Theta Series Approximation with Applications to Compute-and-Forward Relaying

Amaro Barreal, David Karpuk, Member, IEEE,
and Camilla Hollanti Member, IEEE

Abstract

Computing the theta series of an arbitrary lattice, and more specifically a related quantity known as the flatness factor, has been recently shown to be important for lattice code design in various wireless communication setups. However, the theta series is in general not known in closed form, excluding a small set of very special lattices. In this article, motivated by the practical applications as well as the mathematical problem itself, a simple approximation of the theta series of a lattice is derived.

In relation to this, maximum-likelihood decoding in the context of compute-and-forward relaying is studied. Following previous work, it is shown that the related metric can exhibit a flat behavior, which can be characterized by the flatness factor of the decoding function. Contrary to common belief, we note that the decoding metric can be rewritten as a sum over a random lattice only when at most two sources are considered. Using a particular matrix decomposition, a link between the random lattice and the code lattice employed at the transmitter is established, which leads to an explicit criterion for code design, in contrast to implicit criteria derived previously. Finally, candidate lattices are examined with respect to the proposed criterion using the derived theta series approximation.

Index Terms

Lattices, Theta Series, Flatness Factor, Compute-and-Forward, Physical Layer Network Coding.

The authors are with the Department of Mathematics and Systems Analysis, Aalto University, Finland (e-mail: firstname.lastname@aalto.fi). Their work is supported by the Academy of Finland under Grants #268364, #276031, #282938, #283262 and #303819, a grant from the Finnish Foundation for Technology Promotion, as well as a grant from the Foundation for Aalto University Science and Technology.

A preliminary version of the result from Theorem 1 was utilized in [1] and presented in IEEE ITNAC 2016.
I. Introduction

Lattices are mathematical objects which have become indispensable for code design in many areas of wireless communications, as many design criteria for reliable performance rely on the discrete and algebraic structure of lattices. Despite their deceptively simple structure, many computational problems related to lattices are extremely challenging, such as the famous shortest vector problem or related closest vector problem. In particular, as the same lattice can be generated by distinct bases, a natural problem is to find a basis consisting of shortest vectors, a problem so hard that cryptographic protocols have been developed around it. Moreover, even enumerating vectors of certain lengths is very difficult. The generating function for the number of elements in a lattice of a given norm is known as the theta series of the lattice. This is an interesting object in its own right, and it is not surprising that it is only known in closed form for a very small set of highly structured lattices.

From an applications perspective, it has been recently shown that code design in various areas of wireless communications can profit from studying the theta series of certain involved lattices, e.g., in the setting of wiretap coset code design or compute-and-forward relaying. The latter is a promising physical layer network coding protocol proposed in the award-winning paper [2], and exploits the natural effects of interference by decoding linear combinations of the transmitted messages at the intermediate relays to achieve high computation rates.

Originally, a relay operating under the compute-and-forward protocol would first scale the received signal before applying a minimum-distance decoder to obtain an estimate of the desired linear combination of the codewords. The decoding error probability for this decoding procedure was studied in [3]. It was later in [4] where maximum-likelihood (ML) decoding at the relay was first considered. An approach to lattice code design for compute-and-forward was simultaneously derived therein, as well as in [5], and the first efficient decoding algorithm was proposed in dimension 1. The subsequent work [6] builds upon those innovative articles and continues the investigation towards efficient decoding algorithms. The fundamental work carried out in [4], [5] is essential for code design considerations, as it introduces the notion of the flatness factor of a lattice and utilizes it to derive an implicit lattice code design criterion. This criterion is indirect in the sense that it relates to an uncontrollable sum of random lattices and not to the code lattices themselves, where the randomness is enabled by the physical channel. It is also noteworthy that
following the work \[4\], the common belief has been that this sum can be rewritten as a sum over elements of a lattice for any number of transmitters. This is, as shown in this article, only the case at most two sources are considered, the case studied empirically in \[4\], \[5\].

The article is structured as follows. We start by recalling the most important results related to lattices in Section II. The concepts of theta series and flatness factor are subsequently introduced in Section III, wherein we derive a simple approximation of the theta series of a lattice. In Section IV, we summarize the compute-and-forward protocol and, following \[4\], \[5\], investigate the behavior of the ML decoding metric in terms of its flatness factor. Adopting certain restrictions, we establish a link between the resulting random lattice and the code lattice, allowing for an explicit lattice code design criterion. Namely, we show that in order to maximize the flatness factor of the random lattice, it suffices to maximize that of the code lattice. We then make use of the derived theta series approximation to investigate different lattices in varying dimensions with respect to the design criterion. The main contributions are the following.

- In Theorem 1 we derive a simple but accurate approximation of the theta series of a lattice. For a fixed dimension, the approximation is merely a rational function.
- The maximum-likelihood decoding framework is slightly generalized in Proposition 4 to allow for more general lattices than in previous work. While the analysis of the function can become more difficult depending on the matrix decomposition used, the decoding procedure can nonetheless be executed by the relay also in this more general setting.
- In Lemma 3 we note that the decoding metric can be rewritten as a sum over elements of a lattice only for two sources, rectifying the common belief that this holds for any number of transmitters.
- Theorem 2 establishes a link between the code lattice and the random lattice involved in the ML-decoding metric, allowing to state an explicit design criterion for the code lattice.
- Finally, various lattices are examined using the explicit design criterion and the derived theta series approximation.

II. LATTICES

This section is dedicated to acquainting the reader with basic concepts in lattice theory. In this article, a vector is labeled in bold, \(\mathbf{v}\), and is always represented as a column vector.
Definition 1. A lattice $\Lambda \subset \mathbb{R}^n$ is a discrete$^1$ subgroup of $\mathbb{R}^n$ with the property that there exists a basis $\{b_1, \ldots, b_t\}$ of $\mathbb{R}^n$ such that

$$\Lambda = \bigoplus_{i=1}^t b_i \mathbb{Z}. \quad (1)$$

We say that $\{b_1, \ldots, b_t\}$ is a $\mathbb{Z}$-basis of $\Lambda$, thus $\Lambda \cong \mathbb{Z}^t \cong \mathbb{Z}^n$. We call $t = \text{rank}(\Lambda) \leq n$ the rank, and $n$ the dimension of $\Lambda$.

A lattice $\Lambda' \subset \mathbb{R}^n$ such that $\Lambda' \subset \Lambda$ is called a sublattice$^2$ of $\Lambda$.

More conveniently, we can define a generator matrix $M_\Lambda := [b_1 \cdots b_t]$, so that every point $x \in \Lambda$ can be expressed as $x = Mz$ for some vector $z \in \mathbb{Z}^n$. Henceforth we will only consider full lattices, that is, where $t = n$.

Remark 1. Given a pair of full lattices $\Lambda_1 \subseteq \Lambda_2$, we will say that $\Lambda_1$ is nested in $\Lambda_2$. We refer to $\Lambda_2$ as the fine lattice, and to $\Lambda_1$ as the coarse lattice. Similarly, a sequence $\Lambda_1, \ldots, \Lambda_s$ of lattices is nested if $\Lambda_1 \subseteq \Lambda_2 \subseteq \cdots \subseteq \Lambda_s$.

Given a full lattice, $\Lambda \subset \mathbb{R}^n$, the $i$th successive minimum of $\Lambda$, for $i = 1, \ldots, n$, is defined as

$$\lambda_i = \lambda_i(\Lambda) := (\inf \{ r | \dim(\text{span}(\Lambda \cap B_0(r))) \geq i \})^2, \quad (2)$$

where $B_0(r)$ is the sphere of radius $r$ centered at the origin. The first minimum, $\lambda_1 = \min_{x \in \Lambda} ||x||^2$, is referred to as the (square) minimal norm of $\Lambda$, which exists due to the discreteness property of the lattice. If all successive minima are equal, $\lambda_1 = \cdots = \lambda_n$, the lattice is called well-rounded.

Consider now a lattice $\Lambda$ with generator matrix $M_\Lambda = [b_i]_{1 \leq i \leq n}$. The fundamental parallelo-tope of $\Lambda$ is defined as

$$P_\Lambda := \left\{ \sum_{i=1}^n b_i z_i \bigg| 0 \leq z_i < 1 \right\}, \quad (3)$$

and we define the volume $\Lambda$ to be the volume of $P_\Lambda$,

$$\text{vol} (\Lambda) := \text{vol} (P_\Lambda) = |\det(M_\Lambda)|. \quad (4)$$

Note that $\text{vol} (\Lambda)$ is independent of the choice of the generator matrix $M_\Lambda$. We can easily compute the volume of a sublattice $\Lambda' \subset \Lambda$ as $\text{vol} (\Lambda') = \text{vol} (\Lambda) |\Lambda/\Lambda'|$.

---

$^1$By discrete we mean that the metric on $\mathbb{R}^n$ defines the discrete topology on $\Lambda$.

