Rational representations of the Yangian $Y(\mathfrak{gl}_n)$

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Abstract

We construct a series of rational representations of $Y(\mathfrak{gl}_n)$ and intertwining operators between them. We find explicit expressions for the images of highest-weight vectors under the intertwining operators. Finally, we state a conjecture that all irreducible finite-dimensional rational $Y(\mathfrak{gl}_n)$-modules arise as images of the constructed intertwining operators.

0. Introduction

There are two classical approaches to the representation theory of the Lie algebra $\mathfrak{gl}_n$. The first one, known as the theory of highest weight, suggests a parametrization of all complex finite-dimensional irreducible $\mathfrak{gl}_n$-modules by $n$-tuples of complex numbers $\lambda = (\lambda_1, \ldots, \lambda_n)$. These $n$-tuples are called weights and describe the action of the Cartan subalgebra $\mathfrak{h} \in \mathfrak{gl}_n$ on the module. The second approach is based on the Schur-Weyl duality between the representations of the symmetric group $S_m$ and the Lie algebra $\mathfrak{gl}_n$. It produces a set of $\mathfrak{gl}_n$-modules called the Schur modules, which are certain submodules (or quotient modules) of tensor products of the defining $\mathfrak{gl}_n$-module $\mathbb{C}^n$. Let us borrow terminology from the representation theory of the Lie group $\text{GL}_n$, and call a $\mathfrak{gl}_n$-module polynomial (respectively rational) if it is isomorphic to a submodule of $\mathbb{C}^n \otimes (\mathbb{C}^*)^\otimes N_2$. Then, the Schur modules exhaust all the polynomial $\mathfrak{gl}_n$-modules.

Similar approaches can be taken towards the representation theory of the Yangian $Y(\mathfrak{gl}_n)$. The highest weight theory for Yangians was developed by Drinfeld. Let us call two finite-dimensional representations of the algebra $Y(\mathfrak{gl}_n)$ similar if they differ by an automorphism of the form (1.10). Up to similarity the irreducible finite-dimensional $Y(\mathfrak{gl}_n)$-modules were classified in [D3]. Due to this classification every irreducible finite-dimensional $Y(\mathfrak{gl}_n)$-module is described by a set of Drinfeld polynomials, serving the same goals as weights in the representation theory of $\mathfrak{gl}_n$. Later on, analogous results on the representations of shifted Yangians and $W$-algebras were obtained in [BK].

Another approach to the representation theory of $Y(\mathfrak{gl}_n)$ was considered in the works [KN1, KN2] and is based on the $(\text{GL}_m, \mathfrak{gl}_n)$ Howe duality (see [HI, H2]). The main role is played by a certain functor $E_m$ from the category of $\mathfrak{gl}_m$-modules to the category of $Y(\mathfrak{gl}_n)$-modules. This functor appeared as a composition of Drinfeld (see [D2]) and Cherednik (see [C, AST]) functors, and can be regarded as a reformulation of the Olshanski centralizer construction (see [O1, O2]). Then the following steps were taken in [KN1, KN2]. First, the functor $E_m$ was applied to the Verma modules of the algebra $\mathfrak{gl}_n$, which gave rise to a series of standard representations of the Yangian $Y(\mathfrak{gl}_n)$. Second, with the help of the theory of Zhelobenko operators for Mickelsson...
algebras (see \[Z1, Z2, K, KO\]) certain intertwining operators between the standard \(Y(\mathfrak{gl}_n)\)-modules were constructed. Finally, in the works \[KN5, KNP\] it was shown that all irreducible finite-dimensional \(Y(\mathfrak{gl}_n)\)-modules considered up to similarity can be obtained as the images of the intertwining operators constructed in \[KNT, KN2\]. An analogous result for representations of quantum affine algebras appeared earlier in \[AK\].

In analogy with the \(\mathfrak{gl}_n\)-case, let us call a representation of the Yangian \(Y(\mathfrak{gl}_n)\) polynomial if it is a subquotient of a tensor product of vector representations of \(Y(\mathfrak{gl}_n)\). Note that all \(Y(\mathfrak{gl}_n)\)-modules constructed in \[KN1, KN2\] are polynomial. Moreover, any polynomial \(Y(\mathfrak{gl}_n)\)-module appears this way. Let us call a representation of the Yangian \(Y(\mathfrak{gl}_n)\) rational if it is a subquotient of a tensor product of vector and dual vector representations of \(Y(\mathfrak{gl}_n)\). Earlier, the irreducible rational representations of \(Y(\mathfrak{gl}_n)\) associated with skew Young diagrams were investigated by Nazarov in \[N\]. In the present paper we try to generalize the approach of Khoroshkin and Nazarov in order to obtain rational representations of the algebra \(Y(\mathfrak{gl}_n)\).

First, we consider a modification \(E_{p,q}\) of the functor \(E_m\), based on the \((U_{p,q}, \mathfrak{gl}_n)\) Howe duality (see \[EHW, EP, KV\]). Second, we apply the functor \(E_{p,q}\) to the Verma modules of \(\mathfrak{gl}_m\) which gives certain standard rational modules. Third, with the help of technique developed for twisted Yangians \(Y(\mathfrak{so}_{2n}), Y(\mathfrak{sp}_{2n})\) in \[KN3, KN4\] we construct intertwining operators between obtained tensor products and compute the images of highest-weight vectors under intertwining operators. Next, using the results of \[KN5\], we observe that (under some conditions on the parameters of standard rational modules) the image of the certain intertwining operator is an irreducible rational \(Y(\mathfrak{gl}_n)\)-module. Finally, we state as a conjecture that all irreducible rational \(Y(\mathfrak{gl}_n)\)-modules can be obtained by this construction. We return to this problem in the forthcoming publication \[KNS\].

1. Basics

1.1. Yangian \(Y(\mathfrak{gl}_n)\)

The Yangian \(Y(\mathfrak{gl}_n)\) is a deformation in the class of Hopf algebras of the universal enveloping algebra of the Lie algebra \(\mathfrak{gl}_n[u]\) of polynomial current, see for instance \[D1\]. The unital associative algebra \(Y(\mathfrak{gl}_n)\) is generated by the family

\[ T^{(1)}_{ij}, T^{(2)}_{ij}, \ldots \]  

where \(i, j = 1, \ldots, n\).

Consider the generating functions

\[ T_{ij}(u) = \delta_{ij} + T^{(1)}_{ij}u^{-1} + T^{(2)}_{ij}u^{-2} + \cdots \in Y(\mathfrak{gl}_n)[[u^{-1}]] \]  

with formal parameter \(u\). Then the defining relations in \(Y(\mathfrak{gl}_n)\) can be written as

\[ (u - v) \cdot [T_{ij}(u), T_{kl}(v)] = T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u) \]  

where \(i, j, k, l = 1, \ldots, n\).

The above relations imply that for any \(z \in \mathbb{C}\) assignments

\[ \tau_z: T_{ij}(u) \mapsto T_{ij}(u - z) \]  

for \(i, j = 1, \ldots, n\) (1.3)

define an automorphism \(\tau_z\) of the algebra \(Y(\mathfrak{gl}_n)\). Here each of the formal series \(T_{ij}(u - z)\) in \((u - z)^{-1}\) should be re-expanded in \(u^{-1}\), then the assignment (1.3) is a correspondence between the respective coefficients of series in \(u^{-1}\).
Now let $E_{ij} \in \mathfrak{gl}_n$ with $i,j = 1,\ldots,n$ be the standard matrix units. Sometimes $E_{ij}$ will also denote elements of the algebra $\text{End}(\mathbb{C}^n)$ but this should not cause any confusion. The Yangian $Y(\mathfrak{gl}_n)$ contains the universal enveloping algebra $U(\mathfrak{gl}_n)$ as a subalgebra, the embedding $U(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)$ can be defined by the assignments

$$E_{ij} \mapsto T_{ij}^{(1)} \quad \text{for} \quad i,j = 1,\ldots,n.$$  

Moreover, there is a homomorphism $\pi_n : Y(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n)$ which is identical on the subalgebra $U(\mathfrak{gl}_n) \subset Y(\mathfrak{gl}_n)$ and is given by

$$\pi_n : T_{ij}^{(2)}, T_{ij}^{(3)}, \ldots \mapsto 0 \quad \text{for} \quad i,j = 1,\ldots,n.$$  

Let $T(u)$ be an $n \times n$ matrix whose $i,j$ entry is the series $T_{ij}(u)$. The relations (1.2) can be rewritten by means of the Yang $R$-matrix

$$R(u) = 1 \otimes 1 - \sum_{i,j=1}^{n} \frac{E_{ij} \otimes E_{ji}}{u}$$  

where the tensor factors $E_{ij}$ and $E_{ji}$ are regarded as $n \times n$ matrices. Note that

$$R(u) R(-u) = 1 - \frac{1}{u^2}.$$  

Consider two $n^2 \times n^2$ matrices whose entries are series with coefficients in the algebra $Y(\mathfrak{gl}_n)$,

$$T_1(u) = T(u) \otimes 1 \quad \text{and} \quad T_2(v) = 1 \otimes T(v).$$  

Then a collection of relations (1.2) for all possible indices $i,j,k,l$ is equivalent to

$$R(u - v) T_1(u) T_2(v) = T_2(v) T_1(u) R(u - v).$$  

Further, the Yangian $Y(\mathfrak{gl}_n)$ is a Hopf algebra over the field $\mathbb{C}$. We define the comultiplication $\Delta : Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n)^{\otimes 2} \otimes Y(\mathfrak{gl}_n)$ by the assignment

$$\Delta : T_{ij}(u) \mapsto \sum_{k=1}^{n} T_{ik}(u) \otimes T_{kj}(u).$$  

Throughout the article this comultiplication will be used for tensor products of $Y(\mathfrak{gl}_n)$-modules. The counit homomorphism $\varepsilon : Y(\mathfrak{gl}_n) \to \mathbb{C}$ is defined by

$$\varepsilon : T_{ij}(u) \mapsto \delta_{ij} \cdot 1.$$  

The antipode $S$ on $Y(\mathfrak{gl}_n)$ is given by

$$S : T(u) \mapsto T(u)^{-1}$$

and defines an anti-automorphism of the associative algebra $Y(\mathfrak{gl}_n)$.

Let $T'(u)$ be the transpose to the matrix $T(u)$. Then the $i,j$ entry of the matrix $T'(u)$ is $T_{ji}(u)$. Consider $n^2 \times n^2$ matrices

$$T'_1(u) = T'(u) \otimes 1 \quad \text{and} \quad T'_2(v) = 1 \otimes T'(v).$$
Note that the Yang $R$-matrix (1.3) is invariant under applying the transposition to both tensor factors. Hence the relation (1.7) implies

$$T_i'(u)T_j'(v) R(u - v) = R(u - v)T_j'(v) T_i'(u),$$
$$R(u - v)T_i'(-u)T_j'(-v) = T_j'(-v)T_i'(-u) R(u - v).$$

To obtain the latter relation we used (1.9). By comparing the relations (1.7) and (1.9), an involutive automorphism of the algebra $Y(gl_n)$ can be defined by the assignment

$$\omega: T(u) \mapsto T'(-u),$$

understood as a correspondence between the respective matrix entries. For further details on the algebra $Y(gl_n)$ see [MNO] Chapter 1.

1.2. Representations of Yangian $Y(gl_n)$

Let $\Phi$ be an irreducible finite-dimensional $Y(gl_n)$-module. A non-zero vector $\varphi \in \Phi$ is said to be of highest weight if it is annihilated by all the coefficients of the series $T_{ij}(u)$ with $1 \leq i < j \leq n$ and is an eigenvector for all the coefficients of the series $T_{ii}(u)$ for $1 \leq i \leq n$. In that case $\varphi$ is unique up to a scalar multiplier and for $i = 1, \ldots, n - 1$ holds

$$T_{ii}(u)T_{i+1,i+1}(u)^{-1} \varphi = P_i(u + \frac{1}{2})P_i(u - \frac{1}{2})^{-1} \varphi$$

where $P_i(u)$ is a monic polynomial in $u$ with coefficients in $\mathbb{C}$. Then $P_1(u), \ldots, P_{n-1}(u)$ are called the Drinfeld polynomials of $\Phi$. Any sequence of $n - 1$ monic polynomials with complex coefficients arises this way. An irreducible finite-dimensional $Y(gl_n)$-module is defined by the set of eigenfunctions $\Lambda_{ii}(u)$ such that

$$T_{ii}(u)\varphi = \Lambda_{ii}(u)\varphi$$

for $i = 1, \ldots, n$. Thus, an irreducible finite-dimensional $Y(gl_n)$-module is defined by a set of polynomials $P_1(u), \ldots, P_{n-1}(u)$ and some normalizing factor, for example $\Lambda_{nn}(u)$.

Relations (1.2) show that for any formal power series $g(u)$ in $u^{-1}$ with coefficients in $\mathbb{C}$ and leading term 1, the assignments

$$T_{ij}(u) \mapsto g(u)T_{ij}(u)$$

(1.10)

define an automorphism of the algebra $Y(gl_n)$. The subalgebra in $Y(gl_n)$ consisting of all elements which are invariant under every automorphism of the form (1.10), is called the special Yangian of $gl_n$. The special Yangian of $gl_n$ is a Hopf subalgebra of $Y(gl_n)$ and is isomorphic to the Yangian $Y(sl_n)$ of the special linear Lie algebra $sl_n \subset gl_n$ considered in [D2, D3]. For the proofs of the latter two assertions see [M] Subsection 1.8.

Two irreducible finite-dimensional $Y(gl_n)$-modules are called similar if their restrictions to the special Yangian are isomorphic. Therefore, irreducible finite-dimensional $Y(sl_n)$-modules are defined by a set of Drinfeld polynomials, while irreducible finite-dimensional $Y(gl_n)$-modules are parameterized by their Drinfeld polynomials only up to similarity. For further details on representations of $Y(gl_n)$ see [D3, CP].

Now, let us define the fundamental representation $V_z$ and the dual fundamental representation $V_z'$ of $Y(gl_n)$. As a vector space $V_z = V_z' = \mathbb{C}^n$, the corresponding actions are given
by the pull-backs of the defining \( \mathfrak{gl}_n \)-modules through the evaluation and the dual evaluation homomorphisms respectively

\[
\begin{align*}
\pi_z &= \pi_n \circ \tau_z : Y(\mathfrak{gl}_n) \longrightarrow U(\mathfrak{gl}_n), & T_{ij}(u) &\mapsto \delta_{ij} + \frac{E_{ij}}{u + z}, \\
\pi'_z &= \pi_n \circ \tau_z \circ \omega : Y(\mathfrak{gl}_n) \longrightarrow U(\mathfrak{gl}_n), & T_{ij}(u) &\mapsto \delta_{ij} - \frac{E_{ji}}{u + z}.
\end{align*}
\]

Let also \( \Omega_z \) and \( \Omega'_z \) denote one-dimensional representations of \( Y(\mathfrak{gl}_n) \) defined as the pull-backs of the standard action of \( \mathfrak{gl}_n \) on \( \Lambda^n (\mathbb{C}^n) \) through the evaluation and the dual evaluation homomorphisms. Thus,

\[
T_{ij}(u) \mapsto \delta_{ij} \cdot \frac{u + z + 1}{u + z} \quad \text{and} \quad T_{ij}(u) \mapsto \delta_{ij} \cdot \frac{u + z - 1}{u + z}
\]
on \( \Omega_z \) and \( \Omega'_z \) correspondingly.

