Rotation groups

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Abstract

A query, about the orbit $PW$ in real 3-space of a point $P$ under an isometry group $W$ generated by edge rotations of a tetrahedron, leads to contrasting notions, $W$ versus $S$, of “rotation group”. The set $R = \{r_{A_1}, r_{A_2}\}$ of rotations $r_{A_i}$ about axes $A_i$ generates two manifestations of an isometry group on $\mathbb{R}^3$:

(1) In the stationary group $S := S(R)$, all axes $B$ are fixed under a rotation $r_A$ about $A$.

(2) In the peripatetic group $W := W(R)$, each $r_A$ transforms every rotational axis $B \neq A$.

Theorem. If the line $A_1$ is skew to $A_2$, if each $r_{A_i}$ is of infinite order, and if $P \in \mathbb{R}^3$, then both of the orbits $PS$ and $PW$ are dense in $\mathbb{R}^3$.

1 Introduction

Four decades ago, Jan Mycielski posed this question about a regular tetrahedron $T$:

For $G$ the isometry group on 3-space generated by the edge rotations $r_E$ of $T$, where Size($r_E$) is the supplement of the dihedral angle of $T$, what can be said about the orbit, $PG := \{Pf : f \in G\}$, of a point $P$ affixed to $T$?

Mycielski’s response, to our recent answer to his question, led to our study of two manifestations, $W$ and $S$, of the rotation group generated by a set of rotations. We establish sufficient conditions for the orbits $PW$ and $PS$ of a point $P$ to be dense in $\mathbb{R}^d$ for $d \in \{2, 3\}$. We have not studied the case where $d \geq 4$.

2 Technicalities

$\mathbb{R} := (\infty, \infty)$ denotes the set of all real numbers, $\mathbb{Z}$ is the set of all integers, and $\mathbb{N} := \mathbb{Z}^+$ is the set of all positive integers. When $n \in \mathbb{N}$ then $[n] := \{1, 2, \ldots, n\}$. Finally, $\mathbb{Q}$ is the set of all rational numbers.

We use standard interval notation; e.g., $(-3, 7) := \{x : x \in \mathbb{R} \land -3 < x \leq 7\}$. Except where more particularly specified, the dimension $d \geq 2$ of the real $d$-space $\mathbb{R}^d$ of our isometries is arbitrary.

For $d = 2$, the rotational axes are points in $\mathbb{R}^2$. However, when $d \geq 3$, the axes are directed lines in $\mathbb{R}^d$.

$r_A : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a rotation about the axis $A$, and is a directed isometry of $\mathbb{R}^d$. By this we mean that a nonzero rotation has a sign. We deem $r_A$ positive iff we see it as counterclockwise. For $d \geq 3$, we judge $r_A$ to be counterclockwise if we view it as counterclockwise when we look in the direction accorded to line $A$.

Let $F$ be a finite set, and $R := \{r_{A_i} : i \in F\}$ a set of generator rotations, with exactly one generator $r_{A_i}$ per axis $A_i$. For each $i \in F$, let $G_i$ be the cyclic group $\{r_{A_i}^z : z \in \mathbb{Z}\}$ of rotations about the axis $A_i$.

We write $\text{Rad}(r) = \rho \pi$ to indicate the radian measure, modulo $2\pi$, of the angle through which $r$ rotates; here $\rho \in (-1, 1]$ unless otherwise specified. $\text{Size}(r)$ denotes $|\text{Rad}(r)|$. We call an angle $\angle PVQ$ rational iff $\text{Rad}(\angle PVQ)/\pi \in \mathbb{Q}$, and we call $r_A$ rational if it rotates through a rational angle; i.e., if $\rho \in \mathbb{Q}$. Obviously $|G_i| < \infty$ if and only if $r_{A_i}$ is rational or, equivalently, is of finite order. Finally, we call two rotations equal iff they have both the same rotational axis and also the same radian measure modulo $2\pi$.

$f : P \mapsto Pf$ presents the isometry $f$ for $P \in \mathbb{R}^d$. When $M \subseteq \mathbb{R}^d$, we write $f : M \mapsto Mf := \{Vf : V \in M\}$.

