Construction of Liapunov functions for highly nonlinear dynamic systems with feedback

G K Annakulova
Institute of Mechanics and Seismic Stability of Structures of the Academy of Sciences of the Republic of Uzbekistan, Durmonyuli33, Tashkent, 100125, Uzbekistan

E-mail: annaqaulya_g@mail.ru

Abstract. The paper provides stability and qualitative research of oscillations of a highly nonlinear dynamic system with feedback. For a system satisfying Liapunov theorem, definite-positive recurrent functions of order 1 with negative derivatives are constructed. Sufficient stability conditions are established for the considered cases of power nonlinearities. Surface diagrams of Liapunov functions and their derivatives are constructed for various values of the feedback parameter. Using the Liapunov criterion, the behavior of the trajectories of dynamic system on the state planes and near the singular points is investigated. Possible limit cycles are determined based on the Poincaré method of contact curves. Integral curves and phase trajectory diagrams are constructed numerically using the Mathcad 13 software package. A transition process from an unstable focus to self-oscillating and relaxation vibration modes is established, as well as the corresponding limit cycles consistent with analytical definition of rings containing the indicated limit cycles.

1. Introduction
The ability to regulate oscillatory processes, or to minimize them when they are undesirable, is an urgent problem in mechanical engineering. Machine aggregates are complex mechanical systems with nonlinearity, the influence of which on the dynamic characteristics is significant. A large number of issues of fundamental practical interest in the dynamics of systems are of a qualitative nature, i.e. the issues about the presence of stable equilibrium, the existence of stable periodic processes, the issues of transition regimes [1-4].

As is known, the Liapunov function method is a universal method for solving the stability problems of dynamic systems, but its universality extends only to autonomous linear systems. The construction of a function in a defined area of phase space is a more difficult problem. Many mathematicians and mechanical engineers worked in this direction to solve this problem, in particular, the studies given in [5–8]. Determining the Liapunov function of the control system allows evaluating the change in controlled variable, estimating time of transition process (regulation time), assessing the quality of regulation; the domain of attraction can be evaluated [4], the problem of periodic solution existence can be solved, i.e. the issue of the possibility of self-oscillations in the system [5]. To date, there is no reliable, well-developed algorithm to construct the Liapunov function for any nonlinear system. Using the technique of constructing Liapunov functions given in [6, 8, 10], the Liapunov functions for nonlinear self-oscillating systems are obtained.

The behavior of the system trajectories on the state planes is studied in this paper for various power nonlinearity; in particular, at \( k = 1 \), the system under consideration goes into the Van der Pol equation.

2. Statement of problem.
Consider the self-oscillations of a highly nonlinear system in the form [10]

\[
\begin{align*}
  m\ddot{x} + cx^{2k-1} - ax + bx^2\dot{x} &= 0, \\
  k &= 1,2, \ldots, m, \quad c,a,b > 0
\end{align*}
\]  
(1)
where $m$ is a parameter that determines the characteristics of the object; $c$ is the coefficient of the restoring force (at $k = 1$ linear one, at $k = 2$ cubic one, etc.); $a, b$ are the parameters that specify the feedback in the system.

Note that the nonlinear restoring force can be realized using linear elements (due to geometrical nonlinearity).

3. Solution method.

I. Equation (1) contains four-dimensional parameters. Reduce it to a standard form, using notation for dimensionless variables

$$\tau = \gamma t, \ u = \frac{x}{\tau}, \ \gamma = \sqrt{\frac{me}{I}} \frac{1}{2} \left( \frac{a}{b} \right)^{1/2}, \ l = \frac{a}{S} \frac{1}{2}, \ \varepsilon = \frac{a}{(cm)^{1/2}} \frac{e^{k-1}}{2}$$

$$\ddot{u} + u^{2k-1} - \varepsilon (1 - u^2) \dot{u} = 0,$$

where $\gamma, l$ are the transformation parameters; $\varepsilon > 0$ has the meaning of a feedback coefficient.