**Definition 1.1**

a) Representation of Yangian \( Y(\mathfrak{gl}_n) \) is called polynomial if it is isomorphic to a subquotient of tensor product of fundamental representations \( V_z \) with arbitrary values of \( z \).

b) Representation of Yangian \( Y(\mathfrak{gl}_n) \) is called rational if it is isomorphic to a subquotient of tensor product of fundamental and dual fundamental representations \( V_z \) and \( V'_z \) with arbitrary values of \( z \).

Note that representations \( V_z \) and \( \Omega_z \) are polynomial while representations \( V'_z \) and \( \Omega'_z \) are rational. We would also like to point out that the modules \( \Omega_z \) and \( \Omega'_z \) are cocentral, i.e. in tensor products of \( Y(\mathfrak{gl}_n) \)-modules one can permute them with other modules. More precisely, the form of the comultiplication map \( \Delta \) implies that for any \( Y(\mathfrak{gl}_n) \)-module \( M \) there is a pair of canonical isomorphism

\[
M \otimes \Omega_z \cong \Omega_z \otimes M \quad \text{and} \quad M \otimes \Omega'_z \cong \Omega'_z \otimes M
\]
sending \( m \otimes a \mapsto a \otimes m \) where \( m \in M \) and \( a \in \Omega_z \) or \( a \in \Omega'_z \).

1.3. Functor

Let \( E \) be an \( m \times m \) matrix whose \( a, b \) entry is the generator \( E_{ab} \in \mathfrak{gl}_m \), and let \( E' \) be its transpose. Consider a matrix

\[
X(u) = (u + \theta E')^{-1} \quad \text{with} \quad \theta = \pm 1
\]

whose \( a, b \) entry is a formal power series in \( u^{-1} \)

\[
X_{ab}(u) = u^{-1} \left( \delta_{ab} + \sum_{s=0}^{\infty} X^{(s)}_{ab} u^{-s-1} \right).
\]

For \( a, b = 1, \ldots, m \) elements \( X_{ab}^{(s)} \in \mathfrak{gl}_m \), moreover

\[
X_{ab}^{(0)} = -\theta E_{ba} \quad \text{and} \quad X_{ab}^{(s)} = \sum_{c_1, \ldots, c_s=1} E_{c_1 a} E_{c_2 c_1} \ldots E_{c_s c_{s-1}} E_{b c_s} \quad \text{for} \quad s \geq 1.
\]
Consider the ring $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of polynomial functions on $\mathbb{C}^m \otimes \mathbb{C}^n$ with coordinate functions $x_{ai}$ where $a = 1, \ldots, m$ and $i = 1, \ldots, n$. Let $\mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ be the ring of differential operators with polynomial coefficients on $\mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n)$, and let $\partial_{ai}$ be the partial derivation corresponding to $x_{ai}$.

Consider also the Grassmann algebra $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ of the vector space $\mathbb{C}^m \otimes \mathbb{C}^n$. It is generated by the elements $x_{ai}$ subject to the anticommutation relations $x_{ai}x_{bj} = -x_{bj}x_{ai}$ for all indices $a, b = 1, \ldots, m$ and $i, j = 1, \ldots, n$. Let $\partial_{ai}$ be the operator of left derivation on $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ corresponding to the variable $x_{ai}$. Let $\mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ denote the ring of $\mathbb{C}$-endomorphisms of $\mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n)$ generated by all operators of left multiplication $x_{ai}$ and by all operators $\partial_{ai}$.

Let us define

\[
\begin{align*}
\mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n) &= \mathcal{P}(\mathbb{C}^m \otimes \mathbb{C}^n) & \text{and} & & \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n) &= \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n) \quad \text{if} \quad \theta = 1, \\
\mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n) &= \mathcal{G}(\mathbb{C}^m \otimes \mathbb{C}^n) & \text{and} & & \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n) &= \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n) \quad \text{if} \quad \theta = -1.
\end{align*}
\]

Therefore, algebra $\mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is generated by the elements $x_{ai}, \ a = 1, \ldots, m, \ i = 1, \ldots, n$ subject to relations

\[x_{ai}x_{bj} - \theta x_{bj}x_{ai} = 0.\]

Algebra $\mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ is generated by the elements $x_{ai}$ and $\partial_{bj}, \ a, b = 1, \ldots, m, \ i, j = 1, \ldots, n$ subject to relations

\[
\begin{align*}
x_{ai}x_{bj} - \theta x_{bj}x_{ai} &= 0, \\
\partial_{ai}\partial_{bj} - \theta \partial_{bj}\partial_{ai} &= 0, \\
\partial_{ai}x_{bj} - \theta x_{bj}\partial_{ai} &= \delta_{ab}\delta_{ij}. \quad (1.13)
\end{align*}
\]

Note that $\theta = 1$ corresponds to the case of commuting variables, while $\theta = -1$ corresponds to the case of anticommuting variables.

From now on and till the end of the paper we assume $m = p + q$, where $p$ and $q$ are non-negative integers. Let us introduce new coordinates

\[
p_{ci} = \begin{cases} 
-\theta x_{ci}, & \text{for } c = 1, \ldots, p \\
\partial_{ci}, & \text{for } c = p + 1, \ldots, m
\end{cases} \quad q_{ci} = \begin{cases} 
\partial_{ci}, & \text{for } c = 1, \ldots, p \\
x_{ci}, & \text{for } c = p + 1, \ldots, m.
\end{cases} \quad (1.14)
\]

Now relations (1.13) can be rewritten in the following form

\[
\begin{align*}
qu_{ai}q_{bj} - \theta q_{bj}q_{ai} &= 0, \\
p_{ai}p_{bj} - \theta p_{bj}p_{ai} &= 0, \\
p_{ai}q_{bj} - \theta q_{bj}p_{ai} &= \delta_{ab}\delta_{ij}.
\end{align*}
\]

Define the elements $\hat{E}_{ai,bj} \in \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ as

\[\hat{E}_{ai,bj} = q_{ai}p_{bj}. \quad (1.15)\]

Elements $\hat{E}_{ai,bj}$ satisfy relations

\[
\begin{align*}
\left[\hat{E}_{ai,bj}, \hat{E}_{ck,dl}\right] &= \delta_{be}\delta_{jk}\hat{E}_{ai,dl} - \delta_{ad}\delta_{il}\hat{E}_{ck,bj}, \\
\hat{E}_{ai,bj}\hat{E}_{ck,dl} - \theta \hat{E}_{ck,bj}\hat{E}_{ai,dl} &= \delta_{be}\delta_{jk}\hat{E}_{ai,dl} - \theta \delta_{ad}\delta_{il}\hat{E}_{ck,bj} \\
\hat{E}_{ck,dl}\hat{E}_{ai,bj} - \theta \hat{E}_{ck,bj}\hat{E}_{ai,dl} &= \delta_{ad}\delta_{il}\hat{E}_{ck,bj} - \theta \delta_{ab}\delta_{ij}\hat{E}_{ck,dl}. \quad (1.16)
\end{align*}
\]

which also imply

\[\hat{E}_{ck,dl}\hat{E}_{ai,bj} - \theta \hat{E}_{ck,bj}\hat{E}_{ai,dl} = \delta_{ad}\delta_{il}\hat{E}_{ck,bj} - \theta \delta_{ab}\delta_{ij}\hat{E}_{ck,dl}. \quad (1.17)\]
There is an action of the algebra \( \mathfrak{gl}_{m} \) on the space \( \mathcal{H}(\mathbb{C}^{m} \otimes \mathbb{C}^{n}) \) which is defined by homomorphism \( \zeta_{n} : U(\mathfrak{gl}_{m}) \to \mathcal{H}D(\mathbb{C}^{m} \otimes \mathbb{C}^{n}) \), where

\[
\zeta_{n}(E_{ab}) = \theta \delta_{ab} \frac{n}{2} + \sum_{k=1}^{n} \hat{E}_{ak,bb}.
\] (1.19)

The homomorphism property can be verified using the relation (1.16). Hence, there exists an embedding \( U(\mathfrak{gl}_{m}) \to U(\mathfrak{gl}_{m}) \otimes \mathcal{H}D(\mathbb{C}^{m} \otimes \mathbb{C}^{n}) \) defined for \( a, b = 1, \ldots, m \) by the mappings

\[
E_{ab} \mapsto E_{ab} \otimes 1 + 1 \otimes \zeta_{n}(E_{ab}).
\] (1.20)

**Proposition 1.2**  
i) One can define a homomorphism \( \alpha_{m} : Y(\mathfrak{gl}_{m}) \to U(\mathfrak{gl}_{m}) \otimes \mathcal{H}D(\mathbb{C}^{m} \otimes \mathbb{C}^{n}) \) by mapping

\[
\alpha_{m} : \quad T_{ij}(u) \mapsto \delta_{ij} + \sum_{a,b=1}^{m} X_{ab}(u) \otimes \hat{E}_{ai,bj}.
\] (1.21)

ii) The image of \( Y(\mathfrak{gl}_{m}) \) under the homomorphism (1.21) commutes with the image of \( U(\mathfrak{gl}_{m}) \) under the embedding (1.20).

Consider an automorphism of the algebra \( \mathcal{H}D(\mathbb{C}^{m} \otimes \mathbb{C}^{n}) \) such that for all \( a = 1, \ldots, m \) and \( i = 1, \ldots, n \)

\[
x_{ai} \mapsto q_{ai} \quad \text{and} \quad \partial_{ai} \mapsto p_{ai}.
\] (1.22)

A proof of Proposition 1.2 can be obtained by applying the automorphism (1.22) to the results of [KN1] Proposition 1.3 if \( \theta = 1 \) and to the results of [KN2] Proposition 1.3 if \( \theta = -1 \). In the appendix we give an explicit proof of the Proposition 1.2.

Finally, let \( V \) be an arbitrary \( \mathfrak{gl}_{m} \)-module. Let us define a \( U(\mathfrak{gl}_{m}) \otimes \mathcal{H}D(\mathbb{C}^{m} \otimes \mathbb{C}^{n}) \)-module

\[
\mathcal{E}_{p,q}(V) = V \otimes \mathcal{H}(\mathbb{C}^{m} \otimes \mathbb{C}^{n}).
\] (1.23)

The results of proposition 1.2 turns \( \mathcal{E}_{p,q} \) into a functor from the category of \( \mathfrak{gl}_{m} \)-modules to the category of \( \mathfrak{gl}_{m} \) and \( Y(\mathfrak{gl}_{n}) \) bimodules, where the actions of the algebras \( \mathfrak{gl}_{m} \) and \( Y(\mathfrak{gl}_{n}) \) are defined by homomorphisms (1.20) and (1.21) respectively.

### 2. Reduction to standard modules

#### 2.1. Parabolic induction

Let \( l, l_{1}, l_{2} \) be three positive integers such that \( l = l_{1} + l_{2} \). Consider a \( \mathfrak{gl}_{l} \)-module \( U \), then \( \mathcal{E}_{l_{1},l_{2}}(U) \) is a \( Y(\mathfrak{gl}_{n}) \)-module. For any \( z \in \mathbb{C} \) denote by \( \mathcal{E}_{l_{1},l_{2}}^{z}(U) \) the \( Y(\mathfrak{gl}_{n}) \)-module obtained from \( \mathcal{E}_{l_{1},l_{2}}(U) \) via the pull-back through the automorphism \( \tau_{-\theta_{z}} \) of \( Y(\mathfrak{gl}_{n}) \), In other words, the underlying vector space of \( \mathcal{E}_{l_{1},l_{2}}^{z}(U) \) is the same as of \( \mathcal{E}_{l_{1},l_{2}}(U) \), but the action of \( T_{ij}(u) \) on \( \mathcal{E}_{l_{1},l_{2}}^{z}(U) \) is given by the same formula as the action of \( T_{ij}(u + \theta z) \) on \( \mathcal{E}_{l_{1},l_{2}}(U) \). Note that as a \( \mathfrak{gl}_{l} \)-module \( \mathcal{E}_{l_{1},l_{2}}^{z}(U) \) coincides with \( \mathcal{E}_{l_{1},l_{2}}(U) \).

Decomposition \( \mathbb{C}^{m+l} = \mathbb{C}^{m} \oplus \mathbb{C}^{l} \) determines an embedding of the direct sum \( \mathfrak{gl}_{m} \oplus \mathfrak{gl}_{l} \) into \( \mathfrak{gl}_{m+l} \). As a subalgebra of \( \mathfrak{gl}_{m+l} \), the direct summand \( \mathfrak{gl}_{m} \) is spanned by the matrix units \( E_{ab} \in \mathfrak{gl}_{m+l} \) where \( a, b = 1, \ldots, m \), the direct summand \( \mathfrak{gl}_{l} \) is spanned by the matrix units \( E_{ab} \) where \( a, b = m+1, \ldots, m+l \). Let \( q \) and \( q' \) be the Abelian subalgebras of \( \mathfrak{gl}_{m+l} \) spanned respectively by matrix units \( E_{ba} \) and \( E_{ab} \) for all \( a = 1, \ldots, m \) and \( b = m+1, \ldots, m+l \). Define a maximal parabolic subalgebra \( p = \mathfrak{gl}_{m} \oplus \mathfrak{gl}_{l} \oplus q' \) of the reductive Lie algebra \( \mathfrak{gl}_{m+l} \), then \( \mathfrak{gl}_{m+l} = q \oplus p \).
Consider a $\mathfrak{gl}_m$-module $V$ and a $\mathfrak{gl}_l$-module $U$. Let us turn the $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$-module $V \otimes U$ into a $p$-module by letting the subalgebra $q' \subset p$ act by zero. Define $V \otimes U$ to be the $\mathfrak{gl}_{m+l}$-module induced from the $p$-module $V \otimes U$. The $\mathfrak{gl}_{m+l}$-module $V \otimes U$ is called *parabolically induced* from the $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$-module $V \otimes U$.

