Asymptotically stable random walks of index $1 < \alpha < 2$ killed on a finite set

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Abstract

For a random walk on the integer lattice $\mathbb{Z}$ that is attracted to a strictly stable process with index $\alpha \in (1, 2)$ we obtain the asymptotic form of the transition probability for the walk killed when it hits a finite set. The asymptotic forms obtained are valid uniformly in a natural range of the space and time variables. The situation is relatively simple when the limit stable process has jumps in both positive and negative directions; in the other case when the jumps are one sided rather interesting matters are involved and detailed analyses are necessitated.

1 Introduction

Let $S_n = X_1 + \cdots + X_n$ be a random walk on the integer lattice $\mathbb{Z}$ started at $S_0 \equiv 0$, where the increments $X_1, X_2, \ldots$ are independent and identically distributed random variables defined on some probability space $(\Omega, \mathcal{F}, P)$ and taking values in $\mathbb{Z}$. Let $E$ indicate the expectation under $P$ as usual and $X$ be a random variable having the same law as $X_1$. We suppose throughout the paper that the walk $S_n$ is

1) in the domain of attraction of a strictly stable law of index $1 < \alpha < 2$ or, what amounts to the same thing (cf [13]), if $\phi(\theta) := E e^{i\theta X}$, then

$$\lim_{\theta \to \pm 0} \frac{1 - \phi(\theta)}{|\theta|^\alpha L(1/\theta)} = e^{\pm i\pi\gamma/2}$$

(1.1)

where $L(x)$ is a positive even function on the real line $\mathbb{R}$ slowly varying at infinity and $\gamma$ is a real number such that $|\gamma| \leq 2 - \alpha$.

For simplicity we also suppose that

2) the walk is strongly aperiodic in the sense of Spitzer [26], namely for any $x \in \mathbb{Z}$, $P[S_n = x] > 0$ for all sufficiently large $n$.

The essential assumption is of course the condition 1), which entails $EX = 0$ so that the walk is recurrent (see Appendix (A) for an equivalent condition in terms of the tails of distribution function of $X$ and some related facts), the condition 2) giving rise to little loss of generality (see Remark 2.3 and the comment given in the end of Section 7).

Under the assumption (1.1) with $1 \leq \alpha \leq 2$ ($L \equiv 1$ is assumed if $\alpha = 1$) Belkin [11] shows that the law of a normalized sum $S_n$ started at zero and conditioned to avoid a given finite set until time $n$ converges to a probability law. Our result in the present paper is seen as a
local version (including the corresponding ‘conditional local limit theorem’ as a part) of his result with the starting position $x$ allowed to be arbitrary and the convergence uniform in $x$ subject to a certain natural constraint.

Assuming the condition 2) in addition and restricting the exponent to $1 < \alpha < 2$ we shall obtain precise asymptotic forms of the mass function of the hitting time of the origin and of the transition probability for the walk killed when it hits the origin. The estimates obtained is uniform for the space variables within the natural space-time region $x = O(c_n)$, where $c_n$ is a norming sequence associated with the walk. For $\gamma < |2 - \alpha|$ when the limit stable process has jumps in both positive and negative directions, the transition probability of the killed walk will be shown to behave roughly like that of the limit stable process killed at the origin throughout the region. For $\gamma = \pm(2 - \alpha)$ the situation differs and involves interesting matters: we shall identify the space-time region of the same asymptotic behaviour as for the limit stable process and observe that in the remaining part that is unbounded the transition probability for the stable process is negligibly small as compared with the one for the walk, provided that the walk is neither left- nor right-continuous. Here the asymptotics is described by means of the potential function whose property varies in a significant way depending on the distribution of $X$ on its lighter side. We shall extend the results to the case when the walk is killed on hitting a finite set instead of the origin. The corresponding results for the walks with finite variance are obtained by [27], [28].

Among others the following two steps are crucial in the proof of our main results. First we extend Belkin’s result on the conditional law to the case when the starting position is allowed to tend to infinity along with $n$ in a certain reasonable way. Second we deduce a local limit theorem from the integral theorem by Belkin with the help of Gnedenko’s local limit theorem, in which the deduction rests on the idea devised by Denisov and Wachtel [7].

In the classical papers [18] and [17] Kesten and Spitzer, studying a multidimensional lattice walk killed on a finite set, obtained various ratio limit theorems under a mild assumption on the walk; Kesten [17] especially obtains an exact asymptotic result for the ratio of the transition probability to a sum of the mass functions of the hitting time which is directly related to our results (cf. Remark 2.4 at the end of the next section). For a non-lattice walk on $\mathbb{R}^d$ Port and Stone [23] established the existence of the potential operator and applied the result to study the ratio limit theorems similar to those treated in [18]. [15] and [16] study the one dimensional random walk having zero mean and finite variance and killed at zero: in [15] the local limit theorem corresponding to the integral version of [11] is established, while [16] shows that a normalized process of the walk conditioned to be killed at $n$ converges in law to a Brownian excursion. [2] obtains a functional limit theorem for the conditioned walk (on $\mathbb{Z}$) in the same setting as in [1]. In [35] is studied the largest gap within the range of the walk conditioned to avoid a bounded set. Stable or Lévy processes conditioned to avoid zero are studied by [11], [19] and [21]. Our problem is studied by the present author in an article [30] under the restrictive condition that $L \equiv 1$ in 1) by which Fourier analysis is effectively applied, and the results obtained there have provided a guide line for the present work especially in case $\gamma = |2 - \alpha|$.

2 Statements of results

We first introduce fundamental objects that appear in the description of our results and state some well known facts concerning them. Put $p^n(x) = P[S_n = x], p(x) = p^1(x) (x \in \mathbb{Z})$.
and define the potential function

\[ a(x) = \sum_{n=0}^{\infty} [p^n(0) - p^n(-x)]; \]

the series on the RHS is convergent and \( a(x)/|x| \to 1/\sigma^2 \) and \( a(x+y) - a(x) \to \pm y/\sigma^2 \) as \( x \to \pm \infty \), where \( \sigma^2 = EX^2 \) (cf. Spitzer [26]; Sections 28 and 29). To make expressions concise we use the notation

\[ a^\top(x) = 1(x = 0) + a(x), \]

where \(|S(x)| = 1 \) or \( 0 \) according as a statement \( S \) is true or false. If \( S \) is left-continuous (i.e., \( p(x) = 0 \) for \( x \leq 2 \), then \( a(x) = 0 \) for all \( x \geq 0 \) (under \( \sigma^2 = \infty \)), and similarly for right-continuous walks, whereas if it is neither left- nor right-continuous, \( a(x) > 0 \) whenever \( x \neq 0 \). (See Appendices (B) and (C) for additional facts related to \( a \).)

We write \( S_n^x \) for \( x + S_n \), the walk started at \( S_0^x = x \in \mathbb{Z} \). For a subset \( B \subset \mathbb{R} \), put

\[ \sigma_B^x = \inf\{n \geq 1 : S_n^x \in B\}, \]

the time of the first entrance of the walk \( S^x \) into \( B \). To avoid the overburdening of notation we write \( S_{\sigma_B^x}^x \) for \( S_{\sigma_B^x}^x \) and \( S_{\sigma_B^x}^x \) for \( S_{\sigma_B^x}^x \).

When the spatial variables become indefinitely large the asymptotic results are naturally expressed by means of the stable process appearing in the scaling limit and we need to introduce relevant quantities. Let \( Y_t \) be a stable process started at zero with characteristic exponent

\[ \psi(\theta) = e^{i(\text{sgn} \theta)\pi \gamma/2} |\theta|^\alpha \quad (|\gamma| \leq 2 - \alpha, \gamma \text{ is real}) \]

so that \( Ee^{itY_t} = e^{-t\psi(\theta)} \), where \( \text{sgn} \theta = 1 \) if \( \theta > 0 \), \( 0 \) if \( \theta = 0 \) and \(-1 \) if \( \theta < 0 \). (\( \gamma \) has the same sign as the skewness parameter so that the extremal case \( \gamma = 2 - \alpha \) corresponds to the spectrally positive case.) In the assumption (1.1) we can suppose the function \( L \)—of which only asymptotics at infinity is significant—so aptly chosen as to be differentiable and satisfy \( L'(x)/L(x) = o(1/|x|) \) as \( |x| \to \infty \), and then take positive numbers \( c_n, n \geq 0 \) which are increasing and satisfy

\[ nc_n = L(c_n) \to 1. \quad (2.1) \]

It then follows that

\[ \lim_{n \to \infty} \log Ee^{itS_n/c_n} = -\psi(\theta), \]

in other words, the law of \( S_n/c_n \) converges to the stable law whose characteristic function equals \( e^{-\psi(\theta)} \). Denote by \( p_t(x) \) and \( f^x(t) \) the density of the distribution of \( Y_t \) and of the first hitting time to the origin by \( Y_t^x := x + Y_t \), respectively:

\[ p_t(x) = P[Y_t \in dx]/dx, \quad f^x(t) = (d/dt)P[\exists s \leq t, Y_s^x = 0]; \]

there exist jointly continuous versions of these densities (for \( t > 0 \)) and we shall always choose such ones. It follows that \( S_{n\frac{t}{c_n}}/c_n \Rightarrow Y_t \) (weak convergence of distribution) and by Gnedenko’s local limit theorem [14] as \( n \to \infty \)

\[ p^n(x) = \frac{p_1(x/c_n) + o(1)}{c_n}, \quad (2.2) \]

where \( o(1) \) is uniform for \( x \in \mathbb{Z} \) and \( \lfloor b \rfloor \) denotes the integer part of a real number \( b \).

For real numbers \( s, t, s \lor t = \max\{s, t\} \) and \( s \land t = \min\{s, t\} \), \( t_+ = t \lor 0 \), \( t_- = (-t)_+ \) and \( \lfloor t \rfloor \) denotes the smallest integer that is not less than \( t \); for positive sequences \( (s_n) \) and
(t_n), s_n \sim t_n and s_n \asymp t_n mean, respectively, that the ratio s_n/t_n approaches unity and that
s_n/t_n is bounded away from zero and infinity. There will arise an expression like s_n \sim C t_n
with C = 0 which we regard as meaning s_n/t_n \to 0. For two real functions f(x) and g(y)
with g(y) > 0 the expressions
\[ f(x) \ll g(y) \quad \text{and} \quad -g(y) \ll -f(x) \]
mean that either \( f(x) \leq 0 \) or \( |f(x)|/g(y) \) tends to zero as \( x \) or \( y \) (or both) tends to \(+\infty\)
or \(-\infty\) depending on the situation where it occurs. We use the letters \( x, y, z \) and \( w \) to
represent integers which indicate points assumed by the walk when discussing matters on
the random walk, while the same letters may stand for real numbers when the stable process
is dealt with; we shall sometimes use the Greek letters \( \xi, \eta \) etc. to denote the real variables
the stable process may assume. For convenience the normalizing sequence \( c_n \) is extended to
a continuous function on \([0, \infty)\) by linear interpolation.

The rest of this section is divided into four subsections. In the first three we deal with
the special case \( A = \{0\} \), i.e., when the killing takes place at and only at the origin. In
the first subsection \( 2.1 \) we state certain fundamental results. For \( \gamma < |2 - \alpha| \) they provide
the precise asymptotic form of the transition probability of the killed walk within the whole
range \( |x| \vee |y| = O(c_n) \), whereas for \( \gamma = |2 - \alpha| \) the asymptotic form they provide is confined
to a very small part of the range. The results for the remaining region are addressed in the
subsections \( 2.2 \) and \( 2.3 \). The general case of finite sets—closely parallel to the special case
\( A = \{0\} \)—are dealt with in the last subsection \( 2.4 \).

2.1. Fundamental results for the walk killed at the origin.

Let \( f^x(n) \) denote the probability that the walk started at \( x \) visits the origin at \( n \) for the
first time:
\[ f^x(n) = P[\sigma^x_0 = n]. \]

Put
\[ \kappa_{\alpha,\gamma} = \kappa_{\alpha,-\gamma} = \frac{(\alpha - 1) \sin \frac{\pi}{\alpha} \Gamma(\frac{1}{\alpha}) \sin \frac{\pi}{2\alpha}}{\Gamma(\frac{1}{\alpha}) \sin \frac{\pi(\alpha-\gamma)}{2\alpha}} = \frac{(1 - \frac{1}{\alpha}) \sin \frac{\pi}{\alpha}}{p_1(0) \pi} \]
(see \( \text{[5,1]} \) for the second equality); in particular if \( \gamma = |2 - \alpha| \), \( \kappa_{\alpha,\gamma} = (\alpha - 1)/\Gamma(1/\alpha) \).

We know (cf. \( \text{[1]} \) Lemma 2.1]) that as \( n \to \infty \)
\[ P[\sigma^0_{\{0\}} > n] \sim \kappa c_n/n \quad (\kappa = \kappa_{\alpha,\gamma}/(1 - \frac{1}{\alpha})), \tag{2.3} \]
which is shown by the standard method based on Karamata’s Tauberian theorem (a similar
result is given in \( \text{[23]} \) Theorem 5.1 for non-lattice case). Our first theorem is the local
version of \( \text{[23]} \).

**Theorem 1.** For any admissible \( \gamma \), as \( n \to \infty \)
\[ f^0(n) \sim \frac{\kappa_{\alpha,\gamma} c_n}{n^2}. \]

For a non-empty subset \( B \subset \mathbb{Z} \) put
\[ Q^0_B(x, y) := P[S^x_n = y, \sigma^x_B \geq n], \tag{2.4} \]
which entails \( Q^0_B(x, y) = P[S^x_n = y, \sigma^x_B = n] \) \((n \geq 1, y \in B, x \in \mathbb{Z})\) as well as \( Q^0_B(x, y) = 1(x = y) \); and similarly for a closed set \( \Delta \subset \mathbb{R} \)
\[ p_t^\Delta(\xi, \eta) := P[Y_t^\xi \in d\eta, \sigma_{\xi}^t > t]/d\eta. \]
where $Y^\xi_t = \xi + Y_t$ and $\sigma^\Delta$ is the first entrance time of $Y^\xi$ into $\Delta$. By the scaling law for stable processes we have

$$p_n^\Delta(x, y) = n^{-1/\alpha} p_1^{\Delta/n^{1/\alpha}}(x_n, y_n).$$

Let $c_n$ satisfy (2.1) and write $x_n$ for $x/c_n$ and similarly for $y_n$. We are primarily interested in the asymptotic form of $Q^n_B(x, y)$ in the region $|x_n| \vee |y_n| < M$ with an arbitrarily given constant $M$. It differs in a significant way according as $\gamma < 2 - \alpha$ or $|\gamma| = 2 - \alpha$. First we state the result for the former case that is formulated in a rather neat form.

**Theorem 2.** If $|\gamma| < 2 - \alpha$, then for any $M > 1$, uniformly for $|x| \vee |y| < Mc_n$, as $n \to \infty$

$$Q^n_{(1)}(x, y) \sim \begin{cases} 
  a^\dagger(x) f^0(n) a^\dagger(-y) & (|x_n| \vee |y_n| \to 0), \\
  f^{x_n(1)} a^\dagger(-y)/n & (y_n \to 0, |x_n| > 1/M), \\
  a^\dagger(x) f^{-y_n(1)}/n & (x_n \to 0, |y_n| > 1/M), \\
  p^{(1)}_1(x_n, y_n)/c_n & (|x_n| \wedge |y_n| > 1/M).
\end{cases} \quad (2.5)$$

For $\gamma = \pm (2 - \alpha)$ the above formula (2.5) holds in and only in the partial range of $|x_n| \vee |y_n| < M$ that is identified in the next proposition. The range of validity is described by means of the function

$$\Lambda_n(x) := a^\dagger(x)c_n/n.$$

Slightly abusing notation, we write $|x_n| \ll \Lambda_n(x)$ instead of $|x|/a^\dagger(x) \ll c_n^2/n$ with the convention: if the walk is left-continuous (resp. right-continuous) so that $a(x) = 0$ for $x > 0$ (resp. $x < 0$), this condition is understood to be violated for $x > 0$ (resp. $x < 0$). We have only to give results for $\gamma = 2 - \alpha$ when $a(x)/a(-x) \to 0$ ($x \to \infty$), the results for $\gamma = -2 + \alpha$ being obtained from them by exchanging $x$ and $y$.

**Proposition 2.1.** If $\gamma = 2 - \alpha$ (when the limiting stable process has no negative jumps), then for any $M > 1$, (2.3) holds uniformly in $x, y$ under each of the following constraints:

1. $-M < x_n \ll \Lambda_n(x)$ and $-\Lambda_n(-y) \ll y_n < M$;
2. $1/M < x_n < M$ and $-\Lambda_n(-y) \ll y_n < M$;
3. $-M < x_n \ll \Lambda_n(x)$ and $-M < y_n < -1/M$.

[The case (3) is included, although it could be treated as the dual of (2).]

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**Figure 1:** The dotted regions indicate the range of validity for (2.3) described by (1), (2) and (3) in Proposition 2.1, and the striped regions the range that is added by Theorem 3 both in the extreme case $\gamma = 2 - \alpha$. The vertical broken lines roughly indicates the boundary for the regime $x_n \ll \Lambda_n(x)$ and the horizontal one that for $-\Lambda_n(-y) \ll y_n$. 

\[ \gamma = 2 - \alpha \]
By (2.1) together with Remark (2.2) given in the next subsection it follows that

\[ \Lambda_n(\pm c_n) \to \kappa_{\alpha,\gamma,\pm}^a \quad \text{with} \quad \kappa_{\alpha,\gamma,\pm}^a > 0 \] or \( = 0 \) according as \( \pm \gamma \neq 2 - \alpha \) or \( = 2 - \alpha \).

Hence for \( x \geq 0 \), the condition \( x_n << \Lambda_n(x) \) always entails \( x_n \to 0 \) and if \( \gamma < 2 - \alpha \) the converse is true; in case \( \gamma = 2 - \alpha \) it largely depends on the behaviour of \( a(x) \) as \( x \to \infty \) (\( a(x) \) may even be bounded) but still permits \( x \) to grow indefinitely large unless the walk is left-continuous; and similarly for \( x \leq 0 \). Thus for \( \gamma < 2 - \alpha \) and \( x \geq 0 \), the condition \( x << \Lambda_n(x) \) is the same as \( x_n \to 0 \). From Theorem 2 and Proposition 2.1 (see Corollary 1 below) we infer that \( \Lambda_n(x) \sim \kappa^{-1}P[\sigma_{(0)} > n] \) (\( \kappa = (1 - \frac{1}{\alpha})^{-1}\kappa_{\alpha,\gamma} \)) whenever \( |x_n| << \Lambda_n(x) \).

**Remark 2.1.** (a) For fixed \( x, y \), formula (2.5) (reducing to \( Q_{\xi}^n(x, y) \sim a^\dagger(x)a^\dagger(-y)f^0(n) \)) follows from a special case of the result of Kesten [17] (see Remark 2.4 given at the end of this section). When \( \gamma = 0 \) (i.e., the limit stable process is symmetric) and \( L(x) \to 1 \), the asymptotic form of \( f^0(n) \) is derived in [17] in which an asymptotic form for \( \alpha = 1 \) is also obtained, which reads \( f^0(n) \sim \pi/n(\log n)^2 \).

(b) Belkin [1, Theorem 2.1] shows the conditional limit theorem that may read

\[ \lim_{n \to \infty} P[S_n/c_n > \xi | \sigma_{(0)}^0 > n] = \int_{\xi}^{\infty} h(t)dt \quad \text{for} \quad \xi \in \mathbb{R}, \] (2.6)

where \( h \) is a probability density on \( \mathbb{R} \) which is bounded and continuous and whose characteristic function, denoted by \( \hat{h}(\theta) \), is given by

\[ \hat{h}(\theta) = 1 - \psi(\theta) \int_0^1 \xi^{\frac{1}{\alpha} - 1}e^{\psi(\theta)(\xi - 1)}d\xi. \] (2.7)

Our results, Theorem 2 and Proposition 2.1 imply the local version of this limit theorem. Indeed on taking \( x = 0 \) in the third case of (2.5), with the help of (2.3) we find that for \( \xi \in \mathbb{R}, P[S_n = |c_n\xi| | \sigma_{(0)}^0 > n] \sim h(\xi)/c_n(n \to \infty) \); incidentally we have

\[ h(\xi) = [(1 - 1/\alpha)/\kappa_{\alpha,\gamma}]f^{-\xi}(1). \] (2.8)

(This identity is verified directly from (2.7) in our proofs of Theorem 2 and Proposition 2.1)

Proposition 2.1 is included in this section, it being proved by essentially the same arguments that derive Theorems 1 and 2. We state a consequence on \( f^x(n) \) that directly follows from the results given above by virtue of the identity \( f^x(n) = Q_{\xi}^n(x, 0) \).

**Corollary 1.** For each admissible \( \gamma \) and \( M > 1 \), as \( n \to \infty \)

\[ f^x(n) \sim \begin{cases} a^\dagger(x)f^0(n) & \text{if} \ |x_n| << \Lambda_n(x), \\ f^n(1)/n & \text{uniformly for} \ 1/M < |x_n| < M. \end{cases} \] (2.9)

By (2.9) the first three formulae of (2.5) are written as

\[ Q_{\xi}^n(x, y) \sim \begin{cases} f^x(n)a^\dagger(-y) & (y_n \to 0), \\ a^\dagger(x)f^{-y}(n) & (x_n \to 0), \end{cases} \quad \text{uniformly for} \ |x_n| \vee |y_n| < M \]

valid if \( \gamma \neq |2 - \alpha| \) or if \( \gamma = 2 - y \) and \( (x, y) \) is in the range specified by Proposition 2.1.

Taking (2.8) into account, from these formulae one readily deduces the following local conditional limit theorem.
Corollary 2. If $\gamma < |2 - \alpha|$, as $n \to \infty$ uniformly for $|x| \ll c_n$ and $|y| < Mc_n$,

$$P[S^x_n = y | \sigma_{[0]} > n] \sim \left\{ \begin{array}{l} \frac{(1 - \frac{1}{\alpha})\Lambda_n(-y)\alpha}{\sigma_n} \quad (y_n \to 0), \\ h(y_n)/\sigma_n \quad (|y_n| \geq 1/M). \end{array} \right.$$

[For $|y_n| \geq 1$ we have $O(nL(y)/c_n|y|^\alpha)$ as an estimate of the LHS (see Lemma 6.4).]

Corollary 2 entails that the conditional distribution $P[S^x_n/c_n \leq d\xi | \sigma_{[0]} > n]$ converges to $h(\xi)d\xi$ ($n \to \infty$) ‘uniformly’ for $|x| \ll c_n$. If $\gamma = 2 - \alpha$, this remains true for $-1 < x < \Lambda_n(x)$ but does not for $\varepsilon \Lambda_n(x) \leq x \ll 1$ with each $\varepsilon > 0$ any more as is revealed in the next subsection (see (2.16), (2.17)) in which we address the issue for those $x, y$ that satisfy $\varepsilon \Lambda_n(x) < x \ll 1$, $-\Lambda_n(-y) \ll y < M$, hence are excluded in Proposition 2.1 (see Figure 1).

2.2. Case $\gamma = 2 - \alpha$ with $\varepsilon \Lambda_n(x) \leq x \ll 1$, $-\Lambda_n(-y) \ll y < M$.

Let $U_d(x), x = 0, 1, 2, \ldots$ denote the renewal function of the strictly descending ladder height process: $U_d(0) = 1$ and for $x \geq 1$,

$$U_d(x) = 1 + \sum_{z=1}^{x} u(z) \quad \text{and} \quad u(x) = \sum_{n=1}^{\infty} P[\hat{Z}_1 + \cdots + \hat{Z}_n = -x],$$

where $\hat{Z}_1 = S_{\sigma_{[-\infty, -1]}}$ and $\hat{Z}_k$ are independent copies of $\hat{Z}_1$.

Theorem 3. Let $\gamma = 2 - \alpha$. Then for each $M > 1$ uniformly for $0 \leq x < Mc_n$, as $n \to \infty$

$$Q_n^{\alpha}(x, y) \sim \left\{ \begin{array}{l} a^{\alpha}(x) f^{\alpha}(n) + \frac{U_d(x)p_1(-x_n)}{U_d(c_n)n} a^{\alpha}(-y) \quad (-\Lambda_n(-y) \ll y_n \ll 1), \\ a^{\alpha}(x)/(\gamma + 1) + \frac{U_d(x)K_1(y_n)}{U_d(c_n)c_n} \quad (x_n \to 0, y_n \in [1/M, M]), \end{array} \right.$$

(2.10)

where $K_\alpha(\eta) = 0$ for $\eta \leq 0$ and

$$K_\alpha(\eta) = \lim_{\xi \downarrow 0} \frac{1}{\xi} p_\alpha^{(-\infty, 0]}(\xi, \eta) \quad \text{for} \quad \eta > 0. \quad (2.11)$$

We know that if $\gamma = 2 - \alpha$, then

$$f^{\alpha}(t) = xt^{-1}p_t(-x) \quad (x > 0) \quad \text{and} \quad p_t^{(0)}(x, y) = p_t^{(-\infty, 0]}(x, y) \quad (x, y > 0) \quad (2.12)$$

(cf., e.g., [4, Corollary 7.3]). Note that the first case of (2.10) conforms to the second one of (2.5) valid for $x_n \sim 1$ in view of Proposition 2.1.

