TWISTS AND SINGULAR VECTORS IN $\hat{\mathfrak{sl}}(2|1)$ REPRESENTATIONS

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ABSTRACT. We propose new formulas for singular vectors in Verma modules over the affine Lie superalgebra $\hat{\mathfrak{sl}}(2|1)$. We analyze the coexistence of singular vectors of different types and identify the twisted modules $N_{h,k;\theta}^\pm$ arising as submodules and quotient modules of $\hat{\mathfrak{sl}}(2|1)$ Verma modules. We show that with the twists (spectral flow transformations) properly taken into account, a resolution of irreducible representations can be constructed consisting of only the $N_{h,k;\theta}^\pm$ modules.

1. INTRODUCTION

In this paper, we consider elements of representation theory of the affine Lie superalgebra $\hat{\mathfrak{sl}}(2|1)$. The affine $\mathfrak{sl}(2|1)$ symmetry emerges in the models of disordered systems introduced in relation to the integer quantum Hall effect [1]–[3]. The $\hat{\mathfrak{sl}}(2|1)$ algebra is also interesting because in a certain sense, it combines the characteristic features of the affine Lie algebra $\hat{\mathfrak{sl}}(2)$ and the $\mathcal{N}=2$ superconformal algebra. To the latter, it is related via Hamiltonian reduction [4]–[6] and its “inversion” [7]; a relation between the $\hat{\mathfrak{sl}}(2|1)$ and $\hat{\mathfrak{sl}}(2)$ algebras, apart from the obvious subalgebra embedding, was worked out in [8], where $\hat{\mathfrak{sl}}(2|1)$ was shown to be a vertex-operator extension of the sum of two $\hat{\mathfrak{sl}}(2)$ algebras with “dual” levels $k$ and $k'$ such that $(k+1)(k'+1) = 1$. The construction in [8] has led to a decomposition formula for $\hat{\mathfrak{sl}}(2|1)$ representations in terms of $\hat{\mathfrak{sl}}(2)$ representations. A natural class of $\hat{\mathfrak{sl}}(2|1)$ representations can be obtained by taking admissible representations of the two $\hat{\mathfrak{sl}}(2)$ algebras. However, by far not all of these $\hat{\mathfrak{sl}}(2|1)$ representations have been studied before; in particular, the corresponding characters are only known for a subclass of these representations [9].

Characters of a broader class of $\hat{\mathfrak{sl}}(2|1)$ representations can be found by constructing resolutions; this in turn requires analyzing singular vectors in and the mappings between Verma modules. An important role is here played by spectral flows (twists) and other automorphisms of the $\hat{\mathfrak{sl}}(2|1)$ algebra. In this paper, we generalize the “continued” formulas for singular vectors [9], [10] in $\hat{\mathfrak{sl}}(2|1)$ Verma modules such that the new formulas are applicable to the case where degenerations of a certain type occur, under which the previously known formulas gave the (incorrect) vanishing result. These are the degenerations where the so-called “charged”
singular vector exists simultaneously with the MFF singular vectors to be introduced below. This “stability” of singular vectors under degenerations of modules is ensured by incorporating twists into the “continued” formula. These singular vectors can be used in constructing the resolutions and finding character formulas.

2. \(\hat{sl}(2|1)\) Modules and Automorphisms

2.1. The \(\hat{sl}(2|1)\) Algebra and Automorphisms. The affine Lie superalgebra \(\hat{sl}(2|1)\) is spanned by four bosonic currents \(E^{12}, H^-, F^{12}\), and \(H^+\), four fermionic ones, \(E^1, E^2, F^1,\) and \(F^2\), and the central element (which we identify with its eigenvalue \(k\)). For convenience, we give in Fig. 1 the two-dimensional root diagram of the finite-dimensional Lie superalgebra \(sl(2|1)\), represented in Minkowski space with the fermionic roots along the light-cone directions. The \(\hat{sl}(2)\) subalgebra is generated by \(E^{12}, H^-,\) and \(F^{12}\), and it commutes with the \(u(1)\) subalgebra generated by \(H^+\). The nonvanishing commutation relations are given by

\[
\begin{align*}
[H^-_m, E^{12}_n] &= E^{12}_{m+n}, \\
[E^{12}_m, F^{12}_n] &= m\delta_{m+n,0}k + 2H^-_{m+n}, \\
[F^{12}_m, E^1_n] &= F^1_{m+n}, \\
[F^{12}_m, E^2_n] &= -F^2_{m+n}, \\
[H^\pm_m, E^1_n] &= \frac{1}{2}E^1_{m+n}, \\
[H^\pm_m, E^2_n] &= \mp\frac{1}{2}E^2_{m+n}, \\
[H^\pm_m, F^{12}_n] &= -m\delta_{m+n,0}k + H^+_{m+n} - H^-_{m+n}, \\
[E^1_m, F^{12}_n] &= m\delta_{m+n,0}k + H^+_{m+n} + H^-_{m+n}, \\
[E^2_m, F^{12}_n] &= E^{12}_{m+n}, \\
[F^1_m, F^{12}_n] &= F^{12}_{m+n}.
\end{align*}
\]
The Sugawara energy-momentum tensor is given by

\[(2.1) \quad T_{\text{Sug}} = \frac{1}{k+1} \left( H^- H^+ - H^+ H^- + E^{12} F^{12} + E^1 F^1 - E^2 F^2 \right). \]

There are the algebra automorphisms

\[(2.2) \quad \alpha : \quad E^1_n \mapsto F^2_n, \quad E^2_n \mapsto F^1_n, \quad E^{12}_n \mapsto F^{12}_n, \]

\[H^+_n \mapsto H^+_n, \quad H^-_n \mapsto -H^-_n, \]

\[E^1_n \mapsto E^2_n, \quad E^2_n \mapsto E^1_n, \quad E^{12}_n \mapsto E^{12}_n, \]

\[(2.3) \quad \beta : \quad F^1_n \mapsto -F^2_n, \quad F^2_n \mapsto -F^1_n, \quad F^{12}_n \mapsto F^{12}_n, \]

\[H^+_n \mapsto -H^+_n, \quad H^-_n \mapsto H^-_n, \]

and a family of automorphisms for \( \theta \in \mathbb{Z} \),

\[(2.4) \quad U_\theta : \quad E^1_n \mapsto E^1_{n-\theta}, \quad E^2_n \mapsto E^2_{n+\theta}, \quad E^{12}_n \mapsto E^{12}_{n+\theta}, \quad H^+_n \mapsto H^+_n + k \theta \delta_{n,0} \]

(with the \( \hat{s}\ell(2) \) subalgebra remaining invariant). These automorphisms satisfy the relations

\[(2.5) \quad \alpha^2 = 1, \quad \beta^2 = 1, \quad (\alpha \beta)^4 = 1, \quad \alpha U_\theta = U_\theta \alpha, \quad (\beta U_\theta)^2 = 1. \]

The \( \mathbb{Z} \) subgroup of automorphisms \( (U_\theta)_{\theta \in \mathbb{Z}} \) is called the spectral flow and is extensively used in what follows. Another \( \mathbb{Z} \) algebra of automorphisms (a spectral flow affecting the \( \hat{s}\ell(2) \) subalgebra, cf. [1]) acts as

\[(2.6) \quad A_\eta : \quad E^1_n \mapsto F^1_{n+\eta}, \quad E^2_n \mapsto F^2_{n+\eta}, \quad E^{12}_n \mapsto F^{12}_{n+\eta}, \quad H^+_n \mapsto H^+_n + k \eta \delta_{n,0}, \]

\[F^1_n \mapsto F^1_{n-\eta}, \quad F^2_n \mapsto F^2_{n-\eta}, \quad F^{12}_n \mapsto F^{12}_{n-\eta}, \quad H^-_n \mapsto H^-_n. \]

