EFFECTIVE DECAY OF MULTIPLE CORRELATIONS IN SEMIDIRECT PRODUCT ACTIONS

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ABSTRACT. We prove effective decay of certain multiple correlation coefficients for measure preserving, mixing Weyl chamber actions of semidirect products of semisimple groups with $G$-vector spaces. These estimates provide decay for actions in split semisimple groups of higher rank.

1. INTRODUCTION

Ergodicity and notions of mixing constitute the measure theoretic incarnation of statistical uniformity in measurable dynamics. The relationships among them are not completely understood; it is not known, for instance, if every mixing map $T$ on a measure space $X$ is mixing of all orders. For $\mathbb{Z}^d$-actions, $d \geq 2$, Ledrappier gave a counterexample in [14] later generalized by Schmidt (see [19]) using certain algebraic dynamical systems. General $\mathbb{R}^d$-actions can also have a wide range of behaviors depending on the nature of the action and properties of the object acted upon.

In the context of actions of a highly non-commutative group (such as a semisimple group) the situation changes drastically. Many topological and measurable distribution phenomena follow not from the peculiarities of the action, but rather from the nature of the acting group. This is manifest in the following two basic results: the Howe-Moore vanishing theorem [8], which implies that every ergodic action of a simple group is mixing, and Mozes's theorem [16], stating that for well behaved semisimple Lie groups, mixing implies mixing of all orders. It is the rich geometry and non trivial interplay between certain subgroups of semisimple groups that lies at the heart of both these results and accounts for this exceptional behavior of semisimple groups. For example, Mautner's phenomenon (see [2]) in various guises plays a major role in both cited results. A comprehensive survey emphasizing this interplay can be found in [20]. The Howe-Moore vanishing theorem has been made quantitative with significant geometric and arithmetic applications. A natural question that arises in view of this is whether we can give a quantitative form of Mozes's theorem whose asymptotics agree with Howe-Moore for 2-mixing.
Let $G$ be a suitable group with a measure preserving action on a probability measure space $(X, \mu)$ denoted $g \cdot x$. For any $(k + 1)$-tuple $f_i \in L^\infty_0(X)$ (bounded, zero-mean functions) form the correlation integral

$$\int_X f_0(x)f_1(g_1^{-1} \cdot x)\cdots f_k(g_k^{-1} \cdot x)d\mu(x).$$

The task is to bound this integral in terms of data (as explicit as possible) intrinsic to the $f_i$, to the acting group $G$ and to a given notion of growth of the acting tuple $(g_i)$ in $G$. This paper is concerned with the action of the Cartan subgroup of a group $G$, which is the semidirect product of a semisimple group $G$ with a vector space by means of a representation of one on the other. Such groups play an important role as subgroups of semisimple groups (but are not restricted to such a role), and we exploit this inclusion to get quantitative decay for some split semisimple groups of higher rank. Since the precise statement of our result is too technical to give in the introduction, we will give a simplified description here and refer to Section 4 for the full version.

This paper relies heavily on the work of Wang in [23]. References to proofs in [23] abound but an attempt has been made to isolate the parts of [23] most useful to us and present them as standalone lemmas. Whenever we make use of a result or method of [23], we will mention it explicitly and give specific reference within that work.

Following Wang we let $G$ be a connected semisimple almost algebraic (see section 2 for precise definitions) $\mathcal{K}$-group where $\mathcal{K}$ is a local field of characteristic zero together with a $\mathcal{K}$-rational representation $\rho: G \to \text{GL}(V)$ satisfying certain conditions; for the purposes of the introduction, assume the representation is irreducible, faithful (or with finite kernel) and for each almost simple factor $G_i$ the only $\rho(G_i)$-fixed vector in $V$ is 0. Let the group $G \ltimes_\rho V$ act on a probability space $(X, \mu)$ via measure-preserving transformations giving a unitary representation on $L^2(X)$ (assumed separable) that distributes over pointwise (a.e.) products of functions. Restricting it to $G \ltimes_\rho 0$ we get an action of the semisimple group $G$; relative to a Cartan decomposition $G = KD^+FK$ we let $(k + 1)$ distinct elements

$$a^i \in D^+$$

de $G$ act on $(k + 1)$ suitable functions $f_i$ (at least bounded, zero-mean, $K$-finite of bounded spectral support$^1$). For simplicity, assume here that $\mathcal{K} = \mathbb{R}$ and $\text{rank}_\mathcal{K}(G) = 1$.

Under these assumptions the connected component of the identity $D^0$ is isomorphic to the set $\mathbb{R}_{>0}$ and the Cartan elements can be totally ordered as

$$a^0 < a^1 < \cdots < a^k.$$

Let

$$\mathcal{R}(\mathbf{a}) = \left(\sum_{i=0}^{k-1} \frac{a^i}{a^k}\right)\left(\sum_{i=1}^{k} \frac{a^0}{a^i}\right)$$



$^1$ See Section 2 for definitions.
The main theorem of this paper in the rank 1 case takes the following form:

**Theorem 1.1.** Let $a^j$ be ordered as above, and assume that the representation of $1 \ltimes V$ on $L^2_0(X)$ has no invariant vectors; then there exists an $L^2_0(X)$-dense subspace $\mathcal{D}$ of bounded, zero-mean, $K$-finite functions so that if $f_0, \ldots, f_k$ are in $\mathcal{D}$, we have the bound

$$\left| \int_X ((a^0)^{-1} \cdot x) f_0(x) f_1((a^1)^{-1} \cdot x) \cdots f_k((a^k)^{-1} \cdot x) d\mu(x) \right| \leq C \mathcal{R}(a)^q,$$

where $C$ depends on the functions; the exponent $q$ only depends on the action.

One restriction in the bound we presented is the non-uniformity of the bound: the constant $C$ depends on the functions in the dense admissible subspace (this dependence is manageable in many situations; see Section 2.4 for the details). This subspace consists of $K$-finite functions whose spectrum on $V$ is restricted to compact subsets of $V \sim \{0\}$. Using the last condition we can think of the space informally as 'continuous trigonometric polynomials'; their 'degree' features explicitly in the constant $C$.

Another observation is that the space of admissible functions is only dense in $L^2_0(X)$ in general and not in other $L^p(X)$, which precludes a deduction of multiple mixing for all $L^\infty(X)$ functions from the decay of the integral alone by Hölder’s inequality. In the cases where decay holds uniformly in a space of functions dense in all $L^p$ (such as all smooth functions with an appropriate Sobolev norm), we do get multiple mixing from the decay given by the theorem above.

In the higher rank case, the decay is given in terms of the highest and lowest weights of $V$ for irreducible $V$ (and more generally, in terms of extremal weights in each irreducible component of $V$).

For example, consider the case of

$$\mathcal{G} = \text{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3$$

with the standard representation. The highest and lowest weights $\lambda, \varrho$ corresponding to the standard order on the roots are

$$\lambda = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \rightarrow a_1, \quad \varrho = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} \rightarrow a_3.$$

Suppose we want to bound a correlation with four functions $f_0, \cdots, f_3$ acted upon by elements

$$a^0 = \begin{pmatrix} a_1^0 & 0 & 0 \\ 0 & a_2^0 & 0 \\ 0 & 0 & a_3^0 \end{pmatrix}, \quad a^1 = \begin{pmatrix} a_1^1 & 0 & 0 \\ 0 & a_2^1 & 0 \\ 0 & 0 & a_3^1 \end{pmatrix},$$

$$a^2 = \begin{pmatrix} a_1^2 & 0 & 0 \\ 0 & a_2^2 & 0 \\ 0 & 0 & a_3^2 \end{pmatrix}, \quad a^3 = \begin{pmatrix} a_1^3 & 0 & 0 \\ 0 & a_2^3 & 0 \\ 0 & 0 & a_3^3 \end{pmatrix},$$

all in the connected component $a^j_i > 0$. 

Let $a^6$ be the element with the smallest third coordinate and $a^4$ the one with the smallest first coordinate (the notation will become clear in the sequel). The bound can be expressed in terms of the product of the expressions

$$\left( \frac{a_1^4 + a_1^0}{a_1^4} \right)^{q_2} \quad \text{and} \quad \left( \frac{a_1^6}{a_1^3} + \frac{a_2^6}{a_3^2} + \frac{a_3^6}{a_3^3} \right)^{q_2}$$

with $q > 0$ an explicit constant. The first expression is the sum of the highest weight evaluated at the ratios $a^4/a^1$, and the second is the sum of the lowest weight evaluated at $a^6/a^1$.

**Remark 1.2.** In order for either of the two expressions to be small, it is necessary that all these ratios be small; this requirement can be expressed in terms of two pseudometrics involving the weights of the representation. In the rank 1 case, or generally on a one parameter group, this pseudodistance is a distance and the decay gives an effective multiple mixing for the admissible functions.

In the higher rank case, it is possible for a tuple of elements to be far from each other in a norm of the group but have zero pseudodistance: in the previous example, take $a_1^0 = a_1^1$ and $a_2^2 = a_3^3$. Thus for higher rank groups the decay is restricted to tuples with elements far from each other in terms of the highest and lowest weights and does not suffice for the full mixing statement.

Before stating our result for the higher rank case, we need to discuss the nature of the bound we obtain, its strengths and limitations.

As before, let $a = (a^1, \cdots, a^k)$ be a $k$-tuple of Cartan elements of $G$. In Section 2.3 we define a positive quantity $\mathcal{R}(a)$ which has the following property: when there exists an extremal (highest or lowest) weight of the representation of $G$ on $V$ such that all the members of $a$ diverge from each other along that weight, $\mathcal{R}(a)$ decays in a controlled way as well (exponentially fast in a natural pseudometric defined by the corresponding weight).

Note the order of quantifiers, which shows the main difference between the rank one and higher rank case: the condition that

$$(a^1)^a(a^1)^{-1} \rightarrow \infty$$

is not sufficient to ensure divergence because the former does not imply that there exists a single direction along which all elements diverge from each other. The $\text{SL}(3, \mathbb{R})$ example above illustrates this discussion: in that case

$$\mathcal{R}(a) = \left( \frac{a_1^4 + a_1^0}{a_1^4} \right)^{q_2} \left( \frac{a_2^6}{a_3^2} + \frac{a_2^6}{a_3^2} + \frac{a_3^6}{a_3^3} \right)^{q_2};$$

\[\text{If one of the two expressions is larger than 1, it can be replaced by 1.}\]
(replace each factor with 1 if it exceeds it). For \( t \gg 0 \), take
\[
\begin{align*}
\mathbf{a}^0 &= \text{diag}(1,1,1), \\
\mathbf{a}^1 &= \text{diag}(te^{10t}, t^{-2}e^{4t}, te^{-14t}), \\
\mathbf{a}^2 &= \text{diag}(te^{10t}, t^{-1}, e^{-10t}), \\
\mathbf{a}^3 &= \text{diag}(e^{100t}, e^{5t}, e^{-105t}).
\end{align*}
\]
Because \( \mathbf{a}^1 \) and \( \mathbf{a}^2 \) have the same highest weight for the standard representation, the left hand factor in \( \mathfrak{R}(\mathbf{a}) \) above does not contribute any gains. The right hand factor however gives a decay for large \( t \)
\[
e^{-105t} + e^{-95t} + t^{-1}e^{-91t} \sim t^{-\frac{q}{2}}e^{-\frac{91t}{2}}.
\]
On the other hand, take
\[
\begin{align*}
\mathbf{a}^0 &= \text{diag}(e^{105t}, 1, e^{-105t}), \\
\mathbf{a}^1 &= \text{diag}(e^{10t}, 2e^{5t}, \frac{1}{2}e^{-15t}), \\
\mathbf{a}^2 &= \text{diag}(e^{10t}, 1, e^{-10t}), \\
\mathbf{a}^3 &= \text{diag}(e^{100t}, e^{5t}, e^{-105t}).
\end{align*}
\]
These elements diverge from each other as \( t \to \infty \) but there is no single extremal weight along which all pairs diverge; the form of a shown above gives divergence in terms of the slowest diverging pair along the two extremal weights and does not provide a non-trivial bound. This is a limitation that lies at the heart of our method: any bound given as a sum of small contributions of elements evaluated at extremal weights has this problem in the higher rank case.

