Weak values, 'negative probability' and the uncertainty principle.

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Abstract

A quantum transition can be seen as a result of interference between various pathways (e.g. Feynman paths) which can be labelled by a variable \( f \). An attempt to determine the value of \( f \) without destroying the coherence between the pathways produces the weak value of \( \bar{f} \). We show \( \bar{f} \) to be an average obtained with amplitude distribution which can, in general, take negative values which, in accordance with the uncertainty principle, need not contain information about the actual range of \( f \) which contribute to the transition. It is also demonstrated that the moments of such alternating distributions have a number of unusual properties which may lead to misinterpretation of the weak measurement results. We provide a detailed analysis of weak measurements with and without post-selection. Examples include the double slit diffraction experiment, weak von Neumann and von Neumann-like measurements, traversal time for an elastic collision, the phase time, the local angular momentum (LAM) and the 'three-box case' of Aharonov et al.

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I. INTRODUCTION

In his book with Hibbs [1] Feynman formulates the uncertainty principle as follows: 'Any determination of an alternative taken by a process capable of following more than one a alternative destroys the interference between alternatives'. Thus the converse is true: for a quantum system which can reach its final state via a number interfering pathways the uncertainty principle forbids specifying which of the routes has actually been taken. The latter can be quantified as follows: if the interfering pathways are labelled by some variable \( f \), the value of \( f \) must remain, in some sense, indeterminate.

The concepts of interfering pathways provides a convenient description of a general quantum measurement [3], [4]. For a quantum system prepared in a state \( \Psi_0 \) and later observed (post-selected) in a state \( \Psi_1 \), the transition amplitude between the states can be written as a sum over virtual paths traced by some variable \( \hat{A} \) (e.g., Feynman paths, if \( A \) represents the coordinate), which can be arranged according to the value \( f \) of some functional \( F[\text{path}] \), e.g., the value of \( \hat{A} \) at some intermediate time, or its time average. The classes form form a discreet or continuous set of pathways connecting \( \Psi_0 \) and \( \Psi_1 \) and the value of \( f \) can be determined if a system is subjected to a measurement which converts, in full analogy with the double-slit interference experiment [1], interfering pathways into exclusive ones.

One wonders then what would be the result of trying to obtain some information about \( f \) while keeping the interference intact. A straightforward attempt to write down even an average answer fails since no probabilities can be ascribed to the pathways, while an 'average' formally constructed with the probability amplitudes \( \Phi(f) \)

\[
\bar{f} \equiv \frac{\sum f \Phi(f)}{\sum \Phi(f)}
\]

is complex valued and its physical significance is not immediately clear. Alternatively, one can consider a von Neumann measurement with the interaction between the measured system and the meter deliberately reduced in order to minimise the perturbation incurred and analyse the meter’s readings. This general method based on analysing inaccurate, or weak, quantum measurements, was originally formulated by Aharonov et al [5, 6, 7] and is closely related to the attempts to define the duration of a scattering event still debated in literature [8, 9, 10]. The original approach to the problem, which leads to the so-called Wigner-Eisenbad time delay, or the phase time, relies on following the centre of mass of the scattered wavepacket, [8, 9, 11, 12] and has been shown to be equivalent to a weak
measurement experiment 13. A different method was proposed by Baz’ 14, 15, 16, 17 who considered a particle weakly coupled to a Larmor clock. Baz’ criticised the phase time of Smith 12 as generally incorrect and pointed out that elastic collision time must be a sharply defined quantity 15, 17. The Larmor time was shown to be a weak value of the traversal time functional 18 in 19, 20, 21, and its relation to the complex time obtained with the help of Eq.(11) was established in 18, 22. The introduction of such a complex time has often been criticised on the grounds that any observable physical quantity must be real (see, for example, 8, 9).

Finally, the local angular momentum (LAM), a quantity given by an expression broadly similar to the phase time, has recently been employed in 23 in order to identify angular momenta which contribute to an angular distribution at a given scattering angle.

All above examples have a common purpose of determining, in some sense, the value of a physical variable without choosing between the alternatives which contribute to a transition. The purpose of the present paper is to analyse the origin, properties and general usefulness of weak values which occur in various contexts. The term ‘negative probability’ was introduced by Feynman 24 and was used in connection with Wigner functions 25, 26 and the tunnelling time 27. In a slightly different context of this paper we will show that the weak values are related to alternating ‘improper’ distributions which arise because a probability amplitude may take negative values. In agreement with the uncertainty principle, averaging with such distributions effectively ’hides’ the information about interfering pathways. Relative complexity of such analysis owes to fact that while, in the absence of probabilities, certain quantities may exhibit obviously wrong values, they do not have always to do. Thus, there is danger of extrapolating between different cases each of which needs instead to be analysed separately.

The rest of the paper is organised as follows. In Section 2 we list some elementary properties of ‘improper’ non-positive distributions and their moments. In Sect 3 we consider an inaccurate classical meter and show that it is still possible to extract the results of an accurate measurement from the meter’s readings. In Section 4 we consider the quantum version of the classical meter. In Sect. 5 we show that, unlike in the classical case, results of an accurate quantum measurements cannot be extracted from the readings of a quantum meter with a large (quantum) uncertainty in its initial position. Rather the results are expressed in terms of the mean $\bar{f}$ and the higher moments of the improper distribution $\Phi(f)$ in Eq.(11). In
Sect. 6 we consider, as an example, the double-slit experiment and equivalent measurement on a two-level system. In Sect. 7 we show that if the reading of an inaccurate meter are average over the final states of the system, the results are expressed in terms of the mean and variance of obtained with a real distribution \( w_1 \) which is not, in general, non-negative. In Section 8 we show the impulsive von Neumann measurement without post-selection to be a special case where the distribution \( w_1 \) is non-negative and coincides, as in the classical case of Sect. 3, with the probability distribution of an accurate measurement. In Section 9 we use the results of Sect. 7 to show Baz’ conclusion that the elastic collision time is sharply defined \([15, 17]\) to be wrong. In Sect. 10 we show the Wigner-Eisenbad phase time to be a weak value similar to \([11]\) and briefly discuss some of its anomalous properties. In Sect. 11 we show the LAM \([23]\) to be a particular kind of a weak value. In Sect. 12 we briefly discuss the ‘three box case’ considered in \([7, 28]\) and further discussed in \([29, 30]\). We will show that the Aharonov, Leibowitz and Bergmann (ABL) rule to be a simple consequence of Feynman’s rule for ascribing probabilities and suggest an alternative interpretation of the ‘three box paradox’ based on the uncertainty principle. Section 13 contains our conclusions.

II. PROPER, IMPROPER AND COMPLEX DISTRIBUTIONS

Consider a real function \( \rho(f) \), contained within the interval \( 0 \leq x \leq 1 \), which can be used to construct a normalised distribution,

\[
w(f) \equiv \frac{\rho(f)}{\int_{0}^{1} \rho(f) df}.
\]  

(2)

If \( \rho(f) \) does not change sign, \( w(f) \) is non-negative and can, therefore, be used as a proper probability distribution to calculate various moments of the random variable \( f \). In this Section we will assume that \( \rho \) may change sign within the interval and list the consequences for the moments and averages calculated with such an improper distribution.

a) While for \( w(f) \geq 0 \) the expectation value

\[
\langle f \rangle \equiv \frac{\int_{0}^{1} f \rho(f) df}{\int_{0}^{1} \rho(f) df}
\]

(3)

always lies in the region containing the support of \( \rho \), i.e., between 0 and 1, for an alternating \( \rho(f) \) this is no longer true. as the normalisation integral in Eq. (2) can take either sign or vanish. For a simple example consider a function

\[
\rho'(f) \equiv \sin(2\pi f) + \epsilon \quad for \quad 0 \leq f \leq 1, \quad and \quad 0 \quad otherwise
\]

(4)
with the first three moments
\[ \int \rho(f) df = \epsilon, \quad \langle f \rangle = 1/2 - 1/2\pi \epsilon, \quad \langle f^2 \rangle = 1/3 - 1/2\pi \epsilon \] (5)
which yields an improper distribution for \(|\epsilon| \leq 1\). For \(\epsilon \to \pm 0\), the value of \(|\langle f \rangle| \to \infty\) becomes anomalously large. In general, an improper expectation value no longer gives an estimate for the location of the support of the corresponding distribution \(w(f)\).

b) While, for a proper distribution \(w(f) \geq 0\), the equality
\[ \langle f^2 \rangle - \langle f \rangle^2 = 0 \] (6)
forces the conclusion that \(f\) is a sharply defined quantity,
\[ w(f) = \delta(f - \langle f \rangle), \] (7)
the variance (6) may vanish for a broad alternating distribution. For example, for the \(\rho'(f)\) in Eq.(4) this would be the case for
\[ \epsilon = \pm 3^{1/2}/\pi \] (8)
In general, examining the first two moments of a (possibly) improper distribution does allow to establish whether the variable \(f\) is sharply defined.

c) for a proper probability distribution the fact that
\[ \int_a^b w(f) df = 0, \] (9)
where \([a, b]\) lies inside the interval \([0, 1]\), guarantees that the support of \(\rho(x)\) is contained between 0 and \(a\) and \(b\) and 1, while for an improper distribution making such an assumption leads to an obvious contradiction. For example, for
\[ w(f) = 3\pi/2 \sin(3\pi f) \] (10)
we have \(\int_0^{2/3} w(f) df = \int_{1/3}^1 w(f) df = 0\) and adopting the above reasoning we must conclude that \(\rho(f)\) with certainty takes its values in each of two different regions, \(0 \leq f \leq 2/3\) and \(1/3 \leq f \leq 1\). Moreover, there is an infinite number of ways to construct subintervals of \([0, 1]\) in each of which the integral (9) would vanish. Thus, we cannot, in general, uniquely determine which part of the interval \(0 \leq f \leq 1\) contributes to the normalisation integral
\[\int_0^1 \rho(f)df\] even though parts of the interval must be redundant due to the cancellation, and have to conclude that all values of \(f\) between 0 and 1 are equally important. Relation between this observation and the uncertainty principle, already evident, will be discussed further in Sect. 12.

