dx
\]

\[
\leq (1 + O(\lambda_1^{-6} + \lambda_2^{-6})) \int_{I_m} \left( \frac{\epsilon^2}{2 \epsilon_m} \frac{1}{(x^2 - \epsilon_1^2)^2} + \frac{1}{\epsilon_m} \frac{|x^2 - (\epsilon_1^2 + \epsilon_2^2)|}{(x^2 - (\epsilon_1^2 + \epsilon_2^2))^2 + \frac{2}{\epsilon_m}} \right) dx
\]

\[
- (1 - O(\lambda_1^{-6} + \lambda_2^{-6})) \int_{I_m} \left( \frac{1}{\epsilon_m} \frac{|x^2 - (\epsilon_1^2 + \epsilon_2^2)|}{(x^2 - (\epsilon_1^2 + \epsilon_2^2))^2 + \frac{2}{\epsilon_m}} \right) dx
\]

\[\triangleq K_1 - K_2\]

By denoting

\[
F_1(\epsilon_1, \epsilon_2) \triangleq \frac{\sqrt{\epsilon_1^2 + \epsilon_2^2}}{\epsilon_1^2 + \epsilon_2^2} \frac{\sqrt{2(\epsilon_1^2 + \epsilon_2^2)^2}}{\epsilon_1^2 + \epsilon_2^2 + \sqrt{2(\epsilon_1^2 + \epsilon_2^2)^2} + \sqrt{(\epsilon_1^2 + \epsilon_2^2)^4} + \sqrt{(\epsilon_1^2 + \epsilon_2^2)^4}}
\]

and a direct calculation, we have

\[
K_1 - K_2 \leq \frac{\sqrt{2}}{8 \epsilon_2 c_m} \left[ C F_1 \left( \frac{\epsilon_1}{c^*}, \frac{\epsilon_2}{c_m} \right) F_2 \left( \frac{\epsilon_1}{c_m}, \frac{\epsilon_2}{c_m} \right) + \log \sqrt{2} \right] - CF_1 \left( \frac{\epsilon}{c_m}, \frac{\epsilon_2}{c_m} \right) F_2 \left( \epsilon, \frac{\epsilon}{c_m} \right) + C c^{-1},
\]

where \((1 + O(\lambda_1^{-6} + \lambda_2^{-6})) = \tilde{C}\) and \((1 - O(\lambda_1^{-6} + \lambda_2^{-6})) = C\). One can check the details in Appendix A.3.

By a direct calculation, the following estimates hold:

\[
|\tilde{C} - C| \leq O(\lambda_1^{-1} + \lambda_2^{-1});
\]

\[
|\tilde{C} - C| \leq C \left( 1 - \frac{c_m}{c^*} \right)^{-1};
\]

\[
|F_2 \left( \frac{\epsilon_1}{c^*}, \frac{\epsilon_2}{c_m} \right) F_2 \left( \frac{\epsilon_1}{c_m}, \frac{\epsilon_2}{c_m} \right) \right| \leq 1 + c_m^{-1};
\]

\[
|F_1 \left( \frac{\epsilon_1}{c_m}, \frac{\epsilon_2}{c_m} \right) F_2 \left( \frac{\epsilon_1}{c_m}, \frac{\epsilon_2}{c_m} \right) \right| \leq C \epsilon_1^{-1} \log \epsilon_1^{-1};
\]

\[
|\tilde{C} F_2 \left( \frac{\epsilon_1}{c_m}, \frac{\epsilon_2}{c_m} \right) \right| \leq C \log \epsilon_1^{-1}; |\tilde{C} F_1 \left( \frac{\epsilon_1}{c^*}, \frac{\epsilon_2}{c_m} \right) \right| \leq C \epsilon_1^{-1}.
\]

Hence, it follows from the inequality

\[|a_1 a_2 a_3 - b_1 b_2 b_3| \leq |a_1 - b_1||b_2 b_3| + |a_1 b_3||a_2 - b_2| + |a_1 a_2||a_3 - b_3|
\]

that

(62)

\[
K_1 - K_2 \leq O(\lambda_1^{-6} + \lambda_2^{-6}) C \epsilon_1^{-1} \log \epsilon_2^{-1} + C \left( 1 - \frac{c_m}{c^*} \right) \epsilon_1^{-1} \log \epsilon_2^{-1} + (1 + \frac{c_m}{c^*}) C \epsilon_1^{-1} + C \epsilon^{-1},
\]
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which implies

\[(63) \quad R_2 \leq C\epsilon_1^{-1}.\]

Combine (62) with the fact $|1 - \frac{\hat{c}_1}{c_1}| \leq \epsilon_1^{-1} \ll \log \epsilon_2^{-1}$ and $\log \epsilon_2^{-1} \ll \min\{\lambda_1, \lambda_2\}^{-1}$, we have $K_1 - K_2 \leq C\epsilon_1^{-1}$, which implies $R_2 \leq C\epsilon_1^{-1}$.

For $R_3$, by the help of the process above, we have

$$\int_{I_{L+1}^{-I}} C \frac{\delta_t \theta(x)}{\theta(x)} dx \leq 2 \int_{\frac{1}{c_1}}^{2^{L+1}q_{N+L}} \left( \frac{c_2}{2} \tan^2 \theta_1 + \frac{|\cot \theta_1|}{\cot^2 \theta_1 + c_1^2} \right) dx$$

$$+ 2 \int_{0}^{c_1^2} \left( \frac{|\cot \theta_1|}{\cot^2 \theta_1 + c_1^2} \right) dx$$

$$\leq C \int_{\frac{1}{c_1}}^{2^{L+1}q_{N+L}} \left( \frac{c_2}{2c_1^4} \frac{1}{(x^2 - (\frac{\hat{c}_1}{c_1})^2)^2} + \frac{1}{c_1^2} \frac{|x^2 - (\frac{\hat{c}_1}{c_1})^2|}{(x^2 - (\frac{\hat{c}_1}{c_1})^2)^2 + c_1^2} \right) dx$$

$$+ C \int_{0}^{c_1^2} \left( \frac{c_2}{2c_1^4} \frac{1}{(x^2 - (\frac{\hat{c}_1}{c_1})^2)^2} + \frac{1}{c_1^2} \frac{|x^2 - (\frac{\hat{c}_1}{c_1})^2|}{(x^2 - (\frac{\hat{c}_1}{c_1})^2)^2 + c_1^2} \right) dx$$

$$\triangleq \hat{K}_1 + \hat{K}_2.$$

It follows from the process of the estimate for $R_2$ that $\hat{K}_1 \leq C\epsilon_1^{-1} \log(1 + C\epsilon_1^{-2}q_{N+L}^C) + \log(1 + c_1^{-1}) \leq C\epsilon_1^{-1}$ and $\hat{K}_2 \leq C\epsilon_1^{-1} \log(1 + \epsilon_1^{-1}) + C\epsilon_1^{-1}$. Therefore,

\[(64) \quad R_3 \leq C\epsilon_1^{-1}.\]

In summary, (59), (63) and (64) imply

\[(65) \quad \epsilon'' \leq \epsilon'_0 \epsilon_1^{-\frac{1}{2}}\]

with some positive constant $\epsilon'_0 \sim \lambda^{-\frac{1}{2}k(t_0)}$.

**B:** **The case $x \in I_{L+1} - I'$**: For the second part, we divide it into three parts. For simplification, denote $\|A_k\| = \|k\|, I_{L+1,2} - I' := I_1 \cup I_2 \cup I_3$, where

$\quad I_1 = (I_{L+1,2} - I') \cap [c_{L+1,1} + k\alpha - \frac{1}{2L+1} q_{N+L+1-1}^C, c_{L+1,1} + k\alpha - C_l^{-2k}],$

where the image of the angle function is located at $[-\pi + C_l^{-k}, -c_4]$;

$\quad I_2 = [c_{L+1,1} + k\alpha - C''l^{-2k}, c_{L+1,1} + k\alpha + C'l^{-2k}],$

where the image of the angle function is located at $[-c_4, -c_3]$;

$\quad I_3 = [c_{L+1,1} + k\alpha + C''l^{-2k}, c_{L+1,1} + k\alpha + C''l^{-k}],$

where the image of the angle function is located at $[-c_3, maximum]$.

Note that for $x \in I_3$, we may write $\theta_k$ as $c_{k} l^{-2k} x^{-1} + x - c_{2} l^{-k}$ and

$$\left| \frac{\partial \theta_k}{\partial t} \right| \leq c_{3} r_{L+1}^{-2k} x^{-2}.$$
Lemma 38. Let \( n' A_1, A_{n-1} \) and \( \theta_1 \) be defined as in Lemma 37 and \( c \) be as in lemma 26. For any fixed \( t_0 \in EP \) and \( \epsilon > 0 \), there exists some \( N = N(t_0) \), and absolute
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constants $\lambda^{-\frac{1}{2}k(t_0)} \leq c_1^{\prime}, c_2^{\prime} \leq \lambda^{-\frac{1}{2}k(1-\epsilon)k(t_0)}$ with $0 < c_1^{\prime} < c_2^{\prime} \leq c_1^{\prime} + |t_0 - t_1|^h$ such that for $n \geq N, x \in S_2$ and $t_0 + \lambda^{-\Gamma} N + L(n) - 1 \leq t_1 \leq t_0 + \lambda^{-\Gamma} N + L(n) - 2$, we have

$$c_1^{\prime} \sqrt{t_1 - t_0} \leq \left| \int_{T^1 x \in L_{t_1+1}} \frac{d\theta_1}{d\theta_1} \frac{d\|A_n\|}{\|A_n\|} \right| \leq c_2^{\prime} \sqrt{t_1 - t_0}.$$

Proof. For

$$t_0 + \lambda^{-\Gamma} N + L(n) - 1 < t_1 < t_0 + \lambda^{-\Gamma} N + L(n) - 2,$$

all the inferences are similar as above. In fact, if we let $\epsilon_1 := t_1 - t_0$, with the same method we can obtain that

$$\theta_1(t_1, x) \approx c(x - \tilde{x})^2 + \bar{c}_1$$

for $\frac{1}{3} n' \leq l \leq n - \frac{1}{3} n'$.

Then Lemma 39 implies that

$$\int_{T^1 x \in L_{t_1+1}} \frac{d\|A_n\|}{\|A_n\|} \approx c_1^{\prime} \frac{1}{\sqrt{t_1 - t_0}}.$$

Since $1 \ll \epsilon_1^{\frac{1}{h}} \ll \min \{\lambda_1, \lambda_2\}$, we obtain that

$$\sqrt{(1 - \lambda_1^2)^2(\epsilon + c_1^{\prime})^2 + (\frac{1}{\lambda_1} - \lambda_1^2)^2(\epsilon + c_1^{\prime})^2 + 2(1 - \frac{1}{\lambda_1^2})^2(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_1^2})} \leq \frac{\epsilon_1}{\sqrt{t_1 - t_0}}.$$

Note that $\text{sgn}(x - c_{L+1}) \equiv 1$ or $-1$ since the minimum(or maximum) of $g_L$ has been far away from horizontal axis. Thus with the help of Appendix A1, we have

$$\int_{T^1 x \in L_{t_1+1}} \frac{1}{\|A_n\|} \frac{d\|A_n\|}{d\theta} \approx c_1^{\prime} \frac{1}{\sqrt{t_1 - t_0}}.$$

More precisely, there exist some $c_1^{\prime}$ and $c_2^{\prime}$ satisfying $0 < c_1^{\prime} < c_2^{\prime} \leq c_1^{\prime} + |t_0 - t_1|^h$ such that

$$c_1^{\prime} \frac{1}{\sqrt{t_1 - t_0}} \leq \int_{T^1 x \in L_{t_1+1}} \frac{d\|A_n\|}{\|A_n\|} \frac{1}{\|A_n\|} \leq c_2^{\prime} \frac{1}{\sqrt{t_1 - t_0}}.$$

Finally we deal with the case $x \in S_3$. For $\frac{1}{n'} \leq l \leq n - \frac{1}{3} n'$ or $n - \frac{1}{3} n' \leq l \leq n$, $\|\theta_1 - g_{L+1}\|$ may be too large such that $\delta = \theta_1(\tilde{x})$ is larger than $\epsilon_1$ and the argument in the proof of Lemma 38 does not work. For this case, we will prove the following result:

Lemma 39. Let $n' A_1, A_n$ and $\theta_1$ be defined as in Lemma 37 and the constant $c$ is from Lemma 36. For any $t_0 \in E \text{P}$ and $\epsilon > 0$, there exists some $N = N(t_0)$ and absolute constant $\lambda^{-\frac{1}{2}k(t_0)} \leq c_1^{\prime} \leq \lambda^{-\frac{1}{2}k(1-\epsilon)k(t_0)}$ such that for $n \geq N, x \in S_3$ and $\lambda^{-\Gamma} N + L(n) - 1 \leq |t_1 - t_0| \leq \lambda^{-\Gamma} N + L(n) - 2$, we have

$$\int_{T^1 x \in L_{t_1+1}} \frac{d\theta_1}{d\theta_1} \frac{d\|A_n\|}{\|A_n\|} \frac{1}{d\|A_n\|} \left| \frac{1}{dx} \right| \leq c_2^{\prime} n^{\frac{1}{4}} |t_1 - t_0|^{\frac{1}{4}}.$$
Proof. For $1 \leq l \leq \frac{1}{2} n'$ or $n - \frac{1}{2} n' \leq l \leq n$, we only consider the case $t_0 - \lambda^{-\frac{q_0 N + L(n) - 1}{N}} \leq t_1 \leq t_0 - \lambda^{-\frac{q_0 N + L(n) - 1}{N}}$. And for

$$t_0 + \lambda^{-\frac{q_0 N + L(n) - 1}{N}} \leq t_1 < t_0 + \lambda^{-\frac{q_0 N + L(n) - 2}{N}},$$

all the inferences are similar as above.

Let

$$\delta_1 := \delta - \epsilon_1,$$

we have the following estimate.

By lemma 26 and Appendix A1, we have

$$\int_{T^1} \frac{d\theta d||A_n||}{dt} dt \leq \frac{1}{||A_n||} \int_{T^1} \frac{1}{d\theta} dx \leq C \min \{2\lambda_0, \delta_1^{-\frac{5}{8}}\} \leq C \min \{\lambda_0, \delta_1^{-\frac{5}{8}}\} \leq C \min \{\lambda_0, (\lambda_0^{-2} \frac{\log \lambda_0}{\log \lambda})^{\frac{5}{8}}, (\lambda_0^{-2} - \epsilon_1)^{-\frac{5}{8}}\} \leq C \left[ \frac{\log \lambda_0}{\log \lambda} \right]^{\frac{5}{8}} \epsilon_1^{-1} \leq C q^{\frac{5}{8}} N^{L(n) - 1} \epsilon_1^{-0.5} \leq C n^{\frac{5}{8}} \epsilon_1^{-0.5},$$

where $\lambda_0 = \min \{\lambda_1, \lambda_2\}$. See the details of the third inequality in A.1 and the fifth inequality is due to $|\delta| \leq \max \{\lambda_1^{-2}, \lambda_2^{-2}\} = \lambda_0^{-2} \frac{\log \lambda_0}{\log \lambda} \right)^{\frac{5}{8}}, (\lambda_0^{-2} - \epsilon_1)^{-\frac{5}{8}}$ from lemma 36.

For the seventh inequality, we need slightly change the scale of $n$ and corresponding $I_{L(n)}$. Whereas, we won’t specifically change it and just admit it for convenience.

\title{Corollary 40.} Let $c$ be as in lemma 26. For any $t_0 \in EP$, there exists some $N = N(t_0), c_1''$ and $c_2''$ independent on $t_0$ satisfying $0 < c_1'' \leq c_2'' \leq c_3'' + |t_0 - t_1|^h$ such that for $n \geq N$ it holds that

$$|L_n(t_1)| \leq c_2'' |t_0 - t_1|^{-\frac{5}{8}}$$

for $\lambda^{-\frac{q_0 N + L(n) - 1}{N}} \leq |t_0 - t_1| \leq \lambda^{-\frac{q_0 N + L(n) - 2}{N}}$, and

$$c_1'' |t_0 - t_1|^{-\frac{5}{8}} \leq |L_n'(t_1)| \leq c_2'' |t_0 - t_1|^{-\frac{5}{8}}$$

for $\lambda^{-\frac{q_0 N + L(n) - 1}{N}} \leq t_1 - t_0 \leq \lambda^{-\frac{q_0 N + L(n) - 2}{N}}$, where $L(n)$ is determined by $q_0 N + L(n) - 1 \leq n \leq q_0 N + L(n)$. Both $c_1''$ and $c_2''$ independent on $t_0$.

Proof. For any fixed $n \geq N$, since $|I_{L(n)}| \leq \frac{1}{2 q_{N + L(n)}}$, from the Diophantine condition on $\alpha$ and the definition of $L(n)$, for $0 \leq l \leq n$ we have that $|T^1(0)| \geq \frac{n}{48} > \frac{1}{4}$.

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Thus we obtain (69). It implies that there exists at most one $0 \leq l \leq n$ such that $T^l(x) \in I_{L(n)+1}$.

Recall that $\left[ \frac{-\log |t_1-t_0|}{\log \lambda} \right] + R_{N_0}^2 = n'$ and $T = S_1 \cup S_2 \cup S_3$, where

$$S_1 = \{ x \in T | T^l(x) \notin I_{L+1} \text{ for any } 0 \leq l \leq n \},$$

$$S_2 = \{ x \in T | \text{there exists } \frac{1}{4} n' \leq l \leq n - \frac{1}{4} n' \text{ such that } T^l(x) \in I_{L+1} \},$$

$$S_3 = \{ x \in T | \text{there exists } 0 \leq l < \frac{1}{2} n' \text{ or } n - \frac{1}{2} n' < l \leq n \text{ such that } T^l(x) \in I_{L+1} \}.$$

(1) For $x \in S_1$, since there is no $0 \leq l < n$ such that $T^l x \in I_{L+1}$, by corollary 29 it holds that

$$\frac{1}{\| A_n(x, t) \|} \left| \frac{\partial \| A_n(x, t) \|}{\partial t} \right| \leq \max \{ n, q_{N+L-1} \} C \leq n^C,$$

which implies

$$\frac{1}{n} \left| \int_{x \in S_1} \frac{1}{\| A_n \|} \frac{d\| A_n \|}{dt} dx \right| \ll \epsilon_1^{-\frac{3}{4}}.$$

(2) For $x \in S_2$, there exists some $\frac{1}{4} n' \leq l \leq n - \frac{1}{4} n'$ such that $T^l x \in I_{L+1}$. For this case, Lemma 37 and 38 hold true. Thus it holds that

$$\frac{c_1}{n} (n - o(n^{\frac{1}{4}})) \epsilon_1^{-\frac{7}{4}} \leq \frac{1}{n} \left| \int_{x \in S_2} \frac{1}{\| A_n \|} \frac{d\| A_n \|}{dt} dx \right| \leq \frac{c_2}{n} (n - o(n^{\frac{1}{4}})) \epsilon_1^{-\frac{7}{4}},$$

(3) For $x \in S_3$, there exists some $0 \leq l < \frac{1}{4} n', n - \frac{1}{4} n' < l$ such that $T^l x \in I_{L+1}$. For this case, Lemma 39 holds true. Thus it holds from $n'^{3.5} \leq n$ that

$$\frac{1}{n} \left| \int_{x \in S_3} \frac{1}{\| A_n \|} \frac{d\| A_n \|}{dt} dx \right| = \frac{o(n^{\frac{1}{4}})}{n} n^{\frac{3}{4}} \epsilon_1^{-\frac{7}{4}}.$$

Thus on one hand for $\lambda^{-\frac{1}{2}q_{N+L(n)-1}} \leq |t_0 - t_1| \leq \lambda^{-\frac{1}{2}q_{N+L(n)-2}}$, it holds that

$$|L_{\epsilon_1}(t)| = \left| \int_{R/2} \frac{1}{n} \| A_n \| \frac{d\| A_n \|}{dt} dx \right| \leq \left| \int_{S_1} + \int_{S_2} + \int_{S_3} \right| \leq o(\epsilon_1^{-\frac{3}{4}}) + \frac{c_2}{n} (n - o(n^{\frac{1}{4}})) \epsilon_1^{-\frac{7}{4}} + \frac{o(n^{\frac{1}{4}})}{n} n^{\frac{3}{4}} \epsilon_1^{-\frac{7}{4}} \leq \epsilon_1^{-\frac{3}{4}}.$$

Thus we obtain [69].
On the other hand, for $\lambda^{-\frac{n}{n+L(n-1)}} t_1 - t_0 \leq \lambda^{-\frac{n}{n+L(n-2)}}$, by a similar calculation, we obtain

$$|L_{q}^k(t)| = \left| \int_{S_1 + S_2 - S_3} \frac{1}{n+2} \frac{d||A||}{dt} dt \right|$$

$$\geq \left| \int_{S_2} - \int_{S_1} - \int_{S_3} \right|$$

$$\geq \frac{c_2}{n} \left( n - o(n) \right) \epsilon_1^{-\frac{1}{2}} - o(\epsilon_1^{-\frac{1}{2}}) - \frac{o(n)}{n} n \frac{1}{2} \epsilon_1^{-\frac{1}{2}}$$

$$\geq c^m \epsilon_1^{-\frac{1}{2}},$$

with $0 < c^m_1 \leq c^m_2 \leq c^m_1 + |t_1 - t_0|^b$, which together with (69) implies (70).

