The Lagrangian dynamics of thermal tracer particles in Navier-Stokes fluids

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A basic issue for Navier-Stokes (NS) fluids is their characterization in terms of the so-called NS phase-space classical dynamical system, which provides a mathematical model for the description of the dynamics of infinitesimal (or ideal) tracer particles in these fluids. The goal of this paper is to analyze the properties of a particular subset of solutions of the NS dynamical system, denoted as thermal tracer particles (TTPs), whose states are determined uniquely by the NS fluid fields. Applications concerning both deterministic and stochastic NS fluids are pointed out. In particular, in both cases it is shown that in terms of the ensemble of TTPs a statistical description of NS fluids can be formulated. In the case of stochastic fluids this feature permits to uniquely establish the corresponding Langevin and Fokker-Planck dynamics. Finally, the relationship with the customary statistical treatment of hydrodynamic turbulence (HT) is analyzed and a solution to the closure problem for the statistical description of HT is proposed.

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I. INTRODUCTION

A fundamental aspect of theoretical fluid dynamics is represented by the discovery of the thermal tracer particles (TTPs) for Navier-Stokes (NS) fluids recently reported (see Ref. [1]). The latter represent a suitable subset of the so-called ideal tracer particles (ITPs [2]) and are defined in such a way that their states are uniquely dependent, in a sense to be specified below, on the local state of the fluid. A basic implication of the result is that an appropriate statistical ensemble of TTPs should reproduce exactly the dynamics of the fluid. In other words, it should be possible to determine the fluid fields characterizing the fluid state by means of suitable statistical averages on the ensemble of TTPs, and in particular performed so that they satisfy identically a required set of fluid equations. The conclusion is expected to apply, in principle, to arbitrary NS fluids described as mesoscopic, i.e., continuous fluids, which can be either viscous or inviscid, compressible or incompressible, thermal or isothermal, isentropic or non-isentropic.

We shall assume, for this purpose, that the state of these fluids is represented by an ensemble of observables \( \{Z(r, t)\} \equiv \{Z_i(r, t), i = 1, \ldots, n\} \) (with \( n \) an integer \( \geq 1 \)), i.e., fluid fields, which can be unambiguously prescribed as continuous and suitably smooth functions, respectively, in \( \Omega \times I \) and in the open set \( \Omega \times I \), with \( \Omega \subset \mathbb{R}^3 \) and \( I \equiv \mathbb{R} \) being the configuration space and time axis respectively. We intend to show that, as a basic consequence, the Newtonian state of each TTP, namely \( x(t) \equiv x = (r, v) \), with \( r \) and \( v \) denoting respectively the particle position and velocity, is advanced in time in terms of a suitable acceleration field \( F = F(x, t) \) which can be defined in such a way to depend only on the state of the same particle (mean-field acceleration). Remarkably, it is found that \( F \) can be uniquely prescribed in such a way to determine self-consistently the time evolution of the complete set of fluid equations characterizing the fluid. This implies that TTPs must reproduce exactly the dynamics of the fluid. In other words, by means of appropriate statistical averages on the ensemble of the TTPs, it is possible to uniquely determine the time-evolution of the fluid state, in such a way that it satisfies identically the required set of fluid equations.
A. Lagrangian dynamics of ideal tracer particles

A key aspect of fluid dynamics is the proper definition of the phase-space Lagrangian dynamics for continuous fluid systems, whereby possibly all the fluid fields characterizing the actual fluid state \( \{Z(r,t)\} \) can be identified with suitable statistical averages on appropriate ensembles of (fictitious) particles. Thus, for example, in the case of an incompressible NS fluid, this would require to represent both the fluid velocity \( \mathbf{V}(r,t) \) and the fluid pressure \( p(r,t) \) in terms of suitable statistical averages of an appropriate probability density. This goal can be realized by means of the inverse kinetic theory (IKT) developed by Ellero and Tessarotto (see Refs. [3, 4]). This refers, in particular, to the phase-space dynamics of ideal tracer particles, namely rigid extended classical particles immersed in the fluid, all having the same support and infinitesimal size such that during their motion they do not mutually interact and do not perturb the state of the fluid. Depending on their inertial mass \( m_P \), ITPs can belong to different species of particles; thus, in general, their mass can differ from that of the corresponding displaced fluid element \( m_F \). On the other hand, ITPs carrying the mass \( m_P \equiv m_F \) will be denoted as the NS ideal tracer particles (NS-ITPs). In the following, in order to characterize the Lagrangian dynamics of NS fluids, ITPs will be identified only with NS-ITPs. In this framework, it follows that ITPs can undergo, by assumption, only “unary” interactions with external force-fields and with the continuum fluid. Namely, in both cases they are subject only to the action of a continuum mean-field acceleration which depends only on the local state of each particle. As a consequence, ITPs can be treated as Newtonian point-like particles characterized by a Newtonian state \( x = (r, v) \) spanning the phase-space \( \Gamma \equiv \Omega \times U \), with the position \( r \) and the kinetic velocity \( v \) belonging respectively to the configuration space of the fluid \( \Omega \) (in the following to be identified with a bounded subset of \( \mathbb{R}^3 \)) and the velocity space \( U \equiv \mathbb{R}^3 \).

B. The Navier-Stokes dynamical system

By assumption, the state \( x \) of a generic ITP advances in time by means of a Newtonian classical dynamical system (DS) defined in terms of the vector field \( \mathbf{X}(x,t) \equiv \left\{ \frac{1}{m_P} \mathbf{K} \equiv \mathbf{F} \right\} \), with \( \frac{1}{m_P} \mathbf{K} \equiv \mathbf{F} \) a suitable mean-field acceleration. This is identified with the flow \((T_{t_o, t}, x) \equiv x(t) = T_{t_o, t} x_o\) generated by the initial value problem associated to the deterministic equations of motion (Newton’s equations)

\[
\begin{align*}
\frac{dx}{dt} &= \mathbf{X}(x,t), \\
_x(t_o) &= x_o.
\end{align*}
\]

Such flow is referred to as Navier-Stokes dynamical system (NS–DS) and is a homeomorphism in \( \Gamma \) with existence domain \( \Gamma \times I \), of the type

\[
T_{t_o, t} : x_o \rightarrow x(t) = T_{t_o, t} x_o,
\]

with \( t \in I \subseteq \mathbb{R} \) and \( T_{t_o, t} \) being a measure-preserving evolution operator associated to \( \mathbf{X}(x,t) \). Thus, by definition, the NS-DS is uniquely prescribed by the couple \( \{x, \mathbf{X}(x,t)\} \), with \( x(t) \equiv x = (r, v) \) to be identified with the instantaneous Newtonian state of a generic ITP.

C. The relative-dynamics NS-DS

The state of a TTP can be equivalently represented in terms of the relative-dynamics Newtonian state \( y = (r, u) \), with \( u(t) = v(t) - \mathbf{V}(r(t), t) \) denoting the relative kinetic velocity defined with respect to the local fluid velocity \( \mathbf{V}(r,t) \). As a consequence, by introducing the local phase-space diffeomorphism \( x = (r, v) \rightarrow y = (r, u) \), the dynamical system \([2]\) can be cast in terms of \( y = (r, u) \). Then, \( (2) \) can be equivalently represented by the homeomorphism in \( \Gamma \) :

\[
T_{t_o, t}^{(RD)} : y_o \rightarrow y(t) = T_{t_o, t}^{(RD)} y_o,
\]

to be identified with the relative-dynamics NS dynamical system \( (RD-NS-DS) \), \( T_{t_o, t}^{(RD)} \) being the flow generated by the initial-value problem

\[
\begin{align*}
\frac{dr}{dt} &= u + \mathbf{V}(r,t), \\
\frac{dy}{dt} &= F_u(x,t), \\
y(t_o) &= y_o.
\end{align*}
\]
Here, for a prescribed form of the mean-field \( \mathbf{F}(x, t) \) (see below), \( \mathbf{F}_u(x, t) \) denotes the kinetic relative acceleration which is defined as

\[
\mathbf{F}_u(x, t) = \mathbf{F}(x, t) - \mathbf{F}_H(r, t) - \mathbf{u} \cdot \nabla \mathbf{v}(r, t),
\]

with \( \mathbf{F}_H(r, t) \) being the NS fluid acceleration defined by Eq.\( \text{(142)} \) (see Appendix A). To establish on rigorous grounds the connection with the corresponding fluid description, the vector field \( \mathbf{F}(x, t) \) must be suitably determined. For definiteness, we shall consider here the case of a Navier-Stokes thermodiffusion described either by the compressible or incompressible Navier-Stokes-Fourier equations, requiring that the fluid fields \( \{Z(r, t)\} \) are strong solutions either of the compressible or incompressible Navier-Stokes-Fourier problems (CNSFE or INSFE problems respectively; see Appendix A). In the case of CNSFE the state of the fluid is defined by the ensemble of smooth fluid fields

\[
\{Z(r, t)\} = \{\rho(r, t), \mathbf{V}(r, t), p(r, t), T(r, t), S_T(t)\},
\]

with \( \rho(r, t) \geq 0, \mathbf{V}(r, t), p(r, t) \geq 0, T(r, t) \geq 0 \) and \( S_T(t) \) denoting the fluid mass density, the fluid velocity, the fluid scalar pressure, the fluid temperature and the global thermodynamic entropy respectively. In particular, by introducing an arbitrary reference mass \( m > 0 \), for example to be identified with the particle mass \( m_F \) defined above, the notion of fluid number density \( n(r, t) \equiv \frac{1}{m} \rho(r, t) \) can be introduced. As an alternative, in the following the set \( \{Z(r, t)\} \) will be replaced by the reduced set of fluid fields

\[
\{Z_1(r, t)\} = \{\rho(r, t), \mathbf{V}(r, t), p_1(r, t), S_T(t)\}.
\]

Here \( p_1(r, t) \) denotes the kinetic pressure, i.e., a strictly positive scalar observable defined as

\[
p_1(r, t) = p_0(t) + p(r, t) - \phi(r, t) + n(r, t)T(r, t),
\]

with \( p_0(t) \) being the pseudo-pressure and \( \phi(r, t) \) the potential associated to the conservative part of the volume force (see Appendix A)\( \text{[55]} \). For later use we introduce here also the notion of specific kinetic pressure \( \tilde{p}_1(r, t) \):

\[
\tilde{p}_1(r, t) \equiv p_1(r, t)/\rho(r, t).
\]

In particular, we shall assume that in \( \overline{\Omega} \times I \) both \( \phi(r, t), p_0(t) \) and \( S_T(t) \) are uniquely defined. In particular they are such that:

- \( \phi(r, t) \) is bounded and such that, for \( r \equiv \mathbf{R}_o \)
  \[
  \phi(\mathbf{R}_o, t) = 0,
  \]

with \( \mathbf{R}_o \) being a suitable vector belonging to \( \Omega \).

- \( p_0(t) \) is a suitably-prescribed smooth real function defined so that \( p_1(r, t) \) remains strictly positive in \( \overline{\Omega} \times I \) (see Axiom \#2, Section 4).

- The initial value \( p_0(t_o) \) is in principle an arbitrary constant to be prescribed in such a way that \( p_0(t_o) > p_{0\inf} \)
  with \( p_{0\inf} \) being such that \( p_1(r, t_o) \) vanishes locally in \( \overline{\Omega} \) (see Axiom \#2, Section 4).

D. The IKT statistical description

It must be stressed that the vectors \( \mathbf{v} \) and \( \mathbf{u} \) defined above can be interpreted as stochastic variables [see Appendix B], and the corresponding equations of motion, Eqs.\( \text{(11)} \) and \( \text{(14)} \) viewed as Langevin (i.e., stochastic) equations, provided a statistical description in terms of a suitable probability density is introduced for them. Statistical descriptions of this type, based on classical statistical mechanics (CSM), can be adopted in principle both for classical and quantum fluids (see for example Refs.\( \text{[3, 7]} \) and \( \text{[8]} \)) characterized either by deterministic or stochastic flows \( \text{[9, 10]} \). This is realized by introducing an appropriate axiomatic approach denoted as statistical model, represented by a set \( \{f, \Gamma\} \), with \( f(x, t) \) denoting a suitable kinetic distribution function (KDF) - or a probability density function (PDF), to be identified with the so-called 1-point PDF - which is defined in the phase-space \( \Gamma \). Its is worth noting that such a type of approach can be determined in accordance with the GENERIC dynamical model \( \{f, \Gamma\} \) developed by Grmela and Ottinger\( \text{[11, 12]} \). This is defined in such a way to prescribe:

- A bundle structure on \( \Gamma \text{[11]}, \) i.e., a mapping between \( \Gamma \) itself and the ensemble of fluid fields \( \{Z\} \) (or \( \{Z_1\} \)), generated via appropriate statistical averages, i.e., phase-space moments, of \( f(x, t) \).
A functional class \( \{ f \} \) to which \( f(x, t) \) belongs \( \frac{\partial}{\partial x} \). This should be based exclusively on the knowledge of the same set of fluid fields. The prescription of \( \{ f \} \) includes also the initial condition on \( f(x, t) \) at \( t = t_o \).

- A suitable phase-space dynamics: this is introduced in terms of the NS-DS [see Eq. (2)]. This should inherit the basic properties of the fluid system, i.e., in particular, the conservation of mass and momentum, the energy balance equation and the entropy law. Thus, in the case it is described by deterministic, dissipative and irreversible fluid equations, it should be a deterministic, non-conservative and irreversible dynamical system.

The problem of its construction (Frisch, 1995 [13]), i.e., the actual definition of the vector field \( F \), has remained for long time unsolved (see for example, Vishik and Fursikov, 1988 [14] and Ruelle, 1989 [15], where approaches based on kinetic theory were attempted). Nevertheless, in the past various models for the dynamics of tracer particles in incompressible fluids have actually been developed [16–20], which should manifestly apply, at least in principle, also to ITPs. As a consequence, their dynamics should be controlled only by the mean-field force \( F \) produced by the unperturbed fluid fields. Since the original Basset-Boussinesq-Oseen approach [21–23] formulated in the case of a uniform flow, several attempts to evaluate the form of \( F \), as well as the vector field \( F_u \), have appeared [16–20]. All such approaches propose “ad hoc” modifications or corrections of the same equation, exclusively based on phenomenological arguments, in order to adapt it for the treatment of non-uniform flows in NS fluids. A popular form (for \( F \)) frequently adopted for fluid simulations is the one developed by Maxey and Riley (1982 [20]). These treatments appear questionable because of the critical common assumption on which they are based. Precisely, the requirement that the tracer-particle velocity \( v(t) \) (kinetic velocity) remains always suitably close to the local velocity of the fluid \( V(r, t) \) evaluated at the position of the moving particle \( r = r(t) \). This implies that for all ITPs the asymptotic condition

\[
|u(t)| \ll |V(r(t), t)|
\]

should hold at any time \( t \). As a consequence, in the determination of \( F \) all contributions proportional to the relative velocity should be considered negligible. The constraint \( |u(t)| \ll |V(r(t), t)| \) imposes, however, potentially serious limitations on particle dynamics. In fact, it may easily be violated either due to the arbitrariness of the particle initial conditions in Eq. (11) - in fact ITPs can be injected in a fluid with arbitrary initial velocity - or because ITPs initially at rest [in Eq. (1)] - in fact ITPs can be injected in a fluid with arbitrary initial velocity - or because ITPs initially at rest with respect to the fluid might develop finite relative velocities, thus causing Eq. (11) to fail. On the other hand, the determination of the exact ITP dynamics is manifestly of fundamental importance in order to obtain detailed quantitative theoretical and numerical predictions in fluid dynamics.

A first-principle solution of this problem has recently been proposed in Ref. [2] adopting a suitable statistical description for incompressible Navier-Stokes thermo-fluids (see Appendix A), i.e., the representation of the dynamical system \( \{ f(x, t) \} \) in terms of the IKT-statistical model \( \{ f(x, t), \Gamma \} \), obtained in the framework of the so-called inverse kinetic theory (IKT; see Refs. [1, 5]). In such a formulation,

\[
f(t) \equiv f(x, t)
\]

is identified with a KDF whose velocity and phase-space moments are prescribed in terms of a suitable subset of the fluid fields \( \{ Z \} \), to be identified with the ensemble \( \{ Z_1 \} \). Hence, provided \( \int dV f(x, t) > 0 \), the corresponding velocity PDF is

\[
\tilde{f}(x, t) = f(x, t)/\int_U dV f(x, t).
\]

The KDF \( f \) is required to satisfy in \( \Gamma \times I \) the statistical equation

\[
Lf = 0,
\]

denoted as inverse kinetic equation (IKE [3]) in Eulerian form, with \( L \equiv \partial/\partial t + v \cdot \nabla f + \partial/\partial x \cdot (F f) \) being the Liouville streaming operator. The same equation can be equivalently cast in terms of the integral Lagrangian IKE

\[
f(x, t) = f_o(T_{i, t_0}, x) \left| \frac{\partial T_{i, t_0}}{\partial x} \right|.
\]

Here \( f_o(x_o) \) is a suitable initial KDF, while

\[
\left| \frac{\partial T_{i, t_0}}{\partial x} \right| = \exp \left\{ - \int_{t_o}^{t} dt' \frac{\partial}{\partial V(t')} \cdot F(x(t'), t') \right\}
\]

(16)
(Liouville theorem). In addition, the following assumptions are introduced (see points a,b,c,d below):

a) A particular solution of Eq. (14) is provided by the local Gaussian distribution function (kinetic equilibrium)

$$f_M(x, t) = \frac{\rho(x, t)}{(\pi)^{\frac{d}{2}} v_{th}^d} \exp \{-X^2\},$$

where $X^2 = \frac{u^2}{v_{th}^2}$ and $v_{th} = \sqrt{\frac{2kT}{\rho}}$ denotes the thermal velocity.

b) In terms of a suitable set of the velocity and phase-space moments, the complete set of fluid equations are constructed from IKE.

c) Let us assume that $f(x, t)$ is strictly positive in $\Gamma$ and admits for all $(x, t) \in \Gamma \times t$ the Boltzmann-Shannon (BS) statistical entropy [6] (also known as differential entropy). This is defined as the functional

$$S(f(t)) \equiv -\alpha_1^2 \int_\Gamma df(x, t) \ln f(x, t) + c_1,$$

where $\alpha_1 \neq 0$ and $c_1$ are arbitrary real constants independent of $(x, t)$, to be suitably defined. In information theory the Boltzmann-Shannon entropy $S(f(t))$ can be intended as a measure of the ignorance on $f(t)$. Here we remark that, by suitable definition of the constants $\alpha_1$ and $c_1$, $S(f(t))$ can be represented in terms of the corresponding BS entropy for the phase-space PDF $f(t) = f(t)/M$, where $M = \int \Gamma df(x, t)$. In fact, letting

$$S(\overline{f}(t)) \equiv -\int_\Gamma df(x, t) \ln \overline{f}(x, t),$$

it follows that

$$S(\overline{f}(t)) = \frac{1}{M} S(f(t)) + \ln M,$$

which is again of the form (18). In the following we shall set in particular $\alpha_1 = 1$ and $c_1 = 0$ in Eq. (18).

d) The time-derivative of the pseudopressure $dp_0(t)/dt$ is determined by suitably prescribing the entropy production rate $\frac{d}{dt} S(f(t))$ [6].

