A simple integrate-and-fire system and various super-stable periodic orbits

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Received April 10, 2017; Revised July 17, 2017; Published January 1, 2018

Abstract: This paper studies super-stable periodic orbits and related phenomena in a simple switched dynamical system. The system repeats integrate-and-fire behavior between a constant threshold and a periodic base signal. The base signal consists of two components: the fundamental and higher frequency triangular signals. First, we demonstrate “chaos + chaos = order” such that the system exhibits chaos if the base signal is either of the two components whereas the system exhibits a super-stable periodic orbit if the base signal consists of both components. This phenomenon is confirmed experimentally and is explained theoretically. Second, we extract two key parameters and show that the system can exhibit a variety of super-stable periodic orbits. Applying a mapping procedure, existence regions of basic super-stable periodic orbits are calculated precisely.

Key Words: switched dynamical systems, super-stable periodic orbits, chaos

1. Introduction

The bifurcating neuron (BN) is a simple switched dynamical system inspired by spiking neuron models [1–6]. Repeating integrate-and-fire behavior between a constant threshold and a periodic base signal, the BN can exhibit various chaotic/periodic phenomena. On the other hand, integrate-and-fire neuron models have been applied to many engineering systems including associative memories [2], image processors [7], and communication systems [4, 8]. In the spiking neuron models (incl. BNs), analysis of the chaotic/periodic phenomena is important not only as basic study of nonlinear dynamics but also for the engineering applications.

This paper studies dynamics of a class of BNs whose periodic base signal is given by addition of two triangular signals. The first signal is period $T$, the second signal is period $T/3$, and they are synchronized. In order to analyze the dynamics precisely, we derive one-dimensional return map (Rmap). The Rmap is piecewise linear and can be described exactly. For convenience of the analysis, we extract two key parameters from many original parameters. We then present two main results.

First, we demonstrate an interesting phenomenon as the following. If the base signal is either the first or the second signal only, the BN exhibits chaos [9]. If the base signal is given by addition of the first and the second signals, the BN exhibits a super-stable periodic orbit (SSPO) such that a set of initial states fall rapidly into the periodic orbit. Such a phenomenon is referred to as “chaos + chaos
order (CCO)”. Presenting a simple test circuit, the CCO is confirmed experimentally. Using the Rmap, the CCO is explained theoretically. Such phenomena have been observed in coupled BNs [5, 6]. However, the coupled BNs and their Rmap are complicated and analysis of the CCO is not sufficient.

Second, we analyze a variety of SSPOs in a space of the two key parameters. The analysis of the SSPOs is integrated into the analysis of the Rmap including four segments of slope zero. Using the Rmap, existence regions of basic SSPOs can be calculated precisely. The existence regions distribute in complicated shapes in the parameter space. Such complicated existence regions have not been presented in previous works [4–6]. Preliminary results along these lines can be found in our conference paper [10].

2. Circuit model and typical phenomena

Figure 1 shows a circuit model of the BN and Fig. 2 illustrates the circuit dynamics. Integrating a constant current $I > 0$, a capacitor voltage $v$ rises. When $v$ reaches a constant threshold $V_T$, the comparator outputs a spike. The spike closes a switch SW and $v$ is reset to the base signal $B(t)$ with period $T$. Repeating this integrate-and-fire behavior, the circuit outputs a spike-train $Y(t)$. For simplicity, we ignore inner resistors, i.e., $r_1 \to \infty$ and $r_2 \to 0$. In this case, the circuit dynamics is described by

$$\begin{cases}
  C \frac{dv}{dt} = I, & Y(t) = -E \text{ for } v(t) < V_T \\
  v(t+) = B(t+), & Y(t+) = E \text{ if } v(t) = V_T \\
  B(t+T) = B(t), & B(t) < V_T
\end{cases}$$

(1)

where $\pm E$ denotes output voltages of the comparator. The base signal $B(t)$ consists of three components: the first component $E_1(t)$ with period $T$, the second component $E_3(t)$ with period $T/3$, and the dc component $E_0 \geq 0$. The first and second components are triangular signals and are assumed to be synchronized.

Fig. 1. A circuit model.