By Proposition 2.1 we have $Q_n^{\alpha}(x, y) \sim a^{\alpha}(x) f^{\alpha}(n)a^{\alpha}(-y)$ for $-M < x_n \ll \Lambda_n(x)$. Comparing this with the first relation of (2.10) shows that for $x \geq 0$, $x_n \ll \Lambda_n(x)$ implies $U_d(x)/U_d(c_n) \ll \Lambda_n(x)$. The converse implication is obvious because of the bound $U_d(x)/U_d(c_n) \geq x_n\{1 + o(1)\} \quad (x_n \leq 1)$ (see Lemma 8.8(ii)). It therefore follows that for $0 \leq x < Mc_n$,

$$U_d(x)/U_d(c_n) \ll \Lambda_n(x) \quad \text{if and only if} \quad x_n \ll \Lambda_n(x). \quad (2.13)$$

By (2.13) it follows that in both cases of (2.10) the second terms are superfluous if $x_n \ll \Lambda_n(x)$. Taking this into account we reformulate the result for $\gamma = 2 - \alpha$ as the following.
Corollary 3. Let $\gamma = 2 - \alpha$. Then for any $M > 1$,

(i) uniformly for $x$, as $n \to \infty$

$$f^x(n) \sim \begin{cases} \alpha \tilde{a}(x) \tilde{f}(n) & \text{if } -1 \ll n \ll \Lambda_n(x), \\
\alpha \tilde{a}(x) \tilde{f}(n) + \frac{U_d(x)p_1(-x_n)}{U_d(c_n)n} & \text{if } M^{-1} \Lambda_n(x) < x_n < M, \end{cases} \quad (2.14)$$

(ii) uniformly for $-\Lambda_n(-y) \ll y_n \ll 1$ and $|x_n| < M$

$$Q_{\{0\}}^n(x, y) \sim f^x(n)\alpha \tilde{a}(y). \quad (2.15)$$

In view of (2.13) it follows from (2.14) that as $n \to \infty$

$$f^x(n) \sim \begin{cases} \kappa_{\gamma, \alpha} \alpha \tilde{a}(x) c_n/n^2 & \text{if } 0 \leq n \ll \Lambda_n(x), \\
\frac{U_d(x)p_1(-x_n)}{U_d(c_n)n} & \text{if } \Lambda_n(x) \ll x_n < M, \end{cases} \quad (2.16)$$

and the two terms on the RHS are comparable with each other if $\varepsilon \Lambda_n(x) \ll x_n \ll \varepsilon^{-1} \Lambda_n(x)$ for each $\varepsilon > 0$. The same kind of crossover takes place in (2.10): in the both cases of it the crossover occurs around $x \approx c_n \Lambda_n(x)$ as in (2.10). This may be exhibited in a noticeable form of the local limit theorem corresponding to Corollary 2. Let $Q_t(\xi)$ designate the density of the stable meander of length $t$ at time $t$ (see (8.3)). Recall that $h$ is the density in the conditional limit theorem (2.0).

Corollary 4. If $\gamma = 2 - \alpha$, then uniformly for $0 \leq x_n \ll 1$ and $y \in \mathbb{Z}$,

$$P[S_n^z = y \mid \sigma_{\{0\}}^z > n] = \frac{\theta_{n,x}h(y_n) + (1 - \theta_{n,x})Q_1'(y_n)}{c_n} + o\left(\frac{r_{n,x,y} + 1}{c_n}\right), \quad (2.17)$$

where

$$\theta_{n,x} = \frac{P[\sigma_{\{-\infty,0\}}^z < n < \sigma_{\{0\}}^z]}{P[n < \sigma_{\{0\}}^z]} \quad \text{and} \quad r_{n,x,y} = 1(0 \leq y_n < 1) \frac{L(c_n)}{L(y)}.$$
(b) In (2.1) one may replace 1 by any positive constant \( \lambda \), or what effects the same thing, replace \( c_n \) by \( \tilde{c}_n = \lambda^{-1/\alpha} c_n \) so that if \( \tilde{Y}_t = Y_M \) and \( \tilde{L} = \lambda L \), then \( S_n/\tilde{c}_n \xrightarrow{\text{law}} \tilde{Y}_1 \) and \( n\tilde{c}_n^{-1} L(\tilde{c}_n) \rightarrow 1 \). This causes merely the replacements of the functions associated with \( Y_t \) by those with \( \tilde{Y}_t = Y_M \) (along with \( c_n \rightarrow \tilde{c}_n, x_n \rightarrow \tilde{x}_n = \lambda^{1/\alpha} x_n \), e.g., \( f^{\tilde{x}_n}(1) \rightarrow \lambda^{\tilde{x}_n}(\lambda) \), \( p^{(0)}_\lambda(x_n,y_n) \rightarrow p^{(0)}_\lambda(\tilde{x}_n,\tilde{y}_n) \), and so on. (Note that the formula of Theorem 6 becomes \( f^{\tilde{x}_n}(n) \sim \kappa_{\alpha,\gamma} \lambda^{1/\alpha} \tilde{c}_n / n^2 \) since \( p_\lambda(0) = \lambda^{-1/\alpha} p_1(0) \).)

Below we provide some consequences on \( p^{(0)}_1(\xi,\eta) \) that are drawn from Theorems 2 and 3 (and that the present author fails to find explicit statements for them in the existing literature) and also known facts on \( f^{\xi}(t) \).

**Some consequences on \( p^{(0)}_1(\xi,\eta) \) and \( K_1(\xi) \)**

(i) Let \( \gamma = 2 - \alpha \). For the present purpose we may suppose \( L \equiv 1 \) and \( E[|Z|] < \infty \) so that \( c_n \sim n^{1/\alpha} \) and \( U_d(x)/U_d(c_n) \sim x_n \). Formula (2.15) implies that if one takes the successive limit as first \( x_n \rightarrow \xi > 0 \), \( y_n \rightarrow \eta > 0 \) as well as \( n \rightarrow \infty \) and then \( \xi \vee \eta \rightarrow 0 \), then

\[
\frac{Q_n^{(0)}(x,y)}{f^2(n)a(-y)} = \frac{Q_n^{(0)}(x,y)}{p_n^{(0)}(x,y)} \frac{p_n^{(0)}(x,y)}{f^2(n)a(-y)} \rightarrow 1.
\]

Since the first ratio of the middle member approaches 1 by virtue of the last relation of (2.14), it therefore follows from second formula of (2.14) (or (5.1)) that as \( \xi \rightarrow 0 \) and \( \eta = y_n \rightarrow 0 \)

\[
\frac{p^{(0)}_1(\xi, y_n)n^{1-1/\alpha}}{p_1(-\xi)n(-y)} \rightarrow 1. 
\]

(2.20)

On noting \( p^{(0)}_1(\xi, \eta) = p^{(0)}_1(\eta, \xi) \) and using (2.19) this shows that

\[
p^{(0)}_1(\xi, \eta) \sim \frac{p^{(0)}_1(\xi)}{\Gamma(\alpha)} \times \left\{ \begin{array}{ll} \xi \eta^{\alpha-1} & (\xi \downarrow 0, \eta \downarrow 0), \\
(-\xi)^{\alpha-1}(\eta) & (\xi \uparrow 0, \eta \uparrow 0). \end{array} \right.
\]

(2.21)

(See Lemma 8.6 for the case when one of \( \xi \) and \( \eta \) is fixed.) In a similar way we deduce

\[
p^{(0)}_1(\xi, \eta) \sim \kappa_{\alpha,\gamma}[\Gamma(\alpha)]^{-2}(-\xi \eta)^{\alpha-1} & (\xi \uparrow 0, \eta \downarrow 0).
\]

(ii) In the same way as above but with

\[
\frac{p^{(0)}_1(\xi, y_n)n^{1-1/\alpha}}{f^{\xi}(1)a(-y)} \rightarrow 1
\]

(2.22)

in place of (2.20) we deduce that if \( |\gamma| < 2 - \alpha \),

\[
p^{(0)}_1(\xi, \eta) \sim \kappa'_{\alpha,\gamma}\{\sin[\frac{1}{2}\pi(\alpha + (\text{sgn} \xi \gamma))]\{\sin[\frac{1}{2}\pi(\alpha - (\text{sgn} \eta \gamma))]\}!|\xi \eta|^{\alpha-1} \]

as \( |\xi| \vee |\eta| \rightarrow 0 \),

where \( \kappa'_{\alpha,\gamma} = \kappa_{\alpha,\gamma}[\Gamma(1-\alpha)/\pi]^2 \). Similarly \( \lim_{\xi \rightarrow \pm 0} p^{(0)}_1(\xi, \eta)/\xi^{\alpha-1} = \kappa_{\alpha,\gamma,\pm} f^{-\eta}(1) \) for \( \eta > 0 \).

(iii) Let \( \gamma = 2 - \alpha \). Then \( p_1(0) = 1/\alpha \Gamma(1-1/\alpha) \) and by (2.21)

\[
K_1(\eta) \sim \frac{p^{(0)}_1(0)}{\Gamma(\alpha)} \eta^{\alpha-1} & (\eta \downarrow 0).
\]

(2.23)
The scaling relation of $K_t(\eta)$ is given by $K_t(\eta) = K_1(\eta/t^{1/\alpha})/t^{2/\alpha}$.

**Asymptotic properties of $f^\xi(t)$.**

The density function $f^\xi(t)$ satisfies the scaling relation

$$f^\xi(\lambda t) = f^\xi(t^{1/\alpha})(\lambda)/t = f^1((\lambda t)/|x|^{\alpha})/\lambda t (x \neq 0, t > 0, \lambda > 0). \quad (2.24)$$

In case $\gamma = |2 - \alpha|$, expansions of $f^\xi(t)t = f^\xi(t^{1/\alpha})(1)$ into power series of $x/t^{1/\alpha}$ are known. Indeed, if $\gamma = 2 - \alpha$, owing to (2.12) the series expansion for $x > 0$ is obtained from that of $t^{1/\alpha}p_x(-x)$ which is found in [13], while for $x < 0$, the series expansion is derived by Peskir [22]. In the recent paper [19, Theorem 3.14] Kuznetsov et al. obtain a similar series expansion for all cases. Here we state the leading term for $f^1(t)$ and an error estimate (as $t \rightarrow \infty$) that are deduced from these expansions.

As $t \rightarrow \infty$

$$f^1(t) = \begin{cases} 
-1/\Gamma(-1/\alpha)]t^{-1-1/\alpha} \{1 + O(t^{-1/\alpha})\} & \text{if } \gamma = 2 - \alpha, \\
\kappa_{\alpha,\gamma}^f t^{-2+1/\alpha} \{1 + O(t^{-2/\alpha})\} & \text{if } \gamma \neq 2 - \alpha,
\end{cases} \quad (2.25)$$

where

$$\kappa_{\alpha,\gamma}^f = \Gamma(2 - \alpha) \sin(\pi/\alpha) \sin[\pi/2 (\alpha + \gamma)]/\alpha^2 p_1(0) = \sin \pi/\alpha \int_0^{\infty} u^{1-\alpha} p_1(-u)du.$$

[The first expression shows that $\kappa_{\alpha,\gamma}^f$ is positive if $\gamma < 2 - \alpha$ and zero if $\gamma = 2 - \alpha$; see Lemma 8.3 for the second expression.]

In [19, Theorem 3.14] an asymptotic expansion as $t \rightarrow 0$ (see Lemma 8.2) is also obtained which entails

$$f^1(t) = [\alpha^3 \Gamma(\alpha)/\Gamma(2 - \alpha)] \kappa_{\alpha,\gamma}^f t^{1/\alpha} + O(t^{1+1/\alpha}) \quad \text{as } t \rightarrow 0.$$

We observe that the leading term in the second formula of (2.25) is deduced from Corollary 1. To this end we may let $L \equiv 1$, for which if $\gamma \neq 2 - \alpha$,

$$\lim_{\xi \downarrow 0} \frac{f^\xi(1)}{\xi^{\alpha-1}} = \lim_{\xi \downarrow 0} \frac{n f^\xi(n)}{\xi^{\alpha-1}} = \lim_{\xi \downarrow 0} \frac{a(\xi) K_{\alpha,\gamma}}{\xi^{\alpha-1} n^{1-1/\alpha}} = K_{\alpha,\gamma} = \kappa_{\alpha,\gamma}^f.$$

which is the same as what is to be observed because of the scaling property (2.24).

**2.3. Case $\gamma = 2 - \alpha$ with $xy < 0$ and upper bounds outside the principal regime.**

From Theorem 3 or Proposition 2.1 is excluded the regime $x_n > \varepsilon \Lambda_n(x)$, $y_n < -\varepsilon \Lambda_n(-y)$ for any $\varepsilon > 0$, where there arises a difficulty in estimating $p_{\Lambda_n(0)}(x,y)$ in general; below we give a result under an extra assumption on the tail as $t \rightarrow -\infty$ of the distribution function

$$F(t) := P[X \leq t].$$

Suppose that $\gamma = 2 - \alpha$ so that $F(-t) = o(1 - F(t)) \ (t \rightarrow \infty)$. In [31, Theorem 2(iii)] a criterion for the limit

$$C^+ := \lim_{x \rightarrow +\infty} a(x) \leq \infty$$
(which exists) to be finite is obtained. Under the present assumption on $F$ it says that

$$\sum_{y=1}^{\infty} F(-y)[a(-y)]^2 < \infty \quad \text{and} \quad F(-2) > 0 \quad (2.26)$$

is necessary and sufficient for $0 < C^+ < \infty$.

**Theorem 4.** Suppose that $\gamma = 2 - \alpha$ and (2.26) holds. Then, given $M > 1$, uniformly for $-M \leq y_n < 0 \leq x_n < M$,

(i) \[ Q_{\{0\}}^n(x, y) \sim \frac{a^\dagger(x)a^\dagger(-y)f^0(n)}{n} + \frac{a^\dagger(x)|y_n|p_1(y_n) + a^\dagger(-y)x_np_1(-x_n)}{n} \]

$(x_n \land (-y_n) \to 0)$,

(ii) \[ Q_{\{y\}}^n(x, y) \sim \frac{C_+ (x_n - y_n)p_1(y_n - x_n)}{n} = \frac{C_+ x_n - y_n(1)}{n} \]

$(x_n \land (-y_n) > 1/M)$, as $n \to \infty$.

An application of Theorem 3 leads to the next result which exhibits a way the condition $C^+ < \infty$ is reflected in the behaviour of the walk $S^x$, $x > 0$: conditioned on $S^x_n = -x$ it enters $(\infty, -1]$ without visiting the origin ‘continuously’ or by a very long jump for large $x$ according as $C^+$ is finite or not. Exactly the same behaviour of the pinned walk is observed in [28] in the case $E|X|^2 < \infty$ but with the condition (2.26) replaced by $E[|X|^4; X < 0] < \infty$ which is equivalent to $\lim_{x \to \infty}[a(-x) - x/\sigma^2] < \infty$.

**Proposition 2.2.** Let $\gamma = 2 - \alpha$. Then for each $M \geq 1$, as $n \to \infty$ under the constraint $-Mc_n < y < 0 < x < Mc_n$

$$P[S^x_{\sigma(-\infty,0)} < -R | \sigma_{\{0\}} > n, S^x_n = y] \longrightarrow \begin{cases} 0 & \text{as } R \to \infty \quad \text{uniformly for } x, y \quad \text{if } C^+ < \infty, \\ 1 & \text{as } x \land (-y) \to \infty \quad \text{for each } R > 0 \quad \text{if } C^+ = \infty. \end{cases}$$

In case $|x_n| \to \infty$ upper bounds is provided by the following proposition, where we include a reduced version of that for the case $|x_n| < 1$ given above.

**Proposition 2.3.** There exists a constant $C$ such that for all $\gamma$ and $x$,

$$f^x(n) \leq C \left( \frac{a^\dagger(x) + a^\dagger(-x)}{n^2/c_n} \land P[|X| > |x|] \right).$$

**Proposition 2.4.** (i) For all admissible $\gamma$ and $M > 1$, there exists a constant $C_M$ such that for all $n \geq 1$ and $x \in \mathbb{Z}$,

$$Q_{\{0\}}^n(x, y) \leq C_M \left( \frac{|x|^{\alpha - 1} \lor 1}{L(x)n^2/c_n} \land \frac{L(x)}{|x|^\alpha \lor 1} \right) \frac{|y|^{\alpha - 1} \lor 1}{L(y)} (|y_n| < M);$$

(ii) If $\gamma = 2 - \alpha$, there exists a constant $C$ such that for all $x, y \in \mathbb{Z}$,

$$Q_{\{0\}}^n(x, y) \leq C \left[ \frac{a^\dagger(x)a^\dagger(-y)}{n^2/c_n} + \frac{a^\dagger(-y)\{(x_n) \lor 1\} + a^\dagger(x)\{(y_n) \lor 1\}}{n} \right]. \quad (2.27)$$
Proposition 2.3 follows from Proposition 2.4(i) in view of \( f^0(n) = Q^n(x,0) \).

As for Proposition 2.4(i) follows from Lemma 6.1 for \( |x| < c_n \) and from Lemma 6.2 for \( |x| \geq c_n \). (ii) follows from Theorems 2 and 3 in case \( |x| \wedge |y| \leq c_n \) with \( xy \geq 0 \), from Proposition 6.2(i) in case \( y \leq 0 \leq x \), from Lemmas 6.1(i) and its dual (see Proposition 6.2(ii)) in case \( 0 < (-x) \wedge y \leq c_n \), from Lemmas 6.1 and 6.2 in case \( |x| \wedge |y| \leq c_n \leq |x| \wedge |y| \) with \( xy \geq 0 \) (Remark 6.2 is much weaker in this case) and from the bound \( p^n(x) \leq C/c_n \) in case \( |x| \wedge |y| \geq c_n \) with \( xy \geq 0 \) as well as in case \( (-x) \wedge y \geq c_n \).

**Remark 2.3.** When the condition 2) (strong aperiodicity) is not assumed, the results stated above must be subjected to some modifications. Let \( \nu \geq 1 \) denote the period of the walk, which amounts to assume (in addition to (1.1)) that \( p^n(0) > 0 \) and \( p^{n+j}(0) = 0 \) \((1 \leq j < \nu)\) for all sufficiently large \( n \). Then the process \( \tilde{S}_n := S_{vn}/\nu \), \( n = 0,1, \ldots \) is a strongly aperiodic walk on \( \mathbb{Z} \) such that \( 1 - E[e^{i\theta \tilde{S}_n}] = 1 - [\phi(\theta/\nu)]^\nu \sim \nu^{1-\alpha}L(1/\theta)\psi(\theta) \), and \( S^\nu_k \neq 0 \) a.s. for all combinations of \( x \in \nu\mathbb{Z} \) and \( k \notin \nu\mathbb{Z} \); hence in case \( A = \{0\} \) the results restricted on \( \nu\mathbb{Z} \) follow from those of the aperiodic walks and the extension to \( \mathbb{Z} \) is then readily performed by using \( E[a(S^\nu_k)] = a^\dagger(x) \) if it concerns Theorems 1, 2 and Proposition 2.4 this in particular results in

\[
f^0(n) = \nu \mathbf{1}(n \in \nu\mathbb{Z}) f_*(n) \{1 + o(1)\} \quad (n \to \infty),
\]

where \( f_*(n) = \kappa_{\alpha, \gamma} c_n/n^2 \) (with \( c_n \) satisfying (2.1)), and for \( x, y \) such that \( p^n(y-x) > 0 \),

\[
Q^n_{[0]}(x,y) \sim \nu a_\dagger(x) f_*(n) a_\dagger(-y) \quad \text{under} \quad |x_n| \ll \Lambda_n(x) \text{ and } |y_n| \ll \Lambda_n(-y)
\]

(note that \( \tilde{a}(x/\nu) = a(x), \ x \in \nu\mathbb{Z} \) for the last relation), and analogously for the other ranges of \( x, y \). As for the extension of Theorems 3 and 4 we must replace the renewal function \( U_d \) by \( \tilde{U}_d \) say, the corresponding one for the walk \( \tilde{S} \), but the formulae therein are kept unchanged under this replacement except obvious modifications as made above, since

\[
\tilde{U}_d(x)/\tilde{U}_d(c_n) \sim U_d(x)/U_d(c_n) \quad \text{as} \quad n \wedge x \to \infty
\]

(see Lemma 8.1).

**2.4. Extension to an arbitrary finite set.**

Let \( A \) be a finite subset of \( \mathbb{Z} \). Suppose for simplicity that for some \( M > 1 \)

\[
G_A(x,y) > 0 \quad \text{if} \quad |x| \wedge |y| > M,
\]

where for a non-empty \( B \subset \mathbb{Z} \), \( G_B \) denotes the Green function for the walk killed on \( B \):

\[
G_B(x,y) = \sum_{n=0}^\infty Q^n_B(x,y).
\]

Since \( p^1 \) is carried by an unbounded set, (2.29) implies \( G_A(x,y) > 0 \) for all \( x, y \). Under the condition \( E X^2 = \infty \) there exists

\[
u \mathbf{A}(x) = \lim_{|y| \to \infty} G_A(x,y)
\]

[26] T30.1. \( \nu \mathbf{A} \) is positive and harmonic for the killed walk: \( \nu \mathbf{A}(x) = \sum_{z \notin A} p^1(z-x) \nu \mathbf{A}(z) > 0 \) for all \( x \in \mathbb{Z} \). Put

\[
f^\nu_A(n) = P(\sigma^\nu_A = n).
\]
In order to obtain the asymptotic form of \( Q_n^A(x, y) \) and \( f_A^x(n) \) we may simply replace \( a^x(t) \) by \( u_A(x) \) and \( a^x(-y) \) by \( u_{-A}(-y) \) on the RHS of formulae given in Theorems 2 to 4 and Corollaries 1 and 3—and accordingly replace \( C^+ \) by \( C^+_A = \lim_{x \to \infty} u_A(x) \) (positive under (2.29)) in Theorem 1(ii)—with the resulting formulae valid in the same range of variables.

Thus by Corollary 1 we have

\[
\begin{aligned}
f_A^x(n) &\sim \begin{cases} 
\frac{u_A(x) f^0(n)}{f^x(n)} & \text{for } |x_n| << \Lambda_n(x), \\
\frac{1}{f^x(n)} & \text{uniformly for } 1/M < |x_n| < M.
\end{cases}
\end{aligned}
\]

(2.32)

Similarly corresponding to Theorem 3 we have:

**Theorem 5.** If \( \gamma = 2 - \alpha \), for each \( M > 1 \) uniformly for \( |x| < M c_n \), as \( n \to \infty \)

\[
Q_n^A(x, y) \sim \begin{cases} 
\left\{ \frac{u_A(x) f^0(n)}{U_d(x) p_1(-x_n)} \right\} u_{-A}(-y) & (-\Lambda_n(-y) << y_n << 1), \\
\frac{u_A(x) f^0(n)}{n} & (x_n \to 0, y_n \in [1/M, M]).
\end{cases}
\]

[For \(-M < x_n < \Lambda_n(x)\), the second terms on the RHS are negligible as compared to the first.]

Corollary 4 with \( \sigma_A^x \) in place of \( \sigma^x_{01} \) remains in force without any modification.

Restricting to \( y \in A \) the formula above is valid to infer the following

**Corollary 5.** For all admissible \( \gamma \), uniformly for \( |x_n| < M \), as \( n \to \infty \)

\[
Q_n^A(x, y) = P[\sigma_A^x = n, S_n^x = y] = f_A^x(n) u_{-A}(-y) \quad (y \in A).
\]

From the definition of \( G_A(x, y) \) it follows that \( u_A \) is the probability distribution of the hitting place of \( A \) by the dual walk ‘started at infinity’. By (2.32) it therefore follows that

\[
\sum_{x \in A} \sum_{y \in A} Q_n^A(x, y) = \sum_{x \in A} f_A^x(n) \sim f^0(n).
\]

(2.33)

**Remark 2.4.** As was mentioned in Introduction Kesten [17] obtained asymptotic formulae of \( Q_n^A(x, y) \) with \( x, y \) fixed for a large class of random walks on multidimensional lattices \( \mathbb{Z}^d \); in particular Theorem 6a of [17] specialized to one-dimensional walk may read in the present notation

\[
\lim_{n \to \infty} Q_n^A(x, y) \bigg/ \sum_{z \in A} \sum_{w \in A} Q_n^A(z, w) = u_A(x) u_{-A}(-y) \quad (y \notin A),
\]

valid whenever the walk is recurrent, strongly aperiodic and having infinite variance. When \( \gamma = 0 \) and \( L \equiv 1 \) relation (2.33) is also observed in [17].

