Uniform Anderson Localization in One-Dimensional Floquet Maps

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Anderson localization in discrete-time quantum map dynamics shows unique features and provides significant numerical benefits. We propose a minimal single-particle map in one dimension with nearest-neighbor hopping only and a gapless spectrum. We demonstrate that strong disorder in a local phase field leads to the universality of Anderson localization within the spectrum, such that for every possible value of quasi-energy, localization happens in the same way. We build an exact theory for the calculation of the localization length, $1/L_{\text{loc}} = |\ln (|\sin(\theta)|)|$, that is tunable between zero and infinity by variation of a control parameter $\theta$ related to hopping.

I. INTRODUCTION

Anderson localization (AL) is a phenomenon of prohibited wave transport in various non-interacting quantum systems when placed in an uncorrelated random medium$^{1-4}$. This is most notably expressed by all the states in the Anderson phase being exponentially localized with the characteristic length $L_{\text{loc}}$, called the localization length, which is determined by the specifics of the given system. Since its discovery, AL has been studied in applications to physical systems including photonic crystal waveguides$^5$, light$^6,7$, microwaves$^8$, and ultrasound$^{9,10}$. These and many other examples indicate that AL is well understood in Hamiltonian systems, where the spectra of the systems are defined on the real axis. This, generally, leads to every state being uniquely classified by its energy $E$, and the localization length becomes a function of the state $L_{\text{loc}}(E)$. One notable counterexample is the Aubry–André model$^{11}$, where the underlying potential is pseudo-random (quasi-periodic) and, due to self-similarity, the localization length is universal for all eigenstates$^{12}$. Nevertheless, this does not indicate the universality of the localization length within a continuous portion of the energy spectrum, as its structure is fractal in this model.

Another somewhat less studied situation arises if the dynamics is not continuous in time, thus representing a map. Such systems often result from time-periodic Hamiltonians, i.e. their associated Floquet maps. One notable example is the quantum kicked rotor, in which temporal behavior is strongly connected to AL$^{13}$ and the corresponding localization length is uniquely defined by the system parameters$^{14}$. Another example is the discrete-time quantum walk (DTQW)$^{15}$, a class of systems discrete in both space and time. Under the application of a random phase field, which uniformly covers the whole complex circle, the localization length has a universal value for any quasi-energy$^{16}$. In addition to the emergence of such unique properties, quantum maps present benefits for numerical studies, as the application of a map is much less challenging than numerical integration$^{17,18}$.

In this work, we propose a new model representing a Floquet quantum map in one dimension. Unlike previously studied models, it contains only nearest-neighbor coupling and a gapless spectrum, so the suggested model is in a certain sense a minimal one-dimensional quantum map. Using the transfer matrix approach, we prove that there exist regimes with a universal localization length over the whole spectrum. We provide an exact theory for the calculation of this localization length based on the system parameters.

II. MODEL

Consider a single particle on a periodic one-dimensional lattice with length $N$ such that each unit...
cell contains two sites, or in other words, the particle has a spinor degree of freedom. The wave function of such a system reads $\tilde{\Psi} = \{\Psi_{1,+}, \Psi_{1,-}, \Psi_{2,+}, \ldots, \Psi_{N,+}, \Psi_{N,-}\}$. The Hilbert space can be factorized as a product of the direct space and the spinor space, so $\Psi_{\alpha} = |k\rangle \otimes |\alpha\rangle$. We study a class of time-periodic Hamiltonians that lead to specific Floquet maps defined on the chain, and thus,

$$\tilde{\Psi}(t+1) = \hat{S}\tilde{\Psi}(t),$$

where time $t$ is measured in periods of evolution and $\hat{S}$ is the Floquet map operator. Evolution of such a state $\Psi$ that both a) preserves two sites per unit cell, i.e., is invariant under operation $\sum_{k,\alpha=\pm} |k,\alpha\rangle \langle k \pm 1, \alpha|$, and b) only involves nearest-neighbor coupling in terms of the Floquet map, $\hat{S}$, can generally (up to the definition of the unit cell) be written in the following form:

$$\hat{S} = \hat{S}_1 \otimes \hat{S}_2,$$

$$\hat{S}_1 = \sum_{k,\alpha=\pm, \beta=\pm} u_{\alpha\beta} |k,\alpha\rangle \langle k,\beta|,$$

$$\hat{S}_2 = \sum_{k,\alpha=\pm, \beta=\pm} v_{\alpha\beta} |k,\alpha\rangle \langle k + l,\beta|,$$

where $l_{\beta} = 1$ if $\beta = +$ and 0 otherwise. Thus, two 2x2 matrices $u$ and $v$ contain the parameters defining the system. This evolution may be easily understood in the following way. First, all the spinors are mapped by $u$, then the wave function in the linear form is rotated left, after which all the spinors are mapped by $v$, and finally the wave function is rotated back. See the schematic representation in Fig. 1.

