ON THE DISTRIBUTION OF THE DIVISOR FUNCTION
AND HECKE EIGENVALUES

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ABSTRACT. We investigate the behavior of the divisor function in both short intervals and in
arithmetic progressions. The latter problem was recently studied by É. Fouvry, S. Ganguly,
E. Kowalski, and Ph. Michel. We prove a complementary result to their main theorem. We
also show that in short intervals of certain lengths the divisor function has a Gaussian limiting
distribution. The analogous problems for Hecke eigenvalues are also considered.

1. Introduction

The study of the behavior of
\[ \Delta(x) = \sum_{n \leq x} d(n) - x(\log x + 2\gamma - 1), \]
where \( \gamma \) is Euler’s constant is a classical topic in analytic number theory. For instance,
Dirichlet’s divisor problem asks for the smallest \( \alpha \) such that
\[ \Delta(x) \ll x^{\alpha + \varepsilon}, \]
for all \( \varepsilon > 0 \). Dirichlet showed that \( \alpha \leq 1/2 \), which was sharpened by Voronoi \[20\] who proved
\( \alpha \leq 1/3 \). A more recent result of Huxley \[9\] gives \( \alpha \leq 131/416 \). On the other hand, Hardy \[6\]
proved that \( \Delta(x) = \Omega((x \log x)^{1/4} \log \log x) \), and it is conjectured that \( \alpha = 1/4 \).

In this article we study the average behavior of \( d(n) \) on two different sparse sets, namely,
short intervals and arithmetic progressions modulo a large prime number. The similarities
between these two problems are striking and in the analogous problems for function fields
over a finite field there is a fundamental identity that clarifies this connection in that setting
(see Lemma 4.2 of \[17\] for a similar identity).

1.1. The divisor function in arithmetic progressions. The behavior of divisor function
in an arithmetic progression has been studied by numerous authors. For instance, Blomer
\[2\] and Lau and Zhao \[13\] have investigated the variance of sums of the divisor function in
progressions. Notably, Lau and Zhao prove an asymptotic formula for the variance of sums of
the divisor function \( d(n) \) with \( 1 \leq n \leq X \) in arithmetic progressions modulo \( q \), for \( q \) satisfying
\( X^{1/2} < q < X^{1-\varepsilon} \).

Instead of working with a sum of \( d(n) \) over \( 1 \leq n \leq X \) it is technically advantageous to
consider smoothed sums of the form
\[ \sum_{n} d(n)w\left(\frac{n}{X}\right), \]
where \( w \) is a smooth function compactly supported on the positive real numbers. Recently,
É. Fouvry, S. Ganguly, E. Kowalski, and Ph. Michel \[3\] studied the distribution of smoothed

2010 Mathematics Subject Classification. 11N60, 11F11, 11F30, 60F05.
sums of \(d(n)\) over arithmetic progressions modulo a prime number, \(p\) and showed that it has a Gaussian limiting distribution as \(p \to \infty\), for \(p^{2-\epsilon} \ll X = o(p^2)\).

To state their result more precisely, let \(w\) be a real-valued smooth function compactly supported in the positive real numbers. For a prime \(p\), define

\[
S_d(X, p, a; w) = \sum_{n \geq 1 \,(n \equiv a \pmod{p})} d(n)w\left(\frac{n}{X}\right),
\]

and

\[
M_d(X, p; w) = \frac{1}{p} \sum_{n \geq 1} d(n)w\left(\frac{n}{X}\right) - \frac{1}{p^2} \int_0^\infty (\log y + 2\gamma - 2 \log p)w\left(\frac{y}{X}\right) dy.
\]

Also, let

\[
E_d(X, p, a; w) = S_d(X, p, a; w) - M_d(X, p; w)
\]

and

\[
\|w\|_{L^2} \sqrt{2\pi - 2 \left(\frac{X}{p}\right)} \cdot \log \frac{3}{2} \left(\frac{p^2}{X} + 2\right).
\]

Theorem 1.1 of [3] states that for \(X = p^2/\Phi(p)\), where \(\Phi(p) \geq 1\) is a function that tends to infinity with \(p\) in a way such that \(\log \Phi(p) = o(\log p)\), as \(p \to \infty\),

\[
\frac{1}{p-1} \#\left\{ a \in \mathbb{F}_p^\times : \alpha \leq E_d(X, p, a; w) \leq \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_0^\beta e^{-x^2/2} dx + o(1). \tag{1.1}
\]

This theorem is proved by calculating the moments of \(E_d(X, p, a; w)\). These moments are estimated through an application of the Voronoi summation formula along with estimates of moments of Kloosterman sums due to N. Katz.

We are interested in seeing if the smooth weight function can be replaced with a sharp cut-off function in this result. This is because in some cases smoothing completely alters the nature of the problem. For instance, smoothing substantially changes the Dirichlet divisor problem. One can prove for any \(A \geq 1\) and a smooth function \(w\) that is compactly supported on the positive real numbers that

\[
\sum_{n \geq 1} d(n)w\left(\frac{n}{X}\right) = \int_0^\infty (\log y + 2\gamma)w\left(\frac{y}{X}\right) dy + O(X^{-A}),
\]

where the implied constant depends on \(w\) and \(A\). Thus, the remainder term is very small, unlike that of in the original Dirichlet divisor problem so that the smooth weight function cannot be replaced with a sharp cut-off function here. Moreover, something similar happens in the case of smoothed sums of the divisor function in arithmetic progressions modulo a prime number \(p\) when \(p\) is small relative to \(X\). Here one can show that \(E_d(X, p, a; w) \ll X^{-A}\) if \(X^\epsilon < p < X^{1/2-\epsilon}\) (see Lemma 2.6 below). From this we see that understanding the distribution of \(E_d(X, p, a; w)\) in this regime is trivial since it is always smaller than any negative power of \(X\). This is an effect of smoothing (see Theorem 4 of [14]), and it would be interesting to study the distribution of a sharp cut-off analog of \(E_d(X, p, a; w)\) in this regime.

We show that an analog of Theorem 1.1 of [3] holds for sums of the divisor function with sharp cut-offs. The process of removing the weight function is subtle. Our method requires the existence of a compactly supported function \(w_p\), that depends on \(p\), such that: 1) \(w_p\) approximates \(1_{[0,1]}\) as \(p \to \infty\) in a suitably strong sense; 2) a certain integral transform of \(w_p\), arising from the Voronoi summation formula, decays sufficiently rapidly.
Theorem 1.1. For a prime $p$, let

$$S_d(X, p, a) = \sum_{1 \leq n \leq X, \atop n \equiv a \pmod{p}} d(n),$$

and

$$M_d(X, p) = \frac{1}{p} \sum_{1 \leq n \leq X} d(n) \cdot \frac{X}{p^2} \left( \log X - 1 + 2\gamma - 2 \log p \right).$$

Also, let

$$E_d(X, p, a) = S_d(X, p, a) - M_d(X, p).$$

For $X = p^2 / \Phi(p)$, where $\Phi(p) \geq 1$ is a function that tends to infinity with $p$ in a way such that $\log \Phi(p) = o(\log p)$. As $p \to \infty$, we have

$$\frac{1}{p - 1} \# \{ a \in \mathbb{F}_p^\times : \alpha \leq E_d(X, p, a) \leq \beta \} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-x^2/2} \, dx + o(1).$$

Let $d_k(n)$ be the number of ways of writing $n$ as a product of $k$ factors. E. Kowalski and G. Ricotta [13] prove an analog of Theorem 1.1 of [3] for $d_k(n)$ for any integer $k \geq 3$. We have not succeeded in removing the smooth weight for any $k > 2$.

É. Fouvry, S. Ganguly, E. Kowalski, and Ph. Michel also prove an analog of (1.1) for holomorphic Hecke cusp forms of weight $k$ and level one (see Corollary 1.4 of [3]). We prove an analog of that result for a sharp cut-off function as well.

Let $f$ be a primitive (Hecke eigenform) cusp form of even weight $k$ and level 1, and consider its Fourier expansion

$$f(\tau) = \sum_{n=1}^{\infty} \rho_f(n) \frac{n^{k-1}}{\Gamma(k)} e(n\tau),$$

where $f$ is normalized so that $\rho_f(1) = 1$, so $\rho_f(n)$ is the eigenvalue of the (suitably normalized) Hecke operator $T(n)$. Let $c_f = \frac{(4\pi)^k}{k!} \| f \|^2$. Here we used the notation

$$\| f \|^2 = \frac{3}{\pi} \int \int y^k |f(x + iy)|^2 \frac{dx dy}{y^2},$$

where the integral is taken over any fundamental domain for $SL_2(\mathbb{Z})$.

