Distributed Rate Adaptation and Power Control in Fading Multiple Access Channels

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Abstract

Traditionally, the capacity region of a coherent fading multiple access channel (MAC) is analyzed in two popular contexts. In the first, a centralized system with full channel state information at the transmitters (CSIT) is assumed, and the communication parameters like transmit power and data-rate are jointly chosen for every fading vector realization. On the other hand, in fast-fading links with distributed CSIT, the lack of full CSI is compensated by performing ergodic averaging over sufficiently many channel realizations. Notice that the distributed CSI may necessitate decentralized power-control for optimal data-transfer. Apart from these two models, the case of slow-fading links and distributed CSIT, though relevant to many systems, has received much less attention.

In this paper, a block-fading AWGN MAC with full CSI at the receiver and distributed CSI at the transmitters is considered. The links undergo independent fading, but otherwise have arbitrary fading distributions. The channel statistics and respective long-term average transmit powers are known to all parties. We first consider the case where each encoder has knowledge only of its own link quality, and not of others. For this model, we compute the adaptive capacity region, i.e. the collection of average rate-tuples under block-wise coding/decoding such that the rate-tuple for every fading realization is inside the instantaneous MAC capacity region. The key step in our solution is an optimal rate allocation function for any given set of distributed power control laws at the transmitters. This also allows us to characterize the optimal power control for a wide class of fading models. Further extensions are also proposed to account for more general CSI availability at the transmitters.

I. INTRODUCTION

The multiple access channel (MAC) is a fundamental model for many multiple-transmitter single-receiver systems, such as the up-link of a cellular network. It is well known that the achievable data-rates over a fading MAC system depends on the availability of channel state information (CSI). While it is reasonable to assume that the receiver has access to full CSI, the availability of the CSI at the transmitters (CSIT) depends on factors like the coherence-time, admissible feedback overhead etc. In this paper, we consider a slow fading MAC with distributed CSI at the transmitters and full receiver CSI. We call this a distributed CSI MAC, where each encoder has some level of local CSI available.

There has been significant work on fading MAC channels under different CSI assumptions at the encoders. In fast fading channels, coding over a large block spanning many independent fading states is common, and it brings the average behaviour of the channel into play in the same coding block. The resulting capacity region is called the ergodic capacity region. The ergodic capacity region for a fading AWGN MAC has been characterized under perfect CSI at the transmitters and the receiver [1], [2]. A distributed CSI model where each encoder is aware only of its own link quality is considered in [3], where the ergodic sum-capacity is analyzed. Under more generalized CSI availability at the transmitters, [4] characterized the ergodic capacity region as an optimization problem over suitable power-control laws. However, explicit solutions for the optimal power-control are difficult to obtain, and good thumb rules for distributed power control are usually employed [3]. MACs with fast-fading can also be analyzed using the framework of channels with state. Models of discrete memoryless MAC with state have got significant attention under various assumptions on CSI availability, such as causal/non-causal CSIT [5], [6], asymmetric CSIT [7], [8], asymmetric CSI at the transmitters and no CSI at the receiver [9] etc. Notice that a ergodic utility is more suitable in a fast-fading model, where sufficient channel variations are available in the coding block. For slow fading models with full CSI, the results of [1] still apply, and the capacity region is known. A remaining question of interest is on slow-fading models with distributed CSI.

Consider a block fading AWGN MAC, where the fading states remain fixed for a large block length (coherence time), and change in an i.i.d. manner from block to block, a widely used assumption [10]. Unlike in the ergodic framework, coding is allowed only within a single block or coherence time, which is assumed to be large enough. Such within block coding models appear in several practical slow fading contexts [11], [12]. In addition, the transmitters may have varying levels of CSI availability of the CSI at the transmitters (CSIT) depends on factors like the coherence-time, admissible feedback overhead etc. Notice that the distributed CSI may necessitate decentralized power-control for optimal data-transfer. Apart from these two models, the case of slow-fading links and distributed CSIT, though relevant to many systems, has received much less attention.

(i) Safe Mode: In this mode, henceforth also called the ‘outage-free’ mode, the transmitters attempt to play it safe in each block, by choosing rates and powers such that the data can be decoded at the receiver. The challenge is to choose the rates and powers blockwise based on the distributed CSI, while ensuring correct decoding with high enough probability in each block. Such a MAC model was introduced in [13], [14], where the case of distributed state information at the respective encoders

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was considered. Note that, in contrast, the ergodic setup requires the error probability to be low when averaged over a large number of fading realizations. We assume a sufficiently large block length (coherence time), and require that the average error probability decays exponentially to zero in blocklength for every fading vector realization. In other words, the rate-tuple in each block should be within the instantiated MAC capacity region, which is determined by the fading realizations and the chosen transmit powers in that block. We will say that the system remains ‘outage-free’ in each block. The long-term average (over blocks) rate-tuples achievable under this model is known as the adaptive capacity region.

(ii) ARQ mode: Another option in the distributed CSI setting is to adopt a more aggressive rate-choice which allows the effective rate-tuple to be outside the instantaneous capacity region for some combinations of the channel states. The lost data can either be re-transmitted in an ARQ based system with feedback or can be recovered using an inter-block outer erasure code. It may be noted that the inter-block outer erasure code violates the basic framework of within-block coding, and is a special form of coding across fading states. In either case, the achieved rate is calculated by simply discounting the lost data in the outage events. The capacity region under this setup will be inside the ergodic capacity region, but may be bigger \cite{15} than the outage-free capacity region for the ‘safe-mode’. Alternate approaches based on broadcasting to mitigate the lack of CSIT also exists, see \cite{16} for a recent account.

For both safe mode as well as ARQ mode, there are two time-scales of interest. In the terminology of \cite{10}, a ‘short-term’ or per-block average power constraint dictates the choice of codebooks used in a block. The transmitter may have some freedom in adapting the short-term constraint based on the available CSI, however the adaptations should respect a long-term average (over blocks) power constraint imposed by physical considerations. We will use the same nomenclature here, see also \cite{12,11} for the origin and physical significance of these terms. Similarly, the rate-adaptation schemes may change the transmission-rates from block to block, and our utilities capture the long-term average rates.

As in \cite{14,13,15,17}, this paper focuses on outage-free (safe-mode) operations over block fading MAC under distributed CSI. For most parts of this paper, we consider a fading MAC where each transmitter is aware only of its own link quality, we from block to block, and our utilities capture the long-term average rates. For accessibility, we will describe the results for the discrete fading states in detail first, given in Sec. \ref{sec:discrete} (for two users) and Sec. \ref{sec:continuous} (for arbitrary number of users). These are then generalized to cover the corresponding continuous contexts, or we may append the word power to signify power control. Recently, the adaptive sum-capacity under identical fading statistics across users were presented in \cite{15,17}, where the optimal power-allocation was shown to have a water-filling form. It was also shown in \cite{15} that the sum-capacity can be achieved by rate-splitting and a successive cancellation decoder of lower complexity. The main contribution of the current paper is in characterizing the complete adaptive capacity-region of an individual CSI MAC, valid for arbitrary fading statistics and power constraints. Extensions to other local CSI models are also proposed.

A. Contribution and organization of this paper

This work primarily addresses the power controlled adaptive capacity region for an individual CSI MAC under arbitrary fading distributions, independent across links. Section \ref{sec:formulation} presents the system model together with some definitions and notations. We summarize our contribution below with respect to earlier related works.

- For a given set of power control laws at the transmitters, we present an almost closed form solution in Section \ref{sec:power} to the adaptive sum-capacity for the distributed CSI MAC with arbitrary fading distributions, which are independent across links. Presented for both discrete and continuous fading states, these are easily computable for any set of fading distributions. In contrast, earlier works like \cite{14,13} focused more on a single letter characterization for the discrete memoryless case. Evaluating these formulas for the Gaussian case resulted in unsolved optimization problems in terms of power control and rate-adaptation functions. Notice that simple numerical solutions for such problems can only handle channels with very few states and a small number of users. The work in \cite{15,17} provided the solution for some special cases. The approach there critically depends on the assumption of identical channel statistics across users, a limitation which is circumvented in this work using a novel rate-adaptation technique.

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valued fading states, in Sec. III-C (for two users) and in Sec. III-D (for arbitrary number of users). The two users case is presented first for both types of fading states since this clearly illustrates the underlying ideas. The generalization to multiple users also require some new techniques in the proofs. While generalized formulae encompassing both discrete and continuous valued fading states are possible, it will make the presentation a bit awkward. Furthermore, the discrete case carries considerable insight into the solutions, in addition to the chronological order in which the results were obtained.

- Section IV generalizes the results of Section III to find the maximum weighted sum-capacity for any weight vector, thus allowing the computation of the whole adaptive capacity region by taking different weight vectors.

- In section V, we present the power-controlled adaptive capacity region as a convex optimization problem under linear constraints for discrete (finite number of) fading states. The optimization problem is shown to be tractable due to a crucial monotonicity property for an optimal power allocation. It is shown that there is an optimal power allocation function for which the received power is a monotonically non-decreasing function of the fading magnitude. This allows the weighted sum-rate to be expressed as a fixed function of the power allocation, and this leads to an optimization problem where the number of variables (power values) is same as the number of states.

- In section VI, we extend the results to a CSI model where each user knows some partial information about the other users’ fading states. The proposed techniques also easily extend to the case of arbitrary CSIT models provided that the knowledge of its own state at a user is at least as good as other users’ knowledge of the same. In other words, each user is aware of what others know about its fading state. It is worth pointing out that the two user asymmetric CSI MAC model of [7] is an extreme case of the CSI availability that we consider, where one user has full CSIT, and the other knows only its own link. Single letter characterizations for the ergodic region of asymmetric CSI models are available [7], see [8] for generalizations. In contrast, we consider adaptive coding under more generalized versions of CSI availability. Nevertheless, the techniques that we propose in Section VI also allow the numerical evaluation of the capacity region for specific cases like the asymmetric CSI MAC of [7], a result of independent interest. This connection is not further explored in the current paper.

Finally, Section VII concludes the paper with suggestions for some future work.

II. SYSTEM MODEL

Consider a system where $N$ transmitters have independent data-streams to be sent to a common receiver. We use the subscript $i \in \{1, 2, \cdots, N\}$ to represent variables associated with user $i$. The channel is modeled as a block fading MAC where the received symbol is given by

$$Y = \sum_{i=1}^{N} H_i X_i + Z,$$

where $X_i \in \mathbb{R}$ is the symbol transmitted by user $i$, $H_i \in \mathcal{H}_i \subseteq \mathbb{R}$ is the fading state of the channel from user $i$ to the receiver, and $Z \sim \mathcal{N}(0, 1)$ is a real additive white Gaussian noise (the variance is assumed to be 1 without loss of generality) independent of $\{X_i | 1 \leq i \leq N\}$ and $\{H_i | 1 \leq i \leq N\}$. The fading coefficients $H_i$, $1 \leq i \leq N$ are assumed to be independent of each other. The fading vector $\mathbf{H} = (H_1, H_2, \cdots, H_N)$ remains constant within a sufficiently large block of fixed size and varies independently across blocks. We assume that the fading statistics as well as the respective long-term average power constraints are known to all parties.

The transmitters have the freedom to adapt their rates and power according to the available local knowledge of the fading vector. However the choice of rates should ensure that the decoding error probability exponentially decays with blocklength for
every realization of the fading vector. This is different from having an arbitrarily small error probability in the Shannon sense, which may need infinite block-lengths, see [14](page 587). In particular, our target is an acceptably small error probability permitted by the large blocklength, while averaged over the uniform choice of messages in that block.

We assume a CSI model where the $i$-th user has an estimate $\hat{H}^{(i)} := g_{ij}(H_j)$ of $H_j$, where $g_{ij}$ is a function. So the CSI available at the $i$-th user is $\mathbf{H}^{(i)} := (H_1^{(i)}, H_2^{(i)}, \cdots, H_N^{(i)})$. Note that the estimates $\hat{H}_j^{(i)}$ are deterministic functions of $H_j$, and there is no random noise in the estimate. For most parts of this paper (Sec. III till Sec. V) we assume that the $i$-th transmitter knows only its own channel state $H_i$ before transmitting in that block. That is, $\hat{H}_i^{(i)} = H_i \forall i$, and $\hat{H}_j^{(i)} = \emptyset, \forall j \neq i$. In Sec. VI, we will relax our assumptions and equip user $i$ with some additional partial information about the other channel states $H_j; j \neq i$, i.e., $g_{ij}$ is not a constant function for all $j \neq i$. By an abuse of notation, we will denote the image of $g_{ij}$ by $H_j^{(i)}$, and so $\hat{H}_j^{(i)} \in \hat{H}_j^{(i)}$.

A few more comments on notation are in order. We will denote vectors by bold-face, i.e. $\mathbf{u}$ represent a vector with $u_i$ at position $i$, where $u_i$ can be either a scalar or a function. The overbar symbol usually denotes an average quantity. Also, in case of multiple subscripts, we may write $h_{i,j+k}$ as $h_{i(j+k)}$ for clarity.

