Complement for Algebraic differential equations...

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Abstract. We complete the study of some periods of polynomials in $(n+1)$-variables with $(n+2)$-monomials in computing the behavior of these periods in the natural parameter for such a polynomial.

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1 Introduction

This note is a complement to the study in [B.13] of period integrals of non quasi-homogeneous polynomials in $n + 1$ variables with $n + 2$ monomials. We focus here on the dependance of these period integrals on the “natural” parameter $\lambda \in \mathbb{C}^*$ which is the only “free” coefficient of such a polynomial modulo the dilatations of the variables.

For that purpose we recall first the fact that for a polynomial function $f$ depending polynomially of a parameter $\lambda$ we may define a natural “$b$–connection” on the highest $(f, \lambda)$–relative de Rham cohomology group of $f$ which induces the derivation $\frac{\partial}{\partial \lambda}$ on period integrals. The construction for any holomorphic function depending of a holomorphic parameter is precised in the appendix.

Then we show how to compute explicitly this connection in our specific situation and we obtain a simple partial differential equation for the period integrals associated to any monomial in $\mathbb{C}[x_0, \ldots, x_n]$ when we consider a polynomial of the type

$$f_\lambda(x) = \sum_{j=1}^{n+1} x^{\alpha_j} + \lambda x^{\alpha_{n+2}}$$

where $\alpha_j \in \mathbb{N}^{n+1}$, $j \in [1, n + 2]$ with the following assumptions

i) The $(n + 2, n + 2)$–matrix obtained from $M := (\alpha_1, \ldots, \alpha_{n+2})$ by adding a first line of 1 has rank $n + 2$.

ii) The elements $\alpha_1, \ldots, \alpha_{n+1}$ form a basis of $\mathbb{Q}^{n+1}$.

Note that the first condition is equivalent to the fact that $f$ is not quasi-homogeneous, and that the condition ii) is always satisfied assuming i), up to change the order of the monomials (and then to change the parameter $\lambda$ to $c.\lambda^m$ for some $c \in \mathbb{Q}^*$ some $m \in \mathbb{Z}^*$).

2 The $\lambda$–connection.

2.1 The general situation.

We consider here a polynomial $f \in R := C[x_0, \ldots, x_n][\lambda]$ depending polynomially on a parameter $\lambda$. We consider on $R \otimes \Lambda^*(\mathbb{C}^{n+1}) := \Omega_f^*$ the $\lambda$–relative de Rham complex, where $(\mathbb{C}^{n+1})^* := \oplus_{i=0}^n \mathbb{C}.dx_i$, and we denote $d_f$ its differential.

We shall denote by $\mathcal{A}$ the unitary (non commutative) algebra generated by $a$ and $b$ with the commutation relation $ab - ba = b^2$ and by $\mathcal{A}[\lambda] := \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}[\lambda]$ with its natural structure of algebra for which $\lambda$ commutes with $a$ and $b$.

Then the quotient $E_f := \Omega_f^{n+1}/d_f \wedge d_f \Omega_f^{n-1}$ has a natural left $\mathcal{A}[\lambda]$–module structure defined by

1In our hypothesis we may assume that all monomials have coefficient 1 excepted the last one up to a linear diagonal change of variable.
• The action of $a$ is given by the multiplication by $f$.
• The action of $b$ is given by $d/f \wedge d_f^{-1}$.

Remark that for fixed $\lambda$, assuming that $f_\lambda$ has an isolated singularity at 0, the $b-$completion $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$ is the usual (formal) Brieskorn module associated to $f_\lambda$ at 0. For a given monomial $\mu \in \mathbb{C}[x_0, \ldots, x_n]$ the decomposition theorem of [B.13] (theorem 3.1.2) applies to the quotient $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(\mu)$ where $\tilde{\mathcal{A}}$ is the $b-$completion of $\mathcal{A}$ and where $P(\mu)$ is the element in $\mathcal{A}$ constructed in the theorem 1.2.1 of [B.13]. Then $P_d(\mu)$ is a (left-)multiple of the Bernstein element of $\tilde{\mathcal{A}}.\mu dx$ in $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$ and it determines a finite set of possible eigenvalues for the monodromies around $s = 0$ for the period integrals ($\lambda$ fixed)

$$\varphi_\lambda(s) = \int_{\gamma_{\lambda,s}} \frac{\mu dx}{d_f}$$

for any horizontal family $\gamma_{\lambda,s}$ of compact $n-$dimensional cycles in the fibers of $f_\lambda$.

