Energy conservation for point particles undergoing radiation reaction

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Abstract

For smooth solutions to Maxwell’s equations sourced by a smooth charge-current distribution $j_a$ in stationary, asymptotically flat spacetimes, one can prove an energy conservation theorem which asserts the vanishing of the sum of (i) the difference between the final and initial electromagnetic self-energy of the charge distribution, (ii) the net electromagnetic energy radiated to infinity (and/or into a black hole/white hole), and (iii) the total work done by the electromagnetic field on the charge distribution via the Lorentz force. A similar conservation theorem can be proven for linearized gravitational fields off of a stationary, asymptotically flat background, with the second order Einstein tensor playing the role of an effective stress-energy tensor of the linearized field. In this paper, we prove the above theorems for smooth sources and then investigate the extent to which they continue to hold for point particle sources. The “self-energy” of point particles is ill defined, but in the electromagnetic case, we can consider situations where, initially and finally, the point charges are stationary and in the same spatial position, so that the self-energy terms should cancel. Under certain assumptions concerning the decay behavior of source-free solutions to Maxwell’s equations, we prove the vanishing of the sum of the net energy radiated to infinity and the net work done on the particle by the DeWitt-Brehme radiation reaction force. As a by-product of this analysis, we provide a definition of the “renormalized self-energy” of a stationary point charge in a stationary spacetime. We also obtain a similar conservation theorem for angular momentum in an axisymmetric spacetime. In the gravitational case, we argue that similar conservation results should hold for freely falling point masses whose orbits begin and end at infinity. This provides justification for the use of energy and angular momentum conservation to compute the decay of orbits due to radiation reaction. For completeness, the corresponding conservation theorems for the case of a scalar field are given in an appendix.
I. INTRODUCTION

The problem of calculating the motion of an isolated body coupled to fields in curved spacetime is an old one which is currently receiving renewed interest. This recent interest is largely driven by the need for accurate calculations of processes which emit gravitational waves (in anticipation of results from the new generation of gravitational wave detectors) but similar issues arise for bodies coupled to electromagnetic and scalar fields. In order to describe in a simple manner those aspects of a body’s motion which are independent of its detailed internal structure, one often attempts to calculate the motion of a “point particle.” A central problem in all such investigations is to calculate the effects of the particle’s own fields, commonly referred to as “self force” or “radiation reaction” effects. This problem is mathematically ill posed since the fields diverge on the world line of the particle itself.

Nevertheless, there is a long history of attempts to calculate the self-force on a particle directly from the local fields. All such schemes involve some prescription for subtracting away the infinite contributions to the force due to the singular nature of the field on the particle’s world line. In 1938, Dirac produced a force expression for a point charge coupled to an electromagnetic field in Minkowski spacetime by imposing local energy conservation on a tube surrounding the particle’s world line [1]. The infinite contributions to the force were subtracted through a “mass renormalization” scheme. In 1960, Dewitt and Brehme [2] generalized this approach to an arbitrary curved background spacetime. (A trivial computational error in their paper was later corrected by Hobbs [3].) More recently, Mino et al. [4] further adapted this approach to produce an expression for the self force on a massive particle coupled to linearized gravity on a vacuum background spacetime. Recently, we also have derived [5] the formulas for the electromagnetic and gravitational self-forces by using an axiomatic approach which, in effect, regularizes the forces by comparing forces in different spacetimes.

Despite the fact that it is thereby known, in principle, how to calculate the electromagnetic and gravitational self-force on a point particle, serious difficulties arise in practice when one attempts to evaluate this self-force on account of the difficulties in computing the “tail term” contribution. Indeed, the evaluation of the “tail term” is highly nontrivial even in the slow motion, weak field limit [3-4]. Consequently, many researchers have employed the following iterative strategy for calculating the motion of point particles. First, one calculates the motion of the particle in the absence of self-force effects. Then, the energy (and/or angular momentum) radiated to infinity by the resulting fields is calculated. Finally, the effects of this energy (and/or angular momentum) loss are introduced as a perturbation to the particle’s motion. Note that the applicability of this approach is limited in that the energy and angular momentum of the particle are not even defined when the background spacetime fails, respectively, to be stationary and axisymmetric. In addition, even when the energy and/or angular momentum of the particle are well defined, the energy and angular momentum radiated to infinity would be expected to equal the energy and angular momentum loss by the particle only in a time averaged sense [5] Hence, the approach is, in essence, limited to

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1 The example of a point charge in Minkowski spacetime which undergoes a period of uniform acceleration explicitly demonstrates the failure of temporally local conservation of this sort, since
calculating the secular decay of an otherwise stable orbit. Furthermore, in the absence of spherical symmetry, the loss of energy and angular momentum in general is not sufficient to determine even this secular decay (for example, it does not determine the variation of the Carter constant for non-equatorial orbits in the Kerr spacetime). Finally, this approach does not allow one to calculate any so-called “conservative forces”, i.e., contributions to the self-force which are not associated with energy or angular momentum loss. Nevertheless, this approach is extremely simple to apply and has been widely used to estimate the effects of radiation reaction on the motion of point particles.

Although the derivations of the self-force given in [1], [2], and [4] were heuristically motivated by local conservation of energy, it is far from obvious, a priori, that they satisfy the property of “global energy conservation” as assumed in the iterative procedure described above. Specifically, in order to justify the iterative procedure, it is necessary that the total energy and angular momentum radiated to infinity by the electromagnetic or gravitational fields be equal, respectively, to the net work and torque done by the self-force over the world line of the particle. The main purpose of this paper is to investigate the extent to which this is the case.

Our main results are the following: In the electromagnetic case, we consider stationary, globally hyperbolic, asymptotically flat spacetimes. We assume that source-free Maxwell fields satisfy a certain decay property, and that the advanced and retarded solutions with stationary sources have suitable fall-off at infinity. We then prove that if a point charge is asymptotically stationary in the past and future and in the same position, then the net electromagnetic energy radiated to infinity (and/or into a black hole/white hole) is equal to the integral over the particle’s world line of the force expression given by DeWitt and Brehme (as corrected by Hobbs) contracted with the timelike Killing field. This provides some justification for the use of the above global energy conservation method to calculate the effects of radiation reaction. (Alternatively, the fact that the DeWitt-Brehme force gives rise to global energy conservation may be viewed as providing further justification for its own validity.) As byproduct of this theorem, we show that it is possible to consistently define the “renormalized self-energy” of a stationary point charge. We argue that a similar energy conservation result should also hold for a point charge which is in inertial motion near infinity in the asymptotic past and future. We also show that similar results hold for angular momentum conservation in axisymmetric spacetimes. As an additional byproduct of our analysis, we show that our conservation theorems also hold for the force expression used by Gal’tsov [9] and others, in which the Lorentz force associated with the advanced-minus-retarded “radiative” field is used. This result clarifies the relationship between the DeWitt-Brehme and Gal’tsov formulas and demonstrates explicitly that global energy conservation alone is insufficient to determine the local force.

Our analysis of the gravitational case closely parallels that of the electromagnetic case, the radiation reaction force (and, hence, the work done by it) vanishes at retarded times during which a nonzero flux of radiation reaches infinity. Nevertheless, the net work done by the radiation reaction force equals the net energy radiated to infinity if the motion of the particle is static at early and late times [8]. The generalizations of such average conservation results to curved spacetime and to the gravitational case are the main subjects of the present paper.
with the second-order Einstein tensor \([10]\) of the metric perturbation replacing the stress-energy tensor of the electromagnetic field. However, our analysis is hampered by a number of technical obstacles—such as the fact that the energy flux as calculated from the second order Einstein tensor has been proven to agree with the Bondi flux only for perturbations of compact spatial support—and its domain of applicability is limited to freely falling point masses (since if the point mass is not freely falling, an additional stress-energy source for the metric perturbation must be present). Nevertheless, we argue that energy and angular momentum conservation results analogous to those in the electromagnetic case should hold for a point mass which is in geodesic motion near infinity in the asymptotic past and future.

The electromagnetic case is analyzed in Sec. II. We begin by proving energy conservation theorems for the case of smooth charge-current distributions in a spacetime with no black or white holes. The desired theorem for the case of point particle sources is then proven. Generalizations to establish angular momentum conservation and to allow for the presence of black and white holes are then described.

Linearized gravitational perturbations are considered in Sec. III. Again, we begin by proving a conservation theorem for smooth stress-energy sources and then analyze the point particle case.

Finally, for completeness, we present the analogous results for a scalar field in an appendix.

Our notation and conventions throughout the paper follow [11].

**II. ELECTROMAGNETIC CASE**

In this section, we will state and prove our conservation theorems for the case of a Maxwell field in a fixed background spacetime. For simplicity, we shall first consider conservation of energy in a stationary, asymptotically flat spacetime which contains no black or white holes. Generalizations to other symmetries and to allow for the presence of black holes and white holes will be discussed at the end of this section. Thus, until these generalizations are considered, we will restrict consideration to spacetimes satisfying the following conditions:

**Spacetime assumptions:** Let \((M, g_{ab})\) be a globally hyperbolic spacetime that is asymptotically flat at null and spatial infinity in the sense of Ashtekar and Hansen [12] and is stationary in the sense that it possesses a Killing field \(t^a\) that is asymptotically a time translation at infinity. We further assume that no black holes or white holes are present in \((M, g_{ab})\) (i.e., the domain of outer communications is the entire spacetime) and that there exists a smooth, spacelike Cauchy surface, \(\Sigma\) for \((M, g_{ab})\) such that in the unphysical spacetime \((\tilde{M}, \tilde{g}_{ab})\), \(\Sigma \cup i^0\) is compact. (This implies that \(\Sigma\) is of the form of a disjoint union \(\Sigma = \Sigma_{\text{end}} \cup \Sigma'\) where \(\Sigma_{\text{end}}\) has the topology of \(\mathbb{R}^3\) minus a closed ball and \(\Sigma'\) is compact.)

Without loss of generality, we may assume that \(t^a n_a < 0\) everywhere on \(\Sigma\), where \(n^a\) is the future directed normal to \(\Sigma\) (see proposition 4.1 of [13]). We now deform \(\Sigma\) to the future in a small neighborhood of \(i^0\) so that the deformed surface, \(C^+\), satisfies \([C^+ \cap \Sigma] \supset \Sigma'\) and, in the unphysical spacetime, \(C^+\) remains a smooth, spacelike hypersurface, but now intersects \(I^+\).
in a cross-section $S^+$. We similarly deform $\Sigma$ to the past to construct a smooth, spacelike hypersurface $C^-$ such that $[C^- \cap \Sigma] \supset \Sigma'$ and $C^-$ intersects $I^-$ in a cross-section $S^-$. For all $t^+ > 0$, we define $C^+(t^+)$ to be the hypersurface obtained by “time translating” $C^+$ along the orbits of $t^a$ by $t^+$. Similarly, for all $t^- < 0$, we define $C^-(t^-)$ to be the hypersurface obtained by “time translating” $C^-$ along the orbits of $t^a$ by $t^-$. It then follows that for all $t^+ > 0$ and $t^- < 0$ the surface $C^+(t^+) \cup [I^+ \cap J^- (S^+(t^+))] \cup C^-(t^-) \cup [I^- \cap J^+ (S^-(t^-))] \cup t^0$ bounds a compact region $V(t^+, t^-)$, of the unphysical spacetime, as illustrated in Figure 1. Furthermore, it follows that any $p \in M$ lies within $V(t^+, t^-)$ for sufficiently large $t^+, t^-$. In a spacetime $(M, g_{ab})$ satisfying the above properties, we wish to consider solutions to Maxwell’s equations with source $j^a$,

\[
\nabla^a F_{ab} = -4\pi j_b \tag{1}
\]

\[
\nabla_{[a} F_{bc]} = 0. \tag{2}
\]

It follows immediately from Maxwell’s equations that the stress-energy tensor

\[
T_{ab} = \frac{1}{4\pi} \left( F^c_a F_{bc} - \frac{1}{4} g_{ab} F^{cd} F_{cd} \right) \tag{3}
\]

satisfies

\[
\nabla^b T_{ab} = -F_{ab} j^b. \tag{4}
\]

We define the stress-energy current three-form $J_{abc}$ by

\[
J_{abc} = t^e T_{de} \epsilon_{abc} \tag{5}
\]

where $\epsilon_{abcd}$ is the (positively oriented) volume element associated with the spacetime metric. We choose the orientation of all spacelike and null hypersurfaces to be given by $v^a \epsilon_{abcd}$, where $v^a$ is any future directed timelike vector field. Then, the integral of $J_{abc}$ over a spacelike hypersurface represents the total electromagnetic energy on that hypersurface, whereas the integral of $J_{abc}$ over a null hypersurface represents the flux of electromagnetic energy through that hypersurface. Equation (4) implies that

\[
(dJ)_{abcd} = -\nabla^f (t^e T_{fe}) \epsilon_{abcd}
= -t^e \nabla^f T_{fe} \epsilon_{abcd}
= t^e F_{efj} \epsilon_{abcd} \tag{6}
\]

Initially, we shall consider smooth solutions to Maxwell’s equations with a smooth source $j^a$, but later in this section we will consider solutions with point particle sources which are smooth away from the world line of the particle. In all cases we shall consider only solutions which satisfy the following three conditions:

2One way of doing this would be to choose a cross-section $S^+$ of $I^+$ such that $[J^- (S^+) \cap \Sigma] \subset \Sigma_{end}$ and consider the $C^0$, partially null hypersurface $[\Sigma - I^- (S^+)] \cup [J^-(S^+) \cap J^+ (\Sigma)]$; then smooth this hypersurface (see [14]) to a smooth spacelike hypersurface.
Maxwell field assumptions: (1) $j_a$ vanishes in a neighborhood of $I^+ \cup I^- \cup i^0$. (2) The unphysical Maxwell field $F_{ab} = \hat{F}_{ab}$ continuously extends to $I^+$ and $I^-$. (3) The physical energy current three-form $t^a T_{de} \epsilon_{abc}^d$ falls off sufficiently rapidly at spatial infinity that (a) its integral over $\Sigma_{end}$ is finite and (b) its integral over $B_s \cap \mathcal{M}$ vanishes as $r \to 0$, where $B_s$ is a suitably chosen coordinate sphere of radius $r$ around $i^0$ in the unphysical spacetime.

