Spectrum analysis for the relativistic Boltzmann equation

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Abstract

The spectrum structure of the linearized relativistic Boltzmann equation around a global Maxwellian is studied in this paper. Based on the spectrum analysis, we establish the optimal time-convergence rates of the global solution to the Cauchy problem for the relativistic Boltzmann equation.

Key words. relativistic Boltzmann equation, spectrum analysis, optimal time decay rates.

2010 Mathematics Subject Classification. 76P05, 82C40, 82D05.

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1 Introduction

In this paper, we consider the Cauchy problem for the relativistic Boltzmann equation

\[
\begin{aligned}
\partial_t F + \hat{v} \cdot \nabla_x F &= Q(F,F), \\
F(0, x, v) &= F_0(x, v),
\end{aligned}
\]  

(1.1)

where \( F = F(t, x, v) \) is the distribution function with \((t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^3_x \times \mathbb{R}^3_v \) and the relativistic velocity \( \hat{v} \) is defined by

\[
\hat{v} = \frac{v}{v_0}, \quad v_0 = \sqrt{1 + |v|^2}.
\]

The collision operator \( Q(F,G) \) is given by

\[
Q(F,G)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} v_M \sigma(g, \theta)[F(v')G(u') - F(u)G(u)]dud\omega,
\]  

(1.2)
where
\[ \frac{v_M}{u_0v_0} = 2g\sqrt{1 + g^2}, \quad u_0 = \sqrt{1 + |u|^2}, \]
\[ 4g^2 = 2(u_0v_0 - uv - 1) = s - 4, \]
\[ s = 2(u_0v_0 - uv - 1), \]
and \(d\omega\) is surface measure on the unit sphere \(S^2\). The scattering angle \(\theta\) is defined by
\[ \cos \theta = \frac{(U - V)(U' - V')}{(U - V)(U - V)'}, \]
with
\[ U = (u_0, u_1, u_2, u_3), \quad V = (v_0, v_1, v_2, v_3), \]
\[ U' = u_0v_0 - \sum_{k=1}^{3} u_kv_k. \]
Notice that the conservation law of momentum and energy are given by
\[ u + v = u' + v', \quad \sqrt{1 + |u|^2} + \sqrt{1 + |v|^2} = \sqrt{1 + |u'|^2} + \sqrt{1 + |v'|^2}. \]

The method of spectral analysis for the Relativistic Boltzmann equation is similar to that of Boltzmann equation \[11, 25, 26\]. Without loss of generality, a global relativistic Maxwellian takes the form
\[ M(v) = e^{-\sqrt{1 + |v|^2}}. \]
Set the perturbation \(f(t, x, v)\) of \(F(t, x, v)\) around \(M(v)\) by
\[ F = M + \sqrt{M}f. \]
Then the RB (1.1) becomes
\[ \begin{cases} \partial_t f + \hat{v} \cdot \nabla_x f - Lf = \Gamma(f, f), \\ f(0, x, v) = f_0(x, v), \end{cases} \]
where the linearized collision operator \(Lf\) and nonlinear term \(\Gamma(f, f)\) are defined by
\[ Lf = \frac{1}{\sqrt{M}}[Q(M, \sqrt{M}f) + Q(\sqrt{M}f, M)], \]
\[ \Gamma(f, f) = \frac{1}{\sqrt{M}}Q(\sqrt{M}f, \sqrt{M}f). \]
The linearized collision operator \(L\) can be written as \[11, 28\]
\[ Lf = -\nu(v)f + Kf, \]
where \(\nu(v)\) is collision frequency and its expression is
\[ \nu(v) = \int_{\mathbb{R}^3} \int_{S^2} v_M \sigma(g, \theta) \mu(u) d\omega du, \]
and \(K\) is a compact operator on \(L^2(\mathbb{R}^3)\) and its expression is
\[ Kf = \int_{\mathbb{R}^3} \int_{S^2} v_M \sigma(g, \theta) M^{\frac{1}{2}}[M^{\frac{1}{2}}(u')f(u') + M^{\frac{1}{2}}(v')f(u') - M^{\frac{1}{2}}(v)f(u)] d\omega du. \]
The scattering kernel $\sigma$ satisfies the following condition \[ \tag{1.8} \]

$$C_1 \frac{g^{\beta+1}}{1 + g} \sin^g \theta \leq \sigma(g, \theta) \leq C_2 (g^\beta + g^\delta) \sin^\gamma \theta,$$

where $C_1$ and $C_2$ are positive constants, $0 \leq \delta < \frac{1}{2}$, $0 \leq \beta < 2 - 2\delta$, and either $\gamma \geq 0$ or $|\gamma| \leq \min \left\{ 2 - \beta, \frac{1}{2} - \theta, \frac{1}{3} (2 - 2\theta - \beta) \right\}.$

In fact, $L$ is a non-positive and self-adjoint operator and its null space, denoted by $N_0$, is 5 dimensional subspace given by

$$N_0 = \text{span}\{\sqrt{M}, v_j \sqrt{M}, 1 \leq j \leq 3, v_0 \sqrt{M} \}. \tag{1.9}$$

We normalized the elements of $N_0$ as

$$\psi_0 = p_0^{-\frac{1}{2}} \sqrt{M}, \quad \psi_1 = p_1^{-\frac{1}{2}} v_j \sqrt{M}, \quad \psi_4 = p_4^{-\frac{1}{2}} (v_0 - p_2) \sqrt{M}. \tag{1.10}$$

Here the normalized constants are given by

$$p_0 = \int_{\mathbb{R}^3} M(v) dv, \quad p_1 = \int_{\mathbb{R}^3} v_j^2 M(v) dv, \quad p_2 = \int_{\mathbb{R}^3} v_0 M(v) dv, \quad p_3 = \int_{\mathbb{R}^3} v_j^2 M dv - p_2^2.$$

Set $L^2(\mathbb{R}^3)$ be a Hilbert space of complex-value functions $f(v)$ on $\mathbb{R}^3$ with the inner product and the norm

$$(f, g) = \int_{\mathbb{R}^3} f(v) \overline{g(v)} dv, \quad \|f\| = \left( \int_{\mathbb{R}^3} |f(v)|^2 dv \right)^{1/2}.$$

Define $P_0$ be the projection operator from $L^2(\mathbb{R}^3)$ to the null $N_0$ and $P_1 = I - P_0$. Corresponding to the linearized operator $L$, it is shown in \cite{11} that there exists a constant $\mu > 0$ such that

$$(L f, f) \leq -\mu (P_1 f, P_1 f), \quad \forall f \in D(L), \tag{1.11}$$

where $D(L)$ is the domain of $L$ given by

$$D(L) = \{ f \in L^2(\mathbb{R}^3) \mid \nu(v)f \in L^2(\mathbb{R}^3) \}.$$

Note that the solution $f$ can be decomposed into the macroscopical part and microscopical part as

$$f = f_0 + P_1 f, \quad P_0 f = n \psi_0 + \sum_{j=1}^3 m_j \psi_j + q \psi_4, \tag{1.12}$$

where the density $n$, the momentum $m_j = (m_1, m_2, m_3)$ and the energy $q$ are defined by

$$(f, \psi_0) = n, \quad (f, \psi_j) = m_j, \quad (f, \psi_4) = q. \tag{1.13}$$

To obtain the optimal decay rate of global solution to \cite{1.5}, we need to consider Cauchy problem for the linear relativistic Boltzmann equation

$$\begin{cases}
\partial_t f = B f, \quad t > 0, \\
f(0, x, v) = f_0(x, v), \quad (x, v) \in \mathbb{R}^3_x \times \mathbb{R}^3_v,
\end{cases} \tag{1.14}$$

where the operator $B$ is defined by

$$B f = L f - \hat{v} \cdot \nabla_x f. \tag{1.15}$$

We take the Fourier transform in \cite{1.14} with respect to $x$ to get

$$\begin{cases}
\partial_t \hat{f} = \hat{B}(k) \hat{f}, \quad t > 0, \\
\hat{f}(0, k, v) = \hat{f}_0(k, v), \quad (k, v) \in \mathbb{R}^3_k \times \mathbb{R}^3_v,
\end{cases} \tag{1.16}$$
where the operator $\hat{B}(k)$ is defined by
\[ \hat{B}(k)f = (L - ik \cdot \partial)f. \]  

(1.17)

**Notations:** Before stating the main results in this paper, we list some notations. Define the Fourier transform of $f(x, v) = F(\hat{f}(k, v))$ by
\[ \hat{f}(k, v) = \mathcal{F}f(k, v) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} f(x, v)e^{-ix \cdot k}dx, \]
where and throughout this paper we denote $i = \sqrt{-1}$.

Let us introduce a Sobolev space of function $f = f(x, v)$ by $H^N = L^2(\mathbb{R}_x^3; H^N(\mathbb{R}_v^3))$ with the norm
\[ \|f\|_{H^N} = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} (1 + |k|^2)^N|\hat{f}(k, v)|^2dkdv\right)^{1/2}. \]

For $l \in \mathbb{R}$, we define a weight function by
\[ w_l(v) = (\sqrt{1 + |v|^2})^l, \]
and the Sobolev spaces $H_{N,l}$ as
\[ H_{N,l} = \{ f \in L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3) \mid \|f\|_{H_{N,l}} < \infty \}, \]
with the norm
\[ \|f\|_{H_{N,l}} = \sum_{|\alpha| \leq N} \|w_l(v)\partial_x^\alpha f\|_{L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)}. \]

We also define a space of function $f = f(x, v)$ by $Z_q = L^2(\mathbb{R}_x^3; L^q(\mathbb{R}_v^3))$ with the norm
\[ \|f\|_{Z_q} = \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |f(x, v)|^qdx\right)^{2/q}dv\right)^{1/2}. \]

Throughout this paper, denote $\partial_x^\alpha = \partial_x^{\alpha_1} \partial_x^{\alpha_2} \partial_x^{\alpha_3}$ and $k^\alpha = k_1^{\alpha_1} k_2^{\alpha_2} k_3^{\alpha_3}$ with $k = (k_1, k_2, k_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$. We denote by $\| \cdot \|_{L^2_{x,v}}$ and $\| \cdot \|_{L^q_{x,v}}$ the norms of the function spaces $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$ and $L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$, respectively, and denote by $\| \cdot \|_{L^2_{x}}$, $\| \cdot \|_{L^q_{x}}$ and $\| \cdot \|_{L^q_{v}}$ the norms of the function spaces $L^2(\mathbb{R}_x^3)$, $L^q(\mathbb{R}_x^3)$ and $L^2(\mathbb{R}_x^3)$, respectively.

The main results can be stated as follows. First, we have the spectrum set of the linear operator $\hat{B}(k)$ as follows.

**Theorem 1.1.** (1) For any $\tau_1 > 0$, there exists $d = d(\tau_1) > 0$ so that for $|k| > \tau_1$,
\[ \sigma(\hat{B}(k)) \subset \{ \lambda \mid \text{Re}\lambda < -d \}. \]

(2) There exists a small constant $\tau_0 > 0$ so that for $|k| \leq \tau_0$,
\[ \sigma(\hat{B}(k)) \cap \{ \lambda \in \mathbb{C} \mid \text{Re}\lambda > -\mu/2 \} = \{ \lambda_j(|k|) \}_{j=-1}^3, \]
where $\lambda_j(|k|)$ is $C^\infty$ function of $|k|$. In particular, the eigenvalues $\lambda_j(|k|)$ admit the following asymptotic expansion for $|k| \leq \tau_0$,
\[
\begin{align*}
\lambda_{\pm 1}(|k|) & = \pm i\sqrt{a^2 + b^2}|k| + A_{\pm 1}|k|^2 + O(|k|^3), \\
\lambda_0(|k|) & = A_0|k|^2 + O(|k|^3), \\
\lambda_2(|k|) & = \lambda_3(|k|) = A_2|k|^2 + O(|k|^3),
\end{align*}
\]
where $a, b > 0$ and $A_j > 0$, $j = -1, 0, 1, 2$ are constants given by (2.34).

The, we have the time-asymptotical decay rates of global solution to the nonlinear Boltzmann equation (1.5) as follows.
Theorem 1.2. Assume that $f_0 \in H_{N,1} \cap Z_1$ for $N \geq 4$, and $\|f_0\|_{H_{N,1} \cap Z_1} \leq \delta_0$ for a constant $\delta_0 > 0$ small enough. Then, there exists a globally unique solution $f = f(t,x,v)$ to the relativistic Boltzmann equation (1.5), which satisfies
\[
\|\partial^j_t (f, \psi_j)\|_{L^2} \leq C\delta(1+t)^{-\frac{j}{2} - \frac{7}{4}}, \quad j = 0, 1, 2, 3, 4, \tag{1.19}
\]
\[
\|\partial^j_t P_t(f)\|_{L^2} \leq C\delta(1+t)^{-\frac{j}{2} - \frac{7}{4}}, \tag{1.20}
\]
\[
\|P_t f\|_{H_{N,1} + \|\nabla_x P_t f\|_{H^{N-1}} \leq C\delta(1+t)^{-\frac{j}{2}}, \tag{1.21}
\]
for $|\alpha| = 0, 1$. If it further holds that $P_0 f_0 = 0$, then
\[
\|\partial^j_t (f, \psi_j)\|_{L^2} \leq C\delta(1+t)^{-\frac{j}{2} - \frac{7}{4}}, \quad j = 0, 1, 2, 3, 4, \tag{1.22}
\]
\[
\|\partial^j_t P_t(f)\|_{L^2} \leq C\delta(1+t)^{-\frac{j}{2} - \frac{7}{4}}, \tag{1.23}
\]
\[
\|P_t f\|_{H_{N,1} + \|\nabla_x P_t f\|_{H^{N-1}} \leq C\delta(1+t)^{-\frac{j}{2}}, \tag{1.24}
\]
for $|\alpha| = 0, 1$.

We can also prove that the above time-decay rates are indeed optimal in following sense.

