Loop Corrections and Naturalness in a Chiral Effective Field Theory

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Abstract

The loop expansion is applied to a chiral effective hadronic lagrangian; with the techniques of Infrared Regularization, it is possible to separate out the short-range contributions and to write them as local products of fields that are already present in our lagrangian. (The appropriate field variables must be re-defined at each order in loops.) The corresponding parameters implicitly include short-range effects to all orders in the interaction, so these effects need not be calculated explicitly. The remaining (long-range) contributions that must be calculated are nonlocal and resemble those in conventional nuclear-structure calculations. Nonlinear isoscalar scalar ($\sigma$) and vector ($\omega$) meson interactions are included, which incorporate many-nucleon forces and nucleon substructure. Calculations are carried out at the two-loop level to illustrate these techniques at finite nuclear densities and to verify that the coupling parameters remain natural when fitted to the empirical properties of equilibrium nuclear matter. Contributions from the $\omega N$ tensor coupling are also discussed.

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I. INTRODUCTION

In recent years, density functional theory (DFT) has emerged as a powerful tool for describing the structure of nuclei. Based on the seminal works of Kohn, Hohenberg, and Sham [1–3], DFT was originally developed for use in condensed matter physics; interestingly, it was discovered that it also provides a systematic framework for modeling the low-energy sector of the strong interaction [4–10]. Central to DFT is the notion that, if the energy functional is considered a functional of the density, then by minimizing the energy one can acquire the exact ground-state density [1, 3]. Furthermore, if the quantum many-body equations are replaced by single-particle equations with local, classical fields, one can exactly reproduce the total energy, scalar and vector densities, and chemical potential [2, 11]. Thus the many-body problem is reduced to finding the proper form of the energy functional. As it turns out, the lagrangian of quantum hadrodynamics determines just such a functional.

Quantum hadrodynamics (QHD) is a low-energy effective theory of the strong interaction that uses the observed degrees of freedom at this energy scale (hadrons). It has evolved from the early model of Walecka [12] into more modern incarnations that are based on chiral effective field theory (EFT) and DFT [7–10, 13–16]. The QHD energy functional is constructed as an expansion in powers of the mean meson fields, which are in fact Kohn–Sham potentials, over a heavy mass, such as the chiral symmetry breaking scale, and includes all possible terms consistent with the underlying symmetries of QCD. Field redefinitions are used to move the complexity of the problem (e.g., many-body forces and nucleon substructure) into the meson field self-interactions. As this is an effective theory, the terms are characterized by unknown coefficients. Once all the dimensional and combinatorial factors have been removed, however, the remaining dimensionless coefficients are assumed to be natural, or of order unity [7, 8, 17–20]. The long-range dynamics are included explicitly, and short-range physics is contained in the parametrization; therefore, this theory incorporates the natural separation of length scales. While QHD in principle contains all possible terms consistent with the underlying symmetries of the system, in practice it is a perturbative expansion for the energy functional that can be truncated at a manageable level (since the ratio of the mean fields to the heavy mass is small). The now finite number of coefficients are fixed by experimental data, and this theory can then be used for predictive purposes [21–27].

QHD is a strong-coupling theory. Unlike QED, there is no obvious asymptotic expansion to use to obtain results and refine them systematically. It is thus unknown whether QHD permits any expansion for systematic computation and refinement of theoretical results. One possibility is the loop expansion, which was partially explored in [28–31].

Why a loop expansion? The loop expansion is a simple and well-developed expansion scheme in powers of $\hbar$ that is derived from the path integral [28, 32–37]. The mean meson fields are included non-perturbatively, and the correlations are included perturbatively. Therefore, one can analyze the many-body effects order by order. Previous work in QHD has shown that the mean-field terms dominate the nuclear energy, and exchange and correlation effects do not significantly modify the energy or nucleon self-energies, at least for states in the Fermi sea [34]. We stress that we are not certain that the loop expansion is practical; the answer to this question is left for future consideration. However, the loop expansion has the advantage that it is fairly easy to separate the short-range and long-range dynamics and to analyze their structures.

The loop expansion was investigated in the simplest case at the two-loop level [28, 31] (only the most rudimentary couplings were retained). This was a severe truncation of the
EFT lagrangian and implies that we were interested only in the lowest level of accuracy. Here we are also concerned with the loop expansion only up to two-loop order (consideration of higher-loop effects is left for the future), but now nonlinear meson self-couplings and tensor terms are retained. Note that the nonlinear terms incorporate short-range many-body nuclear forces, vacuum dynamics, and the effects of nucleon substructure. The extension of the level of truncation to include these terms allows us to describe nuclear matter (and presumably nuclei) with greater accuracy.

The result of the loop expansion up to the two-loop level is the mean field energy density [7–9] plus a series of integrals. We can categorize these integrals in two groups: baryon loops and purely mesonic loops. The baryon loops are integrals at the two-loop level that contain two factors of the baryon propagator and one factor of a meson propagator. Each nucleon propagator can be separated into a Feynman and a Density part [28, 34]. This is accomplished by incorporating the proper pole structure of the propagator. The Feynman part describes the propagation of a baryon or an antibaryon; the Density part involves only on-shell propagation in the Fermi sea and accounts for the exclusion principle.

As before [31], the two-loop baryon integrals can each be separated into three distinct parts: the exchange, Lamb-shift, and vacuum-fluctuation contributions. The exchange term has two factors of the Density portion of the nucleon propagator. The Lamb-shift term contains both Feynman and Density parts of the nucleon propagator. The vacuum-fluctuation term contains two factors of the Feynman propagator. The Lamb-shift and vacuum-fluctuation parts are short-range physics, and it was shown in the simple case of [31] that they can be expressed as a sum of terms that are already present in the EFT lagrangian. This is an effective theory, so the coefficients of these terms are not yet determined; thus, the two-loop terms are just absorbed into terms already present in the lagrangian and are not to be calculated explicitly [31]. As a result, only the exchange portion of the two-loop integrals must be calculated explicitly.

In addition, a number of purely mesonic loops arise when nonlinear meson field self-interactions are included. These terms, however, can be expressed as a power series in the meson fields with undetermined coefficients. As before, these terms are just absorbed into the lagrangian parameters and should not be calculated explicitly.

It is interesting to note that the natural separation of the length scales in QHD and the parametrization of the short-distance dynamics is similar to Infrared Regularization [38–47]. In Infrared Regularization, loop integrals are separated into infrared regular and singular contributions (corresponding to short- and long-range physics). The regular portion is expanded as a power series in the momentum and is absorbed into the unknown coefficients of the underlying lagrangian; the singular part must be calculated explicitly. This is directly analogous to what was done for the loop expansion in QHD [31].

One question that remains unanswered is how the addition of these many-body correlations to the energy density affects the naturalness assumption, which states that once all the dimensional and combinatorial factors in a given term are accounted for, what remains is a dimensionless coefficient of order unity [7, 8, 17–20]. Once one decides on a level of truncation in the underlying lagrangian, the now finite number of unknown constants can be parametrized by the data. Thus, one purpose of this work is to prove that sets of parameters exist at different levels of truncation in the two-loop energy density in which the parameters are all natural and thereby show that the naturalness assumption holds. For the purposes of this work, we will consider only uniform, symmetric nuclear matter, as in [31]. Here, however, we also include nonlinear meson self-interactions and tensor terms. These
additional terms not only increase the number of two-loop integrals, but also complicate the meson propagators, due to the short-range and many-body physics mentioned earlier. The loop expansion is carried out to the two-loop level, and the resulting two-loop integrals are separated into long-range and short-range physics. The short-range contributions are expressed in forms that already appear in the lagrangian. Since the coefficients of these terms have yet to be determined, they are just redefined, thus incorporating these new contributions into the mean field energy (as in [31]). As a result, they are already present in the one-loop QHD calculation. The long-range physics must be calculated explicitly. In this work, we fit the two-loop energy to empirical properties of nuclear matter. Then we compare parameter sets constructed at the two-loop level with those of previous works obtained at the mean field level and consider the naturalness of the parameters.

