Nonlinearly Charged Black Hole Chemistry with Massive Gravitons in the Grand Canonical Ensemble

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In the context of Black Hole Chemistry (BHC), holographic phase transitions of asymptotically anti-de Sitter (AdS) charged topological black holes (TBHs) in massive gravity coupled to Power Maxwell Invariant (PMI) electrodynamics are discussed in the grand canonical (fixed \(U(1)\) potential, \(\Phi\)) ensemble. Considering the higher-order of graviton’s self-interactions of (dRGT) massive gravitational field theory in arbitrary dimensions, we derive exact TBH solutions. We also calculate the explicit form of the on-shell action and the associated thermodynamic quantities in the Grand Canonical Ensemble (GCE). Besides, we examine the validity of the first law of thermodynamics and Smarr relation in the extended phase space thermodynamics. Regarding this model, we show that a) a van der Waals (vdW) behavior takes place in \(d \geq 4\) dimensions, b) a typical reentrant phase transition can happen in \(d \geq 5\), and c) both anomalous vdW and triple point phenomena are recognized in \(d \geq 6\). Accordingly, we find that the obtained results are quite different from their counterparts in the GCE of Maxwell-massive gravity. Furthermore, we briefly study the critical behaviors of higher-dimensional TBHs with a conformally invariant Maxwell source as a specific subclass of our solutions for Einstein’s theory and the massive gravity in various ensembles.

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I. INTRODUCTION

The subject of Black Hole Chemistry (BHC) \cite{1,2} has received much attention recently due to the similarity between specific holographic phase transitions and those counterparts in the real world. In this regard, van der Waals (vdW) behavior \cite{3}, reentrant phase transition (RPT) with the associated large/intermediate/large black hole (LBH/IBH/LBH) phase transition \cite{4,5}, triple point phenomenon with the associated small/intermediate/large black hole (SBH/IBH/LBH) phase transition \cite{6}, and \(\lambda\)-line black hole phase transition \cite{7} have been observed in various asymptotically AdS black hole configurations. Developments of BHC are deeply rooted in extending the usual thermodynamic phase space (known as the extended phase space \cite{8–10}), but criticality may take place in non-extended phase spaces as well (for these alternatives, see \cite{11–17} and references therein). However, in the extended phase space, there is an exact identification between intensive/extensive quantities of AdS black holes and fluid systems \cite{8} which is absent in non-extended phase space. Thus, it seems to be an important issue to consider the extended phase space thermodynamics and one can learn many things from it along different lines \cite{1,18–20}. At present, a great deal of research has devoted to exploring the \(P – v\) criticality and phase structures of different charged-AdS black hole configurations \cite{21–42}, exhibiting a vast variety of critical phenomena sometimes close to real-world experiments.

Elementally, \(P – v\) criticality can take place in both the canonical and grand canonical ensembles. However, in black hole physics, the case of GCE (fixed \(U(1)\) potential, \(\Phi\)) does not generally consider as an interesting case since only canonical ensemble phase transitions (the ensemble with fixed \(U(1)\) charge, \(Q\), at infinity) are observed for charged-AdS black holes in Einstein gravity \cite{8}. The same statement holds for rotating-AdS black holes; black hole phase transition takes place in the canonical ensemble with fixed angular momentum \((J)\) at infinity but not in the GCE with fixed angular velocity \((\Omega)\) at infinity \cite{23}. Nevertheless, this statement is no longer true in modified theories of gravity and, in some cases such as charged TBHs of massive gravity, the grand canonical phase structure is even richer than the canonical ensemble \cite{38}. So, modified gravity theories provide a more complex environment for TBHs which leads to richer phase structures and subsequently more possibilities for the investigation of BHC. On the other hand, modified gravities provide a good framework to examine BHC in various ensembles.

Among various candidates of the modified theories of gravitation, massive gravity theories (see some excellent reviews in Refs. \cite{43,44}) have received increased attention lately due to a wide variety of motivations such as explaining the current observations related to dark matter \cite{45,46}, describing the accelerating expansion of the

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appears in Fierz-Pauli theory \cite{55}). This theory requires a fiducial reference metric (\( g_{\mu \nu} \)) in which

\[
F = 2 \partial_{\mu} A^\nu - \partial^\mu A_\nu
\]

is defined as \( F_{\mu \nu} = \partial_{\mu} A^\nu - \partial^\nu A_\mu \). In particular, each choice of reference metric leads to a special family of massive gravity \cite{75} and in several studies, it is proved that the dRGT massive gravity is ghost-free by assuming various reference metrics such as Minkowski and degenerate (singular) reference metrics \cite{52, 57}. Therefore, considering different possibilities for the reference metric could lead to a variety of new solutions. Cosmological studies of massive gravity have primarily focused on the Minkowski reference metric \cite{53, 54}. On the other hand, there have been found a number of asymptotically flat and (A)dS black hole solutions in the context of massive gravity which relied either on the flat (Minkowski) reference metric or on a degenerate (spatially) and singular reference metric \cite{63, 70}. A particularly interesting case is the AdS black hole solutions with the degenerate (spatial) reference metric which is singular and has important applications in gauge/gravity duality \cite{70} as well as BHC \cite{57}. This class of solutions are dual to homogenous and isotropic condensed matter systems \cite{70, 72} and can describe different phases of matter with broken translational symmetry \cite{55, 72, 73}. Within the BHC framework, a new range of phase transitions can be found for this class of AdS black holes \cite{37}.

Concentrating on the subject of BHC, it is found that higher-dimensional TBHs in dRGT massive gravity can mimic the critical behavior of everyday substances in nature without the inclusion of any extra or unusual matter fields in the bulk action \cite{23}. In fact, a finite number of graviton's self-interaction potentials \((\mathcal{U}_i)\) exist in each dimension and, naturally, more potential terms contribute to higher dimensions. So, in principle, there are additional degrees of freedom in higher dimensions that enrich the thermodynamic phase space of TBHs. In Ref. \cite{35}, a generalization of this study for charged TBHs has been performed in both the canonical (fixed charge, \( Q \)) and grand canonical (fixed potential, \( \Phi \)) ensembles showing that the nature of phase transitions depends on the spacetime dimensions and also the ensemble one is dealing with. As indicated in Ref. \cite{35}, for charged TBHs within the framework of Maxwell-massive gravity, more interesting (critical) phenomena are observed in the GCE in comparison with the canonical ensemble. In the present paper, we intend to study the chemistry of TBHs in a more complex environment via massive gravity coupled with a nonlinear \( U(1) \) sector arisen from the power Maxwell invariant (PMI) electrodynamics \cite{70, 75} in the GCE.\footnote{In Ref. \cite{78}, the phase space structure of this model in the canonical ensemble has been investigated in detail. In Sect. \textbf{V I} of the present paper, we will compare the outcomes of criticality for both ensembles.} As will be evident in the present work, a massive spin-2 field Lagrangian \( \mathcal{L}_{\text{PMI}} \) minimally coupled to a PMI \( U(1) \) gauge field causes a range of novel black hole phase transitions in both the canonical and grand canonical ensembles. In addition, this model up to four graviton’s self-interactions \((\mathcal{U}_i)\) has already been explored in Ref. \cite{79} in order to investigate the electrical transport behavior of the dual field theory in the presence of a PMI gauge field for the 4- and 5-dimensional black brane solutions.

The theory of PMI electrodynamics as a matter source for AdS black holes has resulted in some interesting consequences \cite{17, 23, 74, 77, 74, 81}. The Lagrangian density of PMI electrodynamics is given by a power of the Maxwell Lagrangian as

\[
\mathcal{L}_{\text{PMI}} = (-\mathcal{F})^s,
\]

in which \( \mathcal{F} = F_{\mu \nu} F^{\mu \nu} \), \( s \) is the nonlinearity parameter and the Faraday tensor \( F_{\mu \nu} \) is defined as \( F_{\mu \nu} = 2 \partial_{\mu} A^\nu - \partial^\mu A_\nu \). In the limit \( s \to 1 \), the standard results of \( U(1) \) Maxwell electrodynamics can be recovered simply. There exist several motivations for considering such a model of nonlinear electrodynamics, listed below:

- This theory allows us to examine various deviations from the Coulomb law \cite{73, 74}. These deviations can be realized by comparing the electric potential in Maxwell’s electrodynamics and PMI theory, i.e.,

\[
\text{Maxwell} : V \propto \frac{1}{r^{d-3}} \quad \rightarrow \quad \text{PMI} : V \propto \frac{1}{r^{d-2s+1}}.
\]

The range of allowed deviations from the Coulomb law is bounded by demanding a finite value for the electric potential at infinity and also by implementing the causality and unitarity principles to the PMI Lagrangian (see more details in Sec. \textbf{19}), yielding \( 1 \leq s < \frac{d-3}{d-1} \). Using the PMI field equations in the vacuum, it is straightforward to show that the relative permittivity and the relative permeability are given by

\[
\frac{\varepsilon}{\varepsilon_0} = \frac{\mu_0}{\mu} = s(-\mathcal{F})^{s-1} = s(E^2 - \epsilon^2 B^2)^{s-1},
\]
which clearly indicates the polarization of vacuum (this result is different from that of Born-Infeld (BI) theory since their effective Lagrangians have different functional forms.) In the limit $s \to 1$, Eq. (1.3) reduces to the case of the absence of vacuum polarization effects ($\varepsilon/\varepsilon_0 = \mu_0/\mu = 1$), which is a familiar result in the Maxwell theory. In comparison with effective field theories inspired by QED \cite{82,55}, this vacuum polarization can be justified by the presence of virtual pair particles as effective dipoles in vacuum (the well-known screening effect). In 4-dimensions, for infinitesimal deviations from Coulomb law, the PMI potential law \cite{12} takes the form $V \propto 1/r$, where $\varepsilon$ is always positive due to the causality and unitarity conditions. As an example, for $r \gg 1$, we find that the PMI potential is slightly larger than the Coulomb law, qualitatively (but not quantitatively) in agreement with the QED Uehling potential \cite{80}. Besides, we should note that PMI is a classical model and setting an energy scale for classical theories is ambiguous. However, with a specific series expansion of the PMI Lagrangian (1.1), one can obtain a BI-type Lagrangian (see the next item) which simulates the results of QED the same as the Euler-Heisenberg theory.

