VIRTUALLY RFRS MAPPING TORI AND COHERENCE

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Abstract. Let \( G \) be a finitely presented group that can be written as an extension
\[
1 \rightarrow K \rightarrow G \rightarrow F_2 \rightarrow 1
\]
where \( K \) is either the finitely generated free group \( F_n, n > 2 \) or the fundamen-
tal group of a closed surface of genus \( g > 1 \). We prove that if the image of the
monodromy map \( \rho : F_2 \rightarrow \text{Out}(K) \) contains an element \( \varphi \in \text{Out}(K) \) such that the
mapping torus \( K \rtimes_\varphi \mathbb{Z} \) is virtually residually finite rationally solvable (for instance
whenever the mapping torus is hyperbolic), then \( G \) is not coherent. This applies,
in particular, when the image is a purely pseudo–Anosov free subgroups of the
mapping class group.

1. Introduction and Main Results

A finitely presented group is called coherent if all its finitely generated subgroups
are finitely presented. Examples of coherent groups are fairly familiar: for instance it
has long been known that free groups and fundamental groups of closed surfaces and
3–manifolds are coherent. In more recent times, Feighn and Handel proved coherence
of free-by-cyclic groups in [FH99]. On the opposite end, \( F_2 \times F_2 \) is not coherent:
the epimorphism \( \phi : F_2 \times F_2 \rightarrow \mathbb{Z} \) that maps all standard generators of \( F_2 \times F_2 \)
to the generator of \( \mathbb{Z} \) is an algebraic fibration, namely its kernel is finitely generated;
however, it is not finitely presented. This is historically and logically the first instance
of noncoherence of group extensions.

Recently, some attention has been devoted to the study of coherence of some classes
of nontrivial extensions (see [FV19a, FV19b, KrWa19]), including free-by-free groups
and surface group-by-free groups. An overarching theme of these papers is that
in presence of excessive homology (which in the case of Eq. (1) amounts to the
condition \( b_1(G) > b_1(F_2) = 2 \)) the group \( G \) is noncoherent (see [KrWa19, Theorem
4.4]). Ideally, we would like to prove that, virtually, the extensions of Eq. (1) have
excessive homology; this would have the pleasant outcome that those group would
be virtually algebraically fibered (see [FV19b, KrWa19]). We fail to do so, but not
miserably: the byproduct of our attempt is that all the groups in questions contain a
subgroup (possibly infinite index) that is an extension with excessive homology. This
guarantees that \( G \) itself is noncoherent.

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The extension of Eq. (1) is determined by a monodromy map $\rho : F_2 \to \text{Out}(K)$. Remember that given an element $\varphi \in \text{Out}(K)$, we can choose a lift $f \in \text{Aut}(K)$ of $\varphi$, well-defined up to conjugation by an element of $K$, and consider the mapping torus of $f$. The isomorphism type of the mapping torus does not depend on the choice of the lift, but only on the monodromy $\varphi$. Because of that we will denote it as $K \rtimes_\varphi \mathbb{Z}$.

Our main result is the following

**Theorem 1.1.** Let $G$ be an extension of $F_2$ by $K$ where $K$ is the free group $F_n$, $n > 2$ or the fundamental group of a closed surface of genus $g > 1$. Assume that the image of the monodromy map $\rho : F_2 \to \text{Out}(K)$ contains an element $\varphi \in \text{Out}(K)$ such that the mapping torus $K \rtimes_\varphi \mathbb{Z}$ is virtually RFRS. Then $G$ is not coherent.

For instance, extensions whose monodromy has image contained in a purely pseudo-Anosov subgroups of the mapping class group of a surface of genus $g > 1$ satisfy the conditions of this Theorem. The interest of this theorem is likely furthered by the following corollaries:

**Corollary 1.2.** Let $N_A, N_B$ be two 3-manifolds that fiber over $S^1$ with fiber $\Sigma$ of genus $g > 1$. Denote $\Pi_A = \pi_1(N_A)$ and $\Pi_B = \pi_1(N_B)$ be their fundamental groups and $K = \pi_1(\Sigma)$. If at least one of the 3-manifolds is nonpositively curved, the amalgamated free product $\Pi = \Pi_A \ast_K \Pi_B$ is not coherent.

Note that the only fibered 3-manifolds with fiber of genus $g > 1$ which are not covered by this statement are non-nonpositively curved graph manifolds. (Here, we adhere to the convention that a graph manifold must admit a nontrivial JSJ decomposition.)

This corollary entails that if $X$ be a surface bundle over a surface with base and fiber of genus at least 2, either $\pi_1(X)$ is not coherent, or the monodromy along any simple curve on the base gives a non-nonpositively curved graph manifold. This further narrows the class of potential nontrivial examples of coherent fundamental groups of aspherical Kähler surfaces with positive irregularity, already reduced to the case of Kodaira fibrations of virtual Albanese dimension 1 with successive work of [Ka98, Ka13, Py16, FV19a].

**Corollary 1.3.** Let $\Pi$ be the free product $\Pi_A \ast_{F_n} \Pi_B$ of two $F_n$-by-$\mathbb{Z}$ groups amalgamated along $F_n$ for $n > 2$, and assume that the monodromy of $\Pi_A$ is atoroidal. Then $\Pi$ is not coherent.

