SETTING HIDDEN SYMMETRIES FREE BY THE NONCOMMUTATIVE VERONESE MAPPING

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Abstract. The note is devoted to the setting free of hidden symmetries in Verma modules over \( \mathfrak{sl}(2, \mathbb{C}) \) by the noncommutative Veronese mappings.

In many cases the behavior of systems is governed not only by their natural (geometric) symmetries but also by hidden ones. The main difficulty to work with hidden symmetries is that they are often "packed", and as a rule can't be "unpacked" to the universal enveloping algebras of Lie algebras, so there exists a problem how to set them free "correctly" (see e.g. [1-3]). That means to find a "correct" algebraic structure, which is represented by them. This is one of the themes of this short note. It maybe considered as preliminary to the second one, which is related to a view of the noncommutative geometry [1] on the setting hidden symmetries free. The most interesting setting free mappings are noncommutative Veronese mappings (certain analogs of classical ones [4]); on such way the problem of the noncommutative "birational" equivalence of the differently obtained set free hidden symmetries is appeared (cf. [5]). Both marked subjects are interacted in the paper on the simplest examples of hidden symmetries in Verma modules over \( \mathfrak{sl}(2, \mathbb{C}) \).

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Definition.

A. Let \( \mathfrak{g} \) be a Lie algebra and \( \mathcal{A} \) be an associative algebra such that \( \mathfrak{g} \subset \text{Der}(\mathcal{A}) \); a linear subspace \( V \) of \( \mathcal{A} \) is called a space of hidden symmetries iff (1) \( V \) is a \( \mathfrak{g} \)-submodule of \( \mathcal{A} \), (2) the Weyl symmetrization defines a surjection \( W : S(V) \hookrightarrow \mathcal{A} \) (the elements of \( V \) are called hidden symmetries with respect to \( \mathfrak{g} \)). An associative
algebra $\mathcal{F}$ such that $\mathfrak{g} \subset \text{Der}(\mathcal{F})$ is called an algebra of the set free hidden symmetries iff (1) $\mathcal{F}$ is generated by $V$, (2) there exists a $\mathfrak{g}$–equivariant epimorphism of algebras $\mathcal{F} \twoheadrightarrow \mathcal{A}$, (3) the Weyl symmetrization defines an isomorphism $S(V) \cong \mathcal{F}$ of $\mathfrak{g}$–modules.

B. Let $V$ be a space of hidden symmetries and $\mathcal{C}$ is some class of associative Ore algebras; the hidden symmetries from $V$ are called $\mathcal{C}$–regular iff the algebras of quotients $D(\mathcal{F})$ are isomorphic for all corresponding to $V$ algebras $\mathcal{F}$ of the set free hidden symmetries from the class $\mathcal{C}$.

C. Let $V$ be a space of hidden symmetries in algebra $\mathcal{A}$ with respect to the Lie algebra $\mathfrak{g}$; a subspace $V_0$ of $V$ is called a coordinate base of $V$ iff (1) $V_0$ is a $\mathfrak{g}$–submodule of $V$, (2) the image of the Weyl symmetrization mapping $W_0 : S(V_0) \rightarrow \mathcal{A}$ contains $V$. Hidden symmetries from $V$ are called of type $(V_0, n)$ iff the image of $\bigoplus_{i \leq n} S^i(V_0)$ under the Weyl symmetrization mapping $W_0$ coincides with $V$; in this case the mapping $\bigoplus_{i \leq n} S^i(V_0) \rightarrow \mathcal{F}$, a composition of $W_0$ and the imbedding of $V$ into $\mathcal{F}$, is called the noncommutative Veronese mapping.

D. Let $\mathfrak{g}$ be a Lie algebra, $V$ be a certain $\mathfrak{g}$–module, $\mathcal{A}_s$ be a family of associative algebras, parametrized by $s \in \mathcal{S}$ such that $\mathfrak{g} \subset \text{Der}(\mathcal{A}_s)$, $\pi_s : V \rightarrow \mathcal{A}_s$ be a family of $\mathfrak{g}$–equivariant imbeddings such that $\pi_s(V)$ is a space of hidden symmetries in $\mathcal{A}_s$ with respect to $\mathfrak{g}$ for a generic $s$ from $\mathcal{S}$. An associative algebra $\mathcal{F}$ is called an algebra of the $\mathcal{A}_{s, s \in \mathcal{S}}$–universally set free hidden symmetries iff $\mathcal{F}$ is an algebra of the set free hidden symmetries corresponding to $V \cong \pi_s(V)$ for generic $\mathcal{A}_s$ ($s \in \mathcal{S}$).