- The series expansion of the PMI Lagrangian at the unknown constant ($\mathcal{F}_0$) yields \cite{23}

$$
\mathcal{L}_{PMI}(\mathcal{F}) = \sum_{n=0}^{\infty} \frac{\mathcal{L}^{(n)}_{PMI}(\mathcal{F}_0)}{n!} (\mathcal{F} - \mathcal{F}_0)^n
\simeq a_1 F + (s - 1) [a_0 + a_2 (-F)^2 + a_3 (-F)^3 + \ldots],
$$

where we have set $a_1 = -1$ to recover the standard linear Lagrangian ($-F$) in the Maxwell limit $s \to 1$. The other constants $a_i$’s depend on $s$ and $\mathcal{F}_0$. So, up to the second order approximation, we end up with a quadratic Maxwell invariant ($F^2$) in addition to the Maxwell Lagrangian in a way reminiscent of BI-type theories of nonlinear electrodynamics since BI-type theories generally have a series expansion as $\mathcal{L}_{BI} \simeq -F + b_1 F^2 + O(F^4)$ \cite{87}. Perhaps, the most important motivation for considering the BI-type nonlinear theories of electrodynamics comes from the fact that one loop QED corrections can be effectively understood via the quadratic Maxwell invariant term of these theories \cite{88,59}. As seen, this is also valid for the expansion of the PMI theory (1.3) up to the second order approximation. In fact, there exist a direct analogy between the relative permittivity and the relative permeability (in BI-type nonlinear models) and the vacuum permittivity and permeability tensors in QED. In such cases, one can establish a connection between the (classical electrodynamics) nonlinear parameter and the QED coefficient in the vacuum permittivity and permeability tensors \cite{90}. For these reasons, nonlinear models of electrodynamics can classically simulate some quantum mechanical effects of QED. In addition, it is shown that nonlinear electrodynamics with certain assumptions contains low-energy QED as a special case \cite{91}. In conclusion, at scales where loop QED corrections have significant contribution, such modifications can be classically considered without requiring any quantum mechanical treatment.

- The Maxwell action enjoys conformal invariance in 4-dimensions but it does not possess this symmetry in higher dimensions. However, the PMI action extends the conformal invariance in higher dimensions if the power is chosen as $s = d/4$ \cite{73}. Therefore, with this choice, the corresponding energy-momentum tensor will be traceless in arbitrary dimensions. In this way, it is possible to have a conformally invariant Maxwell source for black hole systems in arbitrarily higher dimensions.

- It has been recently indicated that, under certain conditions, the PMI theory with the Lagrangian density (1.1) can also remove the singularity of the electric field of point-like charges \cite{92}.

- The low energy limit of $E_8 \times E_8$ heterotic string theory reduces to an effective field theory including a quadratic Maxwell invariant term as well as the Maxwell Lagrangian \cite{93}. Moreover, it has been demonstrated that electromagnetic fields on the world-volumes of D-branes are governed by BI-type theories \cite{01}. In conclusion, inspired by string theory models, one can replace the conventional Maxwell Lagrangian with the Lagrangian of PMI theory which essentially involves higher-derivative corrections of the $U(1)$ gauge field. For this reason, such unconventional coupling to nonlinear source of electrodynamics has been already investigated in the context of AdS/CFT holographic superconductors \cite{02,08}.

Considering the massive gravity on AdS space in the presence of PMI electromagnetic field leads to a new class of TBHs exhibiting a range of phase transitions in both the canonical and the grand canonical ensembles. The nonlinear effect of electromagnetic field arising from the PMI theory introduces new interesting features for black hole thermodynamics beyond the Einstein-Maxwell gravity. Naturally, this property enriches the extended thermodynamic phase space, as proved for AdS black holes in Einstein gravity coupled with PMI electrodynamics in which spherically symmetric black holes exhibit $P - \nu$ criticality in both the canonical and the grand canonical ensembles \cite{23} (Note that the GCE phase transition for charged-AdS black holes is absent in Einstein-Maxwell gravity \cite{24}.). This theory...
also allows us to explore the extended phase space thermodynamics of charged TBHs with a conformally invariant
Maxwell (CIM) source, which has not yet been studied. As a matter of fact, this leads to AdS black hole spacetimes
with the property of the inverse square electric field (the so-called Coulomb law) in higher dimensions [76, 77] and
this is the case for the massive gravity theory, as will be discussed in this paper. It should be noted that the outcomes
of Einstein-Maxwell gravity is naturally recovered by taking the limits \( m_g \to 0 \) (massless graviton limit) and \( s \to 1 \)
(the Maxwell limit).

The chemistry of TBHs in modified gravity coupled to a nonlinear electromagnetic source is a challenging task and
requires further attention since the thermodynamics of TBHs takes a more complicated structure. Therefore, we study
the subjects of BHC and the extended phase space thermodynamics of charged TBHs in the context of PMI-massive
gravity in detail. To do so, this paper is organized as follows: in Sec. II we review the theory of PMI-massive
gravity and find the exact charged TBH solutions. In Sec. III we obtain the semi-classical partition function of
TBHs with the fixed potential at infinity as a boundary condition. Then, in Sec. IV we extract the thermodynamic
quantities using the grand partition function and show that the first law of extended thermodynamics is satisfied. In
Sec. V and Sec. VI we analyze the holographic phase transitions in detail and compare them with the results of the
canonical ensemble, respectively. Afterward, in Sec. VII we investigate briefly the critical behaviors of charged TBHs
in Einstein and massive gravity theories coupled with a CIM source. In Sec. VIII we compute the critical exponents
for the obtained TBHs in both ensembles. Finally, in Sec. IX we summarize the main results.

II. PMI-MASSIVE GRAVITY

Since in the presence of negative cosmological constant, gravitational field equations admit black hole solutions of
constant (positive, zero and negative) curvature for the event horizon, we consider such TBH spacetimes with various
horizon’s geometries. But, Ricci flat and hyperbolically symmetric black holes do not admit phase transitions in
Einstein gravity, so one needs additional degrees of freedom which enrich the thermodynamic phase space. To do
so, one can define an explicit mass term for gravitons, and then see that all kinds of TBHs can experience critical
behavior and phase transition exactly in the same way due to the massive graviton’s self-interaction potentials [37].
In addition, we intend to consider a more general (nonlinear) electrodynamics in terms of a power of the Maxwell
invariant [76, 77], more specifically with the form \((-F)^s\), which recovers the outcomes of Maxwell electrodynamics (if
\( s = 1 \)) and also enjoys conformal invariance in higher dimensions [76, 77] and
the Maxwell limit).

