Coherent dynamics of radiating atomic systems in pseudospin representation

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Abstract
The aim of this review is twofold. Firstly, a general approach is presented which can provide a unified description of the dynamics of radiating systems of different natures. Both atomic systems and spin assemblies can be treated within the same mathematical method, which is based on a pseudospin (or spin) representation of the evolution equations. The approach is applicable to all stages of radiation dynamics, including the most difficult initial quantum stage, where coherence is not yet developed. This makes it possible to study the process of coherent self-organization from the chaotic quantum stage. Secondly, the approach is illustrated by applying it to the description of several coherent phenomena. Different types of superradiance are characterized: pure superradiance, triggered superradiance, pulsing and punctuated superradiance. The theory of such interesting effects as triggering dipolar waves, turbulent photon filamentation, collective liberation of light, pseudospin atomic squeezing, and operator entanglement production is presented.

Keywords: radiating atomic systems, stochastic mean-field quantization, scale separation approach, triggering dipolar waves, pulsing and punctuated superradiance, turbulent photon filamentation, collective liberation of light

1. Introduction

Coherent radiation from atomic and molecular systems, being the basis of laser radiation, has been extensively studied. The theory of this radiation, occurring in optical or infrared wavelengths, has been thoroughly expounded in many books (see, e.g., [1–5]). Collective radiation from spin assemblies has also attracted much recent attention [6]. Atomic and spin systems, being rather different in nature, are usually described by different types of equations. The physical processes in these systems are really quite different, but it is possible to develop a mathematical approach that can provide a similar description of both system types. In this review we present such an approach. We concentrate on the dynamics of atomic systems, illustrating the approach through the description of several coherent phenomena.

A problem of great interest is the self-organization of coherence from initially chaotic quantum fluctuations. Our approach makes it possible to analyze in detail such self-organization. In summary, the basic mathematical points of the approach are: pseudospin representation of evolution equations, stochastic quantization, scale separation, transverse mode expansion, and probabilistic pattern selection.

To illustrate the approach, several nontrivial phenomena are treated, including dipolar wave triggering, different kinds of superradiance (pure superradiance, triggered superradiance, pulsing and punctuated superradiance), turbulent photon filamentation, collective liberation of light, pseudospin atomic squeezing, and operator entanglement production.

The main point, emphasized throughout the paper, is the generality of the mathematical techniques, which can be applied to any kind of evolution equations describing radiating systems. The equations of motion for spin systems are written for the spin degrees of freedom. The pseudospin representation for atomic systems makes it straightforward to apply the same mathematical techniques to both spin and atomic radiating systems.

This review for the special issue is devoted to the memory of Igor V Yevseyev. Igor was my friend for many years and I had the pleasure of discussing with him various scientific and non-scientific problems. My memories of Igor are connected with the problems we discussed. It was exactly this connection...
2. Equations in pseudospin representation

In order to reduce the equations for radiating atomic systems to a form that is also convenient for treating spin systems, it is natural to resort to a pseudospin representation and eliminate the field degrees of freedom [7, 8]. If one also aims to accurately represent self-organization of coherence from chaos, it is necessary to start with a microscopic picture.

Let us consider \( N \) atoms (or molecules) resonantly interacting with an electromagnetic field. The general form of the Hamiltonian is

\[
\hat{H} = \hat{H}_a + \hat{H}_f + \hat{H}_d + \hat{H}_{ \text{int}}. \tag{1}
\]

The first term

\[
\hat{H}_a = \sum_{j=1}^{N} \omega_0 \left( \frac{1}{2} + S_j^z \right) \tag{2}
\]

corresponds to resonant atoms with the transition frequency \( \omega_0 \); the pseudospin operator \( S_j^z \) describes interlevel electronic transitions of the \( j \)th atom. Considering here two-level atoms, we deal with pseudospin operators of spin one-half. The operators are called pseudospin because they satisfy the spin algebra, but do not of higher-order pseudospin operators. The operators are called pseudospin because they satisfy the spin algebra, but do not describe real spins, representing instead interlevel transitions.

Here and in what follows, a system of units is employed in which Planck’s constant is set to unity.

The second term in equation (1) defines the energy of the radiated electromagnetic field

\[
\hat{H}_f = \frac{1}{2 \kappa} \int (E^2 + H^2) \, d\mathbf{r}, \tag{3}
\]

where \( E \) is the electric field and \( H \) is the magnetic field represented through a vector potential \( \mathbf{A} \),

\[
\mathbf{H} = \nabla \times \mathbf{A}. \tag{4}
\]

The vector potential is assumed to satisfy the Coulomb gauge condition

\[
\nabla \cdot \mathbf{A} = 0. \tag{5}
\]

The third term describes the atom–field interaction

\[
\hat{H}_d = - \sum_{j=1}^{N} \left( \frac{1}{c} \mathbf{J}_j \cdot \mathbf{A}_j + \mathbf{P}_j \cdot \mathbf{E}_0 \right), \tag{6}
\]

where dipolar transitions are assumed, \( \mathbf{E}_0 \) is an external electric field, the current operator is

\[
\mathbf{J}_j = - i \omega_0 \left( \mathbf{d}^a S_j^- - \mathbf{d}^a S_j^+ \right) \tag{7}
\]

and the polarization operator is

\[
\mathbf{P}_j = \mathbf{d}^a S_j^- + \mathbf{d}^a S_j^+, \tag{8}
\]

with the ladder pseudospin operators

\[
S_j^\pm = S_j^x \pm i S_j^y,
\]

and \( \mathbf{d} \) is a transition dipole. The notation

\[
A_j \equiv \mathbf{A}(\mathbf{r}_j, t), \quad E_{\mathbf{d}j} \equiv \mathbf{E}_d(\mathbf{r}_j, t)
\]

is used.

The last term in equation (1) describes the interaction of the radiated field with the matter of surrounding atoms. This term is absent when atoms are in a vacuum, but if atoms are immersed into some kind of matter, the interaction term is

\[
\hat{H}_{\text{int}} = - \frac{1}{c} \int \mathbf{j}_{\text{mat}} \cdot \mathbf{A} \, d\mathbf{r}, \tag{9}
\]

where \( \mathbf{j}_{\text{mat}} \) is the density current in the matter.

The evolution equations are prescribed by the Heisenberg equations of motion, with the corresponding commutation relations. The electromagnetic operators satisfy the relations

\[
\begin{align*}
[E^\omega(\mathbf{r}, t), A^\omega(\mathbf{r}', t)] &= 4 \pi i c \delta_{\omega\omega'}(\mathbf{r} - \mathbf{r}'), \\
[A^\omega(\mathbf{r}, t), H^{\beta}(\mathbf{r}', t)] &= 0, \\
[E^\omega(\mathbf{r}, t), H^{\beta}(\mathbf{r}', t)] &= -4 \pi i c \sum_\mathbf{r} \varepsilon_{\omega\gamma \beta} \frac{\partial}{\partial r^\gamma} \delta(\mathbf{r} - \mathbf{r}'),
\end{align*}
\]

in which \( \varepsilon_{\omega\gamma \beta} \) is the unitary antisymmetric tensor, \( c \) is the velocity of light, and the transverse delta function is defined as

\[
\delta_{\omega\omega'}(\mathbf{r}) = \frac{2}{3} \delta_{\omega\omega'}(\mathbf{r}) - \frac{1}{4 \pi} D_{\omega\omega'}(\mathbf{r}),
\]

with the dipolar tensor

\[
D_{\omega\omega'}(\mathbf{r}) \equiv \frac{1}{r^2} \left( \delta_{\omega\omega'} - 3 n^\omega n^{\omega'} \right), \tag{10}
\]

where

\[
r \equiv |\mathbf{r}|, \quad n \equiv \frac{\mathbf{r}}{r}.
\]

The pseudospin operators obey the spin algebra

\[
[S_j^-, S_j^+] = -2 \delta_{\omega\omega'} S_j^z, \quad [S_j^-, S_j^\pm] = \delta_{\omega\omega'} S_j^\pm, \quad [S_j^+, S_j^\pm] = -\delta_{\omega\omega'} S_j^\pm.
\]

The Heisenberg equations for the field variables yield Maxwell’s equations

\[
\frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{H} - \frac{4 \pi}{c} \mathbf{j}, \quad \frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} = - \mathbf{E}, \tag{11}
\]

from which, with the Coulomb calibration (5), the wave equation follows:

\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \mathbf{A} = - \frac{4 \pi}{c} \mathbf{j}, \tag{12}
\]

with the current density

\[
\mathbf{j}^\omega(\mathbf{r}, t) = \sum_\mathbf{r} \sum_{\omega_i} \delta_{\omega\omega'}(\mathbf{r} - \mathbf{r}_i) J_i^\omega(t) + \int \delta_{\omega\omega'}(\mathbf{r} - \mathbf{r}') j_{\text{mat}}(\mathbf{r}', t) \, d\mathbf{r}'. \tag{13}
\]

The solution to equation (11) reads

\[
\mathbf{A}(\mathbf{r}, t) = \mathbf{A}_{\text{vac}}(\mathbf{r}, t) + \frac{1}{c} \int [\mathbf{j}(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)] \, d\mathbf{r}', \tag{14}
\]
where $A_{\text{vac}}$ is the vacuum vector potential that is a solution to the equation
\[
\left( \nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) A_{\text{vac}} = 0.
\]

Heisenberg’s equations for the pseudospin variables give
\[
\frac{dS_i^r}{dt} = -i\omega_0 S_i^r + 2SJ_z(\mathbf{k}_0 \mathbf{d} \cdot \mathbf{A}_j - i\mathbf{d} \mathbf{E}_0),
\]
\[
\frac{dS_j^z}{dt} = -S_i^r(\mathbf{k}_0 \mathbf{d} \cdot \mathbf{A}_j - i\mathbf{d} \mathbf{E}_0) - S_j^z(\mathbf{k}_0 d^+ \mathbf{A}_j \mathbf{d} + i\mathbf{d}^+ \mathbf{E}_0),
\]
where $\mathbf{k}_0 = \omega_0/c$.

The vector potential (13), with the current density (12), can be written, excluding self-action, as
\[
\mathbf{A} = A_{\text{vac}} + A_{\text{rad}} + A_{\text{dip}} + A_{\text{mat}}.
\]

The first term is due to radiating atoms,
\[
A_{\text{rad}}(\mathbf{r}, t) = \sum_{j\neq i} \frac{2}{3c r_j} \left( t - \frac{r_j}{c} \right),
\]
where $r_{ij} = |\mathbf{r}_{ij}|$, $\mathbf{r}_i = \mathbf{r} - \mathbf{r}_j$.

Using the form of the current equation (7), we find
\[
A_{\text{dip}}(\mathbf{r}, t) = -i \sum_{j\neq i} \frac{2\mathbf{k}_0}{3c r_j} \left[ \mathbf{d}^* S_j^r(\mathbf{t} - \frac{r_j}{c}) - \mathbf{d} S_j^r(\mathbf{t} - \frac{r_j}{c}) \right].
\]

The second term in equation (16) is caused by induced atomic dipoles,
\[
A_{\text{rad}}(\mathbf{r}, t) = - \sum_{j\neq i} \int \frac{D_{\text{dip}}(\mathbf{r'} - \mathbf{r}_j)}{4\pi c |\mathbf{r} - \mathbf{r}'|} J^0(\mathbf{r} - \frac{|\mathbf{r} - \mathbf{r}'|}{c}) d\mathbf{r}'.
\]

And the last term
\[
A_{\text{mat}}(\mathbf{r}, t) = \sum_{j\neq i} \int \frac{\delta_{\text{dip}}(\mathbf{r} - \mathbf{r}_j)}{|\mathbf{r} - \mathbf{r}'|} \mathbf{J}^0(\mathbf{r}' - \mathbf{r}, \mathbf{t} - \frac{|\mathbf{r} - \mathbf{r}'|}{c}) d\mathbf{r}' d\mathbf{r}'
\]
corresponds to the vector potential created by the current in the matter.

