Coordinate-free study of Finsler spaces of \( H_p \)-scalar curvature

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Dedicated to Professor Nabil Youssef on the occasion of his 70th birthday

Abstract. The aim of the present paper is to provide an intrinsic investigation of special Finsler spaces of \( H_p \)-scalar curvature and of \( H_p \)-constant curvature. Characterizations of such spaces are shown. Sufficient condition for Finsler space of \( H_p \)-scalar curvature to be of perpendicular scalar curvature is investigated. Necessary and sufficient condition under which a Finsler space of scalar curvature turns into a Finsler space of \( H_p \)-scalar curvature is given. Further, certain conditions under which a Finsler manifolds of \( H_p \)-scalar curvature and of scalar curvature reduce to a Finsler manifold of \( H_p \)-constant curvature are obtained. Finally, various examples are studied and constructed.

Keywords. Berwald connection; \( H_p \)-scalar curvature; \( H_p \)-constant curvature; scalar curvature; constant curvature; projection operator.

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Introduction

In Riemannian geometry, the study of the Riemannian manifolds of scalar curvature was very fruitful. It has been contributed in classifying lots of Riemannian manifolds. For example, the special manifolds of constant curvature $-1, 0, +1$. The concept of scalar curvature was extended to Finsler geometry. Most of the special spaces are derived from the fact that the tensor fields (torsions and curvatures) associated with a linear connection can be given in special forms. In Riemannian geometry there exist a unique linear connection, that is, Levi-Civita connection. But in Finsler geometry there are lots of linear connections, for example, Cartan connection, Berwald connection, etc. Consequently, the special Finsler spaces are more numerous than those of Riemannian geometry. Special Finsler spaces are investigated locally (using local coordinates) by many authors, see for example ([2], [6], [7], [8], [9], [10], [11], [15]). On the other hand, the global (or intrinsic, free from local coordinates) investigation of such spaces is very rare in the literature. Some considerable contributions in this direction are [14], [17] and [19].

In a recent paper [17], some characterizations of a Finsler space of scalar curvature are investigated. Also, necessary and sufficient conditions under which a Finsler space of scalar curvature reduces to a Finsler space of constant curvature are shown.

The present paper is a continuation of [17] and [19], where we provide an intrinsic investigation of some important special Finsler manifolds related to the Berwald curvature tensors namely, Finsler manifold of $H_p$-scalar curvature and of $H_p$-constant curvature.

In [15], Yoshida introduced, locally, the notion of Finsler space of $H_p$-scalar curvature. In this paper, we study, in a coordinate-free fashion, the Finsler spaces of $H_p$-scalar curvature and $H_p$-constant curvature. We give a characterization for any Finsler manifold to be of $H_p$-scalar curvature. We find a sufficient condition under which a Finsler space of $H_p$-scalar curvature is of perpendicular scalar curvature.

Section 3 is devoted to focus on the Finsler spaces of scalar curvature and constant curvature. Necessary and sufficient condition under which a Finsler space of scalar curvature is of $H_p$-scalar curvature is given. We show that every Finsler space of constant curvature is of $H_p$-scalar curvature. The converse is true only in some specific cases, for example, see Theorem 3.6. But, generally, not every Finsler space of $H_p$-constant curvature is of Constant curvature, see Section 4 (Example 6).

In Section 4, we study various examples. Some of these examples are mentioned in the literature, but in different contexts. And some examples are constructed, for instance Example 6.

It should finally be noted that the present work is formulated in a prospective modern coordinate-free form. Moreover, the outcome of this work is twofold. Firstly, the local expressions of the obtained results, when calculated, coincide with the existing local results. Secondly, new coordinates-free proofs have been established.

1 Notations and Preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [11] [13] [14].

Throughout, $M$ is a smooth manifold of finite dimension $n$. The $\mathbb{R}$-algebra of smooth real-valued functions on $M$ is denoted by $C^\infty(M)$; $\mathfrak{X}(M)$ stands for the $C^\infty(M)$-module
of vector fields on $M$. The tangent bundle of $M$ is $\pi_M : TM \rightarrow M$, the subbundle of nonzero tangent vectors to $M$ is $\pi : TM \rightarrow M$. The vertical subbundle of $TTM$ is denoted by $V(TM)$. The pull-back of $TM$ over $\pi$ is $P : \pi^{-1}(TM) \rightarrow TM$.

Elements of $\mathfrak{X}(\pi(M))$ will be called $\pi$-vector fields and will be denoted by barred letters $\overline{X}$. Tensor fields on $\pi^{-1}(TM)$ will be called $\pi$-tensor fields. The fundamental $\pi$-vector field is the $\pi$-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in TM$.

We have the following short exact sequence of vector bundle morphisms:

$$0 \rightarrow \pi^{-1}(TM) \overset{\gamma}{\rightarrow} T(TM) \overset{\rho}{\rightarrow} \pi^{-1}(TM) \rightarrow 0.$$  

Here $\rho := (\pi_\tau, \pi*)$, and $\gamma$ is defined by $\gamma(u, v) := j_u(v)$, where $j_u$ is the canonical isomorphism from $T_{\pi_M(v)}M$ onto $T_u(T_{\pi_M}(v)M)$. Then, $J := \gamma \circ \rho$ is a vector 1-form on $TM$ called the vertical endomorphism. The Liouville vector field on $TM$ is the vector field defined by $C := \gamma \circ \overline{\eta}$, $\overline{\eta}(u) = (u, u)$, $u \in TM$.

