Pseudoscalar decay constants at large $N_c$

R. Kaiser, H. Leutwyler
Institute for Theoretical Physics, University of Bern,
Sidlerstr. 5, CH–3012 Bern, Switzerland
E-mail: kaiser@itp.unibe.ch, leutwyler@itp.unibe.ch

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Abstract

In the large $N_c$ limit, the variables required to analyze the low energy structure of QCD in the framework of an effective field theory necessarily include the degrees of freedom of the $\eta'$. We evaluate the decay constants of the pseudoscalar nonet to one loop within this extended framework and show that, as a consequence of the anomalous dimension of the singlet axial current, some of the effective coupling constants depend on the running scale of QCD. The calculation relies on a simultaneous expansion in powers of momenta, quark masses and $1/N_c$.

1 Introduction

The low energy properties of QCD are governed by an approximate, spontaneously broken symmetry, which originates in the fact that three of the quarks happen to be light. If $m_u, m_d, m_s$ are turned off, the symmetry becomes exact. The spectrum of the theory then contains eight strictly massless pseudoscalar mesons, the Goldstone bosons connected with the spontaneous symmetry breakdown.

If the number of colours is taken large, the quark loop graph which gives rise to the U(1)-anomaly is suppressed [1]. This implies that, in the limit $N_c \to \infty$, the singlet axial current is also conserved: The theory in effect acquires a higher degree of symmetry. Since the operator $\bar{q}q$ fails to be invariant under the extra U(1)-symmetry, the formation of a quark condensate, $\langle 0 | \bar{q}q | 0 \rangle \neq 0$, implies that this symmetry is also spontaneously broken [2]. The spectrum of QCD, therefore, contains a ninth state, the $\eta'$, which becomes massless if not only $m_u, m_d, m_s$ are turned off, but if in addition the number of colours is sent to infinity [3].

Chiral symmetry imposes strong constraints on the properties of the Goldstone bosons. These may be worked out in a systematic manner by means of the effective Lagrangian method, which describes the low energy structure of the theory in terms of an expansion in powers of energies, momenta and quark masses [4, 5]. The fact that, in the large $N_c$ limit, the $\eta'$ also plays the role of a Goldstone boson implies
that the properties of this particle are subject to analogous constraints, which may again be worked out by means of a suitable effective Lagrangian.

The main features of this Lagrangian were studied long ago \[6\]. In particular, it was shown that the mass of the \(\eta'\) is controlled by the topological susceptibility of Gluodynamics and by the pion decay constant \(\pi\):

\[
M_{\eta'}^2 = 6 \frac{\tau_0}{F_{\pi}^2} + \ldots
\]

The topological susceptibility \(\tau_0\) represents the mean square winding number per unit volume of euclidean space,

\[
\tau_0 \equiv \left(\frac{\nu^2}{V}\right)_{GD} = \int d^4x \langle 0 | T \omega(x)\omega(0) | 0 \rangle_{GD},
\]

where \(\omega = \frac{1}{16\pi^2} \text{tr} G_{\mu\nu} \tilde{G}^{\mu\nu}, \quad \nu = \int d^4x \omega\).

In the large \(N_c\) limit, \(\tau_0\) becomes independent of \(N_c\) while \(F_\pi\) grows in proportion to \(\sqrt{N_c}\), so that \(M_{\eta'}\) tends to zero \[3\].

In more recent work, the derivative expansion of the relevant effective Lagrangian was investigated beyond leading order \[7\]-\[13\]. In particular, we have shown that the analysis of the pseudoscalar decay constants in the framework of the effective theory requires two different mixing angles that are related through a low energy theorem \[11, 12\]. The decay constants are defined by

\[
\langle 0 | A_\mu^a | P \rangle = ip_\mu F_\mu^a, \quad A_\mu^a = \gamma_5 \gamma_0 \frac{1}{2} \lambda^a q.
\]

where \(P = (\pi^0, \ldots, \eta')\) labels the pseudoscalar mesons. The index \(a = (0, \ldots, 8)\) refers to a basis of U(3), normalized with \(\text{tr}(\lambda^a \lambda^b) = 2 \delta^{ab}\). The two angles mentioned above specify the \(\eta\) and \(\eta'\) projections of the states \(A_\mu^a | 0 \rangle\) and \(A_\mu^0 | 0 \rangle\), respectively:

\[
F_8^\eta = \cos \vartheta_8 F_8, \quad F_8^{\eta'} = \sin \vartheta_8 F_8, \quad F_0^\eta = - \sin \vartheta_0 F_0, \quad F_0^{\eta'} = \cos \vartheta_0 F_0.
\]

Chiral symmetry implies that, at leading order in a simultaneous expansion in powers of quark masses and \(1/N_c\), the difference between the two angles is determined by \(F_K\) and \(F_\pi\) \[11, 12\]:

\[
\sin(\vartheta_0 - \vartheta_8) = \frac{2\sqrt{2}(F_{K}^2 - F_{\pi}^2)}{4F_{K}^2 - F_{\pi}^2} + \ldots
\]

The \(\eta-\eta'\) mixing pattern has recently attracted much attention in connection with production and decay processes involving these particles \[14\]-\[24\]. As pointed out in \[21, 22\], the analysis in terms of two different mixing angles indeed yields a
more coherent picture than the canonical treatment with \( \vartheta = \vartheta_0 \). The sign of the prediction is confirmed, but the numerical value obtained for the difference between the two angles is somewhat smaller than what is required by (3). The purpose of the present paper is to discuss the corrections which this low energy theorem receives from higher order effects.