Notice that, due to a dependence on the variable $t - r/c$, we have
\[
\left( \frac{1}{c} \frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right) S_i^z(\mathbf{t} - \frac{r}{c}) = 0.
\]

We assume that electromagnetic fields do not strongly disturb atomic level structure, so that
\[
\frac{d|\mathbf{d} \mathbf{E}|}{\omega_0} \ll 1, \quad \frac{d|\mathbf{d} \mathbf{E}_0|}{\omega_0} \ll 1
\]
and the retardation effects can be described in the Born approximation. Under condition (22), from equation (15) it follows
\[
\frac{\partial}{\partial r} S_i^z(\mathbf{t} - \frac{r}{c}) = i\omega_0 S_i^z(\mathbf{t} - \frac{r}{c}).
\]

Setting the retardation condition
\[
S_i^z(t) = 0 \quad (t < 0),
\]
we finally get, in the Born approximation
\[
S_i^z(t - \frac{r}{c}) = S_i^z(t) \Theta(ct - r)e^{i\delta},
\]
\[
S_j^z(t - \frac{r}{c}) = S_j^z(t) \Theta(ct - r).
\]

### 3. Self-action of a radiating atom

Strictly speaking, by substituting the current density (12) into the integral in equation (13), one obtains the terms corresponding to atomic self-action, which can be treated as follows. The vector potential generated by a single atom is
\[
A_i^0(\mathbf{r}, t) = \frac{1}{c} \sum_{\beta} \int \frac{\delta_{\text{dip}}(\mathbf{r} - \mathbf{r}_j)}{|\mathbf{r} - \mathbf{r}'|} J^0(\mathbf{r} - \frac{|\mathbf{r} - \mathbf{r}'|}{c}) d\mathbf{r}',
\]
with the current
\[
J(t - \frac{r}{c}) = i\omega_0 [\mathbf{d} S^z(t)e^{-i\delta - \mathbf{d}^* S^-(t)e^{i\delta}}] \Theta(ct - r),
\]
where $S^z(t) = S^z(0, t)$. At small distances, such that $k_0 r \ll 1$, one may write $e^{i\delta/\omega_0} = 1 + i k_0 r$. Substituting the transverse $\delta$-function into the vector potential (26), we recall that averaging the dipolar tensor over spherical angles gives
\[
\int D_{\text{dip}}(\mathbf{r}) d\Omega(\mathbf{r}) = 0.
\]

Then, for $k_0 r \ll 1$, the vector potential (26) becomes
\[
A_i(\mathbf{r}, t) = \frac{2}{3} k_0^3 [\mathbf{d} S^z(t) + \mathbf{d}^* S^-(t)] + \frac{2 k_0^2}{3r} [\mathbf{d} S^z(t) - \mathbf{d}^* S^-(t)].
\]

To avoid divergence in the term $1/r$, one can average it between the electron wavelength $\lambda_e = 2\pi hmc$, with $m$ being the electron mass, and the radiation wavelength $\lambda_0 = 2\pi k_0$. Since $\lambda_e \ll \lambda_0$, we have
\[
\frac{1}{\lambda_0 - \lambda_e} \int_0^{\lambda_0} \frac{k_0 dr}{r} = \frac{k_0}{2\pi} \ln \left( \frac{mc^2}{\hbar \omega_0} \right).
\]

Then for the self-acting vector potential, we get
\[
A_i(0, t) = \frac{2}{3} k_0^3 [\mathbf{d} S^z(t) + \mathbf{d}^* S^-(t)] + \frac{2 k_0}{3\pi} \ln \left( \frac{mc^2}{\hbar \omega_0} \right)[\mathbf{d} S^z(t) - \mathbf{d}^* S^-(t)].
\]

Substituting this into the evolution equations for the case of a single atom, we employ the properties of spin one-half operators. Then we come to the equations for a single atom
\[
\frac{dS^-}{dt} = -i(\omega_0 - \delta_l - i\gamma_0)S^- + \frac{d^2}{d\mathbf{r}^2} [(\delta_0 + i\delta_s) S^+,
\]
\[
\frac{dS^z}{dt} = -2k_0 \left( \frac{1}{2} + S^z \right).
\]
in which the notations for the natural width
\[ \gamma_0 \equiv \frac{2}{3} |\mathbf{d}|^2 k_{\theta} \] (29)
and the Lamb shift
\[ \delta_L \equiv \frac{\gamma_0}{2\pi} \ln \left( \frac{mc^2}{\hbar \omega_0} \right) \] (30)
are introduced. The solutions to equation (28), remembering that \( \gamma_0 \ll \omega_0 \) and \( \delta_L \ll \omega_0 \), are
\[
S^-(t) = S^-(0) \exp \left\{ -i(\omega_0 - \delta_L)t - \eta \xi \right\}, \\
S^+(t) = -\frac{1}{2} + \left\{ \frac{1}{2} + S^-(0) \right\} \exp \left\{ -2\eta \xi \right\}.
\]
Thus, the self-action of a radiating atom leads to the appearance of attenuation in the dynamics of the pseudospin operators and to the Lamb frequency shift. The latter can always be included in the definition of the transition frequency \( \omega_0 \).

Taking into consideration the attenuation, one usually generalizes the equations of motion by including \( \gamma_2 \), instead of \( \gamma_0 \), for \( S^- \) and inserting \( \gamma_1 \), instead of \( 2\gamma_0 \), for \( S^+ \).

4. Stochastic mean-field quantization

Expressing the vector potential (16) through the pseudospin variables, we obtain the pseudospin equation (15) involving only the pseudospin degrees of freedom. However, these equations are not yet closed, since they contain the products of the pseudospin operators. Employing the semiclassical approximation would eliminate quantum fluctuations, which would make it impossible to describe the initial stage of radiation, when coherence is not yet developed. To simplify the equations through a kind of mean-field approximation, whilst at the same time retaining the influence of quantum fluctuations, we use a stochastic mean-field quantization [8].

We observe that equation (15) contains terms with different properties. The combination
\[ \xi = 2k_0 |\mathbf{d}| (A_{vac} + A_{dip} + A_{mad}) \] (31)
describes short-range fast fluctuations, while the remaining terms containing the pseudospin variables in the radiation vector potential (18), are long-range in nature. Thus, it is possible to distinguish two types of variable. One of them, \( \xi \), can be treated as a stochastic variable, while the remaining set
\[ \tilde{\mathbf{S}} \equiv \{ S_j; j = 1, 2, ..., N \} \] (32)
is treated as a collection of spin operators. Thus all the quantities in equation (15) are functions of two variables, which can be denoted as \( \langle \tilde{f}(\tilde{S}, \xi) \rangle \).

Having two types of variable, it is natural to introduce two different averaging procedures. One is spin averaging
\[ \langle \tilde{f}(\tilde{S}, \xi) \rangle \equiv \text{Tr}_{\tilde{\mathbf{S}}} \tilde{f}(\tilde{S}, \xi), \] (33)
with a statistical operator \( \tilde{\mathbf{S}} \) and the trace over the spin degrees of freedom. The other is stochastic averaging
\[ \langle \tilde{f}(\tilde{S}, \xi) \rangle \equiv \int \tilde{f}(\tilde{S}, \xi) D\xi, \] (34)
with the functional integration over the stochastic variable \( \xi \).

Since the vector potential (18) characterizes long-range interactions, decaying as \( 1/r \), it is possible to use mean-field decoupling with respect to the spin averaging:
\[ \langle S_i^+ S_j^- \rangle = \langle S_i^+ \rangle \langle S_j^- \rangle \] (35)
where the stochastic variable \( \xi \) is kept untouched.

Following the spin averaging of equation (15), we define the transition function
\[ u(r_j, t) \equiv 2\langle S_i^- \rangle, \] (36)
coherence intensity
\[ w(r_j, t) \equiv 2 \sum_{i \neq j} \langle S_i^+ \rangle \langle S_j^- \rangle, \] (37)
and the population imbalance
\[ s(r_j, t) \equiv 2 \langle S_i^- \rangle. \] (38)

To simplify presentation of the resulting equations, we introduce the effective external force, due to the external field,
\[ f_0(r, t) \equiv -2i|\mathbf{d}| E_0(r, t) \] (39)
and the effective radiation force
\[ f_{rad}(r, t) \equiv 2k_0 |\mathbf{d}| A(r, t), \] (40)
caused by atomic interactions through the common radiation field. Passing from summation over atoms to spatial integration by means of the replacement
\[ \sum_{j=1}^{N} \rightarrow \rho \int d\mathbf{r} \] \[ \left( \rho \equiv \frac{N}{V} \right), \]
with the integration over the whole atomic system, we get the effective radiation force
\[ f_{rad}(r, t) = -i\gamma_0 \rho \int \left\{ G(r-r', t) u(r', t) - \mathbf{e}_d G^\theta(r-r', t) u^\theta(r', t) \right\} d\mathbf{r}'. \] (41)
where \( \mathbf{e}_d \equiv d/|\mathbf{d}| \) and the transfer function is
\[ G(r, t) \equiv \Theta(ct-r) \exp \left( i\mathbf{k}\mathbf{r} \right) \frac{\exp (i\mathbf{k}\mathbf{r})}{k_d}. \]
The total effective force, acting on the spin variables, is the sum
\[ f(r, t) \equiv f_0(r, t) + f_{rad}(r, t) + \xi(r, t). \] (42)
Finally, we come to the equations for the transition function,
\[ \frac{\partial u}{\partial t} = -(i\omega_0 + \gamma_2) u + fs, \] (43)
for the coherence intensity,
\[ \frac{\partial w}{\partial t} = -2\gamma_2 w + (u^* f + f^* u)s, \] (44)
and for the population imbalance,
where $\zeta$ is an equilibrium population imbalance for a single atom. As usual, these equations are assumed to be complemented by the corresponding initial conditions and, if necessary, boundary conditions.

5. Scale separation approach

The following analysis of equation (45) can be done using the scale separation approach [7–10] that is a generalization to stochastic differential equations with multiple scales of the Krylov–Bogolubov averaging method [11]. Partial differential equations can also be treated by this method [8, 12]. Here we lay out the principles of the scale separation approach, keeping in mind equations (43)–(45).

The separation of scales is based on the existence of small parameters. Thus, the influence of the external field is small according to equation (22). The attenuations are also assumed to be small, so that

$$\frac{\gamma_0}{\omega_0} \ll 1, \quad \frac{\gamma_1}{\omega_0} \ll 1, \quad \frac{\gamma_2}{\omega_0} \ll 1.$$ (46)

Equations (43)–(45) can be written in the form

$$\frac{\partial u}{\partial t} = f_u, \quad \frac{\partial w}{\partial t} = f_w, \quad \frac{\partial s}{\partial t} = f_s,$$ (47)

in which

$$f_u = f_u(u, w, s, \xi, t) \quad (\alpha = u, w, s).$$

The right-hand sides here are such that, if all small parameters tend to zero, then

$$f_w \to 0, \quad f_s \to 0,$$ (48)

while $f_u$ remains finite. This means that the functions $w$ and $s$ are quasi-integrals of motion, or slow variables, while $u$ is a fast variable. The equation for $u$ is solved, with the slow variables $w$ and $s$ kept constant, which defines $u = u(w, s, \xi, t)$. Then this solution is substituted into the equations for the slow variables, averaging their right-hand sides over the fast variable and over the stochastic variable, according to the rule

$$\overline{f_u}(w, s) \equiv \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \langle f_u(u, w, s, \xi, t) \rangle \, dt.$$ (49)

This results in the guiding center equations for the slow variables:

$$\frac{\partial w}{\partial t} = \overline{f}_w, \quad \frac{\partial s}{\partial t} = \overline{f}_s.$$ (50)

If necessary, it is possible to find corrections to the guiding centers [8–10].

6. Magnetic dipole transitions

In the previous sections, it has been assumed that the resonant atoms experience electric dipole transitions. Now we consider magnetic dipole transitions. We show that, despite their difference, both these transitions lead to the same kind of evolution equations. This consideration is also important in establishing whether spin systems can demonstrate the occurrence of superradiance in the same way as atoms. It turns out that there is a key difference between atomic and spin systems, since the latter, in addition to interactions through the common radiation field, have rather strong direct dipole spin interactions, which destroy coherence [13–16].

