MINIMAL LENGTH ELEMENTS OF FINITE COXETER GROUPS

XUHUA HE AND SIAN NIE

Abstract. We give a geometric proof that minimal length elements in a (twisted) conjugacy class of a finite Coxeter group $W$ have remarkable properties with respect to conjugation, taking powers in the associated Braid group and taking centralizer in $W$.

Introduction

Let $W$ be a finite Coxeter group. Let $\mathfrak{O}$ be a conjugacy class of $W$ and $\mathfrak{O}_{\text{min}}$ be the set of elements of minimal length in $\mathfrak{O}$. In [GP1], Geck and Pfeiffer showed that the elements in $\mathfrak{O}_{\text{min}}$ have remarkable properties with respect to conjugation in $W$ and in the associated Hecke algebra $H$. In [GM], Geck and Michel showed that there exists some element in $\mathfrak{O}_{\text{min}}$ that has remarkable properties when taking powers in the associated Braid group. These properties were later generalized to twisted conjugacy classes. See [GKP] and [H1]. In a recent work [L3], Lusztig showed that the centralizer of a minimal length element in $W$ also has remarkable properties.

These remarkable properties lead to the definition of determination of “character tables” of Hecke algebra $H$. They also play an important role in the study of unipotent representation [L2] and in the study of geometric and cohomological properties of Deligne-Lusztig varieties [OR], [H2], [BR], [L3] and [HL].

The proofs of these properties were of a case-by-case nature and relied on computer calculation for exceptional types.

In this paper, we’ll give a case-free proof of these properties on $\mathfrak{O}_{\text{min}}$ based on a geometric interpretation of conjugacy classes and length function on $W$. Similar ideas will also be applied to affine Weyl groups in a future work.

In [R], Rapoport pointed out to us that this paper together with [OR], [BR] (see also [HL] for a stronger result) gives a computer-free proof of the vanishing theorem [OR, 2.1] on the cohomology of Deligne-Lusztig varieties. This simplifies several steps in Lusztig’ classification of representation of finite groups of Lie type [L1]. More precisely, the

2000 Mathematics Subject Classification. 20F55, 20E45.

Key words and phrases. Minimal length element, good element, finite Coxeter group.
proof of [L1, Proposition 6.10] applies without the assumption of \(q \geq h\) and many arguments in [L1, Chapter 7-9] can then be bypassed.

1. Geometric interpretation of twisted conjugacy class

1.1. Let \(W\) be a finite Coxeter group with generators \(s_i\) for \(i \in S\) and corresponding Coxeter matrix \(M = (m_{ij})_{i,j \in S}\). The elements \(s_i\) for \(i \in S\) are called simple reflections. Let \(\delta : W \to W\) be a group automorphism sending simple reflections to simple reflections. We still denote by \(\delta\) the induced bijection on \(S\). Then \(\delta(s_i) = s_{\delta(i)}\) for all \(i \in S\).

Set \(\tilde{W} = \langle \delta \rangle \ltimes W\). For any \(i \in \mathbb{Z}\) and \(w \in W\), we set \(\ell(\delta^i w) = \ell(w)\), where \(\ell(w)\) is the length of \(w\) in the Coxeter group \(W\).

For any subset \(J\) of \(S\), we denote by \(W_J\) the standard parabolic subgroup of \(W\) generated by \((s_i)_{i \in J}\) and by \(w_J\) the maximal element in \(W_J\). We denote by \(\tilde{W}^J\) (resp. \(\tilde{J}\)) the set of minimal coset representatives in \(\tilde{W}/W_J\) (resp. \(W_J/\tilde{W}\)). We simply write \(\tilde{W}^J\) for \(\tilde{W} \cap \tilde{W}^J\).

For \(\tilde{w} \in \tilde{W}^J\), we write \(\tilde{w}(J) = J\) if the conjugation of \(\tilde{w}\) sends simple reflections in \(J\) to simple reflections in \(J\).

1.2. Two elements \(w, w'\) of \(W\) are said to be \(\delta\)-twisted conjugate if \(w' = \delta(x)wx^{-1}\) for some \(x \in W\). The relation of \(\delta\)-twisted conjugacy is an equivalence relation and the equivalence classes are said to be \(\delta\)-twisted conjugacy classes.

For \(w, w' \in W\) and \(i \in S\), we write \(w \xrightarrow{s_i \delta} w'\) if \(w' = s_{\delta(i)} ws_i\) and \(\ell(w') \leq \ell(w)\). We write \(w \rightarrow_\delta w'\) if there is a sequence \(w = w_0, w_1, \ldots, w_n = w'\) of elements in \(W\) such that for any \(k, w_{k-1} \xrightarrow{s_i \delta} w_k\) for some \(i \in S\).

We write \(w \approx_\delta w'\) if \(w \rightarrow_\delta w'\) and \(w' \rightarrow_\delta w\). It is easy to see that \(w \approx_\delta w'\) if \(w \rightarrow_\delta w'\) and \(\ell(w) = \ell(w')\).

We call \(w, w' \in W\) elementarily strongly \(\delta\)-conjugate if \(\ell(w) = \ell(w')\) and there exists \(x \in W\) such that \(w' = \delta(x)wx^{-1}\) and \(\ell(x) + \ell(w) = \ell(x) + \ell(w)\). We call \(w, w'\) strongly \(\delta\)-conjugate if there is a sequence \(w = w_0, w_1, \ldots, w_n = w'\) such that for each \(i, w_{i-1}\) is elementarily strongly \(\delta\)-conjugate to \(w_i\). We write \(w \sim_\delta w'\) if \(w\) and \(w'\) are strongly \(\delta\)-conjugate.

Now we translate the notations \(\rightarrow_\delta, \sim_\delta, \approx_\delta\) in \(W\) to some notations in \(\tilde{W}\).

By [GKP, Remark 2.1], the map \(w \mapsto \delta w\) gives a bijection between the \(\delta\)-twisted conjugacy classes of \(W\) and the ordinary conjugacy classes of \(\tilde{W}\) that is contained in \(\tilde{W}\).

For \(\tilde{w}, \tilde{w}' \in \tilde{W}\) and \(i \in S\), we write \(\tilde{w} \xrightarrow{s_i \delta} \tilde{w}'\) if \(\tilde{w}' = s_{\delta(i)} \tilde{w}s_i\) and \(\ell(\tilde{w}') \leq \ell(\tilde{w})\). We write \(\tilde{w} \rightarrow \tilde{w}'\) if there is a sequence \(\tilde{w} = \tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_n = \tilde{w}'\) of elements in \(\tilde{W}\) such that for any \(k, \tilde{w}_{k-1} \xrightarrow{s_i \delta} \tilde{w}_k\) for some \(i \in S\). The notations \(\approx\) and \(\sim\) on \(\tilde{W}\) are also defined in a similar way as \(\approx_\delta, \sim_\delta\).
on $W$. Then it is easy to see that for any $w, w' \in W$, $w \rightarrow_{\delta} w'$ iff $\delta w \rightarrow \delta w'$, $w \approx_{\delta} w'$ iff $\delta w \approx \delta w'$ and $w \sim_{\delta} w'$ iff $\delta w \sim \delta w'$.

1.3. Let $V$ be a real vector space with inner product $(,)$ such that there is an injection $\tilde{W} \hookrightarrow GL(V)$ preserving $(,)$ and for any $i \in S$, $s_i$ acts on $V$ as a reflection. By [B, Ch. V], such $V$ always exists. Unless otherwise stated, we regard $\tilde{W}$ as a reflection subgroup of $GL(V)$. We denote by $|| \cdot ||$ the norm on $V$ defined by $||v|| = \sqrt{(v, v)}$ for $v \in V$.

For any subspace $U$ of $V$, we denote by $U^\perp$ its orthogonal complement.

For any hyperplane $H$, let $s_H \in GL(V)$ be the reflection along $H$. Let $\mathfrak{H}$ be the set of hyperplanes $H$ of $V$ such that the reflection $s_H$ is in $W$. Let $V^W$ be the set of fixed points by the action of $W$. Since $W$ is generated by $s_H$ for $H \in \mathfrak{H}$, $V^W = \cap_{H \in \mathfrak{H}} H$.

Even if we start with $V$ with no nonzero fixed points, some pair $(W', V')$ with $(V')^W' \neq \{0\}$ appears in the inductive argument in this paper. This is the reason that we consider some vector space other than the one introduced in [B, Ch. V].

A connected component $A$ of $V - \cup_{H \in \mathfrak{H}} H$ is called a Weyl chamber. We denote its closure by $\bar{A}$. Let $H \in \mathfrak{H}$, if the set of inner points $H_A = (H \cap \bar{A})^0 \subset H \cap \bar{A}$ spans $H$, then we call $H$ a wall of $A$ and $H_A$ a face of $A$.

The Coxeter group $W$ acts simply transitively on the set of Weyl chambers. The chamber containing $C = \{x \in E; (x, e_i) > 0 \text{ for all } i \in S\}$ is called the fundamental chamber which is also denoted by $C$. For any Weyl chamber $A$, we denote by $x_A$ the unique element in $W$ such that $x_A(C) = A$.

Let $K \subset V$ be a convex subset. A point $x \in K$ is called a regular point of $K$ if for each $H \in \mathfrak{H}$, $K \subset H$ whenever $x \in H$. The set of regular points of $K$ is open dense in $K$.

1.4. Given any element $\tilde{w} \in \tilde{W}$ and a Weyl chamber $A$, we define $\tilde{w}_A = x_A^{-1} \tilde{w} x_A$. Then the map $A \mapsto \tilde{w}_A$ gives a bijection from the set of Weyl chambers to the conjugacy class of $\tilde{w}$ in $\tilde{W}$.

