Wiener Filter for Short-Reach Fiber-Optic Links

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Abstract—Analytic expressions are derived for the Wiener filter (WF), also known as the linear minimum mean square error (LMMSE) estimator, for an intensity-modulation/direct-detection (IM/DD) short-haul fiber-optic communication system. The link is purely dispersive and the nonlinear square-law detector (SLD) operates at the thermal noise limit. The achievable rates of geometrically shaped PAM constellations are substantially increased by taking the SLD into account as compared to a WF that ignores the SLD.

Index Terms—Digital dispersion equalization, Wiener filter, LMMSE estimator, intensity modulation/direct detection (IM/DD), geometric shaping, short-haul fiber-optic communication.

I. INTRODUCTION

SHORT-REACH fiber-optic communications systems, e.g., for data-center interconnects, usually use transceivers based on intensity-modulation (IM) and direct detection (DD) (e.g., [1], [2]). Compared to coherent transceivers, IM/DD transceivers offer lower power consumption and hardware complexity, smaller form factors and hence reduced overall costs [1], [3]. To further reduce cost and complexity, short-link communication systems are usually operated without optical amplification and dispersion compensating fiber and require signal distortions to be compensated digitally at the receiver.

In short-reach communication systems, inter-symbol-interference (ISI) caused by chromatic dispersion (CD) is the limiting effect [2]–[4]. CD is described by a complex-valued impulse response. A DD receiver consists of a photodiode which measures the intensity of the imminent electrical field, and hence discards phase information. This complicates CD removal. Due to the absence of amplifiers on short-reach links, the square-law detector (SLD) is the only noise source. Since the receive signal is significantly attenuated, the SLD is assumed to operate at the thermal noise limit and adds white Gaussian noise to the intensity measurements [5, P. 154].

Common CD equalizers include linear feed-forward equalization (FFE) or non-linear methods like decision feedback equalizing (DFE), Volterra series based equalization, and neural network based equalization (e.g., [3, Sec. IV], [6]). In this letter, we consider a linear equalizer, namely the minimum mean square error (MMSE) estimator, also known as the Wiener filter (WF). We derive analytic expressions for the WF coefficients for short-reach IM/DD systems. Due to small transmit signal powers, the Kerr nonlinearity of the link can be neglected [5, P. 65] and the link is purely dispersive.

In [7] the authors compute the WF assuming either real-valued Gaussian transmit symbols and a real-valued channel matrix or circularly symmetric complex Gaussian transmit symbols and a complex-valued channel matrix. We consider real-valued transmit symbols, originating from any symmetric probability density function (PDF), and a complex-valued channel matrix and extend the expressions from [7].

Notation: Bold letters indicate vectors and matrices, non-bold letters express scalars. For a matrix $A$, we denote complex conjugate transpose and Hermitian transpose by $A^*$, $A^T$ and $A^H$, respectively. The Hadamard product and the Kronecker product are denoted by $\otimes$ and $\odot$, respectively. Dirac’s delta is expressed by $\delta(t)$ and $\delta(\cdot)$, respectively. By $\langle a(t), a(t) \rangle = \int_{-\infty}^{\infty} |a(t)|^2 dt$ we denote the energy of the vector $a$. The sinc function is defined as $\text{sinc}(\pi s) = \sin(\pi s) / (\pi s)$. The Nyquist ISI-free property of a pulse $g(t)$ with symbol period $T_s$ reads $g(t)|_{t \equiv kT_s} = g(0)\delta[k], \forall k \in \mathbb{Z}$. A pulse $g(t)$ has the Nyquist property, if $g(t) * g^*(-t)$ has the Nyquist property. The Fourier pair $a(t) = \mathcal{F}^{-1}\{A(f)\}$ and $A(f) = \mathcal{F}\{a(t)\}$, is denoted by $a(t) \leftrightarrow A(f)$. By $\mathbb{R}\{A\}$ we denote that the function $f(\cdot)$ is applied element-wise to the set $A$, i.e., $f(\{a\}) = \{f(a)\}$. By $\mathbb{S}\{A\}$ we denote element-wise real and imaginary part of the complex-valued matrix $A$, respectively.

