QUANTUM LANGLANDS DUALITY AND
CONFORMAL FIELD THEORY

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Abstract. V. Drinfeld proposed conjectures on geometric Langlands correspondence and its quantum deformation. We refine these conjectures and propose their relationship with algebraic conformal field theory.

Introduction

Geometric Langlands correspondence was proposed by V. G. Drinfeld [1] as a geometric analog of the Langlands conjecture relating Galois representations of a global field with representations of adelic algebraic group. In the geometric Langlands correspondence the global field is replaced by the field of functions on a complete nonsingular algebraic curve $C$ defined over the field of complex numbers $\mathbb{C}$; Galois representations are replaced by local systems on the curve $C$; finally, representations of adelic algebraic group are replaced by $D$-modules (or constructible sheaves) on the moduli space of bundles on $C$. Thus, the conjectural geometric Langlands correspondence is a correspondence between (certain) $G^\vee$-local systems on $C$ (here $G^\vee$ is a semisimple algebraic group over $\mathbb{C}$) and between (certain) $D$-modules on the moduli space $\text{Bun}_G$ of principal $G$-bundles on $C$ (here $G$ is the group Langlands dual to $G^\vee$).

The goal of this paper is to state a conjecture on deformation of the geometric Langlands correspondence, and to relate it with constructions of algebraic conformal field theory [2]. This conjecture, originally proposed also by V. G. Drinfeld, looks more simple and symmetric than the conjecture on the geometric Langlands correspondence itself. Its relation with conformal field theory found by the author is a refinement and an argument in favour of these conjectures, because it unifies many notions into a self-consistent picture.

For a more detailed exposition of the material of this paper, see [3].

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1. THE MAIN CONJECTURES

1.1. Let $G$ be a simple algebraic group over $\mathbb{C}$, let $C$ be a complete smooth algebraic curve of genus $g$, and let $\text{Bun}_G$ be the moduli space of principal $G$-bundles on the curve $C$. It is known [3] that for any affine algebraic group $A$ the moduli space $\text{Bun}_A$ of principal $A$-bundles on $C$ is a smooth algebraic stack [4]. Moreover, this stack can be covered by open substacks of the form $X_n/G_n$, where $X_n$ is a smooth algebraic variety, and $G_n$ is an affine algebraic group acting on $X_n$. Hence various objects (functions, $\mathcal{D}$-modules, sheaves, etc.) on the stack $\text{Bun}_A$ can be obtained by glueing corresponding $G_n$-equivariant objects on the varieties $X_n$.

Denote by $G^\vee$ the group Langlands dual to $G$. For a principal $G^\vee$-bundle $P^\vee$ on the curve $C$, the space of (algebraic) connections on $P^\vee$ is an affine space, whose associated vector space is the cotangent space $T^*_C \text{Bun}_{G^\vee} \cong \Gamma(C, \Omega^1_C \otimes \text{ad} P^\vee)$ to the stack $\text{Bun}_{G^\vee}$ at the point $P^\vee$. Hence we call the moduli space of $G^\vee$-local systems on $C$, i.e. principal $G^\vee$-bundles with a connection, by the twisted cotangent bundle to the stack $\text{Bun}_{G^\vee}$, and denote it by $\tilde{T}^* \text{Bun}_{G^\vee}$. It is known [3] that the cocycle of the affine bundle $\tilde{T}^* \text{Bun}_{G^\vee}$ is obtained from the cocycle of the canonical line bundle $\omega_{\text{Bun}_{G^\vee}}$ on the stack $\text{Bun}_{G^\vee}$ via the homomorphism $d \log : H^1(\text{Bun}_{G^\vee}, \mathcal{O}_{\text{Bun}_{G^\vee}}^*) \to H^1(\text{Bun}_{G^\vee}, \Omega^1_{\text{Bun}_{G^\vee}})$.

1.2. Conjecture 1. [6] The derived category of $\mathcal{D}_{\text{Bun}_{G^\vee}}$-modules is equivalent to the derived category of quasicoherent $\mathcal{O}_{\tilde{T}^* \text{Bun}_{G^\vee}}$-modules.

