On the strength of connectedness of a random hypergraph

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Abstract

Bollobás and Thomason (1985) proved that for each $k = k(n) \in [1, n - 1]$, with high probability, the random graph process, where edges are added to $V = [n]$ uniformly at random one after another, is such that the stopping time of having minimal degree $k$ is equal to the stopping time of becoming $k$-(vertex)connected. We extend this result to the $d$-uniform random hypergraph process, where $k$ and $d$ are fixed. Consequently, for $m = \frac{2}{d} \left( \ln n + (k - 1) \ln \ln n + c \right)$ and $p = \left( d - 1 \right)^{\frac{1}{n} n + (k - 1) \ln \ln n + c}$, the probability that the random hypergraph models $H_d(n, m)$ and $H_d(n, p)$ are $k$-connected tends to $e^{-e^{-c}/(k - 1)!}$.

**Keywords:** random hypergraph; vertex connectivity

1 Introduction

Let $H_d(n, p)$ denote the random $d$-uniform hypergraph with vertex set $[n] := \{1, 2, ..., n\}$, where each of the $\binom{n}{d}$ potential (hyper)edges of cardinality $d$ is present with probability $p$, independently of all other potential edges. Likewise, let $H_d(n, m)$ be the random $d$-uniform hypergraph on $[n]$, where $m$ edges are chosen uniformly at random among all sets of $m$ potential edges. The model $H_d(n, m)$ can be gainfully viewed as a snapshot of the random hypergraph process $\{H_d(n, \mu)\}_{\mu \geq 0}$, where $H_d(n, \mu + 1)$ is obtained from $H_d(n, \mu)$ by inserting an extra edge chosen uniformly at random among all $\binom{n}{d} - \mu$ remaining potential edges. For $d = 2$, these models are the typical random graph models, $G(n, p)$, $G(n, m)$ and $\{G(n, \mu)\}_{\mu \geq 0}$.

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As customary, we say that for a given \( m = m(n) \) (\( p \) resp.) some graph property \( \mathcal{Q} \) holds with high probability, denoted w.h.p., if the probability that \( H_d(n, m) \) (\( (H_d(n, p) \) resp.) has property \( \mathcal{Q} \) tends to 1 as \( n \to \infty \). Further, \( m(n) \) is the sharp threshold for \( \mathcal{Q} \) if for each \( \epsilon > 0 \), w.h.p. \( H_d(n, (1 - \epsilon)m) \) does not have \( \mathcal{Q} \) and w.h.p \( H_d(n, (1 + \epsilon)m) \) does have \( \mathcal{Q} \). For the random hypergraph process, the stopping time of \( \mathcal{Q} \), denoted \( \tau(\mathcal{Q}) \) is the first moment that the process has this property \( \mathcal{Q} \); we denote the hypergraph process stopped at this time by \( H_d(n, \tau(\mathcal{Q})) \).

In one of the first papers on random graphs, Erdős and Rényi \cite{erdos_renyi_1960} showed that \( m = \frac{1}{2} \ln n \) is a sharp threshold for connectivity in \( G(n, m) \). Later, Stepanov \cite{stepanov_1979} established the sharp threshold of connectivity for \( G(n, p) \) among other results. More recently, Bollobás and Thomason \cite{bollobas_thomason_1992} proved the stronger result for the random graph process that w.h.p. at the moment the graph process loses its last isolated vertex is also the moment that the process becomes connected; in other words, w.h.p. \( \tau(\text{no isolated vertices}) = \tau(\text{connected}) \).

We prove an analogous result for the the random \( d \)-uniform hypergraph process; a consequence of this result is that \( m = \frac{n}{2} \ln n \) is a sharp threshold of connectivity for \( H_d(n, m) \).

There are various measures for the strength of connectedness of a connected graph, but here we will focus on \( k \)-(vertex)connectivity. For \( k \in \mathbb{N} \), a hypergraph with more than \( k \) vertices is \( k \)-connected if whenever \( k - 1 \) vertices are deleted, along with their incident edges, the remaining hypergraph is connected. Note that the definition of 1-connectedness coincides with connectedness. Necessarily, for a hypergraph to be \( k \)-connected, each vertex must have degree at least \( k \); since, if a vertex has degree less than \( k \), then a neighbor from each incident edge can be deleted which isolates the original vertex. However, as commonly seen in these types of results, the main barrier to \( k \)-connectivity in these random graph models arises from such vertices that can be separated from the rest of the graph by the deletion of their neighbors (see for instance Erdős-Rényi \cite{erdos_renyi_1959}, Ivchenko \cite{ivchenko_1960}, Bollobás \cite{bollobas_1993}, \cite{bollobas_1998}). Here, we extend this idea to random \( d \)-uniform hypergraphs; in particular, we find that if \( m_0 = \frac{n}{d} (\ln n + (k - 1) \ln \ln n - \omega) \) and \( m_1 = \frac{n}{d} (\ln n + (k - 1) \ln \ln n + \omega) \), \( \omega \to \infty \), then w.h.p. \( H_d(n, m_0) \) is not \( k \)-connected and w.h.p. \( H_d(n, m_1) \) is \( k \)-connected; also we find an analogous threshold value for \( H_d(n, p) \).

