Dynamics of products of matrices in max algebra

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Abstract

The aim of this manuscript is to understand the dynamics of matrix products in a max algebra. A consequence of the Perron-Fröbenius theorem on periodic points of a nonnegative matrix is generalized to a max algebra setting. The same is then studied for a finite product associated to a \( p \)-lettered word on \( N \) letters arising from a finite collection of nonnegative matrices, with each member having its maximum circuit geometric mean at most 1.

Keywords

Products of nonnegative matrices · Max-algebras · Boolean matrices · Fröbenius normal form of nonnegative matrices · Circuit geometric mean · Periodic points

1 Introduction

We work throughout over the field \( \mathbb{R} \) of real numbers and use the standard notations \( \mathbb{R}^n \) and \( M_n(\mathbb{R}) \) to denote the \( n \)-dimensional real vector space of \( n \)-tuples of real numbers and the real vector space of \( n \times n \) square matrices with real entries respectively. We concern ourselves with only those matrices whose entries are nonnegative real numbers. This set will be denoted by \( M_n(\mathbb{R}_+). \) Other notations and terminologies used in this work will be introduced later.

Recall that given a self map \( f \) on a topological space \( X \), an element \( x \in X \) is called a periodic point of \( f \) if there exists a positive integer \( q \) such that \( f^q(x) = x \). In such a case, the smallest such integer \( q \) that satisfies \( f^q(x) = x \) is called the period of the periodic point \( x \). The starting point for this work is the following consequence of the Perron-Fröbenius theorem.

**Theorem 1.1** (Theorem B.4.7, Lemmens and Nussbaum (2012)) Let \( A \in M_n(\mathbb{R}_+) \) with spectral radius less than or equal to 1. Then, there exists a positive integer \( q \) such that for
every \( x \in \mathbb{R}^n \) with \( \| A^k x \| \) bounded, we have
\[
\lim_{k \to \infty} A^{kq} x = \xi_x,
\]
where \( \xi_x \) is a periodic point of \( A \) whose period divides \( q \).

In an attempt to generalize Theorem 1.1, when the matrix \( A \) in the above theorem is replaced by a product of the matrices \( A_r \)'s, possibly an infinite one, drawn from the finite collection of nonnegative matrices, \( \{ A_1, A_2, \ldots, A_N \} \), the following result was obtained by the authors. The details may be found in Jayaraman et al. (2022).

**Theorem 1.2** Jayaraman et al. (2022) Let \( \{ A_1, A_2, \ldots, A_N \} \), \( N < \infty \), be a collection of \( n \times n \) matrices with nonnegative entries, each having spectral radius at most 1. Assume that the collection has a nontrivial set of common eigenvectors, \( E \). For any finite \( p \), let \( A_\omega \) denote the matrix product associated to a \( p \)-lettered word \( \omega \) on the letters \( \{ 1, 2, \ldots, N \} \). Suppose \( \mathcal{LC}(E) \) denotes the set of all real linear combinations of the vectors in \( E \). Then, for any \( x \in \mathcal{LC}(E) \), there exists an integer \( q \geq 1 \) (independent of \( \omega \)) such that
\[
\lim_{k \to \infty} A_\omega^{kq} x = \xi_x,
\]
where \( \xi_x \) (independent of \( \omega \)) is a periodic point of \( A_\omega \), whose period divides \( q \).

The aim of this work is to explore the possibilities of extending Theorems 1.1 and 1.2 in the setting of max algebras. By a max algebra, we mean the triple \((\mathbb{R}_+, \oplus, \otimes)\), where \( \mathbb{R}_+ \) denotes the set of nonnegative real numbers, \( \oplus \) denotes the binary operation of taking the maximum of two nonnegative numbers and \( \otimes \) is the usual multiplication of two numbers. There are several abstract examples of max algebras, as may be found in Butkovic (2010). The one given above is more amenable to work with, while dealing with nonnegative matrices. Another example is the set of real numbers, together with \( -\infty \), equipped with the binary operations of maximization and addition, respectively. The latter system is isomorphic to the former one via the exponential map. A good reference on max algebras is the monograph by Butkovic (2010). For Perron-Fröbenius theorem in max algebras, one may refer to Bapat (1998).

## 2 Preliminaries on max algebras

We introduce some preliminary notions from max algebras that we use, in this section. In particular, we introduce the notion of a max eigenvalue, the corresponding max eigenvector and a few properties of the same. We begin with a description of matrix product in max algebra. Recall that for \( a, b \in \mathbb{R}_+ \)
\[
a \oplus b = \max\{a, b\} \quad \text{and} \quad a \otimes b = ab.
\]

Let \( A \) and \( B \) be two \( n \times n \) nonnegative matrices. Then the matrix product under the considered max algebra of \( A \) and \( B \) is denoted by \( AB = A \otimes B \) and is defined by
\[
[AB]_{ij} = [A \otimes B]_{ij} = \max_k \{a_{ik} \otimes b_{kj}\}.
\]

Further, if \( A = B \), then \( A^{\otimes 2} \) denotes the square of \( A \). For simplicity and ease of writing, we shall denote this by \( A^2 \). Further, \( A^k \) shall denote the \( k \)-th power of \( A \) in this algebra. By convention, when \( k = 0 \), we assume \( A^0 \) to be the identity matrix.
Definition 2.1 An $n \times n$ matrix $A$ is said to be reducible if there exists a permutation matrix $P$ such that
\[ PAP^T = P \otimes A \otimes P^T = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \] (2.1)
where $B$ and $D$ are square matrices. Otherwise $A$ is irreducible.

Note that the notion of irreducibility is the same in the max algebra and the Euclidean algebra. If $A$ is reducible and is in the form (2.1), and if a diagonal block is reducible, then this block can be reduced further via a permutation similarity. Continuing this process, we have a suitable permutation matrix $P$ such that $PAP^T$ is in the block triangular form
\[ PAP^T = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{bmatrix}, \] (2.2)
where each block $A_{ii}$ is square and is either irreducible or a $1 \times 1$ null matrix. This block triangular form is called the Frobenius normal form of $A$.

Given an $n \times n$ nonnegative matrix $A$, there is a natural way to associate a simple weighted directed graph $G(A)$ to the matrix as follows: $G(A)$ has $n$ vertices, say 1, \ldots, $n$, such that there is an edge from $i$ to $j$ with weight $a_{ij}$ if and only if $a_{ij} > 0$. By a circuit, we always mean a simple circuit. In contrast, our paths may include a vertex and/or an edge more than once. By the product of a path, we mean the product of the weights of the edges in the path.

