MEAN SQUARE STABILITY OF STOCHASTIC THETA METHOD FOR
STOCHASTIC DIFFERENTIAL EQUATIONS DRIVEN BY FRACTIONAL
BROWNIAN MOTION

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ABSTRACT. In this paper, we study the mean square stability of the solution and its stochas-
tic theta scheme for the following stochastic differential equations driven by fractional
Brownian motion with Hurst parameter $H \in (\frac{1}{2}, 1)$:

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB^H(t).$$

Firstly, we consider the special case when $f(t, X(t)) = -\lambda \kappa t^{\kappa - 1} X$ and $g(t, X(t)) = \mu X$. The solution is explicit and is mean square stable when $\kappa \geq 2H$. It is proved that if the parameter $2H \leq \kappa \leq 3/2$ and

$$\frac{\sqrt{37}}{\sqrt{3/2} + 1} (\approx 0.77) \leq \theta \leq 1 \text{ or } \kappa > 3/2 \text{ and } 1/2 < \theta \leq 1,$$

the stochastic theta method reproduces the mean square stability; and that if $0 < \theta < \frac{1}{2}$, the numerical method does not preserve this stability unconditionally. Secondly, we study the stability of the solution and its stochastic theta scheme for nonlinear equations. Due to presence of long memory, even the problem of stability in the mean square sense of the solution has not been well studied since the conventional techniques powerful for stochastic differential equations driven by Brownian motion are no longer applicable. Let alone the stability of numerical schemes. We need to develop completely new set of techniques to deal with this difficulty. Numerical examples are carried out to illustrate our theoretical results.

1. INTRODUCTION AND MAIN RESULTS

Numerical stability analysis of stochastic differential equations (SDEs) is an important
topic in numerical analysis and scientific computing. In order to get insight into the stabil-
ity behavior of numerical methods for SDEs, Saito and Mitsui [21] studied the mean square
stability of several numerical schemes for the following stochastic test problem driven by
standard Brownian motion (Bm)

$$dX(t) = \lambda X(t)dt + \mu X(t)dB(t), \ \lambda, \mu \in \mathbb{C},$$

(1.1)

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Key words and phrases. Stochastic differential equations driven by fractional Brownian motion; stochastic theta method; mean square stability; confluent hypergeometric functions; Gaussian correlation inequality; law of large numbers.
with initial value $X(0) \neq 0$ with probability 1 and $E \left| X(0) \right|^2 < \infty$, where $dB(t)$ is interpreted in Itô sense. The solution of (1.1) is said to be mean square stable if

$$\lim_{t \to \infty} E \left| X(t) \right|^2 = 0.$$  \hspace{1cm} (1.2)

As is well-known, the mean square stability of (1.1) is characterized by

$$\text{Re}(\lambda) + \frac{1}{2} |\mu|^2 < 0,$$

where $\text{Re}(\lambda)$ denotes the real part of $\lambda$. Higham [7, 8] studied the mean square stability properties of stochastic theta method and stochastic theta Milstein method for the test equation (1.1). The A-stability (which means that the numerical method preserves the stability of the underlying test problem unconditionally) of stochastic theta method (STM) and the stochastic theta Milstein method is proved when $\theta \geq \frac{1}{4}$ and $\theta \geq \frac{3}{2}$, respectively.

Subsequently, the stability of the numerical method for nonlinear SDEs driven by Brownian motion

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB(t)$$  \hspace{1cm} (1.3)

received much attention in the past decades. Assume that the drift coefficient $f$ satisfies certain monotone condition, and the diffusion coefficient satisfies the linear growth condition. The authors in [10, 22] proved that the backward Euler method and the split-step backward Euler method reproduce the exponential mean square stability of the underlying nonlinear problem. More recently, some scholars studied nonlinear stability under a coupled condition on the drift and diffusion coefficients. This condition allows that the diffusion coefficient grows super-linearly. For example, Szpruch and Mao [23] studied the asymptotic stability in this nonlinear setting for the STM. Huang [15] proved that for all given step size $\Delta t > 0$, the STM with $\theta \in \left[ \frac{1}{2}, 1 \right]$ is mean square stable for stochastic delay differential equations under the following coupled condition:

$$\alpha |f(t, u, v)| + \frac{1}{2} |g(t, u, v)|^2 \leq \tilde{\alpha} |u|^2 + \tilde{\beta} |v|^2, \quad \forall t > 0, \ u, v \in \mathbb{R},$$  \hspace{1cm} (1.4)

with $\tilde{\alpha} + \tilde{\beta} < 0$. If there exist positive constant $K_1$ and $K_2$ such that the drift coefficients $f$ also satisfy

$$|f(t, u, v)| \leq K_1 |u|^2 + K_2 |v|^2,$$  \hspace{1cm} (1.5)

then the STM with $\theta \in \left[ 0, 1/2 \right]$ is mean square stable under certain stepsize constraint. For more details for nonlinear stability of numerical method for SDEs we refer to [9] and references therein.

In recent decades long memory processes have been widely studied and applied by mathematicians and statisticians. In particular, the theory of stochastic differential equations driven by fractional Brownian motions have been well-developed and have found applications in various fields (e.g. [1, 19]). For example, the thermal dynamics characterized by a fractional Ornstein-Uhlenbeck process based on empirical observation in [2] is applied in the pricing of weather derivatives; The arbitrage in the financial market is eliminated in the case of geometric fBm in [14, 4] and in the case of fractional Langevin equation in [5]. The readers can also find interesting applications of fBm in modeling anisotropic multidimensional data with self-similarity and long-range dependence in [25] and references therein.

Motivated by the above works, we are concerned with the mean square stability analysis of stochastic theta method for some stochastic test equations driven by fractional Brownian motion (fBm) in $\mathbb{R}^d$. We focus our effort on the stability problem of the numerical scheme and try to avoid the complicate issues of existence and uniqueness of the solution.
when $H < 1/2$. For this reason we shall assume exclusively $H > 1/2$ throughout the paper. We also assume $d = 1$. First, a natural choice of the test equation is the extension of (1.1), namely, we replace the Brownian motion in (1.1) by fractional Brownian motion. However, an easy computation (similar to the one shown below) immediately gives that for any parameters $\tilde{\lambda}$ and $\mu$, the solution to $dX(t) = -\lambda X(t)dt + \mu X(t)dB^H(t)$ with a nonzero initial condition $X(0) = x \in \mathbb{R} \setminus \{0\}$ will never be stable in the mean square sense (or any $L_p$ sense for any finite $p$). So, the first thing we shall do is to modify (1.1) to the following new type of test equations:

$$dX(t) = -\lambda \kappa t^{\kappa - 1}X(t)dt + \mu X(t)dB^H(t), \quad t \geq 0, \ X(0) = X_0, \quad (1.6)$$

with $\lambda, \mu \in \mathbb{R}$ and $\kappa \geq 2H$. Here, for simplicity we assume that $X_0$ is a non-zero constant. Notice that (1.6) has an additional factor $t^{\kappa - 1}$ than (1.1) in the drift term. By the chain rule formula (e.g. [12, Proposition 2.7] or [19, Lemma 2.7.1]), we have $X(t) = X_0 \exp(-\lambda t^\kappa + \mu B^H(t))$ and hence

$$E \mid X(t) \mid^2 = E(X_0)^2 \exp \left[ 2(-\lambda t^\kappa + \mu^2 t^{2H}) \right]. \quad (1.7)$$

This formula implies the mean square stability of the solution to (1.6) if

(i) $\kappa > 2H$ and $\lambda > 0$ or (ii) $\kappa = 2H$ and $-\lambda + \mu^2 < 0$. \quad (1.8)

Otherwise, the solution of (1.6) diverges in mean square sense as $t$ goes to infinity. So we only need to consider (1.6) for the above two parameter regions (1.8).

After we obtain the stability result for the above linear equations (1.6), we shall next focus our effort on the numerical stability of the STM for the following nonlinear SDEs which are long memory version of (1.3)

$$dX(t) = f(t, X(t))dt + g(t, X(t))dB^H(t), \quad (1.9)$$

where $B^H(t)$ is fractional Brownian motion (fBm) with Hurst parameter $H > 1/2$. Inspired by the conditions (1.4), (1.5) and (1.8), we shall assume the coefficients in the SDE (1.9) satisfy the following conditions (we assume $d = 1$):

**Assumption 1.** There exist constants $\kappa \geq 2H$, $\lambda > 0$, $\tilde{\lambda} > 0$ and $\mu > 0$ such that for any $t > 0$ and $x \in \mathbb{R}$

- **Monotone condition**: $xf(t, x) \leq -\lambda \kappa t^{\kappa - 1}x^2, \quad (1.10)$
- **Linear growth**: $|f(t, x)| \leq \tilde{\lambda} \kappa t^{\kappa - 1}|x|, \quad (1.11)$
- **Uniform linear growth**: $|g(t, x)| \leq \mu |x| \quad (1.12)$

**Remark 1.1.** We mention that when $\kappa = 1$, conditions (1.10) and (1.11) reduce to the classical monotone condition and linear growth condition (which is discussed in the Brownian motion case, e.g. [10]).

We mention that it is a difficult problem to give the long-time stability analysis of the solution of (1.9) under Assumption 1 there are only few results about the moment bounds of the solution $X(t)$. For example, the moment bounds is given in [13] when $f(t, X) = 0$ and $g(t, X) = \sigma(X)$. More recently, Fan and Zhang [3] obtained the moment bounds with irregular drift term. We shall show that under the condition (1.10) and (1.11) and when $g(t, X(t)) = c(t)X(t)$, the solution $X(t)$ of (1.9) is mean square stable.

**Remark 1.2.** We believe that this stability result is new. If we there is a bijection function $h(t, \cdot) : \mathbb{R} \to \mathbb{R}$ such that $g(t, x) = h(t, x) = c(t)h(t, x)$ and $Y_t = h(t, X_t)$. Then the chain rule
The numerical scheme that we propose to study is the stochastic theta method (STM) to (1.9), which is some kind of implicit-explicit \( \theta \) Euler-Maruyama scheme:

\[
\begin{cases}
X_{n+1} = X_n + \theta f(t_n, X_n) \Delta t + (1 - \theta) f(t_{n+1}, X_n) \Delta t + g(t_n, X_n) V_n^H, \\
\text{where } t_n = n \cdot \Delta t \text{ and } V_n^H = B^H(t_{n+1}) - B^H(t_n),
\end{cases}
\]

(STM)  

where \( \Delta t > 0 \) is fixed stepsize.

In particular, when \( f(t, X) = -\lambda \kappa t^{\kappa - 1}X \) and \( g(t, x) = \mu X \), (1.13) becomes

\[
X_{n+1} = X_n - \kappa \lambda \theta \cdot (t_{n+1})^{\kappa - 1}X_n \Delta t - \kappa \lambda (1 - \theta) \cdot (t_n)^{\kappa - 1}X_n \Delta t + \mu \cdot X_n V_n^H,
\]

(1.14)

The main stability theorems we shall prove are displayed as follows:

**Theorem 1.1.** Let \( \Delta t > 0 \) be fixed and let \( \lambda, \mu \) satisfy (1.8). For the test equation (1.6) and the STM (1.14) we have the following statements.

(i) If \( \kappa \geq 2H \) and \( \frac{\sqrt{3/2}e}{\sqrt{3/2e+1}} \leq \theta \leq 1 \), then the STM (1.14) is mean square stable for the test equation (1.6), namely, \( \lim_{n \to \infty} \mathbb{E} |X_n|^2 = 0 \).

(ii) If \( \kappa > 3/2 \) and \( \frac{1}{2} \leq \theta \leq 1 \), then the STM (1.14) is mean square stable for the test equation (1.6).

(iii) If \( \kappa \geq 2H \) and \( 0 < \theta < \frac{1}{2} \), then the STM (1.14) is not unconditionally mean square stable for the test equation (1.6).

**Remark 1.3.** We are not clear whether or not the STM (1.14) is mean square stable when \( 2H \leq \kappa \leq \frac{3}{2} \) and \( \frac{1}{2} \leq \theta < \frac{\sqrt{3/2e}}{\sqrt{3/2e+1}} \), which will be a topic for future research.

**Theorem 1.2.** Let \( \Delta t > 0 \) be fixed and let \( \lambda, \mu \) in Assumption 7 satisfy (1.8). For the SDEs with fBm (1.9) and the STM (1.13) we have the following statement.

