Correlation Functions And Multicritical Flows
In $c < 1$ String Theory

Suresh Govindarajan
Theoretical Physics Group
Tata Institute of Fundamental Research
Bombay 400 005 INDIA
T. Jayaraman and Varghese John
The Institute of Mathematical Sciences
C.I.T. Campus, Taramani
Madras 600 113, INDIA

We compute all string tree level correlation functions of vertex operators in $c < 1$
string theory. This is done by using the ring structure of the theory. In order to study
the multicritical behaviour, we calculate the correlation functions after perturbation by
physical vertex operators. We show that the $(2k - 1, 2)$ models can be obtained from the
$(1, 2)$ model and the minimal models can be obtained from the $(1, p)$ model by perturbing
the action by appropriate physical operators. Our results are consistent with known results
from matrix models.

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1 email: suresh@theory.tifr.res.in
2 email: jayaram, john@imsc.ernet.in
1. Introduction

The complete solvability of the matrix models has provided string theory with the perfect scenario in which to study various non-perturbative and stringy effects. However in order to translate this success into the context of a generic string theory for which a matrix model formulation may not exist, one needs to understand the matrix models in the continuum language. Hence, this has led to an extensive study of the continuum formulation of matrix models, i.e., $c < 1$ matter coupled to Liouville gravity \[1\]-\[3\]. The algebraic structure of the $c = 1$ string theory i.e, the presence of the ground ring has been translated for the $c < 1$ case in \[14\]-\[17\]. In this paper, we continue the progress of earlier work and obtain all the correlation functions on the sphere for all minimal models coupled to Liouville gravity. We also reproduce the multicritical behaviour seen in the matrix models.

The analysis of non-critical strings in the conformal gauge (using the DDK ansatz\[1\]) was able to predict the susceptibility exponent and the area scaling exponents of gravitationally dressed primary fields in the theory. Subsequently, the three point functions of vertex operators in the theory were obtained by doing the zero mode integration over the Liouville field and then evaluating the three point functions using an analytic continuation in the number of cosmological constant operator insertions \[3\]-\[6\]-\[7\]. To obtain all the matrix model three point functions the vertex operator states with momenta outside the conformal grid had to be included.

The cohomology analysis required to obtain the physical states of the theory was done by Lian and Zuckerman\[18\] and Bouwknegt et al \[19\]. Their analysis showed that there was an infinite number of physical states of non-trivial ghost number. Half of these states (with $\beta < \beta_0$) had scaling exponents which matched those seen in the matrix models. In \[12\], it was shown that the states of non-trivial ghost number were related to pure vertex operator states (DK states) using descent equations arising from the double cohomology used to obtain the physical states. This explains the presence of states outside the Kac’s table.

For $c = 1$ matter coupled to Liouville gravity, it was shown by Witten\[20\] that there exists a ring of operators of zero ghost number with a multiplication rule given by the usual OPE. In the context of $c < 1$ matter coupled to Liouville gravity, Kutasov et. al. obtained a finite ring of ghost number zero operators\[14\]. This ring was extended in \[17\] to an infinite ring of operators at all ghost numbers. However, in \[14\]-\[17\], the matter
sector was represented by the Verma module of the minimal models. If the matter sector is represented by its free field realisation à la Feigin-Fuchs, the ring of ghost number zero operators is infinite as shown in [16]. The states belonging to the edge of the Kac table whose Liouville scaling exponents are observed in the matrix models do not occur in the formalism of [14][17]. These states do not decouple in correlation functions[4][3] and hence have to be included. When the free field realisation of the minimal models is used, the edge states have to be included. Hence, the infinite ring structure of [16] has all the exponents seen in the matrix models. In addition, the ring is identical to that seen in the case of topological minimal matter coupled to topological gravity.

Having obtained the ring structure of these models, it is natural to see if it is possible to obtain all the correlation functions in the theory using the ring. In this paper, we will discuss in detail how the ring structure can be used to obtain all the genus zero correlation functions in the $c < 1$ models[23]. The approach we adopt here is quite general and can be applied to other problems which have the ring structure such as the non-critical superstring, the W-string, and the $SL(2, \mathbb{R})/U(1)$ (black-hole) coset model. We will also show that it is possible to study the behaviour of the correlation functions under perturbation by physical operators in the theory and determine their behaviour as function of the coupling constants; and that it is possible to obtain multicritical behaviour in these models which generalises the results of Distler[24].

The paper is organised as follows. In section 2, we describe the vertex operator states and the ring of a generic $c < 1$ model coupled to Liouville gravity. In section 3, we compute the correlation functions for the $(2k - 1, 2)$ models. In section 4, we study the $(p + 1, p)$ minimal models coupled to gravity. We also compare the results with known KdV/matrix model results. Finally, in section 5, we reproduce the multicritical behaviour seen in the matrix models using the correlation functions computed in the earlier sections. This completes the proof of equivalence of the matrix models and the continuum formulation at the string tree level.

\[3\] It was also suggested in [15][21][22] that a $SO(2, \mathbb{C})$ rotation of the fields was enough to relate the $c < 1$ and the $c = 1$ models coupled to gravity. As was pointed out in [16], this is not sufficient since the role of the Felder BRST operator must be taken into account. This crucially differentiates the properties of a rotated $c = 1$ model and the $c < 1$ model coupled to Liouville gravity.
2. Physical States And Rings

Let us consider the \((p',p)\) model coupled to gravity. We consider two scalars \(X\) (for matter) and \(\phi\) (for the Liouville mode) with background charges \(\alpha_0\) and \(\beta_0\) respectively at infinity. The corresponding energy-momentum tensors are given by

\[
T^M = -\frac{i}{4} \partial X \partial X + i \alpha_0 \partial^2 X ,
\]
\[
T^L = -\frac{i}{4} \partial \phi \partial \phi + i \beta_0 \partial^2 \phi ,
\]

with central charges \(c_M = 1 - 24 \alpha_0^2\) and \(c_L = 1 - 24 \beta_0^2\). For the \((p',p)\) minimal models

\[
\alpha_0^2 = \frac{(p'-p)}{4pp'} \quad \text{and} \quad \beta_0^2 = -\frac{(p+p')^2}{4p(p+1)} .
\]

The vertex operators \(e^{i\alpha X}\) and \(e^{i\beta \phi}\) have conformal weights \(\alpha(\alpha - 2 \alpha_0)\) and \(\beta(\beta - 2 \beta_0)\) respectively. The gravitationally dressed vertex operators are of the form \(e^{i\alpha X + i\beta \phi}\) with conformal dimensions given by \(\alpha(\alpha - 2 \alpha_0) + \beta(\beta - 2 \beta_0) = 1\). The usual matter screening operators are given by :

\[
Q_+ = \Delta(1 + \frac{p'}{p}) \int e^{i\alpha_+ X}, \quad Q_- = \Delta(1 + \frac{p}{p'}) \int e^{i\alpha_- X}
\]

where

\[
\alpha_+ = \sqrt{\frac{p'}{p}} \quad \text{and} \quad \alpha_- = -\sqrt{\frac{p}{p'}} .
\]

and \(\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)}\). (note the measure factor \(\frac{d^2z}{2\pi}\) is implicit. The normalisation of the operators is explained later.)

