THE CENTRALLY SYMMETRIC $V$-STATES FOR ACTIVE SCALAR EQUATIONS.
TWO-DIMENSIONAL EULER WITH CUT-OFF

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ABSTRACT. We consider the truncated equation for the centrally symmetric $V$-states in the 2d Euler dynamics of patches and prove the existence of solutions $y(x, \lambda), x \in [-1,1], \lambda \in (0, \lambda_0)$. These functions $y(x, \lambda) \in C^\infty([-1,1])$ and $\|y(x, \lambda) - |x||_{C([-1,1])} \to 0, \lambda \to 0$.

1. INTRODUCTION

Consider the following general active-scalar equation

$$\dot{\theta} = \nabla \theta \cdot (\nabla^\perp A\theta + S), \quad \theta(0, x, y) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$$

where

$$Af = \int_{\mathbb{R}^2} D(z - \xi)f(\xi)d\xi, \quad z, \xi \in \mathbb{R}^2$$

and $D(z)$ is a radially symmetric function that is monotonically increasing in $r = |z| \in (0, +\infty)$. The symbol $S(t, z)$ denotes the strain, i.e., an exterior velocity. The case $D(z) = \log |z|$ and $S(x, t) = 0$ corresponds to the evolution of vorticity in 2d non-viscous Euler equation. In this situation, the existence of global solution $\theta(t, x, y) \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ is known thanks to [7]. We consider the case when $\theta(0, z) = \chi_{\Omega(0)}(z) + \chi_{-\Omega(0)}(z)$ and $-\Omega(0) = \{-z, z \in \Omega(0)\}$. Assuming $\Omega(0) \cap -\Omega(0) = \emptyset$, one has $\theta(t, z) = \chi_{\Omega(t)}(z) + \chi_{-\Omega(t)}(z)$, i.e., it represents evolution of the centrally symmetric pair of patches (the preservation of central symmetry is a simple feature of dynamics).

For 2d Euler, there is a numerical evidence (e.g., [2, 6] and references there) that suggests the existence of the parametric curve of centrally symmetric patches $V_\lambda \cup -V_\lambda$, the so-called $V$-states, that rotate with constant angular velocity around the origin without changing the shape (i.e., $\Omega(t) = R_{\omega t}V_\lambda$ where $R_\theta$ denotes the rotation around the origin by the angle $\theta$). Here, dist($V_\lambda, -V_\lambda) = 2\lambda$ and $\lambda \in [0, \lambda_0)$. For $\lambda > 0$, the boundary $\Gamma_\lambda = \partial V_\lambda$ seems to be smooth but the two patches form a sharp corner of 90 degrees and touch at the origin when $\lambda = 0$. Assuming existence of the $V$-states in the contact position and their regularity away from the origin, Overman [4] did a careful analysis around the zero. In particular, he explained why the 90 degrees is the only possible nontrivial angle of the contact.

In this paper, we will prove the existence of those $V_\lambda$ for a simplified model: the model with cut-off. If $\Omega_{sc}(0)$ is a simply connected domain with smooth boundary, the evolution of $\Gamma_{sc}(t) = \partial \Omega_{sc}(t)$ is governed by the following integro-differential equation (see, e.g., [4], formula (8.56)):

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\[ \dot{z}(t, \alpha) = C \int_{\Gamma_{sc}(t)} z'(t, \beta) \log |z(t, \beta) - z(t, \alpha)| d\beta, \quad \alpha \in \Gamma_{sc}(t) \]

where \( C \) is an absolute constant.

If one considers centrally symmetric pair of vortices \( \Omega(t) \cup -\Omega(t) \) and parameterizes \( \Gamma(t) = \partial \Omega(t) \) around the origin (e.g., in the window \( x \in [-\delta, \delta], \mu \in [0, \delta] \)) by \( (x, \mu(t, x)) \), then

\[ \dot{\mu}(t, x) = \int_{-\delta}^{\delta} (\mu'(t, x) - \mu'(t, \xi)) K(x, \xi) d\xi + R(t, x) \]

where

\[ K(x, \xi) = H((\mu(x) + \mu(\xi))^2 + (x + \xi)^2) - H((\mu(x) - \mu(\xi))^2 + (x - \xi)^2) \]

Check, e.g., [5], the formula (2.9) for the analogous calculation in the case of surface-quasigeostrophic equation. Here we can always rescale the time to make the constant in front of the integral equal to 1.

Here \( H(t) = D(\sqrt{t}) \) and \( R \) is the contribution coming from the patch outside the window. This \( R \) is smooth on \((-\delta, \delta)\) as the kernel \( D \) is smooth away from the origin. In this paper, we will perform the “cut-off”, i.e., will take \( R = 0 \) thus effectively restricting attention to the interval \((-\delta, \delta)\). Rescaling \( z \), we can always assume that \( \delta = 1 \).

![Figure 1](image)

In the author’s opinion, the following problem is important for understanding the original question of patch evolution.

**Problem 1.** Is there a smooth solution to

\[ \dot{\mu}(t, x) = \int_{-1}^{1} (\mu'(t, x) - \mu'(t, \xi)) K(x, \xi) d\xi, \quad x \in (-1, 1) \]

such that

\[ \mu(t, x) \to y_0(x) = |x|, \quad t \to \infty \]
If so, what are the estimates (lower and upper) on
\[ d(t) = \min_x |\mu(t, x) - y_0(x)|? \]

The function \(y_0(x)\) is a stationary solution to (2). This problem is motivated by the recent results obtained in [3].

The simple analysis based on the Yudovich theory yields the following bound for the Euler evolution:
\[ \text{dist}(\Omega(t), -\Omega(t)) > e^{-e^{Ct}}, \quad t \geq 0 \]

An important question then is to understand whether this bound is sharp. In [3], this estimate was proved to be optimal (even up to a constant \(C\)) assuming that an odd, incompressible, Lipschitz-regular strain is present. This was performed by constructing an approximate solution to the dynamics. The problem 1 above focuses on proving the local version of the merging without the strain or, possibly, with infinitely smooth strain.

Remark. The equation (2) is not well-posed. Indeed, one should at least assign some initial data and the boundary values \(\mu(t, -1)\) or \(\mu(t, 1)\).

The problem 1 seems hard. The important step in understanding it is to address the question of the stationary states for (2).

Problem 2. Find the family of even functions \(y(x, \lambda) \in C^1[-1, 1]\) such that
\[ \int_{-1}^1 (y'(x, \lambda) - y'(-x, \lambda)) K(x, -x) dx = 0 \] (3)

and
\[ y(0, \lambda) = \lambda, \quad \lambda \in (0, \lambda_0); \quad \|y(x, \lambda) - |x|\|_{C[-1, 1]} \to 0, \quad \lambda \to 0 \]

Quite naturally, we will call them “the even V–states for the model with cut-off”. Since the original problem of the patch evolution is invariant with respect to rotations, we expect the existence of other families of V–states that are not necessarily even.

The main reason we introduce the cut-off is that we want to have a singular stationary solution around which we will build the perturbative analysis. In the case of cut-off, this is \(y_0(x) = |x|\) which satisfies (3) for all \(D\). One can also look at the periodic case when the equation is considered on \(\mathbb{T}^2 = [\pi, \pi]^2\) (e.g., 2d Euler is a canonical example). Then, the “patch configuration” \(\theta(x, y) = \text{sgn} x \cdot \text{sgn} y\) is a stationary solution and one can expect the analysis developed in this paper to be applicable. For that model, no cut-off is needed. However, we don’t consider the periodic setting as the kernel of the Laplacian is easier to write down in \(\mathbb{R}^2\). Also, we do not expect the mathematics involved to be much different except for a few details concerning the injectivity of the linearized operator (see the theorem 4.1 below).

The main result of this paper is the following theorem which contains a solution to the problem 2 for the case of 2d Euler.

Theorem 1.1. There is a family of even functions \(y(x, \lambda) \in C^1[-1, 1]\) such that
\[ \int_{-1}^1 (y'(x, \lambda) - y'(-x, \lambda)) \log \left( \frac{(x + \xi)^2 + (y(x, \lambda) + y(-x, \lambda))^2}{(x - \xi)^2 + (y(x, \lambda) - y(-x, \lambda))^2} \right) d\xi = 0 \] (4)
\[ y(0, \lambda) = \lambda, \quad \lambda \in (0, \lambda_0); \quad \|y(x, \lambda) - |x|\|_{C[-1,1]} \to 0, \quad \lambda \to 0 \]

**Remark.** This result does not immediately imply any progress on problem 1. Meanwhile, the technique developed in the current paper might be useful in establishing the existence of self-similar blow-up solutions to problem 1 for other classes of active scalar equations (e.g., \(\alpha\)-patches where \(\alpha = 1 - \epsilon, \epsilon < 1\)). Moreover, if all stationary solutions are found (not just the even ones), then it is conceivable that some control in the value of \(y(t, -1)\) can allow us to “slowly move” \(y(t, x)\) from one \(V_{\lambda_1}\) to another \(V_{\lambda_2}\) dynamically thus making \(\lambda_2 < \lambda_1\) and thus solving problem 1 at least partially. We are planning to address these questions later.

The symbol \(\dot{C}^1[0,1]\) indicates the following space \(\dot{C}^1[0,1] = \{f \in C^1[0,1], f(0) = 0\}\) equipped with the standard \(C^1[0,1]\) norm. We will also use the following (non-standard) notation
\[
\log^+ x = |\log x| + 1, \quad x > 0
\]
Let \(\omega(x)\) be a smooth function such that \(\omega(x) = 1\) on \(|x| < 1/2\), \(\omega(x) = 0\) on \(|x| > 1\) and \(0 \leq \omega(x) \leq 1\). We will write \(a \ll 1\) as a short hand for “there is a sufficiently small \(a_0\) such that \(0 < a < a_0\)”. For a parameter \(0 < a \ll 1\), we consider \(\omega_a(x) = \omega(x/a)\) and \(\omega_a'(x) = 1 - \omega_a(x)\). Given two positive functions \(F_1\) and \(F_2\), we write \(F_1 \lesssim F_2\) if there is a constant \(C\) such that
\[
F_1 < CF_2, \quad C > 0
\]
We write \(F_1 \sim F_2\) if
\[
F_1 \lesssim F_2 \lesssim F_1.
\]

2. Preliminaries

The main result of this paper is solution to problem 2 in the case of 2d Euler equation with a cut-off. However, we want to do some preliminary calculations in the general case first.

Assume that \(y\) solves the problem 2, then
\[
y'(x, \lambda) \int_0^1 K_1(x, \xi) d\xi = \int_0^1 y'(\xi, \lambda) K_2(x, \xi) d\xi, \quad y(0, \lambda) = \lambda \quad (5)
\]
where
\[
K_1(x, \xi) = K(x, \xi) + K(x, -\xi) =
= H((y(x) + y(\xi))^2 + (x + \xi)^2) - H((y(x) - y(\xi))^2 + (x - \xi)^2) +
+ H((y(x) + y(\xi))^2 + (x - \xi)^2) - H((y(x) - y(\xi))^2 + (x + \xi)^2)
\]
and
\[
K_2(x, \xi) = K(x, \xi) - K(x, -\xi) =
= H((y(x) + y(\xi))^2 + (x + \xi)^2) - H((y(x) - y(\xi))^2 + (x - \xi)^2) -
- H((y(x) + y(\xi))^2 + (x - \xi)^2) + H((y(x) - y(\xi))^2 + (x + \xi)^2)
\]
We suppressed the dependence of \(y\) on \(\lambda\) and just write \(y(x)\).
2.1. The explicit solution for the model case. Let us go back to the equation (1). Instead of taking the singular kernels in the convolution, one can instead consider the smooth bump $D(z)$. The “typical” behavior around the origin then would be, e.g.,

$$D(z) = C + |z|^2 + o(|z|^2), \quad |z| \to 0$$

Keeping only the quadratic part, we get

$$K(x, \xi) = 4(y(x)y(\xi) + x\xi), \quad K_1(x, \xi) = 8y(x)y(\xi), \quad K_2(x, \xi) = 8x\xi$$

The equation (2) takes the following form

$$y'(x)y(x) \int_0^1 y(\xi)d\xi = x \int_0^1 \xi y'(\xi)d\xi$$

which easily integrates to

$$y(x) = \sqrt{\lambda^2 + \frac{B}{A}x^2}$$

where

$$A = \int_0^1 y(x)dx, \quad B = \int_0^1 xy'(x)dx$$

We have the following compatibility equations

$$\begin{align*}
B &= \sqrt{\lambda^2 + \frac{B}{A}x^2} - A \\
A &= \int_0^1 \sqrt{\lambda^2 + \frac{B}{A}x^2}dx
\end{align*}$$

Introduce

$$B/A = u, \quad AB = v$$

Then

$$v = \frac{u(\lambda^2 + u)}{(u + 1)^2}, \quad \sqrt{v} = \lambda^2 \int_0^{\lambda^{-1}\sqrt{BA}} \sqrt{1 + \xi^2}d\xi$$

We assume that $\lambda \in (0, \lambda_0), \lambda_0 \ll 1$ and $|u - 1| \ll 1$ and so $|v - 4| \ll 1$. Therefore, if $u = 1 + \alpha, \quad v = 1/4 + \beta, \quad \alpha, \beta \ll 1$

then

$$\beta = \alpha/4 + \lambda^2/4 + O(\alpha^2 + \lambda^2\alpha)$$

and

$$\alpha = 2\beta - \lambda^2 \log \frac{1}{\lambda} + O(\beta^2 + \lambda^2\alpha)$$

Thus,

$$\beta = -0.5\lambda^2 \log \frac{1}{\lambda} + \frac{\lambda^2}{2} + O(\lambda^4 \log^2 \lambda), \quad \alpha = -2\lambda^2 \log \frac{1}{\lambda} + \lambda^2 + O(\lambda^4 \log^2 \lambda)$$

This calculation shows that $V_\lambda$ exists and the asymptotics in $\lambda \to 0$ can be easily established.

Since

$$y(x) = \sqrt{\lambda^2 + (1 + \alpha)x^2}, \quad \alpha < 0$$
the curve will intersect the line \( y = x \) at the point
\[
x_{\lambda}^* = \frac{\lambda}{|\alpha|^{1/2}} = \left(0.5 \log \frac{1}{\lambda}\right)^{1/2}(1 + o(1))
\]

Let us address the question of self-similarity now. Rescale
\[
\mu(\hat{x}) = \lambda^{-1}y(\hat{x}\lambda) = \sqrt{1 + (1 + \alpha)\hat{x}^2}, \quad |\hat{x}| < \lambda^{-1}
\]
This shows that
\[
\sup_{|\hat{x}|<\lambda^{-1}} |\mu(\hat{x}) - \sqrt{1 + \hat{x}^2}| \to 0
\]
and so the self-similar behavior is global.

The model case we just presented is the situation in which the interaction is substantially long-range and the self-similarity of the stationary state is global. The curve that we have in the limit is hyperbola. That seems like a common feature of many long-range models and 2d Euler in particular as will be seen from the subsequent analysis. However, for 2d Euler this self-similarity will be proved only over \(|x| < C\lambda\) with arbitrary fixed \(C\). Notice also that the analogous calculation is possible if the smooth strains are imposed, e.g., a rotation.

2.2. Properties of the kernels \(K_1\) and \(K_2\). Below, we will write \(K_{1(2)}(x, \xi, y)\) when we want to emphasize the dependence of the kernel on the function \(y\).

**Lemma 2.1.** The following is true
\[
K_1(x, \xi, y) = 4y(x)y(\xi)(H'(\eta_1) + H'(\eta_2)), \quad K_2(x, \xi, y) = 4x\xi(H'(\alpha_1) + H'(\alpha_2))
\]
where
\[
\eta_1 > (x + \xi)^2, \quad \eta_2 > (x - \xi)^2
\]
and
\[
\alpha_{1(2)} > (x - \xi)^2
\]

**Proof.** Apply the mean value theorem to the first and second terms in the expression. This gives
\[
K_1 = \left(H((y(x) + y(\xi))^2 + (x + \xi)^2) - H((y(x) - y(\xi))^2 + (x + \xi)^2)\right)
\]
\[
+\left(H((y(x) + y(\xi))^2 + (x - \xi)^2) - H((y(x) - y(\xi))^2 + (x - \xi)^2)\right)
\]
and
\[
K_2 = \left(H((y(x) + y(\xi))^2 + (x + \xi)^2) - H((y(x) + y(\xi))^2 + (x - \xi)^2)\right)
\]
\[
+\left(H((y(x) - y(\xi))^2 + (x + \xi)^2) - H((y(x) - y(\xi))^2 + (x - \xi)^2)\right)
\]
For the case when $H = \log x$, we have the following

$$K_1(x, \xi, y) = \log \left( \frac{(x + \xi)^2 + (y(x) + y(\xi))^2}{(x - \xi)^2 + (y(x) - y(\xi))^2} \cdot \frac{(x - \xi)^2 + (y(x) + y(\xi))^2}{(x + \xi)^2 + (y(x) - y(\xi))^2} \right)$$

Then, assuming that $y(x) \geq 0$,

$$\frac{(x - \xi)^2 + (y(x) + y(\xi))^2}{(x - \xi)^2 + (y(x) - y(\xi))^2} = 1 + \frac{4y(x)y(\xi)}{(x - \xi)^2 + (y(x) - y(\xi))^2} \geq 1$$

and

$$\frac{(x + \xi)^2 + (y(x) + y(\xi))^2}{(x + \xi)^2 + (y(x) - y(\xi))^2} = 1 + \frac{4y(x)y(\xi)}{(x + \xi)^2 + (y(x) - y(\xi))^2} \geq 1$$

Therefore, we have

$$\log \left( 1 + \frac{4y(x)y(\xi)}{(x - \xi)^2 + (y(x) - y(\xi))^2} \right) \lesssim K_1 \lesssim \log \left( 1 + \frac{4y(x)y(\xi)}{(x - \xi)^2} \right)$$

provided that $y \geq 0$. Similarly, for $K_2$,

$$\log \left( 1 + \frac{4x\xi}{(x - \xi)^2 + (y(x) - y(\xi))^2} \right) \leq K_2 = \log \left( 1 + \frac{4x\xi}{(x - \xi)^2 + (y(x) + y(\xi))^2} \right)$$

$$+ \log \left( 1 + \frac{4x\xi}{(x - \xi)^2 + (y(x) - y(\xi))^2} \right) \lesssim \log \left( 1 + \frac{4x\xi}{(x - \xi)^2} \right)$$

and this holds for all $y$.

The following lemma is trivial.

**Lemma 2.2.** Let $0 \leq a \leq b \leq 10$. Then, $b \sim a + b$ and

$$\frac{1}{b - a} \int_a^b \frac{d\eta}{\eta} \sim O(b^{-1}) \sim O((a + b)^{-1}), \quad a > b/2$$

and

$$\frac{1}{b - a} \int_a^b \frac{d\eta}{\eta} \sim O((a + b)^{-1} \log^+ |a + b|), \quad a < b/2$$

Suppose that $y \in [0, 2]$. Then, applying this lemma to $K_1$ with

$$a = (y(x) - y(\xi))^2 + (x + \xi)^2, \quad b = (y(x) + y(\xi))^2 + (x + \xi)^2$$

and then with

$$a = (y(x) - y(\xi))^2 + (x - \xi)^2, \quad b = (y(x) + y(\xi))^2 + (x - \xi)^2$$

gives

$$\frac{|y(x)y(\xi)|}{y^2(x) + y^2(\xi) + (x - \xi)^2} \lesssim K_1 \lesssim \frac{|y(x)y(\xi)|}{y^2(x) + y^2(\xi) + (x - \xi)^2} \log^+ \left( \frac{(x - \xi)^2 + y^2(x) + y^2(\xi)}{y^2(x) + y^2(\xi) + (x - \xi)^2} \right) \quad (6)$$

For $K_2$, the same reasoning yields

$$\frac{x\xi}{x^2 + \xi^2 + (y(x) - y(\xi))^2} \lesssim K_2 \lesssim \frac{x\xi}{x^2 + \xi^2 + (y(x) - y(\xi))^2} \log^+ \left( \frac{x^2 + \xi^2 + (y(x) - y(\xi))^2}{x^2 + \xi^2 + (y(x) - y(\xi))^2} \right) \leq \frac{x\xi \log^+ (x^2 + \xi^2)}{x^2 + \xi^2}$$
3. The implicit function theorem, the 2d Euler case

In this section, we will apply the scheme of the implicit function theorem to the 2d Euler with cut-off which corresponds to \( H(x) = \log x \). However, we first notice that the problem allows the following scaling.

