Dual conformal symmetry at loop level: massive regularization*

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Abstract
Dual conformal symmetry has had a huge impact on our understanding of planar scattering amplitudes in $\mathcal{N}=4$ super Yang–Mills. At tree level, it combines with the original conformal symmetry generators to a Yangian algebra, a hallmark of integrability, and helps in determining the tree-level amplitudes. The latter are now known in closed form. At loop level, it determines the functional form of the four- and five-point scattering amplitudes to all orders in the coupling constant and gives restrictions at six points and beyond. The symmetry is best understood at loop level in terms of a novel AdS-inspired infrared regularization which makes the symmetry exact, despite the infrared divergences. This has important consequences for the basis of loop integrals in this theory. Recently, a number of selective reviews have appeared which discuss dual conformal symmetry, mostly at tree level. Here, we give an up-to-date account of dual conformal symmetry, focussing on its status at loop level.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

The last few years have seen exciting progress in the understanding of scattering amplitudes in gauge theories, in particular in the maximally supersymmetric $\mathcal{N}=4$ super Yang–Mills (SYM) theory in the planar limit. Many of these developments in the field theory have been driven by the discovery of new symmetries, as well as by exploiting the analytic properties of scattering amplitudes, including their infrared structure. Apart from providing us with exciting new results in $\mathcal{N}=4$ SYM, these advances allow us to gain insights into the structure of loop

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amplitudes in general and also have applications for theories with less or no supersymmetry. This review is part of this special issue [1] that aims to give an up-to-date account of these developments.

This review is organized as follows. We begin by motivating the use of dual coordinates for planar graphs and by showing hints for a dual conformal symmetry of loop integrals contributing to scattering amplitudes in $\mathcal{N} = 4$ SYM in section 2. The symmetry is obscured in part by the presence of infrared divergences. In section 3, we introduce an infrared regulator that is motivated by the AdS/CFT correspondence and that allows us to make dual conformal symmetry exact at loop level. We discuss various features of this setup and its implications for the loop-level integral basis. We comment on recent developments for computing loop integrands using recursion relations. In section 4, we present aspects of loop integrals and their analytical computation with focus on the infrared regularization of section 3. We also give an example of an integral belonging to a special class of dual conformal integrals with certain numerator factors that are relevant for $\mathcal{N} = 4$ SYM and satisfy simple differential equations. We motivate a possible connection between the differential equations and the conformal symmetry of $\mathcal{N} = 4$ SYM by giving an example of a Yangian invariant integral.

2. Hints for dual conformal symmetry

The first hints for a dual conformal symmetry of scattering amplitudes in planar $\mathcal{N} = 4$ SYM came from inspecting the loop integrals contributing to the four-gluon amplitudes [2] (see also [3]). To three loops and up to a trivial tree-level factor, they are given by a linear combination of the integrals shown in figure 1 [4, 5]. Although these integrals superficially look like diagrams obtained from a $\phi^3$ theory, one should keep in mind that, at least in principle, they are the result of summing over a large number of Feynman diagrams. In practice, one often uses methods that are based on the analytic properties of the perturbative S-matrix [6–9] (see also [10, 11] of this special issue) and that do not make explicit use of Feynman diagrams.

The fact that only few integral topologies remain at the end is very remarkable. It was understood over the last few years that it is the consequence of a new symmetry of planar scattering amplitudes in $\mathcal{N} = 4$ SYM, as we discuss presently. As an explicit example, at the one-loop level and for four points, only the scalar box integral shown in figure 1(a) appears. It is given by

$$I^{(1)}_{\text{box}} = \int \frac{d^D k}{i \pi^{D/2}} \frac{(p_1 + p_2)^2 (p_2 + p_3)^2}{k^2 (k + p_1)^2 (k + p_1 + p_2)^2 (k - p_4)^2}$$

1 This section is organized in a rather historical fashion for pedagogical purposes. A better understanding of dual conformal symmetry is now available in terms of the mass regulator that we discuss in section 3.
with the on-shell conditions $p_i^2 = 0$, and where the calculation leading to (1) has been done in dimensional regularization with $D = 4 - 2\epsilon$ and $\epsilon < 0$ to regularize infrared divergences. For a generic theory, also triangle integrals could have appeared. Being a planar integral, we can unambiguously define dual (or region) coordinates $x_i$ by

$$p_\mu^i = x_\mu^i - x_\mu^{i+1}$$

with the cyclicity condition $x_{i+4} \equiv x_i$. The on-shell conditions become $x_{i,i+1}^2 = 0$. For the one-loop box integral (1), this leads to

$$I^{(1)\epsilon} = \int \frac{dDx_0}{i\pi^{D/2}} \frac{x_{13}^2 x_{24}^2}{x_{01}^2 x_{02}^2 x_{03}^2 x_{04}^2}.$$  

Here, the change of variables $k_\mu^i = x_\mu^0 - x_\mu^i$ was done, and the resulting dual graph is shown in figure 2(a). See [12] for a reference on graph theory discussing dual graphs. The use of dual variables for planar integrals is in fact very useful, independently of the symmetry that we are going to discuss. For example, imagine we wish to write down the Feynman parametrization for a generic one-loop diagram. Then, if $\alpha_i$ is the Feynman parameter associated with the propagator $1/x_{0i}^2$, the argument of the denominator appearing in the Feynman parameter integral is simply $\sum_{i<j} x_{ij}^2 \alpha_i \alpha_j$, see e.g. [13].

