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CROSS-RATIO DEGREES AND PERFECT MATCHINGS

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(Communicated by Rachel Pries)

Abstract. Cross-ratio degrees count configurations of points $z_1, \ldots, z_n \in \mathbb{P}^1$ satisfying $n - 3$ cross-ratio constraints, up to isomorphism. These numbers arise in multiple contexts in algebraic and tropical geometry, and may be viewed as combinatorial invariants of certain hypergraphs. We prove an upper bound on cross-ratio degrees in terms of the theory of perfect matchings on bipartite graphs. We also discuss several of the many perspectives on cross-ratio degrees — including a connection to Gromov-Witten theory — and give many example computations.

1. Introduction

1.1. Cross-ratio degrees and main result. Recall that if $z_1, z_2, z_3, z_4$ are distinct points on the projective line $\mathbb{P}^1$, their cross-ratio is

$$\text{CR}(z_1, z_2, z_3, z_4) = \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} \in \mathbb{C} \setminus \{0, 1\},$$

extended as usual in case one of $z_1, z_2, z_3, z_4$ is $\infty$. This paper is concerned with counting configurations of points in $\mathbb{P}^1$ satisfying constraints on their cross-ratios, as follows. Let $n \geq 4$, and for each $i = 1, \ldots, n - 3$, fix a 4-tuple $e_i = (v_{i,1}, \ldots, v_{i,4})$ of distinct elements of $\{1, \ldots, n\}$. Write $T = (e_1, \ldots, e_{n-3})$. Then given distinct points $z_1, \ldots, z_n \in \mathbb{P}^1$, we obtain a tuple of cross-ratios:

$$\text{CR}_T(z_1, \ldots, z_n) = (\text{CR}(z_{v_1,1}, z_{v_1,2}, z_{v_1,3}, z_{v_1,4}),$$

$$\text{CR}(z_{v_2,1}, z_{v_2,2}, z_{v_2,3}, z_{v_2,4}),$$

$$\ldots,$$

$$\text{CR}(z_{v_{n-3,1}}, z_{v_{n-3,2}}, z_{v_{n-3,3}}, z_{v_{n-3,4}}) \in (\mathbb{C} \setminus \{0, 1\})^{n-3}.$$  

Recall that $M_{0,n}$ is the $(n - 3)$-dimensional variety that parametrizes $n$-tuples $(z_1, \ldots, z_n) \in \mathbb{P}^1$ of distinct points, up to Möbius transformation. As cross-ratios are invariant under Möbius transformations, $\text{CR}_T$ defines a map $M_{0,n} \to (\mathbb{C} \setminus \{0, 1\})^{n-3}$. Since $\dim(M_{0,n}) = \dim((\mathbb{C} \setminus \{0, 1\})^{n-3}) = n - 3$, $\text{CR}_T$ has a well-defined degree $d_T$, which we call the cross-ratio degree of $T$, a nonnegative integer. Our main result, Theorem [□] just below, gives an upper bound for $d_T$ in terms of matching theory on bipartite graphs.

Permuting the tuple $(z_1, \ldots, z_n)$, the four elements of $e_i$, and the sets $e_i$ all correspond to pre- or post-composing $\text{CR}_T$ by automorphisms, which does not affect $d_T$. Thus $d_T$ depends only on the isomorphism class of $T = (V, E)$ as a hypergraph.

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with vertices $V$ and hyperedges $E$, where $|V| = n$, $|E| = n - 3$, and $|e| = 4$ for all $e \in E$. (We say $T$ is a 4-uniform hypergraph.)

An alternative interpretation of cross-ratio degrees is that they collectively comprise the multidegree of the natural embedding $i : \overline{M}_{0,n} \to \binom{\mathbb{P}^1}{4}$ recording all cross-ratios, where $\overline{M}_{0,n}$ is the usual Deligne-Mumford-Knudsen compactification. Precisely,

$$i_*([\overline{M}_{0,n}]) = \sum_T d_T \prod_e H_e \in H_{n-3}(\mathbb{P}^1, \mathbb{Q}),$$

where $T = (V, E)$ ranges over hypergraphs of the above form, $e$ ranges over 4-element subsets of $\{1, \ldots, n\}$ not in $E$, and $H_e$ is the hyperplane class on the $e$-th copy of $\mathbb{P}^1$.

Cross-ratio degrees may be computed individually in many ways, see Section 1.2 just below. Perhaps the most notable of these is a recursion due to Goldner (1921, see Section 1.2.4), from which any cross-ratio degree can be recovered from the base case $d_{\{1,2,3,4\}} = 1$. It is desirable, and more challenging, to compute $d_T$ in a way that indicates a clear combinatorial relationship with the structure of the hypergraph $T$.

Recall that a hypergraph $T = (V, E)$ is exactly characterized by its bipartite incidence graph $G_T = (V \cup E, I)$, i.e. the bipartite graph whose vertex set is $V \cup E$, and whose edge set is the incidence set $I = \{(v, e) \in V \times E : v \in e\}$. Recall that a perfect matching in a bipartite graph $G = (A \cup B, I)$ is a bijection $f : A \to B$ with $(a, f(a)) \in I$ for all $a \in A$. We write $P(G)$ for the number of perfect matchings in $G$.

**Theorem 1.1.** Let $T$ be a 4-uniform hypergraph with $n$ vertices and $n - 3$ hyperedges. Then for any three distinct vertices $\{v_1, v_2, v_3\} \subset V$, the cross-ratio degree of $T$ satisfies

$$d_T \leq P(G_T - \{v_1, v_2, v_3\}),$$

where $G_T - \{v_1, v_2, v_3\}$ is the bipartite graph obtained by deleting $v_1, v_2, v_3$.

Note that both partite sets of $G_T - \{v_1, v_2, v_3\}$ have cardinality $n - 3$, hence this graph may admit perfect matchings. Note that $P(G_T - \{v_1, v_2, v_3\})$ is also equal to the number of systems of distinct representatives in the hypergraph $T - \{v_1, v_2, v_3\}$, as well as the permanent of the square matrix obtained by deleting columns $v_1, v_2, v_3$ from the biadjacency matrix of $G_T$.

To prove Theorem 1.1 we first reformulate $d_T$ as a curve-counting invariant. For each $v \in V$, fix a general linear subvariety $Y_v \subseteq (\mathbb{P}^1)^{n-3}$ of type $E_v$ (see Section 2).

**Theorem 2.1.** Exactly $d_T$ rational curves $C \subseteq (\mathbb{P}^1)^{n-3}$ of multidegree $(1, \ldots, 1)$ intersect $Y_v$ for all $v \in V$.

In Section 3 we translate this curve-counting problem into a problem of intersecting incidence subvarieties of a moduli space $M$, and compute the “answer” via a calculation in the cohomology ring of a compactification $\overline{M} \cong (\mathbb{P}^3)^{n-4}$ of $M$. This “answer” is an upper bound only, because the boundary $\overline{M} \setminus M$ may contribute nontrivially (Example 3.6).

Proving Theorem 1.1 requires three steps. First, for each incidence subvariety $Z_v \subseteq M$, we compute the class of its Zariski closure $[Z_v] \subseteq \overline{M}$, by a fairly direct

1In fact, $d_T$ is a Gromov-Witten invariant, see Corollary 2.2 and Section 1.2.6 below.
calculation in the cohomology ring of $(\mathbb{P}^3)^{n-4}$ (Proposition 3.1). Second, we use Kleiman’s Bertini theorem [Kle74] to prove that $d_T$ is bounded above by the intersection product $\prod_v[Z_v]$ (Claims 3.3). Third, we show combinatorially that $\prod_v[Z_v]$ is equal to a count of perfect matchings (Claim 3.4).