Theorem 1.3. Let the assumptions of Theorem 1.2 hold. Assume further that there exist two constants $d_0, d_1 > 0$ and a small constant $\tau_0 > 0$ such that $\inf_{|x| \leq \tau_0} \|f_0, \psi_0\| \geq d_0$, $\sup_{|x| \leq \tau_0} \|f_0, \psi_0\| = 0$ and $\sup_{|x| \leq \tau_0} |(\hat{f}_0, \psi') = 0$ with $\psi' = (\psi_1, \psi_2, \psi_3)$. Then, the global solution $f$ to the relativistic Boltzmann equation (1.5) satisfies
\[
C_1\delta_0(1+t)^{-\frac{j}{2}} \leq \|(f(t), \psi_j)\|_{L^2} \leq C_2\delta_0(1+t)^{-\frac{j}{2}}, \quad j = 0, 4, \tag{1.25}
\]
\[
C_1\delta_0(1+t)^{-\frac{j}{2}} \leq \|P_t f(t), \psi'\|_{L^2} \leq C_2\delta_0(1+t)^{-\frac{j}{2}}, \tag{1.26}
\]
\[
C_1\delta_0(1+t)^{-\frac{j}{2}} \leq \|P_t f(t)\|_{L^2} \leq C_2\delta_0(1+t)^{-\frac{j}{2}}, \tag{1.27}
\]
\[
C_1\delta_0(1+t)^{-\frac{j}{2}} \leq \|f(t)\|_{H_{N,1}} \leq C_2\delta_0(1+t)^{-\frac{j}{2}}, \tag{1.28}
\]
for $t > 0$ large with constants $C_2 > C_1 > 0$. If it holds that $P_0 f_0 = 0$, $\inf_{|x| \leq \tau_0} \|\hat{f}_0, L^{-1} P_1(\psi, \omega)\| \geq d_0$ and $\sup_{|x| \leq \tau_0} |(\hat{f}_0, L^{-1} P_1(\psi, \omega)\psi) = 0$ with $j = 0, 4$ and $\omega = k/|k|$, then
\[
C_1\delta_0(1+t)^{-\frac{j}{2}} \leq \|(f(t), \psi_j)\|_{L^2} \leq C_2\delta_0(1+t)^{-\frac{j}{2}}, \quad j = 0, 4, \tag{1.29}
\]
\[
C_1\delta_0(1+t)^{-\frac{j}{2}} \leq \|P_t f(t), \psi'\|_{L^2} \leq C_2\delta_0(1+t)^{-\frac{j}{2}}, \tag{1.30}
\]
\[
C_1\delta_0(1+t)^{-\frac{j}{2}} \leq \|P_t f(t)\|_{L^2} \leq C_2\delta_0(1+t)^{-\frac{j}{2}}, \tag{1.31}
\]
\[
C_1\delta_0(1+t)^{-\frac{j}{2}} \leq \|f(t)\|_{H_{N,1}} \leq C_2\delta_0(1+t)^{-\frac{j}{2}}, \tag{1.32}
\]
for $t > 0$ large.

There have been many important achievements on the existence and long time behaviors of solutions to the classical Boltzmann equation. In particular, in [3] it has been shown that the global existence of renormalized weak solution subject to general large initial data. The global existence and optimal decay rate $(1 + t)^{-\frac{j}{2}}$ of strong solution near Maxwellian for hard potential was proved in [8, 20, 25, 26, 27, 29]. The global existence of solutions near vacuum was investigated in [1, 12, 14]. The spectrum structure of classical Boltzmann equation has been developed vigorously. The spectrum of the Boltzmann equation with hard sphere and hard potentials was constructed in [26, 27, 28] and the optimal time decay rate based on spectral analysis has been established in the torus [25], and in $\mathbb{R}^n$ [20, 27]. The pointwise behavior of the Green function of the Boltzmann equation was verified in [19, 21] for hard sphere, and in [17, 18] for hard potentials and soft potentials.

There have been a lot of works on the relativistic Boltzmann equation, see [2, 4, 5, 8, 11, 13] and the references therein. The background of relativistic Boltzmann equations is mentioned in [2]. The existence
and uniqueness of the solution to the linearized relativistic Boltzmann equation has been proved in [4]. The global solution of relativistic Boltzmann equation near a relativistic Maxwellian is obtained in [10, 11] for hard potentials, and in [3, 24] for soft potentials. Yang and Yu established in [28] that the global solutions to the relativistic Boltzmann and Landau equations tend to the equilibrium at $(1 + t)^{-\frac{3}{4}}$ in $L^2$-norm by using compensating function and the energy method.

The organization of this paper is as follows. In section 2, we study the spectrum and resolvent sets of the linearized relativistic Boltzmann equation and show asymptotic expansions of eigenvalues and eigenfunctions of the linearized operator $\hat{B}(k)$ at low frequency based on the approach in [26]. In section 3, we study the semigroup $e^{t\hat{B}(k)}$ generated by the linear operator $\hat{B}(k)$, and we establish the optimal time decay rates of the global solution to the linearized relativistic Boltzmann equation in terms of the $e^{t\hat{B}(k)}$ by drawing on the idea of [29]. In section 4, based on the asymptotic behaviors of linearized problem, we establish the optimal time decay rates of the original nonlinear relativistic Boltzmann equation.

2 Spectral analysis

In this section, we study the spectrum and resolvent sets of the linearized collision operator $\hat{B}(k)$ defined by (1.17), which will be applied to study the optimal decay rate of solution to the linear system (1.14).

We review some properties of the operators $\nu(v)$ and $K$.

Lemma 2.1 ([4, 12]). Under the assumption (1.8), the following holds.

(i) There are two constant $c_0, c_1 > 0$ such that

$$c_0 v_0^{\beta/2} \leq \nu(v) \leq c_1 v_0^{\beta/2}, \quad v \in \mathbb{R}^3. \tag{2.1}$$

(ii) The operator $K$ is bounded and compact in $L^2(\mathbb{R}_v^3)$. Furthermore, $K$ is an integral operator

$$Kf(v) = \int_{\mathbb{R}^3} K(u, v)f(u)du \tag{2.2}$$

with the kernel $K(u, v)$ satisfying

1. $\sup_v \int_{\mathbb{R}^3} |K(u, v)|du < \infty$,
2. $\sup_v \int_{\mathbb{R}^3} |K(u, v)|^2du < \infty$,
3. $\int_{\mathbb{R}^3} |K(u, v)|(1 + |u|^2)^{-\alpha/2}du \leq C(1 + |v|^2)^{-\frac{\beta}{2}(\alpha + \eta)}$ for any $\alpha \geq 0$, where

$$\eta = 1 - \frac{1}{2}[3|\beta| + \beta + 2\delta] > 0.$$

Let $\hat{A}(k)$ be the operator obtained by dropping $K$ from $\hat{B}(k)$, we have

$$\hat{A}(k)f = (-i k \cdot \hat{v} - \nu(v))f, \quad f \in D(\hat{A}(k)), \tag{2.3}$$

$$D(\hat{A}(k)) = D(\hat{B}(k)) = \{ f \in L^2(\mathbb{R}_v^3) \mid \nu(v)f \in L^2(\mathbb{R}_v^3) \}.$$

Lemma 2.2. The operator $\hat{A}(k)$ generates a continuous contraction semigroup on $L^2(\mathbb{R}_v^3)$, which satisfies

$$\|e^{t\hat{A}(k)}f\| \leq e^{-c_0 t}\|f\|, \quad t > 0, \quad f \in L^2(\mathbb{R}_v^3).$$

In addition, we have $\sigma(\hat{A}(k)) \subset \{ \lambda \mid \text{Re}\lambda \leq -c_0 \}$. 
Proof. By \((2.1)\) and \((2.3)\), we have for any \(f \in D(\hat{A}(k))\) that
\[
\text{Re}(\hat{A}(k)f, f) = \text{Re}(\hat{A}(-k)f, f) = -(\nu f, f) \leq -c_0\|f\|^2,
\]
which proves that \(\hat{A}(k)\) and \(\hat{A}(-k)\) are dissipative operators on \(L^2(\mathbb{R}_c^3)\).

Note that \(\hat{A}(k)\) is densely defined in \(L^2(\mathbb{R}_c^3)\), and the adjoint operator \(\hat{A}(-k)\) of \(\hat{A}(k)\) is also densely defined in \(L^2(\mathbb{R}_c^3)\). This implies that \(\hat{A}(k)\) is a closed operator in \(L^2(\mathbb{R}_c^3)\) \([23]\). Thus, it follows from Corollary 4.4 on p.15 of \([22]\) that \(\hat{A}(k)\) generates a continuous contraction semigroup \(e^{t\hat{A}(k)}\) on \(L^2(\mathbb{R}_c^3)\). Then, by direct computation, we can show for any \(t \geq 0\) that
\[
\|e^{t\hat{A}(k)}f\| \leq e^{-c_0t}\|f\|.
\]
By the semigroup theory, we find
\[
\{\lambda : \text{Re}\lambda > -c_0\} \subset \rho(\hat{A}(k)).
\]
Thus we have
\[
\sigma(\hat{A}(k)) \subset \{\lambda : \text{Re}\lambda \leq -c_0\}.
\]
This proves the lemma. \(\square\)

**Lemma 2.3.** The operator \(\hat{B}(k)\) generates a continuous contraction semigroup on \(L^2(\mathbb{R}_c^3)\), which satisfies
\[
\|e^{t\hat{B}(k)}f\| \leq \|f\|, \quad t > 0, \quad f \in L^2(\mathbb{R}_c^3).
\]

**Proof.** We can compute that \(\hat{B}(k)\) and \(\hat{B}(-k)\) are dissipative. By \((1.1)\), we have
\[
(\hat{B}(k)f, g) = ((L - ik \cdot \hat{v})f, g) = (f, (L + ik \cdot \hat{v})g) = (f, \hat{B}(-k)g).
\]
Thus
\[
\text{Re}(\hat{B}(k)f, f) = \text{Re}(\hat{B}(-k)f, f) = (Lf, f) \leq 0, \quad \forall f \in D(\hat{B}(k)).
\]
Note that \(\hat{B}(k)\) is a densely defined closed operator in \(L^2(\mathbb{R}_c^3)\). Hence, \(\hat{B}(k)\) generates a continuous contraction semigroup in \(L^2(\mathbb{R}_c^3)\). \(\square\)

**Lemma 2.4.** The following conditions hold for all \(k \in \mathbb{R}^3\).

1. \(\sigma_{ess}(\hat{B}(k)) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda \leq -c_0\}\) and \(\sigma(\hat{B}(k)) \cap \{\lambda \in \mathbb{C} : -c_0 < \text{Re}\lambda \leq 0\} \subset \sigma_d(\hat{B}(k))\).

2. If \(\lambda(k)\) is an eigenvalue of \(\hat{B}(k)\), then \(\text{Re}\lambda(k) < 0\) for any \(|k| \neq 0\) and \(\lambda(k) = 0\) iff \(|k| = 0\).

**Proof.** By \((2.1)\) and \((2.3)\), \(\lambda - \hat{A}(k)\) is invertible for \(\text{Re}\lambda > -c_0\). Since \(K\) is a compact operator on \(L^2(\mathbb{R}_c^3)\), \(\hat{B}(k)\) is a compact perturbation of \(\hat{A}(k)\), and so, thanks to Theorem 5.35 in p.244 of \([15]\), \(\hat{B}(k)\) and \(\hat{A}(k)\) have the same essential spectrum, namely, \(\sigma_{ess}(\hat{B}(k)) = \sigma_{ess}(\hat{A}(k)) \subset \{\lambda \in \mathbb{C} : \text{Re}\lambda \leq -c_0\}\). Thus the spectrum of \(\hat{B}(k)\) in the domain \(\text{Re}\lambda > -c_0\) consists of discrete eigenvalues with possible accumulation points only on the line \(\text{Re}\lambda = -c_0\). This proves (1).

Next, we prove (2) as follows. Indeed, let \(u \neq 0\) be the eigenfunction of \(\hat{B}(k)\) corresponding to the eigenvalue \(\lambda\) so that
\[
\hat{B}(k)u = \lambda u,
\]
that is
\[
(L - ik \cdot \hat{v})u = \lambda u.
\]
Taking the inner product with \(u\) and choosing the real part, we have
\[
\text{Re}\lambda\|u\|^2 = (Lu, u).
\]
We know that $\text{Re}\lambda \leq 0$ from $L$ is non-positive operator. Suppose that there is an eigenvalue $\lambda$ such that $\text{Re}\lambda = 0$ holds, then we have $(Lu, u) = 0$ from (2.11), which implies that $u \in N_0$. Thus, the eigenvalue problem (2.10) is transformed to

$$( -ik \cdot \tilde{v}) u = (\text{Im}\lambda) u.$$ (2.12)

It follow that (2.12) holds if and only if $k$ holds, then we have 

(2.13)

and estimate the right hand terms of (2.13) as follows.

**Lemma 2.5.** For any $\delta > 0$, we have

$$\sup_{\text{Re}\lambda \geq -c_0 + \delta, \text{Im}\lambda \in \mathbb{R}} \| K(\lambda - \hat{A}(k))^{-1} \| \leq C \delta^{-1} + \frac{n}{4\pi} (1 + |k|)^{-\frac{n}{2\pi}}. \quad (2.14)$$

For any $\delta$, $\tau_0 > 0$, there is a constant $\tau_0 = 2\tau_0 > 0$ such that if $|\text{Im}\lambda| \geq \tau_0$, then

$$\sup_{\text{Re}\lambda \geq -c_0 + \delta, |k| \leq \tau_0} \| K(\lambda - \hat{A}(k))^{-1} \| \leq C \delta^{-1} - \frac{\delta}{|k|^2} (1 + |\text{Im}\lambda|)^{-\frac{\eta}{|k|^2}}. \quad (2.15)$$

**Proof.** Choosing $R > 1$, we decompose

$$\|K(\lambda - \hat{A}(k))^{-1} f\|^2 \leq 2 \int_{\mathbb{R}^3} \left( \int_{|u| \leq R} K(v, u)(\lambda + \nu(u) + ik \cdot \hat{u})^{-1} f(u) du \right)^2 dv$$

$$+ 2 \int_{\mathbb{R}^3} \left( \int_{|u| \geq R} K(v, u)(\lambda + \nu(u) + ik \cdot \hat{u})^{-1} f(u) du \right)^2 dv$$

$$= I_1 + I_2. \quad (2.16)$$

For $I_1$, we have by Lemma 2.1 (3)

$$I_1 \leq 2 \int_{\mathbb{R}^3} \int_{|u| \leq R} K^2(v, u) f^2(u) du \int_{|u| \leq R} (\lambda + \nu(u) + ik \cdot \hat{u})^{-2} dudv$$

$$\leq C \int_{|u| \leq R} (\lambda + \nu(u) + ik \cdot \hat{u})^{-2} du \| f \|^2. \quad (2.17)$$
Proof.