(i) If (1.10) and (1.12) in Assumption 7 hold, then the STM (1.13) with \( \theta = 1 \) (i.e., the backward Euler method) is mean square stable for the equation (1.9).

(ii) If (1.10), (1.11) and (1.12) in Assumption 7 hold, then the STM (1.13) with \( \frac{\sqrt{6\kappa/\lambda}}{\sqrt{6\kappa/\lambda+1}} \leq \theta < 1 \) is mean square stable for the equation (1.9).

We shall prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. Before we end this section we would point out the new difficulties we encounter compared with the classical Brownian motion (e.g. see subsection 2.5). We can write (1.14) as

\[
X_{n+1} = \left(1 - \kappa (1 - \theta) \lambda (t_{n+1})^{\kappa - 1} \Delta t \right) X_n + \frac{\mu V_n^H}{1 + \kappa \theta \lambda (t_{n+1})^{\kappa - 1} \Delta t} X_n.
\]

(1.15)
When $H = 1/2$ (i.e. the Brownian motion case), $X_{n+1}$ is the product of independent variables and the corresponding computation is much easier. However, this is no longer true in our fBM setting. We encounter two major difficulties:

1. The increments $B^H(t_{n+1}) - B^H(t_n)$ of the fractional Brownian motion depend on the past history, which makes the stability analysis much more sophisticated.

2. The fractional Brownian motion lacks martingale property or Markov property so that some useful techniques such as conditional expectation seems impossible or at least over-sophisticated.

To get around these difficulties we shall employ some other analysis and computation techniques. In fact, in the proof of different parts of Theorem 1.1 we shall use different techniques. For example, in the proof of part (i) of Theorem 1.1 we use the technique of generalized polarization, raw moments formula of Gaussian distributions and the asymptotic properties of confluent hypergeometric function. On the other hand, the main tool to prove part (ii) is the celebrated Gaussian correlation inequality. Finally, the statement of part (iii) is proved through the strong law of large numbers of dependent random variables. All of these are done in Section 2. Let us mention that the test equation (1.6) has not been previously studied even when the fBM is replaced by the standard Brownian motion and it is interesting to carry out the stability analysis of the corresponding stochastic theta scheme for its own sake and for the comparison purpose. This is also done in Section 2. The proof of Theorem 1.2 is analogous to that of part (i) of Theorem 1.1 and is provided in Section 3. In Section 4, some numerical simulations are presented to validate our theoretical results. Finally, some concluding remarks are given in the last section.

2. STM: Mean square linear stability analysis

In this section we shall prove our main result, i.e., Theorem 1.1. The parts (i), (ii) and (iii) are proved in subsection 2.1, 2.2 and 2.3, respectively.

Obviously, (1.4) is equivalent to the following recurrent equation

$$X_{n+1} = \left(\alpha_n(\theta, \lambda, \Delta t) + \beta_n(\theta, \lambda, \mu, \Delta t) V_n^H\right) X_n,$$

where $\kappa \geq 2H > 1$ and

$$\begin{align}
\alpha_n(\theta, \lambda, \Delta t) &= \frac{1 - \kappa(1 - \theta)\lambda (t_n)\kappa^{-1} \Delta t}{1 + \kappa \theta \lambda (t_n+1)^{\kappa-1} \Delta t} = \frac{1 - \kappa(1 - \theta)\lambda n^{\kappa-1} \Delta t^\kappa}{1 + \kappa \theta \lambda (n+1)^{\kappa-1} \Delta t^\kappa}, \tag{2.1} \\
\beta_n(\theta, \lambda, \mu, \Delta t) &= \frac{\mu}{1 + \kappa \theta \lambda (t_n+1)^{\kappa-1} \Delta t} = \frac{\mu}{1 + \kappa \theta \lambda (n+1)^{\kappa-1} \Delta t^\kappa}. \tag{2.2}
\end{align}$$

For notational simplicity, throughout the remaining part of the paper we denote $\alpha_n(\theta, \lambda, \Delta t)$, $\beta_n(\theta, \lambda, \mu, \Delta t)$ by $\alpha_n$ and $\beta_n$, respectively. Note that (2.1) and (2.2) are well defined if we require the condition (1.8) or otherwise the denominators in the expressions of $\alpha_n$ and $\beta_n$ could be 0.

2.1. Heuristic arguments. Before the proof, we would like to explain why Theorem 1.1 could hold true heuristically, namely, why the STM (1.4) is stable when $\theta > 1/2$ and is unstable when $\theta < 1/2$, formally. Denote

$$Z_n(\Delta t) = \alpha_n + \beta_n V_n^H. \tag{2.3}$$

Then we have

$$X_{n+1} = X_0 \prod_{k=0}^{n} Z_k(\Delta t) = X_0 \prod_{k=0}^{n} (\alpha_k + \beta_k V_k^H), \tag{2.4}$$
Obviously, for fixed $\Delta t$, $\lambda$ and $\mu$,
\[ \lim_{n \to \infty} \alpha_n = \frac{1 - \theta}{\theta}, \quad \lim_{n \to \infty} \beta_n = 0. \]
Notice that this is quite different than the setting with $H = 1/2$ where $\alpha_n$ and $\beta_n$ do not depend on $n$ because of the absence of $(t_n)^{\kappa - 1}$ for $\kappa = 2H = 1$ (see Section 3 for more details). Formally, if we could think \[ \{V_k^H\} \] as a sequence of finite numbers, then by the limits of $\alpha_n$ and $\beta_n$, we would have
\[ |X_{n+1}|^2 = |X_0|^2 \prod_{k=0}^{n} \left( \alpha_k + \beta_k V_k^H \right)^2 \approx \left( \frac{1 - \theta}{\theta} \right)^{2n} \to \begin{cases} 0, & \text{if } \frac{1}{2} < \theta \leq 1; \\ \infty, & \text{if } 0 \leq \theta < \frac{1}{2}, \end{cases} \]
where and through the remaining part of this paper, we use $a_n \approx b_n$ to denote that there are two positive constants $c_1$ and $c_2$, independent of $n$, such that $c_1 a_n \leq b_n \leq c_2 a_n$ for all $n \geq 1$.

However, the random variables $\{V_k^H\}$ in our setting are not uniformly bounded. Even worse, they are long range dependent. Therefore, the above heuristic argument cannot be applied directly to analyze (2.4), especially for the scenario of (mean square) stability. Presumably, there are two ways to break these barriers.

(1) Choose $\theta$ carefully so that the oscillation caused by $\{V_k^H\}$ can still be manageable.

(2) Take $\kappa$ sufficiently large so that $\beta_n \cdot V_k^H$ converges to 0 fast enough so that influences of $\{V_k^H\}$ can be neglected.

Our proof will follow these spirits but with much more sophisticated tricks and computations. For example, we need to use the asymptotics of the confluent hypergeometric functions which comes from the moments of Gaussian variables.

2.2. The case of $\kappa \geq 2H$ and $\frac{\sqrt{a/2}}{\sqrt{b/2} + 1} \leq \theta \leq 1$. In this subsection we prove part (i) of the main theorem, namely, we consider the case when $\kappa \geq 2H$ and $\frac{\sqrt{a/2}}{\sqrt{b/2} + 1} \leq \theta \leq 1$. Firstly, we state a useful lemma, which is a generalization of polarization identity.

**Lemma 2.1.** [16] Lemma 1] Let $x_1, \ldots, x_n$ be real numbers, and let $s_1, \ldots, s_n$ be nonnegative integers and $s = \sum_{i=1}^{n} s_i$. Then, we have
\[ x_1^{s_1} \cdots x_n^{s_n} = \frac{1}{s!} \sum_{\nu_1=0}^{s_1} \cdots \sum_{\nu_n=0}^{s_n} (-1)^{\sum_{i=1}^{n-1} \nu_i} \binom{s_1}{\nu_1} \cdots \binom{s_n}{\nu_n} \cdot \left[ \sum_{i=1}^{n} h_i x_i \right]^s, \]
where $h_i = s_i / 2 - v_i$.

**Proof of part (i) of Theorem 7.7** Our goal is to show
\[ \lim_{n \to \infty} \mathbb{E}[|X_n|^2] = \lim_{n \to \infty} \mathbb{E} \left[ X_n^2 \prod_{k=0}^{n-1} (Z_k(\Delta t))^2 \right] = 0, \quad (2.5) \]
where $Z_k(\Delta t)$ is given by (2.4) and $X_n$ is given by (2.3). To illustrate the idea we assume $\kappa = 2H$. The case $\kappa > 2H$ can be handled analogously and is in fact simpler. We divide our proof into three steps.

**Step 1:** Bound $\mathbb{E}[|X_n|^2]$ by a confluent hypergeometric function.

Applying Lemma 2.1 with $s_1 = \cdots = s_n = 2$ and $s = 2n$ we have
\[ \prod_{k=0}^{n-1} Z_k^2(\Delta t) = \frac{1}{(2n)!} \sum_{\nu_1=0}^{2} \cdots \sum_{\nu_n=0}^{2} (-1)^{\sum_{i=1}^{n-1} \nu_i} \binom{s_1}{\nu_1} \cdots \binom{s_n}{\nu_n} \cdot \left[ \sum_{i=1}^{n} h_i Z_i(\Delta t) \right]^{2n}, \]
with \( h_i = 1 - v_i \). Note that \( Z_t(\Delta t) = \alpha_i + \beta_i \cdot V_i^H \overset{d}{\sim} N(\mu_i, \sigma_i) \) with \( \mu_i = \alpha_i \) and \( \sigma_i^2 = \beta_i^2(\Delta t)^2 \). Thus we have

\[
E \left[ \prod_{\xi=0}^{n-1} Z_\xi^2(\Delta t) \right] \leq \frac{2^n}{(2n)!} \sum_{v_1=0}^{2} \cdots \sum_{v_n=0}^{2} \mathbb{E} \left[ \sum_{i=1}^{n} (1 - v_i) \cdot Z_i(\Delta t) \right]^{2n} =: \frac{2^n}{(2n)!} \sum_{v_1=0}^{2} \cdots \sum_{v_n=0}^{2} \mathbb{E}[Q_n]^{2n},
\]

(2.6)

where \( Q_n = Q_n(v_1, \cdots, v_n; x_1, \cdots, x_n) = \sum_{i=1}^{n} (1 - v_i) \cdot Z_i(\Delta t) \). It is obvious that \( Q_n \) is still a normal random variable, with mean \( \mu_n \) and variance \( \sigma_n^2 \) given by

\[
\mu_n := \mu_n(v_1, \cdots, v_n) = \sum_{i=1}^{n} (1 - v_i) \cdot \mu_i = \sum_{i=1}^{n} (1 - v_i) \cdot \alpha_i,
\]

and

\[
\sigma_n^2 := \sigma_n^2(v_1, \cdots, v_n) = E \left[ \left( \sum_{i=1}^{n} (1 - v_i) \cdot \beta_i \cdot V_i^H \right)^2 \right] = \sum_{i,j=1}^{n} (1 - v_i)(1 - v_j) \cdot \beta_i \beta_j \cdot E[V_i^H V_j^H].
\]

From the raw moment formula (2.4, Eq. (17)) it follows

\[
E[Q_n]^{2n} = \frac{2^n}{\sqrt{\pi}} \sigma_n^{2n} \Gamma \left( \frac{2n + 1}{2} \right) \cdot \Phi \left( -n, \frac{1}{2}, -\frac{\mu_n^2}{2 \sigma_n^2} \right)
\]

\[
= \frac{2^n}{\sqrt{\pi}} \sigma_n^{2n} \Gamma \left( \frac{2n + 1}{2} \right) \cdot \exp \left( -\frac{\mu_n^2}{2 \sigma_n^2} \right) \Phi \left( n + \frac{1}{2}, \frac{1}{2}, \frac{\mu_n^2}{2 \sigma_n^2} \right),
\]

(2.7)

where we used Kummer’s transformation (see e.g. (B.5) of the appendix): \( \Phi(\alpha, \gamma; z) = \exp(z)\Phi(\gamma - \alpha, \gamma; z) \). Here, \( \Phi(\alpha, \gamma; z) \) is Kummer’s confluent hypergeometric function (see (B.4) or Chapter 13 in [20] for more details).

