ON UNIFORM CONSISTENCY OF NONPARAMETRIC TESTS. I

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For widespread nonparametric tests, we point out necessary and sufficient conditions of uniform consistency for nonparametric sets of alternatives. Nonparametric sets of alternatives can be defined both in terms of distribution functions and in terms of density. Such conditions are provided for \( \chi^2 \)-tests with an increasing number of cells, for Cramér – von Mises tests, for tests generated by \( L_2 \)-norms of kernel estimators, and for tests generated by quadratic forms of estimators of Fourier coefficients. Necessary and sufficient conditions on sets of alternatives for the existence of uniformly consistent tests are treated as well. Bibliography: 31 titles.

1. Introduction

Let \( X_1, \ldots, X_n \) be a sample of i.i.d.r.v.'s having c.d.f. \( F \in \mathfrak{F} \). Here \( \mathfrak{F} \) is the set of all distribution functions of random variables with values in the interval \((0,1)\).

We study the problem of testing the hypothesis
\[
\mathbb{H}_0 : F(x) = F_0(x) = x, \quad x \in [0,1],
\]
versus sets of alternatives defined in terms of distribution functions
\[
\mathbb{H}_n : F \in \mathfrak{Y}_n, \quad \mathfrak{Y}_n \subset \mathfrak{F},
\]
or in terms of densities \( p(x) = 1 + f(x) = \frac{dF(x)}{dx} \),
\[
\mathbb{H}_{1n} : f \in \mathfrak{P}_n, \quad \mathfrak{P}_n \subset L_2(0,1).
\]

Here \( L_2(0,1) \) is the Hilbert space of all square integrable functions \( g(t) \), \( t \in (0,1) \), with the \( L_2 \)-norm \( \|g\| = \left( \int g^2(t) \, dt \right)^{1/2} \).

For a part of setups, the problem of goodness of fit testing for a distribution function or density is replaced by the problem of signal detection in Gaussian white noise. This allows us to simplify the technical part of the paper.

We are interested in uniform consistency of nonparametric tests. If tests or test statistics are uniformly consistent for sets of alternatives, we say that these sets of alternatives are uniformly consistent for these tests or test statistics.

For the setups mentioned above, we point out necessary and sufficient conditions of uniform consistency of sets of alternatives (1.2) and (1.3) for test statistics of
- Kolmogorov tests;
- Cramér – von Mises tests;
- \( \chi^2 \)-tests having increasing number of cells with growth of sample size;
- tests generated by quadratic forms of estimators of Fourier coefficients of the orthogonal expansion of signal;
- tests generated by \( L_2 \)-norms of kernel estimators.

The last four of the above-mentioned tests statistics have quadratic structure. The results and proofs for these test statistics are similar. We provide these results in the first part of paper. Results about Kolmogorov tests are provided in the second part of the paper.

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Denote by $\hat{F}_n$ the empirical distribution function of $X_1, \ldots, X_n$.

If sets of alternatives are defined in terms of distribution functions, necessary and sufficient conditions of consistency are provided in the framework of the distance method.

Test statistics can be considered as functionals $T_n(\hat{F}_n)$ depending on empirical distribution functions. The functionals $T_n(F)$ admit interpretation as norms or seminorms defined on the set of differences of distribution functions. The established uniform consistency of tests statistics on sets of alternatives

$$\Upsilon_n(T_n, \rho_n) = \{ F : T_n(F) > \rho_n > 0, F \in \mathcal{F} \}$$

allows us to make a conclusion on uniform consistency of any sequence of sets of alternatives $\Upsilon_n$ in terms of their distances or semidistances

$$\inf_{F \in \Upsilon_n} T_n(F)$$

from the hypothesis.

For specially selected sequences $\rho_n$, $\rho_n \to 0$ as $n \to \infty$, in the papers [7–9] (see also Theorems 6.3, 4.3, and 5.2), we have established the uniform consistency of the sets $\Upsilon_n(T_n, \rho_n)$ of alternatives for $\chi^2$-tests having an increasing number of cells with growth of the sample size, for tests generated by $L_2$-norms of kernel estimators, and for tests generated by quadratic forms of estimators of Fourier coefficients Moreover, the asymptotic minimaxity of tests on these sets has been established. In this part of the paper, we establish the uniform consistency of the sets $\Upsilon_n(T, \rho_n)$ of alternatives for the Cramér–von Mises test (see Theorem 7.1). Some similar results are established for the Kolmogorov test in the second part the of paper. Thus, for the problem of nonparametric hypothesis testing on a distribution function, we substantiate the key argumentation of the distance method.

Proofs of results on uniform consistency of sets of alternatives (1.3) defined in terms of densities or signals are based on these results.

The problem of signal detection is considered for the following setup. We observe a realization of a random process $Y_n(t)$ defined by the stochastic differential equation

$$dY_n(t) = f(t) dt + \frac{\sigma}{\sqrt{n}} dw(t), \quad t \in [0,1], \quad \sigma > 0,$$

where $f \in L_2(0,1)$ is an unknown signal and $dw(t)$ is Gaussian white noise.

The following nonparametric sets of alternatives (see [6, 9, 12, 15–17, 21, 25]) are often explored:

$$\Pi_n : f \in V_n = \{ f : \|f\| \geq \rho_n, f \in U \subset L_2(0,1) \}, \quad (1.5)$$

where $\rho_n \to 0$ as $n \to \infty$. Here $U$ is a convex set.

We answer the four questions listed below.

For which bounded centrally symmetric convex sets $U$ do there exist $\rho_n \to 0$ as $n \to \infty$ such that there is a uniformly consistent sequence of tests for sets $V_n$ of alternatives?

We show that uniformly consistent tests exist if and only if the set $U$ is relatively compact (see Theorems 3.1 and 3.3). A similar result is established for the problem of hypothesis testing on a density (see Theorem 3.2). Note that the necessary and sufficient condition for the existence of a consistent nonparametric estimator on a bounded nonparametric set is the relative compactness of this set [14,18]. The same compactness condition arises in solution of ill-posed inverse problems with deterministic errors [5]. The problem of existence of consistent tests has been explored for different setups. The most complete bibliography can be found in [10].

The answer to the next three questions is provided for an i.i.d.r.v.’s model in the case of Cramér–von Mises tests and chi-squared tests. For test statistics generated by quadratic forms
of estimators of Fourier coefficients or for tests generated by $L_2$-norms of kernel estimators, the answer is provided for the problem of signal detection in Gaussian white noise.

Let $\rho_n = n^{-r}, 0 < r \leq 1/2$, and let $r$ be fixed. How to define the largest bounded sets $U$ such that the sets $V_n$ are uniformly consistent for one of the above-mentioned test statistics?

We call such sets $U$ maxisets. The exact definition of maxisets is given in Sec. 2. For $0 < r < 1/2$ and for test statistics having quadratic structure, we show (see Theorems 4.4, 5.1, 6.1, and 7.2) that maxisets are bodies in the Besov spaces $B^s_{2\infty}(P_0), P_0 > 0$. Here $r = \frac{s}{1+4s}$ for chi-squared test statistics, for test statistics being $L_2$-norms of kernel estimator, and for test statistics being quadratic forms of estimators of Fourier coefficients of a signal. For Cramér–von Mises tests, $r = \frac{s}{2+2s}$.

For $r = 1/2$, we could not find sets satisfying all the requirements of the definition of maxisets. However, we show that bounded convex sets of functions having a fixed finite number of nonzero Fourier coefficients satisfy similar requirements. In further statements of this section for $r = 1/2$, and therefore, in the corresponding theorems, maxisets can be replaced by such sets.

The uniform consistency of chi-squared tests and Cramér–von Mises tests for the above-mentioned Besov bodies has been established by Ingster [15].

For nonparametric estimation, the notion of maxisets has been introduced by Kerkyacharian and Picard [19]. Maxisets of nonparametric estimators have been comprehensively explored in [4,20,27] (see also references therein). For nonparametric hypothesis testing, a completely different definition of maxisets has been introduced by Autin, Clausel, Freyermuth, and Marteau [2].

Let each set $\Psi_n$ be bounded in $L_2(0,1)$. Then Cramér–von Mises tests, chi-squared tests, tests generated by $L_2$-norms of kernel estimators and by quadratic forms of estimators of Fourier coefficients of a signal are uniformly consistent if and only if these sets $\Psi_n$ of alternatives do not contain an inconsistent sequence of simple alternatives $f_n \in \Psi_n$. In other words, sets of alternatives are uniformly consistent if and only if all the sequences of simple alternatives $f_n \in \Psi_n$ are consistent. Thus, the problem of uniform consistency for sets $\Psi_n$ of alternatives is reduced to the problem of consistency of any sequence of simple alternatives $f_n \in \Psi_n$.

How to describe all consistent and inconsistent sequences of simple alternatives having given rate of convergence to the hypothesis?

We treat this problem as the problem of testing the hypothesis

$$\bar{H}_0 : f(x) = 0, \quad x \in [0,1],$$

versus a sequence of simple alternatives

$$\bar{H}_n : f = f_n, \quad cn^{-r} \leq \|f_n\| \leq Cn^{-r},$$

where $0 < r \leq 1/2$ and $0 < c < C < \infty$.

For the above-mentioned test statistics, an answer to this question is given in terms of concentration of Fourier coefficients (Theorems 4.1 and 4.2). In Theorem 4.5, we propose the following interpretation of these results.

A sequence of simple alternatives $f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, is consistent if and only if the functions $f_n$ admit representation as functions $f_{1n}$ from a maxiset with the same rate of convergence to the hypothesis plus functions $f_n - f_{1n}$ orthogonal to the functions $f_{1n}$.

In Theorem 4.6, we show that for any $\varepsilon > 0$ there exists a maxiset and functions $f_{1n}$ from this maxiset such that the differences of type II error probabilities for the alternatives $f_n$ and $f_{1n}$ are smaller than $\varepsilon$ and $f_{1n}$ is orthogonal to $f_n - f_{1n}$.
Thus, each function of a consistent sequence of alternatives with a fixed rate of convergence to the hypothesis contains a sufficiently smooth function as an additive component, and this function carries almost all the information on its type II error probability.

What can we say about properties of consistent and inconsistent sequences of alternatives having a fixed rate of convergence to the hypothesis in the $L_2$-norm?

In Theorem 4.7, we show that the asymptotic of type II error probabilities of sums of alternatives from consistent and inconsistent sequences coincides with the asymptotic for the consistent sequence.

We call a sequence of alternatives $f_n$ purely consistent if there does not exist an inconsistent sequence of alternatives $f_{2n}$ having the same rates of convergence to the hypothesis and such that $f_{2n}$ are orthogonal to $f_n - f_{2n}$.

It is easy to show that any sequence of alternatives from maxisets with fixed rates of convergence to the hypothesis is purely consistent.

In Theorem 4.8, in terms of concentration of Fourier coefficients, we point out an analytic assignment of purely consistent sequences of alternatives.

In Theorem 4.9, we show that for any $\varepsilon > 0$ and for any purely consistent sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, there are a maxiset and some sequence $f_{1n}$ from this maxiset such that $\|f_n - f_{1n}\| \leq \varepsilon n^{-r}$.

The paper is organized as follows. In Sec. 2, we introduce the main definitions. In Sec. 3, the answer to the first question is provided. In Secs. 4–7, for $0 < r < 1/2$, the above-mentioned results are established, respectively, for the test statistics based on quadratic forms of estimators of Fourier coefficients, $L_2$-norms of kernel estimators, $\chi^2$-tests, and Cramér – von Mises tests. In Sec. 8, we focus on the case $r = \frac{1}{2}$.

The proofs of all theorems are given in the Appendix.

We use letters $c$ and $C$ as a generic notation for positive constants. Denote by $\mathbf{1}\{A\}$ the indicator of an event $A$. Denote by $[a]$ the integer part of a real number $a$. For any two sequences of positive real numbers $a_n$ and $b_n$, $a_n \asymp b_n$ means that $c < a_n/b_n < C$ for all $n$ and $a_n = o(b_n)$ means that $a_n/b_n \to 0$ as $n \to \infty$. For any complex number $z$, denote by $\overline{z}$ the conjugate number.

Denote by

$$
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\{-t^2/2\} \, dt, \quad x \in \mathbb{R}^1,
$$

the standard normal distribution function.

Let $\phi_j$, $1 \leq j < \infty$, be an orthonormal system of functions in $L_2(0,1)$. For each $P_0 > 0$ define the set

$$
\bar{B}_s^{2\infty}(P_0) = \left\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j, \quad \sup_{\lambda > 0} \lambda^{2s} \sum_{j > \lambda} \theta_j^2 \leq P_0, \quad \theta_j \in \mathbb{R}^1 \right\}. \tag{1.8}
$$

Under definite assumptions about the basis $\phi_j$, $1 \leq j < \infty$, the functional space

$$
\bar{B}_s^{2\infty} = \left\{ f : f = \sum_{j=1}^{\infty} \theta_j \phi_j, \quad \sup_{\lambda > 0} \lambda^{2s} \sum_{j > \lambda} \theta_j^2 < \infty, \quad \theta_j \in \mathbb{R}^1 \right\}
$$

is the Besov space $B_s^{2\infty}$ (see [27]). In particular, $\bar{B}_s^{2\infty}$ is the Besov space if $\phi_j$, $1 \leq j < \infty$, is the trigonometric basis.
If $\phi_j(t) = \exp\{2\pi jtx\}$, $x \in (0, 1)$, $j = 0, \pm 1, \ldots$, denote

$$\mathbb{B}_{2\infty}(P_0) = \left\{ f : f = \sum_{j=\infty}^{\infty} \theta_j \phi_j, \sup_{\lambda > 0} \lambda^{2s} \sum_{|j| > \lambda} |\theta_j|^2 \leq P_0 \right\}. $$

Since here $\phi_j$ are complex functions, $\theta_j$ are complex numbers as well and $\theta_j = \overline{\theta_{-j}}$ for all $-\infty < j < \infty$.

For the same basis, denote

$$\tilde{\mathbb{B}}_{2\infty}^s(P_0) = \left\{ f : f = \sum_{j=\infty}^{\infty} \theta_j \phi_j, f \in \mathbb{B}_{2\infty}(P_0), \theta_0 = 0 \right\}. $$

Balls in the Nikolskii classes

$$\int (f^{(l)}(x+t) - f^{(l)}(x))^2 \, dx \leq L|t|^{2(s-l)}, \|f\| < C, $$

are Besov balls in $\mathbb{B}_{2\infty}^s$. Here $l = [s]$.

2. Main definitions

2.1. Consistency and $n^{-r}$-consistency. For any test $K_n$ denote by $\alpha(K_n)$ its type I error probability and by $\beta(K_n, f)$ its type II error probability for an alternative $f \in L_2(0, 1)$. A similar notation $\beta(K_n, F)$ is used if the alternative is a c.d.f. $F$.

Definitions of consistency are slightly different in different sections. In Sec. 3, the problem of the existence of uniformly consistent tests and uniform consistency of sets of alternatives is considered among all tests.

In Sec. 4, consistency is considered for a fixed sequence of test statistics $T_n$. For kernel-based tests and chi-squared tests, consistency is studied for the whole family of test statistics depending on kernel widths and number of cells, respectively. In Sec. 7, we have only one test statistic.

We have mentioned that the problem of uniform consistency of sets of alternatives is reduced to the problem of consistency of sequences of simple alternatives. Thus, in Secs. 4–7, we explore this setup.

Below we provide a definition of consistency for the setup of Secs. 4 and 7. In Secs. 5 and 6, the definitions will be different in the sense mentioned above.

We say that a sequence of simple alternatives $f_n$ is consistent if for any $\alpha$, $0 < \alpha < 1$, and for sequence of tests $K_n$, $\alpha(K_n) = \alpha (1 + o(1))$, generated by test statistics $T_n$,

$$\limsup_{n \to \infty} \beta(K_n, f_n) < 1 - \alpha. \quad (2.1)$$

If, in addition, $cn^{-r} < \|f_n\| < Cn^{-r}$, we say that the sequence of alternatives $f_n$ is $n^{-r}$-consistent (see [30]).

We say that a sequence of alternatives $f_n$ is inconsistent if, for each sequence of tests $K_n$ generated by test statistics $T_n$,

$$\liminf_{n \to \infty} (\alpha(K_n) + \beta(K_n, f_n)) \geq 1. \quad (2.2)$$

Assume that we consider the problem of testing hypothesis (1.1) versus alternatives (1.3), where $\Psi_n$ can also be sets of signals.

