Bosonization in Higher Dimensions

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Abstract

Using the recently discovered connection between bosonization and duality transformations, we give an explicit path-integral representation for the bosonization of a massive fermion coupled to a $U(1)$ gauge potential (such as electromagnetism) in $d \geq 2$ space ($D = d + 1 \geq 3$ spacetime) dimensions. We perform this integral explicitly in the limit of large fermion mass. We find that the bosonic theory is described by a rank $d - 1$ antisymmetric Kalb-Ramond-type gauge potential, whose action is local for $d = 2$ (given by a Chern-Simons action), but nonlocal for $d \geq 3$. By coupling to a statistical Chern-Simons field for $d = 2$, we obtain a bosonized formulation of anyons. The bosonic theory may be further dualized to a theory involving purely scalars, for any $d$, and we show this to be a higher-derivative theory for which the scalar decouples from the $U(1)$ gauge potential.

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1. Introduction

The technique of bosonization consists of the replacement of a known system of fermions with a theory of bosons which has a completely equivalent physical content, including identical spectra and interactions [1]. It provides an extremely useful tool for analyzing such fermionic systems, since it permits the application to them of powerful techniques that have been developed for bosonic systems. A major limitation of the bosonization technique, however, is its present utility only in $d = 1$ space dimension (i.e. $D = d + 1 = 2$ spacetime dimensions). This is in spite of a number of efforts [2] to extend the theory of bosonization to higher dimensions.

The purpose of this note is to present a different approach to bosonization in dimensions $d \geq 2$. Our approach is based upon the recently-discovered connection between bosonization and duality transformations [3], together with the observation that duality transformations are not intrinsically restricted to $d = 1$ dimensions [4]. We confine our attention here to the case of the ‘abelian’ bosonization of a single Dirac fermion, although we expect that a higher-dimensional generalization of nonabelian bosonization [5] can be obtained along the same lines by using the analogous connection [6] between nonabelian bosonization and nonabelian duality [7].

Starting with a Dirac fermion in $d + 1$ spacetime dimensions, the dualization approach automatically guarantees the existence of a bosonized version of the theory, with an explicit expression for the bosonic action in terms of a path integral over the fermionic and some auxiliary degrees of freedom. The dual, bosonic, variable which appears in this bosonic theory is a rank $d - 1$, completely antisymmetric Kalb-Ramond gauge potential, $\Lambda_{\mu_1 \cdots \mu_{d-1}}$, which is invariant under the gauge freedom $\Lambda \rightarrow \Lambda + d\omega$, where $\omega_{\mu_1 \cdots \mu_{d-2}}$ is an arbitrary $(d - 2)$-form. If the original fermion is coupled to a $U(1)$ gauge potential, $a_\mu$, through the usual interaction $\mathcal{L}_c = i \bar{\psi} \gamma^\mu \psi a_\mu$, then we find that $\Lambda$ necessarily couples through the interaction term $\epsilon^{\mu_1 \cdots \mu_{d+1}} \partial_{\mu_1} a_{\mu_2} \Lambda_{\mu_3 \cdots \mu_{d+1}}$.

Although the bosonic theory we are led to in this way is guaranteed to exist, it is not required to have many of the usual properties that we tend to take for granted in the $d = 1$
case, such as locality. In order to investigate the properties of the bosonic theory in more
detail we explicitly perform the functional integrals which define the bosonic action in the
limit that the fermion mass, $m$, is much larger than the momenta of the external fields.
This allows us to systematically determine the form for the bosonic action as a series in
$1/m$. We find the leading term in this expansion, and show that although the result is
nonlocal when $d \geq 3$, it turns out to be local for the special case $d = 2$.

The $d = 2$ special case is interesting for a number of reasons, besides the locality of
the bosonic action, due to the possibility in this instance of fractional statistics, or anyons
[8]. These have a now-standard representation in terms of fermions coupled to a statistical
Chern-Simons gauge field [9], and by following this field through the bosonization process
we derive here a bosonized formulation for these particles.

