We obtain several exact solutions for a plane symmetric space–time in the framework of a recently constructed \( f(G, T) \) theory of gravity, where \( f(G, T) \) is a generic function of the Gauss–Bonnet invariant \( G \) and the trace \( T \) of the energy–momentum tensor. To obtain solutions, we consider a power-law \( f(G, T) \) gravity model and analyze the obtained results graphically. Moreover, to justify the method, we reconstruct several well-known cosmological results.

**Keywords:** \( f(G, T) \) gravity, plane symmetric space–time, exact solution

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1. Introduction

Researchers have worked extensively to perfect Einstein’s general theory of relativity (GTR). The reason for the evolution of modified theories of gravity is that the GTR is incapable of sufficiently fully explaining conundrums like the expansion of our universe at an increasing rate, flatness, initial singularities, dark matter, and dark energy (DE) and makes inexplicable predictions. Accordingly, modified theories are believed to allow discovering some concealed reasons for the expansion of the universe. They might also help fill other gaps in the GTR.

Our universe is expanding at an accelerating rate, and to explain this phenomenon, we must use a consistent physical theory. Many modified theories have been created, including \( f(G) \), \( f(G, T) \), \( f(R) \), \( f(R, T) \), and \( f(R, G) \), where \( G \) is the Gauss–Bonnet (GB) invariant, \( T \) is the trace of the energy–momentum tensor, and \( R \) is the Ricci scalar. Of all these theories, the most developed modified theory is GB gravity, also called \( f(G) \) gravity, where \( f(G) \) is a generic function of the GB invariant \([1], [2]\). The GB generalization of the theory of gravity contains quadratic real terms, including second-order curvature invariants of the Einstein–Hilbert Lagrangian \([3], [4]\) (namely, a GB term that is a four-dimensional topological invariant). Possibly, this class of modified theories of gravity, in addition to a unified description of inflation and DE, can help to trace the history of the inflation of the universe from early times to accelerating inflation at late times \([5], [6]\). The consistency of the GB generalization with the solar system bounds was shown in \([7]\).

In \([8]\), Nojiri and Odintsov considered \( f(G) \) gravity as a DE alternative. A unified description of early-time inflation and late-time acceleration, a cosmological reconstruction, and future singularities were considered in the framework of the \( f(G) \) theory in \([9]\). In \([10]\), power-law solutions were obtained in the \( f(G) \) setting. In \([11]\), a way to eliminate finite-time future singularities was proposed based on DE models. Moreover, the consistency of the parameter ranges with energy bounds for several \( f(G) \) gravity
models was taken into account in [11]. Cosmological models in $f(G)$ gravity in an anisotropic framework were developed in [12]. In a similar setting in [13], the dynamics of $f(G)$ gravity was investigated. Also, the Noether symmetry approach was used in the $f(G)$ theory for a Friedmann–Robertson–Walker (FRW) universe in [14].

Another modification of the GTR, the $f(R,G)$ theory of gravity, has attracted the attention of many researchers [15]. The conditions under which the $f(R,G)$ theory is a plausible alternative to the GTR were discussed in [16] using perturbation theory. In [17], cosmological DE solutions were investigated. In [18], Capozziello et al. used the Noether symmetry approach in an FRW universe and obtained several specific models in order to show that this approach can lead to important results because of the existence of conserved quantities and the curtailment of the cosmological dynamical system. A similar approach was used in [19] to study locally rotationally symmetric Bianchi type-I universe, which led to a reconstruction of the Kasner solution and a $\Lambda$CDM cosmology. In [20], the authors considered exact solutions using the Taylor expansion of a general function in the vicinity of diminishing values of the Ricci scalar $R$ and the GB invariant $G$ and investigated the spherical symmetry in this theory. In [21], the energy bounds in $f(R,G)$ setting were studied. In [22], $f(R,G)$ models realizing finite-time future singularities were reconstructed.