As a basic consequence of the previous assumptions it is follows that [6, 7]:

1. The mean-field $F$ is generally functionally dependent on $f$, i.e., it is of the form $F = F(x, t; f)$.
2. $F$ is defined up to an arbitrary real gauge vector-field $\Delta F$ obeying the gauge condition

$$\partial F(x, t; f) = 0.$$

3. The choice of the gauge field $\Delta F$ does not affect the time-evolution of $f(x, t)$. Consequently, the corresponding velocity and phase-space moments of the IKE are in all cases necessarily unique.

4. In the case $f \equiv f_M$ and up to the gauge field $\Delta F$, the functional form of the mean field $F(x, t; f_M)$ is uniquely determined.

5. In the case $f \neq f_M$ the determination of $F$, again up to the gauge field $\Delta F$, requires suitable kinetic closure conditions. In fact, in principle, in such a case $F$ might depend on arbitrary higher-order phase-space moments of $f$ which vanish in the case $f \equiv f_M$. To this end, in the case of incompressible NS fluids, in Refs. [2, 3, 6] it was assumed that $F(x, t; f)$ (and $\Delta F$) can be represented as polynomials of lowest possible degree with respect to the relative kinetic velocity $u$ and depend on the lowest-order and minimal number of velocity moments of the KDF.

We remark that, although in the context of IKT the unique specification of $\Delta F$ (and hence of $F$) is superfluous, its determination is, instead, manifestly required in order to uniquely prescribe the dynamics of ITPs. In particular in Ref. [2], based on the analogy with extended thermodynamics [6], $\Delta F$ was identified with a first-degree polynomial with respect to $u$ of the form $\Delta F = \frac{1}{2} \nabla \cdot u - \frac{1}{4} u \cdot \nabla u$. In the following we intend to propose a generalization of IKT for compressible Navier-Stokes thermodinamics (Section 2) and in which $\Delta F$ is uniquely determined, based on suitable physical assumptions. For this purpose, leaving initially unspecified the form of the gauge field $\Delta F$, we intend to prove that $\Delta F$ is uniquely prescribed imposing the requirements stemming from the following Gedanken (conceptual) experiment (GDE).
II. GEDANKEN EXPERIMENT

For a prescribed continuous fluid system, such as a compressible/incompressible NS thermo-fluid, the problem arises whether there might exist a subset of the ensemble of ideal tracer particles (ITPs) for the dynamical system \( \{ Z \} \) such that their Newtonian state and corresponding time evolution depend only on the state of the fluid \( \{ Z \} \). In the following the subset of ITPs which exhibit these properties are referred to as thermal tracer particles (TTPs). Such a result was reached for incompressible and isothermal NS fluids in Ref.\[1\]. Here we claim that it should be possible to extend the same conclusion to arbitrary compressible and non-isothermal NS fluids by performing a conceptual experiment (\textit{Gedanken experiment}) on such a type of fluid, i.e., looking at the properties of the IKT-statistical models \( \{ f, \Gamma \} \). The conjecture is suggested by the following arguments:

- The state of the fluid is solely dependent on the fluid fields, which in the case of a compressible NS thermo-fluid can be identified with the set \( \{ Z_1 \} \).
- The time-evolution of \( \{ Z_1 \} \) as determined by CNSFE is necessarily independent of the KDF \( f(x, t) \) and of the NS-DS \( \{ 2 \} \). In fact, obviously the CNSFE (or INSE) problem cannot depend on the functional form of \( f(x, t) \).
- On the other hand, in the context of IKT, the time evolution of the KDF is determined by the Liouville operator \( L \), which enters the corresponding Liouville equation. As pointed out in Refs.\[2, 5\], this generally contains a vector field \( F \) whose form can depend functionally also on the same KDF \( f(x, t) \).

A. GDE requirements

On the basis of these considerations, here we conjecture that TTPs should exist as a subset of ITPs and fulfill the following properties:

1. \textit{GDE requirement \#1:} their time evolution, as determined by the vector field \( F \) in terms of the NS-DS \( \{ 2 \} \), should remain at all times \( t \in I \) independent of the particular form of the KDF \( f(x, t) \). As a consequence, for them the form of the mean-field force \( F \) should be also independent of the KDF \( f(x, t) \) [introduced in the IKT-statistical model \( \{ f, \Gamma \} \) ], namely simply of the form \( F = F(x, t) \).

2. \textit{GDE requirement \#2:} for prescribed initial conditions, their Newtonian states \( x(t) \equiv x = (r, v) \), and equivalently also \( y(t) \equiv y = (r, u) \), should depend solely on the fluid fields \( \{ Z_1 \} \).

In addition, one should expect that for all TTPs:

3. \textit{GDE requirement \#3:} \textit{Local magnitude of } \( u(t) \): the magnitude of their instantaneous relative velocity \( |u(t)| \equiv |u(r, t)| \) remains at all times \( t \in I \) proportional to the local thermal velocity \( \nu_{th}(r, t) \), i.e., of the form

\[
|u(t)| = \beta \nu_{th}(r, t),
\]  

with \( p_1(r, t) > 0, r \equiv r(t) \) and \( \beta \) denoting respectively the \textit{kinetic pressure} \( \tilde{p} \), the instantaneous position of the same particle and an appropriate non-vanishing constant, i.e., a function independent of \( (r, t) \). This means that \( \beta \) is necessarily determined by the TTP initial condition (see discussion below, after THM.2);

4. \textit{GDE requirement \#4:} \textit{Kinetic constraint on the local direction of } \( u(t) \): let us introduce for \( u(t) \) the representation

\[
u(t) = \beta \nu_{th}(r, t) \nu(r, t),
\]  

with \( \nu(r, t) \) being the unit vector prescribing the local direction of \( u(t) \). Then, if at a given point the constraint \( \tilde{p} = \text{const.} \) is satisfied, in order to warrant that \( u(t) \) satisfies it also at time \( t + dt \) (with \( dt \) being infinitesimal), it is necessary to require that the unit vector \( \nu(r, t) \) be tangent to the \textit{local isobaric surface} \( \tilde{p}(r, t) = \text{const.} \). Therefore, for a non-uniform kinetic pressure satisfying locally \( \nabla \tilde{p}(r, t) \neq 0 \), the unit vector \( \nu(r, t) \) must satisfy the kinetic constraint:

\[
u(r, t) \cdot \nabla \tilde{p}(r, t) = 0.
\]
As a consequence, the direction of \( u(t) \) is necessarily uniquely determined, once the initial conditions (1) and consequently its initial direction

\[
n(r, t_0) \equiv n_o(r)
\]  

have been set. Hence, for a non-uniform specific kinetic pressure \( \hat{p}_1 \), the unit vector \( n(r, t) \) must be orthogonal to the unit vector

\[
b(r, t) = \frac{\nabla \hat{p}_1(r, t)}{|\nabla \hat{p}_1(r, t)|},
\]

i.e., the kinetic constraint

\[
n(r, t) \cdot b(r, t) = 0
\]

must hold identically for all \((r, t) \in \Omega \times I\).

5. **GDE-requirement #5 - Time evolution of \( n(r, t) \):** the unit vector \( n(r, t) \) satisfies an initial-value problem of the form

\[
\begin{cases}
\frac{dn(r, t)}{dt} = \Omega(r, t) \times n(r, t), \\
n(r(t_0), t_0) = n_o(r, t_0),
\end{cases}
\]  

with \( \Omega(r, t) \) denoting a suitable pseudo-vector. Without loss of generality we shall require that \( \Omega(r, t) \) is a smooth real vector function defined in \( \Omega \times I \) and that it is defined also in the limit \( p_1(r, t) \rightarrow 0^+ \).

6. **GDE-requirement #6 - Rotation dynamics of \( n(r, t) \):** we require that the unit vector \( n(r, t) \) exhibits a rotation motion with respect to the direction \( b(r, t) \) which is determined by the parallel component of the vorticity. In other words, the \( \Omega(r, t) \) is required to satisfy the constraint

\[
\Omega(r, t) \cdot b(r, t) = -\xi(r, t) \cdot b(r, t),
\]

where \( \xi(r, t) \equiv \nabla \times V(r, t) \) is the vorticity field. As a consequence of Eqs.(28) and (29), the particle relative velocity exhibits a rotation caused by two distinct physical mechanisms. The first one is due to the rotation of the unit vector \( b(r, t) \) characterizing the isobaric surfaces, while the second one is due to the intrinsic rotation of the parallel component of \( \Omega(r, t) \) around \( b(r, t) \) as determined by the local vorticity field. In particular, the constraint placed by Eq.(29) establishes a direct connection between TTP dynamics and fluid vorticity and has important implications for the treatment of strong turbulence in the framework of TTP statistics (see related discussion in subsection VII.B below).

### III. GOALS OF THE INVESTIGATION

In this paper we point out that IKT can be determined in such a way to satisfy the requirements dictated by the GDE. For this purpose, first the IKT statistical description \( \{f, \Gamma\} \) earlier pointed out in Refs. [2, 5, 6] is extended to the treatment of compressible and non-isothermal NS fluids. Next, the NS dynamical system is shown to admit particular solutions which are of the form of TTPs, namely ITPs for which the particle state takes the form indicated above [see Eqs.(22) and (27)-(29)]. More precisely, extending the results pointed out in Ref.[1], and holding in the case of incompressible and isothermal NS fluids, here we intend to prove that for compressible, non-isothermal fluids satisfying the CNSFE Problem (see THM.1 in Section 4):

- **Goal #1** - **TTPs are particular solutions of the NS dynamical system** (see Section 5, THM.2).

In such a setting, we claim that the following additional properties are fulfilled:

- **Goal #2** - **For all TTPs a unique realization exists for \( F + \Delta F \) satisfying the requirements of GDE.**

- **Goal #3** - **In terms of the ensemble of TTPs a reduced-dimensional statistical model \( \{f_1, \Gamma_1\} \) (TTP-statistical model) is introduced** (Section 6, THM.3).
Another interesting application concerns the treatment of stochastic flows. For this purpose the fluid fields are assumed to admit, in terms of suitable stochastic variables $\alpha = \{\alpha_i, i = 1, k\} \in V_\alpha \subseteq \mathbb{R}^k$, with $k \geq 1$, a stochastic representation of the form
\[
\{Z_1\} = \{Z_1(r,t,\alpha)\},
\]
to be defined in terms of a suitable stochastic model $\{g(r,t,\alpha), V_\alpha\}$ (see Appendix B). We intend to show that for incompressible fluids:

- **Goal #4** - The Langevin equations associated to TTPs dynamics provides a possible mathematical model for tracer-particle motion in the presence of fluctuating fluid fields (Section 7, subsection 7.1).

- **Goal #5** - The stochastic-averaged KDF of the TTP-statistical model $\{f_1, \Gamma_1\}$ satisfies a Fokker-Planck statistical equation and a $H$-theorem (Section 7, subsections 7.2 and 7.3).

- **Goal #6** - The IKT and TTP statistical models $\{f, \Gamma\}$ and $\{f_1, \Gamma_1\}$ are suitably related to the customary statistical treatment of turbulence due to Hopf, Rosen and Edwards (HRE approach [24–26]) (see Section 7, subsection 7.4).

- **Goal #7** - The TTP-statistical model $\{f_1, \Gamma_1\}$ provides a solution to Closure Problem for the statistical description of hydrodynamic turbulence (HT) (Section 7, subsection 7.4).

### IV. IKT FOR COMPRESSIBLE NS THERMOFLUIDS

#### A. Axiomatic formulation

The basic requirements of the *IKT statistical model* $\{f(x,t), \Gamma\}$ have been discussed elsewhere [2, 3]. In the case of a compressible NS thermofluids, these can be re-formulated as follows. First we require that the KDF $f(t)$, solution of IKE [see Eq. (14)], uniquely determines the complete set of fluid fields $\{Z_1\}$, in terms of suitable phase-space moments of the same KDF. This is obtained imposing the following axiom:

**Axiom #1 - Correspondence principle:** For compressible (or incompressible) NS thermofluids - in the closure of the fluid domain $\Omega$, where by definition $\rho(r,t) > 0$ in $\Omega \times I$ - the following functional constraints hold
\[
\begin{align*}
\int_U \rho \, dv f(x,t) &= \rho(r,t), \\
\int_U \rho \, dv v f(x,t) &= V(r,t), \\
\int_U \frac{1}{2} \rho \, dv u^2 f(x,t) &= p_1(r,t), \\
S(f(t)) &= S_T(t),
\end{align*}
\]
which are referred to as correspondence principle for $\{Z_1\}$. Here we remark that due to the arbitrariness of the constants $\alpha_1$ and $c_1$ appearing in the definition of the BS entropy [see Eq. (13)] the last equation can also be replaced by $S(f(t)) = \alpha_2^2 S_T(t) + c_2$, with $\alpha_2 \neq 0$ and $c_2$ being two arbitrary real constants independent of $(x,t)$. Furthermore, we shall require that $f(t)$ admits also the higher-order moments
\[
\begin{align*}
\int_U Q(r,t) &= \int_U \rho \, dv u^2 f(r,u,t), \\
\Pi(r,t) &= \int_U \rho \, dv uu f(r,u,t),
\end{align*}
\]
to be denoted as extended fluid fields. As a consequence, by an appropriate definition of the mean-field, the correspondence principle and IKE [see Eq. (14)] must deliver the complete set of fluid equations, i.e., respectively CNSFE or INSFE (fluid closure condition).

Second, consistent with CSM and the second principle of thermodynamics, the initial KDF $f(t_o) \equiv f_o(x)$ and the pseudopressure $p_0(t)$ are uniquely prescribed. This is obtained introducing the second axiom:

**Axiom #2 - Entropic principle:**

This consists in the following three requirements:

A) At the initial time $t_o \in I$ the initial KDF $f(t_o) \equiv f_o(x)$ is determined in such a way to maximize the BS-entropy $S(f(t_o))$ in a suitable functional class $\{f(t_o)\}$. This coincides with the axiom of CSM known as *principle of entropy maximization* (PEM, Jaynes 1957 [27]).
B) The time derivative of pseudo-pressure \( p_0(t) \) is prescribed for all \( t \in I \) in such a way that the entropy law [131] [see Appendix A] is identically fulfilled.

C) The initial condition \( p_0(t_o) \) is determined by suitably prescribing the initial value of the BS entropy \( S(f(t_o)) \).

The first requirement is met as follows. Denoting by \( \delta \) the Frechet functional derivative operator, let us introduce the first and second variations of \( S(f(t_o)) \), \( \delta S(f(t_o)) \) and \( \delta^2 S(f(t_o)) \). Then, the initial KDF \( f_o \in \{ f(t_o) \} \) is determined imposing the variational equation

\[
\delta S(f(t_o))|_{f_o} = 0
\]

subject to the inequality

\[
\delta^2 S(f(t_o))|_{f_o} < 0.
\]

The determination of \( dp_0(t)/dt \) is obtained, instead, in such a way to warrant - for consistency with the correspondence principle and the entropy law - that the weak H-theorem

\[
\frac{\partial}{\partial t} S(f(t)) = \frac{\partial}{\partial t} S_T(t) \geq 0
\]

holds for all \( t \in I \). In the case of isentropic flows this reduces to the constant H-theorem:

\[
\frac{\partial}{\partial t} S(f(t)) = \frac{\partial}{\partial t} S_T(t) = 0.
\]

Furthermore, the initial kinetic pressure \( p_0(t_0) \) is prescribed in such a way that the initial BS entropy \( S(f(t_o)) \) vanishes, i.e.,

\[
S(f(t_o)) = 0.
\]

The initial condition [42] on the BS entropy is equivalent to demand that the measure of ignorance \( S(f(t_o)) \) is zero. Since \( S(f(t_o)) \leq S(f_M(t_o)) \) this means that the Gaussian KDF \( f_M(t_o) \) must admit the BS entropy and hence that the kinetic pressure is necessarily \( p_t(\mathbf{r}, t_o) \geq 0 \). Due to the arbitrariness of \( p_0(t_0) \) this requirement can always be satisfied.

Finally, suitable kinetic closure conditions are introduced to determine \( \mathbf{F}(\mathbf{x}, t; f) \) in the non-Gaussian case \( f(\mathbf{x}, t) \neq f_M(\mathbf{x}, t) \):

**Axiom #3 - Kinetic closure conditions:** For this purpose, in analogy with INSE [3], we shall assume that \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \) is a polynomial of lowest possible degree with respect to the relative kinetic velocity \( \mathbf{u} \) and depends on the lowest-order and minimal number of velocity moments of the KDF. In particular:

1. **Axiom #3a** \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \) can depend, besides \( \{ Z_1 \} \), only on the minimal set of extended fluid fields \( \{ \mathbf{Q}, \Pi \} \):

2. **Axiom #3b** \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \) depends only linearly with respect to \( \mathbf{Q} \) and \( \Pi \).

