On the theory of the nonlinear Landau damping

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Abstract. An exact solution of the collisionless time-dependent Vlasov equation is found for the first time. By means of this solution the behavior of the Langmuir waves in the nonlinear stage is considered. The analysis is restricted by the consideration of the first nonlinear approximation keeping the second power of the electric strength. It is shown that in general the waves with finite amplitudes are not subject to damping. Only in the linear approximation, when the wave amplitude is very small, are the waves experiencing damping. It is shown that with the definite resonance conditions imposed, the waves become unstable.

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1. Introduction

A large number of papers and textbooks are devoted to the nonlinear theory of the Langmuir waves and Landau damping \cite{1}. In addition to the standard way of explanation of physics, a large number of diverse approaches and interpretations have been published \cite{2–9}, which are more advanced than the standard one.

In these papers, the solution of the linearized Vlasov equation is used. After finding the zero-order solution of the main equation, the authors construct approximations of any higher order \cite{10–14}. In the present paper, however a new approach is presented, allowing to analyze the problem self-consistently in an arbitrary order of nonlinear approximation.

The exact solution for Vlasov equation is found for the first time. Using this solution, it is shown that the waves with finite amplitude are not exposed to damping. Only waves with small amplitudes, when the oscillation frequency of captured (in the wave-well) electrons is smaller than the damping rate, can damp \cite{15, 16}. It is found that on the fulfillment of the definite resonance condition, the waves with finite amplitudes are unstable. For this necessitates, the fulfillment of the definite resonance conditions, which are similar to conditions with the parametric resonance.
2. Exact solution of the Vlasov equation

We start as usual from the collisionless Vlasov equation and Poisson’s equation,

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + v \frac{\partial f}{\partial z} - E(z, t) \frac{\partial f}{\partial v} = 0, \tag{1}
\]

\[
\frac{\partial E}{\partial z} = 1 - n, \quad E = -\frac{\partial \phi}{\partial z}, \tag{2}
\]

where the dimensionless values for the time, the coordinate, the velocity and the electric potential are used

\[
(\omega_{pe}t) \rightarrow t, \quad (z/\lambda_{De}) \rightarrow z, \quad (v/v_{Te}) \rightarrow v, \quad (e\Phi/T_e) \rightarrow \phi, \tag{3}
\]

\(\omega_{pe}\) is the electron plasma frequency, \(\lambda_{De}\) is the electron Debye length and \(v_{Te}\) is the electron thermal velocity. The dimensionless electric field \(E\) and the electron number density \(n\) are defined as follows

\[
\frac{E}{\sqrt{4\pi n_0 T_e}} \rightarrow E, \quad \frac{n}{n_0} \rightarrow n = \int dv \cdot f(z, t, v). \tag{4}
\]

Here \(T_e\) is the electron temperature in the energetic units. It is assumed that ions stay in the equilibrium with the density \(n_0\), which results in the first term (“1”) on the right-hand side (rhs) of Poisson’s Eq. (2).

The characteristic equations for Eq. (1) reads

\[
\frac{dz}{dt} = v, \quad \frac{dv}{dt} = -E(z, t). \tag{5}
\]

For constants of integrals \(R\) and \(U\) \((dR/dt = 0, \ dU/dt = 0\) we find

\[
R = z - \int^{t} dt' \cdot H' \left[ v + \int_{t'}^{t} dt'' \cdot E \left\{ z(t''), t'' \right\} \right], \tag{6}
\]

\[
U = v + \int^{t} dt' \cdot E \left\{ z(t'), t' \right\}, \tag{7}
\]

where the functions \(z(t')\) are defined with the expressions:

\[
z(t') = z - \int_{t'}^{t} dt'' \cdot H' \left[ v + \int_{t''}^{t} dt''' \cdot E \left\{ z(t'''), t''' \right\} \right], \tag{8}
\]

\[
z(t'') = z - \int_{t''}^{t} dt''' \cdot H' \left[ v + \int_{t'''}^{t} dt'''' \cdot E \left\{ z(t''''), t'''' \right\} \right], \tag{9}
\]