This stands in contrast with the fact that the factors are instead coherent by [FH99].

2. **Proofs**

It is well-known that if the monodromy map $\rho : F_2 \to \text{Out}(K)$ is not injective $G$ is not coherent, so we will focus in the injective case.

A recurring theme of the proofs is the following: we will start with a group $G$ that is given as an extension of the form of Equation (1). We will identify a finite index
subgroup $F_m \leq_f F_2$, from which we will discard all but two free factors, to end up with a new pull-back $K$-by-$F_2$ extension $\Pi$ that fits in the diagram

$$
\begin{array}{cccccc}
1 & \longrightarrow & K & \longrightarrow & \Pi & \longrightarrow & F_2 & \longrightarrow & 1 \\
\approx & & & \downarrow & & & \downarrow & & \\
1 & \longrightarrow & K & \longrightarrow & G & \longrightarrow & F_2 & \longrightarrow & 1 \\
\end{array}
$$

where the vertical arrows are monomorphisms.

Our first goal is to show that whenever the image of the monodromy map defining $G$ contains a nontrivial element $\varphi \in \text{Out}(K)$, there exists a subgroup $\Pi \leq G$ which can be written as an amalgamated free product of the mapping tori of two automorphisms of $K$, of which the first can be assumed to coincide with a lift $f$ of $\varphi$.

**Lemma 2.1.** Let $G$ be an extension of $F_2$ determined by an injective monodromy map $\rho: F_2 \to \text{Out}(K)$. If the image of the monodromy contains a nontrivial element $\varphi \in \text{Out}(K)$, then there exists a subgroup

$$
\Pi := \Pi_A \ast_K \Pi_B \leq G
$$

which is the free product of two mapping tori amalgamated along the common base $K$ and where the factor $\Pi_A$ is the mapping torus $K \rtimes_\varphi \mathbb{Z}$.

**Proof.** Let $\mathbb{Z}_\varphi \leq F_2 \leq \text{Out}(K)$ be the cyclic subgroup generated by $\varphi \in \text{Out}(K)$, where we identify $F_2$ with its image in $\text{Out}(K)$. As observed in [KS68], it was proven (albeit not stated) by M. Hall Jr. in [Ha49] that a fg subgroup of a free group $F_n$ is a free factor of a finite index subgroup of $F_n$. It follows that we can assume that there exists a finite index subgroup of $F_2$ such that $\mathbb{Z}_\varphi \ast F_m \leq F_2 \leq \text{Out}(K)$ with $m \geq 1$. By discarding all but one free factors in $F_m$, we get a monomorphism $\mathbb{Z}_\varphi \ast \mathbb{Z} \to \mathbb{Z}_\varphi \ast F_m$. The pullback of such monomorphism is an extension of $\mathbb{Z}_\varphi \ast \mathbb{Z}$ by $K$, and it can be interpreted as an amalgamated free product of two mapping tori, the first of which has monodromy $\varphi$. By construction, it is a subgroup of $G$. □

We will focus therefore on the amalgamated free product $\Pi := \Pi_A \ast_K \Pi_B$ of the mapping tori of two automorphisms of $K$. For sake of presentation, we will limit ourselves at times to discuss in the detail the proofs pertaining to the case where $K$ is the fundamental group of a surface of genus $g > 1$: the case where $K$ is a fg free nonabelian group, if anything, is slightly less cumbersome.

Choose presentations

$$
F_2 = \langle s, t \rangle, \quad K = \langle \alpha_i, \beta_i \rangle \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1
$$
where here and in what follows the index \( i \) ranges over \( 1, \ldots, g \), and observe that we can write

\[
\Pi_A := \langle \alpha_i, \beta_i, s | \alpha_i^g = f(\alpha_i), \beta_i^g = f(\beta_i), \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \rangle,
\]

\[
\Pi_B := \langle \alpha_i, \beta_i, t | \alpha_i^g = g(\alpha_i), \beta_i^g = g(\beta_i), \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \rangle
\]

where \( f \) and \( g \) are suitable automorphisms of \( K \) and \( x^y := x y^{-1} \). Note that these mapping tori carry natural algebraic fibrations \( \Pi_{A,B} \rightarrow \mathbb{Z} \) with kernel \( K \).

With this notation, we can give a presentation of \( \Pi \) as

\[
(3) \quad \Pi = \langle \alpha_i, \beta_i, s, t | \alpha_i^g = f(\alpha_i), \beta_i^g = f(\beta_i), \alpha_i^t = g(\alpha_i), \beta_i^t = g(\beta_i), \prod_{i=1}^{g} [\alpha_i, \beta_i] = 1 \rangle.
\]

Observe that, by the general theory of amalgamated free products, \( \Pi_A \) and \( \Pi_B \) are naturally subgroups of \( \Pi_A \ast_K \Pi_B \), with the identification of the generators having same symbol.

The criterion we will use to prove noncoherence of a group is contained in [KrWa19, Theorem 4.4]:

**Theorem 2.2.** (Kropholler–Walsh) Let \( G \) be an extension of \( F_2 \) by a group \( K \) that does not algebraically fiber. If \( b_1(G) > 2 \) (“excessive homology”) then \( G \) is not coherent.