For any two chambers $A, A'$, denote by $\mathfrak{H}(A, A')$ the set of hyperplanes in $\mathfrak{H}$ separating $A$ and $A'$. Then $\ell(\tilde{w}) = \sharp \mathfrak{H}(C, \tilde{w}(C))$ for any $\tilde{w} \in \tilde{W}$. In general for any Weyl chamber $A$,

$$\ell(\tilde{w}_A) = \sharp \mathfrak{H}(A, \tilde{w}(A)).$$

Let $A, A'$ be Weyl Chambers with a common face $H_A = H_{A'}$, here $H \in \mathfrak{H}$. Then $x_A^{-1} s_H x_A = s_i$ for some $i \in S$. Now

$$\tilde{w}_{A'} = (s_H x_A)^{-1} \tilde{w}(s_H x_A) = s_i x_A^{-1} \tilde{w} x_A s_i = s_i \tilde{w} s_i$$

is obtained from $\tilde{w}_A$ by conjugation a simple reflection $s_i$. We may check if $\ell(\tilde{w}_{A'}) > \ell(\tilde{w}_A)$ by the following criterion.
Lemma 1.1. We keep the notations as above. Define a map $f_{\tilde{w}} : V \to \mathbb{R}$ by $v \mapsto ||\tilde{w}(v) - v||^2$. Let $h \in H_A$ and $v \in H^\perp$ with $x - \epsilon v \in A$ for sufficiently small $\epsilon > 0$. Set $D_v f(h) = \lim_{t \to 0} \frac{f_{\tilde{w}}(h + tv) - f_{\tilde{w}}(h)}{t}$. If $\ell(\tilde{w}_A) = \ell(\tilde{w}_A) + 2$, then $D_v f(h) > 0$.

Proof. It is easy to see that $\mathcal{J}_\Lambda(A', \tilde{w}A') - \mathcal{J}_\Lambda(A, \tilde{w}A) \subset \{H, \tilde{w}H\}$. By our assumption $\mathcal{J}_\Lambda(A', \tilde{w}A') = 2\mathcal{J}_\Lambda(A, \tilde{w}A) + 2$. Hence

$$\mathcal{J}_\Lambda(A', \tilde{w}A') = \mathcal{J}_\Lambda(A, \tilde{w}A) \cup \{H, \tilde{w}H\}$$

and $H \neq \tilde{w}H$. In particular, $H_A \cap \tilde{w}H_A = \emptyset$ and $h \neq \tilde{w}(h)$.

Let $L(h, \tilde{w}(h))$ be the affine line spanned by $h$ and $\tilde{w}(h)$. Then $L(h, \tilde{w}(h)) - H \cup \tilde{w}(H)$ consists of three connected components: $L_- = \{h + t(\tilde{w}(h) - h); t < 0\}$, $L_0 = \{h + t(\tilde{w}(h) - h); 0 < t < 1\}$ and $L_+ = \{h + t(\tilde{w}(h) - h); t > 0\}$. Note that $\mathcal{J}_\Lambda(A, \tilde{w}A) \cap \{H, \tilde{w}H\} = \emptyset$, $A \cap L_0$ and $\tilde{w}A \cap L_1$ are nonempty. Since $(v, H) = 0$ and $h + v, h + (h - \tilde{w}(h))$ are in the same component of $V - H$, we have $(v, h - \tilde{w}(h)) > 0$. Similarly we have $(\tilde{w}(v), \tilde{w}(h) - h) > 0$. Now

$$D_v f(h) = 2(\tilde{w}(h) - h, \tilde{w}(v) - v) = 2(\tilde{w}(h) - h, \tilde{w}(v)) + 2(h - \tilde{w}(h), v) > 0.$$

\hfill \Box

1.5. Let $\text{grad} f_{\tilde{w}}$ be the gradient of $f_{\tilde{w}}$ on $V$, that is, for any vector field $X$ on $V$, $Xf_{\tilde{w}} = (X, \text{grad} f_{\tilde{w}})$. Here we naturally identify $V$ with the tangent space of any point in $V$. Then it is easy to see that $\text{grad} f_{\tilde{w}}(v) = 2(1 - t\tilde{w})(1 - \tilde{w})v$, where $t\tilde{w}$ is the transpose of $\tilde{w}$ with respect to $(\cdot, \cdot)$. Let $C_{\tilde{w}} : V \times \mathbb{R} \to V$ be the integral curve of $\text{grad} f_{\tilde{w}}$ with $C_{\tilde{w}}(v, 0) = v$ for all $v \in V$. Then

$$C_{\tilde{w}}(v, t) = \exp(2t(1 - t\tilde{w})(1 - \tilde{w}))v.$$

Let $S(V) = \{v \in V; (v, v) = 1\}$ be the unit sphere of $V$. For any $0 \neq v \in V$, set $\overline{v} = \frac{v}{||v||} \in S(V)$. Define $p : V - \{0\} \to S(V)$ by $v \mapsto \lim_{t \to -\infty} C_{\tilde{w}}(v, t)$.

In order to study the map $p$, we need to understand the eigenspace of $\tilde{w}$ on $V$.

1.6. Let $\tilde{w} \in \tilde{W}$. Let $\Gamma_{\tilde{w}}$ be the set of elements $\theta \in [0, \pi]$ such that $e^{i\theta}$ is an eigenvalue of $\tilde{w}$ on $V$.

For $\theta \in \Gamma_{\tilde{w}}$, we define

$$V_{\tilde{w}}^\theta = \{v \in V; \tilde{w}(v) + \tilde{w}^{-1}(v) = 2\cos \theta v\}.$$

Then $V_{\tilde{w}} \otimes_{\mathbb{R}} \mathbb{C}$ is the sum of eigenspaces of $V \otimes_{\mathbb{R}} \mathbb{C}$ with eigenvalues $e^{\pm i\theta}$. In particular, if $\theta$ is not 0 or $\pi$, then $V_{\tilde{w}}^\theta$ is an even-dimensional subspace of $V$ over $\mathbb{R}$ on which $\tilde{w}$ acts as a rotation by $\theta$. 
Since \( \tilde{w} \) is a linear isometry of finite order, we have an orthogonal decomposition

\[
V = \bigoplus_{\theta \in \Gamma_{\tilde{w}}} V_{\tilde{w}}^\theta.
\]

Let \( \theta_0 \) be the minimal element in \( \Gamma_{\tilde{w}} \) with \( V_{\tilde{w}}^\theta \neq V^W \) and \( \tilde{w} = V_{\tilde{w}}^{\theta_0} \cap (V^W)\perp \).

Now for any \( v_\theta \in V_{\tilde{w}}^\theta \),

\[
(1 - \tilde{w})(1 - \tilde{w})v_\theta = (1 - e^{i\theta})(1 - e^{-i\theta})v_\theta = ((1 - \cos \theta)^2 + \sin^2 \theta)v_\theta = 2(1 - \cos \theta)v_\theta.
\]

In particular, let \( v \in (V^W)^\perp \), then \( v = \sum v_\theta \), where \( v_\theta \in V_{\tilde{w}}^\theta \) and the summation is over all \( \theta \in \Gamma_{\tilde{w}} \) with \( \theta \geq \theta_0 \). Then \( C_{\tilde{w}}(v, t) = \sum \exp(4t(1 - \cos \theta))v_\theta \) and \( p(v) = \bar{v}_{\theta_0} \) whenever \( v_\theta_0 \neq 0 \).

Hence \( p((V^W)^\perp - V_{\tilde{w}}^\perp) = S(V_{\tilde{w}}) \) and \( p : (V^W)^\perp - V_{\tilde{w}}^\perp \to S(V_{\tilde{w}}) \) is a fiber bundle.

**Proposition 1.2.** Let \( \tilde{w} \in \tilde{W} \) and \( A \) be a Weyl chamber. Then there exists a Weyl Chamber \( A' \) such that \( A' \) contains a regular point of \( V_{\tilde{w}} \) and \( \tilde{w}_A \to \tilde{w}_{A'} \).

**Proof.** Let \( V_{\tilde{w}}^{\geq 1} \subset V_{\tilde{w}} \) be the complement of the set of regular points of \( V_{\tilde{w}} \). By §1.6, \( p^{-1}(V_{\tilde{w}}^{\geq 1}) \) is a finite union of submanifolds of codimension \( \geq 1 \). Let \( V^{\geq 2} \) be the complement of all chambers and faces in \( V \), that is, the skeleton of \( V \) of codimension \( \geq 2 \). Then \( C_{\tilde{w}}(V^{\geq 2}, \mathbb{R}) \subset V \) is a countable union of images, under smooth maps, of manifolds with dimension at most \( \text{dim } V - 1 \). Let

\[
D_{\tilde{w}} = \{v \in V; v \notin C_{\tilde{w}}(V^{\geq 2}, \mathbb{R}) \cup p^{-1}(V_{\tilde{w}}^{\geq 1}) \cup V_{\tilde{w}}^\perp\}.
\]

Then \( D_{\tilde{w}} \) is a dense subset in the sense of Lebesgue measure.

Choose \( y \in A \cap D_{\tilde{w}} \). Set \( x = p(y) \in V_{\tilde{w}} \). Then \( x \) is a regular point in \( V_{\tilde{w}} \). There exists \( T > 0 \) such that for any chamber \( B, x \in B \) whenever \( C_{\tilde{w}}(y, -T) \in B \).

Now we define \( A_i, H_i, h_i, t_i \) as follows.

Set \( A_0 = A \). Suppose \( A_i \) is defined and \( A_i \neq A' \), then we set \( t_i = \text{sup}\{t < T; C_{\tilde{w}}(y, -t) \in A_i\} \). Then \( t_i \leq T \). Set \( h_i = C_{\tilde{w}}(y, -t_i) \). By the definition of \( D_{\tilde{w}} \), \( h_i \) is contained in a unique face of \( A_i \), which we denote by \( H_i \). Let \( A_{i+1} \neq A_i \) be the unique chamber such that \( H_i \) is a common face of \( A_i \) and \( A_{i+1} \). Then \( C_{\tilde{w}}(y, -t_i - \epsilon) \in A_{i+1} \) for sufficiently small \( \epsilon > 0 \).

Since the chambers appear in the above list are distinct with each other. Thus the above procedure stops after finitely many steps. We obtain a finite sequence of chambers \( A = A_0, A_1, \ldots, A_r = A' \) in this way. Since \( C_{\tilde{w}}(y, -T) \in A' \), we have \( x \in A' \).