\[
z(t'''') = z - \int_{t'''}^{t} dt'''' \cdot H' \left[ v + \int_{t'''''}^{t} dt''''' \cdot E \left\{ z(t''''), t'''' \right\} \right], \tag{10}
\]

\[
\ldots.
\]
The chain (8), (9), (10), . . . , can be continued. Here the values \( z(t') \), \( z(t'') \), \( z(t''') \), . . . , must be substituted into the arguments of the electric fields' expressions \( E \{ z(t'), t' \} \), \( E \{ z(t''), t'' \} \), \( E \{ z(t'''), t''' \} \), . . . , etc. In Eqs. (6)-(8) the function \( H(x) \) is defined as follows

\[
H(x) = \frac{1}{2} x^2, \quad H'(x) = x. \tag{11}
\]

In the following the upper dashes in \( H'(x) \) will denote the derivative with the whole argument of the function \( H(x) \) and in \( E' \{ z(t), t \} \) will denote a derivative only with respect to \( z(t) \), \( E' \{ z(t), t \} = \partial E \{ z(t), t \} / \partial z(t) \). The solution of Eq. (1) can be represented in the form

\[
f = f \{ R(v, z, t), U(v, z, t) \}. \tag{12}
\]

Substituting Eq. (12) into Eq. (1), using the definitions (6) and (7), and successively carrying out the derivatives we obtain

\[
\frac{df}{dt} = \frac{\partial f}{\partial U} \cdot \frac{dU}{dt} + \frac{\partial f}{\partial R} \cdot \frac{dR}{dt} = \left\{ \frac{\partial f}{\partial U} \cdot \int dt' \cdot E' \{ z(t'), t' \} + \frac{\partial f}{\partial R} \right\} (-H'[v] + v) = 0. \tag{13}
\]

In Eq. (13) all other terms cancel each other. According to the second relation from (11) \( H'(v) = v \) and hence the solution (12) satisfies the kinetic equation (1). Defining the initial distribution function we assume that at the initial moment \( t_0 \to -\infty \) the electron distribution function depends only on the velocity

\[
f \{ R(v, z, t_0), U(v, z, t_0) \} = f_0(v)_{t_0 \to -\infty}. \tag{14}
\]

As \( f_0(v) \) we can choose the Maxwell distribution function with a normalizing coefficient, \( 1/\sqrt{2\pi} \cdot \exp(-v^2/2) \). From the relations at the initial time \( t_0 \),

\[
R(v, z, t_0) = R \quad \text{and} \quad U(v, z, t_0) = U \tag{15}
\]

and definitions given by Eqs. (6) and (7) we can find expressions for the velocity \( v \) and the coordinate \( z \):

\[
v = U - \int_{t_0}^{t} dt' \cdot E \{ z(t'), t' \}, \tag{16}
\]

\[
z = R + \int_{t_0}^{t} dt' \cdot H' \left[ v + \int_{t'}^{t_0} dt'' \cdot E \{ z(t''), t'' \} \right]. \tag{17}
\]

Into Eqs. (16) and (17) the explicit expressions (6) and (7) can be again substituted. Hence using the initial condition (14) for the electron distribution function we find

\[
f_0 = f_0 \left\{ v + \int_{t_0 \to \infty}^{t} dt' \cdot E \{ z(t'), t' \} \right\}, \tag{18}
\]
where values $z(t')$, $z(t'')$, . . . , are defined by Eqs. (8), (9), . . . . We can represent Poisson’s Eq. (2) in the following form,

$$\frac{\partial E(z,t)}{\partial z} = 1 - \int_{-\infty}^{\infty} dv \cdot f_0 \left\{ v + \int_{t_0 \rightarrow -\infty}^{t} dt' \cdot E \{ z(t'), t' \} \right\}.$$  \hfill (19)

