SPECTRAL MULTIPLICITY AND NODAL SETS FOR GENERIC TORUS-INVARIENT METRICS

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Abstract. Let a torus $T$ act freely on a closed manifold $M$ of dimension at least two. We demonstrate that, for a generic $T$-invariant Riemannian metric $g$ on $M$, each real $\Delta_g$-eigenspace is an irreducible real representation of $T$ and, therefore, has dimension at most two. We also show that, for the generic $T$-invariant metric on $M$, if $u$ is a non-invariant real-valued $\Delta_g$-eigenfunction that vanishes on some $T$-orbit, then the nodal set of $u$ is a connected smooth hypersurface whose complement has exactly two connected components.

1. Introduction

Karen Uhlenbeck \cite{Uhl76} proved that, for the generic metric $g$ on a smooth closed manifold $M$ of dimension $n > 1$, the associated Laplacian $\Delta_g$ acting on functions has one-dimensional eigenspaces. On the other hand, metrics that are invariant under a group action often have Laplace eigenspaces with dimension greater than one. For example, if a torus acts freely on $M$ and preserves $g$, then each non-invariant, real-valued eigenfunction of $\Delta_g$ lies in an eigenspace of dimension at least two. Here we show that for the generic torus invariant metric, the dimension of each eigenspace is at most two.

\textbf{Theorem 1.1.} Let $M$ be a smooth connected closed manifold of dimension $n > d$ such that the $d$-dimensional torus $T^d$ acts freely and smoothly on $M$. Then for each $\ell \geq 2$, there exists a residual subset of the space of the torus-invariant $C^\ell$ Riemannian metrics on $M$ such that if $g$ lies in this residual set, then:

1. Each (real) eigenspace of $\Delta_g$ has dimension at most two. More precisely, if an eigenfunction $\varphi : M \to \mathbb{R}$ is torus-invariant, then $u$ lies in a one-dimensional eigenspace and otherwise $\varphi$ lies in a 2-dimensional eigenspace on which $T^d$ acts irreducibly.

2. The nodal set $\varphi^{-1}(0)$ of each eigenfunction $\varphi$ is a smooth hypersurface.

We also produce new examples of Riemannian manifolds with infinitely many eigenfunctions having exactly two nodal domains. For example, we prove

\textbf{Theorem 1.2.} Suppose $B$ is a connected smooth closed manifold so that $H_2(B, \mathbb{Z})$ contains no non-trivial element of finite order. Let $M$ be the total space of a non-trivial oriented circle bundle $M \to B$. Then for the generic circle-invariant metric $g$ on $M$, each non-invariant eigenfunction of $\Delta_g$ has exactly two nodal domains.

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Junehyuk Jung and Steve Zelditch [JngZld20] proved both Theorem 1.1 and Theorem 1.2 in the case where $M$ is the unit cotangent bundle of an oriented surface $X$ and the metrics on $M$ are chosen so that the fibers are geodesic with constant length.\footnote{Such metrics are sometimes called Kaluza-Klein metrics by physicists. See e.g. [Wesson].}

Theorem 1.2 is remarkable because it provides a large class of Riemannian manifolds with a sequence $\{\varphi_k\}$ of eigenfunctions that span a space of Weyl density 1 so that each $\varphi_k$ has exactly two nodal domains (see Proposition 2.1). Prior to [JngZld20], both the square and the standard two-dimensional sphere were shown to possess a sequence of eigenfunctions with exactly two nodal domains [Str24] [C-H] [Lwy77].\footnote{See [BrnHlf15] and [BrnHlf16] for interesting discussions of these constructions.} However, these sequences span a space of Weyl density zero and their construction relies on the existence of high-dimensional eigenspaces.

Theorem 1.2 applies to some well-known manifolds. For example, each odd-dimensional sphere $S^{2n+1}$ is the total space of the oriented circle bundle associated to the tautological complex line bundle over the $n$-dimensional complex projective space. When $n = 1$, this is the well-known Hopf fibration of $S^3$. Theorem 1.1 and Corollary 1.2 imply that for a generic circle-invariant metric on $S^{2n+1}$, each eigenvalue has multiplicity at most two and each non-invariant real eigenfunction has exactly two nodal domains. In comparison, the standard metric on $S^{2n+1}$ is invariant, but the dimension of the eigenspace and the number of nodal domains of the typical eigenfunction grows with the eigenvalue [NzrSdn09] [NzrSdn16]. Jung and Zelditch [JngZld21] recently showed, however, that, for the standard metric on $S^3$, the typical eigenfunction within a fixed nontrivial isotypical component has exactly two nodal domains.

The proof of Theorem 1.1 consists of analyzing the restriction, $\Delta_{g,\alpha}$, of the Laplacian to the space, $H^k_\alpha$, consisting of vector-valued Sobolev $H^k$ functions $u : M \to \mathbb{R}^2$ that satisfy $u(\theta^{-1} \cdot x) = R_\alpha \cdot u(z)$ for each $\theta \in T$ where $R_\psi$ is rotation by angle $\psi$ (see §2). The proof begins in §3 where we provide some variational formulas for the Laplacian and its eigenvalues. The variational formula for eigenvalues is used in §4 to show that, for the generic torus-invariant metric, the spectra of $\Delta_{g,\alpha}$ and $\Delta_{g,\beta}$ are disjoint if $\alpha \neq \pm \beta$ (Theorem 4.1). In §5 we use the method of Uhlenbeck [Uhl76] to show that the spectrum of $\Delta_{g,\alpha}$ is simple (Theorem 5.5). At the end of §5, we combine Theorem 4.1 and Theorem 5.5 to prove part (1) of Theorem 1.1. In §6 we use the method of Uhlenbeck to show that, for a generic torus-invariant metric, zero is a regular value of each eigenfunction of $\Delta_{g,\alpha}$. We explain how this implies part (2) of Theorem 1.1 as well as Theorem 1.2.

Our overall approach mirrors the approach in [JngZld20]. However, there are some important differences. Jung and Zelditch use the natural unitary isomorphism between $H^0_\alpha$ and the $L^2$-sections of $\kappa^{\otimes \alpha}$ for $\alpha \neq 0$. Under this isomorphism, the operator $\Delta_{g,\alpha}$ corresponds to a Bochner Laplacian acting on sections of $\kappa^{\otimes \alpha}$. Jung and Zelditch show that the spectra of the associated Bochner Laplacians are generically disjoint for $\alpha \neq \beta$, and they use Uhlenbeck’s method to show that the spectrum of each is simple. Moreover, they prove these genericity results for the Bochner Laplacian in the larger context of holomorphic line bundles with generic base metrics, generic hermitian metrics, and/or generic connections. However, the equivalence of $\Delta_{g,\alpha}$ and the Bochner Laplacian is proven only in the context of canonical bundles (see Lemma 6.6 [JngZld20]).
In this paper we analyse the operators $\Delta_{\gamma, \alpha}$ directly, and we do not consider Bochner Laplacians. We also provide a simpler proof of the fact that $\{ x \in M : u(x) \neq 0 \}$ has two components for a non-invariant real-valued eigenfunction $u$. Whereas the proof in [JungZeld20] uses an intricate combinatorial analysis of the nodal domains, we show directly that each nodal set is connected for the generic invariant metric. Thus, because zero is a regular value for the generic invariant metric, the nodal set is a connected smooth hypersurface and hence its complement has at most two components.

It is natural to ask whether an analogue of Theorem 1.1 holds for other groups $G$. In general, if $G$ preserves a metric $g$ on $M$, then each real eigenspace $E$ of $\Delta_g$ is a $G$-invariant subspace of the space of real-valued $L^2$ functions. Theorem 1.1 is equivalent to the statement that if $G$ is a torus, then each real eigenspace is irreducible for the generic $G$-invariant metric. Similar results have been obtained in the case of finite group actions [Zld90], left-invariant metrics on $SU(2)$ [Sch17], and invariant metrics on trivial $SU(2)$ bundles [MrcGmz19]. These results support the belief in quantum mechanics that the eigenspaces of the typical Hamiltonian should be irreducible. Physicists regard an eigenvalue associated to a reducible eigenspace as being ‘accidentally degenerate’ [Wigner] [Faddeev].

Remark 1.3. Building upon the techniques and results of [JungZeld20], Junehyuk Jung and Steve Zelditch recently posted a preprint [JungZeld22] with results similar to those contained in this article. While Jung-Zelditch and we both use Uhlenbeck’s framework, our implementations are very different.

2. Preliminaries

We will let $H^k$ denote the Sobolev space of real-valued functions on $M$ whose weak derivatives of order up to $k$ are square-integrable in each coordinate chart with respect to Lebesgue measure. In particular $L^2 = H^0$.

Let $T^d := (\mathbb{R}/2\pi \mathbb{Z})^d$ denote the $d$-dimensional torus. Given $\theta \in T^d$, let $\varphi_\theta$ denote the diffeomorphism $\varphi_\theta(x) = \theta \cdot x$. The torus $T^d$ acts on the real vector space $H^k$ by

$$\left( \theta \cdot u \right)(x) = u(\varphi_\theta^{-1} \cdot x) = u \circ \varphi_\theta^{-1}(x).$$

In the language of representation theory, this action is called the (left) regular representation. Let $d\mu_\theta := d\theta_1 \cdots d\theta_d$.

