Stochastic Control of Tidal Dynamics Equation with Lévy Noise

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Abstract. In this work we first present the existence, uniqueness and regularity of the strong solution of the tidal dynamics model perturbed by Lévy noise. Monotonicity arguments have been exploited in the proofs. We then formulate a martingale problem of Stroock and Varadhan associated to an initial value control problem and establish existence of optimal controls.

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1. Introduction

Ocean-tide information has considerably many applications. The data obtained is used to solve vital problems in oceanography and geophysics, and to study earth tides, elastic properties of the Earth’s crust and tidal gravity variations. It is also used in space studies to calculate the trajectories of man-made satellites of the Earth and to interpret the results of satellite measurements. The interaction of tides with deep sea ridges and chains of seamounts give rise to deep eddies which transport nutrients from the deep to the surface. The alternate flooding and exposure of the shoreline due to tides is an important factor in the determination of the ecology of the region.

One of the first mathematical explanation for tides was given by Newton by describing tide generating forces. The first dynamic theory of tides was proposed by Laplace. Here we consider the tidal dynamics model proposed by Marchuk and Kagan [17]. The existence and uniqueness of weak solutions of the deterministic tide equation and that of strong solutions of the stochastic tide equation with additive trace class gaussian noise have been proved in Manna, Menaldi and Srinigharan [16]. In this work, we consider the stochastic tide equation with Lévy noise and prove the existence and uniqueness
and regularity of solution in bounded domains. Control of fluid flow has numerous applications in control of pollutant transport, oil recovery/transport problems, weather predictions, control of underwater vehicles, etc. Unification of many control problems in the engineering sciences have been done by studying the optimal control problem of Navier–Stokes equations (see [22], [23]). Here we consider the initial data optimal control of the stochastic tidal dynamics model. We consider the Stroock-Varadhan martingale formulation [24] of the stochastic model to prove the existence of optimal initial value control.

The organization of the paper is as following. A brief description of the model has been given in Section 2. Section 3 describes the functional setting of the problem and states the monotonicity property of the non-linear operator. In Section 4 we consider the a-priori estimates and prove the existence, uniqueness and regularity of strong solution. In Section 5 we consider the stochastic optimal control problem with initial value control.

In the framework of Gelfand triple $H^1_0(O) \subset L^2(O) \subset H^{-1}(O)$ we consider the following tidal dynamics model with Lévy noise

$$du + [Au + B(u) + g\nabla \hat{z}]dt = f(t)dt + \sigma(t, u(t))dW(t) + \int_Z H(u, z)\tilde{N}(dt, dz),$$

$$\dot{\hat{z}} + \text{Div}(hu) = 0 \quad \text{in } O \times [0, T],$$

$$u(0) = u_0, \quad \hat{z}(0) = \hat{z}_0.$$  

The operators $A$ and $B$ are defined in Sections 2 and 3. $(W(t)_{t \geq 0})$ is an $L^2(O)$-valued Wiener process with trace class covariance. $\tilde{N}(dt, dz)$ is a compensated Poisson random measure, where $N(dt, dz)$ denotes the Poisson counting measure associated to the point process $p(t)$ on $Z$, a measurable space, and $\lambda(dz)$ is a $\sigma$-finite measure on $(Z, B(Z))$.

The following theorem states the main result of Section 4. The functional spaces appearing in the statement of the theorem have been defined in Section 3.

**Theorem 1.1.** Let us consider the above stochastic tide model with $f, u_0$ and $\hat{z}_0$ such that

$$f \in L^2(0, T; H^{-1}(O)), \quad u_0 \in L^2(O), \quad \hat{z}_0 \in L^2(O).$$  \hspace{1cm} (1.1)

Assume that $\sigma$ and $H$ satisfy the following hypotheses:

1. $\sigma \in C([0, T] \times H_0^1(O); L_Q(H_0, L^2)), \quad H \in H^1_\alpha([0, T] \times Z; L^2(O)),$

2. For all $t \in (0, T)$, there exists a positive constant $K$ such that for all $u \in L^2(O)$

$$\|\sigma(t, u)\|^2_{L_Q} + \int_Z \|H(u, z)\|^2_{L^2} \lambda(dz) \leq K(1 + \|u\|^2_{L^2}),$$
3. For all \( t \in (0, T) \), there exists a positive constant \( L \) such that for all \( u, v \in \mathbb{L}^2(\mathcal{O}) \)
\[
\|\sigma(t, u) - \sigma(t, v)\|_{L^2}^2 + \int_{\mathcal{Z}} \|H(u, z) - H(v, z)\|_{L^2}^2 \lambda(dz) \leq L \|u - v\|_{L^2}^2.
\]
Then there exist path-wise unique adapted processes \( u(t, x, \omega) \) and \( \hat{z}(t, x, \omega) \) with the regularity
\[
\begin{align*}
\{ & u \in L^2(\Omega, L^\infty(0, T; \mathbb{L}^2(\mathcal{O}))) \cap L^2(0, T; \mathbb{H}^1_0(\mathcal{O})) \cap \mathcal{D}(0, T; \mathbb{L}^2(\mathcal{O}))), \\
& \hat{z}, \dot{\hat{z}} \in L^2(\Omega; L^2(0, T; \mathbb{L}^2(\mathcal{O})))
\end{align*}
\]
(1.2)
satisfying the above stochastic tide model in the weak sense.

In Section 5 we consider the following stochastic optimal control problem with initial value control,
\[
du + [Au + B(u) + g\nabla \hat{z}]dt = f(t)dt + \sigma(t, u(t))dW(t) + \int_{\mathcal{Z}} H(u, z)\tilde{N}(dt, dz)
\]
in \( \mathcal{O} \times [0, T] \), \( \hat{z} + \text{Div}(hu) = 0 \) in \( \mathcal{O} \times [0, T] \), \( u(0) = u_0 + U \), \( \hat{z}(0) = \hat{z}_0 \),
(1.3) \( (1.4) \) \( (1.5) \)
where \( u_0 \in \mathbb{L}^2(\mathcal{O}), \hat{z}_0 \in \mathbb{L}^2(\mathcal{O}) \) and \( U \in \mathbb{L}^2(\mathcal{O}) \). The regularities on the initial values and the assumptions on \( \sigma \) and \( H \) are the same as considered in 1.1.

The cost functional is given by
\[
J(u, U) = \mathbb{E} \left[ \int_0^T \int_{\mathcal{O}} L(t, u, U)dxdt \right],
\]
(1.6)
where the function \( L \) is defined in Section 5.

The main result of Section 5 is

**Theorem 1.2.** Given \( u_0 \in \mathbb{L}^2(\mathcal{O}) \), there exists a pair
\[
(\hat{u}, \hat{U}) \in (L^2(0, T; \mathbb{H}^1_0(\mathcal{O}))) \cap L^\infty(0, T; \mathbb{L}^2(\mathcal{O})) \cap \mathcal{D}(0, T; \mathbb{H}^{-1}(\mathcal{O})) \times L^2(0, T; \mathbb{L}^2(\mathcal{O})),
\]
which gives a martingale solution to equations \( (1.3) - (1.5) \), and
\[
J(\hat{u}, \hat{U}) = \min\{J(u, U); (u, U) \in (L^2(0, T; \mathbb{H}^1_0(\mathcal{O}))) \cap L^\infty(0, T; \mathbb{L}^2(\mathcal{O}))
\]
\[
\cap \mathcal{D}(0, T; \mathbb{H}^{-1}(\mathcal{O})) \times L^2(0, T; \mathbb{L}^2(\mathcal{O})),
\]
where the pair \( (u, U) \) gives a martingale solution to equations \( (1.3) - (1.5) \).

2. Tidal Dynamics: The Model

Under the assumptions that: (1) Earth is perfectly solid, (2) ocean tides do not change Earth’s gravitational field, and (3) no energy exchange takes place between the mid-ocean and shelf zone, Marchuk and Kagan [17] obtained the following mathematical model
\[
\partial_t w + A_1w - \kappa h \triangle w + \frac{r}{h} |w| w + g \nabla \xi = f,
\]
(2.1)
\[ \partial_t \xi + \text{Div}(hw) = 0, \quad (2.2) \]

in \( \mathcal{O} \times [0, T] \), where \( \mathcal{O} \) is a bounded 2-D domain (horizontal ocean basin) with coordinates \( x = (x_1, x_2) \) and \( t \) represents the time. Here \( \partial_t \) denotes the time derivative, \( \Delta, \nabla \) and \( \text{Div} \) are the Laplacian, gradient and the divergence operators respectively.

The unknown variables \((w, \xi)\) represent the total transport 2-D vector (i.e., the vertical integral of the velocity from the ocean surface to the ocean floor) and the displacement of the free surface with respect to the ocean floor. For details on the coefficients and the domain description see Manna, Menaldi and Sritharan [16].

Denote by \( A \) the following matrix operator
\[
A := \begin{pmatrix} -\alpha \Delta & -\beta \\ \beta & -\alpha \Delta \end{pmatrix},
\]
and the nonlinear vector operator \( v \mapsto \gamma |v|v := \begin{pmatrix} \gamma(x)v_1 \sqrt{v_1^2 + v_2^2} \\ \gamma(x)v_2 \sqrt{v_1^2 + v_2^2} \end{pmatrix} \),

where \( \alpha := \kappa h \) and \( \beta := 2\omega \cos(\varphi) \) are positive constants, \( \gamma(x) := \frac{r}{h(x)} \) is a strictly positive smooth function. In this model we assume the depth \( h(x) \) to be a continuously differentiable function of \( x \), nowhere becoming zero, so that
\[
\min_{x \in \mathcal{O}} h(x) = \epsilon > 0, \quad \max_{x \in \mathcal{O}} h(x) = \mu, \quad \max_{x \in \mathcal{O}} |\nabla h(x)| \leq M,
\]
where \( M \) is some positive constant which equals zero at a constant ocean depth.

To reduce to homogeneous Dirichlet boundary conditions consider the natural change of unknown functions
\[ u(x, t) := w(x, t) - w^0(x, t), \quad (2.6) \]
and
\[ \hat{z}(x, t) := \xi(x, t) + \int_0^t \text{Div}(hw^0(x, s))ds, \quad (2.7) \]
which are referred to the tidal flow and the elevation. The full flow \( w^0 \) which is given a priori on the boundary \( \partial \mathcal{O} \), has been extended to the whole domain \( \mathcal{O} \times [0, T] \) as a smooth function and still denoted by \( w^0 \).

Then the tidal dynamic equation can be written as
\[
\begin{align*}
\partial_t u + Au + \gamma|u + w^0|(u + w^0) + g\nabla \hat{z} &= f' \quad \text{in} \quad \mathcal{O} \times [0, T], \\
\partial_t \hat{z} + \text{Div}(hu) &= 0 \quad \text{in} \quad \mathcal{O} \times [0, T], \\
u &= 0 \quad \text{on} \quad \partial \mathcal{O} \times [0, T], \\
u = u_0, \quad \hat{z} = \hat{z}_0 &\quad \text{in} \quad \mathcal{O} \times \{0\},
\end{align*}
\]
\[
(2.8)
\]
where
\[
f' = f - \frac{\partial w^0}{\partial t} + g \nabla \int_0^t \text{Div}(hw^0) dt - Aw^0, \tag{2.9}
\]
\[
u_0(x) = w_0(x) - w^0(x, 0), \tag{2.10}
\]
\[
\hat{z}_0(x) = \xi_0(x). \tag{2.11}
\]

3. Functional setting

We use the (vector-valued) Sobolev spaces \(H^1_0(\Omega) := H^1_0(\Omega, \mathbb{R}^2)\) and \(L^2(\Omega) := L^2(\Omega, \mathbb{R}^2)\), with:

the norm on \(H^1_0(\Omega)\) as
\[
\|v\|_{H^1_0} := \left( \int_\Omega |\nabla v|^2 dx \right)^{1/2}, \tag{3.1}
\]

and the norm on \(L^2(\Omega)\) as
\[
\|v\|_{L^2} := \left( \int_\Omega |v|^2 dx \right)^{1/2}. \tag{3.2}
\]

Using the Gelfand triple \(H^1_0(\Omega) \subset L^2(\Omega) \subset H^{-1}(\Omega)\), we may consider \(\Delta\) or \(\nabla\) as a linear map from \(H^1_0(\Omega)\) or \(L^2(\Omega)\) into the dual of \(H^1_0(\Omega)\) respectively. The inner product in \(L^2(\Omega)\) or \(L^2(\Omega)\) is denoted by \((\cdot, \cdot)\). So
\[
(u, v)_{L^2} = \int_\Omega u(x) \cdot v(x) dx, \tag{3.3}
\]

for any \(u\) and \(v\) in \(L^2(\Omega)\). The duality pairing between \(H^1_0\) and \(H^{-1}\) is denoted by \(\langle \cdot, \cdot \rangle\).

Now we give the definitions and some properties of Hilbert space valued Wiener processes, Lévy processes and Skorokhod spaces, most of which have been borrowed from the books by Da Prato and Zabczyk [7], Applebaum [3] and Metivier [18].

For a Hilbert space \(U\), let us denote the space of all bounded linear operators from \(U\) to \(U\) by \(L(U)\). We consider two Hilbert spaces \(H\) and \(U\), and a symmetric non-negative operator \(Q \in L(U)\) such that \(Tr(Q) < +\infty\). A \(U\)-valued stochastic process \(\{W(t), t \geq 0\}\), is called a \(Q\)-Wiener process if

(i) \(W(0) = 0\),
(ii) \(W\) has continuous trajectories,
(iii) \(W\) has independent increments,
(iv) \(L(W(t) - W(s)) = N(0, (t-s)Q))\), \(t \geq s \geq 0\).

If a process \(\{W(t), t \in [0, T]\}\) satisfies (i) – (iii) and (iv) for \(t, s \in [0, T]\) the we say that \(W\) is a \(Q\)-Wiener process on \([0, T]\).
Let $Q$ be a symmetric non-negative operator $Q \in \mathbb{L}^2$. Define $H_0 = Q^{1/2}\mathbb{L}^2$. Then $H_0$ is a Hilbert space equipped with inner product $(\cdot, \cdot)_0$,

$$(u, v)_0 = \left( Q^{-1/2}u, Q^{-1/2}v \right)_{\mathbb{L}^2}, \forall u, v \in H_0,$$

where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$, and $Q$ is the covariance operator of the $\mathbb{L}^2$-valued Wiener process $(W(t))_{t \geq 0}$.

Let $L_Q$ denote the space of linear operators $S$ such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from $\mathbb{L}^2$ to $\mathbb{L}^2$. Define the norm on the space $L_Q$ by $\|S\|_{L_Q}^2 = Tr(SQS^*)$.

Let $(\mathcal{S}, \rho)$ be a separable and complete metric space. Let $\mathcal{D}(0, T; \mathcal{S})$ denote the set of $\mathcal{S}$-valued functions on $[0, T]$ which are càdlàg (if $g \in \mathcal{D}(0, T; \mathbb{L}^2)$ then for all $t \in [0, T]$ $g$ is right continuous at $t$ and has a left limit at $t$), endowed with the Skorokhod topology. This topology is metrizable by the following metric $\delta_T$:

$$\delta_T(u, v) := \inf_{\lambda \in \Lambda_T} \left[ \sup_{0 \leq t \leq T} \rho(u(t), v \circ \lambda(t)) + \sup_{0 \leq t \leq T} |t - \lambda(t)| + \sup_{s \neq t} \left| \log \left( \frac{\lambda(t) - \lambda(s)}{t - s} \right) \right| \right],$$

where $\Lambda_T$ is the set of increasing homeomorphisms of $[0, T]$. Moreover, $(\mathcal{D}(0, T; \mathcal{S}), \delta_T)$ is a complete metric space.

**Remark 3.1.** A sequence $(u_n) \subset \mathcal{D}(0, T; \mathcal{S})$ converges to $u \in \mathcal{D}(0, T; \mathcal{S})$ iff there exists a sequence $(\lambda_n)$ of homeomorphisms of $[0, T]$ such that $\lambda_n$ tends to the identity uniformly on $[0, T]$ and $u_n \circ \lambda_n$ tends to $u$ uniformly on $[0, T]$.

**Definition 3.2.** Let $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ be a filtered probability space, and $E$ be a Banach space. A process $(X_t)_{t \geq 0}$ with state space $(E, \mathcal{B}(E))$ is called a Lévy process if

(i) $(X_t)_{t \geq 0}$ is adapted to $(\mathcal{F}_t)_{t \geq 0},$
(ii) $X_0 = 0$ a.s.,
(iii) $X_t - X_s$ is independent of $\mathcal{F}_s$ if $0 \leq s < t,$
(iv) $(X_t)_{t \geq 0}$ is stochastically continuous, i.e., $\forall \epsilon > 0, \lim_{s \to t} P(|X_s - X_t| > \epsilon) = 0,$
(v) $(X_t)_{t \geq 0}$ is càdlàg,
(vi) $(X_t)_{t \geq 0}$ has stationary increments, i.e., $E[X_t - X_s] = E[X_{t-s}], 0 \leq s < t.$

The jump of $X_t$ at $t \geq 0$ is given by $\Delta X_t = X_t - X_{t-}$. Let $Z \in \mathcal{B}(\mathbb{R}^+ \times E)$. We define

$$N(t, Z) = N(t, Z, \omega) = \sum_{s; 0 < s \leq t} \chi_Z(\Delta X_s).$$

So, $N(t, Z)$ is the number of jumps of size $\Delta X_s \in Z$ which occur before or at time $t$. $N(t, Z)$ is called the Poisson random measure of $(X_t)_{t \geq 0}$. The differential form of this measure is written as $N(dt, dz)(\omega)$. 
We call \( \tilde{N}(dt,dz) = N(dt,dz) - \lambda(dz)dt \) a compensated Poisson random measure, where \( \lambda(dz)dt \) is known as the compensator of the Lévy process \((X_t)_{t \geq 0}\). Here \( dt \) denotes the Lebesgue measure on \( B(\mathbb{R}^+) \), and \( \lambda(dz) \) is a \( \sigma \)-finite measure on \((Z,B(Z))\).

**Lemma 3.3.** For any real-valued smooth function \( \varphi \) and \( \psi \) with compact support in \( \mathbb{R}^2 \), the following hold:

\[
\| \varphi \psi \|_{L^2} \leq 4 \| \varphi \partial_1 \varphi \|_{L^2} \| \psi \partial_2 \psi \|_{L^2}, \tag{3.4}
\]

\[
\| \varphi \|_{L^4}^4 \leq 2 \| \varphi \|_{L^2}^2 \| \nabla \varphi \|_{L^2}^2. \tag{3.5}
\]

For proof see [12].

Notice that by means of the Gelfand triple we may consider \( A \), given by (2.3), as a mapping of \( H^{1,0}_0(O) \) into its dual \( H^{-1}_0(O) \).

Define the non-symmetric bilinear form

\[
a(u,v) := \alpha (\partial_1 u_1, \partial_1 v_1)_{L^2} + (\partial_2 u_2, \partial_2 v_2)_{L^2} + \beta (u_1, v_2)_{L^2} - (u_2, v_1)_{L^2}, \tag{3.6}
\]

on \( H^1_0 \). Thus if \( u \) has a smooth second derivative then

\[
a(u,v) = (Au,v)_{L^2},
\]

for every \( v \) in \( H^1_0 \). More over, the bilinear form \( a(\cdot, \cdot) \) is continuous and coercive in \( H^1_0(O) \), i.e.,

\[
|a(u,v)| \leq C_1 \| u \|_{H^1_0} \| v \|_{H^1_0} \quad \forall u,v \in H^1_0, \tag{3.7}
\]

\[
(Au,u)_{L^2} = a(u,u) = \| u \|_{H^1_0}^2, \tag{3.8}
\]

for some positive constant \( C_1 = \alpha + \beta \).

Let us denote the nonlinear operator \( B(\cdot) \) by

\[
v \mapsto B(v) := \gamma |v + w^0|[v + w^0]. \tag{3.9}
\]

Then we have the following lemma:

**Lemma 3.4.** Let \( u \) and \( v \) be in \( L^4(O,\mathbb{R}^2) \). Then the following estimate holds:

\[
\langle B(u) - B(v), u - v \rangle \geq 0. \tag{3.10}
\]

For proof see Lemma 3.3 in [16].

