Generalized Störmer and dynamical behavior of charged particles near magnetic planet.

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Abstract. We study the motion of charged dust grains at arbitrary distances from a rotating magnetic planet, we show that the model treated for dipole field is an excellent approximation in most cases for a generalized Störmer problem, but that fundamentally new physics arises with the inclusion of a quadrupolar magnetic term. In particular, the appearance of a new trajectories for different kinds of charged dust grains, which are meaningful when the individual and a comparable impact of the co-rotational and gravitational fields is considered. Furthermore, a detailed mathematical treatment of equilibrium state for various phase spaces for charged components in prograde or retrograde rotation direction with analytical solutions for a non-linear parametric system.

1. Introduction
Investigation of charged particles dynamics subject to the magnetic dipole field has a large significance in astrophysics, while their tiny fraction of the mass in orbit around a planet [1]. It was first introduced in the pioneering work of Störmer (1955) [2], mainly the theoretical study in planetary magnetospheres and it was first considered by Mendis and Axford (1974) in order to explain the origin of the accelerated mysterious dark spokes in Saturn's B ring in space observed by Voyager 1 and 2 [3]. This treatment has been of the interest to unveil the structure of force fields acting on the particles and their behavior [4]. It's evident that the starting point to describe the motion of charged dust particles in a purely magnetic field orbiting a planet with a magnetosphere is the Classical Störmer problem [5]. This model gives the basic physical explanations of radiation belts surrounding a magnetized planet composed of light particles (ions or electrons) whose motion is affected only by magnetic forces [6]. However, the consideration of the charged dust grain is constrained by the much small ratios between charge-to-mass, which complicate the dynamics for heavier one, where the planetary gravity and the corotational electric field must also be taken into account and the dynamics is more rich and complicated and define the Generalized Störmer problem. This treatment characterized by simplified assumptions of keplerian gravity, specific magnetic field (pure dipole), constant charge without radiation pressure effect. We mention the four distinct Störmer problems: CSP (Classical Störmer Problem) in which a charged particle move in a pure dipole magnetic field, RSP (Rotational Störmer problem) characterized by the electric field due to the planet rotation, GSP (Gravitational Störmer problem) only the keplerian gravity is included and the full interesting system (RGSP) (Rotational Gravitational Störmer Problem) counting both fields.
Many analytical models have been developed to study this problem on the way to get various aspects to answer open questions, much work has been done on dynamics of charged dust grains near non-spherical planets (prolate and oblate) [8] and also near a black holes and compact stars under the influence of a strong magnetic fields [4], Besides this, the study of the stability of charged dust grains orbiting a planet characterized by all charge to mass ratios (from ions to rocks) [9] and the interesting investigation of sub-microns circumplanetary dust grains in multipolar fields case, with the inclusion of non-axisymmetric magnetic field terms which gives more interesting results compared to the aligned axisymmetric one for the motion in a dipole field given by [10] In addition, the behavior of grains near solar corona with the influence of the solar radiation pression [11] and the recent work of point injection of charged particles into a magnetic dipole field [12].

In the present work we introduce the effect of a quadrupolar magnetic field derived from the general case included the dipole magnetic field on the Generalized Störmer Problem. The main goal is to perform a detailed analytical treatment for axisymmetric magnetized planet considering different parameters [13]. We will analyse the dynamics of charged dust particles on our proposed model, and verify the magnetic dipole model approximation [6] and the Relative equilibria to map the trajectories of the new system by giving specify Hamiltonian and effective potential.

2. A Model description

Our starting point is based on [6] approach. We assume a particle of mass $m$ and charge $q$ is orbiting around a rotating magnetic planet (aligned centered planet) [13] with mass $M$ and radius $R$. However, the presence of both gravitational and electromagnetic forces is crucial [14]. The general Hamiltonian of this particle in Gaussian units is done as:

$$H = \frac{1}{2} m (\mathbf{P} - \frac{q}{c} \mathbf{A}(r))^2 + U(r)$$

Where $c$ is a speed of light in a vacuum and where $r = (x, y, z)$ corresponds to the Cartesian coordinates or the particle position, and $\mathbf{P}$ the conjugate momenta of $r$. In addition, the vector potential is represented by $\mathbf{A}$, it describes magnetic forces given by:

$$\mathbf{A}(r) = \frac{\mu}{r^3} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$

