MODELS FOR SUBHOMOGENEOUS C*-ALGEBRAS

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Abstract. A new category of topological spaces with additional structures, called m-towers, is introduced. It is shown that there is a contravariant functor which establishes a one-to-one correspondences between unital (resp. arbitrary) subhomogeneous C*-algebras and proper (resp. proper pointed) m-towers of finite height, and between all *-homomorphisms between two such algebras and morphisms between m-towers corresponding to these algebras.

1. Introduction. Subhomogeneous C*-algebras play a central role in operator algebras. For example, they appear in the definition of ASH algebras, which have been widely investigated for a long time, and in composition series of GCR (and CCR) algebras. Moreover, subhomogeneity is a property which well behaves in almost all finite operations on C*-algebras, such as direct sums and products, tensor products, quotients, extensions and passing to subalgebras (which is in contrast to homogeneity; models for homogeneous C*-algebras were described more than 50 years ago by Fell [10] and Tomiyama and Takesaki [21]). For this reason, subhomogeneous C*-algebras form a flexible and rich category. Taking into account all these remarks, any information about the structure of this category is of importance. Moreover, a deep and long-term research on duals or spectra (or Jacobson spaces) of C*-algebras, carried out by Kaplansky [13], Fell [8, 9, 10], Dixmier [3] (consult also [4, Chapter 3]), Effros [5], Bunce and Deddens [2] and others, led to the conclusion that there is no reasonable way of describing all subhomogeneous C*-algebras by means of their spectra. (There are known partial results in this direction, e.g. for C*-algebras with continuous traces or Hausdorff spectra.)

The main aim of the present paper is to describe all such algebras (as well as *-homomorphisms between them) by means of a certain category of topological spaces, which we call m-towers. Taking this into account, our work
may be seen as a solution of a long-standing problem in operator algebras. Our description resembles the commutative Gelfand–Naimark theorem and is functorial (that is, it establishes a one-to-one correspondence, by means of a contraviariant functor, between \(*\)-homomorphisms of algebras and morphisms of \(m\)-towers). Because of this last property, our concept can be considered more natural than Vasil’ev’s \cite{22}, who proposed a totally different description of subhomogeneous \(C^*\)-algebras. To formulate our main results, we first introduce the necessary notions and notations.

For each \(n > 0\), let \(\mathcal{M}_n\) and \(\mathcal{U}_n\) denote, respectively, the \(C^*\)-algebra of all complex \(n \times n\) matrices and its unitary group. For \(U \in \mathcal{U}_n\) and \(X \in \mathcal{M}_n\), we denote by \(U.X\) and \(d(X)\) the matrix \(UXU^{-1}\) and the degree of \(X\) (that is, \(d(X) := n\)), respectively. Finally, when \(X \in \mathcal{M}_n\) and \(Y \in \mathcal{M}_k\), \(X \oplus Y\) stands for the block matrix \(\left( \begin{array}{cc} X & 0 \\ 0 & Y \end{array} \right) \in \mathcal{M}_{n+k}\).

By an \(n\)-dimensional representation of a \(C^*\)-algebra \(\mathcal{A}\) we mean any (possibly zero) \(*\)-homomorphism from \(\mathcal{A}\) into \(\mathcal{M}_n\).

For every unital \(C^*\)-algebra \(\mathcal{A}\) and each \(n > 0\), we denote by \(\mathfrak{X}_n(\mathcal{A})\) the space of all unital \(n\)-dimensional representations of \(\mathcal{A}\), equipped with the pointwise convergence topology. The concrete tower of \(\mathcal{A}\) is the topological disjoint union \(\mathfrak{X}(\mathcal{A}) := \bigcup_{n=1}^{\infty} \mathfrak{X}_n(\mathcal{A})\), equipped with additional ingredients:

- a degree map \(d: \mathfrak{X}(\mathcal{A}) \rightarrow \mathbb{N} := \{1, 2, \ldots\}\), \(d(\pi) := n\) for any \(\pi \in \mathfrak{X}_n(\mathcal{A})\);
- addition \(\oplus: \mathfrak{X}(\mathcal{A}) \times \mathfrak{X}(\mathcal{A}) \rightarrow \mathfrak{X}(\mathcal{A})\) defined pointwise (that is, \((\pi_1 \oplus \pi_2)(a) := \pi_1(a) \oplus \pi_2(a)\) for each \(a \in \mathcal{A}\));
- a unitary action, that is, a collection of functions \(\mathcal{U}_n \times \mathfrak{X}_n(\mathcal{A}) \ni (U, \pi) \mapsto U.\pi \in \mathfrak{X}_n(\mathcal{A})\) \((n > 0)\) where, for \(U \in \mathcal{U}_n\) and \(\pi \in \mathfrak{X}_n(\mathcal{A})\), \(U.\pi: \mathcal{A} \rightarrow \mathcal{M}_n\) is defined pointwise (that is, \((U.\pi)(a) := U.\pi(a)\)).

Based on the concept of concrete towers (introduced above), one may define abstract \(m\)-towers with analogous ingredients (they need to satisfy some additional axioms, which for concrete towers are automatically fulfilled). For any \(m\)-tower \(\mathfrak{X}\), the space \(C^*(\mathfrak{X})\) is defined as the space of all matrix-valued functions \(f\) defined on \(\mathfrak{X}\) such that:

- \(f(\mathfrak{r}) \in \mathcal{M}_{d(\mathfrak{r})}\) for any \(\mathfrak{r} \in \mathfrak{X}\);
- for each \(n > 0\), \(f|_{\mathfrak{X}_n}: \mathfrak{X}_n \rightarrow \mathcal{M}_n\) is continuous where \(\mathfrak{X}_n := d^{-1}(\{n\})\);
- \(f(U.\mathfrak{r}) = U.f(\mathfrak{r})\) for all \((U, \mathfrak{r}) \in \bigcup_{n=1}^{\infty} (\mathcal{U}_n \times \mathfrak{X}_n)\);
- \(f(\mathfrak{r}_1 \oplus \mathfrak{r}_2) = f(\mathfrak{r}_1) \oplus f(\mathfrak{r}_2)\) for any \(\mathfrak{r}_1, \mathfrak{r}_2 \in \mathfrak{X}\);
- \((\|f\| := \sup_{\mathfrak{r} \in \mathfrak{X}} \|f(\mathfrak{r})\| < \infty)\).

It is readily seen that \(C^*(\mathfrak{X})\) is a unital \(C^*\)-algebra when all algebraic operations are defined pointwise. Further, the height of \(\mathfrak{X}\), denoted by \(ht(\mathfrak{X})\), is defined as follows. An element \(\mathfrak{r}\) of \(\mathfrak{X}\) is irreducible if each unitary operator
$U \in \mathcal{U}_{d(t)}$ for which $U \cdot x = x$ is a scalar multiple of the unit matrix. Then

$$\text{ht}(X) := \sup \{d(x) : x \in X \text{ is irreducible} \} \in \{1, 2, \ldots, \infty\}.$$ 

The m-tower $X$ is called proper if each of the subspaces $X_n$ is compact.

Finally, if $\mathcal{A}$ is (again) an arbitrary unital $C^*$-algebra and $a \in \mathcal{A}$, we define a matrix-valued function $\hat{a}$ on $\mathfrak{X}(\mathcal{A})$ by $\hat{a}(\pi) = \pi(a)$. It is a typical property that $J_{\mathcal{A}} : \mathcal{A} \ni a \mapsto \hat{a} \in C^*(\mathfrak{X}(\mathcal{A}))$ is a correctly defined unital $*$-homomorphism. It is also clear that $J_{\mathcal{A}}$ is one-to-one iff $\mathcal{A}$ is residually finite-dimensional. Our main result on m-towers is

**Theorem 1.1.** For any unital subhomogeneous $C^*$-algebra $\mathcal{A}$, $\mathfrak{X}(\mathcal{A})$ is a proper m-tower, $\text{ht}(\mathfrak{X}(\mathcal{A})) < \infty$ and $J_{\mathcal{A}} : \mathcal{A} \to C^*(\mathfrak{X}(\mathcal{A}))$ is a $*$-isomorphism.

Conversely, if $\mathcal{T}$ is a proper m-tower of finite height, then $C^*(\mathcal{T})$ is subhomogeneous and for every finite-dimensional unital representation $\pi$ of $C^*(\mathcal{T})$ there exists a unique point $t \in \mathcal{T}$ such that $\pi = \pi_t$ where

$$\pi_t : C^*(\mathcal{T}) \ni u \mapsto u(t) \in \mathcal{M}_{d(t)}.$$ 

Moreover, the assignment $t \mapsto \pi_t$ correctly defines an isomorphism between the towers $\mathcal{T}$ and $\mathfrak{X}(C^*(\mathcal{T}))$.

Theorem 1.1 establishes a one-to-one correspondence between unital subhomogeneous $C^*$-algebras and proper m-towers of finite heights. It also enables a characterization of $*$-homomorphisms between such algebras by means of so-called morphisms between m-towers.

To describe models for all (possibly) nonunital subhomogeneous $C^*$-algebras, it suffices to collect all (possibly nonunital) finite-dimensional representations. More precisely, for any $C^*$-algebra $\mathcal{A}$, the concrete pointed tower of $\mathcal{A}$, denoted by $(\mathbf{3}(\mathcal{A}), \theta_{\mathcal{A}})$, is the tower $\mathbf{3}(\mathcal{A})$ of all its (possibly zero) finite-dimensional representations together with a distinguished element $\theta_{\mathcal{A}}$ which is the zero one-dimensional representation of $\mathcal{A}$. An abstract pointed m-tower is any pair $(\mathbf{3}, \theta)$ where $\mathbf{3}$ is an m-tower and $\theta \in \mathbf{3}$ is such that $d(\theta) = 1$. For each such a pair one defines $C^*(\mathbf{3}, \theta)$ as the subspace of $C^*(\mathbf{3})$ consisting of all functions $f \in C^*(\mathbf{3})$ for which $f(\theta) = 0$. For any $C^*$-algebra $\mathcal{A}$, $J_{\mathcal{A}} : \mathcal{A} \to C^*(\mathbf{3}(\mathcal{A}), \theta)$ is defined in the same manner as for unital $C^*$-algebras. Then we have

**Theorem 1.2.** For any subhomogeneous $C^*$-algebra $\mathcal{A}$, $(\mathbf{3}(\mathcal{A}), \theta_{\mathcal{A}})$ is a proper pointed m-tower of finite height and $J_{\mathcal{A}} : \mathcal{A} \to C^*(\mathbf{3}(\mathcal{A}), \theta_{\mathcal{A}})$ is a $*$-isomorphism.

Conversely, if $(\mathcal{G}, \kappa)$ is an arbitrary proper pointed m-tower of finite height, then $C^*(\mathcal{G}, \kappa)$ is subhomogeneous and for every finite-dimensional representation $\pi$ of $C^*(\mathcal{G}, \kappa)$ there is a unique point $s \in \mathcal{G}$ such that $\pi = \pi_s$. 
where
\[ \pi_s : C^*(\mathcal{S}, \kappa) \ni u \mapsto u(s) \in \mathcal{M}_{d(s)}. \]
Moreover, the assignment \( s \mapsto \pi_s \) correctly defines an isomorphism between the pointed towers \((\mathcal{S}, \kappa)\) and \( \mathcal{F}(C^*(\mathcal{S}, \kappa), \theta_{C^*(\mathcal{S}, \kappa)}) \).

From Theorems 1.1 and 1.2 one may easily deduce the commutative Gelfand–Naimark theorem as well as a result on models for homogeneous \( C^* \)-algebras due to Fell [10] and Tomiyama and Takesaki [21] (this is discussed in Remark 5.8). Since \( \ast \)-homomorphisms between subhomogeneous \( C^* \)-algebras may naturally be interpreted, in a functorial manner, as morphisms between \( m \)-towers, our approach to homogeneous (as a part of subhomogeneous) \( C^* \)-algebras may be seen as better than the original one proposed by Fell, and Tomiyama and Takesaki (because using their models it is far from “natural” to describe \( \ast \)-homomorphisms between \( n \)-homogeneous and \( m \)-homogeneous \( C^* \)-algebras for different \( n \) and \( m \)). A one-to-one functorial correspondence between the realms of subhomogeneous \( C^* \)-algebras and (proper) \( m \)-towers gives new tools in investigation of other classes of \( C^* \)-algebras, such as approximately subhomogeneous (ASH), approximately homogeneous (AH) or CCR (which are ASH, due to a result of Sudo [20]) as well as GCR \( C^* \)-algebras (taking into account their composition series; see [13]).

Although Theorem 1.1 is very intuitive, its proof is difficult and involves our recent result [17] on norm closures of convex sets in uniform spaces of vector-valued functions.

The paper is organized as follows. In Section 2 we discuss special kinds of towers, which are very natural and intuitive. Based on their properties, in Section 3 we introduce abstract towers and distinguish \( m \)-towers among them. We establish fundamental properties of \( m \)-towers, which will find applications in further sections. In Section 4 we study so-called essentially locally compact \( m \)-towers and prove for them a variation of the classical Stone–Weierstrass theorem (see Theorem 4.14). Its special case, Theorem 4.2 is crucial for proving Theorem 1.1. We also characterize there all (closed two-sided) ideals (see Corollary 4.8) and finite-dimensional nondegenerate representations of \( C^* \)-algebras associated with essentially locally compact \( m \)-towers (Corollary 4.9). Section 5 is devoted to the proofs of Theorems 1.1 and 1.2 and some of their consequences.

**Notation and terminology.** Representations of \( C^* \)-algebras need not be nonzero and \( \ast \)-homomorphisms between unital \( C^* \)-algebras need not preserve unities. Ideals in \( C^* \)-algebras are, by definition, closed and two-sided. All topological spaces are assumed to be Hausdorff (unless otherwise stated). A **map** is a continuous function. A map \( u : X \to Y \) is **proper** if \( u^{-1}(L) \) is
For any standard tower $X$ functions any standard pointed tower $\theta$ point the form

\[ X \oplus Y \in \mathcal{T} \iff X, Y \in \mathcal{T}. \]

All notation and terminology introduced earlier in this section will be in force throughout. Additionally, we denote by $I_n$ the unit $n \times n$ matrix.

2. Standard towers. This section is devoted to a very special type of m-towers called standard towers (general m-towers will be discussed in the next part), which are intuitive and natural. We start from them in order to elucidate the main ideas and make it easier to assimilate further definitions (which may seem strange or somewhat artificial). Actually, we shall prove in the following that each proper m-tower is isomorphic to a standard tower.

Let $\Lambda$ be an arbitrary nonempty set (of indices). For any integer $n > 0$ let $\mathcal{M}^A_n$ stand for the vector space of all functions from $\Lambda$ into $\mathcal{M}_n$. We define $\mathcal{M}[A]$ as the topological disjoint union $\bigsqcup_{n=1}^{\infty} \mathcal{M}^A_n$ of the spaces $\mathcal{M}^A_n$ each of which is equipped with the pointwise convergence topology. Further, for two members $X = (X_\lambda)_{\lambda \in \Lambda}$ and $Y = (Y_\lambda)_{\lambda \in \Lambda}$ of $\mathcal{M}[A]$, we define $d(X)$ and $X \oplus Y$ as follows: $d(X) := n$ where $n > 0$ is such that $X \in \mathcal{M}^A_n$ and $X \oplus Y := (X_\lambda \oplus Y_\lambda)_{\lambda \in \Lambda}$. If, in addition, $U \in \mathcal{U}_d(X)$, then $U.X$ is defined coordinatewise, that is, $U.X := (U.X_\lambda)_{\lambda \in \Lambda}$. In this way we have obtained functions $d: \mathcal{M}[A] \to \mathbb{N}$ and $\oplus: \mathcal{M}[A] \times \mathcal{M}[A] \to \mathcal{M}[A]$ and a collection

$\mathcal{U}_n \times \mathcal{M}^A_n \ni (U, X) \mapsto U.X \in \mathcal{M}^A_n, \quad n = 1, 2, \ldots,$

of group actions. We call $d$, $\oplus$ and the above collection the degree map, the addition and the unitary action (respectively). For further use, we put $\mathcal{M} := \mathcal{M}[[1]]$ and we shall think of $\mathcal{M}$ as the space of all (square complex) matrices.