Felder\[25\], has shown that the screening operators have a BRST like action which enables one to truncate the large space of Virasoro primaries of a scalar field to the finite set of primaries seen in the minimal models. Before coupling to gravity, the two screening operators play identical roles. However, on coupling to gravity this duality is broken\[23\]. The reason for this is as follows. Suppose we choose to use the screening operator \(Q_+\) to function as the BRST operator\[2\], then \(e^{i\alpha X}\) becomes a physical operator. Hence, one can no longer use \(Q_-\) as a screening operator. Further, the spectrum of physical states in the two resolutions are seen to be different\[23\]. This is not surprising. In the matrix model, the \((p',p)\) model can be obtained in two different ways – as a the \(p\)-th critical point of the

\[\text{This will be referred to as the } Q_+ \text{ resolution. The choice of } Q_- \text{ as the Felder BRST gives the } Q_- \text{ resolution.}\]
matrix model or the $p'$-th critical point of the $(p - 1)$ matrix model. The spectrum of the two resolutions agree with those seen in the two different matrix models. Later, we shall explicitly demonstrate this difference.

It has been shown\[12\][16], that using the descent equations arising from the double cohomology of the string and Felder BRST operators that the physical operators with Liouville charge $\beta < \beta_0$ exist at two ghost numbers:

--- the ghost number 1 operators correspond to pure vertex operators of the form $c\bar{c}\exp(i\alpha X + i\beta \phi)$. These are the DK states\[6\][7]\[12\]. The allowed values of matter momenta take values both outside and inside the conformal grid (Kac Table).

--- the ghost number 0 operators which form the chiral ring\[16\].

2.1. DK States

The DK operators for $(p', p)$ models in the $Q_-$ resolution are given by

$$V_n^\alpha = \Delta(n + \frac{\alpha + 1}{p'}) \exp \left[ \frac{p(n - 1) + \alpha + 1 - p'}{2\sqrt{p'p}} \phi + \frac{-p'(n - 1) - \alpha - (1 + p)}{2\sqrt{p'p}} iX \right]$$

where $\alpha = 0, \ldots, (p - 2)$. As was explained in \[12\], these include all vertex operator states with matter momenta corresponding to values occuring to the left of the central Fock tower $F_{m', m}$ in the Felder complex (with $0 < m < p$, $0 < m' < p'$). The Fock tower labels are $F_{\pm m' + 2jp', m}$ for $j = 1, 2, \ldots$. The states corresponding to the edge of the Kac Table with labels $(jp', m)$ are included since they do not decouple after coupling to gravity\[7\]. However, in the $Q_-$ resolution, the other edge states with labels $(m', jp)$ are not included since they are (Felder)BRST exact. So unlike the states inside the Kac Table, the states which belong to the edge of the Kac table come with the resolution. In the $Q_+$ resolution, edge states with labels $(m', jp)$ are no longer (Felder)BRST exact. However, the other set of edges are now trivial. This leads to the following set of DK states in the $Q_+$ resolution.

$$V_n^\alpha = \Delta(n + \frac{\alpha + 1}{p'}) \exp \left[ \frac{p'(n - 1) + \alpha + (1 - p')\phi + [-p'(n - 1) - \alpha - (1 + p)] iX}{2\sqrt{p'p}} \right]$$

where $\alpha = 0, \ldots, (p' - 2)$.

All the operators are normalised by a momentum-dependant ratio of $\Gamma$ functions of the form $\Delta(n + \frac{\alpha + 1}{p})$ for the operators appearing in the $Q_-$ resolution and by $\Delta(n + \frac{\alpha + 1}{p'})$ for the operators appearing in the $Q_+$ resolution. (Notice that this normalisation is finite for all the allowed values of $\alpha$.) This normalisation removes all the external leg factors.
corresponding to the operators in the scattering amplitudes, and provides the correct normalisation to compare with the matrix models.

In the calculation of the correlation functions, we will use only physical operators that appear in one resolution. This is essential to obtain the right combinatorial factors required to reproduce the matrix model results. Another benefit of not mixing resolutions (as we shall see soon) is that the ring structure is identical to that seen in topological matter coupled to topological gravity. Thus it seems that on coupling to gravity the usual minimal model duality is lost. Furthermore, only one screening operator can be inserted to obtain charge conservation in the matter sector. Another interesting point to note here, is that the DK states on being Lorentz boosted to a compact $c = 1$ model, correspond to tachyons at a particular radius. However, the screening operator corresponds to a tachyon of a different radius indicating that this theory is not a Lorentz boosted $c = 1$ theory.

2.2. Ring Structure

Apart from the vertex operator states there are an infinite set of operators of zero ghost-number in the double cohomology, which form a ring under multiplication. The chiral ring is generated by two elements

\[ x = (b_{-2}c_1 + t(L_L^L - L_M^M)) e^{\frac{t}{\hbar}(-iX + \phi)} \]

\[ y = (b_{-2}c_1 + \frac{1}{t}(L_L^L - L_M^M)) e^{\frac{t}{\hbar}(iX + \phi)} \]  

(2.4)

where $t = \sqrt{\frac{\hbar}{p}}$. Again, just as in the case of the DK states, the ring obtained is dependent on the choice of resolution. For the $Q_-$ resolution, the chiral ring is given by

\[ \{x^m y^n \mid x^{(p'-2)} = 0\} \]  

(2.5)

It can be seen that $x^{(p'-2)}$ is $Q_F$-exact. However, this is no longer true if $Q_+$ is used as the Felder BRST operator. Now, $y^{p-2}$ is $Q_F$ exact. So the chiral ring in the $Q_+$ resolution is given by

\[ \{x^m y^n \mid y^{p-2} = 0\} \]  

(2.6)

The ring structure seen here is identical to that seen in the case of topological matter coupled to topological gravity. In the $Q_-$ resolution, the ring generated by $x$ is identical to that of topological matter with $y$ corresponding to the infinite ring of topological gravity. This is suggestive of an underlying twisted $N = 2$ supersymmetry in this theory.\(^5\)

\(^5\) See a recent paper by Bershadsky, et. al.\(^{[26]}\) where this has been explicitly demonstrated. Also, see \(^{[27]}\).
In [12], an equivalence relation had been obtained by demanding that both resolutions be identical. There, the role of the edge states was unclear and the decoupling of the so-called ‘wrong-edge’ was not obvious. However, by staying in one resolution as in this paper, these questions are easily answered as we have shown. The wrong-edge states are Felder-exact and hence do not belong to the set of physical states. This explains the relations (2.3) (2.6) as opposed to the equivalence relation obtained in [12].

So far we have introduced only the holomorphic part, however keeping in mind the fact that the Liouville boson is non-compact we require that the holomorphic-antiholomorphic Liouville momenta are equal. This leaves us with two elements:

\[ a_+ = x \bar{x}, \quad a_- = y \bar{y}, \]  

(2.7)

with the appropriate ring relation according to the resolution chosen. These ring elements through their action on the DK states provide relations between correlation functions using which we find some operator identities. These operator identities we will show (in the next section) are enough to determine the correlation functions completely.

