Comment on “Success of collinear expansion in the calculation of induced gluon emission”

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Abstract

We show that the arguments against our recent paper on the failure of the collinear expansion in the calculation of the induced gluon emission raised by X.N. Wang are either incorrect or irrelevant.

1. In our recent paper \cite{1} (below referred to as AZZ) we have investigated the relation between the light-cone path integral (LCPI) approach \cite{2} (for reviews, see \cite{3, 4, 5}) to the induced gluon radiation and the higher-twist formalism by Guo, Wang and Zhang (GWZ) \cite{6, 7}. The GWZ approach is based on the Feynman diagram formalism and collinear expansion. It includes only $N = 1$ rescattering contribution. The GWZ formalism has been developed for the gluon emission from a fast quark in $eA$ DIS. The gluon spectrum predicted in \cite{6, 7} contains the logarithmically dependent nucleon gluon density, which is absent in the LCPI calculations \cite{8}. The AZZ analysis \cite{1} has been motivated by this discrepancy between the GWZ gluon spectrum and the $N = 1$ contribution to the LCPI spectrum (which in general accounts for arbitrary number of rescatterings).

In \cite{1} we have demonstrated that the approximations used in \cite{6, 7} really lead to a disagreement with the LCPI approach \cite{2}. However, contrary to the results of \cite{6, 7} the correct use of the collinear expansion gives a zero gluon spectrum. This result is confirmed by the exact calculations of the gluon spectrum within the oscillator approximation in the LCPI \cite{2} and BDMPS approaches \cite{9, 4} which is equivalent to the collinear expansion in momentum space used in \cite{6, 7}. The nonzero spectrum obtained in \cite{6, 7} is a consequence of the unjustified neglect of some important terms in the collinear expansion. In \cite{10} Wang has criticized the AZZ analysis \cite{1}. This comment is our reply to Wang’s criticism.

2. As in \cite{1, 6, 7, 10} we consider the induced gluon radiation from a fast massless quark produced in $eA$ DIS (as usual $q$ will denote the virtual photon momentum). We take the virtual photon momentum in the negative $z$ direction, and describe the 0 and 3 components of the four-vectors in terms of the light-cone variables $y^\pm = (y^0 \pm y^3)/\sqrt{2}$.

In the GWZ approach \cite{6, 7} the gluon emission is described by the set of diagrams like those shown in Fig. 1. The lower soft parts are expressed in terms of the matrix element $\langle A|\bar{\psi}(0)A^+(y_1)A^+(y_2)\psi(y)|A\rangle$, and the upper hard parts, $H$, are calculated perturbatively.
Figure 1.

In the limit of large struck quark energy all the $y^+$ coordinates in the soft part can be set to zero. Due to conservation of the large $p^-$ momenta of fast partons in the Feynman propagators only the Fourier components with $p^- > 0$ are important. It means that the Feynman propagators are effectively reduced to the retarded (in $y^-$ coordinate) ones. The integrations over the $p^+$ momenta of fast partons in the GWZ analysis have been performed with the help of the contour integration using the poles of the retarded propagators. The combinations of different poles leads to the processes with different longitudinal momentum transfers (double-hard, hard-soft, and the interferences in the terminology of [6, 7]). The collinear expansion used in [6, 7] corresponds to replacement of the hard part by its second order expansion in the $t$-channel transverse gluon momentum $\vec{k}_T$

$$H(\vec{k}_T) \approx H(\vec{k}_T = 0) + \left. \frac{\partial H}{\partial k_T^\alpha} \right|_{\vec{k}_T = 0} k_T^\alpha + \left. \frac{\partial^2 H}{\partial k_T^\alpha \partial k_T^\beta} \right|_{\vec{k}_T = 0} \cdot \frac{k_T^\alpha k_T^\beta}{2}.$$

(1)

For evaluation of the gluon emission only the second order term in (1) is important. Using the transverse momenta coming from this term and integrating by parts over the transverse coordinates with the help of the Collins-Soper formula [11] for the gluon density one can combine the vector potentials in the soft element into the unintegrated gluon density. This procedure leads to the gluon spectrum in the form of an integral over the final gluon transverse momentum with an integrand proportional to the nucleon gluon density times $\nabla_{k_T^\gamma}^2 H|_{\vec{k}_T = 0} = 0$.