For tests $K_n$, $\alpha(K_n) = \alpha + o(1)$, $0 < \alpha < 1$, generated by test statistics $T_n$, denote $\beta(K_n, \Psi_n) = \sup_{f \in \Psi_n} \beta(K_n, f)$. We say that a sequence of sets $\Psi_n$ of alternatives is uniformly consistent if

$$\limsup_{n \to \infty} \beta(K_n, \Psi_n) < 1 - \alpha. \quad (2.3)$$
For sets of alternatives $\Upsilon_n$ defined by (1.2), the definition of uniform consistency is the same.

2.2. Purely consistent sequences. We say that an $n^{-r}$-consistent sequence of alternatives $f_n$ is purely $n^{-r}$-consistent if there does not exist a subsequence $f_{n_i}$ such that $f_{n_i} = f_{1n_i} + f_{2n_i}$, where $f_{2n_i}$ is orthogonal to $f_{1n_i}$ and the sequence $f_{2n_i}$, $\|f_{2n_i}\| > c_1 n^{-r}$, is inconsistent.

2.3. Maxisets. The definition of maxisets is given for the problem of signal detection in Gaussian white noise. After that, the differences for the problem of hypothesis testing on a density is pointed out.

Let $\phi_j$, $1 \leq j < \infty$, be an orthonormal basis in $L_2(0,1)$. We say that a set $U$, $U \subset L_2(0,1)$, is orthosymmetric with respect to this basis if the equality $f = \sum_{j=1}^{\infty} \theta_j \phi_j \in U$ implies that $\tilde{f} = \sum_{j=1}^{\infty} \tilde{\theta}_j \phi_j \in U$ for any $\tilde{\theta}_j = \theta_j$ or $\tilde{\theta}_j = -\theta_j$, $j = 1, 2, \ldots$.

For a closed, orthosymmetric, bounded, convex set $U$, $U \subset L_2(0,1)$, denote by $\Xi$ the functional space with unit ball $U$.

For the problem of signal detection, a bounded, orthosymmetric, closed set $U$, $U \subset L_2(0,1)$, is called a maxiset and the functional space $\Xi$ is called a maxispace if

(i) any subsequence of alternatives $f_{n_i} \in U$, $cn_i^{-r} < \|f_{n_i}\| < Cn_i^{-r}$, with $n_i \to \infty$ as $i \to \infty$, is consistent;

(ii) if $f \notin \Xi$, then in any convex orthosymmetric set $V$ that contains $f$ there is an inconsistent subsequence of alternatives $f_{n_i} \in V$ for samples $X_1, \ldots, X_{n_i}$ such that $cn_i^{-r} < \|f_{n_i}\| < Cn_i^{-r}$ with $n_i \to \infty$ as $i \to \infty$.

Condition (ii) implies that $U$ is the largest set satisfying (i).

For the problem of hypothesis testing on a density, we assume that we are given an orthonormal system of functions $1, \phi_1, \phi_2, \ldots$, and, in the definition of maxiset, we make an additional assumption:

(ii) is considered only for functions $f = 1 + \sum_{i=1}^{\infty} \theta_i \phi_i$ (or $f = 1 + \sum_{|i| \geq l}^{\infty} \theta_i \phi_i$) satisfying the following condition.

D. There is an $l_0 = l_0(f)$ such that for all $l > l_0$, the functions $1 + \sum_{|i| \geq l}^{\infty} \theta_i \phi_i$ are nonnegative (are densities).

Condition D allows us to analyze the tails $f_{nj} = \sum_{|i| \geq j} \theta_i \phi_i$ of orthogonal expansions of $f$ to establish (ii). In D, we assume that the functions $1, \phi_1, \phi_2, \ldots$ form an orthonormal basis in $L_2(0,1)$.

It is clear that if $U$ is a maxiset, then $\gamma U$, $0 < \gamma < \infty$, is a maxiset as well.

Simultaneous assumptions of the convexity and orthosymmetry of a set $V$ are rather strong. They imply that if $f \in V$, $f = \sum_{i=1}^{\infty} \theta_i \phi_i$, then any $f_\eta \in V$, where $f_\eta = \sum_{i=1}^{\infty} \eta_i \phi_i$, $|\eta_i| < |\theta_i|$, $1 \leq i < \infty$.

Test statistics of tests generated by $L_2$-norms of kernel estimators and Cramér–von Mises tests admit representations as linear combinations of squares of estimators of Fourier coefficients. Therefore, for these test statistics, the consistency of a sequence $f_n$ implies the consistency of any sequence of orthosymmetric functions $f_n$ generated by $f_n$. Moreover, type II error probabilities of the sequences $f_n$ and $\tilde{f}_n$ have the same asymptotic. Thus, the requirement of
orthosymmetry seems natural for test statistics admitting representations as linear combinations of squares of estimators of Fourier coefficients. For chi-squared tests, by Theorem 6.1 given in what follows, a similar situation takes place.

2.4. Another approach to the definition of maxisets. The requirement of orthosymmetry of a set \( U \) does not allow one to call a maxiset any convex set \( W \) generated by an equivalent norm in \( \Xi \). In the definition of a maxiset given below, we do not make such an assumption.

Let \( \Xi \subset \mathbb{L}_2(0,1) \) be a Banach space with norm \( \| \cdot \|_\Xi \). Denote by \( U = \{ f : \| f \|_\Xi \leq \gamma, f \in \Xi \} \), \( \gamma > 0 \), a ball in \( \Xi \). Assume that the set \( U \) is compact.

Define subspaces \( \Pi_i, 1 \leq k < \infty \), by induction.

Denote \( d_1 = \max\{\| f \|, f \in U \} \) and denote by \( e_1 \) a function \( e_1 \in U \) such that \( \| e_1 \| = d_1 \). Denote by \( \Pi_1 \) the linear subspace generated by the vector \( e_1 \).

For \( i = 2, 3, \ldots \), denote \( d_i = \max\{\rho(f, \Pi_{i-1}), f \in U \} \), where \( \rho(f, \Pi_{i-1}) = \inf\{\| f - g \|, g \in \Pi_{i-1} \} \). Define a function \( e_i \), \( e_i \in U \), such that \( \rho(e_i, \Pi_{i-1}) = d_i \) and \( e_i \) is orthogonal to \( \Pi_{i-1} \). Denote by \( \Pi_i \) the linear subspace spanned by the functions \( e_1, \ldots, e_i \).

For any function \( f \in \mathbb{L}_2(0,1) \) denote by \( f_{\Pi_i} \) the projection of the function \( f \) to the subspace \( \Pi_i \) and denote \( \tilde{f}_i = f - f_{\Pi_i} \).

Thus, we associate with each \( f \in \mathbb{L}_2(0,1) \) a sequence of functions \( \tilde{f}_i, \tilde{f}_i \to 0 \) as \( i \to \infty \).

For the problem of signal detection, we say that a set \( U \) is a maxiset for test statistics \( T_n \) and \( \Xi \) is a maxispace if the following two statements take place:

(i) any subsequence of alternatives \( f_{n_j} \in U \), \( cn_j^{-r} < \| f_{n_j} \| < Cn_j^{-r}, n_j \to \infty \) as \( j \to \infty \), is consistent for samples \( X_1, \ldots, X_{n_j} \);

(ii) for any \( f \in \mathbb{L}_2(0,1) \), \( f \notin \Xi \), there are sequences \( i_n \) and \( j_{i_n} \) with \( i_n \to \infty \) and \( j_{i_n} \to \infty \) as \( n \to \infty \) such that the subsequence of alternatives \( \tilde{f}_{i_{j_{i_n}}} \) is inconsistent for samples \( X_1, \ldots, X_{j_{i_n}} \) and \( c j_{i_{i_n}}^{-r} < \| \tilde{f}_{i_{j_{i_n}}} \| < C j_{i_{i_n}}^{-r} \).

Note that (i) does not hold for a bounded, centrally symmetric, convex set \( U \) if the set \( U \) is not compact (see Theorems 3.1 and 3.2 below).

For the problem of hypothesis testing on a density, we take \( e_1 = 1 \) and carry out the same construction. We verify (ii) for densities \( 1 + f \) such that \( 1 + \tilde{f}_i \) are densities for all \( i > i_0 \).

We provide proofs of our theorems for the definition of maxisets in terms of Sec. 2.3. However, it is easy to see that a slight modification of our reasoning provides proofs for the definition of Sec. 2.4 as well. The basis \( \phi_j, 1 \leq j < \infty \), in Sec. 2.3 coincides in this reasoning with the basis \( e_j \).

3. Necessary and sufficient conditions of uniform consistency

We consider the problem of signal detection in Gaussian white noise discussed in the Introduction. The problem is studied in terms of the sequence model.

Stochastic differential equation (1.4) can be rewritten in terms of the sequence model based on an orthonormal system of functions \( \phi_j, 1 \leq j < \infty \), in the following form:

\[
y_j = \theta_j + \frac{\sigma}{\sqrt{n}} \xi_j, \quad 1 \leq j < \infty,
\]

where

\[
y_j = \int_0^1 \phi_j dY_n(t), \quad \xi_j = \int_0^1 \phi_j dw(t), \quad \text{and} \quad \theta_j = \int_0^1 f \phi_j dt.
\]

Denote \( y = \{ y_j \}_{j=1}^\infty \) and \( \theta = \{ \theta_j \}_{j=1}^\infty \).
We can consider \( \theta \) as a vector in a Hilbert space \( \mathbb{H} \) with the norm \( \| \theta \| = \left( \sum_{j=1}^{\infty} \theta_j^2 \right)^{1/2} \). We implement the same notation \( \| \cdot \| \) for the norm in \( L_2 \) and in \( \mathbb{H} \). The sense of this notation is always clear from the context.

The problem of hypothesis testing can be rewritten in the following form. One needs to test the hypothesis

\[
H_0 : \theta = 0
\]

versus alternatives

\[
H_n : \theta \in V_n = \{ \theta : \| \theta \| \geq \rho_n, \theta \in U, U \subset \mathbb{H} \}. \tag{3.2}
\]

Here \( U \) is a bounded, centrally symmetric, convex set. The set \( U \) is centrally symmetric if the inclusion \( \theta \in U \) implies that \(-\theta \in U\).

**Theorem 3.1.** Assume that \( U \) is a bounded, centrally symmetric, convex set. Then there is a sequence \( \rho_n \to 0 \) as \( n \to \infty \) such that there exists a uniformly consistent sequence of tests for sets of alternatives \( V_n \) with this sequence \( \rho_n > 0 \) if and only if the set \( U \) is relatively compact.

If the set \( U \) is relatively compact, then there is a uniformly consistent estimator \( \hat{\theta}_n \) on the set \( U \) for the loss function \( \| \hat{\theta}_n - \theta \| \) (see [14] and [18]). Therefore, we can choose the \( L_2 \)-norm of the consistent estimator as a uniformly consistent test statistic.

**Remark 3.1.** Let a set \( U \subset \mathbb{H} \) be convex, centrally symmetric, and bounded. Then Theorem 3.1 implies that a uniformly consistent estimator of \( \| \theta \|, \theta \in U \), exists only if the set \( U \) is compact.

A version of Theorem 3.1 is true for the problem of testing the hypothesis on density in the following setup. Let \( P \) be a probability measure on a \( \sigma \)-algebra \( \mathfrak{F} \) defined on a set \( D \). Denote by \( L_2(P) \) the set of measurable functions \( f : D \to \mathbb{R}^1 \) such that

\[
\int_D f^2 dP < \infty.
\]

Let \( X_1, \ldots, X_n \) be i.i.d.r.v.’s having probability measure \( Q \) with density \( q = \frac{dQ}{dP} \in L_2(P) \).

The problem is to test the hypothesis \( H_0 : q(s) = 1 \) for all \( s \in D \) versus the alternatives

\[
H_n : q(s) - 1 \in V_n = \{ f : \| f \| \geq \rho_n > 0, f \in U, U \subset L_2(P) \}. \tag{3.3}
\]

Here \( \| f \| \) denotes the \( L_2(P) \)-norm of a function \( f \) and \( U \) is a bounded convex set in \( L_2(P) \).

Define the function \( 0(s) = 0 \) for all \( s \in D \).

**Theorem 3.2.** Assume that \( U \) is a bounded, centrally symmetric, convex set in \( L_2(P) \). Let the set \( U \) be such that for any function \( f \in U \), the function \( 1 + f \) is a probability density. Then there is a sequence \( \rho_n \to 0 \) as \( n \to \infty \) such that there exists a uniformly consistent sequence of tests for sets of alternatives \( V_n \) with this sequence \( \rho_n \) if and only if the set \( U \) is relatively compact.

The reasoning in the proof of Theorem 3.2 coincides with that in the proof of Theorem 3.1 with a single difference: we implement Theorem 4.1 of [10] instead of Theorem 5.3 of [10]. We omit this reasoning.

A similar theorem holds for the problem of signal detection in a linear inverse ill-posed problem.

In a Hilbert space \( \mathbb{H} \), we observe a realization of a Gaussian random vector

\[
y = A\theta + \epsilon \xi, \quad \epsilon > 0, \tag{3.4}
\]

where \( A : \mathbb{H} \to \mathbb{H} \) is a known linear operator. Here \( \xi \) is a Gaussian random vector having known covariance operator \( R : \mathbb{H} \to \mathbb{H} \) and such that \( E[\xi] = 0 \).
We study the same problem of hypothesis testing $H_0 : \theta = 0$ versus alternatives $H_n : \theta \in V_n$. For any operator $S : H \to H$ denote by $\mathcal{R}(S)$ the range space of $S$.

Assume that the nullspaces of $A$ and $R$ equal zero and $\mathcal{R}(A) \subseteq \mathcal{R}(R^{1/2})$.

**Theorem 3.3.** Let the operator $R^{-1/2}A$ be bounded. Assume that a centrally symmetric bounded set $U$ is convex. Then the statement of Theorem 3.1 holds.

**Remark 3.2.** In the literature, another definition of uniform consistency is often explored (see, for example, [15]). In this definition, (2.3) is replaced by the requirement of the existence of a sequence of tests $K_n$ such that $\alpha(K_n) \to 0$ and $\beta(K_n, V_n) \to 0$ as $n \to \infty$. By the theorem on exponential decay of type I and type II error probabilities (see [23] and [29]), Theorems 3.1–3.3 for this definition of consistency hold as well.

4. Quadratic test statistics

**4.1. General setup.** We study problem (1.4), (1.7) with $0 < r < 1/2$ of signal detection in Gaussian white noise discussed in the Introduction. The problem is provided in terms of the sequence model (3.1).

If $U$ is a compact ellipsoid,

$$U = \left\{ \theta : \sum_{j=1}^{\infty} a_j \theta_j^2 \leq P_0, \theta = \{\theta_j\}_{j=1}^{\infty}, \theta_j \in \mathbb{R}_1 \right\},$$

with $a_j > 0$ and $a_j \to \infty$ as $j \to \infty$, then asymptotically minimax test statistics for sets of alternatives $V_n$ are quadratic forms

$$T_n(Y_n) = \sum_{j=1}^{\infty} \kappa_{nj}^2 y_j^2 - \sigma_n^{-2} n^{-1} \rho_n$$

with some specially defined coefficients $\kappa_{nj}^2$ (see [6]). Here $\rho_n = \sum_{j=1}^{\infty} \kappa_{nj}^2$.

If the coefficients $\kappa_{nj}^2$ satisfy some regularity assumptions, the test statistics $T_n(Y_n)$ are asymptotically minimax (see [9]) for a wider sets of alternatives,

$$\mathbb{H}_n : f \in \Upsilon_n(R_n, c) = \{ f : R_n(f) > c, f \in L_2(0, 1) \},$$

where

$$R_n(f) = A_n(\theta) = \sigma_n^{-4} n^2 \sum_{j=1}^{\infty} \kappa_{nj}^2 \theta_j^2$$

and $f = \sum_{j=1}^{\infty} \theta_j \phi_j$.