Let us now state our result in the higher rank case extrapolating from the above \( \text{SL}(3, \mathbb{R}) \ltimes \mathbb{R}^3 \) example; we will omit some details and refer to Section 2.3 for definitions. Notation is as in the previous theorem except for dropping the \( \mathcal{K} = \mathbb{R} \) and rank 1 assumptions.

**Theorem 1.3.** Let \( \lambda \) and \( \varrho \) be the highest and lowest weights of \( \rho \) respectively. Let \( \mathbf{a}^\lambda \), and \( \mathbf{a}^\varrho \) be the acting elements with the smallest absolute values on the weights \( \lambda \) and \( \varrho \) respectively. There exists an \( L^2_0(X) \)-dense subspace \( \mathcal{D} \) (that depends on \( \rho \)) of bounded, zero-mean, \( K \)-finite functions so that if \( k + 1 \) functions \( f_i \) are in \( \mathcal{D} \), we have the bound
\[
\left| \int_X f_0((\mathbf{a}^0)^{-1} \cdot x)f_1((\mathbf{a}^1)^{-1} \cdot x) \cdots f_{k}(((\mathbf{a}^k)^{-1} \cdot x)d\mu(x) \right| \leq C\mathfrak{R}(\mathbf{a})^{\frac{q}{2}}
\]
for some \( C > 0 \) depending on the functions; the exponent \( q \) only depends on the action and \( \mathfrak{R} \) decays like
\[
k^2 \left| \lambda \left( \frac{\mathbf{a}^\lambda}{\mathbf{a}^{\lambda_{i_\lambda}}} \right) \right| \cdot \left| \varrho \left( \frac{\mathbf{a}^\varrho}{\mathbf{a}^{\varrho_{i_\varrho}}} \right) \right|,
\]
where \( i_\lambda \) and \( i_\varrho \) are the indices different from \( \lambda, \varrho \) respectively that maximize these expressions (i.e., the slowest decay).
Most split semisimple algebraic groups of higher rank have enough isogenous images of semidirect products of the form above to cover the entire Weyl chamber action, so we can extend the bounds to diagonal one parameter subgroups of those groups by a standard restriction process. We discuss how to get such bounds in general in Section 4. Note that Dolgopyat has provided robust decay estimates on multiple correlations for one parameter subgroups in [6] by completely different methods.

**Theorem 1.4.** Let $G$ be the group of $\mathcal{X}$-rational points of a $\mathcal{X}$-split simple group of $\mathcal{X}$-rank at least 2. Consider a measure-preserving, mixing action action $G \curvearrowright X$

on a probability space $X$; let $a : \mathcal{X}^0 \to D^0$ be a one parameter subgroup of a maximal split torus $D$ corresponding to a non-multipliable root $\omega$ of $G$ that is not a short root in the group $G_2$ or a long root in $\text{Sp}(2n)$.

There exists an $L^2_0(X)$-dense space of functions $\mathcal{D}$ so that for every $(k+1)$-tuple $(f) = (f_0, \cdots, f_k)$ in $\mathcal{D}$ and $t_0, \cdots, t_k \in \mathcal{X}$ ordered with increasing absolute values there exists $C = C(f)$ so that

$$
\left| \int_X f_0((a(t_0))^{-1} \cdot x) f_1((a(t_1))^{-1} \cdot x) \cdots f_k((a(t_k))^{-1} \cdot x) d\mu(x) \right| 
\leq C \left[ \left( \sum_{i=1}^{k} \left| \frac{t_0}{t_i} \right| \right) \left( \sum_{i=0}^{k-1} \left| \frac{t_i}{t_k} \right| \right) \right]^{q_\omega} .
$$

The number $q_\omega$ depends on the root dual to the group $a$.

Let us make a few remarks about methods. Mozes’s theorem relies on geometric considerations concerning how the group acts on a certain space of measures. It uses compactness in an essential way precluding an immediate quantitative refinement. Our method was inspired by a proof of quantitative decay of matrix coefficients given in [9]. Initially we had worked out the case $\text{SL}(n, \mathbb{R})$ only, when the paper of Zhenqi Jenny Wang [23] was brought to our attention which generalized computations in [9]; this led us to expand the setting and try to isolate parts of the computation that can work in more general settings. Apart from methods, we have adopted notation from these expositions, so the reader is advised to consult them for further reference. For complete proofs of unproven statements found here and in the book of Howe and Tan, see [13], Chapters 1, 2 and Appendices 1-3 or [1, Appendix D].

While this paper was being written, I was notified that similar results were obtained by Björklund, Einsiedler and Gorodnik in [3]; that work treats the full action of semisimple groups over local fields and adeles acting on suitable homogeneous spaces. It covers actions of rank 1 groups with spectral gap as well as non-split groups, two situations that we cannot treat here, and provides uniform estimates for Sobolev vectors. Their method is dynamical in nature and examines the quantitative properties of the orbit of the correlation measures by the acting elements; in that sense, it is close to the spirit of Mozes’s original
method. In contrast, our method is spectral and builds on consecutive approximations by nicely behaved functions. Since semidirect products are our main focus, for the application of our estimates to semisimple groups we need these groups to contain sufficiently many semidirect products. This excludes groups of split rank at most 1 and almost direct products of such groups.

2. Set up and central notions

In the following sections, we lay down notation, the central objects of study, and the tools we will use in the proofs.

2.1. Semidirect products and excellent representations. Let $\mathcal{K}$ be a local field of characteristic zero, i.e., the real or complex numbers or a finite extension of a $p$-adic field. Having fixed $\mathcal{K}$, whenever we talk about an algebraic group as a functor we will indicate it by boldface, e.g., $G$. Then $G = G(\mathcal{K})$ will denote the group of $\mathcal{K}$-rational points of $G$.

Let $G$ be the group of $\mathcal{K}$-rational points of a connected semisimple algebraic group $G$. Let $D$ be a maximal $\mathcal{K}$-split torus and $B$ a minimal parabolic containing $D$. Write $X(D)$ for the characters of $D$ defined over $\mathcal{K}$. The choice of a parabolic group $B$ determines an ordering of the characters. Let $X^+$ be the set of positive characters with respect to the given ordering.

**Definition 2.1.** Let

$$\mathcal{K}^0 = \{ x \in \mathbb{R} | x \geq 0 \} \quad \text{and} \quad \overline{\mathcal{K}} = \{ x \in \mathbb{R} | x \geq 1 \}$$

when $\mathcal{K}$ is Archimedean. When $\mathcal{K}$ is non-Archimedean, we fix a uniformizer $q$ with $|q|^{-1}$ the cardinality of the residue field of $\mathcal{K}$. Then correspondingly

$$\mathcal{K}^0 = \{ q^n | n \in \mathbb{Z} \} \quad \text{and} \quad \overline{\mathcal{K}} = \{ q^{-n} | n \in \mathbb{N} \}.$$ 

Define subgroups $D^0$ and $D^+$ of $D$ by

$$D^0 = \{ d \in D | \chi(d) \in \mathcal{K}^0 \} \quad \text{for each} \quad \chi \in X(D),$$

$$D^+ = \{ d \in D | \chi(d) \in \overline{\mathcal{K}} \} \quad \text{for each} \quad \chi \in X^+.$$ 

We call $D^+$ the positive Weyl chamber in $D$ (relative to the prescribed data).

Denote the centralizer of $D$ in $G$ by $Z$ and transfer the ordering of $X(D)$ to $X(Z)$ by inclusion. Let

$$Z_+ = \{ z \in Z | |\chi(z)| \geq 1 \} \quad \text{for each} \quad \chi \in X(Z)^+$$

and

$$Z_0 = \{ z \in Z | |\chi(z)| = 1 \} \quad \text{for each} \quad \chi \in X(Z)^+.$$ 

Note that these are $\mathcal{K}$-subgroups of the $\mathcal{K}$-group $Z$. We then have the following decomposition (see [4] and the discussion in [23]):
**Lemma 2.2.** There exists a ‘good’ maximal compact subgroup $K$ of $G$ such that

1. $N_G(D) \subset KD$.
2. We have the decomposition $G = K(Z_+ / Z_0)K$ such that for each $g \in G$, there exists a unique element $z$ of $Z_+$ modulo $Z_0$ so that $g \in KzK$.
3. There exists a finite subset $F \subset C_G(D)$ so that $G = K(D^+ F)K$ and for each $g \in G$ there exist unique $d \in D^+$ and $f \in F$ so that $g \in KdfK$.

Recall that any semisimple group is the almost direct product of its almost simple factors, $G = \prod G_i$. We will assume that no factor is compact, although this can be avoided at the expense of making the statements of the theorems more complicated. We opt for simplicity.

**Definition 2.3.** Now let $\rho : G \to \text{GL}(V)$ be a representation on a finite dimensional $\mathcal{K}$-vector space $V$ with the following properties:

1. $\rho$ is continuous when $\mathcal{K} = \mathbb{R}$ and $\mathcal{K}$-rational in all other cases;
2. for each almost simple factor $G_i$, the only $\rho(G_i)$-fixed point in $V$ is $0$.

Such representations are called excellent. We will assume once and for all that $\ker(\rho) < Z(G)$.

By means of such representations we define the main acting object of this work.

**Definition 2.4.** Let

\[ (2.1) \quad \mathcal{G} = G \rtimes \rho V \]

be the semidirect product of $G$ with $V$ by means of $\rho$. This is the group of $\mathcal{K}$-points of a $\mathcal{K}$-group whose unipotent radical over $\mathcal{K}$ coincides with $V$.

Since $G$ is semisimple and $\text{char}(\mathcal{K}) = 0$, the representation $\rho$ is completely reducible and thus $V$ breaks into irreducible components

\[ V = \bigoplus_{i=1}^{N} V_i. \]

For simplicity we will assume from now on that $V$ is irreducible.