One particular well-studied example of such a system is the DTQW. This can be specifically achieved by setting $u$ as a $U(2)$ matrix that corresponds to a generalized DTQW coin $^{19}$, and setting $v = \sigma^{(x)}$ ($\sigma$ is a Pauli matrix) to realize the so-called shift operation of the DTQW.

In this paper, we are focused on the case when $u = v$. We take them as the most general unitary matrices,

$$u = v = e^{i\varphi} \begin{pmatrix} e^{i\varphi_{1}} \cos \theta & e^{i\varphi_{2}} \sin \theta \\ -e^{-i\varphi_{2}} \sin \theta & e^{-i\varphi_{1}} \cos \theta \end{pmatrix},$$

where $\theta, \varphi, \varphi_1, \varphi_2$ are parameters characterizing the system, with $\varphi$ being a potential-energy-like local field, and $\theta$ a hopping-like parameter that controls coupling. Given translational invariance, we can find the eigenstates as plain waves given by the ansatz $\psi_k(n) = e^{-i\kappa n}|\psi_{k,+}, \psi_{k,-}\rangle$. Defining the corresponding eigenvalue of $\hat{S}$ as $e^{i\omega}$, where $\omega$ is the eigenfrequency, we find the dispersion relation

$$\cos(\omega - 2\varphi) = \cos^2(\theta) + \cos(k - 2\varphi_2) \sin^2(\theta).$$

Note that $\varphi$ shifts the frequency, $\varphi_2$ shifts the wavenumber, and $\varphi_1$ is irrelevant. The band structure is presented in Fig. 2. It generally constitutes two bands touching at $2\varphi_2$, where each is exactly 0 when $\theta = 0$ and straight when $\theta = \pi/2$.

### III. ANDERSON LOCALIZATION

In this section, we study AL in the case of a random phase field applied to each state component. We modify the translationally invariant evolution defined by (3) by additionally multiplying each component $\psi_{k,\alpha}$ by $e^{i\phi_{k,\alpha}}$, where $\phi_{k,\alpha}$ are random numbers drawn uniformly from $[-W/2, W/2]$ where $W$ is the strength of the disorder. This modification is implemented by making the matrix $v$ unit-cell dependent. Note, also, that while we present the result for random phases being independent for each spinor component, the same results apply for unit-cell-only dependent random phases.

The inverse localization length as a function of the eigenfrequency for different values of disorder strength $W$ is depicted in Fig. 3. We notice that for the maximum strength of disorder $W = \pi$, the localization length is exactly the same for all the states in the spectrum. This can be explained by the fact that such disorder smears all the frequencies uniformly over the complex unit circle, making them indistinguishable. We will further study this case.

To calculate the localization length, we use the transfer matrix approach. An eigenstate with an eigenvalue $e^{i\omega}$ admits the following equation:
This map has a single invariant submanifold $|z_1| = 1$, and thus the stationary distribution of $z_1$ is located on this unit circle and it is sufficient to calculate the stationary distribution of $\arg(z_1)$. However, the projection of (9) on $\arg(z_1)$ is highly non-trivial and virtually impossible to work with; it is much more common to work with distributions of unbound variables. To make this possible, we transform the $z_1$ plane using a conformal map,

$$z_1 = \frac{z_2 - i}{z_2 + i},$$

(10)

thus mapping the circle $|z_1| = 1$ onto the real axis. The map $z_2 \rightarrow f_2(z_2)$ reads

$$f_1(z_1) = \frac{z_1 e^{i\theta} - (1 - 2 e^{i\theta_{n+1}} + \cos(2\theta)) - 2(e^{i\theta_{n+1}} + 1) \cos \theta}{2 z_1 e^{i\theta} - (e^{i\theta_{n+1}} + 1) \cos \theta + e^{i\theta_{n+1}} + \cos(2\theta) - 2 e^{i\theta_{n+1}} + 1}. \quad \text{(9)}$$