Theorem 2.6. Following decomposition for $\Re \tau$
there exists $|k| = |\tau_0|$, $|u| \leq R$ and $|\Im \lambda| \geq 2\tau_0$, we have

$$|\Im \lambda + k \cdot \hat{v}| \geq |\Im \lambda| - |k| |\hat{v}| \geq |\Im \lambda| - \frac{\tau_0 R}{\sqrt{1 + R^2}} \geq \frac{|\Im \lambda|}{2}.$$  

(2.20)

Hence

$$I_1 \leq C \int_{|u| \leq R} |\lambda + ik \cdot \hat{u} + \nu(u)|^{-2} du \leq C(\delta^2 + |\Im \lambda|^2)^{-1} R^3.$$  

(2.21)

By taking $R = (|\Im \lambda|/\delta)^{1/2}$, we can verify (2.15).

\(\square\)

Theorem 2.6. (1) For any $\tau_0 > 0$, there exists $d(\tau_0) > 0$ such that when $|k| > \tau_0$,

$$\sigma(\hat{B}(k)) \subseteq \{ \lambda \in \mathbb{C} \mid \Re \lambda < -d(\tau_0) \}.$$  

(2.22)

(2) For any $\delta > 0$ and all $k \in \mathbb{R}^3$, there exists $y_1 = y_1(\delta) > 0$ such that

$$\rho(\hat{B}(k)) \supset \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -c_0 + \delta, |\Im \lambda| \geq y_1 \} \cup \{ \lambda \in \mathbb{C} \mid \Re \lambda > 0 \}.$$  

(2.23)

Proof. We first show that $\sup_{k \in \mathbb{R}^3} |\Im \lambda| < \infty$ for any $\lambda \in \sigma(\hat{B}(k)) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -c_0 + \delta \}$. By Lemma 2.5,

there exists $\tau_1 = \tau_1(\delta) > 0$ large enough so that for $\Re \lambda \geq -c_0 + \delta$ and $|k| \geq \tau_1$,

$$\|K(\lambda - \hat{A}(k))^{-1}\| \leq \frac{1}{2}.$$  

(2.24)

This implies that the operator $I - K(\lambda - \hat{A}(k))^{-1}$ is invertible on $L^2(\mathbb{R}^3)$. Since operator $\lambda - \hat{B}(k)$ has the following decomposition for $\Re \lambda < -c_0$,

$$\lambda - \hat{B}(k) = \lambda - \hat{A}(k) - K = (I - K(\lambda - \hat{A}(k))^{-1})(\lambda - \hat{A}(k)),$$

it follows that $\lambda - \hat{B}(k)$ is also invertible on $L^2(\mathbb{R}^3)$ for $\Re \lambda \geq -c_0 + \delta$ and $|k| \geq \tau_1$, and it satisfies

$$(\lambda - \hat{B}(k))^{-1} = (\lambda - \hat{A}(k))^{-1}(I - K(\lambda - \hat{A}(k))^{-1})^{-1}, \quad |k| \geq \tau_1,$$  

(2.25)

namely,

$$\rho(\hat{B}(k)) \supset \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -c_0 + \delta \}, \quad |k| \geq \tau_1.$$  

(2.26)

For $|k| \leq \tau_1$, by Lemma 2.5, there exists $y_1 = y_1(\tau_1, \delta) > 0$ such that (2.24) holds for $|\Im \lambda| > y_1$. This also implies the invertibility of $\lambda - \hat{B}(k)$ for $|k| \leq \tau_1$ and $|\Im \lambda| > y_1$, namely,

$$\rho(\hat{B}(k)) \supset \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -c_0 + \delta, |\Im \lambda| > y_1 \}, \quad |k| \leq \tau_1.$$  

(2.27)

By (2.25) and (2.28), we obtain (2.26) and hence

$$\sigma(\hat{B}(k)) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -c_0 + \delta \} \subseteq \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -c_0 + \delta, |\Im \lambda| \leq y_1 \}.$$  

(2.28)

Next, we want to show that $\sup_{|k| > \tau_1} \Re \lambda(k) < 0$ for any $\lambda \in \sigma(\hat{B}(k)) \cap \{ \lambda \in \mathbb{C} \mid \Re \lambda \geq -c_0 + \delta \}$. By (2.26), it holds that $\sup_{|k| \geq \tau_1} \Re \lambda(k) < 0$. Hence, it is sufficient to prove that $\sup_{|k| < \tau_1} \Re \lambda(k) < 0$. If it does not hold, there exists a sequence of $\{k_n, \lambda_n, \phi_n\}$ satisfying $|k_n| \in (\tau_0, \tau_1)$, $\phi_n \in D(\hat{B}(k))$ with $\|\phi_n\| = 1$ such that

$$\hat{B}(k_n)\phi_n = (L - i\hat{v} \cdot k_n)\phi_n = \lambda_n\phi_n, \quad \Re \lambda_n \to 0, \quad n \to \infty.$$  

(2.29)
The above equation can be rewritten as

$$K\phi_n = (\lambda_n + \nu + i\hat{v} \cdot \mathbf{k}_n)\phi_n. \quad (2.30)$$

Since $K$ is a compact operator on $L^2(\mathbb{R}^3)$, there exists a subsequence $\phi_{n_j}$ of $\phi_n$ and $g \in L^2(\mathbb{R}^3)$ such that

$$K\phi_{n_j} \to g, \quad \text{as} \quad j \to \infty.$$ 

Since $|\mathbf{k}_n| \in (\tau_0, \tau_1)$, $\sup_{|\mathbf{k}| > \tau_0} |\text{Im}\lambda_n| < \infty$, $\text{Re}\lambda_n \to 0$, there exists a subsequence $(\mathbf{k}_{n_j}, \lambda_{n_j})$ of $(\mathbf{k}_n, \lambda_n)$ such that

$$\mathbf{k}_{n_j} \to \mathbf{k}_0, \quad \lambda_n \to \lambda_0, \quad \text{as} \quad j \to \infty.$$ 

Hence we have

$$\lim_{j \to \infty} \phi_{n_j} = \lim_{j \to \infty} \frac{g}{\lambda_{n_j} + \nu + i\hat{v} \cdot \mathbf{k}_{n_j}} = \frac{g}{\lambda_0 + \nu + i\hat{v} \cdot \mathbf{k}_0} = f_0. \quad (2.31)$$

Then when $n \to \infty$, the eigenvalue problem (2.24) is transformed to

$$\hat{B}(\mathbf{k}_0)f_0 = \lambda_0 f_0. \quad (2.32)$$

Thus $\lambda_0$ is an eigenvalue of $\hat{B}(\mathbf{k}_0)$ with $\text{Re}\lambda_0 = 0$, which contradicts the fact $\text{Re}\lambda(k) < 0$ for $|k| \neq 0$ established by Lemma 2.4.

**Theorem 2.7.** There exist a constant $\tau_0 > 0$ such that the spectrum $\sigma(\hat{B}(k)) \cap \{\lambda \in \mathbb{C} | \text{Re}\lambda \geq -\mu/2\}$ consists of five points $\{\lambda_j(|k|), j = -1, 0, 1, 2, 3\}$ for $|k| \leq \tau_0$. The eigenvalues $\lambda_j(|k|)$ and the corresponding eigenfunctions $e_j = e_j(|k|, \omega)$ with $\omega = k/|k|$ are $C^\infty$ functions of $|k|$ for $|k| \leq \tau_0$. In particular, the eigenvalues $\lambda_j(|k|)$ admit the following asymptotic expansion for $|k| \leq \tau_0$,

$$\begin{align*}
\lambda_{\pm 1}(|k|) &= \pm i \sqrt{a^2 + b^2} |k| + A_{\pm 1} |k|^2 + O(|k|^3), \\
\lambda_0(|k|) &= A_0 |k|^2 + O(|k|^5), \\
\lambda_2(|k|) &= \lambda_3(|k|) = A_2 |k|^2 + O(|k|^3),
\end{align*} \quad (2.33)$$

where $a, b > 0$ and $A_j > 0$, $j = -1, 0, 1, 2$ are constants defined by

$$\begin{align*}
a &= (\hat{v}\psi_1, \psi_4) > 0, & b &= (\hat{v}\psi_0, \psi_1) > 0, \\
A_j &= -(L^{-1} P_1 (\hat{v} \cdot \omega) E_j, (\hat{v} \cdot \omega) E_j) > 0, \\
E_{\pm 1} &= \sqrt{a^2 + b^2} \psi_0 \pm \sqrt{2} \omega \cdot \psi' + \sqrt{\frac{a^2}{2a^2 + 2b^2}} \psi_4, \\
E_0 &= -\sqrt{\frac{a^2}{2a^2 + 2b^2}} \psi_0 + \sqrt{\frac{b^2}{2a^2 + 2b^2}} \psi_4, \\
E_j &= W_j \cdot \psi', \quad j = 2, 3,
\end{align*} \quad (2.34)$$

$\psi' = (\psi_1, \psi_2, \psi_3)$, and $W_j, j = 2, 3$ are orthonormal vectors satisfying $W_j \cdot \omega = 0$.

The eigenfunctions $e_j = e_j(|k|, \omega)$ are orthogonal to each other and satisfy

$$\begin{align*}
(e_i(|k|, \omega), e_j(|k|, \omega)) &= \delta_{ij}, \quad i, j = -1, 0, 1, 2, 3, \\
e_j(|k|, \omega) &= e_{j,0} + e_{j,1} |k| + O(|k|^2), \quad |k| \leq \tau_0,
\end{align*} \quad (2.35)$$

where the coefficients $e_{j,n}$ are given as

$$\begin{align*}
e_{j,0} &= E_j(\omega), \quad j = -1, 0, 1, 2, 3, \\
e_{l,1} &= \sum_{n=-1}^{1} b_n^l E_k + i L^{-1} P_1 (\hat{v} \cdot \omega) E_l, \quad l = -1, 0, 1, \\
e_{n,1} &= i L^{-1} P_1 (\hat{v} \cdot \omega) E_n, \quad n = 2, 3,
\end{align*} \quad (2.36)$$

and $b_n^l$, $j, n = -1, 0, 1$ are defined by

$$\begin{align*}
b_j^0 &= (L^{-1} P_1 (\hat{v} \cdot \omega) E_j, (\hat{v} \cdot \omega) E_0) / i (u_0 - u_j), \quad j \neq n; \\
u_{\pm 1} &= \pm \sqrt{a^2 + b^2}, \quad u_0 = 0.
\end{align*} \quad (2.37)$$
Proof. Since $L$ is invariant with respect to the rotation $\mathbb{O}$ of $v \in \mathbb{R}^3$, it follows that $\mathbb{O} \hat{B}(k) = \hat{B}(\mathbb{O}^{-1}k)$, which applied to $\hat{B}(k)e = \lambda e$ implies that the eigenvalue $\lambda$ depends only on $|k|$. Now we consider eigenvalue problem in the form
\[
\hat{B}(k)e = |k|\beta e,
\]
that is
\[
(L - i|k|(\hat{v} \cdot \omega))e = |k|\beta e.
\]
By macro-micro decomposition, the eigenfunction $e$ of (2.39) can be divided into
\[
e = P_0e + P_1e = g_0 + g_1.
\]
Hence (2.39) gives
\[
|k|\beta g_0 = -P_0[i|k|(\hat{v} \cdot \omega)(g_0 + g_1)],
\]
\[
|k|\beta g_1 = Lg_1 - P_1[i|k|(\hat{v} \cdot \omega)(g_0 + g_1)].
\]
According to (2.41), we have
\[
g_1 = i(L - iP_1(|k|(\hat{v} \cdot \omega)) - |k|\beta)^{-1}P_1(|k|(\hat{v} \cdot \omega))g_0.
\]
By substitute (2.42) into (2.40), we have
\[
|k|\beta g_0 = -i|k|P_0(\hat{v} \cdot \omega)g_0 + |k|P_0(\hat{v} \cdot \omega)(L - iP_1(|k|(\hat{v} \cdot \omega)) - |k|\beta)^{-1}P_1(|k|(\hat{v} \cdot \omega))g_0.
\]
Define the operator $A(\omega) = P_0(\hat{v} \cdot \omega)P_0$. We have the matrix representation of $A(\omega)$ as follows:
\[
\begin{pmatrix}
0 & b\omega & 0 \\
b\omega^T & 0 & a\omega^T \\
0 & a\omega & 0
\end{pmatrix},
\]
where $\omega^T$ denotes the row vector $\omega$'s transpose and
\[
a = (\hat{v}_1\psi_1, \psi_1), \quad b = (\hat{v}_1\psi_0, \psi_1).
\]
It can be verified that the eigenvalues $u_i$ and normalized eigenvectors $E_i$ of $A$ are given by
\[
\begin{align}
\begin{cases}
 u_{\pm 1} = \pm \sqrt{a^2 + b^2}, & u_j = 0, \quad j = 0, 2, 3, \\
 E_{\pm 1} = \sqrt{|b|^2 + |a|^2} \psi_0 \mp \sqrt{2|a|^2 + 2b} \psi_1, \\
 E_0 = -\sqrt{|b|^2 + |a|^2} \psi_0 + \sqrt{2|a|^2 + 2b} \psi_1, \\
 E_j = W_j \cdot \psi', \quad j = 2, 3, \\
 (E_i, E_j) = \delta_{ij}, \quad 1 \leq i, j \leq 3,
\end{cases}
\end{align}
\]
where $\psi' = (\psi_1, \psi_2, \psi_3)$ and $W_j$ are 3-dimensional normalized vectors such that
\[
W_2(\omega) \cdot W_3(\omega) = 0, \quad W_2(\omega) \cdot \omega = W_3(\omega) \cdot \omega = 0.
\]
To solve the eigenvalue problem (2.43), we write $\psi_0 \in N_0$ in terms of the basis $E_j$ as
\[
\psi_0 = \sum_{j=0}^4 C_j E_{j-1} \quad \text{with} \quad C_j = (\psi_0, E_{j-1}), \; j = 0, 1, 2, 3, 4,
\]
with the unknown coefficients $(C_0, C_1, C_2, C_3, C_4)$ to be determined below. Taking the inner product between (2.43) and $E_j$ for $j = -1, 0, 1, 2, 3$ respectively, we have the equations about $\beta$ and $C_j, \; j = 0, 1, 2, 3, 4$ for $\text{Re} \lambda > -\mu$:
\[
\beta C_j = -iu_{j-1} C_j + |k| \sum_{i=0}^4 C_i D_{ij}(\beta, |k|, \omega),
\]
\[
(2.47)
\]
where

\[ D_{ij}(\beta, |k|, \omega) = ((L - i|k|P_1(\hat{v} \cdot \omega) - |k|\beta)^{-1}P_1(\hat{v} \cdot \omega)E_{i-1}, (\hat{v} \cdot \omega)E_{j-1}). \quad (2.48) \]