Written in this form, (3) is reminiscent of integrals appearing in the study of position space correlation functions of protected operators, i.e. operators with zero anomalous dimension, in $\mathcal{N} = 4$ SYM, see e.g. [14]. The difference is that in those correlation functions, $x_i^\mu$ are unconstrained variables (i.e. they do not satisfy the on-shell conditions $x_i^2 = 0$) and that the integration measure is four dimensional. In that case the integrals have an SO(4,2) conformal symmetry. While the Poincaré symmetry is manifest, invariance under special conformal transformations can be best seen by considering inversions. The transformations are

$$x_i^\mu \rightarrow x_i^\mu, \quad x_{ij}^2 \rightarrow \frac{x_{ij}^2}{x_i^2 x_j^2}, \quad d^Dx_0 = d^Dx_0(x_0^2)^{-D}.\n
$$

We see that for $D = 4$, all factors of $x^2$ from (4) cancel precisely in (3), and the integral is indeed (dual) conformal invariant. The dual conformal symmetry of the off-shell ladder integrals was first noted by Broadhurst [15] and used to explain the equivalence of three- and four-point ladder integrals [16], which are related by conformal transformations.

2 Bubble integrals would be UV divergent and are therefore excluded in a UV-finite theory. This does not mean, of course, that UV divergences cannot appear at intermediate stages of calculations (e.g. in a gauge-dependent wavefunction renormalization), especially when calculations are done in component as opposed to superspace formalisms.
Coming back to the scattering amplitudes, we recall that we have \(D = 4 - 2\epsilon\) and \(\epsilon < 0\). One cannot set \(D = 4\) because of infrared divergences. Therefore, the symmetry of this integral is only approximate. We will see in section 3 how this problem can be cured, but for the moment let us discuss the symmetry in this naïve sense. The crucial observation made in [2] is that this integral and all other integrals contributing up to three loops to the four-gluon amplitude, which were obtained in the pioneering work of [4, 5], are invariant ( naïvely) under conformal transformations in the dual space of the \(x_i^\mu\) variables. Integrals having this property are sometimes called 'pseudoconformal'.

The dual diagrams of the four-point integrals up to three loops are shown in figure 2, and it is easy to see that they all have the above property: what one needs to check is that the conformal weight of each dual integration point is cancelled by propagator and numerator factors attached to it. If there is no numerator, this means that exactly four propagators need to be attached to each integration point, which is the case for the integrals shown in figures 2(a)–(c). We remark that triangle subgraphs do not respect dual conformal symmetry. Figure 2(d) is an example of a dual conformal integral with non-trivial, i.e. loop-dependent, numerator factor. The latter is indicated by a dashed line and is in fact required to cancel the conformal weight at the integration point that is joined by five propagators.

Dual conformal symmetry seems to be a property of planar scattering amplitudes in \(\mathcal{N} = 4\) SYM and its presence was confirmed at higher loop orders \([17, 18]\) as well. It is also a useful guiding principle for finding the correct loop integrands for amplitudes at higher loops or with more external legs \([19–21]\). This is of great practical help, especially when computations are done employing (generalized) unitarity. If the basis of loop integrals is known, unitarity cuts can be used to determine the (rational) coefficients of the integrals. We remark that although presently dual conformal symmetry applies to planar amplitudes only, its existence can also be useful for non-planar studies, thanks to relations between planar and non-planar amplitudes, see e.g. \([22, 23]\).

In fact from the discussion above it is easy to find rules for writing down dual conformal integrals. An important restriction comes from the fact that loop integrands have the structure ‘numerator \(\times\) propagators’, where by propagators we mean products of factors like \(1/p^2\). We have already seen that each dual integration point has to be joined by at least four propagators. If there are more than four propagators joining it, the excess in conformal weight has to be cancelled by appropriate numerator factors. The latter can be inverse propagators as in figure 2(d), or in general also suitably defined traces built from dual variables.

