ON THE NUMBER OF MODULI OF PLANE SEXTICS WITH SIX CUSPS

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Abstract. Let \( \Sigma_{6,0} \) be the variety of irreducible sextics with six cusps as singularities. Let \( \Sigma \subset \Sigma_{6,0} \) be one of irreducible components of \( \Sigma_{6,0} \). Denoting by \( M_4 \) the space of moduli of smooth curves of genus 4, we consider the rational map \( \Pi : \Sigma \rightarrow M_4 \) sending the general point \([\Gamma]\) of \( \Sigma \), corresponding to a plane curve \( \Gamma \subset \mathbb{P}^2 \), to the point of \( M_4 \) parametrizing the normalization curve of \( \Gamma \). The number of moduli of \( \Sigma \) is, by definition the dimension of \( \Pi(\Sigma) \). We know that \( \dim(\Pi(\Sigma)) \leq \dim(M_4) + \rho(2, 4, 6) - 6 = 7 \), where \( \rho(2, 4, 6) \) is the Brill-Noether number of linear series of dimension 2 and degree 6 on a curve of genus 4. We prove that both irreducible components of \( \Sigma_{6,0} \) have number of moduli equal to seven.

1. Introduction

Let \( \Sigma_{n,k,d} \subset \mathbb{P}(H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(n))) := \mathbb{P}^N \), with \( N = \frac{n(n+3)}{2} \), be the closure, in the Zariski’s topology, of the locally closed set of reduced and irreducible plane curves of degree \( n \) with \( k \) cusps and \( d \) nodes. Let \( \Sigma \subset \Sigma_{n,k,d} \) be an irreducible component of the variety \( \Sigma_{n,k,d} \). Denoting by \( M_g \) the moduli space of smooth curves of genus \( g \), it is naturally defined a rational map

\[ \Pi_\Sigma : \Sigma \rightarrow M_g, \]

sending the general point \([\Gamma]\) of \( \Sigma \) to the isomorphism class of the normalization of the curve \( \Gamma \subset \mathbb{P}^2 \) corresponding to \([\Gamma]\). We say that \( \Pi_\Sigma \) is the moduli map of \( \Sigma \) and we set

number of moduli of \( \Sigma := \dim(\Pi_\Sigma(\Sigma)) \).

We say that \( \Sigma \) has general moduli if \( \Pi_\Sigma \) is dominant. Otherwise, we say that \( \Sigma \) has special moduli or that \( \Sigma \) has finite number of moduli. By lemma 2.2 of \[^4\] we know that the dimension of the general fibre of \( \Pi_\Sigma \) is at least equal to

\[ \max(8, 8 + \rho - k), \]

where \( \rho := \rho(2, g, n) = 3n - 2g - 6 \) is the number of Brill-Noether of linear series of degree \( n \) and dimension 2 on a smooth curve of genus \( g \). It follows that, if \( \Sigma \) has the expected dimension equal to \( 3n + g - 1 - k \) and \( g \geq 2 \), then

\[ \dim(\Pi_\Sigma(\Sigma)) \leq \min(\dim(M_g), \dim(M_g) + \rho - k). \]

Definition 1.1. We say that \( \Sigma \) has the expected number of moduli if equality holds in \( 1 \).

In particular, we expect that, if \( \rho - k \leq 0 \), then on the normalization curve \( C \) of the curve \( \Gamma \subset \mathbb{P}^2 \) corresponding to the general point \([\Gamma]\) in \( \Sigma \), there exists only a finite number of linear series of degree \( n \) and dimension 2 mapping \( C \) to a plane curve with nodes and \( k \) cusps as singularities and corresponding to a point of \( \Sigma \), (see the proof of lemma 2.2 of \[^4\] ). For a deeper discussion and a list of known results about the moduli problem of \( \Sigma_{n,k,d} \) we refer to sections 1 and 2 of \[^4\] and related references. In particular, in \[^4\] we have found sufficient conditions in order

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Key words and phrases. number of moduli, sextics with six cusps, plane curves, Zariski pairs.
that an irreducible component $\Sigma$ of $\Sigma^6_{k,d}$ has finite and expected number of moduli. If $\Sigma$ verifies these conditions then $\rho(2,n,g) \leq 0$. Finally in [4] we constructed examples of families of plane curves with nodes and cusps with finite and expected number of moduli. In this paper we consider the particular case of the variety $\Sigma^6_{6,0}$ of irreducible sextics with six cusps.

It was proved by Zariski (see [8]) that $\Sigma^6_{6,0}$ has at least two irreducible components. One of them is the parameter space $\Sigma_1$ of the family of plane curves of equation

$$f_3^2(x_0, x_1, x_2) + f_2^3(x_0, x_1, x_2) = 0,$$

where $f_2$ and $f_3$ are homogeneous polynomials of degree two and three respectively. The general point of $\Sigma_1$ corresponds to an irreducible sextic with six cusps on a conic as singularities. Moreover, $\Sigma^6_{6,0}$ contains at least one irreducible component $\Sigma_2$ whose general element corresponds to a sextic with six cusps not on a conic as singularities and containing in its closure the variety $\Sigma^6_{6,0}$ of elliptic sextics with nine cusps. Recently, A. Degtyarev has proved that $\Sigma_1$ and $\Sigma_2$ are the unique irreducible components of $\Sigma^6_{6,0}$, (see [1]).