By employing the differentiation formula (B.6) and then (B.8), we have with the substitution \( \eta = \frac{\mu_n^2}{2 \sigma_n^2} \)

\[
\frac{d}{d\eta} \left[ e^{-\eta} \Phi \left( \frac{n+1}{2}, \frac{1}{2}; \frac{1}{2}, \eta \right) \right]
\]

\[
= n \cdot e^{-\eta} \Phi \left( \frac{n+1}{2}, \frac{3}{2}; \frac{1}{2}, \eta \right)
\]

\[
= n \cdot e^{-\eta} \cdot \frac{2^{n+1} \Gamma \left( \frac{n+1}{2} \right) e^{\frac{1}{2} \eta}}{\sqrt{\pi} 2^n \eta} \times [U(n - 1/2, -\sqrt{2\eta}) - U(n - 1/2, \sqrt{2\eta})]
\]

\[
= n \cdot e^{-\eta} \cdot \frac{2^{n+1} \Gamma \left( \frac{n+1}{2} \right)}{\sqrt{\pi} \mu_n^2 / \sigma_n^2} \times [U \left( n - 1/2, -\sqrt{\mu_n^2 / \sigma_n^2} \right) - U \left( n - 1/2, \sqrt{\mu_n^2 / \sigma_n^2} \right)].
\]
Using the identity (B.9), we have

\[
U \left( n - \frac{1}{2}, -\sqrt{\mu_z^2 / \sigma_z^2} - \sqrt{\mu_z^2 / \sigma_z^2} \right) - U \left( n - \frac{1}{2}, \sqrt{\mu_z^2 / \sigma_z^2} \right) = \frac{e^{-\frac{\mu_z^2}{4\sigma_z^2}}}{\Gamma(n)} \int_0^\infty \eta^{n-1} e^{-\eta^2/2} \left[ e^{\eta \sqrt{\mu_z^2 / \sigma_z^2}} - e^{-\eta \sqrt{\mu_z^2 / \sigma_z^2}} \right] d\eta \geq 0.
\]

This implies \( e^{-\eta} \Phi \left( \frac{a+1}{2}, \frac{1}{2}, \eta \right) \) is an increasing function with respect to the variable \( \eta (= \frac{\mu_z^2}{2\sigma_z^2}) \). Thus, \( \mathbb{E} [Q_n]^{2n} \) can be bounded by the value at \( \beta_n \) with \( \tilde{m}_n := \mu_n(0, \cdots, 0) = \sum_{i=1}^n a_i \) of this function, i.e.,

\[
\mathbb{E} [Q_n]^{2n} \leq \frac{2^n}{\sqrt{n}} \tilde{m}_n^2 \Gamma \left( 2n + \frac{1}{2} \right) \cdot \exp \left( -\frac{\tilde{m}_n^2}{2\sigma_n^2} \right) \Phi \left( n + \frac{1}{2}, \frac{1}{2}, \frac{\tilde{m}_n^2}{2\sigma_n^2} \right). \tag{2.8}
\]

**Step 2:** Analysis of the confluent hypergeometric function in (2.8).

A key ingredient of our proof is to analyze the asymptotic behavior as \( n \to \infty \) of the right-hand side of (2.8) and this is the objective of this step.

We claim that there exists a positive constant \( C \) which might change from line to line (we shall not point out the universal constants \( C \) unless necessary in this article) such that

\[
\Phi \left( \frac{a}{2} + \frac{1}{4}, \frac{1}{2}, \frac{z^2}{2} \right) \leq C \cdot 2^{a/2} \cdot \Gamma \left( \frac{a}{2} + \frac{3}{4} \right) \cdot \frac{e^{\frac{z^2}{2}} \exp \left( \frac{z^2}{4} \right)}{\Gamma \left( \frac{a}{2} + 1 \right)} \leq \frac{e^{\frac{z^2}{2}} \exp \left( \frac{z^2}{4} \right)}{2^{a/2} \Gamma \left( \frac{a}{2} + \frac{1}{4} \right)} \tag{2.9}
\]

We shall show the key asymptotic behaviors of the confluent hypergeometric functions \( \Phi (a, b, z) \). The idea is motivated by the Poincaré-type asymptotic forms (B.10) of confluent hypergeometric function. In our case, we have \( a = 2n + \frac{1}{2} \), so the parameter \( a \) can also goes to infinity. Fortunately, we have \( z^2 = \frac{\tilde{m}_n^2}{2\sigma_n^2} \geq C \cdot n^{2H} \) (see the proof in the Appendix A), the parameter \( a \) is majored by \( z \) since \( H > 1/2 \).

To prove the claim (2.9) we employ the integral representation of the parabolic cylinder functions (B.9). For \( z > 0 \) the parabolic cylinder functions are computed as follows:

\[
U(a, z) = \frac{z^{a+\frac{1}{4}} \exp \left( -\frac{z^2}{4} \right)}{\Gamma \left( \frac{a}{2} + \frac{1}{4} \right)} \int_0^\infty t^{a-\frac{1}{4}} \exp \left( -z^2 \left( \frac{1}{2} t^2 + t \right) \right) dt
\]

\[
= \frac{z^{a+\frac{1}{4}} \exp \left( \frac{z^2}{4} \right)}{\Gamma \left( \frac{a}{2} + \frac{1}{4} \right)} \int_1^\infty (s-1)^{a-\frac{1}{4}} \exp \left( -\frac{s^2 z^2}{2} \right) ds \tag{2.10}
\]

and

\[
U(a, -z) = \frac{z^{a+\frac{1}{4}} \exp \left( -\frac{z^2}{4} \right)}{\Gamma \left( \frac{a}{2} + \frac{1}{4} \right)} \int_0^\infty t^{a-\frac{1}{4}} \exp \left( -z^2 \left( \frac{1}{2} t^2 - t \right) \right) dt
\]

\[
= \frac{z^{a+\frac{1}{4}} \exp \left( \frac{z^2}{4} \right)}{\Gamma \left( \frac{a}{2} + \frac{1}{4} \right)} \int_{-1}^\infty (s+1)^{a-\frac{1}{4}} \exp \left( -\frac{s^2 z^2}{2} \right) ds. \tag{2.11}
\]


The sum of the integrals in (2.10)-(2.11) can be dominated as follows (with $a = 2n + \frac{1}{2}$ and $z^2 = \frac{\tilde{m}_n^2}{\tilde{a}_n}$).

$$\int_{1}^{\infty} (s-1)^{a-\frac{1}{2}} \exp(-\frac{z^2 s^2}{2}) ds + \int_{-1}^{0} (s+1)^{a-\frac{1}{2}} \exp(-\frac{z^2 s^2}{2}) ds$$

$$\leq 2 \int_{-1}^{0} (s+1)^{a-\frac{1}{2}} \exp(-\frac{z^2 s^2}{2}) ds = 2 \int_{-1}^{0} (s+1)^{2n} \exp(-\frac{\tilde{m}_n^2 s^2}{2\tilde{a}_n^2}) ds.$$ 

Basically, we know that $z^2 = \frac{\tilde{m}_n^2}{\tilde{a}_n^2} \geq C \cdot n^{2H} \geq n$ for $n$ large enough. So, for sufficiently large $n$

$$(s+1)^{2(n+1)} \leq \exp(2(n+1)s) \leq \exp\left(\frac{\tilde{m}_n^2 s^2}{4\tilde{a}_n^2}\right)$$

for all $s \geq -1$. Therefore, we can easily obtain

$$\int_{-1}^{0} (s+1)^{2n} \exp\left(-\frac{\tilde{m}_n^2 s^2}{2\tilde{a}_n^2}\right) ds \leq \int_{-1}^{0} \exp\left(-\frac{\tilde{m}_n^2 s^2}{4\tilde{a}_n^2}\right) ds$$

$$\leq \int_{\mathbb{R}} \exp\left(-\frac{\tilde{m}_n^2 s^2}{4\tilde{a}_n^2}\right) ds = C \cdot \tilde{\sigma}_n = C \cdot \frac{1}{z}.$$

Recall the relation between $\Phi(a, b, z)$ and the parabolic cylinder functions $U(a, z)$ given by (B.7). As a result, we get

$$\Phi\left(\frac{a}{2} + \frac{1}{4}, \frac{1}{2}, \frac{z^2}{2}\right) = \frac{2^{a - \frac{1}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{a}{2} + \frac{3}{4}\right) \exp\left(\frac{z^2}{4}\right) \times [U(a, z) + U(a, -z)]$$

$$\leq C \cdot 2^{a - \frac{1}{2}} \Gamma\left(\frac{a}{2} + \frac{3}{4}\right) \exp\left(\frac{z^2}{4}\right) \times \frac{z^{a - \frac{1}{2}}}{\Gamma\left(\frac{1}{2} + a\right)} \times \frac{1}{z}$$

$$= C \cdot 2^{a - \frac{1}{2}} \Gamma\left(\frac{a}{2} + \frac{3}{4}\right) \times \frac{z^{a - \frac{1}{2}}}{\Gamma\left(\frac{1}{2} + a\right)} \times \frac{1}{z}.$$

Thus we finish the proof of our claim (2.9).

**Step 3:** Completion of the proof of part (i) of Theorem 1.1

Applying (2.12) with $a = 2n + \frac{1}{2}$, $z^2 = \frac{\tilde{m}_n^2}{\tilde{a}_n} \geq C \cdot n^{2H} \geq n$ we obtain

$$\mathbb{E}\left[\prod_{k=1}^{n} Z_k^2(\Delta t)\right] \leq \frac{2^{n}}{(2n)!} \sum_{v_1=0}^{2} \cdots \sum_{v_n=0}^{2} \mathbb{E}[Q_n]^{2a}$$

$$\leq \frac{2^{n} \cdot 3^{n}}{(2n)!} \sqrt{\pi} \tilde{a}_n^{2n} \Gamma\left(\frac{2n+1}{2}\right) \cdot \exp\left(-\frac{\tilde{m}_n^2}{2\tilde{a}_n^2}\right) \Phi\left(n + \frac{1}{2}, \frac{1}{2}, \frac{\tilde{m}_n^2}{2\tilde{a}_n^2}\right)$$

$$\leq \frac{2^{n} \cdot 3^{n}}{(2n)!} \sqrt{\pi} \tilde{a}_n^{2n} \Gamma\left(\frac{2n+1}{2}\right) \cdot \exp\left(-\frac{\tilde{m}_n^2}{2\tilde{a}_n^2}\right) \Phi\left(n + \frac{1}{2}, \frac{1}{2}, \frac{\tilde{m}_n^2}{2\tilde{a}_n^2}\right)$$

$$\leq \frac{2^{n} \cdot 3^{n}}{(2n)!} \sqrt{\pi} \tilde{a}_n^{2n} \Gamma\left(\frac{2n+1}{2}\right) \cdot \exp\left(-\frac{\tilde{m}_n^2}{2\tilde{a}_n^2}\right)$$

$$\leq \frac{C}{\Gamma\left(n + \frac{1}{2}\right)} \left(\frac{\tilde{m}_n^2}{2\tilde{a}_n^2}\right)^n \exp\left(\frac{\tilde{m}_n^2}{2\tilde{a}_n^2}\right)$$

$$\leq \frac{6^n \cdot \tilde{m}_n^2}{(2n)!} \frac{\tilde{m}_n^2}{\sqrt{4\pi n} \cdot (2n/e)^{2n}} \left(1 - \frac{\theta}{\tilde{a}_n}\right)^{2n} \leq \frac{(\sqrt{3/2e})^{2n}}{\sqrt{4\pi n} \cdot \left(1 - \frac{\theta}{\tilde{a}_n}\right)^{2n}},$$

(2.13)
by Stirling’s approximation, where we apply the claim (2.9) in the above third inequality and the fact that $\frac{1}{2} < \theta < 1$ in the above last inequality. Now, it is obvious to see from (2.13) that $E\left[\prod_{k=1}^{t} Z_k^2(\Delta t)\right] \rightarrow 0$ as $n \rightarrow \infty$ if

$$\sqrt{\frac{3}{2}} e \cdot \frac{1-\theta}{\theta} \leq 1 \iff \theta \geq \frac{\sqrt{\frac{3}{2}} e}{\sqrt{\frac{3}{2}} e + 1} \approx 0.77,$$

proving part (i) of Theorem 1.1.

