Improved Asymptotics for Zeros of Kernel Estimates via a Reformulation of the Leadbetter-Cryer Integral *

Kurt S. Riedel
Courant Institute of Mathematical Sciences
New York University
New York, New York 10012-1185

Abstract

The expected number of false inflection points of kernel smoothers is evaluated. To obtain the small noise limit, we use a reformulation of the Leadbetter-Cryer integral for the expected number of zero crossings of a differentiable Gaussian process.

Keywords: Kernel smoothers, Derivative estimation, Change points, Zero-crossings

1 Convergence of Kernel Smoothers

For many applications of nonparametric function estimation, obtaining the correct shape of the unknown function is of importance. A consequence of Mammen et al. (1992, 1995) is that kernel smoothers have a nonvanishing probability of having spurious inflection points if the smoothing level is chosen to minimize the mean integrated square error (MISE). In Riedel (1996), we propose a two-stage estimator where the number and location of the change points is estimated using strong smoothing.

In this letter, we evaluate the probability of obtaining spurious inflection points for kernel smoothers in the small noise/heavy smoothing limit. The proofs are based on powerful and seldom used techniques: Koksma’s theorem and the Leadbetter-Cryer integral for the expected number of zeros of a differentiable Gaussian process.

We consider a sequence of kernel smoother estimates, \( \hat{f}_N(t) \), of \( f(t) \), and examine the convergence of the estimate as the number of measurements, \( N \), increases. We believe that our results are slightly stronger than previous theorems on kernel smoothers (Gasser &

*We thank the referee for useful comments. Research funded by the U.S. Department of Energy.
Müller 1984). For each $N$, the measurements occur at $\{t_i^N, i = 1 \ldots N\}$. We suppress the superscript, $N$, on the measurement locations $t_i \equiv t_i^N$. We define the empirical distribution of measurements, $F_N(t) = \sum_{t_i \leq t} 1/N$, and let $F(t)$ be its limiting distribution.

**Assumption A** Consider the sequence of estimation problems: $y_i^N = f(t_i^N) + \epsilon_i^N$, where the $\epsilon_i^N$ are zero mean random variables and $\text{Cov}[\epsilon_i^N, \epsilon_j^N] = \sigma^2 \delta_{i,j}$. Assume that the distribution of measurement locations converges in the sup norm: $D_N^* \equiv \sup_t \{ |F_N(t) - F(t)| \} \to 0$, where $0 < c_F < C_F$.

The star-discrepancy, $D_N^* \equiv \sup_t \{ F_N(t) - F(t) \}$, is useful because it measures how closely a discrete sum over an arbitrarily placed set of points approximates an integral. (See Theorem 2.) For regularly spaced points, $F(t_i) = (i + .5)/N$ and $D_N^* \sim 1/N$, while for randomly spaced points, $D_N^* \sim \sqrt{\ln[\ln[N]]}/N$ by the Glivenko-Cantelli Theorem.

We consider kernel estimates of the form:

$$
\hat{f}(t) = \frac{1}{Nh_N^\ell} \sum_{i} y_i w_i \kappa(t - t_i) / h_N,
$$

where $h_N$ is the kernel halfwidth and $\{w_i\}$ are weights. We need convergence results for kernel estimators, $\hat{f}_N^\ell(t)$, of $f(t)$. Our hypotheses are stated in terms of the star discrepancy while previous results impose stronger/redundant conditions. We define $\sigma_N^2(t) = \text{Var}[\hat{f}_N^\ell(t)]$, $\xi_N^2(t) = \text{Var}[\hat{f}_N^{(\ell+1)}(t)]$, $\mu_N^2(t) = \text{Corr}[\hat{f}_N^\ell(t), \hat{f}_N^{(\ell+1)}(t)]$. We now evaluate the limiting quantities for a class of kernel smoothers. We use the notation $\mathcal{O}_R(\cdot)$ to denote a size of $\mathcal{O}(\cdot)$ relative to the main term: $\mathcal{O}_R(\cdot) = \times [1 + \mathcal{O}(\cdot)]$. We denote $C^\ell$ as the set of $\ell$ times continuously differentiable function, $TV[0, 1]$ as the function of bound variation with the total variation norm, $\| \cdot \|_{TV}$. We define $\|f\|_{BV}$ to be the sum of the $L_\infty$ and total variation norms of $f$ and define $\|f\|$ to be the $L_2$ norm.