The moduli number of $\Sigma_1$ and $\Sigma_2$ can not be calculated by using the result of [4]. Indeed, in this case $\rho(2,4,6) = 4 > 0$ and then the general element of every irreducible component of $\Sigma^6_{6,0}$ does not verify the hypotheses of proposition 4.1 of [4]. On the contrary, it is easy to verify that, if $\Gamma \subset \mathbb{P}^2$ is the plane curve corresponding to the general element of one of the irreducible components of $\Sigma^6_{6,0}$ and $C$ is the normalization curve of $\Gamma$, then the map $\mu_{o,C}$ is injective. But, in contrast with the nodal case, this information is not useful in order to study the moduli problem of $\Sigma^6_{k,d}$, (see [6] and remark 4.2 of [4]). In the proposition 2.2 and corollary 2.4 we prove that $\Sigma_2$ has the expected number of moduli equal to seven. Moreover, we show that there exists a stratification

$$\Sigma^6_{6,0} \subset \Sigma' \subset \tilde{\Sigma} \subset \Sigma_2,$$

where $\Sigma'$ and $\tilde{\Sigma}$ are respectively irreducible components of $\Sigma^6_{8,0}$ and $\Sigma^8_{6,0}$ with expected number of moduli. Finally, in the corollary 2.8 we prove that also $\Sigma_1$ has the expected number of moduli by using that every element of $\Sigma_1$ is the branch locus of a triple plane.

2. ON THE NUMBER OF MODULI OF COMPLETE IRREDUCIBLE FAMILIES OF PLANE SEXTICS WITH SIXCUSPS

First of all we want to find sufficient conditions in order that, if an irreducible component $\Sigma$ of $\Sigma^6_{k,d}$ has the expected number of moduli, then every irreducible component $\Sigma'$ of $\Sigma^6_{k',d'}$, containing $\Sigma$, has the expected number of moduli. In the corollary 4.7 of [4] we considered this problem under the hypothesis that $\Sigma$ has the expected dimension and $\rho(2,n,g) \leq 0$. Now we are interested to the case $\rho > 0$. We need the following local result.

Let

$$D = \{(a, b, x, y) | y^2 = x^3 + ax + b \} \subset \mathbb{C}^2 \times \mathbb{A}^2$$

be the versal deformation family of an ordinary cusp (see [2] for the definition and properties of the versal deformation family of a plane singularity). We recall that the general curve of this family is smooth. The locus $\Delta$ of $\mathbb{C}^2$ of the pairs $(a, b)$ such that the corresponding curve is singular, has equation $27b^2 = 4a^3$. For $(a, b) \in \Delta$ and $(a, b) \neq (0,0)$, the corresponding curve has a node and no other singularities, whereas $(0,0)$ is the only point parametrizing a cuspidal curve.
Lemma 2.1 (3, page 129.). Let $\mathcal{G} \rightarrow \mathbb{C}^2$ be a two parameter family of curves of genus $g \geq 2$, whose general fibre is stable and which is locally given by $y^2 = x^3 + ax + b$, with $(a, b) \in \mathbb{C}^2$ and let $D \subset \mathbb{C}^2$ be a curve passing through $(0, 0)$ and not tangent to the axis $b = 0$ at $(0, 0)$. Then the $j$-invariant of the elliptic tail of the curve which corresponds to the stable limit of $\mathcal{G}(0,0)$, with respect to the curve $D$, doesn’t depend on $D$. Otherwise, for every $j_0 \in \mathbb{C}$, there exists a curve $D_{j_0} \subset \mathbb{C}^2$ passing through $(0, 0)$ and tangent to the axis $b = 0$ at this point, such that the elliptic tail of the stable reduction of $\mathcal{G}(0,0)$ with respect to $D_{j_0}$, has $j$-invariant equal to $j_0$.

Proposition 2.2. Let $\Sigma \subset \Sigma_{k,d}^n$, with $k < 3n$, be an irreducible component of $\Sigma_{k,d}^n$. Let $g$ be the geometric genus of the plane curve corresponding to the general element of $\Sigma$. Suppose that $g \geq 2$, $\rho(2, g, n) - k \leq 0$ and $\Sigma$ has the expected number of moduli equal to $3g - 3 + \rho - k$. Then, every irreducible component $\Sigma'$ of $\Sigma_{k',d'}^n$, with $k' = k - 1$ and $d = d'$ or $k = k'$ and $d' = d - 1$, such that $\Sigma \subset \Sigma'$, has expected number of moduli.

