UNIQUENESS OF K-POLYSTABLE DEGENERATIONS OF FANO VARIETIES

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Abstract. We prove that K-polystable degenerations of $\mathbb{Q}$-Fano varieties are unique. Furthermore, we show that the moduli stack of K-stable $\mathbb{Q}$-Fano varieties is separated. Together with [Jia17, BL18], the latter result yields a separated Deligne-Mumford stack parametrizing all uniformly K-stable $\mathbb{Q}$-Fano varieties of fixed dimension and volume. The result also implies that the automorphism group of a K-stable $\mathbb{Q}$-Fano variety is finite.

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Throughout, we work over an algebraically closed field $k$ of characteristic 0.

1. INTRODUCTION

1.1. Moduli spaces of Fano varieties. To give a general framework for intrinsically constructing moduli spaces of Fano varieties is a challenging question in algebraic geometry, especially if one wants to find a compactification. Unlike the KSBA construction in the canonically polarized case, the Minimal Model Program often provides more than one limit for a family of Fano varieties over a punctured curve. Thus, it is unclear how to find a theory that picks the right limit. In examples, people have obtained a lot of working experience on how to identify the simplest limit. On the negative side, examples such as [PP10, Section 2.2], which gives a family that is trivially degenerates a homogeneous space to a different quasi-homogeneous space (with non-reductive automorphism group), suggest that we should not consider all smooth Fano varieties.

So when the definition of K-stability from complex geometry (see [Tia97]) and its algebraic formulation (see [Don02]), which was introduced to characterize when a Fano
variety admits a Kähler-Einstein metric, first appeared in front of algebraic geometers, it seemed bold to expect such a notion would be a key ingredient in constructing moduli spaces of Fano varieties. However, as the theory has developed, more and more evidence makes such an expectation believable.

We now expect that the moduli functor \( M_{n,V}^{Kss} \) of \( n \)-dimensional K-semistable \( \mathbb{Q} \)-Fano varieties, which sends \( S \in \text{Sch}_k \) to

\[
M_{n,V}^{Kss}(S) = \left\{ \text{Flat proper morphisms } X \to S, \text{ whose fibers are } \right.
\left. n \text{-dimensional K-semistable klt Fano varieties with volume } V, \text{ satisfying Kollár’s condition} \right\}
\]

is represented by an Artin stack \( M_{n,V}^{Kss} \) of finite type and admits a projective good moduli space \( M_{n,V}^{Kss} \to M_{n,V}^{Kps} \) (in the sense of [Alp13]), whose closed points are in bijection with \( n \)-dimensional K-polystable \( \mathbb{Q} \)-Fano varieties of volume \( V \). Here, the Kollár condition means that, for any \( m \in \mathbb{Z} \) the reflexive power \( \omega_{X/S}^{[m]} \) commutes with arbitrary base change (see [Kol09, 24]).

While [Don01] says that smooth Kähler-Einstein Fano manifolds with finite automorphism group are asymptotic Chow stable, examples in [OSY12, LLSW17] show that the GIT approach likely fails to treat those with infinite automorphism groups or singularities (see [WX14] for examples where asymptotic Chow stability fails to construct compact moduli spaces in the KSBA setting). Therefore, we need to take a more abstract approach to constructing \( M_{n,V}^{Kps} \).

The construction of the moduli space reduces to proving a number of concrete statements about families of \( \mathbb{Q} \)-Fano varieties. We list the main ones:

(I) **Boundedness:** There is a positive integer \( N = N(n, V) \) such that if \( X \in M_{n,V}^{Kss}(k) \), then \(-NK_X\) is a very ample Cartier divisor. This is settled in [Jia17] using the results in [Bir16].

(II) **Zariski Openness:** If \( X \to S \) is a family of \( \mathbb{Q} \)-Fano varieties, then the locus where the fiber is K-semistable is an open set.

Together, (I) and (II) show that \( M_{n,V}^{Kss} \) is an Artin stack of finite type and is a global quotient. The following statements are needed to show \( M_{n,V}^{Kss} \) admits a projective good moduli space.

(III) **Good Quotient:** The stack \( M_{n,V}^{Kss} \) has a good moduli space. To prove this, it suffices to show:

(III.a) **Reductive Automorphism Group:** If \( X \) is a K-polystable \( \mathbb{Q} \)-Fano variety \( X \), then \( \text{Aut}(X) \) is reductive.

(III.b) **Gluing of Local Quotients:** Near each K-polystable \( \mathbb{Q} \)-Fano variety \( X \in M_{n,V}^{Kss}(k) \), there exists a local atlas around \([X]\) given by an \( \text{Aut}(X) \) slice. Furthermore, a point in the slice is GIT (poly)stable with respect to \( \text{Aut}(X) \) if and only if the corresponding \( \mathbb{Q} \)-Fano variety is K-(poly)stable. To complete this step, it remains to verify that the local GIT quotient
spaces glue together to get the good quotient $M^{K\text{ps}}_{n,V}$ (e.g. the hypotheses of [AFS17, Theorem 1.2] are satisfied).

(IV) **Separatedness**: Any two K-semistable degenerations of a family of K-semistable $\mathbb{Q}$-Fano varieties over a punctured curve $C^0 = C \setminus \{0\}$ lie in the same $S$-equivalence class, i.e. they degenerate to a common K-semistable $\mathbb{Q}$-Fano variety via special test configurations.

(V) **Properness**: Roughly speaking, any family of K-semistable Fano varieties over a punctured curve $C^0 = C \setminus \{0\}$, can be filled in over 0 to a family of K-semistable Fano varieties over $C$.

(VI) **Projectivity**: A sufficiently divisible multiple of the CM-line bundle yields an ample line bundle on $M^{K\text{ps}}_{n,V}$.

We note that there are subtleties related to the requirement that objects in $\mathcal{M}^{K\text{ss}}_{n,V}(S)$ satisfy Kollár’s condition. Luckily, such issues are of a local nature and have all been addressed in the construction of the moduli space of KSBA stable varieties (see [Kol09,Kol19]).

Strong evidence for the above picture is that, aside from (VI) (the projectivity of $M^{K\text{ps}}_{n,V}$), the problem is completely solved in [LWX14] (also see [SSY16,Oda15]) for smoothable $\mathbb{Q}$-Fano varieties, and some progress on the projectivity was made in [LWX18a]. However, these results rely heavily on the deep analytic tools established in [CDS15,Tia15]. Therefore, a completely algebraic proof is highly desirable. Such a proof would likely allow us to drop the *smoothable* assumption.

The main result in this paper gives a complete solution to (IV). In the smoothable case, this step is solved in [LWX14,SSY16] using analytic tools. The argument in this document is purely algebraic.

1.2. **Separatedness result**. The following statement is our main result.

**Theorem 1.1.** Let $\pi : (X, \Delta) \to C$ and $\pi' : (X', \Delta') \to C$ be $\mathbb{Q}$-Gorenstein families of log Fano pairs over a smooth pointed curve $0 \in C$. Assume there exists an isomorphism $\phi : (X, \Delta) \times_C C^0 \to (X', \Delta') \times_C C^0$ over $C^0 = C \setminus \{0\}$.

1. K-semistable case: If $(X_0, \Delta_0)$ and $(X'_0, \Delta'_0)$ are K-semistable, then they are $S$-equivalent.
2. K-polystable case: If $(X_0, \Delta_0)$ and $(X'_0, \Delta'_0)$ are K-polystable, then they are isomorphic.
3. K-stable case: If $(X_0, \Delta_0)$ is K-stable and $(X'_0, \Delta'_0)$ is K-semistable, then $\phi$ extends to an isomorphism $(X, \Delta) \simeq (X', \Delta')$ over $C$.

**Remark 1.2.**

1. The K-polystable case of Theorem 1.1 follows immediately from the K-semistable case and Definitions 2.5 and 2.6.
2. By [LWX18b], the conclusion of Theorem 1.1.1 can be strengthened to say that $(X_0, \Delta_0)$ and $(X'_0, \Delta'_0)$ have a common K-polystable degeneration.
(3) A special case of Theorem 1.1.3 was proved in [Oda12, 1.4] with the additional assumption that $\alpha(X_0, \Delta_0) > \dim(X_0)/(\dim(X_0) + 1)$ (see also [Che09, 5.7] for a related statement).

Theorem 1.1.3 implies the following special case of Step (III.a).

**Corollary 1.3.** Let $(X, \Delta)$ be log Fano pair. If $(X, \Delta)$ is $K$-stable, then $\text{Aut}(X, \Delta)$ is finite.

### 1.3. Moduli of uniformly $K$-stable Fano varieties

We now specialize our study to the moduli of uniformly $K$-stable Fano varieties. We define the moduli functor $\mathcal{M}_{n,V}^{\text{uKs}}$ that sends $S \in \text{Sch}_k$ to

$$\mathcal{M}_{n,V}^{\text{uKs}}(S) = \left\{ \text{flat, proper morphisms } X \to S, \text{ whose geometric fibers are } n\text{-dimensional uniformly } K\text{-stable } \mathbb{Q}\text{-Fano varieties of volume } V, \text{ satisfying Kollár’s condition} \right\}.$$  

Combining the following recent results

- (I)$^u$ **Boundedness**: Proved in [Jia17];
- (II)$^u$ **Zariski openness**: Proved in [BL18];
- (III)$^u$ **Separatedness as a stack**: Theorem 3.1,

we obtain

**Corollary 1.4.** The functor $\mathcal{M}_{n,V}^{\text{uKs}}$ is represented by a separated Deligne-Mumford stack of finite type, which has a coarse moduli space $M_{n,V}^{\text{uKs}}$ that is a separated algebraic space.

One still missing property is that

- (IV)$^u$ (**quasi-projectivity**) $M_{n,V}^{\text{uKs}}$ is quasi-projective.

**Significant progress on this problem was made in [CP18].**

### 1.4. Summary of the paper

The original definition of $K$-stability in [Tia97, Don02] is defined in terms of the sign of the generalised Futaki invariant on all test configurations or at least special test configurations (see [LX14]). Recently, there has been tremendous progress in reinterpreting $K$-stability in terms of invariants associated to valuations rather than test configurations.

More specifically, in [BHJ17], the data of a test configuration was translated into the data of a filtration and it was shown that a nontrivial special test configuration yields a divisorial valuation. Then in a series of papers [Fuj16a, Fuj16b, Fuj18] of K. Fujita, all divisorial valuations were studied and an invariant $\beta$ was defined for each divisorial valuation. After [Li17], it became more natural to extend the setup to all valuations over the log Fano variety rather than only considering divisorial valuations (see also [LX16, BJ17]). Moreover, a characterization of $K$-stability notions in terms of the sign of $\beta$-invariant for divisorial valuations was proved in [Li17, Fuj16b] and lead to another characterization by $\delta$-invariant in [FO16, BJ17]. These interpretations
of K-stability using valuations have made it easier to apply techniques from birational geometry, especially the Minimal Model Program, to the study of K-stability.

In Section 2, we will have a short discussion on the above materials. More precisely, we will provide information on the language of valuations and filtrations following [BHJ17, Fuj17a, Li17] and the invariants $\beta$ and $\delta$ associated to them following [Fuj17a, FO16, BJ17]. We also discuss the normalized volume function from [Li18] and its relation with the K-stability of Fano varieties (see [Li17, LX16]).

To proceed with our discussion, let us define a couple of these invariants. Let $(X, \Delta)$ be a log Fano pair. Given a divisor $E$ over $X$ (i.e. $E \subset Y$ is a prime divisor, where $Y$ is a normal variety with a proper birational morphism $\pi: Y \to X$), the $\beta$ invariant of $E$ is given by

$$\beta_{X,\Delta}(E) := A_{X,\Delta}(E)(-K_X - \Delta)^n - \int_0^\infty \text{vol}(\pi^*(-K_X - \Delta) - tE)dt$$

where $A_{X,\Delta}(E)$ is the log discrepancy of $E$. This invariant was defined in [Fuj18] and the K-(semi)stability of $(X, \Delta)$ can be phrased in terms of the positivity of $\beta_{X,\Delta}(E)$ [Fuj16b, Li17].

Next, is the $\delta$-invariant of $(X, \Delta)$, which, as defined in [FO16], measures log canonical thresholds of a certain classes of anti-log canonical divisors of $(X, \Delta)$. It is shown in [BJ17] that

$$\delta(X, \Delta) = \inf_E \frac{A_{X,\Delta}(E)(-K_X - \Delta)^n}{\int_0^\infty \text{vol}(\pi^*(-K_X - \Delta) - tE)dt}. \quad (1)$$

(Hence, we say that a divisor $E$ over $X$ computes $\delta(X, \Delta)$ if it achieves the infimum in (1).) The pair $(X, \Delta)$ is uniformly K-stable (resp. K-semistable) if and only if $\delta(X, \Delta) > 1$ (resp. $\delta(X, \Delta) \geq 1$)[FO16, BJ17].

In Section 3, before attacking Theorem 1.1 in full generality, we consider the special case in which we assume $(X_0, \Delta_0)$ is uniformly K-stable and $(X'_0, \Delta'_0)$ is K-semistable. In this case, we provide a short proof of the separatedness result by using properties of the $\delta$-invariant to reduce the question to the well known separatedness statement for the moduli functor of log Calabi-Yau pairs (Proposition 3.2). This argument is more straightforward than the general case and takes a slightly different approach. We hope this perspective can be applied in other cases.

To prove Theorem 1.1 in full generality is more involved. We need to study the case when the $\delta$-invariants of the special fibers equal one. In general, analyzing the valuation computing $\delta = 1$ is quite subtle. For instance, the following statement has been conjectured by experts.

**Conjecture 1.5.** Let $(X, \Delta)$ be a log Fano pair. If $\delta(X, \Delta) \leq 1$ then $\delta(X, \Delta)$ is computed by a divisor over $X$ and any such divisor is dreamy.

The special case of Conjecture 1.5 when $\delta(X, \Delta) = 1$ implies that K-stability is equivalent to the apparently stronger notion of uniform K-stability. This is known for
smooth Fano varieties by [BBJ15], but the proof relies on analytic tools, in particular the existence of Kähler-Einstein metrics.

In Section 4, we will prove some special cases of Conjecture 1.5 which are needed in our proof of Theorem 1.1. The first result is that if \((X, \Delta)\) is a log Fano variety with \(\delta(X, \Delta) = 1\), then any divisor computing \(\delta(X, \Delta)\) is necessarily dreamy and induces a special test configuration of \((X, \Delta)\). The proof relies on the MMP techniques developed in [LWX18b], which are built upon work in [Li17, LX16, LX17]. Specifically, we consider the cone over our log Fano pair and use the calculation in [Li17, LX16] which shows that \(\beta_{X,\Delta}(E)\) equals the derivative of the normalized volume function on the valuation space of the cone along the path given by the interval connecting the divisorial valuation associated to the pull back of \(E\) and the canonical valuation. Then a careful study as in [LWX18b, Theorem 3.2] shows that \(E\) is indeed a dreamy divisor and induces a special configuration. In Section 4.2 and 4.3, we also address the situations when the \(\delta\)-invariant can be calculated by an ideal or a \(\mathbb{Q}\)-divisor. These results may be of independent interest.

Section 5 is the core of this paper and where we prove Theorem 1.1. The majority of the work in this section is to construct the S-equivalence stated in the theorem.

**Step 1:** We first observe that a pair of two different degenerations will induces filtrations on each others section rings. Furthermore, the associated graded rings of the filtrations are isomorphic with a grading shift matching the calculation of \(\beta\) invariant.

Let us explain the above construction in more detail. Assume we have two \(\mathbb{Q}\)-Gorenstein families of log Fano pairs \(\pi : (X, \Delta) \to C\) and \(\pi' : (X', \Delta') \to C\) over a smooth affine curve \(C\) and an isomorphism

\[
\phi : (X, \Delta) \times_C C^\circ \to (X', \Delta') \times_C C^\circ,
\]

that does not extend to an isomorphism over \(C^\circ = C \setminus \{0\}\). Fix \(r\) so that \(-r(K_X + \Delta)\) and \(-r(K_{X'} + \Delta')\) are Cartier. We choose a proper birational model over \(X\) and \(X'\)

\[
\begin{array}{ccc}
\tilde{X} & \xrightarrow{(\psi, \psi')} & \tilde{X}' \\
\downarrow & & \downarrow \\
X & \xrightarrow{\phi} & X'
\end{array}
\]

and write \(\tilde{X}_0\) and \(\tilde{X}_0'\) for the birational transforms of \(X_0\) and \(X_0'\) on \(\tilde{X}\). The divisor \(\tilde{X}_0'\) induces a filtration \(\mathcal{F}\) on the section of \((X_0, \Delta_0)\) defined by

\[
s \in \mathcal{F}^p H^0(X, -mr(K_{X_0} + \Delta_0))\quad \text{if and only if} \quad \text{ord}_{\tilde{X}_0}(\tilde{s}) \geq p
\]

for some (non-unique) extension \(\tilde{s} \in H^0(X, -mr(K_X + \Delta))\). Then we define

\[
\beta := (-K_{X_0} - \Delta_0)^n a - \int_{0}^{\infty} \text{vol}(\mathcal{F}(t)) \, dt
\]

where \(a\) is the log discrepancy of \(\tilde{X}_0'\) with respect to \((X, X_0 + \Delta)\). Similarly, we can define a filtration \(\mathcal{F}'\) of the section ring of \((X_0', \Delta_0')\) and the values \(a'\) and \(\beta'\) using the
divisor $\widetilde{X}_0$. The construction here can be considered as a relative version of the one in [BHJ17, Section 5], where they consider a test configuration and a trivial family.

Next, we observe that there is an isomorphism of the associated graded rings of the filtrations

$$
\bigoplus_{r|m} \bigoplus_{p=0}^{m(a+a')-p} \text{gr}_p^H H^0(X_0, -m(X_0 + \Delta_0)) \overset{\varphi}{\longrightarrow} \bigoplus_{r|m} \bigoplus_{p=0}^{m(a+a')} \text{gr}_p^H H^0(X'_0, -m(X'_0 + \Delta'_0)),
$$

which sends the degree $(m, p)$ summand to the degree $(m, m(a+a)' - p)$ summand. Using this isomorphism, we deduce that

$$
\beta + \beta' = 0.
$$

Now if we assume $(X_0, \Delta_0)$ and $(X'_0, \Delta'_0)$ are K-semistable, then the $\beta$-invariant of any divisor over $X_0$ or $X'_0$ is non-negative [Fuj16b, Li17]. A similar result is extended to filtrations in [BL18].\footnote{This is also independently obtained by Chi Li and Xiaowei Wang in [LW18].} We can then conclude that $\beta = \beta' = 0$.

