The Casimir effect in a wormhole spacetime

Artem R Khabibullin\textsuperscript{1}, Nail R Khusnutdinov\textsuperscript{1} and Sergey V Sushkov\textsuperscript{2}

\textsuperscript{1}Department of Physics, Kazan State Pedagogical University, Mezhlauk 1, Kazan 420021, Russia
\textsuperscript{2}Department of Mathematics, Kazan State Pedagogical University, Mezhlauk 1, Kazan 420021, Russia

E-mail: arty@theory.kazan-spu.ru, nail@theory.kazan-spu.ru and sushkov@kspu.kcn.ru

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Abstract

We consider the Casimir effect for a quantized massive scalar field with non-conformal coupling $\xi$ in a wormhole spacetime whose throat is surrounded by a spherical shell. In the framework of the zeta-regularization approach we calculate the zero-point energy of the scalar field. We find that depending on values of the coupling $\xi$, a mass of field $m$ and/or the throat’s radius $a$, the Casimir force may be both attractive and repulsive, and even equals zero.

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1. Introduction

The central problem of wormhole physics consists of the fact that wormholes are accompanied by unavoidable violations of the null energy condition, i.e. the matter threading the wormhole’s throat has to possess ‘exotic’ properties. The classical matter does satisfy the usual energy conditions, hence wormholes cannot arise as solutions of classical relativity and matter. If they exist, they must belong to the realm of semiclassical or perhaps quantum gravity. In the absence of a complete theory of quantum gravity, the semiclassical approach begins to play the most important role for examining wormholes. Recently, self-consistent wormholes in the semiclassical gravity were studied numerically in [13, 18, 20, 23]. It was shown that the semiclassical Einstein equations allow the existence of wormholes supported by energy of vacuum fluctuations. However, it should be stressed that the natural size of semiclassical vacuum wormholes (say, the radius of a wormhole’s throat $a$) should be of Planckian scale or less. This fact can be easily argued by simple dimensional considerations [12]. In order to obtain semiclassical wormholes with scales greater than Planckian, one has to consider either non-vacuum states of quantized fields (say, thermal states with a temperature $T > 0$) or vacuum polarization (the Casimir effect) which may happen due to some external boundaries (with a typical scale $R$) existing in the wormhole spacetime. In both cases an additional dimensional...
macroscopical parameter (say R) appears, which may result in the enlargement of the wormhole’s size.

In this paper we will study the Casimir effect in a wormhole spacetime. For this aim we will consider a static spherically symmetric wormhole joining two different universes (asymptotically flat regions). We will also suppose that each universe contains a perfectly conducting spherical shell surrounding the throat. These shells will dictate the Dirichlet boundary conditions for a physical field and, as a result, produce a vacuum polarization. Note that this problem is closely related to the known problem investigated by Boyer [8], who studied the Casimir effect of a perfectly conducting sphere in a Minkowski spacetime (see also [4]). However, there is an essential difference which is expressed in different topologies of wormhole and Minkowski spacetimes. A semitransparent sphere as well as semitransparent boundary condition was investigated in [5, 7, 15, 24–26]. The consideration of the delta-like potential which models a semitransparent boundary condition in quantum field theory causes some problems and there is ambiguity in the renormalization procedure (see [7, 15, 24] and references therein). Thermal corrections to the one-loop effective action on a singular potential background were considered recently in [22].

We will adopt a simple geometrical model of wormhole spacetime: the short-throat flat-space wormhole which was suggested and exploited in [20]. The model represents two identical copies of Minkowski spacetime; from each copy a spherical region is excised, and then boundaries of those regions are identified. The spacetime of the model is everywhere flat except the throat, i.e. a two-dimensional singular spherical surface. We will assume that the wormhole’s throat is rounded by two perfectly conducting spherical shells (in each copy of Minkowski spacetime) and calculate the zero-point energy of a massive scalar field on this background. At the end of the calculations, the radius of one sphere will tend to infinity giving the Casimir energy for a single sphere. For the calculations we will use the zeta-function regularization approach [10, 11] developed in [2–4, 6, 19]. In the framework of this approach, the ground-state energy of the scalar field \( \phi \) is given by

