ON SINGULAR SOLUTIONS FOR SECOND ORDER DELAYED DIFFERENTIAL EQUATIONS

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Abstract. Asymptotic properties and estimate of singular solutions (either defined on a finite interval only or trivial in a neighbourhood of \( \infty \)) of the second order delay differential equation with \( p \)-Laplacian are investigated.

1. Introduction

In this paper, we consider the second order nonlinear delay differential equation

\[
(a(t)|y'|^{p-1}y')' + r(t)|y(\varphi(t))|^{\lambda} \text{sgn} \, y(\varphi(t)) = 0
\]

where \( p > 0, \lambda > 0, a \in C^0(\mathbb{R}_+), r \in C^0(\mathbb{R}_+), \varphi \in C^0(\mathbb{R}_+), a(t) > 0, r(t) > 0, \varphi(t) \leq t \) on \( \mathbb{R}_+ \) and \( \lim_{t \to \infty} \varphi(t) = \infty \).

If \( p = \lambda \), it is known as the half-linear equation, while if \( \lambda > p \), we say that equation \( (1) \) is of the super-half-linear type, and if \( \lambda < p \), we will say that it is of the sub-half-linear type.

We begin by defining what is mean by a solution of equation \( (1) \) as well as some basic properties of solutions.

Definition 1. Let \( T \in (0, \infty], \varphi_0 = \inf_{t \in \mathbb{R}_+} \varphi(t), \phi \in C^0[\varphi_0, 0], \) and \( y'_0 \in \mathbb{R} \). We say that a function \( y \) is a solution of \( (1) \) on \([0, T)\) (with the initial conditions \((\phi, y'_0)\)) if \( y \in C^0[\varphi_0, T), y \in C^1[0, T), a|y'|^{p-1}y' \in C^1[0, T), (1) \) holds on \([0, T), y(t) = \phi(t)\) on \([\varphi_0, 0)\), and \( y'_0(0) = y'_0 \).

We assume that solutions are defined on their maximal interval of existence to the right.

Equation \( (1) \) can be written as the equivalent system

\[
\begin{align*}
\dot{y}_1 &= a^{-\frac{1}{p}}(t)|y_2|^{\frac{1}{p}} \text{sgn} \, y_2, \\
\dot{y}_2 &= -r(t)|y(\varphi(t))|^{\lambda} \text{sgn} \, y(\varphi(t)).
\end{align*}
\]

The relationship between a solution \( y \) of \( (1) \) and a solution \((y_1, y_2)\) of the system \( (2) \) is

\[
(3) \quad y_1(t) = y(t) \quad \text{and} \quad y_2(t) = a(t)|y'(t)|^{p-1}y'(t),
\]

and when discussing a solution \( y \) of \( (1) \), we will often use \( (3) \) without mention.

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Definition 2. Let \( y \) be a solution of (1) defined on \([0, T), T \leq \infty\). It is called singular of the 1st kind if \( T = \infty, \tau \in (0, \infty) \) exists such that \( y \equiv 0 \) on \([\tau, \infty)\) and \( y \) is nontrivial in any left neighbourhood of \( \tau \). Solution \( y \) is called singular of the 2nd kind if \( T < \infty \) and put \( \tau = T \). It is called proper if \( T = \infty \) and it is nontrivial in any neighbourhood of \( \infty \). Singular solutions of either 1st or 2nd kind are called singular.

Note, that a solution of (1) is either proper, or singular or trivial on \((\varphi_0, \infty)\).

Remark 1. If \( y \) is a singular solution of (1) of the 2nd kind, then it is defined on \([0, \tau), \tau < \infty\) and it cannot be defined at \( t = \tau \); so, \( \limsup_{t \to \tau} |y_1(t)| + |y_2(t)| = \infty \).

From this and from (2)

\[
\limsup_{t \to \tau} |y_2(t)| = \infty.
\]

Definition 3. Let \( y \) be a singular solution of (1) of the 1st kind (of the 2nd kind). Then it is called oscillatory if there exists a sequence of its zeros tending to \( \tau \) and it is called nonoscillatory otherwise.

Singular solutions of (1) without delay, i.e. of

\[
(a(t)|y'|^{p-1}y')' + r(t)|y|^{\lambda} \text{sgn} y = 0,
\]

have been studied by many authors, see e.g. [1, 5], [9]–[16] and the references therein. Note, that the first existence results are obtained in [12] for \( p = 1, a = 1 \) and \( r \leq 0 \). In the monography of Kiguradze and Chanturia [13] it is a good overview of results for \( p = 1 \) and \( a = 1 \).