**Definition 2.1.** A standard tower is any subspace $\mathcal{T}$ of $\mathcal{M}[A]$ (for some set $A$) which satisfies the following conditions:

1. (ST0) $\mathcal{T}$ is a closed subset of $\mathcal{M}[A]$;
2. (ST1) if $(X_\lambda)_{\lambda \in \Lambda} \in \mathcal{M}[A]$ belongs to $\mathcal{T}$, then $\|X_\lambda\| \leq 1$ for all $\lambda \in \Lambda$;
3. (ST2) for each $X \in \mathcal{T}$ and $U \in \mathcal{U}_d(X)$, $U.X$ belongs to $\mathcal{T}$ as well;
4. (ST3) for any two elements $X$ and $Y$ of $\mathcal{M}[A]$,

\[ X \oplus Y \in \mathcal{T} \iff X, Y \in \mathcal{T}. \]

For any standard tower $\mathcal{T}$ and each integer $n > 0$, the subspace $\mathcal{T}_n$ is defined as $\{X \in \mathcal{T}: d(X) = n\}$. By a standard pointed tower we mean any pair of the form $(\mathcal{T}, \theta_{\mathcal{T}})$ where $\mathcal{T} \subset \mathcal{M}[A]$ is a standard tower which contains the point $\theta_{\mathcal{T}} = (Z_\lambda)_{\lambda \in \Lambda}$ with $Z_\lambda = 0$ for each $\lambda \in \Lambda$. Note that $d(\theta_{\mathcal{T}}) = 1$ for any standard pointed tower $(\mathcal{T}, \theta_{\mathcal{T}})$. 
The following result is an immediate consequence of the Tikhonov theorem (on products of compact spaces) and the definition of standard towers. We therefore omit its proof.

**Proposition 2.2.** Let $\mathcal{T}$ be a standard tower. Then:

(T0) $d : \mathcal{T} \to \mathbb{N}$ is a proper map;

(T1) $\mathcal{T}$ is a topological semigroup and for each $n > 0$, the map $\mathcal{T} \times \mathcal{T} \ni (X,Y) \mapsto X \oplus Y \in \mathcal{T}$ is a closed embedding;

(T2) $d(X \oplus Y) = d(X) + d(Y)$ and $d(U.X) = d(X)$ for any $X,Y \in \mathcal{T}$ and $U \in \mathcal{U}_d(X)$;

(T3) for each $n > 0$, the function $\mathcal{U}_n \times \mathcal{T} \ni (U,X) \mapsto U.X \in \mathcal{T}$ is a continuous group action (that is, $(UV).X = U.(V.X)$ and $I_n.X = X$ for all $U,V \in \mathcal{U}_n$ and $X \in \mathcal{T}$);

(T4) $U.X = X$ if $X$ is an arbitrary member of $\mathcal{T}$ and $U \in \mathcal{U}_d(X)$ is a scalar multiple of the unit matrix;

(T5) for any $X \in \mathcal{T}_n$, $Y \in \mathcal{T}_k$, $U \in \mathcal{U}_n$ and $V \in \mathcal{U}_k$, $(U \oplus V).(X \oplus Y) = (U.X) \oplus (V.Y)$;

(T6) for any two elements $X$ and $Y$ of $\mathcal{T}$, $U_{d(X),d(Y)}.(X \oplus Y) = Y \oplus X$ where, for any positive integers $p$ and $q$, $U_{p,q} \in \mathcal{U}_{p+q}$ is defined by the rule:

$$(w_1 \ldots w_q z_1 \ldots z_p) \cdot U_{p,q} := (z_1 \ldots z_p w_1 \ldots w_q).$$

In particular, the subspaces $\mathcal{T}_n$ are compact and $\mathcal{T}$ is locally compact.

Properties (T1)–(T6) will serve as axioms of towers (see the beginning of the next section), which are more general than m-towers. The latter structures will also have some other properties, which we shall now establish for standard towers. To this end, we introduce

**Definition 2.3.** Let $\mathcal{T}$ be a standard tower. An element $X$ of $\mathcal{T}$ is

- **reducible** if there are $A,B \in \mathcal{T}$ and $V \in \mathcal{U}_{d(X)}$ such that $V.X = A \oplus B$;
- **irreducible** if $X$ is not reducible.

The stabilizer $\text{stab}(X)$ of $X$ is defined as $\text{stab}(X) := \{U \in \mathcal{U}_{d(X)} : U.X = X\}$.

Finally, for two elements $X$ and $Y$ of $\mathcal{T}$ we shall write

- $X \equiv Y$ if $U.X = Y$ for some $U \in \mathcal{U}_{d(X)}$;
- $X \preceq Y$ if either $X \equiv Y$ or there are $A,B \in \mathcal{T}$ such that $X \equiv A$ and $Y \equiv A \oplus B$;
- $X \perp Y$ if there is no $A \in \mathcal{T}$ for which $A \preceq X$ as well as $A \preceq Y$; if this happens, we call $X$ and $Y$ **disjoint**.

Key properties of standard towers are established below.

**Proposition 2.4.** Let $X$, $Y$ and $Z$ be three elements of a standard tower $\mathcal{T}$. Then:

(mT1) if $Z \perp X$ and $Z \perp Y$, then $Z \perp X \oplus Y$;
(mT2) $X \equiv Y$ if $Z \oplus X \equiv Z \oplus Y$;

(mT3) if $X \perp Y$, then $\text{stab}(X \oplus Y) = \{U \oplus V : U \in \text{stab}(X), V \in \text{stab}(Y)\}$;

(mT4) if $X$ is irreducible and $p := d(X)$, then for each $n > 0$,

$$\text{stab}(X \oplus \cdots \oplus X) = \{I_p \otimes U : U \in \mathcal{U}_n\}.$$ 

Before passing to the proof, we recall that if $U = [z_{jk}] \in \mathcal{U}_n$, then $I_p \otimes U$ coincides with the block matrix $[z_{jk}I_p] \in \mathcal{U}_{np}$ (see, for example, [12, Section 6.6]).

**Proof of Proposition 2.4.** All items of the proposition are known consequences of (well-known) facts on von Neumann algebras and may be shown as follows. Let $A$ be a set such that $\mathcal{T} \subset \mathcal{M}[A]$. Further, for any $X = (X_\lambda)_{\lambda \in A} \in \mathcal{T}$ denote by $\mathcal{W}'(X)$ the set of all matrices $A \in \mathcal{M}_{d(X)}$ which commute with both $X_\lambda$ and $X_\lambda^*$ for any $\lambda \in A$. Then $\mathcal{W}'(X)$ is a von Neumann algebra such that

$$\mathcal{W}'(X) \cap \mathcal{U}_{d(X)} = \text{stab}(X).$$

To show (mT1) and (mT2), we consider $\mathcal{W} = (W_\lambda)_{\lambda \in A} := X \oplus Y \oplus Z$ and the projections $P, Q, R \in \mathcal{W}'(W)$ given by $P := I_{d(X)} \oplus 0_{d(Y)} \oplus 0_{d(Z)}$, $Q := 0_{d(X)} \oplus I_{d(Y)} \oplus 0_{d(Z)}$ and $R := 0_{d(X)} \oplus 0_{d(Y)} \oplus I_{d(Z)}$ (where $0_n$ denotes the zero $n \times n$ matrix). Since the restrictions of $\mathcal{W}$ to the ranges of $P$, $Q$ and $R$ are unitarily equivalent to, respectively, $X$, $Y$ and $Z$ (which means, for example, that, for $H$ denoting the range of $P$, $UW_\lambda|_H = X_\lambda U$ for all $\lambda$ and a single unitary operator $U : H \to \mathbb{C}^{d(X)}$), we conclude from [15, Proposition 2.3.1] (see also [7, Proposition 1.35]; these results are formulated for finite tuples of operators and single operators, but the finiteness assumption is superfluous) that

- $X \equiv Y$ iff the projections $P$ and $Q$ are Murray–von Neumann equivalent in $\mathcal{W}'(W)$; similarly, $Z \oplus X \equiv Z \oplus Y$ iff $R + P$ and $R + Q$ are Murray–von Neumann equivalent in $\mathcal{W}'(W)$;

- $Z \perp X$ (resp. $Z \perp Y$, $Z \perp X \oplus Y$) iff the central carriers $c_R$ and $c_P$ of $R$ and $P$ (resp. $c_R$ and $c_Q$, $c_R$ and $c_{P+Q}$) in $\mathcal{W}'(W)$ are mutually orthogonal.

Hence, under the assumption of (mT1), we conclude that $c_RCP = c_RCQ$. So, $c_R(c_P + c_Q) = 0$ and consequently $c_RCP + Q = 0$, which means that $Z \perp X \oplus Y$. This yields (mT1). Further, if $Z \oplus X \equiv Z \oplus Y$, then the projections $R + P$ and $R + Q$ are Murray–von Neumann equivalent in $\mathcal{W}'(W)$. But $\mathcal{W}'(W)$ is a finite von Neumann algebra and therefore the equivalence of $R + P$ and $R + Q$ implies the equivalence of $P$ and $Q$, which translates into $X \equiv Y$. Thus, we have shown (mT2).

To prove (mT3), observe that the inclusion “$\supset$” holds with no additional assumptions on $X$ and $Y$ (which follows e.g. from (T5)). To see the reverse
inclusion (provided \( X \perp Y \)), put \( n := d(X) \), \( k := d(Y) \) and take an arbitrary matrix \( W \in \text{stab}(X \oplus Y) \). Write \( W \) as a block matrix \( (U \quad A) \) where \( U \in \mathcal{M}_n \), \( V \in \mathcal{M}_k \), \( A \) is an \( n \times k \) and \( B \) is an \( k \times n \) matrix. Since \( W.(X \oplus Y) = X \oplus Y \) (and consequently \( W^*. (X \oplus Y) = X \oplus Y \)), we conclude that \( UX_\lambda = X_\lambda U \), \( VY_\lambda = Y_\lambda V \) and

\[
\begin{align*}
AY_\lambda &= X_\lambda A, \\
AY^*_\lambda &= X^*_\lambda A, \\
BX_\lambda &= Y_\lambda B, \\
BX^*_\lambda &= Y^*_\lambda B
\end{align*}
\]

(2.3)

for any \( \lambda \in A \) (where \( (X_\lambda)_{\lambda \in A} := X \) and \( (Y_\lambda)_{\lambda \in A} := Y \)). So, it suffices to check that both \( A \) and \( B \) are zero matrices. One infers from the first two equations in (2.3) and Schur’s lemma on intertwining operators (see, for example, [7, Theorem 1.5]) that the range \( V \) of \( A \) is a reducing subspace for each \( X_\lambda \), the orthogonal complement \( W \) of the kernel of \( A \) is a reducing subspace for each \( Y_\lambda \) and \( QY_\lambda|_W = X_\lambda|_V Q|_W \) where \( Q \) is the partial isometry which appears in the polar decomposition of \( A \). Since \( Q \) sends isometrically \( W \) onto \( V \), the above properties imply that if \( A \neq 0 \), then \( X \not\perp Y \) (one uses here (ST2) and (ST3)), which contradicts the assumption in (mT3). So, \( A = 0 \). In a similar manner, starting from the last two equations in (2.3), one shows that \( B = 0 \). This completes the proof of (mT3).

Finally, the assertion of (mT4) immediately follows from the facts that if \( X \) is irreducible, then \( W'(X) \) consists of scalar multiples of the unit matrix (thanks to (2.2) and (ST2)–(ST3)), and that if a matrix \( W \in \text{stab}(X \oplus \cdots \oplus X) \) is expressed as a block matrix \([T_{jk}]\) with \( T_{jk} \in \mathcal{M}_p \), then \( T_{jk} \in W'(X) \) for any \( j \) and \( k \).

3. M-towers. A tower is a quadruple \( \mathfrak{T} = (\mathcal{T}, d, \oplus, .) \) satisfying the following conditions:

- \( \mathcal{T} \) is a topological space;
- \( d: \mathcal{T} \to \mathbb{N} \) is a map;
- \( \oplus \) is a function from \( \mathcal{T} \times \mathcal{T} \) into \( \mathcal{T} \);
- \( . \) is a function from \( \bigcup_{n=1}^{\infty} (\mathcal{T}_n \times \mathcal{T}_n) \) into \( \mathcal{T} \) where \( \mathcal{T}_n := \{ X \in \mathcal{T} : d(X) = n \} \);
- conditions (T1)–(T6) of Proposition 2.2 hold.

If, in addition, condition (T0) is fulfilled, the tower \( \mathfrak{T} \) is said to be proper. (If this happens, the subspaces \( \mathcal{T}_n \) are compact and \( \mathcal{T} \) is locally compact.) We call \( d, \oplus \) and \( . \) the ingredients of the tower \( \mathcal{T} \).

A pointed tower is a pair \( (\mathfrak{T}, \theta) \) where \( \mathfrak{T} = (\mathcal{T}, d, \oplus, .) \) is a tower and \( \theta \) is a point in \( \mathcal{T} \) such that \( d(\theta) = 1 \). By a morphism between two towers
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$(\mathcal{T}, d, \oplus, \cdot)$ and $(\mathcal{T}', d', \oplus, \cdot)$ we mean any map $v: \mathcal{T} \to \mathcal{T}'$ such that for any $X, Y \in \mathcal{T}$ each of the following conditions is fulfilled:

(M1) $d'(v(X)) = d(X)$;
(M2) $v(U.X) = U.v(X)$ for all $U \in \mathcal{U}_d(X)$;
(M3) $v(X \oplus Y) = v(X) \oplus v(Y)$.

Similarly, a morphism between two pointed towers $(\mathfrak{T}, \theta)$ and $(\mathfrak{T}', \theta')$ is any morphism between the towers $\mathfrak{T}$ and $\mathfrak{T}'$ which sends $\theta$ onto $\theta'$.

It is easy to see that towers (as objects) with morphisms between them form a category. Thus, we can (and will) speak of isomorphisms between towers and isomorphic towers. (It is easy to see that a morphism between two towers is an isomorphism iff it is a homeomorphism between the underlying topological spaces.)

A subtower of a tower $(\mathcal{T}, d, \oplus, \cdot)$ is any closed subset $T$ of $\mathcal{T}$ which fulfills condition (ST2), and $[2.1]$ for all $X, Y \in \mathcal{T}$. It is easy to see that a subtower of $\mathcal{T}$ is a tower when it is equipped with all ingredients inherited from $\mathcal{T}$. Similarly, a pointed subtower of a pointed tower $(\mathfrak{T}, \theta)$ is any pair $(\mathfrak{T}, \theta')$ where $\mathfrak{T}$ is a subtower of $\mathfrak{T}$ containing $\theta'$ and $\theta' = \theta$.

