Stochastic heating of cooling flows

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ABSTRACT

It is generally accepted that the heating of gas in clusters of galaxies by active galactic nuclei (AGN) is a form of feedback. Feedback is required to ensure a long term, sustainable balance between heating and cooling. This work investigates the impact of proportional stochastic feedback on the energy balance in the intracluster medium. Using a generalised analytical model for a cluster atmosphere, it is shown that an energy equilibrium can be reached exponentially quickly. Applying the tools of stochastic calculus it is demonstrated that the result is robust with regard to the model parameters, even though they affect the amount of variability in the system.

Key words:

1 INTRODUCTION

The cooling time of gas in the cores of many galaxy clusters is much shorter than the Hubble time. In the absence of heat sources this gas will cool and flow towards the centre of the cluster. However, high resolution X-ray spectroscopy has shown that the rate at which gas cools to low temperatures is much lower than initially expected (Peterson et al. 2001; Tamura et al. 2001; Peterson et al. 2003), suggesting that the gas is being reheated.

Feedback from the active galactic nucleus (AGN) in the central galaxy of the cluster has been considered the most promising mechanism for the reheating of the cooling flow (see, e.g., review by McNamara & Nulsen 2007, and references therein). Modelling of the heating process is difficult because the physical properties of the AGN-cluster interaction and AGN accretion are far from clear. It has been suggested that the cluster gas is heated by outflows, bubbles, sound waves, thermal conduction, turbulence, and (or) a combination of the above mentioned processes (see, e.g., Heinz et al. 2004, Brüggen & Kaiser 2002, Ruszkowski et al. 2004, Voigt et al. 2002, Fujita et al. 2004, Scannapieco & Brüggen 2008). Even though it is not clear how effective these individual processes are in raising temperature of the centre of the cluster, it is clear that the energy deposited by the AGN must be thermalised so that on average the heating and the cooling rates remain equal (or at least comparable).

Currently, the theoretical description of AGN activity in cooling flow clusters is based primarily on results from numerical models. A typical numerical experiment would assume a (magneto-)hydrodynamical model for the intracluster plasma, and find a detailed numerical solution of the appropriate equations. In this framework the relative importance of various specific physical processes can be evaluated and their role in the heating of cooling flows can be consequently assessed. It is prohibitively expensive, however, to cover a large parameter space in the simulations, and accumulation of statistics is often very difficult. While the simulations shed light on some relevant physical phenomena, they are not well suited for answering questions regarding long-term stability, variability, and statistics of the population of clusters.

In this article, an analytical model for the atmosphere of a cluster is constructed to study the stability and evolution of the energy balance driven by AGN feedback. The model behaviour is expressed in terms of stochastic differential equations, which are widely used in studies of fluctuating phenomena, but can appear somewhat obscure at first glance. Consequently, the model is deliberately chosen to be very simple, so that the interpretation of the solutions of the model equations remains robust. This analytical approach is complementary to the more common detailed numerical simulations. It helps to probe otherwise difficult to reach parameter space, and addresses some of the above mentioned questions regarding AGN feedback in clusters. Our approach to these questions in the current work is closely related to the ideas developed by Pope (2007). The proposed answers are statistical in nature, which can be useful in selecting parameters for simulations as well as interpreting their results.

Methods of stochastic calculus have been used extensively in application to problems of statistical physics, chemistry, computational biology and finance. In astrophysics they are mainly used in astroparticle research (see, e.g., Litvinenko 2009) and are quite suitable for investigating parameter space in the context of phenomenological models, as demonstrated in the present work.

The outline of the article is as follows. Section 2 presents
the main model assumptions and derives the equations for deterministic energy balance in the ICM. Section 2 introduces the concept of the stochastic mass deposition rate, discusses briefly the rules of the stochastic calculus, and derives stochastic differential equation for the heating of the ICM. Section 3 using a further simplification, solves the equation, derives formulae for the mean, the probability density function of the heating process and its asymptotic form. Section 4 discusses possible observational implications and limitations of the presented model.

2 TOY MODEL

Consider a relaxed cluster of galaxies. For simplicity the intracluster medium (ICM) in this model is assumed to be an unmagnetised, high temperature gas, confined by the gravitational potential of the dark matter halo, which preserves its spherically symmetric distribution.