It is convenient to introduce a new variable - $s$, defined by the relation:

$$v + \int_{-\infty}^{t} dt' \cdot E \{ z(t'), t' \} = s,$$  \hfill (20)

which allows to simplify the argument of $f_0$. Then for the explicit expression for the velocity $v$ we have

$$v = s - \int_{t_0 \rightarrow -\infty}^{t} dt' \cdot E [z(t', s)] = s - \int_{t_0 \rightarrow -\infty}^{t} dt' \cdot E [z - s (t - t') +$$

$$+ \int_{t' \rightarrow -\infty}^{t} dt'' \int_{t_0}^{t'} dt''' \cdot E \{ z - s (t - t'') +$$

$$+ \int_{t'' \rightarrow -\infty}^{t'} dt''' \cdot dt'''' \cdot E (z - s (t - t''' + ...) \}, t'''] \tag{21}$$

where by analogy to Eqs. (8), (9), . . . , we have introduced the relations:

$$z(t', s) = z - s \{ t - t' \} + \int_{t' \rightarrow -\infty}^{t} dt'' \int_{t_0}^{t'} dt''' \cdot E [z(t'', s), t'''], \tag{22}$$

$$z(t'', s) = z - s \{ t - t'' \} + \int_{t'' \rightarrow -\infty}^{t'} dt''' \int_{t_0}^{t''} dt'''' \cdot E [z(t', s), t'''] \tag{23}$$

Poisson’s equation can be represented in the form

$$\frac{\partial E(z,t)}{\partial z} = 1 - \int_{-\infty}^{\infty} ds \cdot \frac{dv(s)}{ds} f_0(s) = 1 + \int_{-\infty}^{\infty} ds \cdot \frac{df_0(s)}{ds} \cdot v(s). \tag{24}$$

Substituting Eq. (21) into Eq. (24) for the Maxwell distribution function $f_0(s)$ the first term in (24) is compensated by the positive charge of ions:

$$\frac{\partial E}{\partial z} = - \int_{-\infty}^{+\infty} ds \cdot \frac{df_0(s)}{ds} \cdot \int_{t_0 \rightarrow -\infty}^{t} dt' \cdot E \{ z(t's), t' \}. \tag{25}$$
Taking the first derivative with time and restricting ourselves to keeping the terms up to second powers of the electric strength (such a restriction will keep everywhere throughout the calculations) we find

\[
\frac{\partial}{\partial t} \frac{\partial E}{\partial z} = - \int_{-\infty}^{+\infty} ds \cdot \frac{\partial f_0(s)}{\partial s} \left\{ -s \int_{t_0 \to -\infty}^{t} dt' \cdot E'[z(t', s), t'] \\
+ \int_{t_0 \to -\infty}^{t} dt' \cdot E'[z(t', s), t'] \int_{t_0 \to -\infty}^{t} dt'' \cdot E[z(t'', s), t''] \right\} (26)
\]

In the last term in the rhs of this equation we can change the ordering of integrals in the following manner

\[
\int_{t_0 \to -\infty}^{t} dt' \int_{t' \to -\infty}^{t} dt'' \cdot Q(t', t'') = \int_{t_0 \to -\infty}^{t} dt'' \int_{t_0 \to -\infty}^{t} dt' \cdot Q(t', t''). (27)
\]

Then the second derivative with time of Eq. (26) gives

\[
\frac{\partial^2}{\partial t^2} \frac{\partial E}{\partial z} = - \frac{\partial E}{\partial z} - \int_{-\infty}^{+\infty} ds \cdot \frac{\partial f_0(s)}{\partial s} \left\{ s^2 \int_{t_0 \to -\infty}^{t} dt' \cdot E''[z(t', s), t'] - \\
- s \cdot \frac{\partial}{\partial z} \int_{t_0 \to -\infty}^{t} dt' \cdot E' \{ z - s (t - t'), t' \} \right\} (28)
\]

In Eq. (28) we can transform the last term (with the first derivative with time) as
On the theory of the nonlinear Landau damping

follows

\[
\frac{\partial}{\partial t} \int_{t_0 \to -\infty}^{t} dt' \cdot E' \left[ z \left( t', s \right), t' \right] \int_{t_0 \to -\infty}^{t} dt'' \cdot E \left[ z \left( t'', s \right), t'' \right] \cong \\
\cong \frac{\partial}{\partial z} \cdot E \left( z, t \right) \int_{t_0 \to -\infty}^{t} dt' \cdot E \left\{ z - s \left( t - t' \right), t' \right\} - s \cdot \frac{\partial}{\partial z} \cdot \\
\cdot \int_{t_0 \to -\infty}^{t} dt' \cdot E \left\{ z - s \left( t - t'' \right), t' \right\} \cdot \\
\cdot \int_{t_0 \to -\infty}^{t} dt'' \cdot E \left\{ z - s \left( t - t''' \right), t''' \right\},
\]