First, we transform the space by a shift and rescaling,

$$z = \cot \theta + z_1 / \sin \theta. \quad \text{(8)}$$

The map $z_1 \rightarrow f_1(z_1)$ then reads

$$y_{n+1} = e^{-i(\phi_1 - \phi_2)} \left( y_n e^{i(\phi_1 - \phi_2)} e^{i\phi_{n+1} + \cot \theta (1 - e^{i\phi_{n+1} + \phi_{n-1}})} + \sin^2 \theta (e^{-i\phi_{n-1}} + \cos \theta e^{i\phi_{n+1} + \cos(2\theta)} - \cos^2 \theta (1 + e^{i(\phi_{n+1} + \phi_{n-1})}) \right). \quad \text{(7)}$$

Rescaling the phase with a constant factor $y_n \rightarrow y_n e^{-i(\phi_1 - \phi_2)}$ does not change the localization length, so we get rid of these factors, thereby demonstrating that the localization length depends only on the parameter $\theta$ of the model. Let us call the resulting map $z \rightarrow f(z)$.

Note that the transfer matrix $T_n$ is always unitary but symplectic if and only if $\phi_{n+1} = \phi_{n-1}$. Also note that the transfer matrix is invariant under $\phi \rightarrow \phi - \omega$. It immediately follows that for $W = 2\pi$, the ensemble of transfer matrices is completely independent of $\omega$ as $U(S^1)$ is invariant under rotation. Also, note that as the parameter $\phi$ only enters as a prefactor with an absolute value of 1, it does not affect the localization length. Thus we omit it. We now introduce a transformation,

$$\left( \Psi_{n+1,+} \Psi_{n+1,-} \right) = A_n \left( y_n \begin{array}{c} 1 \end{array} \right). \quad \text{(6)}$$

In order to calculate the localization length, we need to study the behavior of the Riccati variable $y_n$ at $n \rightarrow \infty$. We can exclude $A_n$ and write a closed iterative expression for the Riccati variable ($\omega = 0$ as it can be chosen arbitrarily) as follows:

$$\left( \Psi_{n+1,+} \Psi_{n+1,-} \right) = T_n \left( \Psi_{n,+} \Psi_{n,-} \right),$$

$$T_n = e^{2i\varphi} \left( e^{i(\phi_{n+1} - \phi_{n-1})} \cot \theta (1 - e^{i(\phi_{n+1} + \phi_{n-1})}) \right) \sin^2 \theta e^{-i(\phi_{n-1} - \omega)} + \cos^2 \theta (e^{i(\phi_{n+1} + \omega)} - e^{i(\phi_{n+1} + \phi_{n-1}) - 1}) \right). \quad \text{(5)}$$

This map has a single invariant submanifold $|z_1| = 1$, and thus the stationary distribution of $z_1$ is located on this unit circle and it is sufficient to calculate the stationary distribution of $\arg(z_1)$. However, the projection of (9) on $\arg(z_1)$ is highly non-trivial and virtually impossible to work with; it is much more common to work with distributions of unbound variables. To make this possible,
The stationary distribution of the solution to the integral equation within \(0\) from (17). The first moment of the probability distribution is approximately 0.958 after 10 iterations each. The black line indicates the analytical value within \([0, 2\pi]\).

After additionally taking into account that the stationary distribution is located on the real axis, we may get the map for the real part \(x = \text{Re}(z_2)\),

\[
f_2(z_2) = \frac{\tan^2\left(\frac{\theta}{2}\right) \cos\theta \left(e^{i\phi_1}(z_2 - i) + e^{i\phi_2}(z_2 + i)\right) + (z_2 - i)e^{i(\phi_1 + \phi_2)} + z_2 + i}{-i \cos(\theta) \left(e^{i\phi_1}(z_2 - i) - e^{i\phi_2}(z_2 + i)\right) + (1 + iz_2)e^{i(\phi_1 + \phi_2)} - iz_2 + 1}.
\]  

(11)