II. THEORY

A. Isoscalar Meson Self-Interactions and Mixing

In this section, we investigate the effects of nonlinear meson self-interactions (including isoscalar meson mixing) on the loop expansion. Here we will keep terms up to order $\nu = 4$ in the isoscalar meson sector of the QHD lagrangian [7, 8] (i.e., $\kappa_3, \kappa_4, \eta_1, \eta_2$, and $\zeta_0 \neq 0$), and let $\alpha_1 = \alpha_2 = 0$. Since we will calibrate the parameters using nuclear matter properties only (and not finite nuclei), we want to keep the number of parameters at a manageable level. Therefore, the baryon and pion parts of the lagrangian are unchanged from [31] (except for the addition of the two-pion–nucleon vertex):

$$\mathcal{L} = -\bar{\psi} \left[ \gamma_\mu (\partial_\mu - ig_\nu V_\mu) - i g_A \frac{g_A}{f_\pi} \gamma_\mu \gamma_5 \partial_\mu \pi + \frac{1}{2} f_\pi^2 \gamma_\mu [\pi, \partial_\mu \pi] + (M - g_S \phi) \right] \psi$$

$$-\frac{1}{2} (\partial_\mu \pi_a)^2 - \frac{1}{2} m_{\pi a}^2 \pi_a^2,$$

which follows from the chirally invariant lagrangian in [7] by retaining only the relevant (lowest-order) terms in the pion fields. The neutral scalar and vector meson lagrangian is now

$$\mathcal{L}_{\phi \omega} = -\frac{1}{2} (\partial_\mu \phi)^2 - m_\omega^2 \phi^2 \left( \frac{1}{2} + \frac{\kappa_3 g_S \phi}{3! M} + \frac{\kappa_4 g_S^2 \phi^2}{4! M^2} \right) - \frac{1}{4} V_{\mu \nu} V_{\mu \nu}$$

$$-\frac{1}{2} m_\omega^2 V_{\mu \nu} \left( 1 + \eta_1 \frac{g_S \phi}{M} + \frac{\eta_2 g_S^2 \phi^2}{2 M^2} \right) + \frac{1}{4!} \zeta_0 g_\omega^2 (V_{\mu \nu})^2,$$

where $V_{\mu \nu} = \partial_\mu V_{\nu} - \partial_\nu V_{\mu}$ and $\pi = \frac{1}{2} \pi_a \cdot \tau_a$. Here $\psi$ are the fermion fields and $\phi, V_{\mu}$, and $\pi_a$ are the meson fields (isoscalar-scalar, isoscalar-vector, and isovector-pseudoscalar, respectively). The heavy meson fields are also chiral scalars. Note that in this work, the conventions of [10] are used. As in [31], we are not making a chiral expansion in powers of the pion mass and include it for kinematical purposes only.

1 In Appendix A, we discuss the role of the tensor coupling to the neutral vector meson at the two-loop level.
The generating functional is defined in the usual way by

\[
Z[j, J_\mu] \equiv \exp \{iW[j, J_\mu]/\hbar\} = \mathcal{N}^{-1} \int D(\bar{\psi}) D(\psi) D(\phi) D(V_\mu) D(\pi_a) \times \exp \left\{ \frac{i}{\hbar} \int d^4x \left[ \mathcal{L}(x) + j(x)\phi(x) + J_\mu(x)V_\mu(x) \right] \right\},
\]

where \( \mathcal{N} \) is the normalization factor (in effect, the vacuum subtraction, which is also an expansion in loops), \( j(x) \) and \( J_\mu(x) \) are the external sources corresponding to the meson fields \( \phi \) and \( V_\mu \), respectively, and the connected generating functional is \( W[j, J_\mu] \). The path integrals of interest here are those over the isoscalar quantum fluctuation fields \( \sigma \) and \( \tilde{\eta}_\mu \) (the others are discussed in [31]). With \( \eta_1, \eta_2 \neq 0 \), the integral is bilinear in form, or

\[
I = \int D(\sigma) D(\tilde{\eta}_\mu) \exp \left\{ i \int d^4x \left[ \frac{1}{2} \sigma \Delta^{-1}_S \sigma + \frac{1}{2} \tilde{\eta}_\mu D^{-1}_{\mu\nu} \tilde{\eta}_\nu 
- m^2_V \left( \frac{\eta_1 g_S}{M} V_\mu^0 + \frac{\eta_2 g_S^2}{M^2} \phi_0 V_\mu^0 \right) \sigma \tilde{\eta}_\mu
+ \sigma + U_\mu \tilde{\eta}_\mu \right] \right\},
\]

where the meson propagators in momentum space are

\[
\Delta^{-1}_S(k) = k^2 + m^2_S \left( 1 + \kappa_3 \frac{g_S \phi_0}{M} + \kappa_4 \frac{g_S^2 \phi_0^2}{2 M^2} \right) - m^2_V \eta_1 \frac{g_S}{M} V_\mu^0 V_\mu^0,
\]

\[
D^{-1}_{\mu\nu}(k) = \left[ k^2 + m^2_V \left( 1 + \eta_1 \frac{g_S \phi_0}{M} + \eta_2 \frac{g_S^2 \phi_0^2}{2 M^2} \right) - \frac{\zeta}{6} \frac{g^2 V_\lambda^0 V_\lambda^0}{V_\mu^0 V_\mu^0} \right] \delta_{\mu\nu} - k_\mu k_\nu - \frac{\zeta}{3} \frac{g^2 V_\mu^0 V_\mu^0}{V_\nu^0 V_\nu^0}.
\]

To simplify the equations, we introduce a five-component notation for the meson fields and their corresponding auxiliary sources [48]:

\[
s = \begin{pmatrix} \sigma \\ \bar{\eta}_\mu \end{pmatrix}, \quad t = \begin{pmatrix} u \\ U_\mu \end{pmatrix},
\]

where \( s_\alpha \) and \( t_\alpha \) have \( \alpha = 0, 1, \ldots, 4 \). We can then rewrite the path integral as

\[
\int D(s_\alpha) \exp \left\{ i \int d^4x \left[ \frac{1}{2} s_\alpha A^{-1}_{\alpha\beta} s_\beta + s_\alpha t_\alpha \right] \right\} = \exp \left\{ -\frac{1}{2} \text{tr} \ln [A^{-1}] \right\} \exp \left\{ -\frac{i}{2} t_\alpha A_{\alpha\beta} t_\beta \right\},
\]

where the matrix \( A^{-1} \) is

\[
A^{-1}(k) = \begin{pmatrix} \Delta^{-1}_S(k) & \eta V_\mu^0 \\ \eta V_\mu^0 & D^{-1}_{\mu\nu}(k) \end{pmatrix},
\]

and

\[
\eta \equiv -m^2_V \left( \frac{\eta_1 g_S}{M} + \frac{\eta_2 g_S^2}{M^2} \phi_0 \right).
\]
If \( \eta_1 = \eta_2 = 0 \), then \( A^{-1} \) is diagonal, there is no mixing, and the system reduces to the non-mixing case, a simple example of which was discussed in [31].

