EXISTENCE AND SPECTRAL THEORY FOR WEAK SOLUTIONS OF NEUMANN AND DIRICHLET PROBLEMS FOR LINEAR DEGENERATE ELLIPTIC OPERATORS WITH ROUGH COEFFICIENTS

D. D. Monticelli and S. Rodney

Abstract. In this paper we study existence and spectral properties for weak solutions of Neumann and Dirichlet problems associated to second order linear degenerate elliptic partial differential operators $X$, with rough coefficients of the form

$$X = -\text{div}(P\nabla) + HR + S'G + F$$

in a geometric homogeneous space setting where the $n \times n$ matrix function $P = P(x)$ is allowed to degenerate. We give a maximum principle for weak solutions of $Xu \leq 0$ and follow this with a result describing a relationship between compact projection of the degenerate Sobolev space $QH^{1,p}$ into $L^q$ and a Poincaré inequality with gain adapted to $Q$.

1. Introduction

This paper studies existence of weak solutions and spectral properties for Dirichlet and Neumann problems on bounded domains of $\mathbb{R}^n$ for linear second order degenerate elliptic partial differential operators with rough measurable coefficients of the form

$$(1.1) \quad X = -\text{div}(P\nabla) + HR + S'G + F,$$

where $P = P(x)$ is a nonnegative definite symmetric measurable matrix function, $H, G, F$ are vector valued functions and $R, S$ are collections of first order subunit vector fields. We refer the reader to section 1.6 for a precise description of the constituents of $X$.

Before continuing we briefly describe how the paper is organized. Our results are developed in a general axiomatic framework similar to those used in [CRW, Section 3], [SW1, SW2], [MRW], that we outlined in Sections 1.1 through 1.6. This axiomatic setting includes the definition of geometric homogeneous space, of degenerate Sobolev spaces associated to a nonnegative definite symmetric measurable matrix function comparable to $P$ and also gives Poincaré–Sobolev type inequalities. Existence of weak solutions and spectral properties associated to the Neumann problem for $X$ (referred to as the $X$-Neumann problem) are studied in Section 2 and the main results are Theorems 2.11, 2.12, Corollary 2.17, and Theorem 2.20. We also mention that we develop a helpful example (see Example 2.21) in full detail for the reader’s convenience. We give a spectral result for the Dirichlet problem for $X$ (referred to as the $X$-Dirichlet problem) in Section 3, see Theorem 3.2, that compliments the existence results of [R3]. Section 4 contains a maximum principle for weak solutions of $Xu \leq 0$ and in Section 5 we demonstrate a relationship between compact embeddings of Sobolev spaces and global Poincaré inequalities with gain; we refer the reader to Theorems 4.3 and 5.1 for these results. All of our results are developed in the spirit of [E] and [GT] using ideas presented in [CRW, MI1, MP1, M2, MRW, R1, R2, R3, SW1, SW2], and other related works.

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2Cape Breton University, Sydney, NS Canada. Partially supported by The Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grants program.
1. Geometric Homogeneous Spaces: Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ and $\rho$ a symmetric quasi-metric defined in $\Omega$. More precisely, $\rho : \Omega \times \Omega \to [0, \infty)$ and there is a $\kappa \geq 1$ so that for every $x, y, z \in \Omega$ each of the following are satisfied:

- $\rho(x, y) = 0 \iff x = y$,
- $\rho(x, y) = \rho(y, x)$, and
- $\rho(x, y) \leq \kappa [\rho(x, z) + \rho(z, y)]$.

Given $x \in \Omega$ and $r > 0$ we define the sets

$$B(x, r) = \{ y \in \Omega : \rho(x, y) < r \}, \quad D(x, r) = \{ y \in \Omega : |x - y| < r \}.$$

We always refer to $B(x, r)$ as the quasimetric ball centered at $x$ with radius $r$. $D(x, r)$ is the corresponding Euclidean ball with the same center and radius.

**Definition 1.1.** We say that the collection of quasimetric balls $\mathcal{B} = \{ B(x, r) \}_{r>0, x \in \Omega}$ is locally geometrically doubling if given any compact $K \subset \Omega$ there is a $\delta = \delta(K) > 0$ so that for all $x \in K$ and $0 < r' \leq r < \delta$, the $\rho$-ball $B(x, r)$ may contain at most $C = C(r, r')$ disjoint $\rho$-balls of radius $r'$ where $C$ is independent of both $x$ and $K$.

It is important to note that the notion of local geometric doubling is weaker than local doubling for Lebesgue measure on the collection of quasimetric balls centered in $\Omega$. Recall that a measure $\mu$ is locally doubling for the collection $\mathcal{B}$ if given a compact set $K \subset \Omega$ there is a $\delta_1 = \delta_1(K)$ so that for any $x \in K$ and $0 < r < \delta_1$ we have $|B(x, 2r)|_\mu \leq C|B(x, r)|_\mu$ where $C > 0$ is independent of $x, r, K$. Helpful discussions on geometric doubling and doubling conditions are found in [HyK], and [HyM]. We now define the topological space on which we build our results.

**Definition 1.2.** The pair $(\Omega, \rho)$ is called a geometric homogeneous space if $\rho$ is a symmetric quasi-metric defined in $\Omega$ for which the three following properties hold:

- For all $x \in \Omega$ and $r > 0$ the quasimetric $\rho$-ball $B(x, r)$ is an open set with respect to Euclidean topology.
- For all $x, y \in \Omega$, $|x - y| \to 0$ if $\rho(x, y) \to 0$.
- The collection of $\rho$-balls $\mathcal{B} = \{ B(x, r) \}_{x \in \Omega, r > 0}$ is locally geometrically doubling.

**Remark 1.3.**

1. The converse of (1.3) holds automatically since $\rho$-balls are open sets with respect to Euclidean topology.
2. In many situations it is convenient to work with $\rho$-balls that do not intersect the boundary of $\Omega$ and is the reasoning behind condition (1.3). To see this notice that (1.3) may be restated as: given $x \in \Omega$ and $\epsilon > 0$ there is a positive $\delta'' = \delta''(x, \epsilon)$ so that $B(x, \delta) \subset D(x, \epsilon)$. From this, given $x \in \Omega$ there is a $t_0 > 0$ so that $B(x, r) \subset \Omega$ for all $0 < r < t_0$.

1.2. Functional Spaces. Fix an open $\Theta \subset \Omega$, and a non-negative definite $n \times n$ measurable matrix $Q(x)$ with $|Q(x)| \in L^\infty_0(\Theta)$. The degenerate Sobolev space $W^{1,2}_Q(\Theta)$ is the collection of pairs $(f, \mathbf{\tilde{g}})$ obtained via isomorphism from the space $\mathcal{Q}H^1(\Theta)$ defined as the completion, with respect to the norm

$$||u||_{\mathcal{Q}H^1(\Theta)} = \left( \int_\Theta |u|^2 dx + \int_\Theta |\sqrt{Q} \nabla u|^2 dx \right)^{\frac{1}{2}},$$

(1.4)
of the collection
\begin{equation}
\text{Lip}_Q(\Theta) = \{ \varphi \in \text{Lip}_{loc}(\Theta) : |\sqrt{Q}\nabla \varphi| \in L^2(\Theta) \}
\end{equation}
of locally Lipschitz functions having finite $QH^1(\Theta)$ norm. In all of our developments we will
denote the vector valued function $\vec{g}$ of the pair $(f, \vec{g}) \in QH^1(\Theta)$ by writing $\vec{g} = \nabla f$, and we
will refer to it as the gradient part (or simply the gradient) of $f$. The canonical projection map
$i : QH^1(\Theta) \to L^2(\Theta)$ defined by
\begin{equation}
i((f, \vec{g})) = f
\end{equation}
is obviously continuous, but it need not be injective. It should be kept in mind at all times that $\vec{g}$ need not be uniquely determined by $f$; see [FKS] for a well known example. In order to keep
notation simple we will often abuse notation by writing $f \in \text{Lip}_Q(\Theta)$ when referring to elements of $QH^1(\Theta)$; i.e. we will write $f \in QH^1(\Theta)$ in place of $f \in QH^1(\Theta)$.

The space $QH^1_0(\Theta)$ (respectively $W_{Q,0}^{1,2}(\Theta)$) is obtained in a similar manner, but in this case
we complete the set $\text{Lip}_0(\Theta)$, the set of those Lipschitz functions having compact support in $\Theta$, with respect to the norm $\| \cdot \|_{L^2(\Theta)}$. Notice that all such functions have finite $QH^1(\Theta)$ norm as $|Q| \in L^\infty_{loc}(\Omega)$.

For clarity, we will always write $QH^1(\Theta)$ and $QH^1_0(\Theta)$ in place of $W_{Q,0}^{1,2}(\Theta)$ and $W_{Q,0}^{1,2}(\Theta)$
respectively, taking isomorphism in context. We adopted this notation in lieu of $W_{Q,0}^{1,2}(\Omega)$ and
$W_{Q,0}^{1,2}(\Omega)$, as used in [SW1, SW2, CRW, MRW], in order to agree with classical literature, see
for example [MS], where it is conventional that “$W$” spaces refer to Sobolev spaces defined with
respect to distributional derivatives. We also mention that it is possible to introduce definitions and
make similar considerations for the spaces $QH^{1,p}(\Theta)$, $QH^{1,1,p}(\Theta)$ (or $W_{Q,0}^{1,2}(\Omega)$, $W_{Q,0}^{1,2}(\Omega)$ as
in the above references) for $1 \leq p < \infty$, even in the case where $|Q(x)|$ is locally unbounded. We
invite the interested reader to see [CRW, MRW, SW2] for the construction of these spaces and
related objects.

1.3. Sobolev and Poincaré Inequalities. Essential to most of the arguments to follow are
Poincaré–Sobolev type inequalities adapted to the matrix $Q$. We list these inequalities now
noting that they are not assumed to hold at all times but, rather, called upon when necessary. To
describe these efficiently, we fix a continuous function $r_1 : \Omega \to (0, \infty)$ to be used as a common
radius restriction for quasimetric balls.

The Local Poincaré Inequality. We say that the local Poicaré inequality of order $p$ holds
if there are constants $C_2 > 0$ and $b \geq 1$ so that for every $\rho$-ball $B(y, r)$ centered in $\Omega$ with $br \in (0, r_1(y))$ the inequality
\begin{equation}
\left( \frac{1}{|B_r|} \int_{B_r} |f - f_{B_r}|^p dx \right)^{\frac{1}{p}} \leq C_2 r \left( \frac{1}{|B_{br}|} \int_{B_{br}} |\sqrt{Q}\nabla f|^p dx \right)^{\frac{1}{p}}
\end{equation}
holds for all $f \in \text{Lip}_{loc}(\Omega)$. Notice that a continuity argument allows one to extend \((1.6)\) to hold
for all pairs $(f, \nabla f) \in QH^{1,p}(\Omega)$.

The Global Sobolev Inequality. For an open set $\Theta \subset \Omega$ with $\overline{\Theta} \subset \Omega$, we say that the
global Sobolev inequality on $\Theta$ holds if there are positive constants $C_3 > 0$ and $\sigma > 1$ such that
\begin{equation}
\left( \int_{\Theta} |f|^{2\sigma} dx \right)^{\frac{1}{2\sigma}} \leq C_3 \left( \int_{\Theta} |\sqrt{Q}\nabla f|^2 dx \right)^{\frac{1}{2}}
\end{equation}
holds for all $f \in \text{Lip}_0(\Theta)$. Again, a continuity argument shows that \((1.8)\) also holds for all $(w, \nabla w) \in QH^1_0(\Theta)$ if it holds for all $f \in \text{Lip}_0(\Theta)$. 

Remark 1.4. It is possible to replace (1.7) with a local version of the form
\[
\left( \int_{B(x,r)} |f|^{2\sigma} \, dx \right)^{1/\sigma} \leq C' \left( r^2 \int_{B(x,r)} |\sqrt{Q}\nabla f|^2 \, dx + \int_{B(x,r)} |f|^2 \, dx \right)^{1/2},
\]
holding for all \( f \in \text{Lip}_0(B(x,r)) \) and quasimetric balls \( B(x,r) \) with \( 0 < r < r_1(x) \). In \( \mathbb{R}^2 \), inequality (1.7) is proved for any open \( \Theta \) such that \( \Theta \subset \Omega \) under the hypotheses given in subsections 1.1 and 1.2, provided Lebesgue measure is doubling for the collection of quasimetric balls \( B \) and both (1.6) with \( p = 2 \) and (1.8) hold. One may even replace the doubling assumption in \( \mathbb{R}^2 \) with local geometric doubling; see [CRW, R2].

The Global Poincaré Inequality With Gain \( \omega \). For an open subset \( \Theta \) of \( \Omega \) satisfying \( \Theta \subset \Omega \) we say that the global Poincaré inequality with gain \( \omega > 1 \) holds on \( \Theta \) if there are constants \( C_4 > 0 \) and \( \omega > 1 \) such that
\[
\left( \int_{\Theta} |f - f_\Theta|^{2\omega} \, dx \right)^{1/\omega} \leq C_4 \left( \int_{\Theta} |\sqrt{Q}\nabla f|^2 \, dx + \int_{\Theta} |f|^2 \, dx \right)^{1/2},
\]
holds for all \( f \in \text{Lip}_Q(\Theta) \).

Remark 1.5.

1. If the global Poincaré inequality (1.19) holds, then Hölder’s inequality implies that the Global Weak Poincaré Inequality with gain \( \omega > 1 \):
\[
\left( \int_{\Theta} |f|^{2\omega} \, dx \right)^{1/\omega} \leq C_4 \left( \int_{\Theta} |\sqrt{Q}\nabla f|^2 \, dx + \int_{\Theta} |f|^2 \, dx \right)^{1/2},
\]
also holds for all \( f \in \text{Lip}_Q(\Theta) \).