A stronger result concerns the random graph process where edges are added one after another. Let \( \tau_k := \tau(\min \text{degree at least } k) \) and \( T_k := \tau(\text{k-connected}) \); note that \( \tau_k \leq T_k \).

In \cite{bollobas_thomason_1992}, Bollobás and Thomason showed that for \( d = 2 \) (the graph case) and any \( k = k(n) \in [1, n - 1] \), \( P(\tau_k = T_k) \to 1 \). We extend this result for \( d \)-uniform random hypergraphs albeit for fixed \( d \) and \( k \).

**Theorem 1.1.** W.h.p, at the moment the \( d \)-uniform hypergraph process loses its last vertex with degree less than \( k \), this process becomes \( k \)-connected. Formally, for \( d \geq 3, k \geq 1 \) (both fixed), \( P(\tau_k = T_k) \to 1 \).

To prove this result, we begin by determining the likely range of \( \tau_k \), and further that just prior to this window, at some \( m_0 \) edges, w.h.p. there are not many vertices of degree less than \( k \). Then, we prove that w.h.p. \( H_d(n, m_0) \) is almost \( k \)-connected in the sense that whenever \( k - 1 \) vertices are deleted, there is a massive component using almost all leftover vertices. Third, we show that w.h.p. to isolate a vertex of \( H_d(n, \tau_k) \), you would
have to delete at least \( k \) of its neighbors (this is trivially true for graphs, but not so for \( d \geq 3 \)). Finally, we show that the probability that \( \tau_k < T_k \), but these three previous likely events also hold tends to zero, which completes the proof of the theorem. The following corollary is nearly immediate in light of the theorem.

**Corollary 1.2.** (i) Let \( m = \frac{n}{d} \left( \ln n + (k - 1) \ln \ln n + c_n \right) \), where \( c_n \to c \in \mathbb{R} \). W.h.p. \( H_d(n, m) \) is \((k - 1)\)-connected, but not \((k + 1)\)-connected. Further, the probability that \( H_d(n, m) \) is \( k \)-connected tends to \( e^{-e^{-c/(k-1)!}} \).

(ii) Let \( p = (d - 1)!^{\ln n + (k - 1) \ln \ln n + c_n} n^k \), where \( c_n \to c \in \mathbb{R} \). W.h.p. \( H_d(n, p) \) is \((k - 1)\)-connected, but not \((k + 1)\)-connected. Further, the probability that \( H_d(n, p) \) is \( k \)-connected tends to \( e^{-e^{-c/(k-1)!}} \).

For the remainder of this paper, let \( d \geq 3 \) and \( k \geq 1 \) be fixed numbers.

### 2 Likely range of \( \tau_k \)

**Lemma 2.1.** Let \( \omega = \omega(n) \to \infty \), but \( \omega = o(\ln \ln n) \), \( m_0 = \frac{n}{d} \left( \ln n + (k - 1) \ln \ln n - \omega \right) \) and \( m_1 = \frac{n}{d} \left( \ln n + (k - 1) \ln \ln n + \omega \right) \). Then w.h.p.,

(i) minimum degree of \( H_d(n, m_0) \) is \( k - 1 \) and the number of vertices with degree \( k - 1 \) is in the interval

\[
\left[ \frac{1}{2} \frac{e^\omega}{(k - 1)!}, \frac{3}{2} \frac{e^\omega}{(k - 1)!} \right].
\]

(ii) there are no vertices of degree \( k - 1 \) in \( H_d(n, m_1) \).

Consequently, w.h.p. \( \tau_k \in [m_0, m_1] \).

**Proof.** We prove that the number of vertices with degree \( k - 1 \), denoted by \( X \), is in the interval (1) by Chebyshev’s Inequality. Note that a given vertex can be \( \binom{n-1}{d-1} \) possible edges, so

\[
E[X] = nP[\deg(1) = k - 1] = n \frac{\binom{n-1}{d-1} \binom{n}{d} - \binom{n-1}{d-1}}{\binom{n}{m_0} \binom{n}{m_0 - k + 1}}.
\]

Here and elsewhere in this paper, we use the identity \( \binom{N}{m-\ell} = \binom{N}{m} \binom{m}{N-m+\ell} \), where \((j)_{\ell} = j(j - 1) \cdots (j - \ell + 1)\), and later, we use the inequality \( \binom{N}{m-\ell} \leq \binom{N}{m} \binom{m}{N-m}^{\ell} \). Now

\[
E[X] = (1 + O(1/n)) \frac{n \cdot n^{(d-1)(k-1)}}{(k-1)!((d-1)!)^{k-1}} \frac{\binom{d!}{m_0} \binom{n}{m_0} \binom{n-1}{m_0}}{\binom{n}{m_0} \binom{n}{m_0} \binom{n}{m_0} \binom{n}{m_0}}.
\]
This latter fraction can be sharply approximated.