Finally, we address a potential conundrum. Aficionados of duality will recognize that
a rank $d - 1$ Kalb-Ramond field in $d$ space dimensions is equivalent to a purely scalar field
with derivative interactions [10]. Given these derivative couplings, one might worry that
the scalar degree of freedom should be much lighter than the fermion mass, $m$, and so not
properly reproduce the properties of the fermionic theory. Similarly, one might worry that
the couplings between $\Lambda_{\mu_1 \cdots \mu_{d-1}}$ and the gauge potential, $a_\mu$, might cause the scalar to be
‘eaten’ by the gauge potential, and to thereby always give this field a mass. We show how
this conundrum gets resolved by explicitly performing this duality, where we find that the
scalar completely decouples from the other external fields.

While this work was in progress we received Ref. [11], which takes a somewhat similar
point of view to the one taken here.

2. The Bosonization Algorithm

We take as our starting point a free Dirac fermion, $\psi$, in $D = d + 1$ spacetime
dimensions. We consider explicitly the relativistic case, but our methods apply equally
well for nonrelativistic systems. (More details concerning the treatment of nonrelativistic
systems may also be found in Ref. [11], since these were the principal applications of this
We take the fermionic lagrangian density to be:

$$L_F = -\bar{\psi} \left[ i \gamma^\mu \partial_\mu + m + J_i M_i \right] \psi,$$

where $M_i$ represents the complete set of Dirac matrices that is appropriate to the chosen dimension of spacetime.\(^1\) The $J_i$ are a collection of external fields, and it is the response of the system to these fields which we wish to compute. (For practical applications below, we take $J_i$ to be an applied electromagnetic field, $a_\mu$, for which $\bar{\psi} M_i \psi = i \bar{\psi} \gamma^\mu \psi$.)

To bosonize we follow Ref. [3] and first enlarge the fermion theory by gauging the global $U(1)$ symmetry $\psi \rightarrow e^{i \theta} \psi$, while constraining the field strength of the corresponding gauge potential, $A_\mu$, to vanish. The lagrangian of this enlarged theory is:

$$L_G = L_F + i \bar{\psi} \gamma^\mu A_\mu + \epsilon^{\mu_1 \cdots \mu_{d+1}} \partial_{\mu_1} A_{\mu_2} \Lambda_{\mu_3 \cdots \mu_{d+1}}. \quad (2)$$

The key point is that this extended theory is precisely equivalent to the original system described by eq. (1). This can be seen by first integrating the lagrange-multiplier field, $\Lambda_{\mu_1 \cdots \mu_{d-1}}$, and then integrating over $A_\mu$. Integration over the lagrange-multiplier field imposes a constraint which ensures that $A_\mu$ is gauge-equivalent to zero. This forces eq. (2) to reduce to eq. (1), and establishes the equivalence of these two theories.

The bosonized result is then obtained by starting from eq. (2), but instead integrating out the fields $\psi$ and $A_\mu$. Only $\Lambda_{\mu_1 \cdots \mu_{d-1}}$ is left to play the role of the bosonized variable. The bosonized lagrangian is therefore defined by the following functional integral (with $J_i = a_\mu$):

$$\exp \left[ i S_B (\Lambda, a) \right] = \int \mathcal{D} \psi \mathcal{D} A_\mu \exp \left[ i S_G (\psi, \Lambda, a, A) + \frac{i}{2 \xi} \int d^D x \left( \partial_\mu A_\mu \right)^2 \right]. \quad (3)$$

The final term is a gauge averaging term which we have chosen to use to gauge fix the $A_\mu$ integration.

\(^1\) For instance, when $D = 4$ we have $M_i = 1, \gamma_5, \gamma^\mu, \gamma^\mu \gamma_5$ and $\gamma^{\mu \nu}$. 
This last equation, eq. (3), is our starting point which defines the bosonic theory for spaces of arbitrary dimension. We next turn to the explicit evaluation of the integrals.

3. The Large-$m$ Limit

Although it is not known how to evaluate the functional integrals of eq. (3) in the general case, they can be performed approximately in certain circumstances. In this section we wish to evaluate them using an expansion in powers of the inverse fermion mass, $1/m$. We focus here on the leading behaviour in this limit, but higher orders in $1/m$ can be obtained in a similar way.

