Random Distances Associated with Rhombuses

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Abstract

Parallelograms are one of the basic building blocks in two-dimensional tiling. They have important applications in a wide variety of science and engineering fields, such as wireless communication networks, urban transportation, operations research, etc. Different from rectangles and squares, the coordinates of a random point in parallelograms are no longer independent. As a case study of parallelograms, the explicit probability density functions of the random Euclidean distances associated with rhombuses are given in this report, when both endpoints are randomly distributed in 1) the same rhombus, 2) two parallel rhombuses sharing a side, and 3) two rhombuses having a common diagonal, respectively. The accuracy of the distance distribution functions is verified by simulation, and the correctness is validated by a recursion and a probabilistic sum. The first two statistical moments of the random distances, and the polynomial fit of the density functions are also given in this report for practical uses.

Index Terms

Random distances; distance distribution functions; rhombuses

I. THE PROBLEM

Define a “unit rhombus” as the rhombus with an acute angle of $\theta = \frac{\pi}{3}$ and a side length of 1. Picking two points uniformly at random from the interior of a unit rhombus, or between two adjacent unit rhombuses, the goal is to obtain the probabilistic density function (PDF) of the random distances between these two endpoints, as illustrated in Fig. 1.

There are four cases depending on the geometric locations of these two random endpoints, when rhombuses are adjacent and similarly oriented, as shown in Fig. 1: i.e., AB that are within the same rhombus; RS that are inside two parallel rhombuses sharing a side; PQ and MN that
Fig. 1: Random Points and Distances Associated with Rhombuses.

are inside two rhombuses sharing a common diagonal. Here PQ and MN are two different cases, and in the following, we refer to them as long (long-diag) and short diagonal (short-diag), respectively. The next section gives the explicit PDFs for the above four cases.

II. DISTANCE DISTRIBUTIONS ASSOCIATED WITH RHOMBUSES

A. $|AB|$: Distance Distribution within a Rhombus

The probability density function of the random Euclidean distances between two uniformly distributed points that are both inside the same unit rhombus is

$$f_{D_1}(d) = 2d \begin{cases} \left(\frac{4}{3} + \frac{2\pi}{9\sqrt{3}}\right)d^2 - \frac{16}{3}d + \frac{2\pi}{\sqrt{3}} & 0 \leq d \leq \frac{\sqrt{3}}{2} \\ \frac{8}{\sqrt{3}} \left(1 + \frac{d^2}{3}\right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left(\frac{4}{3} - \frac{10\pi}{9\sqrt{3}}\right)d^2 - \frac{16}{3}d + \frac{10}{3} \sqrt{4d^2 - 3} - \frac{2\pi}{\sqrt{3}} & \frac{\sqrt{3}}{2} \leq d \leq 1 \\ \frac{4}{\sqrt{3}} \left(1 - \frac{d^2}{3}\right) \sin^{-1} \frac{\sqrt{3}}{2d} - \left(\frac{2}{3} - \frac{2\pi}{9\sqrt{3}}\right)d^2 + \sqrt{4d^2 - 3} - \frac{2\pi}{3\sqrt{3}} - 1 & 1 \leq d \leq \sqrt{3} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$
The corresponding cumulative distribution function (CDF) is

\[
F_D(d) = \begin{cases} 
\left(\frac{2}{3} + \frac{\pi}{9\sqrt{3}}\right) d^4 - \frac{32}{9} d^3 + \frac{2\pi}{\sqrt{3}} d^2 & \text{if } 0 \leq d \leq \frac{\sqrt{3}}{2} \\
\frac{2}{\sqrt{3}} \left(2d^4 + \frac{d^4}{3}\right) \sin^{-1}\frac{\sqrt{2}}{2d} + \left(\frac{2}{3} - \frac{5\pi}{9\sqrt{3}}\right) d^4 - \frac{32}{9} d^3 - \frac{2\pi}{\sqrt{3}} d^2 & \text{if } \frac{\sqrt{3}}{2} \leq d \leq 1 \\
\frac{2}{\sqrt{3}} \left(2d^4 - \frac{d^4}{3}\right) \sin^{-1}\frac{\sqrt{3}}{2d} + \left(\frac{\pi}{9\sqrt{3}} - \frac{1}{3}\right) d^4 - \left(\frac{2\pi}{3\sqrt{3}} + 1\right) d^2 & \text{if } 1 \leq d \leq \sqrt{3} \\
0 & \text{otherwise} 
\end{cases}
\]  

