PLANE CURVES CONTAINING A STAR CONFIGURATION

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Abstract. Given a collection of \( l \) general lines \( \ell_1, \ldots, \ell_l \) in \( \mathbb{P}^2 \), the star configuration \( X(l) \) is the set of points constructed from all pairwise intersections of these lines. For each non-negative integer \( d \), we compute the dimension of the family of curves of degree \( d \) that contain a star configuration.

1. Introduction

Throughout this paper, \( S = k[x_0, x_1, x_2] \) with \( k \) an algebraically closed field. Given any linear form \( L \in S \), we let \( \ell \) denote the corresponding line in \( \mathbb{P}^2 \). Given a collection of \( l \) linear forms \( L_1, \ldots, L_l \) in \( S \) such that \( \ell_i \cap \ell_j \cap \ell_k = \emptyset \) for all triples \( \{i, j, k\} \subseteq \{1, \ldots, l\} \), the star configuration \( X(l) \) is the set of \( \binom{l}{2} \) points formed by taking all possible intersections of these lines. The name star configuration arises from the fact that the five lines that contain a \( X(5) \) resembles a star. These special configurations, and their generalizations in \( \mathbb{P}^n \), have risen in prominence due, in part, to the fact that they have nice algebraic properties (e.g., the minimal generators are products of linear forms), but at the same time exhibit some extremal properties (e.g., the work of Bocci and Harbourne \( \mathbb{B} \) which compares symbolic and regular powers of ideals). The papers \( [1, 2, 7, 8, 9, 13] \) are some of the papers that have contributed to our understanding of star configurations.

In this paper we compute the dimension of the family of curves in \( \mathbb{P}^2 \) of degree \( d \) that contain a star configuration \( X(l) \). More precisely, consider the quasi-projective variety \( D_l \subseteq \mathcal{A}^\times \mathcal{D} \) where \( \{\ell_1, \ldots, \ell_l\} \in D_l \) if and only if no three of the lines meet at a point; here \( \mathcal{B}^\times \) denotes the dual projective space. Notice that \( D_l \) can be seen as a parameter space for star configuration set of points obtained by intersecting \( l \) general lines. With a slight abuse of notation, we will often write \( X(l) \in D_l \), thus identifying a star configuration with the unique set of lines defining it.

We construct the following incidence correspondence

\[
\Sigma_{d,l} = \{ (C, X(l)) : C \supseteq X(l) \} \subseteq \mathbb{P}S_d \times D_l.
\]

Letting \( \phi_{d,l} : \Sigma_{d,l} \to \mathbb{P}S_d \) denote the natural projection map, we define the locus of degree \( d \) curves containing a star configuration \( X(l) \), denoted \( S(d, l) \), to be \( S(d, l) = \phi_{d,l}(\Sigma_{d,l}) \).

We then prove the following result about the dimension of the locus.

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Theorem 1.1. Let \( d \geq 0 \) and \( l \geq 2 \) be integers. Then \( S(d, l) = \emptyset \) if \( d < l - 1 \), and

\[
\dim S(d, l) = \begin{cases} 
\binom{d+2}{2} - 1 & \text{if } d \geq l - 1 \text{ and } l = 2, 3, 4 \\
\binom{d+2}{2} - 2 & \text{if } d = 4 \text{ and } l = 5 \\
\binom{d+2}{2} - 1 & \text{if } d \geq 5 \text{ and } l = 5 \\
\binom{d+2}{2} - \binom{l}{2} + 2l - 1 & \text{if } d \geq l - 1 \text{ and } l \geq 6.
\end{cases}
\]

Theorem 1.1 complements our previous work \cite{4,5} which showed that the generic degree \( d \) plane curve contains a star configuration \( X(l) \) if and only if the projection map \( \phi_{d,l} \) is dominant which happens if and only if \( \dim S(d, l) = \binom{d+2}{2} - 1 \). The tuples \((d, l)\) for which \( \dim S(d, l) = \binom{d+2}{2} - 1 \) are therefore precisely the tuples described in \cite{5} Theorem 6.3. Note that \( S(d, l) = \emptyset \) if no curve of degree \( d \) contains a star configuration \( X(l) \). Because the defining ideal of \( X(l) \) is minimal generated in degree \( l - 1 \), it follows that \( S(d, l) = \emptyset \) for \( d < l - 1 \).

It therefore suffices to focus on proving Theorem 1.1 for the pairs \((4, 5)\) and \((d, l)\) with \( d \geq l - 1 \) and \( l \geq 6 \). Our strategy for the pairs \((d, l) \neq (4, 5)\) is to first translate the problem into computing the dimension of a graded ideal constructed from the linear forms \( L_1, \ldots, L_l \) in a particular degree. This enables us to reduce the problem to computing the rank of a particular matrix. We use the notion of Lüroth quartics to deal with the pair \((d, l) = (4, 5)\).

Our paper is structured as follows. In Section 2 we recall the relevant facts about star configurations. We also translate our problem into a new algebraic question, and we compute \( \dim S(4, 5) \). In Section 3 we prove Theorem 1.1 for all tuples \((d, 6)\) with \( d \geq 5 \). The results of this section provide a base case for the arguments of Section 4. We conclude with remarks about the higher dimensional analog of this problem.