Finally, consider a normalised distribution (2) constructed with the help of a complex valued function \(\rho(f)\), whose real and imaginary parts are contained in the interval \(0 \leq f \leq 1\),

\[
\rho(f) = \rho_1(f) + i\rho_2(f),
\]

\[
\int_0^1 \rho(f)df = A_1 + iA_2.
\]

Any such distribution can be written as a sum of its real and imaginary parts

\[
w(f) \equiv w_1(f) + iw_2(f) = \frac{A_1^2(\rho_1(f)/A_1) + A_2^2(\rho_2(f)/A_2)}{A_1^2 + A_2^2} + iA_1A_2 \frac{\rho_2(f)/A_2 - \rho_1(f)/A_1}{A_1^2 + A_2^2},
\]

normalised to unity and zero, respectively,

\[
\int_0^1 w_1(f)df = 1, \quad \int_0^1 w_2(f)df = 0.
\]

It is readily seen that \(w_1(f)\) is a is a proper distribution provided both \(\rho_1\) and \(\rho_2\) do not change sign for \(0 \leq f \leq 1\), while , \(w_1(f)\) must alternate and can never be a valid probability distribution. For and improper \(w_1(f)\) one can expect to obtain anomalously large values of

\[
Re\langle f \rangle = \int_0^1 fw_1(f)df
\]

when both normalisation integrals \(A_1\) and \(A_2\) are small. For example, for

\[
\rho(f) = \rho^{\epsilon_1}(f) + i\rho^{\epsilon_2}(f)
\]

we have

\[
Re\langle f \rangle = \frac{1}{2} - \frac{1}{2\pi} \frac{\epsilon_1 + \epsilon_2}{\epsilon_1^2 + \epsilon_2^2}
\]

which can indeed take very large positive and negative values in the vicinity of \(\epsilon_1 \approx \epsilon_2 \approx 0\), whereas for \(|\epsilon_1|, |\epsilon_2| > 1\), where \(w_1(f)\) in non-negative, \(Re\langle f \rangle\) remains positive and bounded by 1.

In summary, the use of proper probability distributions largely relies on interpreting the
mean and variance as the location and width of the region which contains physically significant values of the random variable $f$. An improper distribution cannot, in general, be used for this purpose. Analysis of this Section may seem an exercise of little practical importance, as it is not immediately clear under which circumstances a physical quantity may be described by an alternating let alone complex-valued alternating distribution. We will, however, show that averages associated with such improper distributions naturally arise when one attempts to obtain an answer to a question conventionally not answered by quantum mechanics, such as determining the slit take by a particle in a diffraction experiment, or obtaining the value of a physical variable without perturbing the particle’s motion. But first we consider the low accuracy limit of a purely classical measurement.

III. INACCURATE CLASSICAL MEASUREMENTS

Consider a classical meter with a pointer position $f$ and a momentum $\lambda$ coupled to a one-dimensional particle moving in a potential $V(x)$. The Hamiltonian for such a system is

$$H(p, x, \lambda) = \frac{p^2}{2m} + V(x) + \beta(t)\lambda A(p, x)$$

(18)

where the switching function $\beta(t)$ determines the strength of the coupling and $A(p, x)$ is the variable to be measured. It is readily seen that if the (conserved) momentum and the initial position of the meter are put to zero, the pointer position time $t$ is given by

$$f(t) = \int_0^t \beta(t')A(p, x)dt'$$

(19)

so that the meter monitors the value of the functional in the r.h.s. of Eq.(19) on the trajectory $\{p(t'), x(t')\}$ which is unaffected by the measurement. Let us assume in addition, that at $t = 0$ the initial momentum and position of the particle, $P$ and $X$, are not known precisely, but rather are random quantities distributed with the probability density $w(P, X)$. Since each trajectory is uniquely labeled by the values $(P, X)$, the value of the functional (19) is a random variable with a probability distribution

$$w(f) = \int dPdX \delta(f - \int_0^t A(p, x)dt')w(P, X) \equiv \langle \delta(f - \int_0^t A(p, x)dt') \rangle_w.$$ 

(20)

and the first two moments of the form

$$\langle f \rangle \equiv \int df w(f) = \int_0^t dt' < A(t') >_w$$

(21)
and
\[ < f^2 > \equiv \int df f^2 w(f) = 2 \int_0^t dt' \int_0^{t'} dt'' < A(t')A(t'') >_w, \tag{22} \]
where, as in Eq. (20), \( < ... >_w \) denotes an average over all initial values \( P \) and \( X \).

Consider next a meter whose final position can be determined accurately, but whose initial position is uncertain and is distributed around \( f = 0 \) with a normalised probability density \( G(f) \) with a zero expectation value and a known variance. The two sources of uncertainty result in a simple convolution formula for the normalised distribution \( W(f) \) of the meter’s readings,
\[ W(f) = \int G(f - f')w(f')df'. \tag{23} \]

Evaluating the generating function
\[ \langle \exp(-i\lambda f) \rangle_W = \int \exp(-i\lambda f)W(f)df \tag{24} \]
for the \( n \)-th moment of \( W(f) \) we find
\[ \langle f^n \rangle_W = 2\pi i^n \partial_\lambda^n \{ \tilde{G}(\lambda)\tilde{w}(\lambda) \} |_{\lambda=0} \tag{25} \]
where the tilde denotes the Fourier transform, e.g.,
\[ \tilde{G}(\lambda) \equiv (2\pi)^{-1} \int \exp(-i\lambda f)G(f)df, \tag{26} \]
\( \partial_\lambda^n \) is the \( n \)-th derivative with respect to \( \lambda \) and we used the convolution property
\[ \tilde{W}(\lambda) = \tilde{G}(\lambda)\tilde{w}(\lambda). \tag{27} \]

Applying the Leibniz product rule to the derivative (25) yields
\[ \langle f^n \rangle_W = \sum_{k=1}^{n} C^n_k \langle f^n \rangle_G \langle f^n \rangle_w, \tag{28} \]
where \( C^n_k \) are the binomial coefficients and, in particular, \( \langle f \rangle_G = 0 \)
\[ \langle f \rangle_W = \langle f \rangle_w, \tag{29} \]
\[ \langle f^2 \rangle_W = \langle f^2 \rangle_G + \langle f^2 \rangle_w. \tag{30} \]

Next we ask what, if anything, can be learned about the mean and the variance of \( w(f) \) in the low accuracy limit, when the initial pointer position becomes highly uncertain,
\[ G(f) \rightarrow \alpha^{-1}G(f/\alpha), \quad \alpha \rightarrow \infty. \tag{31} \]
In this case $W(f)$ is a very broad distribution with the mean equal to that of $w(f)$ and a large variance $D_W \equiv (\langle f^2 \rangle_W - (\langle f \rangle_W)^2)^{1/2} \approx \alpha$. The value $\langle f \rangle_W$ can be obtained by taking the average $\langle f \rangle_N$ of $N$ consecutive measurements. Since the variance of $\langle f \rangle_N$ is given by $D/N$, so that to determine the value of $\langle f \rangle_w$ to an accuracy $\delta << 1$, one would require a very large number of measurements

$$N >> \alpha^2.$$  \hfill (32)

Similarly, if $G(f)$ is known to a sufficient accuracy, one can use Eq.(30) to determine the second moment of $w(f)$ and, therefore, the original variance of $f$. Note, however, that an accurate determination would require an even larger, $N >> \alpha^4$, number of observations.

In summary, a classical meter with an increasingly uncertain initial position is rendered impractical because, although its readings contain the information about the mean and variance of the measured variable, the number of trials required for its extraction becomes prohibitively large.

