Exclusion-type spatially heterogeneous processes in continua

Michael Blank

Russian Academy of Science, Institute for Information Transmission Problems, Russia
E-mail: blank@iitp.ru

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Abstract. We study deterministic discrete time exclusion-type spatially heterogeneous particle processes in continua. A typical example of this sort is a traffic flow model with obstacles: traffic lights, speed bumps, spatially varying local velocities etc. Ergodic averages of particle velocities are obtained and their connections to other statistical quantities, in particular to particle and obstacle densities (the so-called fundamental diagram), are analyzed rigorously. The main technical tool is a ‘dynamical’ coupling construction applied in a nonstandard fashion: instead of proving the existence of the successful coupling (which might even fail to hold) we use its presence/absence as an important diagnostic tool.

Keywords: rigorous results in statistical mechanics, phase diagrams (theory), driven diffusive systems (theory), traffic models
1. Introduction

In 1970, Frank Spitzer introduced the (now classical) simple exclusion process as a Markov chain that describes nearest-neighbor random walks of a collection of particles on the one-dimensional infinite\(^1\) integer lattice. Particles interact through the hard core exclusion rule, which means that at most one particle is allowed at each site. This seemingly very particular process appears naturally in a very broad list of scientific fields starting from various models of traffic flows [13, 9, 7, 2, 3], molecular motors and protein synthesis in biology, surface growth or percolation processes in physics (see [14, 5] for a review), and ranging up to the analysis of Young diagrams in representation theory [6].

From the point of view of the order of particle interactions, there are exclusion processes of two principal different types: with synchronous and asynchronous updating rules. In the latter case, at each moment of time, a.s. at most one particle may move and hence only a single interaction may take place. This is the main model considered in the mathematical literature (see e.g. [12] for a general account and [1, 8, 10] for recent results), and indeed, the assumption about the asynchronous updating is quite natural in the continuous time setting. The synchronous updating means that all particles are trying to move simultaneously and hence an arbitrarily large (and even infinite) number of interactions may occur at the same time. This makes the analysis of the synchronous updating case much more difficult, but this is what happens in the discrete time case\(^2\). This case is much less studied, but there are still a few results describing ergodic properties of such processes [2, 3, 7, 9, 13].

Recently in [4] we have introduced and studied the synchronous updating version of the simplest exclusion process (TASEP) in a continuum\(^3\). Now our aim is to extend these

\(^1\) Or finite with periodic boundary conditions.

\(^2\) If one does not consider some ‘artificial’ updating rules like a sequential or random updating.

\(^3\) Some other continuous space generalizations were considered e.g. in [11, 15].

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results to the case when the medium is heterogeneous, i.e. contains obstacles. The idea is as follows. Consider a (countable) collection of particles performing (random) walks on a real line with interactions of hard core type (we assume that the particles cannot outrun each other). Assume also that on the real line there are a (countable) number of obstacles. To overcome an obstacle, a particle needs to spend some additional time on it. Thus we have interactions of two types: between particles and with static obstacles. The principal novelty here is that the presence of obstacles leads to a spontaneous creation of ‘traffic jams’ (particle clusters) near obstacles. Indeed, it is easy to construct initial particle configurations having no clusters of particles but such that these clusters will be created in front of obstacles under the dynamics. This feature was completely absent in the case without obstacles: a cluster may only disappear and can never be created. We start the formalization with the simplest version of the model and later in sections 6 and 7 consider what happens in a more complicated setting.

By a configuration \( x := \{x_i\}_{i \in \mathbb{Z}} \) we mean an ordered (i.e. \( x_i \leq x_{i+1} \forall i \in \mathbb{Z} \)) bi-infinite sequence of real numbers \( x_i \in \mathbb{R} \). The union \( X \) of all such sequences plays the role of the phase space of the system under consideration. Consider also a special configuration \( z \in X \) such that \( z_i \leq z_{i+1} \forall i \in \mathbb{Z} \). We refer to elements of \( x \) as positions of particles and to elements of \( z \) as positions of obstacles.

Suppose that \( \Delta_i = \Delta_i(x) := x_{i+1} - x_i \) and that

\[
\tilde{\Delta}_i = \tilde{\Delta}_i(x, z) := \min(x_{i+1} - x_i, \min_j (z_j : z_j > x_i) - x_i)
\]

stand for the minimum among the distances from the point \( x_i \) to the point \( x_{i+1} \) and to the next obstacle (see figure 1). We refer to \( \Delta_i \) and \( \tilde{\Delta}_i \) as gaps and modified gaps corresponding to the \( i \)th particle in \( x \).

For a given real \( v > 0 \) (which we refer to as a maximal local velocity) and the configuration \( z \) we define a map \( T : X \to X \) as follows:

\[
(Tx)_i := x_i + \min(\tilde{\Delta}_i(x, z), v) \quad \forall i \in \mathbb{Z}.
\]

The dynamical system \((T, X)\) describes the collective motion of particles discussed above. If \( \tilde{\Delta}_i(x, z) < v \), we say that the \( i \)th particle is blocked (meaning that its motion is blocked by the \( i + 1 \)th particle or an obstacle) at time \( t \) and we say that it is free otherwise. By a cluster of particles in a configuration \( x \in X \) we mean consecutive particles with gaps \( \Delta_i(x) < v \) having no obstacles between them.

Associating the number of iterations of the map \( T \) with time we often use the notation \( T^t x \equiv x^t := \{x^t_i\} \). In these terms the quantity \( \xi_i^t := \min(\Delta_i(x^t, z), v) \) plays the role of the local velocity of the \( i \)th particle in the configuration \( x \equiv x^0 \) at time \( t \geq 0 \) and thus the

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**Figure 1.** TASEP in a heterogeneous continuum.
dynamics can be rewritten as
\[ x_i^{t+1} := x_i^t + \xi_i^t. \] (1.2)

A more general setting including varying/random velocities and waiting times will be considered in sections 6 and 7.

It is of interest that even without obstacles (i.e. if \( z = \emptyset \)) the behavior of the deterministic dynamical system \((T,X)\) is far from being trivial. In [4] it was shown that this system is chaotic in the sense that its topological entropy is positive (and even infinite).

Our main results are concerned with the so-called fundamental diagram describing the connection between average particle velocities \( V(x) \) and particle/obstacle densities \( \rho(x), \rho(z) \) (see section 2 for definitions) and technically the analysis is based on a (somewhat unusual) ‘dynamical’ coupling construction (see section 3 and also [4]).