B. \(|RS|\): Distance Distribution between Two Parallel Rhombuses Sharing a Side

The probability density function of the random distances between two uniformly distributed points, one in each of the two adjacent unit rhombuses that are parallel to each other, is

\[
f_{Dr}(d) = 2d \begin{cases} 
\frac{4}{3} d - \left(\frac{2}{3} + \frac{\pi}{9\sqrt{3}}\right) d^2 & \text{if } 0 \leq d \leq \frac{\sqrt{3}}{2} \\
-\frac{2}{\sqrt{3}} (d^2 + 2) \sin^{-1}\frac{\sqrt{2}}{2d} + \left(\frac{8\pi}{9\sqrt{3}} - \frac{2}{3}\right) d^2 + \frac{4}{3} d - \frac{11}{6} \sqrt{4d^2 - 3} & \text{if } \frac{\sqrt{3}}{2} \leq d \leq 1 \\
\left(\frac{2}{\sqrt{3}} - \frac{d^2}{3\sqrt{3}}\right) \sin^{-1}\frac{\sqrt{2}}{2d} + \left(\frac{2}{\sqrt{3}} + \frac{d^2}{3\sqrt{3}}\right) \sin^{-1}\frac{\sqrt{3}}{d} + \left(\frac{\pi}{9\sqrt{3}} - \frac{1}{3}\right) d^2 & \text{if } 1 \leq d \leq \sqrt{3} \\
\frac{4d^2}{3\sqrt{3}} \sin^{-1}\frac{\sqrt{3}}{2d} + \left(\frac{2}{3} - \frac{2\pi}{9\sqrt{3}}\right) d^2 - \frac{8}{3} d + \frac{\sqrt{4d^2 - 3}}{3} + \frac{2\pi}{3\sqrt{3}} + \frac{1}{2} & \text{if } \sqrt{3} \leq d \leq 2 \\
\left(\frac{2}{\sqrt{3}} - \frac{d^2}{3\sqrt{3}}\right) \left(\sin^{-1}\frac{\sqrt{2}}{2d} + \sin^{-1}\frac{\sqrt{3}}{d}\right) + \left(\frac{\pi}{9\sqrt{3}} - \frac{1}{3}\right) d^2 & \text{if } 2 \leq d \leq \sqrt{7} \\
0 & \text{otherwise} 
\end{cases}
\]  

\( (2) \)
The corresponding CDF is

\[
F_{D_P}(d) = \begin{cases} 
\frac{8}{9} d^3 - \left( \frac{1}{3} + \frac{\pi}{18\sqrt{3}} \right) d^4 & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
- \frac{1}{\sqrt{3}} (4d^2 + d^4) \sin^{-1} \frac{\sqrt{2}}{2d} + \left( \frac{4\pi}{9\sqrt{3}} - \frac{1}{3} \right) d^4 + \frac{8}{9} d^3 + \frac{2\pi}{\sqrt{3}} d^2 & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
\frac{2d^4}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{2}}{2d} + \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^4 - \frac{16}{9} d^3 + \left( \frac{2\pi}{3\sqrt{3}} + \frac{1}{2} \right) d^2 & 1 \leq d \leq \sqrt{3} \\
- \frac{16}{9} d^3 + \left( \frac{2}{3} - \frac{\pi}{9\sqrt{3}} \right) d^2 + \frac{6d^2 + 2}{16} \sqrt{4d^2 - 3} + \frac{13d^2 + 6}{18} \sqrt{d^2 - 3} - \frac{23}{18} & 2 \leq d \leq \sqrt{7} \\
0 & \text{otherwise}
\end{cases}
\]

(4)

