A route to quasi-perfect invisibility cylindrical cloaks without extreme values in the parameters

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The method of coordinate transformation offers a way to realize perfect cloaks, but provides less ability to characterize the performance of a multilayered cloak in practice. Here, we propose an analytical model to predict the performance of a multilayered cylindrical cloak, based on which, the cloak in practice can be optimized to diminish the intrinsic scatterings caused by discretization and simplification. Extremely low scattering or “quasi-perfect invisibility” can be achieved with only a few layers of anisotropic metamaterials without following the transformation method. Meanwhile, the permittivity and permeability parameters of the layers are relatively small, which is a remarkable advantage of our approach.

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Various efforts have been made on the realization of invisibility [1, 2, 3, 4, 5]. Pendry et al. theoretically proposed the perfect invisibility cloak for electromagnetic waves [1], utilizing anisotropic and inhomogeneous media to mimic the space squeezing. Later, the effectiveness of the transformation based cloak was demonstrated by ray tracing [6], full wave finite element simulations [7, 8], and analytical scattering models [9, 10, 11, 12], as well. In practice, the difficulty in construction a perfect invisibility cylindrical cloak is the requirement of continuous inhomogeneity and high anisotropy with extreme values in the parameters. Simplified parameters based on the coordinate transformation were then utilized to facilitate the physical realization [7, 13, 14], in expense of the aroused inherent scatterings [15]. Constraints on the bandwidth were studied as well [16]. The first sample of cylindrical cloak has been created using multilayered metamaterials [13]. Bi-layered isotropic media was also proposed for achieving the effective anisotropy [17], but a lot of thin layers are needed which increases the construction complexity. Moreover, the transformation method provide less ability to predict the performance of a practical construction composed of discontinuous layers of homogeneous anisotropic metamaterials. Therefore, it is very necessary to investigate a better way to design a practical cloak with

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In this paper, in order to get the exact behavior of a multilayered cloak, the analytical model of a cylindrical cloak created with multilayered anisotropic materials is established based on the full wave scattering theory\cite{18, 19}. Our results show that, by using only a few layers of anisotropic materials, a “quasi-perfect invisibility” multilayered cloak with near zero scattering can still be achieved without following the design method of coordinate transformation, and the parameters obtained are relatively small and possible to be realized by metamaterials. The impedances between the adjoined layers do not really match each other but can produce zero reflection, which can be treated as the counterpart in cylindrical geometry of the reflectionless one-dimensional multilayered slab. All of these results provide a second and better way of designing a multilayered cloak.

We use cylindrical cloak as an example. Without losing the generality, the case a TE-polarized plane wave with unit magnitude normally incident onto an \( M \)-layer cylindrical cloak (from region 1 to \( M \)) is considered, as shown in Fig. 1. The TM case can be analyzed similarly. The radiuses of the boundaries of the cloak are denoted by \( R_m \) \((m = 0, 1, \cdots, M)\). The relative constitutive parameters in region \( m \) are assumed to be constants denoted by \( \mu_{\rho m} \), \( \mu_{\phi m} \) and \( \varepsilon_{zm} \) while the region \( m = 0 \) is assumed to be free space and the core region \((\rho < R_M)\) is assumed to be PEC. The electric fields \( E_{zm} \) in the region \( m \) satisfy the following equation,
By applying the method of separation of variables, the general expression for the electric fields in region $m$ can be expressed as

$$E_{zm} = \sum_{n=-\infty}^{\infty} a_{mn} \left( J_{\nu_{mn}}(k_m \rho) + \tilde{r}_{m(m+1)n} H_{\nu_{mn}}(k_m \rho) \right) \exp(i n \phi),$$