The following definitions are given for the general CSI model described above, though we will mostly consider the special case of an individual CSI MAC.

**Definition 1.** A power rate strategy is a collection of mappings $(P_i, R_i): \hat{H}_1^{(i)} \times \hat{H}_2^{(i)} \times \cdots \times \hat{H}_N^{(i)} \to \mathbb{R}^+ \times \cdots \times \mathbb{R}^+$, $1 \leq i \leq N$.

Thus, in the global fading-state $H$, the $i$th user employs a codebook of rate $R_i(\hat{H}^{(i)})$ and power $P_i(\hat{H}^{(i)})$. Let $C_{MAC}(H, \mathbf{P})$ denote the capacity region of a Gaussian multiple-access channel with a fixed fading vector $\mathbf{H}$ and average power-constraint $\mathbf{P}$, for the user $i$, $1 \leq i \leq N$. It is well known [13, 14] that $C_{MAC}(H, \mathbf{P})$ is the collection of all rate-tuples of the form $\mathbf{R} = (R_1, R_2, \cdots, R_N)$ such that

$$\forall S \in \{1, \cdots, N\}, \sum_{i \in S} R_i \leq 1 \cdot \log \left(1 + \sum_{i \in S} h_i^2 P_i \right). \quad (2)$$

**Definition 2.** A power-rate strategy is called feasible if it satisfies the average power constraints of the users, i.e. for $1 \leq i \leq N$, $E_H(\mathbf{P}_i(\hat{H}^{(i)})) \leq P_i^{avg}$, where $P_i^{avg}$ is the long-term average power constraint of user $i$ and $E(\cdot)$ denotes the expectation operator.

**Definition 3.** A power-rate strategy $(P_1(\cdot), R_1(\cdot), \cdots, P_N(\cdot), R_N(\cdot))$ is termed as outage free if

$$\forall h \in \{H_1, \cdots, H_N\}, \left(R_1(\hat{H}^{(1)}), \cdots, R_N(\hat{H}^{(N)})\right) \in C_{MAC}(h, P_1(\hat{H}^{(1)}), \cdots, P_N(\hat{H}^{(N)})).$$

Such an outage-free power-rate strategy ensures that in each block, the rate-tuple chosen distributedly by the users is inside the polymatroid capacity region given in [2], under the distributed choice of powers $P_i(\hat{H}^{(1)}), \cdots, P_N(\hat{H}^{(N)})$. Thus, Gaussian codebooks at these rates can achieve a decoding error probability exponentially decaying to zero with block-length. The long-term average achieved rate of user $i$ for a given power-rate allocation strategy is given by

$$\bar{R}_i := E \left[ R_i(\hat{H}^{(i)}) \right]$$

where the expectation is over $H$. The average rate-tuple achieved by a power-rate strategy is then $\bar{R} = (\bar{R}_1, \bar{R}_2, \cdots, \bar{R}_N)$. Let $\Theta_{MAC}(\mathbf{P}^{avg})$ denote the collection of all feasible power-rate strategies which are outage-free.

**Definition 4.** A rate vector $\bar{R}$ is said to be an achievable rate-tuple under power-rate adaptation if there exists a feasible outage-free power-rate allocation strategy for which the expected rate-tuple is $\bar{R}$. The power-adaptive capacity region is defined as the closure of the set of achievable expected rate-tuples under power-rate adaptation.

The power-adaptive capacity region can be evaluated by computing the power-controlled adaptive weighted sum-capacity for every non-negative weight vector as defined below.

**Definition 5.** The power-controlled adaptive weighted sum-capacity $C_{MAC}^{pc}(\mathbf{w}, \psi)$, for a non-negative vector $\mathbf{w} = (w_1, \cdots, w_N)$ is defined as

$$C_{MAC}^{pc}(\mathbf{w}, \psi) = \max \sum_{i=1}^{N} E \left[ \psi \cdot R_i(\hat{H}^{(i)}) \right]$$

where the maximization is over all feasible outage-free power-rate strategies in $\Theta_{MAC}(\mathbf{P}^{avg})$.

In some settings, the adaptation is limited to the transmit rates in each block, and the power-control law is specified in advance. The corresponding expected rate-region is known as the adaptive capacity region. Such schemes are of interest in situations where good/practical power control laws are already specified based on heuristics or other engineering considerations.
In several other systems, a regulatory transmit spectral cap may force the power-control to take particularly simple forms, for example, a constant power. Rate-adaptation is the only freedom available in such situations [19]. Notice that in the individual CSIT MAC, a pre-specified power allocation $P_i(H_i)$ is equivalent to no power adaptation, as its effect can be absorbed in the fading coefficients by considering the new fading state to be $\sqrt{P_i(H_i)}H_i$ (with an appropriate distribution on the new fading states).

Though our general interest is to find the power-adaptive capacity region, we will first develop techniques for the case of constant power allocation (or no power control). For a given set of power-control laws across users, let $\vartheta_{MAC}(P^{avg})$ denote the collection of all feasible rate-adaptation strategies which are outage free. Similar to Definition 5, the adaptive capacity region can be characterized by an equivalent weighted sum-rate maximization, defined below. Let us consider a fixed transmit power vector $P^{avg}$.

**Definition 6.** The weighted adaptive sum-capacity $C_{sum}(w, \psi)$, for a non-negative vector $w = (w_1, \cdots, w_N)$, is defined as

$$ C_{sum}(w, \psi) = \max \sum_{i=1}^{N} E \left[ w_i R_i(\hat{H}(i)) \right] $$

where the maximization is over all feasible outage-free rate strategies in $\vartheta_{MAC}(P^{avg})$.

When all the weights $w_i$ are identically one, the sum-throughput is known as the adaptive sum-capacity. This case is of special interest, and all our expositions will start with the sum-capacity, and then extended to the weighted sum-capacity.

**Remark 7.** While the weighted sum-rates can be used to characterize the entire capacity region, sometimes a convex hull operation become necessary. However, in the cases that we consider further, the utilities take the form of an ‘expectation of logarithm function’, and the convex-hull becomes superfluous.

Note that for the individual CSI model we consider in most of the paper (from Sec. III to Sec. IV), the power allocation functions $P_i(\cdot)$ and the rate allocation functions $R_i(\cdot)$ are simply functions of $H_i$.

**Remark 8.** Since each transmitter is aware of its link CSI, only fading magnitudes are important in the computation of the rates. Thus without loss of generality, we assume positive valued fading coefficients for the rest of the paper.

The following definition will be very useful for our technical results.

**Definition 9.** The inverse CDF function for user $i$ is

$$ h_i(x) = \psi^{-1}(x) := \begin{cases} \sup \{ h | \psi_i(h) < x \} & \text{for } 0 < x \leq 1 \\ 0 & \text{when } x = 0. \end{cases} \quad (3) $$

Using this definition, we will slightly abuse the notation and express the long term average rate for user $i$ in an individual CSI MAC as

$$ E[R_i(H_i)] = \int_{0}^{1} R_i(h_i(x))dx, \quad 1 \leq i \leq N. \quad (4) $$

In writing the integral, we have implicitly assumed well-behaved fading distributions, which can be discrete, continuous-valued or mixed. We now state a simple lemma which finds multiple applications in this paper.

**Lemma 10.** Let $(u_1, u_2)$ and $(v_1, v_2)$ be two non-negative vectors with $u_1 + u_2 = v_1 + v_2$. If $u_1 \leq v_1 \leq u_2$ and $u_1 \leq v_2 \leq u_2$, then

$$ \log(1 + u_1) + \log(1 + u_2) \leq \log(1 + v_1) + \log(1 + v_2). $$

The lemma follows by the concavity of the logarithm function. The stage is now set for presenting our results, and we will start with the adaptive sum-capacity of an individual CSI MAC in the next section.

### III. Adaptive Sum Capacity Without Power Control

In this section, we consider an individual CSI MAC, where the transmitters adapt their rates based on the knowledge of their own fading coefficients in a distributed manner. We will start with a model where user $i$ has a fixed transmit power of $P_i$. This corresponds to a short-term, per-block, average power constraint of $P_i$ in every block. This model is also considered in [14], where the optimal rate-allocation is unsolved. The significance and applications of blockwise short-term average power constraints in fading models are detailed in [10], see also [12]. Furthermore, employing fixed power constraints are common in models where there is a spectral cap on the transmissions [19]. Apart from the significance of the model, the solution of the adaptive sum-capacity problem for fixed powers illustrates our key techniques, which will later prove useful in computing the full capacity region as well as the optimal power allocation functions. Handling discrete and continuous-valued fading distributions need somewhat different treatments. We will first present the discrete case, generalizations to arbitrary distributions are presented in Sections III-C and III-D.
For simplicity of exposition, we will first consider two user MACs and later generalize to the $N$-users case. The generalizations require somewhat more involved proofs, however they follow the same two user principles.

A. DISCRETE FADING STATES: TWO USERS

In this section, we develop an inductive algorithm to perform the optimal rate allocation for discrete fading states. Let us consider a two-user fading MAC with fading CDFs $\psi_1(h)$ and $\psi_2(h)$. We first consider an example MAC with two states for each link to illustrate the idea behind the optimal rate allocation.

**Example 1.** Let us consider a 2-user MAC, with each link having two states. The weaker of the states is referred to as the bad (B) state and the stronger state is referred to as the good (G) state. For link $i$, these are denoted by respectively $B_i$ and $G_i$. Fig. 2 shows the MAC capacity regions for each pair of states of the links. For example, the inner pentagon is the capacity region for the state-pair $(B_1, B_2)$, and outer pentagon is the capacity region for $(G_1, G_2)$. Our rate-allocation (in Theorem 12) first chooses any point on the dominant face of the pentagon for the $(B_1, B_2)$ state-pair and assigns the respective co-ordinate values to the rates $R_1(B_1), R_2(B_2)$ for the Bad state-pair. This point is marked as $\oplus$. Suppose that $G_1$ has a higher probability than $G_2$. Then, we can prioritize the rate $R_1(G_1)$ over $R_2(G_2)$. Suppose the horizontal line through the point $\oplus$ intersects the pentagon for the $(G_1, B_2)$ state on the dominant face at point $\circ$. The horizontal coordinate of this point is assigned as the rate $R_1(G_1)$ for the state $G_1$ of user 1. Note that this is the maximum $R_1(G_1)$ (given $R_2(B_2)$) that does not cause outage at the state pair $(G_1, B_2)$. Now suppose the vertical line through $\circ$ intersects the pentagon for the $(G_1, G_2)$ state at point $\circledast$ on its dominant face. The vertical coordinate of this point determines the rate $R_2(G_2)$ of user 2 for the state $G_2$. The allocation ensures (as will be shown in Lemma 14) that the operating rate-pair $\circledast$ for the state-pair $(B_1, G_2)$ is also inside the corresponding capacity region, as depicted in Fig. 2.

![Fig. 2: Illustrating the rate-choice for a 2 state MAC](image)

Now we discuss the rate-allocation for arbitrary discrete states. Let user 1 have $k_1$ channel states with probabilities $p_i; 0 \leq i \leq k_1 - 1$ and let user 2 have $k_2$ channel states with probabilities $q_j; 0 \leq j \leq k_2 - 1$. Let us denote the CDF values of the channels as

$$\alpha_i = \sum_{j=0}^{i} p_j, \quad 0 \leq i \leq k_1 - 1,$$  \hspace{1cm} (5)

$$\beta_i = \sum_{j=0}^{i} q_j, \quad 0 \leq i \leq k_2 - 1,$$  \hspace{1cm} (6)

and let $\Gamma = \{\gamma_i | 0 \leq i \leq |\Gamma| - 1\} := \{\alpha_i | 0 \leq i \leq k_1 - 1\} \cup \{\beta_i | 0 \leq i \leq k_2 - 1\}$ be a set with the elements indexed in an ascending order. Here $|\Gamma| \leq k_1 + k_2 - 1$ (as $\alpha_{k_1 - 1} = \beta_{k_2 - 1} = 1$). For clarity, these are illustrated in Figure 3, where $H_1 \in \{g_{i_1}, g_{i_2}, g_{i_3}\}$ and $H_2 \in \{g_{j_1}, g_{j_2}, g_{j_3}\}$. Note that $\alpha_i, 0 \leq i \leq k_1 - 1$ are the horizontal levels in the plot of $\psi_1$ (see Fig. 3), which partition the interval $(0, 1]$. The elements of $\Gamma$ form a partition of $(0, 1]$ into $|\Gamma|$ segments. This is illustrated in Fig. 4 for two CDFs, where the elements $\gamma_i$ are shown as the levels on the $y$-axis. Clearly $\gamma_0 = \alpha_0, \gamma_1 = \beta_0, \gamma_2 = \beta_1, \gamma_3 = \alpha_1, \gamma_4 = 1$ in Fig 5.

