THE BINORMAL FLOW WITH INITIAL DATA BEING POLYGONAL LINES AND NON-UNIQUENESS

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Abstract. In this paper, we consider the evolution of a curve $\chi \in \mathbb{R}^3$ by the binormal flow:

$$\chi_t = \chi_x \wedge \chi_{xx}.$$  

We give a new construction for the solution of the binormal flow with initial data being the polygonal lines, which was previously obtained by V. Benicia and L. Vega in [4]. Our construction is based on the linear Schrödinger equation at fractional time. This gives a simple proof of the existence result. Since it does not rely on the solvability of the cubic nonlinear Schrödinger equation, as an improvement, we do not require the corners of the polygonal line to be located in integer numbers and are able to obtain the convergence $Lipschitz$ in arc length.

The solutions with the same initial datum constructed in [4] are different from the ones given in this paper, hence we have the non-uniqueness of the solution for the binormal flow generated by the polygonal lines. In particular, the convergence rate of solutions, as $t$ goes to 0, can be attained as $|t|^\beta$ at corners for each $\beta \in \left(\frac{3}{5}, 1\right)$.

1. Introduction

In this paper, we consider the binormal flow equation:

$$\chi_t = \chi_x \wedge \chi_{xx},$$

where $\chi = \chi(t, x) \in \mathbb{R}^3$ is a curve, and $x$ is the arclength parameter of $\chi$. The equation arises from studying an approximation of the dynamics of a vortex filament under the Euler equations. It was first derived by Da Rios [7] in 1906 and promoted by Levi-Civita [18] in 1931, see also [17, 1, 6]. These works can be traced back to the celebrated paper of Helmholtz’s [11] in 1858 on the motion of a three-dimensional incompressible fluid in rotation. Recently, Jerrard and Seis in [14] rigorously established the connection between the evolution of a weak solution of Euler flow with approximately-filamentary vorticity and the binormal curvature flow.

Applying the Frenet-Serret formulas, we can write the equation (1.1) as its equivalent form

$$\chi_t = cb,$$

where $c, b$ are the curvature and the binormal vector of the curve, respectively. In [10], Hasimoto observed that the “filament function” $\psi$ defined as

$$\psi(t, x) = c(t, x)e^{i \int_0^x \tau(t', x') \, dx'},$$

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where \( \tau \) is the torsion of the curve, satisfies the cubic nonlinear Schrödinger equation

\[
\partial_t \psi + \partial_{xx} \psi + \frac{1}{2} (|\psi|^2 - A(t)) \psi = 0, \quad \text{for some } A(t) \in \mathbb{R}.
\]  

(1.2)

This links the binormal flow with the nonlinear Schrödinger equation. Using this observation, the existence of the solution of (1.1) with curvature and torsion in high-order Sobolev space was proved in [10, 8, 16, 19]. Later, Jerrard and Smets [15] proposed a weak formulation for the binormal curvature flow, and proved the global existence and the weak-strong uniqueness of the solutions with the initial data in less regular closed curves. Meanwhile, the curves with a corner and curvature in some weighted space were considered by Banica and Vega [3], in which the global well-posedness was established. The curves with one corner are related to the self-similar solutions of (1.1). Indeed, it was proved by Gutiérrez, Rivas and Vega in [9] that the self-similar solutions of the binormal flow can be uniquely generated by polygonal lines with one corner, see also [2] for the stability of the self-similar solution.

The curves with multiple corners which include the closed polygons were recently studied by Banica and Vega in [4], see also the numerical simulations in [12, 13] and the references therein. In [4], the authors proved that if the initial curve \( \chi_0 \) is a polygonal line with more than one corners located at integer numbers, and the curvature angles in some weighted space, then there exists a global weak solution \( \chi \) of the binormal flow, which is smooth for all \( t \neq 0 \). Further, a jump discontinuity of the energy of such solution was proved in [5].

The binormal flow is also connected with the Schrödinger map onto the sphere. Indeed, considering the tangent vector \( T = \chi_x \in S^2 \), then \( T \) satisfies

\[
T_t = T \wedge T_{xx}.
\]  

(1.3)

Based on the solvability of the cubic nonlinear Schrödinger equation (1.2) for some suitable \( A(t) \), the authors in [4] constructed the solutions by considering the evolutions of the matrix \((T, N) = (T, N)(\psi)\) formed by \( T \) and its complex-valued normal vector \( N \). Moreover, the authors introduced a remarkable geometry method to recover the initial curves: it was shown that a \( \mathbb{R}^3 \)-curve can be uniquely determined by the locations \( x_n \in \mathbb{R}^+ \), the curvature angles \( \theta_n \in (0, \pi) \), the torsion angles \( \tau_n \in (0, \pi) \), and the directions \( \delta_n \in \{-1, +1\} \). Further, they introduced a complex valued sequence \( \{\alpha_k\} \) which can be used to uniquely determine all of the parameters \( \{x_n, \theta_n, \tau_n, \delta_n\} \), and then proved the existence of the solutions by the reversible route map: \( \chi_0 \to \{x_n, \theta_n, \tau_n, \delta_n\}_\chi_0 \to \{a_k\} \to \psi \to (T, N) \to \{x_n, \theta_n, \tau_n, \delta_n\}_\chi(0) \to \chi(0) \).

In this paper, we make an attempt to revise the Cauchy problem of the binormal flow (1.1) with the initial data being polygonal lines. We give a new construction for the solution based on the linear Schrödinger equation at fractional time. In comparison, since our construction does not rely on the solvability of the cubic nonlinear Schrödinger equation, the present result does not require the corners of the polygonal line to be located in integer numbers and are able to obtain the convergence Lipschitz in arclength. Moreover, our result implies the non-uniqueness. Nevertheless, the proof of our result is largely inspired by the strategy in [4].

