Conformally flat travelling plane wave solutions
of Einstein equations

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Abstract
We discuss conformally flat plane wave solutions of Einstein equations
depending on the plane wave phase $\xi = \omega \tau - qx$, where $\tau$ is the conformal
time. We show that ideal fluid Einstein equations and scalar fields with
exponential self-interaction have solutions of this form. We consider in
more detail the source depending on $\xi$ with $\omega = |q|$ describing models
of a massless scalar field, electromagnetic field and relativistic particles
with space-time depending mass density. We obtain explicit conformally
flat metrics solving Einstein equations with such a source of the energy-
momentum.

1 Introduction

On a large scale the universe looks isotropic and homogeneous. Then, the
dynamics involves only the expansion scale factor $a$. The resulting $\Lambda$CDM model
describes well [1] the observational data. An assumption of the isotropy and
homogeneity allows to derive explicit solutions of Einstein equations [2]. There is
however some tension concerning the value of the Hubble constant resulting from
CMB and supernova observations. It may be that the problem can be explained
by inhomogeneities observed on a local scale [3][4]. Inhomogeneous cosmological
models with a spherical symmetry have been extensively studied (see the review
in [5]). At an intermediate scale there are some phenomena (voids, walls )
which disturb the homogeneous isotropic picture [6][4]. For a recent review
of anisotropic cosmoologies see [7]. Particular solutions of Einstein equations
may have a cosmological meaning reflecting some observed inhomogeneities and
anisotropies in galaxy distribution and in CMB. In this paper we find anisotropic
and inhomogeneous solutions of Einstein equations resulting from scalar and
electromagnetic fields as a source of plane waves. We make an assumption that
the metric is determined by the energy-momentum evolving like a plane wave in
a direction $\mathbf{q}$, i.e., that it depends on $\xi = \omega \tau - \mathbf{q} \cdot \mathbf{x}$, where $\tau$ is the conformal time. We consider conformally flat metrics (see [8] for their cosmological relevance). Then, its scale factor $a$ also depends on $\xi$. We did not encounter such an explicit assumption in general relativity although the plane-wave Ansatz is a standard tool in classical theory of scalar waves (solitons). Plane-wave solutions of Einstein equations are discussed (and classified) from the group-theoretical in [2] (sec.37). We think that such plane-waves may describe idealized thin walls encountered in astronomical observations [6]. Einstein equations with a given lhs could be treated as a definition of the energy-momentum on the rhs. This is the way the Riemannian geometry is exploited in the plasma physics [9]. However, without a local Lagrangian defining the energy-momentum on the rhs we would in general get non-local and acausal theories. If $\omega^2 > |\mathbf{q}|^2$ then we show that the plane-waves are solutions of Einstein equations with an ideal fluid on the rhs (as the energy-momentum tensor). Such solutions are Lorentz boosts to an arbitrary frame of the well-known homogeneous solutions in the frame moving with the fluid (see [10] for an application of such boosts). Another explicit travelling wave solution results when the source is a free scalar field or a scalar field with an exponential interaction. The exact solutions discussed in this paper contribute to the exploration of gravitational disturbances travelling with a velocity less than the velocity of light.

We discuss in more detail the case $\omega^2 = |\mathbf{q}|^2$. In this case the travelling plane wave moves with the velocity of light. We show that a massless free field or an electromagnetic field of a plane wave can be a source of the gravitational travelling plane wave. As another source we consider a particle body with a space-time dependent mass. We believe that the travelling gravitational waves moving with the velocity of light may be relevant near the strong sources where the linear approximation to Einstein equations is not sufficient (the well-known plane fronted exact gravitational waves are solutions of sourceless Einstein equations [2]). In general, it is not simple (see [11]) to divide the metric resulting from various sources into the radiative and non-radiative parts. Nevertheless, both parts influence the geodesic motion of a test body, hence are measurable.

The plan of the paper is the following. In sec.2 we discuss the conformal flat metrics. In sec.3 we show that if the source is an ideal fluid then there are solutions of Einstein equations in the form of a travelling plane wave. In sec.4 we discuss scalar fields as a source of the energy-momentum tensor. We show that in this case of the travelling wave there is a (dispersion) relation between $\omega$ and $|\mathbf{q}|$. In sec.5 we solve the geodesic equation in a gravitational field of the travelling plane-wave. In sec.6 we discuss Lagrangian models of the energy-momentum leading to plane wave solutions with $\omega = |\mathbf{q}|$. In sec.7 we obtain some explicit solutions for the metric of the plane wave with $\omega = |\mathbf{q}|$. In sec.8 we summarize the results.
2 The conformally flat metric

We consider the conformally flat metric in four space-time dimensions (we set the velocity of light \( c = 1 \))

\[
ds^2 = a(x)^2(dx^2 - d\mathbf{x}^2) = a(x)^2 \eta^{\mu\nu} dx_\mu dx_\nu, \tag{1}
\]

where \( \eta \) is the Minkowski metric. Then, the components of the Einstein tensor

