Centrifugal’ Force around a Black Hole

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Abstract

Besides having some very interesting perturbatively unstable orbits, it seems that for a Schwarzschild black hole, below $r = 3M$, the force always increases inward with increasing angular momentum. Here this previously known result is derived with greater simplicity, and a similar analysis is performed for black holes with angular momentum and charge.

1 Introduction

Much of the time invested in learning physics is dedicated to developing an intuition for the particular topic in hand. This may involve the introduction of such imaginary concepts as force, which may later need to be abandoned (as in quantum theory and Einstein’s theory of general relativity).

Relativity theory, being so far from the Newtonian physics of the everyday world, is filled by what initially seems paradoxical behaviour. General relativity is especially difficult in this respect, and the partial restoration of gravitational forces (as opposed to geodesic motion) can help simplify understanding. However, preconceived notions of force can cause confusion, and it might be better to deal directly with the concept of the rate of change of momentum. Indeed, as we shall see, this notion of force is not, in general, minus the potential gradient.

It was this line of reasoning that led Abramowicz and others [1-3] to investigate the concept of force around a black hole. In the work that follows, a particularly direct derivation of some of those results is presented, and as a consequence of the simplicity, considerable generalization was found to be possible.

It was discovered that, in general, around a black hole there is a radius below which the force increases inward, ever more so with increasing angular momentum. This is in conflict with normal Newtonian intuition, where a spinning object has a natural tendency to fly off. This interpretation augments an understanding of the accretion mechanism of black holes and so may help in identifying certain astronomical observations as being the result of black hole physics.

Since the physics of this effect is present even for the Schwarzschild black hole, the approach is first illustrated for this case before being applied more generally to a black hole with charge and angular momentum.
1.1 Schwarzschild Black Hole

Starting directly from the equation of motion, derived from $p_\mu p^\mu + m^2 = 0$:

$$\left( \frac{dr}{d\tau} \right)^2 = \tilde{E}^2 - \tilde{V}^2 (\tilde{L}, \tilde{r})$$  \hspace{1cm} (1)

one may derive the radial force:

$$F^r \equiv \frac{dp^r}{d\tau}$$ \hspace{1cm} (2)

where:

$$p^r \equiv m \frac{dr}{d\tau}$$

This yields the perhaps unexpected relation:

$$\frac{M}{m} F^r = -\frac{1}{2} \frac{\partial}{\partial \tilde{r}} \tilde{V}^2$$ \hspace{1cm} (3)

which reduces to the usual gradient formula far from the hole, when $\tilde{V} \sim 1$, where:

$$\tilde{V} = \sqrt{\left(1 - \frac{2}{\tilde{r}}\right) \left(1 + \frac{\tilde{L}^2}{\tilde{r}^2}\right)}$$ \hspace{1cm} (4)

Schwarzschild effective potential (dead black hole, $S = Q = 0$)

$$\tilde{E} \equiv \frac{E}{m} \quad \tilde{V} \equiv \frac{V}{m} \quad \tilde{r} \equiv \frac{r}{M} \quad \tilde{L} \equiv \frac{L}{Mm}$$

as given in Misner, Thorne and Wheeler, 1973 [4], where:

$E$ is the total (conserved) energy
$L$ is the (conserved) angular momentum of the test particle around the black hole
$M$ is the mass of the black hole
$V$ is the effective potential
$m$ is the rest mass of the test particle
$r$ is the distance of the test particle from the black hole centre
$F^r$ is the radial force
$p$ is the momentum of the test particle

The effective potential for the Schwarzschild black hole plots out as:
The perturbatively unstable orbit at \( r = 4M \) is especially interesting, since it could, in theory, be visited and stabilized (through feedback) without a large expenditure of energy, and so would be that appropriate for a space-station.

Then there is the storage aspect. One would assemble a ring of matter at the \( r = 4M \) balance, and then, as one sped it up, one would move it into the corresponding balance between \( 3M \) and \( 4M \). To extract the energy later, one would simply perturb part of the ring outward, from where it would then fly out to infinity on its own accord. (There is an energy storage limit of \( Mc^2/2 \) when the event horizon rises to the ring orbit).