Unfortunately, the structure of (12) is too complicated to allow a direct solution of (13). To get a hint, we notice that the ratios of two random variables are themselves Cauchy distributed under a wide range of circumstances\(^{22}\). This is in particular always the case for ratios of elliptically symmetric distributions\(^{23}\). Thus, we may hypothesize that the stationary distribution \(p(x)\) is indeed a Cauchy distribution. To prove this, we note that for any values of random phases \(\phi_1\) and \(\phi_2\), the map from (12) is an element of the real Möbius group \(\text{SL}(2, \mathbb{R})\), and then note that the Cauchy distribution family is closed under this group\(^{24}\). Consequently, the stationary distribution of (12) is necessarily a Cauchy distribution. We choose it as follows,

\[
p(x) = \frac{1}{\pi \gamma (1 + x^2/\gamma^2)}, \quad \gamma = \tan(\theta/2)^2,
\]  

(14)

and we prove that this is the correct choice of the parameter \(\gamma\) by showing that it solves (13). Integrating (13) over \(x\), we get

\[
p(x') = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} p(f^{-1}(x')) / |f'(f^{-1}(x'))|,
\]  

(15)

where \(f^{-1}\) is the inverse function of \(f\). It generally has more than one value, but not in this case. As we have already proven that the solution is a Cauchy distribution, we only need to confirm the parameter choice, for which it is adequate to integrate at \(x' = 0\) only. Substituting (14) into (15), we get

\[
\frac{\sec^4\left(\frac{\theta}{2}\right)}{4} = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \times
\]

\[
\times \frac{1}{\left(\cos(\theta) \cos\left(\frac{\phi_1 - \phi_2}{2}\right) + \cos\left(\frac{\phi_1 + \phi_2}{2}\right)\right)^2 + \cot^4\left(\frac{\theta}{2}\right) \left(\cos(\theta) \sin\left(\frac{\phi_1 - \phi_2}{2}\right) + \sin\left(\frac{\phi_1 + \phi_2}{2}\right)\right)^2},
\]  

(16)

FIG. 4. Localization length for \(W = 2\pi, \theta = \pi/5\). The orange histogram shows the probability distribution of localization length for 50 samples with length \(N = 2\times500\). The red area is the range of values calculated for various eigenfrequencies within \([0, 2\pi]\) using numerical transfer matrices with \(10^6\) iterations each. The black line indicates the analytical value from (17). The first moment of the probability distribution is within 0.025 relative error from the analytic value.

The stationary distribution of \(x\) can now be found as the solution to the integral equation\(^{21}\),

\[
p(x') = \int_0^{2\pi} \frac{d\phi_1}{2\pi} \int_0^{2\pi} \frac{d\phi_2}{2\pi} \int_{-\infty}^{\infty} dx \ p(x) \delta(x' - f(x)).
\]  

(13)
which is indeed a true identity. As a result of all the above findings, we are finally able to calculate the localization length through

$$\frac{1}{L_{\text{loc}}} = \int dz \ p(z) \ln(|z|)$$

$$= \int_{-\infty}^{\infty} dx \ p(x) \frac{1}{2} \ln \left[ \frac{1}{2} \csc(\theta) \left( \frac{4(x^2 - 1) \cot(\theta)}{x^2 + 1} + (\cos(2\theta) + 3) \csc(\theta) \right) \right] = \ln \left( |\sin(\theta)| \right). \quad (17)$$

![Inverse localization length versus the hopping-like parameter θ. The red line is the analytical calculation from (17), and the black dots are the numerical results from the transfer matrix approach with $10^6$ iterations for each value of θ.](image)

**IV. DISCUSSION**

We proposed a minimal one-dimensional Floquet map with only nearest-neighbor coupling of the unit cells and a gapless spectrum. We demonstrated that the Anderson localization length is universal for all the eigenstates and for every value of the quasi-energy on the complex unit circle (in the thermodynamic limit), and depends only on the hopping-like parameter of the model. Furthermore, the localization length can be tuned from 0 to $\infty$.

These facts can be widely used in numerical studies. For example, in a recent paper, the authors employed a similar system in the absence of disorder to study Lyapunov spectrum scaling\(^2\) and conjectured its behavior in the presence of disorder. The universality and ability to control the disordered phase that we showed in this work may benefit subsequent related studies. Further applications may include research into thermalization, ergodization, localization phenomena in the presence of interactions, the interplay between nonlinearity-induced chaos and localization, delocalization due to dissipation, and many others.