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Remembering Jacqueline Bare Grace — 1942 December 01 - 2016 November 22
The identity isometry on $\mathbb{R}$ get ($f$) 2.2 Peripatetic groups composition of isometries; that for a “stationary” group is the conventional $P(f \circ g) = (Pf)g$. But, as we will explain later, a more peculiar relationship, $P(f \ast g) = (Pf)(fg)$, holds for a “peripatetic” group.

The expression $f^−$ denotes the inverse of the isometry $f$, and so $r_i^−$ is the reverse-sense rotation of $r_i$. The identity isometry on $\mathbb{R}^d$ is $\iota$.

### 2.1 Stationary groups

Let $R := \{r_i : i \in F\}$ for $2 \leq |F| < \infty$, where the axis of $r_i$ is $A_i$ for $i \in F$. The following stipulates a specializing name for that entity which most would take to be “the” $R$-generated rotational isometry group.

**Definition 1.** For $(f, h) \subseteq S$ and all $P \in \mathbb{R}^d$, we let $P(f \circ h) = (Pf)h := (P(f \circ h_1))g$, where $h = h_1 \circ g_i$ with $g_i \in G_i$ for some $i \in F$ and some $h_1 \in S$. The stationary group is $(S, o)$ where $S := S(R)$.

**Stationary Example.** To illustrate the compositional algorithm of the group $S := S(R)$, we will compute the stationary product $s := g_1 \circ g_2 \circ g_3 \circ g_4 \in S$ of the rotational sequence $(g_1, r_2, g_1', g_3) \in E_1 \times E_2 \times E_1 \times E_1$. For $P \in \mathbb{R}^d$, we will produce $s : P \mapsto Ps$ in four steps:

1. We enact $g_1 : P \mapsto Pg_1$ by rotating $P$ about the axis $A_1$ with $g_1$.
2. $g_2 : Pg_1 \mapsto (Pg_1)g_2 = (P(g_1 \circ g_2))$ is realized by rotating the point $Pg_1$ about $A_2$ with $g_2$.
3. We realize $g_3' : (P(g_1 \circ g_2)) \mapsto (P(g_1 \circ g_2 \circ g_3'))$ by rotating $(P(g_1 \circ g_2))$ about $A_1$ with $g_3'$.
4. Finally, $g_3$ rotates $P(g_1 \circ g_2 \circ g_3')$ about $A_3$ to reach $P(g_1 \circ g_2 \circ g_3') = Ps$.

**Caveat.** In the stationary context, neither rotational axes, nor points comprising them, are moved by group actions. However, a common point may own the same coordinate address as an (immobile) axis point $U$; i.e., “$P = U$” in the address-sharing sense. But whereas $U$ is unmoved by $f$, we have $f : P \mapsto Pf \neq P$. That is, $P$ and $U$ are distinct entities at the same location in $\mathbb{R}^d$.

We call a product $g_{j_1} \circ g_{j_2} \circ \cdots \circ g_{j_k}$ of $g_{j_i} \in E_{j_i}$ reduced iff $j_i \neq j_{i+1}$ for every $i < k$.

**Lemma 2.1.** Let $f \in S$. Then there exists exactly one reduced sequence $g := (g_{j_1}, g_{j_2}, \ldots, g_{j_k})$ in $\bigcup\{G_i : i \in F\}$ for which $f = g_{j_1} \circ g_{j_2} \circ \cdots \circ g_{j_k}$. So $S$ is a free group on the generator set $R$, module for those $i \in F$ with $|G_i| < \infty$, to congruences $\mod 2\pi$ which select the $[G_i]$ representatives in $(-\pi, \pi] \cap G_i$ from $\{r_i^z : z \in \mathbb{Z}\}$.

**Proof.** The lemma holds for $k = 1$. Suppose for all $i < k$ that, if $h \in S$ is a product $h = g_{j_1} \circ g_{j_2} \circ \cdots \circ g_{j_i}$, then this factorization of $h$ is unique. Let $h := g_{j_1} \circ g_{j_2} \circ \cdots \circ g_{j_{k-1}}$, and let $f := h \circ g_{j_k}$. Pretend that $f = h \circ g_i$ for some $g_i \neq g_{j_k}$. Then, since both $f$ and $h$ are bijection transformations of $\mathbb{R}^d$, we must infer that $g_i = h^− \circ f = g_{j_k}$, a contradiction. The lemma follows.