Condition (2) has the immediate consequence that the unphysical energy current three-form $t^a \hat{T}_{de} \epsilon_{abc}^d$ (where $\epsilon_{abc}^d$ is the metric-compatible volume element associated with the unphysical metric, with index raising used the unphysical metric) continuously extends to $I^+$ and $I^-$. However, for a conformally invariant field such as $F_{ab}$, the energy current three-form is conformally invariant with conformal weight 0, so condition (2) implies that the physical energy current three-form $t^a T_{de} \epsilon_{abc}^d$ also continuously extends to $I^+$ and $I^-$. Our fundamental result is simply a direct application of Stokes’ theorem to the integral of eq. (6) over the region $V(t^+, t^-)$. Taking account of the absence of contributions from spatial infinity as a consequence of assumption (3) above (i.e., initially excluding from $V(t^+, t^-)$ a coordinate ball of radius $r$ around $i^0$ and then letting $r \to 0$), we obtain

$$\int_{C^+(t^+)} t^a T_{ab} \epsilon_{cde}^b - \int_{C^-(t^-)} t^a T_{ab} \epsilon_{cde}^b + \int_{I^+ \cap I^- (S^+(t^+))} t^a T_{ab} \epsilon_{cde}^b - \int_{I^+ \cap I^- (S^-(t^-))} t^a T_{ab} \epsilon_{cde}^b = \int_{V(t^+, t^-)} t^a F_{ab} j^b \epsilon_{cdef}. \quad (7)$$

Taking the limits as $t^+ \to \infty$ and $t^- \to -\infty$, we obtain our desired theorem for the case of smooth sources:

**Theorem 2.1** Let $(M, g_{ab})$ satisfy the properties stated above and let $F_{ab}$ be a smooth solution to Maxwell’s equations with smooth source $j_a$ which satisfies the three properties stated above. Then,

$$\lim_{t^+ \to +\infty} \int_{C^+(t^+)} t^a T_{ab} \epsilon_{cde}^b - \lim_{t^- \to -\infty} \int_{C^-(t^-)} t^a T_{ab} \epsilon_{cde}^b + \int_{I^+} t^a T_{ab} \epsilon_{cde}^b - \int_{I^-} t^a T_{ab} \epsilon_{cde}^b = \int_{\mathcal{M}} t^a F_{ab} j^b \epsilon_{cdef}, \quad (8)$$

provided that each of the limits and integrals in the above formula exist.

The above theorem has a straightforward physical interpretation. The first pair of terms on the left side can be interpreted as being the difference between the initial and final electromagnetic energy of the charge-current distribution. The second pair of terms is just the net electromagnetic energy radiated to infinity. Finally, since $F_{ab} j^b$ is just the Lorentz force acting on the charge distribution, the right side is the negative of the net work done by the electromagnetic field on the charge distribution, or, equivalently, it is the net work done by the charge distribution on the electromagnetic field. This, in turn, has the interpretation of representing the net amount of “mechanical energy” which is converted to electromagnetic energy. Thus, eq. (8) can be interpreted as stating that total energy is conserved.

Unfortunately, this theorem as stated does not have a direct counterpart for point particle sources. It is true that the net electromagnetic energy radiated to infinity is perfectly well defined in the limit of point particle sources. Furthermore, although the integrand on the right side of eq. (8) becomes singular in the point particle limit, one might nevertheless hope that, in a suitable limiting process, the integral on the right side of eq. (8) would converge to
\[ \int t^a f_a \, d\tau \]  

(9)

where \( f^a \) is the DeWitt and Brehme expression for the electromagnetic self-force on the particle (see eq. (B4) below). However, the self-energy terms in eq. (8) become hopelessly divergent in the point particle limit, so, as it stands, eq. (8) will not make sense for point particles.

Nevertheless, we can proceed by considering cases where the self-energy terms in eq. (8) should cancel. In particular, this should occur if at sufficiently early times the charge-current source \( j_a \) is stationary (i.e., \( \dot{t}_a j_a = 0 \)), and at sufficiently late times, \( j_a \) returns to the same stationary state, i.e., if \( j_a \) differs from a stationary distribution only in a compact spacetime region. However, even in this case, in order to ensure cancellation of the self-energy terms, it is necessary to assume suitable decay properties of source free solutions, as well as some properties of stationary solutions.

It order to formulate our decay assumptions, we need to introduce a suitable norm, and it is most convenient to introduce this norm in terms of the unphysical variables. On the hypersurface \( C^+ \), we decompose the unphysical electromagnetic field \( \tilde{F}_{ab} = F_{ab} \) into its electric and magnetic parts with respect to the unphysical unit normal \( \tilde{n}^a = \Omega^{-1} n^a \) to \( C^+ \),

\[ \tilde{E}_a = \tilde{F}_{ab} \tilde{n}^b \]  

(10)

\[ \tilde{B}_a = -\frac{1}{2} \tilde{\varepsilon}^{cd} \tilde{F}_{cd} \tilde{n}^b \]  

(11)

On \( C^+ \), we define

\[ \| F \| = \sup_{C^+} (\tilde{E}^a \tilde{E}_a + \tilde{B}^a \tilde{B}_a)^{1/2} \]  

(12)

where the indices are raised and lowered with respect to the unphysical metric. Note that for any \( F_{ab} \) which is nonsingular in the physical spacetime, we have \( \| F \| < \infty \) provided only that \( \tilde{F}_{ab} \) continuously extends to \( I^+ \), as we have assumed. Note also that since the components of \( t^a \) are bounded in an unphysical orthonormal frame associated with \( \tilde{n}^a \), it follows that everywhere on \( C^+ \) we have \( |\tilde{T}_{ab} t^a \tilde{n}_b| \leq C \| F \|^2 \) for some constant \( C \). Since \( C^+ \) has finite volume with respect to the unphysical metric, it follows that there exists a constant, \( c \), such that

\[ \int_{C^+} t^a \tilde{T}_{ab} \tilde{\varepsilon}^{cde} b \leq c \| F \|^2 \]  

(13)

Since the energy current 3-form is conformally invariant, this is equivalent to

\[ \int_{C^+} t^a T_{ab} \varepsilon^{cde} b \leq c \| F \|^2 \]  

(14)

i.e., the norm we have introduced on \( C^+ \) bounds the energy on \( C^+ \).

In terms of the physical variables, we have

\[ \| F \| = \sup_{C^+} \Omega^{-2} (E^a E_a + B^a B_a)^{1/2} \]  

(15)
where $E_a$ and $B_a$ are now defined with respect to the physical unit normal $n^a$ and indices are now raised and lowered with respect to the physical metric. We could similarly define a norm on $C^+(t^+)$ by eqs. (12) or (15), but this would not be useful on account of the possible “time dependence” of the conformal factor $\Omega$. Thus, we instead define the function $\Omega'$ on $C^+(t^+)$ by Lie transport of $\Omega$ along $t^a$. (Equivalently, we could require that the conformal factor $\Omega$ defining the unphysical spacetime be chosen so as to satisfy $\mathcal{L}_t \Omega = 0$ to the future of $C^+$. ) For all $t^+ \geq 0$, we define

$$\|F\|(t^+) = \sup_{C^+(t^+)} (\Omega')^{-2}(E^a E_a + B^a B_a)^{1/2}$$

We also define $\Omega'$ on all $C^-(t^-)$ by Lie transport of $\Omega$ on $C^-$ and define $\|F\|(t^-)$ for all $t^- \leq 0$ similarly.

In the following, we shall restrict consideration to spacetimes which satisfy the following "decay hypothesis":

**Decay hypothesis:** A spacetime $(M, g_{ab})$ satisfying the conditions stated at the beginning of this section will be said to satisfy the decay hypothesis for Maxwell fields if for every smooth (in the physical spacetime) solution, $F_{ab}$, of the source-free Maxwell equations which satisfies our Maxwell field assumptions, we have

$$\lim_{t^+ \to \infty} \|F\|(t^+) = \lim_{t^- \to -\infty} \|F\|(t^-) = 0$$

It should be noted that although the definitions of $\|F\|(t^+)$ and $\|F\|(t^-)$ depend upon the choice of conformal factor, $\Omega$, on $C^+$ and $C^-$, it is clear that satisfaction of the decay hypothesis does not depend upon this choice. Furthermore, it is not difficult to verify that the satisfaction of the decay hypothesis does not depend upon the choices of $C^+$ and $C^-$. Thus, the decay hypothesis is, indeed, a condition on the spacetime, $(M, g_{ab})$.

If the Killing field $t^a$ is spacelike somewhere in $M$ (i.e., if an ergoregion is present), then solutions of the source-free Maxwell equations with negative total energy can be constructed, and since only positive energy can be radiated to infinity, it is clear that the decay hypothesis cannot hold (see [16]). However, we conjecture that the decay hypothesis holds for all spacetimes satisfying our spacetime assumptions in which $t^a$ is globally timelike.

Note that if $(M, g_{ab})$ satisfies the decay hypothesis, then there can exist at most one stationary solution to Maxwell’s equations with a stationary source $j_a$ which satisfies our Maxwell field assumptions. (Proof: If two such solutions existed, their difference would be a source-free solution which does not decay.) For a stationary $j_a$, both the advanced and retarded solutions are necessarily stationary. If $j_a$ is stationary and has compact spatial support on a Cauchy surface, then condition (1) of the Maxwell field assumptions automatically is satisfied. The final property we shall need is that the advanced and retarded solutions associated with such a $j_a$ satisfy conditions (2) and (3) as well:

**Stationary solution property:** $(M, g_{ab})$ will be said to satisfy the stationary solution property if for any stationary $j_a$ of compact spatial support, the advanced and retarded solutions satisfy conditions (2) and (3) of the Maxwell field assumptions.

It is readily verified that the stationary solution property holds for Minkowski spacetime, and we believe that the stationary solution property holds for all spacetimes satisfying our spacetime assumptions. Indeed, near infinity, the behavior of stationary, source free solutions
is such that in order to prove that this property holds, we would, in essence, need only show that the advanced and retarded solutions for a stationary \( j_a \) of compact spatial support do not blow up at spatial infinity. While this seems undoubtedly true, one would need some bounds on the long distance behavior of the advanced and retarded Green’s in a general spacetime satisfying our assumptions in order to obtain a proof.

Note that, in view of the above uniqueness property, if \((M, g_{ab})\) satisfies the decay hypothesis and the stationary solution property, then for any stationary \( j_a \) of compact spatial support, the advanced and retarded solutions are equal. Note also that for any stationary solution, the flux of energy through \( \mathcal{I}^+ \) between \( i^0 \) and any cross-section \( \mathcal{S}^+ \) (or through \( \mathcal{I}^- \) between \( i^0 \) and any cross-section \( \mathcal{S}^- \)) must either be zero or infinite. The latter possibility is ruled out by our Maxwell field assumptions. Thus, when the stationary solution property holds, the retarded (= advanced) solution for any stationary \( j_a \) of compact spatial support must have vanishing energy flux through \( \mathcal{I}^+ \) and \( \mathcal{I}^- \). From the form of the stress-energy tensor, it can be seen that this is equivalent to the vanishing of the pullback to \( \mathcal{I}^+ \) (or \( \mathcal{I}^- \)) of \( \tilde{F}_{ab} \tilde{n}^b \), where \( \tilde{n}^a \) denotes the normal to \( \mathcal{I}^+ \); equivalently, we have on \( \mathcal{I}^+ \)

\[
\tilde{F}_{ab} \tilde{n}^b = \alpha \tilde{n}_a
\]

for some function \( \alpha \) on \( \mathcal{I}^+ \).