Theorem 1.2. Let $f$ be a Hecke cusp form of weight $k$ and level one. For a prime $p$, let

$$S_f(X, p, a) = \sum_{1 \leq n \leq X, \atop n \equiv a \pmod{p}} \rho_f(n),$$

and

$$M_f(X, p) = \frac{1}{p} \sum_{1 \leq n \leq X} \rho_f(n).$$

Also, let

$$E_f(X, p, a) = \frac{S_f(X, p, a) - M_f(X, p)}{(c_f \cdot X/p)^{1/2}}.$$
For $X = p^2/\Phi(p)$, where $\Phi(p) \geq 1$ is a function that tends to infinity with $p$ in a way such that $\log \Phi(p) = o(\log p)$. As $p \to \infty$, we have

$$\frac{1}{p-1} \# \{a \in \mathbb{F}_p^\times : \alpha \leq E_f(X, p, a) \leq \beta\} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} \, dt + o(1).$$

1.2. **The divisor function in short intervals.** Heath-Brown studied the distribution of the normalized remainder term $x^{-1/4} \Delta(x)$ as $x \to \infty$, and proved that it has a limiting distribution function [7]. The behavior of the remainder term for the divisor problem in short intervals was studied by several authors (cf. [12, 11]). For example, Ivić proved in [11] an asymptotic formula for the second moment of $\Delta(x) - \Delta(x)$, where $T^c \leq U = U(T) \leq T^{1/2-\varepsilon}$:

$$\frac{1}{T} \int_T^{2T} (\Delta(x+U) - \Delta(x))^2 \, dx \sim \frac{8}{\pi^2} U \log^3 \left(\frac{\sqrt{T}}{U}\right).$$

We study the distribution of the sums of the divisor function in short intervals of the form $[x, x + \sqrt{x}/L]$, where $L$ grows to infinity in a way such that $\log L = o(\log x)$.

**Theorem 1.3.** Let $L = L(T) \to \infty$ as $T \to \infty$, with the condition that $\log L = o(\log T)$. For $\alpha < \beta$ as $T \to \infty$ we have

$$\frac{1}{T} \operatorname{meas}\left\{x \in [T, 2T] : \alpha \leq \frac{\Delta(x + \sqrt{x}/L) - \Delta(x)}{x^{1/4} \sqrt{\frac{8}{\pi^2} \log^3 L}} \leq \beta\right\} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} \, dt + o(1).$$

We remark that the analogous problem for the circle problem, i.e., the distribution of lattice points in thin annuli was studied by Hughes and Rudnick [8] – the corresponding normalized remainder in this case has also a Gaussian limiting distribution.

Next, we prove the analogous result for the distribution of the sum of Hecke eigenvalues in short intervals. Let

$$A_f(x) = \sum_{n \leq x} \rho_f(n).$$

**Theorem 1.4.** For fixed $\alpha < \beta$ as $T \to \infty$

$$\frac{1}{T} \operatorname{meas}\left\{x \in [T, 2T] : \alpha \leq \frac{A_f(x + \sqrt{x}/L) - A_f(x)}{x^{1/4} (c_f / L)^{1/2}} \leq \beta\right\} = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta e^{-t^2/2} \, dt + o(1).$$

We can also prove an analog of this result for the Hecke eigenvalues of Maass forms for the full modular group. The proof requires some additional steps since the Ramanujan bound is not known in this case.

2. **The distribution of the divisor function in progressions**

2.1. **Preliminary Lemmas.** For a smooth function $g$ let

$$B_d(g)(\xi) = \begin{cases} -2\pi \int_0^\infty g(u) J_0(4\pi \sqrt{\xi u}) \, du & \text{if } \xi > 0, \\ 4 \int_0^\infty g(u) K_0(4\pi \sqrt{|\xi| u}) \, du & \text{if } \xi < 0. \end{cases}$$

Also, let

$$B_f(g)(\xi) = 2\pi i^k \int_0^\infty g(u) J_{k-1}(4\pi \sqrt{\xi u}) \, du \quad \text{for } \xi > 0.$$
Lemma 2.1. Let \( A \geq 1 \) be an integer. For \( \xi \neq 0 \) we have
\[
\mathcal{B}_d(w_\delta)(\xi) \ll \min \left\{ 1 + |\log |\xi||, |\xi|^{-A/2-1/4}d^{1-A} \right\},
\]
where the implied constant depends on \( A \). Additionally, for \( \xi > 0 \)
\[
\mathcal{B}_f(w_\delta)(\xi) \ll \min \left\{ 1, |\xi|^{-A/2-1/4}d^{1-A} \right\}
\]
where the implied constant depends on \( A \).

Proof. Note that
\[
J_{k-1}(x) \ll 1, \quad Y_0(x) \ll 1 + |\log x|, \quad \text{and} \quad K_0(x) \ll 1 + |\log x|,
\]
which establishes the first bound for both claims.

We will only prove the second bound for \( \mathcal{B}_f(w_\delta) \); the proof of the bound for \( \mathcal{B}_d(w_\delta) \) follows from a similar argument. By the change of variables \( v = 4\pi \sqrt{\xi}u \) we have
\[
\mathcal{B}_f(w_\delta)(\xi) = \frac{ik}{4\pi \xi} \int_0^\infty w_\delta \left( \frac{v^2}{16\pi^2 \xi} \right) v J_{k-1}(v) \, dv.
\]
Next set \( \alpha = (16\pi^2 \xi)^{-1} \) and note that
\[
\frac{d}{dx} \left( x^{\nu+1} J_{\nu+1}(x) \right) = x^{\nu+1} J_{\nu}(x),
\]
(see [5] equation 8.472.3). Thus, integration by parts gives
\[
\mathcal{B}_f(w_\delta)(\xi) = \frac{ik}{4\pi \xi} \int_0^\infty v^{1-k} \alpha w_\delta(\alpha^2) \, dv J_{k}(v) = -\frac{ik}{4\pi \xi} \int_0^\infty (2\alpha v^2 w_\delta(\alpha^2) + (1-k)w_\delta(\alpha^2)) J_{k}(v) \, dv.
\]
Repeatedly integrating by parts, we see that
\[
\mathcal{B}_f(w_\delta)(\xi) \ll \frac{1}{\xi} \int_0^\infty v^{-A+1} |J_{k-1+A}(v)| \sum_{\ell=0}^A (\alpha v^2) \ell |w_\delta(\ell)(\alpha^2)| \, dv.
\]
We now use the bound \( J_{k-1+A}(x) \ll x^{-1/2} \) then make the change of variables \( y = \alpha v^2 \) to get
\[
\mathcal{B}_f(w_\delta)(\xi) \ll \xi^{-A/2-1/4} \int_0^\infty y^{-A/2-1/4} \sum_{\ell=0}^A y^{\ell} |w_\delta(\ell)(y)| \, dy.
\]
Note that \( w_\delta(\ell)(x) \ll 1/\delta^\ell \) and \( w(\ell)(x) \) is supported on the interval \( [\delta, 2\delta] \cup [1-\delta, 1] \), for \( \ell \geq 1 \). Hence, for \( \ell \geq 1 \) we have
\[
\int_0^{2\delta} y^{-A/2-1/4} |w_\delta(\ell)(y)| \, dy \ll \int_\delta^{2\delta} y^{-A/2-1/4} \, dy + \delta^{-\ell} \int_{1-\delta}^1 1 \, dy \ll \delta^{3/4-A/2} + \delta^{1-\ell} \ll \delta^{1-A}.
\]
For \( \ell = 0 \) we have
\[
\int_0^\infty y^{-A/2-1/4} |w_\delta(y)| \, dy \ll \int_\delta^1 y^{-A/2-1/4} \, dy \ll \delta^{3/4-A/2} + 1 \ll \delta^{1-A}.
\]
The result follows by collecting estimates. \( \square \)
Lemma 2.2. We have for \( \xi \neq 0 \):
\[
\left( \mathcal{B}_d(w_\delta)(\xi) \right)' \ll |\xi|^{-5/4}.
\]
Additionally, for \( \xi > 0 \) and \( k \geq 2 \) we have
\[
\left( \mathcal{B}_f(w_\delta)(\xi) \right)' \ll |\xi|^{-5/4}.
\]
Proof. For the first claim differentiate inside the integral and use the formula
\[
\frac{d}{dx}Z_0(x) = -Z_1(x),
\]
for \( Z = Y \) or \( Z = K \) (see [5] equations 8.473.6 and 8.486.18). Now argue as in the previous proof, but integrate by parts just once.

The proof of the last assertion is similar. Here use the relation
\[
\frac{d}{dx}J_{k-1}(x) = k \frac{J_k(x) - J_{k-1}(x) - J(x)}{x}. 
\]
(see [5] equation 8.472.2). \( \square \)

Lemma 2.3. Let \( 0 < \delta < \varepsilon < 1 \) and \( \phi_{\delta,\varepsilon}(x) = w_\delta(x) - w_\varepsilon(x) \). For \( 2 < Y = o(X) \) as \( X \to \infty \), we have for any \( \varepsilon > 0 \) that
\[
\sum_{1 \leq |n| < X} d^2(|n|) \left( \mathcal{B}_d(\phi_{\delta,\varepsilon}) \left( \frac{n}{Y} \right) \right)^2 \ll \varepsilon Y \log^3 Y + Y^{1/2+\varepsilon}
\]
and
\[
\sum_{1 \leq n < X} \rho_f(n)^2 \left( \mathcal{B}_f(\phi_{\delta,\varepsilon}) \left( \frac{n}{Y} \right) \right)^2 \ll \varepsilon Y + Y^{3/5},
\]
where the implied constant depends on \( f \).