The surface \( S_1 \) of the sphere with radius \( r = r_3 \) (see Fig. 1) is considered to be the outer boundary of the ICM, beyond which the gas is no longer gravitationally bound to the cluster. The radial velocities and the densities in the region \( r > r_3 \) are assumed to be negligible, \( v_3 = v(r \geq r_3) \approx 0 \), \( \rho = \rho(r \geq r_3) \approx 0 \). The density of the ICM rises towards the centre of the cluster, and the high temperature ICM cools by emitting X-rays. The cooling of that gas results in the classical cooling flow – a spherically symmetric flow towards the centre of the cluster. The surface \( S_2 \) of the sphere with radius \( r = r_2 \) is located sufficiently far away from the centre of the cluster so that the density and the temperature of the gas in the volume \( V_{outer} \) remains unaffected by the AGN activity. Therefore, the absolute value of the velocity of the cooling flow on this surface \( v_2 = v(r_2) \) is determined only by the cooling rate in the volume \( V_{outer} \), and remains approximately constant. The volume \( V_{inner} \) is determined with the integral rate \( H(t) \neq 0 \), while it also cools with the integral rate \( L(t) \neq 0 \). In general, values of the velocity \( v_1 = v(r_1, t) \), the density \( \rho_1 = \rho(r_1, t) \), and the pressure \( p_1 = p(r_1, t) \) at \( r_1 \) are not constant because of the AGN activity. The region inside \( S_1 \) is excluded from the model.

It is straightforward to write the energy balance equations for the volumes \( V_{outer} \) and \( V_{inner} \) using the generic equation, \[ \frac{dE_V}{dt} = -\int_{S_V} \left( \rho \frac{v^2}{2} + \rho \omega + \rho \psi \right) v \cdot dS, \] where \( E_V \) is the sum of kinetic, thermal and potential energies of the gas inside the volume \( V \), \( \omega \) is the enthalpy of the gas, \( \psi \) is the gravitational potential, \( dS \) is the element of the surface with an external normal. If the velocity of the gas is subsonic almost everywhere, then \( v^2/2 < \omega \), and we can neglect the contribution of the kinetic energy term in the equation (1). Energy balance for the volume \( V_{outer} \) is then given by, \[ -L_{outer} = -(\omega_2 + \psi_2) \dot{M}_2 \] (2) where \( \dot{M}_2 = 4\pi r_2^2 \rho_2 v_2 \) is the constant mass deposition rate through \( S_2 \). For the volume \( V_{inner} \), the energy balance equation can be written as \[ H(t) - L_{inner}(t) = -(\omega_1 + \psi_1) \dot{M}_1(t) + (\omega_2 + \psi_2) \dot{M}_2 \] \[ = -(\omega_1 + \psi_1) \dot{M}_1(t) + L_{outer}, \] (3) where \( \dot{M}_1 = 4\pi r_1^2 \rho_1 v_1(t) \) is the time dependent mass deposition rate through \( S_1 \). Differentiation of both sides of the equation (3) gives, \[ \frac{dH}{dt} + \frac{dL_{inner}}{dt} = -(\omega_1 + \psi_1) \frac{d\dot{M}_1}{dt} + \frac{d\omega_1}{dt} \dot{M}_1(t) \] \[ = (H(t) - L_{inner}(t) - L_{outer}) \frac{1}{\dot{M}_1} \frac{d\dot{M}_1}{dt} \] (4) where, in the last equation, the term \( -(\omega_1 + \psi_1) \) was expressed from the equation (3). It is reasonable to presume that \( H \gg L_{inner} \), as the main contribution to the heating rate integral \( H \) is likely to come from a small volume of gas close to \( S_1 \), in the vicinity of the AGN. Whereas the integral cooling rate function has the entire volume of \( V_{inner} \) as a support, and, therefore, is likely to be less affected by the changes close to \( S_1 \). Making this assumption results in the following differential equation for the heating of the ICM, \[ \frac{dH}{dt} = \left[H(t) - L(t)\right] \frac{1}{M} \frac{d\dot{M}}{dt} - \frac{d\omega}{dt} \dot{M}(t), \] (5) where \( L(t) = L_{outer} + L_{inner}(t) \), and subscripts ‘1’ were dropped to simplify the notation.
If the heating matches the cooling in the inner region $H = L_{\text{inner}} = \text{const}$, then equation (5) reduces to,
\[ -L_{\text{outer}} \frac{1}{M^2} \frac{dM}{dt} = \frac{d\omega}{dt}, \tag{6} \]
which is easily integrable,
\[ \omega(t) = \omega(t_0) + L_{\text{outer}} \left[ \frac{1}{M(t)} - \frac{1}{M(t_0)} \right]. \tag{7} \]
In this case the enthalpy is inversely proportional to the rate of mass inflow. It is also straightforward to get this result directly from the equation (8).