The rest of the paper is organized as follows. In Section 3 we prove some preliminary lemmas that are fundamental for our proofs of Theorems 1 and 2 and Proposition 2.1 that are given in Section 4. The proof of Theorem 3 is given in Section 5. In Section 6 some estimations of \( Q_{n0}^A(x, y) \) are made in case \( xy < 0 \) and, for this purpose, beyond the regime \( |x| \vee |y| = O(n^{1/\alpha}) \): Propositions 6.1 and 6.2 given there provide a lower and upper bound, respectively and Theorem 4 and Proposition 2.2 are proved after them. In Section 7 the results are extended to those for an arbitrary finite set instead of the single point set \( \{0\} \). In Section 8 we deal with the limit stable process and present some properties of \( f^L(t) \), \( p_{10}^L(x, y) \) and \( U_d(x) \). In Appendix we provide (A) condition (1.1) expressed in terms of the tails of \( F \) and some related facts, (B) an relation between \( a(x) \) and the hitting distribution of non-positive half line, and (C) ‘some bounds of escape probabilities’ from the origin.
3 Preliminary lemmas.

In this section we show that the conditional law of $S_n^\gamma/c_n$ given $\sigma_{T(0)}^\gamma > n$ converges as $n \to \infty$ uniformly for $|x_n| << \Lambda_n(x)$ (under a certain assumption if $\gamma = |2 - \alpha|$ that will be removed in the next section), extending Belkin’s result (2.6) for the special case $x = 0$.

**Lemma 3.1.** Put $\kappa_{\alpha,\alpha,\pm}^a = -\Gamma(1 - \alpha)\pi^{-1}\sin[\frac{1}{2}\pi(\alpha \pm \gamma)]$. Then under the assumption above

\[
\lim_{x \to \pm \infty} \frac{a(x)L(x)}{|x|^{\alpha-1}} = \kappa_{\alpha,\alpha,\pm}^a,
\]

(i) \quad \lim_{x \to \pm \infty} \left\{a(x + 1) - a(x)\right\}|x|^{2-\alpha}L(x) = \pm(\alpha - 1)\kappa_{\alpha,\alpha,\pm}^a;

(ii) \quad \kappa_{\alpha,\alpha,\pm}^a > 0 \text{ if } |\gamma| < 2 - \alpha, \text{ and } \kappa_{\alpha,\alpha,\pm}^a = 0 \text{ or } 1/\Gamma(\alpha) \text{ according as } \pm \gamma > 0 \text{ or } < 0 \text{ if } |\gamma| = 2 - \alpha.

**Proof.** (i) is given in [2] (without proof). Here we give a proof, which is partly used in the proof of (ii). The proof is based on

\[
\int_0^\infty \left\{1 - \cos u \over \sin u\right\} du = \int_{-1}^1 du = -\pi \sin \frac{\pi}{2},
\]

(3.1)

(cf. [12, pp.10, 68], [33, p.260]). In the representation $a(x) = \frac{1}{2\pi} \int_{-\pi}^\pi (1 - e^{ix\theta})(1 - \phi(\theta))^{-1} d\theta$ we replace $1 - \phi(\theta)$ by $\psi(\theta)L(1/\theta)$, its principal part about zero, and compute the resulting integral. Changing a variable we have

\[
\int_{-\pi}^\pi \frac{1 - e^{ix\theta}}{\psi(\theta)L(1/\theta)} d\theta = |x|^{\alpha-1} \int_{-\pi}^\pi \frac{1 - \cos u \mp i \sin u}{\cos \frac{\pi}{2} \gamma + i u/|u| \sin \frac{\pi}{2} \gamma} \cdot \frac{|u|^{-\alpha} du}{L(x/u)} (\pm = x/|x|),
\]

(3.2)

which an easy computation with the help of (3.1) shows to be asymptotically equivalent to

\[
-\frac{2\Gamma(1 - \alpha)|x|^{\alpha-1}}{L(x)} \left[\cos \frac{\pi \gamma}{2} \sin \frac{\pi \alpha}{2} \pm \sin \frac{\pi \gamma}{2} \cos \frac{\pi \alpha}{2}\right]
\]

as $|x| \to \infty$. The combination of the sine’s and cosine’s in the square brackets being equal to $\sin[\frac{1}{2}\pi(\alpha \pm \gamma)]$ we find the equality (i), provided that the replacement mentioned at the beginning causes only a negligible term of the magnitude $o(|x|^{\alpha-1})/L(x)$, but this is assured from the way of computation carried out above since the integrand in the RHS integral in (3.2) is summable on $\mathbb{R}$.

For the proof of (ii) it suffices to show that

\[
\int_{-\pi}^\pi e^{ix\theta} (1 - e^{iy\theta}) \left[\frac{1}{1 - \phi(\theta)} - \frac{1}{\psi(\theta)L(1/\theta)}\right] d\theta = o\left(|x|^{\alpha-2}\right),
\]

(3.3)

for $a(x + 1) - a(x) = \frac{1}{2\pi} \int_{-\pi}^\pi e^{ix\theta} (1 - e^{iy\theta})(1 - \phi(\theta))^{-1} d\theta$ and this integral with $1 - \phi(\theta)$ replaced by $\psi(\theta)L(1/\theta)$ is asymptotically equivalent to $\pm(\alpha - 1)\kappa_{\alpha,\alpha,\pm}^a |x|^{\alpha-2}/L(x)$ as one sees by looking at the increment of the RHS of (3.2). Recall $L$ is so chosen as to be smooth enough. Because of the fact that if $\psi(\theta)L(1/\theta) = \{1 - \phi(\theta)\} \{1 + \delta(\theta)\}$ then $\delta'(\theta) \to 0 \text{ (} \theta \to 0 \text{)}$ (cf. (1.6)), the relation (3.3) is shown in a standard way. \(\square\)

**Lemma 3.2.** (i) There exists a constant $C$ such that

\[
|p^n(0) - p^n(x)| \leq C|x|^\gamma_c^2 (x \in \mathbb{Z}, n \geq 1).
\]

(ii) For any $\varepsilon > 0$, \(\frac{1}{a_1(x)} \sum_{k \geq \varepsilon n} |p^k(0) - p^k(x)| \to 0 \text{ (} n \to \infty \text{) under } |x_n| << \Lambda_n(x).\)
Note that if the walk is left-continuous (right-continuous), the assertion (ii) of Lemma 3.2 is void for \( x > 0 \) \( (x < 0) \). The same remark applies to the subsequent lemmas.

**Proof.** By (1.1) \(- \log |\phi(\theta)| \sim (\cos \frac{1}{2} \gamma \pi)|\theta|^\alpha L(1/\theta) \) \( (\theta \to 0) \) and by the Fourier representation

\[
p^n(0) - p^n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\phi(\theta)]^n(1 - e^{-i\theta}) d\theta
\]

as well as the assumed strong aperiodicity that entails \(|\phi(\theta)| < 1 \) \( (0 < |\theta| \leq \pi) \) it follows that for any positive \( \varepsilon > 0 \) small enough

\[
|p^n(0) - p^n(x)| \leq \frac{|x|}{\pi} \left[ \int_0^\varepsilon e^{-\lambda n \theta^\alpha L(1/\theta)} \theta d\theta + O(e^{-Cn}) \right]
\]

with \( \lambda := \frac{1}{2} \cos \frac{1}{2} \gamma \pi \) and a positive constant \( C = C(\varepsilon) \). Bringing in the variable \( t = \theta c_n \) we have

\[
n \theta^\alpha L(1/\theta) = t^\alpha L(c_n/t)/L(c_n)\{1 + o(1)\}
\]

because of (2.1). Hence, on choosing \( \varepsilon > 0 \) so that \( u^{-\alpha/2} L(u) \) is decreasing on \( u \geq 1/\varepsilon \),

\[
t^\alpha L(c_n/t)/L(c_n) = t^{\alpha/2}[\{t/c_n\}^{\alpha/2} L(c_n/t)]/[c_n^{-\alpha/2} L(c_n)] > t^{\alpha/2} \quad \text{for} \quad 1 \leq t \leq \varepsilon c_n.
\]

Hence

\[
\int_0^\varepsilon e^{-\lambda n \theta^\alpha L(1/\theta)} \theta d\theta \leq \int_0^{1/c_n} \theta d\theta + \frac{1}{c_n} \int_1^{(\varepsilon c_n)^{\alpha/2}} e^{-t^{\alpha/2}(\lambda + o(1))} t dt \leq \frac{C'}{c_n^2},
\]

which combined with (3.5) shows the first assertion (i). The second formula (ii) follows from the first by an easy computation. \( \square \)

**Lemma 3.3.** If \( \gamma \neq 2 - \alpha \), then there exists a constant \( C \) such that

\[
\sum_{k=0}^{\infty} |p^k(0) - p^k(-x)| \leq C a(x) \quad \text{for} \quad x \geq 0.
\]

**Proof.** Note that by Lemma 3.2 the infinite series on the LHS is convergent, so that we may consider only large values of \( x \). For each \( x \) large enough let \( j(x) \) be the smallest integer such that \( c_{j(x)} \geq x \), which entails

\[
j(x) \sim x^\alpha / L(x)
\]

in view of the definition of \( c_n \). Since \( p_1(-x/c_k) < \frac{1}{2} p_1(0) \) if \( k/j(x) \) small enough and \( x \) large enough, we can choose, owing to the local limit theorem, positive constants \( \delta \) and \( N \) that do not depend on \( x \) such that

\[
p^k(0) - p^k(-x) > 0 \quad \text{for} \quad N \leq k \leq \delta j(x).
\]

Using Lemma 3.3 we deduce that

\[
\sum_{k>\delta j(x)} |p^k(0) - p^k(-x)| \leq C \delta^{1-2/\alpha} x j(x)/c_{j(x)}^2.
\]

Here \( c_{j(x)} \) may be replaced by \( x \) and because of (3.6) the RHS is dominated by a constant multiple of \( x^{\alpha-1}/L(x) \), hence of \( a(x) \) if \( \gamma \neq 2 - \alpha \), owing to Lemma 3.1. Because of (3.7) this concludes the proof. \( \square \)
Lemma 3.4. Let \( \kappa = \kappa_{\alpha, \gamma}/(1 - \frac{1}{\alpha}) \). If \( \gamma \neq 2 - \alpha \), then

(i) \( P[\sigma_{(0)}^x > n] \sim \kappa a^+(x)c_n/n \quad (n \to \infty) \quad \text{under} \quad 0 \leq x_n \ll \Lambda_n(x), \)

(ii) \( \exists C > 0, \quad P[\sigma_{(0)}^x > n] \leq C\{a^+(x)P[\sigma_{(0)}^0 > n] + x_n\} \quad \text{for} \quad x \geq 0, \quad n \geq 1. \)

Recall that for \( \gamma \neq 2 - \alpha \), the condition \( 0 \leq x_n \ll \Lambda_n(x) \) is the same as \( 0 \leq x_n \ll 1. \)

Proof. For \( x = 0 \) the assertion (i) reduces to (2.3). For the proof of (i) it therefore suffices to show

\[
P[\sigma_{(0)}^x > n] \sim a^+(x)P[\sigma_{(0)}^0 > n] \quad (0 \leq x_n \ll \Lambda_n(x)). \quad (3.8)
\]

We have the identity

\[
p^n(-x) = f^x(n) + \sum_{k=1}^{n-1} p^k(-x)f^0(n-k) \quad (n \geq 1, \quad x \in \mathbb{Z}).
\]

Noting that for \( x = 0 \), the RHS can be written as \( \sum_{k=0}^{n-1} p^k(0)f^0(n-k) \), we find

\[
f^x(n) = p^n(-x) - p^n(0) + \sum_{k=0}^{n-1} [p^k(0) - p^k(-x)]f^0(n-k) \quad (n \geq 1, \quad x \neq 0),
\]

and by the usual method of generating function or by direct computation one can easily derives

\[
P[\sigma_{(0)}^x \geq n] = \sum_{k=0}^{n-1} [p^k(0) - p^k(-x)]P[\sigma_{(0)}^0 \geq n - k] \quad (n \geq 1, \quad x \neq 0). \quad (3.9)
\]

We split the sum on the RHS at \( k = \varepsilon n \) with a small \( \varepsilon > 0 \). Now suppose \( \gamma \neq 2 - \gamma \) and apply Lemmas 3.2 (ii) and 3.3 as well as the result for \( x = 0 \) to see that the sum restricted to \( 0 \leq k \leq \varepsilon n \) is written as

\[
\sum_{k \leq \varepsilon n} = a^+(x)P[\sigma_{(0)}^0 \geq n]\{1 + o_\varepsilon(1)\} \quad (3.10)
\]

where \( o_\varepsilon(1) \) is bounded on \( x \geq 0 \) and tends to zero as \( n \to \infty \) and \( \varepsilon \to 0 \) in this order uniformly for \( 0 \leq x_n \ll \Lambda_n(x) \), whereas for the other sum

\[
\sum_{\varepsilon n \leq k < n} \leq \sum_{\varepsilon n \leq k < n} \frac{Cx}{c_k^2} \cdot \frac{c_{n-k}}{n-k} \sim \frac{Cx}{\varepsilon^{1/\alpha} c_n} \int_{\varepsilon}^{1} \frac{1}{u^{2/\alpha}(1 - u)^{1-1/\alpha}}. \quad (3.11)
\]

These estimates together conclude (3.8) as well as (ii). \( \square \)

The same (rather simplified) proof as above verifies that even for \( \gamma = 2 - \alpha \) the statements (i) and (ii) of Lemma 3.4 hold if \( x \) is confined in an arbitrarily fixed finite interval.

We shall prove a local version of formula (2.3) in the next section. Combined with it with the above remark taken into account the next lemma will remove the restriction \( \gamma \neq 2 - \alpha \) from Lemma 3.4.

Lemma 3.5. Suppose \( f^0(n) \leq Cc_n/n^2 \) for some constant \( C \). Then both (i) and (ii) of Lemma 3.4 hold for all admissible \( \gamma \).
Proof. Put \( a(x, n) = \sum_{k=n}^{\infty} [p^k(0) - p^k(-x)] \). In the identity \((3.9)\) we split the sum on its RHS at \( \varepsilon n > 0 \), which we suppose to be an integer for simplicity. Then summation by parts transforms the sum restricted on \( k \leq \varepsilon n \) into

\[
a(x)P[\sigma_{\{0\}}^0 \geq n] - a(x, \varepsilon n + 1)P[\sigma_{\{0\}}^0 \geq (1 - \varepsilon)n] + \sum_{k=1}^{\varepsilon n} f^0(n - k)a(x, k) \quad (x \neq 0).
\]

By Lemma \([3.2](i)\) we have \( a(x, k) \leq C|x|k/e^2 \), which together with \((2.3)\) and the assumption of the lemma shows that the second and the third terms are dominated in absolute value by a constant multiple of \(|x_n| << a(x)P[\sigma_{\{0\}}^0 \geq n] \) under \(|x_n| << \Lambda_n(x)\). This combined with the bound \((3.11)\) which is valid also for \( \gamma = 2 - \alpha \) concludes the proof.

Now we can prove an extension of the aforementioned result \((2.6)\) of \([1]\).

Lemma 3.6. If \( \gamma \neq 2 - \alpha \), then for any interval \( I \subset \mathbb{R} \), as \( n \to \infty \) under \( 0 \leq x << c_n \)

\[
P[S_{n/c_n}^x \in I \mid \sigma_{\{0\}}^0 > n] \longrightarrow \int_I h(\xi) d\xi.
\]

Proof. This proof will reduce the general case of \( x \) to the case \( x = 0 \) that is treated in \([1]\). As in \([1]\), observe the identity

\[
Q^x_{\{0\}}(x, y) = p^n(y - x) - \sum_{k=1}^{n-1} f^x(k)p^{n-k}(y) \quad (y \in \mathbb{Z}),
\]

and on putting \( r^x_n = P[\sigma_{\{0\}}^0 > n] \) carry out summation by parts, which results in

\[
Q^x_{\{0\}}(x, y) = \sum_{k=0}^{n-1} r^x_k [p^{n-k}(y) - p^{n-k-1}(y)] + r^x_n p^0(y) + p^n(y - x) - p^n(y).
\]

Then we can write the Fourier series \( \sum_{y \in \mathbb{Z}} Q^x_{\{0\}}(x, y)e^{i\theta y} \) as

\[
-\sum_{k=0}^{n-1} r^x_k [\phi^{n-k-1}(\theta) - \phi^{n-k}(\theta)] + r^x_n \phi^n(\theta) (e^{i\theta x} - 1) \quad (-\pi < \theta \leq \pi).
\]

Writing \( \theta_n = \theta/c_n \) we accordingly obtain

\[
E[e^{i\theta x S_{n/c_n}^x} \mid \sigma_{\{0\}}^0 > n] = \frac{1}{r^x_n} \sum_{y \in \mathbb{Z}} Q^x_{\{0\}}(x, y)e^{i\theta_n y} = 1 - \sum_{k=0}^{n-1} \frac{r^x_k}{r^x_n} [1 - \phi(\theta_n)] \phi^{n-k-1}(\theta_n) + \frac{e^{i\theta_n x} - 1}{r^x_n} \phi^n(\theta_n).
\]

Now let \( \gamma < 2 - \alpha \). Then using Lemma \([3.4](i)\) we observe that for each small \( \varepsilon > 0 \)

\[
\frac{r^x_k}{r^x_n} = \frac{r^0_k}{r^0_n} (1 + o(1)) \quad (k \geq \varepsilon n) \quad \text{and} \quad \frac{|e^{i\theta_n x} - 1|}{r^x_n} = |\theta| \times o(1) \quad \text{for} \quad 0 \leq x_n << \Lambda_n(x),
\]

of which the former relation shows that \( x \) can be replaced by 0 for \( k > \varepsilon n \) in the sum on the right-most member of the above equalities and the latter that the last term of it tends to zero. The reduction to the case \( x = 0 \) follows if we can show that the sum over \( k \leq \varepsilon n \)
tends to zero as \( n \to \infty \) and \( \varepsilon \to 0 \) in turn. To this end observe that by (i) and (ii) of Lemma 3.4
\[
\sum_{k \leq n} r_k^2 \leq C \sum_{k \leq n} r_k^0 + C \sum_{k \leq n} |x|/c_k \frac{|a_k(x)c_n^2|}{n},
\]
of which the second sum on the RHS is at most a constant multiple of \( n|x|^2/|a_k(x)c_n^2| = n|x_n|/\Lambda_n(x) = o(n) \). Thus the desired reduction is achieved, for \( 1 - \phi(\theta_n) = O(1/n) \). \( \square \)

**Remark 3.1.** If \( x \geq 0 \) is fixed, (3.12) holds for all \( \gamma \) unless the walk is left-continuous. In case \( \gamma = 2 - \alpha \), if \( f^0(n) = O(c_n/n) \) is assumed in addition, then (3.12) holds under \( 0 \leq x_n << \Lambda_n(x) \). (The same proof applies; see what is remarked before Lemma 3.5.)

In the sequel we let \( \kappa = \kappa_{\alpha,\gamma}/(1 - \frac{1}{\gamma}) \) as in Lemma 3.4.

**Lemma 3.7.**
\[
h(\xi) = f^{-\xi}(1)/\kappa \quad (\xi \in \mathbb{R}).
\] (3.13)

The identity above would be comprehended in view of the identity \( Q_{\{0\}}^n(0, x) = f^{-x}(n) \), it being expected that \( f^{-x}(n) \sim f^{-\xi}(1)/n \) if \( x_n \to \xi \neq 0 \) which entails \( Q_{\{0\}}^n(0, x) \sim h(\xi)P[\sigma^0_{(-\infty,0]} > n]/c_n \sim \kappa h(\xi)/n \) and hence (3.13).

For the proof of (3.13) one may suppose that \( L \equiv 1 \) and the analysis under this assumption that is carried out in [30] yields \( f^{-x}(n) \sim f^{-\xi}(1)/n \) (as \( n \to \xi \)), hence (3.13). We shall see \( Q_{\{0\}}^n(0, x) \sim \kappa h(\xi)/n \) shortly (see (4.3)) and this together with the functional limit theorem also leads to (3.13). We shall provide another more direct proof in Section 8 (Lemma 8.10) by computing the Fourier transform of \( f^{-x} \).

### 4 Proof of Theorems 1, 2 and Proposition 2.1

The proof is made by showing several lemmas and is based on Lemma 3.6 that asserts the convergence—with some uniformity in \( x \)—of the conditional law of \( S_{\tau}^\gamma/c_n \) given \( \sigma_{\{0\}}^\tau > n \). Employing Lemma 3.4(i) we can rephrase this convergence result as follows:

**If** \( \gamma \neq 2 - \alpha \) **and** \( \varphi \) **is a continuous function on** \( \mathbb{R} \) **and** \( I \) **is a finite interval of the real line, then uniformly for** \( |y_n| < M \) **and** \( 0 \leq x_n << 1 \), **as** \( n \to \infty \)
\[
\sum_{w:w/c_n \in I} Q_{\{0\}}^n(x, y + w) \varphi(w/c_n) \sim \kappa a^1(x) \frac{c_n}{n} \int h(y_n + \xi) \varphi(\xi) d\xi.
\] (4.1)

In the sequel \( M \) denotes an arbitrarily fixed constant larger than 1.

**Lemma 4.1.** **If** \( \gamma \neq 2 - \alpha \), **then uniformly for** \( |y_n| \in [1/M, M] \), **as** \( n \to \infty \)
\[
Q_{\{0\}}^n(x, y) \sim a^1(x) f^{-y_n}(1)/n \quad \text{under} \quad 0 \leq x_n << \Lambda_n(x).
\] (4.2)

The comment given to (3.12) in Remark 3.1 is applicable to this relation; in particular
\[
f^\tau(n) = Q_{\{0\}}^n(0, x) \sim f^x(1)/n
\] (4.3)
uniformly for \( |x_n| \in [1/M, M] \) for every admissible \( \gamma \).
As is noted previously the first assertion of the lemma is void for \( x > 0 \) if the walk is left-continuous when the asymptotics is described by means of the stable meander.

**Proof.** (4.2) is essentially the local limit theorem corresponding to (4.1) in view of Lemma 3.7. We are going to derive the former from the integral one with the help of Gnedenko’s theorem. The idea of the following proof is borrowed from [7] (the proof of Theorem 5 in it). Taking \( m = \lfloor \varepsilon^{2\alpha} n \rfloor \) with a small \( \varepsilon > 0 \) we decompose

\[
Q_{\{0\}}^n(x, y) = \sum_{z \in \mathbb{Z} \setminus \{0\}} Q_{\{0\}}^{n-\delta}(x, z)Q_{\{0\}}^m(z, y).
\]

Note \( c_m/c_{n-m} \sim \varepsilon^2/(1 - \varepsilon^{2\alpha})^{1/\alpha} \). We apply (4.1) in the form

\[
n \sum_{|u| < \varepsilon c_{n-m}} Q_{\{0\}}^{n-\delta}(x, y - u)\varphi(u/c_m) \sim \frac{\kappa a^\dagger(x)}{1 - \varepsilon^{2\alpha}} \int_{|\xi| < \varepsilon} h(y_{n-m} - \xi)\frac{\varphi(\xi/\varepsilon^2)}{\varepsilon^2} d\xi \tag{4.4}
\]

valid for each \( \varepsilon > 0 \) fixed, where \( \varepsilon = \varepsilon/(1 - \varepsilon^{2\alpha})^{1/2a} \sim \varepsilon (\varepsilon \to 0) \).

