ON CERTAIN SUBCLASSES OF CLOSE–TO–CONVEX FUNCTIONS RELATED WITH THE SECOND–ORDER DIFFERENTIAL SUBORDINATION

H. MAHZOON AND R. KARGAR

Abstract. Let \( A \) be the family of all analytic and normalized functions in the open unit disc \( |z| < 1 \). In this article we study two certain subclasses of close–to–convex functions, denoted by \( R(\alpha, \beta) \) and \( L_\alpha(b) \) as follows

\[
R(\alpha, \beta) = \left\{ f \in A : \Re \left\{ f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) \right\} > \beta, \ |z| < 1 \right\}
\]

and

\[
L_\alpha(b) = \left\{ f \in A : \left| f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) - b \right| < b, \ |z| < 1 \right\},
\]

where \(-\pi < \alpha \leq \pi\), \(0 \leq \beta < 1\) and \(b > 1/2\). We show that if \( f \in R(\alpha, \beta) \), then \( \Re \{ f'(z) \} \) and \( \Re \{ f(z)/z \} \) are greater than \( \beta \), and if \( f \in L_\alpha(b) \), then \( 0 < \Re \{ f'(z) \} < 2b \). Also, some another interesting properties for the class \( L_\alpha(b) \) are investigated. Finally, we obtain the radius of univalence of 2–th section sum \( f \in R(\alpha, \beta) \) and pose a conjecture about the every section sum of \( f \in R(\alpha, \beta) \).

1. Introduction

Let \( \Delta := \{ z \in \mathbb{C} : |z| < 1 \} \) where \( \mathbb{C} \) is the complex plane. We denote by \( B \) the class of all analytic functions \( w(z) \) in \( \Delta \) with \( w(0) = 0 \) and \( |w(z)| < 1 \), and denote by \( A \) the class of all functions that are analytic and normalized in \( \Delta \). The subclass of \( A \) consisting of univalent functions in \( \Delta \) is denoted by \( S \). For functions \( f \) and \( g \) belonging to the class \( A \), we say that \( f \) is subordinate to \( g \) in the unit disk \( \Delta \), written \( f(z) \prec g(z) \) or \( f \prec g \), if and only if there exists a function \( w \in B \) such that \( f(z) = g(w(z)) \) for all \( z \in \Delta \). In particular, if \( g \) is univalent function in \( \Delta \), then we have the following relation

\[
f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).
\]

Denote by \( S^* \) and \( K \) the set of all starlike and convex functions in \( \Delta \), respectively. A function \( f \in A \) is said to be close-to-convex, if there exists a convex function \( g \) such that

\[
\Re \left\{ \frac{f'(z)}{g'(z)} \right\} > 0 \quad (z \in \Delta).
\]

This class was introduced by Kaplan in 1952 \cite{K} and we denote by \( CK \). It is clear that if we take \( g(z) \equiv z \) in the class \( CK \), then we have the Noshiro–Warschawski class as follows

\[
C := \{ f \in A : \Re \{ f'(z) \} > 0, \ z \in \Delta \}.
\]

By the basic Noshiro–Warschawski lemma \cite[§2.6]{N}, we have \( C \subset S \).

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1}
\caption{Graphical representation of the function.}
\end{figure}

\textbf{2010 Mathematics Subject Classification.} 30C45.

\textbf{Key words and phrases.} Univalent; Positive real part; Circular domains; Close-to-convex; Marx–Strohhäcker problem; Section sum.

*Corresponding Author.
Here, we recall from [15], two certain subclasses of analytic functions as follows

\[ \mathcal{L}_\alpha := \left\{ f \in A : \Re \left\{ f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) \right\} > 0, \ z \in \Delta \right\} \]

and

\[ \mathcal{L}_\alpha(b) := \left\{ f \in A : \left| f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) - b \right| < b, \ z \in \Delta \right\}, \]

where \( \alpha \in (-\pi, \pi] \) and \( b > 1/2 \). We note that if \( b \to \infty \), then \( \mathcal{L}_\alpha(b) \to \mathcal{L}_\alpha \). Also, \( \mathcal{L}_\pi \) contains \( \mathcal{L}_\alpha \) for each \( \alpha \). On the other hand, Trojnar-Spelina [19] showed that \( \mathcal{L}_\alpha(b) \subset \mathcal{L}_\pi \), for every \( \alpha \in (-\pi, \pi] \) and \( b \geq 1 \).