**Remark 11.** We will often refer to the $\Gamma$ defined above as the horizontal cuts of the CDF, in reference to Figure 3.
Theorem 12. For any ρ in the positive interval \([\frac{1}{2} \log(1 + \frac{h_{10}^2 P_1}{1 + h_{20}^2 P_2}), \frac{1}{2} \log(1 + h_{10}^2 P_1)]\), the rate-strategy given by

\[
R_1(h_{10}) = \rho \\
R_2(h_{2i}) = \frac{1}{2} \log(1 + h_{1j}^2 P_1 + h_{2j}^2 P_2) - R_1(h_{1i}) \\
R_1(h_{1j}) = \frac{1}{2} \log(1 + h_{1j}^2 P_1 + h_{2(j-1)}^2 P_2) - R_2(h_{2(j-1)}),
\]

where \(0 \leq i < k\), \(1 \leq j < k\) and \(k = k_1 + k_2 - 1\), is outage-free and achieves the adaptive sum-capacity \(C_{\text{sum}}(1, 1, \psi_1, \psi_2)\).

In the above theorem, the rates for the users are assigned iteratively, alternating between the users. More precisely, they are assigned to \(h_{ij}\), in the lexicographic order of the pair \((i, j)\). At any stage of rate assignment, the sum-rate is maximized with the last state (of the other user), thus guaranteeing the maximum sum-rate in all pairs of consecutive states in this order of the rate assignment. Though the expanded states \(h_{ij}\) repeat, it is easy to see that the mentioned rate assignment is still well defined. That is, if \(h_{ij} = h_{jk}\) for some \(j, i, k\), the rate assignment algorithm in Theorem 12 ensures \(R_j(h_{ij}) = R_j(h_{jk})\). Note that the choice of the parameter \(\rho\) leaves some flexibility in the optimal rate assignment. If either \(h_{10} = 0\) or \(h_{20} = 0\), then \(\rho\) is confined to take a single value.

The sequence of rate assignment is illustrated in Figure 3 for two example CDFs \(\psi_1(h)\) and \(\psi_2(h)\). The iterative rate-assignment is shown at the right, where the rate-choice at the base of each arrow determines the rate for the state at the head/front of the arrow. For example, the rate-choice \(R_2(g_{20})\) as well as \(R_2(g_{2c})\) are determined by the choice of \(R_1(g_{1c})\), that is, the assignment ensures that the rate-pairs \((g_{1b}, g_{2b})\) and \((g_{1b}, g_{2c})\) achieve the respective maximum sum-rates. Similarly, \(R_1(g_{1c})\) is determined by the choice made for \(R_2(g_{2c})\).

Before proving Theorem 12, we first provide two alternate forms of the rate-assignment. The first alternate inductive form is as follows. For any \(\rho \in \left[\frac{1}{2} \log(1 + \frac{h_{10}^2 P_1}{1 + h_{20}^2 P_2}), \frac{1}{2} \log(1 + h_{10}^2 P_1)\right]\),

\[
R_1(h_{10}) = \rho, \quad R_2(h_{20}) = \frac{1}{2} \log(1 + h_{10}^2 P_1 + h_{20}^2 P_2) - \rho \\
R_1(h_{1i}) = R_1(h_{1(i-1)}) + \frac{1}{2} \log(1 + h_{1i}^2 P_1 + h_{2(i-1)}^2 P_2) - \frac{1}{2} \log(1 + h_{1(i-1)}^2 P_1 + h_{2(i-1)}^2 P_2), \\
R_2(h_{2i}) = R_2(h_{2(i-1)}) + \frac{1}{2} \log(1 + h_{1i}^2 P_1 + h_{2i}^2 P_2) - \frac{1}{2} \log(1 + h_{1i}^2 P_1 + h_{2(i-1)}^2 P_2)
\]
for $i \geq 1$. For a given $\rho$, this assignment can also be expressed in closed form as
\begin{equation}
R_1(h_{10}) = \rho, \quad R_2(h_{20}) = \frac{1}{2} \log(1 + h_{10}^2 P_1 + h_{20}^2 P_2) - \rho \tag{12a}
\end{equation}
\begin{equation}
R_1(h_{1i}) = R_1(h_{10}) + \sum_{j=1}^{i} \left( \frac{1}{2} \log(1 + h_{1j}^2 P_1 + h_{2(j-1)}^2 P_2) - \frac{1}{2} \log(1 + h_{1(j-1)}^2 P_1 + h_{2(j-1)}^2 P_2) \right), \tag{12b}
\end{equation}
\begin{equation}
R_2(h_{2i}) = R_2(h_{20}) + \sum_{j=1}^{i} \left( \frac{1}{2} \log(1 + h_{1j}^2 P_1 + h_{2j}^2 P_2) - \frac{1}{2} \log(1 + h_{1j}^2 P_1 + h_{2(j-1)}^2 P_2) \right) \tag{12c}
\end{equation}

**Remark 13.** The rate allocation in Theorem 12 is stated in a simplified, but somewhat specialized, manner to avoid cumbersome presentation. For each $i$, first the rate $R_1(h_{1i})$ is chosen to be the maximum possible without violating the outage condition with the $h_{2(i-1)}$ state, and then the maximum rate for $h_{2i}$ is chosen without violating the outage condition with $h_{1i}$. This gives more priority to the first user. If the priority to the second user is desired, then the order of allocation can be the opposite without affecting the expected sum-rate. More generally, independently for each $i$, the rates for $h_{1i}$ and $h_{2i}$ can be allocated in an arbitrary order. Even more generally, for each $i$, the rates $R_1(h_{1i})$ and $R_2(h_{2i})$ can be chosen inductively from the dominant face of a pentagon, i.e., satisfying

\[ R_1(h_{1i}) + R_2(h_{2i}) = \frac{1}{2} \log(1 + h_{1i}^2 P_1 + h_{2i}^2 P_2) \tag{13a} \]
\[ R_1(h_{1i}) \leq \frac{1}{2} \log(1 + h_{1i}^2 P_1 + h_{2(i-1)}^2 P_2) - R_2(h_{2(i-1)}) \tag{13b} \]
\[ R_2(h_{2i}) \leq \frac{1}{2} \log(1 + h_{1(i-1)}^2 P_1 + h_{2i}^2 P_2) - R_2(h_{2(i-1)}) \tag{13c} \]

It is not difficult to show that the proof of Theorem 12 given below will also hold true for any rate allocation satisfying the general conditions stated in (13).

We will first show that the rates given in Theorem 12 is outage-free (see Definition 2). The next lemma will provide a building-block for the proof.

**Lemma 14.** Let $h_1$ and $h'_1 \geq h_1$ be two channel states of user 1, and let $h_2$ and $h'_2 \geq h_2$ be two channel states of user 2. If
\[ R_1(h_1) + R_2(h_2) \leq \frac{1}{2} \log(1 + h_1^2 P_1 + h_2^2 P_2), \]
\[ R_1(h'_1) + R_2(h'_2) \leq \frac{1}{2} \log(1 + h'_1^2 P_1 + h'_2^2 P_2), \]
and $R_1(h_1) + R_2(h_2) = \frac{1}{2} \log(1 + h_1^2 P_1 + h_2^2 P_2)$,
then
\[ R_1(h'_1) + R_2(h'_2) \leq \frac{1}{2} \log(1 + h'_1^2 P_1 + h'_2^2 P_2). \]

**Proof:** For the fading states given in the statement of the lemma,
\[ R_1(h'_1) + R_2(h'_2) = (R_1(h_1) + R_2(h_2)) + (R_1(h'_1) + R_2(h'_2)) - (R_1(h_1) + R_2(h'_2)) \]
\[ \leq \frac{1}{2} \log(1 + h_1^2 P_1 + h_2^2 P_2) + \frac{1}{2} \log(1 + h'_1^2 P_1 + h'_2^2 P_2) - \frac{1}{2} \log(1 + h_1^2 P_1 + h'_2^2 P_2). \tag{14} \]

Now, let us denote $u_1 = h_1^2 P_1 + h_2^2 P_2$, $u_2 = h_2^2 P_1 + h_2^2 P_2$, $u_3 = h_1^2 P_1 + h_2^2 P_2$, and $u_4 = h_2^2 P_1 + h_2^2 P_2$. By the hypothesis, $u_1 \leq u_i \leq u_2$, $i = 3, 4$. Then, by Lemma 10 we have
\[ \frac{1}{2} \log(1 + u_1) + \frac{1}{2} \log(1 + u_2) \leq \frac{1}{2} \log(1 + u_3) + \frac{1}{2} \log(1 + u_4). \tag{15} \]

The lemma is proved by applying (15) to (14).

**Proof of Theorem 12.** In order to check that a given rate-strategy is outage-free, we need to verify three constraints of the pentagon for each pair of states. Let us first check the sum-rate constraint, followed by the individual rate constraints.

Let $h$ and $h'$ be arbitrary states of user 1 and user 2 respectively. We will show that the chosen rate-pair is inside the corresponding MAC pentagon, and thus outage-free. By the definition in (7), for some $i$ and $j$, $h = h_{1i}$ and $h' = h_{2j}$. To check the sum-rate constraint of (7), let us assume w.l.o.g that $i \leq j$. The proof will be done by induction on $|j-i|$. If $i = j$, then by (9), $R_1(h) + R_2(h') = \frac{1}{2} \log(1 + h^2 P_1 + h'^2 P_2)$. For $j = i+1$, the maximum sum-rate is achieved for state-pairs $(h_{11}, h_{2i})$, $(h_{1j}, h_{2j})$, and $(h_{1j}, h_{2i})$ by (9), (10) and (9) respectively. So, Lemma 14 gives $R_1(h_{1i}) + R_2(h_{2j}) \leq \frac{1}{2} \log(1 + h_{1i}^2 P_1 + h_{2j}^2 P_2)$. Now
suppose for some \( t \geq 2 \), and all \( i, j \) with \( |j - i| < t \), it holds that \( R_2(h_{2j}) + R_1(h_{1i}) \leq \frac{1}{2} \log(1 + h_{2j}^2 P_1 + h_{2j}^2 P_2) \). Then for \( j = i + t \), we have

\[
R_2(h_{2(j - 1)}) + R_1(h_{1(i - 1)}) = \frac{1}{2} \log(1 + h_{2(j - 1)}^2 P_1 + h_{2(j - 1)}^2 P_2) \leq \frac{1}{2} \log(1 + h_{2(i - 1)}^2 P_1 + h_{2(i - 1)}^2 P_2) \quad (16a)
\]

where \((16b)\) follows from \((9)\), and \((16a)\) and \((16c)\) follow from the induction hypothesis. Using this in Lemma \(14\) it follows that \( R_2(h_{2j}) + R_1(h_{1i}) \leq \frac{1}{2} \log(1 + h_{1i}^2 P_1 + h_{2j}^2 P_2) \). This completes the proof by induction.

Having verified the sum-rate constraint, we also lose to \( R_1(h_{iii}) \leq \frac{1}{2} \log(1 + h_{11}^2 P_2) \), for \( j = 1, 2 \). We do this by induction on \( i \). The base case of \( i = 0 \) follows from \((8)\) and \((9)\). Now let us consider \( i > 0 \). We give the proof for \( j = 1 \), and the proof for \( j = 2 \) follows similarly. By \((9)\) and \((10)\),

\[
R_1(h_{1i}) = \frac{1}{2} \log(1 + h_{11}^2 P_1 + h_{11}^2 P_2) - \frac{1}{2} \log(1 + h_{2(i - 1)}^2 P_1 + h_{2(i - 1)}^2 P_2) + R_1(h_{1(i - 1)}) \leq \frac{1}{2} \log(1 + h_{11}^2 P_1 + h_{11}^2 P_2) + \frac{1}{2} \log(1 + h_{2(i - 1)}^2 P_1 + h_{2(i - 1)}^2 P_2) - \frac{1}{2} \log(1 + h_{2(i - 1)}^2 P_1) \leq \frac{1}{2} \log(1 + h_{11}^2 P_1 + h_{11}^2 P_2) - \frac{1}{2} \log \left( \frac{1 + h_{2(i - 1)}^2 P_2}{1 + h_{2(i - 1)}^2 P_1} \right) \leq \frac{1}{2} \log(1 + h_{11}^2 P_1 + h_{11}^2 P_2) - \frac{1}{2} \log \left( \frac{1 + h_{2(i - 1)}^2 P_2}{1 + h_{11}^2 P_1} \right) \leq \frac{1}{2} \log(1 + h_{11}^2 P_1).
\]

Inequality \((17)\) follows from the induction hypothesis, whereas \((18)\) uses the fact that \( h_{1i} \geq h_{1(i - 1)} \).

Let us now prove that our rate-strategy maximizes the expected sum-rate. The key is to notice that, using the inverse CDF definitions of \((5)\), our rate-allocation ensures that for any \( x \in [0, 1] \), \( R_1(h_1(x)) + R_2(h_2(x)) = \frac{1}{2} \log(1 + h_1^2(x) P_1 + h_2^2(x) P_2) \). But any outage-free rate-allocation \((R_1(\cdot), R_2(\cdot))\) satisfies

\[
\mathbb{E}(R_1(H_1)) + \mathbb{E}(R_2(H_2)) = \int_0^1 (R_1(h_1(x)) + R_2(h_2(x))) dx \leq \frac{1}{2} \int_0^1 \log(1 + h_1^2(x) P_1 + h_2^2(x) P_2) dx. \quad (19)
\]

The equality in the first line is by \((4)\), and the inequality above follows from \((2)\). Clearly, the proposed scheme achieves this upper bound and this completes the proof of the theorem.