Let \( |y_n| \in [1/M, M] \). It is easy to see

\[
\sup_{z:|z| \geq \varepsilon c_{n-m}} Q_{\{0\}}^n(z, y) < C\varepsilon^{a+1}/c_m
\]

which combined with the preceding bound yields

\[
Q_{\{0\}}^n(x, y) = \sum_{|z-y| < \varepsilon c_{n-m}} Q_{\{0\}}^{n-\delta}(x, z)P_m(y-z)\{1 + o\varepsilon(1)\} + C\frac{P[\sigma_{\{0\}}^2] > n - m}{c_m}\varepsilon^{a+1}.
\]

On the RHS \( P_m(y-z) \) may be replaced by \( P_1((y-z)/c_m)/c_m \) in the first term (whenever \( \varepsilon \) is fixed) whereas the second term is dominated by a constant multiple of \( a^\dagger(x)|c_n/nc_m|\varepsilon^{a+1} \sim a^\dagger(x)\varepsilon^{a-1}/n \). Hence, after a change of variable

\[
nQ_{\{0\}}^n(x, y) = n \sum_{|u| < \varepsilon c_{n-m}} Q_{\{0\}}^{n-\delta}(x, y - u)\frac{P_1(u/c_m)}{c_m}\{1 + o\varepsilon(1)\} + a^\dagger(x)\times o\varepsilon(1).
\]

Now on letting \( n \to \infty \) and \( \varepsilon \to 0 \) in this order (4.4) shows that the RHS can be written as

\[
\kappa a^\dagger(x)\left\{ \int_{|\xi| < \varepsilon} h(y_n - \xi)\frac{P_1(\xi/\varepsilon^2)}{\varepsilon^2} d\xi + o\varepsilon(1)\right\} = a^\dagger(x)\{\kappa h(y_n) + o\varepsilon(1)\} \tag{4.5}
\]

and we find the formula (4.2) owing to Lemma 3.7.

**Lemma 4.2.** Uniformly for \( |x_n| \in [1/M, M] \) and \( |y_n| \in [1/M, M] \),

\[
Q_{\{0\}}^n(x, y) = \frac{1}{c_n}\{P_1^{(0)}(x_n, y_n) + o(1)\}.
\]

[Note that if \( \gamma = |2 - \alpha| \), then \( P_1^{(0)}(x_n, y_n) = 0 \) for \( \gamma x > 0 \) and \( \gamma y < 0 \).]
Proof. Let $|x_n|, |y_n| \in [1/M, M]$. By (4.3) $f^n(k) \sim f^k(1)/k$ for $k \approx c_n$ and we see

$$Q_{[0]}^n(x, y) = p^n(y - x) - \sum_{k=0}^{n-1} f^k(y)$$

$$= \frac{p_1(y_n - x_n)}{c_n} \{1 + o(1)\} - \frac{(1-\epsilon)n}{k} \frac{f^x(1)}{k} \cdot \frac{p_1(y_n - k)}{c_n} \{1 + o(1)\}$$

$$+ R_1(\epsilon) + R_2(\epsilon),$$

where $R_1(\epsilon)$ and $R_2(\epsilon)$ denote the sums of $f^x(k)p^n(k)$ over $1 \leq k < \epsilon n$ and $(1-\epsilon)n < k \leq n$, respectively. Since $c_k/c_n \sim (k/n)^{1/\alpha}$ uniformly for $\epsilon n < k < (1-\epsilon)n$, on employing scaling property $p_1(\xi) = p_1(\xi/t^{1/\alpha})/t^{1/\alpha}$ and similar one of $f^x(t)$ the last sum above may be written as

$$\frac{1}{nc_n} \sum_{k=\epsilon n}^{(1-\epsilon)n} f^{x}(n/k) p_{1-k/n}(y_n) \sim \frac{1}{c_n} \int_{\epsilon}^{1-\epsilon} f^{x_n}(t) p_{1-t}(y_n) dt.$$

By the local limit theorem (2.2) and (4.3) we see

$$R_1(\epsilon) \leq C \frac{1}{c_n} P[\sigma^{x}_{[0]} < \epsilon n] \leq C \frac{1}{c_n} P[\sigma^{x}_{[-\infty,0]} < \epsilon n] = o_{\epsilon}(1) \times \frac{1}{c_n},$$

and

$$R_2(\epsilon) \leq \left( \sup_{(1-\epsilon)n < k \leq n} f^x(k) \right) \sum_{k=0}^{\epsilon n} p^k(y) \leq C \frac{1}{c_n} \sum_{k=1}^n C_\epsilon^{k^{1-1/\alpha}}/c_n.$$

and similarly $(\int_{\epsilon}^{1-\epsilon} + \int_{1-\epsilon}^{1}) f^{x_n}(t) p_{1-t}(y_n) dt \to 0$ as $\epsilon \to 0$. Collecting these estimates we obtain the relation of the lemma since $p_1(0)(x_n, y_n) = p_1(y_n - x_n) - \int_{0}^{y_n} f^{x_n}(t) p_{1-t}(y_n) dt.$

**Lemma 4.3.** If $\gamma \neq 2 - \alpha$, then uniformly for $0 \leq x_n < \Lambda_n(x)$ and $y_n \in [1/M, M]$, as $n \to \infty$ and $\epsilon \downarrow 0$

$$\sup_{Q_{[0]}^n(x, y)} \{Q_{[0]}^n(x, z) : \epsilon \epsilon n < \epsilon \ or \ |z_n| > 1/\epsilon \} \xrightarrow{Q_{[0]}^n(x, y)} 0.$$

The same comment as given to (3.12) in Remark 3.1 applies to this relation.

**Proof.** For simplicity suppose $m := n/2$ is an integer and decompose

$$Q_{[0]}^n(x, z) = \sum_{w} Q_{[0]}^m(x, w) Q_{[0]}^m(w, z).$$

Observe that $\sup_{|w| \leq c_n/2} Q_{[0]}^n(x, w) \leq c_n^{-1} [\sup_{|w| > c_n/2} p_1(u) + o(1)]$ uniformly for $|z_n| > 1/\epsilon$. If $\gamma \neq 2 - \alpha$, then $\sum_{|w| > c_n/2} Q_{[0]}^n(x, w)/P[\sigma^{x}_{[0]} > n] \to 0$ as $n \to \infty$ and $\epsilon \downarrow 0$ uniformly for $0 \leq x_n < \Lambda_n(x)$ in view of Lemma 3.6 and hence by Lemma 3.4 (ii)

$$\sup_{|z_n| > 1/\epsilon} Q_{[0]}^n(x, z) \leq P[\sigma^{x}_{[0]} > n] c_n^{-1} \times o_{\epsilon}(1) = a^{\uparrow}(x)n^{-1} \times o_{\epsilon}(1),$$

where $o_{\epsilon}(1) \to 0$ as $n \to \infty$ and $\epsilon \downarrow 0$ interchangeably. As for the case $|z_n| < \epsilon$, further making decomposition $Q_{[0]}^n(x, z) = \sum_{w \neq 0} Q_{[0]}^m(x, w') Q_{[0]}^m(w', z)$ we deduce,

$$\sup_{w} Q_{[0]}^n(x, z) \leq \sup_{w, w'} Q_{[0]}^m(x, w') \sum_{u \neq 0} Q_{[0]}^m(u, z) \leq CP[\sigma^{x}_{[0]} > m/2]/c_n = c_n^{-1} \times o_{\epsilon}(1),$$

so that $Q_{[0]}^n(x, z) \leq C' P[\sigma^{x}_{[0]} > m] c_n^{-1} \times o_{\epsilon}(1) = [a^{\uparrow}(x)/n] \times o_{\epsilon}(1)$, which together with (4.7) concludes the proof, since $Q_{[0]}^n(x, y) \sim a^{\uparrow}(x)/n$ for $|y_n| \in [1/M, M]$ owing to Lemma 4.1.
Lemma 4.4. Uniformly under $|x_n| << \Lambda_n(x)$ and $|y_n| << \Lambda_n(-y)$, as $n \to \infty$

$$Q^{n}_{\{0\}}(x, y) \sim a^{\dagger}(x)f^{0}(n)a(-y).$$

Proof. Pick a constant $\lambda$ such that $0 < \lambda < 1$ and put $m = \lfloor \lambda n \rfloor$. Consider the decomposition

$$Q^{n}_{\{0\}}(x, y) = \sum_{z \neq 0} Q^{m}_{\{0\}}(x, z)Q^{n-m}_{\{0\}}(z, y).$$

Let $\gamma \neq |2 - \alpha|$. Then by Lemma 3.6

$$\frac{1}{P[\sigma_{\{0\}}^2 > m]} \left( \sum_{|z_n| < \varepsilon} + \sum_{|z_n| > \varepsilon^{-1}} \right) Q^{m}_{\{0\}}(x, z) \to \left( \int_{|\xi| < \varepsilon/\lambda^{1/\alpha}} + \int_{|\xi| > \varepsilon^{-1}/\lambda^{1/\alpha}} \right) h(\xi)d\xi,$$

which together with Lemma 4.3 applied to $Q^{n-m}_{\{0\}}(z, y) = Q^{n-m}_{\{0\}}(y, -z)$ shows that the sum over $z$ subject to $|z_n| < \varepsilon$ or $|z_n| > \varepsilon^{-1}$ is negligible as $\varepsilon \to 0$. As for the sum over $z$: $\varepsilon \leq |z_n| \leq \varepsilon^{-1}$ we see that its summands are expressed as

$$a^{\dagger}(x)a^{\dagger}(-y)\int_{\lambda^{1/\alpha}/1}^{\infty} (1 + o(1))$$

owing to Lemma 4.1 and, on noting that $z_m$ and $z_{n-m}$ can be replaced by $z_n/\lambda^{1/\alpha}$ and $z_n/(1 - \lambda)^{1/\alpha}$, respectively, and that $f^{\xi/\lambda^{1/\alpha}}(1) = f^{\xi}(t)$, the sum itself is given as

$$\sum_{z: \varepsilon \leq |z_n| \leq \varepsilon^{-1}} \frac{a^{\dagger}(x)a^{\dagger}(-y)\lambda(1 - \lambda)c_n}{m(n - m)} \int_{\varepsilon < |\xi| < 1/\varepsilon} f^{-\xi}(\lambda)f^{\xi}(1 - \lambda)d\xi\{1 + o(1)\}.$$

Since $\varepsilon$ may be arbitrarily small, we conclude

$$Q^{n}_{\{0\}}(x, y) \sim \frac{a^{\dagger}(x)a^{\dagger}(-y)c_n}{n^2} \int_{-\infty}^{\infty} f^{-\xi}(\lambda)f^{\xi}(1 - \lambda)d\xi$$

(under $|x_n| << \Lambda_n(x), |y_n| << \Lambda_n(-y)$). Taking $x = y = 0$ this becomes

$$f^{0}(n) \sim \frac{c_n}{n^2} \int_{-\infty}^{\infty} f^{-\xi}(\lambda)f^{\xi}(1 - \lambda)d\xi.$$

(4.9)

Hence the formula of the lemma follows.

In view of Remark 3.1 (and the comments given at the end of the succeeding lemmas) (4.8) with $x, y$ fixed—in particular (4.9)—holds for all $\gamma$, which in turn shows that the assertion of the lemma is also valid for $\gamma = |2 - \alpha|$. □

Proof of Theorem 4. Comparing (4.9) with $P[\sigma_{\{0\}}^2 > n] \sim (1 - 1/\alpha)\kappa_{\gamma, \alpha}c_n/n$ one obtains the identity

$$\int_{-\infty}^{\infty} f^{-\xi}(\lambda)f^{\xi}(1 - \lambda)d\xi = \kappa_{\alpha, \gamma},$$

(4.10)

which together with (4.9) shows Theorem 4.

Proofs of Theorem 2 and Proposition 2.4. What are asserted is virtually involved in Lemmas 4.1 and 4.4 because of duality. This is immediate for Theorem 2. Proposition 2.4 is verified as follows. Denote by $D_+$ and $D_-$ the diagonals $x - y = 0$ and $x + y = 0$, respectively. The range of $(x, y)$ for validity of the formulae in (2.5) is symmetrical about $D_+$ to itself and about $D_-$ to that for the dual walk. If $\gamma = -2 + \alpha$, then by Lemma 4.1 (2.5)
functions of the stable meander of length $t$ for the definition). Doney [8] obtains an elegant asymptotic formula of the probability notation may be rewritten (by using the duality relation) as

The proof is based on the result of Doney [8, Proposition 11], given below, for the walk ascending ladder height process of the walk $V$ in Figure 2 and $0 \leq n \leq 1/M$. To this region Lemmas 4.2 and 4.4 add the regions $x, |y_n| \in [1/M, M]$ and $0 \leq x, -y < c$, respectively. As a consequence it follows that for $\gamma = -2 + \alpha$, the range of validity contains

$$0 \leq x_n < M, \ y_n \in [-M, 0] \cup [1/M, M].$$

(4.11)

Now let $\gamma = 2 - \alpha$. First note that the assumption on $f^0(n)$ in Remark 3.1 is valid (since Theorem 1 has been shown) so that the formula of [4.1] is available. Then by Lemma 4.4 and (4.1) (2.5) holds for $0 \leq x_n << \Lambda_n(x)$ along with $y$ subject to $y_n \in [-M,-1/M]$ or $-\Lambda_n(y) << y_n < M$ (the darkened region in Figure 2 left). We may add to this the reflection of the region (4.11) about $D_+$, and finally find the symmetrization of the resulting region about $D_-$ agrees with the one described in Proposition 2.1.

5 Proof of Theorem 3

The proof is based on the result of Doney [8, Proposition 11], given below, for the walk killed not at 0 but on the negative half line. Let $V_s$ denote the renewal function of weakly ascending ladder height process of the walk $S$ and $Q_0(\eta)$ and $\hat{Q}_0(\eta)$, $\eta \geq 0$ the distribution functions of the stable meander of length $t$ at time $t$ for $Y$ and $-Y$, respectively (see (8.3) for the definition). Doney [8] obtains an elegant asymptotic formulae of the probability $P[S_n = x - y, \sigma_{[x+1,\infty]}^0 > n] (x \geq 0)$, which under the present assumption and with our notation may be rewritten (by using the duality relation) as

$$Q^n_{(-\infty,-1]}(x, y) \sim \begin{cases} 
U_d(x)V_s(y)p_1(0) 
& \text{if } (x_n \downarrow 0, y_n \downarrow 0), \\
ncn 
& \text{if } (y_n \downarrow 0, x_n > 1/M), \\
V_s(y)P[\sigma_{[0,\infty)}^0 > n]\hat{Q}'_1(x_n) 
& \text{if } (x_n \downarrow 0, y_n > 1/M), \\
cn 
& \text{if } (x_n \downarrow 0, y_n > 1/M), \\
U_d(x)P[\sigma_{(-\infty,-1]}^0 > n]Q'_1(y_n) 
& \text{if } (x_n \downarrow 0, y_n > 1/M), \\
cn 
& \text{if } (x_n \downarrow 0, y_n > 1/M), \\
p_1^{(-\infty,0)}(x_n, y_n)/cn 
& \text{if } (x_n \wedge y_n \geq 1/M).
\end{cases}$$

(5.1)
valid for all admissible $\gamma$ uniformly for $x \vee y < M c_n$.

Let $\gamma = 2 - \alpha$ throughout the rest of this section. It holds that as $n \to \infty$

$$P[\sigma^0_{(-\infty,-1]} > n| U_d(c_n) \to \kappa > 0 \tag{5.2}$$

(B2) (with $\kappa_0 = 1/\Gamma(1 - 1/\alpha)$ by Lemma 8.8) and $U_d(\xi c_n)/U_d(c_n) \to \xi$. Combining these

with the last two equivalences in (5.1) we see that $K_1(\eta) = \lim_{\xi \downarrow 0} p_{1(-\infty,0]}(\xi, \eta)/\xi = \kappa_0 Q'_1(\eta)$, so that the third case in (5.1) is equivalently given as

$$Q^n_{(-\infty,-1]}(x, y) \sim \frac{U_d(x)K_1(y_n)}{U_d(c_n)c_n} \quad (x_n \downarrow 0, y_n \in [1/M, M]). \tag{5.3}$$

Putting $m = [\delta n]$ with a small number $\delta > 0$ we decompose $Q^n_{(0]}(x, y) = I + II$, where

$$I = \sum_{z=1}^{\infty} Q^n_{(-\infty,0]}(x, z)Q^{n-m}_{(0]}(z, y) \quad \text{and} \quad II = \sum_{k=0}^{m} \sum_{z=1}^{\infty} Q^n_{(-\infty,0]}(x, -z)Q^{n-k}_{(0]}(-z, y).$$

[Note that here appears not $Q^n_{(-\infty,-1]}(x, \pm z)$ but $Q^n_{(-\infty,0]}(x, \pm z)$.] Asymptotic forms of

the second factors of the summands are essentially given in Proposition 2.1 (as is observed

shortly), by which, together with (5.1), we can compute the double sum with an appropriate

accuracy. Bringing in the function

$$W_{n,x} = \frac{U_d(x)}{U_d(c_n)c_n}. \quad \text{our main task then is performed by showing that}$$

$$\begin{align*}
(a) \quad I &= Q^n_{(-\infty,0]}(x, y) + o(W_{n,x}) \quad \text{uniformly for} \quad \left\{ \begin{array}{l}
0 < x_n, < 1, \\
y_n \in [1/M, M],
\end{array} \right.
(b) \quad II &= a(x)f^{-y}(n) + W_{n,x} \times o(1)
\end{align*} \tag{5.4}$$

($o(1) \to 0$ as $n \to \infty$ and $\delta \downarrow 0$) by which the formula of Theorem 3 follows if restricted to

this same regime since $Q^n_{(-\infty,0]}(x, y) \sim W_{n,x}(x_n \downarrow 0, y_n \simeq 1)$. Proof of (5.4a).} Let $y_n \in [1/M, M]$. Since for any $\varepsilon > 0$, $Q^n_{(0]}(z, y) \sim Q^n_{(-\infty,0]}(z, y)$ for

$z_n > \varepsilon$ in view of Proposition 2.1 and since $Q^n_{(-\infty,0]}(z, y) \leq C/c_n$, it suffices to show that

$$\frac{1}{c_n} \sum_{1 \leq z < c_n} Q^n_{(-\infty,0]}(x, z) \leq C\varepsilon\alpha W_{n,x}, \tag{5.5}$$

$\varepsilon$ can be made arbitrarily small. By (5.1) the LHS is dominated by

$$\frac{U_d(x)}{nc_n^2} \sum_{z < c_n} V_a(z) \sim \alpha\varepsilon\alpha \frac{U_d(x)V_a(c_n)}{nc_n^2},$$

which implies (5.5) since $V_a(c_n) \sim C_1 n/U_d(c_n)$. The proof of (5.4b) is given after showing a few lemmas. Since $\gamma = 2 - \alpha$ entails that the ascending ladder height has infinite expectation, according to 31 Corollary 1) we have

$$a^1(x) = E[a(S^n_{\gamma(x,-\infty,0)}]) = \sum_{k=0}^{\infty} \sum_{z=1}^{\infty} Q^n_{(-\infty,0]}(x, -z) a(-z) \quad x \in \mathbb{Z} \tag{5.6}$$

(see (9.8)). Put $R(w) = \sum_{z=1}^{\infty} p(-z - w) a(-z)$. Using $G_B$ defined in 2.30 we rephrase

identity (5.6) as

$$a^1(x) = \sum_{w=1}^{\infty} G_{(-\infty,0]}(x, w) R(w). \tag{5.7}$$
Lemma 5.1. There exists a constant $C$ such that for $y \geq 2x \geq 1$,
\[
\sum_{w=y}^{\infty} G_{(-\infty,0]}(x,w)R(w) \leq C \frac{U_d(x)a(y/2)}{U_d(y)}.
\]

Proof. After summation by parts the asymptotic form of $a(-z)$ given in Lemma 3.1 yields

\[
R(w) \sim C_0 \sum_{j=w}^{\infty} r(j), \quad \text{where} \quad r(j) = \sum_{z=1}^{\infty} p(-z - j) z^{\alpha - 2} \frac{L(z)}{U_d(z)}
\]

($C_0 = (\alpha - 1) \kappa_{\alpha,\gamma, -}$). Put $v(x) = V_a(x) - V_a(x - 1)$, $x \geq 1$ and $v(0) = V_a(0)$. Then

\[
G_{(-\infty,0]}(x,w) = \sum_{k=1}^{x\wedge w} u(x - k)v(w - k) \quad (5.8)
\]

(cf. [26]). By summation by parts $\sum_{w=y}^{\infty} v(w - k)R(w) \leq C_1 \sum_{w=y}^{\infty} V(w - k)r(w)$ ($y \geq k$). Hence for $y \geq x$,

\[
\sum_{w=y}^{\infty} G_{(-\infty,0]}(x,w)R(w) \leq C_1 U_d(x) \sum_{w=y}^{\infty} V_a(w)r(w) \quad (5.9)
\]

On the other hand by (5.7) it plainly follows that

\[
a(x) \geq \sum_{k=1}^{x} u(x - k) \sum_{w=x}^{\infty} v(w - k)R(w).
\]

To find a lower bound of the inner sum on the RHS apply the same summation by parts as above and then, noting $V_a(2x - k) > V_a(2x - 2k) \sim 2^{\alpha - 1}V_a(x - k)$, we infer that for $k \leq x$

\[
\sum_{w=x}^{\infty} v(w - k)R(w) \geq (2^{\alpha - 1} - 1) \sum_{w=2x}^{\infty} V_a(w - k)r(w) \{1 + o(1)\} > c \sum_{w=2x}^{\infty} V_a(w)r(w)
\]

with $c > 0$ so that

\[
a(x) \geq cU_d(x) \sum_{w=2x}^{\infty} V_a(w)r(w).
\]

Hence

\[
\sum_{w=2x}^{\infty} V_a(w)r(w) \leq c^{-1}a(x)/U_d(x).
\]

which combined with (5.9) shows that for $y > 2x$,

\[
\sum_{w=y}^{\infty} G_{(-\infty,0]}(x,w)R(w) \leq C' U_d(x)a(y/2)/U_d(y). \quad \square
\]

Lemma 5.2. Uniformly for $1/M < y_n < M$ and $0 \leq x_n < M$, as $n \to \infty$ and $\varepsilon \downarrow 0$

\[
\sup \{Q^n_{(-\infty,0]}(x,z) : 1 \leq z_n < \varepsilon \text{ or } z_n > 1/\varepsilon \}
\]

\[
\frac{Q^n_{(-\infty,0]}(x,y)}{Q^n_{(-\infty,0]}(x,y)} \to 0.
\]

Proof. For $0 \leq x_n < M$, the laws of $S_{x_n}^{\sigma_{n}}/c_n$ conditioned on $\sigma_{(-\infty,0]}^{n} > n$ constitute a tight family, and the proof of Lemma 14.3 is available with obvious modifications. \quad \square
Lemma 5.3. Uniformly for $1 \leq x < c_n$, as $y \to \infty$ under $y < c_n$

$$\sum_{k=m}^{\infty} \sum_{w=1}^{y} Q_{k}^{c}(x, w) R(w) = \frac{U_{d}(x) V_{a}(y)}{c_{m}} \times o(1) \quad (m = \lceil \delta n \rceil).$$

Proof. We split the sum defining $R(w)$ at $z = w$. Recall $F(t) = P[X \leq t]$, $t \in \mathbb{R}$. Performing summation by parts we then deduce

$$\sum_{z=1}^{w} p(-z - w)a(-z) \leq C \int_{0}^{w} F(-t - w) \frac{t^{\alpha-2}}{L(t)} dt = \frac{L(w)}{w^{\alpha}} \int_{0}^{w} \frac{t^{\alpha-2}}{L(t)} dt \times o(1) = o(1/w)$$

as $w \to \infty$, where we have $o(1)$ since $F(-z) = o(L(z)/z^{\alpha})$. The other sum is evaluated in a similar way: as a result we obtain\[R(w) = o(1/w).