A sequence of tests $L_n, \alpha(L_n) = \alpha(1 + o(1)), 0 < \alpha < 1$, is called asymptotically minimax if, for any sequence of tests $K_n, \alpha(K_n) \leq \alpha$,

$$\liminf_{n \to \infty} (\beta(K_n, \Upsilon_n(R_n, c)) - \beta(L_n, \Upsilon_n(R_n, c))) \geq 0.$$

A sequence of test statistics $T_n$ is called asymptotically minimax if the tests generated by the test statistics $T_n$ are asymptotically minimax.

We make the following assumptions.

**A1.** For each $n$, the sequence $\kappa_{nj}^2$ is decreasing.
A2. There are positive constants $C_1$ and $C_2$ such that
\[ C_1 < A_n = \sigma^{-4} n^2 \sum_{j=1}^{\infty} \kappa_{nj}^4 < C_2 \] (4.1)
for each $n$.

A3. There are positive constants $c_1$ and $c_2$ such that $c_1 n^{-2r} \leq \rho_n \leq c_2 n^{-2r}$.

Denote $\kappa_n^2 = \kappa_{nk}^2$, where $k_n = \sup \{ k : \sum_{j<k} \kappa_{nj}^2 \leq \frac{1}{2} \rho_n \}$.

A4. There are $C_1$ and $\lambda > 1$ such that
\[ \kappa_{n,(1+\delta)k_n}^2 < C_1 (1+\delta)^{-\lambda} \kappa_n^2 \]
for any $\delta > 0$ and for each $n$.

A5. The relations $\kappa_{n1}^2 \asymp \kappa_n^2$ hold as $n \to \infty$. For any $c > 1$ there is a $C$ such that $\kappa_{n,[ckn]}^2 \geq C \kappa_n^2$ for all $n$.

Example. Let $\kappa_{nj}^2 = n^{-\lambda} \frac{1}{j^\gamma + cn^\beta}$, $\gamma > 1$, where $\lambda = 2 - 2r - \beta$ and $\beta = (2 - 4r)\gamma$. Then the assumptions A1–A5 hold.

Note that the assumptions A1–A5 imply that $\kappa_{n1}^4 \asymp n^{-2} k_n^{-1}$ and $k_n \asymp n^{2-4r}$.

(4.2)

Theorems 4.1–4.10 given below represent a realization of the program announced in the Introduction.

4.2. Analytic form of necessary and sufficient conditions of consistency. The results are given in terms of Fourier coefficients of the functions $f_n = \sum_{j=1}^{\infty} \theta_{nj} \phi_j$.

Theorem 4.1. Assume that the conditions A1–A5 are satisfied. A sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, is consistent if and only if there are $c_1$, $c_2$, and $n_0$ such that
\[ \sum_{|j| < c_2 k_n} |\theta_{nj}|^2 > c_1 n^{-2r} \] (4.3)
for all $n > n_0$.

Versions of Theorems 4.1, 4.2, and 4.8 hold for setups of other sections as well. In setups of these sections, indices $j$ may take negative values and $\theta_{nj}$ may be complex numbers. For that reason, we write $|j|$ instead of $j$ and $|\theta_{nj}|$ instead of $\theta_{nj}$ in (4.3), (4.4), and (4.9).

Theorem 4.2. Assume that the conditions A1–A5 are satisfied. A sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, is inconsistent if and only if
\[ \sum_{|j| < c_2 k_n} |\theta_{nj}|^2 = o(n^{-2r}) \text{ as } n \to \infty \] (4.4)
for any $c_2$.

Proofs of the theorems are based on Theorem 4.3 on the asymptotic minimaxity of the test statistics $T_n$.

Define a sequence of tests $K_n(Y_n) = 1_{\{n^{-1}T_n(Y_n) > (2A_n)^{1/2}x_{\alpha}\}}$, $0 < \alpha < 1$, where $x_{\alpha}$ is determined by the equation $\alpha = 1 - \Phi(x_{\alpha})$. 811
Theorem 4.3. Assume that the conditions A1–A5 are satisfied. Then a sequence of tests $K_n(Y_n)$ is asymptotically minimax for the sets $\mathcal{Y}_n(R, c)$ of alternatives. In addition, $\alpha(K_n) = \alpha + o(1)$ and

$$\beta(K_n, f_n) = \Phi(x_\alpha - R_n(f_n)(2A_n)^{-1/2})(1 + o(1))$$

(4.5)

uniformly over all sequences $f_n$ such that $R_n(f_n) < C$ for any $C > 0$.

A version of Theorem 4.3 for the problem of signal detection with heteroscedastic white noise has been proved in [9].

Such a form of conditions in Theorems 4.1 and 4.2 can be explained by the concentration of the coefficients $\kappa_{nj}^2$ in the zone $j = O(k_n)$ for test statistics $T_n$ and for $A_n(\theta_n)$.

A version of Theorem 4.3 for the problem of hypothesis testing on distribution function provides necessary and sufficient conditions of uniform consistency of sets of alternatives defined in terms of distribution functions.

4.3. Maxisets. Qualitative structure of consistent sequences of alternatives. Denote $s = \frac{r - 4}{4r}$. Then $r = \frac{2s}{1 + 4s}$.

Theorem 4.4. Assume that the conditions A1-A5 are satisfied. Then the balls $\tilde{B}_{2\infty}(P_0)$, $P_0 > 0$, are maxisets for the test statistics $T_n(Y_n)$.

For maxisets $\tilde{B}_{2\infty}(P_0)$ with a deleted “small” $L_2$-ball, asymptotically minimax tests have been found in [11]. In [16], a similar result has been obtained for Besov bodies in $\mathbb{B}_{2\infty}$ defined in terms of wavelets coefficients.

Theorem 4.5. Assume that the conditions A1–A5 are satisfied. Then a sequence of alternatives $f_n$, $cn^{-r} \leq ||f_n|| \leq Cn^{-r}$, is consistent if and only if there is a maxiset $\tilde{B}_{2\infty}(P_0)$, $P_0 > 0$, and a sequence $f_{1n} \in \tilde{B}_{2\infty}(P_0)$, $c_1n^{-r} \leq ||f_{1n}|| \leq C_1n^{-r}$, such that $f_{1n}$ is orthogonal to $f_n - f_{1n}$, i.e.,

$$||f_n||^2 = ||f_{1n}||^2 + ||f_n - f_{1n}||^2.$$  

(4.6)

Therefore, if we have a maxiset $\tilde{B}_{2\infty}(P_0)$, $P_0 > 0$, a sequence of arbitrary functions $f_{1n} \in \tilde{B}_{2\infty}(P_0)$, $c_1n^{-r} \leq ||f_{1n}|| \leq C_1n^{-r}$, and a sequence of arbitrary functions $f_{2n}$, $c_1n^{-r} \leq ||f_{2n}|| \leq C_1n^{-r}$, orthogonal to $f_{1n}$, then the sequence of simple alternatives $f_n = f_{1n} + f_{2n}$ is consistent.

Theorem 4.6. Assume that the conditions A1–A5 are satisfied. Then for any $\varepsilon > 0$ and for any consistent sequence of alternatives $f_n$, $cn^{-r} \leq ||f_n|| \leq Cn^{-r}$, there is a maxiset $\tilde{B}_{2\infty}(P_0)$, $P_0 > 0$, and a sequence of functions $f_{1n}$, $c_1n^{-r} \leq ||f_{1n}|| \leq C_1n^{-r}$, belonging to the maxiset $\tilde{B}_{2\infty}(P_0)$ such that the function $f_{1n}$ is orthogonal to $f_n - f_{1n}$ and for any $\alpha$, $0 < \alpha < 1$, for the tests $K_n$, $\alpha(K_n) = \alpha(1 + o(1))$ as $n \to \infty$, there is an $n_\varepsilon$ such that

$$|\beta(K_n, f_n) - \beta(K_n, f_{1n})| \leq \varepsilon$$  

(4.7)

for any $n > n_\varepsilon$ and

$$\beta(K_n, f_n - f_{1n}) \geq 1 - \alpha - \varepsilon.$$  

(4.8)

If the functions $f_n = \sum_{j=1}^{\infty} \theta_{nj}\phi_j$ satisfy the inequalities $c_1n^{-r} \leq ||f_n|| \leq C_1n^{-r}$, then for any $c$ there is a $P_0$ such that $f_{1n} = \sum_{j=1}^{[ck_n]} \theta_{nj}\phi_j \in \tilde{B}_{2\infty}(P_0)$ (see Lemma A.3). Since the coefficients $\kappa_{nj}^2$ are relatively small for large $c$, this allows us to prove Theorems 4.5 and 4.6.

The maxiset $\tilde{B}_{2\infty}(P_0)$, $P_0 > 0$, in Theorems 4.5, 4.6, and 4.9 can be replaced by an arbitrary maxiset $cU$ with some positive real number $c$.
4.4. Interaction between consistent and inconsistent sequences of alternatives. Purely consistent sequences

**Theorem 4.7.** Assume that the conditions A1–A5 are satisfied. Let a sequence of alternatives \(f_n\) be consistent. Then for any inconsistent sequence of alternatives \(f_{1n}\) and for tests \(K_n\), \(\alpha(K_n) = \alpha(1 + o(1))\), \(0 < \alpha < 1\), generated by the test statistics \(T_n\), the following relation holds:

\[
\lim_{n \to \infty} (\beta(K_n, f_n) - \beta(K_n, f_n + f_{1n})) = 0.
\]

**Theorem 4.8.** Assume that the conditions A1–A5 are satisfied. A sequence of alternatives \(f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r}\), is purely \(n^{-r}\)-consistent if and only if for any \(\varepsilon > 0\) there is a \(C_1 = C_1(\varepsilon)\) such that

\[
\sum_{|j| > C_1 k_n} |\theta_{nj}|^2 \leq \varepsilon n^{-2r}
\]

for all \(n > n_0(\varepsilon)\).

**Theorem 4.9.** Assume that the conditions A1–A5 are satisfied. Then a sequence of alternatives \(f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r}\), is purely \(n^{-r}\)-consistent if and only if for any \(\varepsilon > 0\) there is a \(\gamma_\varepsilon\) and a sequence of functions \(f_{1n}\) belonging to the maxiset \(B^2_{2\infty}(\gamma_\varepsilon)\) such that \(\|f_n - f_{1n}\| \leq \varepsilon n^{-r}\) and (4.6) holds.

**Theorem 4.10.** Assume that the conditions A1–A5 are satisfied. Then a sequence of alternatives \(f_n, cn^{-r} < \|f_n\| < Cn^{-r}\), is purely \(n^{-r}\)-consistent if and only if for any inconsistent subsequence of alternatives \(f_{1n_i}, cn_i^{-r} < \|f_{1n_i}\| < Cn_i^{-r}\),

\[
\|f_{n_i} + f_{1n_i}\|^2 = \|f_{n_i}\|^2 + \|f_{1n_i}\|^2 + o(n_i^{-r}),
\]

where \(n_i \to \infty\) as \(i \to \infty\).

**Remark 4.1.** Let \(k_{nj}^2 > 0\) for \(j \leq l_n\) and let \(k_{nj}^2 = 0\) for \(j > l_n\), where \(l_n \sim n^{2-4r}\) as \(n \to \infty\). Analysis of proofs of the above theorems shows that Theorems 4.1–4.10 remain valid for this setup if the conditions A4 and A5 are replaced by the following condition.

**A6.** For any \(c, 0 < c < 1\), there is a \(c_1\) such that \(k_{nj}^2 \geq c_1 k_{n1}^2\) for all \(n\).

In all the corresponding reasoning, we put \(k_{nj}^2 = k_{n1}^2\) and \(l_n = l_n\).

Theorems 4.2 and 4.8 hold with the following changes. It suffices to put \(c_2 < 1\) in Theorem 4.2 and to take \(C_1(\varepsilon) < 1\) in Theorem 4.8.

Proofs of the corresponding versions of Theorems 4.1–4.10 are obtained by a simplification of the provided reasoning and are omitted.

5. Kernel-based tests

We continue to explore problem (1.6) and (1.7) of signal detection in Gaussian white noise with \(0 < r < 1/2\). We additionally assume that the signal \(f\) belongs to the set \(L^2_{\text{per}}(\mathbb{R}^1)\) of 1-periodic functions such that \(f(t) \in L_2(0, 1)\). This allows us to extend our model to the real line \(\mathbb{R}^1\) putting \(w(t + j) = w(t)\) for all integer \(j\) and \(t \in [0, 1)\) and to write the forthcoming integrals over the whole real line.

Define the kernel estimator

\[
\hat{f}_n(t) = \frac{1}{h_n} \int_{-\infty}^{\infty} K\left(\frac{t - u}{h_n}\right) dY_n(u), \quad t \in (0, 1),
\]

where \(h_n > 0\) and \(h_n \to 0\) as \(n \to \infty\).
The kernel $K$ is a bounded function such that the support of $K$ is contained in $[-1, 1]$, $K(t) = K(-t)$ for $t \in \mathbb{R}$, and $\int_{-\infty}^{\infty} K(t) dt = 1$.

Denote $K_h(t) = \frac{1}{h} K\left(\frac{t}{h}\right)$, where $t \in \mathbb{R}$ and $h > 0$.

In (5.1), we assume that
\[
\int_{1-v}^{1+v} K_{hn}(t-u) dY_n(u) = \int_{0}^{v} K_{hn}(t-1-u) f(u) du + \frac{\sigma}{\sqrt{n}} \int_{0}^{v} K_{hn}(t-1-u) dw(u)
\]
and
\[
\int_{-v}^{0} K_{hn}(t-u) dY_n(u) = \int_{1-v}^{1} K_{hn}(t-u+1) f(u) du + \frac{\sigma}{\sqrt{n}} \int_{1-v}^{1} K_{hn}(t-u+1) dw(u)
\]
for any $v$, $0 < v < 1$.

Define kernel-based test statistics
\[
T_n(Y_n) = T_{nh_n}(Y_n) = nh_n^{1/2} \sigma^{-2} \gamma^{-1} (\|\hat{f}_n\|^2 - \sigma^2(nh_n)^{-1}\|K\|^2),
\]
where
\[
\gamma^2 = 2 \int_{-\infty}^{\infty} \left( \int K(t-s)K(s) ds \right)^2 dt.
\]

We call a sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, $n^{-r}$-consistent if there is a constant $c_1$ such that (2.1) holds for any tests $K_n$, $\alpha(K_n) = \alpha(1+o(1))$, $0 < \alpha < 1$, generated by the sequence of test statistics $T_n$ with $h_n < c_1 n^{4r-2}$, $h_n \asymp n^{4r-2}$.

We call a sequence of alternatives $f_n$, $cn^{-r} \leq \|f_n\| \leq Cn^{-r}$, $n^{-r}$-inconsistent if the sequence of alternatives $f_n$ is inconsistent for any tests generated by arbitrary test statistics $T_n$ with $h_n \rightarrow 0$ as $n \rightarrow \infty$.

The problem will be explored in terms of the sequence model.

Let us observe a realization of a random process $Y_n(t)$ with $f = f_n$.

For $-\infty < j < \infty$, denote
\[
\hat{K}(jh) = \int_{-1}^{1} \exp\{2\pi ijt\} K_h(t) dt, \quad h > 0,
\]
\[
y_{nj} = \int_{0}^{1} \exp\{2\pi ijt\} dY_n(t), \quad \xi_j = \int_{0}^{1} \exp\{2\pi ijt\} dw(t),
\]
and
\[
\theta_{nj} = \int_{0}^{1} \exp\{2\pi ijt\} f_n(t) dt.
\]

In this notation, we can write the kernel estimator in the following form:
\[
\hat{\theta}_{nj} = \hat{K}(jh_n) y_{nj} = \hat{K}(jh_n) \theta_{nj} + \sigma n^{-1/2} \hat{K}(jh_n) \xi_j, \quad -\infty < j < \infty,
\]
and the test statistics $T_n$ admit the following representation:
\[
T_n(Y_n) = nh_n^{1/2} \sigma^{-2} \gamma^{-1} \left( \sum_{j=-\infty}^{\infty} |\hat{\theta}_{nj}|^2 - n^{-1} \sigma^2 \sum_{j=-\infty}^{\infty} |\hat{K}(jh_n)|^2 \right).
\]
If we put $|\hat{K}(jh_n)|^2 = \sigma_{nj}^2$, we see that the definitions of the test statistics $T_n(Y_n)$ in this section and in Sec. 4 almost coincide. The setup of Sec. 5 differs from that of Sec. 4 only by the presence of a heteroscedastic white noise. Another difference of the setups is that the function $\hat{K}(\omega)$, $\omega \in \mathbb{R}^1$, may have zeros. Since these differences are insignificant, the same results are valid. Denote $k_n = [n^{2-\eta}]$.

