ON OLD AND NEW CLASSES OF FEEBLY COMPACT SPACES

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Abstract. We introduce three new classes of pracompact spaces, consider their basic properties and relations with other compact-like spaces.

1. Definitions and relations

In general topology are often investigated different classes of compact-like spaces and relations between them, see, for instance, basic [8, Chap. 3] and general works [9, 15, 18, 17, 14]. We consider the present paper as a next small step in this quest.

Farther we shall follow the terminology of [8]. By N we shall denote the set of all positive integers.

A subset of a topological space X is called regular open if it equals the interior of its closure.

1.1. Old classes. We recall that a topological space X is said to be

- semiregular if X has a base consisting of regular open subsets;
- compact if each open cover of X has a finite subcover;
- sequentially compact if each sequence \( \{x_n\}_{n \in \mathbb{N}} \) of X has a convergent subsequence in X;
- \( \omega \)-bounded if each countable subset of X has the compact closure;
- totally countably compact if each sequence of X contains a subsequence with the compact closure;
- countably compact if each open countable cover of X has a finite subcover;
- countably compact at a subset \( A \subseteq X \) if every infinite subset \( B \subseteq A \) has an accumulation point \( x \) in X;
- countably pracompact if there exists a dense subset \( D \) in X such that X is countably compact at \( D \);
- feebly \( \omega \)-bounded if for each sequence \( \{U_n\}_{n \in \mathbb{N}} \) of non-empty open subsets of X there is a compact subset \( K \) of X such that \( K \cap U_n \neq \emptyset \) for each \( n \);
- selectively sequentially feebly compact if for each sequence \( \{U_n\}_{n \in \mathbb{N}} \) of non-empty open subsets of X we can choose a point \( x_n \in U_n \) for each \( n \in \mathbb{N} \) such that the sequence \( \{x_n\} \) has a convergent subsequence.

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• **selectively feebly compact**\(^1\), if for each sequence \(\{U_n\}_{n \in \mathbb{N}}\) of non-empty open subsets of \(X\) we can choose a point \(x \in X\) and a point \(x_n \in U_n\) for each \(n \in \mathbb{N}\) such that the set \(\{n \in \mathbb{N} : x_n \in W\}\) is infinite for every open neighborhood \(W\) of \(x\).

• **sequentially feebly compact** \([7, \text{Def. 1.4}]\) if for each sequence \(\{U_n : n \in \mathbb{N}\}\) of non-empty open subsets of the space \(X\) there exist a point \(x \in X\) and an infinite set \(I \subset \mathbb{N}\) such that for each neighborhood \(U\) of the point \(x\) the set \(\{n \in I : U_n \cap U = \emptyset\}\) is finite.\(^2\)

• **feebly compact** if each locally finite open cover of \(X\) is finite.

• **\(k\)-space** if \(X\) is Hausdorff and a subset \(F \subset X\) is closed in \(X\) if and only if \(F \cap K\) is closed in \(K\) for every compact subspace \(K \subset X\).

According to Theorem 3.10.22 of \([8]\), a Tychonoff topological space \(X\) is feebly compact if and only if each continuous real-valued function on \(X\) is bounded. Also, a Hausdorff topological space \(X\) is feebly compact if and only if every locally finite family of non-empty open subsets of \(X\) is finite.

Relations between different classes of compact-like spaces are well-studied. Some of them are presented at Diagram 3 in \([15, \text{p.17}]\), at Diagram 1 in \([5, \text{p. 58}]\) (for Tychonoff spaces), and at Diagram 3.6 in \([17, \text{p. 611}]\).

1.2. **New classes.** In order to refine the stratification of compact-like spaces even more, we introduce the following definitions. In each of them we require that a space \(X\) contains a dense subset \(D\) with a special property. Namely,

- if each sequence of points of the set \(D\) has a convergent subsequence (in \(X\)) then \(X\) is **sequentially pracompact**;
- if each sequence of points of the set \(D\) has a subsequence with the compact closure (in \(X\)) then \(X\) is **totally countably pracompact**;
- if each countable subset of the set \(D\) has the compact closure (in \(X\)) then \(X\) is **\(\omega\)-pracompact**.