C. \(|PQ|\): Distance Distribution between Two Long-Diag Adjacent Rhombuses

The probability density function of the random distances between two uniformly distributed points, one in each of the two adjacent unit rhombuses that have a common long diagonal, is

\[
f_{D_{LD}}(d) = 2d \begin{cases} 
\left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^2 & 0 \leq d \leq 1 \\
- \frac{4d^2}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{\pi}{\sqrt{3}} - 1 \right) d^2 + \frac{8}{3} d - \frac{\sqrt{d^2 - 3}}{3} - 1 & 1 \leq d \leq \sqrt{3} \\
\frac{4}{\sqrt{3}} \left( \frac{d^2}{3} - 2 \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^2 + \frac{8}{3} d - \frac{7}{3} \sqrt{4d^2 - 3} + \frac{4\pi}{3\sqrt{3}} + 1 & \sqrt{3} \leq d \leq 2 \\
\frac{4}{3\sqrt{3}} \left( \frac{d^2}{3} - 2 \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{2d^2}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{d} + \left( 1 - \frac{\pi}{\sqrt{3}} \right) d^2 & 2 \leq d \leq \sqrt{7} \\
- \frac{7}{3} \sqrt{4d^2 - 3} + \frac{2}{3} \sqrt{d^2 - 3} + \frac{4\pi}{3\sqrt{3}} + 3 & \sqrt{7} \leq d \leq 2\sqrt{3} \\
\frac{2}{\sqrt{3}} \left( 4 - \frac{d^2}{3} \right) \sin^{-1} \frac{\sqrt{3}}{d} + \left( \frac{\pi}{9\sqrt{3}} - \frac{1}{3} \right) d^2 + 2\sqrt{d^2 - 3} - \frac{4\pi}{3\sqrt{3}} - 2 & \\
0 & \text{otherwise}
\end{cases}
\]

(5)
The corresponding CDF is

\[
F_{D_{LD}}(d) = \begin{cases} 
\left(\frac{1}{6} - \frac{\pi}{18\sqrt{3}}\right) d^4 & 0 \leq d \leq 1 \\
-\frac{2d^4}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{2d} + \left(\frac{\pi}{6\sqrt{3}} - \frac{1}{2}\right) d^4 + \frac{16}{9} d^3 - d^2 - \frac{10d^2 - 3}{36} \sqrt{4d^2 - 3} + \frac{1}{12} & 1 \leq d \leq \sqrt{3} \\
\frac{2}{\sqrt{3}} \left(d^4 - 4d^2\right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left(\frac{1}{6} - \frac{\pi}{18\sqrt{3}}\right) d^4 + \frac{16}{9} d^3 + \left(1 + \frac{4\pi}{3\sqrt{3}}\right) d^2 - \frac{6d^2 + 3}{4} \sqrt{4d^2 - 3} + \frac{19}{12} & \sqrt{3} \leq d \leq 2 \\
\frac{2}{\sqrt{3}} \left(d^4 - 4d^2\right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{d^4}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{d} + \left(\frac{1}{6} - \frac{\pi}{6\sqrt{3}}\right) d^4 + \left(3 + \frac{4\pi}{3\sqrt{3}}\right) d^2 - \frac{6d^2 + 3}{9} \sqrt{4d^2 - 3} + \frac{5d^2 - 6}{9} \sqrt{d^2 - 3} + \frac{11}{12} & 2 \leq d \leq \sqrt{7} \\
\frac{1}{\sqrt{3}} \left(8d^2 - \frac{d^4}{3\sqrt{3}}\right) \sin^{-1} \frac{\sqrt{3}}{d} + \left(\frac{\pi}{18\sqrt{3}} - \frac{1}{6}\right) d^4 - \left(\frac{4\pi}{3\sqrt{3}} + 2\right) d^2 + \frac{11d^2 + 30}{9} \sqrt{d^2 - 3} - 5 & \sqrt{7} \leq d \leq 2\sqrt{3} \\
0 & \text{otherwise}
\end{cases}
\]

(6)

