Additive self-similar approximants

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Abstract

A novel type of approximants is introduced, being based on the ideas of self-similar approximation theory. The method is illustrated by the examples possessing the structure typical of many problems in applied mathematics. Good numerical convergence is demonstrated for the cases that can be compared with exact solutions, when these are available. The method is shown to be not less and as a rule essentially more accurate than that of Padé approximants. Comparison with other approximation methods is also given.

Keywords: Self-similar approximation theory, Additive approximants, Numerical convergence

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1 Introduction

Suppose we are looking for the solution of a complicated equation that cannot be solved exactly, and the sole thing we are able to accomplish is to find asymptotic expansions near the boundary of the solution domain. Then the problem arises of reconstructing the sought function for its whole domain from the limited knowledge of only its asymptotic expansions. The most often used method for this purpose is that of Padé approximants \[1\]. Although being useful in many cases, this method has a number of known deficiencies that have been repeatedly discussed in literature (see, e.g., \([1–5]\)). Another very efficient method is based on self-similar approximation theory \([6–8]\) resulting in several types of self-similar approximants \([9–11]\).

In the present paper, we introduce a novel type of self-similar approximants and illustrate their effectiveness by examples whose mathematical structure is typical of many calculational problems in a variety of applications, e.g., in mathematical chemistry, statistical physics, non-linear phenomena, and field theory. We demonstrate that the new approximants, that we name *additive self-similar approximants*, to distinguish them from multiplicative factor approximants \([9–11]\), enjoy sufficiently fast numerical convergence and provide good accuracy of approximations. Their accuracy is not worse, and usually much better, than that of Padé approximants.

2 Construction of additive approximants

Let us be looking for a solution that is a real function \(f(x)\) of a real variable \(x\). In general, the function domain can be arbitrary. For concreteness, we consider here the interval \(0 \leq x < \infty\). This does not mitigate the generality of the consideration, since by a change of variables it is practically always possible to reduce a given interval to the ray \([0, \infty)\).

Suppose that the sought function is defined by complicated equations that allow us to find only its asymptotic expansion near one of the domain boundaries, say, for asymptotically small \(x\), where

\[
 f(x) \simeq f_k(x) \quad (x \to 0) ,
\]

with the \(k\)-th order finite series

\[
 f_k(x) = \sum_{n=0}^{k} a_n x^n .
\]

Or the large-variable expansion can be available, such that

\[
 f(x) \simeq f^{(p)}(x) \quad (x \to \infty) ,
\]

with the finite series

\[
 f^{(p)}(x) = \sum_{n=1}^{p} b_n x^{\beta_n} .
\]

The powers in the above series are arranged in the ascending order:

\[
 \beta_n > \beta_{n+1} \quad (n = 1, 2, \ldots, p - 1) .
\]

The standard situation corresponds to the uniform power decrease with the constant difference

\[
 \Delta \beta \equiv \beta_n - \beta_{n+1} \quad (n = 1, 2, \ldots) .
\]
For the problem of interpolation between the small-variable expansion (2) and large-variable expansion (4), we would need the values of the coefficients $b_n$. However, the most interesting and most complicated problem is that of the extrapolation of the small-variable expansion (2) into the large-variable limit, when the coefficients $b_n$ are not known, although the powers $\beta_n$ can be available. In the present paper, we shall pay the main attention exactly to this problem of extrapolation, with unknown coefficients $b_n$. In that procedure, when the small-variable expansion is obeyed by construction, but the large-variable coefficients are not known, the error of an approximation tends to zero, as $x \to 0$, while, vice versa, the error increases when the variable tends to infinity, reaching a maximal value in the limit $x \to \infty$. Therefore, in the problem of extrapolation, the accuracy of the procedure as a whole is defined by the large-variable limit, that is, by the accuracy of the amplitude

$$B \equiv \lim_{x \to \infty} x^{-\beta_1} f(x) = b_1$$

that has to be compared with the large-variable limits of the studied approximations.

The procedure of extrapolating the small-variable series to the whole range of the variable $x \in [0, \infty]$, by employing the self-similar approximation theory has been described in full mathematical details in our previous papers [6–11]. Therefore, here we omit the description of mathematical techniques, only stressing the main steps of the procedure, which results in the novel type of approximants.

