DESCARTES’S “RULE OF SIGNS” AND POINCARÉ’S POSITIVSTELLENSATZ

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Abstract. This is an exposition of Poincaré’s 1883 paper, “Sur les équations algébriques,” which gives an important refinement of Descartes’s rule of signs and was a precursor of Pólya’s Positivstellensatz.

1. A refinement of Descartes rule of signs

In 1637, René Descartes [2] wrote,

An equation can have as many true roots as it contains changes of sign, from + to − or from − to +; and as many false roots as the number of times two + signs or two − signs are found in succession.

In the 17th century, a “true root” was a positive root and a “false root” was a negative root. Descartes’s “rule of signs” is the most famous result in the theory of equations. Descartes did not prove the theorem. In 1828, Gauss [3, 5] gave a beautiful elementary proof.

In this paper we consider only polynomials with real coefficients. In more modern language, Descartes’s theorem states that the number $Z(F)$ of positive roots (counting multiplicity) of a polynomial $F(x)$ is the number $V(F)$ of sign variations in the sequence of coefficients of the polynomial minus a nonnegative even integer $\nu(F)$:

$$Z(F) = V(F) - \nu(F).$$

The number of sign variations in a sequence $(a_0, a_1, a_2, \ldots, a_n)$ of real numbers is the number of pairs $(j, k)$ with $0 \leq j < k \leq n$ such that $a_j a_k < 0$ and $a_i = 0$ if $j < i < k$. The number of sign variations of a polynomial $F(x) = \sum_{i=0}^{n} a_i x^i$ is the number of sign variations in the sequence $(a_0, a_1, a_2, \ldots, a_n)$ of coefficients of $F$. The sign variation function $V(F)$ gives an upper bound for the number of positive roots of $F(x)$, but only if $V(F)$ is odd does it imply that $F(x)$ has a positive root.

The polynomial may or may not have positive root if $V(F)$ is even. For example, if $F(x) = x^2 - 3x + 2$, then $V(F) = Z(F) = 2$ and $\nu(F) = 0$. If $F(x) = x^2 - 3x + 5$, then $V(F) = \nu(F) = 2$ and $Z(F) = 0$. It has been a nagging open problem in the theory of equations to understand the even integer $\nu(F)$. For example, virtual roots of polynomials [1, 4] have been introduced to interpret $\nu(F)$.

Henri Poincaré was evidently also bothered by $\nu(F)$. In a beautiful paper published in 1883, he applied an elementary argument to “remove” $\nu(F)$. Poincaré uses
only algebraic and trigonometric identities to prove the following results. Proofs of these identities are collected in Appendix A.

A polynomial has positive coefficients if all of its nonzero coefficients are positive. Sums and products of polynomials with positive coefficients are polynomials with positive coefficients. If \( G(x) \) is a polynomial with positive coefficients, then \( V(G) = 0 \) and \( G(x) \) has no positive root.

Poincaré [7] proved the following.

Theorem 1. A monic polynomial \( F(x) \) is positive for all positive \( x \) if and only if there exists a polynomial \( G(x) \) with positive coefficients such that the product polynomial \( F(x)G(x) \) has positive coefficients.

Equivalently, \( F(x) > 0 \) for all \( x > 0 \) if and only if there exists a nonzero polynomial \( G(x) \) with positive coefficients such that \( V(FG) = \nu(FG) = 0 \).

In real algebraic geometry, a Positivstellensatz is a theorem that certifies that a polynomial is positive on some subset of its domain. Theorem 1 (Poincaré’s Positivstellensatz) provides a certificate that a monic polynomial is positive on \( \Omega_1 = (0, \infty) \). Pólya [6, 8] cites this result as a precursor of his Positivstellensatz for \( n \)-ary \( m \)-adic forms that are positive on the nonzero nonnegative orthant \( \Omega_n \).

Theorem 2. Let \( F(x) \) be a monic polynomial of degree \( n \) with exactly \( p \) positive roots (counting multiplicity). Thus, \( V(F) = p + \nu(F) \) for some nonnegative even integer \( \nu(F) \). There exists a nonzero polynomial \( K(x) \) such that \( V(FK) = Z(FK) = p \) and \( \nu(FK) = 0 \).