We have the following theorem:

**Theorem 2.2:** Let \((M, g_{ab})\) be a spacetime satisfying the conditions stated at the beginning of this section which, in addition, satisfies the decay hypothesis and stationary solution property. Let \( F_{ab} \) be a smooth solution to Maxwell’s equations satisfying our Maxwell field assumptions such that, in addition, \( j_a - j_a^S \) vanishes outside of a compact region of \( M \), where \( j_a^S \) is a stationary charge-current distribution of compact spatial support. Then, we have

\[
\int_{\mathcal{I}^+} t^a T_{ab} \epsilon_{cde}^b - \int_{\mathcal{I}^-} t^a T_{ab} \epsilon_{cde}^b = \int_M t^a F_{ab} j^b \epsilon_{cdef},
\]

**Proof:** Without loss of generality, we may assume that the support of \( j_a^S \) on \( \Sigma \) is contained in \( \Sigma' \), so that \( j_a^S \) vanishes in the region, \( V(0,0) \), bounded by \( C^+ \) and \( C^- \). Since eq. (19) holds for \( F_{ab} \), it suffices to show that

\[
\lim_{t^+ \to +\infty} \int_{C^+(t^+)} t^a T_{ab} \epsilon_{cde}^b
\]

exists and equals

\[
\lim_{t^- \to -\infty} \int_{C^-(t^-)} t^a T_{ab} \epsilon_{cde}^b
\]

Let \( F_{ab}^S \) be the retarded (= advanced) solution to Maxwell’s equations with source \( j_a^S \), and let

\[
F'_{ab} = F_{ab} - F_{ab}^S
\]

Then, by straightforward estimates similar to those used to obtain eq. (14) above, we obtain for all \( t^+ \geq 0 \),
\[
\left| \int_{C^+} t^a T_{ab} \epsilon_{cde}^b - \int_{C^+} t^a T^S_{ab} \epsilon_{cde}^b \right| \leq K \left[ \|F'(t^+)\| + \|F^S\|(t^+) \right] \|F'(t^+)\| (23)
\]

for some constant \( K \), where \( T^S_{ab} \) denotes the stress-energy tensor of \( F^S_{ab} \). However, by stationarity, we have

\[
\int_{C^+} t^a T_{ab} \epsilon_{cde}^b = \int_{C^+} t^a T^S_{ab} \epsilon_{cde}^b (24)
\]

Furthermore, since \( F'_{ab} \) is source free for sufficiently large \( t^+ \), the decay hypothesis applies to it. Consequently, we obtain

\[
\lim_{t^+ \to +\infty} \int_{C^+} t^a T_{ab} \epsilon_{cde}^b = \int_{C^+} t^a T^S_{ab} \epsilon_{cde}^b (25)
\]

Similarly, we obtain

\[
\lim_{t^- \to -\infty} \int_{C^-} t^a T_{ab} \epsilon_{cde}^b = \int_{C^-} t^a T^S_{ab} \epsilon_{cde}^b (26)
\]

Finally, we apply eq. (27) to \( F^S_{ab} \), choosing \( t^+ = t^- = 0 \). Taking into account the fact that the energy flux integrals through \( \mathcal{I}^+ \) and \( \mathcal{I}^- \) vanish for \( F^S_{ab} \) and that \( j^S_a \) vanishes in \( V(0,0) \), we find that the right sides of eqs. (25) and (26) are equal, as we desired to show.

Applying this theorem to the case of a source-free Maxwell field \( F_{ab} \), we have

\[
\int_{\mathcal{I}^+} t^a T_{ab} \epsilon_{cde}^b - \int_{\mathcal{I}^-} t^a T_{ab} \epsilon_{cde}^b = 0. (27)
\]

That is, for a source-free solution, the energy radiated “into” the spacetime through past null infinity is equal to the energy radiated “out of” the spacetime through future null infinity. This observation, which will be important in our subsequent analysis, demonstrates the intuitive content of our decay hypothesis: no energy remains “trapped” in the spacetime.

We are now prepared to generalize the above theorem to the case of a point particle source with charge \( e \) and world line \( z(\tau) \). In this case, the current \( j_a \) is given by the distribution

\[
j_a(x) = e \int \delta(x,z(\tau)) u_a d\tau. (28)
\]

Since Maxwell’s equations are linear, they are well defined for distributional sources. We shall consider only distributional solutions \( F_{ab} \) which are smooth away from the world line of the particle and which satisfy the Maxwell field assumptions stated near the beginning of this section. In that case, the integrals at \( \mathcal{I}^+ \) and \( \mathcal{I}^- \) appearing in eq. (19) are well defined. However, since \( F_{ab} \) is necessarily distributional in a neighborhood of the world line of the particle the integral on the right hand side of eq. (19), which represents the work done by the source on the field, contains a formal product of distributions and is therefore ill defined. Nevertheless, we now wish to show that, in the point particle case, we obtain a new version of eq. (19) in which the problematic product of distributions on the right hand side is replaced by the integral
\[ \int t^a f_a \, d\tau, \]  
where

\[ f_a = e F_{ab}^\text{in} u^b + \frac{2}{3} e^2 (\dot{a}_a - a^2 u_a) + \frac{1}{3} e^2 (R_{ab} u^b + u_a R_{bc} u^b u^c) \]

\[ + e^2 u^b \int_{-\infty}^{\tau} 2 \nabla^a G_{bc}^R u^c (\tau') \, d\tau', \]

is the expression given by DeWitt and Brehme (as corrected by Hobbs) for the total electromagnetic force on a point particle coupled to a Maxwell field \( F_{ab} \). In this force expression, \( G_{ab}^R \) is the retarded Green’s function for the vector potential, satisfying

\[ \nabla^b \nabla^c G_{ab}^R - R_{ab} G_{cd}^R = -4\pi \bar{g}_{aa'} \delta(x, z), \]

\( F_{ab}^R \) is the retarded solution with source \( (28) \), and

\[ F_{ab}^\text{in} \equiv F_{ab} - F_{ab}^R. \]

In the integral over the particle’s past world line, usually referred to as the “tail term”, there is an implicit limiting procedure: the integral is performed from \(-\infty\) to a point \( \tilde{\tau} < \tau \) and then the limit \( \tilde{\tau} \to \tau \) is taken. One consequence of the Hadamard expansion is that this limit exists; i.e., no singular part of \( G_{aa'} \) is encountered in this limit.

In order to better understand the behavior of eq. (19) in the point particle limit, we write \( F_{ab} \) as the sum of the half-advanced plus half retarded solution

\[ \bar{F}_{ab} \equiv (F_{ab}^R + F_{ab}^A)/2 \]

and a source-free solution

\[ F_{ab}^\text{free} \equiv F_{ab} - \bar{F}_{ab}. \]

We shall consider only particle motions such that \( F_{ab}^R \) and \( F_{ab}^A \) are smooth away from the world line of the particle and such that \( F_{ab}^R \) and \( F_{ab}^A \) satisfy our Maxwell field assumptions. If the stationary solution property holds, this will hold whenever there exist \( t^- \), \( t^+ \) such that the particle is stationary for all \( t < t^- \) and all \( t > t^+ \).

Since \( T_{ab} \) is quadratic in \( F_{ab} \), we have

\[ T_{ab}[F, F] = T_{ab}[F, F] + T_{ab}[F^\text{free}, F^\text{free}] + 2 T_{ab}[F^\text{free}, F], \]

where we have defined

\[ ^3 \text{The sign of the “tail term” in eq. (19) of reference [4] is incorrect due to an error in transcribing DeWitt and Brehme’s expression into our notation. Additionally, a factor of 2 error was introduced in copying the “tail term” to eq. (23). As a result, the “tail term” in eqs. (25) and (26) of reference [5] must be multiplied by a factor of } -2 \text{ in order to produce the correct formula (given in eq. (30) of the present paper).} \]
\[ T_{ab}[F^1, F^2] = \frac{1}{4\pi} \left( F^1_{(a} F^2_{b)c} - \frac{1}{4} g_{ab}(F^1)^{cd} F^2_{cd} \right). \]  

(36)

Now, \( F^\text{free}_{ab} \) is a source-free solution which is smooth away from the world line of the particle and consequently (by the propagation of singularities theorem [17]) is smooth everywhere. Since \( F^\text{free}_{ab} \) also satisfies our Maxwell field assumptions, eq. (27) applies to it, and \( T_{ab}[F^\text{free}, F^\text{free}] \) will make zero contribution to the net energy radiated to infinity (the left hand side of eq. (19) above). Furthermore, since \( T_{ab}[F^\text{free}, F] \) is the product of a distribution and a smooth tensor field and thus is well defined as a distribution, this term will give rise to a well-defined volume integral when we apply Stokes’ theorem; we will evaluate this term in Proposition 2.1 below. Thus, the only term that is mathematically problematic is \( T_{ab}[\bar{F}, \bar{F}] \), which contains a product of distributions. However, we will now show that, if the point particle source is initially and finally stationary (i.e., if there exist \( t^-, t^+ \) such that the particle is stationary for all \( t < t^- \) and all \( t > t^+ \)), this term makes vanishing contribution to the net energy radiated to infinity.

To show this, we write \( T_{ab}[\bar{F}, \bar{F}] \) as

\[ T_{ab}[F, F] = T_{ab}[F^\text{rad}, F^\text{rad}] + T_{ab}[F^R, F^A], \]  

(37)

where

\[ F^\text{rad}_{ab} = (F^R_{ab} - F^A_{ab})/2. \]  

(38)

Like \( F^\text{free}_{ab} \), the radiative solution \( F^\text{rad}_{ab} \) is a source-free solution which is smooth away from the world line of the particle and, consequently, is smooth everywhere. Since \( F^\text{rad}_{ab} \) also satisfies our Maxwell field assumptions, \( T_{ab}[F^\text{rad}, F^\text{rad}] \) makes no contribution to the net energy radiated to infinity. Therefore, we have

\[ \int_{\mathcal{I}^+} t^a T_{ab}[\bar{F}, \bar{F}] \epsilon_{cde} b - \int_{\mathcal{I}^-} t^a T_{ab}[\bar{F}, \bar{F}] \epsilon_{cde} b = \int_{\mathcal{I}^+} t^a T_{ab}[F^R, F^A] \epsilon_{cde} b - \int_{\mathcal{I}^-} t^a T_{ab}[F^R, F^A] \epsilon_{cde} b \]  

(39)

However, since the particle is initially stationary, it follows that \( F^R_{ab} \) satisfies eq. (18) on \( \mathcal{I}^- \). Taking into account the fact that on \( \mathcal{I}^- \), \( t^a \) is proportional to the normal, \( \tilde{n}^a \), to \( \mathcal{I}^- \) and \( F^A_{ab} \) is antisymmetric in its indices, we see that the flux integral over \( \mathcal{I}^- \) of the “advanced-retarded cross-term” vanishes. Similarly, using the fact that the particle is finally stationary, we see that the integral of this cross term over \( \mathcal{I}^+ \) also vanishes, which yields the desired result.

We are now ready to state and prove the following intermediate result:\footnote{It should be noted that a direct analog of this proposition also holds in the case of smooth sources and, indeed, the present proposition could be obtained by taking the point particle limit of the analogous result for smooth sources. We chose not to present this result in the smooth case because Theorem 2.2 (i.e., the smooth source analog of Theorem 2.3 below) could be proven more directly by other means.}

**Proposition 2.1:** Let \((M, g_{ab})\) be a spacetime satisfying the conditions stated at the beginning of this section together with the decay hypothesis and stationary solution property.
Let $z(\tau)$ be a timelike curve which differs from an orbit, $z_0(\tau)$, of the stationary Killing field $t^a$ only over a finite interval. Let $F_{ab}$ be a solution to Maxwell’s equations with source (28) which satisfies our Maxwell field assumptions. Then, we have

$$\int_{I^+} t^a T_{ab} F_{cde}^b - \int_{I^-} t^a T_{ab} F_{cde}^b = \int t^a \hat{f}_a \, d\tau,$$

(40)

where

$$\hat{f}_a = e F_{ab}^\text{free} u^b.$$  

(41)

**Proof:** We apply Stokes’ theorem to the differential of energy current three-form associated with the cross-term $2T_{ab}[F^\text{free}, \bar{F}]$. In analogy with the derivation of eq. (8), we obtain

$$\lim_{t^+ \to +\infty} 2 \int_{C^+(t^+)} t^a T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cde}^b - \lim_{t^- \to -\infty} 2 \int_{C^-(t^-)} t^a T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cde}^b$$

$$+ 2 \int_{I^+} t^a T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cde}^b - 2 \int_{I^-} t^a T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cde}^b = -2 \int_M t^a \nabla^b T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cdef}$$

$$= \int_M t^a F_{ab}^\text{free} \epsilon_{cdef}$$

$$= e \int t^a F_{ab}^\text{free} u^b \, d\tau$$

$$= \int t^a \hat{f}_a \, d\tau.$$  

(42)

However, by the above results, we have

$$2 \int_{I^+} t^a T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cde}^b - 2 \int_{I^-} t^a T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cde}^b = \int_{I^+} t^a T_{ab} \epsilon_{cde}^b - \int_{I^-} t^a T_{ab} \epsilon_{cde}^b$$

(43)

Thus, it suffices to show that

$$\lim_{t^+ \to +\infty} \int_{C^+(t^+)} t^a T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cde}^b = \lim_{t^- \to -\infty} \int_{C^-(t^-)} t^a T_{ab}[F_{cde}^\text{free}, \bar{F}] \epsilon_{cde}^b = 0$$

(44)

To show this, in analogy with eq. (22) we define

$$F''_{ab} = \bar{F}_{ab} - F_{ab}^S$$

(45)

where $F_{ab}^S$ is the stationary solution sourced by a point charge following the orbit, $z_0(\tau)$, of $t^a$. We use eq. (15) to substitute for $\bar{F}$ in eq. (14). Since $F_{cde}^\text{free}$ is a source free solution satisfying our Maxwell field assumptions and, in the asymptotic past and future, so is $F''$, it follows immediately from the decay hypothesis that

$$\lim_{t^+ \to +\infty} \int_{C^+(t^+)} t^a T_{ab}[F_{cde}^\text{free}, F''] \epsilon_{cde}^b = \lim_{t^- \to -\infty} \int_{C^-(t^-)} t^a T_{ab}[F_{cde}^\text{free}, F''] \epsilon_{cde}^b = 0$$

(46)

On the other hand, although $F_{ab}^S$ is singular on the world line of the source, the Hadamard expansion shows that on $C^+(t^+)$ it diverges as $1/\sigma$ as one approaches the point charge, where $\sigma$ denotes squared geodesic distance. Consequently, the unphysical orthonormal frame components of $\bar{F}_{ab}^S$ are $L^1$ functions on $C^+(t^+)$ with respect to the unphysical volume element $\epsilon_{abcd} \hat{n}^a$. Therefore, we obtain

13
\[
\left| \int_{C^+(t^+)} t^a T_{ab}[F^{\text{free}}, F^S] e_{cde}^b \right| = \left| \int_{C^+(t^+)} t^a \tilde{T}_{ab}[\tilde{F}^{\text{free}}, \tilde{F}^S] e_{cde}^b \right| \\
\leq k\|F^S\|_{L^1} \|F^{\text{free}}\|
\]  
for some constant \(k\), which, together with the decay hypothesis applied to \(F^{\text{free}}\), implies
\[
\lim_{t^+ \to +\infty} \int_{C^+(t^+)} t^a T_{ab}[F^{\text{free}}, F^S] e_{cde}^b = 0
\]  
and similarly for the limit as \(t^- \to -\infty\).