More precisely, let \( \{x_k\} \) be a (finite or infinite) sequence in \( \mathbb{R}^+ \), then we define the space \( l^{1,1} \) by the norm

\[
\|a_k\|_{l^{1,1}} = \sum_k (1 + |x_k|) |a_k|.
\]
Theorem 1.1. Let $\chi_0$ be an arclength parametrized polygonal line with corners located at $x_k \in \mathbb{R}^+$, and with the sequence of curvature angles $\theta_k \in (0, \pi)$ such that
\[
\sqrt{-\ln \left( \sin \frac{\theta_k}{2} \right)} \in t^{1+1}.
\]
Then for any $\gamma \in (0, \frac{\pi}{4})$, there exists a global weak solution $\chi(t)$ of the binormal flow, which is smooth for all $t \neq 0$, with
\[
\| \chi(t) - \chi_0 \|_{L^\infty(\mathbb{R})} \lesssim |t|^{1-\gamma}, \quad \forall |t| \leq 1,
\]
in which the equality can be attained at $x_k$. Moreover, let $\beta = \min\{1-2\gamma, 2-5\gamma\}$, then
\[
\| \chi(t) - \chi_0 \|_{Lip(\mathbb{R})} \lesssim |t|^\beta, \quad \forall |t| \leq 1.
\]

The theorem above implies the non-uniqueness of the solution for the binormal flow generated by the polygonal lines. In fact, as proved in Section 2.5, the various solutions with the same initial data constructed by the various Frenet frame (2.2) via (2.1) have the different asymptotic behavior at the corner $x_k$ as follows,
\[
\chi(t, x_k) = \chi(0, x_k) + C_k t^{1-\gamma} + O(t + t^{1-\gamma+\beta}), \quad \text{as } t \to 0,
\]
for some constant $C_k$ and each $\gamma \in (0, \frac{\pi}{4})$. While the solution constructed in [14] behaves as
\[
\chi(t, k) = \chi(0, k) + v_1 \sqrt{t} \sin \left( M \ln \sqrt{t} \right) + v_2 \sqrt{t} \cos \left( M \ln \sqrt{t} \right) + O(t), \quad \text{as } t \to 0,
\]
for some $v_1, v_2, M \in \mathbb{R}$, which is also different from the solutions constructed in the present paper.

Moreover, we notice that the non-unique evolutions were provided in [15] by the initial datum consisting of the sum of two circles of different radii and that have exactly one intersection point; meanwhile, if the initial data is the integral currents as long as no self-intersections, then the solution is weak-strong uniqueness. Our result implies that if the initial curve is not smooth (with corners), then even there is no self-intersection, the corresponding solution of (1.1) is not unique.

2. The proof of Theorem 1.1

2.1. Some notations. We write $X \lesssim Y$ or $X = O(Y)$ to indicate $X \leq CY$ for some constant $C > 0$. Throughout this paper, we use $\phi_{\leq a}$ for $a \in \mathbb{R}^+$ to be the smooth function
\[
\phi_{\leq a}(x) = \begin{cases} 1, & |x| \leq a, \\ 0, & |x| \geq 2a. \end{cases}
\]
Moreover, we denote $\phi_{\geq a} = 1 - \phi_{\leq a}$.

2.2. Reformulation. Firstly, we define the characterizing sequence $\{x_k, \theta_k, \tau_k, \delta_k\}$ of $\chi_0$ as follows. Without loss of generality, we may assume that $x_k < x_{k+1}$. Denote $T_{0k} \in \mathbb{S}^2$ be the unit tangent vector of $\chi_0$ for $x \in (x_k, x_{k+1})$. Define $\{\theta_k, \tau_k, \delta_k\} \in (0, \pi)^2 \times \{\pm 1\}$ by
\[
\cos \theta_k = T_{0k-1} \cdot T_{0k}; \quad \cos \tau_k = \frac{T_{0k-1} \wedge T_{0k}}{|T_{0k-1} \wedge T_{0k}|} \cdot \frac{T_{0k} \wedge T_{0k+1}}{|T_{0k} \wedge T_{0k+1}|};
\]
\[
\delta_k = \text{sgn} \left[ \left( \frac{T_{0k-1} \wedge T_{0k}}{|T_{0k-1} \wedge T_{0k}|} \wedge \frac{T_{0k} \wedge T_{0k+1}}{|T_{0k} \wedge T_{0k+1}|} \right) \cdot T_{0k} \right].
\]
Then we construct the solution of (1.1) with the initial curve $\chi_0$ in the following way. By the time reversibility of the binormal flow, we only consider the positive time. Define the function $u$ to be

$$u(t) := \sum_k \alpha_k e^{it\Delta} \delta(\cdot - x_k),$$

where the parameters $\alpha_k \in \mathbb{C}$, and $\gamma > 0$ will be determined later. By the explicit formula of the linear Schrödinger flow, we have that for some absolute constant $c \in \mathbb{C}$,

$$u(t, x) = c \sum_k \alpha_k t^{-\gamma} e^{-\frac{(x-x_k)^2}{4t^{2\gamma}}}, \quad (2.1)$$

which is smooth for any $t \neq 0$. Given $t_0 > 0$, $P \in \mathbb{R}^3$ and an orthonormal basis $(v_1, v_2, v_3) \in \mathbb{R}^{3 \times 3}$, we define the evolution of the tangent vector $T \in S^2$ and the normal vector $e_1 \in S^2$, $e_2 \in S^2$ by

$$
\begin{pmatrix}
T \\
e_1 \\
e_2
\end{pmatrix}_t = 
\begin{pmatrix}
0 & -\text{Im} u_x & \text{Re} u_x \\
\text{Im} u_x & 0 & 0 \\
-\text{Re} u_x & 0 & 0
\end{pmatrix}
\begin{pmatrix}
T \\
e_1 \\
e_2
\end{pmatrix},
$$

(2.2)

with the initial and boundary datum

$$(T, e_1, e_2)(t_0, 0) = (v_1, v_2, v_3).$$

Then a direct computation gives that $(T, e_1, e_2)(t, x)$ is an orthogonal basis in $\mathbb{R}^3$ for all $x \in \mathbb{R}, t > 0$. Moreover, the equation for $T$ is (1.3).