### 2.2 Peripatetic groups

**Definition 2.** For $g_i \in G_i$ and $f : \mathbb{R}^d \to \mathbb{R}^d$ an isometry, $fg_i$ denotes the rotation about the axis $A_i$, $f$ with $\operatorname{Rad}(fg_i) = \operatorname{Rad}(g_i)$. The binary operation $\ast$ of the group $W := W(R)$ is defined recursively by $P(f \ast g_i) := (Pf)(fg_i)$ for all $P \in \mathbb{R}^d$, and by $i \ast f = f \ast i$. We call the group $(W(R); \ast)$ peripatetic.

$W$ is called “peripatetic” as a reminder that $A_jg_i \neq A_j$ for all $\{i, j\} \subseteq F$ with $i \neq j$ unless $g_i = \iota$.

**Peripatetic Example.** Here $(g_1, g_2, g_1', g_3)$ is the same four-term sequence we employed in the Stationary Example above. Again let $P \in \mathbb{R}^d$. We will illustrate the peripatetic group’s computational algorithm by showing how to obtain $w : P \mapsto Pw$ via the peripatetic group product $w := g_1 \ast g_2 \ast g_1 \ast g_3$.

Our calculation realizing $w : P \mapsto Pw$ proceeds in the following four steps:

1. $g_1 : P \mapsto Pg_1$ swings $P$ about axis $A_1$, while rotating the axis $A_2$ into the position $A_2g_1$; i.e., when $W$ is the group, then $g_1 : A_1 \mapsto A_1g_1$ if $i \in \{2, 3\}$. Of course $A_1g_1 = A_1$.
2. The composite rotation $g_1g_2$ moves $Pg_1$ to $(Pg_1)(g_1g_2) = (P(g_1 \ast g_2))$, by rotating $Pg_1$ about the new axis $A_2g_1$. Simultaneously, $g_1g_2 : (A_1g_1, A_2g_1, A_3g_1) \mapsto (A_1(g_1 \ast g_2), A_2(g_1 \ast g_2), A_3(g_1 \ast g_2))$.
3. The composite rotation $(g_1 \ast g_2)g_1'$ rotates the point $P(g_1 \ast g_2)$ about the axis $A_1(g_1 \ast g_2)$, thus effecting $P((g_1 \ast g_2)) \mapsto (P(g_1 \ast g_2))(g_1') = P((g_1 \ast g_2) \ast g_1') = P(g_1 \ast g_2 \ast g_1')$. Concomitantly moving axes, we get $(g_1 \ast g_2)g_1' : (A_1(g_1 \ast g_2), A_2(g_1 \ast g_2), A_3(g_1 \ast g_2)) \mapsto (A_1(g_1 \ast g_2 \ast g_1'), A_2(g_1 \ast g_2 \ast g_1'), A_3(g_1 \ast g_2 \ast g_1'))$. 

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(4). \((g_1 \ast g_2 \ast g_3^*)g_1 : P(g_1 \ast g_2 \ast g_3^*) \Rightarrow P(g_1 \ast g_2 \ast g_3^* \ast g_3)\). Thus does the group \(W(R)\) produce \(w : P \mapsto Pw\).

We ask our readers to note the difference between this \(Pw\) and the \(Ps\) in the stationary example.

The tumbling \(T\) issue is more naturally treated by \(W\) than by \(S\), since by design \(W\) maintains the axes of rotation inside \(T\). Overall, \(W\) groups may lend themselves more readily to navigational strategies in a spaceship than do \(S\) groups, since a rotational coordinate system rooted in \(W(R)\) travels with the traveler.

We now firm up the foundation on which rest plausible but unproven assumptions about \(W\).

