THE BÉZOUT EQUATION ON THE RIGHT HALF PLANE IN A WIENER SPACE SETTING

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ABSTRACT. This paper deals with the Bézout equation $G(s)X(s) = I_m$, $\Re s \geq 0$, in the Wiener space of analytic matrix-valued functions on the right half plane. In particular, $G$ is an $m \times p$ matrix-valued analytic Wiener function, where $p \geq m$, and the solution $X$ is required to be an analytic Wiener function of size $p \times m$. The set of all solutions is described explicitly in terms of a $p \times p$ matrix-valued analytic Wiener function $Y$, which has an inverse in the analytic Wiener space, and an associated inner function $\Theta$ defined by $Y$ and the value of $G$ at infinity. Among the solutions, one is identified that minimizes the $H^2$-norm. A Wiener space version of Tolokonnikov’s lemma plays an important role in the proofs. The results presented are natural analogs of those obtained for the discrete case in [11].

1. INTRODUCTION AND MAIN RESULTS

In this paper we deal with the Bézout equation $G(s)X(s) = I_m$ on the closed right half plane $\Re s \geq 0$, assuming that the given function $G$ is of the form

$$G(s) = D + \int_0^\infty e^{-st}g(t)\,dt \quad (\Re s \geq 0),$$

(1.1)

where $g \in L^1_{m \times p}(\mathbb{R}_+) \cap L^2_{m \times p}(\mathbb{R}_+)$. In particular, $G$ belongs to the analytic Wiener space $W^{m \times p}_+$. We are interested in solutions $X \in W^{p \times m}_+$, that is,

$$X(s) = DX + \int_0^\infty e^{-st}x(t)\,dt \quad (\Re s \geq 0),$$

where $x \in L^1_{m \times p}(\mathbb{R}_+)$. Throughout $p \geq m$. We refer to the final paragraph of this introduction for a further explanation of the notation.

With $G$ given by (1.1) we associate the Wiener-Hopf operator $T_G$ mapping $L^2_p(\mathbb{R}_+)$ into $L^2_m(\mathbb{R}_+)$ which is defined by

$$(T_G h)(t) = Dh(t) + \int_0^\infty g(t-\tau)h(\tau)d\tau, \quad t \geq 0 \quad (h \in L^2_m(\mathbb{R}_+)).$$

2010 Mathematics Subject Classification. Primary 47A56; Secondary 47A57, 47B35, 46E40, 46E15.

Key words and phrases. Bézout equation, corona problem, Wiener space on the line, matrix-valued functions, minimal norm solutions, Tolokonnikov’s lemma.

The third author gratefully thanks the mathematics department of North-West University, Potchefstroom campus, South Africa, for the hospitality and support during his visit from September 21–October 15, 2015.

This work is based on the research supported in part by the National Research Foundation of South Africa (Grant Number 93406).
For $X$ as in (1.2), we define the Wiener-Hopf operator $T_X$ mapping $L^2_p(\mathbb{R}_+)$ into $L^2_p(\mathbb{R}_+)$ in a similar way, replacing $D$ by $D_X$ and $g$ by $x$. If the Bézout equation
\begin{equation}
(1.4) \quad G(s)X(s) = I_m, \quad \Re s \geq 0.
\end{equation}

has a solution $X$ as in (1.2), then (using the analyticity of $G$ and $X$) the theory of Wiener-Hopf operators (see [1] Section XII.2 or [2] Section 9) tells us that $T_GT_X = T_{G\times X} = I$, where $I$ stands for the identity operator on $L^2_p(\mathbb{R}_+)$. Thus for the Bézout equation (1.4) to be solvable the operator $T_G$ must be surjective or, equivalently, $T_GT_G^*$ must be strictly positive. We shall see that this condition is also sufficient.

To state our main results, we assume that $T_GT_G^*$ is strictly positive. Then $D = G(\infty)$ is surjective, and hence $DD^*$ is strictly positive too. We introduce two matrices $D^+$ and $E$, of sizes $p \times m$ and $p \times (p - m)$, respectively, and a $p \times p$ matrix function $Y$ in $W^{p \times p}_+$, as follows:
\begin{itemize}
  \item[(i)] $D^+ = D^*(DD^*)^{-1}$, where $D = G(\infty)$;
  \item[(ii)] $E$ is an isometry mapping $\mathbb{C}^{p-m}$ into $\mathbb{C}^p$ such that $\text{Im } E = \text{Ker } D$;
  \item[(iii)] $Y$ is the $p \times p$ matrix function given by
\end{itemize}
\begin{equation}
(1.5) \quad Y(s) = I_p - \int_0^\infty e^{-st}y(t)dt, \quad \Re s \geq 0, \quad \text{where } y = T_G^*(T_GT_G^*)^{-1}g.
\end{equation}

From the definitions of $D^+$ and $E$ it follows that the $p \times p$ matrix $[D^+ \ E]$ is non-singular. In fact
\begin{equation}
(1.6) \quad \begin{bmatrix}
  D^+ \\
  E^*
\end{bmatrix}
\begin{bmatrix}
  D^+ & E
\end{bmatrix} = 
\begin{bmatrix}
  I_m & 0 \\
  0 & I_{p-m}
\end{bmatrix}.
\end{equation}

As we shall see (Proposition 2.1 in Section 2 below), the fact that the given function $g \in L^1_{m \times p}(\mathbb{R}_+) \cap L^2_{m \times p}(\mathbb{R}_+)$ implies that a similar result holds true for $y$. In particular, $Y \in W^{p \times p}_+$. In what follows $\Xi$ and $\Theta$ are the functions defined by
\begin{align}
\Xi(s) &= \left(I_p - \int_0^\infty e^{-st}y(t)dt\right)D^+ = Y(s)D^+, \quad \Re s > 0; \\
\Theta(s) &= \left(I_p - \int_0^\infty e^{-st}y(t)dt\right)E = Y(s)E, \quad \Re s > 0.
\end{align}

Since $Y \in W^{p \times p}_+$, we have $\Xi \in W^{p \times m}_+$ and $\Theta \in W^{p \times (p-m)}_+$. Finally, recall that a function $\Omega$ in the analytic Wiener space $W^{k \times r}_+$ is inner whenever $\Omega(s)$ is an isometry for each $s \in i\mathbb{R}$. We now state our main results.

**Theorem 1.1.** Let $G$ be the $m \times p$ matrix-valued function given by (1.1). Then the equation $G(s)X(s) = I_m, \quad \Re s > 0$, has a solution $X \in W^{p \times m}_+$ if and only if $T_G$ is right invertible. In that case the function $\Xi$ defined by (1.7) is a particular solution and the set of all solutions $X \in W^{p \times m}_+$ is given by
\begin{equation}
(1.9) \quad X(s) = \Xi(s) + \Theta(s)Z(s), \quad \Re s > 0,
\end{equation}

where $\Xi$ and $\Theta$ are defined by (1.7) and (1.8), respectively, and the free parameter $Z$ is an arbitrary function in $W^{(p-m) \times m}_+$. Moreover, the function $\Theta$ belongs to $W^{p \times (p-m)}_+$ and is inner. Furthermore, the solution $\Xi$ is the minimal $H^2$ solution.
in the following sense
\[
\|X(u)\|_{H^2}^2 = \|\Xi(u)\|_{H^2}^2 + \|Z(u)\|_{H^2}^2,
\]
(1.10)
where \( u \in \mathbb{C}^m \) and \( Z \in \tilde{W}_+(p-m) \times m \).