Let $\chi$ be defined as

$$\chi(t, x) = P + \int_{t_0}^t (T \wedge T_x)(\tau, 0) d\tau + \int_0^x T(t, x') dx',$$

then $\chi$ is a solution of (1.1). Note that

$$\chi_t = T \wedge T_x = \text{Im}(\bar{u}N),$$

then by (2.2) and (2.1) we obtain that for any $0 < t_1 < t_2 < 1$, there exists $C = C(\|\alpha_k\|_{L^1}) > 0$ such that

$$\|\chi(t_2) - \chi(t_1)\|_{L^\infty(\mathbb{R})} \leq \int_{t_1}^{t_2} \|\bar{u}N\|_{L^\infty(\mathbb{R})} dt \leq Ct_2^{1-\gamma}.$$

Hence, we have the existence of $\chi(0, x) = \lim_{t \to 0} \chi(t, x)$, which is continuous. Moreover,

$$\|\chi(t) - \chi(0)\|_{L^\infty(\mathbb{R})} \leq Ct^{1-\gamma}.
$$

(2.3)

Hence, we can reformulate the problem by considering the evolutions of the matrix $(T, e_1, e_2)$.

2.3. The existence of trace at $t = 0$. Let $N = e_1 + ie_2$. According to the evolution of $(T, N)$ in (2.2), we have

$$T_t = \text{Im}(\bar{u}xN); \quad N_t = -i\text{Re}xT.
$$

(2.4)

**Proposition 2.1.** Let $\gamma \in \left(0, \frac{1}{2}\right)$. Then for any $x \in \mathbb{R}^+$, there uniquely exists $(T(0,x), N(0,x))$, such that

$$\|\left(T(t, \cdot), N(t, \cdot)\right) - \left(T(0, \cdot), N(0, \cdot)\right)\|_{L^\infty(\mathbb{R})} \leq Ct^{\beta}, \quad \forall |t| \leq 1,$$

where $\beta = \min\{1 - 2\gamma, 2 - 5\gamma\}$. 

Proof. Note that for any $0 < t_1 < t_2 < 1$,

$$T(t_2, x) - T(t_1, x) = \int_{t_1}^{t_2} T_t(t, x) \, dt = \int_{t_1}^{t_2} \operatorname{Im}(u_x N) \, dt.$$ 

From the formula (2.1), we have

$$u_x = \frac{c}{2} \sum_k \alpha_k t^{-3\gamma} i(x - x_k) e^{\frac{i(x-x_k)^2}{4t^{2\gamma}}}.$$ 

Hence, we get

$$T(t_2, x) - T(t_1, x) = -\frac{1}{2} \operatorname{Im} \sum_k \alpha_k \int_{t_1}^{t_2} t^{-3\gamma} i(x - x_k) e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \, dt.$$ 

We split the integration as

$$\int_{t_1}^{t_2} t^{-3\gamma} i(x - x_k) e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \, dt = \int_{t_1}^{t_2} t^{-3\gamma} \phi_{\leq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) i(x - x_k) e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \, dt$$

$$+ \int_{t_1}^{t_2} t^{-3\gamma} \phi_{\geq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) i(x - x_k) e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \, dt.$$ 

For the first term, we have

$$\left| \int_{t_1}^{t_2} t^{-3\gamma} \phi_{\leq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) i(x - x_k) e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \, dt \right| \lesssim \frac{1}{1 - 2\gamma} t_2^{1-2\gamma}.$$ 

For the second term, integration by parts we get

$$\int_{t_1}^{t_2} t^{-3\gamma} \phi_{\geq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) i(x - x_k) e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \, dt$$

$$= \frac{2}{\gamma} \int_{t_1}^{t_2} \phi_{\geq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) \frac{t^{1-\gamma}}{x - x_k} N(t) \, dt \left( e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} \right)$$

$$= \frac{2}{\gamma} \phi_{\geq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) \frac{t^{1-\gamma}}{x - x_k} e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \bigg|_{t_1}^{t_2}$$

$$- \frac{2}{\gamma} \int_{t_1}^{t_2} \left( \phi_{\geq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) \frac{t^{1-\gamma}}{x - x_k} \right) e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \, dt$$

$$- \frac{2}{\gamma} \int_{t_1}^{t_2} \phi_{\geq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) \frac{t^{1-\gamma}}{x - x_k} e^{-\frac{i(x-x_k)^2}{4t^{2\gamma}}} N(t) \, dt.$$ 

The first two terms are $O(t_2^{1-2\gamma})$. For the third term, by applying (2.4) it is equal to

$$\frac{c}{\gamma} \sum_j \alpha_j \int_{t_1}^{t_2} \phi_{\geq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) \frac{x - x_j}{x - x_k} t^{-4\gamma} e^{-\frac{i(x-x_k)^2 - i(x-x_j)^2}{4t^{2\gamma}}} T(t) \, dt.$$ 

Note that

$$\phi_{\geq 1} \left( \frac{x - x_k}{t^{\gamma}} \right) \frac{x - x_j}{x - x_k} \lesssim 1 + |x_k - x_j| t^{-\gamma}.$$ 