Let $D$ be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. The connection (or the deflection) map associated with $D$ is defined by

$$K : TTM \rightarrow \pi^{-1}(TM) : X \mapsto DX\overline{\eta}.$$  

A tangent vector $X \in T_u(TM)$ at $u \in TM$ is horizontal if $K(X) = 0$. The vector space $H_u(TM) = \{X \in T_u(TM) : K(X) = 0\}$ is called the horizontal space at $u$. The connection $D$ is said to be regular if

$$T_u(TM) = V_u(TM) \oplus H_u(TM) \quad \forall u \in TM.$$  

Let $\beta := (\rho|_{H(TM)})^{-1}$, called the horizontal map of the connection $D$, then

$$\rho \circ \beta = \text{id}_{\pi^{-1}(TM)}, \quad \beta \circ \rho = \text{id}_{H(TM)} \text{ on } H(TM).$$

For a regular connection $D$, the horizontal covariant derivative $\stackrel{h}{D}$ and the vertical covariant derivatives $\stackrel{v}{D}$ are defined, for a vector $(1)\pi$-form $A$, for example, by

$$\left(\stackrel{h}{D}A\right)(\overline{X}, \overline{Y}) := (D_{\beta\overline{X}}A)(\overline{Y}), \quad \left(\stackrel{v}{D}A\right)(\overline{X}, \overline{Y}) := (D_{\gamma\overline{X}}A)(\overline{Y}).$$

The (classical) torsion tensor $T$ (resp. the curvature tensor $K$) of the connection $D$ are given by

$$T(X, Y) = DX\rho Y - DY\rho X - \rho[X, Y],$$

$$K(X, Y)\rho Z = -DXDY\rho Z + D_Y DX\rho Z + D_X[Y, \rho Z],$$

for all $X, Y, Z \in \mathfrak{X}(TM)$. The horizontal (h-) and mixed (hv-) torsion tensors are defined respectively by

$$Q(\overline{X}, \overline{Y}) := T(\beta\overline{X}, \beta\overline{Y}), \quad T(\overline{X}, \overline{Y}) := T(\gamma\overline{X}, \beta\overline{Y}) \quad \forall \overline{X}, \overline{Y} \in \mathfrak{X}(\pi(M)).$$

and the horizontal (h-), mixed (hv-) and vertical (v-) curvature tensors are defined respectively by

$$R(\overline{X}, \overline{Y})Z := K(\beta\overline{X}, \beta\overline{Y})Z, \quad P(\overline{X}, \overline{Y})Z := K(\beta\overline{X}, \gamma\overline{Y})Z, \quad S(\overline{X}, \overline{Y})Z := K(\gamma\overline{X}, \gamma\overline{Y})Z.$$  

The (v)h-, (v)hv- and (v)v-torsion tensors are defined respectively by

$$\hat{R}(\overline{X}, \overline{Y}) := R(\overline{X}, \overline{Y})\overline{\eta}, \quad \hat{P}(\overline{X}, \overline{Y}) := P(\overline{X}, \overline{Y})\overline{\eta}, \quad \hat{S}(\overline{X}, \overline{Y}) := S(\overline{X}, \overline{Y})\overline{\eta}.$$  

For a Finsler manifold $(M, L)$, we have the Berwald connection $D^o$ on the pullback bundle.
Theorem 1.1. [18] Let \((M, L)\) be a Finsler manifold. There exists a unique regular connection \(D^\circ\) on \(\pi^{-1}(TM)\) such that

(a) \(D^\circ h_X L = 0\),

(b) \(D^\circ\) is torsion-free: \(T^\circ = 0\),

(c) The \((v)h\)-torsion tensor \(\hat{\epsilon} P^\circ\) of \(D^\circ\) vanishes: \(\hat{P}^\circ(X, Y) = 0\).

Such a connection is called the Berwald connection associated with the Finsler manifold \((M, L)\).

2 Finsler Spaces of \(H_p\)-scalar curvature

In this section, we investigate (intrinsically) some important special Finsler spaces related to the Berwald curvature tensors namely, Finsler spaces of \(H_p\)-scalar curvature and of \(H_p\)-constant curvature. Characterizations of such spaces are obtained. Relation between Finsler spaces of \(H_p\)-scalar curvature and of perpendicular scalar curvature is investigated.

Throughout, for given Finsler manifold \((M, L)\), \(g\) denotes the Finsler metric on \(\pi^{-1}(TM)\) and \(\nabla\) denotes the Cartan connection. Also, \(T\) stands for the Cartan tensor, \(R\) and \(\hat{R}\) for the \(h\)-curvature and \((v)h\)-torsion of Cartan connection, \(\hat{\epsilon} R\) and \(\hat{\epsilon} R^\circ\) for the \(h\)-curvature and \((v)h\)-torsion of Berwald connection, and \(H := i^\pi R\) for the deviation tensor. Moreover, \(\ell := L^{-1}i\pi g\), \(\phi(X) := X - L^{-1}\ell(X)\eta\) and \(h(X, Y) := g(\phi(X), Y) = g(X, \overline{Y}) - \ell(X)\ell(Y)\) the angular metric tensor. Finally, \(\hat{D}^\circ\) and \(D^\circ\) will denote respectively the horizontal covariant derivative and the vertical covariant derivative associated with \(D^\circ\).

We begin with the following definitions and results of [17] which are useful for subsequent use.

Definition 2.1. [19] A Finsler manifold \((M, L)\) of dimension \(n \geq 3\) is called of scalar curvature \(k\) if the deviation tensor \(H\) satisfies

\[ H(X) = kL^2\phi(X), \]

where \(k(x, y) \in \mathcal{C}^\infty(TM)\) is a positively homogenous of degree zero in \(y\) \((x \in M\) and \(y \in T_xM)\). Especially, if the scalar curvature \(k\) is constant, then \((M, L)\) is called a Finsler manifold of constant curvature.

Definition 2.2. [19] Let \(\mathcal{P}\) be the projection operator of indicatrix (or simply, projection operator) and \(\phi(X) := X - L^{-1}\ell(X)\eta\). If \(\omega\) is a \(\pi\)-tensor field of type \((1,p)\), then \(\mathcal{P} \cdot \omega\) is a \(\pi\)-tensor field of the same type defined by:

\[ (\mathcal{P} \cdot \omega)(\underbrace{X_1, \ldots, X_p}_p) := \phi(\omega(\phi(X_1), \ldots, \phi(X_p))). \]

Similarly, if \(\omega\) is a \(\pi\)-tensor field of type \((0,p)\), then \(\mathcal{P} \cdot \omega\) is a \(\pi\)-tensor field of the same type defined by:

\[ (\mathcal{P} \cdot \omega)(\underbrace{X_1, \ldots, X_p}_p) := \omega(\phi(X_1), \ldots, \phi(X_p)). \]

