Low-SNR Analysis of Interference Channels under Secrecy Constraints

Junwei Zhang and Mustafa Cenk Gursoy
Department of Electrical Engineering
University of Nebraska-Lincoln, Lincoln, NE 68588
Email: junwei.zhang@huskers.unl.edu, gursoy@engr.unl.edu

Abstract—In this paper, we study the secrecy rates over weak Gaussian interference channels for different transmission schemes. We focus on the low-SNR regime and obtain the minimum bit energy $E_{b_{\text{min}}}$ values, and the wideband slope regions for both TDMA and multiplexed transmission schemes. We show that secrecy constraints introduce a penalty in both the minimum bit energy and the slope regions. Additionally, we identify under what conditions TDMA or multiplexed transmission is optimal. Finally, we show that TDMA is more likely to be optimal in the presence of secrecy constraints.

I. INTRODUCTION

The open nature of wireless communications allows for the signals to be received by all users within the communication range. Thus, wireless communication is vulnerable to eavesdropping. This problem was first studied in [1] where Wyner proposed a wiretap channel model. In this model, a single source-destination communication link is eavesdropped by a wiretapper. The secrecy level is measured by the equivocation rate. Wyner showed that secure communication is possible without sharing a secret key if the eavesdropper’s channel is a degraded version of the main channel. Later, Wyner’s result was extended to the Gaussian channel in [3] and recently to fading channels in [4]. In addition to the single antenna case, secrecy in multi-antenna models is addressed in [5]–[8]. Multiple access channels with confidential messages were considered in [9]. Liu et al. [10] presented inner and outer bounds on secrecy capacity regions for broadcast and interference channels. They also described several transmission schemes for Gaussian interference channels and derived their achievable rate regions while ensuring mutual information-theoretic secrecy. Recently, Bloch et al. in [11] discussed the theoretical aspects and practical schemes for wireless information-theoretic security.

Another important concern in wireless communications is the efficient use of limited energy resources. Hence, the energy required to reliably send one bit is a metric that can be adopted to measure the performance. Generally, energy-per-bit requirement is minimized, and hence the energy efficiency is maximized, if the system operates in the low-SNR regime. In [12], Verdu has analyzed the tradeoff between the spectral efficiency and bit energy in the low-SNR regime for a general class of channels. As argued in [12], two key performance measures in the low-power regime are the minimum energy per bit $E_{b_{\text{min}}}$ required for reliable communication and the slope of the spectral efficiency versus $E_{b}$ curve at $E_{b_{\text{min}}}$. Caire et al. in [13] employed these two measures to study the multiple access, broadcast, and interference channels in the low-power regime. By comparing the performance of TDMA and superposition schemes, they concluded that the growth of TDMA-achievable rates with energy per bit is suboptimal except in some special cases.

In this paper, we study secure transmission over Gaussian weak interference channels in the low-power regime. The organization of the rest of the paper is as follows. In Section II we describe the channel model and obtain the secrecy achievable rate regions for TDMA, multiplexed transmission schemes and artificial noise schemes, and compare their performances in terms of the achievable rates. In Section III we compute the minimum energy per bit and slope at $E_{b_{\text{min}}}$ for TDMA and multiplexed transmission schemes. In Section IV we use results in Section III to evaluate how secrecy constraints affect the performance in the low-power regime and identify optimal transmission schemes. Finally, we provide conclusions in Section V.

II. GAUSSIAN INTERFERENCE CHANNELS WITH CONFIDENTIAL MESSAGES

We consider secure communication over a two-transmitter, two-receiver Gaussian interference channel. The input-output relations for this channel model are given by

\begin{align}
y_1 &= c_{11}x_1 + c_{12}x_2 + n_1, \quad \text{and} \quad (1) \\
y_2 &= c_{21}x_1 + c_{22}x_2 + n_2 \quad (2)
\end{align}

where $x_1$ and $x_2$ are the channel inputs of the transmitters, the coefficients $\{c_{ij}\}$ denote the channel gains and are deterministic scalars, and $n_1$ and $n_2$ are independent, circularly symmetric, complex Gaussian random variables with zero mean and common variance $\sigma^2$. It is assumed that the transmitters are subject to the following average power constraint:

\begin{equation}
E[|x_i|^2] \leq P_i = \text{SNR}_i \sigma^2, \quad i = 1, 2. \quad (3)
\end{equation}

We focus on the weak interference channel i.e., we assume that $|c_{12}|^2 < 1$ and $|c_{21}|^2 < 1$. Over this channel, transmitter $i$ for $i = 1, 2$ intends to send a confidential message by transmitting $x_i$ to the desired receiver $i$, which receives $y_i$, while ensuring that the other receiver does not obtain any information by listening the transmission. Following [10], we next consider three transmission schemes and their corresponding achievable secrecy rate regions.

