Local discontinuous Galerkin method on layer-adapted meshes for singularly perturbed reaction–diffusion problems in two dimensions

Yanjie Mei∗, Yao Cheng†, Sulei Wang†, Zhijie Xu†
March 2, 2021

Abstract. We analyse the local discontinuous Galerkin (LDG) method for two-dimensional singularly perturbed reaction–diffusion problems. A class of layer-adapted meshes, including Shishkin- and Bakhvalov-type meshes, is discussed within a general framework. Local projections and their approximation properties on anisotropic meshes are used to derive error estimates for energy and “balanced” norms. Here, the energy norm is naturally derived from the bilinear form of LDG formulation and the “balanced” norm is artificially introduced to capture the boundary layer contribution. We establish a uniform convergence of order $k$ for the LDG method using the balanced norm with the local weighted $L^2$ projection as well as an optimal convergence of order $k+1$ for the energy norm using the local Gauss–Radau projections. Numerical experiments are presented.

Keywords. local discontinuous Galerkin method, singularly perturbed, reaction-diffusion, layer-adapted meshes, balanced-norm

AMS. 65N30, 65N15, 65N12

1 Introduction

Over the past few decades, singularly perturbed problems have attracted considerable attention in the scientific community. Such problems arise in many applications, including the modelling of viscous fluid flows, semiconductor devices, and more.

∗International Education School, Suzhou University of Science and Technology, 215009 Suzhou, Jiangsu Province, P. R. China. E-mail: yjmey@post.usts.edu.cn.
†School of Mathematical Sciences, Suzhou University of Science and Technology, Suzhou 215009, Jiangsu Province, P. R. China. E-mail: ycheng@usts.edu.cn (corresponding author), slwang@post.usts.edu.cn, zjxu@post.usts.edu.cn.
For reaction-diffusion problems, difficulties arise owing to the presence of boundary layers in the solution. Unless the meshes are sufficiently refined, traditional finite-difference or finite-element methods on uniform or quasi-uniform meshes yield oscillatory and inaccurate numerical solutions. Consequently, three common approaches have been proposed in the literature. The first is to use traditional numerical methods on strongly refined, layer-adapted meshes, such as the Shishkin-type (S-type) or Bakhvalov-type (B-type) mesh \[2,12,13,18\]. Various parameter-uniform convergence results have been established in this way; notably, the order of convergence and error constant are independent of the singular perturbation parameters. The second approach is to use a stabilised numerical method, such as the streamline diffusion finite-element method, interior-penalty discontinuous Galerkin method, or local discontinuous Galerkin (LDG) method \[3,4,10\]; well-behaved local error estimates have been investigated using uniform or quasi-uniform meshes. The third approach is to combine the aforementioned stabilised numerical methods with layer-adapted meshes. From a practical perspective, the third approach is preferable because it is more stable and less sensitive to the choice of transition point on the layer-adapted mesh.

The LDG method is a form of finite-element method; it was first proposed as a generalisation of the discontinuous Galerkin (DG) method for a convection–diffusion problem \[7\]. Later, it was applied to solve the purely elliptic problem \[8\] and other higher-order partial differential equations \[20\]. Because the LDG method shares many advantages of the DG methods and can effectively simulate the acute change of a singular solution, it is particularly suited to solving singularly perturbed problems. For example, Cheng et al. performed double-optimal local error estimates for two explicit, fully discrete LDG methods on quasi-uniform meshes \[3,4\]. Xie et al. established uniform convergence and super-convergence analyses of the LDG method on a standard Shishkin mesh \[19, 21, 22\]. However, few results have been established for the LDG method on general S-type or B-type meshes.

Recently, we analysed the LDG method on several S-type meshes and a B-type mesh for singularly perturbed convection–diffusion problems. Robust error estimates were derived from the energy norm \[5\]. However, the reaction–diffusion case remains unexplored. Despite its simpler appearance, reaction-dominated diffusion without convection differs from convection–diffusion in the following three respects:

- For singularly perturbed reaction–diffusion problems, the boundary layer structure is considerably more complicated. As a result, the regularity of the solution is complex. This adds many difficulties to the theoretical analysis, such as in the construction of layer-adapted meshes and the estimates of various approximation errors.

- If a purely alternating numerical flux is employed in the LDG method, we have no interior boundary jump term in the energy norm. Therefore, it is possible to establish an optimal convergence of order \(k + 1\) for the LDG method in the energy norm. To highlight the influence of the singularly perturbed parameter,
we perform a more elaborate analysis for the two-dimensional Gauss–Radau projections on anisotropic meshes.

- In the reaction–diffusion region, the energy norm is inadequate because it cannot reflect the contribution of the boundary layer component. A balanced norm was introduced in [11] to address this problem. To date, balanced-norm error estimates are available for the Galerkin finite-element method (FEM) [15, 16], mixed FEM [11], and hp-FEM [14], but not for the LDG method. For the first time, we establish the uniform convergence of the LDG method for the balanced norm.

The remainder of this paper is organised as follows: in Section 2, we describe the LDG method; in Section 3, we introduce a class of layer-adapted meshes and state some elementary lemmas for them; in Section 4, we establish uniform convergence for the balanced and energy norms; in Section 5, we present numerical experiments; and in Section 6, we mention a convergence result for the fully discrete LDG $\theta$-scheme for parabolic singularly perturbed problems.

## 2 The LDG method

Consider a two-dimensional singularly perturbed reaction–diffusion problem, expressed as

$$
-\varepsilon \Delta u + b(x, y)u = f(x, y), \quad \text{in } \Omega = (0, 1)^2, \quad (2.1a)
$$

$$
u = 0, \quad \text{on } \partial \Omega, \quad (2.1b)
$$

where $0 < \varepsilon \ll 1$ is a perturbation parameter, and $b(x, y) \geq 2\beta^2 > 0$ for any $(x, y) \in \overline{\Omega}$ and for some positive constant $\beta$. In this section, we present the LDG method for (2.1).

Let $\Omega_N = \{K_{ij}\}_{i=1,2,...,N_x}^{j=1,2,...,N_y}$ be a rectangle partition of $\Omega$ with element $K_{ij} = I_i \times J_j$, where $I_i = (x_{i-1}, x_i)$ and $J_j = (y_{j-1}, y_j)$. We set $h_{x,i} = x_i - x_{i-1}$, $h_{y,j} = y_j - y_{j-1}$, and $h = \min_{K_{ij} \in \Omega_N} \min\{h_{x,i}, h_{y,j}\}$. We let

$$\mathcal{V}_N = \{v \in L^2(\Omega) : v|_K \in Q^k(K), K \in \Omega_N\}, \quad (2.2)$$

be the discontinuous finite-element space, where $Q^k(K)$ represents the space of polynomials on $K$ with a maximum degree of $k$ in each variable. $\mathcal{V}_N$ is contained in a broken Sobolev space, expressed as

$$\mathcal{H}^1(\Omega_N) = \{v \in L^2(\Omega) : v|_K \in H^1(K), \ K \in \Omega_N\}, \quad (2.3)$$

whose function is allowed to have discontinuities across element interfaces. For $v \in \mathcal{H}^1(\Omega_N)$ and $y \in J_j$, $j = 1, 2, \ldots, N_y$, we use $v_{i,y}^\pm = \lim_{x \to x_i^\pm} v(x, y)$ and $v_{x,j}^\pm =$
\[ \lim_{y \to y^+} v(x, y) \] to express the traces evaluated from the four directions. We denote
\[
[v]_{i,j} = v_{i,j}^+ - v_{i,j}^- \quad \text{for} \quad i = 1, 2, \ldots, N_x - 1, \quad [v]_{0,j} = v_{0,j}^+, \quad [v]_{N_x,j} = -v_{N_x,j}^-;
\]
\[
[v]_{x,j} = v_{x,j}^+ - v_{x,j}^- \quad \text{for} \quad j = 1, 2, \ldots, N_y - 1, \quad [v]_{x,0} = v_{x,0}^+, \quad [v]_{x,N_y} = -v_{x,N_y}^-;
\]
as the jumps on the vertical and horizontal edges, respectively.

Rewrite (2.1) into an equivalent first-order system:
\[
-p_x - q_y + bu = f, \quad \varepsilon^{-1}p = u_x, \quad \varepsilon^{-1}q = u_y, \quad \text{in} \ \Omega. \tag{2.4}
\]

Let \( \langle \cdot, \cdot \rangle_D \) be the inner product in \( L^2(D) \). Then, the LDG scheme is defined as follows. Find \( w = (u, p, q) \in V_N^3 \equiv V_N \times V_N \times V_N \) such that in each element \( K_{ij} \), the variational forms
\[
\langle p, v_x \rangle_{K_{ij}} - \langle \hat{p}_{i,y} - v_{i,y}^- \rangle_{J_j} + \langle \hat{p}_{i-1,y} - v_{i,y}^+ \rangle_{J_j} + \langle q, v_y \rangle_{K_{ij}} - \langle \hat{u}_{i,j} - v_{x,j}^- \rangle_{I_i} + \langle \hat{u}_{i,j-1} - v_{x,j}^+ \rangle_{I_i} + \langle b n, v \rangle_{K_{ij}} = \langle f, v \rangle_{K_{ij}}, \tag{2.5a}
\]
\[
\varepsilon^{-1} \langle p, s \rangle_{K_{ij}} + (u, s_x)_{K_{ij}} - \langle \hat{u}_{i,j} - s_{x,y}^- \rangle_{J_j} + \langle \hat{u}_{i,j-1} - s_{x,y}^+ \rangle_{J_j} = 0, \tag{2.5b}
\]
\[
\varepsilon^{-1} \langle q, r \rangle_{K_{ij}} + (u, r_y)_{K_{ij}} - \langle \hat{u}_{x,j} - r_{x,x}^- \rangle_{I_i} + \langle \hat{u}_{x,j-1} - r_{x,x}^+ \rangle_{I_i} = 0, \tag{2.5c}
\]
hold for any \( z = (v, s, r) \in V_N^3 \), where the “hat” terms are numerical fluxes defined by
\[
\hat{p}_{i,y} = \begin{cases} p_{i,y} + \lambda_{i,y}[u]_{i,y}, & i = 0, 1, \ldots, N_x - 1, \\ p_{N_x,y} - \lambda_{N_x,y}[u]_{N_x,y}, & i = N_x, \end{cases} \tag{2.6a}
\]
\[
\hat{u}_{i,j} = \begin{cases} 0, & i = 0, N_x, \\ u_{i,j}, & i = 1, 2, \ldots, N_x - 1, \end{cases} \tag{2.6b}
\]
for \( y \in J_j \) and \( j = 1, 2, \ldots, N_y \). Here, \( \lambda_{i,y} \geq 0(i = 0, 1, \ldots, N_x) \) are stabilisation parameters to be determined later. Analogously, for \( x \in I_i \) and \( i = 1, 2, \ldots, N_x \), we can define \( \hat{q}_{x,j} \) and \( \hat{w}_{x,j} \) for \( j = 0, 1, \ldots, N_y \).

Write \( \langle w, v \rangle = \sum_{K_{ij} \in \Omega_N} \langle w, v \rangle_{K_{ij}} \). Then, we write the above LDG method into a compact form: Find \( w = (u, p, q) \in V_N^3 \) such that
\[
B(w; z) = \langle f, v \rangle, \quad \forall z = (v, s, r) \in V_N^3, \tag{2.7}
\]
where
\[
B(w; z) = \mathcal{T}_1(w; z) + \mathcal{T}_2(u; z) + \mathcal{T}_3(w; v) + \mathcal{T}_4(u; v), \tag{2.8}
\]
with
\[
\mathcal{T}_1(w;z) = \varepsilon^{-1} \langle p, s \rangle + \varepsilon^{-1} \langle q, x \rangle + \langle bu, v \rangle,
\]
\[
\mathcal{T}_2(u;z) = \langle u, s_x \rangle + \sum_{j=1}^{N_y} \sum_{i=1}^{N_x-1} \langle u_{i,y}^+ [s]_{i,y} \rangle J_j + \langle u, x_y \rangle + \sum_{i=1}^{N_x} \sum_{j=1}^{N_y-1} \langle u_{x,j}^- [x]_{x,j} \rangle I_i,
\]
\[
\mathcal{T}_3(w;v) = \langle p, v_x \rangle + \sum_{j=1}^{N_y} \left[ \sum_{i=0}^{N_y} \langle p_{i,y}^+ [v]_{i,y} \rangle J_j - \langle p_{N_x,y}, v_{N_x,y}^- \rangle J_j \right] + \langle q, v_y \rangle + \sum_{i=1}^{N_x} \left[ \sum_{j=0}^{N_y-1} \langle q_{x,j}^+, [v]_{x,j} \rangle I_i - \langle q_{x,N_y}, v_{x,N_y}^- \rangle I_i \right],
\]
\[
\mathcal{T}_4(u;v) = \sum_{j=1}^{N_y} \sum_{i=0}^{N_x} \langle \lambda_{i,y} [u]_{i,y}, [v]_{i,y} \rangle J_j + \sum_{i=1}^{N_x} \sum_{j=0}^{N_y} \langle \lambda_{x,j} [u]_{x,j}, [v]_{x,j} \rangle I_i.
\]

3 Layer-adapted meshes

To introduce the layer-adapted meshes, we extract some precise information from the exact solution of (2.1) and its derivatives [6,9].