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1. Philip W Anderson. Absence of diffusion in certain random lattices. *Physical review*, 109(5):1492, 1958.
2. Patrick A Lee and TV Ramakrishnan. Disordered electronic systems. *Reviews of modern physics*, 57(2):287, 1985.
3. Bernhard Kramer and Angus MacKinnon. Localization: theory and experiment. *Reports on Progress in Physics*, 56(12):1469, 1993.
4. Il’ya M Lifshits, SI Gredeskul, and Leonid Andreevich Pastur. Introduction to theory of non-ordered systems. *Moscow Eds Nauka*, 1982.
5. Luca Sapienza, Henri Thyrrestrup, Søren Stobbe, Pedro David Garcia, Stephan Smolka, and Peter Lodahl. Cavity quantum electrodynamics with anderson-localized modes. *Science*, 327(5971):1352–1355, 2010.
6. Tal Schwartz, Guy Bartal, Shmuel Fishman, and Mordechai Segev. Transport and anderson localization in disordered two-dimensional photonic lattices. *Nature*, 446:52, 03 2007.
7. Yoav Lahini, Assaf Avidan, Francesca Pozzi, Marc Sorel, Roberto Morandotti, Demetrios N. Christodoulides, and Yaron Silberberg. Anderson localization and nonlinearity in one-dimensional disordered photonic lattices. *Phys. Rev. Lett.*, 100:013906, Jan 2008.
8. Rachida Dalichaouch, J. P. Armstrong, S. Schultz, P. M. Platzman, and S. L. McCall. Microwave localization by two-dimensional random scattering. *Nature*, 354:53, 11 1991.
9. R.L. Weaver. Anderson localization of ultrasound. *Wave Motion*, 12(2):129 – 142, 1990.
10 Hefei Hu, A. Strybulevych, J. H. Page, S. E. Skipetrov, and B. A. van Tiggelen. Localization of ultrasound in a three-dimensional elastic network. Nature Physics, 4:945, 10 2008.

11 Serge Aubry and Gilles André. Analyticity breaking and anderson localization in incommensurate lattices. Ann. Israel Phys. Soc., 3(133):18, 1980.

12 Alejandro J Martínez, Mason A Porter, and PG Kevrekidis. Quasiperiodic granular chains and hofstadter butterflies. Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences, 376(2127):20170139, 2018.

13 Shmuel Fishman, DR Grempel, and RE Prange. Chaos, quantum recurrences, and anderson localization. Physical Review Letters, 49(8):509, 1982.

14 Shmuel Fishman, RE Prange, and Meir Griniasty. Scaling theory for the localization length of the kicked rotor. Physical Review A, 39(4):1628, 1989.

15 Yakir Aharonov, Luiz Davidovich, and Nicim Zagury. Quantum random walks. Physical Review A, 48(2):1687, 1993.

16 Ihor Vakulchyk, Mikhail V Fistul, Pinquan Qin, and Sergej Flach. Anderson localization in generalized discrete-time quantum walks. Physical Review B, 96(14):144204, 2017.

17 Merab Malishava, Ihor Vakulchyk, Mikhail Fistul, and Sergej Flach. Floquet anderson localization of two interacting discrete time quantum walks. Physical Review B, 101(14):144201, 2020.

18 Ihor Vakulchyk, Mikhail V Fistul, and Sergej Flach. Wave packet spreading with disordered nonlinear discrete-time quantum walks. Physical review letters, 122(4):040501, 2019.

19 C Madaiah Chandrashekar, Radhakrishna Srikanth, and Raymond Laflamme. Optimizing the discrete time quantum walk using a su (2) coin. Physical Review A, 77(3):032326, 2008.

20 Andrea Crisanti, Giovanni Paladin, and Angelo Vulpiani. Products of random matrices: in Statistical Physics, volume 104. Springer Science & Business Media, 2012.

21 Bernard Derrida and HJ715727 Hilhorst. Singular behaviour of certain infinite products of random 2×2 matrices. Journal of Physics A: Mathematical and General, 16(12):2641, 1983.

22 Natesh S Pillai. Ratios and cauchy distribution. arXiv preprint arXiv:1602.08181, 2016.

23 Barry C Arnold and Patrick L Brockett. On distributions whose component ratios are cauchy. The American Statistician, 46(1):25–26, 1992.

24 Peter McCullagh et al. Möbius transformation and cauchy parameter estimation. Annals of statistics, 24(2):787–808, 1996.

25 Merab Malishava and Sergej Flach. Lyapunov spectrum scaling for classical many-body dynamics close to integrability. Physical Review Letters, 128(13):134102, 2022.