Now consider bimodules $\mathcal{E}_{p,q+r}(V \otimes U)_q$ and $\mathcal{E}_{p+r,q}(V \otimes U)$ over $\mathfrak{gl}_{m+r}$ and $\mathfrak{y}(\mathfrak{gl}_n)$, which are parabolically induced from the $\mathfrak{gl}_p \oplus \mathfrak{gl}_q$ and $\mathfrak{gl}_{p+r} \oplus \mathfrak{gl}_q$-module $V \otimes U$ respectively. The action of $\mathfrak{y}(\mathfrak{gl}_n)$ commutes with the action of the Lie algebra $\mathfrak{gl}_{m+r}$, and hence with the action of the subalgebra $q \subset \mathfrak{gl}_{m+r}$. For any $\mathfrak{gl}_m$-module $W$ denote by $W_q$ the vector space $W/ q \cdot W$ of the coinvariants of the action of the subalgebra $q \subset \mathfrak{gl}_m$ on $W$. Then the vector spaces $\mathcal{E}_{p,q+r}(V \otimes U)_q$ and $\mathcal{E}_{p+r,q}(V \otimes U)_q$ are quotients of the $\mathfrak{y}(\mathfrak{gl}_n)$-modules $\mathcal{E}_{p,q+r}(V \otimes U)$ and $\mathcal{E}_{p+r,q}(V \otimes U)$ respectively. Note that the subalgebras $\mathfrak{gl}_p \oplus \mathfrak{gl}_q$ and $\mathfrak{gl}_{p+r} \oplus \mathfrak{gl}_q$ also act on these quotient spaces.

**Theorem 2.1**

i) The bimodule $\mathcal{E}_{p,q+r}(V \otimes U)_q$ over the Yangian $\mathfrak{y}(\mathfrak{gl}_n)$ and the direct sum $\mathfrak{gl}_p \oplus \mathfrak{gl}_{q+r}$ is isomorphic to the tensor product $\mathcal{E}_{p,q}(V) \otimes \mathcal{E}_q(V)$. 

ii) The bimodule $\mathcal{E}_{p+r,q}(V \otimes U)_q$ over the Yangian $\mathfrak{y}(\mathfrak{gl}_n)$ and the direct sum $\mathfrak{gl}_{p+r} \oplus \mathfrak{gl}_q$ is isomorphic to the tensor product $\mathcal{E}_{p,q}(V) \otimes \mathcal{E}_q(V)$.

**Theorem 2.1** is equivalent to [KN1] Theorem 2.1] under the action of automorphism (1.22) if $\theta = 1$ and to [KN2] Theorem 2.1] under the action of automorphism (1.22) if $\theta = -1$. In both cases Theorem 2.1 was proved by establishing a linear map

$$\chi: \mathcal{E}_m(V) \otimes \mathcal{E}_l^m(U) \rightarrow \mathcal{E}_{m+l}(V \otimes U)_q.$$ 

Both the source and the target are bimodules over the algebras $\mathfrak{gl}_m \oplus \mathfrak{gl}_l$ and $\mathfrak{y}(\mathfrak{gl}_n)$, while $\chi$ is a bijective map intertwining actions of algebras. One can easily show that the map $\chi$ commutes with the automorphism (1.22), hence the intertwining property follows in our case. A proof that the map $\chi$ is bijective can be almost word by word taken from [KN1] or [KN2] though one has to keep in mind, that the automorphism (1.22) alters the filtration of the algebra $H(C^m \otimes C^n)$ described in the papers just mentioned. Thus, one should consider descending filtrations

$$\bigoplus_{N=K}^{\infty} P(C^m) \otimes P(C^r) \quad \text{and} \quad \bigoplus_{N=K}^{\infty} P^N(C^m) \otimes P(C^r) \quad \text{if} \quad \theta = 1,$$

$$\bigoplus_{N=K}^{r} G(C^m) \otimes G(C^r) \quad \text{and} \quad \bigoplus_{N=K}^{m} G^N(C^m) \otimes G(C^r) \quad \text{if} \quad \theta = -1$$

for cases i) and ii) of the Theorem 2.1 respectively. Therefore, the proof of the theorem follows in our case.

Let us consider the triangular decomposition

$$\mathfrak{gl}_m = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}'.$$

of the Lie algebra $\mathfrak{gl}_m$. Here $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{gl}_m$ with the basis vectors $E_{11}, \ldots, E_{mm}$. Further, $\mathfrak{n}$ and $\mathfrak{n}'$ are the nilpotent subalgebras spanned respectively by the elements $E_{ab}$ and $E_{ab}$ for all $a, b = 1, \ldots, m$ such that $a < b$. Denote by $V_\lambda$ the vector space $V/ \mathfrak{n} \cdot V$ of the coinvariants of the action of the subalgebra $\mathfrak{n} \subset \mathfrak{gl}_m$ on $V$. Note that the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$ acts on the vector space $V_\lambda$. Now consider the bimodule $\mathcal{E}_{p,q}(V)$. The action of $\mathfrak{y}(\mathfrak{gl}_n)$ on this bimodule commutes with the action of the Lie algebra $\mathfrak{gl}_m$, and hence with the action of the subalgebra $\mathfrak{n} \subset \mathfrak{gl}_m$. Therefore, the space $\mathcal{E}_{p,q}(V)_\lambda$ of coinvariants of the action of $\mathfrak{n}$ is a quotient
of the $Y(\mathfrak{gl}_n)$-module $E_{p,q}(V)$. Thus, we get a functor from the category of all $\mathfrak{gl}_m$-modules to the category of bimodules over $\mathfrak{h}$ and $Y(\mathfrak{gl}_n)$

$$V \mapsto E_{p,q}(V)_n = (V \otimes \mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n))_n. \quad (2.2)$$

By the transitivity of induction, Theorem 2.1 can be extended from the maximal to all parabolic subalgebras of the Lie algebra $\mathfrak{gl}_m$. Consider the case of the Borel subalgebra $\mathfrak{h} \oplus \mathfrak{n}'$ of $\mathfrak{gl}_m$. Apply the functor (2.2) to the $\mathfrak{gl}_m$-module $V = M_\mu$, where $M_\mu$ is the Verma module of weight $\mu \in \mathfrak{h}^*$. We obtain the $(\mathfrak{h}, Y(\mathfrak{gl}_n))$-bimodule

$$E_{p,q}(M_\mu)_n = (M_\mu \otimes \mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n))_n.$$

Using the basis $E_{11}, \ldots, E_{mm}$ we identify $\mathfrak{h}$ with the direct sum of $m$ copies of the Lie algebra $\mathfrak{gl}_1$. Consider the Verma modules $M_{\mu_1}, \ldots, M_{\mu_m}$ over $\mathfrak{gl}_1$. By applying Theorem 2.1 repeatedly we get

**Corollary 2.2** The bimodule $E_{p,q}(M_\mu)_n$ over $\mathfrak{h}$ and $Y(\mathfrak{gl}_n)$ is isomorphic to the tensor product

$$E_{1,0}(M_{\mu_1}) \otimes E_{1,0}^1(M_{\mu_2}) \otimes \cdots \otimes E_{1,0}^{p-1}(M_{\mu_p}) \otimes E_{0,1}^{p}(M_{\mu_{p+1}}) \otimes \cdots \otimes E_{0,1}^{m-1}(M_{\mu_m}).$$

### 2.2. Standard modules

Let us now describe the bimodules $E_{t,0}^z(M_t)$ and $E_{0,1}^z(M_t)$ over $\mathfrak{gl}_1$ and $Y(\mathfrak{gl}_n)$ for arbitrary $t, z \in \mathbb{C}$. The Verma module $M_t$ over $\mathfrak{gl}_1$ is one-dimensional, and the element $E_{11} \in \mathfrak{gl}_1$ acts on $M_t$ by multiplication by $t$. The vector space of bimodules $E_{1,0}(M_t)$ and $E_{0,1}(M_t)$ is the algebra $\mathcal{H}(\mathbb{C}^1 \otimes \mathbb{C}^n) = \mathcal{H}(\mathbb{C}^n)$. Then $E_{11}$ acts on the bimodules $E_{1,0}(M_t)$ and $E_{0,1}(M_t)$ as differential operators

$$t + \theta \frac{n}{2} - \theta \sum_{k=1}^n \partial_{1k} x_{1k} \quad \text{and} \quad t + \theta \frac{n}{2} + \sum_{k=1}^n x_{1k} \partial_{1k}$$

respectively. The action of $E_{11}$ on $E_{1,0}^z(M_t)$ is the same as on $E_{p,q}(M_t)$. The action of $Y(\mathfrak{gl}_n)$ on $E_{1,0}^z(M_t)$ and $E_{0,1}^z(M_t)$ is given by

$$T_{ij}(u) \mapsto \delta_{ij} - \theta \frac{x_{1j} \partial_{1j}}{u + \theta(t - z)} \quad \text{and} \quad T_{ij}(u) \mapsto \delta_{ij} + \frac{x_{1j} \partial_{1j}}{u + \theta(t - z)} \quad (2.3)$$

respectively, this is what Proposition 1.2 states in the case $m = 1$. Note that both operators $x_{1i} \partial_{1j}$ and $\theta \partial_{1i} x_{1j}$ describe actions of the element $E_{11} \in \mathfrak{gl}_1$ on $\mathcal{H}(\mathbb{C}^1 \otimes \mathbb{C}^n)$.

When speaking about these actions we will omit the first indices and write $x_i$ and $\partial_i$ instead of $x_{1i}$ and $\partial_{1i}$. Hence, actions of the algebra $Y(\mathfrak{gl}_n)$ on $E_{1,0}^z(M_t)$ and $E_{0,1}^z(M_t)$ can be obtained from the actions of $\mathfrak{gl}_n$ on $\mathcal{H}(\mathbb{C}^n)$ by pulling back through the evaluation homomorphism $\pi_{\theta(t-z)}$.

Now, consider $Y(\mathfrak{gl}_n)$-modules $\tilde{\Phi}_z$ and $\Phi_z$ with the underlying vector space $\mathcal{H}(\mathbb{C}^n)$ and Yangian actions defined by

$$T_{ij}(u) \mapsto \delta_{ij} - \theta \frac{x_{1j}}{u + \theta z} \quad \text{and} \quad T_{ij}(u) \mapsto \delta_{ij} + \frac{x_{1j}}{u + \theta z}$$

correspondingly. Therefore, the bimodules $E_{1,0}^z(M_t)$ and $E_{0,1}^z(M_t)$ are respectively isomorphic to $\tilde{\Phi}_{t-z}$ and $\Phi_{t-z}$ as the $Y(\mathfrak{gl}_n)$-modules. Moreover, corollary 2.2 implies that the bimodule $E_{p,q}(M_\mu)_n$ of $\mathfrak{h} \in \mathfrak{gl}_m$ and $Y(\mathfrak{gl}_n)$ is isomorphic as a $Y(\mathfrak{gl}_n)$-module to the tensor product

$$\tilde{\Phi}_{\mu_1 + \rho_1} \otimes \cdots \otimes \tilde{\Phi}_{\mu_p + \rho_p} \otimes \Phi_{\mu_{p+1} + \rho_{p+1}} \otimes \cdots \otimes \Phi_{\mu_m + \rho_m} \quad (2.4)$$
where $\rho_a = 1 - a$.

Let us also define a $Y(\mathfrak{g}_n)$-module $\Phi'_z$ with the underlying vector space $\mathcal{H}(\mathbb{C}^n)$ and the $Y(\mathfrak{g}_n)$-action given by

$$T_{ij}(u) \mapsto \delta_{ij} - \frac{x_i \partial_i}{u + \theta(z - 1)}.$$  

Using commutation relation $\partial_i x_j - \theta x_j \partial_i = \delta_{ij}$ one can verify that

$$\delta_{ij} - \theta \frac{\partial_i x_j}{u + \theta z} = \frac{u + \theta(z - 1)}{u + \theta z} \left( \delta_{ij} - \frac{x_j \partial_i}{u + \theta(z - 1)} \right),$$

which implies the isomorphism of $Y(\mathfrak{g}_n)$-modules

$$\tilde{\Phi}_z \cong \Omega'_z \otimes \Phi'_z \quad \text{if} \quad \theta = 1,$$

$$\tilde{\Phi}_z \cong \Omega_z - \omega \otimes \Phi'_z \quad \text{if} \quad \theta = -1.$$ 

Hence, the bimodule $E_{p,q}(M_\mu)_n$ over $\mathfrak{h} \in \mathfrak{gl}_m$ and $Y(\mathfrak{g}_n)$ is isomorphic as the $Y(\mathfrak{g}_n)$-module to the tensor product

$$\bigotimes_{a=1}^p \Omega^*_{\theta(\mu_a + \rho_a)} \otimes \bigotimes_{a=1}^p \Phi'_{\mu_a + \rho_a} \otimes \bigotimes_{a=p+1}^m \Phi_{\mu_a + \rho_a} \quad (2.5)$$

where

$$\Omega^*_z = \Omega'_z \quad \text{if} \quad \theta = 1 \quad \text{and} \quad \Omega^*_z = \Omega_z \quad \text{if} \quad \theta = -1.$$ 

Note that the $Y(\mathfrak{g}_n)$-modules $\Phi_z$ and $\Phi'_z$ can also be realized as pull-backs of the $\mathfrak{g}_n$-modules $\mathcal{P}(\mathbb{C})$ and $\mathcal{G}(\mathbb{C})$ through the evaluation and the dual evaluation homomorphisms. Moreover, modules $\Phi_z$ and $\Phi'_z$ are rational, and hence so are their subquotients. We call an $Y(\mathfrak{g}_n)$-module a standard rational module if it is a tensor product of modules $\Phi_z$ and $\Phi'_z$ with arbitrary values of $z$.

3. Zhelobenko operators

3.1. Definition

Consider $\mathfrak{S}_m$ as the Weyl group of the reductive Lie algebra $\mathfrak{gl}_m$. Let $E_{11}^*, \ldots, E_{mm}^*$ be the basis of $\mathfrak{h}^*$ dual to the basis $E_{11}, \ldots, E_{mm}$ of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{gl}_m$. The group $\mathfrak{S}_m$ acts on the space $\mathfrak{h}^*$ so that for any $\sigma \in \mathfrak{S}_m$ and $a = 1, \ldots, m$

$$\sigma: E_{aa}^* \mapsto E_{\sigma(a)\sigma(a)}^*.$$ 

If we identify each weight $\mu \in \mathfrak{h}^*$ with the sequence $(\mu_1, \ldots, \mu_m)$ of its labels, then

$$\sigma: (\mu_1, \ldots, \mu_m) \mapsto (\mu_{\sigma^{-1}(1)}, \ldots, \mu_{\sigma^{-1}(m)}).$$

Let $\rho \in \mathfrak{h}^*$ be the weight with sequence of labels $(0, -1, \ldots, 1 - m)$. The shifted action of any element $\sigma \in \mathfrak{S}_m$ on $\mathfrak{h}^*$ is defined by the assignment

$$\mu \mapsto \sigma \circ \mu = \sigma(\mu + \rho) - \rho.$$ 

The Weyl group also acts on the vector space $\mathfrak{gl}_m$ so that for any $\sigma \in \mathfrak{S}_m$ and $a, b = 1, \ldots, m$

$$\sigma: E_{ab} \mapsto E_{\sigma(a)\sigma(b)}.$$
The latter action extends to an action of the group $\mathfrak{S}_m$ by automorphisms of the associative algebra $U(\mathfrak{gl}_m)$. The group $\mathfrak{S}_m$ also acts by automorphisms of the space $\mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$ so that element $\sigma \in \mathfrak{S}_m$ maps

$$p_{ai} \mapsto p_{\sigma(a)i} \quad \text{and} \quad q_{ai} \mapsto q_{\sigma(a)i}.$$  

Note that homomorphisms (1.20) and (1.21) of the algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ into the algebra $U(\mathfrak{gl}_m) \otimes \mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$ are $\mathfrak{S}_m$-equivariant.