**Lemma 2.2.** Let \(f \in W\). Then there is exactly one reduced sequence \(g := \langle g_{j_1}, g_{j_2}, \ldots, g_{j_k} \rangle \) in \(\bigcup \{G_i : i \in F\}\) for which \(f = g_{j_1} \ast g_{j_2} \ast \cdots \ast g_{j_k}\). Therefore \(W\) is a free group on the generator set \(R\), subject to the congruences modulo \(2\pi\) of \(|G_i|\) for those \(i \in F\) with \(|G_i| < \infty\).

**Proof.** Uniformly substituting “\(\ast\)” for “\(\circ\)” in the proof of Lemma 2.1, we obtain a proof of Lemma 2.2. □

**Corollary 3.2.** The groups \(S(R)\) and \(W(R)\) are isomorphic.

**Proof.** This corollary is immediate from the two Lemmas, 2.1 and 2.2. □

Although as abstract algebraic structures the groups \(S(R)\) and \(W(R)\) are indistinguishable, they differ as specific subgroups of the group of all isometries on \(R^d\). Given \(P \in R^d\) and \(R\), we ask our readers for an efficient algorithm that expresses some isomorphism \( \varphi : S(R) \rightarrow W(R) \).

For \(P \in R^d\) what relationships obtain between the sets \(PS\) and \(PW\)?

### 3 The tumbling tetrahedron and peripatetic orbital densities

When \(x \in R\) then the expression \(x\mathbb{Z}/\mathbb{Z}\) denotes the set \([0, 1) \cap \{x/a : \{a, b\} \subseteq \mathbb{Z}\}\).

Our applications of \(W(R)\) to density issues depend upon the following well-known consequence\(^1\) of a theorem of Kronecker. \(^2\) A theorem of Hurwitz. \(^2\) \(^3\) \(^4\) \(^5\), also provides a proof\(^3\).

**Lemma 3.1.** Let \(x \in R \setminus Q\). Then the set \(x\mathbb{Z}/\mathbb{Z}\) is dense in \([0, 1)\).

We need a corollary which is immediate from Lemma 3.1.

**Corollary 3.2.** Let \(P\) be a point on a circle \(C\) with centerpoint \(U\), and let \(r\) be a rotation about \(U\). Then these three assertions are equivalent:

1. The rotation \(r\) is of infinite order.
2. The size of \(r\) is an irrational multiple of \(\pi\) radians.
3. The set \(\{Pr^z : z \in \mathbb{Z}\}\) is dense in \(C\).

The generating sets \(R := \{r_A, r_{A'}\}\) we will be using are two-membered, except when we deal with the tumbling tetrahedron \(T\), for which \(|R_T| = 6\). Under the action of \(W\), a copy of \(R^2\) moves against a fixed \(R^2\) background, whose points are not budged by \(W\) actions.

Recall that in both the stationary and the peripatetic contexts, \(G_i\) is the cyclic subgroup \(\{r_i^z : i \in \mathbb{Z}\}\).

We now state and prove our more easily visualized peripatetic rotational density theorem\(^3\).

**Theorem 3.3.** Let \(U_1 \neq U_2\) be points in \(R^2\). Let \(r_1\) and \(r_2\) be rotations of infinite order about \(U_1\) and \(U_2\) respectively. Let \(W := W(R)\), where \(R := \{r_1, r_2\}\). Let \(P \in R^2\). Then the orbit \(PW\) is dense in \(R^2\).

**Proof.** Take it that \(P \neq U_1\). We argue by contradiction. Pretend there is a (fixed) point \(Q\), and a (fixed) disc \(D \subseteq R^2\) with center at \(Q\), and whose radius \(\epsilon > 0\) is the largest that allows \(D \cap PW = \emptyset\). Since \(\epsilon \leq \|P - Q\| < \infty\), we have \(Pf \in S := \overline{D} \setminus D\) for some \(f \in W\), where \(\overline{D}\) is the closure of \(D\), and \(S\) is a circle.

The points \(Pf, (U_2f)(g_1) := U_2(g \ast g_1)\) and \(Q\) are noncollinear for some \(g_1 \in G_1\). Let \(C_1 \subseteq R^2\) be the circle with centerpoint \(U_1\) and with \(P \in C_1\); by hypothesis, the radius of \(C_1\) is \(\|P - U_1\| > 0\). Let \(C_2\) be the circle with centerpoint \(U_2\) and with \(P \in C_2\); the radius of \(C_2\) is \(\|P - U_2\| \geq 0\).

