On the lattice construction of electroweak gauge theory

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Based on the Ginsparg-Wilson relation, a gauge invariant formulation of electroweak SU(2) × U(1) gauge theory on the lattice is considered. If the hypercharge gauge coupling is turned off in the vacuum sector of the U(1) gauge fields, the theory consists of four left-handed SU(2) doublets and it is possible, as in vector-like theories, to make the fermion measure defined globally in all topological sectors of SU(2). We then try to incorporate U(1) gauge field, following Lüscher’s reconstruction theorem. The global integrability condition is proved for “gauge loops” in the space of the U(1) gauge fields with arbitrary SU(2) gauge field fixed in the background. For “non-gauge loops”, however, the proof is given so far only for the classical SU(2) instanton backgrounds.

1. An approach to electroweak theory

Recently, gauge-covariant and local lattice Dirac operators satisfying the Ginsparg-Wilson relation\cite{1} has been constructed\cite{2,3,4}, and this opens the possibility to formulate chiral gauge theories on the lattice with exact gauge invariance. Indeed, Lüscher has given a constructive proof of the existence of Weyl fermion measure in anomaly-free abelian chiral gauge theories\cite{5}. The same author has also examined the construction of generic non-abelian chiral gauge theories and has formulated the reconstruction theorem for the fermion measure\cite{6}. The purpose of this article is to examine the possible extension of the construction to electroweak SU(2) × U(1) gauge theory.

Electroweak theory is the chiral gauge theory of left-handed leptons and quarks in SU(2) doublet and right-handed quarks in SU(2) singlet. Taking into account of the color degrees of freedom, there are four doublets in each generation. In order to construct this theory on the lattice, we adopt the definition of lattice Weyl fermion based on Ginsparg-Wilson Dirac operator\cite{7,8,9,5,12}:

\[ \gamma_5 D + D \gamma_5 = 0, \quad \gamma_5 \equiv \gamma_5 (1 - aD), \quad (1) \]

\[ \psi_L(x) = \hat{P} - \psi_L(x), \quad \hat{P} = (1 - \gamma_5)/2. \quad (2) \]

The projection for $\bar{\psi}$ is defined as usual with $\gamma_5$. Gauge fields are given by link variables $U(x, \mu) = U^{(2)}(x, \mu) \otimes U^{(1)}(x, \mu)$. We assume the lattice to be finite with periodic boundary conditions in all directions. We require that U(1) and SU(2) components of link variables satisfy the so-called admissibility conditions respectively, which ensure the locality and smoothness of the Dirac operator\cite{4}. Then the space of the admissible SU(2) × U(1) gauge fields is divided into the topological sectors\cite{10}, each one is the product of a U(1) topological sector and an SU(2) topological sector.

Our approach to electroweak theory on the lattice relies on the specific feature of the original theory. In the vacuum sector of the U(1) gauge fields where any configuration can be deformed smoothly to the trivial one $U(x, \mu) = 1$, we can turn off the hypercharge gauge coupling. Then the theory can be regarded as vector-like due to the pseudo reality of SU(2). It is indeed possible to make the fermion measure defined globally in all topological sectors of SU(2) by the following choice of the basis for a pair of doublets $(a, b)\cite{3,4}$:

\[ u^{(a)}_j(x) = u_j(x), \]  \[ (3) \]

\[ u^{(b)}_j(x) = (\gamma_5 C^{-1} \otimes i \sigma_2) [u_j(x)]^*, \]  \[ (4) \]

where $\hat{P} - u_j(x) = u_j(x)$. This fact implies the cancellation of Witten’s SU(2) anomaly (cf.\cite{11}).
Given the basis for the SU(2) doublets defined globally, one may try to extend the fermion measure to incorporate the U(1) gauge field following the reconstruction theorem \[8\], as far as we concern the vacuum sector of the U(1) gauge fields.