**Theorem 1 (Generalized Gasser-Müller (1984))** Let $f(t) \in C^{\ell+1}[0,1] \cap TV[0,1]$ and consider a sequence of estimation problems satisfying Assumption A. Let $\hat{f}_N^\ell(t)$ be a kernel smoother estimate as given in (1), where the halfwidth, $h_N$, and the weights, $\{w_i\}$, satisfy $|w_i| \sim \mathcal{O}(D_N^*/h_N)$. Let the kernel, $\kappa^{(\ell+1)} \in TV[-1,1] \cap C[-1,1]$, satisfy the moment condition: $\int_{-1}^1 \kappa(s)ds = 1$, and the boundary conditions: $\kappa^{(j)}(-1) = \kappa^{(j)}(1) = 0$ for $0 \leq j \leq \ell$. Choose the kernel halfwidths such that $h_N \to 0$, and $D_N^*/h_N^{\ell+2} \to 0$; then

i) $\mathbb{E}[\hat{f}_N^\ell(t)] \to f(t) + \mathcal{O}_R(h_N + D_N^*/h_N^{\ell+1})$,

ii) $\mathbb{E}[\hat{f}_N^{(\ell+1)}(t)] = \int_{-1}^1 f^{(\ell+1)}(t + hs)\kappa(-s)ds + \mathcal{O}(\|f^{(\ell+1)}\|_{BV}D_N^*/h_N^{\ell+2})$,

iii) $\sigma_N^2(t) \to \sigma^2 \|\kappa^{(\ell)}\|^2 / (NF(t)h_N^{2\ell+1}) + \mathcal{O}_R(h_N + D_N^*/h_N)$,

iv) $\xi_N^2(s) \to \sigma^2 \|\kappa^{(\ell+1)}\|^2 / (NF^{(s)}h_N^{2\ell+3}) + \mathcal{O}_R(h_N + D_N^*/h_N)$, and

v) $\mu_N^2(t) \to \mathcal{O}_R(h_N + D_N^*/h_N)$

uniformly in the interval, $[h_N, 1 - h_N]$.

Our proof of Theorem 1 is based on Koksma’s Theorem which bounds the difference between integrals and discrete sum approximates:
Theorem 2 (Generalized Koksma Niederieter (1992)) Let $g$ be a bounded function of bounded variation, $\|g\|_{TV}$, on $[0,1]$: $g \in TV[0,1] \cap L_{\infty}[0,1]$. Let the star discrepancy be measured by a distribution, $F(t) \in C^1[0,1]$ with $0 < c_F < F'(t) < C_F$. If the discrete sum weights, $\{w_i, i = 1, \ldots N\}$, satisfy $|w_i - 1| \leq CD_N$, then

$$\left| \sum_{i=1}^{N} g(t_i)w_i \right| \leq \|g\|_{TV} + C\|g\|_{\infty}D_N^*.$$  

In our version of Koksma’s Theorem, we have added two new effects: a nonuniform weighting, $\{w_i, i = 1, \ldots N\}$, and a nonuniform distribution of points, $dF$. The total variation of $g(t(F))$ with respect to $dF$ is equal to the total variation of $g(t)$ with respect to $dt$. Theorem 2 follows from Koksma’s Theorem by a change of variables.

