Box Covers and Domain Orderings for Beyond Worst-Case Join Processing

Kaleb Alway
University of Waterloo, Canada
kpalway@uwaterloo.ca

Eric Blais
University of Waterloo, Canada
eric.blais@uwaterloo.ca

Semih Salihoglu
University of Waterloo, Canada
semih.salihoglu@uwaterloo.ca

Abstract
Recent beyond worst-case optimal join algorithms Minesweeper and its generalization Tetris have brought the theory of indexing and join processing together by developing a geometric framework for joins. These algorithms take as input an index $B$, referred to as a box cover, that stores output gaps that can be inferred from traditional indexes, such as B+ trees or tries, on the input relations. The performances of these algorithms highly depend on the certificate of $B$, which is the smallest subset of gaps in $B$ whose union covers all of the gaps in the output space of a query $Q$. Different box covers can have different size certificates and the sizes of both the box covers and certificates highly depend on the ordering of the domain values of the attributes in $Q$. We study how to generate box covers that contain small size certificates to guarantee efficient runtimes for these algorithms. First, given a query $Q$ over a set of relations of size $N$ and a fixed set of domain orderings for the attributes, we give a $\tilde{O}(N)$-time algorithm that generates a box cover for $Q$ that is guaranteed to contain the smallest size certificate across any box cover for $Q$. Second, we show that finding a domain ordering to minimize the box cover size and certificate is NP-hard through a reduction from the 2 consecutive block minimization problem on boolean matrices. Our third contribution is an $\tilde{O}(N)$-time approximation algorithm to compute domain orderings, under which one can compute a box cover of size $\tilde{O}(K^r)$, where $K$ is the minimum box cover for $Q$ under any domain ordering and $r$ is the maximum arity of any relation. This guarantees certificates of size $\tilde{O}(K^r)$. Our results allow us to provide several new beyond worst-case bounds, which on some inputs and queries can be unboundedly better than the bounds stated in prior work.

2012 ACM Subject Classification General and reference → General literature; General and reference

Keywords and phrases Beyond worst-case join algorithms, Tetris, Box covers, Domain orderings

Digital Object Identifier 10.4230/LIPIcs.CVIT.2016.23

1 Introduction

Performing the natural join of a set of relational tables is one of the core operations in relational database management systems. After the celebrated result of Atserias, Grohe and Marx [3] that provided a tight bound on the maximum (or worst-case) size of natural join queries, now known as the AGM bound, a new class of worst-case optimal join algorithms were introduced whose runtimes are asymptotically bounded by the AGM bound. More recently, Ngo et al. and Abo Khamis et al., respectively, introduced the Minesweeper [21] algorithm, and its generalization Tetris [1], which adopt a geometric framework for joins and provide beyond worst-case guarantees that are closer to the highest algorithmic goal of instance optimality. Henceforth, we focus on the Tetris algorithm, the more general of these two algorithms.
Let \( Q \) be a query over a set of \( m \) relations \( R \). Unlike traditional join algorithms that operate on input tuples, Tetris takes as input a box cover \( B = \bigcup_{R \in R} B_R \), where each \( B_R \) is a set of gap boxes, i.e., tuple-free regions of the relation \( R \) whose union covers the complement of \( R \). For technical reasons, these boxes need to be of a particular form called dyadic, but we delay this technical discussion to Section 2. These boxes effectively imply regions in the output space of queries where output tuples cannot exist. Tetris operates on these gaps by performing geometric resolutions, which generate new gap boxes. The runtime of Tetris is bounded by \( \tilde{O}(C_{\square}(B)^w + Z) \) where: (i) \( C_{\square}(B) \) is the size of the box certificate for \( B \), which is the smallest subset of boxes in \( B \) that cover the output gaps; (ii) \( w \) is the treewidth of the query; and (iii) \( Z \) is the number of output tuples. Figure 1 shows an example of this geometric framework. The example is on query \( R(A, B) \cap S(A, C) \), purple unit boxes indicate input tuples, the boxes in the box cover are shown with rectangles and the boxes forming the certificate are drawn as red rectangles. This Tetris result is analogous to the data-optimal result of Yannakakis’s algorithm for acyclic queries and its combination with worst-case optimal join algorithms, which yields results of the form \( \tilde{O}(N^{fhtw} + Z) \), where \( fhtw \) is the fractional hypertree width and \( N \) is the number of tuples in the input. The performance of Tetris’s results can be significantly better than Yannakakis-based algorithms, as the certificates are always \( \tilde{O}(N) \) and can be \( o(N) \), e.g., constant size, on some inputs.

In reference [1] (and reference [21]), a box cover was assumed to be given to Tetris by the system and inferred from the available indexes on the relations. Consider a B+ tree index on a relation \( R(A, B) \) with sort order \( (A, B) \) and two consecutive tuples \( (a_1, b_1) \) and \( (a_1, b_2) \). From these two tuples, a system can infer a gap box \( (a_1, [b_1 + 1, b_2 - 1]) \) in the output space of any join query that involves \( R \). The boxes in Figure 1 are inferred from B+ tree indexes on \( R \) and \( S \) with sort orders \( (A, B) \) and \( (A, C) \), respectively. Using different indexes can result in very different box covers and lead to significantly different runtimes of Tetris. This motivates the first question we study in this paper:

**Question 1**: How can a system efficiently generate a good box cover for a set of relations?

Given a query \( Q \), let \( C_{\square}(Q) \) be the minimum certificate size across all possible box covers for the relations in \( Q \). An ideal goal for a system would be to generate a box cover whose

---

1 As in reference [1], throughout this paper \( \tilde{O} \) notation hides polylogarithmic factors in \( N \) as well as query or database schema dependent factors, such as the number of relations or attributes of a query.

2 A second upper bound that depends on the number of attributes instead of \( w \) is also provided in [1].

3 This example is borrowed from reference [1].

4 Note that our use of the notation \( C_{\square}(Q) \) is different from reference [1], where \( B \) was assumed to be given, and \( C_{\square}(B) \) was used to indicate the certificate size for \( B \). Since we drop this assumption, \( C_{\square}(B) \)
Figure 2 Two equivalent queries (up to attribute reordering) with different box certificate sizes. Figure 2b shows an example of this. In the example, the queries \( Q \) and \( Q' \) are both triangle queries joining three binary relations. These queries are equivalent up to reorderings. That is, it is possible to reorder the rows and columns of the grid in Figure 2a to obtain Figure 2b. Let \( \sigma \) be the set of three permutations on the domains of \( A, B, \) and \( C \) which transforms \( Q \) into \( Q' \). Specifically, for each attribute, \( \sigma \) maps the even values to values between 000 and 011, and the odd values to values between 100 and 111. Despite their equivalence up to reorderings, \( Q \) requires a box cover of size 96, as each white grid cell in Figure 2a must have a unit gap box covering it, while \( Q' \) only requires a box cover size 6. The same also applies to the certificate sizes, as every gap box in the box cover must also here denotes the certificate size for \( B \) and \( C_{\square}(Q) \) denotes the certificate size over all possible box covers.
Table 1 Overview of our results. We assume all problems take as input a query $Q$ and the set of tuples in the relations of $Q$. All problems are defined in terms of general boxes. $C_{\square}(Q)$ is the minimum certificate size of any box cover for $Q$. $K$ is the minimum box cover size under any domain ordering for $Q$. $r$ is the maximum arity of any relation in $Q$.

| Name       | Description                                           | Domains | NP-hard | Algorithm                      |
|------------|-------------------------------------------------------|---------|---------|--------------------------------|
| BoxMinC    | Find a box cover with a certificate of size $C_{\square}(Q)$. | Fixed   | Yes [7] | GAMB: $O(1)$-approx. to $C_{\square}(Q)$. |
| DomOrBoxMinC | Find a $\sigma$ s.t. $\text{BoxMinC under } \sigma$ is minimized. | Flexible | Yes (Thm [10]) | Open |
| DomOrBoxMinB | Find a $\sigma$ s.t. the smallest box cover size under $\sigma$ is minimized. | Flexible | Yes (Thm [10]) | ADORA: $\tilde{O}(K^r)$-approx. |

be part of the box certificate in this case. By extending the domains of the attributes in this example, the difference in box cover and certificate sizes can be made arbitrarily large. Therefore, a system could improve the performance of Tetris significantly by reordering the domains of attributes. Ideally, a system should find a domain ordering $\sigma$ such that $C_{\square}(\sigma(Q))$, the certificate size of $Q$ under $\sigma$, is minimized. We refer to this problem as $\text{DomOrBoxMinC}$. However, the size of the certificate is a quantity that is not directly measurable by the system and even when the domain orderings are fixed, we are unaware of a technique to estimate it without actually performing the join. Instead, the box cover size is an upper bound on the certificate size and is a quantity that a system can directly optimize for, which motivates our second question:

**Question 2:** How can a system efficiently reorder the domains to obtain a small box cover?

We refer to the problem of finding a domain ordering $\sigma$ such that the minimum box cover size under $\sigma$ is minimized as $\text{DomOrBoxMinB}$. Let $B^\ast$ be the minimum size box cover for a query under any domain ordering, $K = |B^\ast|$, and $\sigma^\ast$ be the ordering under which $B^\ast$ is achieved. We first provide a hardness result showing that computing $\sigma^\ast$ is NP-hard through a reduction from the 2 consecutive block minimization problem on boolean matrices [14]. We then provide an approximation algorithm, which we refer to as ADORA, to obtain the following result:

**Theorem 2.** Let $r$ be the maximum arity of any relation in $Q$. There is a $\tilde{O}(N)$-time algorithm that computes an attribute ordering $\sigma$ for $Q$, under which one can compute a box cover of size $\tilde{O}(K^r)$, guaranteeing a certificate of size $\tilde{O}(K^r)$.

In practice after $\sigma$ is obtained with ADORA, a system should run GAMB, as there can be smaller certificates under $\sigma$. Our algorithm is based on an intuitive and powerful heuristic that groups the domain values in an attribute that have identical value combinations in the remainder of attributes across the relations and makes the values in each group consecutive. Interestingly, our approximation ratio does not depend on any other parameters of the query, such as different notions of width or the number of relations. For example, on any query over binary relations, we can compute an ordering and a box cover whose certificate is of size $\tilde{O}(K^2)$. Once an ordering is obtained, Tetris can be executed on the reordered query and results converted back to the original domain. There are classes of inputs and queries where this combined algorithm can be unboundedly faster than running Tetris on a fixed but bad domain ordering. For reference, Table 1 gives an overview of our results.
2 Notation and Preliminaries

Throughout the paper, $Q$ refers to an equi-join query on a set of $m$ relations $R$ over a set of $n$ attributes $A$. At times, we will also refer to a database $D$ that consists of $m$ relations over $n$ attributes and consider queries running over any subset of the relations. As in reference [1], for ease of presentation we assume the domains of each attribute $A \in A$ consist of all $d$ bit integers but our results only require domain values to be discrete and ordered. For $R \in R$ and $A \in A$, the attribute set of $R$ is denoted $\text{attr}(R)$ and the domain of $A$ is denoted $\text{dom}(A)$.

Tetris takes as input a box cover $B$ that contains dyadic gap boxes, which are boxes whose span over each attribute is encoded as a binary prefix. Let $R \in R$ be a relation with $n$ attributes. Formally, a dyadic gap box in $B_R$ is an $n$-tuple $b = (s_1, s_2, \ldots, s_n)$ where each $s_i$ is a binary string of length at most $d$. We use $*$ to denote the empty string. For example, if $R$ is over attributes $A_1$ and $A_2$, where $d$ is 3, the dyadic box $(01, 1)$ indicates a box whose $A_1$ and $A_2$ dimensions include all values with prefix 01 and 1, respectively, i.e., the box is the rectangle with sides $[[010 - 011], [100 - 111]]$. The restriction to dyadic boxes has several benefits. First, dyadic boxes allow Tetris to perform geometric resolutions (explained momentarily) efficiently. Second, the runtime analysis of Tetris relies on the observation that there are only $\tilde{O}(1)$ dyadic boxes covering any point in an $n$-dimensional space.