A. Time Division Multiple Access

In TDMA, the transmission period is divided into two nonoverlapping time slots. Transmitters 1 and 2 transmit using
\(\alpha\) and \(1 - \alpha\) fractions of time, respectively. We note that under this assumption, the channel in each time slot reduces to a Gaussian wiretap channel [3], and the following rate region can be achieved with perfect secrecy [10]:

\[
\begin{align*}
R_1 &\geq 0 \\
R_2 &\geq 0 \\
R_1 &\leq \log \left(1 + \frac{c_{11}^2 \text{SNR}_1}{\alpha} \right) - \log \left(1 + \alpha \frac{c_{21}^2 \text{SNR}_1}{1 - \alpha} \right) \\
R_2 &\leq \log \left(1 + \frac{c_{22}^2 \text{SNR}_2}{1 - \alpha} \right) - \log \left(1 + \frac{c_{12}^2 \text{SNR}_2}{\alpha} \right)
\end{align*}
\]  

(4)

over all possible transmitting signal-to-noise-ratio pairs \(\text{SNR}_1 \in [0, P_1/\sigma^2]\), \(\text{SNR}_2 \in [0, P_2/\sigma^2]\) and time allocation parameter \(\alpha\).

B. Multiplexed Transmission

In the multiplexed transmission scheme, transmitters are allowed to share the same degrees of freedom. By the constraint of information-theoretic security, no partial decoding of the other transmitter’s message is allowed at a receiver. Hence, the interference results in an increase of the noise floor. Thus, the following rate region can be achieved with perfect secrecy [10]:

\[
\begin{align*}
R_1 &\geq 0 \\
R_2 &\geq 0 \\
R_1 &\leq \log \left(1 + \frac{c_{11}^2 \text{SNR}_1}{1 + c_{12}^2 \text{SNR}_2} \right) - \log \left(1 + c_{21}^2 \text{SNR}_1 \right) \\
R_2 &\leq \log \left(1 + \frac{c_{22}^2 \text{SNR}_2}{1 + c_{21}^2 \text{SNR}_1} \right) - \log \left(1 + c_{12}^2 \text{SNR}_2 \right)
\end{align*}
\]  

(5)

over all possible transmitting signal-to-noise-ratio pairs \(\text{SNR}_1 \in [0, P_1/\sigma^2]\), \(\text{SNR}_2 \in [0, P_2/\sigma^2]\).

C. Artificial Noise

This scheme allows one of the transmitters (e.g. transmitter 2) to generate artificial noise. This scheme will split the power of transmitter 2 into two parts: \(\lambda P_2\) for generating artificial noise and the remaining \((1 - \lambda)P_2\) for encoding the confidential message. As detailed in [10], the achievable rate region is

\[
\begin{align*}
R_1 &\geq 0 \\
R_2 &\geq 0 \\
R_1 &\leq \log \left(1 + \frac{c_{11}^2 \text{SNR}_1}{1 + c_{12}^2 \text{SNR}_2} \right) - \log \left(1 + c_{21}^2 \text{SNR}_1 \right) \\
R_2 &\leq \log \left(1 + \frac{c_{22}^2 \text{SNR}_2}{1 + c_{21}^2 \text{SNR}_1} \right) - \log \left(1 + c_{12}^2 \text{SNR}_2 \right) - \log \left(1 + c_{12}^2 \lambda \text{SNR}_2 \right)
\end{align*}
\]  

(6)

over all possible transmitting signal-to-noise-ratio pairs \(\text{SNR}_1 \in [0, P_1/\sigma^2]\), \(\text{SNR}_2 \in [0, P_2/\sigma^2]\) and power splitting parameter \(\lambda\). We can further enlarge the rate region by reversing the roles of transmitters 1 and 2.

