Charging of quantum battery with periodic driving

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We explore the charging of a quantum battery based on spin systems through periodic modulation of a transverse-field like Ising Hamiltonian. In the integrable limit, we find that resonance tunnelling can lead to a higher transfer of energy to the battery at specific drive frequencies. When the integrability is broken in the presence of an additional longitudinal field, we find that the effective Floquet Hamiltonian contains terms which may lead to a global charging of the battery. However, we do not find any quantum advantage in the charging power, thus demonstrating that global charging is only a necessary and not sufficient condition for achieving quantum advantage.

I. INTRODUCTION

In recent times, there has been a growing interest towards the understanding of thermodynamic aspects of small-scale quantum systems [1–6]. This has necessitated a closer look at the dynamics of energy conversion at such small scales, with a particular focus on how useful work can be extracted from thermal sources utilizing what are known as quantum thermal machines [7, 8]. As an offshoot of this development, efforts have also been made to explore the possibility of efficiently storing the extracted energy in so-called ‘quantum batteries’ [9,10], with the expectation that quantum effects may lead to a superior performance of such batteries when compared to their classical counterparts [11–17].

The performance of a quantum battery is usually judged on three aspects – the maximum amount of extractable energy, charging/discharging power and the stability of the stored energy. In the simplest of settings where environmental dissipation can be ignored, the charging or the transfer of energy to the battery is carried out using a unitary quench protocol. In other words, given that the quantum system designated as the battery is initially in the ground-state of the ‘battery Hamiltonian’, the charging process involves turning on additional interactions or external fields for a finite time during which the state of the battery evolves unitarily, driven by a net ‘charging Hamiltonian’. At the end of the charging process, the additional interactions/fields are switched off; the energy difference between the final and initial state with respect to the battery Hamiltonian is considered as the energy stored in the battery. However, in the reverse process, it is often not possible to extract all the energy stored in the battery through unitary protocols, and the maximum amount of the useful energy that can be extracted is termed as ergotropy [10]. The rate at which energy is stored (extracted) in the battery is defined as its charging (discharging) power which should ideally be maximized. Finally, a high variance in the charged state of the battery with respect to the battery Hamiltonian implies an undesirable instability in the energised state of the battery [18].

For a battery composed of \( N \) identical quantum systems or ‘cells’, parallel charging operations – where each of the cell is separately charged – can only lead to a linear scaling of the charging power with \( N \). A quantum advantage [11] thus emerges in the scenario when the charging power scales super-linearly. Recently, it has been shown that global charging protocols where the charging Hamiltonian couples multiple cells may lead to a quantum advantage, but this is only a necessary condition and not a sufficient one [15, 10, 19, 20].

In this work, we depart from the traditional way of charging a quantum battery through a quench protocol and consider the case of a periodic driving protocol. To elaborate, we choose the charging Hamiltonian to be a periodic function of time and the energetics of the battery is observed only at stroboscopic instants which are separated by an interval equal to the time period of the charging Hamiltonian. It is well known that under such a periodic driving protocol, the dynamics of the system at stroboscopic instants is dictated by a time-independent Floquet Hamiltonian [21, 22], which can have drastically different properties from that of the charging Hamiltonian. As we shall demonstrate, this leads to the emergence of interesting features in the performance of the quantum battery.

Although quantum batteries can be modelled in a variety of ways, we consider a simple model based on a lattice system of \( N \) non-interacting half-integer spins. Firstly, we focus on the case where the battery is charged using a periodically modulated transverse-field Ising Hamiltonian (TFIH) [23–25] with the nearest-neighbour interactions. We find that the energy transferred to the battery can be enhanced depending on the frequency of the charging Hamiltonian. However, given the integrability of the model, no quantum advantage in terms of the charging power can be expected in this case [15]. Therefore, we also consider the case of periodically driving the battery with the same Hamiltonian but with an additional periodically modulated longitudinal field which breaks the integrability of the model. In this case, we find that even though the effective Floquet Hamiltonian leads to a global charging like scenario at low frequency drives, no quantum advantage is seen in the charging power.
The rest of the paper is organized as follows: in Sec. II we introduce the periodically driven quantum battery and show how the relevant quantities are to be calculated. In Sec. III we investigate a quantum battery driven with the integrable TFIH and show that the asymptotic energy stored in the battery strongly depends on the drive frequency. In Sec. IV we proceed to explore the charging power of the battery in presence of an additional integrability-breaking longitudinal field, having same periodicity as that of the TFIH. Concluding remarks are provided in Sec. V.