Let $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$ be a circuit in $G(A)$. Then $a_{i_1i_2}a_{i_2i_3}\cdots a_{ik_i1}$ is the corresponding circuit product and its $k^{th}$ root is the circuit geometric mean corresponding to the circuit. The maximum among all possible circuit geometric means in $G(A)$ is denoted by $\mu(A)$. Observe that $\mu(A) > 0$ or $\mu(A) = 0$ indicates the presence or absence (respectively) of a circuit in $G(A)$. A circuit with circuit geometric mean $\mu(A) = \mu(G(A))$ is called a critical circuit, and vertices on the critical circuits are critical vertices. The critical matrix of $A$, denoted by $A^C = [a^C_{ij}]$, is formed from a principal sub-matrix of $A$ corresponding to the critical vertices, by setting $a^C_{ij} = a_{ij}$ if $(i, j)$ is in a critical circuit, and $a_{ij} = 0$ otherwise. Thus the critical graph $G(A^C)$ has vertex set $V^C = \{\text{critical vertices}\}$.

Note that the weighted directed graph associated with $A$, namely $G(A)$, and the weighted directed graph associated with $PAP^T$, for a permutation matrix $P$, namely $G(PAP^T)$ are isomorphic. This implies that the corresponding circuit geometric means are equal. For more on circuit geometric means, see Bapat (1998); Berman and Plemmons (1994); Elsner and van den Driessche (1999).

Definition 2.2 Let $A$ be a nonnegative matrix. We say that $\lambda$ is a max eigenvalue of $A$ if there exists a nonzero, nonnegative vector $x$ such that $A \otimes x = \lambda \otimes x$. Further, $x$ is called a max eigenvector associated to $\lambda$.

The following result is referred to as the max version of the Perron-Fröbenius theorem and can be found in Bapat (1998).

Theorem 2.1 (Theorem 2, Bapat (1998)) Let $A$ be an $n \times n$ nonnegative, irreducible matrix. Then there exists a positive vector $x$ such that $A \otimes x = \mu(A) \otimes x$. 

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It can be proved that an irreducible matrix has only one eigenvalue with a corresponding eigenvector in max algebras. Moreover, unlike the eigenvalues in the Euclidean algebra, the max eigenvalue has an explicit formula given by,

$$
\mu(A) = \mu(A^C) = \max_{i} \left( a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{i_1}} \right)^\frac{1}{i} .
$$

We now define a relation between two vertices in $G(A)$.

**Definition 2.3** Let $A$ be a nonnegative square matrix of order $n$. For $1 \leq i, j \leq n$, we say that $i$ has access to $j$ if there is a path from vertex $i$ to vertex $j$ in $G(A)$, and that $i$ and $j$ communicate if $i$ has access to $j$ and $j$ has access to $i$.

Communication is an equivalence relation. Note that in the identity matrix, we assume that every vertex $i$ communicates with itself. The following result concerns the spectrum of a nonnegative matrix in max algebra.

**Theorem 2.2** Bapat (1998) Let $A$ be an $n \times n$ nonnegative matrix in Fröbenius normal form,

$$
\begin{bmatrix}
A_{11} & A_{12} & \cdots & A_{1m} \\
0 & A_{22} & \cdots & A_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{mm}
\end{bmatrix}.
$$

Then $\lambda$ is an eigenvalue with a corresponding nonnegative eigenvector if and only if there exists a positive integer $i \in \{1, 2, \cdots, m\}$ such that $\lambda = \mu(A_{ii})$ and furthermore, class $j$ does not have access to class $i$ whenever $\mu(A_{jj}) > \mu(A_{ii})$.

Writing the Fröbenius normal form of $A$ as the sum of its diagonal blocks and the strict upper triangular block, say

$$
PAP^T = \begin{bmatrix}
A_{11} & 0 & \cdots & 0 \\
0 & A_{22} & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A_{mm}
\end{bmatrix} \oplus \begin{bmatrix}
0 & 0 & \cdots & 0 \\
A_{12} & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
A_{2m} & \cdots & \cdots & 0
\end{bmatrix}
=: DA \oplus NA,
$$

we observe that $NA$ does not contribute in computing the circuit geometric mean of $PAP^T$, and consequently has no role while determining $\mu(A)$. This follows from Theorem 2.2 and the communication relation defined above. Moreover, note that $\mu(A) = \mu(D_A) = \max \{ \mu(A_{11}), \mu(A_{22}), \ldots, \mu(A_{mm}) \}$.

We illustrate Theorem 2.2 below. Similar examples can be found in Bapat (1998).

**Example 2.3** Let

$$
A = \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 2 \\ 0 & 5 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 5 & 2 \\ 0 & 4 \end{bmatrix}.
$$
Note that $A$ has two max eigenvalues, namely, 4 and 5, with corresponding max eigenvectors $(1, 0)$ and $(0, 1)$ respectively. Observe that $B$ has two max eigenvalues namely 4 and 5 with corresponding max eigenvectors $(1, 0)$ and $(2, 5)$ respectively, whereas $C$ has only one max eigenvalue 5.

We now state two useful results due to Elsner. The first one is a $DAD$-type theorem that says that a nonnegative irreducible matrix can be bounded by a matrix of all ones via a $DAD$ transform. The second one concerns the period of a nonnegative irreducible matrix with $\mu(A) = 1$. More about this will be discussed in the section on Boolean matrices.

**Lemma 2.4** Elsner and van den Driessche (2001) Let $A$ be an irreducible matrix with $\mu(A) \leq 1$, $x \in \mathbb{R}_n^+$, $x \neq 0$ and $z = A^*x$, where $A^* = I \oplus A \oplus A^2 \oplus \ldots \oplus A^{n-1}$. Then every coordinate of $z$ is strictly positive. If $D = \text{diag}(z_1, z_2, \ldots, z_n)$, then

$$(D^{-1}AD)_{ij} = (D^{-1} \otimes A \otimes D)_{ij} \leq 1.$$ 

In the hypothesis, suppose we start with $\mu(A) < 1$, then the assertion of the Lemma is also a strict inequality.

**Theorem 2.5** Elsner and van den Driessche (1999); Butkovic (2010) Assume that $A$ is a nonnegative irreducible definite matrix, meaning $\mu(A) = 1$. Then, there exist $q$ and $t_0$ in $\mathbb{Z}_+$ such that for all $t \geq t_0$, we have

$$A^{t+q} = A^t,$$ 

where the powers are taken in the max algebra. \hfill (2.4)

It is interesting to note that the assertion in the above theorem is analogous to the concept of sequences that are “ultimately geometric” and “ultimately periodic”, as introduced by Gaubert (1994). de Schutter (2000) uses these concepts and obtains results pertaining to powers of matrices in the max-plus algebraic setting. The notion of periodicity that we deal with in this paper is the same as found in the assertion of Theorem 2.5; although we make a precise definition of the same in Section 4.

A generalised version of Lemma 2.4 is available in the literature for what is known as the scaling of a matrix $A$. If $D$ is an invertible diagonal matrix, then $D^{-1}AD$ is called a scaling of $A$. For such scaling of $A$, we have the following theorem, as can be found in Butkovic (2010).

