Recognizing \([h,2,1]\) graphs

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Abstract

An \((h,s,t)\)-representation of a graph \(G\) consists of a collection of subtrees of a tree \(T\), where each subtree corresponds to a vertex of \(G\) such that (i) the maximum degree of \(T\) is at most \(h\), (ii) every subtree has maximum degree at most \(s\), (iii) there is an edge between two vertices in the graph \(G\) if and only if the corresponding subtrees have at least \(t\) vertices in common in \(T\). The class of graphs that have an \((h,s,t)\)-representation is denoted \([h,s,t]\).

An undirected graph \(G\) is called a VPT graph if it is the vertex intersection graph of a family of paths in a tree. In this paper we characterize \([h,2,1]\) graphs using chromatic number. We show that the problem of deciding whether a given VPT graph belongs to \([h,2,1]\) is NP-complete, while the problem of deciding whether the graph belongs to \([h,2,1] - [h-1,2,1]\) is NP-hard. Both problems remain hard even when restricted to \(\text{Split} \cap \text{VPT}\). Additionally, we present a non-trivial subclass of \(\text{Split} \cap \text{VPT}\) in which these problems are polynomial time solvable.

Key words: intersection graphs, VPT graphs, representations on trees, recognition problems.

1 Introduction

The intersection graph of a set family is a graph whose vertices are the members of the family, and the adjacency between them is defined by a

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non-empty intersection of the corresponding sets. Classical examples are interval graphs and chordal graphs.

An interval graph is the intersection graph of a family of closed intervals on the real line, or equivalently the intersection graph of a family of subpaths of a path. A chordal graph is a graph without induced cycles of length at least four. Gravril [4] proved that a graph is chordal if and only if it is the intersection graph of a family of subtrees of a tree, considering vertex intersection. Both classes has been widely studied [1].

In order to allow larger families of graphs to be represented by subtrees, several graph classes are defined imposing conditions on trees, subtrees and intersection sizes [11, 12]. An \((h,s,t)\)-representation of a graph \(G\) consists of a collection of subtrees of a tree \(T\), each subtree corresponding to a vertex of \(G\), such that (i) the maximum degree of \(T\) is at most \(h\), (ii) every subtree has maximum degree at most \(s\), (iii) there is an edge between two vertices in the graph \(G\) if and only if the corresponding subtrees have at least \(t\) vertices in common in \(T\). The class of graphs that have an \((h,s,t)\)-representation is denoted \([h,s,t]\). When there is no restriction on the degree of \(T\) or on the degree of the subtrees, we use \(h = \infty\) and \(s = \infty\) respectively. Notice that \([\infty, \infty, 1]\) is the class of chordal graphs; \([2, 2, 1]\) is the class of interval graphs; \([\infty, 2, 1]\) and \([\infty, 2, 2]\) are the well known \(VPT\) and \(EPT\) graphs [14].

In [3], the minimum \(t\) such that a given graph belongs to \([3,3,t]\) is studied. In [9], \([4,4,2]\) graphs are characterized and a polynomial time algorithm for their recognition is given. In [8], the class \([4,2,2]\) is studied. In [6], different aspects of \([\infty,2,t]\) graphs are considered. The relation between the different classes is analyzed in [7]. In [5], it is shown that the problem of recognizing \(VPT\) graphs is polynomial times solvable, but the recognition of \(EPT\) graphs is an NP-complete problem.

In this work we focuses in the classes \([h,2,1]\), all them are subclasses of \(VPT\). The problem is deciding whether a given \(VPT\) graph can be represented as intersection of paths in a tree with maximum degree \(h\). Since \([2,2,1] = \text{Interval} \) and \([3,2,1] = [3,2,2] = \text{EPT} \cap \text{chordal} \) [5], we consider \(h \geq 4\). We characterize \([h,2,1]\) graphs using chromatic number. We show that the problem of deciding whether a given \(VPT\) graph belongs to \([h,2,1]\) is NP-complete, while the problem of deciding whether the graph belongs to \([h,2,1] - [h-1,2,1]\) is NP-hard. Both problems remain hard even when restricted to \(\text{Split} \cap \text{VPT}\). Additionally, we present a non-trivial subclass of \(\text{Split} \cap \text{VPT}\) in which these problems are polynomial time solvable. In Section 2 we provide basics definitions and known results. In Section 3 we characterize \([h,2,1]\) graphs for \(h \geq 3\). In Section 4 we present the results about time complexity. Finally, in Section 5 we present some open questions.
2 Preliminaries