For a given \( v > 0 \) and a configuration of obstacles \( z \), denote by \( \tilde{z} \) the extended configuration of obstacles obtained by inserting between the members of each pair of entries \( z_i, z_{i+1} \) new \( \lfloor (z_{i+1} - z_i)/v \rfloor \) ‘virtual’ obstacles at distances \( v \) between them starting from the point \( z_i \). Here \( \lfloor u \rfloor \) stands for the integer part of the number \( u \).

**Theorem 1 (Fundamental diagram).** Let \( \rho(x), \rho(\tilde{z}) \) be well defined. Then
\[ V(x) = \min\{1/\rho(\tilde{z}), 1/\rho(x)\}. \] (1.3)

Therefore the phase space is divided into two parts: gaseous \( \{(x, \tilde{z}) : \rho(x) \leq \rho(\tilde{z})\} \) (consisting of configurations of eventually non-interacting particles) and liquid \( \{(x, \tilde{z}) : \rho(x) > \rho(\tilde{z})\} \) (where clusters of particles are present at any time).

These results are illustrated in figure 2. It is worth noting that a naive argument tells us that the average velocity of a particle in an infinite system is essentially the minimum of the average inter-particle distance and the average inter-obstacle distance. Theorem 1 shows that this is absolutely not the case: instead of the average inter-obstacle distance one needs to take into account the average distance between the elements of the extended configuration \( \tilde{z} \). As we shall see, the latter is very different from the former, e.g. we always have \( \rho(\tilde{z}) \geq 1/v \) independently of \( \rho(z) \) (which might even not be well defined). Actually the mere fact that the correspondence between the average velocities and particle/obstacle densities is one-to-one comes as a surprise, especially without any regularity assumptions for the positions of obstacles.
The main steps of the proof are as follows. First we show that for given configurations of particles and obstacles, upper/lower average particle velocities are the same for all particles. Now to compare average velocities in different particle configurations with the same configuration of obstacles, we develop a special dynamical coupling construction. Applying this construction, we prove that upper/lower average particle velocities are the same provided that the particle densities are the same. Thus to calculate the dependence of average velocities on densities it is enough to construct for each density a single configuration having this density, whose velocity we are able to calculate explicitly. To this end we consider an auxiliary zero-range lattice process whose trajectories are invariant under the dynamics of our original system without obstacles. To calculate the average velocity in the zero-range lattice process, one uses corresponding results obtained in [4].

Despite various couplings being widely used in the analysis of interacting particle systems (IPS; see e.g. [12]), the application of our approach is very different from the usual method. In particular, we do not prove the existence of the so-called successful coupling (which might even fail to hold); instead use its presence/absence as an important diagnostic tool. Remark also that typically one uses the coupling argument to prove the uniqueness of the invariant measure and to derive later other results from this fact. In our case there might be a very large number of ergodic invariant measures or no invariant measures at all (e.g. the trivial example of a single particle performing a skewed random walk). This example reminds us of another important statistical quantity—the average particle velocity. The dynamical coupling will be used directly to find connections between the average particle velocities and other statistical features of the systems under consideration, in particular with the corresponding particle densities.

It is worthy of note that all approaches used to study discrete time lattice versions of IPS are heavily based on the combinatorial structure of particle configurations. This structure has no counterparts in the continuum setting under consideration. In particular the particle–vacancy symmetry is no longer applicable in our case. This explains the need to develop a fundamentally new technique for the analysis of IPS in continua. The presence of obstacles also prevents the direct application of the scheme developed in [4] for the spatially homogeneous case.

2. Basic properties

Here we shall study questions related to particle densities and velocities.

By the density
\[ \rho(x, I) := \frac{\#\{i \in \mathbb{Z} : x_i \in I\}}{|I|} \]
of a configuration \( x \in X \) in a bounded segment \( I = [a, b] \in \mathbb{R} \) we mean the number of particles from \( x \) whose centers \( x_i \) belong to \( I \) divided by the Lebesgue measure \( |I| > 0 \) of the segment \( I \). If for any sequence of nested bounded segments \( \{I_n\} \) with \( |I_n| \to \infty \) the limit
\[ \rho(x) := \lim_{n \to \infty} \rho(x, I_n) \]
is well defined, we call it the density of the configuration \( x \in X \). Otherwise one considers upper and lower (with respect to all possible collections of nested intervals \( I_n \)) particle densities \( \rho_{\pm}(x) \).

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Remark 1. If $\exists \rho(x) < \infty$, then $|x_n - x_m|/|n - m| \xrightarrow{|n-m| \to \infty} 1/\rho(x)$.

Lemma 2. The upper/lower densities $\rho_{\pm}(x^t)$ are preserved by the dynamics, i.e. $\rho_{\pm}(x^t) = \rho_{\pm}(x^{t+1}) \forall t \geq 0$.

Proof. For a given segment $I \in \mathbb{R}$ the number of particles from the configuration $x^t \in X$ which can leave it during the next time step cannot exceed 1 and the number of particles which can enter this segment also cannot exceed 1. Thus the total change of the number of particles in $I$ cannot exceed 1, because if a particle leaves the segment through one of its ends no other particle can enter through this end. Therefore

$$\left|\rho(x^t, I) - \rho(x^{t+1}, I)\right| \cdot |I| \leq 1$$

which implies the claim. $\square$

By the (average) velocity of the $i$th particle in the configuration $x \in X$ at time $t > 0$, we mean

$$V(x, i, t) := \frac{1}{t} \sum_{s=0}^{t-1} \xi_i^s \equiv (x_i^t - x_i^0)/t.$$ 

If the limit

$$V(x, i) := \lim_{t \to \infty} V(x, i, t)$$

is well defined, we call it the (average) velocity of the $i$th particle. Otherwise one considers upper and lower particle velocities $V_{\pm}(x, i)$.

Lemma 3. Suppose that $x \in X$ and let $\rho(\tilde{z}) < \infty$ be well defined. Then $|V(x, j, t) - V(x, i, t)| \xrightarrow{t \to \infty} 0$ a.s. $\forall i, j \in \mathbb{Z}$.

Proof. It is enough to prove this result for $j = i + 1$. Consider the difference between (average) velocities of consecutive particles:

$$V(x, i + 1, t) - V(x, i, t) = \frac{x_{i+1}^t - x_i^t}{t} - \frac{x_i^0 - x_i^0}{t} = \frac{x_{i+1}^t - x_i^0}{t} - \frac{x_{i+1}^0 - x_i^0}{t} = \Delta_i^t/t - \Delta_i^0/t.$$ 

The last term vanishes as $t \to \infty$ and it is enough to show that the same happens with $\Delta_i^t/t$. Here and in the sequel we use the notation $\Delta_i^t \equiv \Delta_i(x^t)$.