Hence, (2.7) can be controlled by

$$t_2^{2-4\gamma} + t_2^{2-5\gamma} |x_k - x_j|.$$
Lemma 2.2. There exist sequences $\{T_k\} \subset S^2$ and $\{\tilde{N}_k\} \subset C^3$, such that
\[ T(0, x) \equiv T_k, \quad \tilde{N}(0, x) \equiv \tilde{N}_k, \quad \text{for any} \quad x \in (x_k, x_{k+1}). \]

Proof. Note that
\[ T_x = \text{Re}(\bar{u}N); \quad N_x = -uT. \] (2.9)

Then for any two different points $x', x'' \in (x_k, x_{k+1})$, we have that for $t \in (0, 1)$,
\[ T(t, x'') - T(t, x') = \int_{x'}^{x''} T_x(t, x) \, dx = \int_{x'}^{x''} \text{Re}(\bar{u}N)(t, x) \, dx = t^{-\gamma} \text{Re} \sum_j c\alpha_j \int_{x'}^{x''} e^{-\frac{i(x-x_j)^2}{4t^2\gamma}} N(t, x) \, dx. \]

Since $x \neq x_j$, integration by parts, we get
\[ T(t, x'') - T(t, x') = -2t^{2\gamma} \text{Re} \sum_j c\alpha_j \int_{x'}^{x''} \frac{1}{i(x-x_j)} e^{-\frac{i(x-x_j)^2}{4t^2\gamma}} N(t, x) \bigg|_{x'}^{x''} \]
\[ -2t^{2\gamma} \text{Re} \sum_j c\alpha_j \int_{x'}^{x''} \frac{1}{i(x-x_j)^2} e^{-\frac{i(x-x_j)^2}{4t^2\gamma}} N(t, x) \, dx \]
\[ +2t^{2\gamma} \text{Re} \sum_j c\alpha_j \int_{x'}^{x''} \frac{1}{i(x-x_j)} e^{-\frac{i(x-x_j)^2}{4t^2\gamma}} N_x(t, x) \, dx. \]

The first two terms are $O(t^\gamma)$. For the third term, applying (2.9) again, it is equal to
\[ -2|c|^2 \text{Re} \sum_{j,h} \alpha_j \alpha_h \int_{x'}^{x''} \frac{1}{i(x-x_j)} e^{\frac{i(x-x_j)^2-(x-x_j)^2}{4t^2\gamma}} T(t, x) \, dx. \]
When \( j = h \), it vanishes due to “Re”, hence it is further written as
\[
-2|c|^2 \text{Re} \sum_{j \neq h} \alpha_j \alpha_h \int_{x'}^{x''} \frac{1}{i(x - x_j)} e^{\frac{2(x_j - x_h)(x + x_h^2 - x'^2)}{4i\gamma}} T(t, x) \, dx.
\]

Thanks to \( x_j \neq x_h \), we can integrate by parts again, to get
\[
-4|c|^2 t^{2\gamma} \sum_{j \neq h} \alpha_j \alpha_h \int_{x'}^{x''} (x - x_j)(x_h - x_j) e^{\frac{2(x_j - x_h)(x + x_h^2 - x'^2)}{4i\gamma}} T(t, x) \, dx
\]
\[
-4|c|^2 t^{2\gamma} \sum_{j \neq h} \alpha_j \alpha_h \int_{x'}^{x''} (x - x_j)(x_h - x_j) e^{\frac{2(x_j - x_h)(x + x_h^2 - x'^2)}{4i\gamma}} T(t, x) \, dx
\]
\[
+4|c|^2 t^{2\gamma} \sum_{j \neq h} \alpha_j \alpha_h \int_{x'}^{x''} (x - x_j)(x_h - x_j) e^{\frac{2(x_j - x_h)(x + x_h^2 - x'^2)}{4i\gamma}} T_x(t, x) \, dx.
\]

The first two terms are \( O(t^{2\gamma}) \). Further, by (2.9) and (2.1), we have \( T_x = O(t^{-\gamma}) \). Hence, the third term is \( O(t^\gamma) \).

Therefore, together with the estimates above, we obtain that for some constant \( C = C(\alpha_k ||_{L^1}) > 0 \),
\[
|T(t, x'') - T(t, x')| \leq C t^\gamma.
\]

Taking \( t \to 0 \), we get \( T(0, x) = T_k \) for some \( T_k \in S^2 \).

For \( \tilde{N} \), by (2.9) and (2.1), we have that for any two different points \( x', x'' \in (x_k, x_{k+1}), \)
\[
\tilde{N}(t, x'') - \tilde{N}(t, x') = \int_{x'}^{x''} \tilde{N}_x(t, x) \, dx = \int_{x'}^{x''} \left( -e^{\alpha_2} uT + i\Phi_x \tilde{N} \right) \, dx.
\]

Then argued similarly as above, we get
\[
- \int_{x'}^{x''} e^{\alpha_2} uT \, dx = O(t^\gamma) + 2ct\gamma \sum_{j} \alpha_j \int_{x'}^{x''} \frac{1}{i(x - x_j)} \text{Re}(\bar{u}N) e^{\frac{i(x - x_j)^2}{4t\gamma}} \, dx
\]
\[
= O(t^\gamma) + ct\gamma \sum_{j} \alpha_j \int_{x'}^{x''} \frac{1}{i(x - x_j)} \bar{u}N e^{\frac{i(x - x_j)^2}{4t\gamma}} \, dx
\]
\[
+ ct\gamma \sum_{j} \alpha_j \int_{x'}^{x''} \frac{1}{i(x - x_j)} u\tilde{N} e^{\frac{i(x - x_j)^2}{4t\gamma}} \, dx.
\]