This conjecture (as well as definition of the derived categories [5]) is due to Beilinson and Drinfeld. They refine it in the following way. The required equivalence of derived categories should be given by the “kernel” $\mathcal{L}_0$, an object of derived category of $\mathcal{D}_{\text{Bun}_{G^\vee}} \boxtimes \mathcal{O}_{\tilde{T}^* \text{Bun}_{G^\vee}}$-modules. The restriction of this object to the open substack $\tilde{T}^* \text{Bun}_{G^\vee}^\circ$ of irreducible $G^\vee$-local systems should be a $\mathcal{D}_{\text{Bun}_{G^\vee}} \boxtimes \mathcal{O}_{\tilde{T}^* \text{Bun}_{G^\vee}^\circ}$-module, flat as an $\mathcal{O}_{\tilde{T}^* \text{Bun}_{G^\vee}^\circ}$-module, whose fiber over the local system $P^\vee = (P^\vee, \nabla) \in \tilde{T}^* \text{Bun}_{G^\vee}^\circ$ should be the unique, up to isomorphism, holonomic $\mathcal{D}_{\text{Bun}_{G^\vee}}$-module $\mathcal{F}_{P^\vee}$, which is a Hecke eigen-$\mathcal{D}_{\text{Bun}_{G^\vee}}$-module with eigenvalue $P^\vee$. (For
a definition of this notion, see [5].) The correspondence $\mathcal{P}^\vee \to \mathcal{F}_{\mathcal{P}^\vee}$ is called the geometric Langlands correspondence.

1.3. Let us now proceed to the conjecture on deformation of the geometric Langlands correspondence. Denote

$$\xi = \omega_{\text{Bun}_G} \otimes \mathbb{Z} \otimes \mathbb{Q}$$

(Pic is the Picard group), where $h^\vee$ is the dual Coxeter number of the group $G$. In the case when $G$ is simply connected, it is known [5] that $\xi$ is the positive generator of the group $\text{Pic} \text{Bun}_G \simeq \mathbb{Z}$. Similarly, introduce the element

$$\xi^\vee = \omega_{\text{Bun}_G^\vee} \otimes \mathbb{Z} \otimes \mathbb{Q}.$$  

**Conjecture 2.** [6] The derived category of twisted $\mathcal{D}_{\text{Bun}_G}(\xi^\otimes \kappa)$-modules is equivalent to the derived category of twisted $\mathcal{D}_{\text{Bun}_G^\vee}(\xi^v \otimes \kappa^v)$-modules for any $\kappa \in \mathbb{C}$, $\kappa \neq 0$, where $\kappa^v = 1/(r\kappa)$, and $r$ is the maximal multiplicity of an edge in the Dynkin diagram of the group $G$ (or $G^\vee$); $r = 1, 2, \text{ or } 3$.

The idea of this conjecture is due to V. G. Drinfeld. This conjecture also has several refinements. We are going to state them in the rest of this paper.

The required equivalence of derived categories should be given by a kernel $\mathcal{L}_\kappa$ which is an object of derived category of $\mathcal{D}_{\text{Bun}_G}(\xi^\otimes \kappa) \boxtimes \mathcal{D}_{\text{Bun}_G^\vee}(\xi^v \otimes \kappa^v)$-modules. All refinements of Conjecture 2 stated below deal with the properties of this kernel.

1.4. **Property 1. Dependence on the parameter $\kappa$; classical limits.**

Let us first define the notion of asymptotic twisted $\mathcal{D}$-module, or $\mathcal{D}_X^\text{asy}(\xi^\otimes t)$-module, on a smooth variety $X$ with a line bundle $\xi$. This notion is introduced in [3]; let us briefly discuss it.

