On the vorticity of flow in redshift space

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1 INTRODUCTION

Peculiar motions (deviations from pure Hubble flow) could cause significant deviations between the distribution of galaxies in redshift space ($s$-space) and real space ($x$-space) (e.g. Kaiser 1987). Because of these deviations (redshift distortions), the known linear and nonlinear relations between dynamical fields (e.g. peculiar velocity and density) in $x$-space cannot directly be applied in $s$-space. Most analysis of redshift distortions has been done using linear theory (Kaiser 1987, Hamilton 1992, Nusser & Davis 1994, Fisher et al. 1995, Heavens & Taylor 1995). Although various approximations to dynamics in $x$-space have been developed, little progress has been made in extending these approximations to $s$-space (but see Hivon et al. 1995, Fisher & Nusser 1996, Taylor & Hamilton 1996, Hatton & Cole 1998, Scoccimarro et al. 1999). The common belief that the velocity field in $s$-space is not irrotational (e.g. Kaiser 1987) is not true even in the distant observer limit (DOL), has been the main hindrance in the development of nonlinear analysis methods in $s$-space.

According to Kelvin’s circulation theorem (e.g., Landau & Lifshitz 1959), if the initial velocity field is irrotational, i.e., curl-free, then, in the absence of shell-crossing, the nonlinear velocity field is also irrotational. Kelvin’s theorem concerns velocity fields in $x$-space. This means that the $x$-space velocity field, $v(x)$, at any point $x$ can be derived from a potential, $\Phi_v(x)$, i.e.,

$$ v(x) = -\nabla_x \Phi_v(x). $$

This relation has been the corner stone in the majority of large scale structure studies. It is the basis of the POTENT method (cf. Dekel 1994 for a review) and the modal expansion method (Nusser & Davis 1995) for recovering velocity fields from raw data of radial peculiar velocities. It is also the basis of various nonlinear approximations to dynamics (Bernardeau 1992, Gramann 1993, Mancinelli & Yahil 1995, Chodorowski 1997, Chodorowski & Lokas 1997, Chodorowski et al. 1998, Bernardeau et al. 1999). Here we will show that in the DOL and in the absence of multi-values zones (regions with overlapping streams in $s$-space) the $s$-space velocity field associated with an irrotational flow in $x$-space, is curl-free. Although the proof is valid only in the DOL, the limit in which most studies of redshift distortions are done, we show how the POTENT method can be generalized to provide $x$-space velocity potentials as a function of the $s$-space coordinate and thereby avoid Malmquist bias corrections.

In section 2 we prove our claim by showing that the circulation is zero and also by direct calculation of the curl of the $s$-space velocity field. In section 3 we present a relation between the $s$-space and $x$-space velocity potentials and show how POTENT can be modified to provide $x$-space velocity and potential fields as a function of the $s$-space coordinate. In section 4 we summarize the results and briefly discuss possible applications.

2 THE PROOF

Let $v(x)$ be the $x$-space comoving peculiar velocity as a function of the $x$-space comoving coordinate $x$, both expressed

* Vorticity is the curl of a vector field. An irrotational vector field is a field with zero vorticity at any point in space. This is equivalent to having zero circulation around an arbitrary closed path or to being derivable from a scalar potential function.
in km s$^{-1}$. The s-space comoving coordinate, $s$, is then
\[ s = x + \hat{x} \cdot v(x) \hat{x}, \]  
where $\hat{x} = x/x$ is a unit vector in the line of sight direction. Denote by $u(s)$ the s-space peculiar velocity as a function of $s$. Consider a patch of matter at an x-space position $x$ moving with velocity $v(x)$. The s-space position of this patch of matter is given by (4) and its s-space velocity is
\[ u(s) = v(x). \]  
In the DOL, we restrict the analysis to a region of space where the separation between any two points is small compared to the distance of the observer from the region. Therefore the lines of sight to all points in the region can be approximated to have the same direction, which we arbitrarily choose to be the unit vector $\hat{x}_3$ in a Cartesian coordinate system $(x_1, x_2, x_3)$. Therefore, substituting $\hat{x} = \hat{x}_3$ in equation (2), we find
\[ s = x + v_3(x) \hat{x}_3, \]  
where $v_3 = \hat{x} \cdot \hat{x}_3$ is the component of $v$ along the line of sight.