Now we are ready to prove part of theorem 1.

Fix any $\tilde{t} \in EP$ and let $L_0$ sufficiently large. Choose the interval

$$I_\tilde{t} := [\tilde{t} - \lambda^{-\frac{n}{n+L(N+L_0)}}, \tilde{t} + \lambda^{-\frac{n}{n+L(N+L_0)}}]$$

and denote $I^+_{\tilde{t}} := [\tilde{t}, \tilde{t} + \lambda^{-\frac{n}{n+L(N+L_0)}}]$; $I^-_{\tilde{t}} := [\tilde{t} - \lambda^{-\frac{n}{n+L(N+L_0)}}, \tilde{t}]$.

Note that either

$$L_{q}^k(t) > 0 or < 0 for any t \in I_{\tilde{t}} and q^4_{N+l+n} \leq k \leq q^4_{N+l+n+1}.$$

Without loss of generality, we assume that

$$L_{q}^k(t) > 0 for any t \in I_{\tilde{t}} and q^4_{N+l+n} \leq k \leq q^4_{N+l+n+1}.$$

Then for any fixed $\tilde{t}' \in I^+_{\tilde{t}}$ (the case $I^-_{\tilde{t}}$ is similar), there exists some $0 < \tilde{l} \leq L_0$ such that

$$\lambda^{-\frac{n}{n+L(N+l-1)}} \leq \tilde{l} - \tilde{l}' \leq \lambda^{-\frac{n}{n+L(N+l)}}.$$

For $n \in \mathbb{N}^+$, let $t_n := \tilde{l} - \lambda^{-\frac{n}{n+L(N+l+n-1)}}$ and $t_0 := \tilde{l}'$.

It follows from lemma 20 that for $\ell \in N$

$$L(t_\ell) - L(t_{\ell+1}) \leq 2L_2q^4_{N+l+n+1}(t_{\ell+1}) - 2L_2q^4_{N+l+n+1}(t_\ell) + (Lq^4_{N+l+n+1}(t_{\ell+1}) - Lq^4_{N+l+n+1}(t_\ell))$$

$$+ |Lq^4_{N+l+n+1}(t_\ell) + L(t_\ell) - 2Lq^4_{N+l+n+1}(t_\ell)|$$

$$+ |Lq^4_{N+l+n+1}(t_{\ell+1}) + L(t_{\ell+1}) - 2Lq^4_{N+l+n+1}(t_{\ell+1})|$$

$$\leq 3c_2(\sqrt{t_\ell} - \sqrt{t_{\ell+1} - \tilde{l}} + 2\lambda^-\tilde{q}^4_{N+l+n+1})$$

Similarly, we have

$$L(t_\ell) - L(t_{\ell+1}) \geq 2L_2q^4_{N+l+n+1}(t_{\ell+1}) - 2L_2q^4_{N+l+n+1}(t_\ell) - (Lq^4_{N+l+n+1}(t_{\ell+1}) - Lq^4_{N+l+n+1}(t_\ell))$$

$$- |Lq^4_{N+l+n+1}(t_\ell) + L(t_\ell) - 2Lq^4_{N+l+n+1}(t_\ell)|$$

$$- |Lq^4_{N+l+n+1}(t_{\ell+1}) + L(t_{\ell+1}) - 2Lq^4_{N+l+n+1}(t_{\ell+1})|$$

$$\geq (2c_2 - c_1)(\sqrt{t_\ell} - \sqrt{t_{\ell+1} - \tilde{l}} - 2\lambda^-\tilde{q}^4_{N+l+n+1})$$
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Thus, for arbitrary $k \in \mathbb{N}^+$,

$$|L(t') - L(\tilde{t})| \geq \left| \sum_{0 \leq i \leq k-1} (L(t_i) - L(t_{i+1})) \right| - |L(t_k) - L(\tilde{t})|$$

$$\geq (2c_2 - c_1) \sum_{0 \leq i \leq k-1} (\sqrt{t_i - \tilde{t}} - \sqrt{t_{i+1} - \tilde{t}}) - \sum_{0 \leq i \leq k-1} 2\lambda^{-\frac{1}{2}q_{N+i+1}} |L(t_k) - L(\tilde{t})|$$

$$\geq (2c_2 - c_1) \sqrt{t' - \tilde{t}} - \sum_{0 \leq i \leq +\infty} 2\lambda^{-\frac{1}{2}q_{N+i+1}}$$

$$\geq (2c_2 - c_1)(t' - \tilde{t})^{\frac{1}{2}} - \lambda^{-\frac{1}{2}q_{N+i+1}}$$

$$\geq \tilde{c}(t' - \tilde{t})^{\frac{1}{2}}$$

and

$$|L(t') - L(\tilde{t})| \leq \left| \sum_{0 \leq i \leq k-1} (L(t_i) - L(t_{i+1})) \right| + |L(t_k) - L(\tilde{t})|$$

$$\leq 3c_2 \sum_{0 \leq i \leq k-1} (\sqrt{t_i - \tilde{t}} - \sqrt{t_{i+1} - \tilde{t}}) + \sum_{0 \leq i \leq k-1} 2\lambda^{-\frac{1}{2}q_{N+i+1}} + |L(t_k) - L(\tilde{t})|$$

$$\leq 3c_2 \sqrt{t' - \tilde{t}} + \sum_{0 \leq i \leq +\infty} 2\lambda^{-\frac{1}{2}q_{N+i+1}}$$

$$\leq 3c_2 (t' - \tilde{t})^{\frac{1}{2}} + \lambda^{-\frac{1}{2}q_{N+i+1}}$$

$$\leq \tilde{c}'(t' - \tilde{t})^{\frac{1}{2}}$$

for proper constant $\lambda^{-\frac{1}{2}k(t_0)} \leq \tilde{c} < \tilde{c}' < \lambda^{-\frac{1}{2}k(t_0)}$. The third inequality is followed by the continuity of $L(t)$ and the arbitrary choice of $k$.

In conclusion, we have proved that for any $E \in EP$, $L(\cdot)$ is exactly local $\frac{1}{2}$-Hölder continuous.

7. The proof of local Lipschitz continuity

In this section we prove local Lipschitz continuity of $L(E)$ on a full measure set $FR$, which is different from exact $\frac{1}{2}$-Hölder continuity of $L(E)$ on $EP$. The reason why the regularities for these two cases are different lies in the fact that the measure of the set of ‘bad’ $x$ for the former is much less than the one for the latter. It can be found from the degeneration of the function $g(x, \cdot)$ of $t \in EP$ and the nondegeneration of the one of $t \in FR$, respectively.

Roughly speaking, the proof for $(1 - \epsilon)$-Hölder continuity of $L(t)$ for any $\epsilon > 0$ and $t \in FR$ is similar to the one for $\frac{1}{2}$-Hölder continuity of $L(t)$ for $t \in EP$. However, it is quite different to improve the regularity from $(1 - \epsilon)$-Hölder continuity to Lipschitz continuity.

The following result is the key lemma for this section, which shows that $L'_n$ is uniformly bounded in some local sense when $n$ is sufficiently large.

First, we denote $Y(n) \triangleq \{ l | q_{N+l-1}^2 \leq n \leq q_{N+l}^2 \}$. $L_0(t) \triangleq \max \{ n | g_n(t) is of type III \}$

**Lemma 41.** For any fixed $t \in FR$, there exists some $\lambda_0 > 0$ and $n_0 > q_{N+L_0(t)-1}$, such that

$$|L'_n(t')| \leq \frac{q_{N+L_0(t)-1}^{300\gamma^2}}{51}$$
for \( n \geq n_0 \), \( \lambda \geq \lambda_0 \) and \( |t' - t| \leq \lambda^{-\frac{2}{2}}q_{N+Y(n)} \),

Remark 42. The upper bound of \( |L_n'(t')| \) is independent of the scale of \( |I_{L_0}| \) and the best estimate is only dependent on \( t \) and \( \lambda \).

**The proof of the Lipschitz continuity for** \( t \in FR \) **by Lemma [11]**

**Proof.** Fix any \( t \in FR \). By Lemma [11], there exists \( n_0 \) sufficiently large such that

\[
|L_n'(t)| \leq q_{N+L_0(t)-1}^{300\tau^2}
\]

for any \( n \geq n_0 \). Let \( I_0 := [t - \lambda^{-\frac{2}{2}}q_{N+Y(n_0)-1}, t + \lambda^{-\frac{2}{2}}q_{N+Y(n_0)-1}] \). Together with Lemma [26] and the definition of \( Y(n) \), for any \( \lambda^{-\frac{2}{2}}q_{N+Y(n)-1} \leq |t - t'| \leq \lambda^{-\frac{2}{2}}q_{N+Y(n)} \), it holds that

\[
|L(t) - L(t')| \leq |L_n(t) - L_n(t')| + |L_{2n}(t) - L_{2n}(t')| + |L_n(t) + L(t) - 2L_{2n}(t)| + |L_n(t') + L(t') - 2L_{2n}(t')|
\]

\[
\leq C|t - t'| + 2\lambda^{-\frac{2}{2}}q_{N+L(n)-1}
\]

\[
\leq C|t - t'| + \lambda^{-\frac{2}{2}}q_{N+L(n)-1} \leq C|t - t'| + \lambda^{-\frac{2}{2}}q_{N+L(n)-1} \leq (C + 1)|t - t'|
\]

\[
\leq C'|t - t'|.
\]

It is clear that as \( n \) increases from \( n_0 \) to \(+\infty\), \( t' \) goes through \( I_0 \). Thus we prove the local Lipschitz continuity of \( L(\cdot) \) at \( t \) on \( I_0 \) with the constant \( C' \). \( \square \)

We are at the position to prove Lemma [11]. From the definition, for any \( t_0 \in FR \), we have that \( g_{N+j}(t_0) \) is of type I for all \( j \geq L_0 \).

Note that for any \( t \in [\lambda^{-\frac{2}{2}}q_{N+i-1} + t_0, \lambda^{-\frac{2}{2}}q_{N+i} + t_0] \), \( g(t) \) is still of type I for any \( l > L_0 \), which is followed by the definition of \( FR \). And it follows from Lemma [15] that

\[
q_{N+L_0-1}^C \leq \left| \frac{\partial g(t, x)}{\partial t} \right| \leq q_{N+L_0-1}^C,
\]

where \( C \) can be chosen as \( 100\tau^{100} \).

Assume \( A_n(x) = A_{n-l}(x)A_l(x) \) and denote

\[
(71) \quad \|A_{n-l}(x)\| := \lambda_1, \quad \|A_l(x)\| := \lambda_2, \quad \text{and} \quad \theta_l(x) := \frac{\pi}{2} - s(A_{n-l}(x)) + u(A_l(x)).
\]

We need the two following lemmas later.

**Lemma 43.**

1. It holds that

\[
\frac{1}{\|A_n\|} \frac{d\|A_n\|}{d\lambda_i} < C\lambda_i^{-1}, \quad i = 1, 2.
\]

2. If

\[
\lambda_i^2 \leq \lambda_1 (\text{resp.} \lambda_i^2 \leq \lambda_2),
\]

then we have

\[
\frac{1}{\|A_n\|} \frac{d\|A_n\|}{d\lambda_1} = \lambda_1^{-1} + o(\lambda_1^{-1}) \quad (\text{resp.} \frac{1}{\|A_n\|} \frac{d\|A_n\|}{d\lambda_2} = \lambda_2^{-1} + o(\lambda_2^{-1})).
\]
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(3) If

$$|\theta| \geq \lambda^{-\eta}, \text{ where } \lambda_{m} := \min\{\lambda_1, \lambda_2\} \text{ and } \eta \ll 1,$$

Then we have

$$\frac{1}{\|A_n\|} \frac{d\|A_n\|}{d\lambda_1} = \lambda_1^{-1} + o(\lambda_1^{-1}), \quad \frac{1}{\|A_n\|} \frac{d\|A_n\|}{d\lambda_2} = \lambda_2^{-1} + o(\lambda_2^{-1}).$$

**Proof.** Without loss of generality, we only consider the estimates on $\frac{1}{\|A_n\|} \frac{d\|A_n\|}{d\lambda_1}$. From Lemma 6, we have

$$\frac{1}{\|A_n\|} \frac{\partial\|A_n\|}{\partial \lambda_1} = \frac{\text{sgn}(\theta) \cdot \lambda_1^{-1} \left( (1 - \frac{1}{\lambda_1^2}) \cot \theta_i + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan \theta_i \right)}{\sqrt{(1 - \frac{1}{\lambda_1^2})^2 \cot^2 \theta_i + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2})^2 \tan^2 \theta_i + 2(1 + \frac{1}{\lambda_1^2})(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}) - \frac{8}{\lambda_1^4 \lambda_2^4}}} = \frac{a_1}{\lambda_1^4 + a_2},$$

where $a_1 = (1 - \frac{1}{\lambda_1^2}) \cot \theta_i + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan \theta_i$ and $a_2 = \frac{a_1}{\lambda_1^2}(1 - \frac{1}{\lambda_2^2})$.

Thus it is easy to obtain $\left| \frac{1}{\|A_n\|} \frac{d\|A_n\|}{d\lambda_1} \right| < C\lambda_1^{-1}$. Assume the condition (72) holds, then $\frac{1}{\lambda_1^4} \ll \frac{1}{\lambda_2^4}$, which implies two terms of $a_1$ share the same sign. Thus

$$a_1 = \cot \theta_i + \frac{1}{\lambda_2^2} \tan \theta_i + o(a_1) > \frac{1}{2\lambda_2^2}.\]$$

Since $a_2 = \frac{2}{\lambda_1^2} + o(a_2)$, with the help of (72), we have $a_2 \ll |a_1|$. Assume the condition (74) holds, then

$$\left| (1 - \frac{1}{\lambda_1^2 \lambda_2^2}) \cot \theta_i \right| \gg \max\left( \left| \frac{1}{\lambda_2^2} - \frac{1}{\lambda_1^2} \right| \tan \theta_i, \ a_2 \right).$$

Thus (75) holds true. \qed

For any $x \in \mathbb{T}$, let $0 \leq j_1 < j_2 < \cdots < j_s \leq n$ be all the times for $x$ returning to $I_L$, where $s$ depends on $x$, $E$.

Let $\hat{I}_{L+1} = B(C_{L+1}, \frac{1}{q_{N+L}})$ and define $\Omega_{L,1} = \{x \in \mathbb{T} | T^k x \not\in \hat{I}_{L+1}, 0 \leq k \leq n\}$ and $\Omega_{L,2} = \mathbb{T} \backslash \Omega_{L,1}$. From the Diophantine condition, there exists at most one $0 \leq l \leq n$ such that $T^l x \in \hat{I}_{L+1}$ (for simplicity we only consider one interval of $\hat{I}_{L+1}$ and still write it as $I_L$).

It is not difficult to see that for the measure of $\Omega_{L,2}$, we have $\text{mes}(\hat{I}_{L+1}) \leq \frac{2}{q_{N+L}}$, which implies

$$\text{mes}(\Omega_{L,2}) \leq \frac{2n}{q_{N+L}}.$$  

(76) The following lemma shows that the contribution of the ‘bad’ set $\Omega_{L,2}$ for $L'$ can be ignored.
Lemma 44. It holds that

\[ \frac{1}{n} \left| \int_{\Omega_{L,2}} \frac{1}{\| A_i \|} \frac{\partial \| A_n \|}{\partial t} \, dx \right| \leq 2q_{N+L}^{-500C_T} + C \frac{\log q_{N+L+1}}{q_{N+L}}. \]

Before proving Lemma 44, we need some notations. Consider two matrices $A$ and $B$. Assume by the singular value decomposition it holds that $BA = R_{\phi} \cdot \text{diag}(\| B \|, \| B \|^{-1}) \cdot R_{\phi}{\cdot} \text{diag}(\| A \|, \| A \|^{-1}) \cdot R_{\phi}$. Then we denote $\theta(A, B) = \phi_v$. Moreover, we define $I_i = \{ x \in \mathbb{S}^1 \mid \text{dist}(x, C_i) \leq \frac{1}{q_{N+i}} \}$, where $C_i$ is the critical point set at the scale $i$ and dist is the distance. To prove Lemma 44, we need the following estimate.

Lemma 45. For $x \in I_i$ with $L_1 \leq i \leq L-1$, let $\dot{A}_i(x, t) = A_{r+i}(x, t) \cdot A_{-r+i}(T-r_x T, x, t)$, $\lambda_1 = \| A_{r+i}(T-r_x T, x, t) \|$, $\lambda_2 = \| A_{r+i}(x, t) \|$, and $\theta_i = \theta(A_{r+i}(x, t), A_{-r+i}(T-r_x T, x, t))$. Then it holds that

\[ \left| \int_{I_i \setminus I_{i+1}} \frac{1}{\| A_i \|} \frac{\partial \| A_i \|}{\partial t} \frac{\partial \theta_i}{\partial t} \, dx \right| \leq \frac{C}{q_{N+i}} \log \frac{q_{N+i}}{q_{N+i+1}} \]

and

\[ \left| \int_{I_i \setminus I_{i+1}} \frac{1}{\| A_i \|} \frac{\partial \| A_i \|}{\partial t} \frac{\partial \theta_i}{\partial t} \, dx \right| \leq \frac{C}{q_{N+i}} \log \frac{q_{N+i+1}}{q_{N+i+2}}. \]

Proof. From Lemma 45, we have

\[ \frac{1}{\| A_i \|} \frac{\partial \| A_i \|}{\partial t} = \frac{\text{sgn}(\theta_i)(1 - \lambda_1^{-4})(1 - \lambda_2^{-4})}{\sqrt{(1 - \frac{1}{\lambda_1^2 \lambda_2^2}) \cos^2 \theta_i + (\frac{1}{\lambda_1^2} - \frac{1}{\lambda_2^2}) \tan^2 \theta_i + 2(1 + \frac{1}{\lambda_1^2})(\frac{1}{\lambda_1^2} + \frac{1}{\lambda_2^2}) - \frac{8}{\lambda_1^2 \lambda_2^2}}}. \]

Denote $K(i) = \max\{ s < i | g_{s-1} \text{ is type I and } g_s \text{ is type III} \}$, then it follows from Lemma 45 that

O1: $\text{sgn}(\frac{\partial \| A_i \|}{\partial t}) = \text{sgn}(\theta_i)$;

O2: $M_1(i) \leq \frac{\partial \| A_i \|}{\partial t} \leq \lambda^{-\frac{2^4}{k_N^{-\theta(i)}}}$, for any $\lambda \in I_i$ and $|t - t_0| \leq \lambda^{-\frac{2^4}{k_N^{-\theta(i)}}}$, where $M_1$ and $M_2$ are some proper constants with $M_2 - M_1 \leq Cq_{N+i}^{-1000\tau}$. In fact, for any $(x, t), (x', t') \in I_i \times J_i(\frac{1}{2} | t_0 - \lambda^{-\frac{2^4}{k_N^{-\theta(i)}}}, t_0 + \lambda^{-\frac{2^4}{k_N^{-\theta(i)}}})$, by Lagrange Mean Value Theorem, we have

\[ \left| \frac{\partial \theta_i(x, t)}{\partial t} \right| - \left| \frac{\partial \theta_i(x', t')}{{\partial t}^2} \right| \leq \left| (x - x') \frac{\partial^2 \theta_i(x' + (x' - x)q + (t - t')q)}{{\partial t}^2} + (t - t_0) \frac{\partial^2 \theta_i(x' + (x' - x)q + (t - t')q)}{{\partial t}^2} \right| \leq \sqrt{(\frac{\partial^2 \theta_i(x' + (x' - x)q + (t - t')q)}{{\partial t}^2})^2 + (\frac{\partial^2 \theta_i(x' + (x' - x)q + (t - t')q)}{{\partial t}^2})^2} \sqrt{(x - x')^2 + (t - t_0)^2}. \]

Direct calculation shows that $\frac{\partial^2 \theta_i}{\partial t^2} \sim \lambda^{-\frac{2^4}{k_N^{-\theta(i)}}}$. Therefore,

\[ \left| \frac{\partial \theta_i(x, t)}{\partial t} \right| - \left| \frac{\partial \theta_i(x', t')}{{\partial t}^2} \right| \leq \lambda^{-\frac{2^4}{k_N^{-\theta(i)}}} | I_i |, \text{ which implies O2.} \]

O3: $C_1(i) \leq \frac{\partial \theta_i(x, t)}{\partial t} \leq C_2(i) \leq \lambda^{-\frac{2^4}{k_N^{-\theta(i)}}}$, for any $x \in I_i$ and $|t - t_0| \leq \lambda^{-\frac{2^4}{k_N^{-\theta(i)}}}$, where $C_1$ and $C_2$ are some proper constants with $C_2 - C_1 \leq Cq_{N+i}^{-1000\tau}$. The proof is similar to O2.
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O4: For any $x, y \in \hat{I}_i, i = 1, 2 \left| \frac{\lambda(x)}{\lambda(y)} \right| \leq 1 + \frac{\epsilon}{q_{N+i}}$. In fact, it follows from lemma O5 that

$$|\log \lambda_j(x) - \log \lambda_j(y)| \leq \int_{I_i} \left| \frac{d \log \lambda_j}{dx} \right| dx$$

$$\leq \int_{I_i} \frac{1}{\lambda_j} \frac{d \lambda_j}{dx} dx$$

$$\leq |I_i| \cdot \max \{l, q_{N+i}^4 - l, q_{N+i} \} \cdot q_{N+i}^C$$

$$\leq \frac{2}{q_{N+i}^C} \cdot q_{N+i}^C$$

Therefore, $\left| \frac{\lambda(x)}{\lambda(y)} \right| \leq \exp^{\frac{1}{q_{N+i}^C}} \leq 1 + \frac{\epsilon}{q_{N+i}}$.