II. PERIODICALLY DRIVEN QUANTUM BATTERY

The quantum battery composed of $N$ non-interacting half-integer spins is represented by the Hamiltonian,

$$H_B = \hbar_z \sum_{j=1}^{N} \sigma_j^z,$$

where $\sigma_j^z$'s are Pauli matrices. During the charging process, the state of the battery evolves under the charging Hamiltonian, $H_c(t) = H_B + V(t)$, which lasts for a finite time $\tau$, such that $V(0) = V(\tau) = 0$. At $t = 0$, the battery is assumed to be in the ground state $\rho(0) = |\psi_0 \rangle \langle \psi_0|$ of the Hamiltonian $H_B$. At the end of the charging process, the time-evolved state of the battery is given by,

$$\rho(\tau) = U(\tau) \rho_0 U^\dagger(\tau),$$

where $U(\tau) = T \exp(-i \int_0^{\tau} H_c(t') dt')$ with $T$ being the time ordering operator. Note that we have assumed natural units, i.e., $\hbar = 1$. In what follows, we shall consider a periodically driven battery such that,

$$H_c(t) = H_c(t + T),$$

where $T = 2\pi/\omega$ is the time period of the charging Hamiltonian and $\omega$ is the frequency of the drive. Naturally, this requires $V(t + T) = V(t)$. Further, we shall restrict ourselves to the stroboscopic dynamics of the charging process; the charging time $\tau$ therefore shall always satisfy the condition $\tau = nT$, where $n \in \mathbb{Z}^+$. At these instants, the time-evolution operator assumes the form, $U(nT) = (U^F)^n$, where the Floquet operator $U^F$ is given by by [21][22],

$$U^F = T \exp \left( -i \int_0^{T} H_c(t') dt' \right) = \exp(-i H^F T).$$

In other words, when observed only at the stroboscopic instants, the battery evolves under the action of the time-independent Floquet Hamiltonian $H^F_c$.

The energy stored in the battery after $n$ stroboscopic instants is given by,

$$E(n) = \text{Tr} [\rho(nT)H_B] - \text{Tr} [\rho_0 H_B] = \text{Tr} \left[ (U^F)^n \rho_0 (U^{F})^n H_B \right] - \text{Tr} [\rho_0 H_B],$$

and the instantaneous power is obtained as,

$$P(n) = \frac{E(n)}{nT}.$$

Similarly, the variance of the battery Hamiltonian at time $t = nT$ is determined by

$$\Delta H^2_B(n) = \text{Tr} [\rho(nT)H^2_B] - \left( \text{Tr} [\rho(nT)H_B] \right)^2.$$

III. PERIODIC DRIVING WITH INTEGRABLE TRANSVERSE-FIELD ISING HAMILTONIAN

In this section, we consider a periodic modulation to the battery Hamiltonian,

$$V(t) = J(t) \sum_{j=1}^{N-1} \sigma_j^x \sigma_{j+1}^x,$$

so that the charging Hamiltonian $H_c(t) = H_B + V(t)$ assumes the form of the TFIH. For the drive, we choose a square-pulse modulation,

$$J(t) = \begin{cases} J_0, & \text{for } 0 < t < \frac{T}{2} \\ -J_0, & \text{for } \frac{T}{2} < t < T \end{cases}.$$ 