In this paper, all graphs are connected, finite, simple and loopless. Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$, the open neighborhood $N_G(v)$ of a vertex $v$ is the set of all vertices adjacent to $v$. The closed neighborhood $N_G[v]$ is $N_G(v) \cup \{v\}$. The degree of $v$, denoted by $d_G(v)$, is the cardinality of $N_G(v)$. For simplicity, when no confusion arise, we omit the subindex $G$ and simply write $N(v)$, $N[v]$ or $d(v)$.

For $S \subseteq V(G)$, $G[S]$ is the subgraph of $G$ induced by $S$; $G - S$ is a shorthand for $G[V(G) - S]$; and $G - v$ is used for $G - \{v\}$.

A complete set is a subset of vertices inducing a complete subgraph. A clique is a maximal complete set. The set of cliques of $G$ is denoted by $\mathcal{C}(G)$. A stable set is a subset of vertices pairwise non-adjacent.

The graph $G$ is split if $V(G)$ can be partitioned into a stable set $S$ and a clique $K$. The pair $(S,K)$ is the split partition of $G$. The vertices in $S$ are called stable vertices, and $K$ is called the central clique of $G$. We say that a stable vertex $s \in S$ is dominated if there exists $s' \in S$ such that $N(s) \subseteq N(s')$. Notice that if $G$ is split then $\mathcal{C}(G) = \{K,N[s] \text{ for } s \in S\}$.

A VPT-representation of $G$, denoted by $\langle P,T \rangle$, is an $(\infty,2,1)$-representation. This means that $\mathcal{P}$ is a family $(P_v)_{v \in V(G)}$ of subpaths of a host tree $T$ satisfying that two vertices $v$ and $v'$ of $G$ are adjacent if and only if $P_v$ and $P_{v'}$ have at least one vertex in common. If $q$ is a vertex of the host tree $T$, then $P[q]$ denote the set $\{P \in \mathcal{P} \mid q \in V(P)\}$ and $C_q$ denote the complete set $\{v \in V(G) \mid q \in V(P_v)\}$. Notice that for every clique $C$ of $G$, there exists $q \in V(T)$ such that $C = C_q$.

Definition 2.1 [5] Let $C \in \mathcal{C}(G)$. The branch graph of $G$ for the clique $C$ denoted by $B(G/C)$ is defined as follows: the vertex set $V(B(G/C))$ contains the vertices of $V(G) \setminus C$ adjacent to some vertex of $C$. Two vertices $v$ and $w$ are adjacent in $B(G/C)$ if and only if

1. $vw \notin E(G)$;
2. there exists a vertex of $C$ adjacent to both $v$ and $w$; and
3. there exist vertices $v'$ and $w'$ of $C$ such that $v'$ is adjacent to $v$ and non-adjacent to $w$, and $w'$ is adjacent to $w$ and non-adjacent to $v$.

Let $q \in V(T)$, with $N_T(q) = \{y_1,y_2,\ldots,y_h\}$. We call branches of $T$ at $q$ to the connected components of $V(T) - \{q\}$. Observe that each $y_i$ is contained in a different branch which will be called $T_i$. 

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The graph $G$ is $k$-colorable if its vertices can be colored with at most $k$ colors in such a way that no two adjacent vertices share the same color. The chromatic number of $G$, denoted by $\chi(G)$, is the smallest number of colors needed to coloring $G$.