Let us briefly comment on these requirements. In principle the vector field \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \) might depend on arbitrary higher-order moments of the KDF. In fact, due to the arbitrariness in its definition (see discussion above), it is always possible to include an additive contribution which vanishes identically in the case \( f = f_M \) (Gaussian KDF) and which does not contribute the velocity moments of the Liouville equations corresponding to the weight functions \( G = (1, \mathbf{v}, \mathbf{u}^2/3) \). Hence the above closure conditions warrant that such contributions are excluded, so that Axioms #3a and #3b actually realize the minimal requirements when \( f \) is non-Gaussian.

The motivation for the precise choice of Axiom #3 is mathematical simplicity. In fact, within the framework of IKT, the mean-field \( \mathbf{F} \) is not a physical observable, both for Gaussian and non-Gaussian KDFs, and therefore it remains intrinsically non-unique. Its indeterminacy in the case of a non-Gaussian KDF arises because the vector field \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \) might include in principle higher-order velocity moments of the KDF, besides the fluid fields \( \{ Z_1 \} \) which are by construction the only observables. These additional moments have no physical meaning (i.e., they are not observables), and therefore remain completely undetermined in the framework of IKT. Hence, \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \) should depend only on a minimum finite number of fluid fields which are required for the validity of the theory. This means that there must exist a finite subset of moment equations which coincide with CNSFE. To further clarify the issue, we notice that \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \) can always be given a polynomial representation in terms of the relative velocity \( \mathbf{u} \). Such terms would necessarily depend on higher-order velocity moments which vanish in the case of the Gaussian KDF and can always be prescribed in such a way not to contribute to the same moment equations. Unless Axioms #3a and #3b are introduced, such additional contributions to \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \), depending, besides the CNSFE fluid fields, also on \( \mathbf{Q} \) and \( \Pi \), would remain undetermined. Due to the intrinsic freedom of their choice, in the following they will be set identically equal to zero. As clarified below, such an assumption is equivalent to require Axioms #3a and #3b. This choice does not affect the validity of the CNSFE problem and does not constraint in any way its solutions. In conclusion, in view of these considerations, Axiom #3 can be viewed as a set of kinetic closure condition which are needed for the prescription of the mean-field \( \mathbf{F}(\mathbf{r}, \mathbf{u}, t; f) \) and the related kinetic equation [133].
B. The IKT statistical model for compressible thermofluids

Based on the axiomatic formulation given above [Axioms #1-#3], we can now proceed to the explicit determination of the mean-field \( \mathbf{F} = \mathbf{F}(\mathbf{x}, t, f) \) appropriate for a compressible thermofluid satisfying the CNSFE problem [defined by Eqs.\(^{(128)-(131)}\) and the initial-boundary conditions \((163)\); see Appendix A]. It is immediate to show that in such a case the form of the mean-field \( \mathbf{F}(\mathbf{x}, t, f) \) can be determined analytically. The result is summarized by the following theorem:

**Theorem 1 - IKT statistical model for CNSFE** - Let us require that the IKT statistical model \( \{f(\mathbf{x}, t), \Gamma\} \) satisfies Axioms #1-#3 and furthermore that:

1) The CNSFE problem admits a smooth strong solution in \( \Gamma \times I \).

2) The mean-field \( \mathbf{F}(\mathbf{x}, t, f) \) is defined as

\[
\mathbf{F}(\mathbf{x}, t, f) = \mathbf{F}_H(\mathbf{r}, t) + \mathbf{F}_\text{r}(\mathbf{x}, t; f) + \mathbf{F}_a(\mathbf{r}, \mathbf{u}, t; f) + \mathbf{u} \cdot \nabla \mathbf{V} + \Delta \mathbf{F}(\mathbf{x}, t; f). \tag{38}
\]

Here \( \mathbf{F}_H \) is the Navier-Stokes acceleration, given by Eq.\(^{(113)}\) [see Appendix A], \( \mathbf{F}_\text{r} \) is the relative kinetic acceleration defined as

\[
\mathbf{F}_\text{r}(\mathbf{x}, t; f) = \frac{v_1^2}{2} \nabla \ln \rho + \frac{\mathbf{u}}{2p_1} A + \frac{v_2^2}{2} \nabla \ln \left( \frac{p_1}{\rho} \right) \left( X^2 - \frac{1}{2} \right), \tag{39}
\]

with

\[
A = \rho \frac{D}{Dt} \left( \frac{p_1}{\rho} \right) = \rho \frac{D}{Dt} \left( \frac{p_o + \rho - \phi}{\rho} \right) + nK(\mathbf{r}, t), \tag{40}
\]

and with \( K(\mathbf{r}, t) \) being prescribed by Eq.\(^{(130)}\) in Appendix A. Finally, \( \mathbf{F}_a(\mathbf{r}, \mathbf{u}, t; f) \) is defined as

\[
\mathbf{F}_a(\mathbf{r}, \mathbf{u}, t; f) \equiv \frac{1}{\rho} \left[ \nabla \Pi - \nabla p_1 \right] + \frac{\mathbf{u}}{2p_1} \left[ \nabla \cdot \mathbf{Q} - \nabla \ln \left( \frac{p_1}{\rho} \right) \cdot \mathbf{Q} \right]. \tag{41}
\]

3) \( \mathbf{F}(\mathbf{x}, t; f) \) is defined up to an arbitrary real gauge field \( \Delta \mathbf{F}(\mathbf{x}, t; f) \) satisfying the gauge condition \((21)\). We shall require that \( \Delta \mathbf{F}(\mathbf{x}, t; f) \) is a smooth vector field analytic with respect to the Newtonian velocity vector \( \mathbf{v} \in U \).

4) The BS entropy and the velocity moments \( \int_U d\mathbf{v} G(\mathbf{x}, t)f(\mathbf{x}, t) \) evaluated for \( \{G(\mathbf{x}, t)\} = \{1, \mathbf{v}, \frac{1}{3} \mathbf{v}^2, \mathbf{uu}, \frac{1}{2} \mathbf{u}^2 \} \) exist for all \( (\mathbf{r}, t) \in \Omega \times I \).

5) In \( \Omega \times I \) the KDF \( f(\mathbf{x}, t) \) admits the correspondence principle defined by Eqs.\(^{(17)}\).

6) Let us introduce the decomposition

\[
\frac{\partial}{\partial t} S(f(t)) = P_1(f(t)) + \frac{\partial}{\partial t} S_T(t), \tag{42}
\]

\[
P_1(f(t)) \equiv P(f(t)) - \frac{\partial}{\partial t} S_T(t), \tag{43}
\]

where

\[
P(f(t)) = \int_{\Omega} d\mathbf{r} \left[ \frac{3\rho(\mathbf{r}, t)}{2p_1} A + \rho(\mathbf{r}, t) \nabla \cdot \mathbf{V} + \frac{3\rho(\mathbf{r}, t)}{2p_1} \left( \nabla \cdot \mathbf{Q} - \nabla \ln \left( \frac{p_1}{\rho} \right) \cdot \mathbf{Q} \right) \right] \tag{44}
\]

and \( \frac{\partial}{\partial t} S_T(t) \) denotes the global thermodynamic entropy production rate [defined by \(^{(162)}\)]. Then, we require that for all \( t \in I \) the pseudo-pressure \( p_0(t) \) is determined so that

\[
P_1(f(t)) = 0. \tag{45}
\]

It follows that:

T1) The local Gaussian distribution function \((17)\) is a particular solution of the IKE \((14)\) if and only if the fluid fields \( \{Z\} \) satisfy the CNSFE problem [see Appendix A]. For a generic KDF \( f(\mathbf{x}, t) \), introducing the representation

\[
f(\mathbf{x}, t) = f_M(\mathbf{x}, t)h(\mathbf{x}, t), \tag{46}
\]
it follows that the reduced KDF \( h(x, t) \) satisfies the integral IKE

\[
h(x, t) = h_0(T_{t, t}, x) \exp \left\{ - \int_{t_o}^t dt' \frac{\partial}{\partial \nu(t')} \cdot F_u(x(t'), t'; f) \right\},
\]

(47)

\( h_0(x_o) \) being a suitable initial KDF.

**T1.2.** In the case of a general non-Gaussian KDF \( f(x, t) \), the velocity-moment equations obtained by taking the weighted velocity integrals of Eq. (14) with the weights \( \{ G(x, t) = \{ 1, \nu, u^2/3 \} \) deliver identically the fluid equations (128)-(130).

**T1.3.** For all \( t \in I \) the pseudo-pressure \( p_0(t) \) must satisfy the ODE:

\[
\frac{d}{dt} p_0 = \frac{1}{2} \int_\Omega \frac{d}{d\rho} \left[ S_p(t) + Q(t) + \frac{\partial}{\partial T} S_T(t) \right],
\]

(48)

where

\[
S_p(t) \equiv -\frac{3}{2} \int_\Omega \frac{d}{d\rho} \left[ \frac{\partial}{\partial t} (p - \phi + nT) + \nabla \cdot \nabla (p - \phi + nT) \right] - \int_\Omega \frac{d}{d\rho} \nabla \cdot \nabla,
\]

(49)

\[
Q(t) \equiv -\int_\Omega 3 \frac{\partial}{d\rho} \frac{\rho}{2p_1} \left[ \nabla \cdot Q - \nabla \ln \left( \frac{p_1}{\rho} \right) \cdot Q \right].
\]

(50)

**T1.4.** The BS entropy \( S(f(t)) \) satisfies for all \( t \in I \) the H-theorem (51). Instead, for an isothermal fluid the constant H-theorem (52) holds.

**T1.5.** In validity of Eq. (48), for all \( (r, t) \in \Omega \times I \) the kinetic pressure \( p_1(r, t) \) is strictly positive. Furthermore, the initial value \( p_0(t_o) \) is uniquely determined by prescribing the condition of vanishing of the initial BS entropy in the case \( f(t_o) = f_M(t_o) \) [see Axiom #2, entropic principle].

Proof - First, it is immediate to prove that, in validity of Eqs. (38)-(41), \( f_M(x, t) \) is a particular solution of the inverse kinetic equation (14). The proof is analogous to that given in Refs. [2, 5] and it follows by direct substitution of the distribution \( f_M(x, t) \) in the same equation (Proposition T1.1). This implies that \( f_M(x, t) \) satisfies necessarily the integral Liouville equation (15), so that

\[
f_M(x, t) = f_M(T_{t, t}, x, t_o) \exp \left\{ - \int_{t_o}^t dt' \frac{\partial}{\partial \nu(t')} \cdot F_u(x(t'), t'; f) + \frac{\partial}{\partial r(t')} \cdot V(r(t'), t') \right\}.
\]

Therefore, in case of a non-Gaussian KDF the same equation manifestly implies also Eq. (17).

Instead, if we assume that in \( \Gamma \times I \), \( f(x, t) \) is a particular solution of the inverse kinetic equation, it follows that the fluid fields \( \{ \rho, V, p, T \} \) are necessarily solutions of the CNSFE equations. This can be proved either in the case \( f \equiv f_M(x, t) \) by direct substitution in Eq. (14) or, in the general case in which \( f \neq f_M(x, t) \) is an arbitrary smooth and strictly positive particular solution, by direct calculation of the velocity moments of the same equation, evaluated with respect to the weight-functions \( G(x, t) = \{ 1, \nu, u^2/3 \} \) (Proposition T1.2). In fact, in validity of Axiom #2 and Eqs. (38)-(40), the moments equations corresponding to Eqs. (51) yield respectively:

\[
\frac{\partial \rho(r, t)}{\partial t} + \nabla \cdot [\rho(r, t)V(r, t)] = 0,
\]

(51)

\[
\frac{\partial \rho(r, t)}{\partial t} V(r, t) + \nabla \cdot [\rho(r, t)V(r, t)V(r, t)] \equiv \rho \frac{D}{Dt} V(r, t) = \rho F_H,
\]

(52)

\[
\frac{\partial}{\partial t} p_1(r, t) + \nabla \cdot [V(r, t)p_1(r, t) + Q] =
\]

\[
= A + \left[ \nabla \cdot Q - \nabla \ln \left( \frac{p_1}{\rho} \right) \cdot Q \right] + Q \cdot \nabla \ln \left( \frac{p_1}{\rho} \right).
\]

(53)
The first two equations coincide, respectively, with the continuity and Navier-Stokes equations while the third one, thanks to Eq. (14), recovers the Fourier equation [see respectively Eqs. (128), (129) and (130) in Appendix A]. Let us now evaluate the entropy production rate $\frac{\partial}{\partial t} S(f(t))$ (Proposition T1$_3$). First we notice that thanks to the Brillouin Lemma 28:

$$S(f(t)) = - \int \nabla f(x) \ln f(x,t) \leq$$

$$\leq - \int \nabla f(x,t) \ln f_M(x,t) = - \int \nabla f_M(x,t) \ln f_M(x,t) \equiv S(f_M(t)),$$

which implies

$$\frac{\partial}{\partial t} S(f(t)) \equiv \int \nabla f(x,t) \cdot F(x,t,f) \leq \frac{\partial}{\partial t} S(f_M(t)) \equiv \int \nabla f_M(x,t) \cdot F(x,t,f_M),$$  \hspace{1cm} (55)

where respectively:

$$\frac{\partial}{\partial t} \cdot F(x,t,f) = \frac{3}{2p_1} A + \frac{3}{2p_1} \left( \nabla \cdot Q - \nabla \ln \left( \frac{p_1}{\rho} \right) \cdot Q \right) + \nabla \cdot V,$$  \hspace{1cm} (56)

$$\frac{\partial}{\partial t} \cdot F(x,t,f_M) = \frac{3}{2p_1} A + \nabla \cdot V. \hspace{1cm} (57)$$

It follows that $P(f(t)) \leq P(f_M(t))$, with $P(f(t))$ being defined by Eq. (44), implying in turn that the inequality

$$Q(t) \geq 0 \hspace{1cm} (58)$$

is necessarily fulfilled for all $t \in I$. Therefore, introducing the decomposition (12)-(13), the constraint (45) manifestly implies that Proposition T1$_3$ must hold. In addition, the entropy production rate fulfills identically the constraint

$$\frac{\partial}{\partial t} S(f(t)) = \frac{\partial}{\partial t} S_T(t).$$ \hspace{1cm} (59)

Hence, thanks to the entropy law (141), necessarily the BS entropy $S(f(t))$ satisfies the H-theorem (55) (Proposition T1$_4$). Let us now prove Proposition T1$_5$, namely that the constraint (18) requires the kinetic pressure $p_1(r,t)$ to be strictly positive in $\Omega \times I$. For this purpose, we first consider the case $f = f_M$ and impose $\forall t \in I_o \equiv \{ t : t \geq t_o, \forall t \in I \}$ that the constant-entropy condition $S(f_M(t)) = S(f_M(t_o))$ holds, requiring:

$$\frac{\partial}{\partial t} S(f_M(t)) \equiv \int \Omega \frac{3}{2p_1} A = 0. \hspace{1cm} (60)$$

Then, thanks to the identity

$$\int \Omega \frac{3}{2p_1} A = \frac{\partial p_0(t)}{\partial t} \frac{3}{2} \int \Omega \frac{p_1}{p_1} + \frac{3}{2} \int \Omega \frac{p_1}{p_1} \left[ \frac{\partial}{\partial t} (p - \phi + nT) + \nabla \cdot (p - \phi + nT) \right], \hspace{1cm} (61)$$

it follows that $p_0(t)$ must satisfy the ODE

$$\frac{dp_0(t)}{dt} = \frac{1}{\frac{3}{2} \int \Omega \frac{p_1}{p_1}} S_p(t), \hspace{1cm} (62)$$

with $S_p(t)$ being given by Eq. (19). Hence $p_1(r, t)$ is necessarily strictly positive in $\Omega \times I$. The same conclusion manifestly follows imposing instead

$$\frac{\partial}{\partial t} S(f_M(t)) \equiv \int \Omega \frac{3}{2p_1} A = \frac{\partial}{\partial t} S_T(t) \geq 0, \hspace{1cm} (63)$$
\[ \frac{dp_0(t)}{dt} = \frac{1}{2} \int_{\Omega} d\mathbf{r} \left[ S_p(t) + \frac{\partial}{\partial t} S_T(t) \right]. \tag{64} \]

Analogous conclusion holds, thanks to the inequality \((58)\), also in the case \( f \neq f_M \) [see Eq.\((58)\)]. Finally, let us impose the initial condition for the initial kinetic pressure \( p_0(t_0) \). Denoting \( M = \int d\mathbf{r} \), in view of Axiom \#2 this requires

\[ S(f_M(t_0)) = \frac{3}{2} \int_{\Omega} d\mathbf{r} \ln p_1 + M \left[ \frac{3}{2} + \ln (2\pi)^{3/2} \right] - \frac{5}{3} \int_{\Omega} d\mathbf{r} \ln \rho = 0, \]

which uniquely determines \( p_0(t_0) \). \textbf{Q.E.D.}

Here it is worth noting that:

- Eqs. \((58)\)-\((61)\) yield a realization of the mean-field \( \mathbf{F}(\mathbf{x}, t; f) \), of the type indicated above (see subsection 1.4) and holding for compressible NS thermofluids, which satisfies the CNSFE problem [see Appendix A].