3 STOCHASTIC FEEDBACK

AGN feedback process is thought to recycle the rest-mass energy of the infalling material during the course of its accretion onto the central supermassive black hole. Details of the physics of accretion, methods of the energy transfer and thermalisation in the ICM are far from clear. Nevertheless, as the end product of a feedback cycle, the enthalpy of the ICM must rise and $\dot{M}$ must be reduced. The magnitude of the change of the mass flux, $dM$, is proportional to the magnitude of the AGN “response”, which in turn is likely to be proportional to the “input signal” $\dot{M}$,
\[ dM \propto \dot{M}. \tag{8} \]

Such linear relation between the response and the input signal is found, e.g., in transient X-ray binaries, where the power of the outburst was found to be proportional to the amount of the material in the disk (Shabaz et al. 1998). Although the physical processes responsible for this relationship in X-ray binaries are likely to be quite different from the physics of AGN in clusters, it does suggest that the linear relation between response and the input may be a reasonable assumption.

The proportionality coefficient in the equation (8) is unlikely to be a universal constant. In principle, magnitudes of the responses can vary given the same input $\dot{M}$. Across a population of clusters such variation could be due to the differences in masses and spins of the central black holes. In an individual cluster, the constant of proportionality could also vary because of changes in the state of the accretion disk.

In order to account for a range of possible responses, the value of $dM$ (and, therefore, $\dot{M}$) can be considered to be a random variable parameterised by time, i.e., a stochastic process. The equations of the toy model in combination with a suitable distribution for $\dot{M}$ can then be used to infer properties of the AGN heating. The most unbiased choice of the distribution, which still reflects the property of the proportionality (8) is a uniform distribution of $dM/\dot{M}$.

The uniform distribution corresponds to the “white-noise” stochastic process $\zeta(t)dt; \langle \zeta(t) \rangle = 0, \langle \zeta(t)\zeta(t') \rangle = \delta(t-t')$. Using notations accepted in the theory of stochastic differential equations (see, e.g., Gardiner 2003, for an introduction into the theory and applications of stochastic calculus) this choice of the distribution implies the following stochastic differential equation (SDE),
\[ \frac{d\dot{M}}{\dot{M}} = b dW(t), \tag{9} \]
where $b$ is a constant, and $W(t)$ is a Wiener process.

The Wiener process is defined as a continuous time random walk (Brownian motion) in the limit of infinitesimally small step size. According to the rules of stochastic (also called Itô) calculus the equality $\zeta(t)dt = dW(t)$ is interpreted symbolically as defining the integral relation $W(t) - W(t_0) = \int_0^t \zeta(t')dt'$, because strictly speaking the derivative of $W(t)$ does not exist. The simplest way to find an analytical solution to a SDE is to make a suitable variable substitution. If the substitution transforms the term $f(\cdot)dW(t)$, where $f(\cdot)$ is a function, into $cdW(t)$, where $c$ is a constant, the equation can be easily integrated. According to the rules of stochastic calculus the change can be formally done by expanding the new variable to the second order, and using the following identities: $(dW(t))^2 = dt$, $(dt)^2 = 0$, $dW(t)dt = 0$, the first of which simply states that the variance of Brownian motion is equal to the elapsed time.