A coupling has been established between the quadratic curvature invariant and matter (namely, $f(G,T)$) leading to a new theory of gravity [23]. This area is full of possibilities to obtain new results and to develop new approaches for solving various problems. In the same paper in the framework of a flat FRW model, the motion of massive test particles along nongeodesic paths and also energy bounds were investigated. In [24], Sharif and Ikram reproduced various well-known cosmological models in this theory, and in [25], they analyzed the stability of the Einstein universe for both a conserved and a nonconserved tensor $T_{\mu\nu}$ [25] by parameterizing the stability regions. They considered energy conditions in $f(G,T)$ gravity in [26]. Recently in [27], Shamir and Ahmad used the Noether symmetry approach in the $f(G,T)$ theory with two specific gravity models taken into account and reconstructed the de Sitter model. They also obtained some exact solutions in the framework of this model [28]. Details of an anisotropic universe in $f(G,T)$ gravity were recently considered in [29]. Applying this theory in thermodynamics was also recently attempted. In [30], the laws of nonequilibrium thermodynamics were investigated, and a generalized second law of thermodynamics was verified for reconstructed $f(G,T)$ models. The role of $f(G,T)$ gravity in the evolution of relativistic stars was recently discussed [31]. We can predict that for some specific models, applying this theory can lead to important results explaining the accelerated expansion of the universe. This in particular makes this theory interesting to study.

Here, our efforts are directed toward obtaining plane symmetric solutions in $f(G,T)$ gravity. For this, we consider a plane symmetric space–time. A plane symmetric approximation is considered next in accuracy after spherically and cylindrically symmetric models in cosmological problems. Studying plane symmetric solutions in $f(R)$ and $f(R,T)$ theories of gravity has already yielded interesting results [32], [33]. Here, in Sec. 2, we present some preliminary information regarding $f(G,T)$ gravity and also briefly introduce the theory of field equations. In Sec. 3, we give general solutions of the field equations for some specific $f(G,T)$ models. In Sec. 4, we brief summarize our work.

2. The $f(G,T)$ gravity: Elemental structure

The action for $f(G,T)$ gravity is [23]

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} [R + f(G,T)] + \int d^4x \sqrt{-g} L_m, \quad (1)$$

where $\kappa$, $g$, and $L_m$ are the respective coupling constant, determinant of the metric tensor $g^{\mu\nu}$, and standard matter Lagrangian. The GB term and the trace of the energy–momentum tensor are denoted by $G$ and $T$. 

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The function $f(G, T)$ is a generic function of the GB invariant $G$ and the energy–momentum tensor trace $T$. The energy–momentum tensor $T_{\zeta\eta}$ is related to the matter Lagrangian $L_m$ and can be calculated from the relation

$$T_{\zeta\eta} = -\frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_m}{\delta g_{\zeta\eta}}.$$  \hspace{1cm} (2)

But if the matter distribution depends only on the metric tensor $g_{\zeta\eta}$, then Eq. (2) becomes

$$T_{\zeta\eta} = g_{\zeta\eta} L_m - 2 \frac{\partial L_m}{\partial g_{\zeta\eta}}.$$  \hspace{1cm} (3)

Varying action (1) with respect to the metric tensor $g^{\mu\nu}$ now yields the gravitational field equations

$$0 = \partial S = \frac{1}{2\kappa^2} \int d^4x [(R + f(G, T))\delta \sqrt{-g} + \sqrt{-g}(\delta R + f_G(G, T)\delta G + f_T(G, T)\delta T)] + \int d^4x \delta(\sqrt{-g} L_m),$$  \hspace{1cm} (4)

where the subscripts $G$ and $T$ in $f_G$ and $f_T$ respectively represent the partial variations of $f$ with respect to $G$ and $T$. The variation of different quantities yields the relations

$$\delta \sqrt{-g} = -\frac{1}{2} \sqrt{-g} g_{\zeta\eta} \delta g^{\zeta\eta}, \quad \delta R_{\mu\nu}\delta g^{\zeta\eta} = \nabla_\eta (\delta \Gamma^\zeta_{\mu\nu}) - \nabla_\nu (\delta \Gamma^\zeta_{\mu\eta}),$$

$$\delta \Gamma^\zeta_{\mu\nu} = \Gamma^\zeta_{\mu\nu}, \quad \delta R = (R_{\mu\nu} + g_{\mu\nu} \nabla^2 - \nabla_\mu \nabla_\nu) \delta g^{\mu\nu}.$$  \hspace{1cm} (5)

Here, $\nabla_\mu$ operates as covariant differentiation, and $\Gamma^\zeta_{\mu\nu}$ are Christoffel symbols. Similarly,

$$\delta G = 2R \delta R - 4 \delta (R_{\mu\nu} R^{\mu\nu}) + \delta (R_{\zeta\mu\eta} R^{\zeta\mu\eta}), \quad \delta T = (T_{\mu\nu} + \Theta_{\mu\nu}) \delta g^{\mu\nu}. \hspace{1cm} (6)$$