- The expression of the vector field \( \mathbf{F}(\mathbf{x}, t; f) \) in Eqs. \((58)\)-\((61)\) is determined, up to the gauge field \( \Delta \mathbf{F} \), according to the form of the KDF as follows. In the case \( f = f_M \) it is obtained by solving explicitly for \( \mathbf{F} \) the equation

\[ \left( \frac{\partial}{\partial t} + \mathbf{v} \cdot \frac{\partial}{\partial \mathbf{r}} \right) \ln f_M - 2 \mathbf{u}_{\text{vth}} \cdot \mathbf{F} = - \left( \frac{\partial}{\partial \mathbf{v}} \mathbf{F} \right), \quad \tag{65} \]

and imposing the validity of CNSFE. The procedure is analogous to that outlined, for example, in Ref.\,[2]. Instead, in the general case in which \( f \neq f_M \), with \( f \) being a strictly positive KDF satisfying Axioms \#1-\#3, \( \mathbf{F}(\mathbf{x}, t; f) \) is determined by requiring it is of the form \((58)-(59)\) with \( \mathbf{F}_a(\mathbf{x}, t; f) \) to be suitably prescribed. In particular, thanks to Axiom \#3, \( \mathbf{F}_a(\mathbf{x}, t; f) \) is taken to be a polynomial of first degree in the relative velocity \( \mathbf{u} \). Therefore, it is necessarily of the form \( \mathbf{F}_a(\mathbf{r}, \mathbf{u}, t; f) = \mathbf{F}_a^{(0)}(\mathbf{r}, t; f) + \mathbf{u} \mathbf{F}_a^{(1)}(\mathbf{r}, t; f) \), with \( \mathbf{F}_a^{(0)}(\mathbf{r}, t; f) \) and \( \mathbf{F}_a^{(1)}(\mathbf{r}, t; f) \) being respectively two suitable moments of the KDF \( f \). Their precise form is obtained by imposing Axiom \#1 and requiring that the velocity moments of IKE corresponding to the weight functions \( G(\mathbf{x}, t) = (1, \mathbf{v}, u^2/3) \) coincide with CNSFE. For example, \( \mathbf{F}_a^{(0)}(\mathbf{r}, t; f) \) follows by constructing the moment equation with respect to \( G = \mathbf{v} \). From IKE, utilizing the definitions for \( \mathbf{F}_H(\mathbf{r}, t) \) and \( \mathbf{F}_H(\mathbf{x}, t; f) \), it follows that \( \mathbf{F}_a^{(0)}(\mathbf{r}, t; f) \) must be prescribed so that the equation

\[ \rho \frac{D}{Dt} \mathbf{V} + (\nabla \cdot \mathbf{Q} - \nabla p_1) = - \rho \mathbf{F}_H - \int_{\Omega} d\mathbf{v} f(\mathbf{x}, t) \mathbf{F}_a^{(0)} = 0 \tag{66} \]

coincides with the NS equation. This yields the unique solution \( \mathbf{F}_a^{(0)}(\mathbf{r}, t; f) \equiv \frac{1}{\rho} \left[ \nabla \cdot \mathbf{Q} - \nabla p_1 \right] \). Similarly, the expression for \( \mathbf{F}_a^{(1)}(\mathbf{r}, t; f) \) follows from the moment equation with respect to \( G = u^2/3 \). This yields the solution

\[ \mathbf{F}_a^{(1)}(\mathbf{r}, t; f) \equiv \frac{1}{2\rho} \left[ \nabla \cdot \mathbf{Q} - \nabla \ln \left( \frac{\mathbf{v}^2}{\rho} \right) \right] \cdot \mathbf{Q}. \]

The resulting expression for \( \mathbf{F}_a(\mathbf{r}, \mathbf{u}, t; f) \) coincides with Eq.\((61)\).

- If \( f \equiv f_M(\mathbf{x}, t) \) the extended fluid fields \( \mathbf{Q}(\mathbf{r}, t) \) and \( \mathbf{Q}(\mathbf{r}, t; f) \) [see Eqs.\((52)\)] vanish identically. As a consequence, in this case \( \mathbf{F}_a(\mathbf{r}, t) \equiv 0 \).

- Eqs. \((58)\)-\((61)\) apply also in the case of an incompressible thermofluids [INSFE problem], and in particular for isothermal fluids [INSE problem; see Appendix A]. In the case \( f \equiv f_M(\mathbf{x}, t) \) the corresponding mean-field \( \mathbf{F}(\mathbf{x}, t; f_M) \) is consistent with Refs.\,[2, 5]. However, in the case of non-Gaussian KDFs, the linearity condition here imposed as a kinetic closure condition on \( \mathbf{F}(\mathbf{x}, t; f) \) (see Axiom \#3b) actually leads to a representation of the mean-field in terms of the extended fluid fields \( \mathbf{Q}(\mathbf{r}, t) \) and \( \mathbf{Q}(\mathbf{r}, t; f) \) which is different from that adopted previously in Refs.\,[2, 5].

In addition, in agreement with the GENERIC dynamical model \([11, 12]\), it is possible to show that:

- Both in the case of CNSFE and INSFE the IKT-statistical description permits to represent the reduced set of fluid fields \( \{Z_1\} \) [see Eq.\((7)\)] in terms of velocity and phase-space moments (bundle structure on \( \Gamma \)).
V. DYNAMICS OF NS IDEAL TRACER PARTICLES IN DETERMINISTIC FLUIDS

A basic consequence of the previous theorem is that the functional form of $\Delta F$ remains non-unique for ITPs. Its possible unique prescription requires, therefore, the adoption of suitable additional kinetic closure conditions. For this purpose in this section we intend to show that for all TTPs $\Delta F$ can be uniquely determined in such a way to fulfill the requirement of the Gedanken experiment (GDE-requirements #1-#6).

A. Preliminary Lemma

Let us first show that due to the kinetic constraints (27) and (29) for TTPs the pseudo-vector $\Omega(r, t)$ entering the evolution equation for $n(r, t)$ [see Eq.(28)] is actually uniquely determined. In fact the following result holds.

**Lemma to THM.2 - General form of $\Omega(r, t)$**

If $\Omega_1 \equiv \Omega \times I$ denotes the existence domain of the fluid fields $\{Z\}$, let us assume that in the subset $\Omega_1 \subseteq \Omega_1$ in which $\nabla p_1 \neq 0$:

1. the real unit vectors $n(r, t), b(r, t)$ and the pseudo-vector $\Omega(r, t)$ are all differentiable and suitably smooth;
2. the unit vector $n(r, t)$ satisfies the kinetic constraint (27) and the initial-value problem (28);
3. moreover, $\Omega(r, t)$ satisfies the constraint (29) and is defined in the limit $p_1(r, t) \to 0^+$ and also for arbitrary finite values of the kinetic pressure $p_1(r, t)$.

It follows that, in the domain $\Omega_1$, $\Omega(r, t)$ necessarily takes the form:

$$\Omega(r, t) = b(r, t) \times \frac{db(r, t)}{dt} + c(r, t)b(r, t),$$

(67)

where the GDE-requirement #6 implies

$$c(r, t) = b \times n \ \frac{dn}{dt} \equiv \Omega(r, t) \cdot b(r, t) = -\xi(r, t) \cdot b(r, t),$$

(68)

with $\xi(r, t)$ denoting the local fluid vorticity.

**Proof** - In fact, by definition $|n(r, t)| = 1$ and hence $\frac{dn(r, t)}{dt} \cdot n(r, t) = 0$, so that there must exist a pseudo-vector $\Omega(r, t)$ such that Eq.(28) holds identically for arbitrary initial condition $n(r(t_0), t_0) = n(r_o, t_o)$. Let us now impose the validity of the kinetic constraint (27), implying

$$\frac{d}{dt}n(r, t) \cdot b(r, t) = -n(r, t) \cdot \frac{d}{dt}b(r, t).$$

(69)

Hence, necessarily $\Omega(r, t)$ satisfies the equation

$$n(r, t) \cdot \frac{d}{dt}b(r, t) = \nabla [\Omega(r, t) \times n(r, t)] \cdot b(r, t),$$

(70)

i.e., due to the arbitrariness of the unit vector $n(r, t)$, $b(r, t) \times \Omega(r, t) = -\frac{d}{dt}b(r, t)$. This yields for $\Omega(r, t)$ a general solution of the form

$$\Omega(r, t) = b(r, t) \times \frac{db(r, t)}{dt} + c(r, t)b(r, t).$$

(71)

Substituting this solution in Eq.(28) and taking the scalar product of the resulting equation by $b \times n$, in validity of GDE-requirement #6 Eq.(68) follows identically, which uniquely determines the form of $\Omega(r, t)$.

Q.E.D.
B. Construction of TTP solutions

Let us now prove the existence of the TTPs, particular solutions of the initial-value problem (4). More precisely, we intend to prove that for all \( t \in I \) and an appropriate choice of the mean-field \( \mathbf{F} + \Delta \mathbf{F} \), the NS-DS (2) (or equivalent RD-NS-DS) (3), necessarily maps an arbitrary TTP initial state \( \mathbf{x}(t_o) = \mathbf{x}_o \) into a TTP state \( \mathbf{x}(t) = T_{t_o,t} \mathbf{x}_o \). For this purpose we impose that, consistent with GDE-requirements #1-#6, \( \mathbf{F} \) is defined by Eqs. (39)-(40). We intend to show that, as a consequence, for all TTPs both \( \mathbf{F} \) and the gauge field \( \Delta \mathbf{F} \) [see Eq. (24)] are necessarily uniquely determined. On the other hand, in view of GDE, one expects that the vector field \( \mathbf{F} \) which characterizes TTP dynamics should not depend on the form of the KDF. If true, the result would clearly be conceptually important because it would imply the uniqueness of TTP dynamics in all cases. For reference, let us first consider the case of the Gaussian KDF \( f = f_M \), leaving the extension to a non-Gaussian KDF to the discussion below. Then, the following theorem holds.

**THEOREM 2 - Existence and uniqueness of TTP dynamics**

In validity of THM.1, let us require that \( \Delta \mathbf{F} \) and \( \mathbf{F}_u \) are analytic functions in \( \Gamma \) which are defined also in the limit \( p_1(\mathbf{r},t) \to 0^+ \). Then it follows that:

T21) The initial-value problem defined by (4) admits particular solutions fulfilling the GDE-requirements #1-#6, here denoted as TTPs. In particular, for all \( t \in I \) they are characterized by a relative kinetic velocity defined by Eq. (23) and is such that:

- A) the local magnitude of the relative kinetic velocity \( |\mathbf{u}(t)| \) is determined by the equation:

\[
|\mathbf{u}(t)| = u_{th}(\mathbf{r},t) \equiv \beta v_{th}(\mathbf{r},t),
\]

with \( \beta \geq 0 \) being independent of \( (\mathbf{r},t) \);

- B) \( \mathbf{n}(\mathbf{r},t) \) satisfies both the constraint equation (24) and the initial-value problem (25);

- C) \( \Omega(\mathbf{r},t) \) is uniquely determined by Eqs. (67) and (68).

T22) For arbitrary TTPs, the mean-field \( \mathbf{F}(\mathbf{r},u_{th},t;f_M) + \Delta \mathbf{F} \) has necessarily the unique representation

\[
\mathbf{F}(\mathbf{r},u_{th},t;f_M) + \Delta \mathbf{F} = \mathbf{F}_0 + \mathbf{u}_{th} \cdot \nabla \mathbf{V} + \frac{u_{th}}{2} \frac{D}{Dt} \ln (\hat{p}_1) + \beta u_{th} \Omega(\mathbf{r},t) \times \mathbf{n}.
\]

This implies that, in the case of the Gaussian KDF \( f = f_M \), and for arbitrary \( \beta \geq 0 \), the vector field \( \Delta \mathbf{F} \equiv \Delta \mathbf{F}(\mathbf{r},u_{th},t) \) has the unique representation

\[
\begin{cases}
\Delta \mathbf{F} = \Delta \mathbf{F}_0 + \Delta \mathbf{F}_1, \\
\Delta \mathbf{F}_0(\mathbf{r},u_{th},t) = - \left[ \frac{u_{th}^2}{2} \nabla \ln \rho + \frac{u_{th}^2}{2} \nabla \ln (\hat{p}_1) \left( \beta^2 - \frac{1}{2} \right) \right], \\
\Delta \mathbf{F}_1(\mathbf{r},u_{th},t) = \beta u_{th} \Omega(\mathbf{r},t) \times \mathbf{n}.
\end{cases}
\]  

T23) Particular solutions of the form (23) which fulfill requirements A-C must satisfy the initial conditions

\[
\begin{cases}
\mathbf{r}(t_o) = \mathbf{r}_o, \\
\mathbf{u}(t_o) = \beta v_{th}(\mathbf{r}_o,t_o) \mathbf{n}(\mathbf{r}_o,t_o),
\end{cases}
\]  

with \( \beta \geq 0 \) being an arbitrary real constant, \( p_1(\mathbf{r}_o,t_o) \) the initial kinetic pressure and \( \mathbf{n}(\mathbf{r}_o,t_o) \) a unit vector satisfying the orthogonality condition

\[
\mathbf{n}(\mathbf{r}_o,t_o) \cdot \nabla \hat{p}_1(\mathbf{r}_o,t_o) = 0.
\]

**Proof - T21-T22** For generality let us assume that \( |\nabla p_1| \neq 0 \) everywhere in \( \Omega \times I \). Then, it is sufficient to prove the theorem in the subset \( \Omega_1 \subseteq \Omega_1 \equiv \Omega \times I \) in which \( |\nabla p_1| \neq 0 \). Let us show that in \( \Omega_1 \), for an arbitrary non-negative constant \( \beta \in \mathbb{R}^+ \), a particular solution of the initial-value problem (4) of the type (23), which satisfies requirements A-C, exists and is unique. In fact, let us assume that \( |\mathbf{u}(t)| \) is of the form (23), with \( \beta \equiv \beta(\mathbf{r},t) \geq 0 \) denoting now an arbitrary smooth real function of \( (\mathbf{r},t) \) defined in \( \Omega \times I \). It is immediate to show that necessarily \( \beta \) must be everywhere constant with respect to \( (\mathbf{r},t) \) in \( \Omega \times I \). Indeed, Eq. (41) requires

\[
\begin{cases}
\frac{d}{dt} (\beta v_{th}(\mathbf{r},t)) = \frac{v_{th}^2}{2} \mathbf{n} \cdot \nabla \ln \rho + \frac{1}{2} \beta v_{th}(\mathbf{r},t) \frac{\beta}{2} \ln (\hat{p}_1) + \\
\quad + \frac{v_{th}^2}{2} \mathbf{n} \cdot \nabla \ln (\hat{p}_1) \left( \beta^2 - \frac{1}{2} \right) + \mathbf{n} \cdot \Delta \mathbf{F}, \\
\frac{d}{dt} \mathbf{n}(\mathbf{r},t) = \frac{1}{\beta v_{th}} \left[ \frac{v_{th}^2}{2} \nabla \ln \rho + \frac{v_{th}^2}{2} \nabla \ln (\hat{p}_1) \left( \beta^2 - \frac{1}{2} \right) \right] \cdot (\mathbf{1} - \mathbf{n} \mathbf{n}) + \\
\quad + \frac{1}{\beta v_{th}} \mathbf{n} \cdot \Delta \mathbf{F} \cdot (\mathbf{1} - \mathbf{n} \mathbf{n}) \equiv \Omega(\mathbf{r},t) \times \mathbf{n}.
\end{cases}
\]
On the other hand, imposing the constraint \( p_1(r, t) \) requires necessarily, thanks to the Lemma, that \( n(r, t) \) must satisfy the initial-value problem \( \ref{eq:initial-value} \). We require for consistency
\[
\begin{align*}
n \cdot \nabla \ln (\tilde{p}_1) &= 0, \\
\frac{v^2}{2} n \cdot \nabla \ln \rho + n \cdot \Delta F &= 0.
\end{align*}
\]
Hence, since by assumption \( \Delta F \) is defined also in the limit \( p_1(r, t) \to 0^+ \) (or equivalently \( \beta \to 0^+ \)), it follows necessarily that
\[
\begin{align*}
\Delta F &= \Delta F_0(r, u_{th}, t) + \Delta F_1(r, u_{th}, t), \\
\Delta F_1(r, u_{th}, t) \cdot (\mathbb{1} - nn) &= \beta u_{th} \Omega(r, t) \times n(r, t),
\end{align*}
\]
with \( \Omega(r, t) \) being given by the Lemma and
\[
\frac{d\beta(r, t)}{dt} = 0.
\]
Therefore, due to the arbitrariness of \( v \equiv \nabla v(r, t) + u_{th}(r, t)n(r, t), \beta \) is necessarily independent of \( (r, t) \); furthermore, \( \Delta F(r, u_{th}, t; f_M) \) and \( F(r, u_{th}, t; f_M) \) are necessarily of the form \( (\ref{eq:form1}) \) and \( (\ref{eq:form2}) \). Finally, the initial conditions \( (\ref{eq:initial-value}) \) are an immediate consequence of Eq.\( (\ref{eq:initial-value}) \) and the requirements A-C.

Q.E.D.

Let us now consider the extension of the theorem to the case of a non-Gaussian KDF. We notice that the gauge field \( \Delta F \) evaluated for the state of a generic TTP, can always be identified with the vector field
\[
\Delta F = \Delta F_0 + \Delta F_1 - F_a(r, u_{th}, t; f),
\]
where \( \Delta F_0 \) and \( \Delta F_1 \) are still given by Eq.\( (\ref{eq:form1}) \), while \( F_a(r, u_{th}, t; f) \) is prescribed according to Eq.\( (\ref{eq:form3}) \) and computed for \( u = u_{th} \). As a consequence, it is immediate to show that, for TTPs, this prescription of \( \Delta F \) warrants the uniqueness of the vector field \( \mathbf{F}(r, u_{th}, t; f_M) \), in agreement with GDE. Therefore, for TTPs, its form is independent of the form of the KDF, namely \( \mathbf{F}(r, u_{th}, t; f) = \mathbf{F}(r, u_{th}, t; f_M) \).

### C. Implications and physical interpretation

Let us briefly analyze the implications of THM.2. First, we remark that by construction for all TTPs the mean-field acceleration \( \mathbf{F} + \Delta \mathbf{F} \) is unique and independent of the form of the KDF \( f(x, t) \). As a consequence TTP particular solutions [of the initial-value problem \( (\ref{eq:initial-value}) \)] realize a classical dynamical system with existence domain \( \Gamma_1 \times I, \Gamma_1 \) denoting a suitable subset of the phase-space \( \Gamma \) (see related discussion in Section 6). This is defined by a homeomorphism of the form \( T_{t_0,t}^{(TTP)} : y_o \to y(t) = T_{t_0,t}^{(TTP)} y_o \), where for all \( t \in I \),
\[
y(t) = \{ r, u(t) = u_{th}(t) \}
\]
and \( u_{th}(t) \) is given by Eq.\( (\ref{eq:form3}) \). Manifestly \( T_{t_0,t}^{(TTP)} \) is a subset of the RD-NS-DS defined by Eq.\( (\ref{eq:rd-ns}) \), or equivalent of the NS-DS defined by Eq.\( (\ref{eq:ns}) \).

