AN INTRODUCTION TO UPPER HALF PLANE POLYNOMIALS

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Polynomials with all real roots have many interesting and useful properties. The purpose of this article is to introduce a generalization to polynomials in many variables and with complex coefficients [1–6, 10].

Definition 1. $\mathbb{U}_d(\mathbb{C}) = \begin{cases} 
\text{All polynomials } f(x_1, \ldots, x_d) \text{ with complex coefficients such that } f(\sigma_1, \ldots, \sigma_d) \neq 0 \text{ for all } \sigma_1, \ldots, \sigma_d \text{ in the upper half plane. If we don't need to specify } d \\
\text{we simply write } \mathbb{U}(\mathbb{C}).
\end{cases}$

We call such polynomials upper half plane polynomials, or simply upper polynomials.

For example, $x_1 + \cdots + x_d \in \mathbb{U}_d(\mathbb{C})$. This follows from the fact that the upper half plane is a cone, so if $\sigma_1, \ldots, \sigma_d$ are in the upper half plane then so is their sum.

Another example is $x_1 x_2 - 1$. If $\sigma_1$ and $\sigma_2$ are in the upper half plane then $\sigma_1 \sigma_2 \in \mathbb{C} \setminus (0, \infty)$, so $\sigma_1 \sigma_2 - 1$ is not zero.

$\mathbb{U}_1(\mathbb{C})$ is easily described. It is all polynomials in one variable whose roots are either real, or lie in the lower half plane.

It is important to observe that the zero polynomial is not in $\mathbb{U}(\mathbb{C})$. This is unfortunate, since it causes many conclusions to be of the form “... $\in \mathbb{U}(\mathbb{C}) \cup \{0\}$...”.

Conventions: $d$ is a positive integer, $i = \sqrt{-1}$, and $y$ is a variable distinct from $x_1, \ldots, x_d$. We use the following notation:

- $x = (x_1, \ldots, x_d)$
- $y = (y_1, \ldots, y_d)$
- $\partial x = (\partial x_1, \ldots, \partial x_d)$
- $f(x) = f(x_1, \ldots, x_d)$
- $I = (i_1, \ldots, i_d)$
- $x^I = (x_1^{i_1}, \ldots, x_d^{i_d})$
- $J = (j_1, \ldots, j_d)$
- $x^J = (x_1^{j_1}, \ldots, x_d^{j_d})$

1. Complex Coefficients

Fact 1. Suppose $f(x) \in \mathbb{U}_d(\mathbb{C})$.

1. If $\alpha \neq 0$ then $\alpha f(x) \in \mathbb{U}_d(\mathbb{C})$.
2. If $a_1 > 0, \ldots, a_d > 0$ then $f(a_1 x_1, \ldots, a_d x_d) \in \mathbb{U}_d(\mathbb{C})$.
3. If $\Im(\sigma_1) > 0, \ldots, \Im(\sigma_d) > 0$ then $f(x_1 + \sigma_1, \ldots, x_d + \sigma_d) \in \mathbb{U}_d(\mathbb{C})$.

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Proof. These are all immediate from the definition.

\[ \text{Fact 2. If } f(x) \in U(\mathbb{C}) \text{ then } f(a, x_2, \ldots, x_d) \in U(\mathbb{C}) \cup \{0\} \text{ if } a \in \mathbb{R}. \]

Proof. It suffices to assume \( a = 0 \). Let \( g_r(x) = f(x_1/r, x_2, \ldots, x_d) \). Since \( \lim_{r \to \infty} g_r(x) = f(0, x_2, \ldots, x_d) \) the Hurwitz theorem (see below) implies the conclusion, where \( \Omega \) is the upper half plane.

**Theorem 1** (Hurwitz). Let \( (f_n) \) be a sequence of functions which are all analytic and without zeros in a region \( \Omega \). Suppose, in addition, that \( f_n(z) \) tends to \( f(z) \), uniformly on every compact subset of \( \Omega \). Then \( f(z) \) is either identically zero or never equal to zero in \( \Omega \).

\[ \text{Fact 3. } U(\mathbb{C}) \text{ is closed under multiplication and extracting factors.} \]

That is, \( f(x)g(x) \in U(\mathbb{C}) \) iff \( f(x) \in U(\mathbb{C}) \) and \( g(x) \in U(\mathbb{C}) \).

Proof. This is also immediate from the definition of \( U(\mathbb{C}) \).