D. |MN|: Distance Distribution between Two Short-Diag Adjacent Rhombuses

The probability density function of the random distances between two uniformly distributed points, one in each of the two adjacent unit rhombuses that have a common short diagonal, is

\[
f_{D_{SD}}(d) = 2d \begin{cases} 
\left(\frac{1}{3} + \frac{2\pi}{9\sqrt{3}}\right) d^2 & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
\frac{8d^2}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{2d} + \left(\frac{1}{3} - \frac{10\pi}{9\sqrt{3}}\right) d^2 + \frac{2}{3} \sqrt{4d^2 - 3} & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
-\frac{4}{\sqrt{3}} \left(d^2 + 2\right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left(\frac{1}{3} + \frac{2\pi}{9\sqrt{3}}\right) d^2 + \frac{8}{3} d - 3 \sqrt{4d^2 - 3} + \frac{8\pi}{3\sqrt{3}} + 1 & 1 \leq d \leq \sqrt{3} \\
\frac{8}{\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{d} - d^2 + \frac{8}{3} d + \frac{8}{3} \sqrt{d^2 - 3} - \frac{8\pi}{3\sqrt{3}} - 4 & \sqrt{3} \leq d \leq 2 \\
0 & \text{otherwise}
\end{cases}
\]

(7)
The corresponding CDF is

\[
F_{DSD}(d) = \begin{cases}
\left( \frac{1}{6} + \frac{\pi}{9\sqrt{3}} \right) d^4 & 0 \leq d \leq \frac{\sqrt{3}}{2} \\
\frac{4d^4}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{1}{6} - \frac{5\pi}{9\sqrt{3}} \right) d^4 + \frac{16d^2-3}{18} \sqrt{4d^2 - 3} & \frac{\sqrt{3}}{2} \leq d \leq 1 \\
-\frac{2}{\sqrt{3}} \left( \frac{d^3}{3} + 4d^2 \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{1}{6} + \frac{\pi}{9\sqrt{3}} \right) d^4 + \frac{16}{9} d^2 + \left( \frac{8\pi}{9\sqrt{3}} + 1 \right) d^2 - \frac{74d^2 + 21}{36} \sqrt{4d^2 - 3} + \frac{1}{4} & 1 \leq d \leq \sqrt{3} \\
\frac{8d^2}{\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{d} - \frac{d^4}{2} + \frac{16}{9} d^3 - \left( 4 + \frac{8\pi}{3\sqrt{3}} \right) d^2 + \frac{16d^2 + 24}{9} \sqrt{d^2 - 3} + 1 & \sqrt{3} \leq d \leq 2 \\
0 & \text{otherwise}
\end{cases}
\]

(8)

Note that although unit rhombuses are assumed throughout (1)–(8), the distance distribution functions can be easily scaled by a nonzero scalar, for rhombuses of arbitrary side length. For example, let the side length of such rhombuses be \( s > 0 \), then

\[
F_{sD}(d) = P(sD \leq d) = P(D \leq \frac{d}{s}) = F_D\left(\frac{d}{s}\right).
\]

Therefore,

\[
f_{sD}(d) = F'_D\left(\frac{d}{s}\right) = \frac{1}{s} f_D\left(\frac{d}{s}\right).
\]

(9)

III. VERIFICATION AND VALIDATION

A. Verification by Simulation

Figure 2 plots the probability density functions, as given in (1), (3), (5) and (7), respectively, of the four random distance cases shown in Fig. 1. Figure 3 shows a comparison between the cumulative distribution functions (CDFs) of the random distances, and the simulation results by generating 1,000 pairs of random points with the corresponding geometric locations as illustrated in Fig. 1. Figure 3 demonstrates that our distance distribution functions are very accurate when compared with the simulation results.
Fig. 2: Distributions of Random Distances Associated with Rhombuses.

B. Validation by Recursion

When looking at Fig. 1, we find that the four adjacent rhombuses together resemble a larger rhombus, with a side length of 2. According to (9), the distance distribution in the large rhombus is $f_{2D}(d) = \frac{1}{2} f_{D_i}(\frac{d}{2})$. On the other hand, if we look at the two random endpoints of a given link inside the large rhombus, they will fall into one of the four individual cases: both endpoints are inside the same small rhombus, with probability $\frac{1}{4}$; the two endpoints fall into two parallel
rhombuses, with probability \( \frac{1}{7} \); and the two endpoints fall into two diagonal rhombuses (either long or short-diag), both with probability \( \frac{1}{8} \). Thus the distance density function for the large rhombus can be given by a probabilistic sum, \( f_{2D}(d) = \frac{1}{4} f_{D_1}(d) + \frac{1}{2} f_{D_P}(d) + \frac{1}{8} f_{D_{LD}}(d) + \frac{1}{8} f_{D_{SD}}(d) \), where \( f_{D_1}(d) \), \( f_{D_P}(d) \), \( f_{D_{LD}}(d) \) and \( f_{D_{SD}}(d) \) are given in (1), (3), (5) and (7), respectively.