Let $A$ be the associative algebra generated by the algebras $U(\mathfrak{gl}_m)$ and $\mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$ with the cross relations

$$[X, Y] = \zeta_\eta(X, Y)$$  

for any $X \in \mathfrak{gl}_m$ and $Y \in \mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$. The brackets at the left hand side of the relation (3.1) denote the commutator in $A$, while the brackets at the right hand side denote the commutator in the algebra $\mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$ embedded into $A$. In particular, we will regard $U(\mathfrak{gl}_m)$ as a subalgebra of $A$. An isomorphism of the algebra $A$ with the tensor product $U(\mathfrak{gl}_m) \otimes \mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$ can be defined by mapping the elements $X \in \mathfrak{gl}_m$ and $Y \in \mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$ in $A$ respectively to the elements

$$X \otimes 1 + 1 \otimes \zeta_\eta(X) \quad \text{and} \quad 1 \otimes Y$$

in $U(\mathfrak{gl}_m) \otimes \mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$. The action of the group $\mathfrak{S}_m$ on $A$ is defined via the isomorphism of $A$ with the tensor product $U(\mathfrak{gl}_m) \otimes \mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$. Since the homomorphism $\zeta_\eta$ is $\mathfrak{S}_m$-equivariant the same action of $\mathfrak{S}_m$ is obtained by extending the actions of $\mathfrak{S}_m$ from the subalgebras $U(\mathfrak{gl}_m)$ and $\mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)$ to $A$.

For any $a, b = 1, \ldots, m$ put $\eta_{ab} = E^*_{aa} - E^*_{bb} \in \mathfrak{h}^*$ and $\eta_c = \eta_{cc+1}$ with $c = 1, \ldots, m - 1$. Put also

$$E_c = E_{cc+1}, \quad F_c = E_{c+1c} \quad \text{and} \quad H_c = E_{cc} - E_{c+1c+1}.$$  

For any $c = 1, \ldots, m - 1$ these three elements form an $\mathfrak{sl}_2$-triple.

Let $\overline{U(\mathfrak{h})}$ be the ring of fractions of the commutative algebra $U(\mathfrak{h})$ relative to the set of denominators

$$\{E_{aa} - E_{bb} + z \mid 1 \leq a, b \leq m; \ a \neq b; \ z \in \mathbb{Z}\}.$$  

The elements of this ring can also be regarded as rational functions on the vector space $\mathfrak{h}^*$. Then the elements of $U(\mathfrak{h}) \subset \overline{U(\mathfrak{h})}$ become polynomial functions on $\mathfrak{h}^*$. Denote by $\hat{A}$ the ring of fractions of $A$ relative to the set of denominators (3.3), regarded as elements of $\hat{A}$ using the embedding of $\mathfrak{h} \subset \mathfrak{gl}_m$ into $A$. The ring $\hat{A}$ is defined due to the following relations in the algebras $U(\mathfrak{gl}_m)$ and $A$: for $a, b = 1, \ldots, m$ and $H \in \mathfrak{h}$

$$[H, E_{ab}] = \eta_{ab}(H)E_{ab}, \quad [H, p_{ak}] = -\theta E^*_{aa}(H)p_{ak}, \quad [H, q_{bk}] = E^*_{bb}(H)q_{bk},$$  

where $p_{ak}$ and $q_{bk}$ are given by (1.14). Therefore, the ring $\hat{A}$ satisfies the Ore condition relative to its subset (3.3). Using left multiplication by elements of $\overline{U(\mathfrak{h})}$, the ring of fractions $\hat{A}$ becomes a module over $\overline{U(\mathfrak{h})}$.

The ring $\hat{A}$ is also an associative algebra over the field $\mathbb{C}$. The action of the group $\mathfrak{S}_m$ on $A$ preserves the set of denominators (3.3) so that $\mathfrak{S}_m$ also acts by automorphisms of the algebra $\hat{A}$. For each $c = 1, \ldots, m - 1$ define a linear map $\xi_c: A \to \hat{A}$ by setting

$$\xi_c(Y) = Y + \sum_{s=1}^{\infty} (s!H_{c}^{(s)})^{-1}E^*_c F^*_c(Y)$$  

for any $Y \in A$. Here

$$H_{c}^{(s)} = H_c(H_c - 1) \cdots (H_c - s + 1)$$
and $\hat{F}_c$ is the operator of adjoint action corresponding to the element $F_c \in A$, so that

$$\hat{F}_c(Y) = [F_c, Y].$$

For any given element $Y \in A$ only finitely many terms of the sum (3.5) differ from zero, hence the map $\xi_c$ is well defined.

Let $J$ and $\bar{J}$ be the right ideals of the algebras $A$ and $\bar{A}$ respectively, generated by all elements of the subalgebra $\frak{n} \subset \frak{gl}_m$. Let $J'$ be the left ideal of the algebra $A$, generated by the elements $X - \zeta_n(X)$, or equivalently by the elements

$$X \otimes 1 \in U(\frak{gl}_m) \otimes 1 \subset U(\frak{gl}_m) \otimes \mathcal{H}D (\mathbb{C}^m \otimes \mathbb{C}^n),$$

where $X \in \frak{n}'$. Denote $J' = \overline{U(\frak{h})J'}$, then $J'$ is a left ideal of the algebra $\bar{A}$.

Now we give a short observation of some results proved in [KN1]. For any elements $X \in \frak{h}$ and $Y \in A$ we have

$$\xi_a(XY) \in (X + \eta_a(X))\xi_a(Y) + J.'$$

This allows us to define a linear map $\bar{\xi}_a : \bar{A} \to J' \backslash \bar{A}$ by setting

$$\bar{\xi}_a(XY) = Z\xi_a(Y) + \bar{J} \quad \text{for} \quad X \in \overline{U(\frak{h})} \quad \text{and} \quad Y \in A$$

where the element $Z \in \overline{U(\frak{h})}$ is defined by the equality

$$Z(\mu) = X(\mu + \eta_a) \quad \text{for} \quad \mu \in \frak{h}^*$$

and both $X$ and $Z$ are regarded as rational functions on $\frak{h}^*$. The backslash in $J' \backslash \bar{A}$ indicates that the quotient is taken relative to a right ideal of $\bar{A}$.

The action of the group $\mathfrak{S}_m$ on the algebra $U(\frak{gl}_m)$ extends to an action on $\overline{U(\frak{h})}$ so that for any $\sigma \in \frak{S}_m$

$$(\sigma(X))(\mu) = X(\sigma^{-1}(\mu)),$$

when the element $X \in \overline{U(\frak{h})}$ is regarded as a rational function on $\frak{h}^*$. The action of $\mathfrak{S}_m$ by automorphisms of the algebra $A$ then extends to an action by automorphisms of $\bar{A}$. For any $c = 1, \ldots, m - 1$ let $\sigma_c \in \mathfrak{S}_m$ be the transposition of $c$ and $c + 1$. Consider the image $\sigma_c(\bar{J})$, that is again a right ideal of $\bar{A}$. By [KN1 Proposition 3.2] we have

$$\sigma_c(\bar{J}) \subset \ker \bar{\xi}_c.$$

This allows us to define for any $c = 1, \ldots, m - 1$ a linear map

$$\bar{\xi}_c : \bar{J} \backslash \bar{A} \to \bar{J} \backslash \bar{A}$$

as the composition $\bar{\xi}_c \sigma_c$ applied to the elements of $\bar{A}$ taken modulo $\bar{J}$. The operators $\bar{\xi}_1, \ldots, \bar{\xi}_{m-1}$ on the vector space $\bar{J} \backslash \bar{A}$ are called the Zhelobenko operators. By [KN1 Proposition 3.3] the Zhelobenko operators $\bar{\xi}_1, \ldots, \bar{\xi}_{m-1}$ on $\bar{J} \backslash \bar{A}$ satisfy the braid relations

$$\bar{\xi}_c \bar{\xi}_{c+1} \bar{\xi}_c = \bar{\xi}_{c+1} \bar{\xi}_c \bar{\xi}_{c+1} \quad \text{for} \quad c < m - 1,$$

$$\bar{\xi}_b \bar{\xi}_c = \bar{\xi}_c \bar{\xi}_b \quad \text{for} \quad |b - c| > 1.$$ 

Therefore, for any reduced decomposition $\sigma = \sigma_{c_1} \ldots \sigma_{c_k}$ in $\mathfrak{S}_m$ the composition $\bar{\xi}_{c_1} \ldots \bar{\xi}_{c_k}$ of operators on $\bar{J} \backslash \bar{A}$ does not depend on the choice of decomposition of $\sigma$. Finally, for any $\sigma \in \mathfrak{S}_m$, $X \in \overline{U(\frak{h})}$, and $Y \in \bar{J} \backslash \bar{A}$ we have relations

$$\bar{\xi}_\sigma(XY) = (\sigma \circ X)\bar{\xi}_\sigma(Y)$$

(3.10)

$$\bar{\xi}_\sigma(YX) = \bar{\xi}_\sigma(Y)(\sigma \circ X)$$

(3.11)

which follow from [KN1 Proposition 3.1].
3.2. Intertwining properties

Let \( \delta = (\delta_1, \ldots, \delta_m) \) be a sequence of \( m \) elements from the set \( \{1, -1\} \). Let the symmetric group \( \mathfrak{S}_m \) act on the set of sequences \( \{\delta\} \) by

\[
\sigma(\delta) = \sigma \cdot \delta
\]

where

\[
\delta = (\delta_1, \ldots, \delta_p, -\delta_{p+1}, \ldots, -\delta_{p+q}) \quad \text{and} \quad \sigma \cdot \delta = (\delta_{\sigma^{-1}(1)}, \ldots, \delta_{\sigma^{-1}(m)}).
\]

Denote \( \delta^+ = (1, \ldots, 1) \) and \( \delta' = \delta^+ = (1, \ldots, 1, -1, \ldots, -1) \).

For a given sequence \( \delta \) let \( \varpi \) denote a composition of automorphisms of the ring \( \mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n) \) such that

\[
x_{ak} \mapsto -\theta \partial_{ak} \quad \text{and} \quad \partial_{ak} \mapsto x_{ak} \quad \text{whenever} \quad \delta_a = -1.
\]

(3.12)

For any \( \mathfrak{gl}_m \)-module \( V \) we define a bimodule \( E_{\delta}(V) \) over \( \mathfrak{gl}_m \) and \( Y(\mathfrak{gl}_n) \). Its underlying vector space is \( V \otimes \mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n) \) for every \( \delta \). The action of the algebra \( \mathfrak{gl}_m \) on the module \( E_{\delta}(V) \) is defined by pushing the homomorphism \( \xi \) forward through the automorphism \( \varpi \), applied to the second tensor factor \( \mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n) \) of the target of \( \alpha \)

\[
T_{ij}(u) \mapsto \delta_{ij} + \sum_{a,b=1}^{m} X_{ab}(u) \otimes \varpi(\hat{E}_{ai,bj}).
\]

For instance, we have \( E_{p,q}(V) = E_{\delta^+}(V) \).

Let \( \mu \in \mathfrak{h}^* \) be a generic weight of \( \mathfrak{gl}_m \), which means that

\[
\mu_a - \mu_b \not\in \mathbb{Z} \quad \text{for all} \quad a, b = 1, \ldots, m.
\]

(3.13)

In the remaining of this section we show that the Zhelobenko operator \( \tilde{\xi}_\sigma \) determines an intertwining operator

\[
E_{p,q}(M_\mu)_n \to E_{\delta}(M_{\sigma \mu})_n \quad \text{where} \quad \delta = \sigma(\delta^+).
\]

(3.14)

Let \( I_\delta \) be the left ideal of the algebra \( A \) generated by the elements \( x_{ak}, \ k = 1, \ldots, n, \) for \( \delta_a = -1 \) and by the elements \( \partial_{ak}, \ k = 1, \ldots, n, \) for \( \delta_a = 1 \). For instance, ideal \( I_{\delta^+} \) is generated by the elements \( \partial_{ak} \) with \( a = 1, \ldots, m \) and \( k = 1, \ldots, n \). Let \( \overline{I}_\delta \) be the left ideal of the algebra \( \overline{A} \) generated by the same elements as the ideal \( I_\delta \) in \( A \). Occasionally, \( \overline{I}_\delta \) will denote the image of the ideal \( I_\delta \) in the quotient space \( \overline{J} \setminus \overline{A} \).

Proposition 3.1 For any \( \sigma \in \mathfrak{S}_m \) the operator \( \tilde{\xi}_\sigma \) maps the subspace \( \overline{I}_{\delta^+} \) to \( \overline{I}_{\sigma(\delta^+)} \).

Proof: For all \( a = 1, \ldots, m - 1 \) consider the operator \( \hat{F}_a \). Due to (3.1), (3.2), and (1.19) we have

\[
\hat{F}_a(Y) = \sum_{k=1}^{n} [q_{a+1,k}d_{ak}, Y]
\]
for any $Y \in \mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$. The above description of the action of $\hat{F}_a$ with $a = 1, \ldots, m - 1$ on the vector space $\mathcal{HD}(\mathbb{C}^m \otimes \mathbb{C}^n)$ shows that this action preserves each of the two $2n$ dimensional subspaces spanned by the vectors

$$q_{ai} \quad \text{and} \quad q_{a+1i} \quad \text{where} \quad i = 1, \ldots, n; \quad (3.15)$$

$$p_{ai} \quad \text{and} \quad p_{a+1i} \quad \text{where} \quad i = 1, \ldots, n. \quad (3.16)$$

This action also maps to zero the $2n$ dimensional subspace spanned by

$$p_{ai} \quad \text{and} \quad q_{a+1i} \quad \text{where} \quad i = 1, \ldots, n. \quad (3.17)$$

Therefore, for any $\delta$ the operator $\hat{\xi}_a$ maps the ideal $\bar{I}_\delta$ of $\bar{A}$ to the image of $\bar{I}_\delta$ in $\bar{J} \setminus \bar{A}$ unless $\delta_a = \delta'_a$ and $\delta_{a+1} = -\delta'_{a+1}$ for $a = 1, \ldots, m - 1$. Hence, the operator $\hat{\xi}_a = \tilde{\xi}_a \sigma_a$ maps the subspace $\bar{I}_\delta$ to the image of $\bar{I}_\sigma(\delta)$ unless $\delta_a = -\delta'_a$ and $\delta_{a+1} = \delta'_{a+1}$.