In [1] we have demonstrated that the evaluation of the hard parts of the fast partons in terms of Feynman diagrams in the GWZ formalism [6, 7] is equivalent to that in terms the transverse Green’s functions used in the LCPI approach [2]. For this reason before the collinear expansion the hard parts in the GWZ approach should coincide with the $N = 1$ hard parts in the LCPI formalism. The direct comparison performed in [1] shows that this is really the case. However, after the collinear expansion the results of [1] and [6, 7] differ. In [1] we have shown that up to the terms suppressed by the small factor $R_N/L_f$ (hereafter $R_N$ is the nucleon radius, $L_f$ is the gluon formation length) $\nabla_{k_T}^2 H|_{\vec{k}_T = 0} = 0$. The corrections suppressed by the $R_N/L_f$ are beyond the accuracy of the approximations of [6, 7]. For this reason under the approximations used in [6, 7] the $N = 1$ gluon spectrum vanishes. However, according to the GWZ calculations $\nabla_{k_T}^2 H|_{\vec{k}_T = 0}$ is nonzero. In [10] Wang proceeds to claim that this is the case.
3. In [6, 7] the nonzero second derivative of the hard part at \( z \ll 1 \) (hereafter \( z \) is the fractional gluon momentum) comes from the graph shown in Fig. 1b. The authors use for the integration variable in the hard part of this graph the transverse momentum of the final gluon, \( \vec{l}_T \). The \( \vec{l}_T \)-integrated hard part obtained in [7] (Eq. (15) of [7]) reads (up to an unimportant factor)

\[
H(\vec{k}_T) \propto \int \frac{d\vec{l}_T}{(\vec{l}_T - \vec{k}_T)^2} R(\vec{l}_T, \vec{k}_T),
\]

where

\[
R(\vec{l}_T, \vec{k}_T) = \frac{1}{2} \exp \left[ \frac{i \vec{l}_T \cdot (\vec{k}_T - \vec{y}_1 - \vec{y}_2)}{2 q^- (1 - z)} \right] \cdot \left[ 1 - \exp \left( \frac{-i \vec{l}_T \cdot (\vec{k}_T - \vec{y}_1 - \vec{y}_2)}{2 q^- (1 - z)} \right) \right].
\]

Here \( \vec{y}_1, \vec{y}_2 \) correspond to the coordinates of the quark interactions with the virtual photon and \( t \)-channel gluons (our \( z \) equals \( 1 - z \) in [6, 7]). Note that (2) corresponds to the transverse momentum integrated gluon spectrum. Namely this case has been discussed in [6, 7] and [1]. In calculating \( \nabla_{\vec{k}_T}^2 H(\vec{k}_T) \) the authors differentiate only the factor \( 1/(\vec{l}_T - \vec{k}_T)^2 \). In [1] we have argued that the omitted terms from the factor \( R \) are important, and after the \( \vec{l}_T \) integration they almost completely cancel the contribution from the \( 1/(\vec{l}_T - \vec{k}_T)^2 \) term. Indeed, the dominating configurations correspond to \( |\vec{y}_1 - \vec{y}_2| \gg R_N, |\vec{y}_1 - \vec{y}_2| \gg R_N \). Neglecting the small corrections suppressed by \( R_N/L_f \) one can put in (3) \( \vec{y}_1 = \vec{y}_2 \). Then, one can change the variable \( \vec{l}_T \rightarrow (\vec{l}_T + \vec{k}_T) \), and the right-hand part of (2) becomes independent of \( \vec{k}_T \) at all, and one gets \( \nabla_{\vec{k}_T}^2 H|_{\vec{k}_T=0} = 0 \). Note that for the transverse momentum integrated spectrum there is no difference between differentiating the integrand of the hard part with respect to \( \vec{k}_T \) at fixed \( \vec{l}_T \) or \( \vec{l}_T + \vec{k}_T \). We emphasize this fact since in [10] Wang presents the formulas for the fully differential spectrum (in \( \vec{l}_T \) and \( z \)) and says that one should keep the final gluon transverse momentum \( \vec{l}_T \) fixed in the collinear expansion. He claims that namely due to ignoring this fact the incorrect conclusion on the GWZ approach [6, 7] has been done in [1]. However, it is clear misrepresentation of the AZZ analysis [1] since in [1] (as in [6, 7]) only the transverse momentum integrated spectrum has been discussed when the above change of the integration variable can safely be done.

