Mean-field delayed BSDEs in finite and infinite horizon

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Dedicated to Belkacem Selatnia for his birthday

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Abstract

We establish sufficient conditions for the existence and uniqueness of different types of delayed BSDEs in finite time horizon. We consider then infinite horizon, replacing the terminal value condition in the finite horizon case with a condition of strong decay at infinity.

Keywords: Backward stochastic differential equations; Time delayed generator; Mean-field; Poisson random measure.

1 Introduction

Recently, Lasry and Lions in [18], introduced a mathematical mean-field approach for high dimensional systems that involve a large number of particles (if we deal with statistical mechanics, physics, quantum mechanics and quantum chemistry) or agents (if we deal with economics, finance and game theory). After them, a lot of works have been done in mean-field problems especially in optimal control and games theory. In what follows, we have to consider a probability space \((\Omega, \mathcal{F}, P)\) with the filtration \((\mathcal{F}_t)_{t \geq 0}\), generated by the Brownian motion \(B\) and an independent compensated Poisson random measure \(\tilde{N}(dt, d\zeta)\).

In the paper [16], the authors consider a general stochastic optimal control of mean-field problems where the mean-field term appears as a function \(F : L^2(P) \to \mathbb{R}\), which \(F\) is a Fréchet differentiable in \(L^2(P)\). This includes the case of a mean-field function \(F(x) = F(P(x))\) where the function \(F\) is continuously differentiable w.r.t. the measure, this case was studied in the paper [7] and the references there in.

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Buckdahn et al in [6] studied a mean-field backward stochastic differential equation (MFBSDE) driven by a forward stochastic differential equation (FSDE) of Mackean-Vlasov type. They proved that the triplet \((X^N, Y^N, Z^N)\) of \(N\) independent copies, converges in law to the solution of some forward-backward stochastic differential equation of mean-field type, which is not only governed by a Brownian motion but also by an independent Gaussian field.

On the other hand, if we want to find an investment strategy and an investment portfolio which should replicate a liability or meet a target which depend on the applied strategy or the portfolio depends on its past values. In this setting, a managed investment portfolio serves simultaneously as the underlying security on which the liability/target is contingent and as a replicating portfolio for that liability/target. This is usually the case for capital-protected investments and performance-linked pay-offs. The delayed backward stochastic differential equations (DBSDEs) are the best tool to solve this financial problems. DBSDE can also arise in variable annuities, unit-linked products and participating contract. For more applications of such equations, we refer to [10] and [11].

For the more, a possible application of BSDE should be a recursive utility associated to the consumption rate "\(\pi\)". It was introduced by Duffie and Epstein in [9] where they proposed to use the recursive utility process \(Y\) as a part of a solution associated to the following BSDE in the Brownian case:

\[
dY(t) = -f(t, Y(t), Z(t), \pi(t))dt + Z(t)dB(t), \quad t \in [0, T].
\]

(1)

In our setting, we consider a new type of recursive utilities which is a mean-field delayed BSDE (MFDBSDE) with jumps and the time horizon is possibly infinite, then

\[
dY(t) = -f(t, Y_t, Z_t, K_t(\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)], \pi(t))dt + Z(t)dB(t) + \int_{-\delta}^{0} K(t, \zeta)\tilde{N}(dt, d\zeta), t \in [0, \tau] \text{ for } \tau \leq \infty.
\]

(2)

Here

\[
Y_t = Y(t + s), \\
Z_t = Z(t + s), \\
K_t(\cdot) = K(t + s, \cdot),
\]

for a given \(s \in [-\delta, 0]\).

Now, for any stochastic process \(X\) is a solution of a stochastic functional differential equation, let the bold \(X\) denotes

\[
X(t) := \left( \int_{-\delta}^{0} X(t + s)\mu_1(ds), \ldots, \int_{-\delta}^{0} X(t + s)\mu_N(ds) \right),
\]

where each \(\mu_i\) is a bounded Borel measure, which is either absolutely continuous w.r.t. the Lebesgue measure or a Dirac measure \(\delta_{t_0}\) for some \(t_0 \in [-\delta, 0]\). Dealing with the maximum principle, the authors in the paper [2], consider a control problem involving decoupled system of finite and infinite time horizon mean-field...
forward backward stochastic differential equations (MFFBSDE). The backward part of these equations in [2], has the form (together with either a terminal condition or a decay condition, depending on the given time horizon):

The state equation

\[ dY(t) = -f(t, X(t), Y(t), Z(t), K(t, \cdot), \mathbb{E}[X(t)], \mathbb{E}[Y(t)], \pi(t)) dt + Z(t) dB(t) + \int_{\mathbb{R}_0} K(t, \zeta) \tilde{N}(dt, d\zeta), t \in [0, \infty), \]

(3)

The corresponding derivative process

\[ d\mathcal{Y}(t) = \left(-\nabla g(t, \pi) \cdot \left(\mathcal{X}(t), \mathcal{Y}(t), \mathbb{E}[\mathcal{X}(t)], \mathbb{E}[\mathcal{Y}(t)], \mathcal{Z}(t), \mathcal{K}(t, \cdot), \eta(t)\right)\right)^T dt \]

\[ + Z(t) dB(t) + \int_{\mathbb{R}_0} K(t, \zeta) \tilde{N}(dt, d\zeta), t \in [0, \infty), \]

(4)

where \(X\) and \(X\) are solutions of forward equations that does not depend on \(Y, Z, K\) and \(Y, Z, K\), respectively and we denote by \((\cdot)^T\) the transpose.

In this note, we establish sufficient conditions for existence and uniqueness for such equations, in both finite and infinite time horizon.