The fermionic functional integration is comparatively simple to evaluate in the large-$m$ limit since it leads to a local effective lagrangian that is dominated by those interactions which have the lowest scaling dimension. For a perturbative system such as the one considered here, lowest scaling dimension reduces to lowest naive dimension, leading to a very simple expression for the result:

$$
\exp \left[ i \Gamma_f (A) \right] \equiv \int \mathcal{D} \psi \exp \left\{ -i \int d^D x \overline{\psi} \left[ \gamma^\mu (\partial_\mu - i A_\mu) + m \right] \psi \right\} 
\equiv \exp \left\{ \frac{i}{2} \int d^D x A_\mu \Pi^{\mu \nu} A_\nu + \cdots \right\},
$$

(4)

where the ellipses denote terms whose coefficients involve additional powers of $1/m$. We drop here, and elsewhere, any irrelevant field-independent multiplicative constants.\footnote{A technical aside: The fermionic functional integral diverges, and we choose to regularize these divergences using dimensional regularization. All such divergences must be renormalized, as usual, and we do so using \textit{MS}. This scheme has the advantage of making it simple to track the $m$ dependence of the results.}

Notice that the condition of lowest dimension implies that, for all spacetime dimensions, the lowest-dimension operator is just quadratic in the applied field, $A_\mu$.\footnote{This observation is also made for more complicated systems in Ref. [11].} This permits the explicit calculation of the subsequent integrals over the potential $A_\mu$. 


Calculations of the response of a Dirac fermion to applied electromagnetic fields have a distinguished history [12], and the corresponding vacuum polarization, $\Pi_D^{\mu\nu}$, takes different forms in the cases $D = 3$ (i.e. $d = 2$) and $D \geq 4$ ($d \geq 3$). This is the origin of the differences in the properties of their bosonized forms. For $D \geq 4$ one has:

$$\Pi_D^{\mu\nu} = k_D (\Box \eta^{\mu\nu} - \partial^{\mu} \partial^{\nu})$$

(5)

where $k_D$ is a $D$-dependent constant. (For even $D$, $k_D$ is also a divergent constant, and so must be renormalized. As a result, the only unambiguous conclusions that can be drawn about its form is how $k_D(\mu)$ runs as the renormalization point, $\mu$, changes.)

For $D = 3$ ($d = 2$) there is an operator which has lower scaling dimension than that of eq. (5), and its coefficient is known to be finite [13]:

$$\Pi_3^{\mu\nu} = k_3 \epsilon^{\mu\lambda\nu} \partial_\lambda,$$

(6)

with $k_3 = \text{sign}(m)/(8\pi^2)$.

As written, eq. (7) holds even for nonrelativistic $\Pi_D^{\mu\nu}$. This equation simplifies significantly for the relativistic vacuum polarizations of eqs. (5) and (6), however. Using the
following identities (which hold equally well for both $D = 3$ and $D \geq 4$):

\[
(\hat{\Pi}_D^{-1})^{\mu \lambda} (\Pi_D)_{\lambda \nu} = \delta^\mu_{\nu} - \frac{\partial^\mu \partial^\nu}{\Box}; \quad (\Pi_D)^{\mu \lambda} (\hat{\Pi}_D^{-1})_{\lambda \rho} (\Pi_D)^{\rho \nu} = \Pi_D^{\mu \nu},
\]

(8)

eq (7) reduces to

\[
S_B(\Lambda, a) = -\frac{1}{2} \int d^Dx \left[ \Omega_\mu (\hat{\Pi}_D^{-1})^{\mu \nu} \Omega_\nu + 2 \Omega_\mu \left( \eta^{\mu \nu} - \frac{\partial^\mu \partial^\nu}{\Box} \right) a_\nu \right].
\]

(9)

3.1) The Case $d \geq 3$

Consider first the case $D \geq 4$ ($d \geq 3$). Explicitly inverting $\hat{\Pi}_D^{\mu \nu}$ gives:

\[
(\hat{\Pi}_D^{-1})^{\mu \nu} = \frac{1}{k_D} \left[ \eta^{\mu \nu} + (\xi k_D - 1) \frac{\partial^\mu \partial^\nu}{\Box} \right] \frac{1}{\Box}
\]

(10)

This, together with the identity $\partial_\mu \Omega^\mu = 0$, leads to the following expression for the bosonic action:

\[
S_B(\Lambda, a) = -\int d^Dx \left[ \frac{1}{2 k_D} \Omega_\mu \frac{1}{\Box} \Omega^\mu + \Omega_\mu a^\mu \right].
\]

(11)

Notice, as advertised, the nonlocality of the $\Omega_\mu \Box^{-1} \Omega^\mu$ term.