Another construction that preserves upper polynomials is the reversal of one variable. Note the introduction of a minus sign.

\[ \text{Fact 4. If } \sum_0^n f_i(x) y^i \in U(\mathbb{C}) \text{ then } \sum_0^n f_i(x) (-y)^{n-i} \in U(\mathbb{C}). \]

Proof. If \( g(x, y) = \sum f_i(x)y^i \) then
\[
y^n g(x, -1/y) = \sum_0^n f_i(x)(-y)^{n-i}
\]

If \( \Im(\sigma) > 0 \) then \( \Im(-\frac{1}{\sigma}) > 0 \), so \( g(x, -1/y) \) doesn’t vanish on the upper half plane.

**Example 1.** We can reverse some polynomials determined by matrices. If \( X = \text{diag}(x_1, \ldots, x_n) \) and \( A \) is \( n \times n \) then the reverse with respect to \( x_1, \ldots, x_n \) of \( |X + A| \) is \( |-I + XA| \).

\[ \text{Fact 5. } U(\mathbb{C}) \text{ is closed under differentiation.} \]

That is, if \( f(x) \in U(\mathbb{C}) \) then \( \partial_{x_i} f(x) \in U(\mathbb{C}) \cup \{0\} \).

Proof. We will show that if \( f(x) \in U(\mathbb{C}) \) then \( \partial_{x_i} f(x) \in U(\mathbb{C}) \cup \{0\} \). If \( \sigma_2, \ldots, \sigma_d \) are in the upper half plane then it suffices to show \( \frac{d}{dx} g(x) \in U_1(\mathbb{C}) \) where
\[
g(x) = f(x, \sigma_2, \ldots, \sigma_d)
\]

By hypothesis \( g \) has no roots in the upper half plane, so by the Gauss-Lucas theorem all roots of \( g' \) lie in the convex hull of the roots of \( g \), and so do not lie in the upper half plane.

\[ \text{Fact 6. If } \sum f_i(x)y^i \in U(\mathbb{C}) \text{ then all coefficients } f_i(x) \text{ are in } U(\mathbb{C}) \cup \{0\}. \]
Proof. If we differentiate \( i \) times with respect to \( y \) we get

\[
i!f_i(x) + y(i+1)!f_{i+1}(x) + \cdots + \frac{n!}{(n-i)!}y^{n-i}f_n(x) \in \mathbb{U}(\mathbb{C}) \cup \{0\}
\]

Now substitute \( y = 0 \). \( \square \)

**Remark 1.** Coefficients can certainly be zero. Consider \( x^2 - 1 \).

**Definition 2.** We say that \( f(x), g(x) \) interlace, written \( f \leftarrow^U g \), if and only if \( f(x) + yg(x) \in \mathbb{U}(\mathbb{C}) \).

See Remark 4 for the connection with the usual definition of interlacing in terms of roots.

**Remark 2.** Positive constants interlace linear functions. That is, if \( \ell = \sum a_i x_i + b \) where all \( a_i \) are positive and \( c > 0 \) then \( \sum a_i x_i + cy \in \mathbb{U}(\mathbb{C}) \), so \( \ell \leftarrow^U c \).

**Fact 7.** Consecutive coefficients interlace.

That is, if \( \sum_0^n f_i(x)y^i \in \mathbb{U}(\mathbb{C}) \) then \( f_i(x) \leftarrow^U f_{i+1}(x) \) for \( i = 0, \ldots, n-1 \), provided \( f_i \) and \( f_{i+1} \) are not both zero.

Proof. Differentiate \( i \) times with respect to \( y \)

\[
i!f_i(x) + y(i+1)!f_{i+1}(x) + \cdots + \frac{n!}{(n-i)!}y^{n-i}f_n(x) \in \mathbb{U}(\mathbb{C}) \cup \{0\}
\]

Reverse with respect to \( y \)

\[
\frac{n!}{(n-i)!}f_n + \cdots + (-y)^{n-i-1}(i+1)!f_{i+1} + (-y)^{n-i}i!f_i \in \mathbb{U}(\mathbb{C}) \cup \{0\}
\]

Differentiate \( n - i - 2 \) times with respect to \( y \)

\[
(n - i - 1)!(i+1)!(-1)^{n-i-1}f_{i+1} + (n - i)!i!(-1)^{n-i}yf_i \in \mathbb{U}(\mathbb{C}) \cup \{0\}
\]

Reversing again, factoring out constants, and rescaling \( y \) yields the result. \( \square \)

**Fact 8.** If \( f \in \mathbb{U}(\mathbb{C}) \) then \( f \leftarrow^U \partial_{x_i} f(x) \).

**Proof.** Expanding into a Taylor series

\[
f(x_1, \ldots, x_i + y, \ldots, x_d) = f(x) + \frac{\partial f(x)}{\partial x_i}y + \cdots
\]

shows that \( f \leftarrow^U \frac{\partial f}{\partial x_i} \) since they are consecutive coefficients of a polynomial in \( \mathbb{U}(\mathbb{C}) \). \( \square \)

**Fact 9.**

1. \( fg \leftarrow^U fh \) iff \( f \in \mathbb{U}(\mathbb{C}) \) and \( g \leftarrow^U h \).
2. If \( f \leftarrow^U g \) then \( g \leftarrow^U -f \).
3. If \( f \leftarrow^U g \) and \( f \leftarrow^U h \) then \( f \leftarrow^U g + h \).
(4) If \( f \leftarrow U g \) and \( h \leftarrow U g \) then \( f + h \leftarrow U g \).