\textbf{Theorem 5.1.} The statements of Theorems 4.1, 4.2, and 4.7–4.10 hold for this setup as well. The statements of Theorems 4.4–4.6 also hold with $\mathbb{B}_{2,\infty}^8$ replaced with $\mathbb{B}_{2,\infty}^2$.

In the version of Theorem 4.4, item (ii) of the definition of maxisets holds for test statistics $T_n$ having arbitrary values $h_n > 0$, $h_n \to 0$ as $n \to \infty$.

Denote
\[
T_{1n}(f) = T_{1n}(f, h_n) = \frac{1}{h_n} \int_0^1 K \left( \frac{t-s}{h_n} \right) f(s) \, ds \, dt.
\]

For a sequence $\rho_n > 0$, define the sets
\[
\mathcal{Y}_{nh_n}(T_{1n}, \rho_n) = \{ f : T_{1n}(f) > \rho_n, \, f \in \mathbb{L}_2^{per}(\mathbb{R}^1) \}.
\]

Define a sequence of kernel-based tests $K_n = \mathbb{1}_{\{T_n(Y_n) \geq x_\alpha\}}$, $0 < \alpha < 1$, where $x_\alpha$ are determined by the equation $\alpha = 1 - \Phi(x_\alpha)$.

Proofs of our theorems are based on the following Theorem 5.2 on asymptotic minimaxity of kernel-based tests $K_n$ (see Theorem 2.1.1 of [8]).

\textbf{Theorem 5.2.} Let $h_n^{-1/2} n^{-1} \to 0$ and $h_n \to 0$ as $n \to \infty$. Let
\[
0 < \liminf_{n \to \infty} n \rho_n h_n^{1/2} \leq \limsup_{n \to \infty} n \rho_n h_n^{1/2} < \infty.
\]

Then the sequence of kernel-based tests $K_n$ is asymptotically minimax for the sets of alternatives $\mathcal{Y}_{nh_n}(T_{1n}, \rho_n)$. The relations $\alpha(L_n) = \alpha(1 + o(1))$ and
\[
\beta(K_n, f_n) = \Phi(x_\alpha - \gamma^{-1} \sigma^2 n h_n^{1/2} T_{1n}(f_n))(1 + o(1))
\]
hold uniformly over sequences $f_n \in \mathbb{L}_2^{per}(\mathbb{R}^1)$ such that $nh_n^{1/2} T_{1n}(f_n) < C$.

Note that
\[
T_{1n}(f_n) = \sum_{j=-\infty}^{\infty} |\hat{K}(jh_n)|^2 |\theta_{nj}|^2.
\]

Note also that the only difference between the setups of Theorems 5.2 and 4.3 is the presence of a heteroscedastic noise. Thus, a proof of Theorem 5.2 can be obtained by an easy modification of the proof of Theorem 4.3.

6. $\chi^2$-Tests

Let $X_1, \ldots, X_n$ be i.i.d.r.v.'s having c.d.f. $F \in \mathfrak{F}$. Let the c.d.f. $F(x)$ have density $1+f(x) = dF(x)/dx$, $x \in (0, 1)$, $f \in \mathbb{L}_2^{per}(0, 1)$.

We explore the problem of testing hypothesis (1.6) versus alternatives (1.7) with $0 < r < 1/2$ discussed in the Introduction.

For any sequence $m_n$, denote $\hat{p}_{nj} = \hat{F}_n(j/m_n) - \hat{F}_n((j-1)/m_n)$, $1 \leq j \leq m_n$.

The test statistics of $\chi^2$-tests equal
\[
T_n(\hat{F}_n) = n m_n \sum_{j=1}^{m_n} (\hat{p}_{nj} - 1/m_n)^2.
\]

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\[ f_n = \sum_{j=-\infty}^{\infty} \theta_{nj} \phi_j, \quad \phi_j(x) = \exp\{2\pi j x\}, \quad x \in (0, 1). \]

We call a sequence of alternatives \( f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r}, n^{-r}\)-consistent if there is a \( c_1 \) such that (2.1) holds for any tests \( K_n, \alpha(K_n) = \alpha(1 + o(1)), 0 < \alpha < 1 \), generated by a sequence of chi-squared test statistics \( T_n \) with number of cells \( m_n \geq c_1 n^{2-4r}, m_n \asymp n^{2-4r}. \)

We call a sequence of alternatives \( f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r}, n^{-r}\)-inconsistent if the sequence of alternatives \( f_n \) is inconsistent for all tests generated by test statistics \( T_n \) with arbitrary number of cells \( m_n, m_n \to \infty \) as \( n \to \infty \).

Denote \( k_n = \left[ n^{1+4s} \right] \asymp n^{-2r}. \)

The differences in versions of Theorems 4.1–4.10 for this setup are caused only by the requirement that the functions \( f_n, f_{1n}, \) and \( f_{2n} \) must be densities.

**Theorem 6.1.** The statements of Theorems 4.1, 4.2, 4.4–4.6, and 4.8–4.10 hold for this setup with the following differences.

In the versions of Theorems 4.4–4.6, the balls \( \tilde{B}^s_{2\infty} \) are replaced by the bodies \( \tilde{B}^s_{2\infty}. \)

In the version of Theorem 4.4, item (ii) of the definition of maxisets holds for test statistics \( T_n \) with arbitrary choice of number of cells \( m_n, m_n \to \infty \) as \( n \to \infty. \)

In the version of Theorem 4.6, we consider only sequences of alternatives \( f_n \) such that the following assumption holds.

**B.** There is a \( c_0 \) such that for all \( k > c_0 k_n \), the functions

\[ 1 + f_{cn} = 1 + \sum_{|j| > k} \theta_j \phi_j \quad \text{and} \quad 1 + f_n - f_{cn} = 1 + \sum_{|j| < k} \theta_j \phi_j \]

are densities.

We refer to the definition of purely consistent sequences only for sequences \( f_n \) satisfying assumption B.

In the proof of the version of Theorem 4.6 for chi-squared tests, we show that there is a \( C_c = C(\varepsilon, c, C, c_0) \) such that, for the densities \( 1 + f_{1n} = 1 + \sum_{|j| < C_c k_n} \theta_j \phi_j \), relations (4.6), (4.7), and (4.9) hold. By Lemma A.3 given below, there is a \( P_{0c} \) such that \( f_{1n} \in \tilde{B}^s_{2\infty}(P_{0c}). \)

In Theorem 6.2 given below, the definitions of consistency and inconsistency formulated in Sec. 2.1 are treated if the simple alternatives \( f_n \) are replaced by distribution functions \( F_n \) and the hypothesis is \( \mathbb{H}_0 : F(x) = F_0(x) = x, x \in [0, 1]. \)

**Theorem 6.2.** Let a sequence of alternatives \( F_n \) be consistent. Let \( F_{1n} \) be an inconsistent sequence of alternatives such that \( F_{2n} = F_n(x) + F_{1n}(x) - F_0(x) \) are distribution functions. Then for tests \( K_n, \alpha(K_n) = \alpha(1 + o(1)), 0 < \alpha < 1, \) the following relation holds:

\[ \lim_{n \to \infty} (\beta(K_n, F_n) - \beta(K_n, F_{2n})) = 0. \]

Proofs of our theorems are based on Theorem 6.3 on asymptotic minimaxity of chi-squared tests given below. Theorem 6.3 is a summary of results of Theorems 2.1 and 2.4 of [7].

For a sequence \( \rho_n > 0 \) define sets of alternatives

\[ \Upsilon_n(T_n, \rho_n) = \left\{ F : T_n(F) \geq \rho_n, F \in \mathcal{F} \right\}. \]

The definition of asymptotic minimaxity of tests is the same as in Sec. 4.

Define the tests

\[ K_n = \mathbb{1}_{\{2^{-1/2}m_n^{-1/2}(\hat{F}_n - m_n) > x_\alpha\}}, \]

where \( x_\alpha \) is determined by the equation \( \alpha = 1 - \Phi(x_\alpha) \).
Theorem 6.3. Let \( m_n \to \infty \) and \( m_n^{-1} n^2 \to \infty \) as \( n \to \infty \). Let
\[
0 < \liminf_{n \to \infty} m_n^{-1/2} \rho_n \leq \limsup_{n \to \infty} m_n^{-1/2} \rho_n < \infty.
\]
Then \( \chi^2 \)-tests \( K_n, \alpha(K_n) = \alpha + o(1), 0 < \alpha < 1 \), are asymptotically minimax for the sets of alternatives \( \mathcal{Y}_n(\mathbf{T}_n, \rho_n) \). The relation
\[
\beta(K_n, F_n) = \Phi(x_\alpha - 2^{-1/2} m_n^{-1/2} T_n(F_n))(1 + o(1))
\]
holds uniformly over sequences \( F_n \) such that \( T_n(F_n) \leq C m_n^{1/2} \).

Note that to apply Theorem 6.3 in the proofs of Theorems 6.1 and 6.2, we have to make a transition from indicator functions to trigonometric functions. Such a transition is realized in the Appendix.

7. Cramér–von Mises tests

We consider Cramér–von Mises test statistics as the functionals
\[
T^2(\hat{F}_n - F_0) = \int_0^1 (\hat{F}_n(x) - F_0(x))^2 \, dF_0(x)
\]
depending on the empirical distribution function \( \hat{F}_n \). Here \( F_0(x) = x, x \in [0,1] \).

Denote by \( K_n = K_n(X_1, \ldots, X_n) \) a sequence of Cramér – von Mises tests.

A part of the further results holds for setup (1.1) and (1.2) with \( \mathcal{Y}_n = \mathcal{Y}_n(a) = \mathcal{Y}_n(T^2, an^{-1}), a > 0 \).

We say that a Cramér – von Mises test is asymptotically unbiased if
\[
\limsup_{n \to \infty} \sup_{F \in \mathcal{Y}_n(a)} \beta_F(K_n) < 1 - \alpha
\]
for any \( a > 0 \), for any \( \alpha, 0 < \alpha < 1 \), and for tests \( K_n, \alpha(K_n) = \alpha + o(1) \). Nonparametric tests satisfying (7.1) are also called uniformly consistent (see [24, Chap. 14.2]).

Proofs of our results are based on the following Theorem 7.1.

Theorem 7.1. The following three statements hold.

(i) For a sequence of alternatives \( F_n \) there is a sequence of Cramér – von Mises tests \( K_n \) such that
\[
\lim_{n \to \infty} (\alpha(K_n) + \beta_{F_n}(K_n)) = 0
\]
if and only if
\[
\lim_{n \to \infty} n T^2(F_n - F_0) = \infty.
\]

(ii) Cramér – von Mises tests are asymptotically unbiased.

(iii) For any sequence of Cramér – von Mises tests \( K_n \),
\[
\lim_{n \to \infty} (\alpha(K_n) + \beta_{F_n}(K_n)) \geq 1
\]
if and only if
\[
\lim_{n \to \infty} n T^2(F_n - F_0) = 0.
\]

The sufficiency statements in items (i) and (iii) of Theorem 7.1 are well known (see [15]). The necessity statements in items (i) and (iii) easily follow from (ii).

From now on, we explore the problem of testing hypothesis (1.6) versus alternatives (1.7) with \( 0 < r < 1/2 \) discussed in the Introduction.
If a c.d.f. \( F \) has density, we can write the functional \( T^2(F - F_0) \) in the following form (see [28, Chap. 5]):

\[
T^2(F - F_0) = \int_0^1 \int_0^1 (\min\{s, t\} - st) f(t) f(s) \, ds \, dt,
\]

where \( f(t) = d(F(t) - F_0(t))/dt \).

If we consider the orthonormal expansion of a function

\[
f(t) = \sum_{j=1}^{\infty} \theta_j \phi_j(t)
\]

in the trigonometric basis \( \phi_j(t) = \sqrt{2} \cos(\pi j t), 1 \leq j < \infty \), then we get the equality

\[
nT^2(F - F_0) = n \sum_{j=1}^{\infty} \frac{\theta_j^2}{\pi^2 j^2}.
\]

(7.4)

Denote \( k_n = [n^{(1-2r)/2}] \).

In Theorems 7.2 and 7.3 given below, we consider the definition of consistency given in Sec. 2.1.

**Theorem 7.2.** For the orthonormal system of functions \( \phi_j(t) = \sqrt{2} \cos(\pi j t), t \in [0,1), j = 1, 2, \ldots \), the bodies \( \bar{B}_s^\infty(P_0) \) with \( s = \frac{2r}{1-2r}, r = \frac{s}{2+2s} \), are maxisets for Cramér – von Mises test statistics.

In the previous sections, the functionals \( T_n \) depended on \( n \). In the present setup, we explore the same functional \( T \) for all \( n \) and for different values of \( r, 0 < r < 1/2 \). To separate the study of sequences of alternatives for different \( r \), we consider for fixed \( r \) only sequences of alternatives satisfying the following condition G1.

**G1.** For any \( \varepsilon > 0 \) there is a \( c_\varepsilon \) such that

\[
n \sum_{|j| < c_\varepsilon k_n} \theta_{nj}^2 j^{-2} < \varepsilon
\]

for all \( n > n_0(\varepsilon, c_\varepsilon) \).

If the condition G1 does not hold for some \( c_\varepsilon = c_n \to 0 \), where \( c_n k_n \to \infty \) as \( n \to \infty \), and the functions \( 1 + \tilde{f}_n = 1 + \sum_{j < c_n k_n} \theta_{nj} \phi_j \) are densities, then (2.1) holds for some sequence of functions \( \tilde{f}_n, \| \tilde{f}_n \| = o(n^{-r}) \). Thus, this case of consistency can be studied in the framework of a faster rate of convergence of sequence of alternatives.

**Theorem 7.3.** Let a sequence of alternatives \( f_n \) satisfy the condition G1. Then for the sequence \( f_n \), the statements of Theorems 4.1, 4.2, 4.5, 4.6, 4.8, and 4.10 are valid with the following changes.

In the version of Theorem 4.6, it is assumed that condition B holds.

In Theorem 7.3, the definition of pure consistency is considered for sequences of functions \( f_n \) satisfying condition B.

**Theorem 7.4.** The statement of Theorem 6.2 holds for this setup as well.
8. \( n^{-1/2} \)-RATE OF CONVERGENCE

In this section, we extend results of Secs. 4–7 to the case \( r = 1/2 \). We show that for \( r = 1/2 \), the sets

\[
U(l, P_0) = \{ f : f = \sum_{j=1}^{l} \theta_j \phi_j, \|f\| \leq P_0, f \in L_2(0, 1) \}
\]

with \( l = 1, 2, \ldots \) and \( P_0 > 0 \) satisfy item (i) and the linear space \( \Xi = \{ f : f \in U(l, P_0) \} \) for some integer \( l \) and \( P_0 > 0 \) satisfies item (ii) in the definition of maxisets, respectively. Moreover, versions of Theorems 4.5, 4.6, and 4.9 hold if we replace maxisets by the sets \( U(l, P_0) \).

The problems of hypothesis testing in Secs. 4, 5, and 7 are covered by the following setup. We observe a sequence of independent random variables \( y_j = \theta_j + n^{-1/2}\sigma_j \xi_j \), where \( \xi_j, 1 \leq j < \infty \), are Gaussian random variables, \( E\xi_j = 0 \), and \( E[\xi_j^2] = 1 \).

Define the functional

\[
T(\theta) = \sum_{j=1}^{\infty} \kappa_j^2 \theta_j^2, \quad \theta = \{\theta_j\}_1^\infty,
\]

where the coefficients \( \kappa_j^2 \) satisfy the following conditions.

**D1.** The sequence \( \kappa_j^2 \) is decreasing and \( \sum_{j=1}^{\infty} \kappa_j^2 < \infty \).

**D2** There is a \( C > 0 \) such that \( 0 < \sigma_j < C \) for all \( 1 \leq j < \infty \).

The problem is to test the hypothesis

\[
\mathbb{H}_0 : \theta_j = 0, \quad 1 \leq j < \infty,
\]

versus the alternatives

\[
\mathbb{H}_n : \theta_j = \theta_{nj}, \quad 1 \leq j < \infty,
\]

where \( T(\theta_n) \sim n^{-1} \) with \( \theta_n = \{\theta_{nj}\}_1^\infty \).

