HIGH TEMPERATURE SHERRINGTON-KIRKPATRICK MODEL FOR GENERAL SPINS

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ABSTRACT. Francesco Guerra and Fabio Toninelli [3, 1] have developed a very powerful technique to study the high temperature behaviour of the Sherrington-Kirkpatrick mean field spin glass model. They show that this model is asymptotically comparable to a linear model. The key ingredient is a clever interpolation technique between the two different Hamiltonians describing the models.

This paper contribution to the subject are the following:

• The replica-symmetric solution holds for general spins, not just ±1 valued.
• The proof does not involve cavitation but only first order differential calculus and Gaussian integration by parts.

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1. Introduction

In spin glasses models a generic configuration \( \sigma = (\sigma_1, \ldots, \sigma_n) \) represents the position of \( n \) spins \( \sigma_i \in \{\pm 1\} \). In the Sherrington-Kirkpatrick model, the external disorder is given by \( n(n-1)/2 \) iid (independent identically distributed) random variables \( (g_{ij})_{1 \leq i < j \leq n} \) assumed \( \mathcal{N}(0,1) \) that is centered unit Gaussian. The Hamiltonian, for a given inverse temperature \( \beta \) and in some external field of strength \( h \), is given by

\[
H_n^{(SK)}(\sigma) = \frac{1}{\sqrt{n}} \sum_{i < j} g_{ij} \sigma_i \sigma_j + h \sum_{i=1}^n \sigma_i.
\]

The partition function \( Z_n^{(SK)}(\beta) \) and free energy \( \alpha_n^{(SK)}(\beta) \) of the model are given by

\[
Z_n^{(SK)}(\beta) = \sum_{\sigma} e^{H_n(\sigma)} , \quad \alpha_n^{(SK)}(\beta) = \frac{1}{n} \mathbb{E}^{(g)}[\log Z_n(\beta)].
\]

Physicists [3] and mathematicians [4] both proved that for high temperatures, \( \alpha_n^{(SK)}(\beta) \) converges to the Sherrington-Kirkpatrick replica symmetric solution

\[
\alpha_n^{(SK)}(\beta) \to \alpha_\infty^{(SK)}(\beta) = \log 2 + \frac{\beta^2}{4} (1 - q)^2 + \mathbb{E}^{(g)}[\log \cosh(h + g\beta \sqrt{q})],
\]

where \( q \) is the unique solution of the equation \( q = \mathbb{E}^{(g)}[\tanh^2(\beta g \sqrt{q} + \beta h)] \) (here and in the following \( g \) is a standard \( \mathcal{N}(0,1) \) Gaussian random variable).

To introduce the concept of general spins, we need to normalize things

- \( \beta = \sqrt{2t} \).
- We divide the partition function by the number \( 2^n \) of configurations.
- We compensate each weight (Boltzmann factor) \( e^{H_n^{(SK)}(\sigma)} \) so that it has expectation 1 with respect to the external disorder.

With the notation \( H_n(\sigma) = \sqrt{\frac{2}{n}} \sum_{i < j} g_{ij} \sigma_i \sigma_j + \frac{1}{\sqrt{n}} \sum_i g_n \sigma_i^2 \) the partition function is

\[
Z_n(t) = \mathbb{E} \left[ e^{\sqrt{t} H_n(\sigma) + h \sum_{i=1}^{n-1} \sigma_i - \frac{t}{2n}} \right] = 2^{-n} \sum_{\sigma} e^{\sqrt{t} H_n(\sigma) + h \sum_{i=1}^n \sigma_i - \frac{t}{2n}},
\]

where under the probability \( \mathbb{P} \), \( \sigma_i \) are iid random variables with distribution \( \mathbb{P}(\sigma_i = \pm 1) = \frac{1}{2} \). We have now

\[
\alpha_n(t) = \frac{1}{n} \mathbb{E}^{(g)}[\log Z_n(t)] = \alpha_n^{(SK)}(\beta) - \log 2 - \frac{t}{2},
\]

and thus Sherrington-Kirkpatrick’s result can be rephrased

\[
\alpha_n(t) \to \alpha_\infty(t) = \frac{t}{2} q^2 + \mathbb{E}^{(g)} \left[ \log \cosh(h + g \sqrt{2qt}) \right] - t q,
\]

with \( q \) the unique solution of \( q = \mathbb{E}^{(g)}[\tanh^2(\beta g \sqrt{q} + h)] \).
(let us observe that the introduction of a fixed random variable $\sum_i g_{ii} \sigma_i^2 = \sum_i g_{ii}$ does not change the free energy and simplifies the computations).