Proof. For the first assertion we will only consider the sum over \( n > 0 \) since the terms with \( n < 0 \) are handled in the same way. We cite the formula
\[
\sum_{n \leq t} d^2(n) = c_3 t \log^3 t + c_2 t \log^2 t + c_1 t \log t + c_0 t + O(t^{1/2+\varepsilon}),
\]
(2.2)
where \( c_3, c_2, c_1 \) and \( c_0 \) are absolute constants with \( c_3 = 1/\pi^2 \), (see [15] and equation 14.30 of [10]). Let \( Q_3(x) = \sum_{j=0}^3 c_j x^j \). We have
\[
\sum_{1 \leq n < X} d^2(n) \left( \mathcal{B}_d(\phi_{\delta,\varepsilon}) \left( \frac{n}{Y} \right) \right)^2 = \int_{1/2}^{X} \left( \mathcal{B}_d(\phi_{\delta,\varepsilon}) \left( \frac{x}{Y} \right) \right)^2 (Q_3(\log x) + Q_3'(\log x)) dx 
\]
\[
+ \int_{1/2}^{X} \left( \mathcal{B}_d(\phi_{\delta,\varepsilon}) \left( \frac{x}{Y} \right) \right)^2 dR(x) 
\]
\[
= I_1 + I_2,
\]
(2.3)
where \( R(x) \ll x^{1/2+\varepsilon} \).

First we consider \( I_2 \). Integrating by parts we get
\[
I_2 = R(x) \left( \mathcal{B}_d(\phi_{\delta,\varepsilon}) \left( \frac{x}{Y} \right) \right)^2 \bigg|_{1/2}^{X} - \frac{2}{Y} \int_{1/2}^{X} R(x) \mathcal{B}_d(\phi_{\delta,\varepsilon}) \left( \frac{x}{Y} \right) \left( \mathcal{B}_d(\phi_{\delta,\varepsilon}) \right)' \left( \frac{x}{Y} \right) dx.
\]
Applying Lemma 2.1 and Lemma 2.2 we see that
\[ I_2 \ll Y^{1/2+\varepsilon} + Y^{1/4} \log Y \int_{1/2}^{Y} x^{-3/4+\varepsilon} \, dx + Y \int_{Y}^{X} x^{-3/2+\varepsilon} \, dx \ll Y^{1/2+2\varepsilon}. \quad (2.4) \]

Next, observe that
\[
I_1 \ll Y \log^3 Y \int_{0}^{\infty} \left( B_d(\phi_{\delta,\varepsilon})(x) \right)^2 \, dx + Y \left( \int_{1/(2Y)}^{Y^2} + \int_{Y^2}^{\infty} \right) \left( B_d(\phi_{\delta,\varepsilon})(x) \right)^2 Q_3(|\log x|) \, dx
\]
\[ \ll Y \log^3 Y \int_{0}^{\infty} \left( B_d(\phi_{\delta,\varepsilon})(x) \right)^2 \, dx + Y^\varepsilon, \]
by Lemma 2.1. From Proposition 2.3 of [3] it follows that \( \| B_d(\phi_{\delta,\varepsilon}) \|_{L^2}^2 = \| \phi_{\delta,\varepsilon} \|_{L^2}^2 \), where the \( L^2 \) norm is computed in \( \mathbb{R}^\times \) with respect to Lebesgue measure. Also, \( \| \phi_{\delta,\varepsilon} \|_{L^2}^2 \ll \varepsilon \). Thus,
\[ I_1 \ll \varepsilon Y \log^3 Y + Y^\varepsilon. \]

The proof of the first assertion follows by this (2.4) and (2.4).

To prove the other assertion we argue similarly. First, we cite the formula Rankin and Selberg (see [16] and [18])
\[ \sum_{1 \leq n < X} \rho_f^2(n) = c_f X + O_f(X^{3/5}). \quad (2.5) \]

We have
\[ \sum_{1 \leq n < X} \rho_f(n)^2 \left( B_f(\phi_{\delta,\varepsilon})(\frac{n}{Y}) \right)^2 = c_f \int_{1/2}^{X} \left( B_f(\phi_{\delta,\varepsilon})(\frac{x}{Y}) \right)^2 \, dx + \int_{1/2}^{X} \left( B_f(\phi_{\delta,\varepsilon})(\frac{x}{Y}) \right)^2 \, d\mathcal{R}(x), \]
where \( \mathcal{R}(x) \ll x^{3/5} \). The first integral is \( \ll \varepsilon Y \). Integrating by parts, then applying Lemmas 2.1 and 2.2 we see that the second integral is \( \ll Y^{3/5} \).

\[ \square \]

2.2. The proof of Theorems 1.1 and 1.2. We first deduce Theorems 1.1 and 1.2 from the following two lemmas.

**Lemma 2.4.** Let \( 0 < \varepsilon < 1 \) be fixed. Suppose that \( X^{1/2} < p < X^{1-\theta} \) for some \( \theta > 0 \). For \( \delta \) such that \( (p/X)^{3/4-\eta} \ll \delta \ll \varepsilon \), for some \( \eta > 0 \), we have for \( \ast = f \) or \( d \) that
\[ \frac{1}{p-1} \sum_{1 \leq a < p} \left( E_\ast(X,p,a;w_\delta) - E_\ast(X,p,a;w_\varepsilon) \right)^2 = O(\varepsilon). \]

**Lemma 2.5.** Suppose that \( p < X^{1-\theta} \) for some \( \theta > 0 \). If \( 2p/X \leq \delta \ll (p/X)^{1/2}(pX)^{-\eta} \), for some \( \eta > 0 \), we have for \( \ast = f \) or \( d \) that
\[ \frac{1}{p-1} \sum_{1 \leq a < p} \left| E_\ast(X,p,a) - E_\ast(X,p,a;w_\delta) \right| \ll p^{-\eta}. \]

*Proof of Theorems 1.1 and 1.2.* Let \( \varepsilon > 0 \) be fixed. Also, let \( X = p^2/\Phi(p) \) where \( \Phi(p) \) tends to infinity with \( p \) in a way such that \( \log \Phi(p) = o(\log p) \). Corollary 1.4 of [3] gives as \( p \to \infty \) that
\[ \frac{1}{p-1} \# \left\{ a \in \mathbb{P}^\times : \alpha \leq E_\ast(X,p,a;w_\varepsilon) \leq \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-x^2/2} \, dx + o(1), \quad (2.6) \]
for \( \ast = d \) or \( f \).
Take $\delta = p^{-2/3}$. By Lemma 2.4 we have
\[
\frac{1}{p - 1} \# \left\{ a \in \mathbb{F}_p^\times : \left| E_\star(X, p, a; w_\delta) - E_\star(X, p, a; w_\epsilon) \right| > \epsilon^{1/3} \right\} \ll \epsilon^{1/3}.
\]
Since, $\epsilon > 0$ is fixed we see that by (2.6)
\[
\frac{1}{p - 1} \# \left\{ a \in \mathbb{F}_p^\times : \alpha \leq E_\star(X, p, a) \leq \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-x^2/2} dx + O(\epsilon^{1/3}) + o(1).
\]
Additionally, by Lemma 2.5 we have
\[
\frac{1}{p - 1} \# \left\{ a \in \mathbb{F}_p^\times : \left| E_\star(X, p, a; w_\delta) - E_\star(X, p, a; w_\epsilon) \right| > \epsilon^{1/3} \right\} \ll \epsilon^{1/3} p^{-1/18}.
\]
Thus,
\[
\frac{1}{p - 1} \# \left\{ a \in \mathbb{F}_p^\times : \alpha \leq E_\star(X, p, a) \leq \beta \right\} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-x^2/2} dx + O(\epsilon^{1/3}) + O(\epsilon^{-1/3} p^{-1/18}) + o(1).
\]
Since, $\epsilon > 0$ is arbitrary the result follows. \qed

For integers $a, b,$ and $c \geq 1$ the Kloosterman $S(a, b; c)$ is given by
\[
S(a, b; c) = \sum_{x \equiv 1 \mod c} e \left( \frac{xa + \bar{x}b}{c} \right),
\]
where $x\bar{x} \equiv 1 \mod c$ and $e(x) = e^{2\pi ix}$. Let
\[
\text{Kl}_2(a, b; c) = \frac{S(a, b; c)}{c^{1/2}}.
\]

Before proving Lemma 2.4 we require a version of the Voronoi summation formula that is proved in [3].