(5) If \( f \leftarrow U g \) and \( h \leftarrow U g \) then \( f - h \leftarrow U g \).

Proof. If \( fg \leftarrow U fh \) then \( fg + yfh = f(g + yh) \in \mathbb{U}(\mathbb{C}) \) so \( g + yh \in \mathbb{U}(\mathbb{C}) \).

If \( f + yg \in \mathbb{U}(\mathbb{C}) \) then the reverse is \( g - yf \), so \( g \leftarrow U f \).

Next, if \( f \leftarrow U g \) and \( f \leftarrow U h \) then \( f + yg \) and \( f + yh \) are in \( \mathbb{U}(\mathbb{C}) \), so their product
\[
f^2 + yf(g + h) + y^2gh \in \mathbb{U}(\mathbb{C})
\]
Thus \( f^2 \leftarrow U f(g + h) \) which implies that \( f \leftarrow U g + h \).

(4) is similar to (3). For (5), apply (4) to \( f \leftarrow U g \) and \( -h \leftarrow U g \). \(\square\)

The last property is especially useful. For instance, we have recurrences that are just like the recurrences for orthogonal polynomials in one variable.

Fact 10. Suppose \( f_0 = 1, f_1 = \sum a_i x_i + b \) where the \( a_i \) are positive. If all constants \( a_{nk} \) and \( c_k \) are positive and
\[
p_{n+1} = \left( \sum_k a_{nk} x_k + b_n \right) f_n - c_n f_{n-1}
\]
then
\[
\cdots \leftarrow U p_n \leftarrow U p_{n-1} \leftarrow U \cdots p_1 \leftarrow U p_1
\]

Proof. We prove by induction that \( p_n \leftarrow U p_{n-1} \). This follows from the interlacings
\[
\left( \sum a_{nk} x_k + b_n \right) f_n \leftarrow U f_n \leftarrow U c_n f_{n-1}
\]
and Fact \(\square\)

Definition 3. If \( f(x) \) is a polynomial then \( f^H(x) \) is the sum of all terms with highest total degree.

Lemma 1. Suppose that \( f(x) \in \mathbb{U}_d(\mathbb{C}) \) is an upper polynomial of degree \( n \).

1. \( f^H(x) \) is homogeneous, and an upper polynomial.

2. \( f^H \) is the limit of homogeneous upper polynomials such that all monomials of degree \( n \) have non-zero coefficient.

3. All the coefficients of \( f^H \) have the same argument.

Proof. The first part follows from Hurwitz’s theorem and the fact that \( f^H(x) \) equals \( \lim_{\epsilon \to 0} \epsilon^n f(x/\epsilon) \). For the next part, define
\[
f_{\epsilon} = f\left( \sum_{j=1}^d \epsilon_{1j} x_i, \cdots \sum_{j=1}^d \epsilon_{dj} x_i \right)
\]
where all \( \epsilon_{ij} \) are positive. By Fact \(\square\) \( f_{\epsilon} \) is an upper polynomial, and it converges to \( f \) as we let \( \epsilon_{ij} \to 1 \) and \( \epsilon_{ij} \to 0 \), for \( i \neq j \). For index sets \( I, J, K \) of degree \( n \) each non-zero monomial \( x^I \) in \( f_{\epsilon} \) contributes a non-zero coefficient to \( x^J \), which is different from the contribution of \( x^K \), for \( I \neq K \).
Thus, the coefficient of $x^j$ is a non-zero polynomial in the $\epsilon_{ij}$’s, and hence is non-zero for $\epsilon_{ij}$ close to 1, and $\epsilon_{ij}$ close to zero ($i \neq j$).

By the second part we may assume that all coefficients of monomials of degree $n$ are non-zero. For any index set with $|i| = n - 1$ the polynomial $\partial_{x^j} f(x)$ is linear, so all the coefficients have the same argument. It follows that if $i, j$ satisfy $|i - j| = 1$ then the coefficients of $x^j$ and $x^l$ have the same argument. Since all the monomials of degree $n$ have non-zero coefficient, it follows that all these coefficients have the same argument. □

We can determine if a polynomial is an upper polynomial, if two upper polynomials interlace, or if two real upper polynomials are constant multiples of one another by reduction to properties of polynomials of one variable.