O5: For any $x \in \hat{I}_{i+1}$ and $j = 1, 2$, we have $\frac{d}{dx} q_{N+i}^C \equiv \lambda_j \leq \lambda_j \leq \lambda_j^2$. (Recall that $R_N = \max \{r_i^+, r_i^- \}$)

O2 and O3 in the above show that in the domain under consideration, the change of $\theta_i$ is much slower than that of $\frac{1}{\| A_i \|^2} \frac{\partial \| A_i \|}{\partial \theta_i}$ such that it can roughly be regarded as a constant. O4 tells us that the difference between $\lambda_i(x)$ and $\lambda_i(y)$ is small and it implies that $\lambda_i^- \leq \lambda_i(x) \leq \lambda_i^+$, where $(1 \pm \frac{1}{q_{N+i}^C})\lambda_i(0) \equiv \lambda_i^\pm$.

The first inequality: It follows from Lemma O1 and O5 that

$$\left| \int_{I_i \backslash I_{i+1}} \frac{\partial \| A_i \|}{\partial \theta_i} \frac{\partial \theta_i}{\partial t} \frac{d\theta_i}{dx} \right|$$

$$\leq \left| \int_{c \cdot L_i + 1}^{c \cdot L_i + 1} \frac{1}{q_{N+i}^C} \left(1 + O\left(\frac{1}{C_{1/2}}\right)\right) \right| \left| M_2 \cdot \frac{1}{\cot^2 \theta_i + \frac{2}{\lambda_2^2}} dx \right|$$

By O2 and O3, we have

$$\left| \int_{c \cdot L_i + 1}^{c \cdot L_i + 1} \frac{1}{q_{N+i}^C} \left(1 - O\left(\frac{1}{\lambda_2^2}\right)\right) \right| \left| M_1 \cdot \frac{1}{\cot^2 \theta_i + \frac{2}{\lambda_2^2}} dx \right|$$

$$= \left| O\left(\frac{2}{\lambda_2}\right) \int_{c \cdot L_i + 1}^{c \cdot L_i + 1} 2M_2 \cdot \frac{1}{\sqrt{C_{1/2}^2 x^2 + \frac{2}{\lambda_2^2}}} dx \right|.$$
If we denote \( \delta_i \triangleq \frac{1}{q_{N+1}^{1/2}} \), then it’s clear that

\[
\begin{align*}
|C_2 \lambda^{2r_i} \delta_i^2 + 2 + C_1 \lambda^{4r_i} \delta_i| & \leq \frac{M_2}{C_1} \log |U_1(C_1, \lambda, \delta_i) - \frac{C_1 M_1}{C_2} \log |U_2(C_2, \lambda, \delta_i)|, \\
& \triangleq \left| \frac{M_2}{C_1} \log |U_1(C_1, \lambda, \delta_i)| - \frac{C_1 M_1}{C_2} \log |U_2(C_2, \lambda, \delta_i)| \right|,
\end{align*}
\]

where \( U_1 = \log \left| \frac{\delta}{\delta_{i+1}} + o(\lambda^{-3R_i}) \right| \) and \( U_2 = \log \left| \frac{\delta}{\delta_{i+1}} + o(\lambda^{-\frac{r_i}{2}}) \right| \).

Thus

\[
\begin{align*}
\left| \frac{M_2}{C_1} \log q_{N+i+1} + C \log q_{N+i} \right| & \leq \frac{M_2}{C_1} \log q_{N+i+1} + C \log q_{N+i} + O(\lambda^{-\frac{r_i}{2}}).
\end{align*}
\]

**The second inequality:** With the help of lemma 8 and O1, we have

\[
\begin{align*}
\int_{I_L} \frac{1}{|A_i|} \frac{\partial}{\partial \theta_i} \left| \frac{\partial}{\partial \theta_i} \right| \frac{dx}{dx}
\leq \int_{c_{L+1}}^{c_L} \frac{(1 + O(\frac{1}{\lambda^2} + \frac{1}{\lambda} \log \lambda)) \text{sgn}(\theta_i) M_2}{\sqrt{\cot^2 \theta_i + (\frac{1}{\lambda^2} - \frac{1}{\lambda^2} \log \lambda)^2 \tan^2 \theta_i + \frac{2}{\lambda^2} + \frac{2}{\lambda^2}}} dx \\
+ \int_{c_{L+1}}^{c_L} \frac{(1 - O(\frac{1}{\lambda^2} + \frac{1}{\lambda} \log \lambda)) \text{sgn}(\theta_i) M_1}{\sqrt{\cot^2 \theta_i + (\frac{1}{\lambda^2} - \frac{1}{\lambda^2} \log \lambda)^2 \tan^2 \theta_i + \frac{2}{\lambda^2} + \frac{2}{\lambda^2}}} dx.
\end{align*}
\]

By O2 and O3, we have

\[
\begin{align*}
\int_{c_{L+1}}^{c_L} \frac{1}{|A_i|} \frac{1}{\sqrt{C_1^2 x^2 + (\frac{1}{\lambda^2} - \frac{1}{\lambda^2} \log \lambda)^2 C_1^{-2} x^{-2} + \frac{2}{\lambda^2} + \frac{2}{\lambda^2}}} dx
\leq \int_{c_{L+1}}^{c_L} \frac{M_2}{\sqrt{C_1^2 x^2 + (\frac{1}{\lambda^2} - \frac{1}{\lambda^2} \log \lambda)^2 C_1^{-2} x^{-2} + \frac{2}{\lambda^2} + \frac{2}{\lambda^2}}} dx \\
- \int_{c_{L+1}}^{c_L} \frac{M_1}{\sqrt{C_2^2 x^2 + (\frac{1}{\lambda^2} - \frac{1}{\lambda^2} \log \lambda)^2 C_2^{-2} x^{-2} + \frac{2}{\lambda^2} + \frac{2}{\lambda^2}}} dx
\end{align*}
\]

By O4, we only need to consider the following two cases:

**CASE1:** \( \text{dist} \{[\lambda_1^-, \lambda_1^+], [\lambda_2^-, \lambda_2^+] \} \leq \frac{1}{q_{N+1}^{1/2}} (\lambda_1^+ + \lambda_2^+) \).
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In this case, note that

$$0 \leq \left| \frac{1}{\lambda_1^4(x)} - \frac{1}{\lambda_2^4(x)} \right| = \left| \frac{\lambda_1 - \lambda_2}{\lambda_1^4 \lambda_2^4} \right|$$

$$= \max\left\{ |\lambda_1^+ - \lambda_2^-|, |\lambda_2^- - \lambda_1^-| \right\} \left( \lambda_1^+ + \lambda_2^- \right)^2 \right) \right)$$

$$\leq \left( \lambda_1^+ \lambda_2^- \right)^4 \frac{1}{q_{N+L}}$$

$$\leq \left( \lambda_1^+ \lambda_2^- \right)^4 \frac{1}{q_{N+L}}$$

$$\leq \left( \lambda_1^+ \lambda_2^- \right)^4 \frac{1}{q_{N+L}}$$

Hence,

$$\left| \int_0^{\frac{1}{2q_{N+L}}} M_2 \sqrt{\left( \lambda_1^+ \lambda_2^- \right)^4 \frac{1}{q_{N+L}}} dx \right|$$

$$\left| \int_0^{\frac{1}{C_1}} M_1 \sqrt{\left( \lambda_1^+ \lambda_2^- \right)^4 \frac{1}{q_{N+L}}} dx \right|$$

$$\left| \int_0^{\frac{1}{2q_{N+L}}} M_2 \sqrt{\left( \lambda_1^+ \lambda_2^- \right)^4 \frac{1}{q_{N+L}}} dx \right|$$

By a direct calculation, see A.2, we have

$$\left| \frac{1}{2q_{N+L}} \log S_{L,+}(x, C_1, \lambda_1^+, \lambda_2^+) - \frac{1}{2q_{N+L}} \log S_{L,-}(x, C_2, C_1, \lambda_1^-, \lambda_2^-) \right|$$

$$\left| \frac{1}{2q_{N+L}} \log \left( \sqrt{\left( \lambda_1^+ \right)^4 \left( \lambda_2^+ \right)^4 \frac{1}{q_{N+L}}} + \sqrt{\left( \lambda_1^- \right)^4 \left( \lambda_2^- \right)^4 \frac{1}{q_{N+L}}} \right) \right|$$

where

$$S_{L,+}(x, C_1, \lambda_1^+, \lambda_2^+)$$

$$= \log \left| \sqrt{\left( \lambda_1^+ \right)^4 \left( \lambda_2^+ \right)^4 \frac{1}{q_{N+L}}} + \sqrt{\left( \lambda_1^- \right)^4 \left( \lambda_2^- \right)^4 \frac{1}{q_{N+L}}} \right|$$

$$S_{L,-}(x, C_1, C_2, \lambda_1^-, \lambda_2^-)$$

$$= \log \left| \sqrt{\left( \lambda_1^- \right)^4 \left( \lambda_2^- \right)^4 \frac{1}{q_{N+L}}} + \sqrt{\left( \lambda_1^+ \right)^4 \left( \lambda_2^+ \right)^4 \frac{1}{q_{N+L}}} \right|$$
If we denote $\delta_L \triangleq \frac{1}{q_{N+L}}$, then it’s clear that

$$\text{(80)} \quad \leq \frac{M_2}{C_1} \log \frac{S_{L,+}(\delta_L)}{S_{L,+}(0)} - \frac{M_1}{C_2} \log \frac{S_{L,-}(\delta_L)}{S_{L,-}(0)} + o((\max\{\lambda_1^-, \lambda_2^-\}^{-3})(\leq o(\lambda_m^-))).$$

It follows from lemma 15 that

$$\text{(81)} \quad \min\{C_1, C_2\} \geq c_1, \min\{M_1, M_2\} \geq c_2,$$

where $c_1 \sim |L_1|$ and $c_2$ is an absolutely constant from lemma 15.

By a direct calculation, we have

$$\text{(82)} \quad \max \left\{ \left| \frac{S_{L,+}(\delta_L)}{S_{L,+}(0)} \right|, \left| \frac{S_{L,-}(\delta_L)}{S_{L,-}(0)} \right| \right\} \leq C \log((\lambda_1^+)^4(\lambda_2^+)^4) \leq C q^4_{N+L} \log \lambda.$$

On the other hand, from the fact $\lambda_m \gg \delta_q$ and O4 we have

$$\text{(83)} \quad \leq \log \left| \frac{S_{L,+}(\delta_L)}{S_{L,+}(0)} \right| - \log \frac{S_{L,-}(\delta_L)}{S_{L,-}(0)} \leq \log \left( \frac{M_2}{C_1} \right) \frac{C_1}{C_2} \frac{\log q_{N+L}}{\log q_{N-L}} + o(\lambda_m^-) \log \left| \frac{S_{L,+}(\delta_L)}{S_{L,+}(0)} \right| - \log \frac{S_{L,-}(\delta_L)}{S_{L,-}(0)} \leq \left( 1 + \frac{1}{q_{N+L}} \right)^8 \frac{C_1 - C_2}{C_2} \log q_{N+L} \log \lambda + \log \left( 1 + \frac{C_2}{C_1 q_{N+L}} \right) \left( 1 + \frac{1}{q_{N+L}} \right)^4 \leq \left( \frac{M_2}{C_1} - \frac{M_1}{C_2} \right) \log q_{N+L} + o(\lambda_m^-) \leq C \left( \frac{M_2}{C_1} - \frac{M_1}{C_2} \right) \log q_{N+L} + o(\lambda_m^-).$$

In the above, we use the inequality $\log(1 + x) \leq x$ for $x \geq 0$. Hence, combining (81), (82), and (83), we have

$$\text{(84)} \quad \leq \left( \frac{M_2}{C_1} - \frac{M_1}{C_2} \right) \max \left\{ \log \left| \frac{S_{L,+}(\delta_L)}{S_{L,+}(0)} \right|, \log \left| \frac{S_{L,-}(\delta_L)}{S_{L,-}(0)} \right| \right\} + \max \left\{ \left( \frac{M_2}{C_1}, \frac{M_1}{C_2} \right) \right\} \log q_{N+L} + \frac{M_1 \frac{1}{C_1}}{q_{N+L}} + \frac{M_2 \frac{1}{C_2}}{q_{N+L}} \leq C \left( \frac{M_1}{C_1} - \frac{M_2}{C_2} \right) q_{N+L} \log q_{N+L} \log \lambda + C \frac{1}{q_{N+L}^{1/2}} \leq C \frac{q_{N+L}^{1/2}}{q_{N+L}^{1/2}} \leq C q_{N+L} \log q_{N+L}.$$

**CASE2:**\text{ dist}\{[\lambda_1^+, \lambda_2^+], [\lambda_2^+, \lambda_1^-]\} > \frac{1}{q_{N+L}}(\lambda_1^+ + \lambda_2^+).

Without loss of generality, we assume $\lambda_1^- > \lambda_2^-$, which implies $\lambda_1(x) > \lambda_2(x)$ for any $x \in \hat{I}_L$. $\lambda_1^+ \leq 1 - \frac{1}{q_{N+L}}$ and $\lambda_1^- \leq 1 - \frac{1}{q_{N+L}}$. 
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It follows from O4 that

$$\lambda_1(x) - \lambda_2(x) > dist\{[\lambda_1^-, \lambda_1^+], [\lambda_2^-, \lambda_2^+]\}$$

$$> \frac{1}{q_{N+L}^{600q}}(\lambda_1^+ + \lambda_2^-)$$

$$= \frac{1}{q_{N+L}^{600q}}(1 + \frac{1}{q_{N+L}^{1000q}})(\lambda_1(0) + \lambda_2(0))$$

$$> \frac{1}{q_{N+L}^{600q}}(1 + \frac{1}{q_{N+L}^{1000q}})(1 - \frac{1}{q_{N+L}^{1000q}})(\lambda_1(x) + \lambda_2(x)) > \frac{1}{q_{N+L}^{600q}}(\lambda_1(x) + \lambda_2(x)).$$

Hence,

$$\left|\frac{1}{\lambda_2(x)} - \frac{1}{\lambda_1(x)}\right|$$

$$\geq \left|\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right|$$

$$= \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \left(1 \frac{\left(\frac{\lambda_2^+}{\lambda_1^-}\right)^4 - \left(\frac{\lambda_2^-}{\lambda_1^+}\right)^4}{1 - \left(\frac{\lambda_2^-}{\lambda_1^+}\right)^4}\right) = \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \left(1 \frac{\left(\frac{\lambda_2^+}{\lambda_1^-}\right)^4 - \left(\frac{\lambda_2^-}{\lambda_1^+}\right)^4}{1 - \left(\frac{\lambda_2^-}{\lambda_1^+}\right)^4}\right)$$

$$\geq \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \left|\left(1 \frac{\left(\frac{1}{q_{N+L}^{1000q}}\right) - 1 \frac{1}{q_{N+L}^{1000q}}}{1 - \left(\frac{\lambda_2^-}{\lambda_1^+}\right)^4}\right) \geq \left(1 - \frac{1}{q_{N+L}^{1000q}}\right)\right| \frac{1}{\lambda_2} - \frac{1}{\lambda_1}$$

(84)

Similarly, we have

$$\left|\frac{1}{\lambda_2(x)} - \frac{1}{\lambda_1(x)}\right|$$

$$\leq \left|\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right|(1 + \frac{1}{q_{N+L}^{1000q}})$$

(85)

With the help of (84) and (85), we have

$$\frac{1}{\lambda_2(x)} - \frac{1}{\lambda_1(x)} \leq \int_0^{\frac{1}{q_{N+L}^{1000q}}} M_2 \sqrt{C_1^2 x^2 + (1 - \frac{1}{q_{N+L}^{1000q}})^2\left(\frac{1}{(\lambda_1^-)^4} - \frac{1}{(\lambda_2^-)^4}\right)^2 C_2^2 x^{-2} + \frac{2}{(\lambda_1^-)^4} + \frac{2}{(\lambda_2^-)^4} dx} + o(\lambda_m^{-3})$$

$$- \int_0^{\frac{1}{q_{N+L}^{1000q}}} M_1 \sqrt{C_1^2 x^2 + (1 + \frac{1}{q_{N+L}^{1000q}})^2\left(\frac{1}{(\lambda_1^+)^4} - \frac{1}{(\lambda_2^+)^4}\right)^2 C_2^2 x^{-2} + \frac{2}{(\lambda_1^+)^4} + \frac{2}{(\lambda_2^+)^4} dx}$$

By a direct calculation, we have
\[
\left( \frac{M_2}{2C_1} \log S_L(x, C_1, C_2, \lambda_1^+, \lambda_2^+) - \frac{M_1}{2C_2} \log S_L(x, C_2, C_1, \lambda_1^-, \lambda_2^-) \right) \left| \begin{array}{c} \frac{1}{q_{N+L}} \\ 0 \end{array} \right|
\]

where

\[
S_L(x, C_1, C_2, \lambda_1^+, \lambda_2^+) = \log \left( \left( \frac{\lambda_1^+}{\lambda_2^+} \right)^{\lambda_2^+} \right) \frac{C_1^4 x^4 + 2C_2^2 (\lambda_1^+)^4 (\lambda_2^+)^4 + 2(\lambda_1^+)^4}{2 \lambda_1^+ (\lambda_2^+)^4} x^2 + \left( 1 - \frac{1}{q_{N+L}^2} \right)^2 ((\lambda_1^+)^4 - (\lambda_2^+)^4)^2 \right) + (\lambda_2^+)^4 (\lambda_2^+)^4 C_2^2 x^2 + (\lambda_1^+)^4 + (\lambda_2^4) \right|;
\]

By a direct calculation, we have

\[
\max \left\{ \frac{S_L(\delta L, C_1, C_2, \lambda_1^+, \lambda_2^+)}{S_L(0, C_1, C_2, \lambda_1^+, \lambda_2^+)} \right\} \leq C \log((\lambda_1^+)^4 (\lambda_2^+)^4) \leq C q_{N+L}^4 \log \lambda.
\]

It holds from direct calculation that the following two facts holds:

**F1:** For \( x_1, x_2, y_1, y_2 \in \mathbb{R}^+ \) satisfying \( \frac{x_1}{x_2} \geq \frac{y_1}{y_2} \) and \( y_2 \geq x_2 \), the following inequality holds true:

\[
\frac{x_1 + y_1}{x_2 + y_2} \leq \frac{1}{2} \frac{x_1 + y_1}{x_2 + y_2}.
\]

**F2:** For \( a_1, a_2, b_1, b_2 \in \mathbb{R}^+ \) satisfying \( a_2 \neq b_2 \) and \( \frac{a_1}{a_2} = \frac{b_1}{b_2} \), the following inequality holds true:

\[
\frac{a_1 - b_1}{a_2 - b_2} = \frac{a_1 + a_2}{b_1 + b_2}.
\]