Expansion to second order of the function $\mu = \ln \dot{M}$ relating to the stochastic process $\dot{M}$ is given by,
\[ d\mu = \frac{1}{\dot{M}} d\dot{M} - \frac{1}{2\dot{M}^2} (d\dot{M})^2 = b dW(t) - \frac{1}{2} b^2 dt. \tag{10} \]
The last equation can be straightforwardly integrated,
\[ \mu(t) = \mu(t_0) + b[\dot{W}(t) - \dot{W}(t_0)] - \frac{1}{2} b^2 (t - t_0). \tag{11} \]
Changing the variable $\mu$ back to $\dot{M}$ gives the analytical solution of the SDE (9),
\[ \dot{M}(t) = \dot{M}(t_0) e^{b[\dot{W}(t) - \dot{W}(t_0)] - \frac{1}{2} b^2 (t - t_0)/2}. \tag{12} \]
Using the fact that $\dot{W}(t)$ is a normally distributed random variable with zero mean and variance $(\dot{W}(t))^2 = t$, it is easy to calculate the first two moments for $\dot{M}$,
\[ \langle \dot{M}(t) \rangle = \langle \dot{M}(t_0) \rangle, \]
\[ \langle \dot{M}(t)^2 \rangle = \langle \dot{M}(t_0)^2 \rangle e^{b^2 (t - t_0)}. \tag{13} \]
In other words, if $d\dot{M}/\dot{M}$ is a white-noise process, then the mean of $\dot{M}$ remains constant, whereas the variance grows exponentially.

Note that according to equation (12) the mass deposition rate is non-negative, $\dot{M} \geq 0$. The limiting value $\dot{M} = 0$ corresponds to the stable situation, when the heating rate equals the cooling rate exactly. According to the equation (4) if $\dot{M} = 0$ the heating rate is given by $H = L_{\text{inner}} + L_{\text{outer}}$. In the present model, however, the heating ability of AGN is explicitly limited to the inner part of the cluster. The maximum heating rate is, therefore, $H = L_{\text{inner}}$. If $L_{\text{outer}} \neq 0$ the mass deposition rate is always positive, $\dot{M} > 0$, and the cold gas continuously accumulates in the cluster’s centre.

In order to understand how the stochastic behaviour of $\dot{M}$ influences the energy balance in the ICM, it is necessary to make a further assumption about the reaction of the ICM to the change in $\dot{M}$. As was shown above, see (7), in the state of the stable deterministic equilibrium the enthalpy of the ICM, $\omega$, is inversely proportional to $\dot{M}$. By presuming that this is also true in the case of the stochastically behaving $\dot{M}$, the reaction of the enthalpy to the change in $\dot{M}$ can be
calculated using the rules of Itô calculus,
\[
d\omega = -L_{\text{outer}} \frac{1}{M^2} dM + L_{\text{outer}} \frac{1}{M^3} (dM)^2 = -L_{\text{outer}} \frac{1}{M} (bdW - b^2 dt).
\]
Substitution of equations (14) and (11) into equation (5) yields the following SDE for the ICM heating,
\[
dH(t) = [H(t) - L_{\text{inner}}(t)]bdW(t) - b^2 L_{\text{outer}} dt.
\]

4 HEATING PROCESS

Considering for simplicity that \(L_{\text{inner}} \approx \text{const}\), the heating SDE (15) can be rewritten with the dimensionless variables:
\[
h = H/L_{\text{inner}}, \quad \lambda = L_{\text{outer}}/L_{\text{inner}}, \quad \text{and} \quad \tau = b^2 t,
\]
\[
dh(\tau) = [h(\tau) - 1]dW(\tau) - \lambda \, d\tau.
\]