**Fact 11.**

1. $f(x) \in U_l(\mathbb{C})$ iff $f(a + xb) \in U_l(\mathbb{C})$ for all vectors $a$, and all vectors $b$ with all positive coordinates.

2. $f(x) \leftarrow U g(x)$ iff $f(a + xb) \leftarrow U g(a + xb)$ for all vectors $a$, and all vectors $b$ with all positive coordinates.

3. Suppose that $f, g \in U(\mathbb{C})$ have all real coefficients. If for all $a$ and $b > 0$ there is a constant $c_{a,b}$ so that $f(a + xb) = c_{a,b}g(a + xb)$ then $f$ and $g$ are constant multiples of one another.

**Proof.** We begin with (1). The first direction follows from Fact 11. Conversely, suppose that $\sigma_1, \ldots, \sigma_d$ are in the upper half plane. If we choose $\sigma$ to have smaller positive imaginary part than any of $\sigma_1, \ldots, \sigma_d$ then we can find $a_i$ and positive $b_i$ so that $\sigma_i = a_i + b_i\sigma$. Thus $f(\sigma_1, \ldots, \sigma_d) = f(a + \sigma b) \neq 0$ since $f(a + yb) \in U_l(\mathbb{C})$.

For the second one, one direction is trivial. Conversely, assume that $f(a + xb) \leftarrow U g(a + xb)$ for all $a$ and $b$ as before. By definition this means that $f(a + xb) + yg(a + xb) \in U_2(\mathbb{C})$. By Fact 11 we can substitute $a + bx$ for $y$, and thus by the first part $f(x) + yg(x) \in U(\mathbb{C})$.

For (3), first observe that the constant term of $f(a + xb)$ is $f(a)$. The leading coefficient of $f(a + xb)$ is $f_H(b)$. It follows that $f(a) = c_{a,b}g(a)$ and $f_H(b) = c_{a,b}g(b)$

Since all coefficients of $g_H$ have the same argument we can choose vectors $a', b' > 0$ so that $g(a') \neq 0$ and $g_H(b') \neq 0$. It follows that $c_{a,b} = c_{a,b'} = c_{a',b'}$ so that $c_{a,b}$ is constant, which yields the conclusion. □

Interlacing is essentially reflexive.

**Fact 12.** Suppose $f, g \in U(\mathbb{C})$

1. $f^2 + g^2 \in U(\mathbb{C})$ iff $f$ and $g$ are constant multiples of one another.
If $f U \leftarrow g$ and $g U \leftarrow f$ then $f$ and $g$ are constant multiples of one another.

Proof. Choose $a, b > 0$. If $f(a + x b)^2 + g(a + x b)^2$ has all real roots then clearly $f(a + x b)$ and $g(a + x b)$ are constant multiples of each other. It follows that $f$ and $g$ are constant multiples also.

In the second part we know $f + y g$ and $g + y f$ are upper polynomials so $$ (f + y g)(g + y f) = fg + (f^2 + g^2)y + fgy^2 \in U(\mathbb{C}) $$

The coefficient of $y$ is a upper polynomial, so the first part finishes the proof. \qed

The real and complex parts of an upper polynomial are also upper polynomials. In one variable this is the well-known

**Theorem 2 (Hermite-Biehler).** If $f(x)$ is a polynomial with no roots in the upper half plane and we write $f(x) = g(x) + i h(x)$ where $g$ and $h$ have all real coefficients then $g U \leftarrow h$.

**Fact 13.** Suppose $f(x)$ is a polynomial and we write $f(x) = g(x) + i h(x)$ where $g$ and $h$ have all real coefficients. Then $f \in U(\mathbb{C})$ iff $g U \leftarrow h$.

Proof. If $g U \leftarrow h$ then $g + y h \in U(\mathbb{C})$, and if we substitute $y = i$ we find that $f \in U(\mathbb{C})$. Conversely, choose vectors $a$ and positive $b$. We know that $f(a + x b) = g(a + x b) + i h(a + x b) \in U_1(\mathbb{C})$. By the Hermite-Biehler theorem for one variable we conclude that $g(a + x b) U \leftarrow h(a + x b)$, and therefore $g(x) U \leftarrow h(x)$. \qed

## 2. Real coefficients

**Definition 4.** $U_d$ consists of all polynomials in $U_d(\mathbb{C})$ with all real coefficients. $U$ is all polynomials in $U(\mathbb{C})$ with all real coefficients. We call such polynomials real upper polynomials.

The real upper polynomials in one variable are those polynomials with all real coefficients and all real roots.

Interlacing is equivalent to closure under linear combinations. In one variable this is called Obreschkoff’s theorem.[9]

**Fact 14.** Suppose $f, g \in U$. The following are equivalent

1. $\alpha f + \beta g \in U \cup \{0\}$ for $\alpha, \beta \in \mathbb{R}$
2. Either $f U \leftarrow g$ or $g U \leftarrow f$.