Proof. First we consider the case $k' = k - 1$ and $d = d'$. Let $q_1, \ldots, q_k$ be the cusps of $\Gamma$. It is well known that, since $k < 3n$ then $[\Gamma] \in \Sigma_{k-1,d}^n$. In particular, for every fixed cusp $q_i$ of $\Gamma$ there exists an irreducible analytic branch $S_i$ of $\Sigma_{k-1,d}^n$ passing through the point $[\Gamma]$ and whose general point corresponds to a plane curve $\Gamma'$ of degree $n$ with $d$ nodes and $k - 1$ cusps specializing to the singular points of $\Gamma$ different from $q_i$, as $\Gamma'$ specializes to $\Gamma$. Moreover, it is possible to prove that every $S_i$ is smooth at the point $[\Gamma]$, see [7] or chapter 2 of [5]. Let $\Sigma'$ be one of the reducible components of $\Sigma_{k-1,d}^n$ containing $\Sigma$. Notice that the general element of $\Sigma'$ corresponds to a curves of genus $g' = g + 1$. Since $\rho(2, g', n) - k' = 3n - 2g - 2 - 6 - k + 1 = \rho(2, g, n) - k - 1 < 0$, in order to prove the theorem it is enough to show that the general fibre of the moduli map

$$
\Pi_{\Sigma'} : \Sigma' \rightarrow \mathcal{M}_{g+1}
$$

has dimension equal to eight. Let us notice that the map $\Pi_{\Sigma'}$ is not defined at the general element $[\Gamma]$ of $\Sigma$. More precisely, let $\gamma \subset S_i \subset \Sigma'$ be a curve passing through $[\Gamma]$ and not contained in $\Sigma$. Let $\mathcal{C} \rightarrow \gamma$ be the tautological family of plane curves parametrized by $\gamma$. Let $C' \rightarrow \gamma$ be the family obtained from $\mathcal{C} \rightarrow \gamma$ by normalizing the total space. The general fibre of $C' \rightarrow \gamma$ is a smooth curve of genus $g + 1$, while the special fibre $C'_0 := \Gamma'$ is the partial normalization of $\Gamma$ obtained by smoothing all the singular points of $\Gamma$, except the marked cusp $q_i$. If we restrict the moduli map $\Pi_{\Sigma'}$ to $\gamma$, we get a regular map which associates to $[\Gamma]$ the point corresponding to the stable reduction of $\Gamma$ with respect to the family $C' \rightarrow \gamma$, which is the union of the normalization curve $C$ of $\Gamma$ and an elliptic curve, intersecting at the point $q \in C$ which maps to the cusp $q_i \in \Gamma$. Now, let $\mathcal{G} \subset \Sigma' \times \mathcal{M}_{g+1}$ be the graph of $\Pi_{\Sigma'}$, let $\pi_1 : \mathcal{G} \rightarrow \Sigma'$ and $\pi_2 : \mathcal{G} \rightarrow \mathcal{M}_{g+1}$ be the natural projections and let $U \subset \Sigma$ be the open set parametrizing curves of degree $n$ and genus $g$ with exactly $k$ cusps and $d$ nodes as singularities. From what we observed before, if we denote by $\Pi_{\Sigma'}(\Sigma)$ the Zariski closure in $\mathcal{M}_{g+1}$ of $\pi_2|_{\pi_1^{-1}(U)}$, then $\Pi_{\Sigma'}(\Sigma)$ is contained in the divisor $\Delta_1 \subset \mathcal{M}_{g+1}$, whose points are isomorphism classes of reducible curves which are union of a smooth curve of genus $g$ and an elliptic curve, meeting at a point. Denoting by $\Pi_\Sigma : \Sigma \rightarrow \mathcal{M}_g$ the moduli map of $\Sigma$, the rational map

$$
\Delta_1 \rightarrow \mathcal{M}_g
$$

which forgets the elliptic tail, restricts to a rational dominant map

$$
q : \Pi_{\Sigma'}(\Sigma) \rightarrow \Pi_\Sigma(\Sigma).
$$

The dimension of the general fibre of $q$ is at most two. Since, by hypothesis, the dimension of the fibre of the moduli map $\Pi_\Sigma$ is eight, there exists only a finite
number of $g_n^2$ on $C$, ramified at $k$ points, which maps $C$ to a plane curve $D$ such that the associated point $[D] \in \mathbb{P}^{\frac{n(n+3)}{2}}$ belongs to $\Sigma$. In particular, the set of points $x$ of $C$ such that there is a $g_n^2$ with $k$ simple ramification points, one of which at $x$, is finite. So, the dimension of the general fibre of $q$ is at most one. In order to decide if the general fibre of $q$ has dimension zero or one, we have to understand how the $j$-invariant of the elliptic tail of the stable reduction of $\Gamma'$ with respect the family $\mathcal{C}' \to \gamma$, depends on $\gamma$. If $\mathcal{C} \to \mathbb{C}^2$ is the étale versal deformation family of the cusp. By versality, for every fixed cusp $p_i$ of $\Gamma$, there exist étale neighborhoods $U \to \mathbb{P}^{\frac{n(n+3)}{2}}$ of $[\Gamma]$ in $\mathbb{P}^{\frac{n(n+3)}{2}}$, $V \to \mathbb{C}^2$ of $(0,0)$ in $\mathbb{C}^2$ and $U_i$ of $p_i$ in the tautological family $U \to \mathbb{P}^{\frac{n(n+3)}{2}}$ with a morphism $f : U \to V$ such that the family $U_i \to U$ is the pullback, with respect to $f$, of the restriction to $V$ of the versal family. By the properties of the étale versal deformation family of a plane singularity, (see [2]), we have that $f^{-1}((0,0))$ is an étale neighborhood of $[\Gamma]$ in the (smooth) analytic branch $\Sigma_{\Gamma,0}$ whose general element corresponds to an irreducible plane curves with only one cusp at a neighborhood of the cusp $q_i$ of $\Gamma$. So, $\dim(f^{-1}((0,0))) = \frac{n(n+3)}{2} - 2$ and the map $f$ is surjective. Moreover, if $g$ is the restriction of $f$ at $u^{-1}(\Sigma')$, then also $g$ is surjective. Indeed,

