A continuous transformation between non-Hermitian and Lindbladian evolution

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Non-Hermitian Hamiltonians and dynamical semigroups are some of the most important generators of dynamics for describing quantum systems interacting with different kinds of environments. The first type differs from conservative evolution by an anti-Hermitian term that causes particle decay, while the second type differs by a dissipation operator in Lindblad form that allows energy exchange with a reservoir. However, although under some conditions the two types of maps can be used to describe the same observable, they form a disjoint set as there is no continuous transformation between the two. In this work, we propose a generalized generator of dynamics of the form

\[ L_{\text{mixed}}(\rho) = -i[H, \rho] + \sum_i \left( \frac{\Gamma_i}{\hbar} F_i^\dagger \rho F_i + \frac{1}{2} \{F_i^\dagger F_i, \rho\} + \right) \]

that depends on the energy \(z\), and has a tunable parameter \(\Gamma_i\) that determines the degree of particle density lost. It has as its limits non-Hermitian (\(\Gamma_i \to 0\)) and Lindbladian dynamics (\(\Gamma_i \to \infty\)). The intermediate regime evolves density matrices such that \(0 \leq \text{Tr}(\rho) \leq 1\). We derive our generator with the help of an ancillary continuum manifold acting as a sink for particle density. The system is solved with a copy of Liouville space that linearizes a quadratic eigenvalue problem to obtain the evolution operator. It corresponds to a map that can exchange both particle density and energy with its environment, and we describe its main features for a two level system.

Master equations for open quantum systems allow an efficient explicit description of a system interacting with an environment. We consider two possible interactions: exchanging 1) particle density or 2) energy. A system exchanges particle density with its environment in a molecular junction where the electron goes from molecule (system) to lead (environment), when the electron of an atom or molecule (system) photoionizes to lead (environment), when the electron goes from an environment. We consider two possible interactions: 1) particle density or 2) energy. A system exchanges particle density with its environment in a molecular junction where the electron goes from molecule (system) to lead (environment), when the electron of an atom or molecule (system) photoionizes.

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There is no physical continuous transformation between non-Hermitian and Lindbladian dynamics on account of the operator \(F_i^\dagger \rho F_i\). It is either there - preserving the trace - or absent - resulting in a decay. The loss of the density matrix trace for a Hamiltonian with an anti-Hermitian part \(F_i^\dagger F_i\) is \(\frac{\gamma_i}{\hbar} \text{Tr}(\rho(t)\rho(t)) = \frac{\gamma_i}{\hbar} \text{Tr}(\rho(t)\rho(t) F_i^\dagger F_i)\) \[13\] \[14\]. Nonetheless, a trace preserving map for a NH evolution can be obtained by adding a nonlinear term that compensates for this loss of trace \[13\] \[17\]. These nonlinearities give rise to complex behaviour such as anharmonicities in two-level systems \[14\].

We present in this work a generalized generator of dynamics and corresponding evolution map that has as limiting cases a Lindbladian or a non-Hermitian evolution. We begin by a microscopic derivation of the map by means of an ancillary continuum acting as a reservoir for particle density. The projection of the generator of dynamics onto the subset of levels of interest yields an energy-dependent operator that requires the solution of a quadratic eigenvalue problem. We show a possible solution via a linearization and illustrate the main features of this mixed map.

**Main result.** We propose the following dynamical map for describing particle density and energy exchange with an environment:

\[ L_{\text{mixed}}(\rho) = -i[H, \rho] + \sum_i \left[ \Delta_i(z) F_i^\dagger \rho F_i + \frac{1}{2} \{F_i^\dagger F_i, \rho\} + \right] \]

where \(H\) is the (Hermitian) Hamiltonian, \(F_i\) are jump operators in the Lindblad dissipator associated to a rate \(\gamma_i\), \(\{\}+\) is the anti-commutator and we have introduced a new \(z\)-dependent function \(\Delta(z) = \frac{\gamma_i}{\hbar + \gamma_i}\), where \(z\) is a generalized frequency. Here and in the following we set \(\hbar = 1\). The factor \(\Delta(z)\) becomes 1 for Lindblad generators and 0 for non-Hermitian generators. The evolution
operator in Liouville space can be obtained by taking the inverse Laplace transform of \((z - L_{\text{mixed}}(z))^{-1}\) and written compactly as

\[ U(t) = \sum_{i=1}^{K} \chi_i e^{\lambda_i t} \]  

where \(\lambda_i\) and \(\chi_i\) are eigenvalues and projection operators that solve a nonlinear eigenvalue problem. Eq. \((2)\) differs from the exponential map of a normal Lindblad generators in that, here, for an \(N\) level system \(K > N^2\), unlike for a Lindblad or NH operator where \(K = N^2\). Below, we outline the derivation of Eq. \((1)\) and \((2)\).