Introduce the notation $u = x_1$, $\dot{u} = x_2$ then equation (2) takes the form

$$x_1 = x_2,$$

$$\dot{x}_2 = -x_1^{2k-1} + \varepsilon (1 - x_1^2)x_2,$$  \hspace{1cm} (3)

A. Nonlinear system (3) at $k = 1$ has the form

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -x_1 + \varepsilon (1 - x_1^2)x_2.$$  \hspace{1cm} (4)

Consider the stability of system (4); for this, introduce the Liapunov function in the form [6]

$$V_1 = x_2^2 - 2 \int_0^{x_1} \varphi_1(x_1, 0) dx_1,$$  \hspace{1cm} (5)

where

$$\varphi_1(x_1, x_2) = -x_1 + \varepsilon (1 - x_1^2)x_2.$$  \hspace{1cm} (6)

The Liapunov function $V_1$, with relation (6), is represented as

$$V_1 = x_2^2 + x_1^2.$$  \hspace{1cm} (7)

The derivative of function $V_1$, by virtue of equations (4), has the form

$$\dot{V}_1 = 2\varepsilon (1 - x_1^2)x_2.$$  \hspace{1cm} (8)

The sufficient stability conditions for the system of equations (4) have the form:

a) $\varphi_1(x_1, 0)x_1 < 0$, at $x_1 \neq 0$;

b) $\left[ \varphi_1(x_1, x_2) - \varphi_1(x_1, 0) \right] x_2 < 0$, at $x_2 \neq 0$;

c) $\int_0^{x_1} \varphi_1(x_1, 0) dx_1 = x_1^2 \to \infty$, at $x_1 \to \infty$.

B. At $k = 2$, system (3) takes the form

$$\dot{x}_1 = x_2,$$

$$\dot{x}_2 = -x_1^3 + \varepsilon (1 - x_1^2)x_2,$$  \hspace{1cm} (9)

introduce the Liapunov function $V_2$ in the form

$$V_2 = x_2^2 - 2 \int_0^{x_1} \varphi_2(x_1, 0) dx_1,$$  \hspace{1cm} (10)

where

$$\varphi_2(x_1, x_2) = -x_1^3 + \varepsilon (1 - x_1^2)x_2.$$  \hspace{1cm} (11)

Then $V_2$, with relation (11), has the form

$$V_2 = x_2^2 + \frac{x_1^4}{2},$$  \hspace{1cm} (12)

and the derivative of function $V_2$, by virtue of equations (9), takes the form

$$\dot{V}_2 = 2\varepsilon (1 - x_1^2)x_2^3.$$  \hspace{1cm} (13)

The stability conditions for the system of equations (9) have the form:

a) $\varphi_2(x_1, 0)x_1 < 0$, at $x_1 \neq 0$;

b) $\left[ \varphi_2(x_1, x_2) - \varphi_2(x_1, 0) \right] x_2 < 0$, at $x_2 \neq 0$;
\[ c) \int_{0}^{x_1} \varphi_2(x_4, 0) \, dx_1 = \frac{x_1^4}{2} \to \infty, \quad \text{at} \quad x_1 \to \infty. \]

C. At \( k = n \), system (3) has the form

\[ x_1 = x_2, \]
\[ x_2 = -x_1^{2n-1} + \varepsilon(1-x_1^2)x_2. \]  

Introducing the Liapunov function in the form

\[ V_n = x_2^2 - 2 \int_{0}^{x_1} \varphi_n(x_1, 0) \, dx_1, \]

where

\[ \varphi_n(x_1, x_2) = -x_1^{2n-1} + \varepsilon(1-x_1^2)x_2. \]  

Then the functions \( V_n \) and \( \dot{V}_n \) by virtue of equations (14), are represented in the form

\[ V_n = x_2^2 + \frac{x_1^{2n}}{n}, \]
\[ \dot{V}_n = 2\varepsilon(1-x_1^2)x_2^2. \]

