GLOBAL RIGID INNER FORMS AND MULTIPLICITIES OF DISCRETE AUTOMORPHIC REPRESENTATIONS

Tasho Kaletha

Abstract

We study the cohomology of certain canonical Galois gerbes for the absolute Galois groups of number fields with ramification conditions. This cohomology provides a bridge between the refined local endoscopy introduced in [Kal] and classical global endoscopy. As particular applications, we express the canonical adelic transfer factor that governs the stabilization of the Arthur-Selberg trace formula as a product of normalized local transfer factors, we give an explicit construction of the pairing between an adelic $L$-packet and the corresponding $S$-group that is the essential ingredient in the description of the discrete automorphic spectrum of a reductive group, and we give a proof of some expectations of Arthur.

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This research is supported in part by NSF grant DMS-1161489.
Let $F$ be a number field and let $G$ be a connected reductive group defined over $F$. A central question in the theory of automorphic forms is the decomposition of the right regular representation of $G(\mathbb{A})$ on the Hilbert space $L^2_{\text{disc}}(Z(\mathbb{A})G(F) \setminus G(\mathbb{A}))$ or slight variants of it. Work of Labesse-Langlands [LL79], Langlands [Lan83], and Kottwitz [Kot84], on the Arthur-Selberg trace formula and its stabilization has provided a conjectural answer to this question. It is believed that there exists a group $\mathcal{L}_F$ having the Weil group $W_F$ of $F$ as a quotient. Admissible homomorphisms $\phi$ from $\mathcal{L}_F$ into the $G$-group of $G$ with bounded image parameterize $L$-packets of tempered representations of $G(\mathbb{A})$ and each tempered automorphic representation belongs to one of these $L$-packets. However, not all representations in a given $L$-packet $\Pi_\phi$ are automorphic. It is expected that there exists a complex-valued pairing $\langle -,- \rangle$ between a finite group $S_\phi$ (closely related to the centralizer in $\hat{G}$ of the image of $\phi$) and the packet $\Pi_\phi$ corresponding to $\phi$, which realizes each $\pi \in \Pi_\phi$ as (the character of) some representation of the group $S_\phi$. The number

$$m(\phi,\pi) = |S_\phi|^{-1} \sum_{x \in S_\phi} \langle x,\pi \rangle$$

is then a positive integer and is expected to give the contribution of the pair $(\pi,\phi)$ to the $G(\mathbb{A})$-representation $L^2_{\text{disc}}(Z(\mathbb{A})G(F) \setminus G(\mathbb{A}))$. In other words, the multiplicity of $\pi$ in the discrete spectrum is expected to be the sum of the numbers $m(\phi,\pi)$ as $\phi$ runs over all equivalence classes of parameters with $\pi \in \Pi_\phi$. Arthur [Art89, Art90] has extended this conjecture to encompass discrete automorphic representations that are non-tempered. We refer the reader to [BR94] for more details.

One of the main goals of this paper is to give an explicit construction of the pairing $\langle -,- \rangle$ based on the local conjectures formulated in [Kal]. Such an explicit construction was previously known only under the assumption that $G$ is quasi-split. In that case, the pairing $\langle -,- \rangle$ can be given as a product over all places of $F$ of (again conjectural) local pairings $\langle -,- \rangle_v$ between the local analogs of $S_\phi$ and $\Pi_\phi$. Our contribution here is to remove the assumption that $G$ is quasi-split. The main difficulty that arises once this assumption is dropped is that the local pairings $\langle -,- \rangle_v$ stop being well-defined, even conjecturally. This is a reflection of the fact that there is no canonical normalization of the Langlands-Shelstad endoscopic transfer factor. Since the endoscopic character identities tie the values of the local pairing with the values of the transfer factor, we cannot expect to have a well-defined local pairing without having a normalization of the transfer factor. The problem runs deeper, however: Examples as basic as the case of inner forms of $\text{SL}_2$ show that the local analog of $S_\phi$ doesn’t afford any pairing with the local analog of $\Pi_\phi$ that realizes elements of $\Pi_\phi$ as characters of $S_\phi$.

One can attempt to deal with these difficulties in an ad-hoc way. For example, one could fix an arbitrary normalization of the transfer factor. This is certainly sufficient for the purposes of stabilizing the geometric side of the trace formula. However, it is insufficient for the purposes of interpreting the spectral side of the stabilized trace formula, and in particular for the construction of the global pairing. For example, it doesn’t resolve the problem that the local $S_\phi$-group doesn’t afford a pairing with the local $L$-packet. One could also introduce an ad-hoc modification of the local $S_\phi$-group, but then it is not clear what normalizations of the transfer factor correspond to pairings on such a modification.
In this paper, we overcome these difficulties by developing a bridge between
the refined local endoscopy introduced in [Kal, §5] and classical global en-
doscopy. The results of [Kal, §5] provided a resolution to the local problems
listed above. Namely, they provide a natural normalization of the Langlands-
Shelstad transfer factor and a modification of the classical local $S_\phi$-group to-
gether with a precise conjectural description of the internal structure of local
$L$-packets. The latter takes the form of a canonically normalized perfect pair-
ing between the local $L$-packet and the modified $S_\phi$-group. Using these objects,
we obtain in the present paper the following results.

1. We prove that the product of the normalized transfer factors of [Kal, §5.3]
is equal to the canonical adelic transfer factor involved in the stabilization
of the trace formula (Proposition 4.2, resp. Equation (4.8)).

2. We give an explicit formula for the global pairing $\langle -, - \rangle$ in terms of the
conjectural local pairings introduced in [Kal, §5]. Moreover, the global
pairing $\langle -, - \rangle$ is independent of the auxiliary cohomology classes in-
volved in its construction and is thus canonically associated to the group
$G$ (Proposition 4.3, resp. Equation (4.9)).

3. We prove an expectation of Arthur [Art13, Hypothesis 9.5.1] about the ex-
istence of globally coherent collections of local mediating functions (Sub-
section 4.6).

Let us recall that the normalized local transfer factor and the conjectural local
pairings depend on two refinements of the local endoscopic set-up: A rigidi-
fication of the local inner twist datum [Kal, §5.1] and a refinement of the local
endoscopic datum [Kal, §5.3]. If we change either of these refinements, the nor-
malized local transfer factors and local pairings also change. An obvious ques-
tion is then, whether it is necessary to introduce the global analogs of these
refinements into the global theory of endoscopy. The answer to this question
is: no. The global theory of endoscopy needs no refinement. In particular, the
results of the present paper fit seamlessly with the established stabilization of
the Arthur-Selberg trace formula.

However, we do need a bridge between refined local endoscopy and usual
global endoscopy. The basis of refined local endoscopy was the cohomology
functor $H^1(u \to W)$ introduced in [Kal, §3]. In order to relate this object to the
global setting we introduce in the present paper certain Galois gerbes for the
Galois groups of number fields with ramification conditions and study their co-
homology. There are localization maps between the global cohomology groups
introduced here and the local cohomology groups $H^1(u \to W)$. These local-
ization maps are the foundation of the bridge between refined local and global
endoscopy.

The roles of the local and global cohomology groups are thus quite different.
The local cohomology groups influence the normalizations in local endoscopy
and must therefore enter into the statement of the local theorems and conjec-
tures. On the other hand, the global groups serve to produce coherent collec-
tions of local objects, but the global objects obtained from these coherent col-
lections are independent of the global cohomology classes used. This means
that the global cohomology groups are much less present in the statements of
global theorems and conjectures, but they are indispensable in their proofs.

Besides establishing a clear conjectural picture for general reductive groups,
these results provide an important step towards proving these conjectures.
This is due to the fact that the approach developed by Arthur [Art89, Art90] to prove both the local Langlands correspondence and the description of the discrete automorphic spectrum using the stabilization of the trace formula relies on the interplay between the local and the global conjectures. This approach was successfully carried out for quasi-split symplectic and orthogonal groups in [Art13], but its application to non-quasi-split groups was impeded by the lack of proper normalizations of the local objects, both transfer factors and spectral pairings. This paper, in conjunction with [Kal], removes this obstruction.

An example of Arthur’s approach being carried out in the setting of non-quasi-split groups was given in [KMSW], where inner forms of unitary groups were treated. The results of [KMSW] were obtained prior to the present work and use Kottwitz’s theory of isocrystals with additional structure [Kot85], [Kot97], [Kol] to refine the local endoscopic objects instead. That theory works very well for reductive groups that have connected centers and satisfy the Hasse principle. When the center is not connected, however, not all inner forms can be obtained via isocrystals. Moreover, when the Hasse principle fails, we do not see a way to use the global theory of isocrystals to build a bridge between local and global endoscopy. The constructions of the present paper remove these conditions and work uniformly for all connected reductive groups. Moreover, our statement of the local conjecture is very closely related to the work of Adams-Barbasch-Vogan [ABV92] on real groups as well as to Arthur’s statement given in [Art06]. The relationship with [ABV92] was partially explored in [Kal, §5.2] and the relationship with [Art06] is discussed at the end of the current paper.

We will now explain in some detail the construction of the global pairing. While in the body of the paper we treat general connected reductive groups, for the purposes of the introduction we assume that $G$ is semi-simple and simply connected. This case is easier to describe, because the notation is simpler and because we do not need to address the possible failure of the Hasse principle. At the same time, this case is in some sense the hardest from the point of view of local inner forms, so it will serve as a good illustration.

First let us assume in addition that $G$ is quasi-split and briefly recall the situation in this case. One fixes a Whittaker datum for $G$, which is a pair $(B, \psi)$ of a Borel subgroup of $G$ defined over the number field $F$ and a generic character $\psi : U(\mathfrak{b}_F) \rightarrow \mathbb{C}^\times$ of the unipotent radical $U$ of $B$ that is trivial on the subgroup $U(F)$ of $U(\mathfrak{b}_F)$. For each place $v$ of $F$, restriction of $\psi$ to the subgroup $U(F_v)$ of $U(\mathfrak{b}_F)$ provides a local Whittaker datum $(B, \psi_v)$ for $G$. Given a discrete global generic Arthur parameter $\phi : L_F \rightarrow LG$, let $\phi_v : L_{F_v} \rightarrow LG$ be its localization at each place $v$ of $F$. We consider the global centralizer group $S_\phi = \text{Cent}(\phi(L_F), \widehat{G})$ as well as, for each place $v$ of $F$, its local analog $S_{\phi_v} = \text{Cent}(\phi(L_{F_v}), \widehat{G})$. We also set $S_\phi = \pi_0(S_\phi)$ and $S_{\phi_v} = \pi_0(S_{\phi_v})$. In fact, since $\phi$ is discrete and $G$ is semi-simple we don’t need to take $\pi_0$ in the global case, but we do need to take $\pi_0$ in the local case, as $\phi_v$ will usually not be discrete. There is a conjectural packet $\Pi_{\phi_v}$ of irreducible admissible tempered representations of $G(F_v)$ and a conjectural pairing $\langle -, - \rangle_v : S_{\phi_v} \times \Pi_{\phi_v} \rightarrow \mathbb{C}$ such that $\langle -, \pi_v \rangle_v$ is an irreducible character of the finite group $S_{\phi_v}$ for each $\pi_v \in \Pi_{\phi_v}$. Once the $L$-packets are given, this pairing is uniquely determined by the endoscopic character identities taken with respect to the transfer factor that is normalized according to the local Whittaker datum $(B, \psi_v)$. According to a conjecture of Shahidi [Sha90, §9], the local pairing is expected to satisfy $\langle -, \pi_v \rangle_v = 1$ for the unique $(B, \psi_v)$-generic member $\pi_v \in \Pi_{\phi_v}$. The global packet is defined as $\Pi_{\phi} = \{ \pi = \otimes \pi_v | \pi_v \in \Pi_{\phi_v}, \langle -, \pi_v \rangle_v = 1 \text{ for a.a. } v \}$. Via the
map \( S_\phi \to S_{\phi_v} \) each local pairing \( (-, \pi_v) \), can be restricted to \( S_\phi \) and the global pairing
\[
\langle - , - \rangle : S_\phi \times \Pi_\phi \to \mathbb{C}
\]
is defined as the product \( \langle x, \pi \rangle = \prod_v \langle x, \pi_v \rangle \).

We now drop the assumption that \( G \) is quasi-split, while still maintaining the assumption that it is semi-simple and simply connected. There exists a quasi-split semi-simple and simply connected group \( G^* \) and an isomorphism \( \xi : G^* \to G \) defined over \( \mathbb{F} \) such that for each \( \sigma \in \Gamma = \text{Gal}(\overline{\mathbb{F}} / \mathbb{F}) \) the automorphism \( \xi^{-1} \sigma(\xi) \) of \( G^* \) is inner. We fix this data, thereby realizing \( G \) as an inner twist of \( G^* \). For a moment we turn to the quasi-split group \( G^* \) and fix objects as above: a global Whittaker datum \( (B, \psi_v) \) for \( G^* \) with localizations \( (B, \psi_v) \), and a discrete generic Arthur parameter \( \phi : L_F \to \ell^* G^* \) with localizations \( \phi_v \). The groups \( S_\phi \) and \( S_{\phi_v} \) are also defined as above and are subgroups of \( \hat{G}^* \). We also keep the group \( S_\phi = \pi_0(S_{\phi_v}) \), but the conjecture of [Kal] §5.4 tells us that we need to replace the group \( S_{\phi_v} \) by \( \pi_0(S_{\phi_v}^{sc}) \), where \( S_{\phi_v}^{sc} \) is the preimage of \( S_{\phi_v} \) in \( \hat{G}^*_{sc} \), the simply connected cover of the (in this case adjoint) group \( \hat{G}^* \). We will now obtain for each place \( v \) of \( F \) a conjectural local pairing \( (-, -)_{z_v} : \pi_0(S_{\phi_v}^{sc}) \times \Pi_\phi(G) \to \mathbb{C} \), but for this we need to endow \( G \) with the structure of a rigid inner twist of \( G^* \) at the place \( v \), which is a lift \( z_v \in \mathcal{Z}^1(u_v \to W_v, Z(G^*_{sc}) \to G^*_{sc}) \) of the element of \( \mathcal{Z}^1(\Gamma_v, G^*_{ad}) \) given by \( \sigma \mapsto \xi^{-1} \sigma(\xi) \). Here we are using the cohomology functor \( H^1(\mathcal{U}_v \to W_v, -) \) defined in [Kal] as well as the corresponding cocycles. Once \( z_v \) is fixed, according to [Kal] §5.4 there exists a pairing \( (-, -)_{z_v} : \pi_0(S_{\phi_v}^{sc}) \times \Pi_\phi(G) \to \mathbb{C} \) such that for each \( \pi_v \in \Pi_\phi(G) \) the function \( (-, \pi_v)_{z_v} \) is an irreducible character of \( \pi_0(S_{\phi_v}^{sc}) \) and the endoscopic character identities are satisfied with respect to the normalization of the transfer factor given in [Kal] §5.3, which involves both the local Whittaker datum \( (B, \psi_v) \) and the datum \( z_v \). As we noted above, the normalization of the transfer factor is established in [Kal] §5.3 unconditionally and the endoscopic character identities specify the pairing \( (-, \pi_v)_{z_v} \) uniquely, provided it can be shown to exist. We now define the global packet as \( \Pi_\phi(G) = \{ \pi = \otimes \pi_v | \pi_v \in \Pi_\phi(G), (-, \pi_v)_{z_v} = 1 \text{ for a.a. } v \} \). Letting \( S_{\phi_v}^{+} \) be the preimage in \( \hat{G}^*_{sc} \) of the global centralizer group \( S_\phi \) and taking the product of the local pairings over all places \( v \) we obtain a global pairing
\[
\langle - , - \rangle : S_{\phi_v}^{+} \times \Pi_\phi(G) \to \mathbb{C}
\]

We must now discuss the dependence of this global pairing on the choices of \( z_v \). At each place \( v \) there will usually be several choices for \( z_v \). Changing from one choice to another affects both the transfer factor at that place as well as the local pairing \( (-, \pi_v)_{z_v} \). It is then clear that the global pairing that we have defined depends on the choices of \( z_v \) at each place \( v \). What we will show however is that if we impose a certain coherence condition on the collection \( (z_v)_v \) of local cocycles (indexed by the set of places of \( F \)) then the pairing becomes independent of this choice. One instance of this coherence should be that the product over all places of the normalized transfer factors is equal to the canonical adelic transfer factor involved in the stabilization of the Arthur-Selberg trace formula. Another instance of this coherence should be that the global pairing is independent of the collection \( (z_v)_v \), used in its construction, and furthermore descends from the group \( S_{\phi_v}^{+} \) to its quotient \( S_\phi \).

It is this coherence that necessitates the development of the global Galois gerbes and their cohomology, and this work takes up a large part of this paper. We construct a functor \( H^1(P_V \to \mathcal{E}_V) \) that assigns to each algebraic torus \( T \) defined over \( F \) and each finite subgroup \( Z \subset T \) defined over \( F \) an abelian group

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\( H^1(P_V \to E_V, Z \to T) \). At each place \( v \) of \( F \) there is a functorial localization map \( H^1(P_V \to E_V, Z \to T) \to H^1(u_v \to W_v, Z \to T) \). A collection \((z_v)_v\) of local cohomology classes is coherent if it is in the image of the total localization map.

The construction of the global Galois gerbes and the study of their cohomology is more involved than that of their local counterpart. Let us briefly recall that to construct the local gerbe we consider the pro-finite algebraic group

\[
\phi = \lim_{\overset{\longrightarrow}{\mathbb{Z}/\mu_n}} \text{Res}_{E_v/F_v} \mu_n/\mu_n
\]

over the given local field \( F_v \). Here \( E_v \) runs over the finite Galois extensions of \( F_v \) contained in \( F_v \) and \( n \) runs over all natural numbers. We show that \( H^1(\Gamma_v, u_v) \) vanishes and \( H^2(\Gamma_v, u_v) \) is pro-cyclic with a distinguished generator. The isomorphism class of the gerbe is determined by this generator and the gerbe itself is unique up to (essentially) unique isomorphism due to the vanishing of \( H^1(\Gamma_v, u_v) \). Using this one introduces for each torus \( T \) and each finite subgroup \( Z \subset T \), both defined over \( F_v \), a cohomology group denoted by \( H^1(u_v \to W_v, Z \to T) \). Two central features are the surjectivity of the map \( \text{Hom}(u_v, Z)^{F_v} \to H^2(\Gamma_v, Z) \) evaluating a homomorphism at the distinguished generator of \( H^2(\Gamma, u_v) \) for all finite multiplicative groups \( Z \), and the existence of a Tate-Nakayama-type isomorphism for the cohomology group \( H^1(u_v \to W_v, Z \to T) \).

Now consider a number field \( E \). The analog of the groups \( \text{Res}_{E_v/F_v} \mu_n/\mu_n \) is more technical. To describe it, let \( S \) be a finite set of places of \( F \) containing the archimedean places. Fix a finite Galois extension \( E/F \) unramified outside of \( S \) and a natural number \( N \) that is a unit away from \( S \). We then define the finite \( \text{Gal}(E/F)-\)module

\[
M_{E,S,N} = \text{Maps}(\text{Gal}(E/F) \times S_E, \frac{1}{N}\mathbb{Z}/\mathbb{Z})_{0,0}.
\]

Here \( S_E \) is the set of all places of \( E \) lying above \( S \) and we are considering all maps \( \phi \) from the finite set \( \text{Gal}(E/F) \times S_E \) to the finite abelian group \( \frac{1}{N}\mathbb{Z}/\mathbb{Z} \) which satisfy the two conditions

\[
\forall w \in S_E : \sum_{\sigma \in \text{Gal}(E/F)} \phi(\sigma, w) = 0 \quad \text{and} \quad \forall \sigma \in \text{Gal}(E/F) : \sum_{w \in S_E} \phi(\sigma, w) = 0.
\]

Let \( \hat{S}_E \subset S_E \) be a set of lifts for the places of \( S \). We consider the submodule \( M_{E,\hat{S}_E,N} \subset M_{E,S,N} \) consisting of all \( \phi \) that satisfy the condition

\[
\phi(\sigma, w) \neq 0 \Rightarrow \sigma^{-1}w \in \hat{S}_E.
\]

We take this submodule as the character module of a finite multiplicative group defined over \( O_{F,S} \) and consider its \( O_S \)-points

\[
P_{E,\hat{S}_E,N}(O_S) = \text{Hom}(M_{E,\hat{S}_E,N}, O_S^\times),
\]

where \( O_S^\times \) is the group of units in the ring of integers of \( F_S \), the maximal extension of \( F \) unramified outside of \( S \). After placing multiple technical restrictions on the set \( S \) and its lift \( \hat{S}_E \) we are able to show that \( H^2(\hat{\Gamma}_S, \mathcal{V}) \) has a distinguished element (even though it is no longer pro-cyclic). As we vary \( E \), \( S \), and \( N \), the groups \( P_{E,\hat{S}_E,N} \) form an inverse system and the transition maps identify the distinguished cohomology classes. If we let \( \mathcal{V} \) be the inverse limit
of that system, we obtain a map \( \text{Hom}(P_V, Z) \to H^2(\Gamma, Z) \) and it is again surjective. However, unlike in the local case, we do not have a gerbe bound by \( P_V \). The reason is that by taking the limit over \( S \) we have destroyed the setting of restricted ramification, which in turn leads to the failure of certain finiteness and duality properties necessary for the construction. But if we fix \( S \), then we are not allowed to take the limit over \( E \), because one of the technical conditions on \( S \) is that the \( S \)-class group of \( E \) vanishes. Thus we cannot enlarge \( E \) without enlarging \( S \). This forces us to fix both \( E \) and \( S \) and allows us to only take the limit over \( N \). We must therefore work with a gerbe \( E_{E,S,E} \) for a fixed pair of \( S,E \). But then we have to live with the problem that \( H^1(\Gamma_S, P_{E,S,E}) \neq 0 \), i.e. the gerbe \( E_{E,S,E} \) has non-trivial automorphisms. This means that a-priori the cohomology group \( H^1(P_{E,S,E} \to E_{E,S,E}, Z \to T) \) depends not only on the distinguished generator of \( H^2(\Gamma_S, P_{E,S,E}) \), but also on the particular realization of \( E_{E,S,E} \) within its isomorphism class. What saves us is a careful study of the group theoretic and functorial properties of \( H^1(P_{E,S,E} \to E_{E,S,E}) \). These imply that even though the gerbe \( E_{E,S,E} \) may have non-trivial automorphisms, these all act as the identity on the cohomology functor \( H^1(P_{E,S,E} \to E_{E,S,E}) \). This observation allows us furthermore to establish well-defined inflation maps between these cohomology groups for varying \( E \) and \( S \), which in turn lead to a canonical cohomology group \( H^1(P_V \to E_V, Z \to T) \), despite the lack of a gerbe \( E_V \). It comes equipped with a localization map to each local cohomology group \( H^1(u_{F_v} \to W_{F_v}, Z \to T) \) and these localization maps in turn lead to the notion of coherent families of local cohomology classes.

We will now give a brief overview of the contents of this paper. Section 3 is devoted to the construction of the cohomology groups \( H^1(P_V \to E_V, Z \to T) \). We begin in Subsection 3.1 with a review of Tate’s description of the cohomology groups \( \tilde{H}^1(\text{Gal}(E/F), T(O_{E,S})) \) and prove some results about the compatibility of this description with variations of \( E \) and \( S \). In Subsection 3.2 we derive a description of the cohomology group \( H^2(\Gamma, Z) \) for a finite multiplicative group \( Z \) defined over \( F \). This description is the basis of the study of the finite multiplicative groups \( P_{E,S,E,N} \) in Subsection 3.3. After defining these groups we show that \( H^2(\Gamma_S, P_{E,S,E,N}(O_S)) \) has a distinguished element and study how this element varies with \( E \), \( S \), and \( N \), as well as the compatibility of this element with the distinguished elements of \( H^2(\Gamma_v, u_v) \) for all places \( v \). In Subsection 3.4 we introduce the cohomology groups \( H^1(P_{E,S,E,N} \to E_{E,S,E,N}, Z \to T) \). We prove a Tate-Nakayama-type description for them and construct inflation maps that relate these groups for varying \( E \), \( S \), and \( N \). Finally, in Subsection 3.5 we introduce the localization maps \( H^1(P_{E,S,E,N} \to E_{E,S,E,N}, Z \to T) \to H^1(u_{F_v} \to W_{F_v}, Z \to T) \).

Section 4 contains the applications to automorphic forms. Subsections 4.2 and 4.3 constitute the bridge between refined local and global endoscopy. Given a global inner twist we construct a coherent collection of rigid local inner twists, and given a global endoscopic datum we construct a coherent collection of refined local endoscopic data. The local collections depend on some choices, but these choices are ultimately seen to have no influence on the global objects obtained from the local collections. In Subsection 4.4 we show that the product of normalized local transfer factors is equal to the canonical adelic transfer factor, and in Subsection 4.5 we show that the product of normalized local pairings descends to a canonical global pairing. In both cases we use the coherent local collections obtained from the global inner twist and the global endoscopic datum, but the product of normalized local transfer factors and the product of normalized local pairings are independent of these collections. Both product
formulas contain some extra factors that one might not initially expect. These factors are explicit and non-conjectural and their appearance is just a matter of formulation, as we discuss in the next paragraph.

In the final Subsection 4.6 we summarize the results of Section 4 using the language of [Art06]. This subsection carries little mathematical content, but provides a dictionary between the language of [Kal] and that of [Art06]. More precisely, it shows that the local conjecture of [Kal] implies (and is in fact equivalent to) a stronger version of the local conjecture of [Art06]. We believe that this dictionary can be useful, because both languages have their advantages. The formulation of [Art06] is from the point of view of the simply connected cover of the given reductive group and has the advantage that the passage between local and global endoscopy and the global product formulas take a slightly simpler form, in which in particular the extra factors referred to above are not visible, because they become part of the local normalizations. The formulation of [Kal] is from the point of view of the reductive group itself and has the advantage of making the local statements more flexible and transparent. In particular, certain basic local manipulations, like Levi descent, are easier to perform in this setting. Moreover, this language accommodates the notions of pure and rigid inner twists, which have been shown in the work of Adams-Barbasch-Vogan [ABV92] on real groups to play an important role in the study of the local conjecture and in particular in its relationship to the geometry of the dual group.