Ancillary states. We consider the possible microscopic models from which each evolution map discussed in this paper can be derived. One of the simplest ways to induce non-Hermiticity is to connect discrete energy levels to a continuous manifold of states that act as a particle sink and then remove this continuous manifold from the explicit description (for example see [1][10]). We can calculate the effective Hamiltonian for the discrete manifold using Feshbach projectors. We find that for a continuous manifold obeying the wideband approximation [13], this coupling to a continuum is analogous to an analytical continuation that makes the lifetime of the excited state finite, i.e. for a transition frequency \(\omega_e\) connected to a continuum of states through a coupling \(V = \sqrt{\gamma/(2\pi)}\), we get the transformation \(\omega_e \rightarrow \omega_e - i\gamma/2\). The evolution is fully described by a Hamiltonian as \(\psi = -iH_{\text{eff}}\psi\), or alternatively by its density matrix \(\dot{\rho} = -i(H_{\text{eff}} - \rho H_{\text{eff}}^\dagger)\). The Lindblad operator can be obtained microscopically from tracing out a collection of harmonic oscillators in the Markovian limit.

Evolution operator. We switch to Liouville space by means of the isomorphism \(S_n \rho S_n^\dagger \rightarrow S_n^\dagger \otimes S_n \rho\), where \(\rho_c\) is a column-stretched vector built from the density matrix \(\rho\). Defining the Feshbach projectors for the system and ancillary spaces as \(P\) and \(Q\), respectively, the effective Liouvillian in \(P\) is \([13][22][24]\)

\[ L_{\text{mixed}}(z) = PLP + P LQG_0(z)QLP \]  

which from we can obtain the evolution operator

\[ U(t) = \frac{1}{2\pi i} \oint \frac{e^{zt}}{z - L_{\text{eff}}(z)} \]  

as long as \(Q\rho(0) = 0\) [23]. Reminding that \(L_{\text{NH}} = -i[1 \otimes H - H^* \otimes 1] - \frac{1}{2} \{ F^\dagger_i F_i, \rho \}\) for a dissipative channel with rate \(\gamma_i\). Thus, a continuous manifold can act as a particle sink and a bath of harmonic oscillators induce a dissipative transition.

We propose a mixed map where first the particle density is destroyed, by sending it into a continuous manifold of levels, and then recovered by having a dissipative transition from this particle sink back into the system (Figure 1b). This structure is inspired by work on Hamiltonians with continuous manifolds adapted from atomic physics (i.e. Fano Hamiltonians [20]) into dissipative environments [18][21][21][23]. For each dissipative transition \(i\) with a rate \(\gamma_i\) connecting two levels, we intersperse a continuous manifold which is coupled to the decaying state with strength \(\sqrt{\gamma_i/(2\pi)}\) and which decays via a Lindblad dissipator to the final state with rate \(\Gamma_{c,i}\) (Figure 1b).

\[ L_{\text{mixed}}(z) = L_{\text{Lindblad}} - \sum_i \frac{z}{z + \Gamma_{c,i}} J_i \]

where \(J_i = F_i^\dagger \otimes F_i\) is the operator that restores the lost population of the excited state back into the ground state. Because of the nonlinearity of \(L_{\text{mixed}}(z)\) (i.e. its dependence on \(z\)), we cannot express the evolution map as the exponential of \(L_{\text{mixed}}(z)\). The generalized eigenvalue problem \((z - L_{\text{mixed}}(z))v) = 0\) is now nonlinear and in the case where \(\Gamma_{c,i} \equiv \Gamma_c\) for all dissipative transitions \(i\), it is quadratic. Eq. \((4)\) can be numerically integrated to obtain the evolution. Instead, we can move closer to an analytic solution via a linearization of the problem. We write the explicit \(z\)-dependence of \(L_{\text{mixed}}(z)\) in the

\[ \sum_{i=1}^{K} \chi_i e^{\lambda_i t} \]
Combining Eqs. (9) and (10), we arrive at a compact expression for the resolvent now has a factor \((z + \Gamma_c)\) eigenvectors. We decompose the resolvent \(\tilde{G}(z) = (z - \tilde{M})^{-1}\)
into its projectors [25]:

\[
\begin{bmatrix}
D(z) \\
0
\end{bmatrix} = \tilde{E}(z)(\tilde{M} - z\tilde{B})\tilde{F}(z)
\]