Moreover, for any \(\pi\)-tensor field \(\omega\) is called an indicatory tensor if \(\mathcal{P} \cdot \omega = \omega\).
Theorem 2.3. [17] A Finsler manifold \((M, L)\) is of scalar curvature \(k\) if, and only, if the \(h\)-curvature \(R\) satisfies

\[
\hat{R}(\overline{X}, \overline{Y}, \overline{Z}) = \mathbb{A}_{\overline{X}, \overline{Y}}\{ \phi(\overline{Y})\{ \ell(\overline{Z})[k\ell(\overline{X}) + \frac{1}{3} C^k(\overline{X})] + \frac{1}{3} B^k(\overline{Z}, \overline{X}) \}
+ \frac{2}{3} \ell(\overline{X}) C^k(\overline{Z}) + kh(\overline{Z}, \overline{X}) \} + \frac{1}{3} \ell(\overline{X}) C^k(\overline{Y}) \phi(\overline{Z})
+ L^{-1} h(\overline{X}, \overline{Z}) \eta[ k\ell(\overline{Y}) + \frac{1}{3} C^k(\overline{Y}) ],
\]

where \(C^k(\overline{X}) := L(\overline{D}^o k)(\overline{X})\), \(B^k(\overline{X}, \overline{Y}) := L(\mathcal{P} \cdot \overline{D}^o C^k)(\overline{X}, \overline{Y})\), and \(\overline{D}^o\) is the vertical covariant derivative associated with \(D^o\).

Theorem 2.4. [17] A necessary and sufficient condition for a Finsler manifold of scalar curvature \(k\) to be of constant curvature is that the \(\pi\)-scalar form \(C^k\) (or \(B^k\)) vanishes.

Remark 2.5. One easily show that the \(\pi\)-tensor fields \(\hat{P}, T, \phi, h, C^k\) and \(B^k\) are indicatory tensors, and that \(\mathcal{P} \cdot \ell\) vanishes identically.

Now, we are in a position to introduce the definition of Finsler spaces of \(H_p\)-scalar curvature.

Definition 2.6. A Finsler manifold \((M, L)\) of dimension \(n \geq 4\) is called of \(H_p\)-scalar curvature \((H_p-sc)\) \(\varepsilon\) if the \(h\)-curvature tensor of Berwald connection \(\hat{R}\) satisfies

\[
(\mathcal{P} \cdot \hat{R})(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = \varepsilon \mathbb{A}_{\overline{X}, \overline{Y}}\{ h(\overline{X}, \overline{Z}) h(\overline{Y}, \overline{W}) \},
\]

where \(\varepsilon(x, y) \in C^\infty(TM)\) is a positively homogenous of degree zero in \(y\). Especially, if the \(H_p\)-scalar curvature \(\varepsilon\) is constant, then \((M, L)\) is called a Finsler manifold of \(H_p\)-constant curvature \((H_p-cc)\). Moreover, if the \(H_p\)-scalar curvature \(\varepsilon\) vanishes, then \((M, L)\) is called a Finsler manifold of vanishing \(H_p\)-scalar curvature.

Now, we investigate intrinsically a characterization for a Finslers spaces of \(H_p\)-sc.

Theorem 2.7. A Finsler manifold is of \(H_p\)-sc \(k\), if and only if the \(h\)-Berwald curvature \(\hat{R}\) has the form

\[
\hat{R}(\overline{X}, \overline{Y}, \overline{Z}, \overline{W}) = L^{-1}\{ \ell(\overline{Z})\hat{R}(\overline{X}, \overline{Y}, \overline{W}) - \ell(\overline{W})\hat{R}(\overline{X}, \overline{Y}, \overline{Z}) \}
+ \mathbb{A}_{\overline{X}, \overline{Y}}\{ L^{-1}\ell(\overline{X})\{ \hat{R}(\overline{Z}, \overline{W}, \overline{Y}) - T(H(\overline{Z}), \overline{W}, \overline{Y}) + T(H(\overline{W}), \overline{Z}, \overline{Y}) \}
- T(H(\overline{Y}), \overline{Z}, \overline{W}) - (D^o_{\overline{Z} \overline{Y}}\hat{P})(\overline{Z}, \overline{W}, \overline{Y}) \}
- L^{-2}\{ \hat{R}(\overline{Y}, \overline{X}, \overline{Z}) \ell(\overline{Y}) \ell(\overline{W}) + \hat{R}(\overline{Y}, \overline{W}, \overline{X}) \ell(\overline{Z}) - kL^2 h(\overline{X}, \overline{Z}) h(\overline{Y}, \overline{W}) \} \}
\]

where \(\hat{P}\) is the (\(v\))hv-torsion of Cartan connection \(\nabla\), \(\hat{P}(\overline{X}, \overline{Y}, \overline{Z}) := g(\hat{P}(\overline{X}, \overline{Y}), \overline{Z})\)
\(\hat{R}(\overline{X}, \overline{Y}, \overline{Z}) := g(\hat{R}(\overline{X}, \overline{Y}), \overline{Z})\) and \(T(\overline{X}, \overline{Y}, \overline{Z}) := g(T(\overline{X}, \overline{Y}), \overline{Z})\).

To prove this theorem, we need the following two lemmas.

\[\mathbb{A}_{\overline{X}, \overline{Y}}\{ \omega(\overline{X}, \overline{Y}) \} := \omega(\overline{X}, \overline{Y}) - \omega(\overline{X}, \overline{X})\]
Lemma 2.8. [20] The h-curvature tensor \( \hat{R} \) of the Berwald connection has the properties:

(a) \( \hat{R}(X, Y, Z, W) = -\hat{R}(Y, X, Z, W) \),

(b) \( \hat{R}(X, Y) = \hat{R}(X, Y) \),

(c) \( \hat{R}(X, Y, Z, W) + \hat{R}(X, Y, W, Z) = 2\hat{X} \hat{Y}(\{ (D_{\beta\gamma}^\beta) \hat{X}, \hat{Y}, \hat{Z}, \hat{W} \}) - 2T(\hat{R}(X, Y), Z, W) \),

(d) \( \Theta_{X, Y, Z}[\hat{R}(X, Y)Z] = 0 \).