For the second term, applying (2.1) again and treated as \( T \) above (integration by parts), we get
\[
ct\gamma \sum_{j} \alpha_j \int_{x'}^{x''} \frac{1}{i(x - x_j)} \bar{u}N e^{\frac{i(x - x_j)^2}{4t\gamma}} \, dx = |c|^2 \sum_{j \neq h} \alpha_j \alpha_h \int_{x'}^{x''} \frac{e^{\alpha_2} N}{i(x - x_j)} e^{\frac{i(x - x_j)^2 - (x - x_h)^2}{4t\gamma}} \, dx
\]
\[
= -i \int_{x'}^{x''} \Phi_x \tilde{N} \, dx + |c|^2 \sum_{j \neq h} \alpha_j \alpha_h \int_{x'}^{x''} \frac{e^{\alpha_2} N}{i(x - x_j)} e^{\frac{i(x - x_j)^2 - (x - x_h)^2}{4t\gamma}} \, dx
\]
\[
= -i \int_{x'}^{x''} \Phi_x \tilde{N} \, dx + O(t^\gamma).
\]
For the third term, by (2.1) it is equal to
\[
e^2 \sum_{j,h} \alpha_j \alpha_h \int_{x'}^{x''} \frac{e^{i\Phi} \tilde{N}}{i(x - x_j)} e^{i(x - x_h)^2 + (x - x_j)^2 \over 4ix^2} \, dx.
\]
By symmetry, it further turns to
\[
\frac{1}{2} e^2 \sum_{j,h} \alpha_j \alpha_h \int_{x'}^{x''} \left( \frac{1}{i(x - x_j)} + \frac{1}{i(x - x_h)} \right) e^{i\Phi} e^{i(x - x_h)^2 + (x - x_j)^2 \over 4ix^2} \, \tilde{N} \, dx
\]
\[= \frac{1}{2} e^2 \sum_{j,h} \alpha_j \alpha_h \int_{x'}^{x''} \frac{2x - x_j - x_h}{i(x - x_j)(x - x_h)} e^{i(x - x_h)^2 + (x - x_j)^2 \over 4ix^2} e^{i\Phi} \tilde{N} \, dx.
\]
Note that
\[
i(2x - x_j - x_h) e^{i(x - x_h)^2 + (x - x_j)^2 \over 4ix^2} = 2t^2 \gamma \partial_x \left( e^{i(x - x_h)^2 + (x - x_j)^2 \over 4ix^2} \right),
\]
then integration by parts and treated as $T$, we get
\[
e^2 \sum_{j,h} \alpha_j \alpha_h \int_{x'}^{x''} \frac{e^{i\Phi} \tilde{N}}{i(x - x_j)} e^{i(x - x_h)^2 + (x - x_j)^2 \over 4ix^2} \, dx = O(t^\gamma).
\]
Therefore, collecting the estimates above, we obtain
\[
- \int_{x'}^{x''} e^{i\Phi} uT \, dx = -i \int_{x'}^{x''} \Phi_x \tilde{N} \, dx + O(t^\gamma).
\]
This yields that for some constant $C = C(\|\alpha_k\|_{1,1}) > 0$,
\[
|\tilde{N}(t, x'') - \tilde{N}(t, x')| \leq Ct^\gamma.
\]
Taking $t \to 0$, we get $\tilde{N}(0, x) = \tilde{N}_k$ for some $\tilde{N}_k \in \mathbb{C}^3$. \qed

The following results show that the corners of $\chi(0, x)$ appear at $x_k$. In particular, $T_k \neq T_j$ for any $j \neq k$.

**Proposition 2.3.** For $k \in \mathbb{Z}^+$, there exists $(T_k^*(x), N_k^*(x)) \in \mathbb{S}^2 \times \mathbb{C}^3$, such that
\[
(T(n, x_k + t_n^* x), N(n, x_k + t_n^* x)) \to (T_k^*(x), N_k^*(x)), \quad \text{as } t_n \to 0. \tag{2.10}
\]
Moreover, there uniquely exist a rotation $R_k \in \mathbb{M}^3$, $A_k^\pm \in \mathbb{R}^3, B_k^\pm \in \mathbb{C}^3$, which only depend on $|\alpha_k|$ such that
\[
T_k^*(x) \to R_k A_k^+ \epsilon_{|\alpha_k|^2 \ln |x|} e^{i\text{Arg}(\alpha_k)} N_k^+(x) \to R_k B_k^+, \quad \text{as } x \to \pm \infty.
\]
Furthermore, let $A_k^\pm = (A_{k,1}^\pm, A_{k,2}^\pm, A_{k,3}^\pm)$, then $A_k^\pm, \Re B_k^\pm, \Im B_k^\pm$ are orthogonal basises in $\mathbb{R}^3$, and
\[
A_{k,1}^+ = A_{k,1}^- = e^{-\frac{\pi}{2} |\alpha_k|^2}, \quad A_{k,2}^+ = -A_{k,2}^-, \quad A_{k,3}^+ = -A_{k,3}^-.
\]

**Proof.** The proof is split into two steps. For short of notations, we denote
\[
T_n^k(x) = T(n, x_k + t_n^* x), \quad N_n^k(x) = N(n, x_k + t_n^* x).
\]
Step 1, the existence of the limit. By (2.9) and (2.11), we have
\[
\partial_x T^k_n(x) = t_n^\gamma T_n(t_n, x_k + t_n^\gamma x) = \text{Re} \sum_{j} c_{x_j} e^{-\frac{i((x_k - x_j + t_n^\gamma x)^2}{4t_n^\gamma}} N_n^k(x); \quad (2.11)
\]
\[
\partial_x N_n^k(x) = t_n^\gamma N_n(t_n, x_k + t_n^\gamma x) = - \sum_{j} c_{x_j} e^{-\frac{i((x_k - x_j + t_n^\gamma x)^2}{4t_n^\gamma}} T_n^k(x). \quad (2.12)
\]

Since \(|T_n^k(x)| = 1, |N_n^k(x)| = 2, from (2.11) and (2.12) we get
\[
|\partial_x T_n^k(x)|, |\partial_x N_n^k(x)| \lesssim \|\alpha_k\|_{1,\infty}.
\]

Then by Arzela-Ascoli’s theorem, there exists \((T_n^k(x), N_n^k(x)) \in \mathbb{S}^2 \times \mathbb{C}^3, such that (2.10) holds.