To solve the equation (16) it is best to start by letting \(\lambda = 0\). This corresponds to the scenario when the AGN can heat the entire cluster. The change of the variable \(x = \ln(1-h)\) leads to the following equation,
\[
x = -\frac{1}{1-h} dh - \frac{1}{2(1-h)^2} (dh)^2 = dW(\tau) - \frac{1}{2} d\tau,
\]
which can be easily integrated, giving the solution,
\[
h(\tau) = 1 - (1 - h_0) e^{W(\tau) - \tau/2},
\]
where \(\tau_0 = 0, h_0 = h(0), W(0) = 0\). It is important to note, however, that although the change of variable guarantees that \(h < 1\) it also allows an unphysical situation when \(h < 0\). To exclude solutions with the negative heating it is necessary to treat \(h = 0\) as a boundary condition. Clusters that reach this boundary can be reflected at it (\(h\) turns positive), they can be absorbed at the boundary (\(h\) remains zero), or, possibly, a fraction of clusters can be reflected and the remaining part gets absorbed. The reflective boundary seems to be a natural choice, since the case of the absorbing boundary requires permanent extinction of the active nucleus, while generally \(M \neq 0\), leading to continuous accumulation of the cold gas in the centre. According to the equation (18) the heating remains positive in the range \(W(\tau) < W' = \tau/2 - \ln(1 - h_0)\). Solving (18) for \(W\), and reflecting the function around the point \(W'\), gives the equation for the reflected process, valid in the range \(W > W'\). The full solution of the heating SDE, with reflection at \(h = 0\) (\(W = W'\)), is given by,
\[
h(\tau) = \begin{cases} 1 - (1 - h_0) e^{W(\tau) - \tau/2}, & W(\tau) < W', \\ 1 - \frac{h_0}{1 - h_0} e^{-W(\tau) + \tau/2}, & W(\tau) > W'. \end{cases}
\]
Using the solution (19) it is straightforward to calculate the mean value of the heating,
\[
\langle h(\tau) \rangle = 1 - \frac{h_0}{2} \text{erfc} \left( \frac{\tau/2 + \ln(1 - h_0)}{\sqrt{2\tau}} \right) - \frac{h_0 e^{\tau}}{2(1 - h_0)} \text{erfc} \left( \frac{3\tau/2 + \ln(1 - h_0)}{\sqrt{2\tau}} \right),
\]
which shows that the equilibrium \(h \to 1\) (\(H \to L\)) is approached exponentially quickly as \(\tau \to \infty\), see Fig. 3. Using (19) it is also possible to obtain analytical formulae for the variance and the autocorrelation function, but the expressions are excessively long and will not be reproduced here.

The analysis of the solutions for the SDE can be complemented by considering the probability density function (PDF) of the processes. This is of interest for building a framework with which to interpret observations. For example, the most probable value of the observed heating rate is the modal value, where the PDF peaks. However, if the PDF is skewed, the mode might provide a poor estimate for the long term average heating rate, which is given by the mean of the distribution.

The stochastic process described by the SDE (19) has a PDF \(f_{h,\tau}(h, \tau)\), which satisfies the following Fokker-Plank

![Figure 2](Image311x263 to 538x434)

**Figure 2.** Five sample heating curves (dotted lines) produced using equation (20) with \(h_0 = 0.1\), and the theoretical mean of the heating process (black line) as given by equation (20). Note the change of the variability (dispersion) with time, compared to Fig. 3.

![Figure 3](Image311x520 to 538x691)

**Figure 3.** Two sample heating curves (dotted lines) produced using numerical solutions of equation (19) with \(h_0 = 0.1, \lambda = 0.2\), and the mean of the heating process (black line) computed as an average of 10000 sample heating curves. Note that the amount of the variability (dispersion) remains constant in time, unlike in the case that \(\lambda = 0\), see Fig. 2.
The equation (FPE),
\[ \frac{\partial f_{x\tau}}{\partial \tau} = \lambda \frac{\partial f_{x\tau}}{\partial x} + \frac{1}{2} \frac{\partial^2}{\partial x^2} [(h-1)^2 f_{x\tau}] \],
(21)
(see Gardiner 2005, for a general derivation of the relation between FPE and SDE). Substitution of the variable \( x = \ln(1 - h) \) and the corresponding change the PDF function \( f_{x\tau} = f_{x\tau} e^{-x} \) (which preserves the probability measure) transforms the FPE (21) into the equation
\[ \frac{\partial f_{x\tau}}{\partial \tau} = \frac{\partial}{\partial x} \left( \frac{1}{2} - \lambda e^{-x} \right) f_{x\tau} + \frac{1}{2} \frac{\partial^2 f_{x\tau}}{\partial x^2}, \]
(22)
which describes a diffusion process on the interval \( x \in (-\infty, 0] \) in a medium with constant diffusion coefficient, 1/2, in presence of the inhomogeneous force field, 1/2 - \( \lambda e^{-x} \). The reflective boundary condition at \( x = 0 \) can be satisfied automatically by extending the solution interval to the whole real axis, and reflecting the distribution at 0 so that it becomes an even function. In the case where \( \lambda = 0 \), the equation can be easily solved,
\[ f_{x\tau} = \frac{e^{x^2/2 - \tau/8}}{\sqrt{2\pi \tau}} \int_{-\infty}^{\infty} \left( e^{-(x+\xi)^2/2\tau} + e^{-(x-\xi)^2/2\tau} \right) f_{\xi 0} e^{\xi^2/2} d\xi, \]
(23)
where \( f_{\xi 0} \) is the initial \((\tau = 0)\) PDF. An example of how the PDF changes with time, starting from \( f_{\xi 0} \propto \delta(\xi - 0.1) \), is shown in Fig. 4. For large values of \( \tau \), the PDF becomes a narrow peak at \( h = 1 \), in agreement with the SDE result, \((h) \rightarrow 1\), see (20) and Fig. 2.