3. Correlation Functions in \((2k - 1, 2)\) Models

In this section we will calculate the \(n\)-point correlation functions in the \((2k - 1, 2)\) models using the action of the rings on the Vertex operator states. The partition function is given by:

\[ Z = \int \mathcal{D}\phi \mathcal{D}X \mathcal{D}b \mathcal{D}c \ e^{-S_M - S_L - S_{gh} + \mu \int V_0} \]  

(3.1)

where the matter and Liouville theories are defined in terms of free bosons with charges at infinity given by:

\[ \alpha_0^2 = \frac{(2k-3)^2}{8(2k-1)} \quad \text{and} \quad \beta_0^2 = -\frac{(2k+1)^2}{8(2k-1)}. \]

Consider the correlation function

\[ \langle \langle \prod_{i=1}^{L} V_{n_i} \rangle \rangle = \mu^S \Gamma(-S) \langle c \bar{c} V_{n_1}(0) c \bar{c} V_{n_2}(1) c \bar{c} V_{n_3}(\infty) \prod_{i=4}^{L} V_{n_i} \rangle (\int V_0)^S \frac{1}{R_1} (Q_-)^R, \]  

(3.2)

where the average is with respect to the action defined in (3.1) The RHS is obtained after doing the zero mode integration over the Liouville field as was done in (3.1). The values of \(R, S\) are determined by the requirement of charge conservation, in the Liouville and matter
sectors. For the moment we will consider only the correlation functions that require a positive integer number of screening operators. After determining the dependence of the amplitudes on the screening operators we will analytically continue the results to evaluate the correlation functions that require insertion of negative integer or fractional number of screening operators for charge conservation. The vertex operators in the $Q_-$ resolution in this theory are given by:

$$V_n = \Delta(n + \frac{1}{2}) \exp \frac{(n - k)\phi + (n + k - 1)iX}{\sqrt{2(2k - 1)}} ,$$  \hspace{1cm} (3.3)

where $n = 0, 1, \ldots$. These models are simpler than the generic $(p + 1, p)$ models in the sense that there is only one allowed value of $\alpha$ and hence the algebra required to obtain the correlation functions is straightforward and transparent. Note that the “physical” screening operator $Q_+$ is given by $V_k$. The charge conservation relations for the correlation functions given in (3.2) are given by

$$\sum_{i=1}^{L} n_i - kL - Sk = -(2k + 1) ,$$

$$\sum_{i=1}^{L} n_i + kL - L + (k - 1)S + (2k - 1)R = 2k - 3 .$$  \hspace{1cm} (3.4)

The ring elements are generated by a single generator $a_-$, where

$$a_- = -|bc + \frac{1}{2} \sqrt{\frac{2k - 1}{2}} \partial(\phi - iX)|^2 \exp \frac{(\phi + iX)}{\sqrt{2(2k - 1)}} .$$  \hspace{1cm} (3.5)

(Since the Liouville field is treated as a non-compact boson with a charge at infinity the Liouville momenta in the holomorphic and anti-holomorphic sectors is taken to be the same.) The ring elements are $(a_-)^n$. The action of the ring element on the DK states is given by $[14][28][29]$

$$\lim_{z \to w} a_-(z)c\bar{c}V_n(w) \sim c\bar{c}V_{n+1}(w), \quad a_-(z)c\bar{c}e^{i\alpha-X}(w) \sim 0 .$$  \hspace{1cm} (3.6)

Now consider a charge conserving correlation function with one $a_-$ and DK states

$$F(w, \bar{w}) \equiv \langle a_-(w)c\bar{c}V_{n_1}(0)c\bar{c}V_{n_2}(1)c\bar{c}V_{n_3}(\infty) \prod_{i=4}^{N} \int V_{n_i} \frac{1}{R_i!}(Q_-)^{R_i} \rangle .$$  \hspace{1cm} (3.7)
where $\Delta(x)$ is a constant independent of $w$ and $\bar{w}$. Equating $F(0)$ with $F(1)$ and using (3.9), we obtain

\[
\langle c\bar{c}V_{n+1}(0)c\bar{c}V_{n_2}(1)c\bar{c}V_{n_3}(\infty) \prod_{i=1}^{N} \int V_{i} \frac{1}{R!}(Q^-)^R \rangle = \langle c\bar{c}V_{n_1}(0)c\bar{c}V_{n_2+1}(1)c\bar{c}V_{n_3}(\infty) \prod_{i=1}^{N} \int V_{i} \frac{1}{R!}(Q^-)^R \rangle .
\]

This gives us the following recursion relation (similar to the one in [24])

\[
V_{n+1}(z)V_{m}(w) = V_{n}(z)V_{m+1}(w) .
\]

In particular, $V_{n}V_{0} = V_{n-k}V_{k}$. This is sufficient to convert all correlation functions (with positive integer number of Liouville screening) to ones which are of the Dotsenko-Fateev type [30] making them computable. We can now use the recursion relation, to obtain

\[
\langle \prod_{i=1}^{N} \int V_{ni} \rangle = \mu^{S} \Gamma(-S) \langle c\bar{c}V_{0}(0)c\bar{c}V_{0}(1) c\bar{c}V_{k-1}(\infty) \int V_{2}^{0} \frac{1}{R!}(Q^-)^R \rangle \]

\[
= \mu^{S} \Gamma(-S) \mathcal{N} \prod_{i,j=1}^{X} \int d^{2}z_{i}d^{2}w_{j} |z_{i}|^{2(2k-1)}|1-z_{i}|^{2(2k-1)} \prod_{i<j} |z_{i}-z_{j}|^{2(2k-1)}
\]

\[
\times |w_{i}|^{-2(k-1)\rho} |1-w_{i}|^{-2(k-1)\rho} \prod_{i<j} |w_{i}-w_{j}|^{4\rho} \prod_{i,j} |z_{i}-w_{j}|^{-4}
\]

\[
= \mu^{S} \Gamma(-S) \mathcal{N} (X!) (\pi)^{X+R} (\rho)^{4RX} \Delta(1-\rho')^{X} \Delta(1-\rho)^{R}
\]

\[
\times \prod_{i=1}^{X} \Delta(i\rho' - R) \prod_{i=1}^{R} \Delta(i\rho)
\]

\[
\times \prod_{i=0}^{X-1} \Delta(k - R + i\rho) \Delta(k - R + i\rho') \Delta(R - 2k + 1) - (X - 1 + i)\rho'
\]

\[
\times \prod_{i=0}^{R-1} \Delta(1 - (k - 1)\rho + i\rho) \Delta(1 - (k - 1)\rho + i\rho') \Delta(2X - 1 - 2(k - 1)\rho - (R - 1 + i)\rho)
\]

(3.10)

where $\Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)}$; $X = L + S - 3$; $\rho = \frac{2}{2k-1}$; $\rho' = \frac{2k-1}{2}$ and $\mathcal{N}$ is the normalisation factor associated with the operators.

This correlator was evaluated using formula (B.10) of [30]. This gives

\[
\langle \prod_{i=1}^{L} \int V_{ni} \rangle = \rho \mu^{S} \Gamma(-S) (L + S - 3)! \frac{\Gamma(1)}{\Gamma(0)} = \rho \mu^{S} \frac{\Gamma(L + S - 2)}{\Gamma(S + 1)} ,
\]

8
where $\rho = \frac{2}{(2k-1)}$, and using (3.4)

$$S = (2 - L) + \frac{1}{k} \left( \sum_{i=1}^{L} n_i \right) + \frac{1}{k}$$

$$2R = 2 \left( \sum_{i=1}^{L} n_i \right) - (L + S - 4)$$

The cases of non-integer $S$ and $R$ are obtained by analytically continuing the above result to non-integer values. Hence, one obtains

$$\langle \langle \prod_{i=1}^{L} \int V_{n_i} \rangle \rangle = \rho \mu^S \frac{\Gamma \left( \frac{\sum_{i} n_i + 1}{k} \right)}{\Gamma (S + 1)} ,$$

(3.11)

This answer is the same as the matrix model results [31] [32] [24]. The use of the ring to obtain operator relations has simplified the computations enormously. These correlation functions have been subsequently calculated using different techniques in [33]. However, there the role of the Felder BRST charge, especially in the choice of resolution is not clear.