Remark 2.1. We believe our method can also work under the condition that $X_0 = 0$ with probability 0 and $E[|X_0|^2] < \infty$. For example, one can apply Hölder inequality to (2.5) and then follow the same argument there. But this makes the computations much more involved. We are not pursuing the detail along this direction to simplify our presentation.

Remark 2.2. Following the same strategy as in our proof, we can prove more general results: For any integer $p \geq 2$, if $\frac{1}{1+p} \leq \theta < 1$, where $M_p = \frac{2}{e} \frac{1}{(p+1)(p+2)}$, then $\lim_{n \rightarrow \infty} E(X_n^p) \rightarrow 0$.

2.3. The case of $\kappa > 3/2$ and $\frac{1}{2} < \theta \leq 1$. In this subsection we shall prove part (ii) of Theorem 1.1. To begin with, let us recall the celebrated Gaussian correlation inequality.

Lemma 2.2. [18, Theorem 2] Let $n = n_1 + n_2$ and $X$ be an $n$-dimensional centered Gaussian vector. Then for any $t_1, \cdots, t_n > 0$,

$$\text{P}\left\{\left|X_1\right| \leq t_1, \cdots, \left|X_n\right| \leq t_n\right\} \geq \text{P}\left\{\left|X_1\right| \leq t_1, \cdots, \left|X_{n_1}\right| \leq t_{n_1}\right\} \cdot \text{P}\left\{\left|X_{n_1+1}\right| \leq t_{n_1+1}, \cdots, \left|X_n\right| \leq t_n\right\}.$$

Proof of part (ii) of Theorem 1.1. Let us consider

$$X_{n+1}^2 = \prod_{k=1}^{n} (Z_k(\Delta t))^2 = \prod_{k=1}^{p(n)} (Z_k(\Delta t))^2 \cdot \prod_{k=p(n)+1}^{n} (Z_k(\Delta t))^2$$

$$= \prod_{k=1}^{p(n)} (Z_k(\Delta t))^2 \cdot \prod_{k=p(n)+1}^{n} (Z_k(\Delta t))^2 \cdot \mathbb{I}\{Z_k(\Delta t) \leq 0: p(n)+1 \leq k \leq n\}$$

$$+ \left[\prod_{k=1}^{n} (Z_k(\Delta t))^2\right] \cdot \left[1 - \prod_{k=p(n)+1}^{n} \mathbb{I}\{Z_k(\Delta t) \leq 0: p(n)+1 \leq k \leq n\}\right], \quad (2.14)$$

with $p(n) = n/p$, $q(n) = n/q$ and $1/p + 1/q = 1$. Formally, we know $Z_k(\Delta t)$ converges to $c_\theta = -\frac{1-\theta}{\theta} < 0$. Thus, the probability of the event $\{Z_k(\Delta t) \leq 0: p(n)+1 \leq k \leq n\}$ converges to one.

Firstly, applying the following inequality of bounding the geometric mean by the arithmetic one

$$\left\{a_0, a_1, \cdots, a_n\right\} \leq \left(\frac{p_0 a_0 + p_1 a_1 + \cdots + p_n a_n}{p_0 + p_1 + \cdots + p_n}\right)^{p_0 + p_1 + \cdots + p_n}, \quad a_0, \cdots, a_n \geq 0, \quad p_0, p_1, \cdots, p_n \in \mathbb{N}^+, \quad \text{with } p_0 = \cdots = p_n = 2.$$


to the second factor of (2.14) yields
\[
\tilde{X}_{n+1}^2 := \prod_{k=1}^{p(n)} (Z_k(\Delta t))^2 \cdot \prod_{k=p(n)+1}^{n} (-Z_k(\Delta t))^2 \cdot \mathbb{1}_{\{Z_k(\Delta t) \leq 0; \: p(n)+1 \leq k \leq n\}} \leq \prod_{k=1}^{p(n)} (Z_k(\Delta t))^2 \cdot \mathbb{1}_{\{Z_k(\Delta t) \leq 0; \: p(n)+1 \leq k \leq n\}} \] .
\] (2.16)

By the Hölder inequality, we then have
\[
\mathbb{E}\tilde{X}_{n+1}^2 \leq \left( \mathbb{E} \prod_{k=1}^{p(n)} (Z_k(\Delta t))^4 \right)^{\frac{1}{4}} \cdot \left( \mathbb{E} \left[ \frac{1}{n-p(n)} \sum_{k=p(n)+1}^{n} Z_k(\Delta t) \right]^{8n} \right)^{\frac{1}{4}} \cdot \left( \mathbb{P}\{Z_k(\Delta t) \leq 0; \: p(n)+1 \leq k \leq n\} \right)^{\frac{1}{4}} \leq \left( \mathbb{E} \prod_{k=1}^{p(n)} (Z_k(\Delta t))^4 \right)^{\frac{1}{4}} \cdot \left( \mathbb{E} \left[ \frac{1}{n-p(n)} \sum_{k=p(n)+1}^{n} Z_k(\Delta t) \right]^{8n} \right)^{\frac{1}{4}} .
\] (2.17)

By the same methods as in the proof of part (i), there is an \( M > 1 \) such that the first factor of (2.17) can be bounded by
\[
\left( \mathbb{E} \prod_{k=1}^{p(n)} (Z_k(\Delta t))^4 \right)^{\frac{1}{4}} \leq \mathbb{M}^{p(n)} = (M^{1/p})^n .
\] (2.18)

For the second term in (2.17), we have from Kummer’s transformation (B.5)
\[
\mathbb{E} \left[ \frac{1}{n-p(n)} \sum_{k=p(n)+1}^{n} Z_k(\Delta t) \right]^{8n} = \frac{C}{\sqrt{\pi}} \varrho_{p(n)}^{8n} 2^{4n} \Gamma \left( \frac{8n+1}{2} \right) \cdot \Phi \left( -4n, \frac{1}{2}; \frac{\tilde{\mu}_{p(n)}^2}{2\tilde{\sigma}_{p(n)}^2} \right)
\] 
\[
= \frac{C}{\sqrt{\pi}} \varrho_{p(n)}^{8n} 2^{4n} \Gamma \left( \frac{8n+1}{2} \right) \cdot \exp \left( \frac{\tilde{\mu}_{p(n)}^2}{2\tilde{\sigma}_{p(n)}^2} \right) \Phi \left( 4n + \frac{1}{2}, \frac{1}{2}; \frac{\tilde{\mu}_{p(n)}^2}{2\tilde{\sigma}_{p(n)}^2} \right) .
\] (2.19)

where \( \tilde{\mu}_{p(n)} := \frac{1}{n-p(n)} \sum_{k=p(n)+1}^{n} \alpha_k \) and \( \tilde{\sigma}_{p(n)}^2 := \mathbb{E} \left[ \left| \frac{1}{n-p(n)} \sum_{k=p(n)+1}^{n} \beta_k \cdot V_k^H \right| \right]^2 \). Then by the same procedure as in Step 2, we can bound (2.19), namely, the second factor of (2.17) by the following:
\[
C \varrho_{p(n)}^{8n} 2^{4n} \Gamma \left( \frac{8n+1}{2} \right) \exp \left( -\frac{\tilde{\mu}_{p(n)}^2}{2\tilde{\sigma}_{p(n)}^2} \right) \cdot \frac{\Gamma(1/2)}{\Gamma(4n+1/2)} \left( \frac{\tilde{\mu}_{p(n)}^2}{2\tilde{\sigma}_{p(n)}^2} \right)^{4n} \exp \left( \frac{\tilde{\mu}_{p(n)}^2}{2\tilde{\sigma}_{p(n)}^2} \right) \leq C \tilde{\mu}_{q(n)}^{q(n)} \leq C \left( \frac{1-\theta}{\theta} \right)^{q(n)} = C \left( \frac{1-\theta}{\theta} \right)^{n/q} \to 0 ,
\] as \( n \to \infty \).
\] (2.20)
Combining (2.17)–(2.20), we conclude for the first summand (2.14)
\[
\mathbb{E} \left[ X_n^2 \right] \leq M^{p(n)} \cdot |c_\theta|^{q(n)} \to 0
\]
\[
\iff M^{1/p} \cdot |c_\theta|^{1/q} < 1 \iff p > \frac{\ln(|c_\theta|) - \ln(M)}{\ln(|c_\theta|)} > 1.
\]
Next, we treat (2.15). It is easy to see that if \( \beta_n > 0 \) (i.e. \( \lambda < 0, \mu > 0 \)), then
\[
\mathbb{P}\left\{ Z_k(\Delta t) \leq 0 \right\} = \mathbb{P}\left\{ V_k^H \leq -\alpha_k/\beta_k = -C_{\lambda,\mu,\Delta t}(n) \right\} = \mathbb{P}\left\{ V_k^H \geq C_{\lambda,\mu,\Delta t}(n) \right\},
\]
and if \( \beta_n < 0 \) (i.e. \( \lambda < 0, \mu < 0 \)), then
\[
\mathbb{P}\left\{ Z_k(\Delta t) \leq 0 \right\} = \mathbb{P}\left\{ V_k^H \geq -\alpha_k/\beta_k = -C_{\lambda,\mu,\Delta t}(n) \right\} = \mathbb{P}\left\{ V_k^H \leq C_{\lambda,\mu,\Delta t}(n) \right\},
\]
where \( C_{\lambda,\mu,\Delta t}(n) := \frac{1}{\mu} \lambda n^{e-1} \Delta t^k \). Consequently, we have by the classical concentration inequality for normal variable \( V_k^H \)
\[
\mathbb{P}\left\{ Z_k(\Delta t) \leq 0 \right\} \geq \mathbb{P}\left\{ |V_k^H| \leq |C_{\lambda,\mu,\Delta t}(n)| \right\} \geq 1 - 2 \exp\left( -\frac{|C_{\lambda,\mu,\Delta t}(n)|^2}{2(\Delta t)^{2H}} \right). \tag{2.21}
\]
Then, by the Gaussian correlation inequality (Lemma 2.2), we can get
\[
\mathbb{P}\left\{ Z_k(\Delta t) \leq 0 : p(n) + 1 \leq k \leq n \right\} \geq \mathbb{P}\left\{ |V_k^H| \leq |C_{\lambda,\mu,\Delta t}(n)| : p(n) + 1 \leq k \leq n \right\}
\]
\[
\geq \prod_{k=p(n)+1}^{n} \mathbb{P}\left\{ |V_k^H| \leq |C_{\lambda,\mu,\Delta t}(n)| \right\}
\]
\[
\geq \prod_{k=p(n)+1}^{n} \left( 1 - 2 \exp\left( -\frac{|C_{\lambda,\mu,\Delta t}(n)|^2}{2(\Delta t)^{2H}} \right) \right). \tag{2.22}
\]
Denote
\[
\tilde{X}_n := 1 - \prod_{k=p(n)+1}^{n} \mathbb{I}\{ Z_k(\Delta t) \leq 0 : p(n) + 1 \leq k \leq n \}.
\]
By the Weierstrass product inequality:
\[
\prod_{i=1}^{n} (1 - x_i) > 1 - \sum_{i=1}^{n} x_i, \quad \forall \quad x_1, \cdots, x_n \in (0, 1),
\]
we have
\[
\mathbb{E} \left[ \tilde{X}_n \right] \leq 1 - \prod_{k=p(n)+1}^{n} \left( 1 - 2 \exp\left( -\frac{|C_{\lambda,\mu,\Delta t}(n)|^2}{2(\Delta t)^{2H}} \right) \right)
\]
\[
\leq 2 \sum_{k=p(n)+1}^{n} \exp\left( -\frac{|C_{\lambda,\mu,\Delta t}(n)|^2}{2(\Delta t)^{2H}} \right) \lesssim \exp\left( -\frac{|C_{\lambda,\mu,\Delta t}(p(n))|^2}{2(\Delta t)^{2H}} \right), \tag{2.23}
\]
since when \( n \) is sufficiently large that \( 2 \exp\left[ -\frac{|C_{\lambda,\mu,\Delta t}(n)|^2}{2(\Delta t)^{2H}} \right] < 1 \). Because \( \tilde{X}_n \) is either 0 or 1, i.e. \( \tilde{X}_n^2 = \tilde{X}_n \), we have for the second summand (2.15)
\[
\mathbb{E}\left[ \prod_{k=1}^{n} (Z_k(\Delta t))^2 \cdot \tilde{X}_n \right] \leq \left( \mathbb{E}\left[ \prod_{k=1}^{n} (Z_k(\Delta t))^4 \right] \right)^{\frac{1}{4}} \cdot \left( \mathbb{E}(\tilde{X}_n) \right)^{\frac{1}{2}}
\]
\[
\leq C M^{2n} \cdot \exp\left( -\frac{|C_{\lambda,\mu,\Delta t}(p(n))|^2}{2(\Delta t)^{2H}} \right) \leq C M^{2n} \exp\left[ -\frac{\lambda^2}{\mu^2} p(n)^2(\Delta t)^{2\kappa - 2H} \right].
\]
Here we applied \( \mathbb{E} \left[ \prod_{k=1}^{n} (Z_k(\Delta t))^4 \right] \leq M^4 n \) for some constant \( M > 1 \), which can be proved analogously as in the proof of part (i) of the theorem. Hence, it is easy to see if \( \kappa > 3/2 \), \( p(n)^{1/(\kappa-1)} \geq C_p \cdot n^{2/(\kappa-1)} \gg n \), then the above term converges to 0. 

\[ \blacksquare \]

2.4. **The case of** \( 0 < \theta < \frac{1}{2} \). In this subsection we prove part (iii) of Theorem 1.1. First, we state the following strong law of large numbers (SLLN).