When the transmitting power is moderate, neither too high nor too small, as demonstrated in [10], transmission strategy with artificial noise provides the largest achievable rate region while TDMA gives the smallest rate region.

On the other hand, when we consider the two extreme cases of high- and low-SNR regimes, the picture changes. In the high-SNR regime, when we let \(\text{SNR}_1 \to \infty\), \(\text{SNR}_2 \to \infty\) and \(\lim \text{SNR}_1, \text{SNR}_2 = q\) in (4), (5), and (6), we can see that multiplexed transmission can not achieve any positive secrecy rate, while TDMA rates are bounded by

\[
R_1 < \alpha \log \left(\frac{c_{11}^2}{c_{21}^2} \right), \quad R_2 < (1 - \alpha) \log \left(\frac{c_{22}^2}{c_{12}^2} \right).
\]

For the strategy with the artificial noise, rate

\[
R_1 \text{ is bounded by } R_1 < \log \left(\frac{1 + c_{11}^2}{1 + c_{21}^2} \right), \text{ but we can not achieve any secrecy rate for } R_2. \text{ Thus, TDMA is the best choice when we want both users to have secure communication in the high-SNR regime.}
\]

In the low-SNR regime (as SNR approaches zero), TDMA and multiplexed transmission achievable regions become identical. They converge to the following rectangular rate region, as illustrated in Fig. 1:

\[
\begin{align*}
R_1 &\geq 0 \\
R_2 &\geq 0 \\
R_1 &\leq \left|c_{11}\right|^2 \text{SNR}_1 - \left|c_{21}\right|^2 \text{SNR}_1 + o(\text{SNR}_1) \\
R_2 &\leq \left|c_{22}\right|^2 \text{SNR}_2 - \left|c_{12}\right|^2 \text{SNR}_2 + o(\text{SNR}_2)
\end{align*}
\]  

(7)

Thus, these schemes have similar performances at vanishing SNR levels in terms of the asymptotic rates. However, a finer analysis in the next section will provide more insight. We note that in the case of transmission with artificial noise, we have

\[
R_1 \leq \left|c_{11}\right|^2 \text{SNR}_1 - \left|c_{21}\right|^2 \text{SNR}_1 + o(\text{SNR}_1) \quad \text{and} \quad R_2 \leq \left(1 - \lambda\right)(\left|c_{22}\right|^2 \text{SNR}_2 - \left|c_{12}\right|^2 \text{SNR}_2 + o(\text{SNR}_2)) \text{ which is strictly smaller than that in (7). This lets us to conclude that introducing artificial noise is not preferable in the low-SNR regime.}
\]

III. ENERGY EFFICIENCY IN THE LOW-SNR REGIME

The tradeoff of spectral efficiency versus energy per information bit is the key measure of performance in the low-SNR regime. The two major analysis tools in this regime are the
minimum value of the energy per bit $E_k/N_{0_{\text{min}}}$, and the slope $S$ of the spectral efficiency versus $E_k/N_{0_{\text{min}}}$ curve at $E_k/N_{0_{\text{min}}}$ [12]. These can be obtained from

$$E_k/N_{0_{\text{min}}} = \log_2 \frac{2}{C(0)} \quad (8)$$

and

$$S = \frac{2[C(0)^2]}{C(0)} \quad (9)$$

where $C(0)$ and $\dot{C}(0)$ denote the first and second derivatives of the channel capacity with respect to SNR at SNR = 0.

In this section, using these tools, we analyze the performance in interference channels with confidential messages, following an approach similar to that in [13]. Note that in interference channels, we have the achievable rate pairs $(R_1, R_2)$. As the SNRs of both users approach zero in the low-SNR regime, it can be easily seen that $R_1 \to 0$ and $R_2 \to 0$. In this regime, we introduce the parameter $\theta$, and assume that the ratio of the rates is $R_1/R_2 = \theta$ as $R_1$ and $R_2$ both vanish. In both TDMA and multiplexed transmissions, we have

$$\theta = \frac{R_1}{R_2} = \frac{\text{SNR}_1 (|c_{11}|^2 - |c_{21}|^2)}{\text{SNR}_2 (|c_{22}|^2 - |c_{12}|^2)}. \quad (10)$$