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R
\]

are \([12][2]\)

\[
G^{00} = 3(a^{-1}\partial_\tau a)^2 + (a^{-1}\nabla a)^2 - 2a^{-1}\triangle a, \tag{2}
\]

\[
G^{0j} = 4a^{-2}\partial_\tau a\partial_j a - 2a^{-1}\partial_\tau \partial_j a, \tag{3}
\]

\[
G^{jk} = 4a^{-2}\partial_j a\partial_k a + \delta_{jk}(\omega^2 - q^2)(\partial_\tau a)^2 - 2a^{-1}(\partial_j \partial_k a + \delta_{jk}(\partial_\tau^2 - \nabla^2) a). \tag{4}
\]

We assume that the fields as well as the scale factor depend only on the plane wave phase

\[
\xi = \omega \tau - \mathbf{q} \cdot \mathbf{x} = \eta^{\mu\nu} q_\mu x_\nu, \tag{5}
\]

where \((q_\mu) = (\omega, \mathbf{q})\). Then (where \( q^2 = q^2 \))

\[
G^{00} = (3\omega^2 + q^2)a^{-2}\left(\frac{da}{d\xi}\right)^2 - 2a^{-1}\frac{d^2 a}{d\xi^2} \tag{6}
\]

The remaining components of the Einstein tensor as functions of \( \xi \) are

\[
G^{0j} = \omega q_j a^{-2}\left(4\left(\frac{da}{d\xi}\right)^2 - 2a\frac{d^2 a}{d\xi^2}\right), \tag{7}
\]

\[
G^{jk} = a^{-2}\left(4q_j q_k + \delta_{jk}(\omega^2 - q^2)\right)\left(\frac{da}{d\xi}\right)^2 - 2a^{-1}(q_j q_k + \delta_{jk}(\omega^2 - q^2))\frac{d^2 a}{d\xi^2}. \tag{8}
\]

If \( \partial_\mu a \) is not a null vector then Einstein equations can be treated as an identity defining an energy-momentum tensor for a relativistic viscous fluid \([13][14]\)(this way Einstein equations are treated in the quark-gluon plasma \([9]\)). We write \( T_{\mu\nu} = (8\pi G)^{-1} G_{\mu\nu} \) (where \( G \) is the Newton constant) then \( G_{\mu\nu} \) can be expressed as

\[
(8\pi G)^{-1} G_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu} + \Pi_{\mu\nu}, \tag{9}
\]

where the fluid velocity is defined by

\[
u_\mu = \partial_\mu a \left(\partial_\mu a \partial^\mu a\right)^{-\frac{1}{2}}. \tag{10}
\]

The energy density is

\[
8\pi G \rho = 3a^{-2} g^{\mu\nu} \partial_\mu a \partial_\nu a - g^{\mu\nu} \Pi_{\mu\nu} \tag{11}
\]
the pressure

\[ 8\pi Gp = a^{-2} g^{\mu\nu} \partial_\mu a \partial_\nu a + g^{\mu\nu} \Pi_{\mu\nu}, \]

(12)

where

\[ 8\pi G\Pi_{\mu\nu} = -2a^{-1} \partial_\mu \partial_\nu a. \]

(13)
The conservation law \((G^{\mu\nu})_{;\nu} = 0\) can be decomposed into a continuity equation for the fluid \(u_\mu (T^{\mu\nu})_{;\nu} = 0\) and a Navier-Stokes type equation \((g_{\sigma\mu} - u_\sigma u_\mu) (T^{\mu\nu})_{;\nu} = 0\).

For space-like \(q^\mu\) the square root in eq.(10) makes no sense. Then, we define

\[ u_\mu = \partial_\mu a \left( - \partial_\mu a \partial^\mu a \right)^{-\frac{1}{2}}. \]

(14)

\[ 8\pi G\rho = -3a^{-2} g^{\mu\nu} \partial_\mu a \partial_\nu a + g^{\mu\nu} \Pi_{\mu\nu} \]

(15)

\[ 8\pi Gp = a^{-2} g^{\mu\nu} \partial_\mu a \partial_\nu a + g^{\mu\nu} \Pi_{\mu\nu}, \]

(16)

Hence

\[ (8\pi G)^{-1} G_{\mu\nu} = (\rho - p) u_\mu u_\nu - p g_{\mu\nu} + \Pi_{\mu\nu}, \]

(17)

It follows from eq.(9) that for general \(a\) the Einstein tensor \(G_{\mu\nu}\) has the form of the energy-momentum tensor of a viscous fluid. However, if \(a\) depends only on \(\xi\) then we can write \(G_{\mu\nu}\) as the energy-momentum of an ideal fluid (we restrict ourselves to time-like \(q^\mu\))

\[ (8\pi G)^{-1} G_{\mu\nu} = (\tilde{\rho} + \tilde{p}) \tilde{u}_\mu \tilde{u}_\nu - \tilde{p} g_{\mu\nu}, \]