**Lemma 2.3.** [11, Theorem 1]

Let \( \xi_1, \xi_2, \ldots, \xi_n \) be a sequence of square-integrable random variables and suppose that there exists a sequence of constants \( R_k \) such that

\[
\sup_{n \geq 1} \left| \text{Cov}(\xi_n, \xi_{n+k}) \right| \leq R_k, \quad k \geq 1, \quad \sum_{k=1}^{\infty} \frac{R_k}{k^q} < \infty \quad \text{for some } 0 \leq q < 1, \quad (2.24)
\]

and

\[
\sum_{k=1}^{\infty} \frac{\mathbb{V}(\xi_n) \cdot (\log(n))^2}{n^2} < \infty, \quad (2.25)
\]

then the SLLN holds. More precisely, letting \( S_n = \sum_{i=0}^{n} \xi_i \), one has

\[
\lim_{n \to \infty} \frac{S_n - \mathbb{E}(S_n)}{n} = 0 \quad \text{almost surely.} \quad (2.26)
\]

With the help of this lemma we now give the proof of the last part of the theorem.

**Proof of part (iii) of Theorem 1.1**

Denote \( Y_0 = \ln X_0^2, Y_k = \ln (\alpha_k + \beta_k V_k^H)^2 \) and \( S_n = \sum_{k=0}^{n} Y_k \).

In the above definition if \( \alpha_k + \beta_k V_k^H = 0 \), then we put \( Y_k := 0 \). Notice that \((\alpha_k + \beta_k V_k^H)^2\) is positive almost surely, so \( Y_k \) are well defined for \( k \geq 0 \). We shall apply Lemma 2.3 to \( \xi_n = Y_n \). It is easier to verify that (2.25) holds. The main objective is to verify the conditions in (2.24). For \( q \in (2H - 1, 1) \), the second condition of (2.24) holds if \( R_k \gg k^{2H-2} \) for sufficiently large \( k \). Thus, the proof of part (iii) in Theorem 1.1 is completed if we can show for some constant \( C \)

\[
\sup_{n \geq 1} \left| \text{Cov}(Y_n, Y_{n+k}) \right| \leq R_k \leq C \cdot |k|^{2H-2}. \quad (2.27)
\]

In fact, assume (2.27) and recall that if \( 0 < \theta < \frac{1}{2} \), then

\[
\lim_{n \to \infty} \frac{1}{n} \mathbb{E}(S_n) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{E} \left[ \ln(\alpha_k + \beta_k V_k^H)^2 \right] = C \cdot \ln \left( \frac{1 - \theta}{\theta} \right)^2 =: C_\theta > 0.
\]

Therefore, by Lemma 2.3 with \( q \in (2H - 1, 1) \), we get

\[
\frac{S_n}{n} = \frac{\ln(X_n)^2}{n} \xrightarrow{a.s.} C_\theta > 0.
\]

This implies \( (X_n)^2 \xrightarrow{a.s.} \infty \) almost surely. Consequently, by Fatou’s Lemma, one has

\[
\lim_{n \to \infty} \mathbb{E} \left[ (X_n)^2 \right] \geq \mathbb{E} \left[ \lim_{n \to \infty} (X_n)^2 \right] = \infty,
\]

which completes the proof of part (iii) of the theorem.

So, it suffices to show (2.27). We shall show that \( |\text{Cov}(Y_i, Y_j)| \leq i - j |2H-2| \) as \( i - j \to \infty \) which is obviously equivalent to (2.27). In fact,

\[
\text{Cov}(Y_i, Y_j) = \mathbb{E} \left[ \ln(\alpha_i + \beta_i V_i^H)^2 \ln(\alpha_j + \beta_j V_j^H)^2 \right] - \mathbb{E} \left[ \ln(\alpha_i + \beta_i V_i^H)^2 \right] \mathbb{E} \left[ \ln(\alpha_j + \beta_j V_j^H)^2 \right].
\]
Denote the probability densities of normal variables $V_i^H$ and $V_j^H$ by $f_i(x)$ and $f_j(y)$, and denote the corresponding cumulative distributions by $F_i(x)$ and $F_j(y)$ respectively. The symmetric covariance matrix of $V_i^H$ and $V_j^H$ is given by

$$
\Sigma = \begin{pmatrix}
\sigma_i^2, & \rho_i \sigma_i \sigma_j \\
\rho_i \sigma_i \sigma_j, & \sigma_j^2
\end{pmatrix},
$$

where $\sigma_i := \sqrt{\mathbb{E}[(V_i^H)^2]} = |t_{i+1} - t_i|^H$ and $\sigma_j := \sqrt{\mathbb{E}[(V_j^H)^2]} = |t_{j+1} - t_j|^H$ are standard deviations of $V_i^H$ and $V_j^H$, $\rho_{ij} := \frac{\mathbb{E}[V_i^HV_j^H]}{\sigma_i \sigma_j}$ is the correlation coefficient between $V_i^H$ and $V_j^H$. Their joint distribution has the following form

$$
f_{i,j}(x,y) = \frac{1}{\sqrt{(2\pi)^2|\Sigma|}} \exp \left( \frac{-\frac{1}{2}(x^T \Sigma^{-1} x)}{2} \right),
$$

with $X = [x,y]^T$. Without loss of generality, we can assume that $i \geq j + 1$. Then we have using the joint density (2.28):

$$
\text{Cov}(Y_i, Y_j) = \int_{\mathbb{R}^2} \left[ \ln(\alpha_i + \beta_i x)^2 \ln(\alpha_j + \beta_j y)^2 \right] \cdot [f_{i,j}(x,y) - f_i(x)f_j(y)] \, dx \, dy
$$

$$
= \int_{\mathbb{R}^2} \left[ \ln(\alpha_i + \beta_i x)^2 \ln(\alpha_j + \beta_j y)^2 \right] \cdot \exp \left( -\frac{\bar{\rho}_{ij}}{2} \left[ \frac{x^2}{\sigma_i^2} + \frac{y^2}{\sigma_j^2} \right] \right)
$$

$$
\times \left[ -\exp \left( \frac{\bar{\rho}_{ij}}{2} \left[ \frac{x^2}{\sigma_i^2} + \frac{y^2}{\sigma_j^2} \right] \right) + \frac{1}{\sqrt{1 - \rho_{ij}^2}} \exp \left( \rho_{ij} \frac{xy}{\sigma_i \sigma_j} \right) \right] \, dF_i(x) \, dF_j(y),
$$

where $\bar{\rho}_{ij} = \frac{\rho_{ij}}{1 - \rho_{ij}^2}$, $\bar{\rho}_{ij} = \frac{\rho_{ij}}{1 - \rho_{ij}^2}$. By the Hölder inequality, $\text{Cov}(Y_i, Y_j)$ can be bounded by

$$
\left( \int_{\mathbb{R}^2} \left[ \ln(\alpha_i + \beta_i x)^2 \ln(\alpha_j + \beta_j y)^2 \right]^2 \, dF_i(x) \, dF_j(y) \right)^{\frac{1}{2}} \times \left( \int_{\mathbb{R}^2} \exp \left( \frac{\bar{\rho}_{ij}}{2} \left[ \frac{x^2}{\sigma_i^2} + \frac{y^2}{\sigma_j^2} \right] \right) \right)^{\frac{1}{2}} =: A_{ij} \times B_{ij}.
$$

We proceed to estimate $A_{ij}$ and $B_{ij}$. To estimate $A_{ij}$ we only need to consider

$$
\int_{\mathbb{R}} \left[ \ln(\alpha + \beta x)^2 \right]^2 \times \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( -\frac{x^2}{2\sigma^2} \right) \, dx
$$

$$
= \int_{|\alpha + \beta \sigma x| \leq 1} \left[ \ln(\alpha + \beta \sigma x)^2 \right]^2 \times e^{-\frac{\beta^2}{2\sigma^2}} \, dx + \int_{|\alpha + \beta \sigma x| \geq 1} \left[ \ln(\alpha + \beta \sigma x)^2 \right]^2 \times e^{-\frac{\beta^2}{2\sigma^2}} \, dx
$$

$$
\leq \int_{|\alpha + \beta \sigma x| \leq 1} \left[ \ln(\alpha + \beta \sigma x)^2 \right]^2 \, dx + \int_{|\alpha + \beta \sigma x| \geq 1} [\alpha + \beta \sigma x]^4 \times e^{-\frac{\beta^2}{2\sigma^2}} \, dx.
$$
Here, we neglect the subscripts of $\alpha_i$ and $\beta_i$ to simplify the notations. Obviously, there exists a constant $C$ such that $A_{ij} \leq C$ since $\sigma = |\Delta t|^H$ and $\alpha$ and $\beta$ defined by (2.1)-(2.2) are bounded above and bounded below away from 0. Next, for $|i - j| \to \infty$, we deal with $B_{ij}$:

$B_{ij} = \int_{\mathbb{R}^2} \left[ \exp \left( \frac{\hat{\rho}_{ij}}{2} \left[ \frac{x^2}{\sigma_i^2} + \frac{y^2}{\sigma_j^2} \right] \right) - \frac{1}{\sqrt{1 - \rho_{ij}^2}} \exp \left( \frac{\hat{\rho}_{ij} \cdot xy}{\sigma_i \sigma_j} \right) \right]^2 \times \frac{1}{2\pi \sigma_i \sigma_j} \exp \left( -\frac{1}{2} \left[ \frac{x^2}{\sigma_i^2} + \frac{y^2}{\sigma_j^2} \right] \right) dxdy.$

By variable substitutions $x \to \sqrt{\sigma_i}x, y \to \sqrt{\sigma_j}y$, we have

$B_{ij} \leq C \int_{\mathbb{R}^2} \left[ \exp \left( \hat{\rho}_{ij} \left[ x^2 + y^2 \right] \right) + \frac{1}{\sqrt{1 - \rho_{ij}^2}} \exp \left( 2\hat{\rho}_{ij} \cdot xy \right) \right] \times \exp \left( -\left[ x^2 + y^2 \right] \right) dxdy$

$= C \int_{\mathbb{R}^2} \left[ \exp \left( 2\hat{\rho}_{ij} \left[ x^2 + y^2 \right] \right) + \frac{1}{1 - \rho_{ij}^2} \exp \left( 4\hat{\rho}_{ij} \cdot xy \right) \right. \left. - \frac{2}{\sqrt{1 - \rho_{ij}^2}} \exp \left( \hat{\rho}_{ij} \left[ x^2 + y^2 \right] + 2\hat{\rho}_{ij}xy \right) \right] \times \exp \left( -\left[ x^2 + y^2 \right] \right) dxdy.$

The above three integrals can be explicitly evaluated as follows:

$\int_{\mathbb{R}^2} \exp \left( 2\hat{\rho}_{ij} \left[ x^2 + y^2 \right] \right) \times \exp \left( -\left[ x^2 + y^2 \right] \right) dxdy = \frac{\pi}{1 - 2\rho_{ij}},$