2. A choice of the fermion measure

According to the reconstruction theorem, the first step to obtain the Weyl fermion measure is to construct the measure term \( \eta_\mu \equiv \delta U_\mu U_\mu^{-1} \):

\[
\mathcal{L} = a^4 \sum_x \left\{ \eta^{(2)}_\mu(x) j^{(2)}_\mu(x) + \eta^{(1)}_\mu(x) j^{(1)}_\mu(x) \right\},
\]

where \( j^{(n)}_\mu (n = 2, 1) \) should satisfy the anomalous conservation law and the integrability condition, and should be defined smoothly over the space of the gauge fields. This question can be mapped to the equivalent local cohomology problem in the 4+2 dimensions. Two of the authors have examined the cohomology problem for electroweak theory and have shown that the currents \( j^{(n)}_\mu (n = 2, 1) \) with the desired properties can be constructed in the infinite volume [12]. Then the remaining issue is to obtain the currents \( j^{(n)}_\mu (n = 2, 1) \) in the finite volume. The similar problem has already been solved by Lüscher in the case of abelian chiral gauge theories [13], and that procedure can actually be applied to the U(1) part of the current \( j^{(1)}_\mu \) in our case, if we regard the SU(2) gauge field as a background. The resulted current is local and smooth with respect to the SU(2) gauge field and is invariant under the SU(2) gauge transformation.

Given the smooth current \( j^{(1)}_\mu \) and the “global” measure for \( U^{(2)} \otimes 1 \) (the associated Weyl fermion basis \( \{ \psi_j(x) \} \)), we may consider a choice of the fermion measure as follows: let us consider a smooth curve in the space of the gauge fields, along which the U(1) gauge field is interpolated from the trivial configuration \( U^{(1)}_0 = 1 \) to a certain non-trivial configuration \( U^{(1)} = U^{(1)} \) in the vacuum sector, while the SU(2) gauge field is fixed to an arbitrary configuration \( U^{(2)} \) in a given topological sector. Along the curve \( \{ U^{(2)} \otimes U^{(1)}_t \} \) for \( 0 \leq t \leq 1 \), we introduce the Wilson line \( W^{(1)} \) using \( j^{(1)}_\mu \), and the evolution operator \( Q^{(1)}_t \) (0 \( \leq t \leq 1 \)). Then we define a basis for \( U^{(2)} \otimes U^{(1)}_t \) as follows:

\[
v_j = \begin{cases} 
Q^{(1)}_1 w_j W^{(1)}^{-1} & \text{if } j = 1 \\
Q^{(1)}_1 w_j & \text{otherwise}
\end{cases}
\]

We can check directly the local properties of the fermion measure obtained from the basis [8], by evaluating the measure term in the neighbor of \( U^{(2)} \otimes U^{(1)} \). We can see that both currents \( j^{(2)}_\mu \) and \( j^{(1)}_\mu \) are obtained in this case and they indeed satisfy the integrability condition and the anomalous conservation law. Thus this choice of the basis gives a consistent fermion measure at least locally.

3. Global integrability for U(1) loops

For the fermion measure defined with the basis [8] to be consistent globally, it is required that the Wilson line \( W^{(1)} \) should satisfy the global integrability condition

\[
W^{(1)} = \det(1 - P_0 + R_0 Q^{(1)}_1)
\]

(\( P_t = \hat{P} \mid_{U^{(2)} \otimes U^{(1)}} \)) for all closed loops in the space of the U(1) gauge fields with arbitrary SU(2) gauge field fixed in the background. We next examine this global integrability condition in this section.

The admissible U(1) gauge fields in the vacuum sector can be parametrized as

\[
U^{(1)}(x, \mu) = \Lambda(x) w(x, \mu) \Lambda(x + \mu)^{-1},
\]

\[
w(x, \mu) = w_\mu \delta_{x, 0},
\]

up to contractible components, where \( \Lambda(x) \in U(1), w_\mu \in U(1) \). There are two types of non-trivial loops in the space of the U(1) gauge fields: the first one is related to the gauge degrees of freedom \( \Lambda(x) \) and referred as “gauge loops”, and the second one is “non-gauge loops” related to \( w_\mu \). Then we should examine the global integrability condition for such non-trivial loops while the SU(2) gauge field is kept fixed.

For the gauge loops

\[
U^{(1)}_t(x, \mu) = \Lambda_t(x) \Lambda_t(x + \mu)^{-1},
\]

\[
\Lambda_t(x) = \exp(2\pi it \delta_{xy}),
\]

where
we can prove the global integrability in the similar manner as abelian chiral gauge theories: we set \( \eta_\mu^{(1)}(x) = -\partial_\mu \delta_{\bar{x} y} \), and \( \eta_\mu^{(2)}(x) = 0 \), and then the anomalous conservation law implies that the Wilson loop is given by the U(1) part of the gauge anomaly as

\[
W = \exp\{i2\pi A^{(1)}(y)_{t=0}\}. \tag{12}
\]

On the other hand, the twist (the determinant on the r.h.s. of (12)) can be evaluated with the formula

\[
\lim_{n \to \infty} \det (1 - P_{t_0} + P_{t_n} P_{t_{n-1}} \cdots P_{t_0}) \tag{13}
\]

and \( P_t = R[A_t] P_0 R[A_t]^{-1} \) to reproduce the r.h.s. of (12). As we can see, this proof holds with any SU(2) gauge field in the background.