By fixing $\theta$, we can rewrite the achievable rate region of multiplexed transmission in (5) as

$$R_1 \geq 0 \quad R_2 \geq 0 \quad R_1 \leq \log \left( 1 + \frac{|c_{11}|^2 \text{SNR}_{1}}{1 + |c_{21}|^2 \text{SNR}_{1} \left( \frac{|c_{11}|^2 - |c_{21}|^2}{|c_{22}|^2 - |c_{12}|^2} \right) \text{SNR}_1} \right) - \log(1 + |c_{21}|^2 \text{SNR}_1)$$

$$R_2 \leq \log \left( 1 + \frac{|c_{22}|^2 \text{SNR}_{2}}{1 + |c_{21}|^2 \text{SNR}_{2} \left( \frac{|c_{11}|^2 - |c_{21}|^2}{|c_{22}|^2 - |c_{12}|^2} \right) \text{SNR}_2} \right) - \log(1 + |c_{21}|^2 \text{SNR}_2). \quad (11)$$

From (4) and (11), we can see that when SNR diminishes, the bit energy $E_k/N_0 = \frac{S N_0}{R_1}$ for both TDMA and multiplexed transmission schemes monotonically decreases. Furthermore, it can be shown that the rates are concave functions of SNR in the low-SNR regime. Thus, the minimum energy per bit is achieved as SNR $\to 0$. The following theorems provide the minimum energy per bit and the slope at the minimum energy per bit.

**Theorem 1:** For all $\theta = R_1/R_2$, the minimum bit energies in the Gaussian interference channel with confidential messages for both TDMA and multiplexed transmissions are

$$\frac{E_1}{N_{0_{\text{min}}}} = \log_2 \frac{2}{|c_{11}|^2 - |c_{21}|^2}, \quad (12)$$

$$\frac{E_2}{N_{0_{\text{min}}}} = \log_2 \frac{2}{|c_{22}|^2 - |c_{12}|^2}. \quad (13)$$

**Proof:** From (4) and (11), we can for both cases easily compute the derivatives of the achievable rates with respect to SNR as

$$\dot{R}_1(0) = |c_{11}|^2 - |c_{21}|^2 \quad (14)$$

$$\dot{R}_2(0) = |c_{22}|^2 - |c_{12}|^2. \quad (15)$$

Using (3), we get the minimum bit energy expressions. \hfill \Box

From the result of Theorem 1, we see that TDMA and multiplexed transmission achieve the same minimum energy per bit. Next, we consider the wideband slope regions.

**Theorem 2:** Let the rates vanish while keeping $R_1/R_2 = \theta$. Then, for the Gaussian interference channel with confidential messages, the slope region achieved by TDMA is

$$0 \leq S_1 < 2 \quad 0 \leq S_2 < 2 \quad \frac{S_1}{2A} + \frac{S_2}{2B} = 1 \quad (16)$$

and the slope region achieved by multiplexed transmission is

$$0 \leq S_1 < 2 \quad 0 \leq S_2 < 2 \quad \left( \frac{2A}{S_1} - 1 \right) \left( \frac{2B}{S_2} - 1 \right) = \frac{4|c_{11}|^2 |c_{22}|^2 |c_{22}^2 |c_{21}|^2}{(|c_{11}|^4 - |c_{21}|^4)(|c_{22}|^4 - |c_{12}|^4)} \quad (17)$$

where

$$A = \frac{|c_{11}|^2 - |c_{21}|^2}{|c_{11}|^2 + |c_{21}|^2}; \quad (18)$$

$$B = \frac{|c_{22}|^2 - |c_{12}|^2}{|c_{22}|^2 + |c_{12}|^2}. \quad (19)$$

**Proof:** Note again that for both transmission schemes, we have

$$\tilde{R}_1(0) = \frac{|c_{11}|^4 - |c_{21}|^4}{\alpha}, \quad (22)$$

$$\tilde{R}_2(0) = \frac{|c_{22}|^4 - |c_{12}|^4}{(1 - \alpha)}. \quad (23)$$

Then, using (9), we get

$$S_1 = \frac{2\alpha(|c_{11}|^2 - |c_{21}|^2)}{|c_{11}|^2 + |c_{21}|^2}, \quad (24)$$

$$S_2 = \frac{2(1 - \alpha)(|c_{22}|^2 - |c_{12}|^2)}{|c_{22}|^2 + |c_{12}|^2}. \quad (25)$$