Let \( v_i \in V \) such that \( (v_i, h_i - h) = 0 \) for \( h \in H_i \) and \( h_i - \epsilon v_i \in A_i \) for sufficiently small \( \epsilon > 0 \). Since \( C_{\tilde{w}}(y, -t_i - \epsilon') \in A_{i+1} \) for sufficiently
small \( \ell' > 0 \), \( D_{\ell'}f_{\bar{w}}(h_i) = (v_i, (\text{grad}f_{\bar{w}})(h_i)) \leq 0 \). Hence by Lemma 1.1, \( \ell(\bar{w}_{A_{i+1}}) \leq \ell(\bar{w}_{A_i}) \) and \( \bar{w}_{A_i} \rightarrow \bar{w}_{A_{i+1}} \).

Therefore \( \bar{w}_{A_i} \rightarrow \bar{w}_{A'} \) and \( \bar{A}' \) contains a regular point \( x \) of \( V_{\bar{w}} \). \( \square \)

2. LENGTH FORMULA

2.1. The main goal of this section is to give a length formula for the element \( \bar{w}_A \) with \( \bar{A} \) containing a regular point of some subspace of \( V \) preserved by \( \bar{w} \).

Let \( K \subset V \) be a convex subset. Let \( \mathfrak{H}_K = \{ H \in \mathfrak{H}; K \subset H \} \) and \( W_K \subset W \) be the subgroup generated by \( s_H \) (\( H \in \mathfrak{H}_K \)). For any two chambers \( A \) and \( A' \), set \( \mathfrak{H}_K(A,A') = \mathfrak{H}_K(A, A') \cap \mathfrak{H}_K \).

Let \( A \) be a Weyl chamber. We set \( W_{K,A} = x^{-1}_A W_K x_A \). If \( \bar{A} \) contains a regular point \( k \) of \( K \), then we set \( W_{K,\bar{A}} = \bar{W}_{I(K,\bar{A})} \) is the parabolic subgroup of \( W \) generated by simple reflections \( I(K,\bar{A}) = \{ s_H \in S; k \in x_A H \} \).

It is easy to see that \( \bar{A} \) contains a regular point of \( K \) if and only if it contains a regular point of \( K + V^W \). In this case, \( I(K,\bar{A}) = I(K + V^W, A) \).

If \( A' \) is a Weyl chamber such that \( \bar{A}' \) also contains \( k \). Then there exists \( x \in W_K \) with \( x(A) = A' \). We set \( x_{A,A'} = x_A^{-1} x x_A \). Then \( x_{A,A'} \in W_{K,\bar{A}} \) and

\[
\bar{w}_{A'} = (xx_A)^{-1}\bar{w}(xx_A) = (x_A x_{A,A'}^{-1} x_A)^{-1}\bar{w}(x_A x_{A,A'} x_A^{-1} x_{A,A'}^{-1} \bar{w} x_A x_{A,A'}). 
\]

Moreover,

\[
\ell(x_{A,A'}) = \sharp \mathfrak{H}(C, x_{A,A'}(C)) = \sharp \mathfrak{H}(x_A(C), x x_A(C)) = \sharp \mathfrak{H}(A, A'). 
\]

We first consider the follows special case.

**Lemma 2.1.** Let \( \bar{w} \in \bar{W} \) and \( K \subset V_{\bar{w}}^0 \) be a subspace such that \( \bar{w}(K) = K \). Let \( A \) be a Weyl chamber such that \( A \) and \( \bar{w}(A) \) are in the same connected component of \( V - \cup_{H \in \mathfrak{H}_K} H \). Assume furthermore that \( \bar{A} \) contains a nonzero element \( v \in K \) such that for each \( H \in \mathfrak{H}, v, \bar{w}(v) \in H \) implies that \( K \subset H \). Then

\[
\ell(\bar{w}_A) = \sharp \mathfrak{H}(A, \bar{w}(A)) = \frac{\theta}{\pi} \sharp \mathfrak{H}(\mathfrak{H} - \mathfrak{H}_K).
\]

**Proof.** By our assumption, \( \mathfrak{H}(A, \bar{w}(A)) \subset \mathfrak{H} - \mathfrak{H}_K \). Moreover, for any \( H \in \mathfrak{H}(A, \bar{w}(A)) \), the intersection of \( H \) with the closed interval \([v, \bar{w}(v)]\) is nonempty.

If \( \theta = 0 \), then \( \bar{w}(v) = v \). For \( H \in \mathfrak{H}(A, \bar{w}(A)) \), \( v \in H \) and hence \( H \in \mathfrak{H}_K \). That is a contradiction. Hence \( \mathfrak{H}(A, \bar{w}(A)) = \emptyset \) and \( \ell(\bar{w}_A) = \sharp \mathfrak{H}(A, \bar{w}(A)) = 0 \).

If \( \theta = \pi \), then \( \bar{w}(v) = -v \). We see \( \mathfrak{H}(A, \bar{w}(A)) = \mathfrak{H} - \mathfrak{H}_K \). Thus \( \ell(\bar{w}_A) = \sharp (\mathfrak{H} - \mathfrak{H}_K) \).

Now we assume \( 0 < \theta < \pi \) and \( d \) is the order of \( \bar{w} \). Set \( v_i = \bar{w}^i(v) \in S(K) \) for \( i \in \mathbb{Z} \). Since \( \bar{w} \) acts on \( K \) by rotation by \( \theta \), there exists a
2-dimensional subspace of $K$ that contains $v_i$ for all $i$. Let $S^1$ be the unit circle in this subspace. Let $Q_i \subset S^1$ be the open arc of angle $\theta$ connecting $v_i$ with $v_{i+1}$ and $Q'_i = Q_i \cup \{v_i\}$.

Let $H \in \mathfrak{H}(\tilde{W}(A), \tilde{W}^{i+1}(A))$. Then by our assumption, $H \in \mathfrak{H} - \mathfrak{H}_K$. If $v_i \notin H$ and $v_{i+1} \notin H$, then $H \cap Q_i \neq \emptyset$. On the other hand, for any $H \in \mathfrak{H}$, if $H \cap Q_i \neq \emptyset$, then $H \in \mathfrak{H}(\tilde{W}(A), \tilde{W}^{i+1}(A))$.

If $H \in \mathfrak{H} - \mathfrak{H}_K$ and $v_i \in H$, then $v_{i-1}, v_{i+1} \notin H$ and $\{v_i\}$ is the intersection of $H$ with the open arc connecting $v_{i-1}$ with $v_{i+1}$ passing through $v_i$. Hence $H$ belongs to exactly one of the two sets: $\mathfrak{H}(\tilde{W}^{i-1}(A), \tilde{W}(A))$ and $\mathfrak{H}(\tilde{W}(A), \tilde{W}^{i+1}(A))$. Therefore

\[
(*) \quad \sum_{i=0}^{d-1} \mu(i) = \sum_{i=0}^{d-1} \mu(\{H \in \mathfrak{H} - \mathfrak{H}_K; H \cap Q_i \neq \emptyset\}).
\]

Notice that each $H \in \mathfrak{H} - \mathfrak{H}_K$ intersects $S^1$ at exactly 2 points. Hence $H$ appears on the right hand side of $(*)$ exactly $d\theta/\pi$-times. Now

\[
d\ell(\tilde{w}_A) = d\mu(\mathfrak{H}(A, \tilde{W}(A)) = \frac{d\theta}{\pi} \mu(\mathfrak{H} - \mathfrak{H}_K)
\]
and $\ell(\tilde{w}_A) = \frac{\theta}{\pi} \mu(\mathfrak{H} - \mathfrak{H}_K)$. \hfill $\Box$

**Proposition 2.2.** Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}^{\emptyset}$ be a subspace with $\tilde{w}(K) = K$. Let $A$ be a Weyl chamber whose closure contains a regular point $v$ of $K$. Then

\[
\tilde{w}_A = \tilde{w}_{K,A} u
\]
for some $u \in W_{K,A}$ with $\ell(u) = \mu(\mathfrak{H}(A, \tilde{W}(A))_K$ and $\tilde{w}_{K,A} \in T(K,A) \tilde{W}(K,A)$ with $\tilde{w}(I(K,A)) = I(K,A)$ and $\ell(\tilde{w}_{K,A}) = \frac{\theta}{\pi} \mu(\mathfrak{H} - \mathfrak{H}_K)$.

**Proof.** We may assume that $A$ is the fundamental Weyl Chamber $C$ by replacing $\tilde{w}$ by $\tilde{w}_A$. We then simply write $J$ for $I(K,C)$. We have that $\tilde{w} = u'u''$ for some $u', u'' \in W_J$ and $\tilde{w}' \in \tilde{W}'$. Since $\tilde{w}_J \tilde{w}^{-1} = W_J$ and $u', u'' \in W_J$, $\tilde{w}'W_J(\tilde{w}')^{-1} = W_J$. We also have that $\tilde{w}' \in J \tilde{W}'$. Hence $\tilde{w}'(J) = J$. Set $u = (\tilde{w}')^{-1} u' u'' \in W_J$. Then $\tilde{w} = u'u''u' = \tilde{w}'u$.

Since $u$ acts on $K$ trivially, $\tilde{w}'K = K$ and $K \subset V_{\tilde{w}}^{\emptyset}$. By Lemma 2.1 $\ell(\tilde{w}') = \frac{\theta}{\pi} \mu(\mathfrak{H} - \mathfrak{H}_K)$. Also

\[
\ell(u) = \mu(\mathfrak{H}(C, u(C))) = \mu(\mathfrak{H}(C, u(C))_K = \mu(\mathfrak{H}(C, \tilde{w}(C)))_K = \mu(\mathfrak{H}(C, \tilde{w}(C))).
\]
where the third equality is due to the fact that both $\tilde{w}'(C)$ and $C$ belong to $U$. \hfill $\Box$

2.2. Let $\tilde{w} \in \tilde{W}$ and $K \subset V_{\tilde{w}}^{\emptyset}$ be a subspace with $\tilde{w}(K) = K$. Let $U$ be a connected component of $V - \bigcup_{H \in \mathfrak{H}_K} H$. We denote by $\ell(U)$ the number of hyperplanes in $\mathfrak{H}_K$ that separates $U$ and $\tilde{w}(U)$. Then by Proposition 2.2, $\ell(\tilde{w}_A) = \ell(U) + \frac{\theta}{\pi} \mu(\mathfrak{H} - \mathfrak{H}_K)$ for any Weyl chamber $A \subset U$ such that $A$ contains a regular element of $K$. 
Proposition 1.2 and Proposition 2.2, such that \( \ell \) is minimal among all the connected components. By Proposition 1.2 and Proposition 2.2,

1. \( \ell(\tilde{w}_A) \geq \ell(U_0) + \frac{\theta}{\pi}(\mathcal{H} - \mathcal{H}_{V_{\tilde{w}}}) \) for any Weyl chamber \( A \).
2. if \( A \subseteq U_0 \) and \( \tilde{A} \) contains a regular element of \( V_{\tilde{w}} \), then \( \ell(\tilde{w}_A) = \ell(U_0) + \frac{\theta}{\pi}(\mathcal{H} - \mathcal{H}_{V_{\tilde{w}}}) \).