Fig. 2. Integrate-and-fire dynamics. Dimensionless variables $x$, $\tau$, $y$, and $b(\tau)$ are proportional to $v$, $t$, $Y$, and $B(t)$, respectively. The threshold $v = V_T$ is normalized as $x = 1$. 

12
\[ E_1(t) = \begin{cases} -A_1 t & \text{for } -D_1 \leq t < D_1 \\ B_1(t - T/2) & \text{for } D_1 \leq t < T - D_1 \end{cases} \]

\[ D_1 \equiv \frac{B_1 T}{2(A_1 + B_1)}, \quad E_1(t + T) = E_1(t) \]  

\[ E_3(t) = \begin{cases} -A_3 t & \text{for } -D_3 \leq t < D_3 \\ B_3(t - T/6) & \text{for } D_3 \leq t < T/3 - D_3 \end{cases} \]

\[ D_3 \equiv \frac{B_3 T}{6(A_3 + B_3)}, \quad E_3(t + T/3) = E_3(t) \]  

where \( A_1 > 0, B_1 > 0, A_3 > 0, \) and \( B_3 > 0. \) In this paper, we consider three cases:

Case 1 : \( B(t) = E_1(t), \)  
Case 2 : \( B(t) = E_3(t) - E_0, \)  
Case 3 : \( B(t) = E_1(t) + E_3(t) - E_0 \)  

The base signal in Case 3 is given by addition of base signals in Cases 1 and 2. We have performed laboratory experiments and Fig. 3 shows typical phenomena. In the experiments, the current source with its inner resistor \( r_1 \) is implemented by an equivalent circuit of a voltage source \( E \) with its inner resistor \( r_1. \) The parameter \( I \) in Eq. (1) is approximated by \( I \approx \frac{E}{(r_1 - r_1 e^{-1})}. \) In the figure, we can see that the BN exhibits chaotic waveforms in Cases 1 and 2; and that the BN exhibits periodic waveform in Case 3. That is, adding two base signals each of which causes chaotic waveforms, the BN exhibits periodic waveforms. We refer to such phenomena as “chaos + chaos = order” (CCO, [10]).
In the following sections, we consider the CCO and various periodic phenomena\(^1\).

### 3. Dimensionless equations and return map

In order to analyze the dynamics of BN, we derive a dimensionless equation and return map (Rmap). First, we introduce the following dimensionless variables and parameters.

\[
\tau = \frac{t}{T}, \quad x = \frac{v}{V_T}, \quad (\dot{x} \equiv \frac{dx}{dt}) \quad y = \frac{Y + E}{2E}, \quad s = \frac{IT}{CV_T},
\]

\[
b_0 = \frac{E_0}{V_T}, \quad a_1 = \frac{A_1 T}{V_T}, \quad b_1 = \frac{B_1 T}{V_T}, \quad a_3 = \frac{A_3 T}{V_T}, \quad b_3 = \frac{B_3 T}{V_T}.
\]

Using these, Eqs. (1) to (3) are transformed into the following.

\[
\begin{align*}
\dot{x} &= s, \\
y &= 0 \quad \text{for } x < 1 \\
x(\tau_) &= b(\tau_+), \quad y(\tau_+) &= 1 \quad \text{if } x(\tau) = 1 \quad b(\tau + 1) = b(\tau), \quad b(\tau) < 1
\end{align*}
\]

where \(b(\tau) \equiv B(T \tau)/V_T\).

\[
e_1(\tau) = \begin{cases} -a_1 \tau & \text{for } -d_1 \leq \tau < d_1 \\ b_1 (\tau - 1/2) & \text{for } d_1 \leq \tau < 1 - d_1 \end{cases} \quad d_1 = \frac{b_1}{2(a_1 + b_1)}, \quad e_1(\tau + 1) = e_1(\tau)
\]

\[
e_3(\tau) = \begin{cases} -a_3 \tau & \text{for } -d_3 \leq \tau < d_3 \\ b_3 (\tau - 1/6) & \text{for } d_3 \leq \tau < 1/3 - d_3 \end{cases} \quad d_3 = \frac{b_3}{6(a_3 + b_3)}, \quad e_3(\tau + 1/3) = e_3(\tau)
\]

Case 1: \(b(\tau) = e_1(\tau), \quad \text{Case 2: } b(\tau) = e_3(\tau) - b_0, \quad \text{Case 3: } b(\tau) = e_1(\tau) + e_3(\tau) - b_0
\]

This dimensionless equation is piecewise linear and solutions can be calculated exactly. Figure 4 shows waveforms of CCO corresponding to the laboratory measurements in Fig. 3.