**Proposition 3.1.** Assume that the solution \( u \) of (2.1) can be decomposed by
\[
u = S + 4 \sum_{k=1}^{N_x} W_k + 4 \sum_{k=1}^{N_y} Z_k, \quad (x, y) \in \Omega,
\]
where \( S \) is a smooth part, \( W_k \) is a boundary layer part and \( Z_k \) is a corner layer part. More precisely, for \( 0 \leq i, j \leq k + 2 \), there exists a constant \( C \) independent of \( \varepsilon \) such that
\[
|\partial_x^i \partial_y^j S| \leq C, \quad |\partial_x^i \partial_y^j W_1| \leq C \varepsilon^{-\frac{i}{2}} e^{-\frac{\partial_x^i}{\sqrt{\varepsilon}}} , \quad |\partial_x^i \partial_y^j Z_1| \leq C \varepsilon^{-\frac{i+j}{2}} e^{-\frac{\partial_x^i \partial_y^j}{\sqrt{\varepsilon}}} ,
\]
and so on for the remaining terms. Here, \( \partial_x^i \partial_y^j v = \frac{\partial^{i+j} v}{\partial x^i \partial y^j} \).

The layer-adapted mesh is constructed as follows: For notational simplification, we assume that \( N_x = N_y = N \). Let \( N \geq 4 \) be a multiple of four. We introduce the mesh points
\[
0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1, \quad 0 = y_0 < y_1 < \cdots < y_{N-1} < y_N = 1,
\]
and consider a tensor-product mesh with mesh points \((x_i, y_j)\). Because both meshes have the same structure, we only describe the mesh in the \( x \)-direction.

Suppose \( \varphi \) is a function defined in \([0, 1/4] \) with
\[
\varphi(0) = 0, \varphi' > 0, \varphi'' \geq 0.
\]
We define the transition parameter
\[
\tau = \min \left\{ \frac{1}{4}, \frac{\sigma \sqrt{\epsilon}}{\beta} \varphi(1/4) \right\},
\]  
(3.12)
where \( \sigma > 0 \) is determined later. Assume that \( \sqrt{\epsilon} \leq N^{-1} \) means that we are in the singularly perturbed case. Moreover, \( \epsilon \) is sufficiently small that (3.12) is replaced by
\[
\tau = \frac{\sigma \sqrt{\epsilon}}{\beta} \varphi(1/4).
\]

The mesh in the \( x \)-direction is equidistant on \([\tau, 1-\tau]\) with \(N/2\) elements, but it is gradually divided on \([0, \tau]\) and \([1-\tau, 1]\) with \(N/4\) elements. Hence, we set the mesh points as
\[
\begin{align*}
x_i &= \begin{cases} 
\frac{\sigma \sqrt{\epsilon}}{\beta} \varphi(i/N), & \text{for } i = 0, 1, \ldots, N/4, \\
\tau + 2(1-2\tau)(i/N - 1/4), & \text{for } i = N/4 + 1, \ldots, 3N/4 - 1, \\
1 - \frac{\sigma \sqrt{\epsilon}}{\beta} \varphi(1-i/N), & \text{for } i = 3N/4, \ldots, N.
\end{cases}
\end{align*}
\]  
(3.13)

In Table 1, we list three typical layer-adapted meshes [13]: Shishkin (S-mesh), Bakhvalov–Shishkin (BS mesh) and Bakhvalov-type (B-type mesh), together with \( \psi = e^{-\varphi} \) and the important quantity \( \max |\psi'| \), which arises in error estimates. Figure illustrates the divisions of \( \Omega \) and the generated meshes for \( \epsilon = 10^{-2} \) and \( N = 16 \).

Note that for these meshes and under the previous assumption, we always have \( \tau \geq \frac{\sigma \sqrt{\epsilon}}{\beta} \ln N \).

| \hline \hline
| \text{S-mesh} | \text{BS-mesh} | \text{B-type mesh} |
| \hline
| \varphi(t) | 4t \ln N | - \ln [1 - 4(1 - N^{-1})t] |
| \psi(t) | N^{-4t} | 1 - 4(1 - N^{-1})t |
| \max |\psi'| | C \ln N | C |
| \hline

In the following, we state two preliminary lemmas that will be frequently employed in the subsequent analysis. Because \( h_{x,i} = h_{y,i}, i = 1, 2, \ldots, N \), we simply use \( h_i \) to denote one of them.

**Lemma 3.1.** Suppose that
\[
\Theta_i = \min \left\{ \frac{h_i}{\sqrt{\epsilon}}, 1 \right\} e^{-\frac{\beta \varphi(i-1)}{\sigma \sqrt{\epsilon}}}, \quad i = 1, 2, \ldots, N/4.
\]

Then, there exists a constant \( C > 0 \) independent of \( \epsilon \) and \( N \) such that
\[
\max_{1 \leq i \leq N/4} \Theta_i \leq C N^{-1} \max |\psi'|, \quad \text{(3.14a)}
\]
\[
\sum_{i=1}^{N/4} \Theta_i \leq C. \quad \text{(3.14b)}
\]
Figure 1: The division of \( \Omega \)

**Proof.** See [5] for details.

**Lemma 3.2.** We have \( h_{N/4+1} = h_{N/4+2} = \cdots = h_{3N/4} \) and \( h \geq C \sqrt{\varepsilon} N^{-1} \max |\psi'| \).

For the \( S \)-type meshes,

\[
\begin{cases}
1 \geq h_i/h_{i+1} \geq C & i = 1, 2, \cdots, N/4 - 1, \\
1 \geq h_{i+1}/h_i \geq C & i = 3N/4 + 1, 3N/4 + 2, \cdots, N - 1.
\end{cases}
\]  

(3.15)

For the \( B \)-type mesh,

\[
\begin{cases}
1 \geq h_i/h_{i+1} \geq C & i = 1, 2, \cdots, N/4 - 2, \\
1 \geq h_{i+1}/h_i \geq C & i = 3N/4 + 2, 3N/4 + 3, \cdots, N - 1.
\end{cases}
\]  

(3.16)

Moreover,

\[
h_{ly} \equiv \max_{i=1, \cdots, N/4, i=3N/4, \cdots, N} h_i \leq C \rho \equiv C \begin{cases}
\sqrt{\varepsilon}, & \text{for } S\text{-type meshes,} \\
N^{-1}, & \text{for } B\text{-type mesh.}
\end{cases}
\]  

(3.17)

**Proof.** It can be seen that \( h_{N/4+1} = h_{N/4+2} = \cdots = h_{3N/4} \). (3.15) and (3.16) can be verified trivially (see [5]). By (3.13), (3.11), and \( \psi = e^{-\varphi} \), we have

\[
h \geq C \sqrt{\varepsilon} N^{-1} \min |\varphi'| = C \sqrt{\varepsilon} N^{-1} |\varphi'(0)| = C \sqrt{\varepsilon} N^{-1} |\psi'(0)| = C \sqrt{\varepsilon} N^{-1} \max |\psi'|.
\]
Now, we prove (3.17). Combining the fact that \( h_{ly} \leq C N^{-1} \) with the inequality
\[
h_{ly} \leq C \sqrt{\varepsilon} N^{-1} \max |\phi'| \leq C \sqrt{\varepsilon} N^{-1} |\phi'(1/4)|,
\]
we obtain
\[
h_{ly} \leq C \min \left\{ \sqrt{\varepsilon} N^{-1} |\phi'(1/4)|, N^{-1} \right\},
\]
which implies (3.17).

4 Convergence analysis

In this section, we perform uniform convergence analysis for the LDG method on layer-adapted meshes. Two related norms are considered.

The first is the energy norm, which is naturally derived from the formulation of the LDG method; that is,
\[
\|w\|_E^2 = \varepsilon^{-1} \|p\|^2 + \varepsilon^{-1} \|q\|^2 + \|b^{1/2}u\|^2 + \sum_{j=1}^{N} \sum_{i=0}^{N} \langle \lambda_i, [u]_{i,y}^2 \rangle_{J_j} + \sum_{i=1}^{N} \sum_{j=0}^{N} \langle \lambda_x, [u]_{x,j}^2 \rangle_{I_i},
\]
by using integration by parts and some trivial manipulations. Here, \( \|z\|^2 = \sum_{K \in \Omega_N} \|z\|^2_K \) and \( \|z\|^2_K = \langle z, z \rangle_K \).

However, this norm is inadequate for reaction–diffusion problems because the layer contributions are not “seen.” In fact, letting \( u = e^{-\beta(x+y)/\sqrt{\varepsilon}} \), we have \( \|(u, \varepsilon u_x, \varepsilon u_y)\|_E = O(\sqrt{\varepsilon}) \), which vanishes as \( \varepsilon \to 0 \). Thus, the following “balanced” norm is introduced:
\[
\|w\|_B^2 = \varepsilon^{-3/2} \|p\|^2 + \varepsilon^{-3/2} \|q\|^2 + \|b^{1/2}u\|^2 + \sum_{j=1}^{N} \sum_{i=0}^{N} \langle 1, [u]_{i,y}^2 \rangle_{J_j} + \sum_{i=1}^{N} \sum_{j=0}^{N} \langle 1, [u]_{x,j}^2 \rangle_{I_i}. 
\]

In the following subsections, we perform convergence analysis for these two norms. Different projections are introduced, and the related approximation properties are investigated.

4.1 Convergence of balanced norm

First, we analyse the LDG method for the balanced norm (4.20). Let \( \omega \in C^1(\Omega_N) \) and \( \omega \geq \omega_0 > 0 \) be a general weight function. We define the piecewise local weight \( L^2 \) projection \( \Pi_{\omega} \) as follows: For each element \( K \in \Omega_N \) and for any \( z \in L^2(\Omega_N) \), \( \Pi_{\omega}z \in V_N \) satisfies
\[
\langle \omega \Pi_{\omega}z, v \rangle_K = \langle \omega z, v \rangle_K, \quad \forall v \in Q^k(K).
\]
In the special case of $\omega = 1$, this operator reduces to the classical local $L^2$ projection, which is denoted by $\Pi$.

Lemma 4.1. \cite{1} There exists a constant $C > 0$, independent of the element size and $z$, such that

$$\|\Pi_\omega z\|_{L^m(K_{ij})} \leq C \|z\|_{L^m(K_{ij})}, \quad (4.22a)$$

$$\|z - \Pi_\omega z\|_{L^m(K_{ij})} \leq C \left[ h_i^{k+1} \|\partial_x^{k+1} z\|_{L^m(K_{ij})} + h_j^{k+1} \|\partial_y^{k+1} z\|_{L^m(K_{ij})} \right], \quad (4.22b)$$

where $m \in \{2, \infty\}$.

Lemma 4.2. Let $\sigma \geq k + 1.5$. Then, there holds

$$\|u - \Pi_\omega u\| \leq C \left( \sqrt{\varepsilon} (N^{-1} \max |\psi'|)^{k+1} + N^{-k+1} \right), \quad (4.23a)$$

$$\|u - \Pi_\omega u\|_{L^\infty(\Omega_N)} \leq C (N^{-1} \max |\psi'|)^{k+1}, \quad (4.23b)$$

$$\sum_{i=1}^{N} \sum_{i=1}^{N-1} \|(u - \Pi_\omega u)_{i,y}\|_{J_j} \leq C (N^{-1} \max |\psi'|)^{2k+1}, \quad (4.23c)$$

$$\sum_{i=1}^{N} \sum_{i=1}^{N} \langle 1, [u - \Pi_\omega u]_{i,y} \rangle_{J_j} \leq C (N^{-1} \max |\psi'|)^{2k+1}, \quad (4.23d)$$

$$\|p - \Pi_\omega p\| \leq C \varepsilon^{\frac{3}{2}} (N^{-1} \max |\psi'|)^{k+1}, \quad (4.23e)$$

$$\sum_{i=1}^{N} \|(p - \Pi_\omega p)_{N,y}\|_{J_j} \leq C \varepsilon (N^{-1} \max |\psi'|)^{2(k+1)}, \quad (4.23f)$$

$$\sum_{i=1}^{N} \sum_{i=1}^{N-1} \|(p - \Pi_\omega p)_{i,y}\|_{J_j} \leq C \varepsilon (N^{-1} \max |\psi'|)^{2k+1}, \quad (4.23g)$$

where $C > 0$ is independent of $\varepsilon$ and $N$. A similar procedure applies for $u$ and $q$ in other spatial directions.