**Theorem 2.1** [13] Let $G$ be a graph and $k \geq 3$. Deciding whether $G$ is $k$-colorable is an NP-complete problem.

A graph $G$ is perfect if and only if $G$ is $\{C_{2n+1}, \overline{C}_{2n+1}, \text{ with } n \geq 2 \}$-free [2].

**Theorem 2.2** [10] Let $G$ be a perfect graph and $k \geq 3$. Deciding whether $G$ is $k$-colorable is a polynomial time solvable problem.

### 3 Characterization of $[h,2,1]$, for $h \geq 3$

In this section we present a characterization of the $VPT$ graphs that can be represented in a tree with maximum degree at most $h$. The characterization is given in terms of the chromatic number of the branch graphs. The following three lemmas are fundamental tools in the proof of the main Theorems 3.1 and 3.2.

**Lemma 3.1** Let $(\mathcal{P}, T)$ be a $VPT$ representation of $G$. Let $C \in \mathcal{C}(G)$ and $q \in V(T)$ such that $C = C_q$. If $v \in V(B(G/C))$ then $P_v$ is contained in some branch of $T$ at $q$. If $v$ is adjacent to $w$ in $B(G/C)$ then $P_v$ and $P_w$ are not contained in a same branch of $T$ at $q$.

**Proof:** If $v \in V(B(G/C))$ then $v \notin C$. It follows that $q \notin V(P_v)$, thus $P_v$ is contained in some branch $T_i$ of $T$ at $q$. Let $w \in N_{B(G/C)}(v)$ and assume for a contradiction that $P_v$ and $P_w$ are contained in the same branch $T_i$. Let $x$ and $y$ be the vertices of $P_v$ and $P_w$ respectively at minimum distance from $q$. Since there exists a vertex of $C$ adjacent to $v$ and $w$, there exists a path in $T$ containing $q$, $x$ and $y$. We can assume, without loss of generality, that $x$ is between $q$ and $y$ or that $x = y$. In both cases, $N(w) \cap C \subseteq N(v) \cap C$. This contradicts the fact that $v$ and $w$ are adjacent in the branch graph. □

**Lemma 3.2** Let $(\mathcal{P}, T)$ be a $VPT$ representation of $G$. Let $C \in \mathcal{C}(G)$ and $q \in V(T)$ such that $C = C_q$. If $d_T(q) = h$, then $B(G/C)$ is $h$-colorable.
Proof: Let \( T_1, T_2, \ldots, T_h \) be the branches of \( T \) at \( q \). By Lemma 3.1, if we color each vertex \( v \) of \( B(G/C) \) with the index \( i \) of the branch \( T_i \) containing \( P_v \), then we obtain a proper coloring of \( B(G/C) \). Since there are \( h \) branches, \( B(G/C) \) is \( h \)-colorable. \( \square \)

Lemma 3.3 Let \( \langle P, T \rangle \) be a VPT representation of \( G \). Consider \( q \in V(T) \) with \( d_T(q) = h \geq 4 \). Assume there exist \( y_1, y_2 \in N_T(q) \) such that for all \( v \in V(G) \), \( \{ y_1, y_2 \} \not\subseteq V(P_v) \). Then there exists a VPT representation \( \langle P', T' \rangle \) of \( G \) with \( V(T') = V(T) \cup \{ a_q \} \), \( a_q \not\in V(T) \), and

\[
d_{T'}(x) = \begin{cases} 
3, & \text{if } x = a_q \\
h - 1, & \text{if } x = q \\
d_T(x), & \text{if } x \in V(T') \setminus \{ q, a_q \}.
\end{cases}
\]

Proof: We obtain the \( \langle P', T' \rangle \) representation of \( G \) as follows (Please refer to Figure 1): the set of vertices of \( T' \) is \( V(T) \cup \{ a_q \} \), where \( a_q \) is a new vertex not in \( V(T) \). The set of edges is \((E(T) \setminus \{ y_1q, y_2q \}) \cup \{ y_1a_q, y_2a_q, qa_q \}\). Observe that the degree of each vertex \( x \in V(T') \) is the required in the statement of the present lemma.