where $\nu_{mn} = n\sqrt{\mu_{\phi m}/\mu_{pm}}$ and the wave number in region $m$ is $k_m = k_0 \sqrt{\varepsilon_m \mu_{\phi m}}$. Different from the isotropic layered case, here, $\nu_{mn}$ is a fraction. $J_{\nu_{mn}}$, $H_{\nu_{mn}}$ represent the $\nu_{mn}$th order Bessel functions of the first kind and the $\nu_{mn}$th order Hankel functions of the first kind, respectively. $a_{mn}$ is the unknown coefficients and $\tilde{r}_{m(m+1)n}$ is the scattering coefficient on the boundary $R_m$. When a standing wave incident from region $m$ onto the boundary $R_m$, the direct reflection coefficient, which represents the ratio between the directly reflected wave and the incident wave, is $r_{m(m+1)n} = (j'j_1 - \eta_m/\eta_{m+1}j'_1)/(-h'j_1 + \eta_m/\eta_{m+1}h'_1)$, and the direct transmission coefficient, which represents the ratio between the directly transmitted wave in region $m+1$ and the incident wave in region $m$, is $t_{m(m+1)n} = -2i/(\pi k_m R_m)/(-h'j_1 + \eta_m/\eta_{m+1}h'_1)$. Similarly, for an outgoing wave the direct reflection and transmission coefficients on $R_m$ are $r_{(m+1)mn} = (h'j - \eta_{m+1}/\eta_m j_1 h')/(-j'1 + \eta_{m+1}/\eta_m j_1 h')$ and $t_{(m+1)mn} = 2i/(\pi k_{m+1} R_m)/(-j'1 + \eta_{m+1}/\eta_m j_1 h')$. Here $j = J_{\nu_{mn}}(k_m R_m)$, $j' = J'_{\nu_{mn}}(k_m R_m)$, $j_1 = J_{\nu_{m+1}n}(k_{m+1} R_m)$, $j'_1 = J'_{\nu_{m+1}n}(k_{m+1} R_m)$, $h = H_{\nu_{mn}}(k_m R_m)$, $h' = H'_{\nu_{mn}}(k_m R_m)$, $h_1 = H_{\nu_{m+1}n}(k_{m+1} R_m)$, $h'_1 = H'_{\nu_{m+1}n}(k_{m+1} R_m)$, and $\eta_m = \sqrt{\mu_{\phi m}/\varepsilon_m}$. Therefore, the scattering coefficient in layer $m$ ($m=0,1,\ldots,M-1$) can be written as

$$\tilde{r}_{m(m+1)n} = r_{m(m+1)n} + \tilde{t}_{(m+1)mn},$$

where $\tilde{t}_{(m+1)mn} = t_{m(m+1)n} t_{(m+1)mn} r_{(m+1)(m+2)n}/(1 - r_{m(m+1)n} r_{(m+1)(m+2)n})$ represents the wave coming out from $R_m$ due to the multiple reflections and transmissions on the boundaries inside $R_m$. At $R_M$, $\tilde{r}_{M(M+1)n} = -J_{\nu_Mn}(k_M R_M)/H_{\nu_Mn}(k_M R_M)$, therefore all the $\tilde{r}_{m(m+1)n}$ can be derived using Eq. 3 and $a_{mn}$ can also be derived by matching the boundary conditions. The coefficients of the scattering fields in region 0 are $b_{0n} = a_{0n} \tilde{r}_{01n}$. Based on the cylindrical scattering model, the far-field total scattering efficiency or the scattering cross section normalized by the geometrical cross section for the multilayered cylindrical cloak is obtained as

$$Q_{sca} = 2/(\pi R_M) \sum_{n=-\infty}^{\infty} |b_{0n}|^2.$$
FIG. 2: Comparison of the differential scattering efficiencies when a TE-polarized plane wave normally incident on to the PEC cylinder without cloak (dash dotted line) and with the cloak proposed in Ref. [13] (dotted line).

