Abstract

In the present work, we consider the existence and spectral stability of multi-pulse solitary wave solutions to a nonlinear Schrödinger equation with both fourth and second order dispersion terms. We first give a criterion for the existence of a single solitary wave solution in terms of the coefficients of the dispersion terms, and then show that a discrete family of multi-pulse solutions exists which is characterized by the distances between the individual pulses. We then reduce the spectral stability problem for these multi-pulses to computing the determinant of a matrix which is, to leading order, block diagonal. Under an additional assumption, which can be verified numerically, we show that all multi-pulses are spectrally unstable. For double pulses, numerical computations are presented which are in good agreement with our analytical results.

Keywords: nonlinear Schrödinger equation, solitary waves, multi-pulse solutions, nonlinear optics

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1. Introduction

It has been nearly 50 years since the discovery by Zakharov and Shabat [17] of the integrability of the nonlinear Schrödinger equation (NLS) and the corresponding soliton solutions, and 40 years since the first experimental demonstration by Mollenauer, Stolen and Gordon of optical solitons propagating in fibers [7]. At its most fundamental level, the NLS soliton represents the balance of chromatic second order dispersion and the Kerr self-focusing nonlinearity. The robustness of the soliton opened up new directions in both theoretical and experimental fronts that continue to this day. As better fibers were built, and technological advances led to the invention of photonic crystal fibers, that enabled the engineering of the dispersion resulting in new discoveries such as supercontinuum generation realized in regimes far from the NLS equation. In this vein, recent experimental work in silicon photonic crystal waveguides has produced for the first time what is now known as pure quartic solitons (PQS) [2]; the term reflects that for this waveguide the leading order dispersion term is fourth order. In [13], spectral stability of PQS was shown numerically, as well as evolution into PQS from Gaussian initial conditions, and this is extended to a more general model in [14]. Our results presented here provide a more rigorous study of this
general model, including for the first time the existence and spectral stability of multi-pulse solutions.

After a brief background, we first present results on the existence and stability of the primary soliton solution in the more general case where both second and fourth order dispersion are accounted for. This includes the particular case of PQS. Under a standard assumption, which can be verified numerically, the primary pulse solution is orbitally stable. The second part of the paper proves the existence of multi-pulses. These are solutions which resemble multiple, well-separated copies of the primary soliton; neighboring pulses in the multi-pulse can be either in phase or out of phase. We then look at the spectral stability of these multi-pulses. Under a mild assumption, which can be verified numerically, all of these pulse trains are unstable. Numerical examples are then presented, followed by a brief discussion of conclusions and directions for future work. The final section contains the proofs for the spectral stability results.

2. Background

The fourth-order generalization of the nonlinear Schrödinger equation (NLS)

\[ iu_t + \frac{\beta_4}{24} u_{xxxx} - \frac{\beta_2}{2} u_{xx} + \gamma |u|^2 u = 0 \]  

was recently investigated in [14] in a study of the properties of solitary wave solutions under a combination of second and fourth order dispersion. (We use the independent variables \((t, x)\) in place of \((z, \tau)\), which is used in [2, 13, 14] and is common in the optics literature). Ordinary NLS solitons are solutions when \(\beta_2 < 0\) and \(\beta_4 = 0\). Pure quartic solitons (PQS) occur when \(\beta_2 = 0\) and \(\beta_4 < 0\). In that case, \(u(x, t)\) satisfies the equation

\[ iu_t + \frac{\beta_4}{24} u_{xxxx} + \gamma |u|^2 u = 0. \]  

Unlike ordinary NLS solitons, PQS have oscillatory, exponentially decaying tails. There has been much recent interest in PQS due to their discovery in experimental media by Blanco-Redondo et al. in 2016 [2]. The existence and spectral stability of PQS solutions was shown numerically in [13], and the existence of solitary wave solutions to the more general equation (1) in terms of the parameters \(\beta_2, \beta_4,\) and \(\omega\) is discussed in [14].

Real-valued, standing wave solutions, i.e. solutions of the form \(e^{i\omega t} u(x)\), satisfy the ODE

\[ \frac{\beta_4}{24} u_{xxxx} - \frac{\beta_2}{2} u_{xx} + \gamma u^3 - \omega u = 0, \]  

which is a rescaling of [3, (7)]. For PQS, equation (3) can be written in parameter free form by using the rescaling

\[ u(x; \omega) = \sqrt[3]{\frac{\omega}{\gamma}} \tilde{u} \left( \frac{\omega}{|\beta_4|} \right)^{1/4} x \]

to obtain the equation

\[ -\frac{1}{24} \tilde{u}_{xxxx} + \tilde{u}^3 - \tilde{u} = 0. \]  

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We observe that the power or photon number of PQS scales as $\omega^{3/4}$ compared to the $\omega^{1/2}$ scaling of classical NLS solitons. A more general rescaling [14, Section VI] transforms equation (3) into the one-parameter equation

$$\tilde{u}_{xxxx} + 2\sigma \tilde{u}_{xx} + \tilde{u} - |\tilde{u}|^2 \tilde{u} = 0,$$

where

$$\sigma = \sqrt{\frac{3}{2\omega|\beta_4|}} \beta_2$$

is a non-dimensional parameter characterizing the relative strengths of the quadratic and quartic dispersion terms.