$^2$If $\dim(\Lambda) = \dim(\Lambda')$, then the index $|\Lambda/\Lambda'|$ is finite.
A further useful function, not only for coding-theoretic purposes, is a lattice quantizer $Q_\Lambda$, a function that maps every point in $y \in \mathbb{R}^n$ to its closest point in the lattice. This function allows us to define a modulo-lattice operation, $y \ (\text{mod} \ \Lambda) := y - Q_\Lambda(y)$. Given a lattice $\Lambda$ and a lattice quantizer $Q_\Lambda$, we can associate to each lattice point $x \in \Lambda$ its Voronoi cell, the set

$$\mathcal{V}_\Lambda(x) := \{ y \in \mathbb{R}^n | Q_\Lambda(y) = x \}. \quad (5)$$

The Voronoi cell around the origin, $\mathcal{V}(\Lambda) := \mathcal{V}_\Lambda(0)$, is called the basic Voronoi cell of $\Lambda$.

With the above definitions, we can now define the notion of a nested lattice code, an object widely used for code construction in different communications scenarios.

**Definition 2.** Let $\Lambda_C \subset \Lambda_F$ be a pair of nested lattices. We define a nested lattice code $C(\Lambda_C, \Lambda_F)$ as the set of representatives

$$C(\Lambda_C, \Lambda_F) := \{ [x] \in \Lambda_F \ (\text{mod} \ \Lambda_C) | x \in \Lambda_F \} = \Lambda_F \cap \mathcal{V}(\Lambda_C). \quad (6)$$

The code rate of $C(\Lambda_C, \Lambda_F)$ in bits per dimension is

$$R = \frac{1}{n} \log_2 |C(\Lambda_C, \Lambda_F)| = \frac{1}{n} \log_2 \frac{\text{vol}(\Lambda_C)}{\text{vol}(\Lambda_F)} = \frac{1}{n} \log_2 |\Lambda_F/\Lambda_C|. \quad (7)$$

Note that some coset representatives fall on the boundary of $\mathcal{V}(\Lambda_C)$, and need to be selected systematically. We illustrate the introduced concepts in Figure 1 below.

### III. The Theta Series and Flatness Factor of a Lattice

In this section, we introduce the objects of main interest for this article: the theta series, and a related quantity, the flatness factor of a lattice.

**Definition 3.** Let $\Lambda \subset \mathbb{R}^n$ be a full lattice. For each $r \in \mathbb{R}$, define

$$\Omega_\Lambda(r) := \left| \{ x \in \Lambda | \|x\|^2 = r \} \right|. \quad (8)$$

The theta series of $\Lambda$ is the generating function

$$\Theta_\Lambda(q) := 1 + \sum_{r > 0} \Omega_\Lambda(r) q^r = \sum_{x \in \Lambda} q^{\|x\|^2}. \quad (9)$$

**Remark 2.** The theta series converges absolutely if $0 \leq q < 1$. We further note that

$$\arg \min_{r > 0} \{ \Omega_\Lambda(r) > 0 \} = \lambda_1, \quad \text{and} \quad \min_{r > 0} \{ \Omega_\Lambda(r) > 0 \} = \kappa(\Lambda), \quad (10)$$
Fig. 1: Nested lattices \( \Lambda_C \subset \Lambda_F = 3\Lambda_C \) with the Voronoi cells around each lattice point of the coarse (solid) and fine (dashed) lattices. On the left figure we fix \( \Lambda_C = A_2 \), the hexagonal lattice, and on the right figure \( \Lambda_C = \Psi(\mathcal{O}_{\mathbb{Q}(\sqrt{5})}) \), the lattice obtained via the canonical embedding \( \Psi \) of the ring of integers of the algebraic number field \( \mathbb{Q}(\sqrt{5}) \).

The centered Voronoi cell \( V(\Lambda_C) \) (red) contains a set of representatives for a nested lattice code \( \mathcal{C}(\Lambda_C, \Lambda_F) \) of cardinality \( |\mathcal{C}(\Lambda_C, \Lambda_F)| = |\Lambda_F/\Lambda_C| = 9 \).

where \( \kappa(\Lambda) \) is the kissing number of \( \Lambda \). Hence, \( \Theta_\Lambda(q) \) encodes important features of \( \Lambda \).

More generally, the theta series is defined in terms of a complex variable \( q = e^{\pi iz} \), where \( z \in \mathbb{C} \). In this case, \( \Theta_\Lambda(q) \) is a holomorphic function for \( \Im(z) \geq 0 \). For the purposes of this article, however, it suffices to view \( \Theta_\Lambda(q) \) as a formal power series in a real variable \( q \).

Although of great importance, the theta series is unfortunately only known in closed form for a handful of lattices, for example those tabulated in Table I, and is usually given in terms of the Jacobi theta functions

\[
\theta_2(q) = \sum_{k=-\infty}^{\infty} q^{(k+\frac{1}{2})^2}, \quad \theta_3(q) = \sum_{k=-\infty}^{\infty} q^{k^2}, \quad \theta_4(q) = \sum_{k=-\infty}^{\infty} (-q)^{k^2}. \tag{11}
\]

Even so, the Jacobi theta functions are by no means simple functions, but rather hard. The reason for this small set of lattices with known closed form theta series is that efficient counting
of lattice points in domains in arbitrary dimensions is still an open problem. While many results have been obtained over the last two decades, such as the results in [7]–[9], the settings are so general that the upper bounds on the number of lattice points in bounded domains are far from being tight, even for very simple lattices and domains. Thus, being able to efficiently compute even an approximated version of the theta series of an arbitrary lattice is a problem which is interesting in its own right.

As additional motivation, and as we shall see in later parts of this article, recent work on lattice code design in different wireless communication scenarios has led to considering the flatness factor of a lattice, which itself is directly related to the theta series of the lattice.

For convenience, we define the quantity $\Sigma_\Lambda(r) := |\{x \in \Lambda ||x||^2 \leq r\}| = \sum_{0 < i \leq r} \Omega_\Lambda(i)$.

**Theorem 1.** Let $\Lambda \subset \mathbb{R}^n$ be a full lattice with volume $\text{vol}(\Lambda)$ and minimal norm $\lambda_1$. The theta series $\Theta_\Lambda(q)$, where $0 \leq q < 1$, can be expressed as

$$\Theta_\Lambda(q) = (1 - q^{\lambda_1}) - \frac{\log(q)\lambda_1^{\frac{n}{2} + 1}}{\Gamma\left(\frac{n}{2} + 1\right)} \text{vol}(\Lambda) \int_1^\infty t^{\frac{n}{2}} q^{\lambda_1 t} dt + \Xi(\Lambda, n, L, q),$$

where

$$\Xi(\Lambda, n, L, q) = -C(\Lambda, n, L) \log(q)\lambda_1 \int_1^\infty t^{\frac{n-1}{2}} q^{\lambda_1 t} dt.$$  

| Lattice        | Dimension | $\lambda_1$ | $\text{vol}(\Lambda)$ | $\Theta_\Lambda(q)$ |
|----------------|-----------|-------------|------------------------|---------------------|
| $\mathbb{Z}^n$ Integer | $n \geq 1$ | 1           | 1                      | $\theta_n^1(q)$     |
| $D_n$ Checkerboard | $n \geq 3$ | 2           | 2                      | $\frac{1}{2}(\theta_n^3(q) + \theta_n^4(q))$ |
| $A_2$ Hexagonal   | 2         | 1           | $\sqrt{\frac{3}{4}}$ | $\theta_2(q^4)\theta_2(q^3) + \theta_3(q^3)$ |
| $E_8$ Gosset      | 8         | 2           | 1                      | $\frac{1}{2}(\theta_8^3(q) + \theta_8^4(q) + \theta_8^6(q))$ |
| $K_{12}$ Coxeter-Todd | 12       | 4           | 27                    | $\frac{9}{16}\theta_2^5(q^2)\theta_2^5(q^3) + (\theta_2(q^4)\theta_2(q^{12}) + \theta_3(q^4)\theta_3(q^{12}))^6 + \frac{45}{16}\theta_2^4(q)\theta_3^4(q^3)(\theta_2(q^4)\theta_2(q^{12}) + \theta_3(q^4)\theta_3(q^{12}))^2$ |
| $\Lambda_{24}$ Leech | 24       | 4           | 1                      | $\frac{1}{2}(\theta_2^5(q) + \theta_3^5(q) + \theta_4^5(q))^3 - \frac{45}{16}(\theta_2(q^4)\theta_3(q^4)\theta_4(q))^8$ |

**TABLE I:** Various important lattices and their basic attributes.
The constant $C(n, \Lambda, L)$ depends on $n$, $\Lambda$, and a Lipschitz constant $L$.

We will build up the proof using a series of propositions.