In this paper we have restricted ourselves to discussing only tempered local and global parameters. This restriction is made just to simplify notation and because the problems we are dealing with here are independent of the question of temperedness. The extension to general Arthur parameters is straightforward and is done in the same way as described in [Art89].

A remark may be in order about the relationship between the cohomological constructions of this paper and those of [Kot]. While both papers study the cohomology of certain Galois gerbes, the gerbes used in the two papers are very different. The gerbes of [Kot] are bound by split multiplicative groups whose character modules are torsion-free. In the local case they are the so-called Dieudonné- and weight-gerbes of [LR87]. On the other hand, the gerbes of [Kal] and the present paper are bound by non-split multiplicative groups whose character modules are both torsion and divisible. This makes the difficulties involved in studying them and the necessary techniques quite different. It is at the moment not clear to us what the relationship between the two constructions is. In a forthcoming paper we plan to compare the local constructions in order to relate the local conjecture in [Kal] to the cohomology of Rapoport-Zink spaces.

It should be clear to the reader that this paper owes a great debt to the ideas of Robert Kottwitz. The author wishes to express his gratitude to Kottwitz for introducing him to this problem and generously sharing his ideas. The author also thanks James Arthur for his interest and support, as well as Alexander Schmidt for bringing to the authors attention the results in [NSW08] on the cohomology of Galois groups of number fields with ramification conditions.

2 Notation

Throughout the paper, $F$ will denote a fixed number field, i.e. a finite extension of the field $\mathbb{Q}$ of rational numbers. We will often use finite Galois extensions $E$
of $F$, which are all assumed to lie within a fixed algebraic closure $\overline{F}$ of $F$. For any such $E/F$, we denote by $\Gamma_{E/F}$ the finite Galois group $\text{Gal}(E/F)$, which is the quotient of the absolute Galois group $\Gamma_F = \text{Gal}(\overline{F}/F)$ by its subgroup $\Gamma_E$. We will write $I_E = \mathbb{A}_E^\times$ for the idele group of $E$ and $C(E) = I_E/E^\times$ for the idele class group of $E$.

We write $V(E)$ for the set of all places (i.e. valuations up to equivalence) of $E$ and $V(E)_\infty$ for the subset of archimedean valuations. A subset $S \subset V(E)$ will be called full, if it is the preimage of a subset of the set of places $V(Q)$ of $Q$. If $K/E/F$ is a tower of extensions and $S \subset V(E)$, we write $S_K$ for the set of all places of $K$ lying above $S$, and we write $S_P$ for the set of all places of $F$ lying below $S$. We write $p : S_K \to S_E$ for the natural projection map. Given a subset $S \subset V(Q)$, we write $\mathbb{N}_S$ for the set of those natural numbers whose prime decomposition involves only primes contained in $S$. We equip $\mathbb{N}_S$ with the partial order given by divisibility. Given a subset $S \subset V(F)$, we write $\mathbb{N}_S$ for $\mathbb{N}_S$ by abuse of notation.

If $V(F)_\infty \subset S \subset V(F)$ is a set of places, and $E/F$ is a finite Galois extension unramified outside of $S$ we will borrow the following notation from [NSW08, §VIII.3]. Let $O_{E,S}$ be the ring of $S$-integers of $E$, i.e. the subring of $E$ consisting of elements whose valuation at all places away from $S_E$ is non-negative. Let $I_{E,S} = \prod_{w \in S_E} E_w^\times$, $U_{E,S} = \prod_{w \in S_E} O_{E,w}^\times$. Then $J_{E,S} = I_{E,S} \times U_{E,S}$ is the subgroup of $E$-ideles which are units away from $S$. We set $C_{E,S} = I_{E,S}/O_{E,S}^\times$ and $C_S(E) = I_E/E^\times U_{E,S}$.

Taking the limit of these notions over all finite Galois extensions $E/F$ that are unramified away from $S$ we obtain the following. We denote by $F_S$ the maximal extension of $F$ that is unramified away from $S$. We write $\Gamma_S$ for the Galois group of $F_S/F$. We denote by $O_S$ the direct limit of $O_{E,S}$ and by $I_S$ the direct limit of $I_{E,S}$, the limits being taken over all finite extensions $F_S/E/F$. We denote by $C_S$ the direct limit of either $C_{E,S}$ or $C_S(E)$. These two limits are the same [NSW08, Prop 8.3.6] are in turn equal to the quotient $C(F_S)/U_S$, where $C(F_S) = \lim_{\rightarrow E/F} C(E)$ is the idele class group of $F_S$, and $U_S = \lim_{\rightarrow E/F} U_{E,S}$. In the special case $S = V(F)$ we will drop the subscript $S$ from the notation, thereby obtaining for example $I = \mathbb{A}_F^\times$ and $C = \mathbb{A}_F^\times/\overline{F}^\times$.

Given two finite sets $X, Y$, we write Maps$(X, Y)$ for the set of all maps from $X$ to $Y$. If $X, Y$ are endowed with an action of $\Gamma_F$, or a related group, we endow Maps$(X, Y)$ with the action given by $\sigma(f)(x) = \sigma(f(\sigma^{-1}x))$.

Given an abelian group $A$, we write $A[n]$ for the subgroup of $n$-torsion elements for any $n \in \mathbb{N}$, and we write $\exp(A)$ for the exponent of $A$, allowing $\exp(A) = \infty$. We follow the standard conventions of orders of pro-finite groups.

If $B$ is a profinite group acting on a (sometimes abelian) group $X$, we write $H^i(B, X)$ for the set of continuous cohomology classes, where $i = 0$ if $X$ is abelian and $i = 1$ if $X$ is non-abelian. The group $X$ is understood to have the discrete topology unless it is given as an inverse limit, in which case we endow it with the inverse limit topology. When $B$ is finite and $X$ is abelian we also have the modified cohomology groups $\tilde{H}^i(B, X)$ defined for all $i \in \mathbb{Z}$ by Tate. For us usually $B = \Gamma_{E/F}$ for some finite Galois extension $E/F$. In this case we will write $N_E/F : X \to X$ for the norm map of the action of $\Gamma_{E/F}$ on $X$, and we will denote by $X^{N_E/F}$ its kernel. We will also write $\mathbb{Z}[\Gamma_{E/F}]$ for the group ring of $\Gamma_{E/F}$ and $I_{E/F}$ for its augmentation ideal. Then $I_{E/F}X$ is a subgroup of $X$. 


3.1 Some properties of Tate duality over $F$

The purpose of this subsection is to review Tate’s description \cite{Tate66} of the cohomology group $H^1(\Gamma_{E/F}, T(O_E,S))$, where $T$ is an algebraic torus defined over $F$ and split over a finite Galois extension $E/F$ and $S$ is a (finite or infinite) set of places of $F$ subject to some conditions. We will also establish some facts about this description that we have been unable to find in the literature.

We assume that $S$ satisfies the following.

**Conditions 3.1.**  
1. $S$ contains all archimedean places and all places that ramify in $E$.
2. Every ideal class in $C(E)$ has an ideal with support in $S_E$.

Let $T$ be an algebraic torus defined over $F$ and split over $E$. We write $X = X^*(T)$ and $Y = X_s(T)$. The group $T(O_E,S) = \text{Hom}(X, O^\times_{E,S}) = Y \otimes O^\times_{E,S}$ is a $\Gamma_{E/F}$-module. In \cite{Tate66} Tate provides a description of the cohomology groups $\hat{H}^i(\Gamma_{E/F}, T(O_E,S))$. Consider the abelian group $\mathbb{Z}[S_E]_0$ consisting of formal integral linear combinations $\sum_{w \in S_E} n_w[w]$ of elements of $S_E$ having the property $\sum_{w \in S_E} n_w = 0$. Tate constructs a canonical cohomology class $\alpha_3(S) \in H^2(\Gamma_{E/F}, \text{Hom}(\mathbb{Z}[S_E]_0, O^\times_{E,S}))$ and shows that cup product with this class induces for all $i \in \mathbb{Z}$ an isomorphism

$$\hat{H}^{i-2}(\Gamma_{E/F}, Y[S_E]_0 \otimes Y) \to \hat{H}^i(\Gamma_{E/F}, T(O_E,S)). \tag{3.1}$$

We have written here $Y[S_E]_0$ as an abbreviation for $\mathbb{Z}[S_E]_0 \otimes Y$. For our applications, it will be important to know how this isomorphism behaves when we change $S$ and $E$. Before we can discuss this, we need to review the construction of the class $\alpha_3(S)$.

Tate considers the group $\text{Hom}((B_S), (A_S))$ consisting of triples of homomorphisms $(f_3, f_2, f_1)$ making the following diagram commute

![Diagram](https://example.com/diagram.png)

The two sequences $(A_S)$ and $(B_S)$ are exact, with $a$ being the natural projection, which is surjective due to Conditions 3.1 and $b$ being the map $b(\sum_w n_w[w]) = \sum_w n_w$. It is clear that extracting the individual entries of a triple $(f_3, f_2, f_1)$ provides maps

$$\text{Hom}((B_S), (A_S)) \xrightarrow{\text{Hom}(\mathbb{Z}[S_E]_0, O^\times_{E,S})} \text{Hom}(\mathbb{Z}[S_E], J_{E,S}) \xrightarrow{\text{Hom}(\mathbb{Z}, C(E))} \text{Hom}(\mathbb{Z}, C(E))$$

The product of the vertical and right-diagonal maps is an injection and induces an injection on the level of $H^2(\Gamma_{E/F}, -)$. The class $\alpha_3(S)$ is the image of a class...
\( \alpha(S) \in H^2(\Gamma_{E/F}, \text{Hom}(B_S, (A_S))) \) under the left diagonal map. The class \( \alpha(S) \) itself is the unique class mapping to the pair of classes

\[ (\alpha_2(S), \alpha_1) \in H^2(\Gamma_{E/F}, \text{Hom}(\mathbb{Z}[S_E], J_{E,S})) \times H^2(\Gamma_{E/F}, \text{Hom}(\mathbb{Z}, C(E))), \]

which we will now describe.

The class \( \alpha_1 \in H^2(\Gamma_{E/F}, \text{Hom}(\mathbb{Z}, C(E))) \) is the fundamental class associated by global class field theory to the extension \( E/F \). Using the Shapiro-isomorphism

\[ H^2(\Gamma_{E/F}, \text{Hom}(\mathbb{Z}[S_E], J_{E,S})) \cong \prod_{v \in S} H^2(\Gamma_{E_v/F_v}, J_{E,S}), \]

where for each \( v \in S \) we have chosen a lift \( \hat{v} \in S_{E_v} \), Tate defines the class \( \alpha_2(S) \in H^2(\Gamma_{E/F}, \text{Hom}(\mathbb{Z}[S_E], J_{E,S})) \) to correspond to the element \( (\alpha_{E_v/F_v})_{v \in S} \) of the right hand side of the above isomorphism, where each \( \alpha_{E_v/F_v} \in H^2(\Gamma_{E_v/F_v}, J_{E,S}) \) is the image of the fundamental class in \( H^2(\Gamma_{E_v/F_v}, E_v^\times) \) under the natural inclusion \( E_v^\times \to J_{E,S} \).

Having recalled the construction of (3.1) we are now ready to study how it varies with respect to \( S \) and \( E \).

**Lemma 3.2.** Let \( S \subset S' \subset V(F) \). The inclusion \( S_E \to S'_E \) provides maps \( \mathbb{Z}[S_E]_0 \to \mathbb{Z}[S'_{E}]_0 \) and \( O_{E,S} \to O_{E,S'} \) which fit in the commutative diagram

\[
\begin{array}{c}
\tilde{H}^i(\Gamma_{E/F}, T(O_{E,S})) \\
\downarrow \\
\tilde{H}^i(\Gamma_{E/F}, T(O_{E,S'}))
\end{array}
\]

\[
\begin{array}{c}
\tilde{H}^{i-2}(\Gamma_{E/F}, Y[S_E]_0) \\
\downarrow \\
\tilde{H}^{i-2}(\Gamma_{E/F}, Y[S'_E]_0)
\end{array}
\]

**Proof.** The top horizontal map is given by cup product with the class \( \alpha_3(S) \in H^2(\Gamma_{E/F}, \text{Hom}(\mathbb{Z}[S_E]_0, O_{E,S})) \), while the bottom horizontal map is given by cup product with the analogous class \( \alpha_3(S') \). We must relate the two classes. For this, let \( \text{Hom}_S(\mathbb{Z}[S'_E], J_{E,S'}) \) be the subgroup of \( \text{Hom}(\mathbb{Z}[S_E], J_{E,S}) \) consisting of those homomorphisms which map the subgroup \( \mathbb{Z}[S_E] \) of \( \mathbb{Z}[S'_E] \) into the subgroup \( J_{S,E} \) of \( J_{S',E} \). We define analogously \( \text{Hom}_S(\mathbb{Z}[S'_E]_0, O_{E,S'}^\times) \) and \( \text{Hom}_S((B_{S'}), (A_{S'})) \). We have the maps

\[ \text{Hom}(\mathbb{Z}[S_E]_0, O_{E,S}^\times) \leftarrow \text{Hom}_S(\mathbb{Z}[S'_E]_0, O_{E,S'}^\times) \to \text{Hom}(\mathbb{Z}[S'_E]_0, O_{E,S'}^\times), \]

the left one given by restriction to \( \mathbb{Z}[S_E]_0 \) and the right one given by the obvious inclusion. It will be enough to show that there exists a class

\[ \alpha_3(S, S') \in H^2(\Gamma_{E/F}, \text{Hom}_S(\mathbb{Z}[S'_E]_0, O_{E,S'}^\times)) \]

mapping to \( \alpha_3(S) \) via the left map and to \( \alpha_3(S') \) via the right map. In order to do this, we consider

\[ \text{Hom}((B_S), (A_S)) \leftarrow \text{Hom}_S((B_{S'}), (A_{S'})) \to \text{Hom}((B_{S'}), (A_{S'})) \]

and show that there exists a class

\[ \alpha(S, S') \in H^2(\Gamma_{E/F}, \text{Hom}_S((B_{S'}), (A_{S'}))) \]

mapping to \( \alpha(S) \) under the left map and mapping to \( \alpha(S') \) under the right map. Indeed we have the squence

\[ 0 \to \text{Hom}_S((B_{S'}), (A_{S'})) \to \text{Hom}_S(\mathbb{Z}[S'_E], J_{E,S'}) \times \text{Hom}(\mathbb{Z}, C(E)) \to \text{Hom}(\mathbb{Z}[S'_E], C(E)) \to 0 \]
in which the last map sends \((f_2, f_1)\) to \(a \circ f_2 - f_1 \circ b\). This map is surjective, because \(\text{Hom}(\mathbb{Z}[S'_{E}], J_{E, S})\) is a subgroup of \(\text{Hom}_S(\mathbb{Z}[S'_{E}], J_{E, S'})\), and since \(\mathbb{Z}[S'_{E}]\) is free and \(a : J_{E, S} \to C(E)\) is surjective we see that \(f_2 \mapsto a \circ f_2\) is already surjective. The above sequence is thus exact. We obtain maps

\[
\text{Hom}_S((B_{S'}), (A_{S'})) 
\]

\[
\text{Hom}_S(\mathbb{Z}[S'_{E}], J_{E, S'}) \quad \text{Hom}(\mathbb{Z}, C(E))
\]

and the long exact cohomology sequence associated to the above exact sequence shows that the product of the vertical and right-diagonal maps is an injection and induces an injection on the level of \(H^2(\Gamma_{E/F}, -)\). Under the Shapiro isomorphism the group \(\text{Hom}_S(\mathbb{Z}[S'_{E}], J_{E, S'})\) is identified with

\[
\prod_{v \in S} H^2(\Gamma_{E_v/F_v}, J_{E, S}) \times \prod_{v \in S \setminus S} H^2(\Gamma_{E_v/F_v}, J_{E, S'}).
\]

Using Tate’s construction of \(\alpha_2(S')\) in this setting we obtain a class \(\alpha_2(S, S') \in \text{Hom}_S(\mathbb{Z}[S'_{E}], J_{E, S})\) and the pair \((\alpha_2(S, S'), \alpha_1)\) is seen to belong to the group \(\text{Hom}_S((B_{S'}), (A_{S'}))\). This is the class \(\alpha(S, S')\) that we wanted to construct, and the class \(\alpha_3(S, S')\) is the image of \(\alpha(S, S')\) under the left diagonal map above.

\[
\square
\]

We will now discuss the variation of \((3.1)\) with respect to \(E\). This has been studied by Kottwitz. Let \(K/E/F\) be a tower of finite Galois extensions and \(S \subset V(F)\) a finite set of places which satisfies Conditions \((3.4)\) with respect to both \(E\) and \(K\). Consider the map \(\mathbb{Z}[S_K]|_0 \to \mathbb{Z}[S_E]|_0\) sending \(\sum_{u \in S_K} n_u[u]\) to \(\sum_{w \in S_E} \sum_{u | w} n_u[w]\). This map induces a map \(\hat{H}^{-1}(\Gamma_{K/F}, Y[S_K]|_0) \to \hat{H}^{-1}(\Gamma_{E/F}, Y[S_E]|_0)\) and Kottwitz proves \([Ko]\) Lemma 8.4] the following.

**Lemma 3.3** (Kottwitz). The diagram

\[
\begin{array}{ccc}
\hat{H}^{-1}(\Gamma_{E/F}, Y[S_E]|_0) & \longrightarrow & H^1(\Gamma_{E/F}, T(O_{E, S})) \\
\downarrow & & \downarrow \text{inf} \\
\hat{H}^{-1}(\Gamma_{K/F}, Y[S_K]|_0) & \longrightarrow & H^1(\Gamma_{K/F}, T(O_{K, S}))
\end{array}
\]

commutes and both vertical maps are isomorphisms.

We will soon need a variant of this result for finite multiplicative groups instead of tori. In that case, the inflation map on the right will no longer be an isomorphism and in order to state the result, we will need to work with the inverse of the map on the left. We will now discuss this inverse and then re-state the lemma with it. Fix an arbitrary section \(s : S_E \to S_K\) of the natural projection \(p : S_K \to S_E\) and let \(s_1 : \mathbb{Z}[S_E]|_0 \to \mathbb{Z}[S_K]|_0\) be the map sending \(\sum_{w \in S_E} n_w[w]\) to \(\sum_{w \in S_E} n_w[s(w)]\).

**Lemma 3.4.** Assume that for each \(\sigma \in \Gamma_{K/F}\) there exists \(u \in S_K\) such that \(\sigma u = u\). For any \(\Gamma_{E/F}\)-module \(M\) the map \(s_1\) induces a well-defined map \(! : \hat{H}^{-1}(\Gamma_{E/F}, M[S_E]|_0) \to \hat{H}^{-1}(\Gamma_{K/F}, M[S_K]|_0)\) that is functorial in \(M\) and independent of the choice of \(s\). When \(M\) is finite rank and torsion-free, this map is the inverse to the left map in Lemma **3.3**.
Proof. Let \( s, s' \) be two sections and let \( x \in M[S_E]_0 \) be given as \( x = \sum w n_w[w] \) with \( n_w \in M \). We claim that \( s(x) = s'(x) \) in \( H_0(\Gamma_{K/F}, M[S_K]_0) \). The set \( \{ w \in S_E | n_w \neq 0, s(w) \neq s'(w) \} \) is finite. It if is empty, then there is nothing to show. By induction on its size we may assume that it contains a single place \( w_0 \). Then we have
\[
s(x) - s'(x) = n_{w_0} [s(w_0)] - n_{w_0} [s'(w_0)].
\]
Let \( \sigma \in \Gamma_{K/E} \) be such that \( \sigma(s(w_0)) = s'(w_0) \). By assumption there exists \( u_0 \in S_K \) with \( \sigma u_0 = u_0 \). Then we have
\[
s(x) - s'(x) = (n_{w_0} [s(w_0)] - n_{w_0} [u_0]) - \sigma(n_{w_0} [s(w_0)] - n_{w_0} [u_0]),
\]
noting that \( \Gamma_{K/E} \) acts trivially on \( M \), and the claim is proved.

One checks easily that for \( x \in M[S_E]_0 \) and \( \sigma \in \Gamma_{K/F} \) we have \( \sigma(s(x)) = (\sigma s)(\sigma x) \). From this and the claim we just proved it follows that \( s_1 \) maps \( I_{E/F} M[S_E]_0 \) to \( I_{K/F} M[S_K]_0 \). We conclude that \( s_1 \) determines a well-defined map \( H_0(\Gamma_{E/F}, M[S_E]_0) \to H_0(\Gamma_{K/F}, M[S_K]_0) \) and that this map is independent of \( s \).

We now argue that if \( x \in M[S_E]_0 \) satisfies \( N_{E/F}(x) = 0 \), then \( N_{K/F}(s_1(x)) = 0 \). Write \( x = \sum w n_w[w] \). Then \( s_1(x) = \sum u \in S_K \delta_{u,p(u)} n_{p(u)}[u] \), where \( \delta_{u,u'} \) is equal to 1 if \( u = u' \) and to 0 otherwise. A quick computation shows
\[
N_{K/F}(s_1(x)) = \sum u \in S_K \sum \tau \in \Gamma_{E/F} \left| \text{Stab}(s(\tau^{-1} p(u)), \Gamma_{K/E}) \right| \tau n_{\tau^{-1} p(u)}[u].
\]
For a fixed \( \tau \in \Gamma_{E/F} \) let \( \sigma \in \Gamma_{K/F} \) be such that \( \sigma u = s(\tau^{-1} p(u)) \). Then
\[
\text{Stab}(s(\tau^{-1} p(u)), \Gamma_{K/E}) = \sigma \text{Stab}(u, \Gamma_{K/F}) \sigma^{-1} \cap \Gamma_{K/E}.
\]
Since \( E/F \) is Galois, the cardinality of this group is equal to the cardinality of \( \text{Stab}(u, \Gamma_{K/E}) \). We thus conclude that
\[
N_{K/F}(s_1(x)) = \sum u \in S_K \left| \text{Stab}(u, \Gamma_{K/E}) \right| \sum \tau \in \Gamma_{E/F} \tau n_{\tau^{-1} p(u)}[u].
\]
The inner sum vanishes due to the assumption \( N_{E/F}(x) = 0 \).

Assume now that \( M \) is finite rank and torsion-free. It is obvious that the map \( s_1 : Z[S_E]_0 \to Z[S_K]_0 \) is right-inverse to the map \( Z[S_K]_0 \to Z[S_E]_0 \) sending \( \sum u \in S_K n_u[u] \) to \( \sum w \in S_E \left( \sum u \in S_K n_u[u] \right) [w] \). This remains true on the level of \( H^{-1} \), but by Lemma 3.3 the latter map is then a bijection. It follows that ! must be the inverse of this bijection. \( \square \)

Corollary 3.5. Under the assumptions of Lemma 3.4 the diagram
\[
\begin{array}{ccc}
\tilde{H}^{-1}(\Gamma_{E/F}, Y[S_E]_0) & \longrightarrow & H^1(\Gamma_{E/F}, T(O_{E,S})) \\
\downarrow & & \downarrow \text{inf} \\
\tilde{H}^{-1}(\Gamma_{K/F}, Y[S_K]_0) & \longrightarrow & H^1(\Gamma_{K/F}, T(O_{K,S}))
\end{array}
\]
commutes and both vertical maps are isomorphisms.

Proof. This follows immediately from Lemmas 3.3 and 3.4 \( \square \)
We now want to relate the group $H^i(\Gamma_{E/F}, T(O_{E,S}))$ to the groups $H^i(\Gamma_S, T(O_S))$ and $H^i(\Gamma, T(\mathcal{F}))$ in the cases $i = 1, 2$.

**Lemma 3.6.** The inflation map $H^i(\Gamma_{E/F}, T(O_{E,S})) \to H^1(\Gamma_S, T(O_S))$ is bijective for $i = 1$ and injective for $i = 2$.

**Proof.** The inflation-restriction sequence takes the form
\[
1 \to H^1(\Gamma_{E/F}, T(O_{E,S})) \to H^1(\Gamma_S, T(O_S)) \to H^2(\Gamma_{E,F}, T(O_{E,S})) \to H^2(\Gamma_S, T(O_S)).
\]
Since $T$ splits over $E$ we have
\[
H^1(\Gamma_{E/F}, T(O_S)) \cong H^1(\Gamma_{E,S}, O_S^\times) = 0
\]
by [NSW08, Prop. 8.3.11] and Conditions 3.1 and the proof is complete. \(\square\)

**Lemma 3.7.** Assume that for each $w \in V(E)$ there exists $w' \in S_E$ with the property that $\text{Stab}(w, \Gamma_{E/F}) = \text{Stab}(w', \Gamma_{E/F})$. The inflation map $H^i(\Gamma_S, T(O_S)) \to H^i(\Gamma, T(\mathcal{F}))$ is injective for $i = 1, 2$.

**Proof.** We begin by constructing a left-inverse of the natural inclusion $S_E \to V(E)$. For each $v \in V(F)$ choose one $\tilde{v} \in V(E)$ and one $\hat{v} \in S_E$ such that $p(\tilde{v}) = v$ and $\text{Stab}(\tilde{v}, \Gamma_{E/F}) = \text{Stab}(\hat{v}, \Gamma_{E/F})$. If $v \in S$ then we demand $\hat{v} = \tilde{v}$. Define a map $f : V(E) \to S_E$ as follows. Given $w \in V(E)$ let $v = p(w)$ and choose $\sigma \in \Gamma_{E/F}$ such that $w = \sigma v$. Set $f(w) = \sigma \hat{v}$. The place $\sigma \hat{v}$ is independent of the choice of $\sigma$ and so this map is well-defined. It is moreover $\Gamma_{E/F}$-equivariant, because given $\tau \in \Gamma_{E/F}$ we have $p(\tau w) = v, \tau w = \tau \sigma v$, and hence $f(\tau w) = \tau \sigma \hat{v} = \tau f(w)$. By construction $f$ is left-inverse to the natural inclusion $S_E \to V(E)$.