**Theorem 2.6** Butkovic (2010) Let $A$ be a matrix with $\mu(A) = 1$. Then there exists a positive vector $z \in \mathbb{R}_n^+$ such that

$$(D^{-1}AD)_{ij} = (D^{-1} \otimes A \otimes D)_{ij} \leq 1 \text{ where } D = \text{diag}(z_1, z_2, \ldots, z_n).$$

Further, $A$ and its scaling have the same dynamics.

We end this section with bounds and inequalities for $\mu(A)$.

**Lemma 2.7** Bapat et al. (1995) Let $A$ be an $n \times n$ irreducible nonnegative matrix.

1. Suppose there exist $\eta_1 > 0$ and a vector $z^{(1)} \neq 0$ such that $A \otimes z^{(1)} \geq \eta_1 z^{(1)}$. Then $\mu(A) \geq \eta_1$.
2. Suppose there exist $\eta_2 > 0$ and a vector $z^{(2)} \neq 0$ such that $A \otimes z^{(2)} \leq \eta_2 z^{(2)}$. Then $\mu(A) \leq \eta_2$. 

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Consequently, we have:

**Corollary 2.8** Bapat et al. (1995) Let $A$ be an $n \times n$ be nonnegative matrix. Then

$$\min_i \max_j a_{ij} \leq \mu(A) \leq \max_i \min_j a_{ij}.$$

Let $A$ and $B$ be two commuting matrices. The following result gives us the circuit geometric mean of the addition and product of the two matrices in max algebras.

**Theorem 2.9** Katz et al. (2012) Let $A$ and $B$ commute. Then,

1. $\mu(A \otimes B) \leq \mu(A) \otimes \mu(B)$.
2. $\mu(A \oplus B) \leq \mu(A) \oplus \mu(B)$.

Moreover, equality holds in both the above inequalities if the matrices $A$ and $B$ are irreducible.

### 3 Statements of results

We state the main results of this paper, in this section. The first result includes an analogue of Theorem 1.1, in the max algebra setting.

**Theorem 3.1** Let $A$ be a matrix with $\mu(A) \leq 1$. Then, there exists an integer $q$ such that for every $1 \leq j \leq q$, we have

$$\lim_{k \to \infty} A^{kq+j} = \widehat{A}(j),$$

where $\widehat{A}(j)$ is a periodic matrix, whose period divides $q$. Further, for every $x \in \mathbb{R}^n_+$, we have

$$\lim_{k \to \infty} A^{kq+j} \otimes x = \xi_x^{(j)},$$

where $\xi_x^{(j)}$ is a periodic point of the matrix $A$, whose period divides $q$.

The following example illustrates that the period of the periodic matrix $\widehat{A}(j)$ and the period of the periodic point $\xi_x^{(j)}$ have no correlation between them. Consider $A = \begin{bmatrix} 0.5 & 1 \\ 1 & 0.5 \end{bmatrix}$. Then, we obtain $q = 2$. Further, $\widehat{A}(1) = A$ whose period is 2 whereas, $\widehat{A}(2) = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix}$ has period 1. For $x = (1, 1)^T$, we find that the period of the periodic point $\xi_x^{(j)}$ is 1, for $j = 1, 2$, while for $y = (1, 0)^T$, the period of the periodic point $\xi_y^{(j)}$ is 2, for $j = 1, 2$.

Experts working in the area of convergence of iterative schemes in max algebras may be aware of the results due to Heidergott et al. (2006), however in the setting of max-plus algebra. We combine two significant results from Heidergott et al. (2006) and present the same in the max-product algebra as follows:

**Theorem 3.2** Heidergott et al. (2006) Let $A$ be a matrix with $\mu(A) \leq 1$. Then,

1. The infinite series $\bigoplus_{k \geq 1} A^k$ converges in max-algebra.
2. If $\lim_{k \to \infty} \left( A^k \otimes x_0 \right)^{1/k}$ exists, then the limit is independent of the starting point, provided we choose the starting point $x_0$ from the interior of $\mathbb{R}^n_+$. 

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However, we urge the readers to observe through the following example that the two limits namely, \( \lim_{k \to \infty} A^k q + j \), as discussed in Theorem 3.1, and the infinite series \( \bigoplus_{k \geq 1} A^k \), as discussed in Theorem 3.2 are quite different in nature. Consider \( A = \begin{bmatrix} 0.2 & 1 & 4 \\ 1 & 0.5 & 6 \\ 0 & 0 & 0.9 \end{bmatrix} \). Then,

\[
\bigoplus_{k \geq 1} A^k = \begin{bmatrix} 1 & 1 & 6 \\ 1 & 1 & 6 \\ 0 & 0 & 0.9 \end{bmatrix}
\]

while \( \tilde{A}(j)'s \) for this matrix are computed in Section 8. Further, taking \( x_0 = (1, 1, 1)^T \), we obtain

\[
\lim_{k \to \infty} \left( A^k \otimes x_0 \right)^\frac{1}{k} = (1, 1, 0.9)^T,
\]

the limiting point not being a periodic point of any order.

The next result deals with a finite collection of matrices, say \( \{A_1, A_2, \cdots, A_N\} \), with each of them having \( \mu(A_i) \leq 1 \). For any finite \( p \in \mathbb{Z}_+ \), we denote the set of all \( p \)-lettered words on the first \( N \) positive integers by

\[
\Sigma_N^p := \{ \omega = (\omega_1 \omega_2 \cdots \omega_p) : 1 \leq \omega_i \leq N \},
\]

and for any \( \omega \in \Sigma_N^p \), we define the matrix product

\[
A_\omega := A_{\omega p} \otimes A_{\omega p-1} \otimes \cdots \otimes A_{\omega 1}.
\]

We now state the second result of this paper, which is analogous to Theorem 1.2, in the max algebra setting, however under the hypothesis that the collection of matrices \( \{A_1, A_2, \cdots, A_N\} \) commute pairwise.