Our main motivation to introduce the above spaces is possible applications in topological algebra. In particular, we are going to use them in the paper \([12]\).

The following diagram shows relations between different classes of compact-like spaces.

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\(^1\)Selectively sequentially feebly compact Tychonoff spaces were recently introduced and studied by Dorantes-Aldama and Shakhmatov in \([5]\). Also they considered selectively feebly compact Tychonoff spaces under the name **selectively pseudocompact** spaces. An equivalent property appeared a few years earlier in papers by Garcia-Ferreira with Ortiz-Castillo \([9]\) and with Tomita \([10]\) under the name **strong pseudocompactness**, but since the term strongly pseudocompact is used in \([1, 4]\) to denote two different properties, we stick to a name for this property which reflects its selective nature and also matches the name of the previous selective property.

\(^2\)One of the authors introduced this notion a few years ago as a natural property intermediate between feebly and sequential compactness, which may be useful in some applications in topological algebra. Indeed, for instance, Proposition 1.10. by Artico et al. \([2]\) combined with Theorem 1.1 by Lipparini \([14]\) states that that each \(T_0\) feebly compact topological group is sequentially feebly compact. But later we found that it is a known property, even with the same name. The oldest reference which we know (see \([15, \text{p. 15}]\)) is Reznichenko’s paper \([16]\). A similar notion had been given by Artico et al. in \([2, \text{Def. 1.8}]\), where are used pairwise disjoint open sets instead. Lipparini proved in \([14]\) that these notions are equivalent.
2. Basic properties

2.1. Extensions. We recall that an extension of a space $X$ is a space $Y$ containing $X$ as a dense subspace. It is easy to check that countable pracompactness, sequential pracompactness, feeble compactness, sequential feeble compactness, selective feeble compactness, selective sequential feeble compactness, and feeble $\omega$-boundedness is preserved by extensions.

2.2. Continuous images. It is easy to check that sequential compactness, feeble compactness, sequential feeble compactness, countably pracompactness, and sequential pracompactness is preserved by continuous images and total countable compactness, total countable pracompactness, $\omega$-boundedness, and $\omega$-pracompactness is preserved by continuous Hausdorff images.

2.3. Products. The investigation of productivity of compact-like spaces is motivated by the fundamental Tychonoff theorem, stating that a product of a family of compact spaces is compact. On the other hand, there are two countably compact spaces whose product is not feebly compact (see [8], the paragraph before Theorem 3.10.16). The product of a countable family of sequentially compact spaces is sequentially compact [8, Theorem 3.10.35]. But already the Cantor cube $D^c$ is not sequentially compact.
(see [3], the paragraph after Example 3.10.38). On the other hand some compact-like spaces are also preserved by products, see [18] § 3-4 (especially Theorem 3.3, Proposition 3.4, Example 3.15, Theorem 4.7, and Example 4.15) and §7 for the history, and [17], § 5). Among more recent results we note that Dow et al. in Theorem 4.1 of [7] proved that a product of a family of sequentially feebly compact spaces is again sequentially feebly compact, and in Theorem 4.3 that every product of feebly compact spaces, all but one of which are sequentially feebly compact, is feebly compact.

In the next propositions we show that sequentially pracom pact, $T_1$ totally countably compact, and $\omega$-pracompact spaces are preserved by products. Their proofs are easy and straightforward but we provide them because a theorem should have a proof.

Let $X$ be a product of a family $\{X_\alpha: \alpha \in A\}$ of spaces. For each subset $B$ of the set $A$ as $\pi_B$ we denote the projection from $X = \prod \{X_\alpha: \alpha \in A\}$ to $\prod \{X_\alpha: \alpha \in B\}$. If $B = \{\alpha\}$ then $\pi_B$ we shall denote also as $\pi_\alpha$. A space $Y \subset X$ is called a $\Sigma$-product of the family $\{X_\alpha\}$ provided there exists a point $y \in X$ such that $Y = \{x \in X: x_\alpha = y_\alpha \text{ for all but countably many } \alpha \in A\}$. In this case $Y$ is also called the Corson $\Sigma$-subset of $X$ based at $y$.