We discuss some combinatorial implications of Theorem 1.1 in Section 4; in particular, Proposition 4.1 gives a connection to the classical notion of graph surplus in matching theory.

Remark 1.2. The upper bound in Theorem 1.1 appears to often give a good estimate for $d_T$. Figure 1 shows histograms of

$$\delta_T := d_T - \min_{v_1, v_2, v_3 \in V} P(G_T - \{v_1, v_2, v_3\}),$$

as $T$ ranges over several thousand randomly-generated 4-uniform hypergraphs such that $\min_{v_1, v_2, v_3 \in V} P(G_T - \{v_1, v_2, v_3\}) > 0$, where $n = 10$ and $n = 15$. Note that when $n = 10$ the upper bound is equal to $d_T$ around $\approx 75\%$ of the time (and $\approx 32\%$ when $n = 15$). In these examples, the average value of $d_T$ was $\approx 1.9$ when $n = 10$, and $\approx 7.3$ when $n = 15$.

While many combinatorial aspects of $M_{0,n}$ and $\overline{M}_{0,n}$ are well-studied (e.g., the characterization of the cohomology groups of $\overline{M}_{0,n}$ in terms of marked trees [Kee92, KM94]), this is to our knowledge the first appearance of the rich combinatorial theory of matchings. From the perspective of Gromov-Witten theory (see Section 1.2.6 just below), the result is fairly unusual in that proving upper bounds for classes of Gromov-Witten invariants is difficult in general. In the reverse direction, Theorem 1.1 motivates studying cross-ratio degrees from a combinatorial perspective as interesting hypergraph invariants.

1.2. Alternate methods of computing $d_T$. We now discuss several of the many ways of computing cross-ratio degrees. As these are not our main focus, we do not include every detail.

1.2.1. Coordinate computation. In small cases, the cross-ratio constraint equations may be solved directly:

Example 1.3. We now describe the simplest hypergraph $T$ with $d_T \neq 0, 1$. Let $T = (V,E)$ with $V = \{1, \ldots, 6\}$ and $E = \{\{1,2,3,4\}, \{1,2,5,6\}, \{3,4,5,6\}\}$, i.e.
$G_T$ has biadjacency matrix
\[
\begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 1
\end{pmatrix}.
\]

This will be a running example in this paper; we will show $d_T = 2$ in six different ways in this section! First, we calculate $d_T$ directly by counting 6-marked lines $(\mathbb{P}^1, p_1, \ldots, p_6)$ satisfying
\[
\text{CR}(p_1, p_2, p_3, p_4) = a_1 \quad \text{CR}(p_1, p_2, p_5, p_6) = a_2 \quad \text{CR}(p_3, p_4, p_5, p_6) = a_3,
\]
where $a_1, a_2, a_3$ are generically chosen scalars. We choose coordinates on $\mathbb{P}^1$ so that $p_1 = \infty$, $p_2 = 0$, $p_3 = 1$. Then the equations above simplify to
\[
p_4 = a_1 \quad \quad \frac{p_6}{p_5} = a_2 \quad \quad \frac{(p_5 - 1)(p_6 - p_4)}{(p_6 - 1)(p_5 - p_4)} = a_3.
\]
There are two solutions $(p_4, p_5, p_6) = (a_1, b, a_2 b)$, where $b$ is a solution to the nondegenerate quadratic equation
\[
a_2(1 - a_3) b^2 + (a_1 a_2 a_3 - a_1 - a_2 + a_3) b + a_1(1 - a_3) = 0.
\]
Thus $d_T = 2$.

1.2.2. **Geometric computation via Theorem 2.1** Theorem 2.1 says that $d_T$ is equal to the number of multidegree-$1$ curves in $(\mathbb{P}^1)^3$ satisfying $n$ incidence conditions derived from $T$, where $n = |V|$. There are various ways to compute these curve counts in individual cases, e.g. we now see that in our running example, we may compute $d_T$ by carefully specializing the incidence conditions:

**Example 1.4.** Let $T$ be the hypergraph of Example 1.3 By Theorem 2.1 $d_T$ is equal to the number of $(1, 1, 1)$-curves $C \subseteq (\mathbb{P}^1)^3$ that intersect

- two general lines $Y_1, Y_2$ of the form $(a, b, *)$,
- two general lines $Y_3, Y_4$ of the form $(a, *, b)$, and
- two general lines $Y_5, Y_6$ of the form $(*, a, b)$.

(Here e.g. $(a, *, b)$ means the first and third coordinates are fixed and the second coordinate is unconstrained.) We solve this intersection problem by specializing the incidence conditions, omitting some straightforward details. We will need to allow singular $(1, 1, 1)$-curves (connected, as always). We choose
\[
\begin{align*}
Y_1 &= (0, 0, *) & Y_3 &= (0, *, 0) & Y_5 &= (*, \infty, 0) \\
Y_2 &= (\infty, 0, *) & Y_4 &= (\infty, *, \infty) & Y_6 &= (*, \infty, \infty).
\end{align*}
\]

Suppose $C$ is a $(1, 1, 1)$-curve passing through $Y_1, \ldots, Y_6$. As $C$ intersects $Y_1$ and $Y_2$, which both lie in the plane $(*, 0, *)$, $C$ must have an irreducible component $C_1$ contained in this plane. Similarly, as $C$ intersects $Y_5$ and $Y_6$, $C$ must have an irreducible component $C_2$ contained in the plane $(*, \infty, *)$. This implies that we have multidegrees $\text{deg}(C_1) = (1, 0, 0)$ and $\text{deg}(C_2) = (0, 1, 0)$, and that $C$ has a third irreducible component $C_3$ with degree $(0, 0, 1)$. The component $C_3$ is uniquely determined by $C_1$ and $C_2$ (and connectedness of $C$). There are exactly two curves of this form that pass through $Y_3$ and $Y_4$, given by the choices
\[
(C_1, C_2) = ((*, 0, 0), (\infty, \infty, *)) \quad \text{and} \quad (C_1, C_2) = ((*, 0, \infty), (0, \infty, *)).
\]

One can (and must) confirm that both curves contribute to the intersection with multiplicity 1, giving $d_T = 2$. 


1.2.3. Tropical coordinate computation. One may compute $d_T$ via tropical geometry, by tropicalizing $\text{CR}_T$. Recall that $M_{0,n}^{\text{trop}}$ is the polyhedral complex parametrizing finite metric trees with $n$ marked (infinite-length) half-edges, where each vertex has valence at least 3. The map $\text{CR}_T$ has a tropicalization $\text{CR}_T^{\text{trop}}: M_{0,n}^{\text{trop}} \to (M_{0,4}^{\text{trop}})^{n-3}$, and it follows from the algebraic-tropical correspondence theorem of Tyomkin [Tyo17, Thm. 5.1] that this map is “generically finite of degree $d_T$” in the sense that over a dense open subset of $(M_{0,4}^{\text{trop}})^{n-3}$, fibers consist of $d_T$ points counted with multiplicity. Explicitly, we have the following.