The above proposition is a somewhat curious result, because although it provides an expression of the general form we are seeking, the force \(\tilde{f}_a\) is not the force expression given by DeWitt and Brehme (Eq. (30) above). Indeed, we have
\[
\tilde{f}_a = eF^{\text{in}}_{ab} u^b + \frac{2}{3} e^2 (\dot{a}_a - a^2 u_a) + \frac{1}{3} e^2 (R_{ab} u^b + u_a R_{bc} u^c)
\]
\[
+ \frac{1}{2} e^2 u^b \int_{-\infty}^{\infty} 2\nabla_{[a} G^R_{bc]} u^c (\tau') d\tau' - \frac{1}{2} e^2 u^b \int_{\tau}^{+\infty} 2\nabla_{[a} G^A_{bc]} u^c (\tau') d\tau',
\]
which differs from eq. (30) by the form of the tail term. Indeed, for the retarded solution (so that \(F^{\text{in}}\) vanishes), eq. (49) contains causal contributions from the future history of the particle, whereas eq. (30) does not. Thus, although \(\tilde{f}_a\) is the correct force expression in Minkowski spacetime \([1]\) (where the tail term vanishes) and it has been used in curved spacetime, most notably by Gal’tsov \([9]\), apparently because of the calculational simplicity it affords, it is not equivalent to \(f_a\). Indeed, we always have \(\tilde{f}_a = 0\) whenever \(F^{\text{free}}_{ab} = 0\), so that, in particular, \(\tilde{f}_a = 0\) for a static charge in Schwarzschild if there is no incoming radiation.

On the other hand, it is not difficult to verify that the force prescription given by Smith and Will \([18]\) for a static charge in Schwarzschild satisfies the axioms of \([3]\) and thus agrees with the DeWitt-Brehme force. The force calculated by Smith and Will is nonvanishing.

We now calculate the difference between the work done by \(f_a\) and \(\tilde{f}_a\). The difference between \(f_a\) and \(\tilde{f}_a\) is given by
\[
f_a - \tilde{f}_a = \frac{1}{2} e^2 u^b \int_{-\infty}^{\tau} 2\nabla_{[a} \bar{G}^R_{bc]} u^c (\tau') d\tau' + \frac{1}{2} e^2 u^b \int_{\tau}^{+\infty} 2\nabla_{[a} \bar{G}^A_{bc]} u^c (\tau') d\tau'
\]
\[
= e^2 u^b \int_{-\infty}^{+\infty} 2\nabla_{[a} \bar{G}_{bc]} u^c (\tau') d\tau',
\]
where
\[
\bar{G}_{ab'} = (G^A_{ab'} + G^R_{ab'})/2
\]
and where it is understood that the singular point \(\tau' = \tau\) is omitted from the region of integration, i.e., one excludes an interval of size \(\epsilon\) centered at \(\tau\) and then takes the limit as \(\epsilon \to 0\). Therefore, the difference between the total work done by \(f_a\) and the total work done by \(\tilde{f}_a\) during the time interval \([\tau^-, \tau^+]\) is\(^5\)

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\(^5\)Note that we proceed by considering the work done in the finite time interval \([\tau^-, \tau^+]\) and only at the end of the calculation do we take the limit \(\tau^\pm \to \pm\infty\). We do so in order that all of the integrals arising in our calculations will converge suitably well to justify the interchanges of orders of integration.
\[
\int_{\tau^-}^{\tau^+} t^a (f_a - \hat{f}_a) d\tau = e^2 \int_{\tau^-}^{\tau^+} \left[ \int_{-\infty}^{\infty} 2 \nabla_{\bar{a}} \bar{G}_{b\bar{b}} u^{b'} d\tau' \right] u^b d\tau \\
= e^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{\infty} 2 t^a \nabla_{\bar{a}} \bar{G}_{b\bar{b}} u^{b'} d\tau' d\tau \\
= -e^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{\infty} u^b \nabla_b (t^a \bar{G}_{ab'}) u^{b'} d\tau' d\tau + e^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{\infty} \mathcal{L}_t \bar{G}_{b\bar{b}} u^b u^{b'} d\tau' d\tau. \tag{52}
\]

We now analyze the contributions of the two terms appearing on the right side of eq. (52).

Consider the first term. On account of the finite range of the \( \tau \) integration, we may interchange the orders of integration to obtain

\[
e^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{\infty} u^b \nabla_b (t^a \bar{G}_{ab'}) u^{b'} d\tau' d\tau = e^2 \int_{-\infty}^{\infty} \left[ t^a \bar{G}_{ab'} (\tau^+, \tau') - t^a \bar{G}_{ab'} (\tau^-, \tau') \right] u^{b'} d\tau' \\
= e \left[ t^a \bar{A}^\text{tail} (z(\tau^+)) - t^a \bar{A}^\text{tail} (z(\tau^-)) \right], \tag{53}
\]

where \( \bar{A}^\text{tail} \) is the “tail part” of the time-symmetric vector potential for our point particle source. (Recall that the point \( \tau' = \tau^\pm \) is omitted from the integration in eq. (53) so that \( \bar{A}^\text{tail} \) is finite.) Now, since the particle is being held stationary for early and late times, the solution must become stationary at early and late times, so

\[
\lim_{\tau^\pm \to \pm \infty} t^a \bar{A}^\text{tail} (z(\tau^\pm)) = -V^\text{tail} (x^\pm, x^\pm), \tag{54}
\]

where \( V^\text{tail}(x_1, x_2) \) is the tail part of \( -t^a A_a \) as measured at spatial position \( x_1 \) for a point particle source held stationary at spatial position \( x_2 \) for all time and where \( x^\pm = \lim_{\tau^\pm \to \pm \infty} x(\tau^\pm) \). Therefore, we have for the first term

\[
\lim_{\tau^\pm \to \pm \infty} e^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{\infty} u^b \nabla_b (t^a \bar{G}_{ab'}) u^{b'} d\tau' d\tau = -e V^\text{tail}(x^+, x^+) + e V^\text{tail}(x^-, x^-). \tag{55}
\]

Since we further assume that the particle begins and ends in the same spatial position, we have \( x^+ = x^- \) and the right side vanishes.

On the other hand, the second term in eq. (52) can be analyzed as follows. The Green’s function \( \bar{G} \) satisfies the property that it is symmetric in its arguments

\[
\bar{G}_{aa'} (x, x') = \bar{G}_{a'a} (x', x). \tag{56}
\]

Furthermore, the invariance of \( \bar{G} \) under the time translation isometries implies that

\[\text{strictly speaking, the decay hypothesis directly tells us only about the decay of the Maxwell field tensor, } F_{ab}. \text{ In order to obtain eq. (54), we must extend the decay hypothesis to include the assumption that in the Lorentz gauge, the vector potential of any smooth, source free solution goes to zero at late times (say, pointwise along every Killing orbit).}

\[\text{This fact follows from the antisymmetry of } G^A - G^R \text{ together with the support properties of } G^A \text{ and } G^R. \text{ The antisymmetry of } G^A - G^R \text{ follows directly from the generalization to curved spacetime of the Maxwell version of lemma 3.2.1 of [13] (see also [2]).} \]
\[ \mathcal{L}_t G_{aa'} + \mathcal{L}_v G_{aa'} = 0. \]  

Consequently, if we integrate over the same limits in \( \tau \) and \( \tau' \), we have

\[
\int \int \mathcal{L}_t G_{b\alpha}(\tau, \tau') u^b u^\alpha \, d\tau' \, d\tau = \int \int \mathcal{L}_v G_{b\alpha}(\tau, \tau') u^b u^\alpha \, d\tau' \, d\tau
\]

\[
= \int \int \mathcal{L}_v G_{b\alpha}(\tau, \tau') u^b u^\alpha \, d\tau' \, d\tau
\]

\[
= - \int \int \mathcal{L}_t G_{b\alpha}(\tau, \tau') u^b u^\alpha \, d\tau' \, d\tau
\]  

Here the first equality was obtained by interchanging the dummy variables \( \tau \) and \( \tau' \), the second equality was obtained using eq. (52), and the last equality was obtained using eq. (57). This shows that any integral of the form (58) vanishes, provided only that the domain of integration is symmetric in \( \tau \) and \( \tau' \) and that the integral is suitably convergent in this domain to justify the interchange of orders of integration.

Applying eq. (58) to the domain \([\tau_-, \tau_+] \times [\tau_-, \tau_+]\), we find that the second term in eq. (52) becomes

\[
e^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{\infty} \mathcal{L}_t \bar{G}_{b\alpha} u^b u^\alpha \, d\tau' \, d\tau = e^2 \int_{\tau^-}^{\tau^+} \int_{\tau^-}^{\tau^+} \mathcal{L}_t \bar{G}_{b\alpha} u^b u^\alpha \, d\tau' \, d\tau + e^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{\tau^-} \mathcal{L}_t \bar{G}_{b\alpha} u^b u^\alpha \, d\tau' \, d\tau
\]  

Focusing attention on the first term, we find

\[
\lim_{\tau^+ \to +\infty} e^2 \int_{\tau^-}^{\tau^+} \int_{\tau^-}^{\tau^+} \mathcal{L}_t \bar{G}_{b\alpha} u^b u^\alpha \, d\tau' \, d\tau = - \lim_{\tau^+ \to +\infty} e^2 \int_{\tau^-}^{\tau^+} \int_{\tau^-}^{\tau^+} \mathcal{L}_v G_{b\alpha} u^b u^\alpha \, d\tau' \, d\tau
\]

\[
= - \lim_{\tau^+ \to +\infty} e^2 \int_{\tau^-}^{\tau^+} \int_{\tau^-}^{\tau^+} \mathcal{L}_v (u^b G_{b\alpha}) \, d\tau' \, u^b \, d\tau
\]

\[
= \lim_{\tau^+ \to +\infty} e^2 \int_{\tau^-}^{\tau^+} \chi u^b \bar{G}_{b\alpha} (z(\tau), z(\tau^+)) \, u^b \, d\tau
\]

\[
= \lim_{\tau^+ \to +\infty} \frac{e}{2} \chi u^b A_b^p (\tau^+)
\]

\[
= - \frac{e}{2} \mathcal{V}^{\text{tail}}(x^+, x^+),
\]  

where we have used the fact that \( t^a = \chi u^a \) for sufficiently late times. Therefore we have

\[
\lim_{\tau^+ \to +\infty} e^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{\tau^-} \mathcal{L}_t \bar{G}_{b\alpha} u^b u^\alpha \, d\tau' \, d\tau = - \frac{e}{2} \left[ \mathcal{V}^{\text{tail}}(x^+, x^+) - \mathcal{V}^{\text{tail}}(x^-, x^-) \right]
\]  

Again, this term vanishes when \( x^+ = x^- \).

The above result together with the previous proposition yield the main theorem of this section:

**Theorem 2.3** Let \((M, g_{ab})\) be a spacetime satisfying the conditions stated at the beginning of this section together with the decay hypothesis (and its extension indicated in footnote 8) and stationary solution property. Let \( z(\tau) \) be a timelike curve which differs from an orbit, \( z_0(\tau) \), of the stationary Killing field \( t^a \) only over a finite interval. Let \( F_{ab} \) be a solution to
Maxwell’s equations with source \( (28) \) which satisfies our Maxwell field assumptions. Then, we have

\[
\int_{I^+} \xi^a T_{ab} \epsilon^{cde} - \int_{I^-} \xi^a T_{ab} \epsilon^{cde} = \int t^a f_a \, d\tau. \tag{62}
\]

where \( f^a \) is the DeWitt-Brehme force, eq. \( (30) \).

Thus, both the correct force prescription \( f_a \) and the incorrect force prescription \( \hat{f}_a \) exhibit the property of global energy conservation. This underscores the fact that global energy conservation is insufficient to determine a local expression for the force on a point particle.