The TFIH is integrable and becomes analytically tractable after a Jordan-Wigner mapping to an equivalent model of non-interacting spinless-fermions and a further assumption of translational invariance. A detailed analysis of the same is given in Appendix. In the quasi-momentum space, the charging Hamiltonian acquires a simple decoupled structure,

$$H_c(t) = \bigoplus_{k \geq 0} H_{c-k}(t).$$

The Floquet operator is found to be of the form,
The energy stored in the battery when observed at stroboscopic intervals saturates to a steady value when \( n \to \infty \) and \( N \to \infty \). Fig. 1 shows the energy stored after \( n = 100 \) as a function of the drive frequency for a system of size \( N = 200 \). It is clear that certain drive frequencies favour a much higher storage of energy in the battery. This occurs due to resonance tunnelling at these frequencies — the eigenspectrum of the Floquet operator becomes degenerate \([27]\). This is easily seen from Eq. (11), where we note that \( U_{k=0}^F = U_{k=\pi}^F = 1 \), for \( \omega = 2h_z/n \), with \( n \in \mathbb{Z} \). Similarily, \( U_{k=0}^F = U_{k=\pi}^F = -1 \) for \( \omega = 4h_z/(2n+1) \). Thus, the drive frequency in a periodically driven quantum battery provides an additional control over the amount of energy stored in the battery during the charging process.

It is well known that a battery of spins driven with a static TFIH exhibits no quantum advantage in charging power \([15]\). It is easy to see that the same also holds true for the periodically driven case. When observed at stroboscopic instants, the evolution of the battery is equivalent to that of the evolution driven by the static Hamiltonian \( H_c^F \). Thus, following Ref. \([15]\), the instantaneous charging power at stroboscopic instants is upper bounded as,

\[
P(n) \leq \sqrt{\Delta H_B^2(n) I_E(n)},
\]

where \( \Delta H_B^2(t) \) is the instantaneous variance as defined in Eq. (7) and \( I_E(t) \) is the Fisher information in energy space \([15]\). A quantum advantage in the charging power can emerge if the variance scales super-extensively with the system size which requires long-ranged entanglement to be generated between the spins. However, Eq. (10) shows that the spin chain battery driven with the TFIH is equivalent to a chain of non-interacting spinless fermions, with each of the fermionic qubit being charged independently. Hence, the integrability of the TFIIH rules out any possible quantum advantage (see Appendix. B for more details).

IV. PERIODIC DRIVING WITH TRANSVERSE-FIELD ISING HAMILTONIAN AND A LONGITUDINAL FIELD

In this section, we investigate the charging of the same quantum battery but with a non-integrable charging Hamiltonian. Specifically, the charging Hamiltonian is chosen to be that of the TFIIH with an additional longitudinal field; the latter having the same time-periodicity as that of the TFIIH. In other words, we consider a mod-

\[
U^F = \bigotimes_{k \geq 0} U^F_k = \bigotimes_{k \geq 0} e^{-i \omega (2J_0 \sin(k) \sigma_y + (2h_z - 2J_0) \sigma_z) - i \omega (2J_0 \sin(k) \sigma_y + (2h_z + 2J_0 \cos(k)) \sigma_z)}
\]

(11)

FIG. 2. Charging power \( (P) \) as a function of the number \( (N) \) of spin-1/2 systems after \( n = n^* \) for different drive frequencies \((\omega = 2 \text{ and } \omega = 8)\), when charged using a modulation of the form given in Eq. \( (13) \). Here, \( n^* \) is the number of time periods for which \( P \) becomes maximum. The other parameters chosen for numerics are \( J_0 = 0.5, h_z = 2 \) and \( h_0 = 0.3 \). For \( \omega = 2 \) and \( \omega = 8 \), the values of \( n^* \) have been found to be 2 and 3, respectively.

ulation of the form,

\[
V(t) = J(t) \sum_{j=1}^{N-1} \sigma_j^z \sigma_{j+1}^z + h_x(t) \sum_{j=1}^{N} \sigma_j^x,
\]

(13)

where \( J(t) = J(t + T) \) and \( h_x(t) = h_x(t + T) \). Once again, we consider a square-pulse charging protocol with the explicit time-dependence of \( J(t) \) as given in Eq. \( (9) \) and that of \( h(t) \) given by,

\[
h_x(t) = \begin{cases} 
  h_0, & \text{for } 0 < t < \frac{T}{2} \\
  -h_0, & \text{for } \frac{T}{2} < t < T.
\end{cases}
\]

(14)

Clearly the integrable limit discussed in Sec. III is retrieved when \( h_0 = 0 \).