Define $H^k_\alpha$ to be the space of $H^k$ functions that are invariant under the action of $T^d$, that is, $\theta \cdot u = u$ for each $\theta \in T^d$. Integration over the orbits defines a projection $\pi_0 : H^k \to H^k_\alpha$ defined by

$$\pi_0(u)(x) = \int_T (\theta \cdot u)(x) d\mu_\theta.$$

For each nonzero $\alpha \in \mathbb{Z}^d$, define the space $H^k_\alpha$ to be the space of $\mathbb{R}^2$-valued functions $u = (u_1, u_2)^t : M \to \mathbb{R}^2$ such that $u_1, u_2 \in H^k$ and for each $\theta \in T^d$ and $x \in M$

$$\begin{pmatrix} u_1(\theta^{-1} \cdot x) \\ u_2(\theta^{-1} \cdot x) \end{pmatrix} = R_{\alpha, \theta} \cdot \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}$$

where

$$R_\psi = \begin{pmatrix} \cos(\psi) & \sin(\psi) \\ -\sin(\psi) & \cos(\psi) \end{pmatrix}.$$
The space \( H^k \) is invariant under the \( T^d \)-action \( \theta \cdot (u_1, u_2) = (\theta \cdot u_1, \theta \cdot u_2) \). From (2) we see that if \( \theta \cdot \alpha = \pi / 2 \mod 2\pi \), then \( \theta \cdot u_1 = u_2 \). In particular, the map \( \iota_\alpha : H^k \rightarrow H_0^k \) defined by \( \iota_\alpha(u_1, u_2) = u_1 \) is injective and the image is invariant under \( T^d \). For \( \alpha = 0 \), let \( \iota_\alpha : H_0^k \rightarrow H^k \) denote the inclusion map. If \( \alpha = -\beta \), then \( \iota_\alpha(H^k) = \iota_\alpha(H_0^k) \), and if \( \alpha \neq \pm \beta \), then \( \iota_\alpha(H^k) \cap \iota_\alpha(H_0^k) = \{0\} \).

For \( \alpha \neq 0 \), the space \( H^k_\alpha \) is the image of the map \( \pi_\alpha : H^k \rightarrow H^k_\alpha \) defined by
\[
\pi_\alpha(u) = \left( \begin{array}{c}
\int_T (\theta \cdot u)(x) \cdot \cos(\alpha \cdot \theta) \, d\mu_	heta \\
\int_T (\theta \cdot u)(x) \cdot (-\sin(\alpha \cdot \theta)) \, d\mu_	heta 
\end{array} \right).
\]
Using this 'Fourier coefficient map' one finds that \( H^k \) is the direct sum
\[
H^k = H_0^k \oplus \left( \bigoplus_\alpha \iota_\alpha(H^k_\alpha) \right)
\]
where \( \alpha \) ranges over a set of orbit representatives of the action of \( \mathbb{Z}/2\mathbb{Z} \) on \( \mathbb{Z}^d \) given by \( m \cdot \alpha = (-1)^m \cdot \alpha \). Indeed, if \( v \in H^k \) were orthogonal to \( \iota_\alpha(H^k_\beta) \) for each \( \alpha \), then the restriction of \( v \) to almost every \( T^d \) orbit would vanish by the standard theory of Fourier series.

If \( e_j \) is the standard basis vector on \( \mathbb{R}^d \), then the path \( t \mapsto t \cdot e_j \) defines a one-parameter subgroup of \( T^d \). Define the smooth vector field \( \partial_j \) on \( M \) by
\[
(\partial_j u)(x) = \frac{\partial}{\partial t} \bigg|_{t=0} u((t \cdot e_j) \cdot x),
\]
where \( u \in C^\infty \). Note that \( (\varphi_\theta)_* (\partial_j |_x) = \partial_j |_{\theta \cdot x} \), which implies that \( \partial_j \circ \partial_k = \partial_k \circ \partial_j \). Moreover, if \( u \in H^k_\alpha \), then it follows from (4) that
\[
\partial_j u = \alpha_j \cdot \left( \begin{array}{c}
0 \\
-1 \\
0
\end{array} \right) \cdot u
\]
and hence \( -\partial_j^2 u = \alpha_j^2 \cdot u \).

Associated to each metric \( g \) on \( M \), there is a Laplace-Beltrami operator \( \Delta_g : H^k \rightarrow H^{k-2} \) defined implicitly by
\[
\int (\Delta_g u) \cdot v \, dv_g = \int g(\nabla_g u, \nabla_g v) \, dv_g
\]
where \( \nabla_g \) is the Riemannian gradient and \( dv_g \) is the Riemannian measure on \( M \). Because \( M \) is compact and \( \Delta_g \) is elliptic, Rellich’s lemma implies that the resolvent \( (\Delta - \lambda)^{-1} \) is compact and hence the spectrum of \( \Delta_g \) consists of isolated eigenvalues with finite dimensional eigenspaces.

Now suppose that \( g \) is \( T \)-invariant. Then \( \Delta_g \circ \varphi_\theta^* = \varphi_\theta^* \circ \Delta_g \), and it follows that \( \Delta_g \) preserves \( \iota_\alpha(H^k_\alpha) \). In particular, for \( u = (u_1, u_2) \in H^k_\alpha \), we may define \( \Delta_{g,\alpha} : H^k_\alpha \rightarrow H_{\alpha}^k \) by
\[
\Delta_{g,\alpha} \left( \begin{array}{c}
u_1 \\
u_2
\end{array} \right) = \left( \begin{array}{c}
\Delta_g u_1 \\
\Delta_g u_2
\end{array} \right).
\]
If \( W \) is an eigenspace of \( \Delta_{g,\alpha} \) then \( \iota_\alpha(W) \) is of an eigenspace of \( \Delta_g \) with the same eigenvalue. Thus, the spectrum of \( \Delta_{g,\alpha} \) is a subset of the spectrum of \( \Delta_g \) and hence is discrete.

We conclude this section by explaining the claim in the introduction that the Weyl density of the non-invariant eigenfunctions equals one. First, we make precise
the notion of Weyl density. For each \( x > 0 \), let \( E_x \) denote the direct sum of the eigenspaces of \( \Delta_g \) whose associated eigenvalues are at most \( x \). The Weyl density of a subspace \( V \subset L^2 \) equals
\[
\limsup_{x \to \infty} \frac{\dim(V \cap E_x)}{\dim(E_x)}.
\]
The following is a consequence of, for example, [Dni78], but we provide a simple explanation for the convenience of the reader.

**Proposition 2.1.** The invariant subspace \( L^2_0 \) has Weyl density equal to zero, and hence the space of non-invariant functions has Weyl density equal to 1.

**Proof.** Since the torus action is smooth and free, the quotient \( M/T^d \) is a smooth \( n-d \)-dimensional manifold and the quotient map \( \pi : M \to M/T \) is a smooth submersion. Since the action preserves \( g \), the metric \( g \) descends to a metric \( g' \) on \( M/T \) and \( \pi_*(d\nu_g) = d\nu_{g'} \). Let \( d\nu_{g'} \) be the associated Riemannian measure on \( M/T \) and let \( H^0(M/T) \) denote the associated Hilbert space of square-integrable functions. For each \( x \in M \), let \( \text{vol}(T \cdot x) \) denote the \( d \)-dimensional measure of the orbit \( T \cdot x \) with respect to \( g' \). The map \( \Phi : H^0(M/T) \to H^0_0 \) defined by
\[
\Phi(u)(x) = \text{vol}(T \cdot x)^{-\frac{1}{2}} \cdot (u \circ \pi)(x).
\]
is a unitary isomorphism. The operator \( P := \Phi^{-1} \circ \Delta_g \circ \Phi \) is a nonnegative second order elliptic differential operator on \( M/T \), and hence Weyl’s law implies that \( \dim(E_x') \) is \( O(x^{n-1}) \) where \( E_x' \) is the direct sum of the eigenspaces of \( P \) whose eigenvalues of size at most \( x \). We have \( \Phi^{-1}(E_x') = E_x \cap L^2_0 \). On the other hand, Weyl’s law implies that \( \dim(E_x) \sim c \cdot x^{n-\frac{n}{2}} \) where \( c > 0 \).

## 3. Some variational formulas

In this section we derive a general formula for the first variation of the Laplacian under metric perturbations. We then specialize this formula to perturbations associated to a decomposition of the tangent bundle into orthogonal subbundles. At the end of the section, we specialize further to the special case of perturbations of \( T \)-invariant metrics.

### 3.1. General variational formulas

We will let \( \mathcal{M}_\ell \) denote the set of \( C^\ell \) Riemannian metric tensors on \( M \). The space \( \mathcal{M}_\ell \) is an open convex subset of the Banach space \( S_\ell \) of \( C^\ell \) symmetric \((0,2)\) tensors on \( M \).\(^3\) We fix \( \ell \geq 2 \), and so in the sequel we will suppress the subscript \( \ell \) from notation.

Given a differentiable path \( t \mapsto g_t \in \mathcal{M} \), let \( dv_t \), \( \nabla_t \), and \( \Delta_t \) denote respectively the measure, gradient, and Laplacian associated to \( g_t \). In what follows, a dot above a symbol will indicate the first variation \( \partial_t|_{t=0} \). For example, \( \lambda \) will indicate \( \partial_{\lambda}|_{\lambda=0} \) and \( dv \) indicates \( \partial_{\lambda}|_{\lambda=0} dv \).