Normally in the deterministic setting, gaps between particles $\Delta_i^t$ are uniformly bounded (see e.g. [2]–[4]). Surprisingly in the present setting this is not the case and we are only able to show that $\Delta_i^t$ may not grow faster than $o(t)$, which fortunately is enough for our aims.

To prove this estimate, we introduce a new configuration having only two particles $\hat{x} := \{\hat{x}_1, \hat{x}_2\}$ which have the same initial positions as the $i$th and the $(i + 1)$th particles of $x$, i.e. $\hat{x}_1 = x_i^0, \hat{x}_2 = x_{i+1}^0$. Defining $\hat{x}^t := T^t \hat{x}$ and $\hat{\Delta}^t := \hat{x}_2^t - \hat{x}_1^t$, we observe that $\hat{\Delta}^t \geq \Delta_i^t \forall t \geq 0$. 

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Consider the movement of the leading particle in the process $\tilde{x}$. There is the first moment of time $t_2 := t_2(\tilde{x}_2)$ when this particle encounters an obstacle from $z$. Denote by $\tilde{z}_k$ the position of this obstacle in the extended configuration $\tilde{z}$. By the definition of the dynamics, $T^t \tilde{z}_k = \tilde{z}_{k+t} \forall t \geq 0$.

The movement of the trailing particle in the process $\tilde{x}$ is a bit more complicated since additionally to being stopped by obstacles, it may be stopped by the leading particle. Anyway, there is a first moment of time $t_1 := t_1(\tilde{x}_1, \tilde{x}_2) > t_2$ when this particle encounters the obstacle located at $\tilde{z}_k$. For each $t > t_1$ these two particles move synchronously along the ‘chain’ $\tilde{z}$. Thus the growth of $\Delta^t$ is completely determined by the concentration of entries in $\tilde{z}$ (i.e. by $\rho(\tilde{z})$). Due to the assumption of the existence of the density of $\tilde{z}$, these fluctuations cannot exceed $o(t)$ and hence $\Delta^t_n/t \leq \Delta^t/t \xrightarrow{t \to \infty} 0$. □

**Corollary 4.** Under the assumptions of lemma 3, the upper and lower particle velocities $V_{\pm}(x, i)$ do not depend on $i$.

**Remark 5.** (a) If the density $\rho(\tilde{z})$ is not well defined, the distance $\Delta^t_i$ may grow linearly with time and the limit average velocities might differ for different particles. (b) The existence of $\rho(z) < \infty$ does not imply the existence of $\rho(\tilde{z}) < \infty$ but for ‘typical’ $z$ this implication does hold (see section 7).

Let us estimate upper/lower densities of the extended configuration $\tilde{z}$ of obstacles for a given configuration $z$ with $\rho(z) > 0$ and a given local velocity $v > 0$.

**Lemma 6.** $\max\{1/v, \rho(z)\} \leq \rho_{-}(\tilde{z}) \leq \rho_{+}(\tilde{z}) \leq \rho(z) + 1/v$.

**Proof.** Observe that for each $i \in Z$ the spatial segment $[z_i, z_{i+1}]$ contains exactly $\left\lfloor (z_{i+1} - z_i)/v \right\rfloor \in [(z_{i+1} - z_i)/v - 1, (z_{i+1} - z_i)/v]$ elements. Therefore $\forall n \in Z_+$,

$$\rho(\tilde{z}, [z_0, z_n]) = \left( n + \sum_{i=0}^{n-1} \left\lfloor (z_{i+1} - z_i)/v \right\rfloor \right)/(z_n - z_0)$$

$$= \rho(z, [z_0, z_n]) + \left( \sum_{i=0}^{n-1} \left\lfloor (z_{i+1} - z_i)/v \right\rfloor \right)/(z_n - z_0)$$

$$\leq \rho(z, [z_0, z_n]) + 1/v.$$  

Similarly

$$\rho(\tilde{z}, [z_0, z_n]) \geq \rho(z, [z_0, z_n]) - n/(z_n - z_0) + 1/v = 1/v.$$  

Passing to the limit as $n \to \infty$ and using that $\tilde{z}$ contains all elements of $z$, we get the result. □

**3. Dynamical coupling through particle overtaking**

Consider two independent particle processes $x^t, \hat{x}^t$. By $i(x^t)$ we denote the $i$th particle of the process $x^t$ (i.e. $x^t_i$ is the location of the particle $i(x^t)$ at time $t$).

We say that the particle $i(x^t)$ overtakes at time $t > 0$ the particle $j(\hat{x}^t)$ if $x^t_{i-1} < \hat{x}^t_{j-1}$ and $x^t_i \geq \hat{x}^t_j$, and denote this event as $i(x^t) \leftrightarrow j(\hat{x}^t)$.
Lemma 7. If \( i(x^t) \leftrightarrow j(\hat{x}^t) \), then:

(a) \( \exists n \in \mathbb{Z} \setminus \{i\} : n(x^t) \leftrightarrow j(\hat{x}^t) \);

(b) \( \exists m \in \mathbb{Z} : j(\hat{x}^t) \leftrightarrow m(x^t) \) and \( x^t_{m-1} < x^t_m \).

Additionally \( i(x^t) \leftrightarrow j(\hat{x}^t) \) and \( (j - 1)(\hat{x}^t) \leftrightarrow i(x^t) \) might happen only if \( x^t_i = \hat{x}^t_{j-1} \).

Note that there is no contradiction between the two parts of this lemma since the former part concerns a particle being overtaken while the latter concerns an overtaking particle.

Proof. By the construction of the process under consideration and the definition of the overtaking, we have

\[
x_{i-1}^t \leq x_i^t < x_j^t \leq x_j^{t-1} \leq x_{i+1}^t.
\]

(3.1)

Now (a) follows from the observation that by (3.1) neither particles preceding \( i(x^t) \) nor particles succeeding it may overtake the particle \( j(\hat{x}^t) \) at time \( t \).

If (b) were to hold, then we would have \( \hat{x}_j^t < x_m^{t-1} \leq x_m^t \leq \hat{x}_j^t \), which combined with (3.1) would imply that \( m > i \). On the other hand, \( \forall m > i \) we have \( \hat{x}_j^t \leq x_i^t \leq x_m^{t-1} \) and hence \( j(\hat{x}^t) \leftrightarrow m(x^t) \) may happen only if \( x_m^{t-1} = x_m^t = x_i^t \).