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2. Properties of star configurations

We recall the relevant results about star configurations in \( \mathbb{P}^2 \) and prove an upper bound on \( \dim S(d, l) \). Although some proofs are omitted, they can be found in \cite{4,8}.

We continue to use the notation introduced in the introduction. For any \( l \geq 2 \), let \( L_1, \ldots, L_l \) be a collection of \( l \) linear forms of \( S = k[x_0, x_1, x_2] \) that are three-wise linearly independent. We call such a collection a collection of \textit{general linear forms}. We let \( X(l) \) denote the star configuration of \( \binom{l}{2} \) points in \( \mathbb{P}^2 \) which is formed from all pairwise intersections of the \( l \) linear forms. Note that when \( l = 2 \), then \( X(2) \) is simply a point, and if \( l = 3 \), then \( X(3) \) is three non-collinear points.

The following lemma allows us to describe the minimal generators of the ideal associated to \( X(l) \) and the Hilbert function of this ideal.

Lemma 2.1. Let \( L_1, \ldots, L_l \) be \( l \geq 2 \) general linear forms of \( S = k[x_0, x_1, x_2] \), and let \( I_{X(l)} \) denote the defining ideal of \( X(l) \).
(i) For each $i \in \{1, \ldots, l\}$, let $\hat{L}_i = \prod_{j \neq i} L_j$. Then $I_{XX(l)} = (\hat{L}_1, \hat{L}_2, \ldots, \hat{L}_l)$.

(ii) The set of points $X(l)$ has the Hilbert function of $\left(\begin{smallmatrix} l \\ 2 \end{smallmatrix}\right)$ generic points, that is

$$HF(X(l), t) = \dim_k(S/I_{X(l)})_t = \min\left\{\left(\begin{smallmatrix} t + 2 \\ 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} l \\ 2 \end{smallmatrix}\right)\right\} \text{ for } t \geq 0.$$  

Proof. For (i), see [4, Lemma 2.3(iv)]. For (ii), see [4, Theorem 2.5]. \hfill \Box

Remark 2.2. Lemma 2.1 (i) shows that no plane curve of degree $d$ with $d < l - 1$ can contain a star configuration $X(l)$. As a consequence, $S(d, l) = \emptyset$ for all $(d, l)$ with $d < l - 1$.

We give an upper bound on $\dim S(d, l)$.

Lemma 2.3. Let $l \geq 2$ and $d \geq l - 1$ be integers. Then

$$\dim S(d, l) \leq \left(\begin{smallmatrix} d + 2 \\ 2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} l \\ 2 \end{smallmatrix}\right) + 2l - 1.$$  

Proof. Consider the incidence correspondence $\Sigma_{d,l}$ as given in (3), and let

$$\psi_{d,l} : \Sigma_{d,l} \longrightarrow \mathcal{D}_l \text{ and } \phi_{d,l} : \Sigma_{d,l} \longrightarrow \mathbb{P}S_d$$

be the natural projection maps. Note that we are following the standard convention that $\mathbb{P}S_d$ is identified with the projective space $\mathbb{P}^{N_d}$ where $N_d = \left(\begin{smallmatrix} d + 2 \\ 2 \end{smallmatrix}\right) - 1$. Using a standard fibre dimension argument, if $d \geq l - 1$, then

$$\dim \Sigma_{d,l} \leq \dim \mathcal{D}_l + \dim_k(I_{X(l)})_{d-1} = \dim \mathcal{D}_l + \left(\begin{smallmatrix} 2 + d \\ d \end{smallmatrix}\right) - \left(\begin{smallmatrix} l \\ 2 \end{smallmatrix}\right) - 1 = 2l + \left(\begin{smallmatrix} 2 + d \\ d \end{smallmatrix}\right) - \left(\begin{smallmatrix} l \\ 2 \end{smallmatrix}\right) - 1.$$  

Here, we are using Lemma 2.1 (ii) since $\dim_k S_d - \dim_k(I_{X(l)})_d = \left(\begin{smallmatrix} l \\ 2 \end{smallmatrix}\right)$ when $d \geq l - 1$. The desired bound now follows from the fact that $\dim S(d, l) \leq \dim \Sigma_{d,l}$. \hfill \Box

Remark 2.4. As shown in [5, Theorem 3.1], if $l \geq 6$, then the map $\phi_{d,l}$ cannot be dominant.

Inspired by our previous work [4, 5], we can reformulate the problem of computing $\dim S(d, l)$ in terms of computing the dimension of an ideal in a specific degree. In fact, the proof of [5, Lemma 4.3] already contains the result we need. We first perform the following geometric construction. With $d \geq l - 1$ define a map of affine varieties

$$\Phi_{d,l} : S_1 \times \cdots \times S_l \times S_{d-l+1} \times \cdots \times S_{d-l+1} \longrightarrow S_d$$

such that

$$\Phi_{d,l}(L_1, \ldots, L_l, M_1, \ldots, M_l) = \sum_{i=1}^l M_i \hat{L}_i.$$

We then rephrase our problem in terms of the map $\Phi_{d,l}$.