Proof of Theorem 1. We rescale: $s_i = (t_i - t)/h_N$ and apply Koksma’s theorem to $f(t + hs)\kappa^{(l)}(-s) \in TV_s[-1,1]$. The contribution of the weights, $w_i$, is $O_R(D_N^*/Nh_N^{l+1})$. Thus $E[f_N^{(l)}(t)] = \int_{-1}^{1} f(t + hs)\kappa(-s)ds + O(||f\kappa^{(l)}||_{\infty}D_N^*/h_N^{l+1})$. Since $|\kappa^{(l+1)}(-s)|^2/F'(t + hNs)$ is in $TV[-1,1]$, the variance satisfies

$$\xi_N^2(t) = \frac{\sigma^2}{Nh_N^{2l+1}} \int \frac{\kappa^{(l)}(-s)^2}{F'(t + hNs)}ds + O_R(D_N^*/h_N).$$

The result follows from expanding $F'(t)$ in $h_N$. $\Box$

Theorem 1 is one of two ingredients which we need to bound the expected number of change points of $f_N^{(l)}(t)$. Section 2 presents the second ingredient.

2 Asymptotics of Zero Crossings

The Leadbetter-Cryer (L-C) expression evaluates the expected number of zeros of a differentiable Gaussian process, $Z(t)$, in terms of a time history integral involving the first and second moments of $Z(t)$ (Leadbetter and Cryer 1965). We reexpress this integral in terms of the zeros of $E[Z(t)]$ and a remainder term. This alternative expression is particularly useful in the small noise limit when one desires an asymptotic evaluation of the number of noise induced zero crossings.

Theorem 3 (Leadbetter & Cryer (1965), Cramér & Leadbetter, 1967, Sec. 13.2) Let $Z(t)$ be a pathwise continuously differentiable Gaussian process in the time interval $[0,T]$. Denote $m(s) = E[Z(s)], \Gamma(s,t) = \text{Cov}[Z(s), Z(t)], \sigma^2(s) = \text{Var}[Z(s)] = \Gamma(s,s), \xi^2(s) = \text{Var}[Z'(s)], \mu(s) = \text{Corr}[Z(s)Z'(s)]$. Let $N_z$ be the number of zero crossings of $Z(t)$. If $m(t)$ is continuously differentiable, $\Gamma(s,t)$ has mixed second derivatives that are continuous at $t = s$ and $\mu(s) \neq 1$ at any point $s \in [0,T]$, then

$$E[N_z] = \int_0^T \frac{\xi(s)\gamma(s)}{\sigma(s)} \phi \left( \frac{m(s)}{\sigma(s)} \right) Q(\eta(s))ds,$$  

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}. $$  

$$\eta(s) = \frac{m(s)}{\sigma(s)}.$$
where \( Q(z) \equiv 2\phi(z) + z[2\Phi(z) - 1] \), \( \gamma(s)^2 = 1 - \mu(s)^2 \), \( \eta(s) = \frac{m'(s) - \xi(s)\mu(s)m(s)/\sigma(s)}{\xi(s)\gamma(s)} \).

By decomposing (3) into two pieces, we derive the following bounds:

**Theorem 4 (Alternate form)** Let the hypotheses of Theorem 3 hold and define \( M(t) \equiv m(t)/\sigma(t) \). Let \(|M(t)| \) have \( N^0_z \) zeros, \( L_{mx} \) relative maxima, \( M_j, j = 1 \ldots L_{mx} \) and \( L_{mn} \) nonzero relative minima, \( m_j \neq 0, j = 1 \ldots L_{mn} \), where \( M(0) \) and \( M(T) \) are counted as relative extrema. Let \( \nu_j \) equal 1 if \( m_j \) occurs at 0 or \( T \) and \( \nu_j = 2 \) otherwise. Define \( \hat{\nu}_j \) similarly for the \( M_j \). Equation (3) can be rewritten as

\[
E[N_z] - N_z^0 = \sum_{j=1}^{L_{mn}} \nu_j \Phi(-m_j) - \sum_{j=1}^{L_{mx}} \hat{\nu}_j \Phi(-M_j) + \int_0^T \frac{\xi(s)\gamma(s)}{\sigma(s)} \phi\left( \frac{m(s)}{\sigma(s)} \right) \tilde{Q}(\eta(s))ds \ , \quad (4)
\]

where \( \tilde{Q}(z) \equiv 2\int_{|z|}^\infty \phi(s')[s' - |z|]ds' \).