Firstly, we note that the charging Hamiltonian can no longer be mapped to the model of decoupled spinless fermions. Secondly, let us take a look at the explicit form of the corresponding Floquet Hamiltonian \( H_c^F \),

\[
H_c^F = h_z \sum_j \sigma_j^z + T \frac{1}{2} h_z J_0 \sum_j \sigma_j^z \sigma_{j+1}^x + h_z J_0 \sum_j \sigma_j^y \sigma_{j+1}^x \\
+ h_z h_0 \sum_j \sigma_j^y - \frac{T^2}{3} (h_0 J_0 h_z \sum_j \sigma_j^z \sigma_{j+1}^z + h_0 J_0 h_z \sum_j \sigma_j^x \sigma_{j+1}^x) \\
+ h_z (J_0^2 + \frac{h_0^2}{2}) \sum_j \sigma_j^z + J_0^2 h_z \sum_j \sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^x + O(T^3) \]

(15)
Note that in the above Hamiltonian, the terms up to order $O(T^2)$ already contain strings of spin operators longer than that present in the time-dependent charging Hamiltonian $H_c(t)$. To elaborate, terms such as $\sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^x$ in $H^F_c$ (Eq. (15)) couple three nearest-neighbour spins while $H_c(t)$ only has interactions between two nearest-neighbour spins. Likewise, higher order terms in Eq. (15) contain even longer strings of spin operators which couple multiple spins. It is important to realize that the presence of long strings of spin operators spanning the length of the battery in the Floquet Hamiltonian implies a global charging protocol where all the unit cells of the battery are collectively charged. However, it must be ensured that the drive frequency is not too large so that higher order terms (of $T$) become negligible. In other words, the periodically driven quantum battery with the modulation defined in Eq. (14) can potentially lead to a quantum advantage at low drive frequencies. We also note in passing that for $\hbar \omega_0 = 0$, the Floquet Hamiltonian in Eq. (15) becomes similar to the extended Ising Hamiltonian which is reducible to spinless free fermionic system and thereby integrable [26,30].

In Fig. 2, we plot the power, maximized over $n$, as a function of the battery size $N$. Contrary to the expected superlinear scaling at low frequencies, we find that the power scales linearly with the system size for all range of frequencies. Thus, no quantum advantage emerges despite the presence of global charging terms in the driving Hamiltonian $H^F_c$, thus demonstrating that global charging is not a sufficient condition for achieving quantum advantage.

V. SUMMARY

In this work, we introduced a periodically driven quantum battery, which differs from the traditional way of charging through a static Hamiltonian. When observed at stroboscopic instants, the evolution is effectively driven by the Floquet Hamiltonian, thus leading to the possibility of engineering specific conditions that may be relatively difficult to achieve through static charging. As an immediate consequence, we have shown, by considering the periodically driven TFIH as the charging Hamiltonian that, the drive frequency can significantly alter the amount of energy stored in the battery. However, periodically driven integrable spin chain models, similar to their static counterpart, cannot provide any quantum advantage with respect to the charging power. Moving on to the case of a periodically driven non-integrable chain, we first show that the corresponding Floquet Hamiltonian consists of long string of spin operators that theoretically results in a global charging of the battery. However, no quantum advantage is manifested in the charging power in this case as well, thus corroborating previous results that global charging is only a necessary but not a sufficient condition to achieve quantum advantage.

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Appendix A: Jordan-Wigner transformation of the transverse-field Ising Hamiltonian

We use the following Jordan-Wigner (JW) transformations [25,29], which map each spin-1/2 system to a system of spinless Fermions:

\[
c_j^+ = \sigma_j^x \prod_{n=1}^{j-1} \sigma_n^z, \quad (A1a)
\]

\[
c_j = \sigma_j^- \prod_{n=1}^{j-1} \sigma_n^z, \quad (A1b)
\]

where $\sigma_j^{\pm} = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y)$ and $c_j^+$ ($c_j^-$) is Fermionic annihilation (creation) operator for $j$-th site, satisfying usual anti-commutation relations. Using JW transformations as mentioned in Eq. (A1a) and (A1b), the battery Hamiltonian in Eq. (1) and the charging Hamiltonian $H_c(t) = H_B + V(t)$ with $V(t)$ of the form mentioned in Eq. (8) are recast to the following forms:

\[
H_B = h_z \sum_{j=1}^{N} (2c_j^+c_j - 1), \quad (A2)
\]

\[
H_c(t) = h_z \sum_{j=1}^{N} (2c_j^+c_j - 1) - J(t) \sum_{j=1}^{N-1} (c_j^+c_{j+1} + c_{j+1}^+c_j) + J(t) \sum_{j=1}^{N-1} (c_j^+c_{j+1} + c_{j+1}^+c_j). \quad (A3)
\]