In the same way as in Definition $2.3$ we define reducible and irreducible elements of a tower, stabilizers of its elements as well as relations “$\equiv$”, “$\triangleleft$” and “$\perp$”. Additionally, for each $n > 0$ and an element $X$ of a tower (or a matrix), we shall denote $\underbrace{X \oplus \cdots \oplus X}_n$ by $n \circ X$. Also, for simplicity, whenever $j$ runs over a finite set (known from the context) of integers and $X_j$ for each such $j$ denotes an element of a common tower (or each $X_j$ is a matrix), $\bigoplus_j X_j$ will stand for the sum of all $X_j$ arranged in accordance with the natural order of the indices $j$. Similarly, if $\mathcal{G}_j$ are arbitrary subsets of a tower (and $j$ runs over a finite set of integers), $\bigoplus_j \mathcal{G}_j$ will denote the set of all elements of the form $\bigoplus_j s_j$ where $s_j \in \mathcal{G}_j$ for each $j$. Furthermore, for any subset $\mathfrak{A}$ of a tower, $\mathcal{U}. \mathfrak{A}$ will stand for the set of all elements of the form $U.a$ where $a \in \mathfrak{A}$ and $U \in \mathcal{U}_d(a)$.

The core of a tower $\mathfrak{T}$, denoted by $\text{core}(\mathfrak{T})$, is defined as the set of all irreducible elements of $\mathfrak{T}$. Additionally, $\text{core}(\mathfrak{T})$ will stand for the closure of the core of $\mathfrak{T}$. The height of the tower $\mathfrak{T}$ is the quantity

$$\text{ht}(\mathfrak{T}) := \sup\{d(T): T \in \text{core}(\mathfrak{T})\} \in \{0, 1, 2, \ldots, \infty\}$$

(where $\sup(\emptyset) := 0$).

DEFINITION 3.1. A (possibly non-Hausdorff) topological space $X$ is said to have property $(\sigma)$ with respect to $(K_n)_{n=1}^{\infty}$ if each of the following conditions is fulfilled:

$(\sigma 1)$ $K_n \subset K_{n+1}$ for all $n$, and $X = \bigcup_{n=1}^{\infty} K_n$;
(σ2) each $K_n$ is a compact Hausdorff space in the topology inherited from $X$;
(σ3) a set $A \subset X$ is closed (in $X$) iff $A \cap K_n$ is closed for each $n$.

The space $X$ is said to have property $(\sigma)$ if $X$ has property $(\sigma)$ with respect to some $(K_n)_{n=1}^\infty$.

Now we may turn to the main topic of the section.

**Definition 3.2.** An $m$-tower is a tower $\mathcal{T} = (\mathcal{T}, d, \oplus, .)$ such that conditions (mT1)–(mT4) of Proposition 2.4 are fulfilled for all $X, Y \in \mathcal{T}$, and the topological space $\mathcal{T}$ has property $(\sigma)$.

Here the prefix “$m$” is to emphasize strong resemblance of $m$-towers to standard towers (which consist of tuples of matrices). Property $(\sigma)$ will enable us to extend certain matrix-valued functions.

**Lemma 3.3.** Let $X$ be a (possibly non-Hausdorff) topological space that has property $(\sigma)$ with respect to $(K_n)_{n=1}^\infty$. Then:
(a) for any topological space $Y$, a function $f : X \to Y$ is continuous iff $f|_{K_n} : K_n \to Y$ is continuous for any $n$; and
(b) $X$ is a paracompact Hausdorff space.

In particular, each $m$-tower is a paracompact topological space.

**Proof.** Point (a) is immediate and may be shown by testing the closedness of the inverse image under $f$ of a closed set in $Y$. We turn to (b). We see that all finite sets are closed in $X$. It is also clear that each $K_n$ is closed in $X$. To show that $X$ is normal, it suffices to check the assertion of Tietze’s theorem; that is, that every real-valued map defined on a closed set in $X$ is extendable to a real-valued map on $X$. To this end, take an arbitrary closed set $A$ in $X$ and a map $f_0 : A \to \mathbb{R}$. By induction, we may find maps $f_n : K_n \cup A \to \mathbb{R}$ such that $f_n$ extends $f_{n-1}$ for $n > 0$. Indeed, putting $K_0 := \emptyset$ and assuming $f_{n-1}$ is defined (for some $n > 0$), we first extend the restriction of $f_{n-1}$ to $K_{n-1} \cup (A \cap K_n)$ to a map $v_n : K_n \to \mathbb{R}$ and then define $f_n$ as the union of $v_n$ and $f_{n-1}$. Finally, we define the extension $f : X \to \mathbb{R}$ of $f_0$ by the rule $f(x) := f_n(x)$ for $x \in K_n$. Since $f|_{K_n}$ is continuous for each $n > 0$, we infer from (a) that $f$ itself is continuous.

We have shown above that $X$ is a normal space (with finite sets closed). It follows from our assumptions that $X$ is σ-compact (that is, $X$ is a countable union of compact sets) and therefore $X$ has the Lindelöf property (which means that every open cover of $X$ has a countable subcover). So, Theorem 3.8.11 in [6] implies that $X$ is paracompact.

**Proposition 3.4.** Each standard tower is an $m$-tower. More generally, a proper tower is an $m$-tower iff conditions (mT1)–(mT4) hold for any pair $X, Y$ of its elements.
Proof. All we need to show is that for a proper tower \((T, d, \oplus, \cdot)\), the space \(T\) has property \((\sigma)\), which is immediate: it suffices to put \(\mathcal{K}_n := d^{-1}\{1, \ldots, n\}\). (Condition \((\sigma 3)\) holds because \(T\) coincides with the topological disjoint union of all \(\mathcal{K}_n\), by \((T0)\).)

From now on, for simplicity, we shall identify an \(m\)-tower \(T = (T, d, \oplus, \cdot)\) with its underlying topological space \(T\) (and thus we shall write, for example, “\(x \in T\)” instead of “\(X \in T\)”). Additionally, \(T_n\) will stand for \(d^{-1}\{n\}\).

In \(m\)-towers a counterpart of the prime decomposition theorem (for natural numbers) holds, as shown by

**Proposition 3.5.** Let \(T\) be an \(m\)-tower.

(A) For each \(t \in T\) there is a finite system \(t_1, \ldots, t_n\) of irreducible elements of \(T\) such that \(t \equiv \bigoplus_{j=1}^n t_j\).

(B) If \(s_1, \ldots, s_k\) are systems of irreducible elements of \(T\) for which \(\bigoplus_{j=1}^n t_j \equiv \bigoplus_{j=1}^k s_j\), then \(k = n\) and there exists a permutation \(\tau\) of \(\{1, \ldots, n\}\) such that \(s_j \equiv t_{\tau(j)}\) for each \(j\).

**Proof.** To show (A), we mimic the classical proof of the prime decomposition theorem for natural numbers. We use induction on \(d(t)\). It is obvious that \(t\) is irreducible provided \(d(t) = 1\). Further, if \(t\) is irreducible, it suffices to put \(n := 1\) and \(t_1 := t\). Finally, if \(t\) is reducible, then \(t \equiv a \oplus b\) for some \(a, b \in T\). Then \(d(a) < d(t)\) and \(d(b) < d(t)\) and thus by the induction hypothesis, \(a \equiv \bigoplus_{j=1}^p a_j\) and \(b \equiv \bigoplus_{j=1}^q b_j\) for some irreducible \(a_1, \ldots, a_p, b_1, \ldots, b_q \in T\). Now \((T5)\) yields \(t \equiv (\bigoplus_{j=1}^p a_j) \oplus (\bigoplus_{j=1}^q b_j)\) and we are done.

To show (B), we employ axioms \((mT1)\), \((mT2)\) and \((T6)\) and use induction on \(\ell := \max(n, k)\). When \(\ell = 1\), we have nothing to do. Now assume \(\ell > 1\). Notice that for any irreducible elements \(\mathfrak{r}\) and \(\mathfrak{y}\) of \(T\), either \(\mathfrak{r} \equiv \mathfrak{y}\) or \(\mathfrak{r} \perp \mathfrak{y}\). Thus, if there was no \(j\) for which \(s_1 \equiv t_j\), then we would deduce that \(s_1 \perp t_j\) for all \(j\) and \((mT1)\) would imply that \(s_1 \perp t := \bigoplus_{j=1}^n t_j\), which is false, because \(s_1 \not\perp t\). So, there is \(\tau(1) \in \{1, \ldots, n\}\) for which \(s_{\tau(1)} \equiv t_{\tau(1)}\). In particular, \(d(s_1) = d(t_{\tau(1)})\), and consequently \(\sum_{j \neq \tau(1)} d(t_j) = \sum_{j=2}^k d(s_j)\).

Since at least one of these sums is positive, both \(n\) and \(k\) are greater than 1. Further, using \((T6)\), one easily shows that \(t \equiv t_{\tau(1)} \oplus t'\) where \(t' := \bigoplus_{j \neq \tau(1)} t_j\). Finally, \((mT2)\) (combined with \((T5)\)) gives \(t' \equiv \bigoplus_{j=2}^k s_j\), and now the induction hypothesis finishes the proof. ■

The proof of the following consequence of Proposition 3.5 is left as an exercise.

**Corollary 3.6.** Let \(t_1, \ldots, t_n\) be irreducible members of an \(m\)-tower \(T\).

(a) If \(\mathfrak{r} \in T\) is such that \(\mathfrak{r} \preceq \bigoplus_{j=1}^n t_j\), then there exists a nonempty set \(J \subset \{1, \ldots, n\}\) for which \(\mathfrak{r} \equiv \bigoplus_{j \in J} t_j\).
(b) If \( s_1, \ldots, s_p \) are irreducible elements of \( \mathcal{I} \), then \( \bigoplus_{j=1}^{n} t_j \perp \bigoplus_{k=1}^{p} s_k \) iff \( t_j \perp s_k \) for all \( j \) and \( k \).

**Definition 3.7.** Let \( \mathcal{I} \) be a tower. A set \( \mathcal{A} \subset \mathcal{I} \) is said to be

- **unitarily invariant** if \( \mathcal{U} \mathcal{A} \subset \mathcal{A} \);
- a **semitower** if \( \mathcal{A} \) is a closed unitarily invariant set and whenever \( t, s \in \mathcal{I} \) are such that \( t \oplus s \in \mathcal{A} \), then \( t, s \in \mathcal{A} \); equivalently, a closed set \( \mathcal{D} \) is a semitower iff each \( t \in \mathcal{I} \) for which there is \( s \in \mathcal{D} \) with \( t \preceq s \) belongs to \( \mathcal{D} \).

A function \( f : \mathcal{A} \rightarrow \mathcal{M} \) (recall that \( \mathcal{M} \) is the space of all matrices) is called **compatible** if

- \( d(f(a)) = d(a) \) for any \( a \in \mathcal{A} \); and
- \( f(U,(\bigoplus_{j=1}^{n} a_j)) = U,(\bigoplus_{j=1}^{n} f(a_j)) \) whenever \( a_1, \ldots, a_n \in \mathcal{A} \) and \( U \in \mathcal{U}_N \) with \( N = \sum_{j=1}^{n} d(a_j) \) are such that \( U,(\bigoplus_{j=1}^{n} a_j) \in \mathcal{A} \).

Since \( \mathcal{M} \) is a topological space, we may, of course, speak of compatible maps. Additionally, we call a compatible function \( f : \mathcal{A} \rightarrow \mathcal{M} \) **bounded** if

\[
(\|f\| := \sup_{a \in \mathcal{A}} \|f(a)\| < \infty)
\]

(where \( \sup(\emptyset) := 0 \)).

We denote by \( C^*_\mathcal{I}(\mathcal{A}) \) the set of all \( \mathcal{M} \)-valued compatible maps defined on \( \mathcal{A} \) that are bounded. It is easy to see that \( C^*_\mathcal{I}(\mathcal{A}) \) is a \( C^* \)-algebra when all algebraic operations are defined pointwise; and \( C^*_\mathcal{I}(\mathcal{A}) \) is unital provided \( \mathcal{A} \neq \emptyset \) (the unit \( j_\mathcal{A} \) of \( C^*_\mathcal{I}(\mathcal{A}) \) is constantly equal to \( I_n \) on each \( \mathcal{A} \cap \mathcal{I}_n \)).

The \( C^* \)-algebra \( C^*(\mathcal{I}) := C^*_\mathcal{I}(\mathcal{I}) \) is called the **\( C^* \)-algebra over \( \mathcal{I} \)\). Similarly, the \( C^* \)-algebra over a pointed tower \( (\mathcal{I}, \theta) \), denoted by \( C^*(\mathcal{I}, \theta) \), consists of all maps \( f \in C^*(\mathcal{I}) \) that vanish at \( \theta \). It is easy to see that \( C^*(\mathcal{I}, \theta) \) is an ideal in \( C^*(\mathcal{I}) \).

It is readily seen that a subtower of an \( m \)-tower is itself an \( m \)-tower, that a finite union of semitowers (in any tower) is a semitower and that \( C^*(\mathcal{S}) = C^*_\mathcal{I}(\mathcal{S}) \) for any subtower \( \mathcal{S} \) of a tower \( \mathcal{I} \). A strong property of \( m \)-towers is formulated below.

**Theorem 3.8.** Let \( \mathcal{I} \) be an \( m \)-tower, \( \mathcal{A} \) a closed set in \( \mathcal{I} \) such that \( \mathcal{A} \subset \text{core}(\mathcal{I}) \) and let \( \mathcal{S} \) be a semitower of \( \mathcal{I} \). Further, let \( f \in C^*_\mathcal{I}(\mathcal{A}) \) and \( g \in C^*_\mathcal{I}(\mathcal{S}) \) be maps such that \( f|_{\mathcal{A} \cap \mathcal{S}} = g|_{\mathcal{A} \cap \mathcal{S}} \). Then there is \( h \in C^*(\mathcal{I}) \) which extends both \( f \) and \( g \) and \( \|h\| = \max(\|f\|, \|g\|) \).

In the proof of Theorem 3.8, we shall need the next five lemmas, in which we assume \( \mathcal{I} \) is an \( m \)-tower. The proof of the first of them is left as an exercise.

**Lemma 3.9.** Let \( \mathcal{A} \) be a closed set in \( \mathcal{I} \). Then \( \mathcal{U} \mathcal{A} \) is closed and any map \( u \in C^*_\mathcal{I}(\mathcal{A}) \) admits a unique extension to a map \( v \in C^*_\mathcal{I}(\mathcal{U} \mathcal{A}) \). Moreover, \( \|v\| = \|u\| \).
The following is a key lemma.

**Lemma 3.10.** Let $t_1, \ldots, t_n$ be a system of irreducible elements of $\mathfrak{T}$, $\tau$ be a permutation of $\{1, \ldots, n\}$ and let $N := \sum_{j=1}^{n} d(t_j)$. Further, let $A_1, \ldots, A_n$ be any system of matrices such that

$$(ax1) \quad d(A_j) = d(t_j) \text{ for each } j; \text{ and}$$

$$(ax2) \quad \text{whenever } j, k \in \{1, \ldots, n\} \text{ and } V \in \mathcal{U}_{d(t_j)} \text{ are such that } V.t_j = t_k, \text{ then } V.A_j = A_k.$$ 

Then $U.(\bigoplus_{j=1}^{n} A_j) = \bigoplus_{j=1}^{n} A_{\tau(j)}$ for any $U \in \mathcal{U}_N$ for which $U.(\bigoplus_{j=1}^{n} t_j) = \bigoplus_{j=1}^{n} t_{\tau(j)}$.

**Proof.** First we assume $\tau$ is the identity. For any positive integers $p$ and $q$, let $U_{p,q}$ be as specified in (T6). Let $\xi$ be a permutation of $\{1, \ldots, n\}$ such that for some finite increasing sequence $\nu_0, \ldots, \nu_n$ of integers with $\nu_0 = 0$ and $\nu_n = n$ one has (for any $s \in \{1, \ldots, N\}$):

$(A1) \quad t_{\xi(j)} \equiv t_{\xi(\nu(s))}$ whenever $\nu_{s-1} < j < \nu_s$; and

$(A2) \quad t_{\xi(\nu(s))} \perp t_{\xi(\nu(s'))}$ whenever $s' \in \{1, \ldots, N\}$ differs from $s$.