And $U(r)$ is the scalar potential which generates the electric and gravitational interactions. The magnetic field $B$ of the magnetic planet is supposed to be a perfect magnetic dipole of strength $\mu$ aligned to the north-south poles of the planet which reads

$$B = \nabla \times A$$

Otherwise, the magnetosphere surrounding the planet is taken as rigid conducting plasma that rotates with the same angular velocity as the planet, where the charge $q$ is subject to a corotational electric field, in general it is defined as:

$$\mathbf{E} = -\frac{1}{c} (\Omega \times r) \times \mathbf{B}$$

In order to characterize the influence of the charge to mass ratio of the charged particle, we get the gravitational and corotational electric fields influence depending on the two parameters.
\[ \sigma_g \text{(connected to the gravitational field) and } \sigma_r \text{ (for the co-rotational electric field) which can take values 1 or 0 [6]} \]

In our Treatment to get more interesting results for the dynamics of charged dust grains near magnetic planet, The magnetic field expression is more detailed, similar to the geomagnetic field [15] it can be written in spherical coordinates as:

\[ \vec{B}(r, \theta, \varphi, t) = - \nabla V(r, \theta, \varphi, t) \]  

(5)

Where the potential is defined by the formulae:

\[ \vec{E} = - \frac{q}{c} \Omega \nabla \left( \frac{a^2}{r} \left(-g_1^0 \sin^2 \theta\right) + \left(\frac{a^3}{2r^2}\right)(3g_2^0 \sin^2 \theta \cos \theta) \right) \]  

(6)

The parametric equation represents the corotational electric field using spherical harmonics, we can find the treatment [6] as a special case of this expression \((a^2g_1^0 = 1 \text{ and } a^3g_2^0 = 0)\), where besides a magnetic dipole another quadrupolar term is considered. The term proportional to \(g\) is allowed in this study. the main question posed in this context is what is the results of the effect of this quadrupolar term on Strmer problem for dynamics of charged dust grains near magnetic planet.

For our contribution, the dynamics of charged dust grains is described by the Hamiltonian function in spherical coordinates with the effect of a quadrupolar magnetic term we define:

\[ H = \frac{1}{2} \left( P_r^2 + \frac{P_\theta^2}{r} \right) + \frac{1}{2} \omega^2 r^2 \sin^2 \theta - \frac{\sigma_g}{r} + \sigma_r \frac{a^2g_1^0 \sin^2 \theta}{r} + \sigma_\delta \frac{3a^3g_2^0 \sin^2 \theta \cos \theta}{2r^2} \]  

(8)

Here, to provide a comprehensive view of the behavior or the trajectories of the dust grains charged positively or negatively we take \(a^2g_1^0\) (connected to [6] treatment) and \(a^3g_2^0 = 1\) (for quadrupolar term contribution). We can write the Effective potential in the form:

\[ U_{eff} = \frac{1}{2} \omega^2 r^2 \sin^2 \theta + \sigma_r \frac{a^2g_1^0 \sin^2 \theta}{r} + \sigma_\delta \frac{3a^3g_2^0 \sin^2 \theta \cos \theta}{2r^2} - \frac{\sigma_g}{r} \]  

(9)

Where the considered potential is depending on the \((r, \theta)\) spherical variables, and other parameters of the system. Specifically, \(\delta\) the charge of the dust grain and the orbital frequency \(\omega\).

3. The relative equilibria with the effect of a quadrupolar magnetic term

In order to study charged particles equilibrium, it is convenient to define the critical points of the effective potential \(U_{eff}\) which are found as the solutions of the system of equations:

\[ \frac{\partial U_{eff}}{\partial r} = 0 \text{ and } \frac{\partial U_{eff}}{\partial \theta} = 0 \]

The Sign of the particle angular velocity defines the nature motion of the charged dust grains, prograde (positive charge sign) or retrograde (negative charge sign) with respect to the planet motion direction. The critical points of the \(U_{eff}\) are found as the solutions of the system equations

\[ \frac{\partial U_{eff}}{\partial r} = -\omega^2 r \sin^2 \theta + \frac{a^2g_1^0}{r} (\omega - \sigma_r) \sin^2 \theta + \frac{3a^3g_2^0 (\omega - \sigma_r) \sin^2 \theta \cos \theta}{2r^2} = 0 \]  

(10)

\[ \frac{\partial U_{eff}}{\partial \theta} = -\omega^2 r^2 \sin \theta \cos \theta - \frac{2a^2g_1^0}{r} \sin \theta \cos \theta + \frac{3a^3g_2^0 (\sigma_r - \omega) \sin^2 \theta \cos^2 \theta}{r^2} + \frac{3a^3g_2^0 (\omega - \sigma_r) \sin^3 \theta}{2r^2} = 0 \]  

