More on cold dark matter from $q$-theory

F.R. Klinkhamer

Institute for Theoretical Physics,
Karlsruhe Institute of Technology (KIT),
76128 Karlsruhe, Germany

G.E. Volovik

Low Temperature Laboratory, Department of Applied Physics,
Aalto University, PO Box 15100, FI-00076 Aalto, Finland,
and
Landau Institute for Theoretical Physics, Russian Academy of Sciences,
Kosygina 2, 119334 Moscow, Russia

Abstract

We consider the rapidly-oscillating part of a $q$-field in a cosmological context and find that its energy density behaves in the same way as a cold-dark-matter component, namely proportional to the inverse cube of the cosmic scale factor.

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*Electronic address: frans.klinkhamer@kit.edu
†Electronic address: volovik@ltl.tkk.fi
I. INTRODUCTION

In a recent article [1], we have explored the idea that the inferred cold-dark-matter component of the present universe corresponds to the rapidly-oscillating part of a so-called $q$-field [2–6]. The constant (spacetime-independent) $q$-field cancels Planck-scale contributions to the gravitating vacuum energy density, where the cancellation occurs without fine tuning. In this way, there would be a combined solution to the missing-mass problem [7] and the cosmological constant problem [8], together with a possible explanation of the nature of the inferred “dark-energy” component of the present universe [7].

In Ref. [1], we started from the static equilibrium configuration of the $q$-field. Here, we turn to a cosmological context with an evolving universe. The goal is to establish whether or not the rapidly-oscillating $q$-field component really behaves as a cold-dark-matter component.

II. THEORY

The results of $q$-theory do not depend on the particular realization of the variable $q$ which describes the quantum vacuum. Here, we use the 4-form realization of $q$-theory with the following action from Ref. [1]:

$$S = -\int_{\mathbb{R}^4} d^4x \sqrt{-g} \left( \frac{R}{16\pi G_N} + \epsilon(q) + \frac{1}{2} C(q) g^{\alpha\beta} (\nabla_\alpha q) (\nabla_\beta q) + \mathcal{L}^{\text{SM}} \right), \quad (2.1a)$$

$$F_{\alpha\beta\gamma\delta} \equiv \nabla_\alpha A_{\beta\gamma\delta}, \quad F_{\alpha\beta\gamma\delta} = q \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta}, \quad (2.1b)$$

where $A$ is a 3-form gauge field with a corresponding 4-form field strength $F \propto q$ (see Refs. [2, 3] and further references therein), $\epsilon(q)$ is a generic even function of $q$, and $\mathcal{L}^{\text{SM}}$ is the Lagrange density of the fields of the standard model (SM) of elementary particle physics. The possible role a 4-form field for the solution of the cosmological constant problem has been emphasized by Hawking [9] among others. Throughout, we use natural units with $c = \hbar = 1$ and take the metric signature $(-+++)$.

In order to simplify the analysis as much as possible, we use these Ansätze:

$$C(q) = \text{constant} = (q_0)^{-1} > 0, \quad (2.2a)$$

$$q_0 = (E_P)^2 \equiv (G_N)^{-1} \approx (1.22 \times 10^{19} \text{GeV})^2, \quad (2.2b)$$

$$\epsilon(q) = \frac{1}{2} (q_0)^2 \left[ \frac{1}{3} \left( \frac{q}{q_0} \right)^4 - \left( \frac{q}{q_0} \right)^2 \right], \quad (2.2c)$$

$$\mu_0 = -\frac{1}{3} q_0, \quad (2.2d)$$

where $q_0$ is the equilibrium value of the $q$-field in the Minkowski vacuum and $\mu_0$ is the corresponding equilibrium value of the integration constant $\mu$ of the generalized Maxwell
equation. This constant \( \mu \) also enters the definition of the gravitating vacuum energy density,

\[
\rho_V(q) \equiv \epsilon(q) - \mu q ,
\]

(2.3)

which suggests the interpretation of \( \mu \) as a “chemical potential” (see Refs. [2, 3] for further discussion).