Since the integral over $\Lambda_{\mu_1 \cdots \mu_{d-1}}$ is gaussian, it may be performed explicitly to give the large-$m$ limit of the fermion integration of the original theory. We do not perform this integral here, as we do so in a later section, while dualizing the $(d - 1)$ form to a scalar field.

3.2) The Case $d = 2$

The special case $D = 3$ ($d = 2$) proceeds along the same lines. The inverse vacuum polarization is

\[
(\hat{\Pi}_3^{-1})^{\mu \nu} = \frac{1}{k_3} \left[ \epsilon^{\mu \lambda \nu} \partial_\lambda - \xi k_3 \frac{\partial^\mu \partial^\nu}{\Box} \right] \frac{1}{\Box}
\]

(12)
Inserting this expression into eq. (9) then gives the local Chern-Simons result

\[ S_B(\Lambda, a) = - \int d^3 x \epsilon_{\mu\lambda\nu} \left[ \frac{1}{2k_3} \Lambda_\mu \partial_\lambda \Lambda_\nu + a_\mu \partial_\lambda \Lambda_\nu \right]. \] (13)

As for the case of \( D \geq 4 \), it is straightforward to verify the equivalence of the bosonic theory we have obtained with the original fermionic one, by performing the gaussian \( \Lambda_\mu \) integration.

3.3) Anyons

We pause here to briefly consider the dualization of anyons in \( D = 3 \) \((d = 2)\) dimensions. A free anyon having a statistical phase, \( \theta \), has a standard representation [9] in terms of a fermion coupled to a dummy statistics gauge field, \( s_\mu \), with an action of the form:

\[ L_{\text{anyon}} = - \bar{\psi} \left[ \gamma^\mu (\partial_\mu - ia_\mu - is_\mu) + m \right] \psi - \frac{1}{2\theta} \epsilon^{\mu\lambda\nu} s_\mu \partial_\lambda s_\nu. \] (14)

We can clearly bosonize this system, in the limit of large \( m \), as in the previous sections. To apply these results one must simply: (i) shift the external electromagnetic field, \( a_\mu \), by \( a_\mu \to a_\mu + s_\mu \), and (ii) add the Chern Simons action for \( s_\mu \). We find in this way a formulation for long-wavelength \((m \to \infty)\) effects of the anyons, in terms of purely bosonic fields:

\[ L_{\text{anyon}} = - \epsilon^{\mu\lambda\nu} \left[ \frac{1}{2k_3} \Lambda_\mu \partial_\lambda \Lambda_\nu + (a_\mu + s_\mu) \partial_\lambda \Lambda_\nu + \frac{1}{2\theta} s_\mu \partial_\lambda s_\nu \right]. \] (15)

4. Dualizing to a Scalar Variable

We next turn to the dualization of the antisymmetric Kalb-Ramond fields, \( \Lambda_{\mu_1 \cdots \mu_{d-1}} \), into derivatively-coupled scalars — a process which is possible for all \( d \) [10]. This is a useful exercise for many reasons, not least because we have more intuition about the behaviour of scalars than we do about Kalb-Ramond fields. In particular, it provides a simple way of
evaluating the functional integrals over $\Lambda_{\mu_1 \cdots \mu_{d-1}}$ for comparison with the known fermionic result (in the large-$m$ limit).