\[
\frac{\binom{n}{d} - \binom{n-1}{d-1}}{\binom{m_0}{d}} = \prod_{i=0}^{m_0-1} \left( 1 - \frac{\binom{n-1}{d-1}}{\binom{n}{d} - i} \right) = \prod_{i=0}^{m_0-1} \left( 1 - \frac{d}{n} + O \left( \frac{i}{n^{d+1}} \right) \right)
\]

\[
= \exp \left( \sum_{i=0}^{m_0-1} -d \frac{1}{n} + O \left( \frac{1}{n^2} \right) + O \left( \frac{i}{n^{d+1}} \right) \right)
\]

\[
= \left( 1 + O \left( \frac{\ln n}{n} \right) \right) \exp \left( -d m_0 \frac{1}{n} \right) = \left( 1 + O \left( \frac{\ln n}{n} \right) \right) \frac{e^\omega}{n(n \ln n)^{k-1}}.
\]

(2)

Hence

\[
E[X] = \left( 1 + O \left( \frac{\ln n}{n} \right) \right) \frac{e^\omega}{(k-1)!} \left( \frac{dm_0/n}{\ln n} \right)^{k-1}
\]

\[
= \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right) \frac{e^\omega}{(k-1)!}.
\]

For the second factorial moment, we have that

\[
E[X(X - 1)] = n(n-1)P(\deg(1) = \deg(2) = k - 1).
\]

We break this latter probability, over \(i\), the number of edges that include both vertices 1 and 2. In particular, vertex 1 is in \(k - 1 - i\) edges that do not contain vertex 2 and vice versa; further there are \(m_0 - 2(k-1) + i\) edges that include neither vertex 1 or 2. So

\[
P(\deg(1) = \deg(2) = k - 1) = \sum_{i=0}^{k-1} \binom{n-2}{d-2} \binom{n-1}{d-1} \binom{n-2}{d-2} \binom{m_0-2(k-1)+i}{m_0} \frac{\binom{n}{d}}{\binom{m}{d}}.
\]

(3)

One can show that the main contribution to this probability comes from the case where these two vertices are not adjacent \((i = 0)\); namely, using similar methods that we used for the asymptotics of \(E[X]\), the sum of the terms of (3) over \(i \in [1, k - 1]\), is \(O(n^{-3})\). Therefore

\[
P(\deg(1) = \deg(2) = k - 1) = \left( \frac{\binom{n-1}{d-1} - \binom{n-2}{d-2}}{k-1} \right)^2 \frac{\binom{n}{d} \binom{m_0-2(k-1)+i}{m_0}}{\binom{m}{d}} + O(n^{-3})
\]

\[
= \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right) \frac{e^{2\omega}}{(n \ln n)^2};
\]

whence

\[
E[X(X - 1)] = \left( 1 + O \left( \frac{\ln \ln n}{\ln n} \right) \right) E[X]^2.
\]

Consequently,

\[
\text{var}[X] = E[X] + O \left( \frac{E[X]^2 \ln \ln n}{\ln n} \right).
\]
By Chebyshev’s Inequality, $X$ is concentrated around its mean and in particular, w.h.p. $X$ is in the interval (1). To finish the proof of part (i), it remains to show that w.h.p. there are no vertices of degree less than $k - 1$, which can be done by a first moment argument using similar techniques to the asymptotics of $E[X]$. Similarly, for part (ii), one can easily show that the expected number of vertices of degree $k - 1$ in $H_d(n, m_1)$ tends to zero as well.

3 $H_d(n, m_0)$ is almost $k$-connected

Now we will establish that w.h.p. $H_d(n, m_0)$ is almost $k$-connected in the sense that if $k - 1$ vertices are deleted, then there remains a massive component containing almost all left-over vertices. To this end, we prove an analogous statement for the random Bernoulli hypergraph $H_d(n, p)$ and use a standard conversion lemma to obtain the desired result for $H_d(n, m_0)$.

Lemma 3.1. Let $m_0 = \frac{n}{d} (\ln n + (k - 1) \ln \ln n - \ln \ln \ln n)$ and $p = m_0 / \binom{n}{d}$. With high probability,

(i) $H_d(n, p)$ has the property “whichever $k - 1$ vertices are deleted, there remains a giant component which includes all but up to $\ln n$ leftover vertices.”

(ii) $H_d(n, m_0)$ has the property “whichever $k - 1$ vertices are deleted, there remains a giant component which includes all but up to $\ln n$ leftover vertices.”

Proof. (i) Given a set of $k - 1$ vertices, $v = \{v_1, \ldots, v_{k-1}\}$, let $F(v)$ be the event that if the vertices $v$ are deleted from $H_d(n, p)$ along with their incident edges, then there remains no components of size at least $n - (k - 1) - \ln n$. In particular, we wish to show that w.h.p. $H_d(n, p)$ is not in $F(v)$ for any $v$. Using the union bound over all $k - 1$ element sets of $[n]$ as well as symmetry, we find that

$$P \left( \bigcup_v F(v) \right) \leq \binom{n}{k-1} P(F(v^*)),$$

where $v^* = \{n - (k - 1) + 1, \ldots, n - 1, n\}$. Note that the remaining hypergraph left after deleting $v^*$ from $H_d(n, p)$ is distributed as $H_d(n', p)$, $n' := n - (k - 1)$ (this is the primary reason that we consider the Bernoulli hypergraph $H_d(n, p)$ rather than $H_d(n, m)$). Therefore $P(F(v^*))$ is precisely the probability that $H_d(n', p)$ does not have a component of size at least $n' - \ln n$.

