Initial value problem for the time-dependent linear Schrödinger equation with a point singular potential by the uniform transform method

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Abstract

We study an initial value problem for the one-dimensional non-stationary linear Schrödinger equation with a point singular potential. In our approach, the problem is considered as a system of coupled initial-boundary value (IBV) problems on two half-lines, to which we apply the unified approach to IBV problems for linear and integrable nonlinear equations, also known as the Fokas unified transform method. Following the ideas of this method, we obtain the integral representation of the solution of the initial value problem. Since the unified approach is known as providing efficient solutions to both linear and nonlinear problems, the present paper can be viewed as a step in solving the initial value problem for the non-stationary nonlinear Schrödinger equation with a point singular potential.

1 Introduction

The nonlinear Schrödinger (NLS) equation with an external potential $V(x)$ (the so-called Gross-Pitaevskii (GP) equation)

$$iu_t + V(x)u + \Delta u + 2\kappa |u|^2 = 0$$

is used to describe a lot of phenomena in physics. In particular, it describes the evolution of Bose-Einstein condensates in dilute boson gases at low temperatures. Also this equation appears in studying laser beams in Kerr media and focusing nonlinearity (see [11] and the references therein).

For applications it is desirable to get exact solution of the initial value (IV) problem for equation (1), because it is helpful for describing detailed aspects of a particular physical system, when the approximate methods could be inadequate. But even in one dimension, it is hard to obtain such solutions.
A reduction of (1), related to short-range interactions, consists in replacing $V(x)$ by a point singular potential. In the one-dimensional case, $V(x) = q\delta(x)$. Assuming that the initial data $u(x, 0) = u_0(x)$ is an even function, the solution is even for all $t$, and the IV problem

$$\begin{align*}
    &\left\{ \begin{array}{l}
    iu_t + u_{xx} + q\delta(x)u + 2u|u|^2 = 0, \ x > 0, \ t > 0, \\
    u(x, 0) = u_0(x), \ x \geq 0
    \end{array} \right.
\end{align*}$$

(2)

can be reduced to the initial-boundary value (IBV) problem for the NLS equation without the singular term, with the Robin homogeneous boundary condition, see [7]:

$$\begin{align*}
    &\left\{ \begin{array}{l}
    iu_t + u_{xx} + 2u|u|^2 = 0, \ x > 0, \ t > 0, \\
    u_x(0, t) + qu(0, t) = 0, \ t \geq 0, \\
    u(x, 0) = u_0(x), \ x \geq 0
    \end{array} \right.
\end{align*}$$

(3)

In the latter problem, the boundary conditions are called linearizable, because in this case there exists an adaptation of the Inverse Scattering Transform (IST) method, which turns to be as efficient as the IST method for the initial value problem (on the whole line), see, e.g., [6], [9]; namely, solving the problem reduces to solving a sequence of linear problems.

Particularly, in [7] the authors have studied in details the long-time behavior of the solution of (2) by applying the nonlinear steepest-descent method for Riemann–Hilbert problems introduced by Deift and Zhou [8].

However, in the general case, when $u_0(x)$ is not assumed to be even, (2) does not reduce to (3) and thus the approach of [7] cannot be applied directly.

In solving a nonlinear problem, it is natural to solve, first, the associated linearized problem:

$$\begin{align*}
    &\left\{ \begin{array}{l}
    iu_t + u_{xx} + q\delta(x)u = 0, \ x > 0, \ t > 0, \\
    u(x, 0) = u_0(x), \ x \geq 0
    \end{array} \right.
\end{align*}$$

(4)

When trying to reduce (4) to a problem on a half-line, we arrive, instead of one problem (3), at a system of two coupled half-line problems, see (10) and (11) below. Our approach is based on using the Fokas unified transform method (initiated by Fokas [2]) to this system. For a detailed account of this approach, see [3].

Since the unified transform method is known as providing effective solutions to both linear and nonlinear integrable equations, the present paper can be viewed as a step in solving the initial value problem for the non-stationary nonlinear Schrödinger equation with a point singular potential.