Proof. (2) implies (1) follows from Fact[9] Conversely, choose vectors $a, b > 0$ and let $T = a + x b$. Since $\alpha f(x) + \beta g(x) \in U \cup \{0\}$ we know $\alpha f(T) + \beta g(T) \in U_1 \cup \{0\}$. By Obreschkoff’s theorem we know that either $f(T) U \leftarrow g(T)$ or $g(T) U \leftarrow f(T)$. If only one of these possibilities occurs then $f(x)$ and $g(x)$ interlace.
If both of these possibilities occur then by continuity we can find a $T$ for which we have $f(T) \xrightarrow{U} g(T)$ and $g(T) \xrightarrow{U} f(T)$. It follows that $f(T) = \gamma g(T)$ for some real $\lambda$, and hence $(f - \lambda g)(T) = 0$. Since $f - \lambda g \in \mathbb{U} \cup \{0\}$ it follows from Fact [1] that $f - \lambda g = 0$. □

We have a general construction of real upper polynomials. The following lemma is easily proved.

**Lemma 2.** If $A, B$ are symmetric and either one is positive definite then $|A + iB|$ is not zero.

**Fact 15.** If $D_1, \ldots, D_d$ are positive definite $n$ by $n$ matrices, $E$ is positive semi-definite, and $S$ is symmetric then

$$|S + \sum_{i=1}^d x_i D_i| \in \mathbb{U}_d \quad |S + iE + \sum_{i=1}^d x_i D_i| \in \mathbb{U}_d (\mathbb{C})$$

**Proof.** If $\sigma_1, \ldots, \sigma_d$ are in the upper half plane and $\sigma_k = \alpha_k + i\beta_k$ then

$$|S + \sum_{i=1}^d x_i D_k| (\sigma_1, \ldots, \sigma_d) = |S + \sum \sigma_k D_k|$$

$$= |(S + \sum \alpha_k D_k) + i(\sum \beta_k D_k)|$$

and this is non-zero by the lemma since $\beta_k > 0$ and so $\sum \beta_k D_k$ is positive definite. The second part is similar, and uses the fact that $E + \sum \beta_k D_k$ is positive definite. □

**Remark 3.** The Lax conjecture (now solved [8]) gives a converse for $n = 2$. It says that if $f \in \mathbb{U}_2$ then we can find symmetric $A$ and positive semi-definite $D_i$ so that $|A + xD_1 + yD_2| = f(x, y)$.

Next is a generalization of Fact [15] that involves sums of determinants [17]. If $S \subset \{1, \ldots, n\}$ and $A$ is an $n$ by $n$ matrix then $A[S]$ is the submatrix of $A$ whose rows and columns are indexed by $S$.

**Fact 16.** Suppose $L_k = \sum_{i=1}^d x_i D_{ik} + A_k$ where the $D_{ik}$ are $n$ by $n$ positive definite matrices and the $A_k$ are symmetric. The following is a real upper polynomial

$$\left( \sum_{S_1 \cup \cdots \cup S_m = \{1, \ldots, n\}} |L_1[S_1]| \cdots |L_m[S_m]| \right)$$

**Proof.** If $W = \text{diag}(w_1, \ldots, w_n)$ then Fact [15] shows that $|W + L_k|$ is an upper polynomial. Thus the reverse with respect to $w_1, \ldots, w_n$ is $|I - WL_k|$ and is also an upper polynomial. The product

$$\prod_{k=1}^n |I - WL_k| = \prod_{i=1}^n \sum_{S \subset \{1, \ldots, n\}} (-w)^{|S|} |L_k[S]|$$

is an upper polynomial, and the coefficient of $w_1 \cdots w_n$ is [11]. □
If the coefficients are real we can reverse all the variables – without a minus sign.

**Fact 17.** If \( f(x) \in U_d \) and if \( x_i \) has degree \( e_i \) then \( x_1^{e_1} \cdots x_d^{e_d} f(1/x_1, \ldots, 1/x_d) \in U_d \).