By definition of subordination and this fact that the image of the function

\[ \phi_b(z) = \frac{1 + z}{1 + \left( \frac{b}{b} - 1 \right) z} \quad (z \in \Delta, \ b > 1/2), \]

is \( \{ w \in \mathbb{C} : |w - b| < b \} \) (see Figure 1 for \( b = 3/2 \)), we have the following lemma.

**Lemma 1.1.** (see [19]) A necessary and sufficient condition for \( f \) to be in the class \( \mathcal{L}_\alpha(b) \) is

\[ f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) < \phi_b(z) \quad (z \in \Delta), \]

where \( \phi_b \) is given by (1.1).

![Figure 1. The boundary curve of \( \phi_{3/2}(\Delta) \)](image)
\( \Delta \). Indeed, he denoted by \( \mathcal{F}_\gamma \) the class of functions \( f \in \mathcal{A} \) which satisfying the following inequality

\[
\text{Re}\{f'(z) + \gamma zf''(z)\} > 0 \quad (z \in \Delta),
\]

where \( \gamma \geq 0 \), and showed that \( \mathcal{F}_\gamma \subset \mathcal{S} \). Also, he proved that if \( f \in \mathcal{F}_\gamma \) and \( \text{Re}\{\gamma\} \geq 0 \), then \( \text{Re}\{f'(z)\} > 0 \) in \( \Delta \). Recent result, also was obtained by Lewandowski et al. in [7].

On the other hand, Gao and Zhou [3] considered the class \( \mathcal{R}(\beta, \gamma) \) as follows:

\[
\mathcal{R}(\beta, \gamma) = \{ f \in \mathcal{A} : \text{Re}\{f'(z) + \gamma zf''(z)\} > \beta, \quad \gamma > 0, \beta < 1, \ z \in \Delta \}.
\]

They found the extreme points of \( \mathcal{R}(\beta, \gamma) \), some sharp bounds of certain linear problems, the sharp bounds for \( \text{Re}\{f'(z)\} \) and \( \text{Re}\{f(z)/z\} \) and determined the number \( \beta(\gamma) \) such that \( \mathcal{R}(\beta, \gamma) \subset \mathcal{S}^* \), where \( \gamma \) is certain fixed number in \([1, \infty)\).

Also, the class \( \mathcal{R}(\beta, \gamma) \) was studied by Ponnusamy and Singh when \( \text{Re}\{\gamma\} > 0 \), see [11].

Motivated by the above classes, we define the class of all functions \( f \in \mathcal{A} \), denoted by \( \mathcal{R}(\alpha, \beta) \) which satisfy the condition

\[
\text{Re}\left\{ f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z) \right\} > \beta \quad (z \in \Delta),
\]

where \( 0 \leq \beta < 1 \) and \( -\pi < \alpha \leq \pi \). It is obvious that \( \mathcal{R}(\pi, \beta) \) becomes the class \( \mathcal{C}(\beta) \), where

\[
\mathcal{C}(\beta) := \{ f \in \mathcal{A} : \text{Re}\{f'(z)\} > \beta, z \in \Delta, 0 \leq \beta < 1 \}.
\]

The class \( \mathcal{C}(\beta) \) was considered in [4] and \( \mathcal{C}(\beta) \subset \mathcal{S} \) when \( 0 \leq \beta < 1 \). It follows from [2, Theorem 5] that \( \mathcal{R}(\alpha, 0) \subset \mathcal{R}(\pi, 0) \equiv \mathcal{C}(0) \equiv \mathcal{C} \). The class \( \mathcal{R}(0, 0) \) studied by Singh and Singh [12], and they showed that \( \mathcal{R}(0, 0) \subset \mathcal{S}^* \). Also, they found for \( f \in \mathcal{R}(0, 0) \) and \( z \in \Delta \) that \( \text{Re}\{f(z)/z\} > 1/2 \) and \( \mathcal{R}(0, \beta) \subset \mathcal{S}^* \) for \( \beta \geq -1/4 \).