Now by (5.1) $Q_{k}^{c}(x, w) \leq C U_{d}(x) V_{a}(w)/k c_{k}$ for all $1 \leq w \leq c_n$ and $k \geq m$ and the required estimate follows immediately. \[\square\]

Lemma 5.4. Uniformly for $1 \leq x \leq y$, as $y \to \infty$

$$\sum_{z=y}^{\infty} \sum_{w=1}^{y} G_{(-\infty, 0]}(x, w)p(-z - w)a(-z) = \frac{U_{d}(x) V_{a}(y)}{y} \times o(1).$$

Proof. Since $F(-z)/F(z) \to 0$ as $z \to \infty$, uniformly for $1 \leq w \leq y$,

$$\sum_{z=y}^{\infty} p(-z - w)a(-z) \leq F(-y - w)a(-y) + C \sum_{z=y}^{\infty} F(-z - w) \frac{z^{\alpha-2}}{L(z)} = o(1/y)$$

as $y \to \infty$, whereas \(\sum_{w=1}^{y} G_{(-\infty, 0]}(x, w) \leq C U_{d}(x) V_{a}(y)\), showing the asserted equality. \[\square\]

Proof of (5.4b) Since

$$Q_{k}^{c}(x, y) = n^{-1} a(-z) f^{(-\alpha n)(1)}(1 + o_{d}(1)) \{1 + o_{d}(1)\}$$

uniformly for $1 \leq k \leq m$, $1 \leq z < c_{m}(\sim \delta^{1/\alpha} c_{n})$, by virtue of (5.6) and (5.7) it suffices to show that for each $\delta > 0$ fixed,

$$\sum_{k=m}^{\infty} \sum_{z=c_{m}+1}^{w} Q_{k}^{c}(x, w) R(w) = o(n W_{n,x})$$

and

$$\sum_{z=c_{m}+1}^{w} \sum_{w=1}^{y} G_{(-\infty, 0]}(x, w)p(-z - w)a(-z) = o(n W_{n,x}).$$

The former one follows from Lemmas 5.1 and 5.3 while the latter from Lemmas 5.1 and 5.4 for

$$a(c_{m}/2) = o(a(-c_{m})) = o(n/c_{n}), \quad \frac{V_{a}(c_{m})}{c_{m}} \sim C_{d} \frac{n}{U_{d}(c_{n})c_{n}}.$$ \[\square\]
Proof of Theorem 3. We have already shown (5.4a) and accordingly the second formula of (2.10) restricted to $0 < x_n << 1$. The cases $x_n > 1/M$ and $-\Lambda_n(x) << x_n \leq 0$ being included in Proposition 2.1, up to now we have shown

$$Q_n^0(x, y) \sim \frac{a^\dagger(x)}{n} f^{-y_0}(1) + \frac{U_d(x)K_1(y_n)}{U_d(c_n)c_n} (-\Lambda_n(x) << x_n < M, y_n \in [1/M, M]). \quad (5.10)$$

The first formula of (2.10) is derived from (5.10) as in the proof of Lemma 4.4. Indeed by Proposition 2.1 we have $Q_n^0(z, y) \sim f^{\sum n(1) a^\dagger(-y)/n}$ for $z_n \in [M^{-1}, M], -M < y_n << 1$, and on putting $m = \lfloor \lambda n \rfloor$

$$\sum_{z=1}^{Mcn} \frac{U_d(x)K_1(z_m)}{U_d(c_m)c_m} Q_n^{m-1}(z, y) \sim \frac{U_d(x)\alpha a^\dagger(-y)}{nU_d(c_n)} \sum_{z=1}^{Mcn} \frac{K_1(z_n, \lambda^{-1} \alpha f)^{-y_0(n-1-\lambda)/\alpha}(1)}{\lambda^{2/\alpha}(1-\lambda) c_n}.$$

After applying the scaling relations of $K_{\alpha}(\eta)$ and $f^\alpha(t)$ we see that the sum above approaches

$$\int_0^\infty K_{\alpha}(\eta)f^\alpha(1-\lambda)d\eta = \lim_{\xi \downarrow 0} \frac{1}{\xi} f^\xi(1) = p_1(0),$$

and recalling (4.9) and the computation leading to (4.8) we have the first formula of (2.10). This finishes the proof.

Remark 5.1. Let $\gamma = 2 - \alpha$ and put $\ell^*(x) = \int_0^x P[-\hat{Z} > \ell]dt$. Here we observe that as $n \wedge y \rightarrow \infty$ under $\ell^*(y)/\ell^*(c_n) \rightarrow 1$ and $y_n << 1$

$$P[S_n^x = y, \sigma^{\infty}_{(-\infty, -1]} < n \leq \sigma^{\infty}_{\{0\}}] \sim f^0(n) a^\dagger(x) a^\dagger(-y) \quad (x_n \downarrow 0),$$

$$Q_n^{\infty_{(0), 0}}(x, y) \sim \frac{U_d(x-1) p_1(x_n)}{U_d(c_n)c_n} a(-y) \quad (0 < x_n < M). \quad (5.11)$$

The corresponding result for the case $0 \leq x_n << 1, y_n \in [1/M, M]$ that may read

$$P[S_n^x = y, \sigma^{\infty}_{(-\infty, -1]} < n < \sigma^{\infty}_{\{0\}}] \sim \frac{a^\dagger(x)}{n} f^{-y_0}(1) \quad (x_n \downarrow 0, y_n \in [1/M, M]) \quad (5.12)$$

$(U(-1) = e^{-\sum \nu^k(0)/k})$ follows immediately from the manner we have derived (5.10). Thus the sums in the formulae on the RHS of (2.10) (in Theorem 3) correspond to the obvious decomposition of $Q_n^0(x, y)$ as the sum of the probabilities on the LHS of (5.11), (5.12); it in particular follows that the dominant contribution to $Q_n^0(x, y)$ comes from the walk trajectories that enter the negative half line before $n$ if $x_n << \Lambda_n(x)$ and those that do not if $\Lambda_n(x) << x_n$.

For verification of (5.11) first note that the function $\ell^*(x)$ varies slowly at infinity and $U_d(x) \sim x/\ell^*(x))$. (Cf. [32] Lemma 12.) It also follows that $V_a(y) \sim y^{\alpha-1} \ell^*(y)/[L(y)\Gamma(\alpha)]$ (cf. Lemma 8.8). The second relation then is checked by looking at the first formula in (5.11) if $x_n \rightarrow 0$. In case $x_n > 1/M$ one can resorts to the analogue of formula (5.3) which reads

$$Q_n^{\infty_{(-\infty, -1)}}(x, y) \sim \frac{V_a(x,y)p_1(-x_n)}{\Gamma(\alpha) V_a(c_n)c_n} \quad (y_n \downarrow 0, x_n \in [1/M, M]) : \quad (5.13)$$

for the derivation use $P[\sigma_{0, \infty} > n] \tilde{Q}_1(\xi) = \xi p_1(\xi)/[\Gamma(\alpha)V_a(c_n)]$ (cf. Lemmas 8.6 and 8.8). The first formula of (5.11) follows from the second of it together with the first case of (2.10). Thus (5.11) has been verified.
6 Upper and lower bounds of $Q_{\{0\}}^n(x, y)$ and proof of Theorem 4

Here we derive estimates of $Q_{\{0\}}^n(x, y)$ for $x, y$ not necessarily confined in $0 < |x_n|, |y_n| < M$, that lead to Propositions 2.3 and 2.4 and are useful for the proof of Theorem 4. Throughout this section we assume $\gamma = 2 - \alpha$ unless stated otherwise explicitly. Sometimes we suppose $E[\tilde{Z}] < \infty$, which entails $\gamma = 2 - \alpha$.

By (2.13) (and by $x_n \leq U_d(x)/U_d(c_n)\{1 + o(1)\}$ $(0 < x < c_n)$) it follows that

$$\frac{a^\dagger(x)c_n}{n} + \frac{U_d(x)}{U_d(c_n)} \simeq \frac{a^\dagger(x)c_n}{n} + x_n \quad (0 < x < Mc_n), \quad (6.1)$$

and the two expressions on the above two sides will be interchangeable in most places of this section.

**Proposition 6.1.** Let $\gamma = 2 - \alpha$. For each $M > 1$, there exists a positive constant $c$ such that for $0 < x_n < M, -M < y_n \leq 0$

$$Q_{\{0\}}^n(x, y) \geq c[D_n(x, y) \lor D_n(-y, -x)]. \quad (6.2)$$

where

$$D_n(x, y) = \left(\sum_{z=2}^x p(-z)zV(z)a(-z)\right)\frac{U_d(x)}{x}\left[\frac{a^\dagger(-y)}{n^2/c_n} + \frac{|y_n|}{n}\right].$$

**Proof.** The walk is supposed to be not left-continuous, otherwise the result being trivial. Let $j(x)$ denote the smallest integer $j$ such that $c_j \geq x$. This proof employs the obvious lower bound

$$Q_{\{0\}}^n(x, y) \geq \sum_{\delta j(x) \leq k \leq n/2} \sum_{1 \leq w \leq c_j/\delta} \sum_{z=1}^x Q_{(-\infty, 0)}^{k-1}(x, w)p(-z - w)Q_{\{0\}}^{n-k}(-z, y)$$

valid for any constant $\delta > 0$. $\delta$ needs to be chosen so small that for all $n, x$ large enough, $c_{\delta j(x)} < \eta c_n$ for some $\eta < 1$. This is fulfilled with $\eta = 1/2$ by taking $\delta = 1/(3M)^\alpha$, for $c_{\delta j(x)/c_n} \sim \delta^{1/\alpha}x_n < \delta^{1/\alpha}M = 1/3$. Let $0 < x_n, -y_n < M$.

For $k, w, z$ taken from the range of summation above, we have by Theorem 3 (see also Corollary 3)

$$Q_{\{0\}}^{n-k}(-z, y) = Q_{\{0\}}^{n-k}(-y, z) \simeq f^y(n - k)a(-z) \simeq \left\{\frac{a^\dagger(-y)}{n^2/c_n} + \frac{U_d(-y)}{U_d(c_n)n}\right\}a(-z),$$

and, noting that for $k \geq \delta j(x)$ and $x \leq c_{k/\delta} \sim c_k\delta^{-1/\alpha}$, we apply (5.1) to obtain

$$Q_{(-\infty, 0)}^{k-1}(x, w) \simeq U_d(x)V_a(w)/kc_k.$$

Hence, putting

$$m(x) = \sum_{z=1}^x \sum_{w=1}^x p(-z - w)V_a(w)a(-z) \quad (6.3)$$

we have

$$Q_{\{0\}}^n(x, y) \geq c'm(x)U_d(x)\left\{\frac{a^\dagger(-y)}{n^2/c_n} + \frac{|y_n|}{n}\right\} \sum_{\delta j(x) \leq k \leq n/2} \frac{1}{kc_k}.$$
where (6.1) is used. Since \( x \sim c_{j(x)} \) and, by our choice of \( \delta, c_{\delta j(x)} < c_n/2 \sim 2^{1/\alpha - 1}c_n/2 \), the last sum is bounded below by a positive multiple of \( 1/x \). In the double sum in (6.3) restricting its range of summation to \( z + w \leq x \) and making change of variables we see

\[
m(x) \geq \sum_{k=2}^{x} p(-k) \sum_{z=-k+1}^{k-1} V_a \left( \frac{k-z}{2} \right) a \left( - \frac{k+z}{2} \right),
\]

where under the symbol \( \sum^* \) the summation is restricted to \( z \) such that \( k-z \) is even. It is easy to see that \( m(x) \geq \varepsilon^p \sum_{k=2}^{x} p(-k)kV_a(k)a(-k) \), showing the required lower bound in view of duality.

**Remark 6.1.** Let \( E|\hat{Z}| < \infty \) so that \( U_{\delta}(x) \sim x/E|\hat{Z}| \). Suppose that \( F(x) \) is regularly varying as \( x \to -\infty \) of index \( -\beta \) (necessarily \( \beta \geq \alpha \)). Then \( a(x) \gg \sum_{w=1}^{x} \sum_{z=1}^{\infty} p(-w-z)V_a(z)^2 \) (cf. \[B1\] Theorem 2(i), (iii))) and we deduce \( \sum_{w=1}^{x} p(-w)w^{2\alpha-1} \gg a(x) \) so that in view of Proposition 6.2(i) (given shortly) the lower bound (6.2) is exact.

**Lemma 6.1.** (i) If \( \gamma = 2 - \alpha \), then for any \( M > 1 \) there exists a constant \( C \) such that

\[
Q^n_{(0)}(x, y) \leq C \left\{ \frac{a^+(x)}{n/c_n} + (x)_+ \right\} \frac{\left( \frac{a^+(y)}{n} \land \frac{L(y)n/c_n}{y^\alpha \lor 1} \right)}{o\left( \frac{L(y)n/c_n}{|y|^\alpha} \right)} \quad \text{for } |x_n| \leq M, y_n \geq 0,
\]

(ii) If \( |\gamma| < 2 - \alpha \),

\[
Q^n_{(0)}(x, y) \leq C \frac{a^+(x)}{n/c_n} \left( \frac{a^+(y)}{n} \land \frac{L(y)n/c_n}{|y|^\alpha \lor 1} \right) \quad \text{for } |x_n| \leq M.
\]

**Proof.** Let \( |x_n| < M \). By Theorem 3 (cf. also (2.15)) we have as before

\[
Q^n_{(0)}(x, y) \sim f^x(n)a^+(y) \left\{ \frac{a^+(x)c_n}{n} + \frac{U_{\delta}(x)1(x \geq 0)}{U_{\delta}(c_n)} \right\} \frac{a^+(y)}{n} \quad \text{for } 0 \leq y_n \leq 4M.
\]

Because of (6.1), in case \( y \geq 0 \) it therefore suffices to show

\[
Q^n_{(0)}(x, y) \leq C_M \left\{ \frac{a^+(x)c_n}{n} + (x)_+ \right\} \frac{n/c_n}{a(-y)y} \quad \text{for } y_n > 4M.
\]

(6.5)

Putting \( R = \lceil y/3 \rceil, N = \lfloor n/2 \rfloor \) we make the decomposition

\[
Q^n_{(0)}(x, y) = \sum_{k=1}^{N} \sum_{z \geq R} P[S_k^n = z, \sigma_{[R, \infty]}^x = k > \sigma_{[0]}^x] Q^{n-k}_{(0)}(z, y)
\]

\[+ \sum_{z < R} P[\sigma_{[R, \infty]}^x \land \sigma_{[0]}^x > N, S_N^x = z] Q^{n-N}_{(0)}(z, y) = J_1 + J_2 \quad \text{(say).}
\]

(6.6)

By a local large deviation bound (cf. \[B1\] Theorem 1, [B3] Theorem 2.3]) it follows that

\[
P[S_n = z] \leq C_0nL(z)|z|^{-\alpha}/c_n \quad (z \neq 0)
\]

(6.7)
which entails
\[ \sup_{z \leq 2R} \sup_{k \leq N} Q_{\{0\}}^{n-k}(z, y) \leq C_0 \frac{nL(R)}{c_n R^\alpha}. \]  
(6.8)

Rewrite \( J_1 \) as
\[ J_1 = \sum_{k=1}^{N} \sum_{0 \neq w < R} \sum_{z=R}^{\infty} Q_{\{0\} \cup [R, \infty)}^k(x, w)p(z - w)Q_{\{0\}}^{n-k}(z, y). \]

We split the inner most summation at \( z = 2R \). By (6.8) it follows that
\[ \sum_{k=1}^{N} \sum_{0 \neq w < R} \sum_{z=R}^{2R} Q_{\{0\} \cup [R, \infty)}^k(x, w)p(z - w)Q_{\{0\}}^{n-k}(z, y) \leq C_1 \frac{nL(R)}{c_n R^\alpha} P[\sigma_{[R, \infty)}^\gamma < \sigma_{\{0\}}^\gamma]. \]  
(6.9)

The contribution from \( z > 2R \) is at most a constant multiple of
\[ \sum_{z > 2R} p(z - R)/c_n \leq CL(R)/c_n R^\alpha \approx L(y)/c_n y^\alpha. \]  
(6.10)

In Appendix (C) (Lemma 9.2) we prove
\[ P[\sigma_{[R, \infty)}^\gamma < \sigma_{\{0\}}^\gamma] \leq C'[a^+(x)/a(-R) + x_+ R^{-1}]. \]  
(6.11)

Collecting these bounds we can conclude
\[ J_1 \leq C' \left\{ \frac{a^+(x)}{a(-y)} + \frac{x_+ + 1}{n} \right\} \frac{n/c_n}{a(-y)y^\alpha}. \]

On the other hand on employing the bound (6.8)
\[ J_2 \leq P[\sigma_{\{0\}}^\gamma \geq \frac{1}{2} n] \sup_{z \leq R} Q_{\{0\}}^{n-k}(z, y) \leq C[nf^\gamma(n)]nL(y)/y^\alpha c_n \]
\[ \leq C' \left\{ \frac{a^+(x)c_n}{n} + (x_n)_{+} \right\} \frac{L(y)n/c_n}{y^\alpha}, \]

where (6.1) is used for the last inequality. Noting that \( n/c_n a(-y) \) is bounded for \( y > c_n \) and comparing terms we find (5.3) obtained. Thus the assertion has been proved in case \( y \geq 0 \). If \( \gamma < 2 - \alpha \), the same proof shows the result for \( y < 0 \).

Let \( \gamma = 2 - \alpha \) and \( y < -c_n/M \). Because of \( F(y) = o(L(y)/|y|^\alpha) \) we have, instead of (6.7), \( p^\gamma(y) = o(nL(y)/|y|^\alpha c_n) \) \( (y \to -\infty) \) according to [3]. Hence, on putting \( R = [-y/3] \),
\[ \sup_{z \geq -2R} \sup_{k \leq N} Q_{\{0\}}^{n-k}(z, y) \leq o(nL(R)/c_n R^\alpha). \]

On the other hand by Lemma 5.5 of [29]
\[ P[\sigma_{[-\infty, -R]}^\gamma < \sigma_{\{0\}}^\gamma] \leq [a^+(x)/a(-R)]\{1 + o(1)\}. \]

With these bounds we can follow the same lines as above with \( \sigma_{[-\infty, -R]}^\gamma \) in place of \( \sigma_{[R, \infty)}^\gamma \) to obtain the asserted estimate of the lemma. \( \square \)

For the convenience of later citations we write down two simplified and (partly reduced) version of the bounds of Lemma 6.1. In the next one \( \gamma \) may be any admissible constant.
Lemma 6.2. For each $M > 1$, there exists a constant $C$ such that for all $n \geq 1$,

$$Q_{\{0\}}^n(x,y) \leq C\frac{|y|^\alpha L(x)}{(|y| \wedge c_n)|x|^\alpha L(y)} \quad \text{if } |x_n| > 1/M \text{ and } y \neq 0.$$  

Proof. For $|y_n| \geq M$ the asserted bound follows from the large deviation estimate \(6.7\). For $|y_n| < M$, it is rephrased in the dual form which is given as

$$Q_{\{0\}}^n(x,y) \leq C|x|^{\alpha - 1}L(y)/|y|^{\alpha}L(x) \quad \text{for } 0 < |x_n| < M, |y_n| > 1/M, \quad (6.12)$$

and hence follows from Lemma \([6.1]\).

Lemma 6.3. Let $\gamma = 2 - \alpha$.

$$Q_{\{0\}}^n(x,y) \leq C[a^\dagger(-y) \wedge a(-c_n)] \frac{a^\dagger(x)c_n/n + (x_n \wedge 1)}{n} \quad (x \geq 0, y \geq 0). \quad (6.13)$$

Proof. This follows from Lemma \([6.1]\) for $x_n < 1$ and Lemma \([6.2]\) for $x_n \geq 1$.

Lemma 6.4. Suppose $\gamma = \alpha - 2$ and define $\omega_{n,x,y}$ for $x \neq 0$ and $y > 0$ via

$$Q_{\{0\}}^n(x,y) = a(-y)f^\gamma(n)\omega_{n,x,y}. \quad (6.14)$$

Then, $\omega_{n,x,y}$ is dominated by a constant multiple of $1 \wedge y^{-1}$ (in particular uniformly bounded), and tends to unity as $y_n \to 0$ and $n \to \infty$ uniformly for $0 < x_n < M$ for each $M > 1$.

Proof. The convergence of $\omega_{n,x,y}$ to unity follows from Theorems \([2]\) and \([5]\) and the asserted bound of $\omega_{n,x,y}$ follows from Lemma \([6.1]\).

Proposition 6.2. Suppose $\gamma = 2 - \alpha$. Then for some constant $C$

(i) $Q_{\{0\}}^n(x,y) \leq C\left[\frac{a^\dagger(x)a^\dagger(-y)}{n^2/c_n} + \frac{a^\dagger(-y)(x_n \wedge 1) + a^\dagger(x)(|y_n| \wedge 1)}{n}\right] \quad (x \geq 0, y \leq 0),$

(ii) $Q_{\{0\}}^n(x,y) \leq \frac{C}{c_n} \left(\frac{a^\dagger(-y)}{n/c_n} + (y_n) - \left(\frac{a(x)}{n/c_n} \wedge \frac{n/c_n}{a(x)|x_n|}\right)\right) \quad (x \leq -1, |y| \leq M/c_n).$

Proof. For the proof of (i) we apply Lemma \([6.3]\) to have

$$Q_{\{0\}}^{n-k}(z,y) = Q_{\{0\}}^{n-k}(-y,-z) \leq Ca(z)\frac{a^\dagger(-y) + \left(|y_n| \wedge 1\right)n/c_n}{n^2/c_n} \quad (z < 0, k \leq n/2). \quad (6.15)$$

We have $E[S_{[0,\infty]}] = \infty$ so that for $x \geq 0$, $E[a(S_{\sigma_{\{0\}}^x}^z)] = a^\dagger(x)$ (by \([9.8]\)), and from \((6.13)\) we deduce

$$P[\sigma_{\{0\}}^x_{(-\infty,0]} \leq n/2, \sigma_{\{0\}}^x > n, S_n^x = y] \leq \sum_{z < 0} P[S_{\sigma_{\{0\}}^x_{(-\infty,0]}}^z = z] \sup_{k \leq n/2} Q_{\{0\}}^{n-k}(z,y) \leq Ca^\dagger(x)\frac{a^\dagger(-y) + \left(|y_n| \wedge 1\right)n/c_n}{n^2/c_n}.$$

Let $\hat{S}^x$ and $\hat{\sigma}_{\{0\}}^x$ denote the dual walk and its hitting time, respectively. It then follows that

$$Q_{\{0\}}^n(x,y) - P[\sigma_{\{0\}}^x_{[0,\infty]} \leq n/2, \sigma_{\{0\}}^x > n, S_n^x = y] \leq P[\hat{\sigma}_{\{0\}}^y_{(-\infty,0]} \leq n/2, \hat{\sigma}_{\{0\}}^y > n, \hat{S}_{\{0\}}^y = x]. \quad (6.16)$$

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By duality relation the probability on the RHS is the same as what we have just estimated but with $x$ and $y$ replaced by $-y$ and $-x$, respectively, and hence we can conclude (i).

By duality (ii) is immediate from Lemma 6.1(i).

**Proof of Corollary 4.** The decomposition (2.17) follows from what is mentioned in Remark 5.1. Here we observe that it is actually involved in Theorem 3 itself. Indeed, we have $K_1(\xi) = \alpha P_p(0)Q'_1(\xi)$ (Lemma 5.6) and $h(\xi) = f^\xi(1)/\kappa$, which together with the second case of (2.10) shows that for $0 \leq x << c_n$, the conditional probability on the LHS of (2.17) is written as

$$
\left\{ \begin{array}{ll}
\theta''_{n,x} h(y_n) + \theta''_{n,x} Q'_1(y_n) / c_n + o(1/c_n) & (y_n \in [1/M, M]) \\
((1 - 1/\alpha) \theta'_{n,x} + \alpha^{-1} \theta''_{n,x}) o(-y)/n + o(1/c_n) & (0 \leq y_n << 1)
\end{array} \right.
$$

where

$$
\theta'_{n,x} = \frac{(1 - 1/\alpha)^{-1} \kappa_{\alpha, \gamma} \Lambda_n(x)}{P[\sigma^r_{\{0\}} > n]}, \quad \theta''_{n,x} = \frac{\alpha P_p(0) U_d(x - 1)/U_d(c_n)}{P[\sigma^r_{\{0\}} > n]}
$$

On the other hand, according to $P[\sigma^r_{\{0\}} > n] \sim U_d(x - 1)/U_d(c_n) \Gamma(1 - 1/\alpha)$, which together with $\alpha P_p(0) \Gamma(1 - 1/\alpha) = 1$ shows

$$
P[\sigma^r_{\{0\}} > n] = (1 - \theta_{n,x}) P[\sigma^r_{\{0\}} > n] \sim \alpha P_p(0) U_d(x - 1)/U_d(c_n) \quad (x \downarrow 0),$$

so that $1 - \theta_{n,x} \sim \theta''_{n,x}$. By (2.14) we deduce

$$
P[\sigma^r_{\{0\}} > n] \sim \kappa \Lambda_n(x) + \alpha P_p(0) U_d(x)/U_d(c_n)
$$

so that $\theta'_{n,x} + \theta''_{n,x} \rightarrow 1$. Hence $\theta'_{n,x} = \theta_{n,x} + o(1)$, showing that (2.17) holds uniformly for $0 \leq x << c_n$ and $y \in [0, M c_n]$ for each $M > 1$.