To generalize the model we assume now that the spins are not $\pm 1$ valued, but that they are just, under $\mathbb{P}$ symmetric iid random variables with values in $[-1,1]$; for instance uniformly distributed on $[-1,1]$. We introduce, as usual, the mutual overlap between two spins $\sigma, \tau$

$$q_{12} = q_{12}(\sigma, \tau) = \frac{1}{n} \sum_{i=1}^{n} \sigma_i \tau_i.$$  

Then

$$H_n(\sigma) = \sqrt{\frac{2}{n}} \sum_{i<j} g_{ij} \sigma_i \sigma_j + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_{ii} \sigma_i^2,$$

is a centered Gaussian process with covariance

$$\mathbb{E} [H_n(\sigma)H_n(\tau)] = \frac{2}{n} \sum_{i<j} \sigma_i \sigma_j \tau_i \tau_j + \frac{1}{n} \sum_{i} \sigma_i^2 \tau_i^2 = n q_{12}^2(\sigma, \tau).$$

Accordingly,

$$Z_n(t) = \mathbb{E} \left[ e^{\sqrt{t} H_n(\sigma) + h \sum_{i=1}^{n} \sigma_i - \frac{t}{2} n q_{12}^2(\sigma, \sigma)} \right], \quad \alpha_n(t) = \frac{1}{n} \mathbb{E}(g) \left[ \log Z_n(t) \right].$$

Our main result is the following

**Theorem 1.** Let $\phi(u, v)$ and $q^{(\text{lin})}(x)$ be defined by

$$e^{\phi(u, v)} = \mathbb{E} \left[ e^{u \sigma_i + v \sigma_i^2} \right], \quad q^{(\text{lin})}(x) = \mathbb{E}(g) \left[ \partial_u \phi^2(h + g \sqrt{x}, -x/2) \right].$$

There exists a number $t_c > 0$ such that for all $t \leq t_c$

$$\alpha_n(t) \rightarrow \alpha_\infty(t) = \frac{t}{2} q^2 + \mathbb{E}(g) \left[ \phi(h + g \sqrt{2qt}, -qt) \right],$$

where $q = q_c(t)$ is the unique solution of $q^{(\text{lin})}(2qt) = q$.

For the classical Sherrington-Kirkpatrick model, $\phi(u, v) = v + \log \cosh u$ and $\partial_u \phi^2(u, v) = \tanh^2(u)$.

Let us now explain what is the key ingredient of the proof: Francesco Guerra’s interpolation technique. It has been successfully used by Guerra [1], Guerra and Toninelli [2] and Talagrand [5], to show that the replica-symmetric formula holds in a region which probably coincides with the Almeida-Thouless region.

We introduce a simpler model, with a linear random Hamiltonian $\sqrt{x} \Lambda_n(\sigma) + h \sum_i \sigma_i$, with $\Lambda_n(\sigma) = \sum_i J_i \sigma_i$, and $J_i$ are independent $\mathcal{N}(0, 1)$. For this linear model, it is immediate to compute the free energy $\alpha^{(\text{lin})}_n(x)$. 

Then, we consider a two parameter Hamiltonian, $\sqrt{i}H_n(\sigma) + \sqrt{\lambda}n(\sigma) + h\sum_i \sigma_i$ and compare the free energies obtained for $x = 0$, the Sherrington-Kirkpatrick model, and $t = 0$ the linear model. This is easily done for some $x = 2qt$ with $q$ a solution of an equation involving $t$ and $h$. 
2. THE LINEAR MODEL

This model is simpler to study than the Sherrington-Kirkpatrick model, because here the partition function is a product of independent factors.

Recall that the spins $\sigma_i$ are assumed to be independent identically distributed, with values in $[-1, 1]$. The function $\phi$ denotes the mixed Laplace exponent

$$\phi(u, v) = \log \mathbb{E}\left[ e^{u\sigma_i + v\sigma_i^2} \right].$$

The Hamiltonian is a linear random form on the spins $\Lambda_n(\sigma) = \sum_{i=1}^n \sigma_i J_i$ with $(J_i)$ independent $\mathcal{N}(0, 1)$: its covariance is $n$ times the the overlap:

$$\mathbb{E}\left[ \Lambda_n(\sigma) \Lambda_n(\sigma') \right] = nq_{12}(\sigma, \sigma').$$