2 Effective theory

The dynamical variables of the QCD Lagrangian are the quark and gluon fields. The low energy analysis of this system is based on an effective field theory where the dynamical variables are mesonic fields with the quantum numbers of the Goldstone bosons. For \( N_c = 3 \), there are eight Goldstone fields – the \( \eta' \) occurs among the massive states which only show up indirectly, through their contributions to the effective coupling constants. The octet of pseudoscalars may be collected in a unitary \( 3 \times 3 \)-matrix \( U(x) \in SU(3) \).

The standard framework does not cover the large \( N_c \) limit, however. Its domain of validity is restricted by the condition

\[
m_s \langle 0 | \bar{u} u | 0 \rangle < 9 \tau_0 ,
\]

which is violated if the limit \( N_c \rightarrow \infty \) is taken at fixed quark masses. Indeed, some of the standard ChPT formulae become meaningless in that limit [25]. For the effective field theory to properly describe the low energy structure also when \( N_c \) is taken large, the set of dynamical variables needs to be enlarged by adding a field that describes the extra Goldstone boson, the \( \eta' \). Quite generally, the effective fields live on the coset space \( G/H \), where \( G \) is the symmetry group of the Hamiltonian and \( H \) is the subgroup that leaves the vacuum invariant. At large \( N_c \), we have \( G = U(3)_R \times U(3)_L \), \( H = U(3)_V \), so that \( U(x) \in U(3) \). The extension from SU(3) to U(3) shows up in the phase of the determinant

\[
\det U(x) = e^{i\psi(x)} .
\]

The field \( \psi(x) \) describes the \( \eta' \).

The effective Lagrangian contains the meson field \( U(x) \) and its derivatives, \( \mathcal{L}_{\text{eff}} = \mathcal{L}_{\text{eff}}(U, \partial U, \partial^2 U, \ldots) \). In order to analyze the matrix elements of the quark currents, it is convenient to ab initio introduce corresponding external fields, replacing the QCD Lagrangian by

\[
\mathcal{L}_{\text{QCD}} = \mathcal{L}_{\text{QCD}}^0 + \bar{q} \gamma^\mu (v_\mu + \gamma_5 a_\mu) q - \bar{q} (s - i\gamma_5 p) q - \theta \omega .
\]

The term \( \mathcal{L}_{\text{QCD}}^0 \) describes the limit where the masses of the three light quarks and the vacuum angle are set to zero. The external fields \( v_\mu(x) \), \( a_\mu(x) \), \( s(x) \), \( p(x) \) represent hermitean \( 3 \times 3 \) matrices in flavour space. The mass matrix of the three light quarks is contained in the scalar external field \( s(x) \). The vacuum angle \( \theta(x) \) represents the variable conjugate to the winding number density \( \omega(x) \).
Naturally, $\mathcal{L}_{\text{eff}}$ then also depends on the external fields and their derivatives. The dependence is strongly constrained by the symmetries of the underlying theory \cite{[5],[26]}: Apart from the Wess-Zumino-Witten term, the effective Lagrangian is invariant under chiral gauge transformations,

$$
\begin{align*}
    r'_{\mu} &= V_R r_{\mu} V_R^\dagger + i V_R \partial_\mu V_R^\dagger, \\
    l'_{\mu} &= V_L l_{\mu} V_L^\dagger + i V_L \partial_\mu V_L^\dagger, \\
    s' + i p' &= V_R (s + ip) V_L^\dagger, \\
    U' &= V_R U V_L^\dagger, \\
    \theta' &= \theta + i \ln \det V_R - i \ln \det V_L,
\end{align*}
$$

(9)

with $r_{\mu} = v_\mu + a_\mu$, $l_{\mu} = v_\mu - a_\mu$ and $V_R(x), V_L(x) \in U(3)$.

The expansion of the general effective Lagrangian in powers of derivatives and quark masses starts with \cite{[5]}

$$
\begin{align*}
    \mathcal{L}_{\text{eff}} &= -V_0 + V_1 \langle D_\mu U^\dagger D^\mu U \rangle + V_2 \langle (s + ip) U^\dagger \rangle + V_3 \langle (s - ip) U \rangle \\
    &\quad + V_4 D_\mu \psi D^\mu \psi + V_5 D_\mu \psi D^\mu \theta + V_6 D_\mu \theta D^\mu \theta + O(p^4),
\end{align*}
$$