For magnetic dipole transitions, the atom–field interaction is given by the Hamiltonian

$$\hat{H}_d = - \sum_{j=1}^N \mathbf{M}_j \cdot \mathbf{B}_j,$$ (51)

where $\mathbf{B}_j = \mathbf{B}(\mathbf{r}_j, t)$, the magnetic moment can be written [8, 14] as

$$\mathbf{M}_j = \overline{\mu} S_j^+ + \overline{\mu} S_j^- + \overline{\mu} \mathbf{r}_j \cdot \mathbf{S}_j,$$ (52)

and the total magnetic field is the sum

$$\mathbf{B} = \mathbf{H}_0 + \mathbf{H} + \mathbf{H}_{\text{vac}} + \mathbf{H}_{\text{mat}}$$ (53)

of an external field, a radiation field, a vacuum field, and the field of the matter surrounding the atoms.

The Heisenberg equations of motion yield

$$\frac{dS_j^-}{dt} = -i\omega_0 S_j^- + i(\overline{\mu}_0 S_j^- - 2\overline{\mu} S_j^-) \mathbf{B}_j,$$ (54)

$$\frac{dS_j^+}{dt} = i(\overline{\mu} S_j^+ - \overline{\mu} S_j^-) \mathbf{B}_j.$$

The total vector potential is the sum

$$\mathbf{A} = \mathbf{A}_{\text{rad}} + \mathbf{A}_{\text{vac}} + \mathbf{A}_{\text{mat}}.$$ (55)

The first term, caused by atomic radiation, is

$$\mathbf{A}_{\text{rad}}(\mathbf{r}, t) = \frac{1}{c} \int \mathbf{j}(\mathbf{r}', t - \frac{|\mathbf{r} - \mathbf{r}'|}{c}) \, d\mathbf{r'},$$ (56)

with the current density

$$\mathbf{j}(\mathbf{r}, t) = -c \sum_{j=1}^N \mathbf{M}_j(t) \times \nabla \delta(\mathbf{r} - \mathbf{r}_j).$$ (57)

Taking into account magnetic momentum (52) makes it possible to rewrite the vector potential (56) in the form

$$\mathbf{A}_{\text{rad}} = \mathbf{A}_+ + \mathbf{A}_- + \mathbf{A}_z,$$ (58)

in which

$$\mathbf{A}_+(\mathbf{r}, t) = -\frac{1}{c} \sum_{j \neq i} \frac{1}{r_{ij}^2} \left( i \left( \frac{r_{ij}}{c} \frac{\partial}{\partial t} - \frac{r_{ij}}{c} \right) + \overline{\mu} \mathbf{S}_j^+ \right) \delta(\mathbf{r} - \mathbf{r}_j).$$ (59)

$\mathbf{A}_-$ is the Hermitian conjugate to $\mathbf{A}_+$,

$$\mathbf{r}_j \equiv \mathbf{r}_j - \mathbf{r}_j, \quad \mathbf{n}_{ij} \equiv \frac{\mathbf{r}_{ij}}{|\mathbf{r}_{ij}|}, \quad r_{ij} \equiv |\mathbf{r}_{ij}|,$$

and the last term of equation (58) is

$$\mathbf{A}_z(\mathbf{r}, t) = -\frac{1}{c} \sum_{j \neq i} \frac{\mathbf{n}_{ij}}{r_{ij}^2} \times \overline{\mu}_0 \mathbf{S}_j^+ \delta(\mathbf{r} - \mathbf{r}_j).$$ (60)
The radiation magnetic field becomes
\[ \mathbf{H}_{\text{rad}} \equiv \nabla \times \mathbf{A}_{\text{rad}} = \mathbf{H}_+ + \mathbf{H}_- + \mathbf{H}_y, \] (61)
with
\[ \mathbf{H}_+(\mathbf{r}, t) = -\sum_{j(\neq i)} N \left[ \mathbf{\mu}_j - (\mathbf{\mu}_j \cdot \mathbf{n}_j) \mathbf{n}_j \frac{\partial^2}{\partial t^2} \right] S^+_j(t) \frac{r_j}{c} \]
+ \frac{2}{3} \mathbf{\mu} \sum_{j(\neq i)} \frac{1}{c^2} \frac{\partial^2}{\partial t^2} S^+_j(t) \] (62)
and
\[ \mathbf{H}_-(\mathbf{r}, t) = -\sum_{j(\neq i)} N \left[ \mathbf{\mu}_j - 3(\mathbf{\mu}_j \cdot \mathbf{n}_j) \mathbf{n}_j \right] \mathbf{n}_j \frac{\partial}{\partial t} \frac{r_j}{c}. \] (63)

Let us define the effective radiation field
\[ \mathbf{H}_{\text{eff}} \equiv \frac{1}{4\pi} \int \mathbf{H}_{\text{rad}} \mathrm{d}\Omega(\mathbf{n}) \] (64)
as field (61) averaged over spherical angles. In this averaging, we take into account the properties
\[ \frac{1}{4\pi} \int [\mathbf{\mu} - (\mathbf{\mu} \cdot \mathbf{n}) \mathbf{n}] \mathrm{d}\Omega(\mathbf{n}) = \frac{2}{3} \mathbf{\mu}, \]
\[ \frac{1}{4\pi} \int [\mathbf{\mu} - 3(\mathbf{\mu} \cdot \mathbf{n}) \mathbf{n}] \mathrm{d}\Omega(\mathbf{n}) = 0, \]
\[ \frac{1}{4\pi} \int (\mathbf{\mu} \cdot \mathbf{n}) \mathrm{d}\Omega(\mathbf{n}) = \frac{1}{3} \mathbf{\mu}. \]

This gives
\[ \frac{1}{4\pi} \int \mathbf{H}_- \mathrm{d}\Omega(\mathbf{n}) = 0. \] (65)

As a result, we find
\[ \mathbf{H}_{\text{eff}} = \mathbf{H}_{\text{rad}}^+ + \mathbf{H}_{\text{rad}}^-, \] (66)
with
\[ \mathbf{H}_{\text{rad}}^+(\mathbf{r}, t) = -\frac{2}{3} \mathbf{\mu} \sum_{j(\neq i)} \frac{1}{c^2} \frac{\partial^2}{\partial t^2} S^+_j(t) \frac{r_j}{c}. \] (67)
Similarly to equation (22), we assume that
\[ \frac{[\mathbf{\mu}_j \mathbf{B}]}{\omega_0} \ll 1, \quad \frac{[\mathbf{\mu} \mathbf{B}]}{\omega_0} \ll 1, \] (68)
the Born approximation can thus be invoked giving, similarly to section 2,
\[ \mathbf{H}_{\text{rad}}^+(\mathbf{r}, t) = \frac{2}{3} k_0^2 \mathbf{\mu} \sum_{j(\neq i)} N G^+(\mathbf{r}_j, t) S^+_j(t), \] (69)
with the transfer function \( G \) introduced in section 4.

The radiation field (61) can be represented as a sum
\[ \mathbf{H}_{\text{rad}} \equiv \mathbf{H}_{\text{eff}} + \mathbf{H}_{\text{dip}} \] (70)
of the effective field (66) and the remaining part, for which
\[ \int \mathbf{H}_{\text{dip}} \mathrm{d}\Omega(\mathbf{n}) = 0. \]

As in section 4, we introduce the effective forces acting on the atoms, caused by the external field,
\[ f_0(\mathbf{r}, t) \equiv -2i\mathbf{\mu} \cdot \mathbf{H}_0(\mathbf{r}, t), \] (71)
due to radiation,
\[ f_{\text{rad}}(\mathbf{r}, t) \equiv -2i(\mathbf{\mu} \cdot \mathbf{H}_{\text{eff}}(\mathbf{r}, t)), \] (72)
and corresponding to local fluctuations
\[ \xi \equiv -2i\mathbf{\mu} \mathbf{H}_{\text{vac}} + \mathbf{H}_{\text{dip}} + \mathbf{H}_{\text{mat}}. \] (73)

Defining the natural linewidth
\[ \gamma_0 \equiv \frac{2}{3} |\mathbf{\mu}|^2 k_0^3, \] (74)
we obtain the radiation force
\[ f_{\text{rad}}(\mathbf{r}, t) = -i\hbar \rho \int \left[ G(\mathbf{r} - \mathbf{r}'), t) u(\mathbf{r}', t) \right. \]
+ \left. \epsilon_0^2 G^+(\mathbf{r} - \mathbf{r}', t) u^+(\mathbf{r}', t) \right] \mathrm{d}\mathbf{r}', \]
(75)
in which \( \epsilon_0 \equiv \mathbf{\mu}/|\mathbf{\mu}|. \)

This force (75) has the same form as that in equation (41). The following analysis can be done in complete analogy with the case of electric dipole transitions, simply replacing the electric transition dipole \( \mathbf{d} \) by the magnetic dipole \( \mathbf{\mu} \).

7. Coherent and incoherent radiation

Some time ago, there was a discussion in the literature on the possible role of external fields in influencing the intensity of radiation. In particular, this problem arose in the study of gamma radiation from Mössbauer nuclei inside magnetic materials with large magnetic moments [17]. We shall investigate this problem in the next section, but we first need to recall the general form of the radiation intensity.

Below, we keep in mind electric dipole transitions, since, as explained above, the mathematics for both electric and magnetic dipoles is the same. The local radiation intensity is defined as
\[ I(\mathbf{n}, t) = \langle \mathbf{n} : \mathbf{S}(\mathbf{r}, t) : \mathbf{n} \rangle r^2, \] (76)
where the colons denote normal ordering, \( \mathbf{n} \equiv \mathbf{r}/r \), and the Poynting vector is
\[ \mathbf{S} \equiv \frac{c}{8\pi} (\mathbf{E}_\text{rad} \times \mathbf{H}_\text{rad} - \mathbf{H}_\text{rad} \times \mathbf{E}_\text{rad}). \] (77)
The radiation fields are given by the expressions
\[ \mathbf{E}_\text{rad} = -\frac{1}{c} \frac{\partial \mathbf{A}_\text{rad}}{\partial t}, \quad \mathbf{H}_\text{rad} = \nabla \times \mathbf{A}_\text{rad}. \] (78)
In the wave zone, one uses the approximation
\[ |\mathbf{r} - \mathbf{r}'| \simeq r - \mathbf{n} \cdot \mathbf{r}', \quad (r \gg |\mathbf{r}'|). \] (79)
Then the radiation vector potential can be written as
\[ \mathbf{A}_\text{rad} = \mathbf{A}_+ + \mathbf{A}_-, \] (80)
with
\[ \mathbf{A}_+(r, t) \simeq i \frac{k_0}{r} \sum_{j=1}^{N} S_j^+(r - \mathbf{n} \cdot \mathbf{r}_j) \]
and \( \mathbf{A}_- \) being the Hermitian conjugate to \( \mathbf{A}_+ \). Using this, we get the electric radiation field
\[ \mathbf{E}_{\text{rad}} \simeq -ik_0(\mathbf{A}_+ - \mathbf{A}_-) \]
and magnetic radiation field
\[ \mathbf{H}_{\text{rad}} \simeq \mathbf{n} \times \mathbf{E}_{\text{rad}}. \]

The radiation intensity (76) becomes
\[ I(n, t) = \frac{cr^2}{4\pi} \langle \mathbf{E}_{\text{rad}}^2 - (\mathbf{n} \cdot \mathbf{E}_{\text{rad}})^2 \rangle, \]
which, treating the retardation in the Born approximation, is reduced to
\[ I(n, t) = 2\omega_0 \sum_{j=1}^{N} \mathbf{q}_j(n)(S_j^+(t)S_j^-(t)), \]
where the form factor
\[ \mathbf{q}_j(n) \equiv \frac{3}{4\pi} [\mathbf{n} \times \mathbf{e}_j]^2 \exp(ik_0 \mathbf{n} \cdot \mathbf{r}_j) \]
is introduced.