Lemma 2.9. The h-curvature tensor \( \hat{R} \) of the Berwald connection satisfies

\[
\hat{R}(X, Y, Z, W) = \hat{R}(Z, W, X, Y) + (D_{\beta\gamma}^\beta) \hat{R}(X, Z, W) - (D_{\beta\gamma}^\beta) \hat{R}(Z, W, Y) + (D_{\beta\gamma}^\beta) \hat{R}(X, W, Y) \\
-(D_{\beta\gamma}^\beta) \hat{R}(X, Z, Y) + T(\hat{R}(X, W), Z, Y) - T(\hat{R}(Y, W), X, Z) \\
+ T(\hat{R}(Y, Z), X, W) - T(\hat{R}(X, Z), Y, W) + T(\hat{R}(Z, W), Y, X) \\
- T(\hat{R}(X, Y), Z, W).
\]

**Proof.** The proof is a direct consequence of Lemma 2.8. \( \square \)

**Proof of Theorem 2.7:** Applying the projection operator \( \mathcal{P} \) on the h-curvature tensor \( \hat{R} \) taking into account definition \( 2.2 \) and Lemma 2.8 we obtain

\[
(\mathcal{P} \cdot \hat{R})(X, Y, Z, W) = \hat{R}(Z, W, X, Y) - L^{-1}\{ \ell(W)\hat{R}(X, Y, Z, W) + \ell(Z)\hat{R}(X, Y, Z, W) \} \\
+ \ell(Y)\hat{R}(X, Y, Z, W) + \ell(X)\hat{R}(Y, X, Z, W) \\
+ L^{-2}\hat{R}(X, Y, Z, W)\ell(Y)\ell(W) + \hat{R}(X, Y, Z, W)\ell(Y)\ell(Z) \\
+ \hat{R}(Y, Z, X, W)\ell(Y)\ell(W) + \hat{R}(Y, Z, X, W)\ell(Y)\ell(Z) \quad (2.2)
\]

On the other hand, from Lemma 2.8 and Lemma 2.9 we obtain

\[
\hat{R}(\eta, Y, Z, W) = \hat{R}(Z, W, X, Y) - (D_{\beta\gamma}^\beta) \hat{R}(Z, W, Y) + T(H(Z), Z, Y) \\
- T(H(W), Z, Y) - T(H(Y), Z, W). \\
\hat{R}(\eta, Y, Z, W) = -\hat{R}(\eta, Y, Z, W) = g(H(Y), Z). \\
\hat{R}(X, \eta, Y) = -\hat{R}(X, \eta, Y).
\]

By using the above relations, Equation (2.2) becomes

\[
(\mathcal{P} \cdot \hat{R})(X, Y, Z, W) = \hat{R}(Z, W, X, Y) - L^{-1}\{ \ell(Z)\hat{R}(X, Y, Z, W) - \ell(W)\hat{R}(X, Y, Z, W) \} \\
- \hat{X} \hat{Y}\{ L^{-1}\ell(X)\{ \hat{R}(Z, W, Y) - T(H(Z), Z, Y) + T(H(W), Z, Y) \} \\
- T(H(Y), Z, W) - (D_{\beta\gamma}^\beta) \hat{R}(Z, W, Y) \} \\
- L^{-2}\{ \hat{R}(\eta, X, Z)\ell(Y)\ell(W) + \hat{R}(\eta, Y, W)\ell(X)\ell(Z) \} \}.
\]

Now, if \((M, L)\) is a Finsler manifold of \( H_p \)-sc \( k \), then Equation (2.1) is satisfied.

Conversely, suppose that \((M, L)\) is a Finsler manifold satisfying Equation (2.1). Then, by applying the projection operator \( \mathcal{P} \) on Equation (2.1) taking Remark 2.5 into account, one can deduce that \((M, L)\) is of \( H_p \)-sc \( k \). \( \square \)

In view of Theorem 2.7 and Remark 2.5 we have
Corollary 2.10. In a Finsler manifold of $H_p\text{-sc}$, we have

(a) $\hat{R}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) + \hat{R}(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}) = 2L^{-1} \ell(\mathbf{Y}) \mathfrak{L}_{\mathbf{X}} \{ (D^2_{\mathbf{Y}} \hat{P})(\mathbf{Z}, \mathbf{W}, \mathbf{Y}) + T(H(\mathbf{X}), \mathbf{Z}, \mathbf{W}) \}$
(b) $(\mathcal{P} \cdot \hat{R})(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) + (\mathcal{P} \cdot \hat{R})(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}) = 0$

Remark 2.11. In Definition 2.6 if we replace the h-curvature tensor $\hat{R}$ of Berwald connection by the h-curvature tensor $R$ of Cartan connection, then $(M, L)$ is called of parallel scalar curvature $\varepsilon$.

Let us define the following tensor:

$$Q(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) := g(\hat{P}(\mathbf{X}, \mathbf{W}), \hat{P}(\mathbf{Y}, \mathbf{Z})) - g(\hat{P}(\mathbf{X}, \mathbf{Z}), \hat{P}(\mathbf{Y}, \mathbf{W})). \quad (2.3)$$

Theorem 2.12. If $Q(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = \{ h(\mathbf{X}, \mathbf{Z})h(\mathbf{Y}, \mathbf{W}) - h(\mathbf{X}, \mathbf{W})h(\mathbf{Y}, \mathbf{Z}) \}$ holds, then the Finsler manifold of $H_p\text{-sc}$ is of perpendicular scalar curvature $(k - q)$.