From now on we will denote the image of the ideal $\bar{I}_\delta$ in the quotient space $\bar{J} \setminus \bar{A}$ by the same symbol $\bar{I}_\delta$. Put

$$\hat{\delta} = \sum_{a=1}^{p} \delta_a E^*_{aa} - \sum_{a=p+1}^{p+q} \delta_a E^*_{aa}. \quad (3.19)$$

Then for every $\sigma \in \mathfrak{S}_m$ we have the equality $\overline{\sigma(\delta)} = \sigma(\hat{\delta})$ where at the right hand side we use the action of the group $\mathfrak{S}_m$ on $\mathfrak{h}^*$. Let $(\ , \ )$ be the standard bilinear form on $\mathfrak{h}^*$ so that the basis of weights $E^*_{aa}$ with $a = 1, \ldots, m$ is orthonormal. The above remarks on the action of the Zhelobenko operators on $\bar{I}_\delta$ can now be rewritten as

$$\xi_a(\bar{I}_\delta) \subset \bar{I}_{\sigma_a(\delta)} \quad \text{if} \quad (\hat{\delta}, \eta_a) \geqslant 0. \quad (3.18)$$

We will prove the proposition by induction on the length of the reduced decomposition of $\sigma$. Recall that the length $\ell(\sigma)$ of a reduced decomposition of $\sigma$ is the total number of the factors $\sigma_1, \ldots, \sigma_{m-1}$ in that decomposition. This number is independent of the choice of decomposition and is equal to the number of elements in the set

$$\Delta_\sigma = \{ \eta \in \Delta^+ \mid \sigma(\eta) \notin \Delta^+ \}$$

where $\Delta^+$ denotes the set of all positive roots of the Lie algebra $\mathfrak{gl}_m$.

If $\sigma$ is the identity element of $\mathfrak{S}_m$ then the statement of the proposition is trivial. Suppose that for some $\sigma \in \mathfrak{S}_m$

$$\xi_{\sigma}(\bar{I}_{\delta^+}) \subset \bar{I}_{\sigma(\delta^+)}. \quad (3.18)$$

Take $\sigma_a$ such that

$$\ell(\sigma_a \sigma) = \ell(\sigma) + 1. \quad (3.19)$$

Now it is only left to prove that

$$\xi_a(\bar{I}_{\sigma(\delta^+)} \subset \bar{I}_{\sigma_a(\delta)}. \quad (3.19)$$

Due to (3.18), the desired property will take place if

$$\langle \overline{\sigma(\delta^+)}, \eta_a \rangle = \langle \sigma(\hat{\delta}^+), \eta_a \rangle \geqslant 0. \quad (3.19)$$

Note that $\eta_a$ is a simple root of the algebra $\mathfrak{gl}_m$ and hence, $\sigma_a(\eta) \in \Delta^+$ for any $\eta \in \Delta^+$ such that $\eta \neq \eta_a$. Since $\ell(\sigma)$ and $\ell(\sigma_a \sigma)$ are the numbers of elements in $\Delta_\sigma$ and $\Delta_{\sigma_a \sigma}$ respectively,
condition (3.19) implies that \( \eta_a \in \sigma(\Delta^+) \). Therefore, \( \eta_a = \sigma(E_{bb}^* - E_{cc}^*) \) for some \( 1 \leq b < c \leq m \). Thus
\[
\left( \sigma(\delta^+) , \eta_a \right) = \left( \sigma(\delta^+) , \sigma(E_{bb}^* - E_{cc}^*) \right) = \left( \sum_{a=1}^{m} \delta_a E_{aa}^* , E_{bb}^* - E_{cc}^* \right) \geq 0.
\]
\( \square \)

Then following [KN3, Corollary 5.2] we obtain

**Corollary 3.2** For any \( \sigma \in S_m \) the operator \( \tilde{\xi}_\sigma \) on \( \bar{J} \setminus \bar{A} \) maps
\[
\bar{J} \setminus (\bar{J} + I_{\delta^+} + \bar{J}) \to \bar{J} \setminus (\bar{J} + I_{\sigma(\delta^+)} + \bar{J}).
\]

For a generic weight \( \mu \) let \( I_{\mu,\delta} \) be the left ideal of the algebra \( A \) generated by \( I_{\delta} + J' \) and by the elements
\[
E_{aa} - \zeta_n(E_{aa}) - \mu_a \quad \text{where} \quad a = 1, \ldots, m.
\]
Recall that under the isomorphism of the algebra \( A \) with \( U(\mathfrak{gl}_m) \otimes \mathcal{H}D (\mathbb{C}^m \otimes \mathbb{C}^n) \) element \( X - \zeta_n(X) \in A \) maps to the element (3.6) for every \( X \in \mathfrak{gl}_m \). Let \( I_{\mu,\delta} \) denote the subspace \( \overline{(\mathfrak{h})I_{\mu,\delta}} \) of \( \bar{A} \). Note that \( I_{\mu,\delta} \) is a left ideal of the algebra \( \bar{A} \).

**Theorem 3.3** For any element \( \sigma \in S_m \) the operator \( \tilde{\xi}_\sigma \) on \( \bar{J} \setminus \bar{A} \) maps
\[
\bar{J} \setminus (\bar{I}_{\mu,\delta^+} + \bar{J}) \to \bar{J} \setminus (\bar{I}_{\sigma(\delta^+)} + \bar{J}).
\]

**Proof:** Let \( \kappa \) be a weight of \( \mathfrak{gl}_m \) with the sequence of labels \( (\kappa_1, \ldots, \kappa_m) \). Suppose that the weight \( \kappa \) satisfies the conditions (3.13) instead of \( \mu \). Let \( \bar{I}_{\mu,\delta} \) denote the left ideal of \( \bar{A} \) generated by \( I_{\delta} + J' \) and by the elements
\[
E_{aa} - \kappa_a \quad \text{where} \quad a = 1, \ldots, m.
\]
Relation (3.11) and Corollary 3.2 imply that the operator \( \tilde{\xi}_\sigma \) on \( \bar{J} \setminus \bar{A} \) maps
\[
\bar{J} \setminus (\bar{I}_{\mu,\delta^+} + \bar{J}) \to \bar{J} \setminus (\bar{I}_{\sigma(\delta^+)} + \bar{J}).
\]

Now choose
\[
\kappa = \mu - \frac{n}{2} \delta^\prime \tag{3.20}
\]
where the sequence \( \delta^\prime \) is regarded as a weight of \( \mathfrak{gl}_m \) by identifying the weights with their sequence of labels. Then the conditions on \( \kappa \) stated in the beginning of this proof are satisfied. For every \( \sigma \in S_m \) we shall prove the equality of left ideals of \( \bar{A} \),
\[
\bar{I}_{\sigma(\delta^+)\sigma(\delta^+)} = \bar{I}_{\bar{I}_{\sigma(\delta^+)} + \bar{J}}. \tag{3.21}
\]
Theorem 3.3 will then follow. Denote \( \delta = \sigma \cdot \delta^\prime \). Then by our choice of \( \kappa \), we have
\[
\sigma \circ \kappa = \sigma \circ \mu - \frac{n}{2} \delta.
\]

Let the index \( a \) run through \( 1, \ldots, m-1 \), then
\[
\zeta_n(E_{aa}) + \theta \frac{n}{2} = \theta \sum_{k=1}^{n} p_k q_k a_k \in I_{\sigma(\delta^+)} \quad \text{if} \quad \delta_a = 1,
\]
\[
\zeta_n(E_{aa}) - \theta \frac{n}{2} = \sum_{k=1}^{n} q_k p_k a_k \in I_{\sigma(\delta^+)} \quad \text{if} \quad \delta_a = -1.
\]
Consider the quotient vector space $A / I_{\mu,\delta}$ for any sequence $\delta$. The algebra $U(\mathfrak{g}_m)$ acts on this quotient via left multiplication, being regarded as a subalgebra of $A$. The algebra $Y(\mathfrak{g}_n)$ also acts on this quotient via left multiplication, using the homomorphism $\alpha_m : Y(\mathfrak{g}_n) \to A$. Recall that in Section 1 the target algebra of the homomorphism $\alpha_m$ was defined as the tensor product $U(\mathfrak{g}_m) \otimes \mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n)$ isomorphic to the algebra $A$ by means of the cross relations (3.1). Part (ii) of the proposition 1.2 implies that the image of $\alpha_m$ in $A$ commutes with the subalgebra $U(\mathfrak{g}_m) \subset A$. Thus, the vector space $A / I_{\mu,\delta}$ becomes a bimodule over $\mathfrak{g}_m$ and $Y(\mathfrak{g}_n)$.

Consider the bimodule $E_\delta(M_\mu)$ over $\mathfrak{g}_m$ and $Y(\mathfrak{g}_n)$ defined in the beginning of this section. This bimodule is isomorphic to $A / I_{\mu,\delta}$. Indeed, let $Z$ run through $\mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n)$. Then a bijective linear map

$$E_\delta(M_\mu) \to A / I_{\mu,\delta},$$

intertwining the actions of $\mathfrak{g}_m$ and $Y(\mathfrak{g}_n)$, can be determined by mapping the element

$$1_\mu \otimes Z \in M_\mu \otimes \mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n)$$

to the image of

$$\varpi^{-1}(Z) \in \mathcal{H}(\mathbb{C}^m \otimes \mathbb{C}^n) \subset A$$

in the quotient $A / I_{\mu,\delta}$, where $\varpi$ is defined by (3.12). The intertwining property here follows from the definitions of $E_\delta(M_\mu)$ and $I_{\mu,\delta}$. The same mapping determines a bijective linear map

$$E_\delta(M_\mu) \to \bar{A} / \bar{I}_{\mu,\delta}. \quad (3.23)$$

In particular, the space $E_\delta(M_\mu)_n$ of $n$-coinvariants of $E_\delta(M_\mu)$ is isomorphic to the quotient $\bar{J} \setminus \bar{A} / \bar{I}_{\mu,\delta}$ as a bimodule over the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_m$ and over $Y(\mathfrak{g}_n)$. Theorem 3.3 implies that the operator $\xi_\sigma$ on $\bar{J} \setminus \bar{A}$ determines a linear operator

$$\bar{J} \setminus \bar{A} / \bar{I}_{\mu,\delta^+} \to \bar{J} \setminus \bar{A} / \bar{I}_{\sigma \mu,\sigma(\delta^+)} \quad (3.24)$$

The definition (3.5) and the fact that the image of $Y(\mathfrak{g}_n)$ in $A$ under $\alpha_m$ commutes with the subalgebra $U(\mathfrak{g}_m) \subset A$ imply that the latter operator intertwines the actions of $Y(\mathfrak{g}_n)$ on the source and the target vector spaces. We also use the invariance of the image of $Y(\mathfrak{g}_n)$ in $A$ under the action of $\mathfrak{g}_m$. Recall that $E_{p,q}(V) = E_{\delta^+}(V)$. Hence, by using the equivalences (3.23) for the sequences $\delta = \delta^+$ and $\delta = \sigma(\delta^+)$, the operator (3.24) becomes the desired operator $Y(\mathfrak{g}_n)$-intertwining operator (3.14).

As usual, for any $\mathfrak{g}_m$-module $V$ and any element $\lambda \in \mathfrak{h}^*$ let $V^\lambda \subset V$ be the subspace of vectors of weight $\lambda$ so that any $X \in \mathfrak{h}$ acts on $V^\lambda$ via multiplication by $\lambda(X) \in \mathbb{C}$. It now follows from the property (3.10) of $\xi_\sigma$ that the restriction of our operator (3.14) to the subspace of weight $\lambda$ is an $Y(\mathfrak{g}_n)$-intertwining operator

$$E_{p,q}(M_\mu)^\lambda_n \to E_{\delta}(M_{\sigma \mu}^\mu)_n^{\sigma \lambda} \quad \text{where} \quad \delta = \sigma(\delta^+). \quad (3.25)$$

Consider $Y(\mathfrak{g}_n)$-modules $\Phi_z$ and $\Phi_{\bar{z}}$ described at the end of the Section 2. The underlying vector space of each of these modules coincides with the algebra $\mathcal{H}(\mathbb{C}^n)$. Note that the action of $Y(\mathfrak{g}_n)$ on each of these modules preserves the polynomial degree. Now for any $N = 1, 2, \ldots$ denote respectively by $\Psi_z^N$ and $\Psi_{\bar{z}}^N$ the submodules in $\Phi_z$ and $\Phi_{\bar{z}}$ consisting of the polynomial functions of degree $N$. It will also be convenient to denote by $\Psi_0^0$ the vector space $\mathbb{C}$ with the trivial action of $Y(\mathfrak{g}_n)$. 

Hence, the relation (3.22) implies the equality (3.21). 

\[ \square \]
The element \( E_{11} \) acts on \( E_{1,0}(M_t) \) and \( E_{0,1}(M_t) \) by

\[
t - \theta \frac{n}{2} - \text{deg} \quad \text{and} \quad t + \theta \frac{n}{2} + \text{deg}
\]

respectively where \( \text{deg} \) is the degree operator. Therefore, corollary 2.2 yields the isomorphism between the source \( Y(\mathfrak{gl}_n) \)-module in (3.25) and the tensor product

\[
\Psi^{-\nu_1}_{\mu_1+p_1} \otimes \cdots \otimes \Psi^{-\nu_p}_{\mu_p+p_p} \otimes \Psi^{\nu_{p+1}}_{\mu_{p+1}+\rho_{p+1}} \otimes \cdots \otimes \Psi^{\nu_m}_{\mu_m+\rho_m}
\]  

(3.26)

where

\[
\nu_a = -\lambda_a + \mu_a - \theta \frac{n}{2} \quad \text{for} \quad a = 1, \ldots, p,
\]

\[
\nu_a = \lambda_a - \mu_a - \theta \frac{n}{2} \quad \text{for} \quad a = p + 1, \ldots, m.
\]  

(3.27)

Let us now consider the target \( Y(\mathfrak{gl}_n) \)-module in (3.25). For each \( a = 1, \ldots, m \) denote

\[
\tilde{\mu}_a = \mu_{\sigma^{-1}(a)}, \quad \tilde{\nu}_a = \nu_{\sigma^{-1}(a)}, \quad \tilde{\rho}_a = \rho_{\sigma^{-1}(a)}.
\]

The above description of the source \( Y(\mathfrak{gl}_n) \)-module in (3.25) can now be generalized to the target \( Y(\mathfrak{gl}_n) \)-module which depends on an arbitrary element \( \sigma \in S_m \).