Although the details of how Tetris works are not necessary to understand our techniques and contributions, we give a brief overview as background and refer the reader to reference [1] for more details. Assume each box in $B$, say those coming from $B_R$, are extended to every attribute not in $\text{attr}(R)$ with prefix $*$. This allows us to think of $B$ as a single gap box index over the output space. The core of Tetris is a recursive subroutine that determines whether the set of boxes covers the entire $n$-dimensional output space $\langle *, *, \ldots, * \rangle$ and returns either YES or NO with an output tuple $o$ as a witness. The witnesses are inserted into $B$. During the execution of the subroutine, the algorithm performs geometric resolutions that take two boxes that are adjacent in one dimension and construct a new box that consists of the union of the intervals in this dimension (and intersections in others). When boxes are dyadic, geometric resolutions can be done in $\tilde{O}(1)$ time. This recursive subroutine is called as many times as there are output tuples until it finally returns YES.

As long as the dyadic box index $B$ can in $\tilde{O}(1)$ time return all dyadic boxes that contain a given tuple $t$ in the output space, two variants of Tetris, called Tetris-Preloaded and Tetris-LoadBalanced, run in time $\tilde{O}(C_{\square}(B)^{n+1} + Z)$ and $\tilde{O}(C_{\square}(B)^{n/2} + Z)$, respectively (see Theorems 4.9 and 4.11 in reference [1]). Any set of dyadic boxes can easily be indexed to satisfy this requirement using a standard trie structure, referred to as a dyadic tree in reference [1], and checking every combination of prefixes of $t$ in the trie. $C_{\square}(B)$ in Tetris’s runtime is the box certificate size of $B$, which is the smallest subset of $B$ whose union equals $B$ (also the smallest subset that, when extended, covers all of the gaps in the output space). We are unaware of a technique to estimate this quantity without actually performing the join. Therefore, in the second question we study, we focus on generating small size box covers by ordering domains, since a small box cover is guaranteed to contain a small certificate.

We end this section with a note on dyadic vs. general boxes. The notions of certificate, box cover, and the problems we study could be defined in terms of dyadic or general boxes. Throughout the paper, except in Section 3, the term box refers to general boxes, and our optimization problems are always defined over general box covers and certificates. For both certificates and box covers, the minimum size obtained with dyadic boxes and general boxes are within $\tilde{O}(1)$ of each other. This is because a dyadic box is a general box by definition and any general box can be partitioned into $\tilde{O}(1)$ dyadic boxes (Proposition B.14 in reference [1]).
Therefore, our approximation results for general boxes imply approximation results for the dyadic versions of these problems up to $O(1)$ factors. However, a hardness result for one version of a problem does not imply hardness of the other, and our hardness results apply only to general boxes.

3 Related Work

3.1 Box Cover Problems

The complement of a relation $R$ with $k$ attributes can be represented geometrically as a set of axis-aligned, rectilinear polytopes in $k$-dimensional space, which may have holes (the tuples in $R$ form the exteriors of the polytopes). The number of vertices in these polytopes is bounded (up to a constant factor) by the number of tuples in the relation, since each vertex is formed where one or more tuples in $R$ are adjacent to some tuple not in $R$. Therefore our work is closely related to covering rectilinear polytopes with a minimum number of rectangles in geometry. This problem has been previously studied in the 2-dimensional setting, i.e., for polygons. The problem is known to be NP-complete, even when the polygon is hole-free [7] and MaxSNP-hard for polygons with holes [3]. There are several approximation algorithms for the problem. Franzblau [9] designed an algorithm that approximates the optimal solution to a factor of $O(\log n)$, where $n$ is the number of vertices in the polygon. If the polygon is hole-free, the approximation factor improves to 2. Anil Kumar and Ramesh [17] showed a tighter approximation ratio of $O(\sqrt{\log n})$ for the same algorithm on polygons with holes.

Franzblau et al. [10] also showed the problem is solvable in polynomial time in the special case when polygons are vertically convex. All of these results are limited to 2D and little is known about the problem in higher dimensions.

The approximation algorithms above can be used to generate box covers for covering the complement of a binary relation $R$. This is a special case of $\text{BoxMinC}$, where the input is a trivial query that consists of a single binary relation $R$. Besides this limited setting, when there is a second relation $S$ in the query, the connection of covering axis-aligned and rectilinear polygons to $\text{BoxMinC}$ breaks. This is because the certificate of a query in this case is essentially the smallest number of boxes that covers the complement of the output, using boxes from the relations. In this case, because the output is not yet computed, it is not known a priori which polytopes should be covered.

In Table 1, we listed three problems, $\text{BoxMinC}$, $\text{DomOrBoxMinC}$, and $\text{DomOrBoxMinB}$, for two of which we present approximation algorithms in this paper. There is a fourth related problem that is omitted from the table. Let $\text{BoxMinB}$ be the problem of finding the minimum size box cover for a query $Q$ under a fixed domain ordering and let $H$ be the size of this box cover. If each relation in $Q$ is binary, then we can use the approximation algorithms above to generate small size box covers that are of size $O(\sqrt{\log n})$. Although this approach can return a box cover whose size is smaller than the box cover that GAMB returns, it does not allow us to state better upper bounds for Tetris’s runtime. That is because GAMB’s output $B$ is guaranteed to contain the smallest size certificate of any box cover, up to a $O(1)$ factor, i.e., the box certificate of $B$ is of size $O(C_{\square}(Q))$, even though we may have $|B| = \omega(H)$.

There are variants of covering polygons that are less directly related to our problems. Reference [13] studies the more general problem of covering a set $P$ of polygons with only obtuse interior angles, of which rectilinear polygons are a subset, and provides approximation algorithms. Reference [18] studies covering the input polygon with squares instead of rectangles. For a survey of geometric covering and packing problems, including shapes beyond polytopes, such as spheres, we refer the reader to references [6] and [25].
3.2 Orderings in Matrices

There are several problems related to ordering the rows and columns of boolean matrices to achieve different optimization goals. The closest to our optimization goal of minimizing box cover sizes is the consecutive block minimization problem (CBMP) \[16\]. Our hardness results are based on a variant of CBMP, called 2 consecutive block minimization \[14\], which we review in Section 5.1. There are two other ordering problems for matrices, which are less related to our work. Testing a boolean matrix for the consecutive ones property is the problem of determining whether there is a column ordering of the matrix such that each row has only one consecutive block of ones \[5\]. Doubly lexical ordering of boolean matrices is the problem of finding a row and column ordering for a boolean matrix to make it doubly lexical \[19\], i.e., both rows and columns are in lexicographic order. Both problems have polynomial time solutions.

3.3 Worst-Case and Beyond Worst-Case Join Algorithms

A join algorithm is said to be worst-case optimal if it runs in time \(\tilde{O}(\text{AGM}(Q))\), where the AGM bound \[3\] is the worst-case upper bound on the number of output tuples for a query based on its shape and the number of input tuples. Examples of worst-case optimal join algorithms are Leapfrog Triejoin \[26\], the NPRR algorithm \[22\], and Generic Join \[23\]. A survey on worst-case optimal join algorithms can be found in \[20\]. There are several results that consider other properties of the query and provide worst-case upper bounds on the size of query outputs that are better than the AGM bound. Olteanu and Závodný \[24\] show that worst-case sizes of queries in factorized representations can be asymptotically smaller than the AGM bound and provide algorithms that meet these factorized bounds. Joglekar and Ré \[15\] developed an algorithm which provides degree-based worst-case results that assume knowledge of degree information for the values in the query. Similarly, references \[2\] and \[11\] provide worst-case bounds based on information theoretical bounds that take into account, respectively, more general degree constraints and functional dependencies.

There are several results that go beyond worst-case bounds and are closer to the notion of instance optimality. The earliest example is Yannakakis’ data optimal algorithm \[27\] for acyclic queries that runs in time \(O(N + Z)\). This result was later generalized to an algorithm \[8\] for arbitrary queries which runs in time \(\tilde{O}(N^{fhtw} + Z)\), where \(fhtw\) is the fractional hypertree width of the query \[12\]. The Minesweeper algorithm \[21\] developed the measure of comparison certificate \(C_{\text{comp}}\) for comparison-based join algorithms, which captures the minimum number of comparisons needed to prove that the output of a join query is correct. Minesweeper runs in time \(\tilde{O}(|C_{\text{comp}}|^{|w|^+1} + Z)\), where \(Z\) is the number of output tuples and \(w\) is the treewidth of the query. The Tetris algorithm \[1\], which motivates our work, generalizes the comparison certificate to the geometric notion of a box certificate, which we reviewed in Section 1. For every comparison certificate \(C_{\text{comp}}\), there is a corresponding box certificate of size at most \(|C_{\text{comp}}|\). In this sense, box certificates are stronger than comparison certificates, and Tetris subsumes the certificate-based results of Minesweeper. Our results on finding box covers with small certificates and domain orderings with small box covers improve the bounds provided by Tetris.

4 Generating a Box Cover

Since the runtime of Tetris depends on the certificate size of its input box cover, an important preprocessing step for the algorithm is to generate a box cover with a small certificate. Ideally,
a system should generate a box cover that contains a certificate of minimum size, across all box covers. We define this quantity as $C[□](Q)$ in Section [1]. We state two useful facts about dyadic boxes from reference [1] that we will use in our solution to this problem.

Lemma 3. (Propositions B.12 and B.14 [1]) Let $b$ be any dyadic box. Then there are $\tilde{O}(1)$ dyadic boxes which contain $b$. Let $b'$ be any (not necessarily dyadic) box. Then $b'$ can be partitioned into a set of $\tilde{O}(1)$ disjoint dyadic boxes whose union is equal to $b'$.

Let a dyadic gap box $b$ for a relation $R$ be maximal if $b$ cannot be enlarged in any of its dimensions and still remain a dyadic gap box, i.e., not include an input tuple of $R$. Observe that generating a box cover with certificate size $\tilde{O}(C[□](Q))$ can be done by generating a box cover that contains all maximal dyadic gap boxes in the input relations. This is because: (1) any general box can be decomposed into $\tilde{O}(1)$ dyadic boxes by Lemma 3, so decomposing a general box cover into a dyadic one can increase its certificate size by at most a factor of $\tilde{O}(1)$; and (2) expanding any non-maximal dyadic boxes to make them maximal can only decrease the size of the certificate. We show in this section that there are only $\tilde{O}(N)$ many maximal dyadic gap boxes in the relations of any $Q$ and generating a box cover with these is enough to generate a box cover with certificate size $\tilde{O}(C[□](Q))$.

Algorithm 1 shows the pseudocode for our algorithm GAMB that generates all maximal dyadic gap boxes for a relation $R$ in $\tilde{O}(N)$ time. GAMB loops over each dyadic box $b$ covering each tuple $t$ in $R$, explores boxes that are adjacent to $b$ (which may or may not be gap boxes) and inserts these into a set $B$. Then it subtracts the set of all dyadic boxes covering any tuples from $B$ to obtain a set of gap boxes. As we argue, this set contains every maximal dyadic gap box (and possibly some non-maximal ones). To generate all maximal boxes for a query $Q = (R, A)$, we can simply iterate over each $R \in R$ and invoke GAMB.

Algorithm 1 GAMB($R$): Generates all maximal dyadic gap boxes for $R$

1: $B := \emptyset$, $\overline{B} := \emptyset$
2: for $t \in R$ do
3: for every dyadic box $b$ such that $t \in b$ do
4: $B := B \cup \{b\}$
5: for $A \in \text{attr}(R)$ such that $b.A \neq \ast$ do
6: Let $b'$ be obtained from $b$ by flipping the last bit of $b.A$
7: $B := B \cup \{b'\}$
8: return $B \setminus \overline{B}$

Theorem 4. GAMB generates all maximal dyadic gap boxes of a relation $R$ in $\tilde{O}(N)$ time.