There is an external linear field of strength $h \geq 0$ so the partition function is:

$$Z_n^{(\text{lin})}(x) = \mathbb{E}\left[ e^{\sqrt{x} \sum_{i=1}^n \sigma_i J_i - \frac{x}{2} nq_{12}(\sigma, \sigma) + h \sum_{i=1}^n \sigma_i} \right].$$

**Proposition 2.**

1) The mean overlap is given by

$$(1) \quad \mathbb{E}^{(g)}[(q_{12})] = \mathbb{E}^{(g)}[\partial_u \phi^2 (h + \sqrt{x}g, -x/2)] \overset{\text{def}}{=} q^{(\text{lin})}(x).$$

2) There exists two numbers $l_0 > 0$ and $L$ such taht for any $l \leq l_0$, and any $x, h, n$:

$$(2) \quad \mathbb{E}^{(g)}\left[ \log \left\langle \exp \ln \left( q_{12}(\sigma, \tau) - q^{(\text{lin})}(x) \right)^2 \right\rangle \right] \leq L.$$

where the bracket represents a double integral with respect to the Gibbs measure.

**Proof.**

1) Let $\alpha^{(\text{lin})}(x) = \frac{1}{n} \mathbb{E}^{(g)}[\log Z_n(x)]$. On the one hand, integration by parts (see section 4) yields

$$\frac{d\alpha^{(\text{lin})}}{dx} = -\frac{1}{2} \mathbb{E}^{(g)}[(q_{12})].$$

On the other hand, a direct computation using the independence of spins yields

$$Z_n^{(\text{lin})}(x) = \prod_{i=1}^n \mathbb{E}\left[ e^{(h+\sqrt{x}g)\sigma_i - \frac{x}{2} \sigma_i^2} \right] = \exp \sum_{i=1}^n \phi(h + \sqrt{x}J_i, -x/2).$$

Therefore, $\alpha^{(\text{lin})}(x) = \mathbb{E}^{(g)}[\phi(h + \sqrt{x}g, -x/2)]$, and by integration by parts

$$\frac{d\alpha^{(\text{lin})}}{dx} = \mathbb{E}^{(g)}\left[ \frac{1}{2\sqrt{x}} \partial_u \phi(h + \sqrt{x}g, -x/2) \right] - \frac{1}{2} \mathbb{E}^{(g)}\left[ \partial_u \phi(h + \sqrt{x}g, -x/2) \right]$$

$$= \mathbb{E}^{(g)}\left[ \frac{1}{2} \phi^2 (h + \sqrt{x}g, -x/2) \right] - \frac{1}{2} \mathbb{E}^{(g)}\left[ \partial_u \phi(h + \sqrt{x}g, -x/2) \right]$$
Hence,
\[ q^{\text{lin}}(x) = E^{(g)}[q_{12}] = E^{(g)} \left[ (\partial_v - \partial_u^2)(h + \sqrt{x}g, -\frac{x}{2}) \right] = E^{(g)}[\partial_u \phi^2 (h + \sqrt{x}g, -x/2)] \]
because \( \partial_v \phi - \partial_u^2 \phi = (\partial_u \phi)^2 \).

2) This part of the proof can be established via the cavitation techniques introduced by Talagrand. This is a direct consequence of a stronger exponential inequality: there exists a constant \( C > 0 \), such that for all \( x, h \) (\( h \) small enough)
\[ E^{(g)} \left[ \exp \frac{n}{C} (q_{12}(\sigma, \tau) - q^{\text{lin}}(x))^2 \right] \leq C. \]

However, we shall give a more direct proof, which avoids the hassles of cavitation.

Let
\[ V_n(\sigma) = \sqrt{x} \sum_{i=1}^n \sigma_i J_i - n^2 q_{12}(\sigma, \sigma) + h \sum_{i=1}^n \sigma_i, \]
and
\[ U_n(\sigma, \tau) = V_n(\sigma) + V_n(\tau) + l(q - q_{12}(\sigma, \tau))^2. \]

If \( Z_n(x, l) \) denotes the partition function \( Z_n(x, l) = E[U_n(\sigma, \tau)] \) then we shall prove that for \( 0 \leq l \leq \frac{1}{20} \) and \( q = q^{\text{lin}}(x) \),
\[ E^{(g)}[\log Z_n(x, l) - \log Z_n(x, 0)] \leq L, \]
where \( L \) is a number.