*Proof.* If \( \sigma_i \) is in the upper half plane then \( 1/\sigma_i \) is also. Thus
\[
f(1/\sigma_1, \ldots, 1/\sigma_d) = f(1/\sigma_1, \ldots, 1/\sigma_d) \neq 0.\]
\[\square\]

**Fact 18.** If \( a, b, c, d \) are real then \( f(x, y) = a + bx + cy + dxy \in U_2 \) iff 
\[
\begin{vmatrix}
  b & a \\
  d & c
\end{vmatrix} \geq 0.
\]

*Proof.* If the determinant is zero then there is a \( \lambda \) so that 
\[
b\lambda = d \quad \text{and} \quad a\lambda = c,
\]
and so \( f(x, y) = (a + bx)(1 + \lambda y) \) is in \( U_2(C) \). If the determinant is not zero then solving \( f = 0 \) yields
\[
y = \frac{a + bx}{c + dx}.
\]

If \( M \) is the Möbius transformation with matrix 
\[
\begin{vmatrix}
  b & a \\
  d & c
\end{vmatrix},
\]
f \( U_2(C) \) and \( x \) is in the upper half plane then \( -Mx \) is in the complement. Thus \( M \) maps the upper half plane to itself. This happens exactly when the determinant is positive. \( \square \)

This result can be generalized \[3\].

**Fact 19.** Suppose that \( f(x) \) is a polynomial where every variable has degree 1. Then \( f(x) \) is in \( U_d \) iff
\[
\frac{\partial f}{\partial x_i} \cdot \frac{\partial f}{\partial x_j} - f \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \geq 0
\]
for all \( x \geq 0 \) and \( 1 \leq i, j \leq d \).

The next two facts are simple consequences of Fact \[18\].

**Fact 20.** If \( f \xrightarrow{U} g \) and \( f, g \in U_d \) then 
\[
\begin{vmatrix}
  \frac{\partial f}{\partial x_1} & \frac{\partial g}{\partial x_1} \\
  \frac{\partial f}{\partial x_2} & \frac{\partial g}{\partial x_2}
\end{vmatrix} \leq 0.
\]

*Proof.* Since \( f + yg \in U \) so is the Taylor series
\[
f(x_1 + z, x_2, \ldots, x_d) + yg(x_1 + z, x_2, \ldots, x_d) = f(x) + z \frac{\partial f}{\partial x_1} + yg(x) + yz \frac{\partial g}{\partial x_1} + \cdots
\]
Reversing, differentiating, and reversing shows that
\[
f(x) + z \frac{\partial f}{\partial x_1} + yg(x) + yz \frac{\partial g}{\partial x_1} \in U
\]
so we can apply Fact \[18\]. \( \square \)

The next fact can be stated for \( d \) variables, but it is fundamentally a property of two variable polynomials.
Fact 21. If $\sum a_{i,j}x^iy^j \in U_2$ then $\left| \begin{array}{cc} a_{r+1,s} & a_{r+1,s+1} \\ a_{r,s} & a_{r,s+1} \end{array} \right| \leq 0$ for $r, s \geq 0$.

Proof. We differentiate, reverse, and differentiate first with respect to $x$ and then to $y$ so that the $x$ and $y$ degrees equal 1. The result is

$r!s!a_{r,s} + (r+1)!s!a_{r+1,s}x + r!(s+1)!a_{r,s+1}y + (r+1)!(s+1)!a_{r+1,s+1}xy$

By Fact 18

$0 \geq r!s!a_{r,s} + (r+1)!s!a_{r+1,s}x + r!(s+1)!a_{r,s+1}y + (r+1)!(s+1)!a_{r+1,s+1}xy$

Remark 4. Interlacing in $U_1$ is closely related to the usual definition of interlacing in terms of the location of roots. Suppose $f, g \in U_1$, and $f$ has positive leading coefficient. We assume the roots of $fg$ are all distinct and that the roots of $f$ and $g$ alternate, with the largest root belonging to $f$. There are various cases depending on the sign of the leading coefficient of $g$, and the degrees.

| Degree of $f$ | Degree of $g$ | Leading coefficient of $g$ | Interlacing |
|---------------|---------------|----------------------------|-------------|
| $n$           | $n-1$         | $+$                        | $f \leftarrow g$ |
| $n$           | $n-1$         | $-$                        | $f \rightarrow -g$ |
| $n$           | $n$           | $+$                        | $f \leftarrow g$ |
| $n$           | $n$           | $-$                        | $f \rightarrow -g$ |

In one direction this is proved by explicit constructions that $f + yg$ (and its variants) are in $U_2$. In the other direction we use Obreschkoff’s theorem to conclude that they interlace in the traditional sense. The reflexivity of interlacing implies only the answer in the table is possible.

3. Analytic closure

Definition 5. $\widehat{U}_d$ is the uniform closure on compact subsets of $U_d(\mathbb{C})$.

Fact 22. $e^{-xy} \in \widehat{U}_2$ and $e^{-xy} \in \widehat{U}_d$.

Proof. We know $1 - x_1y_1/n \in U_2(\mathbb{C})$, so $\lim_{n \to \infty} (1 - x_1y_1/n)^n = e^{-x_1y_1} \in \widehat{U}_2$. Closure under multiplication implies

$e^{-xy} = e^{-x_1y_1} \cdots e^{-x_dy_d} \in \widehat{U}_d$.