**Proof.** Write \( Q(z) = |z| + \tilde{Q}(z) \). The first term in (3) equals \(- \int \Phi(M)|M'(t)|dt \). Integrating this term yields the weighted sum of the relative extrema of \( \Phi(-|M|) \). We decompose this sum into \( N^0_z \) zeros of \(|M|(t)\) plus the additional relative extrema: \( \sum_{j=1}^{L_{mn}} \nu_j \Phi(-m_j) - \sum_{j=1}^{L_{mx}} \hat{\nu}_j \Phi(M_j) \).

We are unaware of any previous derivation of Theorem 4. The second term on the right hand side of (4) corresponds to the probability that \( Z(t) \) lacks a zero of \( m(t) \) while the first and third terms correspond to extra zeros. Note that \( \tilde{Q}(z) \leq \phi(z) \leq 1/\sqrt{2\pi} \).

**Corollary 5** Under the hypotheses of Theorems 3 & 4, let \( \{ (x_k, w_k), k = 1 \ldots K \} \) be chosen such that \( |t - x_k| \leq w_k \) implies that \( m'(t) > 0 \) and \(|M(t)| \geq cm'(x_k)|t - x_k|/\sigma(x_k) \), where \( c \) is a fixed number, \( 0 < c < 1 \). Define \( \Psi_k \equiv \sup_{|s - x_k| \leq w_k} \{ \tilde{Q}(s)\xi(s)\gamma(s)\sigma(x_k)/\sigma(s) \} \), \( C = \sup\{ \xi(t)/\sigma(t) \} \), and \( m_o \equiv \inf\{ |M(s)| \} \) for \( s \) such that \(|s - x_k| \geq w_k, k = 1 \ldots K \).

The expected number of zeros of the Gaussian process, \( Z(t) \), satisfies

\[
E[N_z] - N_z^0 \leq \sum_{k=1}^K \frac{\Psi_k}{cm'(x_k)} + O(CT + 2L_{mn})\phi(m_o) \ . \quad (5)
\]

**Proof.** The first term in (5) arises from replacing \( \Psi \int_{x_k - w_k}^{x_k + w_k} \phi\left( \frac{m(s)}{\sigma(s)} \right) ds \) by \( \Psi \int_{-\infty}^{\infty} \phi\left( \frac{cm'(x_k)s}{\sigma(x_k)} \right) ds \) and integrating. \( \Box \)

A sufficient additional condition for the existence of a set of \( (x_k, w_k) \) satisfying Corollary 5 is that \( m(s) \) vanishes only at a finite number of points, \( \{ x_k \} \), and at these points, \( m'(x_k) \neq 0 \). Let \( \delta \) be a small parameter related to the weakness of the noise amplitude. In many cases, the \( \{ w_k \} \) can be chosen to be powers of \( \delta \) and the upper bound of (5) reduces to

\[
E[N_z] - N_z^0 \leq \sum_{k=1}^K \frac{\tilde{Q}(x_k)\xi(x_k)\gamma(x_k)}{cm'(x_k)} [1 + o(1)] \ . \quad (6)
\]
In contrast, a similar naive expansion of the original integral (3) yields the asymptotic expression:

\[ E[N_z] - N_z^0 \leq N_z^0 o(1) + \sum_{k=1}^{K} \frac{\hat{Q}(x_k)\xi(x_k)\gamma(x_k)}{cm'(x_k)} [1 + o(1)] . \tag{7} \]

The advantage of (6) over (7) is that the remainder term, \( N_z^0 o(1) \), has been integrated away.