II. SYSTEM MODEL

A. Transmitter Front-End

In Fig. 1, the transmitter is fed with positive and real-valued, random, discrete-time data symbols $s_{\nu}$, where $\nu$ is the discrete time index. We have $s_{\nu} \in \mathbb{S}$ and modulation alphabet $\mathbb{S}$. 1) Digital-to-Analog Converter (DAC): For ideal digital to analog conversion of the $s_{\nu}$, the DAC performs pulse shaping with symbol time $T_s$. The continuous-time DAC output $a(t)$ reads

$$a(t) = s(t) * g_{\text{th}}(t) = \sum_{\nu = -\infty}^{\infty} s_{\nu} g_{\text{th}}(t - \nu T_s)$$

with real-valued pulse-shaping filter $g_{\text{th}}(t)$ and

$$s(t) = \sum_{\nu = -\infty}^{\infty} s_{\nu} \delta(t - \nu T_s).$$
B. Optical Channel

2) Electrical-Optical Converter (EOC): We use an ideal Mach-Zehnder Modulator (MZM) in push-pull mode [8, P. 19] as the EOC, which is shown in Fig. 2. The MZM modulates the amplitude of an electromagnetic carrier wave from a laser. The electric field \( \tilde{q}(t) \) of the laser light reads

\[
\tilde{q}(t) = E_0 \cdot e^{-j \omega_c t}
\]

with \( E_0 = \tilde{E}_0 e^{j \phi E_0(t)} \in \mathbb{C}, \tilde{E}_0 \equiv |E_0| \) and angular frequency \( \omega_c \). Laser fluctuations [5, P. 100] are neglected and we set \( \phi E_0(t) = 0 \). Modulating \( \tilde{q}(t) \) with \( x(t) \) leads to a complex-valued bandpass signal at the fiber input \( z = 0 \) [8, P. 19]:

\[
\tilde{q}(0,t) = \cos \left( x(t) \cdot \frac{\pi}{2V_c} \right) \cdot \tilde{q}(t)
\]

(4)

with hard constant \( V_c \) of the MZM and we set \( V_c = \pi/2 \). We decompose \( x(t) \) into a bias \( \bar{x} = \pi/2 \) and alternating signal \( \tilde{x}(t) \), i.e., \( x(t) = \bar{x} + \tilde{x}(t) \), and use the small angle approximation for \( \cos(\cdot) \) around \( \bar{x} \), i.e., \( \cos(\bar{x} + \tilde{x}(t)) \approx -\tilde{x}(t) \). The approximation error is small for \( |\tilde{x}(t)| \ll \pi/2 \). The bandpass signal \( \tilde{q}(0,t) \), launched into the fiber, reads as

\[
\tilde{q}(0,t) \approx -E_0 \cdot \tilde{x}(t) \cdot e^{-j \omega_c t}
\]

(5)

and we define \( a(t) \equiv -E_0 \tilde{x}(t) \). Neglecting the carrier signal term from (5), we obtain the real-valued baseband signal

\[
q(0,t) = a(t)
\]

(6)

Modulating the amplitude of the baseband electric field modulates the optical intensity \( I(z,t) \) at \( z = 0 \):

\[
I(0,t) = \gamma_{\text{prop}} \cdot |q(0,t)|^2 = \gamma_{\text{prop}} \cdot a(t)^2
\]

(7)

with constant \( \gamma_{\text{prop}} \equiv 1 \). For invertible relationships between amplitude and intensity, i.e., real-valued non-negative \( a(t) \), this scheme is referred to as IM [8, P. 20]. Using a MZM, we require \( a(t) \) to only be real-valued, but justify a non-negativity condition of the transmit symbols in Sec. IV-B.