In the above theorem, for any positive integer \( k \), \( H^2_k = H^2_k(i \mathbb{R}) \) is the Hardy space of \( \mathbb{C}^k \)-valued functions on the right half plane given by \( H^2_k(i \mathbb{R}) = JL^2_k(\mathbb{R}_+) \), where \( J \) is the unitary operator defined by
\[
J = \frac{1}{\sqrt{2\pi}} F: L^2_k(\mathbb{R}) \to L^2_k(i \mathbb{R})
\]
with \( F \) being the Fourier transform mapping \( L^2_k(\mathbb{R}) \) onto \( L^2_k(i \mathbb{R}) \). Moreover, \( Z \in \tilde{W}_+(p-m) \times m \) means that
\[
Z(s) = DZ + \int_0^\infty e^{-st} z(t) \, dt \quad (\Re s \geq 0),
\]
where \( DZ \) is a \((p-m) \times p\) matrix and \( z \in L^1(p-m) \times p(\mathbb{R}_+) \cap L^2(p-m) \times p(\mathbb{R}_+) \).

See the final part of this introduction for further information about the used notation, in particular, see [1,4] for the definition of the Fourier transform \( F \).

The second theorem is a variant of the Tolokonnikov lemma [20] in the present setting. The result emphasizes the central role of the function \( Y \).

**Theorem 1.2.** Assume \( TGT^*_G \) is strictly positive, and let \( Y \) be the matrix function defined by (1.5). Then \( Y \) belongs to the Wiener space \( W^p_{+p}\times p \), \( \det Y(s) \neq 0 \) whenever \( \Re s \geq 0 \), and hence \( Y \) is invertible in \( W^p_{+p}\times p \). Furthermore, the \( p \times p \) matrix function
\[
\begin{bmatrix}
G(s) \\
E^*Y(s)^{-1}
\end{bmatrix}, \quad \Re s \geq 0,
\]
(1.12)
is invertible in the Wiener algebra \( W^p_{+p}\times p \) and its inverse is given by
\[
\begin{bmatrix}
G(s) \\
E^*Y(s)^{-1}
\end{bmatrix}^{-1} = Y(s) [D^+ \quad E] = [\Xi(s) \quad \Theta(s)], \quad \Re s \geq 0.
\]
(1.13)

The literature on the Bézout equation and the related corona problem is extensive, starting with Carleson’s corona theorem [2] (for the case when \( m = 1 \)) and Fuhrmann’s extension to the matrix-valued case [6], both in a \( H^\infty \) setting. The topic has beautiful connections with operator theory (see the books [14, 16, 17, 18], and the more recent papers [21, 22, 23]). Rational matrix equations of the form (1.4) play an important role in solving systems and control theory problems, in particular, in problems involving coprime factorization, see, e.g., [24, Section 4.1], [10, Section A.2], [25, Chapter 21]. For more recent work see [12] and [13], and [15] page 3 where it is proved that the scalar analytic Wiener algebra is a pre-Bézout ring. For matrix polynomials, the equation (1.4) is closely related to the Sylvester resultant; see, e.g., Section 3 in [9] and the references in that paper.

The present paper is inspired by [5] and [11]. The paper [5] deals with equation (1.4) assuming the matrix function \( G \) to be a stable rational matrix function, and the solutions are required to be stable rational matrix functions as well. The comment in the final paragraph of [5, Section 2] was the starting point for our analysis. The paper [11] deals with the discrete case (when the right half plane is replaced by the open unit disc). Theorems 1.1 and 1.2 are the continuous analogue of Theorem
The absence of an explicit formula for the function $Y^{-1}$ in the present setting makes the proofs more complicated than those in [11].

The paper consists of five sections, including the present introduction and an appendix. Section 2, which deals with the right invertibility of the operator $T_G$, has an auxiliary character. Theorem 1.2 is proved in Section 3, and Theorem 1.1 in Section 4. The Appendix, Section A, contains a number of auxiliary results involving the Lebesgue space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and its vector-valued counterpart, which are collected together simply for the convenience of the reader and contains no significantly new material.

Notation and terminology. We conclude this section with some notation and terminology. Throughout, a linear map $A : \mathbb{C}^r \to \mathbb{C}^k$ is identified with the $k \times r$ matrix of $A$ relative to the standard orthonormal bases in $\mathbb{C}^r$ and $\mathbb{C}^k$. The space of all $k \times r$ matrices with entries in $L^1(\mathbb{R})$ will be denoted by $L^1_{k\times r}(\mathbb{R})$. As usual $\hat{f}$ denotes the Fourier transform of $f \in L^1_{k\times r}(\mathbb{R})$, that is,

$$\hat{f}(s) = (\mathcal{F}f)(s) = \int_{-\infty}^{\infty} e^{-st}f(t)dt, \ s \in i\mathbb{R}. \quad (1.14)$$

Note that $\hat{f}$ is continuous on the extended imaginary axis $i\mathbb{R} \cup \{\pm i\infty\}$, and is zero at $\pm i\infty$ by the Riemann-Lebesgue lemma. By $W^{k\times r}$ we denote the Wiener space consisting of all $k \times r$ matrix functions $F$ on the imaginary axis of the form

$$F(s) = D_F + \hat{f}(s), \ s \in i\mathbb{R}, \text{ where } f \in L^1_{k\times r}(\mathbb{R}) \text{ and } D_F \text{ is a constant matrix.} \quad (1.15)$$

Since $\hat{f}$ is continuous on the extended imaginary axis and is zero at $\pm i\infty$, the function $F$ given by (1.15) is also continuous on the extended imaginary axis and the constant matrix $D_F$ is equal to the value of $F$ at infinity. We write $W^{k\times r}$ for the space of all $F$ of the form (1.15) with the additional property that $f$ has its support in $\mathbb{R}_+ = [0, \infty)$, that is, $f$ is equal to zero on $(-\infty, 0]$. Any function $F \in W^{k\times r}_+$ is analytic and bounded on the open right half plane. Thus any $F \in W^{k\times r}_+$ is a matrix-valued $H^\infty$ function. Finally, by $W^{k\times r}_{k\times r}$ we denote the Wiener space consisting of all $F$ of the form (1.15) with the additional property that $D_F = 0$ and $f$ has its support in $(-\infty, 0]$. Thus we have the following direct sum decomposition:

$$W^{k\times r} = W^{k\times r}_+ + W^{k\times r}_{-0}. \quad (1.16)$$

We write $F \in W^{k\times r}_+$ if the function $f$ in (1.15) belongs to $L^1_{k\times r}(\mathbb{R}_+) \cap L^2_{k\times r}(\mathbb{R}_+)$. Similarly, $F \in W^{k\times r}_{-0}$ if $f \in L^1_{k\times r}(\mathbb{R}_-) \cap L^2_{k\times r}(\mathbb{R}_-)$ and $D_F = 0$.