It is important to remark that if $f_\lambda$ has a non isolated singularity at the origin, despite the fact that there is no finiteness for the $\mathbb{C}[[b]]-$module $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$, the conclusion above is still valid because the quotient $\tilde{\mathcal{A}}/\tilde{\mathcal{A}}.P(\mu)$, and so its image in $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$, is a finite type $\mathbb{C}[[b]]-$module. Then the product decomposition $P_d = (a - r_1,b) \ldots (a - r_d,b)$, where $r_1, \ldots, r_d$ are (explicitly computable) rational numbers, gives that the set $\{e^{2\pi i r_1}, \ldots, e^{2\pi i r_d}\}$ contains the spectrum of these monodromies (counting multiplicities).

**Question.** Is it true that $P_d$ is equal to the Bernstein element of the Brieskorn module $\tilde{\mathcal{A}}.\mu dx$ in $E_f \otimes_{\mathbb{C}[b]} \mathbb{C}[[b]]$ when $f_\lambda$ has an isolated singularity at the origin ?

**Proposition 2.1.1** There exists a $\mathbb{C}-$linear operator $\nabla : E_f \rightarrow E_f$ with the following properties :

1. For $\omega = d_f \xi \in \Omega_f^{n+1}$ we have $\nabla ([\omega]) = [d_f f \wedge \frac{\partial \xi}{\partial x} - \frac{\partial f}{\partial x} \omega]$.
2. The map $b^{-1}.\nabla$ well defined on $b.\tilde{E}_f$ where $\tilde{E}_f := E_f/(b-$torsion), with value in $\tilde{E}_f$, commutes with $a$ and $b$ and is a $\lambda-$connection.

---

For a $(a,b)$-module $E$ with one generator as a $\tilde{\mathcal{A}}-$module, the relation between its Bernstein element $P_d \in \tilde{\mathcal{A}}$ and its Bernstein polynomial $B$ is given by the formula (see [B.09])

$$(-b)^d.B(-b^{-1}.a) = P_d$$

where $d$ is the rank of $E$. 

---
3. If \((\gamma_s, \lambda)(s, \lambda) \in S \times \Omega\) is a horizontal family of compact \(n\)-cycles in the fibers of \((f, \lambda)\) over an open set in \(\mathbb{C} \times \mathbb{C}^* \setminus C(f, \lambda)\) where \(C(f, \lambda)\) is the set of critical values of the map \((f, \text{id}) : \mathbb{C}^{n+1} \times \mathbb{C}^* \to \mathbb{C} \times \mathbb{C}^*\), we have for any \(\omega \in \Omega_{n+1}\) the equality

\[
\frac{\partial}{\partial s} \frac{\partial}{\partial \lambda} \left[ \int_{\gamma_{s,\lambda}} \frac{\omega}{df} \right] = \int_{\gamma_{s,\lambda}} \nabla(\omega). \frac{df}{d\lambda}.
\]

**Proof.** Remark first that \(\nabla\) is well defined because for \(\xi = d/\eta\) we have

\[
\nabla(d/\xi) = d/f \wedge \frac{\partial(d/\eta)}{\partial \lambda} = \frac{d}{f} \wedge d \left( \frac{\partial f}{\partial \lambda} \right)
\]

so it induces 0 in \(E_f\).

Let \(d/\xi = d/\eta \in \Omega_{n+1}\) and let \(d/\eta = d/f \wedge \xi\). Then we have

\[
\nabla(b.[\omega]) = \nabla(d/\eta) = d/f \wedge \frac{\partial(d/\eta)}{\partial \lambda} = d/f \wedge d\left( \frac{\partial f}{\partial \lambda} \right)
\]

\[
= d/f \wedge (\frac{\partial(d/\eta)}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \xi) = b\left[ d/f \wedge \frac{\partial f}{\partial \lambda} \right] - \frac{\partial f}{\partial \lambda} \xi
\]