The sufficient stability conditions in large for the system of equations (14) have the form:

\begin{align*}
\text{a) } & \varphi_n(x_1, 0)x_1 < 0, \text{ at } x_1 \neq 0; \\
\text{b) } & [\varphi_n(x_1, x_2) - \varphi_n(x_1, 0)]x_2 < 0, \text{ at } x_2 \neq 0; \\
\text{c) } & \int_{0}^{x_1} \varphi_n(x_1, 0) \, dx_1 = \frac{x_1^{2n}}{n} \to \infty, \quad \text{at} \quad x_1 \to \infty. 
\end{align*}

Thus, the Liapunov functions and their derivatives for nonlinear system (3) of the

\( 2k \)-th order at \( k = 1, k = 2, \ldots, k = n \) are obtained in general form, as recurrent functions of order 1, i.e.

\[ V_1 = x_2^2 + x_1^2, \quad V_2 = x_2^2 + \frac{1}{2} x_1^4, \ldots, \quad V_n = x_2^2 + \frac{1}{n} x_1^{2n}, \]

\[ \dot{V}_1 = 2\varepsilon(1-x_1^2)x_2^2, \quad \dot{V}_2 = 2\varepsilon(1-x_1^2)x_2^2, \ldots, \quad \dot{V}_n = 2\varepsilon(1-x_1^2)x_2^2. \]

**Theorem** (the Liapunov stability Theorem). If for systems (4), (9) and (14) there exist sign-defined functions \( V^{(1)} \), \( V^{(2)} \ldots V^{(n)} \) in domain \( D \), the time derivatives of which \( \dot{V}^{(1)}, \dot{V}^{(2)} \ldots \dot{V}^{(n)} \) by virtue of systems (4), (9) and (14), are sign-constant functions of the sign opposite to the signs of functions \( V^{(1)} \), \( V^{(2)} \ldots V^{(n)} \), then the equilibrium position is stable in the sense of Liapunov.

Thus, according to expressions (19) and (20), we can conclude that for sufficiently small values of \( x_1 \) and \( x_2 \), the derivatives \( \dot{V}^{(1)}, \dot{V}^{(2)} \ldots \dot{V}^{(n)} \) for all cases are negative. This means that the origin is a stable singular point, i.e. with increasing time \( \tau \) any trajectory crosses the surface \( V = const \) from the outside to the inside.

Table 1 shows the surface diagrams of the Liapunov functions and their derivatives for various values of the feedback parameter

| \( k=1 \) | \( k=2 \) | \( k=3 \) |
|---|---|---|
| \( \text{Surface diagrams of Liapunov functions} \) | \( \text{and their derivatives} \) | \( \text{for various values of the} \) |
| \( \text{feedback parameter} \) |
The higher the feedback coefficient and the higher the power nonlinearity, the better the stability of the dynamic system at large.

To determine the possible limit cycle, the Poincaré method of contact curves is used [5]. Consider a class of coaxial circles centered at a singular point of equations (3). On the plane $x_1x_2$ the geometrical locus of the points is defined at which these circles touch the integral curves of equations (3). This geometrical locus of points forms a contact curve. Excluding time $\tau$ from the system of equations (3), and using the differential form of the class of circles, the contact curve is determined in the form

$$\left(1 - x_1^{2^{(k-1)}}\right)x_1 + \varepsilon(1 - x_1^2)x_2 = 0.$$  \hspace{1cm} (21)

Introducing the polar coordinates, equation (21) is transformed to the form

$$\left(1 - r^{2^{(k-1)}}\cos^{2^{(k-1)}}\theta\right)r\cos \theta + \varepsilon(1 - r^2 \cos^2 \theta)r \sin \theta = 0.$$  \hspace{1cm} (22)

Everywhere except the point $r = \theta$ the contact curve is determined by the equation

$$f \rightarrow V, f_1 \rightarrow \dot{V}.$$
\[ r^{2k-1} \cos^{2k-1} \theta + r^2 \cos^2 \theta \sin \theta - (\cos \theta + \varepsilon \sin \theta) = 0, \]  

(23)