To show that the event discussed in the additional part may take place, consider a pair of configurations satisfying for some \( i, j, t \in \mathbb{Z} \) the following inequalities:

\[
x_j^{t-1} < x_i^{t-1} < x_j^t = x_{j+1}^{t-1} = x_{i+1}^t, \quad \max(\Delta_i(x^{t-1}, z), \Delta_{j-1}(\hat{x}^{t-1}, z)) \leq v.
\]

Then \( x_i^t = \hat{x}_{j-1}^t = \hat{x}_j^t \), which implies that \( i(x^t) \leftrightarrow j(\hat{x}^t) \) and \( (j - 1)(\hat{x}^t) \leftrightarrow i(x^t) \). Assume now that, on the contrary, \( x_i^t \neq \hat{x}_{j-1}^t \). If \( x_i^t > \hat{x}_{j-1}^t \), then \( \hat{x}_{j-1}^t < x_i^t < x_j^t \), which contradicts \( (j - 1)(\hat{x}^t) \leftrightarrow i(x^t) \). Similarly \( x_i^t < \hat{x}_{j-1}^t \) implies \( x_i^t < \hat{x}_{j-1}^t \leq \hat{x}_j^t \), which contradicts \( i(x^t) \leftrightarrow j(\hat{x}^t) \).

Let us introduce the dynamical coupling of the processes \( x^t, \hat{x}^t \) which consists in a sequential pairing of particles of opposite processes. The pairing in our deterministic setting is a purely formal action which does not change the dynamics. The idea is that if a particle overtakes some particles from the opposite process it becomes paired with one of them. The construction starts at time \( t = 0 \) and initially all particles are assumed to be unpaired.

Denote by

\[
J_i(x^t) := \{ j \in \mathbb{Z} : i(x^t) \leftrightarrow j(\hat{x}^t) \}
\]

the set of particles being overtaken simultaneously by the particle \( i(x^t) \) at time \( t \), and by

\[
i_\bullet(x^t) := \min \{ \infty, \min \{ j \in J_i(x^t) : j(\hat{x}^{t-1}) \text{ is paired} \} \}
\]

the paired particle with the minimal index among them, and finally define

\[
i_\circ(x^t) := \begin{cases} i_\bullet(x^t) - 1 & \text{if } i_\bullet(x^t) - 1 \in J_i(x^t) \\ \infty & \text{otherwise} \end{cases}
\]

In words, \( i_\bullet(x^t) \) stands for the paired particle in \( J_i(x^t) \) having the minimal index, and \( i_\circ(x^t) \) stands for the unpaired particle in \( J_i(x^t) \) having the maximal index among those.
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Figure 3. Connections between indices and particle positions.

preceding \( i^* (x^t) \). A typical connection between those indices and positions of the \( i \)th particle at times \( t - 1 \) and \( t \) is shown in figure 3.

To simplify the description of the dynamical coupling we shall use a diagramatic representation for coupled configurations, where paired particles are denoted by black circles and unpaired ones by open circles, and use the upper line of the diagram for the \( x \)-particles (i.e. particles from the \( x \)-process) and the lower line for the \( \dot{x} \)-particles. In this representation a typical pairing event looks as follows:

\[
\textcircled{1} \quad \textcircled{2} \quad \textcircled{3} = \quad \textcircled{4} \quad \textcircled{5} \quad \textcircled{6} \quad \textcircled{7} \quad \textcircled{8}. \tag{3.2}
\]

Here \( \Rightarrow \) corresponds to the dynamics and \( \rightarrow \) to the pairing, while \( \bullet \) and \( \circ \) stand for two different mutual pairs of particles created after particle overtaking under the dynamics.

Now we are ready to define the pairing rigorously. We proceed first with all \( x \)-particles overtaking some \( \dot{x} \)-particles at time \( t \), and then with all \( \dot{x} \)-particles overtaking some \( x \)-particles at this time.

Let the overtaking take place for the \( i \)th \( x \)-particle at time \( t \); then if:

(a) \( i(x^{t-1}) \) is paired and \( i_o(x^t) < \infty \) we re-pair these particles:

\[
\textcircled{1} \quad \textcircled{2} \quad \textcircled{3} \quad \textcircled{4} \quad \textcircled{5} \quad \textcircled{6} \quad \textcircled{7} \quad \textcircled{8} \quad \textcircled{9}.
\]

(b) \( i(x^{t-1}) \) is unpaired and:

(b') \( i^* (x^t) < \infty \) and \( x^t_i > \dot{i}^* (x^t) \) we re-pair these particles:

\[
\textcircled{2} \quad \textcircled{3} \quad \textcircled{4} \quad \textcircled{5} \quad \textcircled{6} \quad \textcircled{7} \quad \textcircled{8} \quad \textcircled{9}.
\]

(b'') if, in contrast, \( i_o(x^t) < \infty \), this unpaired particle forms a new pair with \( i(x^t) \):

\[
\textcircled{2} \quad \textcircled{3} \quad \textcircled{4} \quad \textcircled{5} \quad \textcircled{6} \quad \textcircled{7} \quad \textcircled{8} \quad \textcircled{9}.
\]

The pairing rules when the overtaking takes place for the \( i \)th \( \dot{x} \)-particle are exactly the same except for the exchange of \( x \) with \( \dot{x} \) and vice versa.

The complexity of these rules reflects that first a particle may overtake simultaneously several particles from another process, and second an arbitrary number of particles may share the same spatial position. In particular, in the following event we have

\[
\textcircled{2} \quad \textcircled{3} \quad \textcircled{4} \quad \textcircled{5} \quad \textcircled{6} \quad \textcircled{7} \quad \textcircled{8} \quad \textcircled{9} \quad \textcircled{10}.
\]

rather than \( \longrightarrow \). Indeed, the trailing \( x \)-particle overtakes both trailing \( \dot{x} \)-particles and since the re-pairing rules in (b) demand strict inequalities, only the pairing of trailing unpaired particles takes place.

By lemma 7, each overtaking event may be treated separately. Therefore we apply the pairing rules first for all \( x \)-particles overtaking some \( \dot{x} \)-particles, and then for all \( \dot{x} \)-particles overtaking some \( x \)-particles.
According to the definition, particles from the same pair move synchronously until either the movement of one of them is blocked or one of the members of the pair is swapped with an unpaired particle from the same process. It is convenient to think about the coupled process as a ‘gas’ of single (unpaired) particles and ‘dumbbells’ (pairs). A previously paired particle may inherit the role of the unpaired one from one of its neighbors. In order to keep track of positions of unpaired particles we shall refer to them as $x$- and $\hat{x}$-defects depending on the process to which they belong.

We say that a pair of configurations $(x, \hat{x})$ is proper if for each two mutually paired particles the open segment between them cannot exceed $v$, and does not contain either a defect or an obstacle (i.e. the situations $\circ \bullet \bullet$; $\bullet \mid \bullet$ cannot happen), and there are no crossing mutually paired pairs ($\bullet \ast \ast \bullet$).

