Efficient quantum key distribution scheme with nonmaximally entangled states

Peng Xue, Chuan-Feng Li∗, and Guang-Can Guo†
Laboratory of Quantum Communication and Quantum Computation and
Department of Physics, University of Science and Technology of China,
Hefei 230026, P. R. China

We propose an efficient quantum key distribution scheme based on entanglement. The sender
chooses pairs of photons in one of the two equivalent nonmaximally entangled states randomly, and
sends a sequence of photons from each pair to the receiver. They choose from the various bases
independently but with substantially different probabilities, thus reducing the fraction of discarded
data, and a significant gain in efficiency is achieved. We then show that such a refined data analysis
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In next section, we give the detailed description of our efficient QKD scheme with nonmaximally entangled states. By considering a simple biased eavesdropping strategy by Eve, we note that our refined analysis is an essential feature of our scheme in Sec. III. In Sec. IV, the constraint on the probability $\epsilon$ is derived. Finally, we conclude the scheme in Sec. V.

II. EFFICIENT QKD SCHEME WITH NONMAXIMALLY ENTANGLED STATES

In our scheme, there are two parties: the sender, Alice and the receiver, Bob. Alice prepares pairs of photons in the nonmaximally entangled state

$$|AB\rangle = \alpha |H\rangle_A |H\rangle_B + \beta |V\rangle_A |V\rangle_B$$

(1)

where $|\alpha|^2 + |\beta|^2 = 1$, and $H$ and $V$ denote the horizontal and vertical linear polarization, respectively. Then she performs two $\sigma_x$ operations on the two particles respectively to transform the state $|AB\rangle$ to its equivalent state

$$|AB\rangle' = \beta |H\rangle_A |H\rangle_B + \alpha |V\rangle_A |V\rangle_B$$

(2)

with probability $\frac{1}{2}$. Photon B is sent to Bob and photon A is left for Alice. There are two types of measurements that Alice may perform: she may measure along the rectilinear basis, thus distinguishing between horizontal and vertical photons. Alternatively, she may measure along the diagonal basis, thus distinguishing between the $+45^\circ$ and $-45^\circ$ photons. Bob measures the polarizations at the other end. He measures in one of three bases, obtained by rotating rectilinear basis by angles $\phi_1 (\phi'_1) = 0$, $\phi_2 (\phi'_2) = \tan^{-1} \frac{\beta}{\alpha}$, $\phi_3 (\phi'_3) = -\tan^{-1} \frac{\beta}{\alpha}$. The superscript $\prime$ refers to the case in which Alice chooses $|AB\rangle'$ as the original state.

The two users are connected by a quantum channel and a classical public channel. The quantum channel consists usually of an optical fiber. The public channel, however, can be any communication link. So how does this scheme work?

1. Alice and Bob pick a number $0 < \epsilon \leq 1$ and make its value public. The constraint on $\epsilon$ will be discussed in Sec. IV.

2. Alice sends a sequence of photons B from each pair in one of the two nonmaximally entangled states ($|AB\rangle$ and $|AB\rangle'$) chosen randomly and independently, and leaves the corresponding photons A. She also records her choice of $|AB\rangle$ or $|AB\rangle'$.

3. Alice has two types of measurements. One measurement along rectilinear basis (i.e., $\{|H\rangle, |V\rangle\}$) allows her to distinguish between horizontally and vertically polarized photons. The other measurement along diagonal basis (i.e., $\{\frac{1}{\sqrt{2}} (|H\rangle + |V\rangle), \frac{1}{\sqrt{2}} (|H\rangle - |V\rangle)\}$) allows her to distinguish between photons polarized at $+45^\circ$ and $-45^\circ$. Alice chooses between the two types with probabilities $1 - \epsilon$ and $\epsilon$, respectively. If she detects photon A in the state $|H\rangle$ or $\frac{1}{\sqrt{2}} (|H\rangle + |V\rangle)$, the result is 0; else, the measurement can yield the result 1, and potentially reveal one bit of information. She writes down her measurement bases and the results of the measurements.