Lemma 2.5. With notation as above, the image of $\Phi_{d,l}$ is the affine cone over $S(d, l)$. In particular, $\dim S(d, l) = \dim \text{Im}(\Phi_{d,l}) - 1$. 


Proof. Suppose $H$ is a degree $d$ form that defines a curve $C$ that contains a star configuration $X(l)$. So, there exist linear forms $L_1, \ldots, L_l$ such that $H \in (\hat{L}_1, \ldots, \hat{L}_l)$, and hence $H = \sum_{i=1}^l M_i \hat{L}_i$ with $M_i \in S_{d-l+1}$ for each $i$. But this means that $H \in \text{Im}(\Phi_{d,l})$. Viewing elements of $S(d,l)$ as elements of $\mathbb{P}S_d$, this gives $\dim S(d,l) \leq \dim \text{Im}(\Phi_{d,l}) - 1$.

Now consider a generic $F \in \text{Im}(\Phi_{d,l})$. We want to show that there exists $(L_1, \ldots, L_l, M_1, \ldots, M_l) \in \Phi_{d,l}^{-1}(F)$ such that the linear forms define a star configuration. Define $\Delta \subset S_1 \times \cdots \times S_1 \times S_{d-l+1} \times \cdots \times S_{d-l+1}$ as follows:

$$\Delta = \left\{ (L_1, \ldots, L_l, M_1, \ldots, M_l) \mid \text{there exists } a \neq b \neq c \text{ such that } L_a, L_b, L_c \text{ are linearly dependent} \right\}.$$ 

It suffices to show that $\Phi_{d,l}^{-1}(F)$ is not contained $\Delta$ for a generic form $F \in \text{Im}(\Phi_{d,l})$. Suppose for a contradiction that $\Phi_{d,l}^{-1}(F)$ is contained in $\Delta$. Then $\Delta$ would be a component of the domain of $\Phi_{d,l}$. This is a contradiction as the latter is an irreducible variety being the product of irreducible varieties. This completes the proof. ∎

As a consequence of the above lemma, we only need to compute $\dim \text{Im}(\Phi_{d,l})$. As we now show, we can compute $\dim \text{Im}(\Phi_{d,l})$ by the size of its tangent space. In fact, this value will equal the vector space dimension of a graded ideal in a specific degree.

Lemma 2.6. Let $l \geq 2$ and $d \geq l - 1$ be integers, and consider $l$ general linear forms $L_1, \ldots, L_l$ in $S$. Also, let $M_1, \ldots, M_l \in S_{d-l+1}$ be any homogeneous forms of degree $d-l+1$. Set

$$\hat{L}_i = \prod_{j \neq i} L_j \text{ and } \hat{L}_{i,j} = \prod_{h \neq (i,j)} L_h \text{ for } i \neq j.$$ 

Define the following $l$ forms of degree $d - 1$:

$$Q_1 = M_2 \hat{L}_{1,2} + M_3 \hat{L}_{1,3} + \cdots + M_l \hat{L}_{1,l} = \sum_{i \neq 1} M_i \hat{L}_{1,i}$$

$$Q_2 = M_1 \hat{L}_{2,1} + M_3 \hat{L}_{2,3} + \cdots + M_l \hat{L}_{2,l} = \sum_{i \neq 2} M_i \hat{L}_{2,i}$$

$$\vdots$$

$$Q_l = M_1 \hat{L}_{l,1} + M_2 \hat{L}_{l,2} + \cdots + M_{l-1} \hat{L}_{l,l-1} = \sum_{i \neq l} M_i \hat{L}_{l,i}.$$ 

With this notation, form the ideal

$$I = (\hat{L}_1, \cdots, \hat{L}_l) + (Q_1, \ldots, Q_l) = I_{X(l)} + (Q_1, \ldots, Q_l) \subseteq S.$$ 

Then $I_d$ is the affine tangent space to $\text{Im}(\Phi_{d,l})$ at a generic point. In particular,

$$\dim S(d,l) = \dim_k I_d - 1.$$ 

Proof. The statement about $\dim S(d,l)$ follows from the first statement and the previous lemma. We need to determine the tangent space $\text{Im}(\Phi_{d,l})$ in a generic point $q = \Phi_{d,l}(p)$, where $p = (L_1, \ldots, L_l, M_1, \ldots, M_l)$. We denote with $T_q$ this affine tangent space.

The elements of the tangent space $T_q$ are obtained as
when we vary the forms \(L'_i \in S_1\) and \(M'_i \in S_{d-1+1}\). By a direct computation we see that the elements of \(T_q\) have the form

\[
M'_i \hat{L}_1 + \cdots + M'_i \hat{L}_l + L'_1(M_2 \hat{L}_{1,2} + \cdots + M_i \hat{L}_{1,l}) + \cdots + \\
+ L'_j(M_1 \hat{L}_{j,1} + \cdots + M_l \hat{L}_{j,l}) + \cdots + L'_i(M_1 \hat{L}_{i,2} + \cdots + M_{l-1} \hat{L}_{l,l-1}),
\]

where \(\hat{L}_i = \prod_{j \neq i} L_j\) and \(\hat{L}_{i,j} = \prod_{h \notin \{i,j\}} L_h\), for \(i \neq j\).