Lemma 8. If the pair of configurations $(x^{t-1}, \hat{x}^{t-1})$ is proper for some $t > 1$, then the pair $(x^t, \hat{x}^t)$ is proper as well.

Proof. The situation $\circ \bullet \bullet$ may happen if either a defect overtakes the trailing particle in a pair, or a pair is born around this defect. Both of these possibilities contradict the definition of pairing.

Assume that at time $t$ there is a pair of mutually paired particles $i(x^t), j(\hat{x}^t)$ such that the open segment between them contains an obstacle: $x_i^t < z_k < \hat{x}_j^t$. Assume also that this pair was present at time $t - 1$ as well. According to the construction of the dynamics this may happen only if $x_i^{t-1} < z_k - v$ and $z_k = \hat{x}_j^{t-1}$, which contradicts the assumption that the pair $(x^{t-1}, \hat{x}^{t-1})$ is proper: namely the distance between members of the same pair exceeds $v$. On the other hand, if the particles $i(x^t), j(\hat{x}^t)$ were not paired at time $t - 1$ and the pair was just created at time $t$, then one of the particles should overtake another at this time. Hence the distance between these particles cannot exceed $v$.

It remains to show that the last property still holds for ‘old’ pairs. In order to enlarge the distance between the existing mutually paired particles, one of them should be blocked by another particle or by an obstacle. On the other hand, the obstacle cannot belong to the open segment between pair members and hence it may block only the leading particle in the pair, which may only decrease the distance.

The blocking particle might be paired or unpaired. The former case implies that the ‘companion’ of the blocking particle is at distance at most $v$ and hence it will block the enlargement of the distance by more than $v$. In the latter case the non-blocked paired particle overtake the unpaired one and hence the pair under consideration will be repaired.

The observation that in the moment of the creation of a pair the distance between its members cannot exceed $v$ finishes this part of the proof.

The analysis of the absence of crossing mutually paired pairs is completely similar to that of the absence of defects between elements of a pair and therefore we skip this point. □

Denote by $\rho_u(x, I)$ the density of the $x$-defects belonging to a finite segment $I$, and by $\rho_u(x) := \rho_u(x, \mathbb{R})$ the upper limit of $\rho_u(x, I_n)$ taken over all possible collections of nested finite segments $I_n$ whose lengths go to infinity.

We say that a coupling of two Markov particle processes $x^t, \hat{x}^t$ is nearly successful if the upper density of the $x$-defects $\rho_u(x)$ vanishes with time a.s. This definition differs
Lemma 9. Suppose that \( x, \dot{x} \in X \) with \( \rho(x) = \rho(\dot{x}) > 0 \), and let there exist a nearly successful dynamical coupling \((x^t, \dot{x}^t)\). Then
\[
|V(x, 0, t) - V(\dot{x}, 0, t)| \xrightarrow{t \to \infty} 0.
\]

Proof. Consider an integer valued function \( n_t \) which is equal to the index of the \( \dot{x} \)-particle paired at time \( t > 0 \) with the 0th \( x \)-particle. If the 0th \( x \)-particle is not paired at time \( t \), we set
\[
n_t := \begin{cases} 
n_{t-1} & \text{if } t > 0 \\
0 & \text{if } t = 0.
\end{cases}
\]

To estimate the growth rate of \(|n_t|\) at large \( t \) observe that \( n_t \) changes its value only at those moments of time when the 0th \( x \)-particle meets a \( \dot{x} \)-defect. By the assumption of nearly successful coupling at time \( t \gg 1 \), the average distance between the defects at time \( t \) is of order \( 1/\rho_u(\dot{x}^t) \) while the amount of time needed for two particles separated by the distance \( L \) to meet cannot be smaller than \( L/(2v) \). Therefore the frequency of interactions of the 0th \( x \)-particle with \( \dot{x} \)-defects may be estimated from the above by a quantity of order \( \rho_u(\dot{x}^t) \xrightarrow{t \to \infty} 0 \), which implies that \( n_t/t \xrightarrow{t \to \infty} 0 \).

Now we are ready to prove the main claim.
\[
|V(x, 0, t) - V(\dot{x}, 0, t)| = \frac{|(x^t_0 - x^0_0) - (\dot{x}^t_0 - \dot{x}^0_0)|}{t} \\
\leq \frac{|x^t_0 - \dot{x}^t_0|}{t} + \frac{|x^0_0 - \dot{x}^0_0|}{t} \\
\leq \frac{|x^t_0 - \dot{x}^t_0|}{t} + \frac{n_t}{t} \frac{|x^t_{n_t} - \dot{x}^t_0|}{|n_t|} + \frac{|x^0_0 - \dot{x}^0_0|}{t}.
\]

The second addend is a product of two terms \(|n_t|/t\) and \( |x^t_{n_t} - \dot{x}^t_0|/|n_t|\). As we have shown, the first of them vanishes with time. If \( |n_t| \) is uniformly bounded, then the second term is obviously uniformly bounded on \( t \). Otherwise, for large \( |n_t| \), by remark 1 and the density preservation the second term is of order \( 1/\rho(\dot{x}) \), which proves its uniform boundedness as well. Thus the second addend goes to 0 as \( t \to \infty \). Noting finally that the first and the last addend also vanish with time at rate \( 1/t \), we obtain the result. \( \square \)

Lemma 10. Suppose that \( \rho(x) = \rho(\dot{x}) \) and assume that in the coupled process \( \forall i, j \exists \) a (random) moment of time \( t_{ij} < \infty \) such that \( x^t_i > \dot{x}^t_j \) for each \( t \geq t_{ij} \). Then the coupling is nearly successful.

Proof. By the assumption, each \( x \)-particle will eventually overtake each \( \dot{x} \)-particle located originally to the right from its own position and thus will form a pair with it or with one of its neighbors (if they are so close that they were overtaken simultaneously). Thus the creation of pairs is unavoidable. To show that the upper density of defects cannot remain positive, consider how the defects move under our assumptions. Assume that at time \( t \geq 0 \) the \( i \)th \( x \)-particle is paired with the \( j \)th \( \dot{x} \)-particle. Then by lemma 8, in order to overtake, at time \( s > t \), the \( j \)th \( \dot{x} \)-particle, significantly (by a distance larger than \( v \)), the \( i \)th \( x \)-particle necessarily needs to break the pairing with the \( j \)th \( \dot{x} \)-particle. Thus by the definition of the dynamical coupling, either an \( x \)-defect overtakes the \( j \)th \( \dot{x} \)-particle:
or the \(i\)th \(x\)-particle overtakes an \(x\)-defect: \(\circ \circ \rightarrow \circ \circ \rightarrow \circ \circ \). (Otherwise this pair will not be broken.) Therefore during this process the \(x\)-defects move to the right while the \(\hat{x}\)-defects move to the left. Hence they inevitably meet each other and ‘annihilate’. The assumption of the equality of particle densities implies the result.