There exists a natural sheaf $\mathcal{D}_X^\text{asy}(\xi^\otimes t)$ of quasicoherent $\mathcal{O}_{\mathbb{P}^1}$-algebras on the product $\mathbb{P}^1 \times X$, flat as an $\mathcal{O}_{\mathbb{P}^1}$-module, whose fiber at the point $\kappa \in \mathbb{P}^1$, $\kappa \neq \infty$, is isomorphic to the sheaf $\mathcal{D}_X(\xi^\otimes \kappa)$ of twisted differential operators, and the fiber over the point $\infty \in \mathbb{P}^1$ is isomorphic to the sheaf $\pi_*\mathcal{O}_{\tilde{T}^*X}$, where $\pi : \tilde{T}^*X \to X$ is the twisted cotangent affine bundle over $X$, whose cocycle corresponds to the cocycle of the bundle $\xi$ under the homomorphism

$$d \log : H^1(X, \mathcal{O}_X^*) \to H^1(X, \Omega^1_X).$$

Sections of the sheaf $\mathcal{D}_X^\text{asy}(\xi^\otimes)$ are called twisted asymptotic differential operators, cf. [7]. On a sufficiently small open subset $U \subset X$, on
which the bundle $\xi$ is trivial, we have
\[ \Gamma((\mathbb{P}^1 \setminus \{0\}) \times U, \mathcal{D}_{X}^{\text{asym}}(\xi^{\otimes t})) \simeq \mathcal{O}_U[t^{-1}, t^{-1}\partial_1, \ldots, t^{-1}\partial_n], \]
where $\partial_1, \ldots, \partial_n$ is a basis of vector fields on $U$, $t$ is the parameter on the line $\mathbb{P}^1$.

By definition, a $\mathcal{D}_{X}^{\text{asym}}(\xi^{\otimes t})$-module is a sheaf of $\mathcal{D}_{X}^{\text{asym}}(\xi^{\otimes t})$-modules on the product $\mathbb{P}^1 \times X$ quasicoherent as an $\mathcal{O}_{\mathbb{P}^1 \times X}$-module.

Let us return to the kernel $L_\kappa$. The property of dependence of $L_\kappa$ on the parameter $\kappa$ states that the objects $L_0, L_\kappa$ are the fibers of an object $L_t$ of the derived category of $\mathcal{D}_{\text{Bun}}^G(\xi^{\otimes t}) \boxtimes \mathcal{D}_{\text{Bun}}^{G^\vee}(\xi^{\otimes t^\vee})$-modules on the product $\mathbb{P}^1 \times \text{Bun}_G \times \text{Bun}_{G^\vee}$, flat as an $\mathcal{O}_{\mathbb{P}^1}$-module. Here $t^\vee = 1/(rt)$. The fiber of this object at $t = \kappa$ is the object $L_\kappa$, the fiber at $t = 0$ is the object $L_0$ which is the kernel of the geometric Langlands correspondence defined in 1.2 above, and the fiber at $t = \infty$ is the object $L_\infty$ which is the kernel of the geometric Langlands correspondence for the group $G^\vee$, i.e., the object $L_\infty$ is obtained from $L_0$ by exchanging the roles of the groups $G$ and $G^\vee$.

1.5. **Property 2: singular support of the kernel $L_\kappa$.** One has the Hitchin map $[5]$
\[ \chi_G : T^*\text{Bun}_G \to \bigoplus_{i=1}^{\text{rk} G} \Gamma(C, \omega_C^{\otimes d_i}), \]
where $d_i$ are the exponents of the group $G$, $\text{rk} G$ is the rank of $G$. After restriction to certain open dense subset $U \subset \text{Bun}_G \times \text{Bun}_{G^\vee}$, the kernel $L_\kappa$ should be a coherent $\mathcal{D}_{\text{Bun}}^G(\xi^{\otimes \kappa}) \boxtimes \mathcal{D}_{\text{Bun}}^{G^\vee}(\xi^{\otimes \kappa^\vee})$-module whose singular support coincides with the preimage of the diagonal
\[ \Delta_\kappa = \{ v_i, v_i^\vee \in \Gamma(C, \omega_C^{\otimes d_i}) : v_i^\vee = \kappa^{d_i} v_i \}_{i=1}^{\text{rk} G} \]
under the product of Hitchin maps
\[ \chi_G \times \chi_{G^\vee} : T^*\text{Bun}_G \times T^*\text{Bun}_{G^\vee} \to \bigoplus_{i=1}^{\text{rk} G} \Gamma(C, \omega_C^{\otimes d_i}) \] $^{\otimes 2}$. The classical limit of this property as $\kappa \to 0$ yields the conjecture that the singular support of $\mathcal{D}_{\text{Bun}}^G$-modules $\mathcal{F}_{P^\vee}$ from the geometric Langlands correspondence is contained in the global nilpotent cone, which is the preimage of zero under the Hitchin map.