To this end, observe that by introducing an auxiliary unit gaussian random variable \( \gamma \) with associated expectation \( E^{(g)} \), we can write, using the independence of spins,
\[ Z_n(x, l) = E^{(g)} \left[ E \left[ e^{U_n(\sigma) + U_n(\tau) + \sqrt{2ln} (q_{12}(\sigma, \tau) - q)} \right] \right] = E^{(g)} \left[ e^{-q\sqrt{2ln}} \exp \left( \sum_{i=1}^n \psi(h + \sqrt{x}J_i, -x/2, l) \right) \right] \]

with
\[ e^{\psi(u, v, l)} = E \left[ e^{\sigma_1 u + \sigma_1^2 v + \tau_1 u + \tau_1^2 v + l \sigma_1 \tau_1} \right]. \]

It is easily seen that the function \( l \to \psi(u, v, l) \) is convex, twice differentiable, of first derivative at \( l = 0 \)
\[ \frac{\partial \psi}{\partial l}(u, v, l = 0) = (\partial_u \phi)^2(u, v), \]
and satisfies
\[ 0 \leq \frac{\partial^2 \psi}{\partial^2 l^2}(u, v, l) \leq 4. \]

Indeed \( \psi(u, v, l) - \psi(u, v, 0) \) is the logarithm of the Laplace transform in \( l \) of a random variable taking its values in \([-1, 1]\). Therefore, the second
derivative is the variance with respect to a twisted probability measure of a random variable taking its values in \([-1, 1]\), and is bounded by 4. Therefore,

\[ \psi(u, v, l) \leq \psi(u, v, 0) + l(\partial_u \phi)^2(u, v) + 2l^2. \]

Hence,

\[ \frac{Z_n(x, l)}{Z_n(x, 0)} \leq \mathbb{E}^\gamma \left[ \exp \left( -q\gamma \sqrt{2ln} + 4l\gamma^2 + \sum_{i=1}^{n} \gamma \sqrt{2l/n}(\partial_u \phi)^2(h + \sqrt{x}J_i, -x/2) \right) \right]. \]

From now on, \( L \) is a number whose value may change from line to line. One easily shows (e.g. with the help of Hölder’s inequality) that for \( 0 \leq v \leq 1/5 \) and any \( u \)

\[ \mathbb{E}^\gamma \left[ e^{\gamma u + v^2} \right] \leq Le^{u^2}. \]

this entails that for \( 0 \leq l \leq 1/20 \),

\[ \log \frac{Z_n(x, l)}{Z_n(x, 0)} \leq L + \frac{2l}{n} \left( \sum_{i=1}^{n} (\partial_u \phi)^2(h + \sqrt{x}J_i, -x/2) - q \right)^2. \]

The random variables \( X_i = (\partial_u \phi)^2(h + \sqrt{x}J_i, -x/2) - q \) are independent identically distributed. Since \( q = q^{(\text{lin})}(x) \), they are centered, and we observe that they are bounded: \( X_i \in [-1, 1] \). We can now conclude

\[ \mathbb{E}^{(g)} \left[ \log \frac{Z_n(x, l)}{Z_n(x, 0)} \right] \leq L + \frac{2l}{n} \mathbb{E}^{(g)} \left[ (X_1 + \cdots + X_n)^2 \right] \leq L + 2l \leq L'. \]

\[ \Box \]
3. THE SHERRINGTON KIRKPATRICK MODEL

The Hamiltonian is a bilinear random form on the spins

$$H_n(\sigma) = \sqrt{\frac{2}{n}} \sum_{i<j} g_{ij} \sigma_i \sigma_j + \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g_{ii} \sigma_i^2,$$

where \((g_{ij})\) is a family of independent gaussian \(N(0, 1)\). If, as usual, \(q_{12}\) denotes the overlap

$$q_{12} = q_{12}(\sigma, \tau) = \frac{1}{n} \sum_{i=1}^{n} \sigma_i \tau_i,$$

then the covariance of the process \((H_n(\sigma))\) is given by

$$\mathbb{E}^{(g)}[H_n(\sigma)H_n(\tau)] = nq_{12}^2(\sigma, \tau).$$

We want to understand the asymptotic behaviour of the partition function

$$Z_n(t) = \mathbb{E} \left[ e^{\sqrt{t}H_n(\sigma)} - \frac{t}{2} n q_{12}^2(\sigma, \sigma) + h \sum_{i=1}^{n} \sigma_i \right]$$

and so we introduce the mean free energy

$$\alpha_n(t) = \frac{1}{n} \mathbb{E}^{(g)}[\log Z_n(t)].$$

The main idea, taken from Guerra’s papers, is to show that in the presence of a linear external field, the partition function \(Z_n(t)\) can be compared asymptotically to the partition function of a linear model (see section 2) \(Z_n^{(\text{lin})}(x)\) with \(x\) an implicit function of \(t\). Let us first explain this by giving an upper bound.