In Proposition 2.1 and Theorem 2.3, it was required that the particle begin and end in the same stationary state. It is interesting to consider the case where the particle is stationary for \( t < t^- \) and again becomes stationary for \( t > t^+ \), but its final position differs from its initial position. In this case the proof of Proposition 2.1 holds without essential change, and eq. \( (10) \) still applies. From eqs. \( (52), (55), \) and \( (61) \), we obtain

\[
\int_{I^+} t^a T_{ab} \epsilon^{cde} - \int_{I^-} t^a T_{ab} \epsilon^{cde} + \frac{e}{2} V_{\text{tail}}(x^+, x^+) - \frac{e}{2} V_{\text{tail}}(x^-, x^-) = \int t^a f_a \, d\tau. \tag{63}
\]

Comparison with eq. \( (8) \) strongly suggests that we identify \( \frac{e}{2} V_{\text{tail}}(x, x) \) as the renormalized electromagnetic self-energy, \( E_{\text{self}} \), of a stationary point charge at position \( x \)

\[
E_{\text{self}}(x) = \frac{e}{2} V_{\text{tail}}(x, x) \tag{64}
\]

From the analysis of Smith and Will \([18]\), it can be seen that for a static point charge in Schwarzschild spacetime, we have

\[
E_{\text{self}} = \frac{e^2 M}{2r^2} \tag{65}
\]

where \( M \) is the mass of the Schwarzschild spacetime and \( r \) is the Schwarzschild radial coordinate. For sufficiently slow motion (relative to the static Killing field), the static force associated with eq. \( (53) \) should dominate over any “damping force” associated with radiated energy\(^8\). Therefore, in the slow motion limit, we would expect to obtain the dominant self-force correction to the motion of a freely falling test charge in Schwarzschild spacetime by replacing the usual expression for energy,

\[
E \equiv -mg_{ab} t^a u^b = m \left( 1 - \frac{2M}{r} \right) \dot{\ell}, \tag{66}
\]

by

\[
E' = E + E_{\text{self}}. \tag{67}
\]

\(^8\)The static force associated with eq. \( (53) \) is of order \( e^2 M/r^3 \), whereas in the slow motion, weak field limit, the damping force on a circular geodesic orbit is of order \( e^2 \omega^3 r = e^2 M^{3/2}/r^{9/2} \) (see \([3]\)), which is smaller by a factor of \((M/r)^{1/2}\).
since \( E' \) (rather than \( E \)) should be a constant of the motion. Solving for \( \dot{t} \) and plugging into the expression \( g_{ab} u^a u^b = -1 \) with \( L = mr^2 \phi \), we obtain

\[
\frac{1}{2} m r^2 + \left( 1 - \frac{2M}{r} \right) \left( \frac{L^2}{2mr^2} + \frac{m}{2} \right) + \frac{E_{\text{self}} E'}{m} - \frac{E_{\text{self}}^2}{2m} = \frac{E'}{2m}. \tag{68}
\]

which is the equation of motion for a particle of mass \( m \) and energy \( E'^2/2m \) in a one-dimensional potential, \( mU_{\text{eff}} \). To lowest nontrivial order in \( e \), \( U_{\text{eff}} \) is given by

\[
U_{\text{eff}} = \frac{1}{2} - \frac{M}{r} + \frac{L^2}{2m^2 r^2} - \frac{ML^2}{m^2 r^3} + \frac{e^2 ME}{2m^2 r^2}. \tag{69}
\]

The last term in eq. (69) represents the “self-force” correction to the motion of a charged particle in Schwarzschild spacetime. Interestingly, for the case of a nearly extreme Reissner-Nordstrom black hole, a self-energy correction of the form (67) with \( E_{\text{self}} \) given by eq. (65) is of just the right nature to eliminate the possible counterexamples to cosmic censorship recently proposed by Hubeny [19].

How are our arguments and results modified if the particle motion is not stationary in the past and/or future? In our arguments above, we used stationarity in the past and future only to obtain the following four results: (1) To conclude (in conjunction with our stationary solution property) that the advanced and retarded solutions satisfy our Maxwell field assumptions. (2) To conclude that the “advanced-retarded cross-term” makes no contribution to the energy flux through \( I^+ \) and \( I^- \) (see the right side of eq. (39) above). (3) To prove eq. (44). (4) To evaluate the right side of eq. (52). Thus, eq. (62) will continue to hold in all other circumstances where the above results are valid and where the right side of eq. (52) vanishes. In particular, if we consider a particle orbit which comes in from infinity in the asymptotic past in (nearly) geodesic motion and emerges to infinity in the future also in (nearly) geodesic motion, then, by calculations similar to those done in the stationary case, the right side of eq. (52) can be shown to be given by the limit of “tail terms” in the asymptotic past and future, which should vanish. However, it is somewhat more delicate as to whether the other results hold. For example, results (1)-(3) should hold for a point charge in Minkowski spacetime which undergoes exactly inertial motion in the asymptotic past and future. However, for the case of two point charges in Minkowski spacetime which move near infinity under the influence of each other’s Coulomb field in the asymptotic past and future, property (1) does not hold since \( \tilde{F}^R_{ab} \) fails to smoothly extend to \( I^- \) and \( \tilde{F}^A_{ab} \) fails to smoothly extend to \( I^+ \) [20,21]. Nevertheless, the “pullback” of \( \tilde{F}^R_{ab} \) does continuously extend to \( I^- \); i.e., if we contract \( F^R_{ab} \) into smooth two vector fields which, at \( I^- \), are tangential to \( I^- \), then the resulting scalar field continuously extends to \( I^- \). Since only these components of \( \tilde{F}^R_{ab} \) are relevant for our energy flux calculations, it appears that, even though result (1) fails, eq. (52) will hold nevertheless in this case. More generally, we expect eq. (52) to hold for particle orbits in curved spacetime which come in from infinity in the asymptotic past in (nearly) geodesic motion and emerge to infinity in the future also in (nearly) geodesic motion, provided, of course, that the spacetime satisfies our spacetime assumptions, the decay hypothesis, and the stationary solution property.

If the spacetime \((M, g_{ab})\) is axisymmetric as well as stationary, then essentially all of the results leading to Theorem 2.3 carry over straightforwardly if we replace the timelike Killing
field $t^a$ by the axial Killing field $\phi^a$. The only exception is the analysis of the "advanced-retarded cross-term" of eq. (39) above, which now no longer automatically vanishes as a consequence of eq. (18). Nevertheless, under the additional assumption that point charge stationary solutions are axisymmetric on $I^+ \cap I^-$ and that the vector potential of a source-free solution can be chosen to vanish as one approaches timelike infinity along $I^+$ and $I^-$, this cross-term can be seen to vanish by the following argument.

For definiteness, let us consider the integral over past null infinity. In light of our assumption that the point particle source $j^a$ differs from the stationary source $j^S_a$ only in a compact region of $M$, on $I^-$ we can write $F_{ab}^R$ and $F_{ab}^A$ as

$$
F_{ab}^R = F_{ab}^S
$$

$$
F_{ab}^A = F_{ab}^S + F_{ab}^I.
$$

where $F_{ab}^S$ is the retarded (= advanced) solution associated with $j^S_a$. Then $F_{ab}^I$ is source-free outside of a compact region of $M$ and vanishes outside of the causal past of this region. It follows from eq. (18) that $T_{ab}[F^S, F^S]$ radiates no angular momentum through $I^-$, so we have

$$
2 \int_{I^-} \phi^a T_{ab}[F^R, F^A] \epsilon_{cde}^b = 2 \int_{I^-} \phi^a T_{ab}[F^S, F'] \epsilon_{cde}^b
$$

$$
= \int_{I^-} \phi^a \left( F_{af}^S F_{fb}^I + F_{af}^I F_{fb}^S - \frac{1}{2} g_{ab} F_{fg}^S F_{fg}^I \right) \epsilon_{cde}^b.
$$

(72)

Since $\phi^a$ lies in $I^-$, the last term clearly vanishes. Writing out $F_{ab}^I$ and $F_{ab}^S$ in terms of the corresponding vector potentials, we arrive at

$$
2 \int_{I^-} \phi^a T_{ab}[F^R, F^A] \epsilon_{cde}^b = \int_{I^-} \left( \phi^a \nabla^a [A^S_f]^I b_f + \phi^a \nabla^a [A_f^I]^a b_f \right) \epsilon_{cde}^b
$$

$$
= \int_{I^-} \nabla_f (\phi^a A^S_a) F^b f \epsilon_{cde}^b - \int_{I^-} \nabla_f A^S_a F^b f \epsilon_{cde}^b
$$

$$
+ \int_{I^-} \nabla_f (\phi^a A_f^I) (F^S)^b f \epsilon_{cde}^b - \int_{I^-} \nabla_f A_f^I (F^S)^b f \epsilon_{cde}^b
$$

(73)

The second integral is immediately seen to vanish under our assumption that the vector potential of a stationary solution can be chosen to be axisymmetric at $I^-$. Integrating by parts in the fourth integral gives us

$$
\int_{I^-} \nabla_f (\phi^a A^S_a) F^b f \epsilon_{cde}^b = - \int_{I^-} A^I_f \nabla_f (F^S)^b f \epsilon_{cde}^b,
$$

(74)

which vanishes under the same assumption. (The "boundary term" makes no contribution since the orbits of $\phi^a$ are closed). For the first integral, since $F_{ab}^I$ satisfies Maxwell’s equations, we have

$$
\int_{I^-} \nabla_f (\phi^a A^S_a) F^b f \epsilon_{cde} = 3 \int_{I^-} d \left( (\phi \cdot A^S) \ast F' \right),
$$

(75)

where the two-form $\ast F'$ is the Hodge dual, defined by $\ast F'_{ab} = -(1/2) \epsilon_{abcd} F'_{cd}$. Using Stokes’ theorem, we can convert this integral to two boundary integrals, one near $i^0$ and the other near $i^-$. Since the two-form $\ast F'$ is source-free outside of a compact region of $M$, the integrand
vanishes near $i^0$ by causality and vanishes near $i^-$ under the decay hypothesis. Similarly, for the third integral we have

$$
\int_{I^-} \nabla_f (\phi^a \cdot A'_a) (F^S)^{bf} \epsilon_{cdef} = 3 \int_{I^-} d \left[ (\phi \cdot A') \ast F^S \right].
$$

This term vanishes because the vector potential $A'_a$ can be chosen to vanish near $i^0$ by causality and, by our assumption, can be chosen to vanish near $i^-$. By reversing the roles of $F^R_{ab}$ and $F^A_{ab}$ and repeating the above arguments, we also show that the integral over $I^+$ vanishes. Thus, if $(M, g_{ab})$ is axisymmetric as well as stationary and the additional conditions given above are met, then under the hypotheses of Theorem 2.3, eq. (62) continues to hold if we replace $t^a$ by $\phi^a$.

Finally, we note that our results can be generalized to allow for the presence of a black hole and white hole whose horizons, $\mathcal{H}^+$ and $\mathcal{H}^-$, intersect on a compact 2-surface, $S$. We again require $(M, g_{ab})$ be a globally hyperbolic, stationary spacetime that is asymptotically flat at null and spatial infinity. However, we now require that there exist a spacelike hypersurface, $\Sigma$, with boundary, $S$, of the form $\Sigma = \Sigma_{\text{end}} \cup \Sigma'$ with $\Sigma'$ compact, such that $\Sigma \setminus S$ is a Cauchy surface for the domain of outer communications. In this case, we deform $\Sigma$ to the future in small neighborhoods of both $i^0$ and $S$ so that the deformed surface, $\mathcal{C}^+$, intersects both $I^+$ and the future event horizon $\mathcal{H}^+$ in a cross-section. We construct $\mathcal{C}^-$ similarly. With these changes, our previous analysis for the case of no black or white holes can now be repeated wherein in all of our assumptions, arguments, and formulas, we replace $I^+$ by $I^+ \cup \mathcal{H}^+$ and $I^-$ by $I^- \cup \mathcal{H}^-$. In particular, Theorem 2.3 holds for the above class of spacetimes containing black holes and white holes provided only that we modify eq. (62) so as to include contributions from the energy flux through $\mathcal{H}^+$ and $\mathcal{H}^-$ in exact parallel with the contributions from $I^+$ and $I^-$. However, it would appear that some significant differences do occur in the classes of spacetimes for which the decay hypothesis and stationary solution property hold. Recall that when no black hole or white hole is present, we conjectured that the decay hypothesis would hold if and only if no ergoregion is present. However, for a rotating black hole—i.e., if $t^a$ is not normal to $\mathcal{H}^+$ and $\mathcal{H}^-$—the “ergoregion instability” argument no longer applies, since negative energy can be radiated into the black hole. Thus, it seems plausible that the decay hypothesis will hold for many rotating black holes—in particular, for Kerr black holes—despite the presence of ergoregions. On the other hand, for the case of a rotating black hole, it does not seem plausible that the stationary solution property holds (where, with the modifications described above, this property now includes the requirement that the Maxwell field extend continuously to $\mathcal{H}^+$ and $\mathcal{H}^-$). Specifically, for the retarded solution for a non-axisymmetric, stationary source, there should be a steady, nonvanishing flux of angular momentum into the black hole. Consequently, $F^R_{ab}$ should not be continuously

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9The only exception is that for a rotating black hole, the vanishing of the advanced-retarded cross-terms does not follow from the horizon analog of eq. (18) but instead must be shown by the arguments similar to those used above for the case of angular momentum. However, as we shall note shortly, the validity of the stationary solution hypothesis appears to be limited to non-rotating black holes in any case.
extendible to $\mathcal{H}^{-}$. Similarly, $F_{ab}^A$ should fail to be continuously extendible to $\mathcal{H}^{+}$. However, it seems plausible that the stationary solution property always holds for a non-rotating black hole. Thus, it appears likely that the hypotheses of Theorem 2.3 hold precisely for the case of non-rotating black holes that contain no ergoregion. Of course, the conclusion of that theorem may still hold under more general hypotheses.

