CONVERGENCE RATES IN EXPECTATION FOR A NONLINEAR BACKWARD PARABOLIC EQUATION WITH GAUSSIAN WHITE NOISE

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ABSTRACT. The main purpose of this paper is to study the problem of determining initial condition of nonlinear parabolic equation from noisy observations of the final condition. We introduce a regularized method to establish an approximate solution. We prove an upper bound on the rate of convergence of the mean integrated squared error.

Keywords: Quasi-reversibility method; backward problem; parabolic equation; Gaussian white noise regularization.

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1. INTRODUCTION

The forward problem for parabolic equation is of finding the distribution at a later time when we know the initial distribution. In geophysical exploration, one is often faced with the problem of determining the temperature distribution in the object or any part of the Earth at a time $t_0 > 0$ from temperature measurements at a time $t_1 > t_0$. This is the backward in time parabolic problem. The backward parabolic problems can be applied to several practical areas such as image processing, mathematical finance, and physics (See [1, 2].) Let $T$ be a positive number and $\Omega$ be an open, bounded and connected domain in $\mathbb{R}^d, d \geq 1$ with a smooth boundary $\partial \Omega$. In this paper, we consider the question of finding the function $u(x, t), (x, t) \in \Omega \times [0, T]$, satisfying the nonlinear problem

\[
\begin{align*}
\begin{cases}
\frac{\partial u}{\partial t} - \nabla \left( a(x, t) \nabla u \right) = F(x, t, u(x, t)), & (x, t) \in \Omega \times (0, T), \\
u_{|\partial \Omega} = 0, & t \in (0, T), \\
u(x, T) = g(x), & (x, t) \in \Omega \times (0, T),
\end{cases}
\end{align*}
\]

where the functions $a(x, t), g(x)$ are given and the source function $F$ will be given later. Here the coefficient $a(x, t)$ is a $C^1$ smooth function and $0 < \underline{m} \leq a(x, t) < M$ for all $(x, t) \in \Omega \times (0, T)$ for
some finite constants $m, M$. The problem is well-known to be ill-posed in the sense of Hadamard. Hence, a solution corresponding to the data does not always exist, and in the case of existence, it does not depend continuously on the given data. In fact, from small noise contaminated physical measurements, the corresponding solutions will have large errors. Hence, one has to resort to a regularization. In the simple case of the deterministic noise, Problem (1.1) with $a = 1$ and $F = 0$ has been studied by many authors \[11, 12, 15\]. However, in the case of random noise, the analysis of regularization methods is still limited. The problem is to determine the initial temperature function $f$ given a noisy version of the temperature distribution $g$ at time $T$

$$g_{\delta}^{obs}(x) = g(x) + \delta \xi(x)$$

(1.2)

where $\delta > 0$ is the amplitude of the noise and $\xi$ is a Gaussian white noise. In practice, we only observe some finite errors as follows

$$\langle g_\delta, \phi_j \rangle = \langle g, \phi_j \rangle + \delta \langle \xi, \phi_j \rangle, \quad j = 1, 2, 3, \cdots, N.$$  

(1.3)

where the natural number $N$ is the number of steps of discrete observations and $\phi_j$ is defined in (2). The main goal is to find approximate solution $\hat{u}_N(0)$ for $u(0)$ and then investigate the rate of convergence $E \| \hat{u}_N(0) - u(0) \|$, which is called the mean integrated square error (MISE). Here $E$ denotes the expectation w.r.t. the distribution of the data in the model (1.2). The model (1.2)-(1.3) are considered in some recent paper, such as [21, 22, 23, 24, 25].

The inverse problem with random noise has a long history. The simple case of (1.1) is the homogeneous linear parabolic equation of finding the initial data $u_0 := u(x,0)$ that satisfies

$$\begin{cases} 
  u_t - \Delta u = 0, & (x,t) \in \Omega \times (0,T), \\
  u|_{\partial \Omega} = 0, & t \in (0,T), \\
  u(x,T) = g(x), & (x,t) \in \Omega \times (0,T). 
\end{cases}$$

(1.4)

This equation is a special form of statistical inverse problems and it can be transformed by a linear operator with random noise

$$g = Ku_0 + \text{"noise"}. \quad (1.5)$$

where $K$ is a bounded linear operator that does not have a continuous inverse. The problem (1.4) has been studied by well-known methods including spectral cut-off (or called truncation method) [3, 6, 23, 22], the Tiknonov method [10], iterative regularization methods [13], Bayes estimation method [4, 20], Lavrentiev regularization method [26]. In some parts of these works, the authors show that the error $E \| \tilde{u}_N(0) - u(0) \|$ tend to zero when $N$ is suitably chosen according to the value of $\delta$ and $\delta \rightarrow 0$. For more details, we refer the reader to [5].

To the best of our knowledge, there are no results for the backward problem for nonlinear parabolic equation with Gaussian white noise. The difficulty to study the nonlinear model is the fact that we can not transform the solution of (1.1) into the operator equation (1.3). This makes the study for nonlinear problem with random noise more difficult since we can not apply the known methods. Very recently, in [16], we studied the discrete random model for backward nonlinear parabolic problem. However, the problem considered in [16] is in a rectangular domain which is limited in practice. The present paper uses another random model and also gives approximation of the solution in the case of more general bounded and smooth domain $\Omega$. Our task in this paper is to show that the expectation between the solution and the approximate solution converges to zero when $N$ tends to infinity.

This paper is organized as follows. In section 2 we give a couple of preliminary results. In section 3 we give an explanation for ill-posedness of the problem. For ease of the reader, we divide the problem into three cases under various assumptions on the coefficient $a$, and the source function $F$. **Case 1:** $a := a(x,t)$ is a constant and $F$ is a globally Lipschitz function. In section 4 we will...
study this case and give convergence rates in $L^2$ and $H^p$ norms for $p > 0$. The method here is the well-known spectral method. The main idea is to approximate the final data $g$ by the approximate data and use this function to establish a regularized problem by truncation method.

Case 2: $a := a(x, t)$ depends on $x$ and $t$ and $F$ is locally Lipschitz function. This problem is more difficult. In most practical problems, the function $F$ is often a locally Lipschitz function. The difficulty here is the fact that the solution cannot be transformed into a Fourier series and therefore, we cannot apply well-known methods to find an approximate solution. In Section 5, we will study a new form of quasi-reversibility method to construct a regularized solution and obtain convergence rate. Our method is new and very different than the method of Lions and Lattes [17]. First, we approximate the locally Lipschitz function by a sequence of globally Lipschitz functions and use some new techniques to obtain the convergence rate.

Case 3 Various assumptions on $F$. In practice there are many functions that are not locally Lipschitz. Hence our analysis in section 4 can not applied in section 6. Our method in section 6 is also quasi-reversibility method and is very similar to the method in section 4. But in section 6 we don’t approximate $F$ as we do in section 4. This leads to a convergence rate that is better than the one in section 4. One difficulty that occurs in this section is showing the existence and uniqueness of the regularized solution. To prove the existence of the regularized solution, we don’t follow the previously mentioned methods. Instead, we use the Faedo–Galerkin method, and the compactness method introduced by Lions [18]. To the best of our knowledge, this is the first result where $F$ is not necessarily a locally Lipschitz function. Finally, in section 7, we give some specific equations which can be applied by our method.

2. Preliminaries

To give some details on this random model (1.2), we give the following definitions (See [5, 6]):

Definition 2.1. Let $H$ be a Hilbert space. Let $g, g_\delta \in H$ satisfy (1.2). The representation (1.2) is equivalent to

$$
\langle g_\delta, \chi \rangle = \langle g, \chi \rangle + \delta \langle \xi, \chi \rangle, \quad \forall \chi \in H. \tag{2.6}
$$

Here $\langle \xi, \chi \rangle \sim N(0, \|\chi\|_H^2)$. Moreover, given $\chi_1, \chi_2 \in H$ then

$$
\mathbb{E}\left(\langle \xi, \chi_1 \rangle \langle \xi, \chi_2 \rangle \right) = \mathbb{E}\langle \chi_1, \chi_2 \rangle. \tag{2.7}
$$

Definition 2.2. The stochastic error is a Hilbert-space process, i.e. a bounded linear operator $\xi : H \to L^2(\Omega, \mathcal{A}, P)$ where $(\Omega, \mathcal{A}, P)$ is the underlying probability space and $L^2(., .)$ is the space of all square integrable measurable functions.

Let us recall that the eigenvalue problem

$$
\begin{cases}
-\Delta \phi_j(x) = \lambda_j \phi_j(x), & x \in \Omega, \\
\phi_j(x) = 0, & x \in \partial\Omega,
\end{cases} \tag{2.8}
$$

admits a family of eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_j \leq \ldots$ and $\lambda_j \to \infty$ as $j \to \infty$. See page 335 in [14].

Next, we introduce the abstract Gevrey class of functions of index $\sigma > 0$, see, e.g., [7], defined by

$$
\mathcal{W}_\sigma = \left\{ v \in L^2(\Omega) : \sum_{j=1}^{\infty} e^{2\pi \lambda_j} \left| \langle v, \phi_j(x) \rangle \right|_{L^2(\Omega)}^2 < \infty \right\},
$$

which is a Hilbert space equipped with the inner product

$$
\langle v_1, v_2 \rangle_{\mathcal{W}_\sigma} := \left\langle e^{\sigma \sqrt{-\Delta}} v_1, e^{\sigma \sqrt{-\Delta}} v_2 \right\rangle_{L^2(\Omega)}, \quad \text{for all } v_1, v_2 \in \mathcal{W}_\sigma;
$$
its corresponding norm \( \|v\|_{W_\sigma} = \sqrt{\sum_{j=1}^{\infty} e^{2\pi \lambda_j} |\langle v, \phi_j \rangle_{L^2(\Omega)}|^2} < \infty \).

3. THE ILL-POSEDNESS OF THE NONLINEAR PARABOLIC EQUATION WITH RANDOM NOISE

In this section, for a special case of equation (1.1), we show that the nonlinear parabolic equation with random noise is ill-posed in the sense of Hadamard.

Theorem 3.1. Problem (1.1) is ill-posed in the special case of \( a = 1, \Omega = (0, \pi) \).

Proof. Since \( \Omega = (0, \pi) \) and \( a(x, t) = 1 \), then \( \lambda_N = N^2 \). Let us consider the following parabolic equation

\[
\begin{aligned}
\frac{\partial V_{\delta, N}(\delta)}{\partial t} - \Delta V_{\delta, N}(\delta)(t) &= F_0(V_{\delta, N}(\delta)(x, t)), \quad 0 < t < T, x \in (0, \pi) \\
V_{\delta, N}(\delta)(0, t) &= V_{\delta, N}(\delta)(\pi, t) = 0, \\
V_{\delta, N}(\delta)(x, T) &= G_{\delta, N}(\delta)(x),
\end{aligned}
\]  

(3.9)

where \( F_0 \) is

\[
F_0(v(x, t)) = \sum_{j=1}^{\infty} e^{-T \lambda_j} (v(t), \phi_j(x)) \phi_j(x)
\]  

(3.10)

for any \( v \in L^2(\Omega) \), and \( \phi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx) \). Let us choose \( G_{\delta, N}(\delta) \in L^2(\Omega) \) be such that

\[
G_{\delta, N}(\delta)(x) = \sum_{j=1}^{N(\delta)} (g_\delta(x), \phi_j(x)) \phi_j(x)
\]  

(3.11)

where \( g_\delta \) is defined by

\[
\langle g_\delta, \phi_j \rangle = \delta \langle \xi, \phi_j \rangle, \quad j = 1, N.
\]  

(3.12)

By the usual MISE decomposition which involves a variance term and a bias term, we get

\[
E\|G_{\delta, N}(\delta)\|_{L^2(\Omega)}^2 = E\left( \sum_{j=1}^{N(\delta)} (G_{\delta, N}(\delta), \phi_j)^2 \right) = \delta^2 E\left( \sum_{j=1}^{N(\delta)} \xi_j^2 \right) = \delta^2 N(\delta).
\]  

(3.13)

The solution of Problem (3.9) is given by Fourier series (see [29])

\[
V_{\delta, N}(\delta)(x, t) = \sum_{j=1}^{\infty} \left[ e^{(T-t)\lambda_j} \langle G_{\delta, N}(\delta), \phi_j \rangle - \int_t^T e^{(s-t)\lambda_j} \langle F_0(V_{\delta, N}(\delta)(s)), \phi_j \rangle ds \right] \phi_j.
\]  

(3.14)

We show that Problem (3.14) has unique solution \( V_{\delta, N}(\delta) \in C([0, T]; L^2(\Omega)) \). Let us consider

\[
\Phi v := \sum_{j=1}^{\infty} e^{(T-t)\lambda_j} \langle G_{\delta, N}(\delta), \phi_j \rangle - \sum_{j=1}^{\infty} \left[ \int_t^T e^{(s-t)\lambda_j} \langle F_0(v(s)), \phi_j \rangle ds \right] \phi_j.
\]  

(3.15)
For any \( v_1, v_2 \in C([0, T]; L^2(\Omega)) \), using Hölder inequality, we have for all \( t \in [0, T] \)
\[
\|\Phi v_1(t) - \Phi v_2(t)\|^2_{L^2(\Omega)} = \sum_{j=1}^{\infty} \left[ \int_t^T e^{(s-t)\lambda_j} \langle F_0(v_1(s)) - F_0(v_2(s)), \phi_j \rangle ds \right]^2 \\
\leq T \sum_{j=1}^{\infty} \int_t^T e^{2(s-t)\lambda_j} \|F_0(v_1(s)) - F_0(v_2(s))\|^2 ds \\
= \frac{T}{4T^2} \sum_{j=1}^{\infty} \int_t^T e^{2(s-t-T)\lambda_j} \|v_1(s) - v_2(s)\|^2 ds \\
\leq \frac{1}{4T} \sum_{j=1}^{\infty} \int_t^T \|v_1(s) - v_2(s), \phi_j\|^2 ds \leq \frac{1}{4} \|v_1 - v_2\|^2_{C([0, T]; L^2(\Omega))}.
\] (3.16)

Hence, we obtain that
\[
\|\Phi v_1 - \Phi v_2\|_{C([0, T]; L^2(\Omega))} \leq \frac{1}{2} \|v_1 - v_2\|_{C([0, T]; L^2(\Omega))}.
\] (3.17)