Using all these relations in Eq. (4) yields the equation

$$R_{\zeta\eta} - \frac{1}{2} R g_{\zeta\eta} = -[2R g_{\zeta\eta} \nabla^2 - 2R \nabla_\zeta \nabla_\eta - 4g_{\zeta\eta} R^{\mu\nu} \nabla_\mu \nabla_\nu - 4R_{\xi\eta} \nabla^2 + 4R^\mu_{\xi\mu} \nabla_\eta \nabla_\mu +$$

$$+ 4R^\rho_{\eta\rho} \nabla_\zeta \nabla_\mu + 4R_{\eta\mu\rho} \nabla^\nu \nabla^\nu f_G(G, T)] + \frac{1}{2} g_{\zeta\eta} f(G, T) - [T_{\zeta\eta} + \Theta_{\zeta\eta}] f_T(G, T) -$$

$$- [2RR_{\zeta\eta} - 4R^\rho_{\zeta\rho} R_{\mu\eta} - 4R_{\zeta\mu\rho\eta} R^{\mu\rho} + 2R^\alpha_{\zeta\mu} R^{\mu\rho\delta} f_G(G, T) + \kappa^2 T_{\zeta\eta}].$$  \hspace{1cm} (7)

Also here, $\Theta_{\zeta\eta} = g^{\mu\nu} \delta T_{\mu\nu}/\delta g_{\zeta\eta}$. It can be verified that appropriate substitutions reduce Eq. (7) to equations for $f(G)$ and the usual GTR field equations. For convenience, we set $f(G, T) \equiv f_G$, $f_T(G, T) \equiv f_T$, etc. The trace of Eq. (7) is

$$R + \kappa^2 T - (T + \Theta) f_T + 2f + 2G f_G - 2R \nabla^2 f_G + 4R^\zeta_{\eta} \nabla_\zeta \nabla_\eta f_G = 0.$$  \hspace{1cm} (8)

From this relation, we can conclude that in contrast to the GTR, the relation here between $R$, $G$, and $T$ is differential and not algebraic, whence we can suppose the existence of substantially more solutions than in the GTR. The covariant-type divergence of Eq. (7)

$$\nabla^\zeta T_{\zeta\eta} = \frac{f_T}{\kappa^2 - f_T} \left[ (T_{\zeta\eta} + \Theta_{\zeta\eta}) \nabla^\zeta (\log f_T) + \nabla^\zeta \Theta_{\zeta\eta} - \frac{g_{\zeta\eta}}{2} \nabla^\zeta T \right]$$  \hspace{1cm} (9)

is nonvanishing because it contains higher-order derivatives of the energy–momentum tensor. To obtain the standard conservation equation [23], we must impose certain conditions on Eq. (9). This allows minimizing the chance of encountering divergences in applying the theory at larger scales.
To start, we write the expression defining the general static plane symmetric space–time [33]:

\[ ds^2 = A(x) \, dt^2 - dx^2 - B(x) [dy^2 + dz^2], \]

where \( A \) and \( B \) are metric functions of \( x \). Further, we consider the corresponding GB invariant and Ricci scalar

\[ \mathcal{G} = -\frac{1}{2} \left[ \frac{A'^2 B'^2}{A^2 B^2} - 4 \frac{A'B'B''}{A B^2} - 2 \frac{A'' B'^2}{A B^3} + 2 \frac{A'B'^3}{A B^4} \right], \]

\[ R = \frac{1}{2} \left[ 2 \frac{A''}{A} - \frac{A'^2}{A^2} + 2 \frac{A'B'}{A B} + \frac{B''}{B} - \frac{B'^2}{B^2} \right], \]

where the prime denotes variation with respect to \( x \). Assuming that the universe is filled with a perfect fluid, we take the energy–momentum tensor in the form

\[ T_{\mu\nu} = (\rho + p) u_\mu u_\nu - p g_{\mu\nu}, \]

where \( u_\mu, \rho, \) and \( p \) have the usual meanings. The expansion scalar \( \theta \) and shear scalar \( \sigma \) are

\[ \theta = u_\mu^\mu = \frac{A'}{A} + 2 \frac{B'}{B}, \quad \sigma^2 = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} = \frac{1}{3} \left[ \frac{A'}{A} - \frac{B'}{B} \right]^2. \]