First, we subject the variable $x$ to the affine transformation

$$x \to A(1 + \lambda x) ,$$

consisting of a scaling and shift. This transforms the terms of series (2) as

$$a_n x^n \to A_n (1 + \lambda x)^n ,$$

where $A_n = A a_n$. Then the self-similar transformation of series (2) is just the affine transformation of its terms, which yields

$$f_k^*(x) = \sum_i A_i (1 + \lambda x)^{n_i} .$$

(8)

The powers of the first $k$ terms of this series correspond to the powers of series (4),

$$n_i = \beta_i \quad (i = 1, 2, \ldots, k) ,$$

(9)

while all coefficients $A_i$ can be found by the accuracy-through-order procedure, expanding form (8) in powers of $x$ and equating to expansion (2). Expression (8) is the additive approximant, which is named for distinguishing it from the multiplicative factor approximants considered earlier [9–11].

It is clear that in the large-variable limit, approximant (8) will reproduce the terms with the powers of series (4). However, except the terms with the correct powers $\beta_i$, there appear the terms with the powers $\beta_i - 1$. There can exist two situations. It may be that the powers $\beta_i - 1$ do not pertain to the set of the powers $\{\beta_j\}$. Then the terms with the incorrect powers should be canceled by including in approximant (8) correcting terms (counter-terms) with the powers

$$n_j = \gamma_j = \beta_j - 1 \quad (j = 1, 2, \ldots, q) ,$$

(10)
where
\[ \beta_{k+1} < \gamma_j < \beta_1, \] (11)
and the coefficients \( C_j \) are defined by the cancellation of the terms with incorrect powers in the large-variable expansion. In that way, the general form of the additive approximant is
\[ f^*_{k,q}(x) = \sum_{i=1}^{k} A_i (1 + \lambda x)^{\beta_i} + \sum_{j=1}^{q} C_j (1 + \lambda x)^{\gamma_j}. \] (12)

The other possibility is when the powers \( \beta_i - 1 \) turn out to be the members of the set \( \{\beta_i\} \), that is, the set \( \{\beta_i\} \) is invariant under the transformation
\[ \beta_i - 1 = \beta_j. \] (13)
In that case, no correction terms are needed, and the additive approximant is
\[ f^*_k(x) \equiv f^*_{k,0}(x) = \sum_{i=1}^{k} A_i (1 + \lambda x)^{\beta_i}. \] (14)

The coefficients \( A_i \) can be found by the accuracy-through-order procedure, comparing the expansion of the additive approximant with the small-variable expansion, or with the large-variable expansion, or using both of them.

As has been mentioned above, employing the accuracy-through-order procedure at small variables, the asymptotic expansion at \( x \to 0 \) of the additive approximant coincides with the exact asymptotic expansion (2), while the error of the approximation increases for growing \( x \), reaching the maximal values at \( x \to \infty \). It is therefore instructive to compare the exact amplitude (7) with the amplitude of the \( k \)-th approximant
\[ B_k = \lim_{x \to \infty} x^{-\beta_1} f^*_k(x) = A_1 \lambda^{\beta_1}. \] (15)

Of course, not only the leading-order amplitude \( B_k \), representing the coefficient \( b_1 \), can be found, but the subleading amplitudes, representing other coefficients \( b_k \), can also be calculated. However, our primary interest is not in defining particular coefficients, but rather to check the accuracy of the whole additive approximant. This is why we concentrate our attention on the leading amplitude, given by (15), and characterizing the large-variable limit of the additive approximant as a whole.

Defining the coefficients of the additive approximant from the accuracy-through-order procedure, we confront with the nonuniqueness of solutions. Thus, when there are no counter-terms, we have \( k \) solutions in the \( k \)-th order. In the case of \( q \) counter-terms, the \( k \)-th order approximant yields \( k + q - 1 \) solutions. Fortunately, the appearance of multiple solutions is not a serious obstacle, because of the following.

Generally, among the solutions, there can happen real and also complex-valued solutions. The latter come in complex conjugate pairs, so that their sum is real. It turns out that all real solutions and the average sums of the complex conjugate pairs, in each order, are very close to each other. We show this in the examples below. Then there can be two strategies. Either to consider only real solutions, or to take the average sums of all solutions of the given order.
3 Comparison with other methods