The above considerations are very helpful for restricting the loop integrand of scattering amplitudes. In order to make quantitative predictions about the functions obtained after integration, it is important to understand the breaking of the symmetry near four dimensions. Hints for how to do this came from the AdS/CFT correspondence, which suggests a surprising relation between scattering amplitudes and certain light-like Wilson loops \([18, 24–26]\). This conjectured duality is reviewed in \([27, 28]\). The light-like Wilson loops appearing in the duality are defined in coordinate space. They have \(n\) cusps which lie precisely at the positions indicated by the dual coordinates of equation (2). The dual conformal symmetry of the scattering amplitudes is then identified with the conventional conformal symmetry of the Wilson loops. Importantly, the breaking of the latter is controlled to all orders in the coupling constant by anomalous Ward identities. Admitting the duality with the (maximally helicity-violating) scattering amplitudes, the Ward identities can be applied to the latter. Let us now quote the form of the Ward identities. We use \(M_n\) to denote the color-ordered MHV amplitude.

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3 Since scattering amplitudes are gauge invariant, we can assume the Feynman gauge for this discussion. In other gauges, the propagator denominators could be more complicated, or higher powers of \(p^2\) could appear.
with the tree-level term factored out. The universal form of infrared divergences suggests us
to write \( \log M_n = D_n + F_n + O(\epsilon) \) as the sum of a divergent term \( D_n \), a finite term \( F_n \), and \( O(\epsilon) \) corrections. Given the universal form of \( D_n \), the Ward identities can be written for \( F_n \) as

\[
K^\mu F_n = \frac{1}{2} \Gamma_{\text{cusp}}(a) \sum_{i=1}^{n} \left( x_{i,i+1}^\mu \log \frac{x_{i,i+2}^2}{x_{i-1,i+1}^2} \right),
\]

where

\[
K^\mu = \sum_{i=1}^{n} \left[ 2 x_{i,i+1}^\mu x_{i}^\nu \frac{\partial}{\partial x_{i}^\nu} - x_{i}^2 \frac{\partial}{\partial x_{i}^\mu} \right]
\]
is the generator of conformal boosts. The cusp anomalous dimension \( \Gamma_{\text{cusp}} \) [30] is conjectured
to be governed to all loop orders by an integral equation [31]. At four and five points,
equation (5) has a unique solution (to all orders in the coupling constant), which coincides
with the Bern–Dixon–Smirnov ansatz [4, 5] and agrees with the AdS calculation of [24] for
four points. Starting from six points, (5) determines \( F_n \) up to an a priori arbitrary (coupling-
dependent) function of dual conformal cross-rations [26, 29], called the remainder function.

It is important to stress that this Ward identity applies to (the logarithm of) an amplitude,
not to individual integrals. The reason is that the infrared divergences, which are responsible
for the anomaly on the rhs of (5), take a simple form only for that quantity. Since infrared
divergences are universal it is natural to expect that a generalization to non-MHV amplitudes
holds as well [32]. This required the generalization to a dual superconformal space [32],
which is reviewed in [33] of this special issue. These predictions were confirmed in various
cases, at tree level [32, 34, 35], one loop [36–38], and numerically for the six-point NMHV
two-loop amplitude [21].

In the above, dimensional regularization was used to regulate the IR divergences of
the scattering amplitudes. In fact, this regulator is not best suited for exploiting the dual
conformal symmetry. Although the dual conformal anomaly of equation (5) is very simple,
the action of \( K^\mu \) on a generic ‘pseudoconformal’ integral is in general very complicated.
This makes the notion of ‘pseudoconformal’ integrals rather vague, and in fact it is hard to
give a mathematically concise definition for them (one might think that this can be cured by
going off-shell, but that can lead to other problems, such as loss of gauge invariance.) In
section 3, we will introduce an alternative regulator which allows us to realize dual conformal
symmetry at loop level without an anomaly, and thereby can be used to make the notion of
a dual conformally invariant integrals precise. This is obviously of great importance in the
context of the loop integral basis that was alluded to earlier.

3. Scattering amplitudes on the Coulomb branch

In the previous section, we saw that the necessity to regulate the scattering amplitudes obscured
the dual conformal symmetry, and in particular the dimensional infrared regulator breaks the
latter. As we will review presently, it is possible to regulate the infrared divergences in a
different way that allows us to preserve dual conformal symmetry at loop level [39]. We will
first explain how to regulate the IR divergences by introducing certain Higgs masses, and then
discuss how the symmetry manifests itself.

The idea is to start with a gauge group \( U(N+M) \) and to break it to \( U(N) \times U(M) \). Let
the fields associated with the broken part of the gauge group have mass \( m \). If one scatters
\( U(M) \) fields and takes \( N \gg M \), at leading order in \( O(M/N) \) the dominant diagrams are those
where the scattered massless \( U(M) \) fields interact with the fields in the unbroken \( U(N) \) part
of the gauge group via massive particles. The types of diagrams that can occur are most easily seen in the double line notation for color indices, see e.g. figure 3(b). In those color diagrams, only internal loops of $U(N)$ indices are allowed, since other terms would be subleading in $\mathcal{O}(M/N)$. (One might worry that the loop integrals that are dropped are IR divergent, but one could simply employ an auxiliary regulator, i.e. dimensional regularization, before taking the $M/N \ll 1$ limit, and then remove that regulator.) The upshot is that in all diagrams that we need to keep a massive particle runs on the perimeter, and the interior is massless. The masses on the perimeter regulate the infrared divergences. In the limit $m \to 0$, we approach the four-dimensional massless theory. Compared to dimensional regularization, the IR divergences then manifest themselves as $\log^i m^2$ as opposed to $\epsilon^{-i}$, with $i \leq 2L$ and $L$ being the loop order.