Proof. By [20], we have

$$\hat{R}(\mathbf{X}, \mathbf{Y})\mathbf{Z} = R(\mathbf{X}, \mathbf{Y})\mathbf{Z} - T(\hat{R}(\mathbf{X}, \mathbf{Y}), \mathbf{Z}) - \mathfrak{L}_{\mathbf{X}} \mathfrak{L}_{\mathbf{Y}} \{ (\nabla_{\beta_X} \hat{P})(\mathbf{Y}, \mathbf{Z}) + \hat{P}(\mathbf{X}, \hat{P}(\mathbf{Y}, \mathbf{Z})) \}.$$

From which together with (2.3), taking into account the facts that $R(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = -\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}$ and $g((\nabla_{\beta_X} \hat{P})(\mathbf{Y}, \mathbf{Z}), \mathbf{W}) = g((\nabla_{\beta_X} \hat{P})(\mathbf{Y}, \mathbf{W}), \mathbf{Z})$, we obtain

$$R(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = \frac{1}{2} \hat{R}(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) - \hat{R}(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}) - Q(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}).$$

Applying the projection operator $\mathcal{P}$ on both sides of the above relation and using Remark 2.5 one can deduce

$$(\mathcal{P} \cdot R)(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = \frac{1}{2} (\mathcal{P} \cdot \hat{R})(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) - (\mathcal{P} \cdot \hat{R})(\mathbf{X}, \mathbf{Y}, \mathbf{W}, \mathbf{Z}) - Q(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}).$$

Now, if $(M, L)$ is a Finsler manifold of $H_p\text{-s.c.}$ $k$ and using the given assumption for $Q$, we get

$$(\mathcal{P} \cdot R)(\mathbf{X}, \mathbf{Y}, \mathbf{Z}, \mathbf{W}) = (k - q) \{ h(\mathbf{X}, \mathbf{Z})h(\mathbf{Y}, \mathbf{W}) - h(\mathbf{X}, \mathbf{W})h(\mathbf{Y}, \mathbf{Z}) \}.$$

This means that the Finsler manifold $(M, L)$ is of perpendicular scalar curvature $(k - q)$. \qed

3 Finsler spaces of scalar curvature and $H_p\text{-sc}$

Here, the necessary and sufficient condition under which a Finsler manifold of scalar curvature turns into a Finsler manifold of $H_p\text{-scalar curvature}$ is investigated. Moreover, certain conditions under which a Finsler manifolds of $H_p\text{-scalar curvature}$ and of scalar curvature reduces to a Finsler manifold of $H_p\text{-constant curvature}$ or to a Finsler manifold of vanishing $H_p\text{-scalar curvature}$ are obtained.

Theorem 3.1. Let $(M, L)$ be a Finsler manifold of scalar curvature $k$ of dimension $n \geq 4$. Then, $(M, L)$ is a Finsler manifold of $H_p\text{-scalar curvature}$ if, and only if,

$$B^k(\mathbf{X}, \mathbf{Y}) = \alpha h(\mathbf{X}, \mathbf{Y}),$$

where $\alpha = 3(\varepsilon - k)$. 

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Proof. Suppose that \((M, L)\) is a Finsler manifold of scalar curvature \(k\) of dimension \(n \geq 4\). Hence, by Theorem 2.3, the \(h\)-curvature tensor \(\hat{R}\) has the form
\[
\hat{R}(X, Y, Z, W) = A_{X,Y} \{ h(Y, W) \{ \ell(Z) [k\ell(X) + \frac{1}{3} C^k(X)] + \frac{1}{3} B^k(Z, X) \\
+ \frac{2}{3} \ell(X) C^k(Z) + kh(Z, X) \} + \frac{1}{3} \ell(X) C^k(Y) h(Z, W) \\
+ L^{-1} h(X, Z) \ell(W) [k\ell(Y) + \frac{1}{3} C^k(Y)] \}.
\]
Applying the projection operator \(P\) on both sides of the above equation, taking Remark 2.5 into account, we have
\[
(P \cdot \hat{R})(X, Y, Z, W) = A_{X,Y} \{ h(Y, W) \{ \frac{1}{3} B^k(Z, X) + kh(Z, X) \} \}. \quad (3.1)
\]
Now, if \((M, L)\) is a Finsler manifold of \(H_p\)-scalar curvature \(\varepsilon\), then by Definition 2.6
\[
(P \cdot \hat{R})(X, Y, Z, W) = \varepsilon A_{X,Y} \{ h(X, Z) h(Y, W) \}. \quad (3.2)
\]
Hence, from (3.1) and (3.2), we get
\[
\varepsilon h(Z, X) = \frac{1}{3} B^k(Z, X) + kh(Z, X)
\]
Consequently,
\[
B^k(X, Y) = 3(\varepsilon - k)h(X, Y) = \alpha h(X, Y). \quad (3.3)
\]
Conversely, suppose that the \(\pi\)-scalar form \(B^k\) is given in the form (3.3), then, using Equation (3.1), one can show that the \(h\)-curvature tensor \(\hat{R}\) has the form
\[
(P \cdot \hat{R})(X, Y, Z, W) = \varepsilon A_{X,Y} \{ h(X, Z) h(Y, W) \}.
\]
Therefore, from Definition 2.6 \((M, L)\) is a Finsler manifold of \(H_p\)-scalar curvature. \(\square\)

Proposition 3.2. A Finsler manifold of constant curvature \(k\) of dimension \(n \geq 4\) is a Finsler manifold of \(H_p\)-constant curvature \(\varepsilon = k\).

Proof. If \((M, L)\) is a Finsler manifold of constant curvature \(k\) of dimension \(n \geq 4\), then the \(\pi\)-forms \(C^k\) and \(B^k\) vanish. Hence, by Theorem 2.3, the \(h\)-curvature tensor \(\hat{R}\) has the form
\[
\hat{R}(X, Y, Z, W) = A_{X,Y} \{ kh(Y, W) \{ \ell(Z) \ell(X) + h(Z, X) \} + kL^{-1} h(X, Z) \ell(W) \ell(Y) \}.
\]
Applying the projection operator \(P\) on both sides of the above Equation, taking the fact that the angular metric tensor \(h\) is indicator and \(P \cdot \ell = 0\) (Remark 2.5) into account, we obtain
\[
(P \cdot \hat{R})(X, Y, Z, W) = k A_{X,Y} \{ h(X, Z) h(Y, W) \}.
\]
Therefore, \((M, L)\) is a Finsler manifold of \(H_p\)-constant curvature \(\varepsilon = k\). \(\square\)

Remark 3.3. The converse of the above proposition does not hold, in general.
Proposition 3.4. Let \((M, L)\) be a Finsler manifold of scalar curvature \(k\). If \((M, L)\) is of \(H_p\)-scaler curvature \(\varepsilon\), then we have

(a) \(3C^\varepsilon = 2C^k\),

(b) \(3B^\varepsilon = 2B^k\),

(c) \(3A^\varepsilon = 2A^k\),

where \(A^\varepsilon := L\mathcal{P} \cdot \hat{D}^0 B^\varepsilon\), \(A^k := L\mathcal{P} \cdot \hat{D}^0 B^k\). Moreover, the above assertions are equivalent.