Furthermore, let us assume that the initial conditions for Eq.\( (\ref{eq:initial-value}) \) are of the form \( y(t_0) = \{ r_0, u_0 = u_{th}(t_0) \} \), where \( u_{th}(t_0) \) is prescribed by Eq.\( (\ref{eq:form3}) \). While the initial unit vector \( n(r_0, t_0) \) satisfies the constraint \( (\ref{eq:constraint}) \) at \( r = r_0 \). Then, thanks to THM.2, it follows that for all \( t \in I \), \( y(t) \) is necessarily of the form \( (\ref{eq:form3}) \), i.e., it defines, for all \( t \in I \), a TTP. In addition, thanks to Eq.\( (\ref{eq:constraint}) \), it follows that \( \beta \) is constant and therefore is uniquely determined for each TTP by the initial state \( y(t_0) \), namely \( \beta = \frac{\|u_0\|}{v_{th}(r_0, t_0)} \).

We notice that, by construction, TTPs are uniquely associated to the local state of the fluid. As a consequence, this permits to determine also for the remaining NS-ITPs an explicit representation of the mean-field which is necessarily of the form \( \mathbf{F}(r, u; t, f) \), namely it depends explicitly on the KDF. Similarly the gauge-field is of the type \( \Delta \mathbf{F}(r, u; t, f) \). Both hold for arbitrary \( u \in U \) and satisfy at the same time the requirements posed by THM.2, namely that for \( u = u_{th} \), the sum of the two vectors \( \mathbf{F}(r, u; t, f) + \Delta \mathbf{F}(r, u; t, f) \) must reduce to Eq.\( (\ref{eq:form1}) \). For definiteness, in validity of THM.1, let us consider the case of a generally non-Gaussian KDF \( f(r, u, t) \), with \( f(r, u, t) \) denoting a smooth, strictly positive function which satisfies at the same time Axioms \#1-\#6, i.e., is a particular solution of Eq.\( (\ref{eq:gde}) \). Let
us therefore determine $\Delta F(r, u; t; f)$ in such a way to fulfill the constraint equations (73) and the gauge condition (21). For this purpose it is sufficient to let

$$\Delta F(r, u; t; f) = \Delta F(r, u_{th}; t) \frac{f(r, u_{th}, t)}{f(r, u, t)}, \tag{85}$$

with $\Delta F(r, u_{th}; t)$ being defined by Eq. (74).

Finally, an interesting issue concerns the physical interpretation of the evolution equation for the unit vector $n(r, t)$ [i.e., the direction of the particle relative velocity] and related pseudo-vector $\Omega$. In fact, that Eqs. (25) are similar to the Euler equations for a rigid body rotating with angular velocity $\Omega_r \equiv -\Omega$. This suggests that due to the GDE-requirements #5 and #6, two different physical effects contribute to $\Omega$. These are due both to the rotation of the unit vector $n(r, t)$ as determined by Eq. (28) as a consequence of non-uniform specific kinetic pressure, and the contribution of fluid vorticity specified by Eq. (29). Indeed, from Eq. (71), denoting by $\frac{Db(r, t)}{Dt} = \frac{\partial}{\partial t} + V(r, t) \cdot \nabla$ the fluid convective derivative and since by construction $\nabla u = -\nabla V$, it follows

$$\Omega_r \equiv -\Omega = \frac{Db(r, t)}{dt} \times b(r, t) - c(r, t)b(r, t) = \frac{Db(r, t)}{dt} \times b(r, t) + (u \cdot \nabla) b(r, t) \times b(r, t) - c(r, t)b(r, t), \tag{86}$$

where

$$(u \cdot \nabla) b(r, t) \times b(r, t) - c(r, t)b(r, t) = \xi - \frac{1}{|\nabla p_1|} [b \times \nabla (\nabla p_1 \cdot V) + b \times (\nabla p_1 \cdot \nabla V)]. \tag{87}$$

In the last equation the first term on the r.h.s. denotes the vorticity $\xi \equiv \nabla \times V(r, t)$. This means that near a vortex the motion of TTPs is qualitatively similar to that of a rotating rigid body. However, by inspection of the remaining terms in Eqs. (59) and (67), it is evident that more complex particle-acceleration effects may be present, which are driven by time-dependent pressure and velocity-gradients contributions.

VI. THE TTP-STATISTICAL DESCRIPTION

In this section we introduce a statistical description associated to the ensemble of TTPs, denoted as TTP-statistical model, which is represented by the couple $\{f_1, \Gamma_1\}$. We intend to show that, like the IKT-statistical model $\{f, \Gamma\}$, also $\{f_1, \Gamma_1\}$ determines uniquely the time evolution of the complete set of fluid fields $\{\z\}$. However, the result is conceptually important because $\Gamma_1$ is a reduced-dimension subset of $\Gamma$. For this purpose, we notice that if $u_{th}$ denotes the relative velocity of an arbitrary TTP endowed with a relative-Newtonian state $y \equiv (r, u_{th})$, then $u_{th}$ spans the subset of velocity space $U$:

$$U_1 = \{u | u \in U, u = u_{th} = \beta v_{th}(r, t)n(r, t), n(r, t) \cdot b(r, t) = 0, \beta \in \mathbb{R}^+\}. \tag{88}$$

Here by construction $\beta^2 = \frac{u_{th}^2}{\Omega_0^2(r, t)}$ is a constant independent of $(r, t)$. As a consequence, it follows that $\Gamma_1$ is the subset of $\Gamma$, $\Gamma_1 = \Omega \times U_1$, with $U_1 \subset U \equiv \mathbb{R}^3$ and dim($U_1$) = 2.

To define a KDF $f_1(r, u_{th}, t)$ on $\Gamma_1$ let us first consider the PDF defined on $U$ in terms of the KDF $f(r, u, t)$. For a prescribed IKT-statistical model $\{f, \Gamma\}$, the corresponding conditional velocity PDF on $U_1$ is defined as

$$\hat{f}_1(r, u_{th}, t) \equiv \frac{f(r, u_{th}, t)}{\int_{U_1} df(r, u_{th}, t)}. \tag{89}$$

Here, introducing for $u_{th}$ a representation in terms of the spherical coordinates $(u \equiv u_{th}, \varphi, \theta)$ and requiring the $\bar{u}_z \equiv b(r, t)$ it follows by construction that both the conditional PDF and KDF defined by Eqs. (59) and (67) are independent of the angle $\theta$. In particular it follows that by definition $\int_{U_1} df(r, u, t) = \int d\varphi d\theta d\pi/2 f(r, u, t)$ and hence $d\eta = u^2 du d\varphi$, while the corresponding phase-space measure is $d\chi \equiv d^3r d\eta$. Thus, the conditional KDF on $\Gamma_1$ is defined as

$$f_1(r, u_{th}, t) = \rho(r, t)\hat{f}_1(r, u_{th}, t). \tag{90}$$
In particular, if \( f(\mathbf{r}, \mathbf{u}_h, t) \) coincides with the Gaussian KDF \((17)\), it follows that

\[
f_1(\mathbf{r}, \mathbf{u}_h, t) = \frac{2\rho(\mathbf{r}, t)}{\pi^{3/2} \theta_{th}^3} \exp \{-\beta^2\} = f_{1M}(\mathbf{r}, \mathbf{u}_h, t). \tag{91}
\]

The main result can be summarized by the following theorem.

**THEOREM 3 - TTP-statistical model for CNSFE**

Let us require that the IKT-statistical model satisfies THMs. 1 and 2. Then it follows that the conditional KDF \( f_1(t) \equiv f_1(\mathbf{r}, \mathbf{u}_h, t) \) defined by Eqs. \((59)\) and \((90)\) has the following properties:

T31) It is a particular solution of IKE which is independent of the angle \( \vartheta \)

T32) It satisfies the functional constraints:

\[
\begin{align*}
\int_{U_1} \rho f_1(\mathbf{r}, \mathbf{u}_h, t) &= \rho(\mathbf{r}, t), \\
\frac{1}{\rho(\mathbf{r}, t)} \int_{U_1} \rho u f_1(\mathbf{r}, \mathbf{u}_h, t) &= \mathbf{V}(\mathbf{r}, t), \\
\int_{U_1} \rho f_1^{1/2} u^2 f_1(\mathbf{r}, \mathbf{u}_h, t) &= p_1(\mathbf{r}, t), \\
S(f_1(t)) &\equiv \int_{\Gamma_1} \mathbf{U}_1 \cdot \nabla f_1(\mathbf{r}, \mathbf{u}_h, t) \ln f_1(\mathbf{r}, \mathbf{u}_h, t) = S_T(t)
\end{align*}
\]

(correspondence principle).

T33) Its velocity moment equations, determined from IKE in terms of the weight-functions \( \{G(x,t)\} = \{1, \mathbf{v}, \frac{1}{3} \mathbf{u} \mathbf{u} \cdot \mathbf{u} \} \), imply again Eqs. \((91)\) and therefore, together with \((90)\) and the constraint equation \((48)\), they coincide again with CNSFE, so that in particular \( f_1(t) \) satisfies the entropy law.

**Proof - T31** In fact, due to the hypothesis and the definitions \((59)\) and \((90)\), it follows that in \( U_1 \)

\[
f_1(\mathbf{r}, \mathbf{u}_h, t) = 2f(\mathbf{r}, \mathbf{u}_h, t), \tag{93}
\]

and is therefore by construction independent of the angle \( \vartheta \). Hence, in the subset \( \Gamma_1 \times I \), \( f_1(\mathbf{r}, \mathbf{u}_h, t) \) is a particular solution of IKE, in the sense that it satisfies by construction the statistical equation

\[
\frac{\partial}{\partial t} f_1(\mathbf{r}, \mathbf{u}_h, t) + \mathbf{v} \cdot \nabla f_1(\mathbf{r}, \mathbf{u}_h, t) + \mathbf{F}(\mathbf{r}, \mathbf{u}_h, t) \cdot \frac{\partial}{\partial \mathbf{u}_h} f_1(\mathbf{r}, \mathbf{u}_h, t) + f_1(\mathbf{r}, \mathbf{u}_h, t) \left[ \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F}(\mathbf{r}, \mathbf{u}, t, f) f(\mathbf{r}, \mathbf{u}, t) \right]_{\mathbf{u}=\mathbf{u}_h} = 0. \tag{94}
\]

T32) The proof follows by noting that, thanks to Eq. \((93)\) and the fact that \( f(\mathbf{r}, \mathbf{u}_h, t) \) satisfies by construction the correspondence principle (see THM.1), the conditional KDF \( f_1(\mathbf{r}, \mathbf{u}_h, t) \) fulfills identically the correspondence principle \((92)\) too.

T33) Due to T31), T32) and Eq. \((94)\) the moment equations coincide necessarily with CNSFE and hence, in particular, consistent with the second principle of thermodynamics [see Eq. \((131)\)] \( f_1(t) \) satisfies the weak H-theorem

\[
\frac{\partial}{\partial t} S(f_1(t)) \geq 0. \tag{95}
\]

Q.E.D.

We remark that here:

- \( f_1(\mathbf{r}, \mathbf{u}_h, t) \) denotes generally a non-Gaussian conditional KDF. A particular solution is provided by the Gaussian conditional KDF defined by Eq. \((94)\).

- \( f_1(\mathbf{r}, \mathbf{u}_h, t) \) satisfies the inverse kinetic equation \((94)\).

- \( \{f_1, \Gamma_1\} \) is a reduced-dimension statistical model for CNSFE problem. The fluid fields \( \{Z_i\} \) are uniquely advanced in time by means of Eq. \((94)\), or equivalent, by means of the integral Lagrangian IKE \((15)\).

- \( f_1(\mathbf{r}, \mathbf{u}_h, t) \) determines uniquely the time-evolution of the fluid fields \( \{Z_i(\mathbf{r}, t)\} \).

- In view of the discussion presented above after THM.1, \( \{f_1, \Gamma_1\} \) applies also to incompressible fluids described either by the INSFE or INSE problems.

In the following we analyze basic implications of this result.
VII. TTP-DYNAMICS IN STOCHASTIC FLUIDS

A remarkable application of the TTP dynamics concerns the modelling of stochastic tracer-particle dynamics in stochastic fluids, such as for example due to temperature and pressure fluctuations (thermal fluctuations). Thermal fluctuations are important in a wide variety of mesoscopic flows (see for example Refs. [20, 31]). Theoretically, they are usually treated within the framework of fluctuating hydrodynamics, an approach pioneered by Landau and Lifshitz [32, 33]. In this framework extra (stochastic) terms are added to the fluid equations to model possible stochastic effects, so that generally the functional form of the corresponding fluid equations is actually modified with respect to the customary fluid equations. For example, in this case the latter may typically include higher-order spatial derivatives of the fluid fields. The numerical solution of fluctuating hydrodynamic equations may present, as a consequence, serious difficulties (which are nevertheless also present in the case of the incompressible NS equations for isothermal fluids). A possible alternative is represented by particle simulation methods based on kinetic theory. In such an approach, unlike fluctuating hydrodynamics:

1) Requirement #1: the functional form of the corresponding fluid equations and of the related initial-boundary value problem is left unchanged [10, 34], i.e., the differential operators appearing in the fluid stochastic equations are the same ones entering the customary fluid equations in the absence of stochasticity.

2) Requirement #2: the stochastic fluid fields are assumed to be strong solutions of the stochastic CNSFE problem (see Appendix A).

An approach of this type can be achieved by means of the TTP-statistical model. A convenient representation of stochastic fluid fields of this type, and fulfilling Requirements #1 and #2, is provided by Eq. (39). In this case the fluid fields are assumed to depend on a suitable set of stochastic variables \( \alpha = \{\alpha_i, i = 1, k\} \in V_\alpha \subseteq \mathbb{R}^k \), with \( k \geq 1 \), by assumption all independent of \((r, t)\) and endowed with a stochastic probability density \( g(r, t, \alpha) \) on \( V_\alpha \). It must be remarked that “a priori” the parameters \( \alpha \equiv (\alpha_1, ..., \alpha_n) \) and the related probability density \( g(r, t, \alpha) \) can be set arbitrarily. Thus, they can in principle be chosen to provide prescribed mathematical models of stochasticity. In the following we shall assume in particular that the \( \alpha \)'s are also independent of \((r, t)\). As a consequence, introducing the stochastic averaging operator \((172)\), the fluid fields can be represented in terms of the stochastic decomposition

\[
Z(r, t, \alpha) = \langle Z(r, t) \rangle_\alpha + \delta Z(r, t, \alpha),
\]

\( \langle Z(r, t) \rangle_\alpha \) and \( \delta Z(r, t, \alpha) \) denoting respectively the corresponding stochastic-averages and stochastic fluctuations of the fluid fields.

In particular, in contrast to fluctuating hydrodynamics, in the present approach the functional form of the fluid equations is left unchanged. It follows that the precise form of the stochastic-averaged fluid fields \( \langle Z(r, t) \rangle_\alpha \) and of the stochastic fluctuations \( \delta Z(r, t, \alpha) \) depends solely on the model of stochasticity adopted, i.e., the choice of the set \( \{g(r, t, \alpha), V_\alpha\} \). This means that its realization may generally depend on the possible sources of stochasticity adopted, namely:

1) Stochastic initial conditions: in this case the initial fluid fields \( Z(r, t_\alpha) \equiv Z_0(r) \) are assumed stochastic, i.e., of the form, \( Z_0(r, \alpha) = \langle Z_0(r, \alpha) \rangle_\alpha + \delta Z_0(r, \alpha) \), with \( \langle Z_0(r, \alpha) \rangle_\alpha \) and \( \delta Z_0(r, \alpha) \) being suitable vector fields. 2) Stochastic boundary conditions: this occurs if the boundary fluid fields \( Z_{\partial\Omega}(r, t) \big|_{\partial\Omega} \) are prescribed in terms of a suitable stochastic vector field of the form \( Z_{\partial\Omega}(r, t, \alpha) \big|_{\partial\Omega} = \langle Z_{\partial\Omega}(r, t, \alpha) \rangle_\alpha \big|_{\partial\Omega} + \delta Z_{\partial\Omega}(r, t, \alpha) \big|_{\partial\Omega} \). 3) Stochastic forcing: in this case the volume force density acting on the fluid is assumed stochastic, i.e., of the form \( f(r, t, \alpha) = \langle f(r, t, \alpha) \rangle_\alpha + \delta f(r, t, \alpha) \), being \( \langle f(r, t, \alpha) \rangle_\alpha \) and \( \delta f(r, t, \alpha) \) suitable vector fields.