Now we define the paths \( P'_v \) for \( v \in V(G) \): if \( y_1 \) and \( q \) or \( y_2 \) and \( q \) belong to \( V(P_v) \) then \( V(P'_v) = V(P_v) \cup \{ a_q \} \). In any other case, \( V(P'_v) = V(P_v) \). Since \( \{ y_1, q, y_2 \} \not\subseteq V(P_v) \), we have that each \( V(P'_v) \) induces a path in \( T' \). Moreover, since all the paths where vertex \( a_q \) was added had vertex \( q \) in common, it is clear that, for any pair of vertices \( v, w \in V(G) \), \( V(P_v) \cap V(P_w) \neq \emptyset \) if and only if \( V(P'_v) \cap V(P'_w) \neq \emptyset \). It follows that \( \langle P', T' \rangle \) is a VPT representation of \( G \) and the implication is proven. \( \square \)

Figure 1: The degree of \( q \) in the tree \( T \) on the left is \( h \). The degree of \( q \) in the tree \( T' \) on the right is \( h - 1 \).
Theorem 3.1 Let $G \in VPT$ and $h \geq 3$. The graph $G$ belongs to $[h, 2, 1]$ if and only if $B(G/C)$ is $h$-colorable for every $C \in C(G)$. The direct implication is true also for $h = 2$.

Proof: Let $(\mathcal{P}, T)$ be an $(h, 2, 1)$-representation of $G$ with $h \geq 2$. Assume $C \in C(G)$, then there exists $q \in V(T)$ such that $C = C_q$. Since $d_T(q) \leq h$, by Lemma 3.2 $B(G/C)$ is $h$-colorable.

The reciprocal implication for $h = 3$ was proven by Golumbic and Jamison in [5]; then we assume $h \geq 4$.

Let $(\mathcal{P}, T)$ be a $VPT$ representation of $G$. We will prove that $G$ admits an $(h, 2, 1)$-representation.

We proceed by induction on the number $k$ of vertices of $T$ whose degree exceeds $h$. If $k = 0$ we are done.

If $k > 0$, there exists a vertex $q$ of $T$ with degree $d > h$. Say $N_T(q) = \{y_1, y_2, \ldots, y_d\}$ and for every $i, 1 \leq i \leq d$, let $T_i$ be the branch of $T$ at $q$ containing the vertex $y_i$.

If by repeatedly applying the Lemma 3.3 we can obtain a $VPT$ representation $(\mathcal{P}', T')$ of $G$ with $d_{T'}(q) < h$, then the implication is proven by induction since no vertex of $T'$ increases its degree.

In other case, we can assume that for any pair of vertices $y_i, y_i'$ belonging to $N_T(q)$, there exists at least one $v \in V(G)$ such that $\{y_i, y_i'\} \subseteq V(P_v)$.

Notice that this implies that $C_q = \{v \in V(G)/q \in V(P_v)\}$ is a clique of $G$.

We will consider two cases.

Case 1: for every $i, 1 \leq i \leq d$, there exists $v_i \in V(G)$ such that $P_{v_i}$ is totally contained in the branch $T_i$ and $y_i \in V(P_{v_i})$. Observe that each $v_i$ must be a vertex of $B(G/C_q)$. Since $B(G/C_q)$ is $h$-colorable, we can partitioned the set $\{y_1, y_2, \ldots, y_d\}$ in $h$ subsets $D_j, 1 \leq j \leq h$, each one containing the vertices $y_i$ for which the associated vertex $v_i$ has color $j$.

We obtain a new $VPT$ representation $(\mathcal{P}', T')$ of $G$ as follows. The tree $T'$ is obtained from $T$ by means of the following procedure (in Figure 2 we offer an example): 1) remove the edges $qy_i, 1 \leq i \leq d$; 2) add $h$ new vertices $\mu_j, 1 \leq j \leq h$; 3) add the edges $q\mu_j, 1 \leq j \leq h$; and finally, to connect the vertices $\mu_j$ with the vertices $y_i$, 4) add for every $j, 1 \leq j \leq h$, a binary tree rooted at the vertex $\mu_j$ and with the vertices of $D_j$ as leaves. The rest of the tree $T$ remains unchanged.