There are a number of technical advantages associated with this fact [39, 40]. For example, products like $\mathcal{O}(m^2) \times \log m^2$ are evanescent as $m \to 0$, whereas $\mathcal{O}(\epsilon) \times \epsilon^{-1}$ terms in dimensional regularization must be kept when the regulator is sent to zero. Since the IR divergences of the amplitudes produce $\log^i m^2$ behavior, but no polynomial divergences, there will be no cross terms between different loop orders. Of course, individual integrals may diverge linearly in $m^2$ as the mass is taken to zero, and in this case, care is required when they are multiplied by $\mathcal{O}(m^2)$ terms, see e.g. [41–43]. Some further aspects of integrals in this regularization are reviewed in section 4.

A major conceptual motivation for considering the above regulator is that the string theory dual of $\mathcal{N} = 4$ SYM suggests that it is well adapted to the dual conformal symmetry. Indeed, the above is inspired by the string theory setup of [24] (see also [44–46]). In terms of the latter, the Higgs mass corresponds to the inverse radial coordinate in the AdS$_5$ space, see figure 3(a), or equivalently to the radial coordinate in a dual AdS$_5$ space, that is obtained by a T-duality transformation. The isometries of the latter suggest a (dual) conformal symmetry for the scattering amplitudes. The (non-trivial) isometry transformations read

$$K^\mu = \sum_{i=1}^{n} \left[ 2x_i^\mu x_i^\nu \frac{\partial}{\partial x_i^\nu} - x_i^\mu \frac{\partial}{\partial x_i^\mu} - 2x_i^\mu m_i \frac{\partial}{\partial m_i} - m_i^2 \frac{\partial}{\partial x_i^\mu} \right],$$

$$= K^\mu + \sum_{i=1}^{n} \left[ 2x_i^\mu m_i \frac{\partial}{\partial m_i} - m_i^2 \frac{\partial}{\partial x_i^\mu} \right].$$

\[ (7) \]
where we used Poincaré coordinates \( \{ x^\mu_i, m_i \} \) to parametrize AdS_5, denoting the radial coordinate by \( m \). For the \( m = 0 \) we recover the standard form of conformal transformations, equation (6), that we used in the previous section. Similarly, there is a trivial extension of the generator for dual dilatations,

\[
\hat{D} = \sum_{i=1}^{n} \left[ x^\mu_i \frac{\partial}{\partial x^\mu_i} + m_i \frac{\partial}{\partial m_i} \right],
\]

(8)

while the Lorentz generators \( P_\mu \) and \( M_{\mu\nu} \) remain unchanged. Together, \( \hat{K}, \hat{D}, P, M \) generate an \( SO(4,2) \) algebra. Let us stress again that from the point of view of the algebra the only difference with the massless case is that the representation has changed.

When discussing the scattering amplitudes, however, the new representation has profound consequences for the following reason. As we have already seen, the massless amplitudes are infrared divergent, and therefore the discussion of the symmetries in that case is only formal. On the other hand, for non-zero mass \( m \), the amplitudes are infrared finite and we have the realization (7) of the (dual) conformal symmetry. When using (7) it is crucial that we have one parameter \( m_i \) for each dual coordinate \( x^\mu_i \). This can be achieved by refining the above setup by breaking the gauge group further to \( U(N) \times U(1)^M \), thereby introducing several Higgs masses. Given the AdS considerations above, we expect that the scattering amplitudes defined in this way should have an exact dual conformal symmetry, i.e.

\[
\hat{K}^\mu M^\nu = 0.
\]

(9)

Note that the transformations (7) also change the value of the \( m_i \). In fact, the mass can be thought of as a fifth component of the dual coordinates \( x^\mu_i \). This means that we should think of the \( m_i \) as parameters, just like the kinematical variables of the scattering process.