To confirm that the above two definitions of \( f_{2D}(d) \) are equivalent, i.e., \( \frac{1}{4} f_{D_1}(d) + \frac{1}{2} f_{D_P}(d) + \frac{1}{8} f_{D_{LD}}(d) + \frac{1}{8} f_{D_{SD}}(d) \) is equal to \( \frac{1}{2} f_{D_1}(\frac{d}{2}) \), we verify them mathematically as follows.

1) \( 0 \leq d \leq \frac{\sqrt{3}}{2} \):

\[
\frac{1}{4} f_{D_1}(d) = \frac{d}{2} \left[ \left( \frac{4}{3} + \frac{2\pi}{9\sqrt{3}} \right) d^2 - \frac{16}{3} d + \frac{2\pi}{\sqrt{3}} \right],
\]

\[
\frac{1}{2} f_{D_P}(d) = d \left[ \frac{4}{3} d - \left( \frac{2}{3} + \frac{\pi}{9\sqrt{3}} \right) d^2 \right],
\]

\[
\frac{1}{8} f_{D_{LD}}(d) = \frac{d}{4} \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^2,
\]

\[
\frac{1}{8} f_{D_{SD}}(d) = \frac{d}{4} \left( \frac{1}{3} + \frac{2\pi}{9\sqrt{3}} \right) d^2.
\]

Thus,

\[
f_{2D}(d) = \frac{1}{4} f_{D_1}(d) + \frac{1}{2} f_{D_P}(d) + \frac{1}{8} f_{D_{LD}}(d) + \frac{1}{8} f_{D_{SD}}(d)
\]

\[
= \frac{d}{2} \left[ \left( \frac{4}{3} + \frac{2\pi}{9\sqrt{3}} \right) \left( \frac{d}{2} \right)^2 - \frac{16}{3} \left( \frac{d}{2} \right) + \frac{2\pi}{\sqrt{3}} \right] = \frac{1}{2} f_{D_1}(\frac{d}{2}).
\]

2) \( \frac{\sqrt{3}}{2} \leq d \leq 1 \):

\[
\frac{1}{4} f_{D_1}(d) = \frac{d}{2} \left[ \frac{8}{\sqrt{3}} \left( 1 + \frac{d^2}{3} \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{4}{3} - \frac{10\pi}{9\sqrt{3}} \right) d^2 - \frac{16}{3} d + \frac{10}{3} \sqrt{d^2 - 3} - \frac{2\pi}{\sqrt{3}} \right],
\]

\[
\frac{1}{2} f_{D_P}(d) = d \left[ -\frac{2}{\sqrt{3}} (d^2 + 2) \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{8\pi}{9\sqrt{3}} - \frac{2}{3} \right) d^2 + \frac{4}{3} d - \frac{11}{6} \sqrt{d^2 - 3} + \frac{2\pi}{\sqrt{3}} \right],
\]

\[
\frac{1}{8} f_{D_{LD}}(d) = \frac{d}{4} \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^2, \quad \frac{1}{8} f_{D_{SD}}(d) = \frac{d}{4} \left[ \frac{8d^2}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{1}{3} - \frac{10\pi}{9\sqrt{3}} \right) d^2 + \frac{2}{3} \sqrt{d^2 - 3} \right].
\]
Thus,

\[ f_{2D}(d) = \frac{1}{4} f_{D_1}(d) + \frac{1}{2} f_{D_p}(d) + \frac{1}{8} f_{D_{LD}}(d) + \frac{1}{8} f_{D_{SD}}(d) \]

\[ = \frac{d}{2} \left[ \left( \frac{4}{3} + \frac{2\pi}{9\sqrt{3}} \right) \left( \frac{d}{\sqrt{2}} \right)^2 - \frac{16}{3} \left( \frac{d}{\sqrt{2}} \right) + \frac{2\pi}{\sqrt{3}} \right] = \frac{1}{2} f_{D_1}(\frac{d}{\sqrt{2}}). \]