**Proposition 1.** Let $\Lambda \subset \mathbb{R}^n$ be a full lattice with minimal norm $\lambda_1$. Then,

$$
\Theta_\Lambda(q) = (1 - q^{\lambda_1}) - \log(q)\lambda_1 \int_1^\infty \sum_\Lambda(\lambda_1 t) q^{\lambda_1 t} dt. \quad (14)
$$

**Proof.** Using the elementary fact $\int_a^\infty q^t dt = -\frac{q^a}{\log(q)}$ for $a \geq 0$, we write

$$
\Theta_\Lambda(q) = \sum_{x \in \Lambda} q^{||x||^2} = \sum_{x \in \Lambda, ||x||^2} \int_0^\infty - \log(q)q^t dt
$$

$$
= -\int_0^\infty \left| \{ x \in \Lambda : ||x||^2 \leq t \} \right| \log(q)q^t dt \quad (15)
$$

$$
= -\int_0^\infty \sum_\Lambda(t) \log(q)q^t dt. \quad (16)
$$

We observe that $\sum_\Lambda(\lambda_1 t) \equiv 1$ for $t \in [0, 1)$, thus by substituting $t \mapsto \lambda_1 t$ and splitting the integration range, we have

$$
\Theta_\Lambda(q) = -\int_0^1 \sum_\Lambda(\lambda_1 t) \log(q)\lambda_1 q^{\lambda_1 t} dt - \int_1^\infty \sum_\Lambda(\lambda_1 t) \log(q)\lambda_1 q^{\lambda_1 t} dt \quad (17)
$$

$$
= (1 - q^{\lambda_1}) - \log(q)\lambda_1 \int_1^\infty \sum_\Lambda(\lambda_1 t) q^{\lambda_1 t} dt. \quad (18)
$$

The next step is to estimate the quantity $\sum_\Lambda(r)$, which counts the number of lattice points in an $n$-sphere of radius $\sqrt{r}$. To that end, we first need the following technical definition and a related lemma.

**Definition 4.** Let $S \subset \mathbb{R}^n$ be a bounded convex set. We say that $S$ is $(n-1)$-Lipschitz parametrizable, $S \in \text{Lip}(n, T, L)$, if there are $T$ maps $\phi_1, \ldots, \phi_T : [0, 1]^{n-1} \rightarrow S$, the union of images of which cover $S$, and satisfying for all $1 \leq i \leq T$ a Lipschitz condition

$$
|\phi_i(x) - \phi_i(y)| \leq L |x - y|. \quad (20)
$$
Lemma 1. \cite[p. 128, Thm. 2]{10} Let $D \subset \mathbb{R}^n$ be such that $\partial D$ is $(n-1)$-Lipschitz parametrizable, that is, $\partial D \in \text{Lip}(n, T, L)$, and let $\Lambda \subset \mathbb{R}^n$ be a full lattice of volume $\text{vol} (\Lambda)$. Then,

$$\left| \left\{ x \mid x \in \Lambda \cap rD \right\} \right| = \frac{\text{vol}(D)}{\text{vol}(\Lambda)} r^n + O(r^{n-1}),$$

where the constant in $O$ depends on $\Lambda$, $n$, and the Lipschitz constant $L$.

Using the above lemma, we can now prove the next result.

Proposition 2. Let $\Lambda \subset \mathbb{R}^n$ be a full lattice with minimal norm $\lambda_1$ and volume $\text{vol}(\Lambda)$. Let $\Sigma_\Lambda(r) := \left| \left\{ x \in \Lambda \mid \|x\|^2 \leq r \right\} \right|$, $r \in \mathbb{R}_{>0}$. Then,

$$\Sigma_\Lambda(\lambda_1 r) = \frac{(\pi \lambda_1 r)^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right)} \frac{\text{vol}(\Lambda)}{\text{vol}(\Lambda)} + C(\Lambda, n, L)r^{\frac{n-1}{2}},$$

where the constant $C(\Lambda, n, L)$ depends on the lattice, dimension, and a Lipschitz constant $L$.

Proof. We use Lemma 1 with $D_{\lambda_1} := B_0(\sqrt{\lambda_1})$, a sphere of radius $\sqrt{\lambda_1}$ centered at the origin. Since $D_{\lambda_1}$ is bounded and convex, by \cite[Thm. 2.6]{7} we have $\partial D_{\lambda_1} \in \text{Lip}(n, 1, L)$.

We can now write

$$\Sigma_\Lambda(\lambda_1 r) = \left| \left\{ x \in \Lambda \mid \|x\|^2 \leq \lambda_1 r \right\} \right| = \left| \left\{ x \in \left( \Lambda \cap B_0 \left( \sqrt{\lambda_1 r} \right) \right) \right\} \right|$$

$$= \left| \left\{ x \in \Lambda \cap \left( \sqrt{r}D_{\lambda_1} \right) \right\} \right|. \quad (23)$$

Using the relation $\text{vol}(D_{\lambda_1}) = \text{vol}(B_0(\sqrt{\lambda_1})) = \frac{(\pi \lambda_1)^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right)}$, we have

$$\Sigma_\Lambda(\lambda_1 r) = \frac{(\pi \lambda_1 r)^{\frac{n}{2}}}{\Gamma \left( \frac{n}{2} + 1 \right)} \frac{\text{vol}(\Lambda)}{\text{vol}(\Lambda)} + C(\Lambda, n, L)r^{\frac{n-1}{2}}, \quad (25)$$

where by Lemma 1, the constant $C$ depends on $\Lambda$, $n$, and the Lipschitz constant $L$. \hfill \square

We can now prove Theorem 1 using the above results.

Proof of Theorem 1. By Proposition 1 we start by writing

$$\Theta_\Lambda(q) = (1 - q^{\lambda_1}) - \log(q)\lambda_1 \int_1^\infty \Sigma_\Lambda(\lambda_1 t)q^{\lambda_1 t} dt. \quad (26)$$
Using the estimate for $\Sigma_{\Lambda}(r)$ derived in Proposition 2, we can now further manipulate the expression to read

$$\Theta_{\Lambda}(q) + q^{\lambda_1} - 1 = -\log(q)\lambda_1 \int_1^{\infty} \Sigma_{\Lambda}(\lambda_1 t)q^{\lambda_1 t} dt$$

(27)

$$= -\log(q)\lambda_1 \int_1^{\infty} \left( \frac{(\pi \lambda_1 t)^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2} + 1\right) \text{vol} (\Lambda)} + C(\Lambda, n, L)t^{\frac{n}{2} - 1} \right) q^{\lambda_1 t} dt$$

(28)

$$= -\frac{\log(q)\pi^{\frac{n}{2}} \lambda_1^{\frac{n}{2} + 1}}{\Gamma\left(\frac{n}{2} + 1\right) \text{vol} (\Lambda)} \int_1^{\infty} t^{\frac{n}{2}} q^{\lambda_1 t} dt - C(\Lambda, n, L) \log(q)\lambda_1 \int_1^{\infty} t^{\frac{n}{2} - 1} q^{\lambda_1 t} dt$$

(29)

$$= -\frac{\log(q)\pi^{\frac{n}{2}} \lambda_1^{\frac{n}{2} + 1}}{\Gamma\left(\frac{n}{2} + 1\right) \text{vol} (\Lambda)} \int_1^{\infty} t^{\frac{n}{2}} q^{\lambda_1 t} dt + \Xi(\Lambda, n, L, q).$$

(30)

We will henceforth write $\Theta_{\Lambda}^{q}(q)$ for the approximation $\Theta_{\Lambda}(q) - \Xi(\Lambda, n, L, q)$. The following corollary will be of use later.

**Corollary 1.** Let $\sigma^2 \in \mathbb{R}_{>0}$, and $q(\sigma^2) := e^{-\frac{1}{2}\sigma^2}$. Then, as a function of $\sigma^2$, we have

$$\Theta_{\Lambda}^{q}(q(\sigma^2)) = \left(1 - e^{-\frac{\lambda_1}{2\sigma^2}}\right) + \frac{(\lambda_1 \pi)^{\frac{n}{2}} \lambda_1}{2\sigma^2 \Gamma\left(\frac{n}{2} + 1\right) \text{vol} (\Lambda)} \int_1^{\infty} t^{\frac{n}{2}} e^{-\frac{\lambda_1 t}{2\sigma^2}} dt.$$  

(31)

**Remark 3.** While the derived approximation might look involved, for a fixed dimension $n$ it can be computed in closed form as a simple rational function, e.g.,

$$\Theta_{\Lambda}^{q}(q) = \begin{cases} 
1 - q^{\lambda_1} + \frac{q^{\lambda_1} \text{vol}(\Lambda)(\lambda_1 \log(q)-1)}{\log(q)} & \text{if } n = 2, \\
1 - q^{\lambda_1} + \frac{q^{\lambda_1} \pi \text{vol}(\Lambda)(\lambda_1 \log(q)+1)(\lambda_1 \log(q)-2)+2}{2\log(q)^2} & \text{if } n = 4,
\end{cases}$$

(32)

and thus effectively evaluated.

**A. On the Accuracy of the Approximation**

We first depict the accuracy of the approximation $\Theta_{\Lambda}^{q}(q)$ for some of the well-known lattices tabulated in Table I. We choose $q = e^{-\frac{1}{2\sigma^2}}$ and interpret $\Theta_{\Lambda}^{q}(e^{-\frac{1}{2\sigma^2}})$ as a function in the variable $\sigma^2$. The choice of this specific indeterminate $q$ will be clarified in the subsequent sections of this article.
Fig. 2: Comparison of the theta function of various lattices and the derived approximation. The left picture depicts the theta series of low-dimensional, the right picture higher-dimensional lattices as a function of $\sigma^2$.