The only paths of $\mathcal{P}$ which are modified to obtain the paths of $\mathcal{P}'$ are those $Q \in P[q]$. If $Q$ has $q$ as an endpoint, then we obtain $Q'$ by replacing in $Q$ the edge $qy_i$ by the unique subpath of $T'$ linking $q$ and $y_i$. If $Q$ has $q$ as an internal vertex, then there exist $i$ and $i'$ such that $Q$ contains the edges $qy_i$
and \(qy_i\). Notice that the existence of \(Q\) implies that \(v_i\) and \(v'_i\) are adjacent in \(B(G/C_q)\); thus they have different colors, say \(j\) and \(j'\). Therefore, we obtain \(Q'\) by replacing in \(Q\) the edges \(qy_i\) and \(qy_i'\) by the only subpath of \(T'\) linking \(y_i\), \(q\) and \(y_i'\).

It is easy to see that this construction leaves the intersection graph of paths unchanged while reducing the number of tree vertices of degree greater than \(h\). So, by induction, the implication is proven.

Case 2: there exists \(i\), \(1 \leq i \leq d\), such that every path \(P \in \mathcal{P}\) containing \(y_i\) is not contained in the branch \(T_i\). Thus, every path \(P \in \mathcal{P}\) containing \(y_i\) contains also \(q\). Therefore, we can contract the edge \(qy_i\) to obtain a new VPT representation of \(G\) and repeat this as many times as needed to get a representation which is in Case 1. Notice that in this procedure some vertices of \(T\) disappear, and that the degree of \(q\) may increase, but the number of vertices whose degree exceeds \(k\) does not grow, thus the proof follows by induction as in the previous case.

\[\square\]

Figure 2: \(d_T(q) = 7\) and \(B(G/C_q)\) is 4-colorable.

Observe that the reciprocal implication of Theorem 3.1 is false for \(h = 2\); consider, by instance, the graph \(T^3_2\) which consists of one central vertex and 3 edge disjoint paths of 2 edges each intersecting only on the central vertex. It is easy to see that \(T^3_2 \in \text{VPT}\) and \(B(G/C)\) is 2-colorable for all \(C \in \mathcal{C}(G)\), but \(T^3_2 \notin [2, 2, 1]\).

**Theorem 3.2** Let \(G \in \text{VPT}\) and \(h \geq 4\). The graph \(G\) belongs to \([h, 2, 1] - [h - 1, 2, 1]\) if and only if \(\text{Max}_{C \in \mathcal{C}(G)}(\chi(B(G/C))) = h\). The reciprocal implication is also true for \(h = 3\).
Proof: By Theorem 3.1, \( G \in [h, 2, 1] \) if and only if \( \text{Max}_{C \subseteq C(G)} (\chi(B(G/C))) \leq h \). On the other hand, by the same Theorem 3.1, \( G \notin [h-1, 2, 1] \) if and only if \( \text{Max}_{C \subseteq C(G)} (\chi(B(G/C))) > h - 1 \). Therefore, the proof follows. \[ \square \]

4 Complexity

In this Section we prove that the problem of deciding whether a given graph belongs to \([h, 2, 1]\) for \( h \geq 3 \) is NP-complete. We also show that recognizing \([h, 2, 1] - [h-1, 2, 1]\) for \( h \geq 4 \) is NP-hard. Our results prove that both problems remain difficult even when restricted to the class \( VPT \cap \text{Split} \) without dominated stable vertices.

First we state the following fundamental property of \( VPT \cap \text{Split} \) graphs which is used in the proof of Theorems 4.1 and 4.2.

Lemma 4.1 Let \( s \) be a stable vertex of a \( VPT \cap \text{Split} \) graph \( G \). The branch graph \( B(G/N[s]) \) is 1-colorable.