**B. DISCRETE FADEING STATES: MULTIPLE USERS**

The results from the previous sections can be extended to multiple users. We first discuss the rate-allocation achieving the adaptive sum-capacity for arbitrary discrete states for each user. Let user \( i \), \( 1 \leq i \leq N \) have \( k_i \) channel states with probabilities \( p_{ij}; 0 \leq j \leq k_i - 1 \). Let us denote the CDF values of the channels as

\[
\alpha_{il} = \sum_{j=0}^{l} p_{ij}, \quad 0 \leq l \leq k_i - 1,
\]

and let \( \{ \gamma_i | 0 \leq l \leq |\Gamma| - 1 \} = \bigcup_i \{ \alpha_{il} | 0 \leq l \leq k_i \} \) be these values indexed in the ascending order, where \( |\Gamma| \leq \sum_{i=1}^{N} k_i - l + 1 \). These definitions are analogous to the two user ones in \((5)\). The values \( \alpha_{il}, 0 \leq l \leq k_i - 1 \) are all the horizontal levels in the interval \( (0, 1] \) in the discrete CDF \( \psi_i \). \( \gamma_i, 0 \leq l < |\Gamma| \) denote the union of these horizontal levels.

Now for \( 1 \leq i \leq N \), let us define the same number of ‘expanded’ channel states of the users by repeating their individual channel states appropriately using the inverse CDF of the fading states at \( \gamma_i; 0 \leq l \leq |\Gamma| - 1 \):

\[
h_{il} := h_i(\gamma_l) = \sup\{ h | \psi_i(h) < \gamma_l \}. \quad (20)
\]

In this notation, \( h_{i0} \) denotes the fading state of lowest magnitude for user \( i \).

Now, we state the result for discrete fading states. The empty sum is defined to be zero as usual.
Theorem 15. Let $R_i(h_{i0}), 1 \leq i \leq N$ be such that for any $S \subset \{1, 2, \cdots, N\}$

$$\sum_{i \in S} R_i(h_{i0}) \leq \frac{1}{2} \log(1 + \sum_{i \in S} h_{i0}^2 P_i) \quad \text{and} \quad \sum_{i=1}^N R_i(h_{i0}) = \frac{1}{2} \log(1 + \sum_{i=1}^N h_{i0}^2 P_i).$$ (21)

Then, the inductive rate allocation given by

$$R_i(h_{il}) = \frac{1}{2} \log \left( 1 + \sum_{k=1}^i h_{kl}^2 P_k + \sum_{j=i+1}^N h_{j(i-1)}^2 P_j \right) - \left( \sum_{k=1}^{i-1} R_k(h_{kl}) \right) - \left( \sum_{j=1+1}^N R_j(h_{j(i-1)}) \right), \quad 1 \leq i \leq N,$$ (23)

where $1 \leq l < \lfloor \Gamma \rfloor$, is an outage-free strategy achieving the adaptive sum-capacity.

Remark 16. The above rate allocation can also be expressed in an alternate form similar to [11], and also in closed form similar to [12]. However, as discussed in Remark [13] there is a lot more flexibility in the sum-rate optimal rate allocation than what is reflected in Theorem [15]. For each $l$, the rate allocation for the states $h_{1l}, h_{2l}, \cdots, h_{Nl}$ can be done in any order while ensuring the outage-free condition with the already rate-assigned states. Even more generally, inductively for each $l$, any rates can be chosen for the states $h_{1l}, h_{2l}, \cdots, h_{Nl}$ as long as they satisfy

$$\sum_{i=1}^N R_i(h_{il}) \leq \frac{1}{2} \log \left( 1 + \sum_{i \in S} h_{il}^2 P_i \right).$$ (24)

Recall that the fading state $h_{il}$ for user $i$ is defined by (20). Notice that in the special case where $S = \{1, 2, \cdots, N\}$, and $l_i = l, \forall j \in S$, the rate-allocation in (23) guarantees (by taking $i = N$) a rate-tuple on the dominant face of the capacity-region. Let us consider the ordered-pair $(i_l, i)$. Let $(i_k, k)$ be the highest pair in the lexicographical ordering of the states over all the users in $S$. That is, if $l = \max \{l_i : i \in S\}$, then $k = \max \{i \in S : l_i = l\}$.

Let us define, for any $t$, $S_1[t] := \{1, \cdots, t\} \cap S$ and $S_2[t] := \{t, \cdots, N\} \cap S$. Using (23)

$$\sum_{i \in S} R_i(h_{il}) = \sum_{j \in S_1[t-1]} R_j(h_{jl}) + R_k(h_{kl}) + \sum_{j \in S_2[k+1]} R_j(h_{jl}),$$ (27)

$$= \sum_{j \in S_1[k-1]} R_j(h_{jl}) + \frac{1}{2} \log \left( 1 + \sum_{j=1}^k h_{jl}^2 P_j + \sum_{j=k+1}^N h_{j(l-1)}^2 P_j \right) - \sum_{j=1}^{k-1} R_j(h_{jl}) - \sum_{j=k+1}^N R_j(h_{j(l-1)}) + \sum_{j \in S_2[k+1]} R_j(h_{jl})$$

$$= \sum_{j \in S_1[k-1]} R_j(h_{jl}) + \frac{1}{2} \log \left( 1 + \sum_{j=1}^k h_{jl}^2 P_j + \sum_{j=k+1}^N h_{j(l-1)}^2 P_j \right) - \sum_{j=1}^{k-1} R_j(h_{jl}) - \sum_{j=k}^N R_j(h_{j(l-1)}) + \sum_{j \in S_2[k+1]} R_j(h_{jl})$$

$$= \sum_{j \in S_1[k-1]} R_j(h_{jl}) + R_k(h_{kl}) + \sum_{j \in S_2[k+1]} R_j(h_{jl}) - \sum_{j=k}^{k-1} R_j(h_{jl}) - \sum_{j=k}^N R_j(h_{j(l-1)}) + \frac{1}{2} \log \left( 1 + \sum_{j=1}^k h_{jl}^2 P_j + \sum_{j=k+1}^N h_{j(l-1)}^2 P_j \right).$$ (28)
We now provide an inductive argument to show (26). The base case for any \( S \) and \( l_i = 0, \forall i \in S \) holds by (21). We now assume that (26) is true for all \((S, (l_i))_{i \in S}\) with strictly lower \((l_k, k)\). In particular, we assume that

\[
\sum_{j \in S_1[k-1]} R_j(h_{jl}) + R_k(h_{kl}) + \sum_{j \in S_2[k+1]} R_j(h_{jl}) \leq \frac{1}{2} \log \left( 1 + \sum_{j \in S_1[k-1]} h_{jl}^2 P_j + h_{kl}^2 P_k + \sum_{j \in S_2[k+1]} h_{jl}^2 P_j \right).
\]

(29)

By (23), we have

\[
\sum_{j=1}^{k-1} R_j(h_{jl}) + \sum_{j=k}^{N} R_j(h_{jl}) = \frac{1}{2} \log \left( 1 + \sum_{j=1}^{k-1} h_{jl}^2 P_j + \sum_{j=k}^{N} h_{j(l-1)}^2 P_j \right).
\]

(30)

Using Lemma 10 since the arguments of the logarithm sum to the same on both sides, we also have

\[
\frac{1}{2} \log \left( 1 + \sum_{j \in S_1[k-1]} h_{jl}^2 P_j + h_{kl}^2 P_k + \sum_{j \in S_2[k+1]} h_{jl}^2 P_j \right) + \frac{1}{2} \log \left( 1 + \sum_{j=1}^{k-1} h_{jl}^2 P_j + \sum_{j=k}^{N} h_{j(l-1)}^2 P_j \right) \leq \frac{1}{2} \log \left( 1 + \sum_{j \in S_1[k-1]} h_{jl}^2 P_j \right) + \frac{1}{2} \log \left( 1 + \sum_{j=1}^{k-1} h_{jl}^2 P_j \right).
\]

(31)

Now, using (29), (30), and (31) in (28), we have the result, that is, (26).

To complete the proof, we need to check that the rate-allocation is optimal. This follows as in (19), since we have ensured equality to the maximal sum-rate for every horizontal cut (see Remark 11) of the CDFs.

\[\blacksquare\]

C. CONTINUOUS VALUED FADING STATES

When the fading coefficients take continuous values, the rate-allocation algorithm developed in the last section cannot be applied directly. However, one can discretize the channel states with as small a step size as desired and then use the rate-allocation algorithm. This is expected to give a near-optimal rate-allocation. In the limit where the discrete step-size approaches zero, the algorithm provides a closed form elegant solution (Theorem 17 below) to the optimal rate-allocation. Apart from its technical merit, the explicit rate allocation is widely useful, since continuous-valued distributions like Rayleigh are commonly used to model wireless links. Here we will directly provide the rate-allocation formula and prove that it is outage-free and sum-rate optimal. We delegate the details of how the closed form expression was obtained from the algorithm in Theorem 12 to Appendix A. Our results are true for a wide class of distributions including combinations of continuous valued and discrete states.

Consider two continuous valued fading distributions \(\psi_1(h)\) and \(\psi_2(h)\). Recall that \(h_j(x) = \psi_j^{-1}(x)\) is the inverse CDF of user \(j\), as defined in (3).

**Theorem 17.** For a two user Gaussian MAC with fading distributions \(\psi_1(\cdot)\) and \(\psi_2(\cdot)\), and with respective transmit powers \(P_1\) and \(P_2\), the adaptive sum-capacity \(C_{sum}(1, 1, \psi_1, \psi_2)\) with individual CSI is achieved by the rate-allocation

\[
R_i(h) = R_i(h_i(0)) + \int_{h_i(0)}^{h} \frac{yP_i}{1 + y^2 P_i + \sum_{j \neq i} (\psi_j^{-1}(y))^2 P_j} dy, \quad h \geq h_i(0), \quad i \in \{1, 2\},
\]

(32)

for any \(R_1(h_1(0)), R_2(h_2(0))\) satisfying

\[
R_i(h_i(0)) \leq \frac{1}{2} \log(1 + h_i^2(0)P_i), \quad i \in \{1, 2\}\]

\[
\sum_{i=1}^{2} R_i(h_i(0)) = \frac{1}{2} \log(1 + h_1^2(0)P_1 + h_2^2(0)P_2).
\]

**Proof:** Let us first find an upper bound for the expected sum-rate of any achievable scheme.

\[
\sum_{i=1}^{2} E[R_i(H_i)] = \int_{0}^{\infty} R_1(h) d\psi_1(h) + \int_{0}^{\infty} R_2(h) d\psi_2(h).
\]

By the same steps as the discrete-state derivation in [19],

\[
\sum_{i=1}^{2} E[R_i(H_i)] \leq \int_{0}^{\infty} \frac{1}{2} \log(1 + h_1^2(x)P_1 + h_2^2(x)P_2) dx
\]

(33)
To complete the proof, we will show that the rate allocation in (32) is outage-free and it achieves the upper bound in (33).

**Claim 18.** The rate allocation given in (32) is outage-free.

**Proof:** For the rate functions in (32), we will show that \( \forall (h_1, h_2) \) such that \( h_i \geq h_i(0), \ i = 1, 2 \),

\[
R_1(h_1) + R_2(h_2) \leq \frac{1}{2} \log(1 + h_1^2 P_1 + h_2^2 P_2),
\]
and \( R_i(h_i) \leq \frac{1}{2} \log(1 + h_i^2 P_i), \ i = 1, 2 \).

Showing this require a bit of calculus, and is relegated to Appendix B.

Let us now show the optimality of the allocation in (32).