As mentioned in the Introduction, we are not able to show that, virtually, the groups \( G \) that we are considering satisfy such condition. However, it is sufficient to show that \( G \) contains some subgroup of that sort that has excessive homology. This excessive homology will come on the one hand from largeness of all mapping tori of \( K \), and on the other hand from the virtually RFRS factor.

Recall that a f.g. group \( \pi \) is RFRS if there exists a filtration \( \{ \pi_i | i \geq 0 \} \) of finite index normal subgroups \( \pi_i \leq_f \pi_0 = \pi \) with \( \bigcap_i \pi_i = \{1\} \) whose successive quotient maps \( \alpha_i : \pi_i \rightarrow \pi_i / \pi_{i+1} \) factorize through the maximal free abelian quotient:

\[
(4) \quad 1 \rightarrow \pi_i / \pi_{i+1} \rightarrow \pi_i \rightarrow \alpha_i \rightarrow H_1(\pi_i) / \text{Tor} \rightarrow 1
\]

For our purposes, the usefulness of RFRS groups comes from the following property.

**Proposition 2.3.** Let \( \pi \) be a RFRS group \( \pi \) that is a mapping torus of an automorphism of \( K \). Then for any nontrivial \( k \in K \) there exist a finite index subgroup \( \tilde{\pi} \leq_f \pi \) such that \( [k] \in H_1(\tilde{\pi}; \mathbb{Z}) / \text{Tor} \) is nonzero.

**Proof.** By construction, \( \pi \) fits in a s.e.s.

\[
1 \rightarrow K \rightarrow \pi \rightarrow \mathbb{Z} \rightarrow 1.
\]
The Lyndon–Hochschild–Serre spectral sequence (LHSSS) associated to this s.e.s.
yields the well-known fact that
\[ H_1(\pi; \mathbb{Z}) = H_1(K; \mathbb{Z}) \oplus \mathbb{Z} \]
where \( H_1(K; \mathbb{Z}) \) is the group of the coinvariants of \( H_1(\Lambda; \mathbb{Z}) \) under the (monodromy) \( \mathbb{Z} \)-action. As \( \pi \) is RFRS, and \( \bigcap_i \pi_i = \{1\} \), there exist an index \( i \in \mathbb{Z} \) such that \( k \in \pi_i \setminus \pi_{i+1} \). Denote \( S := \pi/\pi_i \) and let \( r: G \to S \) the quotient epimorphism. We have the commutative diagram (with self-defining notation)

\[
\begin{array}{cccccccc}
& & 1 & & 1 & & 1 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \tilde{K} & \longrightarrow & \pi_i & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & K & \longrightarrow & \pi & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
r(K) & \longrightarrow & S & \longrightarrow & S/r(K) & \longrightarrow & 1 & & \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & & 1 & & 1 & & 1 & & 
\end{array}
\]

By assumption \( k \in \tilde{K} = K \cap \pi_i \leq_f K \). As \( k \notin \pi_{i+1} = \ker \alpha_i \), the diagram in Eq. (4) entails that \( \alpha_i(k) \neq 1 \), hence the image of \( k \) in \( H_1(\pi_i; \mathbb{Z})/\text{Tor} \) is nonzero, hence so is \( [k] \in H_1(\tilde{K}; \mathbb{Z})/\text{Tor} \). Applying the LHSSS again, we deduce that \( [k] \neq 0 \in H_1(\tilde{K}; \mathbb{Z})/\text{Tor} \).

(Note that here \( \mathbb{Z} = \pi_i/\tilde{K} \) is the group generated by the stable letter of \( \pi_i \).)

We emphasize that Proposition 2.3 asserts not only that, virtually, the coinvariant homology of \( \pi \) has positive rank, but the stronger fact, afforded by the RFRS condition, that we can make virtually homologically essential any element of \( K \).

We want to use Proposition 2.3 in the context of the group \( \Pi = \Pi_A *_{K} \Pi_B \), with \( \Pi_A \) a vRFRS group and \( \Pi_B \) large. (This latter condition is not restrictive in our setup, although for rather nontrivial reason that we will discuss in what follows.) We will do so by repeatedly passing to finite index subgroups of either \( \Pi_A \) or \( \Pi_B \) that have suitable properties. The problem is that every time that we pick a (normal) finite index subgroup of either \( \Pi_A \) or \( \Pi_B \), the corresponding epimorphism from one of the \( \Pi_A, \Pi_B \) to a finite group will not, in general, extend to the other. This is what prevents us from proving that \( \Pi \) has virtually excessive homology.

However, it does not prevent us from finding an (infinite index) subgroup that does. In what follows, we will flesh out the details of how to prove that.

We will repeatedly make use of the following Lemma.

**Lemma 2.4.** Let \( p: \Pi_A \to \mathbb{Z}_m \) be an epimorphism factorizing through the algebraic fibration \( \Pi_A \to \mathbb{Z} \). Denote \( \tilde{\Pi}_A := \ker p \leq_f \Pi_A \); then \( \Pi = \Pi_A *_{K} \Pi_B \) contains an (infinite index) subgroup isomorphic to \( \tilde{\Pi}_A *_{K} \Pi_B \)
Due to its length, we give first an outline the proof of this Lemma. We will start with the presentation of \( \Pi \) given in Eq. (3) and use the Reidemeister–Schreier rewriting process to write a presentation of a finite index subgroup of \( \Pi \) related to \( \tilde{\Pi} \leq_f \Pi \). This presentation appears as a free product of \( m+1 \) mapping tori amalgamated over \( K \) or, equivalently, a \( K \)-by-\( F_{m+1} \) extension. At that point we will discard all but two of the terms of the decomposition to get the desired subgroup of \( \Pi \).