3) \( 1 \leq d \leq \sqrt{3} \):

\[ \frac{1}{4} f_{D_1}(d) = \frac{d}{2} \left[ \frac{4}{\sqrt{3}} \left( 1 - \frac{d^2}{3} \right) \sin^{-1} \frac{\sqrt{3}}{2d} - \left( \frac{2}{3} - \frac{2\pi}{9\sqrt{3}} \right) d^2 + \sqrt{4d^2-3} - \frac{2\pi}{3\sqrt{3}} - 1 \right], \]

\[ \frac{1}{2} f_{D_p}(d) = d \left[ \frac{4d^2}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{2}{3} - \frac{2\pi}{9\sqrt{3}} \right) d^2 + \sqrt{4d^2-3} - \frac{2\pi}{3\sqrt{3}} + \frac{1}{2} \right], \]

\[ \frac{1}{8} f_{D_{LD}}(d) = d \left[ \frac{-4d^2}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{\pi}{3\sqrt{3}} - 1 \right) d^2 + \frac{2\pi}{3} - 3\sqrt{4d^2-3} + 1 \right], \]

\[ \frac{1}{8} f_{D_{SD}}(d) = d \left[ -\left( \frac{4d^2}{3\sqrt{3}} + \frac{8}{\sqrt{3}} \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{1}{3} + \frac{2\pi}{9\sqrt{3}} \right) d^2 + \frac{8}{3} d - 3\sqrt{4d^2-3} + \frac{8\pi}{3\sqrt{3}} + 1 \right]. \]

Thus,

\[ f_{2D}(d) = \frac{1}{4} f_{D_1}(d) + \frac{1}{2} f_{D_p}(d) + \frac{1}{8} f_{D_{LD}}(d) + \frac{1}{8} f_{D_{SD}}(d) \]

\[ = \frac{d}{2} \left[ \left( \frac{4}{3} + \frac{2\pi}{9\sqrt{3}} \right) \left( \frac{d}{\sqrt{2}} \right)^2 - \frac{16}{3} \left( \frac{d}{\sqrt{2}} \right) + \frac{2\pi}{\sqrt{3}} \right] = \frac{1}{2} f_{D_1}(\frac{d}{\sqrt{2}}). \]

4) \( \sqrt{3} \leq d \leq 2 \): \( f_{D_1}(d) = 0 \), and

\[ \frac{1}{2} f_{D_p}(d) = d \left[ \left( \frac{2}{\sqrt{3}} - \frac{d^2}{3\sqrt{3}} \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{2}{\sqrt{3}} + \frac{d^2}{3\sqrt{3}} \right) \sin^{-1} \frac{\sqrt{3}}{d} + \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^2 - \frac{8}{3} d \right. \]

\[ \left. + \sqrt{d^2-3} + \frac{7}{12} \sqrt{4d^2-3} + \frac{3}{4} - \frac{2\pi}{3\sqrt{3}} \right], \]

\[ \frac{1}{8} f_{D_{LD}}(d) = d \left[ \frac{4d^2}{\sqrt{3}} \left( \frac{d^2}{3} - 2 \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \left( \frac{1}{3} - \frac{\pi}{9\sqrt{3}} \right) d^2 - \frac{7}{3} \sqrt{4d^2-3} + \frac{8}{3} d + \frac{4\pi}{3\sqrt{3}} + 1 \right], \]
\[
\frac{1}{8} f_{D_{SD}}(d) = \frac{d}{4} \left[ \frac{8}{\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{d} - d^2 + \frac{8}{3} d + \frac{8}{3} \sqrt{d^2 - 3} - \frac{8\pi}{3\sqrt{3}} - 4 \right].
\]