This implies that \( \Phi \) is a contraction. Using the Banach fixed-point theorem, we conclude that
the equation \( \Phi(w) = w \) has a unique solution \( V_{\delta,N(\delta)} \in C([0, T]; L^2(\Omega)) \). Using the inequality
\( a^2 + b^2 \geq \frac{1}{2} (a - b)^2, \ a, b \in \mathbb{R} \), we have the following estimate
\[
\left\| V_{\delta,N(\delta)} \right\|^2_{L^2(\Omega)} \geq \frac{1}{2} \left\| \sum_{j=1}^{\infty} e^{(T-t)\lambda_j} \left\langle G_{\delta,N(\delta)}, \phi_j \right\rangle \phi_j \right\|^2_{L^2(\Omega)} \\
- \left\| \sum_{j=1}^{\infty} \left( \int_t^T e^{(s-t)\lambda_j} \left\langle F_0(V_{\delta,N(\delta)}(s)), \phi_j \right\rangle ds \right) \phi_j \right\|^2_{L^2(\Omega)}.
\] (3.18)

First, using Hölder’s inequality, we get
\[
I_2 \leq \sum_{j=1}^{\infty} \left( \int_t^T e^{(s-t)\lambda_j} \left\langle F_0(V_{\delta,N(\delta)}(s)), \phi_j \right\rangle ds \right)^2 \\
\leq T \sum_{j=1}^{\infty} \int_t^T e^{2(s-t)\lambda_j} \left\langle F_0(V_{\delta,N(\delta)}(s)), \phi_j \right\rangle^2 ds \\
\leq \frac{T}{4T^2} \sum_{j=1}^{\infty} e^{2(s-t-T)\lambda_j} \left\langle V_{\delta,N(\delta)}(t), \phi_j \right\rangle^2 ds \leq \frac{1}{4} \left\| V_{\delta,N(\delta)} \right\|^2_{C([0, T]; L^2(\Omega))}.
\] (3.19)

And we have the lower bound for \( I_1 \)
\[
E I_1 = \frac{1}{2} \sum_{j=1}^{\infty} e^{2(T-t)\lambda_j} E \left\langle G_{\delta,N(\delta)}, \phi_j \right\rangle^2 = \frac{1}{2} \sum_{j=1}^{N} \delta^2 e^{2(T-t)\lambda_j} \geq \frac{1}{2} \delta^2 e^{2(T-t)\lambda_N(\delta)}.
\] (3.20)

Combining (3.18), (3.19), (3.20), we obtain
\[
E \left\| V_{\delta,N(\delta)} \right\|^2_{L^2(\Omega)} + \frac{1}{4} E \left\| V_{\delta,N(\delta)} \right\|^2_{C([0, T]; L^2(\Omega))} \geq \frac{1}{2} \delta^2 e^{2(T-t)\lambda_N(\delta)}. \] (3.21)
By taking supremum of both sides on $[0, T]$, we get
\[
\mathbb{E}\left\| V_{\delta,N(\delta)} \right\|_{C([0,T];L^2(\Omega))}^2 \geq \frac{2}{5} \sup_{0\leq t\leq T} \delta^2 e^{2(T-t)\lambda N(\delta)} = \frac{2}{5} \delta^2 e^{2T\lambda N(\delta)} = \frac{2}{5} \delta^2 e^{2TN^2(\delta)}. \tag{3.22}
\]
Choosing $N := N(\delta) = \sqrt{\frac{1}{2T} \ln\left(\frac{1}{\delta}\right)}$, we obtain
\[
\mathbb{E}\left\| G_{\delta,N(\delta)} \right\|_{L^2(\Omega)}^2 = \delta^2 N(\delta) = \delta^2 \sqrt{\frac{1}{2T} \ln\left(\frac{1}{\delta}\right)} \to 0, \text{ when } \delta \to 0. \tag{3.23}
\]
and
\[
\mathbb{E}\left\| V_{\delta,N(\delta)} \right\|_{C([0,T];L^2(\Omega))}^2 \geq \frac{2}{5} \delta^2 e^{2TN^2(\delta)} = \frac{2}{5\delta} \to +\infty, \text{ when } \delta \to 0. \tag{3.24}
\]
From (3.23) and (3.24), we can conclude that Problem (1.1) is ill-posed. \hfill \Box

4. Regularization result with constant coefficient and globally Lipschitz source function

In this section, we consider the question of finding the function $u(x, t)$, $(x, t) \in \Omega \times [0, T]$, that satisfies the problem
\[
\begin{cases}
\begin{aligned}
u_t - \Delta u &= F(x, t, u(x, t)), \quad (x, t) \in \Omega \times (0, T), \\
u|_{\partial \Omega} &= 0, \quad t \in (0, T), \\
u(x, T) &= g(x), \quad (x, t) \in \Omega \times (0, T),
\end{aligned}
\end{cases}
\tag{4.25}
\]
Now we have the following lemma

**Lemma 4.1.** Let $G_{\delta,N(\delta)} \in L^2(\Omega)$ be such that
\[
G_{\delta,N(\delta)} = \sum_{j=1}^{N(\delta)} \langle g_{\delta}^{obs}, \phi_j \rangle \phi_j. \tag{4.26}
\]
Assume that $g \in H^{2\gamma}(\Omega)$. Then we have the following estimate
\[
\mathbb{E}\left\| G_{\delta,N(\delta)} - g \right\|_{L^2(\Omega)}^2 \leq \delta^2 N(\delta) + \frac{1}{\lambda_{N(\delta)}^{2\gamma}} \|g\|_{H^{2\gamma}(\Omega)}^2 \tag{4.27}
\]
for any $\gamma \geq 0$. Here $N$ depends on $\delta$ and satisfies that $\lim_{\delta \to 0} N(\delta) = +\infty$.

**Proof.** For the following proof, we consider the genuine model (1.3). By the usual MISE decomposition which involves a variance term and a bias term, we get
\[
\mathbb{E}\left\| G_{\delta,N(\delta)} - g \right\|_{L^2(\Omega)}^2 = \mathbb{E}\left( \sum_{j=1}^{N(\delta)} \langle g_{\delta}^{obs}, \phi_j \rangle^2 \right) + \sum_{j \geq N(\delta)+1} \langle g, \phi_j \rangle^2
\]
\[
= \delta^2 \mathbb{E}\left( \sum_{j=1}^{N(\delta)} \xi_j^2 \right) + \sum_{j \geq N(\delta)+1} \lambda_j^{-2\gamma} \lambda_j^{2\gamma} \langle g, \phi_j \rangle^2. \tag{4.28}
\]
Since $\xi_j = \langle \xi, \phi_j \rangle \sim N(0, 1)$, it follows that $\mathbb{E} \xi_j^2 = 1$, so
\[
\mathbb{E}\left\| G_{\delta,N(\delta)} - g \right\|_{L^2(\Omega)}^2 \leq \delta^2 N(\delta) + \frac{1}{\lambda_{N(\delta)}^{2\gamma}} \|g\|_{H^{2\gamma}}^2. \tag{4.29}
\]
Using truncation method, we give a regularized problem for Problem (1.1) as follows

\[
\begin{cases}
\frac{\partial}{\partial t}u_N^\delta - \Delta u_N^\delta = J_{\alpha_N(\delta)} F(x, t, u_N^\delta(x, t)), \quad (x, t) \in \Omega \times (0, T), \\
u_N^\delta|_{\partial \Omega} = 0, \quad t \in (0, T), \\
u_N^\delta(x, T) = J_{\alpha_N(\delta)} \overline{\int}_{\delta N(\delta)}(x), \quad (x, t) \in \Omega \times (0, T),
\end{cases}
\]  \tag{4.30}

where \(\alpha_N(\delta)\) is regularization parameter and \(J_{\alpha_N(\delta)}\) is the following operator

\[
J_{\alpha_N(\delta)} v := \sum_{\lambda_j \leq \alpha_N(\delta)} \langle v, \phi_j \rangle \phi_j, \quad \text{for all } v \in L^2(\Omega). \tag{4.31}
\]

Our main result in this section is as follows

**Theorem 4.1.** The problem (4.30) has a unique solution \(u_N^\delta(\delta) \in C([0, T]; L^2(\Omega))\) which satisfies that

\[
u_N^\delta(x, t) = \sum_{\lambda_j \leq \alpha_N(\delta)} \left[ e^{(T-t)\lambda_j} \langle G_{\delta N(\delta)}, \phi_j \rangle - \int_0^T e^{(s-t)\lambda_j} \langle F(u_N^\delta(s)), \phi_j \rangle ds \right] \phi_j. \tag{4.32}
\]

Assume that problem (1.1) has unique solution \(u\) such that

\[
\sum_{j=1}^{\infty} \lambda_j^2 e^{2t\lambda_j} \langle u(\cdot, t), \phi_j \rangle^2 < A', \quad t \in [0, T]. \tag{4.33}
\]

Let us choose \(\alpha_N(\delta)\) such that

\[
\lim_{\delta \to 0} \alpha_N(\delta) = +\infty, \quad \lim_{\delta \to 0} e^{kT\alpha_N(\delta)} = 0, \quad \lim_{\delta \to 0} e^{kT\alpha_N(\delta)} \sqrt{N}\delta = 0 \tag{4.34}
\]

Then the following estimate holds

\[
E\|u(\cdot, t) - u_N^\delta(\cdot, t)\|^2_{L^2(\Omega)} \leq 2e^{k^2(T-t)}e^{-2t\alpha_N(\delta)} \left[ \delta^2 N(\delta) e^{2T\alpha_N(\delta)} + \frac{e^{2T\alpha_N(\delta)}}{\lambda_{N(\delta)}^{2\gamma}} \|g\|_{H^{2\gamma}} + \alpha_{N(\delta)}^{-2\beta} \right] \tag{4.35}
\]

**Remark 4.1.** 1. From the theorem above, it is easy to see that \(E\|u_N^\delta(\cdot, t)\|^2_{L^2(\Omega)}\) is of order

\[
e^{-2t\alpha_N(\delta)} \max \left( \delta^2 N(\delta) e^{2T\alpha_N(\delta)}, \frac{e^{2T\alpha_N(\delta)}}{\lambda_{N(\delta)}^{2\gamma}}, \alpha_{N(\delta)}^{-2\beta} \right). \tag{4.36}
\]

2. Now, we give one example for the choice of \(N(\delta)\) which satisfies the condition (4.34). Since \(\lambda_N \sim N^\gamma\). See [3], we choose \(\alpha_N\) such that \(e^{kT\alpha_N} = |N(\delta)|^a\) for any \(0 < a < \frac{2\gamma}{\gamma}\). Then we have

\[
\alpha_N(\delta) = \frac{a}{kT} \log(N(\delta)). \quad \text{The number } N(\delta) \text{ is chosen as follows}
\]

\[
N(\delta) = \left( \frac{1}{\delta} \right)^{(ba+\frac{\gamma}{2})} \tag{4.37}
\]

for \(0 < b < 1\). With \(N(\delta)\) chosen as above, \(E\|u_N^\delta(\cdot, t)\|^2_{L^2(\Omega)}\) is of order \(\left( \frac{1}{\delta} \right)^{-\frac{(ba+\frac{\gamma}{2})\alpha_T}{kT}}\)

3. The existence and uniqueness of solution of equation (1.1) is an open problem, and we do not investigate this problem here. The case considered in Theorem 4.1 give the existence of the solution of Problem (1.1) in a special case.
Proof of Theorem 4.1. We divide the proof into some smaller parts.

Part 1. The problem (4.30) has a unique solution \( u_{N(\delta)}^\delta \in C([0, T]; L^2(\Omega)) \). The proof is similar to [29] (See Theorem 3.1, page 2975 [29]). Hence, we omit it here.

Part 2. Estimate the expectation of the error between the exact solution \( u \) and the regularized solution \( u_{N(\delta)}^\delta \).

Let us consider the following integral equation

\[
v_{N(\delta)}^\delta(x, t) = \sum_{\lambda_j \leq \alpha_{N(\delta)}} \left[ e^{(T-t)\lambda_j} \langle g, \phi_j \rangle - \int_t^T e^{(s-t)\lambda_j} \langle F(u_{N(\delta)}^\delta(s)), \phi_j \rangle ds \right] \phi_j. \tag{4.37}
\]

We have

\[
\| u_{N(\delta)}^\delta(\cdot, t) - v_{N(\delta)}^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq 2 \sum_{\lambda_j \leq \alpha_{N(\delta)}} e^{2(T-t)\lambda_j} \left\| \mathcal{G}_{\delta, N(\delta)} - g, \phi_j \right\|^2 + 2 \sum_{\lambda_j \leq \alpha_{N(\delta)}} \left[ \int_t^T e^{(s-t)\lambda_j} \left( F_j(u_{N(\delta)}^\delta(s)) - F_j(v_{N(\delta)}^\delta(s)) \right) ds \right]^2
\]

\[
\leq 2e^{2(T-t)\alpha_N} \sum_{\lambda_j \leq \alpha_{N(\delta)}} \left\| \mathcal{G}_{\delta, N(\delta)} - g \right\|^2_{L^2(\Omega)} + 2(T - t) \int_t^T e^{2(s-t)\alpha_{N(\delta)}} \sum_{\lambda_j \leq \alpha_{N(\delta)}} \left( F_j(u_{N(\delta)}^\delta(s)) - F_j(v_{N(\delta)}^\delta(s)) \right)^2 ds
\]

\[
\leq 2e^{2(T-t)\alpha_N} \left\| \mathcal{G}_{\delta, N(\delta)} - g \right\|^2_{L^2(\Omega)} + 2k^2T \int_t^T e^{2(s-t)\alpha_{N(\delta)}} \| u_{N(\delta)}^\delta(\cdot, s) - v_{N(\delta)}^\delta(\cdot, s) \|_{L^2(\Omega)}^2 ds. \tag{4.38}
\]

Taking the expectation of both sides of the last inequality, we get

\[
E\| u_{N(\delta)}^\delta(\cdot, t) - v_{N(\delta)}^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq 2e^{2(T-t)\alpha_N} E\left\| \mathcal{G}_{\delta, N(\delta)} - g \right\|^2_{L^2(\Omega)} + 2k^2T \int_t^T e^{2(s-t)\alpha_{N(\delta)}} E\| u_{N(\delta)}^\delta(\cdot, s) - v_{N(\delta)}^\delta(\cdot, s) \|_{L^2(\Omega)}^2 ds. \tag{4.39}
\]