B. Optical Channel

The propagation of the slowly varying signal \( q \triangleq q(z,t) \) is described by the nonlinear Schrödinger equation [5, P. 65]

\[
\frac{\partial q}{\partial z} = -j \beta_2 \frac{\partial^2 q}{\partial t^2} + j \gamma |q|^2 q - \frac{\alpha}{2} q + n
\]

(8)

where \( \beta_2 \) is the CD coefficient, \( \gamma \) is the Kerr nonlinearity parameter, \( \alpha \) accounts for fiber-loss and \( z \) is the propagated distance. The term \( n \triangleq n(z,t) \) describes noise realizations. The Kerr nonlinearity can be neglected for small optical transmit powers \( P_{\text{tx,opt}} \) [5, P. 65]. With no amplification along the fiber, the dominant noise is added by the SLD at the receiver [1]. We thus set \( n = 0 \) and model electrical noise of the SLD in the following section. Finally, we consider attenuation in the signal-to-noise ratio (SNR) definition at the receiver and therefore simplify (8) to a linear differential equation

\[
\frac{\partial q}{\partial z} = -j \beta_2 \frac{\partial^2 q}{\partial t^2} \frac{\gamma}{2}
\]

(9)

which can be solved analytically in the Fourier domain as

\[
Q(L,\omega) = Q(0,\omega) \cdot e^{j \frac{\beta_2}{2} \omega^2 L}
\]

(10)

with \( Q(\cdot,\omega) \mapsto q(\cdot,\omega) \), frequency response of CD \( H(L,\omega) \triangleq e^{j \frac{\beta_2}{2} \omega^2 L}, H(L,\omega) \mapsto h(L,t) \) and fiber length \( L \).

C. Receiver Front-End

The receiver performs optical to electrical conversion (OEC), digitizes the signal by an ideal analog-to-digital converter (ADC) and recovers the transmitted data by DSP. The receiver in Fig. 3 consists of a p-i-n SLD [5, P. 153], with output current \( r'(t) \) proportional to the intensity of the impinging electrical field, i.e., \( r'(t) = \gamma_{\text{pd}} \cdot |q(L,t)|^2 \), and proportionality constant \( \gamma_{\text{pd}} \equiv 1 \) [5, Eq. (4.1.2-3)]. With no amplifiers on the fiber, \( q(L,t) \) at the fiber end will be significantly attenuated compared to \( q(0,t) \), which allows consideration of the receiver at the thermal noise limit [5, P. 154]. Therefore, \( \eta'(t) \) is described by a white Gaussian random process with two-sided power spectral density (PSD) \( \Phi_{\eta'\eta'}(f) = N_0/2 \), autocorrelation function (ACF) \( \phi_{\eta'\eta'}(\tau) = (N_0/2)\delta(\tau) \), time-lag \( \tau \) and \( N_0 \) as stated in [5, Eq. 4.4.7]. With bandwidth limitation in the electrical filter \( g_{\text{rx}}(t) \), prior to the ADC, the noise energy is finite and communication viable.

III. Discrete-Time System Model

A. Nyquist System

1) Linear Case: A linear communication system (Fig. 4 without SLD) with \( T_s \)-spaced sampling at the receiver has zero ISI, maximum SNR, and additive white Gaussian noise (AWGN) at the sampling times for \( q(t) = g_{\text{rx}}(t) \ast h(L,t) \) being \( \sqrt{\text{Nyquist}} \) and \( g_{\text{rx}}(t) \) the matched filter [9, PP. 175].
2) **Nonlinear Case:** Consider the SLD, zero CD and a single
symbol $s_k$ being pulse-shaped and transmitted, which requires a
$\sqrt{\text{Nyquist}}$ $r'(t)$ and $g_{\text{rx}}(t)$ as the matched filter. How-
ever, [10] showed that nonnegative, *bandlimited* $\sqrt{\text{Nyquist}}$ pulses do not exist and that practical nonnegative $\sqrt{\text{Nyquist}}$
pulses are time-limited to $T_s$. However, zero ISI and AWGN
at the receiver at multiples of $T_s$ is $1/B$ are still viable,
when, e.g., choosing $g_{\text{rx}}(t)$ as a $\text{sinc}$ pulse shaping filter
with bandwidth $B$, and $g_{\text{tx}}(t)$ as a $\text{sinc}$ filter with bandwidth
$2B$. Since $\text{sinc}$ filters fulfill Nyquist and $\sqrt{\text{Nyquist}}$ criterion,
the zero-ISI condition at $T_s$-spaced sampling is met and the
receiver noise samples are uncorrelated, which we show in the
following Sec. IV-B.