The hidden symmetries are called the $\mathcal{A}_{s, s \in \mathcal{S}}$–universally $\mathcal{C}$–regular iff the algebras of quotients $D(\mathcal{F})$ are isomorphic for all algebras $\mathcal{F}$ of the $\mathcal{A}_{s, s \in \mathcal{S}}$–universally set free hidden symmetries from the class $\mathcal{C}$.

If $\mathfrak{g}$ is a Lie algebra and $\mathcal{A}$ is an associative algebra such that $\mathfrak{g} \subset \text{Der}(\mathcal{A})$, $V_0$ is a $\mathfrak{g}$–submodule of $\mathcal{A}$, which elements generate $\mathcal{A}$ as an algebra then in many interesting cases there exists a space of hidden symmetries $V$ of type $(V_0, n)$ in $\mathcal{A}$ for a sufficiently large $n$.

**Theorem.**

A. The tensor operators of type $\pi_1$ and $\pi_2$ in the Verma module $V_h$ over the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ ($\pi_i$ is a finite–dimensional representation of $\mathfrak{sl}(2, \mathbb{C})$ of dimension $2i + 1$) form a space of hidden symmetries of type $(\pi_1, 2)$; the quadratic (non–homogeneous) algebras of the $\text{End}(V_h)$–universally set free hidden symmetries form an one–parametric family $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha)$, where $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); 0)$ is the Racah–Wigner algebra $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}))$ of par.2.2. of ref [6].

B. All the algebras of quotients $D(\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha))$ are isomorphic (hence, the tensor operators of type $\pi_i$ ($i = 1, 2$) form a $\text{End}(V_h)$–universally quadratic–regular scope of hidden symmetries).

C. The central extension $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}))$ of $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}))$ maybe continued to the central extensions $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha)$ of $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha)$ in the class of quadratic (non–homogeneous) algebras.

**Comments on Th.1A,1C.** The statements 1A and 1C are verified by explicit calculations. Here we present the constructions of algebras $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha)$ and $\mathcal{R}\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha)$.

Let $L_i$ be a basis in $\mathfrak{sl}(2, \mathbb{C})$ such that $[L_i, L_j] = (i - j)L_{i+j}$ and $d^k_j (-k \leq j \leq k)$ be basises in $\pi_k$, in which the $\mathfrak{sl}(2, \mathbb{C})$–action has the form $L_i(d^k_j) = (ki - j)d^k_{i+j}$. The corresponding tensor operators in the Verma modules $V_h$ will be denoted by $\hat{T}_i(h)$. These are the elements of quadratic (non–homogeneous) algebra.
the capitals. If the Verma module $V_{\xi}$ is realized in the space $\mathbb{C}[z]$ of polynomials of a complex variable $z$, where the generators of $\mathfrak{sl}(2, \mathbb{C})$ act as $L_{-1} = z$, $L_0 = z\partial_z + h$ and $L_1 = z(\partial_z)^2 + 2h\partial_z$, then the tensor operators $D^k_i$ ($k = 1, 2, 3$) are defined by the formulas

$$D^1_{-1} = z$$
$$D^1_0 = \xi + h$$
$$D^1_1 = (\xi + 2h)\partial_z$$

$$D^2_{-2} = z^2$$
$$D^2_{-1} = z(\xi + h + \frac{1}{2})$$
$$D^2_0 = \xi^2 + 2h\xi + \frac{h(2h+1)}{3}$$
$$D^2_1 = (\xi + 2h)(\xi + h + \frac{1}{2})\partial_z$$
$$D^2_2 = (\xi + 2h)(\xi + 2h + 1)\partial_z^2$$

where $\xi = z\partial_z$.