The unified transform method is based on the so-called Lax pair representation of an equation in question. Recall that the Lax pair of a given (partial differential) equation is a system of linear ordinary differential equations involving an additional (spectral) parameter, such that the given equation is the compatibly condition of this system. Originally, the Lax pair representation was introduced for certain nonlinear evolution equations, called integrable; see, e.g., [1]. This representation plays a key role in solving initial value problems for such equations by
the inverse scattering transform method, where one (spatial) equation from the pair establishes
a change of variables, passing from functions of the spatial variable to functions of the spectral
parameter, whereas the evolution in time turns out to be, due to the second equation, linear.

According to the approach initiated by Fokas for studying initial boundary value problems
for such nonlinear equations, both equations from the Lax pair are considered as spectral
problems.

While only very special (although very important) nonlinear equations have a Lax pair
representation (the NLS equation in (3) is one of them), the Lax pair for linear equations
with constant coefficients can be constructed algorithmically. Particularly, the Lax pair for the
linearized form of the NLS equation,

\[ iu_t + u_{xx} = 0, \]

is as follows:

\[
\begin{align*}
(\mu e^{-ikx+ik^2t})_x &= u e^{-ikx+ik^2t} \\
(\mu e^{-ikx+ik^2t})_t &= (iu_x - ku)e^{-ikx+ik^2t}
\end{align*}
\]

Indeed, direct calculations show that the compatibly condition of (5), which is the equality

\[(ue^{-ikx+ik^2t})_t = ((iu_x - ku)e^{-ikx+ik^2t})_x\]

to be satisfied for all \( k \in \mathbb{C} \), reads \( iu_t + u_{xx} = 0 \).

In this paper we demonstrate how to obtain an integral representation of the solution of
problem (4) using the ideas of the Fokas method. Since this method for both linear and
integrable nonlinear equations uses the similar ingredients — the Lax pair representation and
the analysis of the so-called global relation — we believe that our result can be useful for
studying the initial value problem for the non-stationary nonlinear Schrödinger equation with
a point singular potential.

## 2 Representation of solution

In what follows we will consider the IV problem for the linear time-depended Schrödinger
equation with one-delta potential in the form

\[
\begin{align*}
iu_t + 2q\delta(x-a)u + u_{xx} &= 0 \\
u(x,0) &= u_0(x)
\end{align*}
\]

where \( u_0(x) \) is a smooth function decaying as \( |x| \to \infty \) (below we will specify the meaning
of this). First, let us assume that the solution \( u(x,t) \) of the problem (6) exists, such that
\( u(x,t) \in L_1(\mathbb{R}) \) and \( u_x(x,t) \to 0 \) as \( x \to \pm\infty \) for all \( t \). Under this assumptions we will find the
exact formula for solution \( u(x,t) \). Then we impose some restrictions on the initial value \( u_0(x) \),
which guarantee that the obtained formulas solve the problem (6). Since the delta function
\( 2q\delta(x-a) \) introduces the jump \( u_x(a+0,t) - u_x(a-0,t) + 2qu(a,t) = 0 \), (6) is equivalent to
the following system
\[
\begin{cases}
  iu_t + u_{xx} = 0 \\
  u_x(a+0,t) - u_x(a-0,t) + 2qu(a,t) = 0 \\
  u(x,0) = u_0(x)
\end{cases}
\]  
(7)

Let
\[
u^{(1)}(x,t) := \begin{cases}
  u(x,t), & x \leq a \\
  0, & x > a
\end{cases}, \quad u^{(2)}(x,t) := \begin{cases}
  u(x,t), & x \geq a \\
  0, & x < a
\end{cases}
\]  
(8)

Then the jump relations yield
\[
u_x^{(1)}(a,t) - qu^{(1)}(a,t) = u_x^{(2)}(a,t) + qu^{(2)}(a,t) =: \phi_1(t).
\]

Since \(u(x,t)\) is a continuous function, we can introduce
\[g_1(t) := u^{(1)}(a,t) = u^{(2)}(a,t).
\]

Then the initial value problem (7) is equivalent to the following coupled initial boundary value problems, with the *inhomogeneous* boundary conditions:
\[
\begin{cases}
  iu_t^{(1)} + u_{xx}^{(1)} = 0, & x < a \\
  u^{(1)}(x,0) = u_0^{(1)}(x) \\
  u_x^{(1)}(a,t) - qu^{(1)}(a,t) = \phi_1(t) \\
  u^{(1)}(a,t) = g_1(t)
\end{cases}
\]  
(10)

\[
\begin{cases}
  iu_t^{(2)} + u_{xx}^{(2)} = 0, & x > a \\
  u^{(2)}(x,0) = u_0^{(2)}(x) \\
  u_x^{(2)}(a,t) + qu^{(2)}(a,t) = \phi_1(t) \\
  u^{(2)}(a,t) = g_1(t)
\end{cases}
\]  
(11)

We emphasize that neither \(\phi_1(t)\) nor \(g_1(t)\) are known, but problems (10) and (11) are coupled by the conditions that these functions are the same in the both problems.