3 Number of false change points

We now consider sequences of kernel estimates of \( f^{(\ell)}(t) \), and examine the number of false \( \ell \)-change points. We restrict to independent Gaussian errors: \( \epsilon_i \sim N(0, \sigma^2) \). Thus, \( \hat{f}^{(\ell)}_N(t) \) is a Gaussian process. Mammen et al. (1992, 1995) consider the statistics of change point estimation for kernel estimation of a probability density. We present the analogous result for regression function estimation. In both cases, the analysis is based on the Leadbetter-Cryer formula for zero crossings. The following assumption rules out nongeneric cases:

**Assumption B** Let \( f(t) \in C^{\ell+1}[0,1] \) have \( K \) \( \ell \)-change points, \( \{x_1, \ldots, x_K\} \), with \( f^{(\ell)}(x_k) = 0, f^{(\ell+1)}(x_k) \neq 0, f^{(\ell)}(0) \neq 0 \) and \( f^{(\ell)}(1) \neq 0 \). Consider a sequence of estimation problems with independent, normally distributed measurement errors, \( \epsilon_i^N \), with variance \( \sigma^2 \). Let \( \hat{f}^{(\ell)}_N(t) \) be a sequence of kernel estimates of \( f^{(\ell)} \), on the sequence of intervals, \( [\delta_N, 1 - \delta_N] \).

Gasser and Müller (1984) evaluate the variance of a change point estimate: \( \text{Var}[\hat{x}_k - x_k] = \sigma^2 \hat{\delta}_N(x_k) = \text{Var}[^{\hat{f}^{(\ell)}_N}(x_k)]/|f^{(\ell+1)}(x_k)|^2 \). The following theorem bounds the tail of the empirical change point distribution \( |\hat{x}_k - x_k| >> \sigma_{\hat{\delta}} \). By using the L-C integral, we require weaker conditions than the hypotheses of Gasser and Müller (1984).

**Theorem 6** Let Assumption B hold and consider a sequence of kernel estimators, \( \hat{f}^{(\ell)}_N(t) \), that satisfy the hypotheses of Lemma 1. Choose kernel halfwidths, \( h_N \), and uncertainty intervals, \( w_N \), such that \( h_N/w_N \to 0, w_N \to 0, w^2_{N,k} Nh^2_{N+1} \geq 1 \). The probability, \( p_N(w_N) \), that \( \hat{f}^{(\ell)}_N(t) \) has a false change point outside of a width of \( w_N \) from the actual \( (\ell + 1) \)-change points satisfies

\[ p_N(w_N) \leq \sum_{k=1}^{K} \mathcal{O} \left( \frac{\sigma_{\hat{\delta}}(x_k)}{h_N} \exp \left( \frac{-w^2_N}{2\sigma_{\hat{\delta}}^2(x_k)} \right) \right) , \tag{8} \]

where \( \sigma_{\hat{\delta}}^2(x_k) \to \sigma^2/\|k^{(\ell)}\|^2/|f^{(\ell+1)}(x_k)|^2NF'(x_k)h^2_{N+1} \) on the interval \([h_N, 1 - h_N]\).