4. For each photon, Bob performs measurements and registers the outcome of the measurements in one of three bases, obtained by rotating the rectilinear basis by angles $\phi_1 (\phi'_1) = 0$, $\phi_2 (\phi'_2) = \tan^{-1} \frac{\beta}{\alpha}$, $\phi_3 (\phi'_3) = -\tan^{-1} \frac{\beta}{\alpha}$, i.e., $\{|H\rangle, |V\rangle\}$, $\{\alpha |H\rangle + \beta |V\rangle, \beta |H\rangle - \alpha |V\rangle\}$, and $\{\beta |H\rangle + \alpha |V\rangle, \alpha |H\rangle - \beta |V\rangle\}$, with probabilities $1 - \epsilon$, $\frac{1}{2}$, and $\frac{1}{2}$, respectively. Similar to the measurements performed by Alice, each measurement can yield two results 0 (if he detects photon B in the state $|H\rangle$, $\alpha |H\rangle + \beta |V\rangle$ or $\beta |H\rangle + \alpha |V\rangle$) and 1 (if he detects photon B in the state $|V\rangle$, $\beta |H\rangle - \alpha |V\rangle$ or $\alpha |H\rangle - \beta |V\rangle$). The ensemble of these bits registered by both Alice and Bob is the raw key.

5. After exchanging enough photons, Bob announces on the public channel the sequence of bases he used, but not the results that he obtained.

6. Alice compares this sequence with the states that she originally chose, and the list of polarizations which she measured. Then she tells Bob on the public channel on which occasions his measurements have been done in the correct bases. Whenever Alice and Bob used the compatible basis, they should get perfectly correlated bits. However, due to imperfections in the setup, and to a potential eavesdropper, there will be some errors.

There are two cases in which Alice chooses entangled states $|AB\rangle$ and $|AB\rangle'$, respectively. For either of the two cases, both Alice and Bob are much more likely to choose the rectilinear basis and obtain correlated bits, thus achieving a significant gain in efficiency. If Alice chooses the diagonal basis, in order to generate a sifted key, Bob should choose between the bases $\{\alpha |H\rangle + \beta |V\rangle, \beta |H\rangle - \alpha |V\rangle\}$, and $\{\beta |H\rangle + \alpha |V\rangle, \alpha |H\rangle - \beta |V\rangle\}$ according to the entangled state chosen by Alice and the polarization of photon A. (Otherwise, if he uses the rectilinear basis, he gets
the outcomes 0 and 1 with probabilities $|\alpha|^2$ and $|\beta|^2$, respectively. These results abort.) For example, if Alice chooses the state $|AB\rangle$ and sends photon B to Bob. Then if she detects photon A polarized at $+45^\circ$ by measuring along the diagonal basis, Bob must choose the basis $\{\alpha |H\rangle + \beta |V\rangle, \beta |H\rangle - \alpha |V\rangle\}$ and photon B would be detected in the state $\alpha |H\rangle + \beta |V\rangle$. Therefore, they can generate a key bit “1” with probability $\frac{1}{4} \cdot \epsilon \cdot \frac{\pi}{4}$. The bases used by Alice and Bob agree with probability $(1 - \epsilon)^2 + \frac{\epsilon^2}{2}$ which goes to 1 as $\epsilon$ goes to zero.