Finite time horizon In section 2, we consider different types of delayed backward stochastic differential equation (DBSDE). In the mean-field delay equation (2), we allow the coefficient functionals to depend on the segments of the process in a similar manner as in the SFDEs in [19], [2], and the Backward stochastic functional differential equations (BSFDEs) in [8]. Also related to our equation, are the BSDEs with time delayed generator in [12]. The authors in [12] consider equations similar to ours, only with the Poisson random measures from (2) replaced by the random measure \(\tilde{M}(dt, d\zeta) := \zeta \tilde{N}(dt, d\zeta)\).

Infinite time horizon In section 3, we study the infinite horizon case of the MFDBSDE (2). It can be seen as an extension of Theorem 3.1 in [16] to mean-field with delay.

2 Finite time horizon

Let \(B(t), t \geq 0\) be a Brownian motion, and \(\tilde{N}(dt, d\zeta), t \geq 0\) be an independent compensated Poisson random measure, with compensator \(\nu(\zeta) dt\), on a probability space \((\Omega, \mathcal{F}, P)\). Let \((\mathcal{F}_t)_{t \geq 0}\) denote the natural filtration associated with \(B\) and \(N\). Let \(\delta > 0\), and extend the filtration by letting \(\mathcal{F}_t = \mathcal{F}_0\) for \(t \in [-\delta, 0]\).

Consider an equation of the form

\[ dY(t) = -f(t, Y_t, Z_t, K_t(\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)]) dt + Z(t) dB(t) + \int_{\mathbb{R}_0} K(t, \zeta) \tilde{N}(dt, d\zeta); t \in [0, T], \]

\[ Y(T) = \xi, \]

(5)
where

\[ Y_t(s, \omega) := \begin{cases} Y(t + s, \omega), & s \in [-\delta, 0], t \geq 0, \omega \in \Omega, \\ Y(0), & t < 0 \end{cases} \]

\[ Z_t(s, \omega) := \begin{cases} Z(t + s, \omega), & s \in [-\delta, 0], t \geq 0, \omega \in \Omega, \\ 0, & t < 0 \end{cases} \]

\[ K_t(s, \omega)(\zeta) := \begin{cases} K(t + s, \omega, \zeta), & s \in [-\delta, 0], t \geq 0, \omega \in \Omega, \zeta \in \mathbb{R}_0, \\ 0, & t < 0 \end{cases} \]

and \( \xi \in L^2(\Omega, \mathcal{F}_T) \).

Here, for each \( t \), \((Y_t, Z_t, K_t)\) are assumed to belong to the space \( S^2 \times L^2 \times \mathbb{H}^2 \), of functionals defined below.

i) \( S^2 = S^2(\Omega, \mathcal{D}[-\delta, 0]) \) consisting of the functions \( \alpha : \Omega \rightarrow \mathcal{D}([-\delta, 0], \mathbb{R}) \) such that \( \omega \mapsto \alpha(\omega, s) \) is \( \mathcal{F}_{t+s} \)-measurable for each \( s \in [-\delta, 0] \), and \( s \mapsto \alpha(\omega, s) \) is càdlàg for each \( \omega \in \Omega \). Let \( S^2 \) be equipped with the norm

\[ \| \alpha \|^2_{S^2} := \mathbb{E} \left( \sup_{s \in [-\delta, 0]} |\alpha(s)|^2 \right). \]

We refer to [3] for more on this space in connection with *stochastic functional differential equations*.

ii) \( L^2 \) consisting of the functions

\[ \lambda : \Omega \times [-\delta, 0] \rightarrow \mathbb{R} \]

such that \((s, \omega) \mapsto \lambda(\omega, s)\) is measurable with respect to the predictable \( \sigma \)-algebra \( \mathcal{P} \), generated by the filtration \( \mathcal{F}_{t+s}, s \in [-\delta, 0] \). Now, we equip \( L^2 \) with the norm

\[ \| \lambda \|^2_{L^2} := \mathbb{E} \left( \int_{-\delta}^{0} |\lambda(t)|^2 \, dt \right) < \infty. \]

Now, let \( \mu \) be some bounded Borel measure on \([-\delta, 0]\) which is either the Lebesgue measure or a Dirac point measure.

iii) \( \mathbb{H}^2 := \mathbb{H}^2(\Omega; L^2(\mu)) \) consisting of the functions

\[ \theta : \Omega \times [-\delta, 0] \times \mathbb{R}_0 \rightarrow \mathbb{R} \]

such that \((s, \omega, \zeta) \mapsto \theta(\omega, s, \zeta)\) is measurable with respect to the product \( \mathcal{P} \otimes B(\mathbb{R}_0) \), generated by the filtration \( \{\mathcal{F}_{t+s}\}_{s \in [-\delta, 0]} \), while \( B \) is a Borel set of \( \mathbb{R}_0 \). Now, we equip \( \mathbb{H}^2 \) with the norm

\[ \| \theta \|^2_{\mathbb{H}^2} := \mathbb{E} \left( \int_{\mathbb{R}_0} \int_{-\delta}^{0} |\theta(t, \zeta)|^2 \nu(d\zeta) dt \right) < \infty. \]
Also, define the spaces

• **$S^2$** consisting of the càdlàg adapted processes

$$P : \Omega \times [-\delta, T] \to \mathbb{R}$$

such that $P(s) = P(0)$ for $s \in [-\delta, 0)$ and $\mathbb{E} \left[ \sup_{t \in [0, T]} |P(t)|^2 \right] < \infty$. We equip $S^2$ with equivalent norms

$$\| P \|_{S^2_{\beta}}^2 := \mathbb{E} \left[ \sup_{t \in [0, T]} e^{\beta t} |P(t)|^2 \right], \beta > 0.$$