Lemma 2.6 (Proposition 2.1 of [3]). Let $Y = p^2/X$. Then for any non negative smooth function $w$ compactly supported in the positive reals we have
\[
E_\star(X, p, a; w) = \frac{1}{\sigma_\star(Y)} \sum_{n \neq 0} \tau_\star(n) B_\star(w) \left( \frac{n}{Y} \right) \text{Kl}_2(a, n; p),
\]
where
\[
\sigma^2_\star(Y) = \begin{cases} 
\frac{2}{\pi^2} Y \log^3 Y & \text{if } * = d, \\
cf Y & \text{if } * = f,
\end{cases}
\]
$\tau_d(n) = d(|n|)$ and
\[
\tau_f(n) = \begin{cases} 
\rho_f(n) & \text{if } n > 0, \\
0 & \text{if } n < 0.
\end{cases}
\]

In the proof of Lemma 2.4 we will need to estimate
\[
\frac{1}{p - 1} \sum_{1 \leq a < p} \text{Kl}_2(a, m; p) \text{Kl}_2(a, n; p).
\]
First note that if \( m \) or \( n \) is divisible by \( p \) it is easily seen that this is \( \ll 1/p \). Next, for integers \( \ell \) define

\[
  f(\ell) = \begin{cases} 
    0 & \text{if } \ell \equiv 0 \pmod{p}, \\
    e(\ell/p) & \text{otherwise}.
  \end{cases}
\]

Applying, the discrete Plancherel formula we have for \( m \) and \( n \) not divisible by \( p \) that

\[
  \frac{1}{p-1} \sum_{0 \leq a < p} \kappa_2(a, m; p) \kappa_2(a, n; p) = \frac{1}{p-1} \sum_{0 \leq b < p} f(mb) f(nb)
\]

\[
  = \frac{1}{p-1} \sum_{1 \leq b < p} e\left( \frac{b(m - n)}{p} \right).
\]

We conclude that

\[
  \frac{1}{p-1} \sum_{1 \leq a < p} \kappa_2(a, n; p) \kappa_2(a, m; p) = \begin{cases} 
    1 + O\left( \frac{1}{p} \right) & \text{if } n \equiv m \pmod{p} \text{ and } p \nmid n, \\
    O\left( \frac{1}{p} \right) & \text{otherwise.}
  \end{cases} \tag{2.7}
\]

**Proof of Lemma 2.4** Let \( \phi_{\delta, \varepsilon}(x) = w_{\delta}(x) - w_{\varepsilon}(x) \) and \( Y = p^2/X \). Applying Lemma 2.6 we see that

\[
  \frac{1}{p-1} \sum_{1 \leq a < p} \left( E_*(X, p, a; w_{\delta}) - E_*(X, p, a; w_{\varepsilon}) \right)^2
\]

\[
  \ll \frac{1}{p \cdot \sigma_2^2(Y)} \sum_{1 \leq a < p} \left( \sum_{1 \leq |m| \leq p/2} \tau_*(n) B_*(\phi_{\delta, \varepsilon})(\frac{n}{Y}) \kappa_2(a, n; p) \right)^2
\]

\[
  + \frac{1}{p \cdot \sigma_2^2(Y)} \sum_{1 \leq a < p} \left( \sum_{|n| > p/2} \tau_*(n) B_*(\phi_{\delta, \varepsilon})(\frac{n}{Y}) \kappa_2(a, n; p) \right)^2 = \Sigma_1 + \Sigma_2.
\]

We first consider \( \Sigma_2 \). We have

\[
  \Sigma_2 = \frac{1}{\sigma_2^2(Y)} \sum_{|m|, |n| > p/2} \tau_*(m) \tau_*(n) B_*(\phi_{\delta, \varepsilon})(\frac{m}{Y}) B_*(\phi_{\delta, \varepsilon})(\frac{n}{Y}) \frac{1}{p} \sum_{1 \leq a < p} \kappa_2(a, m; p) \kappa_2(a, n; p).
\]

By (2.7), Lemma 2.1, and the bound \( \tau_*(n) \ll n^{\varepsilon} \) the contribution of the terms with \( m \equiv n \pmod{p} \) to \( \Sigma_2 \) is

\[
  \ll \frac{1}{\sigma_2^2(Y)} \sum_{|m|, |n| > p/2} |\tau_*(m) \tau_*(n) B_*(\phi_{\delta, \varepsilon})(\frac{m}{Y}) B_*(\phi_{\delta, \varepsilon})(\frac{n}{Y})|
\]

\[
  \ll \frac{Y^{5/2}}{\sigma_2^2(Y) \delta^2} \sum_{|m| > p/2} \frac{1}{|m|^{5/4 - \varepsilon}} \sum_{n \equiv m \pmod{p}} \frac{1}{|n|^{5/4 - \varepsilon}} \ll \frac{Y^{5/2}}{p^{3/2 - \varepsilon} \sigma_2^2(Y) \delta^2}.
\]

Similarly, applying (2.7) and Lemma 2.1, the sum of the remaining terms in \( \Sigma_2 \) is

\[
  \ll \frac{1}{p \cdot \sigma_2^2(Y)} \left( \sum_{|n| > p/2} |\tau_*(n) B_*(\phi_{\delta, \varepsilon})(\frac{n}{Y})| \right)^2 \ll \frac{Y^{5/2}}{p \cdot \sigma_2^2(Y) \delta^2} \left( \sum_{|n| > p/2} \frac{1}{|n|^{5/4 - \varepsilon}} \right)^2 \ll \frac{Y^{5/2}}{p^{3/2 - \varepsilon} \sigma_2^2(Y) \delta^2}.
\]
Thus,
\[
\sum_2 \ll \frac{Y^{3/2}}{p^{3/2-\varepsilon}d^2} = \frac{p^{3/2+\varepsilon}}{X^{3/2}d^2}.
\] (2.8)

The estimation of \(\Sigma_1\) is similar. First note that
\[
\sum_1 \ll \frac{1}{p} \cdot \sigma_2^2(Y) \sum_{1 \leq n < p/2} \left( \sum_{1 \leq n \leq p/2} \tau_*(n) B_*(\phi_{\delta,\varepsilon}) \left( \frac{n}{Y} \right) \text{Kl}_2(a, n; p) \right)^2
\]
\[
+ \frac{1}{p} \cdot \sigma_2^2(Y) \sum_{1 \leq n < p} \left( \sum_{-p/2 \leq n \leq -1} \tau_*(n) B_*(\phi_{\delta,\varepsilon}) \left( \frac{n}{Y} \right) \text{Kl}_2(a, n; p) \right)^2.
\] (2.9)

Here if \(m \equiv n \pmod{p}\) then \(m = n\) so that the first sum above is
\[
\ll \frac{1}{\sigma_2^2(Y)} \sum_{1 \leq n \leq p/2} \tau_*(n) B_*(\phi_{\delta,\varepsilon}) \left( \frac{n}{Y} \right) \text{Kl}_2(a, n; p) \ll \varepsilon.
\]

Now, apply the Cauchy-Schwarz inequality to the second sum above. Thus, by Lemma 2.3 the first sum in (2.9) is
\[
\ll \frac{1}{\sigma_2^2(Y)} \cdot \sum_{1 \leq n < p/2} \tau_*(n) B_*(\phi_{\delta,\varepsilon}) \left( \frac{n}{Y} \right) \ll \varepsilon.
\]
The second sum in (2.9) satisfies this bound as well. Thus, by this and (2.8) the proof is completed.

\[\square\]

Proof of Lemma 2.5. Observe that for any \(\varepsilon > 0\)
\[
S_*(X, p, a) - S_*(X, p, a; w_\delta) = \sum_{n \leq X \atop n \equiv a \pmod{p}} \tau_*(n) \left( 1 - w_\delta \left( \frac{n}{X} \right) \right) \ll \frac{\delta X^{1+\varepsilon}}{p},
\]
uniformly in \(a\). Similarly,
\[
\frac{1}{p} \sum_{n \leq X} \tau_*(n) \left( 1 - w_\delta \left( \frac{n}{X} \right) \right) \ll \frac{\delta X^{1+\varepsilon}}{p},
\]
and
\[
\frac{1}{p^2} \int_0^X (\log y + 2\gamma - 2\log p) \left( 1 - w_\delta \left( \frac{y}{X} \right) \right) dy \ll \frac{\delta X^{1+\varepsilon}}{p^2 \log(Xp)}.
\]

Thus,
\[
M_*(X, p, a) - M_*(X, p, a; w_\delta) \ll \frac{\delta X^{1+\varepsilon}}{p}.
\]

It follows that, uniformly in \(a\), we have
\[
\left| E_*(X, p, a) - E_*(X, p, a; w_\delta) \right| \ll \frac{\delta X^{1+\varepsilon}}{p^{1/2}}.
\]

We conclude that
\[
\frac{1}{p-1} \sum_{1 \leq a < p} \left| E_*(X, p, a) - E_*(X, p, a; w_\delta) \right| \ll p^{-\varepsilon}.
\]

for \(2p/X \leq \delta \ll (p/X)^{1/2}(pX)^{-\varepsilon}\).
3. The distribution of $d(n)$ in short intervals

3.1. The Variance of $S(x, L)$. To take averages, instead of working first with Lebesgue measure, we use a smooth average around $T$, so we take a Schwartz function $\omega \geq 0$ supported on the positive real numbers with a unit mass. Define our averages by

$$\langle f \rangle = \int_{-\infty}^{\infty} f(x) \omega\left(\frac{x}{T}\right) \frac{dx}{T} = \int_{-\infty}^{\infty} f(xT) \omega(x) \, dx.$$ 

Also, for $f \in L^1(\mathbb{R})$ let

$$\tilde{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi \sqrt{|x|}} \, dx.$$ 

By repeatedly integrating by parts it follows for any $A \geq 1$ that

$$\tilde{\omega}(\xi) \ll \min \left\{ 1, |\xi|^{-A} \right\}, \quad (3.1)$$

where the implied constant depends on $\omega$.