For ordinary NLS, an analytic solution can be obtained by the inverse scattering transform [17]. For $\beta_4 < 0$ and $\beta_2 < 0$, an analytic solution has been obtained by Karlsson and Höök [6] when $\omega = 24\beta_2^2/25|\beta_4|$.

3. Mathematical setup

Our analysis follows Grillakis, Shatah, and Strauss [4]. Equation (1) can be written in Hamiltonian form as

$$\frac{\partial u}{\partial t} = J\mathcal{E}'(u(t)),$$  \hspace{1cm} (6)

where $J = -i$ and the energy $\mathcal{E}$ is given by

$$\mathcal{E}(u) = \frac{1}{2} \int_{-\infty}^{\infty} \left( \frac{\beta_4}{24} |u_{xx}|^2 + \frac{\beta_2}{2} |u_x|^2 + \frac{\gamma}{2} |u|^4 \right) dx.$$  \hspace{1cm} (7)

The energy $\mathcal{E}$ is invariant under the complex rotation group $T(\theta)$, given by $T(\theta)u = e^{i\theta}u$. The corresponding conserved quantity, often called the charge [4, Section 6.C], is given by

$$\mathcal{Q}(u) = -\frac{1}{2} \int_{-\infty}^{\infty} |u|^2 dx.$$  \hspace{1cm} (8)

We make the following hypothesis regarding the well-posedness of (6), which is the same as [4, Assumption 1].

**Hypothesis 1.** For each initial condition $u_0$, there exists $T > 0$ depending only on $K$, where $\|u_0\| \leq K$, such that the PDE (6) has a solution $u(t)$ on $[0, T]$ with $u(0) = u_0$.

Standing waves are solutions of the form $T(\omega t)u$, where $u$ is independent of $t$. A standing wave solution satisfies the standing wave equation $\mathcal{E}'(u) - w\mathcal{Q}'(u) = 0$ [4, 2.15]. Since $\mathcal{Q}'(u) = -u$, this equation has the form $\mathcal{E}'(u) + wu = 0$, which can be written as

$$\frac{\beta_4}{24} u_{xxxx} - \frac{\beta_2}{2} u_{xx} + \gamma |u|^2 u - \omega u = 0.$$  \hspace{1cm} (9)

The following theorem gives criteria for the existence of real-valued solitary wave solutions to (9) in terms of the parameters $\beta_2$, $\beta_4$, and $\omega$. For the remainder of this paper, we will only consider $\beta_4 < 0$, since that is the physically relevant regime.
Theorem 1. Let $\beta_4 < 0$, and define

$$\omega_c = \frac{3 \beta_2^2}{2 |\beta_4|}.$$  \hfill (10)

Then for either (i) $\omega > \omega_c$, or (ii) $0 < \omega < \omega_c$ and $\beta_2 < 0$, there exists a real-valued, symmetric, exponentially localized solution $\phi(x; \omega) \in H^2(\mathbb{R}) \cap C^5(\mathbb{R})$ of the standing wave equation (9).

Proof. The existence result follows directly from [5], which uses the mountain pass lemma and the concentration-compactness principle. Exponential localization follows from the stable manifold theorem. \qed

Remark 1. For $\omega > \omega_c$, it follows from [5] that there exists a countably infinite family of distinct solitary-wave solutions, which are the multi-pulse solutions we will construct below. In addition, for $\beta_2 = 0$, $\omega_c = 0$, thus PQS exist for all $\omega > 0$.

We make the following standard smoothness assumption (see, for example, [4, Assumption 2]) concerning the solutions $\phi(x; \omega)$ to (3).

Hypothesis 2. The map $\omega \mapsto \phi(x; \omega)$ from $I$ to $H^2(\mathbb{R})$ is $C^1$, where $I$ is the interval for which the primary pulse solution $\phi(x; \omega)$ exists.

Define the scalar

$$d(\omega) = E(\phi(\omega)) - \omega Q(\phi(\omega)).$$  \hfill (11)

By [4, (2.21)],

$$d''(\omega) = \langle Q'(\phi(x; \omega)), \partial_\omega \phi(x; \omega) \rangle = \int_{-\infty}^{\infty} \phi(x; \omega) \partial_\omega \phi(x; \omega) dx,$$  \hfill (12)

where $\partial_\omega \phi(x; \omega)$ is well-defined by Hypothesis 2. By [4, Theorem 3.5], the standing wave $\phi(x; \omega)$ is orbitally stable if $d''(\omega) > 0$. This quantity can be computed numerically, and we take this stability criterion as a hypothesis.

Hypothesis 3. For each $\omega$ such that a primary pulse solution $\phi(x; \omega)$ exists, $d''(\omega) > 0$.