(10)

where $\langle \ldots \rangle$ stands for the trace. The covariant derivatives are defined by

$$
\begin{align*}
    D_\mu U &= \partial_\mu U - i (v_\mu + a_\mu) U + i U (v_\mu - a_\mu) \\
    D_\mu \psi &= \partial_\mu \psi - 2 \langle a_\mu \rangle \\
    D_\mu \theta &= \partial_\mu \theta + 2 \langle a_\mu \rangle.
\end{align*}
$$

(11)

The transformation law (9) implies that the combination

$$
\tilde{\psi} \equiv \psi + \theta
$$

(12)

of the singlet field $\psi$ and the vacuum angle $\theta$ is invariant under chiral transformations. Chiral symmetry does therefore not constrain the dependence of the Lagrangian on this variable: At this stage, the coefficients are arbitrary functions thereof, $V_n = V_n(\tilde{\psi})$. They may be viewed as potentials that control the dynamics of the singlet field $\psi(x)$.

As pointed out in ref. \cite{[5]}, the form of the potentials depends on the choice of field variables: A transformation of the type $U \to U \exp if(\tilde{\psi})$ leaves the structure of the Lagrangian invariant, but modifies the potentials. For our purposes it is convenient to exploit this freedom by eliminating the term \cite{[5]}

$$
V_4 = 0.
$$

(13)

Moreover, we may discard $V_5$, because this term does not contribute to the meson masses, decay constants or photonic transition rates.

In the limit $N_c \to \infty$, the potentials are dominated by the leading terms of their Taylor series in powers of $\tilde{\psi}$ (for a discussion of the large $N_c$ counting rules, see ref.\footnote{The advantage of the convention $V_4 = 0$ is that the quadratic part of the Lagrangian then takes the simple form $\mathcal{L}_{\text{eff}} = \frac{1}{2} A^{ab}(\partial_\mu \phi_a - a_{\mu a})(\partial^\mu \phi_b - a^\mu b) - \frac{1}{2} D^{ab} \phi_a \phi_b + \ldots}$

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Discarding terms that disappear in the large $N_c$ limit, the potentials reduce to a set of constants:

\begin{align}
V_0 &= \text{const.} + \frac{1}{2} \tau \bar{\psi}^2 + O(N_c^{-2}) , \\
V_1 &= \frac{1}{4} F^2 + O(N_c^{-1}) , \\
V_2 &= \frac{1}{2} F^2 B \left(1 - \frac{i}{6} \Lambda_2 \bar{\psi}^2 \right) + O(N_c^{-1}) , \\
V_3 &= \frac{1}{12} F^2 \Lambda_1 + O(N_c^{-2}) .
\end{align}

At this order, the effective Lagrangian thus contains five coupling constants:

\begin{align}
F^2 &= O(N_c) ; \quad B, \tau = O(1) ; \quad \Lambda_1, \Lambda_2 = O(N_c^{-1}) .
\end{align}

The constant $B$ only occurs together with the scalar and pseudoscalar external fields, through the combination

\begin{equation}
\chi \equiv 2B(s + ip) .
\end{equation}

We add a remark concerning the comparison between the extended effective theory based on $U(x) \in U(3)$ and the standard framework, where $U(x) \in SU(3)$. As discussed in detail in ref. [5], the Ward identities obeyed by the Green functions of the vector and axial currents imply that the SU(3) effective Lagrangian is invariant under the full group $U(3)_s \times U(3)_s$ of chiral gauge transformations – up to the Wess-Zumino-Witten term, which accounts for the anomalies occurring in these identities. The transformation law $U' = V_n U V_n^\dagger$, however, violates the constraint $\det U = 1$. Instead of modifying the transformation law, it is more convenient to perform a change of variables, $U \rightarrow U \exp(-\frac{1}{3} \theta)$, i.e. to modify the constraint, which then takes the form $\det U = e^{-i\theta}$. The condition amounts to the gauge invariant relation $\psi(x) = -\theta(x)$, which states that the standard framework arises from the extended one if the extra variable $\psi(x)$ is fixed at the minimum of the potential $V_0 = V_0(\psi + \theta)$.

## 3 Higher orders in the derivative expansion

At order $p^4$, the general effective Lagrangian contains altogether 57 potentials, which are listed explicitly in ref. [10]. Expanding these in powers of $\bar{\psi}$ and retaining only those terms that (a) generate a contribution to the masses or decay constants and (b) do not disappear in the limit $N_c \rightarrow \infty$, the general expression boils down to

\begin{align}
\mathcal{L}^{(p^4)} = L_4(D_\mu U^\dagger D_\mu U) \langle \chi^\dagger U + U^\dagger \chi \rangle + L_5(D_\mu U^\dagger D_\mu U(\chi^\dagger U + U^\dagger \chi)) \\
+ L_6(\chi^\dagger U + U^\dagger \chi)^2 + L_7(\chi^\dagger U - U^\dagger \chi)^2 \\
+ L_8(\chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi) + L_{18} iD_\mu \bar{\psi}(D_\mu \chi^\dagger U - D_\mu U \chi^\dagger) \\
+ L_{25} i \bar{\psi}(U^\dagger \chi U^\dagger - \chi^\dagger U^\dagger \chi) + O(N_c^{-1}) .
\end{align}