The radiation intensity averaged over random fluctuations and the period of fast oscillations
\[ I(n, t) \equiv \frac{\omega_0}{2\pi} \int_{t+2i/\omega_0}^{t} \langle \mathbf{I}(n, t') \rangle dt' \]
can be presented as a sum
\[ I(n, t) = I_{\text{inc}}(n, t) + I_{\text{coh}}(n, t) \]
of the incoherent radiation intensity
\[ I_{\text{inc}}(n, t) = \omega_0 \sum_{j=1}^{N} \mathbf{q}_j(n)[1 + s_j(t)] \]
and the coherent radiation intensity
\[ I_{\text{coh}}(n, t) = \frac{1}{2} \omega_0 \sum_{j \neq j'}^{N} \mathbf{q}_j(n)\mathbf{u}_j^+(t)\mathbf{u}_j^-(t). \]
The diagonal form factor is
\[ \mathbf{q}_j(n) = \frac{3}{8\pi} [\mathbf{n} \times \mathbf{e}_j]^2 \equiv \varphi(n). \]

The total radiation intensity, integrated over the spherical angles,
\[ I(t) \equiv \int I(n, t) d\Omega(n) = I_{\text{inc}}(t) + I_{\text{coh}}(t), \]
consists of the incoherent part
\[ I_{\text{inc}}(t) = \omega_0 \sum_{j=1}^{N} [1 + s_j(t)] \]
and the coherent part
\[ I_{\text{coh}}(t) = \frac{1}{2} \omega_0 \sum_{j \neq j'}^{N} \mathbf{q}_j(t)\mathbf{u}_j^+(t)\mathbf{u}_j^-(t). \]

Here the shape factor is defined as
\[ \varphi_j \equiv i \int \mathbf{q}_j(n)d\Omega(n), \quad \varphi_j \equiv i \int \mathbf{q}(n)d\Omega(n) = 1. \]
Thus we see that to calculate the radiation intensity, we need to find solutions to the evolution equations (43)–(45).

8. Influence of external fields

We shall now analyze how external fields influence the radiation intensity. Because we are interested only in the permanent action of external fields, we shall disregard short temporary coherent effects. The evolution equations (43)–(45) can then be presented as
\[ \frac{\partial u_j}{\partial t} = -\mathbf{a} \cdot \mathbf{E}_0(r, t), \quad \frac{\partial \beta}{\partial t} = -2\mathbf{a} \cdot \mathbf{d} - 2i\mathbf{a} \cdot \mathbf{E}_0(r, t), \quad \frac{\partial s_j}{\partial t} = i(\mathbf{d} \cdot \mathbf{d}) - \mathbf{E}_0(r, t) - \mathbf{a} \cdot (s - \zeta). \]

The external field
\[ \mathbf{E}_0(r, t) = \mathbf{E}_0 + \mathbf{E}_1 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)} + \mathbf{E}_2^* e^{-i(\mathbf{k} \cdot \mathbf{r} - \omega t)} \]
consists of a constant and an alternating field. As usual, there are the following small parameters
\[ \frac{\gamma_1}{\omega_0} \ll 1, \quad \frac{\gamma_2}{\omega_0} \ll 1, \quad \frac{\mathbf{d} \cdot \mathbf{E}_0}{\omega_0} \ll 1, \quad \frac{\mathbf{d} \cdot \mathbf{E}_1}{\omega_0} \ll 1. \]

The alternating field is tuned to resonance, so that the resonance condition
\[ \frac{|\Delta|}{\omega_0} \ll 1, \quad (\Delta \equiv \omega - \omega_0) \]
is valid.

The problem can be solved by the scale separation approach explained in section 5. According to the existing small parameters, the variable \( u \) is treated as fast, while \( \omega \) and \( s \), are slow. For the fast variable, we get
\[ u = u_0 e^{-i(\omega_0 + \gamma_2)\tau} - 2i\mathbf{d} \cdot \mathbf{E}_1 e^{-i(\omega_0 + \gamma_2)\tau} \int_0^t \mathbf{E}_0(r, t' - \tau') e^{i(\omega_0 + \gamma_2)\tau'} d\tau', \]
where \( u_0 \) is an initial value of \( u \). Considering the field described by equation (97), we find
\[ u = \frac{2i\mathbf{d} \cdot \mathbf{E}_0}{\omega_0 - i\gamma_2} \frac{2i\mathbf{d} \cdot \mathbf{E}_1}{\Delta + i\gamma_2} e^{i(\mathbf{k} \cdot \mathbf{r} - \omega_0 \tau)} + \left( u_0 + \frac{2i\mathbf{d} \cdot \mathbf{E}_0}{\omega_0 - i\gamma_2} - \frac{2i\mathbf{d} \cdot \mathbf{E}_1}{\Delta + i\gamma_2} \right) e^{-(i\omega_0 + \gamma_2)\tau}. \]
For the guiding center of the slow variable $s$, we obtain the equation
\[
\frac{ds}{dt} = - \gamma_s^*(s - \zeta^*),
\] (102)
in which
\[
\gamma_s^* = \gamma_1 + 16\gamma_s \left( \frac{|\mathbf{d} E_0|^2}{\omega_0^2 + \gamma_2^2} + \frac{|\mathbf{d} E_0|^2}{\Delta^2 + \gamma_2^2} \right), \quad \zeta^* = \frac{\gamma_s^*}{\gamma_1^*}. \tag{103}
\]

The solution to equation (102) is
\[
s = s_0 e^{-\gamma_s^* t} + (1 - e^{-\gamma_s^* t}) \zeta^*. \tag{104}
\]
Here $s_0$ is an initial value of $s$. For simplicity, we accept a uniform initial condition $s_0(r) \equiv s(r, 0) = s_0$. Then solution (104) is also uniform, $s(r, t) = s(t)$.

Averaging over fast oscillations, we have
\[
\overline{u^n_{s2}} = 4\pi\int \mathbf{d}E_0 |\mathbf{d}E_0|^2 \frac{|\mathbf{d}E_0|^2}{\gamma_0 + \Delta^2 + \gamma_2^2} e^{-ikr},
\]
which can be substituted into the radiation intensity (94). The incoherent radiation intensity is
\[
I_{inc}(n, t) = N\omega_0|\mathbf{d}E_0|^2 [1 + s(t)]. \tag{105}
\]

For the coherent radiation intensity, we find
\[
I_{coh}(n, t) = 2N^2\omega_0|\mathbf{d}E_0|^2 t^2 \times \left[ F(k_0n) \left| \mathbf{d}E_0 \right|^2 + F(k_0n - \mathbf{k}) \left| \mathbf{d}E_0 \right|^2 \right] \frac{\gamma_1^*}{\Delta^2 + \gamma_2^2}, \tag{106}
\]
with the form factor
\[
F(k) \equiv \frac{1}{N} \sum_{j=1}^{N} e^{i\mathbf{k}\cdot\mathbf{r}_j}. \tag{107}
\]

For a cylindrical sample of radius $R$ and length $L$, with $\mathbf{k} = k_0 \mathbf{e}_z$, the form factors in equation (106) read
\[
F(k_0n) = \frac{\lambda^4}{\pi^2R^2L^2 \sin^2 \theta \cos^2 \theta} \left( \frac{2\pi R}{\lambda} \sin \theta \right) \sin^2 \left( \frac{\pi L}{\lambda} \cos \theta \right),
\]
\[
F(k_0n - \mathbf{k}) = \frac{\lambda^4}{\pi^2R^2L^2 \sin^2 \theta (1 - \cos^2 \theta)} \left( \frac{2\pi R}{\lambda} \sin \theta \right) \times \sin^2 \left( \frac{\pi L}{\lambda} (1 - \cos \theta) \right),
\]
where $\mathbf{k} = k_0 \mathbf{e}_z$, $k_0 = 2\pi \lambda$, and $\theta$ is the angle between $\mathbf{n}$ and the axis $z$. The maximum value of the form factor is one. For the form factor $F(k_0n)$, the maximum happens when $k_0n.\mathbf{r}_j = 2\pi n_j$ ($n_j = 0, \pm 1, \pm 2, \ldots$). (108)

For instance, if $n_j = 0$, condition (108) is valid for a chain of atoms with $\mathbf{r}_j$ perpendicular to the direction of $\mathbf{n}$.

The factor $F(k_0n - \mathbf{k})$ is at a maximum, reaching one, when $(k_0n - \mathbf{k}).\mathbf{r}_j = 2\pi n_j$ ($n_j = 0, \pm 1, \pm 2, \ldots$). (109)

We are currently considering radiating atoms, but similar conditions apply in the case of an atomic system scattering external radiation. Thus, the case $n_j = 0$ corresponds to forward or backward scattering. When $n_j \neq 0$, the atoms have to form an ideal lattice, and equation (109) is the condition for Bragg scattering. The increase of scattering intensity from a lattice arrangement of atoms is called the Bornmann effect [18, 19].

The total radiation intensity (92) is the sum of the incoherent intensity
\[
I_{inc}(t) = N\omega_0|\mathbf{d}E_0|^2 [1 + s(t)] \tag{109}
\]
and the coherent radiation intensity
\[
I_{coh}(t) = 2N^2\omega_0|\mathbf{d}E_0|^2 t^2 \left( \frac{\gamma_0}{\omega_0 + \Delta^2 + \gamma_2^2} + \frac{\gamma_1}{\Delta^2 + \gamma_2^2} \right) \tag{110}
\]
In the latter, the following notations are used for the shape factors
\[
\phi_0 \equiv \int \phi(n)F(k_0n) |d\Omega(n), \quad \phi_1 \equiv \int \phi(n)F(k_0n - \mathbf{k}) |d\Omega(n). \tag{111}
\]

The values of the shape factors essentially depend on the shape of the radiating sample and on the type of the dipole transition characterized by the change of the quantum number $\Delta m$. For instance
\[
e_d = \begin{cases} e_z, & (\Delta m = 0) \\ -\frac{1}{\sqrt{2}} (e_x \pm i e_y), & (\Delta m = \pm 1). \end{cases}
\]

Therefore,
\[
|\mathbf{n} \times \mathbf{e}_d|^2 = \begin{cases} 1 - \cos^2 \theta, & (\Delta m = 0) \\ 1 - \frac{1}{2} \sin^2 \theta, & (\Delta m = \pm 1). \end{cases}
\]

In particular, for a pencil-like sample or a disk-like sample and $\Delta m = \pm 1$, one has [1], respectively,
\[
\phi_1 \approx \begin{cases} 3\frac{\lambda}{8L} & (\lambda \frac{\pi R}{2} \ll 1, \frac{R}{L} \ll 1) \\ \frac{3}{8} & (\lambda \frac{\pi R}{2} \ll 1, \frac{L}{R} \ll 1) \end{cases} \tag{112}
\]

It follows from these results that external fields increase the longitudinal attenuation, which accelerates the relaxation of $s$. They also induce coherent radiation, but weakly influence incoherent radiation.

An alternating resonant external field usually produces a stronger effect than a constant field. The latter can play a more important role than an alternating field, if
\[
\phi_1 \frac{|\mathbf{d}E_0|^2}{\gamma_2} < \phi_0 \frac{|\mathbf{d}E_0|^2}{\omega_0} \ll 1. \tag{113}
\]

In other words, when
\[
\phi_0 \frac{|\mathbf{d}E_0|^2}{\phi_1 \omega_0 |\mathbf{E}|} > 1. \tag{114}
\]

The possible influence of a constant external field on the coherent radiation intensity has been discussed [7, 8, 20, 21]
with reference to the so-called Mössbauer magnetic anomaly. The latter is the increase of the spectrum area, when the paramagnetic state changes to the ferromagnetic state [17, 22]. In some papers, such an increase was associated with the influence of the external field on the phonon characteristics of the material. This interpretation, however, was shown to be incorrect [8].

In the case of Mössbauer radiation by 57Fe, typical parameters are \( \omega_0 \sim 10^{19} \text{1/s}, \gamma_2 \sim 10^7 \text{1/s}, \lambda \sim 10^{-8} \text{cm}, H_0 \sim 10^6 \text{G}, \) and \( H_1 \sim 10^5 \text{G}. \) This gives

\[
\frac{\mu_0 |\mathbf{H}_0|}{\omega_0 |\mathbf{H}_1|} \sim 10^{-2}.
\]

Under such typical conditions, if \( \varphi_0 \) and \( \varphi_1 \) are of the same order of magnitude, then an applied constant magnetic field should not play a role. It could play role for a very weak alternating field of the order of \( H_1 \sim 10^{-7} \text{G}. \) The magnetic anomaly in ferromagnetics could be due to the inhomogeneous broadening effect [8].

9. Triggering dipolar waves

When an atomic system is prepared in an excited state, it starts to radiate by spontaneous emission. The radiated field causes correlations to arise between atoms. The semiclassical approximation cannot describe this process. Because it requires that an initial coherence be imposed upon the system, it is applicable only to the coherent stage of radiation.