There also is an automorphism

\[(2.7) \quad \gamma = U_{\frac{1}{2}} \circ A_{-\frac{1}{2}} \]

(while \( U_{\frac{1}{2}} \) and \( A_{-\frac{1}{2}} \) are not automorphisms, but mappings into an isomorphic algebra, their composition is). For \( \theta \in \mathbb{Z} \), its powers \( \mathcal{X}_\theta = \gamma^\theta \) map the generators as

\[(2.8) \quad \mathcal{X}_\theta : \quad E^1_n \mapsto E^1_{n-\theta}, \quad E^2_n \mapsto E^2_{n+\theta}, \quad E^{12}_n \mapsto E^{12}_{n+\theta}, \quad H^+_n \mapsto H^+_n - \frac{k}{2} \theta \delta_{n,0}, \]

\[F^1_n \mapsto F^1_{n+\theta}, \quad F^2_n \mapsto F^2_{n+\theta}, \quad F^{12}_n \mapsto F^{12}_{n+\theta}, \quad H^-_n \mapsto H^-_n + \frac{k}{2} \theta \delta_{n,0}. \]
2.2. **Twisted highest-weight conditions and twisted Verma modules.** Applying algebra automorphisms to modules gives generically non-isomorphic modules. The nilpotent subalgebra is also mapped under the action of automorphisms, and the annihilation conditions satisfied by highest-weight vectors change accordingly. Thus, the existence of an automorphism group entails a certain freedom in choosing the type of annihilation conditions imposed on highest-weight vectors in highest-weight (in particular, Verma) modules. We select a family of nilpotent subalgebras such that the corresponding annihilation conditions read

\[
E_{\geq -\theta}^1 \approx 0, \quad E_{\geq \theta}^2 \approx 0, \\
F_{\geq \theta + 1}^1 \approx 0, \quad F_{\geq 1 - \theta}^2 \approx 0, \quad F_{\geq 1}^{12} \approx 0
\]  

for a fixed \( \theta \in \mathbb{Z} \). These annihilation conditions are called the **twisted highest-weight conditions** for \( \theta \neq 0 \) and the untwisted ones in the particular case where \( \theta = 0 \). The twisted highest-weight conditions with different \( \theta \) are mapped into one another by spectral flow (2.4).

We note that the automorphism \( \gamma \circ \alpha \) maps the twisted highest-weight conditions (2.9) into highest-weight conditions of a different class, namely those where the annihilation conditions are given by

\[
E_{\geq -\theta}^1 \approx 0, \quad E_{\geq \theta + 1}^2 \approx 0, \quad E_{\geq 0}^{12} \approx 0, \\
F_{\geq \theta + 1}^1 \approx 0, \quad F_{\geq -\theta}^2 \approx 0.
\]  

A Verma module generated from a twisted highest-weight state (satisfying conditions (2.9)) can contain a *submodule* generated from a state satisfying (2.10), and it is not a priori guaranteed that the same submodule can be generated from a twisted highest-weight state satisfying annihilation conditions (2.9) for some \( \theta \). This is in contrast with the more familiar case of the \( \hat{\mathfrak{s}} \ell(2) \) algebra, where any submodule in a given Verma module can be generated from vector(s) satisfying highest-weight conditions of the same type as for the highest-weight vector of the module (i.e., not those transformed by automorphisms). Similarly, even within the chosen family (2.9) of twisted highest-weight conditions, a module with the untwisted \((\theta = 0)\) highest-weight vector can have submodules generated from some twisted highest-weight states, but *not* from the one with zero twist.

For any \( \hat{\mathfrak{s}} \ell(2|1) \) module \( \mathcal{R} \), we let \( \mathcal{R}_{\theta} \) denote the twisted module \( \mathcal{U}_{\theta} \mathcal{R} \), see (2.4).

2.3. **The action of the spectral flow transform on characters.** Spectral flow transform (2.4) acts on characters as follows. The character \( \chi_{\mathcal{R}_{\theta}} = \chi_{\mathcal{R}}^\theta \) of a twisted

\[\text{The \( \approx \) sign means that the operators must be applied to a vector; at the moment, we are interested in the list of annihilation operators, rather than in the vector, hence the notation.} \]
module $\mathcal{R}_{\theta}$ is expressed through the character of $\mathcal{R}$ as
\begin{equation}
\chi^\mathcal{R}_{\theta}(q, z, \zeta) = \zeta^{-k\theta} q^{-k\theta^2} \chi^\mathcal{R}(q, z, q^2 \zeta).
\end{equation}

Clearly, in studying characters (as well as other properties of representations), it therefore suffices to consider a representative of each spectral flow orbit.

2.4. Gradings and extremal states. Clearly, each $\hat{\mathfrak{sl}}(2|1)$ module is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$-graded: the gradings are by the charge (the $H_0^-$ eigenvalue), the hypercharge (the $H_0^+$ eigenvalue), and the level (minus the $L_0$ eigenvalue, where $L_n$ are the Virasoro generators corresponding to the Sugawara energy-momentum tensor). Thus, states of the module occupy sites of a three-dimensional lattice. Because of the highest-weight conditions, all the states in the module lie below a certain plane in the three-dimensional space. Those lattice sites that are occupied by at least one state from the module form a (convex) three-dimensional body in $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. The states at the surface of this body are called extremal states, or extremal vectors. In particular, the highest-weight state is an extremal vector. In Fig. [2] we give the extremal diagram of an $\hat{\mathfrak{sl}}(2|1)$ Verma module.

3. $\hat{\mathfrak{sl}}(2|1)$ Verma Modules

In what follows, we assume that $k \in \mathbb{C} \setminus \{-1\}$, $h_-$ and $h_+$ are a priori arbitrary complex numbers, and $\theta \in \mathbb{Z}$.

We define a family of twisted Verma modules related to each other by the spectral flow.

**Definition 3.1.** A twisted Verma module $\mathcal{P}_{h_-, h_+, k; \theta}$ over the level-$k$ $\hat{\mathfrak{sl}}(2|1)$ algebra is freely generated by $E_{\leq \theta - 1}^1$, $E_{\leq \theta - 1}^2$, $E_{\leq \theta}^{12}$, $F_{\leq \theta}^1$, $F_{\leq \theta}^2$, $F_{\leq \theta}^{12}$, $H_{\leq \theta - 1}^-$, and $H_{\leq \theta - 1}^+$ from the twisted highest-weight state $|h_-, h_+, k; \theta\rangle$ satisfying annihilation conditions (2.9) and the conditions
\begin{align}
H_0^- |h_-, h_+, k; \theta\rangle &= h_- |h_-, h_+, k; \theta\rangle, \\
H_0^+ |h_-, h_+, k; \theta\rangle &= (h_+ - k\theta) |h_-, h_+, k; \theta\rangle.
\end{align}

We then have
\begin{equation}
U_{\theta'} |h_-, h_+, k; \theta\rangle = |h_-, h_+, k; \theta + \theta'\rangle
\end{equation}
and, obviously, $U_{\theta'} \mathcal{P}_{h_-, h_+, k; \theta} = \mathcal{P}_{h_-, h_+, k; \theta + \theta'}$ for Verma modules.