\footnote{At leading order of the $1/N_c$–expansion, $\tau$ coincides with the topological susceptibility of Gluodynamics: $\tau = \tau_0 + O(1/N_c)$.}

\footnote{Some of these may be eliminated with a suitable change of variables. In particular, the terms $O_{46}$ and $O_{52}$ – which do generate a contribution to the decay constants – are removed with a transformation of the form $U \rightarrow U \exp[f_3 D_\mu \theta U^\dagger D_\mu U + i f_2 D_\mu D_\mu \theta]$. We exploit this freedom and set $L_{46} = L_{52} = 0$.}
The coupling constants $L_4, \ldots, L_8$ are familiar from the standard framework, where the effective field $U(x)$ is an element of the group SU(3) rather than U(3). Indeed, if the singlet component of the meson field is integrated out, the above expression for the effective Lagrangian does take the standard form. Note, however, that the effective coupling constant $L_7$ then picks up an extra contribution from $\eta'$-exchange, which in the large $N_c$ limit even dominates:

$$L_7^{\text{SU}(3)} = -\frac{F^4 (1 + \Lambda_2)^2}{288 \tau} + L_7 .$$  \hspace{1cm} (18)

In our context, the extension from SU(3) to U(3) thus gives rise to two additional $p^4$-couplings, for which we have retained the numbering used by Herrera-Siklódy et al. (in their notation: $L_{18} \equiv L_{18}(0), L_{25} \equiv L_{25}(0)$). The large $N_c$ counting rules imply

$$L_5, L_8 = O(N_c); \quad L_4, L_6, L_7, L_{18}, L_{25} = O(1) .$$  \hspace{1cm} (19)

It is convenient to order the triple expansion in (i) the number of derivatives, (ii) powers of quark masses and (iii) powers of $1/N_c$ by treating the three expansion parameters as small quantities of order

$$\partial_\mu = O(\sqrt{\delta}) , \quad m = O(\delta) , \quad 1/N_c = O(\delta) .$$  \hspace{1cm} (20)

In this bookkeeping, the fields $U, \psi, \theta$ are of order $\delta^0$, while $v_\mu, a_\mu$ count as terms of $O(\sqrt{\delta})$ and $s, p$ are of $O(\delta)$. The expansion of the effective Lagrangian then takes the form

$$L_{\text{eff}} = L^{(0)} + L^{(1)} + L^{(2)} + \ldots$$  \hspace{1cm} (21)

It starts with the contributions of order $\delta^0$:

$$L^{(0)} = \frac{1}{4} F^2 \langle D_\mu U^\dagger D^\mu U \rangle + \frac{1}{4} F^2 \langle \chi^\dagger U + U^\dagger \chi \rangle - \frac{1}{2} \tau (\psi + \theta)^2 .$$  \hspace{1cm} (22)

The term $L^{(1)} = O(\delta)$ contains the contributions of $O(N_c p^4)$ as well as those of $O(p^2)$ generated by the Okubo-Iizuka-Zweig rule violating couplings $\Lambda_1, \Lambda_2$:

$$L^{(1)} = L_5 \langle D_\mu U^\dagger D^\mu U (\chi^\dagger U + U^\dagger \chi) \rangle + L_8 \langle \chi^\dagger U U^\dagger \chi + U^\dagger \chi U \dagger \chi \rangle$$

$$+ \frac{1}{12} F^2 \Lambda_1 D_\mu \psi D^\mu \psi + \frac{1}{12} F^2 \Lambda_2 i(\psi + \theta) \langle \chi^\dagger U - U^\dagger \chi \rangle .$$  \hspace{1cm} (23)

The remainder of the expression in eq. (17) belongs to $L^{(2)} = O(\delta^2)$:

$$L^{(2)} = L_1 \langle D_\mu U^\dagger D^\mu U \rangle \langle \chi^\dagger U + U^\dagger \chi \rangle + L_6 \langle \chi^\dagger U + U^\dagger \chi \rangle^2 + L_7 \langle \chi^\dagger U - U^\dagger \chi \rangle^2$$

$$+ L_{18} i D_\mu \psi \langle D^\mu U^\dagger \chi - D^\mu \chi^\dagger U \rangle + L_{25} i(\psi + \theta) \langle U^\dagger \chi U \dagger \chi - \chi^\dagger U \chi^\dagger U \rangle$$

$$+ O(N_c p^6) .$$  \hspace{1cm} (24)
4 Anomalous dimensions

The effective theory is constructed in such a way that it reproduces the derivative expansion of the effective action of QCD. The effective action represents the connected vacuum-to-vacuum amplitude in the presence of the external fields,

\[ e^{iS_{\text{eff}}} = \langle 0 \text{ out} | 0 \text{ in} \rangle_{\nu_\mu, a_\mu, s, p, \theta} . \]  

(25)