Now we want to solve for the components of \( A(k) \) using \( A(k)A^{-1}(k) = 1 \):

\[
1 = \begin{pmatrix} A_{00} & A_{0\mu} \\ A_{\nu 0} & A_{\nu \mu} \end{pmatrix} \begin{pmatrix} \Delta_S^{-1}(k) & \eta V^0_\nu \\ \eta V^0_\mu & D^{-1}_{\mu \nu}(k) \end{pmatrix}.
\]

(11)

This leads to the following four equations:

\[
A_{00} \Delta_S^{-1}(k) + A_{0\mu} \eta V^0_\mu = 1,
\]

(12)

\[
A_{00} \eta V^0_\nu + A_{0\mu} D^{-1}_{\mu \nu}(k) = 0,
\]

(13)

\[
A_{\nu 0} \Delta_S^{-1}(k) + A_{\nu \mu} \eta V^0_\mu = 0,
\]

(14)

\[
A_{\nu 0} \eta V^0_\nu + A_{\nu \mu} D^{-1}_{\mu \nu}(k) = \delta_{\nu \nu'}.
\]

(15)

Solving these, we get \( (V^2 = V^0_\mu V^0_\mu) \)

\[
A_{00} = \left( k^2 + M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right) \left[ \left( k^2 + M^2_V - \frac{\zeta_0}{2} g^2_V V^2 \right) M^2_V - \frac{\zeta_0}{6} g^2_V (k \cdot V)^2 \right] / C,
\]

(16)

\[
A_{0\mu} = -\eta \left( k^2 + M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right) \left[ \left( M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right) V^0_\mu + (k \cdot V) k_\mu \right] / C,
\]

(17)

\[
A_{\nu 0} = -\eta \left( k^2 + M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right) \left[ \left( M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right) V^0_\nu + (k \cdot V) k_\nu \right] / C,
\]

(18)

\[
A_{\nu \mu} = \left( k^2 + M^2_S \right) \left\{ \left[ \left( k^2 + M^2_V - \frac{\zeta_0}{2} g^2_V V^2 \right) M^2_V - \frac{\zeta_0}{6} g^2_V (k \cdot V)^2 \right] \delta_{\mu \nu'} + \left( k^2 + M^2_V - \frac{\zeta_0}{2} g^2_V V^2 \right) k_\mu k_{\nu'} + \frac{\zeta_0}{3} g^2_V \left( M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right) V^0_\mu V^0_{\nu'} + \frac{\zeta_0}{3} g^2_V (k \cdot V) \left( k_\mu V^0_{\nu'} + k_{\nu'} V^0_\mu \right) \right\}
\]

\[-\eta^2 \left\{ \left[ \left( M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right) V^2 + (k \cdot V)^2 \right] \delta_{\mu \nu'} + V^2 k_\mu k_{\nu'}
\]

\[-\left( M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right) V^0_\mu V^0_{\nu'} - (k \cdot V) \left( k_\mu V^0_{\nu'} + k_{\nu'} V^0_\mu \right) \right\} / C,
\]

(19)

where

\[
C \equiv \left( k^2 + M^2_V - \frac{\zeta_0}{6} g^2_V V^2 \right)
\]
\[
\left\{ \left( k^2 + M_S^2 \right) \left[ \left( k^2 + M_V^2 - \frac{\zeta_0}{2} g_V^2 V^2 \right) \left( M_V^2 - \frac{\zeta_0}{6} g_V^2 V^2 \right) - \frac{\zeta_0}{3} g_V^2 (k \cdot V)^2 \right] \right.
\]
\[
\left. - \eta^2 \left[ \left( M_V^2 - \frac{\zeta_0}{6} g_V^2 V^2 \right) V^2 + (k \cdot V)^2 \right] \right\},
\]
(20)

and
\[
M_S^2 \equiv m_S^2 \left( 1 + \kappa_3 \frac{g_S \phi_0}{M} + \frac{\kappa_4 g_S^2 \phi_0^2}{2 M^2} \right) - m_V^2 \frac{\eta_2 g_S^2 V^2}{M^2},
\]
(21)
\[
M_V^2 \equiv m_V^2 \left( 1 + \eta_1 \frac{g_S \phi_0}{M} + \frac{\eta_2 g_S^2 \phi_0^2}{2 M^2} \right).
\]
(22)

The corresponding path integral in the vacuum subtraction has a similar structure, except with vanishing classical meson fields; therefore, \( A \) is replaced by \( A_0 \), or
\[
A_0(k) = \left( \frac{\Delta_0^0(k)}{D_{\mu\nu}(k)} \delta_{\mu\nu} \right),
\]
(23)

where \( \Delta_0^0 \) and \( D_{\mu\nu} \), are the free propagators for the scalar and neutral vector mesons, respectively:
\[
\Delta_0^0(k) = \frac{1}{k^2 + m_S^2 - i\epsilon},
\]
(24)
\[
D_{\mu\nu}^0(k) = \frac{1}{k^2 + m_V^2 - i\epsilon} \left( \delta_{\mu\nu} + \frac{k_\mu k_\nu}{m_V^2} \right).
\]
(25)

The pion self-couplings do not contribute to the energy density if there is no pion condensate. Therefore, we simply ignore the pure pion terms in the following calculation. The remaining generating functional is
\[
Z[j, J_\mu] = \mathcal{N}^{-1} \exp \left\{ \frac{i}{\hbar} \int d^4 x \left[ \mathcal{L}_0(x) + j(x) \phi_0(x) + J_\mu(x) V_\mu^0(x) \right] \right\}
\]
\[
\times \exp \left\{ \int d^4 x \int \frac{d^4 k}{(2\pi)^4} \left( \text{tr} \ln \left[ G_{\mu\nu}^0(k) G_{\mu\nu}^{-1}(k) \right] - \frac{1}{2} \text{tr} \ln \left[ A_0(k) A^{-1}_0(k) \right] \right) \right\}
\]
\[
\times \left\{ \exp \left[ i \hbar^{1/2} \int d^4 x \left( \gamma_\mu \gamma_5 \frac{\partial}{\partial u(x)} \right) \left( g_S \frac{\partial}{\partial v(x)} + i g_V \gamma_\mu \left( \gamma_5 \frac{\partial}{\partial u(x)} \right) \right) \right. 
\]
\[
+ i \frac{g_A}{2 f_\pi} \gamma_\mu \gamma_5 \partial_\mu \left( \gamma_5 \gamma_\alpha \frac{\partial}{\partial u(x)} \right) \left( \gamma_5 \gamma_\alpha \frac{\partial}{\partial u(x)} \right) \right)
\]
\[
- i \hbar^{1/2} \int d^4 x \left( m_S^2 \left( \kappa_3 g_S + \frac{\kappa_4 g_S^2 \phi_0}{6 M^2} \right) \left[ \frac{-i\partial}{\partial u(x)} \right] \left[ \frac{-i\partial}{\partial u(x)} \right] \right)
\]
\[
+ m_V^2 \left( \frac{\eta_1 g_S}{2 M} + \frac{\eta_2 g_S^2}{2 M^2} \phi_0 \right) \left[ \frac{-i\partial}{\partial u(x)} \right] \left[ \frac{-i\partial}{\partial u(x)} \right] \right) \right.
\]
\[
\left. - \frac{\zeta_0}{6} g_V^2 V_\mu^0 \left[ \frac{-i\partial}{\partial u(x)} \right] \left[ \frac{-i\partial}{\partial u(x)} \right] \right) + \left[ \frac{-i\partial}{\partial u(x)} \right] \left[ \frac{-i\partial}{\partial u(x)} \right] \right) 
\]
\[ + \hbar \int d^4x \left( \frac{1}{24} \xi^a g^a_\mu \right) \left[ \frac{-i\delta}{\delta U^\mu(x)} \right] \left[ \frac{-i\delta}{\delta U^\nu(x)} \right] \left[ \frac{-i\delta}{\delta U^\nu(x)} \right] - m^2_v \frac{\kappa_4 g^2_\mu}{24 M^2} \left[ \frac{-i\delta}{\delta u(x)} \right]^4 \]

\[ - m^2_v \frac{\eta_2 g^2_\mu}{4 M^2} \left[ \frac{-i\delta}{\delta u(x)} \right]^2 \left[ \frac{-i\delta}{\delta U^\mu(x)} \right] \left[ \frac{-i\delta}{\delta U^\nu(x)} \right] \right) \}

\times \exp \left\{ i \int d^4x \int d^4y \left[ \frac{1}{2} t^a A_{\alpha \beta}(x - y) t^\beta \right. \right.