2. In the elliptic case \( (Q(x) = \text{Id}) \), inequalities of the form (1.9) and (1.10) are proved when the boundary of \( \Theta \) is sufficiently regular. For example, \( \partial\Theta \in C^{0,1} \) is used in [GT] for such purposes. See [GT] Theorem 7.26 and related discussions.

There is a large body of literature discussing the inequalities just mentioned. We refer the reader to [E], [HK], and [J] for helpful discussions and examples.

As a last remark we mention that it is possible to obtain all the theory to follow if Lebesgue measure is replaced by any Radon measure \( \mu \) that is absolutely continuous with respect to Lebesgue measure. The changes required for this are:

- replace \( |E| \) with \( |E|_\mu \) and any almost everywhere considerations shifted to \( \mu \)-a.e.;
- Incorporate \( \mu \) into the definition of Sobolev space by replacing \( ||u||_{QH^1(\Theta)} \) with
\[
||u||_{QH^1_\mu(\Theta)} = ||u||_{L^2(\Theta)} + ||\sqrt{Q}\nabla f||_{L^2(\Theta)};
\]
- replace \( dx \) with \( d\mu \) in all integrals.

1.4. Compact Projections for Degenerate Sobolev Spaces. In our main results, it is essential that we are equipped with a compact mapping from \( QH^1(\Theta) \) (respectively \( QH^1_\mu(\Theta) \)) into \( L^2(\Theta) \) and this serves as our first application of the inequalities just listed. The following results are adapted from [CRW] and we refer the reader to that work for more general statements and weighted results.

Proposition 1.6 ([CRW] Corollary 3.25). Let \((\Omega, \rho)\) be a geometric homogeneous space and fix an open set \( \Theta \) satisfying \( \Theta \subset \Omega \). Suppose that the local Poincaré inequality (1.6) with \( p = 2 \) and
the global Sobolev inequality (1.7) hold. Then the projection map \( i : QH^1_0(\Theta) \rightarrow L^q(\Theta) \) defined by
\[
i((u, \nabla u)) = u
\]
is a compact mapping for all \( q \in [1, 2\sigma) \).

**Proposition 1.7** ([CRW] Theorem 3.14). Let \((\Omega, \rho)\) be a geometric homogeneous space and fix an open set \( \Theta \) satisfying \( \Theta \subset \Omega \). Suppose that the local Poincaré inequality (1.6) with \( p = 2 \) and the global weak Poincaré inequality (1.10) with gain \( \omega > 1 \) hold. Then the projection map \( i : QH^1(\Theta) \rightarrow L^q(\Theta) \) defined by
\[
i((u, \nabla u)) = u
\]
is a compact mapping for all \( q \in [1, 2\omega) \).

1.5. **Notation.** Consider a vector field
\[
W(x) = \sum_{i=1}^{n} w_i(x) \frac{\partial}{\partial x_i} = (w_1(x), \ldots, w_n(x)) \cdot \nabla
\]
If \( u \) is a real valued function on \( \mathbb{R}^n \) and \( \nu \) is a vector in \( \mathbb{R}^n \) we adopt the notation
\[
Wu = \sum_{i=1}^{n} w_i \frac{\partial u}{\partial x_i}, \quad \langle \nu, W \rangle = \sum_{i=1}^{n} w_i \nu_i,
\]
where \( \langle \cdot, \cdot \rangle \) denotes the standard inner product on \( \mathbb{R}^n \). The formal adjoint \( W'(x) \) of the vector field \( W(x) \) is defined by
\[
W'(x)u := -\text{div}(w_1(x)u(x), \ldots, w_n(x)u(x)) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(w_i(x)u(x)).
\]
A vector field \( W(x) \) as above is always identified with the vector valued function \((w_1(x), \ldots, w_n(x))\) and is said to be subunit with respect to the matrix \( Q \) in \( \Theta \) if
\[
\left( \sum_{i=1}^{n} w_i(x)\xi_i \right)^2 \leq \langle \xi, Q(x)\xi \rangle
\]
for every \( \xi \in \mathbb{R}^n \) and almost every \( x \in \Theta \).

**Remark 1.8.** If a vector field \( W(x) \) is subunit with respect to the matrix \( Q = Q(x) \) in \( \Theta \) we will simply refer to it as a “subunit vector field” with the set \( \Theta \) and matrix \( Q \) taken in context.

Given \( N \in \mathbb{N} \), an \( N \)-tuple \( W = (W_1, \ldots, W_N) \) of vector fields and an \( \mathbb{R}^N \)-valued function \( G = (g_1, \ldots, g_N) \), \( WG \) denotes the “inner product” of \( W \) and \( G \), i.e.
\[
WG = \sum_{i=1}^{N} W_i(x)g_i(x).
\]
Lastly, If \( u \) is a real valued function,
\[
GWu = \sum_{i=1}^{N} g_i(x)W_i(x)u(x), \quad W'(Gu) = \sum_{i=1}^{N} W'_i(x)\left(g_i(x)u(x)\right).
\]
1.6. Second Order Linear Degenerate Elliptic Operators with Rough Coefficients.

Let $\Omega$ and $Q(x)$ be as in Chapter 1, $\Theta$ be a bounded domain with $\Theta \subset \Omega$. We consider operators $X$ defined by equations of the form

$$Xu = -\text{div}(P\nabla u) + HRu + S'(Gu) + Fu \text{ in } \Theta,$$

where $P = P(x)$ is a bounded measurable nonnegative definite symmetric $n \times n$ matrix defined in $\Theta$ and comparable with $Q(x)$ in $\Theta$. That is, there exist constants $c_1, C_1 > 0$ so that for every $\xi \in \mathbb{R}^n$ and almost every $x \in \Theta$ one has

$$c_1 \langle \xi, Q(x)\xi \rangle \leq \langle \xi, P(x)\xi \rangle \leq C_1 \langle \xi, Q(x)\xi \rangle.$$

$R$, and $S$ are, for some $N \in \mathbb{N}$, $N$-tuples of subunit vector fields with respect to the matrix $Q(x)$, see (1.13). $H$ and $G$ are $\mathbb{R}^N$-valued measurable functions in $\Theta$ and $F$ is a real-valued measurable function defined in $\Theta$.

Remark 1.9. The sign convention on the principal part of the operator $X$ is as in [E] and [GT]. S. Rodney in [R1], [R2], and [R3] adopted the opposite convention.

An operator $X$ as above will always be referred to as a “second order linear degenerate elliptic operator with rough coefficients”. We will study existence and spectral properties of both Dirichlet and Neumann problems associated to $X$ in the context of two “negativity” conditions that we now introduce.

Definition 1.10. We say that a second order linear degenerate elliptic operator with rough coefficients $X$ satisfies negativity condition (1) if and only if either

i) there exists $\varepsilon > 0$ such that for every $(u, \nabla u), (v, \nabla v) \in QH^1(\Theta)$ satisfying $uv \geq 0$ almost everywhere in $\Theta$ one has

$$\int_{\Theta} Fuv + GS(uv) \, dx \geq \varepsilon \int_{\Theta} uv \, dx$$

or

ii) for every $(u, \nabla u), (v, \nabla v) \in QH^1(\Theta)$ satisfying $uv \geq 0$ almost everywhere in $\Theta$ one has

$$\int_{\Theta} Fuv + HR(uv) \, dx \geq \varepsilon \int_{\Theta} uv \, dx.$$

Definition 1.11. We say that $X$ satisfies negativity condition (2) if and only if either

i) for every $(u, \nabla u), (v, \nabla v) \in QH^1_0(\Theta)$ such that $uv \geq 0$ almost everywhere in $\Theta$ one has

$$\int_{\Theta} Fuv + GS(uv) \, dx \geq 0$$

or

ii) for every $(u, \nabla u), (v, \nabla v) \in QH^1_0(\Theta)$ such that $uv \geq 0$ almost everywhere in $\Theta$ one has

$$\int_{\Theta} Fuv + HR(uv) \, dx \geq 0.$$

It is useful to note that if $X$ satisfies negativity condition (2), part i), then the operator $X$ is a member of the “Nonnegative Class” as described in [R3].
2. Existence and Spectral Results for the X-Neumann Problem

The presentation of the results in this section follows in part the presentation of the analogous results for second order elliptic Dirichlet problems on bounded domains of \( \mathbb{R}^n \) as given in [R3]. Furthermore, many of the arguments used, vis-a-vis existence of weak solutions, follow similar paths as those found in [R3].

Let \( \Omega \) and \( Q = Q(x) \) be as in sections 1.1 and 1.2 respectively. We begin by describing the action of a subunit vector field on elements and products of elements of \( QH^1(\Theta) \).

**Lemma 2.1.** Let \( \Theta \) be a bounded open set such that \( \overline{\Theta} \subset \Omega \), let \( (u, \nabla u), (v, \nabla v) \in QH^1(\Theta) \) and let \( W(x) = (w_1(x), \ldots, w_n(x)) \) be a subunit vector field in \( \Theta \). Assume that the global weak Poincaré inequality with gain \( \omega > 1 \) holds. Then:

1) \( Wu \) is defined as an element of \( L^2(\Theta) \), with \( \|Wu\|_{L^2(\Theta)} \leq \|u\|_{QH^1(\Theta)}, \) and

\[
W(x)\varphi(x) = \sum_{i=1}^{n} w_i(x) \frac{\partial \varphi}{\partial x_i}(x)
\]

for every locally Lipschitz function \( \varphi \) defined on \( \Theta \) having finite \( QH^1(\Theta) \) norm.

2) \( W(uv) \) is defined as an element of \( L^{\frac{2}{\omega+1}}(\Theta) \), with

\[
\|W(uv)\|_{L^{\frac{2}{\omega+1}}(\Theta)} \leq 2C_4 \|u\|_{QH^1(\Theta)} \|v\|_{QH^1(\Theta)},
\]

with \( C_4 > 0 \) as in (1.10), and with \( W(uv) = uWv + vWu \) as elements of \( L^{\frac{2}{\omega+1}}(\Theta) \).

**Proof:** The proof of Lemma 2.1 follows from the same arguments used in [R3, Lemma 3.15]. Here, one simply uses the global weak Poincaré inequality (1.10) with gain \( \omega > 1 \) in place of a global Sobolev inequality as used in [R3] (see condition [R3, (2.11)] or, equivalently, (1.7)).

\[\square\]

**Definition 2.2.** Given a second order linear degenerate elliptic operator with rough coefficients \( X \) as in (1.14), we introduce the associated bilinear form acting on \( QH^1(\Theta) \times QH^1(\Theta) \),

\[
(2.1) \quad L(u, v) = \int_{\Theta} \left[ (\nabla v, P(x)\nabla u) + vHRu + uGSv + Fuv \right] dx.
\]

As was done in [R3] for the X-Dirichlet problem, the bilinear form (2.1) will be used in a moment to define a notion of weak solution for the X-Neumann problem. To begin the study of such objects, we show the boundedness of \( L \) on \( QH^1(\Theta) \) followed by an almost-coercive estimate.

**Proposition 2.3.** Let \( \Theta \) be a bounded domain with \( \overline{\Theta} \subset \Omega \) and assume that the global weak Poincaré inequality (1.10) with gain \( \omega > 1 \) holds. Assume that \( F \in L^t(\Theta) \) with \( t \geq \omega' = \frac{\omega}{\omega - 1} \) and that \( G, H \in [L^q(\Theta)]^N \) with \( q \geq 2 \omega' \). Then there exists a constant \( C_6 = C_6(C_1, C_4, N, \|F\|_{L^{\omega'}(\Theta)}, \|G\|_{L^{2\omega'}(\Theta)} + \|H\|_{L^{2\omega'}(\Theta)}) > 0 \) such that

\[
(2.2) \quad |L(u, v)| \leq C_6 \|u\|_{QH^1(\Theta)} \|v\|_{QH^1(\Theta)}
\]

for every \( u, v \in QH^1(\Theta) \).

**Corollary 2.4.** Under the hypotheses of Proposition 2.3, the bilinear form \( L(\cdot, \cdot) \) introduced in Definition 2.2 is well defined and continuous on \( QH^1(\Theta) \times QH^1(\Theta) \).
Proof of Proposition 2.3. Let $u, v \in QH^1(\Theta)$. Then, by (1.15) and Schwarz’s inequality
\[
\left| \int_\Theta \langle \nabla v, P(x) \nabla u \rangle \, dx \right| \leq \int_\Theta \| P(x) \nabla u \| \sqrt{P(x)} \sqrt{P(x)} \nabla v \, dx \\
\leq \left( \int_\Theta \left| \sqrt{P(x)} \nabla u \right|^2 \, dx \right)^{\frac{1}{2}} \left( \int_\Theta \left| \sqrt{P(x)} \nabla v \right|^2 \, dx \right)^{\frac{1}{2}} \\
\leq C_1^2 \| u \|_{QH^1(\Theta)} \| v \|_{QH^1(\Theta)}. 
\]
Using a generalization of Hölder’s inequality (see [GT, (7.11)] with exponents $2\omega, 2\omega'$, and 2) together with the results of Lemma 2.1 we obtain
\[
\left| \int_\Theta vHRu \, dx \right| \leq \| v \|_{L^{2\omega'}(\Theta)} \| H \|_{L^{2\omega'}(\Theta)} \| Ru \|_{L^{2\omega}(\Theta)} \\
\leq C_4 \sqrt{N} \| H \|_{L^{2\omega'}(\Theta)} \| v \|_{QH^1(\Theta)} \| u \|_{QH^1(\Theta)}. 
\]
Similarly,
\[
\left| \int_\Theta uGSv \, dx \right| \leq C_4 \sqrt{N} \| G \|_{L^{2\omega'}(\Theta)} \| u \|_{QH^1(\Theta)} \| v \|_{QH^1(\Theta)}. 
\]
Thus, (2.2) holds with
\[
C_0 = C_2^2 + C_4 \sqrt{N} \| G \|_{L^{2\omega'}(\Theta)} + \| H \|_{L^{2\omega'}(\Theta)} \| u \|_{QH^1(\Theta)} + C_2^2 \| F \|_{L^{\omega'}(\Theta)}. 
\]

Proposition 2.5. Let $\Theta$ be a bounded domain such that $\overline{\Theta} \subset \Omega$ and assume that the global weak Poincaré inequality (1.10) with gain $\omega > 1$ holds. Assume that $F \in L^t(\Theta)$ with $t > \omega'$ and that $G, H \in L^q(\Theta)$ with $q > 2\omega'$. Then, there exists a constant $C_7 = C_7(c_1, C_4, \omega, N, t, q, \| F \|_{L^t(\Theta)}, \| G \|_{L^q(\Theta)} + \| H \|_{L^q(\Theta)}) > 0$ so that
\[
|\mathcal{L}(u, u)| \geq \frac{c_1}{4} \| u \|_{QH^1(\Theta)}^2 - C_7 \| u \|_{L^2(\Theta)}^2. 
\]
for every $u \in QH^1(\Theta)$.