**Definition 2.5.** We use the following notation concerning $V$ and and the representation $\rho$:

1. $\|\cdot\|$ denotes a $K$-invariant norm on $V$.
2. $\Psi$ is the set of weights of $\rho$ with respect to $D$ on $V$.
3. $\lambda$ resp. $\rho$ are the highest resp. lowest weights of $\rho$.
4. For each weight $w$ of $\rho$, $V_w$ is the corresponding weight subspace of $V$.
5. $\Phi$ is the set of roots of $G$ with respect to $D$.
6. For each $\omega \in \Phi$, denote by $g_\omega$ the root space corresponding to the root.
7. $\{\omega_1, \cdots, \omega_p\} \subset \Phi^+$ is the set of simple roots in $\Phi^+$.
8. $q := \left(\frac{1}{3}\right)^{\Psi - 1}$ if $\dim V_\lambda > 1$, otherwise $q := \left(\frac{1}{3}\right)^{\Psi - 2}$.
More details about the aspects of root systems and weights we will use can be found in [23, Section 2.3] and the references therein. The concepts listed above will play a role in the explicit bounds we will give for the correlations that we will now introduce.

2.2. Actions and unitary representations. Let \((X, \mu)\) be a probability space, and let \(H = L^2_0(X)\) be the Hilbert space of square integrable functions on \(X\) orthogonal to the constants. Let \(\langle \rangle\) be the inner product, \(L = L^\infty_0(X) \subset H\), and \(\sigma\) a measure-preserving action of \(\mathcal{G}\) on \(X\); we always use the notation \(g \cdot x\) for \(\sigma(g)(x)\). The action on \(H\) defined by
\[
(g \cdot f)(x) := f(g^{-1} \cdot x)
\]
is a unitary representation of \(G\); we call \(g \cdot f\) a translate of \(f\) by \(g\), suppressing mention of the action. Note that the representation is multiplicative, i.e., it distributes over pointwise (and a.e. pointwise) products of functions:
\[
g \cdot (f h) = (g \cdot f)(g \cdot h).
\]
Furthermore, we assume that for each irreducible component \(V_i\) of \(V\), the representation \(\sigma|_{V_i}\) has no fixed vectors in \(H\), so that the action of each \(V_i\) is ergodic on \(H\).

**Remark 2.6.** In some cases where ergodicity does not hold, we can still look at the intersections of the orthogonal complements of the constant vectors for \(V\) and get bounds on a smaller Hilbert space. This will be the case, for instance, when \(X\) is a quotient of \(\mathcal{G}\) by a lattice and \(\mathcal{G}\) acting by translation; one can always find non-trivial constant functions for the action of \(1 \ltimes V\): they will be the functions constant on the (toral) fibers. We can then take \(H\) to be the subspace of \(L^2_0\) orthogonal to those functions.

**Definition 2.7.** A function \(f\) in a \(K\)-space \(V\) is called \(K\)-finite if the space
\[
\langle K \cdot f \rangle \subset H
\]
is finite dimensional. For any linear space of functions on \(X\), a subscript \(K\) denotes \(K\)-finite functions in that space.

We will deal especially with the algebra of \(K\)-finite bounded functions \(L_K\) and \(L^2\)-dense subspaces in it. As the representation induced on \(L^2(X)\) by the action is unitary, by the Peter-Weyl theorem \(K\)-finite vectors are dense in \(L^2(X)\).

2.3. Discussion of the bounds. After this setup, we can now describe the kind of correlations we will bound and in what way. Recall the notation in the previous section. Let \(f_i\) be bounded, \(K\)-finite functions on \(X\). We assume that \(V\) has no invariant vectors in \(H\).

**Definition 2.8.** Let \(a = (a^0, \cdots, a^k)\) be a \((k+1)\)-tuple of Cartan elements. Define by abuse of notation \(a^\lambda\) to be the element with the smallest absolute value on \(\lambda\) and \(a^\rho\) the one with the smallest absolute value on \(\rho\). The symbolic expression \(i \neq \lambda\) indicates a sum running over \(a^i\) different from \(a^\lambda\) and similarly for \(\rho\).
Note that for \( i \neq \lambda \),
\[
\left| \lambda \left( \frac{a^i}{a^\lambda} \right) \right| \in (0, 1]
\]
and similarly for \( i \neq \rho \),
\[
\left| \rho \left( \frac{a^\rho}{a^i} \right) \right| \in (0, 1].
\]
Define the sums
\[
L(a) = \sum_{i \neq \lambda} \left| \lambda \left( \frac{a^i}{a^\lambda} \right) \right| \quad \text{and} \quad R(a) = \sum_{i \neq \rho} \left| \rho \left( \frac{a^\rho}{a^i} \right) \right|.
\]
Let
\[
R(a) = \min(1, L(a)) \cdot \min(1, R(a)). \tag{2.2}
\]

To see when these expressions give non-trivial bounds, we make the following definitions:

**Definition 2.9.** Given a tuple \( a \), define
\[
d^\lambda(a) = \min_{i \neq \lambda} \log \left| \lambda \left( \frac{a^i}{a^\lambda} \right) \right| \quad \text{and} \quad d^\rho(a) = \min_{i \neq \rho} \log \left| \rho \left( \frac{a^\rho}{a^i} \right) \right|,
\]
where the logarithm satisfies \( \log e = 1 \) for Archimedean \( K \) and \( \log(|q|^{-1}) = 1 \) for non-Archimedean \( K \) and uniformizer \( q \).

It follows directly from the definitions of \( a^\lambda \) and \( a^\rho \) that
\[
R(a) \leq k^2 e^{-d^\lambda(a) - d^\rho(a)}. \tag{2.3}
\]

**Definition 2.10.** In all our results below, we will assume that the tuple \( a \) satisfies the following universal condition:
\[
\max \left( d^\lambda(a), d^\rho(a) \right) > C_\sigma. \tag{2.4}
\]
Here \( C_\sigma \) is a constant depending only on \( \sigma \) and its action on \( X \). This condition is required because we manipulate objects depending on \( \sigma \) and \( \sigma \) parametrized by \( a \); the manipulation will require these elements to be sufficiently small. The number of manipulations is finite so \( C_\sigma \) is the largest of the implicit constants required for each.

Our main result, Theorem 3.15, implies the following:

**Theorem 2.11.** There exists an \( L^2 \)-dense subspace \( \mathcal{D} \) of bounded, zero-mean, \( K \)-finite functions such that for a \((k + 1)\)-tuple of \( f_i \) in \( \mathcal{D} \) there exists \( C > 0 \) such that we have the bound
\[
\left| \int_X f_0(x) \cdot a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x) \right| \leq C R(a)^{\frac{2}{k}}.
\]

The exponent \( q \) is explicitly given in Definition 2.5.
2.4. Approximate projections. Given a finite dimensional normed vector space \((V, \|\cdot\|)\) over \(K\) with \(K\)-invariant norm, we denote by \(\hat{V}\) the unitary dual, i.e., the topological group of all additive unitary characters of \(V\). Consider a basis of \(V\) given by weight vectors of \(\rho\). For \(x = (x_i), y = (y_i) \in V\) in this basis let 

\[(x, y) = \sum x_i y_i\]

be the standard bilinear form on \(V\). Choosing a fixed non trivial unitary character \(\zeta\) of \(K\), define the map \(V \rightarrow \hat{V}\) by 

\[v \rightarrow \zeta((v, \cdot) = :\zeta_v.\]

This correspondence is a topological group isomorphism between \(V\) and \(\hat{V}\) through which we will usually identify the two. In this situation, given \(v, w \in V\), we denote \([v, w] = \zeta_v(w)\).

Under \((\cdot, \cdot)\) we naturally define the transpose of a linear operator; define \(\rho^*: G \rightarrow \text{GL}(V)\) to be the inverse transpose of \(\rho\), 

\[\rho^*(v) := (\rho^{-1})^T(v).\]

This provides an identification of the dual action of \(G\) on \(V^*\) with the action \(\rho^*\) on \(V\), given the topological isomorphism above. On the highest and lowest weight spaces \(V_\lambda\) and \(V_\rho\) of \(\rho\), \(\rho^*\) acts by 

\[\rho^*(g)V_\lambda = \lambda(g^{-1})V_\lambda\]

and 

\[\rho^*(g)V_\rho = \rho(g^{-1})V_\rho\]

and these are the lowest and highest weights of \(\rho^*\) respectively. Furthermore, if \(\rho\) is irreducible and excellent on \(V\), so is \(\rho^*\); finally, \(\|\cdot\|\) is \(\rho^*(K)\)-invariant as well. See [23, Section 6.1] for these facts.

**Definition 2.12.** For \(f \in L^1(V)\) and \(\chi \in \hat{V}\), define the Fourier transform 

\[(\hat{f})(\chi) = \int_V \overline{\chi(v)} f(v) \, dm(v)\]

where \(dm(v)\) is a Haar measure on \(V\). Using the topological identification of \(V\) and \(\hat{V}\), we can view the Fourier transform as a function on \(V\) by the formula 

\[\hat{f}(w) = \int_V \overline{\zeta(w-v)} f(v) \, dm(v) = \int_V [-w, v] f(v) \, dm(v)\]

in the bracket notation of the pairing.

We will use repeatedly the following theorems (Plancherel, inversion and duality):

**Theorem 2.13.** There is a normalization of the dual Haar measure \(dm(\chi)\) on \(\hat{V}\) so that:

1. The Fourier transform extends to an isometry \(L^2(V) \rightarrow L^2(\hat{V})\).
2. If both \(f\) and \(\hat{f}\) are integrable, then for almost every \(v \in V\) 

\[f(v) = \int_{\hat{V}} \chi(v) \hat{f}(\chi) \, dm(\chi).\]
3. Every \( v \in V \) defines a unitary character of \( \hat{V} \) through the pairing \( (v, \chi) \to \chi(v) \) which furnishes a canonical topological isomorphism between \( V \) and \( \hat{V} \).

The following definition provides a good space of functions to carry out abelian harmonic analysis.

**Definition 2.14.** The Schwartz-Bruhat space \( \mathcal{S}(V) \) is just the usual Schwartz space when \( K \) is Archimedean; in the non-Archimedean case, it consists of compactly supported, locally constant functions on \( V \).

The main properties of \( \mathcal{S}(V) \) are that its functions are dense in \( L^2(V) \) and the Fourier transform furnishes a topological isomorphism \( \mathcal{S}(V) \cong \mathcal{S}(\hat{V}) \). For more details about the Fourier analysis facts we will use [21].

**Definition 2.15.** Given a Schwartz function \( \phi \) on \( \hat{V} \) and \( f \in \mathcal{L}_K \) define

\[
P_\phi(f) := \int_V \hat{\phi}(x) (x \cdot f) \, dm(x)
\]

Here we use the formulation of [13, Chapter 11] for Banach-space valued integrals. In other words, \( P_\phi \) is the Fourier transform of the associated representation \( \pi \) on \( \mathcal{S}(\hat{V}) \).