Let $F \in W^{k\times r}$ be given by (1.15). With $F$ we associate the Wiener-Hopf operator $T_F$ mapping $L^2(\mathbb{R}_+)$ into $L^2(\mathbb{R}_+)$. This operator (see [7, Section XII.2]) is defined by

$$(T_F h)(t) = D_F h(t) + \int_0^\infty f(t - \tau)h(\tau)d\tau, \ t \geq 0 \ (h \in L^2(\mathbb{R}_+)). \quad (1.17)$$

The orthogonal complement of $H^2(\mathbb{R}_+) = JL^2(\mathbb{R}_+)$, with $J$ as in (1.11), in $L^2(\mathbb{R}_+)$ will be denoted by $K^2(\mathbb{R}_+)$. If $F \in W^{k\times r}$, then for each $u \in \mathbb{C}^r$ the function $F(\cdot)u$ belongs to $H^2(\mathbb{R}_+)$. Similarly, $F(\cdot)u$ belongs to $K^2(\mathbb{R}_+)$ if $F \in W^{k\times r}_{-0}$.
Finally, for \( f \in L^1_{k \times r}(\mathbb{R}) \) and \( g \in L^1_{l \times m}(\mathbb{R}) \) the convolution product \( f \ast g \) is the function in \( L^1_{k \times m}(\mathbb{R}) \), see [19, Section 7.13], given by
\[
(f \ast g)(t) = \int_{-\infty}^{\infty} f(t-\tau)g(\tau) \, d\tau \quad \text{a.e. on } \mathbb{R}.
\]

2. Right invertibility of \( T_G \)

In this section \( G \in W^m_{m \times p} \), where \( G \) is given by [1.1] and \( p \geq m \). We already know that the Bézout equation (1.4) having a solution \( X \) in \( W_{m \times m}^p \) implies that \( T_G \) is right invertible, or, equivalently, \( T_GT_G^* \) is strictly positive; see the paragraph containing formula (1.4).

In this section we present an auxiliary result that will be used to prove our main theorems. For this purpose we need the \( m \times m \) matrix-valued function \( R \) on the imaginary axis defined by \( R(s) = G(s)G(s)^* \), \( s \in i\mathbb{R} \). It follows that \( R \in W^m_{m \times m} \).

By \( T_R \) we denote the corresponding Wiener-Hopf operator acting on \( L^2_m(\mathbb{R}_+) \). Thus
\[
(T_Rf)(t) = DD^*f(t) + \int_0^{\infty} r(t-\tau)f(\tau) \, d\tau, \quad 0 \leq t < \infty,
\]
with \( r(t) = Dg^*(t) + g(t)D^* + \int_{-\infty}^{\infty} g(t-\tau)g^*(\tau) \, d\tau, \quad t \in \mathbb{R} \).

Here \( g^*(t) = g(-t)^* \) for \( t \in \mathbb{R} \). It is well-known (see, e.g., formula (24) in Section XII.2 of [1]) that
\[
T_R = T_GT_G^* + HGH_G^*.
\]

Here \( H_G \) is the Hankel operator mapping \( L^2_m(\mathbb{R}_+) \) into \( L^2_m(\mathbb{R}_+) \) defined by \( G \), that is,
\[
(H_Gf)(t) = \int_0^{\infty} g(t+\tau)f(\tau) \, d\tau, \quad f \in L^2_m(\mathbb{R}_+).
\]

We shall prove the following proposition. For the case when \( G \) is a rational matrix function, the first part (of the “if and only if” part) of the proposition is covered by Lemma 2.3 in [5]. The proof given in [5] can also be used in the present setting. For the sake of completeness we include a proof of the first part.

**Proposition 2.1.** Let \( G \) be given by (1.1). Then the operator \( T_G \) is right invertible if and only if \( T_R \) and \( I - H_G^*T_R^{-1}H_G \) are both invertible operators. In that case the inverse of \( T_GT_G^* \) is given by
\[
(T_GT_G^*)^{-1} = T_R^{-1} + T_R^{-1}H_G(I - H_G^*T_R^{-1}H_G)^{-1}H_G^*T_R^{-1}.
\]

Furthermore,
\[
(a) \quad (T_GT_G^*)^{-1} \text{ maps } L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+) \text{ in a one-to-one way onto itself;}
\]
\[
(b) \quad \text{the function } y \text{ defined by } y = T_G^*(T_GT_G^*)^{-1}g \text{ belongs to } L^1_{p \times p}(\mathbb{R}_+) \cap L^2_{p \times p}(\mathbb{R}_+),
\]
in particular, the function \( Y \) given by (1.5) is in \( \tilde{W}^{p \times p} \).

**Proof.** We split the proof into four parts. In the first part we assume that \( T_G \) is right invertible, and we show that \( T_R \) and \( I - H_G^*T_R^{-1}H_G \) are both invertible operators and that the inverse of \( T_GT_G^* \) is given by (2.3). The second part deals with the reverse implication. Items (a) and (b) are proved in the last two parts.

**Part 1.** Assume \( T_G \) is right invertible. Then the operator \( T_GT_G^* \) is strictly positive. According to (2.1) we have \( T_R \geq T_GT_G^* \), and hence \( T_R \) is also strictly positive.
In particular, \( T_R \) is invertible. Rewriting (2.1) as \( T_G T_G^* = T_R - H_G H_G^* \), and multiplying the latter identity from the left and from the right by \( T_R^{-1/2} \) shows that
\begin{equation}
T_R^{-1/2} T_G T_G^* T_R^{-1/2} = I - T_R^{-1/2} H_G H_G^* T_R^{-1/2}.
\end{equation}
Hence \( I - T_R^{-1/2} H_G H_G^* T_R^{-1/2} \) is strictly positive which shows that \( H_G^* T_R^{-1/2} \) is a strict contraction. But then \( H_G^* T_R^{-1} H_G = (H_G^* T_R^{-1/2}) (H_G^* T_R^{-1/2})^* \) is also a strict contraction, and thus the operator \( I - H_G^* T_R^{-1} H_G \) is strictly positive. In particular, \( I - H_G^* T_R^{-1} H_G \) is invertible. Finally, since \( T_G T_G^* = T_R - H_G H_G^* \), a usual Schur complement type of argument (see, e.g., Section 2.2 in [1]), including the well-known inversion formula
\[
(A - BC)^{-1} = A^{-1} + A^{-1} B (I - CA^{-1} B)^{-1} C A^{-1} - 1
\]
then shows that \( (T_G T_G^*)^{-1} \) is given by (2.3).