Proof. Let $\eta_u = u - \Pi_\omega u$, $\eta_p = p - \Pi_\omega p$, and $\eta_q = q - \Pi_\omega q$. We prove (4.23) separately.

(1) Prove (4.23a). Recalling the decomposition $u = S + \sum_{k=1}^{4} W_k + \sum_{k=1}^{4} Z_k$, we have

$$\|\eta_u\| \leq \|\eta_S\| + \sum_{k=1}^{4} \|\eta_{W_k}\| + \sum_{k=1}^{4} \|\eta_{Z_k}\|. \quad (4.24)$$

Using (4.22b) with $m = 2$ and (3.10), we obtain

$$\|\eta_S\| \leq CN^{-(k+1)} \left[ \|\partial_x^{k+1} S\| + \|\partial_y^{k+1} S\| \right] \leq CN^{-(k+1)}, \quad (4.25)$$
because \( h_i \leq CN^{-1} \) for \( i = 1, 2, \ldots, N \).

Moreover, denote \( \Omega_{XL} = \Omega_{11} \cup \Omega_{21} \cup \Omega_{31}, \ \Omega_{XM} = \Omega_{12} \cup \Omega_{22} \cup \Omega_{32}, \) and \( \Omega_{XR} = \Omega_{13} \cup \Omega_{23} \cup \Omega_{33} \); then, we have

\[
\| \eta_{W_1} \|^2 = \sum_{K_{ij} \in \Omega_{XL}} \| \eta_{W_1} \|^2_{K_{ij}} + \sum_{K_{ij} \in \Omega_{XM} \cup \Omega_{XR}} \| \eta_{W_1} \|^2_{K_{ij}} \equiv A_1 + A_2. \tag{4.26}
\]

For \( A_1 \), we use (4.22a) and (4.22b) with \( m = 2 \) to obtain the two estimates

\[
A_1 \leq C \sum_{K_{ij} \in \Omega_{XL}} \| W_1 \|^2_{K_{ij}} \leq C \sum_{K_{ij} \in \Omega_{XL}} \left( e^{-\delta \tau} \right)^2_{K_{ij}}, \tag{4.27}
\]

\[
A_1 \leq C \sum_{K_{ij} \in \Omega_{XL}} \left[ h_i^{2(k+1)} \left\| \partial_{x}^{k+1} W_1 \right\|^2_{K_{ij}} + h_j^{2(k+1)} \right\| \partial_{y}^{k+1} W_1 \|^2_{K_{ij}} \right] \leq C \sum_{K_{ij} \in \Omega_{XL}} \left[ \left( \frac{h_i}{\sqrt{\varepsilon}} \right)^{2(k+1)} + h_j^{2(k+1)} \right] \left\| e^{-\delta \tau} \right\|^2_{K_{ij}}, \tag{4.28}
\]

respectively, where we have used (3.10). Combining (4.27) with (4.28) and using \( h_i/\sqrt{\varepsilon} \geq CN^{-1} \geq C h_j \), \( \sum_{j=1}^{N} h_j = 1 \), \( \sigma \geq k + 1.5 \), and (3.14), we have

\[
A_1 \leq C \sum_{K_{ij} \in \Omega_{XL}} \min \left\{ \left( \frac{h_i}{\sqrt{\varepsilon}} \right)^{2(k+1)}, 1 \right\} \left\| e^{-\delta \tau} \right\|^2_{K_{ij}} \leq C \sqrt{\varepsilon} \max_{1 \leq i \leq N/4} \Theta_i^{2(k+1)} \sum_{i=1}^{N/4} \Theta_i \leq C \sqrt{\varepsilon} (N^{-1} \max |\psi'|)^{2(k+1)}, \tag{4.29}
\]

where we used the trivial inequality

\[
\left\| e^{-\delta \tau} \right\|^2_{K_{ij}} = \frac{\sqrt{\varepsilon}}{2 \beta} h_j e^{-\frac{2\beta x_{i-1}}{\sqrt{\varepsilon}}} (1 - e^{-\frac{2\beta h_i}{\sqrt{\varepsilon}}}) \leq C \sqrt{\varepsilon} h_j e^{-\frac{2\beta x_{i-1}}{\sqrt{\varepsilon}}} \min \left\{ \frac{h_i}{\sqrt{\varepsilon}}, 1 \right\},
\]

because \( 1 - e^{-x} \leq \min\{1, x\} \) for \( x \geq 0 \). For \( A_2 \), we use (4.22a) and (3.10) to obtain

\[
A_2 \leq C \sum_{K_{ij} \in \Omega_{XM} \cup \Omega_{XR}} \| W_1 \|^2_{K_{ij}} \leq C \int_{0}^{1} dy \int_{\tau}^{1} e^{-\frac{2\beta x}{\sqrt{\varepsilon}}} dx \leq C \sqrt{\varepsilon} e^{-\frac{2\beta \tau}{\sqrt{\varepsilon}}} \leq C \sqrt{\varepsilon} N^{-2\sigma}. \tag{4.30}
\]

Inserting (4.29) and (4.30) into (4.26) yields

\[
\| \eta_{W_1} \| \leq C \sqrt{\varepsilon} \left( (N^{-1} \max |\psi'|)^{k+1} + N^{-\sigma} \right). \tag{4.31}
\]
In addition,

\[ \| \eta Z_i \|^2 = \sum_{K_{ij} \in \Omega_{11}} \| \eta Z_i \|^2_{K_{ij}} + \sum_{K_{ij} \in \Omega_N \setminus \Omega_{11}} \| \eta Z_i \|^2_{K_{ij}} = \Xi_1 + \Xi_2. \]

Using (4.22a) and (4.22b) with \( m = 2 \) gives the two estimates

\[ \Xi_1 \leq C \sum_{K_{ij} \in \Omega_{11}} \| Z_1 \|^2_{K_{ij}} \leq C \sum_{K_{ij} \in \Omega_{11}} \bigg\| e^{-\frac{\beta(x+y)}{\sqrt{x}}} \bigg\|_{K_{ij}}^2, \quad (4.32) \]

\[ \Xi_1 \leq C \sum_{K_{ij} \in \Omega_{11}} \left[ h_i^{2(k+1)} \| \partial_x Z_1 \|^2_{K_{ij}} + h_j^{2(k+1)} \| \partial_y Z_1 \|^2_{K_{ij}} \right] \leq C \sum_{K_{ij} \in \Omega_{11}} \left[ \left( \frac{h_i}{\sqrt{\epsilon}} \right)^{2(k+1)} + \left( \frac{h_j}{\sqrt{\epsilon}} \right)^{2(k+1)} \right] \bigg\| e^{-\frac{\beta(x+y)}{\sqrt{x}}} \bigg\|_{K_{ij}}^2, \quad (4.33) \]

respectively. This leads to

\[ \Xi_1 \leq C \sum_{K_{ij} \in \Omega_{11}} \min \left\{ \left( \frac{h_i}{\sqrt{\epsilon}} \right)^{2(k+1)}, 1 \right\} \bigg\| e^{-\frac{\beta(x+y)}{\sqrt{x}}} \bigg\|_{K_{ij}}^2 \]

\[ \leq C \sum_{i=1}^{N/4} \min \left\{ \left( \frac{h_i}{\sqrt{\epsilon}} \right)^{2(k+1)}, 1 \right\} e^{-\frac{2\beta x_{i-1}}{\sqrt{x}}} \min \left\{ \frac{h_i}{\sqrt{\epsilon}}, 1 \right\} \sum_{j=1}^{N/4} e^{-\frac{2\beta y_{j-1}}{\sqrt{y}}} \min \left\{ \frac{h_j}{\sqrt{\epsilon}}, 1 \right\} \]

\[ \leq C \sum_{1 \leq i \leq N/4} \Theta_i^{2(k+1)} \sum_{j=1}^{N/4} \Theta_j + C \sum_{1 \leq j \leq N/4} \Theta_j^{2(k+1)} \sum_{i=1}^{N/4} \Theta_i \]

\[ \leq C \left( N^{-1} \max \| \psi' \| \right)^{2(k+1)}, \quad (4.34) \]

where we used \( \sigma \geq k + 1.5, \) Lemma 3.1, the trivial inequality \( \min\{1, a + b\} \leq \min\{a\} + \min\{b\} , \) and

\[ \bigg\| e^{-\frac{\beta(x+y)}{\sqrt{x}}} \bigg\|_{K_{ij}}^2 = \frac{\epsilon}{4\beta^2} e^{-\frac{2\beta x_{i-1} + y_{j-1}}{\sqrt{x}}} (1 - e^{-\frac{2\beta y_{j-1}}{\sqrt{y}}}) (1 - e^{-\frac{2\beta h_i}{\sqrt{x}}}) \]

\[ \leq C \epsilon e^{-\frac{2\beta(x+y)}{\sqrt{x}}} \min \left\{ \frac{h_i}{\sqrt{\epsilon}}, 1 \right\} \min \left\{ \frac{h_j}{\sqrt{\epsilon}}, 1 \right\}. \]

Similar to before, we have

\[ \Xi_2 \leq C \sum_{K_{ij} \in \Omega_N \setminus \Omega_{11}} \| Z_1 \|^2_{K_{ij}} \leq C \int_{\Omega_N \setminus \Omega_{11}} e^{-\frac{2\beta(x+y)}{\sqrt{x}}} \, dx dy \leq C \epsilon e^{-\frac{2\beta x}{\sqrt{x}}} \leq C \epsilon^{-2\sigma}. \quad (4.35) \]
From (4.34) and (4.35), we obtain
\[ \|\eta Z_1\| \leq C\sqrt{\varepsilon}(N^{-1}\max|\psi'|)^{k+1}. \] (4.36)

Similarly, we can bound the other terms in (4.24) and arrive at (4.23a).

(2) Prove (4.23b) and (4.23c). It can be seen that
\[ \|\eta\|_{\infty} \leq C \|W_1\|_{\infty} + h_j \|\partial_x W_1\|_{\infty} + h_j \|\partial_y W_1\|_{\infty}. \]

From (4.34) and (4.35), we obtain
\[ 1 + \sum_{j=1}^{N} \sum_{i=1}^{N-1} \|\eta_{i,j}\|^2_{\infty} \leq C \sum_{j=1}^{N} \sum_{i=1}^{N-1} h_j\|\eta\|^2_{L^\infty(K_{ij})} \leq C \sum_{j=1}^{N} \sum_{i=1}^{N-1} h_j N^{-(k+1)} \leq CN^{-(2k+1)}. \]

For \( K_{ij} \in \Omega_{XL} \), we obtain from the \( L^\infty \)-stability (4.22a), the \( L^\infty \)-approximation (4.22b), and Lemma 3.1 that
\[ \|\eta_{W_1}\|_{L^\infty(K_{ij})} \leq C \min \left\{ 1, \left( \frac{h_i}{\sqrt{\varepsilon}} \right)^{2(k+1)}, \frac{2(k+1)}{h_j} \right\} e^{-\frac{2\varepsilon i^2}{\sqrt{\varepsilon}}} \]
\[ \leq C \Theta^{2(k+1)} \leq C(N^{-1}\max|\psi'|)^{2(k+1)}, \] (4.37)

where we have used (3.10) and \( \eta_i/\sqrt{\varepsilon} \geq CN^{-1} \geq h_j \). For \( K_{ij} \in \Omega_{XM} \cup \Omega_{XR} \), we obtain from the \( L^\infty \)-stability (4.22a) and \( \sigma \geq k+1 \) that
\[ \|\eta_{W_1}\|_{L^\infty(K_{ij})} \leq C \|W_1\|_{L^\infty(K_{ij})} \leq Ce^{-\frac{2\varepsilon i^2}{\sqrt{\varepsilon}}} \leq Ce^{-\frac{2\varepsilon x}{\sqrt{\varepsilon}}} \leq CN^{-2(k+1)}. \]