**Theorem 8.1.** For \( r = 1/2 \), the sets \( U(l, P_0) \), \( l = 1, 2, \ldots \), \( P_0 > 0 \) and the linear space \( \Xi \) satisfy items (i) and (ii) in the definition of maxisets, respectively.

**Theorem 8.2.** Assume that the conditions D1 and D2 are satisfied. Then Theorems 4.1, 4.2, and 4.4–4.10 are valid with \( k_n = 1 \) and the sets \( \bar{B}_{2\infty}(P_0) \) replaced by the sets \( U(l, P_0) \), where \( l = 1, 2, \ldots \) and \( P_0 > 0 \).

The proof of Theorem 8.2 is based on Theorem 8.3 given below and an obvious modification of item (iii) of Theorem 7.1 for this setup. The reasoning is similar to the proof of theorems in Sec. 4 and is omitted. Note only that to verify item (ii) in the definition of maxisets, we put \( \bar{f}_l = \sum_{j=l}^{\infty} \theta_j \phi_j \) (see the proof of Theorem 4.4). After that, we implement a version of Theorem 4.2 for this setup.

Denote \( z_j = n^{1/2}y_j \) and \( \eta_j = n^{1/2}\theta_j \). Then the problem of hypothesis testing (8.1) and (8.2) is replaced by the following one.

We observe independent random variables \( z_j = \eta_j + \sigma_j \xi_j \). The problem is to test the hypothesis

\[
\mathbb{H}_0 : \eta_j = 0, \quad 1 \leq j < \infty,
\]

versus the alternatives

\[
\mathbb{H}_n : \eta_j = \tau_j, \quad 1 \leq j < \infty,
\]

where \( 0 < T(\tau) < \infty \) with \( \tau = \{\tau_j\}_1^\infty \).
For $a > 0$, define the sets of alternatives
\[ \Upsilon(a) = \{ \eta : T(\eta) > a, \eta = \{\eta_j\}_{1}^{\infty}, \eta_j \in \mathbb{R}^{1} \}. \] (8.5)

We say that a test $K$ is unbiased [24] if
\[ \alpha(K) + \beta(K, \Upsilon(a)) < 1. \] (8.6)

Denote $z = \{z_j\}_{1}^{\infty}$.

**Theorem 8.3.** Assume that the conditions D1 and D2 are satisfied. Then tests $K$, $\alpha(K) = \alpha$, $0 < \alpha < 1$, generated by the test statistics $T(z)$ are unbiased.

A proof of Theorem 8.3 is given in A.5.

For chi-squared tests with number of cells $m_n = m = \text{const}$, a similar theorem holds for $r = 1/2$ with the same definition of consistency as in Sec. 6.

**Theorem 8.4.** For $r = 1/2$ and for chi-squared tests, Theorem 8.1 holds as well. Statements of Theorems 4.1, 4.2, 4.5, 4.6, and 4.8–4.10 hold with the same changes as in Theorem 6.1 and with $k_n = 1$.

Emphasize that the Besov bodies $\overline{B}^{s}_{2,\infty}(F_0)$ in the versions of Theorems 4.5, 4.6, and 4.9 are replaced by the sets $U(l, P_0)$, $l = 1, 2, \ldots$, $P_0 > 0$.

To prove Theorem 8.4, we implement the well-known fact that the inequality $nT_n(F_n) > c$ is a necessary and sufficient condition for the consistency of a sequence of alternatives $F_n \in \mathfrak{S}$ for chi-squared tests with a fixed number of cells.

Theorem 8.3 allows us to obtain versions of Theorems 7.2–7.4 for problem of testing hypothesis (1.6) versus alternatives (1.7) with test statistics $T$ such that $\kappa_j^2 \asymp j^{-2\lambda}$, $2\lambda > 1$.

Such a setup arises, in particular, for test statistics $T$ constructed on the base of the technique of reproducing kernel for Hilbert spaces [13].

**Theorem 8.5.** Let $\kappa_j^2 \asymp j^{-2\lambda}$, $2\lambda > 1$. Then the statements of Theorems 7.2–7.4 hold with $s = \frac{2\lambda r}{1 - 2\lambda r}$ and $k_n \asymp n^{\frac{1-2\lambda}{2\lambda}}$ as $n \to \infty$.

The proof of Theorem 8.5 is akin to that of Theorems 7.2–7.4 and is omitted.

**Appendix A. Proofs of theorems**

**A.1. Proofs of Theorems of Sec. 3.** It suffices to prove only the necessity conditions.

Assume that the set $U$ is closed. The general setup can easily be reduced to this one.

For any vectors $\theta_1 \in \mathbb{H}$ and $\theta_2 \in \mathbb{H}$ define the segment
\[ \text{int}(\theta_1, \theta_2) = \{ \theta : \theta = (1 - \lambda)\theta_1 + \lambda\theta_2, \lambda \in [0, 1] \}. \]

The proof of Theorem 3.1 is based on the following Lemma A.1.

**Lemma A.1.** For any vectors $\theta_1 \in U$ and $\theta_2 \in U$, $\text{int}(\frac{\theta_1 - \theta_2}{2}, \frac{\theta_2 - \theta_1}{2}) \subset U$. The inclusion $0 \in \text{int}(\frac{\theta_1 - \theta_2}{2}, \frac{\theta_2 - \theta_1}{2})$ holds, and the segment $\text{int}(\frac{\theta_1 - \theta_2}{2}, \frac{\theta_2 - \theta_1}{2})$ is parallel to the segment $\text{int}(\theta_1, \theta_2)$.

**Remark 3.1.** Let $\text{int}(\theta_1, \theta_2) \subset U$. Let $\eta$ and $-\eta$ be the points of intersection of the line $L = \{ \theta : \theta = \lambda(\theta_1 - \theta_2), \lambda \in \mathbb{R}^{1} \}$ with the boundary of set $U$. Then, by Lemma A.1, $\|\theta_1 - \theta_2\| \leq 2\|\eta\|$. 

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Proof of Lemma A.1. The segments $\text{int}(\theta_1, \theta_2) \subset U$ and $\text{int}(-\theta_1, -\theta_2) \subset U$ are parallel. For each $\lambda \in [0, 1]$ we have $(1 - \lambda)\theta_1 + \lambda\theta_2 \in \text{int}(\theta_1, \theta_2)$ and $-\lambda\theta_1 - (1 - \lambda)\theta_2 \in \text{int}(-\theta_1, -\theta_2)$. The midpoint $\theta_\lambda = ((1 - 2\lambda)\theta_1 - (1 - 2\lambda)\theta_2)/2$ of the segment $\text{int}((1 - \lambda)\theta_1 + \lambda\theta_2, -\lambda\theta_1 - (1 - \lambda)\theta_2) \subset U$ belongs to the segment $\text{int}\left(\frac{\theta_1 - \theta_2}{2}, \frac{\theta_1 - \theta_2}{2}\right)$ and, for each point $\theta$ of the segment $\text{int}\left(\frac{\theta_1 - \theta_2}{2}, \frac{\theta_1 - \theta_2}{2}\right)$, there is a $\lambda \in [0, 1]$ such that $\theta = \theta_\lambda$. Therefore, $\text{int}\left(\frac{\theta_1 - \theta_2}{2}, \frac{\theta_1 - \theta_2}{2}\right) \subset U$. □

Proof of Theorem 3.1. Define a sequence of orthogonal vectors $e_i$ by induction.

Define a vector $e_1 \in U$, such that $\|e_1\| > \frac{1}{2} \sup\{\|\theta\| : \theta \in U\}$. Denote by $\Pi_1$ the linear subspace generated by $e_1$. Denote by $\Gamma_1$ the linear subspace orthogonal to $\Pi_1$.

Define vectors $e_i$, $i = 2, 3, \ldots$, by induction. We select $e_i \in U \cap \Gamma_{i-1}$ such that $\|e_i\| > \frac{1}{2} d_i = \frac{1}{2} \sup\{\|\theta\| : \theta \in U \cap \Gamma_{i-1}\}$. Denote by $\Pi_i$ the linear subspace spanned by the vectors $e_1, e_2, \ldots, e_i$. Denote by $\Gamma_i$ the linear subspace orthogonal to $\Pi_i$.

The vectors $e_1, e_2, \ldots$ are orthogonal.

Note that $d_i \rightarrow 0$ as $i \rightarrow \infty$. Otherwise, by Theorem 5.3 of [10], there do not exist uniformly consistent tests for the problem of testing the hypothesis $\mathbb{H}_0 : \theta = 0$ versus the alternative $\mathbb{H}_1 : \theta \in \{e_1, e_2, \ldots\}$.

For any $\varepsilon \in (0, 1)$ denote $l_\varepsilon = \min\{j : d_j < \varepsilon, j = 1, 2, \ldots\}$.

Denote by $B_r(\theta)$ the ball having radius $r$ and center $\theta$.

It suffices to show that for any $\varepsilon > 0$ there is a finite cover of the set $U$ by the balls $B_{\varepsilon_1}(\theta)$. Denote $\varepsilon = \varepsilon_1/9$.

Denote by $U_\varepsilon$ the projection of the set $U$ to the subspace $\Pi_\varepsilon$.

Denote by $\tilde{B}_\delta(\theta)$ the ball in $\Pi_\varepsilon$ having radius $r$ and center $\theta \in \Pi_\varepsilon$. There is a ball $\tilde{B}_\delta(0)$ such that $\tilde{B}_\delta(0) \subset U$. Denote $\delta = \min\{\varepsilon, \sigma_1\}$.

Let $\theta_1, \ldots, \theta_k$ be a $\delta$-net in $U_\varepsilon$.

Let $\eta_1, \ldots, \eta_k$ be points of $U$ such that $\theta_i$ is the projection of $\eta_i$ to the subspace $\Pi_{\varepsilon_i}$ for $1 \leq i \leq k$.

Let us show that $B_{\varepsilon_1}(\eta_1), \ldots, B_{\varepsilon_1}(\eta_k)$ is a cover of the set $U$.

Let $\eta \in U$ and let $\theta$ be the projection of $\eta$ to $\Pi_{\varepsilon_i}$. There is an index $i$, $1 \leq i \leq k$, such that $\|\theta_i - \theta\| \leq \delta$. It suffices to show that $\eta \in B_{\varepsilon_1}(\eta_i)$.

By Lemma A.1, $\text{int}\left(\frac{\eta - \eta_1}{2}, \frac{\eta - \eta_1}{2}\right) \subset U$. Since $\theta_i - \theta \in \Pi_{\varepsilon_i}$ and $\theta_i - \theta \in \tilde{B}_\delta(0), (\theta_i - \theta)/2 \in U$.

Since the set $U$ is centrally symmetric and convex, $\frac{1}{2}\|(\eta_i - \eta)/2\| - \frac{1}{2}\|(\theta_i - \theta)/2\| \in U$. Note that the vector $(\eta_i - \theta_i) - (\eta - \theta)$ is orthogonal to the subspace $\Pi_i$. Therefore, $\|((\eta_i - \theta_i) - (\eta - \theta))/4\| \leq 2\varepsilon$. Therefore, $\|\eta - \eta_i\| \leq 8\varepsilon + \|\theta - \theta_i\| < 9\varepsilon$. This implies that $\eta \in B_{\varepsilon_1}(\eta_i)$. □

Proof of Theorem 3.3. The proof of Theorem 3.1 is based on Theorem 5.3 of [10]. For linear inverse ill-posed problems (3.4), Theorem 5.5 of [10] is akin to Theorem 5.3 of [10]. Thus, it suffices to implement Theorem 5.5 of [10] instead of Theorem 5.3 of [10] in the proof of Theorem 3.1. □

A.2. Proofs of Theorems of Sec. 4

Proof of Theorem 4.3. Theorem 4.3 and its version for the setup of Remark 4.1 can be straightforwardly deduced from Theorem 1 of [6].

The lower bound straightforwardly follows from the reasoning of Theorem 1 of [6].

The upper bound follows from the following reasoning. Note that

$$
\sum_{j=1}^{\infty} r^2_{nj} y_j^2 = \sum_{j=1}^{\infty} r^2_{nj} \theta^2_{nj} + 2 \frac{\sigma}{\sqrt{n}} \sum_{j=1}^{\infty} r^2_{nj} \theta_{nj} \xi_j + \frac{\sigma^2}{n} \sum_{j=1}^{\infty} r^2_{nj} \xi_j^2
$$

(A.1)

$$
= n^{-2} A_n(\theta_n) + 2 J_{1n} + J_{2n},
$$

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where

\[ E[J_{2n}] = \frac{\sigma^2}{n} \rho_n, \quad \text{Var}[J_{2n}] = 2\frac{\sigma^4}{n^4} A_n \] (A.2)

and

\[ \text{Var}[J_n] = \frac{\sigma^2}{n} \sum_{j=1}^{\infty} \kappa_{nj}^2 \theta_{nj}^2 \leq \frac{\sigma^2 \kappa_n^2}{n} \sum_{j=1}^{\infty} \kappa_{nj}^2 \theta_{nj}^2 = o(n^{-4}A_n(\theta_n)). \] (A.3)

By the Chebyshev inequality, it follows from A1–A3 that if \( A_n = o(A_n(\theta_n)) \) as \( n \to \infty \), then \( \beta(L_n, f_n) \to 0 \) as \( n \to \infty \). Thus, it suffices to treat the case

\[ A_n \asymp A_n(\theta_n) = n^2 \sum_{j=1}^{\infty} \kappa_{nj}^2 \theta_{nj}^2. \] (A.4)

If A4 holds, then, implementing the reasoning of the proof of Lemma 1 of [6], we see that (4.5) holds.

**Proof of Theorem 4.1.** Let (4.3) hold. Then, by A5 and (4.2),

\[ A_n(\theta_n) = n^2 \sum_{j=1}^{\infty} \kappa_{nj}^2 \theta_{nj}^2 \geq Cn^2 \kappa_n^2 \sum_{j=1}^{\infty} \theta_{nj}^2 \asymp n^2 \kappa_n^2 n^{-2r} \asymp 1. \]

By Theorem 4.3, this implies the sufficiency.

The necessity conditions follow from the sufficiency conditions of Theorem 4.2. \( \square \)

**Proof of Theorem 4.2.** Let (4.4) hold. Then, by (4.2) and A2,

\[ A_n(\theta_n) \leq Cn^2 \kappa_n^2 \sum_{j<c_2k_n} \theta_{nj}^2 + Cn^2 \kappa_n^2 \sum_{j>c_2k_n} \theta_{nj}^2 \asymp o(1) + O(\kappa_n^2/\kappa_n^2). \] (A.5)

By A4,

\[ \lim_{c_2 \to \infty} \lim_{n \to \infty} \frac{\kappa_n^2}{\kappa_n^2} \to 0. \] (A.6)

Combining Theorem 4.3, A5, and A6, we establish the sufficiency. \( \square \)

**Proof of Theorem 4.4.** Statement (i) follows from Theorem 4.1 and Lemma A.2 proved below. \( \square \)

**Lemma A.2.** Let \( f_n \in \mathbb{B}_{2\infty}(c_1) \) and \( cn^{-r} \leq \|f_n\| \leq Cn^{-r} \). Then

\[ \sum_{j=1}^{l_n} \theta_{nj}^2 > \frac{c}{2} n^{-2r} (1 + o(1)), \] (A.7)

where \( l_n = C_1 n^{2-4r} (1 + o(1)) = C_1 n^{\frac{r}{v}} (1 + o(1)) \) with \( C_1^{2s} > 2c_1/c \).

**Proof.** Let \( f_n \in \mathbb{B}_{2\infty}(c_1) \). Then

\[ \sum_{j=l_n}^{\infty} \theta_{nj}^2 = C_1^{2s} n^{2r} \sum_{j=l_n}^{\infty} \theta_{nj}^2 \leq c_1 (1 + o(1)). \]

Hence,

\[ \sum_{j=l_n}^{\infty} \theta_{nj}^2 \leq c_1 C_1^{-2s} n^{-2r} \leq \frac{c}{2} n^{-2r} (1 + o(1)). \] (A.8)

Therefore, A7 holds.
To get a contradiction, assume that (ii) does not hold. Then \( f = \sum_{j=1}^{\infty} \tau_j \phi_j \notin \bar{B}_{2r}^s \). This implies that there is a sequence \( m_l, m_l \to \infty \) as \( l \to \infty \), such that
\[
m_l^{2s} \sum_{j=m_l}^{\infty} \tau_j^2 = C_l, \tag{A.9}
\]
where \( C_l \to \infty \) as \( l \to \infty \).