The nonlinear operator \( B(\cdot) \) is a continuous operator from \( L^4(O) \) to \( L^2(O) \), where

\[
\| B(v) \|_{L^2} \leq C_2 \| v + w^0 \|_{L^4}, \tag{3.11}
\]

\[
\| B(u) - B(v) \|_{L^2} \leq C_2 \| u + w^0 \|_{L^4} + \| v + w^0 \|_{L^4} \| u - v \|_{L^4}, \tag{3.12}
\]

where the constant \( C_2 \) is the sup-norm of the function \( \gamma \).
We define the functional form:

\[
Au + B(u) + g\nabla\hat{z} = f(t) + \sigma(t, u(t)) \mathrm{d}W(t) + \int_{\mathcal{Z}} H(u, z) \tilde{N}(dt, dz),
\]

\[
\dot{\hat{z}} + \text{Div}(hu) = 0,
\]

\[
u(0) = u_0, \quad \hat{z}(0) = \hat{z}_0,
\]

where \(u_0 \in L^2(\mathcal{O})\) and \(\hat{z}_0 \in L^2(\mathcal{O})\). The operators \(A\) and \(B\) are defined through (2.3) and (3.9) respectively. \((W(t)_{t \geq 0})\) is a \(L^2(\mathcal{O})\)-valued Wiener process with trace class covariance. \(H(u, z)\) is a measurable mapping from \(\mathbb{H}_2^0 \times Z\) into \(L^2\).

We define the space \(\mathbb{H}_2^0([0, T] \times Z; L^2(\mathcal{O}))\) as

\[
\mathbb{H}_2^0([0, T] \times Z; L^2(\mathcal{O})) = \{X : \mathbb{E} \left[ \int_0^T \int_{\mathcal{Z}} \|X\|_{L^2(\mathcal{O})}^2 \lambda(dz) dt \right] < \infty\}. \quad (3.16)
\]

We assume that \(\sigma\) and \(H\) satisfy the following hypotheses:

H.1 \(\sigma \in C([0, T] \times \mathbb{H}_1^0(\mathcal{O}); L_Q(H_0, L^2)), \quad H \in \mathbb{H}_2^0([0, T] \times Z; L^2(\mathcal{O}))\),

H.2 For all \(t \in (0, T)\), there exists a positive constant \(K\) such that for all \(u \in L^2(\mathcal{O})\)

\[
\|\sigma(t, u)\|_{L_Q}^2 + \int_{\mathcal{Z}} \|H(u, z)\|_{L^2(\mathcal{O})}^2 \lambda(dz) \leq K(1 + \|u\|_{L^2}^2),
\]

H.3 For all \(t \in (0, T)\), there exists a positive constant \(L\) such that for all \(u, v \in L^2(\mathcal{O})\)

\[
\|\sigma(t, u) - \sigma(t, v)\|_{L_Q}^2 + \int_{\mathcal{Z}} \|H(u, z) - H(v, z)\|_{L^2(\mathcal{O})}^2 \lambda(dz) \leq L(\|u - v\|_{L^2}^2).
\]

4. Energy Estimates and Existence Result

Denote by \(L^2_n(\mathcal{O}) := \text{span}\{e_1, e_2, \ldots, e_n\}\) where \(\{e_j\}\) is any fixed orthonormal basis in \(L^2(\mathcal{O})\) with each \(e_j \in \mathbb{H}_0^1(\mathcal{O})\). Similarly denote by \(L^2_n(\mathcal{O})\) the n-dimensional subspace of \(L^2(\mathcal{O})\). Let \(P_n\) denote the orthogonal projection of \(L^2(\mathcal{O})\) into \(L^2_n(\mathcal{O})\). Define \(u^n = P_n u, W^n = P_n W, \sigma^n = P_n \sigma, H^n = P_n H\) and

\[
\int_{\mathcal{Z}} H^n(u^n(t-), z) \tilde{N}(dt, dz) = P_n \int_{\mathcal{Z}} H(u(t-), z) \tilde{N}(dt, dz).
\]

We define \(u^n\) as the solution of the following stochastic equation in variational form:

\[
(du^n(t), v^n(t))_{L^2} + (u^n(t), v^n(t))_{L^2} + (B(u^n(t)), v^n(t))_{L^2} + (g\nabla\hat{z}^n(t), v^n(t))_{L^2} dt = (f(t), v^n(t))_{L^2} dt + (\sigma^n(t, u^n(t)), v^n(t))_{L^2} dW^n(t)
\]

\[
+ \int_{\mathcal{Z}} (H^n(u^n(t-), z), v^n(t))_{L^2} \tilde{N}(dt, dz) \quad \forall v^n \in L^2_n(\mathcal{O}), \quad (4.1)
\]

\[
(\hat{z}^n(t) + \text{Div}(hu^n(t)), \zeta(t))_{L^2} = 0 \quad \forall \zeta \in L^2_n(\mathcal{O}),\quad (4.2)
\]

\[
u^n(0) = u^n_0, \quad \hat{z}^n(0) = \hat{z}^n_0.\quad (4.3)
\]
Proposition 4.1. Under the above mathematical setting let
\[
\begin{align*}
&\left\{ \begin{array}{l}
w^0 \in L^4(0,T;\mathbb{L}^4(\Omega)), \ f \in L^2(0,T;\mathbb{L}^2(\Omega)), \\
\sigma \in C([0,T] \times \mathbb{H}^1_0(\Omega); L_Q(H_0,\mathbb{L}^2)), \ H \in \mathbb{H}^2_0([0,T] \times Z;\mathbb{L}^2(\Omega)), \\
\end{array} \right. \\
u_0 \in \mathbb{L}^2(\Omega), \ z_0 \in \mathbb{L}^2(\Omega).
\end{align*}
\]
Let \(u^n(t)\) be an adapted process in \(\mathcal{D}(0,T,\mathbb{L}^2_n(\Omega))\) which solves the differential equations (4.11) and (4.2). Then we have the following a priori estimates:
\[
\begin{align*}
\mathbb{E}[\|u^n(t)\|_{L^2}^2 + \|\dot{z}^n(t)\|_{L^2}^2] + 2\alpha \mathbb{E}\left[\int_0^t \|u^n(t)\|_{H^0}^2 dt\right] &\leq C_{1(2)} \quad \forall t \in [0,T], \tag{4.5} \\
\mathbb{E}[\sup_{0 \leq t \leq T} (\|u^n(t)\|_{L^2}^2 + \|\dot{z}^n(t)\|_{L^2}^2)] + 2\alpha \mathbb{E}\left[\int_0^T \|u^n(t)\|_{H^0}^2 dt\right] &\leq C_{2(2)}, \tag{4.6}
\end{align*}
\]
where the constants \(C_{1(2)}\) and \(C_{2(2)}\) depend on the coefficients \(\alpha, g, M, \mu\) and the norms \(\|f\|_{L^2(0,T,\mathbb{L}^1)}\), \(\|w^0\|_{L^4(0,T,\mathbb{L}^4)}\), \(\|u_0^\delta\|_{\mathbb{L}^2}\), \(\|z_0^\delta\|_{\mathbb{L}^2}\), \(T\) and \(\mathcal{D}\).

Proof. Applying Itô’s formula to \(|x|^2\) and the process \(u^n(t)\)
\[
d(\|u^n(t)\|_{L^2}^2) + 2\alpha \|u^n(t)\|^2_{H^0} dt + 2(B(u^n(t)), u^n(t))_{L^2} dt + 2(g\nabla \dot{z}^n(t), u^n(t))_{L^2} dt \\
= 2(f(t), u^n(t))_{L^2} dt + 2(\sigma^n(t, u^n(t)), u^n(t))_{L^2} dW^n(t) + \|\sigma^n(t, u^n(t))\|^2_{L^2} dt \\
+ \int_Z \|H^n(u^n(t), z)\|_{L^2}^2 N(dt, dz) + 2 \int_Z (H^n(u^n(t), z), u^n(t))_{L^2} N(dt, dz). \tag{4.7}
\]
Using the definition of the operator \(B(\cdot)\) and Lemma 3.4,
\[
(B(u^n(t)), u^n(t))_{L^2} \geq \int_\mathcal{O} \gamma(x)|w^0(t)|^2 u^n(t) dx \\
\geq -\frac{r}{\alpha} \|w^0(t)\|^2_{L^4} \|u(t)\|_{L^2} \\
\geq -\frac{r}{2\epsilon} [\|w^0(t)\|^4_{L^4} + \|u(t)\|^2_{L^2}]. \tag{4.8}
\]
Using the divergence theorem and the inequality
\[
2ab \leq \delta a^2 + \frac{1}{\delta} b^2, \tag{4.9}
\]
we obtain,
\[
|g(\nabla \dot{z}^n(t), u^n(t))_{L^2}| = | -g(\dot{z}^n(t), \text{Div}(u^n(t)))_{L^2}| \\
\leq \frac{g}{2} \left[\frac{2g}{\alpha} \|\dot{z}^n(t)\|_{L^2}^2 + \frac{\alpha}{2g} \|\text{Div}(u^n(t))\|_{L^2}^2\right].
\]
Since the divergence is bounded by gradient in \(L^2\)-norm
\[
|g(\nabla \dot{z}^n(t), u^n(t))_{L^2}| \leq \frac{g}{2} \left[\frac{2g}{\alpha} \|\dot{z}^n(t)\|_{L^2}^2 + \frac{\alpha}{2g} \|u^n(t)\|^2_{H^0}\right]. \tag{4.10}
\]
Using Cauchy-Schwarz inequality
\[
|(f(t), u^n(t))_{L^2}| \leq \frac{1}{2} \|f(t)\|_{L^2}^2 + \|u^n(t)\|_{L^2}^2. \tag{4.11}
\]
Hence the energy equality (4.7) yields
\[
d (\|u^n(t)\|_{L^2}^2) + 2\alpha \|u^n(t)\|_{H^1_0}^2 dt \\
\leq (\|f(t)\|_{L^2}^2 + \|u^n(t)\|_{L^2}^2 + \frac{r}{\epsilon} \|u^0(t)\|_{L^4}^4 + \|u(t)\|_{L^2}^2) + \frac{2g^2}{\alpha} \|\hat{z}^n(t)\|_{L^2}^2 dt \\
+ \frac{\alpha}{2} \|u^n(t)\|_{H^1_0}^2 dt + 2(\sigma(t, u^n(t)), u^n(t))_{L^2} dW^n(t) + \|\sigma(t, u^n(t))\|_{L_Q}^2 dt \\
+ \int_Z \|H^n(u^n(t^2, z))\|_{L^2}^2 N(dt, dz) + 2 \int_Z (H^n(u^n(t^2, z), u^n(t^2))_{L^2} \tilde{N}(dt, dz). \\
\]
(4.12)

Using equation (4.2)
\[
\frac{1}{2} \frac{d}{dt} \|\hat{z}^n(t)\|_{L^2}^2 = -(Div(hu^n(t)), \hat{z}^n(t))_{L^2}.
\]

Now
\[
|(Div(hu^n(t)), \hat{z}^n(t))_{L^2}| = |(h \ Div(u^n(t)), \hat{z}^n(t))_{L^2} + (u^n(t) \cdot \nabla h, \hat{z}^n(t))_{L^2}|
\]
\[
\leq |(h \ Div(u^n(t)), \hat{z}^n(t))_{L^2}| + |(u^n(t) \cdot \nabla h, \hat{z}^n(t))_{L^2}|
\]
\[
\leq \|h\|_{L^\infty} \|Div(u^n(t))\|_{L^2} \|\hat{z}^n(t)\|_{L^2} \\
+ \|u^n(t)\|_{L^2} \|\nabla h\|_{L^\infty} \|\hat{z}^n(t)\|_{L^2}.
\]

Using the assumptions on h
\[
|(Div(hu^n(t)), \hat{z}^n(t))_{L^2}| \leq \mu \|u^n(t)\|_{H^1_0} \|\hat{z}^n(t)\|_{L^2} + M \|u^n(t)\|_{L^2} \|\hat{z}^n(t)\|_{L^2}
\]
\[
\leq \frac{\mu}{2} \left[ \frac{\alpha}{2\mu} \|u^n(t)\|_{H^1_0}^2 + \frac{2\mu}{\alpha} \|\hat{z}^n(t)\|_{L^2}^2 \right] \\
+ \frac{M}{2} \|u^n(t)\|_{L^2}^2 + \|\hat{z}^n(t)\|_{L^2}^2.
\]

Thus
\[
d \|\hat{z}^n(t)\|_{L^2}^2 \leq M \|u^n(t)\|_{L^2}^2 dt + \left( \frac{2\mu^2}{\alpha} + M \right) \|\hat{z}^n(t)\|_{L^2}^2 dt + \frac{\alpha}{2} \|u^n(t)\|_{H^1_0}^2 dt. \\
\]
(4.13)

Adding equations (4.12) and (4.13)
\[
d (\|u^n(t)\|_{L^2}^2 + \|\hat{z}^n(t)\|_{L^2}^2) + \alpha \|u^n(t)\|_{H^1_0}^2 dt \\
\leq (1 + M + \frac{r}{\epsilon}) \|u^n(t)\|_{L^2}^2 dt + \left( \frac{2g^2}{\alpha} + \frac{2\mu^2}{\alpha} + M \right) \|\hat{z}^n(t)\|_{L^2}^2 dt \\
+ \frac{r}{\epsilon} \|u^0(t)\|_{L^4}^4 dt + \|f(t)\|_{L^2}^2 dt + \|\sigma(t, u^n(t))\|_{L^2}^2 dt \\
+ 2(\sigma(t, u^n(t)), u^n(t))_{L^2} dW^n(t) + \int_Z \|H^n(u^n(t^2, z))\|_{L^2}^2 N(dt, dz) \\
+ 2 \int_Z (H^n(u^n(t^2, z), u^n(t^2))_{L^2} \tilde{N}(dt, dz). \\
\]
(4.14)

Define
\[
\tau_N = \inf \{ t : \|u^n(t)\|_{L^2}^2 + \|\hat{z}^n(t)\|_{L^2}^2 + \int_0^t \|u^n(s)\|_{H^1_0}^2 ds > N \}. \\
\]
(4.15)
Integrating equation (4.14)

\[ \|u^n(t \wedge \tau_N)\|_{L^2}^2 + \|\tilde{z}^n(t \wedge \tau_N)\|_{L^2}^2 + \int_0^{t \wedge \tau_N} \alpha \|u^n(s)\|_{H^1_0}^2 \, ds \]

\[ \leq (1 + M + \frac{r}{\epsilon}) \int_0^{t \wedge \tau_N} \|u^n(s)\|_{L^4}^2 \, ds + \left( \frac{2g^2}{\alpha} + \frac{2\mu^2}{\alpha} + M \right) \int_0^{t \wedge \tau_N} \|\tilde{z}^n(s)\|_{L^2}^2 \, ds \]

\[ + \frac{r}{\epsilon} \int_0^{t \wedge \tau_N} \|w^0(s)\|_{L^4} \, ds + \int_0^{t \wedge \tau_N} \|f(s)\|_{L^2}^2 \, ds + \int_0^{t \wedge \tau_N} \|\sigma^n(s,u^n(s))\|_{L^4}^2 \, ds \]

\[ + 2 \int_0^{t \wedge \tau_N} (\sigma^n(s,u^n(s)),u^n(s))_{L^2} \, dW^n(s) \]

\[ + \int_0^{t \wedge \tau_N} \int_Z \|H^n(u^n(s),z)\|_{L^2}^2 \tilde{N}(ds,dz) \]

\[ + \int_0^{t \wedge \tau_N} 2 \int_Z (H^n(u^n(s),z),u^n(s))_{L^2} \tilde{N}(ds,dz) \]

\[ + \int_0^{t \wedge \tau_N} \int_Z \|H^n(u^n(s),z)\|_{L^2}^2 \lambda(dz) \, ds + \|u^0_0\|_{L^2}^2 + \|\tilde{z}_0\|_{L^2}^2. \quad (4.16) \]

Let \( C = \max\{1 + M + \frac{r}{\epsilon}, \frac{2g^2}{\alpha} + \frac{2\mu^2}{\alpha} + M\}. \) Since \( H^n(t) \) is strong 2-integrable w.r.t. \( \tilde{N}(dt,dz) \), hence using Hölder’s inequality \( |H^n(t)|^2 \) is strong 1-integrable w.r.t. \( \tilde{N}(dt,dz) \). Then by Theorem 4.12 of [20]

\[ \mathbb{E} \left[ \int_0^t \int_Z \|H^n(u^n(s),z)\|_{L^2}^2 \tilde{N}(ds,dz) \right] \]

\[ \leq 2 \int_0^t \int_Z \mathbb{E} \left[ \|H^n(u^n(s),z)\|_{L^2}^2 \lambda(dz) \, ds \right]. \quad (4.17) \]

Also the stochastic integrals

\[ \int_0^{t \wedge \tau_N} (\sigma^n(s,u^n(s)),u^n(s))_{L^2} \, dW^n(s) \]

\[ \int_0^{t \wedge \tau_N} \int_Z (H^n(u^n(s),z),u^n(s))_{L^2} \tilde{N}(ds,dz) \]

are martingales with zero averages.

Taking expectation of equation (4.16) and using (4.17) and the above property, we get

\[ \mathbb{E} \left[ \|u^n(t \wedge \tau_N)\|_{L^2}^2 + \|\tilde{z}^n(t \wedge \tau_N)\|_{L^2}^2 \right] + \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \alpha \|u^n(s)\|_{H^1_0}^2 \, ds \right] \]

\[ \leq C \mathbb{E} \left[ \int_0^{t \wedge \tau_N} (\|u^n(s)\|_{L^2}^2 + \|\tilde{z}^n(s)\|_{L^2}^2) \, ds \right] + \frac{r}{\epsilon} \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \|w^0(s)\|_{L^4}^4 \, ds \right] \]

\[ + \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \|f(s)\|_{L^2}^2 \, ds \right] + \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \|\sigma^n(s,u^n(s))\|_{L^4}^2 \, ds \right] \]

\[ + 3 \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \int_Z \|H^n(u^n(s),z)\|_{L^2}^2 \lambda(dz) \, ds \right] \]

\[ + \mathbb{E} \left[ \|u^0_0\|_{L^2}^2 + \|\tilde{z}_0^n\|_{L^2}^2 \right]. \quad (4.18) \]
Using the assumption $H.2$

\[
\mathbb{E} \left[ \|u^n(t \wedge \tau_N)\|^2_{L^2} + \|\hat{z}^n(t \wedge \tau_N)\|^2_{L^2} \right] + \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \alpha \|u^n(s)\|^2_{H^1_0} ds \right] \\
\leq C \mathbb{E} \left[ \int_0^{t \wedge \tau_N} (\|u^n(s)\|^2_{L^2} + \|\hat{z}^n(s)\|^2_{L^2}) ds \right] + \frac{r}{\epsilon} \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \|w^0(s)\|_{H^1_0} ds \right] \\
+ \mathbb{E} \left[ \int_0^{t \wedge \tau_N} \|f(s)\|^2_{L^2} ds \right] + 3K \mathbb{E} \left[ \int_0^{t \wedge \tau_N} (1 + \|u^n(s)\|^2_{L^2}) ds \right] \\
+ \mathbb{E} \left[ \|u_0^n\|^2_{L^2} + \|\hat{z}_0^n\|^2_{L^2} \right].
\]

Hence using Gronwall’s inequality and taking limit as $N \to \infty$ we have the desired a priori estimate (4.5). Proceeding similarly but taking supremum over $[0, T \wedge \tau_N]$ over equation (4.16) before taking expectation

\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} (\|u^n(t)\|^2_{L^2} + \|\hat{z}^n(t)\|^2_{L^2}) \right] + \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \alpha \|u^n(s)\|^2_{H^1_0} ds \right] \\
\leq C \mathbb{E} \left[ \int_0^{T \wedge \tau_N} (\|u^n(s)\|^2_{L^2} + \|\hat{z}^n(s)\|^2_{L^2}) ds \right] + \frac{r}{\epsilon} \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \|w^0(s)\|_{H^1_0} ds \right] \\
+ \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \|f(s)\|^2_{L^2} ds \right] + 3K(T \wedge \tau_N) + 3K \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \|u^n(s)\|^2_{L^2} ds \right] \\
+ 2 \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} \left| \int_0^t (\sigma^n(s, u^n(s)), u^n(s))_{L^2} dW^n(s) \right| + \right. \\
+ \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} \left| \int_0^t 2 \int_{\mathbb{R}} (H^n(u^n(s-), z), u^n(s-))_{L^2} \tilde{N}(ds, dz) \right| \right] \\
+ \mathbb{E} \left[ \|u_0^n\|^2_{L^2} + \|\hat{z}_0^n\|^2_{L^2} \right].
\] (4.19)