Multiplying both sides with \( e^{2\alpha_{N(\delta)}} \), we obtain

\[
e^{2\alpha_{N(\delta)}} E\| u_{N(\delta)}^\delta(\cdot, t) - v_{N(\delta)}^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq 2e^{2T\alpha_{N(\delta)}} E\left\| \mathcal{G}_{\delta, N(\delta)} - g \right\|^2_{L^2(\Omega)} + 2k^2T \int_t^T e^{2\alpha_{N(\delta)}} E\| u_{N(\delta)}^\delta(\cdot, s) - v_{N(\delta)}^\delta(\cdot, s) \|_{L^2(\Omega)}^2 ds. \tag{4.40}
\]

Applying Gronwall’s inequality, we get

\[
e^{2T\alpha_{N(\delta)}} E\| u_{N(\delta)}^\delta(\cdot, t) - v_{N(\delta)}^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq 2e^{2T\alpha_{N(\delta)}} e^{2k^2T(T-t)} E\left\| \mathcal{G}_{\delta, N(\delta)} - g \right\|^2_{L^2(\Omega)}. \tag{4.41}
\]

Hence, using Lemma 4.1 we deduce that

\[
E\| u_{N(\delta)}^\delta(\cdot, t) - v_{N(\delta)}^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq 2e^{2k^2T(T-t)} e^{2(T-t)\alpha_{N(\delta)}} E\left\| \mathcal{G}_{\delta, N(\delta)} - g \right\|^2_{L^2(\Omega)} \leq 2e^{2k^2T(T-t)} e^{2(T-t)\alpha_{N(\delta)}} \left( \delta^2 \mathcal{N}(\delta) + \frac{1}{\lambda_{N(\delta)}} \| g \|_{H^2} \right). \tag{4.42}
\]
Now, we continue to estimate \( \| u(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \). Indeed, using Hölder’s inequality and globally Lipschitz property of \( F \), we get
\[
\| u(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \\
\leq 2 \sum_{\lambda_j \leq \alpha_N} \left[ \int_0^T e^{(s-t)} \lambda_j \left( F_j(u)(s) - F_j(v_N^\delta)(s) \right) ds \right]^2 + 2 \sum_{\lambda_j > \alpha_N} \langle u(t), \phi_j \rangle^2 \\
\leq 2 \sum_{\lambda_j > \alpha_N} \lambda_j^{-\beta} e^{-2 \alpha \lambda_j} \lambda_j^{2\beta} e^{2t \lambda_j} \langle u(t), \phi_j \rangle^2 + 2k^2 \int_0^T e^{2(s-t)} \lambda_N \| u(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 ds \\
\leq \alpha_N^{-2\beta} e^{-2t \alpha \lambda_N} \sum_{j=1}^\infty \lambda_j^{2\beta} e^{2t \lambda_j} \langle u(t), \phi_j \rangle^2 + 2k^2 \int_0^T e^{2(s-t) \lambda_N} \| u(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 ds.
\]
Above, we have used the mild solution of \( u \) as follows
\[
u(x, t) = \sum_{j=1}^\infty \left[ e^{(T-t)\lambda_j} \langle g, \phi_j \rangle - \int_0^T e^{(s-t)\lambda_j} \langle F(u(s)), \phi_j \rangle ds \right] \phi_j.
\]
Multiplying both sides with \( e^{2t \alpha \lambda_N} \), we obtain
\[
e^{2t \alpha \lambda_N} \| u(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq \alpha_N^{-2\beta} \sum_{j=1}^\infty \lambda_j^{2\beta} e^{2t \lambda_j} \langle u(t), \phi_j \rangle^2 + 2k^2 \int_0^T e^{2s \alpha \lambda_N} \| u(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 ds \tag{4.43}
\]
Gronwall’s inequality implies that
\[
e^{2t \alpha \lambda_N} \| u(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq e^{2k^2(T-t) \alpha_N^{-2\beta} A'} \tag{4.44}
\]
This together with the estimate \( (4.42) \) leads to
\[
E \| u(\cdot, t) - u_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq 2E \| u_N^\delta(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 + 2\| u(\cdot, t) - v_N^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq 2 e^{2k^2(T-t) \alpha_N} \left( \delta^2 N(\delta) + \frac{1}{\lambda_N^{2\beta}} \| g \|_{H_{2\alpha}} \right) + 2 \alpha_N^{-2\beta} e^{-2t \alpha \lambda_N} e^{2k^2(T-t)} A'
\tag{4.45}
\]
where \( A' \) is given in equation \( (4.33) \). This completes our proof. \( \Box \)

The next theorem provides an error estimate in the Sobolev space \( H^p(\Omega) \) which is equipped with a norm defined by
\[
\| g \|_{H^p(\Omega)}^2 = \sum_{j=1}^\infty \lambda_j^p \langle g, \phi_j(x) \rangle^2. \tag{4.46}
\]
To estimate the error in \( H^p \) norm, we need stronger assumption of the solution \( u \).

**Theorem 4.2.** Assume that problem \( (1.1) \) has unique solution \( u \) such that
\[
\sum_{j=1}^\infty e^{2(t+\tau) \lambda_j} \langle u(\cdot, t), \phi_j \rangle^2 < A'', \quad t \in [0, T]. \tag{4.47} \]
for any $r > 0$. Let us choose $\alpha_{N(\delta)}$ such that
\[ \lim_{\delta \to 0} \alpha_{N(\delta)} = +\infty, \quad \lim_{\delta \to 0} \frac{e^{kT\alpha_{N(\delta)}}}{\lambda_{N(\delta)}} = 0, \quad \lim_{\delta \to 0} e^{kT\alpha_{N(\delta)}} \sqrt{N(\delta)\delta} = 0 \quad (4.48) \]

Then the following estimate holds
\[ E\left\| u_{N(\delta)}^\alpha (\cdot, t) - u_{\alpha(N(\delta))} (\cdot, t) \right\|_{H^p(\Omega)}^2 \leq 2e^{2kT(\delta-t)}e^{-2\alpha_{N(\delta)}} \left[ 2\delta^2 N(\delta) e^{2T\alpha_{N(\delta)}} + 2 \frac{e^{2T\alpha_{N(\delta)}}}{\lambda_{N(\delta)}^2}\|g\|_{H^{2\gamma}} + \alpha e^{-2\alpha_{N(\delta)}} \right] \]
\[ + A^\alpha |\alpha_{N(\delta)}| e^{2r+\alpha_{N(\delta)}} \exp \left( -2(t+r)\alpha_{N(\delta)} \right). \quad (4.50) \]

Proof. First, we have
\[ E\left\| u_{N(\delta)}^\alpha (\cdot, t) - J_{\alpha(N(\delta))} (\cdot, t) \right\|_{H^p(\Omega)}^2 = E \left( \sum_{\lambda_j \leq \alpha_{N(\delta)}} \lambda_j^p \left\langle u_{N(\delta)}^\alpha (x, t) - u(x, t), \phi_j (x) \right\rangle \right)^2 \]
\[ \leq |\alpha_{N(\delta)}| E \left( \sum_{\lambda_j \leq \alpha_{N(\delta)}} \left\langle u_{N(\delta)}^\alpha (x, t) - u(x, t), \phi_j (x) \right\rangle \right)^2 \]
\[ \leq |\alpha_{N(\delta)}| E \left\| u_{N(\delta)}^\alpha (\cdot, t) - u_{\alpha(N(\delta))} (\cdot, t) \right\|_{L^2(\Omega)}^2. \quad (4.51) \]

Next, we continue to estimate $E\left\| u_{N(\delta)}^\alpha (\cdot, t) - u_{\alpha(N(\delta))} (\cdot, t) \right\|_{L^2(\Omega)}^2$ with the assumption (4.47). Let us recall $v_{N(\delta)}^\rho$ from (4.37). The expectation of the error between $u_{N(\delta)}^\alpha$ and $v_{N(\delta)}^\rho$ is given in the estimation (4.32) as
\[ E\left\| u_{N(\delta)}^\alpha (\cdot, t) - v_{N(\delta)}^\rho (\cdot, t) \right\|_{L^2(\Omega)}^2 \leq 2e^{2kT(\delta-t)}e^{2(\delta-t)\alpha_{N(\delta)}} \left( \delta^2 N(\delta) + \frac{1}{\lambda_{N(\delta)}^2} \right) \|g\|_{H^{2\gamma}}. \quad (4.52) \]

Now, we only need to estimate $\|u_{\alpha(N(\delta))} (\cdot, t) - v_{N(\delta)}^\rho (\cdot, t)\|_{L^2(\Omega)}$. Indeed, using Hölder’s inequality and globally Lipschitz property of $F$, we get
\[ \|u_{\alpha(N(\delta))} (\cdot, t) - v_{N(\delta)}^\rho (\cdot, t)\|_{L^2(\Omega)}^2 \leq 2 \sum_{\lambda_j > \alpha_{N(\delta)}} \left( \int_t^T e^{(s-t)\lambda_j} \left| F_j(u(s)) - F_j(v_{N(\delta)}^\rho(s)) \right| ds \right)^2 \]
\[ \leq 2 \sum_{\lambda_j > \alpha_{N(\delta)}} e^{2\alpha_{N(\delta)} \lambda_j} \|u(s) - v_{N(\delta)}^\rho(s)\|_{L^2(\Omega)}^2 \int_t^T e^{2(s-t)\alpha_{N(\delta)}} ds \]
\[ \leq e^{-2(t+r)\alpha_{N(\delta)}} \sum_{j=1}^{\infty} e^{2(t+r)\lambda_j} \left( \|u(s) - v_{N(\delta)}^\rho(s)\|_{L^2(\Omega)}^2 \right) e^{2(\delta-t)\alpha_{N(\delta)}} ds. \]

Multiplying both sides with $e^{2\alpha_{N(\delta)}}$, we obtain
\[ e^{2\alpha_{N(\delta)}} \|u_{\alpha(N(\delta))} (\cdot, t) - v_{N(\delta)}^\rho (\cdot, t)\|_{L^2(\Omega)}^2 \leq \alpha e^{-2\alpha_{N(\delta)}} \]
\[ + 2k^2 T \int_t^T e^{2\alpha_{N(\delta)}} \|u(s) - v_{N(\delta)}^\rho(s)\|_{L^2(\Omega)}^2 ds. \quad (4.53) \]

Gronwall’s inequality implies that
\[ e^{2\alpha_{N(\delta)}} \|u_{\alpha(N(\delta))} (\cdot, t) - v_{N(\delta)}^\rho (\cdot, t)\|_{L^2(\Omega)}^2 \leq e^{2k^2 T(\delta-t)} A^\alpha e^{-2\alpha_{N(\delta)}}. \quad (4.54) \]
This last estimate together with the estimate \((4.52)\) leads to
\[
E\|u(., t) - u^\delta_N(., t)\|_{L^2(\Omega)}^2 \\
\leq 2E\|u^\delta_N(., t) - v^\delta_N(., t)\|_{L^2(\Omega)}^2 + 2\|u(., t) - v^\delta_N(., t)\|_{L^2(\Omega)}^2 \\
\leq 4e^{2k^2(T-t)}e^{2(T-t)\alpha_N(\delta)} \left( \delta^2N(\delta) + \frac{1}{\lambda_{N}^{\gamma}}\|g\|_{H^{\gamma}} \right) + 2e^{2k^2(T-t)}A^\alpha e^{-2t\alpha_N}e^{-2\alpha_N} \\
= 2e^{2k^2(T-t)}e^{-2\alpha_N}\left[ 2\delta^2N(\delta) e^{2T\alpha_N} + 2\frac{e^{2T\alpha_N}}{\lambda_{N}^{2\gamma}}\|g\|_{H^{2\gamma}} + A^\alpha e^{-2\alpha_N} \right]. \tag{4.55}
\]

On the other hand, consider the function
\[
G(\xi) = \xi^p e^{-D\xi}, \quad D > 0. \tag{4.56}
\]

The derivative of \(G\) is \(G'(\xi) = \xi^{p-1} e^{-D\xi} (p - D\xi)\). Hence we know that \(G\) is strictly decreasing when \(D\xi \geq p\). Since \(\lim_{\delta \to 0} \alpha_N(\delta) = +\infty\), we see that if \(\delta\) is small enough then \(2r\alpha_N(\delta) \geq p\). Replace \(D = 2(t + r)\), \(\xi = \alpha_N(\delta)\) into \((4.56)\), we obtain for \(\lambda_j > \alpha_N(\delta)\)
\[
G(\lambda_j) = \lambda_j^p \exp\left( -2(t + r)\lambda_j \right) \leq G(\alpha_N(\delta)) = |\alpha_N(\delta)|^p \exp\left( -2(t + r)\alpha_N(\delta) \right)
\]

The latter equatity leads to
\[
\|u(., t) - J_{\alpha_N(\delta)}u(., t)\|_{H^p(\Omega)}^2 = \sum_{\lambda_j > \alpha_N(\delta)} \lambda_j^p \langle u(x, t), \phi_j(x) \rangle^2 \\
= \sum_{\lambda_j > \alpha_N(\delta)} \lambda_j^p \exp\left( -2(t + r)\lambda_j \right) \exp\left( 2(t + r)\lambda_j \right) \langle u(x, t), \phi_j(x) \rangle^2 \\
\leq |\alpha_N(\delta)|^p \exp\left( -2(t + r)\alpha_N(\delta) \right) \sum_{\lambda_j > \alpha_N(\delta)} \exp\left( 2(t + r)\lambda_j \right) \langle u(x, t), \phi_j(x) \rangle^2 \\
\leq A^p |\alpha_N(\delta)|^p \exp\left( -2(t + r)\alpha_N(\delta) \right) \tag{4.57}
\]

where we use the assumption \((4.47)\) for the last inequality. Combining \((4.51)\), \((4.55)\) and \((4.57)\), we deduce that
\[
E\|u^\delta_N(., t) - u(., t)\|_{H^p(\Omega)}^2 \\
\leq E\|u^\delta_N(., t) - J_{\alpha_N(\delta)}u(., t)\|_{H^p(\Omega)}^2 + \|u(., t) - J_{\alpha_N(\delta)}u(., t)\|_{H^p(\Omega)}^2 \\
\leq 2e^{2k^2(T-t)}e^{-2t\alpha_N}|\alpha_N(\delta)|^p \left[ 2\delta^2N(\delta) e^{2T\alpha_N} + 2\frac{e^{2T\alpha_N}}{\lambda_{N}^{2\gamma}}\|g\|_{H^{2\gamma}} + A^\alpha e^{-2\alpha_N} \right] \\
+ A^p |\alpha_N(\delta)|^p \exp\left( -2(t + r)\alpha_N(\delta) \right) \tag{4.58}
\]
which completes the proof.