Algebra $\mathcal{RW}(\mathfrak{sl}(2, \mathbb{C}); \alpha)$ is generated by $l_i$ ($i = -1, 0, 1$) and $w_i$ ($i = -2, -1, 0, 1, 2$); the action of $\mathfrak{sl}(2, \mathbb{C})$ has the form $L_i(l_j) = (i - j)l_{i+j}$, $L_i(w_j) = (2i - j)w_{i+j}$; the commutation relations are following

$$[l_{-1}, l_0] = -l_{-1} + \alpha(l_1 \circ w_{-2} - 2l_0 \circ w_{-1} + l_{-1} \circ w_0)$$
$$[l_{-1}, l_1] = -2l_0 + 2\alpha(l_1 \circ w_{-1} - 2l_0 \circ w_0 + l_{-1} \circ w_1)$$
$$[l_0, l_1] = -l_1 + \alpha(l_1 \circ w_0 - 2l_0 \circ w_1 + l_{-1} \circ w_2)$$

$$[l_{-1}, w_{-1}] = -w_{-2} + 4\alpha l_{-1}^2 + 4\alpha(w_{-2} \circ w_0 - w_{-1}^2)$$
$$[l_{-1}, w_0] = -2w_{-1} + 8\alpha l_{-1} \circ l_0 + 4\alpha(w_{-2} \circ w_1 - w_{-1} \circ w_0)$$
$$[l_{-1}, w_1] = -3w_0 + 4\alpha(l_{-1} \circ l_1 + 2l_{-1}^2) + 2\alpha(w_{-2} \circ w_2 + 2w_{-1} \circ w_1 - 3w_0^2)$$
$$[l_{-1}, w_2] = -4w_1 + 16\alpha l_0 \circ l_1 + 8\alpha(w_{-1} \circ w_2 - w_0 \circ w_1)$$
$$[l_0, w_{-2}] = 2w_{-2} - 8\alpha l_{-1}^2 - 8\alpha(w_{-2} \circ w_0 - w_{-1}^2)$$
$$[l_0, w_{-1}] = w_{-1} - 4\alpha l_{-1} \circ l_0 - 2\alpha(w_{-2} \circ w_1 - w_{-1} \circ w_0)$$
$$[l_0, w_0] = 0$$
$$[l_0, w_1] = -w_1 + 4\alpha l_0 \circ l_1 + 2\alpha(w_{-1} \circ w_2 - w_0 \circ w_1)$$
$$[l_0, w_2] = -2w_2 + 8\alpha l_1^2 + 8\alpha(w_0 \circ w_2 - w_1^2)$$
$$[l_1, w_{-2}] = 4w_{-1} - 16\alpha l_{-1} \circ l_0 - 8\alpha(w_{-2} \circ w_1 - w_{-1} \circ w_0)$$
$$[l_1, w_{-1}] = 3w_0 - 4\alpha(l_{-1} \circ l_1 + 2l_{-1}^2) - 2\alpha(w_{-2} \circ w_2 + 2w_{-1} \circ w_1 - 3w_0^2)$$
$$[l_1, w_0] = 2w_1 - 8\alpha l_0 \circ l_1 - 4\alpha(w_{-1} \circ w_2 - w_0 \circ w_1)$$
$$[l_1, w_1] = w_2 - 4\alpha l_1^2 - 4\alpha(w_0 \circ w_2 - w_1^2)$$

$$[w_{-2}, w_{-1}] = -2l_{-1} \circ w_{-2}$$
$$[w_{-1}, w_0] = -4(2l_{-1} \circ w_{-2} + l_0 \circ w_{-1})$$
\[ [w_{-2}, w_1] = -3l_{-1} \circ w_0 - 2l_0 \circ w_{-1} - l_1 \circ w_{-2} \]
\[ [w_{-2}, w_2] = -4(l_{-1} \circ w_1 + l_1 \circ w_{-1}) \]
\[ [w_{-1}, w_0] = -\frac{1}{6}(3l_{-1} \circ w_0 + 10l_0 \circ w_{-1} - l_1 \circ w_{-2}) \]
\[ [w_{-1}, w_1] = -\frac{1}{2}(l_{-1} \circ w_1 + 6l_0 \circ w_0 + l_1 \circ w_{-1}) \]
\[ [w_{-1}, w_2] = -l_{-1} \circ w_2 - 2l_0 \circ w_1 - 3l_1 \circ w_0 \]
\[ [w_0, w_1] = \frac{1}{6}(l_{-1} \circ w_2 - 10l_0 \circ w_1 - 3l_1 \circ w_0) \]
\[ [w_0, w_2] = \frac{1}{3}(2l_1 \circ w_1 + l_0 \circ w_2) \]
\[ [w_1, w_2] = -2l_1 \circ w_2 \]

where \( a \circ b = \frac{ab + ba}{2} \).