Using Burkholder-Davis-Gundy inequality, Young’s inequality and assumption $H.2$

\[
2 \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} \left| \int_0^t (\sigma^n(s, u^n(s)), u^n(s))_{L^2} dW^n(s) \right| \right] \\
\leq C_3 \mathbb{E} \left[ \left( \int_0^{T \wedge \tau_N} (\sigma^n(s, u^n(s)), u^n(s))_{L^2} ds \right)^{1/2} \right] \\
\leq C_3 \mathbb{E} \left[ \left( \int_0^{T \wedge \tau_N} \|\sigma^n(s, u^n(s))\|^2_{L^2} \|u^n(s)\|^2_{L^2} ds \right)^{1/2} \right] \\
\leq C_3 K \mathbb{E} \left[ \left( \int_0^{T \wedge \tau_N} (1 + \|u^n(s)\|^2_{L^2}) \|u^n(s)\|^2_{L^2} ds \right)^{1/2} \right] \\
\leq C_3 K \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} \|u^n(s)\|^2_{L^2} \left( \int_0^{T \wedge \tau_N} (1 + \|u^n(s)\|^2_{L^2}) ds \right)^{1/2} \right]
\]
\[
\leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} \| u^n(s) \|^2_{L^2} \right] + (C_3 K)^2 \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \| u^n(s) \|^2_{L^2} ds \right]
+(C_3 K)^2 (T \wedge \tau_N).
\]

(4.20)

Again using Burkholder-Davis-Gundy inequality, Young’s inequality and assumption H.2
\[
2 \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} \left| \int_0^t \int_Z (H^n(u^n(s-), z), u^n(s-))_{L^2} \tilde{N}(ds, dz) \right| \right]
\leq C_4 \mathbb{E} \left[ \left( \int_0^{T \wedge \tau_N} \int_Z \| H^n(u^n(s-), z), u^n(s-))_{L^2} \lambda(dz) ds \right) \right]^{1/2}
\leq C_4 \mathbb{E} \left[ \left( \int_0^{T \wedge \tau_N} \int_Z \| H^n(u^n(s-), z)\|^2_{L^2} \| u^n(s-))_{L^2} \lambda(dz) ds \right) \right]^{1/2}
\leq C_4 K \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} \| u^n(s) \|^2_{L^2} \left( \int_0^{T \wedge \tau_N} (1 + \| u^n(s) \|^2_{L^2}) ds \right) \right]^{1/2}
\leq \frac{1}{4} \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} \| u^n(s) \|^2_{L^2} \right] + (C_4 K)^2 \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \| u^n(s) \|^2_{L^2} ds \right]
+(C_4 K)^2 (T \wedge \tau_N).
\]

(4.21)

Substituting equations (4.20) and (4.21) in equation (4.19) and rearranging the terms we have
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau_N} (\| u^n(t) \|^2_{L^2} + \| \dot{z}^n(t) \|^2_{L^2}) \right] + 2 \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \alpha \| u^n(s) \|^2_{H^1_0} ds \right]
\leq C' \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \sup_{0 \leq s \leq t} (\| u^n(s) \|^2_{L^2} + \| \dot{z}^n(s) \|^2_{L^2}) dt \right] + \frac{2r}{\varepsilon} \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \| w^0(s) \|^4_{L^4} ds \right]
+ 2 \mathbb{E} \left[ \int_0^{T \wedge \tau_N} \| f(s) \|^2_{L^2} ds \right] + C''(T \wedge \tau_N) + 2 \mathbb{E} [\| u^0_0 \|^2_{L^2} + \| \dot{z}^0_0 \|^2_{L^2}],
\]

where \( C' = 2[(C_3 K)^2 + (C_4 K)^2 + 3K] \) and \( C'' = 2[(C_3 K)^2 + (C_4 K)^2 + 3K] \). Now taking limit as \( N \to \infty \) and Gronwall’s inequality we get the desired a priori estimate (4.6).

\( \square \)

4.1. \( L^p \) energy estimate

Let \( 2 < p < \infty \). We assume that \( \sigma \) and \( H \) satisfy the following hypotheses:
Hp.1 \( \sigma \in C([0, T] \times \mathbb{H}^1_0(O); L_Q(H_0, L^2)), H \in \mathbb{H}_0^p([0, T] \times Z; L^2(O)) \),
Hp.2 For all \( t \in (0, T) \), there exists a positive constant \( K \) such that for all \( u \in L^2(O) \)
\[
\| \sigma(t, u) \|^2_{L_Q} + \int_Z \| H(u, z) \|^2_{L^2} \lambda(dz) \leq K (1 + \| u \|^2_{L^2}),
\]
Hp.3 For all \( t \in (0, T) \), there exists a positive constant \( L \) such that for all \( u, v \in L^2(\mathcal{O}) \)
\[
\|\sigma(t, u) - \sigma(t, v)\|_{L^q}^2 + \int_\mathcal{Z} \|H(u, z) - H(v, z)\|_{L^2}^2 \lambda(dz) \leq L(\|u - v\|_{L^2}^2),
\]
Hp.4 For all \( t \in (0, T) \), there exists a positive constant \( M \) such that for all \( 2 < p < \infty \) and \( u \in L^2(\mathcal{O}) \)
\[
\int_\mathcal{Z} \|H(u, z)\|^p_{L^2} \lambda(dz) \leq M(1 + \|u\|_{L^p}^p).
\]

**Proposition 4.2.** Let \( 2 < p < \infty \) and
\[
\begin{cases}
  w^0 \in L^p(0, T; \mathbb{H}_0^1(\mathcal{O})), & f \in L^p(0, T; L^2(\mathcal{O})), \\
  \sigma \in C([0, T] \times \mathbb{H}_0^1(\mathcal{O}); L_Q(H_0, L^2)), & H \in \mathbb{H}_{\lambda}^p([0, T] \times Z; L^2(\mathcal{O})), \\
  u_0 \in L^2(\mathcal{O}), & \hat{z}_0 \in L^2(\mathcal{O}).
\end{cases}
\]

Let \( u^n(t) \) be an adapted process in \( L^p([0, T]; L^2(\mathcal{O})) \cap L^2([0, T]; \mathbb{H}_0^1(\mathcal{O})) \cap D(0, T; L^2(\mathcal{O})) \) which solves the differential equations (4.1) and (4.2). Then we have the following a priori estimate:
\[
\mathbb{E} \left[ \sup_{0 \leq t \leq T} \|u^n(t)\|^p_{L^2} + \|\hat{z}^n(t)\|^p_{L^2} \right] + \alpha p \mathbb{E} \left[ \int_0^T \|u^n(t)\|_{L^2}^{p-2} \|u^n(t)\|_{\mathbb{H}_0^1}^2 dt \right] \leq C_{1(p)},
\]
(4.23)
where the constant \( C_{1(p)} \) depends on the coefficients \( \alpha, g, M, \mu \) and the norms \( \|f\|_{L^p(0, T; L^2)}, \|w^0\|_{L^p(0, T; \mathbb{H}_0^1)}, \|u^n(0)\|_{L^2}, \|\hat{z}^n(0)\|_{L^2} \) and \( T \).

The proposition can be proved using the same ideas used in Proposition 4.1.

### 4.2. Existence Result

**Definition 4.3.** A path-wise strong solution \( u \) is defined on a given filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, P)\) as a \( L^\infty(0, T; L^2(\mathcal{O})) \cap L^2(0, T; \mathbb{H}_0^1(\mathcal{O})) \cap D(0, T; L^2(\mathcal{O})) \) valued function which satisfies the stochastic tide equations (3.13) and (3.14) in the weak sense and also the energy inequalities in Proposition 4.1.

**Theorem 4.4.** Let \( f, u_0 \) and \( \hat{z}_0 \) be such that
\[
f \in L^2(0, T; L^2(\mathcal{O})), \quad u_0 \in L^2(\mathcal{O}), \quad \hat{z}_0 \in L^2(\mathcal{O}).
\]
(4.24)
Suppose \( \sigma \) and \( H \) satisfy the conditions in H.1-H.3, then there exist a unique path-wise strong solution \( u(t, x, \omega) \) and \( \hat{z}(t, x, \omega) \) with the regularity
\[
\begin{cases}
  u \in L^2(\Omega; L^\infty(0, T; L^2(\mathcal{O})) \cap L^2(0, T; \mathbb{H}_0^1(\mathcal{O}))) \cap D(0, T; L^2(\mathcal{O}))), \\
  \hat{z}, \hat{\hat{z}} \in L^2(\Omega; L^2(0, T; L^2(\mathcal{O}))).
\end{cases}
\]
(4.25)

**Proof.** Existence:
Define
\[
F(u) = Au + B(u) - f.
\]
(4.26)
Then
\[ du^n(t) + F(u^n(t))dt + g \nabla \hat{z}^n(t)dt = \sigma^n(t, u^n(t))dW^n(t) + \int_Z H^n(u^n(t), z) \tilde{N}(dt, dz). \]

Using the a priori estimates (4.5)-(4.6), it follows from the Banach-Alaoglu theorem that along a subsequence, the Galerkin approximations \( \{u^n\} \) have the following limits:

- \( u^n \to u \) weakly in \( L^2(\Omega; L^\infty(0, T; L^2(\mathcal{O}))) \cap L^2(0, T; \mathbb{H}_0^1(\mathcal{O})) \),
- \( \hat{z}^n \to \hat{z} \) weakly in \( L^2(\Omega; L^2(0, T; L^2(\mathcal{O}))) \),
- \( F(u^n) \to F_0 \) weakly in \( L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}(\mathcal{O}))) \),
- \( \sigma(\cdot, u^n) \to \sigma_0 \) weakly in \( L^2(\Omega; L^2(0, T; L_Q)) \),
- \( H^n(u^n, \cdot) \to H_0 \) weakly in \( \mathbb{H}^2_\mathcal{Q}([0, T] \times \mathcal{Z}; L^2) \),

where \( u \) has the differential form
\[ du(t) + F_0(t)dt + g \nabla \hat{z}(t)dt = \sigma_0(t)dW(t) + \int_Z H_0(t, z)\tilde{N}(dt, dz), \]
weakly in \( L^2(\Omega; L^2(0, T; \mathbb{H}^{-1}(\mathcal{O}))) \).

Applying Itô's formula to \( e^{-Lt}|\sqrt{h}x|^2 \) and the process \( u^n(t) \)
\[
d[e^{-Lt}\|\sqrt{h}u^n(t)\|_{L^2}^2 + e^{-Lt}g\|\hat{z}^n(t)\|_{L^2}^2] = -Le^{-Lt}\|\sqrt{h}u^n(t)\|_{L^2}^2 dt - Le^{-Lt}g\|\hat{z}^n(t)\|_{L^2}^2 dt
-e^{-Lt}(2F(u^n(t)), hu^n(t))_{L^2} dt
+2e^{-Lt}(\sigma^n(t, u^n(t)), hu^n(t))_{L^2} dW^n(t)
+e^{-Lt}\|\sqrt{h}\sigma^n(t, u^n(t))\|_{L^2}^2 dt
+2e^{-Lt} \int_Z (H^n(u^n(t), z), \sqrt{h}u^n(t, z))_{L^2} N(dt, dz)
+e^{-Lt} \int_Z \|\sqrt{h}H^n(u^n(t), z)\|_{L^2}^2 N(dt, dz).
\]

Integrating from 0 to \( T \) and taking expectation
\[
\mathbb{E} \left[ e^{-LT}\|\sqrt{h}u^n(T)\|_{L^2}^2 + e^{-LT}g\|\hat{z}^n(T)\|_{L^2}^2 - \|\sqrt{h}u^n(0)\|_{L^2}^2 - g\|\hat{z}^n(0)\|_{L^2}^2 \right]
= -\mathbb{E}\left[ \int_0^T Le^{-Lt}\|\sqrt{h}u^n(t)\|_{L^2}^2 dt \right] - \mathbb{E}\left[ \int_0^T Le^{-Lt}g\|\hat{z}^n(t)\|_{L^2}^2 dt \right]
-2\mathbb{E}\left[ \int_0^T e^{-Lt}(F(u^n(t)), hu^n(t))_{L^2} dt \right]
+\mathbb{E}\left[ \int_0^T 2e^{-Lt}(\sigma^n(t, u^n(t)), hu^n(t))_{L^2} dW^n(t) \right]
+\mathbb{E}\left[ \int_0^T e^{-Lt}\|\sqrt{h}\sigma^n(t, u^n(t))\|_{L^2}^2 dt \right]
+\mathbb{E}\left[ \int_0^T 2e^{-Lt} \int_Z (H^n(u^n(t), z), \sqrt{h}u^n(t, z))_{L^2} N(dt, dz) \right]
\]
\[ + \mathbb{E} \left[ \int_0^T e^{-Lt} \int_Z \sqrt{h} H^n(u^n(t), z) \| \|_{L^2}^2 N(dt, dz) \right]. \]

Since
\[ \mathbb{E} \left[ \int_0^T 2e^{-Lt}(\sigma^n(t, u^n(t)), hu^n(t))_{L^2} dW^n(t) \right], \]

and
\[ \mathbb{E} \left[ \int_0^T 2e^{-Lt} \int_Z (H^n(u^n(t), z), \sqrt{h} u^n(t))_{L^2} \tilde{N}(dt, dz) \right], \]

are martingales with zero averages, and since \( \| \sqrt{h} H^n(u^n(t), z) \|^2 \) is strong 1-integrable w.r.t \( \tilde{N}(dt, dz) \),
\[ \mathbb{E} \left[ \int_0^T e^{-Lt} \int_Z \| \sqrt{h} H^n(u^n(t), z) \|^2_{L^2} N(dt, dz) \right] \]
\[ = \mathbb{E} \left[ \int_0^T e^{-Lt} \int_Z \| \sqrt{h} H^n(u^n(t), z) \|^2_{L^2} \lambda(dz) dt \right]. \]

So
\[ \mathbb{E} e^{-LT} \| \sqrt{h} u^n(T) \|_{L^2}^2 + e^{-LT} g \| \tilde{z}^n(T) \|_{L^2}^2 - \| \sqrt{h} u^n(0) \|_{L^2}^2 - g \| \tilde{z}^n(0) \|_{L^2}^2 \]
\[ = -\mathbb{E} \left[ \int_0^T Le^{-Lt} \| \sqrt{h} u^n(t) \|_{L^2}^2 dt \right] - \mathbb{E} \left[ \int_0^T Le^{-Lt} g \| \tilde{z}^n(t) \|_{L^2}^2 dt \right] \]
\[ -2\mathbb{E} \left[ \int_0^T e^{-Lt} (F(u^n(t)), hu^n(t))_{L^2} dt \right] + \mathbb{E} \left[ \int_0^T e^{-Lt} \| \sqrt{h} \sigma^n(t, u^n(t)) \|_{L^2}^2 dt \right] \]
\[ + \mathbb{E} \left[ \int_0^T e^{-Lt} \int_Z \| \sqrt{h} H^n(u^n(t), z) \|^2_{L^2} \lambda(dz) dt \right]. \]

Using the lower semi-continuity of \( L^2 \)-norm
\[ \liminf_n \{-\mathbb{E} \left[ \int_0^T Le^{-Lt} \| \sqrt{h} u^n(t) \|_{L^2}^2 dt \right] - \mathbb{E} \left[ \int_0^T Le^{-Lt} g \| \tilde{z}^n(t) \|_{L^2}^2 dt \right] \]
\[ -2\mathbb{E} \left[ \int_0^T e^{-Lt} (F(u^n(t)), hu^n(t))_{L^2} dt \right] + \mathbb{E} \left[ \int_0^T e^{-Lt} \| \sqrt{h} \sigma^n(t, u^n(t)) \|_{L^2}^2 dt \right] \]
\[ + \mathbb{E} \left[ \int_0^T e^{-Lt} \int_Z \| \sqrt{h} H^n(u^n(t), z) \|^2_{L^2} \lambda(dz) dt \right] \]
\[ = \liminf_n \{ \mathbb{E} e^{-LT} \| \sqrt{h} u^n(T) \|_{L^2}^2 + e^{-LT} g \| \tilde{z}^n(T) \|_{L^2}^2 \]
\[ - \| \sqrt{h} u^n(0) \|_{L^2}^2 - g \| \tilde{z}^n(0) \|_{L^2}^2 \} \]
\[ \geq \mathbb{E} e^{-LT} \| \sqrt{h} u(T) \|_{L^2}^2 + e^{-LT} g \| \tilde{z}(T) \|_{L^2}^2 - \| \sqrt{h} u(0) \|_{L^2}^2 - g \| \tilde{z}(0) \|_{L^2}^2 \]
\[ = -\mathbb{E} \left[ \int_0^T Le^{-Lt} \| \sqrt{h} u(t) \|_{L^2}^2 dt \right] - \mathbb{E} \left[ \int_0^T Le^{-Lt} g \| \tilde{z}(t) \|_{L^2}^2 dt \right] \]
\[ -2\mathbb{E} \left[ \int_0^T e^{-Lt} (F_0(t), hu(t))_{L^2} dt \right] + \mathbb{E} \left[ \int_0^T e^{-Lt} \| \sqrt{h} \sigma_0(t) \|_{L^2}^2 dt \right] \]
\[ + \mathbb{E} \left[ \int_0^T e^{-Lt} \int_Z \| \sqrt{h} H_0(t, z) \|^2_{L^2} \lambda(dz) dt \right]. \] (4.26)
Using the monotonicity property of $F$ and assumption H.3, we have for all $v \in L^2(\Omega; L^\infty(0,T; L^2 (\mathcal{O}))) \cap L^2(0,T; H^1_0(\mathcal{O})) \cap D(0,T; L^2 (\mathcal{O})))$

\[-2\mathbb{E} \int_0^T e^{-Lt}(F(u^n(t)) - F(v(t)), hu^n(t) - hv(t))_{L^2} dt \]

\[+ \mathbb{E} \int_0^T e^{-Lt}||\sqrt{h}\sigma^n(t, u^n(t)) - \sqrt{h}\sigma^n(t, v(t))||_{L^2}^2 dt \]

\[+ \mathbb{E} \int_0^T e^{-Lt} \int_Z ||\sqrt{h}H^n(u^n(t), z) - \sqrt{h}H^n(v(t), z)||_{L^2}^2 \lambda(dz) dt \]

\[-\mathbb{E} \int_0^T Le^{-Lt}||\sqrt{h}u^n(t) - \sqrt{h}v(t)||_{L^2}^2 dt \]

\[-\mathbb{E} \int_0^T Le^{-Lt} g||\dot{z} - \dot{z}||_{L^2}^2 dt \]

\[\leq 0. \quad (4.27)\]

Rearranging the terms

\[-2\mathbb{E} \int_0^T e^{-Lt}(F(u^n(t)), hu^n(t))_{L^2} dt + \mathbb{E} \int_0^T e^{-Lt}||\sqrt{h}\sigma^n(t, u^n(t))||_{L^2}^2 dt \]

\[+ \mathbb{E} \int_0^T e^{-Lt} \int_Z ||\sqrt{h}H^n(u^n(t), z)||_{L^2}^2 \lambda(dz) dt - \mathbb{E} \int_0^T Le^{-Lt}||\sqrt{h}u^n(t)||_{L^2}^2 dt \]

\[-\mathbb{E} \int_0^T Le^{-Lt} g||\dot{z}||_{L^2}^2 dt \]

\[\leq -2\mathbb{E} \int_0^T e^{-Lt}(F(v(t)), hu^n(t) - hv(t))_{L^2} dt - \mathbb{E} \int_0^T e^{-Lt}||\sqrt{h}\sigma^n(t, v(t))||_{L^2}^2 dt \]

\[+ 2\mathbb{E} \int_0^T e^{-Lt}||\sqrt{h}\sigma^n(u^n(t)), \sqrt{h}\sigma^n(v(t))||_{L^2}^2 dt \]

\[-\mathbb{E} \int_0^T e^{-Lt} \int_Z ||\sqrt{h}H^n(v(t), z)||_{L^2}^2 \lambda(dz) dt \]

\[+ 2\mathbb{E} \int_0^T e^{-Lt} \int_Z (\sqrt{h}H^n(u^n(t), z), \sqrt{h}H^n(v(t), z))_{L^2} \lambda(dz) dt \]

\[+ \mathbb{E} \int_0^T Le^{-Lt}||\sqrt{h}v(t)||_{L^2}^2 dt - 2\mathbb{E} \int_0^T Le^{-Lt}||\sqrt{h}u^n(t)||_{L^2}^2 dt \]

\[+ \mathbb{E} \int_0^T Le^{-Lt} g||\dot{z}||_{L^2}^2 dt - 2\mathbb{E} [\int_0^T Le^{-Lt} g||\dot{z}||_{L^2}^2 dt] \]

\[-2\mathbb{E} \int_0^T e^{-Lt}(F(u^n(t)), hv(t))_{L^2} dt \]