The accuracy of the method of additive approximants will be compared with that of other approximation methods. First of all, we consider the usual Padé approximants $P_{M/N}$, as well as the modified $P_{M/N}^\gamma$ Padé approximants, suggested by Baker and Gammel [12], where

$$(M - N)\gamma = \beta_1 .$$

It is also important to compare the additive approximants with other variants of self-similar approximants. As has been mentioned above, additive approximants are distinguished from multiplicative factor approximants [13, 14]. The latter are obtained by reducing the small-variable asymptotic series

$$f_k(x) = f_0(x) \left( 1 + \sum_{n=1}^{k} a_n x^n \right)$$

(16)

to a multiplicative form, accomplishing affine transformations of the variable in the multiplicative factors and realizing self-similar renormalization yielding the factor approximant

$$f^*_k(x) = f_0(x) \prod_{i=1}^{N_k} (1 + A_i x^{n_i}) ,$$

(17)

where

$$N_k = \left\{ \begin{array}{cc} k/2 , & k = 2, 4, \ldots \\ (k + 1)/2 , & k = 3, 5, \ldots , \end{array} \right. \right.$$

with the powers $n_i$ and coefficients $A_i$ being uniquely defined by the accuracy-through-order procedure.

The zero-order term is usually of the form

$$f_0(x) = Ax^\alpha .$$

Then the large-variable limit of the factor approximant gives the power

$$\beta_1 = \alpha + \sum_{i=1}^{N_k} n_i$$

(18)

and the amplitude

$$B_k = A \prod_{i=1}^{N_k} A_i^{n_i} .$$

(19)

Another variant of self-similar approximants is represented by root approximants [5, 15]

having the form

$$f^*_k(x)/f_0(x) = ((\ldots (1 + A_1 x^{n_1}) + A_2 x^{n_2}) + \ldots + A_k x^{n_k})^{n_k} .$$

(20)

There can exist several cases. If the large-variable expansion is known, then all coefficients $A_i$ and powers $\beta_i$ are uniquely defined by the accuracy-through-order procedure at large $x$, giving

$$jn_j = j + 1 - \beta_{k-j} + \beta_{k-j+1} \quad (j = 1, 2, \ldots, k - 1)$$

(21)
and
\[ n_k = \frac{\beta_1 - \alpha}{k} \quad (\alpha \neq \beta_1) . \] (22)

As a rule, the difference
\[ \Delta \beta \equiv \beta_n - \beta_{n+1} = \text{const} \] (23)
is constant. Then the powers up to \( k - 1 \) are defined by the relation
\[ jn_j = j + 1 - \Delta \beta \quad (j = 1, 2, \ldots, k - 1) , \] (24)
with \( n_k \) as in equation (22). Note that if here \( \Delta \beta = 1 \), then \( n_j = 1 \) up to \( j = k - 1 \). When in the large-variable expansion, the powers \( \beta_n \) are known, but the coefficients \( b_n \) are given only for \( n = 1, 2, \ldots, p < k \), then the coefficients \( A_n \) of the root approximant [20] are defined by the accuracy-through-order procedure at small variables, giving \( k - p \) equations and at large variables, giving \( p \) equations. If just the sole large-variable power \( \beta_1 \) is given, then setting
\[ n_j = \frac{j + 1}{j} \quad (j = 1, 2, \ldots, k - 1) , \] (25)
all coefficients \( A_n \) are defined by the accuracy-through-order procedure at small variable.

In the following sections, we illustrate the use of the additive approximants by the examples, whose mathematical structure is typical of many applied problems.

4 Anharmonic partition integral

The structure of the integral
\[ Z(g) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp \left( -z^2 - gz^4 \right) dz \] (26)
is typical for numerous problems in quantum chemistry, field theory, statistical mechanics, and condensed-matter physics dealing with the calculation of partition functions, where \( g \in [0, \infty) \) plays the role of coupling parameter [16]. The integral expansion at small \( g \to 0 \), yields strongly divergent series, with the \( k \)-th order sums
\[ Z_k(g) = k \sum_{n=0} c_n g^n , \] (27)
whose coefficients are
\[ c_n = \frac{(-1)^n}{\sqrt{\pi} \ n! \ \Gamma \left( 2n + \frac{1}{2} \right)} . \]
The coefficients \( c_n \) quickly grow with increasing \( n \) tending to infinity as \( n^n \) for \( n \gg 1 \), which makes the weak-coupling expansion strongly divergent. At strong coupling, we have
\[ Z(g) \simeq b_1 g^{-1/4} + b_2 g^{-3/4} + b_3 g^{-5/4} + b_4 g^{-7/4} \quad (g \to \infty) , \] (28)
where
\[ b_1 = \frac{1}{2\sqrt{\pi}} \Gamma \left( \frac{1}{4} \right) = 1.022765 , \quad b_2 = \frac{1}{8\sqrt{\pi}} \Gamma \left( -\frac{1}{4} \right) = -0.345684 . \]
\[ b_3 = \frac{1}{16\sqrt{\pi}} \Gamma \left( \frac{1}{4} \right) = 0.127846 , \quad b_4 = \frac{1}{64\sqrt{\pi}} \Gamma \left( -\frac{1}{4} \right) = -0.043211 . \]