**Lemma 3.1.** Let \( t \mapsto g_t \in \mathcal{M} \) be a differentiable path of metrics. Then for each \( u, v \in H^2 \) we have
\[
(7) \quad \int_M (\hat{\Delta} u) v \, dv_0 + \int_M \hat{\Delta} (\nabla_0 u, \nabla_0 v) \, dv_0 = -\int_M \hat{\Delta} (\nabla_0 u, \nabla_0 v) \, dv_0 + \int_M (g_0(\nabla_0 u, \nabla_0 v) - (\Delta_0 u)v) \, dv.
\]

\(^3\)A norm on \( S_\ell \) can be constructed using, for example, a choice of a smooth (co)metric on \( M \).
Proof. By (6) we have

\begin{equation}
\int_M (\Delta u) \cdot v \, dv_0 = \int_M g_t(\nabla_i u, \nabla_i v) \, dv_t.
\end{equation}

Since \( g_t(\nabla_i w, X) = X w \) for each \( X \) and \( w \), we find that

\( g(\nabla_i w, X) = -\dot{g}(\nabla_i w, X) \).

Thus by differentiating both sides of (8) with respect to \( t \) and setting \( t = 0 \) we obtain

\[ \int (\dot{\Delta} u) v \, dv_0 + \int (\Delta_0 u) v \, dv = -\int \dot{g}(\nabla_0 u, \nabla_0 v) \, dv_0 + \int g(\nabla_0 u, \nabla_0 v) \, dv_0. \]

The claimed formula follows. \( \square \)

**Corollary 3.2.** Let \( t \mapsto g_t \in \mathcal{M} \) be a differentiable path of metrics. Suppose that \( t \mapsto u_t \) is a differentiable path of \( H^k \) functions such that \( \Delta_t u_t = \lambda_t u_t \). Then

\begin{equation}
\dot{\lambda} \int u_t^2 \, dv_0 = -\int \dot{g}(\nabla_0 u_0, \nabla_0 u_0) \, dv_0 + \int (g_0(\nabla_0 u_0, \nabla_0 u_0) - \lambda_0 \cdot u_t^2) \, dv.
\end{equation}

**Proof.** We have \( \Delta_t u_t = \lambda_t u_t \), and hence differentiation gives \( \Delta u_0 + \Delta_0 \dot{u} = \dot{\lambda} u_t + \lambda_0 \dot{u} \). Integrate both sides of this equation again \( u_0 \) and then apply Lemma 3.1. \( \square \)

### 3.2. Perturbations associated to orthogonal subbundles.

We next construct metric perturbations of a given metric \( g \) associated to a decomposition of \( TM \) into two \( g \)-orthogonal subbundles. That is, we suppose that we are given two smooth subbundles \( \mathcal{V} \) and \( \mathcal{H} \) of the tangent bundle so that \( T_p M = \mathcal{V}_p \oplus \mathcal{H}_p \) is a \( g \)-orthogonal direct sum for each \( p \in M \).

Our first class of perturbations consists of rescaling \( g \) along \( \mathcal{H} \) and \( \mathcal{V} \) independently. To be precise, for each pair, \( a \) and \( b \), of \( C^t \) functions on \( M \), define a new metric \( g_{a,b} \) on \( TM = \mathcal{V} \oplus \mathcal{H} \) by setting

\begin{equation}
g_{a,b}(v + h, v' + h') = a \cdot g(v, v') + b \cdot g(h, h')
\end{equation}

for each \( v, v' \in \mathcal{V} \) and \( h, h' \in \mathcal{H} \). Note that \( \mathcal{V} \oplus \mathcal{H} \) remains a \( g_{a,b} \)-orthogonal direct sum.

Let \( j \) (resp. \( k \)) denote the (constant) dimension of \( \mathcal{V}_p \) (resp. \( \mathcal{H}_p \)). In particular, \( j + k = n \) where \( n \) is the dimension of \( M \). Given \( u \in C^\infty \), let \( \nabla_v^\mathcal{V} u \) (resp. \( \nabla_v^\mathcal{H} u \)) denote the orthogonal projection of \( \nabla_v u \) onto \( \mathcal{V} \) (resp. \( \mathcal{H} \)).

**Lemma 3.3.** Let \( t \mapsto a_t \) and \( t \mapsto b_t \) be differentiable paths in \( C^t \) such that \( b_0 = a_0 = 1 \). Let \( g_t = g_{a_t,b_t} \) denote the associated path of Riemannian metrics defined by (10). For each \( u, v \in C^\infty \), we have

\[ 2 \int (\Delta u) v \, dv_0 = -\int (\Delta_0 u) v \cdot \left( j \cdot \dot{a} + k \cdot \dot{b} \right) \, dv_0 \]

\[ + \int g_0(\nabla_v^\mathcal{V} u, \nabla_v^\mathcal{V} v) \cdot \left( (j - 2) \cdot \dot{a} + k \cdot \dot{b} \right) \, dv_0 \]

\[ + \int g_0(\nabla_v^\mathcal{H} u, \nabla_v^\mathcal{H} v) \cdot \left( j \cdot \dot{a} + (k - 2) \cdot \dot{b} \right) \, dv_0. \]

**Proof.** We will apply Lemma 3.1. First note that \( dv_t = a_t^{j/2} \cdot b_t^{k/2} \cdot dv_0 \), and hence since \( a_0 = b_0 = 1 \) we have \( dv = (j/2)\dot{a} + (k/2)\dot{b} \). Because \( \mathcal{V} \) and \( \mathcal{H} \) are \( g_t \)-orthogonal for each \( t \), we find that

\[ g_0(\nabla_0 u, \nabla_0 v) = g_0(\nabla_0^\mathcal{H} u, \nabla_0^\mathcal{H} v) + g_0(\nabla_0^\mathcal{V} u, \nabla_0^\mathcal{V} v) \]
and
\[ \hat{g}(\nabla_0 u, \nabla_0 v) = \hat{a} \cdot g_0(\nabla_0^V u, \nabla_0^V v) + \hat{b} \cdot g_0(\nabla_0^H u, \nabla_0^H v). \]

The claim now follows from Lemma 3.1. \(\square\)

Next we consider a somewhat different type of metric variation. Let \(X\) be a section of \(\mathcal{H}\) and let \(Y\) be a section of \(\mathcal{V}\). Let \(\xi\) (resp. \(\eta\)) be the dual one form defined by \(\xi(Z) = g(X, Z)\) (resp. \(\eta(Z) = g(Y, Z)\)) for each \(Z\). Define
\[ g_{X,Y} = g + \xi \otimes \eta + \eta \otimes \xi. \]

**Lemma 3.4.** Consider the family \(t \mapsto g_{X,tY}\). For each smooth \(u, v\), we have
\[ \int_M \Delta u \cdot v \, dv_0 = - \int_M ((Xu) \cdot (Yv) + (Yu) \cdot (Xv)) \, dv_0. \]  

**Proof.** We apply Lemma 3.1. Because \(X\) and \(Y\) are \(g\) orthogonal, we find that \(d\hat{\nu} = 0\). We have \(\hat{g} = \xi \otimes \eta + \eta \otimes \xi\). We have \(\xi(\nabla u) = g(\nabla u, X) = Xu\) and \(\eta(\nabla v) = g(\nabla v, Y) = Yv\) and so \(\xi \otimes \eta(\nabla u, \nabla v) = Xu \cdot Yv\). Similarly, \(\eta \otimes \xi(\nabla u, \nabla v) = Yu \cdot Xv\). The claim follows. \(\square\)

### 3.3. Metrics invariant under a torus action

In the following we will specialize to metrics that are invariant under a fixed action of the \(d\)-dimensional torus \(T\). Let \(\mathcal{V}\) be the subbundle consisting of vectors tangent to the \(T\) action. Let \(\mathcal{H}\) be the subbundle consisting of vectors that are \(g\)-orthogonal to \(\mathcal{V}\) where \(g\) is a fixed \(T\)-invariant metric.

Let us specialize Lemma 3.3. We will assume that the smooth functions \(a\) and \(b\) in (10) are \(T\)-invariant. This implies that the metric \(g_{a,b}\) is \(T\)-invariant. The gradient \(\nabla^V\) is a linear combination of the vector fields \(\partial_j\) given by (4), and hence \(\nabla^V\) is well understood. On the other hand, \(\nabla^H\) is less well-understood in general. For this reason we choose \((j-2) \cdot \hat{a} + k \cdot \hat{b} = 0\) so as to eliminate the term in Lemma 3.3 that involves \(\nabla^H\). Noting that \(j = d\) and \(k = n - d\) we obtain the following.

**Lemma 3.5.** If \(g, a, \) and \(b\) are T-invariant and \(d \cdot \hat{a} + (n - d - 2) \cdot \hat{b} = 0\), then
\[ \int_M (\Delta u) v \, dv_0 = - \int_M \hat{b} \cdot (\Delta_0 u) v \, dv_0 + \frac{n-2}{d} \int \hat{b} \cdot g(\nabla_0^V u, \nabla_0^V v) \, dv_0. \]

### 4. The spectra of \(\Delta_{g,\alpha}\) and \(\Delta_{g,\beta}\) are generically disjoint

Recall that \(\Delta_{g,\alpha}\) is the restriction of the Laplacian to the space \(H^k_{\alpha}\) where \(\alpha \in \mathbb{Z}^d\). Let \(\text{spec}(\cdot)\) denote the spectrum of an operator. The purpose of this section is to prove the following:

**Theorem 4.1.** If \(\alpha \neq \pm \beta \in \mathbb{Z}^d\), then the set of metrics \(g\) for which \(\text{spec}(\Delta_{g,\alpha}) \cap \text{spec}(\Delta_{g,\beta}) = \emptyset\) is residual in \(\mathcal{M}^T\).