In practice, the ideal parameters obtained from the transformation method need to be discretized for realization, which will destroy the perfect invisibility of the cloak. Using the proposed method, such effects of discretization and simplification of the transformation based (TB) cloak can be quantitatively analyzed. For example, Ref. [13] proposed a 10-layer simplified cloak with $\mu_\phi = 1$ for the experiment, in which the copper cylinder core with radius $0.709\lambda$ is coated by a multilayered cloak with inner radius $0.768\lambda$ and outer radius $1.670\lambda$. Utilizing the parameters including the losses in the metamaterials in Ref. [13], and assuming the core to be PEC for convenience, for the normal incidence of a TE-polarized plane wave, the forward and backward scatterings have been reduced by about 4.8dB and 4.1dB, respectively, comparing with the bared PEC core, as shown by the dotted line in Fig. 2. The near field distributions can also be calculated using our method. Fig. 3 shows the electric field distributions of the above case. Our analytical model shows some qualitative agreement with the experimental field distributions shown in [13], where both reduced forward and backward scatterings can be observed.

A more interesting thing is that by using the genetic algorithms, a widely used optimization method in engineering and science that enable the individuals of an optimization problem evolves to better solutions [20], our proposed method can realize a general cloak without following the ideal transformation parameters, but still have a quasi perfect performance. Meanwhile, the extreme values exist in the conventional transformation cloak can be avoided. When applying the genetic algorithms, the thickness of each layer of the cloak is fixed and the chromosome is a string of 0s and 1s representing a set of constitutive parameters of each layer, $\{\varepsilon_1, \mu_{\rho 1}, \mu_{\phi 1}, \ldots, \varepsilon_M, \mu_{\rho M}, \mu_{\phi M}\}$. The fitness of an individual is chosen to be $1/Q_{sca}$, where $Q_{sca}$ is the total scattering of this individual shown by Eq. (4). Making
use of the roulette wheel selection, in which individuals with larger fitness have larger chance of going forward to
the next generation, setting the single point crossover probability and the mutation probability to be 0.6 and 0.05,
respectively, and ensuring the fittest individual to propagate to the next generation, evolution is carried out and
optimization is obtained finally. In order to compare the performance of the cloaks created with the TB parameters
and the optimized parameters, the values of the permittivity and permeability in the optimization are confined to be
positive values no larger than the maximums of the TB parameters.

Since in fabrications, it is much easier to create the cloak with fewer layers of materials. We consider the
4-layer cloak as an example. The dimensions of the cloak are \( R_0 = \lambda \) and \( R_4 = \lambda/2 \) with a thickness of \( \lambda/8 \) for each
layer. When a TE-polarized plane wave normally incident onto a PEC cylinder with radius \( \lambda/2 \) the far-field total
scattering efficiency \( Q_{sca} \) is found to be 2.31. When the PEC cylinder is coated by the 4-layer cloak with the TB full
parameters, \( Q_{sca} \) is reduced to be 0.18. While using our proposed method, we design a 4-layer optimized cloak with
the parameters shown in Fig. 4 (a), (b) and (c), the total scattering efficiency \( Q_{sca} \) of this cloak is greatly minimized
to be 0.0025.

Fig. 5 shows the calculated electric field distribution when a TE polarized plane wave normally incident from left
to right onto the optimized cloak. It is shown that in the near region of the cloak, the electric fields stay almost
unperturbed, thus a so called “quasi-perfect” cylindrical multilayered cloak is obtained. In order to see how much
improvement has been achieved through the optimization, the differential scattering efficiency as a function of the

FIG. 3: Electric field distributions for a plane wave incident from left to right onto the cloak proposed in Ref. [13].
scattering angles is calculated as shown in Fig. 6. We see the scatterings are minimized by about 10dB when the TB full parameter 4-layer cloak (dotted line) is used comparing with a bare PEC core (dash dotted line). When the 4-layer cloak is created with the optimized parameters (solid line), about 20dB of reduction are obtained for all the directions comparing with the TB one. Thus a “quasi-perfect” invisibility cloak is achieved although only 4 layers are used.