Proof. As \( p: \Pi_A \to \mathbb{Z}_m \) is trivial when restricted to \( K \), it extends to an epimorphism (that we denote with the same letter) \( p: \Pi_A * K \Pi_B \twoheadrightarrow \mathbb{Z}_m \).

We’ll use the Reidemeister–Schreier rewriting process to find a presentation of \( \ker p \). (The reader may prefer to recast this result in terms of the monodromies of the surface bundles which carry as fundamental group the groups in question.)

We start by observing that a Schreier transversal for \( p \) is given by the collection \( \{1, s, \ldots, s^{m-1}\} \). Looking at the presentation in Eq. (3) we can identify as generating set for \( \ker p \) the set

\[
\alpha_{j,i} := s^j \alpha_i s^{-j} \quad \beta_{j,i} := s^j \beta_i s^{-j} \quad j = 0, \ldots, m-1
\]

\[
w := s^m \quad t_j := s^j t s^{-j} \quad j = 0, \ldots, m-1.
\]

The rewriting process yields the relations

\[(6a) \quad s^{j+1} \alpha_i s^{-1} f(\alpha_i)^{-1} s^{-j} = 1 \quad s^{j+1} \beta_i s^{-1} f(\beta_i)^{-1} s^{-j} = 1 \quad j = 0, \ldots, m-1
\]

\[(6b) \quad s^j t \alpha_i t^{-1} g(\alpha_i)^{-1} s^{-j} = 1 \quad s^j t \beta_i t^{-1} g(\beta_i)^{-1} s^{-j} = 1 \quad j = 0, \ldots, m-1
\]

\[(6c) \quad s^j \prod_{i=1}^{g}[\alpha_i, \beta_i] s^{-j} = 1 \quad j = 0, \ldots, m-1.
\]

The two first sets of \( m \) relations above are quite asymmetric (as they should) in \( A \) and \( B \), and we will use them to simplify the presentation.

Denote by \( \mathcal{X}_j \) the collection of generators \( \{\alpha_{j,i}, \beta_{j,i}, i = 1, \ldots, g\} \), and by \( F(\mathcal{X}_j) \) the free group it generates. It is useful to observe that the isomorphism \( f: K \to K \) determine uniquely a collection of isomorphisms \( f: F(\mathcal{X}_j) \to F(\mathcal{X}_j) \). Explicitly, if for \( k \in K, f(k) = w(\alpha_i, \beta_i) \) is a given word in the generators \( \{\alpha_i, \beta_i\} \) of \( K \), then \( f(s^j k s^{-j}) = w(\alpha_{j,i}, \beta_{j,i}) \) is the same word in the generators \( \{\alpha_{j,i}, \beta_{j,i}\} \) of \( \mathcal{X}_j \). Similarly the isomorphism \( g: K \to K \) determines isomorphisms \( g: F(\mathcal{X}_j) \to F(\mathcal{X}_j) \).

Let’s rewrite the set of relations of Eq. (6a): starting with the case \( j = 0 \) we get

\[s\alpha_i s^{-1} f(\alpha_i)^{-1} = 1 \quad s\beta_i s^{-1} f(\beta_i)^{-1} = 1,
\]

which we rewrite as

\[\alpha_{1,i} = f(\alpha_{0,i}) \quad \beta_{1,i} = f(\beta_{0,i}).\]
(Note that this entails that the generators of $K_1$ could be disposed of.) We continue with $j = 1$ to get
\[
s^2\alpha_is^{-1}f(\alpha_i)^{-1}s^{-1} = s^2\alpha_is^{-2}sf(\alpha_i)^{-1}s^{-1} = s^2\alpha_is^{-2}f(s\alpha_is^{-1})^{-1} = 1
\]
\[
s^2\beta_is^{-1}f(\beta_i)^{-1}s^{-1} = s^2\beta_is^{-2}sf(\beta_i)^{-1}s^{-1} = s^2\beta_is^{-2}f(s\beta_is^{-1})^{-1} = 1
\]
which we rewrite as
\[
\alpha_{2,i} = f(\alpha_{1,i}) \quad \beta_{2,i} = f(\beta_{1,i})
\]
(and again the generators of $K_2$ could be disposed of). This process continues verbatim to yield at the $j$-th step the relations
\[
\alpha_{j+1,i} = f(\alpha_{j,i}) \quad \beta_{j+1,i} = f(\beta_{j,i})
\]
until $j = m - 1$, where we have instead
\[
s^m\alpha_is^{-1}f(\alpha_i)^{-1}s^{-m+1} = s^m\alpha_is^{-m} \cdot s^{m-1}f(\alpha_i)^{-1}s^{-m+1} = s^m\alpha_is^{-m}f(s^{-1}m\alpha_is^{-m+1})^{-1} = 1
\]
\[
s^m\beta_is^{-1}f(\beta_i)^{-1}s^{-m+1} = s^m\beta_is^{-m} \cdot s^{m-1}f(\beta_i)^{-1}s^{-m+1} = s^m\beta_is^{-m}f(s^{-1}m\beta_is^{-m+1})^{-1} = 1
\]
which, using the results above, we rewrite as
\[
w\alpha_{0,i}w^{-1} = f(\alpha_{m-1,i}) \quad w\beta_{0,i}w^{-1} = f(\beta_{m-1,i}) = f^m(\beta_{0,i}).
\]
The set of relations in Eq. (6b) yield a more straightforward outcome: we have, for $j = 0, \ldots, m - 1$,
\[
s^j\alpha_is^{-1}g(\alpha_i)^{-1}s^{-j} = s^jts^{-j}s^j\alpha_is^{-j}s^jt^{-j}s^{-j}s^jg(\alpha_i)^{-1}s^{-j} = 1
\]
\[
s^j\beta_is^{-1}g(\beta_i)^{-1}s^{-j} = s^jts^{-j}s^j\beta_is^{-j}s^jt^{-1}s^{-j}s^jg(\beta_i)^{-1}s^{-j} = 1
\]
which we rewrite as
\[
t_0\alpha_{j,i}t_0^{-1} = g(\alpha_{j,i}) \quad t_0\beta_{j,i}t_0^{-1} = g(\beta_{j,i}).
\]