**PART 2.** In this part we assume that \( T_R \) and \( I - H_G^* T_R^{-1} H_G \) are both invertible operators, and we show that \( T_G \) is right invertible. According to (2.1) the operator \( T_R \) is positive. Since we assume \( T_R \) to be invertible, we conclude that \( T_R \) is strictly positive. Rewriting (2.1) as \( T_G T_G^* = T_R - H_G H_G^* \), and multiplying the latter identity from the left and from the right by \( T_R^{-1/2} \) we obtain the identity (2.4). Hence \( I - T_R^{-1/2} H_G H_G^* T_R^{-1/2} \) is positive which shows that \( H_G^* T_R^{-1/2} \) is a contraction. But then \( H_G^* T_R^{-1} H_G = (H_G^* T_R^{-1/2}) (H_G^* T_R^{-1/2})^* \) is also a contraction, and thus the operator \( I - H_G^* T_R^{-1} H_G \) is positive. By assumption \( I - H_G^* T_R^{-1} H_G \) is invertible. It follows that \( I - H_G^* T_R^{-1} H_G \) is strictly positive, and hence \( T_R^{-1/2} H_G^* T_R^{-1/2} \) is a strict contraction. But then the same holds true for \( T_R^{-1/2} H_G^* T_R^{-1/2} \). This implies that \( I - T_R^{-1/2} H_G H_G^* T_R^{-1/2} \) is strictly positive, and (2.4) shows that \( T_G \) is right invertible.

**PART 3.** In this part we prove item (a). Observe that \( g \in L_m^1 (\mathbb{R}) \) implies that \( T_G \) maps \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \) into \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \). Since \( g^* \in L_m^1 (\mathbb{R}) \) and
\[
(T_G f)(t) = D^* f (t) + \int_0^\infty g^* (t - \tau) f (\tau) \, d\tau, \quad 0 \leq t < \infty,
\]
the operator \( T_G^* \) maps \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \) into \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \). Thus \( T_G T_G^* \) maps \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \) into itself. We have to show that the same holds true for its inverse. To do this we apply Lemmas A.3 and A.4.

Lemma A.3 tells us that \( T_R^{-1} \) maps \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \) in a one-to-one way onto itself. This allows us to apply Lemma A.3 with
\[
Q = T_R^{-1}, \quad H = H_G^* \quad \text{and} \quad \tilde{H} = H_G.
\]
Recall that \( H_G \) is a Hankel operator, see (2.2), and \( H_G^* \) is also a Hankel operator, in fact
\[
(H_G^* f)(t) = \int_0^\infty g (t + \tau)^* f (\tau) \, d\tau, \quad 0 \leq t < \infty.
\]
Since \( I - H_G^* T_R^{-1} H_G \) is invertible, Lemma A.4 then shows that \( I - H_G^* T_R^{-1} H_G \) maps \( L_p^1 (\mathbb{R}_+) \cap L_p^2 (\mathbb{R}_+) \) in a one-to-one way onto itself, and hence the same holds true for its inverse \( (I - H_G^* T_R^{-1} H_G)^{-1} \). To complete the proof of item (a) note that \( H_G^* T_R^{-1} \) maps \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \) into \( L_p^1 (\mathbb{R}_+) \cap L_p^2 (\mathbb{R}_+) \), and \( T_R^{-1} H_G \) maps \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \) into \( L_m^1 (\mathbb{R}_+) \cap L_m^2 (\mathbb{R}_+) \). But then (2.3) shows that \( (T_G T_G^*)^{-1} \) maps \( L_m^1 (\mathbb{R}_+) \cap
Proposition 2.1 (b), and Note that \( g \in W^p_Y \) and (1.8), respectively. From Proposition 2.1, item (b), and (1.8) we know that

\[
\text{Proof. for each } s \in (1.8)
\]

Lemma 3.1. Assume that \( T_G \) is right invertible, and let \( Y \in W^{p \times p}_+ \) be the function defined by (1.10). Then

\[
(3.1) \quad G(s)Y(s) = D, \quad \Re s > 0.
\]

Proof. To prove (3.1) note that \( T_{GY} = T_G T_G^* (T_G T_G^*)^{-1} g = g \). Since the functions \( g \) and \( y \) both have their support in \( \mathbb{R}_+ \), the identity \( T_{GY} = g \) can be rewritten as \( Dy + g \ast y = g \), where \( \ast \) is the convolution product of matrix-valued functions with entries in \( L^1(\mathbb{R}) \); see (1.18). Thus

\[
(3.2) \quad Dy(t) + (g \ast y)(t) = Dy(t) + \int_{-\infty}^{\infty} g(t-\tau) y(\tau) d\tau = g(t), \quad t \in \mathbb{R}.
\]

Next use that the Fourier transform of a convolution product is just the product of the Fourier transforms of the functions in the convolution product. Thus, taking Fourier transforms in (3.2), yields \( D\hat{y} + \hat{g} \hat{y} = \hat{g} \). The latter identity can be rewritten as \( G\hat{y} = \hat{g} \). Hence, using the definition of \( Y \) in (1.1), we obtain

\[
G(s)Y(s) = G(s)(I_p - \hat{y}(s)) = G(s) - \hat{g}(s) = D.
\]

This proves (3.1). \( \square \)

Lemma 3.2. Assume that \( T_G \) is right invertible. Then the function \( \Theta \) defined by (1.8) belongs to \( W^{p \times (p-m)}_+ \) and is an inner function, that is, \( \Theta(s) \) is an isometry for each \( s \in i\mathbb{R} \) and at infinity.

Proof. We already know that \( \Theta \in W^{p \times (p-m)}_+ \). To prove that \( \Theta \) is inner, let \( y = T_G^* (T_G T_G^*)^{-1} g \) as in (1.5), and put \( f = (T_G T_G^*)^{-1} g \). Thus \( f \in L^1_{m \times m}(\mathbb{R}_+) \), by Proposition 2.1 (b), and \( y = T_G^* f \). The latter can be rewritten as

\[
y(t) = D^* f(t) + \int_0^{\infty} g^*(t - \tau) f(\tau) d\tau, \quad t \geq 0.
\]

Note that \( g^*(t) = g(-t)^* \), and hence \( g^* \) has its support in \( (-\infty, 0] \). Therefore

\[
y(t) = D^* f(t) + \int_{-\infty}^{\infty} g^*(t - \tau) f(\tau) d\tau, \quad t \in \mathbb{R}.
\]