\left. + \frac{1}{2} \zeta_\alpha(x) \Delta^{ab}_\pi(x - y) \zeta_\beta(y) - \bar{\xi}(x) G_H(x - y) \xi(y) \right\} \right\}

\text{sources} = 0 . \quad (26)

Here \( \zeta_\alpha \) is the pion source and \( \xi \) is the nucleon source. We define the contribution at the mean field level as

\[ \mathcal{L}_0 = -\frac{1}{2} m^2_v V_\mu^0 V_\mu^0 \left( 1 + \frac{\eta_1 g S \phi_0}{M} + \frac{\eta_2 g^2_\mu \phi_0^2}{2 M^2} \right) - m^2_S \phi_0^2 \left( \frac{1}{2} + \frac{\kappa_3 g S \phi_0}{3 M} + \frac{\kappa_4 g^2_\mu \phi_0^2}{4 M^2} \right) \]

\[ + \frac{1}{4!} \xi^a g^a_\mu (V_\mu^0 V_\mu^0)^2 , \quad (27) \]

and the pion and fermion propagators are given by

\[ \Delta^{ab}_\pi(k) = \frac{1}{k^2 + m^2_\pi - i\epsilon} \delta_{ab} , \quad (28) \]

\[ G^0_F(k) = \frac{1}{i \slashed{k} + M - i\epsilon} , \quad (29) \]

\[ G_H(k) = \frac{1}{i \slashed{k} - ig \gamma_5 \slashed{V}_\mu^0 + (M - g S \phi_0)} . \quad (30) \]

It is straightforward to acquire the connected generating functional at the one-loop order

\[ W_1 = -i\hbar \int d^4x \int \frac{d^4k}{(2\pi)^4} \left\{ \text{tr} \ln \left[ G^0_F(k) G^{-1}_H(k) \right] - \frac{1}{2} \text{tr} \ln \left[ A_0(k) A^{-1}(k) \right] \right\} , \quad (31) \]

and therefore

\[ \Gamma^{(1)} [\phi_e, V^0_e] = \int d^4x \left\{ \mathcal{L}_0 + i\hbar \int \frac{d^4k}{(2\pi)^4} \left( \text{tr} \ln \left[ G^0_F(k) G^{-1}_H(k) \right] 

\left. - \frac{1}{2} \text{tr} \ln \left[ A_0(k) A^{-1}(k) \right] \right) \right\} . \quad (32) \]

The baryon term is identical to the simple case in [31]. To treat the isoscalar terms, we first consider the identity

\[ \text{tr} \ln \left[ A_0(k) A^{-1}(k) \right] \equiv \ln \det \left[ A_0(k) A^{-1}(k) \right] , \quad (33) \]

since it is easy to acquire the determinants:

\[ \det [A_0(k)] = \left[ m^2_v (k^2 + m^2_\pi) (k^2 + m^2_\nu)^3 \right]^{-1} , \quad (34) \]
where $M_i^2$ and $M_j^2$ are defined by Eqs. (21) and (22), and $V_{\mu}^{(1)} = V_\mu^e = i \delta_{\mu i} V_e$.

Apparently, $\det [A^{-1}(k)]$ is an analytic function of $\phi_e$ and $V_e$, and

$$\det [A^{-1}(k)] (\phi_e = 0, V_e = 0) = \det [A_0^{-1}(k)] .$$

Therefore, the meson contribution to the effective action can be categorized by the general form

$$\sum_{i, j = 0}^{\infty} \alpha^{i j} (V_\mu V_\mu)^j ,$$

where the prime indicates that $i + j > 0$. However, terms of this type already appear in the general QHD lagrangian; since the coefficients of these terms have yet to be determined, these meson contributions can be absorbed into the preexisting terms. As a result, they don’t need to be separately considered.

Now we want to determine the connected generating functional at the two-loop level, $W_2$. For this, we have to make a Legendre transformation to get the effective action. The classical meson fields are approximated as [31]

$$\begin{pmatrix} \phi_0(x) \\ V_\mu^0(x) \end{pmatrix} = \begin{pmatrix} \phi_e(x) \\ V_\mu^e(x) \end{pmatrix} + \int d^4 y A(x - y) \begin{pmatrix} \frac{\delta W_1}{\delta \phi_0(y)} e \\ \frac{\delta W_1}{\delta V_\mu^0(y)} e \end{pmatrix} + O(\hbar^2) .$$

One can verify that the tadpole diagrams in $W_2$ eventually cancel, and so we are left with only the one-particle irreducible diagrams:

$$\Gamma^{(2)}[\phi_e, V_\mu^e] = \int d^4 x L_0 [\phi_e(x), V_\mu^e(x)]$$

$$- i \hbar \int d^4 x \int \frac{d^4 k}{(2\pi)^4} \left\{ \text{tr} \ln [G^0_F(k) G^{-1}_H(k)] - \frac{1}{2} \text{tr} \ln [A_0(k) A^{-1}(k)] \right\}$$

$$+ \hbar^2 \int d^4 x \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \left\{ \frac{g_s^2}{2} A_{00}(k - q) \text{tr} [G_H(k) G_H(q)] \right\}$$

$$- \frac{g_A^2}{2} A_{\mu \nu}(k - q) \text{tr} [G_H(k) \gamma_\mu G_H(q) \gamma_\nu]$$

$$- \frac{g_A^2}{2 f_\pi^2} \Delta^{ab}_\pi (k - q) \text{tr} \left[ G_H(k)(\not{k} - \not{q}) \gamma_5 \frac{\tau_a}{2} G_H(q)(\not{k} - \not{q}) \gamma_5 \frac{\tau_b}{2} \right]$$

$$+ i g_s g_V A_{0\mu}(k - q) \text{tr} [\gamma_\mu G_H(k) G_H(q)]$$
+ 3 \left( \kappa_3 \frac{g_S m_S^2}{6M} + \kappa_4 \frac{g_S^2 m_S^2 \phi_0}{6M^2} \right)^2 A_{00}(k) A_{00}(q) A_{00}(k-q)
- \kappa_4 \frac{g_S^2 m_S^2}{8M^2} A_{00}(k) A_{00}(q)
+ \frac{1}{4} \eta^2 [A_{00}(k-q) A_{\mu\nu}(k) A_{\mu\nu}(q) + 2 A_{0\mu}(k) A_{0\nu}(q) A_{\mu\nu}(k-q)]
+ \frac{1}{2} \eta \left( \kappa_3 \frac{g_S m_S^2}{M} + \kappa_4 \frac{g_S^2 m_S^2 \phi_0}{M^2} \right) A_{00}(k-q) A_{0\mu}(k) A_{0\mu}(q)
- \eta^2 \frac{g_S^2 m_V^2}{4M^2} [A_{00}(k) A_{\mu\nu}(q) + 2 A_{0\mu}(k) A_{0\nu}(q)]
+ \frac{1}{24} \zeta_0 g_V^2 [A_{\nu\nu}(k) A_{\mu\mu}(q) + 2 A_{\mu\nu}(k) A_{\mu\nu}(q)]
+ \frac{\eta^2 m_V^4}{2M^4} \eta \zeta_0 g_S^2 g_V^2 [A_{0\lambda}(k) A_{\lambda\mu}(q) V_{\mu}^e A_{\lambda\nu}(k-q) V_{\nu}^e + 2 A_{0\lambda}(k) A_{0\nu}(q) A_{\lambda\nu}(k-q) V_{\nu}^e]
- \frac{m_V^2 \eta \zeta_0 g_S^2 g_V^2}{6M^2} [A_{0\lambda}(k) A_{\lambda\mu}(q) V_{\mu}^e A_{\lambda\nu}(k-q) + 2 A_{0\lambda}(k) A_{0\nu}(k) V_{\mu}^e A_{\mu\nu}(k-q) V_{\nu}^e]
+ \frac{1}{36} \frac{\epsilon^2 g_V^4}{\zeta_0 g_S^2 g_V^2} [A_{\mu\nu}(k) A_{\mu\nu}(q) V_{\lambda}^e A_{\lambda\nu}(k-q) V_{\nu}^e + 2 V_{\mu}^e A_{\mu\nu}(k-q) A_{\lambda\nu}(q) V_{\nu}^e]
- \text{VEV} \bigg\} , \quad (39)

where VEV is the vacuum subtraction. Notice that there are now four distinct fermion loops at the two-loop level. These reduce to the familiar three loops from the non-mixing case when \( \eta_1 = \eta_2 = 0 \).