Proof: Let \( \langle P, T \rangle \) be a \( VPT \) representation of \( G \) such that \( P_s \) is a one vertex path in a leaf \( y \) of \( T \), in other words \( V(P_s) = \{ y \} \) where \( y \) is a leaf of \( T \). Thus \( N[s] \) is the clique \( C_y \). Since \( d_T(y) = 1 \), by Lemma 3.2 \( B(G/N[y]) \) is 1-colorable. \[ \square \]

For the NP-completeness proof, we use a reduction from the chromatic number problem [13].

Given a graph \( G \) we construct in polynomial time a graph \( \hat{G} \in VPT \cap \text{Split} \) without dominated stable vertices, in such a way that \( \chi(G) = h \) if and only if \( \hat{G} \in [h, 2, 1] - [h-1, 2, 1] \).

Let \( V(G) = \{ v_1, v_2, ..., v_n \} \), we define the graph \( \hat{G} \) by means of its \( VPT \) representation \( \langle P, T \rangle \) as follows: the tree \( T \) is a star with central vertex \( q \) and leaves \( y_i \) for \( 1 \leq i \leq n \).

The path family \( P \) contains: a one vertex path \( P_i \) with \( V(P_i) = \{ y_i \} \), for each \( 1 \leq i \leq n \); a path \( P_{ij} \) with \( V(P_{ij}) = \{ y_i, q, y_j \} \), for each \( 1 \leq i < j \leq n \) such that \( v_i v_j \in E(G) \); a path \( P_{iq} \) with \( V(P_{iq}) = \{ q, y_i \} \), for each \( 1 \leq i \leq n \) such that \( d_G(v_i) = 1 \).

We call each vertex of \( \hat{G} \) as the corresponding path of \( P \).

In Figure 3 we offer an example of a graph \( G \), the \( VPT \) representation of \( \hat{G} \) and the graph \( \hat{G} \) obtained.
Figure 3: A graph $G$, the $VPT$ representation of $\hat{G}$ and the graph $\hat{\hat{G}}$.

Notice that $\hat{G}$ is a split graph with the vertex set partitioned in a stable set of size $n = |V(G)| + \{v \in V(G) / d_G(v) = 1\}$ corresponding to the one vertex paths $P_i$; and a central clique of size $|E(G)| + \{v \in V(G) / d_G(v) = 1\}$ corresponding to the remaining paths, all of which contain the vertex $q$ of $T$, thus this clique is $C_q$. The other cliques of $\hat{G}$ are the cliques $C_{y_i}$ for $1 \leq i \leq n$ each one corresponding to the paths containing the vertex $y_i$ of $T$ respectively. The graph $\hat{\hat{G}}$ has no more cliques. In addition, every stable vertex $P_i$ of $\hat{G}$ is non-dominated.

Observe that the branch graphs $B(\hat{\hat{G}}/C_{y_i})$ are described in Lemma 4.1, the following claim does for $B(\hat{\hat{G}}/C_q)$.

**Claim 4.1** If $\hat{G}$ is the graph obtained from $G$ as above, then $B(\hat{\hat{G}}/C_q) = G$.

**Proof:** Notice that $B(\hat{\hat{G}}/C_q)$ has exactly $n$ vertices: $P_i$ for $1 \leq i \leq n$.

We will see that $P_i$ and $P_j$ are adjacent in $B(\hat{\hat{G}}/C_q)$ if and only if $v_i$ and $v_j$ are adjacent in $G$. If $P_iP_j \in E(B(\hat{\hat{G}}/C_q))$ then there exists a vertex of $C_q$ adjacent to both $P_i$ and $P_j$. Then, there exists a path $P_{ij} \in \mathcal{P}$, thus $v_i v_j \in E(G)$. Reciprocally, let $v_i v_j \in E(G)$. Notice that $P_i$ and $P_j$ are non-adjacent in $\hat{\hat{G}}$; and $P_{ij}$ is a vertex of $C_q$ adjacent to $P_i$ and to $P_j$ in $\hat{\hat{G}}$. Let us see that there exists a vertex of $C_q$ adjacent to $P_i$ and non-adjacent to $P_j$. Indeed, if $d_G(v_i) = 1$ then the wanted vertex of $C_q$ is $P_{iq}$. If $d_G(v_i) > 1$ then $v_i$ must have a neighbor $v_l$ with $l \neq j$, thus the wanted vertex of $C_q$ is $P_{il}$. In an analogous way, there exists a vertex of $C_q$ adjacent to $P_j$ and non-adjacent to $P_i$. We have proved that $P_i$ and $P_j$ are adjacent in $B(\hat{\hat{G}}/C_q)$. We conclude that $B(\hat{\hat{G}}/C_q) = G$. \qed
The reduction from chromatic number is complete using the next claim.