It collects the set of all connected Green functions formed with the operators \( \overline{q} \gamma_\mu q \), \( \overline{q} \gamma_\mu \gamma_5 q \), \( \overline{q} q \), \( \overline{q} \gamma_5 q \), \( \omega \) and depends on the running coupling constant \( g \) as well as on the renormalization scale \( \mu \):

\[ S_{\text{eff}} = S_{\text{eff}}(\nu_\mu, a_\mu, s, p, \theta, g, \mu). \]

In principle, all of the effective coupling constants occurring in \( L_{\text{eff}} \) are determined by \( g \), \( \mu \) and by the masses of the heavy quarks \( c \), \( b \), \( t \). We now discuss the manner in which the effective coupling constants depend on the running scale of QCD. In particular, we wish to work out the consequences of the fact that the matrix elements of the singlet axial current depend on the renormalization, because this operator carries anomalous dimension. To our knowledge, this issue is not discussed in the literature, because the matrix elements of the singlet axial current are usually considered only at leading order of the \( 1/N_c \)-expansion.

For the Green functions to remain the same when the renormalization scale is changed, the coupling constant must be adapted,

\[ \mu \frac{dg}{d\mu} = -\beta(g) = -\beta_0 \frac{g^3}{16\pi^2} + O(g^5) , \quad \beta_0 = \frac{1}{3}(11N_c - 2N_f) . \]  

(26)

This does not suffice, however, because some of the operators under consideration carry anomalous dimension. In particular, the scalar and pseudoscalar operators require renormalization for their Green functions to remain unaffected by a change of scale. The same holds for the Green functions containing the singlet axial current \( A_\mu^0 \) – this operator also receives multiplicative renormalization. As it is the case with the quark masses, the values the decay constants associated with the singlet current (\( F_\eta^0 \) and \( F_\eta^0' \)) therefore depend on the scale.

The external fields may be viewed as space-time dependent coupling constants. It is convenient to compensate the scale dependence of the operators by treating the corresponding external fields as scale dependent quantities, in such a way that the effective action becomes scale independent. This requires a renormalization of \( s(x) \), \( p(x) \) and of the singlet component of the axial external field. On the other hand, the algebra obeyed by the charges of \( SU(3) \times SU(3) \) implies that the octet components of the vector and axial currents are of canonical dimension, so that their matrix elements are scale independent. The same is true of the singlet vector current (baryon number). To simplify the renormalization group behaviour of the external fields, it is convenient to replace \( s \) and \( p \) by the combination

\[ m_p = e^{3\theta/4}(s + ip) , \]  

(27)

Note that we are considering the scale dependence within QCD – the one arising from the logarithmic divergences that occur in the effective theory is an entirely different issue (see section §7).

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because this quantity is invariant under the transformations generated by the singlet charges.

The various Green functions are Lorentz invariant expressions formed with the external momenta. There are two categories: those that may be expressed in terms of the metric \( g_{\mu\nu} \) alone (natural parity) and those for which the expression is linear in the tensor \( \epsilon_{\mu\nu\rho\sigma} \) (unnatural parity). Accordingly, the effective action may be decomposed into two parts that are of natural and unnatural parity, respectively. The anomalies only show up in the latter. As we are restricting ourselves to an analysis of the decay constants, we are concerned with the natural parity part, which is invariant under the transformation (9).

The singlet component of \( a_\mu \) is given by the trace \( \langle a_\mu \rangle \). Denoting the octet part by \( \hat{a}_\mu \equiv a_\mu - \frac{1}{3} \langle a_\mu \rangle \) and using the variable \( m_\theta \) instead of \( s, p \), the arguments of the effective action are: \( v_\mu, \hat{a}_\mu, \langle a_\mu \rangle, m_\theta, \theta \). Invariance under the transformation (9) implies that \( \langle a_\mu \rangle \) and \( \theta \) only enter through the gauge invariant combination \( D_\mu \theta = \partial_\mu \theta + 2 \langle a_\mu \rangle \), so that \( S_{\text{eff}} = S_{\text{eff}}(v_\mu, \hat{a}_\mu, D_\mu \theta, m_\theta, g, \mu, \theta) \). The symmetries of the theory thus imply that \( \partial_\mu \theta \) picks up the same renormalization factor as the singlet axial current: The effective action is invariant under a change of the running scale, provided the external fields are subject to the transformation \( m_\theta \rightarrow Z_{\text{M}}^{-1} m_\theta, \quad D_\mu \theta \rightarrow Z_{\text{A}}^{-1} D_\mu \theta \).

\( Z_{\text{M}} \) is the familiar factor that describes the scale dependence of the quark masses. The term \( Z_{\text{A}} \) specifies the one of the singlet decay constants:

\[
F_0^0 \rightarrow Z_{\text{A}} F_0^0 .
\]

It is determined by the anomalous dimension \( \gamma_{\text{A}} \) of the singlet axial current:

\[
\mu \frac{dZ_{\text{A}}}{d\mu} = \gamma_{\text{A}} Z_{\text{A}} , \quad \gamma_{\text{A}} = - \frac{3N_f(N_c^2 - 1)}{8N_c} \left( \frac{g}{2\pi} \right)^4 + O(g^6) .
\]

While the scale dependence of the coupling constant and of the quark masses shows up already at leading order in the \( 1/N_c \)-expansion, the triangle graph responsible for the anomalous dimension of the singlet axial current is suppressed by one power of \( 1/N_c \), so that \( Z_{\text{A}} = 1 + O(1/N_c) \).