$$g^{-1}((0,0)) = f^{-1}((0,0)) \cap u^{-1}(\Sigma') = u^{-1}((\Sigma))$$

and, since $k < 3n$, then $\dim(\Sigma') = 3n + g - 1 - k = \dim(\Sigma') - 2$ and $g$ is surjective. By using lemma 2.1 it follows that the general fibre of the natural map $\Pi_{\Sigma'}(\Sigma) \to \Pi_{\Sigma}(\Sigma)$ has dimension exactly equal to one. Therefore, $\dim(\Pi_{\Sigma'}(\Sigma)) = \dim(\Pi_{\Sigma}(\Sigma)) + 1 = 3g - 3 + \rho(2,g,n) - k + 1 = 3(g+1) - 3 + \rho(2,g+1,n) - k$ By using that

$$\dim(\Pi_{\Sigma'}(\Sigma')) \geq \dim(\Pi_{\Sigma'}(\Sigma)) + 1 = 3(g+1) - 3 + \rho(2,g+1,n) - k + 1.$$ 

and by recalling that, by lemma 2.2 of [4], it is always true that $\dim(\Pi_{\Sigma'}(\Sigma')) \leq 3(g+1) - 3 + \rho(2,g+1,n) - k + 1$, the statement is proved in the case $k' = k - 1$ and $d' = d$.

Suppose, now, that $k = k'$ and $d' = d - 1$. Also in this case $\Sigma$ is not contained in the regularity domain of $\Pi_{\Sigma'}$. More precisely, if $[\Gamma] \in \Sigma$ is general, then $\Pi_{\Sigma'}([\Gamma])$ consists of a finite number of points, corresponding to the isomorphism classes of the partial normalizations of $\Gamma$ obtained by smoothing all the singular points of $\Gamma$, except for a node. Then $\Pi_{\Sigma}(\Sigma)$ is contained in the divisor $\Delta_0$ of $M_{g+1}$ parametrizing the isomorphism classes of the analytic curves of arithmetic genus $g + 1$ with a node and no more singularities. The natural map $\Delta_0 \dashrightarrow M_{g}$ sending the general point $[C']$ of $\Delta_0$ to the isomorphism class of the normalization of $C'$, restricts to a rational dominant map $q: \Pi_{\Sigma'}(\Sigma) \dashrightarrow \Pi_{\Sigma}(\Sigma)$. Since we suppose that $\Sigma$ has the expected number of moduli and $\rho(2,g,n) - k \leq 0$, if $C$ is the normalization of the plane curve corresponding to the general element of $\Sigma$, then the set $S'$ of the linear series of dimension 2 and degree $n$ on $C$ with $k$ simple ramification points, mapping $C$ to a plane curve $D$ such that the associated point $[D]$ in the Hilbert Scheme belongs to $\Sigma$, is finite. We deduce that also the set $S'$ of the pairs of points $(p_1,p_2)$ of $C$, such that there is a $g_n^2 \in S$ such that the associated morphism maps $p_1$ and $p_2$ to the same point of the plane, is finite. So, also $q^{-1}([C])$ is finite and $\dim(\Pi_{\Sigma'}(\Sigma)) = \dim(\Pi_{\Sigma}(\Sigma))$. It follows that

$$\dim(\Pi_{\Sigma'}(\Sigma')) \geq 3g - 3 + 3n - 2g - 6 - k + 1 = 3(g+1) - 3 + 3n - 2(g+1) - 6 - k.$$ 

\[\square\]

**Remark 2.3.** Notice that, the arguments used before to prove lemma [2.1] don’t work if the dimension of the general fibre of the moduli map of $\Sigma$ has dimension bigger than eight. Indeed, in this case, the dimension of the general fibre of the map
\(\Pi_{12}(\Sigma) \to \Pi_2(\Sigma)\) could be bigger than one if \(k' = k - 1\) and \(d = d'\), or than zero if \(k' = k\) and \(d = d' - 1\).