(18)

where

\[ \tilde{u}_\mu = q_\mu a(\omega^2 - q^2)^{-\frac{1}{2}}, \]

(19)

\[ 8\pi G\tilde{p} = (a^{-4} \frac{da}{d\xi})^2 - 2a^{-3} \frac{d^2 a}{d\xi^2}(\omega^2 - q^2), \]

(20)

\[ 8\pi G\tilde{\rho} = 3a^{-4} \left( \frac{da}{d\xi} \right)^2 (\omega^2 - q^2). \]

(21)

We have two equations (20)-(21) for one function \(a\). A solution of eq.(21) for \(\rho\) determines \(p\) in eq.(20). However, with \(p\) and \(\rho\) satisfying eqs. (20)-(21) Einstein equations with \(T_{\mu\nu}\) of eq.(9) (with \(u_\mu\) defined in eq.(19)) will be satisfied.

The energy-momentum \(T_{\mu\nu}\) of the gravitational field proportional to \(G_{\mu\nu}\) is covariantly conserved. We introduce the energy-momentum of the matter field \(Q_{\mu\nu}\) so that

\[ 8\pi Gl_{\mu\nu} = G_{\mu\nu} + Q_{\mu\nu} \]

(22)

satisfies

\[ \partial_\mu \mu^{\mu\nu} = 0. \]

(23)
We obtain
\[ 8\pi Gt_{\mu\nu} = (a^{-2}(\frac{da}{d\xi})^2 - 2a^{-1}\frac{d^2\xi}{d\xi^2})(q_{\mu}q_{\nu} - \eta_{\mu\nu}(\omega^2 - q^2)) \] (24)
and
\[ 8\pi GQ_{\mu\nu} = 3a^{-2}(\frac{da}{d\xi})^2 q_{\mu}q_{\nu}. \] (25)

If \((q^\mu) = (1,0)\) then \(8\pi GQ_{00} = 3(a^{-1}\partial_\tau a)^2\) as in the model with the energy-momentum \(Q_{00}\) of a homogeneous fluid.

### 3 Ideal fluids

We have ten Einstein equations
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi GT_{\mu\nu} \] (26)
for a single conformal factor \(a\). Eqs.(26) are not of the same type. \(G_{jk}\) equations are hyperbolic whereas equations for \(G_{00}\) and \(G_{0j}\) are parabolic (first order in time derivatives). For \(G_{jk}\) we can insert arbitrary initial conditions for time derivatives of \(a\) whereas the initial value for the time derivative of \(a\) in the equations for \(G_{00}\) and \(G_{0j}\) is determined by the initial conditions for \(T_{\mu\nu}\). The distinction between the \(G_{\mu\nu}\) equations is not explicit for the \(\xi\) equations (which are of the second order in \(\xi\)). However, if the \(G_{00}\) equation is to have the \(q = 0\) limit then we have to choose a proper initial condition for \(\partial_\xi a\). If we solve one of Eqs.(26) then the remaining equations determine the other components of the energy-momentum tensor. It remains an open problem whether the energy-momentum tensor defined this way follows from a Lagrangian field theory. As an example let us consider the \(\delta_{jk}\) part of \(G_{jk}\) equation
\[ 8\pi G\bar{p} = (a^{-4}(\frac{da}{d\xi})^2 - 2a^{-3}\frac{d^2\xi}{d\xi^2})(\omega^2 - q^2) \] (27)
Choosing
\[ \bar{p} = p_0a^{-3-3w}(\omega^2 - q^2) \] (28)
we can integrate eq.(27) with the result
\[ a^{-2}(\frac{da}{d\xi})^2 = (k_0 - 8\pi G(3w)^{-1}p_0)a^{-1} + 8\pi Gp_0(3w)^{-1}a^{-1-3w} \] (29)
where \(k_0\) is an arbitrary constant. By differentiation of eq.(29) we obtain
\[ 2a^{-1}\frac{d^2a}{d\xi^2} = (k_0 - 8\pi G(3w)^{-1})a^{-1} + 8\pi Gp_0(3w)^{-1}(1 - 3w)a^{-1-3w} \] (30)
We can then calculate all terms $G_{\mu\nu}$ and $T_{\mu\nu}$ as functions of $a$. In particular, from eq.(21)

\[ \dot{\rho} = 3(\omega^2 - q^2)(8\pi G)^{-1}(k_0 - 8\pi G(3w)^{-1})a^{-3} + \rho_0 a^{-3-3w} \]  

We obtain the standard relation

\[ \dot{\rho} = w\dot{\rho} \]  

if and only if

\[ k_0 = 8\pi G(3w)^{-1} \]  

The initial condition (33) ensures that in the limit $q \to 0$ we obtain the standard homogeneous solution of Einstein equations with the Ansatz (28) and (32).

If we solve eq.(21) with

\[ \rho = \rho_0 a^{-3-3w} \]  

and insert the solution to eq.(20) then we obtain the result (32) and the initial condition (33). This solution is a standard extension $\tau \to \xi$ of the homogeneous solution. We obtain this way a Lorentz transformation of the homogeneous solution if $\tau$ and $\xi$ are related by a Lorentz transformation (see the end of the next section).