3) **Nonlinear and Dispersive Case:** With CD, zero ISI at
multiples of $T_s$ is not viable, as a real-valued $g_{\text{tx}}(t)$ cannot
form an ISI-free filter with the complex-valued CD $h(L, t)$.
Thus, we formulate an optimization problem for the
remaining ISI.

**B. Bandlimited Sampling Receiver**

We choose $g_{\text{tx}}(t) = \text{sinc}(B\pi t)$ as the bandlimited
transmitter pulse shaping filter, with two-sided bandwidth $B = 1/T_s$. With the number of transmit symbols $V$, the average
optical transmit power $P_{\text{tx,opt}}$ per symbol reads

$$P_{\text{tx,opt}} \triangleq \lim_{V \to \infty} \frac{1}{V T_s} \mathbb{E}_a \left[ |a(t)|^2 \right] = \sigma_a^2 + \mu_s^2$$  \hspace{1cm} (11)

We therefore adjust the average fiber launch power with the
symbol variance $\sigma_a^2$ and mean $\mu_s$. The receive filter $g_{\text{rx}}(t)$
has unit frequency gain and sets the receiver bandwidth to $2B$, i.e.,

$$g_{\text{rx}}(t) = 2B \cdot \text{sinc}(2B\pi t) \quad \Rightarrow \quad G_{\text{rx}}(f) = \begin{cases} 1, & |f| \leq B \\ 0, & \text{otherwise} \end{cases}$$   \hspace{1cm} (12)

given that the bandwidth from $q(L, t)$ to $r'(t)$ is doubled by the
$| \cdot |^2$-operation of the SLD. Thus, the receiver is a bandlimited
sampling receiver, sampling $u(t)$ at $N_{\text{os}} \times T_s$, where $N_{\text{os}} \geq 2$ to avoid aliasing. The noise energy in $u(t)$ is thus finite.
With real-valued transmit symbols, the PSD $\Phi_{\eta \eta}(f)$ and ACF
$\phi_{\eta \eta}(\tau)$ of the bandlimited real-valued noise read [9, p. 233],

$$\Phi_{\eta \eta}(f) = \begin{cases} \frac{N_{\text{os}}}{2}, & |f| \leq B \\ 0, & \text{otherwise} \end{cases} \quad \Rightarrow \quad \phi_{\eta \eta}(\tau) = \frac{N_{\text{os}}}{2} \phi_{\text{gs},\text{gs}}(\tau)$$  \hspace{1cm} (13)

with $\phi_{\eta \eta}(\tau) = \phi_{\eta \eta}^t(\tau) \ast \phi_{\text{gs},\text{gs}}(\tau)$, $\phi_{\text{gs},\text{gs}}(\tau) = g_{\text{tx}}(\tau)$.

**C. Discrete-Time Formulation**

Sampling at $t = n T_s'$ with $T_s' = T_s/N_{\text{os}}$ and $N_{\text{os}} \geq 2$ gives

$$u(\kappa T_s') = u[\kappa] = r'[\kappa] + \eta[\kappa]$$  \hspace{1cm} (14)

where the discrete-time instant is $\kappa \in \mathbb{Z}$ and $\eta[\kappa]$ is zero-mean
real-valued Gaussian noise with ACF $\phi_{\eta \eta}(\tau) = \phi_{\eta \eta}(\tau = n T_s')$.
For $N_{\text{os}} = 2$, the noise is white, i.e., $\phi_{\eta \eta}(\tau) = N_0 B [\delta(\tau)]$
and therefore $\eta[\kappa] \sim \mathcal{N}(0, \sigma_\eta^2)$ with $\sigma_\eta^2 = N_0 B$ [9, p. 233].
Choosing $N_{\text{os}} > 2$ gives correlated noise and we thus set $N_{\text{os}} = 2$ for all remaining discussions. The noise-free $r'[\kappa]$ reads

$$r'[\kappa] = \sum_{m=0}^{M-1} \psi[L, m] \cdot s'[\kappa - m]$$  \hspace{1cm} (15)