2. **Relation with conformal field theory**

2.1. For simplicity assume in this Section that the group $G$ is adjoint, and the group $G^\vee$ is simply connected. For the general case, see [3].
Let us fix a Borel subgroup $B \subset G$ with the unipotent radical $N$; let $H = B/N$ be the Cartan group. Consider the diagram

\[
\xymatrix{
\Bun_B \ar[dr]_{\rho} \ar[rr]^\beta & & \Bun_{\omega,H} \ar[dl]_{\alpha} \\
\Bun_B^>0 \ar[ur]_{\sigma} \ar[rr]_{\beta} & & \Bun_H^>0 \ar[ur]_{\alpha}
}
\]

The only thing to be explained in this diagram is what are the spaces $\Bun_{\omega,H}$, $\Conf_{G^\lor}$, and what does the sign $>0$ mean. To explain this, note that

\[
B/[N,N] \simeq \prod_{i=1}^{\text{rk} G} B_i,
\]

where $B_i$ is a copy of the upper triangular Borel subgroup in $\text{PGL}(2)$. Hence $\Bun_{B/[N,N]}$ is identified with the moduli space of sets of exact triples

\[
0 \rightarrow \mathcal{O}_C \rightarrow E_i \rightarrow L_i \rightarrow 0,
\]

where $L_i$ is a line bundle on the curve $C$, $1 \leq i \leq \text{rk} G$. The projection $\Bun_{B/[N,N]} \rightarrow \Bun_H$ takes a set of triples (**) to the set of line bundles $(L_i) \in \Bun_H$. By definition,

\[
\Bun_{H}^>0 = \{(L_i), \deg L_i > 0, 1 \leq i \leq \text{rk} G\};
\]

$\Bun_{B/[N,N]}^>0$ and $\Bun_{B}^>0$ are the preimages of $\Bun_{H}^>0$ under the natural projections. The projection

\[
\beta : \Bun_{B/[N,N]} \rightarrow \Bun_{H}^>0
\]

is a vector bundle whose fiber over a point $(L_i) \in \Bun_{H}^>0$ is the vector space

\[
\oplus \text{Ext}^1_C(L_i, \mathcal{O}) \simeq \oplus H^1(C, L_i^{-1}).
\]

By definition, $\Bun_{\omega,H}$ is the vector bundle dual to the vector bundle $\beta$, and $\Conf_{G^\lor}$ is an open substack of this vector bundle defined as follows:

\[
\Bun_{\omega,H} = \{(L_i, s_i), \deg L_i > 0, s_i \in \Gamma(C, \omega_C \otimes L_i)\},
\]

$\Conf_{G^\lor} = \{(L_i, s_i), s_i \neq 0\} \simeq \{D_i : \deg D_i > 2g - 2\}$,

i. e., $\Conf_{G^\lor}$ is isomorphic to the space of sets of effective divisors $(D_i)$ on the curve $C$, i. e. to the space of divisors with values in the semigroup $\Gamma_+$ of dominant weights of the group $G^\lor$. Let us call these
sets of divisors by coloured divisors. As a variety $\text{Conf}_{G'}$ is the disjoint union of products of symmetric powers of the curve $C$. The fact that an open dense substack of a vector bundle is a disjoint union of projective varieties, is due to the fact that each point of the stack $\text{Bun}_H$ has the group of automorphisms $(\mathbb{C}^*)^{rkG}$.

The space $\text{Conf}_{G'}$ is naturally stratified:

$$\text{Conf}_{G'} = \bigsqcup_{\lambda'} \text{Conf}_{\lambda'}_{G'},$$

where $\lambda' = (\lambda'_1, \ldots, \lambda'_N)$, $\lambda'_i \in \Gamma_+$, and

$$\text{Conf}_{\lambda'}_{G'} = \{\lambda'_1 x_1 + \ldots + \lambda'_N x_N, x_i \in C, x_i \neq x_j \text{ for } i \neq j\}/\mathfrak{S}_\lambda,$$

where $\mathfrak{S}_\lambda$ is the group of permutations of indices $i$ preserving the weights $\lambda'_i$. Denote the inclusion $\text{Conf}_{\lambda'}_{G'} \hookrightarrow \text{Conf}_{G'}$ by $j_{\lambda'}$.