In what follows, we will use the following notations for the direct and inverse Fourier transforms:
\[
\mathcal{F}(f)(k) = \hat{f}(k) = \int_{-\infty}^{+\infty} e^{-ikx} f(x) dx, \quad \mathcal{F}^{-1}(f)(x) = f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{ikx} \hat{f}(x) dx.
\]

Now, let us assume for a moment that the functions \(g_1(t)\) and \(\phi_1(t)\) are given. Then, using the Lax pair for both equations \(u_{it} + u_{xx} = 0, \ i = 1, 2\), we can obtain explicitly the solutions of (10) and (11).
Proposition 1 Let
\[
h_1(k, t) := \int_0^t e^{ik^2(\tau-t)}g_1(\tau) \, d\tau, \quad h_2(k, t) := \int_0^t e^{ik^2(\tau-t)}\phi_1(\tau) \, d\tau.
\] (12)
Then \(u^{(j)}(x, t), \ j = 1, 2\) can be obtained in terms of \(h_1(k, t), h_2(k, t)\), and the Fourier transform of \(u_0(x)\) as follows:
\[
u^{(1)}(x, t) = \mathcal{F}^{-1}\left( e^{-ik^2t}u_0^{(1)}(k) + e^{-ika}(iq_1 + k)h_1(k, t) + ie^{-ika}h_2(k, t) \right),
\] (13)
\[
u^{(2)}(x, t) = \mathcal{F}^{-1}\left( e^{-ik^2t}u_0^{(2)}(k) + e^{-ika}(iq + k)h_1(k, t) - ie^{-ika}h_2(k, t) \right),
\] (14)
where \(\mathcal{F}(u_0^{(i)})(k) = \mathcal{F}(u_0^{(i)})(x), \ i = 1, 2\).

Proof.
Let's consider the Lax pair for equation \(u^{(1)}_t + u^{(1)}_{xx} = 0\):
\[
\begin{aligned}
\begin{cases}
(\mu e^{-ikx+ik^2t})_x &= u^{(1)}(x)e^{-ikx+ik^2t} \\
(\mu e^{-ikx+ik^2t})_t &= (iu^{(1)}_x - ku^{(1)})e^{-ikx+ik^2t}.
\end{cases}
\end{aligned}
\] (15)
Equations (15) determine an exact 1-form \(W_1\), so that (15) can be written as
\[
W_1(x, t, k) = d\left((\mu e^{-ikx+ik^2t})\right), \ x \leq a, t \geq 0.
\] (16)
The fundamental theorem of calculus implies that \(\oint_{\gamma_1} W_1(y, \tau, k) = 0\) for every closed curve \(\gamma_1 \subset \{(x, t) \in \mathbb{R}^2 : x \leq a, t \geq 0\}\). Let \(\gamma_1\) be the rectangle with vertices \((-X, 0)\) (for some large \(X\)) and \((a, t)\). Then the equation \(\int_{\gamma_1} W_1(y, \tau, k) = 0\) takes the form
\[
0 = \int_{\gamma_1} W_1(y, \tau, k) = \oint_\square (u^{(1)}e^{-ikx+ik^2\tau}) \, dx + ((iu^{(1)}_x - ku^{(1)})e^{-ikx+ik^2\tau}) \, d\tau
\] (17)
\[
= \int_{-X}^a e^{-ikx}u^{(1)}_0(x) \, dx + \int_0^t e^{-ika+ik^2\tau}(iu^{(1)}_x(a, \tau) - ku^{(1)}(a, \tau)) \, d\tau
\]
\[
- \int_{-X}^a e^{-ikx+ik^2t}u^{(1)}(x, t) \, dx - \int_0^t e^{ikX+ik^2\tau}(iu^{(1)}_x(-X, \tau) - ku^{(1)}(-X, \tau)) \, d\tau.
\]
Now let \(X \to +\infty\) in (17). Since the last term decays to 0 as \(X \to +\infty\), equation (17) takes the form
\[
\mathcal{F}^{(1)}(k) + \int_0^t e^{-ika+ik^2\tau} \left[ i(\phi_1(\tau) + q_1g_1(\tau)) - kg_1(\tau) \right] \, d\tau - e^{ik^2t}u^{(1)}(k, t) = 0.
\] (17)
Using the notations (12), the last equation can be written as
\[
u^{(1)}(k, t) = e^{-ik^2t}u^{(1)}_0(k) + e^{-ika}(iq_1 - k)h_1(k, t) + ie^{-ika}h_2(k, t), \quad \exists k \geq 0,
\] (18)
which implies (13).
The second IBV problem will be solved in the similar way. Namely,