**Proposition 3.4** For \( \delta = \sigma(\delta^+) \) the \( Y(\mathfrak{gl}_n) \)-module \( E_{\delta}(M_{\sigma\mu})^\sigma \) is isomorphic to the tensor product

\[
\Psi^{-\delta_1 \tilde{\nu}_1}_{\tilde{\mu}_1+p_1} \otimes \cdots \otimes \Psi^{-\delta_p \tilde{\nu}_p}_{\tilde{\mu}_p+p_p} \otimes \Psi^{\delta_{p+1} \tilde{\nu}_{p+1}}_{\tilde{\mu}_{p+1}+\tilde{\rho}_{p+1}} \otimes \cdots \otimes \Psi^{\delta_m \tilde{\nu}_m}_{\tilde{\mu}_m+\tilde{\rho}_m}.
\]  

(3.28)

**Proof:** Consider the bimodule \( E_{p,q}(M_{\sigma\mu})^\sigma \) over \( \mathfrak{h} \) and \( Y(\mathfrak{gl}_n) \). By Corollary 2.2 and the arguments just above this proposition, this bimodule is isomorphic to the tensor product

\[
\Psi^{-\tilde{\nu}_1}_{\tilde{\mu}_1+p_1} \otimes \cdots \otimes \Psi^{-\tilde{\nu}_p}_{\tilde{\mu}_p+p_p} \otimes \Psi^{\tilde{\nu}_{p+1}}_{\tilde{\mu}_{p+1}+\tilde{\rho}_{p+1}} \otimes \cdots \otimes \Psi^{\tilde{\nu}_m}_{\tilde{\mu}_m+\tilde{\rho}_m}
\]  

(3.29)

as a \( Y(\mathfrak{gl}_m) \)-module. The bimodule \( E_\delta(M_{\sigma\mu}) \) can be obtained by pushing forward the actions of \( \mathfrak{h} \) and \( Y(\mathfrak{gl}_n) \) on \( E_{p,q}(M_{\sigma\mu}) \) through the composition of automorphisms

\[
x_a \rightarrow -\theta \partial_a \quad \text{and} \quad \partial_a \rightarrow x_a
\]  

(3.30)

for every tensor factor with number \( a \) such that \( \delta_a = -1 \). These automorphisms exchange \( Y(\mathfrak{gl}_n) \)-modules \( \Phi_{\tilde{z}_a} \) and \( \tilde{\Phi}_{\tilde{z}_a} \) but leave invariant the degree of polynomials. Therefore, they interchange \( \Psi^N_{\tilde{z}_a} \) and \( \Psi^{-N}_{\tilde{z}_a} \) which implies the resulting \( Y(\mathfrak{gl}_n) \)-module to be as in (3.28). \( \square \)

Thus, for any non-negative integers \( \nu_1, \ldots, \nu_m \) we have shown that the Zhelobenko operator \( \xi_\sigma \) on \( \hat{J} \backslash \hat{A} \) defines the intertwining operator between the \( Y(\mathfrak{gl}_n) \)-modules

\[
\Psi^{-\nu_1}_{\mu_1+p_1} \otimes \cdots \otimes \Psi^{-\nu_p}_{\mu_p+p_p} \otimes \Psi^{\nu_{p+1}}_{\mu_{p+1}+\rho_{p+1}} \otimes \cdots \otimes \Psi^{\nu_m}_{\mu_m+\rho_m}
\]

and

\[
\Psi^{-\delta_1 \nu_1}_{\delta_1 \mu_1+p_1} \otimes \cdots \otimes \Psi^{-\delta_p \nu_p}_{\delta_p \mu_p+p_p} \otimes \Psi^{\delta_{p+1} \nu_{p+1}}_{\delta_{p+1} \mu_{p+1}+\delta \rho_{p+1}} \otimes \cdots \otimes \Psi^{\delta_m \nu_m}_{\delta_m \mu_m+\delta \rho_m}.
\]

Moreover, the operator \( \xi_\sigma \) permutes tensor factors of the module (3.26), therefore modules \( E_{p,q}(M_{\rho})^\lambda \) and \( E_\delta(M_{\sigma\mu})^\sigma \), written in the form (2.26), contain similar tensor factors \( \Omega_{\tilde{z}_a} \) or \( \Omega_{\tilde{z}_a}' \). Now recall that modules \( \Omega_{\tilde{z}_a} \) and \( \Omega_{\tilde{z}_a}' \) are cocentral and that the Zhelobenko operator \( \xi_\sigma \) acts
on them as identity operator. These two observations allow us to exclude \( \Omega_{za} \) and \( \Omega'_{za} \) from both source and target modules of the mapping (3.35) and to define an intertwining operator \( \xi'_\sigma \) between tensor products of modules \( \Phi_2 \) and \( \Phi'_2 \). For any integer \( N \) denote respectively by \( \Phi^N_{za} \) and \( \Phi'^N_{za} \) the submodules of \( \Phi_{za} \) and \( \Phi'_{za} \) consisting of polynomial functions of degree \( N \). Note that \( \Phi^N_{za} \) coincides with \( \Psi^N_{za} \) for positive \( N \) but differs for negative \( N \). Hence, the following theorem holds.

**Theorem 3.5** Given a generic weight \( \mu \) and non-negative integers \( \nu_1, \ldots, \nu_m \), the map \( \xi'_\sigma \) intertwines rational \( \mathfrak{gl}_n \)-modules

\[
\Phi^{-\nu_1}_{\mu_1 + \rho_1} \otimes \cdots \otimes \Phi^{-\nu_p}_{\mu_p + \rho_p} \otimes \Phi^{\nu_{p+1}}_{\mu_{p+1} + \rho_{p+1}} \otimes \cdots \otimes \Phi^{\nu_m}_{\mu_m + \rho_m} \tag{3.31}
\]

and

\[
\Phi^{-\delta_1 \tilde{\nu}_1}_{\mu_1 + \rho_1} \otimes \cdots \otimes \Phi^{-\delta_p \tilde{\nu}_p}_{\mu_p + \rho_p} \otimes \Phi^{\delta_{p+1} \tilde{\nu}_{p+1}}_{\mu_{p+1} + \rho_{p+1}} \otimes \cdots \otimes \Phi^{\delta_m \tilde{\nu}_m}_{\mu_m + \rho_m}.
\]

4. Highest weight vectors

4.1. Symmetric case

In this subsection we consider only the case of commuting variables, hence from now on and till the end of the subsection we assume \( \theta = 1 \). Proposition 4.4 below determines the image of the highest weight vector \( v^\lambda_{\mu} \) of the \( \mathfrak{gl}_n \)-module \( \mathcal{E}_{E_{(q)}(M_{\mu})} \) under the action of the operator \( \xi_\sigma \).

The proof of the Proposition 4.4 is based on the following three lemmas. Proofs of the first two of them are similar to the proof of Lemma 5.6 in [KN3]. Proof of the last one is similar to the proof of Lemma 5.7 in [KN3]. Let \( s, t = 0, 1, 2 \ldots \) and \( k, \ell = 1, \ldots, n \).

**Lemma 4.1** For any \( a = 1, \ldots, m - 1 \) the operator \( \xi_a \) on \( \bar{J} \backslash \bar{A} \) maps the image in \( \bar{J} \backslash \bar{A} \) of \( p^a_{ak}p_{a+1}^k \in \bar{A} \) to the image in \( \bar{J} \backslash \bar{A} \) of the product

\[
p^t_{atk}p^s_{a+1} = \prod_{r=1}^s \frac{H_a + r + 1}{H_a + r - t}
\]

plus the images in \( \bar{J} \backslash \bar{A} \) of elements of the left ideal in \( \bar{A} \) generated by \( \bar{J}' \) and (3.15).

**Lemma 4.2** For any \( a = 1, \ldots, m - 1 \) the operator \( \xi_a \) on \( \bar{J} \backslash \bar{A} \) maps the image in \( \bar{J} \backslash \bar{A} \) of \( q^a_{atk}q_{a+1}^k \in \bar{A} \) to the image in \( \bar{J} \backslash \bar{A} \) of the product

\[
q^t_{atk}q^s_{a+1} = \prod_{r=1}^t \frac{H_a + r + 1}{H_a + r - s}
\]

plus the images in \( \bar{J} \backslash \bar{A} \) of elements of the left ideal in \( \bar{A} \) generated by \( \bar{J}' \) and (3.16).

**Lemma 4.3** For any \( a = 1, \ldots, m - 1 \) the operator \( \xi_a \) on \( \bar{J} \backslash \bar{A} \) maps the image in \( \bar{J} \backslash \bar{A} \) of \( p^a_{atk}q_{a+1}^k \in \bar{A} \) to the image in \( \bar{J} \backslash \bar{A} \) of the product

\[
q^t_{atk}p^s_{a+1} = \begin{cases} \prod_{r=1}^s \frac{H_a + r}{H_a + r + t} & \text{if } n = 1 \text{ and } k = \ell = 1, \\ 1 & \text{if } n > 1 \text{ and } k \neq \ell \end{cases}
\]

plus the images in \( \bar{J} \backslash \bar{A} \) of elements of the left ideal in \( \bar{A} \) generated by \( \bar{J}' \) and (3.17).
We keep assuming that the weight $\mu$ satisfies the condition (3.11). Let $(\mu_1^*, \ldots, \mu_m^*)$ be the sequence of labels of weight $\mu + \rho - \frac{r}{2} \delta'$. Suppose that for all $a = 1, \ldots, m$ the number $\nu_a$ defined by (3.27) is a non-negative integer. For each positive root $\eta = E_{bb}^* - E_{cc}^* \in \Delta^+$ with $1 \leq b < c \leq m$ define a number $z_\eta \in \mathbb{C}$ by

$$z_\eta = \begin{cases} 
\prod_{r=1}^{\nu_b} \frac{\mu_b^* - \mu_c^* - r}{\lambda_b^* - \lambda_c^* + r} & \text{if } b, c = 1, \ldots, p, \\
\prod_{r=1}^{\nu_c} \frac{\mu_b^* - \mu_c^* - r}{\lambda_b^* - \lambda_c^* + r} & \text{if } b, c = p + 1, \ldots, m, \\
\prod_{r=1}^{\nu} \frac{\mu_b^* - \mu_c^* - r + 1}{\lambda_b^* - \lambda_c^* + r - 1} & \text{if } b = 1, \ldots, p, \quad c = p + 1, \ldots, m, \quad \text{and } n = 1, \\
1, & \text{if } b = 1, \ldots, p, \quad c = p + 1, \ldots, m, \quad \text{and } n > 1.
\end{cases}$$

Here $\nu = \min(\nu_b, \nu_c)$. Let $v^\lambda_\mu$ denote the image of the vector

$$\prod_{a=1}^{p} p_{aa}^{\nu_a} \prod_{a=p+1}^{m} q_{aa}^{\nu_a} \in \tilde{A}$$

in the quotient space $\bar{J} \backslash \tilde{A} / \bar{I}_{\mu, \delta^+}$.

**Proposition 4.4**

(i) The vector $v^\lambda_\mu$ does not belong to the zero coset in $\bar{J} \backslash \tilde{A} / \bar{I}_{\mu, \delta^+}$.

(ii) The vector $v^\lambda_\mu$ is of weight $\lambda$ under the action of $\mathfrak{h}$ on $\bar{J} \backslash \tilde{A} / \bar{I}_{\mu, \delta^+}$ and is of highest weight with respect to the action of $Y(\mathfrak{gl}_n)$ on the same quotient space.

(iii) For any $\sigma \in \mathfrak{S}_m$ the operator (3.23) determined by $\xi_\sigma$ maps the vector $v^\lambda_\mu$ to the image in $\bar{J} \backslash \tilde{A} / \bar{I}_{\sigma \mu, \sigma(\delta^+)}$ of $\sigma(v^\lambda_\mu) \in \tilde{A}$ multiplied by $\prod_{\eta \in \Delta_\sigma} z_\eta$.

**Proof:** Part (i) follows directly from the definition of the ideal $\bar{I}_{\mu, \delta^+}$. Let us now prove Part (ii). Subalgebra $\mathfrak{h}$ acts on the quotient space $\bar{J} \backslash \tilde{A} / \bar{I}_{\mu, \delta^+}$ via left multiplication on $\tilde{A}$. Let symbol $\equiv$ denote the equalities in $\tilde{A}$ modulo the left ideal $\bar{I}_{\mu, \delta^+}$. Then we have

$$E_{aa} v^\lambda_\mu = v^\lambda_\mu E_{aa} + \left[ \zeta_n(E_{aa}), v^\lambda_\mu \right] = v^\lambda_\mu (E_{aa} \mp \nu_a) \equiv v^\lambda_\mu (\zeta_n(E_{aa}) + \mu_a \mp \nu_a) \equiv v^\lambda_\mu \left( \mp \frac{n}{2} + \mu_a \mp \nu_a \right) = \lambda_a v^\lambda_\mu$$

where one should choose the upper sign for $a = 1, \ldots, p$ and the lower sign for $a = p + 1, \ldots, m$. Next, $Y(\mathfrak{gl}_n)$-modules $\bar{J} \backslash \tilde{A} / \bar{I}_{\mu, \delta^+}$ and $\mathcal{E}_{p,q}(M_\mu)_{n}$ are isomorphic. Recall that $T_{ij}(u)$ acts on $\mathcal{E}_{p,q}(M_\mu)_{n}$ with the help of the comultiplication $\Delta$. Now Corollary 2.2 and formulas 2.3 imply that $T_{ij}(u) v^\lambda_\mu = 0$ for all $1 \leq i < j \leq n$. Moreover, one can check that

$$T_{ii}(u) v^\lambda_\mu = v^\lambda_\mu \cdot \begin{cases} 
\prod_{a=p+1}^{m} \frac{u + \mu_a + \rho_a + \nu_a}{u + \mu_a + \rho_a}, & i = 1 \\
\prod_{a=1}^{p} \frac{u + \mu_a + \rho_a - \nu_a - 1}{u + \mu_a + \rho_a}, & i = n \\
1, & \text{otherwise}.
\end{cases}$$
Therefore, \( v^\lambda_\mu \) is a highest weight vector with respect to the action of \( Y(\mathfrak{g}_m) \) on the quotient space \( \hat{J} \setminus \mathbb{A}/\mathfrak{I}_{\mu, \delta^+} \).