In [10] Wang simply ignores the above transparent argument in favor of vanishing \( \nabla_{\vec{k}_T}^2 H|_{\vec{k}_T=0} \). He claims that the contribution from differentiating of the phase factors entering the hard part “will be linear in \( (\vec{y}_1 - \vec{y}_2)/q^- \) or \( \vec{y}/q^- \) which in general are suppressed by a factor \( \ell_f^2 r_N/q^- \)”, and therefore, they cannot cancel the contribution from differentiating the \( 1/(\vec{l}_T - \vec{k}_T)^2 \) factor. This statement is clearly wrong. It can easily be demonstrated calculating \( \nabla_{\vec{k}_T}^2 H|_{\vec{k}_T=0} \) for \( \vec{y}_1 = \vec{y}_2 = 0 \). Simple calculation gives

\[
\nabla_{\vec{k}_T}^2 H|_{\vec{k}_T=0} \propto 4\pi \int_0^\infty dl_T^2 \left\{ \frac{1 - \cos(al_T^2)}{l_T^4} - \frac{a \sin(al_T^2)}{l_T^2} + a^2 \cos(al_T^2) \right\},
\]

(4)
where $a = y_1^2/2q^-z(1-z)$. The last two terms in the integrand in (4) come from differentiating the factor $R$ (according to Wang’s statement these terms should be absent at $y_1 - y_2 = 0$, $y^- = 0$). Introducing the variable $\tau = a\bar{l}_T^2$ one obtains

$$
\nabla^2_{k_T} H|_{\bar{k}_T=0} \propto 4\pi a \left\{ \int_0^\infty d\tau \frac{1-\cos(\tau)}{\tau^2} - \int_0^\infty d\tau \frac{\sin(\tau)}{\tau} + \int_0^\infty d\tau \cos(\tau) \right\} \quad (5)
$$

which gives $\nabla^2_{k_T} H|_{\bar{k}_T=0} = 0$. Indeed after integrating the first term by parts it cancels the contribution from the second term in (5). The last integral equals zero. It can be obtained treating this integral as $\lim_{\delta \to 0} \int_0^\infty dx \exp(-\delta x) \cos(x)$. In (4), (5) we ignored the kinematical boundaries, and integrated up to infinity. However, accounting for the finite kinematical limit does not change the result. If one introduces a sharp cut-off factor in the gluon emission vertex in the $q \to qg$ transition defined in terms of the invariant mass of the $qg$ state it gives in terms of the integrand of (4) a sharp cut-off in terms of $(\bar{l}_T - \bar{k}_T)^2$.

One can easily show that in this case $\nabla^2_{k_T} H|_{\bar{k}_T=0} = 0$ as well. If one uses a sharp cut-off in terms of the variable $\bar{l}_T^2$ one gets $\nabla^2_{k_T} H|_{\bar{k}_T=0} \sim a \sin(a\bar{l}_T^2,\text{max})$. This strongly oscillating (in $y_1^-$) contribution in the sense of evaluation of the gluon spectrum is equivalent to $\nabla^2_{k_T} H|_{\bar{k}_T=0} \sim 1/\bar{l}_T^2,\text{max}$ which can be safely neglected. Thus the above simple analysis demonstrates that contrary to Wang’s claim the contribution from differentiating the phase factor cancels the contribution from differentiating $1/(\bar{l}_T - \bar{k}_T)^2$. We emphasize that the above arguments concern namely the hard part in the form obtained in [6, 7], and are not related at all to the LCPI approach.

In [10] Wang also gives his interpretation of the relation of the GWZ calculations to the LCPI approach. He gives a “proof” of the fact that the LCPI approach gives the $N=1$ spectrum which agrees with the GWZ result. He writes the hard part of [1] in terms of the variables $\bar{l}_T$ and $\bar{k}_T$ (Eq. (23) of [10]) and expands in $\bar{k}_T$ only the factor $1/(\bar{l}_T - \bar{k}_T)^2$ neglecting the terms which come from the expansion of the phase factor. Of course, this old wrong GWZ prescription leads to the old wrong GWZ result with nonzero gluon spectrum \(^1\). Note that presenting this “proof” Wang does not pay any attention to the evident fact that the collinear expansion in momentum space in the LCPI approach should reproduce the prediction of the oscillator approximation in impact parameter space in which the exact calculations give zero $N=1$ spectrum [8, 1].

4. In summary, we have shown that Wang’s criticism [10] of our recent analysis [1] of the relation between the LCPI [2] and GWZ [6, 7] approaches is unfounded. Using the hard part exactly in the form of [6, 7] by explicit calculations we have demonstrated that the collinear expansion gives a vanishing transverse momentum integrated gluon spectrum. It confirms the conclusion of [1] on the falsity of the GWZ calculations [6, 7] predicting a nonzero gluon spectrum.

\(^1\)Note that the normalization in Wang’s hard part (before the collinear expansion) is incorrect. It is evident since Wang identifies the LCPI hard part with the hard-soft process in GWZ, and claims that the double-hard processes are not included in the LCPI-BDMPS-GLV approaches. It is evidently wrong since the representation of the retarded propagators in terms of the transverse Green’s function obtained in [1] guarantees that in the LCPI calculations all the combinations of the poles (hard-soft, double-hard, and interferences in the GWZ language) of the propagators are automatically included.
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