Let $\beta_4 < 0$ and $\beta_2 \in \mathbb{R}$, and choose $\omega > 0$ such that the primary pulse solution $\phi(x) = \phi(x; \omega)$ exists by Theorem 1. From this point forward, we will suppress the dependence on $\omega$ for simplicity of notation. The linearization of the PDE (1) about $\phi$ is the linear operator $L(\phi) : H^4(\mathbb{R}) \subset L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$, given by

$$L(\phi) = \begin{pmatrix} 0 & L^-(\phi) \\ -L^+(\phi) & 0 \end{pmatrix},$$  \hfill (13)

where

$$L^-(\phi) = -\frac{\beta_4}{24} \partial_{xxxx} + \frac{\beta_2}{2} \partial_{xx} + \omega - \gamma \phi^2$$

$$L^+(\phi) = -\frac{\beta_4}{24} \partial_{xxxx} + \frac{\beta_2}{2} \partial_{xx} + \omega + 3 \gamma \phi^2.$$
It is straightforward to verify that
\[
\begin{align*}
L^- (\phi) \phi &= 0 \\
L^+ (\phi) \partial_x \phi &= 0 \\
L^+ (\phi) (-\partial_\omega \phi) &= \phi. 
\end{align*}
\] (14)

Furthermore, since \( L^- (\phi) \) is self-adjoint and \( \phi' \perp \ker L^- (\phi) \), there exists a function \( z \) such that \( L^- (\phi) z = \phi' \). For the classical NLS equation \( (\beta_4 = 0, \beta_2 \neq 0) \), \( z = \frac{1}{2 \beta_2} x \phi \). Thus \( L(\phi) \) has a kernel with (at least) algebraic multiplicity 4 and geometric multiplicity 2, i.e.
\[
\begin{align*}
L(\phi) \begin{pmatrix} 0 \\ \phi \end{pmatrix} &= 0, \quad L(\phi) \begin{pmatrix} \partial_\omega \phi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \phi \end{pmatrix} \\
L(\phi) \begin{pmatrix} \partial_x \phi \\ 0 \end{pmatrix} &= 0, \quad L(\phi) \begin{pmatrix} 0 \\ z \end{pmatrix} = \begin{pmatrix} \partial_x \phi \\ 0 \end{pmatrix}.
\end{align*}
\] (15)

The spectrum of \( L(\phi) \) can be divided into two disjoint sets: the essential spectrum is the set of \( \lambda \in \mathbb{C} \) for which \( L(\phi) - \lambda I \) is not Fredholm, and the point spectrum is the set of \( \lambda \in \mathbb{C} \) for which \( \ker L(\phi) - \lambda I \) is nontrivial. To find the essential spectrum, which depends only on the background state and is independent of the solution \( \phi \) we are linearizing about, \( L(\phi) \) is exponentially asymptotic to the linear operator \( L(0) \), given by
\[
L(0) = \begin{pmatrix} 0 & L_0 \\ -L_0 & 0 \end{pmatrix}, \quad L_0 = -\frac{\beta_4}{24} \partial_{xxx} + \frac{\beta_2}{2} \partial_{xx} + \omega,
\] (16)

thus the eigenvalue problem \( L(0)v = \lambda v \) is equivalent to \( (L_0 + \lambda^2)p = 0 \). By [15, Theorem 3.1.13], the essential spectrum is given by the curves
\[
\left[ -\frac{\beta_4}{24} (ik)^4 + \frac{\beta_2}{2} (ik)^2 + \omega \right]^2 + \lambda^2 = 0 \quad k \in \mathbb{R},
\]
from which it follows that
\[
\sigma_{\text{ess}} = \left\{ \pm i \left( -\frac{\beta_4}{24} k^4 - \frac{\beta_2}{2} k^2 + \omega \right) : k \in \mathbb{R} \right\}.
\]

If \( \beta_4 < 0 \) and \( \beta_2 \leq 0 \), the essential spectrum is given by
\[
\sigma_{\text{ess}} = \left\{ ki : k \in \mathbb{R}, |k| \geq \omega \right\},
\] (17)

which is purely imaginary, bounded away from the origin, and independent of \( \beta_4 \) and \( \beta_2 \). In particular, this is the case for PQS. If \( \beta_4 < 0, \beta_2 > 0 \), and \( \omega > \omega_c \), the essential spectrum is given by
\[
\sigma_{\text{ess}} = \left\{ ki : k \in \mathbb{R}, |k| \geq \omega - \omega_c \right\},
\] (18)

which is also purely imaginary and bounded away from the origin, but does depend on \( \beta_4 \) and \( \beta_2 \) via \( \omega_c \).

By the stability assumption in Hypothesis 3, no element of the spectrum of \( L(\phi) \) can have a positive real part. Since the PDE (1) is Hamiltonian, all elements of the spectrum of \( L(\phi) \)
must come in quartets $\pm \alpha \pm \beta i$, thus the spectrum of $\mathcal{L}(\phi)$ is contained in the imaginary axis. For PQS, there is an additional pair of imaginary eigenvalues located right before the essential spectrum boundary (approximately $\pm 0.9972 \omega i$), which corresponds to an internal mode of the solitary wave [13]. For $\beta_2 \neq 0$, there can be multiple pairs of internal mode eigenvalues (an example of two pairs internal mode eigenvalues is shown in [14, Figure 9]). By Hypothesis 3, these internal mode eigenvalues must be purely imaginary.

3.1. Generalization to higher order solitons

In a recent conference presentation [1], it was shown this by incorporating an intracavity programmable pulse-shaper in a mode-locked fiber laser, one can manipulate the net cavity dispersion by applying a phase to the pulse so that to leading order, the stationary pulse generated is modeled by the higher-order NLS equation

$$-(i)^k \frac{d^k u}{dx^k} + \omega u - \gamma u^3 = 0,$$

where $k \geq 6$ is a positive, even integer, and the pulse profile satisfies the scaling relation $u(x; \omega) = \sqrt{\frac{\omega}{\gamma}} u(\omega^{1/k} x)$.