Consequently, we obtain from (3.14) that
\[ \sum_{j=1}^{N} \sum_{i=1}^{N-1} \|\eta_{W_1}\|_{\infty}^2_{\infty} \leq C \sum_{i=1}^{N/4} \sum_{j=1}^{N} h_j \Theta_i^{2(k+1)} + C \sum_{i=N/4+1}^{N} \sum_{j=1}^{N} h_j N^{-(2k+1)} \]
\[ \leq C \max_{1 \leq i \leq N/4} \Theta_i^{2k+1} \sum_{j=1}^{N} h_j + CN^{-(2k+1)} \]
\[ \leq C\left( \max_{1 \leq i \leq N} \Theta_i \sum_{j=1}^{N} h_j \right)^{(2k+1)} + N^{-(2k+1)}. \] (4.38)

Moreover, we have \( \|\eta Z_1\|_{L^\infty(K_{ij})}^2 \leq CN^{-2(k+1)} \) for \( K_{ij} \in \Omega_N \setminus \Omega_{11} \). For \( K_{ij} \in \Omega_{11} \), we obtain
\[ \|\eta Z_1\|_{L^\infty(K_{ij})}^2 \leq C \min \left\{ \|Z_1\|_{L^\infty(K_{ij})}^2, h_i^{2(k+1)} \|\partial_x Z_1\|_{L^\infty(K_{ij})}^2 + h_j^{2(k+1)} \|\partial_y Z_1\|_{L^\infty(K_{ij})}^2 \right\} \]
\[ \leq C \min \left\{ 1, \left( \frac{h_i}{\sqrt{\varepsilon}} \right)^{2(k+1)} + \left( \frac{h_j}{\sqrt{\varepsilon}} \right)^{2(k+1)} \right\} e^{-\frac{2\varepsilon (i^2+y^2)}{\sqrt{\varepsilon}}} \]
\[ \leq C(\Theta_i^{2(k+1)} + \Theta_j^{2(k+1)}) \leq C(N^{-1}\max|\psi'|)^{2(k+1)}, \]

12
using \( \sigma \geq k + 1 \). In a similar fashion, we obtain

\[
\sum_{j=1}^{N} \sum_{i=1}^{N-1} \| (\eta_{Z_1})_{i,j}^{-} \|_{J_j}^2 \leq C \left[ (N^{-1} \max |\psi'|)^{(2k+1)} + N^{-(2k+1)} \right]. 
\]

(4.39)

From the solution decomposition and similar arguments for the other terms, we arrive at (4.23b) and (4.23c).

(3) Prove (4.23d). We start from the following inequality:

\[
\sum_{j=1}^{N} \sum_{i=0}^{N} \langle 1, \eta_{u} \rangle_{j} \leq 2 \left[ \sum_{j=1}^{N} \sum_{i=1}^{N} \int_{J_j} [ (\eta_{u})_{i,j}^+]^2 dy + \sum_{j=1}^{N} \sum_{i=1}^{N} \int_{J_j} [ (\eta_{u})_{i,j}^-]^2 dy \right].
\]

For the first term, we notice that

\[
\sum_{j=1}^{N} \sum_{i=1}^{N} \int_{J_j} [ (\eta_{u})_{i,j}^+]^2 dy \leq \sum_{j=1}^{N} \sum_{i=1}^{N} h_j \| \eta_{u} \|_{L^\infty(K_{ij})}^2
\]

and proceed as before. For the second term, we use (4.23c). Thus, (4.23d) follows.

The remaining inequalities of (4.23) can be proved analogously; we omit the details here.

\( \square \)

**Theorem 4.1.** Suppose that \( \lambda_{i,j} = \lambda_{x,j} = \sqrt{\varepsilon} \) for \( i, j = 0, 1, \cdots, N \). Let \( w = (u, p, q) \) be the solution to problem (2.1), satisfying Proposition 3.1; furthermore, let \( w = (u, p, q) \in V_N^3 \) be the numerical solution of the LDG scheme (2.5) on layer-adapted meshes (3.13) when \( \sigma \geq k + 1.5 \). Then, there exists a constant \( C > 0 \), independent of \( \varepsilon \) and \( N \), such that

\[
\| w - w \|_B \leq CN^{-k}(\max |\psi'|)^{k+1/2}. 
\]

(4.40)

**Proof.** Denote \( e = w - w \equiv \eta - \xi \) as

\[
\eta = (\eta_u, \eta_p, \eta_q) = (u - \Pi_b u, p - \Pi_p q - \Pi q), \quad \xi = (\xi_u, \xi_p, \xi_q) = (u - \Pi_b u, p - \Pi_p q - \Pi q) \in V_N^3,
\]

where \( \Pi_b \) is defined in (4.21) with weight \( \omega = b \).

From Proposition 3.1 and the consistency of numerical flux, we obtain the error equation:

\[
B(\xi; z) = B(\eta; z) = T_1(\eta; z) + T_2(\eta_u; z) + T_3(\eta; v) + T_4(\eta_u; v).
\]

(4.42)

It can be seen that \( T_1(\eta; z) = 0 \).
To bound $\mathcal{T}_2(\eta_u; \z)$, we use the Cauchy–Schwarz inequality, inverse inequality, $C\sqrt{\varepsilon}N^{-1} \max |\psi'| \leq h \leq CN^{-1}$, (4.23b), and $\varepsilon^{-1/2} \|\z\|_E$ to obtain
\[
|\langle \eta_u, s_x \rangle| \leq \sum_K \|\eta_u\|_K \|s_x\|_K \leq C \sum_K |K|^{1/2} \|\eta_u\|_{L^\infty(K)} h^{-1}_x \|s\|_K \leq C\sqrt{\varepsilon}N^{-k}(\max |\psi'|)^{k+1/2} \|\z\|_E,
\]
and we used (4.23c) to obtain
\[
\left| \sum_{j=1}^N \sum_{i=1}^{N-1} \langle (\eta_u)^+_i, [s]_i, \psi \rangle \right| \leq C\left( \sum_{j=1}^N \sum_{i=1}^{N-1} \| (\eta_u)^+_i \|_j \right)^{1/2} \left( h^{-1/2} \|s\| \right) \leq C\sqrt{\varepsilon} (N^{-1} \max |\psi'|)^k \|\z\|_E.
\]
Consequently, we have
\[
\mathcal{T}_2(\eta_u; \z) \leq C\sqrt{\varepsilon} N^{-k}(\max |\psi'|)^{k+1/2} \|\z\|_E. \quad (4.43)
\]
Analogously, we have
\[
\mathcal{T}_3(\eta; \psi) = \sum_{j=1}^N \left[ \sum_{i=0}^{N-1} \langle (\eta_p)^+_{i}, [\psi]_{i,y} \rangle \right] \leq C\left( \sum_{j=1}^N \sum_{i=0}^{N-1} \varepsilon^{-1} \| (\eta_p)^+_{i,y} \|_j \right)^{1/2} \left( \sqrt{\varepsilon} h^{-1/2} \|\psi\| \right)
\]
\[
\leq Ch^{-1/2} \sqrt{\varepsilon} (N^{-1} \max |\psi'|)^{k+1/2} \|\psi\| \leq C\sqrt{\varepsilon} (N^{-1} \max |\psi'|)^k \|\z\|_E. \quad (4.44)
\]
Here, (4.23f) and (4.23g) were used.

Finally, we use the Cauchy–Schwarz inequality, (4.23d), and the assumption that $\lambda_{i,j} = \sqrt{\varepsilon}$ for $i, j = 0, 1, \ldots, N$, to obtain
\[
\mathcal{T}_4(\eta_u; \psi) = \sum_{j=1}^N \sum_{i=0}^{N-1} \langle \lambda_{i,j} [\eta_u]_{i,y}, [\psi]_{i,y} \rangle \leq C\left( \sum_{j=1}^N \sum_{i=0}^{N-1} \lambda_{i,j} \langle [\eta_u]_{i,j}, [\psi]_{i,j} \rangle \right)^{1/2} \|\z\|_E \leq C\sqrt{\varepsilon} (N^{-1} \max |\psi'|)^{k+1/2} \|\z\|_E. \quad (4.45)
\]
From (4.43)–(4.45), we have
\[ \|\xi\|_E^2 = B(\xi; \xi) = B(\eta; \xi) \leq C \sqrt{\varepsilon} N^{-k} (\max \vert \psi' \vert)^{k+1/2} \|\xi\|_E, \]
which implies
\[ \|\xi\|_B \leq \varepsilon^{-1/4} \|\xi\|_E \leq C \sqrt{\varepsilon} N^{-k} (\max \vert \psi' \vert)^{k+1/2}. \]
Using (4.23) and a trivial inequality, we derive (4.40).

### 4.2 Improvement of convergence in energy norm

In this subsection, we perform an elaborate analysis and establish an optimal convergence result in the energy norm. The following local Gauss–Radau projections are required.

For each element \( K_{ij} \in \Omega_N \) and for any \( z \in H^1(K_{ij}) \), \( \Pi^- z, \Pi^+_x z, \Pi^+_y z \in Q^k(K_{ij}) \) are defined as

\[
\begin{align*}
\langle \Pi^- z, \psi \rangle_{K_{ij}} &= \langle z, \psi \rangle_{K_{ij}}, \quad \forall \psi \in Q^{k-1}(K_{ij}), \\
\langle \Pi^- z_{i,y}, \psi \rangle_{J_J} &= \langle z_{i,y}, \psi \rangle_{J_J}, \quad \forall \psi \in P^{k-1}(J), \\
\langle \Pi^- z_{x,j}, \psi \rangle_{I_i} &= \langle z_{x,j}, \psi \rangle_{I_i}, \quad \forall \psi \in P^{k-1}(I_i), \\
\Pi^- z(x_i, y_j) &= z(x_i, y_j).
\end{align*}
\]

(4.48a)

\[
\begin{align*}
\langle \Pi^+_x z, \psi \rangle_{K_{ij}} &= \langle z, \psi \rangle_{K_{ij}}, \quad \forall \psi \in P^{k-1}(I_i) \otimes P^{k}(J), \\
\langle \Pi^+_x z_{i,y}, \psi \rangle_{J_J} &= \langle z_{i,y}, \psi \rangle_{J_J}, \quad \forall \psi \in P^{k}(J), \\
\langle \Pi^+_y z, \psi \rangle_{K_{ij}} &= \langle z, \psi \rangle_{K_{ij}}, \quad \forall \psi \in P^{k}(I_i) \otimes P^{k-1}(J), \\
\langle \Pi^+_y z_{x,j}, \psi \rangle_{J_J} &= \langle z_{x,j}, \psi \rangle_{J_J}, \quad \forall \psi \in P^{k}(I_i).
\end{align*}
\]

(4.48b)

(4.48c)

**Lemma 4.3.** \([1,5,22]\) There exists a constant \( C > 0 \), independent of the element size and \( z \), such that

\[
\begin{align*}
\|\Pi^- z\|_{K_{ij}} &\leq C \left[ \|z\|_{K_{ij}} + h_j \|z_{x,j}\|_{I_i} + h_i \|z_{i,y}\|_{J} + h_i h_j |z_{i,j}| \right], \\
\|\Pi^+_x z\|_{K_{ij}} &\leq C \left[ \|z\|_{K_{ij}} + h_i \|z_{i,y}\|_{J} \right], \\
\|\Pi^+_y z\|_{K_{ij}} &\leq C \left[ \|z\|_{K_{ij}} + h_j \|z_{x,j}\|_{I_i} \right], \\
\|\Phi z\|_{L^\infty(K_{ij})} &\leq C \|z\|_{L^\infty(K_{ij})}, \\
\|z - \Phi z\|_{L^m(K_{ij})} &\leq C \left[ h_i^{k+1} \left\| \partial_x^{k+1} z \right\|_{L^m(K_{ij})} + h_j^{k+1} \left\| \partial_y^{k+1} z \right\|_{L^m(K_{ij})} \right],
\end{align*}
\]

where \( \Phi \in \{\Pi^-, \Pi^+_x, \Pi^+_y\} \), \( m \in \{2, \infty\} \) and \( z_{i,j} = z(x_i, y_j) \).
Lemma 4.4. Let $\sigma \geq k + 1.5$. Then, there exists a constant $C > 0$ independent of $\varepsilon$ and $N$ such that

$$
\|u - \Pi^- u\| \leq C\left((\sqrt{\varepsilon} + \sqrt{\varrho})(N^{-1} \max |\psi'|)^{k+1} + N^{-(k+1)}\right),
$$

(4.50a)

$$
\sum_{j=1}^{N}\left\|(u - \Pi^- u)_{N,j}\right\|_{j}^2 \leq C\left((\sqrt{\varepsilon} + \varrho)(N^{-1} \max |\psi'|)^{2(k+1)} + N^{-2(k+1)}\right),
$$

(4.50b)

$$
\varepsilon^{-\frac{1}{2}} \|p - \Pi^+_p p\| \leq C\left((\sqrt{\varepsilon} + \sqrt{\varrho})(N^{-1} \max |\psi'|)^{k+1} + N^{-(k+1)}\right),
$$

(4.50c)

$$
\sum_{j=1}^{N}\varepsilon^{-1}\left\|(p - \Pi^+_p p)_{N,y}\right\|_{j}^2 \leq C(N^{-1} \max |\psi'|)^{2(k+1)},
$$

(4.50d)

where $\varrho$ is given by (3.17). Similarly, we can obtain the same conclusions in another spatial direction.