**Theorem 3.3** Let \( \{A_1, A_2, \cdots, A_N\} \) be a finite collection of pairwise commuting nonnegative matrices with \( \mu(A_i) \leq 1 \), for every \( 1 \leq i \leq N \). Suppose \( \omega \in \Sigma_N^p \) is a \( p \)-lettered word in which the letter \( i \) occurs \( p_i \) many times. Then, there exists an integer \( q \) such that for every \( 1 \leq j \leq q \), we have

\[
\lim_{k \to \infty} A_\omega^{kq + j} = \bigotimes_{i=1}^{N} (A_i^{(j)})^{p_i}.
\]

Before stating our third result, let us assume that the collection \( \{A_1, A_2, \cdots, A_N\} \) of matrices do not necessarily commute, nevertheless, has a set of common max eigenvectors, say, \( E = \{v_1, v_2, \cdots, v_m\} \), (see Definition 2.2). Suppose \( v_i = \left(v_i^{(1)}, \ldots, v_i^{(n)}\right) \). Define

\[
\text{LC}(E) = \left\{ \bigoplus_{i=1}^{m} (\alpha_i \otimes v_i) : \alpha_i \in \mathbb{R}_+ \right\}
\]

\[
= \left\{ u = \left(u^{(1)}, \ldots, u^{(n)}\right) \in \mathbb{R}_+^n : u^{(j)} = \max \left\{ \alpha_1 v_1^{(j)}, \ldots, \alpha_m v_m^{(j)} \right\} : \alpha_i \in \mathbb{R}_+ \right\}.
\]

We now state our third theorem in this article, which is analogous to Theorem 1.2, in the max setting.
Theorem 3.4 Let \( \{A_1, A_2, \cdots, A_N\} \) be a collection of nonnegative matrices with \( \mu(A_i) \leq 1 \). Assume that the collection has a non-trivial set of common eigenvectors, say \( E \). Let \( A_\omega \) be the matrix product associated with the word \( \omega \in \Sigma_N^p \). Then, for any \( x \in \mathcal{LC}(E) \), we have
\[
\lim_{k \to \infty} A_\omega^k \otimes x = \xi_x,
\]
where \( \xi_x \) is a fixed point for every matrix in the collection \( \{A_1, A_2, \cdots, A_N\} \).

Note that Theorem 3.4 is also applicable in the case of commuting matrices. In the case of commuting matrices, Theorem 3.4 however, does not give a complete picture as against Theorem 3.3.

4 Boolean matrices

Recall that \( A \) is a Boolean matrix if all the entries of \( A \) are either 0 or 1. The following fact on Boolean matrices can be found in Pang and Guu (2001); Rosenblatt (1957). Let \( A \) be a Boolean matrix. Then there exist \( c_0 \) and \( q \) such that
\[
A^c + kq = A^c, \quad \text{for all} \quad k \in \mathbb{Z}_+, \quad c \geq c_0 \geq 1. \tag{4.1}
\]
The minimal such \( q \) is called the period of the matrix \( A \). Notice that this definition of the period of a Boolean matrix coincides with the same notion introduced in Theorem 2.5 of Section 2. We mean this same notion of periodicity in Theorem 3.1. We now introduce a few more notions that will help us in understanding more about the period of a matrix. These are taken from Rosenblatt (1957).

Definition 4.1 A Boolean matrix \( A \) is said to be

1. convergent if and only if there exists in the sequence \( \{A^k : k \geq 1\} \) a power \( A^m \) of \( A \) such that \( A^m = A^{m+1} \).
2. oscillatory or periodic if and only if there exists in the sequence \( \{A^k : k \geq 1\} \), a power \( A^m \) of \( A \) such that \( A^m = A^{m+q} \) where \( q \) is the smallest integer for which this holds and \( q > 1 \).

Theorem 4.1 If \( A \) is a Boolean matrix of order \( n \) with graph \( G(A) \), then the following hold:

1. \( A^k \) converges to the zero matrix if and only if \( G(A) \) contains no cyclic nets.
2. \( A^k \) converges to \( J_n \) (the matrix of order \( n \) with all entries being 1) if and only if \( G(A) \) is a universal cyclic net.
3. \( A \) is oscillatory if and only if \( G(A) \) contains at least one maximal cyclic net which is not universal. The period of a sub-matrix corresponding to a specified non-universal maximal cyclic net in \( G(A) \) is given by the greatest common divisor of the order of all simple cyclic nets contained in the maximal net.

Remark 4.2 Cyclic nets are the connected components of \( G(A) \) and the order of a cyclic net is the number of vertices in the cyclic net. A cyclic net is said to be a universal cyclic net if for some positive integer \( q \) every point of the cyclic net is attainable in \( q \) steps from some fixed point in the cyclic net, see Rosenblatt (1957).

We now define asymptotic period of a sequence of matrices, as given in Pang and Guu (2001).
Definition 4.2 A sequence \(\{A_k : k \in \mathbb{Z}_+\}\) of matrices in \(M_n(\mathbb{R})\) is called asymptotic \(q\)-periodic if

\[
\lim_{k \to \infty} A_{j+kq} = \tilde{A}(j) \quad (4.2)
\]

exists for \(j = 1, 2, \ldots, q\). The minimal such \(q\) is called the asymptotic period of the sequence.

We note that the Definitions 4.1 and 4.2 coincide, for Boolean matrices.

Let \(J(m \times n)\) be the \(m \times n\) matrix of all ones. When \(m = n\), we denote the matrix of all ones by \(Jn\). Suppose \(A\) is an \(n \times n\) nonnegative matrix with \(A \leq Jn\) (entry-wise). Consider the following decomposition:

\[
A = B(A) \oplus R(A), \quad (4.3)
\]

where \(B(A) = \begin{cases} a_{ij} & \text{if } a_{ij} = 1 \\ 0 & \text{if } a_{ij} < 1 \end{cases}\) and \(R(A) = \begin{cases} a_{ij} & \text{if } a_{ij} = 1 \\ a_{ij} & \text{if } a_{ij} < 1 \end{cases}\).

The following result is due to Pang and Guu.

Theorem 4.3 Pang and Guu (2001) Let \(A = B(A) \oplus R(A)\). \(B(A)\) has period \(q\) if and only if the sequence \(\{A_k : k \in \mathbb{Z}_+\}\) has asymptotic period \(q\).

Remark 4.4 From the above definitions, results and remarks, it follows that the period of a Boolean matrix is the same as the greatest common divisor of all possible lengths of simple circuits in \(G(A^C)\). Similar results on Max-Plus algebras and Max-Min algebras can also be found in Gavalec (2000a, b) respectively.

5 Two technical lemmas

In this section, we state and prove two lemmas that will be useful in the sequel, while proving the main theorems. We suppose that \(A\) is a reducible matrix with \(\mu(A) \leq 1\), and can be written in its Fröbenius normal form \(A = DA \oplus NA\) (upto a permutation similarity, as stated in Eq. (2.3)), where

\[
D_A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{bmatrix} \quad \text{and} \quad N_A = \begin{bmatrix} 0 & A_{12} & \cdots & A_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}.
\]

Define the matrix

\[
N' = \left( \max_{i,j} (N_A)_{ij} \right) = \begin{bmatrix} 0 & J_{n_1 \times n_2} & \cdots & J_{n_1 \times n_{m-1}} & J_{n_1 \times n_m} \\ 0 & 0 & \cdots & J_{n_2 \times n_{m-1}} & J_{n_2 \times n_m} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & J_{n_{m-1} \times n_m} \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (5.1)
\]

where \(J_{n_i \times n_j}\) is a matrix of size \(n_i \times n_j\), with all of its entries being 1. Suppose \(J = \text{diag} \left( J_{n_1}, J_{n_2}, \ldots, J_{n_m} \right)\), where \(J_{n_i}\) is a square matrix of order \(n_i\), with all of its entries being 1. Observe that \((N')^m = 0\) and \(J^k = J\) for any \(k \in \mathbb{Z}_+\). And, \(J \otimes N' = N' \otimes J = N'\).