Here, we note that Eqs. (5) to (7) are characterized by 6 parameters \(a_1 > 0, b_1 > 0, a_3 > 0, b_3 > 0, b_0 > 0, \text{ and } s > 0\). Since it is extremely hard to consider effects of all the 6 parameters, we reduce the number of parameters. That is, we focus on the case

\[
a_3 \equiv a, \quad b_3 = a, \quad a_1 = a - 1, \quad b_1 = a + 1, \quad s = 1
\]

In this case, Eqs. (5) to (7) depend on the two key parameters

\[
a > 1, \quad b_0 \geq 0 \quad (d_1 = \frac{a+1}{4a}, \quad d_3 = \frac{1}{12})
\]

In order to analyze the dynamics of BN, we derive the spike-position map and Rmap. Let \(\tau_n\) denote the \(n\)-th spike position. Since \(\tau_{n+1}\) is determined by \(\tau_n\), we can define the spike-position map \(F\) from positive reals to itself.

\[
\tau_{n+1} = \tau_n + 1 - b(\tau_n) \equiv F(\tau_n), \quad F(\tau + 1) = F(\tau)
\]

Let \(\theta_n = \tau_n \mod 1\) denote the \(n\)-th spike phase. We can define the Rmap \(f\) of spike-phases from unit interval \([0, 1)\) to itself.

\[
\theta_{n+1} = f(\theta_n) \equiv F(\theta_n) \mod 1, \quad \theta_n \in [0, 1) \equiv I
\]

Figures 5(a) and (b) illustrate the spike-position map and Rmap, respectively. Hereafter, for convenience, we analyze the dynamics based on the Rmap. Since the BN is piecewise linear, the Rmap can be calculated exactly. That is, substituting Eqs. (6) and (7) into Eq. (10), we obtain the Rmaps in the three cases. In Case 1 \(b(\tau) = e_1(\tau))\), the Rmap is described by

\[
f(\theta_n) = F_1(\theta_n) \mod 1, \quad \text{where } F_1 \text{ is the spike-position map given by}
\]

\[
F_1(\tau_n) = \begin{cases} a\tau_n + 1 & \text{for } \tau_n \in [0, d_1) \\ -a(\tau_n - 1/2) + 3/2 & \text{for } \tau_n \in [d_1, 1 - d_1) \\ a(\tau_n - 1) + 2 & \text{for } \tau_n \in [1 - d_1, 1] \end{cases}
\]

In Case 2 \(b(\tau) = e_3(\tau) - b_0\), the Rmap is described by

\(^1\)The system can exhibit such phenomena for higher frequency signal of \(E_3(t)\) (e.g., period \(T/7\)), however, the analysis becomes extremely hard. For simplicity, this paper considers the case \(E_3(t)\) with period \(T/3\).
Fig. 4. Chaos + chaos = order in Eq. (5). \( a_1 = 1.2, b_1 = 3.2, a_3 = 2.2, b_3 = 2.2, (a = 2.2), b_0 = 0.51, s = 1. \) (a) Chaos in Case 1. (b) Chaos in Case 2. (c) Periodic waveform in Case 3.
Fig. 5. (a) Spike-position map, (b) Return map of spike-phases.

Fig. 6. Rmaps for chaos + chaos = order \((a = 2.2, b_0 = 0.51)\). (a) Chaos in Case 1. (b) Chaos in Case 2. (c) SSPO in Case 3.
Fig. 7. Rmaps in Case 3 for $a = 2.25$. (a) $b_0 = 0.75$. SSPO with period 1. (b) $b_0 = 0.5$. SSPO with period 2. (c) $b_0 = 0.25$. SSPO with period 3. (d) $b_0 = 0$. SSPO with period 4.