Define $F(x) = x^{-1/4} \Delta(x)$. Let $L = L(T) \to \infty$ as $T \to \infty$, with the condition that $L \ll T^\varepsilon$ for all $\varepsilon > 0$. Define

$$S(x, L) = F \left( \left( \sqrt{x} + \frac{1}{2L} \right)^2 \right) - F(x).$$

Also, let

$$\sigma^2 = \frac{16 \log^3 L}{\pi^2 L}.$$ 

We first show that the expectation of $S(x, L)$ tends to zero as $T \to \infty$:

**Lemma 3.1.** For all $\varepsilon > 0$, $\langle S(x, L) \rangle = O \left( T^{-1/4+\varepsilon} \right)$.

**Proof.** We use the following formula (see Titchmarsh, [19] (12.4.4) for example)

$$F(x) = \frac{1}{\pi \sqrt{2}} \sum_{n \leq T} \frac{d(n)}{n^{3/4}} \cos \left( 4\pi \sqrt{n}x - \frac{\pi}{4} \right) + O \left( T^{-1/4+\varepsilon} \right)$$

uniformly for $T \leq x \leq 2T$.

We conclude that for such $x$

$$S(x, L) = \frac{-2}{\pi \sqrt{2}} \sum_{n \leq T} \frac{d(n)}{n^{3/4}} \sin \left( \frac{2\pi \sqrt{n}}{L} \right) \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) + O \left( T^{-1/4+\varepsilon} \right),$$

So

$$\langle S(x, L) \rangle = \frac{-2}{\pi \sqrt{2}} \sum_{n \leq T} \frac{d(n)}{n^{3/4}} \sin \left( \frac{2\pi \sqrt{n}}{L} \right) \left\langle \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \right\rangle + O \left( T^{-1/4+\varepsilon} \right).$$
Note that
\[
\left\langle \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \right\rangle = \frac{1}{2i} \left( \tilde{\omega} \left( -2\sqrt{Tn} \right) e^{\pi i \left( \frac{2\pi}{L} \cdot \frac{1}{4} \right)} - \tilde{\omega} \left( 2\sqrt{Tn} \right) e^{-\pi i \left( \frac{2\pi}{L} \cdot \frac{1}{4} \right)} \right),
\]
so from the rapid decay of \( \tilde{\omega} \) we get that for all \( A > 0 \)
\[
\langle S(x, L) \rangle \ll \sum_{n=1}^{\infty} \frac{d(n)}{n^{3/4}} \frac{1}{T^{4/2} n^{4/2}} + O \left( T^{-1/4+\varepsilon} \right) = O \left( T^{-1/4+\varepsilon} \right).
\]

We now compute the variance of \( S(x, L) \):

**Lemma 3.2.** \( \left\langle S(x, L)^2 \right\rangle \sim \frac{16 \log^3 L}{T^2} = \sigma^2 \).

**Proof.** Again, from the formula (12.4.4) in [19], we get that for any (small) \( \delta > 0 \)
\[
S(x, L) = \frac{-2}{\pi \sqrt{2}} \sum_{n \leq T^{1-\delta}} \frac{d(n)}{n^{3/4}} \sin \left( \frac{2\pi \sqrt{n}}{L} \right) \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) + O \left( T^{-1/4+\delta} \right).
\]
Denote the sum in (3.2) by \( P \). By the Cauchy-Schwarz inequality
\[
\left\langle S(x, L)^2 \right\rangle = \left\langle P^2 \right\rangle + O \left( T^{-1/4+\delta} \sqrt{\left\langle P^2 \right\rangle} + T^{-1/2+2\delta} \right)
\]
so (since \( L \ll T^\varepsilon \) for all \( \varepsilon > 0 \)) it is enough to show that \( \left\langle P^2 \right\rangle \sim \frac{16 \log^3 L}{T^2} \).

Indeed, we first see that the contribution of the off-diagonal terms is minor: their contribution equals
\[
\frac{2}{\pi^2} \sum_{n \neq m \leq T^{1-\delta}} \frac{d(n) d(m)}{(nm)^{3/4}} \sin \left( \frac{2\pi \sqrt{n}}{L} \right) \sin \left( \frac{2\pi \sqrt{m}}{L} \right) \times \left\langle \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \sin \left( 4\pi \sqrt{m} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \right\rangle.
\]
Observe that
\[
\sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \sin \left( 4\pi \sqrt{m} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) = -\frac{1}{4} \left( e^{4\pi i (\sqrt{n-1} + \sqrt{n+1} - \sqrt{m-1} + \sqrt{m+1})} + e^{4\pi i (\sqrt{m-1} + \sqrt{m+1} - \sqrt{n-1} + \sqrt{n+1})} - e^{4\pi i (\sqrt{n+1} + \sqrt{m+1} - \sqrt{n-1} + \sqrt{m-1})} - e^{4\pi i (\sqrt{n-1} + \sqrt{m-1} - \sqrt{n+1} + \sqrt{m+1})} \right).
\]
\[
\langle \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \sin \left( 4\pi \sqrt{m} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \rangle \\
= -\frac{1}{4} \left( \tilde{\omega} \left( 2\sqrt{T} (-\sqrt{n} - \sqrt{m}) \right) e^{\pi i \left( -\frac{1}{2} + \frac{4}{T} \sqrt{n} + \sqrt{m} \right)} \right) \\
+ \tilde{\omega} \left( 2\sqrt{T} (\sqrt{n} + \sqrt{m}) \right) e^{\pi i \left( \frac{1}{2} + \frac{4}{T} \sqrt{n} + \sqrt{m} \right)} \\
- \tilde{\omega} \left( 2\sqrt{T} (-\sqrt{n} + \sqrt{m}) \right) e^{\pi i \left( \frac{3}{2} \sqrt{n} - \sqrt{m} \right)} \\
- \tilde{\omega} \left( 2\sqrt{T} (\sqrt{n} - \sqrt{m}) \right) e^{\frac{2\pi i}{T} \left( -\sqrt{n} + \sqrt{m} \right)}.
\]

Since \( \sqrt{m} \pm \sqrt{n} \gg T^{-1/2+\delta/2} \) and \( \tilde{\omega} \) rapidly decays by (3.1), we conclude that for every \( A > 0 \)
\[
\langle \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \rangle \\
\ll \left( \sqrt{T} (\sqrt{m} \pm \sqrt{n}) \right)^{-A} \ll T^{-\delta A/2}.
\]

So the sum in (3.3) is bounded above by
\[
\sum_{n \not= m \leq T^{1-\delta}} T^{-\delta A/2} \leq T^{2-\delta A/2}.
\]

We get that for all \( B > 0 \)
\[
\langle P^2 \rangle = \frac{2}{\pi^2} \sum_{n \leq T^{1-\delta}} \frac{d^2(n)}{n^{3/2}} \sin^2 \left( \frac{2\pi \sqrt{n}}{L} \right) \left\langle \sin^2 \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \right\rangle \\
+ O \left( T^{-B} \right).
\]

Note that
\[
\left\langle \sin^2 \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right) \right\rangle = \\
-\frac{1}{4} \left( \tilde{\omega} \left( -4\sqrt{Tn} \right) e^{\pi i \left( -\frac{1}{2} + \frac{4}{T} \sqrt{n} \right)} + \tilde{\omega} \left( 4\sqrt{Tn} \right) e^{\pi i \left( \frac{1}{2} - \frac{4}{T} \sqrt{n} \right)} - 2 \right) \\
= \frac{1}{2} + O \left( T^{-B} \right).
\]

So actually
\[
\langle P^2 \rangle = \frac{1}{\pi^2} \sum_{n \leq T^{1-\delta}} \frac{d^2(n)}{n^{3/2}} \sin^2 \left( \frac{2\pi \sqrt{n}}{L} \right) + O \left( T^{-B} \right).
\]

To evaluate the main term, write
\[
\sigma^2_{T^{1-\delta}} := \frac{1}{\pi^2} \sum_{n \leq T^{1-\delta}} \frac{d^2(n)}{n^{3/2}} \sin^2 \left( \frac{2\pi \sqrt{n}}{L} \right).
\]
Applying partial summation and using (2.2), we get that
\[
\sigma_{T_1-\delta}^2 \sim \frac{1}{\pi^4} \int_1^{T_1-\delta} \frac{\log^3 x}{x^{3/2}} \frac{\sin^2 \left( \frac{2\pi \sqrt{x}}{L} \right)}{y^2} dy
\]
\[
= \frac{16}{\pi^4 L} \int_{1/L}^{T_1^{1/2-\delta/2}/L} \frac{\log^3 (Ly)}{y^2} \frac{\sin^2 (2\pi y)}{y^2} dy
\]
\[
\sim \frac{16 \log^3 L}{\pi^4} \int_{1/L}^{T_1^{1/2-\delta/2}/L} \frac{\sin^2 (2\pi y)}{y^2} dy
\]
where the last relation holds because
\[
\int_{1/L}^{1/L} \frac{\sin^2 (2\pi y)}{y^2} dy \ll \frac{1}{L}
\]
and
\[
\int_0^{\infty} \frac{\sin^2 (2\pi y)}{y^2} dy \ll \frac{L}{T_1^{1/2-\delta/2}}.
\]
Since \( \int_0^{\infty} \frac{\sin^2 (2\pi y)}{y^2} dy = \pi^2 \), we conclude that
\[
\langle P^2 \rangle \sim \sigma_{T_1-\delta}^2 \sim \frac{16 \log^3 L}{\pi^2 L} = \sigma^2.
\]

For any \( M = O(T^{1-\delta}) \) such that \( L/\sqrt{M} \rightarrow 0 \), define the “short” sum
\[
S(x, L, M) = \frac{-2}{\pi \sqrt{2}} \sum_{n \leq M} d(n) n^{3/4} \sin \left( \frac{2\pi \sqrt{n}}{L} \right) \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right).
\]
By the same arguments as above, we conclude that
\[
\langle S(x, L, M) \rangle = O(T^{-B})
\]
for all \( B > 0 \), with the implied constant independent of \( M \), and
\[
\langle S(x, L, M)^2 \rangle \sim \sigma^2.
\]
The advantage of working with \( S(x, L, M) \) is that we can also calculate its higher moments – this will be done in the next section.