Because of the rapid decay of the Fourier transform \( \hat{\phi} \), \( P_\phi(f) \) retains differentiability properties of \( f \) and the inequality

\[
\| P_\phi(f) \|_p \leq \| \hat{\phi} \|_{L^1(V)} \| f \|_p, \quad 1 \leq p \leq \infty
\]

shows that it is bounded on all the spaces we will consider. Some structural properties of this operator (the case \( \mathcal{H} = \mathbb{R} \) is worked out in [9, Chapter I]; the general case has no new features regarding these properties) include

- \( P \) is self-adjoint (with respect to the inner product of \( \mathcal{H} \)) for real \( \phi \); more generally, \( P_\phi^* = P_{\bar{\phi}} \).
- Operator multiplication transforms to pointwise multiplication of functions:

\[
P_{\phi \psi} = P_\phi \circ P_\psi
\]

This property plus linearity in the subscript shows that \( P \) is a homomorphism from the pointwise algebra of Schwartz functions to self adjoint operators on \( \mathcal{H} \).

- Given the context of Section 2, for \( g \in G \),

\[
\sigma(g) P_\phi \sigma(g^{-1}) = P_{\phi(\rho(g^{-1}) \cdot)}.
\]

When \( \phi \) is \( K \)-invariant, this equation implies that \( P_\phi \) commutes with the \( K \)-action and thus \( K \)-finite vectors are \( L^2 \)-dense in the range of \( P_\phi \).

- We will identify \( \phi \in \mathcal{S}(\hat{V}) \) with \( \phi(\zeta) \in \mathcal{S}(V) \). Under that identification, the action of \( G \) in (2.11) corresponds to the representation \( \rho^* \) on \( \mathcal{S}(V) \), i.e., when we think of \( \phi \) as a function on \( V \), we have

\[
\sigma(g) P_\phi \sigma(g^{-1}) = P_{\phi(\rho^*(g^{-1}) \cdot)}.
\]
Since the projection operators will always be thought as indexed by $S \subset V$, it is this form of the conjugation relation that we will use in the sequel.

- From the last remark, it follows that in terms of sets $S \subset V$ (or supports of Schwartz functions)

\[ \sigma(g)P_S\sigma(g^{-1}) = P_{\rho^*(S)}. \]  

Convergence and limits involving $P$ are obtained using positivity: for $\phi \geq 0$, $P_\phi$ is a positive semidefinite operator. To see this, use (2.10) and self-adjointness:

\[ \langle P_\phi(f), f \rangle = \langle P_{\sqrt{\phi}}(P_{\sqrt{\phi}}(f)), f \rangle = \langle P_{\sqrt{\phi}}(f), P_{\sqrt{\phi}}(f) \rangle \geq 0. \]

This way we see that $P_\phi \geq P_\psi$ and thus $\|P_\phi\|_2 \geq \|P_\psi\|_2$ when $\phi \geq \psi$. Thus, if $\phi_j$ increase or decrease monotonically to a bounded function on $V$, the $P_{\phi_j}$ converge strongly to a bounded, self-adjoint operator on $\mathcal{H}$. Although we will mostly deal directly with the $P_\phi$, since we cannot guarantee control on the $L^\infty$ norm for the limits in general, we will use them as a tool to simplify calculations. Of course, under additional assumptions about the smoothness of the vectors, integration by parts in (2.8) transforms the sequence into one which is $L^p$-convergent for any $p \geq 1$, but since we want to treat $K$-finite vectors that will not be necessarily smooth, we avoid the use of the limit operators as such.

**Definition 2.16.** Let $S$ be a subset of $V$ with the property that its characteristic function $\chi_S$ can be pointwise approximated by a sequence of decreasing compactly supported Schwartz functions; we call such sets admissible. In particular, we will utilize the **annuli**

\[ \text{Ann}(s) := \{ x \in V \mid s^{-1} < \| x \| < s \}. \]

Recall that the norm on $V$ is assumed $K$-invariant. The characteristic function of each annulus $\chi_{\text{Ann}(s)}$ can be approximated by a sequence of smooth functions with the following properties:

\[ \phi_s^k \equiv 1 \text{ on } \text{Ann}(s) \]

\[ \text{supp}(\phi_s^k) \subset \text{Ann}(s + \frac{1}{k}) \]

\[ \phi_s^k \leq \phi_s^l \text{ for } l \leq k \]

The sequence $P_{s,k} := P_{\phi_s^k}$ consists of positive, decreasing, self-adjoint bounded operators on $L^2(X)$ and thus has a strong limit for fixed $s$ as $k$ tends to infinity which by (2.10) is idempotent, since $\phi_s^k \to \chi_{\text{Ann}(s)}$. The image under $P_s = \lim P_{s,k}$ of $L^\infty_0(X)$ is $L^2_0$-dense in $L^\infty_0$ since the $P_s$ form a system of projections that tends to the identity operator in $L^2_0(X)$ as $s$ goes to infinity (see [9] for the computations).

The properties of $P_\phi$ imply the following:

- If $\text{supp}(\phi) \subset S$, then

\[ P_S(P_\phi) = P_\phi. \]
If $S$ is invariant under rotations, then for any $g \in K$,
\begin{equation}
\sigma(g)P_S\sigma(g^{-1}) = P_S.
\end{equation}

By the previous property, when $S$ or $\phi$ are $K$-invariant, $P_S$ or $P_\phi$ commutes with the action of $K$ and thus $K$-finite vectors are dense in the range of $P_S$ or $P_\phi$.

**Definition 2.17.** If $P_S(f) = f$ for some set $S$, we say that the spectral support of $f$ lies in $S$; when $S$ is replaced in the subscript by a Schwartz function $\phi$, the notion will refer to the support of $\phi$.

Intuitively, $P_S$ restricts the spectrum of $f$ to lie in $S$, so a function which is unaffected by this application is justified in being called ‘spectrally supported in $S$’. Note that $P_S(L^2)$ is a closed vector subspace of $L^2$ since the $P_S$ are norm bounded (for fixed $S$).

With this notion in hand, we can define explicitly the dense subspace of $\mathcal{L}_K$ where we will bound the coefficients effectively.

**Definition 2.18.** Define $\mathcal{D}$ to be the union
\[ \mathcal{D} = \bigcup_{s > 0} \bigcup_{k > s^2} P_{\phi_k}(\mathcal{L}_K) \]
and call it the space of spectrally bounded functions in $\mathcal{L}_K$.

It is easy to see that this space is $L^2$-dense in $\mathcal{L}_K$. The specific choice $k > s^2$ is not important; we just need some leeway for approximations, and we do not want $k$ to be so small as to cause problems with stretching annuli.

**Remark 2.19.** Spectrally bounded functions are analogues of trigonometric polynomials in the context of unitary representations. One can ask if we can extend our results to functions with sufficiently fast decay of the Fourier transform on $\hat{V}$, which then covers all smooth $K$-finite vectors in $L^2_0(X)$. The answer is likely yes, but the analysis becomes much more delicate in the Archimedean case. We plan to address this extension in a sequel together with the removal of the $K$-finiteness condition. Inspired by the computation in [11] we try to remove the $K$-finiteness condition using estimates on the Peter-Weyl expansion of a smooth vector with respect to $K$; for the case of multiple correlations, however, $L^2$ convergence of this series is not enough and we need convergence in $C^\infty$. This will complicate the analysis considerably and we postpone this, too, for the technical sequel.

### 3. Main Results

#### 3.1. Projection operators and pointwise multiplication.
First, we need a lemma on how pointwise multiplication of functions behaves with respect to the operators $P_\phi$. Below we identify $\hat{V}$ with $V$ and the two dual Haar measures by the isomorphism in Section 2.3 (compatibility in the computations below is guaranteed by (2.7)). Recall the notation $\langle u, z \rangle = \zeta_u(z)$ for $u, z \in V$ (keep in mind the usual case $\langle u, z \rangle = e^{i\langle (u, z) \rangle}$). The result is the following.
Lemma 3.1. Let \( \phi, \psi \in \mathcal{S}(V) \) and \( f, g \in L^2(X) \) be such that the pointwise (a.e.) product \( P_\phi(f)P_\psi(g) \) is in \( L^2(X) \). Suppose \( \omega \in \mathcal{S}(V) \) is identically equal to 1 on \( \text{supp}(\phi) + \text{supp}(\psi) \); then \( P_\omega(P_\phi(f)P_\psi(g)) = P_\phi(f)P_\psi(g) \).

Proof. Compute:

\[
P_\omega(P_\phi(f)P_\psi(g))
\]

\[
= \int \hat{\omega}(z) \int \hat{\phi}(x) \rho(z + x) f dm(x) \int \hat{\psi}(y) \rho(z + y) g dm(y) \ dm(z)
\]

\[
= \int \hat{\omega}(z) \int \hat{\phi}(x-z) \rho(x) f dm(x) \int \hat{\psi}(y-z) \rho(y) g dm(y) \ dm(z)
\]

\[
= \int \rho(x) f \int \rho(y) g \int \hat{\omega}(z) \hat{\phi}(x-z) \hat{\psi}(y-z) \ dm(z) \ dm(x) \ dm(y).
\]

Now expand the inner integral using the definition of the Fourier transform, valid for \( L^1 \) functions:

\[
\int \hat{\omega}(z) \hat{\phi}(x-z) \hat{\psi}(y-z) \ dm(z)
\]

\[
= \int \int \int \omega(u_3)[-z, u_3] \hat{\phi}(u_1)[-x-z, u_1]
\]

\[
\cdot \psi(u_2)[-y-z, u_2] \ dm(u_2) \ dm(u_1) \ dm(u_3) \ dm(z)
\]

\[
= \int \int \phi(u_1)[-x, u_1] \psi(u_2)[-y, u_2]
\]

\[
\cdot \left( \int \omega(u_3)[-z, u_3] [z, u_3] [z, u_2] \ dm(u_2) \ dm(u_3) \right) \ dm(u_1) \ dm(u_2)
\]

\[
= \int \int \phi(u_1)[-x, u_1] \psi(u_2)[-y, u_2]
\]

\[
\cdot \left( \int [z, u_1 + u_2] \omega(u_3)[-z, u_3] \ dm(u_3) \ dm(z) \right) \ dm(u_1) \ dm(u_2).
\]

The integral in the parentheses is simply

\[
\int [z, u_1 + u_2] \hat{\omega}(z) \ dm(z) = \omega(u_1 + u_2) = 1
\]

by Fourier inversion and the fact that \( u_1 \in \text{supp}(\phi), u_2 \in \text{supp}(\psi) \). Untangling the remaining integrals, we get the required result.

Corollary 3.2. Let \( \text{supp}(\phi) \subset S \) and \( \text{supp}(\psi) \subset T \) for admissible sets \( S \) and \( T \). Then

(3.1) \( P_{S+T}(P_\phi P_\psi) = P_\phi P_\psi \).

The relations (3.1) and (2.15) form the core of the main computation.