Example 1.5. Let $\mathcal{T}$ be the hypergraph of Example 1.3. Consider the cone in $(M_{0,4}^{\text{trop}})^3$ consisting of triples of tropical curves of types

$$\left( \begin{array}{ccc} 1 & 3 & 1 \\ 2 & 4 & 2 \\ 5 & 3 & 5 \\ 6 & 4 & 6 \end{array} \right)$$

This cone is isomorphic to $(\mathbb{R}_{\geq 0})^3$ by recording the edge lengths, and its link is a filled triangle. One may compute that the preimage of this triangle under $\text{CR}_T^{\text{trop}}$ is as depicted in Figure 2. Some fibers contain $d_T = 2$ points, while others contain a single point. The latter points map with multiplicity 2, as the associated piecewise-linear map is locally given by $(x,y,z) \mapsto (x+y, y+z, x+z)$, whose determinant is 2. (The real numbers $(x,y,z)$ are the edge lengths of the metric tree.)

1.2.4. Goldner’s algorithm. Goldner’s algorithm [Gol21, Lem. 3.11], which we mentioned above and now describe, gives a recursion for $d_T$ via tropical geometry. Choose $e = \{v_1, v_2, v_3, v_4\} \in E$, and choose two elements of $e$, say $v_1, v_2$ without loss of generality. Given $V' \subseteq V$ such that $v_1, v_2 \in V'$, $v_3, v_4 \notin V'$, and for all $e' \in E \setminus \{e\}$, $|V' \cap e'| \neq 2$, we may create two new hypergraphs $\mathcal{T}', \mathcal{T}''$ as follows. The vertices of $\mathcal{T}'$ are $V' \cup \{v'\}$ and the edges are $\{e' \in E : V' \cap e' \geq 3\}$, where if $v \in (V \setminus V') \cap e'$, we replace $v$ with $v'$ in $e'$. The vertices of $\mathcal{T}''$ are $(V \setminus V') \cup \{v''\}$, and the edges are $\{e' \in E : (V \setminus V') \cap e' \geq 3\}$, where if $v \in V' \cap e'$, we replace $v$ with $v''$ in $e'$. Then by [Gol21] Lem. 3.11, we have

$$d_T = \sum_{V'} d_{\mathcal{T}'} d_{\mathcal{T}''}$$
where $V'$ runs over subsets of $V$ as above. Note that $T'$ and $T''$ have strictly fewer edges than $T$, as both edge sets are naturally subsets $E \setminus \{e\}$. This implies that any cross-ratio degree can be computed recursively from (1), using the trivial base case $d_T = 1$ when $|V| = \{1, 2, 3\}$ and $E = \emptyset$ — or alternatively the less-trivial base case $d_{\{1,2,3,4\}} = 1$. The algorithm can be interpreted as calculating a fiber of the map $\text{CR}^T_{T'}$ in Section 1.2.3 over a carefully chosen point where all preimage points map with multiplicity 1.

Goldner’s recursion, together with other tropical computational techniques, gives an algorithm for computing genus-zero tropical plane curve counts with cross-ratio constraints; these counts had been the subject of the correspondence theorem of Tyomkin [Tyo17] mentioned in Section 1.2.3. Relatedly but separately, Goldner [Gol20b] generalized Kontsevich’s recursive algorithm [KM94] for rational plane curve counts to allow multiple cross-ratio constraints; cross-ratio degrees appear in this recursion as initial values. (See [Gol20a] for details.)

**Example 1.6.** This example also appears in [Gol21, Ex. 3.10]. Let $T$ be the hypergraph of Example 1.3. Let $(v_1, v_2, v_3, v_4) = \{1, 2, 3, 4\}$. There are two choices for $V'$, namely $\{1, 2, 5\}$ or $\{1, 2, 6\}$. Taking $V' = \{1, 2, 5\}$ gives new hypergraphs $T'$ with vertices $\{1, 2, 5, v'\}$ and the single edge $\{1, 2, 5, v'\}$, and $T''$ with vertices $\{3, 4, 6, v''\}$, and the single edge $\{3, 4, 6, v''\}$. We have $d_{T'} = d_{T''} = 1$ by the base case, giving a contribution of 1 to $d_T$. The other choice $V' = \{1, 2, 6\}$ similarly contributes 1, i.e. $d_T = 1 + 1 = 2$.

### 1.2.5. Intersection theory on $\overline{M}_{0,n}$

A natural intersection theoretic approach to calculating $d_T$ is as follows. Recall that $M_{0,n}$ has a compactification $\overline{M}_{0,n}$ [Knu83], the space of $n$-marked stable rational curves, whose cohomology is relatively well-understood. The cross-ratio degree $d_T$ is equal to the degree of the pullback of the class $\prod_{e \in E} H_e$ of a point in $\mathbb{P}^1)^{n-3}$ under $\text{CR}_T$. That is,

$$d_T = \deg \left( \prod_{e \in E} \text{CR}_T^e(H_e) \right).$$

At first glance, this approach appears similar to the one we use to prove Theorem 1.1. However, the intersection problems are not related in a straightforward way; e.g. here there are $n - 3$ subvarieties being intersected rather than $n$ subvarieties.

Recall [Kee92] that $H^*(\overline{M}_{0,n}, \mathbb{Q})$ is generated as a $\mathbb{Q}$-algebra by the classes of boundary divisors $D_S$ with $S \subseteq \{1, \ldots, n\}$ and $2 \leq |S| \leq n - 2$, with additive relations as characterized in [KM94]. We may express $\text{CR}_T^e(H_e)$ in terms of boundary divisors; e.g. if $e = \{1, 2, 3, 4\}$, then $\text{CR}_T^e(H_e)$ can be written as the sum of all boundary divisors $D_S$ with $1, 2 \in S$ and $3, 4 \notin S$. One may thus express $d_T$ as a truly enormous sum of monomials in boundary divisors. While the class of each monomial can be computed rather straightforwardly, the combinatorics of working with so many terms — which can turn out to be negative — appears prohibitively difficult; computing a single cross-ratio degree this way is feasible, but gaining insight into how $d_T$ depends on the structure of $T$ is probably not.
As \(n\) The five terms of this sum can be evaluated using \([EHKR10, Lem. 5.4]\) to give 
\[ \psi \in \mathbb{Q}_m \] 
the 64 terms of this sum can be evaluated by a routine calculation expressing 
products of boundary divisors in terms of boundary strata and \(\psi\) classes \([Koc01]\); 
50 terms are zero, 8 are equal to 1 (times the class of a point), and 6 are equal to 
\(-1\), giving \(d_T = 2\).

A marginally more promising approach is to use a different generating set for 
\(H^*(\overline{M}_{0,n}, \mathbb{Q})\) introduced in \([Sin04, EHKR10]\), consisting of a divisor \(\Pi_S\) for each 
\(S \subseteq \{2, \ldots, n\}\) with \(|S| \geq 3\). This gives a simpler expression:
\[
\text{CR}_T^+(H_e) = \begin{cases} 
\prod_{e \notin \{1\}} & 1 \in e \\
(\sum_{e \in \{1\}} \prod_{e \notin \{1\}}) - 2\Pi_e & 1 \notin e.
\end{cases}
\]

Again, it is straightforward to evaluate a degree-\((n - 3)\) monomial in the divisors 
\(\Pi_S\) via additive relations, see e.g. \([EHKR10, Lem. 5.4]\). However, the existence of negative signs in the factors of \(\prod_{e \in \{1\}} \text{CR}_T^+(H_e)\) makes it difficult to study how \(d_T\) 
depends combinatorially on \(T\).