• **$L^2$** consisting of the predictable processes

$$Q : \Omega \times [-\delta, T] \to \mathbb{R}$$

such that $Q(s) = 0$ for each $s \in [-\delta, 0)$, $\int_0^T Q(t)^2 dt < \infty$. We equip $L^2$ with the equivalent norms

$$\| Q \|_{L^2_{\beta}}^2 := \int_0^T e^{\beta t} Q(t)^2 dt, \beta > 0.$$

• **$H^2$** consisting of the predictable processes

$$R : \Omega \times [-\delta, T] \times \mathbb{R}_0 \to \mathbb{R}$$

such that $R(s, \zeta) = 0$ for each $s \in [-\delta, 0)$ and $\int_0^T \int_{\mathbb{R}_0} R(u, \zeta)^2 \nu(d\zeta) dt < \infty$. We equip $H^2$ with the equivalent norms

$$\| R \|_{H^2_{\beta}}^2 := \int_0^T \int_{\mathbb{R}_0} e^{\beta t} R(t, \zeta)^2 \nu(d\zeta) dt, \beta > 0.$$

**Definition 1** We say that $(Y, Z, K) \in S^2 \times L^2 \times H^2$ is a solution to (5), if

$$Y(t) = \xi + \int_t^T f(s, Y_s, Z_s, K_s, \mathbb{E}[Y(s)], \mathbb{E}[Z(s)], \mathbb{E}[K(s, \cdot)\| d\tilde{N}_s - \int_t^T K(s, \zeta) dB(s) - \int_t^T \int_{\mathbb{R}_0} K(s, \zeta) \tilde{N}(ds, d\zeta) ; t \in [0, T]. \tag{6}$$

**Assumption (H1)**

Let $f : \Omega \times [0, T] \times S^2 \times L^2 \times H^2 \to \mathbb{R}$, an $\mathcal{F}_t$-adapted and $\xi$ satisfy the following assumptions:

i) $\xi \in L^2(\Omega, \mathcal{F}_T).$
To see why this holds, consider the processes

\[ \int_0^T |f(t, 0, 0, 0, 0, 0)|^2 dt < \infty. \]

**Lemma 2** Suppose that \((Y, f, Z, K, \xi), (\bar{Y}, \bar{f}, \bar{Z}, \bar{K}, \bar{\xi})\) are arbitrary processes in \(S^2 \times \mathbb{R} \times L^2 \times H^2 \times L^2(\mathcal{F}_T)\), that satisfies the equation \(15\). Then under Assumption (H1), there exists a constant \(C_\beta > 0\), such that

\[
\| Y - \bar{Y} \|_{S^2}^2 + \| Z - \bar{Z} \|_{L^2}^2 + \| K - \bar{K} \|_{H^2}^2 \\
\leq C_\beta \left( \mathbb{E} \left[ |\xi - \tilde{\xi}|^2 \right] + \mathbb{E} \left[ \int_0^T e^{\beta t} |f(t) - \bar{f}(t)|^2 dt \right] \right).
\]

For convenience, we have used the simplified notation

\[
f(t) = f(t, Y_t, Z_t, K_t, \xi), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)]
\]

\[
\bar{f}(t) = f(t, \bar{Y}_t, \bar{Z}_t, \bar{K}_t, \xi, \mathbb{E}[\bar{Y}(t)], \mathbb{E}[\bar{Z}(t)], \mathbb{E}[\bar{K}(t, \cdot)]
\]

The constant \(C_\beta\) can be chosen, such that

\[
C_\beta := \max \left( 9e^{\beta T}, 8T + \frac{1}{\beta} \right), \quad \beta > 0.
\]

**Proof.** We proceed as in [12]. Introduce the notations \(\Delta Y := Y - \bar{Y}, \Delta Z := Z - \bar{Z}\) and so on. By applying Itô’s formula to \(e^{\beta t} |\Delta Y(t)|^2\), we find that

\[
d e^{\beta t} |\Delta Y(t)|^2 = \left[ \beta e^{\beta t} |\Delta Y(t)|^2 + e^{\beta t} |\Delta Z(t)|^2 \\
+ e^{\beta t} \int_{\mathbb{R}_0} |\Delta K(t, \zeta)|^2 \nu(d\zeta) - 2e^{\beta t} \Delta Y(t) \Delta f(t) \right] dt \\
+ 2e^{\beta t} \Delta Y(t) \Delta Z(t) dB(t) \\
+ \int_{\mathbb{R}_0} e^{\beta t} \left( |\Delta K(t, \zeta)|^2 + 2e^{\beta t} \Delta Y(t) \Delta K(t, \zeta) \right) \tilde{N}(dt, d\zeta).
\]

We claim now that, the \(dB(t)\) and \(\tilde{N}(dt, d\zeta)\)-integrals above have mean zero. To see why this holds, consider the processes

\[
M_1(t) := \int_0^t 2e^{\beta u} \Delta Y(u) \Delta Z(u) dB(u)
\]

\[
M_2(t) := \int_0^t \int_{\mathbb{R}_0} e^{\beta u} \left( |\Delta K(u, \zeta)|^2 + 2e^{\beta u} \Delta Y(u) \Delta K(u, \zeta) \right) \tilde{N}(du, d\zeta)
\]
separately. Now since the integrands $|\Delta K(u, \zeta)|^2 + e^{\beta u} \Delta Y(u) \Delta K(u)$ is $P \otimes dt \otimes \nu$-integrable, $M_2(t)$ have mean zero (in fact $M_2$ is a martingale, see e.g. [13],[17]).

In order to find an explicit expression for $C_\beta$, we proceed as in Lemma 2.1 in [12]. ■

**Theorem 3** Suppose that Assumption (H1) holds, and that there is some $\beta \geq 0$ such that

$$C_\beta C_f (1 + e^{-\beta s}) \max(1, T) < 1. \tag{9}$$

Then there exists a unique solution $(Y, Z, K) \in S^2 \times L^2 \times H^2$ to equation (3).