5. Regularization result with locally Lipschitz source

Section 4 has addressed a problem in which \(F\) is a global Lipschitz function. In this section we extend the analysis to a locally Lipschitz function \(F\). Results for the locally Lipschitz case are difficult. Hence, we have to find another regularization method to study the problem with locally
Lipschitz source.
Assume that $a$ is noisy by the observation data $a_0^{obs}: \Omega \times [0, T] \rightarrow \mathbb{R}$ as follows

$$a_0^{obs}(x, t) = a(x, t) + \delta \psi(t)$$

(5.59)

where $\delta > 0$ is the amplitude of the noise and $\psi$ is Brownian motion in $t$.

Assume that for each $\mathcal{R} > 0$, there exists $K_\mathcal{R} > 0$ such that

$$|F(x, t; u) - F(x, t; v)| \leq K_\mathcal{R}|u - v|, \text{ if } \max\{|u|, |v|\} \leq \mathcal{R},$$

(5.60)

for $(x, t) \in \Omega \times [0, T]$ and $K_\mathcal{R} := \sup \left\{ \frac{F(x, t; u) - F(x, t; v)}{u - v} : \max\{|u|, |v|\} \leq \mathcal{R}, u \neq v, (x, t) \in \Omega \times [0, T] \right\} < +\infty$.

We note that $K_\mathcal{R}$ is increasing and $\lim_{\mathcal{R} \rightarrow +\infty} K_\mathcal{R} = +\infty$. Now, we outline our idea to construct a regularization for the problem (1.1). For all $\mathcal{R} > 0$, we approximate $F$ by $\mathcal{F}_{\mathcal{R}}$ defined by

$$\mathcal{F}_{\mathcal{R}}(x, t; w) := \begin{cases} F(x, t; -\mathcal{R}), & w \in (-\infty, -\mathcal{R}) \\ F(x, t; u), & w \in [-\mathcal{R}, \mathcal{R}] \\ F(x, t; \mathcal{R}), & w \in (\mathcal{R}, +\infty). \end{cases}$$

(5.61)

For each $\delta > 0$, we consider a parameter $\mathcal{R}(\delta) \rightarrow +\infty$ as $\delta \rightarrow 0^+$. Let us denote the operator $\mathbb{P} = M \Delta$, where $M$ is a positive number such that $M > a_0^{obs}(x, t)$ for all $(x, t) \in \Omega \times (0, T)$. Define the following operator

$$P_\beta^{\mathcal{N}(\delta)} = \mathbb{P} + Q_\beta^{\mathcal{N}(\delta)},$$

where

$$Q_\beta^{\mathcal{N}(\delta)} v(x) = \frac{1}{T} \sum_{j=1}^{\infty} \ln \left( 1 + \beta_{\mathcal{N}(\delta)} e^{MT\lambda_j} \right) \langle v(x), \phi_j(x) \rangle_{L^2(\Omega)} \phi_j(x),$$

(5.62)

for any function $v \in L^2(\Omega)$. Here $\mathcal{N}(\delta)$ is defined in Lemma (1.1).

Therefore, we are going to introduce the main idea to solve the problem (1.1) with a generalized case of source term defined by (5.61), we consider the problem:

$$\begin{cases} \frac{\partial u_\delta^{\mathcal{N}(\delta)}}{\partial t} - \nabla \cdot (a_0^{obs}(x, t) \nabla u_\delta^{\mathcal{N}(\delta)}) - Q_\beta^{\mathcal{N}(\delta)}(u_\delta^{\mathcal{N}(\delta)}(x, t)) = \mathcal{F}_{\mathcal{R}_\delta}(x, t, u_\delta^{\mathcal{N}(\delta)}(x, t)), & (x, t) \in \Omega \times (0, T), \\ u_\delta^{\mathcal{N}(\delta)}|_{\partial \Omega} = 0, & t \in (0, T), \\ u_{\mathcal{N}(\delta)}^{\mathcal{N}(\delta)}(x, T) = G_{\delta, \mathcal{N}(\delta)}(x), & (x, t) \in \Omega \times (0, T), \end{cases}$$

(5.63)

Here $G_{\delta, \mathcal{N}(\delta)}(x)$ is defined in equation (12.20). Now, we introduce some Lemmas which will be useful for our main results. First, we recall the abstract Gevrey class of functions of index $\sigma > 0$, see, e.g., [7], defined by

$$\mathcal{W}_\sigma = \left\{ v \in L^2(\Omega) : \sum_{n=1}^{\infty} e^{2\sigma \lambda_n} \left| \langle v, \phi_n(x) \rangle_{L^2(\Omega)} \right|^2 < \infty \right\},$$

which is a Hilbert space equipped with the inner product

$$\langle v_1, v_2 \rangle_{\mathcal{W}_\sigma} := \langle e^{\sigma \sqrt{-\Delta}} v_1, e^{\sigma \sqrt{-\Delta}} v_2 \rangle_{L^2(\Omega)}, \text{ for all } v_1, v_2 \in \mathcal{W}_\sigma;$$

and corresponding norm $\|v\|_{\mathcal{W}_\sigma} = \sqrt{\sum_{n=1}^{\infty} e^{2\sigma \lambda_n} \left| \langle v, \phi_n(x) \rangle_{L^2(\Omega)} \right|^2 < \infty$. 

Lemma 5.1. For $\mathcal{F}_R \in L^\infty(\Omega \times [0, T] \times \mathbb{R})$, we have

$$|\mathcal{F}_R(x, t; u) - \mathcal{F}_R(x, t; v)| \leq K_R|u - v|, \forall (x, t) \in \Omega \times [0, T], u, v \in \mathbb{R}.$$ 

Proof. See the proof of Lemma 2.4 in [28].

Lemma 5.2. 1. Let $M, T > 0$. For any $v \in \mathcal{W}_{MT}(\Omega)$, we have

$$\|Q_{\beta_N(\delta)}^\delta(v)\|_{L^2(\Omega)} \leq \frac{\beta_N(\delta)}{T^2}\|v\|_{\mathcal{W}_{MT}(\Omega)}.$$  (5.64)

2. Let $\beta_N(\delta) < 1 - e^{-MT\lambda_1}$. For any $v \in L^2(\Omega)$, we have

$$\|P_{\beta_N(\delta)}^\delta v\|_{L^2(\Omega)} \leq \frac{1}{T^2} \ln \left(\frac{1}{\beta_N(\delta)}\right) \|v\|_{L^2(\Omega)}.$$  (5.65)

Proof. Using the inequality $\ln(1 + a) \leq a$, $\forall a > 0$, we have

$$\|Q_{\beta_N(\delta)}^\delta(v)\|_{L^2(\Omega)} = \frac{1}{T^2} \sum_{j=1}^{\infty} \ln^2 \left(1 + \beta_N(\delta) e^{MT\lambda_j}\right) \left|\langle v, \phi_j \rangle_{L^2(\Omega)}\right|^2$$

$$\leq \frac{\beta_N(\delta)}{T^2} \sum_{j=1}^{\infty} e^{2MT\lambda_j} \left|\langle v, \phi_j \rangle_{L^2(\Omega)}\right|^2$$

$$\leq \frac{\beta_N(\delta)}{T^2} \|v\|_{L^2(\Omega)}^2.$$  (5.66)

Since $\beta_N(\delta) < 1 - e^{-MT\lambda_1}$, we know that $\beta_N(\delta) + e^{-MT\lambda_j} < 1$. Using Parseval’s equality, we can easily get

$$\|P_{\beta_N(\delta)}^\delta(v)\|_{L^2(\Omega)} = \frac{1}{T^2} \sum_{j=1}^{\infty} \ln^2 \left(\frac{1}{\beta_N(\delta) + e^{-MT\lambda_j}}\right) \left|\langle v, \phi_j \rangle_{L^2(\Omega)}\right|^2$$

$$\leq \frac{1}{T^2} \ln^2 \left(\frac{1}{\beta_N(\delta)}\right) \sum_{j=1}^{\infty} \left|\langle v, \phi_j \rangle_{L^2(\Omega)}\right|^2$$

$$\leq \frac{1}{T^2} \ln^2 \left(\frac{1}{\beta_N(\delta)}\right) \|v\|_{L^2(\Omega)}^2.$$  

\[\square\]

Theorem 5.1. The problem (5.63) has a unique solution

$$u_{N(\delta)}^\delta \in C\left([0, T]; L^2(\Omega)\right).$$

Assume that the problem (1.1) has a unique solution $u$ satisfying $u(\cdot, t) \in \mathcal{W}_{MT}$. Let us choose $\beta_{N(\delta)}$ such that

$$\lim_{\delta \to 0} \sqrt{N(\delta)} \beta_{N(\delta)}^{-1} = \lim_{\delta \to 0} \beta_{N(\delta)}^{-1} \lambda_N^{-\gamma} = \lim_{\delta \to 0} \beta_{N(\delta)} = 0.$$  (5.67)

Let us choose $R_\delta$ such that

$$\lim_{\delta \to 0} 2\beta_{N(\delta)}^2 e^{2KR_\delta T} = 0, \ t > 0.$$  (5.68)

Then we have the following estimate

$$E \left\|u_{N(\delta)}^\delta(x, t) - u(x, t)\right\|_{L^2(\Omega)}^2 \leq \frac{2}{\beta_{N(\delta)}^2} e^{(2K(R_\delta))T} \tilde{C}(\delta).$$  (5.69)
Here $\tilde{C}(\delta)$ is

$$
\tilde{C}(\delta) = \delta^2 N(\delta) \beta_{N^2}^{-2} + \frac{1}{\lambda^2 N(\delta)^2 N^2} \|g\|_{H^2(\Omega)} + \|u\|^2_{C([0,T];W_{MT}(\Omega))} + \frac{\delta^2 T^3}{b_0 \beta_{N^2}} \|u\|^2_{L^\infty(0,T;H^1(\Omega))}.
$$

Remark 5.1. 1. Under the assumption (5.68), the right hand side of equation (5.69) converges to zero when $t > 0$.
2. Let us choose $\beta_{N(\delta)} = N(\delta)^{-c}$ for any $0 < c < \min\left(\frac{1}{2}, \frac{2\pi}{d}\right)$. And $N(\delta)$ is chosen as follows

$$
N(\delta) = \left(\frac{1}{\delta}\right)^{m(\frac{1}{2} - c)}, \quad 0 < m < 1.
$$

Let us choose $R_\delta$ such that

$$
K(R_\delta) \leq \frac{1}{kT} \ln \left(\ln(N(\delta))\right) = \frac{1}{kT} \ln \left[m\left(\frac{1}{2} - c\right) \ln\left(\frac{1}{\delta}\right)\right].
$$

Then $E \left\| u_{N(\delta)}^\delta(x,t) - u(x,t) \right\|^2_{L^2(\Omega)}$ is of order

$$
\delta^{m(\frac{1}{2} - c)} \frac{1}{T} \ln\left(\frac{1}{\delta}\right).
$$

Proof of Theorem 5.1. The proof is divided into two Steps.

Step 1. The existence and uniqueness of the solution to the regularized problem (5.63). Let $b(x, t)$ be defined by $b(x, t) = M - a(x, t)$. It is obvious that $0 < b(x, t) < M$. Then from (5.63), we obtain

$$
\frac{\partial u_{N(\delta)}^\delta}{\partial t} + \nabla \left( b(x, t) \nabla u_{N(\delta)}^\delta \right) = F \left( x, t, u_{N(\delta)}^\delta(x, t) \right)
$$

$$
- \frac{1}{T} \sum_{j=1}^{\infty} \ln \left( \frac{1}{\beta_{N(\delta)} + e^{-MT\lambda_j}} \right) \left( u_{N(\delta)}^\delta(\cdot, t), \phi_j(x) \right) \phi_j(x),
$$

for $(x, t) \in \Omega \times (0, T)$.

Let $v_{N(\delta)}^\delta$ be the function defined by $v_{N(\delta)}^\delta(x, t) = u_{N(\delta)}^\delta(x, T - t)$. Then we have

$$
\frac{\partial v_{N(\delta)}^\delta}{\partial t}(x, t) = -\frac{\partial u_{N(\delta)}^\delta}{\partial t}(x, T - t), \quad \nabla \left( b(x, t) \nabla v_{N(\delta)}^\delta \right)(x, t) = \nabla \left( b(x, t) \nabla u_{N(\delta)}^\delta \right)(x, T - t)
$$

and

$$
\frac{1}{T} \sum_{j=1}^{\infty} \ln \left( \beta_{N(\delta)} + e^{-MT\lambda_j} \right) \left( v_{N(\delta)}^\delta(x, t), \phi_j(x) \right) \phi_j(x)
$$

$$
= \frac{1}{T} \sum_{j=1}^{\infty} \ln \left( \beta_{N(\delta)} + e^{-MT\lambda_j} \right) \left( u_{N(\delta)}^\delta(x, T - t), \phi_j(x) \right) \phi_j(x).
$$

This implies that $v_{N(\delta)}^\delta$ satisfies the problem

$$
\begin{aligned}
\frac{\partial v_{N(\delta)}^\delta}{\partial t} - \nabla \left( b(x, t) \nabla v_{N(\delta)}^\delta \right) &= \mathcal{G}(x, t, v_{N(\delta)}^\delta(x, t)), \quad (x, t) \in \Omega \times (0, T), \\
v_{N(\delta)}^\delta |_{\partial \Omega} &= 0, \quad t \in (0, T), \\
v_{N(\delta)}^\delta(x, 0) &= \overline{u}_{\delta, N(\delta)}(x), \quad (x, t) \in \Omega \times (0, T),
\end{aligned}
$$

(5.72)
where $G$ is defined by
\begin{equation}
G(x, t, v(x, t)) = -F(x, t, v(x, t)) + \frac{1}{T} \sum_{j=1}^{\infty} \ln \left( \frac{1}{\beta_{N(\delta)} + e^{-MTx_j}} \right) \langle v(\cdot, t), \phi_j \rangle_{L^2(\Omega)} \phi_j(x),
\end{equation}
for any $v \in C([0, T]; L^2(\Omega))$.