From Proposition 6.2(i) and Lemma 6.1(i) it follows that for $0 \leq x < M c_n$,

$$
Q''_{\{0\}}(x, y)/\Lambda_n(x + x_n) \leq C(\Lambda_n(-y) + |y_n|)/c_n = o(1/c_n) \quad (-1 << y_n < 0)
$$

and

$$
Q''_{\{0\}}(x, y)/\Lambda_n(x + x_n) = o(1/c_n) \quad (y_n < -1/M \text{ or } y_n \rightarrow \infty),
$$

respectively. These together show that (2.17) holds also uniformly for $y \notin [0, M c_n]$, since $P[\sigma^r_{\{n\}} > n] \sim \Lambda_n(x) + x_n (0 \leq x < M c_n)$.

In the sequel it is convenient to bring in some notation. For $B \subset \mathbb{Z}$, $H^x_\{B\}(y)$ denote the mass function of hitting distribution to $B$: $H^x_\{B\}(y) = P[S^x_{\sigma_B} = y]$. It follows that

$$
H^x_\{B\}(y) = \sum_{z \notin B} G_B(x, z) p(y - z) \quad \text{for } y \in B.
$$

We write $H^x_\{B\}(\varphi) = \sum_{y \in B} H^x_\{B\}(y) \varphi(y)$ for a function $\varphi$ on $B$. For the special case $B = (-\infty, 0]$ denote by $h^x(n, y)$ the corresponding space-time mass function:

$$
h^x(n, y) = P[\sigma^r_{(-\infty, 0]} = n, S^x_n = y] \quad (y \leq 0),
$$

which is the restriction of $Q_{(-\infty, 0]}(x, y)$ to $y \leq 0$. Write also $H^x_{(-\infty, 0]}(y) = \lim_{x \rightarrow -\infty} H^x_{(-\infty, 0]}(y)$; $H^x_{(-\infty, 0]}$ is a probability on $(-\infty, 0]$ if and only if $E[\tilde{Z}] < \infty$ (26).
Lemma 6.5. Suppose $E|\hat{Z}| < \infty$. Then,

(i) for $M > 1$ and $\varepsilon > 0$, uniformly for $0 < x_n < M$ and $y \leq 0$

\[ h^x(n, y) = \frac{x_n p_1(-x_n)}{n} \left[ H^{\infty}_{(-\infty, 0)}(y) \{1 + o_{\varepsilon}(1)\} + r(n, y) \right] \]

where $o_{\varepsilon}(1)$ is bounded and tend to zero as $n \to \infty$ and $\varepsilon \to 0$ in this order and

\[ |r(n, y)| \leq C_M \sum_{z > \varepsilon c_n} a(-z) p(y - z) \]

for a constant $C_M$ depending only on $M$ and $F$; and

(ii) there exists a constant $C$ such that for all $x \geq 1, y < 0$ and $n \geq 1$,

\[ h^x(n, y) \leq C \left( \frac{1}{n} \wedge \frac{L(x)}{x^\alpha} \right) x_n H^{\infty}_{(-\infty, 0)}(y). \]

Proof. Suppose $E|\hat{Z}| < \infty$, so that $U_d(x) \sim x/E|\hat{Z}|$. Let $\varepsilon > 0$ and in the expression

\[ h^x(n + 1, y) = \sum_{z = 1}^{\infty} Q^n_{(-\infty, 0)}(x, z) p(y - z) \quad (6.19) \]

we divide the sum into two parts, the sum on $z < \varepsilon c_n$ and the remainder which are denoted by $\Sigma_{<\varepsilon c_n}$ and $\Sigma_{\geq\varepsilon c_n}$, respectively. By Doney’s result (5.1) (cf. also (5.13)) together with $\Gamma(\alpha)V_{a}(c_n) \sim n/U_d(c_n)$ it follows that

\[ Q^n_{(-\infty, 0)}(x, z) = \frac{xV_a(z - 1)p_1(-x_n)}{E|\hat{Z}|nc_n} \{1 + o_{\varepsilon}(1)\} \quad \text{uniformly for } z \leq \varepsilon c_n, \]

and substituting this and using

\[ \frac{1}{E|\hat{Z}|} \sum_{z = 1}^{\infty} V_a(z - 1)p(y - z) = H^{+\infty}_{(-\infty, 0)}(y) \quad (y \leq 0), \quad (6.20) \]

(recall (6.17) and (5.8)) we deduce that

\[ \Sigma_{<\varepsilon c_n} = \frac{x_n p_1(-x_n)}{n} \left[ H^{\infty}_{(-\infty, 0)}(y) - \frac{1}{E|\hat{Z}|} \sum_{z > \varepsilon c_n} V_a(z)p(y - z) \right] \{1 + o_{\varepsilon}(1)\}. \]

By Lemma 5.2

\[ \Sigma_{\geq\varepsilon c_n} \leq C \frac{x_n}{n} \sum_{z > \varepsilon c_n} a(-z) p(y - z), \]

and hence the assertion (i) follows. It in particular follows that

\[ h^x(n, y) \leq C n^{-1} x_n H^{\infty}_{(-\infty, 0)}(y) \quad \text{for } 0 \leq x_n \leq M, y \leq 0, \quad (6.21) \]

since $|r(n, y)| \leq C_M H^{+\infty}_{(-\infty, 0)}(y)$ in view of (6.20) and $a(-z) < CV_a(z)$ ($z \geq 1$).

For the proof of (ii) we claim that for $y \leq 0$ and $x \geq 2c_n$,

\[ \sum_{z = 1}^{\infty} Q^n_{(-\infty, 0)}(x, z) p(y - z) \leq C L(x) x^\alpha H^{+\infty}_{(-\infty, 0)}(y) + C c_n F(y - \frac{1}{2}x). \quad (6.22) \]
For verification of (6.22), we break the range of summation into three parts $0 < z \leq c_n$, $c_n < z \leq x/2$ and $z > x/2$, and denote the corresponding sums by $J_1$, $J_2$ and $J_3$, respectively. It is immediate from Lemma 6.2 and (6.21) that $J_1 \leq CL(x) x^{-\alpha} H^{+\infty,-\infty}_t(y)$. By bound (6.7), $Q^n_{(-\infty,0]}(x, z) \leq p^n((x - z) \leq \alpha \, n \, x^\alpha (z < x/2)$ as well as by the bound

$$\sum_{z > c_n} p(y - z) \leq C_1 \sum_{z = 1}^{\infty} \frac{a(-z)}{n/c_n} p(y - z) \leq 2C_1 E|\hat{Z}| \left[ \sup_{z \geq 1} \frac{a(-z)}{n/c_n} \right] H^{+\infty,-\infty}_t(y) \quad (6.23)$$

yields $J_2 = C'H^{+\infty,-\infty}_t(y) L(x)/x^\alpha$. Finally $J_3 \leq CF(y - x/2)/c_n$. These estimates together verify (6.22). As in (6.23) we derive $F(y - \frac{1}{2} x) \leq C_1 H^{+\infty,-\infty}_t(y)/a^1(-x)$. Hence

$$h^x(n, y) \leq C_1 H^{+\infty,-\infty}_t(y) x_n L(x)/x^\alpha \quad \text{for} \quad x > 2c_n, \, y \leq 0, \, n \geq 1,$$

which combined with (6.21) shows the bound in (ii). The proof of Lemma 6.5 is complete. \hfill \Box

**Proof of Theorem 4.** If either $x$ or $y$ remains in a bounded set, the formula (i) of Theorem 4 agrees with that of Theorem 3 so that we may and do suppose both $x$ and $-y$ tend to infinity. Note that the second ratio on the RHS of (i) is then asymptotically equivalent to the ratio in (ii), hence (i) and (ii) of Theorem 4 is written as a single formula. Put

$$\Phi(t; \xi) = t^{-1} \xi p_t(-\xi).$$

Then what is to be shown may be stated as follows: as $n \to \infty$ and $x \vee (-y) \to \infty$

$$Q^n_{\{0\}}(x, y) \sim k_{\alpha, \sigma^2} a(x) a(-y) c_n/n^2 + C^+ \Phi(n; x - y) \quad (6.24)$$

uniformly for $-M < y_n < 0 < x_n < M$, provided $0 < C^+ = \lim_{z \to -\infty} a(z) < \infty$.

We follow the proof in [27] given to the corresponding result for case $\sigma^2 < \infty$. We employ the representation

$$Q^n_{\{0\}}(x, y) = \sum_{k = 1}^{n} \sum_{z < 0} h^x(k, z) Q^n_{\{0\}}(z, y). \quad (6.25)$$

Break the RHS into three parts by partitioning the range of the first summation as follows

$$1 \leq k < \varepsilon n; \quad \varepsilon n \leq k \leq (1 - \varepsilon)n; \quad (1 - \varepsilon)n < k \leq n \quad (6.26)$$

and call the corresponding sums $I, \, II$ and $III$, respectively. Here $\varepsilon$ is a positive constant that will be chosen small. The proof is divided into two cases corresponding to (i) and (ii). Suppose (2.26) to hold (so that $0 < C^+ < \infty$). It then follows that $E[-\hat{Z}] < \infty$, hence $U_4(x) \sim C_1 x$ and $V_4(x) \sim C_2 a(-x) (x \to \infty)$ with some positive constants $C_1, \, C_2$.

**Proof of (i) (case $x_n \wedge |y_n| \to 0$):** By duality one may suppose that $x_n \to 0$. From $ES_{\sigma^2[1, \infty]} = \infty$ and (2.28) it follows [31] Theorems 1 and 2] that

$$H^{+\infty,-\infty}_t\{a\} = a(x) \quad \text{and} \quad C^+ = H^{+\infty,-\infty}_t\{a\} < \infty, \quad (6.27)$$

respectively. From the latter bound above and Lemma 6.5 (or (6.21)) one deduces,

$$\sum_{k \geq \varepsilon n} \sum_{z < 0} h^x(k, z) a(z) \leq C z x_n \quad (6.28)$$
with a constant $C_\varepsilon$ depending on $\varepsilon$. As the dual of (6.14) of Lemma 6.4 we have

$$Q_n^{(0)}(z, y) = a(z)f^{-y}(n)\{1 + r_{n,z,y}\} \quad (z < 0, -Mc_n < y < 0)$$

(6.29)

where $r_{n,z,y}$ is uniformly bounded and tends to zero as $z/c_n \to 0$ and $n \to \infty$ uniformly for $y$, which together with (6.28) shows

$$II \leq C_{\varepsilon ,M} x_n f^{-y}(n).$$

Similarly on using (6.29) above

$$I = \sum_{1 \leq k < n} \sum_{z = -\infty}^{-1} h^z(k, z)a(z)f^{-y}(n - k)\{1 + r_{n-k,z,y}\}.$$ 

For the evaluation of the last double sum we may replace $r_{n-k,z,y}$ by $r_{n,z,y}$, observe that the contribution of $r_{n-k,z,y}$ to the sum is negligible since $\sum_{z > N} H_\infty(z) a(z) \to 0$ ($N \to \infty$) uniformly in $x$ in view of the second relation of (6.27). By (6.28) the summation over $z$ may be extended to the whole half line $k \geq 1$. Now applying the first relation of (6.27) we find

$$I = a(x)f^{-y}(n)\{1 + O(\varepsilon) + o(1)\}.$$

As for III first observe that by (6.29) and Theorem 3

$$\sum_{k=1}^n Q_k^{(0)}(z, y) = G^{(0)}(z, y) - r_n \leq C(a(z) \land a(y)) \quad 0 \leq r_n \leq C_{\varepsilon} a(z)f^{-y}(n) n$$

$(y, z < 0)$. If $y_n$ is bounded away from zero so that $x/y \to 0$, then $III = O(x_n/n) = o(y_n/n)$. On the other hand, applying Lemma 6.5 we see that if $y_n \to 0$,

$$III = x_n p_1(x_n) n^{-1} \sum_{z < 0} H_\infty(z) G^{(0)}(z, y)(1 + O(\varepsilon)) + O(x_n f^{-y}(n)),$$

whereas by (6.27) we infer that $\sum_{z \leq 0} H_\infty(z) G^{(0)}(z, y) \to C^+$ as $y \to -\infty$ (for by subadditivity $|a(-y) - a(z) - y| \leq a(z) + a(-z)$ so that the dominated convergence applies). Hence

$$III = x_n p_1(x_n) n^{-1}(C^+ + o(1) + O(\varepsilon)) + O(x_n f^{-y}(n)).$$

Adding these expressions of $I$, $II$ and $III$ yields the desired formula, because of arbitrariness of $\varepsilon$ as well as the identity $x_n p_1(x_n)/n = \Phi(n; x)$.

**Proof of (ii) (case $x_n \land (-y_n) \geq 1/M$).** By Lemma 6.5(ii) and Corollary 3(i) it follows that in this regime

$$I \leq \sum_{1 \leq k < n} \frac{CL(x)}{c_k x^{a-1}} \sum_{z < 0} H_\infty(z) a(z)f^{-y}(n) \leq C\varepsilon^{1-1/\alpha} \frac{1}{n}.$$

For evaluation of $III$ we change the variable $k$ into $n - k$ and apply Lemma 6.2 to $Q_k^{(0)}(-y, -z)$ (with $(-y, -z)$ in place of $(x, y)$) to see that for any $\delta > 0$

$$\sum_{k=1}^n Q_k^{(0)}(z, y) \leq C \sum_{k \leq \delta(y)\eta(z)} 1/c_k + C_\delta \sum_{\delta(y)\eta(z) < k \leq n} |z|^{a-1} L(y)/|y|^\alpha L(z)$$

$$\leq C\delta^{1-1/\alpha} a(z) + C_\delta(\varepsilon n) a(z) L(y)/|y|^\alpha,$$
where \( j(z) \) is any function such that \( j(z)/c_{j(z)} \sim a(z) \) as \( z \to -\infty \) and \( C_\delta \) may depend on \( \delta \) but \( C \) does not. Then by Lemma 6.3 (ii)

\[
III \leq C'n^{-1}\{C\delta^{1-1/\alpha} + C_\delta \varepsilon\}H^+_{(-\infty,0)}\{a\} \leq C''[C\delta^{1-1/\alpha} + C_\delta \varepsilon]/n,
\]
hence for any \( \varepsilon' > 0 \) we can choose \( \varepsilon > 0 \) and \( \delta > 0 \) so that \( III \leq \varepsilon'/n \).

By Lemma 6.5(i), (6.29) and (6.27)

\[
II = \sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{x_k \p_1(-x_k)}{k} \sum_{x=-x}^{n-k-1} H^+_{(-\infty,0)}(z)Q^{n-k}_{\{0\}}(z,y)(1 + o_\varepsilon(1)) + \frac{o(f^{-y}(n))}{\varepsilon^{1/\alpha}}.
\]

Here (and in the rest of the proof) the estimate indicated by \( o_\varepsilon(1) \) may depend on \( \varepsilon \) but is uniform in the passage to the limit under consideration once \( \varepsilon \) is fixed. Since \(-y_n\) is bounded away from zero as well as infinity, we may replace \( Q^{n-k}_{\{0\}}(z,y) \) by \( a(z)y_{n-k}\p_1(y_{n-k})/(n-k) \) to see that

\[
II = \sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{x_k |y_{n-k}| \p_1(-x_k) \p_1(y_{n-k})}{k(n-k)} \sum_{z=-x}^{n-k-1} H^+_{(-\infty,0)}(z)a(z)(1 + o_\varepsilon(1)) + \frac{o(1/n)}{\varepsilon^{1/\alpha}}.
\]

On noting the identity \( x_k \p_1(-x_k) = x_n \p_k/n(-x_n) = \Phi(k/n; x_n)k/n \) and similarly for \( y_{n-k}\p_1(y_{n-k}) \)

\[
\sum_{\varepsilon n \leq k \leq (1-\varepsilon)n} \frac{x_k |y_{n-k}| \p_1(-x_k) \p_1(y_{n-k})}{k(n-k)} = \frac{1 + o(1)}{n} \int_0^1 \Phi(t; x_n)\Phi(1-t; y_n)dt + O\left(\frac{\varepsilon}{n}\right).
\]

Here we have used the fact that \( \int_0^\varepsilon \p_t(\xi)\xi dt/t = \int_\xi^{\varepsilon^{1/\alpha}} = O(\varepsilon/\xi^\alpha) \). Since for \( \xi > 0 \), \( \Phi(dt; \xi)dt \) is the distribution of the hitting-time to zero by the process \( \xi + Y \), we have

\[
\int_0^1 \Phi(t; x_n)\Phi(1-t; -y_n)dt = \Phi(1; x_n - y_n).
\]

Hence

\[
II = \frac{1}{n}C^+\Phi(1; x_n - y_n)\{1 + o(1)\} + O\left(\frac{\varepsilon}{n}\right) + \frac{o(1/n)}{\varepsilon^{1/\alpha}}. \tag{6.30}
\]

(as well as \( nI + nIII \to 0 \) as \( n \to \infty \) and \( \varepsilon \to 0 \) in this order. Thus (6.24) is obtained, the first term on the RHS of it being negligible in the present regime.

**Proof of Proposition 2.2.** The case \( C^+ = 0 \) is trivial. Let \(-Mc_n < x < 0 < y < Mc_n\). If \( 0 < C^+ < \infty \), on noting that Theorem 4 and Lemma 6.3 (in the dual form (6.15)) together yield

\[
Q^{n-k}_{\{0\}}(z,y)/Q^{n}_{\{0\}}(x,y) \leq C \frac{a(z)[1 + |y|n/c_n^2]}{1 + |y|n/c_n^2 + xn/c_n^2} \leq Ca(z) \quad (z < 0, k < n/2)
\]

and that \( H^x_{(-\infty,0]}(z) \leq (E|\hat{Z}|)H^\infty_{(-\infty,0]}(z) \), we deduce that the conditional probability

\[
P[S^x_{\sigma_{(-\infty,0]} < -R, \sigma_{(-\infty,0]} < n/2 | \sigma_{(0)} > n, S^x_n = y] = \frac{\sum_{k<n/2} \sum_{z<-R} h^x(k,z)Q^{n-k}_{\{0\}}(z,y)}{Q^{n}_{\{0\}}(x,y)}
\]

is at most a constant multiple of \( \sum_{z<-R} H^x_{(-\infty,0]}(z)a(z) \) which approaches zero as \( R \to \infty \).

For the sum over \( n/2 \leq k \leq n \), one uses the bound \( \sum_{n/2 \leq k \leq n} Q^{n-k}_{\{0\}}(z,y) \leq Ca(z) \) as well.
as Lemma 6.3(ii) to obtain the same bound in a similar way. These together verify the first half of the asserted formula.

The second half obviously follows if \( E|\hat{Z}| = \infty \) so that \( H_{(-\infty,0]}^\infty \) vanishes. Let \( E|\hat{Z}| < \infty \). Then we can apply Lemma 6.3(ii) as well as Theorem 3 (in a dual form) to see that the contribution to the sum in (6.25) from \(-R \leq z < 0\) is dominated by a constant multiple of

\[
\sum_{-R \leq z < 0} \sup_{k < n/2} \left[ H_{(-\infty,0]}^x(z)Q_{\{0\}}^{n-k}(z,y) + h^x(n-k,z)G_{\{0\}}(z,y) \right] \leq C \frac{R^{\alpha-1}}{L(R)} \left[ \frac{a(-y)}{n^2/c_n} + \frac{x \lor |y|}{nc_n} \right],
\]

which is negligible as compared with the lower bound of \( Q_{\{0\}}^n(x,y) \) given by Proposition 6.1 provided that \( C^+ = \infty \) or, equivalently, \( \sum_{w \geq 1} p(-w)w^{2\alpha-1}/[L(w)]^2 = \infty \). \( \square \)

7 Extension to general finite sets

Let \( A \) be a finite non-empty subset of \( \mathbb{Z} \). The function \( u_A(x), x \in \mathbb{Z} \) defined in (2.31) may be given by

\[
u_A(x) = G_A(x,y)1(y \notin A) + 1(x = y \in A) + a(x-y) - E[a(S_{\sigma_A}^x - y)] \quad (7.1)
\]

(whether (2.29) is assumed or not), for the RHS is independent of \( y \in \mathbb{Z} \) (cf. [28, Lemma 3.1], [23]) and the difference of the last two terms in it tends to zero as \( |y| \to \infty \). Taking an arbitrary \( w_0 \in A \) for \( y \) it in particular follows that

\[u_A(x) = a^x(x-w_0) - E[a(S_{\sigma_A}^x - w_0)]. \quad (7.2)\]

Hence \( u_A(x) \sim a(x) \) as \( x \to +\infty \) if \( C^+ = \lim_{x \to +\infty} a(x) = \infty \); and similarly for the case \( x \to -\infty \). If \( C^+ < \infty \), then there exists

\[C_A^+ := \lim_{x \to +\infty} u_A(x) = C^+ - H_{\infty}^\infty \{a(\cdot - w_0)\}, \quad (7.3)\]

where \( H_{\infty}^\infty(z) := \lim_{|x| \to \infty} P[S_{\sigma_A}^x = z] \) which exists for every \( z \in A \) ([26, Theorem 30.1]). Combined with (7.2) this shows that \( \lim E[a(S_{\sigma_A}^x - w_0)] \) does not depend on the choice of \( w_0 \).

The function \( u_A \) is harmonic for the walk killed on \( A \) as noted previously, and \( u_A(S_{\sigma_A}^x)1(n < \sigma_A^x) \) is accordingly a martingale for each \( x \in \mathbb{Z} \). It holds that for \( x > m := \max A \)

\[u_A(x) = E[u_A(S_{\sigma_{\{0\}}^x}, S_{\sigma_{\{0\}}^x} \notin A) \mid E[S_{\sigma_{\{0\}}^x}^0 = \infty] \quad (7.4)\]

analogously to the corresponding relation for \( a(x) \) (see Appendix (B)). It follows that \( C_A^+ > 0 \) whenever the assumption (2.29) is satisfied.

For the following two lemmas we do not need to use the assumption 2) (the strong aperiodicity) and to make this clear we restate a result that has been shown up to Lemma 4.4 as follows: Uniformly for \( |x_n| << \Lambda_n(x) \), as \( n \to \infty \)

\[P[\sigma_0^x > n] \sim a^x(x)P[\sigma_0^x > n]. \quad (7.5)\]

This has been shown for \( |\gamma| < 2 - \alpha \) without assuming 2). Since we have \( f^0(n) \leq Cc_n/n^2 \) (see Remark 2.3) the same is true also for \( \gamma = \pm (2 - \alpha) \) (see Lemma 3.5).

According to Theorem 4a of [18] for each \( x \), as \( n \to \infty \)

\[P[\sigma_A^x > n]/P[\sigma_0^0 > n] \longrightarrow u_A(x) \quad (valid \ for \ all \ irreducible \ and \ recurrent \ walks \ on \ \mathbb{Z} \ with \ infinite \ variance \ so \ as \ to \ be \ applicable \ in \ the \ present \ setting), \ of \ which \ we \ need \ to \ have \ the \ following \ uniform \ version.)
Lemma 7.1. Uniformly for $|x_n| \ll \Lambda_n(x)$, as $n \to \infty$

$$P[\sigma^x_A > n] \sim u_A(x)P[\sigma^0_{\{0\}} > n].$$  \hspace{1cm} (7.6)

Proof. We adapt the proof in [18] of its Theorem 4a, which is somewhat simplified due to the explicit asymptotic form of $P[\sigma^0_{\{0\}} > n]$ available for us. The proof is made by induction on the number of points in $A$. Suppose (7.4) holds for the sets of $N$ points and let $A$ consist of $N + 1$ points. Putting $A' = A \setminus \{w\}$ with a point $w \in A$ we have

$$P[\sigma^x_A > n] = P[\sigma^x_{A'} > n] - \sum_{k=1}^{n} Q^k_A(x, w)P[\sigma^w_{A'} > n - k].$$  \hspace{1cm} (7.7)

Let $x_n \ll \Lambda_n(x)$ and taking a constant $\varepsilon \in (0, \frac{1}{2})$ we put $|m| = \varepsilon n$. Then, observing

$$Q^k_A(x, z) \leq f^*_\varepsilon(k) \leq C a^\dagger(x)r^0_m/m \quad \text{for} \quad k \geq m, z \in A$$

(where $r^0_m = P[\sigma^0_{\{0\}} > m]$ as in the proof of Lemma 3.6), we have

$$\sum_{k=m}^{n} Q^k_A(x, w)P[\sigma^w_{A'} > n - k] \leq C' a^\dagger(x)r^0_m \sum_{k=m}^{n} r^0_{n-k} = O\left(a^\dagger(x)[r^0_m]^2\right).$$

As for the other sum, noting that $P[\sigma^w_{A'} > n - k] \sim u_{A'}(w)r^0_{n-k}$ uniformly for $k < m$ owing to the induction hypothesis and that $\sum_{k=m}^{\infty} Q^k_A(x, w) \leq C a^\dagger(x)r^0_m$, we deduce

$$\sum_{k=1}^{m-1} Q^k_A(x, w)P[\sigma^w_{A'} > n - k] = [G_A(x, w) - 1(w = x)]u_{A'}(w)r^0_m(1 + o(1)) + O(a^\dagger(x)[r^0_n]^2)$$

as $n \to \infty$ and $\varepsilon \downarrow 0$ in this order. Observe $G_A(x, w) - 1(w = x) = P[S^x_{\sigma_A} = w]$ on one hand and

$$G_{A'}(x, y) - G_A(x, y) = P[S^x_{\sigma_A} = w]G_{A'}(w, y) \quad \text{for} \quad y \neq w$$

on the other hand, and then letting $y \to \infty$ you obtain

$$u_A(x) = u_{A'}(x) - [G_A(x, w) - 1(w = x)]u_{A'}(w).$$

Now on returning to (7.7) substitution of the estimates obtained above leads to

$$P[\sigma^x_A > n] = u_A(x)r^0_n + a^\dagger(x)r^0_n \times o(1),$$

which shows (7.6), for $a^\dagger(x)/u_A(x)$ is bounded. \hfill \Box

The next lemma, valid for all $\gamma$, extends Lemma 3.6 to the general case of $A$.