Since the \(L'_i \in S_1\) and \(M'_i \in S_{d-1+1}\) can be chosen freely, we obtain \(I_d = T_q\). \(\square\)

**Remark 2.7.** As in [5], we can use the above lemma and appeal to upper-semicontinuity to compute \(\dim S(d,l)\) if we know a (good) upper bound on \(\dim S(d,l)\). Indeed, suppose we know that \(\dim S(d,l) \leq M\). Given \(d\) and \(l\) we construct the ideal \(I\) as in Lemma 2.6 by choosing forms \(L_i\) and \(M_i\). Then we compute \(\dim_k I_d\) using a computer algebra system. If \(\dim_k I_d - 1 = M\), by upper semi-continuity of the dimension (indeed, the dimension can decrease only on a proper closed subset), we have proved \(M = \dim_k I_d - 1 \leq \dim S(d,l) \leq M\), and hence \(\dim S(d,l) = M\) for this pair \((d,l)\). We will require this technique for some small values of \(d\) and \(l\).

It is known that the generic plane quartic does not contain a \(\mathbb{X}(5)\). We call a quartic containing a \(\mathbb{X}(5)\) a Lüroth quartic. These objects were classically studied; see for example [10, 11], and for a modern treatment [12]. Of interest is the following theorem.

**Theorem 2.8 (12 Theorem 11.4).** Lüroth quartics form a hypersurface of degree 54 in the space of plane quartics.

We can now prove \(\dim S(4,5) = 13\) using our techniques.

**Remark 2.9.** By Lemma 2.3 \(\dim S(4,5) \leq \binom{6}{4} - \binom{6}{2} - 2 \cdot 5 - 1 = 14\). However, Theorem 2.8 implies that the projection map \(\phi_{d,l} : \Sigma_{d,l} \to \mathbb{P}S_d\) is not dominant, so \(\dim S(4,5) \leq 13\).

Now consider the following five linear forms

\[ L_1 = x_0; \quad L_2 = x_1; \quad L_3 = x_2; \quad L_4 = x_0 + x_1 + x + 2; \quad \text{and} \quad L_5 = x_0 + 2x_1 + 3x_3. \]

We construct the ideal \(I\) as in Lemma 2.6 where we take \(M_1 = \cdots = M_5 = 1\). Using CoCoA\(^1\), we find that \(\dim_k(I_d) = 14\), whence \(13 = \dim_k(I_d) - 1 \leq \dim S(4,5) \leq 13\), thus giving the desired result via Remark 2.7.

Some additional remarks about computing \(\dim_k I_d\) are given below.

**Remark 2.10.** Consider the ideal \(I\) in Lemma 2.6. We wish to compute \(\dim_k I_d\). Now \(I = I_{X(l)} + (Q_1, \ldots, Q_l) \subseteq S\). Because \(d \geq l - 1\), by Lemma 2.1 \(\dim_k(I_{X(l)})_d = \binom{2 + d}{2} - \binom{l}{2}\). It then follows that to compute \(\dim_k I_d\), it is enough to compute \(\dim_k(\overline{Q_1}, \ldots, \overline{Q_l})_d\) where \((\overline{Q_1}, \ldots, \overline{Q_l}) = I/I_{X(l)} \subseteq S/I_{X(l)}\). So we have

\[
\dim S(d,l) = \dim_k(I_d) - 1 = \dim_k(\overline{Q_1}, \ldots, \overline{Q_l})_d + \left(\frac{2 + d}{2}\right) - \binom{l}{2} - 1.
\]

\(^1\)For our code, see [http://flash.lakeheadu.ca/~avantuyl/research/PlaneCurvesStarConfig_CGVT.html](http://flash.lakeheadu.ca/~avantuyl/research/PlaneCurvesStarConfig_CGVT.html)
By Lemma 2.3 we know \( \dim S(d, l) \leq \binom{2+d}{2} - \binom{l}{2} + 2l - 1 \). So, if we can show that \( \dim_k(Q_1, \ldots, Q_l) = 2l \) for a specific choice of \( Q_i \)'s, then by Remark 2.7 we will in fact have the equality \( \dim S(d, l) = \binom{2+d}{2} - \binom{l}{2} + 2l - 1 \).

We will employ the following strategy in Sections 3 and 4. After fixing some star configuration \( X(l) \), we identify \( 2l \) points \( p_i \) in the star configuration, and determine \( 2l \) linear forms \( H_1, \ldots, H_{2l} \). We then construct a \( 2l \times 2l \) evaluation matrix

\[
\begin{array}{cccc}
H_1Q_1 & H_2Q_1 & \cdots & H_{2l}Q_1 \\
\delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,2l} \\
\delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,2l} \\
\delta_{3,1} & \delta_{3,2} & \cdots & \delta_{3,2l} \\
\vdots & \vdots & \cdots & \vdots \\
\delta_{2l,1} & \delta_{2l,2} & \cdots & \delta_{2l,2l}
\end{array}
\]

where \( \delta_{i,j} \) is the point \( p_i \) evaluated at the degree \( d \) form that indexes column \( j \). If \( M \) denotes the resulting matrix, then \( \text{rk}(M) = \dim_k(Q_1, \ldots, Q_l)_d \) in \( S/I_{X(l)} \).