4. Existence of multi-pulse solitary waves

A multi-pulse is a multi-modal solitary wave resembling multiple, well-separated copies of the primary solitary wave. To prove the existence of multi-pulse solutions to (3), we will reframe the problem using a spatial dynamics approach. From this perspective, the primary solitary wave is a homoclinic orbit connecting the unstable and stable manifolds of a saddle equilibrium. A multi-pulse is a multi-loop homoclinic orbit which remains close to the primary homoclinic orbit. Letting $U = (u_1, u_2, u_3, u_4) = (u, \partial_x u, \partial_x^2 u, \frac{\beta_4}{24} \partial_x^3 u)$, we rewrite equation (3) as the first order system

$$U' = F(U) = \begin{pmatrix} u_2 \\ \frac{u_3}{\beta_4} u_4 \\ \omega u_1 - \gamma u_1^3 \end{pmatrix}.$$

This system has a conserved quantity

$$H(u_1, u_2, u_3, u_4) = -u_4 u_2 - \frac{1}{2} u_3^2 + \frac{\beta_2}{4} u_2^2 - \frac{\gamma}{4} u_4^4 + \frac{1}{2} \omega u_1^2,$$

which we obtain by multiplying (3) by $u_2$ and integrating once. $F(0) = 0$, and the characteristic polynomial of $DF(0)$ is

$$p(t) = t^4 - \frac{12 \beta_2}{\beta_4} t^2 - \frac{24}{\beta_4} \omega,$$

which has a quartet of complex eigenvalues $\pm \alpha \pm \beta i$ when $\omega > \omega_c$. For $\omega > \omega_c$, $U = 0$ is a hyperbolic saddle equilibrium of (20) with two-dimensional stable and unstable manifolds.
which intersect to form a homoclinic orbit. The exponentially localized primary pulse solution corresponding to this homoclinic orbit will have oscillatory tails, with the frequency of oscillations approximately equal to $b$. We have the following result concerning the existence of multi-pulse solutions, which follows immediately from [10, Theorem 3.6].

**Theorem 2.** Assume Hypothesis 1, Hypothesis 2, and Hypothesis 3, and fix $\beta_4 < 0$ and $\omega > \omega_c$. Let $\phi(x)$ be the real-valued, symmetric, exponentially localized primary pulse solution to (3) from Theorem 1, and let $U(x) = (\phi(x), \partial_x \phi(x), \partial_x^2 \phi(x), \partial_x^3 \phi(x))$ be the corresponding homoclinic orbit solution to (20). Let $\pm a \pm bi$ be the eigenvalues of $DF(0)$, with $a > 0$ and $b > 0$. Then for any

(i) $n \geq 2$

(ii) Sequence of nonnegative integers $\{k_1, \ldots, k_{n-1}\}$, with at least one of the $k_j \in \{0, 1\}$

(iii) Sequence of phase parameters $\{\theta_1, \ldots, \theta_n\} \in \{-1, 1\}^n$, with $\theta_1 = 1$

there exists a nonnegative integer $m_0$ such that for any integer $m$ with $m \geq m_0$, there exists a unique $n$-modal solution $U_n(x)$ to (20) which is defined piecewise via

$$U_n \left( x + 2 \sum_{k=1}^{i-1} X_k \right) = \begin{cases} 
\theta_i U(x) + \tilde{U}_i^-(x) & x \in [-X_{i-1}, 0] \\
\theta_i U(x) + \tilde{U}_i^+(x) & x \in [0, X_i] 
\end{cases}$$

for $i = 1, \ldots, n$, where $X_0 = X_n = \infty$. Uniqueness is up to translation and multiplication by $T(\theta)$. The distances between consecutive peaks are given by $2X_i$, where

$$X_i \approx \frac{\pi}{b}(2m + k_i) + \tilde{X},$$

and $\tilde{X}$ is a constant. In addition, we have the estimates

$$\|\tilde{U}_i^\pm\|_\infty \leq Ce^{-aX_{\min}}$$

$$\tilde{U}_i^+(X_i) = \theta_{i+1}U(-X_i) + O(e^{-2aX_{\min}})$$

$$\tilde{U}_i^-(X_i) = \theta_iU(X_i) + O(e^{-2aX_{\min}}),$$

where $X_{\min} = \min\{X_1, \ldots, X_{n-1}\}$, which hold as well for all derivatives with respect to $x$.

**Proof.** Since the spectrum of $DF(0)$ is a quartet of eigenvalues $\pm a \pm bi$ for $\omega > \omega_c$, equation (20) has a conserved quantity (21), and the Melnikov integral $M = \int_{-\infty}^\infty \phi^2_x dx$ is positive, the result follows from [10, Theorem 3.6], with the straightforward modification that the multi-pulse is constructed from copies of $U(x)$ and $-U(x)$. The estimates (24) follow from [11, 12].