For each $s \in \{1, \ldots, N\}$, we put $\beta_s := \nu(s) - \nu(s - 1)$. Employing (T6), we see that there is $W \in \mathcal{U}_N$ that is the product of a finite number of the matrices $U_{p,q}$ and satisfies

$$(3.1) \quad W.(\bigoplus_{j=1}^{n} t_j) = \bigoplus_{j=1}^{n} t_{\xi(j)}.$$ 

The specific form of $W$ also yields

$$(3.2) \quad W.(\bigoplus_{j=1}^{n} A_j) = \bigoplus_{j=1}^{n} A_{\xi(j)}.$$ 

Further, for any $s \in \{1, \ldots, N\}$ and each $j$ with $\nu_{s-1} < j \leq \nu_s$ we choose $V_j \in \mathcal{U}_{d(t_{\xi(\nu(s))})}$ such that $V_j.t_{\xi(j)} = t_{\xi(\nu(s))}$ (see (A1)). Put $V := \bigoplus_{j=1}^{n} V_j$. We infer from (T5) that

$$(3.3) \quad V.(\bigoplus_{j=1}^{n} t_{\xi(j)}) = \bigoplus_{s=1}^{N} (\beta_s \odot t_{\nu(s)})$$

and from (ax2) that

$$(3.4) \quad V.(\bigoplus_{s=1}^{N} A_{\xi(s)}) = \bigoplus_{s=1}^{N} (\beta_s \odot A_{\nu(s)}).$$ 

Further, (A2) and (mT1) imply that $\beta_s \odot t_{\nu(s)} \perp \bigoplus_{s' \neq s} (\beta_{s'} \odot t_{\nu(s')})$ for each $s \in \{1, \ldots, N\}$. So, one deduces from (mT3)–(mT4) that $\text{stab}(\bigoplus_{s=1}^{N} \beta_s \odot t_s) =$
\{ \bigoplus_{s=1}^{N} (I_{d(t_s)} \otimes U_s) : U_s \in \mathcal{U}_{\beta_s} \}. Consequently, it is easy to verify that

\begin{equation}
(3.5) \quad U. \left( \bigoplus_{s=1}^{N} (\beta_s \otimes A_{\nu(s)}) \right) = \bigoplus_{s=1}^{N} (\beta_s \otimes A_{\nu(s)}), \quad U \in \text{stab} \left( \bigoplus_{s=1}^{N} \beta_s \otimes t_{\nu(s)} \right).
\end{equation}

Now, if \( U \in \text{stab}(\bigoplus_{j=1}^{n} t_j) \), then \( VWUW^{-1}V^{-1} \in \text{stab}(\bigoplus_{s=1}^{N} \beta_s \otimes t_{\nu(s)}) \), by (T3), (3.1) and (3.3). A combination of (3.5), (3.4) and (3.2) yields \( U.(\bigoplus_{j=1}^{n} A_j) = (\bigoplus_{j=1}^{n} A_j) \). So, in the case when \( \tau \) is the identity, the proof is finished.

Now assume \( \tau \) is arbitrary. We may find \( W \in \mathcal{U}_N \) that is the product of a finite number of the matrices \( U_{p,q} \) and satisfies \( W.(\bigoplus_{j=1}^{n} t_j) = \bigoplus_{j=1}^{n} t_{\tau(j)} \) (by (T5)) as well as \( W.(\bigoplus_{j=1}^{n} A_j) = \bigoplus_{j=1}^{n} A_j \). We see that if \( U \in \mathcal{U}_N \) is as specified in the lemma, then \( W^{-1}U \in \text{stab}(\bigoplus_{j=1}^{n} t_j) \). So, it follows from the first part of the proof that \( W^{-1}U.(\bigoplus_{j=1}^{n} A_j) = \bigoplus_{j=1}^{n} A_j \) and consequently \( U.(\bigoplus_{j=1}^{n} A_j) = W.(\bigoplus_{j=1}^{n} A_j) \), which completes the proof. \( \blacksquare \)

**Lemma 3.11.** If \( \mathfrak{F} \) is a closed set in \( \mathfrak{T}_N \), then every map \( u \in C^*_T(\mathfrak{F}) \) extends to a map \( v \in C^*_T(\mathfrak{T}_N) \) such that \( ||v|| = ||u||. \)

**Proof.** It follows from Lemma 3.9 that \( u \) extends to a map \( u_1 \in C^*_T(\mathcal{U}.\mathfrak{F}) \) such that \( ||u_1|| = ||u||. \). We conclude from Lemma 3.3 that \( \mathfrak{T}_N \) is a normal topological space. Denote by \( \Omega \) the set of all matrices \( A \in \mathcal{M}_N \) whose norms do not exceed \( ||u||. \). Since \( \Omega \) is convex and compact, it is a retract of \( \mathcal{M}_N. \) So, it follows from Tietze’s extension theorem that there is a map \( w : \mathfrak{T}_N \to \Omega \) which extends \( u_1. \) We define \( v : \mathfrak{T}_N \to \Omega \) by \( v(t) := \int_{\mathcal{U}_N} U^{-1}.w(U.t) \lambda(U) \) where \( \lambda \) is the Haar (probability) measure of \( \mathcal{U}_N \) and the integral is understood entrywise. We leave to the reader the verification of all the desired properties of \( v. \) \( \blacksquare \)

**Lemma 3.12.** Let \( \mathcal{D} \) be a closed set in \( \mathfrak{T} \) that contains all irreducible elements \( t \) of \( \mathfrak{T} \) for which there is \( \mathfrak{d} \in \mathcal{D} \) with \( t \preceq \mathfrak{d}. \) Then the set

\begin{equation}
(3.6) \quad \langle \mathcal{D} \rangle := \left\{ U.(\bigoplus_{j=1}^{n} \mathfrak{d}_j) : n > 0, \mathfrak{d}_1, \ldots, \mathfrak{d}_n \in \mathcal{D} \cap \text{core}(\mathfrak{T}), U \in \mathcal{U}_N \text{ where } N = \sum_{j=1}^{n} d(\mathfrak{d}_j) \right\}
\end{equation}

is a subtower of \( \mathfrak{T} \) that contains \( \mathcal{D}. \)

**Proof.** It follows from the assumptions of the lemma, (T3), (T5) and Corollary 3.6 that \( \langle \mathcal{D} \rangle \) contains \( \mathcal{D} \) and satisfies (ST2) and (2.1). So, the main issue is the closedness of \( \langle \mathcal{D} \rangle \). Since the sets \( \mathfrak{T}_N \) are clopen (because \( d \) is a map), it suffices to show that \( \langle \mathcal{D} \rangle \cap \mathfrak{T}_N \) is closed for every \( N > 0. \) Note that \( \langle \mathcal{D} \rangle \cap \mathfrak{T}_N \) is the union of a finite number of sets of the form
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\( \mathcal{U} \cdot (\bigoplus_{j=1}^{k} (D \cap T_{\nu_j})) \) where \( \sum_{j=1}^{k} \nu_j = N \). But each of the aforementioned sets is closed, thanks to (T1) and Lemma 3.9. 

**Lemma 3.13.** For any semitower \( D \) of \( T \) there exists a subtower \( E \) such that each \( u \in C^*_T(D) \) extends to a map \( w \in C^*(E) \) with \( \|w\| = \|u\| \).

**Proof.** Let \( E := \langle D \rangle \) where \( \langle D \rangle \) is given by (3.6). It follows from Lemma 3.12 that \( E \) is a subtower. We define a function \( w : E \to M \) by the rule

\[
 w\left(U \left( \bigoplus_{j=1}^{n} \varnothing_j \right) \right) := U \left( \bigoplus_{j=1}^{n} u(\varnothing_j) \right)
\]

where \( n > 0 \), each \( \varnothing_j \in D \) is irreducible, and \( U \in \mathcal{U}_N \) with \( N = \sum_{j=1}^{n} d(\varnothing_j) \). The main point is that \( w \) is well defined. To show this, assume (3.7)

\[
 U \left( \bigoplus_{j=1}^{n} t_j \right) = V \left( \bigoplus_{j=1}^{k} s_j \right)
\]

(where \( t_j, s_j \in D \) are irreducible and \( U \) and \( V \) are unitary matrices of respective degrees). We need to prove that

\[
 (3.8) \quad U \left( \bigoplus_{j=1}^{n} u(t_j) \right) = V \left( \bigoplus_{j=1}^{k} u(s_j) \right).
\]

Relation (3.7) means, in particular, that \( \bigoplus_{j=1}^{n} t_j \equiv \bigoplus_{j=1}^{k} s_j \). So, we infer from Proposition 3.5 that \( k = n \) and there is a permutation \( \tau \) of \( \{1, \ldots, n\} \) such that \( s_j \equiv t_{\tau(j)} \). Let \( W_j \in \mathcal{U}_{d(s_j)} \) be such that \( W_j \cdot s_j = t_{\tau(j)} \). Put \( W := \bigoplus_{j=1}^{n} W_j \). We conclude from (T5), (T3) and (3.7) that

\[
 (3.9) \quad (WV^{-1}U) \cdot \left( \bigoplus_{j=1}^{n} t_j \right) = \bigoplus_{j=1}^{n} t_{\tau(j)}.
\]

Further, we claim that for \( A_j := u(t_j) \) conditions (ax1) and (ax2) of Lemma 3.10 are fulfilled. Indeed, (ax1) is immediate, whereas (ax2) follows from the fact that \( u \in C^*_T(D) \). So, Lemma 3.10 combined with (3.9), yields

\[
 (WV^{-1}U) \cdot \left( \bigoplus_{j=1}^{n} u(t_j) \right) = \bigoplus_{j=1}^{n} u(t_{\tau(j)}),
\]

which simply leads us to (3.8).

Further, it follows from the very definition of \( w \) that \( w \) is bounded, \( \|w\| = \|u\| \), \( w \) extends \( u \), and that \( w \in C^*(E) \) provided \( w \) is continuous. Taking these remarks into account, it remains to establish the continuity of \( w \). Notice that \( w \) is continuous iff it is so on each of the sets \( E_N := E \cap T_N \). Arguing as in the proof of Lemma 3.12, we see that it is sufficient that \( w \) be
continuous on each set of the form \( \mathcal{U} \cdot (\bigoplus_{j=1}^{n} \mathcal{E}_{\nu_j}) \) (which is closed), which readily follows from axiom (T1) and Lemma 3.9.

Proof of Theorem 3.8. By induction, we construct sequences \( \mathcal{E}_0, \mathcal{E}_1, \ldots \) and \( h_0, h_1, \ldots \) of subtowers of \( \mathfrak{T} \) and maps (respectively) such that for each \( k \geq 0 \):

1. \( \mathcal{E}_k \supset \mathcal{E}_{k-1} \cup \bigcup_{j=1}^{k} \mathfrak{T}_j \) for \( k > 0 \), and \( \mathcal{E}_0 \supset \mathfrak{A} \cup \mathfrak{S} \);
2. \( h_k \in C^*(\mathcal{E}_k) \) and \( h_k \) extends \( h_{k-1} \) provided \( k > 0 \); and \( h_0 \) extends both \( f \) and \( g \);
3. \( \| h_k \| \leq R := \max(\|f\|, \|g\|) \).

It follows from Lemma 3.9 that there is \( f_1 \in C^*_\mathfrak{T}(\mathcal{U}, \mathfrak{A}) \) which extends \( f \) and has the same norm. Notice that \( \mathcal{U}, \mathfrak{A} \) is a semitower (since \( \mathcal{U}, \mathfrak{A} \subset \text{core}(\mathfrak{T}) \)) and the maps \( f_1 \) and \( g \) agree (because \( \mathfrak{S} \) is unitarily invariant) and their union belongs to \( C^*_\mathfrak{T}(\mathcal{U}, \mathfrak{A} \cup \mathfrak{S}) \). So, \( \mathcal{U}, \mathfrak{A} \cup \mathfrak{S} \) is a semitower as well and we conclude from Lemma 3.13 that there is a subtower \( \mathcal{E}_0 \) of \( \mathfrak{T} \) and a map \( h_0 \in C^*(\mathcal{E}_0) \) that extends both \( f_1 \) and \( g \). We see that conditions (1)-(3) are fulfilled. Now assume \( \mathcal{E}_{k-1} \) and \( h_{k-1} \) are already defined (for some \( k > 0 \)).

First apply Lemma 3.13 with \( N = k \), \( \mathfrak{F} = \mathcal{E}_{k-1} \cap \mathfrak{T}_k \) and \( u = h_{k-1} | \mathfrak{F} \) to obtain a map \( w_k \in C^*_\mathfrak{T}(\mathfrak{F}) \) that agrees with \( h_{k-1} \) on \( \mathcal{E}_{k-1} \cap \mathfrak{T}_k \) and satisfies \( \| w_k \| \leq R \). Observe that \( \mathfrak{D}_k := \mathcal{E}_{k-1} \cup \mathfrak{T}_k \) is a semitower (by (1)-(1)) and the union \( u_k \) of \( w_k \) and \( h_{k-1} \) belongs to \( C^*_\mathfrak{T}(\mathfrak{D}_k) \) (because \( \mathcal{E}_{k-1} \) is a semitower).

Next apply Lemma 3.13 (with \( \mathcal{D} = \mathfrak{D}_k \) and \( u = u_k \)) to obtain \( \mathcal{E}_k \) and \( h_k \).

Finally, we define a function \( h: \mathfrak{T} \to \mathcal{M} \) by \( h(t) := h_n(t) \) for \( t \in \mathfrak{T}_n \) (see (1)).

We leave it to the reader to check that \( h \) is a map we searched for.

**Corollary 3.14.** Let \( \mathfrak{T} \) be an m-tower.

(A) For any sequence \( t_1, t_2, \ldots \) of mutually disjoint irreducible elements of \( \mathfrak{T} \) such that the set \( \{ j : d(t_j) = n \} \) is finite for each \( n \), and for any sequence \( A_1, A_2, \ldots \) of matrices with \( d(A_j) = d(t_j) \) \( (j \geq 1) \) and \( M := \sup_{j \geq 1} ||A_j|| < \infty \), there is \( f \in C^*(\mathfrak{T}) \) such that \( f(t_j) = A_j \) for all \( j \) and \( ||f|| = M \).

(B) If \( \mathfrak{S} \) is a subtower of \( \mathfrak{T} \) and \( t \in \text{core}(\mathfrak{T}) \setminus \mathfrak{S} \), then for any matrix \( A \) with \( d(A) = d(t) \) and each \( u \in C^*(\mathfrak{S}) \) there is \( v \in C^*(\mathfrak{T}) \) that extends \( u \) and satisfies \( v(t) = A \) and \( ||v|| = \max(||A||, ||u||) \).

(C) If \( t_1 \) and \( t_2 \) are distinct elements of \( \mathfrak{T} \), then there exists \( g \in C^*(\mathfrak{T}) \) for which \( g(t_1) \neq g(t_2) \).