For the non-gauge loops

\[
U^{(1)}_t(x, \mu) = \exp\{2\pi i \delta_{\mu\nu} \delta_{\bar{x}_t 0}\}, \tag{14}
\]

we could prove the global integrability condition by using the reflection property of the current \( j^{(1)}_\mu \) in the similar manner as abelian chiral gauge theories:

\[
j^{(1)}_\mu(x)|_{t \to -t} = -j^{(1)}_\mu(-x + \hat{t} - \hat{\mu}), \tag{15}
\]

where the center of reflection is \( \hat{t}/2 \) (\( \hat{t} = (1,1,1,1) \)). With the SU(2) gauge field in the background, however, we need to require that the SU(2) gauge field should have the following reflection symmetry up to gauge transformation,

\[
U^{(2)}(-x + \hat{t} - \hat{\mu}, \mu)^{-1} \equiv U^{(2)}(x, \mu). \tag{16}
\]

For this restricted class of the SU(2) gauge fields, the Wilson line \( W^{(1)} \) turns out to be unity, because of (13). On the other hand, the twist can be evaluated to be unity, using the reflection property of the projection operator: \( P_{t} = \Gamma P_{t-1} \Gamma^{-1} \) where \( \Gamma \psi(x) \equiv \gamma_5 \psi(-x) \).

The classical instantons mapped on to a sufficiently large lattice (centered at \( \hat{t}/2 \), the middle of the lattice sites) indeed possess such reflection invariance, as well as the trivial gauge field \( U^{(2)} = 1 \). It is conceivable that each topological sector of the SU(2) gauge fields has such a reflection invariant configuration. So far, the global integrability condition for the non-gauge loops can be shown only for these restricted SU(2) gauge fields.

4. Discussions

Our approach to electroweak theory on the lattice refers to the trivial U(1) configuration \( U^{(1)} = 1 \), for which the fermion measure can be constructed globally in all topological sectors of SU(2), and therefore is restricted to the vacuum sector of the admissible U(1) gauge fields. It is not clear yet that such a fermion measure exists also in the U(1) magnetic flux sectors.

Our construction of the measure term is incomplete in the sense that the SU(2) part \( j^{(2)}_\mu \) is not constructed explicitly and it is not clear if \( j^{(2)}_\mu \) could be defined globally. The latter condition seems to be equivalent to the global integrability condition of \( j^{(1)}_\mu \) for both gauge and non-gauge loops with any SU(2) background. To show these conditions, it seems necessary to clarify the topological structure of the space of the admissible SU(2) gauge fields and to find the parametrization of the SU(2) link variables.

REFERENCES

1. P.H. Ginsparg, K.G. Wilson, Phys. Rev. D25 (1982) 2649.
2. P. Hasenfratz, Nucl. Phys. B(Proc. Suppl.)63 (1998) 53.
3. H. Neuberger, Phys. Lett. B417 (1998) 141; Phys. Lett. B427 (1998) 353.
4. P. Hernández, K. Jansen, M. Lüscher, Nucl. Phys. B552 (1999) 363.
5. M. Lüscher, Nucl. Phys. B549 (1999) 295.
6. M. Lüscher, Nucl. Phys. B568 (2000) 162.
7. M. Lüscher, Phys. Lett. B428 (1998) 342.
8. R. Narayanan and H. Neuberger, Nucl. Phys. B412 (1994) 574; Phys. Rev. Lett. 71 (1993) 3251; Nucl. Phys. B443 (1995) 305.
9. F. Niedermayer, Nucl. Phys. B(Proc. Suppl.)73 (1999) 105.
10. M. Lüscher, Comm. Math. Phys. 85 (1982) 39.
11. O. Bärr, I. Campos, Nucl. Phys. B581 (2000) 499.
12. H. Suzuki, J. High Energy Phys. 10 (2000) 039.
13. Y. Kikukawa, Y. Nakayama, Nucl. Phys. B597 (2001) 519.