$$\int_{\gamma_2} (u^{(2)} e^{-ik_y + ik^2 \tau}) \, dy + ((iu_y^{(2)} - ku^{(2)}) e^{-ik_y + ik^2 \tau}) \, d\tau = 0$$

(19)

for each closed curve $\gamma_2$, $\gamma_2 \subset \{(x, t) \in \mathbb{R}^2 : x \geq a, t \geq 0\}$. Let $\gamma_2$ be the rectangle with vertices $(a, 0), (X, t)$. Then

$$0 = \int_{\square} (u^{(2)} e^{-ikx + ik^2 \tau}) \, dx + ((iu_x^{(2)} - ku^{(2)}) e^{-ikx + ik^2 \tau}) \, d\tau$$

$$= \int_a^X e^{-ikx} u_0^{(2)}(x) \, dx + \int_0^t e^{-ikx + ik^2 \tau}(iu_x^{(2)}(X, \tau) - ku^{(2)}(X, \tau)) \, d\tau$$

$$- \int_a^X e^{-ikx + ik^2 \tau} u^{(2)}(x, t) \, dx - \int_0^t e^{-ikx + ik^2 \tau}(iu_x^{(2)}(a, \tau) - ku^{(2)}(a, \tau)) \, d\tau.$$  (20)

Now, letting $X \to +\infty$, relation (20) takes the form

$$\hat{u}_0^{(2)}(k) - e^{ik^2 t} \hat{u}^{(2)}(k, t) - \int_a^t e^{-ika + ik^2 \tau} [i(\phi_1(\tau) - \phi_1(\tau)) - k\phi_1(\tau)] \, d\tau = 0.$$  (21)

In terms of (12), the last equation can be written as

$$\hat{w}^{(2)}(k, t) = e^{-ik^2 t} \hat{u}_0^{(2)}(k) + e^{-ika}(iq + k)h_1(k, t) - ie^{-ika}h_2(k, t), \quad \forall k \leq 0,$$  (22)

which implies (14).

\[ \blacksquare \]

**Remark 1** Equations (18) and (22) play a crucial role in our approach. In the framework of the Fokas method, these equations are called the global relations.

Global relations are algebraic relations between the initial and boundary values of a given IBV problem [3]. In our case, only the spectral functions $\hat{u}_0^{(j)}(k), j = 1, 2$, associated with the initial data $u_0(x)$ are given whereas the spectral functions $\hat{u}_j(k, t), j = 1, 2$ are unknown. Although the global relations involve unknown functions ($\hat{u}^{(j)}(k, t)$), they characterize the unknown boundary values in terms of the known (given) boundary and initial values. For single IBV problems for linear equations, see [3, 7] for details. One of the main aims of the present paper is to show that such a characterization can be efficiently used in the case of coupled IBV problems like (10), (11).

An important feature of the global relations (18) and (22) is the properties of analyticity and decay of their left-hand sides. They follow directly from their definitions as the Fourier transforms of functions supported on the associated half-lines.

**Lemma 1** Let

$$C_- = \{z \in \mathbb{C} : \text{Im } z < 0\}, \quad C_+ = \{z \in \mathbb{C} : \text{Im } z > 0\}.$$  (23)

Functions $h_i(k, t)$, see notation (12), are analytic in $C_i, i = 1, 2$; $\hat{u}_1^{(1)}(k, t)$ is analytic in $C_+$ and $\hat{w}_1^{(2)}(k, t)$ is analytic in $C_-$ for all $t \geq 0$. Moreover, $e^{ik(x-2a)}\hat{u}_1^{(1)}(k, t)$ and $e^{ikx}\hat{w}_1^{(2)}(-k, t)$ decay to 0 as $k \to \infty$ in $C_-$ respectively for $x < a$ and $x > a$. 