Proof of Proposition 2.5. Let $u \in QH^1(\Theta)$. Then, (1.15) gives
\[
\mathcal{L}(u, u) \geq c_1 \int_\Theta \langle \nabla u, Q(x) \nabla u \rangle \, dx - \int_\Theta |u| (HRu + GSu) \, dx - \int_\Theta |F| u^2 \, dx. 
\]
Using Hölder’s inequality with exponents $t, t' \geq 1$ we have
\[
\int_\Theta |F| u^2 \, dx \leq \| F \|_{L^{t'}(\Theta)} \| u \|_{L^{2t'}(\Theta)} \leq \left( \varepsilon \| u \|_{L^{2t'}(\Theta)}^2 + \varepsilon^{-\frac{1}{t'-1}} \| u \|_{L^{2t'}(\Theta)}^2 \right) \| F \|_{L^{t'}(\Theta)}, 
\]
for any $\varepsilon > 0$; we used the interpolation inequality [GT, (7.10)]. Now, if $\| F \|_{L^{t'}(\Theta)} = 0$ then $\int_\Theta |F| u^2 \, dx = 0$, otherwise we choose $\varepsilon = \frac{C_1}{8 \| F \|_{L^{t'}(\Theta)}^2}$. So that (1.10) gives
\[
\int_\Theta |F| u^2 \, dx \leq \alpha \| u \|_{L^2(\Theta)}^2 + \frac{C_1}{8} \| u \|_{QH^1(\Theta)}^2, 
\]
with $\alpha = \alpha(c_1, C_4, t, \omega, \| F \|_{L^{t'}(\Theta)}) > 0$. 

In a similar way, we use Lemma [2.1] and [GT] (7.11) to obtain
\[
\int_{\Theta} |u(HRu + GSu)| \, dx \\
\leq \varepsilon \sqrt{N} (||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)}) ||u||_{L^2(\Theta)} ||u||_{L^2(\Theta)} \\
+ \varepsilon \frac{c_1}{\sqrt{N}} (||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)}) ||u||_{QH^1(\Theta)} ||u||_{L^2(\Theta)} \\
+ \varepsilon \frac{c_1}{\sqrt{N}} (||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)}) ||u||_{QH^1(\Theta)} ||u||_{L^2(\Theta)} \\
\leq \varepsilon \sqrt{N} (||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)}) ||u||_{QH^1(\Theta)} ||u||_{L^2(\Theta)} \\
\leq \varepsilon \sqrt{N} (||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)}) (\delta ||u||_{QH^1(\Theta)} + \frac{1}{2}) ||u||_{L^2(\Theta)}.
\]

If \( ||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)} = 0 \) then \( \int_{\Theta} |u(HRu + GSu)| \, dx = 0 \), otherwise we choose
\[
\varepsilon = \frac{c_1}{8 \sqrt{N} C_4 (||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)})}, \\
\delta = \frac{\varepsilon \frac{c_1}{\sqrt{N}}}{\sqrt{N} (||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)})}
\]
so that \([1.10]\) gives
\[
(2.6) \quad \int_{\Theta} |u(HRu + GSu)| \, dx \leq \frac{5c_1}{8} ||u||_{QH^1(\Theta)} + \beta ||u||_{L^2(\Theta)},
\]
where \( \beta = \beta(c_1, C_4, N, q, \omega, ||H||_{L^2(\Theta)} + ||G||_{L^2(\Theta)}) > 0 \). Inserting \([2.5]\) and \([2.6]\) into \([2.4]\) we have
\[
L(u, u) \geq \frac{c_1}{4} ||u||_{QH^1(\Theta)}^2 - (c_1 + \alpha + \beta) ||u||_{L^2(\Theta)}^2,
\]
which is \([2.4]\) with \( C_7 = c_1 + \alpha + \beta > 0 \).

With the boundedness and almost-coercivity of the bilinear form \( L \) concluded, we now formally define the notion of weak solution associated to the X-Neumann problem.

**Definition 2.6.** Let \( X \) be a second order linear degenerate elliptic operator with rough coefficients as in \([1.14]\). Assume that \( \partial \Theta \) is piecewise \( C^1 \) and let \( \nu \) be the unit outward normal vector at each sufficiently regular boundary point. Assume that the global weak Poincaré inequality \([1.10]\) with gain \( \omega > 1 \) holds, let \( G, H \in [L^q(\Theta)]^N \) with \( q \geq 2\omega' \) and let \( F \in L^t(\Theta) \) with \( t \geq \omega' \).

If \( f \in L^2(\Theta), K \in \mathbb{N}, T \) is a \( K \)-tuple of subunit vector fields and \( g \in [L^2(\Theta)]^K \), then a function \( (u, \nabla u) \in QH^1(\Theta) \) is a weak solution of the X-Neumann Problem
\[
(2.7) \quad \begin{cases} X u = f + T g & \text{in } \Theta \\ (\nu, P(x) \nabla u + uG + gT) = 0 & \text{on } \partial \Theta 
\end{cases}
\]
if and only if
\[
(2.8) \quad L(u, v) = \int_{\Theta} f v + gT v \, dx \quad \text{for every } v \in QH^1(\Theta).
\]
Motivation. Assume that $\partial \Theta$ is smooth and that the coefficients of the operator $X$ are $C^1(\Theta)$. If $u \in C^2(\Theta)$ and $v \in C^1(\Theta)$, it’s easy to see using the Divergence Theorem that

$$
\int_\Theta v Xu \, dx = L(u, v) - \int_{\partial \Theta} v \langle \nu, P(x)\nabla u + uG \rangle \, d\sigma,
$$

while

$$
\int_\Theta (f + T^t g)v = \int_\Theta f v + gTv - \int_{\partial \Theta} v \langle \nu, gT \rangle \, d\sigma.
$$

Thus $u$ is a classical solution of problem (2.7) if and only if it satisfies (2.8). Notice that if $G$ and $g$ are identically null and if $P(x)$ is strictly positive definite and continuous on $\Theta$, we recover the usual definition of weak solution of the Neumann Problem related to the (now elliptic) differential operator $X$.

We now come to the first of our existence results - the existence of weak solutions to a modified version of the $X$-Neumann problem in $\Theta$. The theorem is a consequence of Propositions (2.3) and (2.5) with an application of the Lax-Milgram lemma.

**Theorem 2.7.** Let $\Theta$ be a bounded domain such that $\overline{\Theta} \subset \Omega$ and assume that the global weak Poincaré inequality (1.10) with gain $\omega > 1$ holds. Let $X$ be a second order linear degenerate elliptic operator with rough coefficients as in (1.14). Assume that $F \in L^t(\Theta)$ with $t > \omega'$ and $G, H \in [L^q(\Theta)]^N$ with $q > 2\omega'$. Then there is a constant $\gamma > 0$ so that given any $\mu \geq \gamma$, $f \in L^2(\Theta)$ and any $K$--tuple $T$ of subunit vector fields and $g \in [L^2(\Theta)]^K$ there exists a unique weak solution $u \in QH^1(\Theta)$ of the $X$-Neumann Problem

$$
\begin{cases}
Xu + \mu u = f + T^t g & \text{in } \Theta \\
\langle \nu, P(x)\nabla u + uG \rangle = 0 & \text{on } \partial \Theta.
\end{cases}
$$

Moreover, if $u \in QH^1(\Theta)$ is a weak solution of (2.9) then

$$
\|u\|_{QH^1(\Theta)} \leq \frac{4}{c_1} (\|f\|_{L^2(\Theta)} + \sqrt{K} \|g\|_{L^2(\Theta)}).
$$

**Proof of Theorem 2.7.** By Proposition (2.3) there is a $\gamma > 0$ such that for every $\mu \geq \gamma$ one has

$$
L(u, u) + \mu \|u\|_{L^2(\Theta)}^2 \geq L(u, u) + \gamma \|u\|_{L^2(\Theta)}^2 \geq \frac{c_1}{4} \|u\|_{QH^1(\Theta)}^2
$$

for every $u \in QH^1(\Theta)$. Using Proposition (2.5) one also sees that for each $\mu \geq \gamma$

$$
L(u, v) + \mu \int_\Theta uv \, dx \leq (C_0 + \mu)\|u\|_{QH^1(\Theta)}\|v\|_{QH^1(\Theta)}
$$

for every $u, v \in QH^1(\Theta)$. It follows that, for every $\mu \geq \gamma$, the bilinear form $L_\mu$ defined on $QH^1(\Theta) \times QH^1(\Theta)$ by setting

$$
L_\mu(u, v) = L(u, v) + \mu \int_\Theta uv \, dx
$$

for every $u, v \in QH^1(\Theta)$ is both bounded and coercive. Next, we notice that Lemma (2.1) implies that the map $\phi$ defined by

$$
\phi(v) = \int_\Theta f v + gTv \, dx
$$

for every $v \in QH^1(\Theta)$
is linear and continuous. Indeed,
\begin{equation}
\| f \|_{L^2(\Omega)} \| v \|_{L^2(\Omega)} + \| g \|_{L^2(\Omega)} \| T v \|_{L^2(\Omega)} \\
\leq \left( \| f \|_{L^2(\Omega)} + \sqrt{K} \| g \|_{L^2(\Omega)} \right) \| v \|_{QH^1(\Omega)}
\end{equation}
for every $v \in QH^1(\Omega)$. Applying the Lax–Milgram lemma we conclude that there exists a unique $u \in QH^1(\Omega)$ satisfying
\[ \mathcal{L}_\mu(u, v) = \phi(v) \]
for every $v \in QH^1(\Omega)$. Recalling Definition 2.9 we see that $u$ is the unique weak solution in $QH^1(\Omega)$ of the X-Neumann problem (2.9). Moreover, (2.11) and (2.12) indicate that such a $u$ also satisfies
\begin{equation}
\frac{c_1}{4} \| u \|_{QH^1(\Omega)} \leq \left( \| f \|_{L^2(\Omega)} + \sqrt{K} \| g \|_{L^2(\Omega)} \right)
\end{equation}
and inequality (2.10) follows. 

\[ \square \]

**Remark 2.8.** Notice that the same arguments used in the proof of Theorem 2.7 show that for every $\mu \geq \gamma$ and every $\phi \in (QH^1(\Omega))^* \cong QH^1(\Omega)$, the dual space to $QH^1(\Omega)$, there exists a unique $u \in QH^1(\Omega)$ such that
\begin{equation}
\mathcal{L}_\mu(u, v) = \mathcal{L}(u, v) + \mu \int_\Omega uv \, dx = \phi(v) \quad \text{for every } v \in QH^1(\Omega).
\end{equation}
Moreover, the solution map $S_\mu : (QH^1(\Omega))^* \to QH^1(\Omega)$, defined by setting $S_\mu(\phi) = u$ if and only if $u \in QH^1(\Omega)$ satisfies (2.11), is linear and continuous, with
\[ \| u \|_{QH^1(\Omega)} \leq \frac{4}{c_1} \| \phi \|_{(QH^1(\Omega))^*}. \]

**Definition 2.9.** Let $X$ be a second order linear degenerate elliptic operator with rough coefficients as in (1.14). Assume that $\partial \Theta$ is piecewise $C^1$ and let $\nu$ be the outward unit normal to $\partial \Theta$ at each sufficiently regular boundary point. Assume that the global weak Poincaré inequality (1.14) holds, let $G, H \in [L^2(\Omega)]^N$ with $q \geq 2\omega'$ and let $F \in L^1(\Theta)$ with $t \geq \omega'$. If $f \in L^2(\Omega)$, $K \in \mathbb{N}$, $T$ is a $K$-tuple of vector fields and $g \in [L^2(\Theta)]^K$, then the adjoint problem to the X-Neumann problem (2.7) is given by
\begin{equation}
\begin{cases}
X^* u := -\text{div}(P(x) \nabla u) + GSu + R'(H_u) + Fu = f + T'g & \text{in } \Theta \\
(\nu, P(x) \nabla u + uHR - gT) = 0 & \text{on } \partial \Theta.
\end{cases}
\end{equation}
We will say that the X-Neumann Problem (2.7) is self-adjoint if $HR \equiv GS$ on $\Theta$, i.e. if
\[ \int_\Omega vHRu \, dx = \int_\Omega vGSu \, dx \quad \text{for every } u, v \in QH^1(\Omega). \]

**Remark 2.10.** The bilinear form $\mathcal{L}^* : QH^1(\Omega) \times QH^1(\Omega) \to \mathbb{R}$ associated to the adjoint problem (2.15) is
\begin{equation}
\mathcal{L}^*(u, v) = \int_\Omega \langle \nabla v, P(x) \nabla u \rangle + vGSu + uHRv + Fuv \, dx = \mathcal{L}(v, u),
\end{equation}
for every $u, v \in QH^1(\Omega)$.