\section*{IV. QUANTUM METERS AND MEASUREMENTS}

Consider next a similar measurement in the quantum case. A detailed analysis of quantum meters has been given in [3, 4] and here we will limit ourselves to only a brief discussion required for further development. The Schrödinger equation describing a system coupled to a von Neumann-like [31] meter is ($\hbar = 1$)

$$i\partial_t |\Psi(t|f)\rangle = [\hat{H} - i\partial_f \beta(t)\hat{A}]|\Psi(t|f)\rangle.$$  \hfill (33)

Initially, the system is prepared in some state $|\Psi_0\rangle$, and the meter position is set to zero,

$$|\Psi(t = 0|f)\rangle = \delta(f)|\Psi_0\rangle,$$  \hfill (34)

Note that the Heisenberg’s uncertainty principle prevents one from setting the meter position to zero as well, which can be seen as the cause of the perturbation produced by the measurement [3]. After the measurement, at the time $t$, the state of the system is described by the density operator

$$\hat{\rho} = \int df |\Psi(t|f)\rangle\langle \Psi(t|f)|.$$  \hfill (35)
If this state is purified, i.e., after the measurement the system is post-selected in some state $|\Psi_1\rangle$, the probability amplitude $\Phi(f)$ to obtain the meter reading $f$ is given by

$$\Phi(f) = \langle \Psi_1 | \Psi(t|f) \rangle.$$  \hspace{1cm} (36)

A useful representation for $\Phi(f)$ is obtained by solving Eq.(33) by the Fourier transform,

$$\Phi(f) = (2\pi)^{-1} \int d\lambda \exp(i f \lambda) \langle \Psi_1 | \hat{U}_\lambda | \Psi_0 \rangle / \langle \Psi_1 | \hat{U}_0 | \Psi_0 \rangle$$ \hspace{1cm} (37)

$$\hat{U}_\lambda \equiv \exp(-i \int_0^t [\hat{H} + \lambda \beta(t') \hat{A}] dt').$$ \hspace{1cm} (38)

The measurement amplitude (37) can be related to the value of the functional (19) in the following way. Although there is no unique trajectory, as in the classical case, the transition amplitude between the initial and the final states of the system in the absence of the meter can be written as a sum over the virtual paths traced by the variable $A$,

$$\langle \Psi_1 | \exp(-i\hat{H}t) | \Psi_0 \rangle = \lim_{N \to \infty} \sum_{k_1,k_2,...k_N} \langle \Psi_1 | \prod_{j=1}^N |a_{k_j}\rangle$$ \hspace{1cm} (39)

$$\langle a_{k_j} | \exp(-i\hat{H}t/N) | a_{k_{j-1}} \rangle \langle a_{k_{j-1}} | \Psi_0 \rangle$$

$$\equiv \sum_{[a]} \langle \Psi_1 | \Phi[a] \rangle$$

where, as in the following, $a_k$ and $|a_k\rangle$ are the eigenvalues and eigenvectors of the variable of interest $\hat{A}$,

$$\hat{A}|a_k\rangle = a_k |a_k\rangle.$$ \hspace{1cm} (40)

It can be shown \cite{3} that, in the presence of the meter, $\Phi(f)$ in Eq.(37) is given by the restricted path sum

$$\Phi(f) = \sum_{[a]} \delta(f - \int_0^t \beta(t') a(t') dt') \langle \Psi_1 | \Phi[a] \rangle.$$ \hspace{1cm} (41)

The generalisation of Eq.(20) to the quantum case is, therefore, straightforward: a quantum pointer may be shifted by an amount $f$ if among the paths contributing to the transition some give value $f$ to the functional $F[a] = \int_0^t \beta(t') a(t') dt'$. The probability amplitude $\Phi(f)$ for the reading to occur is found by summing the amplitude $\langle \Psi_1 | \Phi[a] \rangle$ over all such paths. The support of $\Phi(f)$ (i.e., the set of $f$ such that $\Phi(f) \neq 0$) yields, therefore, the range
of the values of $f$ which contribute to the transition. It is also obvious that $\Phi(f)$ is a complex amplitude distribution whose normalisation integral is the unperturbed transition amplitude,

$$\int \Phi(f) df = \langle \Psi_1 | \exp(-i\hat{H}t) | \Phi_0 \rangle. \quad (42)$$

Since there are no apriori restrictions on the sign of either $\text{Re}\Phi(f)$ or $\text{Im}\Phi(f)$, we cannot, as discussed in Sect.2(c), in general determine which values within the support of $\Phi(f)$ contribute to the transition, in particular, when $\Phi(f)$ is of order of unity and $\langle \Psi_1 | \hat{U}(t) | \Phi_0 \rangle$ is very small. This is, in essence, the Feynman’s uncertainty principle. One exception is the classical limit, in which a highly oscillatory $\Phi(f)$ has a stationary region near the classical value $f = f_{\text{class}}$, which is the only contributor to the integral (42) [32].

As in Section 3 we proceed with a discussion of a meter whose initial position so that the initial meter state in the position representation is no longer a $\delta$-function but rather some $G(f)$, with a finite width is $\Delta f$. Then the amplitude $\Psi(f)$ to obtain the reading $f$ for the system post-selected in the state $|\Psi_1\rangle$ can be written is a convolution [3]

$$\Psi(f) = \int G(f - f') \langle \Psi_1 | \Phi(t | f') \rangle df' \quad (43)$$

which is similar to Eq.(29) with the important difference that it relates probability amplitudes rather than the probabilities themselves. If $\delta f$ is small, we find the probability to obtain a reading $f$ $\rho(f) \approx |\Phi(f)|^2$ so that an accurate meter measures the value of $F[a]$, and may be used to evaluate the centroid and the width of the range of $f$ values which contribute to the transition amplitude between the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$. There is, however, a price. A measurement perturbs the system, whose state after a measurement yielding $f$ results not equal to that without a meter,

$$\int G(f - f') \Phi(f') df' \neq \langle \Psi_1 | \exp(-i\hat{H}t) | \Psi_0 \rangle = \int \Phi(f') df', \quad (44)$$

where the last equality is obtained by integrating Eq.(11). The perturbation can be minimised by choosing $G(f)$ so broad that it can be replaced by a constant, making the l.h.s. of Eq.(7) proportional to $\exp(-i\hat{H}t) | \Psi_0 \rangle$ with an unimportant overall factor. Whereas an uncertainty in the classical meter’s initial position is clearly undesirable, a similar uncertainty in the quantum case has the advantage of reducing the perturbation a measurement produces on the measured system. One then wishes to know what kind of information about a quantum system can be obtained without affecting its evolution.
V. INACCURATE QUANTUM MEASUREMENTS, WEAK VALUES AND NEGATIVE PROBABILITY

Consider next the moments of a probability distribution for the meter’s readings,

$$\rho(f) = |\Psi(f)|^2 \quad (45)$$

where $\Psi(f)$, given by Eq.(43), is an complex valued and, possibly, alternating amplitude distribution. Representing $\Psi(f)$ as a Fourier integral, calculating the generating function $\langle \exp(-i\lambda f) \rangle_\rho$ and expanding the result in the powers of $k$ yields

$$\langle f^n \rangle \equiv \int f^n \rho(f) df / \int \rho(f) df = \int i^n \frac{\partial^n \Psi(\lambda)}{\Psi(\lambda)} |\tilde{\Psi}(\lambda)|^2 d\lambda / \int |\tilde{\Psi}(\lambda)|^2 d\lambda \quad (46)$$

Recalling that $i^n \partial^n \tilde{\Psi}(\lambda)/\tilde{\Psi}(\lambda) = \int f^n \exp(-i\lambda f) \Psi(f) df / \int \exp(-i\lambda f) \Psi(f) df$ shows that the moments of the proper probability distribution $\rho$ can be expressed via the moments of a family of improper complex distributions $\{\exp(-i\lambda f) \Psi(f)\}$ for all $-\infty < \lambda < \infty$.