By considerations similar to the above, we get that the normalized distance between the short and the long sums tends to zero in the \( L^2 \)-norm:

**Lemma 3.3.** As \( T \rightarrow \infty \)
\[
\left\langle \left( \frac{S(x, L) - S(x, L, M)}{\sigma} \right)^2 \right\rangle = o(1).
\]
Proof. We have for any (small) \( \delta > 0 \)
\[
S(x, L) - S(x, L, M) =
\frac{-2}{\pi \sqrt{2}} \sum_{M < n \leq T^{1-\delta}} \frac{d(n)}{n^{3/4}} \sin \left( \frac{2\pi \sqrt{n}}{L} \right) \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right)
\]
\[+ O\left( T^{-1/4+\delta} \right).\]

Denoting the sum in (3.4) by \( P \), we get by Cauchy-Schwarz’s inequality that
\[
\left\langle \left( \frac{S(x, L) - S(x, L, M)}{\sigma} \right)^2 \right\rangle = \frac{1}{\sigma^2} \langle P^2 \rangle + O \left( \frac{T^{-1/4+\delta}}{\sigma^2} \langle P^2 \rangle + \frac{T^{-1/2+2\delta}}{\sigma^2} \right)
\]
so (since \( L \ll T^\varepsilon \) for all \( \varepsilon > 0 \)) it is enough to show that
\[
\langle P^2 \rangle = o(\sigma^2).
\]

Indeed, by the same reasoning as in the proof of Lemma 3.2
\[
\langle P^2 \rangle \sim \frac{16 \log^3 L}{\pi^4} \int_0^\infty \frac{\sin^2(2\pi y)}{y^2} \frac{dy}{\sqrt{M/L}} \ll \sigma^2 \frac{L}{\sqrt{M}} = o(\sigma^2)
\]
since \( L/\sqrt{M} \to 0 \) as \( T \to \infty \). \( \square \)

3.2. Higher moments. Define
\[
\sigma_M^2 = \frac{1}{\pi^2} \sum_{n \leq M} \frac{d^2(n)}{n^{3/2}} \sin^2 \left( \frac{2\pi \sqrt{n}}{L} \right).
\]

In this section we calculate the moments of \( S(x, L, M)/\sigma_M \) and prove:

Proposition 3.1. Let \( m \geq 2 \) be an integer. Suppose \( M \ll T^{(1-\varepsilon)/(2^{m-1}-1)} \) for some \( \varepsilon > 0 \), \( L \) tends to infinity with \( T \), and \( L/\sqrt{M} \to 0 \). As \( T \to \infty \) we have
\[
\left\langle \left( \frac{S(x, L, M)}{\sigma_M} \right)^m \right\rangle = \begin{cases} m! m^{m/2} & \text{if } m \text{ is even}, \\ o(1) & \text{if } m \text{ is odd}. \end{cases}
\]

(3.6)

We first cite two auxiliary lemmas.

Lemma 3.4 (Theorem 2 of [1]). Let \( q \) denote a square free, positive integer. The set \( \{ \sqrt{q} \} \) is linearly independent over \( \mathbb{Q} \).

Lemma 3.5 (Lemma 3.5 of [3]). For \( j = 1, \ldots, m \), let \( n_j \leq M \) be positive integers and let \( \epsilon_j \in \{-1, 1\} \) be such that
\[
\sum_{j=1}^m \epsilon_j \sqrt{n_j} \neq 0.
\]

Then,
\[
\left| \sum_{j=1}^m \epsilon_j \sqrt{n_j} \right| \geq \frac{1}{(m\sqrt{M})^{2m-1-1}}.
\]

Let \( \{X(q)\}_q \) be a sequence of independent random variables uniformly distributed on the unit circle, where the index \( q \) runs over the square-free numbers.
Lemma 3.6. Let \( a_n, b_n \) be complex numbers such that \( a_n, b_n \ll 1 \). Also, let \( 0 < \epsilon < 1/2 \) and \( k, \ell \geq 0 \) be integers. Suppose \( M \leq T^{(1-\epsilon)/(2k+\ell-1)} \). Write \( n = qf^2 \) where \( q \) is the square-free part of \( n \) and let \( \{X(q)\}_{q} \) be a sequence of independent random variables uniformly distributed on the unit circle. We have for any \( A \geq 1 \) that

\[
\left\langle \left( \sum_{n \leq M} a_n e(2\sqrt{nx}) \right)^k \left( \sum_{n \leq M} b_n e(2\sqrt{nx}) \right) \right\rangle = \mathbb{E} \left( \left( \sum_{qf^2 \leq M} a_{qf^2}(X(q))^f \right)^k \left( \sum_{qf^2 \leq M} b_{qf^2}(X(q))^f \right) \right) + O(T^{-A}),
\]

where the implied constant depends on \( k, \ell, A, \) and \( \epsilon \).

Proof. Integrating term-by-term gives

\[
\int_{\mathbb{R}} \left( \sum_{n \leq M} a_n e(2\sqrt{nx}) \right)^k \left( \sum_{n \leq M} b_n e(2\sqrt{nx}) \right)^\ell \omega \left( \frac{t}{T} \right) dt = \sum_{n_1, \ldots, n_k \leq M} \prod_{s=1}^{k} a_{n_s} \prod_{s=1}^{\ell} b_{m_s} \cdot \bar{\omega} \left( 2\sqrt{T} \left( \sum_{s=1}^{\ell} \sqrt{m_s} - \sum_{r=1}^{k} \sqrt{n_r} \right) \right)
\]

(3.7)

where \( \Sigma_D \) contains the terms where \( \sum_{s=1}^{\ell} \sqrt{m_s} - \sum_{r=1}^{k} \sqrt{n_r} = 0 \) and \( \Sigma_O \) consists of the remaining terms.

For non negative integers \( a \) and \( b \)

\[
\mathbb{E} \left( X(q)^a \overline{X(q)}^b \right) = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{if } a \neq b. \end{cases}
\]

Writing \( n_r = q_r f^2 \) where \( q_r \) is square-free, we have by Lemma 3.4 and the independence of the random variables \( X(q_r) \) that

\[
\Sigma_D = \mathbb{E} \left( \left( \sum_{qf^2 \leq M} a_{qf^2}(X(q))^f \right)^k \left( \sum_{qf^2 \leq M} b_{qf^2}(X(q))^f \right) \right).
\]

Next, note that by Lemma 3.6 for each term in \( \Sigma_O \) we have

\[
\left| \sum_{s=1}^{\ell} \sqrt{m_s} - \sum_{r=1}^{k} \sqrt{n_r} \right| \geq \frac{1}{((k + \ell)\sqrt{M})^{2k+\ell-1-1}}.
\]

Since, by (3.1) \( \bar{\omega} \) decays rapidly and that \( a_n, b_n \ll 1 \) we have for any \( N \geq 1 \)

\[
\Sigma_O \ll \sum_{n_1, \ldots, n_k \leq M} \bar{\omega} \left( 2\sqrt{T} \left( \sum_{s=1}^{\ell} \sqrt{m_s} - \sum_{r=1}^{k} \sqrt{n_r} \right) \right) \ll \left( \frac{((k + \ell)\sqrt{M})^{2k+\ell-1-1}}{\sqrt{T}} \right)^N M^{k+\ell} \ll T^{-A}
\]

by our assumption on \( M \) and since \( N \) is arbitrary. Here the implied constant depends on \( k, \ell, A, \) and \( \epsilon \). \( \square \)
Let
\[ Y(q) = \frac{-2}{q^{3/4} \sqrt{2}} \sum_{f \leq \sqrt{M/q}} \frac{d(qf^2)}{f^{3/2}} e \left( \frac{f \sqrt{q}}{L} - \frac{1}{8} \right) \sin \left( 2\pi \frac{f \sqrt{q}}{L} \right) (X(q))^f. \]

**Lemma 3.7.** Suppose \( m \geq 2 \). For any \( \delta > 0 \) we have
\[
\mathbb{E} \left( \left| \text{Im} Y(q) \right|^m \right) \ll \begin{cases} 
\frac{1}{q^{m/4-m\delta}} & \text{if } q > L^2 \\
\frac{1}{q} \frac{1}{L^{m+1-2m\delta}} & \text{if } q \leq L^2,
\end{cases}
\]
where the implied constant depends on \( \delta \) and \( m \).

**Proof.** Applying the bound \( d(n) \ll n^\delta \) for any \( \delta > 0 \), we have
\[ Y(q) \ll \frac{1}{q^{3/4-\delta}}, \]
The claimed estimate for \( q > L^2 \) follows.

We now assume \( q \leq L^2 \). It suffices to bound the mixed moments
\[ \mathbb{E} \left( Y(q)^k Y(q)^\ell \right), \]
with \( k + \ell = m \). Note that if either \( k \) or \( \ell \) equals zero then the expectation also equals zero.