Let \( \mathcal{O} \) be a rotational transformation in \( \mathbb{R}^3 \) such that

\[ \mathcal{O}^T \omega = (1, 0, 0), \quad \mathcal{O}^T W_2 = (0, 1, 0), \quad \mathcal{O}^T W_3 = (0, 0, 1). \quad (2.49) \]

By changing variable \( v \rightarrow \mathcal{O}v \), we have

\[ D_{ij}(\beta, |k|, \omega) = ((L - i|k|P_1\hat{v}_1 - |k|\beta)^{-1}P_1(\hat{v}_1 F_{i-1}), \hat{v}_1 F_{j-1}) =: R_{ij}(\beta, |k|), \quad (2.50) \]

where

\[
\begin{align*}
F_{\pm 1} &= \sqrt{\frac{b^2}{2a^2 + b^2}} \psi_0 \mp \sqrt{\frac{a^2}{2a^2 + b^2}} \psi_4, \\
F_0 &= -\frac{b}{a} \sqrt{\frac{a^2}{2a^2 + b^2}} \psi_0 + \sqrt{\frac{b^2}{2a^2 + b^2}} \psi_4, \\
F_j &= \psi_j, \quad j = 2, 3, \\
(F_1, F_2) &= \delta_{ij}, \quad -1 \leq i, j \leq 3.
\end{align*}
\]

In particular, \( R_{ij}(\beta, |k|), i, j = 0, 1, 2, 3, 4 \) satisfy

\[
\begin{align*}
R_{ij}(\beta, |k|) &= R_{ji}(\beta, |k|) = 0, \quad i = 0, 1, 2, \quad j = 3, 4, \\
R_{34}(\beta, |k|) &= R_{43}(\beta, |k|) = 0, \\
R_{33}(\beta, |k|) &= R_{44}(\beta, |k|).
\end{align*}
\]

By (2.51) and (2.52), we can divide (2.48) into two systems:

\[ \beta C_j = -iv_{j-1}C_j + |k| \sum_{i=0}^{2} C_i R_{ij}(\beta, |k|), \quad j = 0, 1, 2, \quad (2.53) \]

\[ \beta C_l = -iv_{l-1}C_k + |k|C_l R_{33}(\beta, |k|), \quad l = 3, 4. \quad (2.54) \]

Denote

\[ D_0(\beta, |k|) = \beta - |k|R_{33}(\beta, |k|), \quad (2.55) \]

\[ D_1(\beta, |k|) = \det \begin{vmatrix} \beta + iv_{-1} - |k|R_{00} & -|k|R_{10} & -|k|R_{20} \\ -|k|R_{01} & \beta + iv_{0} - |k|R_{11} & -|k|R_{21} \\ -|k|R_{02} & -|k|R_{12} & \beta + iv_{1} - |k|R_{22} \end{vmatrix}. \quad (2.56) \]

The eigenvalues \( \beta \) can be solved by \( D_0(\beta, |k|) = 0 \) and \( D_1(\beta, |k|) = 0 \). By a direct computation and the implicit function theorem, we can show

**Lemma 2.8.** The equation \( D_0(\beta, s) = 0 \) has a unique \( C^\infty \) solution \( \beta = \beta(s) \) for \( (s, \beta) \in [-\tau_0, \tau_0] \times B_{r_1}(0) \) with \( \tau_0, r_1 > 0 \) being small constants that satisfies

\[ \beta(0) = 0, \quad \beta'(0) = (L^{-1}P_1(\hat{v}_1 F_2), \hat{v}_1 F_2). \]

We have the following result about the solution of \( D_1(\beta, |k|) = 0 \).

**Lemma 2.9.** There exist two small constants \( \tau_0 > 0 \) and \( r_1 > 0 \) so that the equation \( D_1(\beta, s) = 0 \) admits three \( C^\infty \) solutions \( \beta_j(s) \) (\( j = -1, 0, 1 \)) for \( (s, \beta_j) \in [-\tau_0, \tau_0] \times B_{r_1}(-iv_j) \) that satisfy

\[ \beta_j(0) = -iv_j, \quad \beta'_j(0) = (L^{-1}P_1(\hat{v}_1 F_j), \hat{v}_1 F_j). \quad (2.57) \]

Moreover, \( \beta_j(s) \) satisfies

\[ -\beta_j(-s) = \overline{\beta_j(s)} = \beta_{-j}(s), \quad j = -1, 0, 1. \quad (2.58) \]
Proof. From (2.56),

\[
D_1(\beta, 0) = \det \begin{bmatrix}
\beta + iu_1 & 0 & 0 \\
0 & \beta + iu_0 & 0 \\
0 & 0 & \beta + iu_1
\end{bmatrix} = (\beta + iu_1)(\beta + iu_0)(\beta + iu_1).
\]

It follows that \(D_1(\beta, 0) = 0\) has three roots \(\beta_j = -iu_j\) for \(j = -1, 0, 1, 1\). Since

\[
\partial_s D_1(\beta, 0) = -D_{-1}(\beta + iu_0)(\beta + iu_1) - D_0(\beta + iu_0)(\beta + iu_1)
- D_1(\beta + iu_0)\beta,
\]

\[
\partial_\beta D_1(\beta, 0) = (\beta + iu_1)(\beta + iu_0) + (\beta + iu_1)(\beta + iu_1)
+ (\beta + iu_0)(\beta + iu_1),
\]

where

\[
D_j = (L^{-1}P_1(\hat{v}_j F_j), \hat{v}_j F_j) = A_j, \quad j = -1, 0, 1,
\]

it follows that

\[
\partial_\beta D_1(-iu_j, 0) \neq 0.
\]

The implicit function theorem implies that there exist small constants \(\tau_0, r_1 > 0\) and a unique \(C^\infty\) function

\[
\beta_j(s): [-\tau_0, \tau_0] \to B_{r_1}(-iu_j) \text{ so that } D_1(\beta_j(s), s) = 0 \text{ for } s \in [-\tau_0, \tau_0], \text{ and in particular}
\]

\[
\beta_j(0) = -iu_j, \quad \beta_j'(0) = -\frac{\partial D_1(-iu_j, 0)}{\partial_\beta D_1(-iu_j, 0)} = A_j, \quad j = -1, 0, 1.
\]

This proves (2.64).

The eigenvalues \(\lambda_j(|k|)\) and the eigenfunctions \(e_j(|k|, \omega)\), \(j = -1, 0, 1, 2, 3\) can be constructed as follows. For \(j = 2, 3\), we take \(\lambda_j = |k|\beta_j(|k|)\) to be the solution of the equation \(D_0(\beta, |k|) = 0\) defined in Lemma 2.8 and choose \(C_i = 0, i \neq j\). Thus the corresponding eigenfunctions \(e_j(|k|, \omega)\), \(j = 2, 3\) are defined by

\[
e_j(|k|, \omega) = b_j(|k|)E_j(\omega) + i b_j(|k|)|k|L - \lambda_j - i|k|P_1(\hat{v} \cdot \omega|)^{-1}P_1(\hat{v} \cdot \omega)E_j(\omega),
\]

which are orthonormal, i.e., \((e_j(|k|, \omega), e_j(|k|, \omega)) = 0\).

For \(j = -1, 0, 1\), we take \(\lambda_j = |k|\beta_j(|k|)\) to be a solution of \(D_1(\beta, |k|) = 0\) given by Lemma 2.9 and choose \(C_i = 0, i = 2, 3\). Denote by \(\{C_0, C_1, C_2\}\) a solution of system (2.53) for \(\beta = \beta_j(|k|)\). Then we can construct \(e_j(|k|, \omega), j = -1, 0, 1\) as

\[
e_j(|k|, \omega) = P_0 e_j(|k|, \omega) + P_1 e_j(|k|, \omega),
\]

\[
P_0 e_j(|k|, \omega) = C_0(|k|)E_j(\omega) + C_1(|k|)E_0(\omega) + C_2(|k|)E_1(\omega),
\]

\[
P_1 e_j(|k|, \omega) = i|k|L - \lambda_j - i|k|P_1(\hat{v} \cdot \omega|)^{-1}P_1(\hat{v} \cdot \omega)P_0 e_j(|k|, \omega).
\]

We write

\[
(L - i|k|\hat{v} \cdot \omega)e_j(|k|, \omega) = \lambda_j(|k|)e_j(|k|, \omega), \quad -1 \leq j \leq 3.
\]

Taking the inner product \((\cdot, \cdot)\) of the above equation with \(e_j(|k|, \omega)\) and using the facts that

\[
(\hat{B}(k) f, g) = (f, \hat{B}(-k)g), \quad f, g \in D(\hat{B}(k)),
\]

\[
\hat{B}(-k)e_j(|k|, \omega) = \lambda_j(|k|)e_j(|k|, \omega),
\]

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we have
\[(\lambda_j(|k|) - \lambda_l(|k|))(e_j(|k|, \omega), e_l(|k|, \omega)) = 0, \quad -1 \leq j, l \leq 3.\]

For $|k| \neq 0$ being sufficiently small, $\lambda_j(|k|) \neq \lambda_l(|k|)$ for $-1 \leq j \neq l \leq 2$. Therefore, we have
\[(e_j(|k|, \omega), e_l(|k|, \omega)) = 0, \quad -1 \leq j \neq l \leq 3.\]

We can normalize them by taking
\[(e_j(|k|, \omega), e_j(|k|, \omega)) = 1, \quad -1 \leq j \leq 3.\]

The coefficients $b_j(|k|)$ for $j = 2, 3$ defined in (2.64) are determined by the normalization condition as
\[b_j(|k|)^2 (1 - |k|^2 D_j(|k|)) = 1, \quad \text{(2.66)}\]

where
\[D_j(|k|) = ((L - i|k|P_1 \hat{v}_1 - \lambda_j)^{-1} P_1 \hat{v}_1 F_j, (L + i|k|P_1 \hat{v}_1 - \lambda_j)^{-1} P_1 \hat{v}_1 F_j).\]

Substituting (2.33) into (2.64), we obtain
\[b_j(|k|) = 1 + \frac{1}{2} |k|^2 \|L^{-1} P_1 \hat{v}_1 F_j\|^2 + O(|k|^3).\]

This and (2.64) give the expansion of $e_j(|k|, \omega)$ for $j = 2, 3$, stated in (2.36).

To obtain expansion of $e_j(|k|, \omega)$ for $j = -1, 0, 1$ defined in (2.65), we deal with its macroscopic part and microscopic part respectively. By (2.65), the macroscopic part $P_0 e_j(|k|, \omega)$ is determined in terms of the coefficients $\{C_0^j(|k|), C_1^j(|k|), C_2^j(|k|)\}$ that satisfy
\[\beta_j(|k|)C_1^j(|k|) = -i\mu_{l-1} C_1^j(|k|) + |k| \sum_{n=0}^{2} C_n^j(|k|) R_n(\beta_j, |k|), \quad l = 0, 1, 2. \quad \text{(2.67)}\]

Furthermore, we have the normalization condition:
\[1 \equiv (e_j(|k|, \omega), e_j(|k|, \omega)) = C_0^j(|k|)^2 + C_1^j(|k|)^2 + C_2^j(|k|)^2 + O(|k|^2), \quad |k| \leq \tau_0. \quad \text{(2.68)}\]

Assume that
\[C_1^j(|k|) = \sum_{n=0}^{1} C_{1,n}^j |k|^n + O(s^2), \quad l = 0, 1, 2, \quad j = -1, 0, 1.\]

Substituting the above expansion and (2.33) into (2.67) and (2.68), we have their expressions as
\[O(1) \quad \left\{ \begin{array}{l} -i\mu_{l-1} C_{1,0} = -i\mu_{l-1} C_{1,0}^j, \\ (C_{0,0}^j)^2 + (C_{1,0}^j)^2 + (C_{2,0}^j)^2 = 1, \end{array} \right. \quad \text{(2.69)}\]
\[O(|k|) \quad \left\{ \begin{array}{l} -i\mu_{l-1} C_{1,l}^j + A_{1} C_{1,0}^j = -i\mu_{l-1} C_{1,l}^j + \sum_{n=0}^{2} C_{n,0}^j D_{n-1,l-1}, \\ C_{0,0}^j C_{0,1} + C_{1,0}^j C_{1,1}^j + C_{2,0}^j C_{2,1}^j = 0, \end{array} \right. \quad \text{(2.70)}\]

where $j = -1, 0, 1$, $l = 0, 1, 2$, and
\[D_{n,l} = (L^{-1} P_1 (\hat{v}_1 F_n), L^{-1} P_1 (\hat{v}_1 F_l)).\]

By a direct computation, we obtain from (2.69)–(2.70) that
\[\left\{ \begin{array}{l} C_{j+1,0}^j = 1, \\ C_{j,0}^j = 1, \\ C_{j+1,0} = 0, \quad \text{for } j = 0, \quad l \neq j + 1, \end{array} \right. \quad \text{(2.71)}\]

By (2.65) and (2.71), we can obtain the expansion of $e_j(|k|, \omega)$ for $j = -1, 0, 1$ given in (2.36). The proof of this theorem is completed.
Lemma 3.1. Let $\hat{G}$, $j = -1, 0, 1, 2, 3$ be given by (2.31). Thus, the coefficients $A_j > 0$ do not depend on $\omega$.

(2) Since $R_j(\beta, s)$, $j, i = 1, 2, 4$ are analytic in $(s, \beta)$, it follows that $D_j(\beta, s)$ and $D_j(\beta, s)$ are analytic in $(s, \beta)$. By implicit function theorem (section 8 of chapter 0 in [16]), the solution $\beta(s)$ to $D_j(\beta, s) = 0$ is analytic functions of $s$ for $|s| \leq \tau_0$. Thus, the eigenvalues $\lambda_j(|k|)$, $j = -1, 0, 1, 2, 3$ of $\hat{B}(k)$ are analytic function of $|k|$ for $|k| \leq \tau_0$.

Theorem 1.1 directly follows from Theorem 2.6 and Theorem 2.7.

3 Optimal time-decay rates of linearized equation

In this section, we will establish the optimal time-decay rates of global solution for Cauchy problem (1.14).

3.1 Decomposition and asymptotic behaviors of $e^{i\hat{B}(k)}$

In this subsection, we decompose the semigroup $G(t, k) = e^{i\hat{B}(k)}$ and study asymptotic behaviors of this semigroup.