Calculating the Fourier transform of the convolution (33) for the first two moments we obtain

$$\langle f \rangle = i \int \tilde{G}^* \tilde{\Phi}^*[\tilde{G}' \tilde{\Phi} + \tilde{G} \tilde{\Phi}'] d\lambda / \int |\tilde{G}|^2 |\tilde{\Phi}|^2 d\lambda \quad (47)$$

and

$$\langle f^2 \rangle = - \int \tilde{G}^* \tilde{\Phi}^*[\tilde{G}'' \tilde{\Phi} + 2\tilde{G}' \tilde{\Phi}' + \tilde{G} \tilde{\Phi}''] d\lambda / \int |\tilde{G}|^2 |\tilde{\Phi}|^2 d\lambda. \quad (48)$$

It is natural to choose the initial state of the meter to be a real even function with a width of order of unity, $G(f)^* = G(f)$, $G(f) = G(-f)$, e.g., a Gaussian, so that

$$\tilde{G}(\lambda) = \tilde{G}^*(\lambda), \quad \tilde{G}(\lambda) = \tilde{G}(-\lambda). \quad (49)$$

As in the classical case, the accuracy of the quantum measurement can be reduced by increasing the initial uncertainty of the pointer’s position,

$$G(f) \to G(f/\alpha), \quad \tilde{G}(\lambda) \to \tilde{G}(\alpha \lambda), \quad \alpha \to \infty. \quad (50)$$

As the uncertainty increases, the Fourier transform $\tilde{G}(\alpha \lambda)$ becomes sharply peaked around $\lambda = 0$, and the integrals (47) and (48) can be evaluated by expanding the terms containing $\tilde{\Phi}$ in the Taylor series. Estimating

$$\int G^*(\alpha \lambda) \partial^m \tilde{G}(\alpha \lambda) \lambda^m d\lambda = \alpha^{n-m-1} \int G^*(z) \partial^m \tilde{G}(z) z^m dz = O(\alpha^{n-m-1}) \quad (51)$$
and retaining the leading terms in $\alpha^{-1}$, we obtain

$$\langle f \rangle = \text{Re} \bar{f} + O(\alpha^{-1}), \quad (52)$$

and

$$\langle f^2 \rangle = \alpha^2 \frac{\int z^2 G(z)^2 dz}{\int G(z)^2 dz} + C(\text{Re} \bar{f}^2 - |\bar{f}|^2) + |\bar{f}|^2 + O(\alpha^{-1}). \quad (53)$$

where the factor $C$ is given by

$$C \equiv \frac{\int z^2 \tilde{G} \tilde{G}'' dz}{\int \tilde{G}^2 dz} - \frac{\int z^2 \tilde{G}^2 dz \int \tilde{G} \tilde{G}'' dz}{\int G^2 dz \int \tilde{G}^2 dz}. \quad (54)$$

and we have introduced the notation $\bar{f}^n$ for the $n$-th moment of the complex valued amplitude distribution $\Phi(f)$ defined in Eq.(41),

$$\bar{f}^n \equiv \frac{\int f^n \Phi(f) df}{\int \Phi(f) df} = (i)^n \partial^n \tilde{\Phi}(0)/\tilde{\Phi}(0), \quad (55)$$

Expressions similar to Eq.(52) have earlier been obtained in [5, 7] for a weak von Neumann measurement and in [18] for the quantum traversal time.

In summary, Eqs. (52) and (53), obtained here for an inaccurate von Neumann-like measurement of Sect.4, constitute a more general illustration of the uncertainty principle. Whenever probability amplitude for a variable $f$ is obtained by smearing the amplitude for a variable $f'$ with a broad envelope function so that the coherence between different values of $f'$ is not destroyed, evaluating $\langle f \rangle$ and $\langle f^2 \rangle$ does not, in general, reveal the mean and variance obtained in an accurate measurement of $f'$. Rather, the values $\langle f \rangle_w$ and $\langle f^2 \rangle_w$ in the classical Eqs.(29) and (30), are replaced by the weak value $\text{Re} \bar{f}$ and a complicated combination of $\bar{f}$ and $\bar{f}^2$, respectively. Since there is no restriction on the phase of $\Phi(f)$ in Eq.(55), interpretation of these these quantities as averages requires the concept of negative probability. As a result the information about the values of $f'$ which contribute to the transition, may be 'scrambled' by averaging with an improper alternating distribution.

**VI. WHERE WAS THE PARTICLE HALF WAY THROUGH A TRANSITION?**

This unhelpful property of the weak values $\bar{f}$ is most easily illustrated on the double-slit diffraction experiment. Consider a point on the screen such that the amplitudes to reach it via the slit 1 and the slit 2 are $\Phi(1) = 1$ and $\Phi(2) = -1 + \epsilon$, respectively, and attempt to
determine the mean slit number using Eq. (52). The variable \( f \) can only take two values 1 and 2, and the integrals in Eq. (46) are to be replaced by sums, which gives

\[
\bar{f} = 2 - \frac{1}{\epsilon}
\]  

(56)

For \( \epsilon = 0.1 \), Eq. (56) yields \( \bar{f} = -8 \) and it is difficult to interpret the notion that an electron passes on average through the slit number \(-8\) as anything other than a failure of our measurement procedure.

Analysis of Section 5 allows to apply exactly the same reasoning to a more conventional von Neumann measurement of the type considered in [5]. Consider a two level system with a zero Hamiltonian \( \hat{H} = 0 \) and the 'position' operator (c.f. the position operator \( \hat{x} = \int |x\rangle x\langle x|dx \) for a particle in one spatial dimension)

\[
\hat{A} = |1\rangle\langle 1| + |2\rangle\langle 2|,
\]  

(57)

prepared and post-selected in the states (\( N_0 \) and \( N_1 \) are the normalisation constants)

\[
|\Psi_0\rangle = N_0(|1\rangle + |2\rangle) \quad \text{and} \quad |\Psi_1\rangle = N_1(|1\rangle - (1 - \epsilon)|2\rangle),
\]  

(58)

respectively. To determine which state the system was at, say, \( t/2 \) we may employ a von Neumann meter with a Gaussian initial state

\[
G(f) = \exp(-f^2/\alpha^2).
\]  

(59)

Equation (39) shows that for \( \hat{H} = 0 \) only two paths connecting the initial and final states, \( a(t) = 1 \) and \( a(t) = 2 \), have non-zero probability amplitudes

\[
\Phi(1) = \langle \Psi_1 | 1\rangle \langle 1 | \Psi_0 \rangle = 1
\]  

(60)

\[
\Phi(2) = \langle \Psi_1 | 2\rangle \langle 2 | \Psi_0 \rangle = \epsilon - 1,
\]  

respectively (see Fig.1). To obtain the system’s position at \( t/2 \) we choose the switching function in Eq. (35) to be \( \beta(t') = \delta(t' - t/2) \) which yields

\[
\Phi(f) = \delta(f - 1) - (1 - \epsilon)\delta(f - 2)
\]  

(61)

and the average pointer position is given by

\[
\langle f \rangle = \frac{1 + 2(1 - \epsilon)^2 - 3(1 - \epsilon) \exp(-1/2\alpha^2)}{1 + (1 - \epsilon)^2 - 2(1 - \epsilon) \exp(-1/2\alpha^2)}
\]  

(62)
for an arbitrary resolution $\Delta f = \alpha$. In the high accuracy limit, $\alpha \to 0$, for $\epsilon << 1$ we obtain $\langle f \rangle \approx 1.5$ which indicates that the observed system would be found in each of the two states with equal probability. The probability of transition to the state $|\Psi_1\rangle$ would, however, be altered by the measurement,

$$P = N_0^2N_1^2(1 + (1 - \epsilon)^2) \neq N_0^2N_1^2\epsilon^2 = |\langle\Psi_1|\Psi_0\rangle|^2. \quad (63)$$

To keep the transition probability unchanged we may apply a highly inaccurate meter with $\alpha \to \infty$. For $\epsilon << 1$ the initial and final states are nearly orthogonal and, based on the discussion at the end of Sect.2, we expect the weak value obtained as $\alpha \to \infty$ to be without a direct relation to the two actual positions $f = 1$ and $f = 2$, which contribute to the transition. Indeed, in this limit we recover Eq.(56) and for $\epsilon = 0.1$ again find the measured mean position $\langle f \rangle = -8$. The dependence of $\langle f \rangle$ on the resolution $\alpha$ and the parameter $\epsilon$ is shown in Fig.2. Finally we note that, in a similar way, a transition amplitude for a system with three or more discrete states can be mapped onto a diffraction experiment with three or more slits. We will return to this analogy in Sect.7.