**Proof.** Both (A) and (B) are special cases of Theorem 3.8. We turn to (C). First assume that \( t_1 \neq t_2 \) (which is a simpler case). Let \( a_{1}^{(2)}, \ldots, a_{p_1}^{(2)} \) and \( a_{1}^{(2)}, \ldots, a_{p_2}^{(2)} \) be systems of mutually disjoint irreducible elements of \( \mathfrak{T} \) such that, for \( \varepsilon = 1, 2 \), \( t_\varepsilon = U_\varepsilon \cdot (\bigoplus_{j=1}^{p_\varepsilon} (a_{j}^{(\varepsilon)} \oplus a_{j}^{(\varepsilon)})) \) for some unitary matrix \( U_\varepsilon \) and positive integers \( \alpha_1^{(\varepsilon)}, \ldots, \alpha_{p_\varepsilon}^{(\varepsilon)} \) (consult Proposition 3.5). Since \( t_1 \neq t_2 \), one
concludes that either some of \( a_j^{(1)} \) is disjoint from each of \( a_k^{(2)} \) (or conversely), or there are \( k_1 \in \{1, \ldots, p_1\} \) and \( k_2 \in \{1, \ldots, p_2\} \) such that \( a_{k_1}^{(1)} \equiv a_{k_2}^{(2)} \) and \( \alpha_{k_1}^{(1)} \neq \alpha_{k_2}^{(2)} \). Thus, we may assume, with no loss of generality, that either \( a_1^{(1)} \) is disjoint from each of \( a_k^{(2)} \), or \( a_1^{(1)} \equiv a_1^{(2)} \) and \( \alpha_1^{(1)} \neq \alpha_1^{(2)} \). In the former case we conclude that there is \( g \in C^*(\mathcal{T}) \) that vanishes at each \( a_k^{(2)} \) and each \( a_j^{(1)} \) with \( j \neq 1 \), and \( g(a_1^{(1)}) \neq 0 \). Then \( g(t_1) \neq 0 = g(t_2) \) and we are done.

In the latter case we may find \( g \in C^*(\mathcal{T}) \) that vanishes at each \( a_j^{(1)} \) and \( a_k^{(2)} \) with \( j > 1 \) and \( k > 1 \), and \( g(a_1^{(1)}) = g(a_2^{(1)}) = I_n \) where \( n := d(a_1^{(1)}) \). Then, for \( \varepsilon = 1, 2 \), \( g(t_\varepsilon) \) is a projection of rank \( \alpha_\varepsilon n \) and hence \( g(t_1) \neq g(t_2) \). This finishes the proof in the case when \( t_1 \neq t_2 \).

Finally, we assume \( t_1 \equiv t_2 \). Then we can find two unitary matrices \( U_1 \) and \( U_2 \), and finite systems \( a_1, \ldots, a_p \) and \( \alpha_1, \ldots, \alpha_p \) of, respectively, mutually disjoint irreducible elements of \( \mathcal{T} \) and positive integers such that \( t_\varepsilon = U_\varepsilon \cdot (\bigoplus_{j=1}^p (\alpha_j \odot a_j)) \) for \( \varepsilon = 1, 2 \). Since \( t_1 \neq t_2 \), we see that

\[
(3.10) \quad U_2^{-1} U_1 \notin \text{stab}(\bigoplus_{j=1}^p (\alpha_j \odot a_j)).
\]

Axioms (mT1)–(mT4) imply that

\[
(3.11) \quad \text{stab}(\bigoplus_{j=1}^p (\alpha_j \odot a_j)) = \left\{ \bigoplus_{j=1}^p (I_{d(a_j)} \otimes V_j) : V_j \in \mathcal{U}_{\alpha_j} \right\}.
\]

Now take a system \( A_1, \ldots, A_p \) of mutually disjoint (that is, mutually unitarily inequivalent) irreducible matrices such that \( d(A_j) = d(a_j) \). Let \( g \in C^*(\mathcal{T}) \) be such that \( g(a_j) = A_j \) for each \( j \). We claim that then \( g(t_1) \neq g(t_2) \), which follows from (3.10), (3.11) and the relation \( \text{stab}(\bigoplus_{j=1}^p (\alpha_j \odot a_j)) = \mathcal{W}'(\bigoplus_{j=1}^p (\alpha_j \odot A_j)) \cap \mathcal{U}_N \) (where \( N = \sum_{j=1}^p d(A_j) \)). The details are left to the reader.

**Lemma 3.15.** The core of any tower is an open set. If \( \mathcal{T} \) is an \( m \)-tower, then an element \( \mathfrak{x} \in \mathcal{T} \) is irreducible iff \( \text{stab}(\mathfrak{x}) \) consists only of the scalar multiples of the unit matrix.

**Proof.** Let \( \mathcal{S} \) be a tower. Note that \( \text{core}(\mathcal{S}) \) is open iff \( \mathcal{S}_N \setminus \text{core}(\mathcal{S}) \) is closed for any \( N \). To show the latter statement, observe that \( \mathcal{S}_N \setminus \text{core}(\mathcal{S}) \) coincides with the finite union of all sets of the form \( \mathcal{U}_N \cdot (\bigoplus_{j=1}^n \mathcal{S}_{\nu_j}) \) where \( \nu_1, \ldots, \nu_n \) are positive integers which sum to \( N \), and \( n > 1 \). It follows from (T1) and Lemma 3.15 that each of the aforementioned sets is closed, which proves the first claim of the lemma. In order to show the second, we only need to focus on the “if” part (because the “only if” part is included in (mT4)). To this end, assume \( \mathfrak{x} \) is reducible. Then there are a matrix \( U \in \mathcal{U}_{d(\mathfrak{x})} \) and
two elements $t$ and $s$ of $\mathcal{T}$ such that $r = U(t \oplus s)$. One may then easily deduce from (T3) and (T5) that $\text{stab}(r)$ contains matrices different from scalar multiples of the unit matrix. ■

We skip the proofs of the next two results (since they are immediate consequences of Corollary 3.14 and Lemma 3.15).

**Corollary 3.16.** For any element $t$ of an $m$-tower $\mathcal{T}$, the assignment $f \mapsto f(t)$ correctly defines a unital finite-dimensional representation $\pi_t: C^*(\mathcal{T}) \to \mathcal{M}_{d(t)}$. Moreover, $\pi_t$ is irreducible iff $t$ is irreducible, and the assignment $t \mapsto \pi_t$ is one-to-one.

**Corollary 3.17.** For any $m$-tower $\mathcal{T}$, the center of $C^*(\mathcal{T})$ coincides with the set of all $u \in C^*(\mathcal{T})$ such that $u(t)$ is a scalar multiple of the unit matrix for any $t \in \text{core}(\mathcal{T})$.

The next result explains the importance of axioms (mT1)–(mT4).

**Proposition 3.18.** A tower is isomorphic to a standard tower iff it is a proper $m$-tower.

**Proof.** The “only if” part readily follows from Proposition 3.4, so we shall focus on the “if” part. We shall apply Theorem 3.8. Assume $\mathcal{T}$ is a proper $m$-tower. In particular, $\mathcal{T}_n$ is a compact space for each $n$. Denote by $\Lambda$ the closed unit ball of $C^*(\mathcal{T})$ and define $\Phi: \mathcal{T} \to \mathcal{M}[\Lambda]$ by $(\Phi(t))(\lambda) := \lambda(t)$ ($t \in \mathcal{T}$, $\lambda \in \Lambda$). Then $\Phi$ is well defined, continuous and satisfies axioms (M1)–(M3) of a morphism.

We claim that $\Phi$ is a closed embedding. To see that, observe that it suffices that $\Phi$ is one-to-one on each of the sets $\mathcal{T}_n$ (because $\mathcal{T}_n$ is compact and clopen, and $\Phi(\mathcal{T}_n) \subset \mathcal{M}_n[A]$, which is covered by Corollary 3.14).

Finally, let us briefly show that $\mathcal{T} := \Phi(\mathcal{T})$ is a standard tower. Axioms (ST0)–(ST2) and the implication “$\subseteq$” in (ST3) are transparent. To end the proof, assume $\Phi(t) = a \oplus b$ where $t \in \mathcal{T}$, and $a = (A_\lambda)_{\lambda \in \Lambda}$ and $b = (B_\lambda)_{\lambda \in \Lambda}$ are members of $\mathcal{M}[A]$. We only need to prove that $a, b \in \mathcal{T}'$. To this end, we denote by $\mathcal{S}$ the largest standard tower contained in $\mathcal{M}[A]$, that is, $\mathcal{S}$ consists of all $(X_\lambda)_{\lambda \in \Lambda} \in \mathcal{M}[A]$ with $\|X_\lambda\| \leq 1$ for each $\lambda$. (It is easy to check that $\mathcal{S}$ is indeed a standard tower, and $\mathcal{T}' \subset \mathcal{S}$.) It follows from Proposition 3.5 that there are irreducible elements $t_1, \ldots, t_n$ of $\mathcal{T}$ such that $t \equiv \bigoplus_{j=1}^n t_j$. Since $\Phi$ is one-to-one, one easily infers that each of $\Phi(t_j)$ is irreducible in $\mathcal{S}$ (because $\text{stab}(\Phi(x)) = \text{stab}(x)$ for any $x \in \mathcal{T}$). Observe that $(a \oplus b) = \Phi(t) \equiv \bigoplus_{j=1}^n \Phi(t_j)$. So, by Corollary 3.6 (applied to the m-tower $\mathcal{S}$) there are nonempty subsets $J$ and $J'$ of $\{1, \ldots, n\}$ such that $a \equiv \bigoplus_{j \in J} \Phi(t_j) (= \Phi(\bigoplus_{j \in J} t_j))$ and $b \equiv \bigoplus_{j \in J'} \Phi(t_j) (= \Phi(\bigoplus_{j \in J'} t_j))$. These two relations easily imply that both $a$ and $b$ are values of $\Phi$, which is equivalent to $a, b \in \mathcal{T}'$. ■
The following simple result will prove useful later. We skip its proof.

**Lemma 3.19.** Let $\mathfrak{T}$ be an $m$-tower. Then, for every $f \in C^*(\mathfrak{T})$,
\[
\|f\| = \sup \{\|f(t)\| : t \in \text{core}(\mathfrak{T})\}.
\]

### 4. Essentially locally compact $m$-towers

**Definition 4.1.** An $m$-tower $T$ is said to be **essentially locally compact** (briefly, elc) if the space $\text{core}(T)$ is locally compact. If this happens, the $C^*$-algebra $C^*_0(T)$ is defined as the $\ast$-subalgebra of $C^*(T)$ consisting of all maps whose restrictions to $\text{core}(T)$ vanish at infinity. More precisely, a map $f \in C^*(T)$ belongs to $C^*_0(T)$ iff for each $\varepsilon > 0$ there is a compact set $K \subset \text{core}(T)$ such that $\|f(t)\| \leq \varepsilon$ for any $t \in \text{core}(T) \setminus K$.

Similarly, we call a pointed $m$-tower $(\mathfrak{T}, \theta)$ **essentially locally compact** (elc) if the tower $\mathfrak{T}$ is so. In that case we put $C^*_0(\mathfrak{T}, \theta) := C^*(\mathfrak{T}, \theta) \cap C^*_0(\mathfrak{T})$.

Note that $C^*_0(T)$ (respectively, $C^*_0(\mathfrak{T}, \theta)$) is an ideal in $C^*(T)$ (respectively, in $C^*(\mathfrak{T}, \theta)$).

The reader should notice that every proper $m$-tower is elc and that a subtower of an elc $m$-tower is elc as well. Observe also that for an elc tower $\mathfrak{T}$, condition $C^*_0(\mathfrak{T}) = C^*(\mathfrak{T})$ implies $\mathfrak{T}$ has finite height.

For $C^*$-algebras of the form $C^*_0(\mathfrak{T})$ where $\mathfrak{T}$ is an elc $m$-tower all ideals as well as all nondegenerate finite-dimensional representations can be simply characterized. The results on these topics shall be derived from the next result, which is a special case of our **Stone–Weierstrass theorem for elc $m$-towers** (see Theorem 4.14 at the end of the section).

**Theorem 4.2.** Let $\mathfrak{T}$ be an elc $m$-tower and $E$ a $\ast$-subalgebra of $C^*_0(\mathfrak{T})$. Assume that for any irreducible element $t$ of $\mathfrak{T}$, either

1. $g(t) = 0$ for any $g \in E$; or
2. if $s \in \mathfrak{T}$ is irreducible and $s \perp t$, then there is $g \in E$ with $g(t) = I_d(t)$ and $g(s) = 0$.

Then the uniform closure of $E$ in $C^*_0(\mathfrak{T})$ consists of all maps $u \in C^*_0(\mathfrak{T})$ such that

\[\ast\] for any $x \in \text{core}(\mathfrak{T})$ there exists $v \in E$ with $v(x) = u(x)$.

As we have already said, the above result is a special case of Theorem 4.14 but it is also a key step in the proof of the latter.

In the proof of Theorem 4.2 we shall apply the next two results, recently discovered by us. The first of them is another variation of the Stone–Weierstrass theorem.

**Theorem 4.3 ([16]).** Let $K$ be a compact space and $\mathcal{D}$ be a unital $C^*$-algebra. Let $\mathcal{L}$ be a $\ast$-subalgebra of $C(K, \mathcal{D})$ such that for any two points $x$ and $y$ of $K$, either

(SW1) there exists \( u \in \mathcal{L} \) such that \( u(x) \) and \( u(y) \) are normal elements of \( \mathcal{D} \) with disjoint spectra; or

(SW2) the spectra of \( f(x) \) and \( f(y) \) (computed in \( \mathcal{D} \)) coincide for any self-adjoint \( f \in \mathcal{L} \).

Then the (uniform) closure of \( \mathcal{L} \) in \( C(K, \mathcal{D}) \) coincides with the \( * \)-algebra \( \Delta_2(\mathcal{L}) \) of all maps \( u \in C(K, \mathcal{D}) \) such that for any \( x, y \in K \) and each \( \varepsilon > 0 \) there exists \( v \in \mathcal{L} \) with \( \| v(z) - u(z) \| < \varepsilon \) for \( z \in \{x, y\} \).

(The above theorem is a special case of a result formulated and proved in [16].) For a more general result in this direction the reader is referred to the celebrated papers by Longo [14] and Popa [18] (consult also [11], [19, §4.7] and [4, Chapter 11]).

The second tool reads as follows.

**Theorem 4.4** ([17]). Let \( \Omega \) be a locally compact space and \( \mathcal{B} \) a countable collection of pairwise disjoint Borel subsets of \( \Omega \) that cover \( \Omega \). Further, let \( \mathcal{A} \) be a \( C^* \)-algebra and \( V \) a \( * \)-subalgebra of \( C_0(\Omega, \mathcal{A}) \). The (uniform) closure of \( V \) consists of all maps \( f \in C_0(\Omega, \mathcal{A}) \) such that \( f|_L \) belongs to the uniform closure of \( V|_L := \{g|_L : g \in V\} \subset C(L, \mathcal{A}) \) for each \( L \subset \Omega \) such that

(\( ** \)) the set \( L \cap B \) is compact for each \( B \in \mathcal{B} \) and nonempty only for a finite number of such \( B \).

For similar, but much stronger and more general results than Theorem 4.4, consult [17].

For better transparency, let us isolate a part of the proof of Theorem 4.2 in the lemma below.

**Lemma 4.5.** Let \( \mathfrak{T} \) be an elc \( m \)-tower. For a finite increasing sequence \( \nu = (\nu_1, \ldots, \nu_n) \) of positive integers, put

\[
B(\nu) := \left\{ U. \left( \bigoplus_{j=1}^{\nu} t_j \right) : U \in \mathcal{U}_{|\nu|}, t_j \in \text{core}(\mathfrak{T}) \cap \mathfrak{T}_{\nu_j} \right\} \cap \overline{\text{core}}(\mathfrak{T})
\]

where \( |\nu| := \sum_{j=1}^{n} \nu_j \). Then all sets of the form \( B(\nu) \) (where \( \nu \) runs over all finite increasing sequences of positive integers) are pairwise disjoint, Borel and cover \( \overline{\text{core}}(\mathfrak{T}) \).