\[
E(s) = \frac{1}{2} \mu^2 \zeta_L \left( s - \frac{1}{2} \right),
\]

where

\[
\zeta_L(s) = \sum_{(n)} \left( \lambda_n^2 + m^2 \right)^{-s}
\]

is the zeta function of the corresponding Laplace operator. The parameter \( \mu \), having the dimension of mass, makes right the dimension of regularized energy. The \( \lambda_n^2 \) are eigenvalues of the three-dimensional Laplace operator \( L = \Delta - \xi R \), where \( R \) is the curvature scalar (which is singular in our model; see equation (3)). For more details of this approach, see [6].

The organization of the paper is as follows. In section 2, we briefly describe a wormhole spacetime in the short-throat flat-space approximation, analyse a solution to the equation of motion for the massive scalar field and obtain a close expression for zero-point energy. In section 3, we discuss the results obtained and make some speculations.

We use units \( \hbar = c = G = 1 \). The signature of the spacetime, and the sign of the Riemann and Ricci tensors, are the same as in the book by Hawking and Ellis [16].

2. Zero-point energy

To begin, let us briefly discuss the geometry of model. We will take the metric of the static spherically symmetric wormhole in a simple form

\[
d\mathbf{s}^2 = -dt^2 + d\rho^2 + r^2(\rho)(d\theta^2 + \sin^2 \theta \, d\phi^2),
\]
where $\rho$ is the proper radial distance, $\rho \in (-\infty, \infty)$. The function $r(\rho)$ describes the profile of the throat. In this paper we adopt the model suggested in [20], which was called there the short-throat flat-space approximation. In this model, the shape function $r(\rho)$ is

$$r(\rho) = |\rho| + a,$$

with $a > 0$. $r(\rho)$ is always positive and has a minimum at $\rho = 0$: $r(0) = a$, where $a$ is the radius of throat. It is easy to see that in two regions $D_+: \rho > 0$ and $D_-: \rho < 0$ one can introduce new radial coordinates $r_+ = \pm \rho + a$, respectively, and rewrite metric (2) in the usual spherical coordinates

$$ds^2 = -dt^2 + dr_+^2 + r_+^2(d\theta^2 + \sin^2 \theta \, d\phi^2).$$

This form of the metric explicitly indicates that the regions $D_+$ and $D_-$ are flat. However, the spacetime is curved at the wormhole throat with the following singular curvature

$$R = -8 \delta(\rho) \frac{a}{a}.$$

Let us now consider a scalar field $\phi$ in the spacetime with metric (2). The equation for the eigenvalues of operator $L$ is

$$\left(\Delta - \xi R\right)\phi(n) = \lambda^2(n)\phi(n),$$

where $R$ is the scalar curvature, $\xi$ is an arbitrary coupling with $R$ and $\Delta = g^{\alpha\beta} \nabla_\alpha \nabla_\beta, \alpha = 1, 2, 3$. Due to the spherical symmetry of spacetime (2) we consider only the equation for the radial function $u(\rho)$

$$u'' + 2 \frac{r'}{r} u' + \left(\lambda^2 - \frac{l(l + 1)}{r^2} - \xi R\right) u = 0,$$

where a prime denotes the derivative with respect to $\rho$ and $\lambda = \sqrt{\omega^2 - m^2}$. This equation looks like the Schrödinger equation for a massive particle with mass $M$ with total energy $E = \lambda^2/2M$ and potential energy

$$U = \left(\xi R + \frac{r'}{r}\right) \int 2M = \frac{1 - 4\xi}{aM} \delta(\rho).$$

Therefore, $\xi > 1/4$ corresponds to negative potential.