Example 1.8. The product in \([1.7]\) is rewritten as 
\[ \Pi_{\{2,3,4\}} \cdot \Pi_{\{2,5,6\}} \cdot (\Pi_{\{3,4,5\}} + \Pi_{\{3,4,6\}} + \Pi_{\{3,5,6\}} + \Pi_{\{4,5,6\}} - 2\Pi_{\{3,4,5,6\}}). \]
The five terms of this sum can be evaluated using \([EHKR10, Lem. 5.4]\) to give 
\(d_T = 1 + 1 + 1 + 1 - 2 = 2\).

1.2.6. Gromov-Witten theory. We show (Corollary \[2.2\]) which combines Theorem 
\[2.1\] with a result of \([FP97]\) that \(d_T\) is a Gromov-Witten invariant of \((\mathbb{P}^1)^{n-3}\), 
a type of curve-counting invariant known for its relationship to mirror symmetry. 
As \((\mathbb{P}^1)^{n-3}\) is toric, \(d_T\) may be computed via well-known techniques such as torus 
localization \([Kon95]\) or mirror theorems \([Giv98]\). Again, however, it appears difficult 
to find a relationship between \(d_T\) and the combinatorial structure of \(T\) via these 
methods.

1.2.7. Castravet-Tevelev’s hypertree projections. Castravet-Tevelev prove 
\([CT13, Thm. 3.2]\) that \(d_T \leq 1\) in the special case where \(\cap_{e \in E} e \neq \emptyset\). In this case, 
they show that \(d_T = 1\) if and only if \(G_T\) has surplus 3; see Section \[4\] and Question 
\[1.2\] in particular. We note that \([CT13, Thm. 3.2]\) can also be recovered via the 
intersection product calculation in Section \[1.2.5\] using the generators \(\{\Pi_S\}\); in the 
case \(\cap_{e \in E} e \neq \emptyset\), the product in question has a single term, and \([EHKR10, Lem. 
5.4]\) implies that the term is equal to 0 or 1, according to whether or not \(G_T\) has 
surplus 3.

2. Reformulation as a curve-counting problem

In this section we prove Theorem \[2.1\] which recasts \(d_T\) as a curve-counting 
invariant. Let \(S\) be a set with \(|S| = k \geq 1\). Let \(\mathcal{M}_{P^1,S}\) (or \(\mathcal{M}_{P^1,k}\)) denote the moduli 
space of smooth connected curves \([P^1]_S\) of multidegree \((1, \ldots, 1)\), more commonly

\[\text{Such a curve is necessarily rational, as any projection } C \to (\mathbb{P}^1)^S \to P^1 \text{ has degree 1.}\]
denoted by $M_{0,0}((\mathbb{P}^1)^S, (1, \ldots, 1))$. Then $\mathcal{M}_{\mathbb{P}^1, S}$ is a smooth quasiprojective variety of dimension $3(k - 1)$ \cite{FP97}.

Suppose $S' \subseteq S$ is nonempty. The projection of a $(1, \ldots, 1)$-curve along

$$\pi_{S'} : (\mathbb{P}^1)^S \to (\mathbb{P}^1)^{S'}$$

is a $(1, \ldots, 1)$-curve, yielding a natural map $\mathcal{M}_{\mathbb{P}^1, S} \to \mathcal{M}_{\mathbb{P}^1, S'}$. Given $P \in (\mathbb{P}^1)^{S'}$, we define $Y(S', P) = \pi_{S'}^{-1}(P)$. We call a subvariety of this form a \textit{linear subvariety} of $(\mathbb{P}^1)^S$ of type $S'$. For any $S' \subseteq S$, the group $\text{PGL}(2)^S$ acts transitively on linear subvarieties of $(\mathbb{P}^1)^S$ of type $S'$. Note that if $C \in \mathcal{M}_{\mathbb{P}^1, S}$ intersects a proper linear subvariety $Y(S', P)$, then $C \cap Y(S', P)$ is a single reduced point; this follows from the fact that $Y(S', P)$ is contained in a codimension-1 linear subvariety, whose intersection number with $C$ is 1. We denote by $Z(S', P) \subseteq \mathcal{M}_{\mathbb{P}^1, S}$ the \textit{incidence subvariety} $Z(S', P) = \{ C \in \mathcal{M}_{\mathbb{P}^1, S} : C \cap Y(S', P) \neq \emptyset \}$.

Let $\mathcal{T} = (V, E)$ be a 4-uniform hypergraph with vertex set $V = \{1, \ldots, n\}$ and edge set $E$, where $|E| = n - 3$. Let $I = \{(v, e) \in V \times E : v \in e\}$ denote the incidence set, and for $v \in V$, let $E_v = \{e \in E : v \in e\}$.

**Theorem 2.1.** For each $v \in V$, fix a general linear subvariety $Y_v \subseteq (\mathbb{P}^1)^E$ of type $E_v$, i.e. $Y_v = Y(E_v, P_v)$ for a general $P_v \in (\mathbb{P}^1)^E_v$. Then there are exactly $d_T$ curves $C \in \mathcal{M}_{\mathbb{P}^1, E}$ such that $C \cap Y_v \neq \emptyset$ for all $v \in V$. Equivalently, the corresponding incidence subvarieties $Z_v = Z(E_v, P_v) \subseteq \mathcal{M}_{\mathbb{P}^1, E}$ intersect in exactly $d_T$ points.

**Proof.** For $A$ a finite set, let $\mathcal{M}_{\mathbb{P}^1, S, A}$ denote the moduli space of smooth connected curves $C \subseteq (\mathbb{P}^1)^S$ of multidegree $(1, \ldots, 1)$ together with an injection $\iota : A \to C$. This is a smooth quasiprojective variety of dimension $3(|S| - 1) + |A|$, see \cite{FP97}.

There is a natural map $\text{pr}_1 : \mathcal{M}_{\mathbb{P}^1, S, A} \to \mathcal{M}_{\mathbb{P}^1, S}$ that forgets $A$, and when $|A| \geq 3$ there is a natural map $\text{pr}_2 : \mathcal{M}_{\mathbb{P}^1, S, A} \to M_{0, A} = M_{0, |A|}$ that forgets the map $C \to (\mathbb{P}^1)^S$.

We have a commutative diagram:

$$\begin{array}{ccc}
\mathcal{M}_{\mathbb{P}^1, E} & \xrightarrow{\text{pr}_1} & \mathcal{M}_{\mathbb{P}^1, E, V} \\
\downarrow \text{pr}_2 & & \downarrow \text{ev} \\
M_{0, |V|} & \xrightarrow{\text{CR}_T} & (\mathbb{P}^1 \setminus \{\infty, 0, 1\})^E \setminus \Delta_3
\end{array}
$$

where

- $\text{ev} : \mathcal{M}_{\mathbb{P}^1, E, V} \to (\mathbb{P}^1)^V \times E$ is the evaluation map

$$\text{ev}(C \xleftarrow{\varphi}(\mathbb{P}^1)^E, \iota : V \to C) = (\varphi(\iota(v)))_{v \in V},$$

- $\Delta_1 \subseteq (\mathbb{P}^1)^V \times E$ is the set

$$\{(z_{(v, e)})_{(v, e) \in V \times E} : z_{(v, e)} = z_{(v', e)} \text{ for some } e, v \neq v'\},$$