To bound $P(F(v^*))$, we note that any hypergraph on $n'$ vertices without a component of size $n' - \ln n$ has a set of vertices $S$ such that there are no edges between $S$ and $[n'] \setminus S$ where $|S| \in [(\ln n)/2, n' - (\ln n)/2]$. To see this fact, consider a hypergraph $H$ on $n'$ vertices without such a large component and let $L_1, \ldots, L_\ell$ be the vertex sets of the components of $H$. Then there is some minimal $j$ so that

$$\frac{\ln n}{2} \leq | \bigcup_{i=1}^j L_i | < \frac{\ln n}{2} + n' - \ln n = n' - \ln n.$$
Further, there are no edges including a vertex of \( S := \cup_{i=1}^{\ell} L_i \) and \([n'] \setminus S\).

Therefore,

\[
P(\mathcal{F}(\mathbf{v}^*)) \leq \sum_{s=(\ln n)/2}^{n'-(\ln n)/2} P(\exists S \subset [n'], |S| = s, \text{ no edge between } S \text{ and } [n'] \setminus S),
\]

and by symmetry over all such vertex sets \( S \),

\[
P(\mathcal{F}(\mathbf{v}^*)) \leq \sum_{s=(\ln n)/2}^{n'-(\ln n)/2} \binom{n'}{s} P(\text{no edge between } [s] \text{ and } [n'] \setminus [s]).
\]

Further, this latter probability is symmetric about \( s = n'/2 \) (i.e. the probabilities corresponding to \( s \) and \( n' - s \) are equal). Hence

\[
P(\mathcal{F}(\mathbf{v}^*)) \leq 2 \sum_{s=(\ln n)/2}^{[n'/2]} \binom{n'}{s} P(\text{no edge between } [s] \text{ and } [n'] \setminus [s]).
\]

The number of potential edges that contain at least one vertex from \([s]\) and at least one vertex from \([n'] \setminus [s]\) is \( \binom{n'}{d} - \binom{d}{d} - \binom{n' - s}{d} \). Hence

\[
P(\mathcal{F}(\mathbf{v}^*)) \leq 2 \sum_{s=(\ln n)/2}^{[n'/2]} \binom{n'}{s} (1 - p)^{\binom{n'}{d} - \binom{d}{d} - \binom{n' - s}{d}} \leq 2 \sum_{s=(\ln n)/2}^{[n'/2]} \binom{n'}{s} e^{-p\left(\binom{n'}{d} - \binom{d}{d} - \binom{n' - s}{d}\right)}
\]

\[
=: 2E_1 + 2E_2,
\]

where \( E_1 \) and \( E_2 \) are the sums over \( S_1 := [(\ln n)/2, n/(\ln n)] \) and \( S_2 := (n/(\ln n), [n'/2]) \) respectively. We begin with analyzing \( E_2 \) since these bounds will be cruder and simpler.

Trivially,

\[
E_2 \leq \sum_{s \in S_2} \binom{n'}{s} e^{\max -p\left(\binom{n'}{d} - \binom{d}{d} - \binom{n' - s}{d}\right)} \leq 2n' e^{\max -p\left(\binom{n'}{d} - \binom{d}{d} - \binom{n' - s}{d}\right)} = 2n' e^{b_n}, \tag{5}
\]

where this maximum is over \( s \in S_2 \). Now let’s take on these binomial coefficient terms. Trivially \( \binom{n}{d} \leq \frac{d}{d} \), and by Bernoulli’s Inequality,

\[
\frac{\nu^d - d^2 \nu^{d-1}}{d!} \leq \left(\frac{\nu - d}{d}\right)^d \leq \binom{\nu}{d}.
\]

Therefore

\[
b_n \leq \max_{s \in S_2} \frac{-p}{d!} \left( (n')^d - d^2 (n')^{d-1} - s^d - (n' - s)^d \right).
\]

Further the function \( f(x) = x^d + (n' - x)^d \) is decreasing for \( x \in S_2 \), so we have that

\[
b_n \leq -\frac{p}{d!} \left( (n')^d - d^2 (n')^{d-1} - \left(\frac{n}{\ln n}\right)^d - (n' - \frac{n}{\ln n})^d \right).
\]
Therefore,
\[ b_n \leq -\frac{p}{d!} (n')^d \left( 1 - \left( 1 - \frac{n}{n'} \ln n \right)^d \right) + O(p(n/\ln n)^d) + O(pn^{d-1}) \]
\[ = -\frac{n \ln n}{d} \left( \frac{d}{\ln n} \right) + O(n/\ln n) = -n + O(n/\ln n). \]

Using this latter bound, (5) becomes
\[ E_2 \leq \exp (\ln 2 - 1)n + O(n/\ln n)) \rightarrow 0. \tag{6} \]

Now let’s take on the sum \( E_1 \). We begin with taking the leading order terms in the exponent.
\[ E_1 \leq \sum_{s \in S_1} \binom{n'}{s} \exp \left( -p \left( \frac{n'}{d} \right) \left( 1 - \frac{(n'-s)}{(n')} - O \left( \frac{s^d}{n^2} \right) \right) \right). \tag{7} \]