$\frac{1}{1 - \rho_{ij}^2} \int_{\mathbb{R}^2} \exp \left( 4\hat{\rho}_{ij} \cdot xy \right) \times \exp \left( -\left[ x^2 + y^2 \right] \right) dxdy = \frac{\pi}{\sqrt{1 - 4\rho_{ij}^2}},$

and

$\frac{2}{\sqrt{1 - \rho_{ij}^2}} \int_{\mathbb{R}^2} \exp \left( \hat{\rho}_{ij} \left[ x^2 + y^2 \right] + 2\hat{\rho}_{ij}xy \right) \times \exp \left( -\left[ x^2 + y^2 \right] \right) dxdy = \frac{2\pi}{\sqrt{1 - \rho_{ij}^2} \left[ 1 - \rho_{ij}^2 \right]^{\frac{3}{2}}}.$

Thus to bound $B_{ij}$ we need to know the asymptotics of $\rho_{ij}$ and $\hat{\rho}_{ij}$. First, there exists a constant $C_H$ such that

$\mathbb{E}[V^H_iV^H_j] = \mathbb{E} \left[ (B^H(t_{i+1}) - B^H(t_i))(B^H(t_{j+1}) - B^H(t_j)) \right]$

$= \frac{1}{2} \left[ (t_{i+1} - t_j)^{2H} - (t_{i+1} - t_{j+1})^{2H} - (t_i - t_j)^{2H} + (t_i - t_{j+1})^{2H} \right]$

$= \frac{(\Delta t)^{2H}}{2} \left[ (i - j + 1)^{2H} - 2(i - j)^{2H} + (i - j - 1)^{2H} \right]$

$\leq \frac{(\Delta t)^{2H}}{2H(2H-1)} \left[ (i - j - 1)^{2H-2} \wedge 1 \right] \leq C_H \cdot (\Delta t)^{2H} \left[ (i - j)^{2H-2} \wedge 1 \right].$

Hence,

$0 \leq \rho_{ij} = \frac{\mathbb{E}[V^H_iV^H_j]}{\sigma_i \sigma_j} \leq C_H \left[ (i - j)^{2H-2} \wedge 1 \right] \quad \text{as} \quad |i - j| \to \infty.$
Consequently,
\[ \hat{p}_{ij} \leq C \left[ i - j \right]^{2H-2} \land 1 \], \quad \hat{p}_{ij} \leq C \left[ i - j \right]^{4H-4} \land 1 \quad \text{as } |i - j| \to \infty. \quad (2.31)

Therefore, by the relation \((1 + x)^{\alpha} - 1 \sim \alpha x \text{ as } x \to 0\), we have

\[
B_{ij} \leq C \left[ \frac{\pi}{1 - 2\tilde{p}_{ij}} + \frac{1}{1 - \tilde{p}_{ij}^2} \right] \sqrt{1 - 4\tilde{p}_{ij}^2} \frac{2}{\sqrt{\tilde{p}_{ij}^2 - \tilde{p}_{ij}^2}} \leq C \left[ 2\tilde{p}_{ij} - \frac{1}{2} \tilde{p}_{ij}^3 \right] + \frac{1}{2} \left( \tilde{p}_{ij}^2 + 4\tilde{p}_{ij}^2 \tilde{p}_{ij}^2 - 4\tilde{p}_{ij}^2 \tilde{p}_{ij}^2 - \tilde{p}_{ij} \right) \right]
\]

\[
\leq C \left[ \frac{\pi}{1 - 2\tilde{p}_{ij}} \right] \sqrt{1 - 4\tilde{p}_{ij}^2} \frac{2}{\sqrt{\tilde{p}_{ij}^2 - \tilde{p}_{ij}^2}} \leq C \left[ |i - j|^{4H-4} \right],
\]

when \(|i - j|\) is sufficiently large. As a result, we have

\[
\text{Cov}(Y_i, Y_j) \leq C \left[ |i - j|^{2H-2} \land 1 \right]. \quad (2.32)
\]

This completes the proof of (2.27) and hence we finish the proof of part (iii) of Theorem 1.1.

\[ 14 \]

### 2.5. Brownian motion case

In this section we consider the case when \(H = 1/2\), namely, Brownian motion \(B^H = B\). The equation (1.6) becomes

\[
dX(t) = -\lambda \kappa \theta \kappa^{-1} X(t) dt + \mu X(t) dB(t), \quad X(0) = X_0, \quad (2.33)
\]

where \(B\) is standard Brownian motion, \(\lambda, \mu \in \mathbb{R}\) and \(\kappa \geq 2H = 1\). Here, we assume that \(X_0 \neq 0\) with a positive probability and \(|X_0|\) is square integrable. We have \(X(t) = X_0 \exp(\lambda t^\kappa + \mu B(t))\) and

\[
\text{E} |X(t)|^2 = \text{E} |X_0|^2 \exp \left( 2(-\lambda t^\kappa + \mu^2 t) \right). \quad (2.34)
\]

So the solution is stable if (i) \(\kappa > 1\) and \(\lambda > 0\) or (ii) \(\kappa = 1\) and \(-\lambda + |\mu| < 0\). Otherwise, the solution of (2.33) is unstable.

For the above equation we can write (1.14) as

\[
X_{n+1} = \left( \frac{1 - \kappa(1 - \theta)\lambda (t_n)^{k-1} \Delta t}{1 + \kappa \theta \lambda (t_{n+1})^{k-1} \Delta t} + \frac{\mu V_n}{1 + \kappa \theta \lambda (t_{n+1})^{k-1} \Delta t} \right) X_n, \quad (2.35)
\]

with \(t_n = n \cdot \Delta t\) and \(V_n = B(t_{n+1}) - B(t_n)\). Notice that \(V_n\)’s are mutual independent. The equation (2.35) can also be rewritten as follows

\[
X_{n+1} = X_0 \prod_{k=1}^{n} Z_k(\Delta t) = X_0 \prod_{k=1}^{n} \left( \alpha_k + \beta_k V_k \right). \quad (2.36)
\]

where

\[
\left\{
\begin{array}{l}
\alpha_n := \alpha_n(\theta, \lambda, \Delta t) = \frac{1 - \kappa(1 - \theta)\lambda (t_n)^{k-1} \Delta t}{1 + \kappa \theta \lambda (t_{n+1})^{k-1} \Delta t} \quad = \frac{1 - \kappa(1 - \theta)\lambda n^{k-1} \Delta t^k}{1 + \kappa \theta \lambda n^{k-1} \Delta t^k}, \\
\beta_n := \beta_n(\theta, \lambda, \mu, \Delta t) = \frac{\mu}{1 + \kappa \theta \lambda (t_{n+1})^{k-1} \Delta t^k} \quad = \frac{\mu}{1 + \kappa \theta \lambda (n+1)^{k-1} \Delta t^k}.
\end{array}
\right.
\]

\[ 14 \]
Obviously, we have the following.

- If $\kappa > 1$, for every fixed $\Delta t > 0$, $\lambda$ and $\mu$ (even for $\lambda > 0$)
  \[
  \lim_{n \to \infty} \alpha_n = -\frac{1 - \theta}{\theta}, \quad \lim_{n \to \infty} \beta_n = 0.
  \]

Therefore, we have

\[
\mathbb{E}[|X_{n+1}|^2] = \mathbb{E}[|X_0|^2] \prod_{k=1}^n \mathbb{E}[|\alpha_k + \beta_k V_k|^2]
\]

\[
= \mathbb{E}[|X_0|^2] \prod_{k=1}^n \left[ \alpha_k^2 + \beta_k^2 \cdot \Delta t \right] = \left( \frac{1 - \theta}{\theta} \right)^{2n} \rightarrow \begin{cases} 
0, & \text{if } \frac{1}{2} < \theta \leq 1; \\
\infty, & \text{if } 0 \leq \theta < \frac{1}{2}.
\end{cases}
\]

- If $\kappa = 1$, (2.33) is reduced to the standard stochastic test equation (see also [17]).
  Then
  \[
  \alpha_n = \bar{\alpha} = \frac{1 - (1 - \theta)\lambda \Delta t}{1 + \theta \lambda \Delta t}, \quad \beta_n = \bar{\beta} = \frac{-\mu}{1 + \theta \lambda \Delta t}.
  \]

Thus,

\[
\mathbb{E}[|X_{n+1}|^2] = \mathbb{E}(\bar{\alpha} \cdot \bar{\beta} \cdot V_n)^2 \mathbb{E}|X_n|^2.
\]

In this sense, the numerical stability (or non-stability) depends on the condition

\[
\bar{\alpha}^2 + \bar{\beta}^2 \cdot \Delta t < 1 \quad \text{(or} \quad > 1),
\]

\[
\iff (1 - 2\theta)\lambda^2 \Delta t + (-2\lambda + |\mu|^2) < 0 \quad \text{(or} \quad > 0).
\]

Now, we can summarize the discussion above as the following proposition:

**Proposition 2.1.** For the test equation (2.33) and the STM (2.35), we have

(i) When $\kappa > 1$, for any fixed $\lambda$, $\mu$, then the STM (2.35) is mean square stable for the test equation (2.33) if $\frac{1}{2} < \theta \leq 1$ and is not mean square stable if $0 \leq \theta < \frac{1}{2}$;

(ii) When $\kappa = 1$ and $-2\lambda + |\mu|^2 < 0$, then the STM (2.35) is mean square stable for the test equation (2.33) if either $\frac{1}{2} \leq \theta \leq 1$ for all $\Delta t > 0$ or $0 \leq \theta < \frac{1}{2}$ for $\Delta t$ satisfying

\[
0 < \Delta t < \frac{2\lambda - |\mu|^2}{(1 - 2\theta)\lambda^2}.
\]

(iii) When $\kappa = 1$, $-2\lambda + |\mu|^2 > 0$ and $0 \leq \theta < 1/2$, then the STM (2.35) is not mean square stable for the test equation (2.33) for all $\Delta t > 0$.

3. STM: Mean square nonlinear stability analysis

In this section, we shall study the $p$-th moments stability and the numerical stability of the solution to the general SDEs driven by fBm.

3.1. **Mean square stability.** The existence and uniqueness problems of (1.9) have been studied extensively in the last two decades. For precise results, we refer [3] and the references therein. Beyond the well-posedness, as we mentioned in Section 1, it seems too complicated to find the long time asymptotic behavior of (1.9). To the best of our knowledge, there is few results on the convergence of $\mathbb{E}[|X(t)|^2]$ when $t$ goes to infinity. Thus, we focus on the following simplified SDEs with $g(t, X(t)) = c(t)X(t)$ under the assumption (1.11) and (1.12)

\[
dX(t) = f(t, X(t))dt + c(t)X(t)dB^H(t).
\]

In this case, (1.12) means $|c(t)| \leq \mu$ for some $\mu > 0$. 

Theorem 3.1. Let $X(t)$ be the solution to SDE (3.1), and let $p$, $\kappa$ and $\lambda$, $\mu$ satisfy
\begin{align}
(i) \quad \kappa > 2H \text{ and } \lambda > 0 \quad \text{ or } \quad (ii) \quad \kappa = 2H \text{ and } -\lambda + \frac{p}{2} \mu^2 < 0. \tag{3.2}
\end{align}

If $f(t,x)$ in (3.1) satisfies (1.10) in Assumption 7 and $c(t)$ satisfies $|c(t)| \leq \mu$ for $\mu > 0$. Then for any $p \geq 1$, $\mathbb{E}|X(t)|^p \to 0$ as $t \to \infty$ under the condition (1.8).

Remark 3.1. When $p = 2$, i.e. the mean square stable case, the condition (3.2) coincides with (1.8).