**Proposition 2.1.** The ($\Sigma$-) product of a family of sequentially pracompact spaces is sequentially pracompact.

**Proof.** Let $X$ be the non-empty product of a family $\{X_\alpha: \alpha \in A\}$ of sequentially pracompact spaces and $Y \subset X$ be the Corson $\Sigma$-subset of $X$ based at a point $y = (y_\alpha) \in X$. For each index $\alpha \in A$ fix a dense subset $D_\alpha \ni y_\alpha$ of the space $X_\alpha$ such that each sequence of points of the set $D_\alpha$ has a convergent subsequence and fix a point $a_\alpha \in D_\alpha$. Then the set $D = Y \cap \prod_{\alpha \in A} D_\alpha$ is an enumeration of a countable set $\{\alpha \in A: \exists \alpha \in C(x_\alpha \neq y_\alpha)\}$. By induction we can build a sequence $\{x_{\alpha m} \in X_{\alpha m}\}$ of points and a non-increasing sequence $\{S_m\}$ of infinite subsets of $\mathbb{N}$ such that for each neighborhood $U_{\alpha m} \subset X_{\alpha m}$ of the point $x_{\alpha m}$ the set $\{n \in S_m: x_{\alpha n} \notin U_{\alpha m}\}$ is finite. We can easily construct an infinite set $S \subset \mathbb{N}$ such that the set $S \setminus S_m$ is finite for each $m \in \mathbb{N}$. Choose a point $x = (x_\alpha) \in Y$ such that $x_\alpha$ is already defined for $\alpha \in B$ and $x_\alpha = y_\alpha$ for $\alpha \in A \setminus B$. Let $U$ be an arbitrary neighborhood of the point $x$. There exist a finite subset $F$ of the set $A$ and a family $\{U_\alpha: \alpha \in F, U_\alpha \subset X_\alpha\}$ of an open neighborhood of $x_\alpha$ such that $x \in U' = \pi_F^{-1}(\prod \{U_\alpha: \alpha \in F\}) \subset U$. The inductive construction implies that the set $T_\alpha = \{n \in S: x_{\alpha n} \notin U_\alpha\}$ is finite for each $\alpha \in F$. Then $x_\alpha \in U' \subset U$ for each $n \in S \setminus \bigcup \{T_\alpha: \alpha \in F\}$. $\Box$

**Proposition 2.2.** The ($\Sigma$-) product of a family of totally countably pracompact $T_1$ spaces is totally countably pracompact.

**Proof.** Let $X$ be the non-empty product of a family $\{X_\alpha: \alpha \in A\}$ of totally countably pracompact spaces and $Y \subset X$ be the Corson $\Sigma$-subset of $X$ based at a point $y = (y_\alpha) \in X$. For each index $\alpha \in A$ fix a dense subset $D_\alpha \ni y_\alpha$ of the space $X_\alpha$ such that each sequence of points of the set $D_\alpha$ has a subsequence with the compact closure in $X_\alpha$. Put $D = Y \cap \prod_{\alpha \in A} D_\alpha$. Then the set $D$ is a dense subset of the space $X$. Let $C = \{x_n: n \in \mathbb{N}\}$ be a sequence of points of the set $D$ and $B = \{\alpha_m: m \in \mathbb{N}\}$ be an enumeration of a countable set $\{\alpha \in A: \exists x \in C(x_\alpha \neq y_\alpha)\}$. By induction we can build a sequence $\{x_{\alpha m} \in X_{\alpha m}\}$ of points and a non-increasing sequence $\{S_m\}$ of infinite subsets of $\mathbb{N}$ such that for each neighborhood $U_{\alpha m} \subset X_{\alpha m}$ of the point $x_{\alpha m}$ the set $\{n \in S_m: x_{\alpha n} \notin U_{\alpha m}\}$ is finite. We can easily construct an infinite set $S \subset \mathbb{N}$ such that the set $S \setminus S_m$ is finite for each $m \in \mathbb{N}$. Choose a point $x = (x_\alpha) \in Y$ such that $x_\alpha$ is already defined for $\alpha \in B$ and $x_\alpha = y_\alpha$ for $\alpha \in A \setminus B$. Let $U$ be an arbitrary neighborhood of the point $x$. There exist a finite subset $F$ of the set $A$ and a family $\{U_\alpha: \alpha \in F, U_\alpha \subset X_\alpha\}$ of an open neighborhood of $x_\alpha$ such that $x \in U' = \pi_F^{-1}(\prod \{U_\alpha: \alpha \in F\}) \subset U$. The inductive construction implies that the set $T_\alpha = \{n \in S: x_{\alpha n} \notin U_\alpha\}$ is finite for each $\alpha \in F$. Then $x_\alpha \in U' \subset U$ for each $n \in S \setminus \bigcup \{T_\alpha: \alpha \in F\}$. $\Box$