Lemma 3.2. Let $x_0 = -\mu/2$ and $x_1 = -d(\tau_0)$ with $d(\tau_0)$ defined in Theorem 2.6. Then, there exists a constant $C > 0$ such that

$$
\sup_{k \in \mathbb{R}, \gamma \in \mathbb{R}} \| (I - K(-z + iy - \hat{A}(k))^{-1})^{-1} \| \leq C,
$$

where $z = x_0$ for $|k| < \tau_0$ and $z = x_1$ for $|k| \geq \tau_0$.

Proof. Let $\lambda = z + iy$ with $z = x_0$ for $|k| < \tau_0$ and $z = x_1$ for $|k| \geq \tau_0$. By Lemma 2.5 and Theorem 2.7 we have $\lambda \in \rho(\hat{B}(k))$, namely, $\lambda - \hat{B}(k)$ is invertible. Thus

$$
I - K(\lambda - \hat{A}(k))^{-1} = (\lambda - \hat{B}(k))(\lambda - \hat{A}(k))^{-1}
$$
is also invertible. By Lemma 2.5 there exist \( R_0, R_1 > 0 \) large enough such that for either \( |k| \geq R_0 \), or \( |k| \leq R_0 \) and \( |y| \geq R_1 \),
\[
\|K(\lambda - \hat{A}(k))^{-1}\| \leq \frac{1}{2},
\]
which yields
\[
\|\left(I - K(\lambda - \hat{A}(k))^{-1}\right)^{-1}\| \leq 2.
\]

It is sufficient to prove \( \|g\| \leq R \) for \( |y| \leq R_1 \) and \( |k| \leq R_0 \). If it does not hold, then there are sequences \( \{k_n, \lambda_n = z + iy_n\} \) with \( |k_n| \leq R_0 \), \( |y_n| \leq R_1 \), and \( \{f_n, g_n\} \) with \( \|f_n\| \to 0 \), \( \|g_n\| = 1 \) such that
\[
g_n = \left(I - K(\lambda_n - \hat{A}(k_n))^{-1}\right)^{-1}f_n.
\]
This gives
\[
g_n - K(\lambda_n - \hat{A}(k_n))^{-1}g_n = f_n.
\]
Let
\[
w_n = (\lambda_n - \hat{A}(k_n))^{-1}g_n.
\]
We can write (3.5) as
\[
(\lambda_n - \hat{A}(k_n))w_n - Kw_n = f_n.
\]
Since
\[
\|w_n\| \leq \|(\lambda - \hat{A}(k_n))^{-1}\|\|g_n\| \leq C,
\]
and \( K \) is a compact operator on \( L^2(\mathbb{R}^3) \), there exists a subsequence \( w_{n_j} \) of \( w_n \) and \( h_0 \in L^2(\mathbb{R}^3) \) such that
\[
Kw_{n_j} \to h_0, \quad \text{as} \quad j \to \infty.
\]
Since \( |k_n| \leq R_0 \), \( |y_n| \leq R_1 \), there exists a subsequence of (still denoted by) \( \{\lambda_{n_j}, k_{n_j}\} \) and \( (\lambda_0, k_0) \) with \( \lambda_0 = z + iy_0 \), \( |k_0| \leq R_0 \), \( |y_0| \leq R_1 \) such that
\[
\lambda_{n_j} \to \lambda_0, \quad k_{n_j} \to k_0, \quad \text{as} \quad j \to \infty.
\]
Noting that \( \lim_{n \to \infty} \|f_n\| = 0 \), we have by (3.6) and (3.7) that
\[
\lim_{j \to \infty} w_{n_j} = \lim_{j \to \infty} \frac{Kw_{n_j} + f_{n_j}}{\lambda_{n_j} + \nu(v) + i(\hat{v} \cdot k_{n_j})} = \frac{h_0}{\lambda_0 + \nu(v) + i(\hat{v} \cdot k_0)} = w_0,
\]
and hence \( Kw_0 = h_0 \). Thus
\[
Kw_0 = (\lambda_0 + \nu(v) + i(\hat{v} \cdot k_0))w_0.
\]
It follows that \( \lambda_0w_0 = \hat{B}(k_0)w_0 \) and \( \lambda_0 \) is an eigenvalue of \( \hat{B}(k_0) \) with \( \text{Re} \lambda_0 = z \), which contradicts the facts that \( \text{Re}(\lambda)(k) = \lambda_j(|k|), j = -1, 0, 1, 2, 3 \) for \( |k| \leq \tau_0 \) and \( \text{Re} \lambda(k) < -d(\tau_0) \) for \( |k| \geq \tau_0 \).

Then, we have the decomposition of the semigroup \( G(t, k) = e^{t\hat{B}(k)} \) as below.

**Theorem 3.3.** The semigroup \( G(t, k) = e^{t\hat{B}(k)} \) with \( k = |k|\omega \) satisfies
\[
G(t, k)f = G_1(t, k)f + G_2(t, k)f, \quad f \in L^2(\mathbb{R}^3),
\]
where
\[
G_1(t, k) = \sum_{j=-1}^{3} e^{\lambda_j(|k|)t} \left( e_{j}(|k|, \omega) e_{j}(|k|, \omega) \right)_{1_{\{|k| \leq \tau_0\}}},
\]
with \( (\lambda_j(|k|), e_{j}(|k|, \omega)) \) being the eigenvalue and eigenfunction of the operator \( \hat{B}(k) \) given by Theorem 2.7 for \( |\xi| \leq \tau_0 \), and \( G_2(t, k)f = G(t, k)f - G_1(t, k)f \) satisfy for a constant \( \sigma_0 > 0 \) independent of \( k \),
\[
\|G_2(t, k)f\|_{L^2} \leq C e^{-\sigma_0 t} \|f\|_{L^2}.
\]
Proof. The proof method is similar to [20]. We can establish the asymptotic behavior of \( G \)

\[
G(t, k) = e^{t \hat{B}(k)} f = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{\lambda t} (\lambda - \hat{B}(k))^{-1} f d\lambda, \quad x > 0.
\]

(3.11)

We can write the second resolvent equation for \( \hat{A}(k) \) and \( \hat{B}(k) \)

\[
(\lambda - \hat{B}(k))^{-1} = (\lambda - \hat{A}(k))^{-1} + (\lambda - \hat{B}(k))^{-1} K(\lambda - \hat{A}(k))^{-1}.
\]

(3.12)

Combining this and (3.11), we have

\[
(\lambda - \hat{B}(k))^{-1} = (\lambda - \hat{A}(k))^{-1} + Z(\lambda),
\]

where

\[
Z(\lambda) = (\lambda - \hat{A}(k))^{-1} (I - K(\lambda - \hat{A}(k))^{-1})^{-1} K(\lambda - \hat{A}(k))^{-1}.
\]

(3.13)

(3.14)

Substitute this into (3.11), we have

\[
e^{t \hat{B}(k)} = e^{t \hat{A}(k)} + \frac{1}{2\pi i} \lim_{T \to \infty} U_{x,T},
\]

(3.15)

and

\[
U_{x,T} = \int_{-T}^{T} e^{(x+iy)t} Z(x + iy) dy,
\]

(3.16)

where the positive constant \( T > y_1 \) and \( y_1 \) is defined by Theorem 2.6. Set

\[
\sigma_0 = \frac{\mu}{2}, \quad |k| < \tau_0; \quad \sigma_0 = d(\tau_0), \quad |k| \geq \tau_0.
\]

Since \( Z(\lambda) \) is analytic in \( \text{Re}\lambda > -\sigma_0 \) with only finite singularities at the eigenvalues \( \lambda_j(|k|), j = -1, 0, 1, 2, 3, \) we can shift the integration path from the \( \text{Re}\lambda = x > 0 \) to \( \text{Re}\lambda = -\sigma_0 \) where \( \sigma_0 \) is given by Theorem 2.7. Then by the Residue Theorem, we have

\[
U_{x,T} = H_T + 2\pi i \sum_{j=1}^{3} \text{Res} \{ e^{\lambda t} Z(\lambda) f; \lambda_j(|k|) \} + U_{-\sigma_0,T},
\]

(3.17)

where \( \text{Res} \{ f(\lambda); \lambda_j \} \) means the residue of \( f(\lambda) \) at \( \lambda = \lambda_j \) and

\[
H_T = \frac{1}{2\pi i} \left( \int_{-\sigma_0-iT}^{x-iT} - \int_{-\sigma_0+iT}^{x+iT} \right) e^{\lambda t} Z(\lambda) d\lambda.
\]

We can estimate the right side of (3.17) as follows. First, from Lemma 2.5 we verify that

\[
\| H_T \| \to 0, \quad T \to \infty.
\]

(3.18)

Next, we have

\[
\text{Res} \{ e^{\lambda t} Z(\lambda) f; \lambda_j(|k|) \} = \text{Res} \{ e^{\lambda t} (\lambda - \hat{B}(k))^{-1}; \lambda_j(|k|) \} = e^{\lambda_j(|k|)t} P_j(k) f 1_{\{ |k| \leq \tau_0 \}},
\]

(3.19)

where

\[
P_j(k) f = (f, \epsilon_j(|k|, \omega)) \epsilon_j(|k|, \omega).
\]

Define

\[
U_{-\sigma_0,T}(t) = \lim_{T \to \infty} U_{-\sigma_0,T}(t) = \int_{-\infty}^{+\infty} e^{(-\sigma_0+iy)t} Z(-\sigma_0 + iy) dy.
\]

By using Lemma 3.1 and Lemma 3.2 we have for any \( f, g \in L^2(\mathbb{R}_+^3) \),

\[
|\langle U_{-\sigma_0,T}(t)f, g \rangle|
\]

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\[ \leq C e^{-\sigma_0 t} \int_{-\infty}^{+\infty} \|(Z(-\sigma_0 + iy)f, g)\| dy \]
\[ \leq C \|K\| e^{-\sigma_0 t} \int_{-\infty}^{+\infty} \|(\sigma_0 + iy - \hat{A}(k))^{-1}f\| \|(\sigma_0 - iy - \hat{A}(k))^{-1}g\| dy \]
\[ \leq C \|K\| e^{-\sigma_0 t}(c_0 - \sigma_0)^{-1}\|f\|\|g\|. \quad (3.20) \]

This implies that
\[ \|U_{-\sigma_0, \infty}(t)\| \leq C e^{-\sigma_0 t}, \quad t \geq 0. \quad (3.21) \]

Hence it follows from (3.13) and (3.15) that
\[ G(t, k)f = G_1(t, k)f + G_2(t, k)f, \quad (3.22) \]

where
\[ G_1(t, k)f = \sum_{j=-1}^{3} \nu_j e^{\nu_j t} \mathcal{P}_j(k) f \mathbf{1}_{\{|k| \leq \sigma_0\}}, \quad (3.23) \]
\[ G_2(t, k)f = e^{t A(k)} f + U_{-\sigma_0, \infty}(t). \quad (3.24) \]

Moreover,
\[ \|G_2(t, k)f\| \leq \|e^{t A(k)} f\| + \|U_{-\sigma_0, \infty}(t)\| \leq C e^{-\sigma_0 t}\|f\|. \quad (3.25) \]

This proves the theorem.

### 3.2 Optimal time-decay rates of $e^{t \hat{B}(k)}$

For any $f_0 \in L^2(\mathbb{R}_x^3 \times \mathbb{R}_v^3)$, set
\[ e^{t \hat{B}} f_0 = (\mathcal{F}^{-1} e^{t \hat{B}(k)} \mathcal{F}) f_0. \]

By Lemma 2.3, we have
\[ \|e^{t \hat{B}} f_0\|_{L^2}^2 = \int_{\mathbb{R}^3} (1 + |k|^2)^N \|e^{t \hat{B}(k)} f_0\|^2 dk \leq \int_{\mathbb{R}^3} (1 + |k|^2)^N \|f_0\|^2 dk = \|f_0\|^2_{L^2}. \quad (3.26) \]

This implies that the linear operator $B$ generates a strongly continuous contraction semigroup $e^{t B}$ in $H^N$. Therefore $f(t, x, v) = e^{t B} f_0(x, v)$ is a global solution to (1.14) for any $f_0 \in H^N$.

We have the time decay rates of the linearized relativistic Boltzmann equation (1.14) as follows.

**Theorem 3.4.** Assume that $f_0 \in H^N \cap Z_1$ for $N \geq 0$. Then the global solution $f(t, x, v) = e^{t B} f_0(x, v)$ to the linearized relativistic Boltzmann equation (1.14) satisfies for any $\alpha, \alpha' \in \mathbb{N}^3$ with $\alpha' \leq \alpha$ that
\[ \|\partial_{x}^\alpha e^{t \hat{B}} f_0, \psi_j\|_{L^2} \leq C(1 + t)^{-(\frac{\alpha}{2} + \frac{\varsigma}{2})} \|\partial_{x}^\alpha f_0\|_{L^2} + \|\partial_{x}^\alpha f_0\|_{L^2}, \quad j = 0, 1, 2, 3, 4, \quad (3.27) \]
\[ \|P_1(\partial_{x}^\alpha e^{t \hat{B}} f_0)\|_{L^2} \leq C(1 + t)^{-(\frac{\alpha}{2} + \frac{\varsigma}{2})} \|\partial_{x}^\alpha f_0\|_{L^2} + \|\partial_{x}^\alpha f_0\|_{L^2}, \quad (3.28) \]

where $\varsigma = |\alpha - \alpha'| \leq N$. If it further holds that $P_1 f_0 = 0$, then
\[ \|\partial_{x}^\alpha e^{t \hat{B}} f_0, \psi_j\|_{L^2} \leq C(1 + t)^{-(\frac{\alpha}{2} + \frac{\varsigma}{2})} \|\partial_{x}^\alpha f_0\|_{L^2} + \|\partial_{x}^\alpha f_0\|_{L^2}, \quad j = 0, 1, 2, 3, 4, \quad (3.29) \]
\[ \|P_1(\partial_{x}^\alpha e^{t \hat{B}} f_0)\|_{L^2} \leq C(1 + t)^{-(\frac{\alpha}{2} + \frac{\varsigma}{2})} \|\partial_{x}^\alpha f_0\|_{L^2} + \|\partial_{x}^\alpha f_0\|_{L^2}. \quad (3.30) \]