Put

\[
(3.4) \quad \rho(t) = \begin{cases} 0 & \text{when } t \geq 0, \\ (g^* \ast f)(t) & \text{when } t < 0. \end{cases}
\]
Here \( \Omega(s) \) is identically zero, and thus \( \Theta(s)\Theta(s) = E^*Y(s)^*Y(s)E \) for \( s \in i\mathbb{R} \). We have
\[
E^*Y(s)^*Y(s)E = \frac{E^*Y(s)^*E - E^*Y(s)^*G(s)^*\hat{f}(s)E + E^*Y(s)^*\hat{p}(s)E}{(\text{because } G(s)Y(s)E = 0 \text{ by } \Theta(s)\Theta(s) = E^*Y(s)^*Y(s)E)}
\]
\[
= E^*E - E^*\hat{g}(s)^*E + E^*Y(s)^*\hat{p}(s)E
\]
\[
= I_{p-m} - \Omega(s).
\]
Here \( \Omega(s) = E^*\hat{g}(s)^*E - E^*Y(s)^*\hat{p}(s)E \). Note that the functions \( \hat{g}(\cdot)^* \) and \( Y(\cdot)^*\hat{p}(\cdot) \) belong to \( W_{-0}^{p\times p} \), and thus \( \Omega \) belongs to \( W_{-0}^{p\times p} \). On the other hand, the function \( E^*Y(\cdot)^*Y(\cdot)E \) is hermitian on the imaginary axis, and hence the same is true for \( \Omega \). But for any positive integer \( k \) we have
\[
W_{-0}^{k\times k} \cap (W_{-0}^{k\times k})^* = \{0\}.
\]
Thus \( \Omega \) is identically zero, and thus \( \Theta(s)\Theta(s) = E^*Y(s)^*Y(s)E = I_{p-m} \) for any \( s \in i\mathbb{R} \). Moreover, \( \Theta(\infty)^*\Theta(\infty) = E^*E = I \). This proves that \( \Theta \) is inner. \( \square \)

**Lemma 3.3.** Assume that \( T_G \) is right invertible, and let \( Y \in W_+^{p\times p} \) be the function defined by (3.3). Then \( Y \) is invertible in \( W_+^{p\times p} \).

**Proof.** Fix \( s \in i\mathbb{R} \), and assume \( u \in \mathbb{C}^p \) such that \( Y(s)u = 0 \). Then \( G(s)Y(s) = D \) implies that \( Du = 0 \). By definition of \( E \), \( u = Ev \) for some \( v \in \mathbb{C}^{p-m} \). Next use \( \Theta(s) = Y(s)E \). It follows that \( \Theta(s)v = Y(s)Ev = Y(s)u = 0 \). However, \( \Theta(s) \) is an isometry, by Lemma 3.2. So \( v = 0 \), and hence \( u = 0 \). We see that \( \det Y(s) \neq 0 \). Also \( Y(\infty) = I_p \). We conclude that \( T_Y \) is a Fredholm operator; see [7, Theorem XII.3.1].

Next we prove that \( \text{Ker } T_Y = \{0\} \). Take \( h \in \text{Ker } T_Y \). Then \( T_Y h = 0 \), and hence \( Y(s)h(s) = 0 \) for each \( s \in i\mathbb{R} \). But \( \det Y(s) \neq 0 \) for each \( s \in i\mathbb{R} \). Hence \( h = 0 \), and therefore \( h = 0 \).

We want to prove that \( T_Y \) is invertible. Given the results of the preceding first two paragraphs it suffices to show that \( \text{ind } T_Y = 0 \). This will be done in the next step by an approximation argument, using the fact, from [9], that we know the result is true for rational matrix functions.

Let \( g \) be as in (1.1). Note that \( g \) is the limit in \( L^1 \) of a sequence \( g_1, g_2, \ldots \) such that \( G_n(s) = D + \hat{g}_n(s) \) is a stable rational matrix function; cf., Part (v) on page 229 of [7]. Since \( T_G \) is right invertible, \( T_G \) will also be right invertible for \( n \) sufficiently large. In fact, \( T_{G_n}T_{G_n}^* \rightarrow T_YT_Y^* \) in operator norm. Put \( y_n = T_{G_n}^*(T_{G_n}T_{G_n}^*)^{-1}g_n \). Then \( y_n \rightarrow y \) in the \( L^1 \)-norm. Put \( Y_n(s) = I - \hat{y}_n(s) \). Then \( Y_n \rightarrow Y \) in operator

norm. For $n$ sufficiently large the operator $T_{Y_n}$ is invertible (see the paragraph preceding Theorem 1.2 in [5] and formula (2.17) in [5]). In particular, the Fredholm index of $T_{Y_n}$ is zero. But $\text{ind} T_{Y} = \lim_{n \to \infty} \text{ind} T_{Y_n} = 0$. Thus $T_{Y}$ is invertible, and hence $Y$ is invertible in $W^{p \times p}_+$. □

Proof of Theorem 1.2. From Lemma 3.3 we know that $Y \in W^{p \times p}_+$ and that $Y$ is invertible in $W^{p \times p}_+$. Thus we only have to prove the second part of the theorem. Since $Y$ is invertible in $W^{p \times p}_+$, the $p \times p$ matrix function given by (4.12) is well-defined and belongs to $W^{p \times p}_+$. Furthermore, from (4.1) we know that the $p \times p$ matrix $[D^+ \ E]$ is invertible. Hence the function defined by the right hand side of (4.13) belongs to $W^{p \times p}_+$ and is invertible in $W^{p \times p}_+$. Using (3.1) and the identity (4.1) we see that

\[
\begin{bmatrix}
G(s) \\
E^*Y(s)^{-1}
\end{bmatrix} Y(s) 
\begin{bmatrix}
[D^+] \\
\ E
\end{bmatrix} = \begin{bmatrix}
G(s)Y(s) \\
E^*
\end{bmatrix} 
\begin{bmatrix}
[D^+] \\
\ E
\end{bmatrix} 
= \begin{bmatrix}
D \\
E^*
\end{bmatrix} 
\begin{bmatrix}
[D^+] \\
\ E
\end{bmatrix} = \begin{bmatrix}
I_m & 0 \\
0 & I_{p-m}
\end{bmatrix}, \quad \Re s \geq 0.
\]

This proves the first identity (4.13). The second identity is an immediate consequence of the definitions of $\Xi$ and $\Theta$ in (1.7) and (1.8), respectively. □

4. PROOF OF THEOREM 1.1

We begin with a lemma concerning the functions $\Xi$ and $\Theta$.

Lemma 4.1. Assume that $T_G$ is right invertible, and let $\Xi$ and $\Theta$ be the functions defined by (1.7) and (1.8), respectively. Then

\[
(4.1) \quad \text{Ker} T_G = T_\Theta L^2_{p-m}(\mathbb{R}_+) \quad \text{and} \quad \Theta^* \Xi \in \widetilde{W}^{(p-m) \times m}_{0, -}.
\]

Proof. We split the proof into two parts.