Taking limit in $n$ and using (4.26)

\[-2\mathbb{E} \int_0^T e^{-Lt}(F_0(t), hu(t))_{L^2} dt + \mathbb{E} \int_0^T e^{-Lt}||\sqrt{h}\sigma_0(t)||_{L^2}^2 dt \]

\[+ \mathbb{E} \int_0^T e^{-Lt} \int_Z ||\sqrt{h}H_0(t, z)||_{L^2}^2 \lambda(dz) dt - \mathbb{E} \int_0^T Le^{-Lt}||\sqrt{h}u(t)||_{L^2}^2 dt \]
\[
- \mathbb{E}\int_0^T L e^{-Lt} g \|\hat{z}\|_{L^2}^2 dt \\
\leq -2\mathbb{E}\int_0^T e^{-Lt} (F(v(t)), hu(t) - hv(t))_{L^2} dt \\
- \mathbb{E}\int_0^T e^{-Lt} \|\sqrt{h}\sigma(t, v(t))\|_{L^2}^2 dt \\
+ 2\mathbb{E}\int_0^T e^{-Lt} (\sqrt{h} \sigma_0(t), \sqrt{h}\sigma(v(t)))_{L^2} dt \\
- \mathbb{E}\int_0^T e^{-Lt} \int_Z \|\sqrt{h}H(v(t), z)\|_{L^2}^2 \lambda(dz) dt \\
+ 2\mathbb{E}\int_0^T e^{-Lt} \int_Z (\sqrt{h} H_0(t, z), \sqrt{h} H(v(t), z))_{L^2} \lambda(dz) dt \\
+ \mathbb{E}\int_0^T L e^{-Lt} \|\sqrt{h}v(t)\|_{L^2}^2 dt - 2\mathbb{E}\int_0^T L e^{-Lt} (\sqrt{h}u(t), \sqrt{h}v(t))_{L^2} dt \\
+ \mathbb{E}\int_0^T L e^{-Lt} g \|\hat{z}\|_{L^2}^2 dt - 2\mathbb{E}\int_0^T L e^{-Lt} g (\hat{z}(t), \hat{z}(t))_{L^2} dt \\
- 2\mathbb{E}\int_0^T e^{-Lt} (F_0(t), hv(t))_{L^2} dt.
\]

Rearranging the terms
\[
- 2\mathbb{E}\int_0^T e^{-Lt} (F_0(t) - F(v(t)), hu(t) - hv(t))_{L^2} dt \\
+ \mathbb{E}\int_0^T e^{-Lt} \|\sqrt{h}\sigma_0(t) - \sqrt{h}\sigma(t, v(t))\|_{L^2}^2 dt \\
+ \mathbb{E}\int_0^T e^{-Lt} \int_Z \|\sqrt{h}H_0(t, z) - \sqrt{h}H(v(t), z)\|_{L^2}^2 \lambda(dz) dt \\
- \mathbb{E}\int_0^T L e^{-Lt} \|\sqrt{h}u(t) - \sqrt{h}v(t)\|_{L^2}^2 dt \\
\leq 0.
\]

This estimate holds for any \( v \in L^2(\Omega; L^\infty(0, T; L^2_n(\mathcal{O}))) \cap L^2(0, T; \mathbb{H}^1_0(\mathcal{O})) \cap \mathcal{D}(0, T; L^2_n(\mathcal{O}))) \), for any \( n \in \mathbb{N} \). It is obvious from the density argument that the above inequality remains the same for any \( v \in L^2(\Omega; L^\infty(0, T; L^2(\mathcal{O}))) \cap L^2(0, T; \mathbb{H}^1_0(\mathcal{O})) \cap \mathcal{D}(0, T; L^2(\mathcal{O}))) \). In fact, for any \( v \in L^2(\Omega; L^\infty(0, T; L^2(\mathcal{O}))) \cap L^2(0, T; \mathbb{H}^1_0(\mathcal{O})) \cap \mathcal{D}(0, T; L^2(\mathcal{O}))) \) there exists a strongly convergent subsequence \( v_m \in L^2(\Omega; L^\infty(0, T; L^2_m(\mathcal{O}))) \cap L^2(0, T; \mathbb{H}^1_0(\mathcal{O})) \cap \mathcal{D}(0, T; L^2_m(\mathcal{O}))) \) satisfying the above inequality.

Taking \( v = u \), we get \( \sigma_0(t) = \sigma(t, u(t)) \) and \( H_0(t, z) = H(u(t), z) \).

Now we take \( v = u + \lambda w \) where \( \lambda > 0 \) and \( w \) is an adapted process in \( L^2(\Omega; L^\infty(0, T; L^2(\mathcal{O}))) \cap L^2(0, T; \mathbb{H}^1_0(\mathcal{O})) \cap \mathcal{D}(0, T; L^2(\mathcal{O}))) \).
Then we have
\[ \lambda \mathbb{E}[\int_0^T e^{-Lt}(F(u(t) + \lambda w(t)), hw(t)) \, dW_t] \geq \lambda \mathbb{E}[\int_0^T e^{-Lt}(F_0(t), hw(t)) \, dW_t]. \]  
(4.28)

Dividing by \( \lambda \) on both sides on the inequality above and letting \( \lambda \) go to 0, we have by the hemicontinuity of \( F \)
\[ \mathbb{E}[\int_0^T e^{-Lt}(F(u(t)) - F_0(t), hw(t)) \, dW_t] \geq 0. \]
Since \( w \) is arbitrary and \( h \) is a positive, bounded, continuously differentiable function, \( F_0(t) = F(u(t)) \). This proves the existence of a strong solution.

**Uniqueness:**
If \( u \) and \( v \) are two solutions then \( w = u - v \) solves the stochastic differential equation
\[
\begin{align*}
  dw(t) + Aw(t)dt + g\nabla(\hat{z}(t) - \tilde{z}(t))dt \\
  = & B(v(t)) - B(u(t))dt + (\sigma(t, u(t)) - \sigma(t, v(t)))dW_t \\
  & + \int_Z (H(u(t-), z) - H(v(t-), z))\tilde{N}(dt, dz).
\end{align*}
\]

Applying Itô’s formula to \(|x|^2\) and to the process \( w(t) \)
\[
\begin{align*}
  d\left(\|w(t)\|_{L^2}^2\right) + 2\alpha\|w(t)\|_{H^1_0}^2 dt + 2g(\nabla(\hat{z}(t) - \tilde{z}(t)), w(t))_{L^2} dt \\
  = & 2(B(v(t)) - B(u(t)), u(t) - v(t))_{L^2} dt \\
  & + 2(\sigma(t, u(t)) - \sigma(t, v(t)), w(t))_{L^2} dW(t) \\
  & + \|\sigma(t, u(t)) - \sigma(t, v(t))\|^2_{L^2_Q} dt \\
  & + \int_Z \|H(u(t-), z) - H(v(t-), z)\|^2_{L^2} N(dt, dz) \\
  & + \int_Z ((H(u(t-), z) - H(v(t-), z)), w(t-))_{L^2} \tilde{N}(dt, dz).
\end{align*}
\]

Using the result from Lemma 3.4 and equation (4.10)
\[
\begin{align*}
  d\left(\|w(t)\|_{L^2}^2\right) + 2\alpha\|w(t)\|_{H^1_0}^2 dt \\
  \leq & 2g^2\|\hat{z}(t) - \tilde{z}(t)\|^2_{L^2} dt + \frac{\alpha}{2}\|w(t)\|_{H^1_0}^2 dt \\
  & + 2(\sigma(t, u(t)) - \sigma(t, v(t)), w(t))_{L^2} dW(t) + \|\sigma(t, u(t)) - \sigma(t, v(t))\|^2_{L^2_Q} dt \\
  & + \int_Z \|H(u(t-), z) - H(v(t-), z)\|^2_{L^2} N(dt, dz) \\
  & + \int_Z ((H(u(t-), z) - H(v(t-), z)), w(t-))_{L^2} \tilde{N}(dt, dz).
\end{align*}
\]

Notice that
\[
\begin{align*}
  d(\hat{z}(t) - \tilde{z}(t)) + \text{Div}(hw(t))dt = 0.
\end{align*}
\]

(4.29)
Taking inner product with $\tilde{z}(t) - \tilde{z}(t)$, we have as in equation (4.13)
\[
\begin{align*}
d(\|\tilde{z}(t) - \tilde{z}(t)\|_{L^2}^2) \\
&\leq M \|w(t)\|_{L^2}^2 dt + \left(\frac{2\mu^2}{\alpha} + M\right) \|\tilde{z}(t) - \tilde{z}(t)\|_{L^2}^2 dt + \frac{\alpha}{2} \|w(t)\|_{H^0_0}^2 dt. \tag{4.30}
\end{align*}
\]
Let
\[
C = \frac{2\gamma^2}{\alpha} + \frac{2\mu^2}{\alpha} + M.
\]
Adding equations (4.29) and (4.30)
\[
\begin{align*}
d(\|w(t)\|_{L^2}^2 + \|\tilde{z}(t) - \tilde{z}(t)\|_{L^2}^2) + \alpha \|w(t)\|_{H^0_0}^2 dt \\
&\leq C(\|w(t)\|_{L^2}^2 + \|\tilde{z}(t) - \tilde{z}(t)\|_{L^2}^2) dt \\
&+ 2(\sigma(t, u(t)) - \sigma(t, v(t)), w(t))_{L^2} dW(t) + \|\sigma(t, u(t)) - \sigma(t, v(t))\|_{L^0_1}^2 dt \\
&+ \int_Z \|H(u(t-), z) - H(v(t-), z)\|_{L^2}^2 N(dt, dz) \\
&+ \int_Z ((H(u(t-), z) - H(v(t-), z)), w(t-))_{L^2} \tilde{N}(dt, dz).
\end{align*}
\]
Integrating from 0 to $T$ and taking expectation
\[
\begin{align*}
\mathbb{E} \left[\|w(T)\|_{L^2}^2 + \|\tilde{z}(T) - \tilde{z}(T)\|_{L^2}^2\right] + \mathbb{E} \left[\int_0^T \alpha \|w(t)\|_{H^0_0}^2 dt\right] \\
\leq C \mathbb{E} \left[\int_0^T (\|w(t)\|_{L^2}^2 + \|\tilde{z}(t) - \tilde{z}(t)\|_{L^2}^2) dt\right] \\
+ \mathbb{E} \left[\int_0^T \|\sigma(t, u(t)) - \sigma(t, v(t))\|_{L^0_1}^2 dt\right] \\
+ \mathbb{E} \left[\int_0^T \int_Z \|H(u(t-), z) - H(v(t-), z)\|_{L^2}^2 \lambda(dz) \right] \\
+ \mathbb{E} \left[\|w(0)\|_{L^2}^2 + \|\tilde{z}(0) - \tilde{z}(0)\|_{L^2}^2\right].
\end{align*}
\]
Using assumption H.3
\[
\begin{align*}
\mathbb{E} \left[\|w(T)\|_{L^2}^2 + \|\tilde{z}(T) - \tilde{z}(T)\|_{L^2}^2\right] + \mathbb{E} \left[\int_0^T \alpha \|w(t)\|_{H^0_0}^2 dt\right] \\
\leq C \mathbb{E} \left[\int_0^T (\|w(t)\|_{L^2}^2 + \|\tilde{z}(t) - \tilde{z}(t)\|_{L^2}^2) dt\right] + L \mathbb{E} \left[\int_0^T \|w(t)\|_{L^2}^2 dt\right] \\
+ \mathbb{E} \left[\|w(0)\|_{L^2}^2 + \|\tilde{z}(0) - \tilde{z}(0)\|_{L^2}^2\right].
\end{align*}
\]
In particular
\[
\begin{align*}
\mathbb{E} \left[\|w(T)\|_{L^2}^2 + \|\tilde{z}(T) - \tilde{z}(T)\|_{L^2}^2\right] \\
\leq \mathbb{E} \left[\|w(0)\|_{L^2}^2 + \|\tilde{z}(0) - \tilde{z}(0)\|_{L^2}^2\right] \\
+ (C + L) \mathbb{E} \left[\int_0^T (\|w(t)\|_{L^2}^2 + \|\tilde{z}(t) - \tilde{z}(t)\|_{L^2}^2) dt\right].
\end{align*}
\]
Hence the uniqueness of pathwise strong solution follows using Gronwall’s inequality. \qed
4.3. \( \mathbb{H}^1_0 \) regularity

We assume that \( \sigma \) and \( H \) satisfy the following hypotheses:

A.1 \( \sigma \in C([0,T] \times \mathbb{H}^1_0(\mathcal{O}); L_Q(H_0,\mathbb{H}^1_0)), H \in \mathbb{H}^2_\lambda([0,T] \times Z; \mathbb{H}^1_0(\mathcal{O})) \),

A.2 For all \( t \in (0,T) \), there exists a positive constant \( K \) such that for all \( u \in \mathbb{H}^1_0(\mathcal{O}) \)

\[
\| \nabla \sigma(t,u) \|^2_{L_Q} + \int_Z \| \nabla H(u,z) \|^2_{L^2} \lambda(dz) \leq K(1 + \| u \|^2_{\mathbb{H}^1_0}),
\]

A.3 For all \( t \in (0,T) \), there exists a positive constant \( L \) such that for all \( u,v \in \mathbb{H}^1_0(\mathcal{O}) \)

\[
\| \nabla \sigma(t,u) - \nabla \sigma(t,v) \|^2_{L_Q} + \int_Z \| \nabla H(u,z) - \nabla H(v,z) \|^2_{L^2} \lambda(dz) \leq L(\| u - v \|^2_{\mathbb{H}^1_0}).
\]

Proposition 4.5. Let

\[
\begin{aligned}
&\left\{ \begin{array}{l}
w^0 \in L^2(0,T; \mathbb{H}^1_0(\mathcal{O})), f \in L^2(0,T; \mathbb{H}^1_0(\mathcal{O})), \\
\sigma \in C([0,T] \times \mathbb{H}^1_0(\mathcal{O}); L_Q(H_0,\mathbb{H}^1_0)), H \in \mathbb{H}^2_\lambda([0,T] \times Z; \mathbb{H}^1_0(\mathcal{O})), \\
u_0 \in \mathbb{H}^1_0(\mathcal{O}), z_0 \in H_0(\mathcal{O}).
\end{array} \right.
\end{aligned}
\]

Then

\[
\begin{aligned}
&\lim_{T \to 0} \sup_{N} \mathbb{P}( \sup_{0 \leq t \leq T^{\wedge} N} \| u^n(t) \|^2_{\mathbb{H}^1_0} + \| \dot{z}^n(t) \|^2_{H_0} + \frac{\alpha}{2} \int_0^{T^{\wedge} N} \| \triangle u^n(t) \|^2_{L^2} dt \\
&\quad > (N - 1) + \| u_0^n \|^2_{\mathbb{H}^1_0} + \| \dot{z}_0^n \|^2_{H_0}) = 0.
\end{aligned}
\]

Proof. Applying Itô’s formula to \(|x|^2\) and to the process \( \nabla u^n(t) \)

\[
\begin{aligned}
d(\| u^n(t) \|^2_{\mathbb{H}^1_0}) + 2\alpha \| \triangle u^n(t) \|^2_{L^2} dt + 2(\nabla B(u^n), \nabla u^n)_{L^2} dt \\
+ 2(\nabla(g \nabla \dot{z}^n), \nabla u^n)_{L^2} dt \\
= 2(\nabla f, \nabla u^n(t))_{L^2} dt + 2(\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t) \\
+ \| \nabla \sigma^n(t, u^n(t)) \|^2_{L_Q} dt + \int_Z \| \nabla H^n(u^n(t), z) \|^2_{L^2} N(dt, dz) \\
+ 2 \int_Z (\nabla H^n(u^n(t), z), \nabla u^n(t))_{L^2} \tilde{N}(dt, dz).
\end{aligned}
\]

Using divergence theorem

\[
\begin{aligned}
d(\| u^n(t) \|^2_{\mathbb{H}^1_0}) + 2\alpha \| \triangle u^n(t) \|^2_{L^2} dt \\
= 2(B(u^n), \triangle u^n)_{L^2} dt + 2(g \nabla \dot{z}^n, \triangle u^n)_{L^2} dt + 2(\nabla f, \nabla u^n(t))_{L^2} dt \\
+ 2(\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t) + \| \nabla \sigma^n(t, u^n(t)) \|^2_{L_Q} dt \\
+ \int_Z \| \nabla H^n(u^n(t), z) \|^2_{L^2} N(dt, dz) \\
+ 2 \int_Z (\nabla H^n(u^n(t), z), \nabla u^n(t))_{L^2} \tilde{N}(dt, dz).
\end{aligned}
\]
Using Young’s inequality

\[ 2|g(\nabla \hat{z}^n(t), \Delta u^n(t))_{L^2}| \leq \frac{2g^2}{\alpha} \|\hat{z}^n(t)\|_{H_0^1}^2 + \frac{\alpha}{2} \|\Delta u^n(t)\|_{L^2}^2, \quad (4.35) \]

and

\[ (\nabla f, \nabla u^n(t))_{L^2} \leq \frac{1}{2}[\|f(t)\|_{H_0^1}^2 + \|u^n(t)\|_{H_0^1}^2]. \quad (4.36) \]

Using Lemma 3.3, equation (3.11) and Young’s inequality

\[ 2(B(u^n), \Delta u^n)_{L^2} \leq \frac{2}{\alpha} \|B(u^n)\|_{L^2}^2 + \frac{\alpha}{2} \|\Delta u(t)\|_{L^2}^2 \]

\[ \leq C_2(\|u^n(t)\|_{L^2}^4 + \|u^0(t)\|_{L^4}^4) + \frac{\alpha}{2} \|\Delta u(t)\|_{L^2}^2 \]

\[ \leq C_2(2\|u^n(t)\|_{L^2}^2\|u^n(t)\|_{L^4}^2 + \|u^0(t)\|_{L^4}^4) + \frac{\alpha}{2} \|\Delta u(t)\|_{L^2}^2. \]

Now using the a-priori estimate given by equation (4.5) we get

\[ 2(B(u^n), \Delta u^n)_{L^2} \leq 2C_2C_{1(2)}\|u^n(t)\|_{H_0^1}^2 + C_2\|u^0(t)\|_{L^4}^4 + \frac{\alpha}{2} \|\Delta u(t)\|_{L^2}^2. \quad (4.37) \]

Hence

\[ d(\|u^n(t)\|_{H_0^1}^2) + \alpha \|\Delta u^n(t)\|_{L^2}^2 dt \]

\[ \leq (2C_2C_{1(2)} + 1)\|u^n(t)\|_{H_0^1}^2 dt + \frac{2g^2}{\alpha}\|\hat{z}^n(t)\|_{H_0^1}^2 dt + \|f(t)\|_{H_0^1}^2 dt + C_2\|u^0(t)\|_{L^4}^4 \]

\[ + 2(\nabla \sigma^n, \nabla u^n(t))_{L^2} dW^n(t) + \|\nabla \sigma^n(t, u^n(t))\|_{L^Q}^2 dt \]

\[ + \int_{Z} \|\nabla H^n(u^n(t), z)\|_{L^2}^2 N(dt, dz) \]

\[ + 2 \int_{Z} (\nabla H^n(u^n(t), z), \nabla u^n(t) )_{L^2} N(dt, dz). \quad (4.38) \]

Using assumption A.2

\[ d(\|u^n(t)\|_{H_0^1}^2) + \alpha \|\Delta u^n(t)\|_{L^2}^2 dt \]

\[ \leq (2C_2C_{1(2)} + 1)\|u^n(t)\|_{H_0^1}^2 dt + \frac{2g^2}{\alpha}\|\hat{z}^n(t)\|_{H_0^1}^2 dt + \|f(t)\|_{H_0^1}^2 dt + C_2\|u^0(t)\|_{L^4}^4 \]

\[ + 2(\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t) + K(1 + \|u^n(t)\|_{H_0^1}^2) dt \]

\[ + \int_{Z} \|\nabla H^n(u^n(t), z)\|_{L^2}^2 N(dt, dz) \]

\[ + 2 \int_{Z} (\nabla H^n(u^n(t), z), \nabla u^n(t) )_{L^2} N(dt, dz). \quad (4.39) \]