Setting $x = y$ establishes the second part. □

Fact 23. If $f(x, y) \in U_{2d}(\mathbb{C})$ then $e^{-\partial_x \partial_y} f(x, y) \in U_{2d}(\mathbb{C})$. 
Proof. If \( f \in \mathcal{U} (\mathbb{C}) \) then the interlacing of derivatives implies

\[
f \mapsto \partial_{x_i} \left( \frac{1}{n} \partial_{y_i} (\partial_{x_i} f) \right) \]

and from Fact 9

\[
f - \frac{1}{n} \partial_{y_i} \partial_{x_i} f = \left( 1 - \frac{\partial_{x_i} \partial_{y_i}}{n} \right) f \in \mathcal{U} (\mathbb{C})
\]

Iterating this shows that the map \( f \mapsto \left( 1 - \frac{\partial_{x_i} \partial_{y_i}}{n} \right)^n f \) sends \( \mathcal{U} (\mathbb{C}) \) to itself.

Taking limits shows that \( f \mapsto e^{-\partial_{x_i} \partial_{y_i}} f \) also maps \( \mathcal{U} (\mathbb{C}) \) to itself. Composing these transformations for \( i = 1, \ldots, d \) yields

\[
e^{-\partial_{x} \partial_{y}} f = e^{-\partial_{x_1} \partial_{y_1}} \cdots e^{-\partial_{x_d} \partial_{y_d}} f \in \mathcal{U} (\mathbb{C})
\]

\[ \square \]

**Fact 24.** If \( f(x) \in \mathcal{U}_d (\mathbb{C}) \) and \( g(x) \in \mathcal{U}_d (\mathbb{C}) \) then \( f(-\partial_{x}) g(x) \in \mathcal{U}_d (\mathbb{C}) \cup \{0\} \). This also holds for \( f \in \hat{\mathcal{U}}_P^d \).

Proof. Since \( e^{-\partial_{x} \partial_{y}} \) maps \( \mathcal{U}_d (\mathbb{C}) \) to itself the lemma follows from the identity

\[
e^{-\partial_{x} \partial_{y}} g(x) f(y) \bigg|_{y=0} = f(-\partial_{x}) g(x).
\]

By linearity we only need to check it for monomials since \( e^{-\partial_{x} \partial_{y}} g(x) f(y) \in \mathcal{U} (\mathbb{C}) \), and evaluation at \( y = 0 \) is in \( \mathcal{U}(\mathbb{C}) \cup \{0\} \).

\[
e^{-\partial_{x} \partial_{y}} x^I y^J \bigg|_{y=0} = \left( -\frac{\partial_{x} \partial_{y}}{J!} \right)^J x^I \cdot (\partial_{y})^J y^J \bigg|_{y=0} = (-\partial_{x} \partial_{y})^J x^I
\]

All polynomials involved have bounded degree so the passage to the limit presents no problems. \( \square \)

**Fact 25.** Suppose \( T \) is a non-trivial linear transformation defined on polynomials in \( d \) variables. \( T(e^{-x \cdot y}) \in \hat{\mathcal{U}}_P^d \) if and only if \( T \) maps \( \mathcal{U}_d (\mathbb{C}) \cup \{0\} \) to itself.

Proof. Since \( f(-\partial_{y}) \) maps \( \hat{\mathcal{U}} \cup \{0\} \) to itself one direction follows from the identity

\[
f(-\partial_{y}) T(e^{-x \cdot y}) \big|_{y=0} = T(f).
\]

We only need to verify this on monomials:

\[
(-\partial_{y}) T(e^{-x \cdot y}) \big|_{y=0} = (-\partial_{y}) \sum_{I} T(x^I) \frac{(-y)^J}{J!} \bigg|_{y=0} = T(x^I)
\]

The converse can be found in \( [\Pi] \), where \( T \) non-trivial means that \( T(\mathcal{U}(\mathbb{C})) \) has dimension at least 3. \( \square \)
Example 2. The expression $T(x^{-x}y)$ is known as the generating function of $T$. If $H_i(x)$ is the $i$th Hermite polynomial then the linear transformation $T : x_1^i \cdots x_d^i \mapsto H_i(x_1) \cdots H_i(x_d)$ has generating function
\[
\prod_{i=1}^{d} e^{-2x_i y_i y_i^2} = e^{-2x y y}
\]
Since this is in $\mathfrak{U}_{2d}$ it follows that $T$ maps $\mathfrak{U}_d(\mathbb{C})$ to itself.

Fact 26. Suppose that $f(x, y)$ is a polynomial, and define $T(g) = f(x, -\partial_y)g$. The following are equivalent.