**Corollary 2.4.** There exists at least one irreducible component \(\Sigma_2\) of \(\Sigma_{6,0}^6\) having the expected number of moduli equal to \(\dim(\mathcal{M}_4) - 2\) and whose general element corresponds to a sextic with six cusps not on a conic.

**Remark 2.5.** As we already observed in the previous section, \(\Sigma_2\) is the only component of \(\Sigma_{6,0}^6\) parametrizing sextics with six cusps not on a conic by \(\Pi_1\).

**Proof.** Let \(\Sigma_{6,0}^6\) be the variety of elliptic plane curves of degree six with nine cusps and no more singularities. It is not empty and irreducible, because, by the Plücker formulas, the family of dual curves is \(\Sigma_{6,0}^3 \simeq \mathbb{P}^9\), which is irreducible and not empty. Moreover, if we compose an holomorphic map \(\phi : C \to \mathbb{P}^2\) from a complex torus \(C\) to a smooth plane cubic with the natural map \(\phi(C) \to \phi(C)^*\), where we denoted by \(\phi(C)^*\) the dual curve of \(\phi(C)\), we get a morphism from \(C\) to a plane sextic with nine cusps. Therefore, the number of moduli of \(\Sigma_{6,0}^6\) is equal to the number of moduli of \(\Sigma_{8,0}^6\), equal to one. Since \(6 < 3n = 18\), there is at least one irreducible component \(\Sigma'\) of \(\Sigma_{8,0}^6\) containing \(\Sigma_{6,0}^6\). Let \(\Pi_{12} : \Sigma' \to \mathcal{M}_2\) be the moduli map of \(\Sigma'\) and let \(\mathcal{G} \subset \Sigma' \times \mathcal{M}_2\) be its graph. If we denote by \(\pi_1 : \mathcal{G} \to \Sigma'\) and \(\pi_2 : \mathcal{G} \to \mathcal{M}_2\) the natural projection, by \(U\) the open set of \(\Sigma_{6,0}^6\) parametrizing cubics of genus one with nine cusps and by \(\Pi_{12} : (\Sigma_{6,0}^6)\) the Zariski closure in \(\mathcal{M}_2\) of \(\pi_2 \pi_1^{-1}(U)\), then, by arguing as in the first part of the proof of the lemma \([2,2]\) we have a dominant map \(\Pi_{12} : (\Sigma_{6,0}^6) \to \mathcal{M}_1\), whose general fibre has dimension one. We conclude that

\[
\dim(\pi_{12}(\Sigma')) \geq \dim(\pi_{12}(\Sigma_{6,0}^6)) + 1 = 3
\]

and so, the moduli map of \(\Sigma'\) is dominant, as one expects, because \(\rho(2, 2, 6) - 8 = 18 - 4 - 6 - 8 = 0\). Let \(D\) be the plane sextic corresponding to the general point of \(\Sigma'\). By Bezout theorem, the height cusps \(P_1, \ldots, P_6\) of \(D\) don’t belong to a conic and, however we choose five cusps of \(D\), no four of them lie on a line. Then, let \(C_2\) be the unique conic containing \(P_1, \ldots, P_6\). There exists at least a cusp, say \(P_6\), which does not belong to \(C_2\). Since \(8 < 3n = 18\), there exists a family of plane sextics \(D \to \Delta\), whose special fibre is \(D\) and whose general fibre has a cusp at a neighborhood of every cusp of \(D\) different from \(P_7\) and no further singularities. By lemma \([2,2]\) the curve \(\Delta\) is contained in an irreducible component of \(\Sigma_{7,0}^6\) with expected number of moduli. By repeating the same argument for the general fibre of the family \(D \to \Delta\) we get an irreducible component \(\Sigma_2\) of \(\Sigma_{6,0}^6\) with the expected number of moduli and whose general element parametrizes a sextic with six cusps not on a conic.

Now we consider the irreducible component \(\Sigma_1\) of \(\Sigma_{6,0}^6\) parametrizing plane curves of equation \(f_3^2(x_0, x_1, x_2) + f_2^2(x_0, x_1, x_2) = 0\), where \(f_2\) is a homogeneous polynomial of degree two and \(f_3\) is an homogeneous polynomial of degree three. The general element of \(\Sigma_1\) corresponds to an irreducible plane curve of degree six with six cusps on a conic. We want to show that \(\Sigma_1\) has the expected number of moduli equal to \(12 - 3 + \rho(2, 4, 6) - 6 = 7 = \dim(\mathcal{M}_4) - 2\). Equivalently, we want to show that the general fibre of the moduli map

\[
\Sigma_1 \to \mathcal{M}_4
\]

has dimension equal to eight.