We write
\begin{equation}
|X\rangle \doteq |h_-, h_+, k; \theta\rangle
\end{equation}
for any state $|X\rangle$ that satisfies highest-weight conditions (2.9) and (3.1)–(3.2).
The character of $P_{h_-,h_+,k;\theta}$ is given by

\begin{equation}
\chi_{P_{h_-,h_+,k;\theta}}(q, z, \zeta) \equiv \text{Tr}_{P_{h_-,h_+,k;\theta}} \left( q^{zH_0^+ - \zeta H_0^-} \right) =
\end{equation}

\begin{equation}
= z^{h_- - h_+ - (k+1)\theta} q^{\frac{h_-^2 - h_+^2}{k+1} + 2\theta h_+ - (k+1)\theta^2} \frac{\vartheta_{1,0}(q, z^{\frac{1}{2}} \zeta^{\frac{1}{2}}) \vartheta_{1,0}(q, z^{\frac{1}{2}} \zeta^{-\frac{1}{2}})}{\vartheta_{1,1}(q, z) \prod_{m \geq 1} (1 - q^m)^3},
\end{equation}

where the Jacobi theta functions are defined by

\begin{equation}
\vartheta_{1,1}(q, z) = \sum_{m \in \mathbb{Z}} (-1)^m q^{\frac{1}{2}(m^2 - m)} z^{-m} = \prod_{m \geq 0} (1 - z q^m) \prod_{m \geq 1} (1 - q^m) \prod_{m \geq 1} (1 - q^m),
\end{equation}

**Figure 2. Extremal diagram of an $\hat{sl}(2|1)$ Verma module.** The paraboloid surface interpolates between the extremal states. The horizontal axes represent the eigenvalues of $H_0^+ + H_0^-$ and $H_0^+ - H_0^-$ and the vertical axis gives minus the level; the highest-weight vector is conventionally placed at the origin. The sequence of arrows shows a charged singular vector (see Sec. 4); depending on how the identifications are made, this is either $C^(-)(-3)$ (with each odd arrow representing an $F_0^1$ and each even arrow an $E_0^2$ generator) or $C^+(3)$ (respectively, $F_2$ and $E_1$), see Eqs. (4.2) and (4.3). The action of spectral flow (2.4) results in moving the surface over itself such that the origin slides down along the parabolic curve running through the endpoints of every second arrow.
(3.7) \( \vartheta_{1,0}(q, z) = \sum_{m \in \mathbb{Z}} q^{\frac{1}{2}(m^2 - m)} z^{-m} \)

\[
= \prod_{m \geq 0} (1 + z^{-1} q^m) \prod_{m \geq 1} (1 + q^m) \prod_{m \geq 1} (1 - q^m).
\]

We now let \( |h_-, k; \theta \rangle^- \) denote the state satisfying the highest-weight conditions

\[
E_1^\alpha \approx 0, \quad E_2^\alpha \approx 0, \quad F_\alpha \approx 0, \quad F_{1-\alpha} \approx 0
\]

and

\[
H^-_0 |h_-, k; \theta \rangle^- = h_- |h_-, k; \theta \rangle^-.
\]

Comparing the annihilation conditions in Eqs. (2.9) and (3.8), we see that the latter are a strengthened version of the former. In view of the commutation relations of the algebra, it then follows that only one of the two highest-weight parameters \( h_\pm \) remains independent. We let \( \mathcal{N}^-_{h_-, k; \theta} \) be the module freely generated by \( E_{\leq \theta-1}, E_{\leq -1}^\alpha, E_{\leq -1}^{12}, F_{\leq \theta-1}^\alpha, F_{\leq \theta}^2, F_{\leq 0}^{12}, F_{\leq 0}^{1}, H_{\leq -1}, \) and \( H_{\leq -1}^+ \) from \( |h_-, k; \theta \rangle^- \). Its character is given by

\[
\chi_{\mathcal{N}^-_{h_-, k; \theta}}(q, z, \zeta) = \frac{\chi_{\mathcal{P} h_-, h_- k; \theta}(q, z, \zeta)}{1 + q^{-\theta} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}}}. \]

In terms of extremal diagrams of the type shown in Fig. 2, the module \( \mathcal{N}^-_{h_-, k; \theta} \) is more narrow than the Verma module \( \mathcal{P} \). Accordingly, we call \( \mathcal{N}^-_{h_-, k; \theta} \) the narrow Verma modules (even though the term “Verma module” involves a certain abuse in this case).

There is the second type of narrow Verma modules. Namely, let \( |h_-, k; \theta \rangle^+ \) denote the state satisfying the highest-weight conditions

\[
E_1^\alpha \approx 0, \quad E_2^\alpha \approx 0, \quad F_\alpha \approx 0, \quad F_{1-\alpha} \approx 0
\]

and

\[
H^+_0 |h_-, k; \theta \rangle^+ = h_- |h_-, k; \theta \rangle^+.
\]

and let \( \mathcal{N}^+_{h_-, k; \theta} \) be the module freely generated by \( E_{\leq \theta-1}, E_{\leq -1}^\alpha, E_{\leq -1}^{12}, F_{\leq \theta}^1, F_{\leq 0}^{2}, F_{\leq -1}^{12}, F_{\leq 0}^{-1}, H_{\leq -1}, \) and \( H_{\leq -1}^+ \) from \( |h_-, k; \theta \rangle^+ \). The character of \( \mathcal{N}^+_{h_-, k; \theta} \) is given by

\[
\chi_{\mathcal{N}^+_{h_-, k; \theta}}(q, z, \zeta) = \frac{\chi_{\mathcal{P} h_-, h_- k; \theta}(q, z, \zeta)}{1 + q^{\theta} z^{-\frac{1}{2}} \zeta^{-\frac{1}{2}}}. \]
The states introduced above behave as follows under the action of some of the automorphisms listed in Sec. 2.1:

\[(3.14)\quad \beta|h_-, k; \theta^\pm\rangle = |h_-, k; -\theta\rangle, \quad U_{\theta'}|h_-, k; \theta\rangle^\pm = |h_-, k; \theta + \theta'\rangle^\pm.\]

For the corresponding Verma modules, we then have, e.g.,

\[(3.15)\quad \beta N_{h_-, k; \theta}^\pm = N_{h_-, k; -\theta}^\pm.\]

We also note that for any module $\mathcal{M}$, the action of the automorphism $\beta$ on its character is given by $\chi_{\beta \mathcal{M}}(q, z, \zeta) = \chi_{\mathcal{M}}(q, z, \zeta^{-1})$.

### 4. Charged singular vectors

The narrow Verma modules can be obtained from the Verma modules $\mathcal{P}$ by taking the quotient with respect to states that are traditionally called the charged singular vectors.