**Proof.** Lemma 1 shows that \( \xi_N(t)/\sigma_N(t) \to \mathcal{O}(h^{-1}_N) \). Within a neighborhood of \( \sqrt{w_N} \) of \( x_k \), \( \mathbf{E}[\hat{f}^{(\ell)}_N(t)] = f^{(\ell+1)}(x_k)(t-x_k) + \mathcal{O}(\sqrt{w_N} + D^*_N/h^\ell_N) \). Define \( b_N = \inf\{|f(t)| : t \notin \bigcup_{k=1}^K (x_k - \sqrt{w_N}, x_k + \sqrt{w_N})\} \). Note that \( b_N \geq C\sqrt{w_N} \) asymptotically and the integral of (3)
outside of \(\bigcup_{k=1}^{K}(x_k - \sqrt{w_N}, x_k + \sqrt{w_N})\) is bound by \(\exp(-cw_N/\sigma_N^2) << \exp(-w_N^2/2\sigma_N^2(x_k))\). Integrating the \(O(1)\) integrand bound, \(\exp \left( -|f^{(\ell+1)}(x_k)|^2 |t - x_k|^2/2\sigma_N^2(x_k) \right) / h_N\), over the intervals \([x_k \pm \sqrt{w_N}, x_k \pm w_N]\) yields (8). □

Mammen et al. (1992,1995) derived the number of false change points for kernel estimation of a probability density for nonvanishing error probabilities. We now show that these expressions remain valid as the error probability goes to zero. Given Gaussian measurement errors, the sophisticated proof in Mammen (1995) can be simplified in our case.

**Theorem 7 (Analog of Mammen et al. (1992,1995))** Let Assumption B hold. Consider a sequence of kernel smoother estimates \(\hat{f}_N\) which satisfy the hypotheses of Lemma 1 with \(\int_1^1 sk(s)ds = 0\). Let the sequence of kernel halfwidths, \(h_N\), satisfy \(D_N^*N^{1/2}h_N^{2} \to 0\) and \(0 < \liminf_N h_NN^{1/(2\ell+3)} \leq \limsup_N h_NN^{1/(2\ell+3)} < \infty\). The expected number of \(\ell\)-change points of \(\hat{f}_N\) in the estimation region, \([h_N, 1 - h_N]\), is asymptotically

\[
E[\hat{K}] = 2 \sum_{k=1}^{K} H \left( \frac{|f^{(\ell+1)}(x_k)|^2 N F'(x_k) h^{2\ell+3}}{\sigma^2 \|K^{(\ell+1)}\|^2} \right) + o_R(1),
\]

where \(H(z) \equiv \phi(z)/z + \Phi(z) - 1\) with \(\phi\) and \(\Phi\) being the Gaussian density. If \(f^{(\ell+1)}(t)\) has Hölder smoothness of order \(\nu\) for some \(0 < \nu < 1\), and \(h_NN^{1/(2\ell+3)} \to 0\), then (9) remains valid provided that \(h_NN^{1/(2\ell+3+2\nu)} \to 0\).

In Mammen (1992,1995), the correction in (9) is shown to be \(o(1)\) if \(\limsup_N h_NN^{1/(2\ell+3)} < \infty\). We strengthen this result by showing that (9) continues to represent the leading order asymptotics even when \(h_NN^{1/(2\ell+3)} \to \infty\). Our secret is to use (4) instead of (3) because (4) has integrated out the term equal to \(K\).

**Proof of Theorem 7.** Theorem 6 shows that the contribution away from the \(\ell\)-change points is exponentially small for \(|s - x_k| >> \sigma_N(s)\). Lemma 1 shows that \(\frac{\xi_{N(s)\gamma_N(s)}}{\sigma_N(s)} \to \frac{\|\xi^{(\ell+1)}\|}{h_N\|K^{(\ell)}\|}\) and that for \(|s - x_k| << 1\), \(\eta_N(s) \to f^{(\ell+1)}(s)/\sigma_N(s)\).