We extend $f$ linearly to a map $Y[V(E)] \to Y[S_E]$ which restricts to a map $Y[V(E)]_0 \to Y[S_E]_0$ that is $\Gamma_{E/F}$-equivariant and left-inverse to the natural inclusion $Y[S_E]_0 \to Y[V(E)]_0$. We conclude that for all $i \in \mathbb{Z}$ the map $H^i(\Gamma_{E/F}, Y[S_E]_0) \to H^i(\Gamma_{E/F}, Y[V(E)]_0)$ has a left-inverse and hence must be injective.

Applying Lemmas 3.4 and 3.6 we obtain the diagram
\[
\begin{array}{ccc}
\tilde{H}^{-1}(\Gamma_{E/F}, Y[S_E]_0) & \cong & H^1(\Gamma_{E/F}, T(O_{E,S})) \\
\downarrow & & \downarrow \\
\tilde{H}^{-1}(\Gamma_{E/F}, Y[V(E)]_0) & \cong & H^1(\Gamma_{E/F}, T(E)) \\
\end{array}
\]
and the injectivity of the left vertical map implies the injectivity of the right vertical map.

To treat the case $i = 2$ we consider the diagram
\[
\begin{array}{ccc}
0 & \to & H^2(\Gamma_{E/F}, T(O_{E,S})) \\
\downarrow & & \downarrow \\
0 & \to & H^2(\Gamma, T(\mathcal{F}))
\end{array}
\]
\[
\begin{array}{ccc}
0 & \to & H^2(\Gamma_{E/F}, T(O_S)) \\
\downarrow & & \downarrow \\
0 & \to & H^2(\Gamma, T(\mathcal{F}))
\end{array}
\]
\]
\[
\begin{array}{ccc}
0 & \to & H^2(\Gamma_{E/F}, T(E)) \\
\downarrow & & \downarrow \\
0 & \to & H^2(\Gamma, T(\mathcal{F}))
\end{array}
\]
\]
\[
\begin{array}{ccc}
0 & \to & H^2(\Gamma_{E/F}, T(O_S)) \\
\downarrow & & \downarrow \\
0 & \to & H^2(\Gamma, T(\mathcal{F}))
\end{array}
\]
\]
By the five-lemma, we need to show the injectivity of the left and right vertical maps. The injectivity of the left vertical map follows from Lemma 3.4 and the injectivity of \( \bar{H}^0(\Gamma_{E/F}, Y|S_E|_0) \to \bar{H}^0(\Gamma_{E/F}, Y|V(E)|_0) \). Turning to the right vertical map, since \( T \) splits over \( E \) it is enough to prove the injectivity of the inflation map \( H^2(\Gamma_{E,S}, O_S^\times) \to H^2(\Gamma_{E,T}, F^\times) \). For this we consider the exact sequences \( 1 \to O_S^\times \to I_S \to C_S \to 1 \) and \( 1 \to F^\times \to I \to C \to 1 \) and obtain the diagram

\[
\begin{align*}
H^1(\Gamma_{E,S}, C_S) & \longrightarrow H^2(\Gamma_{E,S}, O_S^\times) \longrightarrow H^2(\Gamma_{E,S}, I_S) \\
=0 & \quad =0 \\
H^1(\Gamma_{E,C}) & \longrightarrow H^2(\Gamma_{E,F^\times}) \longrightarrow H^2(\Gamma_{E,I})
\end{align*}
\]

The map on the right is the map

\[
\lim_{\rightarrow F_S/K/E} H^2(\Gamma_{K/E}, I_{K,S}) \to \lim_{\rightarrow F/K/E} H^2(\Gamma_{K/E}, I_K),
\]

where the transition maps in both limits are the usual inflation maps and the map \( H^2(\Gamma_{K/E}, I_{K,S}) \to H^2(\Gamma_{K/E}, I_K) \) is induced by the inclusion \( I_{K,S} \to I_K \).

Since \( I_{K,S} \) is a direct factor of \( I_K \), the map \( H^2(\Gamma_{K/E}, I_{K,S}) \to H^2(\Gamma_{K/E}, I_K) \) is injective. Moreover, the vanishing of \( H^1(\Gamma_{K/E}, I_K) \) and \( H^1(\Gamma_{K/E}, I_{K,S}) \) for all \( K \) shows that the transition maps in the two limits are injective. It follows that the map \( H^2(\Gamma_{E,S}, I_S) \to H^2(\Gamma_{E,I}) \) in the above diagram is also injective, and then so is the map \( H^2(\Gamma_{E,S}, O_S^\times) \to H^2(\Gamma_{E,F^\times}) \). This completes the proof of the case \( i = 2 \).

3.2 A description of \( H^2(\Gamma, Z) \)

Let \( Z \) be a finite multiplicative group defined over \( F \). Write \( A = X^+(Z) \) and \( A^\vee = \text{Hom}(A, \mathbb{Q}/\mathbb{Z}) \). The purpose of this section is to establish a functorial isomorphism

\[
\Theta : \lim_{\rightarrow S} \bar{H}^{-1}(\Gamma_{E/F, S}, A^\vee|S_E|_0) \to H^2(\Gamma, Z). \tag{3.2}
\]

**Fact 3.8.** Let \( n \) be a multiple of \( \exp(Z) \). We have a functorial isomorphism

\[
\Phi_{A,n} : A^\vee \to \text{Hom}(\mu_n, Z), \quad a(\Phi_{A,n}(\lambda)(x)) = x^n(\lambda, a),
\]

where \( a \in A, \lambda \in A^\vee, x \in \mu_n, \) and \( \langle -, - \rangle \) is the natural pairing \( A \otimes A^\vee \to \mathbb{Q}/\mathbb{Z} \). If \( n|m \) and \( x \in \mu_n \) we have

\[
\Phi_{A,m}(x) = \Phi_{A,n}(\frac{x^n}{m}).
\]

Choose a finite Galois extension \( E/F \) splitting \( Z \) and a finite full set \( S \subset V(F) \) satisfying the following conditions.

**Conditions 3.9.**

1. \( S \) contains all archimedean places and all places that ramify in \( E \).
2. Every ideal class in \( C(E) \) contains an ideal supported on \( S_E \).
3. \( \exp(Z) \in \mathbb{N}_S \).
4. For each \( w \in V(E) \) there exists \( w' \in S_E \) such that \( \text{Stab}(w, \Gamma_{E/F}) = \text{Stab}(w', \Gamma_{E/F}) \).

Note that such finite sets exist and if \( S \) satisfies these conditions and \( S' \supset S \), then \( S' \) also satisfies these conditions.

**Fact 3.10.** Let \( \text{Maps}(S_E, A^\vee)_0 \) denote the finite abelian group consisting of all maps \( \phi : S_E \to A^\vee \) satisfying \( \sum_{w \in S_E} \phi(w) = 0 \). Let \( n \) be a multiple of \( \exp(Z) \). We have a functorial isomorphism

\[
\Phi_{A,S,n} : \text{Maps}(S_E, A^\vee)_0 \to \text{Hom}(\text{Maps}(S_E, \mu_n)/\mu_n, Z(O_S)),
\]

which for \( g \in \text{Maps}(S_E, A^\vee)_0, f \in \text{Maps}(S_E, \mu_n), a \in A \) is given by the formula

\[
a(\Phi_{A,S,n}(g))(f) = \prod_{w \in S_E} f(s)^{a(\phi)(w)}.\]

If \( n \mid m \) then \( \Phi_{A,S,m}(g)(f) = \Phi_{A,S,n}(g)(f^\frac{m}{n}) \).

Let \( \alpha_3(S) \in Z^2(\Gamma_{E/F}, \text{Hom}(Z[S_E], O_{E,S}^\times)) \) represent Tate’s class discussed in subsection 4.3. We have

\[
\text{Hom}(Z[S_E], O_{E,S}^\times) = \text{Maps}(S_E, O_{E,S}^\times) / O_{E,S}^\times \to \text{Maps}(S_E, O_{E,S}^\times) / O_S^\times,
\]

where we are identifying \( O_{E,S}^\times \) and \( O_S^\times \) as the subgroups of constant functions. By [NSW08, Proposition 8.3.4] the group \( O_S^\times \) is \( \mathbb{N}_S \)-divisible. For any \( n \in \mathbb{N}_S \) the \( n \)-th power map fits into the exact sequence

\[
1 \to \text{Maps}(S_E, \mu_n) / \mu_n \to \text{Maps}(S_E, O_S^\times) / O_S^\times \to \text{Maps}(S_E, O_{E,S}^\times) / O_{E,S}^\times \to 1.
\]

Fix a co-finite sequence \( n_i \in \mathbb{N}_S \) as well as functions \( k_i : O_S^\times \to O_S^\times \) satisfying \( k_i(x)^{n_i} = x \) and \( k_{i+1}(x)^{-n_i} = k_i(x) \). Then we have

\[
dk_i \alpha_3(S) \in Z^{1,2}(\Gamma_S, \Gamma_{E/F}, \text{Maps}(S_E, \mu_n)/\mu_n),
\]

where we are using the notation \( Z^{i,j} \) from [Kal] 4.3.

We now define a map

\[
\Theta_{E,S} : \tilde{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]) \to H^2(\Gamma_S, Z(O_S))
\]

as follows. We have the obvious identifications \( A^\vee[S_E] = Z[S_E] \otimes A^\vee = \text{Maps}(S_E, A^\vee)_0 \). Given \( g \in H^{-1}(\Gamma_{E/F}, A^\vee[S_E]) \), let

\[
\Theta_{E,S}(g) = dk_i \alpha_3(S) \cup_{E/F} g,
\]

where we have employed the unbalanced cup product of [Kal] 4.3 as well as the pairing

\[
\text{Maps}(S_E, \mu_n)/\mu_n \otimes \text{Maps}(S_E, A^\vee)_0 \to Z(O_S) \quad (3.3)
\]

provided by \( \Phi_{E,S,n} \) of Fact 3.10. Here we must choose \( n_i \) to be a multiple of \( \exp(Z) \), which is possible since \( n_i \) are a co-final sequence in \( \mathbb{N}_S \) and \( \exp(Z) \in \mathbb{N}_S \) by assumption. Moreover, the map \( \Theta_{E,S} \) is independent of the choice of \( n_i \).

**Proposition 3.11.** The map \( \Theta_{E,S} \) is a functorial injection independent of the choices of \( \alpha_3(S) \) and \( k_i \).
The proof is based on the following lemma, which will also have other uses later.

**Lemma 3.12.** Let $T$ be a torus defined over $F$ and split over $E$ and let $Z \to T$ be an injection with cokernel $T$. We write $Y = X_*(T)$ and $\bar{Y} = X_*(\bar{T})$. Then the following diagram commutes and its columns are exact.

\[
\begin{array}{cccc}
\hat{H}^{-1}(\Gamma_{E/F}, Y[S_E]_0) & \xrightarrow{\sim} & H^1(\Gamma_{E/F}, T(O_{E,S})) & \xrightarrow{\sim} & H^1(\Gamma_S, T(O_S)) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^{-1}(\Gamma_{E/F}, \bar{Y}[S_E]_0) & \xrightarrow{\sim} & H^1(\Gamma_{E/F}, \bar{T}(O_{E,S})) & \xrightarrow{\sim} & H^1(\Gamma_S, \bar{T}(O_S)) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) & & \xrightarrow{\Theta_{E,S}} & H^2(\Gamma_S, Z(O_S)) & \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^0(\Gamma_{E/F}, Y[S_E]_0) & \xrightarrow{\sim} & H^2(\Gamma_{E/F}, T(O_{E,S})) & \xrightarrow{\sim} & H^2(\Gamma_S, T(O_S)) \\
\downarrow & & \downarrow & & \downarrow \\
\hat{H}^0(\Gamma_{E/F}, \bar{Y}[S_E]_0) & \xrightarrow{\sim} & H^2(\Gamma_{E/F}, \bar{T}(O_{E,S})) & \xrightarrow{\sim} & H^2(\Gamma_S, \bar{T}(O_S)) \\
\end{array}
\]

**Proof.** We first explain the arrows. The short exact sequence

\[0 \to Y \to \bar{Y} \to A^\vee \to 0\]

remains short exact after tensoring over $\mathbb{Z}$ with the free $\mathbb{Z}$-module $\mathbb{Z}[S_E]_0$, and the left column of the diagram is the associated long exact sequence in Tate cohomology. Writing $X = X_*(T)$ and $\bar{X} = X_*(\bar{T})$, the short exact sequence

\[0 \to \bar{X} \to X \to A \to 0\]

remains short exact after applying $\text{Hom}(-, O_S^\times)$ because $O_S^\times$ is $\mathbb{N}_S$-divisible [NSW08 Prop. 8.3.4] and we are assuming $\exp(Z) \in \mathbb{N}_S$. The right column of the diagram is the corresponding long exact sequence for $H^i(\Gamma_S, -)$. The horizontal maps labelled “TN” are the isomorphisms constructed by Tate and discussed in subsection 3.1. The bottom two that are labelled “-TN” are obtained from these isomorphisms by composing them with multiplication by $-1$. The horizontal maps on the right are the inflation maps discussed in Lemma 3.6.

The only non-obvious commutativity is that of the two squares involving the map $\Theta_{E,S}$. For the first, let $\Lambda \in \hat{Z}^{-1}(\Gamma_{E/F} Y[S_E]_0)$. Then $\text{TN}(\Lambda) = \alpha_3(S) \cup \Lambda \in Z^1(\Gamma_{E/F} T(O_{E,S}))$ and the image of this in $H^2(\Gamma_S, Z(O_S))$ can be computed as follows. Choose $n_i$ to be a multiple of $\exp(Z)$. Then $n_i \Lambda \in \hat{Z}^{-1}(\Gamma_{E/F} Y[S_E]_0)$ and $k_{i,0}(S) \in C_{\mathbb{Q}}^0(\Gamma_S, \Gamma_{E/F}, \text{Maps}(S_E, O_S^\times)/O_S^\times)$. The unbalanced cup product $k_i \alpha_3(S) \cup_{E/F} n_i \Lambda$ belongs to $C^0(\Gamma_S, \Gamma_{E/F}, T(O_S))$ and lifts the 1-cocycle $\text{TN}(\Lambda)$. The differential of this 1-cochain is the image we want. According to [Kal] Fact 4.3 it is equal to

\[d(k_{i,0}(S) \cup_{E/F} n_i \Lambda) = dk_{i,0}(S) \cup_{E/F} n_i \Lambda = \Theta_{E,S}(n_i \Lambda)\]

But the restriction of $n_i \Lambda : \mathbb{G}_m \to T$ to $\mu_n$, takes image in $Z$ and the resulting map $\mu_{n_i} \to Z$ corresponds to the element of $A^\vee$ that is the image of $\Lambda$ under $\bar{Y} \to A^\vee$. This proves the commutativity of the top square involving $\Theta_{E,S}$. 

\[\square\]
For the commutativity of the bottom square let \( g \in \tilde{Z}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) \) and let \( \Lambda \in \tilde{Y} = \tilde{C}^{-1}(\Gamma_{E/F}, \tilde{Y}[S_E]_0) \) be any preimage of \( g \). According to the above discussion we have
\[
\Theta_{E/S}(g) = dk_i \alpha_3(S) \cup F n_i \tilde{\Lambda}.
\]
Mapping this under \( H^2(\Gamma_S, Z(O_S)) \to H^2(\Gamma_S, T(O_S)) \) we can write this using \cite{Kal} Fact 3.4] as
\[
d(k_i \alpha_3(S) \cup n_i \tilde{\Lambda}) = k_i \alpha_3(S) \cup dn_i \tilde{\Lambda}.
\]
The first term is a coboundary in \( C^2(\Gamma_S, T(O_S)) \). Since \( \tilde{\Lambda} \in \tilde{C}^{-1}(\Gamma_{E/F}, \tilde{Y}[S_E]_0) \) was a lift of the \((-1)\)-cocycle \( g \), its differential belongs to \( \tilde{Z}^0(\Gamma_{E/F}, \tilde{Y}[S_E]_0) \). It follows that
\[
k_i \alpha_3(S) \cup dn_i \tilde{\Lambda} = \alpha_3(S) \cup d\tilde{\Lambda}
\]
and this proves the commutativity of the bottom square involving \( \Theta_{E,S} \).

**Proof of Proposition 3.11** In the setting of Lemma 3.12 choose \( \tilde{Y} \) to be a free \( Z[\Gamma_{E/F}] \)-module. Then \( \tilde{Y}[S_E]_0 \) is also a free \( Z[\Gamma_{E/F}] \)-module and consequently both \( \tilde{H}^{-1}(\Gamma_{E/F}, \tilde{Y}[S_E]_0) \) and \( H^1(\Gamma_S, T(O_S)) \) vanish, showing that \( \Theta_{E,S} \) is the restriction of “-TN”. The latter is an injective map and is independent of the choices of \( \alpha_3(S) \) and \( k_i \).

The map \( \Theta_{E,S} \) has a local analog that is implicit in the constructions of \cite{Kal} and that will be useful in subsection 3.3. To describe it, let \( v \in S_F \) and let \( \alpha_v \in Z^2(\Gamma_{E_v/F_v}, E_v^\times) \) represent the canonical class. Let \( n \in \mathbb{N} \) be a multiple of \( \exp(Z) \) and let \( k : F_v \to F_v \) be such that \( k(x)^n = x \). Then we define
\[
\Theta_{E,v} : \tilde{H}^{-1}(\Gamma_{E_v/F_v}, A^\vee) \to H^2(\Gamma_v, Z(F_v)), \quad g \mapsto dk \alpha_v \cup F_{E_v,F_v} \Phi_{A,n}(g).
\]
Similar arguments to those employed for \( \Theta_{E,S} \) show that \( \Theta_{E,v} \) is independent of the choices of \( \alpha_v, k_v, \) and \( n_v \), and fits into the local analog of Diagram 3.4. The following lemma relates the map \( \Theta_{E,S} \) to \( \Theta_{E,v} \).

**Lemma 3.13.** Let \( v \in S_F \). We have the commutative diagram
\[
\begin{array}{ccc}
\tilde{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) & \xrightarrow{\Theta_{E,S}} & H^2(\Gamma, Z(O_S)) \\
\uparrow & & \uparrow \\
\tilde{H}^{-1}(\Gamma_{E_v/F_v}, A^\vee) & \xrightarrow{\Theta_{E,v}} & H^2(\Gamma_v, Z(F_v))
\end{array}
\]
Here the right vertical map is the localization map given by restriction to \( \Gamma_v \) followed by the inclusion \( Z(O_S) \to Z(F_v) \) and the left vertical map is given by restriction to \( \Gamma_{E_v/F_v} \) followed by the projection \( A^\vee[S_E]_0 \to A^\vee \) onto the \( v \)-factor.

**Proof.** We choose \( \tilde{Y} \) to be a free \( Z[\Gamma_{E/F}] \)-module. Then \( \tilde{Y} \) is also a free \( Z[\Gamma_{E_v/F_v}] \)-module and consequently the four cohomology groups \( \tilde{H}^{-1}(\Gamma_{E/F}, \tilde{Y}[S_E]_0), \tilde{H}^{-1}(\Gamma_{E_v/F_v}, \tilde{Y}), H^1(\Gamma_S, T(O_S)), \) and \( H^1(\Gamma_v, T(F_v)) \) all vanish. Looking at Diagram (3.4) and its local analog we see that it is enough to show the commutativity of the following diagram, which follows directly from the construction.
of the Tate-Nakayama isomorphism for tori in [Tat66].

\[
\hat{H}^0(\Gamma_{E/F}, Y [S_E]) \xrightarrow{\Theta_{E,S}} H^2(\Gamma, T(O_S)) \\
\hat{H}^0(\Gamma_{E,v/F_v}, Y) \xrightarrow{\Theta_{E,v}} H^2(\Gamma_v, T(F_v))
\]

We will now study how the map \(\Theta_{E,S}\) behaves when we change \(E\) and \(S\).

**Lemma 3.14.** The inflation map \(H^2(\Gamma_S, Z(O_S)) \to H^2(\Gamma, Z(F))\) is injective.

**Proof.** Choose \(X\) to be a free \(\mathbb{Z}[\Gamma_{E/F}]\)-module. Then so is \(Y\) and the groups \(H^1(\Gamma_S, T(O_S))\) and \(H^1(\Gamma, T(F))\) vanish and we get the diagram

\[
\begin{array}{c}
0 \\
\downarrow \\
0
\end{array}
\xrightarrow{}
\begin{array}{c}
H^1(\Gamma_S, T(O_S)) \\
\downarrow \\
H^1(\Gamma, T(F))
\end{array}
\xrightarrow{}
\begin{array}{c}
H^2(\Gamma_S, Z(O_S)) \\
\downarrow \\
H^2(\Gamma, Z(F))
\end{array}
\]

and the lemma follows from the five-lemma and Lemma 3.7.

**Lemma 3.15.** Let \(K/F\) be a finite Galois extension containing \(E\). Let \(S'\) be a finite set of places satisfying Conditions 3.9 with respect to \(K\) and containing \(S\).

1. The map \(\hat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]) \to H^{-1}(\Gamma_{E/F}, A^\vee[S'_E])\) given by the inclusion \(S \to S'\) is injective and fits in the commutative diagram

\[
\hat{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]) \xrightarrow{\Theta_{E,S}} H^{-1}(\Gamma_{E/F}, Z(O_S)) \\
\hat{H}^{-1}(\Gamma_{E/F}, A^\vee[S'_E]) \xrightarrow{\Theta_{E,S'}} H^{-1}(\Gamma_{E/F}, Z(O_{S'}))
\]

2. The map \(! : \hat{H}^{-1}(\Gamma_{E/F}, A^\vee[S'_E]) \to H^{-1}(\Gamma_{K/F}, A^\vee[S'_{K}])\) of Lemma 3.4 is injective and fits in the commutative diagram

\[
\hat{H}^{-1}(\Gamma_{E/F}, A^\vee[S'_E]) \xrightarrow{\Theta_{E,S'}} H^{-1}(\Gamma_{E/F}, Z(O_{S'})) \\
\hat{H}^{-1}(\Gamma_{K/F}, A^\vee[S'_{K}]) \xrightarrow{\Theta_{K,S'}} H^{-1}(\Gamma_{K/F}, Z(O_{S'}))
\]

**Proof.** Notice first that since for every \(\sigma \in \Gamma_{K/F}\) there exists \(u \in V(K)\) with \(\sigma u = u\), the assumption of Lemma 3.4 follows from part 4 of Conditions 3.9.

According to Proposition 3.11, the maps \(\Theta_{E,S}\) and \(\Theta_{E,S'}\) are injective. According to Lemma 3.14, the inflation map in the first diagram is injective. This proves the injectivity claims.
To prove the commutativity of the first diagram, choose $X$ to be a free $\mathbb{Z}[\Gamma_{E/F}]$-module. Then so is $Y$ as well as $Y[S_E]_0$ and we get $\hat{H}^i(\Gamma_{E/F}, Y[S_E]_0) = 0$ for all $i \leq 1$. Looking at Diagram (3.4), this implies that $\hat{H}^{-1}(\Gamma_{E/F}, Y[S_E]_0) \to \hat{H}^{-1}(\Gamma_{E/F}, A'[S_E]_0)$ is bijective. We consider the following cube.

We want to prove that the right face of this cube commutes. By the bijectivity of the back top map, it is enough to show that all the other faces commute. The back face commutes by functoriality of the map induced by the inclusion $S_E \to S'_E$. The front face commutes by $\delta$-functoriality of $\Theta$. The top and bottom faces commute by Lemma 3.12. The left face commutes by Corollary 3.16. This proves the commutativity of the first of the two diagrams in the statement. The proof of the second is analogous.

**Corollary 3.16.** The maps $\Theta_{E,S}$ splice to a functorial isomorphism

$$\Theta : \lim_{E,S} \hat{H}^{-1}(\Gamma_{E/F}, A'[S_E]_0) \to H^2(\Gamma, Z(\mathcal{F})).$$

**Proof.** According to Proposition 3.11 and Lemmas 3.15 and 3.14 we obtain a functorial injective homomorphism $\Theta$ as displayed. We will now argue that it is also surjective. Let $h \in H^2(\Gamma, Z(\mathcal{F}))$. Choose a finite Galois extension $E/F$ so that $h$ is inflated from $H^2(\Gamma_{E/F}, Z(E))$. Choose $S$ large enough so that it satisfies Conditions (3.9) with respect to $E$ and so that $Z(E) = Z(O_{E,S})$. Thus $h$ is in the image of the inflation $H^2(\Gamma_{E/F}, Z(O_{E,S})) \to H^2(\Gamma, Z(\mathcal{F}))$ and we can pick a preimage $h_{E,S}$.

Choose $\hat{Y}$ to be a free $\mathbb{Z}[\Gamma_{E/F}]$-module and consider Diagram (3.4). Let $h_{T,E,S} \in H^2(\Gamma_{E/F}, T(O_{E,S}))$ be the image of $h_{E,S}$. The image of $h_{T,E,S}$ in the group $H^2(\Gamma_{E/F}, T(O_{E,S}))$ is zero, so the preimage of $h_{T,E,S}$ in $\hat{H}^0(\Gamma_{E/F}, Y[S_E]_0)$ under “$\mathbb{Z}$” lifts to an element $g \in \hat{H}^{-1}(\Gamma_{E/F}, A'[S_E]_0)$. Let $h' \in H^2(\Gamma, Z(\mathcal{F}))$ be the inflation of $\Theta_{E,S}(g) \in H^2(\Gamma, Z(O_{S}))$. Then $h'$ and $h$ have the same image in $H^2(\Gamma, T(\mathcal{F}))$. But since we chose $\hat{Y}$ to be free, the map $H^2(\Gamma, Z) \to H^2(\Gamma, T)$ is injective, so $h' = h$.  

\[\square\]
3.3 The finite multiplicative groups $P_{E,S,E,N}$

Let $E/F$ be a finite Galois extension. Let $S \subset V(F)$ be a finite full set of places and $\hat{S}_E \subset E$ a set of lifts for the places in $S$ (that is, over each $v \in S$ there is a unique $w \in \hat{S}_E$). We assume that the pair $(S, \hat{S}_E)$ satisfies the following.