4. Correlation Functions In $(p + 1, p)$ Models

In this section we will consider the $(p + 1, p)$ models or the unitary minimal models coupled to gravity. Here the number of allowed values of $\alpha$ as introduced in (2.2) are $\alpha = 1, \ldots, p - 2$. (Notice that the number of allowed values of $\alpha$ is the same as the number of primary fields in the corresponding topological sigma models coupled to gravity.) The DK states in the $Q_-$ resolution are

$$V_n^\alpha = \Delta (n + \frac{(\alpha + 1)}{p}) \exp \left[ \frac{p(n - 2) + \alpha}{2} \phi + \frac{pn + \alpha + 2}{2} iX \right] \frac{\sqrt{p}}{p(p + 1)}$$

where $\alpha = 0, \ldots, (p - 2)$.

The ring elements for these models are generated by (4.1)

$$a_+ = -|bc + \frac{1}{2} \sqrt{\frac{p}{p + 1}} \partial (\phi + iX)|^2 \exp \frac{p + 1(\phi - iX)}{2 \sqrt{p(p + 1)}},$$

$$a_- = -|bc + \frac{1}{2} \sqrt{\frac{p + 1}{p}} \partial (\phi - iX)|^2 \exp \frac{p(\phi + iX)}{2 \sqrt{p(p + 1)}}$$

(4.1)

We have restricted our discussion to the unitary minimal model for simplicity. The extension to the arbitrary $(p', p)$ model is straightforward.
The ring elements in the $Q_-$ resolution are

$$(a_-)^n, \ a_+(a_-)^n, \ldots, \ (a_+)^{p-2}(a_-)^n \ . \tag{4.2}$$

with the relation $a_+^{p-1} = 0$. The action of the ring on DK states is

$$\lim_{w \to z} a_-(w)ccV_n^\alpha(z) \sim ccV_n^{\alpha+1}(z) \ ,$$

$$\lim_{w \to z} a_+(w)ccV_n^\alpha(z) \int V_m^\beta(t) \sim ccV_{n+m-1}^{\alpha+\beta+1}(z) \ , \tag{4.3}$$

where we have normalised all the DK states by their appropriate leg-factors. The n-point functions can be obtained as before (after doing the zero mode integration) we have

$$\langle \prod_{i=1}^N \int V_{n_i}^{\alpha_i} \rangle = \mu^S \Gamma(-S) \langle ccV_{n_1}^{\alpha_1}(0)ccV_{n_2}^{\alpha_2}(1)ccV_{n_3}^{\alpha_3}(\infty) \prod_{i=3}^N \int V_{n_i}^{\alpha_i}(\int V_0^0)^S \frac{1}{R!}(Q_-)^R \rangle \tag{4.4}$$

The charge conservation relations are

$$p \sum_{i=1}^N n_i + \sum_{i=1}^N \alpha_i - 2pL - 2pS = -(4p + 2) \ , \text{ and} \tag{4.5}$$

$$p \sum_{i=1}^N n_i + \sum_{i=1}^N \alpha_i + 2L + 2S - 2pR = 2 \ .$$

Now we can use the ring elements to obtain operator relations. If we insert the ring element $a_-$ into a correlation function and use the the fact that the correlation function is independent of the position of the ring element then we have

$$F(w, \bar{w}) \equiv \langle a_-(w)ccV_{n_1}^{\alpha_1}(0)ccV_{n_2}^{\alpha_2}(1)ccV_{n_3}^{\alpha_3}(\infty) \prod_{i=3}^N \int V_{n_i}^{\alpha_i}(\int V_0^0)^S \frac{1}{R!}(Q_-)^R \rangle \tag{4.6}$$

$$\lim_{w \to 0} F(w, \bar{w}) = \lim_{w \to 1} F(w, \bar{w}) \tag{4.7}$$

which implies that

$$\langle ccV_{n_1+1}^{\alpha_1}(0)ccV_{n_2}^{\alpha_2}(1)ccV_{n_3}^{\alpha_3}(\infty) \prod_{i=3}^N \int V_{n_i}^{\alpha_i}(\int V_0^0)^S \frac{1}{R!}(Q_-)^R \rangle \tag{4.8}$$

$$= \langle ccV_{n_1}^{\alpha_1}(0)ccV_{n_2+1}^{\alpha_2}(1)ccV_{n_3}^{\alpha_3}(\infty) \prod_{i=3}^N \int V_{n_i}^{\alpha_i}(\int V_0^0)^S \frac{1}{R!}(Q_-)^R \rangle \ .$$
This gives us the following operator relation

\[ V_{n+1}^{\alpha_1}(z)V_{m}^{\alpha_2}(w) = V_{n}^{\alpha_1}(z)V_{m+1}^{\alpha_2}(w) \]  \hspace{1cm} (4.9)

Using the ring element \( a_+ \) instead of \( a_- \) in (4.6), we get other operator relations. As before, we use the independence of the position of the ring element in the correlation function. Using the second equation of (4.3) and (4.7), we obtain

\[
\sum_{i=3}^{N} \langle c\bar{c}V_{m_1}^{\gamma_i}(0)c\bar{c}V_{n_2}^{\alpha_2}(1)c\bar{c}V_{n_3}^{\alpha}(\infty) \prod_{j=3, j\neq i}^{N} \int V_{n_j}^{\alpha_j} \rangle \\
= \sum_{i=3}^{N} \langle c\bar{c}V_{n_1}^{\alpha_1}(0)c\bar{c}V_{m_2}^{\delta_i}(1)c\bar{c}V_{n_3}^{\alpha}(\infty) \prod_{j=3, j\neq i}^{N} \int V_{n_j}^{\alpha_j} \rangle ,
\]  \hspace{1cm} (4.10)

where \( \gamma_i = (\alpha_1 + \alpha_i + 1) \mod p \), \( \delta_i = (\alpha_2 + \alpha_i + 1) \mod p \), \( m_1 = n_1 + n_i - 1 + a_i \), \( m_2 = n_2 + n_i - 1 + b_i \), \( a_i \equiv \frac{(\alpha_1 + \alpha_i + 1) - \gamma_i}{p} \) and \( b_i \equiv \frac{(\alpha_2 + \alpha_i + 1) - \delta_i}{p} \). Since the DK states chosen in (4.6) were arbitrary (modulo the charge conservation condition), the above relation is valid for all charge conserving correlation functions. This implies that in order that the equality be always maintained, each term in the RHS of (4.10) should be equal to the corresponding term in the LHS. This leads to the following operator relation (on focussing on the \( i \)-th term in (4.10).)

\[ V_{m_1}^{\gamma_i}(0)V_{n_2}^{\alpha_2}(1) = V_{n_1}^{\alpha_1}(0)V_{m_2}^{\delta_i}(1) , \]  \hspace{1cm} (4.11)

Notice that when \( \delta_i, \gamma_i = (p - 1) \) then we have the “wrong edge” momenta appearing, then we use the relations recursively to obtain relations between physical states. These relations involve three operators and are of the form

\[ V_{n}^{p-2}(x)V_{m}^{1}(y)V_{k}^{1}(z) = V_{n+1}^{0}(x)V_{m}^{0}(y)V_{k}^{0}(z) \]  \hspace{1cm} (4.12)