\(^1\)pointed out to us by Wieslaw Dziobiak
\(^2\)We will email a PDF of three very short proofs of this fact to those who request us to do so.
\(^3\)Allan Silbergen suggested Theorem 3.3.
The circles $C_1$ and $C_2$ are mapped by $f$ isometrically onto the respective circles $C_1 f$ with centerpoint $U_1 f$, and $C_2 f$ with centerpoint $U_2 f$. We see that $C_1 f = C_1 (f * g_1)$, since $f * g_1$ merely rotates $C_1 f$ about the point $U_1 f$. However, $C_2 (f * g_1) \neq C_2 f$; for, the rotation $f * g_1$ swings $\mathbb{R}^2$ around $U_1 f$, thus mapping the circle $C_2 f$ isometrically onto the circle $C_2 (f * g_1)$, whose centerpoint is $U_2 f g_1 \neq U_2 f$. If $g_1$ is chosen prudently, then $C_2 (f * g_1) \cap B \neq \emptyset$. Now [3.2] finishes the proof, since $P(f * g_1) = (P f)(f g_1)$ and $f * g_1 \in \mathcal{W}$. \hfill $\square$

The analogous next result will enable us to answer the Mycielski question which inspired this paper.

**Theorem 3.4.** Let $r_1$ and $r_2$ be infinite-order rotations about the respective skew directed lines $X_1$ and $X_2$ in $\mathbb{R}^3$, let $R := \{r_1, r_2\}$, let $\mathcal{W} := \mathcal{W}(R)$, and let $P \in \mathbb{R}^3$. The orbit $P \mathcal{W} := \{P f : f \in \mathcal{W}\}$ is dense in $\mathbb{R}^3$.

**Proof.** We can take it that $P \notin X_1$. For each $i \in \{1, 2\}$ with $P \notin X_i$, let $U_i \in X_i$ be such that $P U_i \perp X_i$, and let $C_i$ be the circle in $\mathbb{R}^3$ of radius $\|P - U_i\|$ and with centerpoint $U_i$. If $P \notin X_i$, then $\{U_i, P, P g_i\}$ is a set of noncollinear points for $i \neq g_1 \in G_1$. But in the event that $P \in X_i$, let $U_i := P$ and $C_i := \{P\}$.

Again arguing by contradiction, we pretend that there is a (fixed) point $Q$, and a $Q$-centered open ball $B \subseteq \mathbb{R}^3$ of radius $\epsilon > 0$, such that $B \cap P \mathcal{W} = \emptyset$, but that if $B'$ is a $Q$-centered ball with radius $\epsilon' > \epsilon$ then $B' \cap P \mathcal{W} \neq \emptyset$. Since $\epsilon \leq \|Q - P\| < \infty$, it follows that $P f \in S := \overline{B'} \setminus B$ for some $f \in \mathcal{W}$, where $\overline{B}$ is the closure of $B$ and $S$ is therefore a 2-sphere.

Since $B$ is open, if $B \cap C_i f \neq \emptyset$ then $B \cap C_i f$ is an arc of positive length in $B$, whence Corollary 3.2 concludes our proof. So we take the circle $C_i f$ to be tangent to the sphere $S$ at the point $P f$. For $i \in \{1, 2\}$, the point $U_i f \in X_i f$ is the centerpoint of the circle $C_i f$.

Since the axes $X_1$ and $X_2$ are skew and since $f$ is an isometry, $X_1 f$ is skew to $X_2 f$. Furthermore, $\text{Order}(r_2) = \infty$ by hypothesis. WeThere are two cases.

**Case**: $P \notin X_2$. Then we can infer from Corollary 3.2 that there exists $g_2 \in G_2$ with $P(f * g_2) := (P f)(f g_2) \in B$. Moreover, $f * g_2 \in \mathcal{W}$. Thus $B \cap P \mathcal{W} \neq \emptyset$, contrary to our choice of $B$.