In order to carry out calculations in the field theory, it is important to have an action that corresponds to the spontaneous symmetry breaking \( U(N+M) \to U(N) \times U(1)^M \) discussed above. The latter and the corresponding Feynman rules were worked out in [39], starting from the component action

\[
\begin{align*}
\tilde{S}^U(N+M)_{N=4} &= \int d^4x \ Tr \left( -\frac{1}{4} \tilde{F}_{\mu\nu}^2 - \frac{1}{2} (D_\mu \Phi_I)^2 + \frac{g^2}{4} [\Phi_I, \Phi_J]^2 \\
&\quad + \frac{i}{2} \tilde{\Psi} \Gamma^\mu D_\mu \tilde{\Phi} + \frac{g}{2} \tilde{\Psi} \Gamma^I [\Phi_I, \tilde{\Phi}] \right),
\end{align*}
\]

(10)

where \( D_\mu = \partial_\mu - ig[A_\mu, \cdot] \), and the range of the indices is \( \mu = 0, 1, 2, 3 \) and \( I = 4, \ldots, 9 \). All fields are Hermitian matrices, which we decompose into blocks as

\[
\lambda_\mu = \begin{pmatrix} (A_\mu)_{ab} & (A_\mu)_{aj} \\ (A_\mu)_{ia} & (A_\mu)_{ij} \end{pmatrix}, \quad \Phi_I = \begin{pmatrix} (\Phi_I)_{ab} & (\Phi_I)_{aj} \\ (\Phi_I)_{ia} & (\Phi_I)_{ij} \end{pmatrix},
\]

\[
\tilde{\Psi} = \begin{pmatrix} (\Psi)_{ab} & (\Psi)_{aj} \\ (\Psi)_{ia} & (\Psi)_{ij} \end{pmatrix},
\]

(11)

where \( a, b = 1, \ldots, N \), \( i, j = N+1, \ldots, N+M \), and we have given a vacuum expectation value \( \langle (\Phi)_{0} \rangle_{ij} = \delta_{ij} m_i / g \) to the scalars in the \( I = 9 \) direction. The shift leads to new quadratic and cubic vertices. The former lead to several types of fields. We have the ‘light’ fields \( O_{ij} \) with masses \( m_i - m_j \), where \( O \) denotes a generic field \( [A_\mu, \Phi_I, \Psi] \), and the heavy fields \( O_{ia} \) of mass \( m_i \). The \( O_{ab} \) and \( O_{ii} \) remain massless. (In the simplest case \( m_i = m_j \) the ‘light’ fields become massless.) Moreover, there are new cubic vertices between scalars, and gluons and scalars proportional to \( m_i \).
Let us now see how the exact dual conformal symmetry appears in practice. A one-loop calculation starting from the action above showed that one obtains the following one-loop four-point amplitude:

$$M_4 = 1 - \frac{a}{2} I_4^{(1)}(s, t, m_i) + O(a^2),$$

(12)

where

$$I_4^{(1)} = \int \frac{d^4 x_0}{17^2} \frac{\left(x_{13}^2 + (m_1 - m_3)^2\right)\left(x_{24}^2 + (m_2 - m_4)^2\right)}{\prod_{i=1}^4 \left(x_{0i}^2 + m_i^2\right)}.$$  

(13)

One can now easily check that $I_4^{(1)}$ is invariant under the extended dual conformal transformations $\hat{K}^\mu$. As before, this is easiest done by noting manifest four-dimensional Poincaré symmetry and applying (dual) conformal inversions to $I_4^{(1)}$, where the masses $m_i$ are treated as higher dimensional components of the dual coordinates $x^\mu$ [39]. Infinitesimally, invariance can be expressed as

$$\hat{K}^\mu M_4 = 0.$$  

(14)

Note that $\mu = 0, 1, 2, 3$, i.e. we still have an $SO(4, 2)$ symmetry as in the massless case, only the representation of the symmetry has changed. In general, equation (9) simply implies that $M_n$, which a priori is a function of the $m_i$ and $p_i \cdot p_j$, depends on a restricted set of variables only. For example, in the four-point case, we have

$$M_4(m_1, m_2, m_3, m_4, x_{13}^2, x_{24}^2) = M_4(u, v),$$

(15)

where

$$u = \frac{m_1 m_3}{x_{13}^2 + (m_1 - m_3)^2}, \quad v = \frac{m_2 m_4}{x_{24}^2 + (m_2 - m_4)^2}.$$  

(16)

In the above example, the fact that the integral $I_4^{(1)}$ depends on $u$ and $v$ only can also be seen directly in the Feynman parametrization of this integral by rescaling the Feynman parameters for each propagator $1/(x_{0i}^2 + m_i^2)$ by $m_i$.

A natural conjecture is that at a given loop level, the amplitude can be written as a linear combination of integrals $I_\sigma$ invariant under extended dual conformal symmetry [40, 41]

$$M_n^{(L)} = \sum_\sigma c_\sigma I_\sigma,$$

(17)

where

$$\hat{K}^\mu I_\sigma = 0,$$

(18)

i.e. the $I_\sigma$ are invariant under the extended dual conformal symmetry, and the $c_\sigma$ are rational coefficients (e.g. numbers in the MHV case or in general dual conformal invariants similar to those that appear in the tree-level amplitude [35]). This is exactly what many authors suspected, using the notion of ‘pseudoconformal’ integrals. The latter can now be replaced by the concise definition (18). As was already explained, equation (17) has important practical consequences, e.g. when computing loop amplitudes through the unitarity method4.