**Theorem 3.** 1) With the notations \(\alpha^{(\text{lin})}(x)\) and \(q^{(\text{lin})}(x)\) introduced in section 2, we have the upper bound:

$$\limsup_{n \to +\infty} \alpha_n(t) \leq \inf_{q>0} \alpha^{(\text{lin})}(2qt) + \frac{t}{2} q^2.$$

2) There exists \(t_c > 0\) such that for \(t \leq t_c\) and \(h > 0\), the infimum is attained at \(q_c(t)\) the unique solution of \(q^{(\text{lin})}(2qt) = q\).

**Proof.** Following Guerra, we introduce an interpolation between the two Hamiltonians \(H_n\) and \(\Lambda_n\). The two Gaussian processes \(H_n(\sigma)\) and \(\Lambda_n(\sigma)\) are assumed independent in the partition function

$$\tilde{Z}_n(t, x) = \mathbb{E} \left[ e^{\sqrt{t}H_n(\sigma) + \sqrt{x} \Lambda_n} \right]$$

According to gaussian integration by parts (see section 4), the partial derivatives of \(\tilde{\alpha}_n(t, x) = \frac{1}{n} \mathbb{E}^{(g)}[\log \tilde{Z}_n(t, x)]\) are given by:

$$\frac{\partial \tilde{\alpha}_n}{\partial t} = -\frac{1}{2} \mathbb{E}^{(g)}[\langle q_{12}^2 \rangle],$$
and (as we have already computed in section 2)

\[ \frac{\partial \tilde{\alpha}_n}{\partial x} = -\frac{1}{2} E^{(g)} [(q_{12})]. \]

Therefore, if we fix \( t, x_0, q > 0 \) and move along the trajectory \( x(s) = x_0 - 2qs \), we have

\[ \frac{d}{ds} \tilde{\alpha}_n(s, x(s)) = \frac{1}{2} \left( q^2 - E^{(g)} [\langle (q_{12} - q)^2 \rangle] \right) \leq \frac{1}{2} q^2. \]

Hence,

\[ \tilde{\alpha}_n(t, x(t)) \leq \tilde{\alpha}_n(0, x_0) + \frac{t}{2} q^2 \leq \alpha^{(\text{lin})}(x_0) + \frac{t}{2} q^2. \]

We now impose the relationship \( x_0 = 2qt \), in order to get \( x(t) = 0 \) and obtain the upper bound:

\[ \alpha_n(t) = \alpha_n(t, 0) \leq \alpha^{(\text{lin})}(2qt) + \frac{t}{2} q^2. \]

We obtain the desired result by taking \( \limsup \) and optimizing in \( q > 0 \).

2) It is just a matter of computing the derivative of \( f(q) = \alpha^{(\text{lin})}(2qt) + \frac{1}{2} q^2 \).

Using the computations of section 2, we get:

\[ f'(q) = 2t(\alpha^{(\text{lin})}(2qt)). \]

Fact 1: since \( h > 0 \), and \( \sigma_i \) is not identically 0, we have \( q^{(\text{lin})}(0) > 0 \).

Indeed, \( \partial_u \phi(u, 0) = \frac{E[\sigma_1 e^{h\sigma_1}]}{E[e^{h\sigma_1}]} \) and therefore, by symmetry

\[ E \left[ \sigma_1 e^{h\sigma_1} \right] = 2 E \left[ \sigma_1 \sinh(h\sigma_1) \right] > 0. \]

Fact 2: if \( q \) is large enough \( f'(q) > 0 \). Indeed, the function \( q^{(\text{lin})} \) is bounded \( q^{(\text{lin})}(x) \leq 1 \) since the spins are themselves bounded by 1.

We have \( f'(0) < 0 \) and \( f'(q) > 0 \) if \( q \) is large enough, therefore, all we have to do now is to prove that the equation \( q = q^{(\text{lin})}(2qt) \) has a unique solution (the infimum of \( f \) will then be attained there).