\[
= b\left[ \frac{\partial d/f}{\partial \lambda} \wedge \xi + d/f \wedge \frac{\partial \xi}{\partial \lambda} \right] - \frac{\partial f}{\partial \lambda} \left( \frac{d/\eta}{\partial \lambda} \right) \wedge \xi
\]

\[
= b\left[ \frac{\partial d/f}{\partial \lambda} \wedge \xi + \frac{\partial \xi}{\partial \lambda} \right] - \frac{\partial f}{\partial \lambda} \left( \frac{d/\eta}{\partial \lambda} \right) \wedge \xi
\]

as \(d/f\) and \(\frac{\partial}{\partial \lambda}\) commute. So we have \(b.\nabla = \nabla.b\).

We have also

\[
\nabla(a.[\omega]) = \nabla(f.d/\xi) = \nabla(d/ f \wedge \xi) - \nabla(d/ f \wedge \xi)
\]

\[
= d/f \wedge \frac{\partial(d/f \wedge \xi)}{\partial \lambda} - \frac{\partial f}{\partial \lambda} d/ f \wedge \xi - \frac{\partial f}{\partial \lambda} d/ f \wedge \xi - \nabla(b.[\omega])
\]

\[
= a.\nabla([\omega]) - b.\nabla([\omega]).
\]

This implies the equality \(a.b^{-1}.\nabla = b^{-1}.\nabla.a\) as \(\mathbb{C}\)-linear maps from \(b.\tilde{E}_f\) to \(\tilde{E}_f\).

Note that the equalities \(\nabla.b = b.\nabla\) and \(\nabla.a = (a - b).\nabla\) as \(\mathbb{C}\)-endomorphisms of \(E_f\) are more precise than the relations above.

Finally let \(\varphi \in \mathbb{C}[\lambda]\) then we have

\[
\nabla(\varphi.d/\xi) = \nabla(d/ (\varphi.\xi)) = d/f \wedge \frac{\partial(\varphi.\xi)}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \varphi.d/\xi
\]

\[
= \frac{\partial \varphi}{\partial \lambda} (d/f \wedge \xi) + \varphi.\nabla(d/\xi) = \frac{\partial \varphi}{\partial \lambda} [d/\xi] + \varphi.\nabla(d/\xi)
\]
and this shows that $b^{-1} \nabla$ is a $\lambda$-connection.

Note again that we proved the equality in $E_f : \nabla(\varphi \omega) = \frac{\partial \varphi}{\partial \lambda} b \omega + \varphi \nabla(\omega)$ valid for $\varphi \in \mathbb{C}[\lambda]$ and $\omega \in E_f$ which is more precise.

To prove the point 3. of the statement consider $\xi \in \Omega^n_f$ and let $d$ be the total de Rham differential (in $x$ and $\lambda$). We have

$$d\xi = d\lambda \wedge \frac{\partial \xi}{\partial \lambda} + d/\xi$$

Assume we can write $d\xi = d\lambda \wedge (v - \frac{\partial f}{\partial \lambda} u) + d/\xi$ with $u, v \in \Omega^n_f$. Then we obtain

$$d\xi = d\lambda \wedge (v - \frac{\partial f}{\partial \lambda} u) + d/\xi$$

If $(\gamma_{s,\lambda})$ is a horizontal family of compact $n$-cycles in the fibers of the map $(f, \text{id}) : \mathbb{C}^{n+1} \times \mathbb{C}^* \to \mathbb{C} \times \mathbb{C}^*$, we shall have

$$d(\int_{\gamma_{s,\lambda}} \xi) = \left[ \int_{\gamma_{s,\lambda}} (v - \frac{\partial f}{\partial \lambda} u) \right] d\lambda + \left[ \int_{\gamma_{s,\lambda}} u \right] ds.$$ 

So, has the chain $\cup_{s,\lambda} \gamma_{s,\lambda}$ is proper and without $\lambda$-relative boundary we obtain

$$\frac{\partial}{\partial s} \int_{\gamma_{s,\lambda}} \xi = \int_{\gamma_{s,\lambda}} \frac{d\xi}{d/\xi}$$

Now consider $\omega \in \Omega^{n+1}_f$ and write $\omega = d\xi$. Then $b[\omega] = [d/\xi f]$ and we have

$$\int_{\gamma_{s,\lambda}} \frac{b[\omega]}{d/\xi f} = \int_{\gamma_{s,\lambda}} \xi$$