From Figure 2 it is visible that the approximation is accurate in the considered cases, even as the dimension increases. An additional common but naive way of approximating the theta series is by simply considering the first term in the power series expression, that is, $\Theta_\Lambda(q) \approx 1 + \kappa(\Lambda)q^{\lambda_1}$. In Figure 3 below, we compare the derived approximation $\Theta^A_\Lambda(q)$ with this truncated sum on the Leech lattice $\Lambda_{24}$. While our approximation accurately approximates the theta series $\Theta_{\Lambda_{24}}(q)$, the truncated sum very quickly diverges from the actual function.

Fig. 3: Comparison of $\Theta^A_\Lambda(q)$ and a truncated sum $1 + \kappa(\Lambda)q^{\lambda_1}$ of the Leech lattice.

The error term in the expression from Theorem $\mathbb{1}$ arises from the estimation of lattice points in an $n$-sphere, i.e., the estimation of $\Sigma_\Lambda(r)$. The original proof of Lemma 4 in [10] is not
constructive, and does not offer any insight into the involved constant. Accurately counting lattice points in more general domains is a topic of the utmost interest lattice theory. In [9], an upper bound on the quantity $|\Lambda \cap P|$, where $\Lambda \subset \mathbb{R}^n$ is a full lattice, and $P \subset \mathbb{R}^n$ an arbitrary polytope of dimension $n' \leq n$, is given. Further, [7] gives an upper bound on $|\Lambda \cap S|$, where $S \subset \mathbb{R}^n$ is a bounded domain, of general narrow class $s \geq 1$. Both mentioned results are however so general, that the upper bounds are not tight, even for low-dimensional, well-conditioned lattices. Hence, accurately estimating the error term involved in our approximation is currently out of reach. However, note that while the approximation can be worse for strongly skewed lattices, due to Theorem 2 proved later in this article, we will not need to deal with lattices that are not well conditioned. Consequently, our approximation will suffice.

**B. The Flatness Factor**

Having introduced the theta series $\Theta_\Lambda(q)$ of a lattice, we now define a related quantity – the flatness factor $\varepsilon_\Lambda(q)$ of $\Lambda$. Consider the usual $n$-dimensional zero-mean Gaussian PDF with variance $\sigma^2$, given by

$$f(t, \sigma^2) = \frac{1}{(\sqrt{2\pi}\sigma^2)^n} e^{-\frac{|t|^2}{2\sigma^2}}.$$  \hfill (33)

We are interested in the case where the variable $t$ ranges over points over a (possibly shifted) full lattice $\Lambda$, yielding for $y \in \mathbb{R}^n$ the sum of Gaussian functions

$$f(\Lambda + y, \sigma^2) := \sum_{x \in \Lambda} f(x + y, \sigma^2).$$  \hfill (34)

As a function of $y$, $f(\Lambda + y, \sigma^2)$ is a $\Lambda$-periodic function, and defines a PDF on the basic Voronoi cell $\mathcal{V}(\Lambda)$ of $\Lambda$, which we refer to as the lattice Gaussian PDF. For the centered sum $f(\Lambda, \sigma^2)$, we have the useful identity

$$f(\Lambda, \sigma^2) = \sum_{x \in \Lambda} f(x, \sigma^2) = \frac{1}{(\sqrt{2\pi}\sigma^2)^n} \sum_{x \in \Lambda} e^{-\frac{|x|^2}{2\sigma^2}}$$  \hfill (35)

$$= \frac{1}{(\sqrt{2\pi}\sigma^2)^n} \Theta_\Lambda \left( e^{-\frac{1}{2\sigma^2}} \right).$$  \hfill (36)

Introduced in [11] as an information theoretic tool in the context of fading wiretap channels, the flatness factor is a quantity which measures the deviation of the lattice Gaussian PDF from the uniform distribution on the Voronoi cell $\mathcal{V}(\Lambda)$. Formally, it can be defined as follows.
Definition 5. Let \( \Lambda \subset \mathbb{R}^n \) be a full lattice, and for \( y \in \mathbb{R}^n \), let \( f(\Lambda + y, \sigma^2) \) denote the lattice Gaussian PDF of the lattice \( \Lambda + y \). The flatness factor of \( \Lambda \) is defined as

\[
\varepsilon_\Lambda(\sigma^2) := \max_{y \in \mathbb{R}^n} \left| \frac{f(\Lambda + y, \sigma^2)}{1/\text{vol}(\Lambda)} - 1 \right|.
\]  

(37)

It is easy to show (see [12]) that the maximum of \( f(\Lambda + y, \sigma^2) \) is achieved for \( y \in \Lambda \). Hence, an explicit representation of \( \varepsilon_\Lambda(\sigma^2) \) is immediate,

\[
\varepsilon_\Lambda(\sigma^2) = \frac{\text{vol}(\Lambda)}{(\sqrt{2\pi}\sigma^2)^n} \Theta_\Lambda \left( e^{-\frac{1}{2\sigma^2}} \right) - 1.
\]  

(38)

If we define the volume-to-noise ratio (VNR) \( \gamma_\Lambda(\sigma^2) := \frac{\text{vol}(\Lambda)}{2\pi\sigma^2} \), then we can equivalently express the flatness factor as [4]

\[
\varepsilon_\Lambda(\sigma^2) = \gamma_\Lambda(\sigma^2) \frac{1}{2\pi} \Theta_\Lambda \left( e^{-\frac{1}{2\sigma^2}} \right) - 1.
\]  

(39)

From the definition of the flatness factor, it is clear that a small flatness factor implies a more uniform distribution.

IV. Theta Series and the Compute and Forward Relaying Strategy

In this section, we consider a protocol known as compute-and-forward relaying [2]. Analyzing the maximum-likelihood (ML) metric in this context, we show how the flatness factor of a certain lattice enters the picture [4], and relate this random lattice to the code lattice at the transmitter. We then utilize the derived theta series approximation to analyze the performance of various lattices with respect to an explicit design criterion. Namely, we show that in order to maximize the flatness factor of the random lattice, it suffices to maximize that of the code lattice.

Assume that \( K > 1 \) transmitters want to communicate to a single destination, aided by \( M \) intermediate relays which, operating under the original compute-and-forward strategy attempt to decode an integer linear combination of the transmitted messages. We assume that each user, relay, and destination is equipped with one antenna only. The model is depicted in Figure 4.

\( ^3 \) The VNR is usually defined without the term \( 2\pi \) in the denominator. Here, the definition is chosen to agree with [4].

\( ^4 \) As shown in [2], a complex channel output can be treated as two separate real equations.
Fig. 4: System model with \( K > 1 \) transmitters and \( M > K \) relays connected to a destination.

The first hop from the transmitters to the relays is modeled as a Gaussian fading channel, while it is usually assumed that the relays are connected to a destination with error-free bit pipes with unlimited capacities. We will henceforth focus on the first hop.

The sources want to communicate messages \( w_k \in \mathbb{F}_p^s \) to the destination, which are encoded into \( n \)-dimensional codewords \( x_k \in \Lambda_{k,F} \subset \mathbb{R}^n \) before transmission. Here, \( \Lambda_{k,F} \) is a full rank lattice employed by transmitter \( k \), acting as the fine lattice in the nested code \( C_k(\Lambda_C, \Lambda_{k,F}) = \{ [x] \in \Lambda_{k,F} (\mod \Lambda_C) \mid x \in \Lambda_{k,F} \} \). We impose the usual symmetric power constraint \( \frac{1}{n} E[||x_k||^2] \leq P \) for all \( k \). We can interpret the coarse lattice \( \Lambda_C \) as the structure imposing the power constraint on the codewords, which allows us to ignore the specific definition of \( \Lambda_C \) in the remainder of this section.

The observed signal at relay \( m \) can be expressed as

\[
y_m = \sum_{k=1}^{K} h_{mk} x_k + n_m, \tag{40}
\]

where \( n_m \) is additive white Gaussian noise with variance \( \sigma^2 \), and the channel coefficients are i.i.d. with normalized unit variance \( \sigma_h^2 = 1 \). Here, the signal-to-noise ratio (SNR) is \( \rho = P/\sigma^2 \).

Channel state information is only available at the relays; more specifically, each relay only knows the channel \( h_{m}^t = (h_{m1}, \ldots, h_{mK}) \) to itself. Operating under the original compute-and-forward protocol, a fixed relay selects a scalar \( \alpha_m \in \mathbb{R} \), as well as an integer vector \( a_m^t = (a_{m1}, \ldots, a_{mK}) \), and attempts to decode a linear combination of the received codewords with
coefficients $a_{mk}$. The channel output is modified to read

$$\alpha_m y_m = \sum_{k=1}^{K} a_{mk} x_k + \sum_{k=1}^{K} (\alpha_m h_{mk} - a_{mk}) x_k + \alpha_m n_m. \quad (41)$$

The so-called effective noise $n_{ef} := \sum_{k=1}^{K} (\alpha_m h_{mk} - a_{mk}) x_k + \alpha_m n_m$ is no longer Gaussian.