#### 4.1. Charged singular vectors: explicit formulas. By definition, charged singular vectors are those singular vectors that occur in the Verma module $\mathcal{P}_{h_-, h_+, k; \theta}$ whenever

\[(4.1)\quad h_+ = \pm h_- - (k + 1)n, \quad n \in \mathbb{Z}.\]

They are given by an explicit construction as follows \[8\], \[10\]. For $h_+ - h_- = -(k + 1)n$, $n \in \mathbb{Z}$, the charged singular vector in the twisted Verma module $\mathcal{P}_{h_-, h_+, k; \theta}$ is given by

\[(4.2)\quad |C^{(-)}(n, h_-, k; \theta)\rangle =
\begin{cases} 
E_{\theta + n}^2 \cdots E_{\theta - 1}^2 \cdot F_{\theta + n}^1 \cdots F_{\theta}^1 |h_+, h_+ - (k + 1)n; k; \theta\rangle, & n \leq 0, \\
E_{\theta - n}^1 \cdots E_{\theta - 1}^1 \cdot F_{\theta - n}^2 \cdots F_{\theta}^2 |h_-, h_- - (k + 1)n; k; \theta\rangle, & n \geq 1.
\end{cases}\]

Similarly, whenever $h_+ + h_- = -n(k + 1)$, $n \in \mathbb{Z}$, the charged singular vector in $\mathcal{P}_{h_-, h_+, k; \theta}$ is

\[(4.3)\quad |C^{(+)}(n, h_-, k; \theta)\rangle =
\begin{cases} 
E_{\theta + n}^2 \cdots E_{\theta - 1}^2 \cdot F_{\theta + n + 1}^1 \cdots F_{\theta}^1 |h_-, -h_+ - (k + 1)n; k; \theta\rangle, & n \leq -1, \\
E_{\theta - n}^1 \cdots E_{\theta - 1}^1 \cdot F_{\theta - n}^2 \cdots F_{\theta}^2 |h_-, -h_+ - (k + 1)n; k; \theta\rangle, & n \geq 0.
\end{cases}\]
It is elementary to verify that vector (4.2) satisfies the annihilation conditions (we omit the singular vector itself)

\[ E_{\theta-n}^1 \approx 0, \quad E_{\theta+n}^2 \approx 0, \quad F_{\theta+n}^1 \approx 0, \quad F_{\theta-n}^2 \approx 0, \]

i.e., conditions (3.8) with the twist parameter \( \theta + n \). It is easy to see that these annihilation conditions imply that this vector does indeed generate a submodule in the corresponding module \( \mathcal{P}_{h-h_+, k; \theta} \). More precisely, we have

\[ |C^(-)(n, h_-, k; \theta)\rangle \equiv \begin{cases} |h_--\frac{1}{2}, k; \theta + n\rangle^-, \quad n \leq 0, \\ |h_-, k; \theta + n\rangle^-, \quad n \geq 1. \end{cases} \]

We here use the notation similar to (3.4) to express the fact that the vector in the left-hand side satisfies the same annihilation conditions and has the same eigenvalues as those determined by Eqs. (3.8), (3.9) for the vector in the right-hand side.

Similarly, vector (4.3) satisfies the annihilation conditions

\[ E_{\theta-n}^1 \approx 0, \quad E_{\theta+n}^2 \approx 0, \quad F_{\theta+n+1}^1 \approx 0, \quad F_{\theta-n}^2 \approx 0, \]

i.e., conditions (3.11) with the twist parameter \( \theta + n \), or more precisely,

\[ |C^+(n, h_-, k; \theta)\rangle \equiv \begin{cases} |h_-, k; \theta + n\rangle^+, \quad n \leq -1, \\ |h_--\frac{1}{2}, k; \theta + n\rangle^+, \quad n \geq 0. \end{cases} \]

In what follows, we write \( C^{(\pm)}(n, h_-, k) = C^{(\pm)}(n, h_-, k) |h_-, h_--n(k+1), k\rangle \) and call \( C^{(\pm)} \) the charged singular vector operator. We now consider the submodules generated from \( |C^-(0, h_-, k; \theta)\rangle \) and \( |C^+(0, h_-, k; \theta)\rangle \) in more detail.

4.2. Charged singular vectors: submodules and quotients. In the Verma module \( \mathcal{P}_{h-h_+, k; \theta} \), there is the charged singular vector \( |C^-(0, h_-, k; \theta)\rangle \) given by

\[ F_{\theta}^1 |h_-, h_-, k; \theta\rangle \equiv |h_--\frac{1}{2}, k; \theta\rangle^- \].

The submodule generated from (4.8) is \( \mathcal{N}_{h--\frac{1}{2}, k; \theta} \subset \mathcal{P}_{h-h_+, k; \theta} \). Moreover, we have the exact sequence

\[ 0 \to \mathcal{N}_{h--\frac{1}{2}, k; \theta} \to \mathcal{P}_{h-h_+, k; \theta} \to \mathcal{N}_{h-h_+, k; \theta} \to 0. \]
The proof follows from the above observations on the highest-weight conditions and the character identity

\[
\chi^{N-}_{h-,\frac{1}{2},k;\theta}(q, z, \zeta) + \chi^{N-}_{h-,k;\theta}(q, z, \zeta) = \chi^{P}_{h-,h-,k;\theta}(q, z, \zeta).
\]

Similarly, in the Verma module \(P_{h-,h-,k;\theta}\), we have the charged singular vector \(|C^+(0, h-, k; \theta)\rangle\) given by

\[
F^2_{\theta}|h-, h-, k; \theta\rangle \doteq |h- - \frac{1}{2}, k; \theta\rangle^+.
\]

The module generated from this state is \(N^+_{h-,\frac{1}{2},k;\theta}\), and we then have the exact sequence

\[
0 \rightarrow N^+_{h-,\frac{1}{2},k;\theta} \rightarrow P_{h-,h-,k;\theta} \rightarrow N^+_{h-,k;\theta} \rightarrow 0.
\]

5. MFF SINGULAR VECTORS IN \(\widehat{sl}(2|1)\) VERMA MODULES

The charged singular vectors considered in the previous section are given by simple explicit formulas because they occur at one of the extremal states of a Verma module. In addition to these singular vectors, there exist those lying in the interior of the module. We study these singular vectors in this section.

5.1. MFF singular vectors as nonvanishing Verma module elements. A proposal for \(\widehat{sl}(2|1)\) singular vectors corresponding to reflections with respect to two bosonic roots has been known for some time \([9] [10]\). The corresponding singular vector formulas are similar to those for affine Lie algebras given in \([12]\) by Malikov, Feigin, and Fuks (hence the acronym MFF), in fact most similar to the \(\widehat{sl}(2)\) singular vector formulas, and we therefore call them the MFF singular vectors even in the \(\widehat{sl}(2|1)\) context. Unlike their \(\widehat{sl}(2)\) counterpart, however, these singular vectors formulas can give the vanishing result for some “degenerate” values of the highest-weight parameters (this was noted in \([10]\)).

Because of this vanishing, it thus appears that the submodule otherwise generated from a chosen MFF singular vector “vanishes” at some degenerate points in the highest-weight parameter space. However, this vanishing in fact pertains to the formula itself and signals only a possible change of the submodule structure. The singular vector formulas in \([9] [10]\) explicitly involve annihilation operators and can therefore lead to the vanishing result in the case of certain degenerations, i.e., on certain subsets in the highest-weight parameter space. The structure of submodules is reorganized at these special points of the parameter space, but the known MFF formula does not provide us with an adequate tool for studying these degenerations.
In what follows, we propose a formula for the singular vectors (which we still call the MFF singular vectors) involving only creation operators and therefore non-vanishing in a given Verma module. Generically, the new expression gives a vector that generates the same submodule as the “old” MFF vector. Unlike the latter, however, the new formulas allow us to see how the structure of submodules is rearranged at the degeneration points in the highest-weight parameter space.

In fact, each of the “old” MFF expressions, \( \text{MFF}^+ \) and \( \text{MFF}^- \), is to be replaced with a pair of formulas, respectively \( (\text{MFF}^+^+, \text{MFF}^+^-) \) and \( (\text{MFF}^-^+, \text{MFF}^-^-) \). Generically, the three vectors \( \text{MFF}^+^+, \text{MFF}^+^- \), and \( \text{MFF}^- \) generate the same submodule, but degeneration can affect the relation between the submodules generated from each of these vectors. The situation with the \( \text{MFF}^-^+ \) and \( \text{MFF}^-^- \) vectors is similar, as we describe in what follows.