Eq. (5) may have singular solutions. Heidel [11] (Coffman, Ulrich [9]) proved the existence of an equation of type (5), \( a \equiv 1, p = 1 \) with singular solutions of the 1st kind (of the 2nd kind) in case \( \lambda < p (\lambda > p) \); in this case \( r \) is continuous but not of locally bounded variation. If \( a \) and \( r \) are smooth enough, then singular solutions of (5) do not exist (see Theorem A below). As concerns to Eq. (1), the existence of singular solutions of the second kind are investigated in [4] in case \( r \leq 0 \). The existence and properties of singular solutions of either the first kind or of the second kind in case \( r \geq 0 \) seem not to be studied at all.

The following theorem sums up results concerning to Eq. (5).

Theorem A. Let \( r \in C^0(\mathbb{R}_+) \) and \( r(t) > 0 \) on \( \mathbb{R}_+ \).

(i) If \( \lambda \geq p \), then there exists no singular solution of (5) of the 1st kind.
(ii) If \( \lambda \leq p \), then there exists no singular solution of (5) of the 2nd kind.
(iii) If \( a^{1/r} \in C^1(\mathbb{R}_+) \), then all solutions of (5) are proper.

Proof. (i), (ii): See Theorems 1.1 and 1.2 in [15]. (iii): It follows from Theorem 2 in [5].

Note that estimates of such kind of solutions are proved by Kvinikadze, see references in [13]. In [1] (for \( p = 1, a = 1, r \leq 0 \)) precise asymptotic formulas of all
solutions are obtained for differential equations of the third and fourth orders, see also [3]. About uniform estimates of solutions of quasi-linear ordinary differential equations see [2]. In [16] estimates of singular solutions of the second kind of a system of second order differential equations (of the form (5)) are derived.

**Theorem B** ([16], Theorem 2). Let \( r \in C^0(\mathbb{R}_+) \) and \( r(t) > 0 \) on \( \mathbb{R}_+ \). Let \( \lambda > p \), \( y \) be a singular solution of (5) of the second kind, \( T \in [0, \tau) \), \( \tau - T \leq 1 \), \( r_0 = \max_{T \leq s \leq \tau} r(s) \), \( C_0 = 2^{\lambda + 2} \) in case \( p > 1 \) and \( C_0 = 2^{2\lambda + 1} \) in case \( p \leq 1 \). Then a positive constant \( C = C(p, \lambda, \tau, r_0) \) exists such that

\[
|y_2(t)| + C_0 r_0 |y(t)|^\lambda \geq C(\tau - t)^{-\frac{p+1}{p(\lambda + 1)}}, \quad t \in [T, \tau).
\]

It is important to study the existence of proper/singular solutions. When studying solutions of (1) and (5), some authors sometimes investigate properties of solutions that are defined on \( \mathbb{R}_+ \) only without proving the existence of them. Moreover, sometimes, proper solutions have crucial role in a definition of some problems, see e.g. the limit-point/limit-circle problem in [6], [8]. Furthermore, noncontinuable solutions appear e.g. in water flow problems (flood waves, a flow in sewerage systems), see e.g. [4].

Our goal is to study properties of singular solutions and to extend Theorems A and B to (1).

For convenience, we define the constants and the function

\[
\delta = \frac{p + 1}{p}, \quad \gamma = \frac{p + 1}{p(\lambda + 1)}, \quad R(t) = a^\frac{1}{p}(t) r(t), \quad t \in \mathbb{R}_+.
\]

If \( y \) is a solution of (1), then we set on its interval of existence

\[
F(t) = R^{-1}(t)|y_2(t)|^\delta + |y(t)|^{\lambda + 1}.
\]

Notice that \( F(t) \geq 0 \) for every solution of (1) and

\[
F'(t) = -\frac{R'(t)}{R^2(t)}|y_2(t)|^\delta + \delta y'(t) e(t)
\]

with

\[
e(t) \overset{\text{def}}{=} \left| y(t) \right|^{\lambda} \text{sgn} y(t) - \left| y(\varphi(t)) \right|^{\lambda} \text{sgn} y(\varphi(t)).
\]

From (6)

\[
|y_2(t)| \leq \left( \gamma^{-1} F(t) \right)^\frac{1}{\lambda + 1}, \quad |y_2(t)| \leq \left[ R(t) F(t) \right]^\frac{1}{\lambda + 1},
\]

\[
|y'(t)| \leq a^{-\frac{1}{p}}(t) R^{\frac{1}{1-\frac{1}{p}}} F^{\frac{1}{1-\frac{1}{p}}}(t).
\]

**2. Singular solutions of the 2nd kind**

The following theorem shows that such solutions do not exist in case \( \lambda \leq p \).