Proof. Let $b'$ be a maximal dyadic gap box for $R$. Let $A$ be an attribute of $R$ for which $b'$ specifies at least one bit (so $b'.A \neq \ast$). Let $b$ be the dyadic box obtained from $b'$ by flipping the last bit of $b.A$. Since $b'$ is maximal, $b$ contains at least one tuple $t \in R$. Since $b$ is a dyadic box containing $t$, some iteration of the for-loop on line 3 will reach box $b$. Then the for-loop on line 5 at some iteration will loop over $A$ and generate exactly $b'$ on line 6. Thus $b'$ is added to $B$ and since $b'$ is a gap box, GAMB will not add it to $\overline{B}$ (which only contains non-gap boxes). Therefore $b'$ will be in the output of GAMB. Note that the returned set

This is not true for general gap boxes. There can be a super-linear number of maximal general boxes in a relation (see Appendix A for an example).
does not contain any non-gap boxes of \(R\), since every box which contains any tuple of \(R\) is added to \(\mathcal{B}\). The outer-most for loop has \(N\) iterations. The for loop on line 3 has \(\tilde{O}(1)\) iterations by Lemma 3. The for-loop on line 5 has \(n\), so \(\tilde{O}(1)\), iterations. Finally, the set difference on line 8 can be done by sorting both \(B\) and \(\mathcal{B}\) and iterating lockstep through the sorted boxes. Therefore, the total runtime of GAMB is \(\tilde{O}(N)\). □

By our earlier observation based on Lemma 3 running GAMB as a preprocessing step is sufficient to generate a box cover with a certificate of size \(\tilde{O}(C_{\Box}(Q))\). Combined with runtime upper bounds of Tetris from reference [1], we can state the following corollary:

**Corollary 5.** Given a database \(D\) of relations, with \(\tilde{O}(N)\) preprocessing time, one can generate a box cover \(\mathcal{B}\) such that running Tetris on \(\mathcal{B}\) yields \(\tilde{O}\left((C_{\Box}(Q))^{w+1} + \tilde{Z}\right)\) or \(\tilde{O}\left((C_{\Box}(Q))^{n/2} + \tilde{Z}\right)\) runtimes for any query \(Q\) over \(D\).

Using the bounds from reference [1], these are the best bounds we can obtain up to \(\tilde{O}(1)\) factors when \(Q\) is fixed, since \(C_{\Box}(Q)\) is the minimum certificate size for \(Q\) under any box cover. To improve on these bounds, we must modify \(Q\) in some way which reduces the box certificate size. We next explore domain orderings as a method to go beyond these bounds.

## 5 Domain Ordering Problems

In this section we will study \(\text{DomOr}_{\text{BoxMinB}}\), an optimization problem over the space of domain orderings. Given a query \(Q\), our goal is to find the minimum size box cover which is possible under any domain ordering for \(Q\) and to find the domain ordering \(\sigma^*\) that yields this minimum possible box cover size. We begin by defining a domain ordering.

**Definition 6 (Domain ordering).** A domain ordering for a query \(Q = (\mathcal{R}, \mathcal{A})\) is a tuple of \(|\mathcal{A}|\) permutations \(\sigma = (\sigma_A)_{A \in \mathcal{A}}\) where each \(\sigma_A\) is a permutation of \(\text{dom}(A)\).

**Example 7.** Let \(A\) and \(B\) be attributes over 2-bit domains. Let \(R(A, B)\) be the following relation presented under the default domain ordering \([00, 01, 10, 11]\) for both \(A\) and \(B\):

\[
R(A, B) = \{(00, 00), (01, 11), (10, 00), (11, 11)\}
\]

Consider the domain ordering \(\sigma\) where \(\sigma_A = \sigma_B = [00 \mapsto 00, 01 \mapsto 10, 10 \mapsto 11, 11 \mapsto 01]\). We write \(\sigma\) as \(\sigma_A = [00, 11, 01, 10]\) to indicate the new “locations” of the previous domain values in the new ordering. Then \(\sigma(R)\) denotes the following relation:

\[
\sigma(R)(A, B) = \{(00, 00), (10, 01), (11, 00), (01, 01)\}
\]

The choice of domain ordering can have a significant effect on box cover sizes and their certificates. Appendix 3 describes families of queries one can generate from arbitrary query instances, that have large box covers and certificates under a default domain ordering, but have unboundedly smaller box covers and certificates under another domain ordering. Although the certificate size is known to directly influence the worst-case runtime of Tetris, even when the domain ordering is fixed, the only technique we are aware of to estimate the certificate size is to compute the join. Appendix B.3 in reference [1] describes how to do this using a variant of the Minesweeper algorithm. Instead, our primary goal in this section is

---

6 Appendix B also shows that our ADORA algorithm (Section 5.2) obtains these better orderings.

7 Appendix C of this paper shows that this can be done with a variant of Tetris as well.
to find a domain ordering which induces a box cover of minimum size because the minimum box cover size is an upper bound on the certificate size and a system can optimize for it directly. Our specific problem is:

**Definition 8**(BoxMinB). Let $K_{\square}^c(\sigma(Q))$ be the minimum box cover size one can obtain for the query $\sigma(Q)$ obtained from $Q$ by ordering the domains according to $\sigma$. Given a query $Q$, output a domain ordering $\sigma^*$ such that $K_{\square}^c(\sigma^*(Q)) = \min_{\sigma} K_{\square}^c(\sigma(Q))$.

In Section 5.1 we show that DomOrBoxMinB is NP-hard. In Section 5.2 we present ADORA, an approximation algorithm for DomOrBoxMinB. Section 5.2.3 combines ADORA and GAMB and states new beyond worst-case bounds our results imply.

### 5.1 DomOrBoxMinB is NP-hard

Our reduction is from the 2 consecutive block minimization problem (2CBMP) on boolean matrices [14]. In a boolean matrix $M$, a consecutive block is a maximal consecutive run of 1-cells in a single row of $M$, which is bounded on the left by either the beginning of the row or a 0-cell, and bounded on the right by either the end of the row or a 0-cell. We use $cb(M)$ to denote the total number of consecutive blocks in $M$ over all rows. 2CBMP, which we define next, was shown to be NP-hard in reference [14].

**Definition 9**(2CBMP). The 2-consecutive block minimization problem (2CBMP) takes as input a boolean matrix $M$ (stored as a 2D dense array) such that each row of $M$ contains at most 2 1-cells. The output is an ordering $\sigma^*_c$ on the columns of $M$ that minimizes the number of consecutive blocks. Formally, the problem is to find an ordering $\sigma^*_c$ on the columns of $M$ such that $cb(\sigma^*_c(M)) = \min_{\sigma} cb(\sigma(M))$.

**Theorem 10.** DomOrBoxMinB is NP-hard.

We give an outline of the proof here and provide the full proof in Appendix [13]. We focus on the special case where $Q$ contains a single relation $R(A,B)$ over exactly 2 attributes, and show that DomOrBoxMinB is NP-hard even in this case. Furthermore, this result implies that DomOrBoxMinB is also NP-hard, because when the input is only one relation, box cover minimization and box certificate minimization are the same problem. Let $M$ be an $n \times m$ boolean matrix input to 2CBMP. Our reduction constructs a $4n \times (m + 2n)$ matrix $M'$ from $M$ and inputs $M'$ to DomOrBoxMinB. In $M'$, the 0-cells correspond to an input tuple in dom($A$) × dom($B$) and 1-cells correspond to gaps. Readers can assume that $M'$ is given to DomOrBoxMinB in tuple format (so by giving only the 0-cells, which are the tuples). An example of $M$ to $M'$ transformation is shown in Figure [5]. For each row $r_i$ of $M$, we create 4 rows in $M'$: $r_i,1, r_i,2, p_i,1,$ and $p_i,2$: $r_i,1$ and $r_i,2$ are duplicates of the original row $r_i$ and $p_i,1$, and $p_i,2$ are the padding rows of $r_i$. We also add 2 padding columns that that contain 1s in the 4 rows of $r_i$ and 2n–2 empty columns that contain only 0s for the 4 rows for $r_i$.

To prove Theorem 10 we prove that there exists an ordering $\sigma_c$ on the columns of $M$ such that $cb(\sigma_c(M)) \leq k$ if and only if there exist orderings $\sigma' = (\sigma'_c, \sigma'_s)$ on the rows and columns of $M'$ such that $\sigma'(M')$ admits a box cover of size at most $k + 2n$. Showing the left to right direction of this claim is simple. That is, if there is a column ordering $\sigma_c$ for

---

This implies DomOrBoxMinB is NP-hard for any number of attributes and relations, since one can duplicate $R$ to another relation $S$ with the same schema, and extend $R$ and $S$ to a third attribute $C$, taking $R' = R \times \text{dom}(C)$. The result is a trivial intersection query $R' \bowtie S'$ for which the ordering that solves DomOrBoxMinB also minimizes the box cover size for the original relation $R$. 
\[ M = \begin{bmatrix} r_1 & 1 & 0 & 0 \\ r_2 & 0 & 0 & 1 \end{bmatrix} \Rightarrow M' = \begin{bmatrix} p_{1,1} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ p_{1,2} & 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ p_{1,3} & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ p_{2,1} & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ p_{2,2} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ p_{2,3} & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ p_{2,4} & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \]

**Figure 3** An example of the 2CBMP input matrix \( M \) and its corresponding \( M' \) matrix.

\( M \) that yields \( k \) consecutive blocks, simply transforming the ordered left \( n \) columns of \( M' \) according to \( \sigma_e \) without modifying \( M' \) in any other way yields a box cover of size \( k + 2n \) boxes. For example in Figure 3 there are 3 consecutive blocks in the default ordering of \( M \) and \( 3 + 2 \cdot 2 = 7 \) highlighted boxes in \( M' \). The converse is significantly more involved. Let \( \sigma' = (\sigma'_c, \sigma'_e) \) be an ordering on the rows and columns of \( M' \) such that \( \sigma'(M') \) admits a box cover \( B \) of size \( k + 2n \). We show in Appendix \( D \) that we can transform \( \sigma'_c \) to exactly match the default ordering of \( M' \) and we can transform \( \sigma'_e \) so that the last \( 2n \) columns exactly match the last \( 2n \) columns in the default ordering of \( M' \), without increasing the number of boxes. This implies a column ordering \( \sigma_e \) for \( M \) that yields \( k \) consecutive blocks.

We note that this result does not imply that finding a domain ordering that minimizes the dyadic box cover size is NP-hard. This problem remains open. Finally, Appendix \( E \) shows that the problem of ordering the domain of only one attribute \( A \) to minimize the box cover (or box certificate) size, when the other orderings are fixed, is also NP-hard.

### 5.2 Approximating \( \text{DomOr}_{\text{BoxMinB}} \)

In this section, we provide an efficient approximation algorithm for \( \text{DomOr}_{\text{BoxMinB}} \). Section 5.2.1 develops some machinery necessary to prove our approximation ratio, and Section 5.2.2 presents our approximation algorithm, ADORA. Section 5.2.3 combines ADORA and GAMB with the results of Tetris to state new beyond worst-case bounds for join processing.

#### 5.2.1 Dividing Relations into Hyperplanes

In the simplest case, suppose that the best domain ordering \( \sigma^* \) for \( Q \) satisfies \( K_{\Delta}(\sigma^*(Q)) = 1 \). Then there is a single gap box \( b \) in some relation \( \sigma^*(R) \) of \( \sigma^*(Q) \) such that \( B = \{b\} \) forms a box cover for \( \sigma^*(Q) \). Fix an arbitrary attribute \( A \in \text{attr}(R) \). We can partition the domain of \( A \) into two sets: values which are in the \( A \)-range spanned by \( b \), and values which are not. Since there is only one box in the cover, this partition is the only meaningful way to differentiate between two values of \( \text{dom}(A) \) in \( Q \). Consider the domain ordering \( \sigma_A \) obtained by placing all the domain values spanned by \( b \) first (in any order), followed by all other values. Doing this for each \( A \in \mathcal{A} \) would obtain a domain ordering \( \sigma = (\sigma_A)_{A \in \mathcal{A}} \) which recovers the box \( b \) and attains the minimum box cover size of 1. Intuitively, any domain values for \( A \) which lie in the span of the same set of boxes in the minimum box cover should be placed adjacent to one another. This intuition can be generalized to an approximation algorithm which works for any minimum box cover size. We begin with several necessary definitions to make this intuition more formal.