Similar to CASE1, it follows from the fact \( \lambda_m \gg \delta_q \) and O4 we have

\[
\left| \log \frac{S_L(\delta L, C_1, C_2, \lambda_1^+, \lambda_2^+)}{S_L(0, C_1, C_2, \lambda_1^+, \lambda_2^+)} \right| - \left| \log \frac{S_L(\delta L, C_2, C_1, \lambda_1^-, \lambda_2^-)}{S_L(0, C_2, C_1, \lambda_1^-, \lambda_2^-)} \right| 
\]
\[
= \log \left( \frac{\lambda_1^+ + \lambda_2^+}{2} \right)^{\lambda_2^+} \frac{C_1(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_1^+)^4 - (\lambda_2^+)^4}{(\lambda_1^+)^4 + (\lambda_2^+)^4} \right)}{C_2(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_2^+)^4 - (\lambda_2^+)^4}{(\lambda_2^-)^4 + (\lambda_2^+)^4} \right)} \right) + \log \left( \frac{\lambda_1^- + \lambda_2^-}{2} \right)^{\lambda_2^-} \frac{C_1(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_1^-)^4 - (\lambda_2^-)^4}{(\lambda_1^-)^4 + (\lambda_2^-)^4} \right)}{C_2(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_2^-)^4 - (\lambda_2^-)^4}{(\lambda_2^-)^4 + (\lambda_2^-)^4} \right)} \right)
\]
\[
\leq \log \left( \frac{\lambda_1^+ + \lambda_2^+}{2} \right)^{\lambda_2^+} \frac{C_1(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_1^+)^4 - (\lambda_2^+)^4}{(\lambda_1^+)^4 + (\lambda_2^+)^4} \right)}{C_2(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_2^+)^4 - (\lambda_2^-)^4}{(\lambda_2^-)^4 + (\lambda_2^-)^4} \right)} \right) + \log \left( \frac{\lambda_1^- + \lambda_2^-}{2} \right)^{\lambda_2^-} \frac{C_1(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_1^-)^4 - (\lambda_2^-)^4}{(\lambda_1^-)^4 + (\lambda_2^-)^4} \right)}{C_2(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_2^-)^4 - (\lambda_2^-)^4}{(\lambda_2^-)^4 + (\lambda_2^-)^4} \right)} \right)
\]
\[
\leq \log \left( \frac{\lambda_1^+ + \lambda_2^+}{2} \right)^{\lambda_2^+} \frac{C_1(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_1^+)^4 - (\lambda_2^+)^4}{(\lambda_1^+)^4 + (\lambda_2^+)^4} \right)}{C_2(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_2^+)^4 - (\lambda_2^-)^4}{(\lambda_2^-)^4 + (\lambda_2^-)^4} \right)} \right) \]n
\[
\leq \log \left( \frac{\lambda_1^- + \lambda_2^-}{2} \right)^{\lambda_2^-} \frac{C_1(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_1^-)^4 - (\lambda_2^-)^4}{(\lambda_1^-)^4 + (\lambda_2^-)^4} \right)}{C_2(1 + \frac{1}{q_{N+L}}) \left( \frac{(\lambda_2^-)^4 - (\lambda_2^-)^4}{(\lambda_2^-)^4 + (\lambda_2^-)^4} \right)} \right)
\]
\[
\leq \left( 1 + \frac{1}{q_{N+L}} \right)^8 \left| \frac{C_1 - C_2}{C_2} \right| + o(\lambda_m^{-2.5}) + \frac{1}{C_1 q_{N+L}} \leq \frac{2}{c_1 q_{N+L}^2},
\]

where the third inequality is due to F1 and F2.
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Therefore, it follows from [81], [86] and [87] that

\begin{equation}
(81) \quad \frac{M_2}{C_1} - \frac{M_1}{C_2} \max \left\{ \log \frac{S_L(\delta_{L, C_1, C_2, \lambda_1^+, \lambda_2^+})}{S_L(0, C_1, C_2, \lambda_1^-, \lambda_2^-)} , \log \frac{S_L(\delta_{L, C_1, C_2, \lambda_1^-, \lambda_2^-})}{S_L(0, C_1, C_2, \lambda_1^+, \lambda_2^+)} \right\} \\
+ \max\{\frac{M_2}{C_1}, \frac{M_1}{C_2}\} \log \frac{S_L(\delta_{L, C_1, C_2, \lambda_1^+, \lambda_2^+})}{S_L(0, C_1, C_2, \lambda_1^-, \lambda_2^-)} - \log \frac{S_L(\delta_{L, C_1, C_2, \lambda_1^-, \lambda_2^-})}{S_L(0, C_1, C_2, \lambda_1^+, \lambda_2^+)} \right\} + o(\lambda_m^{-3})
\end{equation}

\begin{equation}
\leq C \left| \frac{M_2}{C_1} - \frac{M_1}{C_2} \right| q_{N+L}^4 \log \lambda + \frac{M_1}{c_1} \frac{1}{q_{N+L}^4} q_{N+L}^4 \\
\leq C \left( \frac{M_2}{C_1} - \frac{M_1}{C_2} \right)^2 + \frac{C}{q_{N+L}^4} q_{N+L}^4 \\
\leq C \cdot \log q_{N+L} \\
\leq \frac{C}{q_{N+L}^{N+L-1}}.
\end{equation}

In summary,

\begin{equation}
\int_{I_L} \frac{1}{\|A_i\|} \frac{\partial \|A_i\|}{\partial t} \frac{\partial \theta_i}{\partial t} \, dx \leq \frac{C}{q_{N+L}^{N+L-1}} q_{N+L}^4 \log q_{N+L}^{N+L-1}.
\end{equation}

\( \square \)

**Proof of Lemma [42]**

For \( x \in \Omega_{L,2} \), assume \( l \) be the unique time such that \( T^l x \in \hat{I}_{L+1} \). Let \( \lambda_1, \lambda_2 \) and \( \theta_1 \) be defined as in (71). Recall that

\begin{equation}
(88) \quad \frac{1}{\|A_n\|} \frac{\partial \|A_n\|}{\partial t} = \frac{1}{\|A_n\|} \frac{\partial \|A_n\|}{\partial \lambda_1} \frac{\partial \lambda_1}{\partial t} + \frac{1}{\|A_n\|} \frac{\partial \|A_n\|}{\partial \lambda_2} \frac{\partial \lambda_2}{\partial t} + \frac{1}{\|A_n\|} \frac{\partial \|A_n\|}{\partial \theta_1} \frac{\partial \theta_1}{\partial t}.
\end{equation}

With the help of Lemma [43] and (76), we have

\begin{equation}
\left| \int_{\Omega_{L,2}} \frac{1}{\|A_n\|} \frac{\partial \|A_n\|}{\partial \lambda_2} \frac{\partial \lambda_2}{\partial t} \, dx \right| \\
\leq \frac{1}{n} \int_{\Omega_{L,2}} \frac{1}{\|A_n\|} \frac{\partial \lambda_2}{\partial t} \, dx \\
\leq \frac{1}{n} \text{Leb}(\Omega_{L,2}) \cdot C \phi \|A_n\|^{\gamma} \\
\leq \frac{1}{n} \text{Leb}(\Omega_{L,2}) \cdot C \|A_n\|^{\gamma}.
\end{equation}

The second inequality is followed by Lemma [27] as \( l = r_{L+1}^+ \) and \( \|A_n\| = \lambda_2 \). We have a similar estimate on \( \int_{\Omega_{L,2}} \frac{1}{\|A_n\|} \frac{\partial \|A_n\|}{\partial \theta_1} \frac{\partial \theta_1}{\partial t} \, dx \).

From Lemma [45] we have

\begin{equation}
\frac{1}{n} \int_{\Omega_{L,2}} \frac{1}{\|A_n\|} \frac{\partial \|A_n\|}{\partial \theta_1} \frac{\partial \theta_1}{\partial t} \, dx \leq \frac{C}{q_{N+L}^{N+L+1}} \log q_{N+L}^{N+L+1}.
\end{equation}

In conclusion, we obtain (77).
For \( x \in \Omega_{L,1} \), let \( n = j_s \), \( 0 = j_0 \) and \( A_n(x) = A_{j_s-j_s-1}(x)A_{j_s-j_s-2}(x) \cdots A_{j_2}(x) \). Note that \( j_i - j_{i-1} \geq q_{N+L} \) with \( 2 \leq i \leq s \).

Let \( \lambda_2 = \| A_{j_2}(x) \|, \lambda_1 = A_{j_s-j_s-2}(x) \) for \( 4 \leq i \leq s \) and \( \lambda_s = A_{n-j_s-1}(x) \). Define \( \theta_i = \theta(A_{j_s-j_s-1}, A_{n-j_s}), 2 \leq i \leq s - 1 \) and \( \hat{\theta}_n(x) = \Pi_{k=j_s+1}^n A(T^k x) \).

Then by repeatedly applying (88), we have

\[
(90) \quad \frac{1}{\| A_n(x,t) \|} \frac{\partial \| A_n(x,t) \|}{\partial t} = \sum_{i=2}^{s-1} \frac{1}{\lambda_i} \frac{\partial \lambda_i}{\partial t} + \sum_{i=2}^{s} \frac{1}{\lambda_i} \frac{d \lambda_i}{d t}.
\]

The following lemma shows that outside of \( \Omega_{L,2} \), \( \frac{1}{\| A_n \|} \frac{\partial \| A_n \|}{\partial t} \) can roughly be decomposed into the sum of \( \frac{1}{\lambda_i} \frac{d \lambda_i}{d t} \).

**Lemma 46.** It holds that

\[
(91) \quad \frac{1}{n} \int_{\Omega_{L,1}} F_L \, dx \leq \lambda^{-\frac{100}{1000} q_{N+L}} + \frac{1}{100} q_{N+L} - \frac{1}{100} q_{N+L-1}.
\]

**Proof.** Assume \( x \in \Omega_{L,1} \). We write \( A_n = \hat{A}_{n-j_s} \cdot A_{j_2} \) and define \( \lambda_1 = A_{j_2}, \lambda_2 = \hat{A}_{n-j_s} \) and \( \theta = \theta(A_{j_2}, \hat{A}_{n-j_s}) \) from the definition of \( \Omega_{L,1} \), we have

\[
|\theta_i(x,t)| \geq \frac{1}{q_{10000} q_{N+L}} \geq \lambda^{-\frac{100}{1000} q_{N+L}} \geq \left( \min \{ \| \hat{A}_{n-j_s} \|, \| A_{j_2} \| \} \right)^{-\frac{100}{1000}}.
\]

Thus (93) holds true with \( \eta = 1/100 \). Then from Lemma 43 we have

\[
(92) \quad \left| \frac{1}{\| A_n \|} \frac{\partial \| A_n \|}{\partial t} - \frac{1}{\| A_n \|} \frac{\partial \| A_{j_2} \|}{\partial t} \right| \leq \lambda^{-\frac{100}{1000} q_{N+L}}.
\]

With the help of Lemma 43 and a direct computation, we have

\[
(93) \quad \left| \frac{1}{\| A_n \|} \frac{\partial \| A_n \|}{\partial t} - \frac{1}{\| A_{j_2} \|} \frac{\partial \| A_{j_2} \|}{\partial t} \right| \leq \lambda^{-\frac{100}{1000} q_{N+L}}.
\]

Subsequently, we have

\[
(94) \quad \left| \frac{1}{\| A_n \|} \frac{\partial \| A_n \|}{\partial t} - \frac{1}{\| A_{j_2} \|} \frac{\partial \| A_{j_2} \|}{\partial t} \right| \leq \lambda^{-\frac{100}{1000} q_{N+L}}.
\]

Thus to consider \( \int_{T^{-1}(J_{N+L} \cup J_{N+L})} \frac{\partial \| A_n \|}{\partial t} \, dx \), it is sufficient to consider

\[
(95) \quad \int_{T^{-1}(J_{N+L} \cup J_{N+L})} \frac{\partial \| A_{j_2} \|}{\partial t} \, dx.
\]
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with $x \in \Omega_{L,1}$ or equivalently $T^l x \in I_{L+1} \setminus I_{L+1}$ for some $0 \leq l \leq n$.

Similarly, we write $A_{n-j_2} = A_{n-j_2} \cdot A_{j_3-j_2}$ and define $\lambda_1 = A_{j_3-j_2}$ and $\lambda_2 = \hat{A}_{n-j_3}$ and $\theta_1 = \theta(A_{j_3-j_2}, \hat{A}_{n-j_3})$. Then we obtain

$$\frac{1}{\|A_{n-j_2}\|} \frac{\partial \|A_{n-j_2}\|}{\partial \theta} \approx \frac{1}{\|A_{j_3-j_2}\|} \frac{\partial \|A_{j_3-j_2}\|}{\partial \lambda_1},$$

$$\frac{1}{\|A_{n-j_3}\|} \frac{\partial \|A_{n-j_3}\|}{\partial \theta} \approx \frac{1}{\|A_{j_3-j_3}\|} \frac{\partial \|A_{j_3-j_3}\|}{\partial \lambda_2},$$

$$\left| \frac{1}{\|A_{n-j_2}\|} \frac{\partial \|A_{n-j_2}\|}{\partial \theta_1} - \frac{1}{\|A_{n-j_3}\|} \frac{\partial \|A_{n-j_3}\|}{\partial \theta} \right| \leq \lambda^{-\frac{2}{100}qN+L-1}.$$ 

The last inequality implies \((97)\) is applicable to estimate $\int_{T^{-1}(I_{N+L})} \frac{1}{\|A_{n-j_2}\|} \frac{\partial \|A_{n-j_2}\|}{\partial \theta} dx$.

Inductively, we can prove that

$$\frac{1}{\|A_n\|} \frac{\partial \|A_n\|}{\partial t} = \sum_{i=1}^s \frac{1}{\|A_{j_{i+1}-j_i}\|} \frac{\partial \|A_{j_{i+1}-j_i}\|}{\partial t} + F_L$$

with $F_L$ satisfying

$$\left| \int_{\Omega_{L,i}} F_L dx \right| \leq n\lambda^{-\frac{2}{100}qN+L-1} + \sum_{0 \leq i \leq n} \int_{T^{-1}(I_{N+L})} \frac{1}{\|A_{N+L}^{+}+r_{N+L}^{+}(x)\|} \frac{\partial \|A_{N+L}^{+}+r_{N+L}^{+}(x)\|}{\partial t} \left( \frac{\partial \|A_{N+L}^{+}+r_{N+L}^{+}(x)\|}{\partial t} \right) dx,$$

$$\leq n(\lambda^{-\frac{2}{100}qN+L-1} + \frac{1}{100}qN+L-1 - \frac{1}{100}qN+L-2).$$

The first inequality comes from (7) and (93) and the last one comes from (97).

In Lemma 46 $A_n(x)$ is decomposed into a product of matrices according to the returning times of $x$ with respect to $I_L$. In the following, we will repeat the procedure of the decomposition of the product of matrices in a similar method until the original matrix $A_n$ is decomposed into matrices with respect to the returning times of $x$ with respect to $I_{L_0}$.

Lemma 47. For any $L_0 \leq i \leq L$, the following holds true. Let $j_{i,k}$, $1 \leq k \leq s_i$ be all the returning times between $r_i$ and $r_{i+1}$ for $x$ under $T$ with respect to $I_i$. Define $\lambda_{i,k} = \|A_{j_{i,k+1}-j_{i,k}}(T^{j_{i,k+1}}(x))\|$. Then we have

$$\frac{1}{\|A_n(x)\|} \frac{\partial \|A_n(x)\|}{\partial t} = \sum_{k=1}^{s_i} \frac{1}{\lambda_{i,k}} \frac{\partial \lambda_{i,k}}{\partial t} + F_i$$

where $F_i$ satisfies

$$\frac{1}{n} \left| \int_{\Omega_{L,i}} F_i dx \right| \leq \frac{1}{100}qN+L-1 - \frac{1}{100}qN+i + \sum_{k=i}^{L} \lambda^{-\frac{2}{100}qN+i-1}.$$

Proof. The case $i = L$ is just Lemma 46. Inductively, assume the conclusion holds true for $i, \ldots, L$. We will prove (95) holds true for the case $i = 1$.

Consider $\frac{1}{\lambda_{i,k}} \frac{\partial \lambda_{i,k}}{\partial t}$. Let $0 < j_{i,k,0} < j_{i,k,1} < j_{i,k,2} < \cdots < j_{i,k,s(i,k)} = j_{i,k} - j_{i,k-1}$ be all the returning times of $T^{j_{i,k-1}}(x)$ with respect to $I_{i-1}$ under $T$. Define $\lambda_{i,k,u} = \lambda_{i,k} - 1 - j_{i,k-1}$.
∥A_{j_{-1,k}+j_{-1,k}+1}(T^j_{j_{-1,k}+1} x)∥. 1 ≤ u ≤ s(i, k) − 1. Note that $T^j_{j_{-1,k}}(x) ∈ I_{i_{-1}} \setminus I_i$ for $1 ≤ u ≤ s(i, k) − 1$. Thus following the argument in Lemma 46, it holds that
\[
\frac{1}{\lambda_{i,k}} \frac{∂λ_{i,k}}{∂t} = \sum_{u=1}^{s(i,k)-1} \frac{1}{\lambda_{i,k,u}} \frac{∂λ_{i,k,u}}{∂t} + F_{i,k},
\]
which together with the inductive assumption implies that
\[
\frac{1}{n} \int_{Ω_{i-1}} \frac{∂∥A_n(x)∥}{∂t} dx = \sum_{k=1}^{s_{i-1}} \left( \frac{1}{λ_{i-1,k}} \frac{∂λ_{i-1,k}}{∂t} + F_{i,k} \right) + F_i := \sum_{k=1}^{s_{i-1}} \frac{1}{λ_{i-1,k}} \frac{∂λ_{i-1,k}}{∂t} + \tilde{F}_i + F_i,
\]
where $s_{i-1}$ and $λ_{i-1,k}$ are defined similar as above; moreover, we have
\[
\frac{1}{n} \int_{Ω_{i-1}} \tilde{F}_i dx ≤ \frac{1}{n} \int_{Ω_{i-1}} \left| \frac{1}{λ_{i-1,k}} \frac{∂λ_{i-1,k}}{∂t} + F_{i,k} \right| dx + \frac{1}{100} q_{N+L-1} - \frac{1}{100} q_{N+i} ≤ \frac{1}{100} q_{N+i}.
\]
Let $F_{i-1} = \tilde{F}_i + F_i$. Thus from (96), we obtain
\[
\frac{1}{n} \int_{Ω_{i-1}} F_{i-1} dx ≤ \int_{Ω_{i-1} \setminus I_{i-1}} \left| \frac{1}{λ_{i-1,k}} \frac{∂λ_{i-1,k}}{∂t} + F_{i,k} \right| dx + \frac{1}{100} q_{N+i} - \frac{1}{100} q_{N+i-1} \leq \frac{1}{100} q_{N+i} - \frac{1}{100} q_{N+i-1}.
\]
In the opposite direction, we have the same conclusion and similarly we replace $s$ with $u$.

**The proof of Lemma 46**

Clearly, for any given $L_0 ≤ j ≤ L - 1$, we have $Ω_{j,1} ≤ Ω_{j+1,1}$ and $Ω_{j,2} ≥ Ω_{j+1,2}$. And $T$ can be divided into $Ω_{L_0,1}, Ω_{L_0,1} ∩ Ω_{L_0,1+1}, Ω_{L_0,1} ∩ Ω_{L_0,2+1}, \ldots, Ω_{L-1,1} ∩ Ω_{L,1}$ and $Ω_{L,2}$.

On one hand, with the help of Lemma 27, it is clear that
\[
\int_{Ω_{L_0,1}} \left| \frac{1}{∥A_n(x)∥} \frac{∂∥A_n(x)∥}{∂t} \right| dx = \sum_{k=1}^{L_0} \frac{1}{λ_{L_0,k}} \frac{∂λ_{L_0,k}}{∂t} + F_{L_0},
\]
where
\[
\frac{1}{n} \int_{Ω_{L,1}} F_{L_0} dx ≤ \int_{Ω_{L_0,1}} \left| \frac{1}{∥A_n(x)∥} \frac{∂∥A_n(x)∥}{∂t} \right| dx + \frac{1}{100} q_{N+i} - \frac{1}{100} q_{N+i-1}.
\]
Lemma 48 shows \( \left| \frac{1}{\lambda_{t_0,k}} \frac{\partial \lambda_{t_0,k}}{\partial t} \right| \leq q_{N+L_0-1}^{300\tau^2} \). We immediately have
\[
\left| \frac{1}{n} \int_{\Omega_{t_0,1}} \frac{1}{\|A_n(x)\|} \frac{\partial \|A_n(x)\|}{\partial t} \, dx \right| \leq nC(L_0).
\]

For \( \Omega_{j+1,2} \cap \Omega_{j+2,1} \) with \( L - 1 \geq j \geq L_0 \) (\( \Omega_{L+1,1} \triangleq \emptyset \)), by directly applying Lemma 27 we have
\[
\frac{1}{n} \int_{\Omega_{j+1,2} \cap \Omega_{j+2,1}} \frac{1}{\|A_n(x)\|} \frac{\partial \|A_n(x)\|}{\partial t} \, dx \leq 2q_{N+j+1}^{500C\tau} + C\log q_{N+j+2} q_{N+j+2}.
\]

In summary, we obtain
\[
\frac{1}{n} \int_{\Omega_{j+1,2} \cap \Omega_{j+2,1}} \frac{1}{\|A_n(x)\|} \frac{\partial \|A_n(x)\|}{\partial t} \, dx \leq \sum_{j=L_0}^{L} \left[ 2q_{N+j+1}^{500C\tau} + C\log q_{N+j+2} q_{N+j+2} \right] + C(L_0),
\]

which finishes the proof.

All of the above work provide two things, which are as follows:

1. There exists a dense subset \( T := EP \) of \( \Sigma_{\alpha,\lambda} \) with a zero measure such that for any \( E \in T \), \( L(\cdot) \) is exactly \( \frac{1}{2} \) Hölder continuous at \( E \).

2. There exists a subset \( F := FR \) of \( \Sigma_{\alpha,\lambda} \) with a full measure such that for any \( E' \in F \), \( L(\cdot) \) is Lipschitz at \( E' \).