Define a sequence \( \eta_l = \{\eta_lj\}_{j=1}^{\infty} \) such that \( \eta_lj = 0 \) if \( j < m_l \) and \( \eta_lj = \tau_j \) if \( j \geq m_l \).

Since \( V \) is convex and orthosymmetric, \( \tilde{f}_l = \sum_{j=1}^{\infty} \eta_lj \phi_j \in V \).

For the alternatives \( \tilde{f}_l \) we define a sequence \( n_l \) such that
\[
\|n_l\|^2 \propto n_l^{-2r} \propto m_l^{-2s} C_l. \tag{A.10}
\]
Then
\[
n_l \propto C_l^{-1/(2r)} m_l^{s/r} = C_l^{-1/(2r)} m_l^{2-2r}. \tag{A.11}
\]
Therefore,
\[
m_l \propto C_l^{(1-2r)/r} n_l^{2-4r}. \tag{A.12}
\]
By A4, (A.12) implies that
\[
\kappa_{n_l m_l}^2 = O(\kappa_{n_l}^2). \tag{A.13}
\]
Using (4.2), A2, and (A.13), we conclude that
\[
A_{n_l}(\eta_l) = n_l^2 \sum_{j=1}^{\infty} \kappa_{n_l}^2 \eta_lj^2 \leq n_l^2 \kappa_{n_l m_l}^2 \sum_{j=m_l}^{\infty} \theta_{n_lj}^2
\]
\[
\times n_l^{-2r} \kappa_{n_l m_l} = O(\kappa_{n_l m_l}^{-2}) = o(1). \tag{A.14}
\]
By Theorem 4.3, (A.14) implies the \( n_l^{-r} \)-inconsistency of the sequence of alternatives \( \tilde{f}_l \). \( \square \)

**Proof of Theorem 4.5.** Theorem 4.5 follows from Lemmas A.3–A.5.

**Lemma A.3.** For any \( c \) and \( C \) there is a \( \bar{B}_{2r}^s(P_0) \) such that if \( f_n = \sum_{j=1}^{c k_n} \theta_{n_lj} \phi_j \) and \( \|f_n\| \leq C n^{-r} \), then \( f_n \in \bar{B}_{2r}^s(P_0) \).

**Proof.** Let \( C_1 \) be such that \( k_n = C_1 n^{r/s}(1 + o(1)) \). Then
\[
k_n^{2s} \sum_{j=1}^{c k_n} \theta_{n_lj}^2 \leq C_1 n^{2r} \sum_{j=1}^{\infty} \theta_{n_lj}^2 (1 + o(1)) < CC_1 (1 + o(1)).
\]
\( \square \)

**Lemma A.4.** The necessity conditions in Theorem 4.5 are fulfilled.

**Proof.** Let \( f_n = \sum_{j=1}^{\infty} \theta_{n_lj} \phi_j \) and \( f_{1n} = \sum_{j=1}^{\infty} \eta_{n_lj} \phi_j \). Denote \( \zeta_{n_lj} = \theta_{n_lj} - \eta_{n_lj}, 1 \leq j < \infty \).

For any \( \delta > 0, c_1, \) and \( C_2 \) there is a \( c_2 \) such that, for each sequence \( f_{1n} \in \bar{B}_{2r}^s(P_0), \)
\[
\|f_{1n}\| \leq C_2 n^{-r},
\]
\[
\sum_{j>c_2 k_n} \eta_{n_lj}^2 < \delta n^{-2r}. \tag{A.15}
\]
To prove (A.15), it suffices to put \( c_2 k_n = l_n = C_1 n^{2-4r} (1 + o(1)) \) in (A.8) with \( C_1^{2s} > \delta c_1 \).

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Note that
\[ J_n = \left| \sum_{j > c_k n} \theta_{nj}^2 - \sum_{j > c_k n} \zeta_{nj}^2 \right| \leq \sum_{j > c_k n} |\eta_{nj}(2\theta_{nj} - \eta_{jn})| \]
\[ \leq \left( \sum_{j > c_k n} \eta_{nj}^2 \right)^{1/2} \left( 2 \left( \sum_{j > c_k n} \theta_{nj}^2 \right)^{1/2} + \left( \sum_{j > c_k n} \eta_{nj}^2 \right)^{1/2} \right) \leq C \delta^{1/2} n^{-2r}. \tag{A.16} \]

By (4.6), using (A.15) and (A.16), we see that
\[ \sum_{j < c_k n} \theta_{nj}^2 = \sum_{j = 1}^{\infty} \eta_{nj}^2 + \sum_{j = 1}^{\infty} \zeta_{nj}^2 - \sum_{j \geq c_k n} \theta_{nj}^2 \geq \sum_{j < c_k n} \eta_{nj}^2 - J_n \geq \sum_{j < c_k n} \eta_{nj}^2 - C \delta^{1/2} n^{-2r} \geq \|f_1 n\|^2 - \delta n^{-2r} - C \delta^{1/2} n^{-2r}. \tag{A.17} \]

By Theorem 4.1, (A.17) implies the consistency of the sequence \( f_n \). \( \square \)

**Lemma A.5.** Let a sequence of alternatives \( f_n, cn^{-r} \leq \|f_n\| \leq Cn^{-r} \), be consistent. Then (4.6) holds.

*Proof.* By Theorem 4.1, there exist \( c_1 \) and \( c_2 \) such that the sequence \( f_1 n = \sum_{j < c_2 k n} \theta_{nj} \phi_j \) is consistent and \( \|f_1 n\| \geq c_1 n^{-r} \). By Lemma A.3, there is a \( \mathbb{B}_2^\infty(P_0) \) such that \( f_1 n \in \mathbb{B}_2^\infty(P_0) \).

\( \square \)

*Proof of Theorem 4.6.* By A4 and (4.2), for any \( \delta > 0 \) there is a \( c \) such that
\[ n^2 \sum_{j > c_k n} \kappa_{nj}^2 \theta_{nj}^2 \leq \delta. \tag{A.18} \]

By Lemma A.3, there is a \( P_0 \) such that \( f_1 n = \sum_{j < c_k n} \theta_{nj} \phi_j \in \mathbb{B}_2^\infty(P_0) \). By Theorem 4.3 and (A.18), for the sequence of alternatives \( f_1 n \), (4.7) and (4.8) hold.

*Proof of Theorem 4.7.* Let \( f_n = \sum_{j = 1}^{\infty} \theta_{nj} \phi_j \) and \( f_1 n = \sum_{j = 1}^{\infty} \eta_{nj} \phi_j \). Denote \( \eta_n = \{\eta_{nj}\}_{j=1}^{\infty} \).

By the Cauchy inequality,
\[ |A_n(\theta_n) - A_n(\theta_n + \eta_n)| = n^2 \sum_{j = 1}^{\infty} \kappa_{nj}^2 \theta_{nj}^2 - \sum_{j = 1}^{\infty} \kappa_{nj}^2 (\theta_{nj} + \eta_{nj})^2 | \leq 2 A_n^{1/2}(\theta_n) A_n^{1/2}(\eta_n) + A_n(\eta_n). \tag{A.19} \]

By Theorem 4.3, the inconsistency of the sequence \( f_1 n \) implies that \( A_n(\eta_n) = o(1) \) as \( n \to \infty \). Therefore, by (A.19), \( |A_n(\theta_n) - A_n(\theta_n + \eta_n)| = o(1) \) as \( n \to \infty \). Hence, by Theorem 4.3, we get Theorem 4.7.

*Proof of Theorem 4.8.* To prove the sufficiency, assume the opposite. Then there is a sequence \( n_i, n_i \to \infty \) as \( i \to \infty \), such that \( f_{ni} = f_1 n_i + f_{2n_i} \),
\[ \|f_n\|^2 = \|f_1 n_i\|^2 + \|f_{2n_i}\|^2, \tag{A.20} \]
\[ c_1 n_i^{-r} < \|f_1 n_i\| < C_1 n_i^{-r}, \ c_2 n_i^{-r} < \|f_{2n_i}\| < C_2 n_i^{-r}, \] and the sequence \( f_{2n_i} \) is inconsistent.

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Let \( f_n = \sum_{j=1}^{\infty} \theta_{nj} \phi_j \), \( f_{1ni} = \sum_{j=1}^{\infty} \theta_{1nj} \phi_j \), and \( f_{2ni} = \sum_{j=1}^{\infty} \theta_{2nj} \phi_j \).

Then, by Theorem 4.2 and (4.9), we conclude that there exist \( \varepsilon_i, \varepsilon_i \to 0, \) and \( C_i = C(\varepsilon_i), C_i \to \infty \) as \( i \to \infty, \) such that
\[
\sum_{j > C_i k_n} \theta_{nj}^2 = \sum_{j > C_i k_n} (\theta_{nj} + \theta_{2nj})^2 = o(n^{-2r}) \quad \text{and} \quad \sum_{j < C_i k_n} \theta_{2nj}^2 = o(n^{-2r}). \quad \text{(A.21)}
\]

By (A.20) and (A.21),
\[
\sum_{j=1}^{\infty} \theta_{nj}^2 = \sum_{j < C_i k_n} \theta_{nj}^2 + o(n^{-2r}) = \sum_{j < C_i k_n} \theta_{1nj}^2 + o(n^{-2r}). \quad \text{(A.22)}
\]
Hence, by (A.20), \( \|f_{2ni}\| = o(n^{-r}) \). We get a contradiction.

To prove the necessity conditions, assume that (4.9) does not hold. Then there exist \( \varepsilon > 0 \) and sequences \( C_i \to \infty, n_i \to \infty \) as \( i \to \infty, \) such that
\[
\sum_{j > C_i k_n} \theta_{nj}^2 > \varepsilon n_i^{-2r}.
\]
Then, by A4 and (4.2),
\[
n_i^2 \sum_{j > C_i k_n} \kappa_{nj}^2 \theta_{nj}^2 = o(1).
\]
Therefore, by Theorem 4.3, the subsequence \( f_{1ni} = \sum_{j > C_i k_n} \theta_{nj} \phi_j \) is inconsistent. \( \square \)

Proof of Theorem 4.9. To prove the necessity conditions, it suffices to put
\[
f_{1n} = \sum_{j < C(\varepsilon) k_n} \theta_{nj} \phi_j.
\]
By Lemma A.3, there is a \( P_0 > 0 \) such that \( f_{1n} \in \bar{B}_{2\infty}^s(P_0) \). The proof of sufficiency is simple and is omitted. \( \square \)

Proof of Theorem 4.10. The necessity conditions are rather obvious, and the proof is omitted. The proof of sufficiency is also simple.

Lemma A.6. Let for a sequence \( f_n, cn^{-r} < \|f_n\| < Cn^{-r}, \) (4.10) hold. Then the sequence \( f_n \) is purely \( n^{-r} \)-consistent.

Assume that \( f_n = \sum_{j=1}^{\infty} \theta_{nj} \phi_j \) is not purely \( n^{-r} \)-consistent. Then, by Theorem 4.8, there exist \( c_1 \) and sequences \( n_i \) and \( c_{ni}, c_{ni} \to \infty \) as \( i \to \infty, \) such that
\[
\sum_{j > c_{ni} k_{ni}} \theta_{nj}^2 > c_1 n_i^{-r}.
\]
Therefore, if we put \( f_{1ni} = \sum_{j > c_{ni} k_{ni}} \theta_{nj} \phi_j, \) then (4.10) does not hold. \( \square \)
A.3. Proofs of Theorems of Sec. 5

Proof of the version of Theorem 4.1. Since \( \hat{K}(\omega) \) is an analytic function and \( \hat{K}(0) = 1 \), there exists a \( b > 0 \) such that \( |\hat{K}(\omega)| > c > 0 \) for \( |\omega| < b \).

Let (4.3) hold. Then

\[
T_{1n}(f_n) = \sum_{j=-\infty}^{\infty} |\hat{K}(jh_n)|^2 |\theta_{nj}|^2 \geq \sum_{|j|h_n < b} |\hat{K}(jh_n)|^2 |\theta_{nj}|^2
\]

\[
\times \sum_{|j| < c_2 k_n} |\hat{K}(jh_n)|^2 |\theta_{nj}|^2 \times n^{-1} h_n^{-1/2} \times n^{-2r}
\]

for \( c_2 k_n < bh_n^{-1} \). By Theorem 5.2, this implies the consistency. \( \square \)

Proof of the version of Theorem 4.4. We verify only item (ii). Let \( f = \sum_{j=-\infty}^{\infty} \tau_j \phi_j \notin \mathbb{B}^{s}_{2\infty} \).

Then there exists a sequence \( m_l, m_l \to \infty \) as \( l \to \infty \), such that

\[
m_l^{2s} \sum_{|j| \geq m_l} |\tau_j|^2 = C_l, \quad (A.23)
\]

where \( C_l \to \infty \) as \( l \to \infty \).

It is clear that we can define a sequence \( m_l \) such that

\[
m_l^{2s} \sum_{m_l \leq |j| \leq 2m_l} |\tau_j|^2 > \delta C_l, \quad (A.24)
\]

where \( 0 < \delta < 1/2 \), does not depend on \( l \).

Otherwise,

\[
2^{2s(i-1)} m_l^{2s} \sum_{j=2^{i-1}m_l}^{2^i m_l} \tau_j^2 < \delta C_l
\]

for all \( i = 1, 2, \ldots \), which implies that the left hand-side of (A.23) does not exceed \( 2\delta C_l \).

Define a sequence \( \eta_l = \{\eta_{lj}\}_{j=-\infty}^{\infty} \) such that \( \eta_{lj} = \tau_j \) if \( |j| \geq m_l \) and \( \eta_{lj} = 0 \) otherwise.

Denote

\[
\bar{f}_l(x) = \sum_{j=-\infty}^{\infty} \eta_{lj} \exp\{2\pi i j x\}.
\]

For the alternatives \( \bar{f}_l(x) \) we define a sequence \( n_l \) such that \( \|\bar{f}_l(x)\| \asymp n_l^{-r} \).

Then

\[
n_l \asymp C_l^{-1/(2r)} m_l^{s/r}.
\]

We note that \( |\hat{K}(\omega)| \leq \hat{K}(0) = 1 \) for all \( \omega \in \mathbb{R}^1 \) and \( |\hat{K}(\omega)| > c > 0 \) for all \( |\omega| < b \). Hence, if we put \( h_l = h_{ni} = 2^{-1} b^{-1} m_l^{-1} \), then, by (A.24), there is a \( C > 0 \) such that

\[
T_{1n_l}(\bar{f}_l, h_l) = \sum_{j=-\infty}^{\infty} |\hat{K}(jh_l) \eta_{lj}|^2 > C \sum_{j=-\infty}^{\infty} |\hat{K}(jh_l) \eta_{lj}|^2 = C T_{1n_l}(\bar{f}_l, h)
\]

for all \( h > 0 \). Thus, we may choose \( h = h_l \) for further reasoning.

By (A.24),

\[
T_{1n_l}(\bar{f}_l) = \sum_{|j| > m_l} |\hat{K}(jh_l) \eta_{lj}|^2 \times \sum_{j=m_l}^{2m_l} |\eta_{lj}|^2 \asymp n_l^{-2r}, \quad (A.25)
\]

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If we put \( k_l = [h_{n,l}^{-1}] \) and \( m_l = k_l \) in estimates (A.11) and (A.12), then we get the relations
\[
h_{n,l}^{1/2} \asymp C_l (2r-1)^{r-1} n_l^{2r-1}.
\] (A.26)

By (A.25) and (A.26),
\[
m_l T_{n,l} (\tilde{f}_l h_{n,l}^{1/2}) \asymp C_l^{-1} (1-2r)^{r}/2.
\]

By Theorem 5.2, this implies the inconsistency of the sequence of alternatives \( \tilde{f}_l \). \( \square \)

A.4. Proof of Theorems of Sec. 6.