**Conditions 3.17.**

1. $S$ contains all archimedean places and all places that ramify in $E$.
2. Every ideal class in $C(E)$ contains an ideal with support in $\hat{S}_E$.
3. For every $w \in V(E)$ there exist $w' \in \hat{S}_E$ with $\text{Stab}(w, \Gamma_{E/F}) = \text{Stab}(w', \Gamma_{E/F})$.
4. For every $\sigma \in \Gamma_{E/F}$ there exists $\hat{v} \in \hat{S}_E$ such that $\sigma \hat{v} = \hat{v}$.

Pairs $(S, \hat{S}_E)$ that satisfy these conditions exist. Moreover, if $(S', \hat{S}'_E)$ is any pair containing $(S, \hat{S}_E)$ in the sense that $S \subset S'$ and $\hat{S}_E \subset \hat{S}'_E$, and if $(S, \hat{S}_E)$ satisfies these conditions, then so does $(S', \hat{S}'_E)$. If $Z$ is a finite multiplicative group defined over $F$ and split over $E$ and if $\exp(Z) \in \mathbb{N}_S$, then we may apply all results of subsection 3.2 to $Z$.

Fix $N \in \mathbb{N}_S$ and consider the finite abelian group

$$\text{Maps}(\Gamma_{E/F} \times \hat{S}_E, \frac{1}{N}\mathbb{Z} / \mathbb{Z})$$

as well as the following three conditions on an element $\phi$ of it:

1. For every $\sigma \in \Gamma_{E/F}$ we have $\sum_{w \in \hat{S}_E} \phi(\sigma, w) = 0$.
2. For every $w \in \hat{S}_E$ we have $\sum_{\sigma \in \Gamma_{E/F}} \phi(\sigma, w) = 0$.
3. Given $\sigma \in \Gamma_{E/F}$ and $w \in \hat{S}_E$, if $\phi(\sigma, w) \neq 0$ then $\sigma^{-1} w \in \hat{S}_E$.

Define $M_{E,S,N}$ to be the subgroup consisting of elements that satisfy the first two conditions, and $M_{E,\hat{S}_E,N}$ to be the subgroup of $M_{E,S,N}$ consisting of elements satisfying in addition the third condition. Note that both $M_{E,S,N}$ and $M_{E,\hat{S}_E,N}$ are $\Gamma_{E/F}$-stable.

**Lemma 3.18.** Let $A$ be a finite $\mathbb{Z}[\Gamma_{E/F}]$-module.

1. If $\exp(A)|N$, the map

   $$\Psi_{E,S,N} : \text{Hom}(A, M_{E,S,N})^\Gamma \rightarrow \hat{Z}^{-1}(\Gamma_{E/F}, \text{Maps}(\hat{S}_E, A^\vee)_0), \quad h \mapsto h,$$

   defined by $h(w, a) = H(a, 1, w)$ is an isomorphism of finite abelian groups, functorial in $A$. It restricts to an isomorphism

   $$\text{Hom}(A, M_{E,\hat{S}_E,N})^\Gamma \rightarrow \text{Maps}(\hat{S}_E, A^\vee)_0 \cap \hat{Z}^{-1}(\Gamma_{E/F}, \text{Maps}(\hat{S}_E, A^\vee)_0).$$

2. For $N|M$ the isomorphism $\Psi_{E,S,N}$ and $\Psi_{E,S,M}$ are compatible with the natural inclusion $M_{E,\hat{S}_E,N} \rightarrow M_{E,\hat{S}_E,M}$. Setting $M_{E,S} = \varinjt_N M_{E,S,N}$ we thus obtain an isomorphism

   $$\Psi_{E,S} : \text{Hom}(A, M_{E,S})^\Gamma \rightarrow \hat{Z}^{-1}(\Gamma_{E/F}, \text{Maps}(\hat{S}_E, A^\vee)_0).$$
Lemma 3.19. The map

$$\text{Maps}(\hat{\mathcal{S}}_E, A^\vee)_0 \rightarrow \hat{H}^{-1}(\Gamma_{E/F}, \text{Maps}(S_E, A^\vee)_0) \rightarrow \hat{H}^{-1}(\Gamma_{E/F}, \text{Maps}(S_E, A^\vee)_0)$$

is surjective.

Proof. For the first, the inverse of the claimed bijection is given by $H(a, \sigma, w) = (\sigma h)(w, a)$. The second point follows from the trivial equality $\text{Hom}(A, \frac{1}{M}\mathbb{Z}/\mathbb{Z}) = A^\vee = \text{Hom}(A, \frac{1}{M}\mathbb{Z}/\mathbb{Z})$.

For the third point we claim that every class in $\hat{H}^{-1}(\Gamma_{E/F}, \text{Maps}(S_E, A^\vee)_0)$ contains a representative supported on $\hat{S}_E$. Let $h \in \hat{H}^{-1}(\Gamma_{E/F}, \text{Maps}(S_E, A^\vee)_0)$ and write $\text{supp}(h) \subseteq S_E$ for its support. Suppose the set $\text{supp}(h) \setminus \hat{S}_E$ is non-empty and choose a place $w$ in it. Choose $\sigma \in \Gamma_{E/F}$ such that $\sigma w \in \hat{S}_E$ as well as $\nu_0 \in \hat{S}_E$ with $\sigma \nu_0 = \nu_0$, the latter being possible by Conditions 3.17. Consider the element $h(w) \otimes (\delta_w - \delta_{\nu_0}) \in A^\vee \otimes \text{Maps}(S_E, \mathbb{Z})_0 = \text{Maps}(S_E, A^\vee)_0$, where $\delta_w$ is the map with value 1 on $w \in S_E$ and value zero on $S_E \setminus \{w\}$. Then $h = h + \sigma(h(w) \otimes (\delta_w - \delta_{\nu_0})) - h(w) \otimes (\delta_w - \delta_{\nu_0})$ has the same image in $\hat{H}^{-1}(\Gamma_{E/F}, \text{Maps}(S_E, A^\vee)_0)$ as $h$, but $\text{supp}(h') \setminus \hat{S}_E = \text{supp}(h) \setminus \{w\}$. Applying this procedure finitely many steps we obtain a representative $h''$ of the cohomology class of $h$ with $\text{supp}(h'') \subseteq \hat{S}_E$.

Let $P_{E, S, E, N}$ be the finite multiplicative group over $O_{F, S}$ with $X^*(P_{E, S, E, N}) = M_{E, S, E, N}$. Let $A$ be a finite $\mathbb{Z}[\Gamma_{E/F}]$-module with $\text{exp}(A)/N$ and let $Z$ be the finite multiplicative group over $O_{F, S}$ with $X^*(Z) = A$. Composing the map

$$\Psi_{E, S, N} : \text{Hom}(A, M_{E, S, E, N})^\Gamma \rightarrow \hat{H}^{-1}(\Gamma_{E/F}, \text{Maps}(S_E, A^\vee)_0)$$

of the above lemma with $\Theta_{E, S}$ we obtain a map

$$\Theta_{E, S, E, N}^\nu : \text{Hom}(P_{E, S, E, N}, Z)^\Gamma \rightarrow \text{Hom}(A, M_{E, S, E, N})^\Gamma \rightarrow H^2(\Gamma_S, Z(O_S)). \quad (3.5)$$

which is functorial in $Z$.

We may apply this map to the special case $A = M_{E, S, E, N}$. In that case we have the canonical element $\text{id}$ of the source of $(3.5)$ and we let $\xi_{E, S, E, N} \in H^2(\Gamma_S, P_{E, S, E, N}(O_S))$ be its image. We will now study how the classes $\xi_{E, S, E, N}$ vary with $N, S, E$. First, let $N \mid M$. The obvious inclusion $M_{E, S, E, N} \rightarrow M_{E, S, E, M}$ gives rise to the surjection $P_{E, S, E, M} \rightarrow P_{E, S, E, N}$. Let

$$P_{E, S, E} = \lim_{N} P_{E, S, E, N}.$$

Lemma 3.19. We have the equality

$$H^2(\Gamma_S, P_{E, S, E}(O_S)) \rightarrow \lim_{N} H^2(\Gamma_S, P_{E, S, E, N}(O_S)).$$

The elements $\xi_{S, S, E, N}$ form a compatible system and thus lead to an element $\xi_{S, S, E} \in H^2(\Gamma_S, P_{E, S, E}(O_S))$.

Proof. The claimed equality follows from [NSW08 Cor. 2.7.6] and [NSW08 Thm. 8.3.20]. To prove that the classes $\xi_{E, S, E, N}$ form an inverse system, con-
sider the diagram

\[ \text{Hom}(M_{E,S_E,N}, M_{E,S_E,N})^\Gamma \xrightarrow{\Psi_{E,S,N}} \hat{H}^{-1}(\Gamma_{E,F}, \text{Maps}(S_E, M_{E,S_E,N}'))_0 \]
\[ \text{Hom}(M_{E,S_E,M}, M_{E,S_E,M})^\Gamma \xrightarrow{\Psi_{E,S,M}} \hat{H}^{-1}(\Gamma_{E,F}, \text{Maps}(S_E, M_{E,S_E,N}'))_0 \]
\[ \text{Hom}(M_{E,S_E,M}, M_{E,S_E,M})^\Gamma \xrightarrow{\Psi_{E,S,M}} \hat{H}^{-1}(\Gamma_{E,F}, \text{Maps}(S_E, M_{E,S_E,M}'))_0 \]

All vertical maps are induced by the inclusion \( M_{E,S_E,N} \to M_{E,S_E,M} \). The diagram commutes by Lemma 3.18 part 2 being responsible for the top square and the functoriality statement in part 1 for the bottom square. The elements \( \id \in \text{Hom}(M_{E,S_E,N}, M_{E,S_E,N})^\Gamma \) and \( \id \in \text{Hom}(M_{E,S_E,M}, M_{E,S_E,M})^\Gamma \) both map to the natural inclusion in the middle left term, hence the corresponding classes in \( \hat{H}^{-1} \) also meet in the middle right term. The functoriality of the map \( \theta_{E,S} \) now implies the claim that the classes \( \xi_{E,S_E,M} \) form an inverse system.

Next we consider a finite Galois extension \( K/F \) containing \( E \) and a pair \((S', \hat{S}_K')\) satisfying Conditions 3.17 with respect to \( K \) and such that \( S \subset S' \) and \( \hat{S}_E \subset \hat{S}_K' \).

Note that, given \( E, (S, \hat{S}_E) \) and \( K, \) it is always possible to find such a pair \((S', \hat{S}_K')\). We now consider the map

\[ M_{E,S_E,N} \to M_{K,\hat{S}_K',N}; \quad \phi \mapsto \phi^K \]  

(3.6)

defined by

\[ \phi^K(\sigma, u) = \begin{cases} 
\phi(\sigma, p(u)), & \sigma^{-1}u \in \hat{S}_K' \\
0, & \text{else} 
\end{cases} \]

This map is immediately verified to be \( \Gamma_{K/F} \)-equivariant, where \( \Gamma_{K/F} \) acts on \( M_{E,S_E,N} \) via its quotient \( \Gamma_{E/F} \).

**Lemma 3.20.** For any finite \( \Gamma_{E/F} \)-module \( A \) with \( \exp(A)|N \), the following diagram commutes

\[ \text{Hom}(A, M_{E,S_E,N})^\Gamma \xrightarrow{\Theta_{E,S_E,N}^\nu} H^2(G_S, Z(O_S)) \]
\[ \xrightarrow{\text{Inf}} \]
\[ \text{Hom}(A, M_{K,\hat{S}_K',N})^\Gamma \xrightarrow{\Theta_{K,\hat{S}_K',N}^\nu} H^2(G_{S'}, Z(O_{S'})) \]

**Proof.** This follows immediately from Lemma 3.15. Indeed, composing the right map in this diagram with \( \Psi_{K,S'}:N \) gives the same result as composing \( \Psi_{E,S,N} \) with the map \( \hat{H}^{-1}(\Gamma_{E/F}, \text{Maps}(S_E, A')_0) \to \hat{H}^{-1}(\Gamma_{K/F}, \text{Maps}(S_K', A')_0) \) that is the composition of the two left maps in the two diagrams of Lemma 3.15.

**Lemma 3.21.** The image of \( \xi_{K,\hat{S}_K',N} \) under the map \( H^2(\Gamma_{S'}, P_{K,\hat{S}_K',N}(O_{S'})) \to H^2(\Gamma_{S'}, P_{E,S_E,N}(O_{S'})) \) induced by (3.6) is equal to the image of \( \xi_{E,S_E,N} \) under the inflation map \( H^2(\Gamma_S, P_{E,S_E,N}(O_S)) \to H^2(\Gamma_{S'}, P_{E,S_E,N}(O_{S'})) \).
Proof. The proof is similar to that of Lemma 3.19. We consider the diagram

\[
\begin{array}{c}
\text{Hom}(M_{E,S,E,N}, M_{E,S,E,N})^\Gamma \xrightarrow{\Theta_{E,S,E,N}^F} H^2(\Gamma_S, P_{E,S,E,N}(O_S)) \\
\text{Hom}(M_{E,S,E,N}, M_{K,S'_k,E,N})^\Gamma \xrightarrow{\Theta_{K,S'_k,E,N}^F} H^2(\Gamma_{S'}, P_{E,S,E,N}(O_S')) \\
\text{Hom}(M_{K,S'_k,E,N}, M_{K,S'_k,E,N})^\Gamma \xrightarrow{\Theta_{K,S'_k,E,N}^F} H^2(\Gamma_{S'}, P_{K,S'_k,E,N}(O_S'))
\end{array}
\]

All vertical maps except for the inflation map are induced by (3.6). The top square commutes due to Lemma 3.20, while the bottom square commutes by functoriality of \(\Theta_{K,S'_k,E,N}^F\). Since the two elements id of the top and bottom left term meet in the middle left term, the corresponding elements \(\xi_{E,S,E,N}\) and \(\xi_{K,S'_k,E,N}\) of the top and bottom right term meet in the middle right term. 

It is easy to see that the maps (3.6) are compatible with respect to \(N\) and thus splice to a map

\[P_{K,S'_k} \to P_{E,S,E}\]

which maps the class \(\xi_{K,S'_k}\) to the class \(\xi_{E,S,E}\). Let \(E_i\) be an exhaustive tower of finite Galois extensions of \(F\), let \(S_i\) be an exhaustive tower of finite subsets of \(V(F)\), and let \(\dot{S}_i \subset S_{i+1,E_i}\) be a set of lifts for \(S_i\). We assume that \(\dot{S}_i \subset S_{i+1,E_i}\) and that \((S_i, \dot{S}_i)\) satisfies Conditions 3.17 with respect to \(E_i/F\). Then

\[\dot{V} = \lim_{\leftarrow i} \dot{S}_i\]  

(3.7)

is a subset of \(V(F)\) of lifts of \(V(F)\) and

\[P_{\dot{V}} = \lim_{\leftarrow i} P_{E_i,\dot{S}_i}\]  

(3.8)

is a pro-finite algebraic group defined over \(F\). For each finite multiplicative group \(Z\) defined over \(F\) (now without any condition on its exponent) we obtain from \(\Theta_{E_i,\dot{S}_i,N}^F\) a homomorphism

\[\Theta_{\dot{V}}^F : \text{Hom}(P_{\dot{V}}, Z)^\Gamma \to H^2(\Gamma, Z(F))\]  

(3.9)

which is surjective according to Corollary 3.16.

We will now show that this map behaves well with respect to localization. To describe this, we need to recall the local counterpart of \(P_{\dot{V}}\) from [Kal] §3.1. Let \(v \in \dot{V}\). Associated to the local field \(F_v\) there is the pro-finite multiplicative group \(u_v\) introduced in [Kal] §3.1. It is defined as \(\lim_{\leftarrow E_v/F_v,N} u_{E_v/F_v,N}\), the limit being taken over all finite Galois extensions \(E_v/F_v\) and all natural numbers \(N\). Here \(X^*(u_{E_v/F_v,N}) = I_{E_v,F_v,N}\) is the set of those maps \(\phi : \Gamma_{E_v,F_v} \to \mathbb{Z}/\mathbb{Z}\) that satisfy \(\sum_{\tau \in \Gamma_{E_v,F_v}} \phi(\tau) = 0\) and for a tower of finite Galois extensions \(K_v/E_v/F_v\) and \(N[M]\), the transition map \(I_{E_v,F_v,N} \to I_{K_v,F_v,M}\) is given simply by composition with the natural projection \(\Gamma_{K_v,F_v} \to \Gamma_{E_v,F_v}\).

If \(Z\) is a finite multiplicative group defined over \(F_v\) with \(exp(Z)|N\), and we set \(A = X^*(Z)\), then we have the isomorphism

\[\Psi_{E_v,N} : \text{Hom}(A, I_{E_v,F_v,N})^\Gamma \to \mathbb{Z}^{-1}(\Gamma_{E_v,F_v}, A^\vee), \quad H \mapsto \hat{h}\]
defined by $h(a) = H(a, 1)$. The composition of this isomorphism with the map $\Theta_{E,v}$ discussed in subsection 3.2 provides a homomorphism

$$\Theta^u_{E_v,N} : \text{Hom}(u_{E_v/F_v,N}, Z)^\Gamma_v \to H^2(\Gamma_v, Z).$$

Applying this homomorphism to the case $Z = u_{E_v/F_v,N}$ we obtain a distinguished element $\xi_{E_v/F_v,N} \in H^2(\Gamma_v, u_v)$ as the image of the identity map. We can then reinterpret $\Theta^u_{E_v,N}$ as the homomorphism that maps $\phi$ to $\phi(\xi_{E_v/F_v,N})$. The transition map $u_{E_v/F_v,M} \to u_{E_v/F_v,N}$ sends $\xi_{K_v/F_v,M}$ to $\xi_{E_v/F_v,N}$ and the system $\xi_{E_v/F_v,N}$ thus leads to a distinguished element $\xi_v \in H^2(\Gamma_v, u_v)$. We refer to [Kal, Lemma 4.5, Fact 4.6] for more details. We obtain a map

$$\Theta^u_v : \text{Hom}(u_v, Z)^\Gamma_v \to H^2(\Gamma_v, Z)$$

that sends $\phi$ to $\phi(\xi_v)$. It is surjective [Kal, Proposition 3.2].

We now define a localization map

$$loc^P_v : u_v \to P_v$$

for $v \in \hat{V}$ as follows. Fix a finite Galois extension $E/F$, a pair $(S, \hat{S}_E)$ satisfying conditions 3.17, and a natural number $N \in \mathbb{N}_S$, and consider the map

$$loc^M_{E,S_E,N} : M_{E,S_E,N} \to I_{E_v/F_v,N}, \quad H \mapsto H_v,$$

given by $H_v(\tau) = H(\tau, v)$ for $\tau \in \Gamma_{E_v/F_v}$. We have

$$0 = \sum_{\sigma \in \Gamma_{E/F}} H(\sigma, v) = \sum_{\tau \in \Gamma_{E_v/F_v}} H(\tau, v),$$

and this shows that $loc^M_{E,S_E,N}$ is well-defined. It is evidently $\Gamma_{E_v/F_v}$-equivariant. We will write $loc^P_{E,S_E,N} : u_{E_v/F_v,N} \to P_{E,S_E,N}$ for the dual of this map. For varying $N$, these maps splice to a map $loc^P_{E,S_E} : u_v \to P_{E,S_E}$. For varying $i$, the maps $loc^P_{E,S_E}$ in turn splice together to form the map (3.10).

**Lemma 3.22.** Assume that $E/F$ splits $Z$, $(S, \hat{S}_E)$ satisfies Conditions 3.17 and $N \in \mathbb{N}_S$ is a multiple of $\exp(Z)$. We have the commutative diagram

$$\begin{array}{c}
\text{Hom}(P_{E,S_E,N}, Z)^\Gamma_v \xrightarrow{\Theta^P_{E,S_E,N}} H^2(\Gamma, Z) \\
\downarrow{\text{loc}^P_{E,S_E,N}} \\
\text{Hom}(u_{E_v/F_v,N}, Z)^\Gamma_v \xrightarrow{\Theta^u_{E_v,N}} H^2(\Gamma_v, Z)
\end{array}$$

where the right vertical map is the localization map given by restriction to $\Gamma_v$ followed by the inclusion $Z(F) \to Z(F_v)$.

**Proof.** Set $A = X^*(Z)$. According to Lemma 3.13 it is enough to show that the following diagram commutes

$$\begin{array}{c}
\text{Hom}(A, M_{E,S_E,N})^\Gamma_v \xrightarrow{\Phi_{E,S_E,N}} \tilde{H}^{-1}(\Gamma_{E/F}, A^\vee[S_E]_0) \\
\downarrow{\text{loc}^M_{E,S_E,N}} \\
\text{Hom}(A, I_{E_v/F_v,N})^\Gamma_v \xrightarrow{\Phi_{E_v,N}} \tilde{H}^{-1}(\Gamma_{E_v/F_v}, A^\vee)
\end{array}$$

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where the right vertical map is restriction to \( \Gamma_{E_v/F_v} \), followed by projection onto the \( v \)-th component. Explicitly, this map sends \( h \in \text{Maps}(S_E, A^\vee)_0 \) to the element of \( A^\vee \) given by

\[
a \mapsto \sum_{\tau \in \Gamma_{E_v/F_v}^{\Gamma_{E_v/F_v}}} \hat{\tau}h(\hat{\tau}^{-1}v, \hat{\tau}^{-1}a)
\]

where \( \hat{\tau} \in \Gamma_{E/F} \) is any representative of the coset \( \tau \). The resulting class in \( \tilde{H}^{-1}(\Gamma_{E_v/F_v}, A^\vee) \) is independent of the choices of representatives.

The composition of this map with \( \Psi_{E,S,N} \) thus sends \( H \in \text{Hom}(A, M_{E,\dot{S}_E,N})^\Gamma \) to

\[
a \mapsto \sum_{\tau \in \Gamma_{E_v/F_v}^{\Gamma_{E_v/F_v}}} \hat{\tau}H(\hat{\tau}^{-1}a, \hat{\tau}^{-1}v).
\]

Each summand is equal to \( H(a, \hat{\tau}, v) \) and since \( v \in \dot{S}_E \) the definition of \( M_{E,\dot{S}_E,N} \) implies that \( H(a, \hat{\tau}, v) = 0 \) unless \( \hat{\tau}^{-1} \in \Gamma_{E_v/F_v} \). Thus all summands are zero except for the summand indexed by the trivial coset \( \tau \), for which we may take the representative \( \hat{\tau} = 1 \in \Gamma_{E_v/F_v} \). We conclude sending \( H \in \text{Hom}(A, M_{E,\dot{S}_E,N})^\Gamma \) first horizontally and then vertically provides the element of \( \tilde{H}^{-1}(\Gamma_{E_v/F_v}, A^\vee) \) represented by \( a \mapsto H(a, 1, v) \). Recalling the definitions of \( \text{loc}_{v} M_{E,\dot{S}_E,N} \) and \( \Psi_{E_v,N} \) this element also represents the image of \( H \) under the composition of these two maps.

**Corollary 3.23.** For \( v \in \dot{V} \) consider the maps

\[
H^2(\Gamma_S, P_{E,\dot{S}_E}((O_S))) \to H^2(\Gamma_v, P_{E,\dot{S}_E}(\overline{O_v})) \leftrightarrow H^2(\Gamma_v, u_v),
\]

the left one being given by restriction to \( \Gamma_v \) followed by inclusion \( O_S \to \overline{O_v}^\times \), and the right one being given by \( \text{loc}_{v} M_{E,\dot{S}_E} \). The images of \( \xi_{E,\dot{S}_E} \) and \( \xi_v \) in the middle term are equal.

**Proof.** For each \( N \in \mathbb{N}_S \) we have the commutative diagram

\[
\begin{array}{ccc}
\text{End}(P_{E,\dot{S}_E,N})^\Gamma & \longrightarrow & \text{Hom}(u_{E_v/F_v,N}, P_{E,\dot{S}_E,N})^\Gamma \longrightarrow \text{End}(u_{E_v/F_v,N})^\Gamma \\
\downarrow \Theta_{E,\dot{S}_E,N}^\Gamma & & \downarrow \Theta_{E_v,N}^\Gamma \\
H^2(\Gamma_S, P_{E,\dot{S}_E,N}(O_S)) & \longrightarrow & H^2(\Gamma_v, P_{E,\dot{S}_E,N}(\overline{O_v})) \longrightarrow H^2(\Gamma_v, u_{E_v/F_v,N})
\end{array}
\]

where the left square commutes according to Lemma 3.22 applied to \( Z = P_{E,\dot{S}_E,N} \) while the right square commutes by functoriality of \( \Theta_{E_v,N}^\Gamma \) in \( Z \).

The class \( \xi_{E,\dot{S}_E,N} \in H^2(\Gamma_S, P_{E,\dot{S}_E,N}) \) is the image of \( \text{id} \in \text{End}(P_{E,\dot{S}_E,N})^\Gamma \), while the class \( \xi_v \in H^2(\Gamma_v, u_v) \) maps to \( \xi_{E_v/F_v,N} \in H^2(\Gamma_v, u_{E_v/F_v,N}) \), which in turn is the image of \( \text{id} \in \text{End}(u_{E_v/F_v,N})^\Gamma \). The two elements id both map to \( \text{loc}_{v} P_{E,\dot{S}_E,N} \in \text{Hom}(u_{E_v/F_v,N}, P_{E,\dot{S}_E,N})^\Gamma \). This proves the lemma with \( P_{E,\dot{S}_E} \) replaced by \( P_{E,\dot{S}_E,N} \), and the result in general follows by taking the inverse limit over \( N \). □
3.4 The cohomology group $H^1(P_{E,\hat{S}_E} \to \mathcal{E}_{E,\hat{S}_E}, Z \to T)$

We again take a finite Galois extension $E/F$ and a pair $(S, \hat{S}_E)$ satisfying Conditions [3.17]. Let $T$ denote the category whose objects are pairs $(Z, T)$ where $T$ is an algebraic torus defined over $F$ and $Z \subset T$ is a finite subgroup. We will usually write such an object as $[Z \to T]$. A morphism $[Z_1 \to T_1] \to [Z_2 \to T_2]$ is a morphism $T_1 \to T_2$ of algebraic tori defined over $F$ whose restriction to $Z_1$ takes image in $Z_2$. Let $T_{E,S}$ be the full subcategory consisting of those objects $[Z \to T]$ for which $T$ splits over $E$ and $\exp(Z) \in \mathbb{N}_S$.