Proof. The conclusions are more precise than Lemma 4.1 of [5]. The proof proceeds similarly to Lemma 4.2. We mention several differences and use the same notations to prevent confusion.

(1) Prove (4.50a). As before, we have

$$
\|\eta_S\| \leq CN^{-(k+1)}.
$$

(4.51)

To bound $\|\eta_{W_1}\|$, we express it as (4.24). Using (4.49a) and (4.49c) with $m = 2$, we obtain the two estimates

$$
\Lambda_1 \leq C \sum_{K_{ij} \in \Omega_{XL}} \left[ \|W_1\|^2_{K_{ij}} + h_j \|(W_1)_{x,j}\|_{I_i}^2 + h_i \|(W_1)_{i,y}\|_{J_j}^2 + h_i h_j |(W_1)_{i,j}|^2 \right]
$$

\leq C \sum_{K_{ij} \in \Omega_{XL}} \|e^{-\frac{\beta \varepsilon}{\sqrt{\varepsilon}}}\|^2_{K_{ij}},
$$

(4.52)

$$
\Lambda_1 \leq C \sum_{K_{ij} \in \Omega_{XL}} \left[ h_i^{2(k+1)} \|\partial_x^{k+1} W_1\|^2_{K_{ij}} + h_j^{2(k+1)} \|\partial_y^{k+1} W_1\|^2_{K_{ij}} \right]
$$

\leq C \sum_{K_{ij} \in \Omega_{XL}} \left[ \left(\frac{h_i}{\sqrt{\varepsilon}}\right)^{2(k+1)} + \left(\frac{h_j}{\sqrt{\varepsilon}}\right)^{2(k+1)} \right] \|e^{-\frac{\beta \varepsilon}{\sqrt{\varepsilon}}}\|^2_{K_{ij}},
$$

(4.53)

respectively, where we used (3.10) and the monotonic decreasing property of the function $e^{-\beta \varepsilon/\sqrt{\varepsilon}}$. Consequently, we obtain the same estimate for $\Lambda_1$ as before. For $\Lambda_2$, we use the stability (4.49a) and (3.10) to obtain

$$
\Lambda_2 = C \sum_{K_{ij} \in \Omega_{XM} \cup \Omega_{XR}} \left[ \|W_1\|^2_{K_{ij}} + h_j \|(W_1)_{x,j}\|_{I_i}^2 + h_i \|(W_1)_{i,y}\|_{J_j}^2 + h_i h_j |(W_1)_{i,j}|^2 \right]
$$

\leq C \sum_{K_{ij} \in \Omega_{XM} \cup \Omega_{XR}} \|e^{-\frac{\beta \varepsilon}{\sqrt{\varepsilon}}}\|^2_{K_{ij}} = C \int_0^1 dy \int_\tau^1 e^{-\frac{2\beta \varepsilon}{\sqrt{\varepsilon}}}dx \leq C \sqrt{\varepsilon} e^{-\frac{2\beta \varepsilon}{\sqrt{\varepsilon}}} \leq C \sqrt{\varepsilon} N^{-2\sigma}.
$$

(4.54)
As a result, we have
\[ \|\eta W_1\| \leq C\sqrt{\varepsilon}[(N^{-1}\max |\psi'|)^{k+1} + N^{-\sigma}]. \tag{4.55} \]

The term \( \|\eta W_3\| \) must be treated carefully. We decompose it as
\[ \|\eta W_3\|^2 = \sum_{K_{ij}\in\Omega_{XL}} \|\eta W_3\|^2_{K_{ij}} + \sum_{K_{ij}\in\Omega_{XM}} \|\eta W_3\|^2_{K_{ij}} + \sum_{K_{ij}\in\Omega_{XR}} \|\eta W_3\|^2_{K_{ij}} \equiv \Gamma_1 + \Gamma_2 + \Gamma_3. \]

Using \( L^\infty \)-stability and \[ \square \tag{3.10}, \]
we have
\[ \Gamma_1 \leq C \sum_{K_{ij}\in\Omega_{XL}} h_i h_j \|W_3\|^2_{L^\infty(K_{ij})} \leq C \|W_3\|^2_{L^\infty(\Omega_{XL})} \sum_{i=1}^{N/4} h_i \sum_{j=1}^{N} h_j \]
\[ \leq C \|W_3\|^2_{L^\infty(\Omega_{XL})} \leq C e^{-\frac{2\beta(1-\tau)}{\sqrt{\tau}}} \leq C \sqrt{\varepsilon} e^{-\frac{2\beta \tau}{\sqrt{\varepsilon}}} \leq C \sqrt{\varepsilon} N^{-2\sigma}, \tag{4.56} \]
because \( 2(1-\tau) \geq 1 + 2\tau \) for \( 0 \leq \tau \leq 1/4 \), and \( e^{-x} < x^{-1} \) for \( x \geq 1 \).

Using the \( L^2 \)-stability, the uniformity of the mesh in \( \Omega_{XM} \), and Proposition 3.1, we have
\[ \Gamma_2 \leq C \sum_{K_{ij}\in\Omega_{XM}} \left[ \|W_3\|^2_{K_{ij}} + h_i \|\gamma_i(\gamma_i)^\tau\|^2_{L^2} + h_j \|\gamma_j(\gamma_j)^\tau\|^2_{L^2} + h_i h_j \|\gamma_{ij}(\gamma_{ij})^\tau\|^2_{L^2} \right] \]
\[ \leq C \sum_{K_{ij}\in\Omega_{XM}} \left| e^{-\frac{\beta \tau}{\sqrt{\varepsilon}}} \right|^2_{K_{ij}} + C \sum_{j=1}^{N} h_j h_{3N/4} e^{-\frac{2\beta(1-x_{3N/4})}{\sqrt{\varepsilon}}} \]
\[ \leq C \sqrt{\varepsilon} e^{-\frac{2\beta \tau}{\sqrt{\varepsilon}}} + N^{-1} e^{-\frac{2\beta \tau}{\sqrt{\varepsilon}}} \leq C(\sqrt{\varepsilon} + N^{-1}) N^{-2\sigma}. \tag{4.57} \]

To bound \( B_3 \), we decompose it into two parts:
\[ \Gamma_3 = \sum_{i=3N/4+2}^{N-1} \sum_{j=1}^{N} \|\eta W_3\|^2_{K_{ij}} + \sum_{i=3N/4+1,N} \sum_{j=1}^{N} \|\eta W_3\|^2_{K_{ij}} \equiv \Gamma_3^{(1)} + \Gamma_3^{(2)}. \tag{4.58} \]

From \(4.49a) \), \(4.49d) \), and Lemma 3.2, we have
\[ \Gamma_3^{(1)} \leq C \sum_{i=3N/4+2}^{N-1} \sum_{j=1}^{N} \left[ \|W_3\|^2_{K_{ij}} + h_i \|\gamma_i(\gamma_i)^\tau\|^2_{L^2} + h_j \|\gamma_j(\gamma_j)^\tau\|^2_{L^2} + h_i h_j \|\gamma_{ij}(\gamma_{ij})^\tau\|^2_{L^2} \right] \]
\[ \leq C \sum_{K_{ij}\in\Omega_{XR}} \left| e^{-\frac{2\beta(1-\tau)}{\sqrt{\varepsilon}}} \right|^2_{K_{ij}} \tag{4.59} \]
Using \(4.49e\) with \( m = 2 \) yields
\[ \Gamma_3^{(1)} \leq C \sum_{K_{ij}\in\Omega_{XR}} \left[ h_i^{2(k+1)} \|\partial_{x_{k+1}} W_3\|^2_{K_{ij}} + h_j^{2(k+1)} \|\partial_{y_{k+1}} W_3\|^2_{K_{ij}} \right] \]
\[ \leq C \sum_{K_{ij}\in\Omega_{XR}} \left[ \left( \frac{h_i}{\sqrt{\varepsilon}} \right)^{2(k+1)} + h_j^{2(k+1)} \right] \left| e^{-\frac{2\beta(1-\tau)}{\sqrt{\varepsilon}}} \right|^2_{K_{ij}} \tag{4.60} \]
Then, combining (4.59) and (4.60), we obtain
\[ \Gamma^{(1)} \leq C \sum_{i = \frac{3N}{4} + 1}^{N} \min \left\{ \left( \frac{h_i}{\sqrt{\varepsilon}} \right)^{2(k+1)}, 1 \right\} \sqrt{\varepsilon} e^{-\frac{2\beta(1-x_i)}{\sqrt{\varepsilon}}} \min \left\{ \frac{h_i}{\sqrt{\varepsilon}}, 1 \right\} \]
\[ \leq C \sqrt{\varepsilon} (N^{-1} \max |\psi'|)^{2(k+1)}, \] (4.61)
as before.

In a similar manner to (4.37), we have
\[ \| \eta_{W_3} \|_{L^\infty(K_{ij})}^2 \leq C \Theta_i^{2(k+1)} \]
\[ \leq C \left( \sqrt{\varepsilon} + \varrho \right) \left( N^{-1} \max |\psi'| \right)^{2(k+1)}. \] (4.62)

Combining (4.61) with (4.62) leads to
\[ \Gamma_3 \leq C \left( \sqrt{\varepsilon} + \varrho \right) (N^{-1} \max |\psi'|)^{2(k+1)}. \] (4.63)

Collecting up (4.56), (4.57), and (4.63) yields
\[ \| \eta_{W_3} \| \leq C \left[ (\sqrt{\varepsilon} + \varrho) (N^{-1} \max |\psi'|)^{k+1} + (\sqrt{\varepsilon} + N^{-1/2}) N^{-(k+1)} \right]. \] (4.64)

Similarly, we can prove the remainder of (4.24) and arrive at (4.50a).

(2) Note that \((\eta_u)_{N,y} = u_{N,y} - \pi_y(u_{N,y})\), where \(\pi_y\) is a one-dimensional Gauss–Radau projection regarding \(y\) and satisfies analogous stability and approximation conditions to that in Lemma 4.3. From the solution decomposition, we express \(u_{N,y} = S_{N,y} + E_{N,y} + F_{N,y}\), where \(S_{N,y}, E_{N,y}\) and \(F_{N,y}\) are functions of one variable \(y\) and satisfy \(|S_{N,y}^{(j)}| \leq C, |E_{N,y}^{(j)}| \leq C \varepsilon^{-j/2} e^{-\beta y/\sqrt{\varepsilon}}, \) and \(|F_{N,y}^{(j)}| \leq C \varepsilon^{-j/2} e^{-\beta (1-y)/\sqrt{\varepsilon}}\). Following the similar line to that used to prove (4.49a), we obtain
\[ \sum_{j=1}^{N} \| (\eta_{S})_{N,y} \|_{J_j}^2 \leq N^{-2(k+1)}, \]
\[ \sum_{j=1}^{N} \| (\eta_{E})_{N,y} \|_{J_j}^2 \leq C \sqrt{\varepsilon} [(N^{-1} \max |\psi'|)^{2(k+1)} + N^{-2\sigma}], \]
\[ \sum_{j=1}^{N} \| (\eta_{F})_{N,y} \|_{J_j}^2 \leq C \left[ (\sqrt{\varepsilon} + \varrho) (N^{-1} \max |\psi'|)^{2(k+1)} + (\sqrt{\varepsilon} + N^{-1}) N^{-2(k+1)} \right]. \]

Using the triangle inequality leads to (4.50b).

The proofs of (4.50c) and (4.50d) are similar and therefore omitted.
Theorem 4.2. Suppose that \( \lambda_{i,y} = \lambda_{x,j} = 0 \) for \( i, j = 0, 1, \ldots, N-1 \), \( \lambda_{N,y} = \lambda_{x,N} = \sqrt{\varepsilon} \). Let \( w = (u, p, q) \) be the solution to problem (2.1), which satisfies Proposition 3.4. Furthermore, let \( w = (u, p, q) \in V_N^3 \) be the numerical solution of the LDG scheme (2.5) on layer-adapted meshes (3.13) with \( \sigma \geq k+1.5 \). Then, there exists a constant \( C > 0 \) independent of \( \varepsilon \) and \( N \) such that

\[
\| w - \bar{w} \|_E \leq \begin{cases} C \left( \sqrt{\varepsilon}(N^{-1} \ln N)^{k+1} + N^{-(k+1)} \right), & \text{for } S\text{-mesh}, \\ CN^{-(k+1)}, & \text{for } BS-,B\text{-type mesh}. \end{cases}
\] (4.65)

Proof. We follow the proof of Theorem 4.1. Instead of the projection \((\Pi, \Pi, \Pi)\) for \( u, p \) and \( q \), we use \((\Pi^-, \Pi^+, \Pi^+)\) in (4.41).