Last, we consider the relations of Eq. (6c), these yield
\[
s^g\Pi_{i=1}^g[\alpha_{i,i}, \beta_{i,i}]s^{-j} = \Pi_{i=1}^g[s^j\alpha_is^{-j}, s^j\beta_is^{-j}] = 1
\]
which we rewrite in terms of the generators of $X_j$ as $\Pi_{i=1}^g[\alpha_{j,i}, \beta_{j,i}] = 1$.

We are ready to write a presentation (actually, two) for $\ker p$. The first is given by
\[
\ker p = \langle \alpha_{j,i}, \beta_{j,i}, w, t_j | \alpha_{0,i}^w = f^m(\alpha_{0,i}), \beta_{0,i}^w = f^m(\beta_{0,i})
\]
\[
\alpha_{j,0} = g(\alpha_{j,i}), \beta_{j,0} = g(\beta_{j,i}),
\]
\[
\Pi_{i=1}^g[\alpha_{j,i}, \beta_{j,i}] = 1,
\]
\[
\alpha_{j+1,i} = f(\alpha_{j,i}), \beta_{j+1,i} = f(\beta_{j,i})
\]
with the indexes $i = 0, \ldots, g$ and $j = 0, \ldots, m - 1$ except for the last set of relations, where $j = 0, \ldots, m - 2$. This presentation identifies $\ker p$ as result of a construction that we now describe. Take a copy $\hat{\Pi}_A$ having surface subgroup $K_0$ and stable letter
w. Denote by \{α_{0,i}, β_{0,i}, i = 1, \ldots, g\} the generators of \(K_0\). It is well-known (and we implicitly proved it before) that \(\widetilde{Π}_A\) admits a presentation

\[
\widetilde{Π}_A = \langle α_{0,i}, β_{0,i}, w | α_{0,i}^w = f^m(α_{0,i}), β_{0,i}^w = f^m(β_{0,i}), \prod_{i=1}^g (α_{0,i}, β_{0,i}) = 1 \rangle.
\]

Next, take \(m\) copies \(Π_{B,j}, j = 0, \ldots, m - 1\), of \(Π_B\) having surface subgroup \(K_j\) and stable letter \(t_j\), where we denote by \{α_{j,i}, β_{j,i}, i = 1, \ldots, g\} the set of generators of \(K_j\), with monodromy \(g: K_j \rightarrow K_j\) defined in the obvious way. Next, define isomorphisms

\[
φ_{j+1}: K_{j+1} \rightarrow K_j
\]

by setting

\[
φ_{j+1}(α_{j+1,i}) = f(α_{j,i}), \quad φ_{j+1}(β_{j+1,i}) = f(β_{j,i}), \quad j = 0, \ldots, m - 2
\]

We can then identify, by the definition of amalgamated free product,

\[
ker p = \widetilde{Π}_A * _{K_0 \rightarrow K_0} Π_{B,0} * _{K_1 \rightarrow K_0} Π_{B,1} * \cdots Π_{B,m-2} * _{K_{m-1} \rightarrow K_{m-2}} Π_{B,m-1},
\]

where the isomorphism carries generators denoted with the same symbol. There is some important information that we can immediately extract from writing \(ker p\) as amalgamated free product; specifically, each factor is a subgroup of \(ker p\) in the natural way, and so are the subgroups \(K_j\). This entails, in particular, that the subgroups of \(ker p\) generated by \{α_{j,i}, β_{j,i}\} are surface subgroups, isomorphic to \(K\).