At \( k = 1 \) the relationship takes the form

\[ r^2 + r \frac{2}{\sin 2\theta} - \left( \frac{2}{\varepsilon \sin 2\theta} + \frac{1}{\cos^2 \theta} \right) = 0, \]  

(24)

from the quadratic equation (24) we determine

\[ r_{1,2} = -\frac{1}{\sin 2\theta} \pm \left( \frac{1}{\sin^2 2\theta} + \frac{2}{\varepsilon \sin 2\theta} + \frac{1}{\cos^2 \theta} \right)^{1/2}. \]  

(25)

The maximum and minimum values of \( r \) taking into account the angle \( \theta \) are:

\[ r_{\text{max}} = -1 + \left( \frac{3\varepsilon + 2}{\varepsilon} \right)^{1/2}, \quad r_{\text{min}} = -1 - \left( \frac{3\varepsilon + 2}{\varepsilon} \right)^{1/2}. \]  

(26)

For other values of power nonlinearity \( k \), the values of \( r_{\text{max}} \) and \( r_{\text{min}} \) can also be determined. This means that there exists a ring centered at the origin containing all possible limit cycles whose boundaries are the circles of the least \( r_{\text{min}} \) and the greatest \( r_{\text{max}} \) radii of relationships (26) touching the contact curve defined by equation (21).

Table 2 shows the integral curves and diagrams of phase trajectories constructed numerically using the Mathcad 13 software package for cases \( k = 1, k = 2 \) and \( k = 3 \) for various values of parameter \( \varepsilon \).

**Table 2. Diagrams of phase trajectory**
Table 2 shows the transition processes from an unstable focus to self-oscillating and relaxation vibrational modes [9], and the corresponding limit cycles, which are consistent with the analytical definition of rings containing the indicated limit cycles. An increase in the feedback coefficient $\varepsilon$ and the power nonlinearity $k$ of differential equation (3) in the system leads to the appearance of self-oscillating and relaxation vibrational processes.

4. Conclusion
For a highly nonlinear system of the $2k$-1-th order satisfying the Liapunov theorem, definite-positive recurrent functions of order 1 having negative derivatives were obtained. Sufficient stability conditions were established in large for the considered cases of the power nonlinearity. Surface diagrams of Liapunov functions and their derivatives were constructed for various values of the feedback parameter. Using the Liapunov criterion, the behavior of the trajectories of dynamical system on the state planes and near the singular points was investigated. Based on the Poincaré method of contact curves, possible limit cycles were determined. Integral curves and phase trajectory diagrams were constructed numerically using the Mathcad 13 software package. The author presents a transition process from an unstable focus to self-oscillating and relaxation modes of oscillations, as well as the corresponding limit cycles, consistent with the analytical definition of rings containing these limit cycles.

Acknowledgments
The research was carried out under financial support of the Fund of fundamental research (БВ-М-Ф4-001) of the Republic of Uzbekistan.

References
[1] Liapunov A M 1952 General problem of motion stability (Moscow: Gostekhizdat) 472
[2] Barbashin E A 1967 Introduction to stability theory (Moscow: Nauka) 223
[3] Andreev A S 2001 Proc. V International open session Modus Academicus "Computer technologies, science and education in the twenty-first century" (Ulyanovsk) 167-181
[4] Annakulova G K, Lebedev O V 2010 Dynamic modes and chaotic motions of machine drive elements (Tashkent: IMSS AS RUz) 141
[5] Poincaré A 1972 Selected Works vol 2 (Moscow: Nauka) 543
[6] Barbashin E A 1970 Liapunov functions (Moscow: Nauka) 240
[7] Andreev A S, Boykova T A 2002 J. Rigid Body Mechanics 32 109-116
[8] Koval'ev A M 2008 J. Appl. mat. and mechanics 72(2) 266-272
[9] Mishchenko E F 1975 Differential equations with small parameter and the relaxation oscillations (Moscow: Nauka) 248
[10] Annakulova G K, Igamberdiev K A and Abdullaeva M 2016 J. Issues of computational and applied mathematics (Tashkent) 2 71-74