The powers of the strong-coupling expansion,
\[ \beta_n = -\frac{2n - 1}{4} \]

enjoy the uniform difference
\[ \Delta \beta \equiv \beta_n - \beta_{n+1} = \frac{1}{2} . \]

The set \( \{\beta_n\} \) is invariant with respect to transformation (13) because of the property
\[ \beta_n - 1 = \beta_{n+2} . \]

Hence no correction terms are needed. All coefficients \( A_i \) of the additive approximant (14) are obtained from the accuracy-through order procedure at weak coupling. The error of approximants grows as \( g \to \infty \). Therefore the accuracy of the method is defined by the accuracy of the strong-coupling amplitude
\[ B_k = \lim_{g \to \infty} g^{1/4} Z_k^*(g) \]

that has to be compared with the exact value \( b_1 \).

First, we consider only real-valued solutions for \( A_i \). In each odd order, there is just one real solution. Then for the additive approximants (14), we have to third order
\[ Z_3^*(g) = A_1 (1 + \lambda g)^{-1/4} + A_2 (1 + \lambda g)^{-3/4} + A_3 (1 + \lambda g)^{-5/4} , \]

where
\[ A_1 = 1.510761 , \quad A_2 = -0.717990 , \quad A_3 = 0.207229 , \quad \lambda = 7.634834 . \]

This gives the strong-coupling amplitude
\[ B_3 = 0.908858 \quad (Z_3^*) . \]

To fifth order,
\[ Z_5^*(g) = A_1 (1 + \lambda g)^{-1/4} + A_2 (1 + \lambda g)^{-3/4} + A_3 (1 + \lambda g)^{-5/4} + \\
+ A_4 (1 + \lambda g)^{-7/4} + A_5 (1 + \lambda g)^{-9/4} , \]

with the coefficients
\[ A_1 = 1.808031 , \quad A_2 = -1.543729 , \quad A_3 = 1.134917 , \quad A_4 = -0.492745 , \]
\[ A_5 = 0.093526 , \quad \lambda = 12.297696 . \]

The strong-coupling amplitude is
\[ B_5 = 0.965495 \quad (Z_5^*) . \]

Continuing the procedure, we obtain in higher orders
\[ B_7 = 0.992107 \quad (Z_7^*) , \quad B_9 = 1.005760 \quad (Z_9^*) , \quad B_{11} = 1.01312 \quad (Z_{11}^*) , \]
\[ B_{13} = 1.01720 \ (Z_{13}^*), \quad B_{15} = 1.01952 \ (Z_{15}^*), \quad B_{17} = 1.02085 \ (Z_{17}^*), \]
\[ B_{19} = 1.02072 \ (Z_{19}^*). \]

Comparing these amplitudes with the exact \( B = 1.02277 \), we find the corresponding errors 11\%, 6\%, 3\%, 2\%, 0.9\%, 0.5\%, 0.3\%, 0.2\%.

As is seen, the accuracy improves with increasing order, which demonstrates good numerical convergence.

Now let us consider the other way, when, in each order, the average of all solutions is taken. In the second order, the approximant is
\[ Z_2^*(g) = A_1 (1 + \lambda g)^{-1/4} + A_2 (1 + \lambda g)^{-3/4}. \]
The weak-coupling accuracy-through-order procedure for the parameters \( A_i \) gives two complex-conjugate solutions, whose sum is real. The resulting strong-coupling amplitude is
\[ B_2 = 0.858304 \ (Z_2^*). \]

In the third order, there is one real and two complex-conjugate solutions. Summing them up yields the strong-coupling amplitude
\[ B_3 = 0.915248 \ (Z_3^*). \]

The fourth order yields two pairs of complex-conjugate solutions, resulting in the amplitude
\[ B_4 = 0.956250 \ (Z_4^*). \]

In the fifth order, there is one real and two pairs of complex-conjugate solutions, whose average sum gives
\[ B_5 = 0.979861 \ (Z_5^*). \]

The sixth order produces three pairs of complex-conjugate solutions, with the corresponding amplitude
\[ B_6 = 1.000921 \ (Z_6^*). \]

In the seventh order, there appear one real and three pairs of complex-conjugate solutions, giving the amplitude
\[ B_7 = 1.010621 \ (Z_7^*). \]

Comparing the obtained \( B_k \) with the exact strong-coupling amplitude \( B = b_1 = 1.02277 \), we find the errors 16\%, 11\%, 7\%, 4\%, 2\%, 1\%.