Lemma 7.2. For any interval $I \subset \mathbb{R}$, uniformly for $|x_n| \ll \Lambda_n(x)$, as $n \to \infty$

$$P[S^x_n/c_n \in I \mid \sigma^x_A > n] \longrightarrow \int_I h(\xi)d\xi.$$  \hspace{1cm} (7.10)

Proof. As in [11] we make the decomposition

$$Q^n_A(x, y) = p^n(y - x) - \sum_{k=1}^{n} \sum_{z \in A} Q^k_A(x, z)p^{n-k}(y - z) \hspace{1cm} \text{for} \quad y \neq z$$

$$= p^n(y - x) - \sum_{k=1}^{n} f_A(k)p^{n-k}(y)$$

$$+ \sum_{k=1}^{n} \sum_{z \in A} Q^k_A(x, z)\{p^{n-k}(y) - p^{n-k}(y - z)\}$$

$$= J_1(y) + J_2(y),$$

where

$$J_1(y) = p^n(y - x) - \sum_{k=1}^{n} f_A(k)p^{n-k}(y)$$

and

$$J_2(y) = \sum_{k=1}^{n} \sum_{z \in A} Q^k_A(x, z)\{p^{n-k}(y) - p^{n-k}(y - z)\}$$

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where $J_1(y)$ and $J_2(y)$ designates, respectively, the difference of the first two terms and the double sum of the third member above. Let $r_n = P[\sigma_{\{0\}}^n > n]$ as before. We claim that for any finite interval $I$,

$$\sum_{y:y_n \in I} J_2(y) = o(r_n^n),$$

so that $J_2(y)$ is negligible. To this end, splitting the sum that defines $J_2(y)$ at $k = n/2$ and employing the bound $|p^k(y) - p^k(y - z)| \leq C_1|z|/c_k^2$ (Lemma 3.2(i)), we see that

$$\sum_{k=n/2}^n \sum_{z \in A} \leq C'_A P[\sigma_A^x \leq n/2]/c_n^2 \leq C'/c_n^2,$$

so that

$$\sum_{y_n \in I} J_2(y) \leq C_A \left( a_1(x) c_n / n^2 + 1 / c_n^2 \right) \sum_{y_n \in I} 1 = C_{I,A} \left( c_n / n r_n + 1 / c_n \right) = o(r_n^n),$$

showing the claim. Using Lemmas 7.1 in place of Lemma 3.4(i) we can follow the proof of Lemma 3.6 words for words to see

$$\frac{1}{P[\sigma_A^x > n]} \sum_{y=-\infty}^\infty J_1(y)e^{i\alpha y/c_n} \to \hat{h}(\theta),$$

which finishes the proof. □

Proof of the extensions of Theorems 4 to 7. With Lemmas 7.1 and 7.2 at hand the same arguments that prove Theorem 4 and Proposition 2.1 apply to the estimation of $Q^n_A(x,y)$ so as to conclude the extension of them to a general finite set $A$, which in turn allows us to follow the proof of Theorems 3 and 4 to obtain the extensions of them. □

We conclude this section with a comment concerning the deduction of the results for the periodic walks from those for aperiodic ones. Suppose the walk is not strongly aperiodic with period $\nu \geq 2$. When $A = \{0\}$ the problem is addressed in Remark 2.3. As for the general case, recall that our proof of Theorem 4 and Proposition 2.1 is based on Lemmas 7.1 and 7.2 with $A = \{0\}$ that are also valid for periodic walks. Combined with the asymptotic relation

$$\frac{r_{nu+k}(x)}{\nu} = \frac{p_1(x_{nu})}{c_{nu}} 1(x \in D_k) \{1 + o(1)\} \quad (n \to \infty) \quad \text{for} \quad k = 0, \ldots, \nu - 1$$

where $D_k = \{ x : \exists n \geq 1, p^{k+nu}(x) > 0 \}$, this fact allows us to deduce the results for periodic walks from those which are obtained in Theorem 4 and Proposition 2.1. For $\gamma = 2 - \alpha$ we can dispose of the regime $x_n \downarrow 0$ and $y_n \asymp 1$ by using Lemma 7.3 below which applies to periodic walks so that we also obtain the extension of Theorem 3 to the periodic walks. Thus, e.g., if $\gamma = 2 - \alpha$, uniformly for $|x| \lor |y| < Mc_\nu$ satisfying $p^n(y - x) > 0$,

$$\frac{Q^n_A(x,y)}{\nu} = \left\{ u_{A}(x)f_{\gamma}(n) + \frac{U_d(x_+)p_1(x_n)}{U_d(c_n)n} \right\} u_{-A}(-y) \quad (-A_n(-y) \ll y_n \ll 1).$$

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Here \( f_s(n) = \kappa_{\alpha, \gamma} c_n / n^2 \) as in Remark 2.3. Since \( f^x_A(n) = \sum_{y \in A} Q^x_A(x, y) \), it follows immediately that uniformly for \( |x| < Mc_n \) and for \( k = 0, \ldots, \nu - 1 \), as \( n \to \infty \) under \( n \in \nu \mathbb{Z} \)

\[
\frac{f^x_A(n + k)}{\nu} = \kappa^A_{k, x} \left\{ u_A(x) f_s(n) + \frac{U_d(x_+) p_1(x_n)}{U_d(c_n) n} \right\} (1 + o(1)),
\]

where \( \kappa^A_{k, x} = \sum_{y \in A, y - x \in D_k} u_{-A}(-y) \).

The next lemma holds without assuming 2).

**Lemma 7.3.** As \( n \wedge x \wedge y \to \infty \) under \( x \vee y < Mc_n \) and \( x, y, n \in \nu \mathbb{Z} \)

\[
Q^n_{\{0\}}(x, y) - Q^n_A(x, y) = \begin{cases} o(Q^n_{\{0\}}(x, y)) & \text{if } C^+ = \infty, \\ (C^+ - C^+_A) f^{-y}(n) (1 + o(1)) & \text{if } C^+ < \infty. \end{cases} \tag{7.8}
\]

**Proof.** We may suppose \( 0 \in A \). The LHS of (7.8) then equals \( P[\sigma^x_A \leq n < \sigma^x_{\{0\}}, S^x_n = y] \) and is written as

\[
\sum_{k=0}^{n} \sum_{w \in A \setminus \{0\}} Q^k_A(x, w) Q^{n-k}_{\{0\}}(w, y).
\]

Split the outer sum at \( k = \varepsilon n \). Then for each \( \varepsilon > 0 \), the sum over \( k \geq \varepsilon n \) is at most

\[
\sum_{w \in A \setminus \{0\}} \sup_{k \geq \varepsilon n} Q^k_{\{0\}}(x, w) \leq C a^\dagger(x) f^0(n) = o(Q^n_{\{0\}}(x, y))
\]

as \( n \wedge y \to \infty \) when \( f^0(n) / f^{-y}(n) \to 0 \). On the other hand, for the other sum we substitute from the relation \( Q^{n-k}_{\{0\}}(w, y) \sim a^\dagger(w) f^{-y}(n-k) \) to see that as \( n \to \infty \) and \( \varepsilon \downarrow 0 \) in this order,

\[
\sum_{k=0}^{\varepsilon n} \sum_{w \in A \setminus \{0\}} H^x_A(a) f^{-y}(n)
\]

(see (6.17) for \( H^x_A \)). Since \( H^x_A(a) = o(a(x)) \) or \( H^x_A(a) \to H^x_A\infty(a) = C^+ - C^+_A \) \( (x \to \infty) \) according as \( C^+ = \infty \) or \( C^+ < \infty \), we conclude the result. \( \square \)

**8 Some properties of \( f^x_1 \) and \( p^t_1 \)**

In this section we present properties of \( f^x_1 \) and \( p^t_1 \) that are relevant to our estimate of \( Q^n_A(x, y) \). Their proofs are given, which can be easily contrived from the known facts as given in [4] or [25] except, perhaps, for Lemma 8.10.

By specializing the series expansion of \( p_1(x) \) as is found in e.g. [13] Lemma 17.6.1] one deduce

\[
p_1(0) = \frac{\Gamma(1/\alpha)}{\pi \alpha} \sin \frac{\pi (\alpha - \gamma)}{2 \alpha}; \tag{8.1}
\]

if \( |\gamma| = 2 - \alpha \), in particular, \( p_1(0) = -1/\Gamma(-1/\alpha) \).

**Lemma 8.1.**

\[
f^\pm (t) = \frac{\sin(\pi/\alpha)}{\pi p_1(0)} \cdot \frac{\pm 1}{\alpha t^{1+1/\alpha}} \int_0^t (1 - u)^{-1+1/\alpha} u^{-2/\alpha} p_1'(\mp (tu)^{-1/\alpha}) du. \tag{8.2}
\]

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Proof. According to [19] Lemma 8.13

\[ \int_0^t f^\pm(s)ds = \frac{\sin(\pi/\alpha)}{\pi p_1(0)} \int_0^t (t - s)^{-1+1/\alpha} p_s(y)ds. \]

Substitution from \( p_s(y) = s^{-1/\alpha} p_1(y/s^{1/\alpha}) \) and the change of variable \( u = s/t \) transform the integral on the RHS into

\[ \int_0^t (1-u)^{-1+1/\alpha} u^{-1/\alpha} p_1(tu^{-1/\alpha})du. \]

On noting that \( \int_0^1 u^{-1/\alpha} |p_1'(tu^{-1/\alpha})|du = \alpha \int_1^\infty |p_1'(x)|dx < \infty \) differentiation leads to the formula of the lemma. \( \square \)

Lemma 8.2. Let \( \gamma \neq 2 - \alpha \). Then \( f_1(t) \) admits the following asymptotic expansion as \( t \downarrow 0 \):

\[ f_1(t) \sim \frac{\Gamma(1/\alpha) \sin(\pi/\alpha)}{\pi^2 p_1(0)} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{\Gamma(\alpha k + 1)}{\Gamma(k + 1/\alpha)} \sin(\frac{1}{2} k \pi (\alpha + \gamma)) t^{k+1/\alpha}. \]

Proof. The result is obtained in [19] (the normalization therein differs from ours only by the factor \( 1/\alpha \) to the Lévy measure). Here we present a different proof. By the same manner as the asymptotic expansion of \( p_1(x) \) as \( |x| \to \infty \) is derived (cf. [34]) one can show that \( p_1'(x)x^2 \) admits the asymptotic expansion in powers of \( |x|^{-\alpha} \), so that the asymptotic expansion of \( p_1'(x) \) is obtained by formally differentiating the expansion of \( p_1(x) \) (given in [25] Eq(14,34-35)). This results in

\[ p_1'(-x) = \frac{\alpha}{\pi} \sum_{k=1}^{n-1} (-1)^{k-1} \frac{(k\alpha + 1)\Gamma(\alpha k)}{\Gamma(k)} \sin(\frac{1}{2} k \pi (\alpha + \gamma)) x^{-\alpha k - 2} + O(x^{-\alpha n - 2}) \]

as \( x \to \infty \) for any \( n \geq 2 \). After substitution into (8.2) an easy computation leads to the asymptotic expansion of the lemma. \( \square \)

We have stated the asymptotic form of \( f_1(t) \) as \( t \to \infty \) in (2.25) with two expressions of the constant factor \( \kappa_{\alpha, \gamma}^f \). By another application of (8.2) we obtain it with the second expression of \( \kappa_{\alpha, \gamma}^f \) for \( \gamma \neq 2 - \alpha \) in the following lemma.

Lemma 8.3.

\[ \lim_{t \to \infty} t^{2-1/\alpha} f_1(t) = \frac{\sin(\pi/\alpha)}{\pi p_1(0)} \int_0^\infty u^{1-\alpha} p_1'(-u)du. \]

and \( \int_0^\infty u^{1-\alpha} p_1'(-u)du > 0 \) or \( = 0 \) according as \( \gamma < 2 - \alpha \) or \( \gamma = 2 - \alpha \).

Proof. On performing the change of variable \( u = 1/tx^\alpha \) (8.2) becomes

\[ f_1(t) = \frac{\sin(\pi/\alpha)}{\pi p_1(0)} \cdot \frac{1}{t^{2-1/\alpha}} \int_1^\infty \left(1 - \frac{1}{tx^\alpha}\right)^{-1+1/\alpha} x^{1-\alpha} p_1'(-x)dx, \]

which shows the relation of the lemma, for the integral above over \( s \in [1/t^{1/\alpha}, 1/(t\varepsilon)^{1/\alpha}] \) tends to zero for any \( \varepsilon > 0 \), hence is asymptotically equivalent to \( \int_0^\infty x^{1-\alpha} p_1'(-x)dx \) as \( t \to \infty \). The second half follows from (2.25). \( \square \)
Lemma 8.4. If \( \varphi(t) \) is a continuous function on \( t \geq 0 \), then for \( T > 0 \)

\[
\kappa_{a,\gamma,\pm} = \alpha \int_{0}^{\infty} \frac{p_{1}(u) - p_{1}(0)}{u^{a}} du \quad \text{and} \quad \lim_{y \to \pm} \int_{0}^{T} \frac{p_{t}(y) - p_{t}(0)}{|y|^{a-1}} \varphi(t) dt = \kappa_{a,\gamma,\pm} \varphi(0).
\]

(See Lemma 3.1, (2.22) for \( \kappa_{a,\gamma,\pm} \)).

Proof. Let \( w_{y}(t) = [p_{t}(y) - p_{t}(0)]/|y|^{a-1} \). For any \( \varepsilon > 0 \),

\[
\int_{0}^{\varepsilon} w_{y}(t) dt = \int_{0}^{\varepsilon} \left[ p_{1}(y/t^{1/a}) - p_{1}(0) \right] \frac{|y| dt}{|y|^{a} t^{1/a}} = \alpha \int_{|y|/\varepsilon}^{\infty} \frac{p_{1}(u) - p_{1}(0)}{|u|^{a}} du,
\]

where \( \pm \) accords to the sign of \( y \). The last member converges to a constant, say \( b_{a,\gamma}^{\pm} \) and \( w_{y}(t) = O(|y|^{2-a}) \to 0 \) \( (y \to 0) \) uniformly for \( t > \varepsilon \) (since \( p_{t} \) is bounded), and hence the result follows. It remains to show \( b_{a,\gamma}^{\pm} = \kappa_{a,\gamma,\pm} \), which is done in the following lemma. \( \square \)

Lemma 8.5. Uniformly for \( x > 0 \), as \( y \to \pm 0 \)

\[
p_{t}^{(0)}(x, y)/|y|^{a-1} \to \kappa_{a,\gamma,\pm} f^{x}(t).
\]

Proof. Although the result follows from Theorems 2 and 3, we use them only for the identification of the constant \( b_{a,\gamma}^{\pm} = \alpha \int_{0}^{\infty} \frac{p_{1}(u) - p_{1}(0)}{|u|^{a}} du \) in this proof that is based on the identity

\[
p_{t}^{(0)}(x, y) = p_{t}(y - x) - \int_{0}^{t} f^{x}(t - s)p_{s}(y) ds.
\]

On subtracting from this equality that for \( y = 0 \) when the LHS vanishes, and then dividing by \( |y|^{a-1} \)

\[
\frac{p_{t}^{(0)}(x, y)}{|y|^{a-1}} = \frac{p_{t}(y - x) - p_{t}(-x)}{|y|^{a-1}} - \int_{0}^{t} \frac{p_{s}(y) - p_{s}(0)}{|y|^{a-1}} f^{x}(t - s) ds.
\]

As \( y \to 0 \), the first term on the RHS tends to zero and Lemma 8.4 applied to the second term yields the equality of the lemma. The uniformity of the convergence is checked by noting that the above integral restricted to \( s > t/2 \) is negligible. By applying Theorems 2 and 3 with \( L \equiv 1 \) it follows that

\[
b_{a,\gamma}^{\pm} = \frac{1}{f^{1}(1)} \lim_{y_{n} \to 0} \frac{p_{1}^{(0)}(x, y_{n})}{|y_{n}|^{a-1/n^{1-1/a}}} = \lim_{y \to \pm} \frac{a(-y)}{|y|^{a-1}}
\]

(compare with (2.22)), of which the last limit is evaluated in Lemma 3.1(i) as asserted. \( \square \)

Let \( Q_{t}(y) \) denote the distribution function of a stable meander, which may be expressed as

\[
Q_{t}(y) = \lim_{\varepsilon \downarrow 0} P[Y_{t} \leq y \mid \sigma_{(-\infty, -\varepsilon)} > t]
\]

(cf. [4] Theorem 18]) and satisfies the scaling relation \( Q_{t}(y) = Q_{1}(y/t^{1/a}) \).

Lemma 8.6. Let \( \gamma = 2 - \alpha \) and \( t > 0 \). Then for \( y > 0 \)

\[
K_{t}(y) := \lim_{x \downarrow 0} p_{t}^{(0)}(x, y)/x = \alpha p_{t}(0) Q_{t}^{1}(y)
\]

and for \( x > 0 \)

\[
\lim_{y \downarrow 0} \frac{p_{t}^{(0)}(x, y)}{y^{a-1}} = \frac{f^{x}(t)}{\Gamma(\alpha)} = \frac{Q_{t}^{1}(x)}{\Gamma(\alpha) \Gamma(1/\alpha) t^{1-1/\alpha}}.
\]

The convergences in (8.4) and (8.5) are uniform in \( y > 0 \) and \( x > 0 \), respectively.
Combined with (2.23) and (2.25) the equalities above entail that if $\gamma = 2 - \alpha$,
\[ Q'_t(x) \sim [\Gamma(\alpha + 1)]^{-1} x^{\alpha - 1/t^{1+\alpha}} \quad \text{and} \quad \hat{Q}'_t(x) \sim [-\Gamma(1/\alpha)/\Gamma(-1/\alpha)] x/t^{2/\alpha}. \]

**Proof.** For the proof of (8.4) first we show that for any $0 < \delta < y$,
\[
\lim_{x \downarrow 0} \frac{1}{x} \int_{\delta}^{y} p_t^{(0)}(x, z)dz = \alpha p_t(0)[Q_t(y) - Q_t(\delta)].
\]
(8.6)

For $\gamma = 2 - \alpha$, $\sigma^Y_{(-\infty,-\varepsilon]}$ agrees with $\sigma^Y_{(-\varepsilon]} \ a.s.$ Hence for $x > 0$, the integral in (8.6) which equals $P[\delta - x < Y_t \leq y - x, \sigma^Y_{(-\varepsilon]} > \varepsilon]$ (since $\sigma^Y_{(0]} = \sigma^Y_{(-\varepsilon]}$) is expressed as
\[ P[\delta - x < Y_t \leq y - x | \sigma^Y_{(-\infty,-x]} > \varepsilon] P[\sigma^Y_{(-\infty,-x]} > \varepsilon]. \]

The first factor converges to $Q_t(y) - Q_t(\delta)$ as $x \downarrow 0$. For the second one, recalling $f^\ast(s) = x s^{-1} p_s(x) = x s^{-1 - 1/\alpha} p_1(x s^{-1/\alpha})$ and making a change of variable we have
\[ P[\sigma^Y_{(-\infty,-x]} > \varepsilon] = \int_{\varepsilon}^{\infty} f^\ast(s)ds = \alpha \int_{0}^{\varepsilon^{-1/\alpha}} p_1(u)du. \]

Thus dividing by $x$ and passing to the limit conclude the required formula (8.6) since $p_1(0)t^{-1/\alpha} = p_t(0)$. In order to conclude (8.4) it suffices to show that $\lim_{x \downarrow 0} p_t^{(0)}(x, y)/x$ exists and the convergence is uniform in $y$ on any compact set of $(0, \infty)$. To this end we use (5.1) and postpone the proof to that of Lemma 8.8 although the proof can be done directly from the Fourier representation of $p_t^{(0)}(x, y)$.

As for (8.5) we make use of the duality relation and write (8.3) as
\[ \hat{Q}_t(x) = \lim_{\varepsilon \downarrow 0} \int_{0}^{\varepsilon^{-1/\alpha}} \frac{p_t^{[-\infty,\infty]}(\varepsilon, \varepsilon \xi)}{P[\sigma_{[\varepsilon,\infty]} > \varepsilon]} d\xi = \lim_{\varepsilon \downarrow 0} \int_{0}^{\varepsilon^{-1/\alpha}} \frac{p_t^{(-\infty,0]}(\varepsilon, \varepsilon \xi)}{P[\sigma_{[\varepsilon,\infty]} > \varepsilon]} d\xi. \]

The first equality of (8.5) follows from the preceding lemma and is written as $p_t^{[-\infty,0]}(\varepsilon, \varepsilon \xi) = p_t^{(0)}(\varepsilon, \varepsilon \xi) \sim f^\xi(t)e^{\varepsilon \xi - 1/\Gamma(\alpha)} (\xi > 0)$. By $\gamma = 2 - \alpha$ we have $P[Y_t > 0] = 1 - 1/\alpha$ (cf. (9.4)) which entails $P[\sigma^Y_{[\varepsilon,\infty]} > \varepsilon] = P[\sigma^Y_{[1,\infty]} > t/\varepsilon^{1/\alpha}] \sim \mathcal{C}_s(t/\varepsilon^{1/\alpha})^{-1 + 1/\alpha}$ [4, Proposition VIII.2] and accordingly we obtain
\[ \hat{Q}_t(x) = \frac{t^{-1/\alpha}}{\mathcal{C}_s\Gamma(\alpha)} \int_{0}^{x} f^\xi(t) d\xi. \]

We derive $\mathcal{C}_s = 1/\Gamma(\alpha)\Gamma(1/\alpha)$ from $\hat{Q}_t(+\infty) = 1$ with the help of the next lemma (cf. Remark 8.1). Finally differentiation concludes the second equality of (8.5).

The uniformity of convergence in (8.4) is shown by using (2.21) and the fact that $\sup_{0 < x < 1} p_t^{(0)}(x, y)/x \rightarrow 0$ as $y \rightarrow \infty$ (the latter can be shown in the same way as Lemma 6.1), and similarly for the convergence in (8.5). □

**Lemma 8.7.**
\[ \int_{-\infty}^{\infty} p_t(x)|x|dx = \frac{2t^{1/\alpha}}{\pi} \Gamma(1 - 1/\alpha) \sin(\frac{\pi}{2}\pi(\alpha - \gamma)/\alpha), \]

in particular if $\gamma = 2 - \alpha$, $\int_{0}^{\infty} f^\ast(t)dt = t^{-1} \int_{0}^{\infty} p_1(-x)dx = t^{-1 + 1/\alpha}/\Gamma(1/\alpha)$. 