In the pseudospin representation, the process of radiation starts with spontaneous atomic radiation, resulting in self-action, and the atomic correlations which appear are associated with triggering dipolar waves [23]. This initial stage of the atomic state can be described by the pseudospin equations

\[
\frac{d\delta S^0_j}{dt} = -i\omega_0\delta S^0_j - i\langle S^0_j \rangle \sum_{i \neq j}^N (b_{ji}\delta S_i^+ - c_{ji}\delta S_i^-).
\]

\[
\frac{d\delta S_j}{dt} = i \sum_{i \neq j}^N \left[ S^0_j (b_{ji}\delta S_i^+ - c_{ji}\delta S_i^-) - S^0_j (b_{ij}\delta S_i^+ - c_{ij}\delta S_i^-) \right].
\]

(116)
in which

\[
b_{ij} \equiv \frac{k^2_0}{2\pi} \sum_{a\beta} d^aD_{ij}^{a\beta}d^a, \quad c_{ij} \equiv \frac{k^2_0}{2\pi} \sum_{a\beta} d^a(D_{ij}^{a\beta})^*.
\]

\[
D_{ij}^{a\beta} \equiv \int \Theta(ct - | \mathbf{r}_i - \mathbf{r}_j |) \frac{D_{a\beta}(r - r_j)}{| \mathbf{r}_j - \mathbf{r}_i |} \exp (-i\omega_0 | \mathbf{r}_i - \mathbf{r}_j |) d\mathbf{r}.
\]

Triggering dipolar waves correspond to small deviations from the average pseudospin values:

\[
S^0_j = \langle S^0_j \rangle + \delta S^0_j;
\]

(117)

To zero order, we have the equations

\[
\frac{d}{dt}\langle S^0_j \rangle = -i\omega_0 \langle S^0_j \rangle, \quad \frac{d}{dt}\langle S_j \rangle = 0,
\]

whose solutions are

\[
\langle S^0_j(t) \rangle = \langle S^0_j(0) \rangle \exp (-i\omega_0 t), \quad \langle S_j(t) \rangle = \langle S_j(0) \rangle.
\]

(119)

The equations for small deviations are

\[
\frac{d}{dt}\delta S^0_j = -i\omega_0 \delta S^0_j - i\langle S^0_j \rangle \sum_{i \neq j}^N (b_{ji}\delta S_i^+ - c_{ji}\delta S_i^-),
\]

\[
\frac{d}{dt}\delta S_j = i \sum_{i \neq j}^N \left[ (b_{ji}\delta S_i^+ - c_{ji}\delta S_i^-)\langle S^0_j \rangle - (b_{ij}\delta S_i^+ - c_{ij}\delta S_i^-)\langle S^0_j \rangle \right].
\]

(120)

Substituting into equation (120) the Fourier transforms for the deviations,

\[
\delta S^0_j = \sum_k \delta S^0_k e^{ik\mathbf{r}_j}, \quad \delta S_j = \sum_k \delta S_k^+ e^{-ik\mathbf{r}_j},
\]

(121)

and for the coefficients

\[
b_{ij} = \frac{1}{N} \sum_k b_{ij} e^{ik\mathbf{r}_j}, \quad c_{ij} = \frac{1}{N} \sum_k c_{ij} e^{ik\mathbf{r}_j},
\]

we obtain the equations

\[
\frac{d}{dt}\delta S^0_k = -i\mu_k \delta S^0_k - i\lambda_k \delta S^0_k, \\
\frac{d}{dt}\delta S_k^+ = i\mu_k^* \delta S_k^+ + i\lambda_k^* \delta S_k^-. 
\]

(122)
in which

\[
\mu_k \equiv \omega_0 - c_k(S^0_j), \quad \lambda_k \equiv b_k(S^0_j).
\]

The solutions to equations (122) can be represented in the form

\[
\delta S^0_k = \mu_k e^{-i\omega_0 t} + \nu_k e^{i\omega_0 t}, \quad \delta S_k^+ = \mu_k^* e^{i\omega_0 t} + \nu_k^* e^{-i\omega_0 t},
\]

(123)

with the dipolar wave spectrum

\[
\omega_k = \sqrt{\mu_k^2 - |\lambda_k|^2}.
\]

(124)

Because of the inequalities

\[
|b_{ij}| \ll 1, \quad |\omega_0| \ll 1, \quad \lambda_k \ll 1,
\]

the spectrum is positive, so that the dipolar waves are stable. In the long-wave limit, the spectrum reads

\[
\omega_k \approx \omega_0 + \frac{1}{2} \langle S^0_j \rangle \sum_{j \neq i}^N (c_{ij})^2 |\mathbf{k} - \mathbf{k}_j|^2.
\]

(125)

The dipolar waves trigger the process of self-organization in a radiating atomic system.

10. Transverse mode expansion

When the radiation wavelength \( \lambda \) is much shorter than the characteristic sizes of the atomic sample, the radiating beam cannot be uniform, but separates into filaments [23]. The details of such a filamentation will be treated later. Meanwhile, we accept the possible existence of such filaments and describe
the general way of treating them. We consider the situation in which the radiation propagates along the axis \( z \) as a plane wave with the seed frequency \( \omega = c k \).

Let the sample be a cylinder of radius \( R \) and length \( L \) which are both much larger than the radiation wavelength, \[ \frac{\lambda}{R} \ll 1, \quad \frac{\lambda}{L} \ll 1. \] (126)

Suppose that there are \( N_f \) filamentary modes in the sample. Each filament can be surrounded by an enveloping cylinder of volume \( V_f = \pi R^2 L \). The separation of the radiating beam into filaments implies that the solutions to the evolution equations (43)–(45) can be represented as the expansions

\[
\begin{align*}
  u(r, t) &= \sum_{n=1}^{N_f} u_n(r_1, t) e^{ik_n z}, \\
  w(r, t) &= \sum_{n=1}^{N_f} w_n(r_1, t), \\
  s(r, t) &= \sum_{n=1}^{N_f} s_n(r_1, t),
\end{align*}
\]

(127)

over the transverse modes, where \( r_1 \equiv \sqrt{x^2 + y^2} \) is the transverse radial variable. For each filament, one can define the averaged solutions

\[
\begin{align*}
  u(t) &\equiv \frac{1}{V_f} \int u_n(r_1, t) \, dr = \frac{2}{R_f^2} \int_0^{R_f} u_n(r_1, t) r \, dr, \\
  w(t) &\equiv \frac{1}{V_f} \int w_n(r_1, t) \, dr = \frac{2}{R_f^2} \int_0^{R_f} w_n(r_1, t) r \, dr, \\
  s(t) &\equiv \frac{1}{V_f} \int s_n(r_1, t) \, dr = \frac{2}{R_f^2} \int_0^{R_f} s_n(r_1, t) r \, dr,
\end{align*}
\]

(128)

averaged over the related enveloping volumes.

We introduce the effective coupling functions

\[
\alpha(t) \equiv \gamma_\beta \int \Theta(ct - r) \frac{\sin(k_0 r - k_2 z)}{k_0 r} \, dr,
\]

(129)

and

\[
\beta(t) \equiv \gamma_\beta \int \Theta(ct - r) \frac{\cos(k_0 r - k_2 z)}{k_0 r} \, dr.
\]

(130)

We also define the average stochastic variable

\[
\xi(t) \equiv \frac{1}{V_f} \int \xi(r, t) e^{-i k_2 z} \, dr.
\]

(131)

Employing the scale separation approach, we meet nonresonant terms of the type

\[
\begin{align*}
  (\alpha + i \beta) s \mu e_i^2, \\
  s(\alpha + i \beta) (u \mu e_i) + s(\alpha - i \beta) (e \mu u)^2, \\
  \frac{1}{2} (\alpha + i \beta) (u \mu e_i)^2 + \frac{1}{2} (\alpha - i \beta) (e \mu u)^2,
\end{align*}
\]

which contribute of the order of \( \gamma_\beta/\omega_0 \), as compared to resonant terms. Such terms can thus be safely neglected.

For the averaged solutions (128), we obtain the equations

\[
\begin{align*}
  \frac{du}{dt} &= -i(\alpha_0 + \beta s) u - (\gamma_2 - \alpha s) u + \xi s, \\
  \frac{dw}{dt} &= -2(\gamma_2 - \alpha s) w + (\mu^2 \xi + \xi^2 u)s, \\
  \frac{ds}{dt} &= -aw - \frac{1}{2}(\mu^2 \xi + \xi^2 u) s - \gamma(s - \xi).
\end{align*}
\]

(132)

We introduce the effective attenuation

\[
\Gamma \equiv \gamma_2 - \alpha s,
\]

(133)

taking into account collective processes, and the effective frequency

\[
\Omega \equiv \alpha_0 + \beta s,
\]

(134)

including the collective Lamb shift. The solution for the fast variable takes the form

\[
\xi(t) = \Re \lim_{t \to \infty} \int_0^t \xi(t') e^{-i(\Omega + \Gamma) t'} dt'.
\]

(135)

The stochastic variable (131) is assumed to be zero centered,

\[
\langle \langle \xi(t) \rangle \rangle = 0.
\]

(136)

We shall also need the quantity

\[
\gamma_\beta \equiv \Re \lim_{t \to \infty} \int_0^t \int_0^t \langle \langle \xi(t) \xi(t') \rangle \rangle e^{-i(\Omega + \Gamma) t'} dt'.
\]

(137)

Which plays the role of a dynamic attenuation caused by the stochastic variables. It is admissible to set the correlation property

\[
\langle \langle \xi(t) \xi(t') \rangle \rangle = 2 \gamma_\beta \delta(t - t')
\]

(138)

which reduces definition (137) to an identity.

Following the scale separation approach, we substitute the fast variable (135) into the equations for slow variables and average the latter over time and over the stochastic variable \( \xi \). This gives the equation for the coherence intensity

\[
\frac{dw}{dr} = -2(\gamma_2 - \alpha s) w + 2\gamma_\beta s^2,
\]

(139)

and for the population imbalance

\[
\frac{ds}{dr} = -aw - \gamma_\beta s - \gamma(s - \xi).
\]

(140)

These equations are applicable to all stages of atomic radiation.

### 11. Emergence of coherence from chaos

An excited atomic system passes through several qualitatively different dynamic stages, similarly to the relaxation of any statistical system from a nonequilibrium state [24]. The first is the interaction stage,

\[ 0 < t < t_{\text{int}} \quad \text{(interaction stage)}, \]

(141)

during which atoms begin spontaneous radiation, but radiate independently of each other, having had no time yet for the development of mutual interactions. The initial values of the
coherence intensity and population imbalance do not change significantly,
\[ w(t_{int}) \approx w_0, \quad s(t_{int}) \approx s_0. \]  
(142)

The interaction time is very short, being of order \( t_{int} \sim \alpha c \), where \( \alpha \) is the mean interatomic distance.

After the interaction time, atoms start to experience each other through photon exchange, but as there is not yet any correlation between them, they essentially radiate independently. This is the chaotic quantum stage, occupying the interval
\[ t_{int} < t < t_{coh} \quad \text{(chaotic stage)}, \]  
(143)

before the coherence time, when strong correlations between atoms appear. After the interaction time, the effective coupling functions (129) and (130) grow as
\[ \alpha(t) \rightarrow g_2(t), \quad \beta(t) \rightarrow g_2(t) \quad (t > t_{int}), \]

where the dimensionless coupling parameters are
\[ g \equiv \gamma_0 \gamma_2 \int \frac{\sin(k_0 r - k z)}{k_0 r} \, dr \]  
(144)

and
\[ g' \equiv \gamma_0 \gamma_2 \int \frac{\cos(k_0 r - k z)}{k_0 r} \, dr. \]  
(145)

The integration here is over \( V_j \). If no initial coherence is imposed on the excited system, so that
\[ w_0 \equiv w(0) = 0, \quad s_0 \neq 0, \]  
(146)

then the evolution equations are
\[ \frac{dw}{dr} = 2g_2^2 s, \quad \frac{ds}{dr} = - (\gamma_1 + \gamma_3)s + \gamma_5 z. \]  
(147)

The coherence time satisfies the inequality
\[ (\gamma_1 + \gamma_3) t_{coh} \ll 1. \]  
(148)

At this chaotic stage, the solutions to equation (147) are
\[ w \approx 2g_2^2 t, \quad s \approx s_0 - \left[ (\gamma_1 + \gamma_3) s_0 - \gamma_5 z \right] t. \]  
(149)

The coherence time corresponds to the point where the collective term in equation (139) becomes comparable with the chaotic term due to quantum fluctuations, that is, when
\[ \gamma_2 (g s - 1) w = \gamma_5 z^2 \]  
(150)

This may happen under a sufficient initial atomic excitation and a strong coupling, when \( g s_0 > 1 \). Then the coherence time is
\[ t_{coh} = \frac{g s_0}{2 [\gamma_2 (g s_0 - 1) s_0 + \gamma_5 s_0 + \gamma_5 (s_0 - \zeta)]}. \]  
(151)

If the coupling parameter is large, then
\[ t_{coh} \approx \frac{T_2}{2 g s_0} \quad (g s_0 \gg 1), \]  
(152)

where \( T_2 \equiv \gamma_2^{-1} \). We observe that the system can reach the coherence time only if it is initially excited, so that \( s_0 > 0 \).