Using the operator recursion relations (4.9), (4.11) and (4.12), we can reduce the
correlation functions to

\[
\langle \prod_{i=1}^{N} V_{n_i}^{\alpha_i} \rangle = \mu^S \Gamma(-S) \langle c\bar{c}V_0^0(0)c\bar{c}V_0^0(1) c\bar{c}V_{n_3}^{p-2}(\infty) \rangle (\int V_2^0) \frac{1}{R!} (Q_-)^R
\]

\[
= \mu^S \Gamma(-S) \mathcal{N} \prod_{i=1}^{X} \prod_{j=1}^{R} \int d^2 z_i d^2 w_j |z_i|^\frac{4}{p} |1-z_i|^\frac{4}{p} \prod_{i<j} |z_i - z_j|^{-\frac{4(p+1)}{p}}
\times |w_i|^\frac{2}{p+1} [1 - w_i]^{\frac{2}{p+1}} \prod_{i<j} |w_i - w_j|^{\frac{4p}{p+1}} \prod_{i,j} |z_i - w_j|^{-4}
\]

\[
= \mu^S \Gamma(-S) \mathcal{N} (X!)^{X+R}(\rho)^{4RX} \Delta(1 - \rho')^X \Delta(1 - \rho)^R
\prod_{i=1}^{X} \Delta(i\rho' - R) \prod_{i=1}^{R} \Delta(i\rho)
\prod_{i=0}^{X-1} \Delta(1 - R + \frac{2}{p} + i\rho') \Delta(1 - R + \frac{2}{p} + i\rho') \Delta(-1 + R - \frac{4}{p} - (X-1+i)\rho')
\prod_{i=0}^{X-1} \Delta(1 - \frac{2}{p+1} + i\rho) \Delta(1 - \frac{2}{p+1} + i\rho) \Delta(-1 + 2X - \frac{4}{p+1} - (R-1+i)\rho)
\]

(4.13)

where \( \Delta(x) = \frac{\Gamma(x)}{\Gamma(1-x)} \), \( X = L + S - 3 \), \( \rho = \frac{p}{p+1} \), \( \rho' = \frac{p+1}{p} \) and \( \mathcal{N} \) is the normalisation factor associated with the operators. This can be further simplified to obtain

\[
\langle \prod_{i=1}^{L} V_{n_i}^{\alpha_i} \rangle = \rho \mu^S \frac{\Gamma(L + S - 2)}{\Gamma(S + 1)}
\]

(4.14)

\[
= \rho \mu^S \frac{\Gamma\left(\sum_{i} n_i + \frac{(\sum_{i} \alpha_i) + 2}{2p}\right)}{\Gamma(S + 1)}
\]

where \( S = \sum_{i} \left( \frac{n_i}{2} + \frac{\alpha_i}{2p} \right) - L + 2 + \frac{1}{p} \) and \( \rho = \frac{p}{p+1} \).

Having obtained the correlation functions for integer values of \( S \), the results can be analytically continued to include non integer values of \( S \). However one must take care to preserve the \( Z_2 \) selection rule originating from the requirement of charge conservation in the matter sector. The \( Z_2 \) selection rule implies that all correlation functions that are \( Z_2 \) odd cannot be screened by the matter screening operators and so they vanish. Note the \( Z_2 \) charge of the operator \( V_{n}^{\alpha} \) is \((-1)^{n+\alpha}\). In appendix A, we shall indicate explicitly how the correlation functions can be obtained for the Ising Model.

The result in (4.14) is consistent with the known results from matrix model calculations. For the Ising model, the result is in agreement with the \( \langle e^n \rangle \) and \( \langle \sigma \sigma e^n \rangle \) computed
by Dijkgraaf and Witten[32]. As a further proof of the validity of (4.14), we show in the sequel that this result is compatible with the multicritical behaviour seen in the KdV/matrix models.

For later use, we state the results for the \((1, p)\) models.

\[
\langle \prod_{i=1}^{L} V_{n_i}^{\alpha_i} \rangle = \rho \mu^S \frac{\Gamma(L + S - 2)}{\Gamma(S + 1)}
\]

\[
= \rho \mu^S \frac{\Gamma(\sum_i n_i + (\sum_i \alpha_i) + 2)}{\Gamma(S + 1)},
\]

where \(S = \sum_i \left( n_i + \frac{\alpha_i}{p} \right) - L + 2 + \frac{2}{p} \) and \(\rho = \frac{1}{p}\).

5. Perturbed theories and Multicritical behaviour

Having derived all the genus zero correlation functions we can now check to see if we can reproduce the multicritical behaviour seen in the Matrix models. Distler[24] has shown that the \((2k - 1, 2)\) models can be obtained by perturbations of the \((1, 2)\) models, we will first reproduce those results in this formalism, after which we look at the \((1, p)\) models and show that the minimal models \((p + 1, p)\) can be obtained from these models by an appropriate perturbation. Consider the perturbation of the action by the operator \(V_1\) of the form

\[
Z_{\text{pert}}(\lambda_1) = \int \mathcal{D}\phi \mathcal{D}X \mathcal{D}b \mathcal{D}c \ e^{-S_M - S_L - S_{gh} + \mu \int V_0 + \lambda_1 \int V_1} \]  

(5.1)

We expand the exponential as a power series in the coupling \(\lambda_1\) and evaluate each term in the series using the results of the earlier sections. We then sum the series to obtain

\[
Z_{\text{pert}}(\lambda_1) = (1 - \lambda_1)^{-1} Z, \]

(5.2)

where \(Z\) is the unperturbed partition function as defined in (3.1). Now if we set \(\lambda_1 = 0\), the partition function diverges indicating the presence of new multicritical behaviour as was noticed in [24].

Now consider a perturbation of the \((1, 2)\) model by the operator \(V_k\), in this model the operator \(V_1\) is marginal and adding it to the action only rescales all the correlation functions. The perturbed partition function is given by:

\[
Z_{\text{pert}}(\lambda_1, \lambda_k) = \int \mathcal{D}\phi \mathcal{D}X \mathcal{D}b \mathcal{D}c \ e^{-S_M - S_L - S_{gh} + \mu \int V_0 + \lambda_1 \int V_1 + \lambda_k \int V_k} \]  

(5.3)
The correlation functions in the perturbed theory are

\[
\langle \langle \prod_{i=1}^{l} \int V_{n_i} \rangle \rangle_{\text{pert}} = \langle \langle e^{\lambda_1 \int V_1 + \lambda_k \int V_k} \prod_{i=1}^{l} \int V_{n_i} \rangle \rangle
\]  

(5.4)

where the \( \langle \langle \cdots \rangle \rangle_{\text{pert}} \) stands for the perturbed correlation function shown explicitly in the expression on the R.H.S of (5.4). (Note that the expression \( \langle \langle \cdots \rangle \rangle \) stands for the screened correlation function as defined in (3.2).) Now we expand the the exponential interaction as a series in the couplings to obtain

\[
\langle \langle \prod_{i=1}^{l} \int V_{n_i} \rangle \rangle_{\text{pert}} = \sum_{m} \sum_{n} \frac{\mu^s \lambda_1^n \lambda_k^m}{m! n!} \Gamma(-s) \\
\times \langle c\bar{c}V_1(0) c\bar{c}V_2(1) c\bar{c}V_3(\infty) \prod_{i=3}^{l} \int V_{n_i} \int V_1 \int V_k \int V_0 \rangle^{s} \frac{1}{R!} (Q_-)^R
\]  

(5.5)

The Liouville charge conservation relation is

\[
\sum_{i=1}^{l} n_i + n + km - (l + n + m + s) = -3
\]  

(5.6)

which gives us the value for \( s = \sum_{i=1}^{l} n_i + (k-1)m-l+3 \), and \( L + s - 3 = \sum_{i=1}^{l} n_i + n + km \)

where \( L = l + m + n \) is the total number of operators in the correlation function. For ease of notation, we shall use \( \tilde{n} \equiv \sum_{i} n_i \) for the rest of the section.