3. The case \( \ell = 6 \) and \( d \geq 5 \)

In this section we compute \( \dim S(d, 6) \) for all \( d \geq 6 - 1 = 5 \). We make use of the following notion: if \( X(l) \) is the star configuration constructed from \( L_1, \ldots, L_l \), we let \( p_{i,j} \) with \( 1 \leq i < j \leq l \) denote the point formed by intersection of \( \ell_i \) and \( \ell_j \). Thus \( X(l) = \{ p_{i,j} : 1 \leq i < j \leq l \} \).

**Theorem 3.1.** For all \( d \geq 5 \), \( \dim S(d, 6) = \binom{d+2}{2} - 4 \).

**Proof.** We break the proof into three cases: (1) \( d = 5 \), (2) \( d = 6 \), and (3) \( d \geq 7 \). For all three cases, we will use the strategy outlined in Remark 2.10, thus it suffices to construct a \( 12 \times 12 \) evaluation matrix with maximal rank.

(1) For the case \( d = 5 \), consider the six linear forms

\[
L_1 = x_0; \ L_2 = x_1; \ L_3 = x_2; \ L_4 = x_0+x_1+x_2; \ L_5 = x_0+2x_1+3x_2; \text{ and } L_6 = x_0+3x_1+10x_2.
\]

When constructing the \( Q_i \)'s, we set \( M_i = 1 \) for \( i = 1, \ldots, 6 \). We form the evaluation matrix with columns indexed by

\[
L_2Q_4, L_1Q_4, L_3Q_5, L_2Q_5, L_1Q_6, L_3Q_6, L_6Q_1, L_3Q_2, L_6Q_2, L_6Q_3, L_4Q_3, L_5Q_1
\]

and rows are index by

\[
P_{1,4}, P_{2,4}, P_{2,5}, P_{3,5}, P_{3,6}, P_{2,6}, P_{1,5}, P_{2,6}, P_{2,3}, P_{3,4}, P_{1,2}.
\]

We used CoCoA to verify that the resulting matrix has the desired rank of 12 (our code can be found on the third author’s website).

(2) For the case \( d = 6 \), we use the same \( L_i \)'s, but when we construct the \( Q_i \)'s, we first find a linear form \( G \) that does not contain any of the points of \( X(6) \), and set \( M_i = G \) for \( i = 1, \ldots, 6 \). Again, the resulting evaluation matrix (using the same indexing for the rows and columns) has maximal rank.
(3) We now consider the case \( d \geq 7 \). Pick any six general linear forms \( L_1, \ldots, L_6 \) that form a \( \mathbb{X}(6) \), and let \( p_{1,2}, \ldots, p_{4,5} \) be the 15 points of \( \mathbb{X}(6) \). In order to construct the \( Q_i \)'s as defined in Lemma \( \ref{lem:2} \) we need to pick six forms \( M_1, \ldots, M_6 \) in \( S_{d-t+1} \). Since \( d \geq 7 \), each \( M_i \) will have degree at least two. We construct the \( M_i \)'s as follows. First, we pick six linear forms in \( S \) with the following properties:

- \( G \) is a linear form such that the line \( G = 0 \) does not pass through any point of \( \mathbb{X}(6) \);
- \( G_1 \) is a linear form such that line \( G_1 = 0 \) only passes through the point \( p_{1,5} \) of \( \mathbb{X}(6) \);
- \( G_2 \) is a linear form such that the line \( G_2 = 0 \) only passes through the point \( p_{1,2} \) of \( \mathbb{X}(6) \);
- \( G_3 \) is a linear form such that the line \( G_3 = 0 \) only passes through the point \( p_{2,6} \) of \( \mathbb{X}(6) \);
- \( G_4 \) is a linear form such that the line \( G_4 = 0 \) only passes through the point \( p_{3,4} \) of \( \mathbb{X}(6) \);
- \( G_5 \) is a linear form such that the line \( G_5 = 0 \) only passes through the point \( p_{4,6} \) of \( \mathbb{X}(6) \).

We then set

\[
\begin{align*}
M_1 &= G_1 G_2 G^{d-t-1} \\
M_2 &= G_3 G^{d-t} \\
M_3 &= G_4 G^{d-t} \\
M_4 &= G^{d-t+1} \\
M_5 &= G^{d-t+1} \\
M_6 &= G_5 G^{d-t}
\end{align*}
\]

and use these \( M_i \)'s to construct \( Q_1, \ldots, Q_6 \).