We have written 1 and 0 as the identity and null matrix with dimensions \(N^2 \times N^2\) where \(N\) is the number of levels of the subsystem we wish to describe explicitly. This linearization introduces an additional copy of Liouville space. We denote all operators in this extended space by a tilde, while the operators in the original Liouville space have no tilde. If \(\tilde{E}\) and \(\tilde{F}\) have non-zero determinants, the eigenvalues of \(D(z)\) coincide with those of \(\tilde{M}\). The sought after inverse \((z - D(z))^{-1}\)

\[
(z - D(z))^{-1} = S_o(z - \tilde{M})^{-1}S_e
\]

where we define the projection operator onto the original Hilbert space \(S_o = \begin{bmatrix} 1 \\
0 
\end{bmatrix}\) and onto the extended space \(S_e = \begin{bmatrix} 0 \\
1 
\end{bmatrix}\). We decompose the resolvent \(\tilde{G}(z) = (z - \tilde{M})^{-1}\)

\[
\tilde{G}(z) = \sum_{i=1}^{2N^2} \frac{z + \Gamma_c}{z - \lambda_i} \tilde{X}_i
\]

where \(\lambda_i\) are the eigenvalues of \(\tilde{M}\) and \(\tilde{X}_i = \langle v_i \rangle / |\langle v_i \rangle|\) the corresponding projection operators built from the right \((\tilde{M}|v_i) = \lambda_i|v_i)\) and left \((\langle w_i|\tilde{M}) = \langle w_i|\lambda_i\rangle\) eigenvectors. Combining Eqs. (9) and (10), we arrive at a compact expression for the evolution operator (see Eq. 3)

\[
U(t) = \sum_i (\lambda_i + \Gamma_c)e^{\lambda_i t}S_o^T X_i S_e = \sum_i \Lambda_i e^{\lambda_i t}
\]

where we have defined the generalized projection operators \(\Lambda_i = (\lambda_i + \Gamma_c)S_o^T X_i S_e\).

As we introduce an ancillary continuum for each dissipative pathway, we can in principle have a different parameter \(\Gamma_{c,i}\) for each continuum that mediates the relaxation between any two levels. Then the resolvent is:

\[
\mathcal{P}G(z)\mathcal{P} = \frac{1}{z - L_{\text{Lindblad}} + \sum_i \frac{x_i z}{z + \Gamma_{c,i}}}
\]

For every distinct \(\Gamma_{c,i}\), the order of the nonlinear eigenvalue problem increases by one. The problem is not quadratic anymore, and a thorough investigation on the role of choosing different values for the continuum decay will be discussed in subsequent work. We can also choose to have one continuum mediate different dissipative pathways. In this case the ancillary continuum is responsible for generating coherent transitions as well. This commonly occurs in Fano interferences (see for example [18]) and can be handled by the linearization proposed, although its physical meaning is still unclear.

Two-level system. We illustrate the class of dynamics that we obtain with a mixed non-Hermitian Lindblad operator for a two-level system. The generator of the dynamics in Lindblad form is \(H = \delta_e |e\rangle \langle e| + (V_{eg} |e\rangle \langle g| + h.c., \text{ where } \delta_e\) is the detuning, \(V_{eg}\) the coupling between the two discrete states, and a dissipator in Lindblad form with \(F_1 = \sqrt{\pi} |g\rangle \langle e|\). We show the dynamics for different values of \(\Gamma_c\) in two cases: off-resonance \(\delta_{c} > V_{eg}\) (Fig. 2.a.l) for the initial condition \(\rho(0) = \rho_{ee}\).