### 4 Scalar field as a source

Let us consider the Lagrangian for scalar fields

\[ L = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V \]  

The energy-momentum tensor is

\[ T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} L \]  

Let

\[ V = \gamma \exp(\alpha \phi) + V_0, \]  

where $V_0$ is a constant. We assume that the fields depend only on $\xi$. Then, the Lagrange equation for $\phi$ reads

\[ (\omega^2 - q^2) \frac{d}{d\xi} a^2 \frac{d}{d\xi} \phi = -a^4 V'' \]  

We solve the $G_{0j}$ Einstein equations first with (from eq.(36))

\[ T_{0j} = \omega q_j (\frac{d\phi}{d\xi})^2 \]
assuming
\[ \left( \frac{d\phi}{d\xi} \right)^2 = \kappa^2 a^{-r}, \] (40)
where \( \kappa \) is a constant. Let us note that if \( V = V_0 \) (\( \gamma = 0 \) in eq.(37) ) then \( r = 4 \).

Eq.(26) for \( G_{ij} \) can be integrated (with the Ansatz (40); we choose the initial condition \( a(0) = 1 \))
\[ a^{-2} \left( \frac{da}{d\xi} \right)^2 = (k_0 - 8\pi G\kappa^2(r + 2)^{-1})a^2 + 8\pi G\kappa^2(r + 2)^{-1}a^{-r} \] (41)
with an integration constant \( k_0 \). It follows from eq.(41) that
\[ 2a^{-1} \frac{d^2a}{d\xi^2} = 4(k_0 - 8\pi G\kappa^2(r + 2)^{-1})a^2 + 8\pi G\kappa^2(2 - r)(r + 2)^{-1}a^{-r}. \] (42)

We can insert the results (41)-(42) into the remaining Einstein equations and into the Lagrange equations for \( \phi \). If \( \gamma = 0 \) then the remaining Einstein equations have the solution if
\[ 3(k_0 - 8\pi G\kappa^2(r + 2)^{-1})(\omega^2 - q^2) = 8\pi GV_0, \] (43)
whereas the Lagrange equations (38) require \( r = 4 \) in eq.(40) (solutions \( a(\xi) \) of eq.(41) are discussed in sec.7 below eq.(85)) .

When \( \gamma \neq 0 \) then we must have \( V_0 = 0 \) (hence there is zero on the rhs of eq.(43)). Then, the Lagrange equations (38) and the remaining Einstein equations are solved if
\[ a = \sigma \exp(\beta \phi), \] (44)
\[ (r + 2)\beta^2 = 8\pi G, \] (45)
\[ \frac{\alpha}{\beta} = -r - 2. \] (46)

Then, in addition the dispersion relation
\[ \omega^2 = q^2 + \frac{2\gamma}{\kappa^2} \sigma^{r+2} + \frac{2}{4-r} \] (47)
must be satisfied. We obtain
\[ a^{\frac{3}{2}} = 1 \frac{r\kappa \beta \xi}{2} \]

\( G_{\mu\nu} \) is covariant with respect to Lorentz transformations \( L \). Hence, if the energy-momentum tensor on the rhs of eq.(26) is also Lorentz covariant then
\[ (L^{-1})_{\mu}^\alpha (L^{-1})_{\nu}^\beta G_{\alpha\beta}(Lx) = 8\pi G (L^{-1})_{\mu}^\alpha (L^{-1})_{\nu}^\beta T_{\alpha\beta}(Lx). \] (48)

Note that \( \xi = q_{\mu} x_{\nu} \) is Lorentz invariant. In a special Lorentz frame it may take the form \( \xi = \omega_0 \tau \) where \( \omega_0^2 = \eta_{\mu\nu} q_{\mu} q_{\nu} \) if \( q_{\mu} \) is time-like or \( \xi = -q_{10} x^1 \).
where $q_{10}^2 = -\eta^{\mu\nu}q_{\mu}q_{\nu}$ if $q_{\mu}$ is space-like. We may solve Einstein equations (26) in a special frame (with a special choice of $q$). Then, on the basis of eq.(48) they will hold true for a general $\xi$. In eq.(47) $\omega$ and $q$ cannot be connected by a Lorentz transformation. Hence, the solution (44)-(47) does not follow from the well-known homogeneous one [15] (except of the case of $q = 0$).

As an example we may apply the Lorentz transformation when $V = 0$ and $\omega^2 \neq q^2$ in eq.(38). Then

$$\partial_\xi \phi = \kappa a^{-2}$$

(49)

with a certain constant $\kappa$. Choose $\omega_0 = 1$ and $q = 0$. We solve eq.(6) for $G_{00}$ ($a \simeq \tau^{1/2}$). In the resulting solution $a(\tau)$ we replace $\tau$ by $\xi = \gamma_\tau - v_\gamma x_1$ where $(q_\mu) = (\gamma, v_\gamma, 0, 0)$ with $\gamma = (1 - v^2)^{-1/2}$ coming from the Lorentz transformation (an application of such a boost of a homogeneous solution is discussed in [10]).