Silverman in [14] improved this lower bound. He showed that \( \mathcal{R}(0, \beta) \subset \mathcal{S}^* \) for \( \beta \geq -0.2738 \) and also found the smallest \( \beta (\beta \geq -0.63) \) for which \( \mathcal{R}(0, \beta) \subset \mathcal{S} \).

Since the function \( z \mapsto (1 + (1 - 2\beta)z)/(1 - z) \) \((z \in \Delta, 0 \leq \beta < 1)\) is univalent and maps \( \Delta \) onto the right half plane, having real part greater than \( \beta \), we have the following lemma directly. With the proof easy, the details are omitted.

**Lemma 1.2.** A function \( f \in \mathcal{A} \) belongs to the class \( \mathcal{R}(\alpha, \beta) \) if, and only if,

\[
f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad (z \in \Delta, 0 \leq \beta < 1, -\pi < \alpha \leq \pi).
\]

To prove of our main results we need the following lemma.

**Lemma 1.3.** [10] p. 35 Let \( \Xi \) be a set in the complex plane \( \mathbb{C} \) and let \( t \) be a complex number such that \( \text{Re}\{t\} > 0 \). Suppose that a function \( \psi : \mathbb{C} \times \Delta \to \mathbb{C} \) satisfies the condition

\[
\psi(ip, \sigma; z) \notin \Xi
\]

for all real \( p, \sigma \leq -|t - ip|^2/(2\text{Re}t) \) and all \( z \in \Delta \). If the function \( p(z) \) defined by \( p(z) = t + t_1z + t_2z^2 + \cdots \) is analytic in \( \Delta \) and if

\[
\psi(p(z), z\psi'(z); z) \in \Xi,
\]

then \( \text{Re}\{p(z)\} > 0 \) in \( \Delta \).

This paper is organized as follows. In Section 2 some properties of the classes \( \mathcal{R}(\alpha, \beta) \) and \( \mathcal{L}_\alpha(b) \) are studied. In Section 3 we obtain the radius of univalence of 2-th section sum of \( f \in \mathcal{R}(\alpha, \beta) \) and we conjecture that this radius is for every section sum of the function \( f \) that belonging to the class \( \mathcal{R}(\alpha, \beta) \).
2. On the classes $\mathcal{R}(\alpha, \beta)$ and $\mathcal{L}_\alpha(b)$

At first, applying Hergoltz’s Theorem [11 p. 21] we obtain the extreme points of $\mathcal{R}(\alpha, \beta)$ as follows:

\begin{equation}
 f_z(z) = z + 4(1 - \beta) \sum_{n=2}^{\infty} \frac{x^{n-1}}{n[n + 1 + (n-1)\alpha]} z^n \quad (|x| = 1).
\end{equation}

Since the coefficient bounds are maximized at an extreme point, as an application of (2.1), we have

\[ |a_n| \leq \frac{4(1 - \beta)}{n[n + 1 + (n-1)\alpha]} = \frac{4(1 - \beta)}{n\sqrt{2}n^2 + 1 + (n^2 - 1)\cos \alpha} \quad (n \geq 2), \]

where $0 \leq \beta < 1$ and $-\pi < \alpha \leq \pi$. Equality occurs for $f_z(z)$ defined by (2.1).

To prove the first result of this section, i.e. Theorem 2.1, also Theorem 2.2 and Theorem 2.3, we employ the same technique as in [5, Theorem 2.1].

**Theorem 2.1.** Let $0 \leq \beta < 1$ and $-\pi < \alpha \leq \pi$. If $f \in \mathcal{A}$ belongs to the class $\mathcal{R}(\alpha, \beta)$, then

\[ \Re\{f'(z)\} > \beta \quad (0 \leq \beta < 1). \]

This means that $\mathcal{R}(\alpha, \beta) \subset \mathcal{C}(\beta)$.