Lemma 5.1 For any \(k \in \mathbb{Z}_+\) that is bigger than the index of nilpotency of \(N_A\), we have \(A^k \leq D_A^k \oplus N\) where \(N\) is some nilpotent matrix, whose index of nilpotency is at most \(m\).
Proof For a reducible matrix $A = D_A \oplus N_A$, we observe that for any $k \in \mathbb{Z}_+$ with $k \geq m$, we have

$$A^k = (D_A \oplus N_A)^k = D_A^k \bigoplus \left[ D_A^{k-1} N_A \oplus D_A^{k-2} N_A D_A \oplus \cdots \oplus N_A D_A^{k-1} \right] \bigoplus \left[ D_A^{k-2} N_A^2 \oplus D_A^{k-3} N_A D_A N_A \oplus \cdots \oplus N_A^2 D_A^{k-2} \right] \bigoplus \cdots \bigoplus N_A^k. \tag{5.2}$$

Further, $D_A^{k-r} N_A^r \leq J^{k-r} N_A^r = J N_A^r \leq (N')^r$. In fact, for any product that involves $(k - r)$ factors of $D_A$ and $r$ factors of $N_A$, we have

$$\left[ D_A^{k-r} N_A^r \oplus D_A^{k-r-1} N_A^{r-1} D_A N_A \oplus \cdots \oplus N_A^r D_A^{k-r} \right] \leq (N')^r.$$

Thus,

$$A^k \leq D_A^k \oplus N' \oplus (N')^2 \oplus \cdots \oplus (N')^{m-1}.$$

Define $N = N' \oplus (N')^2 \oplus \cdots \oplus (N')^{m-1}$, a nilpotent matrix whose index of nilpotency is at most $m$. \hfill \Box

With the help of Lemma 5.1, we now find the limit of $A^k$ when $\mu(A) < 1$, in the following lemma.

Lemma 5.2 Let $A$ be a matrix with $\mu(A) < 1$. Then, $\lim_{k \to \infty} A^k = 0$.

Proof From the Fröbenius normal form of $A$, we know that we can express $A = D_A \oplus N_A$ where the diagonal blocks are either irreducible or $1 \times 1$ null matrix. Then, applying Lemma 2.4 to $A_{ii}$, there exists a diagonal matrix $D_i$ such that $D_i^{-1} A_{ii} D_i < J_{n_i}$, where $n_i$ is order of $A_{ii}$. However, if $A_{ii}$ is a $1 \times 1$ null matrix, we choose $D_i = [1]$. Now, define $D = \text{diag}(D_1, D_2, \ldots, D_m)$. Then, $D^{-1} D_A D < J$.

Defining

$$D' := \left( \max_{i,j} (D_A)_{ij} \right) J,$$

observe that $D'$ and $N'$ (as defined in Eq. (5.1)) commute. We know from an analogous analysis as in Eq. (5.2) in the proof of Lemma 5.1 that for any $k \in \mathbb{Z}_+$ with $k \geq m$, we have

$$A^k \leq (D' \oplus N')^k = (D')^k \oplus (D')^{k-1} N' \oplus \cdots \oplus (D')^{k-m+1} (N')^{m-1}.$$

As $k \to \infty$, $(D')^{k-r} \to 0$ for every $0 \leq r \leq m - 1$. Thus, $\lim_{k \to \infty} A^k = 0$. \hfill \Box

Remark 5.3 $\mu(A) < 1$ if one of the following conditions holds.

1. $A$ is a nilpotent matrix.
2. $B(A) = 0$.
3. $\mathcal{G}(A)$ does not have any circuits.

Thus, Lemma 5.2 is relevant in all these cases.

6 Proof of Theorem 3.1

In this section, we prove Theorem 3.1. The easy cases are dealt with, in the beginning.
6.1 Case - 1: Suppose $A$ is irreducible

Here, we consider the matrix $A$ to be irreducible. The reducible case will follow after the proof of the irreducible case.

**Proof** Let $A$ be a given irreducible matrix with $\mu(A) \leq 1$. By Lemma 2.4, without loss of generality, we can assume $A \leq J_n$. By equation (4.3), we have $A = B(A) \oplus R(A)$. Since $B(A)$ is a Boolean matrix, there exists $q \geq 1$ such that $B(A)$ is $q$-periodic. Applying Theorem 4.3 to $A$, the sequence $\{A^k : k \in \mathbb{Z}_+\}$ has asymptotic period $q$. That is, for each $j = 1, 2, \ldots, q$, $\lim_{k \to \infty} A^{j+kq} = \tilde{A}^{(j)}$, exists. \hfill \Box

6.2 Case - 2: Suppose $A$ is reducible

Recall from Eq. (2.3) that (upto a permutation similarity), we can express $A = D_A \oplus N_A$. Define $\Lambda = \{i \in \{1, 2, \ldots, m\} : \mu(A_{ii}) = 1\}$. Then, for $i \in \Lambda$, we apply Theorem 2.5 to deduce that $A_{ii}^{t+q(i)} = A_{ii}^{t}$ for all $t \geq t_i$. However, for $i \notin \Lambda$, we appeal to Lemma 5.2 to conclude that $\lim_{k \to \infty} A_{ii}^{t} = 0$, i.e., given $\epsilon > 0$, there exists $t_i$ such that $A_{ii}^{t} < \epsilon J_{n_i}$ for all $t \geq t_i$. In this case, $q(i) = 1$. Now, defining $q = \text{lcm}(q(1), q(2), \ldots, q(m))$ and choosing $t_0$ to be bigger than $\max\{t_1, t_2, \ldots, t_m\}$, we obtain for all $t \geq t_0$,

$$A_{ii}^{t+q} = A_{ii}^{t} \quad \text{for } i \in \Lambda \quad \text{and} \quad A_{ii}^{t} < \epsilon J_{n_i} \quad \text{for } i \notin \Lambda. \quad (6.1)$$

We present the arguments for the two cases, $\mu(A) < 1$ and $\mu(A) = 1$ separately. We urge the readers to observe that the first case has already been dealt with in Lemma 5.2 with the period of the limiting matrix being 1. Thus, we only need to prove the result in the second case, when $\mu(A) = 1$.

From Theorem 2.6, we know that there exists a positive vector $z \in \mathbb{R}_n^+$ such that $D^{-1}AD \leq J_n$, where $D = \text{diag}(z_1, z_2, \cdots, z_n)$. Further, from Equation (4.3), we have that $D^{-1}AD = B(A) \oplus R(A)$. Since $B(A)$ is a Boolean matrix, there exist positive integers $q$ and $t_0$ such that

$$(B(A))^{t+q} = (B(A))^t \quad \forall t > t_0.$$  

Observe that this positive integer $q$ is the same as the one that we obtain in Eq. (6.1). Hence, Theorem 4.3 implies that

$$\lim_{k \to \infty} D^{-1}A^{kq+j}D = D^{-1}\tilde{A}^{(j)}D.$$

Since by Theorem 2.6, any matrix and its scaling have the same dynamics, the proof of the existence of the limiting matrix is complete.