We check that

$$a^2 = \sqrt{\frac{16\pi G}{3} \xi}$$

(50)

solves all of 10 Einstein equations (26) with $T_{\mu\nu}$ of eq.(36) with $V = 0$ transformed to an arbitrary frame by eq.(48). The solution (50) has $\frac{\omega^2}{q^2} = \frac{1}{v^2} > 1$.

We could do the same procedure for the space-like solution of the $G_{00}$ equation (then $\frac{\omega^2}{q^2} = v^2 < 1$). We could also solve another component of the Einstein equations (26) in a special Lorentz frame and subsequently transform the solution to an arbitrary frame with a general $\xi$.

5 Test-body motion

We consider a geodesic motion of a body under the influence of the conformally flat gravity. The geodesic equation is

$$\frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\alpha\nu} \frac{dx^\alpha}{ds} \frac{dx^\nu}{ds} = 0,$$

(51)

where

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta = a^2 d\tau^2 (1 - \left(\frac{dx}{d\tau}\right)^2).$$

Inserting the Christoffel symbols $\Gamma$ we derive a simple equation for $\xi$

$$\frac{d^2 \xi}{ds^2} + 2 \left(\frac{d\xi}{ds}\right)^2 \frac{d}{d\xi} \ln a = -\frac{1}{2} (\omega^2 - q^2) \frac{d}{d\xi} a^{-2}.$$  

(52)

Denoting $\nu = \frac{d\xi}{ds}$ we can integrate eq.(52) with the result

$$\nu^2 = (K_0 - (\omega^2 - q^2)a_0^2)a^{-4} + (\omega^2 - q^2)a^{-2}$$

(53)
where \( K_0 \) and \( a_0 \) are constants of integration. From eq.(53) we can calculate \( a(s) \)

\[
\int da \frac{da}{ds}^{-1} \left( (K_0 - (\omega^2 - q^2)a_0^2)a^{-4} + (\omega^2 - q^2)a^{-2} \right)^{-\frac{1}{2}} = s, \quad (54)
\]

where \( \frac{da}{d\xi} \) is determined by eq.(29) or (41). Subsequently, we can obtain \( \xi(s) \) when \( a(\xi) \) is known.

The velocity \( \frac{d\xi}{ds} \) is expressed by the phase velocity \( \mathbf{u} \)

\[
av = a \frac{d\xi}{ds} = (\omega - q\mathbf{u}) \frac{1}{\sqrt{1 - u^2}}
\]

with

\[
\mathbf{u} = \frac{dx}{d\tau}.
\]

The phase velocity is related to the coordinate velocity in proper time

\[
\frac{dx}{ds} = a^{-1} \mathbf{u} \frac{1}{\sqrt{1 - u^2}}
\]

In order to determine the motion beyond the \( \xi \) plane let us define

\[
v^\mu = \frac{dx^\mu}{ds} - q^\mu (\omega^2 - q^2)^{-1}\frac{d\xi}{ds}
\]

Inserting \( v^\mu \) into the geodesic equations (51) we obtain

\[
\frac{dv^\mu}{ds} + 2v^\mu \frac{d\ln(a)}{ds} = 0
\]

Hence,

\[
v^\mu(s) = v^\mu(0)a(\xi(s)))^{-2}a(\xi(0)))^2
\]

where \( \xi(s) \) is determined from the solution of eq.(52). In principle, we could detect the source of gravity observing its action upon a test body. However, it would be difficult to separate the astronomical sources from the local ones.

### 6 Einstein equations for \( \omega = q \)

When \( \omega = q \) then the travelling plane waves generated by the energy-momentum \( T_{\mu\nu} \) move with the velocity of light. We consider various sources of such waves: massless scalar fields, electromagnetic fields and a relativistic particle with a continuous spectrum of mass. These fields produce a deformation of the space-time which propagates with the velocity of light. The waves will interact with massive test-bodies and charged particles. They carry an energy and momentum which are conserved because if \( \omega = q \) then in eqs.(6)-(8) \( \partial_\mu G^{\mu\nu} = 0 \) hence also \( \partial_\mu T^{\mu\nu} = 0 \).
As a source of the energy-momentum in this section we consider a scalar field with the Lagrangian

$$L_h(\phi) = h(W),$$  \hspace{1cm} (55)

where $h$ is an arbitrary function and

$$W = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi$$

The electromagnetic field with the Lagrangian

$$L_{em} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$  \hspace{1cm} (56)

provides a source moving with a velocity of light. We discuss also a relativistic particle with $x$-dependent energy density $m$ with the Lagrangian

$$L_p(x) = \frac{1}{2} \int ds g_{\mu\nu}(X) \frac{dX^\mu}{ds} \frac{dX^\nu}{ds} m(x) \delta(X(s) - x).$$  \hspace{1cm} (57)