We now consider the 12 \( \times \) 12 evaluation matrix (see below) whose columns and rows are also indexed as above. When determining the entries of the evaluation matrix, note that \( L_r Q_{i,j} = 0 \) if \( i = r \), or \( j = r \), or neither \( i \) and \( j \) equal \( t \). We will also have

\[
\begin{align*}
L_3 Q_5(p_{1,5}) &= 0 & L_1 Q_6(p_{2,6}) &= 0 & L_2 Q_4(p_{3,4}) &= 0 & L_2 Q_4(p_{4,6}) &= 0 & L_3 Q_2(p_{1,2}) &= 0 \\
L_2 Q_5(p_{1,5}) &= 0 & L_3 Q_6(p_{2,6}) &= 0 & L_1 Q_4(p_{3,4}) &= 0 & L_1 Q_4(p_{4,6}) &= 0 & L_6 Q_2(p_{1,2}) &= 0.
\end{align*}
\]

This follows from our choice of \( M_i \)'s. For example,

\[
L_3 Q_5(p_{1,5}) = L_3(M_1 L_2 L_3 L_4 L_6 + M_2 L_1 L_3 L_4 L_6 + M_3 L_1 L_2 L_4 L_6 + M_4 L_1 L_2 L_3 L_6 + M_5 L_1 L_2 L_3 L_4)(p_{1,5})
\]

\[
= M_1 L_2 L_3 L_4 L_6(p_{1,5}) = 0
\]

since \( M_1(p_{1,5}) = 0 \). Again by our choice of \( M_i \)'s, we have

\[
\begin{align*}
L_2 Q_4(p_{1,4}) &\neq 0 & L_1 Q_4(p_{2,4}) &\neq 0 & L_3 Q_5(p_{2,5}) &\neq 0 & L_2 Q_5(p_{3,5}) &\neq 0 & L_1 Q_6(p_{3,6}) &\neq 0 \\
L_3 Q_3(p_{1,6}) &\neq 0 & L_4 Q_3(p_{3,6}) &\neq 0 & L_5 Q_1(p_{1,6}) &\neq 0 & L_1 Q_6(p_{4,6}) &\neq 0 & L_3 Q_6(p_{4,6}) &\neq 0 \\
L_6 Q_1(p_{1,2}) &\neq 0 & L_5 Q_1(p_{1,2}) &\neq 0.
\end{align*}
\]

For example,

\[
L_5 Q_1(p_{1,2}) = L_5(M_2 L_3 L_4 L_5 L_6 + M_3 L_2 L_4 L_5 L_6 + M_4 L_2 L_3 L_5 L_6 + M_5 L_2 L_3 L_4 L_6 + M_6 L_2 L_3 L_4 L_5)(p_{1,2})
\]

\[
= M_2 L_3 L_4 L_5^2 L_6(p_{1,2}) \neq 0
\]

since \( M_2 \) does not vanish at \( p_{1,2} \), and \( p_{1,2} \) does not lie on the lines defined by \( L_3, L_4, L_5 \) or \( L_6 \).
Our evaluation matrix therefore has the form

\[
\begin{array}{cccccccc}
L_2Q_4 & L_1Q_4 & L_3Q_5 & L_2Q_5 & L_1Q_6 & L_3Q_6 & L_6Q_1 & L_3Q_2 & L_6Q_2 & L_6Q_3 & L_4Q_3 & L_5Q_1 \\
p_{1,4} & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{2,4} & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{2,5} & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{3,5} & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{3,6} & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 & 0 \\
p_{1,6} & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\
p_{1,5} & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 \\
p_{2,6} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 \\
p_{3,4} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
p_{4,6} & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \\
p_{1,2} & 0 & 0 & 0 & 0 & 0 & * & 0 & 0 & 0 & 0 & 0 \\
\end{array}
\]

where * denotes a nonzero entry and □ denotes an entry which may or may not be zero. Using Gaussian elimination, we get a matrix in row echelon form (where the nonzero leading coefficients are not necessarily equal to 1), with zero entries down the diagonal. Consequently, the original matrix has maximal rank, as desired.

\[\square\]

### 4. The Case $\ell > 6$ and $d \geq \ell - 1$

We now evaluate $\dim S(d, l)$ for all $l \geq 6$ when $d \geq l - 1$. The key idea is to pick the $l$ linear forms $L_1, \ldots, L_l$ that define $X(l)$ so that the first six forms are as in Theorem 3.1.

**Theorem 4.1.** Let $l \geq 7$ and $d \geq l - 1$. Then $\dim S(d, l) = \binom{d+2}{2} - \binom{l}{2} + 2l - 1$.

**Proof.** As in Theorem 3.1, it suffices to construct a $2l \times 2l$ evaluation matrix of rank $2l$. Let $L_1, \ldots, L_l$ be the $l$ general linear forms that define $X(l)$. If $d = l - 1$ or $d = l$, we let $L_1, \ldots, L_6$ be as in Theorem 3.1. When constructing the $Q_i$'s, we use the following $M_i$'s:

- If $d = l - 1$, let $M_i = 1$ for $i = 1, \ldots, l$.
- If $d = l$, let $M_i = G$, where $G$ is a linear form such that the curve $G = 0$ does not contain any of the points of $X(l)$.
- If $d \geq l + 1$, define $M_1, \ldots, M_6$ as in Theorem 3.1, but with the added condition that each $M_i$ also does not vanish at any other point of $X(l)$. We set $M_7 = \cdots = M_l = G^{d-l+1}$, where again $G$ is a linear form such that the curve $G = 0$ does not contain any points of $X(l)$.