**Lemma 3.1.** If \( y(x) \) solves
\[
\int_{-1}^{1} y'(x) \log \left( \frac{(x + \xi)^2 + (y(x) + y(\xi))^2}{(x - \xi)^2 + (y(x) - y(\xi))^2} \right) \, d\xi = \int_{-1}^{1} y'(\xi) \log \left( \frac{(x + \xi)^2 + (y(x) + y(\xi))^2}{(x - \xi)^2 + (y(x) - y(\xi))^2} \right) \, d\xi
\]
then \( y_\alpha(x) = \alpha y(x/\alpha) \) solves
\[
\int_{-\alpha}^{\alpha} y_\alpha'(x) \log \left( \frac{(x + \xi)^2 + (y_\alpha(x) + y_\alpha(\xi))^2}{(x - \xi)^2 + (y_\alpha(x) - y_\alpha(\xi))^2} \right) \, d\xi = \int_{-\alpha}^{\alpha} y_\alpha'(\xi) \log \left( \frac{(x + \xi)^2 + (y_\alpha(x) + y_\alpha(\xi))^2}{(x - \xi)^2 + (y_\alpha(x) - y_\alpha(\xi))^2} \right) \, d\xi
\]
for every \( \alpha > 0 \).

**Proof.** This is an immediate calculation. \( \square \)

Consider \( y_\lambda(x) \) and take
\[
\hat{y}(\hat{x}, \lambda) = \lambda^{-1} y(\hat{x} \lambda, \lambda), \quad |\hat{x}| < \lambda^{-1}
\]
We will perform this scaling many times in the paper. It allows to reduce the problem to the one on the large interval \( |\hat{x}| < \lambda^{-1} \) with the normalization \( \hat{y}(0, \lambda) = 1 \). The perturbative analysis that follows will be done around the hyperbola \( \hat{y}(\hat{x}) = \sqrt{x^2 + 1} \), not \( |\hat{x}| \).

**Remark 2.** The model case suggests that \( \{\hat{y}(\hat{x}, \lambda)\} \) might have some limiting behavior as \( \lambda \to 0 \). If so, can one guess the asymptotical curve?

Here, let us make very natural assumptions that
\[
\hat{y}(\hat{x}, \lambda) \to f(\hat{x}), \quad \hat{y}'(\hat{x}, \lambda) \to f'(\hat{x})
\]
on \( |\hat{x}| < C \) and that
\[
\hat{y}(\hat{x}, \lambda) = \hat{x}(1 + o(1)), \quad \hat{y}'(\hat{x}, \lambda) = 1 + o(1), \quad |\hat{x}| \gg 1
\]
uniformly in \( \lambda \). For \( |\hat{x}| < C \),
\[
(f'(\hat{x}) + o(1)) \int_{0}^{1/\lambda} \left[ \log \left( 1 + \frac{4\hat{y}(\hat{x}, \lambda)\hat{y}(\hat{\xi}, \lambda)}{\hat{x} - \hat{\xi} + (\hat{y}(\hat{x}, \lambda) - \hat{y}(\hat{\xi}, \lambda))^2} \right) \right. \\
\left. + \log \left( 1 + \frac{4\hat{y}(\hat{x}, \lambda)\hat{y}(\hat{\xi}, \lambda)}{(\hat{x} + \hat{\xi})^2 + (\hat{y}(\hat{x}, \lambda) - \hat{y}(\hat{\xi}, \lambda))^2} \right) \right] \, d\hat{\xi}
\]

\[
= \int_{0}^{1/\lambda} (1 + o(1)) \left[ \log \left( 1 + \frac{4\hat{x}\hat{\xi}}{(\hat{x} - \hat{\xi})^2 + (\hat{y}(\hat{x}, \lambda) - \hat{y}(\hat{\xi}, \lambda))^2} \right) \\
\right. \\
\left. + \log \left( 1 + \frac{4\hat{x}\hat{\xi}}{(\hat{x} + \hat{\xi})^2 + (\hat{y}(\hat{x}, \lambda) - \hat{y}(\hat{\xi}, \lambda))^2} \right) \right] \, d\hat{\xi}
\]
For the l.h.s., the asymptotics of the integrand as $\xi \to \infty$ is
\[
\frac{4\hat{y}(\hat{x}, \lambda)}{\xi} + o(\xi^{-1})
\]
and for the r.h.s., it is
\[
\frac{4\hat{x}}{\xi} + o(\xi^{-1})
\]
Here we assumed that $|\hat{x}| < C$. Taking $\lambda \to 0$, we get
\[
(f'f - \hat{x}) \log 1/\lambda + o(\log 1/\lambda) = 0
\]
This leads to
\[
f(\hat{x}) = (\hat{x}^2 + 1)^{1/2}
\]
This is obtained under strong assumptions so does not imply the self-similarity per se. However, one can take
\[
\tilde{y}(x, \lambda) = (x^2 + \lambda^2)^{1/2}
\]
as an approximate solution. Plugging it into the equation, one can represent the resulting correction as the strain. Similarly to [3], one can show that this strain satisfies the uniform bound
\[
\sup_{|z|<1, \lambda \in (0, 1)} \frac{|S(z, \lambda)|}{|z|} < C
\]
The novelty of this paper is that we will construct the exact solution and thus make $S(z, \lambda) = 0$. It will also be proved that the exact solutions give the hyperbola in the scaling limit but only locally, over $x \in I_\lambda$, where $|I_\lambda| \to 0, \lambda \to 0$.

In the lemma below, we show that all possible solutions $y(x, \lambda)$ have the following common feature.

**Lemma 3.2.** If $y(x)$ solves (5), then there is $x^* \in (0, 1)$ at which $y(x^*) = x^*$. That is, the graph of $y(x)$ intersects the line $y = x$.

**Proof.** Suppose instead that $y(x) > x$ for all $x \in (0, 1)$. Then,
\[
\frac{4y(x)y(\xi)}{(x - \xi)^2 + (y(x) - y(\xi))^2} > \frac{4x\xi}{(x - \xi)^2 + (y(x) - y(\xi))^2}
\]
and
\[
\frac{4y(x)y(\xi)}{(x + \xi)^2 + (y(x) - y(\xi))^2} > \frac{4x\xi}{(x - \xi)^2 + (y(x) + y(\xi))^2}
\]
Therefore, $K_1(x, \xi) > K_2(x, \xi) > 0$. Now, assume that
\[
\max_{x \in [0, 1]} y'(x) = y'(x_1)
\]
Then,
\[
\int_0^1 y'(\xi)K_1(x_1, \xi)d\xi \leq y'(x_1) \int_0^1 K_1(x_1, \xi)d\xi = \int_0^1 y'(\xi)K_2(x_1, \xi)d\xi
\]
and this inequality is strict unless $y'(x) = 0$. \qed
Now that we established what properties the solution $y(x, \lambda)$ needs to possess, we are ready to prove its existence.

Consider small $\delta_1(2) > 0$ and the sets

$$
\Omega = \{ f : \| f(x) - x \|_{C^1[0,1]} < \delta_1 \}, \quad I = \{ \lambda : \lambda \in [0, \lambda_0), \lambda_0 \ll 1 \} \n$$

We will look for $y = \sqrt{\lambda^2 + f^2(x)}$, where $(f, \lambda) \in \Omega \times I$. Notice that $f(x) = \int_0^x f'(t) dt$ and $\| f' - 1 \|_{C[0,1]} \ll 1$. Therefore,

$$
f(x) = x(1 + O(\delta_1))
$$

In particular, $f(x) > 0$ for $x > 0$.

Consider the functional (we specify the dependence of $K$ on $y$ here)

$$
F(f, \lambda) = \frac{f f' \int_0^1 K_1(x, \tau, y) d\tau - \sqrt{\lambda^2 + f^2(x)} \int_0^1 y'(\tau) K_2(x, \tau, y) d\tau}{x \sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2)}
$$

which acts from $\Omega \times I$ to $C[0,1]$. Moreover, $F(x, 0) = 0$.

**Remark.** Here and everywhere in the paper, we define the functions at zero by their limiting values to make them continuous on all of the interval $[0, 1]$.

The equation (5) can be rewritten as

$$
F(f, \lambda) = 0
$$

We will solve it in the following way (this is the inverse function theorem argument). Write

$$
F(f, \lambda) = F(x, \lambda) + \left( D_f F(x, \lambda) \right) \psi + Q(\psi)
$$

where $\psi = f - x$ and this representation defines an operator $Q$. Then, the equation can be rewritten as

$$
\psi = -\left( D_f F(x, \lambda) \right)^{-1} Q(\psi) + \psi_0(\lambda), \quad \psi_0 = -\left( D_f F(x, \lambda) \right)^{-1} F(x, \lambda) \quad (7)
$$

Next, we will show that this equation can be solved by contraction mapping principle in $B_\delta = \{ \| \psi \|_{C^1[0,1]} \leq \delta \}$, $\delta \ll 1$. To this end, we only need to prove:

(a) Linear part:

$$
\| (D_f F(x, \lambda))^{-1} \|_{C[0,1], C^1[0,1]} < C \quad (8)
$$

if $\lambda \in (0, \lambda_1)$ with $\lambda_1 \ll 1$.

(b) Frechet differentiability:

$$
\| Q(\psi) \|_{C[0,1]} = o(1) \| \psi \|_{C^1[0,1]} \quad (9)
$$

and

$$
\| Q(\psi_2) - Q(\psi_1) \|_{C[0,1]} = o(1) \| \psi_2 - \psi_1 \|_{C^1[0,1]} \quad (10)
$$

as $\delta \to 0$; uniformly in $\lambda \in (0, 1)$ provided that $\psi, \psi_1(2) \in B_\delta$. 


(c) Small initial data:
\[ \|\psi_0(\lambda)\|_{\dot{C}^1[0,1]} < \delta/2 \]
where \( \lambda \in [0, \lambda_0], \lambda_0 < \lambda_1. \)

We will first make \( \lambda_1 \) so small that (a) holds. Then, we choose \( \delta \) small enough to have \( o(1) \) in (b) at most \( (10C)^{-1}\delta \) uniformly in \( \lambda \in (0,1) \). Finally, we take \( \lambda_0 \) so small that (c) hold. This will ensure existence and uniqueness of solution in the corresponding complete metric space \( B_\delta \). Its continuous dependence on \( \lambda \) and
\[ \|y(x, \lambda) - x\|_{C^\prime[0,1]} \to 0, \quad \lambda \to 0 \]
will follow from the proof. Notice also that (b) means the Frechet differentiability of \( F \) and the corresponding iterative method will actually converge super-exponentially fast.

4. The analysis of Gateaux derivative for \( H(x) = \log x \)

Taking \( f_t = f + tu, u \in \dot{C}^1[0,1] \), plugging it into \( F \), and computing the derivative in \( t \) at \( t = 0 \) with positive \( x \) fixed, results in
\[ (D_f F(f, \lambda))u = \frac{1}{x\sqrt{x^2 + \lambda^2}\log^+(x^2 + \lambda^2)} (I_1 + \ldots + I_6) \]

We have
\[ I_1 = \left( f' \int_0^1 K_1(x, \tau, y)d\tau \right) u, \quad y = \sqrt{\lambda^2 + f^2} \]
\[ I_2 = \left( f \int_0^1 K_1(x, \tau, y)d\tau \right) u' \]
\[ I_3 = f f' \int_0^1 \delta K_1(x, \tau, y)d\tau \]
where
\[ \delta K_1 = \frac{2(y(x) + y(\xi))(\delta y(x) + \delta y(\xi))}{(x + \xi)^2 + (y(x) + y(\xi))^2} \quad - \frac{2(y(x) - y(\xi))(\delta y(x) - \delta y(\xi))}{(x - \xi)^2 + (y(x) - y(\xi))^2} \]
\[ + \frac{2(y(x) + y(\xi))(\delta y(x) + \delta y(\xi))}{(x - \xi)^2 + (y(x) + y(\xi))^2} \quad - \frac{2(y(x) - y(\xi))(\delta y(x) - \delta y(\xi))}{(x + \xi)^2 + (y(x) - y(\xi))^2} \]

and
\[ \delta y = f u \quad \sqrt{\lambda^2 + f^2} \]
\[ I_4 = - \left( \frac{f}{\sqrt{\lambda^2 + f^2}} \int_0^1 y'K_2(x, \tau, y)d\tau \right) u \]
\[ I_5 = - \sqrt{\lambda^2 + f^2} \int_0^1 \delta y'K_2(x, \tau, y)d\tau \]
where
\[ \delta y' = \frac{f'}{\sqrt{\lambda^2 + f^2}} u + \frac{f}{\sqrt{\lambda^2 + f^2}} u' - \frac{f^2 f' u}{(\lambda^2 + f^2)^{3/2}} \]
\[ I_6 = -\sqrt{\lambda^2 + f^2} \int_0^1 y'(\tau)\delta K_2(x, \tau, y) d\tau \]

where
\[
\delta K_2 = \frac{2(y(x) + y(\xi))(\delta y(x) + \delta y(\xi))}{(x + \xi)^2 + (y(x) + y(\xi))^2} - \frac{2(y(x) - y(\xi))(\delta y(x) - \delta y(\xi))}{(x - \xi)^2 + (y(x) - y(\xi))^2} - \frac{2(y(x) + y(\xi))(\delta y(x) + \delta y(\xi))}{(x - \xi)^2 + (y(x) + y(\xi))^2} + \frac{2(y(x) - y(\xi))(\delta y(x) - \delta y(\xi))}{(x + \xi)^2 + (y(x) - y(\xi))^2}\]

4.1. **The derivative at** \(x\). Define \(L_\lambda = (D_f F)(x, \lambda)\). If \(f = x\) in the previous section, then

\[
L_\lambda u = \frac{1}{x\sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2)} \left( \hat{I}_{1,\lambda} + \ldots + \hat{I}_{6,\lambda} \right)
\]

We again have

\[
\hat{I}_{1,\lambda} = \left( \int_0^1 K_1(x, \tau, y_\lambda) d\tau \right) u
\]

with

\[
y_\lambda(x) = \sqrt{\lambda^2 + x^2}
\]

\[
\hat{I}_{2,\lambda} = x \left( \int_0^1 K_1(x, \tau, y_\lambda) d\tau \right) u'
\]

\[
\hat{I}_{3,\lambda} = x \int_0^1 \delta K_1(x, \tau, y_\lambda) d\tau
\]

where

\[
\delta K_1 = \frac{2(y_\lambda(x) + y_\lambda(\xi))(\delta y_\lambda(x) + \delta y_\lambda(\xi))}{(x + \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} - \frac{2(y_\lambda(x) - y_\lambda(\xi))(\delta y_\lambda(x) - \delta y_\lambda(\xi))}{(x - \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} + \frac{2(y_\lambda(x) + y_\lambda(\xi))(\delta y_\lambda(x) + \delta y_\lambda(\xi))}{(x - \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} - \frac{2(y_\lambda(x) - y_\lambda(\xi))(\delta y_\lambda(x) - \delta y_\lambda(\xi))}{(x + \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2}
\]

and

\[
\delta y_\lambda = \frac{x}{\sqrt{\lambda^2 + x^2}} u
\]

\[
\hat{I}_{4,\lambda} = - \left( \frac{x}{\sqrt{\lambda^2 + x^2}} \int_0^1 y'_\lambda K_2(x, \tau, y_\lambda) d\tau \right) u
\]

\[
\hat{I}_{5,\lambda} = -\sqrt{\lambda^2 + x^2} \int_0^1 \delta y'_\lambda K_2(x, \tau, y_\lambda) d\tau
\]

where

\[
\delta y'_\lambda = \frac{1}{\sqrt{\lambda^2 + x^2}} u + \frac{x}{\sqrt{\lambda^2 + x^2}} u' - \frac{x^2}{(\lambda^2 + x^2)^{3/2}} u
\]

\[
\hat{I}_{6,\lambda} = -\sqrt{\lambda^2 + x^2} \int_0^1 y'_\lambda(\tau)\delta K_2(x, \tau, y_\lambda) d\tau
\]
and
\[ \delta K_2 = \frac{2(y_\lambda(x) + y_\lambda(\xi))(\delta y_\lambda(x) + \delta y_\lambda(\xi))}{(x + \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} - \frac{2(y_\lambda(x) - y_\lambda(\xi))(\delta y_\lambda(x) - \delta y_\lambda(\xi))}{(x - \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} \]
\[ - \frac{2(y_\lambda(x) + y_\lambda(\xi))(\delta y_\lambda(x) + \delta y_\lambda(\xi))}{(x - \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} + \frac{2(y_\lambda(x) - y_\lambda(\xi))(\delta y_\lambda(x) - \delta y_\lambda(\xi))}{(x + \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} \]

4.2. **The operator** $L_\lambda$. For $L_\lambda$, we have the following formula
\[ L_\lambda = A_1 u' + A_2 u + \int_0^1 D_1(x, \xi, \lambda)u(\xi)d\xi + \int_0^1 D_2(x, \xi, \lambda)u'(\xi)d\xi \]
The equation
\[ L_\lambda u = g \]
can be rewritten as
\[ A_1(x, \lambda)u' + A_2(x, \lambda)u + \int_0^1 M(x, \xi, \lambda)u'(\xi)d\xi = g \]
if one assumes $u(0) = 0$ and
\[ M(x, \xi, \lambda) = D_2(x, \xi, \lambda) + \int_\xi^1 D_1(x, \tau, \lambda)d\tau \]
In the calculation above, the substitution
\[ u(0) \int_0^1 D_1(0, \tau, \lambda)d\tau = 0 \]
as follows from the estimate $|u(x)| \lesssim x$ and from the analysis of
\[ \int_0^1 D_1(x, \tau, \lambda)d\tau \]
when $x \to 0$ (see (35) below).
Let us introduce the integral operator $M_\lambda$ with the kernel $M(x, \tau, \lambda)$, e.g.,
\[ M_\lambda f = \int_0^1 M(x, \tau, \lambda)f(\tau)d\tau \]
For the coefficients, we have
\[ A_1 = \frac{\int_0^1 K_1(x, \tau, y_\lambda)d\tau}{\sqrt{x^2 + \lambda^2 \log^+(x^2 + \lambda^2)}} \]
The expression for $A_2$ is more complicated,
\[ A_2 = \frac{1}{x\sqrt{x^2 + \lambda^2 \log^+(x^2 + \lambda^2)}} \left( \int_0^1 K_1(x, \tau, y_\lambda)d\tau - \frac{x}{\sqrt{\lambda^2 + x^2}} \int_0^1 y'_\lambda(\tau)K_2(x, \tau, y_\lambda)d\tau + B_2 \right) \]
where
\[ B_2 = \frac{2x}{\sqrt{x^2 + \lambda^2}} \int_0^1 \left( x - \frac{\xi \sqrt{\lambda^2 + x^2}}{\sqrt{\lambda^2 + \xi^2}} \right) \left( \frac{y_\lambda(x) + y_\lambda(\xi)}{(x + \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} - \frac{y_\lambda(x) - y_\lambda(\xi)}{(x - \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} \right) d\xi \]
\[ + \frac{2x}{\sqrt{x^2 + \lambda^2}} \int_0^1 \left( x + \frac{\xi \sqrt{\lambda^2 + x^2}}{\sqrt{\lambda^2 + \xi^2}} \right) \left( \frac{y_\lambda(x) + y_\lambda(\xi)}{(x + \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} - \frac{y_\lambda(x) - y_\lambda(\xi)}{(x - \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} \right) d\xi \]

For \( D_{1(2)} \), one has
\[ D_2(x, \xi, \lambda) = -\frac{1}{x \log^+(x^2 + \lambda^2)} K_2(x, \xi, y_\lambda) \frac{\xi}{\sqrt{\lambda^2 + \xi^2}} \]
and
\[ D_1(x, \xi, \lambda) = \frac{1}{x \sqrt{\lambda^2 + x^2} \log^+(x^2 + \lambda^2)} \left[ \frac{2x \xi}{\sqrt{\lambda^2 + \xi^2}} \left( \frac{y_\lambda(x) + y_\lambda(\xi)}{(x + \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} - \frac{y_\lambda(x) - y_\lambda(\xi)}{(x - \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} \right) \right. \]
\[ - \frac{2\xi^2 \sqrt{\lambda^2 + x^2}}{\xi^2 + \lambda^2} \left( \frac{y_\lambda(x) + y_\lambda(\xi)}{(x + \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} - \frac{y_\lambda(x) - y_\lambda(\xi)}{(x - \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} \right) \]
\[ + \frac{y_\lambda(x) - y_\lambda(\xi)}{(x - \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} - \frac{y_\lambda(x) + y_\lambda(\xi)}{(x - \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} - \frac{y_\lambda(x) - y_\lambda(\xi)}{(x + \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} - \frac{y_\lambda(x) + y_\lambda(\xi)}{(x + \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} \]
\[ - \frac{\lambda^2 \sqrt{\lambda^2 + x^2}}{(\lambda^2 + \xi^2)^{3/2}} K_2(x, \xi, y_\lambda) \]

In this section, we will obtain estimates/asymptotics of all four terms in the case when \( \lambda \to 0 \). It will be trivial to do that away from 0: e.g., for every \( \delta > 0 \) both \( A_{1(2)}(\lambda) \to A_{1(2)}(0) \) uniformly over \( x \in [\delta, 1] \). The behavior around 0 is delicate and will require more careful treatment.