We compute the derivative

\[ \frac{d q^{(\text{lin})}(x)}{dx} = -\frac{1}{2} E^{(g)} \left[ \left( \partial^4_u + \partial^2_{uv} \right) \phi(h + g\sqrt{x}, -x/2) \right]. \]

Since \( \phi \) and all its partial derivatives are bounded, the derivative of \( q^{(\text{lin})} \) is itself bounded by a constant \( C : \left| \frac{d q^{(\text{lin})}(x)}{dx} \right| \leq C \). Hence, for \( t < \frac{1}{4C} \) the function \( q \rightarrow q^{(\text{lin})}(2qt) \) is Lipschitz with constant at most \( \frac{1}{2} \), and therefore it has a unique fixed point.

\( \square \)

To show that this upper bound yields asymptotically a lower bound, the proof is a little more involved.
Theorem 4. There exists $t_c > 0$ such that for $t < t_c$ we have the convergence
\[
\lim_{n \to +\infty} \alpha_n(t) = \alpha^{(\text{lin})}(2qt) + \frac{t}{2} q^2 ,
\]
where $q = q_c(t)$ is the unique solution of $q^{(\text{lin})}(2qt) = q$.

Proof. We consider now two independent copies $\sigma, \tau$ of the spins, and the partition function
\[
\hat{Z}(t, x, l) = \mathbb{E} \left[ e^{U_n(\sigma, \tau)} \right]
\]
where $l$ is a new positive parameter, and $q$ will be specified later. Accordingly, if $\tilde{\alpha}_n(t, x, l) = \frac{1}{2n} \mathcal{E}^{(g)}[\log \hat{Z}(t, x, l)]$ then
\[
\frac{\partial \tilde{\alpha}_n}{\partial t} = \frac{1}{2} \mathcal{E}^{(g)} \left[ \langle q_{12}^2(\sigma, \tau) - 2q_{12}^2(\sigma, \sigma') \rangle \right] = \frac{1}{2} \mathcal{E}^{(g)} \left[ \langle q_{12}^2 - 2q_{13}^2 \rangle \right]
\]
where we have now four replicas $\sigma, \tau, \sigma', \tau'$ with $\sigma, \tau$ coupled by $l(q - q_{12}(\sigma, \tau))^2$ and $\sigma', \tau'$ coupled by $l(q - q_{12}(\sigma', \tau'))^2$, and $q_{13} = q_{12}(\sigma, \sigma')$.

Fact 1. For every $l \geq 0, t \geq 0, x_0 \geq 2qt, q \geq 0$ we have:
\[
\tilde{\alpha}_n(t, x_0 - 2qt, l) \leq \tilde{\alpha}_n(0, x_0, l + t) + \frac{t}{2} q^2 .
\]

Fix $x_0, t, l_0 > 0$. We shall compute the derivative along the trajectory $x(s) = x_0 - 2qs, l = l_0 - s$:
\[
\frac{d}{ds} \tilde{\alpha}_n(s, x(s), l(s)) = \frac{1}{2} q^2 - \mathcal{E}^{(g)} \left[ \langle (q - q_{13})^2 \rangle \right] \leq \frac{1}{2} q^2 .
\]
Therefore, integrating between 0 and $t$,
\[
\tilde{\alpha}_n(t, x(t), l(t)) - \tilde{\alpha}_n(0, x_0, l_0) \leq \frac{t}{2} q^2 ,
\]
and this is the desired inequality.

Fact 2. We shall now use the following fact, coming from the proof of Theorem 3 (see equation (3)): for every $x_0 \geq 2qt$,
\[
\tilde{\alpha}_n(t, x_0 - 2qt) = \tilde{\alpha}_n(0, x_0) + \frac{t}{2} q^2 - \frac{1}{2} \int_0^t \mathcal{E}^{(g)} \left[ \langle (q - q_{12})^2 \rangle \right] ds .
\]
We can now introduce the function
\[ h_n(t) \overset{\text{def}}{=} \bar{\alpha}_n(0, x_0) + \frac{t}{2} q^2 - \bar{\alpha}_n(t, x_0 - 2qt) = \int_0^t E^{(g)} \left[ \langle (q - q_{12})^2 \rangle_{s, x_0 - 2qs, 0} \right] ds. \]