(11)
The equivalent non linear system reads:

$$\frac{\partial U_{\text{eff}}}{\partial r} = -\omega^2 r \sin^2 \theta + \frac{1}{r} a^2 g_0^1 \delta (\omega - \sigma_r) \sin^2 \theta + \frac{1}{r^3} (3a^3 g_0^2 (\omega - \sigma_r) \sin^2 \theta \cos \theta = 0 \quad (12)$$

$$\frac{\partial U_{\text{eff}}}{\partial \theta} = -\sin \theta \cos \theta r \left( \omega^2 r^4 + 2a^2 g_0^1 \delta (\omega - \sigma_r) r + 3a^3 g_0^2 \delta (\omega - \sigma_r)(\cos \theta - \frac{\sin \theta \tan \theta}{2}) = 0 \quad (13)\right.$$ 

The system of non-linear equations gives a part of the dipolar magnetic field influence parameters and our contribution by terms related to the $a^3 g_0^2$, looking at the system it is evident that it depends on a radial and angular parameter $r$ and $\theta$ for both equations. Besides this to describe the trajectories of charged dust particles in the new conditions we use the methods derived from the model of the generalized Störmer problem [6] Furthermore, if $\theta = \frac{\pi}{2}$ we get equatorial orbits, and when $\theta \neq \frac{\pi}{2}$ we obtain the non-equatorial (Halo) orbits, where their existence is strictly relied on the values of $\omega$ and $\delta$

3.1. Equatorial orbits

Assuming that the motion is in the equatorial plane, such kind of orbits appear when $\theta = \frac{\pi}{2}$, and the second equation of the system vanishes, which satisfy the parametric equation:

$$\omega^2 r^4 + 2a^2 g_0^1 \delta (\omega - \sigma_r) r + 3a^3 g_0^2 \delta (\omega - \sigma_r)(\cos \theta - \frac{\sin \theta \tan \theta}{2}) = 0 \quad (14)$$

The motion of particles is well described in the plane $(\delta, \omega)$ for a generalized Störmer Problem

**Figure 1.** Equatorial orbits existence region for RGSP $a^3 g_0^2 = 1$

**Figure 2.** Equatorial orbits existence region for GSP $a^3 g_0^2 = 1$

cases, and the presence of trajectories is allowed depending on the distance from the center of the planet $r$ with retrograde and prograde orbits, and the situation is similar for the case $a^2 g_0^1 = a^3 g_0^2 = 1$, which proof that the equatorial orbits exists without the quadrupolar perturbations ($a^3 g_0^2 = 0$), all regions for all Störmer Problem Generalizations can be taken into account.
3.2. Halos orbits

Halos orbits (defined as orbits which do not cross the equatorial plane) appear when $\theta \neq \frac{\pi}{2}$ and their description is connected to the solution of the parametric equations depending respectively to the partial radial $r$ and angular $\theta$ parameters.

They are described by solution from a system of equations (12) and (13) given by:

$$Q(r, \omega) = 0$$ (15)

And

$$A(\theta, \omega) = 0$$ (16)

To calculate Halos orbits around magnetic planet, the constraint of the quadrupolar term is also considered, in this investigation we use purely analytical solutions to check the existence/persistence of these trajectories ($d_1 = a^2 g_0^0 \delta$ and $d_2 = a^3 g_2^0 \delta$). Which gives:

$$Q(r, \omega) = 324d_2^3(\sigma_r - \omega)^2 + 4r^4(-\sigma_g + d_1(\sigma_r - \omega) + r^3\omega^2)$$

$$((3\sigma_g - 2d_1(\sigma_r)^2 + 4d_1(3\sigma_g - 2d_1\sigma_r)\omega + 4(d_1^2 - 3\sigma_g r^3)\omega^2 + 8r^6\omega^4)) = 0$$ (17)

$$A(\theta, \omega) = \frac{27}{128}d_2^2(\sigma_r - \omega)^4(7923d_2^2\omega^2 + 1408d_1^4(\sigma_r - \omega)\cos \theta + 13320d_2^2\omega^2 \cos 2\theta$$

$$+ 576d_1^4(\sigma_r - \omega) \cos 3\theta + (7900d_2^2\omega^2 \cos(4\theta) + (64d_1^4(\sigma_r - \omega) \cos 5\theta))$$

$$- (8d_2^2(\sigma_g(\sigma_r - \omega))2\sigma_g \cos \theta(\sigma_g^2 + 18(d_1^2(\sigma_r - \omega)^2 \sin^4 \theta + \sin^2 (8d_1\sigma_g^2(\omega - \sigma_r) \sin(2\theta))$$

$$- (\sigma_g^3 + 2d_1(\sigma_r - \omega) \sin^2 \theta(-5\sigma_g^2 + 9d_1(\sigma_r - \omega) \sin^2 \theta(2\sigma_g + 3d_1(-\sigma_r + \omega) \sin^2 \theta)) \tan \theta = 0$$ (18)