A. Langevin dynamics in fluctuating fluids

As stated above, a fundamental consequence of THM.2 is the uniqueness of the deterministic equations of motion for arbitrary TTPs belonging to a compressible or incompressible, thermal or isothermal NS fluid. It is immediate to show that the dynamics of TTPs is unique also when the same fluids are considered stochastic. It follows that the relative state \( y \) of a generic TTP advances in time by means of a stochastic dynamical system (DS), namely the flow generated by the initial value problem associated to the stochastic equations of motion for TTPs (Langevin equations):

\[
\begin{align*}
\frac{dr}{dt} &= \beta \dot{v}_{th}(r, t, \alpha)n(r, t, \alpha) + V(r, t, \alpha), \\
\frac{dn}{dt} &= F_z(r, \beta \dot{v}_{th}n(r, t, t, \alpha), \\
y(t_\alpha, \alpha) &= y_\alpha(\alpha),
\end{align*}
\]

where the unit vector \( n(r, t, \alpha) \) satisfies the stochastic initial-value problem

\[
\begin{align*}
\frac{dn(r,t, \alpha)}{dt} &= \Omega(r, t, \alpha) \times n(r, t, \alpha), \\
n(r_\alpha, t_\alpha, \alpha) &= n_\alpha(r, \alpha).
\end{align*}
\]
In particular, it follows that the stochastic mean-field $F$ is provided by Eq. (38), with $F_u \equiv F_u(r, \beta v_{th} n(r), t, \alpha)$ being identified with

$$F_u(r, \beta v_{th} n(r), t, \alpha) = \beta v_{th}(r, t, \alpha) \Omega(r, t, \alpha) \times n(r, t, \alpha) + \beta n(r, t, \alpha) \frac{Dv_{th}(r, t, \alpha)}{Dt},$$  \hspace{1cm} (99)$$

where $n(r, t, \alpha)$ is given by Eq. (28) (see also Lemma to THM.2). The flow associated to the Eqs. (97) and (98)

$$T_{t_0, t} : y_0 \rightarrow y(t, \alpha) = T_{t_0, t}y_0,$$  \hspace{1cm} (100)$$
is referred to as stochastic TTP dynamical system (TTP-DS). Here $T_{t_0, t}$ is a measure-preserving evolution operator associated to the Newtonian vector field $X(x, t, \alpha) \equiv \{v, F(x, t, \alpha)\}$ and $F(x, t, \alpha)$ the stochastic mean-field satisfying the initial-value problem (97). Thus, by definition, the TTP-DS is uniquely prescribed by the instantaneous state of a generic TTP $x(t, \alpha) \equiv x = (r, v)$. Therefore, we conclude that:

- The initial-value problem (97) can be viewed as a stochastic model of particle motion in compressible/incompressible NS fluids, describing the dynamics of TTPs in stochastic thermofluids.
- Based on the TTP-statistical model developed in the previous section (see THM.3) the time-evolution of the stochastic KDF $f_1(r, u_{th}, t, \alpha)$ is uniquely prescribed. In particular, independent of the choice of the stochastic model $\{g(r, t, \alpha), V_{\alpha}\}$, the dynamics prescribed by (100) preserves the exact form of the stochastic fluid equations and is unique.
- The stochastic KDF $f_1(r, u_{th}, t, \alpha)$ determined by Eq. (15) is necessarily a particular solution of Eq. (54).
- The stochastic KDF $f_1(r, u_{th}, t, \alpha)$ prescribes uniquely the time-evolution of the stochastic fluid fields $\{Z(r, t, \alpha)\}$.

**B. Fokker-Planck description in strong turbulence**

Let now analyze the time evolution of the stochastic KDF $f_1(r, u_{th}, t, \alpha)$ represented in terms of the stochastic decomposition

$$f_1(r, u_{th}, t, \alpha) = \langle f_1(r, u_{th}, t, \alpha) \rangle_\alpha + \delta f_1(r, u_{th}, t, \alpha),$$  \hspace{1cm} (101)$$

with $\langle f_1(r, u_{th}, t, \alpha) \rangle_\alpha$ denoting the stochastic-average defined by Eq. (172). Then, requiring that the stochastic PDF $g(r, t, \alpha)$ is homogeneous and stationary [see Appendix B], i.e., that $g = g(\alpha)$, it is immediate to obtain from IKE [see Eq. (14)] the stochastic statistical equations advancing in time $\langle f_1(r, u_{th}, t, \alpha) \rangle_\alpha$ and $\delta f_1(r, u_{th}, t, \alpha)$. These are explicitly

$$\langle L \rangle_\alpha \langle f_1 \rangle_\alpha = - \langle L \delta f_1 \rangle_\alpha \equiv \langle C \rangle_\alpha,$$  \hspace{1cm} (102)$$

$$\langle L \rangle_\alpha \delta f_1 = -L \{\langle f_1 \rangle_\alpha + \delta f_1\} + \langle \delta L \delta f_1 \rangle_\alpha,$$  \hspace{1cm} (103)$$

where the streaming operator $L$ has been similarly represented as $L = \langle L \rangle_\alpha + \delta L$. Eqs. (102) and (103) are formally similar to the Vlasov equation arising in the kinetic theory of quasi-linear and strong turbulence for Vlasov-Poisson plasmas [34, 37]. As is well-known, the construction of the precise form of the operator $\langle C \rangle$ appearing in the stochastic-averaged kinetic equation [i.e., Eq. (102)] represents a task of formidable difficulty. The reason is that it requires constructing a formal perturbative solution of the equation (103) for the stochastic perturbation $\delta f$. To obtain a convergent perturbative theory, however, this usually requires the adoption of a suitable renormalization scheme in order to obtain a consistent statistical (kinetic) equation for $\langle f_1 \rangle_\alpha$ [see earlier approaches developed in Refs. [38, 40] which pertain to the statistical treatment of particle dynamics and the specific application to Vlasov-Poisson plasmas]. Nonetheless, in the case of weak-turbulence, the stochastic-averaged kinetic equation [i.e., Eq. (102)] is known to be amenable to an approximate Fokker-Planck kinetic equation advancing in time $\langle f_1 \rangle_\alpha$ alone. Analogous suggestions are provided by phenomenologically-based Markovian Fokker-Planck models of small-scale fluid turbulence [see for example Refs. [41, 43]].

This raises the issue of (the construction of) a possible representation of this type for the stochastic-averaged operator $\langle C \rangle_\alpha$ which has the following properties:

- Property #1: it holds at least locally in the velocity space $U_1$, in a suitable space to be specified.
• Property #2: it holds in the case of "strong turbulence", namely when there are fluctuating quantities such that their stochastic fluctuation is comparable in order of magnitude with their corresponding stochastic averages. In particular, denoting by $\zeta$ a dimensionless infinitesimal parameter, in the following the strong turbulence regime is defined in such a way that

\[ |\delta \mathbf{V}| \sim |\langle \mathbf{V} \rangle_\alpha| [1 + O(\zeta)], \]

\[ |\delta p_1| \sim |\langle p_1 \rangle_\alpha| [1 + O(\zeta)]. \]

The two requirements are mutually consistent and, for arbitrary choices of the velocity stochastic fluctuations $\delta \mathbf{V}$, are required by the Navier-Stokes equation.

• Property #3: it is applicable also in the case in which $\langle f_1 \rangle_\alpha$ is generally non-Gaussian.

• Property #4: it does not rely on renormalization theory.

• Property #5: the KDF is assumed of the form $f_1 = f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, \alpha)$, namely it exhibits slow dependence with respect to the velocity $\mathbf{u}_{th}$.

Notice that the previous properties are assumed to hold for arbitrary smooth non-Gaussian KDFs. In the specific case of a Gaussian KDF Property #5 requires necessarily that $\frac{\delta \mathbf{V}}{v_{th}} \sim O\left(\zeta^{1/2}\right)$. For TTPs this implies $\frac{\delta \mathbf{u}_{th}}{v_{th}} = \beta \mathbf{n} \sim O\left(\zeta^{1/2}\right)$, namely $\beta \sim O\left(\zeta^{1/2}\right)$ and also $\frac{\delta \mathbf{u}_{th}}{v_{th}} \sim O\left(\zeta\right)$. Therefore, the required slow velocity dependence effectively limits the validity of strong turbulence theory to the subset of velocity space in which such an ordering holds.

Regarding, in particular, the form of the KDF $f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, \alpha)$, it must be noted that, even if the latter coincides locally with a Gaussian KDF $f_M(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, \alpha)$, its stochastic average $\langle f_1 \rangle_\alpha$ still remains generally non-Gaussian. Here we intend to prove that an explicit representation of $\langle C \rangle_\alpha$ fulfilling the previous properties can be determined, based on the IKT approach earlier pointed out in Ref. [10]. In this case, the previous problem is exactly solvable provided:

1. $f_1$ depends on the stochastic variables $\alpha$ only through the fluid fields, i.e., $\rho(\mathbf{r}, t, \alpha)$, $p_1(\mathbf{r}, t, \alpha)$ and $\mathbf{V}(\mathbf{r}, t, \alpha)$, and hence also $u_{th} \equiv \beta v_{th}(\mathbf{r}, t, \alpha)$ and the unit vector $\mathbf{n} = \mathbf{n}(\mathbf{r}, t, \alpha)$, namely is of the form

\[ f_1 \equiv f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, \rho, p_1). \]

2. $f_1$ is generally a non-Gaussian KDF which is analytic in $\mathbf{u}_{th}$, $\rho$ and $p_1$.

3. The kinetic pressure $p_1$ and the pseudo-pressure $p_0 \equiv \langle p_0 \rangle_\alpha$ satisfy the ordering

\[ \frac{p_1 - p_0}{p_0} \sim O(\zeta). \]

4. The mass-density perturbations are weak, in the sense

\[ \frac{\delta \rho}{\langle \rho \rangle_\alpha} \sim O(\zeta). \]

5. The stochastic fluctuation of the KDF $\delta f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, \alpha)$ is such that

\[ \frac{\delta f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, \alpha)}{\langle f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, \alpha) \rangle_\alpha} \sim O(\zeta). \]

Invoking for $\delta \mathbf{u}_{th}$ the representation $\delta \mathbf{u}_{th} \equiv \delta \rho \frac{\partial \mathbf{u}_{th}}{\partial (\rho)_{\alpha}} + \delta p_1 \frac{\partial \mathbf{u}_{th}}{\partial (p_1)_{\alpha}} - \delta \mathbf{V}$, as a consequence of the previous requirements, $f_1$ can be Taylor-expanded with respect to $\delta \rho$, $\delta p_1$ and $\delta \mathbf{u}_{th}$ yielding

\[ f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, \rho, p_1) = f_1(\mathbf{r}, \langle \zeta^{1/2} \mathbf{u}_{th} \rangle_\alpha, t, \langle \rho \rangle_\alpha, \langle p_1 \rangle_\alpha) + \sum_{n=1}^{\infty} \left[ \delta \rho \frac{\partial}{\partial \langle \rho \rangle_\alpha} + \delta p_1 \frac{\partial}{\partial \langle p_1 \rangle_\alpha} + \delta \mathbf{u}_{th} \cdot \frac{\partial}{\partial \mathbf{u}} \right]^n f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}, t, \langle \rho \rangle_\alpha, \langle p_1 \rangle_\alpha) \bigg|_{\mathbf{u} = \langle \mathbf{u}_{th} \rangle_\alpha}. \]
Due to the previous ordering assumptions and the requirement of analyticity, the series converges uniformly in $\Gamma_1$ and permits the explicit determination of both $\langle f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, p_1) \rangle_{\alpha}$ and $\delta f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}_{th}, t, p_1)$. As a result, after straightforward algebra the operator $(C)_{\alpha}$ becomes explicitly

$$
(C)_{\alpha} = C_{FP} \equiv \left[ \sum_{i,j,k=1}^{\infty} \frac{\partial}{\partial \mathbf{u}} \left( \mathbf{C}_{i,j,k} \otimes \frac{\partial^n}{\partial \mathbf{p}^n} \delta f_1(\mathbf{r}, \zeta^{1/2} \mathbf{u}, t, \langle p_1 \rangle_{\alpha}) \right) \right]_{\mathbf{u}=(\mathbf{u}_{th})_{\alpha}},
$$

where $\mathbf{C}_{i,j,k}$ are the tensor Fokker–Planck (or Kramers-Moyal) coefficients

$$
\mathbf{C}_{i,j,k} = \frac{1}{n!} \left\langle \delta \mathbf{F} (\delta \mathbf{p})^i (\delta \mathbf{p})^j (\delta \mathbf{u}_{th})^k \right\rangle_{\alpha},
$$

$\otimes$ denotes the tensor product and the summation is carried out on $i, j, k$ from 0 to $\infty$, with $n = i + j + k \geq 2$. Therefore $(C)_{\alpha}$ takes the form of a generalized Fokker-Planck (F-P) operator $(C_{FP})$. Remarkably, Eq.\ref{eq:111} satisfies by construction Properties #2-#5.

As a final point, we comment on the implications of the GDE-requirement #6 which concern TTP dynamics and hold in validity of the strong turbulence formulation developed here. In fact, the kinetic constraint \ref{eq:29} implies that time evolution of the TTP relative velocity depends on the stochastic fluctuations of both kinetic pressure $\delta p_1$ and fluid velocity $\delta \mathbf{V}$. Qualitatively this means that, even in the case in which pressure fluctuations are negligible, TTP dynamics can still exhibit turbulent motion through the fluctuations $\delta \mathbf{V}$ entering in the fluid vorticity.

### C. Statistical irreversibility

We first notice that due to the integral IKE \ref{eq:15}, if $f_1(t_o)$ is strictly positive then, for all $t \geq t_o$ in the existence time interval $I$, also $f_1(t)$ is necessarily so. This manifestly implies, in turn, that in such a case the stochastic-averaged KDF $\langle f_1(\mathbf{x}, t) \rangle_{\alpha} = \langle f_1(\mathbf{r}, \mathbf{u}_{th}, t, \alpha) \rangle_{\alpha}$, solution of the stochastic-averaged statistical equation \ref{eq:102}, is necessarily strictly positive. It is immediate to show that, thanks to THMs.1 and 3, $\langle f_1(t) \rangle_{\alpha}$ must satisfy in $\Gamma_1 \times I$ also a weak H-theorem of the form:

$$
\frac{\partial S(\langle f_1(t) \rangle_{\alpha})}{\partial t} \geq 0.
$$

For definiteness, let us assume that at the initial time $t_o$ both $f_1(\mathbf{x}, t) \equiv f_1(\mathbf{r}, \mathbf{u}_{th}, t, \alpha)$ and $\langle f_1(\mathbf{x}, t) \rangle_{\alpha}$ are strictly positive and admit the BS entropies $S(f_1(\mathbf{x}, t_o))$ and $S(\langle f_1(\mathbf{x}, t_o) \rangle_{\alpha})$. Then, we notice that if $g(\mathbf{x}, t)$ is an arbitrary strictly positive function such that $\int_{\Gamma_1} d\mathbf{x} g(\mathbf{x}, t) = \mu(\Omega)$, the majorization

$$
\int_{\Gamma_1} d\mathbf{x}_1 f_1(\mathbf{x}, t) \ln f_1(\mathbf{x}, t) \geq \int_{\Gamma_1} d\mathbf{x}_1 f_1(\mathbf{x}, t) \ln g(\mathbf{x}, t)
$$

necessarily holds (Brillouin Lemma \ref{eq:28}). On the other hand, assuming that the stochastic-averaging operator $\langle \cdot \rangle_{\alpha}$ commutes with the phase-space integral operator $\int_{\Gamma_1} d\mathbf{x}_1$,

$$
\int_{\Gamma_1} d\mathbf{x}_1 \langle f_1(\mathbf{x}, t) \rangle_{\alpha} = \left\langle \int_{\Gamma_1} d\mathbf{x}_1 f_1(\mathbf{x}, t) \right\rangle_{\alpha},
$$

it follows

$$
\int_{\Gamma_1} d\mathbf{x}_1 f_1(\mathbf{x}, t) \ln f_1(\mathbf{x}, t) \geq \int_{\Gamma_1} d\mathbf{x}_1 f_1(\mathbf{x}, t) \ln \langle f_1(\mathbf{x}, t) \rangle_{\alpha}.
$$

This yields in turn

$$
\left\langle \int_{\Gamma_1} d\mathbf{x}_1 f_1(\mathbf{x}, t) \ln f_1(\mathbf{x}, t) \right\rangle_{\alpha} \geq \int_{\Gamma_1} d\mathbf{x}_1 \langle f_1(\mathbf{x}, t) \rangle_{\alpha} \ln \langle f_1(\mathbf{x}, t) \rangle_{\alpha}.
$$

The last inequality implies manifestly that

$$
S(\langle f_1(\mathbf{x}, t) \rangle_{\alpha}) \geq \langle S(f_1(\mathbf{x}, t)) \rangle_{\alpha}.
$$

Therefore, in view of the entropy inequality \ref{eq:95}, $\langle f_1(t) \rangle_{\alpha}$ satisfies necessarily the weak H-theorem \ref{eq:113}. This assures that for all $t \geq t_o$ in the time interval $I$, $\langle f_1(t) \rangle_{\alpha}$ admits the BS entropy integral $S(\langle f_1(t) \rangle)$, i.e., that $S(\langle f_1(t) \rangle)$ is defined for all $t \geq t_o$ in $I$. As a consequence $\langle f_1(t) \rangle$ exhibits an irreversible behavior.
D. Comparisons with the HRE statistical model and solution of the Closure Problem

In this section we display the relationship between the statistical models obtained here - i.e., both the IKT and TTP statistical models \( \{f, \Gamma \} \) and \( \{f_1, \Gamma_1 \} \), prescribed respectively by means of THMs 1 and 3 - and the customary statistical treatment of turbulence due to Hopf, Rosen and Edwards (HRE approach, \cite{24, 20}; see also \cite{14, 18}; for a review see \cite{49, 50}). Since the latter is usually developed in the case of incompressible fluids, we shall restrict the following analysis to such a case.