Therefore, one should take care of the causality and unitarity criteria in different models of nonlinear electrodynamics.
By implementing the causality and unitarity criteria to the local effective action of nonlinear electrodynamics, the
authors of Ref. [99] have found some requirements in terms of some inequality relations for the Lagrangian density as
\[
\frac{\partial L}{\partial F} \leq 0, \quad \frac{\partial^2 L}{\partial F^2} \geq 0 \quad \text{and} \quad \frac{\partial L}{\partial F} + 2F \frac{\partial^2 L}{\partial F^2} \leq 0.
\]
(2.1)

These conditions are equivalent to the positive convexity of the effective Lagrangian of nonlinear electrodynamics.
For the PMI model, the above requirements are satisfied as
\[
\frac{\partial L_{PMI}}{\partial F} = -s(-F)^{s-1} \leq 0,
\]
(2.2)
\[
\frac{\partial^2 L_{PMI}}{\partial F^2} = s(s-1)(-F)^{s-2} \geq 0,
\]
(2.3)

and
\[
\frac{\partial L_{PMI}}{\partial F} + 2F \frac{\partial^2 L_{PMI}}{\partial F^2} = -s(2s-1)(-F)^{s-1} \leq 0,
\]
(2.4)

which prove that PMI theory satisfies the causality and unitarity conditions for \( s \geq 1 \).
As stated, we are going to look for new phenomena that are not observed in the Einstein-Maxwell, Einstein-PMI and Maxwell-massive gravitational systems. Motivated by this and also following the aforementioned motivations in Sec. 1, we construct the full nonlinear theory of dRGT massive gravity \[49, 50\] minimally coupled with PMI electrodynamics and this is exactly what we need for our purpose. Hence, the bulk action for this theory on AdS is written as

\[ \mathcal{I}_b = - \frac{1}{16\pi G_d} \int d^d x \sqrt{-g} \left[ R - 2\Lambda + m_g^2 \sum_{i=1}^{d-2} c_i \mathcal{U}_i (g, f) + \mathcal{L}_{\text{PMI}} \right], \quad (2.5) \]

where \( \Lambda \) is the cosmological constant related to the AdS radius (\( \ell \)) by \( \Lambda = \frac{-(d-1)(d-2)}{2\ell^2} \). In the bulk action (2.5), \( m_g \) is the graviton’s mass parameter, \( c_i \)'s are arbitrary constants and \( \mathcal{U}_i \)'s are graviton’s self-interaction potentials constructed from the building blocks \( \mathcal{K}^\nu_\nu = \sqrt{g^\mu\alpha f_{\alpha\nu}} \) as

\[ \mathcal{U}_i = \sum_{y=1}^i (-1)^{y+1} (i - y)! \mathcal{U}_{i-y} [\mathcal{K}^y]. \quad (2.6) \]

Varying the bulk action (2.5) with respect to the physical metric \((g_{\mu\nu})\) and \(U(1)\) gauge potential \((A_\mu)\), one arrives at the following relation

\[ \delta\mathcal{I}_b = \frac{1}{16\pi G_d} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{-h} \delta \mathcal{F}_{\mu\nu} \delta g^{\mu\nu} + \frac{1}{8\pi G_d} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{-h} n^\alpha h^{\mu\nu} \delta g_{\mu\nu,\alpha} + \frac{s}{4\pi G_d} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{-h} F_{s-1}^{\mu\nu} \delta A_\nu + \frac{s}{4\pi G_d} \int_{\partial \mathcal{M}} d^{d-1} x \sqrt{-h} \left(-F\right)^{s-1} n_\mu F^{\mu\nu} \delta A_\nu, \quad (2.7) \]

where \( T_{\mu\nu} \) is the stress-energy tensor as

\[ T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} \mathcal{L}(F) - 2 F_{\mu\lambda} F_{\nu}^\lambda \frac{\partial \mathcal{L}(F)}{\partial F}, \quad (2.8) \]

and \( \mathcal{X}_{\mu\nu} \) is the consequence of varying dRGT Lagrangian as

\[ \mathcal{X}_{\mu\nu} = - \sum_{i=1}^{d-2} c_i \left[ \mathcal{U}_i g_{\mu\nu} + \sum_{y=1}^i (-1)^y (i - y)! \mathcal{U}_{i-y} [\mathcal{K}^y] \right]. \quad (2.9) \]

Hence, the gravitational and the electromagnetic field equations are obtained as

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} + m_g^2 \mathcal{X}_{\mu\nu} = \frac{1}{2} g_{\mu\nu} (-F)^s + 2 s F_{\mu\lambda} F_{\nu}^\lambda (-F)^{s-1}, \quad (2.10) \]

\[ \nabla_\mu [F^{s-1} F^{\mu\nu}] = 0. \quad (2.11) \]

In order to find TBH solutions, we make use of the following static ansatz for the physical metric \(g_{\mu\nu}\) in \(d = n + 2\) dimensions as

\[ ds^2 = -V(r) dt^2 + \frac{dr^2}{V(r)} + r^2 h_{ij} dx_i dx_j, \quad (i, j = 1, 2, 3, ..., n), \quad (2.12) \]

where the line element \(h_{ij} dx_i dx_j\) is the metric of \(n\)-dimensional (unit) hypersurface with the constant curvature \(d_1 d_2 k\) and volume \(\omega_n\) with the following forms

\[ h_{ij} dx_i dx_j = \begin{cases} 
  dx_1^2 + \sum_{i=2}^d \prod_{j=1}^{i-1} \sin^2 x_j dx_1^2, & k = 1 \\
  dx_1^2 + \sinh^2 x_1 \left( dx_2^2 + \sum_{i=3}^d \prod_{j=2}^{i-1} \sin^2 x_j dx_2^2 \right), & k = -1, \\
  \sum_{i=1}^d dx_i^2, & k = 0 
\end{cases} \quad (2.13) \]
in which \( d_i = d - i \) (henceforth we will use this notation). For the reference metric, we employ the following singular ansatz \[ 16, 70 \]

\[
f_{\mu \nu} = \text{diag} \left( 0, 0, c_0^2 h_{ij} \right),
\]

that breaks the translational invariance, a typical property of many condensed matter systems in the nature \[ 70, 73, 74 \]. Using this ansatz, the self-interaction potentials are explicitly computed in arbitrary dimensions as

\[
\mathcal{U}_i = \left( \frac{c_0}{r} \right)^{i+1} i! \prod_{j=2}^{d_i} d_j,
\]

indicating a total of \( (d - 2) \) self-interaction potentials exist in a \( d \)-dimensional spacetime.

At this stage, we concentrate on the electromagnetic field equation \[ 2.11 \]. Considering such an equation and using the static ansatz \( A_{\mu} = h(r) \delta_{\mu}^{01} \), the following differential equation is obtained for the scalar potential \( h(r) \)

\[
(2s - 1) r \frac{d^2 h(r)}{dr^2} + d_2 \frac{dh(r)}{dr} = 0,
\]

with the following exact solution

\[
h(r) = \phi - \left( \frac{2s - 1}{d_1 - 2s} \right) qr^{-\left( d_1 - 2s \right)},
\]

in which two integration constants \( q \) and \( \phi \) are related to the total electric charge and potential of spacetime, respectively. The constant \( \phi \) can be found by imposing the regularity condition at the horizon, i.e., \( A_\ell (r = r_+) = 0 \), yielding

\[
\phi = \left( \frac{2s - 1}{d_1 - 2s} \right) qr_+^{-\left( d_1 - 2s \right)}.
\]

Therefore, the electric potential at infinity with respect to the horizon is measured as

\[
\Phi = A_\mu \chi^\mu |_{r \rightarrow \infty} - A_\mu \chi^\mu |_{r \rightarrow r_+} = \left( \frac{2s - 1}{d_1 - 2s} \right) qr_+^{-\left( d_1 - 2s \right)},
\]

where \( \chi = \partial_\ell \) is the temporal Killing vector. Note that the range of nonlinearity parameter \( s \) is constrained by demanding a finite value for the electric potential at infinity \( (r \rightarrow \infty) \), yielding \( \frac{1}{2} < s < \frac{d_1}{2} \). By implementing the causality and unitarity conditions, we already found that the nonlinearity parameter \( s \) must satisfy the range \( s \geq 1 \). So, this together with the asymptotic behavior of the electric potential imply that the acceptable range of \( s \) is

\[
1 \leq s < \frac{d_1}{2}.
\]

As will be seen, all of the critical phenomena presented in this paper meet the above condition. In the limit \( s \rightarrow 1 \), \( A_\mu \) reduces to the gauge potential of the Maxwell case. Besides, the interesting case of conformally invariant Maxwell source is obtained via \( s = d/4 \). These different cases will be separately studied.

TBH solutions of the gravitational field equations \[ 2.10 \] can be obtained by solving its \( rr \)-component, i.e.,

\[
d_2 d_3 (k - V(r)) - d_2 r^{2} \left( \frac{dV(r)}{dr} \right) + \frac{d_1 d_2}{r^2} r^{2} + m^2 g \sum_{i=1}^{d^2} \left( c_0^2 c_i^2 r^{2i-1} \prod_{j=2}^{i+1} d_j \right) - \frac{2^s (2s - 1) q^{2s}}{r^{2(sd_4 + 1)/(2s - 1)}} = 0.
\]