Thus,

\[
f_{2D}(d) = \frac{1}{2} f_{D_{P}}(d) + \frac{1}{8} f_{D_{LD}}(d) + \frac{1}{8} f_{D_{SD}}(d)
\]

\[
= \frac{d}{2} \left[ \frac{8}{\sqrt{3}} \left( 1 + \frac{(d/2)^2}{3} \right) \sin^{-1} \frac{\sqrt{3}}{2(d/2)} + \left( \frac{4}{3} - \frac{10\pi}{9\sqrt{3}} \right) \left( \frac{d}{2} \right)^2 - \frac{16}{3} \left( \frac{d}{2} \right) \\
+ \frac{10}{3} \sqrt{4 \left( \frac{d}{2} \right)^2 - 3 - \frac{2\pi}{\sqrt{3}}} \right] = \frac{1}{2} f_{D_{I}}(\frac{d}{2}).
\]

5) \(2 \leq d \leq \sqrt{7} \): \(f_{D_{I}}(d) = f_{D_{SD}}(d) = 0\), and

\[
\frac{1}{2} f_{D_{P}}(d) = \frac{d}{2} \left[ \frac{2}{\sqrt{3}} \left( \frac{d^2}{3} - 3 \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{\pi}{9\sqrt{3}} \right] d^2 + \frac{7}{12} \sqrt{4d^2 - 3} \\
+ \sqrt{\frac{d^2 - 3}{3}} - \frac{2\pi}{3\sqrt{3}} - \frac{5}{4},
\]

\[
\frac{1}{8} f_{D_{LD}}(d) = \frac{d}{4} \left[ \frac{4}{\sqrt{3}} \left( \frac{d^2}{3} - 2 \right) \sin^{-1} \frac{\sqrt{3}}{2d} + \frac{2d^2}{3\sqrt{3}} \sin^{-1} \frac{\sqrt{3}}{d} + \left( 1 - \frac{\pi}{3\sqrt{3}} \right) d^2 - \frac{7}{3} \sqrt{4d^2 - 3} \\
+ \frac{2}{3} \sqrt{d^2 - 3} + \frac{4\pi}{3\sqrt{3}} + 3 \right].
\]

Thus,

\[
f_{2D}(d) = \frac{1}{2} f_{D_{P}}(d) + \frac{1}{8} f_{D_{LD}}(d)
\]

\[
= \frac{d}{2} \left[ \frac{4}{\sqrt{3}} \left( 1 - \frac{(d/2)^2}{3} \right) \sin^{-1} \frac{\sqrt{3}}{d} - \left( \frac{2}{3} - \frac{2\pi}{9\sqrt{3}} \right) \left( \frac{d}{2} \right)^2 + \sqrt{4 \left( \frac{d}{2} \right)^2 - 3 - \frac{2\pi}{3\sqrt{3}} - 1} \right] \\
= \frac{1}{2} f_{D_{I}}(\frac{d}{2}).
\]
6) $\sqrt{7} \leq d \leq 2\sqrt{3}$: $f_{D_1}(d) = f_{D_p}(d) = f_{D_{SD}}(d) = 0$, and

\[
f_{2D}(d) = \frac{1}{8} f_{D_{LD}}(d) = \frac{d}{4} \left[ \frac{2}{\sqrt{3}} \left( 4 - \frac{d^2}{3} \right) \sin^{-1} \frac{\sqrt{3}}{d} + \left( \frac{\pi}{9\sqrt{3}} - \frac{1}{3} \right) d^2 + 2\sqrt{d^2 - 3} - \frac{4\pi}{3\sqrt{3}} - 2 \right]
\]

\[
= \frac{d}{2} \left[ \frac{4}{\sqrt{3}} \left( 1 - \frac{(d/2)^2}{3} \right) \sin^{-1} \frac{\sqrt{3}}{d} - \left( \frac{2}{3} - \frac{2\pi}{9\sqrt{3}} \right) \left( \frac{d}{2} \right)^2 + \sqrt{\frac{4}{3}} \left( \frac{d}{2} \right)^2 - 3 - \frac{2\pi}{3\sqrt{3}} - 1 \right]
\]

\[
= \frac{1}{2} f_{D_1}(\frac{d}{2}).
\]

In summary, we have $f_{2D}(d) = \frac{1}{2} f_{D_1}(\frac{d}{2})$ by recursion, and the probabilistic sum $\frac{1}{4} f_{D_1}(d) + \frac{1}{2} f_{D_p}(d) + \frac{1}{8} f_{D_{LD}}(d) + \frac{1}{8} f_{D_{SD}}(d)$ is equal to $\frac{1}{2} f_{D_1}(\frac{d}{2})$ in all the cases discussed above. The results are a strong validation of the correctness of the distance distributions that we have derived.