**Theorem 1.** If \( \lambda \leq p \), then all solutions of (1) are defined on \( \mathbb{R}_+ \).

**Proof.** It is proved in Lemma 7 in [6] for \( r < 0 \), for arbitrary \( r \) the proof is the same, it is necessary to replace \( r \) by \(|r|\). \( \square \)

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The following theorem gives us basic properties.

**Theorem 2.** Let $y$ be a singular solution of (1) of the second kind. Then it is oscillatory and $\phi(\tau) = \tau$. If, moreover, $R \in C^1(\mathbb{R}_+)$, then $\phi(t) \not\equiv t$ in any left neighbourhood of $\tau$.

**Proof.** Suppose, contrarily, that $\phi(\tau) < \tau$. Then an interval $I = [\tau_1, \tau)$ exists such that $\tau_1 < \tau$ and $\sup_{t \in I} |\phi(t)| = r(t)|y(\phi(t))|^\lambda \leq \sup_{t \in I} r(t)|y(\phi(t))|^\lambda < \infty$. Hence, $y_2$ is bounded on $I$ that contradicts (4). Hence, $\phi(\tau) = \tau$.

Let $y$ be nonoscillatory. Suppose, for the simplicity, that $y$ is positive in a left neighbourhood of $\tau$. Then, with respect to $\phi(\tau) = \tau$, $\tau_1 < \tau$ exists such that (10)

$$y(\phi(t)) > 0 \quad \text{on} \quad I \overset{\text{def}}{=} [\tau_1, \tau).$$

As according to (2) and (10), $y_2$ is decreasing on $I$ and (4) implies

$$\lim_{t \to \tau^-} y_2(t) = -\infty.$$

From this $\tau_2 \in I$ exists such that (12)

$$y'(t) < 0 \quad \text{on} \quad [\tau_2, \tau)$$

and the integration of (1) and (11)

$$\int_{\tau_2}^{\tau} r(t)y^\lambda(\phi(t)) \, dt = y_2(\tau_2) - \lim_{t \to \tau^-} y_2(t) = \infty.$$

Hence, $\limsup_{t \to \tau^-} y(t) = \infty$ that contradicts (12) and $y$ is oscillatory.

Let $y$ be a singular solution of (1) and $\phi(t) \equiv t$ on a left neighbourhood $J$ on $\tau$. Then $y$ is a singular solution of (5) on $J$. A contradiction with Theorem A(iii) proves that $\phi(t) \not\equiv t$ in any left neighbourhood of $\tau$. \hfill $\Box$

**Remark 2.** According to Theorem 1 there exists no singular solution of (1) of the second kind in case $\phi(t) < t$ on $\mathbb{R}_+$; all solutions are defined on $\mathbb{R}_+$. This fact was used by many authors for special types of (1), see e.g. [10], [4] ($r < 0$).

The following two lemmas serve us for estimate of solutions.

**Lemma 1.** Let $\omega > 1$, $t_0 \in \mathbb{R}_+$, $K > 0$, $Q$ be a continuous nonnegative function on $[t_0, \infty)$ and $u$ be continuous and nonnegative on $[t_0, \infty)$ satisfying

$$u(t) \leq K + \int_{t_0}^{t} Q(s) u^\omega(s) \, ds \quad \text{on} \quad [t_0, T), T \leq \infty.$$

If

$$(\omega - 1)K^{\omega - 1} \int_{t_0}^{\infty} Q(s) \, ds < 1$$

then

$$u(t) \leq K\left[1 - (\omega - 1)K^{\omega - 1} \int_{t_0}^{t} Q(s) \, ds\right]^{1/(1-\omega)}, \quad t \in [t_0, T).$$

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Proof. It is proved in Lemma 2.1 in [14] for \( m = \omega \) and \( p = 1 \). □

**Lemma 2.** Let \( \lambda > p \), \( \int_0^{\infty} r(s) \left( \int_0^s a^{-\frac{\phi}{p}}(\sigma) \, d\sigma \right)^\lambda \, ds < \infty \), \( y \) be a solution of (1) defined on \([0, T] \), \( T \leq \infty \) and let \( t_0 \in [0, T] \). If \( y_* = \max_{\varphi(t_0) \leq s \leq t_0} |y(s)| \) and

\[
(16) \quad |y_2(t_0)| + 2^\lambda y_*^\lambda \int_{t_0}^{\infty} r(s) \, ds \left( \int_{t_0}^{s} a^{-\frac{\phi}{p}}(\sigma) \, d\sigma \right)^\lambda \, ds < 2^{-\lambda} \frac{p}{\lambda - p}.
\]

Then \( T = \infty \) and \( y \) is defined on \( \mathbb{R}_+ \).