Step 2, the profile of the limit. According to (2.11) and (2.12), we rewrite
\[
\partial_x T_n^k(x) = \text{Re}(\alpha_k e^{-\frac{i}{4}x^2 N_n^k(x)}) + f_n^k(x), \quad \partial_x N_n^k(x) = -\alpha_k e^{-\frac{i}{4}x^2 T_n^k(x)} + g_n^k(x), \quad (2.13)
\]
where
\[
f_n^k(x) = \text{Re} \sum_{j \neq k} c_{x_j} e^{-\frac{i((x_k - x_j + t_n^\gamma x)^2}{4t_n^\gamma}} N_n^k(x); \quad g_n^k(x) = - \sum_{j \neq k} c_{x_j} e^{-\frac{i((x_k - x_j + t_n^\gamma x)^2}{4t_n^\gamma}} T_n^k(x).
\]

Note that
\[
e^{\frac{i((x_k - x_j + t_n^\gamma x)^2}{4t_n^\gamma}} = \mp 2i \left(t_n^{-\gamma}(x_k - x_j) + x\right)^{-1} \frac{\partial}{\partial x} \left(e^{\frac{i((x_k - x_j + t_n^\gamma x)^2}{4t_n^\gamma}}\right),
\]
when \(x\) is in a compact set, then \(|(t_n^{-\gamma}(x_k - x_j) + x)^{-1}| \lesssim t_n^{-\gamma}|x_k - x_j|^{-1}\). Hence, integration by parts and combining with the boundedness of \((\partial_x T_n^k(x), \partial_x N_n^k(x))\), we get
\[
f_n^k \to 0, \quad g_n^k \to 0, \quad \text{as } n \to \infty.
\]

Hence, \((T_n^k, N_n^k)\) weakly and thus strongly by the ODE theory obeys the following system
\[
\partial_x T_n^* = \text{Re}(\alpha e^{-\frac{i}{4}x^2 N_n^*}); \quad \partial_x N_n^* = -\alpha e^{-\frac{i}{4}x^2 T_n^*}, \quad (2.14)
\]

Now we need the following result which was established in [9].

**Lemma 2.4.** Let \(\alpha \in \mathbb{C}\), then the system
\[
\partial_x T^* = \text{Re}(\alpha e^{-\frac{i}{4}x^2 N^*}); \quad \partial_x N^* = -\alpha e^{-\frac{i}{4}x^2 T^*},
\]
exists a solution pair \((T^*, N^*) \in \mathbb{S}^2 \times \mathbb{C}^3, which is unique up to a rotation. Moreover, there uniquely exist unitary vectors \(A^\pm \in \mathbb{S}^2, B^\pm \in \mathbb{C}^3\) such that for some rotation \(R\),
\[
(T^*, e^{i|\alpha| \ln |x|} e^{i\text{Arg}(\alpha)} N^*) \to (RA^\pm, RB^\pm), \quad \text{as } x \to \pm \infty.
\]
Moreover, \(A^\pm, B^\pm\) and \(R\) depend only on \(|\alpha|, A^\pm, ReB^\pm, \text{Im}B^\pm\) are orthogonal bases in \(\mathbb{R}^3\), and \(A^\pm = (A^+_1, A^+_2, A^+_3)\) obeys
\[
A^+_1 = A^-_1 = e^{-\frac{\pi}{4}|\alpha|^2}, \quad A^+_2 = -A^-_2, \quad A^+_3 = -A^-_3.
\]

Then the proof of the proposition follows from (2.14) and Lemma 2.4. 

Suitably choosing the amplitude of \(\alpha_k\), we get a consequence of the propositions above.
Corollary 2.5. Let $k \in \mathbb{Z}$, then
\[
T(0, x_k \pm) = R_k A_k^\pm, \quad \hat{N}(0, x_k \pm) = e^{i|c|^2 \sum_{j\neq k} |\alpha_j|^2 \ln |x_k - x_j| - i\text{Arg}(\alpha_k)} R_k B_k^\pm.
\] (2.16)
Moreover, let the parameter $\alpha_k$ satisfy
\[
\cos \frac{\theta_k}{2} = e^{-\frac{\pi}{2}|c|\alpha_k|^2}.
\] (2.17)
Then $T_{k-1} \cdot T_k = \cos \theta_k$.

Proof. We only consider $T(0, x_k^+)$, since $T(0, x_k^-)$ can be treated in the same way. Let $\epsilon > 0$ be an arbitrary small constant, and $x_0 > 0$ be large enough such that by Proposition 2.3
\[
|T_k^*(x_0) - R_k A_k^+| + |e^{i|c|^2 |\alpha_k|^2 \ln |x_0|} e^{i\text{Arg}(\alpha_k)} N_k^*(x_0) - R_k B_k^+| \leq \epsilon.
\]
Note that $T(0, x_k^+) = \lim_{n \to \infty} T(0, x_k + t_n^* x_0)$ Moreover,
\[
T(0, x_k + t_n^* x_0) - R_k A_k^+ = T(0, x_k + t_n^* x_0) - T(t_n, x_k + t_n^* x_0)
+ T(t_n, x_k + t_n^* x_0) - T_k^*(x_0) + T_k^*(x_0) - R_k A_k^+.
\] (2.18)
By Proposition 2.1, there exists $C > 0$ independent of $x_0$, such that
\[
|T(0, x_k + t_n^* x_0) - T(t_n, x_k + t_n^* x_0)| \leq Ct_n^\beta.
\]
By Proposition 2.3 there exists $N = N(x_0) > 0$, such that for any $n \geq N$,
\[
|T(t_n, x_k + t_n^* x_0) - T_k^*(x_0)| \leq \epsilon.
\]
Therefore, the last two estimates above combining with (2.18), we get
\[
|T(0, x_k + t_n^* x_0) - R_k A_k^+| \leq Ct_n^\beta + 2\epsilon.
\] (2.19)
Then by the arbitrary small of $\epsilon$ and letting $t_n \to 0$, we obtain the first equality in (2.16).