**Proposition 2.3.** The product of a family of $\omega$-pracompact spaces is $\omega$-pracompact. Moreover, if all spaces of the family are $T_1$ then a $\Sigma$-product of the family is $\omega$-pracompact too.

**Proof.** Let $X$ be the non-empty product of a family $\{X_\alpha: \alpha \in A\}$ of $\omega$-pracompact spaces and $Y \subset X$ be the Corson $\Sigma$-subset of $X$ based at a point $y = (y_\alpha) \in X$. For each index $\alpha \in A$ fix a dense subset
Let \( D \supset y_\alpha \) of the space \( X_\alpha \) such that each countable subset of the set \( D_\alpha \) has the compact closure in \( X_\alpha \). Put \( D = Y \cap \prod_{\alpha \in A} D_\alpha \). Then the set \( D \) is a dense subset of the space \( X \). Let \( C \) be a countable subset of the set \( D \). Put \( B = \{ \alpha \in A : \exists x \in C(x_\alpha \neq y_\alpha) \} \). Then the set \( B \) is countable, \( C \) is a subset of a closed compact subset \( \prod_{\alpha \in B} \pi_\alpha(\mathcal{C}) \) of the space \( X \). If all spaces \( X_\alpha \) are \( T_1 \) then \( C \) is a subset of a closed compact set \( \prod_{\alpha \in B} \pi_\alpha(\mathcal{C}) \times \prod_{\alpha \in A \setminus B} \{ y_\alpha \} \subset Y \) of the space \( X \). □

Since sequential feebly compactness is preserved by extensions, the next proposition strengthen a bit Theorem 4.1 of [7].

**Proposition 2.4.** The \( \Sigma \)-product of a family of sequentially feebly compact spaces is sequentially feebly compact.