**Proof.** Suppose, contrarily, that \( y \) is singular of the 2nd kind. Then \( T = \tau < \infty \) and denote by

\[
v(t) = \sup_{t_0 \leq s \leq t} |y_2(s)| \quad \text{for} \quad t \in I \overset{\text{def}}{=} [t_0, T).
\]

It follows from (2) that

\[
|y_2(t)| \leq |y_2(t_0)| + \int_{t_0}^{t} r(s) |y(\varphi(s))|^{\lambda} \, ds
\]

and

\[
|y(t)| \leq |y(t_0)| + \int_{t_0}^{t} a^{-\frac{\phi}{p}}(s) |y_2(s)|^{\frac{\lambda}{p}} \, ds, \quad t \in I.
\]

Hence, for \( t_0 \leq s \leq t < T \) we have

\[
|y_2(s)| \leq |y_2(t_0)| + \int_{t_0}^{s} r(z) \left[ y_* + v^{\frac{\phi}{p}}(z) \int_{t_0}^{z} a^{-\frac{\phi}{p}}(\sigma) \, d\sigma \right]^{\lambda} \, dz
\]

\[
\leq |y_2(t_0)| + 2^\lambda y_*^\lambda \int_{t_0}^{\infty} r(\sigma) \, d\sigma + 2^\lambda \int_{t_0}^{t} r(z) \left( \int_{t_0}^{z} a^{-\frac{\phi}{p}}(\sigma) \, d\sigma \right)^{\lambda} v^{\frac{\phi}{p}}(z) \, dz.
\]

From this

\[
(17) \quad v(t) \leq |y_2(t_0)| + 2^\lambda y_*^\lambda \int_{t_0}^{\infty} r(\sigma) \, d\sigma + 2^\lambda \int_{t_0}^{t} r(z) \left( \int_{t_0}^{z} a^{-\frac{\phi}{p}}(\sigma) \, d\sigma \right)^{\lambda} v^{\frac{\phi}{p}}(z) \, dz.
\]

Put \( \omega = \frac{\lambda}{p} > 1 \), \( u = v \), \( K = |y_2(t_0)| + 2^\lambda y_*^\lambda \int_{t_0}^{\infty} r(s) \, ds \)

and \( Q(t) = 2^\lambda r(t) \left( \int_{t_0}^{t} a^{-\frac{\phi}{p}}(\sigma) \, d\sigma \right)^{\lambda} \).

Then (16) and (17) imply (13) and (14), and according to Lemma 1, (15) is valid. As \( T < \infty \), \( y_2 \) is bounded on \( J \). A contradiction with (4) proves the statement. □

**Remark 3.** Note that Lemma 2 is valid even if we suppose \( r \geq 0 \) instead of \( r > 0 \) on \( \mathbb{R}_+ \).

**Remark 4.** The idea of the proof is due to Medved and Pekárková [14] (with \( \varphi(t) \equiv t \)); it is used also in [7] for (1) with \( t - \varphi(t) \leq \text{const.} \) on \( \mathbb{R}_+ \).

The next theorem derives an estimate from below of a singular solution of the second kind.

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Theorem 3. Let $\lambda > p$ and let $y$ be a singular solution of (1) of the 2nd kind. Let $T \in [0, \tau)$, $a_* = \min_{T \leq s \leq \tau} a(s)$, $r_* = \max_{T \leq s \leq \tau} r(s)$ and $y_*(t) = \max_{\varphi(t) \leq s \leq \tau} |y(s)|$ on $[T, \tau)$.

Then
\begin{equation}
|y_2(t)| + 2^{\lambda+1} y_*^{\lambda} r_*(\tau - t) \geq K(\tau - t)^{-\frac{2(\lambda+1)}{\lambda - p}}\tag{18}
\end{equation}
on $[T, \tau)$ with $K = (2^{-2\lambda-1}(\lambda+1)p \frac{\lambda}{\lambda-p} a_*, r_*)^{-\frac{1}{\lambda - p}}$. Especially, a left neighbourhood $I$ of $\tau$ exists such that
\begin{equation}
2^{\lambda+1} y_*^{\lambda} r_*(\tau - t) \geq K_1(\tau - t)^{-\frac{2(\lambda+1)}{\lambda - p}}\tag{19}
on I$ with $K_1 = (2^{-2\lambda-3-\frac{2}{p}(\lambda+1)p} a_*, r_*)^{-\frac{1}{\lambda - p}}$.