- $\Delta_2 \subseteq (\mathbb{P}^1)^I$ is the set

$$\{(z_{(v, e)})_{(v, e) \in I} : z_{(v, e)} = z_{(v', e)} \text{ for some } e, v \neq v'\},$$

and

- $\Delta_3 \subseteq (\mathbb{P}^1 \setminus \{\infty, 0, 1\})^E$ is the large diagonal

$$\{(z_e)_{e \in E} : z_{e_1} = z_{e_2} \text{ for some } e_1 \neq e_2\}.$$
We claim the rectangle in (2) is Cartesian. Indeed commutativity gives a natural map $F: \mathcal{M}_{P^1,E,V} \to M_{0,V} \times_{(P^1)^{\Delta_2}} (P^1)^{\Delta_2}$. We now define a map $G: M_{0,V} \times_{(P^1)^{\Delta_2}} (P^1)^{\Delta_2} \to \mathcal{M}_{P^1,E,V}$ that commutes with $pr_2$ and $\pi_1 \circ ev$, and is thus guaranteed to be an inverse of $F$, as follows. Let $(C, t: V \to C) \in M_{0,V}$ and $(z_{v,e})_{v,e} \in (P^1)^{\Delta_2}$ be points (over a base scheme $S$) such that for all $e = (v_1, v_2, v_3, v_4) \in E$ with $v_1 < v_2 < v_3 < v_4$,

$$CR(z_{v_1,e}, \ldots, z_{v_4,e}) = CR(t(v_1), \ldots, t(v_4)).$$

We define $G((C, t), (z_{v,e})_{v,e} \in I)$ to be the marked curve $(C, t)$, together with the embedding $f = (f_v)_{v \in E}: C \to (P^1)^E$ uniquely determined by

$$(f_v(t(v_1)), f_v(t(v_2)), f_v(t(v_3))) = (z_{v_1,e}, z_{v_2,e}, z_{v_3,e}).$$

The fact that $G$ commutes with $pr_2$ is automatic, and the fact that $G$ commutes with $\pi_1 \circ ev$ follows from $f_v(t(v_4)) = z_{v_4,e}$, which is guaranteed by the assumption (3). Thus (2) is Cartesian, so $\pi_1 \circ ev$ is generically finite of degree $d_T$ by surjectivity of $CR^E$.

The set of curves $C \in \mathcal{M}_{P^1,E}$ that intersect $Y_v$ for all $v \in V$ is precisely

$$pr_1((\pi_1 \circ ev)^{-1}((P_{v,e})_{v \in V})).$$

(Recall $(P_{v,e})_{v \in V}$ is a general point of $(P^1)^E$.) On the other hand, given

$C \in pr_1((\pi_1 \circ ev)^{-1}((P_{v,e})_{v \in V})),

$since $Y_v$ is a linear subvariety of $(P^1)^E$, $C$ intersects $Y_v$ in a single reduced point. Thus $C$ determines a unique point of $\mathcal{M}_{P^1,E,V}$, so there are exactly $d_T$ such curves. □

Corollary 2.2. The cross-ratio degree $d_T$ coincides with the Gromov-Witten invariant.

$$\langle \{Y_v\}_{v \in V}\rangle_{P^1,0,n,(1,1)} = \int_{\overline{\mathcal{M}}_{0,n}((P^1)^E, (1,1,1))} (\pi_1 \circ ev)^*(\{pt\}).$$

Proof. This is immediate from Theorem 2.1 and [FP97] Lem. 14. □

3. Proof of Theorem 1.1

By Corollary 2.2, $d_T$ can be expressed as an intersection number on the Kontsevich compactification $\overline{\mathcal{M}}_{0,n}((P^1)^{n-3}, (1, \ldots, 1))$ of $\mathcal{M}_{P^1,E,V}$. We will prove Theorem 1.1 by computing an intersection number in a similar spirit, but rather on a very simple compactification of $\mathcal{M}_{P^1,E}$ whose cohomology groups are much smaller than those of $\overline{\mathcal{M}}_{0,n}((P^1)^{n-3}, (1, \ldots, 1))$.

3.1. Cohomology classes of incidence subvarieties. For $e \in E$, we define an isomorphism $\rho_e: \mathcal{M}_{P^1,E} \to \text{PGL}(2)^{E \setminus \{e\}}$ as follows. Fix $C \in \mathcal{M}_{P^1,E}$. The projection onto the $e$-th coordinate of $(P^1)^E$ defines an isomorphism $\nu_e: C \to P^1$. For $e' \in E$, $\nu_{e'} \circ \nu_e^{-1}: P^1 \to P^1$ is an automorphism, i.e. $\nu_{e'} \circ \nu_e^{-1} \in \text{PGL}(2)$. Let

$$\rho_e(C) = (\nu_{e'} \circ \nu_e^{-1})_{e' \in E \setminus \{e\}} \in \text{PGL}(2)^{E \setminus \{e\}}.$$  

The map $\rho_e$ has a clear inverse; given $(A_{e'})_{e' \in E \setminus \{e\}} \in \text{PGL}(2)^{E \setminus \{e\}}$, one embeds $P^1$ into $(P^1)^{n-3}$ via the identity map in the $e$-th coordinate, and via $A_{e'}$ in the $e'$-th coordinate for $e' \neq e$. We have an open immersion $\text{PGL}(2) \to P^3$, hence for each $e \in E$, we obtain a compactification $\overline{\mathcal{M}}: \mathcal{M}_{P^1,E} \hookrightarrow (P^1)^{E \setminus \{e\}}$. We now calculate
the class of the Zariski closure of each incidence subvariety \(Z_v\) in the cohomology of \((\mathbb{P}^3)^{E\setminus\{e\}}\).

**Proposition 3.1.** Fix \(v \in V\) and \(e \in E\). For \(P \in (\mathbb{P}^1)^{E_v}\), let \(Z_v = Z(E_v, P) \subseteq \mathcal{M}_{\mathbb{P}^1,E}\) be the incidence subvariety as in Section 3. Let \(\overline{Z}_v\) denote the Zariski closure of \(Z_v\) under the embedding \(\mathbb{P}^*\). For \(e' \in E \setminus \{e\}\), let \(H_{e'}\) denote the pullback of the hyperplane class on \(\mathbb{P}^3\) along the \(e'\)-th projection. Then

\[
[\overline{Z}_v] = \begin{cases} \prod_{e' \in E_v \setminus \{e\}} H_{e'} & v \in e \\ \mathbf{e}_{|E_v| - 1}((\{H_{e'} : e' \in E_v\}) & v \not\in e, \end{cases}
\]

where \(\mathbf{e}_{|E_v| - 1}\) is the elementary symmetric polynomial of degree \(|E_v| - 1\).

**Proof.** Note that the natural \(\text{PGL}(2)^E\)-action on \((\mathbb{P}^1)^E\) induces an action on \(\mathcal{M}_{\mathbb{P}^1,E}\), and that this action extends to \((\mathbb{P}^3)^{E\setminus\{e\}}\). Explicitly, \((A_{e'})_{\nu \in E}\) sends

\[
(B_{e''})_{\nu'' \in E_v \setminus \{e''\}} \to (A_{e''} B_{e''} A_{e''}^{-1})_{\nu'' \in E_v \setminus \{e''\}}.
\]

As observed above, the action is transitive on linear subvarieties, and hence we may assume without loss of generality that \(P = P_0 = (0, \ldots, 0) \in (\mathbb{P}^1)^{E_v}\).