Since
\begin{equation}
\beta_{N(\delta)} \in \left(0, 1 - e^{-MTx_1}\right),
\end{equation}
and using Parseval’s identity, we obtain for any $v_1, v_2 \in L^2(\Omega)$
\begin{equation}
\|G(\cdot, t, v_1(\cdot, t)) - G(\cdot, t, v_2(\cdot, t))\|_{L^2(\Omega)}
\leq \|F(\cdot, t, v_1(\cdot, t)) - F(\cdot, t, v_2(\cdot, t))\|_{L^2(\Omega)}
+ \left| \frac{1}{T} \sum_{j=1}^{\infty} \ln \left( \frac{1}{\beta_{N(\delta)} + e^{-MTx_j}} \right) \langle v_1(x, t) - v_2(x, t), \phi_j(x) \rangle_{L^2(\Omega)} \phi_j(x) \right|_{L^2(\Omega)}
\leq K\|v_1(\cdot, t) - v_2(\cdot, t)\|_{L^2(\Omega)}
+ \left| \frac{1}{T} \sum_{j=1}^{\infty} \ln \left( \frac{1}{\beta_{N(\delta)} + e^{-MTx_j}} \right) \langle v_1(\cdot, t) - v_2(\cdot, t), \phi_n \rangle_{L^2(\Omega)} \phi_n(x) \right|^2
\leq \left[ K + \frac{1}{T} \ln \left( \frac{1}{\beta_{N(\delta)}} \right) \right] \|v_1(\cdot, t) - v_2(\cdot, t)\|_{L^2(\Omega)}.
\end{equation}

So $G$ is a Lipschitz function. Using the results of Theorem 12.2 in [9], we complete the proof of Step 1.

**Step 2. Error estimate**

We pass to the error estimate between the regularized solution of problem (5.63) and the exact solution of problem (1.1).

For $(x, t) \in \Omega \times (0, T)$, we begin by establishing that the functions $b(x, t), b_{\delta}^{\text{obs}}(x, t)$ satisfy
\begin{equation}
0 < b(x, t) \leq M, \quad 0 < b_0 \leq b_{\delta}^{\text{obs}}(x, t) \leq M
\end{equation}
and
\begin{equation}
\begin{pmatrix}
a(x, t) \\
a_{\delta}^{\text{obs}}(x, t)
\end{pmatrix} =
\begin{pmatrix}
M \\
M
\end{pmatrix} - \begin{pmatrix}
b(x, t) \\
b_{\delta}^{\text{obs}}(x, t)
\end{pmatrix}, \quad \forall (x, t) \in \Omega \times (0, T).
\end{equation}

The functions $u_{\delta N(\delta)}^\delta(x, t)$ and $u(x, t)$ solve the following equations
\begin{equation}
\frac{\partial u}{\partial t} + \nabla \left( b_{\delta}^{\text{obs}}(x, t) \nabla u \right) = F(x, t; u(x, t))
\end{equation}
and
\begin{equation}
\frac{\partial u_{\delta N(\delta)}}{\partial t} + \nabla \left( b_{\delta}^{\text{obs}}(x, t) \nabla u_{\delta N(\delta)} \right) = F_{R_\delta} \left( x, t, u_{\delta N(\delta)}(x, t) \right) + P_{\beta_{N(\delta)}}^\delta u_{\delta N(\delta)}.
\end{equation}

For $\rho_{\delta} > 0$, we put
\begin{equation}
V_{\delta N(\delta)}^\delta(x, t) = e^{\rho_{\delta}(t-T)} \left[ u_{\delta N(\delta)}(x, t) - u(x, t) \right].
\end{equation}
Then for \((x, t) \in \Omega \times (0, T)\)
\[
\frac{\partial V_{N(\delta)}^\delta}{\partial t} + \nabla \left( b^\delta_{obs}(x, t) \nabla V_{N(\delta)}^\delta \right) - \rho_\delta V_{N(\delta)}^\delta = P_{\beta N(\delta)}^\delta V_{N(\delta)}^\delta + e^{\rho_\delta(t-T)}Q_{\beta N(\delta)}^\delta u - e^{\rho_\delta(t-T)} \nabla \left( \left( b^\delta_{obs}(x, t) - b(x, t) \right) \nabla u \right) + e^{\rho_\delta(t-T)} \left[ F_{R \delta} \left( x, t, \delta u^\delta_{N(\delta)}(x, t) \right) - F(x, t; u(x, t)) \right],
\]

and
\[
V_{N(\delta)}^\delta \mid_{\partial \Omega} = 0, \quad V_{N(\delta)}^\delta \left( x, T \right) = \overline{G}_{N(\delta)}(x) - g(x).
\]

By taking the inner product of the two sides of equation (5.78) with \(V_{N(\delta)}^\delta\) and noting the equality
\[
\int_\Omega \nabla \left( b^\delta_{obs}(x, t) \nabla V_{N(\delta)}^\delta \right) V_{N(\delta)}^\delta \, dx = -\int_\Omega b^\delta_{obs}(x, t) |\nabla V_{N(\delta)}^\delta|^2 \, dx,
\]
we obtain
\[
\|V_{N(\delta)}^\delta(\cdot, T)\|_{L^2(\Omega)}^2 - \|V_{N(\delta)}^\delta(\cdot, t)\|_{L^2(\Omega)}^2 - 2 \int_t^T \int_\Omega b^\delta_{obs}(x, s)|\nabla V_{N(\delta)}^\delta|^2 \, dx \, ds - 2\rho_\delta \int_t^T \|V_{N(\delta)}^\delta(\cdot, s)\|_{L^2(\Omega)}^2 \, ds
\]
\[
= 2 \int_t^T \left\langle P_{\beta N(\delta)}^\delta V_{N(\delta)}^\delta, V_{N(\delta)}^\delta \right\rangle_{L^2(\Omega)} \, ds + 2 \int_t^T \left\langle e^{\rho_\delta(t-T)}Q_{\beta N(\delta)}^\delta u, V_{N(\delta)}^\delta \right\rangle_{L^2(\Omega)} \, ds
\]
\[
+ 2 \int_t^T \left\langle e^{\rho_\delta(t-T)} \left[ F_{R \delta} \left( x, t, \delta u^\delta_{N(\delta)}(x, t) \right) - F(x, t; u(x, t)) \right], V_{N(\delta)}^\delta \right\rangle_{L^2(\Omega)} \, ds.
\]

First, thanks to inequality (5.65), the expectation of \(\tilde{A}_4\) is estimated as follows
\[
E[\tilde{A}_4] \leq \frac{2}{T} \ln \left( \frac{1}{\rho_\delta} \right) \int_t^T E\|V_{N(\delta)}^\delta(\cdot, s)\|_{L^2(\Omega)}^2 \, ds,
\]

Next, using the inequality (5.64) and the Hölder inequality, we have
\[
E[\tilde{A}_5] \leq \int_t^T e^{2\rho_\delta(s-T)} \frac{\beta N(\delta)}{T} \|u\|_{C([0,T]; W^{1,\infty})}^2 \, ds + \int_t^T E\|V_{N(\delta)}^\delta(\cdot, s)\|_{L^2(\Omega)}^2 \, ds
\]
\[
\leq \frac{\beta N(\delta)}{T} \|u\|_{C([0,T]; W^{1,\infty})}^2 + \int_t^T E\|V_{N(\delta)}^\delta(\cdot, s)\|_{L^2(\Omega)}^2 \, ds,
\]

For estimating the expectation of \(\tilde{A}_6\), we use the Green’s formula to get the equality
\[
\left\langle \nabla \left( (b^\delta_{obs}(x, t) - b(x, t)) \nabla u \right), V_{N(\delta)}^\delta \right\rangle_{L^2(\Omega)} = \left\langle \left( (b^\delta_{obs}(x, t) - b(x, t)) \nabla u \right), \nabla V_{N(\delta)}^\delta \right\rangle_{L^2(\Omega)}
\]
then using Hölder’s inequality and noting the fact that
\[
\int_\Omega |\nabla u(\cdot, s)|^2 \, dx \leq \|u\|_{L^\infty([0,T]; W^{1,\infty}(\Omega))}^2 = \sup_{0 \leq s \leq T} \int_\Omega |\nabla u(\cdot, s)|^2 \, dx,
\]
we obtain

\[
\mathbb{E}[A_0] = 2\mathbb{E} \left| \int_t^T \left\langle e^{\rho_s(s-T)} \left( (b_{\delta}^{obs}(x, t) - b(x, t)) \nabla u, \nabla \mathbb{V}_{N(\delta)}^\delta \right) \right\rangle_{L^2(\Omega)} ds \right|
\]

\[
\leq \mathbb{E} \int_t^T \frac{e^{2\rho_s(s-T)}}{b_0} \int_\Omega \left( (b_{\delta}^{obs}(x, t) - b(x, t))^2 \right) |\nabla u(x, t)|^2 \, dx \, ds + \mathbb{E} \int_t^T \int_\Omega |\nabla \mathbb{V}_{N(\delta)}^\delta|^2 \, dx \, ds
\]

\[
= \delta^2 \int_t^T \mathbb{E} |\psi(s)|^2 \, ds \int_\Omega |\nabla u(., s)|^2 \, dx + \mathbb{E} \int_t^T \int_\Omega |\nabla \mathbb{V}_{N(\delta)}^\delta|^2 \, dx \, ds
\]

\[
\leq \frac{\delta^2 T^2}{2b_0} \|u\|_{L^\infty([0, T]; L^2(\Omega))}^2 + \mathbb{E} \int_t^T \int_\Omega |\nabla \mathbb{V}_{N(\delta)}^\delta|^2 \, dx \, ds. \quad (5.82)
\]

Here in the last inequality, we have used the fact that \( \mathbb{E} |\psi(s)|^2 = s \) since \( \psi \) is Brownian motion. Finally, since \( \lim_{\delta \to 0^+} \mathcal{R}_\delta = +\infty \), for a sufficiently small \( \delta > 0 \), there is an \( \mathcal{R}_\delta > 0 \) such that

\[
\mathcal{R}_\delta \geq \|u\|_{L^\infty([0, T]; L^2(\Omega))}.
\]

For this value of \( \mathcal{R}_\delta \) we have

\[
\mathcal{F}_{\mathcal{R}_\delta} (x; u(x, t)) = F(x, t; u(x, t)).
\]

Using the global Lipschitz property of \( \mathcal{F}_R \) (see Lemma 5.1), one obtains similarly the estimate

\[
\mathbb{E}[A_1] = 2\mathbb{E} \left| \int_t^T \left\langle e^{\rho_s(t-T)} \left[ \mathcal{F}_{\mathcal{R}_\delta} \left( x, t, u_{N(\delta)}^\delta \right) - F(x, t; u(x, t)) \right] \right\rangle_{L^2(\Omega)} ds \right|
\]

\[
\leq 2\mathbb{E} \int_t^T \left| e^{\rho_s(t-T)} \left[ \mathcal{F}_{\mathcal{R}_\delta} \left( x, s, u_{N(\delta)}^\delta(s) \right) - F(x, s; u(x, s)) \right] \right|_{L^2(\Omega)} \| \mathbb{V}_{N(\delta)}^\delta(\cdot, s) \|_{L^2(\Omega)}^2 \, ds
\]

\[
\leq 2K(\mathcal{R}_\delta) \int_t^T \mathbb{E} \| \mathbb{V}_{N(\delta)}^\delta(\cdot, s) \|_{L^2(\Omega)}^2 \, ds. \quad (5.83)
\]

Combining (5.79), (5.80), (5.81), (5.82) and (5.83), we obtain

\[
\mathbb{E} \| \mathbb{V}_{N(\delta)}^\delta(\cdot, T) \|_{L^2(\Omega)}^2 - \mathbb{E} \| \mathbb{V}_{N(\delta)}^\delta(\cdot, t) \|_{L^2(\Omega)}^2
\]

\[
= \int_t^T \left( \frac{\beta_{N_\delta}}{T} \|u\|_{C([0, T]; W_{MT}(\Omega))}^2 + \frac{\delta^2 T^2}{2b_0} \|u\|_{L^\infty([0, T]; \mathcal{H}_\delta^0(\Omega))}^2 \right) ds
\]

\[
\geq 2\mathbb{E} \int_t^T \int_\Omega \left( (b_{\delta}^{obs}(x, s) - b_0) |\nabla \mathbb{V}_{N(\delta)}^\delta|^2 \, dx \, ds
\]

\[
+ \mathbb{E} \int_t^T \left( \frac{2\rho_\delta - 2}{T} \ln \left( \frac{1}{\beta_{N_\delta}} \right) - 2K(\mathcal{R}_\delta) - 1 \right) \| \mathbb{V}_{N(\delta)}^\delta(\cdot, s) \|_{L^2(\Omega)}^2 ds
\]

\[
\geq \int_t^T \left( -\frac{2\rho_\delta}{T} + \ln \left( \frac{1}{\beta_{N_\delta}} \right) + 2K(\mathcal{R}_\delta) + 1 \right) \| \mathbb{V}_{N(\delta)}^\delta(\cdot, s) \|_{L^2(\Omega)}^2 ds. \quad (5.84)
\]

Whereupon,

\[
\mathbb{E} \| \mathbb{V}_{N(\delta)}^\delta(\cdot, t) \|_{L^2(\Omega)}^2 \leq \mathbb{E} \left[ \mathcal{G}_{\mathcal{R}_\delta}(N(\delta) - g) \right]_{L^2(\Omega)}^2
\]

\[
+ \beta_{N_\delta} \|u\|_{C([0, T]; W_{MT}(\Omega))}^2 + \frac{\delta^2 T^3}{b_0} \|u\|_{L^\infty([0, T]; \mathcal{H}_\delta^0(\Omega))}^2
\]

\[
+ \mathbb{E} \int_t^T \left( -\frac{2\rho_\delta}{T} + \frac{2}{T} \ln \left( \frac{1}{\beta_{N_\delta}} \right) + 2K(\mathcal{R}_\delta) + 1 \right) \| \mathbb{V}_{N(\delta)}^\delta(\cdot, s) \|_{L^2(\Omega)}^2 ds. \quad (5.85)
\]