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For solution of this system is given in terms of the Fourier transforms of the initial data, as follows:

\[
\begin{align*}
\hat{u}^{(1)}(k,t) - e^{-ik^2t}\hat{u}_0^{(1)}(-k) &= e^{ika}(iq + k)h_1(k,t) + ie^{ika}h_2(k,t), \\
\hat{u}^{(2)}(k,t) - e^{-ik^2t}\hat{u}_0^{(2)}(k) &= e^{-ika}(iq + k)h_1(k,t) - ie^{-ika}h_2(k,t),
\end{align*}
\]

which holds for all \(k \in \mathbb{C}_-\).

Now suppose for a moment that the functions \(\hat{u}^{(i)}(k,t), i = 1, 2\) are known. Then (24) can be considered as a system of two algebraic equations for two unknown functions \(h_i(k,t)\). The solution of this system is given in

**Lemma 2** For \(3k \leq 0\), functions \(h_1(k,t)\) and \(h_2(k,t)\) can be expressed in terms of \(\hat{u}_0^{(i)}(k)\) and \(\hat{u}^{(i)}(k,t)\) as follows:

\[
\begin{align*}
h_1(k,t) &= \frac{e^{-ika}(\hat{u}^{(1)}(-k,t) - e^{-ik^2t}\hat{u}_0^{(1)}(-k)) + e^{ika}(\hat{u}^{(2)}(k,t) - e^{-ik^2t}\hat{u}_0^{(2)}(k))}{2(iq + k)}, \\
h_2(k,t) &= \frac{e^{-ika}(\hat{u}^{(1)}(-k,t) - e^{-ik^2t}\hat{u}_0^{(1)}(-k)) - e^{ika}(\hat{u}^{(2)}(k,t) - e^{-ik^2t}\hat{u}_0^{(2)}(k))}{2i}.
\end{align*}
\]

Now we are at a position to formulate and prove the main representation result.

**Theorem 1** The solution \(u(x,t)\) of problem (3) depends on the sign of \(q\) and is given, in terms of the Fourier transforms of the initial data, as follows:

- If \(q < 0\), then
  
  \[
  u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \hat{u}_0^{(1)}(k) \, dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \frac{i q e^{-2 i k a} \hat{u}_0^{(1)}(-k) - k \hat{u}_0^{(2)}(k)}{i q + k} \, dk
  \]
  for \(x < a\), and
  
  \[
  u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \hat{u}_0^{(2)}(k) \, dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \frac{k \hat{u}_0^{(1)}(k) + i q e^{-2 i k a} \hat{u}_0^{(2)}(-k)}{i q - k} \, dk
  \]
  for \(x > a\).

- If \(q > 0\), then
  
  \[
  u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \hat{u}_0^{(1)}(k) \, dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \frac{i q e^{-2 i k a} \hat{u}_0^{(1)}(-k) - k \hat{u}_0^{(2)}(k)}{i q + k} \, dk + \frac{q e^{-q(x-a)} + i q^2 t}{2\pi} \left( \int_{-\infty}^{a} e^{q(y-a)} \hat{u}_0^{(1)}(y) \, dy + \int_{a}^{\infty} e^{-q(y-a)} \hat{u}_0^{(2)}(y) \, dy \right)
  \]
  for \(x < a\), and
  
  \[
  u(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \hat{u}_0^{(2)}(k) \, dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \frac{k \hat{u}_0^{(1)}(k) + i q e^{-2 i k a} \hat{u}_0^{(2)}(-k)}{i q - k} \, dk + \frac{q e^{-q(x-a)} + i q^2 t}{2\pi} \left( \int_{-\infty}^{a} e^{q(y-a)} \hat{u}_0^{(1)}(y) \, dy + \int_{a}^{\infty} e^{-q(y-a)} \hat{u}_0^{(2)}(y) \, dy \right)
  \]
  for \(x > a\).
Proof.