Using the Cauchy–Schwarz inequality, (4.50a), and (4.50c), we obtain

\[
T_i(\eta; \bar{w}) \leq C \left( \varepsilon^{-1/2} \| \eta_p \| + \varepsilon^{-1/2} \| \eta_q \| + \left\| \frac{1}{\| \eta_u \|} \right\| \| \bar{w} \|_E \right.
\]

\[
\leq C \left( \sqrt{\varepsilon} \| \eta_p \| + \sqrt{\varepsilon} \| \eta_q \| \right) \| \bar{w} \|_E.
\] (4.66)

From (4.48b), (4.48c), (4.50d), \( \lambda_{N,y} = \lambda_{x,N} = \sqrt{\varepsilon} \), and the Cauchy–Schwarz inequality, we obtain

\[
T_3(\eta; \bar{w}) = \sum_{j=1}^{N} \langle (\eta_p)_{x,y}, \psi_{x,y} \rangle_{J_j} - \sum_{i=1}^{N} \langle (\eta_q)_{x,N}, \psi_{x,N} \rangle_{I_i}
\]

\[
\leq \left[ \sum_{j=1}^{N} \frac{1}{\lambda_{N,y}} \left\| (\eta_p)_{x,y} \right\|^2_{J_j} + \sum_{i=1}^{N} \frac{1}{\lambda_{x,N}} \left\| (\eta_q)_{x,N} \right\|^2_{I_i} \right]^{1/2} \| \bar{w} \|_E
\]

\[
\leq C \sqrt{\varepsilon} \left( \sum_{j=1}^{N} \frac{1}{\lambda_{N,y}} \left\| (\eta_p)_{x,y} \right\|^2_{J_j} + \sum_{i=1}^{N} \frac{1}{\lambda_{x,N}} \left\| (\eta_q)_{x,N} \right\|^2_{I_i} \right) \| \bar{w} \|_E.
\] (4.67)

Similarly, we have

\[
T_4(\eta_u; \bar{w}) = \sum_{j=1}^{N} \langle \lambda_{N,y}(\eta_u), \psi_{N,y} \rangle_{J_j} + \sum_{i=1}^{N} \langle \lambda_{x,N}(\eta_u), \psi_{x,N} \rangle_{I_i}
\]

\[
\leq \left[ \sum_{j=1}^{N} \lambda_{N,y} \left\| (\eta_u)_{x,y} \right\|^2_{J_j} + \sum_{i=1}^{N} \lambda_{x,N} \left\| (\eta_u)_{x,N} \right\|^2_{I_i} \right]^{1/2} \| \bar{w} \|_E
\]

\[
\leq C \sqrt{\varepsilon} \left( \sum_{j=1}^{N} \lambda_{N,y} \left\| (\eta_u)_{x,y} \right\|^2_{J_j} + \sum_{i=1}^{N} \lambda_{x,N} \left\| (\eta_u)_{x,N} \right\|^2_{I_i} \right) \| \bar{w} \|_E,
\] (4.68)

where (4.50b) was used.

To bound \( T_2(\eta_u; \bar{w}) \), we follow \[5\] and investigate the \( \varepsilon^{1/4} \)-factor in the upper-bound. On each element \( K_{ij} \), we define the bilinear forms as

\[
A_{ij}^1(\eta_u, v) = \langle \eta_u, v_{x} \rangle_{K_{ij}} - \langle (\eta_u)_{i,y}, v_{i,y} \rangle_{J_j} + \langle (\eta_u)_{i-1,y}, v_{i-1,y} \rangle_{J_j},
\]

\[
A_{ij}^2(\eta_u, v) = \langle \eta_u, v_{y} \rangle_{K_{ij}} - \langle (\eta_u)_{x,j}, v_{x,j} \rangle_{I_i} + \langle (\eta_u)_{x,j-1}, v_{x,j-1} \rangle_{I_i}.
\]
We have [5]

\[
\begin{align*}
|A_{ij}^1(\eta, v)| & \leq C \sqrt{\frac{h_j}{h_i}} \left[ h_i^{k+2} \left\| \partial_x^{k+2} u \right\|_{L^\infty(K_{ij})} + h_j^{k+2} \left\| \partial_y^{k+2} u \right\|_{L^\infty(K_{ij})} \right] \|v\|_{K_{ij}}, \\
|A_{ij}^1(\eta, v)| & \leq C \sqrt{\frac{h_j}{h_i}} \|u\|_{L^\infty(K_{ij})} \|v\|_{K_{ij}}
\end{align*}
\] (4.69a)

for any \( v \in Q^k(K_{ij}) \). Because \( u = 0 \) on \( \partial \Omega \), we obtain

\[
T_2(\eta, z) = \sum_{K_{ij} \in \Omega} A_{ij}^1(\eta, s) + \sum_{K_{ij} \in \Omega} A_{ij}^2(\eta, z).
\]

By (4.69a), we have

\[
\begin{align*}
& C \sum_{K_{ij} \in \Omega} A_{ij}^1(\eta, s) \\
& \leq C \sum_{K_{ij} \in \Omega} \sqrt{\frac{h_j}{h_i}} \left[ h_i^{k+2} \left\| \partial_x^{k+2} S \right\|_{L^\infty(K_{ij})} + h_j^{k+2} \left\| \partial_y^{k+2} S \right\|_{L^\infty(K_{ij})} \right] \|s\|_{K_{ij}} \\
& \leq C N^{-(k+1)} \|s\|_{\Omega_{22}} + C \left( \sum_{K_{ij} \in \Omega \setminus \Omega_{22}} \left( \sqrt{\varepsilon} \max_{\Omega_{22}} |\psi'| \right)^{-1} N^{-2(k+2)} \right)^{1/2} \|s\|_{\Omega \setminus \Omega_{22}} \\
& \leq C \sqrt{\varepsilon} N^{-(k+1)} \|z\|_{E}
\end{align*}
\]

because \( h_i \geq C \sqrt{\varepsilon} N^{-1} \max_{\Omega_{22}} |\psi'| \geq C \sqrt{\varepsilon} h_j \max_{\Omega_{22}} |\psi'| \) and \( \varepsilon^{-1/2} \|s\| \leq \|z\|_{E} \). Using (4.69a) and \( \sigma \geq k + 2 \), we obtain

\[
\begin{align*}
& \sum_{K_{ij} \in \Omega_{1M} \cup \Omega_{1R}} A_{ij}^1(\eta W_1, s) \leq C \sum_{K_{ij} \in \Omega_{1M} \cup \Omega_{1R}} \sqrt{\frac{h_j}{h_i}} \|W_1\|_{L^\infty(K_{ij})} \|s\|_{K_{ij}} \\
& \leq C \sum_{K_{ij} \in \Omega_{1M} \cup \Omega_{1R}} \varepsilon^{-1/4} e^{-}\frac{|x_j|}{\sqrt{\varepsilon}} \|s\|_{K_{ij}} \\
& \leq C \sqrt{\varepsilon} N^{-(k+1)} \|z\|_{E}.
\end{align*}
\]
Using (4.69), \( \sigma \geq k + 2 \), and Lemma 3.1 yields
\[
\sum_{K_{ij} \in \Omega_X^L} A_{ij}^1(\eta W_1, s) \leq C \sum_{K_{ij} \in \Omega_X^L} \sqrt{h_{ij}} \min \left\{ h_i^{k+2} \| \partial_x^{k+2} W_1 \|_{L^\infty(K_{ij})}, h_j^{k+2} \| \partial_y^{k+2} W_1 \|_{L^\infty(K_{ij})}, \| W_1 \|_{L^\infty(K_{ij})} \right\} \| s \|_{K_{ij}}
\]

\[
\leq C \sum_{K_{ij} \in \Omega_X^L} (\sqrt{\varepsilon} \max |\psi'|)^{-1/2} \Theta_{i}^{k+2} \| s \|_{K_{ij}}
\]

\[
\leq C \sqrt{\varepsilon} (N^{-1} \max |\psi'|)^{k+1} \| \varepsilon \|_E.
\]

Analogously, we can bound \( \sum_{K_{ij} \in \Omega_N} A_{ij}^1(\eta \varphi, s) \) for \( \varphi = W_2, W_3, W_4, Z_1, Z_2, Z_3, Z_4 \). Consequently,
\[
T_2(\eta; z) \leq C \sqrt{\varepsilon} (N^{-1} \max |\psi'|)^{k+1} \| \varepsilon \|_E. \tag{4.70}
\]

Using (4.66), (4.67), (4.68), and (4.70), we obtain
\[
\| \xi \|_E^2 = B(\xi; \xi) = B(\eta; \xi) \leq C \left[ (\sqrt{\varepsilon} + \sqrt{\theta})(N^{-1} \max |\psi'|)^{k+1} + N^{-(k+1)} \right] \| \xi \|_E,
\]
which leads to
\[
\| \xi \|_E \leq C \left[ (\sqrt{\varepsilon} + \sqrt{\theta})(N^{-1} \max |\psi'|)^{k+1} + N^{-(k+1)} \right]. \tag{4.71}
\]

The final assertion follows by repeating similar arguments as before. This completes the proof. \( \square \)

5 Numerical experiments

In this section, we present some numerical experiments. All calculations were conducted in MATLAB R2015B. The system of linear equations resulting from the discrete problems were solved by the lower–upper (LU)-decomposition algorithm. All integrals were evaluated using the 5-point Gauss–Legendre quadrature rule.

The LDG method (2.5) was applied to the layer-adapted meshes presented in Table 1 where \( \sigma = k + 1 \), \( k = 0, 1, 2, 3 \). We let \( e^N \) be the error in either \( \| e \|_E \) or \( \| e \|_B \) for an \( N \)-element. In the former case, we took the flux parameter \( \lambda_i = \sqrt{\varepsilon} \) for \( i = 0, N \) and \( \lambda_i = 0 \) for \( i = 1, 2, \ldots, N - 1 \). In the last case, we took the flux parameter \( \lambda_i = \sqrt{\varepsilon} \) for \( i = 0, 1, \ldots, N \). The corresponding convergence rates were computed by the following formulae:
\[
r_s = \frac{\log e^N - \log e^{2N}}{\log p}, \quad r_2 = \frac{\log e^N - \log e^{2N}}{\log 2}.
\]
Here, $p = 2 \ln N / \ln(2N)$ was used to compute the numerical convergence order with respect to the power of $\ln N / N$.

**Example 1.** Consider a linear reaction–diffusion problem

$$-\varepsilon \Delta u + 2u = f(x, y), \quad \text{in } \Omega = (0, 1)^2, \quad (5.72a)$$

$$u = 0, \quad \text{on } \partial \Omega, \quad (5.72b)$$

where $f(x, y)$ is suitably taken such that the exact solution is $u(x, y) = g(x)g(y)$ with

$$g(v) = \frac{e^{-v/\sqrt{\varepsilon}} - e^{-(1-v)/\sqrt{\varepsilon}}}{1 - e^{-1/\sqrt{\varepsilon}}} - \cos(\pi v). \quad (5.73)$$

We set $\varepsilon = 10^{-8}$, small enough to bring out the singularly perturbed nature of (5.73). In Table 2, we list the balanced-norm errors and their convergence rates. We observed convergence of order $k + 1/2$, which is a half-order superior to the estimate from (4.40). In Table 3, we present the energy norm errors and their convergence rates, which agree with our estimate from (4.65).

We show the relevance of these errors to the small parameter $\varepsilon$. We let $N = 256$, $k = 1$, and varied the values of $\varepsilon$. From Table 4, we see that the errors in the balanced norm are almost unchanged, whereas the errors in the energy norm change slightly. For a visual understanding, we plotted the energy errors via $\varepsilon$ on log–log coordinates. In Figure 2, we observe the subtle influence of the $\varepsilon^{0.25}$-factor on the energy errors; the results agree with our predictions.