3.2. Bounds on functions projected on cones. In this section, we examine how restricting a unit (in the \( L^2 \)-norm) \( K \)-finite vector \( f \) to the image of an approximate projection \( P_\phi \) for suitable \( \phi \) affects its norm. This was accomplished in greater generality in [23], and we refer to that work for complete proofs.
REMARK 3.3. The idea of estimating matrix coefficients by looking at the effect the representation has on their spectral support and then estimating norms of functions with restricted spectral support is a major theme in Chapter 5 of [9]; the non-commutativity of $K$ in our setting increases the complexity of this method considerably. However, the detailed analysis in [23] allows one to carry it out effectively.

In order to state the second main ingredient for bounding norms of projected vectors, we need some additional concepts from [23]. Recall the list of notations from Section 2.1 and recall that $\rho$ is irreducible with highest and lowest weights $\lambda$ and $\varrho$, respectively. For $\psi \in \Psi$, let $\pi_\psi(v)$ be the projection of $v$ on the weight space $V_{\psi}$.

DEFINITION 3.4. Define the cones

\[
\text{Cone}_\varrho(c, s) = \{ v \in V | \| \pi_\varrho(v) \| \leq c \text{ and } \| v \| \geq s \}, \\
\text{Cone}_\lambda(c, s) = \{ v \in V | \| \pi_\lambda(v) \| \leq c \text{ and } \| v \| \geq s \}.
\]

\[\| \pi_\lambda(v) \| \leq c \vline \vline \quad \| v \| \geq s \quad \text{Cone}(c, s) \subset \bigcup_{1 \leq i \leq l} E_i \]

**Figure 1.** Schematic of a cone as the grey-shaded area (extending indefinitely to each horizontal side).

The fundamental properties of these sets are described in [23, Proposition 6.1]. We will not use Proposition 6.1 itself here, but we will follow verbatim the computations in Proposition 7.1, which uses Proposition 6.1 in a crucial way. For this reason, we paraphrase Proposition 6.1 below.

The norm $\| \cdot \|_\infty$ on $V$ defined by

\[\| v \|_\infty = \max_{\psi \in \Psi} \| \pi_\psi(v) \|\]

is equivalent to the given norm since dim($V$) $< \infty$, so in particular

\[C_0^{-1} \| v \| \leq \| v \|_\infty \leq C_0 \| v \|;\]

we will use this constant $C_0$ in the statement of the following proposition.

**Proposition 3.5.** Let $s > 0$ and $c$ small enough to satisfy $cs^{-1} < C$, where $C$ is a constant depending only on $\rho$. There exists a finite open cover of $\text{Cone}_\psi(c, s)$,

\[\text{Cone}_\psi(c, s) \subset \bigcup_{1 \leq i \leq l} E_i \]
so that for each individual \( E_i \) there are at least \([C_0^{-1}(\xi)^{−3}])\) distinct elements \( \tau_j^i \in K \) such that the translates \( \rho(\tau_j^i)E_i \) are pairwise disjoint:

\[
\rho(\tau_j^i)E_i \cap \rho(\tau_k^i)E_i = \emptyset, \quad j \neq k.
\]

**Remark 3.6.** In the case of \( SL(2,\mathbb{R}) \) acting on \( \mathbb{R}^2 \) with highest weight corresponding to the vertical axis and lowest weight to the horizontal axis, these ‘cones’ are contained in actual cones through the origin with a central disk removed. Depending on the angle \( \theta \) of the smallest cone containing them, there exist approximately \( 2\pi/\theta \) rotations in \( K = SO(2,\mathbb{R}) \) that produce disjoint slices. These slices glue together to give the spectrum of a function in \( \mathcal{D} \). The facts that

1. there are many slices,
2. they are disjoint, and
3. the representation \( \pi \) is unitary

together imply that on each slice the norm of the projected function is small. The strategy for the proof of the main estimate is to use the Cartan elements to squash the spectral support into one of these slices, taking care not to accidentally dilate the rest of the functions.

The fact that only the highest and lowest weights feature in Proposition 3.5 is the only constraint that prevents us from using other weights to bound the correlations; however, this restriction on the weights is essential for the proof of this proposition.

The action of the Cartan elements \( a^i \) on annuli is captured in the following definition:

**Definition 3.7.** Let \( Ann(s) \) be the annulus defined in Section 2.4,

\[
X_1(a,s) = Ann(s) \cap \left( \sum_{\ell \neq \rho} \rho^* \left( \frac{a^i}{a^\ell} \right) (Ann(s)) \right)
\]

and

\[
X_2(a,s) = Ann(s) \cap \left( \sum_{\ell \neq \lambda} \rho^* \left( \frac{a^i}{a^\ell} \right) (Ann(s)) \right).
\]

Note the appearance of the contragredient representation \( \rho^* \). The \( X_i \) are the sets in the spectral domain where the correlation integrands will eventually be restricted. The annulus intersecting the sumset corresponds to the action of \( a^0 = I \); the sumset of the squashed annuli will appear after applying Lemma 3.1 in an appropriate manner.

The next proposition states that the \( L^2 \) norms of \( P_{X_i}(f) \) are small. In the case of \( SL(2,\mathbb{R}) \) acting on \( \mathbb{R}^2 \) (and by an easy modification, any irreducible finite dimensional representation of \( SL(2,\mathbb{R}) \)) the proof is given in Chapter 5 of [9]. The general case is much more difficult and lies at the technical heart of [23]. Here we will only sketch the proof.

**Proposition 3.8.** Recall the notation in 2.8 and 2.9. Let \( f \) be \( K \)-finite with \( \| f \|_2 = 1 \), \( \dim(K \cdot f) = d_f \), \( a^i \in D^+ \) for \( i = 0, \cdots, k \) satisfying (2.4), \( Ann(s) \) the
annulus defined in Section 2.4, and \( F_s \in \mathcal{S}(V) \) with compact support inside the set \( X_1(a,s) \). Then for some positive \( C \) independent of \( a, s, \) and \( f \) we have the bounds

\[
\left\| P_{F_s}(f) \right\|_2 \leq C s^q d_f^2 \left( \sum_{i \neq \rho} \left| \varrho \left( \frac{\alpha^i}{\alpha^\rho} \right) \right| \right)^{\frac{q}{2}}.
\]

Similarly, if the support of \( F_s \) is in the set \( X_2(a,s) \), then correspondingly

\[
\left\| P_{F_s}(f) \right\|_2 \leq C s^q d_f^2 \left( \sum_{i \neq \lambda} \left| \lambda \left( \frac{\alpha^i}{\alpha^\lambda} \right) \right| \right)^{\frac{q}{2}}.
\]

We will derive this proposition as a consequence of a sequence of lemmata meant to break down Proposition 7.1 of [23] and extract the parts we need for our purposes. First, we need a crucial result of [5].

**Lemma 3.9.** Let \( \mathcal{Y} \) be a unitary representation of a compact group \( K \), \( \mu \) a \( K \)-finite vector in \( \mathcal{Y} \) with span \( \mathcal{Y}(\mu) \). There exists a unique idempotent function \( e_\mu \) in the convolution algebra \( C(K) \) (i.e., \( e_\mu * e_\mu = e_\mu \)) whose action \( \mathcal{Y}(\hat{e_\mu}) \) fixes \( \mu \) and whose norm satisfies

\[
\left\| e_\mu \right\|_2^2 = d_\mu = \dim(\mathcal{Y}(\mu)).
\]

**Lemma 3.10.** Let \( \overline{P}_F \) be the approximate projection operator associated to the regular representation of \( K \ltimes \hat{V} \). Suppose that \( F \in \mathcal{S}(V) \) non-negative and such that for every left \( K \)-invariant vector \( f \) in \( L^2(K \ltimes V) \), the bound

\[
\left\| \overline{P}_F(f) \right\| < \epsilon(F) \left\| f \right\|_2
\]

holds. Then for any \( K \)-finite \( f \in L^2(X) \) with \( d_f = \dim(\langle K \cdot f \rangle) \) in the setting of Proposition 3.8 the corresponding bound holds:

\[
\left\| P_{F^2}(f) \right\|_2 \leq \epsilon(F^2) d_f \left\| f \right\|_2.
\]

**Remark 3.11.** The content of this useful lemma is a transference principle: when we need a bound for the projection operators \( P_\phi \), we can assume that the projection measure is taken in the regular representation of \( K \ltimes V \) on \( K \)-fixed vectors.

**Proof.** We will only describe the main steps in the reduction; the detailed proof can be found in [23, pp. 47–52]. Unfortunately some of the notation here conflicts with that of Wang.

**Step 1:** Using the amenability of \( K \ltimes V \), one can approximate in the Fell topology the unitary representation \( \pi \) by the left regular representation \( L \). From this approximation we can bound the function

\[
\phi^\pi(x,y) := \langle P_F(\pi(x)f), P_F(f)(\pi(y)f) \rangle
\]

by a convex combination of projected matrix coefficients of the form

\[
\langle \overline{P}_F(\pi(x)f_i), \overline{P}_F(f)(\pi(y)f_i) \rangle.
\]
Convexity implies the bound $\sum \| f_i \|^2 \leq \| f \|^2$.

Furthermore, Lemma 3.9 gives a projection $^3 L(\overline{\sigma_f})$ with the property that

$$e_f \ast \phi^\pi \ast e_f = \phi^\pi,$$

so in fact we can approximate $\phi^\pi$ by a convex combination of projected coefficients of the form

$$e_f \ast \phi^L \ast e_f = \left\langle P_F(L(x)L(\overline{\sigma_f}) f_i), P_F(f)(L(y)L(\overline{\sigma_f}) f_i) \right\rangle.$$

Because $e_f$ is a projection, we can assume that $e_f \ast f_i = f_i$ discarding the orthogonal complement. This procedure reduces the estimation of $\| P_F(f) \|$ to $\phi^\pi(e,e)$ to $\phi^L(e,e)$ for $f_i$ satisfying the conditions listed above.

**Step 2:** Using the invariance of $f_i$ by $e_f$ and the unitarity of the Fourier transform on $L^2(V)$, we get the further invariance $\overline{\sigma_f} \ast \hat{f}_i = \hat{f}_i$ (the Fourier transform is on the $\hat{V}$ factor). Let

$$f_i(k \rtimes v) = \sup_{k' \in K} \hat{f}_i(k' \cdot (k \rtimes v)).$$

Then $f_i$ is a left $K$-invariant function in $L^2(K \rtimes \hat{V})$. Finally let $\hat{f}_i = \mathcal{F}(f_i)$ where $\mathcal{F}$ here is the inverse Fourier transform (so $\hat{f}_i$ is left $K$-invariant function on $K \rtimes V$). The most important estimate for $\hat{f}_i$ is

$$\left\langle P_F(f_i), f_i \right\rangle \leq \left\langle P_F(\hat{f}_i), \hat{f}_i \right\rangle, \quad (3.4)$$

providing a bridge between the original $f$ (approximable by convex combinations of $f_i$) and the nicely behaved $\hat{f}_i$. In the other direction, the $\hat{f}_i$ satisfy

$$\| \hat{f}_i \| \leq (d_f)^{\frac{1}{2}} \| f_i \|; \quad (3.5)$$

this is where the dimension of the $K$-span of the original $f$ appears.