Our proof of Theorem 4.1 is based on analytic perturbation theory [Kato]. In this section \(t \mapsto g_t \in \mathcal{M}^T\) will denote an analytic path of torus-invariant metrics defined for \(t \in (-\delta, \delta)\) where \(\delta > 0\).

**Lemma 4.2.** For each \(t\) there exists a \(g_t\)-orthonormal basis \(\{\varphi_{j,t}^\alpha\}\) of \(H^k_{\alpha}\) consisting of eigenfunctions of \(\Delta_{g_t,\alpha}\) so that \(t \mapsto \varphi_{j,t}^\alpha\) is analytic for each \(j\).
We will let \( \lambda_{j,t}^\alpha \) denote the eigenvalue associated to the eigenfunction \( \varphi_{j,t}^\alpha \). Since \( t \mapsto \varphi_{j,t}^\alpha \) is analytic, the path \( t \mapsto \lambda_{j,t}^\alpha \) is also analytic. Analyticity implies that two eigenvalue branches \( \lambda_{j,t}^\alpha \) and \( \lambda_{k,t}^\beta \) are either equal to each other for countably many \( t \) or they coincide for all values of \( t \). To be precise, let \( \Lambda > 0 \) and define \( I_{\alpha,\beta}(\Lambda) = (-\delta, \delta) \) to be the set of \( t \) such that \( \text{spec}(\Delta_{\alpha,\beta}) \cap [0, \Lambda] \cap \text{spec}(\Delta_{\alpha,\beta}) \neq \emptyset \). Here we assume that \( \alpha \neq \beta \) and hence \( H_0^\alpha \perp H_0^\beta \) for each \( t \).

**Lemma 4.3.** Either \( I_{\alpha,\beta}(\Lambda) \) is countable or there exists \( j,k \) such that \( \lambda_{j,t}^\alpha = \lambda_{k,t}^\beta \) for each \( t \).

**Proof.** Suppose that there does not exist \( j \) and \( k \) so that \( \lambda_{j,t}^\alpha = \lambda_{k,t}^\beta \) for all \( |t| < \delta \). Real-analyticity implies that for each \( j \) and \( k \), the set \( \{ t \in (-\delta, \delta) : \lambda_{j,t}^\alpha = \lambda_{k,t}^\beta \} \) is countable. Hence the union of these sets over \( j,k \) is countable.

**Proof of Theorem 4.1.** Fix \( \alpha \neq \pm \beta \). For each \( \Lambda > 0 \), let \( D(\Lambda) = D_{\alpha,\beta}(\Lambda) \) denote the set of \( g \in \mathcal{M}^T \) so that \( \text{spec}(\Delta_{\alpha,\beta}) \cap [0, \Lambda] \cap \text{spec}(\Delta_{\alpha,\beta}) = \emptyset \). It suffices to show that \( D(\Lambda) \) is open and dense in \( \mathcal{M}^T \) for each \( \Lambda \).

To show that \( D(\Lambda) \) is open will show that the complement is closed. Suppose that there exist a sequence \( g_m \in \mathcal{M}^T \) and a sequence of \( L^2 \) unit norm eigenfunctions \( u_m^\alpha \) of \( \Delta_{\alpha,m} \) and \( u_m^\beta \) of \( \Delta_{\beta,m} \) so that \( u_m^\alpha \) and \( u_m^\beta \) have the same eigenvalue \( \lambda_m \leq \Lambda \). By passing to a subsequence if necessary, we may assume that \( \lambda_m \) converges to \( \lambda \) and that \( u_m^\alpha \) (resp. \( u_m^\beta \)) converges to a unit norm eigenfunction \( u^\alpha \) of \( \Delta_{\alpha} \) (resp. \( u^\beta \) of \( \Delta_{\beta} \)) with (the same) eigenvalue \( \lambda \). Hence \( g \) lies in the complement of \( D(\Lambda) \). Thus, \( D(\Lambda) \) is open.

To show that \( D(\Lambda) \) is dense, we first show that \( D(\Lambda) \) is nonempty. Since \( \alpha \neq \pm \beta \), there exists \( j,k \in \{1, \ldots, d\} \) such that \( \alpha_j \cdot \alpha_k \neq \beta_j \cdot \beta_k \). We consider separately the cases where \( j = k \) and \( j \neq k \).

Suppose \( j = k \). Let \( g_0 \in \mathcal{M}^T \) be a metric on \( M \) such that \( g_0(\hat{e}_i, \hat{e}_j) = \delta_{ij} \) for each \( i, j \). Define the 1-form \( \omega_j \) by \( \omega_j(X) = g_0(X, \hat{e}_j) \). If \( u \in H_1^\alpha \), then by (5)

\[
\omega_j(\nabla_{g_0} u) = \hat{e}_j u = \alpha_j \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot u.
\]

Define a path \( t \mapsto g_t \) of (2, 0) tensors by

\[
g_t = (1 + t) \cdot g_0 - t \cdot n \cdot \omega \otimes \omega_j.
\]

There exists \( \delta > 0 \) so that \( g_t \) is a Riemannian metric for \( |t| < \delta \). A straightforward computation shows that if \( dv_t \) is the measure associated to \( g_t \) then

\[
div = \frac{n}{2} \cdot (1 - g_0(\hat{e}_j, \hat{e}_j)) \cdot dv_0.
\]

If \( D(\Lambda) \) were empty, then we would have \( I(\lambda) = (-\delta, \delta) \), and hence Lemma 4.3 implies that there exist analytic paths of unit norm eigenfunctions \( t \mapsto u_t^\alpha \) and \( t \mapsto u_t^\beta \) with common eigenvalue \( \lambda_t \) so that for each \( t \in (-\delta, \delta) \), the function \( u_t^\alpha \) (resp. \( u_t^\beta \)) is an eigenfunction of \( \Delta_{\alpha,t} \) (resp. \( \Delta_{\beta,t} \)) with (the same) eigenvalue \( \lambda_t \).
For the metric family defined in (13) we have \( \dot{g} = g_0 - n \cdot \omega_j \otimes \omega_j \), and thus, using (12) and the fact that \( \int |u_0|^2 \, dV \equiv 1 \), we would find that formula (9) specializes to
\[
\dot{\lambda} = -\lambda_0 + n \cdot \alpha_j^2.
\]
The same formula would also hold with \( \alpha \) replaced by \( \beta \), and so we would find that \( \alpha_j^2 = \beta_j^2 \), a contradiction. Therefore \( D(\Lambda) \) is not empty.

The proof in the case when \( j \neq k \) is similar. In this case consider the path of metrics
\[
g_t = g_0 - t \cdot (\omega_j \otimes \omega_k + \omega_k \otimes \omega_j).
\]
We have \( \text{div} = 2g_0(\partial_j, \partial_k) \, dt \), and so (9) specializes to
\[
\dot{\lambda} = \alpha_i \cdot \alpha_j
\]
as well as the same equation with \( \alpha \) replaced by \( \beta \). Hence \( \alpha_j \cdot \alpha_k = \beta_j \cdot \beta_k \), a contradiction. Thus \( D(\Lambda) \) is nonempty in this case as well.

To see that \( D(\Lambda) \) is dense, given some other \( g \in \mathcal{M}_T \), consider linear path \( g_t := t \cdot m + (1 - t) \cdot g_0, 0 \leq t \leq 1 \), in \( \mathcal{M}_T \) that joins \( g_0 \) to \( g \). Because \( g_0 \in D(\Lambda) \), Lemma 4.3 implies that the set of \( t \) such that \( g_0 \notin D(\Lambda) \) is countable. Therefore, in every neighborhood of \( g \) there exists a metric \( g' \) that lies in \( D(\Lambda) \).

5. Generic simplicity for \( \Delta_{g,\alpha} \).

In this section, we finish the proof of part (1) of Theorem 1.1. In particular, we adapt the method of K. Uhlenbeck [Uhl76] to prove that there exists a residual subset of metrics \( g \) so that the dimension of \( \ker(\Delta_{g,\alpha} - \lambda I) \) is at most two.

Fix \( \alpha \in \mathbb{Z}^d \setminus \{0\} \). Since \( \alpha \neq 0 \), each eigenspace of \( \Delta_{g,\alpha} \) has dimension at least two. Indeed, if \( u \in \ker(\Delta_{g,\alpha} - \lambda I) \), then
\[
u^* := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot u
\]
also lies in \( \ker(\Delta_{g,\alpha} - \lambda I) \) and \( \int u \cdot u^* = 0 \).

Given a \( T \)-invariant metric \( g \) on \( M \), define \( F_g : H^k_\alpha \times \mathbb{R}^2 \to H^{k-2}_\alpha \) by
\[
F_g(u, a, b) = \Delta_{g,\alpha} u - a \cdot u - b \cdot u^*.
\]
A calculation shows that given any \( u = (u_1, u_2) \in H^k_\alpha \), we have \( F_g(u, a, b) = 0 \) if and only if \( u_1 + iu_2 \) is a complex eigenfunction of \( \Delta_g \) with eigenvalue \( \alpha - bi \). But the eigenvalues of \( \Delta_g \) are real and so the zero locus of \( F_g \) has the following form:
\[
F_g^{-1}(0) = \{(u, \lambda, 0) \mid (\Delta_{g,\alpha} - \lambda I) u = 0 \}.
\]
The following Lemma should be compared with Lemma 2.2 in [Uhl76].