Note that
\[
n_l^{-1} m_l^{-1} T_n (F) = \sum_{l=0}^{m_n-1} \left( \sum_{j=0}^{(l+1)/m_n} f(x) \right)^2.
\]

Using the representation of \( f(x) \) by the Fourier series,
\[
f(x) = \sum_{j=-\infty}^{\infty} \theta_j \exp{2\pi i j x},
\]
we get the equalities
\[
\int_{l/m_n}^{(l+1)/m_n} f(x) dx = \sum_{j=-\infty}^{\infty} \frac{\theta_j}{2\pi i j} \exp{2\pi i j l/m_n} \{ \exp{2\pi i j / m_n} - 1 \}
\]
for \( 1 \leq l < m_n \).

In what follows, we use the following agreement: \( 0/0 = 0 \).

Lemma A.7. The following relations hold:
\[
n_l^{-1} m_l^{-1} T_n (F) = m_n \sum_{k=-\infty}^{\infty} \sum_{j \neq km_n} \frac{\theta_j \bar{\theta}_{j-km_n}}{4\pi^2 j (j - km_n)} (2 - 2 \cos(2\pi j / m_n)).
\] (A.27)

Proof of Lemma A.7. Note that
\[
n_l^{-1} m_l^{-1} T_n (F) = \sum_{l=0}^{m_n-1} \left( \sum_{j \neq 0 \neq \bar{j}} \frac{\theta_j}{2\pi i j} \exp{2\pi i j l/m_n} \{ \exp{2\pi i j / m_n} - 1 \} \right) \times \left( \sum_{j \neq \bar{j}} \frac{-\bar{\theta}_{j}}{2\pi i j} \exp{-2\pi i j l/m_n} \{ \exp{-2\pi i j / m_n} - 1 \} \right) = J_1 + J_2,
\] (A.28)

where
\[
J_1 = \sum_{l=0}^{m_n-1} \sum_{k=-\infty}^{\infty} \sum_{j_1 = j-km_n} \frac{\theta_j \bar{\theta}_{j_1}}{4\pi^2 j j_1} \exp{2\pi i k l} \times \{ \exp{2\pi i j l/m_n} - 1 \} \{ \exp{-2\pi i j_1 l/m_n} - 1 \}
\]
(A.29)

\[
J_2 = \sum_{l=0}^{m_n-1} \sum_{j \neq j_1 \neq \bar{j}} \sum_{j-km_n} \frac{\theta_j \bar{\theta}_{j_1}}{4\pi^2 j j_1} \exp{2\pi i (j - j_1) l/m_n} \times \{ \exp{2\pi i j l/m_n} - 1 \} \{ \exp{-2\pi i j_1 l/m_n} - 1 \} = 0.
\] (A.30)
In the last formula, the condition \(j_1 \neq j - km_n\) indicates that the summation is performed over all \(j_1\) such that \(j_1 \neq j - km_n\) for all integer \(k\).

In the last equality of (A.30), we apply the following identity:
\[
\sum_{l=0}^{m_n-1} \exp\{2\pi i(j - j_1)l/m_n\} = \frac{\exp\{2\pi i(j - j_1)m_n/m_n\} - 1}{\exp\{2\pi i(j - j_1)/m_n\} - 1} = 0
\]
if \(j - j_1 \neq km_n\) for all integer \(k\).

Combining (A.28)–(A.30), we prove (A.27).

For any c.d.f \(F\) and any \(k\), denote by \(\tilde{F}_k\) the function having the derivative
\[
1 + \tilde{f}_k(x) = 1 + \sum_{|j| > k} \theta_j \exp\{2\pi i jx\}
\]
and such that \(\tilde{F}_k(1) = 1\).

Denote \(i_n = [dm_n]\), where \(d > 1\).

Lemma A.8. The following inequalities hold:
\[
n^{-1}m_n^{-2}T_n(\tilde{F}_n) \leq Cm_n^{-1}i_n^{-1} \sum_{|j| > i_n} |\theta_j|^2.
\] (A.31)

Proof. Denote \(\eta_j = \theta_j\) if \(|j| > i_n\) and \(\eta_j = 0\) if \(|j| < i_n\).

We have the inequalities
\[
n^{-1}m_n^{-2}T_n(\tilde{F}_n) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{\eta_j \eta_{j-km_n}}{4\pi^2 j(j-km_n)} (2 - 2\cos(2\pi j/m_n))
\]
\[
\leq C \sum_{|j| > i_n} |\eta_j| \sum_{j=-\infty}^{\infty} \frac{\eta_{j+km_n}}{|j + km_n|}
\]
\[
= C \sum_{j=1}^{m_n} \sum_{k=-\infty}^{\infty} \frac{\eta_{j+km_n}}{|j + km_n|} \sum_{k_1=-\infty}^{\infty} \frac{\eta_{j+(k+k_1)m_n}}{|j + (k + k_1)m_n|}
\]
\[
= C \sum_{j=1}^{m_n} \left( \sum_{k=-\infty}^{\infty} \frac{\eta_{j+km_n}}{|j + km_n|} \right)^2
\]
\[
\leq C \sum_{j=1}^{m_n} \left( \sum_{|k| > d-1} |\eta_{j+km_n}|^2 \right) \left( \sum_{|k| > d-1} (j + km_n)^{-2} \right)
\]
\[
\leq C \sum_{j=-\infty}^{\infty} |\eta_j|^2 \sum_{|k| > d} (km_n)^{-2} \leq C m_n^{-1}i_n^{-1} \sum_{|j| > i_n} |\theta_j|^2.
\] □

Proof of the version of Theorem 4.1. We prove the sufficiency. Assume that (4.3) holds. Denote
\[
\tilde{f}_n = \tilde{f}_{n,c2k_n} = \sum_{|j| > c2k_n} \theta_{nj} \phi_j \quad \text{and} \quad \bar{f}_n = \bar{f}_{n,c2k_n} = f_n - \tilde{f}_n.
\]

Denote by \(\tilde{F}_n\) and \(F_n\) the functions having derivatives \(1 + \tilde{f}_{n,c2k_n}\) and \(1 + \bar{f}_{n,c2k_n}\), respectively, and such that \(\tilde{F}_n(1) = 1\) and \(\bar{F}_n(1) = 1\).

Let \(T_n\) be chi-squared test statistics with number of cells \(m_n = [c_3k_n]\), where \(c_2 < c_3\).

Denote by \(L_{2,n}\) the linear space spanned by the functions \(1_{\{x \in ([j-1]/m_n,j/m_n)\}}, 1 \leq j \leq m_n\).
Denote by $\tilde{h}_n$ the orthogonal projection of $\tilde{f}_n$ to $L_{2,n}$. Denote by $\bar{h}_n$ the orthogonal projection of $\bar{f}_n$ to the line $\{ h : h = \lambda \bar{h}_n, \lambda \in \mathbb{R}^1 \}$.

Note that $n^{-1/2} m_n^{-1} T_n^{1/2}(F_n)$ equals the $L_{2,n}$-norm of the function $f_n$. Hence,

$$n^{-1/2} m_n^{-1} T_n^{1/2}(F_n) \geq \| \bar{h}_n + \tilde{h}_n \|. \quad (A.32)$$

Thus, by Theorem 6.3, it suffices to show that $\| \bar{h}_n + \tilde{h}_n \| \asymp n^{-r}$ if $m_n > c_3 k_n$ for some choice of $c_3$.

Denote $g_n = f_n - \bar{h}_n$ and $\bar{g}_n = \tilde{f}_n - \tilde{h}_n$.

Denote

$$\bar{p}_{jn} = \frac{1}{m_n} \int_{(j-1)/m_n}^{j/m_n} \tilde{f}_n(x) dx, \quad 1 \leq j \leq m_n.$$ 

By Lemmas 3 and 4 of [31, Sec. 7], we have

$$\| \bar{g}_n \|^2 = m_n \sum_{j=1}^{m_n} \int_{(j-1)/m_n}^{j/m_n} (\tilde{f}_n(x) - \bar{p}_{jn})^2 dx \leq 2 \omega^2 \left( \frac{1}{m_n}, \tilde{f}_n \right). \quad (A.33)$$

Here

$$\omega^2(h, f) = \int_0^{1} (f(t + h) - f(t))^2 dt, \quad h > 0,$$

for any $f \in L_{2}^{per}$. If $f = \sum_{j=-\infty}^{\infty} \theta_j \phi_j$, then

$$\omega^2(s, f) = 2 \sum_{j=1}^{\infty} |\theta_j|^2 (2 - 2 \cos(2 \pi js)). \quad (A.34)$$

Since $1 - \cos(x) \leq x^2$, it follows from (A.33) and (A.34) that

$$\| \bar{g}_n \| \leq 4\pi (c_2 k_n / m_n)^{1/2} \| f_n \| = \delta \| f_n \| (1 + o(1)), \quad (A.35)$$

where $\delta = 4\pi (c_2 / c_3)^{1/2}$.

By (4.3), (A.33), and (A.35), there exists a $c_{30}$ such that

$$\| \bar{h}_n \| > \frac{c_1}{2} n^{-r} \quad (A.36)$$

for $c_3 > c_{30}$.

For any functions $g_1, g_2 \in L_2(0, 1)$ denote by $(g_1, g_2)$ the inner product of $g_1$ and $g_2$.

We note that

$$0 = (\tilde{f}_n, \bar{f}_n) = (\tilde{h}_n, \bar{h}_n) + (\bar{g}_n, \bar{f}_n). \quad (A.37)$$

By (A.35),

$$|(\bar{g}_n, \bar{f}_n)| \leq \| \bar{g}_n \| \| \bar{f}_n \| \leq \delta C^2 n^{-2r}.$$ 

Therefore,

$$|(\bar{h}_n, \bar{h}_n)| \leq \delta C^2 n^{-2r}. \quad (A.38)$$

It follows from (A.36) and (A.38) that $\| \bar{h}_n + \tilde{h}_n \| \asymp n^{-r}$ for sufficiently small $\delta > 0$. Hence, using (A.32) and implementing Theorem 6.3, we prove the sufficiency. \hfill \Box
Proof of the version of Theorem 4.2. We prove the sufficiency. Let \( m_n = [c_1k_n] \). We note that
\[
T_n^{1/2}(F_n) \leq T_n^{1/2} (\tilde{F}_n) + T_n^{1/2} (\bar{F}_n).
\]
By Lemma A.8,
\[
n^{-1}m_n^{-2}T_n(\tilde{F}_n) \leq c_2^{-1}m_n^{-1}\|f_n\|^2 \leq c_2^{-1}c_1cn^{-2r}
\]
for \( c_2 > 2c_1 \). We have the inequality
\[
\|\tilde{f}_n\| \geq n^{-1/2}m_n^{-2}T_n^{1/2}(\tilde{F}_n).
\]
Since one can take an arbitrary value \( c_2, c_2 > 2c_1 \), combining Theorem 6.3, (4.4), and (A.39)–(A.41), we get the inconsistency of sequence \( f_n \).

Proof of the version of Theorem 4.4. Let us prove item (ii). Assume the opposite. Then there is a sequence \( i_l, i_l \to \infty \) as \( l \to \infty \), such that
\[
i_l^{2s}||\tilde{f}_{i_l}||^2 = C_l,
\]
where \( C_l \to \infty \) as \( l \to \infty \). Here \( f = \sum_{j=-\infty}^{\infty} \tau_j \phi_j \) and \( \tilde{f}_{i_l} = \sum_{|j|>i_l} \tau_j \phi_j \).

Define a sequence \( n_l \) such that \( n_l^{-r} \|\tilde{f}_{i_l}\| \to \infty \) as \( l \to \infty \).

Applying estimates similar to (A.11) and (A.12), we see that \( n_l^{1/2} \leq C_l^{(2r-1)/2}n_l^{-1} \) as \( l \to \infty \).

If \( m_l = o(i_l) \), then, by Lemma A.8,
\[
m_l^{-1/2}T_{i_l}(\tilde{F}_{i_l}) \leq m_l^{-1/2}i_l^{-1}n_l \sum_{|j|>i_l} |\tau_j|^2 \leq m_l^{-1/2}i_l^{-1}n_l^{1-2r} = o(C_l^{(2r-1)/2}).
\]
Let \( m_l \asymp i_l \) or \( i_l = o(m_l) \). Then
\[
n_l^{-2r} \|\tilde{f}_{i_l}\|^2 \geq n_l^{-1}m_l^{-2}T_{i_l}(\tilde{F}_{i_l}).
\]
Denote by \( \bar{F}_{i_l}(x) \), \( x \in [0, 1] \), the function having derivative \( 1 + \tilde{f}_{i_l} \) and such that \( \bar{F}_{i_l}(1) = 1 \).

Then
\[
m_l^{-1/2}T_{i_l}(\bar{F}_{i_l}) \leq Cm_l^{-1/2}n_l^{1-2r} = Cm_l^{-1/2}i_l^{1/2}C_l^{(2r-1)/2} = o(1).
\]
By Theorem 6.3, (A.42)–(A.44) imply item (ii).

Proof of the version of Theorem 4.6. Let \( \sum_{|j| < c_k} \theta_{nj}\phi_j \). Then, by Lemma A.3, there is a maxiset \( \bar{B}_{2,\infty}^*(P_0) \) such that \( f_{1n} \in \bar{B}_{2,\infty}^*(P_0) \).

Denote by \( F_{1n} \) the function having derivative \( 1 + f_{1n} \) and such that \( F_{1n}(1) = 1 \).

We have the inequality
\[
|T_n^{1/2}(F_n) - T_n^{1/2}(F_{1n})| \leq T_n^{1/2}(F_n - F_{1n} + F_0).
\]
If \( m_n = [c_0k_n] \) and \( c > 2c_0 \), then, by Lemma A.8,
\[
n^{-1}m_n^{-2}T_n(F_n - F_{1n} + F_0) \leq c_0c^{-1}\|f_n - f_{1n}\|^2.
\]
Since the choice of \( c \) is arbitrary, by Theorem 6.3, (A.45) and (A.46) imply (4.7) and (4.8).

The proof of item (i) in the version of Theorem 4.4 and of the versions of Theorems 4.5 and 4.8–4.10 is obtained from Theorem 6.3 and the versions of Theorems 4.1 and 4.2 by the same reasoning as in Sec. A.2. The proof of Theorem 6.2 is akin to that of Theorem 4.7 and is omitted.
A.5. Proofs of Theorems of Sec. 7 and of Theorem 8.3. Lemma A.9 given below allows us to carry over the corresponding reasoning for a Brownian bridge \(b(t), t \in (0, 1)\), instead of empirical distribution functions.

**Lemma A.9.** For any \(x > 0\),
\[
P_{F_n}(nT^2(\hat{F}_n - F_0) < x) - P(T^2(b(t) + \sqrt{n}(F_n(t) - F_0(t))) < x) = o(1) \tag{A.47}
\]
uniformly over sequences of c.d.f.'s \(F_n\) such that \(T(F_n - F_0) < cn^{-1/2}\).

If \(\sqrt{n}(F_n - F_0) \to G\) in the Kolmogorov–Smirnov distance, relation (A.47) had been proved by Chibisov [3] without any statements of uniform convergence.

Lemma A.9 follows from Lemmas A.10 and A.12 given below after implementation of the Hungarian construction (see [28, Chap. 12, Sec. 1, Theorem 3]).

**Lemma A.10.** For any \(x > 0\),
\[
P(T^2(b(F_n(t))) + \sqrt{n}(F_n(t) - F_0(t))) < x)
- P(T^2(b(t) + \sqrt{n}(F_n(t) - F_0(t))) < x) = o(1) \tag{A.48}
\]
uniformly over sequences of c.d.f.'s \(F_n\) such that \(T(F_n - F_0) < cn^{-1/2}\).

Lemma A.10 follows from Lemmas A.11 and A.12 given below.