The connection between the FPE and the SDE also helps to explain the role of the \((h-1)dW\) term in the original SDE (15). If the term was simply \( dW \) the resulting FPE would be an equation of homogeneous diffusion. This would correspond to a case in which the feedback is not scaled with \( M \). The limiting distribution in this case would be a uniform one. The \( h - 1 \) factor creates a force term in the equation, ensuring that, at all times, diffusion in the direction of \( h = 1 \) is preferred over the opposite direction, no matter what kind of the initial distribution is assumed.

The FPE (21) can be rewritten in the form of a conservation law, \( \partial_t f_{x\tau} = \partial_x E_{x\tau} \), the function \( E_{x\tau} \) is called the probability flux function. In the stationary regime, the time derivative is zero, \( \partial_t f_{x\tau} = 0 \). It describes a statistically stable scenario, which is presumably reached in the limit of large \( \tau \). It follows that in this case \( E_{x\tau} = \text{const} \). Because of the reflective boundary at \( h = 0 \) the fluxes on the left and right of this point must have opposite signs \( E_{x\tau} = -E_{x\tau} = 0 \), and therefore in the stationary case the probability flux is zero. \( J_{x\tau} = 0 \) is an ordinary differential equation, which can be readily solved,
\[ f_{\lambda} \propto \begin{cases} \frac{1}{(1-h)^2}, & \lambda = 0, \\ \frac{1}{(1-h)^2} e^{-2\lambda/(1-h)}, & \lambda \neq 0. \end{cases} \]
(24)
In the case \( \lambda \neq 0 \) the normalised distribution is given by,
\[ f_{\lambda} = \frac{2\lambda}{(1-h)^2} e^{-2\lambda/(1-h)}. \]
(25)
In the case \( \lambda = 0 \) the function \( f_{\lambda} \) is not normalisable on \([0, 1]\). However, the solutions (24) are still useful for understanding the qualitative distinction between the two cases. If \( \lambda \neq 0 \) the PDF has a peak at \( h = 1 - \lambda \), see Fig. 5 suggesting that clusters in this case would appear to be under-heated, on average. The change in the character of the PDF is due to the inhomogeneous force 1/2 - \( \lambda/(1-h) \) in the diffusion equation (22). Along with the constant term 1/2, which forces diffusion in the direction of \( h = 1 \), there is a counteracting term, \(-\lambda/(1-h)\), which reverses the diffusion on the interval 1 - 2\( \lambda < h < 1 \).

A physical interpretation of this property becomes transparent in the context of the toy model. The ability of the AGN to heat the cluster is limited to the volume \( V_{\text{inner}} \) (with X-ray luminosity \( L_{\text{inner}} \) or 1). The additional mass deposition from the volume \( V_{\text{outer}} \) (with the luminosity \( L_{\text{outer}} \) or \( \lambda \)) cannot be counterbalanced by the AGN heating. This results in a shift of the balance from the exact match, \( h = 1 + \lambda \), to the lower end with the most likely value \( h = 1 - \lambda \) \((H = L_{\text{inner}} - L_{\text{outer}})\). The shape of the PDF is asymmetric and varies with \( \lambda \), see Fig. 6. Also the mode or the most probable heating rate is different from the mean or the average heating rate in this case, as the distribution is skewed.

It is possible to find an analytical solution to the SDE
in the case \( \lambda \neq 0 \),

\[
h(\tau) = 1 - \varepsilon(\tau) \left( 1 - h_0 + \lambda \int_0^\tau \frac{d\tau'}{\varepsilon(\tau')} \right),
\]

where \( \varepsilon(\tau) = e^{W(\tau) - \tau/2} \) is known as the exponential martingale. This solution must be completed with the reflected process, so that \( h(\tau) > 0 \). However, because the analytical quadrature of the integral term is unknown (see, e.g., Goodman & Kim 2006, for details), this can not be done as easily as in the case \( \lambda = 0 \).