Claim 4.2. Let $\hat{G}$ be the graph obtained from $G$ as above and $h \geq 4$. The graph $\hat{G}$ belongs to $[h, 2, 1] - [h - 1, 2, 1]$ if and only if $\chi(G) = h$.

Proof: By Lemma 4.1 and Claim 4.1, $\max_{C \in \mathcal{C}(\hat{G})} \chi(B(\hat{G}/C)) = \chi(B(\hat{G}/C)) = \chi(G)$. Hence, by Theorem 3.2, $\hat{G}$ belongs to $[h, 2, 1] - [h - 1, 2, 1]$ if and only if $\chi(G) = h$. \qed

We have proved the following theorem.

Theorem 4.1. Let $G \in VPT \cap \text{Split}$ without dominated stable vertices, and $h \geq 4$. Decide whether $G \in [h, 2, 1] - [h - 1, 2, 1]$ is an NP-hard problem.

In addition, since an $(h, 2, 1)$-representation is a polynomial certificate of belonging to $[h, 2, 1]$; using Theorem 3.1 and the construction above, we have proved the following result.

Theorem 4.2. Let $G \in VPT \cap \text{Split}$ without dominated stable vertices, and $h \geq 3$. Decide whether $G \in [h, 2, 1]$ is an NP-complete problem.

We notice that Theorem 4.2 for $h = 3$ has been previously proved in [5].

4.1 A polynomial time solvable subclass

We have proved that deciding whether a given $VPT \cap \text{Split}$ graph without dominated stable vertices admits a representation as intersection of paths of a tree with maximum degree $h$ is an NP-complete problem. In what follows we describe a non-trivial subclass of $VPT \cap \text{Split}$ without dominated stable vertices where the problem is polynomial time solvable.

For $n \geq 4$, a $n$-sun, denoted by $S_n$, is a split graph with stable set $\{s_1, s_2, \ldots, s_n\}$, central clique $\{v_1, v_2, \ldots, v_n\}$, $N(s_i) = \{v_i, v_{i+1}\}$ for $1 \leq i \leq n-1$, and $N(s_n) = \{v_n, v_1\}$. See Figure 4.

Let $G$ be a split graph with partition $(S, K)$. We say that $G$ belongs to $SVS$ (special $VPT$ subclass) whenever

- $G \in VPT$,
- for all $v \in K$, $|N(v) \cap S| \leq 2$, and
- if $S_k$, with $k \in \{4, 2n + 1 \text{ for } n \geq 2\}$, is induced in $G$ then there exists $v \in K$ such that $v$ is adjacent to two non-consecutive vertices of the stable set of $S_k$. 

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The class $SVS$ is not trivial, in the sense that it includes graphs in $[h, 2, 1]$ for all $h \geq 4$.

For example, let $n \geq 4$ and let $A_n$ (see [7]) be a split graph with partition $(S, K)$, where $S = \{s_1, ..., s_n\}$, $K = \{v_{ij} \mid 1 \leq i < j \leq n\}$ and $N(v_{ij}) = \{s_i, s_j\}$, for all $1 \leq i < j \leq n$. It is clear that $A_n$ belongs to $SVS$, and $B(A_n/K) = K_n$ with $V(K_n) = \{s_1, ..., s_n\}$. Hence, by Theorem 3.2, $A_n \in [n, 2, 1] - [n-1, 2, 1]$. (As an example see Figure 5).