In this subsection we are going to define a functor

$$T_{E,S} \to \text{FinAbGrp}, \quad [Z \to T] \mapsto H^1(P_{E,\hat{S}_E} \to \mathcal{E}_{E,\hat{S}_E}, Z \to T).$$

We will moreover prove a Tate-Nakayama-type isomorphism for this functor. We will then show that as $E, S, \hat{S}_E$ vary, these functors fit together to a functor

$$\mathcal{T} \to \text{AbGrp}, \quad [Z \to T] \mapsto H^1(P_V \to \mathcal{E}_V, Z \to T).$$

The definition of $H^1(P_{E,\hat{S}_E} \to \mathcal{E}_{E,\hat{S}_E}, Z \to T)$ is analogous to that of the local cohomology set of [Kal], which was itself inspired by Kottwitz’s notion of algebraic 1-cocycles [Kot97] §8. We take any extension

$$1 \to P_{E,\hat{S}_E}(O_S) \to \mathcal{E}_{E,\hat{S}_E} \to \Gamma_S \to 1$$

corresponding to the distinguished class $\xi_{E,\hat{S}_E} \in H^2(\Gamma_S, P_{E,\hat{S}_E}(O_S))$ discussed in subsection [3.3]. The group $\mathcal{E}_{E,\hat{S}_E}$ is profinite and acts on $T(O_S)$ via its map to $\Gamma_S$. Thus we have the group $H^1(\mathcal{E}_{E,\hat{S}_E}, T(O_S))$ of cohomology classes of continuous 1-cocycles of $\mathcal{E}_{E,\hat{S}_E}$ valued in the discrete group $T(O_S)$. We define $H^1(P_{E,\hat{S}_E} \to \mathcal{E}_{E,\hat{S}_E}, Z \to T)$ to be the subgroup of $H^1(\mathcal{E}_{E,\hat{S}_E}, T(O_S))$ comprised of those classes whose restriction to $P_{E,\hat{S}_E}(O_S)$, which is a well-defined $\Gamma_S$-equivariant continuous homomorphism $P_{E,\hat{S}_E}(O_S) \to T(O_S)$, factors through the inclusion $Z(O_S) \to T(O_S)$ and is an algebraic homomorphism. A bit more precisely, we can describe such a cohomology class as a pair $(v, h)$ consisting of $h \in H^1(\mathcal{E}_{E,\hat{S}_E}, T(O_S))$ and $v \in \text{Hom}(X^*(Z), M_{E,\hat{S}_E})$ such that the homomorphism $h|_{P_{E,\hat{S}_E}} : P_{E,\hat{S}_E}(O_S) \to T(O_S)$ is determined by the composition of $v$ with the projection $X^*(T) \to X^*(Z)$. The datum $v$ is however determined by $h$ and thus need not be kept track of.

We have the inflation-restriction sequence

$$1 \to H^1(\Gamma_S, T(O_S)) \to H^1(P_{E,\hat{S}_E} \to \mathcal{E}_{E,\hat{S}_E}, Z \to T) \to \text{Hom}(P_{E,\hat{S}_E}, Z)^\Gamma \to H^2(\Gamma_S, T(O_S)) \quad (3.11)$$

In which the last map is the composition of $\Theta^P_{E,\hat{S}_E,N}$ with the natural map $H^2(\Gamma_S, Z(O_S)) \to H^2(\Gamma_S, T(O_S))$. The argument for this is the same as for [Kal] Lemma 3.3.

From this definition it is not clear that the resulting cohomology group is independent of the choice of extension $\mathcal{E}_{E,\hat{S}_E}$. We will show that this is the case, but only after we establish the Tate-Nakayama-type isomorphism.

For this we begin by fixing a specific realization of the extension of $\Gamma_S$ by $P_{E,\hat{S}_E}$ corresponding to the class $\xi_{E,\hat{S}_E}$. Let $\alpha_3(S) \in Z^2(\Gamma_{E/F}, \text{Hom}(Z[S_E], O^*_E))$
represent the Tate-class discussed in subsection 3.1. Let $N_i \in \mathbb{N}_S$ be a co-final sequence and $k_i : O^+_S \to O^+_S$ be a system of maps satisfying $k_i(x)^{N_i} = x$ and $k_{i+1}(x)^{N_i+1/N_i} = k_i(x)$. The image of $id \in \text{End}(M_{E,S,E,N_i})^\Gamma$ under the map $\Psi_{E,S,N_i}$ of Lemma 3.18 is the element $\alpha_i \in \hat{\mathbb{Z}}^{-1}(\Gamma_{E/F}, \text{Maps}(S_E, M_{E,S,E,N_i}^\Gamma))$ sending $w \in S_E$ and $\phi \in M_{E,S,E,N_i}$ to the element $(\phi(1), w) \in \frac{1}{N_i}\mathbb{Z}/\mathbb{Z}$.

The class $\xi_{E,S,E,N_i} \in H^2(\Gamma_{S}, P_{E,S,E,N_i}(O_S))$ was defined as $\Theta_{E,S,E,N_i}^\Gamma(id)$ and is thus represented by the 2-cocycle

$$\hat{\xi}_{E,S,E,N_i} = dk_i\alpha_i(S) \sqcup_{E/F} \alpha_i.$$ 

As in subsection 3.2, we are using here the unbalanced cup product of [Kal §4.3] and the pairing (3.3) provided by $\Phi_{E,S,N_i}$ of Fact 3.10. This fact also implies that the projection map $P_{E,S,E,N_i+1} \to P_{E,S,E,N_i}$ maps the 2-cocycle $\tilde{\xi}_{E,S,E,N_{i+1}}$ to the 2-cocycle $\hat{\xi}_{E,S,E,N_i}$. Let $\xi_{E,S,E} \in Z^2(\Gamma_{S}, P_{E,S,E}(O_S))$ be the 2-cocycle determined by the inverse system $(\xi_{E,S,E,N_i})$ and let

$$\hat{\xi}_{E,S,E} = P_{E,S,E}(O_S) \boxtimes_{\xi_{E,S,E}} \Gamma_S.$$ 

This is our explicit realization of the extension $\mathcal{E}_{E,S}$. Pushing out this extension along the projection $P_{E,S,E} \to P_{E,S,E,N_i}$ produces the explicit extension $\hat{\xi}_{E,S,E,N_i} = P_{E,S,E,N_i}(O_S) \boxtimes_{\xi_{E,S,E,N_i}} \Gamma_{S_i}$, so that conversely the explicit extension $\hat{\xi}_{E,S,E}$ is the inverse limit of the explicit extensions $\tilde{\xi}_{E,S,E,N_i}$. We then define $H^1(P_{E,S,E} \to \hat{\xi}_{E,S,E}, Z \to T)$ as described above and see that it is equal to the direct limit of $H^1(P_{E,S,E,N_i} \to \hat{\xi}_{E,S,E,N_i}, Z \to T)$.

We will now formulate the Tate-Nakayama-type isomorphism for these cohomology groups. Fix $[Z \to T] \in \mathcal{A}_{E,S}$ and let $\bar{T} = T/Z$. Write $Y = X_*(T)$ and $\bar{Y} = X_*(\bar{T})$. Then we have the exact sequence

$$0 \to Y \to \bar{Y} \to A^\vee \to 0,$$

which upon applying $\otimes_{\mathbb{Z}}[S_E]_0$ becomes

$$0 \to Y[S_E]_0 \to \bar{Y}[S_E]_0 \to A^\vee[S_E]_0 \to 0.$$

We have $Y[S_E]_0 = \text{Maps}(S_E, Y)_0$ and we have the pairing

$$\text{Maps}(S_E, O^+_S)/O^+_S \otimes \text{Maps}(S_E, Y)_0 \to T(O_S)$$

defined by

$$f \otimes g \mapsto \prod_{w \in S_E} f(w)g(w). \quad (3.12)$$

For $i >> 0$ we have $\exp(Z)|N_i$. If $f \in \text{Maps}(S_E, \mu_{N_i})/\mu_{N_i}$ and $g \in \bar{Y}[S_E]_0$, then $N_i \cdot g \in Y[S_E]_0$ and the image of $f \otimes N_ig$ under (3.12) belongs to $Z(O_S)$ and equals the image of $f \otimes [g]$ under (3.3), where $[g] \in A^\vee[S_E]_0$ is the image of $g$.

We have the subgroup $A^\vee[S_E]_0 \subset A^\vee[S_E]_0$. It is not $\Gamma_{E/F}$-equivariant, but we will make use of the slightly abusive notation $A^\vee[S_E]_0^{NE/F} = A^\vee[S_E]_0^{N_{E/F}} \cap A^\vee[S_E]_0$ to save space. This abelian group is in bijection with $\text{Hom}(P_{E,S,E}, Z)^\Gamma$ via the map $\Psi_{E,S}$ of Lemma 3.18. Let $\bar{Y}[S_E, \hat{S}_E]_0$ be the preimage in $\bar{Y}[S_E]_0$ of $A^\vee[S_E]_0$. We set $\bar{Y}[S_E, \hat{S}_E]_0^{NE/F} = \bar{Y}[S_E]_0^{NE/F} \cap \bar{Y}[S_E, \hat{S}_E]_0$. 

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Proposition 3.24. 1. Given $\hat{\Lambda} \in \hat{Y}[S_E, \hat{S}_E]_{0}^{N_{E/F}}$ and $i > 0$ so that $\exp(Z)|N_i$, the assignment

$$z_{\hat{\Lambda},i} : \hat{\xi}_{E,\hat{S}_E,N_i} \to T(O_S), \quad x \boxtimes \sigma \mapsto \Psi^{-1}_{E,S,N_i}([\hat{\Lambda}]) \cdot (k_i \alpha_3(S) \cup_{E/F} N_i \hat{\Lambda})$$

belongs to $Z^1(P_{E,\hat{S}_E,N_i} \to \hat{\xi}_{E,\hat{S}_E,N_i}, Z \to T)$. Here we have used the unbalanced cup product with respect to the pairing \(\xi_2\).

2. The composition of $z_{\hat{\Lambda},i}$ with the projection $\xi_{E,\hat{S}_E,N_{i+1}} \to \xi_{E,\hat{S}_E,N_i}$ is equal to $z_{\hat{\Lambda},i+1}$. Thus we obtain a well-defined $\hat{\Lambda} \in Z^1(P_{E,\hat{S}_E} \to \hat{\xi}_{E,\hat{S}_E}, Z \to T)$.

3. The assignment $\hat{\Lambda} \mapsto z_{\hat{\Lambda}}$ establishes an isomorphism

$$i_{E,S_E} : \hat{Y}[S_E, \hat{S}_E]_{0}^{N_{E/F}} \to H^1(P_{E,\hat{S}_E} \to \hat{\xi}_{E,\hat{S}_E}, Z \to T)$$

that is functorial in $[Z \to T] \in T_{E,S}$ and fits into the diagram

$$
\begin{array}{ccc}
1 & \xrightarrow{1} & 1 \\
\hat{H}^{-1}(\Gamma_{E/F}, Y[S_E]_0) & \xrightarrow{\text{"TN"}} & H^1(\Gamma_S, T(O_S)) \\
\xrightarrow{i_{E,S_E}} & \xrightarrow{i_{E,S_E}} & \xrightarrow{i_{E,S_E}} \\
\hat{Y}[S_E, \hat{S}_E]_{0}^{N_{E/F}} & \xrightarrow{\Psi^{-1}_{E,S}} & \text{Hom}(P_{E,\hat{S}_E}, Z)^\Gamma \\
\xrightarrow{A^\vee [\hat{S}_E]_{N_{E/F}}} & \xrightarrow{\Psi^{-1}_{E,S}} & H^2(\Gamma_S, T(O_S)) \\
\hat{H}^0(\Gamma_{E/F}, Y[S_E]_0) & \xrightarrow{\text{"TN"}} & H^2(\Gamma_S, T(O_S))
\end{array}
$$

Proof. For the first point it is enough to show that $z_{\hat{\Lambda},i} \in Z^1(\hat{\xi}_{E,\hat{S}_E,N_i}, T(O_S))$, because the defining formula makes it obvious that the restriction of $z_{\hat{\Lambda},i}$ to $P_{E,\hat{S}_E,N_i}$ takes values in $Z(O_S)$. A direct computation shows that the differential is given by

$$dz_{\hat{\Lambda},i}(x \boxtimes \sigma, y \boxtimes \tau) = \Psi^{-1}_{E,S,N_i}([\hat{\Lambda}]) \cdot (k_i \alpha_3(S) \cup_{E/F} N_i \hat{\Lambda})(\sigma, \tau).$$

By [Kat] Fact 4.3 we have $d(k_i \alpha_3(S) \cup_{E/F} N_i \hat{\Lambda}) = dk_i \alpha_3(S) \cup_{E/F} N_i \hat{\Lambda}$. This 2-cocycle takes values in $Z(O_S)$ and is equal to $\Psi^{-1}_{E,S,N_i}([\hat{\Lambda}]) \cdot (\hat{\xi}_{E,\hat{S}_E,N_i})$.

The second point follows from the compatibility of the maps $\Psi_{E,S,N_i}$ with the projections in the system $\xi_{E,\hat{S}_E,N_i}$, which is part 2 of Lemma 3.18 as well as from the fact that $k_{i+1}(x)_{N_{i+1}/N_i} = k_i(x)$.

We now come to the third point. It is clear from the definition that $\hat{\Lambda} \mapsto z_{\hat{\Lambda}}$ is a functorial homomorphism $\hat{Y}[S_E, \hat{S}_E]_{0}^{N_{E/F}} \to H^1(P_{E,\hat{S}_E} \to \hat{\xi}_{E,\hat{S}_E}, Z \to T)$. If $\hat{\Lambda} \in Y[S_E]_{0}^{N_{E/F}}$, then $[\hat{\Lambda}] = 0$ and $k_i \alpha_3(S) \cup N_i \hat{\Lambda} = \alpha_3(S) \cup \hat{\Lambda}$. Thus $z_{\hat{\Lambda},i} = \alpha_3(S) \cup \hat{\Lambda}$. This implies that $\hat{\Lambda} \mapsto z_{\hat{\Lambda}}$ kills $I_{E/F} Y[S_E]_0$ and moreover that the top square in the above diagram commutes. The commutativity of the second
ing this procedure finitely many times yields $S'$.

Since $S$ is a finite abelian group, the isomorphism is not unique – the set of such isomorphisms is a torsor under the $E$. The choice of $X$ gives us an isomorphism of cohomology groups

$$H^1(P_{E,S_E} \to \hat{E}_{E,S_E}, Z \to T) \rightarrow H^1(P_{E,S_E} \to \hat{E}_{E,S_E}, Z \to T),$$

that is functorial in $[Z \to T] \in T_{E,S}$ and is moreover compatible with the inclusion of $H^1(\Gamma_S, T(O_S))$ into both groups as well as with the maps from both groups to Hom($P_{E,S_E}, Z$). We will argue that there is at most one isomorphism with these properties. For this, the following consequence of Proposition 3.24 will be crucial.

**Lemma 3.25.** The group $\lim_{\to Z} H^1(P_{E,S_E} \to \hat{E}_{E,S_E}, Z \to T)$, where $Z$ runs over all finite subgroups of $T$ defined over $F$ and satisfying $\exp(Z) \in \mathbb{N}_S$, is $\mathbb{N}_S$-divisible.

**Proof.** Choosing any isomorphism of extensions $\mathcal{E}_{E,S_E} \to \hat{E}_{E,S_E}$ and applying Proposition 3.24 we see that the claim is equivalent to the $\mathbb{N}_S$-divisibility of

$$\lim_{\to Z} \frac{\hat{Y}[S_E, \hat{S}_E]_0^{N_{E,F}}}{I_{E,F}^* Y[S_E]_0},$$

The colimit can be taken with respect to any co-final sequence $Z_i$ of finite subgroups of $T$ with $\exp(Z_i) \in \mathbb{N}_S$. Such a sequence is obtained by setting $X_i = N_i X$. Then $Y_i = \Hom_Z(X_i, Z) = \frac{Y}{N_i}$. The above direct limit is thus

$$\frac{(S_Q^{-1}Y)[S_E, \hat{S}_E]_0^{N_{E,F}}}{I_{E,F}^* Y[S_E]_0},$$

where $S_Q^{-1}Y = Y \otimes_Z Z[S_Q^{-1}]$ is the localization of $Y$ as $S_Q$. We now claim that

$$\frac{(S_Q^{-1}Y)[\hat{S}_E]_0^{N_{E,F}}}{I_{E,F}^* Y[S_E]_0}$$

is surjective. The argument for this is the same as in the proof of part 3 of Lemma 3.18 Indeed, let $y = \sum_{w \in S_E} y_w[w] \in (S_Q^{-1}Y)[S_E, \hat{S}_E]_0^{N_{E,F}}$. For any $w \in S_E \setminus \hat{S}_E$ we have $y_w \in Y$. If $w \neq 0$, then choose $\sigma \in \Gamma_{E,F}$ with $\sigma w \in \hat{S}_E$ and choose $v_0 \in \hat{S}_E$ with $\sigma v_0 = v_0$. Then

$$y' := y - (y_w[w] - y_w[v_0]) + \sigma(y_w[w] - y_w[v_0]) \in (S_Q^{-1}Y)[S_E, \hat{S}_E]^N_{E,F}$$

represents the same class as $y$ modulo $I_{E,F}^* Y[S_E]_0$, but now $y'_w = 0$. Performing this procedure finitely many times yields $y'' \in (S_Q^{-1}Y)[\hat{S}_E]_0^{N_{E,F}}$.

Since $S_Q^{-1}Y$ is $\mathbb{N}_S$-divisible and torsion-free, the group $(S_Q^{-1}Y)[\hat{S}_E]_0^{N_{E,F}}$ is also $\mathbb{N}_S$-divisible and the proof is complete.
Lemma 3.26. Let $\Delta_1$ be a functor assigning to each torus $T$ defined over $F$ and split over $E$ a finite abelian group $\Delta_1(T)$. Let $\Delta_3$ be a functor assigning to each finite multiplicative group $Z$ defined over $F$ and split over $E$, with $\exp(Z) \in \mathbb{N}_S$, a finite abelian group $\Delta_3(Z)$. Let $\Delta_2$ be a functor assigning to each $[Z \to T] \in T_{E,S}$ a finite abelian group $\Delta_2(Z \to T)$. Assume that for $Z \subset Z'$ the map $\Delta_2(Z \to T) \to \Delta_2(Z' \to T)$ is injective. Assume further that we have a functorial in $[Z \to T]$ exact sequence

$$0 \to \Delta_1(T) \to \Delta_2(Z \to T) \to \Delta_3(Z).$$

If $H^1(\Gamma_S, T(O_S)) \to \Delta_1(Z)$ and $\text{Hom}(P_{E,S_E}, Z)^\Gamma \to \Delta_3(Z)$ are fixed functorial homomorphisms, there exists at most one functorial homomorphism $H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, Z \to T) \to \Delta_2([Z \to T])$ fitting into the commutative diagram

$$\begin{array}{ccc}
1 & \longrightarrow & H^1(\Gamma_S, T(O_S)) \\
| & | & | \\
0 & \longrightarrow & \Delta_1(T) \\
| & | & | \\
& \downarrow & \downarrow & \downarrow \\
& \Delta_2([Z \to T]) & \longrightarrow & \Delta_3(Z)
\end{array}$$

Proof. Let $f_{[Z \to T]}^{(i)} : H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, Z \to T) \to \Delta_2([Z \to T])$ for $i = 1, 2$ be two functorial homomorphisms fitting into the above commutative diagram. Fix a torus $T$ defined over $F$ and split over $E$ and let $Z_i$ be a cofinal sequence of finite subgroups of $T$ defined over $F$ and split over $E$ with $\exp(Z_i) \in \mathbb{N}_S$. Let $H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, T) = \lim_{\longrightarrow} H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, Z_i \to T)$ and let $\text{Hom}(P_{E,S_E}, T)^\Gamma = \lim_{\longrightarrow} \text{Hom}(P_{E,S_E}, Z_i)^\Gamma$. Using the same notation for $\Delta_2$ and $\Delta_3$ we obtain the commutative diagram with exact rows

$$\begin{array}{ccc}
1 & \longrightarrow & H^1(\Gamma_S, T(O_S)) \\
| & | & | \\
0 & \longrightarrow & \Delta_1(T) \\
| & | & | \\
& \downarrow & \downarrow & \downarrow \\
& \Delta_2(T) & \longrightarrow & \Delta_3(T)
\end{array}$$

Let $Q \subset \text{Hom}(P_{E,S_E}, T)^\Gamma$ be the image of $H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, T)$. According to Lemma 3.25 the group $H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, T)$ is $\mathbb{N}_S$-divisible. This property is inherited by its quotient $Q$. At the same time the group $\text{Hom}(P_{E,S_E}, T)^\Gamma$ is $\mathbb{N}_S$-torsion, a property inherited by its subgroup $Q$. The difference $\delta_T = f_{T}^{(2)} - f_{T}^{(1)} \in \text{Hom}(H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, T), \Delta_2(T))$ is a homomorphism $\delta : Q \to \Delta_1(T)$. Then $\delta_T(Q)$ is an abelian group that is $\mathbb{N}_S$-divisible, $\mathbb{N}_S$-torsion, and finite. Hence it is trivial. We conclude $\delta_T = 0$, i.e. $f_{T}^{(1)} = f_{T}^{(2)}$. This implies that $f_{[Z \to T]}^{(1)} = f_{[Z \to T]}^{(2)}$ for all $Z \subset T$, due to the injectivity of $H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, Z \to T) \to H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, T)$ and $\Delta_2([Z \to T]) \to \Delta_2(T)$. \qed

As a first application of this general statement, we obtain.

Corollary 3.27. The group $H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, Z \to T)$ is independent of the choice of extension $\mathcal{E}_{E,S_E}$ up to a unique isomorphism. It comes equipped with a canonical functorial isomorphism $\iota_{E,S_E}$ to the group

$$\frac{\hat{Y}[S_E,S_E]_{N_E/F}}{I_{E/F}Y[S_E]_0}$$

that fits into the commutative diagram of Proposition 3.24.

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Proof. First we apply Lemma \[3.26\] with \( \Delta_2 = H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}) \). We know that any isomorphism of extensions \( \mathcal{E}_{E,S_E} \to \mathcal{E}_{E,S_E} \) provides an isomorphism \( H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}) \to \Delta_2 \) as in the statement of this lemma, and the lemma guarantees that it is unique, i.e. independent of the choice of isomorphism of extensions \( \mathcal{E}_{E,S_E} \to \mathcal{E}_{E,S_E} \).

Next recall that in order to construct the explicit extension \( \mathcal{E}_{E,S_E} \) and the isomorphism \( i_{E,S_E} \) of Proposition \[3.24\] we had to choose the cocycle \( \alpha_3(S) \) representing the Tate-class as well as the sequences \( N_i \) and \( k_i \). Say that we now made different choices for these and obtained an explicit extension \( \mathcal{E}_{E,S_E} \) and isomorphism \( i_{E,S_E} \). In order to complete the proof we must show that the composition \( i_{E,S_E}^{-1} \circ i_{E,S_E} \) coincides with the canonical isomorphism \( H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}) \to H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}) \) established in the above paragraph. This however is another application of Lemma \[3.26\], this time with \( \Delta_2 = H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}) \).

We will now study how the group \( H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, Z \to T) \) changes when we enlarge \( E \) and \( S \). Let \( K/F \) be a finite Galois extension containing \( E \). Let \( (S', \hat{S}_K) \) be a pair satisfying Conditions \[3.17\] with respect to \( K/F \). Assume further that \( S \subset S' \) and \( \hat{S}_E \subset (\hat{S}_K)_E \). Consider any extension \( \mathcal{E}_{K,S'_K} \) corresponding to the class \( \xi_{K,S'_K} \in H^2(\Gamma_{S'}, P_{K,S'_K}(O_{S'})) \) as well as any extension \( \mathcal{E}_{E,S_E} \) corresponding to the class \( \xi_{E,S_E} \).

Given \( [Z \to T] \in T_{E,S} \) we construct an inflation map

\[
\text{Inf}: H^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, Z \to T) \to H^1(P_{K,S'_K} \to \mathcal{E}_{K,S'_K}, Z \to T)
\]

as follows.