We now present our main existence result for the X-Neumann problem.
Theorem 2.11. Let $(\Omega, \rho)$ be a geometric homogeneous space and $\Theta$ be a bounded domain with $\overline{\Theta} \subset \Omega$. Assume that the local Poincaré inequality (1.6) holds with $p = 2$ and that the global weak Poincaré inequality (1.10) holds with gain $\omega > 1$. Let $X$ be a second order linear degenerate elliptic operator with rough coefficients as in (1.14). Assume that $F \in L^t(\Theta)$ with $t > \omega'$ and that $G, H \in [L^q(\Theta)]^N$ with $q > 2\omega'$. Then, we have the following conclusions.

1) One and only one of the following alternatives hold:
   
   (i) given any $f \in L^2(\Theta)$, $K \in \mathbb{N}$, any $K$-tuple of subunit vector fields $T$, and $g \in [L^2(\Theta)]^K$ there exists a unique weak solution $u \in QH^1(\Theta)$ of the $X$-Neumann Problem (2.7) or
   
   (ii) there exist nontrivial weak solutions $u \in QH^1(\Theta) \setminus \{(0,0)\}$ of the homogeneous $X$-Neumann Problem

   \begin{align}
   X u = 0 & \quad \text{in } \Theta \\
   \langle \nu, P(x) \nabla u + u GS \rangle & = 0 \quad \text{on } \partial \Theta.
   \end{align}

2) If alternative 1)—(II) holds, then the dimension of the subspace $N \subset QH^1(\Theta)$ of weak solutions of the homogeneous $X$-Neumann Problem (2.17) is finite and equals the dimension of the subspace $N^* \subset QH^1(\Theta)$ of weak solutions of the adjoint homogeneous $X$-Neumann Problem

   \begin{align}
   X^* w = 0 & \quad \text{in } \Theta \\
   \langle \nu, P(x) \nabla w + w HR \rangle & = 0 \quad \text{on } \partial \Theta.
   \end{align}

3) If alternative 1)—(II) holds, the $X$-Neumann Problem (2.7) admits a weak solution $u \in QH^1(\Theta)$ for given $f \in L^2(\Theta)$, $K \in \mathbb{N}$, $T$ a $K$-tuple of subunit vector fields and $g \in [L^2(\Theta)]^K$ if and only if

   \[ \int_{\Theta} f w + g T w \, dx = 0 \]

   for every $w \in N^*$.

Proof of Theorem 2.11. To begin, we notice that Theorem 2.7 and Remark 2.10 provide a $\gamma > 0$ so that for every $\mu \geq \gamma$ and every $\phi \in (QH^1(\Theta))^*$, there exist unique $v, w \in QH^1(\Theta)$ so that

   \[ L(v, \psi) + \mu \int_{\Theta} w \, dx = \phi(\psi) \quad \text{and} \quad L^*(w, \psi) + \mu \int_{\Theta} v \, dx = \phi(\psi) \]

   for every $\psi \in QH^1(\Theta)$. The corresponding solution maps

   \[ S_\mu, S^*_\mu : (QH^1(\Theta))^* \to QH^1(\Theta) \]

   defined by setting $S_\mu \phi = v$ and $S^*_\mu \phi = w$ are well defined, linear and continuous. Next, we introduce the map $J : QH^1(\Theta) \to (QH^1(\Theta))^*$ defined by setting

   \[ J(u) = \int_{\Theta} u \, dx \]

   for every $u \in QH^1(\Theta)$. The map $J$ is linear, continuous and, by Proposition 1.7 with $q = 2$, compact.

   Now, given $f \in L^2(\Theta)$, $K \in \mathbb{N}$, $T$ a $K$-tuple of subunit vector fields and $g \in [L^2(\Theta)]^K$ we define a continuous linear functional $\varphi$ on $QH^1(\Theta)$:

   \begin{align}
   \varphi(\psi) = \int_{\Theta} f \psi + g T \psi \, dx.
   \end{align}
Notice that \( u \in QH^1(\Theta) \) is a weak solution of the X-Neumann Problem (2.7) if and only if
\[
\begin{align*}
Xu + \gamma u &= f + T'g & \text{in } \Theta \\
\langle \nu, P(x)\nabla u + uG_S - gT \rangle &= 0 & \text{on } \partial \Theta;
\end{align*}
\]
that is, if and only if one has
\[
\mathcal{L}(u, \psi) + \gamma \int_\Theta u\psi\,dx = \varphi(\psi) + \gamma J u(\psi)
\]
for every \( \psi \in QH^1(\Theta) \). This is equivalent to requiring that \( u \in QH^1(\Theta) \) solves (2.20) with
\[
u - Ku = \Phi
\]
where \( \Phi = S_\gamma \varphi \in QH^1(\Theta) \) where \( K = \gamma S_\gamma \circ J : QH^1(\Theta) \to QH^1(\Theta) \); a compact linear operator. Similarly we see that \( u \in QH^1(\Theta) \) solves the homogeneous X-Neumann Problem (2.17) if and only if it is a solution of
\[
u - Ku = 0.
\]
Hence, we can apply the Fredholm Alternative, see for instance [E, theorem 5 (appendix D)], and conclude that either
\[
(A) \text{ for every } \Phi \in QH^1(\Theta) \text{ equation (2.20) admits a unique solution } u \in QH^1(\Theta),
\]
or
\[
(B) \text{ the homogeneous equation associated to (2.20) admits nontrivial solutions } u \in QH^1(\Theta) \setminus \{(0, 0)\}.
\]
If alternative (A) holds, then clearly the X-Neumann Problem (2.7) admits a unique weak solution in \( QH^1(\Theta) \) for any choice of \( f \in L^2(\Theta) \), \( K \in \mathbb{N} \), \( T \) a \( K \)-tuple of subunit vector fields and \( g \in [L^2(\Theta)]^K \); this proves 1)-(I). On the other hand, if alternative (B) holds, the Fredholm Alternative states that there are nontrivial solutions \( u \in QH^1(\Theta) \setminus \{(0, 0)\} \) of the homogeneous equation \( u - Ku = 0 \). This proves 1)-(II) and completes the proof of 1).

Assume now that alternative (B) holds, i.e. that 1)-(II) of the statement of the Theorem holds. Let \( K^* : (QH^1(\Theta))^* \to (QH^1(\Theta))^* \) be the adjoint operator to \( K \) and let \( V^* \subset (QH^1(\Theta))^* \) be the subspace of weak solutions of
\[
\Psi - K^*\Psi = 0.
\]
Then, by the Fredholm Alternative we obtain
\[
1 \leq \dim N = \dim V^* < \infty.
\]
By the properties of adjoint operators one has
\[
K^* = \gamma J^* \circ (S_\gamma)^*,
\]
where \( J^* \), \((S_\gamma)^* \) are the adjoint operators of \( J \) and \( S_\gamma \) respectively. Since \( QH^1(\Theta) \) is reflexive and \( \mathcal{L}(u, v) = \mathcal{L}^*(v, u) \) for every \( u, v \in QH^1(\Theta) \), it is evident that \( J^* = J \) and \((S_\gamma)^* = S_\gamma^* \). Therefore, we have that \( \Psi \in (QH^1(\Theta))^* \) is a solution of equation (2.21) if and only if
\[
\Psi = K^*\Psi = \gamma J(S_\gamma^*\Psi).
\]
Setting \( w = S_\gamma^*\Psi \in QH^1(\Theta) \) and recalling the definitions of the mappings \( J \) and \( S_\gamma^* \), we obtain
\[
\Psi(v) = \gamma \int_\Theta vw\,dx \quad \text{for every } v \in QH^1(\Theta),
\]
where \( w \) is a weak solution of the adjoint homogeneous X-Neumann Problem (2.18). This implies that \( V^* = JN^* \), so that
\[
\dim V^* \leq \dim N^*.
\]
as the map $J$ may fail to be injective (we recall that the map $i : QH^1(\Theta) \to L^2(\Theta)$ of Proposition 1.7 may not be injective). We now conclude that 

$$\dim N \leq \dim N^*.$$ 

If we repeat the above argument, replacing the operator $X$ with $X^*$ and the bilinear form $L(\cdot, \cdot)$ with $L^*(\cdot, \cdot)$, we arrive at the opposite inequality: $\dim N \geq \dim N^*$. Part 2) of the theorem had now been proven.

Suppose again that alternative (B) holds. That is, assume that item 1)–(II) holds. By the previous arguments, $u \in QH^1(\Theta)$ is a weak solution of the $X$-Neumann Problem (2.7) if and only if it is a solution of equation (2.20), with $\Phi = S, \varphi \in QH^1(\Theta)$ and where $\varphi \in (QH^1(\Theta))^*$ is defined by (2.19). The Fredholm Alternative indicates that such an equation admits a solution if and only if $\Psi(\Phi) = 0$ for every $\Psi \in V^* = JN^*$. That is, if and only if 

$$\hat{\Theta} \Phi \, dw = 0$$

for every weak solution $w \in QH^1(\Theta)$ of the homogeneous adjoint $X$-Neumann Problem (2.18).

Notice that such $w$ satisfy 

$$\int_\Theta f w + gT w \, dx = \varphi(w) = L_\gamma(\Phi, w) = L^*(w, \Phi) + \gamma \int_\Theta \Phi w \, dx = \gamma \int_\Theta \Phi w \, dx.$$

Thus, problem (2.7) admits a weak solution $u \in QH^1(\Theta)$ if and only if 

$$\int_\Theta f w + gT w \, dx = 0$$

for every $w \in N^*$. This completes the proof of part 3).

With Theorem 2.11 in hand we now begin our analysis of spectral properties associated to the $X$-Neumann problem.

**Theorem 2.12.** Let $(\Omega, \rho)$ be a geometric homogeneous space and let $\Theta$ be a bounded domain with $\Theta \subset \Omega$. Assume that both the local Poincaré inequality (1.6) with $p = 2$ and the global weak Poincaré inequality (1.10) with gain $\omega > 1$ hold. Let $X$ be a second order linear degenerate elliptic operator with rough coefficients as in (1.14). Assume that $F \in L^t(\Theta)$ with $t > \omega'$ and that $G, H \in [L^q(\Theta)]^N$ with $q > 2\omega'$. Then there exists an at most countable set $\Sigma \subset \mathbb{R}$ so that the problem

$$\tag{2.22} \begin{cases}
    Xu = \lambda u + f + T'g & \text{in } \Theta \\
    (\nu, P(x)\nabla u + uG - gT) = 0 & \text{on } \partial \Theta
\end{cases}$$

has a unique weak solution $u \in QH^1(\Theta)$ for every $f \in L^2(\Theta)$, every $K \in \mathbb{N}$, every $K$-tuple $T$ of subunit vector fields and every $g \in [L^q(\Theta)]^K$ if and only if $\lambda \not\in \Sigma$. Moreover, if $\Sigma$ is infinite, its elements can be arranged in a monotonic sequence that diverges to $+\infty$.

**Definition 2.13.** With $\Sigma$ as in Theorem 2.12 we will call any $\lambda \in \Sigma$ an eigenvalue of the $X$-Neumann Problem (2.22). Any weak solution $u \in QH^1(\Theta) \setminus \{(0,0)\}$ of the Homogeneous $X$-Neumann Problem

$$\tag{2.23} \begin{cases}
    Xu = \lambda u & \text{in } \Theta \\
    (\nu, P(x)\nabla u + uG) = 0 & \text{on } \partial \Theta
\end{cases}$$

will be called an eigenfunction of the $X$-Neumann Problem (2.22) associated to the eigenvalue $\lambda$. 
Proof of Theorem 2.12. By Theorem 2.11 there exists a $\gamma > 0$ such that for every $\mu \geq \gamma$, every $f \in L^2(\Theta)$, every $K \in \mathbb{N}$, every $K$-tuple $T$ of subunit vector fields and every $g \in [L^2(\Theta)]^K$ there exists a unique weak solution to problem (2.21). Thus, problem (2.22) admits a unique weak solution whenever $\lambda \leq -\gamma$.

From now on we will assume that $\lambda > -\gamma$, with $\gamma > 0$. Arguing as in the proof of Theorem 2.11 we see that problem (2.22) admits a unique weak solution for every $f \in L^2(\Theta)$, every $K \in \mathbb{N}$, every $K$-tuple $T$ of subunit vector fields and every $g \in [L^2(\Theta)]^K$ if and only if $(u, \nabla u) = (0, 0)$ is the unique weak solution of (2.23). Moreover, if problem (2.23) admits nontrivial weak solutions, then the subspace generated by those weak solutions has finite dimension.

Now, $u \in QH^1(\Theta)$ is a weak solution of (2.23) if and only if it is a weak solution of

$$
\begin{cases}
Xu + \gamma u = (\lambda + \gamma)u & \text{in } \Theta \\
(\nu, P(x)\nabla u + u GS - gT) = 0 & \text{on } \partial \Theta,
\end{cases}
$$

and this in turn holds if and only if

$$
u = S_\gamma \circ J((\gamma + \lambda)u) = \frac{\gamma + \lambda}{\gamma} \mathcal{K}u,$$

where $\mathcal{K} = \gamma S_\gamma \circ J : QH^1(\Theta) \to QH^1(\Theta)$ is the linear, compact operator defined in the proof of Theorem 2.11. Thus, $u \in QH^1(\Theta) \setminus \{(0, 0)\}$ is a weak solution of problem (2.23) if and only if it is an eigenfunction of the compact linear operator $\mathcal{K}$ associated to the eigenvalue $\frac{\lambda + \gamma}{\gamma}$.