Again, we observe good numerical convergence. The accuracy here is a bit better than in the case of taking only real solutions, although not much different, being of the same order. Thus, the seventh-order real approximant has the error of 3\%, while the sum of all seventh-order approximants gives the error of 1\%. But dealing with only real solutions is simpler.

The accuracy of the additive approximants for the studied problem is much better than that of other approximants. Because of the incompatibility of the powers in the weak-coupling and strong-coupling limits, the standard Padé approximants are not applicable, but the modified Baker-Gammel approximants \( P_{N/(N+1)}^{1/4} \) have to be used. The modified Padé approximant of 19-th order (with \( N = 9 \)) has an error of 10\%, which is much worse than the error of 0.2\% of the additive real approximant. Factor approximants are also less accurate. Thus, the factor approximant of 9-th order yields an error of 11\%, while the additive approximant in this order exhibits an error of 2\%. Root approximants for this problem are not defined, resulting in complex solutions.
Another example that plays the role of a touchstone for any novel approximation method, since it has the structure typical of many applied problems, is the quartic anharmonic oscillator described by the Hamiltonian

\[
\hat{H} = -\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} x^2 + gx^4,
\]

where \(x \in (-\infty, \infty)\) and the anharmonicity strength \(g \in [0, \infty)\).

The ground-state energy is given by the lowest eigenvalue of this Hamiltonian. By perturbation theory [17, 18] with respect to the parameter \(g\), one has

\[
e_k(g) = \sum_{n=0}^{k} c_n g^n,
\]

with the first several coefficients

\[
c_0 = \frac{1}{2}, \quad c_1 = \frac{3}{4}, \quad c_2 = -\frac{21}{8}, \quad c_3 = \frac{333}{16},
\]

\[
c_4 = -\frac{30885}{128}, \quad c_5 = \frac{916731}{256}, \quad c_6 = -\frac{66518401}{1024}, \quad c_7 = \frac{2723294673}{2048}.
\]

The value of these coefficients quickly increases signifying strong divergence of the expansion.

In the large anharmonicity limit, the finite series

\[
e^{(p)}(g) = \sum_{n=1}^{p} b_n g^{\beta_n}
\]

have fractional powers as in the expansion below:

\[
e(g) \simeq b_1 g^{1/3} + b_2 g^{-1/3} + b_3 g^{-1} + b_4 g^{-5/3} + b_5 g^{-7/3} + b_6 g^{-3} + b_7 g^{-11/3} + \ldots,
\]

where \(g \to \infty\) and the first coefficients are

\[
b_1 = 0.667986, \quad b_2 = 0.143669, \quad b_3 = -0.008628,
\]

\[
b_4 = 0.000818, \quad b_5 = -0.000082, \quad b_6 = 0.000008.
\]

The general form of the powers in the large \(g\) expansion

\[
\beta_n = 1 - \frac{2n}{3}
\]

shows that the difference

\[
\Delta \beta \equiv \beta_n - \beta_{n+1} = \frac{2}{3}
\]

is constant. But, contrary to the previous case of the anharmonic partition integral, these powers are not invariant with respect to transformation (13). Really, since

\[
\beta_m - \beta_n = \frac{2}{3} (n - m),
\]
there are no such integers $m$ and $n$ that would give 1 in right-hand side of the above difference. Therefore, correction terms are required, as is explained in Sec. 2.