Using (4.2)

\[ \frac{1}{2} \frac{d}{dt} \|\hat{z}^n(t)\|_{H_0^1}^2 = (Div(h u^n(t)), \Delta \hat{z}^n(t))_{L^2}. \quad (4.40) \]
Now
\[
|\langle \text{Div}(hu^n(t)), \Delta \hat{z}^n(t) \rangle \rangle_{L^2} | \leq \mu \|u^n(t)\|_{H^1_0} \|\Delta \hat{z}^n(t)\|_{L^2} + M \|u^n(t)\|_{L^2} \|\Delta \hat{z}^n(t)\|_{L^2} \\
\leq \frac{\mu}{2} \left( \frac{\alpha}{2 \mu} \|u^n(t)\|^2_{H^1_0} + \frac{2 \mu}{\alpha} \|\Delta \hat{z}^n(t)\|^2_{L^2} \right) \\
+ \frac{M}{2} \|u^n(t)\|^2_{L^2} + \|\Delta \hat{z}^n(t)\|^2_{L^2}.
\]

Using Poincaré’s inequality
\[
\|u^n\|_{L^2} \leq C_F \|u^n\|_{H^1_0}, \quad (4.41)
\]
and rearranging
\[
d(\|\hat{z}^n(t)\|^2_{H^1_0} + \|\hat{z}^n(t)\|^2_{H^1_0}) + \frac{\alpha}{2} \|\Delta u^n(t)\|^2_{L^2} dt \\
\leq (2C_2C_{1(2)} + 1 + MC_F^2 + \frac{\alpha}{2}) \|u^n(t)\|^2_{H^1_0} dt + \frac{2g^2}{\alpha} \|\hat{z}^n(t)\|^2_{H^1_0} dt \\
+ \left( \frac{2\mu^2}{\alpha} + M \right) \|\Delta \hat{z}^n(t)\|^2_{L^2} dt + \|f(t)\|^2_{H^1_0} dt + C_2 \|w^0(t)\|_{L^4}^4 \\
+ 2(\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t) + K(1 + \|u^n(t)\|^2_{H^1_0}) dt \\
+ \int_Z \|\nabla H^n(u^n(t), z)\|^2_{L^2} \tilde{N}(dt, dz) \\
+ 2 \int_Z (\nabla H^n(u^n(t), z), \nabla u^n(t))_{L^2} \tilde{N}(dt, dz). \quad (4.43)
\]
Define
\[
\tau_N := \inf \{ t : \|u^n(t)\|^2_{H^1_0} + \|\hat{z}^n(t)\|^2_{H^1_0} + \int_0^t \frac{\alpha}{2} \|\Delta u^n(s)\|^2_{L^2} ds + \|\Delta \hat{z}^n(t)\|^2_{L^2} > N + \|u^n_0\|^2_{H^1_0} + \|\hat{z}^n_0\|^2_{H^1_0} \}. \quad (4.44)
\]
Integrating and taking supremum over \(0 \leq t \leq T \wedge \tau_N\)
\[
\sup_{0 \leq t \leq T \wedge \tau_N} \left( \|u^n(t)\|^2_{H^1_0} + \|\hat{z}^n(t)\|^2_{H^1_0} \right) + \frac{\alpha}{2} \int_0^{T \wedge \tau_N} \|\Delta u^n(t)\|^2_{L^2} dt \\
\leq S \int_0^{T \wedge \tau_N} (\|u^n(t)\|^2_{H^1_0} + \|\hat{z}^n(t)\|^2_{H^1_0} + \|\Delta \hat{z}^n(t)\|^2_{L^2} + \|f(t)\|^2_{H^1_0} + \|w^0(t)\|_{L^4}^4 + 1) dt \\
+ 2 \sup_{0 \leq t \leq T \wedge \tau_N} \int_0^t (\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t) \\
+ \int_0^{T \wedge \tau_N} \int_Z (\nabla H^n(u^n(t), z), \nabla u^n(t))_{L^2} \tilde{N}(dt, dz) \\
+ \int_0^{T \wedge \tau_N} \int_Z \|\nabla H^n(u^n(t), z)\|^2_{L^2} \tilde{N}(dt, dz) \\
+ \|u^n_0\|^2_{H^1_0} + \|\hat{z}^n_0\|^2_{H^1_0}, \quad (4.45)
\]
where $S = \max\{2C_2C_1(2) + 1 + MC_P^2 + \frac{\alpha}{2} + K, \frac{2g^2}{\alpha}, \frac{2\mu^2}{\alpha} + M, 1, C_2, K\}$. Hence

$$
P \left( \sup_{0 \leq t \leq T \land \tau_N} \|u^n(t)\|_{g_0}^2 + \|\hat{z}^n(t)\|_{g_0}^2 + \frac{\alpha}{2} \int_0^{T \land \tau_N} \|\triangle u^n(t)\|_{L^2}^2 dt \right)

\leq (N - 1) + \|u_0^n\|_{g_0}^2 + \|\hat{z}_0^n\|_{g_0}^2 + \|f(t)\|_{g_0}^2 + \|u_0^0(t)\|_{L^4}^4 + 1 dt > (N - 1)/2

+ 2P \left( \sup_{0 \leq t \leq T \land \tau_N} \int_0^t ((\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t)

+ \int_0^{T \land \tau_N} \int_Z (\nabla H^n(u(t), z), \nabla u^n(t))_{L^2} \tilde{N}(dt, dz)

+ \int_0^{T \land \tau_N} \int_Z \|\nabla H^n(u(t), z)\|_{L^2}^2 N(dt, dz)) > (N - 1)/2 \right).

Now

$$
P \left( \sup_{0 \leq t \leq T \land \tau_N} \int_0^t ((\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t)

+ \int_0^{T \land \tau_N} \int_Z (\nabla H^n(u(t), z), \nabla u^n(t))_{L^2} \tilde{N}(dt, dz)

+ \int_0^{T \land \tau_N} \int_Z \|\nabla H^n(u(t), z)\|_{L^2}^2 N(dt, dz)) > (N - 1)/2 \right)

\leq P \left( \sup_{0 \leq t \leq T \land \tau_N} \int_0^t (\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t) > (N - 1)/6 \right)

+ P \left( \sup_{0 \leq t \leq T \land \tau_N} \int_0^t (\nabla H^n(u(t), z), \nabla u^n(t))_{L^2} \tilde{N}(dt, dz) > (N - 1)/6 \right)

+ P \left( \sup_{0 \leq t \leq T \land \tau_N} \int_0^t \|\nabla H^n(u(t), z)\|_{L^2}^2 N(dt, dz) > (N - 1)/6 \right).$$

Using Doob’s Inequality

$$
P \left( \sup_{0 \leq t \leq T \land \tau_N} \int_0^t (\nabla \sigma^n(t, u^n(t)), \nabla u^n(t))_{L^2} dW^n(t) > (N - 1)/6 \right)

\leq \frac{36}{(N - 1)^2} P \int_0^{T \land \tau_N} \|u^n(t)\|_{g_0}^2 \|\nabla \sigma^n(t, u^n(t))\|_{L^2}^2 dt

\leq \frac{36}{(N - 1)^2} K \mathbb{E} [\int_0^{T \land \tau_N} \|u^n(t)\|_{g_0}^2 (1 + \|u^n(t)\|_{g_0}^2) dt]

\leq \frac{36}{(N - 1)^2} KN(N + 1)(T \land \tau_N).$$
Using Doob’s inequality again

\[
P\left( \sup_{0 \leq t \leq T \wedge \tau_N} \int_0^t \int_Z (\nabla H^n(u(t-), z), \nabla u^n(t-))_{L^2} \, \tilde{N}(dt, dz) > (N - 1)/6 \right)
\]
\[
\leq \frac{36}{(N - 1)^2} \mathbb{E}\left[ \int_0^{T \wedge \tau_N} \|u^n(t)\|_{L^2}^2 \, \lambda(dt) \right]
\leq \frac{36}{(N - 1)^2} K \mathbb{E}\left[ \int_0^{T \wedge \tau_N} \|u^n(t)\|_{L^2}^2 (1 + \|u^n(t)\|_{L^2}^2) \, dt \right]
\leq \frac{36}{(N - 1)^2} K N(N + 1)(T \wedge \tau_N).
\]

Using Doob’s inequality and strong 2-integrability of \(|H^n(u(t-), z)|\)

\[
P\left( \sup_{0 \leq t \leq T \wedge \tau_N} \int_0^t \int_Z \|\nabla H^n(u(t-), z)\|_{L^2}^2 N(dt, dz) > (N - 1)/6 \right)
\]
\[
\leq \frac{6}{N - 1} \mathbb{E}\left[ \int_0^{T \wedge \tau_N} \|\nabla H^n(u(t-), z)\|_{L^2}^2 \lambda(dt) \right]
\leq \frac{6}{N - 1} K \mathbb{E}\left[ \int_0^{T \wedge \tau_N} (1 + \|u(t)\|_{L^2}^2) \, dt \right]
\leq \frac{6}{N - 1} K (N + 1)(T \wedge \tau_N).
\]

Using Chebychev’s inequality

\[
P\left( S \int_0^{T \wedge \tau_N} \left( \|u^n(t)\|_{L^2}^2 + \|\dot{z}^n(t)\|_{L^2}^2 + \|\triangle \dot{z}^n(t)\|_{L^2}^2 \right.ight.
\[
\left. \left. \quad + \|f(t)\|_{L^2}^2 + \|w^0(t)\|_{L^4}^4 + 1 \right) dt > (N - 1)/2 \right)
\]
\[
\leq \frac{2}{N - 1} \mathbb{E}\left[ S \int_0^{T \wedge \tau_N} \left( \|u^n(t)\|_{L^2}^2 + \|\dot{z}^n(t)\|_{L^2}^2 + \|\triangle \dot{z}^n(t)\|_{L^2}^2 \right. \right.
\[
\left. \left. \quad + \|f(t)\|_{L^2}^2 + \|w^0(t)\|_{L^4}^4 + 1 \right) dt \right]
\]
\[
\leq \frac{2}{N - 1} 3 N S(T \wedge \tau_N) + \frac{2}{N - 1} \mathbb{E}\left[ \int_0^{T \wedge \tau_N} \left( \|f(t)\|_{L^2}^2 + \|w^0(t)\|_{L^4}^4 + 1 \right) dt \right].
\]

Hence

\[
\lim_{T \to 0} \sup_N \mathbb{P}\left( \sup_{0 \leq t \leq T \wedge \tau_N} \|u^n(t)\|_{L^2}^2 + \|\dot{z}^n(t)\|_{L^2}^2 + \frac{\alpha}{2} \int_0^{T \wedge \tau_N} \|\triangle u^n(t)\|_{L^2}^2 dt \right.
\[
\left. > (N - 1) + \|u_0^n\|_{L^2}^2 + \|\dot{z}_0^n\|_{L^2}^2 \right) = 0.
\]
\[
\Box
\]
5. Stochastic Optimal Control

5.1. Preliminaries

In this subsection we provide some definitions and known results borrowed from Metivier [18] and Aldous [2].

**Definition 5.1.** Let \((\mathbb{S}, \rho)\) be a separable and complete metric space. Let \(u \in \mathcal{D}(0, T; \mathbb{S})\) and let \(\delta > 0\) be given. A modulus of \(u\) is defined by

\[
    w_{[0,T],\mathcal{S}}(u, \delta) := \inf_{\Pi_\delta} \max_{t_i \in \Pi_\delta} \sup_{t_i \leq s < t_{i+1} \leq T} \rho(u(t)u(s)),
\]

where \(\Pi_\delta\) is the set of all increasing sequences \(\overline{\omega} = \{0 - t_0 < t_1 < \ldots < t_n = T\}\) with the following property

\[
    t_{i+1} - t_i \geq \delta, \quad i = 0, \ldots, n - 1.
\]

**Theorem 5.2.** A set \(A \subset \mathcal{D}(0, T; \mathbb{S})\) has compact closure iff it satisfies the following two conditions:

(a) there exists a dense subset \(I \subset [0, T]\) such that for every \(t \in I\) the set \(\{u(t), u \in A\}\) has compact closure in \(\mathbb{S}\),

(b) \(\lim_{\delta \to 0} \sup_{u \in A} w_{[0,T],\mathcal{S}}(u, \delta) = 0\).

Taking the path space \(Z = \mathcal{D}(0, T; \mathbb{H}^{-1}(\mathcal{O}))_{J} \cap L^\infty(0, T; \mathbb{L}^{2}(\mathcal{O}))_{w^*} \cap L^2(0, T; \mathbb{H}^{1}(\mathcal{O}))_{w} \cap L^2(0, T; \mathbb{L}^{2}(\mathcal{O}))\) into account, we call \(\tau_1 := \mathcal{D}(0, T; \mathbb{H}^{-1}(\mathcal{O}))_{J}\), where \(J\) denotes the extended Skorokhod topology, \(\tau_2 := L^\infty(0, T; \mathbb{L}^{2}(\mathcal{O}))_{w^*}\), where \(w^*\) denotes the weak-star topology and \(\tau_3 := L^2(0, T; \mathbb{H}^{1}(\mathcal{O}))_{w}\), where \(w\) denotes the weak topology and \(\tau_4\) as the strong topology of \(L^2(0, T; \mathbb{L}^{2}(\mathcal{O}))\).

Note that the spaces \(\mathcal{D}(0, T; \mathbb{H}^{-1}(\mathcal{O}))_{J}, L^\infty(0, T; \mathbb{L}^{2}(\mathcal{O}))_{w^*}, L^2(0, T; \mathbb{H}^{1}(\mathcal{O}))_{w}\) are completely regular and continuously embedded in \(L^2(0, T; \mathbb{H}^{-1}(\mathcal{O}))_{w}\). Let \(\tau\) be the supremum of four topologies, that is, \(\tau = \tau_1 \vee \tau_2 \vee \tau_3 \vee \tau_4\).

**Theorem 5.3.** A set \(K \subset Z\) is \(\tau\)-relatively compact if the following three conditions hold

(a) \(\forall u \in K\) and all \(t \in [0,T]\), \(u(t) \in \mathbb{L}^{2}(\mathcal{O})\) and \(\sup_{u \in K} \sup_{s \in [0,T]} \|u(s)\|_{L^2} < \infty\),

(b) \(\sup_{u \in K} \int_{0}^{T} \|u(s)\|^2_{\mathbb{H}^{1}} ds < \infty\),

(c) \(\lim_{\delta \to 0} \sup_{u \in K} w_{[0,T],\mathbb{H}^{-1}(\mathcal{O})}(u, \delta) = 0\).

**Proof.** We can assume that \(K\) is a closed subset of \(Z\). By condition (a) the weak-star topology in \(L^\infty(0, T; \mathbb{L}^{2}(\mathcal{O}))_{w^*}\) induced on \(Z\) is metrizable. Similarly by condition (b) the weak topology in \(L^2(0, T; \mathbb{H}^{1}(\mathcal{O}))_{w}\) induced on \(Z\) is metrizable. Thus the compactness of a subset of \(Z\) is equivalent to its sequential compactness. Let \((u_n)\) be a sequence in \(K\). By the Banach-Alaoglu theorem \(K\) is compact in \(L^\infty(0, T; \mathbb{L}^{2}(\mathcal{O}))_{w^*}\) as well as \(L^2(0, T; \mathbb{H}^{1}(\mathcal{O}))_{w}\).

We need to prove that \((u_n)\) is compact in \(\mathcal{D}(0, T; \mathbb{H}^{-1}(\mathcal{O}))\). By condition (a) for every \(t \in [0,T]\) the set \(\{u_n(t), n \in \mathbb{N}\}\) is bounded in \(\mathbb{L}^{2}(\mathcal{O})\). Since the embedding \(\mathbb{L}^{2}(\mathcal{O}) \subset \mathbb{H}^{-1}(\mathcal{O})\) is compact, \(\{u_n(t), n \in \mathbb{N}\}\) is compact in
$H^{-1}(O)$. Using this and condition (c), Theorem 5.2 implies compactness of the sequence $(u_n)$ in the space $D(0, T; H^{-1}(O))$. Therefore, there exists a subsequence denoted again by $(u_n)$ such that

$$u_n \to u \text{ in } D(0, T; H^{-1}(O)) \text{ as } n \to \infty.$$ 

Since $u_n \to u$ in $D(0, T; H^{-1}(O))$, $u_n \to u$ in $H^{-1}(O)$ for all continuity points of $u$. Hence, by condition (a) and Lebesgue dominated convergence theorem

$$u_n \to u \text{ in } L^2(0, T; H^{-1}(O)) \text{ as } n \to \infty.$$ 

To complete the proof, we need to show that

$$u_n \to u \text{ in } L^2(0, T; L^2(O)) \text{ as } n \to \infty.$$ 

Since embeddings in the Gelfand triple are compact, by Lions [13], for every $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\|u\|_{L^2(O)}^2 \leq \epsilon \|v\|_{H^0(O)}^2 + C_\epsilon \|v\|_{H^{-1}(O)}^2, \quad \forall v \in H^1_0(O).$$ 

Thus

$$\|u_n - u\|_{L^2(0, T; L^2(O))} \leq \epsilon \|u_n - u\|_{L^2(0, T; H^1(O))} + C_\epsilon \|u_n - u\|_{L^2(0, T; H^{-1}(O))}.$$ 

By condition (b)

$$\|u_n - u\|_{L^2(0, T; H^1(O))} \leq 2(\|u_n\|_{L^2(0, T; H^1(O))} + \|u\|_{L^2(0, T; H^1(O))}) \leq 4c,$$ 

where $c = \sup_{u \in K} \|u\|_{L^2(0, T; H^1(O))}$. Hence

$$\limsup_{n \to \infty} \|u_n - u\|_{L^2(0, T; L^2(O))} \leq 4\epsilon + \limsup_{n \to \infty} C_\epsilon \|u_n - u\|_{L^2(0, T; H^{-1}(O))}.$$ 

As $\epsilon > 0$ is arbitrary,

$$\lim_{n \to \infty} \|u_n - u\|_{L^2(0, T; L^2(O))} = 0.$$ 

\[\square\]

**Definition 5.4.** Let $(S, \rho)$ be a separable and complete metric space. Let $(\Omega, F, P)$ be a probability space with the filtration $F := (F_t)_{t \in [0, T]}$ satisfying the usual hypotheses, and let $(X_n)_{n \in \mathbb{N}}$ be a sequence of càdlàg, $F$-adapted and $\mathbb{S}$-valued processes. $(X_n)_{n \in \mathbb{N}}$ is said to satisfy the Aldous condition iff $\forall \epsilon > 0, \forall \eta > 0, \exists \delta > 0$ such that for every sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times with $\tau_n \leq T$

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq \theta \leq \delta} P\{\rho(X_n(\tau_n + \theta), X_n(\tau_n)) \geq \eta\} \leq \epsilon.$$ 

**Lemma 5.5.** Let $(X_n)$ satisfy the Aldous condition. Let $P_n$ be the law of $X_n$ on $D(0, T; S), n \in \mathbb{N}$. Then for every $\epsilon > 0$ there exists a subset $A_\epsilon \subset D(0, T; S)$ such that

$$\sup_{n \in \mathbb{N}} P_n(A_\epsilon) \geq 1 - \epsilon,$$

and

$$\lim_{\delta \to 0} \sup_{u \in A_\epsilon} w_{[0, T]}(u, \delta) = 0.$$
Lemma 5.6. Let \((E, \| \cdot \|_E)\) be a separable Banach space and let \((X_n)_{n \in \mathbb{N}}\) be a sequence of \(E\)-valued random variables. Assume that for every sequence \((\tau_n)_{n \in \mathbb{N}}\) of \(F\)-stopping times with \(\tau_n \leq T\) and for every \(n \in \mathbb{N}\) and \(\theta \geq 0\) the following condition holds
\[
\mathbb{E}[\|X_n(\tau_n + \theta) - X_n(\tau_n)\|_E^\alpha] \leq C\theta^\beta,
\]
for some \(\alpha, \beta > 0\) and some constant \(C > 0\). Then the sequence \((X_n)_{n \in \mathbb{N}}\) satisfies the Aldous condition in the space \(E\).

We use the tightness condition for the Prokhorov-Varadarajan theorem which states that a sequence of measures \((P^n)_{n \in \mathbb{N}}\) is tight on a topological space \(E\) if for every \(\epsilon > 0\) there exists a compact set \(K_\epsilon \subset E\) such that \(\sup_n P^n(E \setminus K_\epsilon) \leq \epsilon\). Hence the tightness of measure in \(Z\) is given by the following theorem.