**Proof.** (To avoid misunderstandings, we recall that a sequence \( (\nu_1, \ldots, \nu_n) \) is increasing if \( \nu_{j-1} \leq \nu_j \) for any \( j \in \{1, \ldots, n\} \) different from 1.) For simplicity, denote by \( \mathcal{B} \) the collection of all sets of the form \( B(\nu) \). It follows from Proposition 3.5 and (T6) that \( \mathcal{B} \) consists of pairwise disjoint sets that cover \( \overline{\text{core}}(\mathfrak{T}) \). So, we only need to show that \( B(\nu) \) is a Borel set. To this end, denote \( D(\nu) = \{ U. \bigoplus_{j=1}^{\nu} t_j : U \in \mathcal{U}_{|\nu|}, t_j \in \text{core}(\mathfrak{T}) \cap \mathfrak{T}_{\nu_j} \} \) and \( F(\nu) = \{ U. \bigoplus_{j=1}^{n} t_j : U \in \mathcal{U}_{|\nu|}, t_j \in \mathfrak{T}_{\nu_j} \} \) (where \( n \) is the number of entries of \( \nu \)) and note that
\textbf{Proof of Theorem 4.2} Our only task is to prove that each map \( u \in C^*_0(\mathfrak{T}) \) that satisfies (*) belongs to the uniform closure of \( \mathcal{E} \). First of all, observe that, under the assumptions of the theorem, (*) is equivalent to

\[(*)' \text{ for any } \mathfrak{r}, \mathfrak{v} \in \text{core}(\mathfrak{T}) \text{ there exists } v \in \mathcal{E} \text{ such that } v(\mathfrak{r}) = u(\mathfrak{r}) \text{ and } v(\mathfrak{v}) = u(\mathfrak{v}).\]

Indeed, if either \( \mathfrak{r} \equiv \mathfrak{v} \) or \((1_\mathfrak{r}) \) (resp. \((1_\mathfrak{v}) \)) holds, the conclusion of \((*)'\) for \( \mathfrak{r} \) and \( \mathfrak{v} \) readily follows from (*) applied to \( \mathfrak{v} \) (resp. to \( \mathfrak{r} \)). On the other hand, if \( \mathfrak{r} \perp \mathfrak{v} \) and both conditions \((2_\mathfrak{r}) \) and \((2_\mathfrak{v}) \) are fulfilled, there are functions \( g_1, g_2, v_1, v_2 \in \mathcal{E} \) with \( g_1(\mathfrak{r}) = I_{d(\mathfrak{r})}, g_1(\mathfrak{v}) = 0, g_2(\mathfrak{r}) = I_{d(\mathfrak{r})}, g_2(\mathfrak{v}) = 0, v_1(\mathfrak{r}) = u(\mathfrak{r}) \) and \( v_2(\mathfrak{v}) = u(\mathfrak{v}) \) (the last two properties are derived from (*)). Then \( v := g_1v_1 + g_2v_2 \) belongs to \( \mathcal{E} \) and satisfies \( v(\mathfrak{r}) = u(\mathfrak{r}) \) and \( v(\mathfrak{v}) = u(\mathfrak{v}) \).

For simplicity, put \( \Omega := \text{core}(\mathfrak{T}) \) and let \( \mathcal{A} \) be the product \( \prod_{n=1}^{\infty} \mathcal{M}_n \) of the C*-algebras \( \mathcal{M}_n \). For each \( A \in \mathcal{M}_k \), we denote by \( A^\# \) the element \((X_n)_{n=1}^{\infty} \) of \( \mathcal{A} \) such that \( X_k = A \) and \( X_n = 0 \) for \( n \neq k \). Further, for any \( u \in C^*_0(\mathfrak{T}) \) we define a function \( \Psi(u) : \Omega \to \mathcal{A} \) by \( \Psi(u)(t) = (u(t))^\# \). In this way one obtains a *-homomorphism \( \Psi : C^*_0(\mathfrak{T}) \to C_0(\Omega, \mathcal{A}) \). What is more, Lemma 3.19 asserts that \( \Psi \) is isometric.

Let \( \mathcal{B} \) be the family of all sets \( B(\nu) \) defined by (4.1) (where \( \nu \) runs over all finite sequences of positive integers). By Lemma 4.5 \( \mathcal{B} \) is a countable collection of pairwise disjoint Borel subsets of \( \Omega \) that cover \( \Omega \). Fix \( u \in C^*_0(\mathfrak{T}) \) for which \((*)'\) holds. Instead of showing that \( u \) belongs to the uniform closure of \( \mathcal{E} \), it suffices to check that \( w := \Psi(u) \) belongs to the uniform closure of \( V := \Psi(\mathcal{E}) \). We now employ Theorem 4.4. Let \( L \subset \Omega \) satisfy \((**') \). Our only task is to show that \( w|_L \) belongs to the uniform closure of \( V|_L \). Equivalently, we only have to show that

\[ (*) \text{ there is a sequence } v_1, v_2, \ldots \text{ of elements of } \mathcal{E} \text{ such that } \lim_{n \to \infty} \sup \{ \|v_n(t) - u(t)\| : t \in L \} = 0. \]

Let \( K \) consist of all \( t \in \text{core}(\mathfrak{T}) \) for which there exists \( s \in L \) with \( t \preceq s \). We
We can prove this as follows. We infer from (**) that there are only a finite number of (finite) sequences \( \nu \) for which \( L \cap B(\nu) \) is nonempty. Let \( \nu^{(1)}, \ldots, \nu^{(p)} \) denote all such sequences. Put \( L_j := L \cap B(\nu^{(j)}) \) for \( j \in \{1, \ldots, p\} \), and denote by \( K_j \) the set of all \( t \in \text{core}(\mathfrak{F}) \) for which there is \( s \in L_j \) with \( t \leq s \). Then \( L = \bigcup_{j=1}^p L_j \) and each \( L_j \) is compact (see (**)).

Observe that \( K = \bigcup_{j=1}^p K_j \) and thus, to deduce (4.2), it suffices to show that the sets \( K_j \) are compact. To this end, fix \( j \), put (for simplicity) \( \nu := \nu^{(j)} \), \( F := \{ U.t: U \in \mathfrak{H}_\nu, t \in L_j \} \) and express \( \nu \) in the form \( (\nu_1, \ldots, \nu_p) \). Note that \( F \) is a compact subset of \( B(\nu) \). Further, let \( \Delta \) consist of all \( (t_1, \ldots, t_p) \in \prod_{\nu_1} \times \cdots \times \prod_{\nu_p} \) for which \( \bigoplus_{k=1}^p t_k \in F \). It follows from (T1) that \( \Delta \) is compact. Finally, since \( F \subset B(\nu) \), Proposition 3.5 implies that \( \Delta \subset (\text{core}(\mathfrak{F}))^p \). So, we conclude that \( K_j = \bigcup_{k=1}^p \text{pr}_k(\Delta) \) (see Corollary 3.6)

where \( \text{pr}_k: \Delta \to \mathfrak{F} \) is the projection onto the \( k \)th coordinate, which finishes the proof of (4.2).

We come back to the main part of the proof. Observe that

\[
(4.3) \quad \|f\|_L = \|f\|_K
\]

for any \( f \in C^*_0(\mathfrak{F}) \) (where, for \( A \subset \mathfrak{F} \), \( \|f\|_A := \sup\{\|f(a)\| : a \in A\} \)). Indeed, for each \( x \in K \) there are \( y, z \in \mathfrak{F} \) such that \( x \oplus y \equiv z \in L \) and then \( \|f(x)\| \leq \|f(z)\| \leq \|f\|_L \) for any such \( f \); conversely, if \( z \in L \), there are irreducible elements \( y_1, \ldots, y_s \in \mathfrak{F} \) for which \( z \equiv \bigoplus_{j=1}^s y_j \), and then \( x_1, \ldots, x_s \in K \) and \( \|f(y_j)\| \leq \max(\|f(x_1)\|, \ldots, \|f(x_s)\|) \leq \|f\|_K \) for all \( f \in C^*_0(\mathfrak{F}) \).

It follows from (4.2) that there is \( N > 0 \) such that \( K \subset \bigcup_{n=1}^N \mathfrak{F}_n \). Put \( R := N! \) and define the \( C^* \)-algebra \( \mathcal{D} \) as \( \mathcal{M}_R \). For each \( f \in C^*_0(\mathfrak{F}) \) let \( \Phi(f): K \to \mathcal{D} \) be given by \( (\Phi(f))(x) := (R/d(x)) \circ f(x) \). It is easy to see that in this way one obtains an \( s \)-homomorphism \( \Phi: C^*_0(\mathfrak{F}) \to C(K, \mathcal{D}) \). Moreover, \( \|\Phi(f)\| = \|f\|_K \) for any \( f \in C^*_0(\mathfrak{F}) \) which, combined with (4.3), yields

\[
(4.4) \quad \|\Phi(f)\| = \|f\|_L \quad (f \in C^*_0(\mathfrak{F})).
\]

The above connection leads us to the conclusion that (\( \clubsuit \)) is equivalent to

(\( \spadesuit \)) \( g \) belongs to the uniform closure (in \( C(K, \mathcal{D}) \)) of \( \mathcal{L} \),

where \( g := \Phi(u) \) and \( \mathcal{L} := \Phi(\mathcal{E}) \). To show (\( \spadesuit \)), we employ Theorem 4.3. To this end, let us check that for any \( x, \eta \in K \) either (SW1) or (SW2) holds.

We know that \( x, \eta \in \text{core}(\mathfrak{F}) \). So, either \( x \equiv \eta \) (and in that case (SW2) holds) or \( x \perp \eta \). In the latter case, either both conditions (1x) and (1y) are fulfilled and then (SW2) holds, or at least one of conditions (2x) and (2y) is satisfied and then (SW1) is fulfilled. So, we infer from Theorem 4.3 that (\( \spadesuit \)) holds provided \( g \in \Delta_2(\mathcal{L}) \). Finally, that \( g \) belongs to \( \Delta_2(\mathcal{L}) \) may simply be deduced from (\( \star ' \)). Thus, the proof is complete.
DEFINITION 4.6. The vanishing tower of an elc m-tower \( \mathcal{I} \), denoted by \( \mathcal{I}(0) \), is defined as the set of all \( t \in \mathcal{I} \) such that \( f(t) = 0 \) for any \( f \in C^*_0(\mathcal{I}) \). It is easy to see that \( \mathcal{I}(0) \) is indeed a (possibly empty) subtower.

Since \( C^*_0(\mathcal{I}) = C^*(\mathcal{I}) \) for proper m-towers of finite height \( \mathcal{I} \), Corollary 3.14 implies that \( \mathcal{I}(0) = \emptyset \) for all such m-towers.

The next consequence of Theorem 4.2 will be used in the proof of the first part of Theorem 1.1.

COROLLARY 4.7. A *-subalgebra \( \mathcal{E} \) of \( C^*_0(\mathcal{I}) \) (where \( \mathcal{I} \) is an elc m-tower) is dense in \( C^*_0(\mathcal{I}) \) iff for any \( t \in \text{core}(\mathcal{I}) \setminus \mathcal{I}(0) \):

(d1) \( \{ f(t) : f \in \mathcal{E} \} = \mathcal{M}_{d(t)} \);

(d2) if \( s \in \text{core}(\mathcal{I}) \setminus \mathcal{I}(0) \) is disjoint from \( t \), then there exists \( g \in \mathcal{E} \) for which \( g(t) = I_{d(t)} \) and \( g(s) = 0 \).

The proof of Corollary 4.7 is left to the reader (to show the necessity of (d1), use Corollary 3.14 and the fact that \( C^*_0(\mathcal{I}) \) is an ideal in \( C^*(\mathcal{I}) \)).

COROLLARY 4.8. For any ideal \( \mathcal{J} \) in \( C^*_0(\mathcal{I}) \) (where \( \mathcal{I} \) is an elc m-tower) there exists a unique subtower \( \mathcal{G} \) of \( \mathcal{I} \) such that \( \mathcal{G} \supseteq \mathcal{I}(0) \) and \( \mathcal{J} \) coincides with the ideal \( \mathcal{J}_{\mathcal{G}} \) of all maps \( f \in C^*_0(\mathcal{I}) \) that vanish at each point of \( \mathcal{G} \).

Proof. Let \( \mathcal{G} \) be the set of all points \( t \in \mathcal{I} \) that are common zeros of all maps from \( \mathcal{J} \). It is easily seen that \( \mathcal{G} \supset \mathcal{I}(0) \), \( \mathcal{G} \) is a subtower, and \( \mathcal{J} \subset \mathcal{J}_{\mathcal{G}} \). We shall now show the reverse inclusion. To this end, observe that \( C^*_0(\mathcal{I}) \) is an ideal in \( C^*(\mathcal{I}) \), and hence \( \mathcal{J} \) is an ideal in \( C^*(\mathcal{I}) \) as well. Fix \( t \in \text{core}(\mathcal{I}) \setminus \mathcal{G} \). We shall check that condition (2) of Theorem 4.2 is fulfilled (with \( \mathcal{J} \) in place of \( \mathcal{E} \)). We know from Corollary 3.14 that the map \( C^*(\mathcal{I}) \ni f \mapsto f(t) \in \mathcal{M}_{d(t)} \) is surjective. Thus, \( \mathcal{J}(t) := \{ f(t) : f \in \mathcal{J} \} \) is an ideal in \( \mathcal{M}_{d(t)} \). Since this ideal is nonzero (because \( t \notin \mathcal{G} \)), we infer that

(4.5) \[ \mathcal{J}(t) = \mathcal{M}_{d(t)}. \]

So, there exists \( g_0 \in \mathcal{J} \) such that \( g_0(t) = I_{d(t)} \). Now let \( s \in \text{core}(\mathcal{I}) \) be disjoint from \( t \). Then it follows from Corollary 3.14 that there exists \( g_1 \in C^*(\mathcal{I}) \) with \( g_1(t) = I_{d(t)} \) and \( g_1(s) = 0 \). It suffices to put \( g := g_0g_1 \) to obtain a map from \( \mathcal{J} \) such that \( g(t) = I_{d(t)} \) and \( g(s) = 0 \). Hence, Theorem 4.2 implies that each map \( u \in C^*_0(\mathcal{I}) \) for which condition (*) holds with \( \mathcal{E} \) replaced by \( \mathcal{J} \) belongs to \( \mathcal{J} \) (which is closed). But (4.5) holds for any \( t \in \text{core}(\mathcal{I}) \setminus \mathcal{G} \) and therefore (*) is (trivially) fulfilled for all \( u \in \mathcal{J}_{\mathcal{G}} \). Consequently, \( \mathcal{J}_{\mathcal{G}} = \mathcal{J} \).

To show the uniqueness of \( \mathcal{G} \), assume \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) are subtowers of \( \mathcal{I} \) that contain \( \mathcal{I}(0) \) and satisfy \( \mathcal{G}_1 \not\subseteq \mathcal{G}_2 \). Then there exists \( t \in \mathcal{G}_1 \setminus \mathcal{G}_2 \) which is irreducible. Now Theorem 3.8 implies that there exists \( u \in C^*(\mathcal{I}) \) such that \( u \) vanishes at each point of \( \mathcal{G}_2 \) and \( u(t) = I_{d(t)} \). Moreover, since \( t \notin \mathcal{I}(0) \), there is \( v \in C^*_0(\mathcal{I}) \) which does not vanish at \( t \). Then \( uv \in \mathcal{J}_{\mathcal{G}_2} \setminus \mathcal{J}_{\mathcal{G}_1} \), and we are done. ■
Corollary 4.9. If $\mathfrak{T}$ is an elc $m$-tower, then for every nondegenerate finite-dimensional representation $\pi$ of $C_0^*(\mathfrak{T})$ there exists a unique element $s \in \mathfrak{T}$ such that $\pi(f) = f(s)$ for any $f \in C_0^*(\mathfrak{T})$. What is more, $s \perp z$ for each $z \in \mathfrak{T}(0)$.

Conversely, for any $t \in \mathfrak{T}$ which is disjoint from any element of $\mathfrak{T}(0)$, the representation $C_0^*(\mathfrak{T}) \ni f \mapsto f(t) \in \mathcal{M}_{d(t)}$ is nondegenerate.