8. Regularity of LE for other \( E \)

To finish the proof of Theorem 1. We need to show \( \frac{1}{2} \)-Hölder continuity for all \( E \in \Sigma_{\alpha,\lambda} \) and find some point \( E'' \in \Sigma_{\alpha,\lambda} \) such that \( L(\cdot) \) is almost \( \beta \)-Hölder (see (100)) continuous at \( E'' \) for any fixed \( \frac{1}{2} < \beta < 1 \).

**Definition.** For some fixed \( \gamma \in (0,1) \), \( x_0 \in \mathbb{R} \) and \( f \in C^0(\mathbb{R}) \), we say
\[
f \text{ is almost } \gamma - \text{-Hölder continuous on } x_0,
\]

if \( f \) is at least \( (\gamma - \delta) \)-Hölder continuous and at most \( (\gamma + \delta) \)-Hölder continuous on \( x_0 \) for any \( \delta > 0 \).

We say
\[
f \text{ is } \gamma^+ - \text{Hölder continuous on } x_0,
\]

if \( f \) is \( (\gamma + \delta) \)-Hölder continuous on \( x_0 \) for any \( \delta > 0 \).

We say
\[
f \text{ is } \gamma^- - \text{Hölder continuous on } x_0,
\]

if \( f \) is \( (\gamma - \delta) \)-Hölder continuous on \( x_0 \) for any \( \delta > 0 \).

We have the following lemma.

**Lemma 48.** For any \( t \in \Sigma \) and \( \frac{1}{2} < \beta(t) < 1 \), \( L(\cdot) \) is almost \( \beta(t) \)-Hölder continuous at \( t \); For any \( t \notin EP \) and \( \beta(t) = \frac{1}{2} \), \( L(\cdot) \) is exactly \( \frac{1}{2}^+ \)-Hölder continuous at \( t \).

**Remark 49.** The proof is divided into three parts. At the first part, we prove the lower bound of the Hölder exponent of \( L(\cdot) \), which essentially follows from the previous proof of Lipschitz continuity and \( \frac{1}{2} \)-Hölder continuity. The second part is to prove the upper bound of the Hölder exponent of \( L(\cdot) \) and we obtain it by
selecting proper sequence \( t_n \to t \) such that \(|L(t) - L(t_n)|\) being too bad to overstep the upper bound we fixed. This is essentially due to the density of \( EP \), where \( L(\cdot) \) is exactly \( \frac{1}{2} \)-Hölder continuous. The third part, also by the density of \( EP \) and the decreasing of the exponent for \( \frac{1}{2} \)-Hölder continuity, we obtain the exactly \( \frac{1}{2} \)-Hölder continuity.

**The proof of at least** \( \beta(t) - \delta \).

If \( t \in I(\beta(t)) \), then there exists some \( N \), such that

\[
|c_{n,1}(t) - c_{n,2}(t) - k_n(t)\alpha| \geq \|A_{k_n}\|(-2(\beta(t))^{2\delta+1})^{-1},
\]

for \( n \geq N \).

Without loss of generality, we assume \( g_{N_1} \) is of type III. Due to \( \text{(107)} \) and the definition of \( \beta(t) \), there exist infinite \( g_i \) s such that \( g_i \) is of type III. Assume that \( \{i_j\}_{j \geq 1} \) are all the time points such that \( g_{i_j} \) is of type III and \( g_{i_j+1} \) is of type I, and \( \{k_j\}_{j \geq 1} \) are all the time points such that \( g_{k_j} \) is of type I and \( g_{k_j+1} \) is of type III. Thus \( g_{i_1}, g_{i_2}, \ldots, g_{i_j}, g_{i_j+1}, g_{k_1}, g_{k_2}, \ldots, g_{k_j} \) are of type I and \( g_{k_j+1} \) is of type III for \( j \geq 1 \). Clearly,

\[
N_1 - 1 = k_1 < i_1 < k_2 < \cdots < i_n < k_{n+1} < \cdots.
\]

Then we have the following lemma:

**Lemma 50.** Assume for some \( t \) and \( l_j - 1 \leq n \leq k_j+1 - 1 \) for some \( j \geq 1 \), \( g_n \) is of type I. Then for any fixed \( \varepsilon > 0 \) and \( t' \in \left[ t - q_{N+n}^{-1000^{-2}}, t - q_{N+n}^{-1000^{-2}} \right] \cup \left[ t + q_{N+n}^{-1000^{-2}}, t + q_{N+n}^{-1000^{-2}} \right], \)

we have

\[
|L(t) - L(t')| \leq |t - t'|^{1-\varepsilon}
\]

for some positive constant \( \varepsilon \ll 1 \).

**Proof.** The proof is similar to the case of Lipschitz continuity. In fact, it follows from Lemmas 11 we have \( |L'_q(t)|, |L'_{q^2}(t)| \leq C(i_j) \leq q_{N+n+1}^{-1000^{2}} \)

for \( t' \in \left[ t - q_{N+n+1}^{-1000^{-2}}, t - q_{N+n+1}^{-1000^{-2}} \right] \cup \left[ t + q_{N+n+1}^{-1000^{-2}}, t + q_{N+n+1}^{-1000^{-2}} \right]. \)

Note that we assume \( i_j \ll k_j \). Then it holds that

\[
|L(t) - L(t')| \\
\leq 2 \left| L_{q_{N+n+1}} - L_{q_{N+n+1}}(t') \right| + \left| L_{2q_{N+n+1}} - L_{2q_{N+n+1}}(t') \right| + \lambda^{-c_{q_{N+n+1}}}
\]

\[
\leq 3\varepsilon |i_j| |t - t'| + \lambda^{-c_{q_{N+n+1}}}
\]

\[
\leq |t - t'|^{1-\varepsilon} + \lambda^{-c_{q_{N+n+1}}}
\]

\[
\leq 2|t - t'|^{1-\varepsilon}.
\]

The inequality second to the last from the fact that \( |t - t'| \leq q_{N+n+1}^{-1000^{-2}} \leq q_{N+n+1}^{-1000^{-2}} \leq q_{N+n+1}^{-1000^{-2}} \). And the last one follows from \( \lambda^{-c_{q_{N+n+1}}} \ll q_{N+n}^{-1000^{-2}} \leq |t - t'|. \)

**Remark 51.** We rearrange the sequence of \( N_1 - 1 = k_1 < i_1 < k_2 < \cdots < i_n < k_{n+1} < \cdots \) into

\[
N_1 - 1 = k_1 < i_1 < l_1 < k_2 < l_1 < \cdots < i_n < l_n < k_{n+1} < l_2 < \cdots
\]

such that \( l_j \gg i_j, l'_j \gg k_j+1 \) and for any \( n \in [i_j, l_j] \) or \( n \in [k_j+1, l'_j] \), \( g_n \) can be considered either as type I or as type III. Moreover, for any \( n \in [l_j, k_{j+1}] \) or \( n \in [l'_j, i_{j+1}] \), \( g_n \) can be considered as type I or as type III, respectively. We will
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apply the argument on Lipschitz continuity on $[l_j, l_j']$ with $l_j \gg i_j$ and the one on $\frac{1}{2}$ Hölder continuity on $[l_j', l_{j+1}]$ with $l_j' \gg k_{j+1}$.

If $k_j \leq n \leq i_j - 2$ for some $j \geq 1$, then we claim that

$$|L_{q_{N+n+1}^k}(t) - L_{q_{N+n+1}^k}(t')| \leq C |t - t'|^{\beta(t) - \delta}.$$  

Actually, as $g_{n+2}$ is of type III. There exists at most one time point $0 \leq l \leq q_{N+n+1}^4$ such that $T^t(x) \in I_{n+2}$. Consider

$$A_{q_{N+n+1}^k}(x) = A_{q_{N+n+1}^k}(T^t x) A_l(x).$$

Let $\lambda_1 = \|A_{q_{N+n+1}^k} - I_l\|$ and $\lambda_2 = A_l$. Similar as the proof of $\frac{1}{2}$-Hölder continuity on EP, it’s not difficult to see that $\|\frac{1}{\|A\|} \frac{d\|A\|}{dt}\|$ is dominated by

$$|1 - \lambda_2^{-4} (1 - \lambda_1^{-4})| \sqrt{\sum \sum (c(x - \tilde{x})^2 - \epsilon_1)^2 + (\frac{1}{x_1} - \frac{1}{x_2})^2 (c(x - \tilde{x})^2 - \epsilon_1)^2 + 2(1 + \frac{1}{x_1 x_2})(\frac{1}{x_1} + \frac{1}{x_2}) - \frac{\tilde{x}}{x_1 x_2}}.$$  

Note that $c \sim \|A_{k_j}\|$, which is followed from A.4 of the [WZ], and $\epsilon_1$ denote the distance between horizontal axis and $g_{n+2}(\epsilon_{n+2})$.

It is followed from Lemma 26, A.1 and straight calculation that

$$\int T^{t} x \in I_{n+2} \frac{d\theta_t}{dt} \frac{d\|A_{q_{N+n+1}^k}\|}{d\theta_t} \frac{1}{\|A_{q_{N+n+1}^k}\|} dx \leq \int T^{t} x \in I_{n+2} \frac{C}{\sqrt{(cx^2 - \delta_1)^2 + (cx^2 - \delta_1)^2 + 2(1 + \frac{1}{x_1})(\frac{1}{x_1} + \frac{1}{x_2})}} dx$$

$$\leq c^{-\frac{1}{2}}(1 - \lambda_1^{-4})^{-1} \leq \|A_{k_j}\|^{-\frac{1}{2}} \epsilon_1^{-\frac{1}{2}}.$$  

Thus

$$|L_{q_{N+n+1}^k}(t)| \leq \|A_{k_j}\|^{-\frac{1}{2}} \epsilon_1^{-\frac{1}{2}},$$

which implies

$$|L_{q_{N+n+1}^k}(t) - L_{q_{N+n+1}^k}(t_0)| \leq \int L_{q_{N+n+1}^k}(t') dt' \leq C \|A_{k_j}\|^{-\frac{1}{2}} \epsilon_1^{-\frac{1}{2}} \leq C \epsilon_1^{\frac{1}{2} - \frac{1}{2} \log \|A_{k_j}\|} \leq C \epsilon_1^{\frac{1}{2} - \frac{1}{2} \log \epsilon_1} \leq C \beta(t) - \delta.$$  

Here $t_0$ denotes the $t$ such that $g_{n+2}(t, \epsilon_{n+2}) = 0$ and $\epsilon_1 = |t_0 - t|$. The last inequality is followed from $t \in I(\beta(t))$. Thus, by the help of Lemma 26, similarly to the case $\frac{1}{2}$–Hölder continuity, it’s not difficult to obtain

$$|L(t) - L(t')| \leq C |t - t'|^{\beta(t) - \delta}$$

for each $t' \in (t_0, t - \eta(n)) \cup (t + \eta(n), t + q_{N+n+1}^C)$ with some $\eta(n) \leq \lambda^{-\frac{1}{2}r_{n+1}}$.

Therefore,

$$|L(t) - L(t')| \leq C |t_0 - t'|^{\beta(t) - \delta},$$

for any $t' \in B_n \triangleq (t_0, t - \eta(n)) \cup (t + \eta(n), t + q_{N+n+1}^C)$.  

(104)
Let $F_i \triangleq \bigcup_{n=1}^{+\infty} B_n \cup \{t\}$, which is a certain interval including $t$. Hence, (104) holds for each $t \in F_i$.

Then the $\beta(t) - \delta$-Hölder continuity on $t$ holds.

**The proof of at most $\beta(t) + \delta$:** By the definition of $\beta(t)$, it holds that for any given $\epsilon > 0$, there exists a sequence of $n_j$ such that

$$|c_{n_j,1}(t) - c_{n_j,2}(t) - k_{n_j}(t)\alpha| < C\|A_{k_{n_j}}\|^{(-2(\beta(t) + \epsilon)+1)^{-1}}.$$  

We define $t_0^j$ as the $x$ such that $g_{n_j}(x)$ has the only zero point on $I_{n_j}$. Then, by the proof of the first part above we know that there exist some $t_0^j \to t$ such that

$$|L_{q_{k_{n_j}}^N}(t) - L_{q_{k_{n_j}}^N}(t_0^j)| \leq C|t - t_0^j|^\gamma$$

with $\gamma \leq \beta(t) + \epsilon$, some proper constant $C > 1$ and $|t - t_0^j| = \lambda^{-c_{nk_{n_j}}} \gg \lambda^{-q_{N_{k_{n_j}}}}$.

Next, we choose a sequence of $t_j$ as follow:

$$t_j := \{x||x - t_0^j| = C^*|t - t_0^j|, \ sgn(x - t_0^j)sgn(t - t_0^j) = -1\},$$

where $C^*$ will be defined later. This means that $t_j$ and $t_0^j$ respectively locates in the two sides of $t_0^j$ and $t_j$ is in some spectrum gap. Due to the $\frac{1}{\gamma}$ Hölder continuity of LE at $t_0^j$, and the exactly $\frac{1}{\gamma}$ Hölder continuity of LE at $t_0^j$ (approximate from the semi-interval located outside the spectrum) with the Hölder exponent $\|A_{k_{n_j}}\|^{-\frac{1}{\gamma}}$, it’s clear that

(105)

$$|L_{q_{k_{n_j}}^N}(t) - L_{q_{k_{n_j}}^N}(t_0^j)| \sim C^*|t - t_0^j|^{\gamma};$$

$$|L_{q_{k_{n_j}}^N}(t_j) - L_{q_{k_{n_j}}^N}(t_0^j)| < C''|t_j - t_0^j|^{\gamma}.$$  

Therefore, by direct calculation, we have

(106)

$$\begin{align*}
(C'(C^*) - C''(C^* + 1)|t - t_j|^{\gamma} \\
(C''(C^*) - C''|t - t_0^j|^{\gamma} - C''|t - t_j|^{\gamma} \\
= C'(C^*)|t - t_0^j|^{\gamma} - C''(t - t_0^j)^\gamma \\
= C'|t_j - t_0^j|^{\gamma} - C''|t - t_j|^{\gamma} \\
\leq |L_{q_{k_{n_j}}^N}(t_j) - L_{q_{k_{n_j}}^N}(t_0^j)| - |L_{q_{k_{n_j}}^N}(t) - L_{q_{k_{n_j}}^N}(t_j)| \\
\leq L_{q_{k_{n_j}}^N}(t) - L_{q_{k_{n_j}}^N}(t_0^j) \\
\leq L_{q_{k_{n_j}}^N}(t) - L_{q_{k_{n_j}}^N}(t_0^j) + |L_{q_{k_{n_j}}^N}(t_j) - L_{q_{k_{n_j}}^N}(t_0^j)| \\
\leq C'|t_j - t_0^j|^{\gamma} + C''|t - t_0^j|^{\gamma} \\
= C'(C^*) + C''(t - t_0^j)^\gamma + C''|t - t_j|^{\gamma} \\
= (C'(C^*) + C'')(C^* + 1)|t - t_j|^{\gamma}.
\end{align*}$$

Then we denote $C_1 \triangleq (C'(C^*) - C'')(C^* + 1)$ and $C_2 \triangleq (C'(C^*) + C'')(C^* + 1)$, and we choose $C^*$ large enough such that $2C_1 > C_2$ (a.e. $C'' > \frac{10C_1}{C_0}$).
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Hence, we obtain
\[
|L(t) - L(t_j)| \geq 2 \left| L_{2q^N_{j+n_1}}(t) - L_{2q^N_{j+n_1}}(t_j) \right| - \left| L_{q^N_{j+n_1}}(t) - L_{q^N_{j+n_1}}(t_j) \right|
- \left| L_{q^N_{j+n_1}}(t) - L(t) \right| + \left| L_{2q^N_{j+n_1}}(t_j) - L_{q^N_{j+n_1}}(t_j) - L(t) \right|
\geq (2C_1 - C_2) |t - t_j| - \lambda^{-q^N_{j+n_1}}
\geq C_3 |t - t_j|^\gamma
\]
with $\gamma \leq \beta(t) + \epsilon$. Which implies the lemma.

The proof of exactly $\frac{1}{2} +$ Hölder continuity: By the definition of $\beta(t)$, for any given $\delta > 0$, there exist a sequence of $n_j \to +\infty$, such that
\[
|c_{n_j,1}(t) - c_{n_j,2}(t) - k_n(t)\alpha| \leq \left\| A_{k_n} \right\|^{-2\delta},
\]
where $g_n(t)$ is of Type III(or II), for $n_{j-1} < l < n_j$, $g_j(t)$ is Type I.

On one hand, we can find $t_j \to t$ such that
\[
|L(t) - L(t_j)| \geq c_{n_j} |t - t_j|^\frac{1}{2},
\]
where $c_{n_j} \sim \|A_n\|^{-\frac{1}{2}}, |t - t_j| \sim \|A_n\|^{-\delta}$. Hence,
\[
|L(t) - L(t_j)| \geq \|A_{n_j}\|^{-\frac{1}{2} + \delta} = |t - t_j|^\frac{1}{2} + \delta
\]
which implies $L(\cdot)$ is at most $\frac{1}{2} + \delta$-Hölder on $t$.

On the other hand, let $M_j$ satisfying $\bigcup_{j=1}^\infty M_j \ni t$ being an open interval, and for any $t_1 \in M_j$, the type of $g_j(t_1)$ are same (Type I, II, or III). Meanwhile, when $g_j$ is Type I on $M_j$, $L(\cdot)$ is $1 - \epsilon$ (for any given $\epsilon > 0$) Hölder continuous on $M_j$; when $g_j$ is of type III (or II) on $M_j$, we have
\[
|L(t) - L(t_j)| \leq c_{n_j} |t - t_j|^\frac{1}{2},
\]
\[
c_{n_j} \sim \|A_{n_j}\|^{-\frac{1}{2}}, |t - t_j| \sim \|A_{n_j}\|^{-\delta}.
\]
\[
|L(t) - L(t_j)| \leq \|A_{n_j}\|^{-\frac{1}{2} - \delta} = |t - t_j|^\frac{1}{2} + \delta
\]
which implies $L(\cdot)$ is at least $\frac{1}{2} + \delta$-Hölder on $t$. □

Proof of the $\frac{1}{2}$-Hölder continuity on $\Sigma$. It’s a obvious corollary of lemma 48. □

Now we are ready to result in the last problem of Theorem 1. For any given $\frac{1}{2} \leq t < 1$, We’d like to find the point $t_0$ such that $L(\cdot)$ is almost $\beta$-Hölder continuous. By lemma 48 it’s enough to find the point $t_0$ such that $t_0 \in I(\beta(t_0))$ with $\beta(t_0) = \beta$. Moreover, it’s sufficient to find $t_0$ satisfying
\[
t_0 \in \bigcup_{k \geq 1} \bigcap_{n \geq k} \left\{ t \left| C_1 \|A_{k_n}\|^{(-2\beta + 1)^{-1}} \leq |c_{n,1}(t) - c_{n,2}(t) - k_n(t)| \leq C_2 \|A_{k_n}\|^{(-2\beta + 1)^{-1}} \right. \right\}
\]
for some constant $C_1 \leq C_2$. We’ll find it by Closed interval set theorem.

Fix $\frac{1}{2} < \beta < 1$ and $t_0 \in EP$. It’s clear that there exists some $N_1 > 0$ such that
\[
|g_n(\tilde{c}_n)| \leq C\lambda^{-\frac{1}{2} + \epsilon}_{n_1 - 1}
\]
for any $n \geq N_1$.
Let \( t_0 \) denotes \( \{ t | g_{N_1}(c_{N_1}, t) = \| A_{k_{N_1}} \|^{-1 + \frac{\rho}{2(\lambda-1)}}, \} \), which implies

\[
|c_{N_1+1,1}(t) - c_{N_1+1,2}(t) - k_n(\alpha)| = \| A_{k_{N_1+1}} \|^{((1-2\beta)^{-1}}.
\]

Clearly, for \( n \geq N_1 \), \( g_n \) cannot be always of type III and it will change into type I at some point. Thus, we denote

\[
N_2 := \min \{ j > N_1 | g_j \text{ is of type I} \}.
\]

Consider \( g_{N_2}(\cdot) \). By the density of EP, there must exist some point \( \tilde{t}_0 \) such that \( \{ g_n \}_{n \geq N_3} \) is of type III and \( N_3 > N_2 \). Besides, we can let \( |t_0' - \tilde{t}_0| < \epsilon_0 \), for any \( \epsilon_0 > 0 \) we want to get, such that \( g_n(\tilde{t}_0') \sim g_n(\tilde{t}_0) \) for any \( n \leq N_2 \). For example, let \( \epsilon_0 := \lambda^{-\frac{q_{N_2}}{2(\lambda-1)}} \). For convenience, we suppose \( \tilde{t}_0 = \tilde{t}_0' \).

Thus, there will exist some \( k_1, k_2, \cdots, k_s \) and \( m_1, m_2, \cdots, m_s \) such that

\[
\text{for } n = k_1 + 1, \cdots, m_1, g_n \text{ is of type I};
\]

\[
\text{for } n = m_1 + 1, \cdots, k_2, g_n \text{ is of type III}.
\]

Here \( m_s + 1 = N_3 \).