**Proof.** By theorem 3.3 and the Plancherel’s equation,
\[ \|\partial_{x}^\alpha e^{t \hat{B}} f_0, \psi_j\|_{L^2} = \|k^\alpha (G(t, k) \hat{f}_0, \psi_j)\|_{L^2} \leq \|k^\alpha (G_1(t, k) \hat{f}_0, \psi_j)\|_{L^2} + \|k^\alpha (G_2(t, k) \hat{f}_0, \psi_j)\|_{L^2}. \quad (3.31) \]
We can estimate the second terms on the right hand of (3.31) as follows:

\[
\|k^\alpha (G_2(t,k)f_0,\psi_j)\|_{L^2_x}^2 \leq \int_{\mathbb{R}^3} (k^\alpha)^2 \|G_2(t,k)f_0\|_{L^2_x}^2 dk
\]

\[
\leq C \int_{\mathbb{R}^3} e^{-2\sigma_0 (k^\alpha)^2} \|f_0\|_{L^2_v}^2 dk \leq C e^{-2\sigma_0 t} \|\partial_\xi^2 f_0\|_{L^2_v}. \tag{3.32}
\]

Next, we establish estimation the first term in the right hand side of (3.31). By (3.8), we have for \(|k| \leq \tau_0,
\]

\[
G_1(t,k)f_0 = \sum_{j=-1}^{3} e^{\lambda_j(|k|)} (\hat{f}_0,\hat{g}_0) + |k|T_j(k)f_0, \tag{3.33}
\]

where \(T_j(k), -1 \leq j \leq 3\) is the linear operator with the norm \(\|T_j(k)\|\) being uniformly bounded for \(|k| \leq \tau_0.\)

Therefore, we have

\[
(G_1(t,k)f_0,\psi_0) = \frac{1}{2} B_1 \sum_{j=\pm 1} e^{\lambda_j(|k|)} (B_1 \hat{n}_0 - j(\hat{m}_0 \cdot \omega) + B_2 \hat{q}_0)
\]

\[
- B_2 e^{\lambda_0(|k|)} (B_1 \hat{n}_0 + B_1 \hat{q}_0) + |k| \sum_{j=-1}^{3} e^{\lambda_j(|k|)} (T_j(k)f_0,\psi_0), \tag{3.34}
\]

\[
(G_1(t,k)f_0,\psi_0') = - \frac{1}{2} \sum_{j=\pm 1} e^{\lambda_j(|k|)} j(B_1 \hat{n}_0 - j(\hat{m}_0 \cdot \omega) + B_2 \hat{q}_0)\omega
\]

\[
+ \sum_{j=2,3} e^{\lambda_j(|k|)} (\hat{m}_0 \cdot W_j) W_j + |k| \sum_{j=-1}^{3} e^{\lambda_j(|k|)} (T_j(k)f_0,\psi'), \tag{3.35}
\]

\[
(G_1(t,k)f_0,\psi_4) = \frac{1}{2} B_2 \sum_{j=\pm 1} e^{\lambda_j(|k|)} (B_1 \hat{n}_0 - j(\hat{m}_0 \cdot \omega) + B_1 \hat{q}_0)
\]

\[
+ B_1 e^{\lambda_0(|k|)} (B_1 \hat{n}_0 - B_2 \hat{q}_0) + |k| \sum_{j=-1}^{3} e^{\lambda_j(|k|)} (T_j(k)f_0,\psi_4), \tag{3.36}
\]

where \(B_1 = \sqrt{\frac{\kappa^2}{\gamma + \omega}}, B_2 = \sqrt{\frac{\kappa^2}{\gamma + \omega}}.\)

\((\hat{n}_0,\hat{m}_0,\hat{q}_0) = (\hat{f}_0,\psi_0), (\hat{f}_0,\psi_0'), (\hat{f}_0,\psi_4))\) is the Fourier transform of the macroscopic density, momentum and energy of the initial data \(f_0, W_j\) is given by theorem (2.7) and

\[
P_t(G_1(t,k)f_0) = \sum_{j=-1}^{3} e^{\lambda_j(|k|)} P_t(T_j(k)f_0). \tag{3.37}
\]

Since

\[
\text{Re}\lambda_j(|k|) = A_j |k|^2 (1 + O(|k|)) \leq -\rho |k|^2, \quad |k| \leq \tau_0, \tag{3.38}
\]

we have

\[
\|k^\alpha (G_1(t,k)f_0,\psi_j)\|_{L^2}^2 \leq C \int_{|k| \leq \tau_0} (k^{\alpha-\alpha'})^2 e^{-2\rho|k|^2} (|k^\alpha(\hat{n}_0,\hat{m}_0,\hat{q}_0)|^2 + |k^2||k^\alpha f_0||_{L^2_x}^2) dk
\]

\[
\leq C (1 + t)^{-\frac{\alpha\alpha'}{1+\omega}} \|\partial_\xi^\alpha (\hat{n}_0,\hat{m}_0,\hat{q}_0)||_{L^2_x}^2 + |\partial_\xi^\alpha f_0||_{L^2_x}^2 \tag{3.39}
\]

where \(j = -1,0,1,2,3.\)

Combining (3.31), (3.32) and (3.39), we verify that (3.27) holds.

For \(P_t(\partial_\xi^\alpha e^{IB} f_0),\) we have estimation that

\[
\|P_t(\partial_\xi^\alpha e^{IB} f_0)\|_{L^2_v}^2 \leq \|k^\alpha P_t(G_1(t,k)f_0)\|_{L^2_v}^2 + \|k^\alpha P_t(G_2(t,k)f_0)\|_{L^2_v}^2. \tag{3.40}
\]

By (3.31) and (3.38), we have

\[
\|k^\alpha P_t(G_1(t,k)f_0)\|_{L^2_v}^2 \leq C \int_{|k| \leq \tau_0} (k^{\alpha-\alpha'})^2 e^{-2\rho|k|^2} |k^\alpha f_0||_{L^2_x}^2 dk
\]

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\[ \leq C(1 + t)^{-\frac{1}{2} + \varepsilon} \| \partial f_0 \|_{L^1}, \]  

(3.41)

Combining (3.31), (3.32), (3.40) and (3.41), we verify that (3.28) holds.

For the case \( P_0 f_0 = 0 \), we have

\[ G_4(t, k) \hat{f}_0 = |k| \sum_{j=-1}^{3} e^{\lambda_j(t,k) t} (f_0, P_j \widetilde{e}_{j,0}) \varepsilon_{j,0} + |k|^2 T_4(t, k) \hat{f}_0, \quad |k| \leq \tau_0, \]  

(3.42)

where \( T_4(t, k) \) is also a linear operator satisfying for \( |k| \leq \tau_0 \) that

\[ \| T_4(t, k) \hat{f} \|_{L^2_x} \leq Ce^{-\theta |k|^2 t} \| \hat{f} \|_{L^2_x}. \]  

(3.43)

Then we have after a direct computation that

\[ (G_4(t, k) \hat{f}_0, \psi_0) = i \frac{1}{2} B_1 |k| \sum_{j=\pm 1} e^{\lambda_j(t,k) t} (B_1 \eta - j(\theta \cdot \omega) + B_2 \gamma) \]  

\[ + i B_2 |k| e^{\lambda_0(t,k) t} (-B_2 \eta + B_1 \gamma) + |k|^2 (T_4(t, k) \hat{f}_0, \psi_0), \]  

(3.44)

\[ (G_4(t, k) \hat{f}_0, \psi') = -i \frac{1}{2} |k| \sum_{j=\pm 1} e^{\lambda_j(t,k) t} (B_1 \eta - j(\theta \cdot \omega) + B_2 \gamma) \]  

\[ + i |k| \sum_{j=2,3} e^{\lambda_j(t,k) t} (\theta \cdot W_j) W_j \]  

\[ + i B_1 |k| e^{\lambda_0(t,k) t} (-B_2 \eta + B_1 \gamma) + |k|^2 (T_4(t, k) \hat{f}_0, \psi'), \]  

(3.45)

\[ (G_4(t, k) \hat{f}_0, \psi_4) = i \frac{1}{2} B_2 |k| \sum_{j=\pm 1} e^{\lambda_j(t,k) t} (B_1 \eta - j(\theta \cdot \omega) + B_2 \gamma) \]  

\[ + i B_1 |k| e^{\lambda_0(t,k) t} (-B_2 \eta + B_1 \gamma) + |k|^2 (T_4(t, k) \hat{f}_0, \psi_4), \]  

(3.46)

where \( \eta, \theta \) and \( \gamma \) are defined by

\[ \eta = (\hat{f}_0, L^{-1} P_1 (\hat{v} \cdot \omega) \psi_0), \quad \theta = (\hat{f}_0, L^{-1} P_1 (\hat{v} \cdot \omega) \psi'), \quad \gamma = (\hat{f}_0, L^{-1} P_1 (\hat{v} \cdot \omega) \psi_4), \]  

(3.47)

and

\[ P_1 (G_4(t, k) \hat{f}_0) = |k|^2 P_1 (T_4(t, k) \hat{f}_0). \]  

(3.48)

Similarly, repeating the above steps, we can establish time-decay estimation (3.29), (3.30). \( \square \)

We can also prove that the above time-decay rates are indeed optimal in following sense.

**Theorem 3.5.** Assume that \( f_0 \in H^N \cap Z_1 \) for \( N \geq 0 \) and there exist positive constants \( d_0, d_1 \) such that \( \hat{f}_0 \) satisfies that \( \inf_{k \leq \tau_0} |(\hat{f}_0, \psi_0)| \geq d_0, \sup_{k \leq \tau_0} |(\hat{f}_0, b \psi_0 - a \psi_0)| = 0 \) and \( \sup_{k \leq \tau_0} |(\hat{f}_0, \psi')| = 0 \) with \( \psi' = (\psi_1, \psi_2, \psi_3) \). Then the global solution \( f(t, x, v) = e^{i B} f_0(x, v) \) to the linear relativistic Boltzmann equation (1.14) satisfies

\[ C_1 (1 + t)^{-\frac{1}{2} + \varepsilon} \leq \| (e^{i B} f_0, \psi_j) \|_{L^2} \leq C_2 (1 + t)^{-\frac{1}{2}}, \quad j = 0, 4, \]  

(3.49)

\[ C_1 (1 + t)^{-\frac{1}{2} + \varepsilon} \leq \| (e^{i B} f_0, \psi') \|_{L^2} \leq C_2 (1 + t)^{-\frac{1}{2}}, \]  

(3.50)

\[ C_1 (1 + t)^{-\frac{1}{2} + \varepsilon} \leq \| P_1 (e^{i B} f_0) \|_{L^2}, \leq C_2 (1 + t)^{-\frac{1}{2}}, \]  

(3.51)

for \( t > 0 \) large and two positive constants \( C_2 \geq C_1 \).

If it holds that \( P_0 f_0 = 0, \inf_{k \leq \tau_0} |(\hat{f}_0, L^{-1} P_1 (\hat{v} \cdot \omega) \psi_j)| \geq d_0 \) and \( \sup_{k \leq \tau_0} |(\hat{f}_0, L^{-1} P_1 (\hat{v} \cdot \omega) \psi_j)| = 0 \) for \( j = 0, 4 \) and \( \omega = k/|k| \), then

\[ C_1 (1 + t)^{-\frac{1}{2} + \varepsilon} \leq \| (e^{i B} f_0, \psi_j) \|_{L^2} \leq C_2 (1 + t)^{-\frac{1}{2}}, \quad j = 0, 4, \]  

(3.52)

\[ C_1 (1 + t)^{-\frac{1}{2} + \varepsilon} \leq \| (e^{i B} f_0, \psi') \|_{L^2} \leq C_2 (1 + t)^{-\frac{1}{2}}, \]  

(3.53)

\[ C_1 (1 + t)^{-\frac{1}{2} + \varepsilon} \leq \| P_1 (e^{i B} f_0) \|_{L^2}, \leq C_2 (1 + t)^{-\frac{1}{2}}, \]  

(3.54)

for \( t > 0 \) large.
Proof. By (3.32), we have
\[
\|e^{iB(t,k)\frac{t}{2}}f_0, \psi\|_{L^2_k} \geq \|(G_1(t,k) f_0, \psi)\|_{L^2_k} - \|G_2(t,k) f_0\|_{L^2_k}.
\]
By (3.31) and
\[
b\hat{q}_0 = a\hat{n}_0, \quad \hat{m}_0 = 0, \quad \lambda_1(|k|) = \overline{\lambda_1(|k|)}, \quad \text{for } |k| \leq \tau_0,
\]
we obtain
\[
|(G_1(t,k) f_0, \psi)|^2 = \frac{b}{a^2 + b^2} (\hat{n}_0 + a\hat{q}_0) e^{Re\lambda_1(|k|)t} \cos(\Im \lambda_1(|k|)t)
+ |k| \sum_{j=-1}^3 \left| e^{i\lambda_1(|k|)} (T_j(k) f_0, \psi) \right|^2
\geq \frac{1}{2} |\hat{n}_0|^2 e^{2Re\lambda_1(|k|)t} \cos^2(\Im \lambda_1(|k|)t) - Ce^{-2\rho|k|^2} |k|^2 \|f_0\|_{L^2_k}^2.
\]
Since
\[
\text{Re}\lambda_j(|k|) = \lambda_j \rho |k|^2 (1 + O(|k|)) \geq -\xi |k|^2, \quad |k| \leq \tau_0,
\]
and
\[
\cos^2(\Im \lambda_1(|k|)t) \geq \frac{1}{2} \cos^2 [\sqrt{a^2 + b^2} |k| t] - O(|k|^3 t^2),
\]
it follows that
\[
\|e^{iB(t,k)\frac{t}{2}}f_0, \psi\|_{L^2_k}^2 \geq \frac{1}{2} b^2 \int_{|k| \leq \tau_0} \frac{1}{2} e^{-2\xi |k|^2} \cos^2(\sqrt{a^2 + b^2} |k| t) dk
- C \int_{|k| \leq \tau_0} e^{-2\rho|k|^2} |k|^2 (|k|^2 \hat{n}_0|^2 + |k|^2 \|f_0\|_{L^2_k}^2) dk
\geq C \int_{|k| \leq \tau_0} e^{-2\xi |k|^2} \cos^2(\sqrt{a^2 + b^2} |k| t) dk - C(1 + t)^{-\frac{1}{2}}.
\]
Set
\[
I_1 = \int_{|k| \leq \tau_0} e^{-2\xi |k|^2} \cos^2(\sqrt{a^2 + b^2} |k| t) dk.
\]
we can estimate for $t \geq t_0 = \frac{L^2}{\xi}$ with the constant $L \geq \sqrt{\frac{a^2}{\xi}}$ that
\[
I_1 = t^{-\frac{1}{2}} \int_{|s| \leq \tau_0 \sqrt{t}} e^{-2\xi s^2} \cos^2(\sqrt{a^2 + b^2} |s| \sqrt{t}) ds
\geq 4\pi (1 + t)^{-\frac{1}{2}} \int_0^1 e^{-2\xi r^2} \cdot r^2 \cos^2(\sqrt{a^2 + b^2} r \sqrt{t}) \, dr
\geq \pi (1 + t)^{-\frac{1}{2}} L^2 e^{-2\xi L^2} \int_0^L \cos^2(\sqrt{a^2 + b^2} r \sqrt{t}) \, dr
\geq \pi (1 + t)^{-\frac{1}{2}} L^2 e^{-2\xi L^2} \int_0^\infty \cos^2 y dy
\geq C(1 + t)^{-\frac{1}{2}}.
\]
By substitute (3.60) and (3.61) into (3.55), we verify that (3.49) holds for $j = 0$.
By (3.55) and (3.56), we can calculate
\[
|(G_1(t,k) f_0, \psi')|^2 = \left| i(B_1 \hat{n}_0 + B_2 \hat{q}_0) \omega \cdot e^{Re\lambda_1(|k|)t} \sin(\Im \lambda_1(|k|)t) \right|^2
+ \left| k \sum_{j=-1}^3 e^{i\lambda_j(|k|)} (T_j(k) f_0, \psi') \right|^2
\]
By (2.36) and (3.37), we have
\[
\geq \frac{1}{2B_t^2}|\hat{n}_0|^2 e^{2\text{Re}\lambda_1(|k|)t} \sin^2(\text{Im}\lambda_1(|k|)t) - C|k|^2 e^{-2\rho|k|^2t} \|f_0\|_{L_x^2}^2. \tag{3.62}
\]