With dimensional regularization, we can shift momentum variables to remove the dependence on the vector field from the fermion propagators and thereby replace \( G_H(k) \) by \( G^*(k) \), where

\[
G^*(k) = \frac{-1}{i \vec{k} + M - g_S \phi_0}
= (i \vec{k} - M^*) \left[ \frac{1}{k^2 + M^*^2 - i\epsilon} - \frac{i\pi}{E^*(k)} \delta[k_4 - E^*(k)] \theta(k_F - |\vec{k}|) \right]
\equiv G^*_F(k) + G^*_D(k) .
\]

Here \( M^* = M - g_S \phi_0 \). We can then separate each of the fermion loops [using \( G^*(k) = G^*_F(k) + G^*_D(k) \)] into three contributions: an exchange term, a Lamb-shift term, and a vacuum-fluctuation term. The exchange terms will be evaluated explicitly later. For now, let us consider the Lamb-shift term for the fermion loop involving scalar–vector mixing:

\[
\int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} A_{0\mu}(k-q) \text{tr} [\gamma_\mu G^*_F(k) G^*_D(q)]
\times \int \frac{d^4 k}{(2\pi)^4} \int \frac{d^4 q}{(2\pi)^4} \delta(q_4 - E^*(q)) \theta(k_F - |q|) \frac{i\pi}{E^*(q)} \left[ \frac{M^*(k+q)\mu}{k^2 + M^*^2} \right]
\]

10
\[ \times [(k - q)^2 + M_V^2] \left[ (k - q) \cdot V \right] (k - q)_\mu + M_V^2 V_\mu) \]
\[ \times \left\{ [(k - q)^2 + M_S^2] [(k - q)^2 + M_V^2] \right\} \]
\[ \times \left( [(k - q)^2 + M_V^2 + \frac{\zeta_0}{3} g_V^2 V^2] M_V^2 - \frac{\zeta_0}{3} g_V^2 [(k - q) \cdot V]_2^2 \right) \]
\[ - \eta^2 \left( [(k - q)^2 + M_V^2] M_V^2 V^2 + [(k - q)^2 + M_V^2 + \frac{\zeta_0}{3} g_V^2 V^2] [(k - q) \cdot V]_2^2 \right) \right\}^{-1} \]
\[ \propto \int \frac{d^4 q}{(2\pi)^4} \int \frac{d^4 p}{(2\pi)^4} \frac{\theta (k_F - |q|)}{E^*(q)} \delta (q_4 - E^*(q)) \left[ p^2 + 2 (p \cdot q) + q^2 + M^2 \right] \]
\[ \times \left\{ [p^2 + M_V^2] [(p \cdot V) \left( p^2 + 2 (p \cdot q) + M_V^2 \right) + 2M_V^2 (p \cdot V)] \right\} \]
\[ \times \left\{ \left( p^2 + M_S^2 \right) (p^2 + M_V^2) \left[ (p^2 + M_V^2 + \frac{\zeta_0}{3} g_V^2 V^2) M_V^2 - \frac{\zeta_0}{3} g_V^2 (p \cdot V)^2 \right] \right\}^{-1} \]
\[ \times \left\{ (p \cdot q)^2 \left\{ (p \cdot V) \left[ p^2 + 2 (p \cdot q) + M_V^2 \right] + 2M_V^2 (p \cdot V) \right\} \right\} \]
\[ \times \left\{ \int \frac{d^4 q}{(2\pi)^4} \frac{\theta (k_F - |q|)}{E^*(q)} \delta (q_4 - E^*(q)) \right. \tilde{F} (M^*, V^2, q \cdot V) \right. , \]
where
\[ M_V^* \equiv M_V^2 - \frac{\zeta_0}{6} g_V^2 V^2 , \]  
\[ k_F \] is the Fermi momentum, and \( E^*(q) \equiv (q^2 + M^*)^{1/2} \). We have made the substitution \( p_\mu = (k - q)_\mu \), which is allowed if we keep \( q \) fixed in the \( k \) integral. The function \( F \) comes from the fractional integrand and is an analytic function of the variables \( p^2, q^2, V^2, \) and \( M^* \). The function \( \tilde{F} \) comes from the integral over \( p \) and is analytic in the variables \( M^*, V^2, \) and \( q \cdot V \), since \( p_\mu \) is always spacelike. The on-shell condition \( q_4 = E^*(q) \) is used to eliminate the dependence on \( q^2 \) in the last line.

Remember that \( V_\mu^0 = iV_0 \delta_{\mu 4} \). Thus, after integrating over \( q_4 \), the remaining expression is an analytic function of \( E^*(q) \). The final result will be complicated, but should be explicitly dependent on some types of baryon densities. These terms can be directly related to terms in the effective lagrangian that are some combination of the nucleon fields (with some derivatives) multiplied by some Lorentz-invariant polynomials in the mean meson fields; as a result, they can be parametrized by local terms in the effective lagrangian. A simple case was illustrated in [31].

The term with the pion propagator is the same as in the non-mixing case. The fermion loops with \( A_{00} \) and \( A_{\mu \nu} \) will, in the case of \( \eta_1 = \eta_2 = 0 \), reduce to the equivalent loops with the pure neutral scalar and vector mesons from the non-mixing case. We expect that they will behave similarly in both the mixing and non-mixing cases, with the former simply
leading to more general local combinations of the fields. *Thus the Lamb-shift terms can be parametrized by terms already present in the lagrangian (before truncation).*

The vacuum-fluctuation terms and all the pure meson loops have no explicit density dependence, so all the conclusions of the previous analysis [31] are still applicable here. The short-range physics which is contained in the Lamb-shift and vacuum-fluctuation terms is absorbed in preexisting terms in the QHD lagrangian. The long-range physics arising from the exchange term must be explicitly calculated; these integrals are considered in the following section.

### B. Two-loop Integral Formulas

We first consider the integral with the propagator from Eq. (16). Therefore, one can write

\[
E_{\Phi-EX}^{(2)} = -\frac{g_5^2}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} A_{00}(k-q) \text{tr}[G_D^*(k)G_D^*(q)]
\]

\[
= \frac{\gamma g_5^2}{32\pi^4} \int_{k_0}^{k_F} \frac{|k|^2d|k|}{E^*(k)} \int_{q_0}^{q_F} \frac{|q|^2d|q|}{E^*(q)} \int_{-1}^{1} d(cos \theta)
\]

\[
\times \left[ E^*(k)E^*(q) - |k||q| \cos \theta + M^*2 \right]
\]

\[
\times \left[ 2E^*(k)E^*(q) - 2|k||q| \cos \theta - 2M^*2 + M_V^*2 \right]
\]

\[
\times \left\{ \left[ 2E^*(k)E^*(q) - 2|k||q| \cos \theta - 2M^*2 + M_V^*2 + \frac{\zeta_0}{3}g_V^2V_0^2 \right] M_V^*2
\]

\[
- \frac{\zeta_0}{3}g_V^2V_0^2 [E^*(k) - E^*(q)]^2 \right\} / B ,
\]

where

\[
B \equiv \left[ 2E^*(k)E^*(q) - 2|k||q| \cos \theta - 2M^*2 + M_S^*2 \right]
\]

\[
\times \left[ 2E^*(k)E^*(q) - 2|k||q| \cos \theta - 2M^*2 + M_V^*2 \right]
\]

\[
\times \left( \left[ 2E^*(k)E^*(q) - 2|k||q| \cos \theta - 2M^*2 + M_V^*2 + \frac{\zeta_0}{3}g_V^2V_0^2 \right] M_V^*2
\]

\[
- \frac{\zeta_0}{3}g_V^2V_0^2 [E^*(k) - E^*(q)]^2 \right)
\]

\[
+ \eta^2V_0^2 \left( \left[ 2E^*(k)E^*(q) - 2|k||q| \cos \theta - 2M^*2 + M_V^*2 \right] M_V^*2
\]

\[
- \left[ 2E^*(k)E^*(q) - 2|k||q| \cos \theta - 2M^*2 + M_V^*2 + \frac{\zeta_0}{3}g_V^2V_0^2 \right]
\]

\[
\times \left[ E^*(k) - E^*(q) \right]^2 \right) .
\]