Unfortunately, in our case it is impossible to find in manifest form the spectrum of the operator $L$ given by equation (4). For this reason, we will use an approach developed in [2–4, 6, 19]. This approach does not need an explicit form of spectrum. The spectrum of an operator is usually found from some boundary conditions which look like an equation $\Psi_1(\lambda) = 0$, where the function $\Psi_1$ is constructed from the solutions of equation (5) and depends additionally on the other parameters of problem. It was shown in [2–4, 6, 19] that the zero-point energy may be represented in the following form:

$$E(s) = -\mu^2 \cos(\pi s) \frac{1}{2\pi} \sum_{(m)} d_n \int_m^\infty dk (k^2 - m^2)^{1/2-s} \frac{\partial}{\partial k} \ln \Psi(ik),$$

with the function $\Psi$ taken on the imaginary axes. The sum is taken over all numbers of the problem and $d_n$ is degenerate of state. This formula also takes into account the possible boundary states. If they exist we have to include them additively at the beginning in equation (1). But integration of the over interval $|k| < m$ (the possible boundary states exist in this domain) will cancel this contribution. For this reason the integration in formula (7) is started from the energy $k = m$. Therefore, hereafter we will consider the solution of equation (5) for

3 For the spherical symmetry case, $(n) = l$ and $d_n = 2l + 1 = 2v$. 
negative energy that is in imaginary axes \( \lambda = ik \). The main problem is now reduced to finding the function \( \Psi \). Thus, now we need no explicit form of the spectrum of the operator \( \mathcal{L} \).

The general solutions of equation (5) were obtained in [20] in terms of the Bessel functions of second type. In contrast, we consider a more general situation. We round the wormhole throat by a sphere of radius \( a + R \) (\( \rho = R \)) in region \( D_+ \), and by a sphere of different radius \( a + R' \) (\( \rho = -R' \)) in region \( D_- \), and impose the Dirichlet boundary condition on both of these spheres, which means that the spheres have perfect conductivity. Therefore, the space of the wormhole is divided by two spheres into three regions: the space of finite volume between the spheres and two infinite volume spaces out of the spheres. Taking into account these conditions we obtain the following relation for function \( \Psi \) which we need for calculation of the energy (7)

\[
\Psi_n = I_n[k(a + R')] \left( \Psi^* \left( \left( \xi - \frac{1}{8} \right) K_{\nu}[ka] + \frac{ka}{4} K'_{\nu}[ka] \right) - \frac{1}{8} K_{\nu}[k(a + R)] \right) - K_n[k(a + R')] \left( \Psi^* \left( \left( \xi - \frac{1}{8} \right) I_{\nu}[ka] + \frac{ka}{4} I'_{\nu}[ka] \right) - \frac{1}{8} I_{\nu}[k(a + R)] \right) = 0, \tag{8a}
\]

with

\[
\Psi^* = I_n[k(a + R)] K_{\nu}[ka] - K_n[k(a + R)] I_{\nu}[ka].
\]

Here \( \nu = l + 1/2 \) and \( I_\nu, K_\nu \) are the Bessel functions of second type. In the case \( R' = R \), the above expression coincides with that obtained in [20]. The solutions of equation (8a) give the spectrum of energies between the spheres \( R \) and \( R' \). The spectra for regions out of these spheres can be found as follows:

\[
\Psi^1_{\text{out}} = K_{\nu}[k(a + R)], \tag{8b}
\]

\[
\Psi^2_{\text{out}} = K_{\nu}[k(a + R')]. \tag{8c}
\]

As expected, this condition coincides with the expression for the space out of the sphere of radius \( a + R \) in Minkowski spacetime [4]. It is obviously because the spacetime out of the sphere (in general out of the throat) is exactly Minkowski spacetime.