The set $\Sigma'$ of real eigenvalues of $\mathcal{K}$ is countable at most and, if it is infinite, its elements can be arranged as a sequence converging to $0$. Consequently, the set $\Sigma \subset \mathbb{R}$ of numbers $\lambda$ such that problem (2.23) has nontrivial weak solutions in $QH^1(\Theta) \setminus \{(0, 0)\}$ is countable at most and, if infinite, it comprises the values of a monotone sequence diverging to $+\infty$.

\[\square\]

Theorem 2.14. Let $(\Omega, \rho)$ be a geometric homogeneous space and let $\Theta$ be a bounded domain with $\overline{\Theta} \subset \Omega$. Assume that both the local Poincaré inequality (1.10) with gain $\omega > 1$ holds. Let $X$ be a second order linear degenerate elliptic operator with rough coefficients as in (1.14). Assume that $F \in L^1(\Theta)$ with $t > \omega'$ and that $G, H \in [L^8(\Theta)]^N$ with $q > 2\omega'$. If $\lambda \notin \Sigma$, there exists a positive constant $C_8 = C_8(\lambda, \Theta, \Omega, c_1, C_1, G, H, F)$ such that if $f \in L^2(\Theta)$, $K \in \mathbb{N}$, $T$ is a $K$-tuple of subunit vector fields, $g \in [L^2(\Theta)]^K$ and $u \in QH^1(\Theta)$ is the unique weak solution of problem (2.22), then one has the estimate

$$
\|u\|_{QH^1(\Theta)} \leq C_8(\|f\|_{L^2(\Theta)} + \sqrt{K}\|g\|_{L^2(\Theta)}).
$$

Proof of Theorem 2.14. We start by showing that under the current hypotheses there exists a constant $\tilde{C} > 0$, independent of $u$, $f$, $K$, $T$, $g$, such that

$$
\|u\|_{L^2(\Theta)} \leq \tilde{C}(\|f\|_{L^2(\Theta)} + \sqrt{K}\|g\|_{L^2(\Theta)}).
$$

To arrive at a contradiction, suppose that (2.25) is false. Then, for every $n \in \mathbb{N}$ there exist $f_n \in L^2(\Theta)$, $K_n \in \mathbb{N}$, $g_n \in [L^2(\Theta)]^{K_n}$, $T_n$ a $K_n$-tuple of subunit vector fields and $u_n \in QH^1(\Theta)$ such that

$$
\begin{cases}
Xu_n = \lambda u_n + f_n + T_n g_n & \text{in } \Theta \\
(\nu, P(x)\nabla u_n + u_n GS - g_n T_n) = 0 & \text{on } \partial \Theta,
\end{cases}
$$

that is

$$
\mathcal{L}(u_n, v) = \int_{\Theta} \lambda u_n v + f_n v + g_n T_n v \, dx
$$

where $\mathcal{L}$ is a linear, compact operator. If $\lambda > 0$, then $u_n$ converges to $0$ in $L^2(\Theta)$ by the Rellich-Kondrachov theorem. If $\lambda < 0$, then $u_n$ converges to a constant $c$ in $L^2(\Theta)$. However, in either case, $\mathcal{L}$ is a compact operator, and hence $\lambda > 0$, which contradicts the assumption that $\lambda \notin \Sigma$. Thus, $\lambda \notin \Sigma$ holds for all $\lambda > 0$, and the proof is complete.
for every $v \in QH^1(\Theta)$, and
\[ \|u_n\|_{L^2(\Theta)} > n(\|f_n\|_{L^2(\Theta)} + \sqrt{K_n}\|g_n\|_{L^2(\Theta)}). \]
Without loss of generality we can assume that $\|u_n\|_{L^2(\Theta)} = 1$ for every $n \in \mathbb{N}$, so that
\[ (\|f_n\|_{L^2(\Theta)} + \sqrt{K_n}\|g_n\|_{L^2(\Theta)}) < \frac{1}{n} \]
for each $n \in \mathbb{N}$. Let $\gamma > 0$ be as in Theorem 2.7. Since $u_n \in QH^1(\Theta)$ is also a weak solution of
\[ \begin{cases} Xu_n + \gamma u_n = (\gamma + \lambda)u_n + f_n + T_n g_n & \text{in } \Theta \\ \langle \nu, P(x)\nabla u_n + u_n G \lambda - g_n T_n \rangle = 0 & \text{on } \partial \Theta, \end{cases} \]
we obtain by inequality (2.10) that
\[ \|u_n\|_{QH^1(\Theta)} \leq \frac{4}{c_1} \left( \|u_n\|_{L^2(\Theta)} + \sqrt{K_n}\|g_n\|_{L^2(\Theta)} \right) \leq \frac{4}{c_1} (\gamma + \lambda + 1) \]
for every $n \in \mathbb{N}$. Since $QH^1(\Theta)$ is a Hilbert space and since $QH^1(\Theta)$ is compactly embedded in $L^2(\Theta)$ by Proposition 1.7, we can assume (up to a subsequence) that
\[ (u_n, \nabla u_n) \rightharpoonup (u, \nabla u) \quad \text{in } QH^1(\Theta), \quad u_n \rightarrow u \quad \text{in } L^2(\Theta). \]
Passing to the limit in (2.27) while exploiting (2.27), (2.28), and the continuity of the bilinear form $\mathcal{L}(\cdot, \cdot)$, we see that for every $v \in QH^1(\Theta)$ we have
\[ \mathcal{L}(u, v) = \int_\Theta \lambda uv \, dx, \]
and, therefore, establishes inequality (2.25).

Now, if $u \in QH^1(\Theta)$ is a weak solution of (2.22) it is also a weak solution of
\[ \begin{cases} Xu + \gamma u = (\gamma + \lambda)u + f + T^* g & \text{in } \Theta \\ \langle \nu, P(x)\nabla u + u G \lambda - g T \rangle = 0 & \text{on } \partial \Theta, \end{cases} \]
where $\gamma > 0$ is as in Theorem 2.7. By inequalities (2.10) and (2.25) we obtain
\[ \|u\|_{QH^1(\Theta)} \leq \frac{4}{c_1} \left( \|\gamma + \lambda\|_{L^2(\Theta)} + \sqrt{K}\|g\|_{L^2(\Theta)} \right) \leq \frac{4}{c_1} (\gamma + \lambda) + 1 \leq \frac{4}{c_1} \left( \|f\|_{L^2(\Theta)} + \sqrt{K}\|g\|_{L^2(\Theta)} \right). \]
which is inequality (2.24) with $C_8 := \frac{4((\gamma + \lambda)\gamma + 1)}{e \epsilon}$.

\[ \]

**Theorem 2.15.** Let $(\Omega, \rho)$ be a geometric homogeneous space and let $\Theta$ be a bounded domain such that $\overline{\Theta} \subset \Omega$. Assume that both the local Poincaré inequality \[ \text{(1.10)} \] with $p = 2$ and the global weak Poincaré inequality \[ \text{(1.10)} \] with gain $\omega > 1$ hold. Let $X$ be a second order linear degenerate elliptic operator with rough coefficients as in \[ \text{(1.14)} \] that satisfies negativity condition (1) as in Definition \[ \text{(1.14)} \]. Assume that $F \in L^1(\Theta)$ with $t > \omega'$ and that $G, H \in [L^q(\Theta)]^N$ with $q > 2\omega'$. Then the only weak solution $(u, \nabla u) \in QH^1(\Theta)$ of the $X$-Neumann Problem

\[
\begin{cases}
Xu = 0 & \text{in } \Theta \\
(\nu, P(x)\nabla u + uGS) = 0 & \text{on } \partial \Theta
\end{cases}
\]
is $(u, \nabla u) = (0, 0)$.

**Proof of Theorem 2.15.** We start by noticing that if $G = H = 0$ almost everywhere in $\Theta$,

\[
0 = \mathcal{L}(u, u) \geq c_1 \int_\Theta \langle \nabla u, Q(x)\nabla u \rangle \, dx + \epsilon \int_\Theta u^2 \, dx \geq \min\{\epsilon, c_1\} \|u\|_{QH^1(\Theta)}^2
\]

where we have used that $X$ satisfies negativity condition (1). Thus, in this case, we see that $(u, \nabla u) = (0, 0)$. From this point onward, we shall assume that $\|G\| + \|H\|_{L^q(\Theta)} \neq 0$.

We now proceed by assuming that there is an $\epsilon > 0$ such that for every $(u, \nabla u), (v, \nabla v) \in QH^1(\Theta)$ satisfying $uv \geq 0$ almost everywhere in $\Theta$ one has

\[ (2.29) \quad \int_\Theta Fuv + GS(uv) \, dx \geq \epsilon \int_\Theta uv \, dx, \]

see Definition \[ \text{(1.10)} \]. Let $(u, \nabla u) \in QH^1(\Theta)$ be a weak solution of \[ (2.29) \]. Then for each $k > 0$,

\[
(v, \nabla v) = ((u - k)^+, \chi_{[u > k]} \nabla u) \in QH^1(\Theta)
\]

is a valid test function for $u$, see \[ \text{[SW2]} \]. Thus,

\[
0 = \mathcal{L}(u, v) = \int_\Theta \langle \nabla v, P(x)\nabla u \rangle \, dx + \int_\Theta vHRu \, dx + \int_\Theta uGSv \, dx + \int_\Theta Fuv \, dx.
\]

Using Lemma \[ \text{2.1} \] and that $uv \geq 0$ almost everywhere in $\Theta$, \[ (2.30) \] gives us:

\[
\int_\Theta \langle \nabla v, P(x)\nabla v \rangle \, dx = - \int_\Theta GS(uv) \, dx + \int_\Theta vGSu \, dx - \int_\Theta vHRu \, dx - \int_\Theta Fuv \, dx
\]

\[
\leq - \epsilon \int_\Theta uv \, dx + \epsilon \int_\Theta v(GSu - HRu) \, dx
\]

\[
\leq - \epsilon \int_\Theta v^2 - \epsilon k \int_\Theta v \, dx + \epsilon \int_\Theta v(GSv - HRv) \, dx.
\]

Setting $\Gamma = \text{supp}(|\sqrt{Q(x)}\nabla v|)$ and $\epsilon_0 = \min\{\epsilon_1, \epsilon\}$, we obtain

\[
\epsilon_0 \|v\|_{QH^1(\Theta)}^2 = \epsilon_0 \left( \int_\Theta |\sqrt{Q(x)}\nabla v|^2 \, dx + \int_\Theta v^2 \, dx \right)
\]

\[
\leq \int_\Theta |\sqrt{P(x)}\nabla v|^2 \, dx + \epsilon \int_\Theta v^2 \, dx
\]

\[
\leq \int_\Theta |v| |H| |Rv| + |G| |Sv| \, dx.
\]

Since \[ \text{[R3] lemma 3.18} \] shows that

\[
|Rv| \leq \sqrt{N} |\sqrt{Q(x)}\nabla v| \quad \text{and} \quad |Sv| \leq \sqrt{N} |\sqrt{Q(x)}\nabla v|
\]
almost everywhere in $\Theta$, we have

$$
\varepsilon_0 \|v\|^2_{QH^1(\Theta)} \leq \int_{\Theta} |v| \left[ |H| |Rv| + |G| |Sv| \right] dx
\leq \sqrt{N} \| (|G| + |H|) |v| \|_{L^2(\Gamma)} \| \sqrt{Q(x)} \nabla v \|_{L^2(\Gamma)}
\leq \sqrt{N} \| (|G| + |H|) |v| \|_{L^2(\Gamma)} \|v\|_{QH^1(\Theta)},
$$

and so H"older’s inequality and (1.6) give

$$
\hat{v} \text{ almost everywhere in } \Theta \text{ one has}
\|v\|_{QH^1(\Theta)} \leq \frac{\sqrt{N} \| (|G| + |H|) |v| \|_{L^2(\Gamma)} \|v\|_{QH^1(\Theta)}}{\varepsilon_0} \leq C_3 \frac{\sqrt{N} \| (|G| + |H|) |v| \|_{L^2(\Theta)} \|v\|_{QH^1(\Theta)}}{\varepsilon_0}.
$$

Dividing (2.31) by $\|v\|_{QH^1(\Theta)}$ we obtain

$$
(2.32) \quad |\Gamma| \geq \left( \frac{\varepsilon_0}{\sqrt{N} C_3 \| (|G| + |H|) |v| \|_{L^2(\Theta)}} \right) \frac{2\varepsilon}{\varepsilon_0} > 0,
$$

independently of $k \in \mathbb{R}$. Now if $l = \sup_{\Theta} u > 0$, we may choose $k > 0$ and let $k \nearrow l$ to arrive at a contradiction. Indeed, $v$ tends to 0 almost everywhere in $\Theta$ as $k \nearrow l$ and also in $L^2(\Theta)$. By (2.31) we obtain that $(v, \nabla v)$ tends to $(0, 0)$ in $QH^1(\Theta)$. But then (2.32) gives

$$
0 = \lim_{k \to l^-} \int_{\Theta} \langle \nabla v, Q(x) \nabla v \rangle dx = \lim_{k \to l^-} \int_{\{u > k\}} \langle \nabla u, Q(x) \nabla u \rangle dx > 0
$$

and we have that

$$
\sup_{\Theta} u \leq 0.
$$

Repeating the above argument, this time with $(v, \nabla v) = ((u + k)^-, \chi_{\{u < -k\}} \nabla u) \in QH^1(\Theta)$, see [SW2], for any $k$ satisfying $0 < k < -\inf_{\Theta} u$ we obtain that

$$
\inf_{\Theta} u \geq 0.
$$

We conclude that $u = 0$ almost everywhere on $\Theta$. Thus, any weak solution of problem (2.29) is of the form $(u, \nabla u) = (0, h) \in QH^1(\Theta)$. By Definition 2.6, since $(u, \nabla u) = (0, h)$ is a solution of (2.29), we must have

$$
0 = \mathcal{L}(u, u) = \int_{\Theta} \langle \nabla u, P(x) \nabla u \rangle dx + \int_{\Theta} uHRu dx + \int_{\Theta} uGSu dx + \int_{\Theta} Fu^2 dx
\geq c_1 \int_{\Theta} \langle h, Q(x) h \rangle dx.
$$

Thus,

$$
\|u\|_{QH^1(\Theta)} = \left( \int_{\Theta} u^2 dx + \int_{\Theta} \langle \nabla u, Q(x) \nabla u \rangle dx \right)^{\frac{1}{2}} = \left( \int_{\Theta} \langle h, Q(x) h \rangle dx \right)^{\frac{1}{2}} = 0.
$$

We conclude that $(u, \nabla u) = (0, 0)$, completing the proof in case (2.30) holds.