Next, consider the case where both \( k \) and \( \ell \) are positive, we have
\[
\mathbb{E} \left( Y(q)^k Y(q)^\ell \right) = \frac{(-1)^{m-2m^2}}{q^{3m/4m^m}} \sum_{f_1, \ldots, f_m \leq \sqrt{M/q}} \prod_{j=1}^m \frac{d(qf_j^2)}{f_j^{3/2}} e \left( \frac{f_j \sqrt{q}}{L} - \frac{1}{8} \right) \sin \left( 2\pi f_j \frac{\sqrt{q}}{L} \right).
\]

Let \( \epsilon_r = 1 \) for \( r = 1, \ldots, k \) and \( \epsilon_r = -1 \) for \( r = k+1, \ldots, m-1 \). By the last condition on the sum we have \( f_m = \sum_r \epsilon_r f_r \). Since we also have \( f_m, f_1 \geq 1 \) it follows that \( f_1 \geq 1 + \max(0, -\sum_{2 \leq j \leq m-1} \epsilon_j f_j) \). Let \( f = (f_2, \ldots, f_m) \in \mathbb{Z}^{m-2} \), and write \( g(f) = \max(0, -\sum_{2 \leq j \leq m-1} \epsilon_j f_j) \). Also, apply the bound \( d(n) \ll n^\delta \). Thus,
\[
\mathbb{E} \left( Y(q)^k Y(q)^\ell \right) \ll \frac{1}{q^{3m/4-m\delta}} \sum_{f_2, \ldots, f_m \geq 1} \sum_{f_1 \geq 1+g(f)} \left| \text{sin} \left( 2\pi \left( \sum_r \epsilon_r x_r \right) \frac{\sqrt{q}}{L} \right) \prod_{j=1}^{m-1} \text{sin} \left( 2\pi x_j \frac{\sqrt{q}}{L} \right) \right| dx_1 dx_2 \ldots dx_{m-1}.
\]

Since \( q \leq L^2 \) the right-hand side is
\[
\ll \frac{1}{q^{3m/4-m\delta}} \int_1^\infty \ldots \int_1^\infty \int_{1+g(x)}^\infty \left| \text{sin} \left( 2\pi \left( \sum_r \epsilon_r x_r \right) \frac{\sqrt{q}}{L} \right) \prod_{j=1}^{m-1} \text{sin} \left( 2\pi x_j \frac{\sqrt{q}}{L} \right) \right| dx_1 dx_2 \ldots dx_{m-1}.
\]

Next, make the change of variables \( u_j = x_j \frac{\sqrt{q}}{L} \) to see that
\[
\mathbb{E} \left( Y(q)^k Y(q)^\ell \right) \ll \frac{1}{q^{(m-1)/2 L^{m+1-2m\delta}}} \int_{\mathbb{R}}^\infty \ldots \int_{\mathbb{R}}^\infty \int_{\mathbb{R}^{m-1}+g(u)}^\infty \prod_{j=1}^{m-1} \left| \text{sin} \left( 2\pi u_j \right) \right|\]
\[
\times \left| \text{sin} \left( 2\pi \left( \sum_r \epsilon_r u_r \right) \right) \right| du_1 du_2 \ldots du_{m-1}.
\]
The multiple integral is $O(1)$. Hence, we have for $q \leq L^2$ that
\[
\mathbb{E}\left(Y(q)^k Y(q)^\ell\right) \ll \frac{1}{q^{\frac{(m-1)}{2}} L^{m+1-2m\delta}}.
\]

**Proof of Proposition 3.1.** By Lemma 3.6 it suffices to estimate

\[
\mathbb{E}\left(\left(\frac{1}{\sigma_M} \cdot \sum_{q \leq M} \text{Im} Y(q)\right)^m\right) = \frac{1}{\sigma_M} \cdot \sum_{q_1, \ldots, q_m \leq M} \mathbb{E}\left(\prod_{j=1}^{m} \text{Im} Y(q_j)\right).
\]

We analyze this sum in the following way. Consider a division of $\{1, \ldots, m\}$ into nonempty disjoint subsets $S_1, \ldots, S_n$ with cardinalities $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 + \cdots + \alpha_n = m$. Then such a division, look at the contribution of the terms in the above sum over $q_1, \ldots, q_m$ such that $q_a = q_b$ if $a, b \in S_j$ for some $j$ and $q_a \neq q_b$ if $a \in S_j$ and $b \in S_i$ with $i \neq j$. Since the random variables $Y(q)$ are independent the sum of these terms equals

\[
\sum_{r_1, \ldots, r_n \leq M} \prod_{j=1}^{n} \mathbb{E}\left(\left(\frac{1}{\sigma_M} \text{Im} Y(r_j)\right)^{\alpha_j}\right).
\]

Thus,

\[
\frac{1}{\sigma_M} \cdot \sum_{q_1, \ldots, q_m \leq M} \mathbb{E}\left(\prod_{j=1}^{m} \text{Im} Y(q_j)\right) = \sum_{n=1}^{m} \sum_{\alpha_j} \frac{m!}{\alpha_1! \cdots \alpha_n!} \cdot \frac{1}{n!} \sum_{r_1, \ldots, r_n \leq M} \prod_{j=1}^{n} \mathbb{E}\left(\left(\frac{1}{\sigma_M} \text{Im} Y(r_j)\right)^{\alpha_j}\right),
\]

where $\sum_{\alpha_j}$ runs over all $n$-tuples of positive integers $(\alpha_1, \ldots, \alpha_n)$ such that $\alpha_1 + \cdots + \alpha_n = m$.

Next note that inside the inner sum in (3.8) if $\alpha_j = 1$ for some $j$ then this term vanishes. Additionally, if $\ell \geq 3$ we have for any $\varepsilon > 0$ that

\[
\mathbb{E}\left(\sum_{q \leq M} \left(\frac{1}{\sigma_M} \text{Im} Y(q)\right)^\ell\right) \ll \frac{1}{L^{1-\varepsilon}}
\]

by Lemma 3.7. Thus, by Lemma 3.7, each term in the inner sum on the right-hand side of (3.8) with $\alpha_j \geq 3$ for some $j$ is $\ll 1/L^{1-\varepsilon}$. The contribution of all such terms is also $\ll 1/L^{1-\varepsilon}$.

The remaining terms have $\alpha_1 = \cdots = \alpha_n = 2$. Since $\alpha_1 + \cdots + \alpha_n = m$ we have that $m$ is even and $n = m/2$. Thus, the sum of these terms equals

\[
\frac{m!}{(2!)^n} \cdot \frac{1}{n!} \sum_{r_1, \ldots, r_n \leq M} \prod_{j=1}^{n} \mathbb{E}\left(\left(\frac{1}{\sigma_M} \text{Im} Y(r_j)\right)^2\right) = \frac{m!}{2^{m/2} (m/2)!} \sum_{r_1, \ldots, r_n \leq M} \prod_{j=1}^{n} \mathbb{E}\left(\left(\frac{1}{\sigma_M} \text{Im} Y(r_j)\right)^2\right).
\]

To complete the proof we estimate the sum on the right-hand side. Note by Lemma 3.7 that

\[
0 \leq \mathbb{E}\left(\left(\frac{1}{\sigma_M} \text{Im} Y(q)\right)^2\right) \leq \begin{cases} \frac{1}{L^{2-\varepsilon}} & \text{if } q > L^2 \\ \frac{1}{q^{1/2}L^{1-\varepsilon}} & \text{if } q \leq L^2. \end{cases}
\]

□
Thus,

\[
\left| \left( \mathbb{E} \left( \sum_{q \leq M} \left( \frac{1}{\sigma_M} \text{Im} Y(q) \right)^2 \right) \right)^n - \sum_{r_1, \ldots, r_n \leq M, r_j \neq r_{n-1}, \ldots, r_1} \prod_{j=1}^n \mathbb{E} \left( \left( \frac{1}{\sigma_M} \text{Im} Y(r_j) \right)^2 \right) \right| \leq \frac{1}{L^{1-\varepsilon}} \mathbb{E} \left( \sum_{q \leq M} \left( \frac{1}{\sigma_M} \text{Im} Y(q) \right)^2 \right)^{n-1}.
\]

Iterating this argument, we see that

\[
\left| \left( \mathbb{E} \left( \sum_{q \leq M} \left( \frac{1}{\sigma_M} \text{Im} Y(q) \right)^2 \right) \right)^n - \sum_{r_1, \ldots, r_n \leq M, r_j \text{ distinct}} \prod_{j=1}^n \mathbb{E} \left( \left( \frac{1}{\sigma_M} \text{Im} Y(r_j) \right)^2 \right) \right| \leq \frac{1}{L^{1-\varepsilon}} \mathbb{E} \left( \sum_{q \leq M} \left( \frac{1}{\sigma_M} \text{Im} Y(q) \right)^2 \right)^{n-1}.
\]

Recalling the definition of \( \sigma_M \), this gives

\[
\sum_{r_1, \ldots, r_n \leq M, r_j \text{ distinct}} \prod_{j=1}^n \mathbb{E} \left( \left( \frac{1}{\sigma_M} \text{Im} Y(r_j) \right)^2 \right) = 1 + O(1/L^{1-\varepsilon}).
\]