Off-resonance (Fig. 2.a.c), the mixing of ground and excited states is minimal and we obtain a decay of the excited state onto the ground state. The trace of the system depends on the value of \(\Gamma_c\). For \(\Gamma_c \ll 1\) we are in the non-Hermitian limit and the systems decays to zero, while in the \(\Gamma_c \rightarrow \infty\) limit we are in the Lindblad limit and the trace is preserved. In between, the particle density can leave the system to later return, and reach a steady-state where \(\text{Tr}(\rho(t \rightarrow \infty)) < 1\). The normalized fidelity \(\mathcal{F}_i = \text{Tr}(\sqrt{\rho_i \rho \sqrt{\rho_i}})/\sqrt{\text{Tr}(\rho_i) \text{Tr}(\rho)}\) is shown for \(i =\text{Lindblad, NH}\) to give a quantitative measure as to the character of the evolution. The pole structure (Figs. 2.d-f) reveals five instead of four poles. One of the poles always corresponds to the steady-state while the fifth pole corresponds to non-Hermitian decay. The two limiting cases are shown in red triangles and blue circles for comparison. The on-resonance case shows a qualitatively similar behavior regarding the trace, with a clear presence of Rabi oscillations and a near equal mixture of ground and excited state in the long-time limit. In contrast with the off-resonant case, the normalized fidelity \(\mathcal{F}_{NH}\) does not represent the system at all times.

It is worth noting that our description of non-Hermitian decay also involves a steady-state \(\lambda = 0\) pole, which can be reconciled if the projector corresponding to this eigenvalue vanishes in the limit \(\Gamma_c \rightarrow 0\). To investigate this in more detail, we study the structure of the eigenvalue manifold as a function of \(\Gamma_c\) as well as the trace of each projector with an (arbitrary) initial state in the upper level \(t_i = \text{Tr}(\Lambda_i \rho_{ee})\). For clarity, we focus on \(t_0(\lambda = 0)\) and \(\sum_i t_i(\lambda_i \neq 0)\) (see Figure 3 c.f.i). As we start from a normalized density matrix at time zero, we have \(\sum_i t_i = 1\), however depending on the value of \(\Gamma_c\), this trace is carried by different projectors, and it is...
For large $\Gamma_c$ the oscillations decay exponentially, while for small enough $\Gamma_c$ they are long-lived and do not decay exponentially. This corresponds to a regime where the coherences decay on the same timescale as the trace of the density matrix and so see their effective lifetime extended from the normalization.

As final remarks, there are limitations regarding our mixed map related to the fundamental distinctions between Hamiltonian and Liouvillian evolution. First, the mixed model does not work for pure decoherence as the populations diverge for any finite value of $\Gamma_c$. This originates in the fact that pure decoherence cannot be captured by mere analytical continuation of the energies and has no analog in non-Hermitian quantum mechanics. Second, $\mathcal{PT}$ symmetric non-Hermitian Hamiltonians also pose problems for $\Gamma_c > 0$. This is because $\mathcal{PT}$ symmetry expressed with Lindblad operators requires the presence of negative dissipative rates. More than a problem with the transformation itself, these problems highlight the fundamental differences of NH processes that have no analog in dissipative dynamics, and vice-versa.

We have assumed throughout that the extended matrix $\hat{M}$ has semisimple eigenvalues, and in particular that the geometric and algebraic multiplicities of $\lambda = -\Gamma_c$ are the same. This is true for the two-level system, however we have not proved it to be the case in general. In a similar fashion we have presented the numerical result that for a two-level system with a single dissipative pathway the poles of the generalized resolvent is five, whereas the dimension of $\hat{M}$ is eight. It can be easily shown that the algebraic multiplicity of $\Gamma_c$ is $N^2$ minus the number of connected dissipative pathways. For a two-level system connected to a finite temperature bath with incoherent pumping and incoherent decay this corresponds to six distinct eigenvalues (see Supplemental Material). Finally, we note that we begin with two maps which are Markovian, however we end up with a non-Markovian map. The expressions developed here assume that $\mathcal{Q}\rho(0) = 0$. Since in general at any given time $t > 0$ this is no longer true, we have that $\hat{U}(t + \tau) \neq \hat{U}(t)\hat{U}(\tau)$.

**Conclusion.** We have proposed a more general generator of dynamics that allows for a continuous transformation between pure decay dynamics obtained from a non-Hermitian Hamiltonian to the trace-preserving dynamics induced by Lindblad operators. This effective operator is rooted in a microscopic derivation using an ancillary continuum. It is energy dependent and so the inverse Laplace transform of its resolvent is not trivial. As long as the decay rates from the ancillary continuum back to the system are the same, the nonlinearities are quadratic and a linearization procedure is proposed to obtain the evolution operator. To this end we resort to a copy of Liouville space whose full usefulness and
meaning remains to be explored. Both Non-Hermitian and Lindblad maps are extensively used in open quantum systems from phase transitions to spectroscopical observables. Our result presents a fundamental connection between them, and also opens a new avenue in the analysis of maps that describe systems exchanging energy and particle density with their surroundings.

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