**Lemma 2.6.** Let \(\Gamma_2 : f_2(x_0, x_1, x_2) = 0\) and \(\Gamma_3 : f_3(x_0, x_1, x_2) = 0\) be a smooth conic and a smooth cubic intersecting transversally. Then, the plane curve

\[
\Gamma : f_3^2(x_0, x_1, x_2) - f_2^2(x_0, x_1, x_2) = 0
\]

is an irreducible sextic of genus four with six cusps at the intersection points of \(\Gamma_2\) and \(\Gamma_3\) as singularities. The curve \(\Gamma\) is projection of a canonical curve \(C \subset \mathbb{P}^3\) from
a point $p \in \mathbb{P}^3$ which is contained in six tangent lines to $C$. Moreover, for every point $q \in \mathbb{P}^3 - C$ such that the projection plane curve $\pi_q(C)$ of $C$ from $q$ is a sextic with six cusps on a conic of equation $g_1^3(x_0, x_1, x_2) - g_2^3(x_0, x_1, x_2) = 0$, where $g_1$ and $g_2$ are two homogeneous polynomials of degree three and two respectively, there exists a cubic surface $S_3 \in |I_C|^{P^3}(3)$, containing $C$, such that the plane curve $\pi_q(C)$ is the branch locus of the projection $\pi_q : S_3 \to \mathbb{P}^2$.

**Remark 2.7.** Notice that, by [1], every irreducible sextic with six cusps on a conic as singularities has equation given by $(f_2(x_0, x_1, x_2))^3 + (f_3(x_0, x_1, x_2))^2 = 0$, with $f_2$ and $f_3$ homogeneous polynomials of degree two and three. In order words, all the sextics with six cusps on a conic as singularities are parametrized by points of $\Sigma_1$.

An other proof of this result as been provided to us by G. Pareschi.

**Proof of lemma 2.6.** Let $f(x_0, x_1, x_2) = f_1^3(x_0, x_1, x_2) - f_2^3(x_0, x_1, x_2) = 0$ be the equation of $\Gamma$. From the relation $f_3(x) = \pm f_2(x) \sqrt{f_2(x)}$, we deduce that $\frac{\partial f_3(x)}{\partial x_i} = \pm 2 \frac{\partial f_2(x)}{\partial x_i} \sqrt{f_2(x)}$ and hence

$$
(2) \quad \frac{\partial f(x)}{\partial x_i} = 2 \frac{\partial f_3(x) f_2(x)}{\partial x_i} - 3 f_2(x) \frac{\partial f_2(x)}{\partial x_i} = -f_2(x) \frac{\partial f_2(x)}{\partial x_i}.
$$

By using that the conic $\Gamma_2 : f_2 = 0$ is smooth, it follows that, if a point $x \in \Gamma$ is singular, then $x \in \Gamma_2$ and hence $x \in \Gamma_3 \cap \Gamma_2$. On the other hand, always from (2), if $x \in \Gamma_2 \cap \Gamma_3$, then $x$ is a singular point of $\Gamma$. Hence, the singular locus of $\Gamma$ coincides with $\Gamma_3 \cap \Gamma_2$. Let $x$ be a singular point of $\Gamma$. If

$$
p_1(x, y) + \text{terms of degree two} = 0
$$

and

$$
q_1(x, y) + \text{terms of degree} \geq \text{two} = 0
$$

are respectively affine equations of $\Gamma_2$ and $\Gamma_3$ at $x$, then, the affine equation of $\Gamma$ at $x$ is given by

$$
p_1(x, y)^2 - q_1(x, y)^3 + \text{terms of degree} \geq \text{four} = 0.
$$

Since $\Gamma_2$ and $\Gamma_3$ intersect transversally, we have that $q_1(x, y)$ does not divide $p_1(x, y)$ and hence $\Gamma$ has an ordinary cusp at $x$. Let now $\phi : C \to \Gamma$ be the normalization of $\Gamma$. We recall that the cubics passing through the six cusps of $\Gamma$ cut out on $C$ the complete canonical series $|\omega_C|$. Since the cusps of $\Gamma$ is contained in the conic $\Gamma_2 \subset \mathbb{P}^2$ of equation $f_2 = 0$, the lines of $\mathbb{P}^2$ cut out on $C$ a subseries $g \subset |\omega_C|$ of dimension two of the canonical series. Moreover, if we still denote by $C$ a canonical model of $C$ in $\mathbb{P}^3$, then the linear series $g$ is cut out on $C$ in $\mathbb{P}^3$ from a two dimensional family of hyperplanes passing through a point $p \in \mathbb{P}^3 - C$. If we project $C$ from $p$ we get a plane curve projectively equivalent to $\Gamma$. Since $\Gamma$ has six cusps as singularities, we deduce that there are six tangent lines to $C$ passing through $p$. To see that $\Gamma$ is the branch locus of a triple plane, let $S_3 \subset \mathbb{P}^3$ be the cubic surface of equation $F_3(x_0, \ldots, x_3) = x_0^3 - 3x_1^2x_2 + 2f_3(x_0, x_1, x_2) = 0$.