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Since

\[ V_{N(\delta)}^\delta(x,t) = e^{\rho_\delta(t-T)} \left( u_{N(\delta)}^\delta(x,t) - u(x,t) \right) \]

and applying Lemma 4.1, we observe that

\[ e^{2\rho_\delta(t-T)} E \left\| u_{N(\delta)}^\delta(\cdot,t) - u(\cdot,t) \right\|^2_{L^2(\Omega)} \leq \delta^2 N(\delta) + \frac{1}{\lambda_{N(\delta)}^2} \| g \|^2_{H^{2\gamma}(\Omega)} \]

\[ + \beta_{N(\delta)} \| u \|^2_{C([0,T]; W^{3r,\infty}(\Omega))} + \frac{\delta^{2T^3}}{b_0} \| u \|^2_{L^\infty(0,T; H^1_{0,\gamma}(\Omega))} \]

\[ + (2K(R_\delta) + 1) \int_0^T e^{2\rho_\delta(s-T)} E \left\| u_{N(\delta)}^\delta(\cdot,s) - u(\cdot,s) \right\|^2_{L^2(\Omega)} ds. \tag{5.86} \]

Gronwall’s lemma allows us to obtain

\[ e^{2\rho_\delta(t-T)} E \left\| u_{N(\delta)}^\delta(x,t) - u(x,t) \right\|^2_{L^2(\Omega)} \leq \delta^2 N(\delta) + \frac{1}{\lambda_{N(\delta)}^2} \| g \|^2_{H^{2\gamma}(\Omega)} \]

\[ + \beta_{N(\delta)} \| u \|^2_{C([0,T]; W^{3r,\infty}(\Omega))} + \frac{\delta^{2T^3}}{b_0} \| u \|^2_{L^\infty(0,T; H^1_{0,\gamma}(\Omega))} \]

\[ + (2K(R_\delta) + 1) \int_0^T e^{2\rho_\delta(s-T)} E \left\| u_{N(\delta)}^\delta(\cdot,s) - u(\cdot,s) \right\|^2_{L^2(\Omega)} ds. \tag{5.87} \]

By choosing \( \rho_\delta = \frac{1}{T} \ln \left( \frac{1}{N_\delta} \right) > 0 \) we have

\[ E \left\| u_{N(\delta)}^\delta(\cdot,t) - u(\cdot,t) \right\|^2_{L^2(\Omega)} \leq \beta_{N(\delta)}^2 e^{2K(R_\delta) + 1)T C(\delta).} \tag{5.88} \]

The proof of Theorem 5.1 is complete. \( \Box \)

6. Regularization result with more general source term

In most of the previous works on backward nonlinear problem the assumption, that the source is global or locally Lipschitz, is required. To the best of our knowledge, this section is the first result on the source term \( F \) is not necessarily a locally Lipschitz source. We will solve the problem (1.1) with a special generalized case of source term defined by (5.61). Our regularized problem is different to the one in section 5 because we don’t approximate the source function \( F \). Indeed, we have the following regularized problem

\[ \left\{ \begin{array}{l}
\frac{\partial u_{N(\delta)}^\delta}{\partial t} - \nabla \left( s_{N(\delta)}^\delta(x,t) x u_{N(\delta)}^\delta \right) - \Delta_{N(\delta)}^\delta (u_{N(\delta)}^\delta)(x,t) = F(x,t,u_{N(\delta)}^\delta(x,t)) , \quad (x,t) \in \Omega \times (0,T), \\
u_{N(\delta)}^\delta |_{\partial \Omega} = 0, \quad t \in (0,T), \\
u_{N(\delta)}^\delta(x,T) = g_{\delta,N(\delta)}(x) , \quad (x,t) \in \Omega \times (0,T), \end{array} \right. \tag{6.89} \]

We make the following assumptions on \( F \in C^0(\mathbb{R}) \) in the following: There exists \( C_1 \) and \( C_1', C_2 \) and \( p > 1 \) and \( \gamma \) such that

\[ z F(x,t,z) \geq C_1 |z|^p - C_1' \]

\[ |F(x,t,z)| \leq C_2 (1 + |z|^{p-1}) \]

\[ (z_1 - z_2)(F(x,t,z_1) - F(x,t,z_2)) \geq -\gamma |z_1 - z_2|^2. \]

It is easy to check that the function \( F(x,t,z) = z^\frac{1}{\gamma} \) satisfies the conditions (6.90), (6.91) and (6.92). Note here that this function is not locally Lipschitz.
Now we have the following result

**Theorem 6.1.** Let us assume that $F$ satisfies (6.90), (6.91) and (6.92). Then, there exists a unique weak solution $u_{N(\delta)}^{\delta}$ of problem (6.89) such that

$$u_{N(\delta)}^{\delta} \in L^2(0, T; H^1) \cap L^\infty(0, T; L^2).$$

Assume that the problem (1.1) has a unique solution $u$ satisfying $u(\cdot, t) \in W_{MT}$. Let us choose $\beta_{N(\delta)}$ as Theorem 5.1. Then, we have the following estimate

$$E \left\| u_{N(\delta)}^{\delta}(x, t) - u(x, t) \right\|_{L^2(\Omega)}^2 \leq \beta_{N(\delta)}^{2t} e^{(27 + 1)t} \tilde{C}(\delta).$$

where $\tilde{C}(\delta)$ is defined in (6.138).

**Remark 6.1.** Our method in this Theorem give the convergence rate (6.93) which better than the error rate in (5.69). Indeed, since $\lim_{\delta \to 0} K(R_X) = +\infty$, we have

\[
\frac{\text{The right hand side of (5.69)}}{\text{The right hand side of (6.93)}} = \frac{\beta_{N(\delta)}^{2t} e^{(2K(R_X)+1)t} \tilde{C}(\delta)}{\beta_{N(\delta)}^{2t} e^{(27 + 1)t} \tilde{C}(\delta)} \to +\infty
\]

when $\delta \to 0$.

6.1. Proof of Theorem 6.1

6.1.1. **Proof of the existence of solution of Problem (6.89).** First, by changing variable $v_{N(\delta)}^{\delta}(x, t) = u_{N(\delta)}^{\delta}(x, T - t)$, we transform Problem (6.89) into the initial value problem

\[
\begin{cases}
\frac{\partial v_{N(\delta)}^{\delta}}{\partial t} - \nabla \left( b_{\delta}^{obs}(x, t) \nabla v_{N(\delta)}^{\delta} \right) = -F(x, t, v_{N(\delta)}^{\delta}(x, t)) + P_{N(\delta)}^{\delta}(v_{N(\delta)}^{\delta}(x, t)), \quad (x, t) \in \Omega \times (0, T), \\
v_{N(\delta)}^{\delta}(x, 0) = \overline{G}_{\delta, N(\delta)}(x), \quad (x, t) \in \Omega \times (0, T).
\end{cases}
\]

where $b_{\delta}^{obs}(x, t) = M - a_{\delta}^{obs}(x, t)$.

The weak formulation of the initial boundary value problem (6.95) can then be given in the following manner: Find $v_{N(\delta)}^{\delta}(t)$ defined in the open set $(0, T)$ such that $v_{N(\delta)}^{\delta}$ satisfies the following variational problem

\[
\int_\Omega \frac{d}{dt} v_{N(\delta), m}^{\delta} \varphi dx + \int_\Omega b_{\delta}^{obs}(x, t) \nabla v_{N(\delta), m}^{\delta} \nabla \varphi dx + \int_\Omega F(v_{N(\delta), m}^{\delta}(t)) \varphi dx = \int_\Omega P_{N(\delta)}^{\delta}(v_{N(\delta), m}^{\delta}(t)) \varphi dx
\]

for all $\varphi \in H^1$, and the initial condition

$$v_{N(\delta)}^{\delta}(0) = \overline{G}_{\delta, N(\delta)}.$$

**Proof of the existence of solution of Problem (6.89).** The proof consists of several steps.

**Step 1:** The Faedo – Galerkin approximation (introduced by Lions [13]). In the space $H^1(\Omega)$, we take a basis $\{e_j\}_{j=1}^{\infty}$ and define the finite dimensional subspace

$$V_m = \text{span}\{e_1, e_2, \ldots, e_m\}.$$ 

Let $\overline{G}_{\delta, N(\delta), m}$ be an element of $V_m$ such that

$$\overline{G}_{\delta, N(\delta), m} = \sum_{j=1}^m d_{mj}^{\delta} e_j \to \overline{G}_{\delta, N(\delta)}$$

strongly in $L^2$. (6.98)
as \( m \to +\infty \). We can express the approximate solution of the problem (6.95) in the form
\[
\nu_{N(\delta),m}(t) = \sum_{j=1}^{m} \epsilon_{mj}(t)e_j,
\] (6.99)
where the coefficients \( \epsilon_{mj} \) satisfy the system of linear differential equations
\[
\int_{\Omega} \frac{d}{dt} \nu_{N(\delta),m}^{\delta} e_i dx + \int_{\Omega} b_{\delta}^{\text{obs}}(x,t) \nabla \nu_{N(\delta),m}^{\delta} \nabla e_i dx + \int_{\Omega} F(\nu_{N(\delta),m}^{\delta}(t)) e_i dx = \frac{1}{P_{\beta N(\delta)}} (\nu_{N(\delta),m}^{\delta})(t)) e_i dx
\] (6.100)
with \( i = 1, m \) and the initial conditions
\[
\epsilon_{mj}(0) = a_{mj}, \quad j = 1, m.
\] (6.101)

The existence of a local solution of system (6.100)–(6.101) is guaranteed by Peano’s theorem on existence of solutions. For each \( m \) there exists a solution \( \nu_{N(\delta),m}^{\delta}(t) \) in the form (6.99) which satisfies (6.100) and (6.101) almost everywhere on \( 0 \leq t \leq T \) for some \( T \), \( 0 < T_m \leq T \). The following estimates allow one to take \( T_m = T \) for all \( m \).

Step 2. A priori estimates.

a) The first estimate. Multiplying the \( i^{\text{th}} \) equation of (6.100) by \( \epsilon_{mj}(t) \) and summing up with respect to \( i \), afterwards, integrating by parts with respect to the time variable from 0 to \( t \), we get after some rearrangements
\[
\left\| \nu_{N(\delta),m}^{\delta}(t) \right\|_{L^2(\Omega)}^2 + 2 \int_{0}^{t} \int_{\Omega} b_{\delta}^{\text{obs}}(x,t) |\nabla \nu_{N(\delta),m}^{\delta}(s)|^2 dx ds + 2 \int_{0}^{t} \int_{\Omega} F(\nu_{N(\delta),m}^{\delta}(s)) \nu_{N(\delta),m}^{\delta}(s) dx ds = \left\| \mathbf{G}_{\delta,N(\delta),m} \right\|^2 + 2 \int_{0}^{t} \int_{\Omega} P_{\beta N(\delta)}(\nu_{N(\delta),m}^{\delta}(s)) \nu_{N(\delta),m}^{\delta}(s) dx ds
\] (6.102)

By (6.98), we have
\[
\left\| \mathbf{G}_{\delta,N(\delta),m} \right\|^2 \leq B_0(\delta), \quad \text{for all} \quad m,
\] (3.8)
where \( B_0(\delta) \) depends on \( \mathbf{G}_{\delta,N(\delta)} \) and is independent of \( m \).

Using the lower bound of \( b_{\delta}^{\text{obs}}(x,t) \), we have the following estimate
\[
2 \int_{0}^{t} \int_{\Omega} b_{\delta}^{\text{obs}}(x,t) |\nabla \nu_{N(\delta),m}^{\delta}(s)|^2 dx ds \geq 2b_0 \int_{0}^{t} \left\| \nu_{N(\delta),m}^{\delta}(s) \right\|_{H^1(\Omega)}^2 ds.
\] (6.103)

Using the assumption on \( F \), we have
\[
2 \int_{0}^{t} \int_{\Omega} F(\nu_{N(\delta),m}^{\delta}(s)) \nu_{N(\delta),m}^{\delta}(s) dx ds \geq 2C_1 \int_{0}^{t} \left\| \nu_{N(\delta),m}^{\delta}(s) \right\|_{L^p(\Omega)}^p ds - 2TC_1^t
\] (6.104)
and
\[
2 \int_{0}^{t} \int_{\Omega} P_{\beta N(\delta)}(\nu_{N(\delta),m}^{\delta}(s)) \nu_{N(\delta),m}^{\delta}(s) dx ds \leq \frac{2}{T} \ln \left( \frac{1}{\beta_{N(\delta)}} \right) \int_{0}^{t} \left\| \nu_{N(\delta),m}^{\delta}(s) \right\|_{L^2(\Omega)}^2 ds.
\] (6.105)

Hence, it follows from (6.103) – (6.105) that
\[
\left\| \nu_{N(\delta),m}^{\delta}(t) \right\|^2_{L^2(\Omega)} + 2b_0 \int_{0}^{t} \left\| \nu_{N(\delta),m}^{\delta}(s) \right\|_{H^1(\Omega)}^2 ds + 2C_1 \int_{0}^{t} \left\| \nu_{N(\delta),m}^{\delta}(s) \right\|_{L^p(\Omega)}^p ds \leq B_0(\delta) + 2TC_1^t + \frac{1}{T} \ln \left( \frac{1}{\beta_{N(\delta)}} \right) \int_{0}^{t} \left\| \nu_{N(\delta),m}^{\delta}(s) \right\|_{L^2(\Omega)}^2 ds.
\] (6.106)
Let us denote
\[
S_m^\delta(t) = \left\| \mathbf{v}_{N(\delta), m}^\delta(t) \right\|^2_{L^2(\Omega)} + 2b_0 \int_0^t \left\| \mathbf{v}_{N(\delta), m}^\delta(s) \right\|^2_{H^1(\Omega)} ds + 2C_1 \int_0^t \left\| \mathbf{v}_{N(\delta), m}^\delta(s) \right\|^p_{L^p(\Omega)} ds. \tag{6.107}
\]

Using the fact that \( \int_0^t \left\| \mathbf{v}_{N(\delta), m}^\delta(s) \right\|^2_{L^2(\Omega)} ds \leq \int_0^t S_m^\delta(s) ds \), we know from (6.106) that
\[
S_m^\delta(t) \leq B_0(\delta) + 2TC_1' + \frac{1}{T} \ln \left( \frac{1}{\beta_{N(\delta)}} \right) \int_0^t S_m^\delta(s) ds \tag{6.108}
\]
Applying Gronwall’s lemma, we obtain
\[
S_m^\delta(t) \leq [B_0(\delta) + 2TC_1'] \exp \left( \frac{t}{T} \ln \left( \frac{1}{\beta_{N(\delta)}} \right) \right) \leq [B_0(\delta) + 2TC_1'] \exp \left( \ln \left( \frac{1}{\beta_{N(\delta)}} \right) \right) = B_1(\delta, T), \tag{6.109}
\]
for all \( m \in \mathbb{N} \), for all \( t, 0 \leq t \leq T_m \leq T \), i.e., \( T_m = T \), where \( C_T \) always indicates a bound depending on \( T \).