(29)

In the first term on the right-hand side of Eq. (29), containing the squared electric field, we can, following Eq. (25), use a simplified linearized expression

\[
\frac{\partial E}{\partial z} \cong - \int_{-\infty}^{+\infty} ds \cdot \frac{\partial f_0 (s)}{\partial s} \int_{t_0 \to -\infty}^{t} dt' \cdot E \left\{ z - s \left( t - t' \right), t' \right\}.
\]

(30)

Using the relations (29) and (30) the fourth derivative with time of Eq. (28) we can represent in the form

\[
\frac{\partial^4 E}{\partial t^4} + \frac{\partial^2 E}{\partial t^2} + 3 \frac{\partial^2 E}{\partial z^2} + \frac{\partial^3 E}{\partial z^3} \cdot \\
\cdot \int_{-\infty}^{+\infty} ds \cdot s^4 \cdot \frac{\partial f_0 (s)}{\partial s} \int_{t_0 \to -\infty}^{t} dt' \cdot E \left\{ z - s \left( t - t' \right), t' \right\} + \\
+ \frac{\partial}{\partial z} \frac{\partial^2}{\partial t^2} \left\{ \int_{t_0 \to -\infty}^{t} dt' \cdot E \left( z, t' \right) \cdot \\
\cdot \int_{t_0 \to -\infty}^{t} dt' \cdot E \left( z, t' \right) - \frac{1}{2} \cdot E \left( z, t \right) \cdot E \left( z, t \right) \right\} = 0.
\]

(31)

On obtaining Eq. (31), the first derivative (\( \partial/\partial z \)) in front of every term has been canceled and the explicit expressions for the integrals from the Maxwell distribution function \(-f_0 (s)\) is used.

3. Waves in the weak nonlinear case

The first three terms of Eq. (31) should describe Landau damping in the linear approximation. By means of the Fourier expansion of these three terms for the electric amplitudes we find

\[
E (\omega, k) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dt \int_{-\infty}^{+\infty} dz \cdot E \left( z, t \right) \cdot \exp \left\{ i (\omega t - k z) \right\},
\]

(32)
which results in the following dispersion relation

$$\omega^4 - \omega^2 - 3k^2 + \int_{-\infty}^{+\infty} ds \cdot \frac{k^3 s^3}{\omega - ks} \cdot s \frac{\partial f_0(s)}{\partial s} = 0. \tag{33}$$

A simple transformation of the last equation gives

$$\omega^4 + \omega^3 \int_{-\infty}^{+\infty} ds \cdot \frac{s}{\omega - ks} \frac{\partial f_0(s)}{\partial s} = 0. \tag{34}$$

After expanding the denominator of (34) in powers of \((ks/\omega)\) for the real and the imaginary parts of the frequency, \(\omega = \omega_0 + i\gamma\), we obtain the equalities

$$\omega_0^2 = 1 + 3k^2 \text{ and } \gamma = -\sqrt{\frac{\pi}{8k^3}} \exp\left\{-\frac{1}{2k^2} - \frac{3}{2}\right\}, \tag{35}$$

which determine the frequency and the Landau damping of the high-frequency Langmuir waves [17].

In fact Landau damping describes the initial stage of the electrons capturing by the wave cavity (the minimum region of the wave), herewith the amplitude must be very small, smaller than the value proportional to \(\gamma\), namely \(\sqrt{eE_0/T_e} \cdot k \ll \gamma/k\) [15] (here \(E_0\) is the amplitude of the wave). In other words, this inequality means that the oscillation frequency of captured electrons in the wave-well, must be much smaller than the damping rate of the wave \(-\gamma\); the electrons are pushed by the back-side wall of the wave-cavity and during the time-interval of passing the cavity width, the wave should be damped [15, 16].