**VII. WEAK MEASUREMENTS WITHOUT POST-SELECTION.**

Until now we have assumed that the system is post selected after a measurement in a known state $|\Psi_1\rangle$ so that the meter reading are sampled only if it is found in $|\Psi_1|\,r_a$, and discarded otherwise. If the system’s final is not controlled and all the reading are kept, the results (52) and (53) must be averaged further with the probabilities $P_m$ to find the system in the state $|m\rangle$ belonging to some orthonormal set. As our measurement is weak, $P_m$ are essentially the same as in the absence of the meter,

$$P_m = |\langle m|\exp(-i\hat{H}_0t)|\Psi_0\rangle|^2 \quad (64)$$

so that we have

$$\langle\langle f^n\rangle\rangle \equiv \sum_m P_m \langle f^n\rangle_m. \quad (65)$$

Here and in the following extra angular bracket denotes average with $P_m$ and the (previously suppressed) subscript $m$ on and average or distribution indicates that it has been evaluated.
for the final state for $|\Psi_1\rangle = |m\rangle$. In particular, for the mean we have

$$\langle \langle f \rangle \rangle = \text{Re} \sum_m P_m \bar{f}_m \equiv \text{Re} \langle \bar{f} \rangle.$$  \hspace{1cm} (66)

where newly introduced average $\langle \bar{f} \rangle$ is the mean calculated with the distribution

$$\langle \Phi(f) \rangle \equiv \sum_m P_m \Phi_m(f) = (2\pi)^{-1} \int d\lambda \exp(i f \lambda) \langle \Psi_0 | \hat{U}_0^{-1} \hat{U}_\lambda | \Psi_0 \rangle \equiv w_1(f) + iw_2(f)$$ \hspace{1cm} (67)

which is a weighted sum of improper distributions $\Phi_m(f)$ and is, for this reason, itself an improper distribution. Recalling the relation (25) between the moments and the Fourier transform of a distribution, Eq.(37), and using the perturbation theory to expand the evolution operator $\hat{U}_\lambda$ in powers of $\lambda$, we find (c.f. the classical Eqs. (21) and (22))

$$\langle \bar{f} \rangle = \int_0^t \beta(t') \langle \Psi(t') | \hat{A} | \Psi(t') \rangle dt'$$ \hspace{1cm} (68)

$$\langle \bar{f}^2 \rangle = \int_0^t dt'' \int_0^t \beta(t') \beta(t'') \langle \Psi(t') | \hat{A} \exp[-i \hat{H}_0 (t'' - t')] \hat{A} | \Psi(t') \rangle,$$ \hspace{1cm} (69)

where $|\Psi(t')\rangle \equiv \exp(-i \hat{H}_0 t') |\Psi_0\rangle$.

To calculate $\langle \langle f^2 \rangle \rangle$ we will require simple sum rules, resulting from the Hermitian nature of the operator $\hat{A}$,

$$I(\lambda) \equiv \sum_m \langle \Psi_0 | \hat{U}_\lambda^{-1} | m \rangle \langle m | \hat{U}_\lambda^{-1} | \Psi_0 \rangle = 1.$$ \hspace{1cm} (70)

Calculating $\partial_\lambda I(0)$ and $\partial^2_\lambda I(0)$ and using Eq.(41) we find

$$\text{Im} \langle \bar{f} \rangle = 0$$ \hspace{1cm} (71)

$$\text{Re} \langle \bar{f}^2 \rangle = \sum_m P_m |\bar{f}_m|^2$$ \hspace{1cm} (72)

The first of these relations confirms that $\langle \bar{f} \rangle$ is real, as is already evident from Eq.(68), and the second helps us average Eq.(53) to obtain

$$\langle \langle f^2 \rangle \rangle = \alpha^2 \int z^2 G(z)^2 dz \int G(z)^2 dz + \text{Re} \langle \bar{f}^2 \rangle$$ \hspace{1cm} (73)

Equations (66) and (73) are the central result of this Section. They have the same form as the classical equations (29) and (30) insofar as the l.h.s. of Eq.(66) and the second term in Eq.(73) are the first two moments of the same distribution $w_1$ in Eq.(67). However, owing to the inaccuracy of the measurement, $w_1$ can, in general, change sign and one must exercise
caution when using these averages. For example, Sect. 2, it has been shown that for such
distributions it is possible to have

$$Re\langle \bar{f}^2 \rangle = (Re\langle \bar{f} \rangle)^2,$$

(74)

while $f$ remains a distributed, rather than a sharply defined quantity. Indeed, the relation
(74) will always take place for $\hat{A}$ and $\hat{H}_0$ such that, regardless of the value of $\lambda$, $\hat{U}_\lambda$ evolves
the initial state $\Psi_0\rangle$ into the same final state $\Psi_1\rangle$, so that in Eq.(67)

$$\langle \Psi_0|\hat{U}_0^{-1}\hat{U}_\lambda|\Psi_0\rangle = \exp[i\phi(\lambda)] \equiv S(\lambda)$$

(75)

where $\phi(\lambda)$ is a real phase, as required by the unitarity. As a result we have

$$Re(\langle \bar{f}^2 \rangle) = Re[-S^{-1}(0)S''(0)] = \phi'(0)^2 = (Re[iS^{-1}(0)S'(0)])^2 = (\langle \bar{f} \rangle)^2.$$

(76)

while, apparently, $Re[(2\pi)^{-1} \int d\lambda \exp(i\lambda f) \exp[i\phi(\lambda) - i\phi(0)] \neq \delta(f - \langle \bar{f}^2 \rangle)$. 

In summary, without post-selection one recovers the classical Eqs.(29) and (30), with the
important difference that both the mean and the variance are obtained with an (possibly)
improper distribution $w_1(f)$. Also, as shown in Sect. 2, anomalously large weak values are
likely to occur for nearly forbidden transitions, whose probability is quite small. For this
reason they do not contribute if the final state of the system is not controlled and an average
is taken over all possible final states. Next we give further examples of (66) and (73), starting
with the conventional von Neumann measurement.

VIII. WEAK VON NEUMANN MEASUREMENTS AS A SPECIAL CASE

An important special case of von Neumann-like measurements described in Sect. 5 are
impulsive von Neumann measurements, already briefly discussed in Sect. 4, whose weak limit
has been first analysed by Aharonov $et$ $al$ in Refs.[5]. The purpose of such a measurement is
to establish the value $f$ of a variable $\hat{A}$ with a discrete spectrum \{a_k\} at some intermediate
time $t_0$ for a system initially prepared in a state $|\Psi_0\rangle$ and then post-selected in a final state $|m\rangle$. The the probability amplitude $\Phi(f)$ is, in this case, the net amplitude on all virtual
eigenpaths in Eq.(39), which at $0 \leq t_0 \leq t$ pass through the value $f = a_k$. Thus putting in
Eq.(3.1)

$$\beta(t') = \delta(t' - t_0)$$

(77)
and evaluating the Fourier transform $\Phi$ we obtain

$$\Phi(f) = \langle m | \Psi(t) \rangle^{-1} \sum_k \delta(f - a_k) \langle n | a_k \rangle \langle a_k | \Psi(t_0) \rangle$$

(78)

Where $|n\rangle$ is the state obtained by evolving $|m\rangle$ back to the time $t_0$, $|n\rangle \equiv \tilde{U}^{-1}(t-t_0)|m\rangle$, and $|\Psi(t)\rangle \equiv \tilde{U}(t_0)|\Psi_0\rangle$. Thus $\Phi(f)$ is, as expected, a complex valued distribution, whose support coincides with spectrum of the operator $\hat{A}$. In general, the distribution is an improper one, as the real and imaginary parts of the complex coefficients $\langle n | a_k \rangle$ and $\langle a_k | \Psi(t_0) \rangle$ which multiply the $\delta$-functions can take either sign.

If no post-selection is made, the distribution (78) needs to be averaged over all final states $|m\rangle$ and Eq.(66) gives

$$\langle \Phi(f) \rangle = \sum_k \delta(f - a_k) |\langle a_k | \Psi(t_0) \rangle|^2$$

(79)

Now the weight multiplying the $\delta$-functions are strictly non-negative and, unlike $\Phi(f)$ the averaged distribution is a proper one. What is more, $\langle \Phi(f) \rangle$ coincides with the probability distribution obtained for and accurate 'strong' measurement of the variable $\hat{A}$ in a state $|\Psi(t_0)\rangle$.

In summary, for weak von Neumann measurements without post selection we recover the classical Eqs. (66) and (73) which allow to extract the mean and variance, obtained in accurate measurements, from a large sample of weak results. Also, finding $\langle \bar{f}^2 \rangle = \langle \bar{f} \rangle^2$ would, in this case, guarantee that the variable is sharply defined, i.e., that $|\Psi(t_0)\rangle$ is one of the eigenstates of $\hat{A}$. However, extending this argument to the case when the measured quantity is not an instantaneous value of an operator can lead to errors, as will be shown in the next Section.