From the Lagrangian $-\frac{R}{8\pi G} + L$ we obtain Einstein equations (26). The scalar field Lagrangian defines the energy-momentum tensor

$$T_{\mu\nu}^h = h'^2 \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} h$$  \hspace{1cm} (58)

From the Lagrangian (56) we obtain

$$T_{\mu\nu} = g^{\alpha\beta} F_{\mu\alpha} F_{\nu\beta} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}.$$  \hspace{1cm} (59)

The energy-momentum of the relativistic particle (57) is defined as

$$T_{\mu\nu} = \int ds \frac{dX^\mu}{ds} \frac{dX^\nu}{ds} \delta(X(s) - x)m(x).$$  \hspace{1cm} (60)

The Lagrangian equations for the scalar field (under the assumption that the fields depend only on $\xi$) are

$$\partial_\mu (h' \sqrt{-g} g^{\mu\nu} \partial_\nu \phi) = (\omega^2 - q^2) \partial_\xi (h' a^2 \partial_\xi \phi) = 0.$$  \hspace{1cm} (61)

It follows from eq.(61) that if $\omega^2 = q^2$ then for every function $h$ any $\phi(\xi)$ satisfies the scalar wave equation. Note that if $\omega^2 = q^2$ then $W = 0$. Hence, $(h(W) = h(0)$ and $h'(W) = h'(0)$ are constants in the energy-momentum (58).

The electromagnetic field satisfies the Maxwell equations

$$(F^{\mu\nu})_{;\mu} = 0$$  \hspace{1cm} (62)

(together with $(\epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta})_{;\mu} = 0$). The Lagrange equations for the particle read

$$\frac{d}{ds} \left( a^{-2}(x) \frac{dX^\mu}{ds} \delta(X(s) - x)m(x) \right) = a(x)^{-2} \eta^{\alpha\beta} \frac{dX_\alpha}{ds} \frac{dX_\beta}{ds} m(x) \frac{\partial}{\partial X^{\alpha}(s)} \delta(X(s) - x)$$  \hspace{1cm} (63)
where \( x = \xi \). If \( q_0 = |q| \) then eq.(63) and the constraint \( x = \xi \) are solved by

\[
X_\mu(s) = q_\mu s + x. \tag{64}
\]

Then, \( \eta^{\mu\nu} X_\mu q_\nu = \eta^{\mu\nu} x_\mu q_\nu = \xi \). Under the assumption that \( a \) depends only on \( \xi \) (5) with \( \omega^2 = q^2 \) eqs.(2)-(4) can be expressed in the form

\[
G_{\mu\nu} = q_{\mu} q_{\nu} \left( 4a^{-2} \left( \frac{da}{d\xi} \right)^2 - 2a^{-1} \frac{d^2 a}{d\xi^2} \right), \tag{65}
\]

where \((q_\mu) = (q, q)\).

The components of the scalar field energy-momentum tensor (on solutions (61)) are

\[
T_{00} = h' \omega^2 \left( \frac{d\phi}{d\xi} \right)^2 - a^2 h, \tag{66}
\]

\[
T_{0j} = h' q_j \omega \left( \frac{d\phi}{d\xi} \right)^2, \tag{67}
\]

\[
T_{jk} = h' q_j q_k \left( \frac{d\phi}{d\xi} \right)^2 + a^2 h \delta_{jk}. \tag{68}
\]

Comparing eqs.(66)-(68) with eq.(65) we can see that if 00 component of Einstein equations is satisfied then the remaining components will be satisfied if

\[
h(0) = 0. \tag{69}
\]

We assume the normalization \( h'(0) = 1 \) as for the free massless scalar field (when \( h(W) = W \)).

After an insertion of the solution (64) the particle energy-momentum (60) is

\[
T^\nu_\mu = q_\mu q^\nu q^{-1} m(\xi). \tag{70}
\]

In the case of the electromagnetic field (56) it is known (see ,e.g.,[16] ) that for the plane wave solutions

\[
T^\nu_\mu(\xi) = \omega^{-2} \rho_{em}(\xi) q_\mu q^\nu, \tag{71}
\]

where \( \rho_{em} \) is the electromagnetic energy density. It follows from eqs.(65) and (66)-(71) that in the Lagrangian models of this section it is sufficient to solve the 00 component of Einstein equations. After the solution of the 00 component the remaining equations will be satisfied.

We have no a priori restriction on the density \( \rho(\xi) q_\mu q_\nu = a^{-2} T^{\mu\nu} \) in the formulas (66)-(71) for \( T^{\mu\nu} \). \( T^{\mu\nu}(\xi) \) can be expressed as a function of \( a \). For a comparison with standard models we choose this \( a \)-dependence in a power-law form as it is obtained for ideal fluids with the equation of state \( p = w p \) (where \( p \) is the pressure and \( w \) is a constant). It is sufficient to restrict ourselves to one
of the models (66)-(71). In the scalar field model the 00 component of Einstein equations reads

$$4a^{-2}\left(\frac{d\phi}{d\xi}\right)^2 - 2a^{-1}\frac{d^2a}{d\xi^2} = 8\pi Gq^{-2}T_{00} = 8\pi Gq^{-2}\left(\frac{d\phi}{d\xi}\right)^2$$  \hspace{1cm} (72)

where we express $\frac{d\phi}{d\xi}$ as a function of $a$.