Proof. Let $t \in \text{core}(\mathfrak{T}) \setminus \mathfrak{T}(0)$. Since $C_0^*(\mathfrak{T})$ is an ideal in $C^*(\mathfrak{T})$, Corollary 3.14 yields

$$\{f(t) : f \in C_0^*(\mathfrak{T})\} = \mathcal{M}_{d(t)}$$

(since the left-hand side of (4.6) is a nonzero ideal in $\mathcal{M}_{d(t)}$). Now combining the above formula again with Corollary 3.14 one infers that for any system $t_1, \ldots, t_n$ of mutually disjoint elements of $\text{core}(\mathfrak{T}) \setminus \mathfrak{T}(0)$,

$$\{(f(t_1), \ldots, f(t_n)) : f \in C_0^*(\mathfrak{T})\} = \mathcal{M}_{d(t_1)} \times \cdots \times \mathcal{M}_{d(t_n)},$$

and consequently

$$\{f\left(\bigoplus_{j=1}^{n} (\alpha_j \circ t_j)\right) : f \in C_0^*(\mathfrak{T})\} = \left\{\bigoplus_{j=1}^{n} (\alpha_j \circ A_j) : A_j \in \mathcal{M}_{d(t_j)}\right\}$$

for any positive integers $\alpha_1, \ldots, \alpha_n$. Thanks to Proposition 3.5 this shows the second claim of the corollary.

Now let $\pi : C_0^*(\mathfrak{T}) \to \mathcal{M}_N$ be a nonzero irreducible representation. Corollary 4.8 implies that there is a subtower $\mathfrak{S}$ of $\mathfrak{T}$ that contains $\mathfrak{T}(0)$ and satisfies

$$\ker(\pi) = \mathcal{J}_\mathfrak{S}.$$ 

Further, since $\mathfrak{T}(0) \subsetneq \mathfrak{S}$ and $\mathfrak{T}(0)$ is a subtower, we conclude that $\mathfrak{S} \setminus \mathfrak{T}(0)$ contains irreducible elements. Further, one deduces from (4.7) that every system of mutually disjoint irreducible elements of $\mathfrak{S} \setminus \mathfrak{T}(0)$ is finite (because the quotient algebra $C_0^*(\mathfrak{T})/\mathcal{J}_\mathfrak{S}$ is finite-dimensional). Let $s_1, \ldots, s_n$ be a maximal such system. We claim that

$$\mathcal{J}_\mathfrak{S} = \{f \in C_0^*(\mathfrak{T}) : (f(s_1), \ldots, f(s_n)) = (0, \ldots, 0)\}.$$ 

The inclusion “$\subset$” is clear. To show the reverse, it suffices to observe that for any element $t$ of $\mathfrak{S}$ there are irreducible elements $t_1, \ldots, t_k \in \mathfrak{T}(0) \cup \{s_1, \ldots, s_n\}$ such that $t \equiv \bigoplus_{j=1}^{k} t_j$.

Further, since $\pi$ is surjective (being irreducible and nonzero), a combination of (4.9), (4.10) and (4.7) yields $\mathcal{M}_N \cong C_0^*(\mathfrak{T})/\mathcal{J}_\mathfrak{S} \cong \prod_{j=1}^{n} \mathcal{M}_{d(s_j)}$, which is possible only when $n = 1$ and $d(s_1) = N$. So,

$$\ker(\pi) = \{f \in C_0^*(\mathfrak{T}) : f(s_1) = 0\}.$$ 

This relation enables us to define correctly a $*$-homomorphism $\kappa : \mathcal{M}_N \to \mathcal{M}_N$ by the rule $\kappa(A) := f(s_1)$ provided $f \in C_0^*(\mathfrak{T})$ is such that $\pi(f) = A$. We infer
from (4.11) that \( \kappa \) is one-to-one. So, \( \kappa: \mathcal{M}_N \to \mathcal{M}_N \) is a \(*\)-isomorphism and hence there is \( U \in \mathcal{H}_N \) for which \( \kappa(A) = U A \) for any \( A \in \mathcal{M}_N \) (in the algebra of matrices this is quite an elementary fact; however, it also follows from [19, Corollary 2.9.32]). But then \( U \pi(f) = \kappa(\pi(f)) = f(s_1) \) and consequently \( \pi(f) = f(U^{-1} s_1) \). So, we have shown that each nonzero irreducible finite-dimensional representation \( \pi \) of \( C^*_0(\mathfrak{T}) \) has the form \( \pi(f) = f(s) \) for some \( s \in \text{core}(\mathfrak{T}) \setminus \mathfrak{T}_0 \).

Finally, when \( \pi \) is an arbitrary nondegenerate finite-dimensional representation of \( C^*_0(\mathfrak{T}) \), there are a finite number of nonzero irreducible finite-dimensional representations \( \pi_1, \ldots, \pi_k \) and a unitary matrix \( U \) (of an appropriate) such that \( \pi(f) = U.\left( \bigoplus_{j=1}^k \pi_j(f) \right) \). It follows from the previous part of the proof that for each \( j \in \{1, \ldots, k\} \) there is \( s_j \in \text{core}(\mathfrak{T}) \setminus \mathfrak{T}_0 \) such that \( \pi_j(f) = f(s_j) \). Then we conclude that \( \pi(f) = f(s) \) where \( s := U.\left( \bigoplus_{j=1}^k s_j \right) \). Noticing that such an \( s \) is disjoint from any element of \( \mathfrak{T}_0 \), it remains to show that \( s \) is unique. To this end, take an arbitrary \( s' \in \mathfrak{T} \) that is different from \( s \). We need to find a map \( f \in C^*_0(\mathfrak{T}) \) such that \( f(s') \neq f(s) \). We conclude from (4.8) that there is \( v \in C^*_0(\mathfrak{T}) \) for which \( v(s) = I_d(s) \). If \( v(s') \neq v(s) \), we are done. Hence, we assume \( v(s') = I_d(s) \). Corollary 3.14 implies that there exists \( u \in C^*(\mathfrak{T}) \) with \( u(s') \neq u(s) \). Then \( f := uv \in C^*_0(\mathfrak{T}) \) is such that \( f(s') \neq f(s) \). \( \blacksquare \)

It turns out that all \( C^* \)-algebras of the form \( C^*_0(\mathfrak{T}) \) (where \( \mathfrak{T} \) is an elc m-tower) are CCR (that is, liminal). Even more is true:

**Theorem 4.10.** Every irreducible representation of \( C^*_0(\mathfrak{T}) \) (where \( \mathfrak{T} \) is an elc m-tower) is finite-dimensional. In particular, \( C^*_0(\mathfrak{T}) \) is CCR.

**Proof.** For each \( n > 0 \), denote by \( \mathcal{J}_n \) the closure of \( \mathfrak{T}_n \cap \text{core}(\mathfrak{T}) \) and by \( \mathcal{A}_n \) the \( C^* \)-algebra of all bounded maps from \( \mathcal{J}_n \) into \( \mathcal{M}_n \). Since \( \mathcal{A}_n \) has sufficiently many \( n \)-dimensional representations, it is subhomogeneous (consult [1 Proposition IV.1.4.6]). Further, denote by \( \mathcal{A} \) the \( c_0 \)-direct product of all \( \mathcal{A}_n \); that is, \( \mathcal{A} \) is the \( C^* \)-algebra of all sequences \( (f_n)_{n=1}^\infty \) for which \( f_n \in \mathcal{A}_n \) (for any \( n > 0 \)) and \( \lim_{n \to \infty} \|f_n\| = 0 \). Since any irreducible representation of \( \mathcal{A} \) is of the form \( (A_n)_{n=1}^\infty \mapsto \pi(A_k) \) where \( k > 0 \) is fixed and \( \pi \) is an irreducible representation of \( \mathcal{A}_k \), we see that each irreducible representation of \( \mathcal{A} \) is finite-dimensional. Finally, the assignment \( f \mapsto (f|\mathcal{J}_n)_{n=1}^\infty \) correctly defines a one-to-one \(*\)-homomorphism of \( C^*_0(\mathfrak{T}) \) into \( \mathcal{A} \) and therefore \( C^*_0(\mathfrak{T}) \) is \(*\)-isomorphic to a subalgebra of \( \mathcal{A} \). So, since irreducible representations of \( C^* \)-subalgebras always admit extensions to irreducible representations of the ambient \( C^* \)-algebra (possibly in greater Hilbert spaces; consult [1 Proposition 2.10.2]), the assertion follows. \( \blacksquare \)

**Corollary 4.11.** Any nonzero irreducible representation of \( C^*_0(\mathfrak{T}) \) (for an elc m-tower \( \mathfrak{T} \)) has the form \( f \mapsto f(s) \) where \( s \in \text{core}(\mathfrak{T}) \setminus \mathfrak{T}_0 \).
Proof. Just apply Theorem 4.10 and Corollary 4.9. □

Another strong consequence of Theorem 4.2 is formulated below.

**Theorem 4.12.** Let \( T \) be an elc m-tower and \( \mathcal{S} \) an arbitrary subtower. A map \( u \in C^*(\mathcal{S}) \) extends to a map \( v \in C^*_0(\Xi) \) iff \( u \in C^*_0(\mathcal{S}) \) and \( u \) vanishes at each point of \( \mathcal{S} \cap \Xi_0 \). What is more, if \( u \in C^*_0(\mathcal{S}) \) vanishes at each point of \( \mathcal{S} \cap \Xi_0 \), then it has an extension \( v \in C^*_0(\Xi) \) with \( \|v\| = \|u\| \).

**Proof.** The necessity is clear (note that \( \text{core}(\mathcal{S}) \subset \text{core}(\Xi) \)). To prove the sufficiency, we employ Theorem 4.2. Define a \(*\)-homomorphism \( \Phi : C^*_0(\mathcal{S}) \to C^*_0(\mathcal{S}) \) by \( \Phi(f) := f|_{\mathcal{S}} \). Since the images of \(*\)-homomorphisms between \( C^*-\)algebras are always closed, it suffices to check that the closure of \( E := \Phi(C^*_0(\Xi)) \) in \( C^*_0(\mathcal{S}) \) coincides with all maps \( u \in C^*_0(\mathcal{S}) \) that vanish at each point of \( \mathcal{S} \cap \Xi_0 \); but this follows easily from Theorem 4.2 (we skip the details). Finally, since each \(*\)-homomorphism sends the closed unit ball onto the closed unit ball (of the range), the additional claim of the theorem follows. □

**Remark 4.13.** To appreciate the strength of Theorem 4.12, the reader may try to construct an extension directly. The first difficulty is to show that the restriction to \( \mathcal{S} \cap \text{core}(\Xi) \) of each map from \( C^*_0(\mathcal{S}) \) that vanishes at each point of \( \mathcal{S} \cap \Xi_0 \) vanishes at infinity. The author has no idea how to prove this directly.

We conclude the section with the variation of the Stone–Weierstrass theorem announced earlier.

**Theorem 4.14 (Stone–Weierstrass Theorem for elc m-towers).** Let \( \mathcal{E} \) be a \(*\)-subalgebra of \( C^*_0(\Xi) \) for some elc m-tower \( \Xi \). The uniform closure of \( \mathcal{E} \) in \( C^*_0(\Xi) \) consists of all functions \( u \in C^*_0(\Xi) \) such that

\[
(u(\xi), u(\eta)) \in \{(v(\xi), v(\eta)) : v \in \mathcal{E}\}
\]

for all \( \xi, \eta \in \text{core}(\Xi) \setminus \Xi_0 \).

**Proof.** Denote by \( \mathcal{F} \) the family of all functions \( u \in C^*_0(\Xi) \) that satisfy (4.12) for all \( \xi, \eta \in \text{core}(\Xi) \setminus \Xi_0 \). It is obvious that \( \mathcal{F} \) is a \( C^*-\)algebra that contains the closure \( \mathcal{E} \) of \( \mathcal{E} \) (note that the right-hand side expression of (4.12) is a closed set, as a subalgebra of a matrix algebra). To show that \( \mathcal{F} = \mathcal{E} \), we shall check that \( \mathcal{E} \) is a rich subalgebra of \( \mathcal{F} \) in the sense of Dixmier (see [4, Definition 11.1.1]). That is, we have to verify the following two conditions:

1. (r1) for every irreducible representation \( \pi \) of \( \mathcal{F} \), \( \pi|_{\mathcal{E}} \) is irreducible as well;
2. (r2) if \( \pi \) and \( \pi' \) are unitarily inequivalent irreducible representations of \( \mathcal{F} \), then \( \pi|_{\mathcal{E}} \) and \( \pi'|_{\mathcal{E}} \) are also inequivalent.

To this end, let \( \pi \) be a nonzero irreducible representation of \( \mathcal{F} \). Since \( \pi \) extends to an irreducible representation of \( C^*_T \), we infer from Corollary 4.11 that there are \( \xi \in \text{core}(\Xi) \setminus \Xi_0 \) and a positive integer \( k \leq d(\xi) \) such that for
any $u \in \mathcal{F}$, the top left $k \times k$ submatrix of $u(\mathfrak{r})$ coincides with $\pi(u)$. Then we infer from (4.12) that $\{u(\mathfrak{r}): u \in \mathcal{F}\} = \{v(\mathfrak{r}): v \in \mathcal{E}\}$ and therefore $\pi|_{\mathcal{E}}$ is a nonzero irreducible representation. So, (r1) holds, and it suffices to check (r2) only for nonzero $\pi$ and $\pi'$. But, if $\pi \not\equiv 0$ and $\pi' \not\equiv 0$ are as specified in (r2), then there is $u_0 \in \mathcal{F}$ for which $\pi(u_0) \neq 0$ and $\pi'(u_0) = 0$. Moreover, the previous argument shows that there are elements $\mathfrak{r}, \eta \in \text{core}(\mathfrak{S}) \setminus \mathfrak{S}(0)$ and positive integers $k \leq d(\mathfrak{r})$ and $\ell \leq d(\eta)$ such that for any $u \in \mathcal{F}$, $\pi(u)$ and $\pi'(u)$ coincide with, respectively, the top left $k \times k$ submatrix of $u(\mathfrak{r})$ and the top left $\ell \times \ell$ submatrix of $u(\eta)$. Now taking into account (4.12) for these $\mathfrak{r}$ and $\eta$, we see that there is $v_0 \in \mathcal{E}$ with $\pi(v_0) = \pi(u_0)$ and $\pi'(v_0) = \pi'(u_0)$. These two relations imply that the restrictions of $\pi$ and $\pi'$ to $\mathcal{E}$ are unitarily inequivalent, which proves (r2).

Finally, $C_0^*(\mathfrak{S})$ is CCR, by Theorem 4.10 and thus also $\mathcal{F}$ is CCR, as a $C^*$-subalgebra of $C_0^*(\mathfrak{S})$. So, $\bar{\mathcal{E}}$ is a rich subalgebra of the CCR $C^*$-algebra $\mathcal{F}$, and hence $\bar{\mathcal{E}} = \mathcal{F}$, thanks to Proposition 11.1.6.

5. Proper m-towers of finite height. In this section we prove Theorems 1.1 and 1.2 and describe *-homomorphisms between subhomogeneous $C^*$-algebras by means of morphisms between the corresponding m-towers. The reader interested in subhomogeneous $C^*$-algebras is referred to Subsection IV.1.4.

The first result of this section summarizes (in a special case) some of the results of the previous section.

PROPOSITION 5.1. Let $\mathfrak{S}$ be a proper m-tower with $n := \text{ht}(\mathfrak{S}) < \infty$ and let $\mathcal{A} = C^*(\mathfrak{S})$.

(A) $\mathcal{A}$ is a unital $n$-subhomogeneous $C^*$-algebra (here “0-subhomogeneous” means “$\mathcal{A} = \{0\}$”).