5.2. The MFF singular vectors: “continued” expressions and the algebraic rules. We recall (see, e.g., \cite{9} and references therein) that singular vectors occur in a Verma module \( \mathcal{P}_{h_-, h_+, k; \theta} \) whenever \( h_- = j^+(r, s, k) \) or \( h_- = j^-(r, s, k) \), where

\[
\begin{align*}
 j^+(r, s, k) &= \frac{r}{2} - \frac{s - 1}{2} (k + 1), & r, s \in \mathbb{N}, \\
 j^-(r, s, k) &= -\frac{r}{2} + \frac{s}{2} (k + 1), & r, s \in \mathbb{N}.
\end{align*}
\]

As noted above, we call the corresponding singular vectors the MFF singular vectors for their obvious similarities with the construction in \cite{12}.

To begin with \( (5.1) \), we now describe the pair of vectors each of which generates a submodule in \( \mathcal{P}_{j^+(r, s, k), h_+, k} \). Writing \( j^+ \) for \( h_- = j^+(r, s, k) \) for brevity, we define

\[
\text{MFF}^+^- (r, s, h_+, k) =
\begin{align*}
&= E_{-s}^2 (F_0^{12})^{2j^++(2s-2)(k+1)} F_{-s+1}^1 (E_{-1}^{12})^{2j^++(2s-3)(k+1)} \\
&\times E_{-s+1}^2 (F_0^{12})^{2j^++(2s-4)(k+1)} F_{-s+2}^1 (E_{-1}^{12})^{2j^++(2s-5)(k+1)} \\
&\vdots \\
&\times E_{-2}^2 (F_0^{12})^{2j^++2(k+1)} F_{-1}^1 (E_{-1}^{12})^{2j^++k+1} \\
&\times E_{-1}^2 (F_0^{12})^{2j^+} F_0^1 \langle j^+(r, s, k), h_+, k \rangle
\end{align*}
\]

and
These expressions can be rewritten as Verma module elements by repeatedly applying the relations

\[(F_0^{12})^n E_m^{12} = (-n(n-1)F_m^{12} - 2nH_m F_0^{12} + E_m^{12} F_0^{12}) (F_0^{12})^{n-2},\]
\[(F_0^{12})^n F_m^{12} = (-nF_m^{12} + E_m^{12} F_0^{12}) (F_0^{12})^{n-1},\]
\[(F_0^{12})^n F_m^{12} = (nF_m^{12} + E_m^{12} F_0^{12}) (F_0^{12})^{n-1},\]
\[(F_0^{12})^n F_m^{12} = (nF_m^{12} + H_m F_0^{12}) (F_0^{12})^{n-1}\]

and

\[(E_{m-1})^n F_m^{12} = (-n(n-1)E_{m-2}^{12} - kn\delta_{m1} E_{m-1}^{12} + 2nH_{m-1} E_{m-1}^{12} + F_{m-1}^{12} E_{m-1}^{12}) (E_{m-1}^{12})^{n-2},\]
\[(E_{m-1})^n F_m^{12} = (nE_{m-1}^{12} + F_{m-1}^{12}) (E_{m-1}^{12})^{n-1},\]
\[(E_{m-1})^n F_m^{12} = (-nE_{m-1}^{12} + F_{m-1}^{12}) (E_{m-1}^{12})^{n-1},\]
\[(E_{m-1})^n F_m^{12} = (-nE_{m-1}^{12} + H_{m} E_{m-1}^{12}) (E_{m-1}^{12})^{n-1},\]

which are valid for \(n \in \mathbb{N}\) and are postulated for \(n \in \mathbb{C}\). Useful consequences of these formulas are given by

\[(E_{-1})^n F_1^{12} F_{-s}^{2} \ldots F_{-1}^{2}(E_{-1}^{12})^\alpha = (-1)^{s+1} E_{-s}^{12} F_{-s}^{2} (E_{-1}^{12})^\alpha E_{-s+1}^{12} F_{-s+1}^{2} \ldots F_{-1}^{2} ,\]

and

\[(E_{-1})^n F_1^{12} F_{-s}^{2} \ldots F_{-1}^{2}(F_0^{12})^\alpha = (F_0^{12})^\alpha E_{-s}^{12} F_{-s}^{2} \ldots F_{-1}^{2} ,\]

and similar relations are valid for the other \(\tilde{sl}(2)\) doublet of currents, \(E_2\) and \(F_1\).

In what follows, these formulas are applied for complex \(\alpha\).

By a direct calculation using (5.5) and (5.6), it is easy to verify the following

Lemma for the vectors \(MFF^+_{(+)}\) and \(MFF^+_{(-)}\).

**Lemma 5.1.** For \(r, s \in \mathbb{N}\), the states \(MFF^+_{(-)}(r, s, h_+, k)\) and \(MFF^+_{(+)}(r, s, h_+, k)\) satisfy the annihilation conditions and eigenvalue formulas such that we can write
(see (3.5))

(5.9) \[ \text{MFF}^{+}_{(-)}(r, s, h_{+}, k) \doteq |j^{+}(r, s, k) - r, h_{+} - s(k + 1), k; -s\rangle, \]

(5.10) \[ \text{MFF}^{+}_{(+)}(r, s, h_{+}, k) \doteq |j^{+}(r, s, k) - r, h_{+} + s(k + 1), k; s\rangle. \]

These conditions imply, in particular, that these vectors are indeed “singular” in the sense that they generate \( \tilde{\mathfrak{sl}}(2|1) \) submodules.

Next, in the case described in (5.2), whenever \( h_{-} = j^{-}(r, s, k) \equiv j^{-} \), we define the pair of singular vectors in \( \mathcal{P}_{j^{-}(r,s,k),h_{+},k} \),

(5.11) \[ \text{MFF}^{-}_{(-)}(r, s, h_{+}, k) = \]

\[ = (E_{-1}^{12})^{(2s-1)(k+1)-2j^{-}} E_{1-s}^{-} (F_{0}^{12})^{(2s-2)(k+1)-2j^{-}} E_{2-s}^{-} F_{1}^{2} \]

\[ \times (E_{-1}^{12})^{(2s-3)(k+1)-2j^{-}} E_{2-s}^{-} (F_{0}^{12})^{(2s-4)(k+1)-2j^{-}} F_{3-s}^{-} \]

\[ \vdots \]

\[ \times (E_{-1}^{12})^{3(k+1)-2j^{-}} E_{-1}^{-} (F_{0}^{12})^{2(k+1)-2j^{-}} F_{0}^{1} \]

\[ \times (E_{-1}^{12})^{k+1-2j^{-}} |j^{-}, h_{+}, k\rangle \]

and

(5.12) \[ \text{MFF}^{-}_{(+)}(r, s, h_{+}, k) = \]

\[ = (E_{-1}^{12})^{(2s-1)(k+1)-2j^{-}} E_{1-s}^{1} (F_{0}^{12})^{(2s-2)(k+1)-2j^{-}} E_{2-s}^{2} F_{1}^{2} \]

\[ \times (E_{-1}^{12})^{(2s-3)(k+1)-2j^{-}} E_{2-s}^{1} (F_{0}^{12})^{(2s-4)(k+1)-2j^{-}} F_{3-s}^{2} \]

\[ \vdots \]

\[ \times (E_{-1}^{12})^{3(k+1)-2j^{-}} E_{-1}^{1} (F_{0}^{12})^{2(k+1)-2j^{-}} F_{0}^{2} \]

\[ \times (E_{-1}^{12})^{k+1-2j^{-}} |j^{-}, h_{+}, k\rangle. \]

Similarly to Lemma 5.1 it is easy to obtain the annihilation conditions satisfied by these vectors.