Equation (9) is an approximation of (4) using Laplace’s method. To prove (9), we must show that \(E[f^{(\ell)}_N(t)] = f^{(\ell)}(t) + o_R(\sigma_N)\) for \(|t - x_k| \sim \sigma_N\). Near the change point, \(x_k\),

\[
E[f^{(\ell)}_N(t)] = f^{(\ell)}(t) + \int_1^1 \kappa(s) \left[ f^{(\ell)}(t + h_N s) - f^{(\ell)}(t) \right] ds + o_R(D_N^*/h_N^{2\ell+1})
\]

\[
= f^{(\ell)}(t) + h_N \int_1^1 \kappa(s) \left[ f^{(\ell+1)}(t + h_N \tau_N(s)) - f^{(\ell+1)}(t) \right] ds,
\]

where \(\tau_N(s)\) lies in \([0, s]\) by the mean value theorem. Since \(f^{(\ell+1)}(t)\) is continuous at \(x_k\), for each \(\delta\), there is a \(\tilde{h}_N(\delta)\) such that \(|f^{(\ell+1)}(t + h_N \tau_N(s)) - f^{(\ell+1)}(t)| < \delta\) for all \(t, t + h_N \tau_N \in [x_k - \tilde{h}_N(\delta), x_k - \tilde{h}_N(\delta)]\). Thus \(E[f^{(\ell)}_N(t)] = f^{(\ell)}(t) + o_R(\delta h_N + D_N^*/h_N^{2\ell+1})\). Here \(\delta\) may be taken arbitrarily small. Applying the Laplace’s method yields (9) with
corrections of $O_R \left( \exp(-\delta h_N / \sigma_{ij}) - 1 \right) + O_R \left( \exp(-D^*_N / h_N^{\ell+2}\sigma_{ij}) - 1 \right)$. The scaling, $h_N \sim N^{-1/(2\ell+3)}$, implies that the first term is $O_R(\delta)$. The discrete sampling effect (the second term) requires the hypothesis that $D_N \sqrt{h_N} \rightarrow 0$ to be $o_R(1)$. When $f^{(\ell+1)}(t)$ is Hölder of order $\nu$, we have the stronger bound: $|f^{(\ell+1)}(t + h_N \tau_N(s)) - f^{(\ell+1)}(t)| < C h^h_N$, and $E[f^{(\ell)}(t)] = f^{(\ell)}(t) + O_R(h^{1+\nu}_N + D^*_N / h^{\ell+1}_N)$. The next order correction in Laplace’s method is $O_R \left( \exp(h^{1+\nu}_N / \sigma_{ij}) \right)$. This term is $o_R(1)$ when $h_N N^{1/(2\ell+3+2\nu)} \rightarrow 0$. $\square$

In Riedel (1996), we propose a two-stage nonparametric function estimator which achieves the correct shape with high probability. In the first stage, we estimate the number and approximate locations of the $\ell$-change point using a pilot estimate with large smoothing. In the second stage, the smoothing is reduced, but we impose the shape restrictions obtained from the pilot estimate. Theorems 6 and 7 imply that if the kernel halfwidth of the pilot estimator satisfies $h_N >\ln[N]N^{-1/(2\ell+3)}$, then spurious inflection points will occur with a probability smaller than $N^c$ for any $c$. To achieve this result, we use an alternate form of the Leadbetter-Cryer integral to remove the $N_o(1)$ from (7).

References

[1] Cramér, H. and Leadbetter, M. R. (1967), *Stationary and related processes*, John Wiley, New York.

[2] Gasser, Th. and Müller, H., Estimating regression functions and their derivatives by the kernel method, *Scand. J. of Stat.* **11** (1984), 171–185.

[3] Leadbetter, M. R. and Cryer J. D. (1965) Curve crossings by normal processes, *Ann. Math. Stat.* **36**, 509-516.

[4] Mammen, E., Marron, J. S. and Fisher, N. J. (1992), Some asymptotics for multimodal tests based on kernel density estimates, *Prob. Th. Rel. Fields* **91** 115-132.

[5] Mammen, E. (1995) On qualitative smoothness of kernel density estimates, *Statistics*, **26** 253-267.

[6] Niederrieter, H. (1992) *Random Number Generators and Quasi-Monte Carlo Methods*, SIAM, Philadelphia, PA.

[7] Riedel, K. S. (1996), Piecewise convex function estimation I: pilot estimators. Submitted for publication.