Take \( \square_1 \) to be the pull-back of \( \mathcal{E}_{E,S_E} \) along the projection \( \Gamma_{S'} \to \Gamma_{S} \), then take \( \square_2 \) to be the push-out of \( \square_1 \) along the inclusion \( P_{E,S_E}(O_{S}) \to P_{E,S_E}(O_{S'}) \), and finally choose a homomorphism of extensions \( \mathcal{E}_{K,S'_K} \to \square_2 \). The latter exists by Lemma \[3.21\]. This results in the commutative diagram

\[
\begin{array}{ccc}
1 & \longrightarrow & P_{K,S'_K}(O_{S'}) \\
\text{inf} & \downarrow & \text{inf} \\
1 & \longrightarrow & P_{E,S_E}(O_{S'}) \\
\end{array}
\]

where the dotted arrow is the one we chose. Now let \( z \in Z^1(P_{E,S_E} \to \mathcal{E}_{E,S_E}, Z \to T) \) and let \( v \in \text{Hom}(P_{E,S_E}, Z) \) be its image. We compose \( z \) with the map \( \square_1 : \mathcal{E}_{E,S_E} \to \mathcal{E}_{E,S_E} \) and with the inclusion \( T(O_{S}) \to T(O_{S'}) \) to obtain a 1-cocycle \( z_1 : \square_1 \to T(O_{S'}) \). Its restriction to \( P_{E,S_E}(O_{S}) \) is still given by \( v \). Now \( v \) determines a homomorphism \( P_{E,S_E}(O_{S'}) \to \mathcal{Z}(O_{S'}) \to T(O_{S'}) \), which glues with \( z_1 \) to a 1-cocycle \( z_2 : \square_2 \to T(O_{S'}) \). We then compose \( z_2 \) with the homomorphism...
depend on the choice of section is also the same as the one given in the proof of Proposition 3.29. The inflation map \( N \) is independent of the choice of dotted arrow in the above diagram. Before doing so, we construct the homomorphism \( \dot{\epsilon} \). We will show momentarily that (3.13) is independent of the choice of dotted arrow in the above diagram. The corresponding homomorphism on the level of cohomology gives (3.13).

We will show momentarily that (3.13) is independent of the choice of dotted arrow in the above diagram. Before doing so, we construct the homomorphism \( \dot{\epsilon} \). It is a homomorphism of finite abelian groups

\[
\frac{\hat{Y}[S_E, \hat{S}_E]|_{\mathbb{N}_{E/F}}}{I_{E/F}Y[E]|_{0}} \rightarrow \frac{\hat{Y}[S'_K, \hat{S}'_K]|_{\mathbb{N}_{K/F}}}{I_{K/F}Y[K]|_{0}}
\]

given as follows. Choose any section \( s : S_E \rightarrow S_K \) of the natural projection subject to the condition \( s(\hat{S}_E) \subset \hat{S}'_K \). Given \( f \in \hat{Y}[S_E, \hat{S}_E]|_{\mathbb{N}_{E/F}} \), which we interpret as an element of Maps(\( S_E, \hat{Y} \)), we define \( sf \in \text{Maps}(S_K, \hat{Y}) \) by

\[
sf(u) = \begin{cases} f(p(u)), & \text{if } sp(u) = u \\ 0, & \text{else} \end{cases}
\]

Note that this is the same definition as the one appearing before Lemma 3.28.

**Lemma 3.28.** The assignment \( f \mapsto sf \) induces a well-defined homomorphism

\[
! : \frac{\hat{Y}[S_E, \hat{S}_E]|_{\mathbb{N}_{E/F}}}{I_{E/F}Y[E]|_{0}} \rightarrow \frac{\hat{Y}[S'_K, \hat{S}'_K]|_{\mathbb{N}_{K/F}}}{I_{K/F}Y[K]|_{0}}
\]

that is independent of the choice of section \( s \).

**Proof.** The argument is essentially the same as the one used in the proof of Lemma 3.28. Indeed, as argued there we see that \( N_{E/F}(sf) = 0 \) provided \( N_{E/F}(f) = 0 \), and furthermore that \( sf(I_{E/F}Y[E]|_{0}) \subset I_{K/F}Y[K]|_{0} \). Since \( f(w) \in Y \) for \( w \in S_E \setminus \hat{S}_E \) we have \( sf(u) \in Y \) for \( u \in S'_K \setminus \hat{S}'_K \). Thus we obtain a well-defined homomorphism as claimed. The argument that it does not depend on the choice of section is also the same as the one given in the proof of Lemma 3.28. One just has to note the following: Since we are only considering sections that map \( \hat{S}_E \) into \( \hat{S}'_K \), if \( s, s' \) are two such sections and \( s(w) \neq s'(w) \), then \( w \in \hat{S}_E \setminus \hat{S}_E \). But for any \( f \in \hat{Y}[S_E, \hat{S}_E]|_{0} \) we then have \( f(w) \in Y \subset \hat{Y} \) and the same argument as before implies \( sf - s'f \in I_{K/F}Y[K]|_{0} \). □

**Proposition 3.29.** The inflation map (3.13) is independent of the choice of dotted arrow in Diagram (3.14). It is injective, functorial in \([Z \rightarrow T] \in T_{E,S} \), and fits into the following commutative diagrams

\[
\begin{array}{ccc}
H^1(P_{E,S_E} \rightarrow \mathcal{E}_{E,S_E}, Z \rightarrow T) & \xrightarrow{\text{Inf}} & H^1(P_{K,S'_K} \rightarrow \mathcal{E}_{K,S'_K}, Z \rightarrow T) \\
\downarrow_{\dot{\epsilon}_{E,S_E}} & & \downarrow_{\dot{\epsilon}_{K,S'_K}} \\
\frac{\hat{Y}[S_E, \hat{S}_E]|_{\mathbb{N}_{E/F}}}{I_{E/F}Y[E]|_{0}} & \xrightarrow{1} & \frac{\hat{Y}[S'_K, \hat{S}'_K]|_{\mathbb{N}_{K/F}}}{I_{K/F}Y[K]|_{0}}
\end{array}
\]
and

\[
\begin{array}{ccc}
1 & \inf & 1 \\
H^1(\Gamma, T(O_S)) & H^1(\Gamma, T(O_{S'})) & \\
H^1(P_{E,\hat{S}E} \to \mathcal{E}_{E,\hat{S}E}, Z \to T) & \inf & H^1(P_{K,\hat{S}K} \to \mathcal{E}_{K,\hat{S}K}, Z \to T) \\
\Hom(P_{E,\hat{S}E}, Z)^\Gamma & \Hom(P_{K,\hat{S}K}, Z)^\Gamma & \\
H^1(P_{\hat{V},Z} \to \mathcal{E}_{\hat{V},Z}, Z \to T) & \Hom(P_{\hat{V},Z})^\Theta & H^2(\Gamma, Z) \\
H^2(\Gamma, T(\overline{F})) & H^2(\Gamma, T(\overline{F})) & \\
\end{array}
\]

**Proof.** The commutativity of the second of the two diagrams is obvious from the construction of the inflation map. The five-lemma then implies that this map is injective.

We will now apply Lemma 3.26 to show that (3.13) does not depend on the choice of dotted arrow in Diagram (3.14), and also that the first diagram in the statement of the proposition commutes. For this we take as \(\Delta_2\) the restriction to \(T_{E,S}\) of \(H^1(P_{K,\hat{S}K} \to \mathcal{E}_{K,\hat{S}K})\). We take as \(f_1\) the map (3.13) constructed using one choice of dotted arrow, as \(f_2\) the map (3.13) constructed using another choice of dotted arrow, and as \(f_3\) the composition \(\iota_{E,\hat{S}E} \circ \iota_{E,\hat{S}E}^{-1}\). Then Lemma 3.26 implies that \(f_1 = f_2 = f_3\).

Recall from subsection 3.3 the exhaustive tower \(E_i\) of finite Galois extensions of \(F\) and the exhaustive tower \(S_i\) of finite full subsets of \(V(F)\). Each \(S_i\) was paired with \(\hat{S}_i \subset S_{i+1,E}\), such that \((S_i, \hat{S}_i)\) satisfies Conditions 3.17 with respect to \(E_i/F\) and moreover \(\hat{S}_i \subset \hat{S}_{i+1,E}\). For each \([Z \to T] \in T\) define

\[
H^1(P_{\hat{V},Z} \to \mathcal{E}_{\hat{V},Z}, Z \to T) = \lim_{\to} H^1(P_{E,\hat{S}E} \to \mathcal{E}_{E,\hat{S}E}, Z \to T),
\]

the colimit being taken with respect to (3.13). This abelian group fits into the following commutative diagram with exact columns

\[
\begin{array}{ccc}
1 & \downarrow & 1 \\
H^1(\Gamma, T(F)) & H^1(\Gamma, T(\overline{F})) & \\
H^1(P_{\hat{V},Z} \to \mathcal{E}_{\hat{V},Z}, Z \to T) & \to & H^1(\Gamma, T(\overline{F})) \\
\Hom(P_{\hat{V},Z})^\Theta & \to & H^2(\Gamma, Z) \\
\to & & \\
H^2(\Gamma, T(\overline{F})) & \to & H^2(\Gamma, T(\overline{F})) \\
\end{array}
\]

Indeed, according to Proposition 3.29 the kernel of \(H^1(P_{\hat{V},Z} \to \mathcal{E}_{\hat{V},Z}, Z \to T) \to \Hom(P_{\hat{V},Z})^\Theta\) is equal to \(\lim_{\to} H^1(\Gamma, T(O_{S_i}))\). All transition maps in this limit
are injective by Lemma 3.7 and the limit itself injects into $H^1(\Gamma, T(\bar{F}))$. On the other hand, every element of $H^1(\Gamma, T(\bar{F}))$ can be represented by a 1-cocycle $z : \Gamma_{E/F} \to T(E)$ of a suitable finite Galois extension $E/F$ and we may then choose a suitably large finite set $S \subset V(F)$ such that all values of $z$ are contained in $T(O_{E,S})$. This shows that the injection $\lim_{\rightarrow} H^1(\Gamma_{S_i}, T(O_{S_i})) \to H^1(\Gamma, T(\bar{F}))$ is also surjective.

**Corollary 3.30.** The map

$$H^1(P_{\hat{V}} \to \mathcal{E}_V, Z \to T) \to H^1(\Gamma, \bar{T}(\bar{F}))$$

is surjective.

**Proof.** This follows from the surjectivity of the map $\Theta_{P_{\hat{V}}}^P$ given by (3.9) and the five-lemma. □

### 3.5 The localization maps

Let $v \in \hat{V} \subset V(\bar{F})$. The decomposition group $\Gamma_v \subset \Gamma$ is identified with the absolute Galois group of the completion $\hat{F}$. In this subsection we are going to define for each $[Z \to T] \in T$ a localization map

$$\text{loc}_v : H^1(P_{\hat{V}} \to \mathcal{E}_V, Z \to T) \to H^1(u_v \to W_v, Z \to T), \quad (3.17)$$

where the target is the cohomology group defined in [Kal, §3] (we are now using the subscript $v$ to emphasize the base field $F_v$).

For this, fix a finite Galois extension $E/F$ and sets $(S, \dot{S}_E)$ satisfying Conditions 3.17 with the property that $[Z \to T] \in T_{E,S}$. Let $\mathcal{E}_{E,\dot{S}_E}$ be any extension corresponding to the class $\xi_{E,\dot{S}_E}$ and let $W_v$ be any extension corresponding to the class $\xi_v$. Let $\Box_1$ be the pull-back of $\mathcal{E}_{E,\dot{S}_E}$ along the map $\Gamma_v \to \Gamma_S$ and let $\Box_2$ be the push-out of $\Box_1$ along $P_{E,\dot{S}_E}(O_S) \to P_{E,\dot{S}_E}(\mathcal{O}_v)$. According to Corollary 3.23, there exits a dotted homomorphism making the following diagram commutative.

1. $1 \to u_v(\bar{F}_v) \to W_v \to \Gamma_v \to 1$ \hspace{1cm} (3.18)

The same procedure that produced the inflation map from Diagram (3.14) now produces a homomorphism

$$H^1(P_{E,\dot{S}_E} \to \mathcal{E}_{E,\dot{S}_E}, Z \to T) \to H^1(u_v \to W_v, Z \to T) \quad (3.19)$$
that is functorial in \([Z \to T] \in T_{E,S}\) and is compatible with \(\text{loc}_v^{P_{E,S_E}}\) in the sense that
\[
H^1(P_{E,S_E} \to E,S) \xrightarrow{\text{loc}_v^{P_{E,S_E}}} \text{Hom}(P_{E,S_E}, Z)^\Gamma \\
H^1(u_v \to W_v, Z) \xrightarrow{\text{loc}_v^{P_{E,S_E}}} \text{Hom}(u_v, Z)^\Gamma
\]

We will soon show that this map is independent of the dotted arrow in Diagram \((\ref{3.18})\) and that it is compatible with the inflation map. Before we do this, we will describe the homomorphism that corresponds to the localization map under the Tate-Nakayama isomorphism \(l_{E,S_E}\).

**Lemma 3.31.** For each coset \(\tau \in \Gamma_{E_v/F_v} \setminus \Gamma_{E/F}\), fix a representative \(\check{\tau} \in \Gamma_{E/F}\) such that \(\check{\tau} = 1\) if \(\tau = \Gamma_{E_v/F_v}\). The map
\[
l_v : \tilde{Y}[S_E, \hat{S}_E]^{N_{E/F}}_{1E/F} \to \tilde{Y}[S_{E_v}, \hat{S}_{E_v}]^{N_{E_v/F_v}}_{1E_v/F_v}
\]
that sends \(f \in \tilde{Y}[S_E, \hat{S}_E]^{N_{E/F}}_{1E/F}\) to \(\sum_{\tau} \check{\tau} f(\check{\tau}^{-1}v) \in \tilde{Y}[S_{E_v}, \hat{S}_{E_v}]^{N_{E_v/F_v}}_{1E_v/F_v}\) is a well-defined homomorphism independent of the choices of \(\check{\tau}\). Given a finite Galois extension \(K/F\), sets \((S', \hat{S}_K')\) satisfying conditions \((\ref{3.12})\) as well as \(S \subset S'\) and \(\hat{S}_E \subset (\hat{S}_K)_E\), we have the commutative diagram
\[
\begin{array}{c}
\tilde{Y}[S_{E_v}, \hat{S}_{E_v}]^{N_{E_v/F_v}}_{1E_v/F_v} \xrightarrow{l_v} \tilde{Y}[S_{E_v}, \hat{S}_{E_v}]^{N_{E_v/F_v}}_{1E_v/F_v} \\
\end{array}
\]
where the right vertical map is induced by the identity on \(\tilde{Y}\) and the left vertical map is that of Lemma \((\ref{3.28})\).

**Proof.** First, we have
\[
N_{E_v/F_v}(\sum_{\tau} \check{\tau} f(\check{\tau}^{-1}v)) = \sum_{\tau} \sum_{\sigma \in \Gamma_{E_v/F_v}} \sigma \check{\tau} f(\check{\tau}^{-1}v) \\
= \sum_{\sigma \in \Gamma_{E/F}} f(\check{\tau}^{-1}v) = (N_{E/F} f)(v) = 0.
\]

Second, if we replace the representative \(\check{\tau}_1\) of some non-trivial coset \(\tau_1 \in \Gamma_{E_v/F_v} \setminus \Gamma_{E/F}\) by \(\sigma \check{\tau}_1\) with \(\sigma \in \Gamma_{E_v/F_v}\), then \(\sum_{\tau} \check{\tau} f(\check{\tau}^{-1}v)\) is replaced by \(\sum_{\tau} \check{\tau} f(\check{\tau}^{-1}v) + \sigma \check{\tau}_1 f(\check{\tau}_1^{-1}v) - \check{\tau}_1 f(\check{\tau}_1^{-1}v).\) Since \(\tau_1\) is non-trivial and \(v \in \hat{S}_E\), we have \(\check{\tau}_1^{-1}v \notin \hat{S}_E\) and consequently \(f(\check{\tau}_1^{-1}v) \in Y\). This shows that \(l_v\) is independent of the choice of \(\check{\tau}_1\).

For the commutativity of the diagram, recall that the map \(!\) is defined in terms of a section \(s : S_E \to S'_K\) having the property \(s(\hat{S}_E) \subset \hat{S}'_K\), which in particular implies that it maps \(v_E \in \hat{S}_E\) to \(v_K \in \hat{S}'_K\) (since we are dealing with different extensions of \(F\), in this argument we are using the subscript \(E\) to emphasize that we mean the image of \(v \in \hat{V} \subset V(\overline{F})\) in \(S_E \subset V(E)\) via the projection.
map \( V(F) \to V(E) \). Via the bijections \( \Gamma_{E/F, E} \backslash \Gamma_{E/F} \to \{ w \in S_E : w|_{V_E} \} \) and \( \Gamma_{K/F, E} \backslash \Gamma_{K/F} \to \{ u \in S'_K : u|_{V_F} \} \), the section \( s \) can be thought of as a section \( \Gamma_{E/F, E} \backslash \Gamma_{E/F} \to \Gamma_{K/F, E} \backslash \Gamma_{K/F} \). We choose a representative \( \hat{\tau} \in \Gamma_{K/F} \) for each coset \( \tau \in \Gamma_{K/F, E} \backslash \Gamma_{K/F} \). Then we have

\[ \sum_{\tau \in \Gamma_{K/F, E} \backslash \Gamma_{K/F}} \hat{\tau} \left[ (s \lambda)(\tau^{-1}v_K) \right] = \sum_{\tau \in \Gamma_{E/F, E} \backslash \Gamma_{E/F}} \hat{\tau} f(\tau^{-1}v_K) = \sum_{\tau \in \Gamma_{E/F, E} \backslash \Gamma_{E/F}} \hat{\tau} f(\tau^{-1}v_E), \]

where in the last term we have taken as a representative in \( \Gamma_{E/F} \) of the coset \( \tau \in \Gamma_{E/F, E} \backslash \Gamma_{E/F} \) the image in \( \Gamma_{E/F} \) of the representative \( \hat{\tau} \in \Gamma_{K/F} \) of \( s(\tau) \).

\[ \square \]

**Proposition 3.32.** The localization map \((3.19)\) is independent of the choice of dotted arrow in Diagram \((3.18)\). It is functorial in \([Z \to T] \in T_{E,S}\) and is compatible with the inflation maps as well as the Tate-Nakayama-type isomorphisms, i.e.

\[ H^1(P_E, S_E \to E_{E,S}) \xrightarrow{loc_v} H^1(u_v \to W_v, Z \to T) \]

\[ \begin{array}{ccc}
Y[S_E, S_E] & \xrightarrow{\delta} & Y[S_E, S_E] \\
\downarrow \delta & & \downarrow \delta \\
I_{E/F}Y[S_E, Y] & \xrightarrow{\delta} & I_{E/F}Y[S_E, Y] \\
\end{array} \\
\begin{array}{ccc}
\xrightarrow{l_{E,S_E}} & & \xrightarrow{l_v} \\
\sum_{v \in V} & & \sum_{v \in V} \\
\xrightarrow{(l_v)_v} & & \xrightarrow{(l_v)_v} \\
\sum_{v \in V} Y[S_E, S_E] & \xrightarrow{(l_v)_v} & \sum_{v \in V} Y[S_E, S_E] \\
\end{array} \]

**Proof.** We let \( \Delta_v \) be the functor that assigns to \([Z \to T] \in T_{E,S}\) the group \( H^1(u_v \to W_v, Z \to T) \). We let \( f_1 \) be \((3.19)\) constructed with respect to one choice of dotted arrow, \( f_2 \) be \((3.19)\) constructed with respect to another choice of dotted arrow, \( f_3 \) be the composition \( l_v \circ l_{E,S_E}^{-1} \) and \( f_4 \) be the composition of \((3.13)\) with the map \((3.19)\) for the extension \( K/F \) (and some choice of dotted arrow). Then Lemma \((3.26)\) implies \( f_1 = f_2 = f_3 = f_4 \).

The compatibility of \((3.19)\) with the inflation map \((3.13)\) implies that the maps \((3.19)\) for different \( E, S \) splice together, and the result is the map \((3.17)\).

The following lemma will be essential for the applications of the group \( H^1(P_V \to E_V, Z \to T) \) in next section.

**Lemma 3.33.** We have the following commutative diagram with exact bottom row

\[ H^1(P_V \to E_V, Z \to T) \xrightarrow{(loc_v)_v} \bigoplus_{v \in V} H^1(u_v \to W_v, Z \to T) \]

\[ \begin{array}{ccc}
Y[E, E] & \xrightarrow{\delta} & Y[E, E] \\
\downarrow \delta & & \downarrow \delta \\
I_{E/F}Y[E, Y] & \xrightarrow{\delta} & I_{E/F}Y[E, Y] \\
\downarrow (l_{E,E}) & & \downarrow (l_{E,E}) \\
\bigoplus_{v \in V} Y[S_E, S_E] & \xrightarrow{\delta} & \bigoplus_{v \in V} Y[S_E, S_E] \\
\downarrow \sum_{v \in V} (l_v) & & \downarrow \sum_{v \in V} (l_v) \\
\sum_{v \in V} Y[S_E, S_E] & \xrightarrow{\delta} & \sum_{v \in V} Y[S_E, S_E] \\
\end{array} \]

**Proof.** The commutativity of the diagram is part of Proposition \((3.32)\). We only need to prove the exactness of the bottom row. The fact that the composition of the two arrows is zero is obvious. Conversely, let \( \bar{\lambda}_v \in Y[N_{E_v, F_v}] \) and with \( \sum_{v} \bar{\lambda} \in I_{E/F}Y \). We can write this element of \( I_{E/F}Y \) as \( (\sigma_1 \lambda_1 - \lambda_1) + \cdots + (\sigma_k \lambda_k - \lambda_k) \) for some \( \lambda_i \in Y \) and \( \sigma_i \in \Gamma_{E/F} \). For each \( i \) let \( v_i \in S_E \) be such that \( \sigma_i v_i = v_i \) and replace \( \bar{\lambda}_{v_i} \) by \( \bar{\lambda}_{v_i} + \lambda_i - \sigma_i \lambda_i \). The new collection \( (\bar{\lambda}_v)_{v \in S_E} \) represents the same element of \( \bigoplus_{v \in S_E} Y[N_{E_v, F_v}] \) as the old collection, but now \( \sum_{v} \bar{\lambda}_v = 0 \), hence \( (\bar{\lambda}_v)_v \) is now in the image of \( (l_v)_v \). \[ \square \]
The reader might wonder if there is a cohomology group that fits above the bottom right term in the above diagram. There is indeed such a cohomology group and it is an enlargement of $H^1(\Gamma, T(\mathcal{M}^r)/T(\mathcal{F}))$. However, since we will not need it for the applications we have in mind, we will not discuss its construction and properties.

4 Applications to the Trace Formula and Automorphic Multiplicities

In this section we are going to use the refined local endoscopic objects introduced in [Kal] to produce two essential global endoscopic objects -- the adelic transfer factor used in the stabilization of the Arthur-Selberg trace formula and the conjectural pairing between an adelic $L$-packet and its global $S$-group. The first of these objects -- the adelic transfer factor -- is already defined, see e.g. [LS87] §6.3 or [KS99] §7.3, where it is denoted by $\Delta_{\gamma}(\gamma, \delta)$. The point here is to show that this factor admits a decomposition as the product of the normalized local transfer factors introduced in [Kal] §5.3. The second global object -- the pairing between an adelic $L$-packet and its global $S$-group -- has so far not been constructed for general non-quasi-split reductive groups. As discussed in [Kot84] §12 and [BR94] §3.4, it is expected that this pairing is canonical and has a decomposition as a product of local pairings. We will construct this pairing here using the refined local pairings of [Kal] §5.4 and then show that the result is canonical.

In the last subsection, we will summarize the results of this section in the language of [Art06]. We will show that the local conjecture of [Kal] implies the local conjecture of [Art06], and we will show that Proposition 4.2 (which is unconditional) implies [Art13] Hypothesis 9.5.1.

4.1 Notation for this section

Let $G$ be a connected reductive group over the number field $F$ with quasi-split inner form $G^*$ and let $\Psi$ be a $G$-conjugacy class of inner twists $\psi : G^* \rightarrow G$. We write $G_{\text{der}}, G_{\text{sc}},$ and $G_{\text{ad}}$, for the derived subgroup of $G$ and its simply connected cover and adjoint quotient, respectively. We use the superscript * to denote the same objects relative to $G^*$. Let $G^*$ be the Langlands dual group of $G^*$ and $L G^* = G^* \times W_F$ the Weil-form of the $L$-group of $G^*$. We will write $\hat{G}_{\text{der}}^*$ for the derived subgroup of $G^*$ and $\hat{G}_{\text{sc}}^*$ and $\hat{G}_{\text{ad}}^*$ for the simply connected cover and adjoint quotient of $\hat{G}_{\text{der}}^*$. We will write $Z_{\text{sc}}$ for the center of $G_{\text{sc}}^*$ and $\tilde{Z}_{\text{sc}}$ for the center of $\hat{G}_{\text{sc}}^*$. We will write $Z_{\text{der}}$ for the center of $G_{\text{der}}^*$.

We set $\bar{G} = G/Z_{\text{der}} = G_{\text{ad}} \times Z(G)/Z_{\text{der}}$ and $\bar{G}_{\text{sc}} = G_{\text{sc}}/Z_{\text{sc}} = G_{\text{ad}}$. Then we have $\hat{G}^* = \hat{G}_{\text{sc}}^* \times Z(\hat{G}^*)^\gamma$ and $\hat{G}_{\text{sc}}^* = \hat{G}_{\text{sc}}$. Recall the notation $Z(\hat{G}^*)^\gamma$ for the elements of $Z(\hat{G}^*)$ which map to $Z(\hat{G}^*)^\gamma$. We will use the analogous notation $Z(\hat{G}_{\text{sc}}^*)^\gamma_v$ for those elements that map to $Z(\hat{G}_{\text{sc}}^*)^\gamma_v$ when $v$ is a place of $\mathcal{F}$.

Just as we did in subsection 3.4, we let $E_i$ be an exhaustive tower of finite Galois extensions of $F$, we let $S_i$ be an exhaustive tower of finite subsets of $V(F)$, and we let $\tilde{S}_i \subset \hat{S}_i$ be a set of representatives for $S_i$ with $\tilde{S}_i \subset \hat{S}_{i+1,E_i}$. We assume that for each $i$ the pair $(\tilde{S}_i, \hat{S}_i)$ satisfies Conditions 3.7 with respect to $E_i/F$. Then $V = \lim_{\rightarrow} \hat{S}_i$ is a set of representatives in $V(\mathcal{F})$ for $V(F)$.
4.2 From global to refined local endoscopic data

Let us first recall the notion of a global endoscopic datum for $G$ following [LS87 §1.2]. It is a tuple $(H, \mathcal{H}, s, \xi)$ consisting of a quasi-split connected reductive group $H$ defined over $F$, a split extension $\mathcal{H}$ of $W_F$ by $\hat{H}$, an element $s \in Z(\hat{H})$, and an $L$-embedding $\xi : \mathcal{H} \to LG$ satisfying the following conditions.