So we conclude that we have

$$\frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \frac{\omega}{d/\xi f} = \frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \xi = \int_{\gamma_{s,\lambda}} \left( \frac{\partial \xi}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \frac{d\xi}{d/\xi f} \right) = \frac{\partial}{\partial \lambda} \int_{\gamma_{s,\lambda}} \nabla[\omega]$$

2.2 The case of a polynomial with $n+2$ monomials in $n+1$ variables.

So we consider now the case were $f := \sum_{j=1}^{n+2} m_j$ where $m_j := x^{\alpha_j} j \in [1, n+1]$ and $m_{n+2} := \lambda x^{\alpha_{n+2}}$ with the following hypotheses (see [B. 13]) : the rank of the square matrix $M' := (\alpha_1, \ldots, \alpha_{n+1})$ is $n+1$ and the rank of the square matrix $\tilde{M}$ obtained by adding a first line of 1 to the matrix $\tilde{M} := (\alpha_1, \ldots, \alpha_{n+2})$ is $n+2$. 

Recall that if we write (with a minimal positive integer $r$) $r.a_{n+2} = \sum_{j=1}^{n+1} p_j.\alpha_j$ where $p_1, \ldots, p_{n+1}$ are in $\mathbb{Z}$, and if we define

$$d = \inf\{r - \sum_{j,p_j \leq 0} p_j, \sum_{j,p_j > 0} p_j\} \quad \text{and} \quad d + h = \sup\{r - \sum_{j,p_j \leq 0} p_j, \sum_{j,p_j > 0} p_j\}$$

there exists an element $P$ in $A[\lambda, \lambda^{-1}]$ of the form

$$P := P_{d+h} + c.\lambda^\pm r.P_d$$

which annihilated the class $[dx]$ in $E_f$, where $P_{d+h}$ and $P_d$ are homogeneous elements in $A$, respectively of degree $d + h$ and $d$ which are monic in $a$ with rational coefficients, and where $c$ is in $\mathbb{Q}^*$. The sign in the exponent of $\lambda$ will be precised in the proof of the proposition 2.2.2.

Recall also that in this situation the $A[\lambda]$ module generated by the class $[dx]$ in $E_f$ is exactly the image in $E_f$ of $C[m_1, \ldots, m_{n+2}]\{\lambda\}dx \subset \Omega^{n+1}_f$ with $m_j = x^{a_j}$ with $j \in [1, n+1]$ and $m_{n+2} = \lambda.x^{a_{n+2}}$.

Our next result uses the following easy lemma:

**Lemma 2.2.1** Let $Q \in A$ a homogeneous element in $(a, b)$ of degree $k$. Then for any $\lambda \in \mathbb{C}$ we have:

$$b.Q.b^{-1}.(a - \lambda.b) = (a - (\lambda + k).b).Q.$$  

**Proof.** Remark first that the map $A \to A$ sending $x \in A$ to $b.x.b^{-1}$ is well defined and bijective thanks to the following facts: $b$ is injective and $b.A = A.b$. We shall prove the lemma by induction $k$. As the case $k = 0$ is obvious, assume that the lemma is proved for $k < k_0$ where $k_0 \geq 1$ and consider an homogeneous element $Q$ of degree $k_0$. We may assume that $Q = b.R$ or that we may find $\mu \in \mathbb{C}$ such that $Q = (a - \mu.b).R$, where $R$ is homogeneous of degree $k_0 - 1$. In the first case we have, using the induction hypothesis:

$$b.b.R.b^{-1}.(a - \lambda.b) = b.(a - (\lambda + k_0 - 1).b).R = (a - (\lambda + k_0).b).b.R = (a - (\lambda + k_0).b).Q.$$  

In the second case we have, using the induction hypothesis:

$$b.(a - \mu.b).R.b^{-1}.(a - \lambda.b) = (a - (\mu + 1).b).b.R.b^{-1}.(a - \lambda.b)$$

$$= (a - (\mu + 1).b).(a - (\lambda + k_0 - 1).R)$$

$$= (a - (\lambda + k_0).b).(a - \mu.b).R$$

$$= (a - (\lambda + k_0).b).Q.$$  

---

$^3$More is proved in [B.13]: $P_{d+h}$ and $P_d$ factorize in product of $(a - r_j.b)$ with $r_j \in \mathbb{Q}$.