If $p = [0, 0, 0, 1]$, then, by using Implicit Function Theorem, the ramification locus of the projection $\pi_p : S_3 \to \mathbb{P}^2$, is given by the intersection of $S_3$ with the quadric $S_2$ of equation $\frac{\partial F_3}{\partial x_3} = x_3^2 - f_2(x_0, x_1, x_2) = 0$. Now, if $x = [x_0, x_1, x_2] \in S_3 \cap S_2$, then $x_3 = \pm \sqrt{f_2(x_0, x_1, x_2)}$. By substituting in the equation of $S_3$, we find that the branch locus of the projection $\pi_p : S_3 \to \mathbb{P}^2$ coincides with the plane curve $\Gamma$. From what we proved before, it follows that the ramification locus of the projection map $\pi_p : S_3 \to \mathbb{P}^2$ is the normalization curve $C$ of $\Gamma$. Finally, if $q \in \mathbb{P}^3 - C$ is an other point such that the plane projection $\pi_q(C)$ is an irreducible sextic with six cusps on a conic parametrized by a point $x_0 \in \Sigma_1 \subset \mathbb{P}^2$, then, up to projective motion, we may always assume that $q = [0 : 0 : 0 : 1]$ and hence, if $g_3^2(x_0, x_1, x_2) - g_2^3(x_0, x_1, x_2) = 0$
is the equation of the plane curve $\pi_q(C)$, then $C$ is the locus of ramification of the projection from $q$ to the plane of the cubic surface of equation

$$x_3^3 - 3g_2(x_0, x_1, x_2)x_3 + 2g_3(x_0, x_1, x_2) = 0.$$ 

\[ \square \]

Corollary 2.8. The irreducible component $\Sigma_1$ of $\Sigma_{6,0}^6$ parametrizing plane curves of equation $f_3^1(x_0, x_1, x_2) + f_3^2(x_0, x_1, x_2) = 0$, where $f_2$ is an homogeneous polynomial of degree two and $f_3$ is an homogeneous polynomial of degree three, has the expected number of moduli equal to $7 = \dim(M_4) + \rho(2, 4, 6) - 6$.

Proof. Let $[\Gamma] \subset \mathbb{P}^2$ be a plane sextic of equation $f_3^1(x_0, x_1, x_2) - f_3^2(x_0, x_1, x_2) = 0$, where the conic $f_2 = 0$ and the cubic $f_3 = 0$ are smooth and they intersect transversally. Let $C \subset \mathbb{P}^3$ be the normalization curve of $\Gamma$ and let $S_C$ be the set of points $[v] = [v_0 : \cdots : v_3] \in \mathbb{P}^3$ such that there exists a cubic surface $S_3 \in |I_C|_{\mathbb{P}^3}(3)$, containing $C$, such that the curve $C$ is the ramification locus of the projection $\pi_x : S_3 \rightarrow \mathbb{P}^2$. By the former lemma, in order to prove that $\Sigma_1$ has the expected number of moduli, it is enough to find a point $[\Gamma]$ of $\Sigma_1$ corresponding to an irreducible plane sextic $\Gamma \subset \mathbb{P}^2$ with six cusps of a conic such that the set $S_C$ is finite. Let $\Gamma_2$ be the smooth conic of equation $f_2(x_0, x_1, x_2) = x_0^2 + x_1^2 - x_2^2 = 0$ and let $\Gamma_3$ be the smooth cubic of equation $f_3(x_0, x_1, x_2) = x_0^3 + x_0x_2^2 - x_1^2x_2 = 0$. If $a_1, a_2$ and $a_3$ are the three different solutions of the polynomial $x^3 + x^2 + x - 1 = 0$, then $\Gamma_2$ and $\Gamma_3$ intersect transversally at the points $[a_i, \sqrt{a_i}, 1], [a_i, -\sqrt{a_i}, 1]$, with $i = 1, 2, 3$. By the former lemma, the plane sextic $\Gamma$ of equation $f_3^1 - f_3^2 = 0$ is irreducible and it has six cusps at the intersection points of $\Gamma_2$ and $\Gamma_3$ as singularities. Moreover, the normalization curve $C$ of $\Gamma$ is the canonical curve of genus 4 in $\mathbb{P}^3$ which is intersection of the cubic surface $S_3 \subset \mathbb{P}^3$ of equation

$$F_3(x_0, x_1, x_2, x_3) = x_3^3 + (x_0^3 + x_0^2 - x_2^2)x_3 + x_0^2x_2^2 - x_1^2x_2 = 0$$

and the quadric $S_2$ of equation

$$\frac{\partial F_3}{\partial x_3} = 3x_3^2 + x_0^2 + x_1^2 - x_2^2 = 0.$$ 

We want to show that $S_C$ is finite. To see this we observe that, since $h^0(\mathbb{P}^3, I_C|_{\mathbb{P}^3}(2)) = 1$ and $h^0(\mathbb{P}^3, I_C|_{\mathbb{P}^3}(3)) = 5$,

the equation of every cubic surface containing $C$ and which is not the union of $S_2$ and an hyperplane is given by

$$G(x_0, \ldots, x_3; \beta_0, \ldots, \beta_3) = F_3(x_0, x_1, x_2, x_3) + \sum_{j=0}^3 \beta_j x_j \frac{\partial F_3(x_0, x_1, x_2, x_3)}{\partial x_3} = 0,$$

with $\beta_j \in \mathbb{C}$, for $i = 0, \ldots, 3$. Now, a point $[\nu] = [v_0, \ldots, v_3] \in S_C$ if and only if there exist $\beta_0, \ldots, \beta_3$ such that $C$ is contained in the intersection of

$$G(x_0, \ldots, x_3; \beta_0, \ldots, \beta_3) = 0 \quad \text{and} \quad \frac{\partial G(x_0, \ldots, x_3; \beta_0, \ldots, \beta_3)}{\partial \nu} = 0.$$ 