We start with the following calculation that will simplify the expressions above.
We write
\[ \sqrt{x^2 + 1} - \sqrt{\xi^2 + 1} = (\hat{x} - \hat{\xi}) r_1(x, \xi) \quad (15) \]
where
\[ r_1 = \frac{\hat{x} + \hat{\xi}}{\sqrt{x^2 + 1} + \sqrt{\xi^2 + 1}} = 1 + O \left( \frac{1}{\hat{x} \hat{\xi}} \right), \quad \text{if} \quad \hat{x}, \hat{\xi} \gg 1 \]
Similarly,
\[ \sqrt{x^2 + 1} + \sqrt{\xi^2 + 1} = (\hat{x} + \hat{\xi}) r_1^{-1} \quad (16) \]
Thus, we have for $K_2$

\[
\frac{(\hat{x} + \hat{\xi})^2 + (\sqrt{x^2 + 1} + \sqrt{\xi^2 + 1})^2}{(\hat{x} - \hat{\xi})^2 + (\sqrt{x^2 + 1} - \sqrt{\xi^2 + 1})^2} \cdot \frac{(\hat{x} + \hat{\xi})^2 + (\sqrt{x^2 + 1} - \sqrt{\xi^2 + 1})^2}{(\hat{x} - \hat{\xi})^2 + (\sqrt{x^2 + 1} + \sqrt{\xi^2 + 1})^2} = \frac{(\hat{x} + \hat{\xi})^2}{(\hat{x} - \hat{\xi})^2} (1 + r_1^{-2}) (\hat{x} + \xi)^2 + (\sqrt{x^2 + 1} + \sqrt{\xi^2 + 1})^2 (\hat{x} - \xi) \frac{\lambda}{(\hat{x} - \xi)^2} f^{-4}
\]

after the unexpected cancelation.

Similarly, for $K_1$

\[
\frac{(\hat{x} + \hat{\xi})^2 + (\sqrt{x^2 + 1} + \sqrt{\xi^2 + 1})^2}{(\hat{x} - \hat{\xi})^2 + (\sqrt{x^2 + 1} - \sqrt{\xi^2 + 1})^2} \cdot \frac{(\hat{x} + \hat{\xi})^2 + (\sqrt{x^2 + 1} - \sqrt{\xi^2 + 1})^2}{(\hat{x} - \hat{\xi})^2 + (\sqrt{x^2 + 1} + \sqrt{\xi^2 + 1})^2} = \frac{(\hat{x} + \hat{\xi})^2}{(\hat{x} - \hat{\xi})^2} r_1^{-4}
\]

Therefore, we have

\[
K_2(x, \tau, y_\lambda) = K_2(x, \tau, y_0)
\]

and

\[
K_1(x, \tau, y_\lambda) = K_1(x, \tau, y_0) - 4 \log r_1
\]

Now, we are ready for the analysis of the asymptotics for the coefficients in $L_\lambda$.

1. The coefficient $A_1$.

Consider $A_1(x, 0)$ first. We have

\[
A_1(x, 0) = \frac{1}{x \log^+(x^2)} \int_0^1 \log \left(\frac{x + \xi}{x - \xi}\right)^2 d\xi = \frac{1}{\log^+(x^2)} \int_0^{1/x} \log \left(\frac{1 + u}{1 - u}\right)^2 \, du = 2 + o(1), \quad x \to 0
\]

and it is smooth in $(0, 1)$.

**Lemma 4.1.** We have

\[
\lim_{\lambda \to 0} \|A_1(x, \lambda) - A_1(x, 0)\|_{C[0,1]} = 0
\]

**Proof.** If $x = \lambda \tilde{x}$, then

\[
\lambda \int_0^{1/\lambda} \log \left(\frac{(\tilde{x} + \tilde{\xi})^2 + (\sqrt{\tilde{x}^2 + 1} + \sqrt{\tilde{\xi}^2 + 1})^2}{(\tilde{x} - \tilde{\xi})^2 + (\sqrt{\tilde{x}^2 + 1} - \sqrt{\tilde{\xi}^2 + 1})^2} \cdot \frac{(\tilde{x} + \tilde{\xi})^2 + (\sqrt{\tilde{x}^2 + 1} + \sqrt{\tilde{\xi}^2 + 1})^2}{(\tilde{x} - \tilde{\xi})^2 + (\sqrt{\tilde{x}^2 + 1} - \sqrt{\tilde{\xi}^2 + 1})^2} d\tilde{\xi}
\]

We have several regimes:
1. $\hat{x} \in [0, 1]$. Then, by computing the asymptotics as $\hat{\xi} \to \infty$, we get

$$
\lambda \int_0^{1/\lambda} \log \left( \frac{\hat{x} + \hat{\xi}}{\hat{x} - \hat{\xi}} \cdot \frac{\hat{x} + \hat{\xi} + \sqrt{x^2 + 1 + \hat{\xi}^2 + 1}}{\hat{x} - \hat{\xi} + \sqrt{x^2 + 1 - \hat{\xi}^2 + 1}} \right) d\hat{\xi}
$$

$$
= 4\lambda \sqrt{x^2 + 1} \log(1/\lambda) + O(\lambda) = 4\sqrt{x^2 + \lambda^2} \log(1/\lambda) + O(\lambda) = 2\sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2) + O(\lambda)
$$

(23)

Given any $\delta \in (0, 1)$, we have:

2. $x \in (\delta, 1)$. We trivially have

$$
\int_0^1 K_1(x, \tau, y_\lambda) d\tau \to \int_0^1 K_1(x, \tau, y_0) d\tau, \quad \lambda \to 0
$$

3. $\lambda < x < \delta$. We substitute (16) to get

$$
\int_0^1 K_1(x, \tau, y_\lambda) d\tau = 4x \log(1/x) + 4\lambda(\sqrt{1 + \hat{x}^2 - \hat{\xi}}) \log(1 + x^{-1}) + O(\lambda)
$$

$$
= 4 \log(1/x)(x + \lambda \sqrt{1 + \hat{x}^2} - \lambda \hat{x}) + O(\lambda)
$$

$$
= 2 \sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2) + O(\lambda)
$$

(24)

Thus, we have

$$
\frac{\int_0^1 K_1(x, \tau, y_\lambda) d\tau}{\sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2)} \to \frac{\int_0^1 K_1(x, \tau, y_0) d\tau}{x \log^+ x^2}, \quad \lambda \to 0
$$

uniformly in $x \in [0, 1]$.

Later, we will need the following result

**Lemma 4.2.** Suppose $\|g(x) - x\|_{C^1[0,1]} \leq \delta_1 \ll 1$. Then,

$$
\left| \frac{\int_0^1 K_1(x, \tau, \sqrt{\lambda^2 + g^2(\tau)}) d\tau}{\sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2)} \right| \lesssim 1
$$

uniformly in $x$, $\lambda$, and $g$.

The proof repeats the argument in the previous lemma (see also the proof of lemma 6.2 below to check how the problems can be reduced to the homogeneous one for which the scaling can be easily performed to get the desired bound). This result can also be obtained by comparing to the case $g = x$ and using the stability estimates established in lemma 6.1 below.

2. The coefficient $A_2$. 

16
Lemma 4.3. For every fixed $\delta > 0$, we have
\[ A_2(x, \lambda) \to A_2(x, 0) = \frac{2 \log(x^{-2} + 1)}{x \log^+(x^2)}, \quad \lambda \to 0 \] (25)
uniformly over $x \in [\delta, 1]$.

Moreover, we have an estimate
\[ A_2(x, \lambda) \sim \frac{1}{x} \] (26)
which holds uniformly in $x$ and $\lambda$.

Proof. The expression for $A_2(x, 0)$ is easy to compute and the first part of the lemma is immediate.

The expression for $A_2$ contains three terms. The first one involves $K_1$ and its asymptotics was established before. Consider the second term now. By (15), we get
\[ \int_0^1 y_\lambda' K_2(x, \tau, y_\lambda) d\tau = \lambda \int_0^{1/\lambda} \frac{\hat{\xi}}{\sqrt{\hat{\xi}^2 + 1}} \log \left( \frac{x + \hat{\xi}}{x - \hat{\xi}} \right)^2 d\hat{\xi} \]
The similar analysis yields:
1. Uniformly in $x \in (\delta, 1]$, we get
\[ \int_0^1 y_\lambda' K_2(x, \tau, y_\lambda) d\tau \to \int_0^1 K_2(x, \tau, y_0) d\tau, \quad \lambda \to 0 \] (27)
2. If $\widehat{x} \in [0, 1]$, then we can split the integral into two:
\[ \int_0^1 \frac{\hat{\xi}}{\sqrt{\hat{\xi}^2 + 1}} \log \left( \frac{x + \hat{\xi}}{x - \hat{\xi}} \right)^2 d\hat{\xi} \]
We have
\[ \int_0^1 \hat{\xi} \log \left( 1 + \frac{2 \hat{\xi} x}{(\hat{\xi} - x)^2} \right) d\hat{\xi} = x^2 \int_0^{x^{-1}} t \log \left( 1 + \frac{2t}{(1-t)^2} \right) dt \sim \widehat{x} \]
So, the integration over $[0, 1]$ amounts to $O(x)$ after multiplication by $\lambda$.

For the integral over $[1, \lambda^{-1}]$, we get
\[ \int_1^{1/\lambda} \frac{\hat{\xi}}{\sqrt{\hat{\xi}^2 + 1}} \log \left( \frac{x + \hat{\xi}}{x - \hat{\xi}} \right)^2 d\hat{\xi} = 4\widehat{x} \log(1/\lambda) + O(\widehat{x}) \]
Multiplying by $\lambda$, we get
\[ \int_0^1 y_\lambda' K_2(x, \tau, y_\lambda) d\tau = x \left( 4 \log(1/\lambda) + O(1) \right) \]
3. If $x \in (\lambda, \delta)$, then the integral over $[0, 1]$ can be handled as before and its contribution is at most $\widehat{x}^{-1}$. The integral over $[1, 1/\lambda]$ gives
\[ \int_1^{1/\lambda} \left( 1 + O(\hat{\xi}^{-2}) \right) \log \left( \frac{x + \hat{\xi}}{x - \hat{\xi}} \right)^2 d\hat{\xi} = \widehat{x} \int_{1/\widehat{x}}^{1/x} \left( 1 + \frac{1}{2t^2} \right) \log \left( \frac{1+t}{1-t} \right)^2 dt = 4\widehat{x} \log(1/x) + O(1) \]
and we have
\[
\int_0^1 y'_\lambda K_2(x, \tau, y_\lambda) d\tau = 4x (\log(1/x) + O(1)), \quad \lambda \to 0
\]
Summarizing, we get the uniform bound
\[
\int_0^1 y'_\lambda K_2(x, \tau, y_\lambda) d\tau = \begin{cases} 
4x (\log(1/\lambda) + O(1)), & x < \lambda \\
4x (\log(1/x) + O(1)), & x > \lambda 
\end{cases}
\quad \text{(28)}
\]
as well as (27).
For the third term in the expression for $A_2$, we have
\[
B_2 = B_2^{(1)} + B_2^{(2)}
\]
\[
\begin{align*}
B_2^{(1)} &= \frac{2x}{\sqrt{x^2 + \lambda^2}} \int_0^1 \left( x - \frac{\xi \sqrt{x^2 + x^2}}{\sqrt{x^2 + \xi^2}} \right) \left( \frac{y_\lambda(x) + y_\lambda(\xi)}{(x + \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} \\
&\quad - \frac{y_\lambda(x) - y_\lambda(\xi)}{(x - \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} \right) d\xi \\
B_2^{(2)} &= \frac{2x}{\sqrt{x^2 + \lambda^2}} \int_0^1 \left( x + \frac{\xi \sqrt{x^2 + x^2}}{\sqrt{x^2 + \xi^2}} \right) \left( \frac{y_\lambda(x) + y_\lambda(\xi)}{(x - \xi)^2 + (y_\lambda(x) + y_\lambda(\xi))^2} \\
&\quad - \frac{y_\lambda(x) - y_\lambda(\xi)}{(x + \xi)^2 + (y_\lambda(x) - y_\lambda(\xi))^2} \right) d\xi
\end{align*}
\]
Rescale the variables and recall the formulas (15) and (16).
One gets
\[
B_2^{(1)} = -\lambda \frac{4\hat{x}}{\sqrt{x^2 + 1}} \int_0^{1/\lambda} \frac{\hat{\xi} r_1}{\sqrt{1 + \hat{\xi}^2 (\hat{x} \sqrt{1 + \hat{\xi}^2} + \hat{\xi} \sqrt{1 + \hat{x}^2}) (1 + r_1^2)}} d\hat{\xi}
\quad \text{(29)}
\]
As usual, we consider two cases.
1. $\hat{x} \in [0, 1]$. For the integral over $[0, 1]$,
\[
0 \leq \int_0^1 \frac{\hat{\xi} r_1}{\sqrt{1 + \hat{\xi}^2 (\hat{x} \sqrt{1 + \hat{\xi}^2} + \hat{\xi} \sqrt{1 + \hat{x}^2}) (1 + r_1^2)}} d\hat{\xi} \lesssim 1
\]
The other integral obeys the bound
\[
\int_1^{1/\lambda} \frac{\hat{\xi} r_1}{\sqrt{1 + \hat{\xi}^2 (\hat{x} \sqrt{1 + \hat{\xi}^2} + \hat{\xi} \sqrt{1 + \hat{x}^2}) (1 + r_1^2)}} d\hat{\xi} \lesssim \log(1/\lambda)
\]
since $r_1$ is uniformly bounded.
2. $\hat{x} \in [1, 1/\lambda]$. We can write
\[
0 \leq \int_0^{1/\lambda} \frac{\hat{\xi} r_1}{\sqrt{1 + \hat{\xi}^2 (\hat{x} \sqrt{1 + \hat{\xi}^2} + \hat{\xi} \sqrt{1 + \hat{x}^2}) (1 + r_1^2)}} d\hat{\xi} \lesssim \frac{\log(1/\lambda)}{\hat{x}}
\quad \text{(30)}
\]
For the $B_2^{(2)}$, we have similarly

$$B_2^{(2)} = \frac{4\hat{x}}{\sqrt{x^2 + 1}} \int_0^{1/\lambda} \frac{\hat{x} \sqrt{\hat{\xi}^2 + 1} + \hat{\xi} \sqrt{\hat{x}^2 + 1}}{\sqrt{\hat{\xi}^2 + 1}} \cdot \frac{\hat{\xi}r_1}{(\hat{x} + \hat{\xi})^2 + r_1^2(\hat{x} - \hat{\xi})^2} d\hat{\xi} \quad (31)$$

1. If $\hat{x} \in [0, 1]$, we get

$$r_1 \lesssim \hat{x} + \hat{\xi}$$

and therefore

$$0 < \int_0^{1/\lambda} \frac{\hat{x} \sqrt{\hat{\xi}^2 + 1} + \hat{\xi} \sqrt{\hat{x}^2 + 1}}{\sqrt{\hat{\xi}^2 + 1}} \cdot \frac{\hat{\xi}r_1}{(\hat{x} + \hat{\xi})^2 + r_1^2(\hat{x} - \hat{\xi})^2} d\hat{\xi} \lesssim 1$$

For the other interval,

$$\int_0^{1/\lambda} \frac{\hat{x} \sqrt{\hat{\xi}^2 + 1} + \hat{\xi} \sqrt{\hat{x}^2 + 1}}{\sqrt{\hat{\xi}^2 + 1}} \cdot \frac{\hat{\xi}r_1}{(\hat{x} + \hat{\xi})^2 + r_1^2(\hat{x} - \hat{\xi})^2} d\hat{\xi} = \frac{\hat{x} + \sqrt{\hat{x}^2 + 1}}{2} \log(1/\lambda) + O(1)$$

2. If $\hat{x} \in [1, 1/\lambda]$, then the asymptotics of $r_1$ yields

$$\int_0^{1/\lambda} \frac{\hat{x} \sqrt{\hat{\xi}^2 + 1} + \hat{\xi} \sqrt{\hat{x}^2 + 1}}{\sqrt{\hat{\xi}^2 + 1}} \cdot \frac{\hat{\xi}r_1}{(\hat{x} + \hat{\xi})^2 + r_1^2(\hat{x} - \hat{\xi})^2} d\hat{\xi} \sim \hat{x} + \frac{1}{\hat{x}} \log x \quad (32)$$

Now, the formulas (29) and (31) imply that $B_2^{(2)} \geq 0$ and $B_2^{(1)} \leq 0$. However, for $B_2 = B_2^{(2)} + B_2^{(1)} \geq 0$. Indeed, (29) and (31) yield

$$\frac{\hat{x} \sqrt{\hat{\xi}^2 + 1} + \hat{\xi} \sqrt{1 + \hat{x}^2}}{(\hat{x} + \hat{\xi})^2 + r_1^2(\hat{x} - \hat{\xi})^2} \geq \frac{1}{(\hat{x} \sqrt{\hat{\xi}^2 + 1} + \hat{\xi} \sqrt{1 + \hat{x}^2})(1 + r_1^2)}$$

since $r_1 \leq 1$. Thus, we have

$$0 \leq B_2 \leq B_2^{(2)} \lesssim x \log(1/\lambda), \quad 0 < x < \lambda$$

and

$$0 \leq B_2 \leq B_2^{(2)} \lesssim x \log^+(1/x), \quad \lambda < x < 1$$

Moreover, (30) and (32) provide a lower bound

$$B_2 \geq C_1 \hat{x} \log^+ x - C_2 \hat{x}^{-1} \log^+ \lambda, \quad x > \lambda$$

and therefore

$$B_2 \geq C_3 \hat{x} \log^+ x \quad (33)$$

for $\hat{x} > C_4$ where $C_4$ is sufficiently large absolute constant.