If instead there exists $\varepsilon > 0$ such that for every $(u, \nabla u), (v, \nabla v) \in QH^1(\Theta)$ satisfying $uv \geq 0$ almost everywhere in $\Theta$ one has

$$
\int_{\Theta} Fuu + HRR(uv) dx \geq \varepsilon \int_{\Theta} uv dx,
$$
see Definition 1.10, then we can repeat the above argument for the adjoint homogeneous $X$–Neumann Problem

\[(2.33)\]

\[
\begin{aligned}
X^* v &= 0 \quad \text{in } \Theta \\
\langle \nu, P(x) \nabla v + v \nabla \rangle &= 0 \quad \text{on } \partial \Theta
\end{aligned}
\]

and conclude that it admits only the trivial weak solution $(v, \nabla v) = (0, 0) \in QH^1(\Theta)$. Now suppose that problem (2.29) has a nontrivial weak solution $(u, \nabla u) \in QH^1(\Theta)$, then by Theorem 2.11 also problem (2.33) must admit a nontrivial weak solution, a contradiction. Thus the only weak solution of problem (2.29) is $(u, \nabla u) = (0, 0)$, and the proof is complete.

\[\square\]

**Corollary 2.16.** Assume the hypotheses of Theorem 2.15 hold. Then the set $\Sigma$ of real eigenvalues of the $X$–Neumann Problem (2.23) satisfies $\Sigma \subset (0, \infty)$.

**Corollary 2.17.** Assume the hypotheses of Theorem 2.15 hold. Then \(0 \notin \Sigma\) and the $X$–Neumann Problem (2.3) admits a unique weak solution $u \in QH^1(\Theta)$ for every $f \in L^2(\Theta)$, every $K \in \mathbb{N}$, every $K$–tuple $T$ of subunit vector fields and every $g \in [L^2(\Theta)]^K$. Moreover there exists a constant $C > 0$, independent of $u$, $f$, $K$, $T$ and $g$, such that

\[
\|u\|_{QH^1(\Theta)} \leq C(\|f\|_{L^2(\Theta)} + \sqrt{K} \|g\|_{L^2(\Theta)})
\]

when $u \in QH^1(\Theta)$ is a weak solution of (2.7).

**Proof of Corollaries 2.16 and 2.17:** These corollaries are simple consequences of Theorems 2.11, 2.12 and 2.15.

\[\square\]

**Remark 2.18.** Let \((\Omega, \rho)\) be a geometric homogeneous space and let $\Theta$ be a bounded domain with $\overline{\Theta} \subset \Omega$. Assume that the local Poincaré inequality (1.6) with $p = 2$ holds, and that the global weak Poincaré inequality (1.10) holds for every $f \in L^p(\Theta)$, every $K \in \mathbb{N}$, every $K$–tuple $T$ of subunit vector fields and every $g \in [L^p(\Theta)]^K$. It was shown in [R3] that negativity condition (2)–i) for the operator $X$, see Definition 1.11, is sufficient for the well–posedness of the Dirichlet Problem

\[
\begin{aligned}
X u &= f + T'g \quad \text{in } \Theta \\
u &= 0 \quad \text{on } \partial \Theta,
\end{aligned}
\]

with $f \in L^2(\Theta)$, $K \in \mathbb{N}$, $T$ a $K$–tuple of subunit vector fields and $g \in [L^2(\Theta)]^K$. As is shown by Example 2.27 below, negativity condition (2) for $X$ is not sufficient for the well–posedness of the corresponding Neumann Problem (2.7).

**Remark 2.19.** All the preceding results easily extend to include complex valued weak solutions and complex eigenvalues/eigenfunctions.

**Theorem 2.20.** Let \((\Omega, \rho)\) be a geometric homogeneous space and $\Theta$ be a bounded domain such that $\overline{\Theta} \subset \Omega$. Assume that the local Poincaré inequality (1.6) holds for every $f \in L^p(\Theta)$, every $K \in \mathbb{N}$, every $K$–tuple $T$ of subunit vector fields and every $g \in [L^p(\Theta)]^K$. As is shown by Example 2.27 below, negativity condition (2) for $X$ is not sufficient for the well–posedness of the corresponding Neumann Problem (2.7).

1) All the eigenvalues of the $X$–Neumann Problem (2.23) are real, infinite and can be ordered in a monotone sequence which diverges to $+\infty$.
2) One has
\[ \lambda_1 = \min \Sigma = \min_{u \in QH^1(\Theta) \setminus \{(0,h)\}} \frac{\mathcal{L}(u,u)}{\int_{\Theta} u^2 \, dx}. \]
Moreover there exists an eigenfunction \((u_1, \nabla u_1) \in QH^1(\Theta)\) of the Neumann Problem (2.23) related to the eigenvalue \(\lambda_1\) such that \(u_1 \geq 0\) a.e. in \(\Theta\).

3) One has that \(\lambda_2 = \min \left\{ \frac{\mathcal{L}(u,u)}{\int_{\Theta} u^2 \, dx} \mid u \in QH^1(\Theta) \setminus \{(0,h)\}, \int_{\Theta} uu_j \, dx = 0 \text{ for all } j = 1, \ldots, k-1 \right\} \)
is an eigenvalue of the Neumann Problem (2.23), with corresponding eigenfunction \((u_2, \nabla u_2) \in QH^1(\Theta)\) whose first component \(u_2\) is orthogonal to \(u_1\) in \(L^2(\Theta)\). Recursively, for every \(k \in \mathbb{N}\)
\[ \lambda_k = \min \left\{ \frac{\mathcal{L}(u,u)}{\int_{\Theta} u^2 \, dx} \mid u \in QH^1(\Theta) \setminus \{(0,h)\}, \int_{\Theta} uu_j \, dx = 0 \text{ for all } j = 1, \ldots, k-1 \right\} \]
is an eigenvalue of the Neumann Problem (2.23), with corresponding eigenfunction \((u_k, \nabla u_k) \in QH^1(\Theta)\) whose first component \(u_k\) is orthogonal to \(u_j\) in \(L^2(\Theta)\) for every \(j = 1, \ldots, k-1\).

4) \(\lambda \in \mathbb{R}\) is an eigenvalue if and only if \(\lambda = \lambda_k\) for some \(k \in \mathbb{N}\).

5) The sequence \(\{u_k\}_{k \in \mathbb{N}} \subset L^2(\Theta)\) forms a complete orthogonal system of \(L^2(\Theta)\). The sequence \(\{(u_k, \nabla u_k)\}_{k \in \mathbb{N}} \subset QH^1(\Theta)\) is an independent system of elements of \(QH^1(\Theta)\), which is also a system of generators of the whole space if and only if the projection map \(i : QH^1(\Theta) \to L^2(\Theta)\) is injective.

6) The \(X\)-Neumann Problem (2.22) is variational, with associated functional defined on \(QH^1(\Theta)\) by
\[ I(u) = \frac{1}{2} \mathcal{L}(u,u) - \frac{\lambda}{2} \int_{\Theta} u^2 \, dx - \int_{\Theta} fu + g^T u \, dx. \]

**Proof of Theorem** 2.20: The proof of this Theorem is a standard application of functional analysis techniques, see for instance chapter 8.12 of [GT].

\[ \square \]

**Example** 2.21. Let \((\Omega, \rho)\) be a geometric homogeneous space and fix a bounded piecewise \(C^1\) domain \(\Theta\) with \(\overline{\Theta} \subset \Omega\). Let \(P = P(x)\) be an \(n \times n\) matrix as in (1.14) and consider the following Neumann Problem

\[ \begin{cases} -\text{div}(P(x)\nabla u) = f + T^T g & \text{in } \Theta \\ \langle \nu, P(x)\nabla u \rangle = 0 & \text{on } \partial \Theta, \end{cases} \]

where \(f \in L^2(\Theta)\), \(K \in \mathbb{N}\), \(T\) is a \(K\)-tuple of subunit vector fields and \(g \in [L^2(\Theta)]^K\). Assume that the following hold:

1) the global weak Poincaré inequality on \(\Theta\) with gain \(\omega > 1\), see (1.10),
2) the local Poincaré inequality \((1.6)\) with \(p = 2\).

Theorems 2.11, 2.12 and 2.14 apply to this problem where
\[ Xu = -\text{div}(P(x)\nabla u) \quad \text{and} \quad \mathcal{L}(u,v) = \int_{\Theta} \langle \nabla v, P(x)\nabla u \rangle \, dx. \]
Moreover, the problem is self adjoint so that Theorem 2.20 also applies. Let \( N \subset QH^1(\Theta) \) be the subspace of weak solutions of
\[
\begin{aligned}
-\text{div}(P(x)\nabla u) &= 0, & \text{in } \Theta \\
\langle \nu, P(x)\nabla u \rangle &= 0, & \text{on } \partial \Theta.
\end{aligned}
\]

Obviously,
\[\{(u, \nabla u) = (c, 0) \in QH^1(\Theta) \mid c \in \mathbb{R}\} \subseteq N.\]

Now suppose \((u, \nabla u) \in N\). Then by Definition 2.6, for every \(v \in QH^1(\Theta)\) one has
\[L(u, v) = \int_\Theta \langle \nabla v, P(x)\nabla u \rangle \, dx = 0.\]

In particular, choosing \((v, \nabla v) = (u, \nabla u)\),
\[\int_\Theta |u - u_{B_r(y)}| \, dx \leq C \left( \int_{B_r(y)} |\nabla u|^2 \, dx \right)^{1/2} = 0,
\]
where \(C > 0 \) is independent of \((u, \nabla u) \in QH^1(\Theta)\). Hence \(u = u_{B_r(y)}\) almost everywhere on \(B_r(y)\). As quasimetric balls are open sets, the function \(u \in L^2(\Theta)\) is locally constant in \(\Theta\). Since \(\Theta\) is connected, \(u\) must therefore be constant in \(\Theta\). This proves that
\[\{(u, \nabla u) = (c, 0) \in QH^1(\Theta) \mid c \in \mathbb{R}\} = N.\]

By Theorem 2.11 problem (2.34) admits a weak solution \(u \in QH^1(\Theta)\) if and only if
\[\int_\Theta cf + gT_0 \, dx = 0 \quad \text{for every } (c, 0) \in N,
\]
i.e. if and only if
\[\int_\Theta f \, dx = 0.
\]

Now it is clear that if \((u, \nabla u) \in QH^1(\Theta)\) is a weak solution of problem (2.34), then so is \((u+c, \nabla u)\) for every \(c \in \mathbb{R}\). Hence it is also clear that for every \(f \in L^2(\Theta)\) satisfying \(\int_\Theta f \, dx = 0\), every \(K \in \mathbb{N}\), every \(K\)-tuple \(T\) of subunit vector fields and every \(g \in [L^2(\Theta)]^K\) there exists a unique weak solution \((u, \nabla u) \in QH^1(\Theta)\) of problem (2.34) satisfying
\[\int_\Theta u \, dx = 0.
\]

**Claim:** We claim that there exists a positive constant \(C\) so that if \(f \in L^2(\Theta)\) satisfies \(\int_\Theta f \, dx = 0\), \(K \in \mathbb{N}\), \(T\) is a \(K\)-tuple of subunit vector fields, \(g \in [L^2(\Theta)]^K\) and \((u, \nabla u) \in QH^1(\Theta)\) is a weak solution of problem (2.34) with \(\int_\Theta u \, dx = 0\), then
\[\|u\|_{QH^1(\Theta)} \leq C \left( \|f\|_{L^2(\Theta)} + \sqrt{K} \|g\|_{L^2(\Theta)} \right).
\]

To see this, notice that Proposition 1.7 applies under our current assumptions and we have that the projection onto the first component is a compact mapping from \(QH^1(\Theta)\) into \(L^2(\Theta)\).
Therefore, we are able to apply Theorem 5.11 with \( p = 2 \), see Section 5, to conclude inequality (2.36). Thus,

\[
\int \Theta u^2 \, dx = \int \Theta |u - u_0|^2 \, dx \leq C_5 \int \Theta \langle \nabla u, Q(x) \nabla u \rangle \, dx.
\]

It is now easy to see that

\[
\|u\|_{QH^1(\Theta)}^2 \leq (C_5 + 1) \int \Theta \langle \nabla u, Q(x) \nabla u \rangle \, dx.
\] (2.36)