**Lemma 5.2.** For \( r, s \in \mathbb{N} \), the states \( \text{MFF}^{-}_{(-)}(r, s, h_{+}, k) \) and \( \text{MFF}^{-}_{(+)}(r, s, h_{+}, k) \) satisfy the annihilation conditions and eigenvalue formulas such that we can write

(5.13) \[ \text{MFF}^{-}_{(-)}(r, s, h_{+}, k) \doteq |j^{-}(r, s, k) + r, h_{+} - (s - 1)(k + 1), k; 1 - s\rangle, \]

(5.14) \[ \text{MFF}^{-}_{(+)}(r, s, h_{+}, k) \doteq |j^{-}(r, s, k) + r, h_{+} + (s - 1)(k + 1), k; s - 1\rangle. \]

### 5.3. MFF singular vectors: relation between submodules.

We first consider the \( \text{MFF}^{+}_{(\pm)} \) vectors. For generic values of \( h_{+} \) and \( k \), the \( \text{MFF}^{+}_{(-)}(r, s, h_{+}, k) \) and \( \text{MFF}^{+}_{(+)}(r, s, h_{+}, k) \) singular vectors generate the same submodule, but this is not
so in several “degenerate” cases, as we now see. The relation between $\text{MFF}_{(-)}^+$ and $\text{MFF}_{(+)}^+$ is described in the following Lemma.

**Lemma 5.3.** In the Verma module $\mathcal{P}_{3^+(r,s,k),h_+,k}$, the following relations are satisfied up to a sign,

\begin{equation}
\text{(5.15)} \quad E_{2s}^1 \cdots E_{2s-2}^1 F_{2s}^1 \cdots F_{2s-1}^1 \text{MFF}_{(-)}^+(r, s, h_+, k) = \\
= \prod_{i=0}^{2s-1} (h_+ - j^+(r, s, k) - i(k + 1)) \text{MFF}_{(+)}^+(r, s, h_+, k)
\end{equation}

and

\begin{equation}
\text{(5.16)} \quad E_{2s}^2 \cdots E_{2s-2}^2 F_{2s}^2 \cdots F_{2s-1}^2 \text{MFF}_{(+)}^+(r, s, h_+, k) = \\
= \prod_{i=0}^{2s-1} (h_+ + j^+(r, s, k) + i(k + 1)) \text{MFF}_{(-)}^+(r, s, h_+, k).
\end{equation}

This is readily proved using Eqs. (5.7). These equations imply that whenever the factor in either (5.15) or (5.16) vanishes, one of the MFF vectors generates a submodule in the module generated from the other.

A similar relation between $\text{MFF}_{(-)}^-$ and $\text{MFF}_{(+)}^-$ is described as follows.

**Lemma 5.4.** In the Verma module $\mathcal{P}_{3^-(r,s,k),h_+,k}$, the following relations are satisfied up to a sign,

\begin{equation}
\text{(5.17)} \quad E_{2s}^1 \cdots E_{2s-2}^1 F_{2s}^1 \cdots F_{2s-1}^1 \text{MFF}_{(-)}^-(r, s, h_+, k) = \\
= \prod_{i=1}^{2s-2} (h_+ - j^-(r, s, k) - i(k + 1)) \text{MFF}_{(+)}^-(r, s, h_+, k)
\end{equation}

and

\begin{equation}
\text{(5.18)} \quad E_{2s}^2 \cdots E_{2s-2}^2 F_{2s}^2 \cdots F_{2s-1}^2 \text{MFF}_{(+)}^-(r, s, h_+, k) = \\
= \prod_{i=1}^{2s-2} (h_+ + j^-(r, s, k) + i(k + 1)) \text{MFF}_{(-)}^-(r, s, h_+, k).
\end{equation}

A number of degenerations that can occur in Verma modules can be studied by analyzing the relative positions of the singular vectors $\text{MFF}_{(+)}^+$ and $\text{MFF}_{(-)}^+$ and, on the other hand, $\text{MFF}_{(+)}^-$ and $\text{MFF}_{(-)}^-$. 
5.4. Coexistence of the MFF and charged singular vectors. Explicit formulas for the charged singular vectors and “almost” explicit ones for the MFF singular vectors allow us to describe the relevant coexistence cases of singular vectors of these two types. We start with the Verma module containing an MFF$^+$ singular vector and a charged singular vector $|C^-(n, h_-, k; \theta)\rangle$ with $n \geq 1$; by the choice of the twisted highest-weight vector from which the module is generated, we can always assume that $n = 1$ (a similar remark applies to other charged singular vectors considered in what follows).

**Lemma 5.5.** In the Verma module $\mathcal{P}_{h_-,h_-(k+1),k}$ with $h_- = j^+(r, s, k)$ for $r \geq 1$ and $s \geq 1$ such that $\frac{r}{k+1} \notin \mathbb{Z}$, there are submodules

$$\mathcal{P}_{h_-,h_-(k+1),k} \hookrightarrow \mathcal{N}^-_{h_-; k; 1}$$

(5.19)

$$\mathcal{P}_{h_-,h_-(k+1),k} \hookrightarrow \mathcal{N}^+_{h_- - \frac{1}{2}; k; s}$$

where the horizontal arrows are the embeddings of the submodules generated from charged singular vectors.

**Proof.** The Verma module $\mathcal{P}_{h_-,h_-(k+1),k}$ has the charged singular vector $C^-(1, h_-, k)$, with the corresponding submodule $\mathcal{N}^-_{h_-; k; 1}$. Under the conditions of the Lemma, $C^-(1, h_-, k)$ is the only charged singular vector in $\mathcal{P}_{h_-,h_-(k+1),k}$. Next, there is a submodule generated from the MFF vectors $\mathcal{MFF}^+_{\pm}(r, s, h_+, k)$. It follows from Eqs. (5.16) and (5.15) that $\mathcal{MFF}^+_{\pm}(r, s, h_+, k)$ and $\mathcal{MFF}^+_{\pm}(r, s, h_+, k)$ generate the same submodule. This submodule, in turn, has a charged singular vector that can be written as

$$\mathcal{C}^+(0, -\frac{r}{2} - \frac{s-1}{2}(k+1), k; s) \mathcal{MFF}^+_{\pm}(r, s, \frac{r}{2} - \frac{s+1}{2}(k+1), k) \vdash$$

$$\vdash | -\frac{r}{2} + \frac{s-1}{2}(k+1), k; s^+ \rangle \equiv | j^+(r, s, k) - r - \frac{1}{2}, k; s^+ \rangle$$