**Proof.** Let $f'(z) \neq 0$ for $z \neq 0$ and $p(z)$ be defined by

\[ p(z) = \frac{1}{1 - \beta} (f'(z) - \beta) \quad (0 \leq \beta < 1). \]

Then $p(z)$ is analytic in $\Delta$, $p(0) = 1$ and

\[ f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z) = (1 - \beta)[p(z) + (1 + e^{i\alpha})zp'(z)/2] + \beta = \phi(p(z), zp'(z); z), \]

where $\phi(r, s; z) := (1 - \beta)[r + (1 + e^{i\alpha})s/2]$. Since $f \in \mathcal{R}(\alpha, \beta)$, we define the set $\Omega_\beta$ as follows:

\begin{equation}
 \{\phi(p(z), zp'(z); z) : z \in \Delta\} \subset \{w : \Re\{w\} > \beta\} =: \Omega_\beta.
\end{equation}

For all real $\rho$ and $\sigma$, that $\sigma \leq -(1 + \rho^2)/2$, we get

\[ \Re\{\phi(i\rho, \sigma; z)\} = \Re\{(1 - \beta)[i\rho + (1 + e^{i\alpha})\sigma/2]\} = (1 - \beta)(1 + \cos \alpha)\sigma/2 + \beta \]

\[ \leq \beta - \frac{(1 - \beta)}{4}(1 + \cos \alpha)(1 + \rho^2) \leq \beta. \]

This shows that $\Re\{\phi(p(z), zp'(z); z)\} \not\in \Omega_\beta$. Thus by Lemma 1.3 we get $\Re\{p(z)\} > 0$ or $\Re\{f'(z)\} > \beta$. This means that $f \in \mathcal{C}(\beta)$ and concluding the proof. \(\square\)

Taking $\beta = 0$ in the above Theorem 2.1, we get.

**Corollary 2.1.** If $f \in \mathcal{L}_\alpha$, then $\Re\{f'(z)\} > 0$ ($z \in \Delta$) and thus $f$ is univalent.

**Remark 2.1.** Since $\Re\{(1 + e^{i\alpha})/2\} = (1 + \cos \alpha)/2 \geq 0$ where $\alpha \in (-\pi, \pi]$, thus the above Theorem 2.1 is a generalization of the results that earlier were obtained by Chichra [2] and Lewandowski et al. [7].

The problem of finding a lower bound for $\Re\{f(z)/z\}$ is called Marx–Strohhäcker inequality. Because, first time Marx and Strohhäcker (8, 17) proved that if $f \in \mathcal{K}$, then $\Re\{f(z)/z\} > 1/2$. In the sequel we consider this problem for the class $\mathcal{R}(\alpha, \beta)$. 
Theorem 2.2. Let $0 \leq \beta < 1$ and $-\pi < \alpha \leq \pi$. If $f \in \mathcal{A}$ belongs to the class $\mathcal{R}(\alpha, \beta)$, then we have

$$\text{Re} \left\{ \frac{f(z)}{z} \right\} > \beta \quad (0 \leq \beta < 1).$$

Proof. Let the function $f \in \mathcal{A}$ belongs to the class $\mathcal{R}(\alpha, \beta)$ where $0 \leq \beta < 1$ and $-\pi < \alpha \leq \pi$. Define the function $p$ as

$$p(z) := \frac{1}{1 - \beta} \left( \frac{f(z)}{z} - \beta \right).$$

Since $f \in \mathcal{A}$, easily seen that $p$ is analytic in $\Delta$ and $p(0) = 1$. The equation (2.3), with a simple calculation implying that

$$f'(z) = \beta + (1 - \beta)p(z) + (1 - \beta)z p'(z)$$

and

$$f''(z) = 2(1 - \beta)p'(z) + (1 - \beta)z p''(z).$$

Now, from (2.4) and multiplying (2.5) by $\frac{1 + e^{i\alpha}}{2} z$, we get

$$f'(z) + \frac{1 + e^{i\alpha}}{2} z f''(z)$$

$$\quad = \beta + (1 - \beta)p(z) + [(2 + e^{i\alpha})(1 - \beta)]z p'(z) + (1 - \beta)\frac{1 + e^{i\alpha}}{2} z^2 p''(z)$$

$$\quad = \psi(p(z), z p'(z), z^2 p''(z); z),$$

where

$$\psi(r, s, t; z) = \beta + (1 - \beta)r + [(2 + e^{i\alpha})(1 - \beta)]s + (1 - \beta)\frac{1 + e^{i\alpha}}{2} t.$$