**Step 2:** At this point, we know that $X_0$ and $X'_0$ have a common degeneration. Indeed, using the Rees construction there exist degenerations

$$
X_0 \leadsto X_0 := \text{Proj} \bigoplus_{r|m} \bigoplus_{p=0}^{m(a+a')} \text{gr}_p^H H^0(X_0, -m(rK_{X_0} + \Delta_0))
$$

and

$$
X'_0 \leadsto X'_0 := \text{Proj} \bigoplus_{r|m} \bigoplus_{r|m} \text{gr}_p^H H^0(X'_0, -m(rK_{X'_0} + \Delta'_0)).
$$

By (2), the degenerations $X_0$ and $X'_0$ are isomorphic.

An immediate concern is that the above graded rings are not necessarily finitely generated. (Note that notions of K-stability have been investigated in the setting of non-finitely generated filtrations [WN12, Szé15].) Since we aim to prove $X_0$ and $X'_0$ have a common degeneration to a K-semistable $\mathbb{Q}$-Fano variety, we must show that the filtrations $\mathcal{F}$ and $\mathcal{F}'$ are finitely generated and induce special test configurations with generalized Futaki invariant zero. By [LWX18b, 3.1], this will imply that the degenerations are naturally K-semistable pairs.

To proceed, we rely on the fact that our filtrations are induced by divisors over our families. More precisely, we use that $\beta = 0$ to show that there exists an extraction $Y \rightarrow X$ of $\widetilde{X}_0$ and the fiber $Y_0 = V \cup W$, where $V$ and $W$ are the birational transforms of $X_0$ and $X'_0$. Now, we set $E = W|_V$ and observe that $E$ induces a filtration $\mathcal{F}_E$ on the section ring of $(X_0, \Delta_0)$. Furthermore, we show that $F := \text{Supp}(E)$ is a prime divisor and $\beta_{X_0, \Delta_0}(F) = 0$. By Theorem 4.1, this yields that $\mathcal{F}_E$ is finitely generated and the corresponding degeneration of $(X_0, \Delta_0)$ is a special test configuration with generalized Futaki invariant zero.
Next, we seek to show that the filtrations $\mathcal{F}$ and $\mathcal{F}_E$ induce the same degeneration of $(X_0, \Delta_0)$. Hence, we must show that $\mathcal{F}$ and $\mathcal{F}_E$ are equal or at least equal on a Veronese subring of the section ring. This statement is equivalent to the surjectivity of certain restriction maps and is non-trivial. To achieve the result, we take a relative cone of $(X, \Delta)$ over $C$ and run an analysis similar to the proof of Theorem 4.1. After completing this argument, we can conclude that the degenerations are naturally $K$-semistable pairs.

Finally, we need to show that the isomorphism $X_0 \simeq X'_0$ sends the degeneration of $\Delta_0$ to the degeneration of $\Delta'_0$, so that we get an isomorphism of pairs. To verify this, we choose a divisor $B \subseteq \text{Supp}(\Delta)$ and write $B' \subseteq \text{Supp}(\Delta')$ for its strict transform. Now, $B_0$ degenerates to a divisor on $X_0$ that corresponds to the initial ideal $\text{in}(I_{B_0})$ in the associated graded ring. Rather than showing that $\varphi(\text{in}(I_{B_0})) = \text{in}(I_{B'_0})$, we introduce auxiliary ideals $I$ and $I'$ such that the equality $\varphi(I) = I'$ is clear. (The ideal $I$ is defined by restricting elements of the relative section ring that vanish to certain orders along $B$ and $X'_0$.) Using the relative cone construction again, we show that $I$ and $I'$ agree with $\text{in}(I_{B_0})$ to $\text{in}(I_{B'_0})$ at codimension one points. We can then conclude that the desired isomorphism of boundaries holds.

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2. Preliminaries on valuations and $K$-stability

2.1. **Conventions.** We will follow the terminologies in [KM98, Kol13]. A variety $X$ is $\mathbb{Q}$-Fano if it is projective, has klt singularities, and $-K_X$ is ample. A pair $(X, \Delta)$ is log Fano if $X$ is projective, $(X, \Delta)$ is klt, and $-K_X - \Delta$ is ample.

**Definition 2.1.** A $\mathbb{Q}$-Gorenstein family of log Fano pairs $\pi : (X, \Delta) \to C$ over a smooth curve $C$ is composed of a flat proper morphism $\pi : X \to C$ and an effective $\mathbb{Q}$-divisor $\Delta$, not containing any fiber of $\pi$, satisfying:

1. $\pi$ has normal, connected fibers (hence, $X$ is normal as well)
2. $-K_X - \Delta$ is a $\pi$-ample $\mathbb{Q}$-Cartier divisor, and
3. $(X_t, \Delta_t)$ is klt for all $t \in C$.

2.2. **Valuations.** Let $X$ be a variety. A valuation on $X$ will mean a valuation $v : K(X)^\times \to \mathbb{R}$ that is trivial on $\mathfrak{k}$ and has center on $X$. Recall, $v$ has center on $X$ if there exists a point $\xi \in X$ such that $v \geq 0$ on $X$ and $v > 0$ on the maximal ideal of
$\mathcal{O}_{X, \xi}$. Since $X$ is assumed to be separated, such a point $\xi$ is unique, and we say $v$ has
center $c_X(v) := \xi$.

Following [JM12, BdFFU15], we write $\text{Val}_X$ for the set of valuations on $X$ and $\text{Val}_X^*$ for the set of non-trivial. To any valuation $v \in \text{Val}_X$ and $p \in \mathbb{N}$, there is an associated valuation ideal $a_p(v)$ . For an affine open subset $U \subseteq X$, $a_p(v)(U) = \{ f \in \mathcal{O}_X(U) \mid v(f) \geq p \}$ if $c_X(v) \in U$ and $a_p(v)(U) = \mathcal{O}_X(U)$ otherwise.

For an ideal $a \subseteq \mathcal{O}_X$ and $v \in \text{Val}_X$, we set

$$v(a) := \min \{ v(f) \mid f \in a \cdot \mathcal{O}_{X, c_X(v)} \} \in [0, +\infty].$$

We can define $v(s)$ when $L$ is a line bundle on $X$ and $s \in H^0(X, L)$. After trivializing $L$ at $c_X(v)$, we set $v(s) = v(f)$, where $f$ is the local function corresponding to $s$ under this trivialization. This is independent of choice of trivialization.

2.2.1. Divisors over $X$. Let $X$ be a variety and $\pi : Y \to X$ be a proper birational morphism, with $Y$ normal. A prime divisor $E \subset Y$ defines a valuation $\text{ord}_E : K(X) \to \mathbb{Z}$ given by order of vanishing at $E$. Note that $c_X(\text{ord}_E)$ is the generic point of $\pi(E)$ and, assuming $X$ is normal, $a_p(v) = \pi_*\mathcal{O}_X(-pE)$.

We identify two such prime divisors on $Y_1$ and $Y_2$ as above if one is the birational transform of the other (equivalently, they induce the same valuation of $K(X)$). A divisor over $X$ is an equivalence class given by this relation.

2.2.2. Log discrepancies. Let $(X, \Delta)$ be a pair. We write

$$A_{X, \Delta} : \text{Val}(X) \to \mathbb{R}_{\geq 0} \cup \{ +\infty \}$$

for the log discrepancy function with respect to $(X, \Delta)$ as in [JM12, BdFFU15] (see [Blu18b] for the case when $\Delta \neq 0$).

When $\pi : Y \to X$ is a proper birational morphism with $Y$ normal and $E \subset Y$ a prime divisor,

$$A_{X, \Delta}(\text{ord}_E) = 1 + \text{coeff}_E (K_Y - \pi^*(K_X + \Delta)).$$

We will often write $A_{X, \Delta}(E)$ for the above value.

The function $A_{X, \Delta}$ is homogenous of degree $1$, i.e. $A_{X, \Delta}(\lambda v) = \lambda \cdot A_{X, \Delta}(v)$ for $\lambda \in \mathbb{R}_{> 0}$ and $v \in \text{Val}_X$. A pair $(X, \Delta)$ is klt (resp., lc) if and only if $A_{X, \Delta}(v) > 0$ (resp., $\geq 0$) for all $v \in \text{Val}_X$.

2.2.3. Graded sequences. A graded sequence of ideals $a_* = (a_p)_{p \in \mathbb{Z}_{> 0}}$ on a variety $X$ is a sequence of ideals on $X$ satisfying $a_p \cdot a_q \subseteq a_{p+q}$ for all $p, q \in \mathbb{Z}_{> 0}$. By convention, $a_0 = \mathcal{O}_X$. We set $M(a_*) := \{ p \in \mathbb{Z}_{> 0} \mid a_p \neq (0) \}$ and will always assume $M(a_*)$ is nonempty. Note that if $v \in \text{Val}_X$, then $a_*(v)$ is a graded sequence of ideals.

Let $a_*$ be a graded sequence of ideals on $X$ and $v \in \text{Val}_X$. It follows from Fekete’s Lemma that the limit

$$v(a_*) := \lim_{M(a_*) \ni m \to \infty} \frac{v(a_m)}{m}$$

exists and equals $\inf_{m \geq 1} \frac{v(a_m)}{m}$; see [JM12, §2.1].
Let \( x \in X \) be a closed point. If \( a_* \) is a graded sequence of ideals on \( X \) and each ideal \( a_p \) is \( m_x \)-primary, we set
\[
\text{mult}(a_*) := \lim_{p \to \infty} \dim_k (O_X/a_p)/p^n/n!.
\]
If \( v \in \text{Val}_X \) has center \( \{x\} \), then \( a_p(v) \) is \( m_x \)-primary for each \( p > 0 \). In this case, we call \( \text{vol}(v) := \text{mult}(a_*(v)) \) the volume of \( v \).

2.2.4. Log canonical thresholds. Let \((X, \Delta)\) be an lc pair. Given a nonzero ideal \( a \subseteq O_X \), the log canonical threshold of \( a \) is given by
\[
\text{lct}(X, \Delta; a) := \sup \{ c \in \mathbb{Q} \geq 0 \mid (X, \Delta + ac) \text{ is lc} \}.
\]
If \( a_* \) a graded sequence of ideals on \( X \), the log canonical threshold of \( a_* \) is given by
\[
\text{lct}(X, \Delta; a_*) := \lim_{M(a_*) \ni m \to \infty} m \cdot \text{lct}(X, \Delta; a_m).
\]
Fekete’s Lemma implies that the above limit exists and equals \( \sup_{m \cdot \text{lct}(X, \Delta; a_m)} \) [JM12, 2.5].

It is straightforward to show
\[
\text{lct}(X, \Delta; a_*) \leq \frac{A_{X, \Delta}(v)}{v(a_*)},
\]
for \( v \in \text{Val}_X^* \) satisfying \( 0 \neq A_{X, \Delta}(v) < +\infty \). Hence, if \( v \in \text{Val}_X^* \) satisfies \( A_{X, \Delta}(v) \neq 0 \), then
\[
\text{lct}(X, \Delta; a_*(v)) \leq A_{X, \Delta}(v), \quad (3)
\]
since \( v(a_*(v)) = 1 \) [Blu18b, 3.4.9].

2.2.5. Extractions. Let \( E \) be a divisor over a normal variety \( X \). We say that \( \mu : X_E \to X \) is an extraction of \( E \) if \( \mu \) is a proper birational morphism with \( X_E \) is normal, \( E \) arises as a prime divisor \( E \subset X_E \), and \( -E \) is \( \mu \)-ample.

Note that if \( \mu : X_E \to X \) is an extraction of \( E \), then \( E \subset \text{Exc}(\mu) \). Indeed, Lemma 4.6 implies that if \( p \in \mathbb{Z}_{>0} \) is sufficiently divisible, then \( \mu \) is the blowup along \( a_p(E) \) and \( a_p(\text{ord}_E) \cdot O_Y = O_Y(-E) \).

The following technical statement gives a criterion for when an exceptional divisor may be extracted. The criterion will be used repeatedly in Section 5.

**Proposition 2.2.** Let \((X, \Delta)\) be a klt pair or a plt pair such that \( [\Delta] = S \) is a non-zero \( \mathbb{Q} \)-Cartier. If \( E \) is a divisor over \( X \) satisfying
\[
c := A_{X, \Delta}(E) - \text{lct}(X, \Delta; a_*(\text{ord}_E)) < 1,
\]
then there exists an extraction \( \mu : X_E \to X \) of \( E \) and \((X_E, \mu^{-1}(\Delta) + (1 - c)E)\) is lc.

The proposition is a consequence of [BCHM10] and properties of the log canonical threshold of a graded sequence of ideals.
Proof. See the argument in [Blu17, 1.5] for the case when \((X, \Delta)\) is klt. If \((X, \Delta)\) is plt, observe that \((X, \Delta_\varepsilon := \Delta - \varepsilon S)\) is klt for \(0 < \varepsilon < 1\). If we set
\[ c_\varepsilon := A_{X,\Delta_\varepsilon}(\text{ord}_E) - \text{lct}(X, \Delta_\varepsilon; a_\bullet(\text{ord}_E)), \]
then \(\lim_{\varepsilon \to 0} c_\varepsilon = c\) and we may reduce to the klt case. \(\Box\)

2.3. Filtrations. Let \((X, \Delta)\) be a \(n\)-dimensional log Fano pair. Fix a positive integer \(r\) such that \(L := -r(K_X + \Delta)\) is Cartier and write
\[ R = R(X, L) = \bigoplus_{m \in \mathbb{N}} R_m = \bigoplus_{m \in \mathbb{N}} H^0(X, \mathcal{O}_X(mL)) \]
for the section ring of \(L\). Set \(M(L) := \{m \in \mathbb{N} \mid H^0(X, \mathcal{O}_X(mL)) \neq 0\}\).

**Definition 2.3.** A filtration \(\mathcal{F}\) of \(R\) we will mean the data of a family of \(k\)-vector subspaces
\[ \mathcal{F}^\lambda R_m \subseteq R_m \]
for \(m \in \mathbb{N}\) and \(\lambda \in \mathbb{R}\), satisfying
1. \(\mathcal{F}^\lambda R_m \subseteq \mathcal{F}^{\lambda'} R_m\) when \(\lambda \geq \lambda'\);
2. \(\mathcal{F}^\lambda R_m = \cap_{\lambda' < \lambda} \mathcal{F}^{\lambda'} R_m\) for all \(\lambda\);
3. \(\mathcal{F}^0 R_m = R_m\) and \(\mathcal{F}^\lambda R_m = 0\) for \(\lambda \gg 0\).
4. \(\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subseteq \mathcal{F}^{\lambda + \lambda'} R_{m+m'}\).

A filtration \(\mathcal{F}\) of \(R\) is a called an \(\mathbb{N}\)-filtration if \(\mathcal{F}^\lambda R_m = \mathcal{F}^{[\lambda]} R_m\) for all \(m \in \mathbb{N}\) and \(\lambda \in \mathbb{R}\). To give a \(\mathbb{N}\)-filtration \(\mathcal{F}\), it suffices to give a family of subspaces \(\mathcal{F}^p R_m \subseteq R_m\) for \(m, p \in \mathbb{N}\) satisfying (1), (3), and (4).

A filtration \(\mathcal{F}\) is linearly bounded if there exists \(C > 0\) so that \(\mathcal{F}^{Cm} R_m = 0\) for all \(m \in \mathbb{N}\).

2.3.1. Rees construction. Let \(\mathcal{F}\) be an \(\mathbb{N}\)-filtration of \(R\). The Rees algebra of \(\mathcal{F}\) is the \(k[t]\)-algebra
\[ \text{Rees}(\mathcal{F}) := \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{Z}} t^{-p} \mathcal{F}^p R_m \subseteq R[t, t^{-1}]. \]
The associated graded ring of \(\mathcal{F}\) is
\[ \text{gr}_\mathcal{F} R := \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{Z}} \text{gr}_\mathcal{F}^p R_m, \quad \text{where} \quad \text{gr}_\mathcal{F}^p R_m = \frac{\mathcal{F}^p R_m}{\mathcal{F}^{p+1} R_m}. \]
Note that
\[ \text{Rees}(\mathcal{F}) \otimes_{k[t]} k[t, t^{-1}] \simeq R[t, t^{-1}] \quad \text{and} \quad \frac{\text{Rees}(\mathcal{F})}{t \text{Rees}(\mathcal{F})} \simeq \text{gr}_\mathcal{F} R. \quad (4) \]
Hence, \(\text{Rees}(\mathcal{F})\) is said to give a degeneration of \(R\) to the associated graded ring of \(\mathcal{F}\).

We say that an \(\mathbb{N}\)-filtration \(\mathcal{F}\) is finitely generated if \(\text{Rees}(\mathcal{F})\) is a finitely generated \(k[t]\)-algebra. Equivalently,
\[ \bigoplus_{(m,p) \in \mathbb{N} \times \mathbb{Z}} \mathcal{F}^p R_m \]
is finitely generated over $k$.

If $\mathcal{F}$ is finitely generated, we set $\mathcal{X} := \text{Proj}_{\mathbb{A}^1} (\text{Rees}(\mathcal{F}))$. By (4),

$$\mathcal{X}_{\mathbb{A}^1 \setminus \{0\}} \simeq X \times (\mathbb{A}^1 \setminus \{0\}) \quad \text{and} \quad \mathcal{X}_0 \simeq \text{Proj}(\text{gr}_F R).$$

We write $D$ for the $\mathbb{Q}$-divisor that is the closure of $\Delta \times (\mathbb{A}^1 \setminus \{0\})$ under the embedding of $X \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{X}$.

The scheme $\mathcal{X}$ can naturally be endowed with the structure of a test configuration of $(X, \Delta)$. The test configuration is special if $(\mathcal{X}, D) \to \mathbb{A}^1$ is a $\mathbb{Q}$-Gorenstein family of log Fano pairs. See [LX14, §3] and [BHJ17, §2] for information on test configurations and the generalized Futaki invariant.

With the above setup, consider a subscheme $Z \subset X$ and write $I_Z \subset R$ for the corresponding homogenous ideal. The scheme theoretic closure of $Z \times (\mathbb{A}^1 \setminus \{0\})$ in $X$, denoted by $Z$, is defined by the ideal

$$\bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{Z}} t^{-p}(F^p R_m \cap I_Z) \subseteq \text{Rees}(\mathcal{F}).$$

Indeed, the corresponding subscheme agrees with $Z \times (\mathbb{A}^1 \setminus \{0\})$ away from $\{0\}$ and is torsion free over 0. The above description of $Z$ yields that its scheme theoretic fiber along 0 is given by the initial ideal

$$\text{in}(I_Z) := \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{Z}} \text{im}(F^p R_m \cap I_Z \to \text{gr}_F R_m) \subset \text{gr}_F R_m.$$

2.3.2. **Volume.** Given a filtration $\mathcal{F}$ of $R$, we set

$$\text{vol}(\mathcal{F} R^{(x)}) := \limsup_{m \to \infty} \frac{\dim(F^{xm} R_m)}{m^n / n!}$$

for $x \in \mathbb{R}_{\geq 0}$. Assuming $\mathcal{F}$ is linearly bounded (which implies $\text{vol}(\mathcal{F} R^{(x)}) = 0$ for $x \gg 0$), we set

$$S(\mathcal{F}) := \frac{1}{r^{n+1} (-K_X - \Delta)^n} \int_0^\infty \text{vol}(\mathcal{F} R^{(x)}) \, dx. \quad \text{(5)}$$

By [BC11] (see also [BHJ17, 5.3]),

$$S(\mathcal{F}) = \lim_{m \to \infty} \left( \frac{1}{r \dim R_m} \int_0^\infty \dim(F^{mx} H^0(X, mL)) \, dx \right).$$

2.3.3. **Base ideals.** To a filtration $\mathcal{F}$ of $R$, we associate a graded sequence of base ideals. For $p, m \geq 0$, set

$$b_{p,m} := \text{im} (F^p R_m \otimes \mathcal{O}_X(-mL) \to \mathcal{O}_X).$$

We set $b_p(\mathcal{F}) = b_{p,m}$ for $m \gg 1$. The ideal $b_p(\mathcal{F})$ is well defined and $b_\bullet(\mathcal{F}) = (b_p(\mathcal{F}))_{p \in \mathbb{N}}$ is a graded sequence of ideals [BJ17, 3.17-3.18].