2. Proofs

Let \( \Lambda \) be the set of real and complex roots of the polynomial \( F(x) \in \mathbb{R}[x] \). For all \( \lambda \in \Lambda \), let \( \mu_\lambda \) be the multiplicity of \( \lambda \) as a root of the polynomial \( F(x) \).

If \( \lambda = \alpha + i\gamma \) is a complex root of \( F(x) \) with \( \gamma > 0 \), then \( \overline{\lambda} = \alpha - i\gamma \) is also a complex root of \( F(x) \). We consider the quadratic polynomial \( f_\lambda(x) \in \mathbb{R}[x] \) defined by

\[
\begin{align*}
f_\lambda(x) &= (x - \lambda)(x - \overline{\lambda}) \\
&= x^2 - 2\alpha x + \alpha^2 + \gamma^2 \\
&= x^2 - 2\alpha x + \beta^2
\end{align*}
\]

where \( \beta = \sqrt{\alpha^2 + \gamma^2} > |\alpha| \).

Consider the following four subsets of \( \Lambda \):

\[
\begin{align*}
\Lambda_1 &= \{ \alpha \in \Lambda : \alpha \leq 0 \} \\
\Lambda_2 &= \{ \alpha + \gamma i \in \Lambda : \alpha \leq 0 \text{ and } \gamma > 0 \} \\
\Lambda_3 &= \{ \alpha + \gamma i \in \Lambda : \alpha > 0 \text{ and } \gamma > 0 \} \\
\Lambda_4 &= \{ \alpha \in \Lambda : \alpha > 0 \}.
\end{align*}
\]
Note that the number of positive roots of $F(x)$ (counting multiplicity) is $\sum_{\alpha \in \Lambda_4} \mu_\alpha$. Associated with these sets are the monic polynomials

$$F_1(x) = \prod_{\alpha \in \Lambda_1} (x - \alpha)^{\mu_\alpha}$$
$$F_2(x) = \prod_{\lambda \in \Lambda_2} f_{\lambda}(x)^{\mu_{\lambda}} = \prod_{\lambda = \alpha + \gamma i \in \Lambda_2} (x^2 - 2\alpha x + \beta^2)^{\mu_{\lambda}}$$
$$F_3(x) = \prod_{\lambda \in \Lambda_3} f_{\lambda}(x)^{\mu_{\lambda}} = \prod_{\lambda = \alpha + \gamma i \in \Lambda_3} (x^2 - 2\alpha x + \beta^2)^{\mu_{\lambda}}$$
$$F_4(x) = \prod_{\alpha \in \Lambda_4} (x - \alpha)^{\mu_\alpha}$$

and

$$F(x) = F_1(x)F_2(x)F_3(x)F_4(x).$$

We have $\alpha \leq 0$ for all roots in $\Lambda_1 \cup \Lambda_2$, and so $F_1(x)$ and $F_2(x)$ are polynomials with positive coefficients.

Let $\lambda = \alpha + \gamma i \in \Lambda_3$. We have $\gamma > 0$ and so $0 < \alpha < \sqrt{\alpha^2 + \gamma^2} = \beta$.

There is a unique number $\varphi$ such that

$$0 < \varphi < \pi/2$$

and

$$\cos \varphi = \frac{\alpha}{\beta}.$$ 

There is a unique integer $n \geq 2$ such that

$$0 < \varphi < 2\varphi < \cdots < n\varphi < \pi \leq (n + 1)\varphi < 3\pi/2.$$ 

This inequality implies that

$$\sin(n + 1)\varphi \leq 0 < \sin k\varphi \quad \text{for all } k \in \{1, 2, \ldots, n\}$$

Define the polynomial

$$g_{\lambda}(x) = \beta^{n-1} \sin \varphi + (\beta^{n-2} \sin 2\varphi) x + (\beta^{n-3} \sin 3\varphi) x^2 + \cdots + (\sin n\varphi) x^{n-1}.$$ 