Theorem 5.7. Let \((X_n)_{n \in \mathbb{N}}\) be a sequence of càdlàg \(F\)-adapted \(\mathbb{H}^{-1}(\mathcal{O})\)-valued processes such that
\[
\begin{align*}
&\text{(a) there exists a positive constant } C_1 \text{ such that } \\
&\quad \sup_{n \in \mathbb{N}} \mathbb{E}[\sup_{s \in [0,T]} \|X_n(s)\|_{L^2}] \leq C_1, \\
&\text{(b) there exists a positive constant } C_2 \text{ such that } \\
&\quad \sup_{n \in \mathbb{N}} \mathbb{E}[\int_0^T \|X_n(s)\|^2_{\mathbb{H}^1} ds] \leq C_2, \\
&\text{(c) } (X_n)_{n \in \mathbb{N}} \text{ satisfies the Aldous condition in } \mathbb{H}^{-1}(\mathcal{O}).
\end{align*}
\]
Let \(P_n\) be the law of \(X_n\) on \(Z\). Then for every \(\epsilon > 0\) there exists a compact subset \(K_\epsilon\) of \(Z\) such that
\[
P_n(K_\epsilon) \geq 1 - \epsilon, \quad (5.3)
\]
and the sequence of measures \(\{P_n, n \in \mathbb{N}\}\) is said to be tight on \((Z, \tau)\).

Proof. Let \(\epsilon > 0\). By the Chebyshev inequality and by (a), we see that for any \(r > 0\)
\[
P(\sup_{s \in [0,T]} \|X_n(s)\|_{L^2} > r) \leq \frac{\mathbb{E}[\sup_{s \in [0,T]} \|X_n(s)\|_{L^2}]}{r} \leq \frac{C_1}{r}.
\]
Let \(R_1\) be such that \(\frac{C_1}{R_1} \leq \frac{\epsilon}{3}\). Then
\[
P(\sup_{s \in [0,T]} \|X_n(s)\|_{L^2} > R_1) \leq \frac{\epsilon}{3}.
\]
Let \(B_1 := \{u \in Z : \sup_{s \in [0,T]} \|u(s)\|_{L^2} \leq R_1\}\).

By the Chebyshev inequality and by (b), we see that for any \(r > 0\)
\[
P(\|X_n\|_{L^2(0,T;\mathbb{H}^1)} > r) \leq \frac{\mathbb{E}[\|X_n\|^2_{L^2(0,T;\mathbb{H}^1)}]}{r^2} \leq \frac{C_2}{r^2}.
\]
Let $R_2$ be such that $\frac{C_1}{R_2} \leq \frac{\epsilon}{3}$. Then

$$P(\|X_n\|_{L^2(0,T;\mathbb{H}^1_0)} > R_2) \leq \frac{\epsilon}{3}.$$  

Let $B_2 := \{u \in \mathcal{Z} : \|u\|_{L^2(0,T;\mathbb{H}^1_0)} \leq R_2\}$.

By Lemma 5.5 there exists a subset $A_{\frac{\epsilon}{3}} \subset \mathcal{D}(0,T;\mathbb{H}^{-1})$ such that $P_n(A_{\frac{\epsilon}{3}}) \geq 1 - \frac{\epsilon}{3}$ and

$$\lim_{\delta \to 0} \sup_{u \in A_{\frac{\epsilon}{3}}} w_{[0,T]}(u,\delta) = 0.$$  

It is sufficient to define $K_\epsilon$ as the closure of the set $B_1 \cap B_2 \cap A_{\frac{\epsilon}{3}}$ in $\mathcal{Z}$. By Theorem 5.3, $K_\epsilon$ is compact in $\mathcal{Z}$.  

\[ \Box \]

5.2. Martingale Problem  

We now consider the stochastic tidal dynamics equation with Lévy forcing as defined in Section 3 with initial value control as

\[
du + [Au + B(u) + g\nabla \hat{z}]dt = f(t)dt + \sigma(t, u(t))dW(t) + \int_{Z} H(u, z)\tilde{N}(dt, dz),
\]

\[
\hat{z} + \text{Div}(hu) = 0, \tag{5.4}
\]

\[
u(0) = u_0 + U, \quad \hat{z}(0) = \hat{z}_0, \tag{5.5}
\]

\[
u(0) = u_0 + U, \quad \hat{z}(0) = \hat{z}_0, \tag{5.6}
\]

where $f \in L^2(0,T;\mathbb{L}^2(\mathcal{O})), \ u_0, U \in \mathbb{L}^2(\mathcal{O})$ and $\hat{z}_0 \in L^2(\mathcal{O})$. We also assume that $\mathbb{E}[\|u\|^2_{\mathbb{L}^2}] < C_c$.

We assume that $\sigma$ and $H$ satisfy the following hypotheses:

S.1 $\sigma \in C([0,T] \times \mathbb{H}^1_0(\mathcal{O}); L_Q(H_0, \mathbb{L}^2)), H \in \mathbb{H}^2_\lambda([0,T] \times \mathbb{Z}; \mathbb{L}^2(\mathcal{O}))$,

S.2 For all $t \in (0,T)$, there exists a positive constant $K$ such that for all $u \in \mathbb{L}^2(\mathcal{O})$

\[
\|\sigma(t, u)\|^2_{L^2_Q} + \int_{Z} \|H(u, z)\|^2_{L^2}\lambda(dz) \leq K(1 + \|u\|^2_{\mathbb{L}^2}),
\]

S.3 For all $t \in (0,T)$, there exists a positive constant $L$ such that for all $u, v \in \mathbb{L}^2(\mathcal{O})$

\[
\|\sigma(t, u) - \sigma(t, v)\|^2_{L^2_Q} + \int_{Z} \|H(u, z) - H(v, z)\|^2_{L^2}\lambda(dz) \leq L(\|u - v\|^2_{\mathbb{L}^2}).
\]

Definition 5.8. A martingale solution of (5.4)-(5.6) is a system $(\hat{\Omega}, \hat{\mathcal{F}}, F, P, \pi, \hat{\nu}, \hat{\sigma}, \hat{\mathcal{N}}, \hat{\mathbb{W}})$, where

- $(\hat{\Omega}, \hat{\mathcal{F}}, F, P)$ is a filtered probability space with a filtration $\hat{F} = \{\hat{F}_t\}_{t \geq 0}$,
- $\hat{\mathcal{N}}$ is a time homogeneous Poisson random measure over $(\hat{\Omega}, \hat{\mathcal{F}}, F, P)$ with the intensity measure $\lambda$,
- $\hat{\mathbb{W}}$ is a cylindrical Wiener process over $(\hat{\Omega}, \hat{\mathcal{F}}, F, P)$,
- For all $t \in [0,T]$

\[
\pi \in \mathcal{D}(0,T;\mathbb{H}^{-1}(\mathcal{O})) \cap L^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap L^2(0,T;\mathbb{H}^1_0(\mathcal{O})), \quad \nu \in L^2(0,T;\mathbb{L}^2(\mathcal{O})),
\]

\[
\hat{\nu} \in L^2(0,T;\mathcal{D}(0,T;\mathbb{H}^{-1}(\mathcal{O}))) \cap L^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap L^2(0,T;\mathbb{H}^1_0(\mathcal{O})),
\]

\[
\hat{\mathbb{W}} \in L^2(0,T;\mathbb{L}^2(\mathcal{O})).
\]
\textbf{Lemma 5.9.} The set of measures \( \mathcal{L}(u^n), n \in \mathbb{N} \) is tight on \((Z, \tau)\).

\textbf{Proof.} We prove the tightness of the measures using the tightness criterion given in Theorem 5.7. By Proposition 4.1, conditions (a) and (b) are satisfied. We use Lemma 5.6 to prove the Aldous condition for the sequence \( (u^n)_{n \in \mathbb{N}} \) in the space \( \mathbb{H}^{-1}(O) \). Let \( (\tau_n)_{n \in \mathbb{N}} \) be a sequence of stopping times where \( 0 \leq \tau_n \leq T \). Then using (5.9)

\[
\begin{align*}
\mathbf{5.3. } & \text{Existence of Martingale Solution} \\
& \text{We prove the existence of a martingale solution using the Galerkin approximations as explained in Section 4. Hence} \\
& u^n(t) = u^n_0 + U^n - \int_0^t (A(u^n(s)) + B(u^n(s)) + g \nabla \hat{z}^n(s) - f(s))ds \\
& + \int_0^t \sigma^n(s, u^n(s))dw^n(s) + \int_0^t \int_{\mathcal{Z}} H^n(u^n(s), z)\hat{N}(ds, dz). \quad (5.9) \\
& \hat{z}^n(t) = \hat{z}^n_0 - \int_0^t \text{Div}(hu^n(s))ds. \quad (5.10)
\end{align*}
\]

For each \( n \in \mathbb{N} \), we use the measure \( \mathcal{L}(u^n) \) defined on \((Z, \tau)\) by the solution \( u^n \) of the Galerkin equation.

\[
\text{Hence}
\]

\[
\begin{align*}
\mathbf{u^n(t) = u^n_0 + U^n - \int_0^t (A(u^n(s)) + B(u^n(s)) + g \nabla \hat{z}^n(s) - f(s))ds} \\
& + \int_0^t \sigma^n(s, u^n(s))dw^n(s) + \int_0^t \int_{\mathcal{Z}} H^n(u^n(s), z)\hat{N}(ds, dz) \\
& = J_1^n + J_2^n + J_3^n(t) + J_4^n(t) + J_5^n(t) + J_6^n(t) + J_7^n(t) + J_8^n(t), \quad t \in [0, T].
\end{align*}
\]

Let \( \theta > 0 \). We need to show that (5.2) holds for each \( J_i^n, \ i \in \{1, 2, \ldots, 6\} \). Since \( J_1^n \) and \( J_2^n \) are independent of time, they clearly satisfy (5.2).

Since \( A : \mathbb{H}^1_0(O) \to \mathbb{H}^{-1}(O) \), for all \( v \in \mathbb{H}^1_0(O) \)

\[
\langle A(u), v \rangle = (A(u), v)_{L^2} = a(\nabla u, \nabla v)_{L^2} \leq C_1 \| u \|_{\mathbb{H}^1_0} \| v \|_{\mathbb{H}^1_0}.
\]

Hence using the above inequality

\[
\| A(u) \|_{\mathbb{H}^{-1}} \leq C_1 \| u \|_{\mathbb{H}^1_0}.
\]
Therefore
\[ E[\|J^n_3(\tau_n + \theta) - J^n_3(\tau_n)\|_{\mathbb{H}^{-1}}] = E[\| \int_{\tau_n}^{\tau_n+\theta} A(u^n(s))ds \|_{\mathbb{H}^{-1}}] \]
\[ \leq C_1 E[\int_{\tau_n}^{\tau_n+\theta} \|u^n(s)\|_{\mathbb{H}^1} ds] \]
\[ \leq C_1 E[\theta^{1/2} \left( \int_0^T \|u^n(s)\|_{\mathbb{H}^1}^2 ds \right)^{1/2}] \]
\[ \leq c_2 \theta^{1/2}. \]

Thus \( J^n_3 \) satisfies (5.2) with \( \alpha = 1 \) and \( \beta = \frac{1}{2} \).

Since \( L^2(O) \hookrightarrow \mathbb{H}^{-1}(O) \), we get for \( J^n_4 \)
\[ E[\|J^n_4(\tau_n + \theta) - J^n_4(\tau_n)\|_{\mathbb{H}^{-1}}] \]
\[ = E[\| \int_{\tau_n}^{\tau_n+\theta} B(u^n(s))ds \|_{\mathbb{H}^{-1}}] \]
\[ \leq c E[\int_{\tau_n}^{\tau_n+\theta} \|B(u^n(s))\|_{L^2} ds] \]
\[ \leq c E[\int_{\tau_n}^{\tau_n+\theta} \|B(u^n(s))\|_{L^2} ds] \]
\[ \leq c C_2 E[\int_{\tau_n}^{\tau_n+\theta} \|u^n + w^0(s)\|_{L^4}^2 ds] \]
\[ \leq \sqrt{2} c C_2 E[\int_{\tau_n}^{\tau_n+\theta} \|u^n + w^0(s)\|_{L^2} \|u^n + w^0(s)\|_{\mathbb{H}^1} ds] \]
\[ \leq c_3 E[\left( \int_{\tau_n}^{\tau_n+\theta} \|u^n + w^0(s)\|_{L^2}^2 ds \right)^{1/2} \left( \int_{\tau_n}^{\tau_n+\theta} \|u^n + w^0(s)\|_{\mathbb{H}^1}^2 ds \right)^{1/2}] \]
\[ \leq c_3 E[\left( \sup_{0 \leq s \leq T} \|u^n + w^0(s)\|_{L^2}^2 \right)^{1/2} \left( \int_0^T \|u^n + w^0(s)\|_{\mathbb{H}^1}^2 ds \right)^{1/2}] \]
\[ \leq c_4 \theta^{1/2}. \]

Thus \( J^n_4 \) satisfies (5.2) with \( \alpha = 1 \) and \( \beta = \frac{1}{2} \).

Consider the operator \( C : L^2(O) \to \mathbb{H}^{-1}(O) \) given by
\[ C(\hat{z}) = g \nabla \hat{z}. \]

For all \( v \in \mathbb{H}^1_0(O) \)
\[ \langle C(\hat{z}), v \rangle = \langle C(\hat{z}), v \rangle_{L^2} \leq -g(\hat{z}, \text{Div}(v))_{L^2} \leq g\|\hat{z}\|_{L^2} \|v\|_{\mathbb{H}^1_0}. \]

Hence
\[ \|C(\hat{z})\|_{\mathbb{H}^{-1}} \leq g\|\hat{z}\|_{L^2}. \]
So we have
\[
\mathbb{E}[\|J^n_5(\tau_n + \theta) - J^n_5(\tau_n)\|_{\mathbb{H}^{-1}}] = \mathbb{E}[\int_{\tau_n}^{\tau_n + \theta} \|g\nabla \hat{z}^n(s)\|_{L^2} ds]
\leq g \mathbb{E}[\|\hat{z}^n(s)\|_{L^2}^{1/2} (T \sup_{0 \leq s \leq T} \|\hat{z}^n(s)\|_{L^2}^{1/2})]
\leq c_5 \theta^{1/2}.
\]
Thus \(J^n_5\) satisfies (5.2) with \(\alpha = 1\) and \(\beta = \frac{1}{2}\).

Since \(L^2(\mathcal{O}) \hookrightarrow \mathbb{H}^{-1}(\mathcal{O})\), by Hölder inequality, we have
\[
\mathbb{E}[\|J^n_6(\tau_n + \theta) - J^n_6(\tau_n)\|_{\mathbb{H}^{-1}}] = \mathbb{E}[\int_{\tau_n}^{\tau_n + \theta} \|f(s)\|_{L^2} ds]
\leq c \mathbb{E}[(\int_0^T \|f(s)\|_{L^2}^2 ds)^{1/2}]
= c_6 \theta^{1/2}.
\]
Thus \(J^n_6\) satisfies (5.2) with \(\alpha = 1\) and \(\beta = \frac{1}{2}\).

Since \(L^2(\mathcal{O}) \hookrightarrow \mathbb{H}^{-1}(\mathcal{O})\), using the hypothesis S.2 and the Itô isometry,
\[
\mathbb{E}[\|J^n_7(\tau_n + \theta) - J^n_7(\tau_n)\|_{\mathbb{H}^{-1}}^2] = \mathbb{E}[\int_{\tau_n}^{\tau_n + \theta} \|\sigma^n(s, u^n(s))dW^n(s)\|_{\mathbb{H}^{-1}}^2]
\leq c \mathbb{E}[\int_{\tau_n}^{\tau_n + \theta} \|\sigma^n(s, u^n(s))dW^n(s)\|_{L^2}^2]
= c \mathbb{E}[\int_{\tau_n}^{\tau_n + \theta} \|\sigma^n(s, u^n(s))\|_{L^2}^2 ds]
\leq cK \mathbb{E}[\int_{\tau_n}^{\tau_n + \theta} (1 + \|u^n(s)\|_{L^2}^2) ds]
\leq cK \theta (1 + \mathbb{E}[\sup_{0 \leq s \leq T} \|u^n(s)\|_{L^2}^2])
\leq c_6 \theta.
\]
Thus \(J^n_7\) satisfies (5.2) with \(\alpha = 2\) and \(\beta = 1\).

Since \(L^2(\mathcal{O}) \hookrightarrow \mathbb{H}^{-1}(\mathcal{O})\), using the hypothesis S.2, and the isometry (5.11)
\[
\mathbb{E}[\int_0^T \int_Z H^n(u^n(s-), z) \tilde{N}(ds, dz)] = \mathbb{E}[\int_0^T \int_Z H^n(u^n(s-), z) ||H^n||_{L^2} \lambda(dz) ds],
\]
(5.11)
we get
\[ \mathbb{E}[||J^n_8(\tau_n + \theta) - J^n_8(\tau_n)||^2_{\mathbb{H}^{-1}}] \leq c \mathbb{E}[\int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{R}^d} H^n(u^n(s-), z) \tilde{N}(ds, dz) \, d\lambda(z) \, ds] \]
\[ = cK \mathbb{E}[\int_{\tau_n}^{\tau_n+\theta} \int_{\mathbb{R}^d} ||H^n(u^n(s), z)||^2_{\mathbb{L}^2} \, d\lambda(z) \, ds] \]
\[ \leq cK \theta (1 + \mathbb{E}[\sup_{0 \leq s \leq T} ||u^n(s)||^2_{\mathbb{L}^2}]) \leq c_6 \theta. \]
Thus \( J^n_8 \) satisfies (5.2) with \( \alpha = 2 \) and \( \beta = 1. \)

Hence by Lemma 5.10, the sequence \((u^n)\) satisfies the Aldous condition in the space \( \mathbb{H}^{-1}(\mathcal{O}) \), which completes the proof. \( \square \)

**Lemma 5.10.** The set of measures \( \{ \mathcal{L}(\tilde{z}^n), n \in \mathbb{N} \} \) is tight on \( L^2(0, T; L^2(\mathcal{O}))_w \).

**Proof.** Using Proposition 4.1, \( \mathbb{E}[||\tilde{z}^n(t)||^2_{L^2}] \leq C_1 \), hence using Fubini’s theorem
\[ \mathbb{E}[||\tilde{z}^n||^2_{L^2(0, T; L^2)}] \leq C_1 T. \]
Using the Chebychev inequality, we see that for any \( r > 0 \)
\[ P(||\tilde{z}^n||_{L^2(0, T; L^2)} > r) \leq \frac{\mathbb{E}[||\tilde{z}^n||^2_{L^2(0, T; L^2)}]}{r^2} \leq \frac{C_1 T}{r^2}. \]
Let \( R_1 \) be such that \( C_1 T \frac{R_1}{R_1^2} \leq \epsilon \). Then
\[ P(||\tilde{z}^n||_{L^2(0, T; L^2)} > R_1) \leq \epsilon. \]
Define
\[ B_\epsilon = \{ \tilde{z} \in L^2(0, T; L^2(\mathcal{O})) : P(||\tilde{z}||_{L^2(0, T; L^2)} > R_1) \leq R_1 \}. \]
Hence \( P(B_\epsilon) \geq 1 - \epsilon. \) \( \square \)

**Lemma 5.11.** The set of measures \( \{ \mathcal{L}(\mathcal{U}^n), n \in \mathbb{N} \} \) is tight on \( L^2(0, T; L^2(\mathcal{O}))_w \).