Below we consider the case when the inverse inequality is fulfilled, that means the predominance of the electrons’ oscillation frequency in the well over the damping rate \(-\gamma\). Then the captured electrons are reflected many times from the cavity walls (getting and losing the energy) and on average the wave keeps its energy – hence at \(\sqrt{eE_0/T_e} \cdot k \cdot k \gg \gamma\) the damping of the wave does not take place [15, 16].

To simplify further calculations, it is convenient to introduce the following auxiliary value,

$$I(z, t) = \int_{t_0 \rightarrow -\infty}^{t} dt' \cdot E(z, t'), \tag{36}$$

for which from Eq. (31) we can obtain the following equation, neglecting the term corresponding to the wave damping:

$$\frac{\partial^4 I}{\partial t^4} + \frac{\partial^2 I}{\partial t^2} + 3 \frac{\partial^2 I}{\partial z^2} + \frac{\partial}{\partial z} \frac{\partial}{\partial t} \left\{ f^2(z, t) - \frac{1}{2} \left( \frac{\partial I(z, t)}{\partial t} \right)^2 \right\} = 0. \tag{37}$$
By means of Eq. (37) we can construct the expression for the value \( \frac{1}{2} \left( \frac{\partial I(z,t)}{\partial t} \right)^2 \). The straightforward calculations give

\[
\frac{1}{2} \left( \frac{\partial I(z,t)}{\partial t} \right)^2 \approx -\frac{1}{2} I^2(z,t) - 3 \int_{-\infty}^{t} dt' \frac{\partial I(z,t')}{\partial t} \frac{\partial^2}{\partial z^2}.
\]

Further we a) hold on to the approximation usually used in the theory of Landau damping – the assumption of the smooth dependence of the electric field on the spatial coordinate. Therefore in Eq. (37) the terms only up to the second derivative with the spatial \( z \) coordinate are kept, and b) use the relation \( \left( \frac{\partial I}{\partial t} \right) = E(z,t) = -\frac{\partial \phi}{\partial z} \), which follows from Eq. (36) and restrict with the quadratic term \( I^2 \) in the last term in the rhs of Eq. (38). Assuming the dependence of all unknown values in the argument \( \xi = z - V \cdot t \), we find that \( I(\xi) = (1/V) \cdot \phi(\xi) \), where \( V \) is the dimensionless velocity of the waves, which is assumed to be large, \( V \gg 1 \). Under restrictions a) and b) the last two terms in the rhs of Eq. (38) give negligibly small contributions. Substituting the remaining first term in the rhs into Eq. (37) after two-times integration we obtain

\[
V^4 \frac{\partial^2 \phi}{\partial \xi^2} + (V^2 + 3) \cdot \phi - \frac{3}{2} \phi^2 = \frac{1}{2} C_1.
\]

Here the constant \( C_1 \) does not depend on time. In the rhs of Eq. (39) we have neglected the term proportional to \( \xi \), leading to the nonphysical result at \( \xi \to \infty \). Multiplying Eq. (39) by \( (\partial \phi/\partial \xi) \) after integration we find:

\[
V^4 \left( \frac{\partial \phi}{\partial \xi} \right)^2 = \phi^3 - (V^2 + 3) \cdot \phi + C_1 \cdot \phi + C_2,
\]

where the constants of integrations \( C_1 \) and \( C_2 \) can be expressed in terms of the minimum \( - \phi_n \) and the maximum \( - \phi_m \) values of the potential, \( (\partial \phi/\partial \xi)|_{\phi_m, \phi_n} = 0 \). As a result we obtain

\[
V^4 \left( \frac{\partial \phi}{\partial \xi} \right)^2 = (\phi_m - \phi) \cdot (\phi_s - \phi) \cdot (\phi - \phi_n).
\]

The constant \( \phi_s \) (together with \( \phi_m \) and \( \phi_n \)) defines the wave velocity \( V \):

\[
V^2 = \phi_m + \phi_s + \phi_n - 3.
\]