$^4$Recall that any homogeneous element in $A$ which is monic in $a$ factorizes as a product of linear factors $(a - r_i.b)$, where the $r_i$ are complex numbers; see [B.09].
Proposition 2.2.2 Let $\mu$ be a monomial of degree $k$ in $\mathbb{C}[x_0, \ldots, x_{n+1}]$. Then we have in $E_f$ the relation
\[
\nabla([\mu]) = -\frac{1}{\lambda}(\sigma.a + (\tau - k.\sigma).b)[\mu]
\]
where $\sigma, \tau$ are defined by the relation $m_{n+2}([\mu]) = (\sigma.a + \tau.b)[\mu]$. Moreover the value of $\sigma$ is $\pm r/h$ so it does not depend on the choice of the monomial $\mu$. As a consequence, if we have on an open set $S \times \Omega$ in $\mathbb{C}^* \times \mathbb{C}^*$, a horizontal family $(\gamma_s, \lambda)_{(s, \lambda) \in S \times \Omega}$ of compact $n$-cycles in the fibers of the map $\mathbb{C}^{n+1} \times \mathbb{C}^* \rightarrow \mathbb{C} \times \mathbb{C}^*$ defined by $(x, \lambda) \mapsto (f_\lambda(x), \lambda)$, the holomorphic function
\[
(s, \lambda) \mapsto \varphi(s, \lambda) := \int_{\gamma_s, \lambda} \frac{\mu.dx}{df}
\]
satisfies the partial differential equation
\[
-\lambda.\frac{\partial}{\partial \lambda} \frac{\partial}{\partial s} \varphi = \sigma.\frac{\partial(s.\varphi)}{\partial s} + (\tau - k).\varphi
\]
on $S \times \Omega$. ■

PROOF. As we have $\lambda.\nabla([1]) = -m_{n+2}$ in $E_f$ and as we know that there exist $\sigma, \tau$ in $\mathbb{Q}$ such that $(\sigma.a + \tau.b)[1] = m_{n+2}$ for the case $\mu = 1$ the only thing to prove is the computation of $\sigma$.

Using the Cramer system with matrix $(n+2, n+2)$ obtained by adding a first line of 1 to the matrix $M := (\alpha_1, \ldots, \alpha_{n+2})$, computing $a[1]$ and the $b_i[1]$ we find that $\sigma$ is the coefficient $(n+2, 1)$ in the matrix $M^{-1}$. Let $M'$ be the principal $(n+1, n+1)$ minor of $M$. This implies that
\[
\sigma = (-1)^{n+1} \frac{\text{det}(M')}{\text{det}(M)}.
\]

But using the relation $\alpha_{n+2} = \sum_{j=1}^{n+1} \frac{p_j}{r}.\alpha_j$ we obtain
\[
\text{det}(M) = (-1)^{n+1}.(1 - \sum_{j=1}^{n+1} \frac{p_j}{r}).\text{det}(M')
\]
so we conclude that
\[
\sigma = \frac{r}{r - \sum_{j=1}^{n+1} p_j}.
\]

Now we have two cases:

i) $r - \sum_{p_j < 0} p_j = d + h > \sum_{p_j > 0} p_j = d$. Then $r - \sum_{j=1}^{n+1} p_j = (d + h) - d = h$.

So $\sigma = r/h$, and the exponent of $\lambda$ in $P$ is $r$.

5The sign is precised in the proof and only depends on $\alpha_1, \ldots, \alpha_{n+2}$.
ii) $\sum_{p_j>0} p_j = d + h > r - \sum_{p_j<0} p_j = d$. Then $r - \sum_{j=1}^{n+1} p_j = d - (d + h) = -h$.

So $\sigma = -r/h$, and the exponent of $\lambda$ in $P$ is $-r$.