**Lemma A.11.** The following inequality holds:
\[
E[|T^2(b(F_n(t))) - T^2(b(t))|] < cT^{1/4}(F_n - F_0). \tag{A.49}
\]

**Proof.** Note that
\[
E^2[|T^2(b(F_n(t))) - T^2(b(t))|] \leq E^2[|(T(b(F_n(t))) - T(b(t)))(T(b(F_n(t))) + T(b(t)))|]
\]
\[
\leq E[|(T(b(F_n(t))) - T(b(t)))^2] E[(T(b(F_n(t))) + T(b(t)))^2]
\]
\[
\leq C E[|(T(b(F_n(t))) - T(b(t)))^2] \leq CE[T^2(b(F_n(t)) - b(t))]
\]
\[
= C \int_0^1 (F_n(t) - F_0^2(t) - 2 \min(F_n(t), F_0(t)) + 2F_n(t)F_0(t) + F_0(t) - F_0^2(t)) dt
\]
\[
= C \int_0^1 F_n(t) + F_0(t) - 2 \min(F_n(t), F_0(t)) - (F_n(t) - F_0(t))^2 dt \tag{A.50}
\]
\[
= C \int_0^1 |F_n(t) - F_0(t)| - (F_n(t) - F_0(t))^2 dt
\]
\[
\leq C \int_0^1 |F_n(t) - F_0(t)| dt \leq T^{1/2}(F_n - F_0).
\]

**Lemma A.12.** The densities of c.d.f.'s \(P(T^2(b(t) + n^{1/2}(F_n(t) - F_0(t)))) \leq x\) are uniformly bounded over the set of all c.d.f.'s \(F_n\) such that \(nT^2(F_n - F_0) < C\), where \(C\) is an arbitrary constant.

**Proof.** A Brownian bridge \(b(t)\) admits the representation
\[
b(t) = \sum_{j=1}^{\infty} \frac{\xi_j}{\pi j} \psi_j(t),
\]

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where \( \psi_j(t) = \sqrt{2} \sin(\pi j t) \) and \( \xi_j, 1 \leq j < \infty \), are i.i.d. Gaussian random variables such that \( \mathbb{E} \xi_j = 0 \) and \( \mathbb{E} \xi_j^2 = 1 \).

Therefore, if \( F_n(t) = \sum_{j=1}^{\infty} \theta_{nj} \psi_j \), then

\[
T^2(b(t) + n^{1/2}(F_n(t) - F_0(t))) = \sum_{j=1}^{\infty} \left( \frac{\xi_j}{\pi j} + n^{1/2} \theta_{nj} \right)^2. \tag{A.51}
\]

The right hand-side of (A.51) is a sum of independent random variables. Thus, it suffices to show that for any \( C \), the random variables

\[
(\xi_1 + n^{1/2} \theta_{n1})^2 + (\xi_2/2 + n^{1/2} \theta_{n2})^2
\]

have uniformly bounded densities with respect to \( n^{1/2} |\theta_{n1}| \leq C \) and \( n^{1/2} |\theta_{n2}| \leq C \).

The densities \((\xi_1 + a)^2\) and \((\xi_2 + b)^2\) have well-known analytical form, and the proof of the uniform boundedness of the densities of \((\xi_1 + a)^2 + \frac{1}{4}(\xi_2 + b)^2\) with \(|a| \leq C\) and \(|b| \leq C\) is obtained by routine technique. We omit these standard estimates. \(\square\)

To prove Theorem 7.1, it suffices to prove item (ii). The Hungarian construction allows us to reduce the reasoning to a proof of the corresponding statement for a Brownian bridge \( b(t) \), \( t \in [0, 1] \). Thus, Theorem 7.1 follows from Theorem 8.3.

**Proof of Theorem 8.3.** Denote \( \zeta = \{\zeta_j\}_1^{\infty}, \zeta_j = \sigma_j \xi_j \).

To get a contradiction, assume that (8.6) is not valid. Then there is a subsequence of vectors \( \eta_j = \{\eta_{nj}\}_1^{\infty} \subset \Upsilon(a) \) such that

\[
\lim_{n \to \infty} P(T(\eta_n + \zeta) \leq x_\alpha) \geq 1 - \alpha. \tag{A.52}
\]

Denote \( \theta_{nj} = \kappa_j \eta_{nj}, 1 \leq j < \infty \).

There exist \( \Theta = \{\theta_j\}_1^{\infty} \) and a subsequence \( n_i \to \infty \) such that \( \theta_{nj} \to \theta_j \) as \( i \to \infty \) for each \( j, 1 \leq j < \infty \).

Therefore, there are sequences \( C_k \to \infty \) and \( i_k \to \infty \) as \( k \to \infty \) such that

\[
\lim_{k \to \infty} \sum_{j < C_k} \theta_{nj}^2 = 1 \tag{A.53}
\]

and

\[
\lim_{k \to \infty} \sum_{j < C_k} (\theta_{nj} - \theta_j)^2 = 0. \tag{A.54}
\]

We consider two cases.

Case (i). The following equality holds:

\[
\lim_{k \to \infty} \sum_{j > C_k} \theta_{nj}^2 = 0.
\]

Case (ii). The following inequality holds:

\[
\sum_{j > C_k} \theta_{nj}^2 > c \quad \text{for all} \quad k > k_0.
\]

If (i) holds, then

\[
\mathbb{E} \left( \sum_{j > C_k} \kappa_j \xi_j \theta_{nj} \right)^2 = \sum_{j > C_k} \kappa_j^2 \sigma_j^2 \theta_{nj}^2 = o(1). \tag{A.55}
\]
By (A.54),
\[
E \left( \sum_{j < C_k} \kappa_j \zeta_j (\theta_{n_k,j} - \theta_j) \right)^2 = \sum_{j < C_k} \kappa_j^2 \sigma_j^2 (\theta_{n_k,j} - \eta_j)^2 = o(1). \tag{A.56}
\]

It follows from (A.55) and (A.56) that
\[
P \left( \sum_{j=1}^\infty (\kappa_j \zeta_j + \theta_{n_k,j})^2 < x_\alpha \right)
= P \left( \sum_{j < C_k} (\kappa_j \zeta_j + \theta_{n_k,j})^2 + \sum_{j > C_k} \kappa_j^2 \zeta_j^2 < x_\alpha (1 + o(1)) \right)
= P \left( \sum_{j < C_k} (\kappa_j \zeta_j + \theta_j)^2 + \sum_{j > C_k} \kappa_j^2 \zeta_j^2 < x_\alpha (1 + o(1)) \right)
< P \left( \sum_{j=1}^\infty \kappa_j^2 \zeta_j^2 < x_\alpha \right) (1 + o(1)),
\]
where the last inequality follows from Lemma A.13 given below.

**Lemma A.13.** Let \( \theta = \{\theta_j\}_1^\infty \) be such that \( \sum_{j=1}^\infty \theta_j^2 > c \). Then
\[
P \left( \sum_{j=1}^\infty (\kappa_j \zeta_j + \theta_j)^2 < x_\alpha \right) > P \left( \sum_{j=1}^\infty (\kappa_j \zeta_j + \theta_j)^2 < x_\alpha \right). \tag{A.57}
\]

**Proof.** To simplify the notation, we give the reasoning for \( \theta_1 \neq 0 \). Implementing the Anderson theorem [1], we see that
\[
P \left( \sum_{j=1}^\infty (\kappa_j \zeta_j + \theta_j)^2 < x_\alpha \right)
= (2\pi)^{-1/2} \int_{-\kappa_1^{-1}\sigma_1^{-1}\sqrt{x_\alpha - \eta_1}}^{\kappa_1^{-1}\sigma_1^{-1}\sqrt{x_\alpha - \eta_1}} \exp \left\{ -\frac{x^2}{2} \right\} \left[ P \left( \sum_{j=2}^\infty (\kappa_j \zeta_j + \theta_j)^2 < x_\alpha - (\kappa_1 \sigma_1 x + \theta_1)^2 \right) \right] dx
\leq (2\pi)^{-1/2} \int_{-\kappa_1^{-1}\sigma_1^{-1}\sqrt{x_\alpha - \eta_1}}^{\kappa_1^{-1}\sigma_1^{-1}\sqrt{x_\alpha - \eta_1}} \exp \left\{ -\frac{x^2}{2} \right\} \left( \sum_{j=2}^\infty \kappa_j^2 \zeta_j^2 < x_\alpha - (\kappa_1 \sigma_1 x + \theta_1)^2 \right) dx
= P \left( (\kappa_1 \zeta_1 + \theta_1)^2 + \sum_{j=2}^\infty \kappa_j^2 \zeta_j^2 < x_\alpha \right) < P \left( \sum_{j=1}^\infty \kappa_j^2 \zeta_j^2 < x_\alpha \right).
\]

To prove the last inequality in (A.58), it suffices to note that \( P(\kappa_1 \zeta_1^2 < x) > P((\kappa_1 \zeta_1 + \theta_1)^2 < x) \) for \( x \in (0, x_\alpha) \) and that for any \( \delta \), \( 0 < \delta < x_\alpha \), there is a \( \delta_1 > 0 \) such that the function \( P(\kappa_1 \zeta_1^2 < x) - P((\kappa_1 \zeta_1 + \theta_1)^2 < x) - \delta_1 \) is positive on the interval \( (\delta, x_\alpha) \).

Assume that (ii) holds. We also assume that \( n_{i_k} = n \), which allows us to simplify the notation. Then
\[
T(\eta_n + \zeta) = \sum_{j < C_n} (\kappa_j \zeta_j + \theta_{n,j})^2 + J_{2n}, \tag{A.59}
\]

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We note that

\[ J_{2n} = \sum_{j \geq C_n} \kappa_j^2 \zeta_j^2 + 2 \sum_{j \geq C_n} \kappa_j \zeta_j \theta_{nj} + \sum_{j \geq C_n} \theta_{nj}^2 = J_{21n} + 2J_{22n} + J_{23n}. \]  

(A.60)

We note that

\[ J_{21n} = o_P(1) \quad \text{and} \quad J_{22n} \leq J_{21n}^{1/2} J_{23n}^{1/2} = o_P(1). \]  

(A.61)

Taking into account (A.59)–(A.61) and using the Anderson theorem [1], we show that

\[ \mathbb{P}\left( \sum_{j=1}^{\infty} (\kappa_j \zeta_j + \theta_{nj})^2 < x \right) \leq \mathbb{P}\left( \sum_{j < C_n} (\kappa_j \zeta_j + \theta_{nj})^2 \leq x - c + o_P(1) \right) \]

\[ \leq \mathbb{P}\left( \sum_{j < C_n} \kappa_j^2 \zeta_j^2 \leq x - c + \delta \right) (1 + o(1)) \]

\[ \leq \mathbb{P}\left( \sum_{j=1}^{\infty} \kappa_j^2 \zeta_j^2 \leq x - c + 2\delta \right) (1 + o(1)) < \mathbb{P}\left( \sum_{j=1}^{\infty} \kappa_j^2 \zeta_j^2 \leq x \right) \]

for any \( 0 < \delta < c/2 \), where the last inequality follows from Proposition 7.1 of [26].

Proof of the version of Theorem 4.1. Let (4.3) hold. Then

\[ n \sum_{j=1}^{\infty} \frac{\theta_{nj}^2}{\pi^2 j^2} \geq n \sum_{j < c_2 k_n} \frac{\theta_{nj}^2}{\pi^2 j^2} \geq c_2^{-2} n k_n^{-2} \sum_{j < c_2 k_n} \theta_{nj}^2 \geq 1. \]

By (7.3), this implies the sufficiency.

\[ \square \]

Proof of the version of Theorem 4.2. Let (4.4) hold. Then

\[ n \sum_{j=1}^{\infty} \frac{\theta_{nj}^2}{\pi^2 j^2} = n \sum_{j < c_2 k_n} \frac{\theta_{nj}^2}{\pi^2 j^2} + n \sum_{j > c_2 k_n} \frac{\theta_{nj}^2}{\pi^2 j^2} \]

\[ \leq o(1) + (c_2 k_n)^{-2} n \sum_{j > c_2 k_n} \theta_{nj}^2 \asymp o(1) + (c_2 k_n)^{-2} n^{-1-2r} = O(c_2^{-2}). \]

(A.63)

Since \( c_2 \) is arbitrary, it follows from (7.3) that (A.63) implies the sufficiency.

\[ \square \]

Proof of Theorem 7.2. The proof of item (i) is similar to that of item (i) of Theorem 4.4. The statement follows from (4.3) and Lemma A.14 provided below.

Lemma A.14. Let \( f_n \in B_{s,2\infty}^s(c_1) \) and let \( cn^{-r} \leq \| f_n \| \leq C n^{-r} \). Then

\[ \sum_{j=1}^{k_n} \theta_{nj}^2 > \frac{c}{2} n^{-2r} \]

for \( k_n = C_1 n^{(1-2r)/2} (1 + o(1)) \) with \( C_1^2 > 2c_1/c \).

The proof of Lemma A.14 is akin to that of Lemma A.2 and is omitted.

The reasoning in the proof of item (ii) is akin to that of item (ii) of Theorem 4.4. Assume the opposite. Then there exists an \( f = \sum_{j=1}^{\infty} \tau_j \phi_j \notin B_{s,2\infty}^s \) and a sequence \( m_l, m_l \to \infty \) as \( l \to \infty \), such that (A.9) holds. Define sequences \( \eta_l, \eta_l, \) and \( \tilde{f}_l \) in the same way as in the proof of Theorem 4.4.

Then

\[ n_l \asymp C_l^{-1/(2r)} m_l^{s/r} = C_l^{-1/(2r)} m_l^{2-2r}. \]
such that for

\[ m_l \asymp C_l^{(1-2r)/(4r)} \frac{1-2r}{n_l}. \]

Hence,

\[ m_l \sum_{j=1}^{\infty} \frac{\eta_{lj}^2}{j} \leq n_l m_l^{-2} \sum_{j=m_l}^{\infty} \eta_{lj}^2 \asymp n_l^{1-2r} m_l^{-2} \asymp C_l^{2r-1} = o(1). \]  \hspace{1cm} (A.64)

By Theorem 7.1, relation (A.64) implies the inconsistency of the sequence of alternatives \( \tilde{f}_l \). \hfill \Box

Proof of Theorem 7.4. By Lemma A.9, it suffices to prove that for any \( \varepsilon > 0 \), there is an \( n_0(\varepsilon) \) such that for \( n > n_0(\varepsilon) \), the following inequality holds:

\[ |\mathbf{P}(T^2(b(F_n(t) + F_{1n}(t) - F_0(t)) + \sqrt{n}(F_n(t) + F_{1n}(t) - 2F_0(t))) > x_\alpha) - \mathbf{P}(T^2(b(F_n(t)) + \sqrt{n}(F_n(t) - F_0(t))) > x_\alpha)| < \varepsilon. \]  \hspace{1cm} (A.65)

Since \( T \) is a norm, by Lemma A.12, the proof of (A.65) is reduced to a proof of the fact that

\[ \mathbf{P}(|T(b(F_n(t) + F_{1n}(t) - F_0(t))) - T(b(F_n(t)))| > \delta_1) = o(1) \]  \hspace{1cm} (A.66)

for any \( \delta_1 > 0 \) and there is a sequence \( \delta_n, \delta_n \to 0 \) as \( n \to \infty \) such that

\[ n^{1/2} |T(F_n(t) + F_{1n}(t) - 2F_0(t)) - T(F_n(t) - F_0(t))| < \delta_n. \]  \hspace{1cm} (A.67)

Note that

\[ |T(b(F_n(t) + F_{1n}(t) - F_0(t))) - T(b(F_n(t)))| \leq T(b(F_n(t)) + F_{1n}(t) - F_0(t)) - b(F_n(t)) \]  \hspace{1cm} (A.68)

and

\[ |T(F_n(t) + F_{1n}(t) - 2F_0(t)) - T(F_n(t) - F_0(t))| \leq T(F_{1n}(t) - F_0(t)). \]  \hspace{1cm} (A.69)

By Lemma A.10,

\[ \mathbf{E} T^2(b(F_n(t) + F_{1n}(t) - F_0(t)) - b(F_n(t))) \leq T^{1/4}(F_{1n} - F_0) = o(1). \]  \hspace{1cm} (A.70)

Relations (A.68) and (A.70) imply (A.66).

Since the sequence of alternatives \( f_{1n} \) is inconsistent,

\[ nT^2(F_{1n}(t) - F_0(t)) = o(1) \]  \hspace{1cm} (A.71)

as \( n \to \infty \). Relations (A.69) and (A.71) imply (A.67). \hfill \Box

Theorem 7.1, combined with the conditions G1 and B reduces the proof of Theorem 7.3 to analysis of the sums \( \sum_{c_kn, c_kn < j < Ck_n} \theta_{lj}^2 n_j \) with \( C > c \). Such an analysis has been provided in detail in Sec. A.2 with another parameters \( r \) and \( s \). We omit the proof of Theorem 7.3.

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