---

*Note that this differs from the definition of $S(\mathcal{F})$ in [BJ17, BL18] by a factor of $1/r$. Since we are interested in the polarization $-K_X - \Delta$, not $L$, such a convention is natural.*
2.3.4. **Filtrations induced by valuations.** Given $v \in \text{Val}_X$, we set
\[
\mathcal{F}_v^\lambda R_m = \{ s \in R_m \mid v(s) \geq \lambda \}.
\]
for each $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$. Equivalently, $\mathcal{F}_v^\lambda R_m = H^0(X, \mathcal{O}_X(mL) \otimes a_\lambda(v))$. Note that $\mathcal{F}_v$ is a filtration $R$.

If $\mathcal{F}_v$ is linearly bounded, we set $S(v) := S(\mathcal{F}_v)$. By [BJ17, 3.1], if $A_{X,\Delta}(v) < +\infty$, then $\mathcal{F}_v$ is linearly bounded.

2.3.5. **Filtrations induced by divisors.** If $E$ is a divisor over $X$, we set $\mathcal{F}_E := \mathcal{F}_{\text{ord}_E}$ and $S(E) := S(\mathcal{F}_E)$. Following [Fuj16b], we say $E$ is *dreamy* if $\mathcal{F}_E$ is a finitely generated filtration of $R$.

When $E$ arises as a prime divisor on a proper normal model $\mu : Y \to X$, $\mathcal{F}_E^\lambda R_m = H^0(Y, \mathcal{O}_Y(mr\mu^*(-K_X - \Delta) - \lceil \lambda \rceil E)).$

Therefore,
\[
S(E) := \frac{1}{(-K_X - \Delta)^n} \int_0^\infty \text{vol}(\mu^*(-K_X - \Delta) - xE) \, dx.
\]

2.4. **K-stability.** Based on the original analytic definition in [Tia97], an algebraic definition of K-(semi,poly)-stability was introduced in [Don02]. Here, we will define these notations for log Fano pairs using valuations.

2.4.1. **$\beta$-invariant.** Let $(X, \Delta)$ be an $n$-dimensional log Fano pair and $E$ a divisor over $X$. Following [Fuj16b], we set
\[
\beta_{X,\Delta}(E) := (-K_X - \Delta)^n (A_{X,\Delta}(E) - S(E))
\]
Equivalently,
\[
\beta_{X,\Delta}(E) = (-K_X - \Delta)^n A_{X,\Delta}(E) - \int_0^\infty \text{vol}(\mu^*(-K_X - \Delta) - xE) \, dx.
\]
where $E$ arises a prime divisor on a proper normal model $\mu : Y \to X$.

More generally, we may define the $\beta$-invariant for valuations and filtrations as follows. If $v \in \text{Val}_X$ with $A_{X,\Delta}(v) < +\infty$, we set
\[
\beta_{X,\Delta}(v) := (-K_X - \Delta)^n (A_{X,\Delta}(v) - S(v)).
\]
If $\mathcal{F}$ is a non-trivial linearly bounded filtration of $R = R(X, L)$, we set
\[
\beta_{X,\Delta}(\mathcal{F}) := (-K_X - \Delta)^n (\text{lct}(X, \Delta; b_\bullet(\mathcal{F})) - S(\mathcal{F})).
\]

**Definition 2.4.** A log Fano pair $(X, \Delta)$ is

1. **$K$-semistable** if $\beta_{X,\Delta}(E) \geq 0$ for all divisors $E$ over $X$;
2. **$K$-stable** if $\beta_{X,\Delta}(E) > 0$ for all dreamy divisors $E$ over $X$;
3. **uniformly $K$-stable** if there exists an $\varepsilon > 0$ such that
\[
\beta_{X,\Delta}(E) \geq \varepsilon A_{X,\Delta}(E)(-K_X - \Delta)^n
\]
for all divisors $E$ over $X$. 
The equivalence of the above definition with the original definitions was addressed in [Fuj16b, Fuj17a, Li17] (see also [BJ18]) and the arguments rely on the special degeneration theory of [LX14]. In Corollary 4.3, we will show that the wordy dreamy may be removed from Definition 2.4.

**Definition 2.5.** A log Fano pair \((X, \Delta)\) is \(K\)-polystable if it is \(K\)-semistable and any special test configuration \((\mathcal{X}, \mathcal{D})\) of \((X, \Delta)\) with \((\mathcal{X}_0, \mathcal{D}_0)\) \(K\)-semistable satisfies \((X_0, \Delta_0) \simeq (\mathcal{X}_0, \mathcal{D}_0)\).

The equivalence of the above definition with the definition in [LX14, 6.2] relies on the following fact: If \((X, \Delta)\) is a \(K\)-semistable log Fano pair and \((X, D)\) is a special test configuration of \((X, \Delta)\), then \(\text{Fut}(X, D) = 0\) if and only if \((X_0, D_0)\) is \(K\)-semistable. Indeed, the forward implication is [LWX18b, 3.1]. The reverse implication follows from the computation of the Futaki invariant in terms of the weight of the \(G_m\) action on \((\mathcal{X}_0, \mathcal{D}_0)\).

**Definition 2.6.** Two \(K\)-semistable log Fano pairs \((X, \Delta)\) and \((X', \Delta')\) are \(S\)-equivalent if they degenerate to a common \(K\)-semistable log Fano pair via special test configurations.

By [LWX18b, 3.2], \(S\)-equivalent log Fano pairs degenerate to a common \(K\)-polystable pair via special test configurations. Furthermore, the \(K\)-polystable pair is uniquely determined up to isomorphism.

**2.4.2. \(\delta\)-invariant.** We recall an interpretation of the above discussion using an invariant introduced in [FO16].

Let \((X, \Delta)\) be a log Fano pair. Fix a positive integer \(r\) so that \(L := -r(K_X + \Delta)\) is a Cartier divisor and \(H^0(X, \mathcal{O}_X(L)) \neq 0\). Given \(m \in r\mathbb{N}\), we say \(D \sim_{\mathbb{Q}} -K_X - \Delta\) is an \(m\)-basis type \(\mathbb{Q}\)-divisor of \(-K_X - \Delta\) if there exists a basis \(\{s_1, \ldots, s_{Nm}\}\) of \(H^0(X, \mathcal{O}_X(m(-K_X - \Delta)))\) such that

\[
D = \frac{1}{mNm} \left(\{s_1 = 0\} + \cdots + \{s_{Nm} = 0\}\right).
\]

We set

\[
\delta_m(X, \Delta) := \min\{\text{lct}(X, \Delta; D) \mid D \sim_{\mathbb{Q}} -K_X - \Delta\text{ is }m\text{-basis type}\}.
\]

The \(\delta\)-invariant (also known as the stability threshold) of \((X, \Delta)\) is

\[
\delta(X, \Delta) = \limsup_{m \to \infty} \delta_{mr}(X, \Delta),
\]

and is independent of the choice of \(L\) [BJ17, 4.5]. The invariant may also be calculated in terms of valuations or filtrations.

**Theorem 2.7** ([BJ17, Theorems A,C]). We have

\[
\delta(X, \Delta) = \inf_E \frac{A_{X, \Delta}(E)}{S(E)} = \inf_v \frac{A_{X, \Delta}(v)}{S(v)},
\]
where the first infimum runs through all divisors $E$ over $X$ and the second through all $v \in \text{Val}_X$ with $A_{X, \Delta}(v) < +\infty$. Furthermore, the limit $\lim_{m \to \infty} \delta_{mr}(X, \Delta)$ exists.

**Proposition 2.8** ([BL18, Proposition 4.10]). We have

$$\delta(X, \Delta) = \inf_{\mathcal{F}} \frac{\text{lct}(X, \Delta; b_\bullet(\mathcal{F}))}{S(\mathcal{F})}$$

where the infimum runs through all non-trivial linearly bounded filtrations of $R = R(X, L)$.

Combining Definition 2.4 and Theorem 2.7, we immediately see

**Theorem 2.9** ([FO16, BJ17]). A log Fano pair $(X, \Delta)$ is uniformly K-stable (resp., K-semistable) if and only if $\delta(X, \Delta) > 1$ (resp., $\geq 1$).

While in Section 3 we will use the definition of the $\delta$-invariant in terms of $m$-basis type divisors, in Section 5 we will rely on its characterization in terms of valuations and filtrations.

### 2.5. Normalized volume.

Now, we discuss an invariant similar to the $\delta$-invariant, but defined in a local setting. This invariant was first introduced in [Li18] and closely related to the K-semistability of log Fano pairs.

Let $(Y, \Gamma)$ be an $n$-dimensional klt pair and $x \in Y$ a closed point. The **non-archimedean link** of $Y$ at $x$ is defined as

$$\text{Val}_{Y,x} := \{ v \in \text{Val}_Y \mid c_Y(v) = \{ x \} \} \subset \text{Val}_Y.$$

**Definition 2.10** ([Li18]). The **normalized volume function**

$$\hat{\text{vol}}_{(Y, \Gamma), x} : \text{Val}_{Y,x} \to (0, +\infty]$$

is defined by

$$\hat{\text{vol}}_{(Y, \Gamma), x}(v) = \begin{cases} A_{(Y, \Gamma)}(v)^n \cdot \text{vol}(v) & \text{if } A_{(Y, \Gamma)}(v) < +\infty; \\ +\infty & \text{if } A_{(Y, \Gamma)}(v) = +\infty. \end{cases}$$

Here, $A_{Y, \Gamma}(v)$ is the log discrepancy of $v$ and $\text{vol}(v)$ is the volume of $v$. The **volume of the singularity** $(x \in (Y, \Gamma))$ is defined as

$$\hat{\text{vol}}(x, Y, \Gamma) := \inf_{v \in \text{Val}_{Y,x}} \hat{\text{vol}}_{(Y, \Gamma), x}(v).$$

The previous infimum is a minimum by the main result in [Blu18a].

See [LLX18] for a survey on the recent study of the normalized volume function, especially the guiding question, the Stable Degeneration Conjecture (see [Li18, Conjecture 7.1] and [LLX18, Conjecture 4.1]).
2.5.1. Relation to K-stability. The connection between the normalize volume function and K-semistability is via the cone construction first studied in [Li17].

Let $(X, \Delta)$ be a log Fano pair and fix a positive integer $r$ so that $L := -r(K_X + \Delta)$ is a Cartier divisor. Let

$$(Y, \Gamma) = C(X, \Delta; L)$$

denote the cone over $X$ and $x \in Y$ denote the vertex. By this we mean, $Y = \text{Spec}(R(X, L))$ is the cone over $X$ with respect to the polarization $L$ and $\Gamma$ is the closure of the pullback of $\Delta$ via the projection map $Y \setminus \{x\} \to X$. By blowing up the vertex, we get an exceptional divisor $X_{\infty} \approx X$. We call $v_0 := \text{ord}_{X_{\infty}}$ the canonical valuation over the cone.

**Theorem 2.11** ([Li17, LL19, LX16]). The canonical valuation $v_0$ is a minimizer of $\hat{\text{vol}}_{(Y, \Gamma), x}$ if and only if $(X, \Delta)$ is K-semistable.

At first sight, using the normalized volume function to study the K-stability of log Fano pairs may seem indirect. However, this approach yields a number of new results (for example, see [LX16, LWX18b]). In this paper, the following key ingredient in the proof of Theorem 2.11 plays an important role in the proof of our main result.

Let $E$ be a divisor over $X$ that arises on a proper normal model $\mu : Z \to X$. Following [Li17, LX16], $E$ gives rise to a ray of valuations

$$\{v_t \mid t \in [0, \infty) \subset \text{Val}_{Y, x}\}.$$ 

Since the blowup of $Y$ at 0 is canonically isomorphic to the total space of the line bundle $\mathcal{O}_X(-L)$, there is a proper birational map from $Z_{L^{-1}} \to Y$, where $Z_{L^{-1}}$ denote the total space of $\mu^*\mathcal{O}_X(-L)$. Now,

$$v_0 = \text{ord}_{X_{\infty}} \quad \text{and} \quad v_\infty = \text{ord}_{E_{\infty}},$$

where $E_\infty$ denotes the pullback of $E$ under the map $Z_{L^{-1}} \to Y$ and $X_{\infty}$ denote the zero section of $Z_{L^{-1}}$. Furthermore, $v_t$ is defined to be the quasimonomial valuation with weights $(1, t)$ along $X_{\infty}$ and $E_\infty$.

We have

$$A_{Y, T}(v_t) = A_{Y, T}(\text{ord}_{X_{\infty}}) + A_{Y, T}(\text{ord}_{E_\infty}) = 1/r + at.$$ 

For $t > 0$,

$$a_p(v_t) = \bigoplus_{m \geq 0} \mathcal{F}_E^{(p-m)/t} R_m \subseteq R \quad \text{and} \quad a_p(v_0) = \bigoplus_{m \geq p} R_m \subseteq R.$$ 

When $k \in \mathbb{N}$, $v_{1/k} = 1/k \text{ord}_{E_k}$, where $E_k$ is a divisor over $X$.

By the calculation in [Li17, (31, 32)] (also see [LX16, proof of Proposition 4.5] or Lemma 2.12), we have

$$\left. \frac{d \hat{\text{vol}}(v_t)}{dt} \right|_{t=0^+} = (n+1)\beta_{X, \Delta}(E). \quad (7)$$
A key input in our proof Theorem 4.1 is the above equation. More specifically, we will follow ideas from [LWX18b] and analyze directions along which the normalized volume function has derivative equal to zero.

2.5.2. C. Li’s derivative formula. In the proof of Theorem 1.1, we will need a more general version of (7). The more general formula follows from the original argument in [Li17].

Let \((X, \Delta)\) be an \(n\)-dimensional log Fano pair and \(r \in \mathbb{Z}_{>0}\) so that \(L := -r(K_X + \Delta)\) is Cartier. Set \(R = R(X, L)\) and fix a linearly bounded filtration \(\mathcal{F}\) of \(R\).

Associated to \(\mathcal{F}\), we define a collection of graded sequences of ideals of \(R\). For \(t \in \mathbb{R}_{\geq 0}\) and \(j \in \mathbb{Z}_{>0}\), set
\[
b_{t,j} := \bigoplus_{m \geq 0} \mathcal{F}^{(j-m)/t} R_m \subset R \quad \text{and} \quad b_{0,j} := \bigoplus_{m \geq j} R_m \subset R.
\]

Note that \(b_{t,\bullet}\) is a graded sequence of ideals of \(R\) for each \(t \in \mathbb{R}_{\geq 0}\). Additionally, \(b_{t,j}\) contains \(\bigoplus_{m \geq j} R_m\).

**Lemma 2.12.** With the above notation, fix \(a > 0\) and set
\[
f(t) = (1/r + at)^{n+1} \text{mult}(b_{t,\bullet})
\]
for \(t \in \mathbb{R}_{\geq 0}\). The following hold:

1. \(\text{mult}(b_{t,\bullet}) = r^n(-K_X - \Delta)^n - (n+1) \int_0^\infty \text{vol}(\mathcal{F}R^{(x)}) \frac{dx}{(1+tx)^{n+2}}
\)
2. \(\left. \frac{df}{dt} \right|_{t=0^+} = (n+1) \left( a(-K_X - \Delta)^n - \frac{1}{r^{n+1}} \int_0^\infty \text{vol}(\mathcal{F}R^{(x)}) dx \right).

**Proof.** This follows from the argument in [Li17, (18)-(25)]. For the reader’s convenience, we give a brief proof. For \(t \in \mathbb{R}_{>0}\), we have
\[
\text{mult}(b_{t,\bullet}) = \lim_{j \to \infty} \frac{(n+1)!}{j^{n+1}} \text{dim}_k(R/b_{t,j})
\]
\[
= \lim_{j \to \infty} \frac{(n+1)!}{j^{n+1}} \sum_{m=0}^\infty \text{dim}_k(R_m/\mathcal{F}^{(j-m)/t} R_m)
\]
\[
= \lim_{j \to \infty} \frac{(n+1)!}{j^{n+1}} \sum_{m=0}^j \left( \text{dim}_k R_m - \text{dim}_k \mathcal{F}^{(j-m)/t} R_m \right)
\]
\[
= \text{vol}(L) - \lim_{j \to \infty} \frac{(n+1)!}{j^{n+1}} \sum_{m=0}^j \text{dim}_k \mathcal{F}^{(j-m)/t} R_m.
\]

Statement (1) now follows from [Li17, (25)], where \(c_1 = 0\), \(\alpha = \beta = \frac{1}{t}\).

For (2), compute
\[
\left. \frac{df}{dt} \right|_{t=0^+} = a(n+1) \left( \frac{1}{r} \right)^n \text{mult}(b_{0,\bullet}) + \frac{1}{r^{n+1}} \left. \frac{d}{dt} \left( \text{mult}(b_{t,\bullet}) \right) \right|_{t=0^+}.
\]
From (1), we know $\mult(b_0, \cdot) = r^n(-K_X - \Delta)^n$ and
\[
\frac{d}{dt} \left( \mult(b_t, \cdot) \right) \bigg|_{t=0^+} = -(n+1) \int_0^\infty \left( \vol(FR^{(x)}) \left( \frac{1-tx(n+1)}{(1+tx)^{n+3}} \right) \right) dx.
\]
Since the latter simplifies to $-(n+1) \int_0^\infty \vol(FR^{(x)}) dx$, (2) is complete. \qed

3. Uniformly K-stable Fanos

In this section, we prove a special case of Theorem 1.1 for uniformly K-stable Fano varieties. We will then apply the result to study the moduli functor $\mathcal{M}^{uK_\varnothing}_{n,V}$.

3.1. Separatedness result. The following result is a special case of Theorem 1.1 and will be reproved in Section 5. We present its proof independently since the following argument is much simpler than the proof in Section 5 and requires significantly less application of tools from the MMP.