We will prove Part (iii) by induction on the length of a reduced decomposition of \( \sigma \). If \( \sigma \) is the identity element of \( \mathfrak{S}_m \), then the required statement is tautological. Now suppose that for some \( \sigma \in \mathfrak{S}_m \) the statement of (iii) is true. Take any simple reflection \( \sigma_a \in \mathfrak{S}_m \) with \( 1 \leq a \leq m - 1 \) such that \( \sigma_a \sigma \) has a longer reduced decomposition in terms of \( \sigma_1, \ldots, \sigma_m \) than \( \sigma \).

Consider the simple root \( \eta_a \), corresponding to the reflection \( \sigma_a \). Let \( \eta = \sigma^{-1}(\eta_a) \) then

\[
\sigma_a \sigma(\eta) = \sigma_a(\eta_a) = -\eta_a \notin \Delta^+.
\]

Thus, \( \Delta_{\sigma_a \sigma} = \Delta_\sigma \cup \{ \eta \} \). Let \( \kappa \in \mathfrak{h}^* \) be the weight with the labels determined by \( \mu, \delta \). Using the proof of Theorem \( \text{3.3} \) we get the equality of two left ideals of the algebra \( A \),

\[
\overline{I}_{(\sigma_a \sigma) \circ \mu, (\sigma_a \sigma)(\delta^+)} = \overline{I}_{(\sigma_a \sigma) \circ \mu, (\sigma_a \sigma)(\delta^+)}.
\]

Modulo the second of these two ideals the element \( H_a \) equals

\[
((\sigma_a \sigma) \circ \kappa)(H_a) = (\sigma_a(\kappa + \rho) - \rho)(H_a) = (\kappa + \rho)(\sigma^{-1}(\sigma_a(H_a))) - \rho(H_a) = - (\kappa + \rho)(\sigma^{-1}(H_a)) - 1 = - (\kappa + \rho)(H_\eta) - 1 = - \mu_b^* + \mu_c^* - 1.
\]

Here \( H_\eta = \sigma^{-1}(H_a) \) is the coroot corresponding to the root \( \eta \), and we use the standard bilinear form on \( \mathfrak{h}^* \).

Let us use the statement of (iii) as the induction assumption. Denote \( \delta = \sigma(\delta^+) \). Consider three cases.

I. Let \( b, c = 1, \ldots, p \), then \( \delta_a = \delta'_a \) and \( \delta_a + 1 = \delta'_a + 1 \). Hence,

\[
\sigma(v^\lambda_\mu) = p^e a_n p^d a_{n+1} Y
\]

where \( Y \) is an element of the subalgebra of \( \mathcal{P}_D(\mathbb{C}^m \otimes \mathbb{C}^n) \) generated by all \( x_\delta k \) and \( \partial_\delta k \) with \( d \neq a, a + 1 \). Now we apply the Lemma \( \text{4.1} \) with \( s = v_b \) and \( t = v_c \). After substitution \( - \mu_b^* + \mu_c^* - 1 \) for \( H_a \), the fraction in the lemma turns into

\[
\prod_{r=1}^{v_b} \frac{- \mu_b^* + \mu_c^* + r}{- \mu_b^* + \mu_c^* + v_b - v_c - r} = \prod_{r=1}^{v_b} \frac{\lambda_b^* - \lambda_c^* + r}{\lambda_b^* - \lambda_c^* + r}.
\]

II. Let \( b, c = p + 1, \ldots, m \), then \( \delta_a = - \delta'_a \) and \( \delta_a + 1 = - \delta'_a + 1 \). Hence,

\[
\sigma(v^\lambda_\mu) = q_a^n q_{a+1} Y
\]

where \( Y \) is an element of the subalgebra of \( \mathcal{P}_D(\mathbb{C}^m \otimes \mathbb{C}^n) \) generated by all \( x_\delta k \) and \( \partial_\delta k \) with \( d \neq a, a + 1 \). Now we apply the Lemma \( \text{4.2} \) with \( s = v_b \) and \( t = v_c \). After substitution \( - \mu_b^* + \mu_c^* - 1 \) for \( H_a \), the fraction in the lemma turns into

\[
\prod_{r=1}^{v_c} \frac{- \mu_b^* + \mu_c^* + r}{- \mu_b^* + \mu_c^* + v_c - v_b - r} = \prod_{r=1}^{v_c} \frac{\lambda_b^* - \lambda_c^* + r}{\lambda_b^* - \lambda_c^* + r}.
\]

III. Let \( b = 1, \ldots, p, c = p + 1, \ldots, m \), then \( \delta_a = \delta'_a \) and \( \delta_a + 1 = - \delta'_a + 1 \). Hence,

\[
\sigma(v^\lambda_\mu) = p^e a_n q_{a+1} Y
\]
where \( Y \) is an element of the subalgebra of \( \mathcal{PD}(\mathbb{C}^m \otimes \mathbb{C}^n) \) generated by all \( x_{dk} \) and \( \partial_{dk} \) with \( d \neq a, a + 1 \). Now we apply the Lemma 4.3 with \( s = \nu_b \) and \( t = \nu_c \). When \( n = 1 \), after substitution \( -\mu^*_b + \mu^*_c - 1 \) for \( H_a \), the fraction in the lemma turns into

\[
\prod_{r=1}^{\nu} \frac{-\mu^*_b + \mu^*_c + r - 1}{-\mu^*_b + \mu^*_c + \nu_b + \nu_c - r} = \prod_{r=1}^{\nu} \frac{\mu^*_b - \mu^*_c - r + 1}{\lambda^*_b - \lambda^*_c + r - 1}.
\]

Thus, in the three cases considered above the operator

\[
\tilde{\xi}_{\sigma_a \sigma} : \bar{J} \setminus \bar{A} / \bar{I}_{\mu, \delta^+} \to \bar{I}_{(\sigma_a \sigma) \circ \mu, (\sigma_a \sigma)(\delta^+)}
\]

maps the vector \( v^\lambda_{\mu} \) to the image of

\[
\sigma_a \sigma(v^\lambda_{\mu}) \cdot \prod_{\eta \in \Delta_{\sigma_a \sigma}} z_{\eta} \in \bar{A}
\]

in the vector space \( \bar{J} \setminus \bar{A} / \bar{I}_{(\sigma_a \sigma) \circ \mu, (\sigma_a \sigma)(\delta^+)} \). This observation makes the induction step. \( \square \)

### 4.2. Skew-symmetric case

In this subsection we consider only the case of anticommuting variables, hence from now on and till the end of this subsection we assume \( \theta = -1 \). Proposition 4.8 below determines the image of the highest weight vector \( v^\lambda_{\mu} \) of the \( Y(\mathfrak{gl}_n) \)-module \( E_{p,q}(M_{\mu})^\lambda \) under the action of the operator \( \tilde{\xi}_\sigma \). The proof of the Proposition 4.8 is based on the following three lemmas. Proofs of the first two of them are similar to the proof of Lemma 5.6 in [KN4]. Proof of the last one is similar to the proof of Lemma 5.7 in [KN4]. Let \( s, t = 1, \ldots, n \), define

\[
f_{as} = p_{an-s+1} \cdots p_{an} \quad \text{and} \quad g_{as} = q_{a1} \cdots q_{as}.
\]

#### Lemma 4.5

For any \( a = 1, \ldots, m - 1 \) the operator \( \tilde{\xi}_a \) on \( \bar{J} \setminus \bar{A} \) maps the image in \( \bar{J} \setminus \bar{A} \) of \( f_{as} f_{a+1} t \in \bar{A} \) to the image in \( \bar{J} \setminus \bar{A} \) of the product

\[
f_{a+1} s f_{at} = \begin{cases} H_a + s - t + 1 & \text{if } s > t, \\ H_a + 1 & \text{if } s \leq t \end{cases}
\]

plus the images in \( \bar{J} \setminus \bar{A} \) of elements of the left ideal in \( \bar{A} \) generated by \( \bar{J}' \) and (3.15).

#### Lemma 4.6

For any \( a = 1, \ldots, m - 1 \) the operator \( \tilde{\xi}_a \) on \( \bar{J} \setminus \bar{A} \) maps the image in \( \bar{J} \setminus \bar{A} \) of \( g_{as} g_{a+1} t \in \bar{A} \) to the image in \( \bar{J} \setminus \bar{A} \) of the product

\[
g_{a+1} s g_{at} = \begin{cases} H_a + t - s + 1 & \text{if } s < t, \\ H_a + 1 & \text{if } s \geq t \end{cases}
\]

plus the images in \( \bar{J} \setminus \bar{A} \) of elements of the left ideal in \( \bar{A} \) generated by \( \bar{J}' \) and (3.16).
**Lemma 4.7** For any \( a = 1, \ldots, m - 1 \) the operator \( \xi_a \) on \( J / \tilde{A} \) maps the image in \( J \setminus \tilde{A} \) of \( f_a g_a + g_a f_a \in \tilde{A} \) to the image in \( J \setminus \tilde{A} \) of the product

\[
\left( f_a g_a + g_a f_a \right) : \begin{cases} 
\frac{H_a + s + t - n + 1}{H_a + 1} & \text{if } s + t > n, \\
1 & \text{if } s + t \leq n.
\end{cases}
\]

plus the images in \( J \setminus \tilde{A} \) of elements of the left ideal in \( \tilde{A} \) generated by \( J' \) and \( B_{\sigma(\lambda)} \).

We keep assuming that the weight \( \mu \) satisfies the condition \( B_{\sigma(\lambda)} \). We also assume that \( \nu_1, \ldots, \nu_m \in \{0, 1, \ldots, n\} \).

Set

\[
\nu'_a = \begin{cases} 
n - \nu_a & \text{if } a = 1, \ldots, p, \\
\nu_a & \text{if } a = p + 1, \ldots, m.
\end{cases}
\]

Let \( \mu_1^*, \ldots, \mu_m^* \) be the sequence of labels of weight \( \mu + \rho + \frac{b}{2} \gamma' \). For each positive root \( \eta = E_{ab}^* - E_{cc}^* \in \Delta^+ \) with \( 1 \leq b < c \leq m \) define a number \( z_\eta \in \mathbb{C} \) by

\[
z_\eta = \begin{cases} 
\frac{\mu_b^* - \mu_c^*}{\mu_b^* \mu_c^*} & \text{if } \nu'_b < \nu'_c, \\
1 & \text{otherwise}.
\end{cases}
\]

Let \( v_\mu^\lambda \) denote the image of the vector

\[
\prod_{a=1}^p f_{\nu_a} \cdot \prod_{a=p+1}^m g_{\nu_a} \in \tilde{A}
\]

in the quotient space \( J \setminus \tilde{A} / \tilde{I}_{\mu, \delta^+} \).

**Proposition 4.8** (i) The vector \( v_\mu^\lambda \) does not belong to the zero coset in \( J \setminus \tilde{A} / \tilde{I}_{\mu, \delta^+} \).

(ii) The vector \( v_\mu^\lambda \) is of weight \( \lambda \) under the action of \( \mathfrak{h} \) on \( J \setminus \tilde{A} / \tilde{I}_{\mu, \delta^+} \) and is of highest weight with respect to the action of \( Y(\mathfrak{gl}_m) \) on the same quotient space.

(iii) For any \( \sigma \in \mathcal{S}_m \) the operator \( B_{\sigma(\lambda)} \) determined by \( \xi_a \) maps the vector \( v_\mu^\lambda \) to the image in \( J \setminus \tilde{A} / \tilde{I}_{\sigma(\mu), \sigma(\delta^+)} \) of \( \sigma(v_\mu^\lambda) \) in \( \tilde{A} \) multiplied by \( \prod_{\eta \in \Delta^+} z_\eta \).

**Proof:** Part (i) follows directly from the definition of the ideal \( \tilde{I}_{\mu, \delta^+} \). Let us now prove Part (ii). Subalgebra \( \mathfrak{h} \) acts on the quotient space \( J \setminus \tilde{A} / \tilde{I}_{\mu, \delta^+} \) via left multiplication on \( \tilde{A} \). Let symbol \( \equiv \) denote the equalities in \( \tilde{A} \) modulo the left ideal \( \tilde{I}_{\mu, \delta^+} \). Then we have

\[
E_{aa}v_\mu^\lambda = v_\mu^\lambda E_{aa} + \left[ \lambda_n(E_{aa}), v_\mu^\lambda \right] = v_\mu^\lambda (E_{aa} \mp \nu_a) \equiv v_\mu^\lambda (\lambda_n(E_{aa}) + \mu_a \mp \nu_a) \equiv v_\mu^\lambda \left( \pm \frac{n}{2} + \mu_a \mp \nu_a \right) = \lambda_a v_\mu^\lambda
\]

where one should choose the upper sign for \( a = 1, \ldots, p \) and the lower sign for \( a = p + 1, \ldots, m \).

Next, \( Y(\mathfrak{gl}_m) \)-modules \( J \setminus \tilde{A} / \tilde{I}_{\mu, \delta^+} \) and \( \mathcal{E}_{p,q}(M_\mu)_n \) are isomorphic. Recall that \( T_{ij}(u) \) acts on \( \mathcal{E}_{p,q}(M_\mu)_n \) with the help of the comultiplication \( \Delta \). Now, Corollary 2.2 and formulas (3.3) imply that \( T_{ij}(u)v_\mu^\lambda = 0 \) for all \( 1 \leq i < j \leq n \). Moreover, one can check that

\[
T_{ii}(u)v_\mu^\lambda = v_\mu^\lambda \cdot \prod_{a: \nu'_a \geq i} \frac{u - \mu_a - \rho_a + 1}{u - \mu_a - \rho_a}.
\]
\[
\sigma \in \mathcal{H} \bigcup \{\eta\}.
\]
Therefore, \(v_\mu^\lambda\) is a highest weight vector with respect to the action of \(Y(\mathfrak{g}_m)\) on the quotient space \(\mathcal{H} \bigcup \mathcal{I}_\mu.\delta^+\).

We will prove Part (iii) by induction on the length of a reduced decomposition of \(\sigma\). If \(\sigma\) is the identity element of \(\mathcal{G} \mathcal{H}_m\), then the required statement is tautological. Now suppose that for some \(\sigma \in \mathcal{G} \mathcal{H}_m\) the statement of (iii) is true. Take any simple reflection \(\sigma_a \in \mathcal{G} \mathcal{H}_m\) with \(1 \leq a \leq m - 1\) such that \(\sigma \circ \sigma_a\) has a longer reduced decomposition in terms of \(\sigma_1, \ldots, \sigma_m\) than \(\sigma\).