Now we consider the integral with the propagator from Eq. (19); as a result, we get

\[
E_{V-EX}^{(2)} = \frac{g_V^2}{2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} A_{\mu\nu}(k-q) \text{tr}[\gamma_\mu G_D^*(k)\gamma_\nu G_D^*(q)]
\]
III. DISCUSSION

In the previous section, we showed that the short- and long-range dynamics of the two-loop contributions can be separated even in the presence of various nonlinearities in the isoscalar meson fields. The short-range physics can be expressed as a series of terms that are already present in the underlying lagrangian; thus, they are implicitly contained in the undetermined coefficients. The long-range exchange contributions must be calculated explicitly. The question remains what effect these new contributions have on the naturalness assumption.

Naturalness [7, 8, 17–20] states that once all the dimensional and combinatorial factors in a given term of the lagrangian are removed, what is left is a dimensionless constant of order unity. To show that naturalness is not violated by the inclusion of the two-loop exchange integrals, we will show that sets of natural parameters exist that adequately reproduce the empirical properties of symmetric nuclear matter at equilibrium [13].

We conduct parameter fits at different levels of truncation in the underlying lagrangian, as in [7], but here we include the two-loop exchange contributions. Note that the two-loop portion of the energy density contains no new parameters. In this work, we fit the parameters to bulk properties of nuclear matter: $E/\rho_B - M = -16.10$ MeV, $k_F = 1.30$
TABLE I: Parameter sets used in this work compared with one-loop parameter sets at the same level of truncation.

|                | Q2 [7] | M1A   | C1 [7] | M2A   | M3A   | M4A   |
|----------------|--------|--------|--------|--------|--------|--------|
| $m_S/M$        | 0.54268| 0.5400 | 0.53874| 0.5400 | 0.5400 | 0.5400 |
| $m_V/M$        | 0.83280| 0.8328 | 0.8328 | 0.8328 | 0.8328 | 0.8328 |
| $g_S/4\pi$     | 0.78661| 0.80762| 0.77756| 0.91847| 0.84973| 0.83407|
| $g_V/4\pi$     | 0.97202| 0.93485| 0.98486| 1.15837| 0.98992| 0.96368|
| $\eta_1$      | —      | —      | 0.29577| 1.9207 | 0.45972| 0.25290|
| $\eta_2$      | —      | —      | —      | —      | —      | 0.19238|
| $\kappa_3$    | 1.7424 | 1.2072 | 1.6698 | 4.3207 | 2.6308 | 2.1978 |
| $\kappa_4$    | -8.4836| -1.8824| —      | —      | -5.6463| -5.3909|
| $\zeta_0$     | -1.7750| 9.9968 | —      | —      | 8.9212 | 9.0313 |

fm$^{-1}$, $M^*/M = 0.610$, and compressibility $K = 250 \pm 50$ MeV. A least-squares fit was conducted to these four pieces of data with respective weights of 0.0015, 0.002, 0.0015, and 0.08. The fitting was performed using the downhill simplex method, with nuclear equilibrium maintained throughout the procedure. Table I lists the results of this fitting for a number of different levels of truncation in the lagrangian (M1A, M2A, M3A, and M4A)—defined below—along with some corresponding parameter sets determined at the one-loop level with the same truncation (Q2 and C1 [7]) for comparison. We hold the meson masses fixed in the new sets ($m_S/M = 0.5400$ and $m_V/M = 0.8328$), since there are only four empirical input data; they accurately reproduce the reduced meson mass in [49]. We let $M = 939$ MeV.

- The mean field lagrangian for the M1A (and Q2) set includes cubic and quartic scalar field self-couplings, as well as the quartic vector field self-coupling. In this case there is no mixing in the meson propagators ($\eta = 0$); as a result, there are only three two-loop exchange integrals ($A_{0\mu}$ drops out).

- The M2A (and C1) set includes a cubic scalar field self-coupling in addition to the third-order mixing term. This case contains mixing in the meson propagators through the inclusion of the $\eta_1$ term; therefore, there are four exchange integrals to calculate.

- Two other mixing sets were constructed (M3A and M4A); however, no comparable mean field set exists in the literature. These sets have the same terms as M1A plus $\eta_1 \neq 0$ and $\eta_1, \eta_2 \neq 0$, respectively. Both sets give rise to four two-loop integrals as in the M2A case.

In the set M1A, the value of $\zeta_0$ was allowed to vary; however, $\zeta_0$ places a strong constraint on the compressibility $K$. One must also ensure that the vector field equation still has an appropriate solution at high density. Confining $\zeta_0$ to positive values is sufficient to reproduce the correct high-density behavior. One can see from Table I that parameter sets exist at the two-loop level in which all the parameters are natural. In all cases the properties of nuclear matter were reproduced accurately while maintaining naturalness. The bulk nuclear properties for the various sets are shown in Table II; note that all the observables in this table except $g_V V_0$ were used in the fit.
TABLE II: Nuclear matter properties for the parameter sets in Table I. The values of $\mathcal{E}/\rho_B - M$, $g_V V_0$, and $K$ are in MeV. The Fermi wavenumber is in fm$^{-1}$ and $M^*/M$ is dimensionless.

|                | Q2 [7] | M1A | C1 [7] | M2A | M3A | M4A |
|----------------|--------|-----|--------|-----|-----|-----|
| $\mathcal{E}/\rho_B - M$ | -16.13 | -16.10 | -16.19 | -16.11 | -16.10 | -16.10 |
| $k_F$         | 1.303  | 1.300 | 1.293  | 1.306 | 1.300 | 1.300 |
| $M^*/M$      | 0.614  | 0.610 | 0.532  | 0.611 | 0.610 | 0.610 |
| $g_V V_0$    | 292    | 234  | 255    | 239  | 231  | 230  |
| $K$           | 279    | 270  | 304    | 288  | 235  | 247  |

TABLE III: Size of two-loop integrals for the parameter sets in Table I. Values are in MeV.

|                | Q2 [7] | M1A | C1 [7] | M2A | M3A | M4A |
|----------------|--------|-----|--------|-----|-----|-----|
| $\mathcal{E}^{(2)}_{\phi - EX}$ | 38.77  | 33.51 | 26.73  | 21.26 | 31.39 | 32.99 |
| $\mathcal{E}^{(2)}_{V - EX}$    | -28.32 | -15.49 | -23.47 | -18.50 | -15.85 | -15.67 |
| $\mathcal{E}^{(2)}_{M - EX}$    | —      | —     | 3.54   | 11.99 | 4.24  | 3.30  |
| $\mathcal{E}^{(2)}_{\pi - EX}$  | 12.87  | 12.76 | 12.73  | 12.79 | 12.71 | 12.68 |

The size of the various two-loop integrals is shown in Table III. We also show the size of these integrals when the constants are taken from the Q2 and C1 sets. Note that there is some cancellation between the scalar and vector contributions.

Next, we compare the magnitude of the two-loop integrals to the mean-field terms. In Fig. 1, the mean field terms for the Q2 and M1A sets are plotted order by order in the power counting, with the two-loop integrals calculated using M1A. The crosses in Fig. 1 represent the expected magnitude per order of terms in the meson sector using the rules of naive dimensional analysis (NDA) [7], with the chiral symmetry breaking scale of $\Lambda = 650$ MeV. Fig. 4 compares the saturation curve of M1A at the two-loop level with the saturation curve of Q2 at the one-loop level and the two-loop binding curve of Q2. Apparently, the net effect of the exchange terms is repulsive.