Similarly, $\hat{N}(0, x_k^+) = \lim_{n \to \infty} \hat{N}(0, x_k + t_n^* x_0)$. Moreover, from (2.8),
\[
\hat{N}(0, x_k + t_n^* x_0) = e^{i|c|^2 \sum_{j\neq k} |\alpha_j|^2 \ln |x_k - x_j| + t_n^* x_0} e^{i|c|^2 |\alpha_j|^2 (\gamma \ln t_n + \ln x_0)} N(0, x_k + t_n^* x_0).
\]
Choose a subsequence, still denoted by $\{t_n\}$, such that $e^{i|c|^2 |\alpha_j|^2 \ln t_n} = 1$, then by Propositions 2.1 and 2.3 and treated as above, we have
\[
\hat{N}(0, x_k + t_n^* x_0) = e^{i|c|^2 \sum_{j\neq k} |\alpha_j|^2 \ln |x_k - x_j| + t_n^* x_0} e^{i|c|^2 |\alpha_j|^2 \ln x_0} N(0, x_k + t_n^* x_0)
\]
\[
= e^{i|c|^2 \sum_{j\neq k} |\alpha_j|^2 \ln |x_k - x_j|} e^{i|c|^2 |\alpha_j|^2 \ln x_0} N(t_n, x_k + t_n^* x_0) + O(t_n^\gamma)
\]
\[
= e^{i|c|^2 \sum_{j\neq k} |\alpha_j|^2 \ln |x_k - x_j|} e^{i|c|^2 |\alpha_j|^2 \ln x_0} N_k^*(x_0) + O(t_n^\gamma + \epsilon)
\]
\[
= e^{i|c|^2 \sum_{j\neq k} |\alpha_j|^2 \ln |x_k - x_j|} e^{-i\text{Arg}(\alpha_k)} R_k B_k^+ + O(t_n^\gamma + \epsilon),
\]
where the implicit constant in $O(\cdot)$ depends on $||\alpha_j||_{l^{1,1}}$, $|x_k - x_{k-1}|$ and $|x_k - x_{k+1}|$. Hence, we get the second equality in (2.16).

Note that $T_k = T(0, x_k^+) = R_k A_k^+$, $T_{k-1} = T(0, x_k^-) = R_k A_k^-$. Then by (2.15),
\[
T_{k-1} \cdot T_k = R_k A_k^- \cdot R_k A_k^+ = A_k^- \cdot A_k^+ = 2e^{-\pi|c|^2|\alpha_k|^2} - 1.
\]
From (2.17), we get $T_{k-1} \cdot T_k = \cos \theta_k$. This proves the lemma. □
Now we get the similar profile of the limit curve \((T(0), N(0))\), hence the same argument in [4] can be used to recover the torsion of the initial data by the similar choice of \(\text{Arg}(\alpha_k)\). For sake of the self-containedness, we present the sketch of the proof in the following.

As \(|\alpha_k|\) has been fixed in (2.17), the parameters \(A_k^\pm, B_k^\pm, R_k\) are determined by Proposition 2.3. In particular, it was proved in Lemma 4.8 in [4], that there exists \(\varphi_k\) which is only dependent on \(|\alpha_k|\) and thus \(\theta_k\), such that

\[
\frac{A_k^- \wedge A_k^+}{|A_k^- \wedge A_k^+|} = \text{Re}(e^{i\varphi_k} B_k^+) = -\text{Re}(e^{i\varphi_k} B_k^-).
\]

Now we determine \(\text{Arg}(\alpha_k)\) by the following system,

\[
\cos \psi_k = -\cos \tau_k, \quad \text{sgn} \sin \psi_k = -\delta_k,
\]

where we set \(\text{Arg}(\alpha_0) = 0, \psi_k = \text{Arg}(\alpha_k) - \text{Arg}(\alpha_{k+1}) + \varphi_k - \varphi_{k+1} - \beta_k\), and

\[
\beta_k = |c|^2 \sum_{j \neq k} |\alpha_j|^2 \ln |x_j - x_j| - |c|^2 \sum_{j \neq k+1} |\alpha_j|^2 \ln |x_{k+1} - x_j|.
\]

Then we have

\[
T_{k-1} \wedge T_k = R_k A_k^- \wedge R_k A_k^+ = R_k (A_k^- \wedge A_k^+)
= |T_{k-1} \wedge T_k| \text{Re}(e^{i\varphi_k} R_k B_k^+) . \tag{2.20}
\]

Similarly,

\[
T_k \wedge T_{k+1} = R_{k+1} A_{k+1}^- \wedge R_{k+1} A_{k+1}^+ = R_{k+1} (A_{k+1}^- \wedge A_{k+1}^+)
= -|T_k \wedge T_{k+1}| \text{Re}(e^{i\varphi_{k+1}} R_{k+1} B_{k+1}^-). \tag{2.21}
\]

Note that \(\tilde{N}(0, x_{k+1}) = \tilde{N}(0, x_{k+1}) = \tilde{N}_k\), by (2.10), we have

\[
R_{k+1} B_{k+1}^- = e^{i\beta_k + i\text{Arg}(\alpha_{k+1}) - i\text{Arg}(\alpha_k)} R_k B_k^+.
\]