Using the computation for the correlation function in section 3, we find that the series in (5.5) can be rewritten as

\[
\langle \langle \prod_{i=1}^{l} \int V_{n_i} \rangle \rangle_{\text{pert}} = \rho \sum_{m} \sum_{n} \frac{\mu^s \lambda_1^n \lambda_k^m}{m! n!} \frac{\lambda_1^n \lambda_k^m}{(\tilde{n} + (k-1)m - l + 3)!} (\tilde{n} + n + km)!
\]  

(5.7)

Now we can do the summation over the index \( n \), to obtain

\[
\langle \langle \prod_{i=1}^{l} \int V_{n_i} \rangle \rangle_{\text{pert}} = \rho \mu^{(\tilde{n} - l + 3)} (\tilde{\lambda}_1)^{-(\tilde{n} + 1)} \sum_{m} \frac{(\mu^{(k-1)} \hat{\lambda}_1^{k-1} \lambda_k)^m}{m!} (\tilde{n} + km)!
\]  

(5.8)

where \( \hat{\lambda}_1 = (1 - \lambda_1) \). Now if we define

\[
z = (\mu^{(k-1)} \hat{\lambda}_1^{k-1} \lambda_k) \left( \frac{k^k}{(k-1)(k-1)} \right)
\]
and replace the factorials with $\Gamma$ functions we obtain

$$
\langle \prod_{i=1}^{l} \int V_{n_i} \rangle_{\text{pert}} = \rho \mu^{(\tilde{n}-l+3)}(\lambda_1)^{-(\tilde{n}+1)}
\times \sum_{m} \frac{z^m}{m!} \left( \frac{k^k}{(k-1)^{(k-1)}} \right)^{-m} \frac{\Gamma(\tilde{n} + km + 1)}{\Gamma(\tilde{n} + (k-1)m - l + 4)}. \tag{5.9}
$$

Now the summation can be expressed in terms of the generalised hypergeometric functions (using the results of Appendix B) as:

$$
\langle \prod_{i=1}^{l} \int V_{n_i} \rangle_{\text{pert}} = \rho \mu^{(\tilde{n}-l+3)}(\lambda_1)^{-(\tilde{n}+1)} \frac{\Gamma(\tilde{n} + 1)}{\Gamma(\tilde{n} - l + 4)} {}_K F_{K-1}(\alpha_i; \gamma_j; z) \tag{5.10}
$$

where

$$
\alpha_i = \frac{\tilde{n} + 1}{k} + \frac{(i - 1)}{k}, \quad i = 1 \ldots k,
$$

and

$$
\gamma_j = \frac{\tilde{n} - l + 4}{(k-1)} + \frac{(j - 1)}{(k-1)}, \quad j = 1 \ldots (k-1).
$$

We can extract the behaviour of the hypergeometric function in limit $|z| \to \infty$ (using the results of Appendix B) to obtain:

$$
\langle \prod_{i=1}^{l} \int V_{n_i} \rangle_{\text{pert}} = \rho \mu^{(\tilde{n}-l+3)}(\lambda_1)^{-(\tilde{n}+1)} \frac{\Gamma(\tilde{n} + 1)}{\Gamma(\tilde{n} - l + 4)} \frac{\Gamma(\tilde{n} + 1)}{\Gamma(\tilde{n} + 1 - l + 3)} \tag{5.11}
$$

where $\tilde{n} = \sum_i n_i$. This is the expression for the correlation functions of the operators $\langle \prod_{i=1}^{l} \int V_{n_i} \rangle$ of the $(2k-1,2)$ model. As we have just seen they can all be obtained by perturbation from the $(1,2)$ model, just as we can obtain the correlation functions of the $k$-th multicritical point in the one matrix model case.

Now we will consider the $(1,p)$ models and study their behaviour under perturbation by physical operators, we will show that the minimal models can be obtained from these models under perturbation. The physical operators in these models are given by

$$
V_{n} \alpha = \Delta(n + \frac{(\alpha + 1)}{p}) \exp \left[ \frac{p(n-1) + \alpha}{2\sqrt{p}} - \frac{[p(n-1) + \alpha + 2]iX}{2\sqrt{p}} \right]
$$

where $\alpha = 0, \ldots, (p-2)$. The correlation functions in these models can be obtained using the ring to provide recursion relations, as was done for the minimal models.
Consider a perturbation of the \((1, p)\) model by the operator \(V_0^0\) and \(V_1^0\). (In this model adding the operator \(V_1^0\) to the action rescales all the correlation functions. This can be shown by a calculation similar to the one in [5.2].) The perturbed partition function is given by:

\[
Z_{\text{pert}} = \int \mathcal{D} \phi \, \mathcal{D} X \, \mathcal{D} b \mathcal{D} c \, e^{-S_M - S_L - S_{gh} + \mu \int V_0^0 + \lambda_1 \int V_1^0 + \lambda_k \int V_k^0}
\]  

(5.12)

The correlation functions in the perturbed theory are

\[
\langle \prod_{i=1}^{l} V_{\alpha_i} \rangle_{\text{pert}} = \langle \exp^{\lambda_1 \int V_1^0 + \lambda_k \int V_k^0} \prod_{i=1}^{l} V_{\alpha_i} \rangle
\]  

(5.13)

where the \(\langle \cdot \cdot \cdot \rangle_{\text{pert}}\) stands for the perturbed correlation function shown explicitly in the expression on the R.H.S of (5.13). (Note that the expression \(\langle \cdot \cdot \cdot \rangle\) stands for the screened correlation function.)

Now we expand the exponential interaction as a series in the couplings to obtain

\[
\langle \prod_{i=1}^{l} V_{\alpha_i} \rangle_{\text{pert}} = \sum_{m} \sum_{n} \mu^{\lambda_1} \lambda_k^{m} \frac{n^{m}}{m! n!} \Gamma(-s) 
\times \langle c \bar{c} V_{n_1}^{\alpha_1}(0) c \bar{c} V_{n_2}^{\alpha_2}(1) c \bar{c} V_{n_3}^{\alpha_3}(\infty) \prod_{i=3}^{l} V_{n_i}^{\alpha_i}(\int V_{k_1}^{0}) n(\int V_{k_2}^{0}) m(\int V_{k_3}^{0}) s \frac{1}{R!}(Q-)^R \rangle
\]  

(5.14)

The Liouville charge conservation relation is

\[
p \tilde{n} + \tilde{\alpha} + (k - 1) m p - (l p + s p) = -(2p + 1)
\]  

(5.15)

which gives us the value for

\[
s = \tilde{n} + (k - 1) m - l + 2 + \frac{\tilde{\alpha} + 2}{p}
\]

and

\[
L + s - 3 = \tilde{n} + n + k m - 1 + \frac{\tilde{\alpha} + 2}{p}
\]

where \(L = l + m + n\) is the total number of operators in the correlation function and \(\tilde{\alpha} = \sum_{i} \alpha_i\). For ease of notation, we shall use \(\tilde{\alpha} \equiv \sum_{i} \alpha_i\) for the rest of the section.
Using (4.15), we find that the series in (5.14) can be rewritten as

\[
\langle \prod_{i=1}^{l} \int V_{n_i}^{\alpha_i} \rangle_{\text{pert}} = \rho \sum_{m} \sum_{n} \mu^{s} \frac{\lambda_{n}^{\alpha} \lambda_{k}^{m}}{m! \; n!} \frac{(\bar{n} + n + km - 1 + \frac{\bar{\alpha}+2}{p})!}{(\bar{n} + (k - 1)m - l + 2 + \frac{\bar{\alpha}+2}{p})!}
\] (5.16)