Consider the sum of the first two terms in (14). We have

$$\int_0^1 K_1(x, \tau, y_\lambda) d\tau - \frac{x}{\sqrt{\lambda^2 + x^2}} \int_0^1 y_\lambda' \tau K_2(x, \tau, y_\lambda) d\tau$$
= 4 \log(1/\lambda)(\sqrt{x^2 + \lambda^2} - x) + O(\lambda), \quad 0 < x < \lambda \\
and \\
= 4 \log(1/x)(\sqrt{x^2 + \lambda^2} - x) + O(x), \quad \lambda < x < \delta 

Adding \ B_2\ to\ this\ expression,\ dividing\ by\ x\sqrt{x^2 + \lambda^2}\log^+ (x^2 + \lambda^2),\ and\ taking\ the\ lower\ bound\ \(3\)\ into\ account\ gives\ the\ statement\ of\ the\ lemma\ on\ the\ small\ interval\ \([0, \delta]\).\ For\ \([\delta, 1]\),\ we\ have\ convergence\ to\ \A_2(x, 0)\ which\ is\ positive.\ \Box

Similar to lemma 4.2, we have

**Lemma 4.4.** Suppose \(\|g(x) - x\|_{C^1[0,1]} \leq \delta_1 \ll 1\). Then,

\[
\left| \int_0^1 \left( \sqrt{\lambda^2 + g^2(\tau)} \right) \frac{K_2(x, \tau, \sqrt{\lambda^2 + g^2(\tau)})d\tau}{x \log^+ (x^2 + \lambda^2)} \right| \lesssim 1
\]

uniformly in \(x, \lambda, \) and \(g\).

This result can be proved directly or by comparison to the case when \(g = x\) if the stability estimates (see (53) below) are used.

**3. The kernel \(M(x, \xi, \lambda)\) and the corresponding operator**

We will show that \(M(x, \xi, \lambda) \to M(x, \xi, 0)\) in a suitable sense when \(\lambda \to 0\).

**Lemma 4.5.** Fix any \(\delta > 0\). Then,

\[
\sup_{x > \delta} \int_0^1 |M(x, \xi, \lambda) - M(x, \xi, 0)|d\xi \to 0, \quad \lambda \to 0
\]

Thus,

\[
\|\omega_\delta^c(x)(M_\lambda - M_0)\|_{C[0,1],C[0,1]} \to 0, \quad \lambda \to 0
\]

**Proof.** We start with

\[
\lim_{\lambda \to 0} \sup_{x > \delta} \int_0^1 |D_2(x, \xi, \lambda) - D_2(x, \xi, 0)|d\xi = 0
\]

By (19),

\[
\int_0^1 |D_2(x, \xi, \lambda) - D_2(x, \xi, 0)|d\xi < C(\delta) \int_0^1 \left( 1 - \frac{\xi}{\sqrt{\xi^2 + \lambda^2}} \right) \log \left| \frac{x + \xi}{x - \xi} \right| d\xi
\]

and the last expression tends to zero uniformly in \(x \in [\delta, 1]\) when \(\lambda \to 0\).

To handle \(D_1\), we only need to show that

\[
\lim_{\lambda \to 0} \sup_{x \in [\delta, 1], \xi \in [0,1]} \left| \int_\xi^1 D_1(x, \tau, \lambda)d\tau - \int_\xi^1 D_1(x, \tau, 0)d\tau \right| = 0 \quad (34)
\]

To this end, we first simplify the expression for \(D_1(x, \tau, \lambda)\) using the formulas (15) and (16).

\[
D_1(x, \xi, \lambda) = D_1^{(1)} + D_1^{(2)} + D_1^{(3)} \quad (35)
\]
where (below \( x = \lambda \hat{x} \) and \( \xi = \lambda \hat{\xi} \))

\[
D_1^{(1)}(x, \xi, \lambda) = \frac{1}{x \sqrt{x^2 + \lambda^2 \log^+(x^2 + \lambda^2)}} \cdot \frac{4 \hat{x} \hat{r}_1}{(1 + r_1^2)(\hat{\xi} \sqrt{1 + \hat{\xi}^2} + \hat{x} \sqrt{1 + \hat{x}^2})(1 + \hat{\xi}^2)}
\]

\[
D_1^{(2)}(x, \xi, \lambda) = \frac{1}{x \sqrt{x^2 + \lambda^2 \log^+(x^2 + \lambda^2)}} \cdot \frac{\hat{x} \sqrt{1 + \hat{\xi}^2 + \hat{\xi} \sqrt{1 + \hat{x}^2}}}{1 + \hat{\xi}^2} \cdot \frac{4 \hat{x} \hat{r}_1}{(\hat{\xi} \sqrt{1 + \hat{\xi}^2} + \hat{x} \sqrt{1 + \hat{x}^2})(\hat{x} + \hat{\xi}^2 + (\hat{x} - \hat{\xi})^2 r_1^2)}
\]

\[
D_1^{(3)}(x, \xi, \lambda) = -\frac{1}{x \log^+(x^2 + \lambda^2)} \cdot \frac{\lambda^2}{(\lambda^2 + \xi^2)^{3/2}} \log \left( \frac{x + \xi}{x - \xi} \right)^2
\]

Since \( D_1^{(3)}(x, \xi, 0) = 0 \), we first show that

\[
\sup_{x > \delta} \int_0^1 |D_1^{(3)}(x, \xi, \lambda)| d\xi \to 0, \quad \lambda \to 0
\]

This follows from

\[
\int_0^1 \frac{\lambda^2}{(\lambda^2 + \xi^2)^{3/2}} \log \left( \frac{x + \xi}{x - \xi} \right)^2 d\xi = \int_0^{\delta/2} \frac{\lambda^2}{(\lambda^2 + \xi^2)^{3/2}} \log \left( \frac{x + \xi}{x - \xi} \right)^2 d\xi
\]

\[
+ \int_{\delta/2}^1 \frac{\lambda^2}{(\lambda^2 + \xi^2)^{3/2}} \log \left( \frac{x + \xi}{x - \xi} \right)^2 d\xi
\]

The second integral goes to zero as \( \lambda \to 0 \) uniformly in \( x > \delta \). The first one is bounded by

\[
\int_0^{\delta/2} \frac{\xi \lambda^2}{(\lambda^2 + \xi^2)^{3/2}} d\xi \lesssim \lambda
\]

All constants involved are \( \delta \) dependent.

Similarly, \( D_1^{(1)}(x, \xi, 0) = 0 \) and we have

\[
\sup_{x > \delta} \int_0^1 D_1^{(1)}(x, \xi, \lambda) d\xi \lesssim \lambda + \lambda \int_1^{\infty} \frac{\hat{\xi} d\hat{\xi}}{(\hat{x} \hat{\xi})(1 + \hat{\xi}^2)} \lesssim \lambda
\]

For \( D_1^{(2)}(x, \xi, 0) \), we have

\[
D_1^{(2)}(x, \xi, 0) = \frac{1}{x^2 \log^+(x^2)} \cdot \frac{4x^2}{x^2 + \xi^2}
\]

To show that

\[
\lim_{\lambda \to 0} \sup_{x > \delta, \xi > 0} \int_\xi^1 |D_2^{(2)}(x, \tau, \lambda) - D_2^{(2)}(x, \tau, 0)| d\tau = 0
\]

it is sufficient to prove

\[
\lim_{\lambda \to 0} \sup_{x > \delta} \int_0^{1/\lambda} \left| \frac{\hat{x} \sqrt{1 + \hat{\xi}^2 + \hat{\xi} \sqrt{1 + \hat{x}^2}}}{1 + \hat{\xi}^2} \cdot \frac{4 \hat{x} \hat{r}_1}{(\hat{\xi} \sqrt{1 + \hat{\xi}^2} + \hat{x} \sqrt{1 + \hat{x}^2})(\hat{x} + \hat{\xi}^2 + (\hat{x} - \hat{\xi})^2 r_1^2)} - \frac{4 \hat{x}^2}{\hat{x} \xi^2} \right| d\hat{\xi} = 0
\]

The integral over any interval \([0, T]\) is uniformly bounded and for large \( \hat{x} \) and \( \hat{\xi} \) we substitute

\[
r_1 = 1 + O \left( \frac{1}{\hat{x} \hat{\xi}} \right)
\]
and
\[ \sqrt{1 + \hat{\xi}^2} = \hat{\xi} + O(\hat{\xi}^{-1}), \sqrt{1 + \hat{x}^2} = \hat{x} + O(\hat{x}^{-1}) \]
Collecting the errors produced by this substitution, we have
\[ \lambda \int_1^{1/\lambda} \frac{\hat{x}^2}{\hat{x}^2 + \hat{\xi}^2} (\hat{\xi}^{-2} + \hat{x}^{-2}) d\hat{\xi} \lesssim \lambda \to 0 \]

The next step is to estimate
\[ \| \omega_\delta(x) M_\lambda \|_{C[0,1],C[0,1]} \]
where \( \delta \) and \( \lambda \) are small.

**Lemma 4.6.** We have
\[ \lim_{\delta \to 0} \lim_{\lambda \to 0} \sup_{x \in [0,\delta]} \int_0^1 \left| D_2(x, \xi, \lambda) + \int_\xi^1 D_1(x, \tau, \lambda) d\tau \right| d\xi = 0 \]

**Proof.** We only need to show that
\[ \lim_{\delta \to 0} \lim_{\lambda \to 0} \sup_{x \in [0,\delta]} \int_0^1 \left| D_2(x, \xi, \lambda) + \int_\xi^1 D_1(x, \tau, \lambda) d\tau \right| d\xi = 0 \]
Indeed, for \( \lambda = 0 \) the calculation is explicit. It gives
\[ \frac{1}{x \log^+ x} \int_0^1 \left( 2 \log \left| \frac{x + \xi}{x - \xi} \right| - \int_\xi^1 \frac{4x}{x^2 + \tau^2} d\tau \right) d\xi \]
\[ = \frac{2}{\log^+ x} \int_0^{1/x} \left( \log \left| \frac{1 + \xi}{1 - \xi} \right| - \int_\xi^{1/x} \frac{2}{1 + \tau^2} d\tau \right) d\xi \]
We have
\[ \int_\xi^{1/x} \frac{2}{1 + \tau^2} d\tau = \int_\xi^{\infty} \frac{2}{1 + \tau^2} d\tau + O(x) = \frac{2}{\xi} + O(\xi^{-2}), \quad \xi \gg 1 \]
and
\[ \log \left| \frac{1 + \xi}{1 - \xi} \right| = \frac{2}{\xi} + O(\xi^{-2}) \]
This gives necessary cancelation and a bound
\[ \frac{1}{x \log^+ x} \int_0^1 \left( 2 \log \left| \frac{x + \xi}{x - \xi} \right| - \int_\xi^1 \frac{4x}{x^2 + \tau^2} d\tau \right) d\xi \lesssim \frac{1}{\log^+ x} \]
The logarithm in the denominator will give a necessary estimate as \( x \to 0 \).

Now, we will need to prove analogous inequalities uniformly in small \( \lambda \). The expression
\[ \int_0^1 \left| D_2(x, \xi, \lambda) + \int_\xi^1 D_1(x, \tau, \lambda) d\tau \right| d\xi \]
will be handled term by term.
We start by proving
\[ \lim_{\delta \to 0} \lim_{\lambda \to 0} \sup_{x \in [0,\delta]} \int_0^1 \int_\xi^1 |D_1^{(3)}(x, \tau, \lambda)| d\tau d\xi = 0 \]
The integral is bounded by
\[
\frac{1}{\hat{x} \log^+ (\lambda^2 \hat{x}^2 + \lambda^2)} \int_0^{1/\lambda} \int_{\xi}^{1/\lambda} \frac{1}{1 + \hat{\tau}^3} \log \left| \frac{\hat{x} + \hat{\tau}}{\hat{x} - \hat{\tau}} \right| d\hat{\tau} d\xi
\]
\[
= \frac{1}{\hat{x} \log^+ (\lambda^2 \hat{x}^2 + \lambda^2)} \int_0^{1/\lambda} \frac{\hat{\tau}}{1 + \hat{\tau}^3} \log \left| \frac{\hat{x} + \hat{\tau}}{\hat{x} - \hat{\tau}} \right| d\hat{\tau}
\]
For the integral, an estimate holds
\[
\int_0^{1/\lambda} \frac{\hat{\tau}}{1 + \hat{\tau}^3} \log \left| \frac{\hat{x} + \hat{\tau}}{\hat{x} - \hat{\tau}} \right| d\hat{\tau} \lesssim \hat{x} + \int_{1/\hat{x}}^{1/\hat{x}} \hat{x}^{-1} u^{-2} \log \left| \frac{1 + u}{1 - u} \right| du
\]
The last integral is bounded by \(C\hat{x}\) for \(\hat{x} < 1\). For \(\hat{x} > 1\), it is estimated by
\[
C \log^+ \frac{\hat{x}}{\hat{x}}
\]
Since
\[
\lim_{\lambda \to 0} \sup_{\hat{x} \in (0, 1)} \frac{\hat{x}}{\hat{x} \log^+ (\lambda^2 \hat{x}^2 + \lambda^2)} = 0
\]
and
\[
\lim_{\lambda \to 0} \sup_{\hat{x} > 1} \frac{\log^+ \hat{x}}{\hat{x}^2 \log^+ (\lambda^2 \hat{x}^2 + \lambda^2)} = 0
\]
we get (37).

Consider the other terms
\[
\int_0^{1} \left| D_2(x, \xi, \lambda) + \int_{\xi}^{1} \left( D_1^{(1)}(x, \tau, \lambda) + D_1^{(2)}(x, \tau, \lambda) \right) d\tau \right| d\xi
\]
\[
\lesssim \frac{1}{\hat{x} \sqrt{\hat{x}^2 + 1} \log^+ (\lambda^2 \hat{x}^2 + \lambda^2)} \left( \int_0^{1/\lambda} \left| \frac{\sqrt{\hat{x}^2 + 1}}{\sqrt{\hat{\xi}^2 + 1}} \log \left( \frac{\hat{x} + \hat{\xi}}{\hat{x} - \hat{\xi}} \right) \right| \right.
\]
\[
- \int_{\hat{\xi}}^{1/\lambda} \left( \frac{4\hat{x}r_1}{(1 + r_1^2)(1 + \hat{\tau}^2)(\hat{x} + 1 + \hat{x}^2 + \hat{x}\sqrt{1 + \hat{x}^2})} \right.
\]
\[
- \frac{\hat{x}\sqrt{1 + \hat{\tau}^2} + \hat{\tau}\sqrt{1 + \hat{x}^2}}{1 + \hat{\tau}^2} \cdot \frac{4\hat{x}r_1}{(\hat{x} + \hat{\tau})^2 + (\hat{x} - \hat{\tau})^2 r_1^2} \left. \right) d\hat{\tau} \bigg| \bigg. d\xi
\]
We consider two cases.
(1). Take \(\hat{\xi} \in (0, 1]\). First, consider the regime \(\hat{\xi} \in (0, 1]\). We get
\[
\int_0^{1} \left| \frac{\sqrt{\hat{x}^2 + 1}}{\sqrt{\hat{\xi}^2 + 1}} \log \left( \frac{\hat{x} + \hat{\xi}}{\hat{x} - \hat{\xi}} \right) \right| \right.
\]
\[
- \int_{\hat{\xi}}^{1/\lambda} \left( \frac{4\hat{x}r_1}{(1 + r_1^2)(1 + \hat{\tau}^2)(\hat{x} + 1 + \hat{x}^2 + \hat{x}\sqrt{1 + \hat{x}^2})} \right.
\]
\[
- \frac{\hat{x}\sqrt{1 + \hat{\tau}^2} + \hat{\tau}\sqrt{1 + \hat{x}^2}}{1 + \hat{\tau}^2} \cdot \frac{4\hat{x}r_1}{(\hat{x} + \hat{\tau})^2 + (\hat{x} - \hat{\tau})^2 r_1^2} \left. \right) d\hat{\tau} \bigg| \bigg. d\hat{\xi}
\]
Thus, this gives \((\log^+ \lambda)^{-1}\) contribution. If \(\hat{\xi} > 1\), we can use the asymptotical formulas for \(r_1 = 1 + O(\hat{\xi}^{-1})\) and \(\sqrt{\hat{\xi}^2 + 1} = \hat{\xi} + O(\hat{\xi}^{-1})\) to get

\[
\int_0^{1/\lambda} \left| \frac{\sqrt{\hat{\xi}^2 + 1}}{\sqrt{\hat{\xi}^2 + 1}} \xi \log \left( \frac{\hat{x} + \hat{\xi}}{\hat{x} - \hat{\xi}} \right) \right| d\hat{x} 
\]

because

\[
\frac{2\hat{x}}{\hat{x} + \sqrt{\hat{\xi}^2 + 1}} + 2\hat{x}(\hat{x} + \sqrt{1 + \hat{x}^2}) = 4\hat{x}\sqrt{1 + \hat{x}^2}
\]

and we have cancelation of the main terms.