We now prove that the limiting matrix $\tilde{A}^{(j)}$ is periodic, the arguments involve a clever usage of Elsner’s Theorem 2.5, a certain splitting of the max sum and comparison arguments. We prove this by induction on the number of diagonal blocks in the Fröbenius normal form of the given matrix $A$. 

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For $1 \leq j \leq q$, we know from the Fröbenius normal form of $A$ that there exists a permutation matrix $P$ such that

$$\lim_{k \to \infty} P^T A^{kj} P = P^T A(j) P = P^T \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{mm} \end{bmatrix} P.$$

Here, for $i \in \Lambda$,

$$\widetilde{A}_{ii}^{(j)} = \lim_{k \to \infty} A_{ii}^{kj} = \lim_{k \to \infty} A_{ii}^{t_0+j+(kj-t_0)}.$$

Suppose $t_0 = rq + s$ where $0 \leq s \leq q-1$, observe that

$$\widetilde{A}_{ii}^{(j)} = \lim_{k \to \infty} A_{ii}^{t_0+j+(kj-rq-s)} = \lim_{k \to \infty} A_{ii}^{t_0+j+(k-r-1)+q-s} = \lim_{k \to \infty} A_{ii}^{t_0+j+(k-r-1)+s'},$$

where $0 \leq s' \leq q-1$. Assuming $t_0$ to be a multiple of $q$, as assured by Eq. (6.1), we have

$$\widetilde{A}_{ii}^{(j)} = \begin{cases} A_{ii}^{t_0+j} & \text{when } \mu(A_{ii}) = 1 \\ 0 & \text{when } \mu(A_{ii}) < 1, \end{cases}$$

(6.2)

where the latter case is given by Lemma 5.2. We continue with the proof, however, only for the case when $j = q$. For any other $1 \leq j < q$, we note from Eq. (6.2) that we merely multiply with a factor of $A_{ii}^j$ to find the appropriate limit. For notational convenience, we supress the superscript in the notation $\widetilde{A}^{(q)}$ and merely write $\widetilde{A}$, the components of the matrix also following suit.

We start with the base case of induction when $m = 2$ and when $\mu(A_{11}) = 1$ and $\mu(A_{22}) < 1$. Then, we have

$$(\widetilde{A})^{t+q} = \begin{bmatrix} A_{11}^{t_0(t+q)} & A_{11}^{t_0(t+q-1)} \widetilde{A}_{12} \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A_{11}^{t_0} & A_{11}^{t_0(t-1)} \widetilde{A}_{12} \\ 0 & 0 \end{bmatrix} = (\widetilde{A})^t \quad \forall t > t' = 1.$$

A similar exercise holds when $\mu(A_{11}) < 1$ and $\mu(A_{22}) = 1$ and when $\mu(A_{11}) < 1$ and $\mu(A_{22}) < 1$. Finally, in the case when $\mu(A_{11}) = \mu(A_{22}) = 1$, we have

$$(\widetilde{A})^{t+q} = \begin{bmatrix} A_{11}^{t_0(t+q)} & A_{11}^{t_0(t+q-1)} \widetilde{A}_{12} \\ A_{11}^{t_0} A_{12} A_{22}^{t_0(t+q-\ell-1)} A_{22} \\ 0 & A_{22}^{t_0(t+q)} \end{bmatrix}. $$
Consider

\[
\bigoplus_{\ell = 0}^{t+q-1} A_{11}^{1\ell} A_{12} A_{22}^{t+q-\ell-1} = \bigoplus_{\ell = 0}^{t-2} A_{11}^{1\ell} A_{12} A_{22}^{t+q-\ell-1} + A_{11}^{1\ell} A_{12} A_{22}^{t+q-(t-1)-1} + A_{11}^{1\ell} A_{12} A_{22}^{t+q-(t-1)-1}
\]

Whenever \( t + 1 \geq q \), observe that for every \( t - 1 \leq \ell' \leq t + q - 2 \), there exists a \( 0 \leq \ell \leq t - 1 \) such that

\[
A_{11}^{1\ell} A_{12} A_{22}^{t+q-\ell-1} = A_{11}^{1\ell'} A_{12} A_{22}^{t+q-\ell'-1}. 
\]

Thus,

\[
(\tilde{A})^{t+q} = \begin{bmatrix} A_{11}^{t+q} & 0 \\ 0 & A_{22}^{t+q} \end{bmatrix} \bigoplus \begin{bmatrix} t+q-1 \\ 0 \end{bmatrix} = \begin{bmatrix} t+q-1 \\ 0 \end{bmatrix}
\]

proving the periodicity of \( \tilde{A} \).

For the general case, we consider the following partition

\[
\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \ldots & \tilde{A}_{1m} \\ \tilde{A}_{21} & \tilde{A}_{22} & \ldots & \tilde{A}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \tilde{A}_{mm} \end{bmatrix} =: \begin{bmatrix} B & \tilde{C} \\ 0 & D \end{bmatrix}
\]

where \( \tilde{B} = \tilde{A}_{11} = \begin{cases} A_{11}^{1d} & \text{when } \mu(A_{11}) = 1 \\ 0 & \text{when } \mu(A_{11}) < 1 \end{cases} \).

Further, owing to the induction hypothesis, \( \tilde{D} \), a block upper triangular matrix of order \( m - 1 \) is a \( q \)-periodic matrix. Thus, there exists a \( t_1 \) (multiple of \( q \)) such that for all \( t \geq t_1 \), \( \tilde{D}^{t+q} = \tilde{D}^{t_1} \). Now, for \( t \geq t_1 + q + 1 \), consider

\[
\bigoplus_{\ell = 0}^{t+q-1} \tilde{B}^t \tilde{C} \tilde{D}(t+q-\ell-1) = \tilde{C} \tilde{D}(t+q-1) \bigoplus_{\ell = 1}^{q} \tilde{B}^t \tilde{C} \tilde{D}(t+q-\ell-1) = \tilde{C} \tilde{D}(t-1) \bigoplus_{\ell = 1}^{q} \tilde{B}^t \tilde{C} \tilde{D}(t+q-\ell-1) + \tilde{B}^t \tilde{C} \tilde{D}(t+q-\ell-1)
\]

\[
= \tilde{C} \tilde{D}(t-1) \bigoplus_{\ell = 1}^{q} \tilde{B}^t \tilde{C} \tilde{D}(t+q-\ell-1) + \tilde{B}^t \tilde{C} \tilde{D}(t+q-\ell-1)
\]

\[
= \tilde{C} \tilde{D}(t-1) \bigoplus_{\ell = 1}^{q} \tilde{B}^t \tilde{C} \tilde{D}(t+q-\ell-1) + \tilde{B}^t \tilde{C} \tilde{D}(t+q-\ell-1)
\]
\[ q \bigoplus_{\ell = 1}^{t-1} \tilde{B}^\ell \tilde{C} \tilde{D}^{(t-\ell-1)} + \bigoplus_{\ell' = 1}^{t-1} \tilde{B}^\ell \tilde{C} \tilde{D}^{(t-\ell'-1)} = \bigoplus_{\ell' = 0}^{t-1} \tilde{B}^\ell \tilde{C} \tilde{D}^{(t-\ell'-1)}. \]