1. $T : \mathfrak{U}_d(\mathbb{C}) \rightarrow \mathfrak{U}_d(\mathbb{C}) \cup 0$.
2. $f(x, y) \in \mathfrak{U}_{2d}(\mathbb{C})$.

Proof. The generating function of $T$ is
\[
f(x, -\partial_y)e^{-x u y v} \big|_{u=v=0} = f(x, v)
\]
so the result follows from Fact 25.

4. Uniqueness results

We end with some interesting uniqueness results that we present with only sketches of the proof.

Fact 27. [1] Suppose that $T$ is a linear transformation that maps $\mathfrak{U}_d \rightarrow \mathfrak{U}_d \cup \{0\}$. If $T(x^i) = a_i x^i$ then $T$ is a composition of one-dimensional transformations. That is, there are linear transformations $T_i : \mathcal{E}_1 \rightarrow \mathcal{E}_1 \cup \{0\}$ of the form $T_i(x_i^k) = a_{ik} x_i^k$ and $T = T_1 \cdots T_d$.

Proof. It suffices to assume $d = 2$, so write $T(x^i y^j) = a_{i,j} x^i y^j$. The inequality of Fact 21 applied to $T(x^i y^j (1 + x) (1 + y))$ and $T(x^i y^j (1 + x) (1 - y))$ yields that $a_{ij} a_{i+1,j+1} = a_{i+1,j} a_{i,j+1}$. We use this identity to prove induc-tively that $a_{ij} = a_{0,0} a_{ij}$. $T$ is now the product of the two transformations $x^i \mapsto a_{i,0} x^i$ and $y^j \mapsto a_{0,j} y^j$.

Fact 28. [2] If $T$ is a linear transformation on $\mathfrak{U}_d$ such that

1. $T$ reduces degree.
2. $f \xrightarrow{U} T f$
then there are $a_1, \ldots, a_d$ of the same sign so that
\[
Tf = a_1 \frac{\partial f}{\partial x_1} + \cdots + a_d \frac{\partial f}{\partial x_d}
\]

Proof. We only discuss $d = 1$. We use the fact that $\alpha(x + b)^{n-1}$ is the only polynomial of smaller degree interlacing $(x + b)^n$. Choosing $f = x^n$ shows that $T(x^n) = a_n x^{n-1}$. Choosing $f = (x + 1)^n$ shows
\[
T(x + 1)^n = \sum a_i \binom{n}{i} x^i = \alpha \sum \binom{n-1}{i} x^i
\]
and equating coefficients yields $T(x^n) = a_1 nx^{n-1}$.

**Fact 29.** If $T$ is a linear transformation such both $T$ and $T^{-1}$ map $U_1$ to itself then $(Tf)(x) = af(bx + c)$ where $ab \neq 0$.

**Proof.** We may assume that $T(1) = 1$ and $T(x) = x$. Since $T(x^n) \xrightarrow{U} T(x^n)'$ we have that $x^n \xrightarrow{U} T^{-1}(T(x^n))'$, so the linear transformation $S(f) = T^{-1}(T(f)')$ satisfies the hypothesis of the previous fact. Consequently $T^{-1}(T(x^n))' = \alpha_0(x^n)'$ and hence $T(f') = (Tf)'$. We use this inductively to show that $T(x^n) = x^n$.

**Fact 30.** If $f(x, y) \in U_2$ has the property that all exponents of $x$ and $y$ are even then $f(x, y) = g(x)h(y)$ with $g, h \in U_1$.

**Proof.** This is a special case of the following result:

If $f(x, y) = \cdots + f_i-1(x)y^{i-1} + \cdots + f_i+1(x)y^{i+1} + \cdots \in U_2$

where $f_{i-1}f_{i+1} \neq 0$ then $f(x, y) = g(x)h(y)$ where $f, g \in U_1$.

To prove this we differentiate, reverse, and differentiate so that we only have $f_{i-1}(x) + f_{i+1}(x)y^2$. It follows that $f_{i+1}$ is a constant multiple of $f_{i-1}$. Continuing this argument concludes the proof.

5. **Questions**

Here are a few unsolved questions.

**Question 1.** Suppose that $f(x), g(x) \in U_d$ have the property that $f + \alpha g \in U_d$ for all positive $\alpha$. Show that there is an $h \in U_d$ so that $h \xrightarrow{U} f$ and $h \xleftarrow{U} g$. This is easy if $d = 1$.

**Question 2.** If $T$ is a bijection on $U_d$ then are there constants $a > 0$, $b, \alpha$ and a permutation $\sigma$ of $\{1, \ldots, d\}$ so that

$$T(f(x)) = \alpha f(a_1 x_{\sigma 1} + b_1, \ldots, a_d x_{\sigma d} + b_d)?$$

**Question 3.**

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