**Theorem 3.1.** Let $\pi : (X, \Delta) \to C$ and $\pi' : (X', \Delta) \to C$ be $\mathbb{Q}$-Gorenstein families of log Fano pairs over a smooth pointed curve $0 \in C$. Assume there exists an isomorphism $\phi : (X, \Delta) \times_C C^o \to (X', \Delta') \times_C C^o$ over $C^o : = C \setminus \{0\}$. If $(X_0, \Delta_0)$ is uniformly K-stable and $(X'_0, \Delta'_0)$ is K-semistable, then $\phi$ extends to an isomorphism $(X, \Delta) \simeq (X', \Delta')$ over $C$.

The proof of Theorem 3.1 follows from properties of the $\delta$-invariant and the following birational geometry fact.

**Proposition 3.2.** Let $\pi : (X, \Delta) \to C$ and $\pi' : (X', \Delta) \to C$ be $\mathbb{Q}$-Gorenstein families of log Fano pairs over a smooth pointed curve $0 \in C$. Assume there exists an isomorphism $\phi : (X, \Delta) \times_C C^o \to (X', \Delta') \times_C C^o$ over $C^o : C \setminus \{0\}$. If there exists effective horizontal $\mathbb{Q}$-divisors $D$ and $D'$ on $X$ and $X'$ satisfying

1. $D \sim_{C, \mathbb{Q}} -K_X - \Delta$ and $D' \sim_{C, \mathbb{Q}} -K_{X'} - \Delta'$,
2. $\phi$ induces an isomorphism $(X, \Delta + D) \times_C C^o \simeq (X', \Delta' + D') \times_C C^o$, and
3. $(X_0, \Delta_0 + D_0)$ is klt and $(X'_0, \Delta'_0 + D'_0)$ is lc,

then $\phi$ extends to an isomorphism $(X, \Delta) \simeq (X', \Delta')$ over $C$.

The above proposition is well known to experts and follows from the separatedness of the moduli functor of klt log Calabi-Yau pairs (e.g. see [Oda12, Theorem 4.3], [LWX14, Theorem 5.2]). For the convenience of the reader, we prove the result.

**Proof.** Fix a birational model $W$ over $X$ and $X'$

\[
\begin{array}{ccc}
W & \leftarrow & X \\
\downarrow^p & \phi & \downarrow^q \\
X' & \rightarrow &
\end{array}
\]
such that $W$ is both a log resolution of $(X, \Delta + D)$ and $(X', \Delta' + D')$. Let $\widetilde{X}_0$ and $\widetilde{X}'_0$ denote strict transforms of $X_0$ and $X'_0$ on $W$.

If $\widetilde{X}_0 = \widetilde{X}'_0$, then $\phi$ extends to an isomorphism. Indeed, if the strict transforms are equal, then $\phi : X \dashrightarrow X'$ is an isomorphism in codimension one. Thus, $\phi$ induces an isomorphism

$$\pi_* O_X(-m(K_X + \Delta)) \simeq \pi'_* O_{X'}(-m(K_{X'} + \Delta'))$$

for all $m \in \mathbb{N}$. Since $-K_X - \Delta$ and $-K_{X'} - \Delta'$ are relatively ample over $C$, we have

$$X = \text{Proj}_C \bigoplus_{m \geq 0} \pi_* O_X(-m(K_X + \Delta)) \text{ and } X' = \text{Proj}_C \bigoplus_{m \geq 0} \pi'_* O_{X'}(-m(K_{X'} + \Delta')).$$

Hence, $(X, \Delta) \dashrightarrow (X', \Delta')$ extends to an isomorphism over $C$.

Now, we assume that the strict transforms of $X_0$ and $X'_0$ are distinct and aim for a contradiction. Write

$$p^*(K_X + \Delta + D) = a_0 \widetilde{X}'_0 + \sum_{i \in I} a_i E_i + K_W + p_*^{-1}(\Delta + D)$$

and

$$q^*(K_{X'} + \Delta' + D') = a'_0 \widetilde{X}_0 + \sum_{i \in I} a'_i E_i + K_W + q_*^{-1}(\Delta' + D'),$$

where the $\{E_i \mid i \in I\}$ is the set of prime divisors on $W$ that are both $p$- and $q$-exceptional. Set $J := \{i \in I \mid \pi(E_i) = \{0\}\}$. By assumption (2), $a_i = a'_i$ for all $i \in I \setminus J$.

Next, by inversion of adjunction and our assumption that $(X_0, \Delta_0 + D_0)$ is klt, we see $(X, \Delta + D + X_0)$ is plt in a neighborhood of $X_0$. Hence,

$$-1 < a_0(\widetilde{X}'_0, X, \Delta + D + X_0) = a(\widetilde{X}'_0, X, \Delta + D) = \text{ord}_{\widetilde{X}'_0}(X_0).$$

Since $\text{ord}_{\widetilde{X}'_0}(X_0) \geq 1$, this implies $a_0 = a(\widetilde{X}'_0, X, \Delta + D) > 0$. Again, by inversion of adjunction and our assumption that $(X'_0, \Delta' + D'_0)$ is lc, we see $(X', \Delta' + D' + X'_0)$ is lc in a neighborhood of $X'_0$. A similar argument as above implies $a'_0 \geq 0$.

Note that

$$a_0 \widetilde{X}'_0 + \sum_{i \in J} a_i E_i \sim_{C, Q} a'_0 \widetilde{X}_0 + \sum_{i \in J} a'_i E_i,$$  \hspace{1cm} (8)

since

$$p^*(K_X + \Delta + D) \sim_{C, Q} q^*(K_{X'} + \Delta' + D') \sim_{C, Q} 0,$$

$$p_*^{-1}(D + \Delta) = q_*^{-1}(D' + \Delta'),$$

and $a_i = a'_i$ for $i \in I \setminus J$. Now, (8) implies there exists $c$ such that

$$a_0 \widetilde{X}'_0 - a'_0 \widetilde{X}_0 + \sum_{i \in J} (a_i - a'_i) E_i = c q^*(X_0).$$

Comparing the coefficients of $\widetilde{X}'_0$ on the two sides implies $c > 0$. Similarly, comparing the coefficients of $\widetilde{X}_0$ gives $c \leq 0$. Hence, a contradiction is reached. \qed
Lemma 3.3. Keep the notation and setup of Theorem 3.1. If \( m \in \mathbb{Z}_{\geq 0} \) is sufficiently divisible, then there exists effective horizontal \( \mathbb{Q} \)-divisors \( B \) and \( B' \) on \( X \) and \( X' \) such that

1. \( B \sim_{\mathbb{C}, \mathbb{Q}} -K_X - \Delta \) and \( B \sim_{\mathbb{C}, \mathbb{Q}} -K_{X'} - \Delta' \),
2. \( \phi \) induces an isomorphism \( (X, \Delta + B) \times_C C^0 \simeq (X', \Delta' + B') \times_C C^0 \), and
3. \( B_0 \) and \( B'_0 \) are \( m \)-basis type with respect to \( (X_0, \Delta_0) \) and \( (X'_0, \Delta'_0) \).

Proof. Fix \( m \in \mathbb{Z}_{\geq 0} \) sufficiently divisible so that \( -m(K_X - \Delta) \) and \( -m(K_{X'} - \Delta') \) are Cartier, and \( \pi_* \mathcal{O}_X(-m(K_X + \Delta)) \) and \( \pi'_* \mathcal{O}_{X'}(-m(K_{X'} + \Delta')) \) are nonzero. Since \( H^i(X_t, \mathcal{O}_{X_t}(-m(K_{X_t} + \Delta_t))) \) and \( H^i(X'_t, \mathcal{O}_{X'_t}(-m(K_{X'_t} + \Delta'_t))) \) are zero for all \( i > 0 \) and \( t \in T \) by Kawamata-Viehweg vanishing, \( \pi_* \mathcal{O}_X(-m(K_X + \Delta)) \) and \( \pi'_* \mathcal{O}_{X'}(-m(K_{X'} + \Delta')) \) are vector bundles. Furthermore, cohomology and base change hold.

Now, the birational map \( \phi \) induces a map from local sections of \( \pi_* \mathcal{O}_X(-m(K_X + \Delta)) \) to rational sections of \( \pi'_* \mathcal{O}_{X'}(-m(K_{X'} + \Delta')) \). After twisting by \( dX'_0 \), where \( d \gg 0 \), we get a morphism

\[
\pi_* \mathcal{O}_X(-m(K_X + \Delta)) \to \pi'_* \mathcal{O}_{X'}(-m(K_{X'} + \Delta') + dX'_0)
\]

that is an isomorphism away from \( 0 \in C \). By the projection formula, \( \pi'_* \mathcal{O}_{X'}(-m(K_{X'} + \Delta') + dX'_0) \) is a vector bundle and cohomology and base change change commute.

Next, tensor the previous map by \( \mathcal{O}_{C, 0} \). This gives a morphism

\[
\varphi: \pi_* \mathcal{O}_X(-m(K_X + \Delta)) \otimes \mathcal{O}_{C, 0} \to \pi'_* \mathcal{O}_{X'}(-m(K_{X'} + \Delta') + dX'_0) \otimes \mathcal{O}_{C, 0}
\]

of locally free \( \mathcal{O}_{C, 0} \) modules that is an isomorphism after tensoring with \( K(C) \). Write \( t \) for the uniformizer of \( \mathcal{O}_{C, 0} \). Since \( \mathcal{O}_{C, 0} \) is a principal ideal domain, we may find bases \( \{s_1, \ldots, s_{N_m}\} \) and \( \{s'_1, \ldots, s'_{N_m}\} \) for the above free modules so that the transformation matrix is diagonal. Thus, we can write \( \varphi(s_i) = ts'_i \) for \( 1 \leq i \leq N_m \), where each \( p_i \in \mathbb{Z}_{\geq 0} \).

For a sufficiently small neighborhood \( 0 \in U \subset C \), we may extend each \( s_i \) to a section \( \tilde{s}_i \in \pi_* \mathcal{O}_X(-m(K_X + \Delta))(U) \) and each \( s'_i \) to a section \( \tilde{s}'_i \in \pi'_* \mathcal{O}_{X'}(-m(K_{X'} + \Delta'))(U) \). Let \( B \) and \( B' \) denote the closures of

\[
\frac{1}{mN_m} \{ \tilde{s}_1 = 0 \} + \cdots + \{ \tilde{s}_{N_m} = 0 \} \quad \text{and} \quad \frac{1}{mN_m} \{ \tilde{s}'_1 = 0 \} + \cdots + \{ \tilde{s}'_{N_m} = 0 \}.
\]

in \( X \) and \( X' \). By construction, \( B_0 \) and \( B'_0 \) are both \( m \)-basis type divisors and \( (X, B) \times_C C^0 \simeq (X', B') \times_C C^0 \).

Proof of Theorem 3.1. Since \( X_0 \) is uniformly K-stable and \( X'_0 \) is K-semistable,

\[
\delta(X_0, \Delta_0) > 1 \quad \text{and} \quad \delta(X'_0, \Delta'_0) \geq 1.
\]

Hence, way choose \( 0 < \varepsilon \ll 1 \) so that

\[
\frac{1 - \varepsilon}{\delta(X_0, \Delta_0)} + \frac{\varepsilon}{\alpha(X_0, \Delta_0)} < 1,
\]

where \( \alpha(X_0, \Delta_0) \) is Tian’s \( \alpha \)-invariant, i.e.

\[
\alpha(X_0, \Delta_0) = \inf \{ \operatorname{lct}(X_0, \Delta_0; D) : 0 \leq D \sim_{\mathbb{Q}} -K_{X_0} - \Delta_0 \}.
\]
Next, choose a positive integer $M$ so that

$$\frac{1 - \varepsilon}{\delta_m(X_0, \Delta_0)} + \frac{\varepsilon}{\alpha(X_0, \Delta_0)} < 1 \quad \text{and} \quad \delta_m(X'_0, \Delta'_0) > 1 - \varepsilon \quad (10)$$

for all positive integers $m$ divisible by $M$. Such a choice is possible by (9), the inequality $\delta(X'_0, \Delta'_0) \geq 1$, and the fact that $\delta$ is a limit.

Now, fix $m$ sufficiently divisible so that: $M$ divides $m$, the conclusion of Lemma 3.3 holds for $m$, and $-m(K'_X + \Delta')$ is relatively base point free over $C$. Hence, after shrinking $C$ in a neighborhood of 0, we may find divisors $B$ and $B'$ on $X$ and $X'$, respectively, satisfying the conclusion of Lemma 3.3. Since $B_0$ and $B'_0$ are $m$-basis type,

$$\lct(X_0, \Delta_0; B_0) \geq \delta_m(X_0, \Delta_0) \quad \text{and} \quad \lct(X'_0, \Delta'_0; B'_0) \geq \delta_m(X'_0, \Delta'_0) > 1 - \varepsilon.$$

Thus, $(X'_0, \Delta'_0 + (1 - \varepsilon)B'_0)$ is lc.

Since $-m(K'_X + \Delta')$ is assumed to be relatively base point free over $C$, after shrinking $C$ in a neighborhood of 0, we may apply [KM98, Lemma 5.17] to find an effective divisor $G' \in |-m(K'_X + \Delta')|$ in general position so that $(X'_0, \Delta'_0 + (1 - \varepsilon)B'_0 + (\varepsilon/m)G'_0)$ remains lc. Write $G$ for the $\mathbb{Q}$-divisor on $X$ such that $\phi$ induces an isomorphism $G \times_C C^\circ \simeq G' \times_C C^\circ$ and $X_0 \not\subset \text{Supp}(G)$. Note that $G \sim_{C, \mathbb{Q}} -m(K_X + \Delta)$, since the statement holds over $C^\circ$. Thus, $\lct(X_0, \Delta_0; (1/m)G_0) \geq \alpha(X_0, \Delta_0)$.

Now, consider the divisors

$$D := (1 - \varepsilon)B + \frac{\varepsilon}{m}G \quad \text{and} \quad D' := (1 - \varepsilon)B' + \frac{\varepsilon}{m}G'.$$

Note that $D \sim_{C, \mathbb{Q}} -K_X$, $D' \sim_{C, \mathbb{Q}} -K_{X'}$, and $(X, D) \times_C C^\circ \simeq (X', D') \times_C C^\circ$. As mentioned above, $(X_0, \Delta'_0 + D'_0)$ is lc. Additionally, the pair $(X_0, \Delta_0 + D_0)$ is klt. Indeed, since

$$1/\lct(D + F) \leq 1/\lct(D) + 1/\lct(F)$$

for any two effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors $D$ and $F$ on a klt pair, we know

$$\frac{1}{\lct(X_0, \Delta_0; D_0)} \leq \frac{1}{\lct(X_0, \Delta_0; (1 - \varepsilon)B_0)} + \frac{1}{\lct(X_0, \Delta_0; (\varepsilon/m)G_0)},$$

which is $< 1$ by (10). Since the $\mathbb{Q}$-divisors $D$ and $D'$ satisfy the hypotheses of Proposition 3.2, we may apply the proposition to see that $\phi$ extends to an isomorphism. $\square$

**Remark 3.4.** If $(X_0, \Delta_0)$ and $(X'_0, \Delta'_0)$ are only assumed to be K-semistable, then they are not necessarily isomorphic (but are $S$-equivalent by Theorem 1.1). Therefore, we do not expect the the above strategy to be useful in this more general case.

Recall, if $(X, \Delta)$ is a log Fano pair, then $\text{Aut}(X, \Delta)$ is the closed subgroup of $\text{Aut}(X)$ defined by

$$\text{Aut}(X, \Delta) := \{ \sigma \in \text{Aut}(X) \mid \sigma^* \Delta = \Delta \}.$$
The following result is an immediate corollary of Theorem 3.1 and a special case of Corollary 1.3.

**Corollary 3.5.** Let \((X, \Delta)\) be a log Fano pair. If \((X, \Delta)\) is uniformly K-stable, then \(\text{Aut}(X, \Delta)\) is finite.

**Proof.** Since \(\text{Aut}(X, \Delta)\) is a linear algebraic group, it is affine. To conclude that \(\text{Aut}(X, \Delta)\) is a finite group, it suffices to show that it is proper. To see the properness, consider a map \(f : C^o \to \text{Aut}(X, \Delta)\), where \(0 \in C\) is a smooth pointed curve and \(C^o = C \setminus \{0\}\). The map \(f\) induces an isomorphism \((X \times C, \Delta \times C) \times_C C^o \to (X \times C, \Delta \times C) \times_C C^o\) over \(C^o\). By applying Theorem 3.1 to the above isomorphism, we see \(f\) extends to a map \(f : C \to \text{Aut}(X, \Delta)\). Hence, \(\text{Aut}(X, \Delta)\) is proper, and the proof is complete. \(\square\)

In [BHJ16, Corollary E], it is shown that the polarized automorphism group of a uniformly K-stable polarized manifold \((X, L)\) is finite. Their proof uses analytic tools.

**Remark 3.6.** Our proofs of Theorem 3.1 and Corollary 3.5 extend to the case of polarized klt pairs \((X, \Delta; L)\) (that is, \((X, \Delta)\) is a projective klt pair and \(L\) an ample \(\mathbb{Q}\)-Cartier divisor on \(X\)) such that \(K_X + \Delta + L\) is nef and \(\delta(X, \Delta; L) > 1\).

### 3.2. Moduli spaces.

**Proof of Corollary 1.4.** As previously mentioned, the result relies on [Jia17, BL18] and Theorem 3.1. Indeed, [Jia17] (see also [Che18] or [LLX18, 6.14]) yields that the set of varieties \(\mathcal{M}_{n,V}^{uKs}(k)\) is bounded. Hence, there exists a positive integer \(M\), so that \(-MK_X\) is a very ample Cartier divisor for all \(X \in \mathcal{M}_{n,V}^{uKs}(k)\). Furthermore, the set of Hilbert functions \(m \mapsto \chi(\omega_X^{[-m]})\) with \(X \in \mathcal{M}_{n,V}^{uKs}(k)\) is finite.

For such a Hilbert function \(h\), consider the subfunctor \(\mathcal{M}_{h}^{uKs} \subset \mathcal{M}_{n,V}^{uKs}\) parameterizing uniformly K-semistable \(\mathbb{Q}\)-Fano varieties with Hilbert function \(h\). Note that \(\mathcal{M}_{n,V}^{uKs} = \coprod_h \mathcal{M}_{h}^{uKs}\). Set \(N := h(M) - 1\) and let \(\text{Hilb}(\mathbb{P}^N)\) be the Hilbert scheme parameterizing closed subschemes of \(\mathbb{P}^N\) with Hilbert polynomial \(h(M \cdot -)\). Write \(X \to \text{Hilb}(\mathbb{P}^N)\) for the corresponding universal family.