Suppose \(v \in e\). Recall that \(Z(E_v, P_0)\) consists of curves \(C \subseteq (\mathbb{P}^1)^E\) passing through \(\pi_{E_v}^{-1}(P_0)\). This incidence condition on \(C\) is equivalent to \(\nu_{e'} \circ \nu^{-1}(0) = 0 \in \mathbb{P}^1\) for \(e' \ni v\), where \(\nu_e : C \to \mathbb{P}^1\) is the isomorphism defined above. In other words,

\[
\rho_e(Z(E_v, P_0)) = \{(B_{e''})_{\nu'' \in E_v \setminus \{e''\}} : B_{e''} \cdot [0 : 1] = [0 : 1]\text{ if } v \in e'\} \subseteq \text{PGL}(2)^{E\setminus\{e\}}.
\]

That is, the upper right entry of the matrix \(B_{e''}\) must be zero for \(e' \ni v\). The same condition holds after taking the Zariski closure, so \([\overline{Z}_v]\) is the product of hyperplane classes pulled back from \((\mathbb{P}^3)^{E_v \setminus \{e\}}\), as claimed.

Suppose \(v \not\in e\). Fix \(e'' \in E_v\). Consider \(\overline{Z}_v \cap J_{Id}\), where

\[
J_{A} = \{(B_{e''})_{\nu'' \in E_v \setminus \{e''\}} : B_{e''} = A\} \subseteq (\mathbb{P}^3)^{E\setminus\{e\}}.
\]

Note that \(J_{Id} \cong (\mathbb{P}^3)^{E\setminus\{e''\}}\), and \([J_{Id}] = H^3_{2, e''}\). Inside \(J_{Id}\), we have a natural identification of \(\overline{Z}_v \cap J_{Id}\) with \(\rho_{e''}(Z(E_v, P))\), where \(Z_{E_v \setminus \{e''\}}(E_v, P) \subseteq \mathcal{M}_{\mathbb{P}^1,E\setminus\{e''\}}\) is the incidence subvariety with ambient space \((\mathbb{P}^1)^{E\setminus\{e''\}}\). By the previous computation, inside \((\mathbb{P}^3)^{E\setminus\{e''\}}\), we have \([\overline{Z}_v \cap J_{Id}] = \prod_{e' \in E_v \setminus \{e''\}} H_{e'}\). We claim the intersection \(\overline{Z}_v \cap J_{Id}\) is transverse, so that in \((\mathbb{P}^3)^{E\setminus\{e\}}\) we have

\[
[\overline{Z}_v \cap J_{Id}] = H^3_{2, e''} \cdot \prod_{e' \in E_v \setminus \{e''\}} H_{e'}.
\]

To prove this, we must show that at any point of \(\overline{Z}_v \cap J_{Id}\), the tangent space to \(\overline{Z}_v\) spans the normal space to the fiber \(J_{Id}\), which is naturally identified with the tangent space to \(\mathbb{P}^3\) at \(Id\). To see this, note that since \(v \not\in e\), \(\overline{Z}_v\) is invariant under the action of \(\text{PGL}(2)\) on \((\mathbb{P}^3)^{E\setminus\{e\}}\) induced by the action on the \(e\)-th coordinate of \((\mathbb{P}^1)^E\). This action respects the projection from \((\mathbb{P}^3)^{E\setminus\{e\}}\) to the \(e''\)-th coordinate \(\mathbb{P}^3\), and its derivative acts transitively on the tangent space to \(\mathbb{P}^3\) at \(Id\); both follow from the explicit form of the action at the beginning of this proof. Thus at any point of \(\overline{Z}_v \cap J_{Id}\), the tangent space of \(\overline{Z}_v\) spans the normal space to \(J_{Id}\), i.e. the intersection is transverse.

The only degree-(\(|E_v| - 1\)) element

\[
\alpha \in H^*((\mathbb{P}^3)^{E\setminus\{e\}}) \cong \mathbb{Z}[(H_{e'})_{\nu \in E_v \setminus \{e''\}}]/(\{H_{e'}\}_{\nu \in E_v \setminus \{e''\}})
\]
satisfying $H^0_{V'} \alpha = H^0_{V'} : \prod_{e' \in E_v \setminus \{e''\}} H_{e'}$ for all $e'' \in E_v$ is $\alpha = e_{|E_v| - 1} \langle \{H_{e'} : e' \in E_v\} \rangle$, concluding the proof.

3.2. Proof of the upper bound. We will need the following, which is also one of the basic observations about the numbers $d_T$:

Lemma 3.2. Let $T = (V, E)$ be a 4-uniform hypergraph with $|V| = n = |E| + 3$, and fix $v_1, v_2, v_3 \in V$. Let $T' = (V', E')$ with $V' = V \cup \{n + 1\}$ and $E' = E \cup \{e'\}$, where $e' = \{v_1, v_2, v_3, n + 1\}$. Then $d_T = d_{T'}$.

Proof. Consider the diagram

$$
\begin{array}{ccc}
M_{0,V} & \xrightarrow{\text{CR}_T} & (\mathbb{P}^1)^{E'} \\
\mu_{n+1} \downarrow & & \downarrow \pi_E \\
M_{0,V} & \xrightarrow{\text{CR}_T} & (\mathbb{P}^1)^{E}
\end{array}
$$

where $\mu_{n+1}$ is the map that forgets the marked point $\mu(n + 1)$. This defines a natural map $F : M_{0,V'} \to M_{0,V} \times (\mathbb{P}^1)^{E'}$. We define a rational inverse $G : M_{0,V} \times (\mathbb{P}^1)^{E'} \to M_{0,V}$ by $G((C, i), P) = (C, i')$, where $i' : V' \to C$ is defined by $i'((v)) = i(v)$ for $v \in V$ and $i'(n + 1)$ is determined by $\text{CR}(\nu_{\{v_1\}}, \nu_{\{v_2\}}, \nu_{\{v_3\}}, \nu((v))) = \nu_{\{e'\}}(P) \in \mathbb{P}^1$.

Then $G$ is well-defined whenever $\nu_{\{e'\}}(P)$ avoids the finite set

$$
\{\text{CR}(\nu_{\{v_1\}}, \nu_{\{v_2\}}, \nu_{\{v_3\}}, \nu((v)))\}_{v \in V} \subseteq \mathbb{P}^1,
$$

hence is a rational map. Where defined, $G$ commutes with $\mu_{n+1}$ by definition, and commutes with $\text{CR}_T$ by the condition $\text{CR}_T(C,i) = \pi_E(P)$. Thus $G$ is a birational inverse to $F$. As $\text{CR}_T$ is generically finite of degree $d_T$, so is $\text{CR}_T \circ G$, hence so is $\text{CR}_T \circ G \circ F = \text{CR}_T$. That is, $d_T = d_{T'}$ as claimed.

Proof of Theorem 1.1. Let $T, V, E,$ and $I$ be as before, and fix $v_1, v_2, v_3 \in V$. By Lemma 3.2, we have $d_T = d_{T'}$, where $T'$ has vertices $V' = V \cup \{n + 1\}$ and hyperedges $E' = E \cup \{e'\}$ (and incidence set $I'$), with $e' = \{v_1, v_2, v_3, n + 1\}$. For each $v \in V'$, fix a general linear subvariety $Y_v \subseteq (\mathbb{P}^1)^{E'}$ of type $E_v$, let $Z_v \subseteq M_{\mathbb{P}^1, E'}$ be the associated incidence variety, and let $\overline{Z_v'}$ denote the Zariski closure of the image of $Z_v$ under the the embedding $\nu_{n+1} : M_{\mathbb{P}^1, E'} \to (\mathbb{P}^3)^{E'} \setminus \{e'\} = (\mathbb{P}^3)^E$.

Claim 3.3. We have $d_T \leq \deg \left( \prod_{v \in V'} \overline{Z_v'} \right)$.