Proof. Let $y$ be a singular solution of (1) of the 2nd kind defined on $[0, \tau)$. Let $t \in [T, \tau)$ be fixed. Define

\begin{align*}
\bar{r}(t) &= r(t) \\
\bar{a}(t) &= a(t) \quad \text{for} \quad t \in [0, \tau], \\
\bar{r}(t) &= \frac{r(\tau)}{\tau - t} (-t + 2\tau - \bar{t}), \quad \bar{a}(t) = \frac{a(\tau)}{\tau - t} (-t + 2\tau - \bar{t}) \quad \text{for} \quad t \in (\tau, 2\tau - \bar{t}) \\
\bar{r}(t) &= 0, \quad \bar{a}(t) = 0 \quad \text{for} \quad t > 2\tau - \bar{t}.
\end{align*}

note that $\bar{r}$ and $\bar{a}$ are continuous on $\mathbb{R}_+$ and are linear on $[\tau, 2\tau - \bar{t}]$. Furthermore, we have
\begin{equation}
\int_t^\infty \bar{r}(s) \left( \int_t^s \bar{a}^{-\frac{1}{\lambda}}(\sigma) d\sigma \right)^{\lambda} ds \leq r_* \int_t^\infty \frac{\bar{a}^{-\frac{1}{\lambda}}}{r_*} (s - \bar{t})^{\lambda+1} ds \leq \frac{2^{\lambda+1}}{r_*} \int_t^\infty \frac{\bar{a}^{-\frac{1}{\lambda}}}{r_*} (s - \bar{t})^{\lambda+1} ds
\end{equation}
and
\begin{equation}
\int_t^\infty \bar{r}(s) ds \leq \int_t^{2\tau - \bar{t}} r_* ds = 2r_*(\tau - \bar{t}).
\end{equation}

Consider an auxiliary equation
\begin{equation}
(\bar{a}(t)|y'|^{p-1}z') + \bar{r}(t)|z|^{\lambda} \text{sgn } z(\varphi) = 0.
\end{equation}

Then $z = y$ is the singular solution of (22) of the second kind defined on $[0, \tau)$. Suppose that (18) is not valid for $t = \bar{t}$, i.e.
\begin{equation}
\left[|y_2(\bar{t})| + 2^{\lambda+1} y_*^{\lambda} r_*(\tau - \bar{t}) \right]^{\lambda} < 2^{-2\lambda-1}(\lambda+1)p \frac{\lambda}{\lambda-p} a_*^{-\frac{1}{\lambda-1}} r_*^{-1}(\tau - \bar{t})^{-\lambda-1}\tag{23}
\end{equation}
holds. We apply Lemma 2 and Remark 3 with $T = \tau$ and $t_0 = \bar{t}$. Then it follows from (20), (21) and (23) that all assumptions of Lemma 2 are valid. Hence, $z$ is defined on $\mathbb{R}_+$ and the contradiction with $z$ to be singular proves that (18) is valid. Furthermore, a left neighbourhood $I$ of $t = \tau$ exists such that

\begin{align*}
2r_{} &\leq r(\tau) \\
\frac{a(\tau)}{2} &\leq a_* \leq 2a(\tau)
\end{align*}
and (20) follows from this and from (18).
Theorem 4. Let \( \tau \) oscillatory and \( \phi \) then a left neighbourhood \( \tau \) unbounded. Hence, an increasing sequence \( \tau \) such that \( \lim_{k \to \infty} t_k = \tau \) and

\[
\left| y(t_k) \right| \geq M(\tau - t_k)^{\frac{\lambda}{\lambda - 1}}, \quad k = 1, 2, \ldots
\]

Proof. Let \( y \) be a singular solution of the 2nd kind. Then according to Lemma 2 and Corollary 2 it is oscillatory and unbounded. Hence, an increasing sequence \( \{t_k\}_{k=1}^{\infty} \) exists such that \( t_k \to \infty \) and \( y \) has the local extreme at \( t_k \) and

\[
\left| y(t_k) \right| \geq |y(t)| \quad \text{for} \quad t \in [\varphi_0, t_k], \quad k = 1, 2, \ldots
\]

Then \( y'(t_k) = 0, \quad \max_{\varphi(t_k) \leq s \leq t_k} |y(s)| = |y(t_k)| \), and the statement follows from (19). \( \square \)

3. Singular solution of the 1st kind

This paragraph begins with some basic properties

Theorem 4. Let \( y \) be a singular solution of (1) of the first kind. Then it is oscillatory and \( \varphi(\tau) = \tau \). Moreover,

(i) if \( R \in C^1(\mathbb{R}_+) \), then \( \varphi(t) \neq t \) in any left neighbourhood of \( \tau \);

(ii) if \( R \in C^1(\mathbb{R}_+) \), \( \lambda \geq p \) and \( \varphi \) is nondecreasing in a left neighbourhood \( J \) of \( \tau \), then a left neighbourhood \( J_1 \) of \( \tau \) exists such that \( \varphi(t) < t \) on \( J_1 \).