Now we can do the summation over the index \( n \), to obtain

\[
\langle \prod_{i=1}^{l} \int V_{n_i}^{\alpha_i} \rangle_{\text{pert}} = \mu^{(\bar{n}-l+2+\frac{\bar{\alpha}+2}{p})} (\hat{\lambda}_{1})^{-(\bar{n}+\frac{\bar{\alpha}+2}{p})} \\
\times \sum_{m} (\mu^{(k-1)} \hat{\lambda}_{1}^{-k} \lambda_{k})^{m} \frac{(\bar{n} + km - 1 + \frac{\bar{\alpha}+2}{p})!}{m! \; (\bar{n} + (k - 1)m - l + 2 + \frac{\bar{\alpha}+2}{p})!}
\] (5.17)

where \( \hat{\lambda}_{1} = (1 - \lambda_{1}) \). Now if we define

\[
z = (\mu^{(k-1)} \hat{\lambda}_{1}^{-k} \lambda_{k}) \left( \frac{k^{k}}{(k - 1)(k-1)} \right)
\]

and replace the factorials with \( \Gamma \) functions we obtain

\[
\langle \prod_{i=1}^{l} \int V_{n_i}^{\alpha_i} \rangle_{\text{pert}} = \rho \mu^{(\bar{n}-l+2+\frac{\bar{\alpha}+2}{p})} (\hat{\lambda}_{1})^{-(\bar{n}+\frac{\bar{\alpha}+2}{p})} \\
\times \sum_{m} z^{m} \frac{k^{m}}{m! \; (k - 1)(k-1)} \frac{\Gamma(\bar{n} + km + \frac{\bar{\alpha}+2}{p})}{\Gamma(\bar{n} + (k - 1)m - l + 3 + \frac{\bar{\alpha}+2}{p})}
\] (5.18)

Now the summation can be expressed in terms of the generalised hypergeometric functions (using the results of Appendix B):

\[
\langle \prod_{i=1}^{l} \int V_{n_i}^{\alpha_i} \rangle_{\text{pert}} = \rho \mu^{(\bar{n}-l+2+\frac{\bar{\alpha}+2}{p})} (\hat{\lambda}_{1})^{-(\bar{n}+\frac{\bar{\alpha}+2}{p})} \\
\times \frac{\Gamma(\bar{n} + \frac{\bar{\alpha}+2}{p})}{\Gamma(\bar{n} - l + 3 + \frac{\bar{\alpha}+2}{p})} \kappa_{F_{K-1}}(a_{i}; b_{j}; z)
\] (5.19)

where

\[
a_{i} = \frac{(\bar{n} + \frac{\bar{\alpha}+2}{p})}{k} + \frac{(i - 1)}{k} \quad i = 1 \ldots (k - 1)
\]

and

\[
b_{j} = \frac{(\bar{n} - l + 3 + \frac{\bar{\alpha}+2}{p})}{(k - 1)} + \frac{(j - 1)}{(k - 1)} \quad j = 1 \ldots (k - 1)
\]

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Notice that in the limit of $z \to \infty$ the correlation functions diverge indicating the presence of new multicritical behaviour. We can extract the behaviour of the hypergeometric function in this limit (using the results of Appendix B) to obtain:

$$\langle \langle \prod_{i=1}^{l} \int V_{n_i} \rangle \rangle_{\text{pert}} = \rho \ k^{-1} \ \mu^{\frac{\bar{n} p + \bar{\alpha} + 2}{pk}} \ \lambda_{k}^{-\frac{\bar{n} + \bar{\alpha} + 2}{p}} \ \frac{\Gamma\left(\frac{\bar{n} p + \bar{\alpha} + 2}{pk}\right)}{\Gamma\left(\frac{\bar{n} p + \bar{\alpha} + 2}{pk} - l + 3\right)}, \quad (5.20)$$

where $\bar{n} = \sum n_i$ and $\bar{\alpha} = \sum \alpha_i$. Now if we choose the perturbing operator to be $V_0^2$ i.e, $k = 2$ then we obtain the minimal model $(p+1,p)$ correlation functions as given in eqn. (4.14). For arbitrary $k$, one obtains the $(pk - p + 1, p)$ models.

6. Conclusions

In this paper we have shown that the correlation functions of $c < 1$ string theory can be obtained using the operator identities provided by the action of the ring inside correlation functions. We have also shown that the multicritical behaviour of the matrix models can be reproduced by perturbing the action by physical vertex operator states. and expanding in powers of the coupling constant. In particular, we have shown that the $(2k - 1, 2)$ models can be obtained as the k-th multicritical point of the $(1, 2)$ models, and the minimal models $(p+1, p)$ can be obtained as the second multicritical point of the $(1, p)$ model.

The techniques introduced in this paper are sufficiently general and can be used to evaluate the correlation functions in other models like the $W_n$ string which have a similar ring structure.

The analytic continuation in the number of operators is relevant for all computations for strings in non-trivial backgrounds which do not have global translation invariance. However, for such generic models it is not clear if the ring structure exists and if it is sufficient to determine all the correlation functions.

Although it has been possible to obtain the correspondence between the matrix models and the continuum picture completely for genus zero and for the genus one partition function the higher genus correlation functions are required to ascertain the equivalence of the two formulations, and remains a challenging problem.

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Appendix A. Correlation Functions In the Ising Model

In this appendix we will indicate explicitly how the correlation functions can be obtained for the Ising model. We shall now use the ring elements to calculate arbitrary N-point functions explicitly for the case of the Ising model which is the (4,3) model. In the Ising model, \( V_0^0 \) corresponds to the identity operator, \( V_0^1 = \sigma \) and \( V_1^1 = \epsilon \) correspond to the spin and energy operators respectively (with appropriate Liouville dressing). \( V_2^0 \) is the “physical” screening operator. One can prove the following shift recursion relation. The arguments are identical to the one used in deriving eqn. (3.9).

\[
V_{n+1}^\alpha (z)V_{m}^\beta (w) = V_n^\alpha (z)V_{m+1}^\beta (w)
\]  
(A.1)

Now consider a charge conserving correlation function with one \( a_+ \) and DK states.

\[
\langle a_+ (w) \mid c \bar{c} V^0_n \rangle \mid c \bar{c} V^1_m \rangle \alpha \beta \prod_{i=1}^{L} \int V_{n_i}^0 \prod_{j=1}^{M} \int V_{m_i}^1 \frac{1}{R!} (Q^-)^R
\]  
(A.2)

Using the \( w \) independence of the above correlation function, we equate the value of the correlator at \( w = 0 \) and \( 1 \). This gives us after using the second equation in (4.3)

\[
\sum_{k=1}^{L} \langle c \bar{c} V^1_{n+n_k-1} \rangle \langle c \bar{c} V^1_m \rangle \alpha \beta \prod_{i=1, i \neq k}^{L} \int V_{n_i}^0 \prod_{j=1}^{M} \int V_{m_i}^1
\]

\[
+ \sum_{k=1}^{M} \langle c \bar{c} V^2_{n+m_k-1} \rangle \langle c \bar{c} V^1_m \rangle \alpha \beta \prod_{i=1, i \neq k}^{L} \int V_{n_i}^0 \prod_{j=1, j \neq k}^{M} \int V_{m_i}^1
\]

\[
\sum_{k=1}^{L} \langle c \bar{c} V^0_n \rangle \langle c \bar{c} V^2_{m+n_k-1} \rangle \alpha \beta \prod_{i=1, i \neq k}^{L} \int V_{n_i}^0 \prod_{j=1}^{M} \int V_{m_i}^1
\]

\[
+ \sum_{k=1}^{M} \langle c \bar{c} V^0_n \rangle \langle c \bar{c} V^0_{m+m_k} \rangle \alpha \beta \prod_{i=1, i \neq k}^{L} \int V_{n_i}^0 \prod_{j=1, j \neq k}^{M} \int V_{m_i}^1
\]