PART 1. In this part we prove the inclusion of (4.1). Take $s \in i\mathbb{R}$. Note that in Proposition 2.2 it was shown for $s \in i\mathbb{R}$ that

\[
Y(s) = I - G(s)^* \hat{f}(s) + \hat{\rho}(s),
\]

where $f = (T_GT_G^*)^{-1} g$ and $\rho$ is defined by (3.4); see (3.5). From (1.7) and (1.8) we then see that

\[
\Theta(s)^* \Xi(s) = E^* Y(s)^* Y(s) D^+ = E^* Y(s)^* \left( I - G(s)^* \hat{f}(s) + \hat{\rho}(s) \right) D^+.
\]

Now use that $G(s)Y(s)E = DE = 0$, and hence $E^* Y(s)^* G(s)^* = 0$. The latter identity and the fact that $E^* D^+ = 0$ and $Y = I - \hat{y}$ imply that

\[
\Theta(s)^* \Xi(s) = - \hat{E}^* \hat{y}(s)^* D^+ + E^* \hat{\rho}(s) D^+ - E^* \hat{\gamma}(s)^* \hat{\rho}(s) D^+
\]

\[
= - A(s) + B(s) - C(s).
\]

From item (b) in Proposition 1.1 we know that $y \in L^2_{p \times p}(\mathbb{R}_+) \cap L^2_{p \times p}(\mathbb{R}_+)$, and thus $\hat{y} \in \widetilde{W}^{p \times p}_+ + \hat{\gamma}(\infty) = 0$, that is, $\hat{y} \in \widetilde{W}^{p \times p}_0$. It follows that

\[
(4.3) \quad A(\cdot) := E^* \hat{y}(\cdot)^* D^+ \in \widetilde{W}^{(p-m) \times p}_{0, -}.
\]

Recall that $\rho$ is given by (3.2) with $f = (T_GT_G^*)^{-1} g$. Since the function $g$ belongs to $L^1_{m \times p}(\mathbb{R}_+) \cap L^2_{m \times p}(\mathbb{R}_+)$, item (b) in Proposition 2.1 tells us that the same holds...
true for $f$. It follows that $g^* \ast f \in L^1_{p \times p}(\mathbb{R}) \cap L^2_{p \times p}(\mathbb{R})$. The latter implies that 
\[ \rho \in L^1_{p \times p}(\mathbb{R}_-) \cap L^2_{p \times p}(\mathbb{R}_-) \]. We conclude that 
\[ B(\cdot) := E^* \hat{\rho}(\cdot) D^+ \in \widetilde{W}^{(p-m)\times p}_{0,-} \].

Finally, note that $y^*(t) = y(-t)^*$ for $t \in \mathbb{R}$ and $(\hat{y}(s))^* = \hat{y}^*(s)$ for $s \in i\mathbb{R}$. Thus 
\[ \hat{y}(s)^* \hat{\rho}(s) = (\hat{y}^* \ast \hat{\rho})(s), \quad s \in i\mathbb{R}, \]

and 
\[ (y^* \ast \rho)(t) = \int_{-\infty}^{\infty} y^*(t - \tau) \rho(\tau) d\tau = \int_{-\infty}^{0} y^*(t - \tau) \rho(\tau) d\tau. \]

Since both $y^*$ and $\rho$ belong to $L^1_{p \times p}(\mathbb{R}_-) \cap L^2_{p \times p}(\mathbb{R}_-)$, it is well known (see, e.g., Section 2 in [1]) that the same holds true for $y^* \ast \rho$. But then 
\[ C(\cdot) := E^* \hat{y}(\cdot)^* \hat{\rho}(\cdot) D^+ \in \widetilde{W}^{(p-m)\times p}_{0,-}. \]

From (4.3), (4.4), (4.5) and (4.2) it follows that $\Theta \Xi \in \widetilde{W}^{(p-m)\times m}_{0,-}$.

**PART 2.** In this part we prove the identity of (4.1). Using (3.1) we see that 
\[ G(s) \Theta(s) = G(s) Y(s) E = D E = 0, \quad s \in i\mathbb{R}. \]

This implies that $T_G T_\Theta = 0$, and hence $\text{Im} T_\Theta \subset \text{Ker} T_G$. To prove the reverse inclusion, take $h \in \text{Ker} T_G$. Thus $h \in L^2_{p \times \mathbb{R}}(\mathbb{R}_+)$ and $T_G h = 0$. It follows that $G(s) \hat{h}(s) = 0$ for $\Re s > 0$. Put $H(s) = \hat{h}(s)$. Then $H(\cdot)$ belongs to $H^2_m(i\mathbb{R})$. Next we apply Theorem 1.2. Using the identities in (1.13) we see that 
\[ H(s) = [\Xi(s) \quad \Theta(s)] \begin{bmatrix} G(s) \\ E^* Y(s)^{-1} \end{bmatrix} H(s) \\
= [\Xi(s) \quad \Theta(s)] \begin{bmatrix} 0 \\ E^* Y(s)^{-1} H(s) \end{bmatrix} = \Theta(s) E^* Y(s)^{-1} H(s). \]

Hence $\hat{h}(s) = \Theta(s) \Psi(s)$, where $\Psi(s) = E^* Y(s)^{-1} \hat{h}(s)$. Since $h \in L^2_{p \times \mathbb{R}}(\mathbb{R}_+)$ and $Y(\cdot)^{-1}$ is a matrix function with $H^\infty$ entries, we conclude that $\Psi \in H^\infty_{p\times m}$, and hence $\Psi = \hat{u}$ for some $u \in L^2_{p \times \mathbb{R}}(\mathbb{R}_+)$. The identity $\hat{h}(s) = \Theta(s) \Psi(s)$ then yields $\hat{h}(s) = \Theta(s) \hat{u}(s)$. This shows that $h = T_\Theta u$, and hence $\text{Ker} T_G \subset \text{Im} T_\Theta$. \[ \square \]

**Proof of Theorem 1.2.** From Lemma 3.2 we know that $\Theta$ is an inner function in $W^1_{p\times m}(\mathbb{R})$. The proof of the other statements is split into three parts.

**PART 1.** In this part we show that the equation $G(s) X(s) = I_m$, $\Re s > 0$, has a solution $X \in W^1_{p\times m}$ if and only if $T_G$ is right invertible. Furthermore, we show that in that case the function $\Xi$ defined by (1.7) is a particular solution. From the one but last sentence of the paragraph containing (1.4) we know that it suffices to prove the “if part” only. Therefore, in what follows we assume that $T_G$ is right invertible. Since $\Xi(s) = Y(s) D^+$ and $Y \in W^1_{p\times p}$, we have $\Xi \in W^1_{p\times m}$. Moreover, using the identity (4.1) we have 
\[ G(s) \Xi(s) = G(s) Y(s) D^+ = DD^+ = I_m. \]

Thus $\Xi$ is a particular solution.
Part 2. This second part deals with the description of all solutions in $\mathcal{W}_+^{p \times m}$. Let $Z$ be an arbitrary function in $\mathcal{W}_+^{p \times m}$, and let $X \in \mathcal{W}_+^{p \times m}$ be defined by (1.9). Then
\[
G(s)X(s) = G(s)\Xi(s) + G(s)\Theta(s)Z(s) = I_m + G(s)\Theta(s)Z(s), \quad Rs \geq 0.
\]
Recall that $G(s)\Theta(s) = G(s)Y(s)E = DE = 0$. Thus $G(s)X(s) = I_m$, $Rs \geq 0$, and thus $X$ is a solution.