2.3. Two chambers \( A \) and \( A' \) are called strongly connected if they have a common face. For any subspace \( K \) of \( V \), \( A \) and \( A' \) are called strongly connected with respect to \( K \) if \( \tilde{A} \cap A' \cap K \) spans a codimension 1 subspace of \( K \) of the form \( H \cap K \) for some \( H \in \mathcal{H} - \mathcal{H}_K \). The following result will also be used in the next section.

**Proposition 2.3.** Let \( \tilde{w} \in \tilde{W} \). Let \( A \) and \( A' \) be Weyl Chambers in the same connected component of \( V - \cup_{H \in \mathcal{H}_K} H \). Assume that \( \tilde{A} \cap A' \cap V_{\tilde{w}} \) spans \( H_0 \cap V_{\tilde{w}} \) for \( H_0 \in \mathcal{H} \) and \( \tilde{w}(H_0 \cap V_{\tilde{w}}) \neq H_0 \cap V_{\tilde{w}} \), where \( H_0 \) is the common wall of \( A \) and \( A' \). Then

\[
\ell(\tilde{w}_A) = \ell(\tilde{w}_{A'}) = \frac{\theta}{\pi}(\mathcal{H} - \mathcal{H}_K).
\]

**Proof.** Set \( K = V_{\tilde{w}} \) and \( P = H_0 \cap K \). Then \( P \) is a codimension 1 subspace of \( K \). Since \( P \neq \tilde{w}(P), K = P + \tilde{w}(P) \). There exists a regular element \( v \) of \( P \) such that \( v \in \tilde{A} \cap A' \). For \( H \in \mathcal{H} \) with \( v, \tilde{w}(v) \in H \), \( P \subseteq H \) and \( \tilde{w}(P) \subseteq H \).

Since \( \tilde{w}(K) = K \), \( \tilde{w} \) permutes the connected components of \( V - \cup_{H \in \mathcal{H}_K} H \). Let \( U \) be the connected component that contains \( A \) and \( A' \). There exists \( u \in W_K \) such that \( u^{-1}\tilde{w}(U) = U \). By Lemma 2.1, \( \frac{\theta}{\pi}(\mathcal{H} - \mathcal{H}_K) = \frac{\theta}{\pi}(\mathcal{H} - \mathcal{H}_K) \).

Now we define a map \( \phi : \mathcal{H}(A, \tilde{w}A) - \mathcal{H}(A, \tilde{w}A) \rightarrow \mathcal{H} \) by

\[
\phi(H) = \begin{cases} 
 u^{-1}(H), & \text{if } \tilde{w}(v) \in H; \\
 H, & \text{otherwise}.
\end{cases}
\]

Notice that \( u\tilde{w}(v) = \tilde{w}(v) \). Thus \( \tilde{w}(v) \in H \) if and only if \( \tilde{w}(v) \in u^{-1}(H) \). Therefore the map \( \phi \) is injective. Let \( H \in \mathcal{H}(A, \tilde{w}A) - \mathcal{H}(A, \tilde{w}A) \). If \( \tilde{w}(v) \in H \), then \( v \notin H \). Hence \( H \) separates \( v \) from \( \tilde{w}A \), hence \( \phi(H) = u^{-1}H \) separates \( u^{-1}(v) = v \) from \( u^{-1}\tilde{w}A \) and \( \phi(H) \in \mathcal{H}(A, u^{-1}\tilde{w}A) \). If \( \tilde{w}(v) \notin H \), then \( \phi(H) = H \) separates \( u^{-1}\tilde{w}(v) = \tilde{w}(v) \) from \( A \) and hence \( \phi(H) \in \mathcal{H}(A, u^{-1}\tilde{w}A) \). Thus the image of \( \phi \) is contained in \( \mathcal{H}(A, h^{-1}\tilde{w}(A)) \).

On the other hand, let \( H \in \mathcal{H}(A, u^{-1}\tilde{w}(A)) \). Since \( A \) and \( u^{-1}\tilde{w}(A) \) are both in \( U \), \( H \notin \mathcal{H}_K \). If \( \tilde{w}(v) \in H \), then \( H \) separates \( v \) from \( u^{-1}\tilde{w}(A) \) and \( u(H) \) separates \( v \) from \( \tilde{w}(A) \). Hence \( u(H) \in \mathcal{H}(A, \tilde{w}(A)) \). If \( \tilde{w}(v) \notin H \), then \( H \) separates \( \tilde{w}(v) \) from \( A \) and hence \( H \in \mathcal{H}(A, \tilde{w}(A)) \).
Therefore the image of $\phi$ is $\mathfrak{H}(A, \bar{w}(A))$. Since $\phi$ is bijective,
\[
\ell(\bar{w}_A) = \sharp\mathfrak{H}(A, \bar{w}(A)) = \sharp\mathfrak{H}(A, \bar{w}(A))_K + \sharp\mathfrak{H}(A, u^{-1}\bar{w}(A))_K = \sharp\mathfrak{H}(A, \bar{w}(A))_K + \frac{\theta}{\pi}(\mathfrak{H} - \mathfrak{H}_K).
\]

Similarly, $\ell(\bar{w}_A') = \sharp\mathfrak{H}(A', \bar{w}(A'))_K + \frac{\theta}{\pi}(\mathfrak{H} - \mathfrak{H}_K)$. Since $A$ and $A'$ are in the same connected component of $V - \cup_{H \in \mathfrak{H}K} H$, $\mathfrak{H}(A, \bar{w}(A))_K = \mathfrak{H}(A', \bar{w}(A'))_K$. The Proposition is proved.

□

3. Strongly conjugacy

The following result is proved in [GP1], [GKP] via a case-by-case analysis.

**Theorem 3.1.** Let $(W, S)$ be a finite Coxeter group and $\delta : W \to W$ be an automorphism sending simple reflections to simple reflections. Let $O$ be a $\delta$-twisted conjugacy class in $W$ and $O_{\min}$ be the set of minimal length elements in $O$. Then

1. For each $w \in O$, there exists $w' \in O_{\min}$ such that $w \to_\delta w'$.
2. Let $w, w' \in O_{\min}$, then $w \sim_\delta w'$.

By §1.2, we may reformulate it as follows.

**Theorem 3.2.** Let $(W, S)$ be a finite Coxeter group and $\delta : W \to W$ be an automorphism sending simple reflections to simple reflections. Set $\tilde{W} = \langle \delta \rangle \rtimes W$. Let $O$ be a $W$-conjugacy class in $\tilde{W}$ and $O_{\min}$ be the set of minimal length elements in $O$. Then

1. For each $w \in O$, there exists $w' \in O_{\min}$ such that $w \to w'$.
2. Let $w, w' \in O_{\min}$, then $w \sim w'$.

The main purpose of this section is to give a case-free proof of this result.

3.1. We first discuss some relation between a conjugacy class in $\tilde{W}$ and in a “smaller” subgroup. This is a special case of “partial conjugation” method in [H1].

Let $J \subset S$. Let $\bar{w} \in \tilde{W}^J$ be an element with $\bar{w}(J) = J$. We denote by $\delta'$ the automorphism on $W_J$ defined by the conjugation of $\bar{w}$. Set $\tilde{W}' = \langle \delta' \rangle \rtimes W_J$. Let $\ell'$ be the length function on $\tilde{W}'$. Then the map
\[
f : \tilde{W}' \to \tilde{W}, \quad \delta' x \mapsto \bar{w}x
\]
is equivariant for the conjugation action of $W_J$ and $\ell(f(\delta' x)) = \ell(x) + \ell(\bar{w}) = \ell_1(\delta' x) + \ell(\bar{w})$. Hence for any $x, x' \in W_J$, $\bar{w}x \to \bar{w}x'$ if and only if $\delta' x \to \delta' x'$ (in $\tilde{W}'$). Similar results hold for $\sim$ and $\approx$. 
3.2. We prove Theorem 3.2 (1). We argue by induction on \( \sharp W \). The statement holds if \( W \) is trivial. Now we assume that the statement holds for any \((W', S', \delta')\) with \( \sharp W' < \sharp W \).

Any element in the conjugacy class of \( \bar{w} \) is of the form \( \bar{w}_A' \) for some Weyl chamber \( A' \). Set \( K = V_{\bar{w}} \). By Proposition 1.2, \( \bar{w}_A' \rightarrow \bar{w}_A \) for some Weyl chamber \( A \) such that \( \bar{A} \) contains a regular element of \( K \).

Set \( J = I(K, A) \). By Proposition 2.2, \( \bar{w}_A = \bar{w}_{K,A}u \), where \( u \in W_J \), \( \bar{w}_{K,A} \in J\bar{W}^J \) with \( \bar{w}_{K,A}(J) = J \) and \( \ell(\bar{w}_{K,A}) = \frac{\delta}{\sharp}(\bar{S}_1 - \bar{S}_K) \).