Table I. Example of the case where Alice chooses $|AB\rangle$ as the original state. The measurement bases are presented as the angles by which the rectilinear basis is rotated (here $\theta = \tan^{-1} \frac{\epsilon}{2}$). The two users choose a basis with certain probability to measure their particles and register the bit value (0 or 1), respectively. The ensemble of these bits is the raw key. Alice tells Bob on the public channel on which occasions his measurements have been done in the correct bases, and they keep only the bits corresponding to the compatible bases. This is the sifted key.

| A basis | 0 | 0 | 0 | 0 | 0 | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ |
|---------|---|---|---|---|---|---|---|---|---|
| A bit value | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| B basis | 0 | $\theta$ | $\theta$ | 0 | $\theta$ | $\theta$ | 0 | $\theta$ | $\theta$ |
| B bit value | 0/1 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 | 1/0 |
| compatible? | y | n | n | y | n | n | y | n | n |
| sifted key | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 |

7. For each of the two cases in which Alice chooses the entangled states $|AB\rangle$ or $|AB\rangle'$, Alice and Bob divide up their polarization data into twelve cases according to the actual bases used and the bit values yielded (shown in Table I), respectively. Then they throw away the eight cases when they have used non-compatible bases. Since the total probabilities for the two users to obtain the results 0 and 1 are equal, the ensemble of these bits of the remaining four cases is a sifted key. Hence, the remaining cases are kept for further analysis and to generate the secret key.

8. Alice and Bob divide up the accepted data into two subsets according to the entangled states originally chosen by Alice. From the subset where Alice chooses $|AB\rangle$ as the prior state, there are three cases. In one case where Alice and Bob both use the rectilinear basis (including two cases shown in Table I, in each of which the bit value is “0” or “1”), they randomly pick a fixed number say $m_1$ photons and publicly compare their polarizations. The number of mismatches $r_1$ (here, mismatch means the polarizations of photons are not correlated) tells them the estimated error rate $\epsilon_1 = \frac{r_1}{m_1}$. In the case where Alice uses the diagonal basis and Bob uses the basis $\{\alpha |H\rangle + \beta |V\rangle, \beta |H\rangle - \alpha |V\rangle\}$, they pick a fixed number say $m_2$ photons and publicly compare their polarizations. The number of mismatches $r_2$ gives the estimated error rate $\epsilon_2 = \frac{r_2}{m_2}$. In the case where Alice uses the diagonal basis and Bob uses the basis $\{\beta |H\rangle + \alpha |V\rangle, \alpha |H\rangle + \beta |V\rangle\}$, they pick a fixed number say $m_3$ photons and publicly compare their polarizations. The number of mismatches $r_3$ gives the estimated error rate $\epsilon_3 = \frac{r_3}{m_3}$. Similarly, from the subset where Alice chose $|AB\rangle'$ as the prior state, there are also three cases. Corresponding to the above discussion, we obtain the error rates $\epsilon'_1 = \frac{r'_1}{m'_1}$, $\epsilon'_2 = \frac{r'_2}{m'_2}$, and $\epsilon'_3 = \frac{r'_3}{m'_3}$.

Note that the test samples $m_1$, $m'_1$, $m_2$, $m'_2$, $m_3$, and $m'_3$ are sufficiently large, the estimated error rates $\epsilon_1$, $\epsilon'_1$, $\epsilon_2$, $\epsilon'_2$, $\epsilon_3$, and $\epsilon'_3$ should be rather accurate. Now they demand that $\epsilon_1$, $\epsilon'_1$, $\epsilon_2$, $\epsilon'_2$, $\epsilon_3$, and $\epsilon'_3 < e_{\max}$ where $e_{\max}$ is a prescribed maximal tolerable error rate. If these independent constraints are satisfied, they proceed to the next steps. Otherwise, they throw away the polarization data and re-start the whole procedure. Notice that the constraints $\epsilon_1$, $\epsilon'_1$, $\epsilon_2$, $\epsilon'_2$, $\epsilon_3$, and $\epsilon'_3 < e_{\max}$ are more stringent than the original naive prescription $\bar{\epsilon} < e_{\max}$ (here $\bar{\epsilon}$ is the average error rate). We will discuss it in detail in Sec. III.