There is another presentation of \(ker p\) that is worth discussing: in fact, we can use the relations in Eq. (7) to get rid of all generators other than those of \(K_0\). Specifically, we can write

\[
α_{j,i} = f^j(α_{0,i}), \quad β_{j,i} = f^j(β_{0,i}), \quad j = 0, \ldots, m - 1
\]

using these relations, we can rewrite the action of the stable letters \(t_0, \ldots, t_{m-1}\) as follows:

\[
\begin{align*}
t_jα_{j,i}t_j^{-1} &= g(α_{j,i}) & t_jβ_{j,i}t_j^{-1} &= g(β_{j,i}) \\
&t_jf^j(α_{0,i})t_j^{-1} = g \circ f^j(α_{0,i}) & t_jf^j(β_{0,i})t_j^{-1} = g \circ f^j(β_{0,i}) \\
f^j(t_jα_{0,i}t_j^{-1}) &= g \circ f^j(α_{0,i}) & f^j(t_jβ_{0,i}t_j^{-1}) = g \circ f^j(β_{0,i}) \\
t_jα_{0,i}t_j^{-1} &= f^{-j} \circ g \circ f^j(α_{0,i}) & t_jβ_{0,i}t_j^{-1} &= f^{-j} \circ g \circ f^j(β_{0,i})
\end{align*}
\]

which, getting rid of the by now unnecessary indices for \(K\) gives the presentation

\[
ker p = \langle α_i, β_i, w, t_j | α_i^w = f^m(α_i), β_i^w = f^m(β_i), α_i^j = f^{-j} \circ g \circ f^j(α_i), β_i^j = f^{-j} \circ g \circ f^j(β_i), \prod_{i=1}^g (α_i, β_i) = 1 \rangle.
\]
Either presentation allows us to interpret \( \ker p \) as extension
\[
1 \longrightarrow K \longrightarrow \ker p \longrightarrow F_{m+1} \longrightarrow 1
\]
were \( F_{m+1} = \langle w, t_0, ..., t_{m-1} \rangle \) and (the images, under a canonical section, of) \( w, t_0, ..., t_{m-1} \) act on \( K \) via the automorphisms \( f^m, g, f^{-1} \circ g \circ f, ..., f^{-m+1} \circ g \circ f^{-1} \in \text{Aut}(K) \) respectively.

To proceed, we will focus on a subgroup of \( \ker p \), which derives from disregarding all but one copy of \( \Pi_B \). This subgroup is the pull-back of the monomorphism \( F_2 \to F_{m+1} \) given as
\[
\langle w, t_0 \rangle \to \langle w, t_0, ..., t_{m-1} \rangle.
\]
This yields the diagram
\[
\begin{array}{cccccc}
1 & \longrightarrow & K & \longrightarrow & \tilde{\Pi}_A * K \Pi_B & \longrightarrow & F_2 & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
1 & \longrightarrow & K & \longrightarrow & \ker p & \longrightarrow & F_{m+1} & \longrightarrow & 1
\end{array}
\]
where all vertical maps are monomorphisms. We stress out that the amalgamation along \( K \), in \( \tilde{\Pi}_A * K \Pi_B \), is the identity because of the choice of the 0-th copy of \( \Pi_B \). (Choices of other copies would have yield, alternatively, a different isomorphism, or a modification of the monodromy in that copy. This is doable, but notationally more cumbersome.) Relabeling the generators for simplicity this group has presentation
\[
\tilde{\Pi}_A * K \Pi_B = \langle \alpha_i, \beta_i, w, t | \alpha_i^w = f^m(\alpha_i), \beta_i^w = f^m(\beta_i), \alpha_i^t = g(\alpha_i), \beta_i^t = g(\beta_i), \prod_{i=1}^g [\alpha_i, \beta_i] = 1 \rangle.
\]

Lemma 2.4 is the tool that allows us to pass some virtual properties of a factor of \( \Pi = \Pi_A * K \Pi_B \) to a factor of some subgroup of \( \Pi \). The properties of the factors that we need are listed in the following well-known Proposition, that summarize some highly nontrivial results if various authors on the mapping tori of \( K \).

**Proposition 2.5.** Let \( \pi = K \rtimes \varphi \mathbb{Z} \) be a mapping torus with monodromy \( \varphi \in \text{Out}(K) \), where \( K \) is the free group \( F_n, n > 2 \) or the fundamental group of a closed surface of genus \( g > 1 \). Then \( \pi \) is large, and

- when \( K = F_n \) and \( \varphi \) is atoroidal, then \( \pi \) is virtually RFRS;
- when \( K \) is the fundamental group of a closed surface of genus \( g > 1 \), and \( \pi \) is the fundamental group of a nonpositively curved 3-manifold, then \( \pi \) is virtually RFRS.
We detail the contributions to this result, well aware that brevity prevents us from doing justice to the authors.

- When $K = F_n$, either $\pi$ contains a $\mathbb{Z} \oplus \mathbb{Z}$ subgroup, or the monodromy is atoroidal. In the first case, $\pi$ is large by [Bu08, Theorem 5.1]. In the second case, $\pi$ is hyperbolic by [Br00], hence combining [?, HW15, HW16] it is virtually RFRS, in particular large;
- When $K$ is the fundamental group of a closed surface of genus $g > 1$, $\pi$ is the fundamental group of a closed 3–manifold that either admits a nontrivial JSJ decomposition, or it is Seifert fibered, or it is hyperbolic. In the first case by [LN91, Theorem 2.1] and in the second case by the fact that the monodromy must be periodic, we have that $\pi$ is large; in the hyperbolic case, by [A13], it is large; moreover, unless the 3-manifold is non-nonpositively curved (in particular a graph manifold) then $\pi$ is virtually RFRS, by [A13, L13, PW18]. (To be precise, these statement are tailor made to the case of a monodromy $\varphi \in \text{Out}^+(K) \leq_f \text{Out}(K)$, but as we are interested only in virtual properties, we don’t need to worry about orientation issues.)