We have seen above that exact dual conformal symmetry reduces the number of variables that a function can depend on. It is important to realize that it is a stronger constraint to require that such a function should come from a loop integral, i.e. built from propagators that are integrated over spacetime. For example, in the Feynman gauge, the propagator denominators

Given equation (17) for amplitudes on the Coulomb branch, one may wonder what its consequences are for amplitudes at the origin of the Coulomb branch and for $D \approx 4$. We caution the reader that switching between IR regularizations at intermediate steps of a calculation is in general very subtle, especially at higher loop orders.

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are always \(1/p^3\) or \(1/(p^2 + m^2)\), and at \(L\) loops we have \(L\)-fold loop integrals built from such propagators and possibly numerator factors as a result of the numerator algebra. This is important in two respects. Firstly, for a given scattering amplitude, one can classify the loop integrals having this property, which are naturally much fewer than the set of generic loop integrals. Secondly, the fact that the functions we are dealing with come from loop integrals means that we can use properties of the latter such as their analytic structure, unitarity cuts, etc [47], to infer properties of the functions (see also the comments in the next section).

The interpretation of the masses as components of higher dimensional momenta motivated several groups to investigate dual conformal symmetry in higher dimensions. It was shown that tree-level (super)amplitudes in six dimensions [48, 49] and ten dimensions [50] have a dual conformal symmetry. In turn, since the higher dimensional amplitudes can be interpreted as the massive four-dimensional Coulomb branch amplitudes of [39], this proves that the latter are indeed dual conformally invariant at tree level.

This also has important consequences for loop-level amplitudes on the Coulomb branch of \(\mathcal{N} = 4\) SYM, as it essentially proves the conjectures made in [39]. Previous evidence in support of these had come from [40, 41, 48, 51]. It was shown in [49] that all unitarity cuts of planar loop amplitudes in that theory have the (extended) dual conformal symmetry. This proves the (extended) dual conformal symmetry of loop amplitudes in \(\mathcal{N} = 4\) SYM, up to potential terms not detected by any unitarity cuts. Similarly one can argue that in theories where tree amplitudes determine the loop integrand, e.g. through recursion relations [52–54], the latter should inherit the (extended) dual conformal symmetry from the trees [50].

As was explained above, the restrictions imposed above from the (extended) dual conformal symmetry on the loop amplitudes are very useful when determining the loop integrand through the (generalized) unitarity method. Recently, it was realized that the BCFW idea [8, 9] of determining tree-level amplitudes from their factorization channels can also be applied to planar loop integrands [52–54]. The loop integrand of a given amplitude can then be iteratively determined starting from tree amplitudes in the forward limit (see also [55] and references therein). In practice, this works extremely well for computing the loop integrand in four dimensions, since the corresponding tree-level amplitudes are known [35]. In order to obtain an integrand that can be safely integrated, one should in principle determine e.g. the \(D\)-dimensional loop integrand (for dimensional regularization), or the integrand on the Coulomb branch. Given the four-dimensional integrand, the extended dual conformal symmetry provides useful guidance for how to ‘translate’ the latter to the Coulomb branch integrand, and it is argued that this should give the correct integrand, up to \(O(m^2)\) corrections [53]. We note another interesting recent approach to loop integrands that is based on a connection to correlation functions [56].

4. Properties of the loop integrals

Here, we make a number of comments on properties of the loop integrals with a special focus on the mass regulator. We comment on their evaluation using Mellin–Barnes (MB) methods, their properties in the Regge limit, and review a type of dual conformal integrals with special numerators. The latter integrals satisfy simple differential equations. We comment on a possible relation to conventional conformal symmetry.

A state of the art tool for evaluating loop integrals is the MB method [13], where one trades Feynman parameter integrals for contour integrals by (repeatedly) using the identity

\[
(X + Y)^{-\lambda} = \frac{1}{\Gamma(\lambda)} \oint_{|\lambda-i\infty} \frac{dz}{2\pi i} \frac{Y^z}{X^{z+\lambda}} \Gamma(-z)\Gamma(z+\lambda),
\]

(19)
with $\beta < 0$. This approach works well for the massless as well as for the massive case. Experience shows [13] that introducing the MB parameters loop by loop is a good strategy. Moreover, in the present case, one can often perform all manipulations while staying in $D = 4$ dimensions. This should be done whenever possible to obtain a low-dimensional MB representation. It is interesting to note that starting from the four-loop level, the massive MB representations tend to involve fewer parameters as compared to the dimensional regularization case. A very detailed derivation of the MB representations for the massive three-loop four-point integrals is given in appendix A of [40].