Eq.(72) is solved with the initial conditions $a(\xi = 0) = a_0$ and $\frac{da}{d\xi}(\xi = 0) = u_0$. Let $K = u^2$ then eq.(72) can be expressed as

$$\frac{dK}{da} - 4a^{-1}K = -8\pi Gq^{-2}aT_{00}.$$  \hspace{1cm} (73)

Eq.(73) can be integrated

$$\left(\frac{da}{d\xi}\right)^2 = K = a^4k_0 - 8\pi Gq^{-2}a^4\int_{a_0}^a b^{-3}T_{00}(b)db,$$  \hspace{1cm} (74)

where we introduced an integration constant $k_0 \geq 0$ related to $u_0 = \pm a_0^2\sqrt{\kappa}$.

If $T_{00} = 0$ then $a = a_0 = const$ is a solution of eq.(74) with $k_0 = u_0 = 0$. However, there is also a non-trivial solution. It can be checked that if $\omega = q$ then

$$\frac{1}{a} = \frac{1}{a_0} \pm \sqrt{k_0}\xi$$  \hspace{1cm} (75)

is a solution of the Einstein equation (26) with the energy-momentum tensor $T_{\mu\nu} = 0$. This is a special case of the plane fronted gravitational waves (flat polarization) when they are conformally flat (as noted in [17][18]).

We assume that $T_{00}$ is a power of $a$ according to the conventional assumption on the ideal fluids in an expanding universe

$$\rho = a^{-2}T_{00} = a^{-2}\kappa^2\left(\frac{d\phi}{d\xi}\right)^2 = \rho_0a^{-3-3w}.$$  \hspace{1cm} (76)

We may assume the $a$-dependence because a function of $\xi$ can be expressed as a function of $a$ (if $a(\xi)$ is invertible). So, we may assume that $\phi(\xi) = \tilde{\phi}(a)$ (we skip "tilde" further on). Then

$$\left(\frac{d\phi}{d\xi}\right)^2 = \left(\frac{d\phi}{da}\right)^2\left(\frac{da}{d\xi}\right)^2.$$  \hspace{1cm} (77)

Hence,

$$\left(\frac{d\phi}{da}\right)^2 = \kappa^2\rho_0a^{-1-3w}K^{-1},$$  \hspace{1cm} (78)

where from eqs.(74) and (76)

$$K(a) = (k_0 - 2\alpha^2(3w + 3)^{-1})a^4 + 2\alpha^2(3w + 3)^{-1}a^{1-3w},$$  \hspace{1cm} (79)
where
\[ \alpha^2 = 4\pi G \rho_0 c^{-2} \] (80)
and we set \( a_0 = 1 \) (\( a_0 > 0 \) has no physical meaning it just rescales coordinates).

Taking the square root in eq.(79) and integrating we can calculate \( \phi \) as a function of \( a \).

Note that if we know \( \phi(a) \) then eq.(77) can be expressed in another form suitable for integration
\[ \left( \frac{da}{d\xi} \right)^2 = K_0 a^4 \exp \left( -8\pi G \int da (\frac{d\phi}{da})^2 a \right). \] (81)

Eq.(74) can be expressed as
\[ \left( \frac{da}{d\xi} \right)^2 = -V(a), \] (82)
where \( V = -K \) is a potential of a particle moving with the kinetic energy \( K \) on the half-line \( a \geq 0 \) with the total energy \( E = 0 \). The motion is possible in the range of \( a \) such that \( -V \geq 0 \). The evolution \( a(\xi) \) based on eqs.(74) and (82) is discussed in the next section.

7 Elementary solutions of Einstein equations

Taking the square root of eq.(79) and integrating we obtain the equation for \( a \)
\[ \int_1^a db K(b)^{-\frac{1}{2}} = \pm \xi. \] (83)

We consider the cases when the integral (83) can be expressed by elementary functions or elliptic functions (such integrals have been discussed in [19][20][21]).

Let \( w = 1 \) in eq.(76) (stiff matter [22]). Denote
\[ 16\sigma_s = k_0 - \frac{1}{3}\alpha^2 \]
and consider \( A = a^2 \). Then
\[ K = a^{-2}(16\sigma_s a^6 + \frac{1}{3}\alpha^2). \] (84)

If \( \sigma_s > 0 \)
\[ \int_1^A dA \left( 4A^3 + \frac{\alpha^2}{48\sigma_s} \right)^{-\frac{1}{2}} = 8\sqrt{\sigma_s}(\xi + \xi_0) \] (85)
then the integral is expressed by the Weierstrass elliptic function \( \mathcal{P}(8\sqrt{\sigma_s}(\xi + \xi_0), 0, -\frac{\alpha^2}{48\sigma_s}) \) [19][20]. We can extend the upper limit in eq.(85) to infinity.
showing that if $\sigma_s > 0$ then $a$ achieves infinity for a finite $\xi$ (then the energy density tends to zero). If $\sigma_s < 0$ (as in eq.(43)) then the range of $a$ is bounded by the requirement $K \geq 0$.