For \( l, t, x_0 \) small enough, and \( q = q^{(\text{lin})}(x_0) \), we have
\[
\frac{d}{dt} h_n(t) = \frac{l}{2} E^{(g)} \left[ \langle (q - q_{12})^2 \rangle_{t, x_0 - 2qt, 0} \right] \quad \text{(from (\ref{eq:energy})�})
\leq \frac{1}{2n} E^{(g)} \left[ \log \langle e^{ln(q - q_{12})^2} \rangle_{t, x_0 - 2qt, 0} \right] \quad \text{(Jensen’s inequality)}
= \bar{\alpha}_n(t, x_0 - 2qt, l) - \bar{\alpha}_n(t, x_0 - 2qt, 0)
\leq \bar{\alpha}_n(0, x_0, l + t) + \frac{t}{2} q^2 - \bar{\alpha}_n(t, x_0 - 2qt, 0) \quad \text{(from (\ref{eq:energy})}}
\leq \bar{\alpha}_n(0, x_0, 0) + \frac{1}{2n} \log L \frac{t}{2} q^2 - \bar{\alpha}_n(t, x_0 - 2qt, 0) \quad \text{(from Proposition (\ref{eq:energy}))}
= h_n(t) + \frac{1}{n} L.
\]
From this differential inequality, we easily get that for \( t, x_0, l \) small enough, and \( q = q^{(\text{lin})}(x_0) \), we have
\[ h_n(t) \leq e^{t/l} \frac{1}{n} L. \]
Since \( h_n \) is positive and \( h_n(0) = 0 \), by taking \( l = t \) we obtain that \( h_n(t) \to 0 \).

We now impose the constraint, \( x_0 = 2qt \) and that is possible because \( q = q_c \) is the solution of the equation \( q_c = q^{(\text{lin})}(2qt) \), and then \( h_n(t) \to 0 \) means exactly for \( q = q_c(t) \)
\[ \bar{\alpha}_n(0, 2qt) + \frac{t}{2} q^2 - \bar{\alpha}_n(t, 0) = \alpha^{(\text{lin})}(2qt) + \frac{t}{2} q^2 - \alpha_n(t) \to 0. \]
\[ \square \]
4. A FRAMEWORK FOR GAUSSIAN INTEGRATION BY PARTS

Let $X$ be a random variable defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in the configuration space $(\Gamma, \mathcal{G})$.

We assume given a gaussian environment, that is another probability space $(\Omega^{(g)}, \mathcal{F}^{(g)}, \mathbb{P}^{(g)})$ and a bimeasurable Hamiltonian $H : \Omega^{(g)} \to \mathbb{R}$ such that $(H(\gamma), \gamma \in \Gamma)$ is a centered Gaussian process with covariance

$$
\mathbb{E}^{(g)}[H(\gamma_1)H(\gamma_2)] = R(\gamma_1, \gamma_2).
$$

For example, in the classical Sherrington-Kirkpatrick model, $\Gamma = \{-1, 1\}^n$ is the space of configurations of n spins $\sigma = (\sigma_1, \ldots, \sigma_n)$ with $\sigma_i = \pm 1$ and under $\mathbb{P}$ the spins are independent Bernoulli ($\pm 1$) random variables, and $H_n(\sigma) = \sqrt{2/n} \sum_{i<j} \sigma_i \sigma_j g_{ij}$ with $g_{ij}$ iid standard Gaussian:

$$
R(\sigma, \tau) = n(q^2_{12}(\sigma, \tau) - \frac{1}{n} q_{12}(\sigma^2, \tau^2)).
$$

To use the replica technique, we assume given iid configurations $(X_i)_{i \geq 1}$ defined on the same probability space. We can then consider the Gibbs measure of one or several independent replicas, with respect to the same environment:

$$
\langle \phi(X) \rangle = \frac{1}{Z(t)} \mathbb{E} \left[ \phi(X) e^{\sqrt{t} H(X) - \frac{1}{2} R(X, X)} \right]
$$

$$
\langle \psi(X_1, X_2) \rangle = \frac{1}{Z(t)^2} \mathbb{E} \left[ \psi(X_1, X_2) e^{\sqrt{t} (H(X_1) + H(X_2) - \frac{1}{2} (R(X_1, X_1) + R(X_2, X_2)))} \right]
$$

where $Z(t)$ is the partition function

$$
Z(t) = \mathbb{E} \left[ e^{\sqrt{t} H(X) - \frac{1}{2} R(X, X)} \right].
$$

**Proposition 5.** If $\alpha(t) = \mathbb{E}^{(g)}[\log Z(t)]$ then

$$
\frac{d\alpha}{dt} = -\frac{1}{2} \mathbb{E}^{(g)}[\langle R(X_1, X_2) \rangle].
$$