**Example 2.** Consider the nonlinear–reaction-diffusion problem

$$-\varepsilon \Delta u + [2 + xy(1-x)(1-y)]u = f(x, y), \quad \text{in } \Omega = (0, 1)^2, \quad (5.74a)$$

$$u = 0, \quad \text{on } \partial \Omega, \quad (5.74b)$$

where $f(x, y)$ is suitably taken such that the exact solution is $u(x, y) = h(x)h(y)$ and

$$h(v) = 1 + (v - 1)e^{-v/\sqrt{\varepsilon}} - ve^{-(1-v)/\sqrt{\varepsilon}}. \quad (5.75)$$

We let $\varepsilon = 10^{-8}$. In Table 5 and Table 6, we list the error and convergence rates for the balanced and energy norms, respectively. We still observed convergence of orders $k + 1/2$ and $k + 1$ for the balanced-norm and energy norm errors.

Moreover, we tested the dependence of these two types of errors on $\varepsilon$. We clearly observed in Table 7 that the errors in the balanced norm were almost constant, whereas the errors in the energy norm changed slightly. In Figure 3, we confirmed the influence of the factor $\varepsilon^{0.25}$ on the energy norm errors obtained for the three layer-adapted meshes. Note that for this example, the $\varepsilon^{0.25}$-factor is clearly observed on both the S-type and B-type meshes. This may be due to the fact that the regular part of the exact solution belongs to $V_N$, as described in [21].
Figure 2: Energy norm error from $\varepsilon$ in Example 1.

Figure 3: Energy norm error from $\varepsilon$ in Example 2.
Table 2: Balanced error and convergence rates for Example 1.

|   | N  | S-mesh | BS-mesh | B-mesh |
|---|----|--------|---------|--------|
|   | Balanced error | $r_s$ | Balanced error | $r_2$ | Balanced error | $r_2$ |
| 0 | 8   | 1.37e+00 | - | 1.36e+00 | - | 1.55e+00 | - |
|   | 16  | 1.09e+00 | 0.57 | 1.04e+00 | 0.39 | 1.10e+00 | 0.49 |
|   | 32  | 8.36e-01 | 0.57 | 7.47e-01 | 0.47 | 7.67e-01 | 0.52 |
|   | 64  | 6.31e-01 | 0.55 | 5.30e-01 | 0.50 | 5.36e-01 | 0.52 |
|   | 128 | 4.73e-01 | 0.54 | 3.74e-01 | 0.50 | 3.76e-01 | 0.51 |
|   | 256 | 3.52e-01 | 0.53 | 2.64e-01 | 0.50 | 2.65e-01 | 0.51 |
| 1 | 8   | 3.67e-01 | - | 2.48e-01 | - | 3.86e-01 | - |
|   | 16  | 2.22e-01 | 1.25 | 9.83e-02 | 1.33 | 1.22e-01 | 1.66 |
|   | 32  | 1.19e-01 | 1.32 | 3.75e-02 | 1.39 | 4.17e-02 | 1.55 |
|   | 64  | 5.83e-02 | 1.40 | 1.39e-02 | 1.43 | 1.46e-02 | 1.51 |
|   | 128 | 2.68e-02 | 1.44 | 5.04e-03 | 1.46 | 5.16e-03 | 1.50 |
|   | 256 | 1.18e-02 | 1.46 | 1.81e-03 | 1.48 | 1.83e-03 | 1.49 |
| 2 | 8   | 1.61e-01 | - | 7.24e-02 | - | 1.45e-01 | - |
|   | 16  | 7.40e-02 | 1.92 | 1.58e-02 | 2.20 | 2.26e-02 | 2.69 |
|   | 32  | 2.68e-02 | 2.16 | 3.11e-03 | 2.35 | 3.71e-03 | 2.61 |
|   | 64  | 8.19e-03 | 2.32 | 5.83e-04 | 2.42 | 6.34e-04 | 2.55 |
|   | 128 | 2.23e-03 | 2.42 | 1.06e-04 | 2.45 | 1.11e-04 | 2.52 |
|   | 256 | 5.62e-04 | 2.46 | 1.91e-05 | 2.48 | 1.95e-05 | 2.51 |
| 3 | 8   | 7.16e-02 | - | 2.16e-02 | - | 5.87e-02 | - |
|   | 16  | 2.52e-02 | 2.57 | 2.52e-03 | 3.10 | 4.22e-03 | 3.80 |
|   | 32  | 6.29e-03 | 2.96 | 2.55e-04 | 3.30 | 3.29e-04 | 3.68 |
|   | 64  | 1.21e-03 | 3.23 | 2.42e-05 | 3.40 | 2.74e-05 | 3.59 |
|   | 128 | 1.96e-04 | 3.38 | 2.22e-06 | 3.45 | 2.35e-06 | 3.54 |
|   | 256 | 2.85e-05 | 3.45 | 2.01e-07 | 3.47 | 2.06e-07 | 3.51 |

Acknowledgements

This study was supported by the National Natural Science Foundation of China (No. 11801396), and the Natural Science Foundation of Jiangsu Province (No. BK20170374).
Table 3: Energy error and convergence rates for Example 1.

| k  | N  | S-mesh |     | BS-mesh |     | B-mesh |     |
|----|----|--------|-----|---------|-----|--------|-----|
|    |    | Energy error | r₂  | Energy error | r₂  | Energy error | r₂  |
| 0  | 8  | 2.22e-01     | -   | 2.22e-01     | -   | 2.21e-01     | -   |
| 16 |    | 1.13e-01     | 1.67| 1.13e-01     | 0.96| 1.13e-01     | 0.98|
| 32 |    | 5.67e-02     | 1.46| 5.66e-02     | 0.94| 5.66e-02     | 0.99|
| 64 |    | 2.84e-02     | 1.35| 2.83e-02     | 0.99| 2.83e-02     | 1.00|
| 128|    | 1.43e-02     | 1.28| 1.42e-02     | 1.00| 1.42e-02     | 1.00|
| 256|    | 7.15e-03     | 1.23| 7.08e-03     | 1.00| 7.08e-03     | 1.00|
| 1  | 8  | 2.30e-02     | -   | 2.29e-02     | -   | 2.30e-02     | -   |
| 16 |    | 6.06e-03     | 3.29| 5.77e-03     | 1.99| 5.81e-03     | 1.98|
| 32 |    | 1.73e-03     | 2.67| 1.45e-03     | 1.99| 1.46e-03     | 1.99|
| 64 |    | 5.40e-04     | 2.27| 3.64e-04     | 2.00| 3.66e-04     | 2.00|
| 128|    | 1.77e-04     | 2.07| 9.12e-05     | 2.00| 9.17e-05     | 2.00|
| 256|    | 5.81e-05     | 1.99| 2.29e-05     | 1.99| 2.30e-05     | 1.99|
| 2  | 8  | 2.20e-03     | -   | 1.66e-03     | -   | 2.23e-03     | -   |
| 16 |    | 7.06e-04     | 2.80| 2.26e-04     | 2.87| 2.89e-04     | 2.95|
| 32 |    | 2.24e-04     | 2.44| 3.05e-05     | 2.89| 3.72e-05     | 2.96|
| 64 |    | 6.01e-05     | 2.58| 4.04e-06     | 2.92| 4.77e-06     | 2.96|
| 128|    | 1.39e-05     | 2.72| 5.30e-07     | 2.93| 6.08e-07     | 2.97|
| 256|    | 2.85e-06     | 2.83| 6.90e-08     | 2.94| 7.74e-08     | 2.97|
| 3  | 8  | 7.11e-04     | -   | 2.19e-04     | -   | 6.89e-04     | -   |
| 16 |    | 2.31e-04     | 2.77| 2.03e-05     | 3.43| 4.37e-05     | 3.98|
| 32 |    | 5.27e-05     | 3.14| 1.60e-06     | 3.66| 2.75e-06     | 3.99|
| 64 |    | 9.15e-06     | 3.43| 1.16e-07     | 3.79| 1.73e-07     | 3.99|
| 128|    | 1.29e-06     | 3.63| 8.05e-09     | 3.85| 1.08e-08     | 3.99|
| 256|    | 1.56e-07     | 3.77| 5.43e-10     | 3.89| 6.80e-10     | 3.99|

Appendix

In this appendix, we mention several results for parabolic singularly perturbed reaction–diffusion problems:

\[
\begin{align*}
    &u_t - \varepsilon \Delta u + b(x, y)u = f(x, y, t) & \text{in } \Omega \times (0, T], \quad \text{(A.1a)} \\
    &u|_{t=0} = u_0(x, y), & \text{in } \bar{\Omega} \quad \text{(A.1b)} \\
    &u|_{\partial \Omega} = 0, & \text{for } t \in (0, T). \quad \text{(A.1c)}
\end{align*}
\]

Let \( M \) be a positive integer and \( 0 = t^0 < t^1 < \cdots < t^M = T \) be an equidistant partition of \([0, T]\). We define the time interval \( K^m = (t^{m-1}, t^m], \ m = 1, 2, \ldots, M \), with a mesh width \( \Delta t = t^m - t^{m-1} \), which satisfies \( M\Delta t = T \). We write \( v^m = v(t^m) \) and \( v^{m,\theta} = \theta v^m + (1 - \theta)v^{m-1} \).
Table 4: Energy norm and balanced-norm errors for different $\varepsilon$.

$$
\begin{array}{cccccc}
\varepsilon & \text{Energy error} & \text{Balanced error} \\
 & S\text{-mesh} & BS\text{-mesh} & B\text{-mesh} & S\text{-mesh} & BS\text{-mesh} & B\text{-mesh} \\
10^{-6} & 1.71e-04 & 2.94e-05 & 3.02e-05 & 1.18e-02 & 1.85e-03 & 1.87e-03 \\
10^{-7} & 9.79e-05 & 2.42e-05 & 2.45e-05 & 1.18e-02 & 1.83e-03 & 1.84e-03 \\
10^{-8} & 5.81e-05 & 2.29e-05 & 2.30e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
10^{-9} & 3.76e-05 & 2.26e-05 & 2.26e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
10^{-10} & 2.81e-05 & 2.25e-05 & 2.25e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
10^{-11} & 2.44e-05 & 2.25e-05 & 2.25e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
10^{-12} & 2.31e-05 & 2.25e-05 & 2.25e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
10^{-13} & 2.27e-05 & 2.25e-05 & 2.25e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
10^{-14} & 2.25e-05 & 2.25e-05 & 2.25e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
10^{-15} & 2.25e-05 & 2.25e-05 & 2.25e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
10^{-16} & 2.25e-05 & 2.25e-05 & 2.25e-05 & 1.18e-02 & 1.81e-03 & 1.83e-03 \\
\end{array}
$$

Assume $1/2 \leq \theta \leq 1$. The fully discrete scheme for (A.1) is constructed by the LDG in space and by the implicit $\theta$-scheme in time. It reads as follows: Let $U^0 = \Pi u_0$ be the local $L^2$ projection of $u_0$. For any $m = 1, 2, \ldots, M$, find the numerical solution $W^m = (U^m, P^m, Q^m) \in V^3_N$ such that

$$
\left\langle \frac{U^m - U^{m-1}}{\Delta t}, v \right\rangle + B(W^{m,\theta}; z) = 0 \quad (A.2)
$$

holds for any $z = (v, s, r) \in V^3_N$, where $B(W^{m,\theta}; z)$ is defined in (2.8). Here, we use the abbreviations $t^{m,\theta} = \theta t^m + (1 - \theta)t^{m-1}$ and $g^{m,\theta} = \theta g^m + (1 - \theta)g^{m-1}$ for functions $g = g(t)$ and $g^m = g(m\Delta t)$. Following the traditional energy analysis and concept of [5], we have

**Theorem A.1.** Let $w = (u, p, q)$ be the solution to problem (A.1) and satisfy an analogous decomposition as the stationary case. Let $W^m = (U^m, P^m, Q^m) \in V^2_N$, $m = 1, 2, \ldots, M$ be the numerical solution of the fully discrete scheme (A.2), where $V^2_N$ is composed of piecewise polynomials of degree $k \geq 0$ on layer-adapted meshes (3.13) with $\sigma \geq k + 2$. Then, there exists a constant $C > 0$ independent of $\varepsilon, N$ and $M$ such that

$$
\|u^M - U^M\|^2 + \Delta t \sum_{m=1}^{M} \|w^{m,\theta} - W^{m,\theta}\|_L^2 \\
\leq C(1 + T) \left[ (\sqrt{\varepsilon} + \varrho)(N^{-1} \max |\psi'|)^2(k+1) + N^{-2(k+1)} + (\Delta t)^{2r} \right], \quad (A.3)
$$

where $r = 1$ if $1/2 < \theta \leq 1$ and $r = 2$ if $\theta = 1/2$. 