Here,

\[ \sigma_{\mu\nu} = \frac{1}{2} \left( u_{\mu;\alpha} h^\alpha_\nu + u_{\nu;\alpha} h^\alpha_\mu \right) - \frac{1}{3} \theta h_{\mu\nu}, \quad h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu. \]

With (10), field equations (7) now become

\[ 2 \left( \frac{B''}{B} + \frac{B'^2}{4 B^2} \right) + 2 \left( \frac{B'^3}{B^3} - \frac{2 B'B''}{B^2} \right) f''_\nu - 2 \frac{B'^2}{B^2} f''_\nu - 2(\rho + p) f_T + \mathcal{G} f_\nu + f = 2 \kappa^2 \rho, \]

\[ 2 \left( \frac{B'^2}{4 B^2} + \frac{A'B'}{4 A B} \right) + 3 \frac{A'^2 B'^2}{AB^2} f''_\nu - \mathcal{G} f_\nu + f = 2 \kappa^2 p, \]

\[ 2 \left( \frac{A'B'}{4 AB} + \frac{A''}{2 A} + \frac{B''}{2 B} - \frac{A'^2}{4 A^2} - \frac{B'^2}{4 B^2} \right) + \left( \frac{A'B''}{AB} - \frac{A'' B'}{AB} + \frac{A'^2 B''}{AB^2} - \frac{A'' B'}{A^2 B} \right) f''_\nu + \]

\[ + 2 \frac{A'B'}{AB} f''_\nu - \mathcal{G} f_\nu + f = 2 \kappa^2 p. \]

To obtain solutions of these highly nonlinear, very complicated differential equations with five unknowns, we must impose some additional constraints. The proportionality of the shear scalar \( \sigma \) and the expansion scalar \( \theta \) gives

\[ A = B^n, \quad n \in \mathbb{R}. \]

According to recent observations, the cosmological expansion can become isotropic if the ratio of the shear scalar to the expansion scalar is constant; this physical condition is therefore important in a cosmological theory [34]. Many researchers acted similarly in solving the field equations [35]–[40]. With condition (19),

\[ 1848 \]
field equations (16)–(18) become

\[ 2 \left[ -\frac{B''}{B} + \frac{B'^2}{4B^2} \right] + 2 \left[ \frac{B'^3}{B^3} - \frac{2B'B''}{B^2} \right] f'_G - \frac{2B'^2}{B^2} f''_G - 2(\rho + p)f_T + Gf_G - f = 2\kappa^2 \rho, \]  \hspace{1cm} (20)\]

\[ 2(2n + 1) \frac{B'^2}{4B^2} + 3n \frac{B'^3}{B^3} f'_G - Gf_G + f = 2\kappa^2 p, \]  \hspace{1cm} (21)\]

\[ 2(n^2 - n - 1) \frac{B'^2}{4B^2} + 2(n + 1) \frac{B''}{2B} + \left[ 4n \frac{B'B''}{B^2} + n(n - 3) \frac{B'^3}{B^3} \right] f'_G + \]

\[ + 2n \frac{B'^2}{B^2} f''_G - Gf_G + f = 2\kappa^2 p. \]  \hspace{1cm} (22)\]

In the next section, we try to find some general solutions of these truncated field equations.

3. Modified field equations: Solutions

We consider the \( f(G, T) \) model of the form [29]

\[ f(G, T) = \alpha f_1(G) + \beta f_2(T), \]  \hspace{1cm} (23)\]

where \( \alpha \) and \( \beta \) are arbitrary constants. Further, we take a well-known \( f_1(G) \) model as the choice for \( f(G) \):

\[ f_1(G) = G^{m+1}, \]  \hspace{1cm} (24)\]

where \( m \) is an arbitrary constant. This choice is important because it avoids the Big-Rip singularity, is viable, and lacks irregular spin-2 ghosts [2], [41]–[43]. For this choice, we have a dominating curvature that implies the Einstein term dominating \( f(G) \) terms, which prevents the Big-Rip singularity [44]. Subtracting Eq. (21) from Eq. (22) gives a general ordinary differential equation

\[ (n^2 - 3n - 2) \frac{B'^2}{4B^2} + (n + 1) \frac{B''}{2B} + \frac{1}{2} \left[ 4n \frac{B'B''}{B^2} + n(n - 3) \frac{B'^3}{B^3} \right] f'_G + n \frac{B'^2}{B^2} f''_G = 0. \]  \hspace{1cm} (25)\]