5. Spectrum of multi-pulse solitary waves

Let $U(x) = (\phi(x), \partial_x \phi(x), \partial_x^2 \phi(x), \partial_x^3 \phi(x))$ be the primary homoclinic orbit corresponding to the primary pulse $\phi(x)$, and let $U_n = (\phi_n(x), \partial_x \phi_n(x), \partial_x^2 \phi_n(x), \partial_x^3 \phi_n(x))$ be a multi-loop homoclinic orbit solution to (20) constructed according to Theorem 2. The first component $\phi_n$ of $U_n$ is a multi-pulse solitary wave solution to (3). As in [8, 11], we will locate the eigenvalues near the origin of the linearized operator $\mathcal{L}(\phi_n)$. Both $\mathcal{L}(\phi)$ and $\mathcal{L}(\phi_n)$ have two
eigenfunctions in the kernel. Since the kernel of $\mathcal{L}(\phi)$ is 2-dimensional, we expect that $\mathcal{L}(\phi_n)$ will have $4(n-1)$ additional eigenvalues near 0. Since these arise from nonlinear interactions between the tails of neighboring pulses in the multi-pulse structure, we call them interaction eigenvalues. Once again using a spatial dynamics approach, we rewrite the eigenvalue problem $\mathcal{L}(\phi_n)v = \lambda v$ as the first order system

$$V'(x) = K(\phi_n)V(x) + \lambda B_1 V(x),$$

(25)

where

$$K(\phi_n) = \begin{pmatrix} K^+(\phi_n) & 0 \\ 0 & K^-(\phi_n) \end{pmatrix}, \quad B_1 = \begin{pmatrix} 0 & B \\ -B & 0 \end{pmatrix},$$

$$K^-(\phi_n) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 24 \beta_4 \\ \omega - \gamma \phi_n^2 & 0 & \frac{\beta_2}{2} \\ 0 & \omega - 3\gamma \phi_n^2 & 0 & \frac{\beta_2}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K^+(\phi) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 24 \beta_4 \\ \omega & 0 & \frac{\beta_2}{2} \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

The associated variational equation

$$V'(x) = K(\phi)V(x)$$

(26)

has two linearly independent, exponentially decaying solutions $\tilde{Q}(x) = (U'(x), 0)^T$ and $Q(x) = (0, U(x))^T$. The corresponding adjoint variational equation

$$W'(x) = -K(\phi)^*W(x)$$

(27)

has two linearly independent, exponentially decaying solutions $\tilde{Q}^*(x) = (\Psi'(x), 0)^T$ and $Q^*(x) = (0, \Psi(x))^T$, where

$$\Psi(x) = \left(-\frac{\beta_1}{24} \partial^3_x \phi(x) + \frac{\beta_2}{2} \partial_x \phi(x), \frac{\beta_1}{24} \partial^3_x \phi(x) - \frac{\beta_2}{2} \phi(x), -\frac{\beta_1}{24} \partial_x \phi(x), \phi(x) \right).$$

(28)

The following theorem, which is analogous to [11, Theorem 2], reduces the problem of locating the eigenvalues of $\mathcal{L}(\phi_n)$ in a ball around the origin in the complex plane to finding the determinant of a $2n \times 2n$ matrix which is, to leading order, block diagonal. The proof is given in subsection 8.1.

**Theorem 3.** Assume Hypothesis 1, Hypothesis 2, and Hypothesis 3. Let $U(x)$ be the primary homoclinic orbit from Theorem 1, and let $U_n(x)$ be an $n$-pulse solution constructed according to Theorem 2 with phase parameters $\{\theta_1, \ldots, \theta_n\}$ and pulse distances $X_1, \ldots, X_{n-1}$. Let $\pm a \pm bi$ be the eigenvalues of $DF(0)$, with $a > 0$ and $b > 0$. Then there exists $\delta > 0$ with the following property. There exists a bounded, nonzero solution $V(x)$ of (25) for $|\lambda| < \delta$ if and only if

$$E(\lambda) = \det S(\lambda) = 0,$$

(29)

where $S(\lambda)$ is the $2n \times 2n$ block matrix

$$S(\lambda) = \begin{pmatrix} A + \lambda^2 MI & 0 \\ 0 & (a^2 + b^2) A - \lambda^2 MI \end{pmatrix} + R(\lambda).$$

(30)
The tri-diagonal matrix $A$ is defined by

$$
A = \begin{pmatrix}
-a_1 & a_1 & & & & \\
 & -a_1 - a_2 & a_2 & & & \\
 & & -a_2 - a_3 & a_3 & & \\
& & & \ddots & \ddots & \\
& & & & -a_{n-1} & -a_{n-1}
\end{pmatrix},
$$

where

$$
a_i = \theta_i \theta_{i+1} \langle \Psi(X_i), U(-X_i) \rangle,
$$

and $\Psi(x)$ is defined by (28). The constants $M$ and $\tilde{M}$ are given by

$$
M = \int_{-\infty}^{\infty} \phi(x) \partial_\omega \phi(x) dx = d''(\omega) > 0, \quad \tilde{M} = \int_{-\infty}^{\infty} \partial_x \phi(x) z(x) dx,
$$

where $z(x)$ is defined in section 3 after (14). The remainder term $R(\lambda)$ is analytic in $\lambda$ and has uniform bound

$$
|R(\lambda)| \leq C \left( |\lambda| (|\lambda| + e^{-\alpha X_{\text{min}}^2} + e^{-(2\alpha + \gamma) X_{\text{min}}}) \right),
$$

where $\gamma > 0$.

If $\tilde{M} > 0$, which is supported by numerical computation, then all multi-pulse solutions are unstable by the following corollary.