2.2. Property 3 of the kernel $L_\kappa$. This property describes the object of derived category of twisted $\mathcal{D}_{\text{Conf}_{G'} \times \text{Bun}_{G'}}$-modules

$$(***) \quad j_{\lambda'}^! F \rho \sigma^! L_\kappa,$$

where all the spaces in the diagram (*) are multiplied by $\text{Bun}_{G'}$; $F$ denotes the Fourier–Laplace transform of a $\mathcal{D}$-module on the vector bundle $\text{Bun}_{B/[N,N]}^0$. The object $(* *)$ should be isomorphic to the twisted $\mathcal{D}_{\text{Conf}_{G'} \times \text{Bun}_{G'}}$-module $KZ^\lambda_{G',G'}$ constructed in conformal field theory. As an $O_{\text{Conf}_{G'} \times \text{Bun}_{G'}}(\xi^{\otimes \kappa^\vee})$-module, $KZ^\lambda_{G',G'}$ coincides with the induced module from the $O_{\text{Conf}_{G'} \times \text{Bun}_{G'}}$-module whose fiber

at the point

$$(\lambda'_1 x_1 + \ldots + \lambda'_N x_N, P^\vee) \in \text{Conf}_{G'} \times \text{Bun}_{G'}$$

is the tensor product $$(V^\lambda_{P^\vee})_{x_1} \otimes \ldots \otimes (V^\lambda_{P^\vee})_{x_N},$$

where $V^\lambda_{P^\vee}$ is the vector bundle on $C$ associated with the principal $G'$-bundle $P^\vee$ and with the $G'$-module $V^\lambda$ with highest weight $\lambda'_i$. Further, this $O_{\text{Conf}_{G'} \times \text{Bun}_{G'}}(\xi^{\otimes \kappa^\vee})$-module has a natural structure of a twisted $\mathcal{D}_{\text{Conf}_{G'}}$-module [1], which is a direct generalization of the Knizhnik–Zamolodchikov connection to curves of genus $g$. A direct check with the use of Riemann–Roch theorem shows that the object $(* *)$ has the same twist. The property 3 of the kernel $L_\kappa$ states that these two objects are isomorphic.

2.3. Classical limits of the property 3.

a) The limit $\kappa \to 0$ amounts to the geometric analog of the Casselman–Shalika–Shintani formula for the Whittaker function of an automorphic function.
form. This statement is that the $\mathcal{D}_{\text{Conf}^{\lambda \vee}_G}$-module
\[ j_{\lambda \vee}^! t^! F_p \sigma^! F_{P_{\rho}} \]
is isomorphic to the local system whose fiber over the point $\lambda_1^\vee x_1 + \ldots + \lambda_N^\vee x_N \in \text{Conf}^{\lambda \vee}_G$ equals $(V_{P_{\rho}}^{\lambda \vee})_{x_1} \otimes \ldots \otimes (V_{P_{\rho}}^{\lambda \vee})_{x_N}$.

b) The limit $\kappa \to \infty$ amounts to the construction of the $\mathcal{D}_{\text{Bun}_G^{\lambda \vee}}$-module $F_P$, for $P \in \tilde{T}^* \text{Bun}_G$, by means of a $G$-oper with regular singularities with trivial monodromy [1]. For details, see [3].

2.4. The following natural question arises: what is the result of applying the same operation $j_{\lambda \vee}^! t^! F_p \sigma^!$ to the twisted $\mathcal{D}$-module $KZ^{\lambda \vee}_G$ on the product $\text{Conf}^{\lambda \vee}_G \times \text{Bun}_G$? The arising twisted $\mathcal{D}$-module on the product $\text{Conf}^{\lambda \vee}_G \times \text{Conf}^{\lambda \vee}_G$ should be related with the $W$-algebra $W^\kappa_G \simeq W^\kappa^{\lambda \vee}_G$ [8]. In the case $G = SL(2)$ it is the Virasoro algebra.

For a statement of this kind for a curve $C$ of genus $g = 0$ with marked points and for $G = SL(2)$, see [9].

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