**Lemma 19.** For \( x \in [0, 1] \) and the rate allocation in (32),

\[
R_1(h_1(x)) + R_2(h_2(x)) = \frac{1}{2} \log(1 + h_1^2(x) P_1 + h_2^2(x) P_2).
\]

**Proof:** From the rate allocation in (32), it follows that

\[
R_1(h_1(0)) + R_2(h_2(0)) = \frac{1}{2} \log(1 + h_1^2(0) P_1 + h_2^2(0) P_2).
\]

Also, for \( x > 0 \),

\[
\sum_{i=1}^{2} R_i(h_i(x)) = R_1(h_1(0)) + R_2(h_2(0)) + \int_{h_1(0)}^{h_1(x)} \frac{yP_1}{1 + y^2 P_1 + (\psi^{-1}(\psi_1(y)))^2 P_2} dy + \int_{h_2(0)}^{h_2(x)} \frac{yP_2}{1 + y^2 P_2 + (\psi^{-1}(\psi_2(y)))^2 P_1} dy
\]

Substituting \( \psi_1^{-1}(\psi_2(y)) = z \) in the second integral, we get

\[
\sum_{i=1}^{2} R_i(h_i(x)) = R_1(h_1(0)) + R_2(h_2(0)) + \int_{h_1(0)}^{h_1(x)} \frac{yP_1}{1 + y^2 P_1 + (\psi^{-1}(\psi_1(y)))^2 P_2} dy + \int_{h_1(0)}^{h_1(x)} \frac{P_2 \psi^{-1}(\psi_1(z))(\psi^{-1}(\psi_1(z)))' + zP_1}{1 + z^2 P_1 + (\psi^{-1}(\psi_1(z)))^2 P_2} dz
\]

\[
= R_1(h_1(0)) + R_2(h_2(0)) + \int_{h_1(0)}^{h_1(x)} \frac{1}{1 + h_1^2(x) P_1 + h_2^2(x) P_2} dp \quad \text{(by substituting } 1 + z^2 P_1 + (\psi^{-1}(\psi_1(z)))^2 P_2 = p \text{)}
\]

\[
= \frac{1}{2} \log(1 + h_1^2(0) P_1 + h_2^2(0) P_2) + \frac{1}{2} \log(1 + h_2^2(x) P_1 + h_2^2(x) P_2) - \frac{1}{2} \log(1 + h_1^2(0) P_1 + h_2^2(0) P_2)
\]

This proves the lemma. We have thus shown that the rate allocation in (32) is optimal for achieving the adaptive sum-capacity. This completes the proof of Theorem 17.

The rate-allocation in Theorem 17 reduces to the optimal rate-allocation formula (12) for the discrete fading states as a special case (with \( \rho = R_1(h_1(0)) \)). The formula also extends to more users than two, presented in the next subsection.

**D. CONTINUOUS CASE: MULTIPLE USERS**

For \( N \) users with continuous valued fading states, the rate allocation in (32) is generalized in the following theorem.

**Theorem 20.** The rate allocation given by:

\[
R_i(h) = R_i(h_i(0)) + \int_{h_i(0)}^{h} \frac{yP_i}{1 + y^2 P_i + \sum_{j \neq i} (\psi_j^{-1}(\psi_j(y)))^2 P_j} dy, \ h \geq h_i(0), \ 1 \leq i \leq N,
\]

for any \( R_i(h_i(0)) \) satisfying

\[
\sum_{i \in S} R_i(h_i(0)) \leq \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i), \ S \subset \{1, 2, \cdots, N\},
\]

achieves the adaptive sum-capacity \( C_{\text{sum}}(1, \psi) \) of an \( N \)-user individual CSI MAC, where \( 1 \) is a vector of all ones.

The proof of this theorem is similar to the two-users case, and is relegated to Appendix C.
E. SIMULATION STUDY

We demonstrate the advantage of our solution by an example. Let $\psi_1(h)$ be the normalized Rayleigh CDF, and $\psi_2(h)$ be uniformly distribution in $[0, \sqrt{3}]$. Thus $\mathbb{E}[H_1^2] = \mathbb{E}[H_2^2] = 1$. Figure 5 shows the adaptive sum-capacity when the transmit power is varied while maintaining $P_1 = P_2$. For comparison, we also show the sum-rate achieved by the conventional strategy of time division multiplexing (TDMA), where the time is divided into equal-sized slots. The same cap on transmit power is imposed in both cases. Clearly, the proposed solution outperforms the conventional strategy.

![Fig. 4: Two User Adaptive Sum-capacity with $P_1 = P_2$ for independent Rayleigh and uniform fading states](image)

An astute reader will quickly point out that a generalized TDM scheme can also employ more power while transmitting, and still maintain the short-term average power constraint. We can go even one step further and employ the best power-adaptation scheme for TDMA. Nevertheless, even in this case, we will show in Section V that our rate-adaptation strategy will perform significantly better. In fact, an improved performance can be achieved by employing the TDMA power-adaptation itself, which is in general suboptimal for achieving the power-controlled adaptive sum-capacity, this is demonstrated in Figure 9.

IV. ADAPTIVE CAPACITY REGION

Recall that the adaptive capacity region is the collection of all rate-tuples of the form, $\left( \mathbb{E}_{H_1}[R_1(H_1)], \ldots, \mathbb{E}_{H_L}[R_L(H_L)] \right)$, where the rate-allocation strategies do not lead to outage in any block. The adaptive capacity region in the presence of individual CSI can be characterized by maximizing the weighted sum-rate $\sum_{i=1}^{L} w_i \mathbb{E}_{H_i}[R_i(H_i)]$ for all non-negative vectors $w$. For the economy of space, we present the adaptive capacity region for the case of $L = 2$, extending to more users is reasonably straightforward. We also assume in this section that the transmitter $i$ uses a fixed transmit power $P_i$ for all fading states, i.e $P_i(h) = P_i$, $\forall h, i = 1, 2$. The general case where power control is allowed will be addressed in Section V.

Without loss of generality, let us describe the solution for $w_1 = 1$ and $w_2 = \alpha \leq 1$, the opposite case will follow by a simple renaming of the variables. In terms of the notation in Section IV (see Definition 6), we have to evaluate $C_{\text{sum}}(1, \alpha, \psi_1, \psi_2)$, where $\psi_1, i = 1, 2$ are the respective CDFs of the two links. Using the definition of inverse in (3), we can write

$$ ER_1(H_1) + \alpha ER_2(H_2) = \int_0^\infty R_1(h_1) d\psi_1(h_1) + \alpha \int_0^\infty R_2(h_2) d\psi_2(h_2) $$

$$ = \int_0^1 (R_1(h_1(x)) + \alpha R_2(h_2(x))) \, dx. \quad (36) $$

When $\alpha = 1$, the sum of terms inside the integral of (36) is maximized by the corresponding sum-rate. This will suggest choosing a suitable operating point for every pair $(h_1(x), h_2(x))$ on the dominant face of the corresponding capacity pentagon. This concept was already explained in the example shown in Figure 3 for discrete fading states. An analogous picture for the continuous case is shown in Figure 5 with the respective CDFs $\psi_1$ and $\psi_2$. For every horizontal cut there, the proposed rate-allocation chooses a point on the dominant face of the corresponding pentagon. Figure 5 shows the rate-allocation $(R_1(h_1), R_2(h_2))$ for a particular cut which corresponds to a CDF value of 0.75.

For $\alpha < 1$, a similar point-wise maximization of the weighted sum-rate at all horizontal levels will end up choosing the right corner-point at such state-pairs. This does not ensure outage-free operation for state pairs $(h_1, h_2)$ for which $\psi_1(h_1) > \psi_2(h_2)$. This is because, if $h'_1$ is such that $\psi_1(h'_1) = \psi_2(h_2)$ (this means that $h'_1 < h_1$), then the sum-rate at $(h_1, h_2)$ is

$$ \frac{1}{2} \log(1 + h'_1^2 P_1) + \frac{1}{2} \log \left( 1 + \frac{h'_2^2 P_2}{1 + h'_1^2 P_1} \right) > \frac{1}{2} \log(1 + h_1^2 P_1) + \frac{1}{2} \log \left( 1 + \frac{h_2^2 P_2}{1 + h_1^2 P_1} \right) $$

$$ \frac{1}{2} \log(1 + h_1^2 P_1) + \frac{1}{2} \log \left( 1 + \frac{h_2^2 P_2}{1 + h_1^2 P_1} \right). $$

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$$ \frac{1}{2} \log(1 + h'_1^2 P_1) + \frac{1}{2} \log \left( 1 + \frac{h'_2^2 P_2}{1 + h'_1^2 P_1} \right) > \frac{1}{2} \log(1 + h_1^2 P_1) + \frac{1}{2} \log \left( 1 + \frac{h_2^2 P_2}{1 + h_1^2 P_1} \right) $$

$$ \frac{1}{2} \log(1 + h_1^2 P_1) + \frac{1}{2} \log \left( 1 + \frac{h_2^2 P_2}{1 + h_1^2 P_1} \right). $$
For Theorem 21. For $0 < \alpha < 1$, we have $C_{\text{sum}}(1, 0, \psi_1, \psi_2) = C_{\text{sum}}(1, 1, \phi_1, \phi_2)$ where $\phi_1$ and $\phi_2$ are two derived CDFs given by

\begin{align}
\phi_1(h_1) &= \psi_1(h_1)
\phi_2(h_2) &= \alpha \psi_2(h_2) + (1 - \alpha), \ h_2 \geq 0.
\end{align}

Before we prove this result, a few remarks are in order. First of all, we already know an optimal rate-allocation achieving the sum-capacity for any given set of CDFs from the results of the previous section. Thus, evaluating the sum-capacity over $\psi_i, i = 1, 2$ is straightforward. Second, only one of the CDFs need to be transformed to obtain the solution. The transformation first scales the CDF and then shifts it appropriately to maintain its maximum height at unity, ensuring a valid CDF after the transformation. This is illustrated in Fig. 6, where $\phi_2$ is derived from $\psi_2$.

Proof of the Achievability: Since an outage-free rate-allocation does not lead to outage in any fading block, it remains outage-free even if we change the underlying fading distribution, provided the respective supports of the distributions do not enlarge. Thus the optimal sum-capacity achieving rate-allocation for $\phi_i, i = 1, 2$ is also an outage-free rate allocation under $\psi_i, i = 1, 2$, however this may not be an optimal sum-capacity achieving rate allocation for the latter when $\alpha < 1$. Nevertheless, our interest is in achieving the $(1, \alpha)$-weighted sum-capacity for $\psi_i, i = 1, 2$, and for that the optimal sum-capacity achieving rate allocations for $\psi_i, i = 1, 2$ suffice. These rate-allocations are given by

\begin{align}
R_i(h_i) &= \int_0^{h_i} \frac{y P_i}{1 + y^2 P_i + (\phi_i^{-1}(y))^2 P_j} dy,
\end{align}

where $i = 1, 2, j = 1, 2, i \neq j$, and $\phi_1(\cdot), \phi_2(\cdot)$ are as defined in (37) and (38).

Let us now show that $C_{\text{sum}}(1, \alpha, \psi_1, \psi_2) \geq C_{\text{sum}}(1, 1, \phi_1, \phi_2)$. Using the rate allocations (39), albeit in channels $\psi_1$ and $\psi_2$, we get

\begin{align*}
\mathbb{E}[R_1 + \alpha R_2] &= \int_0^1 R_1(\psi_1^{-1}(x)) dx + \alpha \int_0^1 R_2(\psi_2^{-1}(x)) dx \\
&= \int_0^1 R_1(\phi_1^{-1}(x)) dx + \alpha \int_0^1 R_2(\phi_2^{-1}(x)) dx \\
&= \int_0^1 R_1(\phi_1(x)) dx + \int_0^1 R_2(\phi_2^{-1}(y)) dy \\
&= \int_0^1 R_1(\phi_1(x)) dx + \int_0^1 R_2(\phi_2^{-1}(x)) dx \\
&= C_{\text{sum}}(1, 1, \phi_1, \phi_2).
\end{align*}
where the second last equality followed from the fact that $\phi_2^{-1}(x) = 0$, for $x < 1 - \alpha$. Notice that $R_i(\cdot)$, $i = 1, 2$ are chosen in (39) as the sum-capacity achieving rate-allocation for the CDFs $\phi_i$, $i = 1, 2$. This completes the achievability proof.

**Proof of the converse:** We will now show that

$$C_{\text{sum}}(1, 1, 1, \psi_1, \psi_2) \leq C_{\text{sum}}(1, 1, \phi_1, \phi_2).$$

Using the definitions in (3)

$$\mathbb{E}[R_1] + \alpha \mathbb{E}[R_2] = \int_0^1 R_1(h_1(x))dx + \alpha \int_{\bar{\alpha}}^1 R_2\left(h_2\left(\frac{y - \bar{\alpha}}{\alpha}\right)\right)\frac{dy}{\alpha}$$

$$= \int_0^\bar{\alpha} R_1(h_1(x))dx + \int_{\bar{\alpha}}^1 R_1(h_1(x))dx + \int_{\bar{\alpha}}^1 R_2\left(h_2\left(\frac{y - \bar{\alpha}}{\alpha}\right)\right)dy$$

$$\leq \int_0^\bar{\alpha} \frac{1}{2} \log(1 + h_1(x)^2 P_1)dx$$

$$+ \int_{\bar{\alpha}}^1 \frac{1}{2} \log\left(1 + h_1^2(x) P_1 + h_2^2\left(\frac{x - \bar{\alpha}}{\alpha}\right) P_2\right).$$

This completes the proof of the converse, and thus Theorem 21 is also proved.

**A. Numerical Example**

It is of interest to characterize the adaptive capacity region for some practical models. Consider a slow-fading MAC with independent and identical Rayleigh distributed links. Figure 7 sketches the capacity region for a transmit power $P_1 = P_2 = 1$. The variance of the fading coefficient is taken to be 0.214 (second moment = 1). Notice that the results known so far in literature were only successful in identifying a sum-capacity achieving rate-pair [20], [14], whereas our current result obtain the full-capacity region. For comparison, we also show the full CSI capacity region under no power-control. Note that even for maximizing the sum-rate, the full-CSI scheme is different from the one where only the best user transmits [1] since we do not allow power control. The best scheme for full CSI can be numerically determined, we omit the details.

