Solenoidal scaling laws for compressible mixing

John Panickacheril John and Diego A. Donzis
Department of Aerospace Engineering, Texas A&M University, College Station, Texas 77843, USA

Katepalli R. Sreenivasan
Department of Mechanical and Aerospace Engineering, Department of Physics and Courant Institute of Mathematical Sciences, New York University, New York, New York 10012, USA

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Mixing of passive scalars in compressible turbulence does not obey the same classical Reynolds number scaling as its incompressible counterpart. We first show from a large database of direct numerical simulations that even the solenoidal part of the velocity field fails to follow the classical incompressible scaling when the forcing includes a substantial dilatational component. Though the dilatational effects on the flow remain significant, our main results are that both the solenoidal energy spectrum and the passive scalar spectrum scale assume incompressible forms, and that the scalar gradient aligns with the most compressive eigenvalue of the solenoidal part, provided that only the solenoidal components are used for scaling in a consistent manner. Minor modifications to this statement are also pointed out.

A defining feature of turbulence is the ability to mix substances with orders of magnitude greater effectiveness than molecular mixing. The subject has been studied extensively [1] when the mixing agent is incompressible turbulence because it is a fundamentally important problem in its own right and a good paradigm for many practical circumstances. However, there are critically important applications from astrophysics to high-speed aerodynamics in which compressibility needs to be explicitly considered. Including compressibility renders inapplicable the Reynolds number scaling laws [2, 3] that are used extensively in incompressible turbulence. This paper shows one successful way of incorporating compressibility explicitly. We show by three examples that the standard incompressible laws work in the compressible case by rescaling the appropriate variables.

The initial attempt to include compressibility was through a suitably defined Mach number as an additional parameter. For the ideal case of homogeneous isotropic turbulence in a cubic box with periodic boundary conditions, this Mach number, \( M_t = u'/\langle c \rangle \), where \( u' = \langle u_i'^2 \rangle^{1/2} \), \( u_i \) being the velocity in the Cartesian direction \( i \), and the angular brackets indicate a suitable average. However, as has been pointed out by Ni [4], \( M_t \) is not adequate when the velocity field has a strong dilatational component. Indeed, DNS data with different types of large scale forcing, such as pure solenoidal forcing [5, 6], homogeneous shear forcing [7], dilatational forcing [8, 10] and thermal forcing [11], have revealed that the dilatational flow field characteristics depend on the details of forcing, even for fixed \( M_t \). Further progress has been made recently [12] by adding yet another parameter, namely \( \delta = u_d'/u'_s \), which is the ratio of root-mean-square (rms) dilatational to solenoidal velocity. These two components can be readily obtained for homogeneous compressible turbulence, by utilizing the Helmholtz decomposition of the velocity field, \( \mathbf{u} = \mathbf{u}_s + \mathbf{u}_d \), where the solenoidal part, \( \mathbf{u}_s \), represents vortical contribution and satisfies the incompressibility condition \( \nabla \cdot \mathbf{u} = 0 \). The dilatational part, \( \mathbf{u}_d \), represents the irrotational component and satisfies \( \nabla \times \mathbf{u}_d = 0 \). The improved physical understanding that arises from [12] can be used to assess the scaling of the passive scalars in compressible turbulence.

The data for the current work come from direct numerical simulations (DNS) of compressible Navier-Stokes equations in a periodic box yielding homogeneous and isotropic turbulence, and span the following conditions: the macroscale Reynolds number \( R_\lambda \equiv \langle \rho \rangle u \lambda / \mu \), where \( \langle \rho \rangle \) is the mean density, \( \lambda \) is the Taylor microscale and \( \mu \) the mean dynamic viscosity, ranges from 38 to 165; the turbulent Mach number, \( M_t \), varies between 0 and about 0.6; the Schmidt number \( Sc = \mu / [\langle \rho \rangle D] \), where \( D \) is the diffusivity of the scalar, is unity. The forcing at low wavenumbers contains a strong dilatational component as well, with \( \delta \) ranging from 0 to 7.5. Figure 1 shows the wide range of compressibility conditions covered for the scalar field in the parameter spaces of \( R_\lambda \), \( \delta \) and \( M_t \).