**Step 3:** Since (3.4) implies that

$$\| P_{\sqrt{\mathcal{F}}}(f_i) \| \leq \| P_{\sqrt{\mathcal{F}}}(\hat{f}_i) \|,$$

it is sufficient to bound the last norm by an expression of the form

$$e(\sqrt{\mathcal{F}}) \| \hat{f}_i \| \leq e(\sqrt{\mathcal{F}}) d_f^{\frac{1}{2}} \| f_i \|$$

and use Step 1 to bound the original norm. \hfill \Box

---

Note that $L(g)$ is precisely the action of convolution by $g$. 

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Journal of Modern Dynamics Volume 10, 2016, 81–111
**Lemma 3.12.** The set $X_1(a, s)$ satisfies

$$ X_1(a, s) \subseteq \text{Cone}_\varrho \left( C_0 s \sum_{i \neq \varrho} \varrho \left( \frac{a^\varrho}{a^i} \right), s^{-1} \right). $$

Similarly,

$$ X_2(a, s) \subseteq \text{Cone}_\lambda \left( C_0 s \sum_{i \neq \lambda} \lambda \left( \frac{a^\lambda}{a^i} \right), s^{-1} \right). $$

**Proof.** By definition, $v \in X_1(a, s)$ satisfies the following:

1. $s^{-1} \leq \|v\| \leq s$,
2. there exist $v_1, \cdots, \widehat{v_i}, \cdots, v_n$ (the hatted factor is omitted) such that $v = v_1 + \cdots + v_n$ (the sum ranges over $i \neq \varrho$), and
3. for each $i \neq \varrho$, $v_i = \rho^* \left( \frac{a^i}{a^\varrho} \right) w_i$ for some $w_i$ satisfying $s^{-1} \leq \|w_i\| \leq s$.

For each $i$, then, 3 shows that

$$ s^{-1} \leq \left\| \rho^* \left( \frac{a^\varrho}{a^i} \right) v_i \right\| \leq s. $$

Noting that $\rho^*$ acts on the weight space of $\rho$ as $\varrho^{-1}$, we have

$$ \left\| \varrho \left( \frac{a^i}{a^\varrho} \right) \right\| \| \pi_\varrho(v_i) \| = \left\| \pi_\varrho \left( \rho^* \left( \frac{a^\varrho}{a^i} \right) v_i \right) \right\| \leq \left\| \rho^* \left( \frac{a^\varrho}{a^i} \right) v_i \right\|_\infty. $$

From the equivalence of $\|\cdot\|$ and $\|\cdot\|_\infty$ and the definition of $\|\cdot\|_\infty$ we get

$$ \left\| \rho^* \left( \frac{a^i}{a^\varrho} \right) v_i \right\|_\infty \leq C_0 \left\| \rho^* \left( \frac{a^i}{a^\varrho} \right) v_i \right\| \leq C_0 s $$

and using the previous inequality, we get

$$ \left\| \varrho \left( \frac{a^i}{a^\varrho} \right) \right\| \| \pi_\varrho(v_i) \| \leq C_0 s $$

giving

$$ \| \pi_\varrho(v_i) \| \leq \left\| \rho \left( \frac{a^\varrho}{a^i} \right) \right\| \| \pi_\varrho(v_i) \|. $$

Now $v = \sum_{i \neq \varrho} v_i$, so the triangle inequality gives

$$ \| \pi_\varrho(v) \| = \left\| \sum_{i \neq \varrho} \pi_\varrho(v_i) \right\| \leq C_0 s \sum_{i \neq \varrho} \left\| \rho \left( \frac{a^\varrho}{a^i} \right) \right\|. $$

Together with inequality

$$ s^{-1} \leq \|v\| \leq s, $$

this shows that

$$ v \in \text{Cone}_\varrho \left( C_0 s \sum_{i \neq \varrho} \left\| \rho \left( \frac{a^\varrho}{a^i} \right) \right\|, s^{-1} \right). $$

The proof for $X_2$ is identical. \hfill $\square$
Now we can aggregate everything together to get Proposition 3.8.

*Proof of Proposition 3.8.* We will prove only inequality (3.2), the next one having an identical proof.

In the notation of Lemma 3.10, let $\tilde{f}$ be a left $K$-invariant element in $L^2(K \ltimes V)$. Let $F_\xi$ be non-negative, $F_\xi \leq 1$ and supported in $X_1(a,s)$. By monotonicity of $P$ and Lemma 3.12 we have

$$\| \tilde{P}_{F_\xi}(\tilde{f}) \| \leq \| \tilde{P}_{\text{Cone}}(\tilde{f}) \| \leq \sum_{j=1}^l \| \tilde{P}_{w_j}(\tilde{f}) \|,$$

where

$$\text{Cone} = \text{Cone}_{\rho} \left( C_0 s \sum_{i \neq \rho} \left| \rho \left( \frac{a^\rho}{a_i} \right) \right|, s^{-1} \right)$$

and the $w_j$ are supported on the decomposition of Cone into $E_j$ given by Proposition 3.5.

Using monotonicity, left $K$-invariance and the elementary Parseval identity, the norm $\| \tilde{f} \|^2$ satisfies for each $j$

$$\| \tilde{f} \|^2 \geq N \times \| \tilde{P}_{w_j}(\tilde{f}) \|^2$$

where $N$ is the number of disjoint orbits of $E_j$; by Proposition 3.5, this number is at least as large as

$$[C_0^{-1}(\frac{c}{s^{-1}})^{-q}] \sim C_1 s^{-2q} \left( \sum_{i \neq \rho} \left| \rho \left( \frac{a^\rho}{a_i} \right) \right|^q \right)$$

where $C_1$ only depends on the action $^4 \rho$. Note the $s^{-1}$ in the description of Cone.

This implies the bound

$$\| \tilde{P}_{F_\xi}(\tilde{f}) \|^2 \leq IC_2 s^{2q} \left( \sum_{i \neq \rho} \left| \rho \left( \frac{a^\rho}{a_i} \right) \right|^q \right) \| \tilde{f} \|^2.$$

Since $F_\xi$ was arbitrary non-negative, using Lemma 3.10 we get inequality (3.2) for the original $f$.

Since admissible sets can be approximated from above by Schwartz functions and the operators $P_\xi$ are monotone, we get the corollary

**Corollary 3.13.** With notation as in Proposition 3.8 and $S$ an admissible set contained in one of the $X_1(a,s)$, the same bound holds for $\| P_\xi(f) \|_2$.

**Remark 3.14.** In the case of $G = \text{SL}(2,\mathbb{R})$, $K$ is commutative and a much easier proof of Lemma 3.8 follows from [9, Chapter V, Theorem 3.3.1].

To help the reader navigate Propositions 6.1 and 7.1 of [23] we list a correspondence of notation between the two papers:

1. Our operator $P_\phi(f)$ is given as $\Pi(\phi)$.

---

4 The full details on $C_1$ can be found in [23, pp.52-53].
2. Our annulus $\text{Ann}(s)$ is denoted $X_s$.
3. Our $F_i$ is denoted $f_i$.
4. Cartan elements $a$ are denoted $a$; for us, $\omega = 1$ throughout as the action is living on a single Weyl chamber.
5. The behavior of $P_\phi(f)$ under conjugation

$$\sigma(g)P_\phi\sigma(g^{-1}) = P_\phi(\rho(g^{-1})\cdot)$$

is expressed by

$$\Pi(g)\hat{\Pi}(\phi)\Pi(g^{-1}) = \hat{\Pi}(g \ast \phi).$$

3.3. **Main theorem.** Now consider distinct elements $a^i \in D^+$, $i = 0, \ldots, k$ as above and $k + 1$ functions $f_i \in \mathcal{D}$, $i = 0, \ldots, k$. We want to bound the correlation integral

$$\int_X a^0 \cdot f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x).$$

Since $P_{s,k}(f) \rightarrow f$ for any $f \in \mathcal{L}$ and we only have finitely many $f_i$, we can assume that all $f_i$ are in the image of $P_{s,l}$ for some $s$, $l$ (and thus certainly in the image of $P_{s'}$ where $s' = s + \frac{1}{2}$).

For notational convenience, we will denote the action $\rho^* (a) \rightarrow \phi^k (\rho^* (a)^{-1})$ by $\rho^* (a) \cdot \phi^k$ in accordance with the action on its support.

**Theorem 3.15.** Let $a^i, f_i, s$ be as above. Let $f_\lambda$ and $f_\theta$ be the factors acted upon by $a^k$ and $a^0$ respectively. Let

$$d_i = \dim \langle K \cdot f_i \rangle.$$

There exists a positive constant $C'$ independent of the $f_i$ such that if

$$\max \left\{ d^k (a), d^0 (a) \right\} > C_\sigma,$$

we have the bound

$$\left| \int_X a^0 \cdot f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x) \right|$$

$$\leq s^{2k} d_k^{\frac{1}{2}} \| f_\lambda \|_2 d_\theta \| f_\theta \|_2 \left( \prod_{i \neq \lambda, \theta} \| f_i \|_\infty \right) \mathcal{R}(a)^{\frac{k}{2}}.$$

The proof is based on an examination of the effect on the spectrum of $P_\phi(f)$ of Cartan elements, of taking pointwise products, and finally of correlating with other $P_\phi^2(f)$. In Figure 2 we see a spectral picture of an approximate annulus $\phi$ applied to the spectrum of a sample $f$; as Cartan elements act on this spectrum it is distorted\(^5\) as in Figure 3. Observe the gradual degeneration of the spectrum into a long, thin strip with a central ball removed.