The reduced Maxwell equation (4) in Ref. [1], with integration constant \( \mu \), is now precisely a Klein–Gordon equation,

\[
\square q = q_0 \frac{d\rho_V(q)}{dq} ,
\]

(2.4)

with \( \rho_V(q) \) from (2.3). As mentioned already in Ref. [6], Eq. (2.4) describes, in the equilibrium vacuum, the spectrum of a massive particle with mass-square \( M^2 = q_0 \). Note that the right-hand-side of (2.4) would contain further nonlinear \( q \) terms if \( C(q) \) were nonconstant.

The standard Einstein equation,

\[
R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = -8\pi G_N \left( T_{\alpha\beta}^{(q)} + T_{\alpha\beta}^{(SM)} \right) ,
\]

(2.5)

has, with the above assumptions, the following \( q \)-field energy-momentum tensor [1]:

\[
T_{\alpha\beta}^{(q)} = -g_{\alpha\beta} \left[ \rho_V(q) + \frac{1}{2} (q_0)^{-1} \nabla_\alpha q \nabla^\alpha q \right] + (q_0)^{-1} \nabla_\alpha q \nabla_\beta q .
\]

(2.6)

### III. COSMOLOGICAL MODEL

Consider the spatially-flat \( (k = 0) \) Robertson–Walker (RW) metric for standard comoving coordinates. The \( q \)-field is taken to be homogeneous, so that \( q = q(t) \). In the present article, we omit the matter described by the SM fields, but their relativistic and nonrelativistic components can easily be added to the dynamic equations below.

In fact, the reduced Maxwell equation [2.4] and the standard Einstein equation [2.5] with the \( q \)-field energy-momentum tensor [2.6] take the following form in a spatially-flat RW universe:

\[
\ddot{q} + 3 \left( \frac{\dot{a}}{a} \right) \dot{q} = -q_0 \frac{d\rho_V(q)}{dq} ,
\]

(3.1a)

\[
\frac{\dot{a}}{a} = -\frac{8\pi G_N}{3} \left[ \left( q_0 \right)^{-1} \left( \partial_t q \right)^2 - \left( q_0 \right)^{-1} \rho_V(q) \right] ,
\]

(3.1b)

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G_N}{3} \left[ \frac{1}{2} \left( q_0 \right)^{-1} \left( \partial_t q \right)^2 + \left( q_0 \right)^{-1} \rho_V(q) \right] ,
\]

(3.1c)

where the dot stands for differentiation with respect to the cosmic time \( t \) and \( a(t) \) is the cosmic scale factor.

Now introduce the dimensionless \( q \)-field perturbation \( \xi \) by

\[
q(t)/q_0 = 1 + \xi(t)
\]

(3.2)
and define Planck-scale dimensionless units with

\[ q_0 = (E_P)^2 = 1. \] (3.3)

For example, the dimensionless cosmic time is given by \( \tau \equiv E_P t \) and a generic dimensionless energy density by \( r_X \equiv (E_P)^{-4} \rho_X \). In these units, the dimensionless mass of the \( q \)-field perturbation is \( m = 1 \), which is also the oscillation frequency of the \( \xi(\tau) \) field in a RW universe (the dimensionless period being \( 2\pi \)).

With \( \rho_V(q) \) from (2.3) for general chemical potential \( \mu \) and the \( \epsilon(q) \) Ansatz from (2.2c), the dimensionless ordinary differential equations (ODEs) and the dimensionless vacuum energy density \( r_V \) are

\[
\ddot{\xi} + 3 \left( \frac{\dot{a}}{a} \right) \dot{\xi} = -\frac{d r_V}{d \xi},
\] (3.4a)

\[
\frac{\dot{a}}{a} = -\frac{8\pi}{3} \left[ \dot{\xi}^2 - r_V \right],
\] (3.4b)

\[
\left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} \left[ \frac{1}{2} \dot{\xi}^2 + r_V \right],
\] (3.4c)

\[
r_V(\xi) = \frac{1}{2} \xi^2 + \frac{2}{3} \xi^3 + \frac{1}{6} \xi^4 - (u - u_0) (1 + \xi),
\] (3.4d)

\[
u_0 = -\frac{1}{3},
\] (3.4e)

where the dot now stands for differentiation with respect to the dimensionless cosmic time \( \tau \). The equilibrium value of the dimensionless chemical potential \( u \) is denoted by \( u_0 \).