IX. IS THE ELASTIC COLLISION TIME SHARPLY DEFINED?

It is possible then that someone not familiar with the analysis of Sect. 7 and implicitly assuming the values (74) obtained with a weak von Neumann-like meter to be proper probabilistic averages, might incorrectly conclude that the value $f$ of a functional $F[a]$ is sharply defined, i.e., has a unique precise value. One such example is the distribution of the elastic collision time studied by Baz’ with the help of a weakly coupled semiclassical Larmor clock [14, 15]. In Baz’ approach, a small constant magnetic field along the $z$-axis is created in a
sphere containing the target and a particle, described in the distant past by an incoming plane wave \( \exp(-ikr) \) is equipped with large nearly classical spin \( j >> 1 \) initially polarised along the \( x \) axis. The spin rotates for as long as the particle remains inside the sphere, and after the collision the spin of the outgoing particle is rotated in the \( xy \) plane. The mean collision time \( \bar{\tau} \), and its mean square \( \bar{\tau}^2 \) are then defined as

\[
\bar{T} = (\omega j)^{-1} \bar{j}_y \\
\bar{T}^2 = (\omega j)^{-2}[\bar{j}_y^2 - j/2]
\]

where \( \bar{j}_y \) and \( \bar{j}_y^2 \) are the expectation values of the spin’s \( y \)-component and its square, respectively and \( \omega \) is the Larmor frequency. A simple calculation shows that

\[
\bar{T}^2 = (\bar{T})^2
\]

which led Baz’ to conclude that ‘for given energy \( E \) and angular momentum \( l \) the time interval during which the colliding particles are inside a sphere of radius \( R \) is a sharply defined quantity’ [13, 17]. The matter was further discussed in Refs. [33] and briefly mentioned in [27]. The purpose of this Section is to show that for \( j >> 1 \) \( \bar{\tau} \) and \( \bar{\tau}^2 \) in Eqs. (80) are just the weak values calculated for the traversal time functional [18] (\( \theta_R(\vec{r}) = 1 \) for \( r < R \) and 0 otherwise)

\[
\tau[\vec{r}(.)] = \int_{-\infty}^{\infty} \theta_R(\vec{r}) dt'
\]

which computes the net duration spent by a Feynman path \( \vec{r}(t) \) inside the sphere of the radius \( R \) and that these values obey Eq. (76). Indeed, it can be shown [22] that the final state of the clock’s spin, \( |M_F> \), is just a superposition of rotations of its initial state \( |M_I> \) around the \( z \)-axis by the angles \( \omega \tau \) each weighted by the amplitude distribution \( \Phi(\tau) \) with which the duration \( \tau \) contributes to the collision. Thus, expanding in the eigenstates \( |m> \), \( m = -j, ..., j \) of the \( z \)-component of the spin, \( \hat{j}_z \) we have

\[
\langle m|M_F> = \int d\tau \Phi(\tau) \exp(-im\omega \tau)\langle m|M_I>.
\]

which shows that the Larmor clock is similar to von Neumann like meter of a kind described in Sect.4. For a large spin polarised along the \( x \)-axis Baz’ wrote

\[
\langle m|M_I> = C \exp(-m^2/2j),
\]
which restricts $|m| \leq j^{1/2}$. The matrix $\langle m'|j_y|m \rangle$ has two non-zero off-diagonal elements $\langle m+1|j_y|m \rangle = -\langle m|j_y|m+1 \rangle = -i(j+m)^{1/2}(j-m+1)^{1/2}/2$. With the restriction on $|m|$, for a large $j$ we may write $\hat{j}_y \approx -ij\partial_m$ so that in the continuous limit,

$$j \to \infty, \quad \omega \to 0, \quad \omega j \to \infty \quad \omega^2 j \to 0,$$

after introducing $\lambda \equiv m\omega$ we have

$$\bar{j}^n / j^n \omega^{n+1} \equiv \sum_{m,m'} \langle M_F|m'| \langle m'|j_y^n|m \rangle \langle m|M_F \rangle / j^n \omega^{n+1} =$$

$$\int d\lambda \exp(-\lambda^2/2\omega^2 j) \tilde{\Phi}^*(\lambda) \partial^{\alpha}_{\lambda} \{\exp(-\lambda^2/2\omega^2 j) \tilde{\Phi}(\lambda)\} / \int d\lambda \exp(-\lambda^2/\omega^2 j)|\tilde{\Phi}(\lambda)|^2$$

in which we recognise Eqs.(46) and (50) with $\alpha^2 = 1/\omega^2 j \to \infty$ and $\tilde{G}(\alpha\lambda) \equiv \exp(-\lambda^2/2\omega^2 j)$ and Eqs. (80) are seen to be equivalent to Eqs.(66) and (73). Thus, for a small Larmor frequency the first of Eqs.(80) gives the improper weak value of the traversal time Eq.(82). Finally, for the functional (81) the evolution operator $\hat{U}_\lambda = \exp[-i(j^2/2m + V(r) + \lambda R(r))]$ contains an additional constant potential inside the sphere of interest which modifies the scattering phase $\phi(k)$, so that (cf. Eq.(75))

$$\hat{U}_\lambda \exp(-ikr) = \exp[i\phi(k,\lambda)] \exp(ikr)$$

and therefore, according to Eq.(76),

$$\bar{T}^2 = Re\bar{\tau}^2 = (\bar{\tau})^2 = \bar{T}^2.$$ 

For a rectangular potential, $V(r) = \Omega \theta_R(\vec{r})$, the traversal time amplitude distribution $\Phi(\tau)$ in Eq.(83) and the weak value $\bar{\tau}$ vs. $\Omega$ are shown in Figs. 3a and 3b, respectively.

In summary, the suggestion that the collision time has a precise value in elastic scattering is shown to be incorrect. Rather, Baz’ result demonstrates that, for such a single-channel collision, the real part of the traversal time amplitude, $Re\{\Phi(\tau) / \int \Phi(\tau) d\tau\}$ is a broad improper distribution with vanishing variance. As was also observed by Baz’ [16], this is no longer true if a particle is post-selected in one of several channels, e.g., for transmission across a potential barrier where both reflection and transmission are possible.

X. TIME DELAY IN TRANSMISSION AND THE PHASE TIME

A different type of the time delay variable, not directly related to the traversal time functional (82) or indeed to any other functional of the particle’s Feynman paths can be
constructed as follows. Consider a classical particle with a unit mass in one dimension crossing from left to right a potential $V(x)$ which vanishes everywhere outside the region $-a < x < a$. Inside the region the particle will experience a time delay or a speed up depending on whether $V(x)$ is a barrier or a well. This time delay $\tau$ can be evaluated by taking a snapshot of the particle’s at some large time $t$ and comparing it with the position of a particle that has been moving freely along the trajectory with the same initial conditions. If the distance between the two is $x'$, we have ($p$ is the particle’s initial momentum)

$$\tau(p) = -x'/p$$

(89)

which is positive (delay) if the particle lags behind, or negative (speed up) if it lies ahead of the free one. This is a measurement which differs from the one discussed in Sect. 4 in that the role of the pointer is played by the particle’s own position, but a measurement nevertheless. It is not surprising, therefore that a quantum extension of such a procedure is a quantum measurement. Initially one represents a particle by a wavepacket

$$\Psi(x, t = 0) = G(x) \exp(ipx) = \int A(k) \exp(ikx) dk.$$  
(90)

The transmitted part, is then given by

$$\Psi^T(x, t) = \int T(k) A(k) \exp(ikx - ik^2t/2) dk$$

(91)

where $T(k)$ is the transmission amplitude. Rewriting the Fourier transform (91) as a convolution and neglecting the spreading of the wavepacket yields [13]

$$\Psi^T(x, t) = \exp(ipx - ip^2t/2) \int G(x - pt - x')\Phi_p(x')dx',$$

(92)

where

$$\Phi_p(x) \equiv (2\pi)^{-1} \exp(-ipx) \int T(k) \exp(ikx) dk.$$  
(93)

and

$$\int \Phi_p(x) dx = T(p).$$  
(94)

On sees that the transmitted wavepacket is constructed from the freely propagating envelopes each shifted by $x'$ and weighted by the probability amplitude $\Phi_p(x')$. Associating with each spatial shift $x'$ a time delay $\tau$ with the help of Eq. (89) shows that transmission of a particle with a momentum $p$ involves not one but many time delays, whose amplitude distribution.
is given by the Fourier transform (93) of $T(p)$. Moreover, observing transmitted particle at
a location $x$ amounts to measuring $\tau$ to the accuracy determined by the coordinate spread
of the particle’s initial wavepacket.