26
Table 5: Balanced error and convergence rates for Example 2.

| $k$ | $N$ | S-mesh Balanced error | $r_s$ | BS-mesh Balanced error | $r_2$ | B-mesh Balanced error | $r_2$ |
|-----|-----|-----------------------|------|------------------------|------|-----------------------|------|
|     | 0   | 1.32e+00              | -    | 1.30e+00               | -    | 1.63e+00              | -    |
|     | 16  | 1.09e+00 0.48         | 9.81e-01 0.41 | 1.10e+00 0.57 |
|     | 32  | 8.83e-01 0.45         | 7.07e-01 0.47 | 7.47e-01 0.56 |
|     | 64  | 6.99e-01 0.46         | 5.01e-01 0.50 | 5.15e-01 0.54 |
|     | 128 | 5.42e-01 0.47         | 3.54e-01 0.50 | 3.59e-01 0.52 |
|     | 256 | 4.13e-01 0.49         | 2.50e-01 0.50 | 2.52e-01 0.51 |
| 1   | 8   | 5.07e-01              | -    | 3.33e-01               | -    | 5.41e-01              | -    |
|     | 16  | 3.12e-01 1.20         | 1.37e-01 1.28 | 1.72e-01 1.65 |
|     | 32  | 1.68e-01 1.32         | 5.27e-02 1.38 | 5.88e-02 1.55 |
|     | 64  | 8.25e-02 1.40         | 1.96e-02 1.43 | 2.06e-02 1.51 |
|     | 128 | 3.79e-02 1.44         | 7.11e-03 1.46 | 7.29e-03 1.50 |
|     | 256 | 1.67e-02 1.47         | 2.55e-03 1.48 | 2.58e-03 1.50 |
| 2   | 8   | 2.27e-01              | -    | 1.01e-01               | -    | 2.05e-01              | -    |
|     | 16  | 1.05e-01 1.91         | 2.22e-02 2.19 | 3.18e-02 2.69 |
|     | 32  | 3.80e-02 2.16         | 4.37e-03 2.34 | 5.22e-03 2.61 |
|     | 64  | 1.16e-02 2.32         | 8.19e-04 2.42 | 8.93e-04 2.55 |
|     | 128 | 3.15e-03 2.42         | 1.50e-04 2.45 | 1.56e-04 2.52 |
|     | 256 | 7.95e-04 2.46         | 2.69e-05 2.47 | 2.74e-05 2.51 |
| 3   | 8   | 1.01e-01              | -    | 3.06e-02               | -    | 8.30e-02              | -    |
|     | 16  | 3.57e-02 2.57         | 3.56e-03 3.10 | 5.96e-03 3.80 |
|     | 32  | 8.90e-03 2.96         | 3.61e-04 3.30 | 4.66e-04 3.68 |
|     | 64  | 1.71e-03 3.23         | 3.43e-05 3.40 | 3.87e-05 3.59 |
|     | 128 | 2.78e-04 3.38         | 3.15e-06 3.45 | 3.33e-06 3.54 |
|     | 256 | 4.03e-05 3.45         | 2.84e-07 3.47 | 2.91e-07 3.51 |

References

[1] Apel, T. *Anisotropic finite elements: local estimates and applications*. Advances in Numerical Mathematics, B.G. Teubner, Stuttgart (1999).

[2] Bakhvalov, N.: The optimalization of methods of solving boundary value problems with a boundary layer. USSR Comput. Math. Math. Phys 9(4), 139-166 (1969)

[3] Cheng, Y.; Zhang, F. and Zhang, Q. Local analysis of local discontinuous Galerkin method for the time-dependent singularly perturbed problem. J. Sci. Comput., 63, 452-477 (2015).
Table 6: Energy error and convergence rates for Example 2.

| $k$ | $N$ | S-mesh | BS-mesh | B-mesh |
|-----|-----|--------|---------|--------|
|     |     | Energy error | $r_S$ | Energy error | $r_2$ | Energy error | $r_2$ |
| 0   | 8   | 1.06e-02   | -      | 9.30e-03   | -      | 1.51e-02   | -      |
|     | 16  | 8.15e-03   | 0.65   | 5.57e-03   | 0.74   | 7.65e-03   | 0.98   |
|     | 32  | 5.79e-03   | 0.73   | 3.12e-03   | 0.84   | 3.96e-03   | 0.95   |
|     | 64  | 8.15e-03   | 0.80   | 1.69e-03   | 0.89   | 2.05e-03   | 0.95   |
|     | 128 | 5.79e-03   | 0.86   | 8.95e-04   | 0.92   | 1.05e-03   | 0.96   |
|     | 256 | 8.15e-03   | 0.91   | 4.69e-04   | 0.95   | 5.36e-04   | 0.97   |
| 1   | 8   | 5.07e-03   | -      | 3.29e-03   | -      | 6.42e-03   | -      |
|     | 16  | 2.86e-03   | 1.41   | 1.12e-03   | 1.56   | 1.76e-03   | 1.86   |
|     | 32  | 1.37e-03   | 1.57   | 3.35e-04   | 1.74   | 4.68e-04   | 1.91   |
|     | 64  | 5.73e-04   | 1.70   | 9.48e-05   | 1.82   | 1.23e-04   | 1.93   |
|     | 128 | 1.97e-04   | 1.80   | 2.60e-05   | 1.87   | 3.18e-05   | 1.95   |
|     | 256 | 7.59e-05   | 1.88   | 6.97e-06   | 1.91   | 8.19e-06   | 1.96   |
| 2   | 8   | 2.27e-03   | -      | 9.73e-04   | -      | 2.34e-03   | -      |
|     | 16  | 9.63e-04   | 2.11   | 1.75e-04   | 2.47   | 3.11e-04   | 2.91   |
|     | 32  | 3.15e-04   | 2.37   | 2.71e-05   | 2.69   | 4.08e-05   | 2.93   |
|     | 64  | 8.49e-05   | 2.57   | 3.89e-06   | 2.80   | 5.31e-06   | 2.94   |
|     | 128 | 1.96e-05   | 2.72   | 5.36e-07   | 2.86   | 6.86e-07   | 2.95   |
|     | 256 | 4.04e-06   | 2.82   | 7.23e-08   | 2.89   | 8.80e-08   | 2.96   |
| 3   | 8   | 1.00e-03   | -      | 2.91e-04   | -      | 9.59e-04   | -      |
|     | 16  | 3.27e-04   | 2.76   | 2.80e-05   | 3.38   | 6.15e-05   | 3.98   |
|     | 32  | 7.46e-05   | 3.14   | 2.23e-06   | 3.65   | 3.87e-06   | 3.99   |
|     | 64  | 1.29e-05   | 3.43   | 1.62e-07   | 3.78   | 2.43e-07   | 3.99   |
|     | 128 | 1.83e-06   | 3.63   | 1.13e-08   | 3.85   | 1.53e-08   | 3.99   |
|     | 256 | 2.21e-07   | 3.77   | 7.62e-10   | 3.89   | 9.56e-10   | 4.00   |

[4] Cheng, Y.; Zhang, Q. Local analysis of the local discontinuous Galerkin method with the generalized alternating numerical flux for one-dimensional singularly perturbed problem. J. Sci. Comput., 72, 792-819 (2017).

[5] Cheng,Y.; Mei,Y.J.; Roos,H.G.: The local discontinuous Galerkin method on layer-adapted meshes for time-dependent singularly perturbed convection-diffusion problems. arXiv:2012.03560, http://arxiv.org/abs/2012.03560

[6] Clavero, C., Gracia, J.L., O’Riordan, E.: A parameter robust numerical method for a two dimensional reaction-diffusion problem. Math. Comp.74(252), 1743-1758 (2005).
Table 7: Energy norm and balanced-norm errors for different $\varepsilon$.

| $\varepsilon$ | S-mesh | BS-mesh | B-mesh | S-mesh | BS-mesh | B-mesh |
|---------------|---------|---------|--------|---------|---------|--------|
| $10^{-6}$     | 2.40e-04 | 2.20e-05 | 2.37e-05 | 1.67e-02 | 2.55e-03 | 2.57e-03 |
| $10^{-7}$     | 1.35e-04 | 1.24e-05 | 1.40e-05 | 1.67e-02 | 2.55e-03 | 2.58e-03 |
| $10^{-8}$     | 7.59e-05 | 6.97e-04 | 8.19e-06 | 1.67e-02 | 2.55e-03 | 2.58e-03 |
| $10^{-9}$     | 4.27e-05 | 3.92e-06 | 4.71e-06 | 1.67e-02 | 2.55e-03 | 2.59e-03 |
| $10^{-10}$    | 2.40e-05 | 2.21e-06 | 2.69e-06 | 1.67e-02 | 2.55e-03 | 2.59e-03 |
| $10^{-11}$    | 1.35e-05 | 1.24e-06 | 1.52e-06 | 1.67e-02 | 2.55e-03 | 2.59e-03 |
| $10^{-12}$    | 7.58e-06 | 6.98e-07 | 8.55e-07 | 1.67e-02 | 2.55e-03 | 2.59e-03 |
| $10^{-13}$    | 4.27e-06 | 3.92e-07 | 4.79e-07 | 1.67e-02 | 2.55e-03 | 2.59e-03 |
| $10^{-14}$    | 2.40e-06 | 2.21e-07 | 2.69e-07 | 1.67e-02 | 2.55e-03 | 2.59e-03 |
| $10^{-15}$    | 1.35e-06 | 1.24e-07 | 1.50e-07 | 1.67e-02 | 2.55e-03 | 2.59e-03 |
| $10^{-16}$    | 7.58e-07 | 6.98e-08 | 8.40e-08 | 1.67e-02 | 2.55e-03 | 2.59e-03 |

[7] Cockburn, B., Shu, C.W.: The local discontinuous Galerkin method for time-dependent convection-diffusion systems. SIAM J. Numer. Anal. 35(6), 2440-2463 (1998).

[8] Cockburn, B.; Kanschat, G.; Perugia, I. and Schötzau, D. Superconvergence of the local discontinuous Galerkin method for elliptic problems on cartesian grids. SIAM. J. Numer. Anal., 39, 264-285, (2001).

[9] Han, H.; Kellogg, R.B. Differentiability properties of solutions of the equation $-\varepsilon \Delta u + ru = f(x,y)$ in a square. SIAM J. Math. Anal., 21, 394-408 (1990).

[10] Johnson, C., Nävert, U., Pitkäranta, J.: Finite element methods for linear hyperbolic problems. Comput. Methods Appl. Mech. Engrg. 45, 285-312 (1984)

[11] Lin, R., Stynes, M.: A balanced finite element method for singularly perturbed reaction-diffusion problems. SIMA J. Numer. Anal. 50(5), 2729-2743 (2012)

[12] Linß, T.; Stynes, M. Numerical methods on Shishkin meshes for convection-diffusion problems. Comput. Methods Appl. Mech. Engrg., 190, 3527-3542(2000).

[13] Linß, T.: Layer-adapted meshes for convection-diffusion problems, Comput. Methods Appl. Mech. Engrg. 192(9-10), 1061-1105 (2003)

[14] Melenk, J.M, Xenophontos, C.: Robust exponential convergence of $hp$-FEM in balanced norms for singularly perturbed reaction-diffusion equations. Calcolo. 53, 105-132 (2016)
[15] Roos, H.G., Schopf, M.: Convergence and stability in balanced norms of finite element methods on Shishkin meshes for reaction-diffusion problems. ZAMM Z. Angew. Math. Mech. 95(6), 551-565 (2015)

[16] Roos, H.G: Error estimates in balanced norms of finite element methods on layer-adapted meshes for second order reaction-diffusion problem. Proc. of BAIL, Beijing (2016)

[17] Roos, H.G, Stynes, M., Tobiska, L.: Robust Numerical Methods for Singularly Perturbed Differential Equations. Springer, Berlin (2008)

[18] Shishkin, G.: Grid approximation of singularly perturbed elliptic and parabolic equations (Second doctorial thesis). Keldysh Institute, Moscow (in Russian) (1990)

[19] Xie, Z., Zhang, Z.: Uniform superconvergence analysis of the discontinuous Galerkin method for a singularly perturbed problem in 1-D. Math. Comp. 79(269), 35-45 (2010)

[20] Xu, Y., Shu, C.W.: Local discontinuous Galerkin methods for high-order time-dependent partial differential equations. Commun. Comput. Phys 7, 1-46 (2010)

[21] Zhu, H., Tian, H., Zhang, Z.: Convergence analysis of the LDG method for singularly perturbed two-point boundary value problems. Comm. Math. Sci. 9(4), 1013-1032 (2011)

[22] Zhu, H., Zhang, Z.: Uniform convergence of the LDG method for a singularly perturbed problem with the exponential boundary layer. Math. Comp. 83(286), 635-663 (2014)