IV. ANALYTIC SOLUTIONS

The ODEs (3.4) have special solutions given by certain constant \( \xi(\tau) \) functions. These solutions correspond to Minkowski spacetime for \( u = u_0 \) and to de-Sitter spacetime for \( u < u_0 \). Specifically the solutions are given by

\[
\xi(\tau) = \xi_{\text{const}, n},
\] (4.1a)

\[
h^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi}{3} r_V(\xi_{\text{const}, n}),
\] (4.1b)

with the constant value \( \xi_{\text{const}, n} \) being a solution of the following equation:

\[
\left. \frac{d r_V}{d \xi} \right|_{\xi=\xi_{\text{const}, n}} = 0,
\] (4.1c)

where the discrete index \( n \) labels different solutions. For the particular \( r_V \) Ansatz (3.4d), Eq. (4.1c) is a cubic equation.
The initial boundary conditions leading to the special solution (4.1) include $\dot{\xi}(1) = 0$. For generic initial boundary conditions, the solution can be written as follows:

$$\xi(\tau) = \xi_{\text{const}, \pi} + \xi_{\text{oscill}}(\tau),$$

for a particular index $\bar{n}$ so that $\xi_{\text{oscill}}(\tau) \to 0$ for $\tau \to \infty$. Generic solutions (4.2) can be obtained numerically, as will be shown in Sec. V.

For later use, we already define the dimensionless $q$-dark-matter energy density and pressure,

$$r_{q-\text{DM}} \equiv \frac{1}{2} \dot{\xi}^2 + [r_V(\xi) - r_V(\xi_{\text{const}, \pi})],$$

$$p_{q-\text{DM}} \equiv \frac{1}{2} \dot{\xi}^2 - [r_V(\xi) - r_V(\xi_{\text{const}, \pi})],$$

with $\xi_{\text{const}, \pi}$ corresponding to the asymptotic value of the solution $\xi(\tau)$. Note that the constant part of $r_V$ corresponds to an effective cosmological constant,

$$r_V - CC = p_V - CC \equiv r_V(\xi_{\text{const}, \pi}).$$

The square bracket on the right-hand-side of the Friedmann equation (3.4c) then contains precisely the combination $r_{q-\text{DM}} + r_V - CC$.

V. NUMERIC SOLUTIONS

The numeric solutions are obtained from the two second-order ODEs (3.4a) and (3.4b), with initial boundary conditions obeying (3.4c).

Numerical results for the case of an equilibrium value of the chemical potential (dimensional $\mu = \mu_0$ and dimensionless $u = u_0$) are given in Fig. 1. The asymptotic value of this $\xi(\tau)$ solution is given by

$$\xi_{\text{const}, 1} = 0.$$  (5.1)

The numerical results of Fig. 1 show that

$$\rho_{q-\text{DM}} \propto 1/a^3,$$  (5.2a)

$$\langle a^3 P_{q-\text{DM}} \rangle \sim 0,$$  (5.2b)

where the bracket $\langle \ldots \rangle$ in (5.2b) denotes a time average over a time interval very much larger than the Planck-scale oscillation period of $\xi(\tau)$, which is given by $2\pi$ in our units. Note that $h(\tau)$ and $a(\tau)$ oscillate with the double frequency ($\omega = 2M$) or the half period (given by $\pi$ in our units), and precisely these oscillations may lead to particle production [10]. The behavior (5.2) corresponds to what is expected for a cold-dark-matter component. For the case considered, the effective cosmological constant from (4.4) vanishes, $r_V - CC = -p_V - CC = 0$. 