(B) The assignment $\mathfrak{S} \mapsto \mathfrak{S}(\mathfrak{S})$ establishes a one-to-one correspondence between all subtowers of $\mathfrak{S}$ and all ideals in $\mathcal{A}$.

(C) Every unital finite-dimensional representation of $\mathcal{A}$ has the form

$$\pi_s: \mathcal{A} \ni f \mapsto f(s) \in M_{d(s)}$$

where $s \in \mathfrak{S}$. The assignment $t \mapsto \pi_t$ is one-to-one and $\pi_t$ is irreducible iff $t \in \text{core}(\mathfrak{S})$.

Proof. As noted in the previous section, $\mathfrak{S}$ is elc and $C^*(\mathfrak{S}) = C_0^*(\mathfrak{S})$. Moreover, $\mathfrak{S}(0)$ is empty (according to Corollary 3.14). Consequently, (B) follows from Corollary 4.8 (C) from Corollaries 4.9 and 4.11, whereas (A) is an immediate consequence of (C) (and the definition of $n$).

We recall that for any (unital) $C^*$-algebra $\mathcal{A}$, the concrete tower $\mathfrak{S}(\mathcal{A})$ (resp. $\mathfrak{S}(\mathcal{A})$) was defined in the Introduction and it consists of all (resp. all unital) finite-dimensional representations of $\mathcal{A}$.
Lemma 5.2. For any (unital) $C^*$-algebra $\mathcal{A}$, $\mathcal{Z}(\mathcal{A})$ (resp. $\mathcal{X}(\mathcal{A})$) is a proper $m$-tower.

Proof. Denote by $A$ the closed unit ball of $\mathcal{A}$. Put $\Xi := \mathcal{Z}(\mathcal{A})$ (resp. $\mathcal{S} := \mathcal{X}(\mathcal{A})$). To explain that $\Xi$ is a proper $m$-tower, it suffices to show that $\Xi$ is isomorphic to a standard $m$-tower. Consider the function $\Psi : \mathcal{S} \ni \pi \mapsto \pi|_A \in M[\Lambda]$ and observe that $d(\Psi(\pi)) = d(\pi)$, $\Psi(U.\pi) = U.\Psi(\pi)$ and $\Psi(\pi \oplus \pi') = \Psi(\pi) \oplus \Psi(\pi')$ for any $\pi, \pi' \in \Xi$ and $U \in \mathcal{U}_{d(\pi)}$. What is more, since the space of all (resp. all unital) $n$-dimensional representations of $\mathcal{A}$ is compact in the pointwise convergence topology, we see that condition (ST0) for $\mathcal{T} := \Psi(\mathcal{S})$ is fulfilled. Further, (ST1) follows from the definition of $\Lambda$ (and the fact that each representation has norm no greater than 1), whereas (ST2) and (ST3) are straightforward. So, $\mathcal{T}$ is a standard tower, and $\Psi$ is a homeomorphism (which follows from the definitions of the topologies on $\mathcal{S}$ and $\mathcal{T}$). Thus, $\mathcal{S}$ is a proper $m$-tower.

Proof of Theorem 1.1. It follows from the previous lemma that $\mathcal{S} := \mathcal{X}(\mathcal{A})$ is a proper $m$-tower. Moreover, a finite-dimensional representation $\pi$ belongs to the core of $\mathcal{S}$ if and only if $\pi$ is irreducible, which implies that $\text{ht}(\mathcal{S}) < \infty$.

To show the main part of the theorem, namely, that $J_\mathcal{A}$ is surjective, it suffices to employ Corollary 4.7 (recall that $\mathcal{S}$ is elc and $C^*(\mathcal{S}) = C^*_0(\mathcal{S})$) for $\mathcal{E} := J_\mathcal{A}(\mathcal{S})$. Conditions (d1)–(d2) transform into:

(d1') each unital irreducible representation $\pi$ of $\mathcal{A}$ is surjective;
(d2') if $\pi_1$ and $\pi_2$ are inequivalent unital irreducible representations of $\mathcal{A}$, then $\pi_1(a) = I$ and $\pi_2(a) = 0$ for some $a \in \mathcal{A}$ (where $I$ denotes the unit matrix of the appropriate degree).

But both conditions (d1') and (d2') are already known (they are special cases of [4, Proposition 4.2.5]) and thus the proof of this part is finished.

Now assume $\mathcal{S}$ is an arbitrary proper $m$-tower of finite height. According to Proposition 5.1 we only need to prove that the assignment $t \mapsto \pi_t$ defines a homeomorphism between $\mathcal{S}$ and $\mathcal{X}(C^*(\mathcal{S}))$. But this simply follows from the fact that each of $\mathcal{S}_n$ is compact (and this assignment is bijective).

Proof of Theorem 1.2. This result is actually an immediate consequence of Theorem 1.1 and may be shown as follows. Let $\mathcal{A}$ be an arbitrary subhomogeneous $C^*$-algebra (with or without unit). Let $\mathcal{A}_1$ denote the unitization of $\mathcal{A}$ (which may be constructed even for unital $\mathcal{A}$). Notice that $\mathcal{A}_1$ is a subhomogeneous $C^*$-algebra in which $\mathcal{A}$ is an ideal such that the quotient space $\mathcal{A}_1/\mathcal{A}$ is one-dimensional. Moreover, there is a natural one-to-one correspondence between all finite-dimensional representations of $\mathcal{A}$ (including the zero representation) and all unital finite-dimensional representations of $\mathcal{A}_1$. 

This means that $\mathcal{Z}(\mathcal{A})$ may naturally be identified with $\mathfrak{X}(\mathcal{A}_1)$. Under such an identification, the restriction of the $*$-isomorphism $\mathcal{J}_{\mathcal{A}_1} : \mathcal{A}_1 \to C^*(\mathcal{Z}(\mathcal{A}))$ to $\mathcal{A}$ coincides with $\mathcal{J}_{\mathcal{A}}$ and sends $\mathcal{A}$ onto a subspace $\mathcal{E}$ of $C^*(\mathcal{Z}(\mathcal{A}), \theta_{\mathcal{A}})$ such that $C^*(\mathcal{Z}(\mathcal{A}))/\mathcal{E}$ is one-dimensional. This implies that $\mathcal{E} = C^*(\mathcal{Z}(\mathcal{A}), \theta_{\mathcal{A}})$ and we are done. (The details are left to the reader.)

The second part of the theorem may be proved in the same manner (using the fact that $C^*(\mathcal{Z})$ is the unitization of $C^*(\mathcal{Z}, \theta)$) and is left to the reader. ■

Our next aim is to characterize $*$-homomorphisms between subhomogeneous $C^*$-algebras. As before, we start from unital algebras.

**Theorem 5.3.** Let $\mathfrak{T}$ and $\mathfrak{S}$ be two proper $m$-towers of finite height. For every unital $*$-homomorphism $\Phi : C^*(\mathfrak{T}) \to C^*(\mathfrak{S})$ there exists a unique morphism $\tau : \mathfrak{S} \to \mathfrak{T}$ such that $\Phi = \Phi_\tau$ where

\[
\Phi_\tau : C^*(\mathfrak{T}) \ni f \mapsto f \circ \tau \in C^*(\mathfrak{S}).
\]

Conversely, if $\tau : \mathfrak{S} \to \mathfrak{T}$ is an arbitrary morphism, then (5.1) correctly defines a unital $*$-homomorphism $\Phi_\tau$.

**Proof.** For any $s \in \mathfrak{S}$, the function $C^*(\mathfrak{T}) \ni f \mapsto (\Phi(f))(s) \in \mathcal{M}_{d(s)}$ is a unital finite-dimensional $*$-representation of $\mathcal{A}$. So, it follows from Corollary 1.9 that there is a unique point $(\tau(s) := t) \in \mathfrak{T}$ such that $(\Phi(f))(s) = f(t)$ for any $f \in C^*(\mathfrak{T})$. In this way one obtains a function $\tau : \mathfrak{S} \to \mathfrak{T}$ such that $\Phi(f) = f \circ \tau$. Moreover, it is immediate that $\tau$ is unique and satisfies all axioms of a morphism, apart from continuity, which in turn follows from the fact that $\mathfrak{T}$ is naturally isomorphic (as described in Theorem 1.1) to $\mathfrak{X}(C^*(\mathfrak{T}))$. The second claim is much simpler and is left as an exercise. ■

Under the notation introduced in (5.1), it is obviously seen that $\Phi_{\tau'} \circ \Phi_{\tau'} = \Phi_{\tau' \circ \tau}$ for any morphisms $\tau : \mathfrak{T} \to \mathfrak{T}'$ and $\tau' : \mathfrak{T}' \to \mathfrak{T}''$ (and $\Phi_{\text{id}} = \text{id}$ where $\text{id}$ stands for the identity map on an appropriate space). In particular, $*$-isomorphisms between unital subhomogeneous $C^*$-algebras correspond to isomorphisms between $m$-towers.

**Corollary 5.4.** Two unital subhomogeneous $C^*$-algebras are $*$-isomorphic iff their $m$-towers of unital finite-dimensional representations are isomorphic.

**Corollary 5.5.** Let $(\mathfrak{T}, \theta)$ and $(\mathfrak{S}, \kappa)$ be proper $m$-towers of finite height. For every $*$-homomorphism $\Phi : C^*(\mathfrak{T}, \theta) \to C^*(\mathfrak{S}, \kappa)$ there exists a unique morphism $\tau : (\mathfrak{S}, \kappa) \to (\mathfrak{T}, \theta)$ such that $\Phi = \Phi_\tau$ where

\[
\Phi_\tau : C^*(\mathfrak{T}, \theta) \ni f \mapsto f \circ \tau \in C^*(\mathfrak{S}, \kappa).
\]

Conversely, if $\tau : (\mathfrak{S}, \kappa) \to (\mathfrak{T}, \theta)$ is an arbitrary morphism, then (5.2) correctly defines a $*$-homomorphism $\Phi_\tau$. 
Proof. Use the fact that each *-homomorphism from $C^*(\mathcal{S}, \theta)$ into $C^*(\mathcal{G}, \kappa)$ extends uniquely to a unital *-homomorphism between $C^*(\mathcal{S})$ and $C^*(\mathcal{G})$ and then apply Theorem 5.3.

**Proposition 5.6.** Let $\tau: \mathcal{G} \to \mathcal{S}$ be a morphism between two proper m-towers of finite height. The *-homomorphism $\Phi_\tau$ (given by (5.1)) is surjective iff $\tau$ is one-to-one. If this happens, $\tau(\mathcal{S})$ is a subtower of $\mathcal{T}$ and $\tau$ is an isomorphism from $\mathcal{S}$ onto $\tau(\mathcal{S})$.

Proof. If $\Phi_\tau$ is surjective, we infer from Corollary 3.14 that $\tau$ is one-to-one (because maps from $C^*(\mathcal{S})$ separate points of $\mathcal{S}$). Conversely, if $\tau$ is one-to-one, then $\mathrm{stab}(\tau(s)) = \mathrm{stab}(s)$ for any $s \in \mathcal{S}$. Consequently,

$$\tau(\mathrm{core}(\mathcal{S})) \subset \mathrm{core}(\mathcal{T})$$

(by Lemma 3.15) and $\tau$ is a closed embedding (because $\mathcal{S}$ is proper). So, to check that $\mathcal{S}' := \tau(\mathcal{S})$ is a subtower of $\mathcal{T}$, it suffices to show that if $\tau(s) = a \oplus b$ for some $s \in \mathcal{S}$ and $a, b \in \mathcal{T}$, then both $a$ and $b$ belong to the image of $\tau$. This can be shown as follows (cf. the proof of Proposition 3.18). We infer from Proposition 3.5 that there are $s_1, \ldots, s_n \in \mathrm{core}(\mathcal{S})$ such that $s \equiv \bigoplus_{j=1}^n s_j$. Then also $a \oplus b = \tau(s) \equiv \bigoplus_{j=1}^n \tau(s_j)$ and $\tau(s_j) \in \mathrm{core}(\mathcal{T})$ (by (5.3)). So, it follows from Corollary 3.6 that there are nonempty subsets $J_1$ and $J_2$ of $\{1, \ldots, n\}$ such that $a \equiv \bigoplus_{j \in J_1} \tau(s_j)$ and $b \equiv \bigoplus_{j \in J_2} \tau(s_j)$, from which one deduces that $a, b \in \tau(\mathcal{S})$.

Further, since $\tau$ is an embedding, we see that $\tau$ is an isomorphism from $\mathcal{S}$ onto $\mathcal{S}'$. So, if $u \in C^*(\mathcal{S})$, then $u \circ \tau^{-1} \in C^*(\mathcal{S}')$. Now Theorem 3.8 implies that there is $v \in C^*(\mathcal{T})$ which extends $u \circ \tau^{-1}$. Then $\Phi_\tau(v) = u$ and hence $\Phi_\tau$ is surjective.

**Remark 5.7.** Under the notation of Proposition 5.6 one may also (easily) check that $\Phi_\tau$ is one-to-one iff $\mathcal{T}$ coincides with the smallest subtower of $\mathcal{T}$ that contains $\tau(\mathcal{S})$. However, if $\Phi_\tau$ is one-to-one but not surjective, then $\tau(\mathcal{S})$ may not be a subtower of $\mathcal{T}$, which causes problems in studying ASH algebras by means of m-towers.

**Remark 5.8.** If $\mathcal{A}$ is a unital $n$-homogeneous $C^*$-algebra, then

$$\mathcal{H} := \mathrm{core}(\mathcal{X}(\mathcal{A}))$$

is compact and coincides with $\mathcal{X}_n(\mathcal{A})$. It is then easy to show that the assignment $f \mapsto f|_{\mathcal{H}}$ defines a *-isomorphism from $C^*(\mathcal{X}(\mathcal{A}))$ onto $C^*(\mathcal{H},.) := \{f \in C(\mathcal{H}, \mathcal{M}_n): f(Uh) = U.f(h) \ (U \in \mathcal{M}_n, h \in \mathcal{H})\}$ (to establish surjectivity, apply Lemma 3.13). Similarly, if $\mathcal{A}$ is nonunital $n$-homogeneous $C^*$-algebra, then $\mathcal{H} := \mathrm{core}(\mathcal{X}(\mathcal{A}))$ is locally compact and $\mathcal{X}_n(\mathcal{A}) = \mathcal{H} \cup \{n \circ \theta_{\mathcal{A}}\}$ ($\theta_{\mathcal{A}}$ denotes the zero one-dimensional representation of $\mathcal{A}$). In that case the assignment $f \mapsto f|_{\mathcal{H}}$ defines a *-isomorphism from $C^*(\mathcal{X}(\mathcal{A}), \theta_{\mathcal{A}})$ onto $C^*(\mathcal{H},.)$ where $C^*(\mathcal{H},.)$ is the set of all maps $f: \mathcal{H} \to \mathcal{M}_n$ that vanish at
infinity and satisfy \( f(U, \mathfrak{h}) = U \mathfrak{h} \) for all \( U \in \mathcal{U}_n \) and \( \mathfrak{h} \in \mathfrak{h} \). In this way one obtains less complicated (and more transparent) models for all homogeneous \( C^* \)-algebras, which was first shown in [10] and [21] (see also [16]). The main disadvantage of these models is that there is no “bridge” between \( n \)-homogeneous and \( m \)-homogeneous \( C^* \)-algebras for different \( n \) and \( m \) (and hence these models are hardly applicable in investigations of approximately homogeneous \( C^* \)-algebras).

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A reader interested in further development of the theory of \( m \)-towers or in their applications to other \( C^* \)-algebras than subhomogeneous (called by us shrinking) is referred to the extended version of this paper available at [arXiv:1310.5595].

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