Let \( \delta_1 \) be the automorphism on \( W_J \) defined by the conjugation of \( \bar{w}_{K,A} \). Set \( \bar{W}_1 = \langle \delta_1 \rangle \times W_J \). Since \( K \) is not contained in \( V_{\bar{w}}^W \), there exists \( H \in \bar{S}_J \) such that \( K \not\subseteq H \). Thus \( W_J \not\subseteq W \). Now by induction hypothesis on \( \bar{W}_1 \), there exists \( u' \in W_J \) such that \( \delta_1 u' \) is a minimal length element in its conjugacy class in \( \bar{W}_1 \) and \( \delta_1 u' \). Then \( \bar{w}_{K,A}u \rightarrow \bar{w}_{K,A}u' \).

Let \( U \) be the connected component of \( V - \cup_{H \in \bar{S}_K} H \) that contains \( A \). Let \( x \in W_J \) with \( \delta_1 u' = x^{-1} \delta_1 u x \). Set \( B = x_A x x_A^{-1} \) and \( U' = x_A x x_A^{-1}(U) \). Then \( \bar{w}_B = x^{-1} \bar{w}_A x = \bar{w}_{K,A}u' \). Since \( \delta_1 u' \) is a minimal length element in its conjugacy class in \( \bar{W}_1 \), \( \ell(U') \) is minimal among all the connected component of \( V - \cup_{H \in \bar{S}_K} H \). Hence by Proposition 2.2 and \( \S 2.2 \), \( \bar{w}_B = \bar{w}_{K,A}u' \) is a minimal length element in the conjugacy class of \( \bar{w} \). Part (1) of Theorem 3.2 is proved.

To prove Theorem 3.2 (2), we need the following result.

**Lemma 3.3.** Assume that Part (2) of Theorem 3.2 holds for \((W', S', \delta')\) with \( \sharp W' < \sharp W \). Let \( \bar{w} \in \bar{W} \) and \( K \subset V_{\bar{w}} \) be a nonzero subspace with \( \bar{w}(K) = K \). Let \( A, A' \) be two chambers whose closures contain a common regular point \( x \) of \( K \). Assume further that \( \bar{w}_A \) and \( \bar{w}_A' \) are of minimal length in their conjugacy class of \( \bar{W} \). Then \( \bar{w}_A \sim \bar{w}_A' \).

**Proof.** Set \( J = I(K, A) \). By Proposition 2.2, \( \bar{w}_A = \bar{w}_{K,A}u \), where \( u \in W_J \), \( \bar{w}_{K,A} \in J\bar{W}^J \) with \( \bar{w}_{K,A}(J) = J \). We define \( \delta_1, \bar{W}_1 \) as in \( \S 3.2 \). Let \( \ell_1 \) be the length function on \( \bar{W}_1 \). Let \( x \in W_K \) with \( x(A) = A' \). Set \( y = x_A^{-1} x x_A \). Then \( y \in W_J \) and \( \bar{w}_{A'} = y^{-1} \bar{w}_{K,A} uy \). Since \( \ell(\bar{w}_{A'}) = \ell(\bar{w}_A), \ell_1(y^{-1} \delta_1 uy) = \ell_1(\delta_1 u) \). Hence by induction hypothesis on \( \bar{W}_1 \), \( y^{-1} \delta_1 uy \sim \delta_1 u \). By \( \S 3.1 \), \( \bar{w}_A' \sim \bar{w}_A \). \( \square \)

3.3. Now we prove Theorem 3.2 (2). As in \( \S 3.2 \), we assume that the statement holds for any \((W', S', \delta')\) with \( \sharp W' < \sharp W \). Set \( K = V_{\bar{w}} \). Let \( \bar{w}_A, \bar{w}_A' \in \bar{w}_{\text{min}} \). By Proposition 1.2, it suffices to consider the case where \( \bar{A} \) and \( \bar{A}' \) both contain regular elements of \( K \). Let \( U \) (resp. \( U' \)) be the connected component of \( V - \cup_{H \in \bar{S}_K} H \) that contains \( A \) (resp. \( A' \)). Then by \( \S 2.2 \), \( \ell(U) = \ell(U') \) are minimal among all the connected component of \( V - \cup_{H \in \bar{S}_K} H \).
We define \( \delta_1, \tilde{W}_1 \) as in § 3.2. Let \( x \in W_K \) with \( x(U) = U' \). Set \( y = x_{A,x(A)} \). Then \( \tilde{w}_{x(A)} = y^{-1} \tilde{w}_A y \) and

\[
\ell(\tilde{w}_{x(A)}) = \ell(U') + \frac{\theta}{\pi} \sharp(\bar{\mathcal{S}} - \mathcal{S}_K) = \ell(U) + \frac{\theta}{\pi} \sharp(\bar{\mathcal{S}} - \mathcal{S}_K) = \ell(\tilde{w}_A).
\]

Hence \( y^{-1} \delta_1 uy \) is a minimal length element in the conjugacy class of \( \tilde{W}_1 \) that contains \( \delta_1 u \). By induction hypothesis on \( \tilde{W}_1 \), \( y^{-1} \delta_1 uy \sim \delta_1 u \).

Hence \( \tilde{w}_{x(A)} = y^{-1} \tilde{w}_{K,A} uy \sim \tilde{w}_{K,A} u = \tilde{w}_A \).

Thus to prove part (2), it suffices to prove that \( \tilde{A} \) and \( \tilde{A}' \) are in the same connected component of \( V - \cup_{H \in \bar{\mathcal{S}}_K} H \) and \( \tilde{w}_A, \tilde{w}_A' \in \mathcal{O}_{\text{min}} \), then \( \tilde{w}_A \sim \tilde{w}_{A+i} \) for any \( i \).

By definition, \( \tilde{A}_i \cap \tilde{A}_{i+1} \) spans \( H_0 \cap \tilde{K} \) for some \( H_0 \in \tilde{\mathcal{S}} - \tilde{\mathcal{S}}_K \). Set \( P = H_0 \cap \tilde{K} \). Then \( \tilde{A}_i \) and \( \tilde{A}_{i+1} \) contains a common regular element \( v \) of \( P \).

Case 1: \( \tilde{w}(P) \neq P \). There is a sequence of chambers \( A_i = B_0, \ldots, B_t = A_{i+1} \) in the same component of \( V - \cup_{H \in \bar{\mathcal{S}}_K} H \) such that for any \( j \), \( v \in B_j, B_j \) and \( B_{j+1} \) share a common wall. By Proposition 2.3, \( \ell(\tilde{w}_{B_0}) = \ell(\tilde{w}_{B_1}) = \cdots = \ell(\tilde{w}_{B_t}) \). Since \( B_j \) and \( B_{j+1} \) are strongly connected, \( \tilde{w}_{B_j} \approx \tilde{w}_{B_{j+1}} \). In particular, \( \tilde{w}_{A_i} \approx \tilde{w}_{A_{i+1}} \).

Case 2: \( P = \tilde{w}(P) \) and \( \dim(K) \geq 2 \). Then \( \dim(P) \geq 1 \) is a nonzero subspace of \( V' \). Apply Lemma 3.3 for \( P \), we obtain that \( w_A \sim \tilde{w}_{A_i} \).

Case 3: \( \dim(K) = 1 \). Then \( P = \{0\} \). By §1.6, \( \theta_0 = 0 \) or \( \pi \). If \( \theta_0 = \pi \), then \( \tilde{w} \) acts as \( -\text{id} \) on \( (V^W)^{\perp} \), hence \( \tilde{w}_{A_i} = \tilde{w}_{A_{i+1}} \) acts as \( -\text{id} \) on \( (V^W)^{\perp} \). Now assume that \( \theta_0 = 0 \). Let \( v \) be a regular element of \( K \) with \( v \in \tilde{A}_i \). Then \( -v \in \tilde{A}_{i+1} \). Since \( \tilde{w}(v) = v \), then

\[
\mathcal{S}(A_i, \tilde{w}(A_i)) = \mathcal{S}(A_{i+1}, \tilde{w}(A_{i+1})) \subset \tilde{\mathcal{S}}_K.
\]

So

\[
\mathcal{S}(A_i, \tilde{w}(A_{i+1})) - \mathcal{S}(A_i, \tilde{w}(A_{i+1}))_K = \mathcal{S}(A_i, A_{i+1}) - \mathcal{S}(A_i, A_{i+1})_K = \mathcal{S}(A_i, A_{i+1}).
\]

Let \( x \in W \) with \( x(A_{i+1}) = A_i \). By §2.1, \( \tilde{w}_{A_i} x_{A_i,A_{i+1}} = x_{A_i}^{-1} \tilde{w}_{A_i} x_{A_i} \) and

\[
\ell(\tilde{w}_{A_i} x_{A_i,A_{i+1}}) = \sharp(\mathcal{S}(C, \tilde{w}_{A_i} x_{A_i,A_{i+1}}(C))) = \sharp(\mathcal{S}(x_{A_i}(C), \tilde{w}_{A_i} x_{A_i}(C)))
\]

\[
= \sharp(A_i, \tilde{w}(A_{i+1})) = \sharp(\mathcal{S}(A_i, \tilde{w}(A_{i+1})))_K = \sharp(\mathcal{S}(A_i, A_{i+1}))_K
\]

\[
= \ell(\tilde{w}_{A_i}) + \ell(x_{A_i,A_{i+1}}).
\]

Hence \( \tilde{w}_{A_i} \sim \tilde{w}_{A_{i+1}} \).

4. Elliptic Conjugacy Class

4.1. We call a conjugacy class $\mathcal{C}$ of $\tilde{W}$ (or an element of it) elliptic if for some (or equivalently, any) element $\tilde{w} \in \mathcal{C}$, points in $V$ fixed by $\tilde{w}$ are contained in $V^W$. By [H1, Lemma 7.2], $\mathcal{C}$ is elliptic if and only if $\mathcal{C} \cap (\langle \delta \rangle \ltimes W_J) = \emptyset$ for any proper subset $J$ of $S$ with $\delta(J) = J$. In particular, the definition of elliptic conjugacy class/element is independent of the choice of $V$.

We’ve shown in the previous section that any two minimal length element in a conjugacy class of $\tilde{W}$ are strongly conjugate. In this section, we’ll obtained a stronger result for elliptic conjugacy classes.