**Definition 11** \( (A\text{-hyperplane}) \). Let \( R \in \mathcal{R} \) be over a set of attributes \( \text{attr}(R) \). Let \( A \in \text{attr}(R) \) and \( a \in \text{dom}(A) \). The \( A\)-hyperplane of \( R \) defined by \( a \) is the relation \( H(R, A, a) = \pi_{\text{attr}(R) \setminus \{A\}}(\sigma_{A=a}(R)) \).
Let $n_R = |\text{attr}(R)|$. The $A$-hyperplane defined by $a$ in $R$ can be thought of as the “slice” of the $n_R$-dimensional space occupied by $R$ containing only the $(n_R - 1)$-dimensional subspace where the attribute $A$ is fixed to the value $a$. This is a natural generalization of “rows” and “columns” which were useful for discussing 2-dimensional relations in Section 5.1.

**Definition 12 (Equivalent domain values).** Let $Q = (R,A)$ be a query, let $A \in \mathcal{A}$, and let $a_1, a_2 \in \text{dom}(A)$. $a_1$ and $a_2$ are equivalent in $Q$ if for all $R \in \mathcal{R}$ we have $H(R,A,a_1) = H(R,A,a_2)$. In this case, we write $a_1 \sim a_2$.

For $a \in \text{dom}(A)$, the subset of domain values $\text{Eq}(a) = \{a' \in \text{dom}(A) : a \sim a'\} \subseteq \text{dom}(A)$ is called the equivalence class of $a$. The equivalence classes for all of the values in $\text{dom}(A)$ form a partition of $\text{dom}(A)$. The next lemma bounds the size of this partition, i.e. the number of equivalence classes, as a function of the minimum box cover size of any domain ordering $\sigma$.

**Lemma 13.** Let $\sigma$ be a domain ordering for $Q = (R,A)$. Let $A$ be an attribute in $A$ and $h$ be the number of equivalence classes of the values in $\text{dom}(A)$. Then $h \leq 2 \cdot K_{\square}(\sigma(Q)) + 1$.

**Proof.** Let $A \in \mathcal{A}$ and let $a_1, a_2 \in \text{dom}(A)$ be such that $a_1$ directly precedes $a_2$ in $\sigma_A$, i.e., $a_1$ is immediately to the left of $a_2$, and $a_1 \not\sim a_2$. We refer to the $a_1, a_2$ boundary as a “switch” along the $A$ attribute. First observe that there are at least $h - 1$ switches along $A$. This minimum is attained when the values in each equivalence class are placed in a single consecutive run in $\sigma_A$. Since $a_1 \not\sim a_2$, there is some relation $R \in \mathcal{R}$ such that $H_1 = H(R,A,a_1) \neq H(R,A,a_2) = H_2$. Then there is some tuple $t$ which is in $H_1$ but not $H_2$ or vice versa. Without loss of generality, assume $t \in H_1$ and $t \not\in H_2$. Let $t_1 = (a_1,t)$ and $t_2 = (a_2,t)$ be the tuples that extend $t$ to the $A$ attribute with values $a_1$ and $a_2$, respectively. This means that $t_1 \in R$ and $t_2 \not\in R$. Let $\mathcal{B}$ be a box cover for $\sigma(Q)$ with $K_{\square}(\sigma(Q))$ boxes. Let $\mathcal{B}_R$ be the set of boxes in $\mathcal{B}$ that are from $R$ and cover the complement of $R$ (so $|\mathcal{B}_R| \leq K_{\square}(\sigma(Q))$). Let $b \in \mathcal{B}_R$ be a box covering $t_2$ (and not $t_1$ since $b$ is a gap box). Since $t_1$ and $t_2$ are adjacent in $\sigma_A$, one face of $b$ along the $A$ axis is exactly the $(a_1,a_2)$ switch. More formally, one face of $b$ lies exactly on the $H_1$, $H_2$ hyperplane boundary. Note that every box $b$ has exactly two faces on dimension $A$, so there are at most $2K_{\square}(\sigma(Q))$ faces of boxes in $\mathcal{B}$ along the $A$ axis. Note also that two different switches cannot correspond to the same face of the same box. As an example, Figure 4 shows the switches in attribute $A$ and the faces of gap boxes that these switches correspond to, which are highlighted in colour. This completes the argument that each $(a_1,a_2)$ switch corresponds to a (distinct) face of some box along the $A$ axis. There are at least $h-1$ switches and at most $2K_{\square}(\sigma(Q))$ different box faces, so $h \leq 2K_{\square}(\sigma(Q)) + 1$. ▶
Algorithm 2 ADORA\((Q = (\mathcal{R}, \mathcal{A})):\) Computes a domain ordering

1: for \(A \in \mathcal{A}\) do
2: \(\sigma_A := \text{ORDERAttr}(Q, A)\)
3: return \(\sigma = \{\sigma_A\}_{A \in \mathcal{A}}\)

Algorithm 3 ORDERAttr\((Q, A)\): Groups equivalence classes for \(A\) into consecutive runs

1: \(\phi := \) any attribute ordering of \(\mathcal{A}\) which places \(A\) first
2: \(S := \{R \in \mathcal{R} : A \in \text{attr}(R)\}, D := \bigcup_{R \in S} \pi_A(R), T := \emptyset\)
3: for \(R \in S\) do
4: Sort \(R\) lexicographically according to \(\phi\)
5: for \(a \in D\) do
6: \(T[a] := []\)
7: for \(R \in S\) in a fixed order do
8: \(T[a].append(H(R, A, a))\)
9: Sort \(D\) by ordering \(a_i\) and \(a_j\) according to the lexicographic order of \(T[a_i]\) and \(T[a_j]\)
10: return \(\sigma_A = D\) (append \(a \notin D\) to \(\sigma_A\) in arbitrary order)

Lemma 13 inspires an approximation algorithm for \(\text{DomOrBoxMinB}\), which we present next.

5.2.2 ADORA

Let \(\sigma^*\) be the optimal domain ordering for \(\text{DomOrBoxMinB}\) on \(Q\). Let \(K = K_\varnothing(\sigma^*(Q))\) throughout this section. Algorithm 2 presents the pseudocode for our Approximate Domain Ordering Algorithm (ADORA). ADORA uses Algorithm 3 as a subroutine to produce an ordering \(\sigma_A\) for \(\text{dom}(A)\) that contains each equivalence class in \(\text{dom}(A)\) as a consecutive run. In the next theorem, we argue that \(Q\) under the ordering output by ADORA yields a box cover of size \(\tilde{O}(K^r)\) and that the runtime of ADORA is \(\tilde{O}(N)\).

\(\blacktriangleright\) Theorem 14. Let \(Q = (\mathcal{R}, \mathcal{A})\) be a query and \(\sigma^*\) be an optimal domain ordering for \(\text{DomOrBoxMinB}\) on \(Q\). Let \(K = K_\varnothing(\sigma^*(Q))\). Then ADORA produces a domain ordering \(\sigma\) in \(\tilde{O}(N)\) time such that \(K_\varnothing(\sigma(Q)) = \tilde{O}(K^r)\), where \(r\) is the maximum arity of a relation in \(\mathcal{R}\).

Proof. We begin by arguing that given an attribute \(A\), the ordering returned by Algorithm 3 places every equivalence class of \(\text{dom}(A)\) in a single consecutive run. The for-loop beginning on line 5 iterates over each \(a\) value in \(\text{dom}(A)\) that appears somewhere in \(Q\) and constructs an array \(\mathcal{T}[a]\). \(\mathcal{T}[a]\) is the result of appending the \(A\)-hyperplanes \(H(R, A, a)\) for each \(R \in S\) in a fixed order. Furthermore, line 3 sorts each relation lexicographically starting with \(A\) (notice that the order \(\phi\) is defined to place \(A\) first). These two facts ensure that after the for-loop beginning on line 5 has finished, \(\mathcal{T}[a_1] = \mathcal{T}[a_2]\) if and only if \(a_1 \sim a_2\). The final sort of \(D\) on line 8 sorts values of \(\text{dom}(A)\), say \(a_i\) and \(a_j\), according to the lexicographic order of \(\mathcal{T}[a_i]\) and \(\mathcal{T}[a_j]\), which ensures that all \(A\) values that are in the same equivalence class will be in a single consecutive run. This sorted \(D\) is the output of Algorithm 3.

The output of ADORA is a domain ordering \(\sigma\), which orders each attribute \(A \in \mathcal{A}\) according to the \(\sigma_A\) returned by Algorithm 3. We next prove that there exists a box cover for \(\sigma(Q)\) of size \(\tilde{O}(K^r)\). Let \(R \in \mathcal{R}\). Suppose \(\text{attr}(R) = n_R\) and note that \(n_R \leq r\). Let \(A \in \text{attr}(R)\). Lemma 13 states that \(\text{dom}(A)\) contains at most \(2K + 1\) equivalence classes, which we proved are placed consecutively in \(\sigma_A\). By definition, if \(a_1 \sim a_2\), then \(H(R, A, a_1) = H(R, A, a_2)\). These facts imply that \(\sigma_A\) consists of a sequence of at most \(2K + 1\)
consecutive runs of \(A\)-values where the values in each run have identical \(A\)-hyperplanes in \(R\). This holds for all \(A \in \text{attr}(R)\). The runs of identical hyperplanes partition the \(n_R\)-dimensional space of \(\sigma(R)\) into at most \((2K + 1)^{n_R}\) many \(n_R\)-dimensional grid boxes. Each dimension of a grid box is formed by one of the (at most) \(2K + 1\) runs from one attribute. Note that by construction, these grid boxes form a partition of the \(n_R\)-dimensional space as each grid box is a distinct combination of equivalence classes for the attributes and the orderings returned by Algorithm 3 cover all the values in \(\text{dom}(A)\). Figure 4b shows the example grid boxes implied by the equivalence classes in the orderings of a relation. In the figure, there are two equivalence classes for attribute \(A\) and three for \(B\), dividing the relation into 6 grid boxes.

We next argue that each grid box is completely full of either gaps or tuples (as can be verified for the example in Figure 4b). Let \(t\) be a tuple in \(R\), let \(g\) be the grid box containing \(t\), and let \(t'\) be another point in \(g\). Consider moving from \(t\) to \(t'\) through any sequence of adjacent points in \(g\). Each time we pass through a point we are moving from one \(A\)-hyperplane to an identical \(A\)-hyperplane for some attribute \(A\). Thus every point along this path must also be a tuple in \(R\). A similar argument for gaps implies that every point in a grid box that contains one gap must also be a gap. Since the grid boxes partition the entire space of \(R\), constructing one box for each gap grid box results in a box cover \(B_R\) for \(R\). Since there are at most \((2K + 1)^{n_R}\) grid boxes, \(|B_R| \leq (2K + 1)^{n_R}\). We can construct such a box cover for each \(R \in \mathcal{R}\) to obtain a box cover for \(\sigma(Q)\) of size \(\sum_{R \in \mathcal{R}} (2K + 1)^{n_R} \leq m(2K + 1)^r = \tilde{O}(K^r)\), completing the proof of ADORA’s approximation ratio.

Finally, we analyze the runtime of ADORA, which calls Algorithm 3, \(n\), so \(\tilde{O}(1)\), times. In Algorithm 3, the sorting of \(m\) relations according to \(\phi\) on line 4 takes \(\tilde{O}(N)\) time. The for-loop beginning on line 5 iterates over each domain value \(a \in D\) and each \(R \in \mathcal{S}\) and appends \(H(R, A, a)\) to \(T[a]\). Since \(R\) was sorted lexicographically according to \(\phi\), which places \(A\) as the first relation, all tuples with the same \(A\)-value are now consecutive in \(R\). Therefore, with a single linear pass through \(R\), we can compute all of the hyperplanes \(H(R, A, a)\). We do this for each relation, so the runtime is bounded by \(O(mN) = \tilde{O}(N)\). For the final sorting of \(D\) on line 8, observe that the total size of the array \(T\), summed over all domain values \(a\), is at most \(N\). Thus, we are sorting an array of arrays where the total amount of data is of size \(\tilde{O}(N)\), which can be done in \(\tilde{O}(N)\) time (e.g., with a merge-sort algorithm that merges two sorted sub-arrays in \(\tilde{O}(N)\) time), completing our proof.