Upon observing the faded superposition of transmitted codewords, each relay proceeds in the same fashion in order to decode a linear combination. We can hence focus on a single relay and, for ease of notation, drop the subscript $m$ henceforth. The focused system model, now resembling a $K$-user multiple-access channel, is illustrated in Figure 5.

![Fig. 5: System model focused on the first hop, with $K > 1$ transmitters and a fixed relay.](image)

An important performance metric of the compute-and-forward protocol is the so-called computation rate. If $R_M(k) = \frac{s}{n} \log p$ denotes the message rate at transmitter $k$, then the relay is able to decode a linear combination involving the codewords whose corresponding message rates are smaller than the computation rate $R_C(h, a)$ achieved by the relay, that is, which satisfy $R_M \leq R_C(h, a)$. The main results on the computation rate are shortly summarized below.

**Lemma 2.** [2], [13] For a relay employing the original compute-and-forward strategy under a real-valued channel model, the computation rate region is maximized by choosing $\alpha$ as the MMSE estimate

$$\alpha_{MMSE} = \frac{\rho h' a}{1 + \rho \|h\|^2}, \quad (42)$$

resulting in the computation rate region

$$R_C(h, a) = \frac{1}{2} \log^+ \left( \left( \frac{\|a\|^2 - \rho (h'a)^2}{1 + \rho \|h\|^2} \right)^{-1} \right). \quad (43)$$
Moreover, the optimal coefficient vector is the solution to the minimization problem

$$a_{\text{opt}} = \arg\min_{a \in \mathbb{Z}^K \setminus \{0\}} a^t G a,$$  \hspace{1cm} (44)

where $G = I_K - \frac{\rho h h^t}{1 + \rho \|h\|^2}$. Hence, $a_{\text{opt}}$ corresponds to the coefficient vector of the shortest vector in the lattice with Gram matrix $G$.

**Remark 4.** The lattice shortest vector problem is in general a computationally hard problem. However, it has been shown recently that in certain instances in the context of compute-and-forward, e.g., for solving (44), it can be solved in polynomial time \[14\].

A low-complexity approach assuming no cooperation between the relays has also been proposed in \[15\].

### A. Decoding Linear Equations

For each $k$, let $C_k := C_k(\Lambda_C, \Lambda_{k,F})$ denote the nested lattice code employed by transmitter $k$. Assume that the fine lattices, possibly after reordering the indexes, are nested, $\Lambda_{1,F} \supseteq \Lambda_{2,F} \supseteq \cdots \supseteq \Lambda_{K,F}$. Since the codebook is finite for each transmitter, the codewords can be assumed to be equiprobable in $C_k$.

A relay attempts to decode $y = \sum_{k=1}^K h_k x_k + n$ to a lattice point $[\lambda] = \sum_{k=1}^K a_k x_k \pmod{\Lambda_C}$ in two steps:

i) Scale the received signal by a scalar $\alpha$, compute an equation coefficient vector $a^t = (a_1, \ldots, a_K)$ by solving the shortest vector problem (44), and decode an estimate $\hat{\lambda}$ of

$$\lambda = \sum_{k=1}^K a_k x_k \in \Lambda_F := \sum_{k=1}^K a_k \Lambda_{k,F}. \hspace{1cm} (45)$$

ii) Apply the modulo-lattice operation to shift the received signal back into $\mathcal{V}(\Lambda_C)$,

$$[\lambda] = \lambda \pmod{\Lambda_C}. \hspace{1cm} (46)$$

The requirement $\Lambda_{1,F} \supseteq \Lambda_{2,F} \cdots \supseteq \Lambda_{K,F}$ guarantees\footnote{Note that nesting is not necessary, but sufficient; more generally, it suffices to fix a common superlattice for all transmitters. We adopt the nested assumption to be consistent with \[2\].} that

$$\Lambda_F = \sum_{k=1}^K a_k \Lambda_{k,F}. \hspace{1cm} (47)$$
is a lattice. The crucial step is the first one, estimating \( \hat{\lambda} \in \Lambda_F \). Originally, a nearest neighbor decoder is used for this estimation. As this method is only optimal at high SNR, we employ ML decoding at the relay instead.

### B. The ML Decoding Metric

Let \( \Lambda_F = \sum_{k=1}^{K} a_k \Lambda_k \) be the lattice defined above. By the imposed norm constraint on the codewords, the desired lattice point \( \lambda = \sum_{k=1}^{K} a_k x_k \) is contained in a finite subset \( L_F \subseteq \Lambda_F \), which is determined by the norm restriction of the original codewords as well as the coefficient vector \( a \). Thus, a relay can restrict its search space to \( L_F \). We make this more precise in the following straightforward proposition.

**Proposition 3.** For a fixed coefficient vector \( a^t = (a_1, \ldots, a_K) \), the lattice point \( \lambda \) is contained in the set

\[
L_F = \left\{ \lambda \in \Lambda_{k_{\min},F} \left| \|\lambda\| \leq \sum_{k=1}^{K} |a_k| \max_{x \in C_{k_{\min}}} \{\|x\|\} \right. \right\},
\]

where \( k_{\min} := \arg \min_{1 \leq k \leq K} \{a_k \neq 0\} \).

**Proof.** By definition, \( k_{\min} \) is the index of the first non-zero entry in the coefficient vector \( a \), hence, the index of the first codeword to be included in the targeted linear combination. As \( \Lambda_{1,F} \supseteq \cdots \supseteq \Lambda_{K,F} \), we have that \( x_k \in C_{k_{\min}} \) for all \( k \geq k_{\min} \). Consequently, each of the codewords involved in the linear combination satisfies \( \|x_k\| \leq \max_{x \in C_{k_{\min}}} \{\|x\|\} \). We conclude

\[
\|\lambda\| = \sum_{k=1}^{K} a_k \|x_k\| \leq \sum_{k=1}^{K} |a_k| \|x_k\| \leq \sum_{k=1}^{K} |a_k| \max_{x \in C_{k_{\min}}} \{\|x\|\}.
\]

In this context, ML decoding amounts to maximizing the conditional probability

\[
\hat{\lambda} = \arg \max_{\lambda \in L_F} \mathbb{P} [\alpha y \mid \lambda] = \arg \max_{\lambda \in L_F} \sum_{(x_1, \ldots, x_K) \in (C_1, \ldots, C_K)} \mathbb{P} [\alpha y \mid (x_1, \ldots, x_K)] \mathbb{P} [(x_1, \ldots, x_K) \mid \sum_{k=1}^{K} a_k x_k = \lambda]
\]

(50)
The former factor in the above expression behaves as

$$P[\alpha y \mid (x_1, \ldots, x_K)] \propto \exp \left\{ -\frac{1}{2\sigma^2 \alpha^2} \left\| \alpha y - \sum_{k=1}^{K} \alpha h_k x_k \right\|^2 \right\}.$$  \hfill (51)

Note that this is independent of $\alpha$. We define the function

$$\varphi(\lambda) := \sum_{(x_1, \ldots, x_K) \in (C_1, \ldots, C_K), \sum_{k=1}^{K} a_k x_k = \lambda} \exp \left\{ -\frac{1}{2\sigma^2} \left\| y - \sum_{k=1}^{K} h_k x_k \right\|^2 \right\},$$  \hfill (52)

and using the assumption that the codewords are equiprobable in $(C_1, \ldots, C_K)$, we conclude that the estimate $\hat{\lambda}$ of $\lambda$ can be computed by solving

$$\hat{\lambda} = \arg \max_{\lambda \in L_F} \varphi(\lambda).$$  \hfill (53)

**Remark 5.** We are not proposing a decoding algorithm, but rather elucidating the behavior of the decoding metric and deriving a code design criterion. It has been shown in [5] that in dimension $n = 1$ decoding based on Diophantine approximation is optimal, and in the same article it was conjectured to be optimal for $n \geq 2$ as well. However, how to treat simultaneous Diophantine equations is a mathematically open problem, which would be needed for implementing the Diophantine decoder in higher dimensions. While other optimal decoding methods may be derived, related work, such as [6] have to date only proposed efficient decoding algorithms in arbitrary dimensions for Gaussian channels.

Our goal in the remainder of this section is to study the behavior of $\varphi(\lambda)$. To analyze the decoding metric, we first need to express the function $\varphi(\lambda)$ in terms of the lattice point $\lambda$. This is achieved in the following proposition, whose proof we include as important quantities will be defined within. We follow a similar procedure described in [4], [5], but in more generality.