The first instance of the inadequacy of incompressible scaling is the energy spectrum which, according to [13], follows the relation \( E(k) = C\langle \epsilon \rangle^{2/3} k^{-5/3} \) in the inertial range, where \( C \) is the Kolmogorov constant, \( k \) is the wavenumber, and \( \langle \epsilon \rangle \) is the mean total energy dissipation. The energy spectrum has the property that \( \int_0^\infty E(k) \, dk = \langle u_s^2 \rangle / 2 = \langle u_d^2 \rangle / 2 = \langle u_s u_d \rangle \). In Fig. 2(a) we see that, unlike in incompressible turbulence, there is no collapse of spectral data when normalized according to [13]. This is not surprising: it has been pointed out already in theories [14, 15] and simulations [16, 17] that the dilatational component of energy can take on a wide range of values without violating the incompressibility assumption.

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range of behaviors and can depart from the classical Kolmogorov scaling.

As an improvement, it has been suggested that the solenoidal part of the energy spectra \((\nabla \cdot \mathbf{u} = 0)\) does scale according classical Kolmogorov scaling; the basis for this claim comes from solenoidally forced DNS \([3, 4]\). However, this result does not hold when the forcing has a strong dilatational component, as shown in Fig. 2(b), where the Kolmogorov-compensated solenoidal energy spectra \(E_s(k)\), defined such that \(\int_0^\infty E_s(k) \, dk = \langle u_r^2 \rangle / 2\), does not scale when the forcing includes a dilatational part.

The second instance of this inadequacy is the scalar spectrum. In incompressible turbulence, its behavior is reasonably well understood at the phenomenological level \([1, 17, 23]\). For unity Schmidt number, the appropriate normalization for the passive scalars is the Obukhov-Corrsin normalization \(E_\phi(k) = C' \langle \epsilon \rangle k^{-1/3} \rho^{-5/3} \) where \(E_\phi\) is defined such that \(\int_0^\infty E_\phi(k) \, dk = \langle \phi'^2 \rangle / 2\) and \(\langle \epsilon \rangle\) is the mean scalar dissipation; \(C'\) is the Obukhov-Corrsin constant. In Fig. 3 we plot the Obukhov-Corrsin compensated scalar spectra for all cases. There is no collapse of the data, and so compressibility appears to have a first order effect on the scalar spectra.

As a third quantity, consider the alignment of the scalar gradient with the directions of the eigenvectors of the strain field. In incompressible turbulence, the turbulent velocity field plays an important role in the stirring action of passive scalars where the different isosurfaces of the scalars are brought together \([1, 21, 25]\). This stirring action results in high scalar gradients across the flow field, ultimately enabling molecular diffusion to act. Batchelor’s theory \([19]\), initially proposed for large Schmidt numbers, shows that the scalar gradient aligns itself with the most compressive eigenvalue. DNS studies \([26]\) have shown that this aspect of the theory is valid, perhaps surprisingly, even for Schmidt numbers of order unity; see also Vedula et al. \([27]\). Danish et al. \([28]\) studied this alignment for decaying compressible turbulence and found that the topology and alignment were universal for a range of Reynolds and Mach numbers, though their studies were confined to a narrow range of initial
The eigenvectors of this tensor, called here $e_\alpha$, $e_\beta$, and $e_\gamma$, correspond respectively to the maximum, intermediate, and minimum eigenvalues with $\alpha > \beta > \gamma$; incompressible turbulence is constrained by $\alpha + \beta + \gamma = 0$. The previous observations by Blaisdell et al., and more recently by Ni, that contributions from the dilatational field to the scalar flux are negligible compared to the solenoidal part alone, correspond to a narrow range of conditions.