**Case**: $P = U_2$. By Corollary 3.2 there exists $g_2 \in G_2$ with $B \cap C_1 (f * g_2) \neq \emptyset$. So here too $B \cap P \mathcal{W} \neq \emptyset$.

In both cases we reach a contradiction.

**Remark.** If the hypotheses of Theorem 3.4 allowed $X_1 \cap X_2 = \{V\}$, then $P \mathcal{W} \subseteq \mathcal{Y}$ where $\mathcal{Y} \subseteq \mathbb{R}^3$ is the sphere of radius $\|P - V\|$ and centerpoint $V$; if also $P \neq V$ then $\|P - V\| > 0$, and $P \mathcal{W}$ would be dense in $\mathcal{Y}$.

In order that assure that $P \mathcal{W}(R_T)$ is dense in $\mathbb{R}^3$, where $R_T$ is the set of six edge rotations $r_E$ of $T$, Theorem 3.4 requires only two of the six to be irrational. But $\text{Size}(r_E) := \pi - \theta$ for each $r_E \in R_T$, where $\theta$ is the size of each dihedral angle of $T$. So we will need to prove that $\theta$ is irrational.

**Quiz.** Show that $\sin(\theta) = 2\sqrt{2}/3$, and that equivalently $\cos\theta = 1/3$.

**Lemma 3.5.** Let $0 < \theta < \pi/2$. If $\cos(\theta) = 1/3$ then $\theta$ is irrational.

**Proof.** By Corollary 3.12 of [3], both the angle $\pi/2 - \phi$ and the real number $\cos(\phi) = \sin(\pi/2 - \phi)$ are rational if and only if $\phi = \pi/3$. So, since $\cos(\pi/3) = 1/3 \neq 1/3 = \cos(\theta)$, we see that $\theta$ is irrational. \hfill $\square$

The answer to Mycielski’s half-century-old query about tumbling $T$ is now obvious from 3.4 with 3.6

**Corollary 3.6.** Let $T$ be a regular tetrahedron in $\mathbb{R}^3$, and let $\mathcal{W} := \mathcal{W}(R_T)$ be the peripatetic rotational isometry group determined by the set $R_T$ of six generating rotations $r_E$ around the edges $E$ of $T$, where each $\text{Size}(r_E)$ is the supplement of the dihedral angle of $T$. Then the orbit $P \mathcal{W}$ is dense in $\mathbb{R}^3$ for each $P \in \mathbb{R}^3$.

Steve Silverman asks whether there exists a tetrahedron, all six of whose dihedral angles are rational. He provides an example in which four of the six are rational. Is four best possible? If “yes”, then the tumbling of an arbitrary tetrahedron $T$, each edge $E$ of which is assigned a rotation $r_E$ whose size is the supplement of the dihedral angle at $E$, will trace out an orbit $P \mathcal{W}$ that is dense in $\mathbb{R}^3$ for each $P \in \mathbb{R}^3$.

4 Orbital density for stationary groups

The following may be a duplication of a decades-old unpublished result of Jan Mycielski.
Theorem 4.1. Let $d \in \{2,3\}$. Let the generating set $\mathbb{R}$ be as in Theorem 3.3 or 3.4, and let both of the rotations $r_1$ and $r_2$ be of infinite order. Let $S := S(\mathbb{R})$ be the stationary group determined $\mathbb{R}$. Then the orbit $VS$ is dense in $\mathbb{R}^d$ for every $V \in \mathbb{R}^d$.

Proof. Since the proof for $d = 3$ is essentially the same as that for $d = 2$, we will argue only the $d = 2$ case. The axes of $r_1$ and $r_2$ are $U_1$ and $U_2$ respectively. However, unlike in Theorem 3.3, here the points $U_i$ are fixed, and are unaffected by actions of $S$. On the other hand, the point $V$ is not fixed. Suppose without loss of generality that $V \neq U_2$; i.e., that the points $V$ and $U_2$ have different addresses in $\mathbb{R}^2$.