Such an accumulator would be well suited if one wished to take up the energy from a spinning black hole, which might then be used to kick a space-craft across space. Of course, another such black hole would be needed to stop or turn about! Gravitational acceleration is also the only known way to achieve large speed changes in small times without experiencing destructive forces.

It is an amazing design of nature. Such orbits, being perturbatively unstable, would be clear of debris and thus clear for use by people – the ultimate sling-shot!

One may now derive the central force from:

\[
\frac{M}{m} F^r = -\frac{1}{2} \frac{\partial}{\partial r} \vec{V}^2
\]

to yield:

\[
\frac{M}{m} F^r = -\frac{1}{r^2} + \frac{\text{Static}}{3 - \frac{\vec{r}}{r^4} L^2}
\]

\[\text{Schwarzschild force}\] (5)

For orbits above \( 3M \), the radial force always finally gets to point outward, for high enough angular momentum, which is like the usual Newtonian behaviour. For orbits below \( r = 3M \) the ‘Abramowicz force’ [1, 2] always increases inward with rotation. Here
we also see the original discovery [1] that at \( r = 3M \), the force is independent of the motion.

So the phenomena of reversed centripetal force is seen below \( 3M \). The results generalise quite simply for a general black hole, when care is taken to derive the force correctly.

When investigating accretion, it is perhaps better to contemplate the situation of conserved angular momentum, rather than fixed radius (which would require rocket intervention below \( r = 3M \)).

1.2 General, Kerr-Newman Black Hole (Charged and Spinning):

Begin from the equation of equatorial motion [4], derived from \( p_\mu p^\mu + m^2 = 0 \):

\[
(p^r)^2 = \frac{\alpha E^2 - 2\beta E + \gamma}{r^4}
\]

(6)

where:

\[
p^r = m \frac{dr}{d\tau}
\]

\[
\alpha \equiv (r^2 + a^2)^2 - a^2 \Delta
\]

\[
\beta \equiv (La + eQr)(r^2 + a^2) - La\Delta
\]

\[
\gamma \equiv (La + eQr)^2 - L^2 \Delta - m^2 r^2 \Delta
\]

\[
\Delta \equiv r^2 - 2Mr + a^2 + Q^2
\]

As before, the force is the rate of change of momentum, and as such is not always minus the potential gradient. In this case the force has no direct relation to the effective potential, but is given by:

\[
F^r = \frac{1}{2m} \frac{\partial}{\partial r} \left( \frac{\alpha E^2 - 2\beta E + \gamma}{r^4} \right)
\]

(7)

and yields the radial force components:

\[
M \frac{F^r}{m} (\hat{a}, \hat{q}) =
\]

\[
\begin{aligned}
&- \frac{1}{\tilde{r}^2} \tilde{E} \hat{q}^2 + \left( 1 - \tilde{E}^2 \right) \hat{a}^2 + \left( 1 - \tilde{E}^2 \right) \hat{q}^2 + 3 \tilde{E} \hat{q} \hat{E} \hat{a}^2 + 2 \hat{E} \hat{q} \hat{E} \hat{a}^2 \\
&\text{Centripetal}
\end{aligned}
\]

\[
\begin{aligned}
&- \frac{1}{\tilde{r}^3} \hat{E} \hat{q}^2 + \left( 3 - \frac{2\hat{q}^2}{\tilde{r}^4} \right) \hat{a} \hat{E} + \left( 3 - \frac{2\hat{q}^2}{\tilde{r}^5} \right) \hat{a} \hat{q} \hat{E} \hat{a}^2
\end{aligned}
\]

\[
\text{Coriolis}
\]

\[
- \left( \frac{3}{\tilde{r}^4} + \frac{4\hat{q}^2}{\tilde{r}^5} \right) \hat{a} \hat{E} \hat{a}^2
\]

Equatorial radial force
The centri‘fugal’ component changes sign at the radius:

\[
\frac{3 + \sqrt{9 - 8q^2}}{2}
\]

the Coriolis force at:

\[
\frac{4 \tilde{E}q^2}{3(2\tilde{E} - \tilde{c}q)}
\]