Finally, in the axisymmetric case, we may consider conservation of angular momentum in spacetimes containing a non-rotating black hole and white hole. As in the case where no black hole or white hole is present, all of our arguments carry through straightforwardly when $t^{a}$ is replaced by $\phi^{a}$ except for the vanishing of the advanced-retarded cross term. However, the presence of a black and white hole now produces a significant difference because, even in Schwarzschild spacetime, the field of a stationary point charge fails to be axisymmetric on the horizon. Thus, the symmetry argument used above to obtain the vanishing of the advanced-retarded cross terms at $I^{+}$ and $I^{-}$ is inapplicable to $\mathcal{H}^{+}$ and $\mathcal{H}^{-}$. We have not been able to show that these cross terms vanish (or that the cross term contribution from $\mathcal{H}^{+}$ cancels that from $\mathcal{H}^{-}$). It would be of interest to determine if any violations of angular momentum conservation can occur in this case and, if so, how large the violations could be.

III. GRAVITATIONAL CASE

In this section, we shall give an analysis of conservation results in the gravitational case in close parallel with our analysis of the electromagnetic case given in the previous section. Although there are many similarities between the gravitational and electromagnetic cases, there also are some key differences as well as some additional technical difficulties occurring in the gravitational case. Consequently, with the exception of Theorem 3.1 below, we shall not attempt to prove theorems based upon analogs of our Maxwell field and decay hypothesis assumptions, but rather will merely sketch how it should be possible to obtain similar results. In this sense, our results for point particles in the gravitational case will be considerably weaker than in the electromagnetic case.

To begin, we wish to analyze energy conservation for solutions to the linearized Einstein equation on a spacetime, $(\mathcal{M}, g_{ab})$, which satisfies the spacetime assumptions stated at the beginning of the previous section. However, unlike the electromagnetic case, in order to have well defined linearized field equations, it is necessary here that the background spacetime be a solution to Einstein’s equation. Although there should not be any difficulty in principle in allowing the background spacetime to possess suitable matter fields, in order to directly apply the results of [3] in the case of point particles, we must restrict to the case where the spacetime is vacuum at least in a neighborhood of the worldline of the particle. This

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10In particular, in the static case, the equality of $F_{ab}^{R}$ and $F_{ab}^{A}$ for static sources throughout the domain of outer communications follows from the time reflection symmetry. Since $F_{ab}^{R}$ is regular on $\mathcal{H}^{+}$ and $F_{ab}^{A}$ is regular on $\mathcal{H}^{-}$ it follows that both must have continuous limits to $\mathcal{H}^{+} \cup \mathcal{H}^{-}$, in accord with the stationary solution property. Note, however, that $F_{ab}^{R}$ will be discontinuous across $\mathcal{H}^{-}$ and $F_{ab}^{A}$ will be discontinuous across $\mathcal{H}^{+}$ in a manner similar to the behavior of the retarded and advanced solutions for a uniformly accelerating charge in Minkowski spacetime [22].
restriction would still allow us to consider, for example, a fluid star solution in which the particle remains exterior to the star, but the inclusion of fluid or other matter fields would significantly complicate the analysis. Hence, for simplicity, we shall restrict attention to the case where the background spacetime satisfies $G_{ab} = 0$. Unfortunately, this restricts the background spacetime to be Minkowski spacetime.\footnote{11} More generally, when black holes are admitted, the stationary black hole uniqueness theorems restrict the background spacetime to being Kerr. However, we shall not make explicit use of any special properties of the background spacetime in our analysis below, and we believe that our analysis should generalize to cases where suitable matter is present in the background spacetime.

On $(M, g_{ab})$, we initially wish to consider smooth solutions, $\gamma_{ab}$, to the linearized Einstein equation with a smooth source $T_{ab}$

$$G^{(1)}_{ab}[\gamma_{cd}] = 8\pi T_{ab} \quad (77)$$

Our basic strategy here will be to repeat the analysis of the previous section, replacing the electromagnetic energy current three-form $\epsilon^{a}T_{ab} \epsilon_{cde}^{b}$ with the “effective gravitational energy current” three-form $\epsilon^{a}\tau_{ab} \epsilon_{cde}^{b} \equiv -\frac{1}{8\pi}T_{ab}^{(2)} \epsilon_{cde}^{b}$ constructed from the second order Einstein tensor $G^{(2)}_{ab}$ (see \cite{10}) associated with the linearized solution. In doing so, we will rely on the results of Habisohn \cite{10}, which show that the net flux from this effective energy current at null infinity agrees with the net Bondi flux to second order. However, in order to apply Habisohn’s results, we need to assume that $T_{ab}$ vanishes in a neighborhood, $U$, of $\mathcal{I}^{+} \cup \mathcal{I}^{-} \cup i^{0}$ that contains complete Killing orbits and that, in $U$, $\gamma_{ab}$ satisfies both the weak and strengthened fall-off conditions of \cite{10}. Here, the weak fall-off conditions of \cite{11} require that $\gamma_{ab}$ vanish in a neighborhood of $i^{0}$ and satisfy the decay properties in $M \cap U$ stated in \cite{10}. The strengthened fall-off conditions require that, when transformed to the Geroch-Xanthopoulos gauge \cite{23}, the unphysical metric perturbation $\tilde{\gamma}_{ab} = \Omega^{2}\gamma_{ab}$ (which smoothly extends to $\mathcal{I}^{+}$ and $\mathcal{I}^{-}$ in this gauge) satisfies similar decay properties in $\tilde{M} \cap U$, where $\tilde{M}$ denotes the conformally completed spacetime. The requirement that $\gamma_{ab}$ vanish in a neighborhood of $i^{0}$ is much stronger than condition (3) imposed in the electromagnetic case, and, indeed, is too strong to enable us to use it directly for our purposes below. Undoubtedly, Habisohn’s conditions could be weakened considerably, but this would require further analysis that we do not wish to undertake here.

Let $g_{ab}(\lambda) = g_{ab} + \lambda\gamma_{ab}$, where $G_{ab}[g_{cd}] = 0$ and $\gamma_{ab}$ satisfies eq. (77). The Bianchi identity to second order in $\lambda$ yields

$$\frac{1}{2} \frac{d^{2}}{d\lambda^{2}} \left[ \nabla^{(1)}_{a}G^{(2)}_{ab}(\lambda) \right]_{\lambda=0} = G^{(2)}_{ab}[\gamma_{cd}] + \nabla^{(1)}_{a}G^{(1)}_{ab}[\gamma_{cd}] = 0; \quad (78)$$

i.e.,

$$\nabla_{a}G^{(2)ab}[\gamma_{cd}] = -C^{a}_{ad}[\gamma_{cd}]G^{(1)db}[\gamma_{cd}] - C^{b}_{ad}[\gamma_{cd}]G^{(1)ad}[\gamma_{cd}] \quad (79)$$

\footnote{11}This follows immediately from the positive energy theorem together with the fact that the (Komar) mass of a vacuum spacetime satisfying the spacetime assumptions stated at the beginning of Sec. II is easily seen to vanish.
where

\[ C^{a\,bc} = \frac{1}{2} g^{ad} [\nabla_b \gamma_{cd} + \nabla_c \gamma_{bd} - \nabla_d \gamma_{bc}] \] (80)

and where \( \nabla_a \) denotes the derivative operator associated with \( g_{ab} \). Since \( \gamma_{ab} \) satisfies the linearized Einstein equation (77), we find

\[ \nabla_a G^{(2)ab} = -8\pi [C^{a\,ac} T_{cb} - C^{b\,ac} T_{ac}] \] (81)

Equation (81) is a direct analog of eq. (4) in the electromagnetic case. Furthermore, although in this case the physical “effective energy current three-form”

\[ t^a \tau_{ab} \epsilon_{cde} b \equiv -\frac{1}{8\pi} t^a G^{(2)ab} \epsilon_{cde} b \] (82)

need not continuously extend to null infinity, Habsohn [10] has shown that if his weak and strengthened fall-off conditions hold, its pullback to certain timelike 3-surfaces has a continuous limit as these surfaces approach null infinity. Consequently, in exact parallel with the derivation of eq. (8), we obtain the following theorem:

**Theorem 3.1** Let \((M, g_{ab})\) be a vacuum solution of Einstein’s equation satisfying the space-time assumptions\(^{12}\) of Sec. II. Let \( \gamma_{ab} \) be a smooth solution to the linearized Einstein equation with smooth source \( T_{ab} \) such that \( T_{ab} \) vanishes in a neighborhood, \( U \), of \( \mathcal{I}^+ \cup \mathcal{I}^- \cup i^0 \) that contains complete Killing orbits and such that, in \( U \), \( \gamma_{ab} \) satisfies both the weak and strengthened fall-off conditions of [10]. Then,

\[
\lim_{t^+ \to +\infty} \int_{C^+ (t^+)} t^a \tau_{ab} \epsilon_{cde} b = \lim_{t^- \to -\infty} \int_{C^- (t^-)} t^a \tau_{ab} \epsilon_{cde} b + \int_{\mathcal{I}^+} t^a \tau_{ab} \epsilon_{cde} b - \int_{\mathcal{I}^-} t^a \tau_{ab} \epsilon_{cde} b = -\int_M t^a [C^{b\,bc} T_{ca} + C^{a\,be} T_{bc}] \epsilon_{defg}. \] (83)

provided that the integral over \( M \) and the limits of the integrals over \( C^+ (t^+) \) and \( C^- (t^-) \) exist. Here, the integrals over \( \mathcal{I}^+ \) and \( \mathcal{I}^- \) are understood to mean the limits of the integrals over suitable timelike surfaces in the physical spacetime which approach \( \mathcal{I}^+ \) and \( \mathcal{I}^- \); these limits were proven by Habsohn [10] to exist.

The second pair of terms in eq. (83) was proven by Habsohn [10] to agree, to second order in \( \gamma_{ab} \), with the net Bondi energy radiated to infinity. In analogy with the electromagnetic case, it would be natural to attempt to interpret the first pair of terms as representing the difference between the initial and final gravitational energy of the matter distribution, and, similarly, to interpret the right side as representing the work done by the matter on the gravitational field. However, unlike the electromagnetic case, these terms are, in general, gauge dependent. Thus, unless some further restrictions and/or gauge conditions

\(^{12}\)As previously noted, these assumptions actually imply that \((M, g_{ab})\) is Minkowski spacetime. However, as also previously noted, the theorem can be generalized to allow for black holes (in which case the Kerr family of solutions would be admissible) and, presumably, also could be generalized to allow solutions to the nonvacuum Einstein equation with suitable forms of matter.

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are imposed, such an interpretation would not, in general, be meaningful. Nevertheless, under the hypotheses of the theorem, the second pair of terms is gauge independent—provided, of course that one stays in a gauge where the weak fall-off conditions hold.

In order to proceed further, in analogy with the analysis given for the electromagnetic case, we need to restrict consideration to situations where the first pair of terms in eq. (83) cancel. As in the electromagnetic case, we consider the case where $T_{ab}$ differs from a stationary matter distribution only in a compact region of spacetime. However, in order that $\gamma_{ab}$ satisfy the decay properties needed for cancellation of the first pair of terms in eq. (83), it also is necessary to impose gauge conditions on $\gamma_{ab}$. Note that if a choice of gauge compatible with the Habisohn fall-off conditions [10] has been made so that the first pair of terms in eq. (83) cancel, the right side of that equation automatically must be gauge invariant with respect to any remaining restricted gauge freedom. Therefore, in that case it would seem appropriate to interpret the right side of that equation as corresponding to the work done by the matter on the gravitational perturbation, $\gamma_{ab}$, from the “viewpoint” of the background spacetime, $(M, g_{ab})$.

A choice of gauge which appears to be suitable for the above purpose is the Lorentz gauge condition,

$$\nabla^a \tilde{\gamma}_{ab} = 0$$

where

$$\tilde{\gamma}_{ab} = (\gamma_{ab} - \frac{1}{2} \gamma g_{ab})$$

In the Lorentz gauge, $\tilde{\gamma}_{ab}$ satisfies a wave equation,

$$\nabla^c \nabla_c \tilde{\gamma}_{ab} - 2 R^c_{\ ab \ cd} \tilde{\gamma}^{cd} = -16\pi T_{ab}$$

and it seems highly plausible that, in a wide class of spacetimes, suitable decay properties at late and early times will hold for source free solutions with suitable fall-off near $i^0$. In particular, it seems plausible that for source free solutions with suitable fall-off near $i^0$, in this gauge the supremum over $C^+(t^+)$ of the components of $\gamma_{ab}$ and its first two derivatives will go to zero as $t^+ \to \infty$, and that similar decay properties will hold as $t^- \to -\infty$. It also seems plausible that, for such solutions in this gauge, the stronger fall-off conditions of [11] near $I^+$ and $I^-$ will hold as $t^+ \to \infty$, and $t^- \to -\infty$. Finally, it appears highly plausible that, in the Lorentz gauge, an analog of the stationary solution property (see Sec. II) will hold. Consequently, a direct analog of theorem 2.2 should hold in the gravitational case when we impose the Lorentz gauge condition. However, we shall not attempt to state and prove such a theorem here, since it would require considerable further analysis to make a judicious choice of mathematically precise formulations of the gravitational versions of the decay hypothesis and the other relevant assumptions needed for the proof of such a theorem.