**Proof.** For simplicity in notation, assume $a^0 = a^0$. Apply $(a^0)^{-1}$ to all factors in the correlation. To ease notation, denote the ratios $(a^0)^{-1} a^i$ by

$$a^*_i := (a^0)^{-1} a^i.$$

\(^5\) Artifacts of the numerical approximation exist in Figure 3, but the limited band of the support is accurate.
Note $a_s^0 = I$. The correlation can be written as

\[(3.9) \quad \int_X P_{s,1}(f_0) a_s^1 \cdot P_{s,1}(f_1) \cdots a_s^k \cdot P_{s,1}(f_k) \, d\mu\]
which by (2.11) becomes

\[(3.10) \quad \int_X P_{s,l}(f_0) P_{\rho^*(a^j_1)}(a^1_1 \cdot f_1) \cdots P_{\rho^*(a^j_k)}(a^k_1 \cdot f_k) \, d\mu. \]

We now use Lemma 3.1 repeatedly to conclude that

\[P_{\rho^*(a^j_1)}(a^1_1 \cdot f_1) \cdots P_{\rho^*(a^j_k)}(a^k_1 \cdot f_k) \in P_\Sigma(\mathcal{Z}) \]

where \(\Sigma\) is the iterated sum set \(\sum_{i=1}^k \rho^*(a^j_i)\) (Ann(\(s'))\); thus, in particular, if

\[z := P_{\rho^*(a^j_1)}(a^1_1 \cdot f_1) \cdots P_{\rho^*(a^j_k)}(a^k_1 \cdot f_k), \]

then \(P_\Sigma(z) = z\). Thus the integral above becomes

\[(3.11) \quad \int_X P_{s,l}(f_0) P_\Sigma(z) \, d\mu. \]

Now \(P_\Sigma\) is an orthogonal projection so we can transfer \(P_\Sigma\) from \(z\) to \(P_s(f_0)\), getting

\[P_\Sigma(P_{s,l}(f_0)) = P_{\chi_s \phi_j}(f_0). \]

Here we are abusing notation a little bit, since the last expression need not be a bounded function; we will take this shortcut to mean that we have an arbitrary Schwartz function \(\phi\) dominating the function \(\chi_\Sigma\) and we are applying \(P_{\phi}\) to both terms of the ‘inner product’; the rightmost term is unaffected, while the leftmost has spectral support approximately equal to that of \(\chi_s \phi_j\), since \(\phi\) is arbitrary and the support of \(\chi_s \phi_j\) is easily seen to be an admissible set (also see Corollary 3.13). Thus the integral becomes

\[\int_X P_{\chi_s \phi_j}(f_0) z \, d\mu = \int_X P_{\chi_s \phi_j}(f_0) P_{\rho^*(a^1_1)}(a^1_1 \cdot f_1) \cdots P_{\rho^*(a^k_1)}(a^k_1 \cdot f_k) \, d\mu \]

\[= \int_X P_{\chi_s \phi_j}(f_0) a^1_1 \cdot P_{s,l}(f_1) \cdots a^k_1 \cdot P_{s,l}(f_k) \, d\mu \]

\[= \int_X a^0 \cdot P_{\chi_s \phi_j}(f_0) a^1_1 \cdot P_{s,l}(f_1) \cdots a^k_1 \cdot P_{s,l}(f_k) \, d\mu. \]

In the last two steps, we unwrapped the correlation back into the original form, but we gained the factor

\[P_{\chi_s \phi_j}(f_0) \]

whose norm we know how to bound. Write

\[U_0 := \chi_s \phi_j. \]

Now we will repeat the procedure with \(a^j\). If it happens that \(a^j = a^0\), there is no content to this step; we can skip directly to the last bound and the left hand factor of \(\Re(a)\) is 1. Otherwise, assume for convenience in notation that \(a^j = a^k\).

Apply \((a^k)^{-1}\) to all factors in the correlation and denote the resulting ratios by \(a^*_k\). We get

\[(3.12) \quad \int_X a^*_k \cdot P_{U_0}(f_0) P_{s,l}(f_k) \prod_{i=1}^{k-1} a^*_i \cdot P_{s,l}(f_i) \, d\mu. \]
By unitarity, the value of the integral is not affected. So, now we can repeat the reasoning above, summing the indices for all factors except for $P_s, l(f_k)$, and conclude that this integral is equal to

$$\int_X a_0^0 \cdot P_{U_0}(f_0) P_{X,l}(f_k) \prod_{i=1}^{k-1} a_i^{i,i} \cdot P_{s,l}(f_i) d\mu$$

(3.13)

$$= \int_X P_{s,l}(f_k) P_{s,k}(z_k) d\mu$$

$$= \int_X P_{s,k}(f_k) z_k d\mu$$

(3.14)

$$= \int_X a^0 \cdot P_{U_0}(f_0) a^k \cdot P_{U_k}(f_k) \prod_{i=1}^{k-1} a^{i,i} \cdot P_{s,l}(f_i) d\mu$$

where

$$\Sigma_k = \sum_{j=0}^{k-1} a_j^{i,i} (\text{Ann}(s')), \quad z_k = \prod_{i=0}^{k-1} a_i^{i,i} \cdot P_{s,l}(f_i),$$

and $U_k = \chi_{\Sigma_k} \phi_s^k$.

Denote by $U_0$ and $U_k$ respectively also the supports of the corresponding functions (which are bounded above by 1 and thus by the characteristic functions of the supports). We now apply Proposition 3.8 with $U_0$ and $U_k$ for $F_s$ to bound

$$\|a^0 \cdot P_{U_0}(f_0)\|_2 = \|P_{U_0}(f_0)\|_2 \leq C s_s^d (\sum_{i \neq 0} |\theta(\frac{a^0}{a^i})|)^{\frac{q}{2}}$$

and if $a^i \neq a^0$,

$$\|a^k \cdot P_{U_k}(f_k)\|_2 = \|P_{U_k}(f_k)\|_2 \leq C s_s^d (\sum_{i \neq k} \lambda(\frac{a_k}{a^i})|)^{\frac{q}{2}}.$$
weights; the roots of \( SL(2, \mathcal{K}) \) are
\[
\text{diag}(a, a^{-1}) \rightarrow a^\pm 2
\]
and the weights of the standard representation on \( \mathcal{K}^2 \) are
\[
\text{diag}(a, a^{-1}) \rightarrow a^\pm 1
\]
with the obvious weight spaces \( V_1 = \{(v_1, 0) \in \mathcal{K}^2 \} \) and \( V_2 = \{(0, v_2) \in \mathcal{K}^2 \} \). Take the weight
\[
\text{diag}(a, a^{-1}) \rightarrow a
\]
to be positive, so this is the highest weight. The highest weight space is one dimensional, so the exponent \( q \) defined in Section 2 is in our case 1. Write
\[
a^i = \begin{pmatrix} a_i & 0 \\ 0 & a_i^{-1} \end{pmatrix}
\]
for \((k + 1)\) Cartan elements with \( a_0 < a_1 < \cdots < a_k \) and if \( i > j \), \( \frac{a_i}{a_j} > C_0 \) for some \( C_0 \) depending on the action of \( G \) on \( X \). Applying Theorem 3.15, we get

**Corollary 3.17.** In the setting of Theorem 3.15 and \( G = SL(2, \mathcal{K}) \ltimes \mathcal{K}^2 \) with the standard action and \( \|f_i\|_\infty \) normalized to 1, we get the bound
\[
\int_X a^0 \cdot f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x) \leq C \sigma^2 \frac{1}{d_0} \left( \sum_{i=0}^{k-1} \frac{|a_i|}{a_k} \right) \frac{1}{2} \left( \sum_{i=1}^{k} \frac{|a_0|}{a_i} \right) \frac{1}{2}.
\]

In the case \( k = 1 \) we recover the bound from Chapter 5 of [9].

For the second example, consider the action of \( SL(2, \mathcal{K}) \) on its Lie algebra over \( \mathcal{K} \), denoted by \( g \); it is equivalent to \( S^2(\mathcal{K}^2) \), the second symmetric power of \( \mathcal{K}^2 \) (in the case \( \text{char}(\mathcal{K}) = 0 \) that we are considering). The weights and weight spaces in this case coincide with the roots and the highest weight space (pick \( \text{diag}(a, a^{-1}) \rightarrow a^2 \) as positive) is again one dimensional. Therefore, by the same procedure as above, we have

**Corollary 3.18.** In the setting of Theorem 3.15 and \( G = SL(2, \mathcal{K}) \ltimes g \) with the adjoint action on the Lie algebra and \( \|f_i\|_\infty \) normalized to 1, we get the bound
\[
\int_X a^0 \cdot f_0(x) a^1 \cdot f_1(x) \cdots a^k \cdot f_k(x) \, d\mu(x) \leq C \sigma^2 \frac{1}{d_0} \left( \sum_{i=0}^{k-1} \frac{|a_i|}{a_k} \right) \frac{1}{2} \left( \sum_{i=1}^{k} \frac{|a_0|}{a_i} \right) \frac{1}{2}.
\]

4. **Higher rank split simple groups**

4.1. **Semidirect products in split groups.** In this section we describe how to use the results obtained so far to get effective multiple mixing of correlations of one parameter subgroups in simple split groups of higher rank and restricted effective multiple mixing for general tuples in a Weyl chamber. We do this by locating semidirect products enveloping the one parameter subgroup in question to which we can apply the main results. We keep notation from previous sections when referring to the functions \( f_i \) in the definition of the multiple correlation, the Cartan elements \( a^i \), etc. In this section we will make heavy use of results from [17].
Our setting involves a simple algebraic group split over $\mathcal{K}$ of rank greater than or equal to 2, a maximal $\mathcal{K}$-split torus $D$, root system $\Phi = \Phi(G,D)$ and ordering $\Phi^+$. Consider a mixing action $\sigma$ of $G$ on a standard probability space $(X, \mu)$. We want to apply the results above to bound multiple correlation coefficients for a given one parameter group $\sigma(\mathfrak{a}(t))$ on $X$. In order to achieve this, following the proof of Proposition 1.6.2 in [15] we do the following: given the root $\omega \in \Phi^+$ corresponding to $\mathfrak{a}$, we choose another positive root $\omega'$ that is not orthogonal to $\omega$. Then from the Dynkin diagram this pair of roots corresponds to either an $A_2$ system, $G_2$ system or $C_2$ system, so we get a surjective morphism

$$\text{SL}(3) \to \langle U_{\pm \omega}, U_{\pm \omega'} \rangle =: G_{\omega, \omega'},$$

$$\text{Sp}(4) \to \langle U_{\pm \omega}, U_{\pm \omega'} \rangle =: G_{\omega, \omega'}$$

or

$$G_2 \to \langle U_{\pm \omega}, U_{\pm \omega'} \rangle =: G_{\omega, \omega'}$$

with finite central kernel. Furthermore, if $\omega$ corresponds to the root $\tilde{\omega}$ in $G_{\omega, \omega'}$ then its kernel in $D \cap G_{\omega, \omega'}$ corresponds to the kernel in the diagonal $A \subset \text{SL}(3)$ (resp. $A \subset \text{Sp}(4), G_2$) of $\tilde{\omega}$. The unipotent groups $U_{\pm \omega}$ along with $D \cap G_{\omega}$ generate a copy of $\text{SL}(2)$ which is situated inside the rank 2 group $G_{\omega, \omega'}$ in one of four ways described by [17, Lemma 3.6]. From that lemma, we see that whenever $\omega$ is not conjugate to a long simple root in $\text{Sp}(2n)$ or a short simple root in $G_2$, the group $\langle U_{\pm \omega} \rangle$ comes with a linear action on a unipotent abelian group forming a semidirect product of one of the types treated at the end of Section 3.4, plus the symplectic action in the case of $G_2$, which we omit for brevity; note that the $G_2$ system does not appear in any higher rank system so the parameters for that semidirect product are only relevant if our group $G$ is locally isomorphic to $G_2$.

Thus we get an isogeny from $\text{SL}(2, \mathcal{K}) \ltimes V$ to its image in $G_{\omega, \omega'}$ with the positive diagonal in $\text{SL}(2)$ going to the one parameter semigroup $\mathfrak{a}$ in the positive Weyl chamber of $G$. We will denote the copy of $\text{SL}(2, \mathcal{K}) \ltimes V$ corresponding to the root $\omega$ by $\text{SL}(2, \mathcal{K})_\omega \ltimes V_\omega$, and from now on any quantity defined in previous sections for semidirect products subscripted with $\omega$ will refer to its definition over $\text{SL}(2, \mathcal{K})_\omega \ltimes V_\omega$.