Uniformly over \( s \in S_1 \), we have that
\[ \frac{(n'-s)}{(n')} = \left( 1 - \frac{s}{n'} + O \left( \frac{s^2}{n^2} \right) \right)^d = 1 - \frac{ds}{n'} + O \left( \frac{s^2}{n^2} \right). \]

Consequently, the exponent in (7) is
\[ p \left( \frac{n'}{d} \right) \left( \frac{ds}{n} + O \left( \frac{s^2}{n^2} \right) \right) = (\ln n + (k - 1) \ln \ln n - \ln \ln \ln n) \left( s + O(s^2/n) \right) + O(1); \]
whence there is some (fixed) \( \gamma > 0 \) such that for all \( s \in S_1 \),
\[ p \left( \frac{n'}{d} \right) \left( \frac{ds}{n} + O \left( \frac{s^2}{n^2} \right) \right) \geq (\ln n - \ln \ln \ln n)(s - \gamma s^2/n) - \gamma. \tag{8} \]

Using the bound \( \binom{n'}{s} \leq (en/s)^s \) as well as (8), (7) becomes
\[ E_1 \leq \sum_{s \in S_1} \left( \frac{en}{s} \right)^s \exp \left( -(\ln n - \ln \ln \ln n)(s - \gamma s^2/n) + \gamma \right) \]
\[ = e^\gamma \sum_{s \in S_1} \exp \left( s \left( 1 - \ln s + \ln \ln n + \gamma s(\ln n - \ln \ln \ln n) \right) \right). \]

However, for \( s \in S_1 \), we have that
\[ s \leq \frac{n}{\ln n} \leq \frac{2n}{\ln n - \ln \ln \ln n} \implies s(\ln n - \ln \ln \ln n)/n \leq 2. \]

Therefore
\[ E_1 \leq e^\gamma \sum_{s \in S_1} \exp \left( s \left( 1 - \ln s + \ln \ln n + 2\gamma \right) \right). \]
The first term in this latter sum dominates (ratios of consecutive terms uniformly tends to zero) and we have that

\[ E_1 = O \left[ \left( \frac{2e^{1+2\gamma \ln \ln n}}{\ln n} \right)^{(\ln n)/2} \right] = O \left[ \exp \left( -\frac{1}{2} \ln n \ln \ln n + o(\ln n \ln \ln n) \right) \right]. \quad (9) \]

Using our bounds for \( E_1 \) and \( E_2 \) ((9) and (6), respectively), we have that

\[ P(\mathcal{F}(v^*)) \leq \exp \left( -\frac{\ln n \ln \ln n}{2} + o(\ln n \ln \ln n) \right), \quad (10) \]

which most definitely is \( o(n^{-(k-1)}) \) and so by the bound (4), part (i) of the lemma is proved.

Part (ii) is established by using a standard conversion technique between \( H_d(n, m) \) and \( H_d(n, p) \). For any collection of hypergraphs \( \mathcal{A} \), we have that

\[ P(H_d(n, p) \in \mathcal{A}) = \sum_{m=0}^{\binom{n}{d}} P(H_d(n, m) \in \mathcal{A}) P(e(H_d(n, p)) = m), \]

where \( e(H) \) is the number of edges of \( H \). Therefore, for any (possible) \( m \),

\[ P(H_d(n, p) \in \mathcal{A}) \geq P(H_d(n, m) \in \mathcal{A}) P(e(H_d(n, p)) = m), \]

whence

\[ P(H_d(n, m) \in \mathcal{A}) \leq \frac{P(H_d(n, p) \in \mathcal{A})}{P(e(H_d(n, p) = m))}. \]

For \( m = \Theta(n \ln n) \) and \( p = m/\binom{n}{d} \), one can show that

\[ P(e(H_d(n, p)) = m) = \binom{n}{d}/m \cdot p^m (1-p)^{(n-d)-m} = \Theta(m^{-1/2}). \]

Hence in our case,

\[ P(H_d(n, m_0) \in \cup_v \mathcal{F}(v)) = O \left( \sqrt{n \ln n} P(H_d(n, p) \in \cup_v \mathcal{F}(v)) \right). \]

In the proof of part (i), we found that this latter probability tends to zero superpolynomially fast. \( \square \)

4 Quasi-disjoint Edges

For the random graph process \((d = 2)\), it was found that the main barrier to \( k \)-connectivity is the presence of vertices of degree less than \( k \), which could be isolated with the deletion of their neighbors (see Erdős-Rényi [5], Ivchenko [6], Bollobás [1],[2]). We will find a similar situation for the random hypergraph process.
However, we run into an additional issue here for hypergraphs. Even if the degree of a vertex \( v \) is \( k \), we could isolate \( v \) with the deletion of less than \( k \) vertices. For instance, if all of \( v \)'s edges also include vertex \( w \), then the deletion of just \( w \) from the hypergraph (along with its incident edges) will isolate \( v \) from the rest of the hypergraph. Our ultimate goal in this section is to show that w.h.p. each vertex of \( H_d(n, \tau_k) \) has at least \( k \) edges whose pairwise intersections are precisely \( \{v\} \); in this case, for any vertex, you would need to delete at least \( k \) of its neighbors to isolate it. To this end, we first prove that w.h.p. \( H_d(n, m_0) \) has this property for vertices with degree at least \( k \) and as nearly as could be expected for vertices with degree \( k - 1 \).