Proof.

(a) Suppose that \((M, L)\) is a Finsler manifold of scalar curvature \(k\) and of \(H_p\)-scaler curvature \(\varepsilon\). Then, by Theorem 3.1, we have

\[ B^k(\overline{X}, \overline{Y}) = 3(\varepsilon - k)h(\overline{X}, \overline{Y}). \]

Taking the vertical covariant derivative on both sides of the above equation, we get

\[ (D^0 B^k)(\overline{Z}, \overline{X}, \overline{Y}) = 3(D^0 \varepsilon - D^0 k)(\overline{Z})h(\overline{X}, \overline{Y}) + 3(\varepsilon - k)(D^0 h)(\overline{Z}, \overline{X}, \overline{Y}). \]

Applying the projection operator \(\mathcal{P}\) on both sides of the above equation and multiplying the resulting by \(L\), we obtain

\[ L(\mathcal{P} \cdot D^0 B^k)(\overline{Z}, \overline{X}, \overline{Y}) = 3L(\mathcal{P} \cdot D^0 \varepsilon - \mathcal{P} \cdot D^0 k)(\overline{Z})(\mathcal{P} \cdot h)(\overline{X}, \overline{Y}) \]

\[ + 6L(\varepsilon - k)\{\mathcal{P} \cdot (T - L^{-1} h \otimes T)\}(\overline{Z}, \overline{X}, \overline{Y}). \]

In view of Remark 2.5, the above equation reduces to

\[ A^k(\overline{Z}, \overline{X}, \overline{Y}) = 3\{C^\varepsilon(\overline{Z}) - C^k(\overline{Z})\}h(\overline{X}, \overline{Y}) + 6L(\varepsilon - k)T(\overline{Z}, \overline{X}, \overline{Y}). \quad (3.4) \]

On the other hand, using Lemma 3.1 of [17], we have

\[ \mathfrak{A}_{\overline{X}, \overline{Y}} \{ A^k(\overline{X}, \overline{Y}, \overline{Z}) + C^k(\overline{X})h(\overline{Y}, \overline{Z}) \} = 0, \]

Now, from which together with (3.4), it follows that

\[ \mathfrak{A}_{\overline{X}, \overline{Y}} \{ \{3C^\varepsilon(\overline{X}) - 2C^k(\overline{X})\} \phi(\overline{Y}) \} = 0, \]

Taking the contracted trace with respect to \(\overline{Y}\), we obtain

\[ (n - 2)\{3C^\varepsilon(\overline{X}) - 2C^k(\overline{X})\} = 0. \]

Hence,

\[ 3C^\varepsilon = 2C^k \quad \text{(as } n \geq 4). \quad (3.5) \]

(b) Taking the vertical covariant derivative on both sides of (3.5), we get

\[ 3D^0 C^\varepsilon = 2D^0 C^k. \]
Applying the projection operator $\mathcal{P}$ on both sides of the above equation and then multiplying by $L$, we obtain

$$3B^\varepsilon = 2B^k.$$  

(c) The proof can be done in a similar manner as the proof of (b).

Now, we prove the equivalence.

(a) $\Rightarrow$ (b) $\Rightarrow$ (c): The proof is obvious.

(c) $\Rightarrow$ (a): Suppose that $3A^\varepsilon = 2A^k$ holds. Hence, using Lemma 3.1 of [17], we have

$$\frac{2}{3} \left\{ \{3C^\varepsilon(X) - 2C^k(X)\}h(Y, Z) \right\} = 0$$

From which, the result follows.

**Lemma 3.5.** [17] A Finsler manifold of scalar curvature $k$ is of constant curvature if, and only if, $\mathcal{P} \cdot F = 0$, where $F$ is the $\pi$-form defined by

$$F(X, Y) := \frac{1}{3} \left\{ B(X, Y) + 2C(X)\ell(Y) \right\}.$$  

**Theorem 3.6.** Let $(M, L)$ be a Finsler manifold of scalar curvature $k$. If $(M, L)$ is of $H_p$-scaler curvature $\varepsilon$, then, the following properties are equivalent:

(a) $\varepsilon = k$.

(b) $(M, L)$ is of constant curvature $k$.

(c) $(M, L)$ is of $H_p$-constant curvature $\varepsilon$.

(d) $\mathcal{P} \cdot F = 0$

**Proof.** Assume that $(M, L)$ is a Finsler manifold of scalar curvature $k$ and of $H_p$-scaler curvature $\varepsilon$. If $(M, L)$ is of constant curvature $k$, then, by Theorem 3.6, it follows that $\varepsilon = k$. Hence, $(M, L)$ is of $H_p$-constant curvature.

(b) $\iff$ (c): Firstly, suppose that $(M, L)$ is a Finsler manifold of scalar curvature $k$ and of $H_p$-scaler curvature $\varepsilon$. If $(M, L)$ is of constant curvature $k$, then, by Theorem 3.6, it follows that $\varepsilon = k$. Hence, $(M, L)$ is of $H_p$-constant curvature.

Conversely, If $(M, L)$ is a Finsler manifold of $H_p$-constant curvature $\varepsilon$. Then, $C^\varepsilon$ vanishes. Hence, by Proposition 3.4, $C^k$ vanishes. Consequently, by Theorem 2.4 $(M, L)$ is of constant curvature $k$.

(b) $\iff$ (d): This is a direct consequence of Lemma 3.5.  