It follows from Eq. (24) that

\[ f_G = \alpha(m + 1)G^m. \]  \hspace{1cm} (26)\]

For simplicity, we take \( \alpha = 1/(m + 1) \), and Eq. (25) hence becomes

\[ (n^2 - 3n - 2) \frac{B'^2}{4B^2} + (n + 1) \frac{B''}{2B} + \]

\[ + \frac{1}{2} n m G^m \left[ 4 \frac{B'B''}{B^2} + (n - 6) \frac{B'^3}{B^3} \right] \frac{G'}{G} + 2 \frac{B'^2}{B^2} \left( (m - 1) \frac{G'^2}{G^2} + \frac{G''}{G} \right) = 0. \]  \hspace{1cm} (27)\]

This general differential equation can have many viable solutions. Here, we consider only two of them.

Power-law solutions are believed to play an important role in cosmology. We consider the power law [29]

\[ B(x) = \gamma x^k, \]  \hspace{1cm} (28)\]
where $\gamma$ and $k$ are arbitrary constants. Using this relation in Eq. (27), we obtain the constraints

$$k = \frac{2(n + 1)}{n(n - 1)}, \quad m(2n^3m + n^3 + 1 - 2nm) = 0. \quad (29)$$

The $f(G)$ gravity models corresponding to the two roots of the second relation in (29) are

$$f(G) = G + c_1, \quad f(G) = \frac{2n(n - 1)}{n^2 - n - 1}G^{(n^2-n-1)/(2n(n-1))} + c_2,$$

where $c_1$ and $c_2$ are integration constants. The first trivial model corresponds to the case where $c_1 = 0$ and $f_2(T) = 0$. The second model can lead to important viable results. Considering different forms of $f_2(T)$ leads to reconstructing different known solutions.

### 3.1. The $f_2(T)$ model: Linear case

We consider the linear function $f_2(T) = T$, which allows significantly reducing the computational complexity. Equations (20)–(22) yield expressions for the energy density and pressure of universe:

$$\rho = -\frac{n + 1}{(m + 1)(n - 1)^4n^3x^4\kappa^2(\kappa^2 + 1)} \left[2^{3+4m}m(1 + n)(-1 + \kappa^2(3 + 8m^2(n - 1)^2)n + n(2 + n(7n - 18)) + 2m(n - 1)(-2 + n(7n - 9)) + n(4m^2(n - 1)^2n + n(2 + n(2n - 9)) + 2m(2 + n(4 + 3(n - 3)))\left(-\frac{(1 + n)^3(-1 + n(n - 3))}{(n - 1)^4n^3x^4}\right)^m + (1 + m)n(n - 1)^2(-1 + \kappa^2(n - 3) - n)(2n + 1)x^2 \right] - \frac{\beta f_T(T)}{2\kappa^2}, \quad (31)$$

$$p = \frac{1}{(1 + m)(n - 1)^4n^3x^4\kappa^2} \left[2^{3+4m}m(n + 1)^2(-1 + n(4m^2(n - 1)^2)n + n(2 + n(2n - 9)) + 2m(2 + n(4 + 3(n - 3)))\left(-\frac{(n + 1)^3(-1 + n(n - 3))}{(n - 1)^4n^3x^4}\right)^m + (1 + m)n(2n + 1)(n^2 - 1)^2x^2 \right] + \frac{\beta f_T(T)}{2\kappa^2}. \quad (32)$$

It obviously follows from Eqs. (35) and (36) that these expressions are defined for $n \in \mathbb{R} \setminus \{0, 1\}$ if $m = (-1/2)(n^2 - n + 1)/(n(n - 1))$. It follows from the relations obtained for the energy density and pressure that the equation of state parameter $\omega$ dominates (its value is close to $-1$), which implies that the Big-Rip singularity does not appear with time [2]. Plots of the change of the energy density and the pressure for $0 < n < 1$ and $1 < n$ are shown in Figs. 1 and 2.