\(\Box\)

4. The auxiliary lattice zero-range process

Consider now a lattice process which we shall need in the following. This process is defined on an integer lattice \(Z\) occupied by a bi-infinite configuration of particles \(y\) ordered with respect to their positions \(y_i\), i.e. \(y_i \leq y_{i+1} \forall i\). The dynamics is defined as follows. For each \(i \in Z\) consider all particles occupied the site \(i\); choosing the one with the largest index (say \(k_i\)) among them, we move this particle one position to the right, i.e.

\[
y_{k_i} := y_{k_i} + 1.
\]

This is a deterministic version of the so-called zero-range process on \(Z\) with parallel updating rules illustrated in figure 4.

Setting \(v = 1\), \(z = \emptyset\) and assuming that \(x_i \in Z \forall i \in Z\), we observe that in this case (without obstacles) the trajectory \(T^t x\) coincides with the trajectory of the zero-range process. Therefore according to [4] the average particle velocity for the zero-range process is equal to

\[
V(y) = \min\{1, 1/\rho(y)\}.
\]

(4.1)

Assume now that the lattice under consideration is not uniform (e.g. \(Z\)) but the distance between the \(i\)th and \((i + 1)\)th sites is equal to a nonnegative number \(\ell_i\) for each \(i \in Z\). Considering the zero-range process on this heterogeneous lattice, but assuming that the site’s occupation is the same in both cases, we are able to calculate the corresponding statistical quantities.

Denote by \(\hat{y}\) a configuration on a heterogeneous lattice \(\hat{Z}\) in which distances between the \(i\)th and \((i + 1)\)th sites are given by the sequence of numbers \(\{\ell_i\}\). Let \(y\) be a configuration on \(Z\) such that the \(i\)th particle of the configuration \(\hat{y}\) is located on the site whose index coincides with the index of the site occupied by the \(i\)th particle of the configuration \(y\).

\textbf{Lemma 11.} The deterministic zero-range processes on \(Z\) and \(\hat{Z}\) starting with configurations \(y, \hat{y}\) described above for each \(t \geq 0\) preserve the connection between the particle configurations. Therefore the particle velocities in these processes satisfy the relation

\[
\hat{V}(\hat{y}) = V(y)/\rho(\hat{Z}),
\]

(4.2)

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where $\rho(\hat{Z})$ is assumed to be positive and is defined as the density of the particle configuration having exactly one particle at each site of $\hat{Z}$.

**Proof.** The first claim follows from the definition of the zero-range process. Define $L(t) := y_0^t - y_0^0 \forall t$. Then

$$
(y_0^t - y_0^0)/t = \frac{1}{t} \sum_{j=0}^{y_0^t-y_0^0-1} \ell_j = \frac{1}{L(t)} \sum_{j=0}^{L(t)-1} \ell_j \xrightarrow{t \to \infty} V(y)/\rho(\hat{z}).
$$

$\square$

### 5. Proof of theorem 1

**Lemma 12.** Suppose that $x, \hat{x} \in X$, $\rho(x) = \rho(\hat{x})$ and let $V(x)$ be well defined for given $v, z$. Then $V(\hat{x})$ is also well defined and $V(\hat{x}) = V(x)$.

**Proof.** Let $x, \hat{x} \in X_\rho := \{ z \in X : \rho(z) = \rho \}$ be two admissible configurations of the same particle density. If one assumes that the dynamical coupling procedure leads to the nearly successful coupling of particles in these configurations, then by lemma 8 the assumptions of lemma 9 are satisfied and hence $|V(x, 0, t) - V(\hat{x}, 0, t)| \xrightarrow{t \to \infty} 0$, which by lemma 3 implies the claim. In general the assumption about the nearly successful coupling may not hold; however, as we demonstrate below, the pairing construction is still useful.

Define random variables

$$W_{ij}^t := x_i^t - \hat{x}_j^t, \quad i, j \in \mathbb{Z}, \quad t \in \mathbb{Z}_0.$$

Then

$$V(x, i, t) - V(\hat{x}, j, t) = W_{ij}^t / t - W_{ij}^0 / t.$$

Since by lemma 3 the differences between average velocities of different particles belonging to the same configuration vanish with time, it is enough to consider only the case $i = j = 0$. For $W_{00}^t$ there might be three possibilities which we study separately:

(a) $\lim_{t \to \infty} W_{00}^t / t = 0$. Then $|V(x, 0, t) - V(\hat{x}, 0, t)| \leq |W_{00}^t| / t + |W_{00}^0| / t \xrightarrow{t \to \infty} 0$, which by corollary 4 implies that the sets of limit points of the average velocities coincide.

(b) $\lim \sup_{t \to \infty} W_{00}^t / t > 0$. Then $\forall i \in \mathbb{Z}$ the $i$th particle of the $x$-process will overtake eventually each particle of the $\hat{x}$-process located at time $t = 0$ to the right from the point $x_i^0$. This together with the assumption of the equality of particle densities allows us to apply lemma 10, according to which the coupling is nearly successful. On the other hand, by lemma 8 the distance between mutually paired particles cannot exceed $v$. Therefore by lemma 9 we have $|V(x, 0, t) - V(\hat{x}, 0, t)| \xrightarrow{t \to \infty} 0$, which contradicts the assumption (b).

4 Consider e.g. the setting with $1/\rho > 5v$ and the configurations $x_i := i/\rho$ and $\hat{x}_i := i/\rho + 2v$, and assume that there are no obstacles. Then $\rho(x) = \rho(\hat{x}) = \rho$ and $V(x) = V(\hat{x}) = v$ but no pair will be created.
(c) \( \limsup_{t \to \infty} W_{00}^t/t < 0 \). Changing the roles of the processes \( x^t, \hat{x}^t \), one reduces this case to the case (b).

Thus only the case (a) may take place. \( \Box \)

To apply this result to prove theorem 1 one needs to construct for each particle density \( \alpha \) a configuration having this density whose velocity we are able to calculate explicitly. There are two possibilities for realizing this idea. In both cases we use the auxiliary lattice zero-range process \( y^t \) constructed in section 4 on the heterogeneous lattice \( \tilde{Z} \) whose \( i \)th site coincides with the location of the \( i \)th element in the extended configuration of obstacles \( \tilde{z} \).