In terms of the fact that
\[
\sin^2(\text{Im}\lambda_1(|k|)t) \geq \frac{1}{2} \sin^2(\sqrt{a^2 + b^2}|k|t) - O([|k|^3t]^2), \tag{3.63}
\]
we have
\[
\|\hat{G}_1(t,k)\hat{f}_0, \psi')\|_{L_x^2}^2 \geq \frac{1}{2B_t^2} d_0^2 \int_{|k| \leq t_0} e^{-2\xi |k|^2 t} \sin^2(\sqrt{a^2 + b^2}|k|t) dk
- C \int_{|k| \leq t_0} e^{-2\rho|k|^2 t}(|\hat{n}_0|^2 + |k|^2 \|\hat{f}_0\|_{L_x^2}^2) dk
\geq \frac{1}{2B_t^2} d_0^2 \int_{|k| \leq t_0} e^{-2\xi |k|^2 t} \sin^2(\sqrt{a^2 + b^2}|k|t) dk - C(1 + t)^{-\frac{5}{2}}. \tag{3.64}
\]

Set
\[
I_2 = \int_{|k| \leq t_0} e^{-2\xi |k|^2 t} \sin^2(\sqrt{a^2 + b^2}|k|t) dk.
\]

It holds for \( t \geq t_0 = \frac{t^2}{t_0} \) with the constant \( L > 4\pi \) that
\[
I_2 = t^{-\frac{5}{2}} \int_{|s| \leq t_0} e^{-2\xi |s|^2} \sin^2(\sqrt{a^2 + b^2}|s|\sqrt{t}) ds
\geq 4\pi(1 + t)^{-\frac{3}{2}} \int_0^1 e^{-2\xi r^2} \cdot r^2 \sin^2(\sqrt{a^2 + b^2 r \sqrt{t}}) dr
\geq \pi(1 + t)^{-\frac{3}{2}} L e^{-2\xi L^2} \int_0^L \sin^2(\sqrt{a^2 + b^2 r \sqrt{t}}) dr
\geq \pi(1 + t)^{-\frac{3}{2}} L e^{-2\xi L^2} \int_0^\infty \sin^2 y dy
\geq C(1 + t)^{-\frac{5}{2}}. \tag{3.65}
\]

Substitute (3.64) and (3.65) into (3.59), we prove (3.60).

By (3.60) and (3.59), we can calculate
\[
\|\hat{G}_1(t, k, \psi_4)\|_{L_x^2}^2 = \left| \frac{a}{a^2 + b^2} (b \hat{n}_0 + a \hat{q}_0) e^{\text{Re}\lambda_1(|k|)t} \cos(\text{Im}\lambda_1(|k|)t) \right|
+ |k| \sum_{j=-1}^1 e^{\lambda_j(|k|)}(T_j(\hat{f}_0, \psi_4))^2. \tag{3.66}
\]

Then, we can get the same conclusion as the estimate term (3.60) that
\[
\|\hat{G}_1(t,k, \psi_4)\|_{L_x^2}^2 \geq C_1(1 + t)^{-\frac{5}{2}} - C_2(1 + t)^{-\frac{5}{2}}. \tag{3.67}
\]

and the proof process is the same as (3.49) for \( j = 0 \). Thus, by substitute (3.67) into (3.54), we verify (3.49) holds for \( j = 4 \). In conclusion, (3.39) holds for \( t > 0 \) large.

Next, we prove (3.51). By (3.32), we have
\[
\|P_1(e^{tB}f_0)\|_{L_x^2} \geq \|P_1(G_1(t,k)\hat{f}_0)\|_{L_x^2} - \|P_1(G_2(t,k)\hat{f}_0)\|_{L_x^2}
\geq \|P_1(G_1(t,k)\hat{f}_0)\|_{L_x^2} - C e^{-\sigma t} \|f_0\|_{L_x^2}. \tag{3.68}
\]

By (3.35) and (3.37), we have
\[
P_1(G_1(t,k)\hat{f}_0)
\]
By direct calculation, we have

\[ P(\hat{f}_0, e_{j,0}) P_l(e_{j,1}) + |k|^2 T_5(t,k) \hat{f}_0 \]

\[ = -i \sqrt{2} |k| \sum_{j=\pm 1} e^{\lambda_j(|k|) t} (B_1 \hat{\eta}_0 + B_2 \hat{\rho}_0) L^{-1} P_l (\hat{v} \cdot \omega) e_{j,0} + |k|^2 T_5(t,k) \hat{f}_0 \]

\[ = -i |k| B_1^{-1} \hat{\eta}_0 e^{\Re \lambda_1(|k|) t} \cos(\Im \lambda_1(|k|) t) L^{-1} P_l (\hat{v} \cdot \omega) (B_1 \hat{\psi}_0 + B_2 \hat{\psi}_4) \]

\[ - i B_1^{-1} \hat{\eta}_0 e^{\Re \lambda_1(|k|) t} \sin(\Im \lambda_1(|k|) t) L^{-1} P_l (\hat{v} \cdot \omega) (\omega \cdot \psi') + |k|^2 T_5(t,k) \hat{f}_0, \]

(3.69)

where \( T_5(t,k) \) satisfies

\[ ||T_5(t,k) \hat{f}_0||^2 \leq C e^{-2|\rho| \tau^2} ||\hat{f}_0||^2_{L^2}. \]

(3.70)

Therefore,

\[ ||P_l(G_1(t,k) \hat{f}_0)||^2 \leq \frac{1}{2 B_1^2} ||L^{-1} \hat{v} \hat{\psi}_1 \hat{\psi} ||^2_{L^2_{\tau}} \int_{|k| \leq \tau_0} |k|^2 e^{-2|\rho| \tau^2} \sin^2(\sqrt{\alpha^2 + \beta^2}) \text{d}k \]

\[ - C \int_{|k| \leq \tau_0} e^{-2|\rho| \tau^2} (|k|^4 t^2 \hat{\eta}_0^2) \text{d}k - C \int_{|k| \leq \tau_0} |k|^4 e^{-2|\rho| \tau^2} ||\hat{f}_0||^2_{L^2_{\tau}} \text{d}k \]

\[ \geq C(1 + t)^{-\frac{3}{2}} - C(1 + t)^{-\frac{5}{2}}. \]

(3.72)

Further, by direct calculation, we have

\[ ||P_l(G_1(t,k) \hat{f}_0)||^2 \]

\[ \geq \frac{1}{2 B_1^2} ||L^{-1} \hat{v} \hat{\psi}_1 \hat{\psi} ||^2_{L^2_{\tau}} \int_{|k| \leq \tau_0} |k|^2 e^{-2|\rho| \tau^2} \sin^2(\sqrt{\alpha^2 + \beta^2}) \text{d}k \]

\[ - C \int_{|k| \leq \tau_0} e^{-2|\rho| \tau^2} (|k|^4 t^2 \hat{\eta}_0^2) \text{d}k - C \int_{|k| \leq \tau_0} |k|^4 e^{-2|\rho| \tau^2} ||\hat{f}_0||^2_{L^2_{\tau}} \text{d}k \]

By substitute (3.72) into (3.68), we verify (3.51) holds for \( t > 0 \) large.

For the case \( P_l \hat{f}_0 = 0 \), when \( |k| \leq \tau_0 \) and \( f_0, L^{-1} P_l (\hat{v} \cdot \omega) \psi_j = 0 \) for \( j = 0, 4 \), we have

\[ (G_1(t,k) \hat{f}_0, \psi_0) = \frac{1}{2} B_1 i |k| \sum_{j=\pm 1} e^{\lambda_j(|k|) t} j h + |k|^2 (T_4(t,k) \hat{f}_0, \psi_0), \]

(3.73)

\[ (G_1(t,k) \hat{f}_0, \psi') = \frac{1}{2} i |k| \sum_{j=\pm 1} e^{\lambda_j(|k|) t} j h \omega + i |k| \sum_{l=2,3} e^{\lambda_l(|k|) t} j W_j + |k|^2 (T_4(t,k) \hat{f}_0, \psi'), \]

(3.74)

\[ (G_1(t,k) \hat{f}_0, \psi_4) = \frac{1}{2} B_2 i |k| \sum_{j=\pm 1} e^{\lambda_j(|k|) t} j h + |k|^2 (T_4(t,k) \hat{f}_0, \psi_4), \]

(3.75)

\[ P_l(G_1(t,k) \hat{f}_0) = \frac{1}{2} |k|^2 \sum_{j=\pm 1} e^{\lambda_j(|k|) t} h (L^{-1} \hat{v} \hat{\omega}) (\omega \cdot \psi') + L^{-1} P_l (\hat{v} \cdot \omega) (B_1 \hat{\psi}_0 + B_2 \hat{\psi}_4)) \]

\[ + |k|^2 \sum_{l=2,3} e^{\lambda_l(|k|) t} j L^{-1} P_l (\hat{v} \cdot \omega) (W_j \cdot \psi') + |k|^3 T_6(t,k) \hat{f}_0, \]

(3.76)

where \( h = (f_0, L^{-1} P_l (\hat{v} \cdot \omega) (\omega \cdot \psi')) \), \( I_j = (f_0, L^{-1} P_l (\hat{v} \cdot \omega) (W_j \cdot \psi')) \), \( W_j \) is given by theorem 2.4 and \( T_6(t,k) \) satisfies

\[ ||T_6(t,k) \hat{f}_0||^2_{L^2_{\tau}} \leq C e^{-2|\rho| \tau^2} ||\hat{f}_0||^2_{L^2_{\tau}}. \]

(3.77)

By direct calculation, we have

\[ ||(G_1(t,k) \hat{f}_0, \psi_0)||^2 \geq \frac{1}{2} B_2^2 |k|^2 h^2 e^{2 \Re \lambda_1(|k|) t} \sin^2(\Im \lambda_1(|k|) t) - C |k|^4 e^{-2|\rho| \tau^2} ||\hat{f}_0||^2_{L^2_{\tau}}, \]

(3.78)

\[ ||(G_1(t,k) \hat{f}_0, \psi')||^2 \geq \frac{1}{2} |k|^2 h^2 e^{2 \Re \lambda_1(|k|) t} \sin^2(\Im \lambda_1(|k|) t) - C |k|^4 e^{-2|\rho| \tau^2} ||\hat{f}_0||^2_{L^2_{\tau}}, \]

(3.79)
Thus, we can estimate the nonlinear term \( \Gamma(f,f) \) as
\[
\|G_1(t,k)\hat{f}_0, \psi_0\|^2 \geq \frac{1}{2} C_\lambda^2 \|k|^2 \|h^2 e^{2Re \lambda (|k|) t} \sin^2(\Im \lambda (|k|) t) - C|k|^4 e^{-2\rho|k|^2t}\|\hat{f}_0\|^2_{L_k^2},
\]
(3.80)
\[
\|P_1(G_1(t,k)\hat{f}_0)\|^2_{L_k^2} \geq \frac{1}{2} C|k|^4 e^{-2\rho|k|^2t}\|\hat{f}_0\|^2_{L_k^2}.
\]
(3.81)

Then, similarly, repeating the above steps, we can verify (3.52)–(3.54). This proves the theorem.

\[\square\]

4 The original nonlinear problem

In this section, we prove the long time decay rates of the solution to the Cauchy problem for relativistic Boltzmann equation with help of the asymptotic behaviors of linearized problem established in Section 3.

**Proof of Theorem 1.2** Let \( f \) be a solution to RB equation (1.5) for \( t > 0 \). We can represent this solution in terms of the semigroup \( e^{tB} \) as
\[
f(t) = e^{tB} f_0 + \int_0^t e^{(t-s)B} \Gamma(f,f) ds.
\]
(4.1)

For this global solution \( f \), we define two functionals \( Q_1(t) \) and \( Q_2(t) \) for any \( t > 0 \) as
\[
Q_1(t) = \sup_{0 \leq s \leq t} \sum_{|\alpha| = 0}^1 \{(1 + s)^{\frac{5}{2} + \frac{|\alpha|}{2}} \sum_{j=0}^4 \|\partial_x^\alpha (f(s), \psi_j)\|_{L_k^2} + (1 + s)^{\frac{5}{2} + \frac{|\alpha|}{2}} \|\partial_x^\alpha P_1 f(s)\|_{L_k^2} + \}
\]
(4.2)
\[
+ (1 + s)^{\frac{5}{2}} \|P_1 f(s)\|_{H_{N+1}} + \|\nabla_x P_0 f(s)\|_{H_{N-1}}\},
\]
and
\[
Q_2(t) = \sup_{0 \leq s \leq t} \sum_{|\alpha| = 0}^1 \{(1 + s)^{\frac{5}{2} + \frac{|\alpha|}{2}} \sum_{j=0}^4 \|\partial_x^\alpha (f(s), \psi_j)\|_{L_k^2} + (1 + s)^{\frac{5}{2} + \frac{|\alpha|}{2}} \|\partial_x^\alpha P_1 f(s)\|_{L_k^2} + \}
\]
(4.3)
\[
+ (1 + s)^{\frac{5}{2}} \|P_1 f(s)\|_{H_{N+1}} + \|\nabla_x P_0 f(s)\|_{H_{N-1}}\}.
\]

We claim that it holds under the assumptions of Theorem 1.2 that
\[
Q_1(t) \leq C \delta_0,
\]
(4.2)
and if it is further satisfied \( P_0 f_0 = 0 \) that
\[
Q_2(t) \leq C \delta_0.
\]
(4.3)

It is easy to verify that the estimates \( (1.19) \)–(1.21) and \( (1.22) \)–(1.24) from \( (4.2) \) and \( (4.3) \) respectively.