Thus,

\[
(A)^{(t+q)} = \left[ \begin{array}{cc} \tilde{B}^{(t+q)} & 0 \\ 0 & \tilde{D}^{(t+q)} \end{array} \right] \bigoplus \left[ \begin{array}{cc} 0 & \tilde{B}^{t+q-1} \tilde{C} \tilde{D}^{(t+q-\ell-1)} \\ 0 & 0 \end{array} \right] = \left( \hat{A} \right)^t \text{ for all } t \geq t_1 + q + 1
\]

proving the periodicity of \( \hat{A} \).

Finally, for any \( x \in \mathbb{R}^n_+ \), define \( \xi^{(j)} x := \hat{A}^{(j)} \otimes x \). Now we show that \( \xi^{(j)} x \) is a periodic point of \( A \) with its period dividing \( q \).

Hence \( \xi^{(j)} x \) is a periodic point of \( A \) with its period dividing \( q \).

7 Proofs of Theorems 3.3 and 3.4

We start this section with the hypothesis that \( \{A_1, A_2, \ldots, A_N\} \) is a collection of pairwise commuting nonnegative matrices with \( \mu(A_i) \leq 1 \). Consider a \( p \)-lettered word \( \omega \in \Sigma^p \), possibly with all letters present at least once. Let \( A_\omega \) be the matrix associated with the \( p \)-lettered word \( \omega \), as defined in Eq. (3.1). From Theorem 2.9, we know that

\[
\mu(A_\omega) \leq \mu(A_{\omega_1}) \times \mu(A_{\omega_2}) \times \cdots \times \mu(A_{\omega_p}) \leq 1.
\]

**Proof** [of Theorem 3.3] Applying Theorem 3.1 to \( A \), we obtain an integer \( q_i \) such that for every \( 1 \leq j \leq q_i \), we have

\[
\lim_{k \to \infty} A_{i}^{kq_i+j} = \hat{A}_i^{(j)}.
\]

Define \( q := \text{lcm}(q_1, q_2, \ldots, q_N) \). As the matrices commute, we see from the associativity of \( \otimes \) that

\[
A_\omega = A_1^{p_1} \otimes A_2^{p_2} \otimes \cdots \otimes A_N^{p_N},
\]

where \( p_i \) is the number of occurrences of \( i \) in the \( p \)-lettered word \( \omega \) with \( p = p_1 + p_2 + \cdots + p_N \). Observe that \( A_\omega^k = A_1^{kp_1} \otimes A_2^{kp_2} \otimes \cdots \otimes A_N^{kp_N} \) from which, it follows that

\[
\lim_{k \to \infty} A_{\omega}^{kq+j} = \bigotimes_{i=1}^{N} \left( A_i^{(j)} \right)^{p_i}.
\]
Finally, by our definition of $q$, we observe that it is independent of the choice of the word $\omega$ as well as its length. For any word $\omega$ that may not contain all the letters, one makes an analogous argument with a set that has fewer letters.

The following is an obvious corollary to the above theorem.

**Corollary 7.1** Let $A_1$ and $A_2$ be two commuting matrices with all the max eigenvalues being either $0$ or $1$. Suppose $\omega \in \Sigma_1^p$ is such that the letter $i$ occurs $p_i$ many times and has the associated matrix product $A_\omega$. Then, there exist $q$ and $t_0$ such that for any $1 \leq j \leq q$, we have

$$\lim_{k \to \infty} A_\omega^{kj} = L_\omega^{(j)},$$

where $L_\omega^{(j)} \in \{A_1^{p_{i_0}} \otimes A_2^{p_{i_0} + 1}, A_1^{p_{i_0} + 1} \otimes A_2^{p_{i_0} + 1}, \ldots, A_1^{p_{i_0} + q - 1} \otimes A_2^{p_{i_0} + q - 1}\}$.

We now prove Theorem 3.4 with the supposition that the collection $\{A_1, A_2, \ldots, A_N\}$ is not necessarily pairwise commuting. A key hypothesis in the proof of Theorem 3.4 is the existence of a set of common max eigenvectors for the collection $\{A_1, A_2, \ldots, A_N\}$.

**Proof** [of Theorem 3.4] Let $E = \{v_1, v_2, \ldots, v_m\}$ be the set of common max eigenvectors for the collection $\{A_1, A_2, \ldots, A_N\}$. We rearrange the eigenvectors (and rename them, if necessary) as

$$\{v_1, v_2, \ldots, v_\kappa, v_{\kappa+1}, v_{\kappa+2}, \ldots, v_m\}$$

so that $A_i \otimes v_j = v_j$ for all $1 \leq i \leq N$ and $1 \leq j \leq \kappa$. Assuming that the word $\omega \in \Sigma_1^p$ contains all the letters at least once, it follows that for every $\kappa + 1 \leq j \leq m$, $\lim_{k \to \infty} A_\omega^{kj} v_j = 0$. Now, for any $x = \alpha_1 v_1 \oplus \alpha_2 v_2 \oplus \cdots \oplus \alpha_m v_m \in \mathcal{LC}(E)$, consider

$$\lim_{k \to \infty} A_\omega^{kj} \otimes x = \left[\alpha_1 \lim_{k \to \infty} (A_\omega^{kj} \otimes v_1) \right] \oplus \left[\alpha_2 \lim_{k \to \infty} (A_\omega^{kj} \otimes v_2) \right] \oplus \cdots \oplus \left[\alpha_m \lim_{k \to \infty} (A_\omega^{kj} \otimes v_m) \right]$$

$$= \xi_\chi,$$

with $\xi_\chi$ being a fixed point for every matrix in the collection $\{A_1, A_2, \ldots, A_N\}$. □

8 Examples

In this section, we illustrate the theorems in this article with a few examples. We begin with an example that concerns Theorem 3.1.