**Lemma 5.1.** Let \( u \neq 0 \) be in \( \ker(\Delta_{g,\alpha} - \lambda I) \). Then the dimension of \( \ker(\Delta_{g,\alpha} - \lambda I) \) equals two if and only if \( (dF_g)(u, \lambda, 0) \) is surjective.

**Proof.** We have \( F_g(u, \lambda, 0) = 0 \) if and only if \( u \in \ker(\Delta_{g,\alpha} - \lambda I) \). From (15) we find that for each \( (v, \mu, \nu) \in H^k_\alpha \times \mathbb{R}^2 \)
\[
(dF_g)(u, \lambda, 0)(v, \mu, \nu) = (\Delta_{g,\alpha} - \lambda I) \cdot v - \mu \cdot u - \nu \cdot u^*.
\]
Thus
\[
\text{im}((dF_g)(u, \lambda, 0)) = \text{im}(\Delta_{g,\alpha} - \lambda I) + \mathbb{R} \cdot u + \mathbb{R} \cdot u^*.
\]
Because $\Delta_{g,\alpha} - \lambda I$ is self-adjoint, the space $\ker(\Delta_{g,\alpha} - \lambda I)$ is the orthogonal complement of $\operatorname{im}(\Delta_{g,\alpha} - \lambda I)$. Hence the sums on the right hand side of (18) are direct, and so the quotient space $\operatorname{im}((dF_g)_{(u,\lambda,0)})/\operatorname{im}(\Delta_{g,\alpha} - \lambda I)$ is two dimensional.

On the other hand, $H^k_\alpha/\operatorname{im}(\Delta_{g,\alpha} - \lambda I)$ is isomorphic to $\ker(\Delta_{g,\alpha} - \lambda I)$. The claim follows. \hfill \Box

Lemma 5.1 reduces generic irreducibility to the assertion that $(dF_g)_{(u,\lambda,0)}$ is surjective for the generic $g$ and each eigenpair $(u, \lambda)$ of $\Delta_{g,\alpha}$. The next step in Uhlenbeck’s method is to apply ‘parametric transversality’ to the family $(u, a, b, g) \mapsto F_g(u, a, b)$.

Let $S^T$ denote the real Hilbert space of real-valued Sobolev $H^s$ symmetric (0, 2)-tensors on $M$ that are $T$-invariant. The space $\mathcal{M}^T$ of Riemannian metrics on $M$ is the open cone in $S^T$ consisting of positive symmetric (0, 2) tensors. Define $F : H^k_\alpha \times \mathbb{R}^2 \times \mathcal{M}^T \rightarrow H^{k-2}_\alpha$ by
\begin{equation}
F(u, a, b, g) := F_g(u, a, b) = \Delta_{g,\alpha}u - a \cdot u - b \cdot u^*.
\end{equation}

The following is a variant of parametric transversality. Let $\pi : H^k_\alpha \times \mathbb{R}^2 \times \mathcal{M}^T \rightarrow \mathcal{M}^T$ denote the natural projection, for the remainder of this section.

**Proposition 5.2.** Let $\mathcal{U} \subset \mathcal{M}^T$ be an open set. If for each $x \in F^{-1}(0) \cap \pi^{-1}(\mathcal{U})$, the differential $dF_x$ is surjective, then there exists a residual subset of $\mathcal{R} \subset \mathcal{U}$ so that for each $g \in \mathcal{R}$ and each $(u, \lambda, 0) \in F^{-1}(0)$, the differential $(dF_g)_{(u,\lambda,0)}$ is surjective.

**Proof.** Let $x \in F^{-1}(0) \cap \pi^{-1}(\mathcal{U})$. By hypothesis, $dF_x$ is surjective, and hence the implicit function theorem\(^4\) implies that there exists an open neighborhood $\mathcal{V}_x \subset H^k_\alpha \times \mathbb{R}^2 \times \mathcal{M}^T$ of $x$ such that $F^{-1}(0) \cap \mathcal{V}_x$ is a submanifold. Thus, there exists an open subset $\mathcal{W} \subset H^k_\alpha \times \mathbb{R}^2 \times \mathcal{M}^T$ containing $\mathcal{U}$ so that $S := F^{-1}(0) \cap \mathcal{W}$ is a submanifold of $H^k_\alpha \times \mathbb{R}^2 \times \mathcal{M}^T$. The tangent space to $S$ at $x \in S$ is the kernel of $dF_x$. The map $F_g$ is the restriction of $F$ to $\pi^{-1}(g)$ and
\begin{equation}
(dF_g)_{(u,\lambda,0)}(v, \mu, \nu) = (\Delta_{g,\alpha} - \lambda I) \cdot v - \mu \cdot u - \nu \cdot u^*.
\end{equation}

Let $\pi^S : S \rightarrow \mathcal{M}^T$ denote the restriction of $\pi$ to $S$. We claim that if $x = (u, \lambda, 0, g)$ belongs to $S$, then

(a) $d\pi^S_x$ is Fredholm, and

(b) the differential $(dF_g)_{(u,\lambda,0)}$ is surjective if and only if $(d\pi^S)_x$ is surjective.

Indeed, let $x \in S$, and note that $(d\pi^S)_x$ is the restriction of $d\pi_x$ to $\ker(dF_x)$ and $(dF_g)_x$ is the restriction of $dF_x$ to $\ker(d\pi_x)$. We have
\begin{equation}
\ker(dF_x|_{\ker(d\pi_x)}) = \ker(dF_x) \cap \ker(d\pi_x) = \ker(d\pi_x|_{\ker(dF_x)}).
\end{equation}

Because $\Delta_{g,\alpha} - \lambda I$ is self-adjoint, the operator $\Delta_{g,\alpha} - \lambda I$ is Fredholm with index equal to zero. It follows that $(dF_g)_x$ given by (20) is Fredholm. Thus it follows from (21) that $\ker((d\pi^S)_x)$ is finite. Since $dF_x$ and $d\pi_x$ are both surjective, the following maps of quotient spaces are isomorphisms:
\begin{equation}
\frac{S^T}{d\pi_x(\ker(dF_x))} \quad \frac{H^k_\alpha \times \mathbb{R}^2 \times S^T}{(\ker(dF_x) + \ker(d\pi_x))} \quad \frac{H^{k-2}_\alpha}{dF_x(\ker(d\pi_x))}.
\end{equation}

\(^4\)Note that since $H^k_\alpha \times \mathbb{R}^2 \times S^T$ is a Hilbert space, the kernel of $dF_x$ is complemented.
Proof. Let \( \Pi \) be the residual set \( R \subset \mathcal{U} \) so that for each \( g \in R \) and \( x \in (\Pi^S)^{-1}(g) \), the differential \( (d\Pi^S)_x \) is Fredholm and (a) is proven. Part (b) follows directly from the isomorphisms in (22).

Part (a) implies that we may apply the Smale-Sard theorem to \( \Pi^S \) to obtain a residual set \( R \subset \mathcal{U} \) so that for each \( g \in R \) and \( x \in (\Pi^S)^{-1}(g) \), the differential \( (d\Pi^S)_x \) is isomorphic in \( (\mathcal{M}^T)^{\I} \), and therefore \( (d\Pi^S)_x \) is Fredholm and (a) is proven. Part (b) follows directly from the isomorphisms in (22).

Let \( u \in H^k_\alpha \) be an eigenfunction of \( \Delta_{g,\alpha} \). Define \( G_u : \mathcal{M}^T \to H^{k-2}_\alpha \) by setting
\[
G_u(g) = \Delta_{g,\alpha} u.
\]

**Proposition 5.3.** Let \( \mathcal{U} \) be an open subset of \( \mathcal{M}^T \). Suppose that for each \( g \in \mathcal{U} \) and each eigenfunction \( u \) of \( \Delta_{g,\alpha} \), the subspace \( (dG_u)_g(S^T) \) is dense in the orthogonal complement of \( u^* \). If \( g \in \mathcal{U} \) and \( x \in F^{-1}(0) \cap \pi^{-1}(g) \), then the differential \( dF_x \) is surjective.

**Proof.** Let \( g \in \mathcal{U} \) and \( x \in F^{-1}(0) \cap \pi^{-1}(g) \). Then by (16), there exists \( u \) and \( \lambda \) such that \( x = (u,\lambda,0,g) \) and \( (\Delta_{g,\alpha} - \lambda I)u = 0 \). Moreover, from (19) we find that
\[
dF_x(v,\mu,\nu,h) = (\Delta_{g,\alpha} - \lambda) v + (dG_u)_g(h) - \mu \cdot u - \nu \cdot u^*.
\]
In particular, \( dF_x(0,0,0) = (dG_u)_g(h) \), and so the image of \( (dG_u)_g \) is contained in the image of \( dF_x \). Therefore, the hypothesis implies that the image of \( dF_x \) is dense in \( (u^*)^\perp \). Since \( dF_x(0,0,-1,0) = u^* \), we have \( u^*\in \text{im}(dF_x) \), and therefore the image of \( dF_x \) is dense in \( H^k_\alpha \).