Proof of Claim 3.3 Let $G = \text{PGL}(4)^E$, let

$$
\overline{Z_v'} := \{(g, g \cdot C) \in G \times (\mathbb{P}^3)^E : g \in \text{PGL}(4), C \in \overline{Z_v'}\}
$$

denote the universal translate of $\overline{Z_v'}$, let $Z = \bigcap_{v \in V'} \overline{Z_v'}$, and let $\epsilon : Z \to G$ denote the projection. As $G$ acts transitively on $(\mathbb{P}^3)^E$, Kleiman’s Bertini theorem [Kle74] implies $\epsilon$ is generically finite of degree $\deg \left( \prod_{v \in V'} \overline{Z_v'} \right)$.

Let $Z' \subseteq Z$ denote the Zariski closure of $\epsilon^{-1}(\eta)$, where $\eta \in G$ is the generic point. Thus $Z'$ consists of all components of $Z$ that map dominantly to $G$. Note that $\epsilon|_{Z'}$ is generically finite of degree $\deg \left( \prod_{v \in V'} \overline{Z_v'} \right)$. By [GD66 Propt. 15.5.3], for any $g \in G$, $\epsilon^{-1}(g) \cap Z'$ has at most $\deg \left( \prod_{v \in V'} \overline{Z_v'} \right)$ connected components.
Note that the fiber $\epsilon^{-1}((\text{Id})_{e \in E})$ is precisely the intersection $\bigcap_{e \in V'} Z^c_v$. In particular, Theorem 2.1 and the genericity of the linear subvarieties $Y_v$ imply that $M_{\epsilon^Z(E') \cap \epsilon^{-1}((\text{Id})_{e \in E})}$ is a finite set of exactly $d_T$ reduced points $Q_1, \ldots, Q_{d_T}$. Thus if we show $Q_1, \ldots, Q_{d_T} \subseteq Z'$, it will imply the Claim 3.3.

To see that $Q_1, \ldots, Q_{d_T} \subseteq Z'$, note that $Z$ is the intersection of subvarieties of total codimension

$$\sum_{e \in V'} (|E'_v| - 1) = |I'| - |V'| = 3 |E| = \dim((P^3)^E).$$

Thus every irreducible component of $Z$ has dimension at least $\dim(G)$, hence either maps dominantly to $G$, or maps to $G$ with positive relative dimension. If $Z_0$ is a component of the latter type, upper semicontinuity of fiber dimension [CD86, Thm. 13.1.3] implies that every fiber of $\epsilon|_{Z_0}$ has positive dimension (locally at any point). We therefore must have $Q_i \notin Z_0$ for all $i$, as $Q_i$ is an isolated point of its fiber. We conclude $Q_1, \ldots, Q_{d_T} \subseteq Z'$, and Claim 3.3 follows.

Theorem 1.1 now follows from:

Claim 3.4. We have $\deg \left( \prod_{e \in V'} Z^c_v \right) = P(G_T - \{v_1, v_2, v_3\})$.

Proof of Claim 3.4. We compute $\prod_{e \in E'} Z^c_v$ using Proposition 3.1 in the ring $\mathbb{Z}[H_e]_{e \in E}/\langle \{H^3_e\} \rangle$. As $\prod_{e \in V'} Z^c_v$ has codimension

$$\sum_{e \in V'} (|E'_v| - 1) = |I'| - |V'| = 3 |E|,$$

and every monomial in the expansion of $\prod_{e \in V'} Z^c_v$ appears with coefficient $1$ by Proposition 3.1 we must simply count occurrences of $\prod_{e \in E} H^3_e$.

The monomials in the expansion of $\prod_{e \in V'} Z^c_v$ are indexed by choices of a term of $e|_{E_v} \setminus \{H_e : e \in E'_v = E_v\}$ for each $v \in V \setminus e' = V \setminus \{v_1, v_2, v_3\}$. As a term of $e|_{E_v} \setminus \{H_e : e \in E'_v = E_v\}$ contains all but one hyperplane class (and is therefore determined by the missing hyperplane class), there is a bijection $\phi$ from the set of monomials in the expansion of $\prod_{e \in V'} Z^c_v$ to $\prod_{e \in V \setminus \{v_1, v_2, v_3\}} E_v$. As each $e \in E$ has cardinality $4$, such a monomial $m$ is equal to $\prod_{e \in E} H^3_e$ if and only if $\phi(m)$ contains each $e \in E$ exactly once, i.e. if and only if $\phi(m)$ determines a perfect matching $V \setminus \{v_1, v_2, v_3\} \to E$.

This completes the proof of Theorem 1.1. □

Example 3.5. As in Example 1.3, let $T$ be the hypergraph with $V = \{1, \ldots, 6\}$ and $E = \{e_1, e_2, e_3\}$, where $e_1 = \{1, 2, 3, 4\}$, $e_2 = \{1, 2, 5, 6\}$, and $e_3 = \{3, 4, 5, 6\}$. Deleting the vertices $\{1, 2, 3\}$ yields the hypergraph with incidence graph shown on the left in Figure 3 with its two matchings shown on the right. Explicitly, these two matchings arise from the proof of Theorem 1.1 as follows. Let $V' = V \cup \{7\}$ and $E' = E \cup \{e'\}$, where $e' = \{1, 2, 3, 7\}$. We are intersecting incidence subvarieties $Z^c_1, \ldots, Z^c_7 \subseteq (P^3)^E$, with classes

$$\begin{align*}
[Z^c_1] &= H_{e_1}, & [Z^c_2] &= H_{e_1} H_{e_2}, & [Z^c_3] &= H_{e_1} H_{e_3}, & [Z^c_4] &= H_{e_1}, & [Z^c_5] &= H_{e_2} + H_{e_3}, & [Z^c_6] &= 1.
\end{align*}$$

As in Example 1.3, let $T$ be the hypergraph with $V = \{1, \ldots, 6\}$ and $E = \{e_1, e_2, e_3\}$, where $e_1 = \{1, 2, 3, 4\}$, $e_2 = \{1, 2, 5, 6\}$, and $e_3 = \{3, 4, 5, 6\}$. Deleting the vertices $\{1, 2, 3\}$ yields the hypergraph with incidence graph shown on the left in Figure 3 with its two matchings shown on the right. Explicitly, these two matchings arise from the proof of Theorem 1.1 as follows. Let $V' = V \cup \{7\}$ and $E' = E \cup \{e'\}$, where $e' = \{1, 2, 3, 7\}$. We are intersecting incidence subvarieties $Z^c_1, \ldots, Z^c_7 \subseteq (P^3)^E$, with classes

$$\begin{align*}
[Z^c_1] &= H_{e_1}, & [Z^c_2] &= H_{e_1} H_{e_2}, & [Z^c_3] &= H_{e_1} H_{e_3}, & [Z^c_4] &= H_{e_1}, & [Z^c_5] &= H_{e_2} + H_{e_3}, & [Z^c_6] &= 1.
\end{align*}$$
The product is
\[ H_{e_1}^3 H_{e_2}^2 H_{e_3} (H_{e_1} + H_{e_3})(H_{e_2} + H_{e_3})^2 \]
\[ = H_{e_1}^3 H_{e_2}^4 H_{e_3} + 2H_{e_1}^3 H_{e_2}^3 H_{e_3}^2 + H_{e_1}^3 H_{e_2}^2 H_{e_3}^3 \]
\[ + H_{e_1}^4 H_{e_2}^3 H_{e_3} + 2H_{e_1}^3 H_{e_2}^3 H_{e_3}^3 + H_{e_1}^3 H_{e_2}^2 H_{e_3}^4. \]

The monomial \( H_{e_1}^3 H_{e_2}^3 H_{e_3} \) appears with coefficient 2, so \( d_T \leq 2 \) as we already knew. The left perfect matching

\[(\phi^{-1}(4), \phi^{-1}(5), \phi^{-1}(6)) = (e_1, e_2, e_3)\]

in Figure 3 corresponds to choosing the product of the terms \( H_{e_3} \) in \( [Z_4' \cup Z_5'] \), \( H_{e_2} \) in \( [Z_5'] \), and \( H_{e_3} \) in \( [Z_6'] \). The other matching corresponds to choosing the product of the terms \( H_{e_3} \) in \( [Z_4'^c] \), \( H_{e_2} \) in \( [Z_5'^c] \), and \( H_{e_3} \) in \( [Z_6'^c] \).