Therefore, the regularized total energy (7) reads

\[
E(s) = -\mu^2 \cos(\pi s) \sum_{l=0}^{\infty} \nu \int_{m}^{\infty} dk (k^2 - m^2)^{1/2-s} \frac{\partial}{\partial k} \left[ \ln \Psi_{\text{in}} + \ln \Psi^1_{\text{out}} + \ln \Psi^2_{\text{out}} \right]. \tag{9}
\]

Re-grouping terms, we can rewrite the above formula in a form having clear physical sense of each term

\[
E(s) = \Delta E(s) + E^M_R(s) + E^M_R(s), \tag{10}
\]

where

\[
E^M_R(s) = -\mu^2 \cos(\pi s) \sum_{l=0}^{\infty} \nu \int_{m}^{\infty} dk (k^2 - m^2)^{1/2-s} \frac{\partial}{\partial k} \ln I_n[k(a + R)] K_{\nu}[k(a + R)], \tag{11}
\]

\[
E^M_R(s) = -\mu^2 \cos(\pi s) \sum_{l=0}^{\infty} \nu \int_{m}^{\infty} dk (k^2 - m^2)^{1/2-s} \frac{\partial}{\partial k} \ln I_n[k(a + R')] K_{\nu}[k(a + R')], \tag{12}
\]

\[
\Delta E(s) = -\mu^2 \cos(\pi s) \sum_{l=0}^{\infty} \nu \int_{m}^{\infty} dk (k^2 - m^2)^{1/2-s} \frac{\partial}{\partial k} \ln \Psi \tag{13}
\]
and

$$\Psi = \frac{\Psi_m}{I_0[k(a + R)]I_0[k(a + R)']}$$.

The term $E^M_R(s)$ in formula (10) is nothing but the zero-point energy of a sphere of radius $a + R$ in Minkowski spacetime with Dirichlet boundary condition on the sphere [4]; note that the term $E^M_R(s)$ has an analogous sense.

Now we are ready to calculate the Casimir energy for two spherical boundaries by using expression (9). Then let us consider Boyer’s problem. We consider a gedanken experiment: we take a single conducting sphere and measure the Casimir force in this situation. For this reason we have to take a limit $R' \to \infty$. In this case the energy (12) tends to zero, and so the term $\Delta E(s)$ in equation (13) represents the difference between the Casimir energies of a sphere rounding the wormhole and a sphere of the same radius in Minkowski spacetime without a wormhole. In the limit $R' \to \infty$, we find

$$\Psi = \left( K_0[ka] - I_0[ka] \frac{K_0[k(a + R)]}{I_0[k(a + R)]} \right) \left( \left( \xi - \frac{1}{8} \right) K_0[ka] + \frac{ka}{4} K_0'[ka] - \frac{1}{8} \frac{K_0[k(a + R)]}{I_0[k(a + R)]} \right)$$

(14)

If one turns $R \to \infty$, then the energy $E^M_R$ tends to zero and so

$$\Psi \to K_0[ka] \left( \left( \xi - \frac{1}{8} \right) K_0[ka] + \frac{ka}{4} K_0'[ka] \right)$$.

(15)

This expression coincides exactly with that obtained in [20] and describes the zero-point energy for the whole wormhole spacetime without any additional spherical shells.

A comment is in order. As already noted, the case $\xi > 1/4$ corresponds to the attractive potential and therefore the boundary states may appear (see equation (6)). The appearance of boundary states with delta-like potential has been observed in [21]. Thus, we have to take into account the boundary states at the beginning. Nevertheless, the final formula (9) contains these boundary states, as was noted in [3]. But it is necessary to note that in this paper we consider $\xi < 1/4$. As noted in [21] in the opposite case one cannot use the present theory. The same boundary for $\xi$ was noted in [20].

The general strategy of the subsequent calculations is the following (for more details see [2–4, 6, 19]). To single out in manifest form the divergent part of the regularized energy we subtract from and add to the integrand in equation (9) its uniform expansion over $1/\nu$. It is obvious that it is enough to subtract the expansion up to $1/\nu^2$, the next terms will give the convergent series. We may set $s = 0$ in the part from which we subtracted the uniform expansion because it is now finite (see equation (19)). The divergent singled out part will contain the standard divergent terms and some finite terms which we calculate in manifest form (all terms except $A$ in (16)).