## 5 Scaling laws for the effective coupling constants

We now translate these properties into corresponding scaling laws for the coupling constants of the effective Lagrangian. This may be done by working out the first few terms in the chiral expansion of suitable observables. The leading term, \( F \), is evidently scale independent: It represents the pion decay constant in the chiral limit.

\(^6\)The statement represents a generalization of the familiar fact that the quark mass matrix \( M = \text{diag}(m_u, m_d, m_s) \) and the vacuum angle only enter in the combination \( \epsilon^{*\theta} M \).
On the other hand, the lowest order result for the pion mass,
\[ M_\pi^2 = B(m_u + m_d) + O(m^2) , \]
shows that \( B \) transforms contragrediently to the quark masses:
\[ B \rightarrow Z_M B . \] (31)

Next, consider the chiral limit and set \( v_\mu = \hat{a}_\mu = \theta = 0 \), where the classical solution (tree graphs) takes the form \( U = e^{i\psi} \mathbf{1} \). The terms linear and quadratic in \( \psi \),
\[ \mathcal{L}_{\text{eff}} = \frac{1}{16\pi^2} F^2 (1 + \Lambda_1) D_\mu \psi D^\mu \psi - \frac{1}{2} \tau \psi^2 + \ldots \] (32)
yield the singlet decay constant and the mass of the \( \eta' \) at tree level
\[ F^0_{\eta'} = \sqrt{1 + \Lambda_1} F , \quad M^{2}_{\eta'} = \frac{6\tau}{(1 + \Lambda_1) F^2} . \] (33)

Hence \( \tau \) and \( \Lambda_1 \) scale with
\[ \tau \rightarrow Z_\tau^2 \tau , \quad 1 + \Lambda_1 \rightarrow Z_\Lambda^2 (1 + \Lambda_1) . \] (34)

The coupling constants \( L_4, \ldots, L_8 \) are related to Taylor coefficients occurring in the expansion of \( F_\pi, F_K, M_\pi^2, M_K^2 \) in powers of the quark masses and are thus independent of the QCD-scale. Finally, the evaluation of the decay constants and masses of \( \eta \) and \( \eta' \) shows that the remaining coupling constants scale according to
\[ 1 + \Lambda_2 \rightarrow Z_\Lambda (1 + \Lambda_2) , \quad \]
\[ 2L_5 + 3L_{18} \rightarrow Z_\Lambda (2L_5 + 3L_{18}) , \quad \]
\[ 2L_8 - 3L_{25} \rightarrow Z_\Lambda (2L_8 - 3L_{25}) . \] (35)

As a check, we note that the scaling laws for \( \tau \) and \( \Lambda_2 \) are consistent with the scale-independence of the coupling constant \( L_7^{SU(3)} \) specified in (18).

The net result of this analysis is the following. Let us replace the external fields \( s \) and \( p \) by the variable \( m_\theta \), also in the effective Lagrangian and use the scale independent quantity
\[ \chi_\theta = e^{\frac{i}{2} \psi} \chi \] (36)
instead of \( \chi = 2B(s + ip) \). Furthermore, to explicitly display the dependence on the singlet component of the effective field, we set \( U = e^{i\psi} \tilde{U} \), so that \( \tilde{U} \in SU(3) \). The effective Lagrangian then depends on the fields \( \tilde{U}, \psi + \theta, v_\mu, \hat{a}_\mu, D_\mu \theta, \chi_\theta \) and their derivatives. The vacuum angle only enters together with \( \psi - \) this is what becomes of \( \theta \)-independence at the level of the effective theory. The above scaling laws amount to the statement that \( \mathcal{L}_{\text{eff}} \) is invariant under a change of the \( \psi \) – this is what becomes of \( \theta \)-independence at the level of the effective theory. The above scaling laws amount to the statement that \( \mathcal{L}_{\text{eff}} \) is invariant under a change of the QCD scale, provided the singlet component of the effective field is renormalized according to
\[ \psi + \theta \rightarrow Z^{-1}_\Lambda (\psi + \theta) . \] (37)

Conversely, this property ensures that the effective action generated by the effective theory is scale independent.

\[ ^7 \text{The scale-dependence of } \tau \text{ only shows up at nonleading order – since the triangle graph responsible for } Z_\Lambda \text{ does not occur in Gluodynamics, the topological susceptibility } \tau_0 \text{ is independent of the running scale.} \]

\[ ^8 \text{In particular, we have } D_\mu \psi = \partial_\mu (\psi + \theta) - D_\mu \theta. \]
6 Loops and chiral logarithms

If the low energy expansion is ordered according to powers of $\delta$, graphs containing $\ell$ loops yield contributions that are at most of order $\delta^{2\ell}$. Up to and including $O(\delta^2)$, only the one-loop graphs generated by the leading term $\mathcal{L}^{(0)}$ in eq. (22) contribute.