Collecting estimates, we have

\[
\mathbb{E} \left( \left( \frac{1}{\sigma_M} \cdot \sum_{q \leq M} \text{Im} Y(q) \right)^m \right) = \begin{cases} \frac{m!}{2^{m/2}(m/2)!} + O \left( \frac{1}{L^{1-\varepsilon}} \right) & \text{if } m \text{ is even}, \\ O \left( \frac{1}{L^{1-\varepsilon}} \right) & \text{if } m \text{ is odd}. \end{cases}
\]

\[\square\]

3.3. The Proof of Theorem 1.3

Let

\[
\mathbb{P}_{\omega,T} \left( f(x) \in [\alpha, \beta] \right) = \int_{\mathbb{R}} 1_{[\alpha, \beta]}(f(x)) \omega \left( \frac{x}{T} \right) \frac{dx}{T},
\]

where \( \omega \geq 0 \) is a Schwartz function supported on the positive real numbers with unit mass. By Proposition 3.1 it follows that for \( M \) such that \( M \ll T^\varepsilon \) for all \( \varepsilon > 0 \) as \( T \to \infty \)

\[
\mathbb{P}_{\omega,T} \left( \frac{1}{\sigma_M} S(x, L, M) \in [\alpha, \beta] \right) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt + o(1).
\]

Next note that by (3.5) we have \( \sigma_M^2 = \sigma^2 + o(\sigma^2) \) if \( L/\sqrt{M} \to 0 \). Thus, it follows that

\[
\mathbb{P}_{\omega,T} \left( \frac{1}{\sigma} S(t, L, M) \in [\alpha, \beta] \right) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt + o(1).
\]

By Lemma 3.3 we have for any fixed \( \varepsilon > 0 \) that

\[
\mathbb{P}_{\omega,T} \left( \frac{1}{\sigma} \left( S(x, L) - S(x, L, M) \right) > \varepsilon \right) \leq \frac{1}{\varepsilon^2 \sigma^2} \left( \left( S(x, L) - S(x, L, M) \right)^2 \right) = o(1)
\]

as \( T \to \infty \). Thus, since \( \varepsilon > 0 \) is arbitrary we have as \( T \to \infty \)

\[
\mathbb{P}_{\omega,T} \left( \frac{1}{\sigma} S(x, L) \in [\alpha, \beta] \right) = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-t^2/2} dt + o(1).
\]
Let \( \varepsilon > 0 \). We now choose \( \omega \) such that \( \omega(x) = 1 \) for \( x \in [1 + \varepsilon, 2 - \varepsilon] \). Since, \( \omega \geq 0 \) and has unit mass we conclude that

\[
\int_{\mathbb{R}} 1_{[\alpha, \beta]} \left( \frac{1}{\sigma} S(x, L, M) \right) \left( 1_{[1, 2]} \left( \frac{x}{T} \right) - \omega \left( \frac{x}{T} \right) \right) \frac{d\sigma}{T} \leq \left( \int_{-\infty}^{1+\varepsilon} + \int_{2-\varepsilon}^{\infty} \right) \left( 1_{[1, 2]}(x) + \omega(x) \right) dx \ll \varepsilon.
\]

Hence, as \( T \to \infty \)

\[
\frac{1}{T} \operatorname{meas} \left\{ x \in [T, 2T] : \frac{1}{\sigma} S(x, L) \in [\alpha, \beta] \right\} = \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\beta} e^{-x^2/2} dx + o(1).
\]

To complete the proof first note that for \( 2 < h < x \) we have \( \Delta(x + h)/(x + h)^{1/4} = \Delta(x + h)/x^{1/4} + O(h/x^{3/4}) \), since \( \Delta(x) = O(x^{1/2}) \). Next, observe that for \( 0 < h \ll 1 \) we have \( \Delta(x + h) = \Delta(x) + O(h \log x) + O(x^{\delta}) \) for any \( \delta > 0 \), by the bound \( d(n) \ll n^{\delta} \). Thus, as \( T \to \infty \)

\[
\frac{1}{\sigma} S(x, 2L) = \frac{\Delta \left( x + \frac{x}{T} \right) - \Delta(x)}{x^{1/4} \sqrt{\frac{8 \log^3 L}{L}}} \left( 1 + o(1) \right) + o(1).
\]

### 3.4. The proof of Theorem 1.4

Let \( f \) be a primitive cusp form of even weight \( k \) and level 1 with the previous notation. By Deligne’s bound we have \( |\rho_f(n)| \leq d(n) \); the corresponding Dirichlet series

\[
\varphi(s) = \sum_{n=1}^{\infty} \frac{\rho_f(n)}{n^s}
\]

is absolutely convergent in the strip \( \Re(s) > 1 \), and the classical theory shows that it has an analytic continuation to the whole complex plane, with the functional equation

\[
\varphi(s) = \chi(s) \varphi(1-s),
\]

where

\[
\chi(s) = (-1)^{k/2} \frac{(2\pi)^{2s-1}}{\Gamma(s + \frac{k+1}{2})} \frac{\Gamma \left( \frac{k+1}{2} - s \right)}{\Gamma \left( s + \frac{k+1}{2} \right)}.
\]

We will use the following analog of formula (12.4.4) in [19], which is a special case of Theorem 1.2 in [4]:

**Theorem 3.1** (Theorem 1.2 of [4] for \( a(n) = \rho_f(n) \)). For \( x \geq 1 \), \( N \leq x \) and \( \varepsilon > 0 \),

\[
\sum_{n \leq x} \rho_f(n) = \frac{x^{1/4}}{\pi \sqrt{2}} \sum_{n \leq N} \rho_f(n) n^{-3/4} \cos \left( 4\pi \sqrt{\frac{n x}{N} - \frac{\pi}{4}} \right) + O_{\varepsilon,k} \left( \frac{x^{1/2+\varepsilon}}{\sqrt{N}} \right).
\]

We can now deduce the analogous results regarding the distribution of the sum of the Hecke eigenvalues \( \rho_f(n) \) in short intervals: recall that

\[
A_f(x) = \sum_{n \leq x} \rho_f(n),
\]

and define \( F_f(x) = x^{-1/4} A_f(x) \), \( S_f(x, L) = F_f \left( \left( \sqrt{x} + \frac{1}{L} \right)^2 \right) - F_f(x) \), with the condition that \( L \ll T^{\varepsilon} \) for all \( \varepsilon > 0 \). Note that the formula in Theorem 3.1 and the classical formula (12.4.4) in
[19] are the same except for the coefficients $\rho_f(n)$ which replace $d(n)$. The calculation for the expectation of $S_f(x, L)$ goes line by line as in the divisor’s case (i.e. $\langle S_f(x, L) \rangle \ll T^{-1/4+\varepsilon}$), and so is the calculation for the variance, until the part which uses formula (2.2) which is now replaced by Rankin’s result (2.5) We conclude that

$$\langle S_f(x, L) \rangle \sim \frac{1}{\pi^2} \sum_{n \leq T^{1-\delta}} \frac{\rho_f^2(n)}{n^{3/2}} \sin^2 \left( \frac{2\pi \sqrt{n}}{L} \right)$$

$$\sim \frac{c_f}{\pi^2} \int_1^{T^{1-\delta}} \frac{1}{x^{3/2}} \sin^2 \left( \frac{2\pi \sqrt{x}}{L} \right) \, dx$$

$$= \frac{2c_f}{\pi^2 L} \int_{1/L}^{T^{1/2-\delta/2}/L} \frac{1}{y^2} \sin^2(2\pi y) \, dy$$

$$\sim \frac{2c_f}{\pi^2 L} \int_0^\infty \frac{1}{y^2} \sin^2(2\pi y) \, dy = \frac{2c_f}{L}.$$ 

Denote by

$$\sigma_f^2 = \frac{2c_f}{L}$$

the variance we calculated.

Again, for any $M = O(T^{1-\delta})$ such that $L/\sqrt{M} \to 0$, we define the “short” approximation to $S_f(x, L)$ by

$$S_f(x, L, M) = \frac{-2}{\pi \sqrt{2}} \sum_{n \leq M} \frac{\rho_f(n)}{n^{3/4}} \sin \left( \frac{2\pi \sqrt{n}}{L} \right) \sin \left( 4\pi \sqrt{n} \left( \sqrt{x} + \frac{1}{2L} \right) - \frac{\pi}{4} \right).$$

By similar considerations $\langle S_f(x, L, M)^2 \rangle \sim \sigma_f^2$; the proof that $\left\langle \left( \frac{S_f(x, L) - S_f(x, L, M)}{\sigma_f} \right)^2 \right\rangle \to 0$ as $T \to \infty$ is again similar, and so is the proof of the rest of the analogous claims, including the calculation of the higher moments, which does not use any special property of the divisor function except for the property $d(n) \ll n^\varepsilon$, which is also true for $\rho_f(n)$ by Deligne’s bound. We conclude that $\frac{S_f(x, L)}{\sigma_f}$ has a standard Gaussian limiting distribution as $T \to \infty$, from which Theorem [1.4] follows in the same manner as before (the bound $A_f(x) \ll x^{1/2}$ easily follows from Theorem [3.1]).

**Acknowledgments**

We would like to thank Zeév Rudnick for suggesting this problem as well as for numerous helpful discussions and remarks.

**Funding**

The research leading to these results has received funding from the European Research Council under the European Union’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement n° 320755.
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