Let \( \tilde{t} := \min \{ 1 \leq j \leq s | g_{m_j+1} \text{ is of type III and } |g_{m_j+1}(c_{m_j+1})| \leq \| A_{m_j+1} \|^{-1 + \frac{2q}{2(\lambda-1)}} \} \).

Now consider \( g_{m_j+1} \). Let \( \tilde{t}_1 \) denotes

\[
\tilde{t}_1 := \{ t | g_{m_j+1}(\tilde{c}_{m_j+1}, t) = \| A_{k_{m_j+1}} \|^{-1 + \frac{\rho}{2(\lambda-1)}} \}.
\]

Clearly,

\[
|\tilde{t}_1 - \tilde{t}_0| \leq \lambda^{-\frac{q_{m_j+1}}{2(\lambda-1)}} - \lambda^{-\frac{q_{N+m_j-1}}{2(\lambda-1)}}
\]

\[
\leq \lambda^{-\frac{q_{N+m_j-1}}{2(\lambda-1)}} + \lambda^{-2Cq_{N+m_j-1}}
\]

\[
\leq \| A_{k_{m_j+1}} \|^{-2(1 + \frac{\rho}{2(\lambda-1)})}
\]

\[
= o(\| A_{k_{m_j+1}} \|^{-1 + \frac{\rho}{2(\lambda-1)}}).
\]

The second inequality is followed from

\[
\lambda^{-\frac{q_{N+m_j-1}}{2(\lambda-1)}} - q_{N+k_j-1} \leq q_{N+m_j-1}.
\]

So we find some \( \tilde{t}_1 \) satisfies the following property:

\[
|g_{N_1}(c_{N_1}, \tilde{t}_1)| = \| A_{k_{N_1}} \|^{-1 + \frac{\rho}{2(\lambda-1)}} - o(\| A_{k_{m_j+1}} \|^{-1 + \frac{\rho}{2(\lambda-1)}});
\]

\[
|g_{m_j+1}(c_{m_j+1}, \tilde{t}_1)| = \| A_{k_{m_j+1}} \|^{-1 + \frac{\rho}{2(\lambda-1)}} - o(\| A_{k_{m_j+1}} \|^{-1 + \frac{\rho}{2(\lambda-1)}});
\]

\[
|g_{m_j+1}(\tilde{c}_{m_j+1}, \tilde{t}_1)| = \| A_{k_{m_j+1}} \|^{-1 + \frac{\rho}{2(\lambda-1)}}.
\]

This implies

\[
|c_{N_1+1,1}(t) - c_{N_1+1,2}(t) - k_{N_1+1}(\alpha)| = C_{N_1} \| A_{k_{N_1+1}} \|^{((1-2\beta)^{-1}}.
\]
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$$|c_{m_1+1,1}(t) - c_{m_1+1,2}(t) - k_{m_1+1}(\alpha)| > Cm_1 \| A_{k_{N_1+1}} \|^{(1-2\beta)^{-1}};$$

$$\cdots$$

$$|c_{m_j+1,1}(t) - c_{m_j+1,2}(t) - k_{m_j+1}(\alpha)| = Cm_j \| A_{k_{N_1+1}} \|^{(1-2\beta)^{-1}}.$$  

Here each constant can be chosen as some number between $\frac{1}{4}$ and $\frac{3}{4}$. Besides, for any $1 \leq i \leq s$, $g_{k_i+1}, \cdots, g_{m_i}$ are of type I since

$$\|A_{k_{N_1+1}}\|^{-1+\frac{2}{\beta}} \ll q_{N_1+1}$$

for $N_1 \leq l \leq m_j$.

Repeat the process as above, it’s not difficult to obtain a sequence $\{t_n\}_{n \geq 0}$ such that

$$\sum_{i \geq 1} |t_i - \tilde{t}_{i-1}| < \infty.$$  

We denote $\tilde{t}$ the accumulation point.

For $\{g_n(\tilde{t})\}_{n \geq N_1}$, we have the following property:

1. Type I and III alternately appear.
2. There exists a sequence $\{n_k\}_{k \geq 1}$ such that $g_{n_k}$ is of type III and

$$|g_{n_k}(c_{n_k}, \tilde{t})| = C_{n_k} \| A_{k_{n_k}} \|^{-1+\frac{2}{\beta}}.$$  

$$\frac{3}{4} < C_{n_k} < \frac{5}{4}$$

for $k \geq 1$. Thus

$$\frac{3}{4} \| A_{k_{n_k}} \|^{-C(\beta)} \leq |c_{n_k,1} - c_{n_k,2} - k_{n_k} \alpha| \leq \frac{5}{4} \| A_{k_{n_k}} \|^{(1-2\beta)^{-1}}.$$  

While, between $n_k$ and $n_{k+1}$, either $g_n$ is of type I or type III with $|g_n(c_n)| \geq C \| A_n \|^{-1+\frac{2}{\beta}}$, which implies

$$|c_{n,1} - c_{n,2} - k_n \alpha| \geq \| A_n \|^{(1-2\beta)^{-1}}.$$  

It’s easy to check that

$$\tilde{t} \in \bigcup_{k \geq 1} \bigcap_{n \geq k} \left\{ \tilde{t}; \frac{3}{4} \| A_{k_n} \|^{(1-2\beta)^{-1}} \leq |c_{n,1}(t) - c_{n,2}(t) - k_n(\alpha)| \leq \frac{5}{4} \| A_{k_n} \|^{(1-2\beta)^{-1}} \right\},$$

which completes the proof.  

9. The $\frac{1}{2}$-Hölder continuity of LE

In the previous section we deal with the local $\frac{1}{2}$-Hölder continuity of LE at each endpoints, local Lipschitz continuity of LE at each $FR$ and local $\beta$-Hölder continuity for other points with some $0.5 < \beta < 1$. In this section, we try to obtain the global $\frac{1}{2}$-Hölder continuity of LE. Actually, as one can see, the ‘worst’ condition happen only at each $EP$.

We divide the proof into two parts.

As is know to all that, a continuous function defined on a closed interval is uniformly continuous. Similarly, we have the same property for Hölder continuity, which can be seen in the following lemma.

Recall that we have already obtain the following results, and all the results are under the condition $\lambda$ is large enough.
Lemma 52. For any given $t_1 < t_2$ in the $\frac{1}{\pi} \Sigma$ with sufficiently large $\lambda$, we have $|L(t_1) - L(t_2)| \leq C'|t_1 - t_2|^{0.5}$, where $C'$ is an absolute constant.

Remark 53. In fact, as one will see the main part has already been proved in previous lemmas’ proof and the rest is only to do some simple classifications and calculations.

Proof. Clearly, we can assume $\lambda^{-\frac{q}{N+n-1}} t_2 - t_1 \leq \lambda^{-\frac{q}{N+n-1}}$, for some sufficiently large $n \in \mathbb{N}$. We denote $K := q_{N+n-1}^2$. Note that for sufficiently large $\lambda$ and $n$, the following inequalities always hold:

\[(108)\quad |I_n|^{10000} \gg \lambda^{-\frac{q}{N+n-2}}; \]
\[(109)\quad \lambda^{-cK} \gg \lambda^{-\frac{q}{N+n-1}} \gg \lambda^{-q_{N+n-1}^2}; \]

We claim that only the following two cases may occur:

**Case 1** $g_n([t_1, t_2])$ is essentially of type I;

**Case 2** $g_n([t_1, t_2])$ is essentially of type III.

The reason is as follow. For the first case, if $g_n(t_1)$ is of type I, then

\[|c_{n-1,1}(t_1) - c_{n-1,2}(t_1) - s\alpha| > |I_n|,\]

for all $s \leq q_{N+n-1}^2$. Note here $|I_n|$ can be seen as a constant independent on $t$ since the error $\delta \ll |I_n|^{C}$. Combining with

\[\frac{\partial(c_{n-1,1}(t) - c_{n-1,2}(t))}{\partial t} \leq C \leq |I_n|^{-2},\]

we obtain

\[|c_{n-1,1}(t) - c_{n-1,2}(t) - s\alpha| > |I_n|^5 > |I_n|^{200},\]

for all $s \leq q_{N+n-1}^2$ and $t \in [t_1, t_2]$. Thus the first one holds.

For the second one, we assume $g_n(t_1)$ is of type III. Clearly, there exists some $l \leq q_{N+n-1}^2$ and $s \leq n$ such that $|c_{n-1,1}(t_1) - c_{n-1,2}(t_1) - l\alpha| < \frac{1}{4}|I_n|$. Note that
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we can assume $g_{s-1}(t_1)$ is of type I (otherwise select the biggest one). Due to the previous case implies $g_{s-1}([t_1, t_2])$ is essentially of type I, we have

$$\left| \frac{\partial (c_{s-1,1}(t) - c_{s-1,2}(t))}{\partial t} \right| \leq C \leq |I_s|^{200},$$

which implies $|c_{s-1,1}(t) - c_{s-1,2}(t) - l\alpha| < \frac{3}{2} |I_n|$ and hence

$$|c_{n-1,1}(t) - c_{n-1,2}(t) - l\alpha| < \sum_{i=s}^{n-1} \|c_{i-1,1}(t) - c_{i,1}(t)\| + \sum_{i=s}^{n-1} \|c_{i-1,2}(t) - c_{i,2}(t)\| + \frac{3}{2} |I_n| \leq 2|I_n|.$$

Therefore the second case holds.

So we only need consider such two cases as above.

For the **Case 1**, $g_n([t_1, t_2])$ is essentially of type I. This case is easy. Note that

$$|L_K(t_1) - L_K(t_2)| \leq |t_1 - t_2|^{0.9} \leq |t_1 - t_2|^{0.5};$$

$$|L_{2K}(t_1) - L_{2K}(t_2)| \leq |t_1 - t_2|^{0.9} \leq |t_1 - t_2|^{0.5}.$$

Therefore

$$|L(t_1) - L(t_2)| \leq |L_K(t_1) - L_K(t_2)| + |L_{2K}(t_1) - L_{2K}(t_2)| + |L_K(t_1) + L(t_1) - 2L_{2K}(t_1)|$$

$$\leq 5|t_1 - t_2|^{0.5} + 2\lambda^{-5}\pi^K$$

$$\leq 10|t_1 - t_2|^{0.5}.$$

For the **Case 2**, $g_n([t_1, t_2])$ is essentially of type III. We denote the nearest gap $[t_1^0, t_2^0]$ such that $g_n(t_1^0)$ or $g_n(t_2^0)$ is essentially of type III. Consider the following three cases:

1. $t_1^0 < t_1, t_2^0 \geq t_2$.
   This is trivial because $[t_1, t_2] \subset [t_1^0, t_2^0]$ implies the uniformly hyperbolic case.
2. $t_1^0 \leq t_1 \leq t_2^0 \leq t_2$.
   Note we have already obtained the following three things:

   $$|L(t_1)| - L(t_2)| \leq C'|t_1^0 - t_2^0|^{0.5};$$

   $$|L(t_2)| - L(t_1)| \leq C'|t_2^0 - t_1^0|^{0.5};$$

   $$x^{0.5} + y^{0.5} \leq \sqrt{2}(x + y)^{0.5} \text{ for any } x, y \geq 0.$$

   Therefore,

   $$|L(t_1) - L(t_2)| \leq C'|t_2 - t_2^0|^{0.5} + C'|t_1^0 - t_1|^{0.5} \leq 2C'|t_1 - t_2|.$$  

3. $t_1^0 < t_2^0 \leq t_1 \leq t_2$.
   If $t_1 \geq t_1^0 + \lambda^{-q_0+n-1}$, then by the second fact in the beginning and

   $$x^{0.5} - y^{0.5} \leq (x - y)^{0.5} \text{ for any } x > y \geq 0,$$

   we have:

   $$|L(t_1) - L(t_2)| \leq C'|t_2 - t_2^0|^{0.5} - C'|t_1 - t_2^0|^{0.5} \leq C'|t_1 - t_2|.$$. 

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If \( t_1 < t_0 + \lambda^{-\frac{n}{2(n+1)}} := t' \), then we have:

\[
|L(t_1) - L(t_2)| \leq |L(t_1) - L(t')| + |L(t_2) - L(t')|
\]

\[
\leq |L(t_2) - L(t')| + |L(t_2) - L(t_1)| + |L(t_1) - L(t')|
\]

\[
\leq C'|t_2 - t'|^{0.5} + 2C'|t_1 - t_2|^{0.5} + 2C'|t' - t_2|^{0.5}
\]

\[
\leq 10C'|t_1 - t_2|,
\]

where the last inequality is due to the previous case for \( t' = t_2 \) and the fact

\[
|t_1 - t_2| \geq |t_2 - t'| \geq \lambda^{-\frac{n}{2(n+1)}} - \lambda^{-\frac{n}{2(n+1)}} \geq \lambda^{-\frac{n}{2(n+1)}} \geq |t' - t_2| \geq |t' - t_1|.
\]

The remaining cases are similarly by symmetry.

\[
\square
\]

## 10. Appendix

**A1.** Given \( \epsilon_1, \epsilon_2 \ll 1 \), \( 1 \ll q \ll \min\{\frac{1}{\epsilon_1}, \frac{1}{\epsilon_2}\} \), then we have we have

\[
\int_0^\frac{1}{\epsilon_1} \frac{x^2 - \epsilon_1^2}{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2} dx = C_1 \frac{(\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}} - \epsilon_1^{\frac{1}{2}}}{(\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}}} - C_2q \frac{(\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}} - \epsilon_2^2}{(\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}}}
\]

\[
\int_0^\frac{1}{\epsilon_1} \frac{x^2 + \epsilon_1^2}{(x^2 + \epsilon_1^2)^2 + \epsilon_2^2} dx = C_3 \frac{(\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}} + \epsilon_1^{\frac{1}{2}}}{(\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}}} - C_4q \frac{(\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}} - \epsilon_2^2}{(\epsilon_1^2 + \epsilon_2^2)^{\frac{1}{2}}}
\]

\[
\int_0^\frac{1}{\epsilon_1} \frac{\text{sgn}(x^2 - \epsilon_1^2)}{\sqrt{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2}} dx \leq C_5 \frac{1}{\epsilon_1}, \text{ when } \epsilon_1 \gg \epsilon_2;
\]

\[
\int_0^\frac{1}{\epsilon_1} \frac{1}{\sqrt{(x^2 + \epsilon_1^2)^2 + \epsilon_2^2}} dx = C_6 \frac{1}{\epsilon_1}, \text{ when } \epsilon_1 \gg \epsilon_2.
\]

for some proper constants \( C_i > 0, 1 \leq i \leq 5 \).

Moreover,

\[
\int_0^\frac{1}{\epsilon_1} \frac{x^2 - \epsilon_1^2}{(x^2 - \epsilon_1^2)^2 + \epsilon_2} dx = \begin{cases} 
-Cq, \epsilon_1 \geq \frac{1}{q} \\
O(1) \frac{\epsilon_2}{\epsilon_1^2}, \epsilon_2 \frac{1}{q} > \epsilon_1 > \epsilon_2 \\
O(1) \frac{1}{\epsilon_2}, \epsilon_1 \leq \epsilon_2
\end{cases}
\]

\[
\int_0^\frac{1}{\epsilon_1} \frac{x^2 + \epsilon_1^2}{(x^2 + \epsilon_1^2)^2 + \epsilon_2} dx = \begin{cases} 
O(1) \frac{1}{\epsilon_1}, \epsilon_1 > \epsilon_2 \\
O(1) \frac{1}{\epsilon_2}, \epsilon_1 \leq \epsilon_2
\end{cases}
\]

for some constant \( C > 0 \).

**Proof.** Note that

\[
\frac{x^2 - r}{(x^2 - px + q)(x^2 + px + q)} = \frac{(r+q)x - x}{2pq} - \frac{- (r+q)x}{2pq} + \frac{r}{x^2 - px + q} + \frac{- r}{x^2 + px + q}
\]
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Thus,
\[
\int \frac{x^2 - r}{(x^2 - px + q)(x^2 + px + q)} \, dx = \frac{r + q}{4pq} \log(x^2 - px + q) + \frac{q - r}{2q\sqrt{4q - p^2}} \arctan \frac{2x - p}{\sqrt{4q - p^2}} \\
+ \frac{-r - q}{4pq} \log(x^2 + px + q) + \frac{q - r}{2q\sqrt{4q - p^2}} \arctan \frac{2x + p}{\sqrt{4q - p^2}} + C
\]

Let
\[
p = \sqrt{2} \left( \epsilon_1^2 + \sqrt{\epsilon_1^2 + \epsilon_2^2} \right), q = \sqrt{\epsilon_1^2 + \epsilon_2^2}, r = \epsilon_1^2,
\]

by straight calculation, we have
\[ \int_0^\frac{1}{2} \frac{x^2 - e_1^2}{(x^2 - e_1^2)^2 + e_2^2} \, dx = \frac{\sqrt{2}}{8} \frac{\sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{e_1^2 + e_2^2}} \log(x^2 - \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2} x + e_1^2 + e_2^2}) \bigg|_0^{\frac{1}{2}} + \frac{\sqrt{2}}{8} \frac{\sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{e_1^2 + e_2^2}} \log(x^2 + \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2} x + e_1^2 + e_2^2}) \bigg|_0^{\frac{1}{2}} \\
- \frac{\sqrt{2}}{8} \frac{\sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{e_1^2 + e_2^2}} \log(x^2 + \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2} x + e_1^2 + e_2^2}) \bigg|_0^{\frac{1}{2}} + \frac{2x - \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{2} (\sqrt{e_1^2 + e_2^2} - e_1^2)} \bigg|_0^{\frac{1}{2}} \\
+ \frac{\sqrt{2}}{4} \frac{\sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{e_1^2 + e_2^2}} \arctan \frac{2x - \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{2} (\sqrt{e_1^2 + e_2^2} - e_1^2)} \bigg|_0^{\frac{1}{2}} \\
+ \frac{\sqrt{2}}{4} \frac{\sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{e_1^2 + e_2^2}} \arctan \frac{2x + \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{2} (\sqrt{e_1^2 + e_2^2} - e_1^2)} \bigg|_0^{\frac{1}{2}} \\
+ \arctan \frac{2x + \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{2} (\sqrt{e_1^2 + e_2^2} - e_1^2)} \bigg|_0^{\frac{1}{2}} \\
= \frac{\sqrt{2}}{8} \frac{\sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{e_1^2 + e_2^2}} \log(1 - \frac{2\sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{x^2 + \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2} x + e_1^2 + e_2^2}}) \bigg|_0^{\frac{1}{2}} \\
+ \frac{\sqrt{2}}{4} \frac{\sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{e_1^2 + e_2^2}} \arctan \frac{2x - \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{2} (\sqrt{e_1^2 + e_2^2} - e_1^2)} \bigg|_0^{\frac{1}{2}} \\
+ \frac{\sqrt{2}}{4} \frac{\sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{\sqrt{e_1^2 + e_2^2}} \{(\frac{2\sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}}}{x^2 + \sqrt{2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2} x + e_1^2 + e_2^2}} \bigg|_0^{\frac{1}{2}} \\
- \arctan \frac{\sqrt{e_1^2 + e_2^2}}{\sqrt{e_1^2 + e_2^2} - e_1^2} \bigg|_0^{\frac{1}{2}} \bigg) \arctan \frac{\sqrt{e_1^2 + e_2^2}}{\sqrt{e_1^2 + e_2^2} - e_1^2} \bigg|_0^{\frac{1}{2}} \bigg) \}
- C_2 q \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}} \frac{- \sqrt{e_1^2 + e_2^2}}{\sqrt{e_1^2 + e_2^2} - e_1^2} \sqrt{e_1^2 + \sqrt{e_1^2 + e_2^2}} + C_1 e_1^2 + e_2^2 - e_1^2 \sqrt{e_1^2 + e_2^2} \sqrt{e_1^2 + e_2^2} \sqrt{e_1^2 + e_2^2}

. which implies \[1110\]. Similarly, \[1111\] holds if we denote

\[ p = \sqrt{2} \sqrt{- e_1^2 + \sqrt{e_1^2 + e_2^2}, q = \sqrt{e_1^2 + e_2^2}, r = - e_2^2}. \]

The \[1113\] is obvious if one note the following estimate,

\[ \frac{C'}{\sqrt{e_1^2 + e_2^2}} = \int_0^{\frac{1}{2}} \frac{1}{x^2 + e_1^2 + e_2^2} \, dx \leq \int_0^{\frac{1}{2}} \frac{1}{(x^2 + e_1^2)^2 + e_2^2} \, dx \leq \int_0^{\frac{1}{2}} \frac{1}{x^2 + e_1^2} \, dx = C' \frac{1}{e_1}. \]
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For the (114), it’s also clear if we use the following variable substitution.

\[
\int_0^\frac{\lambda}{2} \frac{\text{sgn}(x^2 - \epsilon_1^2)}{\sqrt{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2}} \, dx \leq \left| \int_0^\sqrt{2\epsilon_1} \frac{\text{sgn}(x^2 - \epsilon_1^2)}{\sqrt{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2}} \, dx \right| + \int_\sqrt{2\epsilon_1}^{\frac{\lambda}{2}} \frac{\text{sgn}(x^2 - \epsilon_1^2)}{\sqrt{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2}} \, dx.
\]

For the second items in the right, it’s obvious by direct calculation.

\[
\left| \int_\sqrt{2\epsilon_1}^{\frac{\lambda}{2}} \frac{\text{sgn}(x^2 - \epsilon_1^2)}{\sqrt{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2}} \, dx \right| \leq \int_\sqrt{2\epsilon_1}^{\frac{\lambda}{2}} \frac{1}{(x^2 - \epsilon_1^2)} \, dx = C \frac{1}{\epsilon_1}.
\]

For the first one, we have

(117) \[
\left| \int_0^{\frac{\lambda}{2}} \frac{\text{sgn}(x^2 - \epsilon_1^2)}{\sqrt{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2}} \, dx \right| = \left| \int_0^{\epsilon_1^2} \frac{\text{sgn}(x^2 - \epsilon_1^2)}{\sqrt{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2}} \, dx + \int_{\epsilon_1^2}^{\frac{\lambda}{2}} \frac{\text{sgn}(x^2 - \epsilon_1^2)}{\sqrt{(x^2 - \epsilon_1^2)^2 + \epsilon_2^2}} \, dx \right|.
\]

For the two items in the right side, separately let $\epsilon_1^2 - x^2 = t$ and $x^2 - \epsilon_1^2 = t$.