As soon as coherence develops in the system, the coherent stage comes into play, exhibiting superradiance during the interval
\[ t_{coh} < t < T_2 \quad \text{(coherent stage)}. \]  
(154)

At this stage, collective effects are dominant, so that \( \gamma_1 \ll g_2 \) and \( g s \ll g_2 \). The dynamics are described by the equations
\[ \frac{dw}{dr} = - 2g_2 (1 - g s) w, \quad \frac{ds}{dr} = - g_2^2 s, \]  
(155)

whose solutions are
\[ w = \left( \frac{g}{g_2} \right)^2 \mathrm{sech}^2 \left( \frac{t - t_0}{\tau_p} \right), \]
\[ s = \frac{1}{g} - \frac{g}{g_2} \tanh \left( \frac{t - t_0}{\tau_p} \right). \]  
(156)

Where the following notations are used:
\[ g_s^2 = g_2^2 + 2g_2^2 \gamma_5 \gamma_2^3 \gamma^2_{coh}, \quad \gamma_s = (g s_0 - 1) g_2. \]  
(157)

The delay time, when the coherence intensity is maximised, reads
\[ t_0 = t_{coh} + \frac{\tau_p}{2} \ln \left( \frac{\gamma_p + \gamma_s}{\gamma_p - \gamma_s} \right). \]  
(158)

The superradiant pulse time is
\[ \tau_p \equiv \frac{1}{\gamma_p} = \frac{T_2}{g s_0 - 1} \left[ 1 - \frac{g_2^2 \gamma_5^2 \gamma^2_{coh}}{(g s_0 - 1)^2} \right]. \]  
(159)

If the atomic coupling is strong, such that \( g s_0 \ll 1 \), then
\[ t_0 \approx 5 T_{coh}, \quad \tau_p \approx 2 T_{coh}, \]
and the delay time becomes
\[ t_0 = t_{coh} \left( 1 + \ln \left( \frac{2}{g_2^2 \gamma_{coh}} \right) \right). \]  
(160)

Coherence dies out at the scale of \( T_2 \), after which the system relaxes (the relaxation stage)
\[ T_2 < t < T_1 \quad \text{(relaxation stage)}, \]  
(161)

with the solutions for decay
\[ w \approx \left( \frac{2 \gamma_p}{g_2^2} \right)^2 \exp \left( -\frac{2 r}{\tau_p} \right), \]
\[ s \approx \frac{g s_0 - 1}{g_2^2} + \frac{2 \gamma_p}{g_2^2} \exp \left( -\frac{2 r}{\tau_p} \right). \]  
(162)

The longitudinal relaxation time is \( T_1 = 1/\gamma_1 \).

Finally, for longer times
\[ t > T_1 \quad \text{(quasi-stationary stage)}, \]  
(163)

a quasi-stationary stage emerges, described by the equations
\[ \frac{dw}{dr} = - 2g_2 (1 - g s) w + 2g_2 s^2, \]
\[ \frac{ds}{dr} = - g_2^2 s - \gamma_2 s + \gamma_5 (s - \zeta). \]  
(164)
Stationary solutions are defined by the zeros of the right-hand sides of these equations, and small oscillations around the stationary solutions describe the remnants of radiation [8].

Thus, when no initial coherence has been imposed upon the atomic system, its evolution passes through the following qualitatively different time intervals: interaction stage, chaotic stage, coherent stage, relaxation stage, and quasi-stationary stage. Coherent radiation during the coherent stage corresponds to pure superradiance. Here the term pure stresses that superradiance develops as a self-organized spontaneous process, without being forced by external fields. The self-organization of atomic radiation, due to mutual correlations through the common radiation field, is the essence of the Dicke effect [25].

However, if at the start, a coherent pulse is imposed on the atomic system, then the chaotic stage can be shortened, depending on the pulse intensity. This regime is called triggered superradiance. If the initial coherent pulse is strong, then the chaotic stage may disappear completely, so that the coherent stage immediately follows the interaction stage. The superradiant pulse time then becomes

$$\tau_p = \frac{1}{\sqrt{\tau_0^2 + (g\gamma)^2 w_0}}. \quad (165)$$

Evolution after the coherent stage is the same as for pure superradiance.

In all cases, the transitions between different dynamic stages are not absolutely sharp, but rather are gradual crossovers.

12. Pulsing and punctuated superradiance

If the atomic system is subject to non-resonant pumping, supporting the condition

$$g\gamma \gg 1,$$

then a series of superradiant pulses appears. This regime can thus be named pulsing superradiance [8]. After a series of superradiant bursts, the solutions tend to the stationary state

$$w^* \approx \frac{\gamma_0}{g^*}, \quad s^* \approx \frac{1}{g} \left( 1 - \frac{\gamma_0}{\gamma g^*} \right),$$

which is a stable focus. The intervals between the superradiant pulses are approximately given by the period

$$T_{eff} = \frac{\pi}{2\sqrt{\gamma_0 / g^*}}.$$

A regime of punctuated superradiance can be created by subjecting the atomic system to x pulses, which invert the atomic population at the chosen instants [24].

13. Turbulent photon filamentation

When nonlinear media interact with electromagnetic fields different spatiotemporal structures appear, that are analogous to structures arising in many other complex non-equilibrium systems [8, 26–30]. The best known of such electromagnetic structures are the optical filaments which can be formed in passive nonlinear matter [28–30] and in active laser media [8, 26, 29].

The behaviour and characteristics of optical filaments, arising in laser media, essentially depend on the value of the Fresnel number $$F \equiv R^2/\lambda L$$, in which $$R$$ and $$L$$ are the internal radius (aperture radius) and effective length, respectively, of a cylindrical laser sample, and $$\lambda$$ is the optical wavelength. There are two types of optical filament, regular and turbulent, corresponding respectively to low and high Fresnel numbers.

Here we consider turbulent photon filamentation. The theory of turbulent filamentation in laser media was initially advanced on the basis of stationary models [31–34], invoking the notion of an effective time-averaged energy. A more elaborate approach, based on realistic evolution equations, was later developed [8, 35–38].

As noted above, the two types of optical filament, regular and turbulent, are related to the value of the laser Fresnel number. The latter plays for optical systems the same role as the Reynolds number for moving fluids. As the Reynolds number increases, laminar flow becomes turbulent. In a similar manner, increasing the Fresnel number makes a regular filamentary structure turbulent. Optical turbulence implies, by analogy with the fluid turbulence, that the spatiotemporal dynamics are chaotic. This means that the radiating filaments are randomly distributed in space and are not correlated with each other.

Experimentally, optical filaments are usually observed in the near-field cross-section of lasers. A typical picture, when varying the Fresnel number, is as follows. At very small Fresnel numbers $$F \ll 1$$, a sole transverse central mode exists, uniformly filling the laser medium. When the Fresnel number is around $$F \sim 1$$, the laser cavity can house several transverse modes, seen as a regular arrangement of bright spots in the transverse cross-section. Each mode corresponds to a filament extending through the cylindrical volume. This filamentary structure is regular in space, forming ordered geometric arrays, such as polygons. The transverse structure is imposed by the cavity geometry, being prescribed by the empty-cavity Gauss–Laguerre modes. Such regular structures are well understood theoretically; their description is based on field expansion over modal Gauss–Laguerre functions related to the cylindrical geometry [26]. For Fresnel numbers up to $$F = 5$$, the number of bright filaments follows the $$F^2$$ law as $$F$$ increases. Such regular filamentary structures have been observed in several lasers, such as CO₂ and Na₂ lasers [26]. Similar structures also appear in many passive nonlinear media, e.g. in Kerr media and in active nonlinear media, such as the photorefractive crystal Bi₁₂SiO₂₅ pumped by a laser [28, 29].

As soon as the Fresnel number reaches $$F = 10$$, a qualitative change occurs in the features of the filamentary structure: The regular filaments become turbulent. This transition takes place gradually, with intermittent behaviour in the region $$5 < F < 15$$. The character of this change is again common in active nonlinear media [28, 29] as well as in lasers [39, 40].

At Fresnel numbers $$F > 15$$, the filamentary structures which arise become different in principle from those
existing at low Fresnel numbers. The spatial structures now have no relation to the empty-cavity modes. Modal expansion over the geometrically prescribed Gauss–Laguerre modes is no longer relevant and the boundary conditions have no importance. The laser medium houses a large number of parallel independent filaments exhibited as a set of bright spots randomly distributed in the transverse cross-section. The number of these random filaments is proportional to \( F \), in contrast with the case of low Fresnel numbers, where the number of filaments is proportional to \( F^2 \). The chaotic filaments, being randomly distributed in space, are not correlated with each other. Such chaotic spatio-temporal behaviour is characteristic of hydrodynamic turbulence. The similar phenomenon in optics is thus commonly called \textit{optical turbulence}. In contrast with the low \( F \) regime, where the regularity of spatial structures is imposed by the cavity geometry and symmetry constraints of boundary conditions, turbulent optical filamentation is strictly self-organized. Its organization emerges from the intrinsic properties of the medium. Since optical turbulence is accompanied by the formation of bright filaments with a high density of photons, this phenomenon can be named \cite{35} the \textit{turbulent photon filamentation}. This phenomenon is common in lasers and photorefractive crystals \cite{8, 28, 29}.

To my knowledge the turbulent filamentary structures in lasers, were first observed in a series of experiments \cite{41–45} with resonatorless superluminescent lasers on the vapours of Ne, Ti, Pb, N\(_2\), and N\(_2^*\). In these experiments, the typical characteristics were as follows: \( \lambda = 5 \times 10^{-3} \text{ cm} \), \( R = 0.1\text{–}0.3 \text{ cm} \), \( L = 20\text{–}50 \text{ cm} \), and \( F = 10\text{–}100 \). The number of filaments was \( N_f \approx 10^5 \text{–}10^6 \), with a typical radius \( r_f \approx 0.1 \text{ cm} \).

Filamentary structures in large-aperture optical devices have been observed in several lasers, as reviewed in \cite{39, 40}, and in photorefractive crystals \cite{28, 29}. Numerical simulations have been made \cite{46}. Experimental reports have mainly been from CO\(_2\) lasers \cite{39, 40, 47, 48}, dye lasers \cite{49}, and semiconductor lasers \cite{50, 51}.

The turbulent nature of filamentation in high Fresnel number lasers was carefully studied in a series of experiments \cite{52–58} with CO\(_2\) lasers and dye lasers. Irregular temporal behaviour was observed in local field measurements. It was found that the transverse correlation length was rather short. Randomly distributed transverse patterns generated in short times were observed, being shot-to-shot nonreproducible. For intermediate Fresnel numbers \( F \approx 10 \), instantaneous transverse structures were randomly distributed in space, but after time averaging, they displayed a kind of regularity related to the geometrical boundary conditions. This combination of irregular instantaneous patterns with an averaged or stationary pattern showing the remnant ordering, is understandable for the intermediate regime in the crossover region \( 5 < F < 15 \). Fully developed optical turbulence is reached as the Fresnel number increases up to \( F \approx 100 \).