We need to fix some notation to handle (normal) finite index subgroups of the factor groups of $\Pi = \Pi_A \ast_K \Pi_B$.

Let $\pi$ be the mapping torus of an automorphism of $K$, and let $\hat{\pi} \unlhd_f \pi$ be a normal finite index subgroup. Denote by $q: \pi \to Q$ the epimorphism such that $\hat{\pi} = \ker q$.

We have a diagram, with self-describing notation

\[
\begin{array}{cccccc}
1 & 1 & 1 & \\
\downarrow & & & \\
1 & \longrightarrow & \hat{K} & \longrightarrow & \hat{\pi} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow & & & & & & & & \\
1 & \longrightarrow & K & \longrightarrow & \pi & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
\downarrow q & & & & & & & & \\
1 & \longrightarrow & q(K) & \longrightarrow & Q & \longrightarrow & Q/q(K) & \longrightarrow & 0 \\
\end{array}
\]

We have the following theorem, that in light of Lemma 2.1 and Theorem 2.2 completes the proof of Theorem 1.1.

**Theorem 2.6.** Let $\Pi = \Pi_A *_K \Pi_B$ the free product of the mapping tori of two automorphisms of $K$, where $K$ is the free group $F_n$, $n > 2$ or the fundamental group of a closed surface of genus $g > 1$, amalgamated along $K$. Assume that $\Pi_A$ is virtually RFRS. Then there exists a subgroup $\hat{\Pi}_A *_K \hat{\Pi}_B \leq \Pi_A *_K \Pi_B$ where $\hat{\Pi}_A \leq_f \Pi_A$ and $\hat{\Pi}_B \leq_f \Pi_B$ are mapping tori of two automorphisms of $\hat{K} \leq_f K$, and with $b_1(\hat{\Pi}_A *_K \hat{\Pi}_B) \geq 3$.

**Proof.** We break down this argument in three claims, whose proof is very similar. The template is the following: one of the free factors of $\Pi = \Pi_A *_K \Pi_B$ has a finite index.
subgroup with a suitable property. Using twice Lemma 2.4 we can get a (perhaps infinite index) subgroup of \( \Pi \) that is again an amalgamated free product, with one factor being the aforementioned subgroup.

\textit{Claim.} There exists a subgroup \( \hat{\Pi}_A \ast_R \hat{\Pi}_B \leq \Pi \) where \( \hat{\Pi}_A \leq_f \Pi_A \) and \( \hat{\Pi}_B \leq_f \Pi_B \) are mapping tori of two automorphisms of \( \hat{K} \leq_f K \) such that the factor \( \hat{\Pi}_A \) is RFRS and \( \hat{\Pi}_B \) is large.

\textit{Proof of Claim.} Let \( \hat{\Pi}_A \leq_f \Pi_A \) be the a RFRS subgroup of \( \Pi_A \). It is convenient to break down the inclusion \( \hat{\Pi}_A \leq_f \Pi_A \) in two steps: first, following the notation of Eq. (9), we define the subgroup \( \hat{\Pi}_A \leq_f \Pi_A \) determined by the epimorphism \( p: \Pi_A \to \mathbb{Z} \to Q/q(K) \cong \mathbb{Z}_m \); second, we interpret \( \hat{\Pi}_A \) as kernel of the epimorphism \( q: \hat{\Pi}_A \to q(K) \) obtained by restricting to \( \hat{\Pi}_A \) the epimorphism \( q: \Pi_A \to Q \).

\[
\begin{align*}
1 & \longrightarrow \hat{K} \longrightarrow \hat{\Pi}_A \longrightarrow \mathbb{Z} \longrightarrow 0 \\
1 & \longrightarrow K \longrightarrow \hat{\Pi}_A \longrightarrow \mathbb{Z} \longrightarrow 0 \\
1 & \longrightarrow K \longrightarrow \Pi_A \longrightarrow \mathbb{Z} \longrightarrow 0.
\end{align*}
\]

We focus on \( \hat{\Pi}_A \) first: we can apply Lemma 2.4 to deduce that there exists a subgroup \( \hat{\Pi}_A \ast_K \hat{\Pi}_B \leq \Pi \). Next, we want to get a subgroup containing \( \hat{\Pi}_A \). There is an obstacle to do so right away. Namely, there is no reason for the epimorphism \( q: \hat{K} \to q(K) \) to extend to \( \Pi_B \). However, as is well-known, there exist some finite index subgroup \( \hat{\Pi}_B \leq \Pi_B \) (that we obtain by taking some suitable power of the automorphism \( g \in \text{Aut}(K) \)) determined by an epimorphism \( r: \Pi_B \to \langle t \rangle \to \mathbb{Z}_m \) such that that \( q \) extends to an epimorphism \( \hat{\Pi}_B \to q(K) \) that is trivial on the stable letter. We denote \( \hat{\Pi}_B \leq_f \hat{\Pi}_B \) the resulting subgroup; this is a mapping torus of \( \hat{K} \). So to proceed we first apply Lemma 2.4 again (with the role of \( A \) and \( B \) interchanged) to find a subgroup \( \hat{\Pi}_A \ast_K \hat{\Pi}_B \leq \hat{\Pi}_A \ast_K \Pi_B \leq \Pi \). At this point, \( q \) is defined on the entire \( \hat{\Pi}_A \ast_K \hat{\Pi}_B \), and trivial on the stable letters. Then \( \text{ker} q = \hat{\Pi}_A \ast_R \hat{\Pi}_B \) is the required subgroup, with the first factor RFRS, and the second still large.