In Fig. 2, the mean field terms for the C1 and M2A sets are plotted order by order in the power counting, with the two-loop integrals calculated using M2A. A comparison of the saturation curve of M2A at the two-loop level with the saturation curve of C1 at the one-loop level and the two-loop binding curve of C1 is shown in Fig. 5.

Lastly, the terms (mean field and two-loop) for both M3A and M4A are plotted in Fig. 3. Fig. 6 is a graph of the saturation curves for both M3A and M4A at the two-loop level.

One can see from Figs. 1, 2, and 3 that the magnitudes of the mean field terms are consistent with the NDA estimates to within a factor of two, which supports the naturalness of the coupling parameters.\(^2\) Moreover, the magnitude of the two-loop terms is shown

\(^2\) The determination of naturalness is a tricky business, and the relevant observation here is that the hierarchy of successive terms in Figs. 1, 2, and 3 is maintained with the new parameter sets. A more precise discussion of the naturalness would require an examination of the linearly independent combinations of parameters, as in [13].
FIG. 1: Comparison of the magnitudes of the mean field terms in the meson sector with the two-loop exchange integrals. The inverted triangles represent, from top to bottom, the scalar, vector, and pion two-loop integrals. The abscissa denotes the order $\nu$ in the power counting at the mean field level [7].

to be roughly equal to the third order ($\nu = 3$) in the power counting in the mean field energy. While this is not large, they cannot be neglected in a description of nuclear matter properties, particularly in view of the nearly complete cancellation of scalar and vector terms at order $\nu = 2$. Nevertheless, when the parameters are properly renormalized to reproduce empirical equilibrium properties, exchange corrections modify the energy only slightly, and the two-loop binding curves for nuclear matter are similar to the one-loop results.

Exchange interactions are two-body, so they go like $O(\rho^2)$ in $\mathcal{E}$. This is the same dependence as the two-body mean field terms at $O(\nu = 2)$. Because of the lack of a spin–isospin sum in the exchange terms, the exchange terms are numerically smaller and contribute at the same level as the $O(\nu = 3)$ terms. A deeper understanding of how loops fit into the finite-density power counting scheme will require a look at higher-order terms in the loop expansion, which is currently under investigation [50].

Exchange contributions from the the vector meson two-loop diagrams with tensor couplings at one or both vertices were also considered. The analytic expressions for these integrals are given in the Appendix, and the numerical results are shown in Table IV. The vector–tensor cross terms (XT) are roughly equal to fourth order ($\nu = 4$) in the power counting, and the tensor–tensor terms (TT) are negligible. This is not surprising considering that the tensor coupling on a vertex introduces an additional factor of $\nabla/M \approx 1/4$ (see [7] for a description of the power counting). A value of $f_V = 0.5$ was used here, which is comparable to values obtained from fits to properties of finite nuclei [7]. The tensor terms are small enough that their effect on the naturalness of the parameters is negligible.

In conclusion, we have shown that the results of [31] are still valid when various nonlinearities in the isoscalar meson fields are included in the lagrangian. The short- and long-range physics at the two-loop level in the loop expansion can be separated. The short-range (local) dynamics is parametrized by the undetermined coefficients of the lagrangian, like those in
FIG. 2: Comparison of the magnitudes of the mean field terms in the meson sector with the two-loop exchange integrals. The inverted triangles represent, from top to bottom, the scalar, vector, pion, and mixed two-loop integrals. The abscissa denotes the order $\nu$ in the power counting at the mean field level [7].

| $\mathcal{E}_{TT-EX}^{(2)}$ | Q2 [7] | M1A | C1 [7] | M2A |
|-------------------------------|--------|-----|--------|------|
| $\mathcal{E}_{XT-EX}^{(2)}$   | -0.0241| -0.0189| -0.0210| -0.0195|
|                              | -3.94  | -2.95| -3.43  | -3.12|

TABLE IV: Size of two-loop integrals involving the tensor coupling to the vector meson for the parameter sets in Table I. Values are in MeV.

the nonlinear meson self-interactions, and is either removed by field redefinitions or fitted to empirical data. The long-range (nonlocal) physics must be calculated explicitly. The assumption of naturalness is shown to hold in the presence of the two-loop exchange integrals. The size of the two-loop contributions is roughly equal to third order in the mean field power counting. The inclusion of tensor contributions was also considered and found to be small.

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FIG. 3: Comparison of the magnitudes of the mean field terms in the meson sector with the two-loop exchange integrals. The inverted triangles represent, from top to bottom, the scalar, vector, pion, and mixed two-loop integrals. The abscissa denotes the order $\nu$ in the power counting at the mean field level [7].

FIG. 4: Comparison of the nuclear binding curves for the set Q2 (both at the one-loop and two-loop level) with the set M1A (two-loop level).
FIG. 5: Comparison of the nuclear binding curves for the set C1 (both at the one-loop and two-loop level) with the set M2A (two-loop level).

FIG. 6: Comparison of the saturation curves for the sets M3A and M4A (both at the two-loop level).

APPENDIX A: TENSOR PORTION OF TWO-LOOP CALCULATION

In this section, we present the corrections to the theory arising from the tensor coupling to the neutral vector meson at the two-loop level, where

\[ \mathcal{L}_T = \frac{f_V g_N}{4M} \bar{\psi} \sigma_{\mu\nu} V_{\mu\nu} \psi. \]  

(A1)
After working out the variational derivatives, we acquire the energy density. Now we can replace the Hartree propagator by $G_H \rightarrow G^* = G_F^* + G_D^*$, as before. The Lamb-shift and vacuum fluctuation terms are absorbed by the same process as before. (The only difference here is more gamma matrices and they don’t influence the result.) Now we are left with only the exchange terms

\[
\mathcal{E}_{TT-EX}^{(2)} = \frac{f_V^2 g_V^2}{8M^2} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} A_{\nu\rho}(k-q) \\
\times \text{tr} \left[ \sigma_{\mu\nu}(k-q) \mu G_D^*(k) \sigma_{\rho\sigma}(k-q) \rho G_D^*(q) \right] \\
= -\frac{f_V^2 g_V^2}{64\pi^4 M^2} \int_0^{k_F} \frac{|k|^2 d|k|}{E^*(k)} \int_0^{k_F} \frac{|q|^2 d|q|}{E^*(q)} \int_{-1}^1 d(\cos \theta) \\
\times \left[ 2E^*(k) E^*(q) - 2|k||q| \cos \theta - 2M^{*2} + M_S^2 \right] \\
\times \left\{ 2 \left[ 2E^*(k) E^*(q) - 2|k||q| \cos \theta - 2M^{*2} + M_V^2 + \frac{\zeta_0}{2} g_V^2 V_0^2 \right] \right\} \\
\times \left[ M_V^2 + \frac{\zeta_0}{6} g_V^2 V_0^2 \right] \\
\times \left[ E^*(k) E^*(q) - |k||q| \cos \theta - M^{*2} \right] \\
+ \frac{\zeta_0}{3} g_V^2 V_0^2 \left[ M_V^2 + \frac{\zeta_0}{6} g_V^2 V_0^2 \right] \\
\times \left( 2 \left[ E^*(k) E^*(q) - |k||q| \cos \theta - M^{*2} \right] - M^{*2} \left[ E^*(k) - E^*(q) \right]^2 \right) \\
- \eta^2 V_0^2 \left[ 2 \left[ \left( E^*(k) - E^*(q) \right)^2 - \left( M_V^2 + \frac{\zeta_0}{6} g_V^2 V_0^2 \right) \right] \right] \\
\times \left[ E^*(k) E^*(q) - |k||q| \cos \theta - M^{*2} \right] \\
+ \left( M_V^2 + \frac{\zeta_0}{6} g_V^2 V_0^2 \right) \left\{ 2 \left[ E^*(k) E^*(q) - |k||q| \cos \theta - M^{*2} \right] \\
\times \left[ E^*(k) E^*(q) - M^{*2} \right] - M^{*2} \left[ E^*(k) - E^*(q) \right]^2 \right\} \right\} / B ,
\]