First of all, we prove the claim \( (1.2) \) as follows. From [28], it holds that for any \( \alpha \in [0,1] \),
\[
\|v^{-\alpha} \Gamma(f,g)\|_{L_k^2} \leq C (\|v^{1-\alpha} f\|_{L_k^2} \|g\|_{L_k^2} + \|f\|_{L_k^2} \|v^{1-\alpha} g\|_{L_k^2}).
\]
(4.4)
Thus, we can estimate the nonlinear term \( \Gamma(f,f) \) for \( 0 \leq s \leq t \) in the terms of \( Q_1(t) \) as
\[
\|\Gamma(f,g)\|_{L_k^2} \leq C \|v f\|_{L_k^2} \|g\|_{L_k^2} \leq C (1 + s)^{-2} Q_1(t)^2,
\]
\[
\|\Gamma(f,f)\|_{L_k^2} \leq C \|v f\|_{L_k^2} \|g\|_{L_k^2} \leq C (1 + s)^{-2} Q_1(t)^2,
\]
and
\[
\|\nabla_x \Gamma(f,f)\|_{L_k^2} \leq C \|v \nabla_x f\|_{L_k^2} \|g\|_{L_k^2} + \|v f\|_{L_k^2} \|\nabla_x g\|_{L_k^2} \leq C (1 + s)^{-2} Q_1(t)^2,
\]
\[
\|\nabla_x \Gamma(f,f)\|_{L_k^2} \leq C \|v \nabla_x f\|_{L_k^2} \|g\|_{L_k^2} + \|v f\|_{L_k^2} \|\nabla_x g\|_{L_k^2} \leq C (1 + s)^{-2} Q_1(t)^2.
\]
Since the nonlinear term $\Gamma(f, f)$ satisfies $P_0\Gamma(f, f) = 0$, we obtain the long decay rate of the macroscopic part by (3.27) and (3.29) as follow:

$$
\|f(t, \psi_j)\|_{L^2_x} \leq C(1 + t)^{-\frac{3}{2}}(\|f_0\|_{L^2_{x,v}} + \|f_0\|_{Z_1})
+ C \int_0^t (1 + t - s)^{-\frac{3}{2}}(\|\Gamma(f, f)\|_{L^2_{x,v}} + \|\Gamma(f, f)\|_{Z_1})ds
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C \int_0^t (1 + t - s)^{-\frac{3}{2}}(1 + s)^{-\frac{3}{2}}Q_1(t)^2ds
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C(1 + t)^{-\frac{3}{2}}Q_1(t)^2,
$$

and

$$
\|\nabla_x(f(t), \psi_j)\|_{L^2_x} \leq C(1 + t)^{-\frac{3}{2}}(\|\nabla_x f_0\|_{L^2_{x,v}} + \|f_0\|_{Z_1})
+ C \int_0^t (1 + t - s)^{-\frac{3}{2}}(\|\nabla_x \Gamma(f, f)\|_{L^2_{x,v}} + \|\Gamma(f, f)\|_{Z_1})ds
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C \int_0^t (1 + t - s)^{-\frac{3}{2}}(1 + s)^{-\frac{3}{2}}Q_1(t)^2ds
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C(1 + t)^{-\frac{3}{2}}Q_1(t)^2.
$$

Then, we estimate the time-decay rate of microscopic part $P_1 f(t)$ as follows. We obtain from (4.5) and (4.30) that

$$
\|P_1 f(t)\|_{L^2_{x,v}} \leq C(1 + t)^{-\frac{3}{2}}(\|f_0\|_{L^2_{x,v}} + \|f_0\|_{Z_1})
+ C \int_0^t (1 + t - s)^{-\frac{3}{2}}(\|\Gamma(f, f)\|_{L^2_{x,v}} + \|\Gamma(f, f)\|_{Z_1})ds
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C \int_0^t (1 + t - s)^{-\frac{3}{2}}(1 + s)^{-\frac{3}{2}}Q_1(t)^2ds
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C(1 + t)^{-\frac{3}{2}}Q_1(t)^2,
$$

and

$$
\|\nabla_x P_1 f(t)\|_{L^2_{x,v}} \leq C(1 + t)^{-\frac{3}{2}}(\|\nabla_x f_0\|_{L^2_{x,v}} + \|f_0\|_{Z_1})
+ C \int_0^t (1 + t - s)^{-\frac{3}{2}}(\|\nabla_x \Gamma(f, f)\|_{L^2_{x,v}} + \|\nabla_x \Gamma(f, f)\|_{Z_1})ds
+ C \int_0^t (1 + t - s)^{-\frac{3}{2}}(\|\nabla_x \Gamma(f, f)\|_{L^2_{x,v}} + \|\nabla_x \Gamma(f, f)\|_{Z_1})ds
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C \int_0^t (1 + t - s)^{-\frac{3}{2}}(1 + s)^{-\frac{3}{2}}Q_1(t)^2ds
\leq C\delta_0(1 + t)^{-\frac{3}{2}} + C(1 + t)^{-\frac{3}{2}}Q_1(t)^2.
$$

With the help of the priori estimates (4.25)–(4.28), we can verify the claim (4.2). Indeed, by direct computation (cf. [28]), there are two functionals $H_1$ and $D_1$ related to the global solution $f$:

$$
H_1(f) \sim \sum_{|\alpha| \leq N} \|\nu^{|\alpha|} P_1 f\|_{L^2_{x,v}}^2 + \sum_{|\alpha| \leq N-1} \|\partial^\alpha \nabla_x P_0 f\|_{L^2_{x,v}}^2,
$$

$$
D_1(f) \sim \sum_{|\alpha| \leq N} \|\nu^{|\alpha|+1} \partial^\alpha f\|_{L^2_{x,v}}^2 + \sum_{|\alpha| \leq N-1} \|\partial^\alpha \nabla_x P_0 f\|_{L^2_{x,v}}^2.
$$

(4.9)
such that
\[ H_1(f) \leq CD_1(f), \] (4.10)
and
\[ \frac{d}{dt} H_1(f(t)) + \mu D_1(f(t)) \leq C \| \nabla_x P_0 f(t) \|_{L^2_{x,v}}^2. \] (4.11)
This together with (4.6) leads to
\[ H_1(f(t)) \leq e^{-C\mu t} H_1(f_0) + \int_0^t e^{-C\mu (t-s)} \| \nabla_x P_0 f(s) \|_{L^2_{x,v}}^2 \, ds \]
\[ \leq C\delta_0 e^{-C\mu t} + \int_0^t e^{-C\mu (t-s)} (1+s)^{-\frac{7}{5}} (\delta_0 + Q_1(t)^2)^2 \, ds \]
\[ \leq C(1+t)^{-\frac{7}{5}} (\delta_0 + Q_1(t)^2)^2. \] (4.12)
Making summing to (4.5)–(4.8) and (4.12), we obtain
\[ Q_1(t) \leq C\delta_0 + CQ_1(t)^2, \]
from which the claim (4.2) can be verified provided that \( \delta_0 > 0 \) is small enough.
Next, we turn to prove the claim (4.3) for the case \( P_0 f_0 = 0 \) as follows. Indeed, if it holds \( P_0 f_0 = 0 \), the macroscopic density, momentum, energy and their spatial derivatives can be estimated as follows:
\[ \| (f(t), \psi_j) \|_{L^2_{x,v}} \leq C(1+t)^{-\frac{7}{5}} (\| f_0 \|_{L^2_{x,v}} + \| f_0 \|_{Z_1}) \]
\[ + C \int_0^t (1+t-s)^{-\frac{7}{5}} (\| \Gamma(f,f) \|_{L^2_{x,v}} + \| \Gamma(f,f) \|_{Z_1}) \, ds \]
\[ \leq C\delta_0 (1+t)^{-\frac{7}{5}} + C \int_0^t (1+t-s)^{-\frac{7}{5}} (1+s)^{-\frac{7}{5}} Q_2(t)^2 \, ds \]
\[ \leq C\delta_0 (1+t)^{-\frac{7}{5}} + C(1+t)^{-\frac{7}{5}} Q_2(t)^2, \] (4.13)
and
\[ \| \nabla_x (f(t), \psi_j) \|_{L^2_{x,v}} \leq C(1+t)^{-\frac{7}{5}} (\| f_0 \|_{L^2_{x,v}} + \| f_0 \|_{Z_1}) \]
\[ + C \int_0^t (1+t-s)^{-\frac{7}{5}} (\| \nabla_x \Gamma(f,f) \|_{L^2_{x,v}} + \| \Gamma(f,f) \|_{Z_1}) \, ds \]
\[ \leq C\delta_0 (1+t)^{-\frac{7}{5}} + C \int_0^t (1+t-s)^{-\frac{7}{5}} (1+s)^{-\frac{7}{5}} Q_2(t)^2 \, ds \]
\[ \leq C\delta_0 (1+t)^{-\frac{7}{5}} + C(1+t)^{-\frac{7}{5}} Q_2(t)^2, \] (4.14)
where we have used
\[ \| \Gamma(f,f) \|_{L^2_{x,v}} + \| \Gamma(f,f) \|_{Z_1} \leq C(1+s)^{-\frac{7}{5}} Q_2(t)^2, \]
\[ \| \nabla_x \Gamma(f,f) \|_{L^2_{x,v}} + \| \Gamma(f,f) \|_{Z_1} \leq C(1+s)^{-\frac{7}{5}} Q_2(t)^2. \]
In terms of (3.30) the microscopic part and its spatial derivative can be estimated as
\[ \| P_1 f(t) \|_{L^2_{x,v}} \leq C(1+t)^{-\frac{7}{5}} (\| f_0 \|_{L^2_{x,v}} + \| f_0 \|_{Z_1}) \]
\[ + C \int_0^t (1+t-s)^{-\frac{7}{5}} (\| \Gamma(f,f) \|_{L^2_{x,v}} + \| \Gamma(f,f) \|_{Z_1}) \, ds \]
\[ \leq C\delta_0 (1+t)^{-\frac{7}{5}} + C \int_0^t (1+t-s)^{-\frac{7}{5}} (1+s)^{-\frac{7}{5}} Q_2(t)^2 \, ds \]
\[ \leq C\delta_0 (1+t)^{-\frac{7}{5}} + C(1+t)^{-\frac{7}{5}} Q_2(t)^2, \] (4.15)
and
\[ \| \nabla_x P_t f(t) \|_{L^2_{x,v}} \leq C(1 + t)^{-\frac{5}{2}}(\| \nabla_x f(0) \|_{L^2_{x,v}} + \| f_0 \|_{Z_1}) \]
\[ + C \int_0^t (1 + t - s)^{-\frac{5}{2}}(\| \nabla_x \Gamma(f,f) \|_{L^2_{x,v}} + \| \Gamma(f,f) \|_{Z_1}) \, ds \]
\[ \leq C \delta_0 (1 + t)^{-\frac{5}{2}} + C \int_0^t (1 + t - s)^{-\frac{5}{2}}(1 + s)^{-\frac{5}{2}} Q_2(t)^2 \, ds \]
\[ \leq C \delta_0 (1 + t)^{-\frac{5}{2}} + C(1 + t)^{-\frac{5}{2}} Q_2(t)^2. \] (4.16)

Therefore, with the help of (4.14), we can obtain by (4.14) that
\[ H_1(f(t)) \leq e^{-C\mu t} H_1(f_0) + \int_0^t e^{-C\mu(t-s)} \| \nabla_x P_0 f(s) \|_{L^2_{x,v}}^2 \, ds \]
\[ \leq C \delta_0^2 e^{-C\mu t} + \int_0^t e^{-C\mu(t-s)}(1 + s)^{-\frac{5}{2}}(\delta_0 + Q_2(t))^2 \, ds \]
\[ \leq C(1 + t)^{-\frac{5}{2}}(\delta_0 + Q_2(t))^2. \] (4.17)

This together with (4.19) yields
\[ Q_2(t) \leq C \delta_0 + C Q_2(t)^2, \]
which implies the claims (4.23) provided that \( P_0 f_0 = 0 \) and \( \delta_0 > 0 \) is small enough.

**Proof of Theorem 1.5** By (4.11), Theorem 5.4 and Theorem 1.2, we can establish the lower bounds of the time decay rates of the macroscopic part and the microscopic part of the global solution \( f \) when \( t > 0 \) large enough that
\[ \| (f(t), \psi_j) \|_{L^2_x} \geq \| (e^{tB} f_0, \psi_j) \|_{L^2_x} - \int_0^t \| (e^{(t-s)B} \Gamma(f,f), \psi_j) \|_{L^2_x} ds \]
\[ \geq C_1 \delta_0 (1 + t)^{-\frac{5}{2}} - C_2 \delta_0^2 (1 + t)^{-\frac{5}{2}}, \] (4.18)
and
\[ \| P_1 f(t) \|_{L^2_{x,v}} \geq \| P_1(e^{tB} f_0) \|_{L^2_{x,v}} - \int_0^t \| P_1(e^{(t-s)B} \Gamma(f,f)) \|_{L^2_{x,v}} ds \]
\[ \geq C_1 \delta_0 (1 + t)^{-\frac{5}{2}} - C_2 \delta_0^2 (1 + t)^{-\frac{5}{2}}, \] (4.19)
from which and Theorem 1.2, we can obtain
\[ \| f(t) \|_{H_{N,1}} \geq \| P_0 f(t) \|_{L^2_x} - \| \nu P_1 f(t) \|_{L^2_{x,v}} - \sum_{1 \leq |\alpha| \leq N} \| \nu \partial_\alpha^\nu f(t) \|_{L^2_{x,v}} \]
\[ \geq 5C_1 \delta_0 (1 + t)^{-\frac{5}{2}} - 5C_2 \delta_0^2 (1 + t)^{-\frac{5}{2}} - C_3 \delta_0 (1 + t)^{-\frac{5}{2}}. \] (4.20)
This gives rise to (1.25)–(1.28) for sufficiently \( t > 0 \) and \( \delta_0 > 0 \) small enough. By repeating similar arguments, we can prove (1.29)–(1.32), the detail are omitted.

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