When we form our evaluation matrix, we label the first twelve columns as in Theorem 3.1 and we label the remaining $2l-12$ columns with $L_2Q_7, L_1Q_7, L_2Q_8, L_1Q_8, \ldots, L_2Q_l, L_1Q_l$. We label the first twelve rows as in Theorem 3.1 and the remaining rows are labelled with
For example above. Then, for every nonzero entry in this sub-matrix, we have $Q_\star$ where $p_1$, $p_2$, $p_3$, $p_4$, $p_5$, $p_6$, $p_7$, $p_8$, and $p_9$, $p_{10}$, $p_{11}$, $p_{12}$. Our evaluation matrix then has the form:

$$
\begin{array}{cccccccccc}
  & L_2Q_4 & L_1Q_4 & \cdots & L_4Q_3 & L_5Q_1 & L_2Q_7 & L_1Q_7 & L_2Q_8 & L_1Q_8 & \cdots & L_2Q_l & L_1Q_l \\
p_{1,4} & * & 0 & \cdots & 0 & \Box & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
p_{2,4} & 0 & * & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
p_{3,4} & \vdots & \vdots & \vdots & \vdots & \vdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
p_{4,6} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
p_{1,2} & 0 & 0 & \cdots & 0 & \Box & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
p_{1,7} & 0 & 0 & \cdots & 0 & \Box & \star & 0 & 0 & 0 & \cdots & 0 & 0 \\
p_{2,7} & 0 & 0 & \cdots & 0 & \Box & 0 & \star & 0 & 0 & \cdots & 0 & 0 \\
p_{3,8} & 0 & 0 & \cdots & 0 & \Box & 0 & 0 & \star & 0 & \cdots & 0 & 0 \\
p_{4,8} & 0 & 0 & \cdots & 0 & \Box & 0 & 0 & 0 & \star & \cdots & 0 & 0 \\
p_{1,1} & 0 & 0 & \cdots & 0 & \Box & 0 & 0 & 0 & 0 & \cdots & \star & 0 \\
p_{2,1} & 0 & 0 & \cdots & 0 & \Box & 0 & 0 & 0 & 0 & \cdots & 0 & \star \\
\end{array}
$$

where $\star$ denotes a nonzero entry and $\Box$ denotes an entry which may or may not be zero.

Consider the $12 \times 12$ sub-matrix formed by the first 12 rows and 12 columns. Let $Q'_i$ denote the form constructed as in Lemma 2.6 using $L_1, \ldots, L_6$ and the same $M_i$'s as above. Then, for every nonzero entry in this sub-matrix, we have

$$
L_i Q_t(p_{i,j}) = L_i Q'_t L_7 L_8 \cdots L_i(p_{i,j}) = [L_i Q'_t(p_{i,j})][L_7 L_8 \cdots L_i(p_{i,j})].
$$

For example

$$
L_2 Q_4(p_{1,4}) = L_2(M_1 \tilde{L}_{4,1} + M_2 \tilde{L}_{4,2} + \cdots + M_l \tilde{L}_{4,l})(p_{1,4}) = L_2 M_1 \tilde{L}_{4,1}(p_{1,4}) = L_2(M_1 L_2 L_3 L_5 L_6 L_7 \cdots L_i)(p_{1,4}) = [L_2(M_1 L_2 L_3 L_5 L_6)(p_{1,4})][L_7 \cdots L_i(p_{1,4})].
$$

We can factor our evaluation matrix as $AB$ where

$$
A = \begin{bmatrix}
L_7 \cdots L_i(p_{1,4}) & 0 & \cdots & 0 & 0 \\
0 & L_7 \cdots L_i(p_{2,4}) & & 0 & 0 \\
\vdots & \ddots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & L_7 \cdots L_i(p_{4,6}) & 0 \\
0 & 0 & \cdots & 0 & L_7 \cdots L_i(p_{1,2}) \\
\end{bmatrix}
$$

and

$$
B = I_{2l-12}.
$$
where $\mathbf{0}$ denotes an appropriate sized zero matrix, and $I_{2l-12}$ is the identity matrix, and 
where $B$ is the matrix given by

\[
\begin{array}{cccccccc}
L_2Q_1' & L_1Q_1' & \cdots & L_3Q_1' & L_3Q_1 & L_2Q_7 & L_1Q_7 & L_2Q_8 & \cdots & L_2Q_l & L_1Q_l \\
\hline
p_{1,4} & * & 0 & \cdots & 0 & \square & 0 & 0 & 0 & \cdots & 0 & 0 \\
p_{2,4} & 0 & * & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p_{4,6} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
p_{1,2} & 0 & 0 & \cdots & 0 & * & 0 & 0 & 0 & \cdots & 0 & 0 \\
B = & p_{1,7} & 0 & 0 & \cdots & 0 & \square & * & 0 & 0 & \cdots & 0 & 0 \\
p_{2,7} & 0 & 0 & \cdots & 0 & \square & 0 & * & 0 & \cdots & 0 & 0 \\
p_{1,8} & 0 & 0 & \cdots & 0 & \square & 0 & 0 & * & \cdots & 0 & 0 \\
p_{2,8} & 0 & 0 & \cdots & 0 & \square & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
p_{1,l} & 0 & 0 & \cdots & 0 & \square & 0 & 0 & 0 & \cdots & * & 0 \\
p_{2,l} & 0 & 0 & \cdots & 0 & \square & 0 & 0 & 0 & \cdots & 0 & * \\
\end{array}
\]

But now the matrix $B$ has the property that the $12 \times 12$ sub-matrix in the upper left hand corner is exactly the same as the matrix as in Theorem 3.1. As a result, this sub-matrix has rank 12. Furthermore, the lower $(2l-12) \times (2l-12)$ sub-matrix clearly has maximal rank, and so $B$ has maximal rank. Finally, since none of the points indexing the first 12 rows vanish at $L_7, \ldots, L_l$, the matrix $A$ also has maximal rank, so our original evaluation matrix has the desired rank of $2l$. \hfill \Box

For completeness, we now put together all the pieces to prove our main theorem.