Since \( (dF_g)_g(u,\lambda) \) is Fredholm, the space
\[
Z := dF_x(H^k_\alpha \times \{0\} \times \{0\}) = (dF_g)_g(u,\lambda,0) \cdot (H^k_\alpha \times \{0\})
\]
has finite codimension in \( H^k_\alpha \). In particular, there exists a finite dimensional subspace \( Z^c \subset H^k_\alpha \) so that \( Z \oplus Z^c = H^k_\alpha \). Because the image of \( dF_x \) is dense in \( H^k_\alpha \), the projection of \( \text{im}(dF_x) \) onto \( Z^c \) is dense in \( Z^c \). But \( Z^c \) is finite dimensional, and so the projection of \( \text{im}(dF_x) \) equals \( Z^c \). On the other hand, by (25), the image of \( dF_x \) contains \( Z^c \). It follows that \( \text{im}(dF_x) \supset Z \oplus Z^c = H^k_\alpha \). □

So far, our exposition of Uhlenbeck’s method has been somewhat general in the sense that we have not yet used any specific properties of torus invariant metrics. To verify that the image of \( (dG_u)_g \) is dense in the orthogonal complement of \( u^* \), we will need to use properties of these metrics.

For each \( g \in \mathcal{M}^T \) and \( \alpha \in \mathbb{Z}^d \), define \( b_{g,\alpha} : M \to \mathbb{R} \) by
\[
b_{g,\alpha}(x) := \frac{n+2}{d} \sum_{i,j=1}^d g^{ij}(x) \cdot \alpha_i \cdot \alpha_j.
\]

Note that \( b_{g,\alpha} \) is \( T \)-invariant. Define \( \mathcal{U}_\alpha \) to be the set of metrics \( g \in \mathcal{M}^T \) such that \( b_{g,\alpha} \) is nonconstant on each open subset of \( M \). The set \( \mathcal{U}_\alpha \) is open and dense in \( \mathcal{M}^T \).

**Proposition 5.4.** If \( g \in \mathcal{U}_\alpha \) and \( u \) is an eigenfunction of \( \Delta_{g,\alpha} \), then \( (dG_u)_g(S^T) \) is dense in the orthogonal complement of \( u^* \).

**Proof.** To prove the proposition, it suffices to show that \( (dG_u)_g(S^T)^\perp \) is contained in \( \text{span}(u^*) \).
Let $u = (u_1, u_2)^t$ be an eigenfunction of $\Delta_{g, \alpha}$ and $v = (v_1, v_2)^t \in (dG_u)_g(S^T)^\perp$. We use two perturbation formulae to show that $v$ is a constant multiple of $u^s$.

First consider the perturbation defined in Subsection 3.3. Let $\{\hat{\partial}_i\}$ be the natural frame for $V$ defined in (4). If $g_{jk} := g(\hat{\partial}_j, \hat{\partial}_k)$ is the Gram matrix associated to this frame, then $\nabla^V \xi = \sum_{j,k} g_{jk} \hat{\partial}_j \xi \cdot \hat{\partial}_k$. In particular, for each $u = (u_1, u_2)^t$, $v = (v_1, v_2)^t \in H^k_{\alpha}$, we find that

\begin{equation}
 g(\nabla^V u_1, \nabla^V v_1) = \sum_{j,k} g^{jk} \hat{\partial}_j u_1 \cdot \hat{\partial}_k v_1 = u_2 \cdot v_2 \cdot \sum_{j,k} g^{jk} \cdot \alpha_j \cdot \alpha_k
\end{equation}

and

\begin{equation}
 g(\nabla^V u_2, \nabla^V v_2) = \sum_{j,k} g^{jk} \hat{\partial}_j u_2 \cdot \hat{\partial}_k v_2 = u_1 \cdot v_1 \cdot \sum_{j,k} g^{jk} \cdot \alpha_j \cdot \alpha_k.
\end{equation}

Recall the construction of the analytic path $g_{a_t, b_t}$ in Subsection 3.3. For each choice of smooth $T$-invariant function $f : M \rightarrow \mathbb{R}$, we set $b_t = 1 + t \cdot f$ and $a_t := 1 - \frac{n-2}{d} \cdot t \cdot f$ and define $g_t := g_{a_t, b_t}$. Then $d \cdot \dot{a} + (n - d - 2) f = 0$ and so using (27) and (28) we find that Lemma 3.5 implies that for each $v \in H^k_{\alpha}$, we have

\begin{equation}
 \int_M \Delta u \cdot v \, dv_g = 2 \int_M f \cdot (u_1 v_1 + u_2 v_2) \cdot (b_{g, \alpha} - \lambda) \, dv_g.
\end{equation}

Note that $\Delta u = (dG_u)_g(\dot{g})$ and so for each $(v_1, v_2)^t \in (dG_u)_g(S^T)^\perp$ and each $T$-invariant function $f : M \rightarrow \mathbb{R}$,

\begin{equation}
 0 = \int_M [(dG_u)_g(\dot{g})] \cdot v \, dv_g = 2 \int_M f \cdot (u_1 v_1 + u_2 v_2) \cdot (b_{g, \alpha} - \lambda) \, dv_g.
\end{equation}

Since $(u_1 v_1 + u_2 v_2) \cdot (b_{g, \alpha} - \lambda)$ is $T$-invariant, it must vanish on $M$ almost everywhere. By the definition of $U_\alpha$, the zero locus of the smooth function $b_{g, \alpha} - \lambda$ has measure zero. Therefore, $u_1 v_1 + u_2 v_2$ vanishes on a set of full measure.

Take $k$ in $H^k_{\alpha}$ to be large enough such that $v_1$ and $v_2$ are in $C^1$. Then $u_1 v_1 + u_2 v_2 = 0$ on the entire manifold. On points such that $u_2$ is nonzero, define $h := -v_1/u_2$. Then $h$ is a $C^1$, $T$-invariant function away from the zero locus of $u_2$ and $(v_1, v_2) = h \cdot (-u_2, u_1) = h \cdot u^s$.

Now we use another perturbation formula to show that $h$ is constant. Let $X$ be a vector field that is invariant under the torus action and is orthogonal to each $T$-orbit. Let $Y = \hat{\partial}_j$ be the vector field defined in (4). The corresponding perturbation $g_{X, Y}$ that is defined in Lemma 3.4 is $T$-invariant for each $t$ close to zero.

By (11), for each $v = (v_1, v_2)^t \in (dG_u)_g(S^T)^\perp$ we have

\begin{equation}
 0 = \int_M [(dG_u)_g(\dot{g})] \cdot v \, dv_g = \int_M \Delta u_1 \cdot v_1 + \Delta u_2 \cdot v_2 \, dv_0
\end{equation}

\begin{equation}
 = - \sum_{i=1}^2 \int_M ((X u_i) \cdot (Y v_i) + (Y u_i) \cdot (X v_i)) \, dv_0.
\end{equation}

Since $(v_1, v_2) = h \cdot (-u_2, u_1)$, (30) can be simplified using (5) to

\begin{equation}
 0 = \alpha_j \int_M (u_1^2 + u_2^2) \cdot X(h) \, dv_0.
\end{equation}

Since $\alpha_j \neq 0$ for some $j$, the integral equals zero. Moreover, observe that both $(u_1^2 + u_2^2)$ and $X(h)$ are $T$-invariant and $(u_1^2 + u_2^2)$ vanishes on a set of measure zero.
Because the vector field $X$ is an arbitrary $T$-invariant vector field that is orthogonal to the orbits of $T$, the $T$-invariant function $h$ is a constant function.

Therefore, $(dG_u)_g(S^T)^\perp$ is contained in the span of $u^*$ and the proposition is proved.

\[ \Box \]

**Theorem 5.5.** Let $\alpha \neq 0$. There exists a residual subset $\mathcal{Y}_\alpha$ of $\mathcal{M}^T$ so that for each $g \in \mathcal{Y}_\alpha$, each eigenspace of $\Delta_{g,\alpha} - \lambda I$ is two dimensional.

**Proof.** Combine Lemma 5.1, Proposition 5.2, Proposition 5.3, and Proposition 5.4.

\[ \Box \]

Now we have enough to prove the first part of Theorem 1.1.

**Proof of part (1) of Theorem 1.1.** By Theorem 4.1, there exists a residual subset of $\mathcal{M}^T$ such that for each $g$ in this set we have $\text{spec}(\Delta_{g,\alpha}) \cap \text{spec}(\Delta_{g,\beta}) = \emptyset$ unless $\alpha = \pm \beta$. If $\alpha = \pm \beta$, then $H^k_{\alpha} = H^k_{\beta}$ and so the eigenspaces coincide.

If $\alpha \neq 0$, then by Theorem 5.5, there exists a residual set $\mathcal{Y}_\alpha$ such that the eigenspaces of $\Delta_{g,\alpha}$ are two-dimensional. If $\alpha = 0$, then as in the proof of Proposition 2.1, one may identify $H^k_\alpha$ with the Sobolev space of $H_\alpha^k$ functions on the quotient $M/T$. The operator $\Delta_{g,\alpha}$ corresponds to a nonnegative second order elliptic operator on $M/T$, and it follows from Theorem 8 in [Uhl76] that for a residual set of $g$, this operator has simple spectrum.