**Proof.** Let \( X \) be a non-empty product of a family \( \{ X_\alpha : \alpha \in A \} \) of sequentially feebly compact spaces, \( Y \subset X \) be the Corson \( \Sigma \)-subspace of \( X \) based at a point \( y = (y_\alpha) \in X \), and \( \{ V_n : n \in \mathbb{N} \} \) be a sequence of non-empty open subsets of the space \( Y \). For each index \( n \) choose a finite subset \( B_n \) of the set \( A \) and a family \( \{ U_{n\alpha} : \alpha \in B_n, U_{n\alpha} \) is a non-empty open subset of \( X_\alpha \} \) such that \( U_n \cap Y \subset V_n \), where \( U_n = \prod_{n\alpha \in B_n} \{ U_{n\alpha} : \alpha \in B_n \} \). Put \( B = \bigcup B_n \). By Theorem 4.1 of [7], the space \( X' = \{ X_\alpha : \alpha \in B \} \) is sequentially feebly compact. Since \( \{ \pi_B(U_n) \} \) is a sequence of its non-empty open subsets, there exist a point \( x' \in X' \) and an infinite set \( I \subset \mathbb{N} \) such that for each neighborhood \( U' \) of the point \( x' \) the set \( \{ n \in I : \pi_B(U_n) \cap U' = \emptyset \} \) is finite. Define a point \( x = (x_\alpha)_{\alpha \in A} \in Y \) by putting \( x_\alpha = x'_\alpha \) for each \( \alpha \in B \) and \( x_\alpha = y_\alpha \) for each \( \alpha \in A \setminus B \). Let \( V \) be an arbitrary neighborhood of the point \( x \) in the space \( Y \). Pick a canonical neighborhood \( U \) of the point \( x \) in the space \( X \) such that \( U \cap Y \subset V \). Then there exists a subset \( I' \) of the set \( I \) such that a set \( I \setminus I' \) is finite and \( \pi_B(U_n) \cap \pi_B(U) \neq \emptyset \) for each \( n \in I' \). Fix any such \( n \) and pick a point \( z' = (z'_\alpha)_{\alpha \in B} \in \pi_B(U_n) \cap \pi_B(U) \). Define a point \( z = (z_\alpha)_{\alpha \in A} \in Y \) by putting \( z_\alpha = z'_\alpha \) for each \( \alpha \in B \) and \( z_\alpha = y_\alpha \) for each \( \alpha \in A \setminus B \). It is easy to check that \( z \in U_n \cap U \cap Y \subset V_n \cap V \).

### 3. Backward implications

Banakh and Zdomskyy in [3] defined a topological space \( X \) to be an \( \alpha_7 \)-space if for any family \( \{ S_n : n \in \mathbb{N} \} \) of countable infinite subsets of the space \( X \) such that a set \( S_n \setminus U \) is finite for any \( n \) and any neighborhood \( U \) of \( x \) there exist a countable infinite subset \( S \) of the space \( X \) and a point \( y \in X \) such that a set \( S \setminus V \) is finite for any neighborhood \( V \) of \( y \) and \( S_n \cap S \neq \emptyset \) for infinitely many \( n \).

**Proposition 3.1.** Let \( X \) be a Fréchet-Urysohn feebly compact space. Then \( X \) is sequentially feebly compact. Moreover, if \( X \) is quasiregular or \( \alpha_7 \) then \( X \) is selectively sequentially feebly compact.

**Proof.** Let \( X \) be a Fréchet-Urysohn feebly compact space and \( \{ V_n : n \in \mathbb{N} \} \) be a sequence of non-empty open subsets of the space \( X \). For each \( n \) choose a non-empty open set \( U_n \subset V_n \) such that \( \overline{U_n} \subset V_n \) provided the space \( X \) is quasiregular. Since the space \( X \) is feebly compact, there exists a point \( x \in X \) such that each neighborhood of the point \( x \) intersects infinitely many sets of the sequence \( \{ U_n \} \). Put \( I_0 = \{ n \in \mathbb{N} : x \in \overline{U_n} \} \).