Summing up these estimates, we get

\[
\int_0^1 \left| D_2(x, \xi, \lambda) \right| + \int_\xi^{1/\lambda} \left( D_1^{(1)}(x, \tau, \lambda) + D_1^{(2)}(x, \tau, \lambda) \right) d\xi \lesssim \frac{1}{\log^+ \lambda}, \quad x \in (0, \lambda)
\]

(2). Consider the case when \(\hat{x} > 1\). First, take \(\hat{\xi} \in (0, 1)\). We get

\[
\int_0^1 \left| \frac{\sqrt{\hat{\xi}^2 + 1}}{\sqrt{\hat{\xi}^2 + 1}} \xi \log \left( \frac{\hat{x} + \hat{\xi}}{\hat{x} - \hat{\xi}} \right) \right| d\hat{x} \lesssim 1
\]

and

\[
\int_0^1 \int_\xi^{1/\lambda} \left( \frac{2\hat{x}\hat{\xi}r_1}{(1 + \hat{\xi}^2)(1 + \hat{\tau}^2)(\hat{x} + \hat{x} + \hat{\tau}x) + \hat{\tau}(\hat{x} + \hat{\xi} + \hat{\tau}x)} \right) d\hat{\tau} d\hat{\xi} \lesssim 1 + \hat{x}
\]

Thus, this gives the contribution bounded by

\[
\sup_{\hat{x} > 1} \frac{1}{\hat{x} \log^+ (\lambda^2 \hat{x}^2 + \lambda^2)} \to 0, \quad \text{as} \ \lambda \to 0
\]
For the interval \( \hat{\xi} \in (1, \lambda^{-1}) \), we again use asymptotics for \( r_1 = 1 + O((\hat{x}\hat{\xi})^{-1}), \sqrt{x^2 + 1}, \) and \( \sqrt{\hat{\xi}^2 + 1} \).

\[
\int_{1}^{1/\lambda} \sqrt{x^2 + 1}(1 + O(\hat{\xi}^{-2})) \log \left( \frac{\hat{x} + \hat{\xi}}{\hat{x} - \hat{\xi}} \right)^2 - \\
- \int_{\xi}^{1/\lambda} \frac{2\hat{x}}{\tau} \left( \frac{1}{\tau \sqrt{1 + \hat{x}^2 + \hat{x}\tau}} + \frac{\hat{\tau} \sqrt{1 + \hat{x}^2 + \hat{x}\tau}}{\hat{x}^2 + \hat{\tau}^2} \right) (1 + O(\hat{\tau}^{-1}\hat{x}^{-1} + \hat{\tau}^{-2})) \, d\tau \bigg| d\hat{\xi}
\]

The errors are bounded by

\[
C(\log^+ \hat{x} + \hat{\xi})
\]

and the change of variables in the integrals gives

\[
\int_{1/\hat{x}}^{1/x} \hat{x} \sqrt{1 + \hat{x}^2} \log \left( \frac{1 + u}{1 - u} \right)^2 - \int_{u}^{1/x} 2\hat{x}\tau^{-1} \left( \frac{1}{\tau (\sqrt{\hat{x}^2 + 1} + \hat{\tau})} + \frac{\tau (\sqrt{\hat{x}^2 + 1} + \hat{\tau})}{\tau^2 + 1} \right) \, d\tau \bigg| du \lesssim \hat{x}^2
\]

In the last term, we get

\[
\hat{x} \int_{1/\hat{x}}^{1/x} \int_{1/x}^\infty 2\tau^{-1} \left( \frac{1}{\tau (\sqrt{\hat{x}^2 + 1} + \hat{\tau})} + \frac{\tau (\sqrt{\hat{x}^2 + 1} + \hat{\tau})}{\tau^2 + 1} \right) \, d\tau \bigg| du \lesssim \hat{x}^2 + \log^+ \hat{x}
\]

Then,

\[
\int_{1/\hat{x}}^{1} \hat{x} \sqrt{1 + \hat{x}^2} \log \left( \frac{1 + u}{1 - u} \right)^2 - \hat{x} \int_{u}^{\infty} 2\tau^{-1} \left( \frac{1}{\tau (\sqrt{\hat{x}^2 + 1} + \hat{\tau})} + \frac{\tau (\sqrt{\hat{x}^2 + 1} + \hat{\tau})}{\tau^2 + 1} \right) \, d\tau \bigg| du \lesssim \hat{x}^2 \\
\]

and

\[
\int_{1}^{1/x} \hat{x} \sqrt{1 + \hat{x}^2} \log \left( \frac{1 + u}{1 - u} \right)^2 - \int_{u}^{\infty} 2\hat{x}\tau^{-1} \left( \frac{1}{\tau (\sqrt{\hat{x}^2 + 1} + \hat{\tau})} + \frac{\tau (\sqrt{\hat{x}^2 + 1} + \hat{\tau})}{\tau^2 + 1} \right) \, d\tau \bigg| du \lesssim \hat{x}^2
\]

after the cancelation of the main terms in the asymptotics. Collecting all bounds, we get

\[
\int_{0}^{1} \left| D_2(x, \xi, \lambda) + \int_{\xi}^{1} \left( D_1^{(1)}(x, \tau, \lambda) + D_1^{(2)}(x, \tau, \lambda) \right) \, d\tau \right| d\xi \leq \frac{\hat{x}}{\sqrt{x^2 + 1} \log^+(x^2 + \lambda^2)}
\]

and that finishes the proof. \( \square \)

We get the following

**Corollary 4.1.**

\[
\lim_{\lambda \to 0} \| \mathcal{M}_\lambda - \mathcal{M}_0 \|_{C[0,1], C[0,1]} = 0
\]

**Proof.** It is sufficient to apply lemma 4.5 and lemma 4.6. \( \square \)

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25
4.3. **Inverting** $L_\lambda$. Divide the equation

$$L_\lambda u = g$$

by $A_1(x, \lambda)$ to rewrite it as

$$u' + pu + \int_0^1 M_2(x, \xi, \lambda) u'(\xi) d\xi = g_1$$

where

$$p(x, \lambda) = \frac{A_2(x, \lambda)}{A_1(x, \lambda)}$$

and

$$M_2(x, \xi, \lambda) = \frac{M(x, \xi, \lambda)}{A_1(x, \lambda)}, \quad g_1 = \frac{g(x)}{A_1(x, \lambda)}$$

Due to (21) and (22), this is a minor change as far as inversion of $L_\lambda$ is concerned.

The equation

$$u' + pu = F, \quad u(0) = 0$$

has the solution

$$u = \int_0^x \exp \left( - \int_\xi^x p(t) dt \right) F(\xi) d\xi$$

and therefore

$$u' = F - p \int_0^x \exp \left( - \int_\xi^x p(t) dt \right) F(\xi) d\xi$$

This is the same as

$$u'(x) = g_2(x) - \int_0^1 M_2(x, \xi, \lambda) u'(\xi) d\xi \quad (38)$$

$$+ p(x) \int_0^x \exp \left( - \int_\xi^x p(t) dt \right) \int_0^1 M_2(t, \xi, \lambda) u'(\xi) d\xi dt$$

and

$$g_2(x) = g_1(x) - p(x) \int_0^x \exp \left( - \int_\xi^x p(t) dt \right) g_1(\xi) d\xi$$

We can rewrite it as

$$u' + O_\lambda u' = B_\lambda g, \quad u' = (I + O_\lambda)^{-1} B_\lambda g \quad (39)$$

provided that $O_\lambda$ is invertible.

The expressions for $O_\lambda$ and $B_\lambda$ are as follows

$$B_\lambda g = g_2 = \frac{g(x)}{A_1(x, \lambda)} - \frac{A_2(x, \lambda)}{A_1(x, \lambda)} \int_0^x \exp \left( - \int_\xi^x \frac{A_2(t, \lambda)}{A_1(t, \lambda)} dt \right) \frac{g(\xi)}{A_1(\xi, \lambda)} d\xi$$

and

$$O_\lambda f = \frac{1}{A_1(x, \lambda)} (M_\lambda f)(x) -$$

$$- \frac{A_2(x, \lambda)}{A_1(x, \lambda)} \int_0^x \exp \left( - \int_\xi^x \frac{A_2(t, \lambda)}{A_1(t, \lambda)} dt \right) (M_\lambda f)(\xi) \frac{A_1(t, \lambda)}{A_1(\xi, \lambda)} d\xi \quad (40)$$
Lemma 4.7. We have
\[ \|B_\lambda\|_{C[0,1],C[0,1]} \lesssim 1 \] (41)
uniformly in \( \lambda \in (0, \lambda_0) \).

Proof. Since both \( A_1 \) and \( A_2 \) are positive, we have
\[ |B_\lambda g(x)| \leq C \left( \|g\|_{C[0,1]} + \frac{1}{x} \int_0^x |g(\xi)| d\xi \right) \]
uniformly in \( \lambda \in (0, \lambda_0) \) and \( x \in [0,1] \) as follows from the analysis of \( A_1 \) and \( A_2 \). This gives \( \square \).

Consider \( O_\lambda \). We have

Lemma 4.8.
\[ \|O_\lambda - O_0\|_{C[0,1],C[0,1]} \to 0, \quad \lambda \to 0 \]

Proof. For the first term,
\[ \left\| \frac{1}{A_1(x, \lambda)} M_\lambda f - \frac{1}{A_1(x, 0)} M_0 f \right\|_{C[0,1]} = o(1) \|f\|_{C[0,1]}, \]
and \( o(1) \) is when \( \lambda \to 0 \), uniformly in \( f \). Indeed, this follows from the corollary \( 4.1 \) and the properties of \( A_1(x, \lambda) \).

The second term can be written as
\[ \omega_\delta(x) \cdot \frac{A_2(x, \lambda)}{A_1(x, \lambda)} \int_0^x \exp \left( - \int_0^x \frac{A_2(t, \lambda)}{A_1(t, \lambda)} \frac{(M_\lambda f)(\xi)}{A_1(\xi, \lambda)} d\xi \right) \]
\[ + \omega_\delta(x) \cdot \frac{A_2(x, \lambda)}{A_1(x, \lambda)} \int_0^x \exp \left( - \int_0^x \frac{A_2(t, \lambda)}{A_1(t, \lambda)} \frac{(M_\lambda f)(\xi)}{A_1(\xi, \lambda)} d\xi \right) \]
where \( \delta > 0 \). If we denote the first/second expressions by \( S_1(2) \), then
\[ |S_1| \lesssim \frac{1}{x} \int_0^x \chi_{\xi < \delta} \left| \frac{(M_\lambda f)(\xi)}{A_1(\xi, \lambda)} \right| d\xi \]
and
\[ \lim_{\delta \to 0} \lim_{\lambda \to 0} \sup_{\xi \in [0,1]} |S_1| = 0 \]
The last equality follows from \( 36 \).

For \( S_2 \), one can write similarly
\[ S_2 = \omega_\delta(x) \cdot \frac{A_2(x, \lambda)}{A_1(x, \lambda)} \int_0^x \chi_{\xi > \delta} \exp \left( - \int_0^x \frac{A_2(t, \lambda)}{A_1(t, \lambda)} dt \right) \frac{(M_\lambda f)(\xi)}{A_1(\xi, \lambda)} d\xi \]
\[ + \omega_\delta(x) \cdot \frac{A_2(x, \lambda)}{A_1(x, \lambda)} \int_0^x \chi_{\xi > \delta} \exp \left( - \int_0^x \frac{A_2(t, \lambda)}{A_1(t, \lambda)} dt \right) \frac{(M_\lambda f)(\xi)}{A_1(\xi, \lambda)} d\xi \]
The first expression can be handled in the same way. For the second, we consider
\[ \left\| \omega_\delta(x) \cdot \frac{A_2(x, \lambda)}{A_1(x, \lambda)} \int_0^x \chi_{\xi > \delta} \exp \left( - \int_0^x \frac{A_2(t, \lambda)}{A_1(t, \lambda)} dt \right) \frac{(M_\lambda f)(\xi)}{A_1(\xi, \lambda)} d\xi \right\|_{C[0,1]} \]
If $\delta > 0$ is fixed, this expression is bounded by $o(1)\|f\|_{C[0,1]}$ as $\lambda \to 0$ (with constant depending on $\delta$). That follows directly from the properties of $A_{1(2)}$ and $M_\lambda$. □

This lemma implies that inversion of $I + O_\lambda$ can be reduced to showing that $I + O_0$ is invertible. In the next section, we will do that.

4.3.1. The operator $O_0$ and its properties.

**Theorem 4.1.** The operator $I + O_0$ is invertible in $C[0,1]$.

**Proof.** For the case $\lambda = 0$, the formulas are very simple. We recall that

$$K_1(x, \xi, y_0) = K_2(x, \xi, y_0) = \log \left(\frac{x + \xi}{x - \xi}\right)^2$$

Then, (21) and (25) imply that

$$A_1(x, 0) = \frac{1}{\log^+ x^2} \int_0^{1/x} \log \left(\frac{1 + \xi}{1 - \xi}\right)^2 d\xi = 2 + o(1), \quad x \to 0$$

and

$$A_2(x, 0) = \frac{2}{x \log^+ x^2} \log(x^{-2} + 1) = \frac{2}{x} + o(x), \quad x \to 0$$

Thus,

$$p(x, 0) = \frac{A_2(x, 0)}{A_1(x, 0)} = \frac{1}{x} + o(x), \quad x \to 0$$

Then,

$$D_2(x, \xi) = -\frac{1}{x \log^+ x^2} \log \left(\frac{x + \xi}{x - \xi}\right)^2$$

and

$$D_1(x, \xi) = \frac{4}{(x^2 + \xi^2) \log^+ x^2}$$

Therefore,

$$M(x, \xi, 0) = \frac{1}{x \log^+ x^2} \left( -\log \left(\frac{x + \xi}{x - \xi}\right)^2 + \int_\xi^1 \frac{4x}{x^2 + \tau^2} d\tau \right)$$

and

$$M_2(x, \xi, 0) = \frac{M(x, \xi, 0)}{A_1(x, 0)}$$

where $A_1 \sim 1$ on all of $[0,1]$.

**Lemma 4.9.** The operator

$$G_2f = \int_0^1 M_2(x, \xi, 0)f(\xi)d\xi$$

is compact in $C[0,1]$. Moreover, $(G_2f)(0) = 0$. 

28
Proof.

\[ |G_2 f| \lesssim \|f\|_{C[0,1]} \frac{1}{\log^+ x} \int_0^{1/x} \left| \log \left( \frac{1 + \xi}{1 - \xi} \right) \right|^2 d\xi \lesssim \|f\| \log^+ x \]

and thus \( G_2 \) is bounded in \( C[0,1] \).

The compactness follows by the standard approximation argument. Let us write a partition of unity \( 1 = \phi_\delta + \phi_{\delta}^c \). Then, \( \|\phi_\delta G_2\| \to 0 \) as \( \delta \to 0 \) and \( \phi_{\delta}^c G_2 \) is compact since the kernel has a weak singularity on the diagonal and is smooth away from it.

For \( O_0 \), one gets

\[ O_0 f = \frac{1}{A_1(x,0)} M_0 f - \frac{A_2(x,0)}{A_1(x,0)} \int_0^x \exp \left( - \int_\xi^x \frac{A_2(t,0)}{A_1(t,0)} dt \right) (M_0 f)(\xi) d\xi \]

Since the operator \( G_3 \) defined as

\[ G_3 f = \frac{A_2(x,0)}{A_1(x,0)} \int_0^x \exp \left( - \int_\xi^x \frac{A_2(t,0)}{A_1(t,0)} dt \right) f(\xi) d\xi \]

is bounded in \( C[0,1] \), we get the compactness for \( O_0 \). Therefore, the Fredholm theory is applicable to \( I + O_0 \). In particular, to prove invertibility of \( I + O_0 \), we only need to check that its kernel is trivial.

Consider the equation

\[ (I + O_0) f = 0 \]

and suppose that \( f \in C[0,1] \). Recall (40) to write

\[ L_0 u = 0, \quad u \in \hat{C}[0,1] \]

is equivalent to

\[ (I + O_0) u' = 0 \]

Let \( u(x) = \int_0^x f(t) dt \). Thus, we only need to check that \( L_0 \) has zero kernel in \( \hat{C}[0,1] \).

The equation (43) is equivalent to

\[ \int_0^1 (u'(x) - u'(\xi)) K_1(x,\xi,y_0) d\xi + 8 \int_0^1 H'(2x^2 + 2\xi^2)(\xi u(x) + xu(\xi)) d\xi = 0, \quad u \in \hat{C}[0,1] \]

where

\[ K_1(x,\xi,y_0) = H(2(x + \xi)^2) - H(2(x - \xi)^2) = \log \left( \frac{x + \xi}{x - \xi} \right)^2 \]

since

\[ H(x) = \log x \]

in our case.

Multiply the both sides by \( u \) and integrate over \([0,1]\). For the general \( H \), we have

\[ 0.5 \int_0^1 (u(1) - u(\xi))^2 \left( H(2(1 + \xi)^2) - H(2(1 - \xi)^2) \right) d\xi \]

\[ -2 \int_0^1 \int_0^1 (u(x) - u(\xi))^2 \left( H'(2(x + \xi)^2)(x + \xi) - H'(2(x - \xi)^2)(x - \xi) \right) dx d\xi \]
\[
+8 \int_0^1 u^2(x) \int_0^1 \xi H'(2x^2 + 2\xi^2) d\xi dx + 8 \int_0^1 \int_0^1 u(x)u(\xi) xH'(2\xi^2 + 2x^2) dx d\xi = I_1 + \ldots + I_4
\]

Let us study this expression term by term.

If \( u_1(x) = u(1) - u(x) \), then
\[
I_1 = \int_0^1 u_1^2(x) \log \left| \frac{1+x}{1-x} \right| dx \geq 0
\]

This is actually true for generic \( H \) that are monotonically increasing.

Now, take \( H(x) = \log x \). Then, using the symmetrization of the integrals, we get the following expressions
\[
I_2 = -\int_0^1 \int_0^1 \frac{(u(x) - u(\xi))^2}{x + \xi} dx d\xi = -2 \int_0^1 u^2(x) \log \left( \frac{1+x}{x} \right) dx
\]
\[
+ 2 \int_0^1 \int_0^1 \frac{u(x)u(\xi)}{x + \xi} dx d\xi
\]
\[
I_3 = 2 \int_0^1 u^2(x) \log(1 + x^{-2}) dx
\]
\[
I_4 = 4 \int_0^1 \int_0^1 u(x)u(\xi) \frac{x}{x^2 + \xi^2} dx d\xi = 2 \int_0^1 \int_0^1 u(x)u(\xi) \frac{x + \xi}{x^2 + \xi^2} dx d\xi
\]

Notice now that the sum of the first term in \( I_2 \) and \( I_3 \) is
\[
2 \int_0^1 u^2(x) \log \left( \frac{x + x^{-1}}{1 + x} \right) dx \geq 0
\]
because
\[
x + x^{-1} \geq x + 1
\]
if \( x \in (0, 1] \).

In the calculations that follow, the condition \( u(x) = O(x) \), \( x \to 0 \) will ensure the convergence of all integrals involved.

The Hilbert matrix is nonnegative so
\[
G(u) = \int_0^\infty \frac{u(\xi)}{x + \xi} d\xi
\]
is positive definite operator. Thus,\[
G_1(u) = \int_0^\infty \frac{u(\xi)}{x^2 + \xi^2} d\xi
\]
is positive definite as well, as the change of variables in the quadratic form shows. Also,\[
\frac{x + \xi}{x^2 + \xi^2} = \frac{1}{x + \xi} + \frac{2x\xi}{(x^2 + \xi^2)(x + \xi)}
\]
So, we only need to establish that
\[
G_2 u = \int_0^1 \frac{x\xi u(\xi)}{(x^2 + \xi^2)(x + \xi)} d\xi
\]
is positive definite. That, however, is the corollary of the Schur’s theorem for the Hadamard product of the positive matrices, written for the integral operators by the Riemann sum approximation. Indeed, it is sufficient to notice that
\[
\frac{x\xi}{x^2 + \xi^2}
\]
is a positive definite kernel (again, by the change of variables in the quadratic form). □

Summing up the results of this section, we obtain (8).

5. \(\|\psi_0\|_{\mathcal{C}([0,1])}\) IS SMALL

In this section, we will prove (11), the smallness on initial data for the contraction mapping.

**Lemma 5.1.** We have
\[
\|\psi_0\|_{\mathcal{C}([0,1])} = o(1), \quad \lambda \to 0
\]

**Proof.** We only need to show that
\[
\|F(x,\lambda)\|_{\mathcal{C}([0,1])} = o(1), \quad \lambda \to 0
\]

Recall the definition of \(F\),
\[
F(x,\lambda) = \frac{1}{x\sqrt{\lambda^2 + x^2 \log^+(x^2 + \lambda^2)}} \left( x \int_0^1 K_1(x,\tau,\tau_{\lambda})d\tau - \sqrt{\lambda^2 + x^2} \int_0^1 y'_{\lambda}(\tau) K_2(x,\tau,\tau_{\lambda})d\tau \right)
\]
For any given \(\epsilon > 0\), we clearly have
\[
\|\omega_\epsilon \cdot F(x,\lambda)\|_{\mathcal{C}([0,1])} \to 0, \quad \lambda \to 0
\]
For \(x < \epsilon\), we can use the asymptotics established above (e.g., (23), (24), and (28)).

This gives
\[
\|\omega_\epsilon \cdot F(x,\lambda)\|_{\mathcal{C}([0,1])} \lesssim \frac{1}{\log^+ \epsilon}
\]
uniformly in \(\lambda\). □

6. THE FRECHET DIFFERENTIABILITY

In this section, we study \(Q(u)\) given by
\[
Q(u) = F(f,\lambda) - F(x,\lambda) - D_f F(x,\lambda)u, \quad f = x + u
\]
and prove (9) and (10). We assume in this section that \(\lambda \in (0,1)\). Notice first that \(Q(0) = 0\) and therefore (9) follows from (10). Let us prove (10).

We write
\[
Q(u_2) = F(x + u_2,\lambda) - F(x,\lambda) - D_f F(x,\lambda)(x + u_2)
\]
\[
Q(u_1) = F(x + u_1,\lambda) - F(x,\lambda) - D_f F(x,\lambda)(x + u_1)
\]
We subtract and write pointwise
\[
|Q(u_2) - Q(u_1)| \leq |F(x + u_2,\lambda) - F(x + u_1,\lambda) - D_f F(x + u_1,\lambda)(u_2 - u_1)|
\]
\[
+ |D_f F(x + u_1,\lambda) - D_f F(x,\lambda)| (u_2 - u_1)\]
Thus, we only have to prove two bounds:

\[\|F(x + u_2, \lambda) - F(x + u_1, \lambda) - D_f F(x + u_1, \lambda)(u_2 - u_1)\|_{C[0,1]} = o(1)\|u_2 - u_1\|_{C^1[0,1]}\]  \hspace{1cm} (45)

and

\[\|D_f F(x + u, \lambda) - D_f F(x, \lambda)\|_{C^1[0,1],C[0,1]} = o(1), \quad \|u\|_{C^1[0,1]} \leq \delta, \quad \delta \rightarrow 0\]  \hspace{1cm} (46)

### 6.1. The proof of (45).

We start with proving (45).