By hypothesis, for each $i \in \{1,2\}$ we have that $\text{Size}(r_i) = \rho_i \pi$ for an irrational $\rho_i$. We can take it that $\rho_i \in (0,1)$. Let $k_i$ be the positive integer for which $k_i \rho_i < 1 < (k_i + 1) \rho_i$.

Each set $C_{i,V} := C_{i,V} \cap VS$ is dense in the unique circle $C_{i,V}$ with centerpoint $U_i$ and with $V \in C_i$, provided only that $C_i \neq \{U_i\}$. Let $O_1$ be the finite set $\{Vr_i^j : 0 \leq j \leq 2k_1\} \subset C_{1,V}$.

For each $A \in O_1$, we create the finite set $\{Ar_i^j : 0 \leq j \leq 4k_2\} \subset C_{2,A}$, where $C_{2,A}$ is the unique circle having centerpoint $U_2$ and such that $A \in C_{2,A}$. Define $O_2 := \bigcup\{\{Ar_i^j : 0 \leq j \leq 4k_2\} : A \in O_1\}$.

Similarly, using the points $A \in O_2$, define $O_3 := \bigcup\{\{Ar_i^j : 0 \leq j \leq 2^3k_1\} : A \in O_2\}$.

Next, define $O_1 := \bigcup\{\{Ar_i^j : 0 \leq j \leq 2^3k_2\} : A \in O_3\}$. Continue thus, back and forth between the fixed rotational axes $U_1$ and $U_2$. This recursion produces an infinite sequence $(O_n)_{n=1}^{\infty}$ of finite subsets of $VS$ with $O_b \subset O_{b+1}$, where $O_b$ approximate elliptical regions whose minimal diameters increase without bound, and such that the meshes of the sets $O_b$ diminish to zero.

Ultimately we obtain a denumerable subset $\bigcup\{O_b : b \in \mathbb{N}\} \subseteq VS$, where $\bigcup\{O_b : b \in \mathbb{N}\}$ is dense in $\mathbb{R}^2$. Therefore, the orbit $VS$ of $V$ under the actions of the stationary rotational isometry group $S(\mathbb{R})$ is itself dense in $\mathbb{R}^2$.

The three-dimensional case involves a reiteration of the argument above, modulo these five observations:

1. The axes of rotation are now two fixed directed skew lines instead of two fixed points.
2. The circle $C_{1,V}$ is not coplanar with the circle $C_{2,V}$.
3. For $d = 3$, the convex closures of the $O_b$ approximate ellipsoids or spheroids rather than ellipses.
4. Re “mesh”: In $\mathbb{R}^3$ we talk of $\epsilon$-radius balls instead of the $\epsilon$-radius discs that make sense in $\mathbb{R}^2$.
5. In $\mathbb{R}^2$ the convex closure of $VS$ is two-dimensional. But it is three-dimensional in $\mathbb{R}^3$.

5 Related issues

The classes of stationary and peripatetic rotational groups in $\mathbb{R}^d$ for $2 \leq d < \infty$ are antipodal in regards to the modes of their rotational generators $r_1$. This suggests the possible fruitfulness of studying rotational groups of sorts that lie between those antipodal classes; i.e., rotational groups some of whose given axes are stationary while others are peripatetic.

We believe Theorem 3.4 extends in a natural way to $\mathbb{R}^d$ for $d \in \{4,5,6,\ldots\}$, given an appropriate finite collection $X$ of lines $X_i \subseteq \mathbb{R}^d$ and of infinite-order rotations $r_i$ of 2-planes about those lines. Our readers will expect that the claim in such a generalization of Theorem 3.4 would be that the orbit $P\mathbb{V}$ is dense in $\mathbb{R}^d$ for every $P \in \mathbb{R}^d$.

We ask this question of our readers: Which, if any, of the following three hypotheses about the collection $X$ is both necessary and sufficient to guarantee the density in $\mathbb{R}^d$ of $P\mathbb{V}$ for all $P$?

1. $|X| = d - 1$ and no proper hyperplane in $\mathbb{R}^d$ is a superset of $\bigcup X$.
2. $|X| = d - 1$ and $X$ is pairwise skew.
3. $X = \{X_1, X_2\}$ is a set of skew lines.