To quantify the mean time delay associated with the latter one often chooses [8, 9, 10] the
shift of the centre of mass of the transmitted pulse relative to that of the free propagation
divided by its mean velocity.

\[
< \tau(p) > \equiv p^{-1} < x - pt > = \int (x - pt)|\Psi^T(x, t)|^2 dx / \int |\Psi^T(x, t)|^2 dx,
\]

(95)
i.e., the expectation value of the time delay for a particle with the momentum $p$ measured
with the ‘apparatus function’ $G$ determined by the envelope of the pulse (cf. Eq.(43)). Just
as in the case of a von Neumann like measurement (43), an improvement in the accuracy
increases the ‘perturbation’ on the measured system, as a wavepacket narrow in the coordinate
space has a large momentum spread. As a result, the transmission probability $P^T$ is
not equal to that for a plane wave with the momentum $p$,

\[
P^T = \int |T(k)|^2 |A(k)|^2 dk \neq |T(p)|^2.
\]

(96)

In order to minimise this perturbation one can choose $A(k)$ so narrow that the inequality
(96) becomes an approximate equality, and the envelope $G(x)$ becomes very broad. As was
shown in Sect.4, such a measurement is weak and the mean time delay is given by the real
part of the improper weak value

\[
< \tau(p) > \approx p^{-1} Re\bar{\tau}(p) \equiv p^{-1} Re \int x'\Phi_p(x')dx' / \int \Phi_p(x')dx'.
\]

(97)

With the help of Eq.(94) it is easy to show that Eq.(97) can also be written as (cf. Eq.(52))

\[
< \tau(p) > \approx p^{-1/2} Re[-i\partial_p lnT(p)] = \partial_E\phi(p) \equiv \tau_{phase},
\]

(98)

where $\phi(p)$ is the phase of the transmission coefficient $T(p)$, $T(p) = |T(p)| \exp[i\phi(p)]$. Equation
(98) is the standard definition of the ‘phase time’ [8, 9, 10]. The purpose of the above
analysis has been to clarify its origin as an weak value and relate it to some of its ‘anoma-
lous’ properties. One such property property is that for tunnelling across a potential barrier
$\tau_{phase}$ predicts a speed up as if the classically forbidden region had been crossed infinitely
fast. Indeed, for tunnelling across a high rectangular barrier of a height $V > p^2/2$ and a
width $a$ the transmission coefficient can be approximated as (we neglect the pre-exponential factor)

$$T(p) \approx \exp \left[ -(V - p^2)a - ipa \right]$$  \hspace{1cm} (99)

so that

$$\tau_{\text{phase}} \approx -a/p.$$  \hspace{1cm} (100)

At first glance this result appears to contradict the relativistic restriction that the speed of a particle or a photon may not exceed the speed of light, but only if $\tau_{\text{phase}}$ is taken to be the time delay in the classical sense. In reality, it is just a spectacular example of an improper average lying outside the region of support of a continuous oscillating distribution \([93]\). Indeed, as the barrier potential does not have bound states and, therefore, poles in the upper half of the complex $k$-plane, $\Phi_p(x')$ in Eq.\([93]\) vanishes for $x' > 0$ so that only positive time delays contribute to tunnelling of a particle with a momentum $p$. Thus the causality is not violated and the ‘anomalous’ negative value \([100]\) simply indicates the possibility that below the barrier destructive interference between the delayed envelopes may produce a significantly reduced advanced pulse which builds up from their front tails (more details of this analysis can be found in \([13]\)). For a zero-width barrier, $V(x) = \Omega \delta(x)$, the amplitude distribution $\Phi_p(x)$ in Eq.\([93]\) and the phase time \([98]\) vs. $\Omega$ in Figs. 4a and 4b, respectively.

Note here the principal difference between $\tau_{\text{phase}}$ and the collision (traversal) time of the previous Section. While the traversal time represented by the functional \([82]\) vanishes with the size of the region of interest , $\tau_{\text{phase}}$ which relates to the poles of $T(k)$ in the complex $k$-plane remains finite for an infinitely narrow barrier. Thus the traversal time and the phase time are essentially different quantities which share the same classical limit and Baz’ assertion that the former is correct the latter is wrong \([17]\) cannot be sustained.

XI. THE LOCAL ANGULAR MOMENTUM (LAM)

A different example of a weak value is the local angular momentum, designed and applied in \([23]\) to analyse elastic, inelastic and reactive differential cross-sections (DCS). Typically, several angular momenta contribute to the scattering amplitude $f(\theta)$, which is given by a coherent sum over partial waves,

$$f(\theta) \equiv (ik)^{-1/2} \sum_{J=0}^{\infty} (J + 1/2) P_J(\cos(\theta)) S^J(E)$$  \hspace{1cm} (101)
where $\theta$ is the scattering angle, $k$ is the wavevector, $J$ is the total angular momentum, $P_J(\cos(\theta))$ is the Legendre polynomial, and $S^J$ is the $S$-matrix element. In order to estimate the angular momentum which contributes to a particular angle $\theta$ the authors of [23] suggested the quantity with the units of angular momentum

$$LAM(\theta) \equiv d\phi(\theta)/d\theta = \text{Re}\left[-i\partial_\theta \ln f(\theta)\right],$$

(102)

which can be seen to give the correct answer in the semiclassical limit and in the forward glory scattering [23]. As no probabilities can be assigned to the individual terms in Eq.(101) we expect the proposed estimate (102) to be an improper average of some kind. Using the analogy with Eq.(98) of the previous Section we can rewrite Eq.(102) as

$$LAM(\theta) = \sum_{L=\ldots,-2,0,2\ldots} Lw_L(\theta)$$

(103)

where the normalised distribution $w_L(\theta)$ is given by

$$w_L(\theta) = \text{Re}\left\{\Phi_L(\theta)/\sum_{L'=\ldots,-2,0,2\ldots} \Phi_{L'}(\theta)\right\}, \quad L = -2, 0, 2\ldots$$

(104)

and

$$\Phi_L(\theta) \equiv \pi^{-1} \exp(iL\theta) \int_0^\pi f(\theta') \exp(-iL\theta')d\theta'.$$

(105)

Note that the newly introduced quantity with the units of angular momentum $L$ takes even integer values and is not identical to the total angular momentum $J$. As there are no apriori restrictions on the phase of $\Phi_L(\theta)$, $w_L(\theta)$ may change sign and the $LAM(\theta)$ is not, in general, required to take value within the range of the partial waves which contribute to the scattering amplitude.

A detailed discussion of the $LAM$ and its application to the analysis of angular scattering will be given elsewhere. As an illustration, we show in Fig. 3 $LAM(\theta)$ for the ($v,j$ and $K$ are the vibrational, rotational and helicity quantum numbers, respectively)

$$v' = 2, \quad j' = 0, \quad K' = 0 \rightarrow v = 0, \quad j = 0, \quad K = 0$$

(106)

transition for the $F + H_2 \rightarrow FH + H$ reaction [36] at the collision energy $E \approx 38meV$. Figure 3a shows the differential cross-section $\sigma(\theta) \equiv |f(\theta)|^2$ obtained by summing over 12 partial waves, $0 \leq J \leq 12$, while $LAM(\theta)$ is plotted in Fig.3b. and Fig. 3c shows the distribution $w_L(\theta)$ in Eq.(104) near the minimum of the DCS at $\theta \approx 50^o$. 

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The last example given in this Section, although not directly related to weak measurements or values, fits well within the general context of this paper. Consider a three-slit experiment, in which an electron or a photon may reach a detector through slits 1, 2 and 3. Let the amplitudes for the three pathways be

\[ A(1) = A(2) = 1 \quad \text{and} \quad A(3) = -1, \]  

respectively. It is obvious now that since the contributions from the routes 2 and 3 cancel each other, one may plug the two slits without affecting the detector count. It would be wrong, however, to conclude that the particle always travels the route 1, as the count would not be affected if the slits 1 and 3 were close instead. In fact, this is the situation discussed in Section 2 (c). The amplitude distribution for the slits can be written as

\[ \Phi(f) = \delta(f - 1) + \delta(f - 2) - \delta(f - 3), \]  

and the integral \( \int \Phi(f) df \) gives the probability amplitude to arrive at the detector. The distribution alternates, we cannot decide in a unique manner which two parts of the integral cancel each other, and must only conclude that it is not possible to determine through which slit the particle actually went. The same gedankenexperiment can presented in a slightly more intriguing form. Suppose that an meter determines whether an electron goes through the slit 1 but does not distinguish between the two other slits. Then each electron arriving at the screen will also be registered at the slit 1. Similarly, if the slit two is watched, the electron will always be found passing through it. All this is easily explained in terms of interfering and exclusive alternatives (see Chapt.1, Sect.3 of Ref. [1]). By watching the the slit one we produce two alternative routes the the detector: one (I) is through slit 1 itself, and the other (II) through both the slits 2 and 3, which remain interfering alternatives and cannot be distinguished. Now we can use Feynman’s prescription for assigning probabilities: all interfering amplitudes must be added coherently, and then the moduli of the sums must be squared,

\[ P(I) = |A(1)|^2 / (|A(1)|^2 + |A(2) + A(3)|^2) = 1 \]  
\[ P(II) = |A(2) + A(3)|^2 / (|A(1)|^2 + |A(2) + A(3)|^2) = 0 \]  

which explicitly show that the route (II) is not travelled due to the destructive interference. Even though an observation is conducted in such a way that it does not change the detector
count, it changes the situation and fails to provide a clue as to what 'actually' happens to an unobserved particle. Note that if all three amplitude are chosen to be positive, shutting any two slits would always effect the detector count.