The HRE approach, which in its original form applies only to incompressible and isothermal NS fluids, is based on the introduction of a suitable statistical model \( \{f_H, \Gamma \} \), here referred to as the HRE statistical model, with \( \Gamma \equiv \Omega \times U \). In this case, denoting by \( u \equiv u(t) \equiv v - V(r, t) \), \( f_H \) is identified with the velocity PDF

\[
 f_H(r, u, t) \equiv \rho_0 \delta(u) \tag{119}
\]

(HRE velocity KDF), \( \rho_0 \) denoting the constant mass density characterizing an incompressible fluid. Upon identifying the mean-field \( F(x, t) \) with the fluid acceleration \( F_H(r, t) \) [see definition given by Eq.(112) in Appendix A], it follows that by construction \( f_H \) is a particular solution of Eq.(14). As a consequence, its velocity moment equations, evaluated with respect to the weight functions \( \{G \} = \{1, v\} \), coincide respectively with the continuity and the NS equations \( \{128\}, \{129\} \) (see Appendix A). It is important to remark, instead, that by construction the fluid pressure \( \hat{p} \) is a particular solution of Eq.(14) obtained by prescribing the mean-field \( u \equiv u(t) \). As a result, for an incompressible fluid the HRE velocity KDF, corresponding to Eq.(122) becomes

\[
 f_H(r, u - \Delta V, t) \equiv \rho_0 \delta(u - \Delta V), \tag{122}
\]

with \( \Delta V \equiv (\Delta V_1, \Delta V_2, \Delta V_3) \in U \equiv \mathbb{R}^3 \) denote an arbitrary particular solution of the related initial-boundary value problems [i.e., Eqs.(137)-(139) with the constant-entropy equation (146)] and an arbitrary stochastic vector independent of \( (r, t) \) respectively. In particular, the NS equation for \( V_1(r, t) \) is manifestly

\[
 \frac{\partial}{\partial t} V_1(r, t) + V_1(r, t) \cdot \nabla V_1(r, t) \equiv \frac{\partial}{\partial t} V(r, t) + (V(r, t) + \Delta V) \cdot \nabla V(r, t) = F_H + \Delta F_H, \tag{122}
\]

with \( \Delta F_H \) being defined as the stochastic vector field \( \Delta F_H = \Delta V \cdot \nabla V(r, t) \). As a result, for an incompressible fluid the HRE velocity KDF, corresponding to Eq.(122) becomes \( f_H(r, u - \Delta V, t) \equiv \rho_0 \delta(u - \Delta V) \), which proves the statement. In view of these considerations let us now introduce the stochastic model defined by the set \( \{\Delta V \in U, g(r, t, \Delta V)\} \), with \( g(r, t, \Delta V) \) being a suitable stochastic PDF [see Appendix B]. Due to its arbitrariness, it can always be identified with

\[
 g(r, \Delta V, t) \equiv \hat{f}(r, \Delta V, t), \tag{123}
\]

with \( \hat{f} \) being defined by Eq.(13) in terms of the IKT-statistical model \( \{f, \Gamma\} \). This means that the corresponding KDF \( f \) is a particular solution of Eq.(114) obtained by prescribing the mean-field \( F \) in accordance with THM.1. As a result the following identity holds:

\[
 \langle f_H(r, u - \Delta V, t) \rangle_{\Delta V} = f(r, u, t), \tag{124}
\]

where \( \langle \cdot \rangle_{\Delta V} \) is the stochastic average defined in Appendix B [see Eq.(172)] with the stochastic vector \( \alpha \) being identified with \( \Delta V \). In particular this implies, thanks to the correspondence principle \( \delta \), that the variance of the stochastic velocity \( \Delta V \) is prescribed as

\[
 \left\langle \frac{1}{3} \Delta V^2 \right\rangle_{\Delta V} = \hat{\rho}_1(r, t). \tag{125}
\]
As a further implication, when Eq. (124) is evaluated in the subspace of TTPs $U_1$, in view of Eq. (123) and THM.3, it requires
\[ \langle f_H(r, u_{th} - \Delta V, t) \rangle_{\Delta V} = f(r, u_{th}, t) = \frac{1}{2} f_1(r, u_{th}, t). \] (126)

For all $(r, u_{th}, t) \in \Gamma_1 \times I$, this yields also the relationship between the HRE velocity KDF and the conditional KDF $f_1(r, u_{th}, t)$ characterizing the statistical model of TTPs $\{f_1, \Gamma_1\}$.

A preliminary summary is in order. In the case of a NS fluid obeying the INSE problem, the following conclusions are reached:

- In view of the constraint (125), $g(r, \Delta V, t)$ can be interpreted as the stochastic PDF taking into account the stochastic "thermal" motion of ITPs produced in a compressible thermal fluid by the kinetic pressure $p_1(r, t)$.

- Eqs. (124) and (126) permit a comparison between the HRE, IKT and TTP statistical models, $\{f_H, \Gamma\}$, $\{f, \Gamma\}$ and $\{f_1, \Gamma_1\}$ respectively. We remark that such a comparison is always possible (and hence it holds also in the case of INSE). In particular, Eq. (123) determines the relationship between the KDFs $f_H$ and $f$ prescribed by the statistical model $\{f, \Gamma\}$. In addition, Eq. (126) yields the analogous relationship with the KDF $f_1$ characterizing the TTPs statistics.

- Remarkably, both conclusions follow by invoking a single suitable stochastic model. In both cases, in fact, the stochastic-averaging operator [see Eq. (122)] which enters the l.h.s. of Eqs. (124) and (126) is defined with respect to the same stochastic probability density $g(r, \Delta V, t)$ prescribed according to Eq. (123).

- Eqs. (123) and (124) - or equivalent (126) - do not imply any restriction on the flow dynamics, i.e., on the (strong) solutions of the INSE problem.

- Denoting $\langle \cdot \rangle_{\Omega}$ the configuration-space average $\langle \cdot \rangle_{\Omega} \equiv \frac{1}{m(\Omega)} \int_{\Omega} d\mathbf{r}$, Eq. (126) uniquely prescribes also the relationship between the corresponding spatial averages, i.e.,
\[ \langle\langle f_H(r, u_{th} - \Delta V, t) \rangle_{\Delta V} \rangle_{\Omega} = \langle f(r, u_{th}, t) \rangle_{\Omega} = \frac{1}{2} \langle f_1(r, u_{th}, t) \rangle_{\Omega}. \] (127)

Hence, assuming that the initial frequency $\langle\langle f_H(r, u_{th} - \Delta V, t_0) \rangle_{\Delta V} \rangle_{\Omega}$ is prescribed, for the initial velocity PDF $f_1(r, u_{th}, t_0)$ the average $\langle f_1(r, u_{th}, t) \rangle_{\Omega}$ remains uniquely determined too.

Let us now address in detail the issues of the comparison between the HRE, IKT and TTP statistical approaches according to Eqs. (124) and (126) and the related closure condition problem arising in the HRE approach.

For this purpose it is worth recalling that the aim of the HRE approach is actually to predict the time evolution, in the presence of turbulence, of suitable ensemble-averages of the KDF $f_H$ and of the NS fluid fields $\{V, p\}$, i.e., respectively $(f_H(r, u, t))$ and $(V(r, t))$, $\langle p(r, t) \rangle$. Here the brackets $(\cdot)$ denote a suitable ensemble-averaging operator (see for example [50]). In the case of so-called homogeneous, isotropic and stationary turbulence (HIST), this is required to commute with the differential and integral operators $\{Q\} \equiv \left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial r \cdot \partial r}, \frac{\partial}{\partial r \cdot \partial \nu}, \int_{\Omega} dr \int_{\mathbb{V}} d\mathbf{v} \right\}$. As pointed out above, in the context of the statistical description of turbulence, the operator $(\cdot)$ may be equivalently intended as a mean value in the probabilistic sense [57], namely $(\cdot) \equiv \langle \cdot \rangle_{\alpha}$. Here $\langle \cdot \rangle_{\alpha}$ denotes again the stochastic average of the form (172) [see Appendix B], prescribed in terms of a suitable stochastic probability density defined on the space $V_\alpha$ of the stochastic parameters $\alpha$. Thus, in this context the fluid fields are considered as stochastic functions dependent on $\alpha$. In the case of INSE this requires letting $(Z_{1}(r, t, \alpha)) \equiv (V(r, t, \alpha), p_1(r, t, \alpha), S_T(\alpha))$. In particular, in validity of HIST this implies that necessarily the stochastic PDF $g(r, t, \alpha)$ must be taken of the form $g \equiv g(\alpha)$. The corresponding statistical evolution equation for $\langle f_H \rangle$ is well-known and has been investigated by several authors (see for example [38]). In the case of the unforced NS equation, its explicit solution involves the construction of an infinite set of continuous many-point PDFs, coupled via the fluid pressure, which obey a hierarchy of statistical equations, the so-called ML (Monin-Lundgren [51, 52]) hierarchy.

The search of possible “closure conditions” for the ML hierarchy (Closure Problem for the statistical description of HT) remains - to date - one of the outstanding unsolved theoretical problems in fluid dynamics. Its solution involves in principle the search of possible alternative statistical models with the following features:

Requirement #1: it should be characterized by a finite number of (multi-point) PDFs.

Requirement #2: it should be determined in such a way that the complete set of fluid fields can be uniquely represented in terms of the same PDFs.
Requirement #3: the time evolution of the said PDFs is solely determined by a finite number velocity moments of the same PDFs (closure conditions).

It is immediate to show that a possible candidate satisfying all of these features is provided by the TTP statistical model \( \{ f_1, \Gamma_1 \} \). In fact, on the basis of the theory developed above (see in particular THMs. 1 and 3), we conclude that, in the case of INSE:

- The statistical set \( \{ f_1, \Gamma_1 \} \) is realized in terms of the stochastic 1-point conditional PDF \( \hat{f}_1 (r, u_{th}, t, \alpha) \).
- The fluid fields \( \{ Z_i (r, t, \alpha) \} \) are all uniquely prescribed in terms of the same stochastic PDF \( \hat{f}_1 (r, u_{th}, t, \alpha) \) (correspondence principle).
- The time-evolution of the stochastic PDF \( \hat{f}_1 (r, u_{th}, t, \alpha) \) is uniquely prescribed by means of Liouville statistical equation [see the inverse kinetic equation (94)].
- By assumption, such a kinetic equation depends functionally [see Axiom #3 - Kinetic closure conditions] only on a finite number of velocity moments of the same PDF.

Let us analyze how, in practice, the previous conclusions can be implemented in order to avoid the closure problem. In the customary HRE approach one is faced with the formidable issue of prescribing the multi-point PDFs which enter the Monin-Lundgren hierarchy. It is immediate to recognize that this problem arises specifically because of the treatment adopted for the fluid pressure in such approaches. In fact, because the pressure is not represented by a PDF velocity moment, but rather its contribution enters by means of a Green-function convolution integral, it follows that non-local (i.e., multi-point) contributions are necessarily introduced. The precise prescription of such contributions, however, remains undetermined, giving rise to the closure problem. Although several attempts have been suggested (see for example Ref. [53]), no definite solution exists to date.

In contrast, based either on the IKT or TTP statistical descriptions, the problem can be given a consistent solution. This is reached as follows:

1) By replacing the ensemble-averaged HRE-KDF \( \langle f_H (r, v, t) \rangle \equiv \langle f_H (r, v, t, \alpha) \rangle \), either with the corresponding IKT or TTP KDFs, namely \( \langle f (r, u, t) \rangle \equiv \langle f (r, u, t, \alpha) \rangle \) or \( \langle f (r, u_{th}, t) \rangle \equiv \langle f (r, u_{th}, t, \alpha) \rangle \).

2) In both cases the kinetic pressure is determined as a velocity moment of the corresponding KDF. This permits to overcome the closure problem arising in the HRE approach. In fact, non-local contributions due to multi-point PDFs do not appear anymore.

3) By requiring that the KDF \( f (r, u, t, \alpha) \), and respectively \( f_1 (r, u_{th}, t, \alpha) \), satisfy the Liouville equations (14) and (94). These equations are determined imposing the requirement that they depend functionally only on a finite set of velocity moments of the same KDFs. Their numerical solution involves, at most, the determination of the local gradients of the corresponding fluid fields. For comparison, the HRE-KDF obeys, instead, a Fokker-Planck statistical equation which contains non-local contributions due to the fluid pressure which depend by higher-order multi-point PDFs (see for example Ref. [54]).

4) Unlike the HRE approach, both IKT and TTP statistical approaches allow the stochastic fluid pressure to be uniquely determined as a velocity moment of the relevant KDFs. Its stochastic average and fluctuating part follow simply by applying the stochastic average operator on the resulting expression.

5) The choice of the stochastic variables \( \alpha \) remains in principle arbitrary. Thus, they can be identified in accordance with the specific stochastic model adopted (e.g., stochastic initial conditions, stochastic boundary conditions or stochastic volume force).

6) Let us consider the comparison between the customary HRE statistical evolution equation (see again for example Ref. [54]) and the corresponding stochastic-averaged Liouville equations following from Eqs. (14) and (94) upon applying the averaging operator \( \langle \cdot \rangle \). As pointed out in Section 6 and in Ref. [54], the latter equation can be approximated, locally in velocity space, in terms of a Fokker-Planck equation. The corresponding Kramers-Moyal coefficients however are different from those entering in the HRE equation. The remarkable features of our equation is, first, that unlike the HRE one, it recovers exactly the correct velocity moment equations obtained for the weight functions \( G = (1, v) \) (see Ref. [54] on this issue) which also follow from the corresponding stochastic-averaged Liouville equation (Eqs. 14) and (94). Second, the Kramers-Moyal coefficients depend explicitly also on the stochastic pressure fluctuations \( \delta p \), while the strict positivity of the KDF is warranted by the weak H-theorem (113) following from the exact stochastic-averaged Liouville equations (14) or (94).

7) Another remarkable difference with respect to the HRE statistical equation is that the Liouville equations as well as the corresponding Fokker-Planck approximations presented here hold both for Gaussian and suitably-smooth non-Gaussian KDFs. Therefore, both IKT and TTP statistical models appear suitable to describe the non-Gaussian behavior arising in the statistical description of turbulence.
VIII. CONCLUSIONS

A fundamental issue for Navier-Stokes fluids, is their characterization in terms of the dynamics of ideal tracer particles (ITPs), and in particular of the sub-set of thermal tracer particles (TTPs). Based on the formulation of an inverse kinetic theory for compressible/incompressible NS thermofluids, in this paper properties of TTP dynamics and a mathematical model for their description have been investigated. It is found that TTP dynamics can be uniquely determined both for incompressible (isothermal or non-isothermal) and compressible NS fluids described respectively by the INSE, INSFE and CNSFE problems. In addition it has been proved that the discovery of TTP dynamics allows for the construction of a reduced-dimension statistical model, to be identified with the TTP-statistical model. The latter is defined by the set \( \{ f_1, \Gamma_1 \} \), in terms of which the self-consistent time evolution of the fluid fields is determined.

Basic consequences of the theory mentioned in this paper concern the treatment of stochastic fluid fields arising in compressible/incompressible NS fluids by means of the TTP-statistical model. Here we have pointed out in particular:

1. The formulation of TTP dynamics for the stochastic CNSFE problem, represented in two possible forms. The first one is provided by Langevin equations, which describe the dynamics of TTPs in fluctuating compressible/compressible NS fluids. These provide a mathematical model of tracer-particle motion in stochastic fluids. The second one is given by the corresponding Fokker-Planck description.

2. The comparison with the statistical treatment of turbulence due to Hopf, Rosen and Edwards [24–26], which holds in the case of incompressible isothermal NS fluids, has been carried out. As a result, the relationship between the HRE, IKT and TTP velocity PDFs has been pointed out.

3. Finally, based on the statistical model \( \{ f_1, \Gamma_1 \} \), a solution of the closure problem for the statistical description of HT has been proposed.

Applications of the present theory are in principle several. They concern, in general, the dynamics of small particles (such as solid particles or droplets, commonly found in natural phenomena and industrial applications) in compressible/incompressible thermofluids. The accurate description of particle dynamics, as they are pushed along erratic trajectories by fluctuations of the fluid fields, is essential, for example, in combustion processes, in the industrial production of nanoparticles as well as in atmospheric pollutant transport, cloud formation and air-quality monitoring of the atmosphere.

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IX. APPENDIX A: DETERMINISTIC/STOCHASTIC NS THERMOFLUIDS

Let us consider for definiteness a viscous and generally non-isentropic thermofluid either incompressible or compressible, described in both cases by the fluid fields \( \Omega \). In the following, we shall assume that the fluid fields are defined and suitably smooth in the existence domain \( \Omega \times I \), with \( \Omega \) and \( I \) denoting respectively an open subset of the 3-dimensional Euclidean space \( \mathbb{R}^3 \) and a subset of the real axis \( \mathbb{R} \).
A. A.1 - Case of a compressible fluid: CNSFE

Let us consider the case of a compressible thermofluid. Denoting by $\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla$ the fluid convective derivative, its fluid equations are identified with the so-called compressible Navier-Stokes-Fourier equations (CNSFE)

$$\frac{D \rho}{Dt} + \rho \nabla \cdot \mathbf{V} = 0,$$

$$\frac{D \mathbf{V}}{Dt} = \mathbf{F}_H,$$

$$\frac{D T}{Dt} = K,$$

$$\frac{\partial}{\partial t} S_T \geq 0,$$

$$p(\mathbf{r}, t) = p(\mathbf{r}, t, \rho, V, T),$$

where Eqs. (128)-(130) denote the mass continuity, forced Navier-Stokes and Fourier equations respectively; furthermore, the inequality (131) identifies the entropy law, customarily known as the 2nd principle of thermodynamics and Eq. (132) is the equation of state for the fluid pressure. Here the notation is standard. Thus, in particular in Eq. (129) $\mathbf{F}_H$ denotes the NS acceleration

$$\mathbf{F}_H \equiv -\frac{1}{\rho} [\nabla p - f] + \nabla \cdot \mathbf{g}'(\mathbf{r}, t),$$

where the viscous stress tensor $\mathbf{g}'(\mathbf{r}, t)$ is assumed of the form

$$\mathbf{g}'(\mathbf{r}, t) = \mu \left( \nabla \mathbf{V} + (\nabla \mathbf{V})^T - \frac{2}{3} \nabla \mathbf{V} \right) + \lambda \nabla \mathbf{V},$$

with $\mu, \lambda > 0$ to be denoted as first and second viscosity coefficients. As a consequence

$$\nabla \cdot \mathbf{g}'(\mathbf{r}, t) = \nabla \cdot \left[ \mu \left( \nabla \mathbf{V} + (\nabla \mathbf{V})^T \right) \right] + \nabla \left[ \left( \lambda - \frac{2}{3} \mu \right) \nabla \mathbf{V} \right].$$

In addition, $f$ denotes the volume force density, which is assumed to be a suitable smooth vector field of the general form $f = -\nabla \phi + f_R$, with $\phi(\mathbf{r}, t)$ and $f_R(\mathbf{r}, t; \{ \mathbf{Z}(\mathbf{r}, t) \})$ denoting respectively a scalar function (potential) and an additional non-conservative vector generally dependent of the fluid fields and in particular the temperature $T(\mathbf{r}, t)$. As an example, $f$ can be identified with $f = -\rho \mathbf{g} \cdot \left[ 1 - k_p T \right] + f_{R1}$, where $\mathbf{g}$ and $k_p$ are real constants denoting respectively the local acceleration of gravity and the density thermal-dilatation coefficient, $-\rho g k_p T$ is the temperature-dependent gravitational force density and $f_{R1}(\mathbf{r}, t; \{ \mathbf{Z}(\mathbf{r}, t) \})$ is a possible additional volume force density. Finally, the source term $K$ in Eq. (130) denotes the heat production rate, defined as

$$K(\mathbf{r}, t) = \frac{1}{n} \left( c_p - \frac{\alpha}{\beta_T} \right) \left( n J_T - (\beta_T n - \alpha T) \frac{D \rho}{Dt} - p \nabla \cdot \mathbf{V} + \frac{\mu}{2} \left( \frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i} - \frac{2}{3} \delta_{ij} \frac{\partial V_i}{\partial r_k} \right)^2 + \lambda \left( \nabla \cdot \mathbf{V} \right)^2 + \nabla \cdot [k \nabla T] \right),$$

where $c_p$ is the heat capacity at constant pressure and $\alpha, \beta_T$ are suitable (dimensional) phenomenological parameters.