The first construction is as follows. Choose an arbitrary initial configuration of particles \( y \) of a given density for the zero-range process. By lemma 11 and the relation (4.1) we obtain the relation (1.3). Observing that the trajectory of the zero-range process \( y^t \) coincides with the trajectory of our original process \( T^t y \), we get the desired result.

An alternative way to derive the relation (1.3) is to construct a specific initial configuration of the zero-range process, for which we can calculate the corresponding average velocity directly. The key observation here is that for \( x \) configuration of the zero-range process, for which we can calculate the corresponding velocity directly. The key observation here is that for \( x \) configuration of the zero-range process, for which we can calculate the corresponding velocity directly. The key observation here is that for \( x \) configuration of the zero-range process, for which we can calculate the corresponding velocity directly. The key observation here is that for \( x \) configuration of the zero-range process, for which we can calculate the corresponding velocity directly.

Thus only the case (a) may take place.

For a given \( x \in X \) for which \( \exists 0 < \rho(x) < \infty \) and \( \alpha \in \mathbb{R}_+ \), we define the configuration \( \alpha x \in X \) in three steps depending on arithmetic properties of \( \alpha \):

(a) \( kx \) for \( \alpha = k \in \mathbb{Z}_+ \);
(b) \( (k/n)x \) for \( \alpha = k/n \), \( k = mn + k' \) and \( m, k' \in \mathbb{Z}_0 \), \( k' < n \in \mathbb{Z}_+ \);
(c) \( (k/n)x \xrightarrow{n \to \infty} \alpha x \) if \( k/n \xrightarrow{n \to \infty} \alpha \).

Here by \( kx \) we mean a configuration in which each particle in \( x \) is replaced by \( k \) particles. To get \( y := (k/n)x \) we divide \( x \) into blocks having \( n \) consecutive particles each, and construct \( y := \{y_i\} \) as follows. For each \( i \) at location \( x_i \) we set exactly \( m \) particles at positions \( y_j = x_i \). Besides that, at the positions of each of the first \( k' \) particles in each of the blocks we add an additional particle and enumerate the particles according to their positions.

If \( \alpha \neq k/n \), one considers a sequence of its rational approximations: \( k/n \xrightarrow{n \to \infty} \alpha \). For each \( n \), according to the rule (b) we construct a configuration \( x^{(n)} := (k/n)x \). Observe that for \( \ell > 1 \) the configuration \( x^{(\alpha,\ell)} \) differs from \( x^{(\alpha)} \) only by the presence of some additional particles in each of the blocks of length \( n\ell \). Using the existence of \( \rho(x) \), one proves that the limit configuration exists, and we denote the latter by \( \alpha x \).

Now for any \( \alpha \in \mathbb{R}_+ \) we construct \( x := \alpha \tilde{z} \) and calculate \( V(\alpha \tilde{z}) \) as follows. We start with the case \( \alpha = 1 \), i.e. \( x = \tilde{z} \). In this case obviously \( T x \) coincides with the left shift of the configuration \( x \). Therefore (\( \tilde{z}_t - \tilde{z}_0 \cdot \rho(\tilde{z}, [\tilde{z}_0, \tilde{z}_t]) \equiv t \forall t \geq 0 \) and thus

\[
V(x, 0, t) = (\tilde{z}_t - \tilde{z}_0)/t = 1/\rho(\tilde{z}, [\tilde{z}_0, \tilde{z}_t]) \xrightarrow{t \to \infty} 1/\rho(\tilde{z}).
\]

If \( \alpha = k/n \), the configuration \( x := \alpha \tilde{z} \) is ‘almost’ spatially periodic: it consists of blocks of the configuration \( \tilde{z} \) having equal numbers, \( n \), of particles. The spatial periodicity immediately implies the existence of the average velocity \( V(x) \). Observe now that for any rational \( \alpha < 1 \) the particles in the configuration \( \alpha \tilde{z} \) on average move exactly as in the
original configuration \( \tilde{z} \) (i.e. \( V(\alpha \tilde{z}) \equiv V(\tilde{z}) \)). If \( \alpha = k/n \geq 1 \), then the direct calculation gives \( V(x) = (n/k)V(\tilde{z}) \). Passing to the limit as \( k/n \rightarrow 0 \) \( \alpha \) we get eventually

\[
V(\alpha \tilde{z}) := \begin{cases} 
\frac{1}{\rho(\tilde{z})} & \text{if } \alpha \leq 1 \\
\frac{1}{\alpha \rho(\tilde{z})} & \text{otherwise.}
\end{cases}
\]

Here one uses that \( V((k/n)\tilde{z}) \geq V((k/n\ell)\tilde{z}) \forall \ell \in \mathbb{Z}_+ \).

6. Varying waiting times and local velocities

In this section we consider a more general model which includes both varying waiting times and local velocities. Namely we assume that the waiting time at the \( j \)th obstacle is equal to \( \tau_j \in \{1, 2, \ldots, \tau\} \), and the local velocity for particles in the spatial segment \((z_j, z_{j+1})\) is equal to \( v_j \in (0, v] \). It is useful to think about this model as a road divided into parts separated by obstacles (e.g. traffic lights) and having different qualities of the road surface, which we describe by varying the local velocities from part to part.

If \( \tau_j > 1 \), it might be possible that a new particle is coming to an obstacle when some preceding particles are still waiting there. We assume that for the given particle the waiting time \( \tau_j \) at the \( j \)th obstacle starts only when the succeeding particle leaves it. The movement of a particle waiting at a certain obstacle in the next moment of time depends on the time that this and preceding particles have already spent there. Therefore this model is non-Markovian. To cure this pathology, using methods developed in [3] (for a very different situation) one adds a new variable type (for each particle) whose value is equal to the amount of time for which the particle will wait at an obstacle if it is located on the obstacle and zero otherwise.

Strictly speaking, in order to use the notion of the coupling this is important (since it is defined for a pair of Markov processes). Nevertheless in the deterministic setting which we consider in this paper, this issue is not crucial (as we shall see) and we proceed without this Markovian extension.

Denote by \( \tilde{\tau}, \tilde{v} \) the collections of waiting times and local velocities corresponding to corresponding obstacles. If \( \tau_j \equiv \tau \) and \( v_j \equiv v \), we recover the previous setting. Surprisingly the analysis of this general setting is very similar to that of the previous one. Therefore we shall discuss only the points where the analysis differs.

First we refine the configuration of obstacles \( z \), exchanging the original \( i \)th obstacle with \( \tau_i + 1 \) subsequent obstacles located at the same position \( z_i \) but having 0 waiting times, i.e. on leaving one of these new obstacles the particle immediately moves to the next one. The resulting ordered collection of obstacles is again called \( z \). To take care of the change of indices we also change the collection of local velocities, inserting \( \tau_i \) unit velocities (before the original element \( v_i \)) corresponding to new obstacles, and re-enumerating them.