A set of edges \( E \) incident to vertex \( v \) is \textit{quasi-disjoint} if all pairwise intersections of these edges are \( \{v\} \); formally, if \( e, f \in E, e \neq f \), then \( e \cap f = \{v\} \).

**Lemma 4.1.** Let \( m_0 = \frac{2}{d}(\ln n + (k - 1) \ln \ln n - \omega) \), where \( \omega \to \infty \), but \( \omega = o(\ln \ln n) \). W.h.p., \( H_d(n, m_0) \) is such that

(i) the incident edges of a degree \( k - 1 \) vertex form a quasi-disjoint set,

(ii) vertices with degree at least \( k \) have a quasi-disjoint set of incident edges with size at least \( k \).

**Proof.** Note that both parts of this lemma are trivially true for \( k = 1 \) and part (i) is also trivially true for \( k = 2 \). Let \( X(j, \ell) \) be the number of vertices whose maximum quasi-disjoint set has size \( j \) and whose degree is \( j + \ell \). To prove this lemma, it suffices to show that w.h.p. for \( j \leq k - 1, \ell \geq 1 \), we have that \( X(j, \ell) = 0 \), which is shown by a first moment argument. Now

\[
E[X(j, \ell)] = nP(j, \ell),
\]

where \( P(j, \ell) \) is the probability that a generic vertex has a maximum quasi-disjoint set of size \( j \) and whose degree is \( j + \ell \). To bound this probability, note that this generic vertex has a set of \( j \) quasi-disjoint edges and each of the remaining \( \ell \) edges must have at least one vertex from the \( j(d - 1) \) neighbors from the quasi-disjoint edges; further the remaining \( m_0 - j - \ell \) edges do not include this generic vertex. Hence

\[
P(j, \ell) \leq \binom{(n - 1)}{j, (d - 1), (n - 2)} \cdot \binom{\binom{n}{d} \cdot \binom{n - 1}{d - 1} \cdot \binom{\binom{n}{d}}{m_0 - j - \ell}}{m_0} \\
\leq n^{j(d - 1)} \times \binom{e}{\ell} \times \frac{d - 2 \ell}{(d - 2)!} \times \binom{\binom{n}{d} \cdot \binom{n - 1}{d - 1} - m_0}{m_0}^{j + \ell} \times \binom{\binom{n}{d} \cdot \binom{n - 1}{d - 1} - m_0}{m_0}.
\]

We gave sharp asymptotics for the last fraction in (2). Here and throughout the rest of the paper, we will use \( f \leq_b g \) for \( f = O(g) \) when the formula for \( g \) becomes too bulky. Therefore

\[
P(j, \ell) \leq_b (\ln n)^j \times \frac{e^\omega}{\ln n} \times \binom{e j (d - 1)n^{d - 2}m_0}{\ell (d - 2)!} \times \binom{\binom{n}{d} \cdot \binom{n - 1}{d - 1} - m_0}{m_0}^{j + \ell};
\]
whence
\[ P(j, \ell) \leq_b e^\omega \frac{C n^{d-2} m_0}{n \binom{n}{d} - \binom{n-1}{d-1} - m_0} \ell, \]
for some constant \( C > 0 \) (independent of \( j \leq k - 1 \) and \( \ell \geq 1 \)). Thus
\[
\sum_{j=0}^{k-1} \sum_{\ell=1}^{\omega n} E[X(j, \ell)] \leq_b e^\omega \sum_{\ell=1}^{\omega n} \left( \frac{C n^{d-2} m_0}{n \binom{n}{d} - \binom{n-1}{d-1} - m_0} \right)^\ell \leq_b e^\omega \ln n \to 0,
\]
which completes the proof of the lemma.

\[ \square \]

**Lemma 4.2.** W.h.p. each vertex of \( H_d(n, \tau_k) \) has a quasi-disjoint set of incident edges with size at least \( k \).

**Proof.** This lemma is trivially true for \( k = 1 \). Suppose that \( k \geq 2 \). Let \( A_n \) be the event that \( H_d(n, \tau_k) \) has a vertex that does not have a quasi-disjoint set of edges with size at least \( k \); we wish to show that \( P(A_n) \to 0 \). We have proved that w.h.p. \( \tau_k \in [m_0, m_1] \) and that the number of degree \( k - 1 \) vertices in \( H_d(n, m_0) \) is less than \( \frac{3e^\omega}{2(k-1)!} \) (Lemma 2.1) and further that \( H_d(n, m_0) \) has the two properties of the previous lemma (Lemma 4.1). Let \( B_n \) be the intersection of these three likely events. To prove the lemma, it suffices to show that \( P(A_n \cap B_n) \to 0 \).