At \( \phi_m > \phi_s > \phi_n \) the solution of Eq. (41) represents a periodic wave, described by the expression (see the Fig. 1)

\[
\phi = \phi_m - (\phi_m - \phi_n) \cdot dn^n \{x, s\},
\]

where \( dn \{x, s\} \) is a Jacobi elliptic function with the module \( -s \)

\[
dn \{x, s\} = \frac{\pi}{2 \cdot K(s)} + \frac{2\pi}{K(s)} \sum_{n=1}^{\infty} \frac{q^n}{1 + q^{2n}} \cos \left\{ \frac{\pi \cdot n \cdot x}{K(s)} \right\},
\]
On the theory of the nonlinear Landau damping

\[ q = \exp \left[ -\pi \frac{K(s')}{K(s)} \right], \quad s' = \sqrt{1-s^2}, \]

\[ x = \frac{\sqrt{\phi_m - \phi_n}}{2 \cdot V^2} \xi, \quad s^2 = \frac{\phi_s - \phi_n}{\phi_m - \phi_n} \]

(45)

The function \( dn^2\{x, s\} \) is a periodic function with the period \( 2K(s) \), where \( K(s) \) is the complete elliptic integral of the first kind,

\[ K(s) = \frac{\pi/2}{\int_0^\pi \frac{d\alpha}{\sqrt{1-s^2 \cdot \sin \alpha}}}, \]

(46)

therefore the wave length of the periodic solution can be defined according to the relation

\[ \lambda = 4 \frac{4}{\sqrt{\phi_m - \phi_n}} \cdot V^2 K(s) \]

(47)

![Figure 1. Sketch of the dependence of \( dn^2\{x, s\} \) on \( x \) at \( s = \text{const} \); \( \Delta \phi = (1-a^2) \cdot (\phi_m - \phi_n) \), \( a \) is the minimum value of the curve \( dn\{x, s\} \)](image)

4. Instability of waves

We can now analyze the stability of the waves. Starting from Eq. (39) we introduce the potential perturbation \( \delta \phi \) according to the relation \( \phi \rightarrow \phi + \delta \phi \). For the unperturbed part of the potential—\( \phi \) remains the expression and for \( \delta \phi \) we obtain

\[ \frac{\partial^2 \delta \phi}{\partial \xi^2} + \frac{1}{V^2} \left[ 1 + \frac{3 - \phi}{V^2} + \frac{3}{V^2} \left( 1 - dn^2(z, s^2) \right) \right] \cdot \delta \phi = 0. \]

(48)

An explicit solution of this equation with the periodic coefficient can be found applying a Hill’s method. This method is rather cumbersome with the long and difficult calculations. Here we simplify the equation recalling the condition used above, \( V \gg 1 \), and assuming that the parameter \( s \) is small, \( s \ll 1 \). Then Eq. (48) can be
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reduced to Mathieu’s equation \[19, 20\], which describes the phenomenon known as a
parametric resonance,

\[
\frac{\partial^2 \delta \phi}{\partial \xi^2} + \frac{1}{V^2} \{1 + h \cdot \cos k \xi\} \cdot \delta \phi = 0,
\tag{49}
\]

where \( h \approx \frac{12}{V^2} \phi_m - \phi_n \) and \( k \approx \sqrt{\frac{\phi_m - \phi_n}{V^2}} \). Applying a standard method at the
fulfilment of the resonance condition \( k = 2/V \) Eq. (49) gives the following expression
for the rate of the instability \( \gamma = \frac{1}{4V} \) \[20\].

5. Summary

Finding the exact analytic solution for the collisionless Vlasov’s equation, the nonlinear
stage of the Langmuir waves is analyzed in the first non-vanishing (quadratic) nonlinear
approximation. The Langmuir waves with the finite amplitude, and with the oscillation
frequency (of electrons in the wave-well) larger than the damping rate (found in the
linear approximation), do not damp and tend to keep the periodic structure. On the
fulfilment of the definite resonance conditions, the waves are unstable. The finding of
the corresponding rate is quite similar to the procedure applied in the investigation of
the parametric instability.

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