**Proof.** Using the assumption \( \mathbb{E}[||U^n(t)||^2_{L^2}] \leq C_c \), hence using Fubini’s theorem
\[ \mathbb{E}[||U^n||^2_{L^2(0, T; L^2)}] \leq C_c T. \]
Using the Chebychev inequality, we see that for any \( r > 0 \)
\[ P(||U^n||_{L^2(0, T; L^2)} > r) \leq \frac{\mathbb{E}[||U^n||^2_{L^2(0, T; L^2)}]}{r^2} \leq \frac{C_c T}{r^2}. \]
Let \( R_1 \) be such that \( C_c T \frac{R_1}{R_1^2} \leq \epsilon \). Then
\[ P(||U^n||_{L^2(0, T; L^2)} > R_1) \leq \epsilon. \]
Define
\[ B_\epsilon = \{ U \in L^2(0, T; L^2(\Omega)) : P(\|U\|_{L^2(0, T; L^2)} \leq R_1) \}. \]
Hence \( P(B_\epsilon) \geq 1 - \epsilon. \)

**Theorem 5.12.** There exists a martingale solution of (5.4)-(5.6) provided the assumptions S.1-S.3 are satisfied.

**Proof.** By Lemma 5.9, the set of measures \( \{L(u^n), n \in \mathbb{N}\} \) is tight on the space \( (\mathcal{Z}, \tau) \), by Lemma 5.10, the set of measures \( \{L(\hat{z}^n), n \in \mathbb{N}\} \) is tight on the space \( L^2(0, T; L^2(\Omega))_w \), and by Lemma 5.11, the set of measures \( \{L(U^n), n \in \mathbb{N}\} \) is tight on the space \( L^2(0, T; L^2(\Omega))_w \). Define \( N^n = N, \forall n \in \mathbb{N} \). The set of measures \( \{L(N^n), n \in \mathbb{N}\} \) is tight. Define \( W^n = W, \forall n \in \mathbb{N} \). The set of measures \( \{L(W^n), n \in \mathbb{N}\} \) is tight. Thus the set \( \{L(u^n, \hat{z}^n, U^n, N^n, W^n), n \in \mathbb{N}\} \) is tight. By the Skorokhod Embedding theorem [14], there exists a subsequence \( (n_k)_{k \in \mathbb{N}} \), a probability space \( (\Omega, \mathcal{F}, P) \), and on this space random variables \( (u^*, z^*, U^*, N^*, W^*) \), \( (\overline{\pi}^k, \overline{z}^k, \overline{U}^k, \overline{N}^k, \overline{W}^k), k \in \mathbb{N} \) such that

1. \( L((\overline{\pi}^k, \overline{z}^k, \overline{U}^k, \overline{N}^k, \overline{W}^k)) = L((u^{nk}, \hat{z}^{nk}, U^{nk}, N^{nk}, W^{nk})) \) for all \( k \in \mathbb{N} \),
2. \( (\overline{\pi}^k, \overline{z}^k, \overline{U}^k, \overline{N}^k, \overline{W}^k) \) \( \rightarrow \) \( (u^*, z^*, U^*, N^*, W^*) \) with probability 1 on \( (\Omega, \mathcal{F}, P) \) as \( k \rightarrow \infty \),
3. \( (\overline{N}^k(\overline{\pi}), \overline{W}^k(\overline{\pi})) = (N^*(\overline{\pi}), W^*(\overline{\pi})) \) for all \( \overline{\pi} \in \overline{\Omega} \).

We denote these sequences again by \( ((u^n, \hat{z}^n, U^n, N^n, W^n))_{n \in \mathbb{N}} \) and \( ((\overline{\pi}^n, \overline{z}^n, \overline{U}^n, \overline{N}^n, \overline{W}^n))_{n \in \mathbb{N}} \). Using the definition of the space \( \mathcal{Z} \), we have \( P - a.s. \)

\[ \pi^n \rightarrow u^* \text{ in } L^2(0, T; H_0^1(\Omega)) \cap L^2(0, T; L^2(\mathcal{D}(0, T; H^{-1}(\Omega)) \cap L^\infty(0, T; L^2(\mathcal{O}))_w), \]
\[ \overline{z}^n \rightarrow z^* \text{ in } L^2(0, T; L^2(\mathcal{O}))_w, \]
and
\[ \overline{U}^n \rightarrow U^* \text{ in } L^2(0, T; L^2(\mathcal{O}))_w \ (5.14) \]
Since the random variables \( \pi^n \) and \( u^n \) are identically distributed, we have
\[ \sup_{n \geq 1} \mathbb{E}[ \sup_{0 \leq s \leq T} \|\pi^n(s)\|_{L^2}^2 ] \leq C_{1(2)}, \]
and
\[ \sup_{n \geq 1} \mathbb{E}[ \int_0^T \|\pi^n(s)\|_{L^2}^2 ds ] \leq C_{2(2)}. \]
Since the random variables \( \overline{z}^n \) and \( \hat{z}^n \) are also identically distributed, we have
\[ \sup_{n \geq 1} \mathbb{E}[ \sup_{0 \leq s \leq T} \|\overline{z}^n(s)\|_{L^2}^2 ] \leq C_{1(2)}. \]
Using the assumption \( \mathbb{E}[\|U\|_{L^2}^2] < C_c \), we have
\[ \sup_{n \geq 1} \mathbb{E}[ \sup_{0 \leq s \leq T} \|\overline{U}^n(s)\|_{L^2}^2 ] \leq C_c. \]
Define for all \( v \in \mathbb{H}^1_0(\Omega) \)

\[
K^1(\bar{u}^n, \bar{z}^n, U^n, N^n, \bar{W}^n, v)(t) = (\bar{u}_0^n, v)_{L^2} + (U^n, v)_{L^2} + \int_0^t (A(\bar{u}^n(s)), v)_{L^2} ds
+ \int_0^t (B(\bar{u}^n(s)), v)_{L^2} ds - \int_0^t (\nabla \bar{z}^n(s), v)_{L^2} ds
+ \int_0^t (\sigma^n(s, \bar{u}^n(s)), v)_{L^2} d\bar{W}^n(s)
+ \int_0^t \int_Z (H^n(\bar{u}^n(s), z), v)_{L^2} \bar{N}^n(ds, dz),
\]

and for all \( w \in L^2(\Omega) \)

\[
K^2(\bar{u}^n, \bar{z}^n, w)(t) = (\bar{z}_0^n, w)_{L^2} - \int_0^t (\text{Div}(h\bar{u}^n(s)), w)_{L^2} ds.
\]

Hence for all \( v \in \mathbb{H}^1_0(\Omega) \)

\[
K^1(u^*, z^*, U^*, N^*, W^*, v)(t) = (u_0^*, v)_{L^2} + (U^*, v)_{L^2} + \int_0^t (A(u^*(s)), v)_{L^2} ds
+ \int_0^t (B(u^*(s)), v)_{L^2} ds - \int_0^t (\nabla z^*(s), v)_{L^2} ds
+ \int_0^t (\sigma(s, u^*(s)), v)_{L^2} dW^*(s)
+ \int_0^t \int_Z (H(u^*(s), z), v)_{L^2} \bar{N}^*(ds, dz),
\]

and for all \( w \in L^2(\Omega) \)

\[
K^2(u^*, z^*, w)(t) = (z_0^*, w)_{L^2} - \int_0^t (\text{Div}(hu^*(s)), w)_{L^2} ds.
\]

We need to prove that for all \( v \in \mathbb{H}^1_0(\Omega) \)

\[
\lim_{n \to \infty} \|(\bar{u}^n, v)_{L^2} - (u^*, v)_{L^2}\|_{L^2(0, T \times \Omega)} = 0,
\]  

(5.21)

and

\[
\lim_{n \to \infty} \|K^1(\bar{u}^n, \bar{z}^n, U^n, N^n, \bar{W}^n, v) - K^1(u^*, z^*, U^*, N^*, W^*, v)\|_{L^2(0, T \times \Omega)} = 0.
\]  

(5.22)

We also need to show that for all \( w \in L^2(\Omega) \)

\[
\lim_{n \to \infty} \|(\bar{z}^n, w)_{L^2} - (z^*, v)_{L^2}\|_{L^2(0, T \times \Omega)} = 0,
\]  

(5.23)

and

\[
\lim_{n \to \infty} \|K^2(\bar{u}^n, \bar{z}^n, w) - K^2(u^*, z^*, w)\|_{L^2(0, T \times \Omega)} = 0.
\]  

(5.24)

To prove (5.21) we see that

\[
\lim_{n \to \infty} \int_0^T |(\bar{u}^n(t) - u^*(t), v)_{L^2}|^2 dt \leq \|v\|_{L^2}^2 \lim_{n \to \infty} \int_0^T \|\bar{u}^n(t) - u^*(t)\|_{L^2}^2 dt.
\]
Since $\pi^n \to u^*$ in $L^2(0, T; \mathbb{L}^2(\mathcal{O}))$,
\[
\lim_{n \to \infty} \int_0^T |(\pi^n(t) - u^*(t), v)_{\mathbb{L}^2}|^2 dt = 0. \tag{5.25}
\]
Since both $\pi^n$ and $u^*$ satisfy the inequality (5.15), by (5.25) and the Vitali theorem
\[
\lim_{n \to \infty} \| (\pi^n, v)_{\mathbb{L}^2} - (u^*, v)_{\mathbb{L}^2} \|_{L^2(0, T \times \overline{\Omega})} = \lim_{n \to \infty} \mathbb{E} \int_0^T |(\pi^n(t) - u^*(t), v)_{\mathbb{L}^2}|^2 dt
\]
\[
= 0.
\]
To prove (5.22), we see by Fubini’s theorem
\[
\| K^1(\pi^n, z^*, U^*, N^*, W^*, v) - K^1(u^*, z^*, U^*, N^*, W^*, v) \|_{L^2(0, T \times \overline{\Omega})}
\]
\[
= \mathbb{E} \int_0^T |K^1(\pi^n, z^*, U^n, N^n, W^n, v) - K^1(u^*, z^*, U^n, N^n, W^n, v)|^2 dt
\]
\[
= \int_0^T \mathbb{E} [\|K^1(\pi^n, z^*, U^n, N^n, W^n, v) - K^1(u^*, z^*, U^n, N^n, W^n, v)\|^2] dt.
\]
We show the term by term convergence of the above equation in $L^2(0, T \times \overline{\Omega})$.

Since $\pi^n \to u^*$ in $L^\infty(0, T; \mathbb{L}^2(\mathcal{O}))_{w^*}$, we have $\mathcal{F}$-a.s.
\[
(\pi^n(0), v)_{\mathbb{L}^2} \to (u^*(0), v)_{\mathbb{L}^2}.
\]
Hence by (5.15) and the Vitali theorem
\[
\lim_{n \to \infty} \mathbb{E} [(\pi^n(0) - u^*(0), v)_{\mathbb{L}^2}] = 0.
\]
Hence
\[
\lim_{n \to \infty} \| (\pi^n(0) - u^*(0), v)_{\mathbb{L}^2} \|^2_{L^2(0, T \times \overline{\Omega})} = 0. \tag{5.26}
\]
Since $U^n \to U^*$ in $L^2(\mathcal{O})_{w}$, we have $\mathcal{F}$-a.s.
\[
(U^n, v)_{\mathbb{L}^2} \to (U^*, v)_{\mathbb{L}^2}.
\]
Since $\|U^n\|_{\mathbb{L}^2}^2 < \infty$, by the Vitali theorem
\[
\lim_{n \to \infty} \mathbb{E} [\|U^n - U^*\|^2_{\mathbb{L}^2}] = 0.
\]
Hence
\[
\lim_{n \to \infty} \| (U^n - U^*, v)_{\mathbb{L}^2} \|^2_{L^2(0, T \times \overline{\Omega})} = 0. \tag{5.27}
\]
Since $\pi^n \to u^*$ in $L^2(0, T; \mathbb{H}^1_0(\mathcal{O}))_{w}$,
\[
\lim_{n \to \infty} \int_0^t \langle A(\pi^n(s)), v \rangle_{\mathbb{L}^2} ds = \lim_{n \to \infty} \int_0^t \langle \pi^n(s), v \rangle_{\mathbb{H}^1_0} ds
\]
\[
= \int_0^t \langle u^*(s), v \rangle_{\mathbb{H}^1_0} ds
\]
\[
= \int_0^t \langle A(u^*(s)), v \rangle_{\mathbb{L}^2} ds.
\]
By (5.16)
\[
\mathbb{E}[\int_0^t (A(\overline{u}^n(s)), v)_{L^2} ds]\leq \mathbb{E}[\int_0^t |(\overline{u}^n(s), v)_{H^1_0}|^2 ds]
\leq \|v\|_{L^2}^2 \mathbb{E}[\int_0^t \|\overline{u}^n(s)\|_{H^1_0}^2 ds]
< \infty.
\]

Therefore by Vitali theorem, for all \( t \in [0, T] \)
\[
\lim_{n \to \infty} \mathbb{E}|\int_0^t (A(\overline{u}^n(s)) - A(u^*(s)), v)_{L^2} ds|^2 = 0.
\]
Hence by the dominated convergence theorem
\[
\lim_{n \to \infty} \int_0^T \mathbb{E}[\int_0^t (A(\overline{u}^n(s)) - A(u^*(s)), v)_{L^2} ds|^2] = 0.
\]
Since \( \overline{u}^n \to u^* \) in \( L^2(0, T; L^2(\mathcal{O})) \), we have by the continuity of the non-linear term \( B(\cdot) \) given by (3.12)
\[
\lim_{n \to \infty} \int_0^t (B(\overline{u}^n(s)) - B(u^*(s)), v)_{L^2} ds \leq \lim_{n \to \infty} \int_0^t \|B(\overline{u}^n(s)) - B(u^*(s))\|_{L^2} \|v\|_{L^2} ds \leq 0.
\]
By (5.15) and (5.16)
\[
\mathbb{E}[\int_0^t (B(\overline{u}^n(s)), v)_{L^2} ds] \leq \|v\|_{L^2} \mathbb{E}[\int_0^t \|B(\overline{u}^n(s))\|_{L^2} ds] \leq C_2 \|v\|_{L^2} \mathbb{E}[\int_0^t \|\overline{u}^n(s) + w^0(s)\|_{L^2}^2 ds] \leq \sqrt{2} C_2 \|v\|_{L^2} \mathbb{E}[\int_0^t \|\overline{u}^n(s) + w^0(s)\|_{L^2} \|\overline{u}^n(s) + w^0(s)\|_{H^1_0} ds] \leq \sqrt{2} C_2 \|v\|_{L^2} \mathbb{E}[\left( \int_0^t (\overline{u}^n(s) + w^0(s))^2_{L^2} ds \right)^{1/2} \left( \int_0^t \|\overline{u}^n(s) + w^0(s)\|_{H^1_0}^2 ds \right)^{1/2}] \leq \sqrt{2} C_2 \|v\|_{L^2} \mathbb{E}[\left( t \sup_{0 \leq t \leq T} \|\overline{u}^n(s) + w^0(s)\|_{L^2}^2 \right)^{1/2} \left( \int_0^t \|\overline{u}^n(s) + w^0(s)\|_{H^1_0}^2 ds \right)^{1/2}] < \infty.
\]
Hence by Vitali theorem, for all \( t \in [0, T] \)
\[
\lim_{n \to \infty} \mathbb{E}[\int_0^t (B(\overline{u}^n(s)) - B(u^*(s)), v)_{L^2} ds|^2] = 0.
\]
Hence by the dominated convergence theorem
\[
\lim_{n \to \infty} \int_0^T \mathbb{E}[\int_0^t (B(\overline{u}^n(s)) - B(u^*(s)), v)_{L^2} ds|^2] dt = 0.
\]
Since $\mathfrak{x}^n \to z^*$ in $L^2(0; L^2(O))$, we have for all $t \in [0, T]$
\[
\lim_{n \to \infty} \int_0^t (g(\mathfrak{x}^n(s) - z^*(s)), v)_{L^2} ds \leq \lim_{n \to \infty} \int_0^t (g(\mathfrak{x}^n(s) - z^*(s)), \nabla v)_{L^2} |ds| \leq 0.
\]

By (5.17)
\[
\mathbb{E}[\|\int_0^t (g(\mathfrak{x}^n(s), v)_{L^2} ds|] \leq g\mathbb{E}[\int_0^t (|\mathfrak{x}^n(s), \nabla v)_{L^2}|^2 ds]
\]
\[
\leq g\|v\|_{H^1_0}^2 \mathbb{E}[\int_0^t (|\mathfrak{x}^n|_{L^2})^2 ds]
\]
\[
\leq g\|v\|_{H^1_0}^2 \mathbb{E}[\sup_{0 \leq s \leq T} (|\mathfrak{x}^n|_{L^2})^2] < \infty.
\]

Hence by Vitali theorem, for all $t \in [0, T]$
\[
\lim_{n \to \infty} \mathbb{E}[\|\int_0^t (g(\mathfrak{x}^n(s) - z^*(s)), v)_{L^2} ds|] = 0.
\]

Hence by the dominated convergence theorem
\[
\lim_{n \to \infty} \int_0^T \mathbb{E}[\|\int_0^t (g(\mathfrak{x}^n(s) - z^*(s)), v)_{L^2} ds|] dt = 0.
\]

Let $v \in H^1_0(O)$. Then using the hypothesis S.3 and that $\mathfrak{x}^n \to u^*$ in $L^2(0; L^2(O))$,
\[
\lim_{n \to \infty} \int_0^t (|\sigma^n(s, \mathfrak{x}^n(s)) - \sigma(s, u^*(s)), v)_{L^2} ds
\]
\[
\leq \|v\|_{L^2}^2 \int_0^t (|\sigma^n(s, \mathfrak{x}^n(s)) - \sigma(s, u^*(s))|_{L^2} ds
\]
\[
\leq L\|v\|_{L^2}^2 \int_0^t (|\mathfrak{x}^n(s) - u^*(s)|_{L^2}^2 ds
\]
\[
= 0.
\]

Using hypothesis S.2 and (5.15)
\[
\mathbb{E}[\|\int_0^t (|\sigma^n(s, \mathfrak{x}^n(s)) - \sigma(s, u^*(s)), v)_{L^2} ds|]
\]
\[
\leq \|v\|_{L^2}^2 \mathbb{E}[\int_0^t (|\sigma^n(s, \mathfrak{x}^n(s))|_{L^2}^2 + |\sigma(s, u^*(s))|_{L^2}^2) ds]
\]
\[
\leq K\|v\|_{L^2}^2 \mathbb{E}[\int_0^t (2 + |\mathfrak{x}^n(s)|_{L^2}^2 + |u^*(s)|_{L^2}^2) ds]
\]
\[
\leq KT\|v\|_{L^2}^2 \mathbb{E}[\int_0^t (2 + \sup_{0 \leq s \leq T} (|\mathfrak{x}^n(s)|_{L^2}^2 + \sup_{0 \leq s \leq T} |u^*(s)|_{L^2}^2)] < \infty.
\]

Thus by Vitali’s theorem
\[
\lim_{n \to \infty} \mathbb{E}[\|\int_0^t (|\sigma^n(s, \mathfrak{x}^n(s)) - \sigma(s, u^*(s)), v)_{L^2} ds|] = 0. \quad (5.28)
\]
Using the Itô isometry and $\overline{W}^n = W^*$,
\[
\mathbb{E}[|\int_0^t (\sigma^n(s, \overline{w}^n(s)) - \sigma(s, u^*(s)))dW^*(s), v)_{L^2}|^2] = \mathbb{E}[\int_0^t |(\sigma^n(s, \overline{w}^n(s)) - \sigma(s, u^*(s)), v)_{L^2}|^2 ds].
\]
Hence by the dominated convergence theorem
\[
\lim_{n \to \infty} \int_0^T \mathbb{E}[|\int_0^t (\sigma^n(s, \overline{w}^n(s)) - \sigma(s, u^*(s)))dW^*(s), v)_{L^2}|^2]dt = 0. \quad (5.29)
\]
Let $v \in \mathbb{H}_0^1(\mathcal{O})$. Then using the hypothesis S.3 and that $\overline{w}^n \to u^*$ in $L^2(0, T; \mathbb{L}^2(\mathcal{O}))$,
\[
\lim_{n \to \infty} \int_0^t \int_0^T |(H^n(\overline{w}^n(-), z) - H(u^*(s, z), v)_{L^2}|^2 \lambda(dz)ds
\leq \lim_{n \to \infty} \|v\|_{L^2}^2 \int_0^t \int_0^T |(H^n(\overline{w}^n(-), z) - H(u^*(s, z), v)_{L^2}|^2 \lambda(dz)ds
\leq \lim_{n \to \infty} L\|v\|_{L^2}^2 \int_0^T \|\overline{w}^n(s) - u^*(s)\|^2_{L^2} ds
= 0.
\]
Using hypothesis S.2 and (5.15)
\[
\mathbb{E}[|\int_0^t \int_0^T |(H^n(\overline{w}^n(-), z) - H(u^*(s, z), v)_{L^2}|^2 \lambda(dz)ds] \leq \|v\|^2_{L^2} \mathbb{E}[\int_0^T (2 + \|\overline{w}^n(s)\|^2_{L^2} + \|u^*(s)\|^2_{L^2})ds]
\leq KT\|v\|^2_{L^2}(2 + \sup_{0 \leq s \leq T} \|\overline{w}^n(s)\|^2_{L^2} + \sup_{0 \leq s \leq T} \|u^*(s)\|^2_{L^2})]
< \infty.
\]
Thus by Vitali’s theorem
\[
\lim_{n \to \infty} \mathbb{E}[\int_0^t \int_0^T |(H^n(\overline{w}^n(-), z) - H(u^*(s, z), v)_{L^2}|^2 \lambda(dz)ds] = 0. \quad (5.30)
\]
Using the isometry (5.11) and $\overline{N}^n = N^*$,
\[
\mathbb{E}[\int_0^t \int_0^T |(H^n(\overline{w}^n(-), z) - H(u^*(s, z), v)_{L^2} \overline{N}^*(ds, dz)|^2] = \mathbb{E}[\int_0^t \int_0^T |(H^n(\overline{w}^n(-), z) - H(u^*(s, z), v)_{L^2}|^2 \lambda(dz)ds].
\]
Hence by the dominated convergence theorem
\[
\lim_{n \to \infty} \int_0^T \mathbb{E}[\int_0^t \int_0^T |(H^n(\overline{w}^n(-), z) - H(u^*(s, z), v)_{L^2} \overline{N}^*(ds, dz)|^2]dt = 0. \quad (5.31)
\]
To prove (5.23) we see that since $\bar{z}^n \to z^*$ in $L^2(0, T; L^2(\mathcal{O}))$, 
\[
\lim_{n \to \infty} \int_0^T |(\bar{u}^n(t) - u^*(t), w)_{L^2}|^2 dt = 0. \tag{5.32}
\]
Since both $\bar{z}^n$ and $z^*$ satisfy the inequality (5.17), by the Vitali theorem 
\[
\lim_{n \to \infty} \|\bar{z}^n - z^*, w\|_{L^2(0, T \times \bar{\Omega})} = \lim_{n \to \infty} \mathbb{E}\left[\int_0^T |(\bar{z}^n(t) - z^*(t), w)_{L^2}|^2 dt\right] = 0.
\]
To prove (5.24), we see by Fubini's theorem 
\[
\|K^2(\bar{u}^n, \bar{z}^n, w) - K^2(u^*, z^*, w)\|_{L^2(0, T \times \bar{\Omega})} 
= \mathbb{E}\left[\int_0^T |K^2(\bar{u}^n, \bar{z}^n, w) - K^2(u^*, z^*, w)|^2 dt\right] 
= \int_0^T \mathbb{E}[|K^2(\bar{u}^n, \bar{z}^n, w) - K^2(u^*, z^*, w)|^2] dt.
\]
We show the term by term convergence of the above equation in $L^2(0, T \times \bar{\Omega})$.