Considering different values of $\alpha$ leads to the region in (16). Similarly, for multiplexed transmission, we can obtain

$$\tilde{R}_1(0) = |c_{11}|^4 - |c_{21}|^4 + \frac{2|c_{11}|^2 |c_{22}|^2 (|c_{11}|^2 - |c_{21}|^2)}{\theta(|c_{22}|^2 - |c_{12}|^2)}, \quad (26)$$

$$\tilde{R}_2(0) = |c_{22}|^4 - |c_{12}|^4 + \frac{2|c_{22}|^2 |c_{21}|^2 \theta(|c_{22}|^2 - |c_{12}|^2)}{|c_{11}|^2 - |c_{21}|^2}. \quad (27)$$
From the above expression, we can easily see that
\[
S_1 = \frac{2(|c_{11}|^2 - |c_{21}|^2)}{|c_{11}|^2 + |c_{21}|^2 + \frac{2|c_{11}|^2|c_{12}|^2}{\theta(|c_{22}|^2 - |c_{12}|^2)}}, \tag{28}
\]
\[
S_2 = \frac{2(|c_{22}|^2 - |c_{12}|^2)}{|c_{22}|^2 + |c_{12}|^2 + \frac{2|c_{22}|^2|c_{21}|^2}{\theta(|c_{11}|^2 - |c_{21}|^2)}}. \tag{29}
\]

Considering different values of \(\theta\) leads to the slope region given in (17).

IV. THE IMPACT OF SECRECY ON ENERGY EFFICIENCY

For comparison, we provide below the minimum energy per bit and slope region when there are no secrecy constraints [13]. The minimum bit energies for both TDMA and multiplexed transmission are
\[
\frac{E_1}{N_0^{\text{min}}} = \frac{\log_2 2}{|c_{11}|^2}, \tag{30}
\]
\[
\frac{E_2}{N_0^{\text{min}}} = \frac{\log_2 2}{|c_{22}|^2}. \tag{31}
\]
The achievable slope region for TDMA is
\[
0 \leq S_1 < 2
\]
\[
0 \leq S_2 < 2
\]
\[
S_1 + S_2 = 2, \tag{32}
\]
while for multiplexed transmission, we have
\[
0 \leq S_1 < 2
\]
\[
0 \leq S_2 < 2
\]
\[
\left(\frac{2}{S_1} - 1\right)\left(\frac{2}{S_2} - 1\right) = 4 \frac{|c_{12}|^2 |c_{21}|^2}{|c_{22}|^2 |c_{11}|^2}. \tag{33}
\]

We can immediately note that the minimum bit energies in (30) and (31) are strictly smaller than those given in (12) and (13). Thus, there is an energy penalty associated with secrecy. Moreover, comparing the slope regions in (16) and (17) with those in (32) and (33), and noting that
\[
A < 1
\]
\[
B < 1
\]
\[
\frac{4|c_{12}|^2 |c_{21}|^2}{|c_{22}|^2 |c_{11}|^2} < \frac{4|c_{11}|^2 |c_{12}|^2 |c_{22}|^2 |c_{21}|^2}{(|c_{11}|^4 - |c_{21}|^4)(|c_{22}|^4 - |c_{12}|^4)}, \tag{34}
\]
we can easily verify that the slope region of Gaussian weak interference channel is strictly larger than the slope region of Gaussian weak interference channel with confidential messages for both TDMA and multiplexed transmission schemes. Thus, in addition to the increase in the minimum energy per bit, secrecy introduces a penalty in terms of the achievable wideband slope values. In Figs. 2 and 3, we plot the slope regions for TDMA and multiplexed transmissions, respectively, under secrecy constraints. We note that regions become smaller as \(|c_{12}|^2\) and \(|c_{21}|^2\) increase. This is due to the fact that for fixed \(|c_{11}|^2\) and \(|c_{22}|^2\), the larger values of \(|c_{12}|^2\) and \(|c_{21}|^2\) mean that channel of the unintended receiver gets stronger and we have to use more energy to achieve the same secrecy transmission rate.