In the case of an incompressible fluid, denoting $\rho_o$ the constant mass density, $\nu \equiv \mu / \rho_o$ the kinematic viscosity and requiring $\lambda = \alpha = \beta_T = 0$, the previous equations reduce to the incompressible Navier-Stokes-Fourier equations (INSFE):

$$\rho = \rho_o > 0,$$

$$\nabla \cdot \mathbf{V} = 0,$$

$$\frac{D \mathbf{V}}{Dt} = \mathbf{F}_H,$$

$$\frac{D T}{Dt} = K,$$

$$\frac{\partial}{\partial t} S_T \geq 0,$$
where now
\[ F_H \equiv -\frac{1}{\rho_o} [\nabla \rho - f] + \nu \nabla^2 \mathbf{V}, \] (142)

\[ K = \frac{1}{nc_p} \left[ nJ_T + \mu \left( \frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i} - \frac{2}{3} \delta_{ij} \frac{\partial V_i}{\partial r_k} \right)^2 + \lambda (\nabla \cdot \mathbf{V})^2 + \nabla \cdot [k \nabla T] \right]. \] (143)

Furthermore the equation of state (132) is replaced by the Poisson equation for the fluid pressure. This is obtained by taking the divergence of the NS equation (139), yielding
\[ \nabla^2 p = -\rho \nabla \cdot (\mathbf{V} \cdot \nabla \mathbf{V}) + \nabla \cdot \mathbf{f}, \] (144)
with \( p \) to be assumed non negative and bounded in \( \overline{\Omega} \times I \). Finally, we remark that Eqs. (137)-(141) include, as a particular case, the treatment of isothermal fluids. This is obtained assuming an initial spatially uniform temperature
\[ T(\mathbf{r}, t_0) = T_o, \] (145)
requiring that for all \((\mathbf{r}, t) \in \overline{\Omega} \times I\) the heat production rate \( K(\mathbf{r}, t) \) is identically zero in \( \overline{\Omega} \times I \) and imposing, at the same time, the isentropic law
\[ \frac{\partial}{\partial t} S_T = 0. \] (146)
Eqs. (137)-(141) with the constraints (145) and (146) are denoted as (isothermal and) incompressible NS equations (INSE).

1. Equivalent forms of the Fourier equation

To prove Eq. (130) with (136) let us start from the law of energy conservation equation. For a compressible viscous fluid this is [32]:

\[ \frac{\partial}{\partial t} \left( n\varepsilon + \frac{1}{2} \rho \mathbf{V}^2 \right) + \nabla \cdot \left[ \mathbf{V} \left( n\varepsilon + \frac{1}{2} \rho \mathbf{V}^2 + p \right) - \mathbf{V} \cdot \sigma' - k \nabla T \right] = 0, \] (147)
with \( n(\mathbf{r}, t) \) and \( \rho(\mathbf{r}, t) \) denoting respectively the number and mass densities, while \( \varepsilon(\mathbf{r}, t), k > 0 \) and \( T(\mathbf{r}, t) \geq 0 \) are the internal energy density, the thermal conductivity and the temperature. In terms of the convective derivative this delivers
\[ n \frac{D\varepsilon}{Dt} = -\rho \frac{D}{Dt} \left( \frac{1}{2} \mathbf{V}^2 \right) + \nabla \cdot [\mathbf{V} p + \mathbf{V} \cdot \sigma' + k \nabla T]. \] (148)
On the other hand from the NS equation [see Eq. (129)] it follows
\[ \rho \frac{D}{Dt} \left( \frac{1}{2} \mathbf{V}^2 \right) = -\mathbf{V} \cdot \nabla p + \mathbf{V} \cdot \mathbf{f} + \mathbf{V} \cdot \nabla \cdot \mathbf{g'}(\mathbf{r}, t), \] (149)
so that Eq. (148) recovers immediately the internal energy-transfer equation
\[ \frac{D\varepsilon}{Dt} = S_\varepsilon, \] (150)
\[ S_\varepsilon \equiv J_T + \frac{1}{n} \left\{ -p \nabla \cdot \mathbf{V} + \nabla \cdot [k \nabla T] + (\nabla \mathbf{V}) : \mathbf{g'}(\mathbf{r}, t) \right\}, \]
where
\[ J_T \equiv -\frac{1}{n} \mathbf{V} \cdot \mathbf{f}, \] (151)
\[
(\nabla V) \cdot \sigma'(r, t) = \frac{\mu}{2} \left( \frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i} - \frac{2}{3} \delta_{ij} \frac{\partial V_i}{\partial r_j} \right)^2 + \lambda (\nabla \cdot V)^2 \tag{152}
\]

carry respectively the contributions to the due to the \(\varepsilon\)-production rate generated by the volume force \(f\) (i.e., external sources) and viscous energy dissipation.

This equation, thanks to Eqs. (153)-(149), can be equivalently cast into an equation for the temperature \(T\) of the form (130). In fact, for generally non-isothermal and compressible fluids \(\varepsilon(r, t)\) can be taken such that

\[
\frac{D}{Dt} \varepsilon = \left( c_p - \frac{\alpha}{n} \right) \frac{DT}{Dt} + \left( \beta_T - \frac{\alpha T}{n} \right) \frac{Dp}{Dt}, \tag{153}
\]

Eq. (153) requires

\[
n \left( c_p - \frac{\alpha}{n} \right) \frac{DT}{Dt} = -n \left( \beta_T - \frac{\alpha T}{n} \right) \frac{Dp}{Dt} + \rho \frac{D}{Dt} \left( \frac{1}{2} V^2 \right) + \nabla \cdot [-Vp + V \cdot \sigma' + k \nabla T]. \tag{154}
\]

Introducing the definition (151) and invoking Eq. (152) this yields

\[
n \left( c_p - \frac{\alpha}{n} \right) \frac{DT}{Dt} = nJ_T - (\beta_T n - \alpha T) \frac{Dp}{Dt} - p \nabla \cdot V + \frac{\mu}{2} \left( \frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i} - \frac{2}{3} \delta_{ij} \frac{\partial V_i}{\partial r_j} \right)^2 + \lambda (\nabla \cdot V)^2 + \nabla \cdot [k \nabla T], \tag{155}
\]

and hence Eqs. (130) and (136). If we introduce, instead, the representation of \(\varepsilon(r, t)\) in terms of the local entropy \(s(r, t)\):

\[
\frac{D}{Dt} \varepsilon = \frac{T}{n} \frac{Ds}{Dt} + \frac{p}{n} \frac{D \ln n}{Dt}. \tag{156}
\]

Eq. (148) delivers the local entropy equation:

\[
nT \frac{Ds}{Dt} = S_s, \tag{157}
\]

\[
S_s(r, t) \equiv nJ_T + \nabla \cdot [k \nabla T] + \frac{\mu}{2} \left( \frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i} - \frac{2}{3} \delta_{ij} \frac{\partial V_i}{\partial r_j} \right)^2 + \lambda (\nabla \cdot V)^2, \tag{158}
\]

with \(S_s\) denoting the local entropy production rate.

2. **Entropy law - Externally heated thermofluid**

Denoting the global thermodynamic entropy \(S_T(t)\) as

\[
S_T(t) \equiv \int_{\Omega} d\mathbf{r} ns, \tag{159}
\]

it follows identically

\[
\frac{\partial}{\partial t} S_T(t) = \int_{\Omega} d\mathbf{r} \frac{\partial}{\partial t} (ns) = \int_{\Omega} d\mathbf{r} \left[ \frac{\partial}{\partial t} s + \frac{\partial}{\partial t} n \right] = \int_{\Omega} d\mathbf{r} [-n \mathbf{V} \cdot \nabla s + S - s \nabla \cdot (n \mathbf{V})] \equiv \int_{\Omega} d\mathbf{r} S_s. \tag{160}
\]
Hence, subject to the requirement:

$$\int_{\Omega} \frac{1}{T} J_{T} \geq 0,$$

usually referred to as (condition of) *externally heated thermofluid* - the entropy law follows, with

$$\frac{\partial}{\partial t} S_{T}(t) = \int_{\Omega} \left[ \frac{1}{T} J_{T} + \frac{k}{T^{2}} (\nabla T)^{2} + \frac{\mu}{2T} \left( \frac{\partial V_{i}}{\partial r_{k}} + \frac{\partial V_{k}}{\partial r_{i}} - \frac{2}{3} \delta_{ij} \frac{\partial V_{i}}{\partial r_{i}} \right)^{2} + \frac{\lambda}{T} (\nabla \cdot V)^{2} \right]$$

denoting the *global thermodynamic entropy production rate*.

**B. A.2 - Deterministic and stochastic initial-boundary value problems**

The fluid equations defined by CNSFE are required to satisfy initial-boundary value problems defined by appropriate initial and Dirichlet boundary conditions

$$\begin{cases}
Z(r, t_{o}) = Z_{o}(r), \\
Z_{w}(r, t)|_{\partial\Omega} = Z_{w}(r, t)|_{\partial\Omega}.
\end{cases}$$

In particular, denoting $I \subseteq \mathbb{R}$ a suitable subset of the real axis, we shall require that $S_{T}(t)$ is defined and smooth for all $t \in I$ and that a smooth (strong) solution exists for the previous initial-boundary value problem in $\Gamma \times I$ (existence domain).

Finally, we shall assume that the fluid fields $\{Z\}$, together with the volume force density $f$ and the initial and boundary fields $Z_{o}$ and $Z_{w}$ are *all stochastic functions* of the form (see Appendix B)

$$\begin{cases}
Z = Z(r, t, \alpha), \\
f = f(r, t, \alpha), \\
Z_{o} = Z_{o}(r, \alpha), \\
Z_{w} = Z_{w}(r, t, \alpha)|_{\partial\Omega},
\end{cases}$$

with $\alpha \in V_{\alpha}$ stochastic variables independent of $(r, t)$. Depending whether the previous functions (164) are considered deterministic or stochastic, the previous initial-boundary-value problems defined either by:

1. Eqs. (128)-(131) with the initial-boundary conditions (163),

or:

2. Eqs. (137)-(141) with (144) and the initial-boundary conditions (163),

3. Eqs. (137)-(139) with (144) and together with the constraints (145), (146) and the initial-boundary conditions (163),

will be denoted, respectively, as deterministic or stochastic CNSFE, INSFE and INSE problems.

**C. A.3 - Equivalent stochastic fluid equations**

For a prescribed stochastic model $\{g, V_{\alpha}\}$, with $g = g(r, t, \alpha)$ being a stochastic probability density on $V_{\alpha}$, in terms of stochastic decomposition (96) it is immediate to obtain the equations for the average and stochastic fluid fields $\langle Z \rangle_{\alpha}$, $\delta Z$ and the corresponding initial-boundary value problem. For example, in the case of INSE, requiring that $\langle \cdot \rangle_{\alpha}$ commutes with the nabla operator $\nabla$, Laplacian $\nabla^{2}$ and partial time derivative $\frac{\partial}{\partial t}$ operators, the fluid equations for $\langle Z \rangle_{\alpha}$ and $\delta Z$, to be referred to as *stochastic incompressible NS equations*, become respectively

$$\begin{align*}
\langle \frac{D}{Dt}V \rangle_{\alpha} &= \langle F_{H} \rangle_{\alpha} - \frac{1}{\rho_{o}} \left[ \nabla \langle p \rangle_{\alpha} - \langle f \rangle \right] + \nu \nabla^{2} \langle V \rangle_{\alpha}, \\
\nabla \cdot \langle V \rangle_{\alpha} &= 0,
\end{align*}$$

$$\begin{align*}
\frac{D}{Dt} V - \langle \frac{D}{Dt} V \rangle_{\alpha} &= \delta F_{H} - \frac{1}{\rho_{o}} \left[ \nabla \delta p - \delta f \right] + \nu \nabla^{2} \delta V, \\
\nabla \cdot \delta V &= 0.
\end{align*}$$

In particular Eqs. (165)-(166) identify the so-called *stochastic-averaged INSE*. 
X. APPENDIX B - STOCHASTIC/DETERMINISTIC VARIABLES AND STOCHASTIC MODELS

Let \((S, \Sigma, P)\) be a probability space; a measurable function \(\alpha : S \rightarrow V_\alpha\), where \(V_\alpha \subseteq \mathbb{R}^k\) and \(k \geq 1\), is called stochastic (or random) variable.

A stochastic variable \(\alpha\) is called continuous if it is endowed with a continuous stochastic model \(\{\alpha \in V_\alpha, g(r,t,\alpha)\}\), namely a real continuous function \(g\), called PDF on the set \(V_\alpha\), such that:

1) \(g\) is measurable, non-negative and of the form
\[
g = g(r,t,\cdot);
\]
2) if \(A \subseteq V_\alpha\) is an arbitrary Borelian subset of \(V_\alpha\) (written \(A \in \mathcal{B}(V_\alpha)\)), the integral
\[
P_\alpha(A) = \int_A d\alpha g(r,t,\alpha)
\]
exists and is the probability that \(\alpha \in A\); in particular, since \(\alpha \in V_\alpha, g\) admits the normalization
\[
\int_{V_\alpha} d\alpha g(r,t,\alpha) = P_\alpha(V_\alpha) = 1.
\]

The set function \(P_\alpha : \mathcal{B}(V_\alpha) \rightarrow [0,1]\) defined by \((170)\) is a probability measure on \(V_\alpha\). Consequently, if a function \(f : V_\alpha \rightarrow V_f \subseteq \mathbb{R}^m\) is measurable, \(f\) is a stochastic variable too.

Then we define the stochastic-averaging operator \((\cdot)_\alpha\) (see also Refs.\([2,34]\)) as
\[
(f)_\alpha = (f(y,\cdot))_\alpha \equiv \int_{V_\alpha} d\alpha g(r,t,\alpha) f(y,\alpha),
\]
for any \(P_\alpha\)-integrable function \(f(y,\cdot) : V_\alpha \rightarrow \mathbb{R}\), where the vector \(y\) is some parameter.

The ensemble \(\{g(r,t,\alpha), V_\alpha\}\) is denoted as stochastic model. Examples of stochastic model are represented by statistical models. In such a case the stochastic variables \(\alpha\) are identified with hidden variables, i.e., variables from which the fluid fields \(\{Z\}\) depend only implicitly. For example, in the IKT statistical models \(\{f,\Gamma\}\) and \(\{f_1,\Gamma_1\}\) (see THMs.1 and 3, in Sections 4 and 6) the stochastic variables and PDF are prescribed letting \(\alpha \equiv \mathbf{u}\) and \(g(r,t,\alpha) \equiv f(r,\mathbf{u},t)\) or \(g(r,t,\alpha) \equiv f_1(r,\mathbf{u},t)\), with \(f(r,\mathbf{u},t)\) and \(f_1(r,\mathbf{u},t)\) being defined, respectively, by Eqs. \([13]\) and \([89]\). Furthermore, in the two cases the set \(V_\alpha\) is identified with the velocity spaces \(U\) or \(U_1\).

A classification of stochastic models can be given in terms of the defining PDF \(g\) as follows.

**Definition - Homogeneous, stationary, deterministic and stochastic PDF**

The PDF \(g\) is denoted:

a) **homogeneous** if \(g\) is independent of \(r\), namely \(g = g(t,\alpha)\);

b) **isotropic** if \(g\) is a function of the form \(g = g(|r|, t, \alpha)\);

c) **stationary** if \(g\) is independent of \(t\), i.e., \(g = g(r,\alpha)\);

d) **deterministic** if \(g\) is a distribution on \(V_\alpha\) of the form \(g(r,t,\alpha) = \delta^{(k)}(\alpha - \alpha_0)\), with \(\delta^{(k)}(\alpha - \alpha_0)\) denoting the \(k\)-dimensional Dirac delta on the space \(V_\alpha\);

e) **stochastic** if \(g\) is an ordinary function on the space \(V_\alpha\).

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We notice that, in principle, alternative possible definitions of $p_1$ might be obtained replacing the contribution $nT$ on the r.h.s. of Eq. (8) (thermal energy density) either with $n(r,t)e(r,t)$ (internal energy density) or $n(r,t)T(r,t)s(r,t)$ (thermod. energy density), where $e(r,t)$ and $s(r,t)$ are respectively the internal energy and the local entropy (see Appendix A). However, contrary to $T(r,t)$, both $e(r,t)$ and $s(r,t)$ are not observables.

Notice that in the case of homogeneous, isotropic and stationary turbulence (HIST) the precise definition of the stochastic fluctuations of the relevant fluid fields is independent of the specific definition adopted for the stochastic variables $\alpha$.

This viewpoint is also adopted to describe turbulence in plasmas (see for example [35, 37]).