So far our results targeted a fixed transmit power. We will extend this in the next section to incorporate power control by the transmitters.

**V. POWER CONTROLLED ADAPTIVE CAPACITY REGION**

The adaptive sum-capacity of a fading Gaussian MAC with Individual CSI was described in section III, where for each user, the transmit power was fixed across blocks. However, it is well known that a power control strategy which adapts the transmit powers based on the fade values can significantly improve the transmission rates for many systems, for example a MAC with full CSIT [21]. Similar improvements are also expected in the distributed CSI MAC. In this section, we allow power control, and compute the so called power controlled adaptive capacity region of a two user MAC with individual CSI. The optimal power control law for the special case of identical fading statistics across users were already derived in [17].
Lemma 22. Theorem 21. The following lemma is immediate from this discussion. For such a fixed transmit power system, we already know the weighted sum-rate from fading MAC with fading vector $g$. Assume that for $i = 1, 2$, user $i$ employs a power allocation function $P_i(h_i)$ which also meets the long-term average power constraint $P_i^{\nu g}$. Let $C_{sum,P_1,P_2}(1, \alpha, \psi_1, \psi_2)$ denote the adaptive weighted sum-capacity under the given pair of power allocation functions $P_i(h_i), i = 1, 2$ at the respective transmitters. Using this notation, for fixed transmit powers (as in Section III-A) we will write $C_{sum,c_1,c_2}(1, \alpha, \psi_1, \psi_2)$ where $c_i$ is the power employed by user $i$ across fading states.

The quantity $C_{sum,P_1,P_2}(1, \alpha, \psi_1, \psi_2)$ can be evaluated as follows. Let us define $g_i = \sqrt{h_i^2 P_i(h_i)}$ and consider a new block fading MAC with fading vector $g$ and fixed transmit powers of unity across all fading realizations, i.e. there is no power adaptation in this new MAC model. For such a fixed transmit power system, we already know the weighted sum-rate from Theorem 21. The following lemma is immediate from this discussion.

**Lemma 22.**

$$C_{sum,P_1,P_2}(1, \alpha, \psi_1, \psi_2) = C_{sum,1,1}(1, 1, \phi_1, \phi_2)$$

with

$$\phi_1(g_1) = \nu_1(g_1); \quad \phi_2(g_2) = \alpha \nu_2(g_2) + (1 - \alpha), \quad g_2 \geq 0, \quad (43)$$

and $\nu_i(g_i)$ is the CDF of $\sqrt{H_i^2 P_i(H_i)}$ for $i \in \{1, 2\}$.

While the above lemma is simple, it is extremely useful in the sense of separating the power-control and rate maximization. In particular, our results in Section V can easily specify the optimal rate-adaptation for any given set of power-control laws. The question now is about optimizing the power-allocation. Unfortunately we do not have a closed form solution for optimal power-control, except for identical statistics across users [17]. Therefore, one needs to resort to numerical techniques to evaluate the optimal laws. This may appear formidable due to the polymatroidal constraints imposed on the possible rate-choices. An alternate way is to identify some thumb-rule for power-allocation (see [3]) and then choose the optimal rate-adaptation. Iterative techniques based on gradient based search is a widely used technique to identify optimal laws.

Once the power control laws are given, the proposed optimal scheme does rate-allocation in the increasing order of $h_i^2 P_i(h_i)$. If the order among $h_i^2 P_i(h_i)$ is preserved while any algorithm searches for an optimal power control then searching time and effort can be considerably reduced. The reason for this can be explained better for the evaluation of sum-rate. Notice that while evaluating the sum-rate, changing the power-allocation for a particular fading state will have a localized effect on the sum-rate. This can be visualized with the aid of Figure 4.

Suppose the power allocation of any state of user 1 is changed in such a way that the horizontal cuts of the CDF still stay the same. Then the sum-rate of those states of user 2 which shares a horizontal cut on the respective CDFs will be affected by the new rate-allocation. All other sum-rate values stay the same. Similar arguments apply when a power allocation is changed for a pair of states, while preserving the average transmit power. This allows the numerical solutions to proceed by localized searches, a very powerful advantage in locating the optimal power allocation. The complexity of the search becomes of the order of $|\Gamma|$, which is the number of distinct state-pairs intersected by
Lemma 23. There exists an optimal power-allocation in which $h_i^2 P_i(h_i)$ is a non-decreasing function of $h_i$ for $i = 1, 2$.

Proof: Let there be $k_i$ values for $H_i$ with probabilities $p_{ij}, 0 \leq j \leq k_i - 1$. Let $P_1(\cdot)$, and $P_2(\cdot)$ be two power allocation functions at the respective users. Denote $H_i^2 P_i(h_i)$ by $G_i$.

In order to avoid the notations from blowing up, we assume that $G_i$ has its mass on $k_i$ distinct values, say $\{u_{ij}, 0 \leq j \leq k_i - 1\}$ ordered in the ascending fashion. The assumption of distinct values is simply to create a bijection from $H_i$ to $G_i$, the exposition becomes simpler by this. Nevertheless, the proof applies more generally, with some renaming of the indices.

We proceed by contradiction. Consider two fade values $h_i', h_i''$ such that $g_i' = h_i^2 P_i(h_i) > g_i'' = h_i^2 P_i(h_i'')$, though $h_i' < h_i''$. Assume that $g_i'$ and $g_i''$ are the first pair of adjacent values to have this property, i.e. there is no value for $h_i^2 P_i(h)$ in the open interval $(g_i', g_i'')$. There is no generality lost here, if there is a value in the middle, we can redefine $g_i'$ or $g_i''$ appropriately and choose the corresponding $h_i$ values to pair with it.

Let the probabilities of $g_i'$ and $g_i''$ be $p'$ and $p''$ respectively, which are assumed to be positive. Denote the probability mass function of $G_i$ by $Q_i(\cdot)$. We will scale and shift the CDF of $G_i$ to take care of the weighted averages, as in Section [IV]. For the $(1, \alpha)$ weighted sum-rate with $\alpha \in [0, 1]$, let us define $q_{20} := 1 - \alpha + \alpha Q_2(u_{20})1[u_{20}=0]$, and for $1 \leq j \leq k_2 - 1$,

$$q_{2j} = \begin{cases} \alpha Q_2(u_{2(j-1)}) & \text{if } u_{20} = 0 \\ \alpha Q_2(u_{2j}) & \text{otherwise.} \end{cases}$$

(44)

The second operation above shifts the mass function to the right so as to accommodate a mass at zero, if this is not already in the support. In this case, we should also define $q_{2k_2}$, but since $q_{2k_2} = 1 - \sum_{j < k_2} q_{2j}$, a new definition will turn out redundant in our rate-allocation scheme. Let $q_{1j} := Q_1(u_{1j}), 0 \leq j \leq k_1 - 1$. Similar to (5) – (6), we can define

$$\alpha_k = \sum_{j=0}^{k} q_{1j}, \ 0 \leq k \leq k_1 - 1,$$

(45)

$$\beta_k = \sum_{j=0}^{k} q_{2j}, \ 0 \leq k \leq k_2 - 1,$$

(46)

and let $\Gamma = \{\gamma_i|0 \leq i \leq \Gamma| - 1\} := \{\alpha_i|0 \leq i \leq k_1 - 1\} \cup \{\beta_i|0 \leq i \leq k_2 - 1\}$ be an ordered set with the elements following an ascending order.

Let $\nu_{i}, i = 1, 2$ denote the CDFs of the respective mass functions $\{q_{ij}, 1 \leq j \leq k_i - 1\}$. Now for $i = 1, 2$, let us define the inverse CDF values for user $i$ as

$$g_{ij} = \sup\{g|\nu_i(g) < \gamma_j\}.$$

(47)

This definition implies that

$$\sum_{k:g_{ik} = h_{ik}^2 P_i(h_{ik})}[\gamma_{k+1} - \gamma_k] = q_{ik}.$$

(48)

Since we are interested in the weighted sum-capacity, let us assume an optimal rate-allocation according to Theorem 21. Equivalently, by Lemma 22, we can use the allocations (9)–(10) for the CDFs $\nu_1, \nu_2$ and unit power at the transmitters. Thus, the optimum weighted sum-rate with the power allocation $P_1(\cdot), P_2(\cdot)$ is given by

$$\mathbb{E}(R_1) + \alpha \mathbb{E}(R_2) = \sum_{k=0}^{\vert\Gamma\vert-1} (R_1(g_{1k}) + R_2(g_{2k})) [\gamma_{k+1} - \gamma_k]$$

$$= \sum_{k \in A'} \frac{1}{2} \log (1 + g_{1k}'' + g_{2k}) [\gamma_{k+1} - \gamma_k] + \sum_{k \in A'} \frac{1}{2} \log (1 + g_{1k}' + g_{2k}) [\gamma_{k+1} - \gamma_k] + R_r$$

$$= \sum_{k \in A''} \frac{1}{2} \log (1 + h_{1k}' P_1(h_{1k})' + g_{2k}) [\gamma_{k+1} - \gamma_k] + \sum_{k \in A'} \frac{1}{2} \log (1 + h_{1k}' P_1(h_{1k}) + g_{2k}) [\gamma_{k+1} - \gamma_k] + R_r,$$

(49)

where $A' := \{k : g_{1k} = h_{1k}^2 P_1(h_{1k})\}$, $A'' := \{k : g_{1k} = h_{1k}'' P_1(h_{1k})\}$, and $R_r$ denotes the rest of the summation in (49). Let $P_{1e}(\cdot)$ be a new power allocation such that

$$P_{1e}(h_{1k})' = P_1(h_{1k})' + \epsilon; \ P_{1e}(h_{1k})'' = P_1(h_{1k})' - \frac{\epsilon g_{1k}''}{p'}; \ P_{1e}(h_1) = P_1(h_1), \text{ for } h_1 \neq h_1', h_1''.$$


Thus, we have

\[
S(\epsilon) := \mathbb{E}(R_1) + \alpha \mathbb{E}(R_2) = \sum_{k \in A''} \frac{1}{2} \log \left( 1 + h_{1}^{\prime \prime} (h_{1}^{'}) + \frac{\epsilon p''}{p'} h_{1}^{\prime} + g_{2k} \right) \left[ \gamma_{k+1} - \gamma_{k} \right]
+ \sum_{k \in A'} \frac{1}{2} \log \left( 1 + h_{1}^{\prime} P_{1}(h_{1}^{'}) - \frac{\epsilon p''}{p'} h_{1}^{\prime} + g_{2k} \right) \left[ \gamma_{k+1} - \gamma_{k} \right] + R_f.
\]  

(51)

Taking derivative, we have

\[
S'(\epsilon) = \sum_{k \in A''} \frac{h_{1}^{\prime \prime}}{2(1 + h_{1}^{\prime \prime} P_{1}(h_{1}^{'}) + \epsilon p'' + g_{2k})} \left[ \gamma_{k+1} - \gamma_{k} \right]
- \sum_{k \in A'} \frac{p''}{p'} \frac{h_{1}^{\prime}}{2(1 + h_{1}^{\prime} P_{1}(h_{1}^{'}) + \epsilon p' + g_{2k})} \left[ \gamma_{k+1} - \gamma_{k} \right].
\]  

(52)

At \( \epsilon = 0 \),

\[
S'(0) \geq \sum_{k : g_{1k} = g_{1}^{'}} \frac{h_{1}^{\prime \prime}}{2(1 + h_{1}^{\prime \prime} P_{1}(h_{1}^{'}) + g_{2k})} \left[ \gamma_{k+1} - \gamma_{k} \right]
- \sum_{k : g_{1k} = g_{1}^{'}} \frac{p''}{p'} \frac{h_{1}^{\prime}}{2(1 + h_{1}^{\prime} P_{1}(h_{1}^{'}) + g_{2k})} \left[ \gamma_{k+1} - \gamma_{k} \right].
\]

It follows from the definition of \( \gamma_{k} \) that,

\[
\sum_{k \in A'} [\gamma_{k+1} - \gamma_{k}] = p' \quad \text{and} \quad \sum_{k \in A''} [\gamma_{k+1} - \gamma_{k}] = p''.
\]  

(53)

Hence,

\[
S'(0) \geq \frac{p'' h_{1}^{\prime \prime}}{2(1 + h_{1}^{\prime} P_{1}(h_{1}^{'}) + g_{2k})} - \frac{p'' h_{1}^{\prime}}{2(1 + h_{1}^{\prime} P_{1}(h_{1}^{'}) + g_{2k})}.
\]

Since \( h_{1}^{\prime} P_{1}(h_{1}^{'}) > h_{1}^{\prime \prime} P_{1}(h_{1}^{'}) \), and \( h_{1}^{\prime} > h_{1}^{'} \), we conclude that \( S'(0) > 0 \).