Since $f \in \mathcal{R}(\alpha, \beta)$ we consider the following inclusion relation

$$\{ \psi(p(z), z p'(z), z^2 p''(z); z) : z \in \Delta \} \subset \Omega_\beta,$$

where $\Omega_\beta$ is defined in (2.2). Let $\rho, \sigma, \mu$ and $\nu$ be real numbers such that

$$\sigma \leq -\frac{1}{2}(1 + \rho^2), \quad \mu + \nu \leq 0 \quad \text{and} \quad 2\mu + \nu \leq 0.$$

From (2.6) and by [9] (see also [10] Theorem 2.3b), since

$$\psi(i\rho, \sigma, \mu + i\nu; z) = \beta + (1 - \beta)i\rho + [(2 + e^{i\alpha})(1 - \beta)]\sigma + (1 - \beta)\frac{1 + e^{i\alpha}}{2}(\mu + i\nu),$$

we get

$$\text{Re}\{\psi(i\rho, \sigma, \mu + i\nu; z)\} = \beta + (1 - \beta)(2 + e^{i\alpha})\sigma + \frac{1 - \beta}{2}[\mu(1 + \cos \alpha) - \nu \sin \alpha]$$

$$\quad \leq \beta - (1 - \beta)(1 + \rho^2)(1 + \cos \alpha) + \frac{1 - \beta}{2}[\mu(1 + \cos \alpha) - \nu \sin \alpha]$$

$$\quad = F(\alpha, \beta, \rho) + G(\alpha, \beta, \mu),$$

where

$$F(\alpha, \beta, \rho) := \beta - (1 - \beta)(1 + \rho^2)(1 + \cos \alpha)$$

and

$$G(\alpha, \beta, \mu) := \frac{1 - \beta}{2}[\mu(1 + \cos \alpha) - \nu \sin \alpha].$$

It is easy to see that $F(\alpha, \beta, \rho) \leq \beta$. Also because $2\mu + \nu \leq 0$, we have $G(\alpha, \beta, \mu) \leq 0$. Thus $\text{Re}\{\psi(i\rho, \sigma, \mu + i\nu; z)\} \leq \beta$ and this means that

$$\text{Re}\{\psi(p(z), z p'(z), z^2 p''(z); z)\} \not\in \Omega_\beta.$$
Therefore we obtain \( \text{Re}\{p(z)\} > 0 \) where \( p \) is given by (2.3), or equivalently
\[
\text{Re}\left\{ \frac{f(z)}{z} \right\} > \beta \quad (0 \leq \beta < 1).
\]
This completes the proof.

If we put \( \beta = 0 \) in the above Theorem 2.2 we get.

**Corollary 2.2.** If \( f \in \mathcal{L}_\alpha \), then \( \text{Re}\{f(z)/z\} > 0 \) in the open unit disc \( \Delta \).

We shall require the following lemma in order to prove of the next result.

**Lemma 2.1.** Let \( \phi_b(z) \) be defined by (1.1) for \( b > 1/2 \). Then \( \phi_b(\Delta) = \Omega_b \) where
\[
\Omega_b := \{ w \in \mathbb{C} : 0 < \text{Re}\{w\} < 2b \}.
\]

*Proof.* If \( b = 1 \), then we have \( 0 < \text{Re}\{\phi_b(z)\} = \text{Re}\{1 + z\} < 2 \). For \( b > 1/2 \) and \( b \neq 1 \), the function \( \phi_b(z) \) does not have any poles in \( \Delta \) and is analytic in \( \Delta \). Thus looking for the \( \min\{\text{Re}\{\phi_b(z)\} : |z| < 1\} \) it is sufficient to consider it on the boundary \( \partial \phi_b(\Delta) = \{ \phi_b(e^{i\varphi}) : \varphi \in [0, 2\pi] \} \). A simple calculation gives us
\[
\text{Re}\left\{ \phi_b(e^{i\varphi}) \right\} = \frac{(1/b)(1 + \cos \varphi)}{1 + 2(1/b - 1) \cos \varphi + (1/b - 1)^2} \quad (\varphi \in [0, 2\varphi]).
\]
So we can see that \( \text{Re}\{\phi_b(z)\} \) is well defined also for \( \varphi = 0 \) and \( \varphi = 2\pi \). Define
\[
h(x) = \frac{(1/b)(1 + x)}{1 + 2(1/b - 1)x + (1/b - 1)^2} \quad (-1 \leq x \leq 1).
\]
Thus for \( b > 1/2 \) and \( b \neq 1 \), we have \( h'(x) > 0 \). Therefore, we get
\[
0 = h(-1) \leq h(x) \leq h(1) = 2b.
\]
This completes the proof.