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Proof. Put $\chi_\lambda(x) = |x|e^{-\lambda|x|}$ $(\lambda > 0, -\infty < x < \infty)$. By Parseval equality

$$\int_{-\infty}^{\infty} p_t(x)|x|dx = \lim_{\lambda \downarrow 0} \int_{-\infty}^{\infty} p_t(x)\chi_\lambda(x)dx = \frac{1}{\pi} \lim_{\lambda \downarrow 0} \int_{-\infty}^{\infty} e^{-t\psi(\theta)}C_\lambda(\theta)d\theta,$$

where $C_\lambda(\theta) = \int_0^\infty \chi_\lambda(x)d\theta$. To prove it is known [32, Eq(15) and Eq(31)] that for some positive constant

$$C_\lambda(\theta) = \frac{\lambda^2 - \theta^2}{(\lambda^2 + \theta^2)^2}.$$

Observing $\int_0^\infty C_\lambda(\theta)d\theta = 0$, we infer that as $\lambda \downarrow 0$

$$\int_0^\infty e^{-t\psi(\theta)}C_\lambda(\theta)d\theta = \int_0^\infty [e^{-t\psi(\theta)} - 1]C_\lambda(\theta)d\theta \rightarrow \int_0^\infty \frac{1 - \exp\{|-e^{i\gamma\pi/2}/\theta^\alpha\|}{\theta^2}d\theta.$$

The last integral equals $(te^{i\gamma\pi/2})^{1/\alpha}\Gamma(1 - 1/\alpha)$ [12, p.313 (18)], and we find the first formula of the lemma obtained. If $\gamma = 2 - \alpha$, then $\Gamma(1 - 1/\alpha)\sin[\frac{\pi}{2}(\alpha - \gamma)/\alpha] = \pi/(\Gamma(1/\alpha))$, which together with $f^x(t) = xt^{-1}p_t(-x)$ and $\int_{-\infty}^{\infty} p_t(x)dx = 0$ shows the second formula. 

**Remark 8.1.** We have used Lemma [8.7] for identification of the constant factor in (8.5). Alternatively we could have applied either the exact formula for $P[\sup_{s \leq t} Y_s \in d\xi]/d\xi$ obtained in [5] (cf. also [10] for related results) or a known expression of the absolute first moment of the stable law (cf. [22], [20]) from which the formula of the lemma is derived by elementary algebraic manipulations.

The results given in the next lemma are well known except for the expressions of some of constants involved. Put $t^* (x) = \int_0^x P[-\tilde{Z} > t]dt$ as in Remark [5.1].

**Lemma 8.8.** Let $\gamma = 2 - \alpha$. Then

(i) $U_d(c_n)V_a(c_n) \sim n/\Gamma(\alpha)$, $P[\sigma^0\infty] > n]U_d(c_n) \rightarrow 1/\Gamma(1 - 1/\alpha)$ and

$P[\sigma^0_{[0,\infty]} > n]V_a(c_n) \rightarrow 1/\Gamma(\alpha)\Gamma(1/\alpha)$ as $n \rightarrow \infty$;

(ii) $U_d(x) \sim x/t^*(x)$ and $V_a(x) \sim [\Gamma(\alpha)]^{-1}x^{\alpha-1}t^*(x)/L(x)$ as $x \rightarrow \infty$.

**Proof.** It is known [32, Eq(15) and Eq(31)] that for some positive constant $b$,

$$P[\sigma^0\infty] > n] \sim b/U_d(c_n); \text{ and }$$

$$P[\sigma^0\infty] > n]P[\sigma^0_{[0,\infty]} > n] \sim \beta/n \text{ with } \beta := 1/\Gamma(\rho^+)\Gamma(1 - \rho^+),$$

where $\rho^+ = \frac{1}{2}(1 - \gamma/\alpha)$ (cf. [9.4]). Let $\gamma = 2 - \alpha$. Combined with (8.7) the third and fourth cases of (5.1) show that locally uniformly for $\eta > 0$, $p_1^{(n)}(x_n, \eta)/x_n \sim U_d(x)Q_1(\eta)/U_d(c_n)$ as $n \rightarrow \infty$ along with $x_n \rightarrow \xi > 0$ and then $\xi \downarrow 0$. Thus, locally uniformly for $y > 0$,

$$p_1^{(n)}(x, y)/x \rightarrow bQ_1(y) \text{ as } x \downarrow 0.$$

This completes the proof of (8.6) as is notified in the proof of Lemma [8.6] and accordingly shows $b = \alpha p_1(0)$, hence the second relation of (ii) since $\alpha p_1(0) = 1/\Gamma(1 - 1/\alpha)$.

In a similar way employing (8.5) and the second formula of (5.1) we obtain the third relation of (i). The first one of (i) then follows from (8.5).

The first relation of (ii) is shown by Rogozin [24] and then the second follows from the first of (i).

In the following lemma the condition 2) is not assumed.
Lemma 8.9. Let $\nu$ be the period of $S$. Denote by $\hat{Z}^*$ the first descending ladder height of the walk $S_{\nu n}, n = 1, 2, \ldots$ and by $U_{\alpha}^*(x)$ the renewal function for $-\hat{Z}^*$. Then $U_{\alpha}^*(x)/U_{\alpha}(x)$ tends to a positive constant as $x \to \infty$.

Proof. If $E[-\hat{Z}] < \infty$, then $E[-\hat{Z}^*] < \infty$ and the assertion is obvious. Let $E[-\hat{Z}] = \infty$. Then noting that

$$P[|\hat{Z}^* - \hat{Z}| < M | \hat{Z} < -x] \geq 1 - P[|S_k - S_\nu| > M \text{ for } 0 < k < \nu] \to 1$$

as $x \wedge M \to \infty$ under $M < x$, we deduce that $P[\hat{Z}^* < -x]/P[\hat{Z} < -x] \to 1$, which shows the lemma in view of Lemma 8.8 if $\gamma = 2 - \alpha$ and of the equality $\lim U_{\alpha}(x)P[\hat{Z} < -x] = (\sin \frac{\alpha+\gamma}{2})/[(\alpha+\gamma)]$ otherwise (cf. [13, XIV(3.4)] and [24, Theorem 9]).

The next lemma verifies the identity $h(\xi) = \hat{f}^{-1}(1)/\kappa$ asserted in Lemma 3.7.

Lemma 8.10. Let $\kappa = \kappa_{\alpha, \gamma}/(1 - 1/\alpha) = (\sin \frac{\pi}{\alpha})/p_1(0)\pi$. Then

$$\frac{1}{\kappa} \int_{-\infty}^{\infty} \hat{f}^{-x}(1)e^{itx}dx = 1 - \psi(\theta) \int_{0}^{1} (1 - t)\frac{1}{\alpha} e^{-\psi(\theta)t}dt \quad (\theta \in \mathbb{R}). \quad (8.10)$$

Recall that the RHS is the characteristic function of $h$ which is denoted by $\hat{h}(\theta)$.

Proof. From the identity $\hat{f}(1) = \int \text{sgn}(x)|x|^{-\alpha}|x|^{-\alpha}$ and the integral representation of $\hat{f}^\pm(1)$ given in (8.2) we deduce

$$\hat{f}^{-x}(1) = \frac{-x}{\alpha} \int_{0}^{1} (1 - u)^{-1+1/\alpha} u^{-2/\alpha} p'_1(u^{-1/\alpha} x)du. \quad (8.11)$$

After a change of variable the Fourier transform of this identity is written as

$$\int_{-\infty}^{\infty} \frac{\hat{f}^{-x}(1)}{\kappa} e^{itx}dx = \frac{1}{\alpha} \int_{0}^{1} (1 - u)^{-1+1/\alpha} \omega(u^{1/\alpha} \theta)du, \quad (8.12)$$

where

$$\omega(\zeta) = - \int_{-\infty}^{\infty} x p'_1(x)e^{i\zeta x}dx \quad (\zeta \in \mathbb{R}).$$

On integrating by parts

$$\omega(\zeta) = \int_{-\infty}^{\infty} (1 + i\zeta x)p'_1(x)e^{i\zeta x}dx = e^{-\psi(\zeta)} + \zeta \frac{d}{d\zeta} e^{-\psi(\zeta)} = e^{-\psi(\zeta)} - \alpha \psi(\zeta)e^{-\psi(\zeta)},$$

and by substitution the RHS of (8.12) becomes

$$\frac{1}{\alpha} \int_{0}^{1} (1 - u)^{-1+1/\alpha} e^{-\psi(\theta)u}du + [-\psi(\theta)] \int_{0}^{1} u(1 - u)^{-1+1/\alpha} e^{-\psi(\theta)u}du.$$

Decomposing $u = 1 - (1 - u)$, we can write the second term as

$$\hat{h}(\theta) - \left[1 - \psi(\theta) \int_{0}^{1} (1 - u)^{1/\alpha} e^{-\psi(\theta)u}du\right],$$

of which the quantity in the square brackets equals the first term as is inferred by integration by parts again. This results in the identity of the lemma. \qed
9 Appendix

(A) Here we state condition \((1.1)\) in terms of the tails of the distribution function \(F(t) := P[X \leq t]\), and provide the explicit expressions for the constants relevant to the present paper and an estimate of the derivative \(\phi'(\theta)\).

The assumption \((1.1)\) on the characteristic function \(\phi(\theta)\) is equivalent to the condition

\[
P[X > x] \sim q^+ B x^{-\alpha} L(x) \quad \text{and} \quad P[X < -x] \sim q^- B x^{-\alpha} L(x)
\]

as \(x \to \infty\) with a positive constant \(B\) and two non-negative constants \(q^+\) and \(q^-\) such that \(q^+ + q^- = 1\) (\(L\) is the same slowly varying function as in \((1.1)\)). The characteristic exponent and Lévy measure \(\kappa\) that the constant \(\rho\) according to Zolotarev \([34]\) (cf. \([6, \text{Section 8.9.2}], [4, \text{Section VIII.1}]\)) Spitzer’s constant \(\phi\) paper and an estimate of the derivative \(\phi'(\theta)\).

\[
\psi(\theta) = |\theta|^{\alpha} B \Gamma(1 - \alpha) \{\cos \frac{1}{2} \alpha \pi - i (\operatorname{sgn} \theta) (q^+ - q^-) \sin \frac{1}{2} \alpha \pi\}
\]

and

\[
M\{(-x, 0]\} = B q^- x^{-\alpha}, \quad M\{(0, x]\} = B q^+ x^{-\alpha} \quad (x > 0)
\]

respectively (cf. \([13]\) Section XVII.3); the first equality is immediate from the asymptotic form of \(Ee^{i\theta X} (\theta \to 0)\) as given by \([1, \text{Theorem 1.3}], [29, \text{Eq(6.5)}]\)). From the above expression of \(\psi(\theta)\) we read off

\[
B \Gamma(1 - \alpha) [\cos \frac{1}{2} \alpha \pi] / (\cos \frac{1}{2} \gamma \pi) = 1 \quad \text{and} \quad \tan \frac{1}{2} \gamma \pi = (q^+ - q^-)/(\tan \frac{1}{2} \alpha \pi)
\]

which reduce to \(B = -1/\Gamma(1 - \alpha)\) and \(q^+ = 1\), respectively, if \(\gamma = 2 - \alpha\) and hence

\[
\psi(\theta) = (\cos \frac{1}{2} \gamma \pi)|\theta|^{\alpha}\{1 - i (\operatorname{sgn} \theta) (q^+ - q^-) \tan \frac{1}{2} \alpha \pi\}.
\]

(9.3)

According to Zolotarev \([34]\) (cf. \([6, \text{Section 8.9.2}], [4, \text{Section VIII.1}]\)) Spitzer’s constant \(\rho^+ := \lim_{n \to \infty} n^{-1} \sum_{k=1}^n P[S_k \geq 0]\) is given by

\[
\rho^+ = \frac{1}{2} (1 - \gamma/\alpha).
\]

(9.4)

On putting \(\rho^- = 1 - \rho^+\) we deduce from \((9.2)\) that

\[
B q^+ = \pi^{-1} \Gamma(\alpha) \sin(\alpha \rho^+ \pi).
\]

By the second equality of \((9.2)\) \(\sin[\frac{1}{2} \pi (\alpha \pm \gamma)] = (\sin \frac{1}{2} \alpha \pi)(\cos \frac{1}{2} \gamma \pi)[1 \pm (q^- - q^+)]\) so that the constant \(\kappa_{\alpha, \gamma, \pm}\) defined in Lemma \([3.1]\) is expressed as follows:

\[
\kappa_{\alpha, \gamma, \pm} = -\pi^{-1} \Gamma(1 - \alpha) (\sin \frac{1}{2} \alpha \pi)(\cos \frac{1}{2} \gamma \pi)[1 \pm (q^- - q^+)]
\]

\[
= -\cos \frac{1}{2} \gamma \pi \frac{1}{\Gamma(\alpha) \cos \frac{1}{2} \alpha \pi} q^+.
\]

(9.5)

From \((9.1)\) it follows that

\[
\phi'(\theta)/L(1/|\theta|) \sim -\psi'(\theta) = \mp \alpha e^{\pm i \pi \gamma /2}|\theta|^\alpha - 1 \quad \text{as} \quad \theta \to \pm 0
\]

(9.6)

(which is used in the proof of Lemma \([3.1]\) (ii)). Indeed, on writing \(\phi'(\theta) = i \int_{-\infty}^{\infty} (e^{i \theta t} - 1) t F(t) dt\) the integration by parts yields

\[
\phi'(\theta) = i \int_{-\infty}^{\infty} \{e^{i \theta t} - 1 + i \theta t e^{i \theta t}\} [-F(t)1(t < 0) + (1 - F(t))1(t > 0)]dt;
\]

(9.7)
the integral on the RHS restricted to \( \{|t| < \varepsilon/|\theta|\} \cup \{|t| > 1/|\theta|\} \) tends to zero as \( \theta \to 0 \) and \( \varepsilon \downarrow 0 \) in turn and, scaling by the factor \( 1/|\theta| \), we find that \( \phi'(\theta) \sim \pm \zeta|\theta|^\alpha - 1L(1/|\theta|) \), where
\[
\zeta = iB \int_{-\infty}^{\infty} \left\{ 1 - e^{\pm iu} \mp iue^{\pm iu} \right\} \frac{|u|^\alpha - 1(u < 0) - q^+1(u > 0)}{|u|^\alpha} du.
\]
Since \( \zeta \) depends on the regularity of tails of \( F(x)/L(|x|) \) only and \( -\psi'(\theta) \) is given by the above integral with \( dF \) replaced by the Levy measure \( M\{dx\} \), \( \pm \zeta|\theta|^\alpha - 1 \) must be equal to \( -\psi'(\theta) \).

(B) Let \( Z \) be the ascending ladder height: \( Z = S_{\sigma(1,\infty)} \). Recall \( H_B^T\{\varphi\} = \sum_{y \in B} H_B^T(y)\varphi(y) \) for any function \( \varphi \geq 0 \) on \( B \) (see (6.1) for \( H_B^T \)). Then
\[
H_{B_{\infty,0}}^T\{a\} = a^+(x) \quad (x \geq 0) \quad \text{if} \quad EZ = \infty. \quad (9.8)
\]
This is shown in [31] Corollary 1 for every recurrent walk irreducible on \( \mathbb{Z} \). Under the present setting the proof is much simplified as given below. By standard arguments, one can see that for \( x > 0 \), the process \( M^T(n) := a(S^T_{\infty,0}) \) is a martingale and \( h(x) := a(x) - H_{B_{\infty,0}}^T\{a\} \) is harmonic on \([1, \infty)\), i.e., \( E[h(S^T_1); S^T_1 \geq 1] = h(x) \) (cf. [27] Lemma 2.1). Hence \( H_{B_{\infty,0}}^T\{a\} \leq \lim E[M^T(n)] = a(x) \) so that \( h \geq 0 \), entailing that \( h \) is a constant multiple of \( U_d(x) \) by uniqueness. This concludes (9.3), for we know that \( a(x)/U_d(x) \to 0 \) because of the growth property of \( U_d(x) \) as \( x \to \infty \) that varies regularly with index \( \alpha \rho^- \) (see Lemma 8.8 in case \( |\gamma| = 2 - \alpha \); by (9.4) \( \alpha \rho^- = \frac{1}{2}(\alpha + \gamma) > \alpha - 1 \) if \( \gamma > -2 + \alpha \).

The same argument as above applies to \( u_A(x) \) for verification of (7.4).

(C) Here we give an estimate of \( P[\sigma^T_{\{R,\infty\}} < \sigma^T_{\{0\}}] \) as \( R \to \infty \) valid uniformly for \( x < R \). The problem is much easier than the classical exit problem for intervals. By [26] Proposition 29.4
\[
G_{\{0\}}(x, y) = a^+(x) + a^+(y) - a(x - y),
\]
which entails the subadditivity \( a(x + y) \leq a(x) + a(y) \) and
\[
P[\sigma^T_{\{y\}} < \sigma^T_{\{0\}}] = \frac{G_{\{0\}}(x, y)}{G_{\{0\}}(y, y)} = \frac{a^+(x) + a^+(y) - a(x - y)}{a(y) + a(-y)} \quad (y \neq 0, x). \quad (9.9)
\]

**Lemma 9.1.** If \( \gamma > -2 + \alpha \),
\[
\liminf_{R \to \infty} \inf_{x \in \mathbb{Z}} P[\sigma^T_{\{R\}} < \sigma^T_{\{0\}} | \sigma^T_{\{R,\infty\}} < \sigma^T_{\{0\}}] =: q > 0 \quad (9.10)
\]
with \( q = 1 \) for \( \gamma = 2 - \alpha \).

**Proof.** In view of (9.9) and the decomposition
\[
P[\sigma^T_{\{R\}} < \sigma^T_{\{R,\infty\}} < \sigma^T_{\{0\}}] = \sum_{z \geq R} P[S^T_{\sigma(0,\infty)} = z | \sigma^T_{\{R,\infty\}} < \sigma^T_{\{0\}}] P[\sigma^T_{\{R\}} < \sigma^T_{\{0\}}],
\]
where \( \sigma^T_{\{R\}} \) is defined to be zero if \( z = N \) and agree with \( \sigma^T_{\{R\}} \) otherwise, for the first half of the lemma it suffices to show that
\[
\liminf_{R \to \infty} \inf_{z \geq R} \frac{a(z) + a(-R) - a(z - R)}{a(R) + a(-R)} = \frac{\kappa_{\alpha, \gamma, -} - \kappa_{\alpha, \gamma, +}}{\kappa_{\alpha, \gamma, -} + \kappa_{\alpha, \gamma, +}},
\]
the last ratio being positive if \( \gamma > -2 + \alpha \) and equals unity if \( \gamma = 2 - \alpha \). If \( \gamma < 2 - \alpha \), by Lemma 3.1(ii) \( a(z) - a(z - R) > 0 \) for \( R \) large enough and the equality above follows immediately from Lemma 3.1(i). The case \( \gamma = 2 - \alpha \) also follows from Lemma 3.1(i) and (ii), the latter showing \( \sup_{z \geq R} |a(z) - a(z - R)| = o(R^{\alpha - 1}) \).
Lemma 9.2. For any $\gamma$ there exists a constant $C$ such that for $R > 1$,

$$P[\sigma^x_{[R,\infty)} < \sigma^x_{[0,\infty)}] \leq C\left[\frac{a^\dagger(x)L(R)}{R^{\alpha-1}} + \frac{x_+}{R}\right] (x \leq R).$$

Proof. For $\gamma > -2 + \alpha$, on using Lemma 3.1(ii) the result is derived from the preceding lemma: indeed for $x < R$,

$$P[\sigma^x_{[R,\infty)} < \sigma^x_{[0,\infty)}] \leq CP[\sigma^x_{[R]} < \sigma^x_{[0]}] = \frac{a^\dagger(x) + a(-R) - a(x - R)}{a(R) + a(-R)} \leq C'[a^\dagger(x)/a(-R) + x_+R^{-1}],$$

where Lemma 9.1 is used for the first inequality and Lemma 3.1(ii) is applied to estimate the increment of $a$ for the last inequality (as for the equality see (9.9)). In case $\gamma = -2 + \alpha$ see [29, Lemma 5.5] (use the fact that $M_n^x = a(S^x_{\sigma_{[0]} \cup [R,\infty)})$ is a martingale).

References

[1] B. Belkin, A limit theorem for conditioned recurrent random walk attracted to a stable law, Ann. Math. Statist., 42 (1970), 146-163.

[2] B. Belkin, An invariance principle for conditioned recurrent random walk attracted to a stable law, Zeit. Wharsch. Verw. Gebiete 21 (1972), 45-64.

[3] Q. Berger, Notes on random walks in the Cauchy domain of attraction. Probab. Theor. Rel. Field, In press, i10.1007/s00440-018-0887-0, ihal-01576409v3

[4] J. Bertoin, Lévy Processes, Cambridge Univ. Press, Cambridge (1996).

[5] V. Bernyk, R. C. Dalang and G. Peskir, The law of the supremum of a stable Lévy process with no negative jumps, Ann. Probab. 36(5), (2008), 1777-1789.

[6] N. H. Bingham, G.M. Goldie and J.L. Teugels, regular variation, Cambridge Univ. Press, Cambridge, 1989.

[7] D. Denisov and V. Wachtel, Random walks in cones, Ann. Probab. 43 (2015), 992-1044.

[8] R.A. Doney, Local behaviour of first passage probabilities, Probab. Theor. Rel. Fields, 152, (2012), 559-588.

[9] R.A. Doney, The strong renewal theorem with infinite mean via local large deviations, arXiv:1507.06790, (2015)

[10] R.A. Doney and M. S. Savov, The asymptotic behavior of densities related to the supremum of a stable process. Ann. Probab. 38, (2010), 316-326

[11] L. Doring, A. Kyprianou, and P. Weissmann, Stable processes conditioned to avoid zero. (preprint available at: http://arxiv.org/abs/1802.07223)

[12] A. Erdélyi, Tables of Integral Transforms, vol. I, McGraw-Hill, Inc. (1954)

[13] W. Feller, An Introduction to Probability Theory and Its Applications, vol. 2, 2nd edn. John Wiley and Sons, Inc. NY. (1971)
[14] V. Gnedenko and A. N. Kolmogorov, Limit distributions for sums of independent random variables, Addison-Wesley, Reading, Mass. (1954) [Russian original 1949]

[15] W. D. Kaigh, A conditional local limit theorem for recurrent random walk, Ann Probab. 3 (1975), 883-888.

[16] W. D. Kaigh, An invariance principle for random walk conditioned by a late return to zero. Ann Probab. 4 (1976), 115-121.

[17] H. Kesten, Ratio limit theorems II, Journal d’Analyse Math. 11, (1963), 323-379.

[18] H. Kesten and F. Spitzer, Ratio limit theorems I, Journal d’Analyse Math. 11, (1963), 285-322.

[19] A. Kuznetsov, A.E. Kyprianou, J.C. Pardo and A.R. Watson, The hitting time for a stable process, Electron. J. Probab. 19 no. 30 (2014), 1-26.

[20] M. Muneya and Z. Pawles, Fractional absolute means of heavy-tailed distributions, Braz. J. Probab. Stat. (2016), 272-299.

[21] H. Panti, On Lévy processes conditioned to avoid zero. Lat. Am. J. Probab. Math. Stats. (2016), 657-690.

[22] G. Peskir, The law of the hitting times to points by a stable Lévy process with no negative jumps, Electronic Commun. Probab. 13 (2008), 653-659.

[23] S. C. Port and C. J. Stone, Hitting time and hitting places for non-lattice recurrent random walks, J. Math. Mech., 17 (1967), 35-57.

[24] B. A. Rogozin, On the distribution of the first ladder moment and height and fluctuations of a random walk, Theory Probab. Appl. 16 (1971), 575-595.

[25] K. Sato, Lévy processes and infinitely divisible distributions, Cambridge Univ. Press, Cambridge (1999).

[26] F. Spitzer, Principles of Random Walks, Van Nostrand, Princeton, 1964.

[27] K. Uchiyama, One dimensional lattice random walks with absorption at a point / on a half line. J. Math. Soc. Japan, 63 (2011), 675-713.

[28] K. Uchiyama, One dimensional random walks killed on a finite set, Stoch. Proc. Appl. 127 (2017), 2864-2899.

[29] K. Uchiyama, Estimates of Potential functions of random walks on Z with zero mean and infinite variance and their applications. (preprint available at: http://arxiv.org/abs/1802.09832)

[30] K. Uchiyama, Recurrent random walks on Z with infinite variance: transition probability of them killed on a finite set. (preprint: available at: http://arxiv.org/abs/1808.01484)

[31] K. Uchiyama, The potential function and ladder variables of a recurrent random walk on Z with infinite variance. (preprint available at: http://arxiv.org/abs/1805.03971)

[32] V. A. Vatutin and V. Wachtel, Local probabilities for random walks conditioned to stay positive, Probab. Theory Rel. Fields, 143 (2009), 177-217.

[33] E. T. Whittaker and G. N. Watson, A course of modern analysis, 4th ed. Cambridge Univ. Press, Cambridge, (1927).
[34] V. M. Zolotarev, One dimensional stable distributions. Amer. Math. Soc. Providence RI (1986).

[35] V. Vysotsky, Limit theorems for random walks that avoid bounded sets, with applications to the largest gap problem, Stoch. Proc. Appl. 125 (2015), 1886-1910.