It should be noted that all the terms in the charging Hamiltonian $H_c(t)$ in Eq. (A3) are local as the Hamiltonian in Eq. (A3) involves only nearest neighbour interactions.

Resorting to the Fourier space, we have,

\[
H_B = \bigoplus_{k \geq 0} H_{B,k}, \quad (A4)
\]

\[
H_c(t) = \bigoplus_{k \geq 0} H_{c,k}(t), \quad (A5)
\]
where $H_{B,k}$ and $H_{c,k}(t)$ with momentum $k \in [0, \pi]$ are given by,
\[
H_{B,k} = 2\hbar z \tau_z,
\]
(A6)

\[
H_{c,k}(t) = 2J(t) \sin(k) \tau_y + (2\hbar z - 2J(t) \cos(k)) \tau_z,
\]
(A7)

where $\tau_z$ and $\tau_y$ are Pauli matrices. The Floquet operator for $k$-th mode is determined by
\[
U_k^F = T \exp\left(-i \int_0^T H_{c,k}(t) dt\right) = \exp(-iH_k^F T),
\]
where $H_k^F$ is the Floquet Hamiltonian for $k$-th mode. One can find out the explicit form of $U_k^F$ as:
\[
U_k^F = \exp\left(-\frac{iT}{2} (2J_0 \sin(k) \sigma_y + (2\hbar z - 2J_0 \cos(k)) \sigma_z)\right)
\]
\[
\times \exp\left(-\frac{iT}{2} (-2J_0 \sin(k) \sigma_y + (2\hbar z + 2J_0 \cos(k)) \sigma_z)\right).
\]
(A9)

Interestingly, using Eq. (11), we note that $U_k^F = U_{k=\pi}^F = \mathbb{1}$, when $\omega = 2\hbar z/n$, where $n \in \mathbb{Z}$. On the other hand, $U_k^F = U_{k=\pi}^F = -\mathbb{1}$ when $\omega = 4\hbar z/(2n+1)$, where $n \in \mathbb{Z}$. At these values of $\omega$, Floquet quasi-energy gap closes or reopens, respectively (i.e., eigenvalues of the Floquet Hamiltonian become zero and $\pi$, respectively) and thus the stored energy is expected to attain peaks [27].

**Appendix B: Stored energy ($E$) and $\Delta H^2_B$ for periodic driving with the integrable transverse-field Ising Hamiltonian**

For periodic driving with integrable transverse-field Ising Hamiltonian, the energy stored at stroboscopic time $t = nT$ is given by,
\[
E(n) = \sum_{k \geq 0} E_k(n)
\]
(B1)

where $E_k(n)$ for $k \in [0, \pi]$ is evaluated as,
\[
E_k(n) = \langle \psi_k(nT)|H_{B,k}|\psi_k(nT)\rangle - \langle \psi_k(0)|H_{B,k}|\psi_k(0)\rangle,
\]
(B2)

with the state $|\psi_k(nT)\rangle$ at the stroboscopic time $t = nT$ determined by,
\[
|\psi_k(nT)\rangle = (U_k^F)^n |\psi_k(0)\rangle.
\]
(B3)

Similarly, the variance of the battery Hamiltonian ($\Delta H^2_B$) at time $t = nT$ is given by,
\[
\Delta H^2_B(n) = \sum_{k \geq 0} \Delta H^2_{B,k}(n)
\]
(B4)

where $\Delta H^2_{B,k}(n)$ for $k \in [0, \pi]$ is determined by,
\[
\Delta H^2_{B,k}(n) = \langle \psi_k(nT)|H_{B,k}^2|\psi_k(nT)\rangle
\]

\[
- \langle \psi_k(nT)|H_{B,k}|\psi_k(nT)\rangle^2.
\]
(B5)

Thus, from Eq. (B1) and Eq. (B4), it can be clearly seen that both the stored energy and the variance of the battery Hamiltonian can be written as the sum of decoupled momentum modes, leading to the linear scaling with the system size $N$. Therefore, super-extensive scaling behaviours of charging power and the variance of the battery Hamiltonian are impossible for periodic driving with integrable transverse field Ising Hamiltonian.