Now, let \(U \subset \text{Hilb}(\mathbb{P}^N)\) denote the open subscheme parameterizing normal, Cohen Macaulay varieties. By [HK04, 3.11], there is a locally closed subscheme \(V \subset U\) such that a map \(T \to U\) factors through \(V\) if and only if there is an isomorphism \(\omega_{X_T/T}^{[-M]} \cong \mathcal{L}_T \otimes \mathcal{O}_{X_T}(1)\), where \(\mathcal{L}_T\) is the pullback of a line bundle from \(T\). Now, we may apply [BL18] to the normalization of \(V\) to see that the set

\[V' := \{ t \in V \mid X_T \text{ is a uniformly K-stable \(\mathbb{Q}\)-Fano variety}\}\]

is open \(V\). Finally, we apply [Kol09, 25] or [AH11] to find a locally closed decomposition \(W \to V'\) such that a morphism \(T \to V'\) factors through \(W\) if and only if \(X_T \to X\) satisfies Kollár’s condition.

As a consequence of the above discussion, \(\mathcal{M}_h^{uKs} = [W/\text{PGL}(N + 1)]\). Theorem 3.1 implies \(\mathcal{M}_h^{uKs}\) is a separated Deligne-Mumford stack. Furthermore, we may apply
[KM97] to see $\mathcal{M}_{\text{hKs}}$ has a coarse moduli space $M_{\text{hKs}}$, which is a separated algebraic space. □

4. Places computing the $\delta$-invariant

In this section, we will study the cases when valuations, ideals, and $\mathbb{Q}$-divisors compute the $\delta$-invariant. The results proved here are related to Conjecture 1.5 and will be used in the proof of Theorem 1.1.

4.1. Divisors computing $\delta = 1$.

Theorem 4.1. Let $(X, \Delta)$ be a $K$-semistable log Fano pair and $E$ a divisor over $X$. If

$$1 = \delta(X, \Delta) = \frac{A_{X, \Delta}(E)}{S(E)},$$

then $E$ is dreamy and induces a non-trivial special test configuration $(\mathcal{X}, \mathcal{D})$ such that $\text{Fut}(\mathcal{X}, \mathcal{D}) = 0$. In particular, $(X, \Delta)$ is not $K$-stable.

The proof follows an argument in [LWX18b, Section 3.1]. The argument will be used again in the proof of Lemma 5.11 in a relative setting.

Proof. The argument use the construction in Section 2.5. Fix a positive integer $r$ so that $L := -r(K_X + \Delta)$ is a Cartier divisor and set $R = R(X, L)$. Consider the cone $(Y, \Gamma) := C(X, \Delta; L)$ with vertex $x$, and let $v_0 := \text{ord}_{X, x}$ denote the canonical valuation over the cone and $E_\infty$ the divisor over $Y$ defined via pulling back $E$. We consider the ray of valuations

$$\{v_t \mid t \in [0, \infty)\} \subset \text{Val}_{X, x},$$

where $v_t$ is the quasi-monomial valuation with weights $(1, t)$ along

$$v_0 = \text{ord}_{X, \infty} \quad \text{and} \quad v_\infty = \text{ord}_{E_\infty}.$$

Recall that, for $k \in \mathbb{Z}_{>0}$, $v_k = \frac{1}{k} \text{ord}_{E_k}$ where $E_k$ is a divisor over $X$. By (7), we have

$$\left. \frac{d \hat{\text{vol}}(v_t)}{dt} \right|_{t=0^+} = (n + 1)\beta_{X, \Delta}(E).$$

Since $A_{X, \Delta}(E) - S(E) = 0$, we know $\beta(E) = 0$. Defining $f(t) := \hat{\text{vol}}(v_t)$, a Taylor expansion gives

$$f(t) = f(0) + O(t^2) \quad \text{for} \quad 0 \leq t \ll 1.$$

For a fixed positive integer $k$, set

$$a_k := a(\text{ord}_{E_k}) \quad \text{and} \quad c_k := \text{lct}(Y, \Gamma; a_k).$$

Note that $c_k \leq A_{Y, \Gamma}(E_k)$ by (3). This implies

$$f(0) \leq c_k^n \cdot \text{mult}(a_k) \leq f \left( \frac{1}{k} \right) = A_{Y, \Gamma}(E_k)^n \cdot \text{mult}(a_k),$$
where the first inequality follows from [Liu18, 7] and the assumption that \((X, \Delta)\) is K-semistable. Therefore,

\[
\left( \frac{f(0)}{f(1/k)} \right)^{1/n} \leq \frac{c_k}{A_{Y, \Gamma}(E_k)} \leq 1.
\]

Since \((1 + O(1/k^2))^{1/n}\) is also of the order \(1 + O(1/k^2)\), we see

\[
1 - O\left( \frac{1}{k^2} \right) = \frac{c_k}{A_{Y, \Gamma}(E_k)} \leq 1.
\]

We also know \(A_{Y, \Gamma}(E_k) = k A_{Y, \Gamma}(v_1/k) = k A_{Y, \Gamma}(v_0) + A_{Y, \Gamma}(E_\infty)\). Hence,

\[
\lim_{k \to \infty} \left( A_{Y, \Gamma}(E_k) - c_k \right) = \lim_{k \to \infty} \left( A_{Y, \Gamma}(E_k) \left( 1 - \frac{c_k}{A_{Y, \Gamma}(E_k)} \right) \right) = 0.
\]

By the previous limit, we may apply Proposition 2.2 to extract \(E_k\) for \(k \gg 0\). Specifically, there exists a proper birational morphism \(\mu_k : Y_k \to Y\) such that \(E_k \subset Y\) and \(-E_k\) is ample over \(Y\). Hence, \(\bigoplus_{p \in \mathbb{N}} \mu_k^* (O_{Y_k}(-pE_k))\) is finitely generated. Since \(\mu_k^* (O_{Y_k}(-pE_k)) = a_p(kv_1/k)\), we see

\[
\bigoplus_{p \in \mathbb{N}} \mu_k^* (O_{Y_k}(-pE_k)) = \bigoplus_{p \in \mathbb{N}} \left( \bigoplus_{m \in \mathbb{N}} F_E^{p-mk} R_m \right).
\]

Therefore, \(\text{Rees}(F_E)\) is finitely generated as well and \(E\) is dreamy.

An immediate corollary to Theorem 4.1 is the following strengthening of [Fuj16b, 1.6] and [Li18, 3.7], which was expected as in the arXiv version of [Li18].

**Corollary 4.3.** A log Fano pair \((X, \Delta)\) is K-stable if and only if \(\beta_{X, \Delta}(E) > 0\) for any divisor \(E\) over \(X\).

**Proof.** Theorem 4.1 implies the forward implication. The reverse implication was shown in [Fuj16b, 1.6] and [Li18, 3.7].
4.2. **Ideals computing** $\delta$. Let $(X, \Delta)$ be a log Fano pair and $a \subsetneq \mathcal{O}_X$ a nonzero ideal. Write $\pi : Y \to X$ for the normalized blowup of $X$ along $a$ and $E$ for the effective Cartier divisor on $Y$ such that $a \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$. We set

$$S(a) := \frac{1}{\text{vol}(-K_X - \Delta)} \int_0^{+\infty} \text{vol}(\pi^*(-K_X - \Delta) - tE) \, dt. \quad (11)$$

**Proposition 4.4.** If $(X, \Delta)$ is a log Fano pair and $a \subsetneq \mathcal{O}_X$ a nonzero ideal, then

$$\lct(X, \Delta; a) \geq \delta(X, \Delta).$$

Furthermore, write $\pi : Y \to X$ for the normalized blowup of $a$ and $E$ for the Cartier divisor on $Y$ such that $a \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$. If (12) is an equality, then $\text{Supp}(E)$ is a prime divisor and computes $\delta(X, \Delta)$.

The above proposition is an analog of [LX16, Theorem 3.11] for the $\delta$-invariant and is similar to [Fuj17a, Corollary 3.22].

**Proof.** Choose a divisor $F$ over $X$ computing $\lct(X, \Delta; a)$. By [BCHM10], there is an extraction $\rho : X_F \to X$ of $F$. Set $p := \text{ord}_F(a)$. Hence, $A_{X, \Delta}(F)/p = \lct(X, \Delta; a)$ and $a^k \cdot \mathcal{O}_{X_F} \subseteq \mathcal{O}_{X_F}(-kpF)$ for all $k \in \mathbb{N}$.

By the previous inclusion, if we set $L := -K_X - \Delta$, then

$$\text{vol}(\pi^*L - tE) \leq \text{vol}(\rho^*L - tpF)$$

for all $t \in \mathbb{R}_{\geq 0}$. Hence, $S(a) \leq p^{-1}S(F)$, and we see

$$\frac{\lct(a)}{S(a)} \geq \frac{A_{X, \Delta}(F)}{S(F)}.$$ 

Since $A_{X, \Delta}(F)/S(F) \geq \delta(X, \Delta)$, (12) holds.

Now assume (12) is an equality. In this case, the above argument implies $F$ computes $\delta(X, \Delta)$. To finish the proof, it suffices to show $Y = X_F$ and $\text{Supp}(E) = F$.

Fix a positive integer $k$ so that $-kpF$ is Cartier and choose an ideal $c \subset \mathcal{O}_{X_F}$ such that

$$a^k \cdot \mathcal{O}_{X_F} = c \cdot \mathcal{O}_{X_F}(-kpF).$$

Write $\tau : Z \to X_F$ for the normalized blowup of $X_F$ along $c$ and $G$ for the Cartier divisor on $Z$ such that $c \cdot \mathcal{O}_Z = \mathcal{O}_Z(-G)$. Since $Z$ is normal and

$$a^k \cdot \mathcal{O}_Z = (c \cdot \mathcal{O}_{X_F}(-kpF)) \cdot \mathcal{O}_Z = \mathcal{O}_Z(-pk\tau^*(F) - G)$$

is locally free, $\rho \circ \tau$ factors through $\pi$.
Additionally, $\sigma^*(E) = \tau^*(pF) + k^{-1}G$.

If we can show $c = O_{X_F}$, the proof will be complete. Indeed, if $c = O_{X_F}$, then $\tau$ is an isomorphism and $\sigma^*E = pF$. But, since $\sigma^*E = pF$ is anti-ample over $X$, $\sigma$ must also be an isomorphism and we are done.

We claim that if $c \not\subseteq O_{X_F}$, then
\[
\text{vol}(\pi^*L - tE) < \text{vol}(\rho^*L - tpF)
\]
for $0 < t \ll 1$ and, thus, $S(a) < (1/p)S(F)$. Since, we will then have
\[
\delta(X, \Delta) \leq \frac{A_{X,\Delta}(F)}{S(F)} < \frac{lct(a)}{S(a)} = \delta(X, \Delta),
\]
a contradiction will be reached.

To prove the above claim, fix $0 < \varepsilon \ll 1/k$ so that
\[
H := p\tau^*F + \varepsilon G
\]
is anti-ample over $X$. Note that by our choice of $\varepsilon$, we also have
\[
\text{vol}(\pi^*L - tE) = \text{vol}(\tau^*(\rho^*L) - t\sigma^*E)) \leq \text{vol}(\tau^*(\rho^*L) - tH).
\]
Therefore, it suffices to show
\[
\text{vol}(\tau^*(\rho^*L) - tH) < \text{vol}(\rho^*L - tpF)
\]
for $0 < t \ll 1$.

Fix $0 < t \ll 1$ so that both $A_t := \rho^*L - tpF$ and $B_t := \tau^*(\rho^*L) - tH$ are both ample. Following [Fuj17a], note that for $0 \leq i \leq n - 1$,
\[
0 \leq \varepsilon tG \cdot (\sigma^*A_t)^i \cdot B_t^{n-i-1}
\]
\[
= (\tau^*A_t - B_t) \cdot (\tau^*A_t)^i \cdot B_t^{n-i-1},
\]
since $G$ is effective, $\tau^*A_t$ is nef, and $B_t$ is ample. Additionally,
\[
0 < (\tau^*A_t - B_t) \cdot B_t^{n-1}.
\]
We now see
\[
0 < \sum_{i=0}^{n-1} ((\tau^*A_t - B_t) \cdot (\sigma^*A_t)^i \cdot B_t^{n-i-1})
\]
\[
= (\tau^*A_t)^n - (B_t)^n
\]
\[
= \text{vol}(\rho^*L - tpF) - \text{vol}(\sigma^*(\rho^*L) - tH),
\]
and conclude $\text{vol}(\rho^*L - tpF) < \text{vol}(\sigma^*(\rho^*L) - tH)$ for $0 < t \ll 1$. \qed
4.3. \(\mathbb{Q}\)-divisors computing \(\delta\). Let \((X, \Delta)\) be a log Fano pair, \(\mu : Y \rightarrow X\) a proper birational morphism with \(Y\) normal, and \(E\) an effective \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \(Y\) such that \(-E\) is \(\mu\)-ample. We set

\[ a_m(E) := \mu_* \mathcal{O}_Y(-mE) \subseteq \mathcal{O}_X \]

and

\[ S(E) := \frac{1}{(-K_X - \Delta)^m} \int_0^{\infty} \text{vol}(\mu^*(-K_X - \Delta) - tE) \, dt. \]

Proposition 4.5. With the above notation, we have

\[ \frac{lct(X, \Delta; a_\bullet(E))}{S(E)} \geq \delta(X, \Delta). \] (13)

Furthermore, if (13) is an equality, then \(\text{Supp}(E)\) is a prime divisor.

The statement is a consequence Proposition 4.4 and the following elementary lemma.

Lemma 4.6. Let \(\mu : Y \rightarrow X\) be a proper birational morphism of normal varieties and \(E\) an effective \(\mathbb{Q}\)-Cartier \(\mathbb{Q}\)-divisor on \(Y\) such that \(-E\) is \(\mu\)-ample. Set

\[ a_p(E) := \mu_* \mathcal{O}_Y([-pE]) \subseteq \mathcal{O}_X. \]

If \(p \in \mathbb{Z}_{>0}\) is sufficiently divisible, then the following hold:

1. \(Y\) is the blowup of \(X\) along \(a_p(E)\);
2. \(a_p(E) \cdot \mathcal{O}_Y = \mathcal{O}_Y(-pE)\);
3. \((a_p(E))^\ell = a_{p\ell}(E)\) for all \(\ell \in \mathbb{Z}_{>0}\).

Proof. Since \(-E\) is ample over \(X\), \(\bigoplus_{m \in \mathbb{N}} a_m(E)\) is a finitely generated \(\mathcal{O}_X\)-algebra and

\[ Y \cong \text{Proj}_X \left( \bigoplus_{m \in \mathbb{N}} a_m(E) \right). \]

The former statement implies that if \(p \in \mathbb{Z}_{>0}\) is sufficiently divisible, then the \(p\)-th Veronese, \(\bigoplus_{m \in \mathbb{N}} a_{pm}(E)\), is finitely generated in degree 1. Hence, (1) and (3) are complete. For (2), observe that the natural map \(\mu^* \mu_* \mathcal{O}_X(-pE) \rightarrow \mathcal{O}_Y(-pE)\) is surjective for \(p \in \mathbb{Z}_{>0}\) sufficiently divisible, since \(-E\) is \(\mu\)-ample. \(\square\)

Proof of Proposition 4.5. Fix \(p \in \mathbb{Z}_{>0}\) satisfying (1)-(3) of Lemma 4.6 and set \(a := a_p(E)\). By (1) and (2), \(p \cdot S(a) = S(E)\). By (3)

\[ lct(X, \Delta; a_\bullet(E)) := \lim_{m \rightarrow \infty} (mp \cdot lct(X, \Delta; a_{pm}(E))) = p \cdot lct(X, \Delta; a). \]

The result now follows immediately from Proposition 4.4. \(\square\)
5. Constructing the S-equivalence

In this section, we prove Theorem 1.1. In Section 5.1 we will construct filtrations of
\[ R = \bigoplus_{m \in \mathbb{N}} H^0(X_0, -mr(K_{X_0} + \Delta_0)) \quad \text{and} \quad R' = \bigoplus_{m \in \mathbb{N}} H^0(X'_0, -mr(K_{X'_0} + \Delta'_0)), \]
whose associated graded rings are isomorphic. Then in Section 5.2, we concentrate on proving that these filtrations and their associated graded rings are finitely generated.

5.1. Filtrations induced by degenerations. Let \( \pi : (X, \Delta) \to C \) and \( \pi' : (X', \Delta') \to C \) be \( \mathbb{Q} \)-Gorenstein families of log Fano pairs over a smooth pointed curve \( 0 \in C \). Assume there exists an isomorphism \( \phi : (X, \Delta) \times_C C^0 \to (X', \Delta') \times_C C^0 \)
over \( C^0 : C \setminus \{0\} \) that does not extend to an isomorphism \( (X, \Delta) \simeq (X', \Delta') \) over \( C \).

After shrinking \( 0 \in C \), we may assume \( C \) is affine and there exists \( t \in \mathcal{O}(C) \) so that \( \text{div}_C(t) = \{0\} \).

From this setup, we will construct filtrations on the section rings of the special fibers. Set
\[
L := -r(K_X + \Delta) \quad \text{and} \quad L' := -r(K_{X'} + \Delta'),
\]
where \( r \) is a positive integer so that \( L \) and \( L' \) are Cartier. To minimize bulky notation, for each non-negative integer \( m \), set
\[
R_m := H^0(X, \mathcal{O}_X(mL)) \quad \text{and} \quad R'_m := H^0(X', \mathcal{O}_X(mL')).
\]
Additionally, set
\[
\mathcal{R} := \oplus_m R_m, \quad R := \oplus_m R_m, \quad \mathcal{R}' := \oplus_m R'_m, \quad \text{and} \quad R' := \oplus_m R'_m.
\]

Observe that the natural maps
\[
\mathcal{R} \otimes k(0) \to R \quad \text{and} \quad \mathcal{R}' \otimes k(0) \to R'
\]
are isomorphisms. Indeed, Kawamata-Viehweg applied to the fibers of \( \pi \) and \( \pi' \) implies \( R^i\pi_*\mathcal{O}_X(mL) \) and \( R^i\pi'_*\mathcal{O}_X(mL) \) vanish for all \( i > 0 \) and \( m \geq 0 \). Hence, \( \pi_*\mathcal{O}_X(mL) \) and \( \pi'_*\mathcal{O}_X(mL) \) are vector bundles and their cohomology commutes with base change.

Since \( C \) is affine, \( \mathcal{R}_m \) and \( \mathcal{R}'_m \) can be identified with the \( \mathcal{O}_C \)-module \( \pi_*\mathcal{O}_X(mL) \) and \( \pi'_*\mathcal{O}_X(mL') \), and the statement follows.