Thus for a sufficiently large $\lambda > 0$, combining with the fact $\epsilon_1 \gg \epsilon_2$ we have

(117) \[
= \int_0^{\epsilon_1^2} \frac{t \, dt}{\sqrt{t^2 + \epsilon_1^2 \sqrt{\epsilon_1^2 - t} + \sqrt{\epsilon_1^2 - t + \epsilon_1^4 - t^2}}} + \int_{\epsilon_1^2}^{\frac{\lambda}{2}} \frac{t \, dt}{\sqrt{t^2 + \epsilon_1^2 \sqrt{\epsilon_1^2 - t} + \sqrt{\epsilon_1^2 - t + \epsilon_1^4 - t^2}}} = \frac{\epsilon_1^2}{2} \int_0^{\sqrt{\epsilon_1^2 - \epsilon_1^2}} \frac{1}{\sqrt{1 - \frac{t^2}{\epsilon_1^2}}} \, dt.
\]

Let $\sqrt{\epsilon_1^2 - t} = x$, we’ll obtain

(117) \[
\leq \frac{1}{2 \lambda \epsilon_1} \int_0^{\sqrt{1 - \epsilon_1^4 - \epsilon_1^2}} \frac{1}{x \sqrt{2 \epsilon_1^2 - x^2 + 2 \epsilon_1^2 - x^2}} \, dx.
\]

Then let $x = \sqrt{2\epsilon_1} \sin \theta$

(117) \[
\leq \frac{1}{2 \lambda \epsilon_1} \left[ \frac{1}{\epsilon_1} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\arcsin \left( \frac{1 - \epsilon_1^4 - \epsilon_1^2}{2 \epsilon_1^2} \right) + \frac{\pi}{4}}{\sin \theta} \, d\theta \right].
\]

which implies (114).

\[\square\]

**A2.** For any $a, b, c, d, e \in \mathbb{R}$ satisfying $(b^4 + a^4)d - a^4b^4c > 0$, consider the following integral:

\[
\int \frac{1}{\sqrt{c^2x^2 + e^2(d^2x^2 - 2^2 + 2 \frac{2}{2}}} \, dx.
\]

We try to calculate it out.

At first, apply linearity, we have

\[
= a^4b^4 \int \frac{1}{\sqrt{a^4b^4c^2d^2x^2 + a^4b^4 \frac{a^4b^4c^2d^2x^2}{a^4b^4c^2d^2} + (2a^4b^4 + 2a^4b^4) \frac{a^4b^4c^2d^2x^2}{a^4b^4c^2d^2}}} \, dx
\]
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Second, we substitute \( u = x^2 \), it holds that

\[
\frac{a^4 b^4 d}{2} \int \frac{1}{\sqrt{a^4 b^4 d^2 (a^4 b^4 c^2 u^2 + 2 (b^4 + a^4) u) + e a^4 b^4}} \, du.
\]

Next, by substituting \( v = \frac{d (a^4 b^4 c^2 u + b^4 + a^4)}{(b^4 + a^4)^2 d^2 - (a^4 b^4 c^2)^2} \), we have

\[
= \frac{1}{2c} \int \frac{1}{\sqrt{v^2 - 1}} \, dv = \frac{1}{2c} \log |\sqrt{v^2 - 1} + v|.
\]

Eventually, it’s not difficult to figure out the answer as follow:

\[
\log \left| \sqrt{d^2 c^4 a^8 b^8 x^4 + 2d^2 c^2 a^4 b^4 (a^4 + b^4) x^2 + (a^4 b^4 c^2)^2 + d (a^4 b^4 c^2 x^2 + b^4 + a^4)} \right| + C
\]

\[
\frac{2c}{2c_m} \sqrt{\frac{(\frac{\epsilon}{c_m})^2 + \sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}}{\sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}}} \log(1 - \frac{\epsilon}{c_m}) + 2\frac{\sqrt{2}}{4c_m} \sqrt{\frac{(\frac{\epsilon}{c_m})^2 + \sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}}{\sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}}} \Bigg|_{1}^{1} + \sqrt{2} \frac{\epsilon}{c_m} \arctan \left( \frac{2x - \sqrt{2} \sqrt{\frac{(\epsilon}{c_m})^2 + \sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}}}{\sqrt{2(\frac{(\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4 - (\epsilon^2)}}} \right) + \arctan \left( \frac{2x + \sqrt{2} \sqrt{\frac{(\epsilon}{c_m})^2 + \sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}}}{\sqrt{2(\frac{(\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4 - (\epsilon^2)}}} \right) \leq (1 + O(\lambda_1^{-6} + \lambda_2^{-6})) (\frac{\epsilon}{c_m} \log |\epsilon|) + \epsilon^{-\frac{1}{2}} + \epsilon_{-1}^{-1}) + (1 + O(\lambda_1^{-6} + \lambda_2^{-6})).
\]

\[
\frac{\sqrt{2}}{8c_m^2} \frac{(\frac{\epsilon}{c_m})^2 + \sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}}{\sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}} - 2\log \left| \frac{(\frac{\epsilon}{c_m})^2}{\epsilon_{-1}^2 + \sqrt{2} \frac{(\frac{\epsilon}{c_m})^2 + \sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4} \sqrt{\epsilon_{-1}^2 + \epsilon_{-1}^2}}{\sqrt{(\frac{\epsilon}{c_m})^4 + (\frac{\epsilon}{c_m})^4}}} \right| + C \epsilon_{-1}^{-1}
\]

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For $K_2$, similarly we have

$$K_2 \geq \frac{\sqrt{2}}{8c_{m}} \left\lfloor \frac{2\sqrt{\left(\frac{c_{m}}{c_{m}}\right)^2 + \left(\frac{c_{m}}{c_{m}}\right)^4 + \left(\frac{c_{m}}{c_{m}}\right)^4}}{\left(\frac{c_{m}}{c_{m}}\right)^4 + \left(\frac{c_{m}}{c_{m}}\right)^4} \right\rfloor \sqrt{\frac{c_{m}}{c_{m}}^2 + \left(\frac{c_{m}}{c_{m}}\right)^4 + \left(\frac{c_{m}}{c_{m}}\right)^4}.$$
to the function $g_{L+1}$, which can be seen in the following estimate:

$$||\tilde{g}_l - g_{L+1}||_{C^2(I_{L+1})} \leq (\min\{||A_l||, ||A_{i-l}||\})^{-1.5} \leq \lambda \frac{q}{q_{N+L-1}} \leq \lambda \frac{q}{q_{N+L-1}}.$$  

Thus, by direct calculation, we have

$$||A_l(x)|| \geq \cos(\frac{\pi}{2} - \tilde{g}_l(T^l x)) \cdot ||A_l(x)|| \cdot ||A_{i-l}(T^l x)||$$

$$\geq \lambda \frac{q}{q_{N+L-1}} \cdot \lambda \frac{q}{q_{N+L-1}}$$

$$\geq \lambda \frac{q}{q_{N+L-1}}.$$

for $x \notin \bigcup_{0 \leq j \leq l} B(c_{L+1} + j \alpha, \lambda \frac{q}{q_{N+L-1}})$.

**a2:** $i - l$ or $l$ is less than $\frac{1}{50}q_{N+n-1}$. Then it holds that

$$||A_l(x)|| \geq \max\{||A_l(x)|| \cdot ||A_{i-l}(T^l x)||^{-1}, ||A_l(x)||^{-1} \cdot ||A_{i-l}(T^l x)||\} \geq \lambda \frac{q}{q_{N+L-1}}.$$

**CASE B:** There exists $0 \leq l_1, l_2 \leq i$ such that $T^{l_1} x, T^{l_2} x \in I_{L+1}$.

In this case, the resonance may occur and we consider

$$A_l(x) = A_{l_1} A_{l_2} A_{l_1} A_{l_2}.$$

The following several subcases need to be thought about.

**b1:** $\min\{i - l_2, l_2 - l_1, l_1\} > \frac{1}{50}q_{N+n-1}.$  

In this situation, no resonance occur, hence it’s absolutely same as the first case.

For $x \notin \bigcup_{0 \leq j \leq i} \bigcup_{0 \leq m \leq 4} B(c_{L+1,m} + j \alpha, \lambda \frac{q}{q_{N+L-1}})$, it is clear that

$$||A_l(x)|| \geq \cos(\frac{\pi}{2} - \tilde{g}_l(T^l x)) \cdot \cos(\frac{\pi}{2} - \tilde{g}_l(T^{l_1} x)) \cdot ||A_l|| \cdot ||A_{l_2}|| \cdot ||A_{l_2}|| \cdot ||A_{l_1}||$$

$$\geq \lambda \frac{q}{q_{N+L-1}} \cdot \lambda \frac{q}{q_{N+L-1}} \cdot \lambda \frac{q}{q_{N+L-1}} \cdot \lambda \frac{q}{q_{N+L-1}} \cdot \lambda \frac{q}{q_{N+L-1}}$$

$$\geq \lambda \frac{q}{q_{N+L-1}}.$$

Here $c_{L+1,1}, c_{L+1,2}, c_{L+1,3}$ and $c_{L+1,4}$ are the possible four critical points when $g_{L+1}$ is of type III.

**b2:** $\min\{i - l_2, l_2 - l_1, l_1\} \leq \frac{1}{50}q_{N+n-1}.$  

Clearly, $\max\{i - l_2, l_2 - l_1, l_1\} > \frac{1}{50}q_{N+n-1}$ and the maximum of the three cannot be equal to the minimum of the three in this case. Hence by the symmetry, we only need to consider the following three cases:

**b2.1:** If $i - l_2 \leq \frac{1}{50}q_{N+n-1}$, and $l_2 - l_1 \geq \frac{1}{50}q_{N+n-1}$, then either $l_1 > \frac{1}{50}q_{N+n-1}$, which implies

$$||A_l|| \geq ||A_{l_2}|| \cdot ||A_{l_1}|| \cdot ||A_{l_2-l_1}||$$

$$\geq \lambda \frac{q}{q_{N+n-1}} \cdot \lambda \frac{q}{q_{N+n-1}} \cdot \lambda \frac{q}{q_{N+n-1}} \geq \lambda \frac{q}{q_{N+n-1}} \geq \lambda \frac{q}{q_{N+n-1}}.$$

or $l_1 > \frac{1}{50}q_{N+n-1}$, which implies the angle between $A_{l_2-l_1}$ and $A_{l_1}$, says $\tilde{g}_l$, is similar to $g_{L+1}$. Thus we can deal with $A_{l_2-l_1} A_{l_1}$ in the same way as the “no resonance” case. By straight calculation, we have

$$||A_l(x)|| \geq ||A_{l_2}|| \cdot ||A_{l_2-l_1}|| \cdot ||A_{l_1}|| \cdot \cos(\frac{\pi}{2} - \tilde{g}_l(T^{l_1} x))$$

$$\geq \lambda \frac{q}{q_{N+n-1}} \cdot \lambda \frac{q}{q_{N+n-1}} \cdot \lambda \frac{q}{q_{N+n-1}} \cdot \lambda \frac{q}{q_{N+n-1}} \geq \lambda \frac{q}{q_{N+n-1}}.$$

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for $x \notin \bigcup_{0 \leq j \leq i} \left( B(e^{(L+1)} + j \alpha, \lambda^{-\frac{2m}{2n}}) \cup B(e^{(L+1,2)}, \lambda^{-\frac{2m}{2n}}) \right) \triangleq \tilde{B}_3$.

**b2.2:** If $i - l_2 < \frac{1}{50} q^4_{N+L-1}$ and $l_1 > \frac{i}{4}$, then either $l_2 - l_1 \geq \frac{1}{50} q^4_{N+L-1}$, which implies the angle between $A_{l_2-l_1}$ and $A_{l_1}$, says $\bar{g}_{l_1}$, is similar to $g_{L+1}$ and we have

$$\|A_i(x)\| \geq \|A_{l_2-l_1}\|^{-1} \cdot \|A_{l_2-l_1}\| \cdot \|A_i\|^{-1} \geq \lambda^{-\frac{2m}{2n}}(i-(l_2-l_1)) \cdot \lambda^{-\frac{2m}{2n}}(l_2-l_1) \cdot \lambda^{-\frac{2m}{2n}} \geq \lambda^{-\frac{2m}{2n}},$$

for $x \notin \bar{B}_3$;

or $l_2 - l_1 \leq \frac{1}{50} q^4_{N+L-1}$, which implies which implies

$$\|A_i\| \geq \|A_{l_2-l_1}\|^{-1} \cdot \|A_{l_2-l_1}\| \cdot \|A_i\|^{-1} \geq \lambda^{-\frac{2m}{2n}}(i-(l_2-l_1)) \cdot \lambda^{-\frac{2m}{2n}}(l_2-l_1) \cdot \lambda^{-\frac{2m}{2n}} \geq \lambda^{-\frac{2m}{2n}}.$$

**b2.3:** If $l_2 - l_1 < \frac{1}{50} q^4_{N+L-1}$ and $l_1 > \frac{i}{4}$, then the resonance occur and only the following two cases is possible.

**b2.3.1:** $i - l_2 < \frac{1}{50} q^4_{N+L-1}$. This implies

$$\|A_i\| \geq \|A_{l_2-l_1}\|^{-1} \cdot \|A_{l_2-l_1}\| \cdot \|A_i\|^{-1} \geq \lambda^{-\frac{2m}{2n}}(i-(l_2-l_1)) \cdot \lambda^{-\frac{2m}{2n}}(l_2-l_1) \cdot \lambda^{-\frac{2m}{2n}} \geq \lambda^{-\frac{2m}{2n}},$$

**b2.3.2:** $i - l_2 \geq \frac{1}{50} q^4_{N+L-1}$. In this case, by the idea of [WZ1], we need to consider $A_i = A_{l_2-l_1} A_{l_2-l_1} A_{l_2}$. We denote $\bar{g}_{l_2}$ the angle between $A_{l_2-l_1}$ and $A_{l_2}$. It’s clear that $\bar{g}_{l_2}$ is similar to the $g_{L+1}$, which has four critical points $\bar{c}_1$ and $\bar{c}_2$, which locate at $I_{L+1,1}$; and $\bar{c}_2$ and $\bar{c}_1$, which locate at $I_{L+1,2}$. It holds from the direct calculation that

$$\|A_i(x)\| \geq \|A_{l_2-l_1}\|^{-1} \cdot \|A_{l_2-l_1}\| \cdot \|A_i\| \cdot \cos \left( \frac{\pi}{2} - \bar{g}_{l_2}(T^l x) \right) \geq \lambda^{-\frac{2m}{2n}}(l_2-l_1) \cdot \lambda^{-\frac{2m}{2n}} \geq \lambda^{-\frac{2m}{2n}},$$

for $x \notin \bigcup_{0 \leq j \leq i} B(\bar{c}_i + j \alpha, \lambda^{-\frac{2m}{2n}}) \cup B(\bar{c}_i + j \alpha, \lambda^{-\frac{2m}{2n}}) \cup B(\bar{c}_i, \lambda^{-\frac{2m}{2n}}) \cup B(\bar{c}_i, \lambda^{-\frac{2m}{2n}}) \triangleq \bar{B}_4$.

**CASE C:** There is no $0 \leq l \leq i$ such that $T^l x \in I_{L+1}$. It is obvious that $\|A_i\| \geq \lambda^{-\frac{2m}{2n}} \geq \lambda^{-\frac{2m}{2n}}.$

In conclusion, in any case for sufficient large $i$ and $\lambda$, there exists a set $K = \bigcup_{i=1}^{4} \bar{B}_i$ and a constant $c$ such that $Leb\{K\} < \lambda^{-\alpha}$ and $\|A_i\| \geq \lambda^{\frac{2m}{2n}}$ for any $x \notin K$.

$\hfill \square$

**A.5**

**Proof.** We process it by induction.

**Case** $n = 1 \to 2$: The beginning step $n = 1$ is trivial. When we come to the case $n = 2$, it follows from [WZ1] that the following three conditions may occur.
Type $I_1 \rightarrow I_2$: In this case, we write $A_{m_2}$ as $A_{s_t-s_{t-1}} \cdots A_{s_1}$ where $s_i$ are all returning times to $I_1$ of $x$. If $t = 1$, then we come back to the case $n = 1$. Thus we only need to consider $t > 1$. We focus on the following two cases:

$s_t - s_{t-1} < q_{N+1}$: Note that $s_t - s_{t-1} \ll q_{N+1}^2 \ll q_{N+1}^2 - q_{N+1}^2 \leq s_t - s_{t-1}$. By the help of lemma\[\text{III}\] we have

\[
\|A_{m_2}\| \geq \frac{\|A_{s_t-s_{t-1}}\|}{\|A_{s_t-s_{t-1}}\|} \geq \lambda^{1-\epsilon} m_2^2;
\]

Therefore

\[
|s(A_{m_2}) - s(A_{s_t-s_{t-1}})|_{C^2} \leq \|A_{s_t-s_{t-1}}\|^{-1.5}.
\]

Similarly,

\[
\|A_{m_2}\| \geq \frac{\lambda^{1-\epsilon} m_2^2}{\|A_{s_t-s_{t-1}}\|};
\]

\[
|g_2 - u(A_{m_2})|_{C^2} \leq \lambda^{-0.5r_1}.
\]

In conclusion, $|g_{m_1^2, m_2^2} - g_2|_{C^2} \leq \lambda^{-0.5r_1}$.

$s_t - s_{t-1} \geq q_{N+1}$: By the definition of $m_2^2$, it’s clear that $T^{s_t} x \notin I_2$, which implies $|T^{s_t} x - c_2| \geq \frac{C}{q_{N+1}^2}$ and $g_2(T^{s_t} x) \geq \frac{C}{q_{N+1}^2}$. Denote $\tilde{g}_2 \triangleq s(A_{s_t-s_{t-1}}) - u(A_{s_t-s_{t-1}})$ and $\tilde{c}_2$ be the zero point of $\tilde{g}_2$. It follows from the conclusion of the previous step that $|\tilde{g}_2(T^{s_t} x) - g_2(T^{s_t} x)|_{C^2} \leq \lambda^{-\frac{1}{2}} \ll \frac{1}{q_{N+1}^2}$. Therefore, it follows from lemma\[\text{III}\] that $\|A_{m_2}\| \geq \frac{\|C\|}{q_{N+1}^2} \|A_{s_t-s_{t-1}}\| \geq \lambda^{1-\epsilon} m_2^2$. Similarly, we have $\|A_{m_2}\| \geq \lambda^{1-\epsilon} m_2^2$. Again from lemma\[\text{III}\] it’s not difficult to see that $|g_{m_1^2, m_2^2} - g_2|_{C^2} \leq \lambda^{-0.5r_1}$.

Type $I_1 \rightarrow III$: We assume that $I_{2,1} + k\alpha \cap I_{2,2} \neq \emptyset$, for some $k \leq q_{N+1}$.

$x \in I_{2,1}$: In this case, note that $r_2(x) = k \ll q_{N+1}$. So no resonance happen in this case and all the estimates are quite similar to type I to type I.

$x \in I_{2,2}$: Also no resonance happen in this case, thus it’s same as the above case.

Type $III \rightarrow I_2, II_2$ or $II_2$: This is same as Type I$\rightarrow$ Type I and Type I$\rightarrow$ Type III.

Now we assume that for all the case $n \leq q$ the conclusion holds.