Consider now the case of a degree $k$ monomial $\mu \in \mathbb{C}[x_0, \ldots, x_n]$. Then there exists again $\sigma', \tau'$ in $\mathbb{Q}$ such that $(\sigma'.a + \tau'.b)[\mu] = [m_{n+2}.\mu]$ in $E_f$. As $a[\mu], (\beta_i + 1).b[\mu], i \in [0, n]$, where $\beta_i$ is the degree in $x_i$ of $\mu := x_\beta$, are again given from the $[m_j.\mu], j \in [1, n + 2]$ by the same Cramer system, we conclude that $\sigma' = \sigma$. To conclude the proof it is enough to apply the proposition 2.1.1.

Note that in the case i) above $P := P_{d+h} + c.\lambda^r.P_d$ annihilated $[\mu]$ in $E_f$ and in the case ii) we have $P := P_{d+h} + c.\lambda^{-r}.P_d$.

The lemma 2.2.1 gives that $\lambda.\nabla(P.[\mu]) = -(\sigma.a + (\tau' - k.\sigma).b).P[\mu]$ which makes explicit the fact that $\lambda.\nabla$ is well defined on $A[\lambda].[\mu] \subset E_f$.

Remark. Recall that in [B.13] we have built in an explicit way a differential equation in $s \in S$, depending in a very simple and concrete way on $\lambda \in \mathbb{C}^*$ which is satisfied by $\varphi$. So it is easy to see that the knowledge of a formal asymptotic expansion when $s$ goes to 0 in $S^\varnothing$ for a given $\lambda_0$, of the type

$$\varphi(\lambda_0, s) \simeq \sum_{i,j} C_{i,j}s^{\rho_i}.(\text{Logs})^j$$

where $\rho_1, \ldots, \rho_f$ are in $-1 + \mathbb{Q}^+, j \in [0, n]$ are integers and $C_{i,j}$ are in $\mathbb{C}[[s]]$, determines (uniquely) via the partial differential equation above, a formal expansion of the same type for each given $\lambda \in \Omega$, whose coefficients $C^\lambda_{i,j}$ are polynomials in $\text{Log}\lambda$ easily computable from the coefficients $C_{i,j}^{\lambda_0} := C_{i,j}$ of the asymptotic expansion at the initial value $\lambda_0$ of $\lambda$. This is described in the following lemma.

Lemma 2.2.3 Let $\Omega$ be a simply connected domain in $\mathbb{C}^*$. Let $(\rho_i)_{i \in I}$ be a finite collection of rational numbers strictly bigger than $-1$. Assume that the formal power serie

$$\varphi_\lambda := \sum_{k=0}^{N} \sum_{i \in I} \sum_{m \geq 0} c_{m}^{i,k}(\lambda).s^{m+\rho_i}.(\text{Logs})^k / k!$$

where $c_{m}^{i,k}$ are holomorphic functions in $\Omega$, satisfies the partial differential equation

$$\lambda.\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial s} \varphi_\lambda = \alpha.s \frac{\partial (\varphi_\lambda)}{\partial s} + \beta.\varphi_\lambda$$

for each $\lambda \in \Omega$. Then for each $i, k$ fixed, the function $c_{m}^{i,k}$ is a polynomial in $\text{Log}\lambda$ of degree $\leq m$ for each $m$. Moreover the collection of numbers $c_{m}^{i,k}(\lambda_0)$ for a given $\lambda_0 \in \Omega$ determines uniquely these polynomials.

---

6This is always the case when $S$ contains an open sector with edge at the origin.
Proof. The partial differential equation implies the following recursion relation for each \(i, k, m\):

\[
(m + \rho_i + 1) \lambda \frac{\partial c_{m+1}^{i,k}(\lambda)}{\partial \lambda} + \lambda \frac{\partial c_m^{i,k+1}(\lambda)}{\partial \lambda} = (\alpha(m + \rho_i) + \beta)c_m^{i,k}(\lambda) + \alpha c_m^{i,k+1}(\lambda)
\]

We shall make a descending induction on \(k\). For \(k = N\) the recursion relation reduces to

\[
(m + \rho_i + 1) \lambda \frac{\partial c_N^{i,N}(\lambda)}{\partial \lambda} = (\alpha(m + \rho_i) + \beta)c_N^{i,N}(\lambda)
\]

and an easy induction on \(m \geq 0\) gives our assertion for \(k = N\).