Denote \(\rho(x) = x + u_1(x)\). By our assumptions we have

\[\|\rho'(x) - 1\|_{C[0,1]} \leq \delta \ll 1, \quad \rho(0) = 0\]

Therefore,

\[\rho(x) = x \left(1 + \int_0^1 (\rho'(xt) - 1)dt\right) = x(1 + O(\delta))\]

**Remark.** We will use the following property many times in the arguments below. Given arbitrary \(M > 0\), the scaled function \(\rho_M(\tilde{x}) = M\rho(M^{-1}\tilde{x})\) satisfies:

\[\rho_M(0) = 0, \quad \|\rho_M'(\tilde{x}) - 1\|_{C[0,M^{-1}]} \leq \delta\]

Moreover, if \(\|h - g\|_{C^1[0,1]} \leq \epsilon\), then \(\|h_M - g_M\|_{C^1[0,M^{-1}]} \leq \epsilon\) after scaling.

Take \(t \in \mathbb{R}\) with \(|t| < t_0 = \|u_2 - u_1\|_{C[0,1]}\) and \(f : \|f\|_{C^1[0,1]} \leq 1\). Consider \(f_t(x) = \rho(x) + tf(x)\). We only need to show that

\[\|F(f_t, \lambda) - F(f_0, \lambda) - tD_f F(f_0, \lambda)f\|_{C[0,1]} = o(1), \quad t \rightarrow 0\]  \hspace{1cm} (47)

uniformly in \(f\) and \(\rho\).

Fix arbitrary \(x \in (0, 1]\) and apply the mean-value formula to \(F(f_t, \lambda) - F(f_0, \lambda)\),

\[F(f_t, \lambda) - F(f_0, \lambda) = t \frac{P_1 + \ldots + P_6}{{x^3}}\]

where \(t_1(x) \in [0, t]\). Introducing \(Y_{\lambda,t_1}(x) = \sqrt{\lambda^2 + f_t^2(x)}\), we get

\[P_1 = f(\rho' + t_1f') \int_0^1 K_1(x, \tau, Y_{\lambda,t_1})d\tau\]

\[P_2 = (\rho + t_1f)f' \int_0^1 K_1(x, \tau, Y_{\lambda,t_1})d\tau\]

\[P_3 = 2(\rho + t_1f)(\rho' + t_1f') (X_1 + \ldots + X_4)\]

where

\[X_1 = \int_0^1 \frac{Y_{\lambda,t_1}(x) + Y_{\lambda,t_1}(\tau)}{(x + \tau)^2 + (Y_{\lambda,t_1}(x) + Y_{\lambda,t_1}(\tau))^2} \left(\frac{f_t(x)f(x)}{Y_{\lambda,t_1}(x)} + \frac{f_t(\tau)f(\tau)}{Y_{\lambda,t_1}(\tau)}\right) d\tau\]

\[X_2 = \int_0^1 \frac{Y_{\lambda,t_1}(x) + Y_{\lambda,t_1}(\tau)}{(x - \tau)^2 + (Y_{\lambda,t_1}(x) + Y_{\lambda,t_1}(\tau))^2} \left(\frac{f_t(x)f(x)}{Y_{\lambda,t_1}(x)} + \frac{f_t(\tau)f(\tau)}{Y_{\lambda,t_1}(\tau)}\right) d\tau\]

\[X_3 = \int_0^1 \frac{Y_{\lambda,t_1}(x) - Y_{\lambda,t_1}(\tau)}{(x - \tau)^2 + (Y_{\lambda,t_1}(x) - Y_{\lambda,t_1}(\tau))^2} \left(\frac{f_t(x)f(x)}{Y_{\lambda,t_1}(x)} - \frac{f_t(\tau)f(\tau)}{Y_{\lambda,t_1}(\tau)}\right) d\tau\]
and
\[
X_4 = -\int_0^1 \frac{Y_{\lambda,t_1}(x) - Y_{\lambda,t_1}(\tau)}{(x + \tau)^2 + (Y_{\lambda,t_1}(x) - Y_{\lambda,t_1}(\tau))^2} \left( \frac{f_{t_1}(x)f(x)}{Y_{\lambda,t_1}(x)} - \frac{f_{t_1}(\tau)f(\tau)}{Y_{\lambda,t_1}(\tau)} \right) d\tau
\]
and
\[
P_4 = -\frac{f_{t_1}f}{Y_{\lambda,t_1}} \int_0^1 Y_{\lambda,t_1}'(\tau)K_2(x, \tau, Y_{\lambda,t_1}) d\tau
\]
\[
P_5 = -Y_{\lambda,t_1} \int_0^1 \left( \frac{f_{t_1}f}{Y_{\lambda,t_1}} \right)' K_2(x, \tau, Y_{\lambda,t_1}) d\tau
\]
\[
P_6 = -2Y_{\lambda,t_1}(L_1 + \ldots + L_4)
\]
Similarly, for \( \{L_j\} \) we have
\[
L_1 = \int_0^1 Y_{\lambda,t_1}'(\tau) \frac{Y_{\lambda,t_1}(x) + Y_{\lambda,t_1}(\tau)}{(x + \tau)^2 + (Y_{\lambda,t_1}(x) + Y_{\lambda,t_1}(\tau))^2} \left( \frac{f_{t_1}(x)f(x)}{Y_{\lambda,t_1}(x)} + \frac{f_{t_1}(\tau)f(\tau)}{Y_{\lambda,t_1}(\tau)} \right) d\tau
\]
\[
L_2 = -\int_0^1 Y_{\lambda,t_1}'(\tau) \frac{Y_{\lambda,t_1}(x) + Y_{\lambda,t_1}(\tau)}{(x - \tau)^2 + (Y_{\lambda,t_1}(x) + Y_{\lambda,t_1}(\tau))^2} \left( \frac{f_{t_1}(x)f(x)}{Y_{\lambda,t_1}(x)} + \frac{f_{t_1}(\tau)f(\tau)}{Y_{\lambda,t_1}(\tau)} \right) d\tau
\]
\[
L_3 = -\int_0^1 Y_{\lambda,t_1}'(\tau) \frac{Y_{\lambda,t_1}(x) - Y_{\lambda,t_1}(\tau)}{(x - \tau)^2 + (Y_{\lambda,t_1}(x) - Y_{\lambda,t_1}(\tau))^2} \left( \frac{f_{t_1}(x)f(x)}{Y_{\lambda,t_1}(x)} - \frac{f_{t_1}(\tau)f(\tau)}{Y_{\lambda,t_1}(\tau)} \right) d\tau
\]
\[
L_4 = \int_0^1 Y_{\lambda,t_1}'(\tau) \frac{Y_{\lambda,t_1}(x) - Y_{\lambda,t_1}(\tau)}{(x + \tau)^2 + (Y_{\lambda,t_1}(x) - Y_{\lambda,t_1}(\tau))^2} \left( \frac{f_{t_1}(x)f(x)}{Y_{\lambda,t_1}(x)} - \frac{f_{t_1}(\tau)f(\tau)}{Y_{\lambda,t_1}(\tau)} \right) d\tau
\]
We need to show that
\[
\left\| (P_1 + \ldots + P_6) - (P_1^0 + \ldots + P_6^0) \right\|_{L^\infty[0,1]} = o(1)
\]
as \( t \to 0 \) uniformly in \( f \) and \( \rho \). Here \( P_j^0 \) are the similar expressions taken with \( t_1 = 0 \).

(1) We start with \( P_1 - P_1^0 \):

\[
\frac{1}{x\sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2)} \left| f(\rho' + t_1 f') \int_0^1 K_1(x, \tau, Y_{\lambda,t_1}) d\tau - f \rho' \int_0^1 K_1(x, \tau, Y_{\lambda,0}) d\tau \right|
\]
\[
\leq \frac{t}{\sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2)} \int_0^1 K_1(x, \tau, Y_{\lambda,t_1}) d\tau
\]
\[
+ \frac{1}{\sqrt{x^2 + \lambda^2} \log^+(x^2 + \lambda^2)} \left| \int_0^1 K_1(x, \tau, Y_{\lambda,t_1}) d\tau - \int_0^1 K_1(x, \tau, Y_{\lambda,0}) d\tau \right|
\]
To handle the first term, we use the lemma 4.2 and the lemma 6.1 below, which takes care of the second term as well.

**Lemma 6.1.** We have

\[
\left\| \int_0^1 K_1(x, \tau, Y_{\lambda,t_1}) d\tau - \int_0^1 K_1(x, \tau, Y_{\lambda,0}) d\tau \right\|_{L^\infty[0,1]} = o(1), \quad t \to 0
\]
uniformly in \( \lambda, f, \) and \( \rho \).
Proof. By the mean-value formula we have
\[ \int_0^1 K_1(x, \tau, Y_{\lambda,t}) d\tau - \int_0^1 K_1(x, \tau, Y_{\lambda,0}) d\tau = t_1(x)(\hat{X}_1 + \ldots + \hat{X}_4) \]
where the expressions \( \hat{X}_j \) are different from \( X_j \) defined above only by \( t_1 \) replaced with \( t_2 \).

The bound
\[ \left\| \frac{\hat{X}_1 + \ldots + \hat{X}_4}{\sqrt{x^2 + \lambda^2 \log^+ (x^2 + \lambda^2)}} \right\|_{L^\infty[0,1]} \lesssim 1 \]
follows from the theorem \[ \text{(1)} \] below. \[ \square \]

(2). The term \( P_2 - P_2^0 \) can be handled in exactly the same way.

(3). The term \( P_3 - P_3^0 \) is more complicated.

Arguing similarly to \( P_1 \), we only need to prove the following theorem.

**Theorem 6.1.**
\[ \left\| \frac{(X_1 + \ldots + X_4) - (X_1^0 + \ldots + X_4^0)}{\sqrt{x^2 + \lambda^2 \log^+ (x^2 + \lambda^2)}} \right\|_{L^\infty[0,1]} = o(1), \quad t \to 0 \]

and
\[ \left\| \frac{X_1^0 + \ldots + X_4^0}{\sqrt{x^2 + \lambda^2 \log^+ (x^2 + \lambda^2)}} \right\|_{L^\infty[0,1]} \lesssim 1 \]

uniformly in \( \lambda, f, \) and \( \rho \).

**Proof.** Let us introduce \( x = \lambda \hat{x} \) and \( \tau = \lambda \hat{\tau} \). Notice that
\[ Y_{\lambda,t}(\lambda \hat{x}) = \lambda \sqrt{1 + (\lambda^{-1} f_\lambda(\lambda \hat{x}))^2} \]

The rescaled function, say, \( f_\lambda(\hat{x}) = \lambda^{-1} f(\lambda \hat{x}) \) satisfies the similar properties:
\[ f_\lambda(\hat{x}) = \hat{x} \epsilon(\lambda \hat{x}), \quad \| f_\lambda \| \leq 1 \]
as we already mentioned above.

Let us focus of \( X_1 + X_3 \) first. We are going to prove the following general result. Once we do that, it suffices to apply it to the scaled \( X_1 + X_3 \) by taking \( y_1(x) = f(x) \) and \( y_2(x) = f_{t_1}(x) \).

**Lemma 6.2.** Suppose \( y_1, y_2, \tilde{y}_2 \in C[0, \lambda^{-1}] \) and
\[ \| y'_1 \|_\infty \leq 1, \quad \| y'_2 - \tilde{y}_2' \|_\infty \leq \epsilon, \quad \| y'_2 - 1 \|_\infty \ll 1 \]

If one defines
\[ H = \frac{1}{\sqrt{x^2 + \lambda \log^+ (\lambda^2 (x^2 + 1))}} \int_0^{1/\lambda} \left( \frac{y_1(\hat{x}) y_2(\hat{x})}{1 + y'_2(\hat{x})} + \frac{y_1(\hat{\tau}) y_2(\hat{\tau})}{1 + y'_2(\hat{\tau})} \right) \times \]
\[ \frac{\sqrt{1 + y'_2(\hat{x})} + \sqrt{1 + y'_2(\hat{\tau})}}{(\hat{x} + \hat{\tau})^2 + (\sqrt{1 + y'_2(\hat{x})} + \sqrt{1 + y'_2(\hat{\tau})})^2} \]
\[ - \left( \frac{y_1(\hat{x}) y_2(\hat{x})}{\sqrt{1 + y'_2(\hat{x})}} - \frac{y_1(\hat{\tau}) y_2(\hat{\tau})}{\sqrt{1 + y'_2(\hat{\tau})}} \right) \frac{\sqrt{1 + y'_2(\hat{x})} - \sqrt{1 + y'_2(\hat{\tau})}}{(\hat{x} - \hat{\tau})^2 + (\sqrt{1 + y'_2(\hat{x})} - \sqrt{1 + y'_2(\hat{\tau})})^2} \right) d\hat{\tau} \]
and

\[
\tilde{H} = \frac{1}{\sqrt{x^2 + 1} \log^+(\lambda^2(x^2 + 1))} \int_0^{1/\lambda} \left( \frac{y_1(x)\tilde{y}_2(x)}{1 + \tilde{y}_2^2(x)} + \frac{y_1(\tilde{\tau})\tilde{y}_2(\tilde{\tau})}{1 + \tilde{y}_2^2(\tilde{\tau})} \right) \times
\]

\[
\frac{\sqrt{1 + \tilde{y}_2^2(x) + \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}}}{(\tilde{x} + \tilde{\tau})^2 + (\sqrt{1 + \tilde{y}_2^2(x)} + \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})})^2} -
\]

\[
\left( \frac{y_1(x)\tilde{y}_2(x)}{1 + \tilde{y}_2^2(x)} - \frac{y_1(\tilde{\tau})\tilde{y}_2(\tilde{\tau})}{1 + \tilde{y}_2^2(\tilde{\tau})} \right) \frac{\sqrt{1 + \tilde{y}_2^2(x) - \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}}}{(\tilde{x} - \tilde{\tau})^2 + (\sqrt{1 + \tilde{y}_2^2(x)} - \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})})^2} d\tilde{\tau}
\]

then

\[
\|H - \tilde{H}\|_{L^\infty[0,1/\lambda]} = o(1), \quad \epsilon \to 0
\]

and

\[
\|H\|_{L^\infty[0,1/\lambda]} \lesssim 1
\]

uniformly in \( \lambda \in (0, 1) \), \( y_1 \), \( y_2 \), and \( \tilde{y}_2 \).

**Proof.** We will study \( H \) in detail and, in particular, its stability in \( y_2 \). That will give the necessary bounds. Notice first that

\[
\left| \frac{y_2(\hat{x})}{\sqrt{\tilde{y}_2^2(x) + 1}} - \frac{\tilde{y}_2(x)}{\sqrt{\tilde{y}_2^2(x) + 1}} \right| \lesssim \left\{ \begin{array}{ll}
\epsilon \hat{x}, & \hat{x} < 1 \\
\epsilon \hat{x}^{-2}, & \hat{x} > 1
\end{array} \right.
\]

(49)

Then, the second term in the formula for \( H \) has the singularity of the type \((\hat{x} - \hat{\tau})^2\) in the denominator. However, this is compensated by the zero in the numerator and

\[
\sup_{\hat{x}} \left| \int_{|\hat{\tau} - \hat{x}| < 1} \left( \frac{y_1(x)\tilde{y}_2(x)}{1 + \tilde{y}_2^2(x)} \right) \frac{\sqrt{1 + \tilde{y}_2^2(x) - \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}}}{(\hat{x} - \hat{\tau})^2 + (\sqrt{1 + \tilde{y}_2^2(x)} - \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})})^2} d\hat{\tau} \right| = o(1)
\]

as follows from the lemma 8.2.1 in Appendix. Indeed,

\[
\left| \left( \frac{y_1(x)\tilde{y}_2(x)}{1 + \tilde{y}_2^2(x)} \right) ' \right| \lesssim \begin{cases} \epsilon \hat{x}, & \hat{x} < 1 \\ \epsilon, & \hat{x} > 1 \end{cases}
\]

and

\[
\left| \left( \frac{\sqrt{1 + \tilde{y}_2^2(x)}}{x + 1} \right) ' \right| \lesssim \frac{\hat{x}}{\hat{x} + 1}, \quad \left| \left( \sqrt{1 + \tilde{y}_2^2(x)} \right) ' \right| \lesssim \frac{\hat{x}}{\hat{x} + 1}
\]

Notice also that, in the expression for \( H \), the integral over every finite interval gives the bounded contribution after division, which is stable in \( y_2 \). Therefore, we can focus on \( \hat{\tau} : |\hat{x} - \hat{\tau}| > 1 \) only.

We consider two cases: \( \hat{x} \in (0, 1) \) and \( \hat{x} \in [1, \lambda^{-1}] \).

1. Let \( \hat{x} \in (0, 1] \). Clearly, we can assume that \( \hat{\tau} \gg 1 \). Let

\[
H = \frac{B_1 + B_2}{\sqrt{x^2 + 1} \log^+(\lambda^2(x^2 + 1))}
\]

35
where
\[
B_1 = \frac{y_1(\hat{x})y_2(\hat{x})}{\sqrt{1 + y_2^2(\hat{x})}} \int_0^{1/\lambda} \sqrt{1 + y_2^2(\hat{x}) + \sqrt{1 + y_2^2(\tilde{\tau})}} \frac{d\tilde{\tau}}{(\hat{x} - \tilde{\tau})^2 + (\sqrt{1 + y_2^2(\hat{x})} - \sqrt{1 + y_2^2(\tilde{\tau})})^2}
\]

and
\[
B_2 = \int_0^{1/\lambda} \frac{y_1(\tilde{\tau})y_2(\tilde{\tau})}{\sqrt{1 + y_2^2(\tilde{\tau})}} \left( \sqrt{1 + y_2^2(\hat{x}) + \sqrt{1 + y_2^2(\tilde{\tau})}} \right) \frac{d\tilde{\tau}}{(\hat{x} - \tilde{\tau})^2 + (\sqrt{1 + y_2^2(\hat{x})} - \sqrt{1 + y_2^2(\tilde{\tau})})^2}
\]

We only need to handle integration over \(\tilde{\tau} \in [2, 1/\lambda]\).

Consider \(B_2\) first. The integrand has asymptotics
\[
y_1(\tilde{\tau}) \left( 2\sqrt{1 + y_2^2(\hat{x})}(\tau^2 + y_2^2(\tilde{\tau}))^{-1} - 4y_2(\tilde{\tau})\frac{\hat{x}\tilde{\tau} + \sqrt{1 + y_2^2(\tilde{\tau})\sqrt{1 + y_2^2(\hat{x})}}}{(\tau^2 + y_2^2(\tilde{\tau}))^2} \right) (1 + O(\tilde{\tau}^{-1}))
\]

Thus, we immediately have a bound
\[
|B_2| \lesssim \log^+ \lambda
\]

Comparing the integral with the one where \(y_2\) is replaced by \(\tilde{y}_2\) gives us the necessary stability
\[
\int_0^{1/\lambda} \frac{y_1(\tilde{\tau})}{\tilde{\tau}^2 + y_2^2(\tilde{\tau})} d\tilde{\tau} - \int_1^{1/\lambda} \frac{y_1(\tilde{\tau})}{\tilde{\tau}^2 + y_2^2(\tilde{\tau})} d\tilde{\tau} = O(\epsilon) \log^+ \lambda
\]

and the same estimates are valid for other integrals involved. For the remainder \(O(\tilde{\tau}^{-1})\), the corresponding function is bounded by \(C\tilde{\tau}^{-2}\) and this decay is integrable and thus the integral over \([T, \lambda^{-1}]\) is small uniformly as \(T \to +\infty\). The integral over any finite interval \(\tilde{\tau} \in [0, T]\) can be made small by taking \(\epsilon \to 0\). Thus, we first take \(T\) large and then send \(\epsilon \to 0\) to ensure the stability of the errors in \(y_2\).