(A.3)

Using eqn. (A.1), one can see that every term inside each of the sums are the same. One can now see that the following operator relations provides a consistent solution to the above equality.

\[
V_{n+1}^2 (z)V_n^1 (w) = V_n^0 (z)V_{n+1}^0 (w)
\]

\[
V_{m+1}^2 (z)V_m^0 (w) = V_m^1 (z)V_{m+1}^1 (w)
\]

(A.4)

However, \( V_m^2 \) belongs to the “wrong-edge” and is not physical. However by using this relation twice, we obtain the following recursion which involves only physical operators.

\[
V_m^1 (z)V_n^1 (w)V_r^1 (t) = V_{m+1}^0 (z)V_n^0 (w)V_r^0 (t)
\]

(A.5)
Equations (A.4) and (A.5) are sufficient to convert all charge conserving correlation functions to those which are of the Dotsenko-Fateev type and hence are computable. We shall now demonstrate this. Consider the following correlation function

\[ \langle \langle \prod_{i=1}^{L} \int V_{n_i}^{\alpha_i} \rangle \rangle = \mu^S \Gamma(-S) \langle c \bar{c} V_{n_1}^{\alpha_1}(0) c \bar{c} V_{n_2}^{\alpha_2}(1) c \bar{c} V_{n_3}^{\alpha_3}(\infty) \prod_{i=4}^{L} \int V_{n_i}^{\alpha_i}(\int V_0^0)^{S} \frac{1}{R!} (Q_-)^R \rangle, \]  

(A.6)

where \( S = \sum_i \frac{n_i}{2} - L + 2 + \frac{(\sum_i \alpha_i + 2)}{6} \) and \( R = \sum_i (\frac{n_i}{2} + \frac{\alpha_i}{6}) + \frac{S}{3} \). When, \( S \) and \( R \) are positive integers, using eqns. (A.4) and (A.3) in eqn.(A.6) we obtain

\[ \langle \langle \prod_{i=1}^{L} \int V_{n_i}^{\alpha_i} \rangle \rangle = \mu^S \Gamma(-S) \langle c \bar{c} V_{0}^{0}(0) c \bar{c} V_{0}^{0}(1) c \bar{c} V_{1}^{1}(\infty)(\int V_2^0)^{(L+S-3)} \frac{1}{R!} (Q_-)^R \rangle. \]  

(A.7)

The correlation function in the RHS can be explicitly computed using the formula of Dotsenko and Fateev. We obtain

\[ \langle \langle \prod_{i=1}^{L} \int V_{n_i}^{\alpha_i} \rangle \rangle = \rho \mu^S \frac{(L+S-3)!}{S!}, \]  

(A.8)

where \( \rho = \frac{3}{4} \). This is in agreement with matrix model results. For the cases when \( S \) and \( R \) are not positive integers, the results are obtained by analytic continuation. Of course, one has to take care that the \( Z_2 \) invariance of the minimal models is not violated. One imposes this by setting all non-\( Z_2 \) invariant correlators to zero by hand. For the example of Ising model, the \( Z_2 \) charge of the operator \( V_n^\alpha \) is \((-1)^{n+\alpha}\). So the correlation function in (A.6) is non-zero provided \( \sum_i (n_i + \alpha_i) \) is even. For such cases, the result one obtains after analytic continuation in both \( S \) and \( R \) is

\[ \langle \langle \prod_{i=1}^{L} \int V_{n_i}^{\alpha_i} \rangle \rangle = \rho \mu^S \frac{\Gamma(\sum_i \frac{n_i}{2} + \frac{\sum_i \alpha_i + 2}{6})}{\Gamma(S+1)}. \]  

(A.9)

To compare with the matrix model results in [32], we now exhibit some results which follow form (A.9)

\[ \langle \langle \epsilon^n \rangle \rangle = \rho \left( \frac{\partial}{\partial \mu} \right)^{n-3} \left( \mu \frac{2n}{3} - \frac{2}{3} \right), \]  

\[ \langle \langle \sigma \sigma \epsilon^n \rangle \rangle = \rho \left( \frac{\partial}{\partial \mu} \right)^{n-1} \left( \mu \frac{2n}{3} - \frac{1}{3} \right). \]  

(A.10)

This agrees with the result given in [32] up to trivial normalisation factors. We have also checked that the results (for the above two sets of correlation functions) in section 5, for the \((3k-1, 3)\) models agree with the two-matrix model results with \( k \) labelling the various critical points.
Appendix B. Some relevant results on Generalised Hypergeometric Functions

In this appendix we present the relevant details and the notations used in the text to describe the generalised hypergeometric functions. One can find more details in the following references [34]. The definition of the generalised hypergeometric function is

\[
p_{\alpha, \gamma}(\alpha_i, \gamma_i; z) = \frac{\Gamma(\gamma_1) \Gamma(\gamma_2) \cdots \Gamma(\gamma_q)}{\Gamma(\alpha_1) \Gamma(\alpha_2) \cdots \Gamma(\alpha_p)} \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + n) \cdots \Gamma(\alpha_p + n) \cdot z^n}{\Gamma(\gamma_1 + n) \cdots \Gamma(\gamma_q + n) \cdot n!}
\]

The domain of validity of the above expression is for \(|z| < 1\). There is also an identity satisfied by products of Gamma functions which is

\[
\prod_{i=0}^{p-1} \frac{\Gamma(\alpha + \frac{i}{p})}{\Gamma(p\alpha)} = (2\pi)^{\frac{p-1}{2}} (p)^{-p\alpha + 1/2} \Gamma(p\alpha)
\]

Using the product identity we can write

\[
p_{\alpha, \gamma}(\alpha_i, \gamma_i; z) = \frac{\Gamma(q\alpha)}{\Gamma(p\alpha)} \sum_{n=0}^{\infty} \left( \frac{(q)^q}{(p)^p} \right)^n \frac{z^n \Gamma(p\alpha + np)}{n! \Gamma(q\gamma + nq)}
\]

There is also an integral representation for the generalised hypergeometric function

\[
p_{\alpha, \gamma}(\alpha_i, \gamma_i; z) = \prod_{j=1}^{p} \frac{\Gamma(\gamma_j)}{\Gamma(\alpha_j + 1) \Gamma(\gamma_j - \alpha_{j+1})} \times \int_{0}^{1} \prod_{k=1}^{P} dt_k (t_k)^{\alpha_{k+1}-1} (1 - t_k)^{\gamma_j - \alpha_{j+1} - 1} (1 - z t_1 t_2 \cdots t_p)^{-\alpha_1} + \text{perms.}
\]

Using the integral representation, we extract the behaviour of the hypergeometric function in the limit \(|z| \to \infty\)

\[
\lim_{|z| \to \infty} p_{\alpha, \gamma}(\alpha_i, \gamma_i; z) \sim (-z)^{-\alpha_1} \prod_{j=1}^{p} \frac{\Gamma(\gamma_j) \Gamma(\alpha_{j+1} - \alpha_1)}{\Gamma(\alpha_j + 1) \Gamma(\gamma_j - \alpha_{j+1})},
\]

where we choose \(\alpha_1 = \min\{\alpha_i\}\). We have used the results given above in deriving the behaviour of the correlation functions at the multicritical points.
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