This differential equation admits the following black hole solution

\[
V(r) = \frac{k + \frac{r^2}{\ell^2} - \frac{m}{\ell d_5} + m^2 g \sum_{i=1}^{d^2} \left( c_0^2 c_i^2 r^{2i-2} \prod_{j=2}^{i+1} d_j \right)}{\frac{2^s (2s - 1) q^{2s}}{d^2 (d_1 - 2s)^{r^{2(sd_4 + 1)/(2s - 1)}}}},
\]

in which \( m \) is an integration constant related to the finite mass of black hole. It is straightforward to check that the obtained solution satisfies other components of Eq. \[ 2.11 \] as well. If we take the massless graviton limit \( (m_g \rightarrow 0) \), the above metric function reduces to the case of Schwarzschild-AdS-PMI black hole spacetime reported in Refs. \[ 23, 70, 77 \]. Also, the charged black hole solution of Maxwell-massive gravity \[ 38 \] is recovered in the limit \( s \rightarrow 1 \). For the solution obtained, the existence of an essential singularity at the origin is simply confirmed by examining the behavior of the curvature scalars such as the Kretschmann scalar, which diverges only at \( r = 0 \). This singularity is
covered by the event horizon \((r_+)\), i.e., the largest root of metric function \((2.21)\) with a positive slope. Using the Euclidean trick \([100, 101]\), the Hawking temperature of this spacetime is given by

\[
\beta^{-1} = T = \frac{1}{4\pi} \left. \frac{dV(r)}{dr} \right|_{r=r_+} = \frac{1}{4\pi d_2 r^+} \left[ d_2 d_3 k + \frac{d_1 d_2}{\ell^2} r^2_+ + m_2^2 \sum_{i=1}^{d_2} \left( c_0^i c_1^i r^2_{i,-i} \prod_{j=2}^{i+1} d_j \right) - \frac{2^s (2s-1) q^{2s}}{r^{2s}(d_2+1)/(2s-1)} \right],
\]

which is in agreement with the definition of surface gravity \([102, 103]\). In addition, following Ref. \([10]\), the ADM mass formula can be obtained through the Hamiltonian approach as

\[
M = \frac{d_2 \omega_n}{16\pi} m,
\]

where \(m\) is determined from \(V(r_+) = 0\) as

\[
m = k r^d_3 + \frac{d_1}{\ell^2} + m_2^2 \sum_{i=1}^{d_2} \left( c_0^i c_1^i r^2_{i,-i} \prod_{j=2}^{i+1} d_j \right) + \frac{2^s (2s-1) q^{2s}}{r^{2s}(d_2+1)/(2s-1)}.
\]

It is worth mentioning that although the asymptotic symmetry group of black hole solution is not necessarily that of pure AdS, the validity of relation \((2.23)\) confirms that the Ashtekar-Magnon-Das (ADM) mass formula \([104, 105]\) still holds for black holes solutions of dRGT massive gravity. In the next section, we explicitly confirm the above mass formula by evaluating the black hole partition function and extract all the thermodynamic quantities.

### III. SEMI-CLASSICAL BLACK HOLE PARTITION FUNCTION

It is believed that all thermodynamic information of a given black hole system encodes in the associated semi-classical partition function. Thus, evaluating the black hole’s partition function may be the starting point for studying the BHC at the critical point. In this section, we are going to evaluate the semi-classical partition function of the obtained TBHs in the GCE via the Euclidean path integral formalism \([100]\).

According to Eq. \((2.18)\), the Hawking temperature \((2.22)\) can be rewritten in terms of fixed potential as

\[
\beta^{-1} = T = \frac{1}{4\pi d_2 r^+} \left[ d_2 d_3 k + \frac{d_1 d_2}{\ell^2} r^2_+ + m_2^2 \sum_{i=1}^{d_2} \left( c_0^i c_1^i r^2_{i,-i} \prod_{j=2}^{i+1} d_j \right) - \frac{2^s (d_2-2s) q^{2s}}{(2s-1)^2 s_{d_2-1}/(2s-1)} \right].
\]

The partition function in the GCE can be defined by a Euclidean path integral over the tensor field \(g_{\mu\nu}\) and vector gauge field \(A_\mu\) as

\[
\mathcal{Z}_{\text{GCE}} = \int \mathcal{D}[g, A] e^{-I_E[g, A]} \approx e^{-I_{\text{on-shell}}(\beta, r_+)} e^\Phi,
\]

where \(\mathcal{D}\) denotes integration over all paths and \(I_E\) represents the Euclidean version of the Lorentzian action \(I_Q\) by implementing the Wick rotation, \(t \rightarrow it\) \([106–108]\). In the GCE, the electric charge \((Q)\) fluctuates, but the associated potential \((\Phi)\) is fixed at infinity \([109]\). Using the boundary condition \(\delta A_\nu |_{\partial M} = 0\) which removes the last surface term in Eq. \((2.7)\), the electric potential is fixed at infinity. It leads to the fixed potential boundary condition necessary for the GCE. In what follows, we implement the Hawking-Witten prescription (the so-called subtraction method \([107, 110]\)) to evaluate the finite on-shell action. To do this, we only need to evaluate the on-shell bulk action \((2.5)\) for the TBH configurations and the associated AdS backgrounds. It should be noted that these background solutions are given by setting \(m = Q = 0\) in Eq. \((2.21)\), i.e.,

\[
ds^2 = V_0(r) dt^2_E + \frac{dr^2}{V_0(r)} + r^2 dx_i dx_j, \quad V_0(r) = k + r^2 + m_2^2 \sum_{i=1}^{d_2} \left( \frac{c_0^i c_1^i}{d_2 r^2_{i,-i}} \prod_{j=2}^{i+1} d_j \right)
\]

which are not pure AdS in usual sense; it will later become evident that this choice is the appropriate background, as proved for massive gravity theories in Refs. \([37, 38]\).

In order to find the finite on-shell action, we need to simplify the form of bulk Lagrangian. Considering the gravitational field equations \((2.10)\), the Ricci scalar is obtained as

\[
R = \frac{1}{d_2^2} \left( -\frac{d_1 d_2 d_3}{\ell^2} + 2 m_2^2 \chi - (d-4s)(-\mathcal{F})^s \right),
\]

\[(3.4)\]
where $\chi \equiv g^{\mu \nu} \chi_{\mu \nu}$. Regarding the above relation, the bulk Lagrangian in the action (2.6) may be rewritten as

$$\mathcal{L}_{\text{bulk}} = R + \frac{d_1 d_2}{\ell^2} + m_g^2 \sum_{i=1}^{d_2} c_i \mathcal{U}_i(g, f) + (-\mathcal{F})^s = \frac{2}{d_2} \left( -\frac{d_1 d_2}{\ell^2} + m_g^2 \sum_{i=1}^{d_2} (i-2) \frac{c_i^2}{r_i^2} \prod_{j=3}^{i+1} d_j + (2s-1)(-\mathcal{F})^s \right). \tag{3.5}$$

After tediously long calculation, the on-shell action for the obtained TBHs is computed as follows

$$\mathcal{I}_{\text{BH}} = \frac{\beta \omega_n}{16\pi G_d} \left[ \frac{2}{\ell^2} R^{d_1} - m_g^2 \sum_{i=1}^{d_2} \frac{(i-2) c_i^2}{d_1 - 1} \prod_{j=3}^{i+1} d_j - \frac{2^{s+1}(d_1 - 2s)^{2s-1}}{d_2(2s-1)^{2(s-1)} r_+^2} \mathcal{F}^{2s} \right], \tag{3.6}$$

in which $\beta$ is presented in Eq. (3.1) and $"R"$ is an upper cutoff regularizing the on-shell action. Doing the same computation for the case of AdS thermal background, we arrive at the following relation

$$\mathcal{I}_{\text{AdS}} = \frac{\beta_0 \omega_n}{16\pi G_d} \left[ \frac{2}{\ell^2} R^{d_1} - m_g^2 \sum_{i=1}^{d_2} \frac{(i-2) c_i^2}{r_i^2} \prod_{j=3}^{i+1} d_j - \frac{2^s(d_1 - 2s)^{2s-1}}{d_2(2s-1)^{2(s-1)} r_+^2} \mathcal{F}^{2s} \right]. \tag{3.7}$$

where $\beta_0$ is the period of thermal background. Finally, we should subtract AdS background action (3.7) from on-shell (black hole) action (3.6). To do so, both the black hole spacetime and the thermal background must have the same geometry at $r = R$, i.e., setting $\beta_0 \sqrt{\mathcal{V}_0(r)}|_{r=R} = \beta \sqrt{\mathcal{V}(r)}|_{r=R}$ which leads to $\beta_0 = \beta \left( 1 - \frac{m_g^2}{2 R^2} + O(R^{-2(d_1)}) \right)$. Now, applying the mentioned subtraction yields

$$\mathcal{I}_{\text{on-shell}} = \lim_{R \to \infty} (\mathcal{I}_{\text{BH}} - \mathcal{I}_{\text{AdS}}) \quad \text{or} \quad \mathcal{I}_{\text{on-shell}} = \frac{\beta \omega_n}{16\pi G_d} \left[ k - \frac{r_+^2}{r_i^2} + m_g^2 \sum_{i=1}^{d_2} \frac{(i-1) c_i^2}{r_i^2} \prod_{j=3}^{i+1} d_j - \frac{2^s(d_1 - 2s)^{2s-1}}{d_2(2s-1)^{2(s-1)} r_+^{2s}} \mathcal{F}^{2s} \right]. \tag{3.8}$$

Applying appropriate adjustments, this relation can be examined in various limits, e.g., one can obtain the finite on-shell action of Einstein-PMI gravity (as $m_g \to 0$) or that of Maxwell-massive gravity (as $s \to 1$) or those of Einstein theory or massive gravity coupled with conformally invariant Maxwell source (as $s \to d/4$).