b) The second estimate. Multiplying the \( i \)th equation of (6.100) by \( t^2 \frac{d}{dt} \mathbf{v}_{N(\delta), m}^\delta(t) \) and summing up with respect to \( i \), we have
\[
\left\| t \frac{d}{dt} \mathbf{v}_{N(\delta), m}^\delta(t) \right\|^2_{L^2(\Omega)} + 2t^2 \int_\Omega \mathbf{b}_{N(\delta), m}^{obs}(x, t) \nabla \mathbf{v}_{N(\delta), m}^\delta(t) \nabla \left( \frac{d}{dt} \mathbf{v}_{N(\delta), m}^\delta(t) \right) dx + \int_\Omega t^2 F(\mathbf{v}_{N(\delta), m}^\delta(t)) \frac{d}{dt} \mathbf{v}_{N(\delta), m}^\delta(t) dx = \int_\Omega t^2 \mathbf{P}_{\beta_{N(\delta)}} \left( \mathbf{v}_{N(\delta), m}^\delta(t) \right) \frac{d}{dt} \mathbf{v}_{N(\delta), m}^\delta(t) dx. \tag{6.110}
\]

It is easy to check that for any \( u \in H^1(\Omega) \)
\[
\frac{d}{dt} \left[ \int_\Omega \mathbf{b}_{\delta}^{obs}(x, t) |\nabla u(t)|^2 dx \right] = 2 \int_\Omega \mathbf{b}_{\delta}^{obs}(x, t) \nabla u(t) \nabla u'(t) dx + \int_\Omega \frac{\partial}{\partial t} \mathbf{b}_{\delta}^{obs}(x, t) |\nabla u(t)|^2 dx. \tag{6.111}
\]

The equality (6.110) is equivalent to
\[
2 \left\| t \frac{d}{dt} \mathbf{v}_{N(\delta), m}^\delta(t) \right\|^2_{L^2(\Omega)} + \frac{d}{dt} \left[ t^2 \int_\Omega \mathbf{b}_{\delta}^{obs}(x, t) |\mathbf{v}_{N(\delta), m}^\delta(t)|^2 dx \right] + 2 \int_\Omega t^2 F(\mathbf{v}_{N(\delta), m}^\delta(t)) \frac{d}{dt} \mathbf{v}_{N(\delta), m}^\delta(t) dx = 2t \int_\Omega \mathbf{b}_{\delta}^{obs}(x, t) |\nabla \mathbf{v}_{N(\delta), m}^\delta(t)|^2 dx + t^2 \int_\Omega \frac{\partial}{\partial t} \mathbf{b}_{\delta}^{obs}(x, t) |\nabla \mathbf{v}_{N(\delta), m}^\delta(t)|^2 dx \tag{6.112}
\]
By integrating the last equality from 0 to $t$, we get
\[
2 \int_0^t \left| s \frac{d}{ds} \nu^\delta_{N(m)}(s) \right|^2_{L^2(\Omega)} ds + t^2 \int_\Omega b^{obs}_\delta(x,t) |\nabla \nu^\delta_{N(m)}(t)|^2 dx \\
+ 2 \int_0^t \int_\Omega \frac{d}{ds} \nu^\delta_{N(m)}(s) ds \int_\Omega s^2 F(\nu^\delta_{N(m)}, s) ds ds
\]
\[
= 2 \int_0^t \int_\Omega s b^{obs}_\delta(x,s) |\nabla \nu^\delta_{N(m)}(s) |^2 ds ds + \int_0^t \int_\Omega \frac{d}{ds} \nabla b^{obs}_\delta(x,s) ds ds
\]
\[
+ \int_0^t \int_\Omega \frac{d}{ds} \nu^\delta_{N(m)}(s) ds ds .
\]

Estimate $I_1$. Since the assumption $b^{obs}_\delta(x,t) \geq b_0$, we know that
\[
I_1 = t^2 \int_\Omega s b^{obs}_\delta(x,t) |\nabla \nu^\delta_{N(m)}(t)|^2 dx \geq b_0 \left\| tv^\delta_{N(m)}(t) \right\|^2_{H^1} .
\]

Estimate $I_2$. To estimate $I_2$, we need the following Lemma

Lemma 6.1. Let $\mu_0 = \left( \frac{x^p}{x} \right)^{1/p}$, $\underline{m} = \int_{-\mu_0}^{+\mu_0} F(x) dx$, $\tilde{F}(z) = \int_0^z F(y) dy$, $z \in \mathbb{R}$. Then we get
\[
-\underline{m} \leq \tilde{F}(z) \leq C_2 \left( |z| + \frac{1}{p} |z|^p \right) , \quad z \in \mathbb{R}.
\]

The proof of Lemma (6.1) is easy and we omit it here. Now we return to estimate $I_2$. By a simple computation and then using Lemma (6.1), we have
\[
I_2 = 2 \int_0^t s^2 \frac{d}{ds} \left[ \int_\Omega \int_0^s F(x,s) dx dy \right]
\]
\[
= 2 \int_0^t s^2 \frac{d}{ds} \left[ \int_\Omega \int_0^s F(x,s) dx dy \right]
\]
\[
= 2 \int_0^t \left[ \frac{d}{ds} \left( s^2 \int_\Omega \tilde{F}(\nu^\delta_{N(m)}, x,s) dx \right) - 2s \int_\Omega \tilde{F}(\nu^\delta_{N(m)}, x,s) dx \right]
\]
\[
= 2t^2 \int_\Omega \tilde{F}(\nu^\delta_{N(m)}, x,t) dx - 4 \int_0^t s ds \int_\Omega \tilde{F}(\nu^\delta_{N(m)}, x,s) dx
\]
\[
\geq -2T^2 \underline{m} |\Omega| - 4C_1 \int_0^t s \left[ \left\| \nu^\delta_{N(m)}(s) \right\|_{L^1} + \frac{1}{p} \left\| \nu^\delta_{N(m)}(s) \right\|_{L^p}^p \right] ds
\]
\[
\geq -2T^2 \underline{m} |\Omega| - 4TC_2 \left[ T \left\| \nu^\delta_{N(m)} \right\|_{L^\infty(0,T;L^2)} + \frac{1}{p} \frac{1}{2C_1} S^\delta_{m}(t) \right]
\]
\[
\geq -B_2(\delta, T) .
\]

Estimate $I_3$. Using (6.107), we have the following estimate
\[
I_3 \leq 2T b_1 \int_0^t \left\| \nu^\delta_{N(m)}(s) \right\|^2_{H^1} ds \leq \frac{2T b_1}{2b_0} S^\delta_{m}(t) .
\]
Estimate $I_4$. Let us set
\[
\bar{a}_T = \sup_{(x,t) \in [0,1] \times [0,T]} \partial_t b_{h_\delta}^{obs}(x,t)
\]
then $I_4$ is bounded by
\[
I_4 \leq \bar{a}_T \int_0^t \left\| s\mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds + 2 \left\| b^{obs}(x,t) \right\|_{H^1}^2 ds \leq T^2 \bar{a}_T \int_0^t \left\| \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds \leq \frac{T^2 \bar{a}_T}{a_0} S_m^\delta(t).
\] (6.118)

Estimate $I_5$. Using Lemma [5.21], we obtain the following estimate for $I_5$
\[
I_5 \leq 2 \int_0^t \left\| sP_{\delta, N(\delta)}^{\beta} (\mathbf{v}^{N(\delta),m}(s)) \right\|_{L^2(\Omega)}^2 ds + t \left\| \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds
\leq \ln^2 \left( \frac{1}{\beta_{N(\delta)}} \right) \int_0^t \left\| \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds + \int_0^t \left\| s \frac{d}{ds} \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds
\leq \ln^2 \left( \frac{1}{\beta_{N(\delta)}} \right) S_m^\delta(t) + \int_0^t \left\| s \frac{d}{ds} \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds. \] (6.119)

Combining (6.114), (6.116), (6.117), (6.118), we obtain
\[
2 \int_0^t \left\| s \frac{d}{ds} \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds + b_0 \left\| t \mathbf{v}^{N(\delta),m}(t) \right\|_{H^1}^2 \leq B_2(\delta, T) + \frac{T^2 b_1}{2b_0} S_m^\delta(t) + \frac{T^2 \bar{a}_T}{a_0} S_m^\delta(t)
\]
\[
+ \ln^2 \left( \frac{1}{\beta_{N(\delta)}} \right) S_m^\delta(t) + \int_0^t \left\| s \frac{d}{ds} \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds. \] (6.120)

Let us set
\[
R_m^\delta(t) = \int_0^t \left\| s \frac{d}{ds} \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds + \left\| t \mathbf{v}^{N(\delta),m}(t) \right\|_{H^1(\Omega)}^2.
\]
then since
\[
\int_0^t R_m^\delta(s) ds \geq \int_0^t \left\| s \frac{d}{ds} \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds
\]
together with (6.120), we deduce that
\[
R_m^\delta(t) \leq \frac{B(3, \delta)}{\min(2, b_0)} + \frac{1}{\min(2, b_0) \int_0^t R_m^\delta(s) ds}. \] (6.121)

where
\[
B(3, \delta) = B_2(\delta, T) + \frac{T^2 b_1}{2b_0} B(2, \delta) + \frac{T^2 \bar{a}_T}{a_0} B(2, \delta) + \ln^2 \left( \frac{1}{\beta_{N(\delta)}} \right) B(2, \delta).
\]

Applying Gronwall’s inequality, we obtain that
\[
\int_0^t \left\| s \frac{d}{ds} \mathbf{v}^{N(\delta),m}(s) \right\|_{L^2(\Omega)}^2 ds + \left\| t \mathbf{v}^{N(\delta),m}(t) \right\|_{H^1(\Omega)}^2 \leq B_4(\delta, T), \] (6.122)

where $B(4, \delta)$ depends only on $\delta, T$ and does not depend on $m$.

Step 3. The limiting process.
Combining (6.107), (6.109) and (6.122), we deduce that, there exists a subsequence of \( \{v^\delta_{N(\delta),m}\} \) still denoted by \( \{v^\delta_{N(\delta),m}\} \) such that (see [18])

\[
\begin{align*}
  v^\delta_{N(\delta),m} &\to v^\delta_{N(\delta)} & \text{in } L^\infty(0,T;L^2) \text{ weak*}, \\
  v^\delta_{N(\delta),m} &\to v^\delta_{N(\delta)} & \text{in } L^2(0,T;H^1) \text{ weak}, \\
  tv^\delta_{N(\delta),m} &\to tv^\delta_{N(\delta)} & \text{in } L^\infty(0,T;H^1) \text{ weak*}, \quad (6.123) \\
  (tv^\delta_{N(\delta),m})' &\to (tv^\delta_{N(\delta)})' & \text{in } L^2(Q_T) \text{ weak}, \\
  v^\delta_{N(\delta),m} &\to v^\delta_{N(\delta)} & \text{in } L^p(Q_T) \text{ weak}.
\end{align*}
\]

Using a compactness lemma ([18, Lions, p. 57] applied to (6.123), we can extract from the sequence \( \{v^\delta_{N(\delta),m}\} \) a subsequence still denoted by \( \{v^\delta_{N(\delta),m}\} \) such that

\[
(tv^\delta_{N(\delta),m})' \to (tv^\delta_{N(\delta)})' \quad \text{strongly in } L^2(Q_T). \quad (6.124)
\]

By the Riesz-Fischer theorem, we can extract from \( \{v^\delta_{N(\delta),m}\} \) a subsequence still denoted by \( \{v^\delta_{N(\delta),m}\} \) such that

\[
v^\delta_{N(\delta),m}(x,t) \to v^\delta_{N(\delta)}(x,t) \quad \text{a.e.} \quad Q_T = \Omega \times (0,T). \quad (3.40)
\]

Because \( F \) is continuous, then

\[
F \left( x, t, v^\delta_{N(\delta),m}(x,t) \right) \to F \left( x, t, v^\delta_{N(\delta)}(x,t) \right) \quad \text{a.e.} \quad Q_T = \Omega \times (0,T). \quad (6.125)
\]

On the other hand, using (6.91), (6.107), (6.109), we obtain

\[
\left\| F \left( v^\delta_{N(\delta),m}(x,t) \right) \right\|_{L^{p'}(Q_T)} \leq B_5(\delta, T), \quad (6.126)
\]

where \( B_5(\delta, T) \) is a constant independent of \( m \). We shall now require the following lemma, the proof of which can be found in [18] (see Lemma 1.3).