Since \( h_i(k,t) = h_i(-k,t) \), the first equation in (24) gives the system

\[
\begin{align*}
\left\{ \begin{array}{l}
e^{-ika}(\hat{u}^{(1)}(-k,t) - e^{-ik^2t_0^{(1)}}(-k)) = (iq + k)h_1(k,t) + ih_2(k,t) \\
e^{ika}(\hat{u}^{(1)}(k,t) - e^{-ik^2t_0^{(1)}}(k)) = (iq - k)h_1(k,t) + ih_2(k,t)
\end{array} \right.
\]
\tag{27}
\]

that allows getting rid of \( h_2(k,t) \):

\[
e^{-ika}\hat{u}^{(1)}(-k,t) - e^{ika}\hat{u}^{(1)}(k,t) - e^{-ika-ik^2t_0^{(1)}}(-k) + e^{ika-ik^2t_0^{(1)}}(k) = 2kh_1(k,t). \tag{28}
\]

Applying the inverse Fourier transform, one gets

\[
-u^{(1)}(a + x,t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-a)-ik^2t_0^{(1)}}(k) \, dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-a)-ik^2t_0^{(1)}}(-k) \, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-2kh_1(k,t)} \, dk, \quad x < 0,
\]

which is equivalent to

\[
-u^{(1)}(x,t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx-ik^2t_0^{(1)}}(k) \, dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-2a)-ik^2t_0^{(1)}}(-k) \, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-a)}2kh_1(k,t) \, dk, \quad x < a.
\tag{30}
\]

Now, in order to obtain \( u^{(1)}(x,t) \), we have to express \( I_1(x,t) := \int_{-\infty}^{\infty} e^{ik(x-a)}2kh_1(k,t) \, dk \) for \( x < a \) in terms of the given data, i.e., in terms of \( u_0^{(i)}(x) \). First, we notice that by the definition of \( h_1(k,t) \), it decays to 0 as \( k \to \infty \) in the quadrant \( \Re k > 0, \Im k < 0 \). Thus, by Jordan’s lemma,

\[
I_1(x,t) = \frac{1}{2\pi} \int_{\gamma} e^{ik(x-a)}2kh_1(k,t) \, dk,
\tag{31}
\]

where the contour \( \gamma \) is shown in Fig. 1. Then, using (25) in the r.h.s. of (31), we have

\[
I_1(x,t) = I_2(x,t) - I_3(x,t),
\]

where

\[
I_2(x,t) = \frac{1}{2\pi} \int_{\gamma} e^{ik(x-a)}k e^{-ika}\hat{u}^{(1)}(-k,t) + e^{ika}\hat{u}^{(2)}(k,t) \, dk
\]

and

\[
I_3(x,t) = \frac{1}{2\pi} \int_{\gamma} e^{ik(x-a)-ik^2t}k e^{-ika}\hat{u}^{(1)}(-k) + e^{ika}\hat{u}^{(2)}(k) \, dk.
\]

Then Lemma 1 implies that \( I_2(x,t) = 0 \).

Finally, in order to evaluate \( I_3(x,t) \), we deform, again using Lemma 1, the integration contour back to the real axis:

\[
I_3(x,t) = I_{31}(x,t) + I_{32}(x,t),
\]

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where
\[ I_{31}(x, t) = -i \text{res}_{-i q} e^{i k(x-a) - i k^2 t} e^{-i k a \gamma_0^{(1)}(-k)} + e^{i k a \gamma_0^{(2)}(k)} \]
and
\[ I_{32}(x, t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k(x-a) - i k^2 t} e^{-i k a \gamma_0^{(1)}(-k)} + e^{i k a \gamma_0^{(2)}(k)} \frac{dk}{i q + k} \]

Obviously, in the case \( q < 0 \) we have \( I_{31}(x, t) = 0 \) whereas in the case \( q > 0 \),
\[ I_{31}(x, t) = q e^{q(x-a) + i q^2 t} \left( \int_{-\infty}^{a} e^{q(y-a)} u_0^{(1)}(y) dy + \int_{a}^{\infty} e^{-q(y-a)} u_0^{(2)}(y) dy \right) \]

Substituting this into (30) we arrive at the statements of Theorem 1 concerning \( u(x, t) \) for \( x < a \).