Defining all parameters of the additive approximants from the small $g$ expansion, we again have the situation, where the error increases with growing $g$. Therefore again the accuracy of the approximants is defined by that of the large $g$ amplitude

$$B_k = \lim_{g \to \infty} g^{-1/3} e_k^*(g) .$$  

(35)

In order to illustrate the influence of counter-terms, we, first, consider the additive approximants $e_{k,0}^*$, with real solutions. For instance, to third order, we have

$$e_{3,0}^* = A_1 (1 + \lambda g)^{1/3} + A_2 (1 + \lambda g)^{-1/3} + A_3 (1 + \lambda g)^{-1} ,$$

for which the accuracy-through-order procedure gives

$$A_1 = 0.324485 , \quad A_2 = 0.212357 , \quad A_3 = -0.036842 , \quad \lambda = 10.105351 .$$

The large $g$ amplitude (35) is

$$B_{3,0} = 0.701528 \quad (e_{3,0}^*) .$$

Continuing calculations to higher orders, we get the amplitudes

$$B_{5,0} = 0.681609 \quad (e_{5,0}^*) , \quad B_{7,0} = 0.675129 \quad (e_{7,0}^*) , \quad B_{9,0} = 0.672345 \quad (e_{9,0}^*) ,$$

$$B_{11,0} = 0.670931 \quad (e_{11,0}^*) , \quad B_{13,0} = 0.670022 \quad (e_{13,0}^*) , \quad B_{15,0} = 0.669619 \quad (e_{15,0}^*) ,$$

$$B_{17,0} = 0.669283 \quad (e_{17,0}^*) , \quad B_{19,0} = 0.669041 \quad (e_{19,0}^*) , \quad B_{21,0} = 0.668765 \quad (e_{21,0}^*) .$$

Comparing the accuracy of the approximants, with respect to the exact amplitude $B = b_1 = 0.667986$, we obtain the errors

$$5\% , \quad 2\% , \quad 1\% , \quad 0.7\% , \quad 0.4\% , \quad 0.3\% , \quad 0.24\% , \quad 0.19\% , \quad 0.16\% , \quad 0.12\% .$$

Therefore, numerical convergence is achieved even without counter-terms.

Now, let us take into account the required counter-terms. In the first nontrivial order, we have

$$e_{2,1}^* = A_1 (1 + \lambda g)^{1/3} + A_2 (1 + \lambda g)^{-1/3} + C_1 (1 + \lambda g)^{-2/3} .$$

The counter-term, with the power $-2/3$, is introduced for cancelling the incorrect terms in the large $g$ expansion, as is explained in Sec. 2. The accuracy-through-order procedure displays two real solutions giving the amplitudes 0.699953 and 0.668733, whose average is

$$B_{2,1} = 0.684343 \quad (e_{2,1}^*) .$$

The next approximant

$$e_{3,1}^* = A_1 (1 + \lambda g)^{1/3} + A_2 (1 + \lambda g)^{-1/3} + A_3 (1 + \lambda g)^{-1} + C_1 (1 + \lambda g)^{-2/3}$$

possesses one real solution and a complex-conjugate pair of solutions for the parameters $A_i$ and $C_1$. The real solution gives the amplitude 0.682509, while the average of the conjugate pair yields 0.677471. The average of all solutions results in the amplitude

$$B_{3,1} = 0.679990 \quad (e_{3,1}^*) .$$
The higher approximant
\[ e_{3,2}^* = A_1(1 + \lambda g)^{1/3} + A_2(1 + \lambda g)^{-1/3} + A_3(1 + \lambda g)^{-1} + C_1(1 + \lambda g)^{-2/3} + C_2(1 + \lambda g)^{-4/3} \]
contains two counter-terms. For its parameters, we get four solutions composing two complex-conjugate pairs. The averages of each pair give the amplitudes 0.672316 and 0.675572, which leads to the average amplitude
\[ B_{3,2} = 0.673944 \quad (e_{3,2}^*) . \]

Continuing in this way, with averaging over all solutions of the same order, we obtain the amplitudes
\[ B_{4,2} = 0.670643 \quad (e_{4,2}^*) , \quad B_{4,3} = 0.668888 \quad (e_{4,3}^*) , \quad B_{5,3} = 0.668109 \quad (e_{5,3}^*) . \]

Summarizing, for the approximants with counter-terms, we get the errors
\[ 2.4\% , \quad 1.8\% , \quad 0.9\% , \quad 0.4\% , \quad 0.1\% , \quad 0.02\% . \]

This shows numerical convergence and good accuracy.