The uniform asymptotic expansions, both (14) and (15), are the same for $R \neq 0$. Indeed, in this case the ratios

$$\frac{I_0[ka]}{K_0[ka]} \frac{K_0[k(a + R)]}{I_0[k(a + R)]} \approx e^{-2\nu \ln(1+\frac{R}{a})}$$

and

$$\frac{1}{K_0^2[ka]} \frac{K_0[k(a + R)]}{I_0[k(a + R)]} \approx 2\nu e^{-2\nu \ln(1+\frac{R}{a})}$$

are exponentially small and we may neglect them. The well-known uniform expansions of the Bessel functions [1] were used in these expressions. For this reason we may disregard this fraction in equation (14) and arrive at equation (15). This is a key observation for the next calculations. Due to this observation the divergent part which we have to subtract for renormalization from (13) has been already calculated in [20]. By using the results of this
Figure 1. The plots of renormalized zero-point energy $E_{\text{ren}}/m$ as a function of $x = R/a$ for $\beta = 0.04, 0.5$ and for various values of $\xi$ and fixed mass $m$. We observe that increasing $\xi$ leads to the appearance of a maximum and/or minimum. For subsequent increasing $\xi$, the curve will turn over and the extremum disappears. If the radius of the spherical shell exceeds ten times the radius of the throat, the zero-point energy takes on a value which equals the zero-point energy in the whole wormhole spacetime.

In this section we will discuss the results of the numerical calculations of zero-point energy given by formula (16). The renormalized zero-point energy is represented in figures 1 and 2 as a function of $x = R/a$ for various values of $\beta = ma$ and $\xi$. (Note that the value $x = R/a$ characterizes the position of the sphere surrounding the wormhole; $x = 0$ corresponds to where the sphere’s radius equals the throat’s radius.) In figure 1, we only show the full energy $E$. 

$$\Delta E = -\frac{1}{32\pi^2a}(b \ln \beta^2 + \Omega), \quad (16)$$

$$\Omega = A + \sum_{k=-1}^{3} \omega_k(\beta), \quad (17)$$

$$b = \frac{1}{2}b_0\beta^4 - b_1\beta^2 + b_2, \quad (18)$$

where

$$A = 32\pi \sum_{l=0}^{\infty} v^2 \int_{\beta/\nu}^{\infty} dy \sqrt{y^2 - \beta^2} \frac{\partial}{\partial y} \left( \ln \Psi + 2v\eta(y) + \frac{1}{\nu} N_1 - \frac{1}{\nu^2} N_2 + \frac{1}{\nu^3} N_3 \right), \quad (19)$$

$$\Psi = \left( K_v[vy] - I_v[vy] \frac{K_v[vy(1+x)]}{I_v[vy(1+x)]} \right) \left( \left( \xi - \frac{1}{8} \right) K_v[vy] + \frac{vy}{4} K_v'[vy] \right)$$

$$\frac{1}{8} I_v[vy(1+x)], \quad (20)$$

where $b_k$ are the heat kernel coefficients, $\beta = ma$ is a dimensionless parameter of mass, and $x = R/a$ is a dimensionless parameter of the sphere’s radius. The explicit form of the heat kernel coefficients $b_k$ and also expressions for $\omega_k, N_k, \eta$ are given in [20]. Note that they do not depend on the radius of sphere $R$. The only dependence on $R$ is contained in the coefficient $A$ which has to be calculated numerically. The expression for the contribution of the sphere in Minkowski spacetime (11) may be found in [4]. We only have to make a change $R \rightarrow a + R$. 

3. Discussion and conclusion

In this section we will discuss the results of the numerical calculations of zero-point energy given by formula (16). The renormalized zero-point energy is represented in figures 1 and 2 as a function of $x = R/a$ for various values of $\beta = ma$ and $\xi$. (Note that the value $x = R/a$ characterizes the position of the sphere surrounding the wormhole; $x = 0$ corresponds to where the sphere’s radius equals the throat’s radius.) In figure 1, we only show the full energy $E$. 