$$M_t \ (0.50 - 0.70) \text{ and } R_\lambda \ (18 - 24).$$

For the wider range of compressible turbulent states considered here, in terms of $R_\lambda$, $M_t$, and $\delta$, Fig. 3 shows that the scalar gradient, $\nabla \phi = \partial \phi / \partial x_i$, does not align uniquely with the symmetric part of the velocity gradient tensor, $S_{ij}$, where

$$S_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

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We now come to the orientation of the scalar gradient with respect to the velocity strain field. Following the observations above, we assess the effect of the solenoidal component of the tensor, $S_{ij}^{(s)}$. In particular, we examine the statistics of the normalized eigenvalues ($\hat{\beta}_s$) given by $\hat{\beta}_s = \sqrt{6 \beta_s / (\alpha^2 + \beta^2 + \gamma^2)}$, such that $-1 \leq \hat{\beta}_s \leq 1$.

In Fig. 5(a) is plotted the probability density function (PDF) of $\hat{\beta}_s$ for a wide range of compressibility conditions. Excellent collapse is observed (curve (i)), indicating that the ratio of the PDF of the eigenvalues is unaffected by compressibility. Similar universal behavior is observed for the ratio of $\beta_\alpha / \gamma_\alpha$ shown as curve (ii) in the same figure. We also note that the maximum probable value of $\beta_s / \gamma_s$ is approximately 0.28 which corresponds to the ratio of $\gamma_s / \beta_s = 3.7$, close to the situation suggested for incompressible turbulence and consistent with results for solenoidal forcing. This...
feature suggests that, while compressibility may change the solenoidal field itself, it does not alter its mixing capability and would remain as efficient as incompressible turbulence.

Figure 6(b) plots the alignment of the scalar gradient with the solenoidal frame of reference. One finds that the behavior of the scalar gradient is very similar to that of incompressible turbulence, with a high probability for the scalar gradient to align with the most compressive direction. There are, however, some weak compressibility effects. To understand them qualitatively, we show in Fig. 7 the PDF values for \( \cos(\nabla \phi, e_\gamma) \in [0.995, 1] \)—that is, when the two vectors are almost perfectly aligned—as a function of turbulent Mach number, \( M_t \). The figure shows that \( R_\alpha \) is the major effect, though a weaker decreasing trend with \( M_t \) is also seen. In order to completely understand compressible turbulent mixing, one has to include these secondary compressibility effects on the fine scale structure of turbulence.

In summary, using high fidelity DNS data, we have shown that the interaction between passive scalar and solenoidal velocity field is universal under a wide range of compressibility conditions, for both the velocity and the scalar field, if both the velocity field and the energy dissipation are taken from the solenoidal part of the velocity.

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FIG. 5. Kolmogorov-compensated solenoidal energy spectra (a) and Obukhov-Corrsin compensated scalar spectra (b) using solenoidal dissipation, \( \langle \epsilon_s \rangle \) and solenoidal Kolmogorov length scale, \( \eta_s \). Dashed line in the bottom figure is for the incompressible case.

FIG. 6. (a) Normalized eigenvalues of the solenoidal symmetric velocity gradient tensor, \( S_{ij}^s \): i) \( \beta_s = \sqrt{\alpha_s^2 + \beta_s^2 + \gamma_s^2} \); ii) \( \beta_s / \gamma_s \). (b) Alignment of scalar gradient (\( \nabla \phi \)) with \( e_\gamma, e_\beta, e_\alpha \), the eigenvectors of \( S_{ij}^s \).

*donzis@tamu.edu*
FIG. 7. (a) Probability of scalar gradient ($\nabla \phi$) being perfectly aligned with the eigendirection $e_\gamma$ corresponding to the most compressive eigenvalue. Symbols in the figure correspond to different percentages of dilatational forcing, $\sigma$: incompressible simulations (diamonds), $\sigma = 0$ (circles), 10-30 (triangles), 30-65 (squares), and 65-100 (stars).

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