*Theorem 2.3.* Let \( f \in \mathcal{A} \) be a member of the class \( \mathcal{L}_\alpha(b) \) where \( b > 1/2 \) and \(-\pi < \alpha \leq \pi \). Then
\[
0 < \text{Re}\{f'(z)\} < 2b \quad (z \in \Delta).
\]

*Proof.* Let us \( f \in \mathcal{L}_\alpha(b) \). Then by Lemma 2.1 and by definition of the subordination principle we have
\[
0 < \text{Re}\left\{ f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) \right\} < 2b \quad (z \in \Delta, b > 1/2, -\pi < \alpha \leq \pi).
\]
First, we assume that
\[
\text{Re}\left\{ f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) \right\} > 0.
\]
Then by Corollary 2.2 we have \( \text{Re}\{f'(z)\} > 0 \). Now we let
\[
\text{Re}\left\{ f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) \right\} < 2b.
\]
Put \( \xi = 2b \) and so \( \xi > 1 \). Let \( f'(z) \neq 0 \) for \( z \neq 0 \). Consider
\[
q(z) = \frac{1}{1 - \xi} \left( f'(z) - \xi \right) \quad (\xi > 1, z \in \Delta).
\]
Then \( q(z) \) is analytic in \( \Delta \) and \( q(0) = 1 \). A simple check gives us
\[
f'(z) + \frac{1 + e^{i\alpha}}{2} zf''(z) = (1 - \xi)[q(z) + (1 + e^{i\alpha})zq'(z)/2] + \xi = \eta(q(z), zq'(z); z),
\]
where \( \eta(x, y; z) = (1 - \xi)[x + (1 + e^{i\alpha})y/2] + \xi \). Now we define
\[
\{ \eta(q(z), zq'(z); z) : z \in \Delta \} \subset \{ w : \text{Re}\{w\} < \xi \} =: \Omega_\xi.
\]
Again with a simple calculation we deduce that
\[ \text{Re}\{i\rho, \sigma; z}\} = \text{Re}\left\{ (1 - \xi)[i\rho + (1 + e^{i\alpha})\sigma/2]\right\}
\geq (1 - \xi)(1 + \cos\alpha)\sigma/2 + \xi
\geq \frac{(\xi - 1)}{4}(1 + \cos\alpha)(1 + \rho^2) + \xi \quad (\sigma \geq (1 + \rho^2)/2)
\]
This shows that \( \text{Re}\{i\rho, \sigma; z}\} \not\in \Omega_{\xi} \) and therefore \( \text{Re}\{q(z)\} > 0 \), or equivalently \( \text{Re}\{f'(z)\} < \xi \). It is the end of proof. \( \square \)

**Theorem 2.4.** Assume that \( b > 1/2, -\pi < \alpha \leq \pi \) and \( f \in \mathcal{L}_0(b) \). Then for each \( |z| = r < 1 \) we have
\[ 1 - \frac{(2b - 1)r}{b + (b - 1)r} \leq \text{Re}\left\{ f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z)\right\} \leq 1 + \frac{(2b - 1)r}{b + (b - 1)r}. \]