We may now attempt ask in the gravitational case whether, for a point particle source whose motion differs from a Killing orbit for only a finite time interval, the analog of eq. (B2) holds, i.e., whether

$$\int_{I^+} t^a \tau_{ab} \epsilon^{bcde} - \int_{I^-} t^a \tau_{ab} \epsilon^{bcde} = \int t^a f_a \ d\tau.$$  (87)
where the integrals over $I^+$ and $I^-$ have the same meaning as in theorem 3.1, and where $f^a$ now denotes the gravitational self-force \[ f^a = m \left( \frac{1}{2} \nabla^a \gamma^b_{ac} - \nabla^b (\gamma^a_{bc})_c \right) u^b u^c - m^2 \left( \frac{11}{3} \dot{a}^a + \frac{1}{3} a^2 u^a \right) \\
+ m^2 u^b u^c \int_{-\infty}^{\tau} \left( \frac{1}{2} \nabla^a G^R_{bca'b'} - \nabla^b (G^R_{c'a'b'})_a \right) u^{a'} u^{b'} d\tau' \]

(88)

Unfortunately, however, this question as posed has only a trivial domain of applicability: As emphasized in \[5\], in order for a solution to the linearized Einstein equation with a point particle source to exist, it is necessary that the particle move on a geodesic. Therefore, a point particle cannot follow a Killing orbit for a finite time interval unless that Killing orbit is a geodesic, in which case the particle’s motion can never deviate from the Killing orbit. In other words, the constraints on the motion of a particle imposed by the linearized Einstein equation preclude the possibility of having a particle which is nonstationary for only a finite time interval. Note that this contrasts sharply with the situation in the electromagnetic case, where Maxwell’s equations impose no constraints on the motion of a charged particle.

Nevertheless, it is worth noting that it is not manifestly inconsistent to have general distributional sources containing point masses which are stationary outside of a compact spacetime region. In other words, we may have a system of point masses connected by devices such as rods, springs, or strings such that the point masses are initially and finally in the same stationary position, but are set into motion at intermediate times. Under decay assumptions of the type expected to hold in the Lorentz gauge as described above, it then should be possible to prove a direct analog of Proposition 2.1. Namely, if $T_{ab}$ now denotes a distributional stress-energy tensor which is stationary outside of a compact spacetime region, and if $\gamma_{ab}$ denotes a (distributional) solution to the linearized Einstein equation with source $T_{ab}$ such that $\gamma_{ab}$ is smooth near infinity and satisfies suitable decay properties near $i^0$, then it should follow that

$$\int_{I^+} t_a \tau_{ab} \epsilon_{cde} \frac{b}{b} - \int_{I^-} t_a \tau_{ab} \epsilon_{cde} \frac{b}{b} = - \int_M t_a \left[ (C^\text{free})_{bc} T^{ca} \right. + \left. (C^\text{free})_{bc} T^{bc} \right] \epsilon_{defg}. \quad (89)$$

where $(C^\text{free})_{bc}$ is given by eq. \[80\], with $\gamma_{ab}$ replaced by $\gamma^\text{free}_{ab}$, where

$$\gamma^\text{free}_{ab} \equiv \gamma_{ab} - \bar{\gamma}_{ab} \quad (90)$$

with

$$\bar{\gamma}_{ab} = \frac{1}{2} [\gamma^A_{ab} + \gamma^R_{ab}]. \quad (91)$$

Note that in eq. \[89\] it is, in general, necessary to take into account the effect of the rods and other devices both directly with regard to their contribution to $T_{ab}$ and indirectly with regard to their contribution to the gravitational perturbation $\gamma_{ab}$.

Although we cannot even pose a meaningful question for a system composed purely of point masses which are stationary in the past and future, we can consider energy conservation for a freely falling point mass which follows a trajectory which starts at infinity and returns
to infinity. In a general, curved spacetime such a particle of mass \( m \) will radiate energy of order \( m^2 \) to infinity, and one may ask if this agrees, to lowest order, with minus the net work done on the particle by the gravitational self-force. On account of this gravitational self-force, the particle will fail to move on a geodesic, resulting in self-consistency issues with regard to treating its motion that were discussed in \[4\]. However, in order to calculate the work done by the gravitational self-force to order \( m^2 \), the deviations from geodesic motion produced by the gravitational self-force can be neglected. Thus, we may ask whether eq. (87) holds for a particle moving on a geodesic.

In parallel with our discussion in the electromagnetic case (see the paragraph below eq. (69) of Sec. II), if suitable decay properties at late and early times hold for source free solutions, if the advanced and retarded solutions for the particle moving on the given geodesic are suitably well behaved, and if the “advanced-retarded cross-term” makes no contribution to the flux of \( t^a \tau_{ab} \epsilon_{cde} \) through \( \mathcal{I}^+ \) and \( \mathcal{I}^- \), then eq. (89) should hold, i.e.,

\[
\int_{\mathcal{I}^+} t^a \tau_{ab} \epsilon_{cde} - \int_{\mathcal{I}^-} t^a \tau_{ab} \epsilon_{cde} = \int t^a \hat{f}_a \, d\tau. \tag{92}
\]

where

\[
\hat{f}^a = m \left( \frac{1}{2} \nabla^a \gamma_{bc} - \nabla_b (\gamma^a_{\ \ c})_c \right) u^b u^c + \frac{1}{2} m^2 u^b u^c \int_{-\infty}^{\tau} \left( \frac{1}{2} \nabla^a G^R_{bca'b'} - \nabla_b (G^R)_{c \ \ a'b'} \right) u^{a'} u^{b'} \, d\tau', \tag{93}
\]

However, the difference between \( f^a \) and \( \hat{f}^a \) is given by

\[
f^a - \hat{f}^a = m^2 u^b u^c \int_{-\infty}^{+\infty} \left( \frac{1}{2} \nabla^a G^-_{bca'b'} - \nabla_b (G^A)_{c \ \ a'b'} \right) u^{a'} u^{b'} \, d\tau'. \tag{94}
\]

Our task is to show that, just as in the electromagnetic case, this difference between the two forces does no net work over the world line of the particle.

In analogy with eq. (52), we have

\[
\int_{\tau^-}^{\tau^+} t^a (f_a - \hat{f}_a) \, d\tau = m^2 \int_{\tau^-}^{\tau^+} t^a \left[ \int_{-\infty}^{+\infty} \left( \frac{1}{2} \nabla^a \epsilon_{bca'b'} - \nabla_b \epsilon_{c \ \ a'b'} \right) u^{a'} u^{b'} \, d\tau' \right] u^b u^c \, d\tau
\]

\[
= \frac{m^2}{2} \left[ \int_{\tau^-}^{\tau^+} \int_{-\infty}^{+\infty} \epsilon_{bca'b'} - \epsilon_{ac'b'} \nabla_b t^a - t^a \nabla_b \epsilon_{c \ \ a'b'} \right] u^b u^c u^{a'} u^{b'} \, d\tau' \, d\tau
\]

\[
= \frac{m^2}{2} \left[ \int_{\tau^-}^{\tau^+} \int_{-\infty}^{+\infty} \epsilon_{bca'b'} u^b u^c u^{a'} u^{b'} \, d\tau' \, d\tau
\]

\[
- m^2 \left[ \int_{\tau^-}^{\tau^+} \int_{-\infty}^{+\infty} \nabla_b (t^a u^c \epsilon_{a'b'}) u^{a'} u^{b'} \, d\tau' \, d\tau \right] \tag{95}
\]

\[13\] As already mentioned, our above spacetime assumptions restrict us to flat spacetime, but the remarks here are relevant for generalizations to allow for the presence of black holes as well as generalizations to non-vacuum cases.
By arguments similar to those which produced eqs. (55) and (61), both of these integrals can be reduced to expressions involving “tail contributions” on the worldline of the particle for \( \tau \geq \tau^+ \) and \( \tau \leq \tau^- \). Since here we restrict consideration to geodesic particle trajectories which start at infinity and return to infinity in asymptotically flat spacetimes, these tail contributions should vanish in the limit \( \tau^\pm \to \pm\infty \). Thus, eq. (87) should hold in this case, although the reader surely will have noted the numerous gaps in our arguments that would have to be filled in order to convert our arguments into a theorem.

Finally, we comment that, in parallel with the electromagnetic case, our arguments should generalize to yield a conservation of angular momentum result in the case of a spacetime that is axisymmetric as well as stationary. Similarly, our arguments should generalize straightforwardly to the case where black holes and white holes are present provided that the fluxes through the black hole and white hole horizons are included in the manner described near the end of Sec. II.

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APPENDIX A: MASSLESS KLEIN-GORDON SCALAR CASE

In this appendix, we outline the derivation of conservation results for the case of a massless Klein-Gordon field, satisfying

\[
\nabla^a \nabla_a \phi = -j. \tag{A1}
\]

To a large extent, the analysis of energy and angular momentum conservation for this case closely parallels the electromagnetic case analyzed in Sec. II, although there are some complications arising from the lack of conformal invariance of the theory and the important difference that the Klein-Gordon force need not be orthogonal to the four-velocity.

The stress-energy tensor of the massless Klein-Gordon scalar field is given by

\[
T_{ab} = \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} g^{cd} \nabla_c \phi \nabla_d \phi \tag{A2}
\]

By virtue of eq. (A1), it satisfies

\[
\nabla_b T_{ab} = -j \nabla_a \phi. \tag{A3}
\]

In precise analogy with the electromagnetic case, we define the energy three-current three-form by

\[
J_{cde} = t^a T_{ab} \epsilon_{cde} \tag{A4}
\]

It follows that

\[
(dJ)_{cdef} = j t^a \nabla_a \phi \epsilon_{cdef}. \tag{A5}
\]
The task of determining falloff conditions on $\phi$ at null infinity is less trivial here as compared with the Maxwell case since the field equation (A1) is not conformally invariant. However, the standard conformally invariant modification of eq. (A1), namely $\nabla^a \nabla_a \phi - \frac{1}{6} R \phi = -j$, is equivalent to eq. (A1) in vacuum spacetimes. This suggests that we define an “unphysical Klein-Gordon scalar field”, $\tilde{\phi}$, by using the same conformal weight as in the conformally invariant case, i.e.,

$$\tilde{\phi} = \Omega^{-1} \phi$$

and require that $\tilde{\phi}$ extend smoothly to $I^+$ and $I^-$. We adopt this condition as the analog of the Maxwell field assumption (2) of Sec. II.

In terms of unphysical variables, the physical energy three-current takes the form

$$J_{cde} = t^a \left[ \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} g^{cd} \nabla_c \phi \nabla_d \phi \right] g^{bf} \epsilon_{cdef}$$

$$= t^a \left[ \nabla_a \tilde{\phi} \nabla_b \tilde{\phi} + 2 \Omega^{-1} \tilde{\phi} \nabla_a \tilde{\phi} \nabla_b \Omega + \Omega^{-2} \tilde{\phi}^2 \nabla_a \Omega \nabla_b \Omega 
- \frac{1}{2} g_{ab} g^{cd} \left( \nabla_c \tilde{\phi} \nabla_d \tilde{\phi} + 2 \Omega^{-1} \tilde{\phi} \nabla_c \tilde{\phi} \nabla_d \Omega + \Omega^{-2} \tilde{\phi}^2 \nabla_c \Omega \nabla_d \Omega \right) \right] g^{bf} \epsilon_{cdef}$$

where $\nabla_a$ denotes the derivative operator associated with the unphysical metric $\tilde{g}_{ab}$ (although it should be noted that in this formula, only derivatives of scalar fields are taken, so there actually is no distinction between $\nabla_a$ and $\nabla_a$). Since $n_a \equiv \nabla_a \Omega \neq 0$ on $I \equiv I^+ \cup I^-$ we see that $J_{cde}$ fails to extend continuously to $I$ even when $\tilde{\phi}$ extends smoothly to $I$. However, the pull-back, $\tilde{J}_{cde}$, of $J_{cde}$ to a surface of constant $\Omega$ is given by

$$\tilde{J}_{cde} = - \left[ t^a \nabla_a \tilde{\phi} n_b \nabla_b \tilde{\phi} + f \tilde{\phi} t^a \nabla_a \tilde{\phi} + \frac{1}{2} h \tilde{\phi}^2 - \frac{1}{2} \Omega h g^{cd} \nabla_c \tilde{\phi} \nabla_d \tilde{\phi} \right] (3) \tilde{\epsilon}_{cde}$$

where the functions $f \equiv \Omega^{-1} \tilde{g}_{ab} n_a n_b$ and $h \equiv \Omega^{-1} t^a n_a$ extend smoothly to $I$, and $(3) \tilde{\epsilon}_{abc}$ is defined (up to a three-form with vanishing pullback) by

$$\tilde{\epsilon}_{abcd} = n_{[a} (3) \tilde{\epsilon}_{bcd]}$$

Consequently, although $J_{cde}$ fails to extend continuously to $I$, it can be seen by inspection of eq. (A8) that $\tilde{J}_{cde}$ extends smoothly to $I$. Equivalently, if we choose any three smooth vector fields in the unphysical spacetime which are everywhere tangent to surfaces of constant $\Omega$ and we contract these vector fields with the free indices of $J_{cde}$, then the resulting function will extend smoothly to $I$.

However, it is not difficult to verify that, in general, the limit of $\tilde{J}_{cde}$ to $I$ will depend upon the choice of conformal factor, $\Omega$, used to define the surfaces to which $J_{cde}$ is pulled back. Nevertheless, we argue now that the integral over $I^+$ or $I^-$ of $\tilde{J}_{cde}$ cannot depend upon the choice of $\Omega$, provided only that the scalar source, $j$, vanishes in a neighborhood of $I \cup i^0$ and that $\tilde{\phi}$ has suitable fall-off properties at $i^0$ and at timelike infinity. Namely, let $\Omega$ and $\Omega'$ be two choices of conformal factor. Integrate $(dJ)_{cdef}$ over the region bounded by a surface of constant $\Omega$ and a surface of constant $\Omega'$, with suitable “endcaps” inserted near $i^0$ and future timelike infinity. Now apply Stokes’ theorem—using eq. (A4), the vanishing of $j$ near $I$, and the decay of $\phi$ near $i^0$ and timelike infinity—to conclude that, in the limit
as one approaches $I^+$, the integral of $J_{cde}$ over a surface of constant $\Omega$ equals the integral of $J_{cde}$ over a surface of constant $\Omega'$. From this it follows that the integral of $\mathcal{J}_{cde}$ over $I^+$ is independent of the choice of $\Omega$. Similar results apply, of course, to integrals over $I^-$. Thus, if $\tilde{\phi}$ is smooth at $\mathcal{I}$ and satisfies suitable fall-off properties at $i^0$ and at timelike infinity, and if $j$ vanishes in a neighborhood of $\mathcal{I} \cup i^0$, then the total energy flux through $I^+$ and $I^-$ is well defined.