**Proposition 4.** Let $\varphi(\lambda)$ be the decoding metric defined in (52). Then, $\varphi(\lambda)$ can be expressed in terms of the lattice point $\lambda$ as

$$\varphi(\lambda) = \sum_{t \in S \subset \mathbb{Z}^{nK}} \exp \left\{ -\frac{1}{2\sigma^2} \left\| \omega(\lambda) - M_L \hat{U} \hat{t} \right\|^2 \right\},$$  \hfill (54)

where $S \subset \mathbb{Z}^{nK}$ is finite, $\omega(\lambda)$ is explicitly given in terms of $\lambda$, $\hat{U} \in \text{Mat}(n(K-1) \times nK, \mathbb{R})$ and $M_L \in \text{Mat}(n \times n(K-1), \mathbb{R})$. 
Proof. For each transmitter $1 \leq k \leq K$, let $M_k \in \text{Mat}(n, \mathbb{R})$ denote the generator matrix of $\Lambda_{k,F}$, and write $x_k = M_k z_k$ for some $z_k \in \mathbb{Z}^n$. We define the matrix $M := [a_1 M_1 \ldots a_K M_K] \in \text{Mat}(n \times nK, \mathbb{R})$, where $a^t = (a_1, \ldots, a_K)$ is the solution to (44), and express $\lambda$ as

$$\lambda = \sum_{k=1}^{K} a_k x_k = \sum_{k=1}^{K} a_k M_k z_k = \begin{bmatrix} a_1 & M_1 & \cdots & a_K & M_K \end{bmatrix} = M z. \quad (55)$$

Let now $U \in \text{GL}_{nK}(\mathbb{R})$ be an invertible matrix such that

$$\hat{B} := MU = \begin{bmatrix} 0_{n \times n(K-1)} & B \end{bmatrix}, \quad (56)$$

where $B \in \text{Mat}(n, \mathbb{R})$ is invertible. We proceed by decomposing the matrix $U$ into blocks $V_i \in \text{Mat}(n, \mathbb{R})$ and $U_i \in \text{Mat}(n \times n(K-1), \mathbb{R})$, as

$$U = \begin{bmatrix} U_1 & V_1 \\ \vdots & \vdots \\ U_K & V_K \end{bmatrix}. \quad (57)$$

Let now $\tilde{r} := U^{-1} z = (r^t, r_n^t)^t$, where $r_n$ denotes the last $n$ components of $\tilde{r}$, and write

$$\lambda = Mz = \hat{B}U^{-1}z = \hat{B}\tilde{r} = Br_n. \quad (58)$$

Note that $r_n = B^{-1} \lambda$. To describe $r$, the first $n(K-1)$ components of $\tilde{r}$, let $\hat{U}$ be composed of the first $n(K-1)$ rows of $U^{-1}$. Then $r = \hat{U} z$. We can now write

$$z = U\tilde{r} = \begin{bmatrix} U_1 & V_1 \\ \vdots & \vdots \\ U_K & V_K \end{bmatrix} \begin{bmatrix} r \\ B^{-1} \lambda \end{bmatrix} = \begin{bmatrix} U_1 r + V_1 B^{-1} \lambda \\ \vdots \\ U_K r + V_K B^{-1} \lambda \end{bmatrix} = \begin{bmatrix} z_1 \\ \vdots \\ z_K \end{bmatrix}, \quad (59)$$

and consequently rewrite the codewords $x_k$ in terms of $\lambda$ as

$$x_k = M_k z_k = M_k U_k r + M_k V_k B^{-1} \lambda = M_k U_k (\hat{U} z) + \mu_k(\lambda), \quad (60)$$

where $\mu_k(\lambda) := M_k V_k B^{-1} \lambda$ is explicitly given in terms of $\lambda$. For $\nu_k := M_k U_k (\hat{U} z)$, the exponent of $\varphi(\lambda)$ now takes the form

$\mu_k(\lambda)$.

$^6$We will later choose a specific decomposition. However, any decomposition of this form suffices for decoding purposes.
\[ \|y - \sum_{k=1}^{K} h_k x_k\|^2 = \|y - \sum_{k=1}^{K} h_k (\mu_k(\lambda) + \nu_k)\|^2 = \left\| \left( y - \sum_{k=1}^{K} h_k \mu_k(\lambda) \right) - \sum_{k=1}^{K} h_k \nu_k \right\|^2. \] (61)

To further simplify the expression, define the matrix
\[ M_L := \sum_{k=1}^{K} h_k M_k U_k, \] (62)
which allows us to rewrite \( \varphi(\lambda) \) explicitly in terms of \( \lambda \) as
\[ \varphi(\lambda) = \sum_{t \in S \subset \mathbb{Z}^{nK}} \exp \left\{ -\frac{1}{2\sigma^2} \left\| \omega(\lambda) - M_L \hat{U} t \right\|^2 \right\}. \] (63)

Here \( S \subset \mathbb{Z}^{nK} \) is finite and we have defined \( \omega(\lambda) := y - \sum_{k=1}^{K} h_k \mu_k(\lambda) \).

We state a lemma related to the structure defined by the matrix \( M_L \) for future reference, and quickly discuss the consequences.

**Lemma 3.** Let \( M_L = \sum_{k=1}^{K} h_k M_k U_k \) be the matrix defined in (62). Then \( M_L \) defines a subgroup \( \mathcal{L} \) of \( \mathbb{R}^n \) of rank \( n(K-1) \), which can only be discrete for \( K = 2 \). Hence, for \( K \geq 3 \), \( \mathcal{L} \) is not a lattice almost surely, i.e., with probability one.

**Remark 6.** We remark that the authors in [4], [5] are not aiming at analyzing the behavior of \( \varphi(\lambda) \) for actual resulting lattice sums \( \mathcal{L} \). The structure of \( \mathcal{L} \) has only been studied in the case \( K = 2 \), and consequently, \( \mathcal{L} \) has been commonly believed to be a lattice for any number of transmitters. By Lemma 3 \( \mathcal{L} \) is a lattice for \( K = 2 \), but lacks a discrete structure when \( K > 2 \).

The main problem is the effect of the random channel coefficients \( h_k \) and, as an important implication, the function \( \varphi(\lambda) \) does not converge if the sum ranges over all of \( \mathcal{L} \). This fact has dramatic consequences, as it implies that the tools developed in [4] for analyzing the behavior of \( \mathcal{L} \) can only be applied in the case \( K = 2 \).

In general \( \mathcal{L} = \sum_{i=1}^{K-1} \mathcal{L}_i \) is a sum of \((K-1)\) lattices, i.e., consists of vectors of the form \( \mathbf{q} = \sum_{i=1}^{K-1} \mathbf{q}_i \), where \( \mathbf{q}_i \in \mathcal{L}_i \). An example of a finite subset \( \overline{\mathcal{L}} \subset \mathcal{L} \) for a sum of 2 lattices \( \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \) (\( K = 3 \)) for \( n = 2 \) and a fixed channel vector is depicted in Figure 6 for illustrative purposes.
Fig. 6: Sum of \((K-1) = 2\) lattices \(\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2\) in dimension \(n = 2\). The depicted points correspond to coefficient vectors \(z \in [-p, p]^4\) with \(p = 1\), and the density increases rapidly as \(p\) grows. The employed code lattices are \(\mathbb{Z}^2\) on the left, and \(\Psi \left( \mathcal{O}_{\sqrt{5}} \right)\) on the right figure.

In the proof of Proposition 4, we assumed the existence of a matrix \(U \in \text{GL}_{nK}(\mathbb{R})\) which yields the desired decomposition (56). For a general matrix \(U \in \text{GL}_{nK}(\mathbb{R})\), its inverse is a matrix with coefficients in \(\mathbb{R}\), and hence \(r = \hat{U}z\) is not an integer vector. Thus, \(M_L \hat{U}z\) cannot be interpreted as an element of the lattice sum \(\mathcal{L}\).

In [4]–[6], the authors propose a decomposition based on the Hermite normal form (HNF) of \(M\). While the use of this specific decomposition has certain disadvantages, for example it only allows to consider integer lattices at the transmitter, it also allows to further simplify the decoding expression. Using the HNF, the matrix \(U\) is unimodular, i.e., \(U \in \text{GL}_{nK}(\mathbb{Z})\). In this special situation, we have that \(\hat{U} \in \text{Mat}(n(K-1) \times nK, \mathbb{Z})\), and consequently \(r = \hat{U}z \in \mathbb{Z}^{n(K-1)}\).

This allows to further simplify the ML decoding decision (63) to read

\[
\hat{\lambda} = \arg \max_{\lambda \in \mathcal{L}} \sum_{q \in \mathbb{Z}} \exp \left\{ \frac{1}{2\sigma^2} ||\omega(\lambda) - q||^2 \right\},
\]

(64)

where \(q = M_Lz\) with \(z \in \mathbb{Z}^{n(K-1)}\) ranges over a finite subset \(\overline{\mathcal{L}} \subset \mathcal{L}\).

Nonetheless, any decomposition yielding a matrix in the form (56) allows for ML decoding at the relay.
C. The behavior of $\varphi(\lambda)$

We move on to analyze the behavior of the function

$$\varphi(\lambda) = \sum_{t \in S \subset \mathbb{Z}^n K} \exp \left\{ \frac{1}{2\sigma^2} \left\| \omega(\lambda) - M_L \hat{U} t \right\|^2 \right\},$$

(65)

which, as indicated in [4], can be flat for certain parameters leading to ambiguous decoding decisions and ultimately resulting in decoding errors. We begin by illustrating the behavior of $\varphi(\lambda)$ in Figure 7 for dimensions $n = 1$ and $2$. In order to show that the flatness behavior of $\varphi(\lambda)$ prevails when using a decomposition other than the HNF, as well as when employing non-integer lattices, we use the $LQ$-decomposition of $M$. Here $M = LQ$, where $L$ is lower triangular and $Q$ unitary, and we choose $U := Q$, cf. [56].