Proof. We can assume that $p \geq 2$ is an even positive number. Denote $F_t = \exp \left[- \int_0^t c(s) dB^H_s \right]$ and $Y_t = F_t \cdot X(t)$. Then by the chain rule formula (e.g. [12, Proposition 2.7] or [19, Lemma 2.7.1]) we have
\[
\frac{d}{dt} Y_t = F_t \cdot f(t, X(t)) = F_t \cdot f(t, (F_t)^{-1} Y_t).
\]

Note that it is a deterministic ordinary differential equation for the function $t \to Y_t(\omega)$ for every $\omega \in \Omega$. Then by the condition (1.10) in Assumption 1, we get
\[
\frac{d}{dt} Y_t^p = p Y_t^{p-1} \frac{d}{dt} Y_t = p Y_t^{p-1} (F_t)^{-1} F_t^2 f(t, (F_t)^{-1} Y_t) \leq -p \lambda \kappa^2 Y_t^p.
\]

Thanks to Gronwall’s inequality, we have
\[
Y_t^p \leq Y_0^p \exp \left( -p \lambda \int_0^t \kappa^2 ds \right) = X_0^p \exp(-p \lambda t^2),
\]

and
\[
X(t)^p \leq X_0^p \exp \left( -p \lambda t^2 + p \int_0^t c(s) dB^H_s \right).
\]

Therefore, letting $C_H = H(2H - 1)$, we have
\[
\mathbb{E}|X(t)|^p \leq C \exp \left( -p \lambda t^2 + p^2 C_H \int_0^t \int_0^s c(r) \cdot (s - r)^{2H - 2} c(r) dr ds \right) \leq C \exp \left( -p \lambda t^2 + \frac{p^2}{2} \mu^2 t^{2H} \right).
\]

So we have that $\mathbb{E}|X(t)|^p$ converges to 0 under the condition (3.2). \hfill \Box

3.2. Numerical stability. For the numerical stability of SDEs driven by fBm, we consider a more general diffusion coefficient. More precisely, instead of $g(t, X) = c(t) X$ we allow the diffusion term $g$ to be generally nonlinear satisfying (1.12). We hope this will shed light to the stability of the original solution.

Now, we give the proof of Theorem 1.2.

Proof of Theorem 1.2. From STM (1.13), we have
\[
X_{n+1} - \theta f(t_n, X_n) \Delta t = X_n + (1 - \theta) f(t_n, X_n) \Delta t + g(t_n, X_n) V_n^H. \tag{3.3}
\]

By the condition (1.10), we have
\[
|f(t, X)|^2 \geq (\lambda \kappa^2)^2 |X|^2.
\]
We bound the square of left hand side of (3.3) as

\[
|X_{n+1} - \theta f(t_{n+1}, X_{n+1}) \Delta t|^2 \\
= (X_{n+1})^2 + \theta^2 |f(t_{n+1}, X_{n+1})|^2 (\Delta t)^2 - 2\theta \Delta t X_{n+1} f(t_{n+1}, X_{n+1}) \\
\geq (X_{n+1})^2 + (\theta \lambda \cdot \kappa(t_{n+1}) \Delta t)^2 (X_{n+1})^2 + 2\theta \lambda \cdot \kappa(t_{n+1}) \Delta t (X_{n+1})^2
\]

(3.4)

Therefore, we have from (3.4) and (3.5)

Thus, we get that

Thus, we get that

\[
\text{Step 1 (} \theta = 1\text{): With the condition (1.12), it is clear that the square of right hand side of (3.3) can be bounded by}
\]

\[
|X_n + \theta (t_n, X_n) V_n^H|^2 \leq 2 \left( (X_{n+1})^2 + \mu^2 X_n^2 (V_n^H)^2 \right).
\]

(3.5)

Therefore, we have from (3.4) and (3.5)

\[
|X_n|^2 \leq 2 \left( \alpha_n^2 + \beta_n^2 (V_n^H)^2 \right) |X_{n-1}|^2 \leq 2^n \prod_{j=1}^{n} \left( \alpha_j^2 + \beta_j^2 (V_j^H)^2 \right) X_0^2,
\]

(3.6)

where

\[
\alpha_n = \frac{1}{1 + \lambda \cdot \kappa(t_n) \Delta t}, \quad \beta_n = \frac{\mu}{1 + \lambda \cdot \kappa(t_n) \Delta t}.
\]

Let us rewrite \( \prod_{j=1}^{2n} \left[ \alpha_j^2 + \beta_j^2 (V_j^H)^2 \right] \) in (3.6) as

\[
\prod_{j=1}^{2n} Z_j = \prod_{j=1}^{n} (\beta_j V_j^H + \iota \alpha_j)(\beta_j V_j^H - \iota \alpha_j),
\]

with \( \iota \) being the imaginary number and

\[
Z_{2j-1} = \beta_j V_j^H + \iota \alpha_j, \quad Z_{2j} = \beta_j V_j^H - \iota \alpha_j.
\]

Applying Lemma 2.1 with \( s_1 = \cdots = s_{2n} = 1, s = \sum_{j=1}^{2n} s_j = 2n, \) one has

\[
\prod_{j=1}^{2n} Z_j = \frac{1}{2^n} \frac{\iota}{v_1} \cdots \frac{\iota}{v_{2n}} \left( 1 \over v_1 \right) \cdots \left( 1 \over v_{2n} \right) \left( -1 \right)^{\Sigma_{j=0}^{2n-1} v_j} \left[ \sum_{j=1}^{2n} h_j Z_j \right]^{2n},
\]

(3.7)

where \( h_j = \frac{1}{2} - \frac{v_j}{2} = \frac{(-1)^j v_j}{2} \). Note that

\[
\sum_{j=1}^{2n} h_j Z_j = \sum_{j \text{ odd}} h_j (\beta_j V_j^H + \iota \alpha_j) + \sum_{j \text{ even}} h_j (\beta_j V_j^H - \iota \alpha_j)
\]

\[
= \sum_{j=1}^{2n} h_j \beta_j V_j^H + \iota \left( \sum_{j \text{ odd}} h_j \alpha_j - \sum_{j \text{ even}} h_j \alpha_j \right).
\]

Thus, we get that

\[
\left| \sum_{j=1}^{2n} h_j Z_j \right|^{2n} \leq \left[ \left( \sum_{j=1}^{2n} h_j \beta_j V_j^H \right)^2 + \left( \sum_{j \text{ odd}} h_j \alpha_j - \sum_{j \text{ even}} h_j \alpha_j \right)^2 \right]^{2n}.
\]
Therefore, taking expectation on both sides of (3.6), we obtain
\[
E[|X_n|^2] \leq 2^n \cdot E \left[ \prod_{j=1}^{2n} Z_j \right]
\]
\[
\leq \frac{2^n}{2n!} \sum_{v_1=0}^{2n} \sum_{v_2n=0}^{2n} \left( \frac{1}{v_1} \right) \ldots \left( \frac{1}{v_2n} \right) \sum_{j=1}^{2n} h_j Z_j
\]
\[
\leq \frac{2^n}{2n!} \sum_{v_1=0}^{2n} \sum_{v_2n=0}^{2n} \frac{1}{2n} \left[ E \left( \sum_{j=1}^{2n} (-1)^j \beta_j \cdot V_j^H \right)^2 + \left( \sum_{j=0}^{2n} \alpha_j \right)^2 \right]
\]
\[
= I_1 + I_2.
\]

Denote \( R = \sum_{j=1}^{2n} (-1)^j \beta_j \cdot V_j^H \). Then \( R \) is a Gaussian random variable with mean zero and variance \( \sigma_R^2 \) given by
\[
\sigma_R^2 = E \left( \sum_{j=1}^{2n} (-1)^j \beta_j \cdot V_j^H \right)^2 \leq C \cdot n^{2H-2\kappa}
\]
(see the computation in the appendix A). Thus, we have
\[
E \left( \sum_{j=1}^{2n} (-1)^j \beta_j \cdot V_j^H \right)^2 = 2^n \Gamma(n+1/2) \cdot (\sigma_R)^{2n}
\]
\[
\leq C^n \cdot 2^n \Gamma(n+1/2) \cdot n^{(2H-2\kappa)n}.
\]

By Stirling’s formula, we further have
\[
I_1 \leq \frac{2^n}{2n!} \cdot \frac{2^n}{2n} \cdot E \left( \sum_{j=1}^{2n} (-1)^j \beta_j \cdot V_j^H \right)^2
\]
\[
\leq \frac{4^n C^n}{2n!} \cdot \Gamma(n+1/2) \cdot n^{(2H-2\kappa)}
\]
\[
\approx \frac{1}{(2n)^{2n}} \cdot n^{(2H-2\kappa)} \cdot \left( \frac{n}{e} \right)^{n-\frac{1}{2}}
\]
\[
\approx C^n \cdot n^{(1+2H-2\kappa)} \to 0,
\]
as \( n \to \infty \) since \( \kappa \geq 2H > 1 \). For the term \( I_2 \), as \( n \to \infty \), we also have
\[
I_2 \leq \frac{2^n}{2n!} \cdot \frac{2^n}{2n+1} \left( \sum_{j=1}^{2n} \alpha_j \right)^2 \leq \frac{C^n \cdot n^{(2-\kappa)n}}{\sqrt{2\pi(2n)} \cdot (2n)^{2n}} \approx C^n \cdot n^{(2-\kappa)n} \to 0.
\]

Step 2 (\( \theta < 1 \)): Similar to (3.5), it follows with additional condition (1.1) that the right hand side of (3.5) can be bounded by
\[
|X_n + (1 - \theta) f(t_n, X_n) \Delta t + g(t_n, V_n)|^2
\]
\[
\leq 3 \left( X_n^2 + |(1 - \theta) f(t_n, X_n) \Delta t |^2 + |g(t_n, V_n)|^2 \right)
\]
\[
\leq 3 \left( X_n^2 + (1 - \theta)^2 \lambda \cdot \kappa \cdot t_n^{\kappa-1} \Delta t)^2 \right) X_n^2 + \mu^2 X_n^2 (V_n^H)^2.
\]

Combining (3.4) and (3.8), we have
\[
[1 + \theta \lambda \cdot \kappa \cdot t_n^{\kappa-1} \Delta t]^2 (X_{n+1})^2 \leq 3 \left( 1 + (1 - \theta)^2 \lambda \cdot \kappa \cdot t_n^{\kappa-1} \Delta t \right) X_n^2 + \mu^2 (V_n^H)^2 (X_n)^2.
\]
We can the above inequality as
\[(X_n)^2 \leq 3 \left[ \alpha_n^2 + \beta_n^2 \cdot (V_n^H)^2 \right] (X_{n-1})^2,\]
where \(\alpha_n\) and \(\beta_n\) are given by
\[
\alpha_n^2 = \frac{1 + (1 - \theta)^2 (\lambda t_n^{\kappa - 1} \Delta t)^2}{[1 + \theta \lambda t_n^{\kappa - 1} \Delta t]^2}, \quad \beta_n^2 = \frac{\mu^2}{[1 + \theta \lambda t_n^{\kappa - 1} \Delta t]^2}. \tag{3.9}
\]
Therefore,
\[
X_n^2 \leq 3^n \prod_{j=1}^{n} (\alpha_j^2 + \beta_j^2 (V_j^H)^2) X_0^2
= 3^n \prod_{j=1}^{n} (\beta_j V_j^H + \iota \alpha_j)(\beta_j V_j^H - \iota \alpha_j). \tag{3.10}
\]
Thus by the same procedure as in Step 1 \((\theta = 1)\), taking expectation on both sides of (3.10) gives
\[
E[|X_n|^2] \leq 3^n \cdot E \left[ \prod_{j=1}^{2n} Z_j \right]
\leq \frac{3^n}{2n!} \cdot \frac{1}{2^n} \cdot \frac{1}{2^n} \cdot E \left[ \left( \sum_{j=1}^{2n} (-1)^j \beta_j \cdot V_j^H \right)^{2n} + \left( \sum_{j=1}^{2n} \alpha_j \right)^{2n} \right]
=: \tilde{I}_3 + \tilde{I}_4.
\]
We can prove that the term \(\tilde{I}_3\) converges to 0 by the same technique used for the term \(\tilde{I}_1\) in Step 1 \((\theta = 1)\). For the term \(\tilde{I}_4\), we further have
\[
\tilde{I}_4 \leq \frac{3^n}{(2n)!} \cdot \frac{2^{2n}}{2n+1} \left( \sum_{j=1}^{2n} \alpha_j \right)^{2n}
\leq \frac{6^n}{\sqrt{2\pi} \cdot 2n^{\frac{2n}{e}} \cdot (2n)^{2n} \left( 1 - \theta \lambda \right)^{2n}}.
\]
Obviously, we should require that
\[
\sqrt{6} \cdot e \cdot \frac{1 - \theta \lambda}{\lambda} < 1 \tag{3.11}
\]
to ensure \(\tilde{I}_4 \to 0\) as \(n \to \infty\). The inequality (3.11) is equivalent to \(\theta > \frac{\sqrt{6} \cdot e \cdot \frac{1}{\lambda}}{\sqrt{6} \cdot e \cdot \frac{1}{\lambda + 1}} \approx 0.87\) since \(\tilde{\lambda} \geq \lambda\). This completes the proof of Theorem 1.2. \(\square\)

4. Numerical Experiments

We shall carry simulations for the following three equations.