9. Reconciliation and privacy amplification (see Ref. [1,13]).

III. REFINED ERROR ANALYSIS

For each photon, the eavesdropper, Eve does not know which nonmaximally entangled state it is chosen from. So for Eve, each photon is in an entangled mixed state. She has eavesdropping attack as below:

i). with a probability $p_1$ measures polarization of each photon along the rectilinear basis and re-sends a photon according to the result of her measurement to Bob;

ii). with a probability $p_2$ measures polarization of each photon along the basis $\{\alpha |H\rangle + \beta |V\rangle, \beta |H\rangle - \alpha |V\rangle\}$ and re-sends a photon according to the result of her measurement to Bob;

iii). with a probability $p_3$ measures polarization of each photon along the basis $\{\beta |H\rangle + \alpha |V\rangle, \alpha |H\rangle - \beta |V\rangle\}$ and re-sends a photon according to the result of her measurement to Bob;
Therefore, Alice and Bob will find that, for the biased eavesdropping attack, the average error rate occurs only if Eve is measuring along the rectilinear basis or the basis \{p_{1}, p_{2}, p_{3}\}. This happens with a conditional probability \( p_{1} + p_{3} \) and when it happens, the photon polarization is randomized. Thus, the error rate for this case is
\[
e_{1}(e'_{1}) = 2\alpha^{2}\beta^{2}(p_{2} + p_{3}).
\]

Similarly, errors for the case where Alice uses the diagonal basis and Bob uses the basis \{\alpha |H\rangle + \beta |V\rangle, \beta |H\rangle - \alpha |V\rangle\} occur only if Eve uses the other two bases. This happens with a conditional probability \( p_{1} + p_{3} \).
\[
e_{2}(e'_{3}) = 2\alpha^{2}\beta^{2}p_{1} + 8\alpha^{2}\beta^{2}(\alpha^{2} - \beta^{2})^{2}p_{3}.
\]

Therefore, Alice and Bob will find that, for the biased eavesdropping attack, the average error rate
\[
\bar{\tau} = \frac{(1 - \epsilon)^{2}(e_{1} + e'_{1}) + e_{2} + e_{3} + e'_{2} + e'_{3}}{2(1 - \epsilon)^{2} + \frac{e'_{2}}{2}}
\]
\[
= \alpha^{2}\beta^{2}\left[2\sqrt{(1 - \epsilon)^{2}(p_{2} + p_{3}) + \epsilon^{2}p_{1} + 2\epsilon^{2}(\alpha^{2} - \beta^{2})^{2}(p_{2} + p_{3})}\right]
\[
= \frac{\alpha^{2}\beta^{2}e_{1}^{2}}{(1 - \epsilon)^{2} + \frac{e'_{2}}{2}} \rightarrow 0
\]

Suppose Eve always eavesdrops only along rectilinear basis (i.e., \( p_{1} = 1, p_{2} = p_{3} = 0 \)), then
\[
\tau = \frac{\alpha^{2}\beta^{2}e_{1}^{2}}{(1 - \epsilon)^{2} + \frac{e'_{2}}{2}} \rightarrow 0
\]
as \( \epsilon \) tends to 0, which is similar with the result of Ref. [13]. This means that if Eve is always eavesdropping along the dominant basis, with a naive error analysis prescribed as \( \tau < e_{\text{max}} \), Alice and Bob will fail to detect eavesdropping by Eve.

To ensure the security of our scheme, it is crucial to employ a refined data analysis: the accepted data are further divided into various subsets according to the actual basis used by Alice and Bob and the error rate of each subset is computed separately. In Sec. II, we have already computed the error rates \( e_{1}, e'_{1}, e_{2}, e'_{2}, e_{3}, \) and \( e'_{3} \), where \( e_{\text{max}} \) is a prescribed maximal tolerable error rate. From Eqs. (3, 4, 5), we can see that these error rates \( e_{1}, e'_{1}, e_{2}, e'_{2}, e_{3}, \) and \( e'_{3} \) depend on Eve’s eavesdropping strategy and the degree of entanglement of the original state, but not on the value of \( \epsilon \). So the refined data analysis guarantees the security of the present scheme.