In Case 3 ($b(\tau) = e_1(\tau) + e_3(\tau) - b_0$), the Rmap is described by

$$f(\theta_n) = F_2(\theta_n) \mod 1,$$ where $F_2$ is the spike-position map given by

$$F_2(\tau_n) = \begin{cases} 
(1 + a)\tau_n + 1 + b_0 & \text{for } \tau_n \in [0, \frac{1}{12}) \\
(1 - a)(\tau_n - \frac{7}{12}) + \frac{14}{12} + b_0 & \text{for } \tau_n \in \left(\frac{1}{12}, \frac{5}{12}\right) \\
(1 + a)(\tau_n - \frac{1}{12}) + \frac{16}{12} + b_0 & \text{for } \tau_n \in \left[\frac{5}{12}, \frac{9}{12}\right) \\
(1 - a)(\tau_n - \frac{9}{12}) + \frac{18}{12} + b_0 & \text{for } \tau_n \in \left[\frac{5}{12}, \frac{9}{12}\right) \\
(1 + a)(\tau_n - \frac{6}{12}) + \frac{20}{12} + b_0 & \text{for } \tau_n \in \left[\frac{7}{12}, \frac{9}{12}\right) \\
(1 - a)(\tau_n - \frac{10}{12}) + \frac{22}{12} + b_0 & \text{for } \tau_n \in \left[\frac{7}{12}, \frac{9}{12}\right) \\
(1 + a)(\tau_n - 1) + 2 + b_0 & \text{for } \tau_n \in \left[\frac{11}{12}, 1\right)
\end{cases}$$

(13)
Fig. 8. Rmaps in Case 3. (a) $a = 2.75$, $b_0 = 0.25$. Coexisting SSPO with period 4 and SSPO with period 1. (b) $a = 2.25$, $b_0 = 0.25$. Coexisting SSPO with period 3 and SSPO with period 1. (c) $a = 2.5$, $b_0 = 0.45$. Coexisting SSPO with period 4 and SSPO with period 1. (d) $a = 2.35$, $b_0 = 0.84$. Coexisting two SSPOs with period 2 and SSPO with period 1.

$f(\theta_n) = F_3(\theta_n) \mod 1$, where $F_3$ is the spike-position map given by

$$F_3(\tau_n) =
\begin{cases}
2a\tau_n + 1 + b_0 & \text{for } \tau_n \in [0, \frac{1}{12}] \\
\frac{5}{12}a + 1 + b_0 & \text{for } \tau_n \in \left[\frac{1}{12}, \frac{3}{12}\right] \equiv S_1 \\
2a\tau_n - \frac{9}{12}a + 1 + b_0 & \text{for } \tau_n \in \left[\frac{3}{12}, d_1\right] \\
\frac{7}{12}a + \frac{3}{2} + b_0 & \text{for } \tau_n \in \left[d_1, \frac{5}{12}\right] \equiv S_2 \\
-2a\tau_n + a + \frac{3}{2} + b_0 & \text{for } \tau_n \in \left[\frac{5}{12}, \frac{7}{12}\right] \\
-\frac{7}{12}a + \frac{3}{2} + b_0 & \text{for } \tau_n \in \left[\frac{7}{12}, 1 - d_1\right] \equiv S_3 \\
2a\tau_n - \frac{29}{12}a + 2 + b_0 & \text{for } \tau_n \in \left[1 - d_1, \frac{9}{12}\right] \\
-\frac{5}{12}a + 2 + b_0 & \text{for } \tau_n \in \left[\frac{9}{12}, \frac{11}{12}\right] \equiv S_4 \\
2a\tau_n - 2a + 2 + b_0 & \text{for } \tau_n \in \left[\frac{11}{12}, 1\right]
\end{cases}
$$

where $d_1 < \frac{5}{12}$ is assumed. Figure 6 shows Rmaps for chaos + chaos = order.
4. Super-stable periodic orbits

We analyze the CCO and various super-stable periodic orbits (SSPOs). First, we introduce several concepts. Let $Df_1(\theta)$, $Df_2(\theta)$, and $Df_3(\theta)$ denote slope of Rmaps $f_1$, $f_2$, and $f_3$, respectively. If an Rmap $f$ is expanding, $|Df| > 1$, then the Rmap exhibits chaos [5].