Obviously for any configuration of particles \( x \), the movements of its elements for the original configurations of obstacles and velocities and for the refined ones are exactly the same.

To construct the extended configuration of obstacles \( \tilde{z} \) we proceed as follows: between the obstacles \( z_i \) and \( z_{i+1} \) we insert \([(z_{i+1} - z_i)/v_i]\) virtual obstacles at distance \( v_i \) starting
from the point \( z_i \) for each \( i \in \mathbb{Z} \). If \( v_i \equiv v \), this construction boils down to the one described in section 1.

It is straightforward to check that all constructions and results obtained in sections 2–4 remain valid in this more general setting except just one point. In the definition of the proper pair of configurations, one changes to saying that the open segment between two mutually paired particles belonging to the interval \([z_i, z_{i+1}]\) for some \( i \in \mathbb{Z} \) cannot exceed \( v_i \) (instead of \( v \) as in the original definition).

Therefore using the same arguments as in the proof of theorem 1, we get its generalization.

**Theorem 2 (Fundamental diagram).** Let \( z, \tau_j \in \{1, 2, \ldots, \tau \}, v_j \in (0, v] \) be given and let \( \rho(x), \rho(\bar{z}) \) be well defined and positive. Then
\[
V(x) = \min \{1/\rho(\bar{z}), 1/\rho(x)\}.
\]

(6.1)

7. Discussion and generalizations

(1) The density of a configuration in the way in which it was defined in section 2 depends sensitively on the statistics of both left and right tails of the configuration. A close look reveals that in fact if all particles move in the same direction, say right, one needs only the information about the corresponding (right) tail, which allows one to expand significantly the set of configurations having densities and for which our results can be applied.

For a configuration \( x \in X \), by a one-sided particle density we mean the limit
\[
\hat{\rho}(x) := \lim_{\ell \to \infty} \rho(x, [0, \ell]).
\]

(7.1)

The upper and lower one-sided densities correspond to the upper and lower limits.

**Theorem 3.** All previous results formulated in terms of ‘two-sided’ densities remain valid if one replaces the usual particle density \( \rho \) by the one-sided density \( \hat{\rho} \).

**Proof.** The key observation here is that the movement of a given particle in a configuration \( x^t \in X \) depends only on particles with larger indices. Therefore if one changes the positions of all particles with negative indices, the particles with positive indices will still have the same average velocity. On the other hand, by lemma 3 the average velocity does not depend on the particle index. This allows us to apply the following trick.

With each configuration \( x \in X \) of density \( \rho(x) \) we associate a new configuration \( \hat{x} \in X \) defined by the relation
\[
\hat{x}_i := \begin{cases} x^t_i & \text{if } i \geq 0 \\ x_0 + i/\rho(x) & \text{otherwise.} \end{cases}
\]

Then obviously \( \hat{\rho}(x) = \rho(\hat{x}) = \rho(x) \).

Therefore for all purposes related to the average velocities, all results valid for the configuration \( \hat{x} \) remain valid for \( x \) as well. \( \square \)

(2) A close look at the proof of theorem 1 may lead to the hypothesis that for a given configuration of obstacles \( z \), ‘typical’ trajectories eventually become supported only by
the locations belonging to the extended configuration $\tilde{z}$. Indeed, the calculation of the average velocity is given exactly for the configurations of this type. Let us show that this is not the case even in the simplest setting when the configuration of obstacles has density 1 and is supported by integer points (i.e. a single obstacle at each point of $\mathbb{Z}$). Suppose that $\rho(x) \geq 2$ and let the initial configuration have at least two particles at each half-integer point $\frac{1}{2} \mathbb{Z}$. Then for each $v \geq 1/2$ and $t \geq 1$ the configuration $x^t$ is supported by the lattice $\mathbb{Z} \cup \frac{1}{2} \mathbb{Z}$.

(3) We consider the question of the existence of $\rho(\tilde{z})$. Let $\rho(z) > 0$ be well defined. For each given $v > 0$ we say that the configuration $z$ is regular if the density of the corresponding extended configuration $\tilde{z}$ is well defined. In the topology induced by the uniform metric in the space of sequences, there exists an open set containing irregular configurations $z$. Nevertheless, assuming a reasonable model for the creation of the configuration of obstacles, one can show that a ‘typical’ configuration of obstacles is regular.

Indeed, assume that the sequence $z$ is a realization of an ergodic stochastic process with stationary increments. Then the Birkhoff Ergodic Theorem implies that the Cesaro means for the sequence of fractional parts $\{(z_{i+1} - z_i)/v\}$ converge to the limit and thus our claim follows.

If $\rho_+ (\tilde{z}) \neq \rho_- (\tilde{z})$, then one may consider upper/lower particle velocities and the corresponding statement for the fundamental diagram may be rewritten as follows.

**Theorem 4.** Let $v, \rho(z) > 0$ be given. Then

$$V_\pm (x) = \min \{1/\rho_\pm (\tilde{z}), \ 1/\rho_\pm (x)\}.$$  

The proof of this result follows basically the same arguments as in the case of theorem 1 but some additional technical estimates related to partial limits are necessary.

(4) Assume now that we consider our original setting (i.e. $\tau_j \equiv 0 \ \forall j \in \mathbb{Z}$) and the local velocity of the $i$th particle does not depends on space (as in section 6) but does depend on time according to a given collection of local velocities $\{v^t_i\}$, i.e. $v^t_i$ stands for the local velocity of the $i$th particle at time $t$. Thus using the notation introduced in section 1 we get $\xi^t_i := \min (\Delta_i (x, z), v^t_i)$ and $x^t_{i+1} := x^t_i + \xi^t_i$.

**Theorem 5.** For any given configuration of obstacles $z$ such that $z_k \xrightarrow{k \to \infty} \infty$ and two one-particle configurations $x := \{x_0\} \neq \hat{x} := \{\hat{x}_0\}$, there exists a sequence of local velocities $\{v^t_0\}$ such that $|x^t_0 - \hat{x}^t_0| \geq Ct$ for all $t \in \mathbb{Z}_+$ and some $C > 0$. Hence average particle velocities cannot coincide.

The proof is based on the observation that one can choose $\{v^t_0\}$ such that each time $t$ when the $x$-particle meets an obstacle, it makes a ‘full’ step equal to $\{v^t_0\}$, while the $\hat{x}$-particle which meets a different obstacle makes only half a step.

This result shows that a direct generalization of our pure deterministic setting to the random case is not possible.

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Exclusion-type spatially heterogeneous processes in continua

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