Fix a common log resolution \( \hat{X} \) of \( (X, \Delta) \) and \( (X', \Delta') \)

\[
\begin{array}{c}
\hat{X} \\
\downarrow \psi \\
X \quad \xrightarrow{\phi'} \quad X'
\end{array}
\]
and write $\tilde{X}_0$ and $\tilde{X}_0'$ for the birational transforms of $X_0$ and $X_0'$ on $\hat{X}$. Set
\[ a := A_{X, \Delta + X_0}(\tilde{X}_0') \quad \text{and} \quad a' := A_{X', \Delta + X_0}(\tilde{X}_0). \]
Equation (14)

Observe that $\tilde{X}_0 \neq \tilde{X}_0'$, since otherwise $\phi$ would extend to an isomorphism over $C$ by the second paragraph of the proof of Proposition 3.2.

5.1.1. Definition of filtrations. For each $p \in \mathbb{Z}$ and $m \in \mathbb{N}$, set
\[ \mathcal{F}^p R_m := \{ s \in R_m | \text{ord}_{\tilde{X}_0}(s) \geq p \}, \]
and
\[ \mathcal{F}^p R'_m := \{ s \in R_m | \text{ord}_{\tilde{X}_0}(s) \geq p \}. \]
We define $\mathbb{N}$-filtrations of $R$ and $R'$ by setting
\[ \mathcal{F}^p R_m := \text{im}(\mathcal{F}^p R_m \to R_m) \]
and
\[ \mathcal{F}^p R'_m := \text{im}(\mathcal{F}^p R'_m \to R'_m), \]
where the previous maps are given by restriction of sections. It is straightforward to check that $\mathcal{F}$ and $\mathcal{F}'$ are filtrations of $R$ and $R'$.

Note that a section $s \in R_m$ lies in $\mathcal{F}^p R_m$ if and only if there exists an extension $\tilde{s} \in R_m$ of $s$ such that $\tilde{s} \in \mathcal{F}^p R_m$. The analogous statement holds for $\mathcal{F}'$.

5.1.2. Relating the filtrations. We aim to show that $\text{gr}_{\mathcal{F}} R$ and $\text{gr}_{\mathcal{F}'} R$ are isomorphic as graded rings.

Since $p^*(X_0) = q^*(X_0')$ have multiplicity one along $\tilde{X}_0$ and $\tilde{X}_0'$, we may write
\[ K_{\tilde{X}} + \psi_*^{-1}(\Delta) = \psi^*(K_X + \Delta) + a\tilde{X}_0 + F \]
and
\[ K_{\tilde{X}} + \psi'_{-1}(\Delta') = \psi'^*(K_{X'} + \Delta') + a'\tilde{X}_0 + F', \]
where the components of $\text{Supp}(F) \cup \text{Supp}(F')$ are both $\psi$- and $\psi'$-exceptional. Now,
\[ \mathcal{F}^p R_m \simeq H^0\left( \tilde{X}, \mathcal{O}_{\tilde{X}}(m\psi^* L - p\tilde{X}_0) \right) \]
\[ = H^0\left( \tilde{X}, \mathcal{O}_{\tilde{X}}(m\psi'^* L' + (mra - p)\tilde{X}_0' - mra'\tilde{X}_0 + mr(F - F')) \right). \]
Hence, for $s \in \mathcal{F}^p R_m$, multiplying $\psi^* s$ by $t^{mra-p}$ gives an element of
\[ H^0\left( \tilde{X}, \mathcal{O}_{\tilde{X}}(m\psi'^* L' - (m(a + a') - p)\tilde{X}_0) \right), \]
which can be identified with $\mathcal{F}^{m(a+a')-p} R'_m$.

As described above, for each $p \in \mathbb{Z}$ and $m \in \mathbb{N}$, there is a map
\[ \varphi_{p,m} : \mathcal{F}^p R_m \to \mathcal{F}^{m(a+a')-p} R'_m. \]
which, when \( \mathcal{R}_m \) and \( \mathcal{R}'_m \) are viewed as submodules of \( K(X) \) and \( K(X') \), sends \( s \in \mathcal{F}^p\mathcal{R}_m \) to \( t^{mr-a-p}(\phi^{-1})^*(s) \). Similarly, there is a map

\[
\varphi'_{p,m} : \mathcal{F}^p\mathcal{R}_m \longrightarrow \mathcal{F}^{mr(a+d')-p}\mathcal{R}_m,
\]

which sends \( s' \in \mathcal{F}^p\mathcal{R}'_m \) to \( t^{mr-a-p}\phi^*(s') \).

**Lemma 5.1.** The map \( \varphi_{p,m} \) is an isomorphism. Furthermore, given \( s \in \mathcal{F}^p\mathcal{R}_m \)

1. \( s \) vanishes on \( X_0 \) if and only if \( \varphi_{p,m}(s) \in \mathcal{F}^{mr(a+a')-p+1}\mathcal{R}'_m \), and
2. \( \varphi_{p,m}(s) \) vanishes on \( X'_0 \) if and only if \( s \in \mathcal{F}^{p+1}\mathcal{R}_m \).

**Proof.** The map \( \varphi'_{mr(a+a')-p,m} \) is the inverse to \( \varphi_{p,m} \), since \( \varphi_{mr(a+a')-p,m} \circ \varphi_{p,m} \) is multiplication by \( t^{mr-(mr(a+a')-p)}t^{m-a}=1 \). Hence, \( \varphi_{p,m} \) is an isomorphism.

For (1), fix \( s \in \mathcal{F}^p\mathcal{R}_m \) and note that \( s \) vanishes on \( X_0 \) if and only if

\[
\psi^*s \in H^0\left( \tilde{X}, \mathcal{O}_{\tilde{X}}(m\psi^*L - p\tilde{X}_0 - \tilde{X}_0) \right).
\]

The latter holds precisely when

\[
t^{mr-a-b}\psi^*s \in H^0\left( \tilde{X}, \mathcal{O}_{\tilde{X}}(m\psi^*L' - (mr(a + a') - p + 1)\tilde{X}_0) \right),
\]

which is identified with \( \mathcal{F}^{mr(a+a')-p+1}\mathcal{R}'_m \). Statement (2) follows from a similar argument. \( \square \)

**Proposition 5.2.** The maps \( \langle \varphi_{p,m} \rangle \) induce an isomorphism of graded rings

\[
\bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{Z}} \text{gr}_F^p\mathcal{R}_m \xrightarrow{\varphi} \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{Z}} \text{gr}_F^p\mathcal{R}'_m,
\]

that sends the degree \((m, p)\)-summand on the left to the degree \((mr(a + a') - p)\)-summand on the right. Hence, \( \text{gr}_F^p\mathcal{R}_m \) and \( \text{gr}_F^p\mathcal{R}'_m \) vanish for \( p > mr(a + a') \).

**Proof.** Consider the map

\[
\mathcal{F}^p\mathcal{R}_m \longrightarrow \text{gr}_F^{mr(a+a')-p}\mathcal{R}'_m, \tag{15}
\]

defined as follows. Given an element of \( s \in \mathcal{F}^p\mathcal{R}_m \), choose \( \tilde{s} \in \mathcal{F}^p\mathcal{R}_m \) such that \( \tilde{s} \) is an extension of \( s \). Now, send \( s \) to the image of \( \varphi_{p,m}(\tilde{s}) \) under the composition of maps

\[
\mathcal{F}^{mr(a+a')-p}\mathcal{R}'_m \rightarrow \mathcal{F}^{mr(a+a')-p}\mathcal{R}'_m \rightarrow \text{gr}_F^{mr(a+a')-p}\mathcal{R}'_m.
\]

To see that this map is well defined, let \( \hat{s} \in \mathcal{F}^p\mathcal{R}_m \) be a different extension of \( s \). Since \( \hat{s} - \hat{s} \) vanishes on \( X_0 \), Lemma 5.1.1 implies

\[
\varphi_{p,m}(\hat{s}) - \varphi_{p,m}(\tilde{s}) = \varphi_{p,m}(\hat{s} - \hat{s}) \in \mathcal{F}^{mr(a+a')-p+1}\mathcal{R}'_m.
\]

Therefore, the image of \( \varphi_{p,m}(\hat{s}) - \varphi_{p,m}(\tilde{s}) \) in \( \text{gr}_F^{mr(a+a')-p}\mathcal{R}'_m \) is zero.

Using Lemma 5.1, we see that (15) is surjective and has its kernel equal to \( \mathcal{F}^{p+1}\mathcal{R}_m \). Indeed, the surjectivity follows from the fact that \( \varphi_{p,m} \) is an isomorphism. The description of the kernel is a consequence of Lemma 5.1.2. Therefore,

\[
\text{gr}_F^p\mathcal{R}_m \rightarrow \text{gr}_F^{mr(a+a')-p}\mathcal{R}'_m
\]
is an isomorphism. The previous isomorphism induces an isomorphism of graded rings, since \( \varphi_{p_1,m_1}(s_1) \varphi_{p_2,m_2}(s_2) = \varphi_{p_1+p_2,m_1+m_2}(s_1 s_2) \) for \( s_1 \in \mathcal{F}^{p_1} R_{m_1} \) and \( s_2 \in \mathcal{F}^{p_2} R_{m_2} \).

To see the vanishing statement, observe that \( \text{gr}_p^R R_m \) and \( \text{gr}_p^R R_m' \) vanish for \( p < 0 \). Hence, the isomorphism of graded rings yields the vanishing for \( p > mr(a + a') \). \( \square \)

5.1.3. Properties of the filtrations.

**Proposition 5.3.** We have \((a - S(\mathcal{F})) + (a' - S(\mathcal{F}')) = 0\).

**Proof.** Applying Proposition 5.2, we see

\[
\dim \mathcal{F}^p R_m = \sum_{j=p}^{mr(a+a')} \text{gr}_j^\mathcal{F} R_m = \sum_{j=0}^{mr(a+a')-p} \text{gr}_j^\mathcal{F} R'_m
\]

for \( p \in \{0, \ldots, mr(a + a') + 1\} \). Therefore,

\[
\sum_{p=0}^{mr(a+a')} \dim \mathcal{F}^p R_m + \sum_{p=0}^{mr(a+a')} \dim \mathcal{F}^p R'_m = mr(a + a') \dim R_m.
\]

The previous equation and (5) implies \( S(\mathcal{F}) + S(\mathcal{F}') = a + a' \). \( \square \)

**Remark 5.4.** It also natural to rescale the above values and set

\[
\beta := (-K_{X_0} - \Delta_0)^n(a - S(\mathcal{F})) \quad \text{and} \quad \beta' := (-K_{X_0}' - \Delta_0')^n(a' - S(\mathcal{F}')).
\]

In this language, Proposition 5.3 says

\[
\beta + \beta' = 0.
\]

**Lemma 5.5.** For all \( p \geq 0 \), we have

1. \( b_p(\mathcal{F}) = a_p(\text{ord}_{\tilde{X}_0}) \cdot O_{X_0} \) and
2. \( b_p(\mathcal{F}') = a_p(\text{ord}_{\tilde{X}_0}) \cdot O_{X_0}' \).

**Proof.** Recall that

\[
b_p(\mathcal{F}) := \text{im}(\mathcal{F}^p R_m \otimes O_{X_0}(-mL_0) \rightarrow O_{X_0})
\]

for \( m \gg 0 \). Since \( \mathcal{F}^p R_m := \text{im}(\mathcal{F}^p R_m \rightarrow R_m) \), we see

\[
b_p(\mathcal{F}) = \text{im}(\mathcal{F}^p R_m \otimes O_X(-mL) \rightarrow O_X) \cdot O_{X_0}.
\]

Therefore, (1) reduces to showing that

\[
a_p(\text{ord}_{\tilde{X}_0}) = \text{im}(\mathcal{F}^p R_m \otimes O_X(-mL) \rightarrow O_X)
\]

for \( m \gg 0 \). Since \( \mathcal{F}^p R_m = H^0(X, O_X(mL) \otimes a_p(\text{ord}_{\tilde{X}_0})) \) and \( L \) is \( \pi \)-ample, the latter statement holds. (2) follows from the same argument. \( \square \)

**Proposition 5.6.** We have

1. \( a \geq \text{lct}(X, \Delta + X_0; a_p(\text{ord}_{\tilde{X}_0})) = \text{lct}(X_0, \Delta_0; b_\bullet(\mathcal{F})) \), and
(2) \( a' \geq \lct(X', \Delta' + X'_0; \mathbf{a}_\bullet(\ord_{\tilde{X}'_0})) = \lct(X'_0, \Delta'_0; \mathbf{b}_\bullet(\mathcal{F}')) \).

**Proof.** The first pair of inequalities is Lemma (3). The second pair follow immediately from Lemma 5.5 and inversion of adjunction. \( \square \)

### 5.2. Proof of Theorem 1.1

The goal of this subsection is to prove Theorem 1.1. To do so, we consider the filtrations defined in Section 5.1. Under the hypothesis that \((X_0, \Delta_0)\) and \((X'_0, \Delta'_0)\) are K-semistable, will show that the filtrations are induced by dreamy divisors.

Furthermore, we will prove that these dreamy divisors induce special test configurations \((\mathcal{X}, \mathcal{D})\) and \((\mathcal{X}', \mathcal{D}')\) of \((X_0, \Delta_0)\) and \((X'_0, \Delta'_0)\) with generalized Futaki invariant zero. Hence, the log Fano pairs cannot be K-stable. Proposition 5.2 will then be used to show that \((X'_0, \mathcal{D}_0) \simeq (\mathcal{X}'_0, \mathcal{D}'_0)\) and allow us to conclude that \((X_0, \Delta_0)\) and \((X'_0, \Delta'_0)\) are S-equivalent.

**Proof of Theorem 1.1.** Assume \((X_0, \Delta_0)\) and \((X'_0, \Delta'_0)\) are both K-semistable and \(\phi\) does not extend to an isomorphism. We must show \((X_0, \Delta_0)\) and \((X'_0, \Delta'_0)\) are S-equivalent and not K-stable. To do so, we use the filtrations \(\mathcal{F}\) and \(\mathcal{F}'\) constructed in Section 5.1.

Since \((X_0, \Delta_0)\) and \((X'_0, \Delta'_0)\) are K-semistable, Proposition 2.8 implies
\[
\lct(X_0, \Delta_0; \mathbf{b}_\bullet(\mathcal{F})) \geq S(\mathcal{F}) \quad \text{and} \quad \lct(X'_0, \Delta'_0; \mathbf{b}_\bullet(\mathcal{F}')) \geq S(\mathcal{F}').
\]

Combining the previous inequalities with Propositions 5.3 and 5.6, we see
\[
a = \lct(X, \Delta + X_0; \mathbf{a}_\bullet(\ord_{\tilde{X}_0})) = \lct(X_0, \Delta_0; \mathbf{b}_\bullet(\mathcal{F})) = S(\mathcal{F}) \quad (16)
\]
and
\[
a' = \lct(X', \Delta' + X'_0; \mathbf{a}_\bullet(\ord_{\tilde{X}'_0})) = \lct(X'_0, \Delta'_0; \mathbf{b}_\bullet(\mathcal{F}')) = S(\mathcal{F}'). \quad (17)
\]
Furthermore, \(\delta(X_0, \Delta_0) = \delta(X'_0, \Delta'_0) = 1\).

By the first pair of equalities in (16) and (17), we may apply Proposition 2.2 to extract \(\tilde{X}'_0\) over \(X\) and \(\tilde{X}_0\) over \(X'\). Specifically, there exist proper birational morphisms \(\mu\) and \(\mu'\):

\[
\begin{array}{ccc}
V \cup W \subset Y & & Y' \supset V' \cup W' \\
\downarrow & & \downarrow \mu' \\
X_0 \subset X & \overset{\phi}{\longrightarrow} & X' \supset X'_0 \\
\uparrow \pi & & \pi' \\
C
\end{array}
\]

such that the following hold:
(1) the fibers of \(Y\) (resp., \(Y'\)) over 0 contains two components \(V\) and \(W\) (resp., \(V'\) and \(W'\)) and they are the birational transforms of \(X_0\) and \(X'_0\);
(2) \(-W\) and \(-V'\) are ample over \(X\) and \(X'\);
(3) \((Y, V + W + \mu^{-1}_*\Delta)\) and \((Y', V' + W' + \mu'^{-1}_*\Delta')\) are lc.
We write $\mu_0 : V \to X_0$ and $\mu'_0 : W' \to X'_0$ for the restrictions of $\mu$ and $\mu'$ to $V$ and $W'$. Clearly, $\mu_0$ and $\mu'_0$ are proper birational morphisms.

**Lemma 5.7.** The prime divisors $V$ and $W'$ are normal.

*Proof.* We claim that $(Y, V + \mu^{-1}_0(\Delta))$ is plt. Indeed, $(Y, V + \mu^{-1}_0(\Delta))$ is plt away from $\text{Exc}(\mu)$, since $(X, X_0 + \Delta)$ is plt by inversion of adjunction. Since $(Y, V + \mu^{-1}_0(\Delta))$ is lc, $(Y, V + \mu^{-1}_0(\Delta))$ cannot have lc centers in $W$ and the claim follows. Hence, we may apply [KM98, 5.52] to see $V$ is normal. The normality of $W'$ follows from the same argument. \qed

Now, consider the restrictions of $W$ and $V'$ to the birational transforms of $X_0$ and $X'_0$:

$$E := W|_V \quad \text{and} \quad E' := V'|_{W'}.$$  

Since $W$ and $V'$ are $\mathbb{Q}$-Cartier, but not necessarily Cartier, $E$ and $E'$ may have fractional coefficients.

The $\mathbb{Q}$-divisors $E$ and $E'$ induce filtrations on $R$ and $R'$ defined by

$$\mathcal{F}^p E R_m := H^0 \left( V, \mathcal{O}_V(\mu_0^*(L_0) - \lceil pE \rceil) \right) \subseteq R_m$$

and

$$\mathcal{F}^p E' R'_m := H^0 \left( W', \mathcal{O}_{W'}(\mu'_0^*(L'_0) - \lceil pE' \rceil) \right) \subseteq R'_m$$

for $p, m \geq 0$. Note that

$$\mathcal{F}^p R_m \subseteq \mathcal{F}^p E R_m \quad \text{and} \quad \mathcal{F}^p R'_m \subseteq \mathcal{F}^p E' R'_m.$$  

Therefore,

$$S(\mathcal{F}) \leq S(\mathcal{F}_E) \quad \text{and} \quad S(\mathcal{F}') \leq S(\mathcal{F}_E').$$

**Lemma 5.8.** There exist positive integers $d, d'$ and prime divisors $F \subset V$ and $F' \subset W'$ such that:

1. $E = \frac{1}{d} F$ and $F$ computes $\delta(X_0, \Delta_0)$;
2. $E' = \frac{1}{d'} F'$ and $F'$ computes $\delta(X'_0, \Delta'_0)$.