A point $p \in I$ is said to be a periodic point with period $k$ if an orbit started from $p$ returns to itself after $k$ iterations: $f^k(p) = p$ and $f^l(p) \neq p$ for $0 < l < k$ where $f^k$ is the $k$-fold composition of $f$. A sequence of periodic points $\{f(p), \cdots, f^k(p)\}$ is said to be a periodic orbit. A periodic point $p$ is said to be super-stable if $Df(p) = 0$. A sequence of super-stable periodic points is said to be a super-stable periodic orbit (SSPO)\(^2\).

Referring to Eqs. (12) and (13), the slope $Df_1$ is $a$ or $-a$ in Case 1 and the slope $Df_2$ is $1 + a$ or $1 - a$ in Case 2. Hence, if $2 < a < \infty$ then the Rmap is expanding and exhibits chaos [5] in Cases 1

\(^2\)A set of initial states fall rapidly into the SSPO. Examples of SSPOs in simplified models of switching power converters can be found in [11].
Fig. 10. Existence region of SSPO with period 2. (a) Red region: SSPO through $S_1$, (b) Blue region: SSPO through $S_2$, (c) Pink region: SSPO through $S_3$, (d) Orange region: SSPO through $S_4$.

and 2. Referring to Eq. (12), in Case 3, the slope is given by

$$
Df_3(\theta) = \begin{cases} 
2a & \text{for } \theta \in [0, \frac{1}{12}] \text{ or } \theta \in [\frac{3}{12}, d_1] \text{ or } \theta \in [1 - d_1, \frac{9}{12}] \text{ or } \theta \in [1, 1 - d_1, 1] \\
-2a & \text{for } \theta \in [\frac{5}{12}, \frac{7}{12}] \\
0 & \text{for } \theta \in [\frac{1}{12}, \frac{3}{12}] \equiv S_1 \text{ or } \theta \in [d_1, \frac{5}{12}] \equiv S_2 \text{ or } \theta \in [\frac{7}{12}, 1 - d_1] \equiv S_3 \text{ or } \theta \in [\frac{9}{12}, \frac{11}{12}] \equiv S_4
\end{cases}
$$

(Note that $|Df_3| = 0$ on four intervals $S_1$ to $S_4$ and $|Df_3| = 2a$ on the other parts. If $2 < a < \infty$ then the Rmap exhibits a SSPO that passes through either (or all) of the four intervals $S_1$ to $S_4$. If the Rmap does not exhibit a SSPO then the Rmap must exhibit chaotic attractor that includes neither of $S_1$ to $S_4$. However, it contradicts a result in [12]: a chaotic attractor must include at least one discontinuity point. In our Rmap, the discontinuity points are end points of the four intervals $S_1$ to $S_4$. That is, if $2 < a < \infty$ then the Rmap does not exhibit chaos but SSPOs in Case 3. Since the Rmap exhibits chaos in Cases 1 and 2, $2 < a < \infty$ guarantees the CCO.)
Here, we consider SSPOs in Case 3 for two key parameters $a$ and $b_0$. As $a$ and $b_0$ vary, the Rmap can exhibit a variety of SSPOs. Figure 7 shows several SSPOs as $b_0$ varies. Some of SSPOs can co-exist and Rmap exhibits either SSPO depending on initial condition. Figure 8 shows several examples of co-existing SSPOs. Since all the SSPOs must pass through either of the four intervals $S_1$ to $S_4$ on which the slope is 0, the four initial points $f(S_1)$ to $f(S_4)$ are sufficient to consider the SSPOs. That is, for some values of parameters $a$ and $b_0$, the Rmap exhibits SSPO with period $k$ if $f^k(\theta_0) = \theta_0$ for $\theta_0 \in \{f(S_1), f(S_2), f(S_3), f(S_4)\}$. That is, we can calculate existence regions of SSPOs precisely as the following.