Comparing these results with other approximation methods, we see that again the standard Padé approximants are not applicable, because of the incompatibility of powers in the small \( g \) and large \( g \) expansions. The modified Padé approximants \( \tilde{P}^{N/(N+1)}_{N} \) of Baker-Gammel [12] can be employed, although their accuracy is much worse than that of the additive approximants. For example, the rather high-order modified Padé approximant of 24-th order has the accuracy of 4\%. Factor approximants are more accurate than Padé approximants, but less accurate than additive approximants. Thus, the factor approximant of 9-th order gives the amplitude \( B_9 = 0.704391 \), which has the error of 5\%. The accuracy of root approximants is between that of modified Padé and factor approximants.

6 Electron correlation energy

It is also instructive to illustrate the efficiency of the method for the problems, where not many expansion terms are available, but rather just a few. As an example of such a problem, let us consider the calculation of correlation energy for electron gas. The correlation energy of electron gas is usually expressed in dimensionless units as a function of the Seitz radius \( r_s \). Then the limit of high density corresponds to \( r_s \to 0 \), while that of low density, to \( r_s \to \infty \).

For one-dimensional electron gas of high density [19], one has
\[ e(r_s) \simeq -\frac{\pi^2}{360} + 0.00845r_s \quad (r_s \to 0) . \quad (36) \]

And the low-density expansion gives
\[ e(r_s) \simeq \frac{b_1}{r_s} + \frac{b_2}{r_s^{3/2}} \quad (r_s \to \infty) , \quad (37) \]

where
\[ b_1 = -\left( \ln \sqrt{2\pi} - \frac{3}{4} \right) = -0.168939 , \quad b_2 = 0.359933 . \]
The powers in expansion (37) have the general form
\[ \beta_n = -\frac{n + 1}{2}. \tag{38} \]
The power difference
\[ \Delta \beta \equiv \beta_n - \beta_{n+1} = \frac{1}{2} \tag{39} \]
is constant. The set of the powers \( \{ \beta_j \} \) is invariant under transformation (13), since
\[ \beta_m - \beta_n = \frac{1}{2}(n - m), \tag{40} \]
which yields \( \beta_m - \beta_n = 1 \) for \( m = n - 2 \). Hence there is no need for counter-terms.

Here we can construct only the low-order additive approximants whose parameters are defined by the asymptotic expansions. For instance
\[ e^*_2(r_s) = A_1(1 + \lambda r_s)^{-1} + A_2(1 + \lambda r_s)^{-3/2}. \]
There are two complex-conjugate solutions for the parameters, so that the average of the related forms \( e^*_2 \) is real.

The next approximant is
\[ e^*_3(r_s) = A_1(1 + \lambda r_s)^{-1} + A_2(1 + \lambda r_s)^{-3/2} + A_3(1 + \lambda r_s)^{-2}. \]
There exist three solutions, a pair of complex-conjugate and one real. In the latter case, the parameters are
\[ A_1 = -0.040184, \quad A_2 = 0.041756, \quad A_3 = -0.028987, \quad \lambda = 0.237864. \]

The accuracy of the additive approximants can be compared with Monte Carlo numerical data [19] available for the range \( 0 < r_s < 20 \). Thus, the maximal error of the real additive approximant \( e^*_3 \) is 7% at the point \( r_s \approx 15 \). The two-point Padé approximants yield the errors in the range between 2% and 9% at the approximately same point \( r_s \). For instance, \( P_{1/2}(\sqrt{r_s}) \) has an error of 2%, while \( P_{0/3}(\sqrt{r_s}) \) gives an error of 9%. Factor approximants are rather accurate, with an error for \( e^*_3 \) of about 1%. But the root approximants are slightly less accurate, giving in third order an error of 8%.

7 Conclusion

We have introduced a novel type of approximants, whose derivation is based on the ideas of self-similar approximation theory. These approximants enjoy the same asymptotic expansion as the exact small-variable asymptotic expansion, and possess correct powers in the large-variable expansion. The efficiency of the method is illustrated by the examples possessing the structure typical of many problems in applied mathematics. Good numerical convergence is demonstrated for the cases that can be compared with exact solutions, when a number of terms in the small-variable asymptotic expansion are available. It is also shown that the approach is applicable to the problems having just a few terms in their asymptotic expansions.

Together with other approximation methods, in the frame of self-similar approximation theory, additive approximants provide a very efficient new approach for constructing accurate approximate solutions for different complicated problems.
The accuracy of additive approximants is not less and, as a rule, is essentially higher than that of Padé approximants, including modified Baker-Gammel Padé approximants [12]. Note that the accuracy of Padé approximants can be strongly improved by combining the method of self-similar approximants with that of Padé approximants, as is demonstrated in Ref. [20].

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