The algebra \( \widetilde{RW}(\mathfrak{sl}(2, \mathbb{C}); \alpha) \) admits a representation by tensor operators in the Verma module \( V_h \) over \( \mathfrak{sl}(2, \mathbb{C}) \) by \( l_i \mapsto \delta^{-1} D_i^1, w_i \mapsto \delta^{-1} D_i^2 \) \((\delta = 1 - \frac{(2h+1)(2h+3)}{3} \alpha)\).

The algebra \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha) \) is generated by \( l_i (i = -1, 0, 1) \), \( w_i (i = -2, -1, 0, 1, 2) \) and the central element \( \rho \). The commutation relations coincide with ones for \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha) \) up to subsidiary terms for \([w_i, w_j]\). Namely, the improved commutators have the form

\[ [w_{-2}, w_{-1}] = -2l_{-1} \circ w_{-2} \]
\[ [w_{-2}, w_0] = -\frac{4}{3}(2l_{-1} \circ w_{-1} + l_0 \circ w_{-2}) \]
\[ [w_{-2}, w_1] = -3l_{-1} \circ w_0 - 2l_0 \circ w_{-1} - l_1 \circ w_{-2} - \rho l_{-1} \]
\[ [w_{-2}, w_2] = -4(l_{-1} \circ w_1 + l_1 \circ w_{-1} + \rho l_0) \]
\[ [w_{-1}, w_0] = -\frac{1}{6}(3l_{-1} \circ w_0 + 10l_0 \circ w_{-1} - l_1 \circ w_{-2} + 3\rho l_{-1}) \]
\[ [w_{-1}, w_1] = -\frac{1}{2}(l_{-1} \circ w_1 + 6l_0 \circ w_0 + l_1 \circ w_{-1} + \rho l_0) \]
\[ [w_{-1}, w_2] = -l_{-1} \circ w_2 - 2l_0 \circ w_1 - 3l_1 \circ w_0 - \rho l_1 \]
\[ [w_0, w_1] = \frac{1}{6}(l_{-1} \circ w_2 - 10l_0 \circ w_1 - 3l_1 \circ w_0) \]
\[ [w_0, w_2] = -\frac{4}{3}(2l_1 \circ w_1 + l_0 \circ w_2) \]
\[ [w_1, w_2] = -2l_1 \circ w_2 \]

**Sketch of the proof of Th.1B.** The statement 1B is proved by an explicit construction of isomorphism in 4 steps. 1st step: operators \( L_i \) are represented in the form \( L_i = \text{ad}(1_i) = \text{ad}(l_i) + \sum_{j \geq 1} \alpha^j \text{ad}(X_{i,j}) \), \( X_{i,j} \in \mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha) \). 2nd step: there exists the unique operators \( w_i = w_i + \sum_{j \geq 1} \alpha^j Y_{i,j}, Y_{i,j} \in \mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha) \) such that \([1_i, w_j] = (2i - j)w_{i+j}\). 3rd step: \( 1_i \) and \( w_j \) obey the commutation relations of \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C})) \). By these three steps we proved that the deformation \( \{\mathcal{W}(\mathfrak{sl}(2, \mathbb{C}); \alpha) : \alpha \in \mathbb{C}\} \) of \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C})) \) is formally trivial. 4th step: formal series for \( 1_i \) and \( w_j \) are rational.

**Remark.** \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C})) \) is a Hopf algebra \([1, 7]\) with the co–commutative comultiplication \( l_i \mapsto l_i \otimes 1 + 1 \otimes l_i, w_i \mapsto w_i \otimes 1 + 1 \otimes w_i + c_j^k l_j \otimes l_k \), where \( c = \{c_i^j\} \in [\pi_2 \otimes S^2(\pi_1)]^{\mathfrak{sl}(2, \mathbb{C})} \) and the antipode \( 1 \mapsto 1, l_i \mapsto -l_i, w_j \mapsto w_j \). The structure of Hopf algebra on \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C})) \) is a deformation of such structure on \( \mathcal{U}(\mathfrak{sl}(3, \mathbb{C})) \) (with non–standard antipode corresponding to the Cartan anti–automorphism). This deformation is realized by the fixing of structures of Hopf algebras on \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C}))/\rho = \rho_0 \) \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C}))/\rho = \rho_0 \rightarrow m \rightarrow \mathcal{U}(\mathfrak{sl}(3, \mathbb{C})) \) par.2.2 of ref [6]. Unfortunately, I do not know a way to make a Hopf algebra from \( \mathcal{W}(\mathfrak{sl}(2, \mathbb{C}))/\rho \).
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