Proof. Let \( y \) be a singular solution of (1) of the first kind. Then

(24) \( y(t) = 0 \quad \text{for} \quad t \geq \tau \)

and

(25) \( y(t) \neq 0 \quad \text{in any left neighbourhood of} \quad \tau \).

Suppose, contrarily, that \( \varphi(\tau) < \tau \). Then \( \lim_{t \to \infty} \varphi(t) = \infty \) implies the existence of \( \tau_1 \) such that \( \tau_1 > \tau \) and \( \varphi(t) > \tau \) for \( t \geq \tau_1 \). Denote \( I = [\tau, \tau_1] \). Then according to (1) and (24)

(26) \( y(\varphi(t)) = -r^{\frac{1}{\lambda}}(t) \left| a(t) |y'(t)|^{p-1} y'(t) \right|^{1/\lambda} \text{sgn} \left( a(t) |y'(t)|^{p-1} y'(t) \right) = 0 \)

for \( t \in I \). As \( \varphi(\tau_1) > \tau \) we have

\[ [\varphi(\tau), \tau] \subset [\varphi(\tau), \varphi(\tau_1)] \subset \{ \varphi(t) : t \in I \} \]

From this and from (26), \( y(t) = 0 \) on \( [\varphi(\tau), \tau] \) that contradicts (25). Hence, \( \varphi(\tau) = \tau \).

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We prove that \( y \) is oscillatory. Suppose, contrarily, that \( y(t) > 0 \) in a left neighbourhood of \( \tau \); case \( y(t) < 0 \) can be studied similarly. From this and from
\[
\varphi(\tau) = y \quad \text{an interval} \quad I_1 = [\tau_2, \tau], \quad \tau_2 < \tau \text{ exists such}
\]
(27)
\[
y(\varphi(t)) > 0 \quad \text{for} \quad t \in I_1.
\]
As, according to (2), \( y_2 \) is decreasing on \( I_1 \) and (24) implies \( y_2(\tau) = 0 \) we have
\( y_2 > 0 \) on \( I_1 \); hence, \( y' > 0 \) on \( I_1 \). The contradiction with (27) and (24) proves that
\( y \) is oscillatory.
Case (i). The proof follows from Theorem A(iii) by the same way as in the proof of Theorem 1.
Case (ii). Let \( \lambda \geq p \) and \( R \in C^1(\mathbb{R}_+) \). Then (i) implies \( \varphi \) is nontrivial in any
left neighbourhood of \( \tau \). Suppose that an increasing sequence \( \{\tau_k\}_{k=1}^{\infty} \) exists such that
\[
\lim_{k \to \infty} \tau_k = \tau \quad \text{and} \quad \varphi(\tau_k) = \tau_k.
\]
As \( \varphi \) is nondecreasing in \( J \), \( \{\tau_k\} \) may be choosen such that
(28)
\[
\varphi(t) \in [\tau_k, \tau] \quad \text{for} \quad t \in [\tau_k, \tau] .
\]
It follows from (24) and (25) that \( y_2(\tau) = 0 \) and \( F(\tau) = 0 \). Denote \( \tilde{F}_k = \max_{\tau_k \leq s \leq \tau} F(s) \). Then (28), (7) and (9) imply
\[
F(s) = -\int_{\tau_k}^{\tau} F'(\sigma) \, d\sigma \leq \tilde{F}_k \int_{\tau_k}^{\tau} \frac{R'(\sigma)}{R(\sigma)} \, d\sigma
\]
\[
+ 2\delta \gamma^{-\lambda} \tilde{F}_k \int_{\tau_k}^{\tau} a^{-\frac{1}{p+1}}(\sigma) R^{\frac{1}{p+1}}(\sigma) \, d\sigma
\]
for \( s \in [\tau_k, \tau] \) where \( \omega = \frac{1}{p+1} + \frac{\lambda}{\gamma+1} \geq 1 \) due to \( \lambda \geq p \). Hence,
(29)
\[
\tilde{F}_k \leq \tilde{F}_k \int_{\tau_k}^{\tau} \frac{R'(\sigma)}{R(\sigma)} \, d\sigma + 2\delta \gamma^{-\lambda} \tilde{F}_k \int_{\tau_k}^{\tau} a^{-\frac{1}{p+1}}(\sigma) R^{\frac{1}{p+1}}(\sigma) \, d\sigma
\]
k = 1, 2, \ldots. As \( \lim_{k \to \infty} \tilde{F}_k = F(\tau) = 0 \) and
\[
\lim_{k \to \infty} \int_{\tau_k}^{\tau} \frac{R'(\sigma)}{R(\sigma)} \, d\sigma = 0, \quad \lim_{k \to \infty} \int_{\tau_k}^{\tau} a^{-\frac{1}{p+1}}(\sigma) R^{\frac{1}{p+1}}(\sigma) \, d\sigma = 0
\]
we obtain the contradiction in (29) for large \( k \). Hence, \( \{\tau_k\} \) does not exists and the
statement holds in this case. \( \square \)

The following result is a consequence of Theorem 2 and Theorem 4.