Since an intersection of residual sets is residual, the claim follows. \[ \Box \]

6. Nodal sets

In this section we prove part (2) of Theorem 1.1 and we prove Theorem 1.2.

**Proposition 6.1.** Let $\alpha \in \mathbb{Z}^d \setminus \{0\}$. There exists a residual subset of $\mathcal{M}^T$ so that if $g$ belongs to this set, then $0 \in \mathbb{R}^2$ is a regular value of each $u \in \ker(\Delta_{g,\alpha} - \lambda I)$.

The idea of the proof originates in [Uhl76], and an analogue appears as Proposition 4.8 in [JngZld20]. The proof is based on an analysis of the set $Q$ consisting of $(u, \lambda, g) \in H^2_\alpha \times \mathbb{R} \times \mathcal{M}^T$ such that the dimension of $\ker(\Delta_{g,\alpha} - \lambda I)$ equals two. By the discussion surrounding (16), if $(u, \lambda, \lambda', g)$ lies in the submanifold $F^{-1}(0) \subset H^2_\alpha \times \mathbb{R} \times \mathcal{M}^T$ then $\lambda' = 0$, and hence $Q$ is naturally embedded in $F^{-1}(0)$. For each $g \in \mathcal{M}^T$, each eigenspace of $\Delta_{g,\alpha}$ has dimension at least two, and so the continuity of the spectrum implies that the embedding of $Q$ is open in $F^{-1}(0)$.

To prove Proposition 6.1 we will use the following lemma.

**Lemma 6.2.** Let $(u, \lambda, g) \in Q$, and let $E$ be the subspace of $\ker(\Delta_{g,\alpha} - \lambda I)^\perp$ consisting $v$ such that there exists $(\mu, h) \in \mathbb{R} \times T^T$ so that $(v, \mu, h)$ lies in the tangent space $T_{(u, \lambda, g)}Q$. Then $E$ is dense in $\ker(\Delta_{g,\alpha} - \lambda I)^\perp$.

**Proof.** The tangent space $T_{(u, \lambda, g)}Q$ is the kernel of $dF_{(u, \lambda, 0, g)}$ which by (24) is the space of $(v, \mu, 0, h) \in H^2_\alpha \times \mathbb{R} \times \mathcal{M}$ such that

\begin{equation}
(\Delta_{g,\alpha} - \lambda I)v + (dG_u)(h) - \mu \cdot u = 0.
\end{equation}

For simplicity of notation, let $V := \ker(\Delta_{g,\alpha} - \lambda I)^\perp$. The operator $\Delta_{g,\alpha} - \lambda I$ is self-adjoint, and hence $V = \text{im}(\Delta_{g,\alpha} - \lambda I)$. Let $p : H^2_\alpha \to H^k_\alpha$ be the orthogonal projection onto $V$. Let $L := (\Delta_{g,\alpha} - \lambda I)$ denote the restriction of $\Delta_{g,\alpha}$ to $V$. The map $L$ is an isomorphism onto $V$ with bounded inverse $L^{-1}$.
We claim that the image of \( -L^{-1} \circ p \circ (dG_u)_g \) is contained in \( E \). Indeed, suppose that
\[
v = -L^{-1} \circ p \circ (dG_u)_g(h).
\]
Then
\[
Lv = -p \circ (dG_u)_g(h),
\]
and so
\[
(\Delta_{g,\alpha} - \lambda I)v = -(dG_u)_g(h) + w
\]
for some \( w \in \ker(\Delta_{g,\alpha} - \lambda I) = \mathbb{R} \cdot u \oplus \mathbb{R} \cdot u^* \). Since the image of \( (dG_u)_g \) lies in \((u^*)^\perp\) and \( Lv \in V \subset (u^*)^\perp \), we have \( w = c \cdot u \) for some \( c \in \mathbb{R} \). Thus,
\[
(\Delta_{g,\alpha} - \lambda I)v + (dG_u)_g(h) - c \cdot u = 0,
\]
and therefore \( v \in E \) as claimed.

By Proposition 5.4, the image of \( (dG_u)_g \) is dense in \((u^*)^\perp\), and hence the image of \( L^{-1} \circ p \circ (dG_u)_g \) is dense in \( V \). Therefore \( E \) is dense in \( V \). \( \square \)

**Proof of Proposition 6.1.** Assume that \( k \) is large enough so that \( H^k_\alpha \subset C^1 \) by the Sobolev embedding theorem. Then the map \( f : H^k_\alpha \times \mathbb{R} \times \mathcal{M}^T \times M \to \mathbb{R}^2 \) by
\[
f(u, \lambda, g, x) = u(x)
\]
is \( C^1 \). The differential is given by
\[(32)\]
\[
df_{(u,\lambda,g,x)}(v,\mu,h,z) = du_x(z) + v(x).
\]
We now restrict \( f \) to the submanifold \( Q \times M \), and, abusing notation, we denote the restriction by \( f \). The differential of \( f \) is still given by \((32)\) where now \((v,\mu,h,z) \) belongs to the tangent space \( T_{(u,\lambda,g,x)}(Q \times M) \).

We claim that \( df_{(u,\lambda,g,x)}(v,\mu,h,z) = 0 \). If not, then there exists \((u, \lambda, g, x) \in f^{-1}(0) \) and \( a \in \mathbb{R}^2 \) so that for each \((v, \mu, h, z) \in T_{(u,\lambda,g,x)}(Q \times M) \) we have
\[
a \cdot df_{(u,\lambda,g,x)}(v,\mu,h,z) = 0.
\]
In particular, \( a \cdot df_{(u,\lambda,g,x)}(v,\mu,h,0) = a \cdot v(x) = 0 \). For each \( v \in E \) where the set \( E \)—defined in Lemma 6.2— is dense in \( V = \ker(\Delta_{g,\alpha} - \lambda I)^\perp \). We have chosen \( k \) so that the \( H^k \) topology is stronger than the \( C^0 \) topology, and so \( a \cdot v(x) = 0 \) for each \( v \in E \) implies that \( a \cdot v(x) = 0 \) for each \( v \in V \). Separately, since \( f(u, \lambda, g, x) = 0 \), we have \( u(x) = 0 \) and hence \( u^*(x) = 0 \). Thus \( a \cdot w(x) = 0 \) for each \( w \in \ker(\Delta_{g,\alpha} - \lambda I) \), and in sum we find that \( a \cdot w(x) = 0 \) for each \( w \in H^k_\alpha = V \oplus \ker(\Delta_{g,\alpha} - \lambda I) \).

Let \( w \in H^k_\alpha \) be such that \( w(x) \neq 0 \). If \( a \neq 0 \), then there exists a rotation \( R \in SO(2) \) and a positive real number \( c \) so that \( a = c \cdot Rw \). But \( c \cdot Rw \in H^k_\alpha \) and so using the above we have \( 0 = a = c \cdot Rw(x) = c \cdot a \cdot a \), a contradiction. Hence \( a = 0 \) and \( df_{(u,\lambda,g,x)} \) is surjective for each \((u, \lambda, g, x) \in f^{-1}(0) \).

Let \( \pi : Q \times M \to Q \) denote the natural projection. Given \( q = (u, \lambda, g) \in Q \) let \( f_q \) denote the restriction of \( f \) to \( \pi^{-1}(q) \). Because, \( df_{(u,\lambda,g,x)} \) is surjective for each \((u, \lambda, g, x) \in f^{-1}(0) \), the parametric transversality theorem applies to provide a residual subset \( P \subset Q \), so that if \( q = (u, \lambda, g) \in P \) then \( df_q(x) = du_x \) is surjective for each \( x \in \pi^{-1}(0) \).

Let \( \mathcal{R} \) be the set of \( g \in \mathcal{M}^T \) such that \( \ker(\Delta_{g,\alpha} - \lambda I) \) is two dimensional for each \( \lambda \in \text{spec}(\Delta_{g,\alpha}) \). The first part of Theorem 1.1 implies that \( \mathcal{R} \) is residual in \( \mathcal{M}^T \). Since \( \pi : Q \to \mathcal{M}^T \) is open, the set \( \pi(P) \) is residual in \( \mathcal{M}^T \), and hence \( \pi(P) \cap \mathcal{R} \) is residual in \( \mathcal{M}^T \). \( \square \)
Proof of part (2) of 1.1. If an eigenfunction is $T$-invariant, then it descends to an eigenfunction of an elliptic operator on $M/T$, and it follows from Theorem 8 in [Uhl76] that, for a residual set of $g$, the nodal set of each $T$-invariant eigenfunction of $\Delta_g$ is smooth.

If an eigenfunction is not $T$-invariant and $g$ lies in the residual subset obtained in the proof of part (1) of Theorem 1.1, then the eigenfunction is the first coordinate $u_1$ of a vector-valued function $u = (u_1, u_2)^t \in H^2_{\alpha}$. If $(u_1(x), u_2(x))^t = 0$, then Proposition 6.1 implies that $d(u_1)_x$ is surjective. If $u_1(x) = 0$ but $u_2(x) \neq 0$, then from (4) we have $(\partial_x u)(x) = \alpha_1 \cdot u_2$, and since $\alpha \neq 0$, the differential $d(u_1)_x$ is surjective. Therefore, the implicit function theorem implies that $u_1^{-1}(0)$ is smooth.

The following generalizes parts (2) and (3) of Theorem 1.5 in [JngZhi20].