Typical laser parameters are as follows. A pulsed CO\(_2\) laser, of wavelength \( \lambda = 1.06 \times 10^{-3} \text{ cm} \) and frequency \( \omega = 1.78 \times 10^{14} \text{ s}^{-1} \), emits pulses of \( r_f = 0.7 \times 10^{-7} \text{ s} \) or \( 10^{-6} \text{ s} \). The aperture radius is \( R = 1 \text{ cm} \) and laser length \( L = 100 \text{ cm} \). The inversion and polarization decay rates are \( \gamma_1 = 10^7 \text{ s}^{-1} \) and \( \gamma_2 = 3 \times 10^6 \text{ s}^{-1} \). The CO\(_2\) density is \( \rho = 2 \times 10^{18} \text{ cm}^{-3} \). The Fresnel number is \( F = 10 \). The characteristic filament radius is \( r_f \approx 0.1 \text{ cm} \).

A pulsed dye laser, of wavelength \( \lambda = 0.6 \times 10^{-4} \text{ cm} \) and frequency \( \omega = 3.14 \times 10^{15} \text{ s}^{-1} \), produces pulses of \( r_f = 0.5 \times 10^{-6} \text{ s} \). The decay rates are \( \gamma_1 = 4 \times 10^8 \text{ s}^{-1} \) and \( \gamma_2 = 10^{12} \text{ s}^{-1} \). The cavity length is \( L = 20 \text{ cm} \). By varying the aperture radius between 0.3 cm and 0.8 cm, the Fresnel number can be changed by an order, between \( F = 15 \) and \( F = 110 \). The typical filament radius is \( r_f \approx 0.01 \text{ cm} \).

The theory of turbulent photon filamentation has been developed in \cite{8, 35–38, 59}.

Filaments are randomly distributed in the transverse cross-section of the laser cavity, evolving in space and time independently of each other. The characteristics of each filament essentially depend on the value of the related coupling parameter \( g \). For cylindrical symmetry the latter can be presented in the form

\[
g = 2\pi \rho_0 \int_0^{R_f} \int_0^{L} \sin \left( \frac{k_0 \sqrt{r_f^2 + z^2} - kz}{v} \right) dz \text{d}r_f. \tag{166}
\]

Keeping in mind the resonance condition \( k_0 = k \) and introducing the variable \( x = \sqrt{k_0^2 (r_f^2 + z^2) - z^2} \), we have

\[
g = 2\pi \rho_0 \int_0^{R_f} \int_0^{L} \sin x \left( \frac{kr_f}{z} \right) dz \text{d}r_f. \tag{167}
\]

Since \( \lambda \ll L \), the upper limit \( kL \) in the integral \((167)\) can be replaced by \( kL \to \infty \). This gives

\[
g = 2\pi \rho_0 \int_0^{R_f} \int_0^{\infty} \sin \left( \frac{kr_f}{z} \right) dz \text{d}r_f. \tag{168}
\]

Introducing the notation

\[
\varphi \equiv \frac{\pi R_f^2}{\lambda L}, \tag{169}
\]

varying over the range \( 0 \leq \varphi \leq \pi F \) and playing the role of an effective Fresnel number for a given filament, we transform equation \((168)\) to

\[
g(\varphi) = \pi \rho_0 \int_0^{2\varphi} \sin x \text{d}x = \left[ \pi \varphi - \int_0^{2\varphi} \sin x \text{d}x \right]. \tag{170}
\]

In the same manner, the coupling parameter \( g(\varphi) \) can be reduced to

\[
g'(\varphi) = - \frac{\pi \rho_0 L}{k_r \tau_2} \int_0^{2\varphi} \cos x \text{d}x, \tag{171}
\]

with the integral cosine

\[
\cos x \equiv \int_0^\infty \frac{\cos u}{u} du.
\]
Performing the integration we find
\[ g(\varphi) = \frac{\rho \gamma_{L} L}{2 \gamma_{T}^{2}} \left( 2 \varphi - 2 \varphi \sin(2\varphi) + 1 - \cos(2\varphi) \right), \]
\[ g'(\varphi) = \frac{\rho \gamma_{L} L}{2 \gamma_{T}^{2}} \left( 2 \sin(2\varphi) - 2 \varphi \cos(2\varphi) \right). \]

Thus, the coupling parameters are functions of the effective variable \( \varphi \), which, in turn, depends on the enveloping radius \( R_{f} \) related to the effective filament radius \( r_{f} \). If we assume that the radiation intensity in the transverse cross-section of a filament is distributed according to the Gaussian law and if we define the effective filament radius as the mean-square deviation from the filament axis, then we get the relation \( r_{f} = 0.55 R_{f} \).

In general, filaments of different radii can arise. However, because some of them are more stable than others, the overwhelming majority will possess radii close to a typical value. The distribution of filaments with respect to their radii, and hence, the typical radius, can be found by invoking the general method of probabilistic pattern selection \([8, 36, 60, 61]\). Following this approach, we define the probability distribution
\[ p(\varphi, t) = \frac{1}{Z(t)} \exp \left\{ -X(\varphi, t) \right\} \]
for a filament characterized by the variable \( \varphi \) at time \( t \). Here
\[ X(\varphi, t) = \text{Re} \int_{0}^{t} \text{Tr} \hat{J}(\varphi, t') \text{d}t' \]
is the expansion exponent, expressed through the Jacobian matrix \( \hat{J} \) of the evolution equations, and
\[ Z(t) = \int \exp \left\{ -X(\varphi, t) \right\} \text{d}\varphi \]
is the normalizing factor. The expansion exponent \( X(\varphi, t) \) defines the local expansion rate
\[ A(\varphi, t) \equiv \frac{1}{t} X(\varphi, t). \]

The latter can be represented as the sum of the local Lyapunov exponents. The partial sum of only positive Lyapunov exponents defines the entropy production rate \([62]\), which does not coincide with the local expansion rate \( A(\varphi, t) \).

Thus, the probability for the appearance of filaments, characterized by the parameter \( \varphi \), is given by the probability distribution \( p(\varphi, t) \). The most probable is evidently the filament with a typical \( \varphi \) satisfying the principle of minimal expansion \([8, 36, 60, 61]\)
\[ \max_{\varphi} p(\varphi, t) \Leftrightarrow \min_{\varphi} X(\varphi, t) \Leftrightarrow \min_{\varphi} A(\varphi, t). \]

This general principle follows from the minimization of the pattern information and can be employed for arbitrary dynamic systems.

The dynamics of turbulent photon filamentation is described by the general evolution equation \((164)\). Calculating the corresponding Jacobian matrix gives
\[ \text{Tr} \hat{J}(\varphi, t) = -\gamma_{1} - \gamma_{3} - 2 \varphi_{L}(1 - g s), \]
with \( g = g(\varphi) \) and \( s = s(t) \). For \( t \gg T_{1} \), the expansion rate can be presented as
\[ A(\varphi, t) \approx -\gamma_{1} - \gamma_{3} - 2 \varphi_{L}(1 - g s^*). \]

Defining the stationary state \( s^{*} \), we get
\[ A(\varphi, t) \approx -\gamma_{1} - \gamma_{3} - 2 \varphi_{L}(1 + | g_{L}^{*} |) \quad (g_{L}^{*} \ll -1), \]
\[ A(\varphi, t) \approx -\gamma_{1} - \gamma_{3} - 2 \varphi_{L}(1 - \frac{g_{L}^{*}}{\gamma_{L}^{*}}) \quad (| g_{L}^{*} | \ll 1), \]
\[ A(\varphi, t) \approx -\gamma_{1} - \gamma_{3} - 2 \varphi_{L} \gamma_{L}^{*} \quad (g_{L}^{*} \approx 1). \]

The stationary pumping parameter \( \gamma \) lies in the range \(-1 \leq \gamma \leq 1 \), depending on the level of pumping. When there is no stationary pumping, \( \gamma = 0 \). If \( -1 < \gamma < 0 \), the pumping is considered weak, and if \( 0 < \gamma < 1 \), it is strong. Bearing in mind that the coupling parameter \( g \) is positive, we see that two different cases exist. When the stationary pumping is weak or absent, \( \gamma < 0 \), and when it is strong, \( \gamma > 0 \). According to the principle of minimal expansion \((175)\), the minimum of the expansion rate corresponds to the maximum of \( g(\varphi) \) if \( \gamma < 0 \), and to the minimum of \( g(\varphi) \), if \( \gamma > 0 \). The extreme values of \( g(\varphi) \) are given by the equation
\[ \text{Si}(2\varphi) = \frac{\pi}{2}. \]

In the standard case of absent or weak pumping, where \( \gamma < 0 \), we find the absolute maximum of \( g(\varphi) \). Then equation \((178)\) gives \( g = 0.96 \). Therefore \( R_{f} = 0.55 \sqrt{\lambda L} \), and the typical filament radius is
\[ r_{f} = 0.3 \sqrt{\lambda L}. \]

The number of filaments can be estimated as \( N_{f} \approx R^{2}/R_{f}^{2} \), which yields
\[ N_{f} \approx 3.3 F. \]

The linear dependence of the filament number on the Fresnel number is characteristic of turbulent photon filamentation.

Note that under strong stationary pumping \( (\gamma > 0) \), when we need to find the minimum of \( g(\varphi) \), we would have \( g = 2.45 \), and hence obtain \( R_{f} = 0.88 \sqrt{\lambda L} \) and \( r_{f} = 0.5 \sqrt{\lambda L} \).

Formula \((179)\) for the typical filament can be compared with the radii observed in experiments. Thus, in different vapour lasers \([41–45]\), one has \( r_{f} = 0.01 \text{ cm} \). For CO\(_{2}\) and dye lasers, it was found \([52–58]\) that \( r_{f} = 0.1 \text{ cm} \) and \( r_{f} = 0.01 \text{ cm} \), respectively. All these data are in good agreement with formula \((179)\).

14. Collective liberation of light

Photonic band-gap materials possess a prohibited band gap, where light cannot propagate. The spontaneous radiation of atoms with a frequency inside the prohibited band gap is strongly suppressed \([63]\). This means that the equation for the population difference of a single atom can be effectively represented as
\[ \frac{\text{d} \gamma_{s}}{\text{d} t} = -\gamma (s - s_{0}). \]
A single initially excited atom remains excited, so that \( s = s_0 \), for all times \( t \to 0 \).

However, if the density of doped atoms is sufficiently high, coherent interactions may develop \([8]\). Atoms can then start to radiate, even inside the prohibited band gap. This collective phenomenon for atoms with an atomic frequency increase, of light collective phenomenon for atoms with an atomic frequency start to radiate, even inside the prohibited band gap. This high, coherent interactions may develop \([8\, \text{ and } 187]\). Atoms can then

\[
\frac{\partial u}{\partial t} = -(i\Omega + \Gamma)u + s\xi,
\]

\[
\frac{\partial w}{\partial t} = -2\Gamma w + (u^*\xi + \xi^*u)s,
\]

\[
\frac{\partial s}{\partial t} = -gs^2w - \frac{1}{2}(u^*\xi + \xi^*u)s - \gamma(s - s_0),
\]

where the effective collective width and collective frequency are

\[
\Gamma = \gamma_2(1 - gs), \quad \Omega = \omega_0 + g'\gamma_s.
\]

These equations are to be understood as describing the radiation dynamics in a separate coherent filament of volume \( V_f = \pi R^2 L \approx 2L^2 \). When the stationary population imbalance of a single atom is \( \zeta = s_0 \), then the evolution equations for the ensemble of atoms, after the interaction stage, are

\[
\frac{\partial u}{\partial t} = -(i\Omega + \Gamma)u + s\xi,
\]

\[
\frac{\partial w}{\partial t} = -2\Gamma w + (u^*\xi + \xi^*u)s,
\]

\[
\frac{\partial s}{\partial t} = -gs^2w - \frac{1}{2}(u^*\xi + \xi^*u)s - \gamma(s - s_0),
\]

where the effective collective width and collective frequency are

\[
\Gamma = \gamma_2(1 - gs), \quad \Omega = \omega_0 + g'\gamma_s.
\]

These equations are to be understood as describing the radiation dynamics in a separate coherent filament of volume \( V_f = \pi R^2 L \approx 2L^2 \). Again employing the scale separation approach, we solve the equation for the fast variable \( u \), substitute the solution and average the latter equations over fast oscillations and random variables. Using notation \((137)\), we obtain the equations for the slow variables

\[
\frac{\partial w}{\partial t} = -2\Gamma w + 2\gamma_s^2s^2,
\]

\[
\frac{\partial s}{\partial t} = -g\gamma_s^2w - \gamma s - \gamma(s - s_0).
\]