IV. PRACTICAL RESULTS

A. Statistical Moments of Random Distances

The distance distribution functions given in Section II can conveniently lead to all the statistical moments of the random distances associated with rhombuses. Given $f_{D_1}(d)$ in (1), for example, the first moment (mean) of $d$, i.e., the average distance within a single rhombus, is

\[
M_{D_1}^{(1)} = \int_0^{\sqrt{3}} x f_{D_1}(x) dx = \frac{\sqrt{3}}{8} + \frac{3}{40} \left[ 7 \ln \left( 2\sqrt{3} + 3 \right) - 6 \ln \left( 2\sqrt{3} - 3 \right) \right] \approx 0.5123783359,
\]

and the second raw moment is

\[
M_{D_1}^{(2)} = \int_0^{\sqrt{3}} x^2 f_{D_1}(x) dx = \frac{1}{3},
\]

from which the variance (the second central moment) can be derived as

\[
Var_{D_1} = M_{D_1}^{(2)} - \left[ M_{D_1}^{(1)} \right]^2 \approx 0.0708017742.
\]

When the side length of a rhombus is scaled by $s$, the corresponding first two statistical moments given above then become

\[
M_{D_1}^{(1)} = 0.5123783359s, \quad M_{D_1}^{(2)} = \frac{8}{3} \quad \text{and} \quad Var_{D_1} = 0.0708017742s^2.
\]
Table I lists the first two moments, and the variance of the random distances in the four cases given in Section II and the corresponding simulation results for verification purposes.

| Endpoint Geometry | PDF/Sim | \( M_{D}^{(1)} \) | \( M_{D}^{(2)} \) | \( Var_D \) |
|-------------------|---------|------------------|------------------|----------|
| Within a Single Rhombus | \( f_{D1}(d) \) | 0.5123783359s | 0.3333333333s | 0.0708017742s² |
| Between two Parallel Rhombuses | Sim | 0.5137344650s | 0.3356749448s | 0.0715184432s² |
| Between two Long-Diag Adjacent Rhombuses | \( f_{D_{LB}}(d) \) | 1.0750863337s | 1.331823503s | 0.1773717254s² |
| Between two Short-Diag Adjacent Rhombuses | Sim | 1.0749140141s | 1.3318514164s | 0.1764112787s² |

Table II: Coefficients of the Polynomial Fit and the Norm of Residuals (NR)

| PDF | Polynomial Coefficients | NR |
|-----|-------------------------|----|
| \( f_{D1}(d) \) | \(10^8 \times [0.0000166 0.002995 - 0.024296 0.122501 - 0.423700 \) | 0.095901 |
| \( f_{D_{PF}}(d) \) | \(10^8 \times [0.0000019 - 0.000513 0.006194 - 0.045754 0.231329 \) | 0.059485 |
| \( f_{D_{LB}}(d) \) | \(10^4 \times [0.0000002 - 0.0000074 0.00001242 - 0.0012723 0.0089355 \) | 0.011340 |
| \( f_{D_{SB}}(d) \) | \(10^4 \times [0.0000022 - 0.0000493 - 0.0051165 0.032591 - 0.142436 \) | 0.147836 |

B. Polynomial Fits of Random Distances

Table II lists the coefficients of the degree-20 polynomial fits of the original PDFs given in Section II from \( d^{20} \) to \( d^0 \), and the corresponding norm of residuals. Figure 4 (a)–(d) plot the polynomials listed in Table II with the original PDFs. From the figure, it can be seen that all
the polynomials match closely with the original PDFs. These high-order polynomials facilitate further manipulations of the distance distribution functions, with a high accuracy.

V. CONCLUSIONS

In this report, we gave the closed-form probability density functions of the random distances associated with rhombuses. The correctness of the obtained results has been validated by a recursion and a probabilistic sum, in addition to simulation. The first two statistical moments, and the polynomial fits of the density functions are also given for practical uses.
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