For a spacetime satisfying the assumptions listed at the beginning of Sec. II, we integrate eq. (A5) over the compact region with “endcaps” given by $C^+(t^+)$ and $C^-(t^-)$ but now bounded on the “sides” by a surface of constant, nonzero $\Omega$ (rather than by $\mathcal{I} \cup i^0$). We apply Stokes’ theorem, then take the limit as $t^\pm \to \pm \infty$, and, finally, take the limit as $\Omega \to 0$. The result is the analog of Theorem 2.1, namely

$$\lim_{t^+ \to +\infty} \int_{C^+(t^+)} t^a T_{ab} \epsilon_{cde} b - \lim_{t^- \to -\infty} \int_{C^-(t^-)} t^a T_{ab} \epsilon_{cde} b$$

$$+ \int_{I^+} t^a T_{ab} \epsilon_{cde} b - \int_{I^-} t^a T_{ab} \epsilon_{cde} b = \int_M j^a \nabla_a \phi \epsilon_{cde},$$

(A10)

where the integrals over $I^+, I^-, C^+(t^+)$, and $C^-(t^-)$ are defined by the limiting procedure described above. Assuming suitable analogs of the decay hypothesis and stationary solution property of Sec. II, we may then obtain an analog of Theorem 2.2.

The analog of the Lorentz force on a particle of scalar charge $q$ in an external Klein-Gordon field $\phi$ is

$$f_a = q \nabla_a \phi$$

(A11)

(This equation can be derived by integrating eq. (A3) over a small body and neglecting the self-field of the body; the Lorentz force in electromagnetism can be similarly derived by integration of eq. (4).) It should be noted that, in sharp contrast with the electromagnetic case, the scalar force (A11) fails, in general, to be perpendicular to the 4-velocity, $u^a$, of the particle. Consequently, the rest mass, $m$, of the particle will vary with time, i.e., in general, the particle will necessarily gain or lose rest mass as a result of its interactions with the scalar field. Indeed, we have

$$\frac{dm}{d\tau} = -u^a f_a = -q u^a \nabla_a \phi$$

(A12)

Thus, the 4-momentum of a particle of scalar charge $q$ in an external Klein-Gordon field $\phi$ is given by

$$p_a = (m_0 - q \phi) u_a$$

(A13)

where $m_0$ is a constant.

---

14It also should be noted that—in sharp contrast with the electromagnetic case—conservation of charge is not required for consistency of the Klein-Gordon equation, so there is no obstacle to allowing $q$ to vary with time as well. However, we shall assume throughout this Appendix that $q$ is constant.
When self-field effects are taken into account, the formula for the total force on a test particle (analogous to the DeWitt Brehme formula in the electromagnetic case) is

\[ f_a = q\nabla_a \phi^{\text{in}} + q^2 \left( \frac{1}{3} \dot{a}_a - \frac{1}{3} a^2 u_a + \frac{1}{6} R_{ab} u^b + \frac{1}{6} u_a R_{bc} u^b u^c - \frac{1}{12} R u_a \right) \]

\[ + q^2 \int_{-\infty}^{\tau} \nabla_a G^R \, d\tau'. \]  
\[ \text{(A14)} \]

We wish to establish the analog of Theorem 2.3 for the scalar case, i.e., we wish to show that if \( z(\tau) \) is a timelike curve which differs from an orbit, \( z_0(\tau) \), of the stationary Killing field \( t^a \) only over a finite interval, then

\[ \int_{I^+} t^a T_{ab} \epsilon_{cde} b - \int_{I^-} t^a T_{ab} \epsilon_{cde} b = \int t^a f_a \, d\tau. \]  
\[ \text{(A15)} \]

As in the electromagnetic case, we proceed by first establishing the analog of Proposition 2.1, namely that

\[ \int_{I^+} t^a T_{ab} \epsilon_{cde} b - \int_{I^-} t^a T_{ab} \epsilon_{cde} b = \int t^a \hat{f}_a \, d\tau \]  
\[ \text{(A16)} \]

where

\[ \hat{f}_a = q\nabla_a \phi^{\text{free}}. \]  
\[ \text{(A17)} \]

The only significant difference from the electromagnetic case occurring here is in the analysis of the vanishing of the “advanced-retarded cross-term” in the integrals representing the energy radiated through \( \mathcal{I} \). This cross-term vanished trivially in the electromagnetic case, but here is given by a nontrivial expression. However, we can simplify this expression by choosing \( \Omega \) so that \( \mathcal{L}_t \Omega = t^a n_a = 0 \). For the integral of the cross term over \( I^+ \), we have

\[ 2 \int_{I^+} \mathcal{J}_{cde} [\phi^A, \phi^R] = - \int_{I^+} \left[ t^a \nabla_a \tilde{\phi}^A n^b \bar{\nabla}_b \tilde{\phi}^R + t^a \nabla_a \tilde{\phi}^R n^b \bar{\nabla}_b \tilde{\phi}^A \right. \]

\[ + \left. f \tilde{\phi}^A t^a \nabla_a \tilde{\phi}^R + f \tilde{\phi}^R t^a \nabla_a \tilde{\phi}^A \right] \]  
\[ \text{(3)\,} \Omega_{cde} \]

\[ = - \int_{I^+} \left[ t^a \nabla_a \tilde{\phi}^R n^b \bar{\nabla}_b \tilde{\phi}^A + f \tilde{\phi}^A t^a \nabla_a \tilde{\phi}^R \right] \]  
\[ \text{(3)\,} \Omega_{cde} \]

\[ = - \int_{I^+} \mathcal{L}_t \left[ \tilde{\phi}^R n^b \bar{\nabla}_b \tilde{\phi}^A + f \tilde{\phi}^A \tilde{\phi}^R \right] \]  
\[ \text{(3)\,} \Omega_{cde} \]

\[ = 0. \]  
\[ \text{(A18)} \]

Here, we have used the fact that \( \tilde{\phi}^A \) is a stationary solution near \( I^+ \) so that \( \mathcal{L}_t \tilde{\phi}^A = t^a \nabla_a \tilde{\phi}^A = 0 \). In the last line, we have assumed that suitable falloff conditions on \( \tilde{\phi}^R \) near \( i^0 \) and timelike infinity lead to the vanishing of the boundary terms. Therefore, we see that the “advanced-retarded cross-term” makes no contribution to the energy radiated through \( \mathcal{I} \).

The remainder of the derivation of eq. \((A13)\) follows in close parallel with the electromagnetic case.

To compare the work done by \( f_a \) with that done by \( \hat{f}_a \), we note that \( \hat{f}_a \) is given by

\[ \hat{f}_a = q\nabla_a \phi^{\text{in}} + q^2 \left( \frac{1}{3} \dot{a}_a - \frac{1}{3} a^2 u_a + \frac{1}{6} R_{ab} u^b + \frac{1}{6} u_a R_{bc} u^b u^c - \frac{1}{12} R u_a \right) \]

\[ + q^2 \frac{1}{2} \int_{-\infty}^{\tau} \nabla_a G^R \, d\tau' - q^2 \frac{1}{2} \int_{\tau}^{+\infty} \nabla_a G^R \, d\tau'. \]  
\[ \text{(A19)} \]
Hence, the difference between the total work done by \( f_a \) and the total work done by \( \hat{f}_a \) between \( \tau = \tau^- \) and \( \tau = \tau^+ \) is given by

\[
\int_{\tau^-}^{\tau^+} t^a (f_a - \hat{f}_a) \, d\tau = q^2 \int_{\tau^-}^{\tau^+} t^a \left[ \int_{-\infty}^{+\infty} \nabla_a G \, d\tau \right] \, d\tau
\]

\[
= q^2 \int_{\tau^-}^{\tau^+} t^a \nabla_a G \, d\tau' \, d\tau + q^2 \int_{\tau^-}^{\tau^+} t^a \nabla_a G \, d\tau' \, d\tau
\]

\[
= -q^2 \int_{\tau^-}^{\tau^+} \int_{\tau^+}^{+\infty} t^a \nabla_a G \, d\tau' \, d\tau - q^2 \int_{\tau^-}^{\tau^+} \int_{-\infty}^{-\tau^-} t^a \nabla_a G \, d\tau' \, d\tau
\]

where the scalar analogs of eqs. (A17) and (A18) were used. If we now take the limit \( \tau \to \pm \infty \), then \( t^a = \chi(x^+)a^a \) in the \( \tau' \) region of integration of the first integral, where \( x^+ \) denotes the final spatial position of the particle and \( \chi \equiv (-t^a t_a)^{1/2} \). Thus, the integrand is a total derivative in \( \tau' \). Assuming that \( G \) falls off so that \( G(z(\tau), z(\pm \infty)) = 0 \), we therefore have

\[
\lim_{\tau \to \pm \infty} \int_{\tau^-}^{\tau^+} t^a (f_a - \hat{f}_a) \, d\tau = \lim_{\tau \to \pm \infty} \frac{q^2}{2} \int_{\tau^-}^{\tau^+} \left[ \chi(x^+) G(z(\tau), z(\tau^+)) - \chi(x^-) G(z(\tau), z(\tau^-)) \right] \, d\tau
\]

\[
= \lim_{\tau \to \pm \infty} \frac{q^2}{2} \int_{\tau^-}^{\tau^+} \left[ \chi(x^+) G^A(z(\tau), z(\tau^+)) - \chi(x^-) G^R(z(\tau), z(\tau^-)) \right] \, d\tau
\]

\[
= \frac{q}{2} [\chi(x^+)] \phi^{tail}(x^+, x^+) - \chi(x^-) \phi^{tail}(x^-, x^-),
\]

where \( \phi^{tail}(x_1, x_2) \) is the tail part of \( \phi \) as measured at spatial position \( x_1 \) for a point particle source held stationary at spatial position \( x_2 \) for all time. Thus, if the particle is initially and finally stationary at the same spatial position, then the work done by \( f_a \) is equal to the work done by \( \hat{f}_a \), which is in turn equal to the energy radiated to infinity by the Klein-Gordon field.

As in the electromagnetic case, eq. (A21) suggests that for a particle held stationary at position \( x \), the renormalized self-energy stored in the scalar field is given by

\[
E_{self}(x) \equiv \frac{q}{2} \chi(x) \phi^{tail}(x, x).
\]

However, in the scalar case, additional energy also is stored in the mass of the particle. Indeed, from eq. (A14), we obtain

\[
\frac{dm}{d\tau} = -u^a f_a
\]

\[
= -q \frac{d\phi^{in}}{d\tau} - \frac{q^2}{12} R - q^2 \int_0^\tau \frac{d}{d\tau} G^R(\tau, \tau') \, d\tau'
\]

\[
= -q \frac{d\phi^{in}}{d\tau} - \frac{q^2}{12} R - q^2 \frac{d}{d\tau} \int_0^\tau G^R(\tau, \tau') \, d\tau' + q^2 G^R(\tau, \tau)
\]

where in the last line, \( G^R(\tau, \tau) \) should be understood to mean the limit as \( \epsilon \to 0 \) of \( G^R(\tau, \tau - \epsilon) \). The Hadamard analysis of \[7\] shows that \( G^R(\tau, \tau) = R/12 \), so we obtain

\[
\frac{dm}{d\tau} = -q \frac{d}{d\tau} (\phi^{in} + \phi^{tail})
\]
Thus, the mass of the particle is now given by

\[ m = m_0 - q(\phi^{\text{in}} + \phi^{\text{tail}}) \]  

(A25)

and, hence, the total energy of a stationary particle is

\[
E' = -mt^a u_a + E_{\text{self}} \\
= [m_0 - q(\phi^{\text{in}} + \phi^{\text{tail}})]\chi + E_{\text{self}} \\
= E - \frac{q}{2}\chi(x)\phi^{\text{tail}}(x, x)
\]  

(A26)

where \( E \equiv (m_0 - q\phi^{\text{in}})\chi \) is the energy that the particle would have had in the absence of self-force effects. As in the derivation of eq. (69) in the electromagnetic case, in general this would lead to a corresponding self-energy correction to the effective potential. However, in Schwarzschild spacetime, Wiseman [24] has shown that \( \phi^{\text{tail}}(x, x) \) actually vanishes for all \( x \), so, to order \( q^2 \), there is no scalar self-energy correction to the motion of particle in Schwarzschild spacetime.

Finally, we note that the analysis of this Appendix can be extended straightforwardly (in the manner explained in Sec. II) to treat the case of angular momentum and to allow for the presence of black holes.
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FIGURES

FIG. 1. The unphysical spacetime \((\tilde{M}, \tilde{g}_{ab})\) with the hypersurfaces \(C^+ (t^+)\) and \(C^- (t^-)\) used in our analysis.
\[ I^+ \]

\[ C^+(t^+) \]

\[ \Sigma \]

\[ C^-(t^-) \]

\[ I^- \]