Fig. 7: Behavior of $\varphi(\lambda)$ for $K = 2$ transmitters in dimension $n = 1$ (top) with $\Lambda = \mathbb{Z}$, and $n = 2$ (bottom) with $\Lambda = \Psi \left( O_{Q(\sqrt{5})} \right)$.

In order to decode the lattice point $\lambda$, the relay needs to solve the maximization problem (65). We adopt two necessary restrictions.
i) The definition of the flatness factor involves the volume of the considered lattice. Hence, the analysis of \( \varphi(\lambda) \) in terms of this quantity only makes sense when the volume \( \text{vol}(L) \) is defined. We thus require that \( L \) is a lattice, i.e., \( K = 2 \) (cf. Lemma 3).

ii) Secondly, while any decomposition yielding the desired form allows the relay to solve (63), the matrix \( \hat{U} \) may not be an integer matrix. The fractional part \( \text{frac}(\hat{U}t) = \hat{U}t - \text{int}(\hat{U}t) \) may complicate the analysis of \( \varphi(\lambda) \). To overcome this problem, we henceforth restrict to integer lattices, i.e., lattices with integer generator matrices. This allows us to choose the HNF as the employed decomposition, and consider the simplified expression (64).

**Remark 7.** An extension to the case \( K > 2 \) seems necessary, as numerical results suggest that the flat behavior prevails for more than two transmitters. A natural first step is to study the average flatness factor restricted to finite sets of the lattices constituting \( L \), as a straightforward generalization of the flatness factor for a sum of lattices \( L \) is not obvious. However, the relevance of such an approach needs to be verified, and numerical simulations are currently too expensive.

If the intermediate relay aims to decode a linear combination of \( K = 2 \) codewords, the ML decoding metric (63) is a sum over lattice points, as repeatedly remarked previously. This allows us to characterize the behavior of \( \varphi(\lambda) \) in terms of the flatness factor of the lattice \( L \) (cf. (39)).

**Definition 6.** Let \( K = 2 \). The flatness factor of \( \varphi(\lambda) \) is defined as the flatness factor of \( L \),

\[
\varepsilon_{\varphi(\lambda)}(\sigma^2) := \varepsilon_L(\sigma^2). \tag{66}
\]

**Remark 8.** The description of \( \varepsilon_L(\sigma^2) \) in (39) allows to study the flatness factor as a function of the noise variance \( \sigma^2 \). In the context of compute-and-forward, we need \( \varepsilon_{\varphi(\lambda)}(\sigma^2) \) to be as large as possible, as by the definition large values imply a distinctive maximum, which inhibits a flat behavior of the related function \( \varphi(\lambda) \).

Initially, studying the lattice flatness factor \( \varepsilon_{\varphi(\lambda)}(\sigma^2) \) boils down to studying the flatness factor of the random lattice \( L \) which results at the relays. In order to have a reliable performance in the considered setting, we should choose lattices at the transmitter which are good for reliable communications, i.e., protect against noise and fading, while maximizing the flatness factor of the resulting lattice \( L \). By adopting the two restrictions listed above, it turns out that \( L \) can be related to the lattices employed at the transmitter, a link which we make explicit in Theorem 2.
The consequences of the theorem are that maximizing the flatness factor of \( L \) amounts to maximizing the flatness factor of the original lattice.

**Theorem 2.** Let \( K = 2 \), and let \( \Lambda_1, \Lambda_2 \subset \mathbb{R}^n \) be full integer lattices such that if \( M_\Lambda \) is the generator matrix of \( \Lambda_1 \), then there exists \( c \in \mathbb{Z} \setminus \{0\} \) such that \( cM_\Lambda \) is the generator matrix for \( \Lambda_2 \). Hence, \( \Lambda_1 \supseteq \Lambda_2 \) are nested. Then, employing the Hermite normal form decomposition, the lattices \( L \) and \( \Lambda_1 \) are equivalent.

**Proof.** We determine the generator matrix \( M_L \) of the lattice \( L \). Assume that \( a^t = (a_1, a_2) \) is the coefficient vector determining the linear combination to be decoded. As \( a \) is the solution to a shortest vector problem, we have \( \gcd(a_1, a_2) = 1 \). Define the matrix

\[
M := \begin{bmatrix} a_1M_\Lambda & a_2cM_\Lambda \end{bmatrix}.
\]  

(67)

Since we have \( a^t \neq (0, 0) \), the matrix \( M \) has full-rank. Hence, there always exist \( U \in \text{GL}_{2n}(\mathbb{Z}) \) and \( B \in \text{Mat}(n, \mathbb{Z}) \) invertible, such that \( MU = \begin{bmatrix} 0_n & B \end{bmatrix} \) is in HNF. If we write \( A_1 := \text{diag} \{a_i\}_{i=1}^n \), \( A_2 := \text{diag} \{ca_i\}_{i=1}^n \), and decompose the matrix \( U \) into \( n \times n \) blocks as

\[
U = \begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \end{bmatrix}.
\]  

(68)

We can write

\[
MU = \begin{bmatrix} a_1M_\Lambda & a_2cM_\Lambda \end{bmatrix} U = \begin{bmatrix} M_\Lambda & M_\Lambda \end{bmatrix} \begin{bmatrix} A_1 & 0_n \\ 0_n & A_2 \end{bmatrix} \begin{bmatrix} U_1 & V_1 \\ U_2 & V_2 \end{bmatrix} = \begin{bmatrix} M_\Lambda & M_\Lambda \end{bmatrix} \begin{bmatrix} A_1U_1 & A_1V_1 \\ A_2U_2 & A_2V_2 \end{bmatrix} = \begin{bmatrix} 0_n & B \end{bmatrix}.
\]  

(69)

(70)

As \( M_\Lambda \) generates a full lattice, it is invertible. We multiply by \( M_\Lambda^{-1} \) from the left to get

\[
M_\Lambda^{-1} \begin{bmatrix} M_\Lambda & M_\Lambda \end{bmatrix} \begin{bmatrix} A_1U_1 & A_1V_1 \\ A_2U_2 & A_2V_2 \end{bmatrix} = \begin{bmatrix} I_n & I_n \\ I_n & I_n \end{bmatrix} \begin{bmatrix} A_1U_1 & A_1V_1 \\ A_2U_2 & A_2V_2 \end{bmatrix} = \begin{bmatrix} (A_1U_1 + A_2U_2) & (A_1V_1 + A_2V_2) \\ 0_n & M_\Lambda^{-1}B \end{bmatrix}.
\]  

(71)

(72)

(73)
which yields the equations $A_1 U_1 + A_2 U_2 = 0_n$ and $A_1 V_1 + A_2 V_2 = M^{-1}_A B$. We can rewrite the first equation to read

$$\begin{bmatrix} A_1 & A_2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} a_1 & 0 & \cdots & 0 & ca_2 & 0 & \cdots & 0 \\ 0 & a_1 & 0 & 0 & ca_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_1 & 0 & \cdots & ca_2 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = 0_n.$$  

(74)

This equation is satisfied if and only if

$$\text{colspan} \left( \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \right) \subseteq \ker \left( \begin{bmatrix} A_1 & A_2 \end{bmatrix} \right) = \text{span} \left\{ \begin{bmatrix} ca_2 \cdot e_i \\ -a_1 \cdot e_i \end{bmatrix} \right\},$$

where $e_i$ is the $i^{th}$ standard vector. In particular, we can always choose

$$U_1 = -\text{diag} \{ca_2\}_{i=1}^n; \quad U_2 = \text{diag} \{a_1\}_{i=1}^n.$$  

(75)

With this choice, the generator matrix of $\mathcal{L}$ simplifies to

$$M_\mathcal{L} = h_1 M_A U_1 + h_2 c M_A U_2 = -h_1 M_A \text{diag} \{ca_2\}_{i=1}^n + h_2 c M_A \text{diag} \{a_1\}_{i=1}^n$$

(77)

$$= (h_2 c \text{diag} \{a_1\}_{i=1}^n - h_1 \text{diag} \{ca_2\}_{i=1}^n) M_A = r M_A$$

(78)

for some $r \in \mathbb{R}$. 

This result motivates the study of the flatness factor of the code lattices. As these should be picked to be well conditioned for coding purposes, we only need to compute the flatness factor of reasonably conditioned lattices. Thus, the derived approximation $\Theta^q_A(q)$ will suffice for that purpose.