Appendix B shows that our analysis of ADORA’s approximation factor is asymptotically tight by showing a family of queries over binary relations which have orderings with \(K\) box covers, whereas the orderings that ADORA returns have \(\Omega(K^2)\) boxes.

### 5.2.3 \(\text{DomOr}_{\text{BoxMinB}}\) and Join Processing

We put together ADORA, GAMB and Tetris in a new join algorithm we call \(\text{TetrisReordered}\) to obtain new beyond worst-case optimal results for join queries. Algorithm 4 presents the pseudocode for TetrisReordered.

**Corollary 15.** Let \(Q = (\mathcal{R}, A)\) be a join query. Let \(w\) be the treewidth of \(Q\), \(n = |A|\), and \(r\) the maximum arity of a relation in \(\mathcal{R}\). Let \(\sigma^*\) be an optimal solution to \(\text{DomOr}_{\text{BoxMinB}}\) on \(Q\) and let \(K = K_{\mathcal{R}}(\sigma^*(Q))\). \(\text{TetrisReordered}\) computes \(Q\) in \(\tilde{O}(N + K^{r + 1/2} + Z)\) time by using \(\text{Tetris-Reloaded}\) or in \(\tilde{O}(N + K^{rn/2} + Z)\) time by using \(\text{Tetris-LoadBalanced}\) as a subroutine.

Corollary 15 immediately follows from Theorems 4.9 and 4.11 from reference [1]. As a technical detail, we note that once TetrisReordered computes a query under the reordered domain \(\sigma\), it needs to convert the results back to
Algorithm 4 \textsc{TetrisReordered}($Q$):

1: $\sigma := \text{ADORA}(Q)$ (Algorithm 2)
2: for $R \in R$ do
3: \hspace{1em} $B_R := \text{GAMB}(\sigma(Q))$ (Algorithm 1)
4: return $\sigma^{-1}(\text{Tetris}(B = \{B_R\}_{R \in R}))$

the original domain ordering using $\sigma^{-1}$. By keeping a sorted map of the orderings, this can be done in $\tilde{O}(Z)$ time. For many queries, these bounds represent an improvement on the prior bounds provided in reference [1] where the domain ordering was assumed to be fixed. Appendix B presents classes of queries for which these bounds are unboundedly smaller than any prior bound with a fixed domain ordering. However, these bounds are not always guaranteed to be better than the bounds provided by Tetris on a fixed domain ordering of $Q$. One can easily run \textsc{TetrisReordered} in parallel with Tetris under the original domain ordering, one step of computation from each algorithm at a time, and return the output of the first algorithm that finishes. This avoids an asymptotic slow down of Tetris but speeds up Tetris unboundedly on classes of inputs that benefit significantly from reordering domains.

6 Conclusions

For queries with fixed domain orderings, we established a $\tilde{O}(N)$-time algorithm GAMB, which a system can use to create a single globally good box cover index that is guaranteed to contain a certificate that is at most a $\tilde{O}(1)$ factor away from the minimum size certificate across any box cover. We then studied the $\text{DomOrBoxMinB}$ problem. Given a query $Q$, $\text{DomOrBoxMinB}$ is the problem of finding a domain ordering that yields the smallest possible box cover size. We proved that $\text{DomOrBoxMinB}$ is NP-hard and presented a $\tilde{O}(N)$-time approximation algorithm ADORA that can compute an ordering which yields a box cover of size $\tilde{O}(K^r)$, where $K$ is the minimum box cover size under any ordering and $r$ is the maximum arity of any relation in $Q$. We combined ADORA, GAMB, and Tetris in an algorithm we call \textsc{TetrisReordered} and stated new beyond worst-case optimal runtimes for join processing in Corollary 15. \textsc{TetrisReordered} can improve the known performance bounds of prior versions of Tetris in cases where the input has a bad domain ordering.

Our work leaves several interesting problems unsolved, which are avenues for future work. First, whether or not our $\tilde{O}(K^r)$ approximation ratio can be improved to say a $\tilde{O}(1)$ ratio is unknown. Recall that we referred to Appendix F to show examples where ADORA’s approach, which is based on making equivalent domain values adjacent to one another (based on $A$-hyperplane equivalence), can yield $\Omega(K^r)$ boxes. This implies that if there is a better approximation ratio it will need to use a different algorithmic step than ADORA. Second, very little is known about the $\text{DomOrBoxMinC}$ problem, that aims to find an ordering that minimizes the certificate size. This is a very important question since the box certificate size directly influences the runtime of Tetris. The notion of certificate, however, seems very difficult to estimate from the properties of the input relations or box covers. Even when the domain orderings are fixed, the only approach we know to compute the size of the certificate is to perform the join. A good first step toward solving $\text{DomOrBoxMinC}$ would be to develop techniques to estimate the certificate size without running the join under a fixed ordering.
References

1. Mahmoud Abo Khamis, Hung Q. Ngo, Christopher Ré, and Atri Rudra. Joins via geometric resolutions: Worst case and beyond. *ACM Transactions on Database System*, 41(4), December 2016.

2. Mahmoud Abo Khamis, Hung Q. Ngo, and Dan Suciu. Computing join queries with functional dependencies. In *Proceedings of the 35th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, 2016.

3. Albert Atserias, Martin Grohe, and Dániel Marx. Size bounds and query plans for relational joins. *SIAM Journal on Computing*, 42(4), 2013.

4. Piotr Berman and Bhaskar DasGupta. Complexities of efficient solutions of rectilinear polygon cover problems. *Algorithmica*, 17(4), April 1997.

5. Kellogg S. Booth and George S. Lueker. Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *Journal of Computer and System Sciences*, 13(3), December 1976.

6. John Horton Conway and Neil James Alexander Sloane. *Sphere Packings, Lattices and Groups*, volume 290. Springer Science & Business Media, 2013.

7. Joseph C. Culberson and Robert A. Reckhow. Covering polygons is hard. *Journal of Algorithms*, 17(1), July 1994.

8. Rina Dechter and Judea Pearl. Tree-clustering schemes for constraint-processing. In *Proceedings of the Seventh AAAI National Conference on Artificial Intelligence*, 1988.

9. Deborah S. Franzblau. Performance guarantees on a sweep-line heuristic for covering rectilinear polygons with rectangles. *SIAM Journal on Discrete Mathematics*, 2(3), August 1989.

10. Deborah S. Franzblau and Daniel J. Kleitman. An algorithm for covering polygons with rectangles. *Information and Control*, 63(3), December 1984.

11. Georg Gottlob, Stephanie Tien Lee, Gregory Valiant, and Paul Valiant. Size and treewidth bounds for conjunctive queries. *Journal of the ACM*, 59(3), June 2012.

12. Martin Grohe and Dániel Marx. Constraint solving via fractional edge covers. *ACM Transactions on Algorithms*, 11(1), October 2014.

13. Joachim Gudmundsson and Christos Levcopoulos. Close approximations of minimum rectangular coverings. *Journal of combinatorial optimization*, 3(4), December 1999.

14. Salim Haddadi. A note on the NP-hardness of the consecutive block minimization problem. *International Transactions in Operational Research*, 9, November 2002.

15. Manas R. Joglekar and Christopher M. Ré. It’s all a matter of degree: Using degree information to optimize multiway joins. In *19th International Conference on Database Theory*, 2016.

16. Lawrence T. Kou. Polynomial complete consecutive information retrieval problems. *SIAM Journal on Computing*, 6(1), March 1977.

17. V.S. Anil Kumar and H. Ramesh. Covering rectilinear polygons with axis-parallel rectangles. *SIAM Journal on Computing*, 32(6), October 2003.

18. Christos Levcopoulos and Joachim Gudmundsson. Approximation algorithms for covering polygons with squares and similar problems. In *International Workshop on Randomization and Approximation Techniques in Computer Science*, 1997.

19. Anna Lubiw. Doubly lexical orderings of matrices. *SIAM Journal on Computing*, 16(5), October 1987.

20. Hung Q. Ngo. Worst-case optimal join algorithms: Techniques, results, and open problems. In *Proceedings of the 37th ACM SIGMOD-SIGACT-SIGAI Symposium on Principles of Database Systems*, 2018.

21. Hung Q. Ngo, Dung T. Nguyen, Christopher Ré, and Atri Rudra. Beyond worst-case analysis for joins with Minesweeper. In *Proceedings of the 33rd ACM SIGMOD-SIGACT-SIGART Symposium on Principles of Database Systems*, 2014.

22. Hung Q Ngo, Ely Porat, Christopher Ré, and Atri Rudra. Worst-case optimal join algorithms. *Journal of the ACM*, 65(3), March 2018.
Figure 5 Example of a relation $R_8(A, B)$ with $\omega(N)$ maximal general gap boxes.

A Example with $\omega(N)$ Maximal General Gap Boxes

In Section 4, GAMB is able to generate all maximal dyadic gap boxes in $\tilde{O}(N)$ time. The choice to use dyadic boxes instead of general boxes in GAMB is necessary, because there are relations for which the number of maximal general gap boxes is asymptotically greater than the number of tuples in the relation.

Our construction generalizes the example in Figure 15 in Appendix B.3 of reference [1]. Let $N$ be an even number, let $A$ and $B$ be attributes over domains of size $N$, and let $R_N$ be the following relation.

$R_N(A, B) = \{(i, N/2-i-1) : 0 < i < N/2\} \cup \{(N/2+i, N-i-1) : 0 < i < N/2\}$

Consider the following sets of tuples which are not in $R_N$.

$T_N = \{t_i = (i, N/2-i) : 0 < i < N/2\}$

$S_N = \{s_i = (N/2+i-1, N-i-1) : 0 < i < N/2\}$

Figure 5 depicts $R_8$ and all of the tuples in $T_8$ and $S_8$. In this diagram, a set of 5 maximal general gap boxes with their bottom left corners located at $t_2$ is depicted. The top right corners of these boxes correspond to the 5 tuples in $S_8$. In fact, for each $0 \leq i \leq 4$, there are either 4 or 5 maximal general gap boxes in $R_8$ with their bottom left corner at $t_i$.

This property generalizes from $R_8$ to any value of $N$. For each $0 \leq i \leq N/2$, there are at least $N/2$ maximal general gap boxes in $R_N$ with their bottom left corner at $t_i$. Since there are $N/2 + 1$ tuples in $T_N$, the total number of maximal general gap boxes in $R_N$ is at least $(N/2 + 1)(N/2) = \Theta(N^2) = \omega(N)$. 

---

23 Hung Q Ngo, Christopher Ré, and Atri Rudra. Skew strikes back: New developments in the theory of join algorithms. *ACM SIGMOD Record*, 42(4), February 2014.

24 Dan Olteanu and Jakub Závodný. Size bounds for factorised representations of query results. *ACM Transactions on Database Systems*, 40(1), March 2015.

25 Gábor Fejes Tóth and Wlodzimierz Kuperberg. A survey of recent results in the theory of packing and covering. In *New Trends in Discrete and Computational Geometry*, 1993.

26 Todd L. Veldhuizen. Leapfrog triejoin: A simple, worst-case optimal join algorithm. In *Proceedings of the 17th International Conference on Database Theory*, 2014.

27 Mihalis Yannakakis. Algorithms for acyclic database schemes. In *Proceedings of the 7th International Conference on Very Large Data Bases*, 1981.
B Classes of Queries That Benefit From Domain Reordering

The choice of domain ordering can have a significant effect on the box cover (and box certificate) size for \( Q \), and therefore on the runtime of Tetris on \( Q \). We begin by giving an example, which is a generalization of Figure 2 and is adapted from the proof of Lemma G.5 in reference [1], to demonstrate this on a clique query. We then show how to take an arbitrary query \( Q \) and a default domain ordering for \( Q \) and generate families of query instances that can benefit significantly from domain orderings with ADORA.