1. The homomorphism $W_F \to \text{Out}(\hat{H}) = \text{Out}(H)$ determined by the extension $\mathcal{H}$ coincides with the homomorphism $\Gamma \to \text{Out}(H)$ determined by the rational structure of $H$, in the sense that both homomorphisms factor through some finite quotient $W_F/W_E = \Gamma_F/\Gamma_E$ and are equal.

2. $\xi$ induces an isomorphism of complex algebraic groups $\hat{H} \to \text{Cent}(t, \hat{G}^*)^\circ$, where $t = \xi(s)$.

3. The image $\bar{s} \in Z(\hat{H})/Z(\hat{G}^*)$ of $s$ is fixed by $W_F$ and maps under the connecting homomorphism $H^0(W_F, Z(\hat{H})/Z(\hat{G}^*)) \to H^1(W_F, Z(\hat{G}^*))$ to a locally trivial element.

In condition 3 we have used the fact that conditions 1 and 2 provides a $\Gamma$-equivariant embedding $Z(\hat{G}^*) \to Z(\hat{H})$. Recall that a locally trivial element of $H^1(W_F, Z(\hat{G}^*))$ is one whose image in $H^1(W_F, Z(\hat{G}^*))$ is trivial for all $v \in V(\overline{F})$. Up to equivalence, the datum $(H, \mathcal{H}, s, \xi)$ depends only on the image of $s$ in $\pi_0((Z(\hat{H})/Z(\hat{G}^*))^\Gamma)$.

On the other hand, a refined endoscopic datum is a tuple $(H, \mathcal{H}, \bar{s}, \xi)$ where $H, \mathcal{H},$ and $\xi$ are as above (but now over a local field $F$), and $\bar{s}$ is an element of $Z(\hat{H})^+$ (whose image in $Z(\hat{H})^\Gamma$ we denote by $s$) such that conditions 1 and 2 are satisfied. Condition 3 is unnecessary. We have used here the fact that conditions 1 and 2 provide an $F$-embedding $Z(G) \to Z(H)$ which allows us to form the quotient $\hat{H} = H/Z_{\text{der}}$ and hence obtain $\hat{H}$. Recall that $Z(\hat{H})^+$ is the preimage of $Z(\hat{H})^\Gamma$ under the isogeny $\hat{H} \to \hat{H}$. Up to equivalence, the datum $(H, \mathcal{H}, \bar{s}, \xi)$ depends only on the image of $\bar{s}$ in $\pi_0(Z(\hat{H})^+)$. Note that, unlike in the global case, we are not allowing to change $\bar{s}$ by elements of $Z(\hat{G}^*)$.

We will now show how a global endoscopic datum $s = (H, \mathcal{H}, s, \xi)$ gives rise to a collection, indexed by $V$, of refined local endoscopic data. Using the embedding $Z(G) \to Z(H)$ we form $\hat{H} = H/Z_{\text{der}}$ and obtain from $\xi$ the embedding $\hat{H} \to \hat{G}^*$ whose image is equal to $\text{Cent}(t, \hat{G}^*)^\circ$. In order to simplify the notation, we will use the same letter for elements of $\hat{H}$ or $\hat{H}$ and their image under $\xi$, so in particular we have $s = t$. Let $s_{\text{sc}} \in \hat{G}_{\text{sc}}^*$ be a preimage in $\hat{G}_{\text{sc}}^*$ of the image $s_{\text{ad}} \in \hat{G}_{\text{ad}}^*$ of $s$. According to condition 3 above, given a place $v \in V$ there exists $y_v \in Z(\hat{H})^\Gamma$ such that $s_{\text{der}} \cdot y_v \in Z(\hat{H})^\Gamma$, where $s_{\text{der}} \in \hat{G}_{\text{der}}^*$ is the image of $s_{\text{sc}}$. Write $y_v = y_v' \cdot y_v''$ with $y_v' \in Z(\hat{G}_{\text{der}}^*)$ and $y_v'' \in Z(\hat{G}^*)^\circ$ and choose a lift $\tilde{y}_v \in \hat{Z}_{\text{sc}}$. Then $\hat{s}_v := (s_{\text{sc}} \cdot \tilde{y}_v', y_v'')$ belongs to $Z(\hat{H})^\Gamma$ and $\hat{\epsilon}_v = (H, \mathcal{H}, \hat{s}_v, \xi)$ is a refined local endoscopic datum at the place $v$. We thus obtain the collection $(\hat{\epsilon}_v)_{v \in V}$ of refined local endoscopic data.

We emphasize that we use the same $s_{\text{sc}}$ to form all $\hat{\epsilon}_v$. The collection $(\hat{\epsilon}_v)_{v}$ depends on the choice of $s_{\text{sc}}$ and each member $\hat{\epsilon}_v$ depends furthermore on the
choices of \( \psi' \) and \( \psi'' \). We shall see that these choices do not influence the resulting global objects, which means that they depend only on the equivalence class of the global datum \( \epsilon \).

### 4.3 Coherent families of local rigid inner twists

We begin by recalling a well-known lemma about the Galois cohomology of reductive groups, whose proof we include since we weren’t able to locate a convenient reference.

**Lemma 4.1.** Let \( h \in H^1(\Gamma, G(\overline{F})) \). There exists a maximal torus \( T \subset G \) such that \( h \) is in the image of \( H^1(\Gamma, T(\overline{F})) \rightarrow H^1(\Gamma, G(\overline{F})) \).

**Proof.** We first reduce to the case that \( G \) is semi-simple. Let \( h_{ad} \in H^1(\Gamma, G_{ad}) \) be the image of \( h \). Let \( T \subset G \) be a maximal torus such that \( h_{ad} \) is the image of \( h_{T, ad} \in H^1(\Gamma, T_{ad}) \). The image of \( h_{T, ad} \) in \( H^2(\Gamma, Z(G)) \) is equal to the image of \( h_{ad} \), which is trivial. Hence there exists \( h_T \in H^1(\Gamma, T) \) mapping to \( h_{T, ad} \). Both \( h_T \) and \( h \) have the same image in \( H^1(\Gamma, G_{ad}) \), so there exists \( h_Z \in H^1(\Gamma, Z(G)) \) such that \( h_Z \cdot h_T \in H^1(\Gamma, T) \) maps to \( h \).

We assume from now on that \( G \) is semi-simple and let \( G_{sc} \rightarrow G \) be its simply connected cover with kernel \( C \). Let \( S \subset V(F) \) be a finite set of places containing all \( v \in V(F) \) for which the image of \( h \) in \( H^1(\Gamma_v, G) \) is non-trivial. Assume in addition that \( S \) contains all archimedean places and at least one non-archimedean place. Let \( T \subset G \) be a maximal torus which is fundamental [Kot86, §10] at all \( v \in S \). Such a maximal torus exists by [PR94, §7.1, Cor. 3].

We consider the diagram

\[
\begin{array}{c}
H^1(\Gamma, G_{sc}) \longrightarrow & H^1(\Gamma, G) \longrightarrow & H^2(\Gamma, C) \\
| & | & |
H^1(\Gamma, T_{sc}) \longrightarrow & H^1(\Gamma, T) \longrightarrow & H^2(\Gamma, C) \longrightarrow & H^2(\Gamma, T_{sc})
\end{array}
\]

We claim that the image of \( h \) in \( H^2(\Gamma, T_{sc}) \) is trivial. Indeed, it has the property that for \( v \notin S \) its image in \( H^2(\Gamma_v, T_{sc}) \) is trivial due to the definition of \( S \), and for \( v \in S \) its image in \( H^2(\Gamma_v, T_{sc}) \) is also trivial because of [Kot86, Lemma 10.4]. The claim follows from a theorem of Kneser [PR94, §6.3, Prop 6.12] that asserts that \( T_{sc} \) satisfies the Hasse-principle in degree 2.

Let \( h_T' \in H^1(\Gamma, T) \) be a preimage of the image of \( h \) in \( H^2(\Gamma, C) \) and let \( h' \in H^1(\Gamma, G) \) be the image of \( h_T' \). It may not be the case that \( h = h' \). Let \( G' \) be the twist of \( G \) by \( h' \). Then \( T \) is a maximal torus in \( G' \) and we have the bijection

\[ H^1(\Gamma, G) \rightarrow H^1(\Gamma, G'), \quad y \mapsto y \cdot h'^{-1}. \]

The image of \( h \cdot h'^{-1} \in H^1(\Gamma, G') \) in \( H^2(\Gamma, C) \) is trivial. Let \( h''_T \in H^1(\Gamma, G'_{sc}) \) be a preimage of \( h \cdot h'^{-1} \). We claim that there exists \( h''_{T, sc} \in H^1(\Gamma, T_{sc}) \) mapping to \( h''_T \). Granting this, let \( h_T'' \in H^1(\Gamma, T) \) be the image of \( h''_{T, sc} \). Then the image of \( h_T'' \cdot h_T' \) in \( H^2(\Gamma, G) \) is equal to \( h \) and the proof is complete.
We will now prove the outstanding claim. For this, consider the diagram

\[
\begin{array}{c}
H^1(\Gamma, G'_G) \\
\downarrow \\
H^1(\Gamma, T_{sc}) \quad \longrightarrow \quad \prod_{v \in V(F)_{\infty}} H^1(\Gamma_v, G'_{G_sc}) \quad \longrightarrow \\
H^1(\Gamma, T_{sc}) \quad \longrightarrow \quad \prod_{v \in V(F)_{\infty}} H^1(\Gamma_v, T_{sc})
\end{array}
\]

By work of Kneser, Harder, and Chernousov [PR94, §6.1, Thm 6.6] the top map is bijective. The right map is surjective by [Kot86, 10.2] because \(T_{sc} \cap_{F} \) is fundamental for all \( v \in V(F)_{\infty} \subset S \).

Let \((t_v)_{v \in V(F)_{\infty}}\) be an element of the bottom right term whose image in the top right term corresponds to \( h''_{sc} \). We claim that \((t_v)_{v \in V(F)_{\infty}}\) has a lift \( h''_{T_{sc}} \in H^1(\Gamma, T_{sc}) \). For this we inspect the image of \((t_v)_{v \in V(F)_{\infty}}\) in \( H^1(\Gamma, T_{sc}(\mathfrak{T}_{sc})/T_{sc}(\mathfrak{T})) \). Let \( \tilde{\pi}_0 \) be non-archimedean. Then \( T_{sc,\tilde{\pi}_0} \) is a maximal torus \( T \), and thus also \( h''_{sc} \), is finite. In this turn implies that the map \( \tilde{\pi}_0(\mathfrak{T}_{sc}) \to h''_{sc} \) is injective. This map is dual to the composition

\[
H^1(\Gamma, T_{sc}) \overset{\pi}{\longrightarrow} H^1(\Gamma, T_{sc}(F_{\tilde{\pi}_0} \otimes_{F} \mathfrak{T})) \to H^1(\Gamma, T_{sc}(\mathfrak{T}_{sc})/T_{sc}(\mathfrak{T}))
\]

which therefore must be surjective.

Hence we may augment the collection \((t_v)_{v \in V(F)_{\infty}}\) to a collection \((t_v)_{v \in V(F)_{\infty} \cup \{\tilde{\pi}_0\}}\) so that the image of the new collection in \( H^1(\Gamma, T_{sc}(\mathfrak{T}_{sc})/T_{sc}(\mathfrak{T})) \) vanishes. But then this new collection lifts to an element \( h''_{T_{sc}} \in H^1(\Gamma, T_{sc}) \). The image of this element in the top right term of the above diagram is the same as the image of the old collection \((t_v)_{v \in V(F)_{\infty}}\), hence the same as the image of \( h''_{sc} \). By the bijectivity of the top map this means that the image of \( h''_{T_{sc}} \) in \( H^1(\Gamma, G'_{G_sc}) \) equals \( h''_{sc} \).

With this lemma at hand we return to our discussion. We are given an equivalence class \( \Psi \) of inner twists \( G^* \to G \) over \( F \). At each place \( v \in \tilde{V} \) this leads to an equivalence class \( \Psi_v : G^* \to G \) of inner twists over \( F_v \) and we would like to enrich it to an equivalence class \( \tilde{\Psi}_v : G^* \to G \) of rigid inner twists such that the collection \((\tilde{\Psi}_v)_{v \in \tilde{V}}\) is in some sense coherent. We do this as follows.

The class \( \Psi \) provides an element of \( H^1(\Gamma, G_{ad}(\mathfrak{T})) \). Using the above lemma, we find a maximal torus \( T \subset G^* \) such that this element lifts to an element \( h_{sc} \in H^1(\Gamma, T_{sc}(\mathfrak{T})) \). At each place \( v \) we choose \( i_v \) large enough so that \( E_i \) splits \( T \) and there is \( h_{sc} \in H^1(\Gamma, T_{sc}(\mathfrak{T})) \) lifting \( h_{sc} \). This is possible by Corollary 3.30. Choose \( z_{sc} \in Z^1(P_{E_i, \hat{s}_i} \to \mathfrak{E}_{E_i, \hat{s}_i}, Z_{sc} \to T_{sc}) \) representing \( h_{sc} \), and let \( z_{ad} \in Z^1(\Gamma, T_{ad}(\mathfrak{T})) \subset Z^1(\Gamma, G_{ad}(\mathfrak{T})) \) be its image. There exists \( \psi \in \Psi \) such that \( \psi^{-1} \sigma(\psi) = \text{Ad}(z_{ad}(\sigma)) \).

For any place \( v \in \tilde{V} \) we choose \( i_v \) large enough so that \( v_{E_i} \in S_{\hat{s}_i} \), let \( z_{sc,i_v} \in Z^1(P_{E_{i_v}, \hat{s}_{i_v}} \to \mathfrak{E}_{E_{i_v}, \hat{s}_{i_v}}, Z_{sc} \to T_{sc}) \) be obtained from \( z_{sc} \) via the inflation map (3.13), and let \( z_{sc,\psi} \in Z^1(u_{E_i} \to W, Z_{sc} \to T_{sc}) \) be obtained from \( z_{sc,\psi} \) via the localization map (3.19). Let \( z_{sc,i_v} \in Z^1(u_{E_{i_v}} \to W_{i_v}, Z_{sc,\psi} \to T_{der,\psi}) \subset Z^1(u_{E_{i_v}} \to W_{i_v}, Z_{der} \to G) \) be the image of \( z_{sc,\psi} \). Then \((\psi, z_{sc,\psi}) : G^* \to G \) is a rigid inner twist over \( F_{i_v} \).

We have thus obtained a collection \((\psi, z_{sc,i_v})_v\) of local rigid inner twists indexed by \( v \in \tilde{V} \). The collection depends on the choices of \( T \) and \( z_{sc} \), as well as on the choices of inflating and localizing diagrams (3.14) and (3.18) (the dotted arrow in these diagrams needs to be chosen). We shall see that these choices do not influence the resulting global objects, which means that they depend only on \( \Psi \).
It is interesting to note the parallel of choices made in this subsection and in subsection 4.2. The choice of global objects $T$ and $z_{sc}$ lifting $\Psi$ here corresponds to the choice of $s_{ac}$ lifting $s_{ad}$ there. These choices will influence the local endoscopic objects, but not the global objects built from them.

4.4 A product formula for the adelic transfer factor

In this subsection we will show that the adelic transfer factor admits a decomposition as a product of normalized local transfer factors. We are given an equivalence class $\Psi$ of inner twists $G^* \rightarrow G$, a Whittaker datum $w$ for the quasi-split group $G^*$, an endoscopic datum $\epsilon = (H, \mathcal{H}, s, \xi)$ for $G^*$ and a $z$-pair $\tilde{z} = (H_1, \xi_{H_1})$ for $\epsilon$. The adelic transfer factor is a function

$$\Delta_{\tilde{h}} : H_{1, G, sr}(\tilde{A}) \times G_{sr}(\tilde{A}) \rightarrow \mathbb{C},$$

which associates to a pair of semi-simple and strongly $G$-regular elements $\gamma_1 \in H_{1, G, sr}(\tilde{A})$ and $\delta \in G_{sr}(\tilde{A})$ a complex number $\Delta_{\tilde{h}}(\gamma_1, \delta) \in \mathbb{C}$. This factor is defined in [LS87, §6.3], but see also [KS99, §7.3], where $z$-pairs are explicitly used. It is identically zero unless there exists a pair of related elements $\gamma_{1,0} \in H_{1, G, sr}(F)$ and $\delta_0 \in G_{sr}(F)$, which we now assume.

We let $(\hat{\epsilon}_v)_{v \in \hat{V}}$ be the collection of refined local endoscopic data obtained from $\epsilon$ as in Subsection 4.2. We obtain in a straightforward way a collection $(\hat{\delta}_v)_{v \in \hat{V}}$ of local $z$-pairs from the global $z$-pair $\tilde{z}$. We furthermore let $(\psi, z_v)_{v \in \hat{V}}$ be the collection of local rigid inner twists obtained from $\Psi$ as in Subsection 4.3. Note here that $\psi \in \Psi$ was chosen during that construction. For each $v \in \hat{V}$ we have the normalized local transfer factor

$$\Delta_{[w_v, \hat{\epsilon}_v, \hat{\delta}_v, \psi, z_v]} : H_{1, G, sr}(F_v) \times G_{sr}(F_v) \rightarrow \mathbb{C}$$

defined in [Kal] [§5.3].

**Proposition 4.2.** For any $\gamma_1 \in H_{1, G, sr}(\tilde{A})$ and $\delta \in G_{sr}(\tilde{A})$ we have

$$\Delta_{\tilde{h}}(\gamma_1, \delta) = \prod_{v \in \hat{V}} (z_{sc,v}, y_v') \Delta_{[w_v, \hat{\epsilon}_v, \hat{\delta}_v, \psi, z_v]}(\gamma_1, \delta) \cdot \Delta_{[w_v, \hat{\epsilon}_v, \hat{\delta}_v, \psi, z_v]}(\gamma_1, \delta).$$

For almost all $v \in \hat{V}$, the corresponding factor in the product is equal to 1. For all $v$, the factor in the product is independent of the choices of $y_v'$ and $y_v''$ made in subsection 4.2 as well as from the choices of a maximal torus $T$ and inflation and localization diagrams made in subsection 4.3.

Before we give the proof, we want to make a remark on the statement. At first sight it may seem surprising that we had to include the factor $(z_{sc,v}, y_v')$ in the formula. This is an explicitly given complex number and can of course be hidden by including it into the definition of the normalized transfer factor. This seems natural from the global point of view and is part of the reformulation of our results given in subsection 4.6. However, from the local point of view this appears less natural to us, which is why we have kept the formulation as it is. Note that when $G_{det}$ is simply connected, which is often assumed in the literature when dealing with the passage from global to local endoscopy, we may choose $y_v' = 1$ and this factor disappears.

**Proof.** For each $\hat{\epsilon}_v$, let $\epsilon_v$ be the corresponding usual (non-refined) local endoscopic datum. Then the collection $(\epsilon_v)_v$ is associated to $\epsilon$ as in [LS87] [§6.2]. It
is shown in [LS87] Corollary 6.4.B that if $\Delta^{(v)} : H_{1,G}c(F_v) \times G_{sc}(F_v) \to \mathbb{C}$ is an absolute transfer factor normalized so that $\Delta^{(v)}(\gamma_1, \delta_0) = 1$ for almost all $v$ and such that $\prod_{v \in \mathcal{V}} \Delta^{(v)}(\gamma_1, \delta_0) = 1$, then

$$\Delta_{\mathcal{V}}(\gamma_1, \delta) = \prod_{v \in \mathcal{V}} \Delta^{(v)}(\gamma_1, v).$$

We will show that our normalized transfer factors $\Delta[w, e, \hat{v}, s, \psi, z_v]$ satisfy these properties. First off, they are indeed absolute transfer factors, according to [Kal] Proposition 5.6]. Now let $\gamma_0 \in H(F)$ be the image of $\gamma_1,0$ and let $T_0^H \subset H$ be the centralizer of $\gamma_0$. Choose an admissible embedding $T_0^H \to G^*$ and let $\delta^*_0$ be the image of $\gamma_0$. Then $\delta^*_0$ and $\delta_0$ are stably conjugate, i.e. there exists $g \in G^*(\mathbb{F})$ such that $\psi(g\delta_0^*g^{-1}) = \delta_0$. We recall [Kal] (5.1) that the factor $\Delta[w, e, \hat{v}, s, \psi, z_v](\gamma_1, 0)$ is defined as the product

$$\Delta[w, e, \hat{v}, s, \psi, z_v](\gamma_1, 0, \delta^*_0) : \langle \text{inv}(\delta_0^*, (G, \psi, z_v, \delta_0)), \hat{s}_v(\gamma_1, 0) \rangle,$$

where $\Delta[w, e, \hat{v}, s, \psi, z_v]$ is the Whittaker normalization of the transfer factor for $e_v$ and the quasi-split group $G^*$. It is the product of terms $\epsilon_v$, $\Delta_B$, $\Delta_{IF}$, and $\Delta_{IV}$ (we have arranged that $\Delta_B = 1$ by taking $\delta^*_0$ to be the image of $\gamma_0$ under the admissible embedding $T_0^H \to G^*$ that we are using to construct the factors). Here $\epsilon_v$ is the local $\epsilon$-factor defined in [KS99] §5.3 and the other terms are defined in [LS87] Theorem 6.4.A]. It will thus be enough to show that almost all of the terms

$$\langle z_{sc,v}, y'_{v,0} \rangle^{-1} \langle \text{inv}(\delta_0^*, (G, \psi, z_v, \delta_0)), \hat{s}_v(\gamma_1, 0) \rangle$$

are equal to 1 and that their product is equal to 1. For this, recall that the class $\langle \text{inv}(\delta_0^*, (G, \psi, z_v, \delta_0)), \hat{s}_v(\gamma_1, 0) \rangle \in H^1(u_v \to W_v, Z_{der} \to T_0)$ is represented by the 1-cocycle

$$x_v : W_v \to T_0(\mathbb{F}), \quad w \mapsto g^{-1}z_v(w)\sigma_w(g),$$

where $\sigma_w \in \Gamma_v$ is the image of $w$. The class of $x_v$ does not depend on the choice of $g$ and it will be convenient to assume that $g$ is the image of $g_{sc} \in G_{sc}^*(\mathbb{F})$. Then $x_{sc,v}(w) = y_{sc}^{-1}z_{sc,v}(w)\sigma_w(y_{sc})$ is a lift of $x_v$ to an element of $Z^1(u_v \to W_v, Z_{sc} \to T_{0,sc})$.

On the other hand, $\hat{s}_v(\gamma_1, 0) \in \pi_0(T_0^+=\mathbb{Z}(\mathbb{H})^{+})$ is the image of $\hat{s}_v \in \pi_0(Z(\mathbb{H})^{+})$ under the map $\vec{\varphi}_{\gamma_0,0}^+ : Z(\mathbb{H}) \to \mathbb{Z}_{der} \to \mathbb{Z}_0$ coming from the chosen admissible embedding. We have $T_0^+ = [\mathbb{Z}_{der} \times Z(\mathbb{G})^0$ and by construction $\hat{s}_v(\gamma_0,0) \in \langle \vec{\varphi}_{\gamma_0,0}(z_{sc}, y'_{v,0}) \rangle$. The map $[Z_{sc} \to T_{0,sc}]$ is dualized to the map $[Z_{der} \to T_0]$, and $\mathbb{Z}_{der}$ is given by projection onto the first factor. It follows that

$$\langle z_{sc,v}, y'_{v,0} \rangle^{-1} \langle x_{sc,v}, \vec{\varphi}_{\gamma_0,0}(z_{sc})y'_{v,0} \rangle.$$
$G^*(O_F, S_j)$ and $g \in G^*_\infty(O_E, S_j, E_j)$. Then

$$x_{sc} : \mathcal{E}_{E_j, S_j} \to T_{0,sc}(O_{E_j, S_j, E_j}), \quad e \mapsto g_{sc}^{-1} z_{sc}(e) \sigma_e(g_{sc})$$

is an element of $Z^1(P_{E_j, S_j} \to \mathcal{E}_{E_j, S_j}, Z_{sc} \to T_{0,sc})$, where we have identified $z_{sc} \in Z^1(P_{E_j, S_j} \to \mathcal{E}_{E_j, S_j}, Z_{sc} \to T_{0,sc})$ with its image in $Z^1(P_{E_j, S_j} \to \mathcal{E}_{E_j, S_j}, Z_{sc} \to T_{sc})$ under the inflation map \ref{eq:inflation}. The elements $x_{sc,v}$ and $loc_v(x_{sc})$ of $H^1(u_v \to W_v, Z_{sc} \to T_{0,sc})$ are equal, where $loc_v$ is the localization map \ref{eq:localization}. We now apply Proposition \ref{prop:localization} and Lemma \ref{lem:inflation}. They tell us first that for all $v \in \hat{V}$ that do not lie above $S_j$, we have $loc_v(x_{sc}) = 0$ and hence the corresponding term \ref{eq:product} is equal to 1. Moreover, they tell us that if we restrict each character $\langle loc_v(x_{sc}), - \rangle$ of $\pi_0(T_{0,sc}^*)$ to the group $\pi_0(T_{0,sc}^*)$ and take the product over all $v$, the result is the trivial character. But since the image of $s$ in $Z(\hat{H})/Z(\hat{G})$ is $\Gamma$-fixed, $\hat{\varphi}_{\gamma_0, s^0}(s_{sc})$ belongs to $\pi_0(T_{0,sc}^*)$ and the proof of the product formula is complete.