2.1 The circulation: $\oint u \cdot ds = 0$

If $v$ is irrotational then its circulation around an arbitrary closed path in x-space is zero. So we have,
\[ \oint v(x) \cdot dx = 0. \]  
We will show that, in the DOL, the velocity field $u(s)$ is also irrotational. Consider the circulation
\[ C = \oint u(s) \cdot ds. \]  
In the absence of multi-valued zones, we can use (3) and (4) to express the integral in this equation in terms of $v$ and $x$,
\[ C = \oint v(x) \cdot \left[ x + v_3(x) \hat{x}_3 \right] dx, \]  
where the integration is over a closed path in x-space, obtained by applying the transformation (4) on every point on the closed path in s-space. Because of (3) the relation (7) reduces to
\[ C = \oint v(x) \cdot \hat{x}_3 dv_3(x) = \int v_3 dv_3 = 0. \]  
Therefore the circulation around an arbitrarily chosen path in s-space is zero. According to Stokes theorem this implies that curl of $u$, or its vorticity, is zero. Nevertheless, in the next subsection, we show, by direct calculation, that if $\nabla_x \times v(x) = 0$ then $\nabla_s \times u(s) = 0$.

2.2 The curl: $\nabla_s \times u(s) = 0$

The vorticity of the s-space velocity field is
\[ V = \nabla_s \times u(s). \]  
In the absence of multi-values zones the mapping between $s$ and $x$ is one-to-one and therefore the relation (3) and the chain rule can be used to yield
\[ V = \nabla_s \times u(s) = \frac{\partial x}{\partial s} \nabla_x \times v(x). \]  
Hence, in a Cartesian coordinate system, the vorticity’s $i$-th component is
\[ v_i = \epsilon^{ijk} \frac{\partial x_k}{\partial s_j} \frac{\partial v_j}{\partial x_l} v_l, \]  
where $\epsilon^{ijk}$ is Levy-Civita anti-symmetric symbol. From equation (11) we have
\[ \frac{\partial s_k}{\partial x_j} = \delta_{kj} + \delta_{3j}v_{3,k}, \]  
where $\delta_{ij}$ is the Kronecker delta function and $f_{ij} = (\partial/\partial x_j)f$. To compute the vorticity we need to know the inverse matrix to $\partial s/\partial x$. Its computation is straightforward and the result is
\[ \frac{\partial x_i}{\partial s_k} = \delta_{ik} - J^{-1} \delta_{3i}v_{3,k}, \]  
where $J = 1 + v_{3,3}$ is the exact Jacobian of the transformation from real to redshift space in the DOL.

Using equation (13) in (12) yields
\[ V_i = \epsilon^{ijk} \left( \delta_{ij} - J^{-1} \delta_{3j}v_{3,k} \right) \frac{\partial v_k}{\partial x_l} v_l \]  
\[ = \epsilon^{ijk} \left( \frac{\partial}{\partial x_j} - J^{-1} v_{3,j} \frac{\partial}{\partial x_3} \right) v_k \]  
\[ = \epsilon^{ijk} v_{k,j} - J^{-1} \epsilon^{ijk} v_{3,j} v_{3,k}. \]  

3  \textbf{RELATION BETWEEN S-SPACE AND X-SPACE VELOCITY POTENTIALS}

The s-space velocity potential $\Phi_u(s)$ is defined by
\[ u(s) = -\nabla_s \Phi_u(s), \]  
and can be written, up to an additive constant, as
\[ \Phi_u(s) = -\int^s u(\tilde{s}) \cdot d\tilde{s}, \]  
in which the integration is along any path connecting $s$ to an arbitrary fixed reference point. Using (3) and (4), the last equation yields
\[ \Phi_u(s) = \Phi_v[x(s)] - \frac{1}{2}v_3^2[x(s)], \]  
in which $v_3[x(s)]$ can be replaced by $u_3(s)$.