However it is perhaps better to locate the zero point of the complete dynamical component of the force, that is Coriolis + centri‘fugal’. This stands at the radius:

\[
\frac{3}{2} \left( 1 - \left( 2\tilde{E} - \tilde{c}q \right) \frac{\hat{a}}{L} \right) + \sqrt{\frac{9}{4} \left( 1 - \left( 2\tilde{E} - \tilde{c}q \right) \frac{\hat{a}}{L} \right)^2 - 2 \left( 1 - 2\tilde{E} \frac{\hat{a}}{L} \right) q^2}
\]

(9)

the position of the event horizon being given by:

\[
\hat{r}_{\text{event horizon}} = 1 + \sqrt{1 - \hat{a}^2 - \hat{q}^2}
\]

where:

\[
\tilde{E} \equiv \frac{E}{m}, \quad \tilde{c} \equiv \frac{e}{m},
\]

\[
\hat{r} \equiv \frac{r}{M}, \quad \hat{q} \equiv \frac{Q}{M}, \quad \hat{a} \equiv \frac{a}{M},
\]

\[
\tilde{L} \equiv \frac{L}{Mm}
\]

as given in Misner, Thorne and Wheeler [4], where:

\(E\) is the total (conserved) energy
\(L\) is the (conserved) angular momentum of the test particle around the black hole
\(M\) is the mass of the black hole
\(Q\) is the electric charge of the black hole
\(S\) is the angular momentum of the black hole \((a \equiv S/M)\)
\(e\) is the electric charge of the test particle
\(m\) is the rest mass of the test particle
\(r\) is the distance of the test particle from the black hole centre
\(F^r\) is the radial force
\(p\) is the momentum of the test particle

1.3 Appendix

Much of the simplicity so far depended upon the symmetries present. Time symmetry yielded constant energy \((E)\), and \(\phi\) symmetry yielded conserved angular momentum \((L)\). However a further (Carter) symmetry has been located, and this is equivalent to having solved the final equation of motion (cf. the Hamilton-Jacobi approach). The now decoupled equations of motion are given by:

\footnote{This is not the usual Coriolis force, but an analogous concept.}
\[ p^r = \frac{\sqrt{R}}{\rho^r} \quad p^\phi = \frac{1}{\rho^r} \left( \frac{a}{\Delta} P - aE + \frac{L}{\sin^2 \theta} \right) \quad p^\theta = \frac{\sqrt{\Theta}}{\rho^r} \]

where
\[ R \equiv P^2 - \left( m^2 r^2 + (L - aE)^2 + \varphi \right) \Delta \]
\[ P \equiv E \left( r^2 + a^2 \right) - La - eQr \]
\[ \Theta \equiv \varphi - \left( a^2 \left( m^2 - E^2 \right) + \frac{L^2}{\sin^2 \theta} \right) \cos^2 \theta \]
\[ \rho^r \equiv r^2 + a^2 \cos^2 \theta \]
\[ \Delta \equiv r^2 - 2Mr + a^2 + Q^2 \]
\[ p^r \equiv m \frac{dr}{d\tau} \quad p^\theta \equiv m \frac{d\theta}{d\tau} \quad p^\phi \equiv m \frac{d\phi}{d\tau} \]

and \( \varphi \) is the Carter constant.

As before, these expressions were extracted from Misner, Thorne and Wheeler, 1973 [4]. From these arise the ‘forces’:
\[ F^r = \frac{\partial p^r}{\partial r} \frac{p^r}{m} + \frac{\partial p^\phi}{\partial \theta} \frac{p^\phi}{m} + \frac{\partial p^\theta}{\partial \phi} \frac{p^\theta}{m} = \frac{\partial p^\phi}{\partial r} \frac{p^r}{m} + \frac{\partial p^r}{\partial \theta} \frac{p^\theta}{m} \]

Since the results lack the former simplicity and embody no new physics, it is felt that there is little purpose in listing the solutions.

2 Acknowledgements

The algebraic facilities made available by the scientific computing section of ICTP made possible much of the exploration, although the final results are simple enough to be easily obtained by hand.

It is also a tribute to Misner Thorne and Wheeler’s ‘Gravitation’ book that such a calculation can so easily be picked out and performed.

Thanks also to Marek Abramowicz, for his patience with the many varied, and often incorrect attempts at a simplified derivation of the effect he discovered. That it reduced so beautifully could not have been possible without the pioneering work of him and others.

3 References

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