Example 16. Let \( B^n = \{1^{i-1}0 : i \in [n-1]\} \cup \{1^{n-1}\} \). This set is useful because it contains \( n \) binary prefixes that partition the set binary strings of length \( n-1 \), \{0, 1\}^n. We will use this set to define our query. For any integers \( n \geq 2 \) and \( d \geq 0 \), define \( Q_{n,p} = (R, A) \) by \( A = \{A_1, \ldots, A_n\} \) and \( R = \{R_{i,j}(A_i, A_j) : i \neq j \in [n]\} \). This means \( Q_{n,p} \) is a clique query over \( n \) attributes and \( \binom{n}{2} \) binary relations. Each attribute \( A_i \in A \) has a \((p+n-2)\)-bit domain, so \( \text{dom}(A_i) = \{0, 1\}^{p+n-2} \). For each \( i \neq j \in [n] \), the relation \( R_{i,j} \) is defined as

\[
R_{i,j}(A_i, A_j) = \left\{(p_1 s_1, p_2 s_2) : (p_1, p_2) \in \{0, 1\}^p \land s_1, s_2 \in \{0, 1\}^{n-2} \land (s_1 \neq s_2 \lor s_1 \notin B^{n-1})\right\}
\]

Each tuple in \( R_{i,j} \) has the form \((p_1 s_1, p_2 s_2)\), where \( p_1 \) and \( p_2 \) are arbitrary binary prefixes of length \( p \), and \( s_1 \) and \( s_2 \) are binary prefixes of length \( n-2 \). The only condition on \( s_1 \) and \( s_2 \) is that if \( s_1 = s_2 \), then \( s_1 \notin B^{n-1} \). Informally, this definition ensures that only the last \( n-2 \) bits in each attribute matter, while the first \( d \) bits vary over all possible values for any fixed value of the last \( n-2 \) bits. When \( n = 3 \), \( Q_{3,2} \) is a triangle query in which each relation contains the Cartesian products of all the even numbers with all the even numbers and all the odd numbers with all the odd numbers. \( Q_{3,2} \) is the specific instance of this query which was illustrated in Figure 2.

The relations in \( R \) were defined so that a box cover for \( Q \) would take the following form. For each \( i, j \in [n] \), the set of unit boxes

\[
B_{i,j} = \{(p_1 s, p_2 s) : p_1, p_2 \in \{0, 1\}^d \land s \in B^{n-1}\}
\]

is the minimum size box cover for \( R_{i,j} \) under the default domain ordering. Every box in \( B_{i,j} \) has the form \((p_1 s, p_2 s)\) where \( p_1 \) and \( p_2 \) are arbitrary binary prefixes of length \( p \), and \( s \in B^{n-1} \). It is worth noting that the output of \( Q_{n,d} \) is empty, and furthermore, every box \( b \in B_{i,j} \) must be part of the certificate. If any one of these boxes is removed, the query is no longer empty. This means the optimal certificate size for this domain ordering is \( \Omega(2^p) \). Given these gap boxes, Tetris must perform \( \Omega(2^{2p}) \) geometric resolutions to compute this join, as shown in Lemma G.5 of reference [1].

However, under a different domain ordering, we can obtain a much better runtime for Tetris. Consider a domain ordering \( \sigma^* \) where for each attribute \( A_i \) and each prefix \( s \in B^{n-1} \), the domain values with their last \( n-2 \) bits equal to \( s \) are placed consecutively in \( \sigma^*_i \). Under this ordering, for each \( i, j \in [n] \) and each \( s \in B^{n-1} \), the gap tuples

\[
\{(sp_1, sp_2) : p_1, p_2 \in \{0, 1\}^p\}
\]

can be covered by a single gap box, \((s, s)\). Then each relation requires only \( n-1 = O(1) \) gap boxes to form a box cover. The query \( Q_{3,2} \) under such an improved ordering \( \sigma^* \) was also shown in Figure 2. Again, each of these boxes must be part of the certificate. The certificate size is \( O(1) \) since it depends only on \( n \). Under this domain ordering, and given this box cover, Tetris is able to compute \( Q_{n,d} \) in \( O(1) \) time.
The remainder of this section assumes that the readers have read Section 5.2 that describes ADORA. Binary clique queries with empty outputs, like Example 16, are not the only families of queries we can construct for which running Tetris after running ADORA can be arbitrarily faster than running Tetris with a fixed domain ordering. We can actually generate families of query instances from an arbitrary query \( Q \), with any number of output tuples and any certificate size, and a default domain ordering that would benefit from ADORA. Take an arbitrary query \( Q = (R, \mathcal{A}) \) with \( N \) input tuples and \( Z \) output tuples. Suppose the certificate for \( Q \) under its default ordering is \( C_{\square}(Q) \). Let \( \sigma_{ADR} \) be the ordering ADORA generates on \( Q \) and let the number of grid boxes generated by ADORA on \( Q \), under \( \sigma_{ADR} \), be \( K \), which is not necessarily smaller than \( C_{\square}(Q) \). Recall that \( K \) is the product of the number of equivalence classes that ADORA finds in each attribute and is an upper bound on the box cover size and certificate for \( Q \) under \( \sigma_{ADR} \) ordering. We will generate a family of queries \( Q_p \) from \( Q \) for \( p = 1, 2, ... \), whose certificate will increase to \( 2^p C_{\square}(Q) \) but the number of grid cells that ADORA generates for \( Q_p \) will remain at \( K \). Our approach is to make \( Q_p \) more like the “checkobox” example in Example 15 (and Figure 2) with increasing \( p \).

Let \( A_p \) be the attribute set obtained from \( A \) by adding, for each \( A \in A \), an additional \( p \) bits as a prefix to the \( d \) bits of \( A \). For every relation \( R \in \mathcal{R} \), construct a relation \( R_p \) with \( \text{attr}(R_p) \subseteq A_p \) corresponding to \( \text{attr}(R) \) as appropriate. For each \( t \in R \), add the following tuples to \( R_p \).

\[
\{ p_A t.A | A \in \text{attr}(R) : p_A \in \{0, 1\}^p \ \forall A \in \text{attr}(R) \}
\]

Essentially, the extra \( p \) bits added to each attribute do not affect the structure of the query, since these bits vary over all possible valuations for each tuple from the original query. For each attribute \( A \), these bits effectively create \( 2^p \) “copies” of each \( A \)-hyperplane. This increases the size of the query’s input, output, and box certificate. The query \( Q_p = (R_p, A_p) \) has input size \( 2^p N \), output size \( 2^p Z \), and minimum box certificate size \( 2^p C_{\square}(Q) \), where \( r \) is the maximum arity of a relation in \( \mathcal{R} \) and \( n = |A| \). Note however that this construction does not affect the number of equivalence classes on any dimension. Instead it only increases the sizes of each equivalence on each dimension by \( 2^p \). To see this, consider two values of an attribute, say \( A, a_1 \) and \( a_2 \), that were in the same equivalence class in \( Q \). That is, they had the same \( A \)-hyperplanes for every relation \( R \), whose schema contained \( A \). After adding the \( p \) bits, there will be \( 2^p \) “copies” of \( a_1 \) and \( a_2 \), one for each \( 2^p \) prefix that got appended to tuples that contained \( a_1 \) and \( a_2 \), and each copy will still have the same (but larger) \( A \)-hyperplanes. Therefore the number of equivalence classes on each attribute will remain the same, so the number of grid cells generated by ADORA will remain at \( K \). This shows that as \( p \) increases, the performance of our combined algorithm TetrisReordered can be made to be unboundedly faster than prior versions of Tetris.

### C Generating a Certificate with Tetris

It is possible to modify Tetris so that it computes an approximately minimum size box certificate for \( Q \) as it computes the output for \( Q \). Given input box cover \( B \), this simple modification to Tetris will compute a box certificate for \( B \) of size \( \tilde{O}(C_{\square}(B)) \).

We reviewed Tetris briefly in Section 2. In particular, in this section we will focus on the TetrisReloaded variant, which initializes its knowledge base of boxes to be empty, then adds boxes to the knowledge base whenever its subroutine TetrisSkeleton performs a geometric resolution or returns a witness tuple not covered by a box in the knowledge base. We defer to reference [1] for a detailed description of TetrisReloaded.
Let $B$ be the original box cover input to Tetris, and let $K$ be the knowledge base of gap boxes that Tetris initializes as empty. As our modification to Tetris, we will add a new set of boxes $C$ which we initialize as empty. TetrisSkeleton returns YES if the current knowledge base covers the entire output space, or it returns a witness tuple $o$ otherwise. Tetris then checks if $o$ is an output tuple by querying $B$ for any gap boxes which contain $o$. If $B_o \subseteq B$ is the set of boxes in $B$ which contain $o$, and $B_o \neq \emptyset$, then $o$ is a gap tuple, so Tetris sets $K := K \cup B_o$. At this point, we modify Tetris once again by also setting $C := C \cup B_o$. If $B_o = \emptyset$, then $o$ is an output tuple, so Tetris outputs $o$ and inserts $o$ as a unit gap box into $K$.

This process repeats until the boxes in $K$ cover the entire output space.

After our modified Tetris finishes executing, the resulting set $C$ must form a certificate for $B$, because if there is any gap tuple not covered by $C$, Tetris would have encountered it as a witness before finishing. Let $W$ be the set of witness gap tuples Tetris encountered which resulted in adding one or more boxes to $C$. Then every pair of witnesses $o_1, o_2 \in W$ must be independent in the sense that there is no box $b$ in $B$ that covers both $o_1$ and $o_2$. Otherwise, if $o_1$ was encountered first, then $b$ would have been in $K$ already when $o_2$ was returned by TetrisSkeleton, which is a contradiction. This implies that any certificate for $B$ must have size at least $|W|$. By Lemma 3, we also have that $C$ has size at most $\tilde{O}(|W|)$, since $|B_o| = \tilde{O}(1)$ for each $o \in W$. Therefore $|C| = \tilde{O}(C_2(B))$, i.e., $C$ is a $\tilde{O}(1)$ factor approximation of the minimum certificate for $B$.

## D Proof that $\text{DomOr}_{\text{BoxMinB}}$ is NP-hard

This section contains the full proof for Theorem 10.

**Theorem 17 (Theorem 10 Restated).** $\text{DomOr}_{\text{BoxMinB}}$ is NP-hard.

**Proof.** We focus on the special case where $Q$ contains a single relation $R(A, B)$ over exactly 2 attributes, and show that $\text{DomOr}_{\text{BoxMinB}}$ is NP-hard even in this case. For the purposes of the proof, we will model $R$ as a boolean matrix $M'$, with a row for each value in $\text{dom}(B)$ and a column for each value in $\text{dom}(A)$. Each cell of the matrix corresponds to a possible tuple in $\text{dom}(A) \times \text{dom}(B)$. The matrix $M'$ contains a 0-cell in column $i$ and row $j$ if the tuple $t = \langle i, j \rangle \in R$, and a 0-cell otherwise. This means that a box cover $B$ for $R$ corresponds directly to a set of rectangles which cover all of the 1-cells of $M'$, and vice-versa. Readers can assume $M'$ is given to $\text{DomOr}_{\text{BoxMinB}}$ as a dense matrix or a list of tuples, i.e., $(i, j)$ indices for the 0 cells. Irrespective of the format, the $M'$ we will input to $\text{DomOr}_{\text{BoxMinB}}$ will be of polynomial size in the input size of 2CBMP problem, which we reduce to $\text{DomOr}_{\text{BoxMinB}}$.

Let $M$ be an $n \times m$ boolean matrix input to 2CBMP. We will construct a $(4n) \times (m + 2n)$ matrix $M'$ to use as input to $\text{DomOr}_{\text{BoxMinB}}$. For each row $r_i$ ($i \in [n]$) in $M$, we will insert four rows into $M'$ in top to bottom order. We will refer to this as the *default row ordering* of $M'$. Let $S_i$ be the set of columns which contain 1-cells in row $r_i$ of $M$. Let $e_S$ be the row vector of length $m + 2n$ with value 1 on all indices in $S \subseteq [m + 2n]$, and value 0 everywhere else.

\[
\begin{align*}
p_{i, 1} &= e_{\{m + 2i - 1\}} \\
r_{i, 1} &= e_{S_i \cup \{m + 2i - 1\}} \\
r_{i, 2} &= e_{S_i \cup \{m + 2i\}} \\
p_{i, 2} &= e_{\{m + 2i\}}
\end{align*}
\]

We will refer to the column ordering of $M'$ after this transformation as the *default column ordering* of $M'$. A visualization of an example transformation from $M$ to $M'$ is shown in
Figure 3. To prove this theorem, it suffices to prove that there exists an ordering $\sigma_\epsilon$ on the columns of $M$ such that $cb(\sigma_\epsilon(M)) \leq k$ if and only if there exists orderings $\sigma' = (\sigma'_c, \sigma'_r)$ on the rows and columns of $M'$ such that $\sigma'(M')$ admits a box cover of size at most $k + 2n$.