### 3.2. The $f_2(T)$ model: Starobinsky-like case

We consider the model $f_2(T) = T + \epsilon T^2$, where $\epsilon$ is an arbitrary constant. Manipulation of field equation (20)–(22) leads to the relation

$$21\epsilon p^2 - \rho(3 + 2\kappa^2 + 5\epsilon \rho) - p(5 + 22\epsilon \rho) =$$

$$= \frac{1}{(1 + m)(-1 + n)^4n^3x^4} \left[2(1 + n)(2^{3+4m}m(1 + n)(-3 + 8m^2(-1 + n)^3)n + n(-6 + n(26 + n(-15 + 4n)) + 2m(-1 + n)(2 + n(11 + n(-13 + 6n)))) - \left(-\left(\frac{(1 + n)^3(-1 + (3 + n)n)}{((-1 + n)^4n^3x^4)}\right)^m + (1 + m)(-3 + n)(-1 + n)^2n(1 + 2n)x^2 \right) \right]. \quad (33)$$
It has different solutions under different physical conditions. Because this relation is complicated, for different values of equation of state parameter $\omega$, we can obtain a quadratic equation in either the energy density $\rho$ or the pressure $p$ of the universe. Possibly, it will lead to important insights about the cosmological structure of the universe, the DE issue, the cosmological expansion conundrum, and other gaps in the existing literature devoted to the GTR. The solution metric as a rule has the form

$$ds^2 = \gamma^n x^{(2(n+1))/(n-1)} dt^2 - dx^2 - \gamma x^{(2(n+1))/(n(n-1))} [dy^2 + dz^2].$$  \hfill (34)

### 3.3. A few important cosmological parameters.

The Ricci scalar $R$ and the GB invariant $\mathcal{G}$ become

$$R = -\frac{16(1+n)^2(-1-4n-2n^2+n^3)}{n^3(-1+n)^4x^4},$$  \hfill (35)

$$\mathcal{G} = \frac{2(9n^2+2n^3+3+10n)}{n^2(n-1)^2x^2}. \hfill (36)$$

The expansion and shear scalars are

$$\theta = \frac{2(n+1)(n+2)}{n(n-1)x}, \quad \sigma^2 = \frac{1}{3} \left( \frac{2(n+1)}{xn} \right)^2. \hfill (37)$$

In the considered case, the isotropy condition is also satisfied as $\sigma^2/\theta \to 0$ for $x \to \infty$. The behavior of the isotropy parameter for $0 < n < 1$ and $1 < n$ is plotted in Fig. 3.

### 4. Reconstruction of a few well-known solutions

Many researchers have previously considered a plane symmetry from various standpoints [45]–[50]. We have similarly obtained some solutions for a plane symmetric space–time in the $f(\mathcal{G})$ setting. Solution
Fig. 3. Behavior of the isotropy parameter $\sigma^2/\theta$ for $m = -(n^2 - n + 1)/2n(n - 1)$.

The expansion of the universe is believed to have undergone a decelerating power-law expansion followed by late-time acceleration. This makes power-law solutions extremely important in cosmology because they describe matter-dominated phases subsequently changing to an acceleration phase. Here, we have used a power-law approach to the $f(G,T)$ theory of gravity for a static plane symmetric space–time. We believe that it is the first attempt to investigate such solutions in the $f(G,T)$ gravity setting. To obtain solutions of the field equations, we used a relation between the metric coefficients, in particular, the proportionality between the shear scalar $\sigma$ and the expansion scalar $\theta$.

We first considered a $f(G)$ model in the form of a linear combination of $f(G)$ and $f(T)$ models. Further, as $f(G)$, we took a power-law model known in the literature [2],[41]–[43]. The chosen power law avoids the Big-Rip singularity. We note that there is a difference between Big-Bang and Big-Rip singularities. The Big Bang is an initial singularity based on the hypothesis that the universe initially had an infinitely concentrated density and then during the Big Bang abruptly expanded and continues to expand in our days. In contrast, the Big Rip is a final singularity predicting the ultimate fate of the universe, i.e., our constantly accelerating universe will eventually be ripped apart. For $f(T)$, we considered two cases. We derived general differential equation (27), which can have many solutions. To obtain the sought solutions, we used methods based on power and exponential laws. Two $f(G)$ gravity models correspond to the solution
obtained by the power-law approach. One of them is trivial for $c_1 = 0$ and $f_2(T) = 0$, but the other deserves attention. We plotted the obtained results for the possible parameter ranges. We recovered the well-known Taub metric [51], and the GB term and the Ricci scalar are constant in this case. The solutions obtained using the exponential law allowed reconstructing another well-known anti-de Sitter space–time [52].

We here note that the obtained results and the plots of their behavior agree well with already known facts [53] for the same space–time in $f(G)$ gravity and can be recovered by setting $\beta = 0$ in Eq. (23).

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