**Theorem 5.** If \( \varphi(t) < 0 \) on \( \mathbb{R}_+ \), then all solutions of (1) are proper.

**Lemma 3.** Let \( y \) be a singular solution of the 1st kind, let \( T \in [0, \tau] \) be such that
(30)
\[
\int_{T}^{\tau} R^{-1}(t)|R'(t)| \, dt \leq \frac{1}{2} ,
\]
\( I = [T, \tau], \ K > 0, \ \omega \geq 0 \) and \( |\epsilon(t)| \leq K(\tau - t)^{\omega} \) on \( I \). Then
\[
F(t) \leq K_1(\tau - t)^{\delta(\omega + 1)} , \quad t \in I
\]
where \( K_1 = \left[ 2\delta(\omega + 1)^{-1} K \max_{0 \leq \sigma \leq \tau} a^{-\frac{1}{p+1}}(\sigma) R^{\frac{1}{p+1}}(\sigma) \right]^{\delta} .
\]

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Proof. Let $y$ be a singular solution of the 1st kind. Then (9) implies

$$R^{-1}(t)|y_2(t)|^{\frac{\delta}{2}} \leq F(t), \quad |y'(t)| \leq CF^{-1/\omega}(t)$$
on$I$ with $C = \max_{t \in I} a^{-\frac{1}{\omega}}(t)R^{-1/\omega}(t) > 0$. Define $	ilde{F}(t) = \max_{s \in [t, \tau]} F(s)$ for $t \in I$. From this and from (7), (8) and (30)

$$F(s) = -\int_t^\tau F'(\sigma) d\sigma \leq \int_t^\tau R^{-1}(\sigma)|R'(\sigma)|F(\sigma) d\sigma + \delta \int_t^\tau |y'(\sigma)e(\sigma)| d\sigma$$

$$\leq \tilde{F}(t) \int_t^\tau R^{-1}(\sigma)|R'(\sigma)|F(\sigma) d\sigma + C_1 \int_t^\tau F^{-1/\omega}(\sigma)(\tau - \sigma)^{\omega} d\sigma$$

$$\leq \frac{\tilde{F}(t)}{2} + \frac{C_1}{\omega + 1} \tilde{F}^{1/\omega}(t)(\tau - t)^{\omega + 1}$$

for $t \in I$ and $t \leq s \leq \tau$ where $C_1 = \delta KC$. Hence,

$$\tilde{F}(t) \leq \frac{\tilde{F}(t)}{2} + \frac{C_1}{\omega + 1} \tilde{F}^{1/\omega}(t)(\tau - t)^{\omega + 1}$$

or

$$F(t) \leq \tilde{F}(t) \leq K_1(\tau - t)^{\delta(\omega + 1)} \quad \text{on } I.$$

The following theorem gives us an estimate from above of singular solutions of the 1st kind.

**Theorem 6.** Let $y$ be a singular solution of (1) of the 1st kind and $M > 0$ be such that $\varphi'(t) \leq M$ in a left neighbourhood $S$ of $\tau$.

(i) Let $\lambda \geq p$ and $m > 0$. Then a positive constant $K$ and a left neighbourhood $J$ of $\tau$ exist such that

$$|y(t)| \leq K(\tau - t)^m, \quad |y_2(t)| \leq K(\tau - t)^{\frac{(\lambda + 1)m}{p + 1}} \quad \text{on } J.$$

(ii) Let $\lambda < p$ and $\varepsilon > 0$. Then a positive constant $K$ and a left neighbourhood $J$ of $\tau$ exist such that

$$|y(t)| \leq K(\tau - t)^{\frac{p + 1}{p + 1 - \varepsilon}}, \quad |y_2(t)| \leq K(\tau - t)^{\frac{p(\lambda + 1)}{p - \varepsilon}} \quad \text{on } J.$$

**Proof.** Let $y$ be a singular solution of the 1st kind. According to Theorem 4 $\varphi(\tau) = \tau$. Moreover, $\lim_{t \to \tau^-} y(t) = \lim_{t \to \tau^-} y_2(t) = 0$ and an interval $I = [T, \tau] \subset S, 0 \leq T_1 < T$ exists such that (30) and

$$|y(t)|^{\lambda} \leq \frac{1}{2}, \quad |y(\varphi(t))|^\lambda \leq \frac{1}{2} \quad \text{for } t \in I.$$