Hence, we further get

\[
T_k \wedge T_{k+1} = -|T_k \wedge T_{k+1}| \text{Re}(e^{i\varphi_k - i\psi_k} R_k B_k^+). \tag{2.22}
\]

Combining with (2.20) and (2.21), and using \(\text{Re} B_k^+ \cdot \text{Im} B_k^+ = 0, \text{Re} B_k^+ \in S^2\), we get

\[
\frac{T_{k-1} \wedge T_k}{|T_{k-1} \wedge T_k|} \cdot \frac{T_k \wedge T_{k+1}}{|T_k \wedge T_{k+1}|} = -\text{Re}(e^{i\varphi_k} R_k B_k^+) \cdot \text{Re}(e^{i\varphi_k - i\psi_k} R_k B_k^+)
= - \left[ \cos \varphi_k \text{Re} B_k^+ - \sin \varphi_k \text{Im} B_k^+ \right] \cdot \left[ \cos(\varphi_k - \psi_k) \text{Re} B_k^+ - \sin(\varphi_k - \psi_k) \text{Im} B_k^+ \right]
= - \cos \psi_k = \cos \tau_k;
\]

moreover, using \(\text{Re} B_k^+ \cdot \text{Im} B_k^+ = A_k^+\), we get

\[
\left(\frac{T_{k-1} \wedge T_k}{|T_{k-1} \wedge T_k|} \wedge \frac{T_k \wedge T_{k+1}}{|T_k \wedge T_{k+1}|}\right) \cdot T_k = \left(\text{Re}(e^{i\varphi_k} R_k B_k^+) \wedge \text{Re}(e^{i\varphi_k - i\psi_k} R_k B_k^+))\right) \cdot T_k
= - R_k \left[ \cos \varphi_k \text{Re} B_k^+ - \sin \varphi_k \text{Im} B_k^+ \right] \wedge R_k \left[ \cos(\varphi_k - \psi_k) \text{Re} B_k^+ - \sin(\varphi_k - \psi_k) \text{Im} B_k^+ \right] \cdot T_n
= - \sin \psi_k.
\]

Thus,

\[
\text{sgn} \left[ \left(\frac{T_{k-1} \wedge T_k}{|T_{k-1} \wedge T_k|} \wedge \frac{T_k \wedge T_{k+1}}{|T_k \wedge T_{k+1}|}\right) \cdot T_k \right] = \delta_n.
\]
These two equalities combining with Corollary 2.5 give that the curvature, the torsion angles and the direction sequences \( \{\theta_k, \tau_k, \delta_k\} \) of \( \chi(0) \) are the same as the ones of \( \chi_0 \). Therefore, \( \chi(0) \) and \( \chi_0 \) coincide up to a rotation and a translation. Suitably choosing \((v_1, v_2, v_3)\) and \(P\), we obtain that \( \chi(0) = \chi_0 \). This establishes the existence of the solution of the binormal flow (1.1).

2.5. Optimal convergence rate. In this subsection, we prove that the convergence rate in (2.3) is optimal. Note that by (2.1),

\[
\chi(t, x_k) - \chi(0, x_k) = \int_0^t \text{Im} (\bar{u}N)(s, x_k) \, ds
\]

\[
= \text{Im} \sum_j \bar{c}_j \int_0^t s^{-\gamma} e^{-\frac{i(x_k-x_j)^2}{4s^{2\gamma}}} N(s) \, ds
\]

\[
= t^{1-\gamma} \frac{1}{1-\gamma} \left( \text{Im} \bar{c}_k N(0, x_k) \right) + R_k(t), \tag{2.22}
\]

where

\[
R_k(t) = \text{Im} \bar{c}_k \int_0^t s^{-\gamma} \left( N(s, x_k) - N(0, x_k) \right) \, ds + \text{Im} \sum_{j \neq k} \bar{c}_j \int_0^t s^{-\gamma} e^{-\frac{i(x_k-x_j)^2}{4s^{2\gamma}}} N(s, x_k) \, ds.
\]

From Proposition 2.1, we get

\[
\left| \text{Im} \bar{c}_k \int_0^t s^{-\gamma} \left( N(s, x_k) - N(0, x_k) \right) \, ds \right| \lesssim t^{1-\gamma+\beta}.
\]

Further, integration by parts,

\[
\int_0^t s^{-\gamma} e^{-\frac{i(x_k-x_j)^2}{4s^{2\gamma}}} N(s) \, ds = 2\gamma \frac{1}{i(x_k-x_j)^2} e^{-\frac{i(x_k-x_j)^2}{4s^{2\gamma}}} N(t, x_j)
\]

\[
- \frac{2}{\gamma} \int_0^t \frac{1}{i(x_k-x_j)^2} e^{-\frac{i(x_k-x_j)^2}{4s^{2\gamma}}} \frac{\partial}{\partial s} \left( s^{1+\gamma} N(s, x_k) \right) \, ds.
\]

Noting that \( N_s = O(s^{-\gamma}) \), we have

\[
\int_0^t s^{-\gamma} e^{-\frac{i(x_k-x_j)^2}{4s^{2\gamma}}} N(s) \, ds = O(t).
\]

Hence, we get \( R_k(t) = O(t + t^{1-\gamma+\beta}) \). Thus, by (2.22),

\[
\chi(t, x_k) = \chi(0, x_k) + \frac{1}{1-\gamma} \left( \text{Im} \bar{c}_k N(0, x_k) \right) t^{1-\gamma} + O(t + t^{1-\gamma+\beta}).
\]

This gives the optimality of (2.3). Hence, we finish the proof of Theorem 1.1.

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