**Proof.** We recall the integration by parts formula (see e.g. Talagrand [3]): if $g$ is centered normal, if the function $f$ is $C^1$, and for some constant $C$, $|f(x)| \leq e^{C|x|}$, then

$$
\mathbb{E}[gf(g)] = \mathbb{E}[g^2] \mathbb{E}[f'(g)].
$$

This can be easily extended to functions of several variables. Let $F : \mathbb{R}^n \to \mathbb{R}$ be $C^1$ and such that for a constant $C$ : $|F(x)| \leq e^{C|x|}$. Then if $(u, u_1, \ldots, u_n)$ is a centered Gaussian vector:

$$
\mathbb{E}[uF(u_1, \ldots, u_n)] = \sum_{i=1}^n \mathbb{E}[u_i] \mathbb{E} \left[ \frac{\partial F}{\partial x_i}(u_1, \ldots, u_n) \right].
$$
Differentiating with respect to $t$ yields
\[
\frac{d\alpha}{dt} = \frac{1}{2\sqrt{t}} \mathbb{E}^{(g)}[\langle H(X) \rangle] - \frac{1}{2} \mathbb{E}^{(g)}[\langle R(X,X) \rangle].
\]
Since,
\[
\langle H(X) \rangle = \int \mathbb{P}(X \in dx) \mathbb{E}^{(g)}[H(x) e^{\sqrt{t}H(x)} - \frac{1}{2} R(x,x) \frac{1}{Z(t)}] = \int \mathbb{P}(X \in dx) \mathbb{E}^{(g)}[H(x) F(H(\gamma), \gamma \in \Gamma)]
\]
for a $C^1$ function $F$ with at most exponential growth, we have
\[
\mathbb{E}^{(g)}[\langle H(X) \rangle] = \int \mathbb{P}(X_1 \in dx_1) \mathbb{E}^{(g)}[H(x_1) e^{\sqrt{t}H(x_1)} - \frac{1}{2} R(x_1,x_1) \frac{1}{Z(t)}]
\]
\[
- \int \mathbb{P}(X_1 \in dx_1) \int \mathbb{P}(X_2 \in dx_2) \sqrt{t} R(x_1,x_2) \mathbb{E}^{(g)}[e^{\sqrt{t}(H(x_1)+H(x_2)) - \frac{1}{2} (R(x_1,x_1)+R(x_2,x_2)) \frac{1}{Z(t)^2}}]
\]
\[
= \sqrt{t} \mathbb{E}^{(g)}[\langle R(X,X) \rangle - \langle R(X_1,X_2) \rangle].
\]

With the same type of computations, we obtain

**Proposition 6.** Let
\[
\hat{Z}(t) = \mathbb{E} \left[ e^{\sqrt{t}(H(X_1)+H(X_2)) - \frac{1}{2} (R(X_1,X_1)+R(X_2,X_2)) + c(x_1,x_2)} \right]
\]
where $c(x_1,x_2)$ is a coupling function, and let $\hat{\alpha}(t) = \mathbb{E}^{(g)}[\log \hat{Z}(t)]$. Then
\[
\frac{d\hat{\alpha}}{dt} = \mathbb{E}^{(g)}[\langle R(X_1,X_2) - 2R(X_1,X_3) \rangle],
\]
where in this expression $X_1, \ldots, X_4$ are independent copies of $X$, considered under the same environment, with couplings between $X_1, X_2$ and $X_3, X_4$.

**Proof.** It is mutatis mutandis the same proof, this time for the Gaussian process $K(\gamma_1, \gamma_2) = H(\gamma_1) + H(\gamma_2)$. The expression of the derivative uses symmetry between the $X_i$’s. \qed
REFERENCES

[1] F. Guerra, *Sum rules for the free energy of the spin glass model*. Conference given at Les Houches, Jan. 2000.

[2] F. Guerra and F. L. Toninelli, *Quadratic replica coupling in the Sherrington Kirkpatrick mean field spin glass model*, Journal of Mathematical Physics, 43 (2002). arXiv:cond-mat/0201091 v2 5 Mar 2002.

[3] M. Mezard, G. Parisi, and M. Virasoro, *Spin glass theory and beyond.*, no. 9 in Lecture Notes in Physics, World Scientific, Singapore, 1987. ISBN 9971-50-115-5/hbk; ISBN 9971-50-116-3/pbk.

[4] M. Talagrand, *A first course on spin glasses*. Ecole d’été de Probabilités de Saint Flour XXX, 2000.

[5] ———, *On the high-temperature phase of the Sherrington-Kirkpatrick model*, The Annals of Probability, 30 (2002), pp. 364–381.

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