Similarly we can find \( u^{(2)}(x, t) \). Indeed, the second equation in (24) implies
\[ e^{i k a \gamma} u^{(2)}(k, t) - e^{-i k a \gamma} u^{(2)}(-k, t) + e^{-i k a - i k^2 t} \hat{\gamma}^{(2)}(-k) - e^{i k a - i k^2 t} \hat{\gamma}^{(2)}(k) = 2 k h_1(k, t) \quad (32) \]

Applying the inverse Fourier transform, one gets
\[ -u^{(2)}(a - x, t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x - i k^2 t} \hat{u}_0^{(2)}(-k) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k (x+a) - i k^2 t} \hat{u}_0^{(2)}(k) dk = \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i k x} 2 k h_1(k, t) dk, \quad x < 0 \quad (33) \]
or
\[ -u^{(2)}(x, t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i k x - i k^2 t} \hat{u}_0^{(2)}(-k) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i k (x-a) - i k^2 t} \hat{u}_0^{(2)}(k) dk = \]
\[ = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i k (x-a)} 2 k h_1(k, t) dk, \quad x > a \quad (34) \]

Taking into account that the last term in (34) has already been calculated above, from (34) we obtain
\[ u^{(2)}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i k x - i k^2 t} \hat{u}_0^{(2)}(-k) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i k x - i k^2 t} \frac{i q e^{2 i k a \gamma} u_0^{(2)}(k) - k \hat{u}_0^{(1)}(-k)}{i q + k} dk \quad (35) \]
in the case $q < 0$ and

$$u^{(2)}(x, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - ik^2t} \hat{u}_0^{(2)}(-k) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx - ik^2t} \frac{i\hat{q}e^{2ika} \hat{u}_0^{(2)}(k) - k\hat{u}_0^{(1)}(-k)}{i\hat{q} + k} dk$$

$$+ qe^{-q(x-a) + it^2} \left( \int_{-\infty}^{a} e^{q(y-a)} \hat{u}_0^{(1)}(y) dy + \int_{a}^{\infty} e^{-q(y-a)} \hat{u}_0^{(2)}(y) dy \right)$$

(36)

in the case $q > 0$. Substituting $k$ by $-k$ in (35) and (36) we arrive at the statements of Theorem

**Proposition 2** Let $u_0(x) \in C^2(\mathbb{R} \setminus \{a\})$. Suppose, that $u_0(x), xu_0(x), x^2u_0(x) \in L_1(\mathbb{R})$, $u_0'(x), xu_0'(x), x^2u_0'(x) \in L_1(\mathbb{R})$ and $u_0''(x), xu_0''(x), x^2u_0''(x) \in L_1(\mathbb{R} \setminus [a - \varepsilon, a + \varepsilon]), \varepsilon > 0$. Then the unique solution of problem (6) is given by the formulas in Theorem 7.

**Proof**

Since $u_0(x), u_0'(x) \in L_1(\mathbb{R}), u_0''(x) \in L_1(\mathbb{R} \setminus [a - \varepsilon, a + \varepsilon]),$ we have $\hat{u}_0(k) = O(\frac{1}{k^2}), k \to \infty$ and thus the integrals in the formula for solution in Theorem 7 exist.

In order to prove that $u(x, t)$ given in Theorem 7 solves the Schrödinger equation $iu_t + u_{xx} = 0$, we observe that

$$\int_{-\infty}^{\infty} e^{ikx - ik^2t} k\hat{u}_0(k) dk = \frac{1}{2it} \int_{-\infty}^{\infty} e^{ikx - ik^2t} (ix\hat{u}_0(k) + \hat{u}_0'(k)) dk$$

(37)

$$\int_{-\infty}^{\infty} e^{ikx - ik^2t} k^2\hat{u}_0(k) dk = \frac{1}{2it} \int_{-\infty}^{\infty} e^{ikx - ik^2t} (ixk\hat{u}_0(k) + \hat{u}_0(k) + k\hat{u}_0'(k)) dk$$

(38)

Since $xu_0(x), (xu_0)'(x) \in L_1(\mathbb{R}), (xu_0)''(x) \in L_1(\mathbb{R} \setminus [a - \varepsilon, a + \varepsilon]),$ it follows that $\hat{u}_0'(k) = O(\frac{1}{k^2}), k \to \infty$ and since $x^2u_0(x), (x^2u_0)'(x) \in L_1(\mathbb{R}), (x^2u_0)''(x) \in L_1(\mathbb{R} \setminus [a - \varepsilon, a + \varepsilon]),$ we have $\hat{u}_0''(k) = O(\frac{1}{k^2}), k \to \infty$. Therefore, (37) and (38) imply that the integrals $\int_{-\infty}^{\infty} e^{ikx - ik^2t} k\hat{u}_0(k) dk$ and $\int_{-\infty}^{\infty} e^{ikx - ik^2t} k^2\hat{u}_0(k) dk$ converge, uniformly in $x$ and $t$, in the domain $D = \{(x, t) \in \mathbb{R}^2 | x \in (-X, X), t \in (t_0, T_0)\}$ for each $X > 0, t_0 > 0, T_0 > t_0$. Thus, $u(x, t)$ solves the Schrödinger equation.