**Lemma 6.2.** Let \( Q \) be a bounded open subset of \( \mathbb{R}^N \) and \( G_m, G \in L^q(Q), 1 < q < \infty, \) such that

\[
\|G_m\|_{L^q(Q)} \leq C, \quad \text{where } C \text{ is a constant independent of } m \quad (6.127)
\]

and

\[
G_m \to G \quad \text{a.e.} \quad (x,t) \in Q.
\]

Then

\[
G_m \to G \quad \text{in } L^q(Q) \text{weakly.}
\]

Applying Lemma (6.2) with \( q = p' = \frac{p}{p-1}, \) \( G_m = F \left( v^\delta_{N(\delta),m}(x,t) \right), \) \( G = F \left( v^\delta_{N(\delta)}(x,t) \right), \) we deduce from (6.125) and (6.126) that

\[
F \left( v^\delta_{N(\delta),m} \right) \to F \left( v^\delta_{N(\delta)} \right) \quad \text{in } L^{p'}(Q) \text{ weakly}. \quad (6.128)
\]

Passing to the limit in (6.100) and (6.98) by (6.123) and (6.128), we have established equation (6.116).
6.1.2. Proof of the uniqueness of solution of Problem [6.89]. Assume that the Problem [6.89] has two solutions \( \mathbf{v}_N^\delta \) and \( \mathbf{w}_N^\delta \). We have to show that \( \mathbf{v}_N^\delta = \mathbf{w}_N^\delta \). We recall that
\[
\begin{align*}
\frac{\partial \mathbf{v}_N^\delta}{\partial t} + \nabla \left( \mathbf{b}_\delta \nabla \mathbf{v}_N^\delta \right) &= F \left( x, t, \mathbf{v}_N^\delta(x, t) \right) + \mathbf{p}_N^\delta \nabla \mathbf{v}_N^\delta, \\
\frac{\partial \mathbf{w}_N^\delta}{\partial t} + \nabla \left( \mathbf{b}_\delta \nabla \mathbf{w}_N^\delta \right) &= F \left( x, t, \mathbf{w}_N^\delta(x, t) \right) + \mathbf{p}_N^\delta \nabla \mathbf{w}_N^\delta, \\
u_N^\delta(x, t) &= \mathbf{w}_N^\delta(x, t) = c_t \delta_N(x),
\end{align*}
\]
For \( \mathbf{R}_\delta > 0 \), we put
\[
\mathbf{W}_N^\delta(x, t) = e^{\mathbf{R}_\delta(t-T)} \left[ \mathbf{v}_N^\delta(x, t) - \mathbf{w}_N^\delta(x, t) \right].
\]
Then for \( (x, t) \in \Omega \times (0, T) \), we get
\[
\frac{\partial \mathbf{W}_N^\delta}{\partial t} + \nabla \left( \mathbf{b}_\delta \nabla \mathbf{W}_N^\delta \right) - \mathbf{R}_\delta \mathbf{W}_N^\delta = \mathbf{P}_N^\delta \mathbf{W}_N^\delta + e^{\mathbf{R}_\delta(t-T)} \left[ F \left( x, t, \mathbf{v}_N^\delta(x, t) \right) - F \left( x, t, \mathbf{w}_N^\delta(x, t) \right) \right],
\]
and
\[
\mathbf{W}_N^\delta|_{\partial \Omega} = 0, \quad \mathbf{W}_N^\delta(x, T) = 0.
\]
By taking the inner product of the two sides of (6.130) with \( \mathbf{W}_N^\delta \), then taking the integral from \( t \) to \( T \) and noting the equality
\[
\int_\Omega \nabla \left( \mathbf{b}_\delta \nabla \mathbf{W}_N^\delta \right) \cdot \mathbf{W}_N^\delta dx = - \int_\Omega \mathbf{b}_\delta \nabla \mathbf{W}_N^\delta \cdot \mathbf{W}_N^\delta dx,
\]
we deduce
\[
\begin{align*}
\| \mathbf{W}_N^\delta(., T) \|_{L^2(\Omega)}^2 - \| \mathbf{W}_N^\delta(., t) \|_{L^2(\Omega)}^2 \\
= 2 \int_t^T \int_\Omega \mathbf{P}_N^\delta \mathbf{W}_N^\delta(x, s) ds dx + \int_t^T \int_\Omega \mathbf{b}_\delta \nabla \mathbf{W}_N^\delta \cdot \mathbf{W}_N^\delta dx ds + 2 \mathbf{R}_\delta \int_t^T \| \mathbf{W}_N^\delta(., s) \|_{L^2(\Omega)}^2 ds \\
& + 2 \int_t^T \int_\Omega e^{\mathbf{R}_\delta(t-T)} \left[ F \left( x, s, \mathbf{v}_N^\delta(x, s) \right) - F \left( x, s, \mathbf{w}_N^\delta(x, s) \right) \right] \mathbf{W}_N^\delta(x, s) dx ds \\
& + 2 \int_t^T \int_\Omega e^{\mathbf{R}_\delta(t-T)} \left[ F \left( x, s, \mathbf{w}_N^\delta(x, s) \right) - F \left( x, s, \mathbf{w}_N^\delta(x, s) \right) \right] \mathbf{W}_N^\delta(x, s) dx ds \\
& \geq 2 \int_t^T \int_\Omega \mathbf{P}_N^\delta \mathbf{W}_N^\delta(x, s) ds dx + 2 \mathbf{R}_\delta \int_t^T \| \mathbf{W}_N^\delta(., s) \|_{L^2(\Omega)}^2 ds \\
& + 2 \int_t^T \int_\Omega e^{\mathbf{R}_\delta(t-T)} \left[ F \left( x, s, \mathbf{v}_N^\delta(x, s) \right) - F \left( x, s, \mathbf{w}_N^\delta(x, s) \right) \right] \mathbf{W}_N^\delta(x, s) dx ds
\end{align*}
\]
By the assumption we have
\[
\begin{align*}
& \int_t^T \int_\Omega e^{\mathbf{R}_\delta(s-T)} \left[ F \left( x, s, \mathbf{v}_N^\delta(x, s) \right) - F \left( x, s, \mathbf{w}_N^\delta(x, s) \right) \right] \mathbf{W}_N^\delta(x, s) dx ds \\
= & \int_t^T \int_\Omega e^{\mathbf{R}_\delta(s-T)} \left[ F \left( x, s, \mathbf{v}_N^\delta(x, s) \right) - F \left( x, s, \mathbf{w}_N^\delta(x, s) \right) \right] e^{\mathbf{R}_\delta(s-T)} \left[ \mathbf{v}_N^\delta(x, s) - \mathbf{w}_N^\delta(x, s) \right] dx ds \\
& \geq - \mathbf{R}_\delta \int_t^T \| \mathbf{W}_N^\delta(., s) \|_{L^2(\Omega)}^2 ds.
\end{align*}
\]
Using the inequality (5.55), we get the following estimate
\[ \int_t^T \int_\Omega P^\delta_{\beta_N(\delta)} W^\delta_{N(\delta)}(x, s) dx ds \geq -\frac{2}{T} \ln \left( \frac{1}{\beta_T} \right) \int_t^T \| W^\delta_{N(\delta)}(\cdot, s) \|^2_{L^2(\Omega)} ds. \]  
(6.133)

Combine equations (6.131), (6.135), (6.133) and choose
\[ \mathcal{R}_\delta = \frac{1}{T} \ln \left( \frac{1}{\beta_T} \right) + \gamma \]
to obtain
\[ \| W^\delta_{N(\delta)}(\cdot, T) \|^2_{L^2(\Omega)} - \| W^\delta_{N(\delta)}(\cdot, t) \|^2_{L^2(\Omega)} \geq 0 \]
This implies that for all \( t \in [0, T] \) then \( \| W^\delta_{N(\delta)}(\cdot, t) \|^2_{L^2(\Omega)} = 0 \) since \( W^\delta_{N(\delta)}(x, T) = 0 \). The proof is completed.

6.1.3. Convergence estimate. Our analysis and proof is short and similar to the proof of Theorem [5.1]. Indeed, let us also set
\[ V^\delta_{N(\delta)}(x, t) = e^{\rho_s(s-T)} \left[ u^\delta_{N(\delta)}(x, t) - u(x, t) \right]. \]
By using some of steps as above, we obtain
\[ \| V^\delta_{N(\delta)}(\cdot, T) \|^2_{L^2(\Omega)} - \| V^\delta_{N(\delta)}(\cdot, t) \|^2_{L^2(\Omega)} = \tilde{A}_4 + \tilde{A}_5 + \tilde{A}_6 + 2 \int_t^T \left\langle e^{\rho_s(s-T)} \left[ F \left( x, s, u^\delta_{N(\delta)}(x, s) \right) - F(x, s; u(x, s)) \right], V^\delta_{N(\delta)}(x, s) \right\rangle_{L^2(\Omega)} ds \]
\[ =: \tilde{A}_8 \]
(6.134)
The terms \( \tilde{A}_4, \tilde{A}_5, \tilde{A}_6 \) is similar to (5.79). Now, we consider \( \tilde{A}_8 \). By assumption (6.92), we have
\[ \int_t^T \int_\Omega e^{\mathcal{R}_\delta(s-T)} \left[ F \left( x, s, u^\delta_{N(\delta)}(x, s) \right) - F(x, s; u(x, s)) \right] V^\delta_{N(\delta)}(x, s) dx ds \]
\[ = \int_t^T \int_\Omega e^{\mathcal{R}_\delta(s-T)} \left[ F \left( x, s, u^\delta_{N(\delta)}(x, s) \right) - F(x, s; u(x, s)) \right] e^{\mathcal{R}_\delta(s-T)} \left[ u^\delta_{N(\delta)}(x, s) - u(x, s) \right] dx ds \]
\[ \geq -\gamma \int_t^T \int_\Omega e^{2\mathcal{R}_\delta(s-T)} \left[ u^\delta_{N(\delta)}(x, s) - u(x, s) \right]^2 dx ds \]
\[ = -\gamma \int_t^T \| V^\delta_{N(\delta)}(\cdot, s) \|^2_{L^2(\Omega)} ds. \]
(6.135)
After using the results of the proof of Theorem [5.1] we get
\[ E \| V^\delta_{N(\delta)}(\cdot, t) \|^2_{L^2(\Omega)} \leq E \| G^\delta_{N(\delta)}(x) - g(x) \|^2_{L^2(\Omega)} \]
\[ + \beta_N \| u \|^2_{C([0,T];W^{1,\infty}(\Omega))} + \frac{\delta^2 T^3}{b_0} \| u \|^2_{L^\infty(0,T;H^1(\Omega))} + E \int_t^T \left( -2\rho_\delta + \frac{2}{T} \ln \left( \frac{1}{\beta_N} \right) + 2\gamma + 1 \right) \| V^\delta_{N(\delta)}(\cdot, s) \|^2_{L^2(\Omega)} ds. \]
(6.136)
Since
\[ V^\delta_{N(\delta)}(x, t) = e^{\rho_s(t-T)} \left[ u^\delta_{N(\delta)}(x, t) - u(x, t) \right] \]
We consider the problem

7.1. Ginzburg-Landau equation. Here we consider a special source function \( F(u) = u - u^3 \) for Problem (1.1). This is called Ginzburg-Landau equation. This function satisfies the condition of section 4 and does not satisfy that the condition in section 5. For all \( R > 0 \), we approximate \( F \) by \( F_R \) defined by

\[
F_R(x, t; w) := \begin{cases} 
R^3 - R, & w \in (-\infty, -R) \\
u - u^3, & w \in [-R, R], \\
R - R^3, & w \in (R, +\infty).
\end{cases}
\] (7.140)

We consider the problem

\[
\frac{\partial u_N^\delta(x, t)}{\partial t} - \nabla \left( a_\delta \nabla u_N^\delta(x, t) \right) - Q_{N(\delta)}(u_N^\delta(x, t)) = F_R(u_N^\delta(x, t)), \quad (x, t) \in \Omega \times (0, T),
\]

\[
u_N^\delta(\partial u_N^\delta(x, t) / \partial \Omega) = 0, \quad t \in (0, T),
\]

\[
u_N^\delta(x, T) = \tau_N^\delta(x, T), \quad (x, t) \in \Omega \times (0, T),
\]

(7.141)

It is easy to see that \( K(R_\delta) = 1 + 3R_\delta^2 \). Let us choose \( \beta_{N(\delta)} = N(\delta)^{-c} \) for any \( 0 < c < \min\left(\frac{1}{2}, \frac{2}{3} \right) \). And \( N(\delta) \) is chosen as follows

\[
N(\delta) = \left( \frac{1}{\delta} \right)^{m \left( \frac{1}{2} - c \right)} , \quad \beta_{N(\delta)} = \left( \frac{1}{\delta} \right)^{-mc \left( \frac{1}{2} - c \right)} \quad 0 < m < 1.
\] (7.142)
and choose $\mathcal{R}_\delta$ such that

$$\mathcal{R}_\delta = \sqrt{\frac{K(\mathcal{R}_\delta) - 1}{3}} = \sqrt{\frac{1}{k^T} \ln \left( m \left( \frac{1}{2} - c \right) \ln \left( \frac{1}{\delta} \right) \right) - 1}.$$ 

Then applying Theorem (5.1), the error $\mathbb{E} \left\| u_{\mathcal{N}(\delta)}^\delta(x, t) - u(x, t) \right\|_{L^2(\Omega)}^2$ is of order

$$\ln^2 \left( \frac{1}{\delta} \right) \left( \frac{1}{\delta} \right)^{2mc(\frac{1}{2} - c)}.$$ 

7.2. The nonlinear Fisher–KPP equation. In this subsection, we are concerned with the backward problem for a nonlinear parabolic equation of the Fisher–Kolmogorov–Petrovsky–Piskunov type in the following

$$u_t - \nabla \left( a(x, t) \nabla u \right) = \gamma(x) u^2 - \mu(x) u, \quad (x, t) \in \Omega \times (0, T),$$

with the following condition

$$\begin{cases}
  u(x, T) = g(x), & (x, t) \in \Omega \times (0, T), \\
  u|_{\partial\Omega} = 0, & t \in (0, T),
\end{cases}$$

(7.144)

By Skellam [19], the equation (7.143) has many applications in population dynamics and periodic environments. In these references, the quantity $u(x, t)$ generally stands for a population density, and the coefficients $a(x, t)$, $\gamma(x)$, $\mu(x)$ respectively, correspond to the diffusion coefficient, the intrinsic growth rate coefficient and a coefficient measuring the effects of competition on the birth and death rates.

7.3. The second equation. Taking the function $F(u) = u^\frac{1}{2}$. It is easy to check that $F$ satisfy (6.90), (6.91) and (6.92). Moreover, we can show that $F$ is not locally Lipschitz function. So, we can not regularize problem in this case by Problem (5.63). We consider the problem

$$\begin{cases}
  \frac{\partial u_{\mathcal{N}(\delta)}^\delta}{\partial t} - \nabla \left( a_{\mathcal{N}(\delta)}^\delta(x, t) \nabla u_{\mathcal{N}(\delta)}^\delta \right) - Q_{\mathcal{N}(\delta)}^\delta(x, t) = g(x, t), & (x, t) \in \Omega \times (0, T), \\
  u_{\mathcal{N}(\delta)}^\delta(x, t) = 0, & t \in (0, T), \\
  u_{\mathcal{N}(\delta)}^\delta(x, T) = g(x, T), & (x, t) \in \Omega \times (0, T),
\end{cases}$$

(7.145)

Let us choose $\beta_{\mathcal{N}(\delta)}$ and $\mathcal{N}(\delta)$ be as in subsection 6.1. Applying Theorem (5.1) the error between the solution of Problem (7.145) and $\mathbb{E} \left\| u_{\mathcal{N}(\delta)}^\delta(x, t) - u(x, t) \right\|_{L^2(\Omega)}^2$, is of order $\delta^{2mc(\frac{1}{2} - c)^\frac{1}{2}}$.

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