We consider the lattices $\Lambda^{(3)}_i$ and $\Lambda^{(4)}_i$, $1 \leq i \leq 4$, tabulated in Table II in Appendix A. As the well-known lattices $\mathbb{Z}^n$, $D_n$ and the dual $D^*_n$ are all examples of well-rounded lattices, we consider additional lattices $\Lambda^{(3)}_4$, $\Lambda^{(4)}_3$ and $\Lambda^{(4)}_4$ which are well-rounded as well, for sake of consistency. These are found via computer search.

**Remark 9.** Note that, as it should, the flatness factor of $\varphi(\lambda)$ is independent of the size of constellation, as it is simply the flatness factor of the unconstrained lattice $\mathcal{L}$. For a meaningful comparison, however, we fix a finite codebook for each of the considered lattices, and illustrate their flatness factor with respect to the power-dependent SNR, $\rho = P/\sigma^2$. 

February 20, 2017 DRAFT
The average power $P$ for the employed constellation is also found in Table II. We compare the considered lattices in Figure 8.

![Graph showing the flatness factors of $\Lambda$ for lattices of dimensions $n = 3$ (left) and $n = 4$ (right).]

Both in dimension $n = 3$ and $n = 4$, it is visible that the integer lattice $\Lambda_1^{(n)} = \mathbb{Z}^n$ performs best among the considered lattices with respect to the flatness factor criterion. This is in agreement with the observation in [5] that the lattice $L$ should not be dense. However, the density is not the only factor that plays a role, as visible from the plot in dimension $n = 3$. There, the best quantizer, $\Lambda_3^{(3)} = D_3^*$ exhibits the smallest flatness factor, even below the densest packing $\Lambda_2^{(3)} = D_3$. In dimension $n = 4$, the lattice $\Lambda_2^{(4)} = D_4$ is both the best quantizer and densest packing, and exhibits the smallest flatness factor.

The quintessential statement, however, is not that the lattice $\mathbb{Z}^n$ is the one that should always be used. Indeed, the code lattice should firstly be chosen to perform well in compute-and-forward, and additionally exhibit a large flatness factor. This yields a potential trade-off in code design.

V. CONCLUSIONS

The main goal of this article was to derive a simple approximation of the theta series of a lattice. Our approximation can be shown to be a simple rational function.

We then studied maximum-likelihood decoding in the context of compute-and-forward relaying, and showed that partial code design criteria can be derived based on the so-called flatness factor of certain involved lattices. Using a particular matrix decomposition for manipulating the decoding metric, and adopting two important restrictions, we further prove that the code lattice
at the transmitter and the random lattice at the relay are similar. This allows for a direct design criterion for the code lattice, rather than for the random lattice. Namely, the flatness factor of the code lattice should be maximized.

As the flatness factor is directly related to the theta series of a lattice, it is hence crucial to be able to efficiently compute the latter quantity. Hence, for the purposes of empirically analyzing different lattices at the transmitter, the theta series approximation proves to be crucial, both in this context as in the context of wiretap coset code design, e.g. the results obtained in [1].

This work allows to extend the framework in a variety of directions. First, as noted in this article, the decoding metric is only a sum over lattice points for $K = 2$ transmitters, and the analysis of its behavior becomes more complicated when $K \geq 3$, though numerical results show that the flatness behavior prevails. On the other hand, the used decomposition only allows for integer lattices and integer linear combinations. Following related work [16], [17] where the linear combinations are allowed to be over the ring of integers of an algebraic number field, it would be of benefit to examine the decoding metric in this generalized setting. The Hermite normal form decomposition over the integers $\mathbb{Z}$ is only a special case, and the algorithm has been extended to arbitrary Dedekind domains. Thus using this generalized decomposition would allow to study algebraic lattices for code construction at the transmitters.

REFERENCES

[1] A. Barreal, A. Karrila, D. Karpuk, and C. Hollanti, "Information Bounds and Flatness Factor Approximation for Fading Wiretap MIMO Channels", International Telecommunication Networks and Applications Conference, 2016.
[2] B. Nazer and M. Gastpar, "Compute-and-Forward: Harnessing Interference Through Structured Codes", IEEE Transactions on Information Theory, vol. 57, no. 10, pp. 6463–6486, 2011.
[3] C. Feng, D. Silva, and F. R. Kschischang, "An Algebraic Approach to Physical-Layer Network Coding", IEEE Transactions on Information Theory, vol. 59, no. 11, pp. 7576–7596, 2013.
[4] J.-C. Belfiore, "Lattice Codes for the Compute-and-Forward Protocol: The Flatness Factor", IEEE Information Theory Workshop, 2011.
[5] J.-C. Belfiore and C. Ling, "The Flatness Factor in Lattice Network Coding: Design Criterion and Decoding Algorithm", Zurich Seminar on Communications, 2012.
[6] A. Mejri and G. Rekaya-Ben Othman, "Efficient Decoding Algorithms for the Compute-and-Forward Strategy", IEEE Transactions on Communications, vol. 63, no. 7, pp.2475–2485, 2015.
[7] M. Widmer, "Lipschitz Class, Narrow Class, and Counting Lattice Points", Proceedings of the American Mathematical Society, vol. 140, no. 2, pp. 677–689, 2011.
[8] M. Henk, "Successive Minima and Lattice Points", arXiv:math/0204158, 2002.
[9] L. Fukshansky and A. Schrmann, "Bounds on Generalized Frobenius Numbers", \textit{European Journal of Combinatorics}, vol. 42, no. 3, pp. 361–368, 2011.

[10] S. Lang, "Algebraic Number Theory", Springer-Verlag, 1970.

[11] C. Ling, L. Luzzi, J.-C. Belfiore, and D. Stehle, "Semantically Secure Lattice Codes for the Gaussian Wiretap Channel", \textit{IEEE Transactions on Information Theory}, vol. 60, no. 10, pp. 6399–6416, 2014.

[12] D. Micciancio and O. Regev, "Worst-Case to Average-Case Reductions Based on Gaussian Measures", \textit{SIAM Journal on Computing}, vol. 37, no. 1, pp. 267–302, 2007.

[13] A. Osmane and J.-C. Belfiore, "The Compute-and-Forward Protocol: Implementation and Practical Aspects", arXiv: 1107.0300v1, 2011.

[14] S. Sahraei and M. Gastpar, "Polynomially Solvable Instances of the Shortest and Closest Vector Problems with Applications to Compute-and-Forward", [arXiv:1512.06667] 2015.

[15] A. Barreal, J. Pääkkönen, D. Karpuk, C. Hollanti, and O. Tirkkonen, "A low-complexity message recovery method for Compute-and-Forward relaying", \textit{IEEE Information Theory Workshop}, 2015.

[16] N. E. Tunali, K. R. Narayanan, J. J. Boutros, and Yu-Chih Huang, "Lattices over Eisenstein Integers for Compute-and-Forward", \textit{50th Annual Allerton Conference on Communication, Control, and Computing}, 2012.

[17] Yu-chih Huang, K. R. Narayanan, and Ping-Chung Wang, "Adaptive Compute-and-Forward with Lattice Codes over Algebraic Integers", \textit{IEEE International Symposium on Information Theory}, 2012.
### APPENDIX

This table serves as a summary of the characteristics of the lattices used for simulations, and introduces the employed notation.

| $n = 3$ | Notation | $M_{\Lambda}$ | $\lambda_1$ | $\text{vol}(\Lambda)$ | $\Theta_{\Lambda}(q)$ | $P\ (|C| = 343)$ |
|---------|-----------|---------------|-------------|---------------------|---------------------|-----------------|
| $\Lambda_1^{(3)} = \mathbb{Z}^3$ | $I_3$ | 1 | 1 | $\theta_3^3(q)$ | 4 |  |
| $\Lambda_2^{(3)} = D_3 \cong A_3$ | $\begin{bmatrix} -1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ | 2 | 2 | $\frac{1}{2}(\theta_3^3(q) + \theta_2^3(q))$ | 8 |  |
| $\Lambda_3^{(3)} = D_3^* \cong A_3^*$ | $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ | 3 | 4 | $\theta_2(4q)^3 + \theta_3(4q)^3$ | 16.6667 |  |
| $\Lambda_4^{(3)}$ | $\begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & -1 & -2 \end{bmatrix}$ | 5 | 10 | - | 20 |  |

| $n = 4$ | Notation | $M_{\Lambda}$ | $\lambda_1$ | $\text{vol}(\Lambda)$ | $\Theta_{\Lambda}(q)$ | $P\ (|C| = 2401)$ |
|---------|-----------|---------------|-------------|---------------------|---------------------|-----------------|
| $\Lambda_1^{(4)} = \mathbb{Z}^4$ | $I_4$ | 1 | 1 | $\theta_4^4(q)$ | 4 |  |
| $\Lambda_2^{(4)} = D_4$ | $\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ | 2 | 2 | $\frac{1}{2}(\theta_3^2(q) + \theta_4^2(q))$ | 8 |  |
| $\Lambda_3^{(4)}$ | $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$ | 3 | 8 | - | 12 |  |
| $\Lambda_4^{(4)}$ | $\begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \\ 1 & 1 & -2 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix}$ | 5 | 20 | - | 20 |  |

**TABLE II**: Summary of the lattices employed for simulation results.