Proving one direction of this claim is simple. If there exists an ordering $\sigma_\epsilon$ on the columns of $M$ such that $cb(\sigma_\epsilon(M)) = k$, then set $\sigma'_c$ equal to the default row ordering of $M'$. Also, set the last $2n$ columns in $\sigma'_c$ equal to the default column ordering of the last $2n$ columns of $M'$. Then, set the first $m$ columns in $\sigma'_c$ equal to $\sigma_\epsilon$. Then, the 1-cells in the first $m$ columns of $\sigma'(M')$ can be covered by $k$ boxes, and the 1-cells in the last $2n$ columns can be covered by $2n$ boxes, for a total box cover size of $k + 2n$.

Proving the converse is significantly more involved. Let $\sigma' = (\sigma'_c, \sigma'_r)$ be an ordering on the rows and columns of $M'$ such that $\sigma'(M')$ admits a box cover $B$ of size $k + 2n$. We will show through a sequence of 6 steps that we can transform $\sigma'_r$ to exactly match the default row ordering of $M'$ and we can transform $\sigma'_c$ so that the last $2n$ columns exactly match the last $2n$ columns in the default column ordering of $M'$. Each step modifies the box cover $B$ to cover $\sigma'(M')$ under the modified $\sigma'$, without increasing the total number of boxes in $B$.

Before we list the steps of this process, we need two definitions. Two rows $r_{i,j}$ and $r_{k,\ell}$ in $M'$ ($i, k \in [n]$ and $j, \ell \in \{1, 2\}$) are equivalent if $r_i$ and $r_k$ are equal rows in $M$ (ie. $r_i$ and $r_k$ have 1-cells in the same columns in $M$). A run of equivalent rows is a sequence $E$ of one or more $r_{i,j}$ rows which are consecutive in $\sigma'_r$ such that all rows in $E$ are equivalent to one another.

Below are the steps we will take to reorder $\sigma'$. For each step, we will prove that we can reorder $\sigma'$ such that the claim is true of $\sigma'(M')$ without increasing the number of boxes, assuming that all of the previous claims hold.

1. Every $r_{i,j}$ row can be made adjacent to some equivalent $r_{k,\ell}$ row.
2. Every run of equivalent $r_{i,j}$ rows can be made to have even length.
3. Every run of equivalent $r_{i,j}$ rows can be made to have length 2.
4. The padding rows $p_{i,j}$ can be made adjacent to their matching $r_{i,j}$ rows.
5. The row order $\sigma'_r$ can be made to exactly match the default row order of $M'$.
6. The column order $\sigma'_c$ can be made to exactly match the default column order of $M'$ on the last $2n$ columns.

### Step 1

**Claim.** Every $r_{i,j}$ row can be made adjacent to some equivalent $r_{k,\ell}$ row.

Let $r_1 := r_{i,j}$ be a row which is not adjacent to any equivalent row. Let $r_2 := r_{k,\ell}$ be any row equivalent to $r_1$ (note that at least one such row exists because we duplicate each row of $M$ when constructing $M'$). Since $r_1$ is not adjacent to any equivalent row, and there are an even number of rows equivalent to $r_1$, there must be some run $E$ of rows equivalent to $r_1$ with odd length. If $E$ has length 1, we assume $r_2$ is the one row in $E$, and therefore $r_2$ is not adjacent to any equivalent row. If $E$ has length at least 3, we assume $r_2$ is the second row in $E$, and therefore $r_2$ is not adjacent to $p_{k,\ell}$. Let $p_1 := p_{i,j}$ and let $p_2 := p_{k,\ell}$. Let $c_{p_1}$ be the column where $p_1$ has a 1-cell, and let $c_{p_2}$ be the column where $p_2$ has a 1-cell. Let $c_1$ and $c_2$ be the columns where $r_1$ and $r_2$ both have 1-cells. Let $b_1 \in B$ be the box covering the padding column in $r_1$ with greatest width. Let $b_2 \in B$ be the box covering the padding column in $r_2$ with greatest width. Let $b_3 \in B$ be the box covering the padding column in $p_1$. Let $b_4 \in B$ be the box covering the padding column in $p_2$.

Our approach in this step will be to remove the rows $r_1, r_2, p_1,$ and $p_2$ from $M'$, then insert them in the order $(p_1, r_1, r_2, p_2)$ at the bottom of $M'$. In this order, the 1-cells of these 4 rows can be covered by at most 4 boxes, regardless of the column ordering. A box of width
1 and height 2 can be used to cover the two 1-cells in each of the columns in \( \{c_1, c_2, c_{p1}, c_{p2}\} \). To show that this modification does not increase the number of boxes in \( B \), it suffices to show that there are at least 4 boxes which can be removed from \( B \) when we remove these 4 rows from \( M' \). We split our analysis into four cases.

1. \( b_1 \neq b_3 \) and \( b_2 \neq b_4 \). In this case, all of \( \{b_1, b_2, b_3, b_4\} \) are distinct and all 4 of these boxes are removed when we remove the rows \( r_1, r_2, p_1, p_2 \).

2. \( b_1 \neq b_3 \) and \( b_2 = b_4 \). Since \( b_2 = b_4, r_2 \) is adjacent to \( p_2 \). By our previous assumptions about \( r_2 \), this means \( r_2 \) is not adjacent to any equivalent row. Without loss of generality, assume that \( p_2 \) is directly below \( r_2 \). Let \( r_3 \) be the row directly above \( r_2 \). \( r_3 \) is not equivalent to \( r_2 \), so there exists a box \( b_5 \) covering at least one of \( c_1 \) or \( c_2 \) in \( r_2 \) which has height 1, since it cannot extend vertically to either \( p_2 \) or \( r_3 \). \( b_5 \) is not equal to \( b_2 \), because \( b_2 \) has height 2. Now, the set of boxes \( \{b_1, b_2, b_3, b_5\} \) is a set of 4 distinct boxes which are removed when we remove the rows \( \{r_1, r_2, p_1, p_2\} \).

3. \( b_1 = b_3 \) and \( b_2 \neq b_4 \). Since \( b_1 = b_3, r_1 \) is adjacent to \( p_1 \). Suppose without loss of generality that \( p_1 \) is directly above \( r_1 \). Let \( r_3 \) be the row directly below \( r_1 \). Since \( r_1 \) is not adjacent to any equivalent rows, \( r_3 \) is not equivalent to \( r_1 \). Therefore, there is a box \( b_6 \in B \) covering at least one of \( c_1 \) or \( c_2 \) in \( r_1 \) which has height 1, since it cannot extend vertically to either \( p_1 \) or \( r_3 \). \( b_6 \) is not equal to \( b_1 \), since \( b_1 \) has height 2. Now the set of boxes \( \{b_1, b_2, b_4, b_6\} \) is a set of 4 distinct boxes which are removed when we remove the rows \( \{r_1, r_2, p_1, p_2\} \).

4. \( b_1 = b_3 \) and \( b_2 = b_4 \). This case can be proven by combining the arguments from the previous two cases. Since \( b_2 = b_4 \), we can define the box \( b_6 \) exactly as in case 2. Since \( b_1 = b_3 \), we can define the box \( b_6 \) exactly as in case 3. Then, \( \{b_1, b_2, b_5, b_6\} \) is a set of 4 distinct boxes which are removed from \( B \) when we remove the rows \( \{r_1, r_2, p_1, p_2\} \).

**Step 2**

**Claim.** *Every run of equivalent \( r_{i,j} \) rows can be made to have even length.*

Let \( E_1 \) be a run of equivalent \( r_{i,j} \) rows of odd length. By the claim of step 1, \( E_1 \) has length at least 3. Let \( r_1 \) be the second row in \( E_1 \). Since \( E_1 \) has odd length and there are an even number of total rows equivalent to \( r_1 \), there exists another run \( E_2 \) of rows equivalent to \( r_1 \) with odd length. \( E_2 \) also has length at least 3.

Let \( c_{p1} \) be the column which has a 1-cell only in \( r_1 \) and its corresponding padding row. Let \( b \in B \) be the box which covers \( c_{p1} \) in \( r_1 \). Since \( r_1 \) is not adjacent to its padding row, \( b \) has height 1. If we remove \( r_1 \) from \( M' \), \( b \) can be removed. By inserting \( r_1 \) directly below the first row in \( E_2 \), a unit box can be used to cover \( c_{p1} \) in \( r_1 \).

Let \( r_2 \) be the first row in \( E_2 \). Let \( c_1 \) and \( c_2 \) be the two columns of \( M' \) where \( r_1 \) and \( r_2 \) share 1-cells. To cover these other two 1-cells in \( r_1 \), we can extend vertically the boxes covering \( c_1 \) and \( c_2 \) in \( r_2 \). We may assume these boxes can be extended vertically, because at most two of the rows in \( E_2 \) have their \( c_1 \) (or \( c_2 \)) cell covered by a box which stretches horizontally from a padding column. That is, there is some row in \( E_2 \) where the box covering the \( c_1 \) (or \( c_2 \)) cell can be extended vertically to cover the \( c_1 \) (or \( c_2 \)) cell of \( r_1 \). This ensures that this transformation can be made without increasing the number of boxes in \( B \). After this, both \( E_1 \) and \( E_2 \) have even length. We continue this process until every run of equivalent \( r_{i,j} \) rows have even length.

**Step 3**

**Claim.** *Every run of equivalent \( r_{i,j} \) rows can be made to have length 2.*
Let $E$ be a run of equivalent $r_{i,j}$ rows of even length greater than 2. So $E$ has a length of at least 4. Let $r_1$ be the second row in $E$ and let $r_2$ be the third row in $E$. Since $E$ has length at least 4, neither $r_1$ nor $r_2$ are adjacent to their respective padding rows, $p_1$ and $p_2$. Furthermore, we would like to claim the boxes covering the padding columns in $r_1$ and $r_2$ have width 1. We split our analysis into two cases. Below, $c_1$ and $c_2$ are the two columns where $r_1$ and $r_2$ both have 1-cells.

1. $c_1$ and $c_2$ are adjacent. At most 2 of the rows in $E$ have their padding columns adjacent to $(c_1, c_2)$ on either side. This means there is some row $r_3$ in $E$ for which the box $b$ covering $c_1$ and $c_2$ does not also cover its padding column. $b$ can be extended vertically to cover $c_1$ and $c_2$ in all rows of $E$. Then, any boxes covering padding columns for rows in $E$ can be replaced with boxes of width 1, and all of the 1-cells in the rows of $E$ remain covered.

2. $c_1$ and $c_2$ are not adjacent. At most 2 rows in $E$ have their padding columns adjacent to $c_1$ on either side. This means there is some row $r_3$ in $E$ for which the box $b$ covering $c_1$ does not also cover its padding column. $b$ can be extended vertically to cover $c_1$ in all rows of $E$. The same argument can be applied for $c_2$. Then, any boxes covering padding columns for rows in $E$ can be replaced with boxes of width 1, and all 1-cells in the rows of $E$ remain covered.

Now, removing $p_1$ and $p_2$ removes two boxes from $B$, since unit boxes must be covering the single 1-cells in $p_1$ and $p_2$. Inserting $(p_1, p_2)$ in order in between $r_1$ and $r_2$, we can cover the 1-cells in $(c_{p1}, p_1)$ and $(c_{p2}, p_2)$ by extending vertically the width 1 boxes covering $(c_{p1}, r_1)$ and $(c_{p2}, r_2)$. This splits any boxes which vertically stretched from $r_1$ to $r_2$ into two. There were at most two such boxes, so the total number of boxes in $B$ does not increase. Now $E$ is split into two distinct runs of equivalent rows, one of length 2 and one of length $|E| - 2$. This process can be repeated until all runs have length exactly 2.