Consider further a three level system with a zero Hamiltonian $\hat{H} \equiv 0$ is prepared and post-selected in the states

$$|\Psi_0\rangle = (3)^{-1/2}(|1\rangle + |2\rangle + |3\rangle)$$ (111)

and

$$|\Psi_1\rangle = (3)^{-1/2}(|1\rangle + |2\rangle - |3\rangle),$$ (112)

respectively. Between the preparation and the post-selection, the projector on the first state, $\hat{P}_1 \equiv |1\rangle\langle 1|$ is accurately measured. According to Eq.(39), there are only three paths connecting the initial and final states $|\Psi_0\rangle$ and $|\Psi_1\rangle$,

$$a_1(t') = 1, \quad a_2(t') = 2 \quad and \quad a_3(t') = 3,$$ (113)

with the corresponding amplitudes given by

$$A(n) = \langle \Psi_1 | n \rangle \langle n | \Psi_0 \rangle, \quad , n = 1, 2, 3.$$ (114)

The operator $\hat{P}_1$ has one simple and one doubly degenerate eigenvalues of 1 and 0, respectively, so that, as was shown in Sect. 6 of Ref.[3], its measurement destroys coherence between the paths in the same way as observing the particle passing through the first slit in the three-slit experiment. Thus, inserting Eq.(114) into Eq.(109) yields

$$P(1) = \frac{|\langle \Phi_1 | 1 | \Psi_0 \rangle|^2}{(|\langle \Psi_1 | 1 | \Psi_0 \rangle|^2 + |\langle \Psi_2 | 2 | \Psi_0 \rangle + \langle \Psi_1 | 3 | \Psi_0 \rangle|^2) = 1.}$$ (115)

Similarly, one always finds the particle in the second 'box’ if the projector $\hat{P}_2 \equiv |2\rangle\langle 2|$ is measured instead,

$$P(2) \equiv 1/3/(1/3 + 1/3 - 1/3) = 1$$ (116)

Equation (115) is the Aharonov, Leibowitz and Bergmann (ABL) rule [37] for an operator with degenerate eigenvalues (see Eq.(5) of Ref.[28]). Note that in our analysis the ABL rule is a simple consequence of Feynman’s prescription for adding probability amplitudes (see Sect. 1-7 of [39]) and does not rely a time symmetric formulation of quantum mechanics employed in [28]. Also the Feynman’s uncertainty principle suggests a different interpretation of just described 'three box case’. The authors of [7] note that since the measurement of $\hat{P}_1$ and $\hat{P}_2$,
('opening boxes 1 and 2' in the terminology of [7]) always yield positive results, a particle subjected to the boundary conditions (111)-(112) exists, at any intermediate time in two 'boxes' simultaneously. Alternatively, it can be argued that the measurements of the two projectors correspond to two distinct physical situations which, in turn, provide no clue as to where the particle actually is when no measurement is conducted and all three pathways remain interfering alternatives.

XIII. CONCLUSIONS AND DISCUSSION

In summary, quantum mechanics can be seen to operate by assigning probability amplitudes to scenarios or pathways which can be interpreted as classical outcomes. Some of the scenarios are exclusive by nature, some are normally interfering but can be made exclusive by coupling the system to a meter and some, it appears, cannot be made exclusive at all, i.e., because a suitable meter cannot be constructed. In general one wishes then to know how many outcomes are there and what is the likelihood of the realisation of a particular one. A quantum measurement can be seen as performing this task by labelling the pathways by some variable $f$ and then analysing the moments of its distribution. For exclusive scenarios, e.g., different values of a variable $\hat{A}$ in the presence of an accurate von Neumann meter, a proper probability distribution exists \textit{apriori}. However, some phenomena such as the interference pattern in a double slit experiment or tunnelling transmission across a potential barrier rely on constructive or destructive interference between the relevant pathways. According to the Feynman’s uncertainty principle, interfering scenarios cannot be told apart and form, therefore, a single indivisible pathway connecting the initial and final states of a system. Mathematically, the principle arises form the alternating nature of the probability amplitudes responsible for cancellation between the pathways, which, in turn, forbids the identification of the main contributor(s) to the transition. Accordingly, we have observed, that an attempt to assign a mean value to $f$ when it labels interfering alternatives, be it by performing a weak von Neumann-like measurement, by extending to the quantum context a suitable classical procedure, as in the case of the Wigner-Eisenbad phase time, or by postulating of an expression with an appropriate classical limit, as in the case of the LAM, leads to an improper complex weak value $\bar{f}$ (1). This can, indeed, be expected, as in the absence of probabilities, $\bar{f}$ is the only average one can construct from a real variable and
complex probability amplitudes. We note further that, contrary to what has been claimed by several authors \[8, 9\], the complexity of \( \bar{f} \) is not in itself an obstacle to its interpretation, as the experiment always dictates which part (s) \( \bar{f} \) (in our case, \( Re\bar{f} \), for \( Im\bar{f} \) and \( |\bar{f}| \) see, for example \[38\]) should be used to produce the required real answer.

A far more serious problem is that \( Re\bar{f} \) in an improper average obtained, in general, with an alternating distribution and has a number of undesirable properties discussed in Sect.2. In particular it may lie outside the region containing the support of the amplitude distribution, e.g., the spectrum of the measured variable in the case of an impulsive von Neumann measurement or the range of the time delays prescribed by the causality in the case of the phase time, and take anomalous large valued of either sign even when this support is bounded.

Just because improper averages can take values which appear unreasonable does not mean that they always do that. In particular the amplitude distribution employed in their construction may or may not be improper for all transitions, just as for some selected states the Wigner function \( W(p, x) \) does not always take negative values. In general, quantum interference hides the information about the range of the values contributing to the transition in a way that one can never 'trust' a weak value to represent the centroid of the range without a detailed inspection of the distribution itself. Of course, if such an inspection is possible, there is no longer a need to evaluate the mean \( \langle \rangle \).

In the end one cannot avoid asking of whether the weak values should be treated as 'true' properties of a system in the presence of interference, or a manifestation of a failure of a measurement designed to defy the uncertainty principle. Both points of view are, in principle, possible. The former, expressed in \[5, 7\] is reinforced by the notion that a weak value may be obtained in an act of measurement and, therefore, provides the only answer to the question about the value taken by a variable in the presence of interference. There is also no other 'correct' answer to refute it. However, a suggestion that with only two slits present an electron passes on average through the slit number \(-8\), or that a tunnelling particle spends on average a zero time within a barrier thereby defying the relativity, clearly requires further clarification. The explanation that the weak value is not actually tied to the range of the values contributing to the transition (numbers 1 and 2 of the slits or purely non-negative time delays in the case of the barrier) seriously diminishes the value of the information a weak measurement can provide.

An alternative view can be summarised as follows. Interfering pathways cannot be told apart
without destroying coherence between them and, with it, the studied transition. With the information destroyed by interference, a suitable answer to the above question simply does not exist. If one insists, e.g., by employing an extremely inaccurate ‘weak’ meter, both the theory and experiment provide, much like a politician or a manager, a kind of non-answer, not necessarily related to what has been asked.

In a similar manner Feynman’s uncertainty principle can be used to ‘resolve’ the three box paradox of Sect. 7. If no measurements are conducted, the particle cannot be said to be in either particular box. Opening one of the box creates a new physical situation and two exclusive pathways to which one can now assign probabilities which, however tell us nothing about the case when no measurements are made. The world ‘resolve’ is put in quotes because the pathway analysis does not explain the 'logical difficulties’ [1] associated with quantum interference, but simply compacts them into the Feynman’s formulation of the uncertainty principle. We conclude by quoting Feynman on the double-slit experiment:[39]: "We choose to examine a phenomenon which is impossible, absolutely impossible, to explain in any classical way... In reality, it contains the only mystery.”

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FIG. 1:
Various virtual paths in Eq.(39) contributing to the transition between $|\Psi_0\rangle$ and $|\Psi_1\rangle$ for two-level system. For $\dot{H} = 0$ only the two constant paths (thick solid) have non-zero probability amplitudes.
FIG. 2:

The mean intermediate position of a two-level system in in the transition between the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ in Eq. (58) measured by a Gaussian meter with and accuracy $\alpha$ vs. $\epsilon$ and $\alpha$. 
FIG. 3:
a) Real and imaginary parts of the collision (traversal) time distribution $\Phi(\tau)$ in Eq.(83) (smeared with a Gaussian with the width $\Delta(\tau)/R^2 = 0.03$) for a rectangular potential $V(r)$ of a height $\Omega$ and radius $R$.

b) The weak value $\bar{\tau}$ vs. $\Omega$ for the above potential.
FIG. 4:

a) Real and imaginary parts of the amplitude distribution $\Phi(x)$ in Eq. (93) for a thin barrier $V(x) = \Omega \delta(x)$.

b) The phase time $\tau_{\text{phase}}$ vs. $\Omega/p$
FIG. 5:

a) The $F + H_2$ reactive DCS for the transition (106)

b) $LAM(\theta)$ for the transition (106)

c) The improper distribution $w_L(\theta)$ in Eq. (104) vs. $L$ for $\theta \approx 50^\circ$