Note that \( S'(\epsilon) \) defined in (52) is a continuous, monotonically decreasing and differentiable function for \( \epsilon \geq 0 \). This shows that in an optimum power allocation, \( P_{1}(\cdot) \) is a non-decreasing function.

We should now show that the second user’s power allocation leads to a non-decreasing \( h_{1}^{\prime} P_{2}(h) \). This can be obtained by similar arguments as above, the main change is that the probabilities for non-zero values of \( G_{2} \) have to be scaled by \( \alpha \) in the computations. In particular, the values \( p' \) and \( p'' \) in (53) will be scaled by \( \alpha \), without affecting the overall sign of the quantities. We do not repeat all the arguments here. This completes the proof of Lemma 23. 

The advantage of Lemma 23 can be gleaned by considering the sum-rate evaluation. The lemma implies that under any optimal power allocation, \( h_{1k}^{\prime} P_{1}(h_{1k}) \) is increasing. This ensures that the horizontal levels \( \gamma_{i} \) of the CDF of \( G_{i} \) remain fixed for any optimal power allocation for sum-rate, and these are same as the horizontal levels in the CDF \( \psi_{i} \) (see Remark 11). Hence, the sum-rate expression defined with these \( \gamma_{i} \) in (19) and \( \alpha = 1 \) is a valid objective function for maximization over the
set of all possible power allocations which are candidates for optimality. The constraint set is defined by the average power constraints and the conditions

\[ h_{ik}^j P_i(h_{ij}) \geq h_{ij}^j P_i(h_{ij}) \text{ if } h_{ik} \geq h_{ij}, \quad k, j \in \{1, 2, \ldots, n_i\}, \quad i = 1, 2. \]

It is easy to see that the objective function is a concave function of the power variables and the constraints are linear (hence convex) in the power variables \( P_i(.) \). Hence, standard results in non-linear programming can be used to guarantee the convergence of a gradient based search algorithm for finding the power controlled adaptive sum-capacity, in which the power is modified in each step of the iteration depending on the direction of the gradient of the objective function. These algorithmic aspects are outside the purview of the current paper. However, for illustration, we compute the power controlled capacity region of a two-state fading model, which is an example studied in [14]. Notice that, the same procedure can easily identify the capacity region for several discrete models. While the numerical study in [14] only targets the sum-capacity for a two-state model, the full power-controlled adaptive capacity-region for channels with several fading states can be computed by the techniques presented here.

In the remaining of the section, let us compare the optimal power-controlled TDMA with the proposed schemes here. Notice that the optimal TDMA power control can be sub-optimal when used in conjunction with other rate-allocation mechanisms, however it still serves as a benchmark for performance comparison. In particular, given a time-sharing parameter, the optimal TDMA power control follows a single user water-filling structure, with appropriate water-levels chosen to respect the average power constraints at the users. To illustrate the performance, let us consider a Rayleigh fading link with second moment of \( \sqrt{P} \) and another link uniformly distributed in \([0, \sqrt{3}]\). Under equal power constraints, the optimal sum-rate for generalized TDMA is plotted in Figure 8 against the sum-power. Now, for the same power-control law, we can use the rate-adaptation mechanism given by Lemma 22. This is easily achieved by defining the \( \sqrt{P}(h_i)H_i \) as the new fading coefficient, where \( P_i(H_i) \) is the optimal TDMA water-filling power-control function. It is clear from Figure 9 that the schemes proposed here outperform the best TDMA schemes. Furthermore, employing the best power control schemes can make the rates even better, showing the suboptimality of TDMA in such distributed settings.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig8.png}
\caption{Power-Adaptive capacity region, Users identical with two fading states \( h_1 = 1, h_2 = 2, P(h_b) = 0.2 \) and \( P_{\text{avg}} = 10 \).}
\end{figure}

VI. ADDITIONAL CSI ON THE OTHER LINKS

In this section, we assume that each transmitter also has some partial CSI of the other links, in addition to the complete knowledge of its own link. Let us again consider a two user block fading MAC for simplicity. To start with, we also assume that the additional partial CSI from the other link is generated by a quantizer.

Let \( \hat{h}_1 \) denote the quantized value of \( h_1 \) which is known to user 2, and similarly \( \hat{h}_2 \) as the quantized value of \( h_2 \) available at user 1. Consider a pair of power allocation functions \( P_1(h_1, \hat{h}_2) \) and \( P_2(\hat{h}_1, h_2) \) for the users 1 and 2 respectively. As in the last section, let \( \psi_i(h_i) \) denote the fading CDF of the \( i^{th} \) user and \( \psi(h_1, h_2) = \psi_1(h_1)\psi_2(h_2) \) denote their joint CDF (i.e. independently fading links).

Imagine that the values of \( h_1 \) are partitioned into \( B_1 \) non-overlapping sets \( S_1, \ldots, S_{B_1} \) which are mapped to different output values by the quantizer \( (\hat{h}_1) \). We denote the minimum fading magnitude in the set \( S_l \) by \( m_j \). Similarly, let \( T_1, \ldots, T_{B_2} \)
represent the $B_2$ partitions of $h_2$, and $n_k = \min T_k$. We define $q_{1i} := Pr(H_1 \in S_i)$ for $1 \leq i \leq B_1$ and $q_{2j} := Pr(H_2 \in T_j)$ for $1 \leq j \leq B_2$. We can now write,

$$E(R_1(H_1, H_2) + \alpha R_2(H_1, H_2)) = \sum_{i,j} \int_{S_i} \int_{T_j} R_1(h_1, h_2) d\psi(h_1, h_2) + \alpha \sum_{i,j} \int_{S_i} \int_{T_j} R_2(h_1, h_2) d\psi(h_1, h_2)$$

$$= \sum_{i,j} \left[ q_{2j} \int_{S_i} R_1(h_1, n_j) d\psi(h_1) + \alpha q_{1i} \int_{T_j} R_2(m_i, h_2) d\psi(h_2) \right]$$

$$= \sum_{i,j} q_{1i} q_{2j} R_{\text{sum}}^{(i,j)}(1, \alpha)$$

where

$$R_{\text{sum}}^{(i,j)}(1, \alpha) = \int_{S_i} R_1(h_1, n_j) \frac{d\psi_1(h_1)}{q_{1i}} + \alpha \int_{T_j} R_2(m_i, h_2) \frac{d\psi_2(h_2)}{q_{2j}}$$

(54)

is the weighted sum-rate under the condition $H_1 \in S_i, H_2 \in T_j$. Note that both the users know the values of $i$ and $j$. Thus for different values of $(i, j)$, the pairs of functions $(R_1(\cdot, n_j), R_2(m_i, \cdot))$ can be optimized independently. Notice that

$$\int_{T_j} d\psi(h_2) = q_{2j} \text{ and } \int_{S_i} d\psi(h_1) = q_{1i}.$$ 

So each integral in (54) is evaluated with respect to a conditional distribution. Hence (54) is of the same form as (36), and for each $i,j$, the expression in (54) can be maximized using Theorem 21 this will in turn maximize the overall weighted sum-rate. Let us demonstrate the utility of additional CSI by numerical comparisons.

### A. Numerical Example

In this subsection, we consider the same example setup in Section IV-A however 1 bit of partial CSI from the other link is additionally made available at each transmitter. The single bit is obtained by comparing the CSI against a known threshold. Figure 10 compares the enlargement of the adaptive capacity region with 1 bit additional partial CSI. The threshold for the quantizer was arbitrarily taken to be 0.4 for demonstration purpose. In an application where the threshold can be chosen by the designer/users, the best choice of this threshold is an important question that deserves further investigation.

Under any given power-allocation schemes, we can write each integral in terms of the received powers and use the result described in Section V to maximize the weighted sum-rate. Thus the adaptive capacity region under additional partial CSI can be computed in an efficient manner. Extensions to multiple users and other models where the additional CSI is obtained by deterministic functions of the fading coefficients etc follow along similar principles. At the extreme case, where one user knows both the channels and other knows only its own, the model becomes an asymmetric CSIT MAC [7]. The notions of adaptive and ergodic capacity coincides here and our techniques can numerically solve the capacity region for this case. Let us also mention about power control and additional CSI.
For discrete fading states we can also extend our discussion to the power controlled adaptive capacity. For any fixed power allocation \(P_1(h_1, h_2)\) and \(P_2(h_1, h_2)\) for the users 1 and 2 respectively, the adaptive sum capacity with partial CSI can be obtained using the distribution on received powers, as explained above. By applying Lemma 23, an optimal power allocation can be shown to be such that, for fixed \(h_2\), \(h_1^2P_1(h_1, h_2)\) is monotone increasing in \(h_1\) in each of the regions \(s_k\). Likewise, for fixed \(h_1\), \(h_2^2P_2(h_1, h_2)\) is monotone increasing in \(h_2\) in each of the regions \(t_k\). This leads to an expression for the expected sum rate involving the powers \(P_1(h_1, h_2)\) and \(P_2(h_1, h_2)\) as the variables, similar to the individual CSI case. The expression is a concave function of these power variables and the power constraints are also linear (hence convex). The power allocation which maximizes this expression can be found by any of the methods for solving convex optimization problems, thus giving an algorithm for finding the power controlled adaptive sum capacity with partial CSI about the other link.

VII. Conclusion

In this work, we presented the adaptive capacity region of fading MACs with arbitrary fading statistics for varying amounts of channel state information available at the transmitters. The techniques also provided the power-controlled adaptive capacity region for channels with discrete states.

For the case of individual CSI, the solution for the adaptive sum capacity (without power control) was presented in an elegant closed form for continuous valued fading distributions, and as an iterative rate allocation expression for discrete fading states. These formulas work for any number of users and link statistics. Finding the adaptive capacity region with individual CSI amounts to finding the rate allocations that maximize the expected weighted sum rate of the users. We have presented an outage free rate allocation strategy which can achieve any point on the boundary of the adaptive capacity region. Thus we have characterized the entire adaptive capacity region.

We also presented a result which reduces the problem of finding the power controlled adaptive weighted sum-capacity to a much simpler convex optimization problem with linear constraints. The power controlled adaptive capacity when transmitters have additional partial CSI about the other links was discussed in section VI. While it is of interest to characterize the adaptive capacity region when the individual channel knowledge is not perfect, this can be handled by our techniques in several interesting cases. In particular, any scenario where a user has more information than others about its link, in the sense that user capacity region when the individual channel knowledge is also not perfect, this can be handled by our techniques in several interesting cases.

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We have characterized the entire adaptive capacity region. We also presented a result which reduces the problem of finding the power controlled adaptive weighted sum-capacity to a much simpler convex optimization problem with linear constraints. The power controlled adaptive capacity when transmitters have additional partial CSI about the other links was discussed in section VI. While it is of interest to characterize the adaptive capacity region when the individual channel knowledge is not perfect, this can be handled by our techniques in several interesting cases. In particular, any scenario where a user has more information than others about its link, in the sense that user capacity region when the individual channel knowledge is also not perfect, this can be handled by our techniques in several interesting cases. In particular, any scenario where a user has more information than others about its link, in the sense that user capacity region when the individual channel knowledge is also not perfect, this can be handled by our techniques in several interesting cases. In particular, any scenario where a user has more information than others about its link, in the sense that user capacity region when the individual channel knowledge is also not perfect, this can be handled by our techniques in several interesting cases. In particular, any scenario where a user has more information than others about its link, in the sense that user capacity region when the individual channel knowledge is also not perfect, this can be handled by our techniques in several interesting cases.
VIII. APPENDIX

APPENDIX A

DERIVATION OF THE RATE EXPRESSIONS IN THEOREM 17 FOR CONTINUOUS VALUED DISTRIBUTIONS

We now show that the rate expressions given in theorem 17 for the continuous fading distribution can indeed be obtained by discretizing the CDF in the probability space and applying the rate allocation algorithm for the discrete fading state case given in theorem 12. Consider fig. 11 which shows the CDF of the fading magnitudes $\psi_1(h_1)$ and $\psi_2(h_2)$ of user 1 and user 2 respectively. Let $\delta$ be the interval between the uniform consecutive cuts as shown in the figure. Let $\psi_i^{-1}(.)$ denote the inverse CDF function defined by

$$\psi_i^{-1}(x) = h_i(x) = \sup\{g|\psi_i(g) < x\}$$

(55)
We apply the rate allocation given by theorem 12 as follows. For each horizontal cut \(i\) shown in the figure, user 2 selects rate \(R_2(\psi_2^{-1}(i\delta))\) such that
\[
R_2(\psi_2^{-1}(i\delta)) + R_1(\psi_1^{-1}((i-1)\delta)) = \frac{1}{2} \log(1 + (\psi_2^{-1}(i\delta))^2)P_2 + (\psi_1^{-1}((i-1)\delta))^2P_1 \tag{56}
\]
In other words, user 2 selects the rate \(R_2(\psi_2^{-1}(i\delta))\) such that its sum with \(R_1(\psi_1^{-1}((i-1)\delta))\) achieves the sum rate constraint. In a similar manner, user 1 selects its rate \(R_1(\psi_1^{-1}((i-1)\delta))\) such that
\[
R_1(\psi_1^{-1}((i-1)\delta)) + R_2(\psi_2^{-1}((i-1)\delta)) = \frac{1}{2} \log(1 + P_2(\psi_2^{-1}((i-1)\delta))^2 + P_1(\psi_1^{-1}((i-1)\delta))^2) \tag{57}
\]
This iterative assignment of rates to the users can be used to obtain a closed form expression for the rates in the limit \(\delta\) tends to 0 as we show below.