For \(B_1\), the estimates are very similar. The estimate \([49]\) gives the stability for the first factor
\[
\frac{y_1(\hat{x})y_2(\hat{x})}{\sqrt{1 + y_2^2(\hat{x})}}
\]

and the asymptotics of the integrand is
\[
= \frac{2y_2(\tilde{\tau})}{\tilde{\tau}^2 + y_2^2(\tilde{\tau})} + O(\tilde{\tau}^{-2})
\]

and we can use the estimate similar to \([50]\).

2. Consider the case \(\hat{x} > 1\) now and assume that \(|\hat{x} - \tilde{\tau}| > 1\) in the integration. For \(\tilde{\tau} > 1\) and \(\hat{x} > 1\), we can write
\[
\sqrt{1 + y_2^2(\hat{x})} = y_2(\hat{x})(1 + O(\hat{x}^{-2}))
\]

and
\[
\sqrt{1 + y_2^2(\tilde{\tau})} + \sqrt{1 + y_2^2(\tilde{\tau})} = (y_2(\hat{x}) + y_2(\tilde{\tau}))R_1^{-1}
\]

(51)
\[
\sqrt{1 + y_2^2(x)} - \sqrt{1 + y_2^2(\hat{\tau})} = (y_2(x) - y_2(\hat{\tau}))R_1
\]

and
\[
R_1 = 1 + O\left(\frac{1}{x^\tau}\right)
\]

Let us control how the integral will change if we replace \(\sqrt{1 + y_2^2(x)}\) by \(y_2(x)\) and \(\sqrt{1 + y_2^2(x)} + \sqrt{1 + y_2^2(\hat{\tau})}\) by \(y_2(x) + y_2(\hat{\tau})\). The errors produced in \(B_2\), for example, are at most
\[
C_1 + C_2 \int_1^{\lambda^{-1}} \left(\frac{1}{x} + \frac{1}{\hat{x}}\right) \frac{1}{|x - \hat{x}| + 1} d\hat{\tau} \lesssim 1 + \frac{\log \hat{x} + \log^+ \lambda}{\hat{x}}
\]
The estimate for \(B_1\) is the same. Now, notice that
\[
\sup_{\hat{x} > T, \lambda \in (0, 1)} \sup_{\hat{x} > T, \lambda \in (0, 1)} \frac{\log \hat{x} + \log^+ \lambda + \hat{x}}{\hat{x}^2 \log^+(\lambda^2(\hat{x}^2 + 1))} \to 0, \quad T \to \infty
\]
Since on every finite interval of integration \(\hat{\tau} \in [0, T]\) we have stability in \(y_2\), we only need to handle
\[
\int_{\hat{\tau} \in [0, \lambda^{-1}]} \left|\frac{(y_1(x) + y_1(\hat{\tau}))(y_2(x) + y_2(\hat{\tau}))}{(x + \hat{\tau})^2 + (y_2(x) + y_2(\hat{\tau}))^2} - \frac{(y_1(x) - y_1(\hat{\tau}))(y_2(x) - y_2(\hat{\tau}))}{(x - \hat{\tau})^2 + (y_2(x) - y_2(\hat{\tau}))^2}\right| d\hat{\tau}
\]
Let us change the variable \(\hat{\tau} = \alpha \hat{x}\) and introduce two functions:
\[
f(\alpha, \hat{x}) = \hat{x}^{-1}y_1(\alpha \hat{x}), \quad g(\alpha, \hat{x}) = \hat{x}^{-1}y_2(\alpha \hat{x})
\]
As before, we have \(f(0, \hat{x}) = g(0, \hat{x}) = 0\),
\[
|\partial_\alpha f(\alpha, \hat{x})| = |y_1'(\alpha \hat{x})| \leq 1, \quad |f(\alpha, \hat{x})| \leq \alpha
\]
and
\[
|\partial_\alpha g(\alpha, \hat{x}) - 1| = |y_2'(\alpha \hat{x}) - 1| \lesssim 1,
\]
Moreover, if \(\tilde{g}\) is the scaling of \(\tilde{y}_2\), then
\[
\|g' - \tilde{g}'\|_\infty \leq \epsilon
\]
These estimates are uniform in \(\hat{x}\). The integral takes the form
\[
\hat{x} \int_0^{1/\alpha} \left|\frac{(f(1) + f(\alpha))(g(1) + g(\alpha))}{(1 + \alpha)^2 + (g(1) + g(\alpha))^2} - \frac{(f(1) - f(\alpha))(g(1) - g(\alpha))}{(1 - \alpha)^2 + (g(1) - g(\alpha))^2}\right| d\alpha
\]
We can rewrite
\[
\frac{(f(1) - f(\alpha))(g(1) - g(\alpha))}{(1 - \alpha)^2 + (g(1) - g(\alpha))^2} = \frac{f(1) - f(\alpha)}{1 - \alpha} \cdot \frac{g(1) - g(\alpha)}{1 - \alpha} \cdot \frac{1}{1 + \left(\frac{g(1) - g(\alpha)}{1 - \alpha}\right)^2}
\]
and the lemma \(8.1\) proves stability for the interval \(|\alpha - 1| < 1\). Then, the stability in \(g\) can be easily seen for every interval \(\hat{\tau} \in [0, T]\) given fixed \(T\) as the corresponding error is \(o(1)\hat{x}\) and
\[
\sup_{\hat{x} > 1} \frac{o(1)\hat{x}}{\sqrt{\hat{x}^2 + \log^+(\lambda^2(\hat{x}^2 + 1))}} = o(1)
\]
For large $\alpha$, we get the asymptotics
\[
\frac{(f(1) + f(\alpha))(g(1) + g(\alpha))}{(1 + \alpha)^2 + (g(1) + g(\alpha))^2} - \frac{(f(1) - f(\alpha))(g(1) - g(\alpha))}{(1 - \alpha)^2 + (g(1) - g(\alpha))^2} = \frac{-4f(\alpha)g(\alpha)(\alpha + g(1)g(\alpha))}{(\alpha^2 + g^2(\alpha))^2} + \frac{2(f(1)g(\alpha) + g(1)f(\alpha))}{\alpha^2 + g^2(\alpha)} + O(\alpha^{-2})
\]
This leads to the error of the size
\[
o(1) \int_1^{1/x} \frac{d\alpha}{\alpha} = o(1) \log^+ x
\]
uniformly in $\lambda$ and $x > \lambda$.

Now, we need to handle the other combination: $X_2 + X_4$. The analysis here is nearly identical and is based on the following lemma.

**Lemma 6.3.** Suppose $y_1, y_2, \tilde{y}_2 \in C[0, \lambda^{-1}]$ and
\[
\|y'_1\|_{\infty} \leq 1, \quad \|y'_2 - \tilde{y}'_2\|_{\infty} \leq \epsilon, \quad \|y'_2 - 1\|_{\infty} \ll 1
\]
If one defines
\[
H^{(1)} = \frac{1}{\sqrt{x^2 + 1 + \log^+(\lambda^2(x^2 + 1))}} \int_0^{\lambda} \left( \frac{y_1(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y_2^2(x)}} + \frac{y_1(\tilde{\tau})y_2(\tilde{\tau})}{\sqrt{1 + y_2^2(\tau)}} \right) \times
\frac{\sqrt{1 + y_2^2(\tilde{x})} + \sqrt{1 + y_2^2(\tilde{\tau})}}{(\tilde{x} - \tilde{\tau})^2 + (\sqrt{1 + y_2^2(\tilde{x})} + \sqrt{1 + y_2^2(\tilde{\tau})})^2} - \left( \frac{y_1(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y_2^2(x)}} - \frac{y_1(\tilde{\tau})y_2(\tilde{\tau})}{\sqrt{1 + y_2^2(\tau)}} \right) \frac{\sqrt{1 + y_2^2(\tilde{x})} - \sqrt{1 + y_2^2(\tilde{\tau})}}{(\tilde{x} - \tilde{\tau})^2 + (\sqrt{1 + y_2^2(\tilde{x})} - \sqrt{1 + y_2^2(\tilde{\tau})})^2} d\tilde{\tau}
\]
and
\[
\tilde{H}^{(1)} = \frac{1}{\sqrt{x^2 + 1 + \log^+(\lambda^2(x^2 + 1))}} \int_0^{\lambda} \left( \frac{y_1(\tilde{x})\tilde{y}_2(\tilde{x})}{\sqrt{1 + \tilde{y}_2^2(x)}} + \frac{y_1(\tilde{\tau})\tilde{y}_2(\tilde{\tau})}{\sqrt{1 + \tilde{y}_2^2(\tau)}} \right) \times
\frac{\sqrt{1 + \tilde{y}_2^2(\tilde{x})} + \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}}{(\tilde{x} - \tilde{\tau})^2 + (\sqrt{1 + \tilde{y}_2^2(\tilde{x})} + \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})})^2} - \left( \frac{y_1(\tilde{x})\tilde{y}_2(\tilde{x})}{\sqrt{1 + \tilde{y}_2^2(x)}} - \frac{y_1(\tilde{\tau})\tilde{y}_2(\tilde{\tau})}{\sqrt{1 + \tilde{y}_2^2(\tau)}} \right) \frac{\sqrt{1 + \tilde{y}_2^2(\tilde{x})} - \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}}{(\tilde{x} - \tilde{\tau})^2 + (\sqrt{1 + \tilde{y}_2^2(\tilde{x})} - \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})})^2} d\tilde{\tau}
\]
then, uniformly in $y_1, y_2, \tilde{y}_2$ and $\lambda \in (0, 1)$, we have
\[
\|H^{(1)} - \tilde{H}^{(1)}\|_{L^\infty[0,1/\lambda]} = o(1), \quad \epsilon \to 0
\]
and
\[
\|H^{(1)}\|_{L^\infty[0,1/\lambda]} \lesssim 1
\]
The proof of this lemma repeats the argument for the previous one word for word. The only minor change is contained in how we handle the singularity in the denominator of $X_4$ when both $x$ and $\tau$ go to zero. After the rescaling, we have an integral

$$\left| \int_0^1 \left( \frac{y_1(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y_2^2(\tilde{x})}} - \frac{y_1(\tilde{\tau})y_2(\tilde{\tau})}{\sqrt{1 + y_2^2(\tilde{\tau})}} \right) \frac{\sqrt{1 + y_2^2(\tilde{x})} - \sqrt{1 + y_2^2(\tilde{\tau})}}{\tilde{x} + \tilde{\tau} + (\sqrt{1 + y_2^2(\tilde{x})} - \sqrt{1 + y_2^2(\tilde{\tau})})^2} d\tilde{\tau} \right| \lesssim \int_0^1 \frac{|\tilde{x} - \tilde{\tau}|^2}{\tilde{x}^2 + \tilde{\tau}^2} d\tilde{\tau} \lesssim 1$$

by the application of mean-value theorem. The stability of this expression in $y_2$ follows from the lemma \ref{thm:stability}. \hfill \Box

This finishes the proof of theorem \ref{thm:stability}. \hfill \Box

We continue now with the other terms: $P_4$, $P_5$ and $P_6$.

(4). Consider the term $P_4 - P_4^0$.

To study the stability in $t$, it is more convenient to rescale by $\lambda$ and consider $y_1(\tilde{x}) = \lambda^{-1}f(\tilde{x}\lambda)$ and $y_2(\tilde{x}) = \lambda^{-1}f_t(\tilde{x}\lambda)$. Then, the problem is reduced to proving the stability of

$$P_4 = \frac{1}{\tilde{x}\sqrt{1 + \tilde{x}^2\log^+(\lambda^2(\tilde{x}^2 + 1))}} \int_0^{1/\lambda} \frac{y_1(\tilde{\tau})y_2(\tilde{\tau})}{\sqrt{1 + y_2^2(\tilde{\tau})}} K_2(\tilde{x}, \tilde{\tau}, \sqrt{1 + y_2^2(\tilde{\tau})}) d\tilde{\tau}$$

in $y_2$. As before, we will be taking $\tilde{y}_2$ with $\|y_2 - \tilde{y}_2\|_\infty \leq \epsilon$ and making a comparison. By \eqref{eq:approx} and lemma \ref{lem:approx}, we have

$$\left| \int_0^{1/\lambda} \frac{y_1(\tilde{\tau})y_2(\tilde{\tau})}{\sqrt{1 + y_2^2(\tilde{\tau})}} K_2(\tilde{x}, \tilde{\tau}, \sqrt{1 + y_2^2(\tilde{\tau})}) d\tilde{\tau} \right| \leq \epsilon \tilde{x}^3 \log(1/\lambda), \quad \tilde{x} \in (0, 1)$$

and

$$\leq \epsilon \log(1/\tilde{x}), \quad \tilde{x} > 1$$

Thus, after division, it gives an error at most $\epsilon$.

For the next term, \eqref{eq:approx} again gives

$$\left| \int_0^{1/\lambda} \left( \frac{y_2(\tilde{\tau})\tilde{y}_2(\tilde{\tau})}{\sqrt{1 + y_2^2(\tilde{\tau})}} - \frac{\tilde{y}_2(\tilde{\tau})y_2(\tilde{\tau})}{\sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}} \right) \frac{\sqrt{1 + y_2^2(\tilde{\tau})} - \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}}{\tilde{x} + \tilde{\tau} + (\sqrt{1 + y_2^2(\tilde{\tau})} - \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})})^2} K_2(\tilde{x}, \tilde{\tau}, \sqrt{1 + y_2^2(\tilde{\tau})}) d\tilde{\tau} \right|$$

$$\lesssim \epsilon \tilde{x} \int_0^1 \frac{\tilde{\tau}}{\tilde{x}} K_2(\tilde{x}, \tilde{\tau}, \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}) d\tilde{\tau} + \epsilon \tilde{x} \int_1^{1/\lambda} \left| K_2(\tilde{x}, \tilde{\tau}, \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}) \right| d\tilde{\tau}$$

and

$$\lesssim \epsilon \tilde{x}^2 \log(1/\lambda), \quad \tilde{x} < 1$$

and

$$\lesssim \epsilon \tilde{x}^2 \log(1/\tilde{x}), \quad \tilde{x} > 1$$

After division, it gives the error at most $O(\epsilon)$.

For the last term

$$\frac{y_1(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y_2^2(\tilde{x})}} \int_0^{1/\lambda} \frac{y_2(\tilde{\tau})\tilde{y}_2(\tilde{\tau})}{\sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}} \left( K_2(\tilde{x}, \tilde{\tau}, \sqrt{1 + y_2^2(\tilde{\tau})}) - K_2(\tilde{x}, \tilde{\tau}, \sqrt{1 + \tilde{y}_2^2(\tilde{\tau})}) \right) d\tilde{\tau} \tag{53}$$
we can apply the mean value theorem and the resulting derivative of the kernel can be handled by the theorem [6.2] below. As the result, the expression above can be bounded by
\[ \lesssim \hat{x}^2 \log(1/\lambda) o(1), \quad \hat{x} < 1 \]
and
\[ \lesssim \hat{x}^2 \log(1/x) o(1), \quad \hat{x} > 1 \]
Upon division by
\[ \hat{x} \sqrt{x^2 + 1} \log^+(\lambda^2(\hat{x}^2 + 1)) \]
this is at most \( o(1) \).

(5). The term \( P_5 - P_6^0 \) can be estimated similarly.

Indeed, after scaling we have the following expression
\[ \sqrt{1 + y_2^2(\hat{x})} \int_0^{1/\lambda} \left( y_1' \frac{y_2}{1 + y_2} + y_1 y_2' (1 + y_2^2)^{-1.5} \right) K_2(\hat{x}, \hat{\tau}, \sqrt{1 + y_2^2(\hat{\tau})}) d\hat{\tau} \]
and we can repeat the steps from the previous argument.

(6). We are left to handle \( P_5^1 - P_6^0 \).

This analysis is very similar to the one performed for \( P_3 \).

**Theorem 6.2.**
\[
\left\| \frac{Y_{\lambda,t}(L_1 + \ldots + L_4) - Y_{\lambda,0}(L_1^0 + \ldots + L_4^0)}{x \sqrt{x^2 + \lambda^2 \log^+(x^2 + \lambda^2)}} \right\|_{\infty} = o(1), \quad t \to 0
\]
uniformly in \( \lambda \).

**Proof.** Rescale by \( \lambda \) and rewrite the problem for \( y_1 \) and \( y_2 \), as before. Notice first that
\[ |\sqrt{1 + y_2^2} - \sqrt{1 + \tilde{y}_2^2}| \leq \epsilon \hat{x}^2, \quad \hat{x} < 1 \]
and
\[ |\sqrt{1 + y_2^2} - \sqrt{1 + \tilde{y}_2^2}| \leq \epsilon \hat{x}, \quad \hat{x} > 1 \]
so we only need to show that
\[
\left\| \frac{(L_1 + \ldots + L_4) - (L_1^0 + \ldots + L_4^0)}{x \log^+(x^2 + \lambda^2)} \right\|_{\infty} = o(1), \quad t \to 0
\]
and
\[
\left\| \frac{L_1^0 + \ldots + L_4^0}{x \log^+(x^2 + \lambda^2)} \right\|_{\infty} \lesssim 1
\]
(54)

We group \( (L_1 + L_2) - (L_1^0 + L_2^0) \) and \( (L_3 + L_4) - (L_3^0 + L_4^0) \) and start with the following lemma which handles \( L_3 + L_4 \).
Lemma 6.4. Suppose $y_1, y_2, \tilde{y}_2 \in \hat{C}[0, \lambda^{-1}]$ and
\[
\|y'_1\|_\infty \leq 1, \quad \|y'_2 - \tilde{y}'_2\|_\infty \leq \epsilon, \quad \|y'_2 - 1\|_\infty \ll 1
\]
If one defines
\[
U = \frac{1}{x \log^+(\lambda^2(x^2 + 1))} \int_0^{1/\lambda} \left| \frac{y_2(\tilde{x})y'_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} \right| \left( \frac{y_1(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} - \frac{y_1(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} \right) \times
\]
\[
\left( \frac{\sqrt{1 + y'_2(\tilde{x})} - \sqrt{1 + y'_2(\tilde{x})}}{(\tilde{x} - \tilde{x})^2 + (\sqrt{1 + y'_2(\tilde{x})} - \sqrt{1 + y'_2(\tilde{x})})^2} \right)
\]
then, uniformly in $y_1, y_2, \tilde{y}_2$ and $\lambda \in (0, 1)$, we have
\[
\|U\|_{L^\infty[0, \lambda^{-1}]} = o(1), \quad \epsilon \to 0
\]
Notice that in this lemma we take an absolute value inside the integration.