Consider the simple root \(\eta_a\), corresponding to the reflection \(\sigma_a\). Let \(\eta = \sigma^{-1}(\eta_a)\) then
\[
\sigma_a \sigma(\eta) = \sigma_a(\eta_a) = -\eta_a \notin \Delta^+.
\]
Thus \(\Delta_{\sigma \sigma} = \Delta_{\sigma} \cup \{\eta\}\). Let \(\kappa \in h^*\) be the weight with the labels determined by \((3.20)\) with \(\theta = 1\). Using the proof of Theorem 3.3, we get the equality of two left ideals of the algebra \(A\),
\[
\mathcal{I}_{(\sigma, \sigma)\circ \mu, (\sigma, \sigma)(\delta^+)} = \mathcal{I}_{(\sigma, \sigma)\circ \nu, (\sigma, \sigma)(\delta^+)}.
\]
Modulo the second of these two ideals the element \(\mathcal{H}_a\) equals
\[
((\sigma \circ \sigma) \circ \kappa)(\mathcal{H}_a) = (\sigma \circ \sigma(\kappa + \rho) - \rho)(\mathcal{H}_a) = (\kappa + \rho)(\sigma^{-1}(\mathcal{H}_a)) - \rho(\mathcal{H}_a) = -\rho + \rho - 1 = -\rho - 1.
\]
Here \(\mathcal{H}_\eta = \sigma^{-1}(\mathcal{H}_a)\) is the coroot corresponding to the root \(\eta\), and we use the standard bilinear form on \(h^*\).

Let us use the statement of (iii) as the induction assumption. Denote \(\delta = \sigma(\delta^+).\) Consider three cases.

I. Let \(b, c = 1, \ldots, p\), then \(\delta_a = \delta_a'\) and \(\delta_{a+1} = \delta_{a+1}'.\) Hence,
\[
\sigma(v_\mu^\lambda) = f_{av_b}f_{a+1 \nu_c}Y
\]
where \(Y\) is an element of the subalgebra of \(GD(Cm \otimes Cn)\) generated by all \(x_{dk}\) and \(\partial_{dk}\) with \(d \neq a, a + 1\). Now we apply the Lemma 1.5 with \(s = \nu_b\) and \(t = \nu_c\). After substitution \(-\mu_b + \mu_c - 1\) for \(\mathcal{H}_a\), the fraction in the lemma turns into
\[
-\mu_b^* + \mu_c^* + \nu_b - \nu_c = \frac{\lambda_b^* - \lambda_c^*}{\mu_b^* - \mu_c^*}.
\]
II. Let \(b, c = p + 1, \ldots, m\), then \(\delta_a = -\delta_a'\) and \(\delta_{a+1} = -\delta_{a+1}'.\) Hence,
\[
\sigma(v_\mu^\lambda) = g_{av_b}g_{a+1 \nu_c}Y
\]
where \(Y\) is an element of the subalgebra of \(GD(Cm \otimes Cn)\) generated by all \(x_{dk}\) and \(\partial_{dk}\) with \(d \neq a, a + 1\). Now we apply the Lemma 1.6 with \(s = \nu_b\) and \(t = \nu_c\). After substitution \(-\mu_b^* + \mu_c^* - 1\) for \(\mathcal{H}_a\), the fraction in the lemma turns into
\[
-\mu_b^* + \mu_c^* + \nu_b - \nu_c = \frac{\lambda_b^* - \lambda_c^*}{\mu_b^* - \mu_c^*}.
\]
III. Let \(b = 1, \ldots, p, c = p + 1, \ldots, m\), then \(\delta_a = \delta_a'\) and \(\delta_{a+1} = -\delta_{a+1}'.\) Hence,
\[
\sigma(v_\mu^\lambda) = f_{av_b}g_{a+1 \nu_c}Y
\]
where \( Y \) is an element of the subalgebra of \( \mathcal{GD}(\mathbb{C}^m \otimes \mathbb{C}^n) \) generated by all \( x_{dk} \) and \( \partial_{dk} \) with \( d \neq a, a + 1 \). Now we apply the Lemma 4.7 with \( s = \nu_b \) and \( t = \nu_c \). When \( n = 1 \), after substitution \(-\mu_b^* + \mu_c^* - 1\) for \( H_a \), the fraction in the lemma turns into

\[
\frac{-\mu_b^* + \mu_c^* + \nu_b + \nu_c - n}{-\mu_b^* + \mu_c^*} = \frac{\lambda_b^* - \lambda_c^*}{\mu_b^* - \mu_c^*}.
\]

Thus, in the three cases considered above the operator

\[
\tilde{\xi}_{\sigma_0} : \bar{J} \setminus \bar{A} / \bar{I}_{\mu, \delta^+} \to \bar{I}_{(\sigma_0 \sigma) \circ \mu, (\sigma_0 \sigma)(\delta^+)}
\]

maps the vector \( v^\lambda_\mu \) to the image of

\[
\sigma_0 \sigma(v^\lambda_\mu) \cdot \prod_{\eta \in \Delta_{\sigma_0 \sigma}} z_\eta \in \bar{A}
\]

in the vector space \( \bar{J} \setminus \bar{A} / \bar{I}_{(\sigma_0 \sigma) \circ \mu, (\sigma_0 \sigma)(\delta^+)} \). This observation makes the induction step. \( \square \)

5. Conjecture

The formula (3.5) yields that for any vector \( v \in \bar{J} \setminus \bar{A} / \bar{I}_{\mu, \delta^+} \) the image of \( v \) under the operator \( \tilde{\xi}_\sigma \) is well defined unless \((H_a + 1 - r)v = 0\) for some integer \( r \). As we have shown in the previous section \( H_a \) acts as \(-\mu_b^* + \mu_c^* - 1\) for some \( 1 \leq b < c \leq m \) on the target module in the mapping (3.25). It follows from relations (3.4) that after commuting \( H_a \) with \( v \), we will get \( v(\lambda_b^* - \lambda_c^* + r) \). Therefore, the operators \( \tilde{\xi}_\sigma \) are well defined unless

\[
\lambda_b^* - \lambda_c^* = -1, -2, \ldots \quad \text{for some} \quad 1 \leq b < c \leq m. \tag{5.1}
\]

In the latter case it can be shown that a factor of module (3.26) by the kernel of operator \( \tilde{\xi}_\sigma \) is an irreducible (and non-zero under certain condition on the numbers \( \nu_1, \ldots, \nu_m \)) \( Y(\mathfrak{gl}_n) \)-module, where \( \sigma_0 \) is the longest element of the Weyl group (see [KN5]).

Recall the operator \( \xi'_\sigma \) defined right before Theorem 3.5. Now, we state

**Conjecture 5.1** Every irreducible finite-dimensional rational module of the Yangian \( Y(\mathfrak{gl}_n) \) may be obtained as the factor of the module (3.31) over the kernel of the intertwining operator \( \xi'_{\sigma_0} \) for some weights \( \mu \) and \( \lambda \), with \( \lambda \) satisfying condition (5.1).

**Appendix**

Let us first prove some properties of the matrix \( X(u) \) defined by (1.11).

**Proposition A.1** i) The following relation holds:

\[
(u - v) \cdot X(u)X(v) = X(v) - X(u); \tag{A.1}
\]

ii) The elements \( X_{ab}(u) \) satisfy the Yangian relation:

\[
(u - v) \cdot [X_{ab}(u), X_{cd}(v)] = \theta(X_{cb}(u)X_{ad}(v) - X_{cb}(v)X_{ad}(u)). \tag{A.2}
\]
Proof: The part i) follows from equality

\[(u - v) = (u + \theta E') - (v + \theta E')\]

multiplied by \(X(u)\) from the left and by \(X(v)\) from the right. Let us start the proof of the part ii) with equality

\[
\left[ (u + \theta E')_{ef}, (v + \theta E')_{gh} \right] = \delta_{eh} E_{fg} - \delta_{fg} E_{he}.
\]

We multiply the above equality by \(X_{ae}(u)\) from the left, by \(X_{fb}(u)\) from the right, and take a sum over indices \(e, f\):

\[
\sum_{e,f=1}^{m} \left( (u + \theta E')_{ef} (v + \theta E')_{gh} X_{fb}(u) - X_{ae}(u) \left( (v + \theta E')_{gh} (u + \theta E')_{ef} X_{fb}(u) \right) \right) = \sum_{e,f=1}^{m} (X_{ae}(u) \delta_{eh} E_{fg} X_{fb}(u) - X_{ae}(u) \delta_{fg} E_{he} X_{fb}(u)).
\]

Thus, we get

\[
\left[ (v + \theta E')_{gh}, X_{ab}(u) \right] = \sum_{e,f=1}^{m} (X_{ah}(u) E_{fg} X_{fb}(u) - X_{ae}(u) E_{he} X_{gb}(u)).
\]

Using that

\[
\sum_{f=1}^{m} E_{fg} X_{fb}(u) = \theta (\delta_{bg} - u X_{gb}(u)) \quad \text{and} \quad \sum_{e=1}^{m} X_{ae}(u) E_{he} = \theta (\delta_{ah} - u X_{ah}(u)),
\]

we obtain

\[
\left[ (v + \theta E')_{gh}, X_{ab}(u) \right] = \theta \left( \delta_{bg} X_{ah}(u) - \delta_{ah} X_{gb}(u) \right).
\]

Multiplying the above equation by \(X_{cg}(v)\) from the left, by \(X_{hd}(v)\) from the right, and taking a sum over indices \(g, h\), we arrive to equality

\[
[X_{ab}(u), X_{cd}(v)] = \theta \sum_{g,h=1}^{m} [X_{cb}(v) X_{ah}(u) X_{hd}(v) - X_{cg}(v) X_{gb}(u) X_{ad}(v)].
\]

Now using the result of the part i), we get

\[
(u - v) \cdot [X_{ab}(u), X_{cd}(v)] = \theta \left( X_{cb}(v) (X_{ad}(v) - X_{ad}(u)) - (X_{cb}(v) - X_{cb}(u)) X_{ad}(v) \right)
\]

and the statement of the part ii) follows. \(\Box\)

Proof of Proposition 1.2:

We prove the part i) by direct calculation. During the proof we will write \(T_{ij}(u)\) instead of its image under \(\alpha_m\) in the algebra \(U(\mathfrak{gl}_m) \otimes \mathcal{H}D(\mathbb{C}^m \otimes \mathbb{C}^n)\). Using relations (1.17), (1.18), we
get

\[(u - v) \cdot [T_{ij}(u), T_{kl}(v)] =\]

\[= \sum_{a,b,c,d=1}^m (u - v) \cdot \left( X_{ab}(u)X_{cd}(v) \otimes (\theta \hat{E}_{ck,bj} \hat{E}_{ai,dl} + \delta_{bc} \delta_{jk} \hat{E}_{ai,dl} - \delta_{ab} \delta_{ij} \hat{E}_{ck,dl}) - X_{cd}(v)X_{ab}(u) \otimes (\theta \hat{E}_{ck,bj} \hat{E}_{ai,dl} + \delta_{ad} \delta_{il} \hat{E}_{ck,bj} - \delta_{ab} \delta_{ij} \hat{E}_{ck,dl}) \right) =\]

\[= \sum_{a,b,c,d=1}^m (u - v) \cdot \left( [X_{ab}(u), X_{cd}(v)] \otimes \theta (\theta \hat{E}_{ck,bj} \hat{E}_{ai,dl} - \delta_{ab} \delta_{ij} \hat{E}_{ck,dl}) + X_{ab}(u)X_{cd}(v) \otimes \delta_{bc} \delta_{jk} \hat{E}_{ai,dl} - X_{cd}(v)X_{ab}(u) \otimes \delta_{ad} \delta_{il} \hat{E}_{ck,bj} \right).\]

Next, using relations (A.1) and (A.2), we obtain

\[(u - v) \cdot [T_{ij}(u), T_{kl}(v)] =\]

\[= \sum_{a,b,c,d=1}^m \left( (X_{cb}(u)X_{ad}(v) - X_{cb}(v)X_{ad}(u)) \otimes (\hat{E}_{ck,bj} \hat{E}_{ai,dl} - \delta_{ab} \delta_{ij} \hat{E}_{ck,dl}) + (u - v) \cdot \left( X_{ab}(u)X_{cd}(v) \otimes \delta_{bc} \delta_{jk} \hat{E}_{ai,dl} - X_{cd}(v)X_{ab}(u) \otimes \delta_{ad} \delta_{il} \hat{E}_{ck,bj} \right) \right).\]

Now, using the definition of the homomorphism \(\alpha_m\), we get

\[\left( T_{kj}(u) - \delta_{jk} \right) (T_{il}(v) - \delta_{il}) - \left( T_{kj}(v) - \delta_{jk} \right) (T_{il}(u) - \delta_{il}) - \delta_{ij} \sum_{c,d=1}^m \left( (X(u)X(v))_{cd} - (X(v)X(u))_{cd} \right) \otimes \hat{E}_{ck,dl} + \]

\[+ \delta_{jk} \sum_{a,d=1}^m (u - v) ((X(u)X(v))_{ad}) \otimes \hat{E}_{ai,dl} + \delta_{il} \sum_{b,c=1}^m (u - v) ((X(v)X(u))_{cb}) \otimes \hat{E}_{ck,bj},\]

It follows from relation (A.1) that \(X(u)X(v) = X(v)X(u)\). Therefore, we finish the proof of the part i) by the following calculations:

\[(u - v) \cdot [T_{ij}(u), T_{kl}(v)] = T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u) - \]

\[- \delta_{jk} (T_{il}(v) - T_{il}(u)) - \delta_{il} (T_{kj}(u) - T_{kj}(v)) + \]

\[+ \delta_{jk} (T_{il}(v) - T_{il}(u)) + \delta_{il} (T_{kj}(u) - T_{kj}(v)) =\]

\[= T_{kj}(u)T_{il}(v) - T_{kj}(v)T_{il}(u).\]

The part ii) is also proved by straightforward verification:

\[\left[ E_{cd} \otimes 1 + 1 \otimes \zeta_n(E_{cd}), T_{ij}(u) \right] =\]

\[= \left[ E_{cd} \otimes 1, \sum_{a,b=1}^m X_{ab}(u) \otimes \hat{E}_{ai,bj} \right] + \left[ 1 \otimes \sum_{k=1}^n \hat{E}_{ck,dk}, \sum_{a,b=1}^m X_{ab}(u) \otimes \hat{E}_{ai,bj} \right] =\]

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\[
\sum_{a,b=1}^{m} \left( \delta_{bd} X_{ac}(u) - \delta_{ac} X_{db}(u) \right) \otimes \hat{E}_{ai,bj} + \sum_{k=1}^{n} \sum_{a,b=1}^{m} \left( X_{ab}(u) \otimes \left( \delta_{ad} \delta_{ik} \hat{E}_{ck,bj} - \delta_{bc} \delta_{jk} \hat{E}_{ai,dk} \right) \right) = \\
\sum_{a=1}^{m} X_{ac}(u) \otimes \hat{E}_{ai,dj} - \sum_{b=1}^{m} X_{db}(u) \otimes \hat{E}_{ci,bj} + \sum_{b=1}^{m} X_{db}(u) \otimes \hat{E}_{ci,bj} - \sum_{a=1}^{m} X_{ac}(u) \otimes \hat{E}_{ai,dj} = 0.
\]

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