**Case** $n = p - p + 1$: The following conditions need be considered.

Type $I_p \rightarrow I_{p+1}$: This similar to the original case but we still give it out for completeness. We write $A_{m_1^{p+1}}$ as $A_{s_{t'}-s_{t'-1}} \cdots A_{s_1}$ where $s_i$ are all returning times to $I_1$ of $x$. If $t' = 1$, then we do nothing. Thus we only need to consider $t' > 1$. We focus on the following two cases:

$s_{t'} - s_{t'-1} < q_{N+p}$: Note that $s_{t'} - s_{t'-1} \ll q_{N+p}^2 \ll q_{N+p}^2 - q_{N+p}^2 \leq s_{t'-1}$. By the help of lemma\[\text{III}\] we have

\[
\|A_{m_1^{p+1}}\| \geq \frac{\|A_{s_{t'}-s_{t'-1}}\|}{\|A_{s_{t'}-s_{t'-1}}\|} \geq \lambda^{1-\epsilon} m_1^{p+1};
\]

Therefore

\[
|g_{p+1} - s(A_{m_1^{p+1}})|_{C^2} \leq |s(A_{m_1^{p+1}}) - s(A_{s_{t'}-s_{t'-1}})|_{C^2} + |g_{p+1} - s(A_{s_{t'}-s_{t'-1}})|_{C^2} \leq \|A_{s_{t'}-s_{t'-1}}\|^{-1.5} + \lambda^{-0.5r_1} \leq \lambda^{-0.5r_1}.
\]
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Similarly,
\[
\| A_{m_2^{p+1}} \| \geq \lambda^{(1-\epsilon)m_2^{p+1}};
\]
\[
| g_{p+1} - u(A_{m_2^{p+1}}) |_{C^2} \leq \lambda^{-0.5r_p}.
\]

In conclusion, $| g_{m_1^{p+1},m_2^{p+1},2} - g_{p+1} |_{C^2} \leq \lambda^{-0.5r_p}$.

$s_i - s_{i-1} \geq q_N + p$: By the definition of $m_1^{p+1}$, it’s clear that $T^{s_i}x \notin I_{p+1}$, which implies $| T^{s_i}x - c_{p+1} | \geq \frac{1}{q_N^{m_2^{p+1}}} \geq \frac{C}{q_N^{m_2^{p+1}}}$. Denote $\tilde{g}_{p+1} \triangleq s(A_{s_i - s_{i-1}}) - s(A_{s_i - s_{i-1}})$ and $\tilde{c}_{p+1}$ be the zero point of $\tilde{g}_{p+1}$. It follows from the induction hypothesis that $| \tilde{g}_{p+1}(T^{s_i}x) - g_{p+1}(T^{s_i}x) | \leq \lambda^{-2r_p} \ll \frac{C}{q_N^{m_2^{p+1}}}$. Therefore, it follows from lemma 3 that $\| A_{m_1^{p+1}} \| \geq \| A_{s_i - s_{i-1}} \|_{q_N^{m_2^{p+1}}} \| A_{s_i - s_{i-1}} \| \geq \lambda^{(1-\epsilon)m_2^{p+1}}$. Similarly, we have $\| A_{m_1^{p+1}} \| \geq \lambda^{(1-\epsilon)m_2^{p+1}}$.

Again from lemma 3 it’s not difficult to see that $| g_{m_1^{p+1},m_2^{p+1},2} - g_{p+1} |_{C^2} \leq \lambda^{-0.5r_p}$.

Type $III_p \rightarrow I_{p+1}$: If $\ell = 1$, then we need do nothing. For $\ell \geq 1$, we can also write $A_{m_1^{p+1}}$ as $A_{s_i - s_{i-1}} \cdots A_{s_1}$, where $s_i, \ i = 1, 2, \cdots, \ell - 1$ are all second returning times to $I_p$ of $x$, where $I_p \triangleq I_{p+1} \bigcup I_{p+2}$ and $I_{p+1} \bigcup k' \alpha I_{p+2} \neq \emptyset$ for some $k' \leq q_N^{m_2^{p+1}}$. And for the last returning (or not returning) time $s_{\ell} - s_{\ell-1}$, there are two cases should be concerned.

$s_i - s_{i-1} < q_N^{m_2^{p+1}}$: Note that $q_N^{m_2^{p+1}} \ll q_{N+p-1} \ll s_{i-1} = m_1^{p+1} - (s_i - s_{i-1})$. Thus lemma 3 implies
\[
\| A_{m_1^{p+1}} \| \geq \frac{\| A_{s_i - s_{i-1}} \|}{\| A_{s_i - s_{i-1}} \|} \geq \lambda^{(1-\epsilon)m_1^{p+1}};
\]
\[
| s(A_{m_1^{p+1}}) - g_{p+1} |_{C^2} \leq | g_{p+1} - s(A_{s_i - s_{i-1}}) |_{C^2} + | s(A_{s_i - s_{i-1}}) - s(A_{m_1^{p+1}}) |_{C^2} \leq \lambda^{-1.5r_p} + \| A_{s_i - s_{i-1}} \|^{-1.5} \leq \lambda^{-0.5r_p}.
\]

Similarly, $| u(A_{m_2^{p+1}}) - g_{p+1} |_{C^2} \leq \lambda^{-0.5r_p}$, which implies $| g_{m_1^{p+1},m_2^{p+1},2} - g_{p+1} |_{C^2} \leq \lambda^{-0.5r_p}$ as desire.

$s_i - s_{i-1} \geq q_N^{m_2^{p+1}}$: In this case, note that $k' \leq q_N^{m_2^{p+1}} \ll q_{N+p-1} \ll s_i - s_{i-1}$, thus it either return once to $I_p$ or not. So the following two cases should be concerned.

$r_p^+(T^{s_i-1}x) \neq k'$: This implies $r_p^+(T^{s_i-1}x) \geq q_{N+p-1}^m - q_{N+p-1}$, which means $x$ doesn’t return to any critical intervals $I_{p+1}$ or $I_{p+2}$. On one hand, the induction hypothesis tell us that $\| A_{s_i - s_{i-1}} \| \geq \lambda^{(1-\epsilon)(s_i - s_{i-1})}$; and for $\theta_{s_i - s_{i-1}} \triangleq s(A_{s_i - s_{i-1}}) - u(A_{s_i - s_{i-1}})$ we have $| \theta_{s_i - s_{i-1}} - g_{p+1} |_{C^2} \leq \lambda^{0.5r_p} \ll \frac{1}{q_N^{m_2^{p+1}}}$. On the other hand, it follows from the definition of $m_1^{p+1}$ that $T^{s_i-1}x \notin I_{p+1}$, which implies $\theta_{s_i - s_{i-1}}(s_i - s_{i-1}x) \geq \frac{1}{q_N^{m_2^{p+1}}}$. According to lemma 3 it’s clear that
\[
\| A_{m_1^{p+1}} \| \geq \| A_{s_i} \| \frac{1}{q_N^{m_2^{p+1}}} \| A_{s_i - s_{i-1}} \| \geq \lambda^{(1-\epsilon)m_2^{p+1}};
\]
\[
| s(A_{m_1^{p+1}}) - g_{p+1} |_{C^2} \leq | g_{p+1} - s(A_{s_i - s_{i-1}}) |_{C^2} + | s(A_{s_i - s_{i-1}}) - s(A_{m_1^{p+1}}) |_{C^2} \leq \lambda^{0.5r_p}.
\]

Similarly,
\[
| u(A_{m_2^{p+1}}) - g_{p+1} |_{C^2} \leq \lambda^{-0.5r_p},
\]
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which implies

\[ |g_{m_1^{p+1},m_2^{p+1}} - g_{p+1}|_{C^2} \leq \lambda^{-0.5r_p} \]

as desired.

\( r_p^+(T^{s_{i-1}}x) = k' \): Due to \( k' \leq q_{N-p-1}^2 \leq q_N^3 \leq s_i - s_{i-1}, \) \( x \) must return once back to \( I_{p+1}. \) We are concern about the difference between \( g_{p-1} \) and \( \theta'(A_s) - u(A_{s_{i-1}}). \) It follows from [WZ1] that

\[ |g_p - (s(A_s) - u(A_{s_{i-1}}))|_{C^2} \leq C\lambda^{-\frac{3}{4}q_N^{p-1}}. \]

And by the definition of \( m_1^{p+1} \), we have \( T^{s_{i-1}}x \notin I_{p+1}. \) Therefore, \( \theta'(T^{s_{i-1}}x) \geq \frac{1}{q_N^2+p} \), and the following holds:

\[ \|A_{m_1^{p+1}}\| \geq \|A_{s_{i-1}}\| \frac{1}{q_N^2+p} \|A_{s_{i-1}}\| \geq \lambda^{(1-\epsilon)m_1^{p+1}}; \]

\[ |s(A_{m_1^{p+1}}) - g_{p+1}|_{C^2} \leq |g_{p+1} - u(A_{s_{i-1}})|_{C^2} + |s(A_{s_{i-1}}) - s(A_{m_1^{p+1}})| \leq \lambda^{-0.5r_p} \|A_{s_{i-1}}\|^{-1.5} \leq \lambda^{-0.5r_p}. \]

Similarly,

\[ |u(A_{m_2^{p+1}}) - g_{p+1}|_{C^2} \leq \lambda^{-0.5r_p}, \]

which implies

\[ |g_{m_1^{p+1},m_2^{p+1}} - g_{p+1}|_{C^2} \leq \lambda^{-0.5r_p}, \]

as desired.

Type \( III_p \to III_{p+1} \): Without loss of generality, there exist some \( k_1 \leq q_{N+p-1}^2 \) and \( k_1 \leq k_2 \leq q_{N+p}^2 \) such that \( I_{p,1} + k_1 \alpha \cap I_{p,2} \neq \emptyset \) and \( I_{p+1,1} + k_2 \alpha \cap I_{p+1,2} \neq \emptyset. \)

C1, \( k_1 = k_2 : \) We denote \( k_1 = k_2 = k''. \)

\( x \in I_{p+1,1} : \) Note that \( m_1^{p+1} = r_{p+1}^+ = k'' \) and \( x \in I_{p+1,1} \in I_p. \) Thus we come back to the case \( n = p. \)

\( x \in I_{p+1,2} : \) This is quite similar to the case Type \( III_p \to I_{p+1} \) and we omit it.

C2, \( k_1 < k_2 : \) Note that either \( x \in I_{p+1,1} \) or \( I_{p+1,2}, \) the definition of \( m_1^{p+1} \) implies that \( x \) come back to \( I_{p+1} \) at most once. Thus we only need to deal with the each non-resonance part. By labeling each second returning times to \( I_p, \) we can also divide \( A_{m_1^{p+1}} \) in the same way we just did in the case Type \( III_p \to I_{p+1}. \) The rest is exactly the same.

The following three types are similar to the case \( n = 0 \to 1 \) and there are no new things need be present.

\[ Type I_p \to III_{p+1} \]

\[ Type II_p \to I_{p+1,1}, II_{p+1} \text{ or } III_{p+1}. \]

\[ \square \]

A.6

Proof. We prove it by induction.

For \( m = n, \) the conclusion is trivial.

For \( m = n + 1, \) the following several cases need to be considered.

Type \( I_m \to I_{m+1} : \) We write \( A_{m+1} = A_{m}^{s_m} - s_{m-1} \cdots A_{s_1}, \) where \( \{s_m\}_{i=1}^m \) are all the returning times of \( x \) to \( I_m. \) Then the conclusion holds from [WZ1] or [Y].
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Type $I_m \to III_{m+1}$: Recall that $r_{m+1}^+(x)$ is the first (not the second) returning time of $x$ to $I_{m+1}$. Thus this case is equivalent to Type $I_m \to I_{m+1}$.

Type $III_m \to I_{m+1}$: Assume $I_{m+1} + k_m \alpha \bigcap I_{m+2} \neq \emptyset$ and $x \in I_{m+1}$.

(a) If $r_{m+1}^+(x) = k_m$, then $r_{m+1}^+ = r_{m+1}^+$ and all the conclusion is trivial;

(b) If $r_{m+1}^+(x) \neq k_m$, then we write $A_{r_{m+1}^+} = A_{s_{m+1}^m - s_{m+1}^m} \cdots A_{r_{m+1}^+}$, where

$$\{s_{i}^m\}_{i=1}^{t_{m+1}}$$

are all the returning time of $x$ to $I_m$. Note that $t_m \geq 1$. If $t_m$ is even, then we put each twice returning time points together and the conclusion holds from [WZ1]: If $t_m$ is odd, then the last returning time $s_{t_m}^m - s_{t_m-1}^m = k_m \ll s_{t_m-1}^m$ and the conclusion holds from lemma 3.

Type $III_m \to III_{m+1}$: This is similar to the previous case due to the definition of $r_{m+1}^+$.

The left possible cases are as follow and we omit the proof since it’s quite similar to the above cases which we have stated.

$$Type II_m \rightarrow I_{m+1}, I_{m+1}, or III_{m+1}.$$ 

Assume that for any $s \geq 1$ and $|n - m| \leq s$, the conclusion hold.

Let’s focus on the case $n = m + s + 1$. We still consider the following possible cases.

Type $I_{m+s} \to I_{m+s+1}$: By the Induction hypothesis, for fixed $x_m$ from $I_{m} \to I_{m+s}$ and $I_{m+s}$ to $I_{m+s+1}$, the norms have good growth. We write $A_{r_{m+s+1}^+} = A_{s_{m+s}^m - s_{m+s}^m} \cdots A_{r_{m+s+1}^+}$, where

$$\{s_{i}^{m+s}\}_{i=1}^{t_{m+s+1}}$$

are all the returning times of $x$ to $I_{m+s}$. If $t_{m+s} = 1$, then we can see that $r_{m+s}^+ = r_{m+s+1}^+ > q_{N+m+s} > q_{N+m+s-1}$. By the Induction hypothesis, we have obtained the conclusion. For $t_{m+s} \geq 2$, the following cases should be considered.

(a) If $s_{1}^{m+s} < q_{m+s}$, then $r_{m+s+1}^+ \geq q_{m+s} \gg q_{m+s}$ and holds from lemma 3.

(b) If $s_{1}^{m+s} \geq q_{m+s}$, then we have $r_{m+s} \geq q_{m+s}$. On one hand, Induction hypothesis implies that $\|A_{s_{1}^{m+s}}\| \geq \lambda^{1-e} s_{1}^{m+s}$ and $|\theta_2(x_{m})| \geq C$.

where $\theta_2 \triangleq s(A_{s_{1}^{m+s} - s_{m+s-1}} \cdots A_{s_{2}^{m+s}}) - u(A_{s_{1}^{m+s}})$.

On the other hand, [WZ1] implies $A_{s_{1}^{m+s} - s_{m+s-1}} \cdots A_{s_{2}^{m+s}} \geq \lambda^{1-e}(r_{m+s}^+ s_{m+s+1} - s_{1}^{m+s})$.

Therefore,

$$\|A_{r_{m+s+1}^+}\| \geq \lambda(1-e)(r_{m+s+1}^+ s_{m+s+1} - s_{1}^{m+s}) \geq C \frac{\lambda^{1-e} s_{m+s}^m}{q_{N+m+s}^2} \lambda^{1-e} s_{m+s}^m \geq \lambda^{1-e} r_{m+s+1}^+.$$

Type $I_{m+s} \to III_{m+s+1}$: This is similar to the case Type $I_{m+s} \to I_{m+s+1}$.

Type $III_{m+s} \to I_{m+s+1}$: In this case, we write $A_{r_{m+s+1}^+} = A_{s_{m+s}^m - s_{m+s}^m} \cdots A_{r_{m+s+1}^+}, \{s_{i}^{m+s}\}_{i=1}^{t_{m+s+1}}$ are defined as above case. If $t_{m+s} = 1$, then by the analysis in Type $I_{m+s} \to I_{m+s+1}$ we obtain the conclusion. For $t_{m+s} \geq 2$, note that the short chain ( with the length $k_{m+s} \leq q_{N+m+s+1}^2$) and the long chain ( with the length $\geq q_{N+m+s+1}^2$) alternately occur. We need consider the following cases.

(a) If $t_{m+s} \geq 3$, then we have $r_{m+s+1}^+ - s_{1}^{m+s} \geq q_{N+m+s+1}^2$. We should consider the following two cases. For $s_{1}^{m+s} \leq q_{N+m+s+1}$, it’s clear that $r_{m+s+1}^+ - s_{1}^{m+s} \geq q_{N+m+s+1}^2 \geq s_{1}^{m+s}$ and the conclusion holds from lemma 3. For $s_{1}^{m+s} \geq q_{N+m+s+1}$, by the help of the Induction hypothesis, we have $|u(A_{s_{1}^{m+s}}) - u r_{m+s}^+| C_2 \leq C \lambda^{-\frac{1}{4}} q_{N+m+s+1}$, which...
implies $\theta_1(x_m) \geq \frac{C}{q_{N+2m+1}}$, where $\theta_1 \triangleq s(A_{m+s+1} - s_{m+s}) - u(A_{m+s})$; and $\|A_{m+s}\| \geq \lambda^{-1/2} K_1$. Again due to lemma 3 we obtain the proof.

(b) If $t_{m+s} = 2$, then $r_{m+s}$ is the second returning time point. We rewrite $A_{r_{m+s+1}}$ as $A_{K_1} A_{K_2}$.

(b1): If $\min\{K_1, K_2\} \leq q^5_{N+m+s}$, then the growth of the norm is obvious. For the angle, if $K_2 \leq q^5_{N+m+s}$, which implies $K_1 \geq q^5_{N+m+s} - q^5_{N+m+s} \gg K_2^2$, then $|u(A_{m+s}) - u(A_{K_1})|^C_2 \leq C \lambda^{-5} K_1$; if $K_1 \leq q^5_{N+m+s}$, which implies $K_2 \geq q^5_{N+m+s} - q^5_{N+m+s} \gg K_2^2$, then we go on to divide $A_{K_2}$ into $A_{m+s+1} - s_{m+s+1} \cdots A_{m+1}$, without loss of generality, we assume $t_{m+s-1} \geq 2$ (otherwise it holds from $K_2 \geq \frac{1}{2} q^5_{N+m+s}$ that we can go on to divide it until $t_{m+s-i} \geq 2$ for some $i \geq 2$). For the case $t_{m+s-1} \geq 3$, it’s similar to the previous case (a). For the case $t_{m+s-1} = 2$, we write $A_{m+s+1} = A_{K_1} A_{K_2}$, and we still need to consider the following sub-cases.

(b1,1): If the resonance distance $K_{m+s-1} = K_{m+s}$, then it’s easy to see that $K_{2,1} \gg K_{2,2}$. Therefore, on one hand, it follows from lemma 3 that

$$|u(A_{K_1} A_{K_2}) - u(A_{K_1} A_{K_2}, A_{K_2})|^C_2 \leq \lambda^{-1/5} q^5_{N+m+s-2};$$

on the other hand,

$$|u(A_{K_1} A_{K_2}) - u_{r_{m+s}}|^C_2 \leq \|A_{K_1} A_{K_2}\|^{-1/5} \leq \lambda^{-1/5} q^5_{N+m+s-2}.$$

Thus,

$$|u(A_{r_{m+s+1}}) - u_{r_{m+s}}|^C_2 \leq \lambda^{-1/5} q^5_{N+m+s-2}.$$

(b1,2): If the resonance distance $k_{m+s-1} \neq k_{m+s}$, then of course $k_{m+s-1} < k_{m+s} \leq q^5_{N+m+s}$. This is impossible since $K_1 > K_2$ in this case, which has contradiction with $K_2 \gg K_2^2$.

(b2): If $\min\{k_1, k_2\} \geq q^5_{N+m+s}$, similar to (b1) we write $A_{r_{m+s+1}} = A_{K_1} A_{K_2}$, and consider the following possible two cases.

(b2,1): $k_{m+s-1} = k_{m+s}$. This is same as (b1,1).

(b2,2): $k_{m+s-1} \neq k_{m+s}$. This means $k_{m+s-1} < k_{m+s}$ and we can write $A_{K_1}$ as $A_{p_{u-1}} \cdots A_{p_1}$, where $\{p_i\}_{i=1}^u$ are all the second returning time points to $I_{m+s-1}$. Note that $u \geq 2$, which means $K_1 \geq q^5_{N+m+s-1}$. Therefore, if $K_2 \leq q^5_{N+m+s-1}$, then $K_1 \geq q^5_{N+m+s-1} + q^5_{N+m+s-1} \geq K_{2,1} + K_{2,2} = K_2$. It holds from lemma 3 that the conclusion holds true. If $K_{2,2} \geq q^5_{N+m+s-1}$, then by the help of the Induction hypothesis it’s clear that $A_{K_{2,1}} \cdot A_{K_{2,2}}$ has a good growth and $u(A_{K_1}) - s(A_{K_2}) \triangleq \theta(x_m) \geq \frac{C}{q^5_{N+m+s-1}}$. Hence, combining it with the fact $\|A_{K_1}\| \geq \lambda^{-1/2} K_1$, lemma 3 implies the proof.

Type $III_{m+s} \rightarrow III_{m+s+1}$: This is similar to the case Type $III_{m+s} \rightarrow I_{m+s+1}$. The left possible cases are all similar as the above cases.
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