Assuming the statement proved for \(k + 1\) a simple quadrature in \(\lambda\) implies the case \(k\).

3 Two families of examples with \(d = 2\) and \(h = 1\).

3.1 The family \(x^{2u} + y^{2v} + z^{2w} + \lambda.x^u.y^v.z^w\).

The condition to be in our situation is \(u.v.w > 0\). Then we have the relation

\[m_4^2 = \lambda^2.m_1.m_2.m_3\]

and it shows that \(d = 2\) and \(h = 1\).

Note that the only singularity of \(f\) in \(\{f = 0\}\) is the origine.

To compute \(P := P_3 + c.\lambda^{-2}.P_2\) which annihilates \([1]\) is not difficult. We find

\[
P = (a - (2 + \frac{u + v}{2u.v})b)(a - (1 + \frac{u + w}{2u.w})b)(a - (\frac{v + w}{2v.w})b) + 4\lambda^{-2}.(a - (\frac{u.v + v.w + w.u}{2u.v.w})b)(a - \frac{u.v + v.w + w.u}{2u.v.w}b).
\]

In this case we have

\[
\lambda.\nabla([1]) = 2(a - (\frac{u.v + v.w + w.u}{2u.v.w})b)[1] = -m_4.
\]

Here we are in the case ii) above (so \(\sigma = -2\)).

Let me illustrate this family on a simple example: \(f = x^4 + y^4 + z^2 + \lambda.x^2.y^2.z\)

corresponding to \(u = v = 2, w = 1\). In this case we find

\[
P := (a - \frac{5}{2}b)[(a - \frac{7}{4}b)(a - \frac{3}{4}b) - 4\lambda^{-2}.(a - b)] \quad \text{and} \quad \lambda.\nabla([1]) = 2(a - b)[1].
\]

3.2 The family \(x^{2p}.z^u + y^{2q}.z^v + z^{u+v} + \lambda.x^p.y^q\).

The condition to be in our situation is \(p.q.(u + v) > 0\). Note that the singularity at the origine is not isolated in general in these cases. We have here the equality

\[2.\alpha_4 = \alpha_1 + \alpha_2 - \alpha_3\]
The relation which determines $P$ annihilating $[1]$ is given by $m_2^2.m_3 = \lambda^2.m_1.m_2$, so $r = 2, d = 2, h = 1$ and we are in the case i).

The computation of $P$ gives

$$P = (a - (2 + \frac{p + q}{2p.q}).b)(a - \frac{1}{2} + \frac{p.u + q.v + 2p.q}{2p.q.(u + v)}).b)(a - \frac{p.u + q.v + 2p.q}{2p.q.(u + v)}.b) +$$

$$- 4\lambda^2.(a - (1 + \frac{p.u + q.v + 2p.q + p.(u + v)}{2p.q.(u + v)}).b)(a - \frac{p.u + q.v + 2p.q + q.(u + v)}{2p.q.(u + v)}.b).$$

The computation of $(\sigma, \tau)$ such that $(\sigma.a + \tau.b)[1] = m_4$ is easy and it gives

$$\lambda.\nabla([1]) = -(2.a - \frac{p.u + q.v + 2p.q}{p.q.(u + v)}b) = -m_4.$$ 

Again let me illustrate by an example: for $p = q = 2$ and $u = v = 1$ so for $f = x^4.z + y^4.z + z^2 + \lambda.x^2.y^2$. We find

$$P := (a - \frac{5}{2}.b)(a - \frac{3}{4}.b) - 4\lambda^2.(a - 2b)(a - b) \quad \text{and} \quad \lambda.\nabla([1]) = -2(a - \frac{3}{4}.b)[1]$$

4 Appendix

It is interesting to remark that the proposition 2.1.1 is a special case in a specific algebraic setting of a general result on the filtered Gauss-Man connexion of a holomorphic function depending holomorphically on a parameter. This is the goal of this appendix to precise this point.

Let $M$ be a complex manifold, $D$ an open disc in $\mathbb{C}$ and let $f : D \times M \to \mathbb{C}$ be a holomorphic function. Denote $K_p^\lambda := \text{Ker}[d_f \wedge ; \Omega_D^p \to \Omega_D^{p+1}]$ for $p \geq 2$ and $K_1^\lambda := \text{Ker}[d_f \wedge : \Omega_1^0 \to \Omega_2^0]/\mathcal{O}.d_f$ where $d_f$ is the $\lambda-$relative differential of $f$ and $\Omega_1^0$ the sheaf of $\lambda-$relative holomorphic $p-$forms (compare with [B.08]).