\[
\mathcal{E}_{XT-EX}^{(2)} = \frac{f_V^2 g_V^2 M^*}{2M} \int \frac{d^4k}{(2\pi)^4} \int \frac{d^4q}{(2\pi)^4} A_{\nu\rho}(k-q) \\
\times \text{tr} \left[ \gamma_{\mu} G_D^*(k) \sigma_{\nu\rho}(k-q) \rho G_D^*(q) \right] \\
= -\frac{f_V^2 g_V^2 M^*}{32\pi^4 M} \int_0^{k_F} \frac{|k|^2 d|k|}{E^*(k)} \int_0^{k_F} \frac{|q|^2 d|q|}{E^*(q)} \int_{-1}^1 d(\cos \theta) \\
\times \left[ 2E^*(k) E^*(q) - 2|k||q| \cos \theta - 2M^{*2} + M_S^2 \right] \\
\times \left\{ 8 \left[ 2E^*(k) E^*(q) - 2|k||q| \cos \theta - 2M^{*2} + M_V^2 + \frac{\zeta_0}{2} g_V^2 V_0^2 \right] \right\} \\
\times \left[ M_V^2 + \frac{\zeta_0}{6} g_V^2 V_0^2 \right] - \frac{\zeta_0}{3} g_V^2 V_0^2 \left[ E^*(k) - E^*(q) \right]^2 \right) \\
\times \left[ E^*(k) E^*(q) - |k||q| \cos \theta - M^{*2} \right] \\
\times \left[ E^*(k) E^*(q) - |k||q| \cos \theta - M^{*2} \right]
\]
\[
+ \frac{\zeta}{3} g_0^2 V_0^2 \left[ M^2 V + \frac{\zeta}{6} g_0^2 V_0^2 \right] \\
\times \left\{ \left[ 2E^*(k)E^*(q) - 2|k||q| \cos \theta - 2M^{*2} \right] + \left[ E^*(k) - E^*(q) \right]^2 \right\} \\
- \eta^2 V_0^2 \left[ 8 \left\{ \left[ E^*(k) - E^*(q) \right]^2 - \left( M^2 V + \frac{\zeta}{6} g_0^2 V_0^2 \right) \right\} \right] \\
\times \left[ E^*(k)E^*(q) - |k||q| \cos \theta - M^{*2} \right] \\
+ \left( M^2 V + \frac{\zeta}{6} g_0^2 V_0^2 \right) \left\{ 2 \left[ E^*(k)E^*(q) - |k||q| \cos \theta - M^{*2} \right] \\
+ \left[ E^*(k) - E^*(q) \right]^2 \right\} \right\} / B ,
\]

which were reduced in the same manner as in the previous discussion. Here \( B \) is defined in Eq. (44).

[1] P. Hohenberg and W. Kohn, Phys. Rev. 136B (1964) 864.
[2] W. Kohn and L. J. Sham, Phys. Rev. A140 (1965) 1133.
[3] W. Kohn, Rev. Mod. Phys. 71 (1999) 1253.
[4] R. M. Dreizler and E. K. U. Gross, Density Functional Theory, Springer, Berlin (1990).
[5] C. Speicher, R. M. Dreizler, and E. Engel, Ann. Phys. 213 (1992) 312.
[6] R. N. Schmid, E. Engel, and R. M. Dreizler, Phys. Rev. C 52 (1995) 164.
[7] R. J. Furnstahl, B. D. Serot, and H.-B. Tang, Nucl. Phys. A615 (1997) 441.
[8] R. J. Furnstahl, B. D. Serot, and H.-B. Tang, Nucl. Phys. A640 (1998) 505 (E).
[9] B. D. Serot and J. D. Walecka, Int. J. Mod. Phys. E 6 (1997) 515.
[10] J. D. Walecka, Theoretical Nuclear and Subnuclear Physics, second ed., World Scientific, Singapore (2004).
[11] C. O. Almbladh and U. von Barth, Phys. Rev. B 31 (1985) 3231.
[12] J. D. Walecka, Ann. Phys. 83 (1974) 491.
[13] R. J. Furnstahl and B. D. Serot, Nucl. Phys. A671 (2000) 447.
[14] R. J. Furnstahl, Int. J. Mod. Phys. B 17 (2003) 5111.
[15] B. D. Serot, in: Lecture Notes Phys. 641, G. A. Lalazissis, P. Ring, and D. Vretenar, eds., Springer, Berlin, Heidelberg (2004), p. 31.
[16] R. J. Furnstahl, in: Lecture Notes Phys. 641, G. A. Lalazissis, P. Ring, and D. Vretenar, eds., Springer, Berlin, Heidelberg (2004), p. 1.
[17] A. Manohar and H. Georgi, Nucl. Phys. B234 (1984) 189.
[18] H. Georgi, Phys. Lett. B298 (1993) 187.
[19] J. L. Friar, D. G. Madland, and B. W. Lynn, Phys. Rev. C 53 (1996) 3085.
[20] J. L. Friar, Few Body Syst. 22 (1997) 161.
[21] J. McIntire, Phys. Rev. C 66 (2002) 064319.
[22] J. McIntire, Acta Phys. Pol. B 35 (2004) 2261.
[23] J. McIntire, arXiv:nuc-th/0507006.
[24] M. A. Huertas, Phys. Rev. C 66 (2002) 024318.
[25] M. A. Huertas, Phys. Rev. C 67 (2003) 019901 (E).
[26] M. A. Huertas, Acta Phys. Pol. B 34 (2003) 4269.
[27] M. A. Huertas, Acta Phys. Pol. B 35 (2004) 837.
[28] R. J. Furnstahl, R. J. Perry, and B. D. Serot, Phys. Rev. C 40 (1989) 321.
[29] K. Wehrberger, R. Wittman, and B. D. Serot, Phys. Rev. C 42 (1990) 2680.
[30] B. D. Serot and H.-B. Tang, Phys. Rev. C 51 (1995) 969.
[31] Y. Hu, J. McIntire, and B. D. Serot, arXiv:0706.1547 [nucl-th].
[32] P. A. M. Dirac, Phys. Z. Sowjetunion 3 (1933) 64.
[33] C. Itzykson and J.-B. Zuber, Quantum Field Theory, McGraw–Hill, New York (1980).
[34] B. D. Serot and J. D. Walecka, Adv. Nucl. Phys. 16 (1986) 1.
[35] S. Coleman and E. Weinberg, Phys. Rev. D 7 (1973) 1888.
[36] J. Iliopoulos, C. Itzykson, and A. Martin, Rev. Mod. Phys. 47 (1975) 165.
[37] S. Coleman, New Phenomena in Subnuclear Physics, Plenum, New York (1977).
[38] H.-B. Tang, arXiv:hep-ph/9607436.
[39] P. J. Ellis and H.-B. Tang, Phys. Rev. C 57 (1998) 3356.
[40] T. Becher and H. Leutwyler, Eur. Phys. J. C 9 (1999) 643.
[41] T. Becher, arXiv:hep-ph/0011079.
[42] S. Scherer, Adv. Nucl. Phys. 27 (2003) 277.
[43] T. Fuchs, M. R. Schindler, J. Gegelia, and S. Scherer, Phys. Lett. B575 (2003) 11.
[44] M. R. Schindler, J. Gegelia, and S. Scherer, Phys. Lett. B586 (2004) 258.
[45] M. R. Schindler, J. Gegelia, and S. Scherer, Nucl. Phys. B682 (2004) 367.
[46] S. Scherer, Int. J. Mod. Phys. A21 (2006) 881.
[47] D. Djukanovic, J. Gegelia, and S. Scherer, Eur. Phys. J. A29 (2006) 337.
[48] S. A. Chin, Ann. Phys. 108 (1977) 301.
[49] R. J. Furnstahl, Nucl. Phys. A706 (2002) 85.
[50] J. McIntire, arXiv:0708.2860.