Let $\tilde{w} \in \tilde{W}$ be a minimal length element in its conjugacy class. Let $P_{\tilde{w}}$ be the set of sequences $i = (i_1, \ldots, i_t)$ of $S$ such that

$$\tilde{w} \xrightarrow{i_1} s_{i_1} \tilde{w} s_{i_1} \xrightarrow{i_2} \cdots \xrightarrow{i_t} s_{i_t} \tilde{w} s_{i_t} \cdots s_{i_t}.$$

Since $\tilde{w}$ is a minimal element element, all the elements above are of the same length. We call such $i$ a path from $\tilde{w}$ to $s_{i_t} \cdots s_{i_1} \tilde{w} s_{i_t} \cdots s_{i_1}$. Let $P_{\tilde{w}, \tilde{w}}$ be the subset of $P$ consisting of all paths from $\tilde{w}$ to itself.

Let $W_{\tilde{w}} = \{ w \in W : \ell(w^{-1}\tilde{w}w) = \ell(\tilde{w}) \}$ and $Z_{\tilde{w}}(w) = \{ w \in W : w\tilde{w} = \tilde{w}w \} \subset W_{\tilde{w}}$. Then we have a natural map

$$\tau_{\tilde{w}} : P_{\tilde{w}} \to W_{\tilde{w}}, \quad (i_1, \ldots, i_t) \mapsto s_{i_t} \cdots s_{i_1}.$$

Let $C_{\tilde{w}}$ be the set of all Weyl chambers $A$ with $\ell(\tilde{w}A) = \ell(\tilde{w})$. Then the map $A \mapsto x_A$ gives a bijection between $C_{\tilde{w}}$ and $W_{\tilde{w}}$.

We call an element $v \in V$ subregular if it is either regular in $V$ or regular in $H$ for some $H \in \mathfrak{H}$. Let $V^\text{subreg} \subset V$ be the set of all subregular element. Then $V - V^\text{subreg}$ is a finite union of codimension 2 subspaces.

Lemma 4.1. Let $A$, $A'$ be Weyl chambers in $C_{\tilde{w}}$. Then there is a path from $w_A$ to $w_{A'}$ if and only if $A$ and $A'$ are in the same connected component of $(\cup_{A \in C_{\tilde{w}}} A) \cap V^\text{subreg}$.

Proof. If $A$ and $A'$ are in the same connected component of $(\cup_{A \in C_{\tilde{w}}} A) \cap V^\text{subreg}$, then there is a sequence of Weyl chambers $A = A_0, A_1, \ldots, A_r = A'$ in $C_{\tilde{w}}$ such that $A_i$ and $A_{i+1}$ are strongly connected. Let $H_i$ be the common wall of $A_i$ and $A_{i+1}$. Then $x_{A_i}^{-1}s_{H_i}x_{A_i} = s_i$ for some $i \in S$ and

$$\tilde{w}_A \xrightarrow{i_0} \tilde{w}_A s_{i_1} \xrightarrow{i_2} \cdots \xrightarrow{i_{r-1}} \tilde{w}_{A'}.$$

Therefore $(i_0, \ldots, i_{r-1}) \in P_{\tilde{w}}$.

On the other hand, any path $(i_0, \ldots, i_{r-1})$ from $\tilde{w}_A$ to $\tilde{w}_{A'}$ gives a sequence $A = A_0, A_1, \ldots, A_r = A'$ in $C_{\tilde{w}}$ such that $A_i$ and $A_{i+1}$ are strongly connected. Hence $A$ and $A'$ are in the same connected component of $(\cup_{A \in C_{\tilde{w}}} A) \cap V^\text{subreg}$. \[ \square \]

Our main result in this section is
Theorem 4.2. Let $\mathcal{O}$ be an elliptic conjugacy class of $\tilde{W}$ and $\tilde{w} \in \mathcal{O}_{\min}$. Then the map $\tau_{\tilde{w}} : \mathcal{P}_{\tilde{w}} \to W_{\tilde{w}}$ is surjective.

Proof. We argue by induction on $\mathfrak{z}W$. The statement holds if $W$ is trivial. Now assume that the statement holds for $(W', S', \delta')$ with $\mathfrak{z}W' < \mathfrak{z}W$.

Set $K = V_{\tilde{w}}$ and $Z = (\cup_{A \in C_{\tilde{w}}} A) \cap V_{\text{subreg}}$. By Lemma 4.1, it suffice to show that $Z$ is connected.

Let $A \in C_{\tilde{w}}$. Then by the proof of Proposition 1.2, there exists a Weyl Chamber $A'$ such that $A'$ contains a regular element of $K$ and there is a curve in $Z$ connecting $A$ and $A'$. Now it suffices to show that for any $A, B \in C_{\tilde{w}}$ such that $A$ and $B$ contain regular element of $K$, $A$ and $B$ are in the same connected component of $Z$.

Let $U$ be the connected component of $V - \cup_{H \in \delta_K} H$ that contains $A$. Let $x \in W_K$ with $x(B) \subset U$. Let $J = I(K, B)$ and $\delta_1$ be the automorphism on $W_J$ defined by the conjugation of $\tilde{w}_{K, B}$. Set $\tilde{W}_1 = \langle \delta_1 \rangle \ltimes W_J$ and $V_1 = \sum_{H \in \delta, \delta K \in W_J} H^\perp$. The action of $W$ on $V$ induces an injection $W_J \to GL(V_1)$. Also $\delta_1(V_1) = V_1$. Hence we may regard $\tilde{W}_1$ as a reflection subgroup of $V_1$. By Proposition 2.2, $\tilde{w}_B = \tilde{w}_{K, B}u$ for some $u \in W_J$. Let $v \in V_1$ with $\tilde{w}_B(v) = v$. Then $v \in V_1 \cap V^W \subset V_1^{W_J}$. Thus $\delta_1 u$ is elliptic in $\tilde{W}_1$.

By §2.2, $\ell(\tilde{w}_{x(B)}) = \ell(\tilde{w}_B)$. Hence by §3.1, $\delta_1 u$ and $x^{-1}_{B, x(B)} \delta_1 u x_{B, x(B)}$ are both of minimal length in their conjugacy class in $\tilde{W}_1$. Thus by induction hypothesis on $\tilde{W}_1$, $\delta_1 u \approx x^{-1}_{B, x(B)} \delta_1 u x_{B, x(B)}$. Hence by §3.1, $\tilde{w}_B \approx \tilde{w}_{x(B)}$. Hence by Lemma 4.1, $B$ and $x(B)$ are in the same connected component of $Z$.

Now $A$ and $x(B)$ are in the same connected component $U$ of $V - \cup_{H \in \delta_K} H$. By §3.3, there exists a sequence of chambers $A = A_0, \cdots, A_r = x(B)$ in $C_{\tilde{w}}$ such that for any $i$, $A_i$ and $A_{i+1}$ are strongly connected with respect to $K$. By definition, $\tilde{A}_i \cap \tilde{A}_{i+1} \cap K$ spans $H_0 \cap K$ for some $H_0 \in \delta_1 - \delta_1 K$. If $\theta_0 = \pi$, then $\tilde{w}_{A_i} = \tilde{w}_{A_{i+1}}$ acts as $-\text{id}$ on $(V^W)^\perp$. If $\theta_0 \neq \pi$, then any $\tilde{w}$-stable subspace of $K$ is even-dimensional and $\tilde{w}(H_0 \cap K) \neq H_0 \cap K$. Thus we are in case 1 of §3.3. Hence $\tilde{w}_{A_i} \approx \tilde{w}_{A_{i+1}}$. Therefore $\tilde{w}_A \approx \tilde{w}_{x(B)}$. By Lemma 4.1, $A$ and $x(B)$ are in the same connected component of $Z$.

Therefore $A$ and $B$ are in the same connected component of $Z$. □

The following results follows easily from Theorem 4.2. Both results are known but was proved by a case-by-case analysis.

Corollary 4.3. Let $\mathcal{O}$ be an elliptic conjugacy class of $\tilde{W}$. Let $\tilde{w}, \tilde{w}' \in \mathcal{O}_{\min}$. Then $\tilde{w} \approx \tilde{w}'$. 
Remark. This result was first proved by Geck and Pfeiffer in [GP2, 3.2.7] for $W$ and then by Geck-Kim-Pfeiffer [GKP] for twisted conjugacy classes in exceptional groups and by the first author [H1] in the remaining cases.

Corollary 4.4. Let $O$ be an elliptic conjugacy class of $\bar{W}$ and $\bar{w} \in O_{\min}$. Then $\tau_{\bar{w}} : \mathcal{P}_{\bar{w}, \bar{w}} \to Z_W(\bar{w})$ is surjective.

Remark. This was first conjectured by Lusztig in [L3, 1.2]. He also proved the case where $W$ is of classical type and $\delta$ is trivial in [L3]. The twisted conjugacy classes in a classical group were proved by him later (unpublished). The verification of exceptional groups was due to J. Michel.

5. Good Elements

5.1. Let $B^+$ be the braid monoid associated with $(W, S)$. Then there is a canonical injection $j : W \to B^+$ identifying the generators of $W$ with the generators of $B^+$ and $j(w_1w_2) = j(w_1)j(w_2)$ for $w_1, w_2 \in W$ if $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$.

Now the automorphism $\delta$ induces an automorphism of $B^+$, which is still denoted by $\delta$. Set $\bar{B}^+ = \langle \bar{\delta} \rangle \rtimes B^+$. Then $j$ extends in a canonical way to an injection $\bar{W} \to \bar{B}^+$, which we still denote by $j$. We will simply write $\bar{w}$ for $j(\bar{w})$.

Following [GM], we call $\bar{w} \in \bar{W}$ a good element if there exists a strictly decreasing sequence $S_0 \supset S_1 \supset \cdots \supset S_l$ of subsets of $S$ and even positive integers $d_0, \ldots, d_l$ such that

$$(\bar{w})^d = w_0^{d_0} \cdots w_l^{d_l}.$$ 

Here $d$ is the order of $\bar{w}$ and $w_i$ is the maximal element of the parabolic subgroup of $W$ generated by $S_i$.

Moreover, if $d$ is even, we call $\bar{w} \in \bar{W}$ very good if

$$(\bar{w})^{\frac{d}{2}} = \gamma w_0^{\frac{d}{2}} \cdots w_l^{\frac{d}{2}}$$

for some $\gamma \in \langle \bar{\delta} \rangle$.