We are also interested in determining which transmission scheme performs better in the low-SNR regime. TDMA achievable rate regions converge to those of multiplexed transmission scheme as power decreases. Furthermore, TDMA and multiplexed transmission has the same minimum energy per bit values. Therefore, we should consider the slope regions. From Theorem 2 we know that when
\[
\frac{4|c_{11}|^2 |c_{12}|^2 |c_{22}|^2 |c_{21}|^2}{(|c_{11}|^4 - |c_{21}|^4)(|c_{22}|^4 - |c_{12}|^4)} < 1, \tag{35}
\]
the slope region of multiplexed transmission is strictly larger than the slope region of TDMA, thus in this case, multiplexed transmission is preferred. On the other hand, when
\[
\frac{4|c_{11}|^2 |c_{12}|^2 |c_{22}|^2 |c_{21}|^2}{(|c_{11}|^4 - |c_{21}|^4)(|c_{22}|^4 - |c_{12}|^4)} > 1, \tag{36}
\]
the slope region of TDMA is larger than the slope region of multiplexed transmission. Hence, TDMA should be used in this scenario. Finally, when
\[
\frac{4|c_{11}|^2 |c_{12}|^2 |c_{22}|^2 |c_{21}|^2}{(|c_{11}|^4 - |c_{21}|^4)(|c_{22}|^4 - |c_{12}|^4)} = 1, \tag{37}
\]
the slope regions of TDMA and multiplexed transmission converge to the same triangular region. In this case, TDMA should
still be preferred due to its implementational advantages. These results show parallels to those obtained in [13] in the absence of secrecy constraints. In [13], the function that is compared with one is \( \frac{4|c_{12}|^2|c_{21}|^2}{|c_{11}|^2|c_{22}|^2} \). From (34), we see that when we vary the channel parameters, \( \frac{4|c_{12}|^2|c_{21}|^2}{|c_{11}|^2|c_{22}|^2} \) is more likely to be greater than one than \( \frac{|c_{11}|^2}{|c_{21}|^2} \). This observation lets us conclude that under secrecy constraints, TDMA is more likely to be the optimal transmission scheme. In particular, when

\[
\left( \frac{|c_{11}|^2}{|c_{21}|^2} - \frac{|c_{21}|^2}{|c_{11}|^2} \right) \left( \frac{|c_{22}|^2}{|c_{12}|^2} - \frac{|c_{12}|^2}{|c_{22}|^2} \right) < 4 \left( \frac{|c_{11}|^2}{|c_{21}|^2} \right) \left( \frac{|c_{22}|^2}{|c_{12}|^2} \right)
\]

(38)

TDMA is preferred in secure transmissions while multiplexed transmission is preferred when there are no secrecy limitations. In Fig. 4, we plot the slope regions when the channel parameters are \( |c_{11}|^2 = |c_{22}|^2 = 1, |c_{12}|^2 = 0.4, |c_{21}|^2 = 0.5 \). As explained above, secrecy slope regions are inside the slope regions of Gaussian interference channel with no secrecy constraints. For secure transmissions, the region of TDMA is larger than that of multiplexed transmission while for transmissions without secrecy, the region of multiplexed transmission is larger. In Fig. 5 we plot the slope regions when the channel parameters are \( |c_{11}|^2 = |c_{22}|^2 = 1, |c_{12}|^2 = 0.1, |c_{21}|^2 = 0.2 \). Here, we note that multiplexed transmission scheme is superior to TDMA scheme with and without secrecy constraints.

V. CONCLUSION

In this paper, we have studied the achievable secrecy rates over Gaussian interference channel for TDMA, multiplexed and artificial noise schemes. Although usually TDMA has the worst performance [10], we have noted that only TDMA can achieve positive secrecy rates for both users in the high-SNR regime. In the low-power regime, we have shown that TDMA is optimal when \( \frac{4|c_{11}|^2|c_{12}|^2|c_{22}|^2|c_{21}|^2}{(|c_{11}|^2-|c_{12}|^2)(|c_{22}|^2-|c_{21}|^2)} \geq 1 \). We have also shown that secrecy constraints introduce penalty in both the minimum bit energy and slope. Finally, we have shown that TDMA is more likely to be optimal in the presence of secrecy limitations.