Let \( V_0 \) be the vertex set of vertices of degree \( k - 1 \) in \( H_d(n, m_0) \). Note that
\[
P(A_n \cap B_n) = \sum_{V_0 \subseteq [n], |V_0| \leq 3e^\omega/(2(k-1)!)} P(A_n \cap B_n \cap \{V_0 = V_0\}).
\]

On the event that \( A_n \) and \( B_n \) occur and \( V_0 = V_0 \), necessarily some edge \( e_m \) is added in the hypergraph process at some step \( m \in [m_0, m_1] \) such that \( e_m \) includes both a vertex \( v \in V_0 \) and one of \( v \)’s \( (k-1)(d-1) \) neighbors in \( H_d(n, m_0) \). Thus
\[
P(A_n \cap B_n \cap \{V_0 = V_0\}) \leq \sum_{m=m_0}^{m_1} \frac{|V_0|}{1} \binom{k-1}{d-1} \binom{n-2}{d-2} \frac{n!}{m!} P(V_0 = V_0)
\leq_b (m_1 - m_0) e^\omega \frac{n^{d-2}}{d-2} P(V_0 = V_0) \leq_b \frac{\omega e^\omega}{n} P(V_0 = V_0).
\]

Therefore
\[
P(A_n \cap B_n) \leq_b \frac{\omega e^\omega}{n} \sum_{V_0 \subseteq [n], |V_0| \leq 3e^\omega/(2(k-1)!)} P(V_0 = V_0) \leq \frac{\omega e^\omega}{n} \to 0.
\]

\[ \square \]
5 W.h.p. \( H_d(n, \tau_k) \) is \( k \)-connected

Now that we have sufficient knowledge about the structure of \( H_d(n, m_0) \) and low-degree vertices in \( H_d(n, \tau_k) \), we can prove that w.h.p. \( H_d(n, \tau_k) \) is \( k \)-connected.

**Theorem 5.1.** W.h.p. \( H_d(n, \tau_k) \) is \( k \)-connected. In short, w.h.p. \( \tau_k = T_k \).

**Proof.** Let \( m_i = n_0 \cdot \frac{1}{2} \left( \frac{\ln n + (k - 1) \ln \ln n + (-1)^i \ln \ln n}{\ln \ln \ln n} \right) \), for \( i = 0, 1 \). By Lemma 2.1, we have shown that w.h.p. \( \tau_k \in [m_0, m_1] \) and by Lemma 4.2, each vertex of \( H_d(n, \tau_k) \) has a quasi-disjoint edge set of size at least \( k \). Further, the property “whichever \( k - 1 \) vertices are deleted, there remains a giant component which includes all but up to \( \ln n \) leftover vertices,” denoted \( Q \), is an increasing property. Therefore, by Lemma 3.1, \( H_d(n, \tau_k) \) has property \( Q \) as well. To prove this theorem, it suffices to show that the probability that these three likely events hold yet \( \tau_k < T_k \) tends to zero.

To this end, for \( m \in [m_0, m_1] \), let \( C_m \) be the event that \( H_d(n, m) \) is not \( k \)-connected, but each vertex has a quasi-disjoint edge set of size at least \( k \) and \( H_d(n, m) \) has property \( Q \). To prove this theorem, it suffices to prove that \( P(\cup C_m) \to 0 \). We will in fact show that

\[
P(C_m) \leq b \frac{(\ln n)^{dk+k+1}}{n^{d-1}},
\]

uniformly over \( m \in [m_0, m_1] \). In this case, \( P(\cup C_m) \leq b \frac{(\ln n)^{dk+k+1}}{n^{d-2}} \to 0 \), as desired. All that remains is to prove the bound (11).

On the event \( C_m \), there are \( k - 1 \) vertices, \( w_1, \ldots, w_{k-1} \) such that upon their deletion, there is a component of size \( n' - s \) for some \( s \in [1, \ln n] \). In fact, since each remaining vertex must have at least one incident edge, we must have that \( s \geq d \). Let \( S \) be the set of vertices not in this large component. By the union bound over all \( k - 1 \) element sets of \( [n] \) and sets \( S, |S| = s \), as well as symmetry, we have that

\[
P(C_m) \leq \binom{n}{k-1} \sum_{s=d}^{\ln n} \binom{n-(k-1)}{s} P_s,
\]

where \( P_s \) is the probability that each vertex of \( H_d(n, m) \) has a quasi-disjoint edge set of size at least \( k \) and that after the deletion of \( \{n' + 1, \ldots, n' + (k - 1) = n\} \) from \( H_d(n, m) \), the vertices \( \{n' - s\} \) form a component; in this case, \( S = \{n' - s + 1, \ldots, n'\} \). We now turn to showing that \( P_s \) tends to zero sufficiently fast.