**Proof.** It is a classical result of existence and uniqueness of solutions for BSDEs with jumps, proved for $\beta = 0$ in [4], Theorem 2.1 and Proposition 2.2. We also refer to [12], [22] and [21] for related results, and to [5] for generalizations using conditional expectation with respect to stopping times and with non-homogeneous compensators replacing $\nu \otimes dt$. Now since the $\beta$-norms are equivalent for every $\beta$, the results hold for arbitrary $\beta$.

We show the existence of a solution following [12]. Define the maps $\Upsilon (Y, Z, K) := (\bar{Y}, \bar{Z}, \bar{K})$, for $S^2_{\beta} \times L^2_{\beta} \times H^2_{\beta} \to S^2_{\beta} \times L^2_{\beta} \times H^2_{\beta}$, by

$$d\bar{Y}(t) = -f(t, Y_t, Z_t, K_t(\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)])dt$$

$$+ \bar{Z}(t) dB(t) + \int_{R_0} \bar{K}(t, \zeta) \tilde{N}(dt, d\zeta), \quad 0 \leq t \leq T,$$

$$\bar{Y}(T) = \xi \quad \text{and} \quad \bar{Z}(t) = \bar{K}(t, \cdot) = 0 \quad t \in [-\delta, 0].$$

By assumption, the integrands $f(t, Y_t, Z_t, K_t(\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)])$ has a version in $L^2(P \otimes \mu)$ whenever $Y, Z, K$ satisfy the Lipschitz condition (ii) of Assumption (H1), and hence by Lemma 2, the processes $(\bar{Y}, \bar{Z}, \bar{K})$ are well defined. Now, it suffices to show that $(\bar{Y}, \bar{Z}, \bar{K}) =: \Upsilon(Y, Z, K)$ is a contraction for every $\beta \geq 0$ such that (11) is satisfied. For that, we consider $(Y^1, Z^1, K^1), (Y^2, Z^2, K^2) \in S^2_{\beta} \times L^2_{\beta} \times H^2_{\beta}$ with

$$(Y^1, \bar{Z}^1, \bar{K}^1) =: \Upsilon(Y^1, Z^1, K^1),$$

$$(Y^2, \bar{Z}^2, \bar{K}^2) =: \Upsilon(Y^2, Z^2, K^2).$$

We proceed by showing some useful inequalities. Suppose that $t \in [0, T], P \in S^2, Q \in L^2$ and $R \in H^2$. We have for $s \in [-\delta, 0]$:

$$\mathbb{E} \left[ \int_0^T e^{\beta u} |P(u + s)|^2 du \right] = e^{-\beta s} \mathbb{E} \left[ \int_0^T e^{\beta u} |P(u + s)|^2 du \right] \leq e^{-\beta s} \mathbb{E} \left[ \int_0^T \sup_{0 \leq u \leq T} e^{\beta u} |P(u + s)|^2 du \right] \tag{10}$$

$$= T e^{-\beta s} \| P \|_{S^2},$$

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and
\[ \mathbb{E} \left[ \int_0^T e^{\beta u} (\mathbb{E}[|P(u)|])^2 \, du \right] \leq \mathbb{E} \left[ \int_0^T \sup_{0 \leq u \leq T} e^{\beta u} |P(u)|^2 \, du \right] = T \| P \|_{S_\beta}^2. \]  

Choosing \( v := u + s \), we get
\[ \mathbb{E} \left[ \int_0^T e^{\beta |Q(u + s)|^2} \, du \right] = \mathbb{E} \left[ \int_{s}^{T+s} e^{\beta (v-s)} |Q(v)|^2 \, dv \right] \leq e^{-\beta s} \mathbb{E} \left[ \int_0^T e^{\beta v} |Q(v)|^2 \, dv \right] \]
\[ \leq e^{-\beta s} \| Q \|_{L_2^\beta}, \]  

and
\[ \mathbb{E} \left[ \int_0^T e^{\beta u} (\mathbb{E}[|Q(u)|])^2 \, du \right] \leq \mathbb{E} \left[ \int_0^T e^{\beta u} |Q(u)|^2 \, du \right] \]
\[ = \| Q \|_{L_2^\beta}^2. \]  

Similar estimates \((12) - (13)\), for \( R \in \mathbb{H}^2 \).

Now, by Lemma 2, the Lipschitz Assumption and by the inequalities \((10) - (13)\), we find that
\[ \| Y_1 - Y_2 \|_{S_\beta}^2 + \| Z_1 - Z_2 \|_{L_2^\beta}^2 + \| K_1 - K_2 \|_{H_3}^2 \]
\[ \leq C_\beta C_f (1 + e^{-\beta \delta}) \max(1, T) \left( \| Y_1 - Y_2 \|_{S_\beta}^2 + \| Z_1 - Z_2 \|_{L_2^\beta}^2 + \| K_1 - K_2 \|_{H_3}^2 \right). \]

Whenever \((9)\) is satisfied, the proof is completed. ■

**Remark 4** If the Lipschitz condition (ii) is replaced by stronger Lipschitz condition (A3) in \([12]\), we may choose
\[ C_\beta C_f \left( 2 + \int_{-\delta}^0 e^{-\beta \delta} \mu(ds) \right) \max(1, T) < 1, \]

instead of inequality \((9)\) in Theorem \([3]\) above.

**2.1 Special case**

We consider now a mean-field equation on the form
\[ Y(t) = \xi + \int_0^T f(u, Y(u), Z(u), K(u, \cdot), \mathbb{E}[Y(u)], \mathbb{E}[Z(u)], \mathbb{E}[K(u, \cdot)]) \, du \]
\[ + \int_0^T Z(u) dB(u) + \int_{\mathbb{R}^\delta} K(u, \zeta) \tilde{N}(du, d\zeta); \quad t \in [0, T], \]  

where for a given positive constant \( \delta \).
The boldface processes from equation (14) are defined by

\[ Y(t) := (Y(t), Y(t - \delta)), \]
\[ Z(t) := (Z(t), Z(t - \delta)), \]
\[ K(t, \cdot) := (K(t, \cdot), K(t - \delta, \cdot)), \]

and a generator \( f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{S}^2 \times \mathbb{R} \times \mathbb{L}^2 \times L^2(\nu) \times \mathbb{H}^2 \times \mathbb{R}^3 \to \mathbb{R}. \)

For \( f \) and \( \xi \), we impose the set of assumptions.