Still using that $h^0(\mathbb{P}^3, I_C|_{\mathbb{P}^3}(2)) = 1$, a point $[\nu] \in \mathbb{P}^3$ belongs to $S_C$ if and only if

$$\frac{\partial G(x_0, \ldots, x_3; \beta_0, \ldots, \beta_3)}{\partial \nu} = \lambda \frac{\partial F_3(x_0, \ldots, x_3)}{\partial x_3}$$

for some $\lambda \in \mathbb{R} - 0$, or, equivalently,

$$\sum_{i=0}^2 v_i \frac{\partial F_3}{\partial x_i} + \sum_{i=0}^3 v_i (\sum_{j=0}^3 \beta_j x_j) \frac{\partial F_3}{\partial x_3 \partial x_i} = (\lambda - \sum_{i=0}^3 v_i \beta_i - v_3) \frac{\partial F_3}{\partial x_3}.$$
The previous equality of polynomials is equivalent to the following bilinear system of ten equations in the variables $v_0, \ldots, v_3$ and $\beta_0, \ldots, \beta_3$

\[ \begin{cases} (1 + \beta_3)v_0 + 3\beta_0v_3 = 0 & (x_0x_3) \\ (1 + \beta_3)v_1 + 3\beta_1v_3 = 0 & (x_1x_3) \\ (1 + \beta_3)v_2 - 3\beta_2v_3 = 0 & (x_2x_3) \\ \beta_1v_0 + \beta_0v_1 = 0 & (x_0x_1) \\ \beta_2v_0 + (1 - \beta_0)v_2 = 0 & (x_0x_2) \\ (1 - \beta_2)v_1 + \beta_1v_2 = 0 & (x_1x_2) \\ 2\beta_1v_1 - v_2 = \lambda - \sum_{j=0}^{3}\beta_jv_j - v_3 & (x_1^2) \\ -v_0 + 2\beta_2v_2 = \lambda - \sum_{j=0}^{3}\beta_jv_j - v_3 & (x_2^2) \\ (3 + 2\beta_0)v_0 = \lambda - \sum_{j=0}^{3}\beta_jv_j - v_3 & (x_0^2) \\ 2\beta_3v_3 = \lambda - \sum_{j=0}^{3}\beta_jv_j - v_3 & (x_3^2) \end{cases} \]

The points of $S_C$ are the solutions $v$ of the previous linear system, as a linear system whose coefficients depend on $\beta_0, \ldots, \beta_3$. It is easy to prove that it has only a solution equal to $(v_0, v_1, v_2, v_3) = (0, 0, 0, \lambda)$ if $\beta_0 = \beta_1 = \beta_2 = \beta_3 = 0$ and it has not solutions otherwise, (see [3], page 98). By the previous lemma, we conclude that the point $[0 : 0 : 0 : 1] \in \mathbb{P}^3$ is the only point which belongs to six tangent lines to the canonical curve $C \subset \mathbb{P}^3$ which is intersection of the cubic surface of equation

\[ F_3(x_0, x_1, x_2, x_3) = x_3^3 + (x_0^2 + x_1^2 - x_2^2)x_3 + x_0^3 + x_0x_2^2 - x_1^2x_2 = 0 \]

and the quadric of equation

\[ \frac{\partial F_3}{\partial x_3} = 3x_3^2 + x_0^2 + x_1^2 - x_2^2 = 0. \]

It follows that, on the normalization curve $D$ of the plane curve $\Gamma'$ corresponding to the general point of $\Sigma_1 \subset \Sigma_{6,0}$ there exists only a finite number of linear series of dimension two with six ramification points.

\[ \square \]

**Remark 2.9.** By using the notation introduced in the proof of corollary 2.8 we observe that in this corollary we have proved that if $C$ is a general canonical curve of genus four such that the set $S_C$ is not empty, then $S_C$ is finite. Actually, C. Ciliberto pointed out to our attention that it is possible to show, with a very simple argument, that for every canonical curve $C$ of genus four such that $S_C$ is not empty, we have that $S_C$ is finite. Finally, we observe that, by remark 2.7, for every canonical curve $C$ of genus four, the set $S_C$ coincides with the set of points of $\mathbb{P}^3$ which are contained in six tangent lines to $C$.

**Acknowledgment.** The results of this paper are contained in my PhD-thesis. I would like to thank my advisor C. Ciliberto for introducing me into the subject and for providing me very useful suggestions. I have also enjoyed and benefited from conversation with G. Pareschi and my colleague M. Pacini.

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