For $w = \frac{1}{3}$ (relativistic matter)

$$K = (k_0 - \frac{1}{2} \alpha^2)a^4 + \frac{1}{2} \alpha^2. \quad (86)$$

Hence, again the integral (83) is expressed by an elliptic function. If $k_0 - \frac{1}{2} \alpha^2 > 0$ then there is an explosion at finite $\xi$, whereas if $k_0 - \frac{1}{2} \alpha^2 < 0$ then $a$ varies in a bounded interval.

Let us consider now negative $w$. First, $w = -\frac{1}{3}$ (cosmic string or coasting cosmology)

$$K(a) = a^4(k_0 - \alpha^2) + \alpha^2 a^2. \quad (87)$$

Let

$$4\sigma = \alpha^2 - k_0. \quad (88)$$

Then, the integral (83) gives

$$a = \alpha^2 \exp(-\alpha(\xi + \xi_0)) \left( \alpha^2 \exp(-2\alpha(\xi + \xi_0)) \right)^{-1}, \quad (89)$$

where $\xi_0$ is chosen such a way as to satisfy the initial condition $a(0) = 1$. If $\sigma < 0$ then $a \to \infty$ at finite $\xi$ (then the energy density tends to zero). If $\sigma > 0$ then $a$ is a bounded function ($a \to 0$ when $\xi \to \pm \infty$).

If $w = -\frac{2}{3}$ (domain wall) then

$$K = (k_0 - 2\alpha^2)a^4 + 2\alpha^2 a^3. \quad (90)$$

Eq.(83) gives

$$a = 2\alpha^2 \left( \alpha^2(\xi + \xi_0)^2 - k_0 + 2\alpha^2 \right)^{-1}. \quad (91)$$

If $2\alpha^2 < k_0$ then $a$ explodes at finite $\xi$ (the energy density tends to 0). If $2\alpha^2 > k_0$ then $a$ is a bounded function.

Let $w = -\frac{4}{3}$ (phantom matter)

$$K = (k_0 + 2\alpha^2)a^4 - 2\alpha^2 a^5. \quad (92)$$

From the integral (83) we obtain

$$\begin{align*}
\xi + \xi_0 &= -(k_0 + 2\alpha^2)^{-1}a^{-1} \sqrt{k_0 + 2\alpha^2 - 2\alpha^2 a} \\
&+ \alpha^2(k_0 + 2\alpha^2)^{-\frac{3}{2}} \ln \left( \left( \frac{\sqrt{k_0 + 2\alpha^2} - \sqrt{k_0 + 2\alpha^2 - 2\alpha^2 a}}{\sqrt{k_0 + 2\alpha^2} + \sqrt{k_0 + 2\alpha^2 - 2\alpha^2 a}} \right)^{-1} \right) \quad (93)
\end{align*}$$

$\xi_0$ is chosen to satisfy the initial condition $a(\xi = 0) = 1$. $a$ varies in the interval $[1, 1 + \frac{k_0}{2\alpha^2}]$. 

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If \( w = -\frac{5}{3} \) then
\[
K = (k_0 + \alpha^2)a^4 - \alpha^2 a^6.
\] (94)

The integral is again an elementary function. This problem is similar to the previous one so we do not write down (rather complicated) explicit formula. If \( w = -2 \) then \( a \) varies in a bounded interval, it can be expressed by an elliptic function. If \( w < -2 \) then the integral (83) can be expressed neither by an elementary nor an elliptic function.

8 Summary

We have discussed some exact solutions of Einstein equations in the form of conformally flat travelling plane waves. Such gravitational deformations of space-time can propagate with the velocity smaller than the velocity of light. We can test their source on the basis of a geodesic motion of a test body but it may be difficult to distinguish the astronomical sources from the local ones. If the source is an ideal fluid then the plane waves can be considered as a transformation of the homogeneous expanding solution to the moving frame. However, general solutions cannot be reduced by a Lorentz transformation to the well-known homogeneous case. In the case when the phase velocity is equal to the velocity of light the travelling waves may contain gravitational waves produced by a massless scalar field or an electromagnetic field. The metric is not asymptotically flat. Hence, the gravitational field is not of the type of pure radiation (it may contain a non-radiation background). The electromagnetic field which is produced when neutron stars merge can be a source of plane conformally flat gravitational plane waves. These waves could possibly be detected in a different experimental set-up than the one so far prepared for a detection of the transverse (TT) gravitational waves.

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