In fact, in this example the same bound arises from any choice of \( v_1, v_2, v_3 \).

Example 3.6. It is illuminating to see how the intersection calculation above can fail to give the exact answer for \( d_T \). This may happen even in small examples; we now describe the simplest case where it does happen. Let \( T \) be the hypergraph with \( \mathcal{V} = \{1, \ldots, 6\} \) and \( \mathcal{E} = \{\{1, 2, 3, 4\}, \{1, 2, 3, 4\}\} \). We certainly have \( d_T = 0 \) as the map \( CR_T : \mathcal{M}_{0,5} \to (\mathbb{C}^* \setminus \{0, 1\})^2 \) factors through the diagonal. On the other hand, \( G_T \setminus \{3, 4, 5\} \) is the complete bipartite graph \( K_{2,2} \), which has two perfect matchings.

To see what is going on, we modify the hypergraph as in the proof of Theorem 1.1, letting \( T' \) have vertices \( \mathcal{V}' = \mathcal{V} \cup \{6\} \) and hyperedges \( \mathcal{E}' = \mathcal{E} \cup \{e'\} \), where \( e' = \{3, 4, 5, 6\} \). Then \( \mathcal{M}_{\mathcal{P}1, \mathcal{E}'} \) is the space of multidegree-(1, 1, 1) curves in \( (\mathbb{P}^1)^3 \), and \( d_{T'} = 0 \) is the number of such curves passing through two general points \( P_1, P_2 \) and two vertical lines \( L_1, L_2 \). (One may see this directly by considering the projection onto the first two factors, where the constraints give a codimension-4 condition on the 3-dimensional moduli space \( \mathcal{M}_{\mathcal{P}1, \mathcal{E}'} \)). Without loss of generality we may choose \( P_1 = (0, 0, 0) \) and \( P_2 = (\infty, \infty, \infty) \). Under the embedding \( \rho_{e'} : \mathcal{M}_{\mathcal{P}1, \mathcal{E}'} \to (\mathbb{P}^3)^E \), the curves passing through \( P_1 \) and \( P_2 \) map to the matrices

\[
\begin{pmatrix}
a & 0 \\
0 & b \\
0 & c \\
0 & d
\end{pmatrix}
\]

The conditions that such a curve pass through \( L_1 \) and \( L_2 \) are respectively of the forms \( ad + \alpha_1 bc = 0 \) and \( ad + \alpha_2 bc = 0 \), where \( \alpha_1 \neq \alpha_2 \). Together these imply \( bc = 0 \) and \( ad = 0 \), giving two intersection points in \( (\mathbb{P}^3)^E \setminus \mathcal{M}_{\mathcal{P}1, \mathcal{E}'} \):

\[
\begin{pmatrix}
1 & 0 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\text{ and }
\begin{pmatrix}
0 & 0 \\
0 & 1 \\
0 & 0
\end{pmatrix}
\]

These two points contribute to the intersection product, but not to \( d_T \). Note that the incidence subvarieties may be deformed in \( (\mathbb{P}^3)^E \) (e.g. via the \( \text{PGL}(4)^E \) action).
so that the intersection lies inside $\mathcal{M}_{P^1,E}$; however, the moved subvarieties will no longer be incidence subvarieties for this enumerative problem.

Remark 3.7. It would be interesting to know whether Theorem 1.1 can be proved directly from [Gol21].

4. Some analysis of the upper bound

Recall that for a bipartite graph $G = (A \cup B, I)$, the neighborhood of a subset $A' \subseteq A$ is

$$N(A') = N_G(A') = \{b \in B : (a, b) \in I \text{ for some } a \in A'\},$$

and the surplus [LP09] Sec. 1.3] of $G$ (with respect to $A$) is

$$\sigma(G) := \min_{A' \subseteq A \atop A' \neq \emptyset} (|N(A')| - |A'|).$$

Note that $\sigma(G_T) \leq 3$ as $T$ is 4-uniform. Also, recall Hall’s Theorem on matchings, which states that $G$ has a perfect matching if and only if $\sigma(G) \geq 0$.

**Proposition 4.1.** The upper bound $\min_{v_1, v_2, v_3} P(G_T - \{v_1, v_2, v_3\})$ is nonzero if and only if $\sigma(G_T) = 3$.

**Proof.** Suppose $\sigma(G_T) < 3$. Then there exists a nonempty subset $E' \subseteq E$ such that $|N(E')| < |E'| + 3$. Let $v_1, v_2, v_3 \in N(E')$ be distinct. Then in $G_T - \{v_1, v_2, v_3\}$, we have $|N(E')| < |E'|$, so by (the easy direction of) Hall’s theorem,

$$P(G_T - \{v_1, v_2, v_3\}) = 0.$$

Now suppose $\sigma(G_T) = 3$, and fix $v_1, v_2, v_3 \in V$. For any $E' \subseteq E$, we have $N(E') \geq |E'| + 3$ in $G_T$, so $N(E') \geq |E'|$ in $G_T - \{v_1, v_2, v_3\}$. By Hall’s Theorem,

$$P(G_T - \{v_1, v_2, v_3\}) > 0. \quad \square$$

**Question 4.2.** Proposition 4.1 implies that if $\sigma(G_T) < 3$, then $d_T = 0$. Does the converse hold? That is, if $\sigma(G_T) = 3$, is $d_T$ necessarily nonzero? Experimentally, this appears true. The special case where $\bigcap_{e \in E} \not \in \emptyset$ follows from [CT13], see Section 12.2.

**Remark 4.3.** One obtains a very simple upper bound for $P(G_T - \{v_1, v_2, v_3\})$, and thus for $d_T$, via the Brègman-Minc inequality [Br73], which in this case says

$$P(G_T - \{v_1, v_2, v_3\}) \leq \prod_{e \in E} (|e \setminus \{v_1, v_2, v_3\}|!!)^{1/|e \setminus \{v_1, v_2, v_3\}|}.$$  

For example, in Example 3, the bound (5) is $(1!)^{1/1} \cdot (2!)^{1/2} \cdot (3!)^{1/3} \approx 2.5698$. This is sharp in this case in the sense that $d_T = 2$, but (5) appears to be very far from sharp in general. Note also that (5) and $|e \setminus \{v_1, v_2, v_3\}| \leq 4$ imply the (very loose) uniform bound $d_T \leq 24(n-3)/4 = (2.2133\ldots)^{n-3}$. In fact, the approach via intersection theory on $\overline{M}_{0,n}$ mentioned in Section 1.2.5 can be used to slightly strengthen this uniform bound to $d_T \leq 2^{n-5}$ for $n \geq 5$.

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