In order to prove that $u(x, t) \in L_1(\mathbb{R})$, we use the standard stationary phase method. We have

$$\int_{-\infty}^{\infty} e^{ikx - ik^2t} \hat{u}_0(k) dk = \int_{-\infty}^{\infty} e^{i\xi S(k)} \hat{u}_0(k) dk$$

(39)

where $S(k) = k - \frac{\xi^2}{2}$. Observe that $S'(k) = 0$ if and only if $k = \frac{x}{2\xi}$. Let $l = k - \frac{x}{2\xi}$, then

$$\int_{-\infty}^{\infty} e^{ikx - ik^2t} \hat{u}_0(k) dk = e^{i\frac{x^2}{2\xi}} \int_{-\infty}^{\infty} e^{-ilt^2} \hat{u}_0(l + \frac{x}{2\xi}) dl$$

(40)

Since $\hat{u}_0(k) = O(\frac{1}{k^2}), k \to \infty$, the last integral behaves like $\frac{1}{x^2}$ for large $x$ and thus we have

$$\int_{-\infty}^{\infty} e^{ikx - ik^2t} \hat{u}_0(k) dk \sim \frac{C_1(t)}{x^2}, x \to \infty$$

(41)
Therefore, \( u(x, t) \in L_1(\mathbb{R}) \). Similarly, one gets

\[
\int_{-\infty}^{\infty} e^{ikx-ik^2t} \hat{k}u_0(k) \, dk \sim \frac{C_2(t)}{x}, \quad x \to \infty
\]

and thus \( u_x(x, t) \to 0 \) as \( x \to \infty \).

Now we have to prove that the solution \( u(x, t) \) satisfies the jump condition in (7). This is equivalent to the fact that \( u^{(1)}(x, t) \) and \( u^{(2)}(x, t) \) satisfy the boundary conditions in (10), (11) respectively, which in turn is equivalent to the global relation for \( u^{(1)}(x, t) \) and \( u^{(2)}(x, t) \), see remark 1. But the last holds true, because the formulas for \( u^{(1)}(x, t) \) and \( u^{(2)}(x, t) \) are obtained from the global relation using only the equivalent transformations.

\[\square\]

3 Long-time asymptotics

Having explicit formulas for the solution \( u(x, t) \) of problem (6), the long-time asymptotics of \( u(x, t) \) follows by applying the standard stationary phase method.

**Proposition 3** Let \( u_0(x) \), \( xu_0(x) \) \( \in L_1(\mathbb{R}) \), \( u'_0(x) \), \( xu'_0(x) \) \( \in L_1(\mathbb{R}) \) and \( u''_0(x) \), \( xu''_0(x) \) \( \in L_1(\mathbb{R} \setminus [a-\varepsilon, a+\varepsilon]) \). Then, depending of the sign \( q \), for all fixed \( x \in \mathbb{R} \) the long-time asymptotics for \( u(x, t) \) is as follows:

- if \( q < 0 \), then
  \[
u(x, t) = O\left(\frac{1}{t^{3/2}}\right);
  \]

- if \( q > 0 \), then
  \[
  u(x, t) = q e^{q(x-a)+iq^2t} \left( \int_{-\infty}^{a} e^{q(y-a)} u^{(1)}_0(y) \, dy + \int_{a}^{\infty} e^{-q(y-a)} u^{(2)}_0(y) \, dy \right) + O \left(\frac{1}{t^{3/2}}\right)
  \]
  for \( x \leq a \) and
  \[
  u(x, t) = q e^{-q(x-a)+iq^2t} \left( \int_{-\infty}^{a} e^{q(y-a)} u^{(1)}_0(y) \, dy + \int_{a}^{\infty} e^{-q(y-a)} u^{(2)}_0(y) \, dy \right) + O \left(\frac{1}{t^{3/2}}\right)
  \]
  for \( x > a \).

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