**Example 8.1** Let $A = \begin{bmatrix} 0.2 & 1 & 4 \\ 1 & 0.5 & 6 \\ 0 & 0 & 0.9 \end{bmatrix}$ with corresponding graph given by

![Graph](https://example.com/graph.png)

\[ Springer\]
Recall that the critical circuit is the one where the maximum circuit geometric mean is attained in $G(A)$; in this case, it is $(2, 1)(1, 2)$, that yields $\mu(A) = 1$ and using Remark 4.4, we also obtain $q = 2$. Then, for any $k \in \mathbb{Z}_+$, we have

$$A^{2k+1} = \begin{bmatrix} 0.5 & 1 & 5.4 \\ 1 & 0.5 & 6 \\ 0 & 0 & (0.9)^{2k+1} \end{bmatrix} \quad \text{while} \quad A^{2k} = \begin{bmatrix} 1 & 0.5 & 6 \\ 0.5 & 1 & 5.4 \\ 0 & 0 & (0.9)^{2k} \end{bmatrix}.$$  

Thus,

$$\hat{A}^{(1)} = \begin{bmatrix} 0.5 & 1 & 5.4 \\ 1 & 0.5 & 6 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{while} \quad \hat{A}^{(2)} = \begin{bmatrix} 1 & 0.5 & 6 \\ 0.5 & 1 & 5.4 \\ 0 & 0 & 0 \end{bmatrix}. $$

Hence, for any $x = (x_1, x_2, x_3)^t \in \mathbb{R}_+^3$, it is easy to verify that

$$\lim_{k \to \infty} A^{2k+1} \otimes x = \begin{pmatrix} 0.5x_1 + x_2 + 5.4x_3 \\ x_1 + 0.5x_2 + 6x_3 \\ 0 \end{pmatrix} \quad \text{whereas} \quad \lim_{k \to \infty} A^{2k} \otimes x = \begin{pmatrix} x_1 + 0.5x_2 + 6x_3 \\ 0.5x_1 + x_2 + 5.4x_3 \\ 0 \end{pmatrix}.$$

The following example illustrates Theorem 3.3 and Corollary 7.1.

**Example 8.2** Consider the collection $\{A_1, A_2, A_3\}$ of pairwise commuting $5 \times 5$ nonnegative matrices given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 8 & 5 \\ 1 & 0 & 1 & 5 & 8 \\ 0 & 1 & 0 & 8 & 5 \\ 0 & 0 & 0 & 1 & 0.5 \\ 0 & 0 & 0 & 0.5 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 & 1 & 9 & 9 \\ 1 & 0 & 0 & 9 & 9 \\ 0 & 1 & 0 & 9 & 9 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad A_3 = \begin{bmatrix} 1 & 0 & 0 & 8 & 9 \\ 0 & 1 & 0 & 8 & 9 \\ 0 & 0 & 1 & 8 & 9 \\ 0 & 0 & 0 & 8 & 1 \\ 0 & 0 & 0 & 1 & 0.8 \end{bmatrix}$$

with $\mu(A_i) = 1$ for all $i$. Observe that all the max eigenvalues of the three matrices are equal to 1. Then, owing to the Theorem 3.1, we find that the matrices $A_1$, $A_2$ and $A_3$ have asymptotic periods $q_1$, $q_2$ and $q_3$ given by 3, 3 and 2 respectively, i.e., $\lim_{k \to \infty} A_i^{q_i+1} = \tilde{A}_i^{(j)}$.

We now discuss all possible cases of $\omega \in \Sigma_5^2$, with distinct letters.

(i) Let $A_\omega = A_1 \otimes A_2$. In this case, we find that $\lim_{k \to \infty} A_\omega^{q_\omega+1} = A_\omega$ (irrespective of $j$) is a matrix with period 1, while $q_\omega = 3$.

(ii) Let $A_\omega = A_2 \otimes A_3$. Then, $\lim_{k \to \infty} A_\omega^{q_\omega+1} = L^{(j)}_{\omega}$, where $L^{(j)}_{\omega}$ has period 3 while $q_\omega = 6$.

Further, $L^{(j)}_{\omega} \in \{A_2 \otimes A_3, A_2^2 \otimes A_3, A_2^3 \otimes A_3^3\}$.

(iii) Let $A_\omega = A_1 \otimes A_3$. In this case, $\lim_{k \to \infty} A_\omega^{q_\omega+1} = L^{(j)}_{\omega}$, where $L^{(j)}_{\omega}$ has period $q_\omega = 6$.

Further, $L^{(j)}_{\omega} \in \{A_1 \otimes A_3, A_1^2 \otimes A_3^2, \ldots, A_1^5 \otimes A_3^5\}$.

For a general $p$-lettered word $\omega$ that contains each of the letters at least once, suppose $p_i$ counts the number of occurrences of the letter $i$ in $\omega$, we find that $\lim_{k \to \infty} A_{\omega}^{q_{\omega}+j} = L^{(j)}_{\omega}$. One obtains the period of $L^{(j)}_{\omega}$ to be either 1 or 3 while $q = 6$, here. This is a special occasion since the second diagonal block of the matrices $A_2$ and $A_3$, expressed in their Frobenius normal form, when multiplied yields $J_2$. Thus, the period of $A_3$ makes no contribution in the computation of the period of $L^{(j)}_{\omega}$. Hence,

$$L^{(j)}_{\omega} \in \{A_1^{p_1} \otimes A_2^{p_2} \otimes A_3^{p_3}, A_1^{2p_1} \otimes A_2^{2p_2} \otimes A_3^{2p_3}, A_1^{3p_1} \otimes A_2^{3p_2} \otimes A_3^{3p_3}\}.$$
We conclude this section and the article with an example in the non-commuting case, that explains Theorem 3.4.

Example 8.3 Consider the matrices $A_1$ and $A_2$, where

$$A_1 = \begin{bmatrix} 0.9 & 0.45 & 5 & 6 & 27 \\ 0.45 & 0.9 & 1 & 23 & 8 \\ 0 & 0 & 0.9 & 1 & 0 \\ 0 & 0 & 0 & 0.2 & 1 \\ 0 & 0 & 0 & 0 & 0.1 \\ \end{bmatrix} \quad \text{and} \quad A_2 = \begin{bmatrix} 1 & 1 & 27 & 6 & 2 \\ 0.5 & 1 & 17 & 3 & 23 \\ 0 & 0 & 0.4 & 1 & 0 \\ 0 & 0 & 1 & 0.8 & 1 \\ 0 & 0 & 0.9 & 1 & 0.2 \\ \end{bmatrix}. $$

The max eigenvalues for the matrix $A_1$ are 1 and 0.9, while those of $A_2$ are 1 and 1, with the corresponding common max eigenvectors for the matrices being $u = (27, 23, 1, 1, 1)^T$ and $v = (2, 1, 0, 0, 0)^T$ respectively. Suppose $\omega \in \Sigma_2^P$ with the presence of both the letters at least once. For $x \in \mathcal{LC}(E)$ with $x = \alpha u \oplus \beta v$ for some $\alpha, \beta \geq 0$, we have

$$\lim_{k \to \infty} A_1^k \otimes x = \xi_x = \alpha u,$$

where $\xi_x$ is a common fixed point of $A_1$ and $A_2$.

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