**IV. EXTENDED PHASE SPACE THERMODYNAMICS**

This section is devoted to calculating the conserved and thermodynamic quantities, and examining the first law of thermodynamics in the extended phase space. First, we can get the free energy ($F$) from the (finite) on-shell action by using the relation $\mathcal{I}_{\text{on-shell}} = \beta F$. But, in the extended phase space ($\Lambda = -\frac{d_1 d_2}{2 \ell^2} = -8\pi P$), the free energy ($F$) is identified as the Gibbs free energy (referred to as $G$). Hence, using Eq. (3.3), the Gibbs free energy is computed as

$$G_{\Phi} = -\beta^{-1} \ln \mathcal{Z}_{\text{GCE}}(T, P, \Phi) = \frac{\omega_n r_+^{d_1}}{16\pi} \left[ k - \frac{16\pi P r_i^2}{d_1 d_2} + m_g^2 \sum_{i=1}^{d_2} \frac{(i-1) c_i^2}{r_i^2} \prod_{j=3}^{i+1} d_j - \frac{2^s(d_1 - 2s)^{2s-1}}{d_2(2s-1)^{2(s-1)} r_+^{2s}} \mathcal{F}^{2s} \right]. \tag{4.1}$$

Also, the entropy (as an extensive quantity) conjugate to the temperature (as an intensive quantity) satisfies the so-called area law ($S = A/4$) in the Einstein and massive gravities (this can be easily verified by use of the relation $S = \frac{1}{4} \int d^d x \sqrt{\bar{g}}$, where $\bar{g}$ is the induced metric on the horizon). Now, using the Legendre transformation ($G_{\Phi} = H - TS - Q\Phi$), we can find the corresponding enthalpy ($H$) in the extended phase space. The ADM mass of TBHs ($M$) which is interpreted as the enthalpy ($H$), can be obtained in terms of fixed potential ($\Phi$) as follows

$$M = G_{\Phi} + TS + Q\Phi = \frac{d_2 \omega_n}{16\pi} \left[ kr_+^{d_1} + \frac{16\pi P r_i^2}{d_1 d_2} + m_g^2 \sum_{i=1}^{d_2} \frac{c_i^2}{d_2} \prod_{j=3}^{i+1} d_j + \frac{2^s(d_1 - 2s)^{2s-1}}{d_2(2s-1)^{2(s-1)} r_+^{2s}} \mathcal{F}^{2s} \right]. \tag{4.2}$$

Considering $P = -(d_1 d_2)/(16\pi \ell^2)$ and Eq. (2.18), we find that the above relation is in complete agreement with the obtained ADM mass formula in Eqs. (2.29) and (2.24). The enthalpy is a function of $S$ (through the dependency of $r_+$
and \( S \), \( P \) and \( \Phi \), and therefore, the other thermodynamic quantities can be simply extracted. The thermodynamic volume and \( U(1) \) charge of TBHs are respectively given by
\[
V = \left( \frac{\partial M}{\partial P} \right)_{S,\Phi} = \frac{\omega_n d_1}{d_1} + , \tag{4.3}
\]
and
\[
Q = \left( \frac{\partial M}{\partial \Phi} \right)_{S,P} = \frac{\omega_n 2^{s-1}s}{4\pi} q^{2s-1}. \tag{4.4}
\]
It is easy to check out the finite electric charge of TBHs is in line with the result of the usual method of calculating the flux of the electric field at infinity. In addition, from the enthalpy (ADM mass) formula \(4.2\), the validity of Hawking temperature in the GCE \(4.1\) is verified through the use of the standard thermodynamic relation \( T = (\partial M/\partial S)_{P,\Phi} \).

To sum up, we deduce that all intensive (\( P, \Phi, T \)) and corresponding extensive (\( V, Q, S \)) quantities satisfy the extended first law of thermodynamics in the enthalpy representation with the following form
\[
dM = \left( \frac{\partial M}{\partial S} \right)_{P,\Phi} dS + \left( \frac{\partial M}{\partial P} \right)_{S,\Phi} dP + \left( \frac{\partial M}{\partial \Phi} \right)_{S,P} d\Phi = T dS + V dP + Q d\Phi. \tag{4.5}
\]

One can naturally implement the Legendre transform \( G_\Phi = M - TS - Q\Phi \) to write down the extended first law in the Gibbs energy (grand potential) representation as \( dG_\Phi = -SdT + VdP - Qd\Phi \).

At this stage, we can discuss the Smarr formula for these types of TBHs according to the scaling argument. It is confirmed that these thermodynamic quantities obey the Smarr formula, which for this class of solutions is given by
\[
d_3M = d_2TS - 2PV + \sum_{i=1}^{d_2} (i-2)C_i c_i + \frac{(sd_1 + 1)}{s(2s - 1)} \Phi Q, \tag{4.6}
\]
where
\[
C_i = \left( \frac{\partial M}{\partial c_i} \right)_{S,P,Q,c_j \neq i} = \frac{\omega_n 2^s}{16\pi} \frac{m_{g_1}^2 e_1 r_i^{d_i+1}}{d_i+1} \prod_{j=2}^i d_j. \tag{4.7}
\]
The variable \( C_i \) is a quantity conjugate to the massive couplings \( (c_i) \), which is necessary for the consistency of both the extended first law of thermodynamics and the corresponding Smarr formula. Note that the pair \( C_2c_2 \) does not appear in the Smarr relation (since \( c_2 \) has zero scaling \(3.1\)), so one cannot define a conjugate potential \( C_2 \) in the first law. To conclude, Eq. \(4.6\) suggests the extended first law should be written as
\[
dM = TdS + VdP + Qd\Phi + \sum_{i=1}^{d_2} C_i d c_i. \tag{4.8}
\]
However, based on AdS/CFT duality, the holographic interpretation of these new pairs \( (C_i c_i) \) remains an open question.\(^2\)

V. HOLOGRAPHIC PHASE TRANSITIONS

In order to study BHC at the critical point (or equivalently holographic phase transitions), we need to examine the equation of state (EoS) of TBHs. To do so, we first obtain the associated EoS by inserting \( \Lambda = -8\pi P \) into the Hawking temperature \(3.1\) as follows
\[
P = \frac{d_2}{4r_+} - \frac{d_2 d_3 k_{\text{eff}}}{16\pi r_+^2} - \frac{m_{g_1}^2}{16\pi} \sum_{i=3}^{d_2} \left( \frac{e_1 c_1}{r_i^2} \prod_{j=2}^{i+1} d_j \right) + \frac{2s(d_1 - 2s)^{2s}}{16\pi(2s - 1)^{2s-1}} \left( \frac{\Phi}{r_+} \right)^{2s}, \tag{5.1}
\]
\(^2\) For an example, see Ref. \(111\) for the holographic interpretation of couplings in the extended phase space thermodynamics of Lovelock gravity theories.
where $k_{\text{eff}}$ is the effective topological factor \(^{(37)}\)

$$k_{\text{eff}} \equiv [k + m_2^2c_2^2]$$  \hspace{1cm} (5.2)

and $\tilde{T}$ is the shifted Hawking temperature (first proposed in \(^{(30)}\))

$$\tilde{T} = T - \frac{m_2^2c_0c_1}{4\pi}.$$  \hspace{1cm} (5.3)

Comparing Eq. (5.1) with the expansion of vdW EoS implies that the event horizon radius (not the thermodynamic volume, $V$) is associated with the vdW fluid specific volume ($v$) as \(^{3}\)

$$v = \frac{4r_+d_2^2}{d_2}$$  \hspace{1cm} (5.4)

where $\ell_p$ is the Planck length (here, $\ell_p = 1$). The possible critical point(s) can be obtained by finding the inflection point(s) of isotherms in the $P - v$ (or $P - r_+$) diagrams, i.e.,

$$\left( \frac{\partial P}{\partial v} \right)_T = 0 \quad \leftrightarrow \quad \left( \frac{\partial P}{\partial r_+} \right)_T = 0$$

$$\left( \frac{\partial^2 P}{\partial v^2} \right)_T = 0 \quad \leftrightarrow \quad \left( \frac{\partial^2 P}{\partial r_+^2} \right)_T = 0.$$  \hspace{1cm} (5.5)

The above criteria lead to the following equation for the possible critical horizon radius (radii)

$$2d_2d_3k_{\text{eff}}r_+^{d_4} + m_2^2\sum_{i=3}^2 \left( i(i-1)c_0c_1r_+^{d_2-i} \prod_{j=2}^{i+1} d_j \right) - \frac{2s+1}{(2s-1)^{2(s-1)}} \frac{\Phi^{2s}}{(2s-1)^{2(s-1)}} = 0.$$  \hspace{1cm} (5.6)

In what follows, we investigate the physical solutions of the above equation and then analyze the associated holographic phase transitions case by case with details. However, we should note that our results in the next sections are generic for all types of TBHs. In fact, the effective topological factor $k_{\text{eff}}$ presented in Eq. (5.2) implies that all kinds of TBHs can exhibit criticality in the same way provided that the same value is set for $k_{\text{eff}}$ while keeping other parameters of the theory fixed (this is always possible by varying the massive coupling $c_2$). Hence, THBs do exhibit the critical behavior and the associated phase transition exactly in the same way, even with the same critical points (first indicated in Ref. \(^{37}\)).