It’s important to remark that a family of curves for the four known Generalized Störme cases, with the inclusion of the gravitational and co-rotational fields, with the individual effect too which is not the case in the work [6]
Our analysis has shown the possible families of curves defining different particles trajectories related to their nature on a one hand and the planet rotation direction for the four Störmer problem on the other hand. For the (CSP) (Classical Störmer Problem) in which a charged particle moves in pure dipole magnetic field. Figure (5) exhibits the orbits created in the [6] treatment and the case with the effect of the quadrupolar magnetic term, it appears a new orbits in regions not allowed in the literature model. Although the absence of both fields co-rotational and gravitational, orbits exist for constant radius at constant θ. For the individual effect of the gravitation force, the existence region is different from the Störmer generalizations and the orbits are more in different for positive radius from the center of the magnetic planet and fixed values of θ. Figure (6) In the case of the presence of both the electric and magnetic elds, i.e., adding the rotation to the magnetic eld(RGSP), the non-equatorial orbits exists in all regions for different charges, which the most interesting case in the dipolar magnetic field because it introduce simultaneous forces, and the original case appear modified and may be not circular for all cases and charges. Figure (7) In the case of the presence of both the electric and magnetic elds, i.e., adding the rotation to the magnetic eld(RGSP), the non-equatorial orbits exists in all regions for different charges, which the most interesting case in the dipolar magnetic field because it introduce simultaneous forces, and the original case appear modified and may be not circular for all cases and charges. Figure (8) In the case of the presence of both the electric and magnetic elds, i.e., adding the rotation to the magnetic eld(RGSP), the non-equatorial orbits exists in all regions for different charges, which the most interesting case in the dipolar magnetic field because it introduce simultaneous forces, and the original case appear modified and may be not circular for all cases and charges.
trajectories related to their nature on a one hand and the planet rotation direction for the four Störmer problem on the other hand.

3.3. Dynamical behavior and 3d plots

Now, we study the behaviour of the dynamics with the quadrupolar perturbation appears in Figures [9-16], and it’s evident that the quadrupolar has an important effect on a dipolar treatment. The bottom plots treat the original study near a pure dipole, and the top behavior is related to the perturbed cases.

Figure 9. 3D plot for GSP with $a^3g_2^0 = 1$

Figure 10. 3D plot for RSP with $a^3g_2^0 = 1$

Figure 11. 3D plot for CSP with $a^3g_2^0 = 1$

Figure 12. 3D plot for RGSP with $a^3g_2^0 = 1$
It is useful to treat all Generalized proposed cases for different charged particles (negative or positive). In figures 9–12 the radius from a center of the magnetic planet is constant, contrarily to the fourth other plots in figures 13–16 which are related to a fixed angular parameter $\theta$. We summarize that the individual and simultaneous effect of the gravitational and a co-rotational electric fields modify the dynamics behavior of charged dust grains particles located near a rotating magnetic planet and the quadrupolar perturbations gives interesting results.

4. Conclusion
We have studied the dynamics of charged dust particles around spherical magnetic planet subject to the effect of a quadrupolar magnetic term perturbation include the gravitation and corotational electric fields influence. Our analysis gives new assumptions compared to the work considered and axisymmetric combination of gravitational, and co-rotational electric fields for typical charge to mass ratio. The quadrupolar perturbation is interesting, and the dynamics is more developed, mainly by the influence on the dipolar model on one hand, and the occurrence of the orbits in the plane $\delta, \omega$ on the other hand. For equilibrium state the existence of equatorial orbits of charged dust grains is connected to the dipolar magnetic field conditions for physical reasons for all Störmer cases. For Halos (non-equatorial) orbits the situation is more
considerable, and their existence is meaningful with the simultaneous influence of co-rotational and gravitational fields, besides, the cases where the individual effects of gravity and electric field are not neglected gives supplementary trajectories in regions not allowed by the literature treatment, and the behavior is more complicated with interesting physics Furthermore, for small charge to mass constraint with the simultaneous and individual forces can give Halos orbits at fixed \((r, \theta)\)parameters where the gravity is not essential to create these kind of orbits. Our approach can be helpful for dynamics near spherical rotating magnetic planets, it exhibits the nature prograde or retrograde direction motion which could be of interest in experiment point of view.

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