The reason why quasi-perfect invisibility can still be achieved in a cloak with only a few layers can be physically...
FIG. 6: Comparison of the differential scattering efficiency of a TE wave incidence for the PEC cylinder with radius $\lambda/2$ (dotted line), the same PEC cylinder coated by the TB full parameter multilayered cloak (dashed line) and by the quasi-perfect cloak (solid line).

explained as follows: The scattering of such kind of multilayered cloak is determined by the recurrence equation Eq. 3. When $m = 0$, Eq. 3 indicates that the $n$th order scattering is the sum of the direct scattering at $R_0$ denoted by $r_{01n}$ and the wave coming out from $R_0$ caused by the multiple reflections and transmissions on the inner boundaries, denoted by $\tilde{t}_{10n}$. In order to minimize the total scattering, the parameters of the cloak should be chosen so that $r_{01n}$ and $\tilde{t}_{10n}$ cancel each other. And as denoted by Eq. 3, $r_{01n}$ are actually determined simultaneously by $\nu_{mn}$, $k_m$ and $\eta_m$ in all the layers. Thus the match of impedances between the conjoined layers does not assure a small total scattering, while on the contrary, the mismatch of the impedances, as shown by Fig. 4 (d), can be utilized to form multiple reflections and transmissions among the inner layers, which eventually produce a transmission to the free space being able to destructively interfere with the direct reflection occurs at the outer boundary. In our proposed quasi-perfect cloak, the 0th direct reflection coefficient at $R_0$ is $r_{010} = -0.10741 + 0.3096i$, while the 0th transmission coefficient from $R_0$ is $\tilde{t}_{010} = 0.10739 - 0.3052i$, therefore, the 0th scattering coefficient is $\tilde{r}_{010} = -0.00002 + 0.0044i$ which is very small, due to the destructive interference. This is also similar to the one-dimensional multilayered case, where the impedances of each layer are not necessary to be matched in order to get zero reflection.

It’s interesting to see from Fig. 4 that the optimized parameters (solid lines) are quite different from the parameters obtained by the coordinate transformation (dotted lines). Taking the $\rho$ component as an example, as shown in Fig. 4 (a), the value of relative permeability in the most inner layer is optimized to be 0.58, instead of a close-to-zero value as suggested by the transformation method. We also see that the parameters achieved here are relatively small and
within the limit of metamaterials, which shows the possibility of realizing such a “quasi-perfect” cloak. This is a very important contribution to the implementation of the cloak in practice. As we know, for an ideal cylindrical cloak, the φ component of the parameters will go infinity near the inner boundary, as shown by the dashed line in Fig. 3 (b). Such kind of extreme value near the inner boundary is very difficult to realize. A truncation method [11] at the inner boundary can be used to avoid the extreme value of the inner boundary of the cloak, however, scattering will be aroused and it has been shown that the performance of such a cloak is sensitive to the perturbations on the inner boundary [11]. In order to get a performance as good as our proposed cloak, here $Q_{sca}$ is 0.0025, only about $\lambda/10^8$ truncation is allowed on the inner boundary, which means huge values of permeability about $5 \times 10^7 \mu_0$ is needed in the φ direction near the inner boundary, this is a disadvantage of the transformation based cloak in practical implementation.

In conclusion, the analytical model for the multilayered cylindrical cloak has been well established. Based on this model, the effects of discretization and simplification, as well as losses, of the transformation based cloak can be quantitatively characterized. By utilizing the genetic algorithms, we further show that, it is not the best way to manually assign the transformation parameters for the multilayered cloak. A 4-layer “quasi-perfect” cylindrical cloak is proposed, whose parameters are relatively small, and possible for realization. The parameters obtained do not follow the method of coordinate transformation. Our method was shown to be effective in analyzing a multilayered cloak and provides a robust way of designing a practical cloak.

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