It should be noted that the factor $k_{\text{eff}}$ is introduced for both canonical and grand canonical ensembles in PMI-massive gravity. However, in the context of Maxwell-massive gravity, the effective factor in the GCE is given by $k_{\text{eff}} \equiv [k + m_2^2c_2^2 - 2(d_3/d_2)\Phi^2]$, which means that the $U(1)$ potential ($\Phi$) is absorbed into the effective topological factor \(^{(38)}\). The situation is different when the Maxwell theory is generalized to the PMI theory. Indeed, the nonlinearity parameter ($s$) does not permit, in general, the electrostatic potential sector to be absorbed into $k_{\text{eff}}$. So, the effective topological factor in the GCE is the same as the canonical ensemble in PMI-massive gravity ($s \neq 1$).

### A. vdW behavior and SBH/LBH phase transition

The standard vdW behavior takes place when EoS admits only one physical critical point \(^{3}\). Evidently, considering the EoS (5.1) up to two interaction potentials $O(U_2)$, a critical point may exist if $s \neq 1$. So, there is a possible critical point given by the following critical radius in $d \geq 4$ dimensions

$$r_c = \left( \frac{2s+1}{d_2d_3(2s-1)^{2s-2k_{\text{eff}}}} \frac{\Phi^{2s}}{(2s-1)^{2(s-1)}} \right)^{1/(2s-2)},$$  \hspace{1cm} (5.7)

in which $k_{\text{eff}} > 0$. Obviously, in the massless limit ($m_2 = 0 \rightarrow k_{\text{eff}} = k$), the vdW phase transition takes place only for spherically symmetric AdS black holes ($k = 1$). But, the graviton’s mass generates additional degrees of freedom which may lead to the existence of criticality for all types of TBHs completely in the same way.

To confirm that phase transition takes place with this critical radius for all types of TBHs, we have depicted Fig. 1. In the left panel of Fig. 1, the characteristic behavior of pressure as a function of the event horizon radius is displayed. As it is observed, for isotherms in the range of $T < T_C$, an (unphysical) oscillatory part exists which
implies the two phase-behavior (this oscillatory part is replaced by a line of constant pressure according to Maxwell’s equal-area law). For isotherms with \( T > T_C \), the behavior of ideal gas is detected. In addition, the subcritical isobars of \( T - r_+ \) diagrams (i.e., isobars with \( P < P_C \)) support the two phase-behavior in agreement with the result of \( P - r_+ \) diagrams as well (for more details on this issue see Ref. [21]). Furthermore, the \( G - T \) diagram in the right panel of Fig. 1 confirms the characteristic swallowtail behavior of \( \text{vdW} \) (first-order) phase transition for isobaric curves with \( P < P_C \), in a way reminiscent of \( \text{vdW} \) fluid [112]. Interestingly, at the critical point, the thermodynamic quantities up to two interaction potentials \( O(U_2) \) satisfy the following universal ratio

\[
\rho_c = \frac{P_C v_a}{T_C} = \frac{2s - 1}{4s},
\]

in terms of the shifted Hawking temperature \([5.3]\). Thus, respecting the constraint on the nonlinearity parameter Eq. \([2.19]\), we always can recover the well-known \( \text{vdW} \) ratio as \( \rho_c = 3/8 \) if we set \( s = 2 \) in arbitrary \( d \geq 6 \) dimensions.

![FIG. 1: \( \text{vdW} \) phase transition for PMI-massive TBHs in the GCE: \( P - r_+ \) (left), \( T - r_+ \) (middle) and \( G - T \) (right) diagrams; we have set \( k_{\text{eff}} = 1, d = 4, m_0 = 1, c = c_1 = 1, \Phi = 1 \) and \( s = 1.2 \)

Critical data: \( (T_C = 0.6912, P_C = 1.1998 \) and \( r_c = 0.07434)\).](image)

### B. RPT and LBH/SBH/LBH phase transition

The black hole version of reentrant phase transition (RPT), first reported in Refs. [4, 5], can be observed in a black hole configuration whose EoS possesses two possible critical points, i.e., satisfying criteria in Eq. \([5.5]\). However, the pressure and the temperature associated with these critical points are positive definite, but, one of them cannot minimize the Gibbs free energy and consequently, the global thermal stability fails at this point. So, only one (physical) critical point \((T_C, P_C)\) which satisfies criteria \([5.5]\) remains in the phase space. Further investigation shows that \([4]\) for a certain range of pressure, three separate phases of black holes (small, intermediate and large) emerge, indicating the existence of two new critical points in the phase space. These points are usually referred to as \((T_Z, P_Z)\) and \((T_{1T}, P_{1T})\) at the virtual triple point \((T_{1T}, P_{1T})\), the first-order and the zeroth-order coexistence lines join together. Both the first-order and zeroth-order coexistence lines are terminated at critical points \((T_C, P_C)\) and \((T_Z, P_Z)\), respectively.

We examined the mentioned conditions for our TBH setups and also for the other cases in the literature (for example see \([30, 33]\) indicating this is typical of any RPT in AdS black hole physics. In order to have RPT phenomenon in PMI-massive gravity, we have to consider this model at least up to three graviton’s self-interaction terms \( O(U_3) \) and assume that the other massive couplings vanish in higher dimensions. This assumption leads to the following equation governing the critical points for \( d \geq 5 \) dimensions

\[
d_{2d_3}k_{\text{eff}} r_+ + 3d_2d_3d_4m_0^2 c_0^3 c_3 - \frac{2s(d_1 - 2s)^2 s^{2s}}{(2s - 1)^{2s - 2} r_+^{2s - 3}} = 0. \tag{5.9}
\]

By appropriately fine tuning the parameters, the above equation could admit two positive roots, indicating the existence of two possible critical points. So, we have adjusted them to observe the reentrant behavior of phase
In order to have a triple point phenomenon, two of those possible critical points, referred to as (C3, P3) and (C2, P2), necessarily have to be physical (minimizing the Gibbs free energy). It turns out that another critical point is always unphysical which cannot minimize the Gibbs free energy. We confirm that this is typical behavior of any triple point phenomenon in AdS black holes.

Solving the above equation, numerically, for a suitable set of parameters, we have presented an example for this phenomenon, depicted in Fig. 3. It is obvious that for the isobars in the range of \( P_{C2} < P < P_{C1} \), the standard swallowtail behavior takes place indicating a first-order phase transition between large and small black holes. By decreasing the pressure \( (P_{r}\ T < P < P_{C2}) \), the appearance of two swallowtails is observed which indicates the emergence of third-phase (intermediate) and consequently SBH/IBH/LBH phase transition. By further decreasing the pressure, the mentioned swallowtails merge at \( (T_{r}, P_{r}) \), implying a black hole version of the triple point. For \( P < P_{r} \), we only observe the standard vdW behavior again. In conclusion, the resulting phase structure for TBHs looks almost like those phase structures of many materials in nature [110].
D. Anomalous vdW phase transition

As conjectured in Ref. [38], the anomalous vdW phase transition (also known as vdW-type phase transition) can elementally happen by varying the parameter space of the theory in spacetime dimensions that triple point phenomenon takes place. In fact, for the case of the triple point, the EoS admits three possible critical points in which two of them are physical and the other one is unphysical. So, we can alter at least one of the parameters in order to change this situation to the case that the EoS admits three possible (second-order) critical points in which the only one of them is physical. To do so, here, we vary the parameters of the nonlinear electromagnetic sector of the previous example related to the triple point in Sect. V C (see Fig. 3). We choose the new nonlinearity parameter $s$ and the new potential $\Phi$, respectively, as $s = 2.2$ and $\Phi = 4.5$. For this example, the corresponding $G - T$ and $P - T$ diagrams are shown in Fig. 4. Evidently, the resultant holographic phase transition is of SBH/LBH (or vdW) type with an anomaly in the standard swallowtail behavior. This anomaly (illustrated in Fig. 4) takes place for $P_{C_2} < P < P_{C_3}$ and does not lead to any new phase transition since it cannot minimize the Gibbs free energy. In addition, the corresponding $P - T$ diagram confirms the two-phase behavior, indicating the well-known SBH/LBH phase transition. Since the triple point phenomenon is observed in higher dimensions ($d \geq 6$), so the anomalous vdW behavior takes place in $d \geq 6$ dimensions as well.

VI. COMPARISON WITH THE RESULTS OF CANONICAL ENSEMBLE

It is instructive to compare the outcomes of grand canonical criticality with those counterparts in the canonical (fixed charge) ensemble. We will use the results of Ref. [78], in which the chemistry of TBHs in the canonical ensemble was extensively explored within the framework of PMI-massive gravity. Regarding the canonical ensemble, it was proved that van der Waals phase transition shows up in $d = 6$ dimensions as well. However, it was shown that this claim can analytically be proved for $s = 1$ in $d = 5$ and 6 dimensions (by using numerical investigation in $d = 7, 8, 9$ dimensions, this phenomenon did not observe too). So the existence of RPT phenomenon is not ruled out completely in canonical ensemble since only an analytical proof can judge about this problem. Whether or not such a phenomenon exist in the canonical ensemble remains an open question. Besides, we observe that the gravitational version of triple point and also anomalous vdW behavior can take place in $d \geq 6$ dimensions for both canonical and grand canonical ensembles.