Suppose the set \( I_0 \) is infinite. Then \( U \cap U_n \neq \emptyset \) for each \( n \in I_0 \) and each neighborhood \( U \) of the point \( x \). If the space \( X \) is quasiregular then \( x \in V_n \) for each \( n \in I_0 \), so a constant sequence \( \{ x_n = x : n \in I_0 \} \) converges to \( x \). Assume that \( X \) is an \( \alpha_7 \)-space. Since the space \( X \) is Fréchet-Urysohn, for each \( n \in I_0 \) there exists a sequence \( S'_n = \{ x_k^n \setminus k \in \mathbb{N} \} \) of points of \( U_n \) convergent to a point \( x \). Considering its subsequence, if necessary, we can assume that the sequence \( S'_n \) consists of distinct points or it is constant. In the latter case we have \( x_k^n = x \) for each \( k \) for some point \( x \) in \( U_n \) such that \( x \in \overline{x} \). Put \( I'_0 = \{ n \in I_0 : S'_n \) is constant \}. If the set \( I'_0 \) is finite then a sequence \( \{ x^n : n \in I'_0 \} \) converges to the point \( x \). So we suppose the set \( I'_0 \) is finite. Since \( X \) is an \( \alpha_7 \)-space, there exist a countable infinite
subset $S$ of the space $X$ and a point $y \in X$ such that a set $S \setminus V$ is finite for any neighborhood $V$ of $y$ and a set $I_0'' = \{ n \in I_0 \setminus I_0': \text{there exists a natural } k(n) \text{ such that } x_{k(n)}^n \in S \}$ is infinite. For each $n \in I_0''$ put $x_n = x_{k(n)}^n \in U_n$. If there exists a point $z \in X$ such that a set $I_1 = \{ n \in I_0'' : x_n = z \}$ is infinite then a sequence $\{ x_n : n \in I_1 \}$ converges to the point $z$. Otherwise a sequence $\{ x_n : n \in I_0'' \}$ converges to the point $y$. Indeed, let $V$ be an arbitrary neighborhood of the point $y$. Then a set $S \setminus V$ is finite and $x_n \in V$ for each $n \in I_0'' \setminus \{ n : x_n \in S \setminus V \}$.

Suppose the set $I_0$ is finite. Since $x \in \bigcup \{ U_n : n \in \mathbb{N} \setminus I_0 \}$ and $X$ is a Fréchet-Urysohn space, there exists a sequence $\{ x_m^r : m \in \mathbb{N} \}$ of points of the set $\bigcup \{ U_n : n \in \mathbb{N} \setminus I_0 \}$ converging to the point $x$. For each index $m \in \mathbb{N}$ choose an index $n(m) \in \mathbb{N} \setminus I_0$ such that $x_m^r \in U_{n(m)}$. Put $I_1 = \{ n(m) : m \in \mathbb{N} \}$. Since $x \notin \bigcap I_n$ for each $n \in \mathbb{N} \setminus I_0$, the set $I_1$ is infinite. For each $r \in I_1$ pick a point $x_r = x_{n(r)}^r$, where $n(m(r)) = r$. Then $x_r \in U_r$ and a sequence $\{ x_r : r \in I_1 \}$ converges to the point $x$. Indeed, let $U$ be an arbitrary neighborhood of the point $x$. Since the sequence $\{ x_r \}$ converges to the point $x$, there exists $N \in \mathbb{N}$ such that $x_m^r \in U$ for each $m > N$. Then $x_r \in U$ for each $r \in I_1 \setminus \{ n(m) : 0 \leq m \leq N \}$. □

Proposition 3.2. Each sequential countably pracompact space is sequentially pracompact.

Proof. Let $X$ be a sequential countably pracompact space. There exists a dense subset $D$ of the space $X$ such that each infinite subset of the set $D$ has an accumulation point in $X$. Let $\{ x_n : n \in \mathbb{N} \}$ be a sequence of points of the set $D$. If there exists a point $x \in X$ such that $x \notin \{ x_n \}$ for infinitely many indices $n \in \mathbb{N}$ then the sequence $\{ x_n : x_n = x \}$ is a convergent subsequence of the sequence $\{ x_n : n \in \mathbb{N} \}$. So we suppose that there is no such point $x$. Then the set $B = \{ x_n : n \in \mathbb{N} \}$ is infinite. The set $B$ has an accumulation point $y$ in $X$. Then $y \in \bigcup \{ U_n \setminus \{ y \} \}$. Therefore the set $B \setminus \{ y \}$ is not sequentially closed and there exists a sequence $\{ z_m : m \in \mathbb{N} \}$ of points of the set $B \setminus \{ y \}$ converging to a point $z \notin B \setminus \{ y \}$. Then the sequence $\{ z_m : m \in \mathbb{N} \}$ contains infinitely many different points of the set $B \setminus \{ y \}$. □

Proposition 3.3. Each countably pracompact $k$-space $X$ is totally countably pracompact.