*Proof.* Since $-W$ is ample over $X$, the restriction map $\mu_0^*\mathcal{O}_Y(-pW) \to \mu_0^*\mathcal{O}_V(-pE)$ is surjective for all positive integers $p$ sufficiently divisible. Hence, if we set $a_p(E) := \mu_0^*\mathcal{O}_V(-pE) \subseteq \mathcal{O}_{X_0}$, then $a_p(E) = a_p(\text{ord}_W) \cdot \mathcal{O}_{X_0}$ for such $p$ and inversion of adjunction yields

$$\text{lct}(X_0, \Delta_0; a_*(E)) = \text{lct}(X, \Delta + X_0; a_*(\text{ord}_W)).$$

Since

$$\text{lct}(X, \Delta + X_0; a_*(\text{ord}_W)) = S(\mathcal{F}) \leq S(\mathcal{F}_E)$$
by (16) and (18), Proposition 4.5 yields that $S(\mathcal{F}) = S(\mathcal{F}_E)$ and $F := \text{Supp}(E)$ is a prime divisor that computes $\delta(X_0, \Delta_0)$. To see that $E = \frac{1}{d} F$, localize $Y$ at the generic point of $F$. The statement then follows from a surface calculation similar to [Kol13, 3.45.3] and the observation that $(Y, V + W + \mu^{-1}_* (\Delta))$ is lc. Hence, the proof of (1) is complete and (2) follows from an identical argument. □

**Lemma 5.9.** For all but finitely many $x \in \mathbb{R}_{\geq 0}$,

$$\text{vol}(\mathcal{F}_E \mathcal{R}^{(x)}) = \text{vol}(\mathcal{F} \mathcal{R}^{(x)}) \quad \text{and} \quad \text{vol}(\mathcal{F}_E' \mathcal{R}^{(x)}) = \text{vol}(\mathcal{F}' \mathcal{R}^{(x)}).$$

**Proof.** As shown in the proof of Lemma 5.8, $S(\mathcal{F}) = S(\mathcal{F}_E)$. Hence,

$$\frac{1}{n} \int_0^\infty \text{vol}(\mathcal{F} \mathcal{R}^{(x)}) \, dx = S(\mathcal{F}) = S(\mathcal{F}_E) = \frac{1}{n} \int_0^\infty \text{vol}(\mathcal{F}_E \mathcal{R}^{(x)}) \, dx.$$

Since $\text{vol}(\mathcal{F} \mathcal{R}^{(x)}) \leq \text{vol}(\mathcal{F}_E \mathcal{R}^{(x)})$ by (18) and the two functions are continuous at all but one value [BHJ17, 5.3.ii], the result follows. □

**Proposition 5.10.** There exists a positive integer $M$ so that

$$\mathcal{F}_p \mathcal{R}_m = \mathcal{F}_E \mathcal{R}_m \quad \text{and} \quad \mathcal{F}'_p \mathcal{R}'_m = \mathcal{F}'_E \mathcal{R}'_m$$

for all $p \geq 0$ and $m \geq M$.

Proving this key proposition amounts to showing that the restriction map

$$H^0\left( Y, \mathcal{O}_Y (m\mu^* L - p W) \right) \rightarrow H^0\left( V, \mathcal{O}_V (m\mu^*_0 L_0 - [p E]) \right)$$

is surjective. Since such a statement is quite subtle, we will not study this restriction map directly. Instead, we use a construction that originated in [Li17] (with a refining analysis from [LWX18b]) and work on the cone over our family of log Fano pairs.

Consider the relative cone over $(X, \Delta) \rightarrow C$ with polarization $L$ given by

$$Z := C(X/C, L) = \text{Spec}(\mathcal{R}) \rightarrow C.$$ 

Write $\sigma : C \rightarrow Z$ for the section of cone points and $\Gamma$ for the closure of the inverse image of $\Delta$ under the projection $Z \setminus \sigma(C) \rightarrow X$. Note that the fiber of $(Z, \Gamma)$ over 0, denoted $(X_0, \Delta_0)$, is the cone over $(X_0, \Delta_0)$ and $Z_0 = \text{Spec}(\mathcal{R})$.

The blowup of $Z$ along $\sigma(C)$ is naturally identified with the total space of the line bundle $\mathcal{O}_X (-L)$. Hence, there is a proper birational map $Y_{L^{-1}} \rightarrow Z$, where $Y_{L^{-1}}$ denotes the total space of $\mu^* \mathcal{O}_X (-L)$. We write $X_\infty$ for the zero section of $Y_{L^{-1}}$ and $W_\infty$ for the inverse image of $W$ under the projection map $Y_{L^{-1}} \rightarrow Y$.

Associated to the divisor $W$ over $X$, there is a ray of valuations

$$\{ w_t \mid t \in [0, \infty) \} \subset \text{Val}_Z,$$

where $w_t$ is defined as the quasi-monimal valuation with weights $(1, t)$ along $X_\infty$ and $W_\infty$. Hence, for each positive integer $k$, there is a divisor $W_k$ over $X$ such that
\[ w_{1/k} = \frac{1}{k} \text{ord}_{W_k}. \] Note that
\[
A_{Z, \Gamma + Z_0}(w_t) = A_{Z, \Gamma + Z_0}(\text{ord}_{X_\infty}) + t A_{Z, \Gamma + Z_0}(\text{ord}_{W_\infty}) = A_{Z, \Gamma + Z_0}(\text{ord}_{X_\infty}) + t A_{X, \Delta + X_0}(W) = r^{-1} + t a
\]
and
\[
a_p(w_t) = \bigoplus_{m \in \mathbb{N}} \mathcal{F}^{(p-m)/t} R_m \subseteq \mathcal{R}. \tag{19}
\]

Along the fiber $Z_0$, there is a similar picture. There is a proper birational map $V_{L_0^{-1}} \to Z_0$. We write $X_{0, \infty}$ for the zero section of $V_{L_0^{-1}}$ and $F_\infty$ for the inverse image of $W$ under the projection $V_{L_0^{-1}} \to V$. Note $F_\infty$ is the support of $W_\infty$ intersected with $V_{L_0^{-1}}$.

Associated to the divisor $F$ over $X_0$, there is a ray of valuations
\[
\{v_t \mid t \in [0, \infty)\} \subset \text{Val}_{Z_0},
\]
where $v_t$ is the the quasi-monimal valuation with weights $(1, td)$ along $X_{0, \infty}$ and $F_\infty$. Note that
\[
A_{Z_0, \Gamma_0}(v_t) = A_{Z, \Gamma}(\text{ord}(X_{0, \infty})) + t A_{Z_0, \Gamma_0}(\text{ord}_{F_\infty}) = r^{-1} + t a
\]
and
\[
a_p(v_t) = \bigoplus_{m \in \mathbb{N}} \mathcal{F}_{E}^{(p-m)/t} R_m \subseteq \mathcal{R}. \tag{20}
\]

For each positive integer $k$, there is a divisor $F_k$ over $Z_0$ such that $w_{1/k} = \frac{d}{k} \text{ord}_{F_k}$.

Since $F$ computes $\delta(X_0, \Delta_0) = 1$, the proof of Theorem 4.1 yields that for $k \gg 0$ there is an extraction of $F_k$, which we denote by $\rho_k : Z_0, F_k \to Z_0$.

By modifying the argument used to prove Theorem 4.1, we may extract $W_k$ over $Z$ for $k \gg 0$ as following.

**Lemma 5.11.** For $k \gg 0$, there exists an extraction $\tau_k : Z_{W_k} \to Z$ of $W_k$ over $Z$ such that $(Z_{W_k}, W_k + \tau_k^{-1}(\Gamma + Z_0))$ is lc.

**Proof.** For $t \in \mathbb{R}_{\geq 0}$, let $a_\bullet(w_t) \cdot \mathcal{O}_{Z_0}$ denote the restriction of $a_\bullet(w_t)$ to a graded sequence of ideals on $Z_0$. By (20) and the definition of $\mathcal{F}$, we have
\[
a_p(w_t) \cdot \mathcal{O}_{Z_0} = \bigoplus_{m \in \mathbb{N}} \mathcal{F}_{E}^{(p-m)/t} R_m.
\]
Hence, Lemmas 2.12.1 and 5.9 imply
\[
\text{mult}(a_\bullet(w_t) \cdot \mathcal{O}_{Z_0}) = \text{mult}(a_\bullet(v_t)) \tag{21}
\]
for each $t \in \mathbb{R}_{\geq 0}$.

Set
\[
f(t) := (r^{-1} + at)^n \text{mult}(a_\bullet(w_t) \cdot \mathcal{O}_{Z_0}).
\]
Applying Lemma 2.12, we see
\[ f(0) = (-K_{X_0} - \Delta_0)^n \quad \text{and} \quad f'(0) = (-K_{X_0} - \Delta_0)^n(a - S(F)) = 0. \]
Hence, a Taylor expansion gives \( f(t) = f(0) + O(t^2) \) for \( 0 < t \ll 1 \).

For each positive integer \( k \), define
\[ c_k := \text{lct}(Z, \Gamma + Z_0; a_\bullet(\text{ord}_{W_k})). \]
Note that
\[ c_k \leq A_{Z,\Gamma + Z_0}(W_k) = kr^{-1} + a \]
by (3). Additionally,
\[ c_k = k \cdot \text{lct}(Z, \Gamma + Z_0; a_\bullet(w_{1/k})) = k \cdot \text{lct}(Z_0, \Gamma_0; a_\bullet(w_{1/k}) \cdot \mathcal{O}_{Z_0}) \]
by inversion of adjunction and the relation \( a_{\bullet k}(\text{ord}_{W_k}) = a_\bullet(w_{1/k}) \). Therefore,
\[
\begin{align*}
  f(0) &\leq \text{lct}(Z_0, \Gamma_0; a_\bullet(w_{1/k}) \cdot \mathcal{O}_{Z_0})^n \text{mult}(a_\bullet(w_{1/k}) \cdot \mathcal{O}_{Z_0}) \\
  &\leq \left( \frac{c_k}{k} \right)^n \text{mult}(a_\bullet(w_{1/k}) \cdot \mathcal{O}_{Z_0}) \\
  &\leq f\left( \frac{1}{k} \right),
\end{align*}
\]
where the first inequality follows from [Liu18, 7] and the assumption that \((X_0, \Delta_0)\) is K-semistable (see [Li17, Theorem 3.1] and [LX16, Theorem A]).

For each positive integer \( k \), define
\[ a_k := A_{Z,\Gamma + Z_0}(W_k) - c_k. \]
As in the proof of Theorem 4.1, the previous inequality implies \( \lim_k a_k = 0 \). Hence, if \( k \gg 0 \), Proposition 2.2 yields an extraction \( \tau_k : Z_{W_k} \to Z \) of \( W_k \) such that the pair
\[ (Z_{W_k}, \tau_k^{-1}(\Gamma + Z_0) + (1 - a_k)W_k) \]
is lc. Since \( \lim_{k \to \infty} (1 - a_k) = 1 \), the ACC for log canonical thresholds [HMX14] implies \( (Z_{W_k}, \tau_k^{-1}(\Gamma + Z_0) + W_k) \) must be lc for \( k \gg 0 \). \( \square \)

From now on, we fix a positive integer \( k \) so that \( d \) divides \( k \); there exist extractions \( \rho_k : Z_{0,F_k} \to Z_0 \) and \( \tau_k : Z_{W_k} \to Z \), and \( (Z_{W_k}, \tau_k^{-1}(\Gamma + Z_0) + W_k) \) is lc.

**Lemma 5.12.** We have a diagram
\[
\begin{array}{ccc}
F_k & \subset & Z_{0,F_k} \\
\downarrow \rho_k & & \downarrow \tau_k \\
Z_0 & \subset & Z
\end{array}
\]
(The birational transform of \( Z_0 \) on \( Z_{W_k} \) is the extraction of \( F_k \).) Additionally,

(i) \( W_k|_{Z_0,F_k} = \frac{1}{d} F_k \) and

(ii) \( dW_k \) is Cartier at the generic point of \( F_k \).
Proof. Observe that \( \tau_{k*}^{-1}(Z_0) \), which is the birational transform of \( Z_0 \) on \( Z_{W_k} \), is normal. Indeed, since \((Z_{W_k}, W_k + \tau_{k*}^{-1}(Z_0 + \Gamma))\) is lc, this follows from the same argument used to prove Lemma 5.7.

Since \(-W_k\) and \(-F_k\) are ample over \( Z \) and \( Z_0 \), we may find a positive integer \( p \) so that

\[
a_{pd}((ord_{W_k}) \subseteq \mathcal{O}_Z \quad \text{and} \quad a_p((ord_{F_k}) \subseteq \mathcal{O}_{Z_0}\]

satisfy the conclusions of Lemma 4.6. Hence, \( Z_{W_k} \) is the blowup of \( Z \) along \( a_{pd}((ord_{W_k}) \) and \( (Z_0)_{F_k} \) is the blowup of \( Z_0 \) along \( a_p((ord_{F_k}) \). The former statement implies \( \tau_{k*}^{-1}(Z_0) \) is the blowup of \( Z \) along \( a_p((ord_{W_k}) \cdot \mathcal{O}_{Z_0} \).

We claim that \( a_p((ord_{W_k}) \cdot \mathcal{O}_{Z_0} \) and \( a_p((ord_{F_k}) \) have the same integral closure. Since \( \tau_{k*}^{-1}(Z_0) \) and \( Z_{0,F_k} \) are both normal, the claim implies that \( \tau_{k*}^{-1}(Z_0) = Z_{0,F_k} \).

To prove the above claim, observe that \( a_{pd}((ord_{W_k}) \cdot \mathcal{O}_{Z_0} \subseteq a_p((ord_{F_k}) \). Indeed,

\[
a_{pd}((ord_{W_k}) \cdot \mathcal{O}_{Z_0} = \bigoplus_{m \in \mathbb{N}} \mathcal{F}^{pd-mk}R_m \subseteq \bigoplus_{m \in \mathbb{N}} \mathcal{F}^{pd-mk}R_m = a_p((ord_{F_k}) \).
\]

Therefore, if we can show \( a_{pd}((ord_{W_k}) \cdot \mathcal{O}_{Z_0} \) and \( a_p((ord_{F_k}) \) have the same multiplicity, a theorem of Rees [Ree61] will imply the claim.

To compute the multiplicities of the two ideals, note that

\[
mult(a_p((ord_{F_k}) = p^n \cdot mult(a_{*}(ord_{F_k}) = (pd/k)^n mult(a_{*}(v_1/k)),
\]

since \( a_{p\ell}(ord_{F_k}) = a_p(ord_{F_k})^{\ell} \) for all \( \ell \geq 0 \) and the relation \( d/ord_{F_k} = v_1/k \). By similar reasoning,

\[
mult(a_{pd}(ord_{W_k}) \cdot \mathcal{O}_{Z_0}) = (pd/k)^n mult(a_{*}(w_1/k) \cdot \mathcal{O}_{Z_0}.
\]

Therefore, (21) implies the multiplicities are equal and the claim holds.

To see (i), note that

\[
a_{pd}(ord_{W_k}) \cdot \mathcal{O}_{Z_{W_k}} = \mathcal{O}_{Z_{W_k}}(-pdW_k), \quad \text{and} \quad a_p(ord_{F_k}) \cdot \mathcal{O}_{Z_{0,F_k}} = \mathcal{O}_{Z_{0,F_k}}(-pF_k)
\]

by our choice of \( p \). Furthermore, \( dpW_k \) and \( pF_k \) are both Cartier. Since \( a_{pd}(ord_{W_k}) \cdot \mathcal{O}_{Z_0} \) and \( a_p(ord_{F_k}) \) have the same integral closure,

\[
\mathcal{O}_{Z_{W_k}}(-pdW_k) \cdot \mathcal{O}_{Z_{0,F_k}} = \mathcal{O}_{Z_{0,F_k}}(-pF_k),
\]

which yields (i).

For (ii), the statement may be checked after localizing \( Z_{W_k} \) at the generic point of \( F_k \). The result then follows from (i), the fact that \((Z_{W_k}, W_k + Z_{0,F_k} + \tau_{k*}^{-1}(\Gamma))\) is lc, and the classification of lc surface singularities. \( \square \)

Proof of Proposition 5.10. With the above results, the equality of the two filtrations is now a statement concerning valuation ideals on \( Z \) and \( Z_0 \). Indeed,

\[
a_{pd}(ord_{W_k}) = \bigoplus_{m \in \mathbb{N}} \mathcal{F}^{pd-mk}R_m \quad \text{and} \quad a_p(ord_{F_k}) = \bigoplus_{m \in \mathbb{N}} \mathcal{F}^{pd-mk}R_m.
\]

Let us consider the restriction sequence

\[
0 \to \mathcal{O}_{Z_{W_k}}(-pdW_k - Z_{0,F_k}) \to \mathcal{O}_{Z_{W_k}}(-pdW_k) \to \mathcal{O}_{Z_{0,F_k}}(-pF_k) \to 0. \quad (24)
\]
where \( p \) is a positive integer. We claim that the sequence is exact. Indeed, by the proof of [KM98, 5.26], it suffices to show that if

\[ T := \{ z \in Z \mid dW_k \text{ is not Cartier at } z \}, \]

then \( Z_{0,F_k} \cap T \) has codimension at least two in \( Z_{0,F_k} \). Since \( dW_k \) is Cartier at the generic point of \( F_k \) by Lemma 5.12, the condition holds.

Since (24) is exact and \(-W_k\) is ample over \( Z \), there exists a positive integer \( p_0 \) so that the push forward of the sequence

\[ 0 \to \tau_{k*}\mathcal{O}_{Z_{W_k}}(-pdW_k - Z_{0,F_k}) \to \mathfrak{a}_{pd}(\text{ord}_{W_k}) \to \mathfrak{a}_p(\text{ord}_{F_k}) \to 0, \]

remains exact for \( p \geq p_0 \). Hence, the restriction map

\[ \mathcal{F}^{pd-mk}R_m \to \mathcal{F}^{pd-mk}_{E}R_m \]

is surjective for all \( p \geq p_0 \) and \( m \geq 0 \), and this yields that \( \mathcal{F}^{pd-mk}R_m = \mathcal{F}^{pd-mk}_{E}R_m \) for such \( p \) and \( m \). Since \( k \) was chosen to be a multiple of \( d \), the previous equality implies \( \mathcal{F}^{pd}R_m = \mathcal{F}^{pd}_{E}R_m \) for all \( p \geq 0 \) and \( m \geq p_0 \).

Now, note that

\[ \mathcal{F}^{pd}_{E}R_m = \mathcal{F}^{pd-j}_{E}R_m \quad \text{for all } j \in \{0, \ldots, d-1\}, \]

since \( E = \frac{1}{d}F \). Therefore, the stronger statement \( \mathcal{F}^{p}R_m = \mathcal{F}^{p}_{E}R_m \) for all \( p \geq 0 \) and \( m \geq p_0 \) also holds. \( \square \)

Returning to the proof of Theorem 1.1, it is now clear that \((X_0, \Delta_0)\) is not K-stable. Indeed, Lemma 5.8 states that \( F \) computes \( \delta(X_0, \Delta_0) \). Since \( \delta(X_0, \Delta_0) = 1 \), Theorem 4.1 yields that \((X_0, \Delta_0)\) is not K-stable. We are left to show that \((X_0, \Delta_0)\) and \((X'_0, \Delta'_0)\) are S-equivalent.