Theorem 3.7. Let \((M,L)\) be a Finsler manifold of scalar curvature \(k\). Then, \((M,L)\) is of vanishing \(H_p\)-scalar curvature if, and only if, \(\mathcal{P} \cdot N = 0\), where \(N\) is the \(\pi\)-form defined by

\[
N(X, Y) := k \left( g(X, Y) + \ell(X)\ell(Y) \right) + \frac{1}{3} \left( B(X, Y) + 2\ell(X)C(Y) + 2C(X)\ell(Y) \right).
\]

Proof. If \((M,L)\) is a Finsler manifold of scalar curvature \(k\). Then, by Theorem 2.3, we have

\[
(\mathcal{P} \cdot \tilde{R})(X, Y, Z, W) = \mathcal{A}_{XY}[h(Y, W)\left\{ \frac{1}{3} B^k(Z, X) + kh(Z, X) \right\}]. \tag{3.7}
\]

On the other hand, by using the definition of \(N\), we obtain

\[
(\mathcal{P} \cdot N)(X, Y) = \frac{1}{3} B^k(Z, X) + kh(Z, X).
\]

From which together with (3.7), we get

\[
(\mathcal{P} \cdot \tilde{R})(X, Y, Z) = \mathcal{A}_{XY}\left\{ \phi(Y)(\mathcal{P} \cdot N)(Z, X) \right\}. \tag{3.8}
\]

Now, if \((M,L)\) is of vanishing \(H_p\)-scalar curvature, then, from (3.8), it follows that

\[
\mathcal{A}_{XY}\left\{ \phi(Y)(\mathcal{P} \cdot N)(Z, X) \right\}.
\]

Taking the contracted trace with respect to \(Y\), the above relation reduces to

\[
(n - 2)(\mathcal{P} \cdot N)(Z, X) = 0.
\]

Consequently, as \(n \geq 3\), \(\mathcal{P} \cdot N\) vanishes.

Conversely, if \((M,L)\) is a Finsler manifold of scalar curvature \(k\) such that \(\mathcal{P} \cdot N = 0\). Hence, from (3.8), \(\mathcal{P} \cdot \tilde{R}\) vanishes. Consequently, \((M,L)\) is of vanishing \(H_p\)-scalar curvature.

4 Examples

In this section, we give various examples. Although the definition of the \(H_p\)-scalar curvature manifold requires that \(n \geq 4\), we give special cases of Example 6 with lower dimensions. This only is to show that the converse of some implications are not, generally, true. For instance, in Example 6, when \(n = 2\) we show that the converse of Proposition 3.2 is not generally true. That is, not every Finsler manifold of \(H_p\)-constant curvature is of constant curvature. Also, in dimension 3, we show that not every Finsler manifold of \(H_p\)-scalar curvature is of scalar curvature.

Example 1. \([4]\) The family of Riemannian metrics

\[
L_\mu = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y \rangle^2)}}{1 + \mu|x|^2}, \quad x \in \mathbb{B}^n(r_\mu), \quad y \in T_x\mathbb{B}^n(r_\mu) \cong \mathbb{R}^n,
\]

where \(|.|\) and \(\langle ., . \rangle\) are the standard Euclidean norm and inner product in \(\mathbb{R}^n\), \(r_\mu = 1/\sqrt{-\mu}\) if \(\mu < 0\) and \(r_\mu = \infty\) if \(\mu \geq 0\). This class is of constant curvature \(\mu\). So it is also of
Hp-constant curvature $\mu$. As a verification, we have the following. The spray coefficients $G^i$ of $L_\mu$ are given by

$$G^i = -\frac{\mu(x, y)}{1 + \mu|x|^2} y^i.$$ 

From now on, we denote by $\partial_i$ and $\dot{\partial}_i$ the partial differentiation with respect to the coordinates $x^i$ and $y^i$ respectively and by $(x^i, y^i)$ the induced coordinates on the $TM$.

Firstly, let us compute the non linear connection $G^i_j$. Since, $G^i_j = \dot{\partial}_j G^i$, one has

$$G^i_j = -\mu \frac{y^i x^j \delta_{ij} + \langle x, y \rangle \delta^i_j}{1 + \mu|x|^2}.$$ 

The Berwald connection $G^i_{jk} = \dot{\partial}_k G^i_j$, are given by

$$G^i_{jk} = -\mu \frac{x^j \delta_{ik} \delta^j_k + x^i \delta_{lk} \delta^l_j}{1 + \mu|x|^2}.$$ 

To calculate the h-curvature, $\hat{R}^i_{jk}$, of Berwald connection, we have

$$\hat{R}^i_{jk} = -\mu \frac{(1 + \mu|x|^2)(\delta_{ik} \delta_{jh} + \delta_{jh} \delta_{ij}) - 2\mu(x^i \delta_{lj} \delta^j_k + x^j \delta_{lk} \delta^l_i) x^m \delta_{mb}}{(1 + \mu|x|^2)^2}.$$ 

Now, $\dot{\partial}_k G^i_j = 2 \delta_{jk} G^i_j + G^i_{mk} G^m_{kj}$, $\delta_{jk} = \partial_j - N^j_k \dot{\partial}_j$, we get

$$\dot{\partial}_k G^i_j = \mu(g_{jk} \delta^i_j - g_{ij} \delta^i_k).$$

Applying the projection operator on $\dot{\partial}_k G^i_j$, we get

$$\mathcal{P} \cdot \hat{R}^i_{jk} = \mu(h_{jk} h^b_a h^l_b h^k_c h^d_l) = \mu(h_{ac} h^d_c - g_{ab} h^d_c),$$

where $h_{ij}$ is the angular metric. This gives $\varepsilon = \mu$.

**Example 2.** [12] The Funk metric on the unit ball $\mathbb{B}^n$ in $\mathbb{R}^n$ is given by

$$L = \sqrt{|y|^2 - (|x|^2 |y|^2 - \langle x, y \rangle^2)} + \frac{\langle x, y \rangle}{1 - |x|^2}.$$ 

The Funk metric has constant curvature $k = -\frac{1}{4}$ and hence has Hp-constant curvature $-\frac{1}{4}$.