Proof. There exists a dense subset $D$ of the space $X$ such that each infinite subset of the set $D$ has an accumulation point in $X$. Let $\{ x_n : n \in \mathbb{N} \}$ be a sequence of points of the set $D$. Put $B = \{ x_n : n \in \mathbb{N} \}$. If the set $B$ is finite then there exists a point $x \in X$ such that $x_n = x$ for infinitely many indices $n \in \mathbb{N}$. Then a subsequence $\{ x_n : x_n = x \}$ of the sequence $\{ x_n : n \in \mathbb{N} \}$ has the compact closure $\{ x \}$ in $X$. So we suppose that the set $B$ is infinite. The set $B$ has an accumulation point $y$ in $X$. Then $y \in \bigcup \{ U_n \setminus \{ y \} \}$. Therefore the set $B \setminus \{ y \}$ is not closed and there exists a compact subset $K$ of the space $X$ such that $B \cap K$ is not closed in $K$. Then the set $B \cap K$ is infinite, the sequence $\{ x_n : x_n \in B \cap K \}$ is infinite too and $\{ x_n : x_n \in B \cap K \} \subset K$. □

Proposition 3.4. Each sequentially feebly compact space containing a dense set $D$ of isolated points is sequentially pracompact.

Proof. It is easy to check that each sequence of points of the set $D$ has a convergent subsequence. □

4. Examples

Example 4.1. Let $X_0$ be a non-empty $T_1$ space. Determine a topology on the set $X = (X_0 \times \omega) \cup \{ y_0 \}$, where $y_0 \notin X_0 \times \omega$ by the following base

$$
\mathcal{B} = \{ U \times \{ n \} : U \text{ is an open subset of the space } X_0, n \in \omega \} \cup \\
\bigcup_{m \geq n} \{ y_0 \} \cup \bigcup_{m \geq n} X_0 \times \{ m \} \setminus F_m : n \in \omega, F_m \text{ is a finite subset of } X_0 \text{ for each } m \in \omega \text{ such that } m \geq n \}.
$$
It is easy to check the following:

- the space $X$ is Hausdorff provided the space $X_0$ is Hausdorff;
- the space $X$ is feebly compact provided the space $X_0$ is a feebly compact space without isolated points;
- the space $X$ is sequentially feebly compact provided the space $X_0$ is a sequentially feebly compact space without isolated points.

Now we take the standard unit segment $[0; 1]$ as $X_0$. Then $X$ is a sequentially feebly compact space, containing a closed discrete infinite subspace $\{1\} \times \omega$. Now for each $n \in \omega$ put $U_n = X_0 \times \{n\}$. Let $\{x_n\}$ be a sequence of points of the space $X$ such that $x_n \in U_n$. Then the set $\{x_n\}$ has no accumulation points, so the space $X$ is not selectively feebly compact.

We recall that the Stone-Čech compactification of a Tychonoff space $X$ is a compact Hausdorff space $\beta X$ containing $X$ as a dense subspace so that each continuous map $f : X \to Y$ to a compact Hausdorff space $Y$ extends to a continuous map $\overline{f} : \beta X \to Y$.

**Example 4.2.** (see [8] Exer. 3.6.I, cf. [5] Ex. 2.6.) Let $\{N_\alpha\}_{\alpha \in A}$, where $A \cap \mathbb{N} = \varnothing$, be an infinite family of infinite subsets of $\mathbb{N}$ such that the intersection $N_\alpha \cap N_\beta$ is finite for every pair $\alpha, \beta$ of distinct elements of $A$ and that $\{N_\alpha\}_{\alpha \in A}$ is maximal with respect to the last property. Generate a topology on the set $X = \mathbb{N} \cup S$ by the neighborhood system $\{B(x)\}_{x \in X}$, where $B(x) = \{\{n\}\}$, if $x = n \in \mathbb{N}$ and $B(x) = \{(\alpha) \cup (N_\alpha \setminus \{1, 2, \ldots, n\})\}$ for every pair $x = \alpha \in A$.