The divergences occurring in the one-loop graphs are absorbed in a renormalization of the effective coupling constants. The relevant coefficients have been worked out in ref. [10]. The main difference to the standard framework is that loops involving the propagation of an $\eta'$ require a renormalization of the constant $B$: In dimensional regularization this constant contains a pole at $d = 4$. Expressed in terms of the factor

$$\lambda = \frac{\mu^{d-4}}{16\pi^2} \left\{ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4\pi + \Gamma'(1) + 1 \right) \right\}$$

the renormalization reads (to distinguish it from the running scale $\mu$ of QCD, we denote the scale used in the renormalization of the effective theory by $\mu_\chi$)

$$B = B^r \left\{ 1 + \frac{4\tau}{F^4} \lambda \right\}.$$  (39)

Since $\lambda$ depends on the renormalization scale $\mu_\chi$, the same is true of the renormalized coupling constant (note that the running scale $\mu$ of QCD is kept fixed):

$$\mu_\chi \frac{\partial B^r}{\partial \mu_\chi} = - \frac{\tau B^r}{4\pi^2 F^4}.$$  (40)

The remaining couplings occurring at order $p^0$ and $p^2$ ($F$, $\tau$, $\Lambda_1$ and $\Lambda_2$) do not pick up renormalization, while the one for the terms of order $p^4$ is of the standard form, but the coefficients $\Gamma_6$ and $\Gamma_8$ differ from those relevant for the SU(3)-framework [10]

$$L_n = L^r_n + \Gamma_n \lambda,$$

$$\Gamma_4 = \frac{1}{8}, \quad \Gamma_5 = \frac{3}{8}, \quad \Gamma_6 = \frac{1}{16}, \quad \Gamma_7 = 0, \quad \Gamma_8 = \frac{3}{16}, \quad \Gamma_{18} = -\frac{1}{4}, \quad \Gamma_{25} = 0.$$  (41)

The evaluation of the loops yields the following expressions for the decay constants (we disregard isospin breaking, setting $m_u = m_d$):

$$F_\pi = F \left\{ 1 + 4(2M_K^2 + M_\pi^2)F^{-2}L^r_4 + 4M_\pi^2 F^{-2}L^r_5 - 2\mu_\pi - 2\mu_K - O(p^4) \right\},$$

$$F_K = F \left\{ 1 + 4(2M_K^2 + M_\pi^2)F^{-2}L^r_4 + 4M_K^2 F^{-2}L^r_5 - \frac{3}{2}(\mu_\pi + 2\mu_K + \cos^2 \mu_\eta + \sin^2 \mu_\eta) - O(p^4) \right\},$$

where $\mu_P$ is the standard chiral logarithm,

$$\mu_P = \frac{M_P^2}{32\pi^2 F^2} \ln \frac{M_P^2}{\mu_\chi^2}.$$  (43)
One readily checks that the scale dependence of the chiral logarithms cancels the one of the effective coupling constants $L_4^r$, $L_5^r$, provided the masses and the mixing angle $\vartheta$ are evaluated at leading order of the expansion, where

\[
\cos^2 \vartheta M_\eta^2 + \sin^2 \vartheta M_{\eta'}^2 = \frac{1}{3} \left(4M_K^2 - M_\pi^2\right),
\]

\[
\sin 2\vartheta = -\frac{4\sqrt{2}}{3} \frac{M_K^2 - M_\pi^2}{M_{\eta'}^2 - M_\eta^2} .
\]

These lowest order relations amount to a constraint on the pseudoscalar masses, which the observed values do not obey [28]. For definiteness, we evaluate the one loop formulae with the physical values of $M_\pi$, $M_K$ and $M_{\eta'}$. The above relations then yield $\vartheta \simeq -20^\circ$, $M_\eta \simeq M_K$.

7 Results

The result for $F_\pi$ is identical with the one found in the framework of SU(3), where loops involving the propagation of an $\eta'$ do not occur. In the case of $F_K$, however, there is a contribution from an $\eta'$-loop. The relation

\[
\frac{F_K}{F_\pi} = 1 + 4(2M_K^2 - M_\pi^2)F^{-2}L_5^r + \frac{5}{3} \mu_\pi - \frac{1}{2} \mu_K - \frac{3}{4}(\cos^2 \vartheta \mu_\eta + \sin^2 \vartheta \mu_{\eta'})
\]

differs from the corresponding SU(3) formula: $\mu_\eta$ is replaced by the term $(\cos^2 \vartheta \mu_\eta + \sin^2 \vartheta \mu_{\eta'})$. Numerically, the difference is not significant, however: The two terms differ by less than 0.02.