Using that \( u \in QH^1(\Theta) \) is a weak solution of (2.36) together with the subuniticity of \( \Phi \), we have

\[
\int \Theta \langle \nabla u, Q(x) \nabla u \rangle \, dx \leq \frac{1}{c_1} \int \Theta \langle \nabla u, P(x) \nabla u \rangle \, dx
\]

\[
= \frac{1}{c_1} \int \Theta f u + g \Phi u \, dx
\]

\[
\leq \frac{1}{c_1} \left( \|f\|_{L^2(\Theta)} \|u\|_{L^2(\Theta)} + \|g\|_{L^2(\Theta)} \|T u\|_{L^2(\Theta)} \right)
\]

\[
\leq \frac{1}{c_1} \left( \|f\|_{L^2(\Theta)} + \sqrt{K} \|g\|_{L^2(\Theta)} \right) \|u\|_{QH^1(\Theta)}.
\] (2.37)

Thus (2.36) and (2.37) together yield (2.35), proving our claim with \( C = \frac{C_5 + 1}{c_1} \). Furthermore, (2.36) also shows that

\[
\int \Theta \langle \nabla u, Q(x) \nabla u \rangle \, dx \leq \|u\|_{QH^1(\Theta)}^2 \leq (C_5 + 1) \int \Theta \langle \nabla u, Q(x) \nabla u \rangle \, dx
\]

for every \( u \in QH^1(\Theta) \) with \( \int \Theta u \, dx = 0 \). Therefore, the norm

\[
\|u\|_{QH^1(\Theta)} = \int \Theta \langle \nabla u, Q(x) \nabla u \rangle \, dx,
\]

is equivalent to \( \|u\|_{QH^1(\Theta)} \) on the subspace \( QH^1_\perp(\Theta) \subset QH^1(\Theta) \) defined by

\[
QH^1_\perp(\Theta) = \left\{ u \in QH^1(\Theta) \mid \int \Theta u \, dx = 0 \right\} = N^\perp.
\]

We also mention that Theorem 2.18 shows that the problem

\[
\begin{cases}
-\text{div}(P(x) \nabla u) = \lambda u + f + T^* g & \text{in } \Theta,

(u, P(x) \nabla u) = 0 & \text{on } \partial \Theta,
\end{cases}
\] (2.38)

with \( \lambda < 0 \) admits a unique weak solution \( u \in QH^1(\Theta) \) for every \( f \in L^2(\Theta) \), every \( K \in \mathbb{N} \), every \( K \)-tuple \( T \) of subunit vector fields and every \( g \in [L^2(\Theta)]^K \). Hence, all eigenvalues of problem (2.38) must be nonnegative with \( \lambda_1 = \min \Sigma = 0 \) where \( \Sigma \subset \mathbb{R} \) is the set of eigenvalues of problem (2.38). Furthermore, the eigenvalue \( \lambda_1 \) is simple. Since problem (2.38) is self-adjoint, its eigenvalues form a monotone sequence diverging to \( +\infty \). The corresponding eigenfunctions \( \{u_k, \nabla u_k\}_{k \in \mathbb{N}} \subset QH^1(\Theta) \) form an independent system of elements such that \( \{u_k\}_{k \in \mathbb{N}} \subset L^2(\Theta) \) is a complete orthogonal system. Moreover, one can choose \( (u_1, \nabla u_1) = (1, 0) \in QH^1(\Theta) \). If the projection map \( i : QH^1(\Theta) \to L^2(\Theta) \) is injective, then the eigenfunctions \( \{u_k, \nabla u_k\}_{k \in \mathbb{N}} \) are also a system of generators of \( QH^1(\Theta) \).

One has that \( (u, \nabla u) \in QH^1(\Theta) \) is a solution of problem (2.38) if and only if it is a critical point of the functional \( I : QH^1(\Theta) \to \mathbb{R} \) defined by

\[
I(u) = \frac{1}{2} \int \Theta \langle \nabla u, P(x) \nabla u \rangle \, dx - \frac{\lambda}{2} \int \Theta u^2 \, dx - \int \Theta f u + g \Phi u \, dx.
\]
Finally, if \( P(x) = Q(x) \) almost everywhere in \( \Theta \) one can show that \( \{(u_k, \nabla u_k)\}_{k \in \mathbb{N}} \subset QH^1(\Theta) \) is an orthogonal system. Indeed, for every \( k \neq j, k, j \in \mathbb{N} \), one has

\[
\int_{\Theta} (\nabla u_k, Q(x) \nabla u_j) \, dx = \mathcal{L}(u_j, u_k) = \lambda_j \int_{\Theta} u_j u_k \, dx = 0.
\]

Hence for every \( k \neq j, k, j \in \mathbb{N} \), we obtain

\[
(u_j, u_k)_{QH^1(\Theta)} = \int_{\Theta} (\nabla u_j, Q(x) \nabla u_k) \, dx + \int_{\Theta} u_j u_k \, dx = 0,
\]
as claimed.

The results on the Neumann Problem \([23,31]\) described above extend and complete previous results on such problems obtained in \([R1]\).

3. Spectral Results for the X-Dirichlet Problem

In this section we give a spectral theorem related to X-Dirichlet problems (that is, Dirichlet problems associated to the operator \( X \) with rough coefficients as in \([1.14]\)). Existence results for X-Dirichlet problems are given in \([R3]\) and we refer the interested reader there for statements and proofs. We begin by recalling the definition of weak solution related to the X-Dirichlet Problem on a bounded domain with homogeneous boundary data as given in \([R3]\).

**Definition 3.1.** Let \( X \) be a second order linear degenerate elliptic operator with rough coefficients as in \([1.14]\). Assume that the global Sobolev inequality with gain \( \sigma > 1 \) holds, see \([1.7]\). Let \( G, H \in [L^q(\Theta)]^N \) with \( q \geq 2\sigma' \) and let \( F \in L^t(\Theta) \) with \( t \geq \sigma' \). If \( f \in L^2(\Theta) \), \( K \in \mathbb{N} \), \( T \) is a \( K \)-tuple of subunit vector fields and \( g \in [L^2(\Theta)]^K \), a function \( (u, \nabla u) \in QH_0^1(\Theta) \) is a weak solution of the Dirichlet Problem

\[
\begin{cases}
Xu = f + T'g & \text{in } \Theta \\
\nabla u = 0 & \text{on } \partial \Theta
\end{cases}
\]

if and only if

\[
\mathcal{L}(u, v) = \int_{\Theta} fv + gT'v \, dx \quad \text{for all } v \in QH_0^1(\Theta).
\]

**Theorem 3.2.** Let \((\Omega, \rho)\) be a geometric homogeneous space and let \( \Theta \) be a bounded domain such that \( \overline{\Theta} \subset \Omega \). Assume that the local Poincaré inequality \([1.9]\) holds with \( p = 2 \) and that the global Sobolev inequality \([1.7]\) with gain \( \sigma > 1 \) holds. Let \( X \) be a second order linear degenerate elliptic operator with rough coefficients as in \([1.14]\). Assume that \( F \in L^t(\Theta) \) with \( t > \sigma' \) and that \( G, H \in [L^q(\Theta)]^N \) with \( q > 2\sigma' \). Then each of the following hold.

1) There exists an at most countable set \( \Sigma \subset \mathbb{R} \) such that the X-Dirichlet problem

\[
\begin{cases}
Xu = \lambda u + f + T'g & \text{in } \Theta \\
\nabla u = 0 & \text{on } \partial \Theta
\end{cases}
\]

admits a unique weak solution \( u \in QH_0^1(\Theta) \) for every \( f \in L^2(\Theta) \), every \( K \in \mathbb{N} \), every \( K \)-tuple \( T \) of subunit vector fields and every \( g \in [L^2(\Theta)]^K \) if and only if \( \lambda \notin \Sigma \).

2) If \( \Sigma \) is infinite, its elements can be arranged in a monotone sequence diverging to \( +\infty \).

3) If \( \lambda \notin \Sigma \) there exists a constant \( C = C(\lambda, \Theta, \Omega, c_1, C_1, G, H, F) > 0 \) such that

\[
\|u\|_{QH_0^1(\Theta)} \leq C(\|f\|_{L^2(\Theta)} + \sqrt{K} \|g\|_{L^2(\Theta)})
\]
whenever \( f \in L^2(\Theta), K \in \mathbb{N}, T \) is a \( K \)-tuple of subunit vector fields, \( g \in [L^2(\Theta)]^K \) and \( u \in QH^1_0(\Theta) \) is a weak solution of (3.1).

4) If \( \lambda \in \Sigma \), let \( N \subset QH^1_0(\Theta) \) be the subspace of weak solutions of the \( X \)-Dirichlet Problem

\[
\begin{aligned}
Xu &= \lambda u \quad \text{in } \Theta \\
u &= 0 \quad \text{on } \partial \Theta,
\end{aligned}
\]

and let \( N^* \subset QH^1_0(\Theta) \) be the subspace of weak solutions of the adjoint problem

\[
\begin{aligned}
X^*u &= \lambda u \quad \text{in } \Theta \\
u &= 0 \quad \text{on } \partial \Theta.
\end{aligned}
\]

Then \( 1 \leq \dim N = \dim N^* < \infty \) and problem (3.1) admits a weak solution \( u \in QH^1_0(\Theta) \) if and only if

\[
\int_\Theta f v + gT_v dx = 0 \quad \text{for all } v \in N^*.
\]

5) If \( X \) satisfies negativity condition (2), see Definition 1.11, then \( \Sigma \subset (0, \infty) \).

6) If \( X \) is self-adjoint (that is, if \( HR = GS \) almost everywhere in \( \Theta \)), then all eigenvalues of \( X \) are real, \( \Sigma \) is infinite and we have the following variational characterization of the eigenvalues of \( X \):

\[
\lambda_1 = \min \Sigma \min_{u \in QH^1_0(\Theta) \setminus \{0\}} \frac{\mathcal{L}(u, u)}{\int_\Theta u^2 \, dx},
\]

and there exists an eigenfunction \((u_1, \nabla u_1) \in QH^1_0(\Theta)\) of the \( X \)-Dirichlet Problem (3.1) related to the eigenvalue \( \lambda_1 \) for whom \( u_1 \geq 0 \) a.e. in \( \Theta \). Furthermore,

\[
\lambda_2 = \min \left\{ \frac{\mathcal{L}(u, u)}{\int_\Theta u^2 \, dx} \middle| u \in QH^1_0(\Theta) \setminus \{(0, h)\}, \int_\Theta u_1 \, dx = 0 \right\},
\]

with corresponding eigenfunction \((u_2, \nabla u_2) \in QH^1_0(\Theta)\) where \( u_2 \) is orthogonal to \( u_1 \) in \( L^2(\Theta) \). Recursively, for every \( k \in \mathbb{N} \)

\[
\lambda_k = \min \left\{ \frac{\mathcal{L}(u, u)}{\int_\Theta u^2 \, dx} \middle| u \in QH^1_0(\Theta) \setminus \{(0, h)\}, \int_\Theta u_j \, dx = 0 \text{ for all } j = 1, \ldots, k-1 \right\},
\]

with corresponding eigenfunction \((u_k, \nabla u_k) \in QH^1_0(\Theta)\), where \( u_k \) is orthogonal to \( u_j \) in \( L^2(\Theta) \) for every \( j = 1, \ldots, k-1 \). Moreover, \( \lambda \in \mathbb{R} \) is an eigenvalue if and only if \( \lambda = \lambda_k \) for some \( k \in \mathbb{N} \). The sequence \( \{u_k\}_{k \in \mathbb{N}} \subset L^2(\Theta) \) forms a complete orthogonal system of \( L^2(\Theta) \). The sequence \( \{(u_k, \nabla u_k)\}_{k \in \mathbb{N}} \subset QH^1_0(\Theta) \) is an independent system of elements of \( QH^1_0(\Theta) \), which is also a system of generators of \( QH^1_0(\Theta) \) if and only if the projection map \( i : QH^1_0(\Theta) \to L^2(\Theta) \) is injective. Finally, problem (3.3) is variational with associated functional defined on \( QH^1_0(\Theta) \) by

\[
I(u) = \frac{1}{2} \mathcal{L}(u, u) - \frac{\lambda}{2} \int_\Theta u^2 \, dx - \int_\Theta fu + gT_u \, dx.
\]

**Proof of Theorem 3.2** The proof of this theorem is similar to the proofs of the preceding section where the global Poincaré inequality (1.10) with gain \( \omega > 1 \) is replaced with the global Sobolev inequality (1.7) when necessary. We therefore omit the proof.

\[\Box\]
4. A Maximum Principle for Second Order Linear Degenerate Elliptic Equations with Rough Coefficients

This section contains a maximum principle for weak solutions of the differential inequality $Xu \leq 0$ for second order degenerate elliptic operators $X$ with rough coefficients. To this end, we fix a geometric homogeneous space $(\Omega, \rho)$ with $\Omega$ as in Section 1. We also fix an $n \times n$ matrix $Q(x)$ as in Section 1 and let $\Theta$ be a bounded domain satisfying $\overline{\Theta} \subset \Omega$. Furthermore, in order to avoid confusion we will refer to an element of $QH^1(\Theta)$ by writing $u \in QH^1(\Theta)$ where $u = (u, \nabla u)$.

We begin by giving the definition of weak solution of $Xu \leq 0$ in $\Theta$.

**Definition 4.1.** We say that $u \in QH^1(\Theta)$ is a weak solution of

$$Xu \leq 0 \quad \text{in} \quad \Theta$$

if and only if $L(u, v) \leq 0$ for every $v = (v, \nabla v) \in QH^1_0(\Theta)$ satisfying $v \geq 0$ almost everywhere in $\Theta$.

In order to state our maximum principle, we define a notion of non-positivity for the first component $u$ of an element $u = (u, \nabla u) \in QH^1(\Theta)$ in terms of membership in the space $QH^1_0(\Theta)$.