Our proof of potential flow is valid only in the DOL. Suppose however that we would like to obtain the x-space velocity potential from all sky measurements of radial peculiar velocities of galaxies in our cosmological neighborhood. The POTENT machinery works in x-space. It first smooths the measured velocities in x-space, then it corrects for spatial (homogeneous and inhomogeneous) Malquist biases and integrates the smooth radial velocities along the radial direction to obtain the x-space potential field. We now show how to obtain the x-space velocity potential from the measured radial velocities as a function of the galaxies redshifts.
This is worthwhile as the x-space velocity field, \( v \), presented as function of the s-space coordinate does not suffer from the spatial Malmquist biases (e.g., Schechter 1980, Tully 1988).

Working with spherical coordinates, \( x = (x, \theta, \phi) \), we write (Bertschinger & Dekel 1989)

\[
\Phi_v(x, \theta, \phi) = -\int_0^r v_{rad}(\tilde{x}, \theta, \phi) \, d\tilde{x},
\]

(18)

where \( v_{rad} \) is the smoothed radial velocity field and the integration is along the radial direction. Using (18) and the general relation (3) between x-space and s-space coordinates the last equation becomes,

\[
\Phi_v[s(x), \theta, \phi] = -\int_0^s u_{rad}(\tilde{s}, \theta, \phi) \, d[\tilde{s} - u_{rad}(\tilde{s})],
\]

(19)

where \( s(x) = x + v_{rad} \) and \( u_{rad} \) is the radial component of the s-space velocity \( u \). In arriving at (19) we have assumed that velocities are measured relative to the Local Group motion and so \( s(x = 0) = 0 \). Therefore the x-space potential as a function of the s-space coordinate can be expressed in terms of the s-space radial velocities as

\[
\Phi_v(s) = -\int_0^s u_{rad}(\tilde{s}, \theta, \phi) \, d[\tilde{s} - u_{rad}(\tilde{s})] \frac{1}{2} u_{rad}^2(s). \]

(20)

The full x-space velocity field in terms of the s-space coordinate can be derived from the x-space potential in the following way

\[
v = -\nabla_s \Phi_v(s) = -\frac{\partial s}{\partial x} \nabla_s \Phi_v(s),
\]

(21)

where \( \partial s/\partial x \) is computed by rewriting (3) in the form \( x = s - u_{rad}(s)\hat{s} \).

4 SUMMARY AND DISCUSSION

We have shown that the s-space velocity field in the distant observer limit and in the absence of shell crossing is irrotational. The result can have important implications on methods of nonlinear reconstruction of velocity from density and vice versa in redshift space. It is also relevant to estimating cosmological parameters from the apparent anisotropies of clustering in redshift space (cf. Hamilton 1998, for a review). For example, Nusser & Davis (1994) have presented a relation, based on the Zel’dovich approximation (Zel’dovich 1970), between the velocity and density in s-space. By expressing the s-space velocity in terms of a potential, this relation can now be solved iteratively to obtain the velocity field associated with a given redshift galaxy distribution. Furthermore, our result can simplify calculations of redshift space dynamical relations based on perturbation theory (e.g., Chodorowski 1999).

The distant observer limit is achieved by restricting the analysis to far away regions with sufficiently small opening angle. Future redshift surveys such as the Anglo-Australian 2dF and the Sloan Digital Sky Survey (Gunn & Knapp 1993) will be sufficiently deep that the distant observer limit can be easily achieved for large fractions of the surveys.

Finally, we have shown that the POTENT method can be modified to derive the full x-space velocity field as a function of the s-space coordinate, thus avoiding spatial Malmquist biases. This result is general, i.e. its validity is not restricted to the limit of the distant observer.

**ACKNOWLEDGMENTS**

MC thanks Román Scoccimarro for stimulating discussions. MC also acknowledges partial support by the Polish State Committee for Scientific Research grants No. 2.P03D.008.13 and 2.P03D.004.13.

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