**IV. THE CONSTRAINT ON \( \epsilon \)**

From the above discussion, we know the value of \( \epsilon \) should be small but can not be zero. If \( \epsilon \) were actually zero, the scheme would be insecure. The main constraint on \( \epsilon \) is that there should be enough photons for an accurate estimation of the six error rates \( e_{1}, e'_{1}, e_{2}, e'_{2}, e_{3}, \) and \( e'_{3} \). We assume that \( N \) entangled pairs are chosen by Alice, i.e., \( N \) photons are transmitted from Alice to Bob. On average, for \( |AB\rangle \) or \( |AB'\rangle \), only \( N\epsilon^{2}/8 \) photons belong to each of the two cases where Alice uses the diagonal basis and Bob uses the basis \{\alpha |H\rangle + \beta |V\rangle, \beta |H\rangle - \alpha |V\rangle\} or the basis \{\beta |H\rangle + \alpha |V\rangle, \alpha |H\rangle - \beta |V\rangle\}. To estimate \( e_{2}, e'_{2}, e_{3}, \) and \( e'_{3} \) reasonably accurately, the number \( N\epsilon^{2}/8 \) should be larger than some fixed number say \( m = \max(m_{2}, m'_{2}, m_{3}, m'_{3}) \). The numbers \( m_{2}, m'_{2}, m_{3}, \) and \( m'_{3} \) are the photon number needed for the refined error analysis, which can be computed from classical statistical analysis. So
\[
N\epsilon^{2}/8 \geq m,
\]
\[
\epsilon \geq 2\sqrt{2m/N}.
\]
As \( N \) tends to \( \infty \), \( \epsilon \) can tend to zero, but never reach it. And the efficiency of this scheme is asymptotic 100%.
V. DISCUSSION AND CONCLUSION

From the refined error analysis, we find the error rates depend not only on Eve’s eavesdropping strategy but also on the degree of entanglement of the original state. For the biased eavesdropping attack, the error rates \( e_i \) and \( e'_i \) are functions of \( \alpha \beta \), the probability \( \epsilon \) and the eavesdropping strategy of Eve (seeing Eqs. (3-5)). If Alice uses a product state as the original state, i.e., \( \alpha \beta = 0 \), whatever the probability \( \epsilon \) and Eve’s eavesdropping strategy are, the error rates \( e_i \) and \( e'_i \) equal to zero. That is, Alice and Bob will never detect eavesdropping by Eve whatever she does. In other words, if \( \alpha \beta = 0 \), the scheme is easily broken by an eavesdropper. The security of our scheme is relying on the degree of the entanglement of the original state. If \( |\alpha \beta| = \frac{1}{2} \), this scheme is equivalent to an efficient “simplified EPR scheme” [3].

Of course, the QKD with nonmaximally entangled states may also be completed in another way. At first, the nonmaximally entangled state \( |AB\rangle = \alpha |00\rangle + \beta |11\rangle \) (here \( |\beta| < |\alpha| \)) can be concentrated to an EPR state [16,17] with probability \( 2|\beta|^2 \) [18]. If the concentration fails, EPR pairs are abandoned; else, they are used in an efficient “simplified EPR scheme” [3]. Obviously, the total efficiency of this QKD process should be no more than \( 2|\beta|^2 \).

In summary, we propose a quantum key distribution (QKD) scheme based on entanglement, where Alice and Bob choose between various bases independently with substantially different probabilities. Since two parties are much more likely to be using the same basis, thus reducing the fraction of discarded data, a significant gain in efficiency is achieved. The efficiency can be tend to 100%, as the value of \( \epsilon \) tends to zero (but can not reach it accurately).

To make the scheme secure against the dominant basis eavesdropping attack, it is crucial to have a refined error analysis in place of a naive error analysis. We separate the accepted data into various subsets according to the basis employed and estimate an error rate for each subset separately. It is only when all error rates are small enough that the security of transmission is accepted.

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