A puzzle immediately presents itself once it is realized that the scalar which is dual to a Kalb-Ramond field must automatically be derivatively coupled, since how can it be possible that such a boson can reproduce the properties of a very massive fermion? Furthermore, since $\Lambda_{\mu_1 \cdots \mu_{d-1}}$ typically couples to an external electromagnetic potential, $a_\mu$, we might also expect the bosonic theory to always describe a medium for which electromagnetic gauge invariance is spontaneously broken (such as in a superconductor). While this may happen for some systems, it should not appear as a general result of the large-$m$ limit. We resolve this puzzle here by explicitly performing this duality transformation, thereby constructing the dual scalar theory.

In this case dualization is based on recognizing that the functional integral over $\Lambda_{\mu_1 \cdots \mu_{d-1}}$ can be rewritten as an integral over its field strength, $\Omega^\mu = \epsilon^{\mu \nu_2 \cdots \nu_{d+1}} \partial_{\nu_2} \Lambda_{\nu_3 \cdots \nu_{d+1}}$, subject to the constraint that $\partial_{\mu} \Omega^\mu \equiv 0$. For an arbitrary functional, $F(d\Lambda)$, of the field strength, $d\Lambda$, we therefore write:

$$
\int D\Lambda \ F(d\Lambda) = \int D\Omega \ F(\Omega) \delta(\partial \cdot \Omega) = \int D\Omega \ D\varphi \ F(\Omega) \exp\left[i \int d^D x \varphi \partial_{\mu} \Omega^\mu\right]. \quad (16)
$$

The dual theory is then obtained by performing the (now unconstrained) integration over $\Omega_\mu$. Since for the cases of interest here, the integrals over $\Omega_\mu$ are all gaussian, they may be performed explicitly, and they give the same result for all $D$, including $D = 3$:

$$
S_B(\varphi,a) = \frac{1}{2} \int d^D x \left[ a_\mu \Pi^{\mu \nu}_D a_\nu + \frac{1}{\xi} \varphi \Box \varphi \right]. \quad (17)
$$

Although the $\varphi$ action turns out to be local, it is unorthodox in that it contains higher derivatives. As a result, $\varphi$ does not propagate like a normal scalar. This turns out to be largely irrelevant in this formulation of the theory, however, since $\varphi$ also completely decouples from the low-energy electromagnetic potential. Its functional integral may therefore be completely ignored, contributing as it does just a field-independent overall constant. Notice also that the remaining $a_\mu$-dependent term is precisely the large-$m$ limit of the
fermionic theory (c.f. eq. (4)), allowing us to see in this way that the decoupling of the scalars, \( \varphi \), is just what is required to reproduce the original fermionic result.

5. Conclusions

In summary, we have presented here a way of bosonizing fermions in arbitrary dimensions. When applied to a system of space dimension \( d \), and so spacetime dimension \( D = d + 1 \), it leads to a bosonic variable which is a rank \((d - 1)\) Kalb-Ramond field. When specialized to \( D = 1 + 1 \) dimensions, this construction produces a scalar field, and reproduces all of the known results of abelian and nonabelian bosonization [3], [6].

We give the action for the bosonic form of the theory in general as an explicit integral over the fermionic, as well as some auxiliary, degrees of freedom. These integrals can be performed in the limit of very massive fermions, for which an expansion in powers of \( 1/m \) can be set up. The leading term for this expansion gives gaussian integrals which can be performed explicitly. Although the ability to perform all of the integrals in this case obviates much of the necessity for bosonization, the real utility of the method comes in once interactions are included, since these can be recast into bosonic form and analyzed there. (A very nice example along these lines is given in Ref. [11], where a contact four-fermion interaction becomes a gaussian integration when written in the bosonic formulation.)

We find that in the general case, the bosonic theory that is obtained in this way is nonlocal. The case \( D = 3 \) \((d = 2)\) is special, however, and leads to a local Chern-Simons lagrangian (in the large-\( m \) limit) for the bosonic field \( \Lambda_\mu \). Anyons may also be rewritten using our techniques, through the artifice of introducing a Chern Simons statistics field.

Finally, we show how the bosonic theory can again be dualized and thereby reexpressed (for any \( d \)) as a scalar field theory. The resulting scalar has a higher-derivative propagator, and decouples from the other low-energy degrees of freedom in all the cases we considered.

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