**Step 4**

**Claim.** The padding rows $p_{i,j}$ can be made adjacent to their matching $r_{i,j}$ rows.

Let $r_1 := r_{i,j}$ be a row which is not adjacent to its padding row $p_1 := p_{i,j}$. By the claim of step 3, we know $r_1$ is adjacent to exactly one row, $r_2$, that is equivalent to $r_1$. Let $p_2$ be the padding row matching $r_2$. Let $c_1$ and $c_2$ be the columns where $r_1$ and $r_2$ share 1-cells. Let $c_{p1}$ be the column which has 1-cells only in $r_1$ and $p_1$. Let $c_{p2}$ be the column which has 1-cells only in $r_2$ and $p_2$. Let $b_1$ be the box which covers the 1-cell in row $r_1$ and column $c_{p1}$ of greatest width. Let $b_2$ be the box which covers the 1-cell in row $r_2$ and column $c_{p2}$ of greatest width. Let $b_3$ be the box which covers the 1-cell in $p_1$. Let $b_4$ be the box which covers the 1-cell in $p_2$. We split our analysis into two cases.

1. $r_2$ is adjacent to $p_2$. In this case, similar to our argument in step 1, there exists a box $b_5 \in B$ with height 1 which covers $c_1$ or $c_2$ (or both) in $r_2$. By removing the rows \{r_1, r_2, p_1, p_2\}, the 4 distinct boxes \{b_1, b_2, b_3, b_5\} are all removed from $B$. By inserting the rows $(p_1, r_1, r_2, p_2)$ in order at the bottom of the matrix, we can cover their 1-cells with at most 4 boxes, so the total number of boxes in $B$ does not increase.

2. $r_2$ is not adjacent to $p_2$. In this case, $r_1$ is not adjacent to $p_1$ and $r_2$ is not adjacent to $p_2$, so \{b_1, b_2, b_3, b_4\} is a set of 4 distinct boxes in $B$ which are removed if we remove rows \{r_1, r_2, p_1, p_2\}. By inserting the rows $(p_1, r_1, r_2, p_2)$ in order at the bottom of the matrix, we can cover their 1-cells with at most 4 boxes, so the total number of boxes in $B$ does not increase.

We can repeat this process until all $r_{i,j}$ rows are adjacent to their matching $p_{i,j}$ rows.
Step 5

Claim. The row order \( \sigma'_r \) can be made to exactly match the default row order of \( M' \).

By the claims of steps 3 and 4, all of the rows are now divided into separate 4-row units containing a run of two equivalent \( \sigma_i,j \) rows surrounded by their two matching padding rows. There are no boxes in \( B \) which can stretch vertically across two or more of these separate units, because there are no two \( p_{i,j} \) rows which share a 1-cell. Thus, we are free to reorder these units arbitrarily. Order the units so that for all \( i \), the \( i \)-th unit contains two \( \sigma_i,j \) rows which correspond to the \( i \)-th row of the original matrix \( M \). The resulting row order \( \sigma'_r \) is then equal to the default row ordering of \( M' \), modulo any equivalent rows which are swapped from their default positions.

Since equivalent rows are equal up to reordering the columns of \( M' \), there exists an ordering on the columns of \( M' \) that transforms \( \sigma'_r(M') \) back to the original matrix \( M' \). In other words, the row ordering \( \sigma'_r \) is now equivalent to the default row ordering of \( M' \) up to a relabelling of the rows. This is sufficient for our purposes, since we can relabel the rows accordingly and move on to modifying the column ordering only.

Step 6

Claim. The column order \( \sigma'_c \) can be made to exactly match the default column order of \( M' \) on the last \( 2n \) columns.

For each padding row \( p_{i,j} \), the box \( b \) covering the single 1-cell in \( p_{i,j} \) has width 1. By step 4, each padding row is adjacent to its corresponding \( \sigma_i,j \) row. This means \( b \) extends vertically to also cover the only other 1-cell in its column. Therefore, by moving this column to the right side of the matrix, we do not increase the total number of boxes in \( B \).

Once all of these padding columns have been moved to the right, the boxes covering all of their 1-cells all have width 1. Thus, we can reorder them to exactly match the last \( 2n \) columns in the default column ordering of \( M' \) without modifying any boxes in \( B \).

After these 6 steps, the only difference between \( M' \) and \( \sigma'(M') \) is the ordering of the first \( m \) columns. In \( \sigma'(M') \), the last \( 2n \) columns contain an independent set of 1-cells of size \( 2n \), by taking the single 1-cell from each of the \( p_{i,j} \) rows. All of these \( 2n \) 1-cells are independent from all of the 1-cells in the first \( m \) columns of \( \sigma'_c \).

Let \( \sigma_c \) be the ordering of the first \( m \) columns in \( \sigma'_c \). We claim that the first \( m \) columns contain an independent set of 1-cells of size \( \text{cb}(\sigma_c(M)) \). First, any two 1-cells in separate 4-row units are independent from one another, because the padding rows between them contain only 0-cells on the first \( m \) columns. If a row of \( \sigma_c(M) \) has only one consecutive block, then add a 1-cell from the corresponding 4-row unit to the independent set. If a row of \( \sigma_c(M) \) contains two consecutive blocks, then there are two 1-cells in the first \( m \) columns of the corresponding 4-row unit which are independent from one another. Add both of these to the independent set. Combining the independent sets from the first \( m \) columns and the last \( 2n \) columns, we obtain an independent set of size \( \text{cb}(\sigma_c(M)) + 2n \).

Furthermore, there exists a box cover for \( \sigma'(M') \) of size \( \text{cb}(\sigma_c(M)) + 2n \). All of the 1-cells in the last \( 2n \) columns can be covered by \( 2n \) boxes. For row \( r_i \) in \( M \), if \( \sigma_c(r_i) \) contains 1 consecutive block, then the 1-cells in the first \( m \) columns of the corresponding 4-row unit in \( \sigma'(M') \) can be covered by a single \( 2 \times 2 \) box. If \( \sigma_c(r_i) \) contains 2 consecutive blocks, then the 1-cells in the first \( m \) columns of the corresponding 4-row unit in \( \sigma'(M') \) can be covered by two \( 2 \times 1 \) boxes. In total, this yields a box cover of size \( \text{cb}(\sigma_c(M)) + 2n \).

Since our initial assumption was that \( \sigma'(M') \) has a box cover of size \( k + 2n \), this implies that \( \text{cb}(\sigma_c(M)) \leq k \), which completes the reduction. \( \blacktriangleleft \)
E Ordering One Attribute is NP-hard

Let $R(A, B)$ be a relation over 2 attributes. In this section, we will prove that finding a domain ordering $\sigma^*_A$ for $\text{dom}(A)$ that minimizes $K_{\Box}(\sigma^*_A(R))$ is NP-hard, even when the ordering of $\text{dom}(B)$ is fixed. This means the variant of $\text{DomOr}_{\Box_{\text{2dBMP}}}$ in which all attributes except for one have a fixed domain ordering is NP-hard. Similar to the proof of Theorem 10, this proof will also be use a reduction from 2CBMP.

**Theorem 18.** Given $R(A, B)$ and a fixed domain ordering $\sigma^*_B$ for $\text{dom}(B)$ as input, it is NP-hard to compute a domain ordering $\sigma^*_A$ for $\text{dom}(A)$ that minimizes $K_{\Box}(\sigma^*(R))$.

**Proof.** We will prove this via a polynomial-time reduction from 2CBMP. Let $M$ be an $m \times n$ boolean matrix input to 2CBMP. Define a transformed matrix $M'$ as follows. Add all the rows of $M$ to $M'$. Between each pair of consecutive rows, add one additional row containing all 0-cells. A small example of this transformation is seen below.

\[
M = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} \Rightarrow M' = \begin{bmatrix}
1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix}
\]

Note that $M'$ is a $(2m - 1) \times n$ matrix, which is polynomial size with respect to $M$. It suffices to prove that for any column ordering $\sigma_c$, $\text{cb}(\sigma_c(M)) = \leq k$ if and only if $\sigma_c(M')$ has a minimum box cover size of at most $k$.

Suppose that $\text{cb}(\sigma_c(M)) = k$. Every consecutive block in $\sigma_c(M)$ also appears as a consecutive block in the corresponding row of $\sigma_c(M')$. This block $c$ in $\sigma_c(M')$ can be covered by a single box $b$ of height 1 and length equal to the length of $c$. Every 1-cell in $\sigma_c(M')$ corresponds to some consecutive block of $\sigma_c(M)$, so the set of all $k$ of these boxes $b$ forms a box cover of $\sigma_c(M')$ of size $k$.

Conversely, suppose that $B$ is a minimum size box cover for $\sigma_c(M')$ and $|B| \leq k$. Since there is no 1-cell in $M'$ that has another 1-cell immediately above or below it, each box $b \in B$ has height 1. Furthermore, we may assume without loss of generality that the length of $b$ is maximal, bounded on the left by the beginning of the matrix or a 0-cell, and bounded on the right by the end of the matrix or a 0-cell. In other words, the 1-cells covered by $b$ are exactly a consecutive block in $\sigma_c(M')$. Additionally, no two boxes $b_1, b_2 \in B$ can cover the same consecutive block, or $B$ is not a minimal box cover. Thus, $\sigma_c(M')$ has exactly $k$ consecutive blocks, and so $\sigma_c(M)$ also has exactly $k$ consecutive blocks.

F The ADORA Bound Is Tight

Theorem 13 proved that ADORA produces a domain ordering $\sigma$ for a query $Q$ such that $K_\Box(\sigma(Q)) = O(K^r)$, where $K$ is the minimum box cover size for $Q$ under any domain ordering and $r$ is the maximum arity of a relation in $Q$. In this section, we demonstrate that this bound is tight by presenting a class of 2-dimensional relations $R_d$ for which ADORA returns a domain ordering $\sigma$ such that $K_\Box(\sigma(R_d)) = \Omega(K^2)$, where $K$ is the minimum box cover size for $R_d$ under any domain ordering.

For any positive integer $d$, let $R_d(A, B)$ be the relation over 2 $d$ bit attributes $A$ and $B$ given by
Figure 6 A relation $R_3$ for which the bound of Theorem 14 is tight.

$$R_d(A,B) = \{ (0a, 0b) : a, b \in \{0, 1\}^{d-1}, a \neq b \} \cup \{ (1a, 1b) : a, b \in \{0, 1\}^{d-1}, a \neq b \}$$

The relation $R_3$ is depicted in Figure 6 (left). The minimum size box cover for $R_3$ consists of the 2 boxes which cover the top left and bottom right quadrants, the $2 \times 2$ box which covers the middle 4 empty cells, as well as the 6 unit boxes which cover the diagonal line of gaps from the bottom left to the top right, for a total box cover size of 9. This happens to be the minimum box cover size for $R_3$ under any domain ordering. The relation $\sigma(R_3)$ depicted in Figure 6 (right) is what $R_3$ looks like under a different domain ordering $\sigma$. This ordering $\sigma$ is obtained by moving all of the even domain values to the range $[000-011]$ and all of the odd domain values to the range $[100-111]$ in both $A$ and $B$. The minimum box cover for $\sigma(R_3)$ consists of the 18 unit boxes covering the gap cells which are surrounded by tuples, plus the 7 $2 \times 2$ boxes which can be tiled to cover the remaining diagonal stretch of gaps, for a total box cover size of 25. $R_3$ generalizes to any instance of the relation $R_d$. The default ordering of $R_d$ has a minimum box cover size $K = 2^d + 1$. However, there exists a bad ordering $\sigma_d$ for $R_d$ such that $\sigma_d(R_d)$ has a minimum box cover size of $2^d \cdot 2^{d-1} - 2^d + 1 = \Omega(2^{2d}) = \Omega(K^2)$. The key observation about this class of examples is that no rows or columns in $R_d$ are equal to one another, so ADORA may return $\sigma_d$ as a solution. Since $R_d$ has arity 2, the bound of Theorem 14 is asymptotically tight in this case.