**Proof.** (of Theorem 1.1) By Remark 2.2, $S(d,l) = \emptyset$ if $d < l - 1$. The value for $\dim S(4,5)$ comes from Remark 2.9. The main theorem of [5] determines when $\dim S(d,l) = \binom{d+2}{2} - 1$. Theorems 3.1 and 4.1 cover the remaining cases. \hfill \Box

5. Concluding remarks

It is natural to ask whether the work of this paper can be generalized to star configurations set of points $X(l)$ in $\mathbb{P}^n$; see [4] for more on this. Indeed, let $\Sigma_{n,d,l}$ be the incidence correspondence

$$
\Sigma_{n,d,l} = \{(H, X(l)) : H \supseteq X(l) \} \subseteq \mathbb{P}S_d \times D_l,
$$

where now $S = k[x_0, \ldots, x_n]$ and $D_l \subseteq \mathbb{P}^n \times \cdots \times \mathbb{P}^n$ ($l$ times). Letting $\phi_{n,d,l} : \Sigma_{n,d,l} \to \mathbb{P}S_d$ denote the natural projection map, we wish to compute the dimension of the corresponding locus, that is, $S(d,l,n) = \phi_{n,d,l}(\Sigma_{n,d,l})$.

The proofs of Section 2 extend naturally to this case, thus giving us the upper bound

$$
\dim S(d,l,n) \leq \min \left\{ \binom{d+n}{n} - 1, \binom{d+n}{n} - \binom{l}{n} + nl - 1 \right\} \text{ for all } d \geq l - 1.
$$

Computer experiments suggest that this inequality is an equality for all $d \geq l - 1$ with $n \geq 3$. The results of [4] already verifies part of this claim when the minimum is $\binom{d+n}{n} - 1$. We expect that the approach used in this paper will verify this question; however, the difficulty is now finding the correct evaluation matrix and determining its rank.
As an interesting aside, if this equality holds, this would imply that the Lüroth case is the only time \( \dim S(d,l,n) \) is not the expected value.

Also notice that the case of Lüroth quartic is the only one in the plane for which the locus of star configurations is an hypersurface, and it is hence defined by a single equation. Moreover, the locus of a star configurations is never zero dimensional.

**References**

[1] J. Ahn, Y.S. Shin, The minimal free resolution of a star-configuration in \( \mathbb{P}^n \) and the Weak Lefschetz Property. J. Korean Math. Soc. **49** (2012), no. 2, 405–417.

[2] C. Bocci, S. Cooper, B. Harbourne, Containment results for ideals of various configurations of points in \( \mathbb{P}^N \). To appear J. Pure Appl. Algebra (2013).

[3] C. Bocci, B. Harbourne, Comparing Powers and Symbolic Powers of Ideals. J. Algebraic Geom. **19** (2010), no. 3, 399–417

[4] E. Carlini, E. Guardo, A. Van Tuyl, Star configurations on generic hypersurfaces. Preprint (2012). 
[arXiv:1204.0475v2](http://arxiv.org/abs/1204.0475v2)

[5] E. Carlini, A. Van Tuyl, Star configuration points and generic plane curves. Proc. Amer. Math. Soc. **139** (2011), no. 12, 4181–4192.

[6] CoCoATeam, CoCoA: a system for doing Computations in Commutative Algebra. Available at [http://cocoa.dima.unige.it](http://cocoa.dima.unige.it)

[7] G. Fatabbi, B. Harbourne, A. Lorenzini, Incics, galaxies, star configurations and Waldschmidt constants. Preprint (2013). [arXiv:1304.2217v2](http://arxiv.org/abs/1304.2217v2)

[8] A.V. Geramita, B. Harbourne, J. Migliore, Star configurations in \( \mathbb{P}^n \). J. Algebra **376** (2013) 279–299.

[9] A.V. Geramita, J. Migliore, L. Sabourin, The first infinitesimal neighborhood of a linear configuration of points in \( \mathbb{P}^2 \). J. Algebra **298** (2006), no. 2, 563–611.

[10] J. Lüroth, Einige Eigenschaften einer gewissen Gattung von Curven vierter Ordnung. Math. Ann. **1** (1869), no. 1, 37–53.

[11] F. Morley, On the Lüroth Quartic Curve. Amer. J. Math. **41** (1919), no. 4, 279–282.

[12] G. Ottaviani, E. Sernesi, On the hypersurface of Lüroth quartics. Michigan Math. J. **59** (2010), no. 2, 365–394.

[13] Y.S. Shin, Some applications of the union of star-configurations in \( \mathbb{P}^n \). J. Chungcheong Math. Soc. **24** (2011), no. 4, 807–824.

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