The statement of the independence of choices is immediate from equation \ref{eq:product}. Indeed, neither $y'_v$ nor $y''_v$ appear in this formula. Moreover, only the cohomology class of $x_{sc}$ and its localizations are involved, but these are independent of the particular inflation and localization diagrams used according to Propositions \ref{prop:inflation} and \ref{prop:localization}. The choice of $T$ is also irrelevant: if we chose another $T'$ and another $h_{ad}' \in H^1(\Gamma, T_{ad}'(\hat{F}))$ mapping to $\Psi$, then $h_{ad}'(\sigma) = d^{-1} h_{ad}(\sigma) \sigma(d)$ for some $d \in G^*$. But then $x_{sc}'(e) = d^{-1} z_{sc}(e) \sigma(d)$ is an element of $Z^1(P_{E_j, S_j} \to \mathcal{E}_{E_j, S_j}, Z_{sc} \to T_{sc})$, where we may potentially have to increase $i$ and use the inflation map. But none of this changes the class of $x_{sc}$.

The remaining two choices that we made – of the element $s_{sc}$ in subsection \ref{sec:choices}, which we may multiply by any element of $\hat{Z}_{sc}$, and of the lift $h_{sc}$ of $h_{ad}$, which we may multiply by any element of $H^1(\mathcal{E}_{E_j, S_j}, Z_{sc})$, – cannot influence the individual factors in Proposition \ref{prop:inflation}. However, the proposition implies that they do not influence the product.

### 4.5 The multiplicity formula for discrete automorphic representations

In this subsection we will recall the conjectural multiplicity formula for tempered discrete automorphic representations due to Kottwitz \cite{Kottwitz:1984} \S 12 and then, assuming the validity of the local conjecture stated in \cite{Kalceff:2011} \S 5.4, we will construct the global pairing that occurs in this formula.

Let $\varphi$ be a generic global Arthur parameter for $G^*$. This can either take the form of an $L$-homomorphism $\varphi : L_F \to ^L G^*$ with bounded image, where $L_F$ is the hypothetical Langlands group of $F$, or it can be a formal global parameter modelled on the cuspidal spectrum of $GL_N$, as in \cite{Arthur:2013} \S 1.4. We will assume here that we are dealing with an $L$-homomorphism, since the arguments needed in the case of formal parameters are independent of the problem we are discussing. At each place $v \in \hat{V}$, the parameter $\varphi$ has a localization, which is a parameter $\varphi_v : L_{F_v} \to ^L G^*$. The validity of the local conjecture ensures the existence of an $L$-packet $\Pi_{\varphi_v}$ of tempered representations of rigid inner twists of $G^*$ together with a bijection

$$\iota_{\varphi_v} : \Pi_{\varphi_v} \to \text{Irr}(S^+_{\varphi_v}).$$
The set $\Pi_\varphi$ consists of equivalence classes of tuples $(G'_v, \psi'_v, z'_v, \pi'_v)$, where $(\psi'_v, z'_v) : G^* \to G'_v$ is a rigid inner twist over $F_v$ and $\pi'_v$ is an irreducible tempered representation of $G'_v(F_v)$. The group $S^+_{\varphi_v}$ is the preimage in $\widehat{G}^*$ of the group $S_{\varphi_v} = \text{Cent}(\varphi_v, \widehat{G}^*)$.

We are interested in the particular group $G$. We choose a family of local rigid inner twists $(\psi, z_v, \pi_v) : G^* \to G$ indexed by $v \in \hat{V}$ as in subsection 10.4. Then the subset of $\Pi_\varphi$ consisting of tuples $(G, \psi, z_v, \pi_v)$ is the $L$-packet of representations of $G(F_v)$ corresponding to $\varphi_v$. We consider the adelic $L$-packet

$$\Pi_\varphi = \{ \pi = \otimes_v \pi_v | (G, \psi, z_v, \pi_v) \in \Pi_\varphi, \ell_{\varphi_v}((G, \psi, z_v, \pi_v)) = 1 \text{ for almost all } v \}.$$ 

It consists of irreducible admissible tempered representations of $G(\mathbb{A})$. It is expected that every tempered discrete automorphic representation of $G(\mathbb{A})$ belongs to such a set $\Pi_\varphi$ for a suitable $\varphi$ that is discrete (elliptic in the language of [Kot84, §10.3]). Moreover, Kottwitz has given a conjectural formula [Kot84 (12.3)] for the multiplicity in the discrete spectrum of $G$ of any $\pi \in \Pi_\varphi$. For this, consider the exact sequence

$$1 \to Z(\widehat{G}^*) \to \widehat{G}^* \to \widehat{G}_{\text{ad}}^* \to 1$$

of complex algebraic groups with action of $L_F$ given by $\text{Ad} \circ \varphi$. It leads to the connecting homomorphism

$$\text{Cent}(\varphi, \widehat{G}_{\text{ad}}^*) \to H^1(L_F, Z(\widehat{G}^*)).$$ 

Let $S^\text{ad}_{\varphi}$ be the subgroup of $\text{Cent}(\varphi, \widehat{G}_{\text{ad}}^*)$ consisting of elements whose image in $H^1(L_F, Z(\widehat{G}^*))$ is locally trivial, i.e., has trivial image in $H^1(L_{F_v}, Z(\widehat{G}^*))$ for every place $v \in \hat{V}$. This group is denoted by $S_{\varphi}/Z(\widehat{G}^*)$ in [Kot84 (10.2.3)]. Let $S_{\varphi} = \pi_0(S^\text{ad}_{\varphi})$. Kottwitz conjectures that there exists a pairing

$$(\cdot, \cdot) : S_{\varphi} \times \Pi_{\varphi} \to \mathbb{C} \quad (4.4)$$

realizing each element of $\Pi_{\varphi}$ as the character of a finite-dimensional representation of $S_{\varphi}$, so that the integer

$$m(\varphi, \pi) = |S_{\varphi}|^{-1} \sum_{\pi \in S_{\varphi}} (x, \pi)$$

gives the $\varphi$-contribution of $\pi$ to the discrete spectrum of $G$. In other words, the multiplicity with which $\pi$ occurs in the discrete spectrum of $G$ should be the sum $\sum_{\pi} m(\varphi, \pi)$, where $\varphi$ runs over the set of equivalence classes [Kot84 §10.4] of discrete generic global Arthur parameters with $\pi \in \Pi_{\varphi}$.

We will now give a construction of (4.4). Let $s_{\text{ad}} \in S^\text{ad}_{\varphi}$ and choose a lift $s_{\text{sc}} \in S^\text{sc}_{\varphi}$, where $S^\text{sc}_{\varphi}$ is the preimage in $\widehat{G}_{\text{sc}}^*$ of $S^\text{ad}_{\varphi}$. We follow the argument of subsection 4.2 to obtain from $s_{\text{sc}}$ an element $s_v \in S^+_{\varphi_v}$ for each $v \in \hat{V}$. Namely, by assumption there exists an element $y_v \in Z(\widehat{G}^*)$ such that $s_{\text{der}} \cdot y_v \in S_{\varphi_v}$ for each $v \in \hat{V}$. Decompose $y_v = y'_v \cdot y''_v$ with $y'_v \in Z(\widehat{G}_{\text{der}}^*)$ and $y''_v \in Z(\widehat{G}^*)$ and choose a lift $y'_v \in Z(\widehat{G}_{\text{sc}}^*)$ of $y'_v$. Then $(s_{\text{sc}} \cdot y'_v, y''_v) \in S^+_{\varphi_v}$. Let us write $(s_{\text{sc}} \cdot y'_v, y''_v, (G, \psi, z_v, \pi_v))$ for the character of the representation $\ell_{\varphi_v}((G, \psi, z_v, \pi_v))$ of $\pi_0(S^+_{\varphi_v})$ evaluated at the element $(s_{\text{sc}} \cdot y'_v, y''_v)$.

**Proposition 4.3.** Almost all factors in the product

$$\langle s_{\text{ad}}, \pi \rangle = \prod_{v \in \hat{V}} \langle z_{\text{sc}, v}, y'_v \rangle^{-1} \langle (s_{\text{sc}} \cdot y'_v, y''_v, (G, \psi, z_v, \pi_v)) \rangle$$

...
are equal to 1 and the product is independent of the choices of \( s_{sc}, y_v', y''_v \), and the collection \( \{ z_{sc,v} \}_v \). The function \( s_{ad} \mapsto (s_{ad}, \pi) \) is the character of a finite-dimensional representation of \( S_F \).

**Proof.** By construction of \( \Pi_F \) we have \( \langle (s_{sc} \cdot y'_v, y''_v), (G, \psi, z_v, \pi_v) \rangle = 1 \) for almost all \( v \). Recall from subsection 4.2 that the class of \( z_{sc,v} \) in \( H^1(u_v \to W_v, Z_{sc} \to G_{sc}^\circ) \) is the image of \( loc_v([z_{sc}]) \in H^1(u_v \to W_v, Z_{sc} \to T_{sc}) \), where \( [z_{sc}] \in H^1(P_{E_i, S_i} \to E_{E_i, S_i}, Z_{sc} \to T_{sc}) \). According to Proposition 3.24 and Lemma 3.33 \( loc_v([z_{sc}]) \) is trivial for all \( v \in V \) that do not lie over \( S_i \).

We have established that almost all factors in the product are equal to 1. Now we will show that each factor is the character of a finite-dimensional representation of the group \( S_{sc}^\circ \). Recall from subsection 4.1 that \( \hat{G}^\circ = \hat{G}_{sc}^\circ \times Z(\hat{G})^\circ \). We have the maps \( Z(\hat{G})^{\circ^+} \to S_{sc}^{\circ^+} \) and \( Z(\hat{G})^{\circ^+} \to Z(\hat{G}_{sc}^\circ) \), the second given by projection onto the first factor of \( Z(\hat{G})^\circ = Z(\hat{G}_{sc}^\circ) \times Z(\hat{G})^\circ \). We claim that we have a well-defined homomorphism

\[
S_{sc}^\circ \to S_{sc}^{\circ^+} \times Z(\hat{G})^{\circ^+}, Z(\hat{G}_{sc}^\circ), \tag{4.5}
\]

sending an element \( s_{sc} \in S_{sc}^\circ \) to the element \([ (s_{sc} \cdot y'_v, y''_v), (y'_v)^{-1}] \). Indeed, the tuple \((y'_v, y''_v)\) can only be replaced by \((\hat{y}'_v, \hat{y}''_v)\), \((\hat{y}'_v, \hat{y}''_v)\) is the image of \( \hat{y}'_v \), \( \hat{y}'_v \) is inflated. Second, if we either choose a different lift \( z_{sc} \) of \( \Psi \) or a different lift \( s_{sc} \) of \( s_{ad} \), then some individual factors in the product might change. However, each such change will simultaneously occur in the same factor of the product in Proposition 4.2 because the two factors are related by the endoscopic character identities [Kal (5.11)], which are part of the local conjecture whose validity we are assuming in our construction. However, we know from Proposition 4.2 that the product over all places equals the adelic transfer factor and thus remains unchanged. This means that the product over all places of the changes in the factors is equal to 1. \( \square \)
4.6 Relationship to Arthur’s framework

In [Art06], Arthur states a version of the local Langlands conjecture and endoscopic transfer for general connected reductive groups, motivated by the stable trace formula. His formulation is different from the one given in [Kal] §5.4. For one, it uses a modification of the classical local $S$-group that is different from ours. Moreover, besides the conjectural local pairings between $L$-packets and $S$-groups, it introduces further conjectural objects – the mediating functions $\rho(\Delta, \tilde{s})$ and the spectral transfer factors $\Delta(\phi', \pi)$. Their purpose is to encode the lack of normalization of the (geometric) transfer factors of Langlands-Shelstad as well as the non-invariance of these transfer factors under automorphisms of local endoscopic data, both of which are problems in local endoscopy that arise in the case of non-quasi-split groups. The local pairing is then supposed to be given as the product of the mediating function and the spectral transfer factor.

In this subsection, we will show that the local conjecture formulated in [Kal] §5.4 implies a stronger version of Arthur’s conjecture of [Art06]. The strengthening is due to the fact that the conjecture in [Kal] §5.4 implies that all objects in Arthur’s formulation are canonically specified, and that furthermore the mediating functions have a simple explicit formula which in fact allows them to be eliminated if so desired. The reason we can achieve this is that the main local problems introduced by non-quasi-split groups – namely the lack of canonical normalization of the Langlands-Shelstad transfer factor and its non-invariance under automorphisms of endoscopic data – were resolved in [Kal] §5.3 based on the cohomology of local Galois gerbes.

We feel that the translation between our statement and that of Arthur, even though not difficult, allows one to combine the strengths of both formulations of the local conjectures. Arthur’s framework makes the application of the local conjecture to global questions somewhat simpler. In particular, it makes the extraneous factors in the product expansions of Propositions 5.2 and 5.3 disappear. On the other hand, our local conjecture is more intrinsic to the group $G$ and makes many local operations, for example descent to Levi subgroups, more transparent. It is also closely related (and in some sense extends to arbitrary local fields) some of the discussion of [ABV92].

We begin by recalling Arthur’s conjecture from [Art06] §3. It is stated for non-archimedean local fields of characteristic zero. We let $F$ be such a field and we let $\Psi : G^* \to G$ be an equivalence class of inner twists with $G^*$ quasi-split. Let $\tau$ be a Whittaker datum for $G^*$ and let $\varphi : L_F \to \mathbb{G}_m$ be a tempered Langlands parameter. Let $S_\varphi = \text{Cent}(\varphi, \hat{G}^*)$ be its centralizer, $S^\text{ad}_\varphi$ the image of $S_\varphi$ in $\hat{G}^*_\text{ad}$, and $S^{\text{sc}}_\varphi$ the preimage of $S^\text{ad}_\varphi$ in $\hat{G}^*_\text{sc}$. From $\Psi$ we obtain an element of $H^1(I, G^*_\text{ad})$ and hence by Kottwitz’s isomorphism [Kot86] §1 a character $\zeta_\Psi : \hat{Z}_\text{sc} \to \mathbb{C}^\times$. Arthur proposes to choose an arbitrary extension $\tilde{\zeta}_\Psi : Z_\text{sc} \to \mathbb{C}^\times$ of this character and to consider the set $\text{Irr}(\pi_0(S^{\text{sc}}_\varphi), \tilde{\zeta}_\Psi)$ of irreducible representations of the finite group $\pi_0(S^{\text{sc}}_\varphi)$ that transform under $\tilde{Z}_\text{sc}$ by $\tilde{\zeta}_\Psi$. Arthur expects this set to be in non-canonical bijection with the $L$-packet $\Pi_\varphi(G)$ of representations of $G(F)$ corresponding to $\varphi$. This non-canonical bijection depends on the choice of a normalization $\Delta$ of the Langlands-Shelstad transfer factor as well as on the (arbitrary) choice of a mediating function $\rho(\Delta, \tilde{s})$. Part of the conjecture is the expectation that these two choices can be made in such a way that $\tilde{s} \mapsto (\tilde{s}, \pi) = \rho(\Delta, \tilde{s})^{-1} \Delta(\phi', \pi)$ is the character of the irreducible representation of $\pi_0(S^{\text{sc}}_\varphi)$ corresponding to $\pi$, see [Art06] §3. The endoscopic character identities are supposed to match the stable character of...
an endoscopic parameter $\varphi'$ with the virtual character of the $L$-packet $\Pi_{\varphi'}(G)$ in which the character of $\pi \in \Pi_{\varphi'}(G)$ is weighted by the scalar $\Delta(\varphi' , \pi)$. For global applications, Arthur formulates the following hypothesis [Art13, Hypothesis 9.5.1]: If $F$ is a number field and at each place $v$ of $F$ one has fixed a normalization $\Delta_v$ of the Langlands-Shelstad transfer factor such that $\prod_v \Delta_v = \Delta$, then the mediating functions $\rho(\Delta_v, s_v)$ can be chosen coherently so as to satisfy the formula $\prod_v \rho(\Delta_v, s) = 1$ whenever $\varphi$ is a global parameter and $s \in S^sc_c$.

We will now discuss how the local conjecture of [Kal] and the global results of the current paper imply a stronger form of Arthur’s local and global expectations. Let $F$ be a number field, and $\Psi : G^* \to G$ an equivalence class of inner twists of connected reductive groups defined over $F$ with $G^*$ quasi-split, and $\omega$ a global Whittaker datum for $G^*$.

To discuss the local conjecture, we focus on a non-archimedean place $v \in \bar{V}$. Fix a lift $\hat{\Psi}_{v,sc} \in H^1(u_v \to W_v, Z_{sc} \to G^*_{sc})$ of the localization $\Psi_v \in H^1(\Gamma_v, G^*_{ad})$ of $\Psi$. It exists by [Kal Corollary 3.8]. The Tate-Nakayama-type duality theorem of [Kal] §4 interprets $\hat{\Psi}_{v,sc}$ as a character $\zeta_{\hat{\Psi}_{v,sc}} : \hat{Z}_{sc} \to \mathbb{C}^\times$ that extends $\zeta_{\Psi}$ and thus naturally provides a version of the character $\zeta_{\Psi}$ in Arthur’s formulation. However, we are now in a considerably stronger position, because besides providing the character $\zeta_{\hat{\Psi}_{v,sc}}$, the cohomology class $\hat{\Psi}_{v,sc}$ also normalizes the Langlands-Shelstad transfer factor and the pairings between $L$-packets and $S$-groups. In fact, since in the non-archimedean case $\hat{\Psi}_{v,sc}$ and $\zeta_{\hat{\Psi}_{v,sc}}$ determine each other [Kal Theorem 4.11 and Proposition 5.3], this implies in Arthur’s language that the choice of $\zeta_{\hat{\Psi}_{v,sc}}$ normalizes the transfer factor and local pairings. Note however that in the archimedean case $\hat{\Psi}_{v,sc}$ cannot be recovered from $\zeta_{\hat{\Psi}_{v,sc}}$ and is thus the more fundamental object.

More precisely, let $\hat{\Psi}_v \in H^1(u_v \to W_v, Z_{\text{der}} \to G^*)$ be the image of $\hat{\Psi}_{v,sc}$, which in turn gives a character $\zeta_{\hat{\Psi}_v} : \pi_0(Z(\hat{G}^*)^{+v}) \to \mathbb{C}^\times$. We recall that since we are working with $Z_{\text{der}}$, we have $\hat{G}^* = \hat{G}^*_{sc} \times Z(\hat{G}^*)^\circ$. The character $\zeta_{\hat{\Psi}_v}$ that we obtain is the pull-back of $\zeta_{\hat{\Psi}_{v,sc}}$ under the map

$$\pi_0(Z(\hat{G}^*)^{+v}) \to \hat{Z}_{sc}$$

given by projection onto the first factor. The conjecture of [Kal] §5.4 then states that given a tempered parameter $\varphi_v : L_{F_v} \to L^G$ there is a canonical bijection

$$\Pi_{\varphi_v}(G) \to \text{Irr}(\pi_0(S^+_{\varphi_v}), \zeta_{\hat{\Psi}_v}),$$

where $\Pi_{\varphi_v}(G)$ is the $L$-packet on $G$ corresponding to $\varphi_v$. This packet is related to the compound $L$-packet $\Pi_{\varphi'_v}$ of [Kal] §5.4 by $\Pi_{\varphi'_v}(G) = \{ \pi | (G, \psi, z_v, \pi) \in \Pi_{\varphi_v} \}$, where $(\psi, z_v) \in \hat{\Psi}_v$ is any representative. Here, as well as below, we find it convenient to think of $\hat{\Psi}_v$ as an equivalence class of rigid inner twists $G^* \to G$.

In order to translate this into Arthur’s formulation we need to relate the two sets $\text{Irr}(\pi_0(S^+_{\varphi_v}), \zeta_{\hat{\Psi}_v})$ and $\text{Irr}(\pi_0(S^sc_{\varphi_v}), \zeta_{\hat{\Psi}_{v,sc}})$. We claim that there is in fact a canonical bijection between these sets. For this, we consider the homomorphism

$$S^sc_{\varphi_v} \to S^+_v \times_{Z(\hat{G}^*)^{+v}} Z(\hat{G}^*_{sc}),$$

defined in the same way as (4.5), the only difference being that the source is $S^sc_{\varphi_v}$ instead of the global $S^sc_{\varphi}$ used there. Namely, to $s_{sc} \in S^sc_{\varphi_v}$ we choose $y \in Z(\hat{G}^*)^{+v}$ such that $s_{sc} = y$.
In order to discuss endoscopic transfer we fix a representative virtual character to the pull-back under \((4.6)\) of \(\rho \otimes \zeta_{\psi,v,sc}\).

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denote it by \( \langle s_{sc}, \pi \rangle \) for short, then the map \( s_{sc} \mapsto \langle s_{sc}, \pi \rangle \) is the character of an irreducible representation of \( S_{sc}^{\varphi} \), and the map \( \pi \mapsto \langle -, \pi \rangle \) is the bijection \( \Pi_{\varphi_v} \to \text{Irr}(S_{sc}^{\varphi_v}, \zeta_{\Psi_v,sc}) \) given by (4.7). Proposition 4.3 implies that for any \( s_{ad} \in S_{ad}^{\varphi} \) the product
\[
\langle s_{ad}, \pi \rangle = \prod_{v \in V} \langle s_{sc}, \pi \rangle \tag{4.9}
\]
is independent of the lift \( s_{sc} \in S_{ad}^{\varphi} \) of \( s_{ad} \) and provides the character of a finite-dimensional representation of \( S_{ad}^{\varphi} \).

Arthur’s mediating function is now given by the simple formula
\[
\rho(\Delta[s_{sc}, v], s_{sc}) = 1,
\]
where \( \Delta[s_{sc}, v] \) is of course the transfer factor corresponding to \( s_{sc} \in S_{sc}^{\varphi_v} \) defined above. This formula, together with equation (4.8), which itself is a consequence of Proposition 4.2, imply the validity of [Art13, Hypothesis 9.5.1].

References

[ABV92] Jeffrey Adams, Dan Barbasch, and David A. Vogan, Jr., *The Langlands classification and irreducible characters for real reductive groups*, Progress in Mathematics, vol. 104, Birkhäuser Boston, Inc., Boston, MA, 1992. MR 1162533 (93j:22001)

[Art89] James Arthur, *Unipotent automorphic representations: conjectures*, Astérisque (1989), no. 171-172, 13–71, Orbites unipotentes et représentations, II. MR 1021499 (91f:22030)

[Art90] ———, *Unipotent automorphic representations: global motivation*, Automorphic forms, Shimura varieties, and \( L \)-functions, Vol. I (Ann Arbor, MI, 1988), Perspect. Math., vol. 10, Academic Press, Boston, MA, 1990, pp. 1–75. MR 1044818 (92a:11059)

[Art06] ———, *A note on \( L \)-packets*, Pure Appl. Math. Q. 2 (2006), no. 1, Special Issue: In honor of John H. Coates. Part 1, 199–217. MR 2217572 (2006k:22014)

[Art13] ———, *The endoscopic classification of representations*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013, Orthogonal and symplectic groups. MR 3135650

[BR94] Don Blasius and Jonathan D. Rogawski, *Zeta functions of Shimura varieties*, Motives (Seattle, WA, 1991), Proc. Sympos. Pure Math., vol. 55, Amer. Math. Soc., Providence, RI, 1994, pp. 525–571. MR 1265563 (95e:11051)

[Kal] Tascho Kaletha, *Rigid inner forms of real and \( p \)-adic groups*, arXiv:1304.3292.

[KMSW] Tascho Kaletha, Alberto Minguez, Sug Woo Shin, and Paul-James White, *Endoscopic classification of representations: Inner forms of unitary groups*, arXiv:1409.3731.

[Kot] Robert E. Kottwitz, *\( B(G) \) for all local and global fields*, arXiv:1401.5728.
[Kot84] ______, Stable trace formula: cuspidal tempered terms, Duke Math. J. 51 (1984), no. 3, 611–650. MR 757954 (85m:11080)

[Kot85] ______, Isocrystals with additional structure, Compositio Math. 56 (1985), no. 2, 201–220. MR 809866 (87i:14040)

[Kot86] ______, Stable trace formula: elliptic singular terms, Math. Ann. 275 (1986), no. 3, 365–399. MR 858284 (88d:22027)

[Kot97] ______, Isocrystals with additional structure. II, Compositio Math. 109 (1997), no. 3, 255–339. MR 1485921 (99e:20061)

[KS99] Robert E. Kottwitz and Diana Shelstad, Foundations of twisted endoscopy, Astérisque (1999), no. 255, vi+190. MR 1687096 (2000k:22024)

[Lan83] R. P. Langlands, Les débuts d’une formule des traces stable, Publications Mathématiques de l’Université Paris VII [Mathematical Publications of the University of Paris VII], vol. 13, Université de Paris VII, U.E.R. de Mathématiques, Paris, 1983. MR 697567 (85d:11058)

[LL79] J.-P. Labesse and R. P. Langlands, L-indistinguishability for SL(2), Canad. J. Math. 31 (1979), no. 4, 726–785. MR 540902 (81b:22017)

[LR87] R. P. Langlands and M. Rapoport, Shimuravarietäten und Gerben, J. Reine Angew. Math. 378 (1987), 113–220. MR 895287 (88i:11036)

[LS87] R. P. Langlands and D. Shelstad, On the definition of transfer factors, Math. Ann. 278 (1987), no. 1-4, 219–271. MR 909227 (89c:11172)

[NSW08] Jürgen Neukirch, Alexander Schmidt, and Kay Wingberg, Cohomology of number fields, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 323, Springer-Verlag, Berlin, 2008. MR 2392026 (2008m:11123)

[PR94] Vladimir Platonov and Andrei Rapinchuk, Algebraic groups and number theory, Pure and Applied Mathematics, vol. 139, Academic Press, Inc., Boston, MA, 1994, Translated from the 1991 Russian original by Rachel Rowen. MR 1278263 (95b:11039)

[Sha90] Freydoon Shahidi, A proof of Langlands’ conjecture on Plancherel measures; complementary series for p-adic groups, Ann. of Math. (2) 132 (1990), no. 2, 273–330. MR 1070599 (91m:11095)

[Tat66] J. Tate, The cohomology groups of tori in finite Galois extensions of number fields, Nagoya Math. J. 27 (1966), 709–719. MR 0207680 (34 #7495)