**Definition 4.2.**

(1) We say that $u = (u, \nabla u) \in QH^1(\Theta)$ satisfies $u \leq 0$ on $\partial \Theta$ if and only if $u^+ = (u^+, \chi_{\{u > 0\}} \nabla u) \in QH^1_0(\Theta)$.

(2) We say that $u, v \in QH^1(\Theta)$ satisfy $u \leq v$ on $\partial \Theta$ if and only if $u - v \leq 0$ on $\partial \Theta$ in the sense of item (1).

(3) For each $k \in \mathbb{R}$ recall that $k = (k, 0) \in QH^1(\Theta)$ as $\Theta$ is bounded. Thus, for $u = (u, \nabla u) \in QH^1(\Theta)$ we define

$$\sup_{\partial \Theta} u = \inf \left\{ k \in \mathbb{R} \mid u \leq k \text{ on } \partial \Theta \right\}, \text{ and } \inf_{\partial \Theta} u = - \sup_{\partial \Theta} (-u).$$

**Theorem 4.3.** Noting the first paragraph of this section, assume that the local Poincaré inequality (1.6) with $p = 2$ holds and that the global Sobolev inequality (1.7) with gain $\sigma > 1$ holds. Let $X$ be a second order linear degenerate elliptic operator with rough coefficients as in (1.14) that satisfies negativity condition (2)–i), see Definition 1.11. Assume that $G \subset \Omega$ and fix a geometric homogeneous space $\Omega$ as in Section 1 and let $\Theta$ be a bounded domain satisfying $\overline{\Theta} \subset \Omega$. Furthermore, in order to avoid confusion we will refer to an element of $QH^1(\Theta)$ by writing $u \in QH^1(\Theta)$ where $u = (u, \nabla u)$. We begin by giving the definition of weak solution of $Xu \leq 0$ in $\Theta$.

**Definition 4.4.**

(1) We say that $u = (u, \nabla u) \in QH^1(\Theta)$ satisfies $u \leq 0$ on $\partial \Theta$ if and only if $u^+ = (u^+, \chi_{\{u > 0\}} \nabla u) \in QH^1_0(\Theta)$.

(2) We say that $u, v \in QH^1(\Theta)$ satisfy $u \leq v$ on $\partial \Theta$ if and only if $u - v \leq 0$ on $\partial \Theta$ in the sense of item (1).

(3) For each $k \in \mathbb{R}$ recall that $k = (k, 0) \in QH^1(\Theta)$ as $\Theta$ is bounded. Thus, for $u = (u, \nabla u) \in QH^1(\Theta)$ we define

$$\sup_{\partial \Theta} u = \inf \left\{ k \in \mathbb{R} \mid u \leq k \text{ on } \partial \Theta \right\}, \text{ and } \inf_{\partial \Theta} u = - \sup_{\partial \Theta} (-u).$$

**Theorem 4.3.** Noting the first paragraph of this section, assume that the local Poincaré inequality (1.6) with $p = 2$ holds and that the global Sobolev inequality (1.7) with gain $\sigma > 1$ holds. Let $X$ be a second order linear degenerate elliptic operator with rough coefficients as in (1.14) that satisfies negativity condition (2)–i), see Definition 1.11. Assume that $F \subset L^1(\Theta)$ with $t > \sigma'$ and that $G, H \in [L^q(\Theta)]^N$ with $q > 2\sigma'$. If $u \in QH^1(\Theta)$ is a weak solution of (4.1) then

$$\sup_{\partial \Theta} u \leq \sup_{\partial \Theta} u^+. $$

**Proof of Theorem 4.3.** We argue by contradiction. Let $l = \sup_{\partial \Theta} u$, let $m = \sup_{\partial \Theta} u^+$ and suppose that $l > m$. Fix $k \in \mathbb{R}$ with $m < k < l$ and set $v_k = (v_k, \nabla v_k) = ((u - k)^+, \chi_{\{u > k\}} \nabla u) \in QH^1(\Theta)$. Since $k > m$ and since $m \geq 0$, it is not difficult to see that $v_k \in QH^1_0(\Theta)$ by Definition 4.2. Moreover, $v_k \geq 0$ almost everywhere in $\Theta$. Thus, $uv_k \geq 0$ almost everywhere in $\Theta$, and an application of Lemma 2.1 implies that $GS(uv_k) = uGSv_k + v_kGSu$ and $HR(uv_k) = uHRv_k + v_kHRu$.

Consider now the case where $G = H = 0$ almost everywhere on $\Theta$. Using that $X$ satisfies negativity condition (2)–i) and that $\nabla v_k = \nabla u$ a.e. on the support of $\nabla v_k$, the definition of
Recalling the result of [R3, Lemma 3.18] and setting $\Gamma = \text{supp}(2) – i$), we argue as in the proof of Theorem 2.15 to obtain

$$0 \geq L(u, v_k) = \int_{\Theta} \langle \nabla v_k, P(x) \nabla u \rangle \, dx + \int_{\Theta} F uv_k \, dx \geq \int_{\Theta} \langle \nabla v_k, Q(x) \nabla v_k \rangle \, dx.$$ 

By the global Sobolev inequality (1.7) we see that $v_k = (u - k)^+ = 0$ in $L^2(\Theta)$ and hence that $u \leq k$ almost everywhere in $\Theta$, contradicting our assumption $k < \sup_{\Theta} u$.

We now focus on the case where $|G| + |H| \not\equiv 0$. Since $X$ satisfies negativity condition (2–1), we argue as in the proof of Theorem 2.15 to obtain

$$\int_{\Theta} \langle \nabla v_k, P(x) \nabla v_k \rangle \, dx = \int_{\Theta} \langle \nabla v_k, P(x) \nabla u \rangle \, dx \leq \int_{\Theta} |v_k|(|G||sv_k| + |H|Rv_k) \, dx.$$ 

Recalling the result of [R3, Lemma 3.18] and setting $\Gamma = \text{supp}(|Q(x)\nabla v_k|)$, we see that

$$\|v_k\|^2_{QH^1_0(\Theta)} \leq \frac{1}{c_1} \int_{\Theta} \langle \nabla v_k, P(x) \nabla v_k \rangle \, dx \leq \frac{\sqrt{N}}{c_1} \|(|G| + |H|)v_k\|_{L^2(\Gamma)} \left( \int_{\Theta} \langle \nabla v_k, Q(x) \nabla v_k \rangle \, dx \right)^\frac{1}{2} \leq \frac{\sqrt{N}}{c_1} \|G\|_{L^q(\Theta)} |\nabla v_k|_{L^p(\Theta)} \|v_k\|_{QH^1_0(\Theta)}$$

$$\leq \frac{\sqrt{N}}{c_1} \|G\|_{L^q(\Theta)} |\nabla v_k|_{L^p(\Theta)} \|v_k\|_{QH^1_0(\Theta)}$$

$$\leq \frac{\sqrt{N}}{c_1} \|G\|_{L^q(\Theta)} |\nabla v_k|_{L^p(\Theta)} \|v_k\|_{QH^1_0(\Theta)}$$

Dividing through by $\|v_k\|^2_{QH^1_0(\Theta)}$ we have

$$|\Gamma| \geq \left( \frac{c_1}{\sqrt{N} \tilde{C} \||G| + |H|\|_{L^q(\Theta)}} \right)^\frac{2q \sigma'}{q + 2 \sigma'} > 0$$

independently of $k$, for $m < k < l$. As in Theorem 2.15, sending $k \to l^-$ gives a contradiction and we conclude that

$$\sup_{\Theta} u \leq \sup_{\Theta} u^+.$$

\[\square\]

5. Poincaré Inequalities and Compact Projection of Sobolev Spaces

In this section we give a result demonstrating a global Poincaré inequality with gain as a consequence of a compact “embedding”-type property for degenerate Sobolev spaces. We say that the Compact Projection Property from $QH^{1,p}(\Theta)$ into $L^q(\Theta)$ holds if and only if the projection map $i : QH^{1,p}(\Theta) \to L^q(\Theta)$ defined by $i(u, \nabla u) = u$ is a compact mapping. Recall that the space $QH^{1,p}(\Theta)$ is defined as the closure in the norm

$$\|w\|_{QH^{1,p}(\Theta)} = \|w\|_{L^p(\Theta)} + ||\sqrt{Q(x)}\nabla w||_{L^q(\Theta)}$$
of the collection $\text{Lip}_{Q,p}(\Theta)$ of locally Lipschitz functions defined in $\Theta$ with finite $QH^{1,p}(\Theta)$ norm. Note that $QH^{1,p}(\Theta)$ is denoted by $W_{Q}^{1,p}(\Theta)$ in [MRW], [CRW], and as $W_{Q}^{1,p}(\Theta)$ in [SW2].

**Theorem 5.1.** Let $(\Omega, \rho)$ be a geometric homogeneous space, $\Theta$ be a bounded domain such that $\overline{\Theta} \subset \Omega$ and fix $p, q$ with $1 \leq p \leq q < \infty$. Assume that the local Poincaré inequality (1.6) of order $p$ holds. If the Compact Projection Property from $QH^{1,p}(\Theta)$ to $L^q(\Theta)$ holds, there exists a positive constant $C_5 > 0$ such that

\begin{equation}
(\int_{\Theta} |w - w_{\Theta}|^r \, dx)^{\frac{1}{r}} \leq C_5 \left( \int_{\Theta} |\sqrt{Q} \nabla w|^p \, dx \right)^{\frac{1}{p}}
\end{equation}

for every $r \in [1, q]$ and pair $(w, \nabla)$ in $QH^{1,p}(\Theta)$.

**Corollary 5.2.** Assume that the conditions of Theorem 5.1 hold with $p = 2$. Then, the global Poincaré inequality (1.9) holds (and also the global weak Poincaré inequality (1.10)) with gain $\omega = \frac{1}{2}$.

**Remark 5.3.** When put in context with Remark 1.5, Proposition 1.7 (when $p = 2$) and related results of [CRW], one can see a clear relationship, vis-a-vis almost necessity and sufficiency, between the compact projection property and the Poincaré inequality (5.1) in the setting of a geometric homogeneous space where a local Poincaré inequality of the form (1.9) holds.

**Proof of Theorem 5.1.** We argue by contradiction and begin with the case where $r = q \geq p$. If the result does not hold, then for every $k \in \mathbb{N}$ there is a $w_k \in QH^{1,p}(\Theta)$ so that

\begin{equation}
\left( \int_{\Theta} |w_k - (w_k)_{\Theta}|^q \, dx \right)^{\frac{1}{q}} > \frac{1}{k} \left( \int_{\Theta} |\sqrt{Q} \nabla w_k|^p \, dx \right)^{\frac{1}{p}},
\end{equation}

where $(w_k)_{\Theta} = \frac{1}{|\Theta|} \int_{\Theta} w_k \, dx$. Thus, $\|w_k - (w_k)_{\Theta}\|_{L^q(\Theta)} > 0$ and we are able to define

$$v_k = \frac{w_k - (w_k)_{\Theta}}{\|w_k - (w_k)_{\Theta}\|_{L^q(\Theta)}}.$$ 

It is clear that both

$$\|v_k\|_{L^q(\Theta)} = 1, \quad (v_k)_{\Theta} = 0,$$

and

$$\left( \int_{\Theta} |\sqrt{Q} \nabla v_k|^p \, dx \right)^{\frac{1}{p}} < \frac{1}{k}$$

hold. Since $q \geq p$ and $\Theta$ is bounded, the sequence $\{(v_k, \nabla v_k)\}_{k \in \mathbb{N}}$ is bounded in $QH^{1,p}(\Theta)$. By the Compact Projection Property there exists $v \in L^q(\Theta)$ so that, up to a subsequence, $v_k \to v$ pointwise a.e. in $\Theta$ and also in $L^q(\Theta)$. It is not difficult to see that the limit $v$ satisfies

\begin{equation}
\|v\|_{L^q(\Theta)} = 1, \quad v_{\Theta} = 0.
\end{equation}

Applying the local Poincaré inequality (1.9) to $v_k$ gives

\begin{equation}
\frac{1}{|B_r|} \int_{B_r} |v_k - (v_k)_{B_r}| \, dx \leq C_2 r \left( \frac{1}{|B_{br}|} \int_{B_{br}} |\sqrt{Q} \nabla v_k|^p \, dx \right)^{\frac{1}{p}} < \frac{C_2 r}{|B_{br}|} \frac{1}{k}
\end{equation}

for every $k \in \mathbb{N}$ and every $B_r = B(y, r$) such that $br \in (0, r_1(y))$ and $\overline{B_{br}} \subset \Theta$. Passing to the limit as $k$ tends to $\infty$ we get

$$\|v - v_{B_r}\|_{L^1(B_r)} = 0$$

and so $v$ is a.e. constant on $B_r$. Since quasimetric balls are open, $v$ is locally constant in the connected set $\Theta$ and we conclude that

$$v \equiv \text{const.} \quad \text{a.e. in } \Theta.$$
This contradicts (5.3) and proves (5.1) when \( r = q \). Since \( \Theta \) is bounded, we can recover inequality (5.1) in the case \( r \in [1, q) \) by a simple application of Hölder’s inequality.

\[ \square \]

**Remark 5.4.** Theorem 5.1 is improved by replacing the local Poincaré inequality of order \( p \geq 1 \) with its weaker \( L^1 \rightarrow L^p \) counterpart obtained by replacing \( p \) with 1 on the left-hand side of (1.6). The proof of this improved version is identical to the one just given n.b. (5.4).

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E-mail address: dario.monticelli@gmail.com (Dario Daniele Monticelli)
E-mail address: scott.rodney@gmail.com (Scott Rodney)