Suppose that \( H \) is some hypergraph in the event corresponding to \( P_s \). After the deletion of the \( k - 1 \) vertices, we know that vertex \( \{n'\} \) has at least one incident edge that must reside within \( S \). Further, before deletion, any incident edge to \( \{n'\} \) must be completely contained within \( S \) or this edge must contain one of the \( k - 1 \) to-be-deleted vertices. Moreover, there are at least \( k \) edges incident to \( \{n'\} \) before the deletion. Therefore

\[
P_s \leq \sum_{i=1}^{k} P_s(i),
\]
where \( P_s(i) \) is the corresponding probability to when there are (at least) \( i \) incident edges to \( \{n’\} \) contained within \( S \) and (at least) \( k - 1 - i \) incident edges to \( \{n’\} \) that contain at least one of the to-be-deleted vertices. Note that there can be no edges that contain \( \{n’\} \) and precisely \( d - 1 \) vertices of \([n’ - s]\). Therefore,

\[
P_s(i) \leq \binom{s}{d}^i \left( \binom{1}{1} \binom{k-1}{k-i} \binom{n-2}{d-2} \right) \left( \binom{n}{d} - \binom{s}{d-1} \binom{n’-s}{d-1} \right) \frac{1}{\binom{m}{k}}.
\]

First, we use trivial bounds on the first two binomial terms. Then we use the inequality \( \binom{N-j}{j} \leq \binom{N-j}{j} e^{-j\ell/N} \). Namely, note that

\[
P_s(i) \leq s^d k^k n^{(d-2)(k-i)} \left( \frac{m}{\binom{n}{d}} - s \frac{m}{\binom{s-d-1}{d}} \right)^k \frac{\binom{n}{d-s}{d-1}^{n’-s}}{\binom{m}{d}}.
\]

Further, for \( s \leq \ln n \), we have that

\[
\frac{s^{d-1}}{\binom{n}{d}} = \frac{s^d m}{n} + O \left( \frac{(\ln n)^2}{n} \right) \geq \frac{s^d m_0}{n} + o(1).
\]

Hence

\[
P_s(i) \leq s^d k^k n^{(d-2)(k-1)} \left( \frac{\ln n}{n^{d-1}} \right)^k \left( \frac{\ln \ln n}{n(\ln n)^{k-1}} \right)^s \]

\[
\leq (\ln n)^{dk+k} n^{2-d-k} \left( \frac{\ln \ln n}{n(\ln n)^{k-1}} \right)^s ,
\]

which no longer depends on \( i \). Therefore

\[
P_s \leq (\ln n)^{dk+k} n^{2-d-k} \left( \frac{\ln \ln n}{n(\ln n)^{k-1}} \right)^s ,
\]

and

\[
P(C_m) \leq n^{k-1} \sum_{s=d}^{\ln n} \frac{n^s}{s!} (\ln n)^{dk+k} n^{2-d-k} \left( \frac{\ln \ln n}{n(\ln n)^{k-1}} \right)^s .
\]

Now taking on this sum, we find that

\[
P(C_m) \leq (\ln n)^{dk+k} \sum_{s=d}^{\ln n} \frac{1}{s!} \left( \frac{\ln \ln n}{(\ln n)^{k-1}} \right)^s \leq \exp \left( \frac{\ln \ln n}{(\ln n)^{k-1}} \right) ;
\]

and we find that \( P(C_m) \leq \binom{\ln n}{d}^{dk+k+1} n^{d-1} \), as desired.
6 Sharp Threshold of $k$-connectivity

As a consequence of Theorem 5.1, for any $m$, we have that
\[ P(H_d(n, m) \text{ is } k\text{-connected}) = P(T_k \leq m) = P(\tau_k \leq m) + o(1) = P(\text{min-deg } H_d(n, m) \geq k) + o(1). \] (12)

We use this fact to determine the probability that $H_d(n, m)$ and $H_d(n, p)$ is $k$-connected in the critical window.

**Corollary 6.1.** (i) Let $m = \frac{n}{d} \left( \ln n + (k - 1) \ln \ln n + c_n \right)$, where $c_n \to c \in \mathbb{R}$. W.h.p. $H_d(n, m)$ is $(k - 1)$-connected, but not $(k + 1)$-connected. Further the probability that $H_d(n, m)$ is $k$-connected tends to $e^{-e^{-c}}/(k-1)!$.

(ii) Let $p = (d - 1)! \frac{\ln n + (k - 1) \ln \ln n + c_n}{n - 1}$, where $c_n \to c \in \mathbb{R}$. W.h.p. $H_d(n, p)$ is $(k - 1)$-connected, but not $(k + 1)$-connected. Further the probability that $H_d(n, p)$ is $k$-connected tends to $e^{-e^{-c}}/(k-1)!$.

**Proof.** (i) First, note that w.h.p. $\tau_{k-1} < m$ and $\tau_{k+1} > m$ by Lemma 2.1. Therefore, by Theorem 5.1, w.h.p. $H_d(n, m)$ is $(k - 1)$-connected, but not $(k + 1)$-connected. One can show that $X$, the number of vertices of degree $k - 1$ in $H_d(n, m)$ is asymptotically Poisson with parameter $e^{-c}$, by finding $X$’s factorial moments in a manner similar to Lemma 2.1. Thus
\[ P(\text{min-deg } H_d(n, m) \geq k) = P(\text{Poi}(e^{-c})/(k - 1)! = 0) + o(1) = e^{-e^{-c}}/(k-1)! + o(1). \]

Using the equation (12) finishes off the proof.

(ii) This part can be proved from (i) using a similar conversion technique to that used in Lemma 3.1.

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