The long-term behaviour of the process in the case \( \lambda \neq 0 \) was already studied using the approach based on the FPE (22). The sample heating curves can be calculated using numerical solutions of the SDE (17). In Fig. 3, two sample curves are plotted for processes with \( \lambda = 0.2 \), starting from \( h_0 = 0.1 \). The mean of the heating rapidly approaches the value \( \langle h \rangle \approx 0.55 \) (note that \( h = 0.8 \) is the most probable value) while contrary to the \( \lambda = 0 \) case the standard deviation remains finite, and the sample curves exhibit a lot of variation.

Note that the balance of heating and cooling in the case when \( \lambda \neq 0 \) is given by \( h = 1 + \lambda \) (not \( h = 1 \)) and therefore it is never reached. This is a direct consequence of the imposed limit on the AGN “fuel supply” (i.e., its heating capability, see section 2) to the region \( V_{\text{inner}} \). The mass deposited from the \( V_{\text{outer}} \) is not a part of the AGN feedback cycle, and accumulates in the centre. In practice, this sets a limit to the value of \( \lambda \), which could be determined from future cluster surveys.

5 DISCUSSION

In the context of the model developed here, it is not surprising that most observed clusters are found to have a very tight correspondence between the heating power and the radiative cooling rate (Dunn & Fabian 2008). While it would be naive to equate the heating rate in the current model with the power input measured from the size of the cavities in the clusters, it is reasonable to think that the two energy rates are positively correlated.

In the presented model the self-tuning of the energy balance is rapid, exponentially quick, in fact \( \propto e^\tau \) (see equation 20). It works even in the case when the AGN is able to heat only a fraction of the cluster volume (see Fig. 3). This shows that the correspondence of the cooling power and the AGN heating seen in the observations may indeed be a general rule, which is characteristic of the feedback process. Since the underlying PDF of the heating can be skewed (see Fig. 6 and equation (23)) it is worth noting that the heating rates inferred from the observations most frequently must lie near the mode of the distribution, and can not be used directly to find the mean or the average heating rate. The true picture can only be uncovered in a complete survey.

As was shown above, the time-dependent PDF becomes very close to the static solution for times \( \tau > 10 \), see Figs. 2 and 3. The physical time is proportional to \( \tau \) with the proportionality constant \( b^{-2} \) (see the normalisation of variables just before the equation (14)). The physical time limit is, therefore, \( \tau > 10b^{-2} \), which puts the lower limit on the model parameter \( b \). If the physical time is of the order of the Hubble time the limit on \( b \) is given by \( b \geq 3 \times 10^{-5} \, \text{yr}^{-1/2} \).

In principle, this value can be verified observationally using the relation between the variance of \( M \) at different times, as given by the equation (15). Given sufficient statistics for the mass deposition rates across the population of clusters in different redshift bins, the prediction of the exponential growth of the \( (M(t))^2 \) can be tested. The values of the model parameters \( b \) and \( \lambda \) can also be found. It is unlikely that, over the entire lifetime of a cluster, the real distribution of the heating is very close to the \( \lambda = 0 \) case. Since in this case the AGN would, on average, prevent any cooling flow from developing and also prevent any star formation, which, in reality, is observed to be enhanced in some brightest cluster galaxies (Rafferty et al. 2006). This leaves the single parameter distribution (23) as a plausible heating PDF for the population of “mature” (of age \( \tau > 10 \)) clusters.

The current model sketches just one possible scenario for the feedback. In order to make the model robust and simple, many assumptions were made: the insignificance of the magnetic field and the kinetic energy, rapid thermalisation of the AGN energy, and the inverse proportionality between the enthalpy and the mass flow. They have to be tested and verified in the future using numerical simulations and analysis of the observations. In addition, it is quite possible, for example, that there can be a delay in the AGN response to the cooling flow (Pope 2007). During this delay an excess of cold material could accumulate around the AGN. This would be equivalent to effectively growing the cluster in size from the point of view of the feedback loop. AGN heating of such a larger cluster could then result in the overheating of the ICM, and possibly act as a destabilising factor.

Despite the limitations of the model, this work demonstrates that the employment of stochastic calculus appears to be particularly well suited for the analysis of AGN feedback. The combination of Itô calculus and the Fokker-Plank equations provides a simple, yet powerful, way of investigating behaviour of the time-dependent variability of AGN feedback in clusters.

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