**Example 3.** [5] The class $L_\phi = \frac{\langle a, y \rangle}{(1 + \langle a, x \rangle)^2} \phi(z^1, z^2, ..., z^n)$, $z^i = \frac{(1 + \langle a, x \rangle) y^i - \langle a, y \rangle x^i}{\langle a, y \rangle}$, where $a = (a^1, a^2, ..., a^n)$ is a constant vector in $\mathbb{R}^n$ and $\phi$ is an arbitrary function in the sense that $L_\phi$ is a Finsler function. The spray coefficients are given by

$$G^i = -\frac{\langle a, y \rangle}{1 + \langle a, x \rangle} y^i.$$
The class $L_\phi$ is a class of Finsler metric of zero constant curvature and hence it is also of zero $H_p$-curvature. As in Example 1, straightforward calculations lead to the following.

$$G^i_j = -\frac{y^i a^\ell \delta_{\ell j} + \langle a, y \rangle \delta^i_j}{1 + \langle a, x \rangle}, \quad G^i_{jk} = -\mu \frac{a^\ell \delta_{\ell j} \delta^i_k + a^\ell \delta_{\ell k} \delta^i_j}{1 + \langle a, x \rangle},$$

$$\partial_h G^i_{jk} = \frac{(a^\ell \delta_{\ell j} \delta^i_k + a^\ell \delta_{\ell k} \delta^i_j) a^m \delta_{mh}}{(1 + \langle a, x \rangle)^2}, \quad \hat{R}^i_{hk} = 0.$$ Consequently, applying the projection operator on $\hat{R}^i_{hk}$ gives

$$\mathcal{P} \cdot \hat{R}^i_{hk} = 0,$$

which means that $\varepsilon = 0$.

As a special case of the above example, we choose $\phi$ as the following example.

**Example 4.** Let $M = \mathbb{R}^n$, and

$$L = \frac{\langle a, y \rangle}{(1 + \langle a, x \rangle)^2} \left( |z|^2 + e^{-|z|^2} \right).$$

$F$ is a Finsler metric of zero constant curvature and hence zero $H_p$-constant curvature.

**Example 5.** Let $M = \mathbb{B}^n(1/\sqrt{|a|})$.

$$L = \sqrt{(1 - |a|^2 |x|^4)} |y|^2 + \left( |x|^2 \langle a, y \rangle - 2 \langle a, x \rangle \langle x, y \rangle \right)^2 - (|x|^2 \langle a, y \rangle - 2 \langle a, x \rangle \langle x, y \rangle).$$

$L$ is a Finsler metric of scalar curvature $k = 3 \langle a, y \rangle_F + 3 \langle a, x \rangle^2 - 2|a|^2 |x|^2$.

Now to decide whether the space $(M, L)$ is of $H_p$-scalar curvature, we have to calculate the tensor $k_{ij} := F \dot{\partial}_j (F \dot{\partial}_i k)$.

$$\dot{\partial}_j k = 3 \frac{F a_i - \langle a, y \rangle \ell_i}{F^2}.$$ Hence,

$$k_{ij} = -3a_i \ell_j - 3 \frac{\langle a, y \rangle}{F^2} h_{ij}.$$ Applying the projection operator on $k_{ij}$, we have

$$B_{ij} = \mathcal{P} \cdot k_{ij} = k_{ab} h^a_i h^b_j = -3 \frac{\langle a, y \rangle}{F} h_{ij}.$$ Therefore, by using Theorem 3.1, the space $(M, L)$ is of $H_p$-scalar curvature. Here the $H_p$-scalar curvature $\varepsilon$ is given by

$$\varepsilon = \frac{2 \langle a, y \rangle}{F} + 3 \langle a, x \rangle^2 - 2|a|^2 |x|^2.$$
Example 6. Let $M = \mathbb{R}^n$ and $L = f(x^1)|y|$, where $f(x^1)$ is an arbitrary smooth positive function on $\mathbb{R}$ and $|y|$ is the standard norm on $\mathbb{R}^n$. The metric tensor is given by

$$g_{ij} = f(x^1)^2\delta_{ij}, \quad g^{ij} = f(x^1)^{-2}\delta^{ij}, \quad h_{ij} = f(x^1)^2(|y|\delta_{ij} - y_iy_j).$$

The Berwald connection is given by

$$G^h_{ij} = \phi(x^1)(\delta_{1i}\delta^h_j + \delta_{1j}\delta^h_i - \delta_{ij}\delta^h_1), \quad \phi(x^1) := \frac{f'(x^1)}{f(x^1)}, \quad f'(x^1) := \frac{df(x^1)}{dx^1}.$$

By differentiating $G^h_{ij}$ in the following sense

$$\partial_k G^h_{ij} = \phi'(x^1)(\delta_{1i}\delta^h_k + \delta_{1j}\delta^h_k - \delta_{ij}\delta^h_1) + \phi^2(\delta^h_k\delta_{ij} + \delta_{1i}\delta^h_k\delta^h_j + \delta_{ij}\delta^h_1\delta^h_k),$$

the h-curvature of Berwald connection is given by

$$\hat{\mathcal{R}}^h_{ijk} = \mathfrak{A}_{j,k}\{\phi'(x^1)(\delta_{1i}\delta^h_k + \delta_{1j}\delta^h_k - \delta_{ij}\delta^h_1) + \phi^2(\delta^h_k\delta_{ij} + \delta_{1i}\delta^h_k\delta^h_j + \delta_{ij}\delta^h_1\delta^h_k)\}.$$

In dimension 3, by using Maple program and NF-package [16], one can see that the space $(M, L)$ is not of scalar curvature. But it satisfies the condition of the $H^p$-scalar curvature. Namely,

$$\mathcal{P} \cdot \hat{\mathcal{R}}^h_{ijk} = -\frac{1}{L^2}(\phi'(y^2)^2 + (y^3)^2) + \phi(y^1)^2)(h^a_i h^d_j - h^a_d h^j_i).$$

Hence, $\varepsilon = -\frac{1}{L^2}(\phi'(y^2)^2 + (y^3)^2) + \phi(y^1)^2)$. In dimension 2, by using Maple program and NF-package, one can see that the space $(M, L)$ is of scalar curvature $k = \frac{f'^2}{f^2}$. But applying the projection operator on the h-curvature gives

$$\mathcal{P} \cdot \hat{\mathcal{R}}^h_{ijk} = 0.$$  

Hence, $\varepsilon = 0$. This case shows that the converse of Proposition 3.2 is not generally true.

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