Since $\bar{z}^n \to z^*$ in $L^\infty(0, T; L^2(\mathcal{O}))$, we have $\bar{\mathcal{P}}$-a.s.  
\[
(\bar{z}^n_0, w)_{L^2} \to (z^*_0, w)_{L^2}.
\]
Hence by (5.15) and the Vitali theorem 
\[
\lim_{n \to \infty} \mathbb{E}[|\bar{z}^n_0 - z^*_0, w\|_{L^2}] = 0.
\]
Hence 
\[
\lim_{n \to \infty} \|\bar{z}^n_0 - z^*_0, w\|_{L^2(0, T \times \bar{\Omega})}^2 = 0. \tag{5.33}
\]
Since $\bar{u}^n \to u^*$ in $L^\infty(0, T; \mathbb{H}^1_0(\mathcal{O}))$, 
\[
\lim_{n \to \infty} \int_0^t (\text{Div}(h\bar{u}^n(s)), w)_{L^2} ds = \int_0^t (\text{Div}(hu^*(s)), w)_{L^2} ds.
\]
By (5.16) 
\[
\mathbb{E}[|\int_0^t (\text{Div}(h\bar{u}^n(s)), w)_{L^2} ds|^2] \leq \|w\|_{L^2}^2 \mathbb{E}\left[\int_0^t \|\bar{u}^n(s)\|_{\mathbb{H}^1}^2 ds\right] < \infty.
\]
Therefore by Vitali theorem, for all $t \in [0, T]$ 
\[
\lim_{n \to \infty} \mathbb{E}[|\int_0^t (\text{Div}(h\bar{u}^n(s)), w)_{L^2} - (\text{Div}(hu^*(s)), w)_{L^2} ds|^2] = 0.
\]
Hence by the dominated convergence theorem 
\[
\lim_{n \to \infty} \mathbb{E}\left[\int_0^t (\text{Div}(h\bar{u}^n(s)), w)_{L^2} - (\text{Div}(hu^*(s)), w)_{L^2} ds|^2\right] = 0.
\]
Since $u^n$ is a solution of the Galerkin equation, we have for all $t \in [0, T]$ 
\[
(u^n(t), v)_{L^2} = K^1(u^n, \hat{z}^n, U^n, N^n, W^n, v)(t) \quad \text{P-a.s.}
\]
Hence
\[ \int_0^T \mathbb{E}[|(u^n(t), v)\|_2^2 - K^1(u^n, \hat{z}^n, U^n, N^n, W^n, v)(t)|^2]dt = 0. \]

Since \( \mathcal{L}((\pi^n, \pi^n, U^n, N^n, W^n)) = \mathcal{L}((u^n, \hat{z}^n, U^n, N^n, W^n)) \)
\[ \int_0^T \mathbb{E}[|(\pi^n(t), v)\|_2^2 - K^1(\pi^n, \hat{z}^n, U^n, N^n, W^n, v)(t)|^2]dt = 0. \]

Using (5.21) and (5.22)
\[ \int_0^T \mathbb{E}[|(u^*(t), v)\|_2^2 - K^1(u^*, z^*, U^*, N^*, W^*, v)(t)|^2]dt \]
\[ \leq \int_0^T \mathbb{E}[|(u^*(t), v)\|_2^2 - (\pi^n(t), v)\|_2^2]dt \]
\[ + \int_0^T \mathbb{E}[|(\pi^n(t), v)\|_2^2 - K^1(\pi^n, \hat{z}^n, U^n, N^n, W^n, v)(t)|^2]dt \]
\[ + \int_0^T \mathbb{E}[K^1(\pi^n, \hat{z}^n, U^n, N^n, W^n, v)(t) - K^1(u^*, z^*, U^*, N^*, W^*, v)(t)|^2]dt = 0. \]

Hence for all \( t \in [0, T] \)
\[ (u^*(t), v)\|_2^2 - K^1(u^*, z^*, U^*, N^*, W^*, v)(t) = 0 \quad P\text{-a.s..} \]

Similarly we get for all \( t \in [0, T] \)
\[ (z^*(t), w)\|_2^2 - K^2(u^*, z^*, w)(t) = 0 \quad P\text{-a.s..} \]

Taking \( \pi = u^*, \pi = z^*, U = U^*, N = N^* \) and \( W = W^* \). we see that 
\((\Omega, \mathcal{F}, T, F, \pi, z, U, N, W)\) is a martingale solution of (5.4)-(5.6). \( \Box \)

The pathwise uniqueness of the martingale solution is obvious from proof of uniqueness in Theorem 4.4.

5.4. Existence of Optimal Control

We define the cost functional as
\[ F(u, U) = \mathbb{E} \left[ \int_0^T \int_\mathcal{O} L(t, u, U)dxdt \right], \quad (5.34) \]
where \( L(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{H}_0^1(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O}) \to \mathbb{R} \) is such that

1. \( L(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{H}_0^1(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O}) \) is measurable,
2. \( L(t, \cdot, \cdot) : \mathbb{H}_0^1(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O}) \) is lower semicontinuous \( \forall t \in [0, T] \) where
\( \mathbb{H}_0^1(\mathcal{O})_w \) is the space \( \mathbb{H}_0^1(\mathcal{O}) \) endowed with the weak topology, and
3. \( L(t, u, U) \geq k(U), \forall (t, u, U) \in [0, T] \times \mathbb{H}_0^1(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O}) \) where \( k(U) \) is non-negative and
\[ \int_0^T \int_\mathcal{O} k(U)dxdt \to \infty \text{ as } ||U||_{\mathbb{L}^2} \to \infty. \quad (5.35) \]
The task here is to find the optimal control $U$ which minimizes the cost functional $J(u, U)$.

**Lemma 5.13.** For $u^n \to u$ in the $\tau$-topology and $U^n \to U$ in $L^2(0, T; \mathbb{L}^2(\mathcal{O}))$-weak

\[
\liminf_{n \to \infty} \int_0^T \int_{\mathcal{O}} L(t, u^n, U^n) dx dt \geq \int_0^T \int_{\mathcal{O}} L(t, u, U) dx dt. \tag{5.36}
\]

**Proof.** Let $L_N(t, u, U) = L(t, u, U) \wedge N$. Then

\[
\liminf_{n \to \infty} \int_0^T \int_{\mathcal{O}} L_N(t, u^n, U^n) dx dt \geq \liminf_{n \to \infty} \int_0^T \int_{\mathcal{O}} L_N(t, u, U^n) dx dt \\
\geq -\limsup_{n \to \infty} \int_0^T \int_{\mathcal{O}} (L_N(t, u^n, U^n) - L_N(t, u, U^n))^- dx dt \\
+ \liminf_{n \to \infty} \int_0^T \int_{\mathcal{O}} L_N(t, u, U^n) dx dt, \tag{5.37}
\]

where $f(x)^- = (-f(x)) \wedge 0$. The first integral on the right-hand side is zero due to the lemma given below. Due to the lower semicontinuity of $L_N$, we have

\[
\liminf_{n \to \infty} \int_0^T \int_{\mathcal{O}} L_N(t, u^n, U^n) dx dt \geq \int_0^T \int_{\mathcal{O}} L_N(t, u, U) dx dt. \tag{5.38}
\]

So

\[
\liminf_{n \to \infty} \int_0^T \int_{\mathcal{O}} L(t, u^n, U^n) dx dt \geq \int_0^T \int_{\mathcal{O}} L_N(t, u, U) dx dt. \tag{5.39}
\]

Now using the Beppo-Levi theorem on the bounded measurable functions $L_N$ we get the required semicontinuity. $\Box$

**Lemma 5.14.** Let $u^n \to u$ in the $\tau$-topology and $U^n \to U$ in $L^2(0, T; \mathbb{L}^2(\mathcal{O}))$-weak. Let $\varphi(\cdot, \cdot, \cdot) : [0, T] \times H^1_0(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O}) \to \mathbb{R}_+$ be a bounded measurable function such that, $\forall t \in (0, T), \varphi(t, \cdot, \cdot, \cdot) : H^1_0(\mathcal{O}) \times \mathbb{L}^2(\mathcal{O}) \to \mathbb{R}_+$ is lower semicontinuous. Then

\[
\lim_{n \to \infty} \int_0^T \int_{\mathcal{O}} (\varphi(t, u^n, U^n) - \varphi(t, u, U^n))^- dx dt = 0. \tag{5.40}
\]

**Proof.** Define $\Theta(t, z, U) := \varphi(t, z, U) - \varphi(t, u, U)$. For $\gamma > 0$ and $y \in H^{-1}(\mathcal{O})$, define

\[
Y^m := \left\{ (t, U) \in [0, T] \times \mathbb{L}^2(\mathcal{O}); \frac{\inf_{\langle y, u \rangle - \langle y, z \rangle \leq 1/m} \Theta(t, z, U) \leq -\gamma} \right\}, \tag{5.41}
\]

and

\[
Y^m_n := \left\{ (t, U^n) \in [0, T] \times \mathbb{L}^2(\mathcal{O}); \frac{\inf_{\langle y, u \rangle - \langle y, z \rangle \leq 1/m} \Theta(t, z, U) \leq -\gamma} \right\}. \tag{5.42}
\]
Clearly, $Y^{m+1} \subseteq Y^m$.

As $u^n \to u$, we have

$$\liminf_{n \to \infty} \Theta(t, u^n, U) \geq 0. \quad (5.43)$$

Hence

$$\cap_m Y^m = \emptyset. \quad (5.44)$$

The lower semicontinuity of $\varphi$ implies that each $t$-section of $Y^m$ is closed. Hence we have

$$\lim_{n \to \infty} \limsup_{n \to \infty} \int_0^T \int_{Y^m_n} 1 \, dx \, dt \leq \int_0^T \int_{\cap_m Y^m} 1 \, dx \, dt = \int_0^T \int_0 1 \, dx \, dt = 0. \quad (5.45)$$

Define

$$\hat{Y}^n := \{(t, U); \Theta(t, u^n(t), U) > -\gamma\} = \{(t, U); \Theta(t, u^n(t), U) < -\gamma\}. \quad (5.46)$$

Since $u^n \to u$ in $L^2(0, T; \mathbb{H}_0^1(\mathcal{O}))$, we have for large enough $n$

$$\int_0^T \int_{\hat{Y}^n} 1 \, dx \, dt \leq \int_0^T \int_{Y^m} 1 \, dx \, dt. \quad (5.47)$$

Thus,

$$\limsup_{n \to \infty} \int_0^T \int_{\hat{Y}^n} 1 \, dx \, dt. \quad (5.48)$$

Therefore by \[10\]

$$\lim_{n \to \infty} \int_0^T \int_{\hat{Y}^n} \Theta(t, u^n, U^n) 1 \, dx \, dt = 0, \quad (5.49)$$

which proves (5.40). \[\square\]

**Theorem 5.15.** Given $u_0 \in \mathbb{L}^2(\mathcal{O})$, there exists a pair

$$(\hat{u}, \hat{U}) \in (L^2(0, T; \mathbb{H}_0^1(\mathcal{O})) \cap L^\infty(0, T; L^2(\mathcal{O}) \cap \mathcal{D}(0, T; \mathbb{H}^{-1}(\mathcal{O})))) \times L^2(0, T; \mathbb{H}^{-1}(\mathcal{O}))) \times L^2(0, T; \mathbb{L}^2(\mathcal{O})),$$

which gives a martingale solution to equations (5.4) - (5.6), and

$$J(\hat{u}, \hat{U}) = \min\{J(u, U); (u, U) \in (L^2(0, T; \mathbb{H}_0^1(\mathcal{O})) \cap L^\infty(0, T; L^2(\mathcal{O})) \cap \mathcal{D}(0, T; \mathbb{H}^{-1}(\mathcal{O}))) \times L^2(0, T; \mathbb{L}^2(\mathcal{O})),$$

where the pair $(u, U)$ gives a martingale solution to equations (5.4) - (5.6).
Proof. We restrict ourselves to admissible pairs \((u,U) \in (L^2(0,T;\mathbb{H}_0^1(\mathcal{O})) \cap L^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathcal{D}(0,T;\mathbb{H}^{-1}(\mathcal{O}))) \times L^2(0,T;\mathbb{L}^2(\mathcal{O}))\) which satisfy (5.4)-(5.6) in the martingale sense such that \(F(u,U) < +\infty\).

We have \(F(u,U) \geq 0\) and
\[
F(u,U) \rightarrow +\infty \quad \text{as} \quad \|U\|_{L^2} \rightarrow \infty.
\]
(5.50)

Now, \(F(u^n,U^n) \leq R\), implies that
\[
\mathbb{E}\left[\|U^n\|_{L^2(\mathcal{O})}\right] \leq c(R),
\]
for some constant \(c(R)\). Hence by Lemma 5.11 there exists a sequence of tight measures \(L(U^n)\) in \(L^2(0,T;L^2(\mathcal{O}))_w\). By Theorem 5.12 there exists a corresponding sequence \(u^n \in L^2(0,T;\mathbb{H}_0^1(\mathcal{O})) \cap L^\infty(0,T;\mathbb{L}^2(\mathcal{O})) \cap \mathcal{D}(0,T;\mathbb{H}^{-1}(\mathcal{O}))\) such that the pair \((u^n,U^n)\) gives a martingale solution to (5.4)-(5.6), where \(u^n \rightarrow \hat{u}\) in the \(\tau\)-topology and \(U^n \rightarrow \hat{U}\) weakly in \(L^2(\mathcal{O})\). By the same theorem we also have that \((\hat{u},\hat{U})\) solves (5.4)- (5.6).

By Lemma 5.13 \(\int_0^T \int_{\mathcal{O}} L(t,u^n,U^n) dx dt\) is lower semi-continuous and hence by Theorem 55 in Chapter III of [8], so is \(F\). So, \(F(\hat{u},\hat{U}) \leq \lim \inf_n F(u^n,U^n) \leq \lim_n F(u^n,U^n) = \inf F(u,U)\).

\[\square\]

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References

[1] Adams, R. A. (1975). Sobolev Spaces, Academic Press, New York.
[2] Aldous, D. (1978). Stopping times and tightness; Ann. Probab., Vol. 6, pp. 335–340.
[3] Applebaum, D. (2009). Lévy Processes and Stochastic Calculus, Cambridge University Press, Second edition.
[4] Aubin, J. P. and Ekeland, I. (1984). Applied Nonlinear Analysis, Wiley-Interscience, New York.
[5] Brzeźniak, Z., Hausenblas, E. and Razafimandimby, P. A. (2014). Stochastic reaction diffusion equation driven by jump processes; arXiv preprint arXiv:1010.5933
[6] Brzeźniak, Z. AND Liu, W. AND Zhu, J. (2014). Strong solutions for SPDE with locally monotone coefficients driven by Lévy noise; *Nonlinear Anal. Real World Appl.*, Vol. 17, pp. 283–310.

[7] Da Prato, G. AND Zabczyk, J. (1992). *Stochastic Equations in Infinite Dimensions*, Cambridge University Press.

[8] Dellacherie, C. AND Meyer, P. A. (1975). *Probabilités et Potential*, Hermann, Paris.

[9] Glatt-Holtz, N. AND Ziane, M. (2009). Strong pathwise solutions of the stochastic Navier-Stokes system; *Advances in Differential Equations*, Vol. 14 (5/6), pp. 567–600.

[10] Jacod, J. AND Mémin, J. (1981). Sur un type de convergence intermédiaire entre la convergence en loi et la convergence en probabilité; *Séminaire de Probabilités XV 1979/80*, pp. 529–546.

[11] Kesavan, S. (2004). *Nonlinear Functional Analysis; A First Course*, Hindustan Book Agency, New Delhi.

[12] Ladyzhenskaya, O. A. (1969). *The Mathematical Theory of Viscous Incompressible Flow*, Gordon and Breach, New York.

[13] Lions, J. L. (1969). *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Paris.

[14] Motyl, E. (2011). Martingale Solutions to the 2D and 3D Stochastic Navier-Stokes Equations Driven by the Compensated Poisson Random Measure; Preprint 13. Department of Mathematics and Computer Sciences, Lodz University.

[15] Manna, U. AND Mohan, M. T. (2011). Shell model of turbulence perturbed by Lévy noise; *Nonlinear Differential Equations and Applications NoDEA*, Vol. 18(6), pp. 615–648.

[16] Manna, U. AND Menaldi, J. L. AND Srisrathan, S. S. (2008). Stochastic Analysis of Tidal Dynamics Equation; *Infinite Dimensional Stochastic Analysis, Special Volume in honor of Professor HH. Kuo*, Edited by A. Sengupta and P. Sundar, World Scientific Publishers.

[17] Marchuk, G. I. AND Kagan, B. A. (1989). *Dynamics of Ocean Tides*, Kluwer Academic Publishers, Dordrecht/Boston/London.

[18] Metivier, M. (1988). *STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS IN INFINITE DIMENSIONAL SPACES*, Scuola Normale Superiore, Pisa.

[19] Motyl, E. (2013). Stochastic Navier–Stokes Equations Driven by Lévy noise in Unbounded 3D Domains; *Potential Analysis*, Vol. 38(3), pp. 863–912.

[20] Rüdiger, B. (2004). Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces; *Stochastics and Stochastics Reports*, Vol. 76(3), pp. 213–242.

[21] Sakhivel, K. AND Srisrathan, S. S. (2012). Martingale solutions for stochastic Navier-Stokes equations driven by Lévy noise; *Evolution Equations and Control Theory*, Vol. 1(2), pp. 355–392.

[22] Sritharan, S. S. (2000). Deterministic and Stochastic Control of Navier–Stokes Equation with Linear, Monotone and Hyperviscosities; *Applied Mathematics and Optimization*, Vol. 41(2), pp. 255–308.

[23] Sritharan, S. S. (1998). *Optimal control of viscous flow*, SIAM.
[24] Stroock, D. and Varadhan, S. R. S. (1979). Multidimensional Diffusion Processes, Springer-Verlag, New York.

[25] Temam, R. (1984). Navier-Stokes Equation: Theory and Numerical Analysis, AMS Chelsea Publishing.

[26] Vishik, M. J. and Fursikov, A. V. (1988). Mathematical Problems of Statistical Hydromechanics, Kluwer Academic Publishers, Dordrecht.

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