The constants $F_8$ and $F_0$ are defined through the relations (4): 

\[
(F_8)^2 = (F_{8\eta})^2 + (F_{8\eta'})^2 , \quad (F_0)^2 = (F_{0\eta})^2 + (F_{0\eta'})^2 .
\]

For $F_8$, the evaluation of the loop graphs yields

\[
F_8 = F \left\{ 1 + 4(2M_K^2 + M_\pi^2)F^{-2}L_4^r + \frac{4}{3}(4M_K^2 - M_\pi^2)F^{-2}L_5^r - 3\mu_K + O(p^4) \right\} .
\]

Eliminating the coupling constant $L_5^r$ with (45), we obtain a parameter free prediction for $F_8$:

\[
\frac{F_8}{F_\pi} = 1 + \frac{4}{3} \Delta_F + \frac{1}{3} \mu_\pi - \frac{4}{3} \mu_K + \cos^2 \vartheta \mu_\eta + \sin^2 \vartheta \mu_{\eta'} ,
\]

with $\Delta_F \equiv F_K/F_\pi - 1 = 0.22$. Numerically, this yields $F_8 = 1.34 F_\pi$.

Since $F_0$ depends on the running scale of QCD, chiral symmetry does not predict its numerical value. In the chiral limit, we have $F_0 = \sqrt{1 + \Lambda_1} F$, in agreement with the scaling laws for $F_0$ and $\Lambda_1$. The corrections of order $m_s$ are determined by the coupling constants $L_4 = O(1)$, $L_5 = O(N_c)$ and $L_{18} = O(1)$. In contrast to $F_8$, the
constant $F_0$ does not receive contributions from loops. The result of the tree graph calculation may be written in scale invariant form:

$$F_0 = \sqrt{1 + \Lambda} \tilde{F}_0 ,$$
$$\tilde{F}_0 = F \left\{ 1 + \frac{4}{3} (2M_K^2 + M_\pi^2)F^{-2} (3L_4^r - L_5^r + L_A) + O(p^4) \right\} . \tag{49}$$

The term $L_A$ stands for the combination

$$L_A \equiv \frac{2L_5^r + 3L_{18}^r}{\sqrt{1 + \Lambda_1}} , \tag{50}$$

which is independent of the running scale $\mu$ of QCD as well as of the scale $\mu_\chi$ used in the renormalization of the effective theory ($2\Gamma_5 + 3\Gamma_{18} = 0$). In view of $3\Gamma_4 - \Gamma_5 = 0$, this also holds for $\tilde{F}_0$. In the ratio $\tilde{F}_0/F_\pi$, the coupling constant $L_4^r$ drops out. Hence $\tilde{F}_0$ may be expressed in terms of the scale invariant quantities $F_\pi, F_K$ and $L_A$.

In the large $N_c$ limit, $L_A$ is dominated by the contribution from $L_{18}^r$ -- both $L_{18}^r$ and $\Lambda_1$ represent OZI-violating corrections. The coupling constant $L_A$ also determines the slope of the scalar form factor $\langle \eta(p')|\bar{u}_u+d_d-2S_8|\eta(p)\rangle$, but there is no experimental information about it and, to our knowledge, a dispersive calculation has not been performed, either. The resonance exchange calculations described in [30] yield $L_{18} = \Lambda_1 = 0$, because the resonance couplings used there obey the OZI-rule. Inserting this estimate at the scale $\mu = M_\rho$, we obtain $L_A \simeq 2L_5^r(M_\rho)$, so that $\tilde{F}_0 \simeq F_\pi$. Note, however, that the result is very sensitive to the scale at which the OZI-rule ($|L_{18}^r(\mu_\chi)| \ll |L_5^r(\mu_\chi)|, |\Lambda_1| \ll 1$) is assumed to be valid. The phenomenological determination of $\tilde{F}_0$ on the basis of the photonic decays of $\eta$ and $\eta'$ should yield a more reliable value [31].

The above effective Lagrangian also allows us to calculate the two angles $\vartheta_8, \vartheta_0$ as well as $M_\pi, M_K, M_\eta, M_{\eta'}$ in terms of the coupling constants occurring therein. We will report about this calculation elsewhere [31] and restrict ourselves to the result for the difference between the two angles, which is related to

$$F_\eta F_\eta^0 + F_{\eta'} F_{\eta'}^0 = -F_8 F_0 \sin(\vartheta_0 - \vartheta_8) . \tag{51}$$

Since this combination of decay constants does not receive contributions from loops, it suffices to work out the tree graphs of the effective Lagrangian. The result may be written in the scale invariant form

$$\sin(\vartheta_0 - \vartheta_8) = \frac{8\sqrt{2}(M_K^2 - M_\pi^2)L_A}{3F_8 F_0} + O(p^4) . \tag{52}$$

The formula includes all corrections of order $1/N_c$, but accounts for the symmetry breaking effects generated by the quark masses only to leading order. With the rough estimate for $L_A$ discussed above, the result amounts to $\vartheta_0 - \vartheta_8 \simeq 14^\circ$, to be compared with the number that follows from the leading order formula (3): $\vartheta_0 - \vartheta_8 \simeq 16^\circ$. This may indicate that the higher order effects tend to slightly reduce the difference.
between the two angles, but phenomenological input is required to arrive at a reliable result. In this context, the relations (49) and (52) are very useful, because, even if the coupling constant $L_A$ is treated as a free parameter, they correlate the difference between the two angles with the magnitude of $\bar{F}_0$.

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