Now suppose an $\mathfrak{a}(t)$ acts on a $K$-finite function $f$. Through the isogeny we get a corresponding action on $f$ of $\text{SL}(2)_\omega \ltimes V_\omega$ which we denote again simply by $\mathfrak{a}(t) \cdot f$, and $f$ retains $K$-finiteness for the action of the maximal compact subgroup of the $\text{SL}(2)_\omega$ part. This last remark is crucial, because our main bound requires finiteness under the maximal compact subgroup of the acting group, termed $K_\omega$ here.

Thus $f$ affords an action of $\text{SL}(2)_\omega \ltimes V_\omega$ with no invariant vectors for $V_\omega$ on $L^2_0(X)$ (mixing descends to subgroups and an isogeny has finite kernel, so we get no invariant vectors for the $V_\omega$ factor).

Using this reduction, a correlation

$$\int_X f_0((\mathfrak{a}(t_0))^{-1} \cdot x) f_1((\mathfrak{a}(t_1))^{-1} \cdot x) \cdots f_k((\mathfrak{a}(t_k))^{-1} \cdot x) d\mu(x)$$

(4.1)
of $K$-finite vectors $f_i$ can be viewed as a correlation for the action of $\text{SL}(2)_\mathbb{R} \times V_\omega$. Let $\mathcal{D}_\omega$ be the space of functions in the image of the projections $P_\omega$ corresponding to $V_\omega$. One sees that this space is $L^2$-dense by commutativity with the maximal compact subgroup $K_\omega$.

Applying the appropriate bound from Corollaries 3.17 and 3.18 to this correlation, we get

**Theorem 4.1.** Let $G$ be the group of $\mathcal{K}$-rational points of a $\mathcal{K}$-split simple group of $\mathcal{K}$-rank at least 2. Consider a measure-preserving, mixing action action

$$G \curvearrowleft X$$

on a probability space $X$; let $a : \mathcal{K}^0 \to D^0$ be a one parameter subgroup of a maximal split torus $D$ corresponding to a non-multipliable root $\omega$ of $G$ that is not conjugate to a short simple root in the group $G_2$ or a long simple root in $\text{Sp}(2n)$.

Given a $(k + 1)$-tuple $(f) = (f_0, \cdots, f_k)$ in $\mathcal{D}_s$ and $t_0, \cdots, t_k \in \mathcal{K}$ ordered in increasing absolute values, we have

$$\left| \int_X f_0((a(t_0))^{-1} \cdot x) f_1((a(t_1))^{-1} \cdot x) \cdots f_k((a(t_k))^{-1} \cdot x) d\mu(x) \right| \leq C s^{2q} \dim(K_\omega \cdot f_0)^{\frac{1}{2}} \|f_0\|_2 \dim(K_\omega \cdot f_k)^{\frac{1}{2}} \|f_k\|_2 \left( \prod_{i=1}^{k-1} \|f_i\|_\infty \right) \left( \sum_{i=1}^{k-1} \left| t_0 \right| \left( \sum_{i=0}^{k-1} \left| t_i \right| \left| t_k \right| \right) \right) \epsilon$$

Here the constant $C$ only depends on the action; see Remark 3.16.

**Remark 4.2.** We can obtain similar results for any semisimple $G$ without compact factors and one parameter subgroups corresponding to non-multipliable roots.

Theorem 4.1 also implies the exponential decay bounds for any one parameter subgroup $a(t)$ in the interior of a Weyl chamber. To see this in the real case, split the one parameter subgroup as an orthogonal sum $r(t)C(t)$ corresponding to a root $\omega \in \Phi^+$ and the orthogonal complement $C(t) \in \ker(\omega)$. Form the new functions $\tilde{f} = C(t) \cdot f$ and apply 4.1 to this correlation. From the fact that $\text{SL}(2, \mathbb{R})$ (and thus $K_\omega$) commutes with $C$, we have

$$\dim(K_\omega \cdot \tilde{f}) \leq \dim(K_\omega \cdot f) \leq \dim(K \cdot f);$$

since the action is measure preserving, $\|\tilde{f}\|_\infty = \|f\|_\infty$, which gives the bound.

This gives an effective bound for the arbitrary one parameter group $a(t)$. It is not optimal, since the contribution from $C(t)$ in the correlation is ignored, but it is still exponential with a suboptimal exponent.

The results of the next section elaborate on this argument, extend it to the non-Archimedean case and go beyond one parameter groups.

A similar bound for one parameter subgroups had been obtained previously by T-H. Hui for real semisimple groups $G$ and homogeneous $X = G/\Gamma$ satisfying certain spectral gap properties. Under the hypotheses in that paper the space

Journal of Modern Dynamics | Volume 10, 2016, 81–111
of functions contained all smooth zero-mean functions on \( X \); see [22, Chapter 4]. Finally, see [6] for effective multiple mixing in a wide variety of partially hyperbolic systems.

### 4.2. Beyond one parameter subgroups

The previous result requires the acting elements to be confined on a one parameter subgroup. In this section we state an extension allowing us to get effective restricted multiple mixing. Simultaneously the results here show how to bound an arbitrary one parameter subgroup, not limited to those arising from co-roots.

From the proof of [17, Lemma 5.2], we can decompose \( D^0 \) in the Archimedean case as

\[
D^0 = \ker(\omega)D^0_\omega
\]

and in the non-Archimedean case

\[
2D^0 \subset \ker(\omega)D^0_\omega,
\]

where \( 2D^0 = \{ d^2 \mid d \in D^+ \} \) and in both cases \( D^+_\omega \) corresponds to the positive diagonal of \( \text{SL}(2) \). We will see how to go from \( 2D^0 \) to the full \( D^0 \) below.

The image of the \( \text{SL}(2) \) in \( G_{\omega,\omega'} \) commutes with \( \ker(\omega) \) (this follows from the observation that their Lie algebras commute). Therefore, any maximal compact subgroup \( K_\omega \) of that image commutes with \( \ker(\omega) \). This fact plus the \( K \)-finiteness of the \( f_i \) imply the \( K_\omega \)-finiteness of the translates of the \( f_i \) by elements in \( \ker(\omega) \); note that these translates are no longer necessarily \( K \)-finite when \( \mathcal{K} \) is Archimedean. We need the \( K_\omega \)-finiteness in order for Theorem 3.15 to be applicable to the action of \( \text{SL}(2, K_\omega) \rtimes V_\omega \).

Now suppose \( A \in D^+ \) acts on a \( K \)-finite function \( f \); we can write \( A = SaC \) where \( aC \in 2D^+ \) (\( S = 1 \) if \( \mathcal{K} \) is Archimedean), \( a \) being (the image in \( D^+ \) of) an element of the diagonal group of \( \text{SL}(2) \), \( C \) centralized by the maximal compact of that \( \text{SL}(2) \). Then \( A \cdot f = a \cdot \tilde{f} \) where \( \tilde{f} \) is \( K_\omega \)-finite by the remarks above; in the non-Archimedean case, \( S \) is not necessarily in \( \ker(\omega) \), but in this case translates of \( K \)-finite vectors are still \( K \)-finite, see [17, Lemma 5.6]. Thus the translate \( \tilde{f} \) affords an action of \( \text{SL}(2)_\omega \rtimes V_\omega \) on \( L_2^0(X) \).

Doing this for all terms in a correlation \( A^i \cdot f_i \) we get \( K \)-finite vectors \( f \) for an action of \( \text{SL}(2, \mathcal{K})_\omega \rtimes V_\omega \). Then, given a correlation

\[
\int_X f_0((A^0)^{-1} \cdot x)f_1((A^1)^{-1} \cdot x) \cdots f_k((A^k)^{-1} \cdot x)d\mu(x)
\]

of \( K \)-finite vectors \( f_i \), we obtain a correlation

\[
\int_X \tilde{f}_0((a ^0)^{-1} \cdot x)\tilde{f}_1((a ^1)^{-1} \cdot x) \cdots \tilde{f}_k((a ^k)^{-1} \cdot x)d\mu(x)
\]

for the Cartan action of \( \text{SL}(2, \mathcal{K})_\omega \rtimes V_\omega \). In order to apply the results obtained so far to this context we assume the hypotheses of Theorem 3.15. From the reductions above, we need to understand how the hypotheses on the functions \( f_i \) are affected when we pass to the \( \tilde{f}_i \). The following lemma follows from [17, Lemma 5.6] and the preceding discussion.
LEMMA 4.3. The $\bar{f}_i$ are $K_\omega$-finite functions with
\[
\dim(K_\omega \cdot \bar{f}_i) \leq \max_{t \in D^0 \cap D^0} |K_\omega : tK_\omega t^{-1} \cap K_\omega| \dim(K_\omega \cdot f_i).
\]

DEFINITION 4.4. Let $\Phi_+^+$ be the subset of positive roots $\omega$ not locally conjugate to a long simple root in $\text{Sp}(2m)$ or a short root in $G_2$. Let $\mathcal{D}_s(G)$ be the space of bounded $K$-finite functions that have restricted spectral support in $\text{Ann}(s)$ with respect to $V_\omega$ for all $\omega \in \Phi^+$. This space is $L^2(X)$-dense.

For a Cartan element $A$ and a root $\omega$ denote the part of $A$ corresponding to $\omega$ by $a_\omega$. Since these elements are in a rank 1 Cartan group, we can order them in increasing absolute value; denote $a_\omega^+$ the maximum and $a_\omega^-$ the minimum.

Finally, if the representation corresponding to $\omega$ is the standard representation let $e_\omega = 1$ and if it is the adjoint representation let $e_\omega = 2$.

Now we are ready to state the most general bound for arbitrary Cartan elements.

THEOREM 4.5. Let $G$, $X$ as in Theorem 4.1. Let $A$ be a $(k + 1)$-tuple of Cartan elements in a maximal split torus $D$ and $\Phi$ the root system of $G$. For every $(k + 1)$-tuple $(f_i)$ in $\mathcal{D}_s(G)$ we have
\[
\left| \int_X f_0((A^0)^{-1} \cdot x)f_1((A^1)^{-1} \cdot x) \cdots f_k((A^k)^{-1} \cdot x) d\mu(x) \right| \leq C s^{2e} \min_{\Phi^+} \left( d_{i_0}^{-\frac{1}{2}} \| \bar{f}_{i_0} \|_2 d_{i_1}^{-\frac{1}{2}} \| \bar{f}_{i_1} \|_2 \cdots \prod_{i \neq i_0, i_M} \| \bar{f}_i \|_\infty \right) \left\{ \left( \sum_{i \neq +} \left| \frac{a_+^i}{a_0} \right| e_\omega \right) \left( \sum_{i \neq -} \left| \frac{a_-^i}{a_0} \right| e_\omega \right) \right\}^e.
\]
As before $C$ and $e > 0$ only depend on the action.

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