5.2. Let $\bar{w} \in \bar{W}$. Let $\underline{\theta} = (\theta_1, \theta_2, \cdots, \theta_r)$ be a sequence of elements in $\Gamma_{\bar{w}}$ with $\theta_1 < \theta_2 < \cdots < \theta_r$. We set $F_i = \sum_{j=1}^{r} V^\theta_j$ for $0 \leq i \leq r$. We say that $\underline{\theta}$ is admissible if $F_r$ contains a regular point of $V$. Then we have a filtration

$$0 = F_0 \subset \cdots \subset F_r \subset V.$$ 

Set $W_i = W_{F_i}$. Then $W = W_0 \supset W_1 \supset \cdots \supset W_r = \{1\}$. There exists $0 = i_0 < i_1 < i_2 < \cdots < i_k \leq r$ such that for $0 \leq j < k$, $W_{i_j} = W_{i_{j+1}} = \cdots = W_{i_{j+1}-1} \neq W_{i_{j+1}}$. We then write $r(\underline{\theta}) = (\theta_{i_1}, \theta_{i_2}, \cdots, \theta_{i_k})$ and call it the irredundant sequence associated to $\underline{\theta}$.

For $0 \leq i \leq r$, let $C_i$ be the connected component of $V - \cup_{H \in S_{F_i}} H$ containing $A$. We say that a Weyl chamber $A \subset V$ is in good position
with respect to \((\tilde{w}, \theta)\) if for any \(i\), \(\tilde{C}_i\) contains some regular point of \(F_{i+1}\). It is easy to see that such \(A\) always exists. Moreover, \(A\) is in good position with respect to \((\tilde{w}, \theta)\) if and only if the fundamental chamber \(C\) is in good position with respect to \((\tilde{w}_A, \theta)\).

Let \(\theta_0\) be the sequence consisting of all the elements in \(\Gamma_{\tilde{w}}\). We say that a Weyl chamber \(A \subset V\) is in good position with respect to \(\tilde{w}\) if it is in good position with respect to \((\tilde{w}, \theta_0)\).

**Lemma 5.1.** Let \(\tilde{w} \in \tilde{W}\) and \(0 \leq \theta \leq \pi\). If \(C\) and \(\tilde{w}(C)\) are in the same connected component of \(V - \bigcup_{H \in \mathcal{Y}_{\tilde{w}}} H\) and \(C\) contains a regular point of \(V^\theta_{\tilde{w}}\), then for any \(d \in \mathbb{N}\) with \(d\theta/2\pi \in \mathbb{N}\), we have that
\[
\tilde{w}^d = \sigma(w_1 w_0 w_0 w_0)^{d\theta/2\pi}.
\]
Here \(w_1\) is the maximal element in \(W_{V^\theta_{\tilde{w}}}\) and \(\sigma \in \langle \delta \rangle\) with \(\sigma(w_1) = w_1\).

If moreover, \(d\) is even and \(d\theta/2\pi\) is an odd number, then
\[
\tilde{w}^{d/2} = \sigma' w_0 w_1 (w_1 w_0 w_0 w_0)^{(d\theta/2\pi-1)/2}.
\]
Here \(\sigma' \in \langle \delta \rangle\) with \(\sigma'(w_0 w_1) = w_1 w_0\).

**Proof.** We simply write \(K\) for \(V^\theta_{\tilde{w}}\) and \(J\) for \(I(K, C)\). Assume that \(\tilde{w} \in \tau W\) for \(\tau \in \langle \delta \rangle\). Let \(v \in C\) be a regular point of \(K\). Assume \(\theta = \frac{2q}{p} \pi\) with integers \(p, q\) coprime and \(0 \leq 2p \leq q\). Choose \(s, t \in \mathbb{Z}\) such that \(sp - 1 = tq\). Then \(\tilde{w}^{sq} = \tilde{w}^t q \tilde{w}\). Since \(\tilde{w}^q(v) = v\) and \(\tilde{w}^q\) fixes the connected component of \(V - \bigcup_{H \in \mathcal{Y}_K} H\) containing \(C\), we have that \(\tilde{w}^q(C) = C\). Therefore \(\tilde{w}^q = \tau^q\). Moreover \(\tau^q(w_1) = w_1\).

Set \(x = \tilde{w}^s\). Then \(x\) acts on \(K\) by rotating \(\frac{4s}{q} \pi\) and \(K \subset V_x^{\frac{2\pi}{q}}\). Also \(x(K) = K\). Now by Lemma 2.1, \(\ell(x^k) = \frac{2k}{q} \ell(\mathcal{S} - \mathcal{S}_K)\) for any \(k \in \mathbb{N}\) with \(2k \leq q\).

If \(2 \nmid q\), then \(2 \nmid p\) and \(\ell(x^{q/2}) = \frac{q}{2}(\mathcal{S} - \mathcal{S}_K)\). Also \(x \in \tilde{W}^J\) with \(x(J) = J\). Thus \(x^{q/2} = w_1 w_0 \tau^{sq/2} = \tau^{sq/2} w_0 w_1\). Hence \(\tau^{sq/2}(w_0 w_1) = w_1 w_0\). Notice that \(\tilde{w} = x^p \tau^{tq} \text{ and } \tau^{tq}(x) = x\). Therefore
\[
\tilde{w}^{q/2} = \tau^{tq} x^{q/2} \tau^{-sq/2} (x^{q/2})^p = \tau^{tq/2} \tau^{sq/2} w_0 w_1 (w_1 w_0 w_0 w_1)^{(p-1)/2}.
\]
Since \(\tau^q(w_1) = w_1\) and \(\tau^{sq/2}(w_0 w_1) = w_1 w_0\), we have that \(\tau^{sq/2}(w_0 w_1) = w_1 w_0\) and \(\tilde{w}^q = \tau^q(w_1 w_0 w_0 w_1)^p\).

If \(2 \mid q\), then we set \(k = \frac{q-1}{2}\). Then \(x^k \in \tilde{W}^J\) and \(\ell(x^k) = \frac{2k}{q} \ell(\mathcal{S} - \mathcal{S}_K) = \ell(w_0 w_1) - \frac{\ell(w_0 w_1)}{2} \ell(\mathcal{S} - \mathcal{S}_K)\). We have that \(\tau^{-sk} x^k \in W^J\) and \(\tau^{-sk} x^k w_1 = y^{-1} w_0\) for some \(y \in W\) with \(\ell(y^{-1} w_0 w_1) = \ell(w_0 w_1) - \ell(y)\) and \(\ell(y) = \frac{1}{q} \ell(\mathcal{S} - \mathcal{S}_K)\).

Similarly, \(x^k = w_1 w_0 (y')^{-1} \tau^{sk}\) for some \(y' \in W\) with \(\ell(w_1 w_0 (y')^{-1}) = \ell(w_1 w_0) - \ell(y')\) and \(\ell(y') = \frac{1}{q} \ell(\mathcal{S} - \mathcal{S}_K)\).
Since \( x^q = (\tilde{w})^q = \tau^q \), we have that
\[
x = x^q x^{-k} x^{-k} = \tau^q (w_1 w_0 (y')^{-1} \tau^{sk})^{-1} (\tau^{sk} y^{-1} w_0 w_1)^{-1}
= \tau^{s(q-k)} y' y^{-sk}.
\]
Since \( \ell(x) = 2 \frac{d}{d} (\mathfrak{f} - \tilde{\mathfrak{T}}_K) = \ell(y) + \ell(y') \), we have that
\[
x = \tau^{s(q-k)} y' y^{-sk}.
\]
Moreover,
\[
\tau^q = x^k x^{-k} = x^k (\tau^{s(q-k)} y' y^{-sk} (\tau^{sk} y^{-1} w_0 w_1)) = x^k \tau^{s(q-k)} y' w_0 w_1
= x^k \tau^q (x^{-k}).
\]
Hence
\[
\tilde{w}^q = (\tau^{-tq} x^p)^q = \tau^{-tq} (\tau^q)^p = \tau^{-tq} \tau^{s(p)} (w_1 w_0 w_0 w_1)^p = \tau^q (w_1 w_0 w_0 w_1)^p.
\]

Now we prove the existence of good and very good elements.

**Theorem 5.2.** Let \( \tilde{w} \in \tilde{W} \) and \( \theta \) be an admissible sequence with \( r(\theta) = (\theta_1, \ldots, \theta_k) \). If the fundamental chamber \( C \) is in good position with respect to \( (\tilde{w}, \tilde{\theta}) \), then
\[
\tilde{w}^d = \sigma \tilde{w}_0^{d\theta_1/\pi} \tilde{w}_1^{d\theta_2-\theta_1/\pi} \cdots \tilde{w}_k^{d\theta_k-\theta_{k-1}/\pi},
\]
here \( d \in \mathbb{N} \) with \( d\theta_j/2\pi \in \mathbb{Z} \) for all \( j \), \( w_j \) is the maximal element in \( W_i \), and \( \sigma \in \langle \delta \rangle \).

If moreover, \( d \) is even, then
\[
\tilde{w}^{d/2} = \sigma' \tilde{w}_0^{d\theta_1/2\pi} \tilde{w}_1^{d\theta_2-\theta_1/2\pi} \cdots \tilde{w}_k^{d\theta_k-\theta_{k-1}/2\pi}
\]
for some \( \sigma' \in \langle \delta \rangle \).

**Proof.** We argue by induction on \( \sharp W \). Assume that the statement holds for any \( (W', S', \delta') \) with \( \sharp W' < \sharp W \).

We assume that \( \theta \) is irredundant by replacing \( \theta \) by \( r(\theta) \) if necessary. By assumption, \( \tilde{C} \) contains a regular point of \( F_1 \). Hence by Proposition 2.2, \( \tilde{w} = \tilde{w}' u \), where \( u \in W_{F_1} \), \( \tilde{w}' \in I(F_1, C) \tilde{w}^{-1} I(F_1, C) \) with \( \tilde{w}' (I(F_1, C)) = I(F_1, C) \).