**Example 4.1.** Consider the following linear SDEs driven by fBm
\[
dX(t) = \kappa \cdot \lambda \cdot t^{\kappa - 1} X(t) dt + \mu X(t) dB^H(t), \tag{4.1}
\]
with initial value \(X(0) = 3\).
Example 4.2. Consider the following nonlinear SDEs driven by fBm
\[ dX(t) = -\lambda \cdot \kappa \cdot t^{\kappa - 1}X(t) - X^3(t)dt + \mu X(t)dB_H(t), \] with initial value \( X(0) = 3 \).

Example 4.3. Consider the following nonlinear SDEs driven by fBm
\[ dX(t) = -\lambda \cdot \kappa \cdot t^{\kappa - 1}X(t) - X^3(t)dt + (X(t) + \sin(X(t)))dB_H(t), \] with initial value \( X(0) = 3 \).

For the test for Example 4.2, we choose \( \lambda = 3, \kappa = 2, \mu = 4 \) and \( H = 0.6 \). It is easy to verify that the coefficients of the equation satisfy (1.10) and (1.12) in Assumption 1. We take \( \theta = 1 \) and \( \Delta t = 0.5 \) and 1, respectively. The mean square of the numerical solutions are displayed in Figure 5. As expected, the stable behavior of the numerical solution is in agreement with Theorem 1.2.

Moreover, we consider a more general diffusion term \( g(t, X) = X(t) + \sin(X(t)) \) instead of \( g(t, X) = c(t)X(t) \) with \( c(t) \leq \mu \), in Example 4.3. We take \( \lambda = 3, \kappa = 2, \) and \( H = 0.8 \). Figure 6 depicts the mean square of the numerical solutions with \( \Delta t = 0.5 \) and 1.
numerical results illustrate that the STM with $\theta = 1$ is also stable in this case. We hope the numerical results can shed some light on the asymptotic property of the solution of the nonlinear equations $dX(t) = f(t, X(t))dt + g(t, X(t))dB^H(t)$.

5. CONCLUDING REMARKS

This work first focuses on the mean square stability of the stochastic theta method for the time non-homogeneous linear test equation driven by fBm,

$$dX(t) = \lambda \kappa^{\kappa-1}X(t)dt + \mu X(t)dB^H(t), \ X(0) = 3,$$

whose solution is stable in mean square sense. For $\kappa \geq 2H$, it is proved that the mean square A-stability of STM (1.14) is achieved for $\theta \geq \sqrt{\frac{3/2}{2e+1}}$, and the stochastic theta method cannot preserve the stability property of the test equation for $\theta < 0.5$ in the sense of almost surely and mean square. Moreover, if $\kappa > 3/2$ and $\frac{1}{2} < \theta \leq 1$, then the STM (1.14) is mean square stable for the above test equation. To illustrate our theoretical results, we give some simulation results for the equation (4.1) with $\lambda = -9, \mu = 2, \Delta t = 0.5$ with different $H$ and $\theta$. The simulation results coincide well with our theoretical claims. On the other hand, we currently are not able to use our methods to deal with the case $\frac{1}{2} \leq \theta < \sqrt{\frac{3/2}{2e+1}}$ when $2H \leq \kappa \leq 3/2$. In this case, we simulate the equation (4.1) with $\lambda = -9, \mu = 2, \kappa = 1.4$ to test the stability by applying the stochastic theta method with $\theta = 0.5, \Delta t = 0.5$ over the time interval $0 \leq t \leq 2^{13}$, the numerical results in Figure 4 show that the method is still stable for $\theta = \frac{1}{2}$. Thus, we conjecture that when $2H \leq \kappa \leq 3/2$
and \( \frac{1}{2} \leq \theta < \frac{3/2 e}{\sqrt{3/2 e + 1}} \), the stochastic theta method is still mean square stable and this is our future research. Finally, we also study the stability of the STM for nonlinear non-autonomous case

\[
dX(t) = f(t, X(t)) \, dt + g(t, X(t)) \, dB(t).
\]

Under some conditions on the coefficients \( f \) and \( g \), it is proved that the STM method is stable when \( \theta = 1 \). Moreover, under a stronger condition on the coefficient of drift term \( f \), the STM method is stable when \( \theta > \frac{\sqrt{6(\lambda + \tilde{\lambda})}}{\sqrt{6(\lambda + \tilde{\lambda}) + 1}} \) (where \( \lambda \) and \( \tilde{\lambda} \) are defined (1.10) and (1.11) in Assumption 1, respectively).

**Appendix A. Proof of** \( z^2 = \frac{m_n^2}{2\delta_n^2} \gg n \)

In what follows, we show that \( z^2 = \frac{m_n^2}{2\delta_n^2} \geq \frac{C n^2}{n^{2H}} \asymp n^{2H} \gg n \) as \( n \to \infty \). Note that

\[
\hat{\mu}_n := \hat{\mu}_n(v_1, \cdots, v_n) = \sum_{i=1}^n (1 - v_i) \cdot \mu_i = \sum_{i=1}^n (1 - v_i) \cdot \alpha_i, \quad \hat{m}_n := \hat{m}_n(0, \cdots, 0) = \sum_{i=1}^n \alpha_i,
\]

\[
\hat{\sigma}_n^2 := \hat{\sigma}_n^2(v_1, \cdots, v_n) = \mathbb{E} \left[ \sum_{i=1}^n (1 - v_i) \cdot \beta_i \cdot V_i^H \right]^2 = \sum_{i,j=1}^n (1 - v_i)(1 - v_j) \cdot \beta_i \beta_j \cdot \mathbb{E}[V_i^H V_j^H].
\]
By the property of fBm one can get with notation $\tilde{\beta}_j(\Delta t) := (1 - v_j)\beta_j(\Delta t)$

$$\tilde{\sigma}_n^2 = \sum_{m,j=0}^{n} \tilde{\beta}_m(\Delta t) \tilde{\beta}_j(\Delta t) E(\overline{V}_m^H \overline{V}_j^H)$$

$$= \frac{(\Delta t)^{2H}}{2} \sum_{m,j=0}^{n} \tilde{\beta}_m(\Delta t) \tilde{\beta}_j(\Delta t) \left[ |m - j + 1|^{2H} + |m - j - 1|^{2H} - 2 |m - j|^{2H} \right].$$

(A.1)

When $n$ and $|m - j|$ are large enough, we have

$$|m - j|^{2H} + |m - j - 1|^{2H} - 2 |m - j|^{2H}$$

$$= |m - j|^{2H} \cdot \left[\left|1 + \frac{1}{m-j}\right|^{2H} + \left|1 - \frac{1}{m-j}\right|^{2H} - 2 \right] \approx |m - j|^{2H-2}.$$

Therefore, we can bound (A.1) by

$$\tilde{\sigma}_n^2 = \frac{(\Delta t)^{2H}}{2} \sum_{m,j=0}^{n} \tilde{\beta}_m(\Delta t) \tilde{\beta}_j(\Delta t) \left[ |m - j + 1|^{2H} + |m - j - 1|^{2H} - 2 |m - j|^{2H} \right]$$

$$\leq C(\Delta t)^{2H} \sum_{m,j=0}^{n} \tilde{\beta}_m(\Delta t) \tilde{\beta}_j(\Delta t) |m - j|^{2H-2} \leq C(\Delta t)^{2H} \left( \sum_{m=1}^{n} |\tilde{\beta}_m(\Delta t)|^2 \right)^{2H},$$

(A.2)

where we have used the discrete type Hardy-Littlewood-Sobolev inequality (see e.g. Theorem 381 in [6]) in the last step.
For any given $\Delta t > 0$ (and $\lambda + |\mu| < 0$), one observes that from (A.2)

$$\left( \sum_{m=0}^{n} |\tilde{\beta}_m(\Delta t)|^{\frac{1}{\kappa}} \right)^{2H} \leq \left( \sum_{m=0}^{n} \frac{|\mu|}{1 - \kappa \theta \lambda (m+1)^{\kappa-1} \Delta t^\kappa} \right)^{2H}$$

$$= \left( \frac{\mu}{\kappa \theta \lambda (\Delta t)^\kappa} \right)^2 \left( \sum_{m=0}^{n} \left| \frac{1}{(m+1)^{\kappa-1} \frac{1}{\kappa \theta \lambda (\Delta t)^\kappa}} \right| \right)^{2H}$$

$$\leq C \left( \frac{\mu}{\kappa \theta \lambda (\Delta t)^\kappa} \right)^2 \cdot n^{2(1-\kappa)+2H} = \left( \frac{\mu}{\kappa \theta \lambda (\Delta t)^\kappa} \right)^2 \cdot n^{2+2H-2\kappa}.$$  

(A.3)

Thus, $\sigma_n^2 \leq C \cdot n^{2+2H-2\kappa} (\leq n^{2-2H}) \to 0$ as $n \to \infty$. Consequently,

$$\zeta^2 = \frac{\overline{\sigma_n^2}}{2\sigma_n^2} \geq \frac{C \cdot n^2}{n^{2-2H}} \left( \frac{1 - \theta}{\theta} \right)^2 \approx (n+1)^{2H}.$$  

APPENDIX B. CONFLUENT HYPERGEOMETRIC FUNCTIONS

In this section, we gather some important properties of Kummer’s confluent hypergeometric functions $\Phi(a, b, z)$ that are used in the main body of this work. The reader can find more details in Chapter 13 of [20]. Kummer’s confluent hypergeometric functions $\Phi(a, b, z)$ is defined as

$$\Phi(a, b, z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_{k!}} z^k = 1 + \frac{a}{b} z + \frac{a(a+1)}{b(b+1)2!} z^2 + \cdots.$$  

(B.4)
The following identity is called Kummer’s transformation (see e.g. 13.2.29 in [20])

\[ \Phi(a, b, z) = e^z \Phi(b - a, b, -z). \]  

(B.5)

A differentiation formula related to \( \Phi(a, b, z) \) is helpful to us (13.3.20 in [20]):

\[ \frac{d^n}{dz^n} [e^{-z} \Phi(a, b, z)] = (-1)^n \frac{(b - a)_n}{(b)_n} \Phi(a, b + n, z). \]  

(B.6)

Kummer’s confluent hypergeometric functions \( \Phi(a, b, z) \) can be represented by the so-called parabolic cylinder functions \( U(a, z) \) (13.6.14 and 13.6.15 in [20]):

\[ \Phi(a/2 + 1/4, 1/2, z^2/2) = \frac{2^{\frac{5}{4}} \Gamma(\frac{a}{2} + \frac{1}{4}) e^{\frac{z^2}{4}}}{\sqrt{\pi}} \times [U(a, z) + U(a, -z)]; \]  

(B.7)

\[ \Phi(a/2 + 3/4, 3/2, z^2/2) = \frac{2^{\frac{7}{4}} \Gamma(\frac{a}{2} + \frac{3}{4}) e^{\frac{z^2}{4}}}{\sqrt{\pi z}} \times [U(a, -z) - U(a, z)]. \]  

(B.8)

Recall the integral representation of the parabolic cylinder function \( U(a, z) \) by 12.5.1 in [20]

\[ U(a, z) = \frac{\exp(-\frac{z^2}{4})}{\Gamma(\frac{a}{2} + a)} \int_0^\infty w^{a-\frac{1}{2}} \exp(-w^2/2 - zw) dw, \text{ Re}(a) > -\frac{1}{2}. \]  

(B.9)

Lastly, the Poincaré-type asymptotic forms of confluent hypergeometric function hold (see 13.2.23 in [20]):

\[ M(a, b, z) = \frac{1}{\Gamma(b)} \Phi(a, b, z) \approx \frac{e^{a-b}}{\Gamma(a)} \exp(z), \text{ as } z \to \infty. \]  

(B.10)
Funding

M. Li is supported by the Fundamental Research Funds for the Central Universities, China University of Geosciences (Wuhan, Grant number: CUG2106127 and CUGST2), China. C. Huang is supported by the National Science Foundation of China, No. 11771163 and 12011530058. Y. Hu is supported by Natural Sciences and Engineering Research Council of Canada discovery fund and by a startup fund of University of Alberta.

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