(we remind the reader that the $\equiv$ sign is a statement about the annihilation and eigenvalue conditions satisfied by this state). Explicitly, we have (writing $j^+$ for $j^+(r, s, k)$ for brevity)

$$\mathcal{C}^+(0, -\frac{r}{2} - \frac{s-1}{2}(k+1), k; s) \mathcal{MFF}^+_{\pm}(r, s, \frac{r}{2} - \frac{s+1}{2}(k+1), k) =$$

$$= F_{-s}^2 \cdot E_{-s}^1 (F_{0}^{12})^2 j^+ + (2s-2)(k+1) F_{-s+1}^2 (E_{12}^{12})^2 j^+ + (2s-3)(k+1) \times$$

$$\times E_{-s+1}^1 (F_{0}^{12})^2 j^+ + (2s-4)(k+1) F_{-s+2}^2 (E_{12}^{12})^2 j^+ + (2s-5)(k+1) \times$$

$$\cdots \cdots \cdots \cdots$$

$$\times E_{-2}^1 (F_{0}^{12})^2 j^+ + 2(k+1) F_{-1}^2 (E_{-1}^{12})^2 j^+ + k+1 E_{-1}^1 (F_{0}^{12})^2 j^+ F_0^2 | j^+, j^+ - (k+1), k \rangle.$$
We now have $F_{-s}^2 E_{-s}^1 (F_0^{12})^{2j+2s-2(k+1)} E_{-s}^1 F_{-s}^1$ in accordance with the second formula in (5.3). This brings the operators $E_{-s}^1$ and $F_{-s+1}^1$ together in (5.21), and we can then use the commutation property
\[
E_{-s}^1 F_{-s+1}^2 (E_{-1}^{12})^{2j+2s-3(k+1)} = E_{-s}^1 (E_{-1}^{12})^{2j+2s-3(k+1)} F_{-s+1}^2,
\]
This, in its turn, brings the operators $F_{-s+1}^2$ and $E_{-s+1}^1$ together, and the process is continued by induction, until we arrive at $F_{-s}^2 E_{-s}^1 (F_0^{12})^{2j+2s} F_0^s = F_{-s}^2 (F_0^{12})^{2j+2s} E_{-s+1}^1 F_0^s$, which means that the state (5.21) is in the submodule $N_{h_-,k;1}$ generated from the charged singular vector $C(-)(1, h_-, k) = E_{-1}^1 F_0^s |h_-, h_- - (k+1), k\rangle$. This shows $N_{h_-,r-\frac{1}{2},k;s}^+ \rightarrow N_{h_-,k;1}^-$.

Similarly to (4.9) and (4.12), we have (as before, with $h_- = j^+(r, s, k)$)
\[
\begin{align*}
0 & \leftarrow N_{h_-,r-\frac{1}{2},k;s}^- \\
& \leftarrow \mathcal{P}_{h_-,h_--(k+1),k} \\
& \leftarrow N_{h_-,k;1}^- \leftarrow 0 \\
0 & \leftarrow N_{h_-,r,k;1}^+ \\
& \leftarrow \mathcal{P}_{h_-,r-h_--(k+1),k} \\
& \leftarrow N_{h_-,r-\frac{1}{2},k;s}^+ \leftarrow 0.
\end{align*}
\]
In the quotient with respect to $N_{h_-,k;1}^-$, the submodule generated from the MFF$^+$ vector is therefore $N_{h_-,r-\frac{1}{2},k;s}^+$.

We thus see that whenever an $N^-$ submodule is generated from a charged singular vector in $\mathcal{P}_{j^+(r,s,k),h_+,k}$, an $N^+$ submodule appears in the next generation of “MFF descendants.” We now describe this latter case separately, i.e., describe the simultaneous occurrence of a $C(+) (0, h_-, k)$ singular vector and an MFF vector.

**Lemma 5.6.** In the Verma module $\mathcal{P}_{h_-,h_-}^+$ with $h_- = j^+(r, s, k)$ for $r \geq 1$ and $s \geq 1$ such that $\frac{r}{k+1} \notin \mathbb{Z}$, there are submodules
\[
\begin{align*}
\mathcal{P}_{h_-,h_-} & \leftarrow N_{h_-,r,k;0}^+ \\
\mathcal{P}_{h_-,r-h_--(k+1),k;-s} & \leftarrow N_{h_-,r-\frac{1}{2},k;-s}^-
\end{align*}
\]

**Proof.** The structure of submodules is slightly more interesting in this case. The right-hand side of (5.16) vanishes for the current values of the parameters, while the factor on the right-hand side of (5.15) is nonvanishing under the conditions of the Lemma. We thus conclude that MFF$^+(r, s, -h_-, k)$ generates a submodule in the submodule generated from MFF$^+(-)(r, s, -h_-, k)$. From (5.22), we have
\[
\begin{align*}
MFF^+(-)(r, s, -h_-, k) & \equiv |h_- - r, -h_- - s(k+1), k; -s\rangle = \\
& = |\frac{r}{2} - \frac{s-1}{2}(k+1), \frac{r}{2} - \frac{s+1}{2}(k+1), k; -s\rangle,
\end{align*}
\]
and the submodule generated from \( \text{MFF}^+_{(-)}(r, s, -h_-, k) \) can also be generated from the corresponding charged singular vector

\[
C^-(1; h_--r, k; -s) \text{MFF}^+_{(-)}(r, s, -h_-, k) = \left\lfloor \frac{-r}{2} - \frac{s-1}{2}(k+1), k; 1-s \right\rfloor ,
\]

which gives the submodule \( \mathcal{N}^-_{h_-,-r,k,1-s} \). The quotient is then generated from

\[
F^2_s \text{MFF}^+_{(-)}(r, s, -h_-, k) = \left\lfloor \frac{-r+1}{2} - \frac{s-1}{2}(k+1), k; 1-s \right\rfloor .
\]

At the same time, obviously, the module \( \mathcal{P}_{h_-,-h_-} \) has the submodule \( \mathcal{N}^+_{h_-,-h_-,k; 0} \) generated from the charged singular vector \( C^+(0, h_-, k) \). We thus arrive at (5.23).

A key observation for the construction of the resolution is that the submodule in the bottom line in (5.22) satisfies the conditions of Lemma 5.6.

**Remark 5.7.** We note a subtlety in properly identifying submodules in (5.23). We used both the \( \text{MFF}^+_{(-)} \) and \( \text{MFF}^+_{(+)} \) vectors, with the first one generating the entire submodule \( \mathcal{P}_{h_-,-r,-h_-,s(k+1)} = \mathcal{P}_{-\frac{r}{2}-\frac{s-1}{2}(k+1), -\frac{s}{2}-\frac{s+1}{2}(k+1), k; s} \) and the second generating only \( \mathcal{N}^-_{h_-,-r} = \mathcal{N}^-_{-\frac{r}{2}-\frac{s-1}{2}(k+1), k; 1-s} \) (as before, we assume \( h_-= j^+(r, s, k) \)). Recalling that an MFF singular vector satisfies twisted highest-weight conditions (5.10), we thus see that the submodule generated from \( \text{MFF}^+_{(+)}(r, s, -h_-, k) \) is not freely generated in \( \mathcal{P}_{h_-,-h_-} \). On the other hand, the module freely generated from the twisted highest-weight state satisfying the same highest-weight conditions as \( \text{MFF}^+_{(+)}(r, s, -h_-, k) \) is \( \mathcal{P}_{-\frac{r}{2}-\frac{s-1}{2}(k+1), -\frac{s}{2}-\frac{s+1}{2}(k+1), k; s} \), and it has the charged singular vector \( C^-(1-2s, -\frac{r}{2}-\frac{s-1}{2}(k+1), k; s) \), with the corresponding submodule \( \mathcal{N}^-_{-\frac{r}{2}-\frac{s-1}{2}(k+1), k; 1-s} \). Thus, comparing the module \( \mathcal{P}_{-\frac{r}{2}-\frac{s-1}{2}(k+1), -\frac{s}{2}-\frac{s+1}{2}(k+1), k; s} \) with the actual submodule \( \mathcal{P}_{-\frac{r}{2}-\frac{s-1}{2}(k+1), -\frac{s}{2}-\frac{s+1}{2}(k+1), k; s} \) occurring in (5.23), we see that these are non-isomorphic modules differing by transposing the submodule and the quotient (we note in passing that the two modules have the same character).