Since $A$ is a closed discrete infinite subset of $X$, $X$ is not countably compact. On the other hand, the set $D = \mathbb{N}$ is dense in $X$. Let $\{x_n : n \in \mathbb{N}\}$ be an arbitrary sequence of points of the set $D$. If the set $S = \{x_n : n \in \mathbb{N}\}$ is finite then the sequence $\{x_n : n \in \mathbb{N}\}$ has a constant subsequence. If the set $S$ is infinite then by maximality of $A$ there exists $\alpha \in A$ such that $N_\alpha \cap S$ is infinite. Note that the enumeration $\{x_{n_k} : k \in \mathbb{N}\}$ of $N_\alpha \cap S$ in the increasing order is a subsequence of the sequence $\{x_n : n \in \mathbb{N}\}$ converging to the point $\alpha$. Thus the space $X$ is sequentially precompact.

**Example 4.3.** Endow the set $\mathbb{N}$ with the discrete topology. Let $\mathcal{A}(\mathbb{N}) = \mathbb{N} \cup \{\infty\}$ be a one-point Alexandroff compactification of $\mathbb{N}$ with the remainder $\infty$. We define on $\mathcal{A}(\mathbb{N}) \times \mathbb{N}$ the product topology $\tau_p$ and extend the topology $\tau_p$ onto $X = \mathcal{A}(\mathbb{N}) \times \mathbb{N} \cup \{a\}$, where $a \notin \mathcal{A}(\mathbb{N}) \times \mathbb{N}$, to a topology $\tau^*$ in the following way: bases of the topologies $\tau_p$ and $\tau^*$ coincide at $a$ for any $x \in \mathcal{A}(\mathbb{N}) \times \mathbb{N}$ and the family

$$\mathcal{B}^*(a) = \{U_a(i_1, \ldots, i_n) : i_1, \ldots, i_n \in \mathbb{N}\},$$

where

$$U_a(i_1, \ldots, i_n) = X \setminus ((\{\infty\} \times \mathbb{N}) \cup (\mathcal{A}(\mathbb{N}) \times \{i_1, \ldots, i_n\})),$$

determines a base of the topology $\tau^*$ at the point $a$.

The definition of the topology $\tau^*$ on $X$ implies that $\mathbb{N} \times \mathbb{N}$ is the maximum discrete subspace of $(X, \tau^*)$ and $\mathbb{N} \times \mathbb{N}$ is dense in $(X, \tau^*)$. Hence every dense subset $D$ of $(X, \tau^*)$ contains $\mathbb{N} \times \mathbb{N}$. But $\mathbb{N} \times \mathbb{N} = X$ is not compact, and hence $(X, \tau^*)$ is not an $\omega$-pracompact space.

Now we shall show that $(X, \tau^*)$ is totally countably pracompact. Especially we shall prove that $\mathbb{N} \times \mathbb{N}$ is the requested dense subset of the space $(X, \tau^*)$. Fix an arbitrary square sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N}$. If there exists a positive integer $i$ such that the set $\{x_n\}_{n \in \mathbb{N}} \cap (\mathcal{A}(\mathbb{N}) \times \{i\})$ is infinite then the subsequence $\{x_{ij}\}_{j \in \mathbb{N}} = \{x_n\}_{n \in \mathbb{N}} \cap (\mathcal{A}(\mathbb{N}) \times \{i\})$ with the corresponding renumbering has the compact closure in $(X, \tau^*)$. In the other case the set $\{x_n\}_{n \in \mathbb{N}} \cap (\mathcal{A}(\mathbb{N}) \times \{i\})$ is finite for any positive integer $i$. Then the definition of $(X, \tau^*)$ implies that $\{x_n\}_{n \in \mathbb{N}} = \{a\} \cup \{x_n\}_{n \in \mathbb{N}}$ is a compact subset of $(X, \tau^*)$.

We observe that by Proposition 19 of [11], $(X, \tau^*)$ is Hausdorff non-semiregular countably pracompact non-countably compact space, and hence $(X, \tau^*)$ is not totally countably compact.
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