**Corollary 1.** Let $U_n(x)$ be a $n$-pulse constructed using Theorem 2. Then there are $2(n-1)$ pairs of interaction eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ and $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_{n-1}$, given by

$$
\lambda_i = \sqrt{\frac{\mu_i}{M}} + O \left( e^{-(2\alpha + \gamma) X_{\text{min}}} \right) \quad i = 1, \ldots, n - 1
$$

$$
\tilde{\lambda}_i = -\sqrt{(a^2 + b^2) \frac{\mu_i}{M}} + O \left( e^{-(2\alpha + \gamma) X_{\text{min}}} \right) \quad i = 1, \ldots, n - 1,
$$

where $\{\mu_1, \ldots, \mu_{n-1}, 0\}$ are the real, distinct eigenvalues of $A$. If $\tilde{M} > 0$, then one of each pair $\lambda_i$ is real and the other is purely imaginary, thus there are $n - 1$ positive real eigenvalues.

Finally, we compute the interaction eigenvalues of a 2-pulse solution $U_2(x)$.

**Corollary 2.** Let $U_2(x)$ be a 2-pulse constructed using Theorem 2 with pulse distance $X_1$ and phase parameters $\theta_1, \theta_2$. Then there are four interaction eigenvalues associated with $U_2(x)$, which are, to leading order, given by

$$
\lambda = \pm \sqrt{\frac{2a_1}{M}}, \quad \tilde{\lambda} = \pm \sqrt{-\frac{2(a^2 + b^2)a_1}{M}},
$$

where $a_1 = \theta_1 \theta_2 \langle \Psi(X_1), U(-X_1) \rangle$. If $\tilde{M} > 0$, then one pair is real and one pair is purely imaginary.

**Remark 2.** In addition, the internal mode eigenvalues of the primary pulse will duplicate as pulses are added to the multi-pulse structure. For example, for the pure quartic solitary wave, the 2-pulse will have two pairs of internal mode eigenvalues. If $M > 0$, the 2-pulse is unstable, and these internal mode eigenvalues have no additional effect on stability.
6. Numerical Results

To construct the primary pulse solution \( \phi(x) \), we start with the known solitary wave solution for NLS and gradually modify the parameters \( \beta_2 \) and \( \beta_4 \), solving for the new solitary wave solution at each step using a Newton conjugate-gradient method [16, Chapter 7.2.4] implemented in MATLAB. To obtain the pure quartic solitary wave for \( \beta_4 = -1 \) (Figure 1), we perform this procedure along the line segment connecting \((\beta_2, \beta_4) = (-2, 0)\) and \((\beta_2, \beta_4) = (0, -1)\).

Figure 1: Pure quartic solitary wave solution \( \phi(x) \) to (3) with \( \beta_2 = 0, \beta_4 = -1, \omega = 1 \). (left panel). Plot of \( \log \phi(x) \) vs \( x \) (right panel) showing exponentially-decaying oscillatory tails. Spatial discretization is a uniform grid with \( N = 1024 \) grid points, and we use periodic boundary conditions.

To determine the spectrum of the linearization about the primary pulse, we construct the linear operator \( \mathcal{L}(\phi) \) using Fourier spectral differentiation matrices and compute the eigenvalues using Matlab’s eigenvalue solver \texttt{eig} (Figure 2, left panel). We note that the essential spectrum is discrete, which is a consequence of the spatial discretization, as well as the presence of a pair of internal mode eigenvalues on the imaginary axis.
For the primary solitary wave solutions, we can compute the stability criterion (12) from \cite{4}. In all cases, $M = d''(\omega) > 0$, which suggests that primary pulse solution is orbitally stable. In addition, numerical computation suggests that $\tilde{M} > 0$. To construct double pulses, we glue together two copies of the primary pulse at the pulse distances predicted by Theorem 2 and solve for the double pulse solution using the same Newton conjugate-gradient method we used above. The first eight double pulse solutions are shown in Figure 3. Arbitrary multi-pulses can similarly be constructed.

For the spectrum of $\mathcal{L}(\phi_2)$, the linearization about the double pulse solution $\phi_2$, there is (for both in phase and out of phase double pulses) a pair of purely imaginary interaction eigenvalues and a pair of real interaction eigenvalues, thus the double pulse solutions are all unstable (Figure 4), which verifies Corollary 2. There is also a duplication of the internal
mode eigenvalues (Figure 5); these appear to be purely imaginary, but they do not affect stability.

Figure 4: Eigenvalues for first in-phase double pulse (left panel) and first out-of-phase double pulse (right panel). Interaction eigenvalues shown in red, and kernel eigenvalues in black. Eigenvalues in blue correspond to the discrete essential spectrum; as expected, these eigenvalues are on the imaginary axis and have magnitude \(|\lambda| \geq \omega\). Internal mode eigenvalues are not shown. \(\beta_2 = 0, \beta_4 = -1, \omega = 1\).

Figure 5: Close-up of spectrum showing pair of internal mode eigenvalues (red) and eigenvalues corresponding to the essential spectrum (blue) for third in-phase double pulse. \(\beta_2 = 0, \beta_4 = -1, \omega = 1\).

Finally, we verify the formulas for the interaction eigenvalues from Corollary 2 by plotting the log of the relative error between the leading order term in (32) and the eigenvalues computed by Matlab versus the pulse separation distance \(X\) (Figure 6).