To prove the S-equivalence, we consider the filtrations \( \mathcal{F} \) and \( \mathcal{F}' \) of \( R \) and \( R' \). Observe that the filtrations \( \mathcal{F} \) and \( \mathcal{F}' \) are finitely generated. To see the statement for \( \mathcal{F} \), note that Proposition 5.10 implies \( \mathcal{F} \) is finitely generated if and only if \( \mathcal{F}_E \) is finitely generated. Since \( E = \frac{1}{d}F \) and \( F \) computes \( \delta(X_0, \Delta_0) = 1 \) by Lemma 5.8, Theorem 4.1 yields that \( \mathcal{F}_E \) is finitely generated.

Let \((\mathcal{X}, \mathcal{D})\) and \((\mathcal{X}', \mathcal{D}')\) denote the test configuration of \((X_0, \Delta_0)\) and \((X'_0, \Delta'_0)\) associated to the filtrations \( \mathcal{F} \) and \( \mathcal{F}' \). We claim that these test configurations are special and have generalized Futaki invariant zero. Indeed, \( \mathcal{F} \) and \( \mathcal{F}' \) induce the same test configurations as \( \mathcal{F}_E \) and \( \mathcal{F}'_E \). The second statement of Theorem 4.1 then yields the claim. Hence, [LWX18b, 3.1] implies \((\mathcal{X}_0, \mathcal{D}_0)\) and \((\mathcal{X}'_0, \mathcal{D}'_0)\) are K-semistable.

To prove the S-equivalence, we are left to show that there is an isomorphism \((\mathcal{X}_0, \mathcal{D}_0) \simeq (\mathcal{X}'_0, \mathcal{D}'_0)\). Note that

\[ \mathcal{X}_0 = \text{Proj} \left( \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{N}} \text{gr}^p_{\mathcal{F}}R_m \right) \quad \text{and} \quad \mathcal{X}'_0 = \text{Proj} \left( \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{N}} \text{gr}^p_{\mathcal{F}'}R'_m \right). \]
Therefore, the isomorphism $\varphi : \text{gr}_F R \to \text{gr}_F R'$ in Proposition 5.2 induces an isomorphism $X_0 \simeq X'_0$. This proves Theorem 1.1 in the case when the boundaries $\Delta$ and $\Delta'$ are trivial.

In general, we claim that $\varphi$ also induces an isomorphism of pairs $(X_0, D_0) \simeq (X'_0, D'_0)$. However, to prove $D_0$ and $D'_0$ match under the isomorphism $X_0 \cong X'_0$ is quite delicate, unless

$$\Delta \sim_{\pi, \mathbb{Q}} -cK_X \quad \text{and} \quad \Delta' \sim_{\pi', \mathbb{Q}} -cK_{X'},$$

for some rational number $c$.

Fix a prime divisor $B \subset \text{Supp}(\Delta)$, and let $B' \subset \text{Supp}(\Delta')$ denote its birational transform on $X'$. Write $\mathcal{B} \subset \text{Supp}(\mathcal{D})$ and $\mathcal{B}' \subset \text{Supp}(\mathcal{D})$ for the degenerations of $B_0 \subset X_0$ and $B'_0 \subset X'_0$ on $X$ and $X'$. To complete the proof it suffices to show that the isomorphism $X_0 \simeq X'_0$ sends $B_0$ to $B'_0$.

Recall that the scheme theoretic fibers of $\mathcal{B}$ and $\mathcal{B}'$ over 0 are defined by the ideals

$$\text{in}(I_{B_0}) \subset \text{gr}_F R \quad \text{and} \quad \text{in}(I_{B'_0}) \subset \text{gr}_F R',$$

where $I_{B_0} \subset R$ and $I_{B'_0} \subset R$ denote the ideals defining $B_0$ and $B'_0$. Observe that $\text{in}(I_{B_0})$ and $\text{in}(I_{B'_0})$ are homogenous with respect to the gradings by $m$ and $p$. Furthermore, the graded components may be expressed as

$$\text{in}(I_{B_0})_{p,m} := \text{in}(I_{B_0}) \cap \text{gr}_F^{p} R_{m} = \text{im}(\mathcal{F}^p R_m \cap I_{B_0} \to \text{gr}_F^{p} R_{m}) \simeq \frac{\mathcal{F}^p R_{m} \cap I_{B_0}}{\mathcal{F}^{p+1} R_{m} \cap I_{B_0}},$$

and

$$\text{in}(I_{B'_0})_{p,m} := \text{in}(I_{B'_0}) \cap \text{gr}_F^{p} R'_m = \text{im}(\mathcal{F}^p R'_m \cap I_{B'_0} \to \text{gr}_F^{p} R'_m) \simeq \frac{\mathcal{F}^p R'_m \cap I_{B'_0}}{\mathcal{F}^{p+1} R'_m \cap I_{B'_0}}.$$

Rather than showing that the isomorphism $\text{gr}_F R \to \text{gr}_F R'$ sends $\text{in}(I_{B_0})$ to $\text{in}(I_{B'_0})$, we introduce auxiliary ideals defined using sections of the relative section rings that vanish along $\mathcal{B}$ and $\mathcal{B}'$. For $p, m \geq 0$, set

$$I_{p, m} := \text{im}(\mathcal{F}^p R_m \cap I_B \to \text{gr}_F^{p} R_{m}) \quad \text{and} \quad I'_{p, m} := \text{im}(\mathcal{F}^p R'_m \cap I_{B'} \to \text{gr}_F^{p} R'_m),$$

where $I_B \subset \mathcal{R}$ and $I_{B'} \subset \mathcal{R}'$ are the ideals defining $B$ and $B'$. It is straightforward to check that

$$I := \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{N}} I_{p, m} \subset \text{gr}_F R \quad \text{and} \quad I' := \bigoplus_{m \in \mathbb{N}} \bigoplus_{p \in \mathbb{N}} I'_{p, m} \subset \text{gr}_F R'$$

are ideals and are contained in $\text{in}(I_{B_0})$ and $\text{in}(I_{B'_0})$.

The following two propositions complete our result. Indeed, Proposition 5.13 says that the isomorphism sends $X \simeq X'$ also sends $V(I)$ to $V(I')$. Since $V(I)$ and $V(I')$ agree with $\mathcal{B}_0$ and $\mathcal{B}'_0$ away from codimension two subsets by Proposition 5.14, we can conclude the isomorphism $X_0 \simeq X'_0$ sends $B_0$ to $B'_0$. \hfill \box

We are left to prove the following two propositions used in the above proof.

**Proposition 5.13.** The isomorphism $\varphi : \text{gr}_F R \to \text{gr}_F R'$ sends $I$ to $I'$.
Proof. Observe that for \( \bar{s} \in \mathcal{F}^p \mathcal{R}_m \),
\[
\bar{s} \in \mathcal{F}^p \mathcal{R}_m \cap \mathcal{I}_B \quad \text{if and only if} \quad \varphi_{p,m}(\bar{s}) \in \mathcal{F}^{mr(a+a')-p} \mathcal{R}_m \cap \mathcal{I}_{B'}.
\]
Indeed, \( \bar{s} \) and \( \varphi_{p,m}(\bar{s}) \) differ by a unit away from \( 0 \in C \) and membership in the ideals \( \mathcal{I}_B \) and \( \mathcal{I}_{B'} \) may be tested away from \( 0 \in C \), since \( B \) and \( B' \) are horizontal. Therefore, \( \varphi \) sends \( I_{p,m} \) to \( I_{mr(a+a')-p,m} \) and the result follows. \( \square \)

The next proposition is more difficult to prove.

**Proposition 5.14.** The subschemes defined by

1. \( \text{in}(I_{B_0}) \) and \( I \) on \( \mathcal{X}_0 \);
2. \( \text{in}(I_{B_0}') \) and \( I' \) on \( \mathcal{X}_0' \)

agree away from codimension 2 subsets.

To prove the statement for (1), it suffices to show that

\[
\dim \left( \bigoplus_{p \geq 0} \text{in}(I_{B_0}) \mathcal{P}_m \right) = O(m^{n-2}).
\]

To bound the dimension of the previous module, we return to the cone construction argument used earlier in this section.

Consider the relative cone \((Z, \Gamma)\) and the extractions

\[
\tau_k : Z_{W_k} \to Z \quad \text{and} \quad \rho_k : Z_{0,F_k} \to Z_0
\]

used in the proof of Proposition 5.10. Let \( G \subseteq \text{Supp}(\Gamma) \) denote the prime divisor defined via pulling back \( B \subseteq \text{Supp}(\Delta) \). Write \( \widetilde{G} \) and \( \widetilde{G}_0 \) for the birational transforms of \( G \) and \( G_0 \) on \( Z_{W_k} \) and \( Z_{0,F_k} \).

Observe that for \( j \geq 0 \)
\[
a_j(\text{ord}_{W_k}) \cap \mathcal{I}_G = \bigoplus_{m \geq 0} \left( \mathcal{F}^{jd-mk} \mathcal{R}_m \cap I_B \right)
\]
and
\[
a_j(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0} = \bigoplus_{m \geq 0} \left( \mathcal{F}^{jd-mk} \mathcal{R}_m \cap I_{B_0} \right).
\]

Therefore,
\[
\frac{a_j(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0}}{(a_j(\text{ord}_{W_k}) \cap \mathcal{I}_G) \cdot \mathcal{O}_{Z_0} + (a_{j+1}(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0})} \simeq \bigoplus_{m \geq 0} \frac{\text{in}(I_{B_0}) \mathcal{P}_{-km,m}}{I_{jd-km,m}}.
\]

**Lemma 5.15.** We have
\[
\frac{a_j(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0}}{(a_j(\text{ord}_{W_k}) \cap \mathcal{I}_G) \cdot \mathcal{O}_{Z_0} + (a_{j+1}(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0})} = O(j^{n-2}).
\]

A key subtlety in proving this lemma is that the divisors \( G \) and \( G_0 \), as well as there birational transforms, may fail to be \( \mathbb{Q} \)-Cartier.
Proof. Fix an integer $q$ such that $qdW_k$ is Cartier. For each $r \in \{0, \ldots, q - 1\}$, set

$$Q_r := \text{coker} \left( \mathcal{O}_{Z_{W_k}}(\tilde{G} - rdW_k) \to \mathcal{O}_{Z_{0,F_k}}(\tilde{G}_0 - rF_k) \right),$$

where the previous map is defined via restriction.

Claim: The support of $Q_r$ is contained in the intersection of $Z_{0,F_k}$ and the locus where $G + rdW_k$ is not $Q$-Cartier.

To prove the claim, consider the closed set

$$T := \{ z \in Z_{W_k} \mid \tilde{G} + rdW_k \text{ is not Cartier at } z \}.$$

We will show the following:

- $\text{codim}_{Z_{0,F_k}}(T \cap Z_{0,F_k}) \geq 2$;
- $\tilde{G}_0 + rF_k$ agrees with the restriction of $\tilde{G} + rdW_k$ to $Z_{0,F_k} \setminus T$.

Assuming these statements, the proof of [KM98, 5.26] implies

$$\mathcal{O}_{Z_{W_k}}(\tilde{G} - rdW_k) \to \mathcal{O}_{Z_{0,F_k}}(\tilde{G}_0 - rF_k)$$

is surjective on the locus where $\tilde{G} + rdW_k$ is $Q$-Cartier and the claim will be complete.

To prove the first bullet point, we show $T$ does not contain codimension one points of $Z_{0,F_k}$. Recall that $(Z, Z_{0,F_k} + W_k + \tau_{k*}^{-1}(\Gamma))$ is lc. Since $\tilde{G} \subseteq \tau_{k*}^{-1}(\Gamma)$, [Kol13, 2.32.2] yields $F_k = Z_{0,F_k} \cap W_k$ is not contained in $\tilde{G}$. Since we also know that $dW_k$ is Cartier at the generic point of $F_k$, we can conclude that $T$ does not contain the generic point of $F_k$.

To see $T$ does not contain the remaining codimension one points of $Z_{0,F_k}$, observe that $Z_{W_k} \setminus W_k \simeq Z \setminus \sigma(0)$. Now, the locus where $\tilde{G}|_{Z \setminus \sigma(0)}$ is not Cartier is contained in $Z_{\text{sing}}$. Since $Z$ has normal fibers and $C$ is regular, $\text{codim}_{Z_0}(Z_0 \cap Z_{\text{sing}}) \geq 2$. Hence, $T$ does not contain any codimension one point of $Z_{0,F_k} \setminus F_k$.

The second statement follows from the previous two paragraphs. Indeed, the previous paragraph implies the equality away from $F_k$. The equality at $F_k$ is a consequence of the previous observation that $F_k$ is not contained in $\tilde{G}$ and Lemma 5.12.ii. Hence, the claim is complete.

We now return to the proof of the lemma. Given a positive integer $j$, write $j = bq + r$ where $r \in \{0, \ldots, q - 1\}$. Hence, we get an exact sequence

$$\mathcal{O}_{Z_{W_k}}(\tilde{G} - jdW_k) \to \mathcal{O}_{Z_{0,F_k}}(\tilde{G}_0 - jE_k) \to Q_r(-bqdW_k) \to 0.$$

Pushing forward the sequence by $\tau_{k*}$, we see

$$a_j(\text{ord}_{W_k}) \cap \mathcal{I}_G \to a_j(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0} \to \tau_{k*}Q_r(-bqdW_k) \to 0,$$
is exact for $p \gg 0$, since $-W_k$ is $\tau_k$-ample. Hence,
\[
\dim \left( \frac{a_j(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0}}{(a_{jd}(\text{ord}_{W_k}) \cap \mathcal{I}_G) \cdot \mathcal{O}_{Z_0} + (a_{j+1}(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0})} \right) \\
\leq \dim \left( \frac{a_j(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0}}{(a_{jd}(\text{ord}_{W_k}) \cap \mathcal{I}_G) \cdot \mathcal{O}_{Z_0} + (a_{j+q}(\text{ord}_{F_k}) \cap \mathcal{I}_{G_0})} \right) \\
\leq \dim(\tau_{k*}\mathcal{O}_r(-qjdW_k)/\tau_{k*}\mathcal{O}_r(-(b+1)qdW_k)).
\]

We are now reduced to showing that the last term equals $O(b^{n-2})$.

To reduce bulky notation, let $D$ denote the effective Cartier divisor $qdW_k$. Hence, we have a short exact sequence
\[
0 \to \mathcal{O}_r(-(b+1)D) \to \mathcal{O}_r(-bD) \to \mathcal{O}_r(-bD)|_D \to 0
\]

After pushing forward by $\tau_k$, we see
\[
0 \to \tau_{k*}\mathcal{O}_r(-(b+1)D) \to \tau_{k*}\mathcal{O}_r(-bD) \to H^0(D, \mathcal{O}_r(-bD)|_D) \to 0.
\]
is exact for $b \gg 0$. Since $\mathcal{O}_r$ is supported on the locus of $Z_{0,F_k}$ where $G$ is not $\mathcal{Q}$-Cartier, Lemma 5.16 implies $\mathcal{O}_r|_D$ has dimension at most $\dim(Z_{0,F_k}) - 3 = n - 2$. Therefore,
\[
H^0(D, \mathcal{O}_r(-bD)|_D)) = O(b^{n-2})
\]
and the lemma is complete. \[\square\]

The previous proof used the following property of lc pairs.

**Lemma 5.16.** Let $(X, \Delta + E_1 + E_2)$ be an lc pair such that $E_1$ and $E_2$ are $\mathcal{Q}$-Cartier prime divisors. If $x \in X$ is a codimension three point and $x \in E_1 \cap E_2 \cap \text{Supp}(\Delta)$, then $X$ is $\mathcal{Q}$-factorial at $x$.

**Proof.** After taking appropriate index one covers, we can assume $E_1$ and $E_2$ are Cartier. By cutting, we can assume $\dim(X) = 3$ and $x$ is a closed point. Now, we know $(X, \Delta)$ is klt in a neighborhood of $E_1 \cup E_2$. Therefore, $E_1$ is CM (Cohen-Macaulay) at $x$ [KM98, 5.25].

We claim that $E_1$ is normal at $x$. If not, since $E_1$ is $S_2$ at $x$, it cannot be $R_1$ by Serre’s Theorem. Hence, $E_1$ is singular on a curve $C$ passing through $x$. Note that $C \not\subseteq \text{Supp}(\Delta) \cup E_2$ by [Kol13, 2.32]. If we consider the normalization $E_1' \to E$, we see $\text{Diff}_{E_1'}(\Delta)$ has coefficient one along the divisors in the preimage of $C$ and positive coefficient along the preimage of $\text{Supp}(\Delta)$. By [Kol13, 2.31], this implies that $(E_1'|_E, \text{Diff}_{E_1'}(\Delta) + E_2|_{E_1'})$ is not lc, which contradicts adjunction.

After shrinking around $x$, we may assume $E_1$ is normal. Adjunction gives that $(E_1, \Delta|_{E_1} + E_2|_{E_1})$ is lc. Since $E_2|_{E_1}$ is Cartier, $(E_1, \Delta|_{E_1})$ is canonical at $x$. Using that $x \in \text{Supp}(\Delta|_{E_1})$, [Kol13, 2.29.2] yields that $E_1$ is smooth at $x$. \[\square\]

**Proof of Proposition 5.14.** To prove the statement for (1), it suffices to show
\[
\sum_{p \geq 0} \dim N_{p,m} = O(m^{n-2}), \quad (27)
\]
where \( N_{p,m} := \text{in}(I_{B_0})_{p,m}/I_{p,m} \). The previous estimate is a consequence of Lemma 5.15.

Indeed, by Lemma 5.15 and (26)

\[
\sum_{m \geq 0} \dim N_{jd-mk,m} = O\left(j^{n-2}\right).
\]

Since \( \text{gr}_F R \) is finitely generated, there exists a positive integer \( C \) so that \( \text{gr}_F^p R_m = 0 \) for all \( p > mC \). Hence, \( N_{p,m} = 0 \) for \( p \geq mC \) and we see

\[
\sum_{m=0}^M \sum_{p \geq 0} \dim N_{pd,m} \leq \sum_{j=0}^{M(c+k)/d} \sum_{m \geq 0} \dim N_{jd-mk,m} = O\left(M^{n-1}\right)
\]

Therefore,

\[
\sum_{p \geq 0} \dim N_{pd,m} = O(m^{n-2}).
\]

Observe that \( \text{gr}_F^p R_m = 0 \) for all \( p \) not divisible by \( d \) and \( m \gg 0 \) by Proposition 5.10 and the fact that \( E = \frac{1}{d} F \). Therefore, the previous equation implies (27) and the proof of Proposition 5.14.1 is complete. The statement for Proposition 5.14.2 follows from an identical argument.

\[ \square \]

**Proof of Corollary 1.3.** The proof is the same as the proof of Corollary 3.5, but with Theorem 3.1 replaced by Theorem 1.1.3. \[ \square \]

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