**Assumption (H2)**

i) \( \xi \in L^2(\Omega, \mathcal{F}_T). \)

ii) The generator function \( f \) is \( \mathcal{F}_t \)-adapted and Lipschitz in the sense that there exists a constant \( \tilde{C}_f > 0 \) such that

\[
\begin{align*}
&\left| f(t, y_1, y_2, z_1, z_2, k_1(\cdot), k_2(\cdot), z_3, k_3(\cdot)) - f(t, \tilde{y}_1, \tilde{y}_2, \tilde{z}_1, \tilde{z}_2, \tilde{k}_1(\cdot), \tilde{k}_2(\cdot), \tilde{y}_3, \tilde{z}_3, \tilde{k}_3(\cdot)) \right|^2 \\
&\leq \tilde{C}_f \left( |y_1 - \tilde{y}_1|^2 + |y_2 - \tilde{y}_2|^2 + |z_1 - \tilde{z}_1|^2 + |z_2 - \tilde{z}_2|^2 \\
&\quad + \int_{\mathbb{R}_0} \left| k_1(\zeta) - \tilde{k}_1(\zeta) \right|^2 \nu(d\zeta) + \int_{\mathbb{R}_0} \left| k_2(\zeta) - \tilde{k}_2(\zeta) \right|^2 \nu(d\zeta) \\
&\quad + |y_3 - \tilde{y}_3| + |z_3 - \tilde{z}_3| + \int_{\mathbb{R}_0} \left| k_3(\zeta) - \tilde{k}_3(\zeta) \right|^2 \nu(d\zeta) \right) \\
&\quad \text{a.e. } t, y_1, y_2, y_3, z_1, z_2, z_3, k_1(\cdot), k_2(\cdot), k_3(\cdot), \tilde{k}_1(\cdot), \tilde{k}_2(\cdot), \tilde{k}_3(\cdot).
\end{align*}
\]

iii)

\[ \mathbb{E} \left[ \int_0^T |f(t, 0, 0, 0, 0, 0, 0, 0, 0)|^2 dt \right] < \infty. \]

**Theorem 5** We say that \( (Y, Z, K) \in \mathbb{S}^2 \times \mathbb{L}^2 \times \mathbb{H}^2 \) is a unique solution of (14) if \( \xi \) and \( f \) satisfy the Assumption (H2), and if there is \( \beta \geq 0 \), such that

\[ C_\beta \tilde{C}_f (2 + e^{\beta\delta}) \max(1, T) < 1. \]

**Proof.** The proof is closely related to the proof of Theorem 4. We just need some estimations instead of estimations (10) and (12).

Suppose that \( t \in [0, T], P \in \mathbb{S}^2, Q \in \mathbb{L}^2 \) and \( R \in \mathbb{H}^2 \). We have that

\[
\mathbb{E} \left[ \int_0^T e^{\beta u} |P(u - \delta)|^2 du \right] = e^{\beta\delta} \mathbb{E} \left[ \int_0^T e^{\beta(u-\delta)} |P(u - \delta)|^2 du \right] \leq e^{\beta\delta} \mathbb{E} \left[ \int_0^T \sup_{0 \leq u \leq T} e^{\beta(u-\delta)} |P(u - \delta)|^2 du \right] \leq T e^{\beta\delta} \| P \|_{\mathbb{S}_3^2},
\]

and that

\[
\mathbb{E} \left[ \int_0^T e^{\beta u} |Q(u - \delta)|^2 du \right] = \mathbb{E} \left[ \int_{-\delta}^{T-\delta} e^{\beta(u+\delta)} |Q(v)|^2 dv \right] \leq e^{\beta\delta} \| Q \|_{\mathbb{L}_2^2},
\]

similarly for \( R \in \mathbb{H}^2 \).
Remark 6  Related to the mean field BSDEs with delay (2), (5) and (??) are the BSDEs (3). This fully coupled BSDEs allows for the delay generator by the FSDEs. We can solve the forward equation separately to obtain the process \( X \) and then plugging the solution \( X \) into the backward equation to find the solution \((Y, Z, K)\).

3  Infinite horizon case

We deal now, with the existence and uniqueness of a solution \((Y, Z, K)\) of the following MFDBSDE, in infinite time horizon:

\[
\begin{aligned}
&\frac{dY(t)}{dt} = -f(t, Y_t, Z_t, K_t(\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)])dt + Z(t)dB(t) \\
&+ \int_{\mathbb{R}_0} K(t, \zeta)N(dt, d\zeta) ; \ t \geq 0, \\
&\lim_{t \to \infty} Y(t) = 0.
\end{aligned}
\]

Here the generator \( f : \Omega \times [0, \infty) \times \mathbb{S}^2 \times \mathbb{L}^2 \times \mathbb{H}^2 \times \mathbb{R}^3 \to \mathbb{R} \), depends on previous values of the solution, such as

\[
\begin{align*}
Y_t &:= Y(t + r), \\
Z_t &:= Z(t + r), \\
K_t(\cdot) &:= K(t + r, \cdot),
\end{align*}
\]

for some constant \( r \in [-\delta, 0] \).