One advantage of the Higgs setup is that it is natural to consider the amplitudes and integrals for finite values of $m^2$. In this spirit, one can consider the Regge limit, e.g. $s \gg t, m^2$ in the four-particle case. In [40, 41] it was shown that the integrals contributing to the amplitudes behave very nicely in this limit. One can show that to all loop orders, the leading-log (LL) and next-to-leading-log (NLL) contributions to the Regge limit are given by the two infinite classes of integrals shown in figure 4.

For example, the contribution of $I_{L,H}$ to NLL accuracy is given by (we use the notation $u = s/m^2$ and $v = t/m^2$)

$$I_{L,H} = \frac{(-1)^{L-1}}{(L-1)!} \log^{L-1} u \times K(v)^{L-2} \times K'(v) + \mathcal{O}(\log^{L-2} u),$$  \hspace{1cm} (20)

where $K(v)$ and $K'(v)$ correspond to the two-dimensional bubble and two-loop bubble diagrams shown in figure 4 (see [40] for further discussion). Taking $v$ small, we have

$$K(v) = -2 \log v + \mathcal{O}(v),$$
$$K'(v) = -\frac{4}{3} \log^3 v - \frac{4}{3} \pi^2 \log v + \mathcal{O}(v),$$  \hspace{1cm} (21)

which, combined with the result for $I_{L,a}(v, u)$, gives the correct Regge behavior at LL and NLL [41]. Note that the fact that the $I_{L,H}$ starts contributing at NLL and not NNLL is only possible, thanks to its non-trivial numerator factor, whose presence in turn is required by dual conformal invariance. It is interesting to note that such non-trivial (loop-momentum-dependent) numerator factors are also important when discussing UV properties of scattering amplitudes in $\mathcal{N} = 4$ SYM and $\mathcal{N} = 8$ supergravity [57].

Dual conformal symmetry can also lead to interesting insights about the asymptotic behavior of integrals/amplitudes in certain limits. Recall that the integrals can depend on
the masses only in specific combinations with the kinematical variables, see e.g. (16) in the four-point case. This implies that certain small mass limits are equivalent to Regge limits. See [40, 41] for more details.

Recently, it has become apparent that it is particularly advantageous to introduce dual conformal integrals with certain non-trivial numerator factors [53]. A guiding principle in defining these numerator factors is (potential) infrared divergences. The latter can arise from specific integration regions where loop propagators go on-shell. If the appropriately defined numerator factors vanish in those regions they will soften the infrared divergences of the integral, or even make the integral finite. Let us give a simple example of the latter type.

Consider the following pentagon integral with two off-shell and three on-shell legs. In dual coordinates, it is defined by

$$x_{14} x_{15} x_{36} := \frac{d^4 x}{i \pi^2} \frac{x_{2a}^2}{x_{1a} x_{3a} x_{4a} x_{5a} x_{6a}},$$

where $x_{34}^2 = x_{45}^2 = x_{56}^2 = 0$ and where the ‘magic point’ $x_{2a}^0$, which is denoted by a dashed line on the lhs of (22), is defined as one of the solutions to the four-cut condition

$$x_{3a}^2 = x_{4a}^2 = x_{5a}^2 = x_{6a}^2 = 0.$$

Hence, we see that the numerator factor vanishes in the regions that would otherwise produce infrared divergences, and the integral is finite. The above example is sufficiently simple that we will not need to write out the explicit solution to (23), and we can compute it using e.g. Feynman parameters. (In general the explicit definition of the numerator factors can be written very conveniently using momentum twistor variables [58]. At the loop level, the latter are ideally used in combination with the above mass regularization, as they are intrinsically four dimensional. See [59, 60] for more details.) Being dual conformally invariant, the answer is a function of the cross ratios $u_1 = (x_{13}^2 x_{26}^2) / (x_{14}^2 x_{36}^2)$ and $u_2 = (x_{16}^2 x_{35}^2) / (x_{15}^2 x_{36}^2)$. Multiplying for convenience by $(1 - u_1 - u_2)$, one obtains the simple formula

$$\Psi^{(1)}(u_1, u_2) = (1 - u_1 - u_2) \times \text{equation (22)}$$

$$= \log u_1 \log u_2 + \text{Li}_2(1 - u_1) + \text{Li}_2(1 - u_2) - \zeta_2.$$  

It is important to note that this integral is related to standard integrals by simple integral reduction identities [43, 53]. In the present case, one could represent the pentagon integral above by a linear combination of five (IR-divergent) one-mass box integrals.

Then, the idea is that the integrals of the type discussed above can be used, thanks to the numerator identities, to write loop amplitudes in a simpler form. For example, a good strategy could be to trade the most complicated integrals in a given calculation (say, double pentagon integrals) for the integrals discussed here, and simpler integrals. For example, when applying these ideas to the $n$-point two-loop MHV amplitudes in $\mathcal{N} = 4$ SYM, one obtains very compact expressions [43, 53]. In fact, only one integral topology appears with different arrangements of external legs. In total, only 36 distinct integrals are needed to fully describe the two-loop MHV amplitudes with arbitrary number of external legs. Many of these integrals are interrelated by soft limits.