Proof. We first prove that
\[
\left\| \frac{1}{x \log^+(\lambda^2(x^2 + 1))} \int_0^{1/\lambda} \left| \frac{y_2(\tilde{x})y'_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} \right| \left( \frac{y_1(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} - \frac{y_1(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} \right) \times
\]
\[
\left( \frac{\sqrt{1 + y'_2(\tilde{x})} - \sqrt{1 + y'_2(\tilde{x})}}{(\tilde{x} - \tilde{x})^2 + (\sqrt{1 + y'_2(\tilde{x})} - \sqrt{1 + y'_2(\tilde{x})})^2} \right) \right\|_{L^\infty[0, \lambda^{-1}]} = o(1)
\]
as $\epsilon \to 0$, uniformly in parameters. Let us observe that
\[
\left| \frac{y_2(\tilde{x})y'_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} - \frac{\tilde{y}_2(\tilde{x})\tilde{y}'_2(\tilde{x})}{\sqrt{1 + \tilde{y}'_2(\tilde{x})}} \right| \lesssim \epsilon \tilde{x}, \quad \tilde{x} < 1
\]
and
\[
\left| \frac{y_2(\tilde{x})y'_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} - \frac{\tilde{y}_2(\tilde{x})\tilde{y}'_2(\tilde{x})}{\sqrt{1 + \tilde{y}'_2(\tilde{x})}} \right| \lesssim \epsilon, \quad \tilde{x} > 1
\]
Therefore, to show (55) it is sufficient to use an estimate (57), proved below, and the following inequality
\[
\left| \frac{\tilde{x}}{x \log^+(\lambda^2(\tilde{x}^2 + 1))} \right| \int_0^{1/\lambda} \left| \frac{\tilde{x}}{\sqrt{\tilde{x}^2 + 1}} \right| \left( \frac{\tilde{x}^2(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} - \frac{\tilde{x}^2(\tilde{x})y_2(\tilde{x})}{\sqrt{1 + y'_2(\tilde{x})}} \right) \times
\]
\[
\left( \frac{\sqrt{1 + y'_2(\tilde{x})} - \sqrt{1 + y'_2(\tilde{x})}}{(\tilde{x} - \tilde{x})^2 + (\sqrt{1 + y'_2(\tilde{x})} - \sqrt{1 + y'_2(\tilde{x})})^2} \right) \right\|_{L^\infty[0, \lambda^{-1}]} = o(1)
\]
The latter can be achieved by following the argument in (57).

Now, consider

$$U^{(1)} = \frac{1}{x \log^+ (\lambda^2 (x^2 + 1))} \int_0^{1/\lambda} \sqrt{\frac{\hat{x}}{\hat{\tau}}} |F(\hat{x}, \hat{\tau}) - F_0(\hat{x}, \hat{\tau})| d\hat{\tau}$$

where

$$F = \left( \frac{y_1(\hat{x}) y_2(\hat{x})}{\sqrt{1 + y^2_2(\hat{x})}} - \frac{y_1(\hat{\tau}) y_2(\hat{\tau})}{\sqrt{1 + y^2_2(\hat{\tau})}} \right) \left( \frac{\sqrt{1 + y^2_2(\hat{x})} - \sqrt{1 + y^2_2(\hat{\tau})}}{(\hat{x} - \hat{\tau})^2 + (\sqrt{1 + y^2_2(\hat{x})} - \sqrt{1 + y^2_2(\hat{\tau})})^2} \right) -$$

$$\left. \frac{\sqrt{1 + y^2_2(\hat{x})} - \sqrt{1 + y^2_2(\hat{\tau})}}{(\hat{x} - \hat{\tau})^2 + (\sqrt{1 + y^2_2(\hat{x})} - \sqrt{1 + y^2_2(\hat{\tau})})^2} \right)$$

and

$$F_0 = \left( \frac{y_1(\hat{x}) y_2(\hat{x})}{\sqrt{1 + y^2_2(\hat{x})}} - \frac{y_1(\hat{\tau}) y_2(\hat{\tau})}{\sqrt{1 + y^2_2(\hat{\tau})}} \right) \left( \frac{\sqrt{1 + \hat{y}^2_2(x)} - \sqrt{1 + \hat{y}^2_2(\hat{\tau})}}{(\hat{x} - \hat{\tau})^2 + (\sqrt{1 + \hat{y}^2_2(x)} - \sqrt{1 + \hat{y}^2_2(\hat{\tau})})^2} \right) -$$

$$\left. \frac{\sqrt{1 + \hat{y}^2_2(x)} - \sqrt{1 + \hat{y}^2_2(\hat{\tau})}}{(\hat{x} - \hat{\tau})^2 + (\sqrt{1 + \hat{y}^2_2(x)} - \sqrt{1 + \hat{y}^2_2(\hat{\tau})})^2} \right)$$

We are going to prove that

$$\|U^{(1)}\|_{L^\infty([0,1])} = o(1), \quad \epsilon \to 0$$

(57)

Consider the case $\hat{x} \in [0,1]$. The regime $\hat{x} \to 0$ is what makes the difference when compared to the same analysis for $P_3$.

Consider $F$ and rewrite it as follows

$$F = -4\hat{x} \tau \left( \frac{y_1(\hat{x}) y_2(\hat{x})}{\sqrt{1 + y^2_2(\hat{x})}} - \frac{y_1(\hat{\tau}) y_2(\hat{\tau})}{\sqrt{1 + y^2_2(\hat{\tau})}} \right) \frac{\sqrt{1 + y^2_2(\hat{x})} - \sqrt{1 + y^2_2(\hat{\tau})}}{(\hat{x} - \hat{\tau})^2 + (\sqrt{1 + y^2_2(\hat{x})} - \sqrt{1 + y^2_2(\hat{\tau})})^2}$$

$$\times \frac{1}{\sqrt{1 + y^2_2(\hat{x})} - \sqrt{1 + y^2_2(\hat{\tau})}^2}$$

The lemma 8.1 yields

$$\sup_{\hat{x} \in [0,1]} \int_0^{\tau} |F - F_0| d\hat{\tau} = \hat{x} o(1), \quad \epsilon \to 0$$

for every fixed $T > \sigma > 0$. For the integration over $[0, \sigma]$, we get

$$\sup_{\hat{x} \in [0,1]} \int_0^\sigma (|F| + |F_0|) d\hat{\tau} \lesssim \sup_{\hat{x} \in [0,1]} \hat{x} \int_0^\sigma \frac{(\hat{x} + \hat{\tau}) \hat{\tau}}{\hat{x}^2 + \hat{\tau}^2} d\hat{\tau} \lesssim \sigma \hat{x}$$

This gives

$$\sup_{\hat{x} \in [0,1]} \int_0^{\tau} |F - F_0| d\hat{\tau} = \hat{x} o(1), \quad \epsilon \to 0$$
Now, for $\hat{x} \in [0, 1]$, the asymptotics for large $\tilde{\tau}$ are

$$ F = -\frac{4\hat{x}\tau y_1(\tilde{\tau}) y_2(\tilde{\tau})}{(\tilde{\tau}^2 + y_2^2(\tilde{\tau}))^2} + O(\tilde{\tau}^{-2}), \quad F_0 = -\frac{4\hat{x}\tau y_1(\tilde{\tau}) \tilde{y}_2(\tilde{\tau})}{(\tilde{\tau}^2 + \tilde{y}_2^2(\tilde{\tau}))^2} + O(\tilde{\tau}^{-2}) $$

and therefore

$$ \sup_{\tilde{x} \in [0, 1]} \int_T^{1/\lambda} |F - F_0| d\tilde{\tau} = o(1) \hat{x} \log^+ \lambda + CT^{-1} $$

That shows $U_1$ is small uniformly in $\lambda$ and $\hat{x} \in [0, 1]$. Similarly, we can handle an interval $\tilde{x} \in [0, L]$ with arbitrary large fixed $L$. In case of $\tilde{x} > L$, we can treat the interval $|\tilde{\tau} - \tilde{x}| < 1$ using lemma 8.1 as before. Outside this interval, we again use (51) to get (compare with (52))

$$ \int_1^{1/\lambda} |F - F_0| d\tilde{\tau} \lesssim \hat{x} \int_0^{1/\lambda} u |f(1) - f(u)| \overline{g(1) - g(u)} \frac{((1 + u)^2 + (g(1) - g(u))^2)((1 - u)^2 + (g(1) - g(u))^2)}{(1 + u)^2 + (g(1) - g(u))^2} - \frac{((1 + u)^2 + (\tilde{g}(1) - \tilde{g}(u))^2)((1 - u)^2 + (\tilde{g}(1) - \tilde{g}(u))^2)}{(1 + u)^2 + (\tilde{g}(1) - \tilde{g}(u))^2} \right| du + \log^+ \lambda $$

Computing the asymptotics at infinity, we obtain that the last quantity is

$$ o(1) \hat{x} \log^+ x, \quad \epsilon \to 0 $$

Then,

$$ \sup_{\hat{x} > L} \frac{o(1) \hat{x} \log^+ \hat{x} + \log^+ \lambda}{\hat{x} \log^+(\lambda^2(x^2 + 1))} = o(1) + L^{-1/4}, \quad L < \lambda^{-1/2} $$

and that proves that $U_1$ is small.

The combination $L_1 + L_2$ is handled similarly. We need the following lemma for that.

**Lemma 6.5.** Suppose $y_1, y_2, \tilde{y}_2 \in \hat{C}[0, \lambda^{-1}]$ and

$$ \|y'_1\|_\infty \leq 1, \quad \|y'_2 - \tilde{y}'_2\|_\infty \leq \epsilon, \quad \|y'_2 - 1\|_\infty \ll 1 $$

If one defines

$$ V = \frac{1}{\hat{x} \log^+(\lambda^2(x^2 + 1))} \int_0^{1/\lambda} \left| y_2(\tilde{\tau}) y'_2(\tilde{\tau}) \left[ \left( \frac{y_1(\hat{x}) y_2(\tilde{x})}{\sqrt{1 + y_2^2(\hat{x})}} + \frac{y_1(\hat{x}) y_2(\tilde{x})}{\sqrt{1 + y_2^2(\tilde{x})}} \right) \times \left( \frac{\sqrt{1 + y_2^2(\hat{x})} + \sqrt{1 + y_2^2(\tilde{x})}}{(\hat{x} + \tilde{x})^2 + (\sqrt{1 + y_2^2(\hat{x})} + \sqrt{1 + y_2^2(\tilde{x})})^2} - \frac{\sqrt{1 + y_2^2(\hat{x})} + \sqrt{1 + y_2^2(\tilde{x})}}{(\hat{x} - \tilde{x})^2 + (\sqrt{1 + y_2^2(\hat{x})} + \sqrt{1 + y_2^2(\tilde{x})})^2} \right) \right] \right| d\tilde{\tau} $$

then, uniformly in $y_1, y_2, \tilde{y}_2$ and $\lambda \in (0, 1)$, we have

$$ \|V\|_{L^\infty[0, 1/\lambda]} = o(1), \quad \epsilon \to 0 $$

43
Proof. The proof of this lemma is nearly identical. It is actually even simpler as the singularities in the denominator are absent.

The bound \( (54) \) follows easily from the arguments given in the proofs of lemmas 6.4 and 6.5. The proof of the theorem 6.2 is now finished.

6.2. The bound (46). The estimate (46) was in fact already proved in the previous subsection. Indeed, recall (12). The derivative of \( F \) involves six terms: \( I_1 + \ldots + I_6 \).

For instance, \( I_2 \) gives the following operator
\[
\left( \frac{1}{x \sqrt{x^2 + \lambda^2 \log^+(x^2 + \lambda^2)}} f \int_0^1 K_1(x, \tau, \sqrt{\lambda^2 + f^2}) d\tau \right) v'
\]
from \( \dot{C}^1[0, 1] \) to \( C[0, 1] \). We take \( f = x + u \) where \( \|u\|_{\dot{C}^1[0, 1]} \leq \epsilon \) and one needs to show that
\[
\sup_{\lambda \in (0, 1], \|v\|_{\dot{C}^1[0, 1]} \leq 1, \|v\|_{\dot{C}^1[0, 1]} \leq \epsilon} \left\| \frac{1}{x \sqrt{x^2 + \lambda^2 \log^+(x^2 + \lambda^2)}} \left( f \int_0^1 K_1(x, \tau, \sqrt{\lambda^2 + f^2}) d\tau - x \int_0^1 K_1(x, \tau, \sqrt{\lambda^2 + x^2}) d\tau \right) v' \right\|_{C[0, 1]} = o(1), \quad \epsilon \to 0
\]
The proof of that, however, repeats the one of (48) where \( \rho = x \). All other terms corresponding to \( \{I_j\}_{j \neq 2} \) can be handled similarly and that gives (46).

7. The proof of the main theorem

In this section, we will prove the theorem 1.1.

Proof. We can rewrite the equation (7) as
\[
\psi = \mathcal{O} \psi
\]
and the items (a), (b), and (c) stated on the same page were all justified. In particular, we can choose sufficiently small \( \delta \) and \( \lambda_0 \) such that for every \( \lambda \in (0, \lambda_0) \) the operator \( \mathcal{O} \) has the unique fixed point in \( B_{\delta} = \{ \psi : \|\psi\|_{\dot{C}^1[0, 1]} \leq \delta \} \). It follows from the construction and (44) in particular that the solution
\[
y(x, \lambda) = \sqrt{\lambda^2 + (x + \psi(x, \lambda))^2}
\]
converges to \( |x| \) as \( \lambda \to 0 \). Moreover, one immediately has \( y(x, \lambda) \in C^1[-1, 1] \).

Remark. The self-similar behavior around the origin is an immediate corollary of (44).

Let us prove now that the solution \( y(x, \lambda) \) is actually infinitely smooth.

Theorem 7.1. For every \( \lambda \in (0, \lambda_0) \) we have \( y(x, \lambda) \in C^\infty(-1, 1) \).

Proof. The bound (6) implies that \( K_1(x, \xi, y) > 0 \) and thus \( \int_{-1}^1 K(x, \xi, y) d\xi > 0 \) as well. We have
\[
y'(x, \lambda) = \frac{\int_{-1}^1 y'(\xi, \lambda) K(x, \xi, y) d\xi}{\int_{-1}^1 K(x, \xi, y) d\xi}
\]
and one might want to differentiate this expression consecutively hoping to use the standard bootstrapping argument. Notice that the integrals can be written as

$$\int_{-1}^{1} g(\xi) \log((x - \xi)^2 + (y(x) - y(\xi)^2)d\xi$$

where $g$ is either equal to 1 or to $y'(\xi)$. The logarithm can be split as

$$\log((x - \xi)^2 + (y(x) - y(\xi)^2) = 2 \log|x - \xi| + \log\left(1 + \left(\frac{y(x) - y(\xi)}{x - \xi}\right)^2\right)$$

and only the second term can present the difficulty for bootstrapping the smoothness to $C^\infty(-1, 1)$ due to lemma 8.2.

Suppose we fix $\lambda$ so small that the contraction mapping works. We take $H_\delta(x) = \log(\sqrt{\delta^2 + x^2})$ instead of $H(x) = \log x$ and denote the corresponding kernel by $K_\delta$. Then, in a similar way, one can prove the existence of $y_\delta(x, \lambda)$ and $y_\delta(x, \lambda) \to y(x, \lambda), \delta \to 0$ uniformly over $[-1, 1]$. Since $H_\delta \in C^\infty(-1, 1)$, we immediately get $y_\delta(x, \lambda) \in C^\infty(-1, 1)$. We want to obtain estimates on $\|y_\delta\|_{C^n[-a,a]}$ that are uniform in $\delta$.

To this end, proceed by induction. Differentiate

$$y_\delta'(x) \int_{-1}^{1} K_\delta(x, \xi, y_\delta)d\xi = \int_{-1}^{1} y_\delta'(\xi) K_\delta(x, \xi, y_\delta)d\xi$$

$n$ times consecutively. This gives

$$y^{(n+1)}_\delta(x) \int_{-1}^{1} K_\delta(x, \xi, y_\delta)d\xi + \Omega_n(x) = \delta^{(n)} \int_{-1}^{1} y_\delta'\delta(x, \xi, y_\delta)d\xi$$

Our inductive assumption is that $\|y_\delta\|_{C^n[-b,b]} < C(n, b), b < 1$, uniformly in $\delta$. Therefore, the estimates on the uniform norm for $\Omega_n$ and the right-hand side can be established by application of lemmas 8.2 and 8.4 in Appendix. That gives

$$\|y^{(n+1)}_\delta\|_{C^n[-a,a]} < C_n(\epsilon)\|y_\delta\|_{C^{n+\epsilon-1,-b,b]} + \tilde{F}(\|y_\delta\|_{C^n[-b,b]}), \quad a < b$$

This inequality implies that

$$\|y_\delta\|_{C^{n+1,-b,b]} < C(n + 1, b), \quad b < 1$$

uniformly in $\delta$. Indeed, one only has to take $\epsilon = 1/2$ and use the interpolation, e.g.

$$\|f\|_{C^{1/2}[-1,1]} \lesssim \sqrt{\|f\|_{C^{1}[-1,1]} \|f\|_{C[-1,1]}}$$

for the term $\|y_\delta\|_{C^{n+0.5,-b,b]}$.

That gives $y(x, \lambda) \in C^n[-a, a]$ for every $n$. Indeed, there is a sequence $\{y_\delta\} \to u$ in $C^n[-a, a]$ by Arzela-Ascoli and so $u \in C^n[-a, a]$. However this includes the uniform convergence so $y = u$. Since $n$ is arbitrary, we get the statement of the theorem. \qed
8. Appendix

Lemma 8.1. If \( \| f' - g' \|_{C[0,1]} \leq \delta \), then
\[
\left| \frac{f(x) - f(y)}{x - y} - \frac{g(x) - g(y)}{x - y} \right| \leq \delta
\]
uniformly in \( x, y, f \), and \( g \).

Proof. Indeed, it follows from the following representation
\[
\mathcal{Y}_f(x, y) = \frac{f(x) - f(y)}{x - y} = \int_0^1 f'(y + (x - y)t) dt
\]  
(58)

The next lemmas are needed to show that the solution \( y(x, \lambda) \) is infinitely smooth.

Lemma 8.2. Suppose \( f \in C^\infty(-1, 1) \) and \( 0 < a < b \leq 1 \). Then,
\[
\left\| \int_{-1}^1 f(\xi) \log |x - \xi| d\xi \right\|_{C^n[-a,a]} < C(n,a,b,\epsilon) \| f \|_{C^{n+1+a-b,b}}
\]

Proof. The displacement structure of the kernel implies that it is sufficient to prove the
statement for \( n = 1 \) only. This amounts to checking that
\[
\left\| \int_{-1}^1 \frac{f(x) - f(\xi)}{x - \xi} d\xi \right\|_{C^{1,-[b,b]}} \lesssim \| f \|_{C^{1,-b,b}}
\]
which is trivial.

Lemma 8.3. Suppose \( y(x) \in C^\infty[-1, 1] \). Then
\[
\left\| \int_{-1}^1 \log \left( 1 + \left( \frac{y(x) - y(\xi)}{x - \xi} \right)^2 \right) d\xi \right\|_{C^{1,-[-1,1]}} < C_\epsilon \| y \|_{C^{\epsilon+[-1,1]}}
\]
with \( C_\epsilon \) independent of \( y \).

Proof. We write (58) and differentiate to get
\[
\left| \int_{-1}^1 \frac{2\mathcal{Y}_f(x, \xi)}{1 + \mathcal{Y}_f^2(x, \xi)} \left( \int_0^1 f''(\xi + (x - \xi)t) dt \right) d\xi \right| \lesssim \int_{-1}^1 \left| \int_0^1 \frac{\partial_t(f'/(x - \xi)t - f'(\xi)}{x - \xi} t dt \right| d\xi
\]
\[
\lesssim \int_{-1}^1 \frac{\| f' \|_{C^\epsilon[-1,1]}|x - \xi|^\epsilon}{|x - \xi|} d\xi \lesssim \epsilon^{-1} \| f \|_{C^{1+\epsilon[-1,1]}}
\]

By consecutive differentiation, one gets

Lemma 8.4. Suppose \( y(x) \in C^\infty[-1, 1] \). Then
\[
\left\| \int_{-1}^1 \log \left( 1 + \left( \frac{y(x) - y(\xi)}{x - \xi} \right)^2 \right) d\xi \right\|_{C^{n,-[-1,1]}} < C_n(\epsilon) \| y \|_{C^{n+\epsilon[-1,1]}} + F_n(\| y \|_{C^n[-1,1]})
\]
where \( F_n \) is a certain function of \( \| y \|_{C^n[-1,1]} \) only.
Remark. The lemmas 8.2 and 8.4 will hold true if we replace $\log x$ by $\log \sqrt{x^2 + \delta^2}$. The resulting estimates will be $\delta$ independent.

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