Definition 7  We denote by \( \mathcal{L} \), the space of all \( \mathcal{F}_t \)-adapted processes satisfying:

\[
\|(Y, Z, K)\|^2 := E\left[ \sup_{t \geq 0} e^{\beta t} |Y(t)|^2 + \int_0^\infty e^{\beta t} \left( |Z(t)|^2 + \int_{\mathbb{R}_0} |K(t, \zeta)|^2 \nu(d\zeta) \right) dt \right] < \infty.
\]

We set now assumptions proving existence and uniqueness of the solution of equation (15).

Assumption (H3)

On the generator \( f \):

- The function \( f \) is \( \mathcal{F}_t \)-adapted.

- Integrability condition:

\[
E\left[ \int_0^\infty e^{\beta t} |f(t, 0, 0, 0, 0, 0)|^2 dt \right] < \infty.
\]

- Lipschitz condition: There exists a constant \( C > 0 \), such that

\[
\begin{align*}
|f(t, y_1, z_1, k_1(\cdot), y_2, z_2, k_2(\cdot)) - f(t, \tilde{y}_1, \tilde{z}_1, \tilde{k}_1(\cdot), \tilde{y}_2, \tilde{z}_2, \tilde{k}_2(\cdot))| &
\leq C \left( |y_1 - \tilde{y}_1| + |y_2 - \tilde{y}_2| + |z_1 - \tilde{z}_1| + \int_{\mathbb{R}_0} |k_1(\zeta) - \tilde{k}_1(\zeta)| \nu(d\zeta) \\
&+ \int_{\mathbb{R}_0} |k_2(\zeta) - \tilde{k}_2(\zeta)| \nu(d\zeta) \right) \\
&\text{a.e. } t, y_1, y_2, z_1, z_2, k_1(\cdot), \tilde{k}_1(\cdot), k_2(\cdot), \tilde{k}_2(\cdot).
\end{align*}
\]
There are real numbers $\beta, C$ for sufficiently small $\epsilon$, we have that
\[
\beta > \frac{6C^2}{\epsilon} + \frac{1}{2}.
\]

We give now, the main Theorem of this section which is the extension to the delay case and to mean-field of Theorem 3.1 in [15].

**Theorem 8** Under the above Assumption (H3), there exists a unique solution $(Y, Z, K)$ of a MFDBSDE (15), such that
\[
\mathbb{E} \left[ \sup_{t \geq 0} e^{\beta t} |Y(t)|^2 \times \int_0^\infty e^{\beta t} \left( |Y(t)|^2 + |Z(t)|^2 + \int_{\mathbb{R}_0} |K(t, \zeta)|^2 \nu(d\zeta) \right) dt \right] \\
\leq c \mathbb{E} \left[ \int_0^\infty e^{\beta t} |f(t, 0, 0, 0, 0, 0, 0)|^2 dt \right] < \infty.
\]

(17)

**Proof. 1. Uniqueness**
We assume that we have two solutions, $(Y_1, Z_1, K_1)$, $(Y_2, Z_2, K_2)$, and we note that
\[
(\bar{Y}, \bar{Z}, \bar{K}) = (Y_1 - Y_2, Z_1 - Z_2, K_1 - K_2).
\]
In what follows, we use the simplified notation:
\[
f(t) = f(t, Y_t, Z_t, K_t(\cdot), \mathbb{E}[Y(t)], \mathbb{E}[Z(t)], \mathbb{E}[K(t, \cdot)])
\]
Applying Itô’s formula to $e^{\beta t} |\bar{Y}(t)|^2$, we get for all $0 < t < T$ (constant) $< \infty$,
\[
e^{\beta t} |\bar{Y}(T)|^2 - e^{\beta t} |\bar{Y}(t)|^2 \leq \int_t^T e^{\beta s} \left\{ \beta |\bar{Y}(s)|^2 + |\bar{Z}(s)|^2 + \int_{\mathbb{R}_0} |K(s, \xi)|^2 \nu(d\xi) \right\} ds \\
- 2 \int_t^T e^{\beta s} |\bar{Y}(s)| \cdot |\bar{f}(s)| ds + 2 \int_t^T e^{\beta s} |\bar{Y}(s)| \cdot |\bar{Z}(s)| dB(s) \\
+ 2 \int_t^T \int_{\mathbb{R}_0} e^{\beta s} |\bar{Y}(s)| \cdot |K(s, \xi)| \tilde{N}(ds, d\xi).
\]
(18)
After taking expectation, using Lipschitz condition and the fact that $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$. 

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for all $a, b \in \mathbb{R}$, we obtain

$$
\begin{align*}
\mathbb{E} \left[ e^{\beta t} |\bar{Y}(t)|^2 \right] + \mathbb{E} \left[ \int_t^T e^{\beta s} \left\{ \beta |\bar{Y}(s)|^2 + |\bar{Z}(s)|^2 + \int_{\mathbb{R}_0} |\bar{K}(s, \xi)|^2 \nu(d\xi) \right\} ds \right] \\
= \mathbb{E} \left[ e^{\beta T} |\bar{Y}(T)|^2 \right] + 2\mathbb{E} \left[ \int_t^T e^{\beta s} |\bar{Y}(s)| \cdot |\bar{f}(s)| ds \right] \\
\leq \mathbb{E} \left[ e^{\beta T} |\bar{Y}(T)|^2 \right] + \frac{6C^2}{\epsilon} \mathbb{E} \left[ \int_t^{T+r} e^{\beta s} |\bar{Y}(s)|^2 ds \right] \\
+ 6(1 + e^{-\beta r})\mathbb{E} \left[ \int_t^{T+r} e^{\beta s} |\bar{Y}(s)|^2 ds \right] + 6\epsilon \mathbb{E} \left[ \int_t^{T+r} e^{\beta s} \left( \mathbb{E} \left[ |\bar{Y}(s)| \right] \right)^2 ds \right] \\
+ 6(1 + e^{-\beta r})\mathbb{E} \left[ \int_t^{T+r} e^{\beta s} |\bar{Z}(s)|^2 ds \right] + 6\epsilon \mathbb{E} \left[ \int_t^{T+r} e^{\beta s} \left( \mathbb{E} \left[ |\bar{Z}(s)| \right] \right)^2 ds \right] \\
+ 6(1 + e^{-\beta r})\mathbb{E} \left[ \int_t^{T+r} \int_{\mathbb{R}_0} e^{\beta s} |\bar{K}(s, \xi)|^2 \nu(d\xi) ds \right] \\
+ 6\epsilon \mathbb{E} \left[ \int_t^{T+r} e^{\beta s} \left( \mathbb{E} \left[ |\bar{K}(s, \xi)| \right] \right)^2 \nu(d\xi) ds \right],
\end{align*}
\tag{19}$$