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7. Conclusions and future directions

In this paper, we studied single and multi-pulse solitary wave solutions to a general nonlinear Schrödinger equation with both second and fourth order dispersion terms. We gave criteria for the existence of a primary soliton solution in terms of the parameters of the system, and provided numerical verification that the primary soliton is orbitally stable. We then constructed \( n \)-pulse solutions by splicing together multiple copies of the primary pulse, and reduced the problem of finding the small eigenvalues resulting from interaction between neighboring pulses to that of computing the determinant of a \( 2n \times 2n \) block matrix. Under a mild assumption, which can be verified numerically, we showed that all multi-pulse solutions are unstable. For future research, we could investigate solitons and multi-pulses in higher order NLS equations, as discussed in [1]. We expect that these results would hold for these higher order variants, and that all multi-pulse solutions would be unstable. We could also study generalizations to other nonlinearities. It would be interesting to perform numerical time-stepping starting with perturbed multi-pulses to investigate how the instability evolves in time. Although all multi-pulse solutions to this fourth order model are unstable, this equation represents an idealization of the experimental situation since energy is always conserved. A more realistic model might incorporate gains and losses of energy in the laser cavity, and it is possible that stable multi-pulses could exist in such a scenario.

8. Proof of stability results

8.1. Proof of Theorem 3

The proof is adapted from [8, Section 3.4] and the proof of [11, Theorem 2], and uses an implementation of the Lyapunov-Schmidt reduction known as Lin’s method. It follows from (15) that

\[
\begin{align*}
[Y(x)]' &= K(\phi_n)Y(x), & [Z(x)]' &= K(\phi_n)Z(x) + B_1 Y(x) \\
[\tilde{Y}(x)]' &= K(\phi_n)\tilde{Y}(x), & [\tilde{Z}(x)]' &= K(\phi_n)\tilde{Z}(x) + B_1 \tilde{Y}(x),
\end{align*}
\]

Figure 6: Log of the relative error for the eigenvalues \( \lambda_\pm \) versus the pulse separation \( X \) for the first five in-phase double pulses. \( \beta_2 = 0, \beta_4 = -1, \omega = 1 \).
where
\[ Y(x) = (0, U_n(x))^T, \quad Z(x) = (\partial_x U_n(x), 0)^T \]
\[ Y(x) = (\partial_x U_n(x), 0)^T, \quad Z(x) = (0, Z_n(x))^T, \]
\[ Z_n(x) = (\sigma_n(x), \partial_x \sigma_n(x), \partial^2_x \sigma_n(x), \frac{\beta_1}{\beta_2} \partial^2_x \sigma_n(x)), \] and the first component \( \sigma_n(x) \) solves \( L^- (\phi_n) \sigma_n = \phi'_n \). The analysis is identical to that of [8], except the piecewise ansatz for the eigenfunction also involves \( Z(x) \) and \( \dot{Z}(x) \). Writing the functions (33) in piecewise form as with (22), we take the ansatz
\[ V_i^\pm(x)' = d_i(\bar{Y}_i^\pm(x) + \lambda \bar{Z}_i^\pm(x)) + \bar{d}_i(\bar{Y}_i^\pm(x) + \lambda \bar{Z}_i^\pm(x)) + W_i^\pm \quad i = 1, \ldots, n, \] (35)
where \( V_i^- \in C^0([-X_{i-1}, 0], \mathbb{C}^8) \) and \( V_i^+ \in C^0([0, X_i], \mathbb{C}^8) \). Substituting (35) into (25) and simplifying using (33), the remainder functions \( W_i^\pm(x) \) solve the equation
\[ W_i^\pm(x)' = K(\phi_n) W_i^\pm(x) + \lambda^2 d_i BZ_i^\pm(x) + \lambda^2 \bar{d}_i B \bar{Z}_i^\pm(x) \quad i = 1, \ldots, n. \] (36)
Following [8, 11], we obtain a unique piecewise solution \( W_i^\pm(x) \) which generically has \( n \) jumps at \( x = 0 \) in the direction of \( Q^*(0) \oplus Q^*(0) \). Using the definitions of \( Q^*(x) \) and \( \bar{Q}^*(x) \) together with (24) and [8, (3.19)], these jumps are given by
\[ \xi_i = \theta_{i+1} \langle \Psi(X_i), U(-X_i) \rangle (d_{i+1} - d_i) + \theta_{i-1} \langle \Psi(-X_{i-1}), U(X_{i-1}) \rangle (d_i - d_{i-1}) \]
\[ + \lambda^2 \theta_i d_i \int_{-\infty}^{\infty} \langle \Psi(y), B \partial_x U(y) \rangle dy + \mathcal{O}(|x| + e^{-\alpha x_{\min}^3}) \]
\[ \bar{\xi}_i = \theta_{i+1} \langle \bar{\Psi}(X_i), \bar{U}'(-X_i) \rangle (\bar{d}_{i+1} - \bar{d}_i) + \theta_{i-1} \langle \bar{\Psi}'(-X_{i-1}), \bar{U}'(X_{i-1}) \rangle (\bar{d}_i - \bar{d}_{i-1}) \]
\[ - \lambda^2 \theta_i \bar{d}_i \int_{-\infty}^{\infty} \langle \Psi(y), B \bar{Z}(y) \rangle dy + \mathcal{O}(|x| + e^{-\alpha x_{\min}^3}). \] (37)
where \( Z(x) = (z(x), \partial_x z(x), \partial^2_x z(x), \frac{\beta_1}{\beta_2} \partial^2_x z(x)) \). By symmetry,
\[ \Psi(-x) = - R \Psi(x), \quad U(-x) = R U(x), \] (38)
where \( R \) is the standard reversor operator
\[ R(u_1, u_2, u_3, u_4) = (u_1, -u_2, u_3, -u_4), \]