In other words, while taking the quotients in (5.23) gives

\[
0 \leftarrow \mathcal{N}^+_{h_-,-r,k; 0} \leftarrow \mathcal{P}_{h_-,-h_-}, k \leftarrow \mathcal{N}^-_{h_-,-r,k; 0} \leftarrow 0
\]

(5.25)

\[
0 \leftarrow \mathcal{N}^-_{h_-,-r,-h_-,k; 1-s} \leftarrow \mathcal{P}_{h_-,-r,-h_-,s(k+1), k; s} \leftarrow \mathcal{N}^-_{h_-,-r,k; 1-s} \leftarrow 0,
\]

there is a different short exact sequence

\[
0 \leftarrow \mathcal{N}^-_{-\frac{r}{2}-\frac{s-1}{2}(k+1), k; 1-s} \leftarrow \mathcal{P}_{-\frac{r}{2}-\frac{s-1}{2}(k+1), -\frac{s}{2}-\frac{s+1}{2}(k+1), k; s} \leftarrow \mathcal{N}^-_{-\frac{r}{2}-\frac{s-1}{2}(k+1), k; 1-s} \leftarrow 0
\]
In particular, the throughout mapping

$$\mathcal{P}^+_1(r,s,k), -j^+ (r,s,k), k \leftarrow \mathcal{P}^-_{-\frac{r}{2} - \frac{s}{2} + \frac{3s-1}{2}(k+1), k; s}$$

is not an embedding (i.e., has a kernel). As can be easily verified, there is the isomorphism

$$\mathcal{P}^-_{-\frac{r}{2} - \frac{s}{2} + \frac{3s-1}{2}(k+1), k; s} \approx \mathcal{P}^+_1(r,s,k), -j^+ (r,s,k), k,$$

and this untwisted Verma module is therefore not embedded into $\mathcal{P}^+_1(r,s,k), -j^+ (r,s,k), k$.

**Remark 5.8** (the $r = 0$ “degeneration” of the MFF vectors). The exponents in the MFF formulas, e.g., in Eqs. (5.11)–(5.12) (which we now rewrite for the spin $j^-(r', s', k)$ for the future convenience) are given by $i(k + 1) - 2j^-(r', s', k)$, where $i$ ranges from $1$ to $2s' - 1$ (and we count the different factors from right to left). For $i = s'$, the corresponding factor is raised to the power $r'$. We refer to this operator as the center of the corresponding MFF formula. For definiteness, let now $s'$ be even; then the MFF formula (for example, (5.11)) has the following structure around its center:

$$\cdots E^{-2} (F_{0}^{12})^{r+2(k+1)} F_{E_{-1}}^{12} E^{r+k+1} F_{-\frac{r}{2}}^{1} (E_{-1}^{12})^{r+(k+1)} E^{2} (F_{0}^{12})^{r} F_{1-\frac{r}{2}}^{1} \times$$

$$\times (E_{-1}^{12})^{r-(k+1)} E_{1-\frac{r}{2}}^{1} (F_{0}^{12})^{r-2(k+1)} F_{2}^{1} \cdots.$$

We recall that $r' \geq 1$ in all the MFF formulas; however, we now formally continue this expression to $r' = 0$. We then see that $E_{-\frac{r}{2}}^{1} F_{1-\frac{r}{2}}^{1}$ commutes with $E_{-1}^{12}$, and therefore, the two $(E_{-1}^{12})^{\pm(k+1)}$ factors cancel each other. Up to an overall sign, therefore,

(5.26)

$$\cdots F_{-\frac{1}{2}}^{1} (E_{-1}^{12})^{k-1} E_{-\frac{1}{2}}^{2} F_{1-\frac{1}{2}}^{1} (E_{-1}^{12})^{-(k+1)} E_{1-\frac{r}{2}}^{2} \cdots = \cdots F_{-\frac{1}{2}}^{1} E_{-\frac{1}{2}}^{2} F_{1-\frac{1}{2}}^{1} E_{1-\frac{1}{2}}^{2} \cdots \cdots$$

and the same argument can be applied to other factors in the singular vector formula, leading to the cancellation of all powers of the bosonic generators. Thus, evaluating (5.11) for $r' = 0$, we find

(5.27)\hspace{1cm} MFF_{(-)}^-(0, s, h_+, k) = E_{1-s} F_{2-s} \cdots F_{0}^{1} \bigg|_{\frac{s}{2}(k+1), h_+, k}.

This is an extremal state of the module; depending on the value of $h_+$, however, it can become a charged singular vector.

6. Conclusions

The above statements on mutual positions of singular vectors in $\widehat{\mathfrak{s}l}(2|1)$ Verma modules give an efficient tool for analyzing the structure of modules and constructing resolutions of irreducible representations. Thus, a repeated application
of the lemmas in Sec. 5.4 gives the submodule grid of the module $N^-_{j-\frac{1}{2},k}$ (where $j = j^r(s, k), k + 1 = p/u$, and $1 \leq r \leq p - 1, 1 \leq s \leq u$, with coprime positive integers $p$ and $u$) that entirely consists of narrow Verma modules, see Fig. 3. This gives the resolution of the irreducible quotient of the module

$$P_{j,(k+1),k} = P_{\frac{j}{2} - \frac{s+1}{u} - \frac{s+1}{u}, \frac{j}{2} - \frac{s+1}{u} - \frac{s+1}{u} - 1}$$

through narrow Verma modules. The construction of the resolution for the irreducible representation therefore starts with taking the quotient in diagrams (5.22) and (5.25). The details of the construction and the resulting character formula will be considered elsewhere.

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REFERENCES

[1] D. Bernard, (Perturbed) Conformal field theory applied to 2D disordered systems: an introduction, hep-th/9509137.
[2] Z. S. Bassi and A. LeClair, Nucl. Phys. B578 (2000) 577, hep-th/9911105.
[3] D. Bernard and A. LeClair, *Quasi-spin-charge separation and the copin quantum Hall effect*, cond-mat/0003075.

[4] M. Bershadsky and H. Ooguri, Phys. Lett. B229 (1989) 374.

[5] M. Bershadsky, W. Lerche, D. Nemeschansky, and N. P. Warner, Nucl. Phys. B401 (1993) 304.

[6] K. Ito and H. Kanno, Mod. Phys. Lett. A9 (1994) 1377.

[7] A. M. Semikhatov, Nucl. Phys. B478 (1996) 209, hep-th/9604105.

[8] P. Bowcock, B. L. Feigin, A. M. Semikhatov, and A. Taormina, Commun. Math. Phys. 214 (2000) 495 hep-th/9907171.

[9] P. Bowcock and A. Taormina Commun. Math. Phys. 185 (1997) 467.

[10] A. M. Semikhatov, Theor. Math. Phys. 112 (1997) 949, hep-th/9610084.

[11] B. L. Feigin, A. M. Semikhatov, V. A. Sirota, and I. Yu. Tipunin, Nucl. Phys. B536 (1999) 617, hep-th/9701043.

[12] F. G. Malikov, B. L. Feigin, and D. B. Fuchs Funct. Anal. Appl. 20 (1986) 103.

[13] P. Bowcock, M. R. Hayes, and A. Taormina Nucl. Phys. B510 (1998) 739.

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