where $s'_{\kappa} = s'[(\kappa T_s')]$ by (2), i.e., the, $N_{\text{os}}$-times
upsampled version of the sequence \{$s_{\nu-1}, s_{\nu}, \ldots \}$, and
$\psi[L, m] = \psi(\kappa T_s')$ is the sampled, length $M$, combined
impulse response (CIR) $\psi(L, t)$ of the transmitter pulse
shaping filter and CD, i.e., $\psi(L, t) \leftrightarrow \mathcal{H}(L, \omega) \cdot G_{\text{tx}}(\omega)$.
Henceforth, we omit the argument $L$. Arranging $r'[\kappa] = \left[r'_{\kappa}, \ldots, r'_{\kappa + K - 1}\right]_T$ gives

$$r'[\kappa] = \left| \Psi' \cdot s'_{\kappa} \right|^2 \in \mathbb{R}^K$$  \hspace{1cm} (16)

with standard linear convolution matrix $\Psi' \in \mathbb{C}^{K \times N}$, $N = K + M - 1$ with (right) shifted versions of the CIR vector
$\psi = [\psi_{M-1}, \ldots, \psi_0]^T$ arranged as its rows. The vector
$s'_{\kappa} = [s'_{\kappa-M+1}, \ldots, s'_{\kappa}, \ldots, s'_{\kappa + K - 1}]_T \in \mathbb{R}^N$ contains
the upsampled transmit symbols. Note that by zero-insertion of
upsampling,

$$s_{\kappa + \Delta} = \begin{cases} s_{\kappa}, & \text{if } (\kappa + \Delta) \text{ is even} \\ 0, & \text{otherwise} \end{cases}$$  \hspace{1cm} (17)

with $\nu = \frac{\pi}{2} \Delta$ and $\Delta \in \mathbb{Z}$, following from $s'_{\kappa} = s_{\kappa T_s'}$ by (2). We thus remove entries with odd index from the vector $s'_{\kappa}$
and denote the result as $s[\kappa] \in \mathbb{R}^{N_{\text{os}}}$ in $\mathbb{N}_{\text{os}}$. We also discard the according columns of $\Psi'$ to get $\Psi \in \mathbb{R}^{K \times N_{\text{os}}}$.
Note that $| \cdot |^2$ is applied element-wise. The input-output relationship reads as

$$u[\kappa] = r'[\kappa] + \eta[\kappa] = \left| \Psi \cdot s[\kappa] \right|^2 + \eta[\kappa] \in \mathbb{R}^K$$  \hspace{1cm} (18)

and $u[\kappa] = \left[u_{\kappa}, \ldots, u_{\kappa + K - 1}\right]^T$ and $\eta[\kappa] = \left[\eta_{\kappa}, \ldots, \eta_{\kappa + K - 1}\right]^T$.

**IV. WF Problem Statement**

We now formulate the optimization problem to obtain the
Wiener Filter [11, p. 382]:

$$\min_{\hat{\boldsymbol{\theta}} \in \mathbb{R}^{g_{\text{tx}}}} \text{MSE} = \mathbb{E}_{u[N], \eta} \left| \hat{\eta}_{\kappa} - s'_{\kappa} \right|^2 \text{MSE} \in \mathbb{R}^{g_{\text{tx}}}$$  \hspace{1cm} (19)

where the MSE is formulated between a single estimate $\hat{\eta}_{\kappa}$ of
$\hat{\eta}_{\kappa} = \mathbf{g}^T u[\kappa] \in \mathbb{R}^{g_{\text{tx}}}$. filter vector $g \in \mathbb{R}^{\mathbb{R}^{K \times 1}}$, filter mean
$g_{\text{m}} \in \mathbb{R}$ and a single transmitted symbol $s'_{\kappa}$. Hence, $s'_{\kappa}$ is linearly estimated from a vector of $K$ measurements $u[\kappa]$.

By (17), optimization (19) needs to only be solved for even $\kappa$. 