One of the interesting results of Refs. [78] is that BHC for the case of massive gravity coupled to both the Maxwell and PMI electrodynamics yields, qualitatively, the same results in the canonical ensemble. However, because of
nonlinearity parameter \( s \), there exist more possibilities (and consequently more parameter space) for phase transition in PMI-massive gravity. But, comparing the GCE of TBHs in Maxwell-massive gravity and PMI-massive gravity models exhibits that each critical phenomenon commences appearing in diverse dimensions depending on the gravity model one is dealing with. So, in PMI-massive gravity, these phenomena can show up in lower dimensions with respect to Maxwell-massive gravity. We have summarized these results in Table I. In conclusion, BHC in the GCE in comparison with canonical ensemble for both models is completely different and seemingly richer.

### VII. GRAVITY COUPLED WITH CONFORMALLY INvariant MAXwell SOURCE

In this section, we briefly discuss the case of (massive) gravity coupled with conformally invariant Maxwell source, i.e., setting \( s = d/4 \) for the PMI Lagrangian \([11]\). The matter sector consists of a conformally invariant extension of Maxwell Lagrangian as \( L_{\text{CIM}} = -(\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})^{d/4} \) (for more discussions see Refs. \([76, 77]\)) and one can check out that the black hole solutions exist in this model as well. The bulk action of this model can be written as

\[
\mathcal{I}_b = -\frac{1}{16\pi G_d} \int_{\mathcal{M}} d^dx \sqrt{-g} \left[ R - 2\Lambda + m^2 \sum_{i=1}^{d_2} c_i \mathcal{U}_i(g, f) + (-\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu})^{d/4} \right].
\]
It is easy to show that the matter action enjoys conformal invariance (i.e., \( g_{\mu\nu} \to \Omega^2 g_{\mu\nu} \) and \( A_\mu \to A_\mu \)) and the associated energy-momentum tensor is given by

\[
T_{\mu\nu} = \frac{1}{2} g_{\mu\nu} (-F)^2 + \frac{d}{2} F_{\mu\lambda} F^\nu_{\lambda} (-F)^2 - 1,
\]

which is traceless \( T^\mu_\mu = 0 \). The (static) charged TBH solutions with a purely radial electric field are easily found by using our previous line element ansatz in Eq. (2.12) with the singular (degenerate) reference metric defined in Eq. (2.14). Also the metric function \( V(r) \) is simply obtained by inserting \( s = d/4 \) into the relation (2.21). So, like the case of Einstein gravity coupled with the CIM electromagnetic source, TBH solutions appear for the context of CIM-massive gravity model in arbitrary dimensions as well. Subsequently, the entire thermodynamic quantities such as temperature, entropy, electric potential and charge, ADM mass, volume, etc can be obtained by the use of these substitutions for the case of (massive) gravity with a conformally in variance Maxwell source. We do not intend to repeat them here and just mention that these quantities satisfies the Smarr relation with the following explicit form

\[
d_3 M = d_2 TS - 2PV + \sum_{i=1}^{d_2} (i-2)c_i c + \frac{2d_2}{d} \Phi Q.
\]

Now, we concentrate on the black hole EoS in both canonical (fixed charge) and grand canonical (fixed potential) ensembles. EoS relations in the canonical and grand canonical ensembles are respectively given by

\[
P_{\text{can}} = \frac{d_2 T}{4r_+} - \frac{d_2 d_3 k_{\text{eff}}}{16\pi r_+^2} - \frac{m_g^2}{16\pi} \sum_{i=3}^{d_2} \left( \frac{c_i^2 c + i+1}{r_+} \prod_{j=2}^{d_2} d_j \right) + \frac{2d_4/4}{32\pi} \frac{d_2}{\sqrt{q} r_+^d},
\]

and

\[
P_{\text{grand}} = \frac{d_2 T}{4r_+} - \frac{d_2 d_3 k_{\text{eff}}}{16\pi r_+^2} - \frac{m_g^2}{16\pi} \sum_{i=3}^{d_2} \left( \frac{c_i^2 c + i+1}{r_+} \prod_{j=2}^{d_2} d_j \right) + \frac{2d_4/4}{32\pi} \frac{d_2}{\sqrt{q} r_+^d} \Phi^{d/2}. \tag{7.5}
\]

Let us first discuss the case of GCE. It is inferred from Eq. (7.5) that the final results of the TBH phase transitions in the fixed potential ensemble are qualitatively the same as the neutral TBHs of pure massive gravity \([37]\) or charged TBHs in the fixed potential ensemble of Maxwell-massive gravity \([38]\) in even dimensions \((d = 4, 6, 8, \ldots)\). Indeed, the electromagnetic sector in EoS (7.5) is absorbed into the \(i = d/2\)th massive term. For instance, in \(d = 6\), the vdW and RPT phenomena are observed, exactly the same as the \(P - v\) criticality of (electrically) neutral TBHs in pure massive gravity \([37]\) or charged TBHs in Maxwell-massive gravity in the GCE \([38]\). However, there is no criticality for \(d = 4\) dimensions, but the vdW phase transition could take place in \(d = 5\) dimensions. The other critical phenomena, i.e., anomalous vdW and triple point, will show up in \(d \geq 7\) dimensions.

In the case of canonical ensemble, we observe the standard vdW phase transition in \(d \geq 4\) dimensions but this time for all kinds of TBHs in the context of massive gravity. For \(d \geq 6\), besides the standard vdW, the anomalous vdW and triple point phenomena are also observed. However, by the numerical investigation, we could not find evidence for the RPT phenomenon.

In addition, the vanishing of graviton’s mass \((m_g \to 0)\) leads to Einstein gravity coupled to a CIM electromagnetic source, and regarding this model, one can straightforwardly deduce the following items: the vdW critical behavior occurs for \(d \geq 4\) dimensions in the canonical ensemble and for \(d \geq 5\) dimensions in the GCE. Obviously, criticality takes place only for spherically symmetric AdS black holes. We have summarized the results of criticality in table \([11]\) for convenience.

### VIII. CRITICAL EXPONENTS

In this section, we intend to compute the independent critical exponents \((\alpha, \beta, \gamma, \delta) \) \([116, 118]\) associated to the obtained nonlinearly charged TBHs in PMI-massive gravity. We show that these exponents match the vdW fluid system, i.e., \((\alpha, \beta, \gamma, \delta) = (0, 1/2, 1, 3)\). These independent exponents are listed in the following relations \([3]\)

\[
C_v = T \left( \frac{\partial S}{\partial T} \right)_v \propto |r|^{-\alpha}, \tag{8.1}
\]

\[
\eta = v_t - v_g \propto |r|^{\beta}, \tag{8.2}
\]
TABLE II: Holographic phase transitions in Einstein and massive gravity theories with a CIM source for the canonical and the grand canonical ensembles.

| Theory | Canonical ensemble | GCE |
|--------|--------------------|-----|
| Einstein gravity coupled to CIM source (s = d/4) | vdW for d ≥ 4 (only spherical BHs) | vdW for d ≥ 5 (only spherical BHs) |
| Massive gravity coupled to CIM source (s = d/4) | vdW for d ≥ 4 Anomalous vdW for d ≥ 6 Triple point for d ≥ 6 | vdW for d ≥ 5 RPT for d ≥ 6 Anomalous vdW for d ≥ 7 Triple point for d ≥ 7 |

\[ \kappa_T = -\frac{1}{v} \frac{\partial v}{\partial P} \bigg|_T \propto |\tau|^{-\gamma}, \]  

\[ |P - P_c| \propto |v - v_c|^{\delta}, \]  

in terms of the following reduced variables

\[ p = \frac{P}{P_C}, \quad \tau = \frac{T - T_C}{T_C}, \quad w = \frac{v - v_c}{v_c}, \quad \rho_c = \frac{P_C v_c}{T_C}. \]