Subtracting (57) from (56), we get
\[
R_2(\psi_2^{-1}(i\delta)) - R_2(\psi_2^{-1}(0)) = \frac{1}{2} \log(1 + P_2(\psi_2^{-1}(j\delta))^2 + P_1(\psi_1^{-1}((j-1)\delta))^2) - \frac{1}{2} \log(1 + (P_2\psi_2^{-1}((j-1)\delta))^2 + (P_1\psi_1^{-1}((j-1)\delta))^2)).
\]

Summing over \(i\), we get
\[
R_2(\psi_2^{-1}(i\delta)) - R_2(\psi_2^{-1}(0)) = \sum_{j=1}^{i} \left( \frac{1}{2} \log(1 + P_2(\psi_2^{-1}(j\delta))^2 + P_1(\psi_1^{-1}((j-1)\delta))^2) - \frac{1}{2} \log(1 + (P_2\psi_2^{-1}((j-1)\delta))^2 + (P_1\psi_1^{-1}((j-1)\delta))^2)) \right) \tag{58}
\]

Approximating the difference term in the summation in (58) by partial derivatives, we get
\[
R_2(\psi_2^{-1}(i\delta)) - R_2(\psi_2^{-1}(0)) = \sum_{j=1}^{i} \frac{P_2\psi_2^{-1}((j-1)\delta)}{1 + (P_2\psi_2^{-1}((j-1)\delta))^2 + (P_1\psi_1^{-1}((j-1)\delta))^2} d(\psi_2^{-1}((j-1)\delta))
\]
where \(d(\cdot)\) denotes the differential. Now letting \(\psi_2^{-1}((j-1)\delta) = y\) and \(\psi_2^{-1}(i\delta) = h\) and taking limit \(\delta\) tends to 0, we get the expression for \(R_2(h)\). In a similar manner, \(R_1(h)\) is also obtained.

**APPENDIX B**

**PROOF OF CLAIM 18**

\[
R_s(h_i) = R_s(h_i(0)) + \int_{h_i(0)}^{h_i} \frac{yP_2}{1 + y^2P_1 + (\psi_1^{-1}(\psi_1(y)))^2P_1} \, dy
\]

\[
\leq R_s(h_i(0)) + \int_{h_i(0)}^{h_i} \frac{yP_1}{1 + y^2P_1} \, dy
\]

\[
= R_s(h_i(0)) + \int_{1 + h_i^2(0)P_1}^{1 + h_i^2P_1} \frac{1}{2} dp
\]

\[
= R_s(h_i(0)) + \frac{1}{2} \log(1 + h_i^2P_1) - \frac{1}{2} \log(1 + h_i^2(0)P_1)
\]

\[
\leq \frac{1}{2} \log(1 + h_i^2P_1) + \frac{1}{2} \log(1 + h_i^2(0)P_1) - \frac{1}{2} \log(1 + h_i^2(0)P_1)
\]

Now, let \(R_{12}\) denote the sum-rate \(R_1(h_1) + R_2(h_2)\), where the rates are chosen as in (32). Under the transformation \(\psi_2^{-1}(\psi_1(y)) = z\), we have
\[
R_{12} = R_1(h_1(0)) + R_2(h_2(0)) + \int_{h_2(0)}^{h_2} \frac{P_1\psi_1^{-1}(\psi_2(z))(\psi_1^{-1}(\psi_2(z)))' \, dz}{1 + z^2P_2 + (\psi_1^{-1}(\psi_2(z)))^2P_1} + \int_{h_2(0)}^{h_2} \frac{yP_2}{1 + y^2P_2 + (\psi_1^{-1}(\psi_2(y)))^2P_1} \, dy.
\]

Consider the case when \(\psi_2^{-1}(\psi_1(h_1)) < h_2\). Combining terms of the two integrals above,
\[
R_{12} = R_1(h_1(0)) + R_2(h_2(0)) + \int_{h_2(0)}^{\psi_2^{-1}(\psi_1(h_1))} \frac{P_1\psi_1^{-1}(\psi_2(z))(\psi_1^{-1}(\psi_2(z)))' \, dz}{1 + z^2P_2 + (\psi_1^{-1}(\psi_2(z)))^2P_1} + \int_{h_2(0)}^{\psi_2^{-1}(\psi_1(h_1))} \frac{yP_2}{1 + y^2P_2 + (\psi_1^{-1}(\psi_2(y)))^2P_1} \, dy.
\]
We now substitute $1 + z^2 P_2 + (\psi_1^{-1}(\psi_2(z)))^2 P_1 = p$ in the first integral. We also upper bound the second integral by replacing $z$ in the third term of the denominator by the lower limit of the integral. This gives an upper bound since $(\psi_1^{-1}(\psi_2(y)))^2$ is a non-decreasing function of $z$. By denoting $h^* = (\psi_2^{-1}(\psi_1(h_1)))^2 P_2 + h_1^2 P_1 + 1$, we then get

$$R_{12} \leq R_1(h_1(0)) + R_2(h_2(0)) + \int_{1 + h^*_2(0) P_2 + h_1^2(0) P_1}^{h^*} \frac{y P_2}{1 + y^2 P_2 + (\psi_1^{-1}(\psi_2(y)))^2 P_1} dy + \int_{\psi_1^{-1}(\psi_2(1))}^{h^*_2(0) P_2 + h_1^2(0) P_1} \frac{1}{2p} dp + \int_{\psi_1^{-1}(\psi_2(1))}^{h^*_2(0) P_2 + h_1^2(0) P_1} \frac{y P_2}{1 + y^2 P_2 + h_1^2 P_1} dy$$

$$= \frac{1}{2} \text{log}(1 + h^*_1(0) P_1 + h^*_2(0) P_2) + \left( \frac{1}{2} \text{log}(h^*) - \frac{1}{2} \text{log}(1 + h^*_1(0) P_1 + h^*_2(0) P_2) \right)$$

$$+ \left( \frac{1}{2} \text{log}(1 + h^*_2 P_2 + h_1^2 P_1) - \text{log}(h^*) \right)$$

For the case $\psi_2^{-1}(\psi_1(h_1)) \geq h_2$, the proof follows in a similar fashion.

**Appendix C**

**Proof of Theorem 20**

Let us first find an upper bound for the expected sum-rate of any achievable scheme.

$$\sum_{i=1}^{N} \mathbb{E}[R_i(H_i)] = \sum_{i=1}^{N} \int_0^\infty R_i(h_i) d\psi_i(h_i)$$

(59)

Using similar steps as in the discrete-state derivation in (19), we get by the sum-rate bound of the $N$ user MAC capacity region (see [14]),

$$\sum_{i=1}^{N} \mathbb{E}[R_i(H_i)] \leq \int_0^\infty \frac{1}{2} \text{log}(1 + \sum_{j=1}^{N} h_j^2(x) P_j) dx,$$

thus obtaining an upper bound to the achievable sum-rate.

We will also show that this upper bound is in fact achieved by the rate-allocations prescribed in Theorem 15. The rest of the proof follows from Lemmas 24 and 25 presented below.

**Lemma 24.** For $x \in [0, 1]$ and the rate allocation in (35),

$$\sum_{i=1}^{N} R_i(h_i(x)) = \frac{1}{2} \text{log}(1 + \sum_{j=1}^{N} h_j^2(x) P_j).$$

**Proof:** From the rate allocation in theorem 15 it follows that

$$\sum_{i=1}^{N} R_i(h_i(0)) = \frac{1}{2} \text{log}(1 + \sum_{j=1}^{N} h_j^2(0) P_j).$$

Also, for $x > 0$,

$$\sum_{i=1}^{N} R_i(h_i(x)) = \sum_{i=1}^{N} R_i(h_i(0)) + \sum_{i=1}^{N} \int_{h_i(0)}^{h_i(x)} \frac{y P_i dy}{1 + \sum_{j=1}^{N} (\psi_j^{-1}(\psi_j(y)))^2 P_j}$$

$$= \frac{1}{2} \text{log}(1 + \sum_{j=1}^{N} h_j^2(0) P_j) + \sum_{i=1}^{N} \int_{h_i(0)}^{h_i(x)} \frac{y P_i dy}{1 + \sum_{j=1}^{N} (\psi_j^{-1}(\psi_j(y)))^2 P_j}.$$
Denoting, \( \psi^{-1}(\psi_i(y)) = z \) in the \( i \)th term in the summation and simplifying as discussed in lemma [19]

\[
\sum_{i=1}^{N} R_i(h_i(x)) = \frac{1}{2} \log(1 + \sum_{j=1}^{N} h_j^2(0) P_j) + \int_{h_i(0)}^{h_i(x)} \frac{\sum_{k=1}^{N}(\psi_k^{-1}(\psi_j(y)))P_k}{1 + \sum_{j=1}^{N}(\psi_j^{-1}(\psi_j(y)))^2 P_j} dp
\]

\[
= \frac{1}{2} \log(1 + \sum_{j=1}^{N} h_j^2(0) P_j) + \int_{1}^{1+\sum_{j=1}^{N} h_j^2(x) P_j} \frac{1}{2p} dp
\]

\[
= \frac{1}{2} \log(1 + \sum_{k=1}^{N} h_k^2(x) P_k)
\]

This completes the proof of the lemma.

\[\Box\]

**Lemma 25.** The rate allocation given in (35) is outage-free.

**Proof:** We will show that \( \forall (h_1, h_2, \cdots, h_N) \in \mathcal{H} \) such that \( h_i \geq h_i(0) \) and \( \forall S \subseteq \{1, 2, \cdots, N\} \)

\[
\sum_{i \in S} R_i(h_i) \leq \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2 P_i).
\]

Let \( |S| \) denote the cardinality of the set \( S \) and let \( L \) be the vector formed by reading from left to right the indices of \( \psi_i(h_i) \), \( i \in S \) arranged in increasing order. Let \( L(l) \) denote the value of the \( l \)th component of \( L \). Hence, \( \psi_{L(1)}(h_{L(1)}) \leq \psi_{L(2)}(h_{L(2)}) \leq \cdots \leq \psi_{L(|S|)}(h_{L(|S|)}) \).

\[
\sum_{i \in S} R_i(h_i) = \sum_{i \in S} R_i(h_i(0)) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(y)) dy P_i dy
\]

\[
\leq \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(y)) dy P_i dy
\]

\[
\leq \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(y)) dy P_i dy
\]

As discussed in the two user case, substitute \( \psi_i^{-1}(\psi_i(y)) = z \) in the \( i \)th term in (62) and simplify.

\[
\sum_{i \in S} R_i(h_i) \leq \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(h_i(k))) dy P_i dy
\]

\[
= \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(h_i(k))) dy P_i dy
\]

\[
= \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(h_i(k))) dy P_i dy
\]

where \( \psi_{L(k)}^{-1}(\psi_{L(k)}(h_{L(k)})) = h_{L(k)}(0) \). Also for \( y \in [\psi_{L(k)}^{-1}(\psi_{L(k)}(h_{L(k)})), \psi_{L(k)}^{-1}(\psi_{L(k)}(h_{L(k)}))] \),

\[
\psi_{L(k)}^{-1}(\psi_{L(k)}(h_{L(k)})) \leq \psi_{L(k)}^{-1}(\psi_{L(k)}(h_{L(k)})) \leq y, \forall j \leq k - 1.
\]

Hence in the \( k \)th term in the outer summation, \( \forall j \leq k - 1 \), substituting \( \psi_{L(k)}^{-1}(\psi_{L(k)}(h_{L(k)})) \) for \( y \) in the \( j \)th term in the summation in the denominator of equation (63), we get

\[
\sum_{i \in S} R_i(h_i) \leq \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(h_i(k))) dy P_i dy
\]

\[
= \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(h_i(k))) dy P_i dy
\]

\[
= \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0) P_i) + \sum_{i \in S} \int_{h_i(0)}^{h_i(x)} \psi_i^{-1}(\psi_i(h_i(k))) dy P_i dy
\]

where the empty summation is defined to be 0, i.e. \( \sum_{j=1}^{0} h_j^2 P_j = 0 \).
Simplifying (65),

\[
\sum_{i \in S} R_i(h_i) \leq \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0)P_i) + \frac{1}{2} \log(1 + \sum_{i=1}^{\lvert S \rvert} h_{L(i)}^2 L(i)) - \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2(0)P_i)
\]

\[= \frac{1}{2} \log(1 + \sum_{i \in S} h_i^2 P_i)\]

This proves the claim.

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