A. WF Estimate

Letting $\kappa$ be even, and omitting $\kappa$ for vectors, the solution of (19) reads [11, P. 382],

$$
\begin{equation}
\hat{s}_k = (c_{s_k}^T u C_{uu}^{-1}) \cdot u + \left( \mu_s - c_{s_k}^T u C_{uu}^{-1} \cdot \mu_u \right) \triangleq g^T u + g_m
\end{equation}
$$

where the covariance matrices and mean vector compute as

$$
\begin{equation}
c_{s_k} u = 2 \sigma^2 \mu_s \cdot \Re \{(\Psi \cdot e_{M'}) \circ w^*)^T
\end{equation}
$$

$$
\begin{equation}
\mu_u = \sigma^2 \text{diag}(\Psi \Psi^H) + \mu_2^T |w|^2
\end{equation}
$$

$$
\begin{equation}
C_{uu} = \left( \mu_s - 3 \sigma^2 \left[ \begin{array}{c} 1 \end{array} \right] \right) \cdot \Psi^2 + \mu_2
+ \mu_4 \cdot |w|^4 + \sigma_0^2 I_{K \times K} - \mu_u \mu_u^T
\end{equation}
$$

and $\Psi \in \mathbb{C}^{K \times N'}$, $N' = M' + K' + 1$, with $M' = \left[ \frac{M}{N'} \right]$, $K' = \left[ \frac{K}{N'} \right]$ and vectors $w = \Psi \cdot 1_{N'}$, $z = \Psi \circ w$. For computation of $C_{uu}$ we assumed the transmit symbols $s_v$ to be independent and identically distributed (iid) by a symmetric (around the mean) PDF with mean $\mu_s$ and variance $\sigma^2$ and denote the fourth-order central moment of $s_v$ by $\mu_4$. All quantities (21)–(23) are real-valued, $C_{uu}$ is positive definite and invertible. The noise is iid with $\eta[k] \sim \mathcal{N}(0, \Sigma_{N})$, where $\Sigma_{N} = \sigma_0^2 I_{K \times K}$ and $\sigma_0^2 = N_0 B$. The WF can be pre-computed offline and efficiently applied using the FFT and overlap-add processing. We note that the WF is the optimal affine estimator in the MSE sense. However, it is only Bayesian MSE optimal for jointly Gaussian $u$ and $s_v$, which is not the case here [11, P. 382] and hence better nonlinear estimators will exist.

B. Mismatched WF

The $s_v$ are iid by a symmetric probability mass function (PMF) with support $S$. Considering Fig. 4 without CD, zero noise, and Nyquist property of $g_k(t)$ gives

$$
\begin{equation}
u[k] = \left( \sum_{\nu=-\infty}^{\infty} s_v g_k(\kappa T_k - \nu T_s) \right)^2 = \begin{cases} s_v^2 : \text{even } \kappa \\ 0 : \text{odd } \kappa \end{cases}
\end{equation}
$$

where for even $\kappa$, $\nu = \frac{\pi}{2}$ by (17). The ISI corresponds to $u[k]$ with odd $\kappa$ and is discarded in this simple model. At even $\kappa$, we get the squared transmitted data. Choosing $s \in \mathbb{R}^+$ allows to unambiguously recover the transmitted symbols (already incorporated in Figs. 1, 4). In addition, the receive symbols PMF has support $S' = \{ s \in S \}$, which makes it involved for a linear WF to map $S'$ onto $S$. With AWGN, we also notice that symbols with smaller amplitudes are more affected by noise, which is undesired. From now on we include a pre-distortion block (cf. Fig. 4) in our discussions, which yields

$$
\begin{equation}
s_v = \sqrt{s_v}, \quad \text{with } s_v \in S, \quad s_v \in \sqrt{S}
\end{equation}
$$

where $s_v, s_v$ have mean $\mu_s$, $\mu_s$ and variance $\sigma^2, \sigma^2$, respectively, leading to $s_v^2 \in S$. Note that (25) is applied on the electrical side, i.e., in the transmitter DSP (cf. Fig. 4). Considering (25), the WF needs to be recomputed based on

$$
\begin{equation}
\text{MSE'} = \mathbb{E}_{s,v} \left| s_v - s_v^* \right|^2
\end{equation}
$$

where now in analogy to (17), for even $\kappa$ and $\nu = \frac{\pi}{2}$, we get $s_v^* = s_v \in S$ and $\sqrt{s_v}^* = \sqrt{S} \in \sqrt{S}$. Note that the predistortion is incorporated in $s_v$. To facilitate analytic expressions, we calculate a mismatched WF by approximating $\sqrt{\sqrt{\nu}}$ with a first-order Taylor series around the mean $\mu_s$ of $s_v$.