The random variable, representing the matter, can be written \([68, 69]\) in the form

\[
\xi(t) = \frac{1}{\sqrt{N_0}} \sum_k \xi_k(b_k e^{-i\omega_k t} + b_k^* e^{i\omega_k t}),
\]

where \( b_k \) are bosonic degrees of freedom describing the matter and \( \omega_k \) is the related spectrum of collective excitations. For the bosonic operators the averages are

\[
\langle\langle b_k^* b_k \rangle\rangle = \delta_{kp}n_k, \quad n_k \equiv \langle\langle b_k^* b_k \rangle\rangle,
\]

\[
\langle\langle b_k b_k^* \rangle\rangle = \delta_{kp}(1 + n_k), \quad \langle\langle b_k b_k \rangle\rangle = 0,
\]

and the normalization condition is

\[
\sum_k n_k = N_{\text{mat}}.
\]

The matter is characterized by a frequency gap \( \Delta_p \equiv \omega_2 - \omega_1 \), inside which collective excitations are suppressed, so that

\[
\gamma_k = \begin{cases} 
0, & \text{for } \omega_k \in (\omega_1, \omega_2) \\
\gamma, & \text{for } \omega_k \notin (\omega_1, \omega_2). 
\end{cases}
\]

Then equations \((137)\) and \((184)\) yield the dynamic attenuation

\[
\gamma_3 = \frac{\Gamma}{N_{\text{mat}}} \sum_k |\xi_k|^2 \left[ \frac{n_k}{(\omega_k - \Omega)^2 + \Gamma^2} + \frac{1 + n_k}{(\omega_k + \Omega)^2 + \Gamma^2} \right].
\]

For a narrow gap, such that \( \Delta_p \gg 1 \) \( (\Delta_p \equiv \omega_2 - \omega_1) \), the attenuation \((188)\) simplifies to

\[
\gamma_3 \approx \frac{4\gamma^2\gamma_s(1 - gs)}{\Delta_p^2 + 4\gamma_s^2(1 - gs)^2},
\]

When the coupling and initial excitation are sufficiently weak, such that

\[
|gs_s| \ll 1,
\]

then the stationary solutions for the slow variables are

\[
w^s \approx \frac{\gamma_3 s_0}{\gamma + \gamma_3}, \quad s^s \approx \frac{s_0}{\gamma + \gamma_3}
\]

and the dynamic attenuation \((190)\) is

\[
\gamma_3 \approx \frac{4\gamma^2\gamma_s}{\Delta_p^2 + 4\gamma_s^2(1 - gs)^2}
\]

In the usual situation, when \( \gamma \sim \omega_1 < \gamma_2, \delta_p \gg \gamma_2 \), we have \( \gamma_3 \ll \gamma_1 \). Therefore, in the case of weak coupling \((191)\), the stationary excitation remains virtually unchanged, \( s^s = s_0 \), which means that there is no radiation, and one can say that the light is locked.

If the coupling is strong, but the initial excitation is weak, such that

\[
|gs_s| \gg 1, \quad gs_s < 0,
\]

then the stationary solutions are

\[
w^s \approx \frac{\gamma_3 s_0}{\gamma 2|gs_s|}, \quad s^s \approx s_0 \left(1 - \frac{\gamma_3}{\gamma |gs_s|}\right)
\]

and the attenuation is

\[
\gamma_3 \approx \frac{4\gamma^2\gamma_s|gs_s|}{\Delta_p^2 + 4\gamma_s^2(1 - gs)^2}.
\]

Again, there is no radiation and the light remains locked, since \( s^s = s_0 \).

When the initial excitation and atomic coupling are sufficiently strong, so that

\[
gs_s \gg 1,
\]

then the stationary solutions are

\[
w^s \approx \frac{\gamma_3 s_0}{\gamma 2g}, \quad s^s \approx \frac{1}{g} \left(1 - \frac{\gamma_3}{\gamma |gs_s|}\right)
\]
and the attenuation is
\[ \gamma_s \approx - \frac{4\gamma \gamma_s^* g_s^*}{\Delta_p^2 + 4\gamma_s^2 (g_s^*)^2} \approx - \frac{4\gamma \gamma_s}{\Delta_p^2 + 4\gamma_s^2}. \]  
(199)

In this case, atoms radiate and de-excite to a low population imbalance
\[ \lim_{t \to \infty} s(t) = s^* \approx \frac{1}{g} \ll s_0. \]  
(200)

That is, collective liberation of light occurs.

### 15. Pseudospin atomic squeezing

The effect of squeezing allows one to reduce the noise level when measuring a required quantity. Generally, the notion of squeezing is introduced for two operators, say \( \hat{A} \) and \( \hat{B} \). The uncertainty in measuring the observable quantity, corresponding to the operator \( \hat{A} \), is characterized by the operator dispersion or variance
\[ \text{var}(\hat{A}) \equiv \langle \hat{A}^\dagger \hat{A} \rangle - |\langle \hat{A} \rangle|^2. \]  
(201)

The squeezing of \( \hat{A} \) with respect to \( \hat{B} \) is defined by the squeezing factor
\[ Q(\hat{A}, \hat{B}) \equiv \frac{2 \text{var}(\hat{A})}{|\langle [\hat{A}, \hat{B}] \rangle|^2}. \]  
(202)

The Heisenberg uncertainty relation
\[ \text{var}(\hat{A}) \text{var}(\hat{B}) \geq \frac{1}{4} |\langle [\hat{A}, \hat{B}] \rangle|^2 \]  
(203)

can be written as
\[ Q(\hat{A}, \hat{B})Q(\hat{B}, \hat{A}) \geq 1. \]  
(204)

One says that \( \hat{A} \) is squeezed with respect to \( \hat{B} \) if
\[ \text{var}(\hat{A}) < \frac{1}{2} |\langle [\hat{A}, \hat{B}] \rangle|, \]  
(205)

which implies that
\[ Q(\hat{A}, \hat{B}) < 1. \]  
(206)

Using the pseudospin operators
\[ S^i_N \equiv \sum_{j=1}^N S^i_j, \quad S^N \equiv \sum_{j=1}^N S^N_j, \]  
(207)

we derive the evolution equations for the variables
\[ u \equiv \frac{2}{N} \sum_{j=1}^N (S^+_j) = \frac{2}{N} (S^+_N), \]  
(208)

Taking into account the properties
\[ \text{var}(S^i_N) = \frac{N}{4} (1 - s^2), \quad \langle [S^i_N, S^j_N] \rangle = \frac{N}{2} |u|, \]  
(209)

we obtain the squeezing factor
\[ Q(S^i_N, S^j_N) = \frac{1 - s^2}{\sqrt{w}}. \]  
(210)

which expresses the squeezing of \( S^i_N \) with respect to \( S^j_N \). If this factor is less than one, the atomic imbalance \( s \) can be measured with a better accuracy than measuring transition quantities, such as, for example, coherence intensity \( w \).

Different regimes of atomic evolution produce various temporal behaviours of the squeezing factor (210). Its behaviour also depends on whether the vacuum, describing the material, is squeezed. The level of squeezing can be regulated in the process of punctuated superradiance [70].

### 16. Operator entanglement production

The notion of entanglement is nowadays widely studied because of its role in quantum information processing and quantum computing [71–73]. It is necessary to distinguish between entanglement and entanglement production [74, 75]. Entanglement describes the state of a bipartite, or more generally, a many-body system [71–73], while entanglement production describes the amount of entanglement produced by an operation associated with an operator [74, 75]. We are here considering entanglement production, that plays an important role in both quantum information processing and quantum measurements [76].

Suppose we need to find out how much entanglement is produced by an operator \( \hat{A} \). The operator acts on a multidimensional Hilbert space
\[ \mathcal{H} = \otimes_{i=1}^N \mathcal{H}_i. \]  
(211)

Such spaces are typical of many-body systems [77, 78]. The operator, acting on a non-entangled state, such as
\[ \varphi = \otimes_{i=1}^N \varphi_i \quad (\varphi_i \in \mathcal{H}_i), \]  

can transfer it to an entangled state.

The action of an operator has to be compared with the action of its non-entangling counterpart, that is defined as
\[ \hat{A}^\otimes \equiv \frac{\otimes_{i=1}^N \hat{A}}{(\text{Tr}_{\mathcal{H}} \hat{A})^{N-1}}, \]  
(212)

where
\[ \hat{A}_i \equiv \text{Tr}_{\mathcal{H}_i} \hat{A} \]  
(213)

is a-factor-operator obtained by tracing out from \( \hat{A} \) all degrees of freedom except one associated with the subspace \( \mathcal{H}_i \). The denominator in equation (212) is chosen so as to preserve the normalization condition
\[ \text{Tr}_{\mathcal{H}} \hat{A}^\otimes = \text{Tr}_{\mathcal{H}} \hat{A}. \]  
(214)
The measure of entanglement production is defined as
\[
\epsilon(\hat{A}) \equiv \log \frac{\|\hat{A}\|}{\|\hat{A}^\otimes\|},
\]
where \(\|\hat{A}\|\) implies an operator norm. It is possible to choose different definitions of the norm. One possibility would be the Hilbert–Schmidt norm
\[
\|\hat{A}\| \equiv \sqrt{\text{Tr}(\hat{A}^\dagger \hat{A})},
\]
which has the advantage that it does not depend on the basis used for calculating the trace.

Generally, the operator \(\hat{A}\) can depend on some parameter, so that one can study the dependence of measure (215) on this parameter. In particular, the parameter can be time \([79]\). It is hence possible to consider the temporal behaviour of entanglement production; for instance in the process of atomic collective radiation \([80]\).

Another way to investigate evolulutional entanglement pro-duction is to consider the entanglement production due to the evolution operator
\[
\hat{U}(t) = e^{-i\hat{H}t},
\]
in which \(\hat{H}\) is the system Hamiltonian. Following the usual method, we define the partial factor operator
\[
\hat{U}(t) \equiv \text{Tr}_i \hat{U}(t)
\]
and construct the corresponding non-entangling evolution operator
\[
\hat{U}^\otimes(t) = \left(\prod_{i=1}^N \hat{U}(t)\right)_{\text{Tr}_i \hat{U}(t)}^{N-1}.
\]
The measure of the evolutional entanglement production becomes
\[
\epsilon(t) \equiv \epsilon(\hat{U}(t)) = \log \frac{\|\hat{U}(t)\|}{\|\hat{U}^\otimes(t)\|}.
\]
For the Hilbert–Schmidt norms, we have
\[
\|\hat{U}(t)\|^2 = \prod_{i=1}^N d_i \quad (d_i \equiv \text{dim } \mathcal{H}_i),
\]
\[
\|\hat{U}^\otimes(t)\|^2 = \prod_{i=1}^N \|\hat{U}(t)\|_{\text{Tr}_i \hat{U}(t)}^{N-1}.
\]
Therefore the evolutional entanglement production is quantified by the measure
\[
\epsilon(t) = \frac{1}{2} \log \left\{ \text{Tr}_i \hat{U}(t) \left(\prod_{i=1}^N \|\hat{U}(t)\|_{\text{Tr}_i \hat{U}(t)}^{N-1} \right) \right\},
\]
This measure shows to what extent the evolution operator produces entanglement.

17. Conclusion

A general approach that can be used for both atomic and spin systems has been presented. The approach is based on a pseudospin representation of evolution equations and employs the methods of stochastic quantization, scale separation, transverse mode expansion, and probabilistic pattern selection. The generality of the approach makes it possible to treat all stages of evolution, which in turn makes it possible to study the self-organization of coherence from initial chaotic fluctuations. The qualitatively different temporal intervals are: interaction stage, chaotic stage, coherent stage, relaxation stage, and quasi-stationary stage. The regimes of pure superradiance, triggered superradiance, pulsing superradiance, and punctuated superradiance are analyzed.

This approach has been used in the past to describe non-equilibrium coherent phenomena in spin systems \([6–9, 14, 74, 75]\). The present paper demonstrates that the same mathematical techniques are applicable to the description of radiation processes in atomic systems. The approach has been illustrated by describing several interesting effects, such as triggering dipolar waves, turbulent photon filamentation, collective liberation of light, pseudospin atomic squeezing, and operator entanglement production.

Although physical processes in spin and atomic systems are rather different \([6, 13–16]\), the mathematical methods of investigation turn out to be the same, which makes the suggested approach advantageous.

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