**Appendix C: Jordan-Wigner transformation of the transverse-field Ising Hamiltonian in presence of a longitudinal field**

We use JW transformations mentioned in Eq. (A1a) and (A1b), by which the charging Hamiltonian $H_c(t) = H_B + V(t)$ with $V(t)$ of the form mentioned in Eq. (13) is transformed into the following equation:
\[
H_c(t) = h_z \sum_{j=1}^N (2c_j^\dagger c_j - 1) - J(t) \sum_{j=1}^{N-1} (c_j^\dagger c_{j+1}^\dagger c_{j+1} c_j)
\]
\[
+ J(t) \sum_{j=1}^{N-1} (c_j c_{j+1} + c_{j+1}^\dagger c_j^\dagger)
\]
\[
+ h_x(t) \sum_{j=1}^N \left( c_j^\dagger \prod_{n=1}^{j-1} (2c_n^\dagger c_n - 1) + \prod_{n=1}^{j-1} (2c_n^\dagger c_n - 1) c_j \right),
\]
(C1)

In contrast to Eq. (A3), the term containing $h_z(t)$ in Eq. (C1) renders the charging Hamiltonian non-local.

**Appendix D: Approximated Floquet Hamiltonian for periodic driving with the transverse-field Ising Hamiltonian in presence of a longitudinal field**

Using Magnus expansion, the Floquet Hamiltonian $H_k^F$ in the high frequency limit (i.e., $\omega >> 1$ or $T << 1$) is approximately found to be:
\[
H_k^F \approx \frac{1}{T} \int_0^T dt_1 H_c(t_1)
\]
\[
- \frac{i}{2T} \int_0^T dt_1 \int_0^{t_1} dt_2 [H_c(t_1), H_c(t_2)]
\]
\[
- \frac{1}{6T} \int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 [H_c(t_1), [H_c(t_2), H_c(t_3)]]
\]
\[
+ [H_c(t_3), [H_c(t_2), H_c(t_1)]] + \mathcal{O}(t^3).
\]
(D1)
The transverse-field Ising Hamiltonian in presence of a longitudinal field can be recast in the following form:

\[
H_c(t) = \begin{cases} 
H_1, & \text{for } 0 < t < \frac{T}{2} \\
H_2, & \text{for } \frac{T}{2} < t < T,
\end{cases}
\]

where

\[
H_1 = h_z \sum_j \sigma_j^z + J_0 \sum_j \sigma_j^x \sigma_{j+1}^x + h_0 \sum_j \sigma_j^x,
\]

and

\[
H_2 = h_z \sum_j \sigma_j^z - J_0 \sum_j \sigma_j^x \sigma_{j+1}^x - h_0 \sum_j \sigma_j^x.
\]

Using Eq. (D2), we obtain,

\[
\frac{1}{T} \int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \{[H_c(t_1), [H_c(t_2), H_c(t_3)]] + [H_c(t_3), [H_c(t_2), H_c(t_1)]]\} = 2T^2 (h_0 J_0 h_z \sum_j \sigma_j^x \sigma_{j+1}^x + J_0^2 h_z \sum_j \sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^x).
\]

Similarly, the term of \(H_c^{\mathcal{F}}\) of the order \(T^2\) is determined by:

\[
\frac{1}{T} \int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \{[H_c(t_1), [H_c(t_2), H_c(t_3)]] + [H_c(t_3), [H_c(t_2), H_c(t_1)]]\} = 2T^2 (h_0 J_0 h_z \sum_j \sigma_j^x \sigma_{j+1}^x + J_0^2 h_z \sum_j \sigma_j^x \sigma_{j+1}^x \sigma_{j+2}^x).
\]

Using Eq. (D3), (D4) and (D4), we finally obtain Eq. (15).

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