To prove the converse implication, let $X \in \mathcal{W}_+^{p \times m}$ be a solution of the equation $G(s)X(s) = I_m$. Put $H = X - \Xi$. Then $H \in \mathcal{W}_+^{p \times m}$ and $G(s)H(s) = 0$. Using the identities in (1.13), we obtain
\[
H(s) = \begin{bmatrix} \Xi(s) & \Theta(s) \end{bmatrix} \begin{bmatrix} G(s) \\ E^*Y(s)^{-1} \end{bmatrix} H(s)
= \begin{bmatrix} \Xi(s) & \Theta(s) \end{bmatrix} \begin{bmatrix} 0 \\ E^*Y(s)^{-1}H(s) \end{bmatrix} = \Theta(s)E^*Y(s)^{-1}H(s).
\]
Thus $H(s) = \Theta(s)Z(s)$, where $Z(s) = E^*Y(s)^{-1}H(s)$. Since $Y$ is invertible in $\mathcal{W}_+^{p \times p}$, the function $Y(\cdot)^{-1}$ is in $\mathcal{W}_+^{p \times p}$. Together with the fact that $H \in \mathcal{W}_+^{p \times m}$, this yields $Z \in \mathcal{W}_+^{(p-m) \times m}$. It follows $X$ has the desired representation (1.9).

Part 3. In this part we prove the identity (1.10). Assume $Z \in \mathcal{W}_+^{(p-m) \times m}$, and let $X$ be the function defined by (1.9). Fix $u \in \mathbb{C}^m$. Then $Z(\cdot)u \in H^2_{p-m}(i\mathbb{R})$, and
\[
(4.6) \quad \Theta(\cdot)Z(\cdot)u = M_\Theta Z(\cdot)u \in H^2_{p-m}(i\mathbb{R}).
\]
Here $M_\Theta$ is the operator of multiplication by $\Theta(\cdot)$ mapping $H^2_{m}(i\mathbb{R})$ into $H^2_{p-m}(i\mathbb{R})$. Furthermore, since $M_\Theta$ is an isometry, we also see that
\[
(4.7) \quad \|Z(\cdot)u\| = \|\Theta(\cdot)Z(\cdot)u\|.
\]
The fact that $y \in L^1_{p\times p}(\mathbb{R}_+) \cap L^2_{p\times p}(\mathbb{R}_+)$ implies $Y \in \mathcal{W}_+^{p \times p}$. But then $\Xi(s) = Y(s)D^+$ yields $\Xi(\cdot)u \in H^2_{m}(i\mathbb{R})$. Using the identity (1.9) we conclude that $\Xi(\cdot)u$ also belongs to $H^2_{m}(i\mathbb{R})$. It follows that all norms in (1.10) are well defined, and in order to prove the identity (1.10) it suffices to show that in $H^2_{m}(i\mathbb{R})$ the function $\Theta(\cdot)Z(\cdot)u$ is orthogonal to the function $\Xi(\cdot)v$ for any $v \in \mathbb{C}^m$. The latter fact follows from the inclusion in the second part of (1.1). Indeed, this inclusion tells us that $M_\Theta \Xi(\cdot)v = 0$, and hence
\[
\langle \Xi(\cdot)v, \Theta(\cdot)Z(\cdot)u \rangle_{H^2_{m}(i\mathbb{R})} = \langle \Xi(\cdot)v, M_\Theta Z(\cdot)u \rangle_{H^2_{m}(i\mathbb{R})} = \langle M_\Theta \Xi(\cdot)v, Z(\cdot)u \rangle_{H^2_{m}(i\mathbb{R})} = 0.
\]
This completes the proof. □

Appendix A. The Lebesgue space $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$

The material in this section is standard and is presented for the convenience of the reader. Throughout we deal with the Lebesgue spaces of complex-valued functions on the real line $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, their vector-valued counterparts $L^1_{m}(\mathbb{R})$ and $L^2_{m}(\mathbb{R})$, and the intersection of the latter two spaces: $L^1_{m}(\mathbb{R}) \cap L^2_{m}(\mathbb{R})$. The
norms on these spaces are given by
\[
\|f\|_1 = \int_{-\infty}^{\infty} |f(t)| \, dt \quad \text{for } f \in L^1(\mathbb{R}), \\
\|f\|_2 = \left( \int_{-\infty}^{\infty} |f(t)|^2 \, dt \right)^{1/2} \quad \text{for } f \in L^2(\mathbb{R}), \\
\|f\|_1 = \left( \sum_{i=1}^{m} \|f_i\|_1^2 \right)^{1/2} \quad \text{for } f = (f_1, \ldots, f_m) \in L_{m1}(\mathbb{R}), \\
\|f\|_2 = \left( \sum_{i=1}^{m} \|f_i\|_2^2 \right)^{1/2} \quad \text{for } f = (f_1, \ldots, f_m) \in L_{m2}(\mathbb{R}), \\
\|f\|_0 = \max\{\|f\|_1, \|f\|_2\} \quad \text{for } f \in L_{m1}(\mathbb{R}) \cap L_{m2}(\mathbb{R}).
\]

Let \( k \in L_{m \times p}^1(\mathbb{R}) \). Thus \( k \) is an \( m \times p \) matrix function of which the \((i, j)\)-th entry \( k_{ij} \in L^1(\mathbb{R}) \). For each \( \varphi \in L^1_p(\mathbb{R}) \) and \( \psi \in L^2_p(\mathbb{R}) \) the convolution products \( k \ast \varphi \) and \( k \ast \psi \), see \([1, 2] \), are well defined, \( k \ast \varphi \) belongs to \( L^1_m(\mathbb{R}) \) and \( k \ast \psi \) belongs to \( L^2_m(\mathbb{R}) \). In particular, if \( f \in L^1_p(\mathbb{R}) \cap L^2_p(\mathbb{R}) \), then \( k \ast f \) belongs to \( L^1_m(\mathbb{R}) \cap L^2_m(\mathbb{R}) \). It follows that for a given \( k \in L_{m \times p}^1(\mathbb{R}) \) the convolution product induces linear maps from the space \( L^1_p(\mathbb{R}) \) into \( L^1_m(\mathbb{R}) \), from the space \( L^2_p(\mathbb{R}) \) into \( L^2_m(\mathbb{R}) \), and from the space \( L^1_p(\mathbb{R}) \cap L^2_p(\mathbb{R}) \) into \( L^1_m(\mathbb{R}) \cap L^2_m(\mathbb{R}) \). The resulting operators will be denoted by \( K_1, K_2 \) and \( K_0 \), respectively. The proof of the following lemma is standard (see, e.g., page 216 in \([6]\)) and therefore it is omitted.