We perform this analysis for both canonical and grand canonical ensembles. First of all, we determine the exponent \( \alpha \). For static TBHs, we have \( C_v = 0 \), so the exponent \( \alpha \) is simply equal to zero (\( \alpha = 0 \)). Now, by studying the equation of state near the critical point, we can determine the rest of them. Expanding the equation of states in the vicinity of critical point yields

\[ p = 1 + \frac{\tau}{\rho_c}(1 - w) + h(v_c, c_i, s)w^3 + O(\tau w^2, w^4), \]

where the details of function \( h(v_c, c_i, s) \) does not alter the final result. Schematically, this kind of expansion for any black hole setup ensures that the critical exponents match the standard critical exponents of vdW fluid, as will be clarified in a moment. The explicit form of the function \( h(v_c, c_i, s) \) in the canonical ensemble is obtained as

\[ h = \frac{m_g^2G_0c_1 - 4\pi T C}{4\pi P_Cv_c} + \frac{4d_3k_{\text{eff}}}{d_2\pi P_Cv_c^2} \frac{2^{s-9} \times 4^{2(s-1)/(2s-1)}s(2s - 3)}{3\pi P_C(s - 1/2)^2d^{(2s-1)/2}(2s-1)^2v_c^{1/(2s-1)}} \]

\[ + \frac{40m_g^2}{d_2\pi P_Cv_c^3} \frac{c_1c_3c_4d_1d_2}{c_5d_4d_5} + O \left( \frac{1}{v_c^5} \right), \]

and in the GCE is given by

\[ h = \frac{m_g^2G_0c_1 - 4\pi T C}{4\pi P_Cv_c} + \frac{4d_3k_{\text{eff}}}{d_2\pi P_Cv_c^2} \frac{s(s + 1)(s^2 - 1/4)2^{5s}}{6\pi P_C} \left( \frac{(d_1 - 2s)\Phi}{(2s - 1)d_2v_c} \right)^{2s} \]

\[ + \frac{40m_g^2}{d_2\pi P_Cv_c^3} \frac{c_1c_3c_4d_1d_2}{c_5d_4d_5} + O \left( \frac{1}{v_c^5} \right). \]

To obtain the exponent \( \beta \), which determines the behavior of the order parameter \( \eta \) on the isotherms, we have to differentiate the expansion \[ (8.6) \] for a fixed \( \tau < 0 \), yielding

\[ dp = \left( -\frac{\tau}{\rho_c} + 3h(v_c, c_i, s)w^2 \right) dw. \]

\[ ^3 \text{In the limit } s \to 1, \text{ the function } h(v_c, c_i, s) \text{ should reduce to the Maxwell-massive case in Ref. 38. For the electromagnetic part of this function, there is a mismatch between the coefficients of Eq. (8.7) here (as } s \to 1 \text{) and its counterpart in Ref. 38. Actually, to correctly match the coefficients we note that there is a minor typo in Ref. 38. However, as mentioned, the details of this function are not important at all and does not change the final result.} \]
Inserting this into the Maxwell's equal area law \((\oint v dP = 0)\) leads to
\[
\int_{w_l}^{w_s} w dp = -\frac{\tau}{2\rho_c}(w_s^2 - w_l^2) + \frac{3h}{4}(w_s^4 - w_l^4) = 0,
\] (8.10)
where \(w_l\) and \(w_s\) denote the volume of large and small black holes, respectively. The pressure of different black hole phases keeps unchanged at the critical point, and therefore, for the expansion of EoS \((8.6)\) one arrives at
\[
1 + \frac{\tau}{\rho_c}(1 - w_s) + h(v_c, c_i, s)w_s^3 = 1 + \frac{\tau}{\rho_c}(1 - w_l) + h(v_c, c_i, s)w_l^3,
\] (8.11)
Considering Eqs. \((8.10)\) and \((8.11)\), simultaneously, one can obtain a nontrivial solution as
\[
w_s = -w_l = \sqrt{-\frac{\tau}{\rho_c}h},
\] (8.12)
implying
\[
\eta = v_l - v_s = 2v_c\sqrt{-\frac{\tau}{\rho_c}h} \propto |\tau|^{1/2}.
\] (8.13)
According to Eq. \((8.2)\), this result confirms that \(\beta = 1/2\) for both canonical and grand canonical ensembles.

Now, from the definition of the isothermal compressibility near the critical point, Eq. \((8.3)\), we can find the exponent \(\gamma\). To do this, considering both ensembles, we differentiate the expansion of EoS \((8.6)\) to get
\[
\frac{\partial P}{\partial v} \bigg|_T = -\frac{P_c \tau}{\rho_c v_c} + O(\tau w, w^2).
\] (8.14)
Using \(\frac{\partial v}{\partial P} \bigg|_T = \left(\frac{\partial P}{\partial v} \bigg|_T\right)^{-1}\) and the above relation, it is confirmed that
\[
\kappa_T = -\frac{1}{v} \frac{\partial v}{\partial P} \bigg|_T \propto \frac{\rho_c v_c}{P_c} \frac{1}{\tau},
\] (8.15)
yielding \(\gamma = 1\) for both canonical and grand canonical ensembles.

Finally, we compute the exponent \(\gamma\), Eq. \((8.3)\). Putting \(\tau = 0\) in the expansion \((8.6)\) leads to
\[
P - P_c = \frac{P_c h(v_c, c_i, s)}{v_c^3}(v - v_c)^3,
\] (8.16)
which specifies \(\delta = 3\) for both the ensembles. So, the results are the same as those computations from the mean field theory.

**IX. CONCLUSION**

We have investigated the thermodynamic properties of charged-AdS TBHs in (dRGT) massive gravity coupled to PMI electrodynamic source in the GCE. First, using the appropriate boundary condition, we have fixed the electrostatic potential at the AdS boundary and then written down all the thermodynamic quantities in terms of the potential. We have evaluated the (finite) Euclidean on-shell action and, by having it, we could obtain the semi-classical partition function of TBHs. It was shown that the obtained quantities extracted from the partition function satisfy the first law of black hole thermodynamics in the extended phase space. These quantities also obey the Smarr formula, Eq. \((4.4)\), in agreement with the method of scaling argument developed by Kastor \([8, 9]\). As expected, our results, Eqs. \((4.5)\) and \((4.6)\), proved that the variation of massive couplings \((c_i)\) as well as cosmological constant \((\Lambda = -8\pi P)\) are required for consistency of the extended first law of thermodynamics with the Smarr formula.

Considering the grand canonical (fixed potential, \(\Phi\)) ensemble, we have seen various thermodynamic phenomena, such as vdW behavior in \(d \geq 4\), RPT in \(d \geq 5\), anomalous vdW behavior in \(d \geq 6\) and also triple points in \(d \geq 6\) dimensions. As discussed in Sect. [VI], the RPT phenomenon is observed only in the GCE and probably there does not exist such a critical phenomenon in the canonical ensemble (but, we emphasize that the existence of RPT phenomenon is not ruled out completely in the canonical ensemble since our speculation has relied on the numerical investigation). However, the rest of the critical phenomena show up in the same dimensions in both ensembles. In comparison to
the TBHs in the GCE of Maxwell-massive gravity, it has been observed that the holographic critical phenomena in PMI-massive gravity start appearing in one dimension lower. Interestingly, the critical ratio up to two interaction potentials $O(U_2)$ has a universal behavior as $\rho_c = \frac{\rho C}{T^c} = \frac{2s-1}{4s}$ in terms of the shifted Hawking temperature. In $d \geq 6$ dimensions, one can always set $s = 2$ to obtain the standard critical ratio of vdW fluid, $\rho_c = 3/8$, in higher dimensions.

The effects of massive graviton and PMI electromagnetic field corrections are encoded in the deformation parameters $m_g$ and $s$, respectively. The variation of the nonlinearity parameter ($s$) and also the graviton’s mass ($m_g$) could produce or ruin a specific phase transition. For example, in the limit $s \to 1$, all the critical phenomena that presented in Sect. VII disappear suddenly, but they can be found in higher dimensions as we expected. In fact, in this case ($s \to 1$), the outcomes of Maxwell-massive gravity are recovered straightforwardly. In addition, for all the critical phenomena except the $d$-behavior in the present work, there is a lower bound for the graviton’s mass parameter ($m_g$), referred to as $m_g^*$, in which no phase transition takes place for $m_g < m_g^*$. In table II we have summarized the results of critical phenomena in both ensembles for different limits. We have also discussed black hole criticality in (Einstein and massive) gravity coupled with conformally invariant Maxwell source in Sect. VII. In this limit, the final results were presented in table III. Finally, we have computed the critical exponents in both ensembles and found that they match the standard critical exponents of vdW fluid. These are independent of spacetime dimensions as well.

There are several proposals to extend this work. One of the possible extensions is to study other types of holographic phase transitions which could appear only in $d \geq 7$ or $d \geq 8$ dimensions if one considers the higher order self-interaction potentials, $U_i$ (which means considering more massive couplings ($c_i$) in the EoS). Furthermore, thermodynamic stability properties of the obtained TBHs can be investigated by introducing some thermodynamic coefficients such as isobaric expansivity or adiabatic compressibility. These quantities can be realized in this paper from $T - r_+$ and $P - r_+$ diagrams, respectively. Also, following [120, 121], it would be interesting to explore the microscopic structure of massive-PMI TBHs from the thermodynamic viewpoint by introducing the number density of the black hole molecules. On the other hand, the exact massive-PMI TBHs constructed here could potentially have applications in gauge/gravity duality since massive couplings ($c_i$) in the extended phase space are not fixed a priori, so each massive coupling has a specific interpretation on the gauge field theory side. However, holographic interpretation of both the massive couplings and thermodynamic phase transitions remains an open question.

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