This is a polynomial of degree $n - 1$ with positive coefficients. From Lemma 1 in Appendix A we have the trigonometric identity

$$f_{\lambda}(x)g_{\lambda}(x) = \beta^{n+1} \sin \varphi - (\beta \sin(n + 1)\varphi) x^n + (\sin n\varphi) x^{n+1}.$$ 

It follows that $f_{\lambda}(x)g_{\lambda}(x)$ has positive coefficients. Let

$$G(x) = \prod_{\lambda \in \Lambda_3} g_{\lambda}(x)^{\mu_{\lambda}}.$$ 

The polynomial

$$F_3(x)G(x) = \left( \prod_{\lambda \in \Lambda_3} f_{\lambda}(x)^{\mu_{\lambda}} \right) \left( \prod_{\lambda \in \Lambda_3} g_{\lambda}(x)^{\mu_{\lambda}} \right) = \prod_{\lambda \in \Lambda_3} (f_{\lambda}(x)g_{\lambda}(x))^{\mu_{\lambda}}$$

is a product of polynomials with positive coefficients, and so $F_3(x)G(x)$ has positive coefficients and

$$L(x) = F_1(x)F_2(x)F_3(x)G(x)$$

also has positive coefficients.
The set \( \Lambda_4 \) is the set of positive roots of \( F(x) \). If \( \Lambda_4 = \emptyset \), that is, if \( F(x) \) has no positive root and \( F_4(x) = 1 \), then \( F(x) = F_1(x)F_2(x)F_3(x) \) and \( L(x) = F(x)G(x) \) has positive coefficients. This proves Theorem 1.

Suppose that \( \Lambda_4 \neq \emptyset \). Let \( q - 1 \) be the degree of the polynomial \( L(x) \). For \( \alpha \in \Lambda_4 \), define the polynomial

\[
h_\alpha(x) = \frac{x^q - \alpha^q}{x - \alpha} = \sum_{i=0}^{q-1} \alpha^{q-1-i}x^i.
\]

and let

\[
H(x) = \prod_{\alpha \in \Lambda_4} h_\alpha(x)^{\mu(\alpha)}.
\]

The product polynomial

\[
M(x) = F_4(x)H(x) = \prod_{\alpha \in \Lambda_4} ((x - \alpha)h_\alpha(x))^{\mu(\alpha)} = \prod_{\alpha \in \Lambda_4} (x^q - \alpha^q)^{\mu(\alpha)}
\]

is a monic polynomial of degree \( pq \) that is a sum of powers of \( x^q \). Moreover,

\[
V(M) = \sum_{\alpha \in \Lambda_4} \mu(\alpha) = p.
\]

by Lemma 2.

We have

\[
F(x)\!G(x)\!H(x) = F_1(x)\!F_2(x)\!F_3(x)\!G(x)\!F_4(x)\!H(x) = L(x)\!M(x).
\]

Setting \( K(x) = F(x)G(x) \) and applying Lemma 2 gives

\[
V(FK) = V(FG\!H) = V(L\!M) = V(M) = p.
\]

This completes the proof.

\section*{Appendix A. Proofs of the identities}

We prove the following trigonometric identity.

\begin{lemma}
Let \( f(x) = x^2 - (2\beta \cos \varphi)x + \beta^2 \)
and
\[
g(x) = \beta^{n-1}\sin \varphi + (\beta^{n-2}\sin 2\varphi)x + (\beta^{n-3}\sin 3\varphi)x^2 + \cdots + (\sin n\varphi)x^{n-1}.
\]
Then
\[
f(x)g(x) = \beta^{n+1}\sin \varphi - (\beta \sin(n+1)\varphi)x^n + (\sin n\varphi)x^{n+1}.
\]
\end{lemma}

\begin{proof}
The trigonometric identities

\[
sin(k+1)\varphi = \sin k\varphi \cos \varphi + \cos k\varphi \sin \varphi
\]
and
\[
sin(k-1)\varphi = \sin k\varphi \cos \varphi - \cos k\varphi \sin \varphi
\]
imply

\[
(1) \quad \sin(k+1)\varphi + \sin(k-1)\varphi = 2 \sin k\varphi \cos \varphi.
\]
Also,

\[
(2) \quad \sin 2\varphi = 2 \sin \varphi \cos \varphi
\]
and
\[\sin(n+1)\varphi = \sin n\varphi \cos \varphi + \cos n\varphi \sin \varphi\]
\[= 2 \sin n\varphi \cos \varphi - (\sin n\varphi \cos \varphi - \cos n\varphi \sin \varphi)\]
\[= 2 \sin n\varphi \cos \varphi - \sin(n-1)\varphi\]