**Lemma A.1.** The operators \( K_1, K_2 \) and \( K_0 \) are bounded linear operators, and

(A1) \[ \|K_\nu\| \leq \kappa \ (\nu = 1, 2, 0), \quad \text{where } \kappa = \left( \sum_{i=1}^{m} \sum_{j=1}^{p} \|k_{ij}\|_1^2 \right)^{1/2}. \]

With \( k \in L_{m \times 1}^1(\mathbb{R}_+) \) we also associate the Wiener-Hopf operator \( W \) and the Hankel operator \( H \) defined by

\[
(Wf)(t) = \int_{0}^{\infty} k(t - \tau)f(\tau) \, d\tau, \quad 0 \leq t < \infty, \\
(Hf)(t) = \int_{0}^{\infty} k(t + \tau)f(\tau) \, d\tau, \quad 0 \leq t < \infty.
\]

Using the classical relation between the convolution operator defined by \( k \) and the operators \( W \) and \( H \) (see, e.g., Section XII.2 in \([2]\)) it is easy to see that \( W \) and \( H \) map the space \( L^1_p(\mathbb{R}_+) \) into \( L^1_m(\mathbb{R}_+) \), the space \( L^2_p(\mathbb{R}_+) \) into \( L^2_m(\mathbb{R}_+) \), and the space \( L^1_p(\mathbb{R}_+) \cap L^2_p(\mathbb{R}_+) \) into \( L^1_m(\mathbb{R}_+) \cap L^2_m(\mathbb{R}_+) \). We denote the resulting operators by \( W_1, W_0, W_2, H_1, H_2, H_0 \), respectively. Lemma \([A.1]\) shows that these operators are bounded and

(A2) \[ \|W_\nu\| \leq \kappa \text{ and } \|H_\nu\| \leq \kappa \ (\nu = 1, 2, 0), \quad \text{where } \kappa = \left( \sum_{i=1}^{m} \sum_{j=1}^{p} \|k_{ij}\|_1^2 \right)^{1/2}. \]

Furthermore, using the line of reasoning in Lemma XX.2.4 in \([7]\), we have the following corollary.

**Corollary A.2.** The Hankel operators \( H_1, H_2, \) and \( H_0 \) are the limit in operator norm of finite rank operators, and hence compact.
Next we present an auxiliary result that is used in the proof of Proposition 2.1. Put

\[(A3) \quad R(s) = D_R + \int_{-\infty}^{\infty} e^{-st}r(t)\,dt \quad \text{where} \quad r \in L_{m \times m}^1(\mathbb{R}).\]

By \(T_R\) we denote the Wiener-Hopf operator on \(L_{m}^2(\mathbb{R}^+)\) defined by \(R\), that is,

\[(A4) \quad (T_Rf)(t) = D_Rf(t) + \int_0^\infty r(t-\tau)f(\tau)\,d\tau, \quad 0 \leq t < \infty.\]

As we know from the first paragraph of this section, the fact that \(r \in L_{m \times m}^1(\mathbb{R})\) implies that \(T_R\) maps \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) into itself.

**Lemma A.3.** If \(T_R\) is invertible as an operator on \(L_{m}^2(\mathbb{R}^+)\), then \(T_R^{-1}\) maps the space \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) in a one-to-one way onto itself.

**Proof.** Since \(T_R\) is invertible, \(R\) admits a canonical factorization (see Section XXX.10 in [S]), and hence we can write \(T_R^{-1} = LU\), where \(L\) and \(U\) are Wiener-Hopf operators on \(L_{m}^2(\mathbb{R}^+)\),

\[(A5) \quad (Lf)(t) = D_Lf(t) + \int_0^t \ell(t-\tau)f(\tau)\,d\tau, \quad 0 \leq t < \infty,\]

\[(A6) \quad (Uf)(t) = D_uf(t) + \int_t^\infty u(t-\tau)f(\tau)\,d\tau, \quad 0 \leq t < \infty.\]

Here \(\ell\) and \(u\) both belong to \(L_{m \times m}^1(\mathbb{R})\), with support of \(\ell\) in \(\mathbb{R}^+\) and support of \(u\) in \(\mathbb{R}^-\). The fact that both \(\ell\) and \(u\) belong to \(L_{m \times m}^1(\mathbb{R})\) implies that both \(L\) and \(U\) map \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) into itself. Hence \(T_R^{-1}\) has the same property. Since \(T_R^{-1}\) is one-to-one on \(L_{m}^2(\mathbb{R}^+)\), it is also one-to-one on \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\). For \(f \in L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\), we have \(g = T_Rf \in L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) and \(f = T_R^{-1}T_Rf = T_R^{-1}g\). This shows that \(T_R^{-1}\) maps \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) onto \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\).

**Lemma A.4.** Let \(k \in L_{p \times m}^1(\mathbb{R}^+)\) and \(k \in L_{m \times p}^1(\mathbb{R}^+)\), and let \(H\) and \(\hat{H}\) be the corresponding Hankel operators acting from \(L_{m}^2(\mathbb{R}^+)\) into \(L_{m}^2(\mathbb{R}^+)\) and from \(L_{m}^2(\mathbb{R}^+)\) into \(L_{m}^2(\mathbb{R}^+)\), respectively. Let \(Q\) be any operator on \(L_{m}^2(\mathbb{R}^+)\) mapping \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) into itself, and assume that the restricted operator \(Q_0\) acting on \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) is bounded. If the operator \(I - \hat{H}QH\) is invertible on \(L_{m}^2(\mathbb{R}^+)\), then \(I - \hat{H}QH\) maps the space \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) in a one-to-one way onto itself.

**Proof.** We know that \(H\) maps \(L_{p}^1(\mathbb{R}^+) \cap L_{p}^2(\mathbb{R}^+)\) into \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\). Furthermore the same holds true for \(\hat{H}\) with the role of \(p\) and \(m\) interchanged. Hence our hypothesis on \(Q\) implies that \(I - \hat{H}QH\) maps the space \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) into itself. Let \(M_0\) be the corresponding restricted operator. We have to prove that \(M_0\) is invertible. Note that Corollary A.2 implies that \(M_0\) is equal to the identity operator minus a compact operator, and hence \(M_0\) is a Fredholm operator of index zero. Therefore, in order to prove that \(M_0\) is invertible, it suffices to show that \(\ker M_0\) consists of the zero element only. Assume not. Then there exists a non-zero \(f\) in \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) such that \(M_0f = 0\). The fact that \(f\) belongs to \(L_{m}^1(\mathbb{R}^+) \cap L_{m}^2(\mathbb{R}^+)\) shows that \(0 = M_0f = (I - \hat{H}QH)f\). But \(I - \hat{H}QH\) is assumed to be invertible. Hence \(f\) must be zero. Thus \(M_0\) is invertible.
Acknowledgement

This work is based on the research supported in part by the National Research Foundation. Any opinion, finding and conclusion or recommendation expressed in this material is that of the authors and the NRF does not accept any liability in this regard.

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