To illustrate the contention between the hypotheses, 1 and 2, we offer for $4 \leq d \in \mathbb{N}$ this example:

For each $i \in \mathbb{N}$, let $Y_i := \{(i, ti, ti^2, 0, 0, \ldots, 0) : t \in \mathbb{R}\} \subseteq \mathbb{R}^d$. Then $Y := \{Y_i : i \in \mathbb{N}\}$ is an infinite pairwise skew set of lines for which $\bigcup Y$ is a subset of a copy of $\mathbb{R}^3$ that occurs as a proper subspace of $\mathbb{R}^d$.

Both Theorem 3.4 and Corollary 3.4 provide orbits $P\mathbb{V}$, each of which is dense in $\mathbb{R}^d$. But the orbit encountered in 3.4 is plainly richer than that encountered in 3.4. This invites an observation.

4It is natural to define the mesh of a subset $E \subseteq VS$ as the minimum $\epsilon > 0$ for which no disc $D$ of radius $\epsilon$ inside the convex closure of $E$ can fail to contain points in $E$. 

5
Let the regular tetrahedron $T$ have an edge length of $\sqrt{6}$, let $P$ be its barycenter, let $\mathcal{G}$ be the peripatetic rotational isometry group generated by the six edge rotations $r_{UV}$ of $T$, let $f \in \mathcal{G}$, and let the axes of rotations $r_{AB}$ and $r_{AC}$ share a single vertex, $A$. Then $H \cap P\mathcal{G}$ contains the vertices of a tiling of $H$ by hexagons of edge length 1, where $H$ is the plane determined by the point set $\{Pf, Pr_{AB}, Pr_{AC}\}$.

Conjecture 1. If plane $H \supseteq \{Pf, Pr_{AB}, Pr_{AC}\}$ then $H \cap P\mathcal{G}$ is the vertex set of a hexagonal tiling of $H$.

Let $X$ be the $x$-axis of $\mathbb{R}^3$. Let $Y$ be parallel to the $y$-axis and through $\langle 0, 0, 1 \rangle$. Let $W := W(R)$ be the peripatetic group generated by $R := \{r_X, r_Y\}$, where $\text{Order}(r_X) = \text{Order}(r_Y) = 4$. Let $P \in \mathbb{R}^3 \setminus (X \cup Y)$. Then $PW$ is an infinite discrete set of vertices of rectangular parallelopipeds, and $PW$ is nowhere dense.

This suggests a shift of focus from rotations to their axes of rotation.

For $d \geq 2$, say that nonparallel lines $X$ and $Y$ in $\mathbb{R}^d$ conform rationally iff some translate of $Y$ intersects $X$ to form a rational angle.

Let $X_1$ and $X_2$ be skew lines in $\mathbb{R}^3$, and let both of their respective rotations $r_1$ and $r_2$ be of finite order. Let $F$ be the peripatetic rotational isometry group generated by $F := \{r_1, r_2\}$.

Conjecture 2. $PF$ is nowhere dense for each $P \in \mathbb{R}^3$ if and only if $X_1$ conforms rationally to $X_2$.

Conjecture 3. If $X_1$ does not conform rationally to $X_2$, then $PF$ is dense in $\mathbb{R}^3$ for all $P \in \mathbb{R}^3 \setminus (X_1 \cup X_2)$.

Suppose Conjecture 2 is true. Let $X_1$ and $X_2$ be rationally conforming skew lines in $\mathbb{R}^3$, and let both $r_1$ and $r_2$ be rotations of finite order on those respective axes. Let $F := \{r_1, r_2\}$ be the generator set for the peripatetic group $F := F(F)$.

We call a polyhedron $K \subseteq \mathbb{R}^3$ a $\langle P, F \rangle$-chamber iff $K$ is maximal with respect to these four conditions:

1. $K$ is convex and of finite diameter.
2. Every vertex of $K$ is an element in $PF$.
3. No element in $PF$ is in the interior of $K$.
4. No three vertices of $K$ are collinear.

What can one say about these $\langle P, F \rangle$-chambers?

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Special language: arc-rational, conform rationally, peripatetic, stationary.