Set \( V_1 = F_1^\perp, W_1 = W_{F_1} \) and \( \tilde{W}_1 = \langle \delta_1 \rangle \rtimes W_1 \), where \( \delta_1 \) is the automorphism on \( W_1 \) defined by the conjugation of \( \tilde{w}' \). Then we may naturally regard \( \tilde{W}_1 \) as a reflection subgroup of \( GL(V_1) \). Set \( C' = C_1 \cap V_1 \). Then \( C' \subset V_1 \) is the fundamental Weyl chamber of \( W_1 \). Since \( C \) is in good position with respect to \( \tilde{w} \), \( C' \) is in good position with respect to \( \delta_1 u \in \tilde{W}_1 \).
By induction hypothesis on \( \tilde{W}_1 \),
\[
(\delta_1 u)^d = (\delta_1)^d \tilde{w}_1^{d(\theta_2)/\pi} \cdots \tilde{w}_{k-1}^{d(\theta_{k-1})/\pi}
\]
in \( \langle \delta_1 \rangle \rtimes B_1^+ \), here \( B_1^+ \) is the Braid monoid associated with \( W_1 \). By Lemma 5.1,
\[
\tilde{w}^d = (\tilde{w}')^d \tilde{w}_1^{d\theta_2/\pi} \cdots \tilde{w}_{k-1}^{d(\theta_{k-1})/\pi}
\]
\[
= \sigma(w_1 w_0 w_1)^{d\theta_1/\pi} \tilde{w}_1^{d\theta_2/\pi} \cdots \tilde{w}_{k-1}^{d(\theta_{k-1})/\pi}.
\]

Since \( \tilde{w}_2^2 \) commutes with \( \tilde{w}_1 \), we have that \( (w_1 w_0 w_1)^2 \tilde{w}_2^2 = \tilde{w}_2^2 \).
Hence \( \tilde{w}^d = \sigma w_0^{d\theta_1/\pi} \tilde{w}_1^{d(\theta_2-\theta_1)/\pi} \cdots \tilde{w}_{k-1}^{d(\theta_{k-1})/\pi} \).

The “moreover” part can be proved in the same way. \( \square \)

5.3. It was proved in [GM], [GKP] and [H1] that for any conjugacy class of \( \tilde{W} \), there exists a good minimal length element. Below we give a case-free proof. We’ll also see that it provides a practical way to construct good minimal length element.

**Proposition 5.3.** Let \( \tilde{w} \in \tilde{W} \) and \( A \) be a Weyl chamber. If \( A \) is in good position with respect to \( \tilde{w} \), then \( \tilde{w}_A \) is a good element and is of minimal length in its conjugacy class.

**Proof.** We argue by induction on \( \sharp W \). The statement is obvious if \( W \) is trivial. Now assume that the statement holds for any \((W', S', \delta')\) with \( \sharp W' < \sharp W \).

The fundamental alcove \( C \) is in good position with respect to \( \tilde{w}_A \). Hence by Theorem 5.2, \( \tilde{w}_A \) is good. Set \( F = V^W + V^{\tilde{w}} \). By definition, \( \bar{C} \) contains some regular point of \( V^{\tilde{w}} \). By §2.1, \( I(F, C) = I(V^{\tilde{w}}, C) \).

By Proposition 2.2, \( \tilde{w}_A = \tilde{w}' u \), where \( u \in W_F \), \( \tilde{w}' \in I(F, C) \tilde{W} I(F, C) \) with \( \tilde{w}'(I(F, C)) = I(F, C) \). Set \( V_1 = F_{\delta_1} \), \( W_1 = W_F \) and \( \tilde{W}_1 = \langle \delta_1 \rangle \rtimes W_1 \),
where \( \delta_1 \) is the automorphism on \( W_1 \) defined by the conjugation of \( \tilde{w}' \). The fundamental chamber \( C_1 \cap V_1 \) of \( W_1 \) is in good position with respect to \( \delta_1 u \in \tilde{W}_1 \).

By induction hypothesis on \( \tilde{W}_1 \), \( \delta_1 u \) is of minimal length in its conjugacy class in \( \bar{W}_1 \). Hence \( \tilde{w}_A = \tilde{w}' u \) is of minimal length in its conjugacy class in \( \tilde{W} \). \( \square \)

5.4. Let \( w_0 \) be the maximal element in \( W \). Then \( w_0^2 \) is a central element in \( \tilde{B}^+ \). Now we discuss some good element \( \tilde{w} \) such that \( \tilde{w}^d \in w_0^2 B^+ \), where \( d \) is the order of \( \tilde{w} \).

We’ve shown in the above proposition that for any elliptic conjugacy class in \( \tilde{W} \), there exists a good minimal length element \( \tilde{w} \) such that \( \tilde{w}^d \in w_0^2 B^+ \), where \( d \) is the order of \( \tilde{w} \).

Another example is the conjugacy class of \( d \)-regular element. We call an element \( \tilde{w} \in \tilde{W} \) \textit{d-regular} if it has a regular \( \xi \)-eigenvector, here \( \xi \) is a root of unity of order \( d \). By [S, 4.10] and [BM, Proposition 3.11],
if $\mathcal{O}$ is a conjugacy class of $\tilde{W}$ contains $d$-regular elements, then there exists $\tilde{w} \in \mathcal{O}$ such that $\tilde{w}^d = w^2_0$.

5.5. We call a conjugacy class $\mathcal{O}$ of $\tilde{W}$ quasi-elliptic if for some (or equivalently, any) $\tilde{w} \in \mathcal{O}$, $(V^{\tilde{w}})^\perp$ contains a regular point of $V$. Here $V^{\tilde{w}}$ is the set of points fixed by $\tilde{w}$. Then an elliptic conjugacy class is quasi-elliptic. Also a conjugacy class of $d$-regular elements is also quasi-elliptic.

Now we have that

**Corollary 5.4.** Let $\mathcal{O}$ be a quasi-elliptic conjugacy class of $\tilde{W}$. Then there exists $\tilde{w} \in \mathcal{O}$ such that $\tilde{w}$ is good and $\tilde{w}^d \in w^2_0 B^+$, here $d$ is the order of $\tilde{w}$.

**Proof.** Let $\tilde{w} \in \mathcal{O}$ and $\tilde{\theta}$ be the sequence consisting of all nonzero elements in $\Gamma_{\tilde{w}}$. Since $\mathcal{O}$ is quasi-elliptic, $\tilde{\theta}$ is admissible. Let $A$ be a Weyl chamber in good position with respect to $(\tilde{w}, \tilde{\theta})$. Then $C$ is in good position with respect to $(\tilde{w}_A, \tilde{\theta})$. By Theorem 5.2, $\tilde{w}_A$ is good and $\tilde{w}^d_A \in w_{d\theta_1/\pi} B^+$. Here $\theta_1$ is the minimal element in $\tilde{\theta}$. Since $d\theta_1/2\pi \in \mathbb{Z}$ and $\theta_1 > 0$, we have that $d\theta_1/\pi \geq 2$. Hence $\tilde{w}^d_A \in w^2_0 B^+$. \qed

**Acknowledgement**

We are grateful to G. Lusztig for helpful discussion on the centralizer of minimal length element and useful comments on this paper. We thank M. Rapoport for useful comments on Deligne-Lusztig varieties and representation of finite groups of Lie types and thank O. Dudas for helpful discussion on $d$-regular elements.

**References**

[B] N. Bourbaki, *Éléments de mathématique. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines*, Actualités Scientifiques et Industrielles, No. 1337, Hermann, Paris, 1968.

[BM] M. Bröne and J. Michel, *Sur certains éléments réguliers des groupes de Weyl et les variétés de Deligne-Lusztig associées*, Finite reductive groups (Luminy, 1994), Progr. Math., vol. 141, Birkhäuser Boston, Boston, MA, 1997, pp. 73–139.

[BR] C. Bonnafé and R. Rouquier, *Affineness of Deligne-Lusztig varieties for minimal length elements*, J. Algebra 320 (2008), no. 3, 1200–1206.

[GM] M. Geck and J. Michel, *“Good” elements in finite Coxeter groups and representations of Iwahori-Hecke algebras*, Proc. London. Math. Soc. (3) 74 (1997), 275–305.

[GKP] M. Geck, S. Kim, and G. Pfeiffer, *Minimal length elements in twisted conjugacy classes of finite Coxeter groups*, J. Algebra 229 (2000), no. 2, 570–600.

[GP1] M. Geck and G. Pfeiffer, *On the irreducible characters of Hecke algebras*, Adv. Math. 102 (1993), no. 1, 79–94.
[GP2] ______. Characters of finite Coxeter groups and Iwahori-Hecke algebras, London Mathematical Society Monographs. New Series, vol. 21, The Clarendon Press Oxford University Press, New York, 2000.

[H1] X. He, Minimal length elements in some double cosets of Coxeter groups, Adv. Math. 215 (2007), no. 2, 469–503.

[H2] ______. On the affineness of Deligne-Lusztig varieties, J. Algebra 320 (2008), no. 3, 1207–1219.

[HL] X. He and G. Lusztig, A generalization of Steinberg’s cross-section, arxiv:1103.1769.

[L1] G. Lusztig, Characters of reductive groups over a finite field, Annals of Mathematics Studies, 107. Princeton University Press, Princeton, NJ, 1984.

[L2] ______, Rationality properties of unipotent representations, J. Algebra 258 (2002), no. 1, 1–22, Special issue in celebration of Claudio Procesi’s 60th birthday.

[L3] ______. On certain varieties attached to a Weyl group element, arxiv:1012.2074.

[OR] S. Orlik and M. Rapoport, Deligne-Lusztig varieties and period domains over finite fields, J. Algebra 320 (2008), no. 3, 1220–1234.

[R] M. Rapoport, private communication.

[S] T. A. Springer, Regular elements of finite reflection groups, Invent. Math. 25 (1974), 159–198.

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY, CLEAR WATER BAY, KOWLOON, HONG KONG
E-mail address: maxhhe@ust.hk

INSTITUTE OF MATHEMATICS, CHINESE ACADEMY OF SCIENCES, BEIJING, 100190, CHINA
E-mail address: niesian@gmail.com