The plots of renormalized zero-point energy $E_{\text{ren}}/m$ as a function of $x = R/a$ for $\beta = 0.04, 0.5$ and for various values of $\xi$ and fixed mass $m$. We observe that increasing $\xi$ leads to the appearance of a maximum and/or minimum. For subsequent increasing $\xi$, the curve will turn over and the extremum disappears. If the radius of the spherical shell exceeds ten times the radius of the throat, the zero-point energy takes on a value which equals the zero-point energy in the whole wormhole spacetime. 

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Note that $\Delta E$ differs just slightly from the full energy $E$. For the same reason we reproduce in figure 2 the $\Delta E$ only.

Characterizing the result of the calculations we should first of all stress that the value of the zero-point energy $E_{\text{ren}}$ in the limit $R \to \infty$ tends to some constant value obtained in [20] for the case of a wormhole spacetime without any spherical shells. In the limit $R \to 0$ (i.e., when the sphere radius $a + R$ tends to the throat’s radius $a$) the zero-point energy $E_{\text{ren}}$ is infinitely decreasing for all $\beta$ and $\xi$. This means that the Casimir force acting on the spherical shell and corresponding to the Casimir zero-point energy $E_{\text{ren}}$ is ‘attractive,’ i.e. it is directed inward to the wormhole’s throat, for sufficiently small values of $R$. In the interval $0 < R/a < \infty$, there are three qualitatively different cases of behaviour of $E_{\text{ren}}$ depending on values of $\beta$ and $\xi$. Namely, (i) the zero-point energy $E_{\text{ren}}$ is monotonically increasing in the whole interval $0 < R/a < \infty$. There are neither maxima nor minima in this case. Hence the Casimir force is attractive for all positions of the spherical shell. (ii) $E_{\text{ren}}$ is first increasing and then decreasing. A graph of the zero-point energy has the form of a barrier with some maximal value of $E_{\text{ren}}$ at $R_1/a$. The Casimir force is attractive for the sphere’s radius $R < R_1$ and repulsive for $R > R_1$. The value $R = R_1$ corresponds to the point of unstable equilibrium. (iii) The zero-point energy $E_{\text{ren}}$ is increasing for $R/a < R_1/a$, decreasing for $R_1/a < R/a < R_2/a$ and then finally increasing for $R/a > R_2/a$, so that a graph of $E_{\text{ren}}$ has a maximum and minimum. In this case the Casimir force is directed outward provided the sphere’s radius $R_1 < R < R_2$, and inward provided $R < R_1$ or $R > R_2$. Now the value $R = R_2$ corresponds to the point of stable equilibrium, since the zero-point energy $E_{\text{ren}}$ has here a local minimum.

It is worth noting that the Casimir force is attractive in the whole interval $0 < R/a < \infty$ for sufficiently small values of $\xi$ and/or large values of $\beta$. Otherwise, it can be both attractive and repulsive depending on the radius of the sphere surrounding the wormhole’s throat. A similar situation appears for the delta-like potential on the spherical or on the cylindrical boundaries [25, 26]. The repulsive Casimir force was also observed in [17] for a scalar field in the Einstein static universe.

For the model considered, let us speculate on Casimir’s idea of an electron as a charged spherical shell [9]. Casimir assumed that such a configuration should be stable due to equilibrium between the repulsive Coulomb force and the attractive Casimir force. However, as is known, this idea does not work in Minkowski spacetime, since the Casimir force for a sphere turns out to be repulsive [8]. Now one can revive Casimir’s idea by considering a spherical shell surrounding the wormhole. In this paper we have shown that the Casimir force can now be both attractive and repulsive. Moreover, there exists stable configurations.
for which the Casimir force equals zero; the radius of the spherical shell in this case depends on the throat’s radius $a$ as well as the field’s mass $m$ and coupling constant $\xi$. Thus, one may try to realize Casimir’s idea by taking a sphere surrounding a wormhole. Of course, our consideration was based on the very simple model of a wormhole spacetime. However, we believe that main features of the above consideration remain the same for more realistic models.

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