Then
\[f(x)g(x) = (x^2 - (2\beta \cos \varphi) x + \beta^2) \left(\sum_{k=0}^{n-1} x^k \beta^{n-1-k} \sin(k+1)\varphi\right)\]
\[= \beta^{n+1} \sin \varphi + \beta^n (\sin(2\varphi - 2 \sin \varphi \cos \varphi) x\]
\[+ \sum_{k=2}^{n-1} \beta^{n+1-k} (\sin(k+1)\varphi - 2 \sin k \varphi \cos \varphi + \sin(k-1)\varphi) x^k\]
\[- x^n (2 \sin n\varphi \cos \varphi - \sin(n-1)\varphi) + x^{n+1} \sin n\varphi.\]

Applying identities (1), (2), and (3), we obtain
\[f(x)g(x) = \beta^{n+1} \sin \varphi - x^n \beta \sin(n+1)\varphi + x^{n+1} \sin n\varphi.\]

This completes the proof. □

**Lemma 2.** If \(a_i > 0\) for all \(i \in \{1, 2, \ldots, n\}\), then
\[V \left(\prod_{i=1}^{m} (x - a_i)\right) = m.\]

**Proof.** Because \(a_i > 0\) for all \(i \in \{1, 2, \ldots, n\}\), we obtain
\[V \left(\prod_{i=1}^{m} (x - a_i)\right) = V \left(\prod_{i=1}^{m} (x - \text{sign}(a_i))\right)\]
\[= V ((x - 1)^m) = V \left(\sum_{j=0}^{m} (-1)^{m-j} \binom{m}{j} x^j\right)\]
\[= V ((-1)^m, (-1)^{m-1}, \ldots, 1, -1, 1) = m.\]

This completes the proof. □

**Lemma 3.** Let \(M(x) = \sum_{i=0}^{p} a_i x^i\) be a monic polynomial of degree \(pq\) in which only powers of \(x^i\) occur with nonzero coefficients. Let \(L(x) = \sum_{j=0}^{q-1} s_j x^j\) be a polynomial of degree \(q-1\) with positive coefficients. Then
\[V(LM) = V(M).\]

**Proof.** The sequence of length \((p+1)q\) of coefficients of the polynomial \(M(x)\) is
\[
\begin{pmatrix}
r_0, 0, 0, \ldots, 0, r_q, 0, 0, \ldots, 0, r_{2q}, \ldots, r_{(p-1)q}, 0, 0, \ldots, 0, r_{pq}, 0, 0, \ldots, 0 \cr
q - 1 \text{ roots} & q - 1 \text{ roots} & q - 1 \text{ roots} & q - 1 \text{ roots}
\end{pmatrix}.
\]

For all \(i \in \{0, 1, 2, \ldots, p\}\), we see the subsequence of coefficients of length \(q\)
\[
\begin{pmatrix}
r_{iq}, 0, 0, \ldots, 0 \cr
q - 1 \text{ roots}
\end{pmatrix}.
\]
In the sequence of coefficients of the product polynomial $L(x)M(x)$, this subsequence is replaced by 
\[(r_{iq}s_0, r_{iq}s_1, r_{iq}s_2, \ldots, r_{iq}s_{q-1})\].

For all $j \in \{0, 1, 2, \ldots, q-1\}$ we have $s_j > 0$ and so $\text{sign}(r_{iq}s_j) = \text{sign}(r_{iq})$. It follows that
\[V(r_{iq}s_0, r_{iq}s_1, r_{iq}s_2, \ldots, r_{iq}s_{q-1}) = 0\]
and
\[V(LM) = V(M).\]

This completes the proof. \(\square\)

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