where we have used by changing of variables: $u := s + r$

$$
\int_t^T e^{\beta s} |\bar{Y}(s + r)|^2 ds = \int_{t+r}^{T+r} e^{\beta(u-r)} |\bar{Y}(u)|^2 du \tag{20}
$$

and similarly for $\bar{Z}$ and $\bar{K}$.

The fact that

$$
\mathbb{E} \left[ \int_t^{T+r} e^{\beta s} \left( \mathbb{E} \left[ |\bar{Y}(s)| \right] \right)^2 ds \right] \leq \mathbb{E} \left[ \int_t^{T+r} e^{\beta s} |\bar{Y}(s)|^2 ds \right], \quad \text{where } \bar{Y} = \bar{Y}, \bar{Z}, \bar{K},
\tag{21}
$$
gives that
\[
E\left[e^{\beta t}|\tilde{Y}(t)|^2\right] + E\left[\int_t^T e^{\beta s}\left\{\beta \left|\tilde{Y}(s)\right|^2 + \left|\tilde{Z}(s)\right|^2 + \int_{\mathbb{R}_0^+} |\tilde{K}(s, \xi)|^2 \nu(\,d\xi)\right\}\,ds\right]
\leq E\left[e^{\beta T}|\tilde{Y}(T)|^2\right] + \left(\frac{6C^2}{\epsilon} + 6\epsilon(2 + e^{-\beta r})\right) E\left[\int_t^{T+r} e^{\beta s} |\tilde{Y}(s)|^2 \,ds\right]
+ 6\epsilon(2 + e^{-\beta r}) E\left[\int_t^{T+r} e^{\beta s} |\tilde{Z}(s)|^2 \,ds\right]
+ 6\epsilon(2 + e^{-\beta r}) E\left[\int_t^{T+r} \int_{\mathbb{R}_0^+} e^{\beta s} |\tilde{K}(s, \xi)|^2 \nu(\,d\xi) \,ds\right].
\] (22)

Taking \(\epsilon\) such \(6\epsilon(2 + e^{-\beta r}) < \frac{1}{2}\), then since \(\beta > \frac{6C^2}{\epsilon} + \frac{1}{2}\), we have for \(t < T\),
\[
E\left[e^{\beta t}|\tilde{Y}(t)|^2\right] \leq E\left[e^{\beta T}|\tilde{Y}(T)|^2\right].
\] (23)

Replacing \(\beta\) by \(\beta'\) with, \(\beta > \beta' > \frac{6C^2}{\epsilon} + \frac{1}{2}\), then
\[
E\left[e^{\beta' t}|\tilde{Y}(t)|^2\right] \leq e^{(\beta' - \beta)T} E\left[e^{\beta T}|\tilde{Y}(T)|^2\right].
\] (24)

The second factor on the right hand side is bounded by condition (17) as \(T \to \infty\), while the first factor tends to 0.

The uniqueness is proved.

2. Existence

The existence proof is given in two steps, for all \(n \in \mathbb{N}\). Let us construct the solution \((Y^n, Z^n, K^n)\) of the infinite horizon DBSDE

\[
Y^n(t) = \int_t^n f(s, Y^n_s, Z^n_s, K^n_s, \cdot) \,ds + \int_t^n \mathbb{E}[Y^n(s), \mathbb{E}[Z^n(t), \mathbb{E}[K^n(s, \cdot)]\,ds
- \int_t^n Z^n(s)dB(s) - \int_t^n \int_{\mathbb{R}_0^+} K^n(s, \xi)\tilde{N}(\,d\xi, \,ds)\,; \ t \geq 0,
\]

as follows:

- **Step 1:** It is decomposed into two cases according to \(t\).

**Case 1:** Let \(0 \leq t \leq n\), then \((Y^n, Z^n, K^n)\) is defined as a solution of the DBSDE in \([0, n]\) as follows:

\[
Y^n(t) = \int_t^n f(s, Y^n_s, Z^n_s, K^n_s, \cdot) \,ds + \int_t^n \mathbb{E}[Y^n(s), \mathbb{E}[Z^n(t), \mathbb{E}[K^n(s, \cdot)]\,ds
- \int_t^n Z^n(s)dB(s) - \int_t^n \int_{\mathbb{R}_0^+} K^n(s, \xi)\tilde{N}(\,d\xi, \,ds)\,; 0 \leq t \leq n.
\]

**Case 2:** For \(t \geq n\), \((Y^n, Z^n, K^n)\) is defined by

\[
Y^n(t) := 0, \quad t > n,
Z^n(t) := 0, \quad t > n,
K^n(t, \cdot) := 0, \quad t > n.
\]
We will first establish an a priori estimate for the sequence \((Y^n, Z^n, K^n)\).