**Theorem 6.3.** Suppose that $\dim(M/T) \geq 2$ and let $\alpha \in \mathbb{Z}^d \setminus \{0\}$. There exists a residual subset $\mathcal{Y}_\alpha$ in $\mathcal{M}^T$ so that if $g$ lies in $\mathcal{Y}_\alpha$ and $u = (u_1, u_2)^t$ is an eigenfunction of $\Delta_{g, \alpha}$ whose values include zero, then the nodal set of $u_1$ (resp. $u_2$) is a connected smooth manifold, and its complement consists of exactly two connected components.

**Proof.** Let $u = (u_1, u_2)^t$ be an eigenfunction of $\Delta_{g, \alpha}$ and let $\pi : M \to M/T$ be the standard submersion. By Proposition 6.1 and the implicit function theorem, we find that the set $u^{-1}(0)$ is a smooth submanifold of $M$ with dimension $n - 2$. Moreover, since $u(\theta^{-1} \cdot x) = R_{\alpha \cdot \theta} \cdot u(x)$ for each $\theta$ in the $d$-dimensional torus $T$, the action of $T$ on $M$ preserves $u^{-1}(0)$. In particular, $\pi(u^{-1}(0))$ is of real codimension two, and hence $\pi(M \setminus \pi(u^{-1}(0)))$ is connected. Moreover, if $x \in \pi(u^{-1}(0))$, then $\pi^{-1}(x)$ is a connected torus embedded in $u^{-1}(0)$.

Define $N := u_1^{-1}(0) \setminus u^{-1}(0)$. We claim that the restriction of $\pi$ to $N$ is a submersion onto its image $\pi(N) = \pi(M) \setminus \pi(u^{-1}(0))$. Indeed, since $\alpha \neq 0$, there exists $i$ such that $\alpha_i \neq 0$. Recall from §2 that $\partial_i u = \alpha_i \cdot u^*$ and in particular $\partial_i u_1 = \alpha_1 \cdot u_2$. Thus, we have $\partial_i u_1(x) \neq 0$ and $u_1(x) = 0$ if and only if $u(x) \neq 0$. In particular, the vector field $\partial_i$ is transverse to $N$ and since $\pi$ is a submersion, the restriction $\pi|_N$ is also a submersion. Each fiber of $\pi|_N$ is diffeomorphic to the disjoint union of $2m$ copies of the $d - 1$ dimensional torus where $m = \gcd(\alpha_1, \ldots, \alpha_d)$.

By hypothesis, $u^{-1}(0)$ is nonempty and hence contains some $y_0$. Because $\alpha \neq 0$, the eigenfunction $u$ vanishes on the fiber $\pi^{-1}(x_0)$ where $x_0 = \pi(y_0)$. The fiber $\pi^{-1}(x_0)$ is a torus and hence is connected. Thus, to show that $u_1^{-1}(0)$ is connected, it suffices to show that, for each $y \in N$, the connected component $K_y$ of $u_1^{-1}(0)$ that contains $y$ also contains a point in $u^{-1}(x_0)$.

Since $\pi(M)$ is connected and $\pi(u^{-1}(0))$ is a smooth closed submanifold of codimension 2, there exists a path $\gamma : [0, 1] \to \pi(M)$ such that $\gamma(0) = \pi(y)$, $\gamma(1) = x_0$ and $\gamma(t) \notin \pi(u^{-1}(0))$ for each $t < 1$. Choose a horizontal distribution for $\pi : N \to \pi(N)$. Let $\tilde{\gamma} : [0, 1] \to N$ denote the horizontal lift of $\gamma|_{[0, 1]}$.

Suppose that $t_k \in [0, 1)$ converges to 1. Since $M$ is compact, the sequence $\tilde{\gamma}(t_k)$ has a convergent subsequence that converges to some $z \in M$. On the other hand, $\pi(\tilde{\gamma}(t_k)) = \gamma(t_k)$ converges to $x_0$, and so $z \in \pi^{-1}(x_0)$. Hence $K_y$ intersects $\pi^{-1}(x_0)$. Therefore, $u_1^{-1}(0)$ is connected.

Finally, since $\alpha \neq 0$, the eigenfunction $u_1$ takes on both positive and negative values, and hence the set $M \setminus u_1^{-1}(0)$ has at least two components. In particular, $u_1^{-1}(0)$
separates \( M \) and a classical result in differential topology gives that \( M \setminus u_1^{-1}(0) \) has exactly two connected components. \( \square \)

Let \( \mathcal{V} = \bigcap_{\alpha \neq 0} \mathcal{Y}_\alpha \).

**Corollary 6.4.** Suppose \( \dim(M/T) \geq 2 \). Let \( g \) belong to the residual subset \( \mathcal{Y} \subset \mathcal{M}^T \). If \( \varphi : M \to \mathbb{R} \) is a non-invariant eigenfunction of \( \Delta_g \) such that \( \varphi \) vanishes on some \( T \)-orbit, then the nodal set \( \varphi^{-1}(0) \) is a connected smooth hypersurface and its complement consists of two components.

**Proof.** Since \( g \in \mathcal{Y} \) and \( \varphi \) is non-invariant, there exists \( \alpha \in \mathbb{Z} \setminus \{0\} \) and \( u = (u_1, u_2)^t \in H^k_\alpha \) such that \( \varphi = u_1 \). If \( \varphi(x) = u_1(x) = 0 \) for each \( x \) in some orbit of \( T \), then \( \partial_j u_1(x) = 0 \) vanishes for each \( x \) in this orbit. Thus, if \( \alpha_j \neq 0 \), then from (5) we find that \( u_2(x) = \alpha_j^{-1} \cdot \partial_j u_1(x) \) for \( x \) in the orbit. The claim now follows from Theorem 6.3. \( \square \)

## 7. Circle bundles

In this section, we explain how the space \( H^k_\alpha, \alpha \neq 0 \), may be identified with sections of an oriented rank two real vector bundle, and we translate the results of §6 into this language. Then we briefly discuss principal \( SO(2) \) bundles and the natural group structure on the set such bundles over a fixed base. Finally, we prove Theorem 1.2.

Let \( \alpha \neq 0 \), and define the vector bundle \( \pi : E^\alpha \to M/T \) as the equivariant quotient of the trivial bundle \( M \times \mathbb{R}^2 \to M \) where the action of \( T \) on \( M \times \mathbb{R}^2 \to M \) is given by

\[
(\theta \cdot (x, w)) := (\theta \cdot x, R_{\alpha, \theta} \cdot w)
\]

with \( R_{\alpha, \theta} \) as in (3). A section \( x \to (x, u(x)) \) of the trivial bundle \( M \times \mathbb{R}^2 \to M \) descends to a section \( \sigma_u \) of \( E^\alpha \) if and only if

\[
(\theta \cdot x, u(\theta \cdot x)) = (\theta \cdot x, R_{\alpha, \theta} \cdot u(x))
\]

for each \( x \in M \) and \( \theta \in T \). In other words, the map \( u \mapsto \sigma_u \) is an isomorphism from \( H^k_\alpha \) onto the space of \( H^k \) sections of \( E^\alpha \).

Note that the action of \( SO(2) \) provides an orientation of each fiber of \( E^\alpha \). By removing the zero section of \( E^\alpha \) and quotienting each fiber by the action of \( \mathbb{R}^+ \), we obtain an oriented circle bundle \( M^\alpha \to M/T \).

**Proposition 7.1.** If the oriented circle bundle \( M^\alpha \) is nontrivial, then each eigenfunction \( u \) of \( \Delta_{g, \alpha} \) vanishes on some torus orbit.

**Proof.** Let \( u \in H^k_\alpha \) be an eigenfunction of \( \Delta_{g, \alpha} \). By elliptic regularity, the function \( u \) is smooth, and it corresponds to a smooth section \( \sigma_u \) of \( E^\alpha \). If \( u(x) \neq 0 \) for all \( x \in M \), then \( \sigma_u(b) \neq 0 \) for each \( b \in M/T \). Thus \( \sigma_u(b) \) would define a global section of \( M^\alpha \). But then \( M^\alpha \) would be trivial, contradicting our assumption. \( \square \)

**Proposition 7.2.** If each oriented circle bundle \( M^\alpha \to M/T \) associated to the torus bundle \( M \to M/T \) is nontrivial, then there exists a residual subset of \( \mathcal{M}^T \) such that for each \( g \) in this set and each non-invariant eigenfunction \( \varphi \) of \( \Delta_g \), the nodal set \( \varphi^{-1}(0) \) is a connected smooth submanifold whose complement has exactly two components.

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\(^5\)See, for example, Lemma 4.4.4 in [Hirsch].
Proof. Combine Corollary 6.4 and Proposition 7.1.

Proof of Theorem 1.2. The Euler class \( e(N) \in H_2(B, \mathbb{Z}) \) is a complete invariant for oriented circle bundles \( N \) over \( B \) [Morita]. In particular, if \( M \) is a nontrivial oriented circle bundle, then \( e(M) \neq 0 \). Thus by hypothesis, the class \( e(M) \) has infinite order in \( H_2(B, \mathbb{Z}) \). If \( \alpha \in \mathbb{Z} \), then \( e(M^\alpha) = \alpha \cdot e(M) \). Thus, if \( \alpha \neq 0 \), then \( e(M^\alpha) \neq 0 \) and so \( M^\alpha \) is not trivial. Thus, the claim follows from Proposition 7.2.

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