Denote by $(K^\bullet_d, d_f)$ the topological restriction of the $\lambda-$relative de Rham complex (defined above) for the map $(\lambda, x) \mapsto (\lambda, f(\lambda, x))$, to the analytic subset

$$Z := \{d_f = 0\}$$

and let $\mathcal{H}$ the $p-$th cohomology sheaf of this complex. Recall that these cohomogy sheaves have a natural structure of left $\mathcal{A}[\lambda]-$modules with the action of $a$ given by the multiplication by $f$ and with the action of $b$ defined by $d_f \wedge d_f^{-1}$.

**Proposition 4.0.1** There exists a natural graded map $\nabla^\bullet : \mathcal{H}^\bullet \to \mathcal{H}^\bullet$ with the following properties:

1. For $\omega = d_f \xi \in K_1^{p+1} \cap \text{Ker } d_f$ we have $\nabla([\omega]) = [d_f \wedge \frac{\partial f}{\partial \lambda} - \frac{\partial f}{\partial \lambda} \omega].$
2. The map $b^{-1}_*\nabla$ well defined on $b_*\mathcal{H}^\bullet$ where $\mathcal{H}^\bullet := \mathcal{H}^\bullet/(b-\text{torsion})$, with value in $\tilde{\mathcal{H}}^\bullet$, commutes with $a$ and $b$ and is a $\lambda-$connection.

3. If $(\gamma_{s,\lambda})(s,\lambda) \in S \times \Omega$ is a horizontal family of compact $p-$cycles in the fibers of $(f, \lambda)$ over an open set in $D \times M \setminus C(f, \lambda)$ where $C(f, \lambda)$ is the set of critical values of the map $(\text{id}, f): D \times M \to D \times \mathbb{C}$, we have for any $\omega \in K_p^{p+1} \cap \ker d/f$ the equality

$$\frac{\partial}{\partial s} \frac{\partial}{\partial \lambda} \left[ \int_{\gamma_{s,\lambda}} \frac{\omega}{d/f} \right] = \int_{\gamma_{s,\lambda}} \frac{\nabla(\omega)}{d/f}.$$ 

**Proof.** First remark that if $\omega = d/\xi \in K_p^{p+1}$ with $d/f \wedge \xi = 0$, we have $d/\left(\frac{\partial f}{\partial \lambda}\right) \wedge \xi + d/f \wedge \frac{\partial \xi}{\partial \lambda} = 0$ so $\nabla(d/\xi) = -d/\left(\frac{\partial f}{\partial \lambda}\right)$ is in $d/f K^p$.

Now for $\omega = d/\xi \in K_p^{p+1}$ we have $d/f \wedge \nabla(d/\xi) = 0$ and

$$d_f(\nabla(d/\xi)) = -d/f \wedge d_f\left(\frac{\partial \xi}{\partial \lambda}\right) - d_f\left(\frac{\partial f}{\partial \lambda}\right) \wedge d/\xi$$

and we obtain that

$$d_f(\nabla(d/\xi)) = -\frac{\partial}{\partial \lambda}(d/f \land \omega) = 0$$

using the fact that $\frac{\partial}{\partial \lambda}(d/f \land \omega) \equiv 0$.

The proof of the other statements are analogous to the corresponding ones in proposition 2.1.1. 

**Remarks.**

1. As above we have a more precise formulation for the properties in assertion 2. of the proposition above with the following relations in $\mathcal{H}^\bullet$

$$\nabla(\varphi.\omega) = \frac{\partial \varphi}{\partial \lambda} b.\omega + \varphi.\nabla(\omega) \quad \text{for} \quad \varphi \in \mathcal{O}_\lambda \quad \text{and} \quad \omega \in \mathcal{H}^\bullet$$

$$b.\nabla = \nabla b \quad \text{and} \quad \nabla a = (a - b).\nabla.$$ 

2. The generalization of this proposition to several holomorphic parameters is immediate.

**Bibliography.**

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