Thus
\[ \langle \Psi(-X_{i-1}), U(X_{i-1}) \rangle = - \langle \Psi(X_{i-1}), U(-X_{i-1}) \rangle \]
\[ \langle \Psi'(-X_{i-1}), U'(X_{i-1}) \rangle = - \langle \Psi'(X_{i-1}), U'(-X_{i-1}) \rangle. \] (39)
Finally, we relate \( \langle \Psi(X_i), U(-X_i) \rangle \) and \( \langle \Psi(X_i), U'(-X_i) \rangle \). Since \( DF(\phi) = K^+(\phi), \Psi(x) \) is the unique bounded solution to the adjoint equation \( W'(x) = -DF(\phi)^* W(x) \). Thus by [11, Lemma 6.1], with \( \Psi' \) in place of \( \Psi \), \( p \) in place of \( \phi \), and no parameter \( \mu \),
\[ \langle \Psi(x), U(-x) \rangle = \langle \Psi'(-x), U(x) \rangle = se^{-2ax} \sin(2bx + p) + \mathcal{O}(e^{-(2\alpha + \gamma)x}) \]
\[ \langle \Psi'(x), U'(-x) \rangle = \langle \Psi'(x), U'(-x) \rangle = -se^{-2ax} (b \cos(2bx + p) - a \sin(2bx + p)) + \mathcal{O}(e^{-(2\alpha + \gamma)x}), \] (41)
where \( s > 0 \) and \( \gamma > 0 \). Differentiating \( \langle \Psi(-x), U(x) \rangle \) with respect to \( x \), since the operator \( \partial_x \) is skew symmetric,

\[
\frac{d}{dx} \langle \Psi(x), U(-x) \rangle = 2 \langle \Psi'(x), U(-x) \rangle,
\]

thus we can integrate (40) by parts to get

\[
\langle \Psi(x), U(-x) \rangle = -\frac{1}{a^2 + b^2} s e^{-2a x} (b \cos(2bx + p) + a \sin(2bx + p)) + \mathcal{O}(e^{-(2a+\gamma)x}).
\]

In the proof of [11, Theorem 3], the distances \( X_i \) are chosen to solve \( s e^{-2a X_i} \sin(2b X_i + p) = \mathcal{O}(e^{-(2a+\gamma)X_i}) \), thus for \( x = X_i \) we have

\[
\langle \Psi'(X_i), U'(-X_i) \rangle = (a^2 + b^2) \langle \Psi(X_i), U(-X_i) \rangle + \mathcal{O}(e^{-(2a+\gamma)X_i}). \tag{42}
\]

Using (42) and (39), multiplying by \( \theta_i \), and using the definition of \( B \), equations (37) simplify to the jump conditions

\[
\xi_i = \theta_i \theta_{i+1} \langle \Psi(X_i), U(-X_i) \rangle (d_{i+1} - d_i) - \theta_{i-1} \theta_i \langle \Psi(X_{i-1}), U(-X_{i-1}) \rangle (d_i - d_{i-1})
+ \lambda^2 d_i \int_{-\infty}^{\infty} \phi(y) \partial_x \phi(y) dy + \mathcal{O}(|\lambda|||\lambda| + e^{-\alpha X_{\min}}|^2 + e^{-(2a+\gamma)X_{\min}}))
\]

\[
\tilde{\xi}_i = (a^2 + b^2) \theta_i \theta_{i+1} \langle \Psi(X_i), U(-X_i) \rangle (\tilde{d}_{i+1} - \tilde{d}_i) - (a^2 + b^2) \theta_{i-1} \theta_i \langle \Psi(X_{i-1}), U(-X_{i-1}) \rangle (\tilde{d}_i - \tilde{d}_{i-1})
- \lambda^2 \tilde{d}_i \int_{-\infty}^{\infty} \partial_y \phi(y) z(y) dy + \mathcal{O}(|\lambda||\lambda| + e^{-\alpha X_{\min}})^2 + e^{-(2a+\gamma)X_{\min}}),
\]

which we write in matrix form as in the statement of the theorem.

8.2. Proof of Corollary 1 and Corollary 2

For Corollary 1, let \( \{\mu_1, \ldots, \mu_{n-1}, 0\} \) be the eigenvalues of \( A \), which are real and distinct as in the proof of [9, Theorem 5]. Following the steps in that proof and using the rescaling in [11, Theorem 3], there are \( 2(n-1) \) pairs of interaction eigenvalues, given by (31), which are either real or purely imaginary by Hamiltonian symmetry. Since \( M > 0 \) by Hypothesis 3, if \( \tilde{M} > 0 \) as well, then one of each pair \( \lambda_i, \tilde{\lambda}_i \) is real and the other is purely imaginary. Corollary 2 is the specific case \( n = 2 \), where the nonzero eigenvalue of \( A \) can be computed directly.

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