Step 1: Give a value of key parameters $(a, b_0)$.
Step 2: Set an initial point on $S_i$, $i = 1 \sim 4$.
Step 3: If an orbit returns to $S_i$ after $k$ iterations, then the Rmap exhibits a SSPO with period $k$.

Figures 9 to 12 show existence regions of SSPOs with period 1 to 4. In the figures, four kinds of SSPOs passing through $S_1$ to $S_4$ are classified by four colors: red, blue, pink, and orange, respectively. As period increases, shape of existence regions becomes complex. Even in the case of period 4, the
Fig. 12. Existence region of SSPO with period 4. (a) Red region: SSPO through $S_1$, (b) Blue region: SSPO through $S_2$, (c) Pink region: SSPO through $S_3$, (d) Orange region: SSPO through $S_4$.

shape is extremely complicated as shown in Fig. 12.

5. Conclusions
Basic dynamics of a class of BNs has been considered and two main results are presented in this paper. First, presenting a simple test circuit, the CCO is confirmed experimentally. Using the Rmap, the generation of CCO is explained theoretically. Second, a variety of SSPOs are demonstrated from the BNs in case 3. Extracting two key parameters and calculating slopes of Rmap, existence regions of SSPOs with period 1 to 4 are calculated precisely. The existence regions are complicated shape in the parameter space.

Future problems include analysis of bifurcation phenomena for the SSPOs and engineering applications of the SSPOs such as encoders in communication systems [8]. It should be noted that SSPOs with longer period are hard to be observed in actual circuits because zero-slope ($Df_3 = 0$) is impossible in a noisy hardware. Careful laboratory experiments are required.
Acknowledgments
The authors wish to thank Mr. Yusaku Yanase for his advises in the laboratory experiments. This work is supported by JSPS KAKENHI 15K00350.

References

[1] R. Perez and L. Glass, “Bistability, period doubling bifurcations and chaos in a periodically forced oscillator,” Phys. Lett., vol. 90A, no. 9, pp. 441–443, 1982.
[2] G. Lee and N.H.G., Farhat, “The bifurcating neuron network 1,” Neural networks, vol. 14, pp. 115–131, 2001.
[3] E.D.M. Hernandez, G. Lee, and N.H. Farhat, “Analog realization of arbitrary one-dimensional maps,” IEEE Trans. Circuits Syst. I, vol. 50, no. 12, pp. 1538–1547, 2003.
[4] H. Torikai, T. Saito, and W. Schwarz, “Synchronization via multiplex pulse-train,” IEEE Trans. Circuits Syst. I, vol. 46, no. 9, pp. 1072–1085, 1999.
[5] Y. Kon’no, T. Saito, and H. Torikai, “Rich dynamics of pulse-coupled spiking neurons with a triangular base signal,” Neural Networks, vol. 18, pp. 523–531, 2005.
[6] S. Kirikawa and T. Saito, “Filter-induced bifurcation of simple spike-train dynamics,” IEICE Trans. Fundamentals, vol. E97-A, no. 7, pp. 1508–1515, 2014.
[7] S.R. Campbell, D. Wang, and C. Jayaprakash, “Synchrony and desynchrony in integrate-and-fire oscillators,” Neural Comput., vol. 11, pp. 1595–1619, 1999.
[8] N.F. Rulkov, M.M. Sushchik, L.S. Tsimring, and A.R. Volkovskii, “Digital communication using chaotic-pulse-position modulation,” IEEE Trans. Circuits Syst. I, vol. 48, no. 12, pp. 1436–1444, 2001.
[9] E. Ott, Chaos in dynamical systems, Cambridge, 1993.
[10] R. Takahashi and T. Saito, “A variety of super-stable periodic orbits in a simple dynamical system with integrate-and-fire switching,” Proc. NOLTA, pp. 226–229, 2016.
[11] T. Saito, T. Kabe, Y. Ishikawa, Y. Matsuoka, and H. Torikai, “Piecewise constant switched dynamical systems in power electronics,” Int’l J. of Bifurcation and Chaos, vol. 17, no. 10, pp. 3373–3386, 2007.
[12] T.Y. Li and J.A. Yorke, “Ergodic transformation from an interval into itself,” Trans. Amer. Math. Soc., vol. 235, pp. 183–192, 1978.