**Proof.** Let \( f \in \mathcal{L}_0(b) \). Then from the definition of subordination and by Lemma [1.1] there exists a \( \omega \in \mathcal{B} \) such that
\[ f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z) = \frac{1 + \omega(z)}{1 + \left(\frac{1}{b} - 1\right)\omega(z)} \quad (z \in \Delta). \]
We define
\[ W(z) = \frac{1 + \omega(z)}{1 + \left(\frac{1}{b} - 1\right)\omega(z)}, \]
which readily yields
\[ W(z) - 1 = \frac{(2 - \frac{1}{b})\omega(z)}{1 + \left(\frac{1}{b} - 1\right)\omega(z)}. \]
For \( |z| = r < 1 \), using the known fact that (see [I]) \( |\omega(z)| \leq |z| \) we find that
\[ |W(z) - 1| \leq \frac{(2b - 1)r}{b + (b - 1)r}. \]
Hence, \( W(z) \) maps the disk \( |z| < r < 1 \) onto the disk which the center \( C = 1 \) and the radius \( \delta \) given by
\[ \delta = \frac{(2b - 1)r}{b + (b - 1)r}. \]
Therefore,
\[ 1 - \frac{(2b - 1)r}{b + (b - 1)r} \leq |W(z)| \leq 1 + \frac{(2b - 1)r}{b + (b - 1)r}. \]
Now, the assertion follows from (2.9) and this fact that \( \text{Re}\{z\} \leq |z| \). \( \square \)

**Remark 2.2.** We obtained two lower and upper bounds for
\[ \text{Re}\left\{ f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z)\right\}, \]
when \( f \in \mathcal{L}_0(b) \). From [2.7], we have
\[ 0 < \text{Re}\left\{ f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z)\right\} < 2b \quad (z \in \Delta, b > 1/2, -\pi < \alpha \leq \pi), \]
while by [2.8]
\[ G(r) := 1 - \frac{(2b - 1)r}{b + (b - 1)r} \leq \text{Re}\left\{ f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z)\right\} \leq U(r) := 1 + \frac{(2b - 1)r}{b + (b - 1)r}. \]
It is easy to check that \( U(r) < 2b \) if \( b \geq 1 \) (or \( b \to 1^+ \)) while \( G(r) \geq 0 \) for \( 1/2 < b \leq 1 \) (or \( b \to 1^- \)).
Corollary 2.3. Let $f \in \mathcal{L}_\alpha(1)$. Then we have
\[ 1 - r < \text{Re} \left\{ f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z) \right\} < 1 + r \quad (|z| = r < 1). \]

Corollary 2.4. By a simple geometric observation and applying (2.9) and (2.10), we have
\[ |\arg \left\{ f'(z) + \frac{1 + e^{i\alpha}}{2}zf''(z) \right\}| < \arcsin \frac{(2b - 1)r}{b + (b - 1)}r \quad (|z| = r < 1, b > 1/2). \]

3. Conjecture

In this section, we obtain the radius of univalence of 2-th section sum of $f \in \mathcal{R}(\alpha, \beta)$. We recall that the Taylor polynomial $s_k(z) = s_k(f)(z)$ of $f$ defined by
\[ s_k(z) = s_k(f)(z) = z + a_2z^2 + \cdots + a_kz^k, \]
is called the $k$-th section/partial sum of $f$. In [18], proved that every section $s_k(z)$ of a $f \in \mathcal{S}$ is univalent in the disk $|z| < 1/4$ and the number 1/4 is best possible as the second partial sum of the Koebe function $k(z) = z/(1 - z)^2$ shows. Next, we find the radius of univalence of the 2-th section sum of $f \in \mathcal{R}(\alpha, \beta)$.

Theorem 3.1. The 2-th section sum of $f \in \mathcal{R}(\alpha, \beta)$ is univalent in the disc
\[ |z| < \frac{\sqrt{10 + 6\cos \alpha}}{4(1 - \beta)} \quad (-\pi < \alpha \leq \pi, 0 \leq \beta < 1). \]
The number $\frac{\sqrt{10 + 6\cos \alpha}}{4(1 - \beta)}$ cannot be replaced by a greater one.