The following two lemmas are used in the proof of the main Theorem 4.3 which proves that in the class $SVS$ the graphs belonging to $[h, 2, 1]$ can be recognized efficiently.

**Lemma 4.2** Let $G \in VPT \cap Split$ with partition $(S, K)$ such that for all $v \in K$, $|N(v) \cap S| \leq 2$, and let $n \geq 4$. If $B(G/K)$ has an induced $C_n$ then $G$ has an induced $S_n$. 

Figure 4: The sun graphs $S_4$, $S_5$ and $S_7$.

Figure 5: The graph $A_4$ belongs to $SVS$ and $A_4 \in [4, 2, 1] - [3, 2, 1]$.
Proof: Let \( \langle P, T \rangle \) be a VPT representation of \( G \) and \( q \in V(T) \) such that \( K = C_q \). Let \( C_n \) be an induced cycle of \( B(G/K) \) with vertices \( s_1, s_2, ..., s_n \). It is clear that every \( s_i \in S \). Since \( s_i \) is adjacent to \( s_{i+1} \) in \( B(G/K) \), there exists \( v_i \in K \) such that \( v_i \) is adjacent to \( s_i \) and to \( s_{i+1} \) in \( G \). Since, for all \( v \in K, |N(v) \cap S| \leq 2 \), if \( i \neq i' \) then \( v_i \neq v_i' \), thus \( s_1, s_2, ..., s_n, v_1, v_2, ..., v_n \) induce a \( n \)-sun in \( G \) and the proof is completed.

\[ \Box \]

**Lemma 4.3** If \( G \in SVS \) then every branch graph of \( G \) is perfect.

Proof: Let \( (S, K) \) be a split partition of \( G \). By Lemma 4.1, if \( s \in S \) then \( B(G/N[s]) \) is perfect. Assume for a contradiction that \( B(G/K) \) is not perfect, then \( B(G/K) \) has induced an odd cycle or the complement of an odd cycle. Since the complement of \( C_5 \) is \( C_5 \); and the complement of any odd cycle of size 7 or more has an induced \( C_4 \), it follows that \( B(G/K) \) has an induced \( C_k \), for some \( k \in \{4, 2n + 1 \text{ for } n \geq 2\} \). Therefore, by Lemma 4.2 \( G \) has an induced \( S_k \). Since \( G \in SVS \), there exists \( v \in K \) such that \( v \) is adjacent to two non-consecutive vertices \( s \) and \( s' \) of the stable set of \( S_k \). Notice that the existence of \( v \) implies that the vertices \( s \) and \( s' \) are adjacent in \( B(G/K) \). This contradicts the fact that \( C_k \) is an induced cycle of \( B(G/K) \).

\[ \Box \]

**Theorem 4.3** Let \( G \in SVS \) and \( h \geq 4 \). Decide whether \( G \) belongs to \( [h, 2, 1] - [h - 1, 2, 1] \) is polynomial time solvable.

Proof: Given \( G \in SVS \), in order to determinate if \( G \in [h, 2, 1] - [h - 1, 2, 1] \), by Theorem 3.1, it is enough to calculate the chromatic number of \( B(G/K) \), where \( K \) is the central clique of \( G \). Notice that the branch graph \( B(G/K) \) can be obtained in polynomial time. On the other hand, by Lemma 4.3 \( B(G/K) \) is perfect. Thus, by Theorem 2.2, its chromatic number can be calculated in polynomial time.

\[ \Box \]

5 Future work

In this paper we give a characterization of the \( [h, 2, 1] \) graphs, with \( h \geq 3 \). In addition, we prove that recognizing this class is NP-complete and show a family, called \( SVS \), in which this problem is polynomial time solvable.

We are working in describing a larger subclass of VPT graphs where this
problem remains polynomial. On the other hand, we are analyzing the possibility of extending the techniques used in the present paper to characterize the classes \([h, 2, 2]\).

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