Moreover, it was found in [43] that the new integrals, when evaluated, lead to rather simple functions, just as in the one-loop example above. This allowed e.g. the analytical computation of the six-point remainder function in kinematical limits [43]. This is the first time that this
Figure 5. The integral of equation (26) in momentum space (a) and in dual notation (b). The position space variables $y_\mu^i$ are related to the momenta $p_\mu^i$ by the Fourier transform, and the dual coordinates $x_\mu^\nu$ are defined by equation (2). The original and the dual diagram are both built from quartic vertices only, so that the integral has both a conformal and a dual conformal symmetry.

was achieved directly from the loop integrals (previous analytical results were available from Wilson loop calculations [61]). The simplicity of the integrals is explained (in part) by the fact that they satisfy simple differential equations [62]. For example, for the pentagon example discussed above one can show that

$$u_2 \delta_\mu, u_1 \delta_\mu, \Psi^{(1)}(u_1, u_2) = 1.$$  \hspace{1cm} (25)

It was found in [62] that the integrals relevant for planar MHV amplitudes in $\mathcal{N} = 4$ SYM satisfy similar differential equations, which relate in general $L$-loop to $(L-1)$-loop integrals. Apart from helping in finding analytical solutions, see [62] for several non-trivial examples at the two-loop level, the simple nature of the equations also suggests that their solutions cannot have an arbitrarily complicated structure.

It would be interesting to understand to what extent this is a manifestation of the underlying Yangian symmetry [53, 63] of scattering amplitudes in planar $\mathcal{N} = 4$ SYM. As we have seen, the dual conformal symmetry is under full control at loop level thanks to the mass regulator, so the question is whether one can put to use the underlying conformal symmetry of the massless scattering amplitudes. As a motivation, we note that Yangian symmetric quantities at loop level do exist. Here, we use the term Yangian symmetry in a somewhat loose way, meaning conformal and dual conformal symmetry. Consider for example the following integral, see figure 5(a), which could appear in an eight-particle scattering amplitude in scalar $\phi^4$ theory in four dimensions:\n
$$\int \frac{d^4x_r}{16\pi^2} \frac{1}{x_1^2 x_2^2 x_3^2 x_4^2}.$$  \hspace{1cm} (26)

Since two on-shell legs enter each corner of the box integral, the corresponding momenta, e.g. $p_1^\mu + p_2^\mu$ are off-shell, i.e. $(p_1 + p_2)^2 = x_{12}^2 \neq 0$, and the integral is finite in four dimensions. By the analysis of section 2, it is also dual conformally covariant, as can be seen from its dual graph in figure 5(b). Moreover, because of its origin as a finite graph in $\phi^4$ theory, it is also conformally invariant. This is easiest seen in position space. The conformal symmetry leads to second-order homogeneous differential equations for the integral. In this example, the integral is effectively off-shell, and there are no IR divergences at all, whereas in $\mathcal{N} = 4$ SYM one would first have to separate IR-divergent and IR-finite pieces in a convenient way.

5 Related discussions with J Drummond and J Plefka are gratefully acknowledged.
way. Depending on how this is done, it is plausible that one could find homogenous or inhomogeneous differential equations as a manifestation of the underlying symmetry. In this spirit, it would be interesting if the differential equations found in [62] could be related to, or understood more systematically in, terms of the underlying conformal symmetry of $\mathcal{N} = 4$ SYM.

5. Conclusion

In this review, we have summarized the status of dual conformal symmetry at loop level in planar $\mathcal{N} = 4$ SYM. The best way to understand this symmetry at loop level is on the Coulomb branch of $\mathcal{N} = 4$ SYM and by using a representation that is suggested by the isometries of AdS$_5$. The Coulomb branch amplitudes have an exact dual conformal symmetry. The latter leads to powerful constraints for the loop integrand of the scattering amplitudes.

New recursion relations for loop integrands provide a powerful practical tool for determining the latter. The *four-dimensional* loop integrand can be easily obtained, and dual conformal symmetry helps convert the latter to the correct Coulomb branch integrand, up to $O(m^2)$ corrections [53]. Given this, the main task for solving planar $\mathcal{N} = 4$ SYM lies in the evaluation of the (dual conformal) loop integrals. Here, the formulation in terms of momentum twistor integrals, where necessary in combination with the mass regulator, seems very promising.

The ultimate goal is to obtain results that can interpolate between weak and strong coupling. In fact, there are integrals closely related to the ones discussed here for which all-loop results are available, and where a resummation is possible [64]. The differential equations found in [62] provide hope that this may be possible for the integrals directly relevant to $\mathcal{N} = 4$ SYM as well.

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