Proof. Let $f(z) = z + \sum_{n=2}^{\infty} a_nz^n \in \mathcal{R}(\alpha, \beta)$ and $s_2(z) = z + a_2z^2$ be its second section. By a simple calculation and since $|a_2| \leq \frac{2(1 - \beta)}{\sqrt{10 + 6\cos \alpha}}$, we have
\[ \text{Re}\{s_2'(z)\} = \text{Re}\{1 + 2a_2z\} \geq 1 - 2|a_2||z| \geq 1 - \frac{4(1 - \beta)|z|}{\sqrt{10 + 6\cos \alpha}}, \]
which is positive provided $|z| < \frac{\sqrt{10 + 6\cos \alpha}}{4(1 - \beta)}$. Therefore, $s_2(z)$ is close-to-convex (univalent) in the disk $|z| < \frac{\sqrt{10 + 6\cos \alpha}}{4(1 - \beta)}$. To show that this bound is sharp, we consider the function $f_x$ defined by (2.1). The second partial sum $s_2(f_x)(z)$ of $f_x$ is $z + \frac{4(1 - \beta)}{2(3 + e^{i\alpha})}z^2$. Thus we get
\[ s_2'(z) = 1 + \frac{4(1 - \beta)}{(3 + e^{i\alpha})}z. \]
Hence $\text{Re}\{s_2'(z)\} = 0$ when $z = -\frac{(3 + e^{i\alpha})}{4(1 - \beta)}$. This completes the proof. \qed

Finally, we pose a conjecture as follows:

Conjecture. Every section of $f \in \mathcal{R}(\alpha, \beta)$ is univalent in the disc $|z| < \frac{\sqrt{10 + 6\cos \alpha}}{4(1 - \beta)}$.

References

[1] P.L. Duren, *Univalent Functions*, Springer–Verlag, New York, 1983.
[2] P.N. Chichra, *New subclasses of the class of close-to-convex functions*, Proc. Amer. Math. Soc. **62** (1977), 37–43.
[3] C.–Y. Gao and S.–Q. Zhou, *Certain subclass of starlike functions*, Appl. Math. Comput. **187** (2007), 176–182.
[4] A.W. Goodman, *Univalent Functions*, Vols. 1 and 2, Mariner, Tampa, FL, 1983.
[5] R. Kargar, A. Ebadian and J. Sokół, *On subordination of some analytic functions*, Sib. Math. J. **57** (2016), 599–604.
[6] W. Kaplan, *Close-to-convex schlicht functions*, Michigan Math. J. **1** (1952), 169–185.
[7] Z. Lewandowski, S. Miller and E. Złotkiewicz, *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc. **56** (1976), 111–117.
ON CERTAIN SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

[8] A. Marx, *Untersuchungen iiber schlichte Abbildungen*, Math. Ann. 107 (1932/33), 40–65.

[9] S.S. Miller and P.T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. 28 (1981), 157–171.

[10] S.S. Miller and P.T. Mocanu, *Differential Subordinations, Theory and Applications*, Series of Monographs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York / Basel 2000.

[11] S. Ponnusamy and V. Singh, *Convolution properties of some classes of analytic functions*, J. Math. Sci. 89 (1988), 1008–1020.

[12] R. Singh and S. Singh, *Convolution properties of a class of starlike functions*, Proc. Amer. Math. Soc. 106 (1989), 145–152.

[13] R. Singh and S. Singh, *Starlikeness and convexity of certain integrals*, Ann. Univ. Mariae Curie–Sklodowska Sect A 35 (1981), 45–47.

[14] H. Silverman, *A class of bounded starlike functions*, Internat. J. Math. Math. Sci. 17 (1994), 249–252.

[15] H. Silverman and E.M. Silvia, *Characterizations for subclasses of univalent functions*, Math. Japonica 50 (1999), 103–109.

[16] H.M. Srivastava, D. Răducanu, P. Zaprawa, *A certain subclass of analytic functions defined by means of differential subordination*, Filomat 30:14 (2016), 3743–3757.

[17] E. Strohhäcker, *Beiträge zur Theorie der schlichten Funktionen*, Math. Z. 37 (1933), 356–380.

[18] G. Szegő, *Zur Theorie der schlichten Abbildungen*, Math. Ann. 100 (1928), 188–211.

[19] L. Trojnar–Spelina, *Subclasses of univalent functions related with circular domains*, J. Math. Appl. 34 (2011), 109–116.

Department of Mathematics, Islamic Azad University, Firoozkouh Branch, Firoozkouh, Iran

E-mail address: mahzoon hesam@yahoo.com (H. Mahzoon)

Young Researchers and Elite Club, Ardabil Branch, Islamic Azad University, Ardabil, Iran

E-mail address: rkargar1983@gmail.com (R. Kargar)