Adding and subtracting \(\mathbb{E} \left[ \int_t^\infty e^{\beta s} |Y(s)| \cdot |f(s, 0, 0, 0, 0, 0)| ds \right] \) in (16) and by using \(2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2\), for all \(a, b \in \mathbb{R}\), we get

\[
\mathbb{E} \left[ e^{\beta t} |Y^n(t)|^2 \right] + \mathbb{E} \left[ \int_t^\infty e^{\beta s} \left\{ \tilde{\beta} |Y^n(s)|^2 + \tilde{\alpha} |Z^n(s)|^2 + \tilde{\gamma} \int_{\mathbb{R}_0} |K(s, \xi)|^2 \nu(d\xi) \right\} ds \right]
\leq \frac{1}{\epsilon} \mathbb{E} \left[ \int_t^\infty e^{\beta s} |f(s, 0, 0, 0, 0, 0)|^2 ds \right]
\]

with

\[
\tilde{\beta} := \beta - 6C^2 / \epsilon - 1 / 2 > 0, \\
\tilde{\alpha} := 1 - 6\epsilon(2 + e^{-\beta r}) > 0, \\
\tilde{\gamma} := 1 - 6\epsilon(2 + e^{-\beta r}) > 0,
\]

where we have taking that \(6\epsilon(2 + e^{-\beta r}) < 1\).

Using (25) and the martingale inequality, we have

\[
\mathbb{E} \left[ \sup_{1 \leq s \leq m} e^{\beta s} |Y^n(t)|^2 \right] + \mathbb{E} \left[ \int_s^m e^{\beta r} \left\{ |Y^n(r)|^2 + |Z^n(r)|^2 + \int_{\mathbb{R}_0} |K^n(r, \xi)|^2 \nu(d\xi) \right\} dr \right]
\leq \frac{1}{\epsilon} \mathbb{E} \left[ \int_s^m e^{\beta r} |f(r, 0, 0, 0, 0, 0)|^2 dr \right].
\]

**Step 2:** Let \(m > n\), define

\[
\Delta Y(t) = Y^m(t) - Y^n(t), \\
\Delta Z(t) = Z^m(t) - Z^n(t), \\
\Delta K(t, \cdot) = K^m(t, \cdot) - K^n(t, \cdot).
\]

According to \(t\), we have also two cases.

**Case 1:** For \(n \leq t \leq m\), we have

\[
\Delta Y(t) = \int_t^m f(s, Y^m_s, Z^m_s, K^m_s, \cdot) \mathbb{E} [Y^m(s)] + \mathbb{E} [Z^m(s)] + \mathbb{E} [K^m(s, \cdot)] ds \\
- \int_t^m \Delta Z(s) dB(s) - \int_t^m \int_{\mathbb{R}_0} \Delta K(s, \xi) \tilde{N}(ds, d\xi).
\]

Then by Itô’s formula, we get

\[
\mathbb{E} \left[ e^{\beta t} |\Delta Y(t)|^2 \right] + \mathbb{E} \left[ \int_t^m e^{\beta s} \left\{ \beta |\Delta Y(s)|^2 + |\Delta Z(s)|^2 + \int_{\mathbb{R}_0} |K^n(s, \xi)|^2 \nu(d\xi) \right\} ds \right]
\leq 2\mathbb{E} \left[ \int_t^m e^{\beta s} |\Delta Y(s)| \right] + 2\mathbb{E} \left[ \int_t^m e^{\beta s} |\Delta Z(s)| \right]
\]

We deduce, by the same argument used before, that

\[
\mathbb{E} \left[ \sup_{1 \leq s \leq m} e^{\beta s} |\Delta Y(t)|^2 \right] + \mathbb{E} \left[ \int_m^t e^{\beta s} \left\{ |\Delta Y(s)|^2 + |\Delta Z(s)|^2 + \int_{\mathbb{R}_0} |\Delta K(s, \xi)|^2 \nu(d\xi) \right\} ds \right]
\leq \frac{1}{\epsilon} \mathbb{E} \left[ \int_{\mathbb{R}_0} e^{\beta s} |\Delta Y(s)| ds \right].
\]

\[
\mathbb{E} \left[ \sup_{1 \leq s \leq m} e^{\beta s} |\Delta Y(t)|^2 \right] + \mathbb{E} \left[ \int_t^m e^{\beta s} \left\{ |\Delta Y(s)|^2 + |\Delta Z(s)|^2 + \int_{\mathbb{R}_0} |K^n(s, \xi)|^2 \nu(d\xi) \right\} ds \right]
\leq \frac{1}{\epsilon} \mathbb{E} \left[ \int_{\mathbb{R}_0} e^{\beta s} |\Delta Y(s)| ds \right].
\]
The last inequality goes to zero as $n$ goes to infinity.

**Case 2:** For $t \leq n$, we have

$$
\Delta Y(t) = \Delta Y(n) + \int_t^n \left\{ f(s, Y^n_s, Z^n_s, K^n_s(\cdot), \mathbb{E}[Y^n(s)], \mathbb{E}[Z^n(s)], \mathbb{E}[K^n(s, \cdot)])
- f(s, Y^n_s, Z^n_s, K^n_s(\cdot), \mathbb{E}[Y^n(s)], \mathbb{E}[Z^n(s)], \mathbb{E}[K^n(s, \cdot)]) \right\} ds
- \int_t^n \Delta Z(s) dB(s) - \int_t^n \int_{R_0} \triangle K(s, \xi) \tilde{N}(ds, d\xi).
$$

We proceed as in (19) − (24), in the proof of uniqueness, we obtain

$$
\mathbb{E}\left[e^{\beta t} |\Delta Y(t)|^2 \right] \leq \mathbb{E}\left[e^{\beta n} |\Delta Y(n)|^2 \right]
\leq \frac{1}{\epsilon} \mathbb{E}\left[g \int_{2n}^{\infty} e^{\beta s} |\triangle f(s, 0, 0, 0, 0, 0)|^2 ds \right].
$$

Then the sequence $(Y^n, Z^n, K^n)$ forms a Cauchy sequence for the norm of the space $\mathcal{L}$ in (19) and that the limit $(Y, Z, K)$ is a solution of a MFDBSDE (15).

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