Resetting the notation for our groups for notational convenience we proceed then with the assumption that the first factor of \( \Pi = \Pi_A \ast_K \Pi_B \) is RFRS. We will use next the fact that \( \Pi_B \) is large.

\textit{Claim.} There exists a subgroup \( \hat{\Pi}_A \ast_R \hat{\Pi}_B \leq \Pi \) where \( \hat{\Pi}_A \leq_f \Pi_A \) and \( \hat{\Pi}_B \leq_f \Pi_B \) are mapping tori of two automorphisms of \( \hat{K} \leq_f K \) such that there exists an element \( k \in \hat{K} \) that has nontrivial image in \( H_1(\hat{\Pi}_B; \mathbb{Z})/\text{Tor} \) and \( \hat{\Pi}_A \leq_f \Pi_A \) is RFRS.

\textit{Proof of Claim.} As \( \Pi_B \) is large, there exists a normal subgroup \( \hat{\Pi}_B \leq_f \Pi_B \), in the notation of Eq. (9), which surjects onto \( F_2 \), hence \( b_1(\hat{\Pi}_B) > 1 \). As \( H_1(\hat{\Pi}_B; \mathbb{Z}) = \)}
there exists an element \( k \in \hat{\mathcal{K}} \) that has nontrivial image in \( H_1(\hat{\mathcal{P}}_B;\mathbb{Z})/\text{Tor} \). Now, using twice Lemma 2.4 as in the proof of the previous Claim we obtain the desired result.

We can then assume that the first factor of \( \mathcal{P} = \mathcal{P}_A \ast \mathcal{K} \mathcal{P}_B \) is RFRS and that there exist an element \( k \in \hat{\mathcal{K}} \) that has nontrivial image in \( H_1(\hat{\mathcal{P}}_B;\mathbb{Z})/\text{Tor} \).

**Claim.** There exists a subgroup \( \hat{\mathcal{P}}_A \ast \hat{\mathcal{K}} \hat{\mathcal{P}}_B \leq \mathcal{P} \) where \( \hat{\mathcal{P}}_A \xrightarrow{f} \mathcal{P}_A \) and \( \hat{\mathcal{P}}_B \xrightarrow{f} \mathcal{P}_B \) are mapping tori of two automorphisms of \( \hat{\mathcal{K}} \xrightarrow{f} \mathcal{K} \) such that \( b_1(\hat{\mathcal{P}}_A \ast \hat{\mathcal{K}} \hat{\mathcal{P}}_B) \geq 3 \).

**Proof of the Claim.** As \( \mathcal{P}_A \) is RFRS, we can use Proposition 2.3 to find a subgroup \( \hat{\mathcal{P}}_A \xrightarrow{f} \mathcal{P}_A \), in the notation of Eq. (9), where that specific \( k \in \hat{\mathcal{K}} \xrightarrow{f} \mathcal{K} \) has nontrivial image in \( H_1(\hat{\mathcal{P}}_A;\mathbb{Z})/\text{Tor} \). As before, using twice Lemma 2.4 we can identify a subgroup \( \hat{\mathcal{P}}_B \xrightarrow{f} \mathcal{P}_B \), a mapping torus of an automorphism of \( \hat{\mathcal{K}} \xrightarrow{f} \mathcal{K} \), such that \( \hat{\mathcal{P}}_A \ast \hat{\mathcal{K}} \hat{\mathcal{P}}_B \leq \mathcal{P} \). As \( k \) has nontrivial image in \( H_1(\mathcal{P}_B;\mathbb{Z})/\text{Tor} \), it has nontrivial image in \( H_1(\hat{\mathcal{P}}_B;\mathbb{Z})/\text{Tor} \) by transfer. A Mayer–Vietoris argument, or alternatively the fact that \( k \) has nontrivial image in the coinvariant homology \( H_1(\hat{\mathcal{K}};\mathbb{Z})_F^2/\text{Tor} \) of the \( \hat{\mathcal{K}} \)-by–\( F_2 \) extension \( \hat{\mathcal{P}}_A \ast \hat{\mathcal{K}} \hat{\mathcal{P}}_B \).

This completes the proof of the Theorem. □

It is a bit embarrassing that, in the case where \( \mathcal{K} \) a surface group, we are not capable to extend the proof to the case where both factors are fundamental groups of non-nonpositively curved fibered 3–manifolds. (Such embarrassment is only mildly assuaged by the fact that such class of manifold is referred to as “last frontier” in [AFW15].) In fact, it is possible with a bit of effort to build examples of amalgamated free product of two groups of such type where the same strategy of Theorem 2.6 yields a proof of noncoherence, but we failed to come up with a general proof, or to convince ourselves that the strategy must fail.

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