$$
\begin{equation}
\sqrt{s_v} \approx t_{\alpha} \cdot s_v + t_{\beta}
\end{equation}
$$

with constants $t_{\alpha} = 1/(2\sqrt{\mu_s})$, $t_{\beta} = \sqrt{\mu_s}/2$. We find the mismatched WF by substitutions for all quantities (21)–(23): $\mu_s \rightarrow \mu_s$, $t^2_{\alpha} \rightarrow \sigma_0^2$, $t^2_{\beta} \rightarrow \sigma_0^2$. We obtain the modified $g^T$ and $g_m$ as a function of $\mu_s, \sigma_0^2$ and $\mu_4$.

$$
\begin{equation}
g^T = t_{\alpha} \cdot c_{s_k}^T u C_{uu}^{-1}, \quad g_m = \mu_s - t_{\alpha} \cdot c_{s_k}^T u C_{uu}^{-1} \cdot \mu_u
\end{equation}
$$

C. SNR Definition

Since noise is added on the electrical side, we define the SNR in the electrical domain [5, P. 153] and get the average electrical receive power as

$$
\begin{equation}
\text{P}_{\text{rx,el}} = \frac{1}{K} \sum_{i=1}^{K-1} E_s |r_i'\rho'(T_i)^T|^2 = \frac{\text{tr} (C_{\rho'} r') + \left| \mu_{r'} \right|^2}{N' N_0}
\end{equation}
$$

with $C_{\rho'} = C_{uu} - C_{\eta \eta} \in \mathbb{R} \times \mathbb{R}$. We find (21)–(23) are computed numerically. The electrical SNR after sampling is then given as

$$
\begin{equation}
\text{SNR}_{\text{el}} = \frac{\text{P}_{\text{rx,el}}}{\sigma_0^2}
\end{equation}
$$

For constant $P_{\text{tx,el}} = \sigma^2 + \mu_2^2$, we remark that by (22)–(23), $\text{SNR}_{\text{el}}$ depends on the particular choice of $\sigma^2, \mu_2^2$.

D. Geometric Shaping

The achievable rate for different SNRs depends on the constellation mean $\mu_s$ and its variance $\sigma^2$, with $\sigma^2 + \mu_2^2 = P_{\text{tx,el}}$. In the following, we let $s_v \in S$, where

$$
\begin{equation}
S = \left\{ P_{\text{tx,el}} - \frac{D}{Q-1} \right\}^{Q-1}_{i=0}
\end{equation}
$$

has equal-distance spaced elements and $D \leq 2P_{\text{tx,el}}$ denotes the constellation span. Though not equivalent to rate, we use the error to signal power ratio (ESR),

$$
\begin{equation}
\text{ESR} = \frac{\text{MSE'}}{\sigma_0^2} = 1 - \frac{c_{s_k}^T u C_{uu}^{-1} \cdot c_{s_k}^T u}{\sigma_0^2}
\end{equation}
$$

as a proxy to find good values for $\sigma^2$ and $\mu_s$ for a particular SNR. The ESR expression in closed form enables the solution of a corresponding optimization problem of low computational complexity. Varying $D$ for a fixed $P_{\text{tx,el}}$, we obtain constellations $\sqrt{S}$ with varying spacing and distance from zero (cf. Fig. 5 (b)), thus varying $\mu_s$ and $\sigma_0^2$. At low SNR, larger constellation spacing helps mitigate the effects of AWGN; at high SNR, constellations further away from zero lead to fewer SLD ambiguities.
PAM. We transmit with shaping and the receiver applies the.