Hence, (8) implies $|e(t)| \leq 1$ on $I$ and it follows from Lemma 3 (with $I = I, K = 1, \omega = 0$)

$$F(t) \leq K(T - t)^{\delta}, \quad t \in I$$

with

$$K = \left[2\delta \max_{0 \leq \sigma \leq T} a^{-\frac{1}{\omega}}(\sigma)R^{-1/\omega}(\sigma)\right]^{\delta}.$$
Let \( \{ I_n \}_{n=1}^\infty \) be such that \( I_1 = I, I_n = [T_n, \tau], T_n < T_{n+1} < \tau \) and \( \varphi(t) \in I_n \) for \( t \in I_{n+1}, n = 1, 2, \ldots \); this sequence exists due to \( \varphi(t) \leq t \) and \( \varphi(\tau) = \tau \).

We prove the estimate

\[
F(t) \leq K_n(\tau - t)^\omega \quad \text{on } I_n
\]

by the mathematical induction, where

\[
\omega_1 = \delta, \quad \omega_{n+1} = \delta \left[ \frac{\lambda}{\lambda + 1} \omega_n + 1 \right], \quad n = 1, 2, \ldots
\]

and

\[
K_1 = K, \quad K_{n+1} = K \left[ \gamma^{1/\lambda+1} \left( 1 + \frac{\lambda}{\lambda + 1} \omega_n \right)^{-1} \left( 1 + M^{\omega} \right) \right] K^{1/\lambda+1} \delta^\omega, \quad n = 1, 2, \ldots
\]

For \( n = 1 \) (33) follows from (31) and (32). Suppose the validity of (33) for \( n \). Then (6) and (33) imply

\[
|y(t)|^\lambda \leq (\gamma^{-1} F(t))^{1/\lambda} \leq \gamma^{-1/\lambda+1} K^{1/\lambda+1} \delta \omega_n, \quad t \in I_n
\]

and

\[
|y(\varphi(t))|^\lambda \leq \gamma^{-1/\lambda+1} K^{1/\lambda+1} M \delta \omega_n (\tau - t)^{\omega}, \quad t \in I_{n+1}
\]

as

\[
0 \leq \tau - \varphi(t) = \varphi(\tau) - \varphi(t) = \varphi'(\xi)(\tau - t) \leq M(\tau - t), \quad \xi \in [t, \tau].
\]

From this and from (8)

\[
|e(t)| \leq \gamma^{-1/\lambda+1} \left[ 1 + M^{\omega} \delta \omega_n \right] (\tau - t)^{\omega} = L_n(\tau - t)^{\omega},
\]

where

\[
w_n = \frac{\lambda}{\lambda + 1} \omega_n \quad \text{and} \quad L_n = \gamma^{-1/\lambda+1} K^{1/\lambda+1} \delta \omega_n.
\]

Now, we use Lemma 3 with \( I = I_{n+1}, K = L_n \) and \( \omega = w_n \) and we obtain

\[
F(t) \leq K_{n+1}(\tau - t)^{\omega_{n+1}}.
\]

Hence, (33) holds for all \( n = 1, 2, \ldots \). Denote by

\[
\lambda = \frac{(p+1)}{(\lambda + 1)p},
\]

We prove that

\[
\omega_n \leq \frac{1 - z^n}{1 - z}, \quad n = 1, 2, \ldots \quad \text{for } z \neq 1
\]

\[
\omega_n = \delta n \quad \text{for } z = 1.
\]

If \( v_n = \frac{\omega_n}{z} \), then (34) implies \( v_1 = 1, v_{n+1} = zv_n + 1, n = 1, \ldots \). Hence, \( v_n = 1 + z + z^2 + \ldots + z^{n-1} = \frac{z^n - 1}{z - 1} \) in case \( z \neq 1 \) and \( v_n = n \) in case \( z = 1 \). Now, (36) follows from this.

We have from (35) that

\[
z > 1 \Leftrightarrow \lambda > p, \quad z = 1 \Leftrightarrow \lambda = p, \quad z < 1 \Leftrightarrow \lambda < p.
\]

Furthermore, from this and from (36) \( \lim_{n \to \infty} \omega_n = \infty \) in case \( \lambda \geq p \) and \( \lim_{n \to \infty} \omega_n = \frac{\delta}{1 - \frac{p+1}{p \lambda}} \) in case \( \lambda < p \). Hence, the statement follows from (33) and (6). □

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