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FIG. 1: Numeric solution of the ODEs (3.4) with $u = u_0 = -1/3$. The boundary conditions at $\tau = 1$ are $\{a(1), \dot{a}(1), \xi(1)\} = \{1, 1, 1/10\}$ with $\dot{\xi}(1) = 0.476829$ from (3.4c). The corresponding effective cosmological constant from (4.4) vanishes, with $r_{V-CC} = -p_{V-CC} = 0$ for $\xi_{\text{const},1} = 0$. Shown are the two basic functions $a(\tau)$ and $\xi(\tau)$, together with the dimensionless Hubble parameter $h \equiv \dot{a}/a$ and the dimensionless $q$-dark-matter energy density and pressure from (4.3).

FIG. 2: Same as Fig. 1 but now with $u = -1/3 - 1/10000$. The boundary conditions for $\{a(1), \dot{a}(1), \xi(1)\}$ are the same as in Fig. 1, but the derived value of the $\xi$ derivative from (3.4c) is somewhat different, $\dot{\xi}(1) = 0.476598$. The corresponding effective cosmological constant from (4.4) is nonvanishing, with $r_{V-CC} = -p_{V-CC} \approx 10^{-4}$ for $\xi_{\text{const},1} \approx -0.00010002$. 
Numerical results for the case of a nonequilibrium value of the chemical potential (dimensionless $u = u_0 - 10^{-4}$) are given in Fig. 2. The asymptotic value of this $\xi(\tau)$ solution is given by

$$\xi_{\text{const,}1} = \sqrt{2} \cos \left( \frac{1}{3} \arccos \left[ \frac{-10003}{10000 \sqrt{2}} \right] \right) - 1 \approx -0.00010002,$$

(5.3)

where the arccosine function for real arguments in $[-1, 1]$ takes values in $[0, \pi]$. The top-right panel for $(1.5 \tau) \times h$ in Fig. 2 illustrates the onset of de-Sitter-type expansion ($h = \text{const}$.) for $\tau \gtrsim 20$. Indeed, the effective cosmological constant from (4.4) is nonvanishing, $r_{V-\text{CC}} = -p_{V-\text{CC}} \approx 10^{-4}$, and starts to dominate for $\tau \gtrsim 20$ (bottom right panel in Fig. 2). For $\tau \gg 20$, the expansion is driven by the effective cosmological constant, but the bottom-left and bottom-middle panels of Fig. 2 still show the behavior (5.2).

VI. DISCUSSION

The model universe of Fig. 2 contains already the two main ingredients of our present universe: a nonvanishing effective cosmological constant with $\rho_{V-\text{CC}} = -p_{V-\text{CC}} > 0$ and a cold-dark-matter component with the behavior (5.2). The behavior (5.2) has been established numerically in Sec. V but can also be shown analytically.

With the standard Einstein gravity from Eqs. (2.5) and (2.6), the effective pressureless fluid from the rapidly-oscillating part of the $q$-field can be expected to cluster gravitationally in the same way as a hypothetical cold-dark-matter particle would do (see, e.g., Chap. 15 of Ref. [11]). Remaining issues are the addition of “standard” matter (as mentioned in the first paragraph of Sec. IIII) and the effects from particle production induced by the ultra-rapid (and initially large-amplitude) oscillations of the $q$-field [10].

A more realistic description than Fig. 2 requires a very much smaller (but nonzero) value of $|\mu - \mu_0|/q_0$, so that the cross-over from Friedmann–Robertson–Walker-type expansion (Hubble parameter $H \equiv \dot{a}/a \propto 1/t$) to de-Sitter-type expansion ($H = \text{constant}$) occurs at a cosmic age of the order of $10^9$ years, instead of $20 \times \hbar/E_P \sim 10^{-42}$ s as in Fig. 2.

The main challenge is to derive the correct nonzero value of $|\mu - \mu_0|/q_0$. One mechanism would be a kick of the vacuum energy density $\rho_V(q)$ by TeV-mass particle decays [12], now in a theory with $G = \text{const}$ and $q$-derivatives. But it is very well possible that another mechanism operates to give a nonzero asymptotic value of $\rho_V(q)$, that is, under the assumption that the $q$-theory approach is relevant to dark energy and dark matter.

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