Comparisons are made to the capacity, which is a lower bound on the mutual information of the channel.

\[ C = \max_{\Phi} I(\mathbf{X}, \mathbf{Y}) = I(\mathbf{X}, \mathbf{Y}) \leq \frac{1}{2} \log_2 (1 + I) \]

in analogy to (20) with \( C_{uu} = \frac{1}{2} \mathbf{\Psi} \mathbf{\Psi}^H + C_{\eta \eta} \), \( c_{uu} = \hat{\mu}_2 \mathbf{\Psi} \mathbf{e}_M^H \mathbf{\Psi} \mathbf{e}_M^H \), \( \mathbf{\mu} = \hat{\mu}_2 \mathbf{\Psi} \mathbf{1}_N \), and we keep only the real part of its estimate. For both WFs, we set the number of considered observations to the length of the CIR, i.e., \( K = M \). The sampled CIR \( \psi(L, \kappa T) \) is truncated and only elements larger than \( \frac{1}{100} \cdot \max \{ \psi(L, \kappa T) \} \) are kept in the vector \( \mathbf{\psi} \).

Fig. 5 (a) shows achievable rates in bits per channel use (bpcu) for \( L_{\text{bbo}} = 0 \) km and \( L = 20 \) km plotted against the SNR, when using geometric shaping with optimal normalized constellation span \( D_{\text{norm}} = D/(2\kappa T_{\text{opt}}) \) as given in Fig. 5 (c). Related constellations \( \sqrt{3} \) for \( L = 20 \) km are shown in Fig. 5 (b). For moderate fiber lengths \( L \), the WF achieves the maximum mutual information with finite input alphabets asymptotically in \( \text{SNR}_{\text{el}} \), as it is able to compensate CD and SLD-nonlinearity in the high-SNR regime. In addition, the WF significantly outperforms the suboptimal WF, which saturates to the same rate for all used modulation formats. The achievable rate for the WF and \( L_{\text{bbo}} = 0 \) km has a smaller slope compared to the AWGN channel, which is caused by the ISI of the pulse shaping filter (cf. (24)) and \( N_{\text{os}} = 2 \). Furthermore, for \( L = 20 \) km, the slope decreases further, as the derived WF is only Bayesian MSE optimal for jointly Gaussian observations \( u \) and transmit symbols \( s_y \) (cf. Sec. IV-A).

We also observe that the optimal constellation span in Fig. 5 (c) decreases in the SNR, as predicted in Sec. IV-D. For verification, Fig. 5 (d) shows the empirical ESR, where the MSE' (33) is calculated through simulations. Applying WF with geometric shaping results in a monotonically decreasing ESR, whereas the ESR for WF exhibits an error floor. For reproducible results, all simulations are available on [13].

V. NUMERICAL RESULTS

We consider a standard single-mode fiber (SSMF) at wavelength 1550 nm, with \( \beta_2 = -2.168 \times 10^{-23} \) s^2/km, \( \alpha = 0.046 \) km^{-1}, \( \gamma = 1.27 \) W^{-1}m^{-1}, link length \( L = 20 \) km, receiver oversampling \( N_{\text{os}} = 2 \), symbol rate \( B = f_{\text{sym}} = 27 \) Gbit/s and modulation alphabets \{4,8,16\} unipolar PAM. We transmit \( 100 \times 10^3 \) symbols and set \( \gamma_{\text{eff}} = \phi_{\max}/(1 - \exp(-\alpha L)) \), with maximal phase rotation \( \phi_{\max} = 0.1 \) and \( \gamma_{\text{eff}} = 1 - \exp(-\alpha L) \). For future work, imperfections in the transceiver, especially impairments of the MZM can be addressed. In addition, the impact of the Kerr nonlinearity could be considered at higher transmit powers.

VI. CONCLUSION

We derived the WF, the optimal affine estimator in the MSE sense, for purely dispersive short-haul fiber-optic links with SLD. Together with a transmit constellation optimization, the WF compensates CD and SLD-nonlinearity and achieves the maximum rates for transmission over 20 km. For future work, imperfections in the transceiver, especially impairments of the MZM can be addressed. In addition, the impact of the Kerr nonlinearity could be considered at higher transmit powers.

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