Damping in 2D and 3D dilute Bose gases

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Abstract. Damping in two-dimensional (2D) and 3D dilute gases is investigated using both the hydrodynamical approach and the Hartree–Fock–Bogoliubov (HFB) approximation. We found that both methods are good for Beliaev damping at zero temperatures and Landau damping at very low temperatures. However, at high temperatures, the hydrodynamical approach overestimates the Landau damping and HFB gives a precise asymptotic behavior. This result shows that the comparison of the theoretical calculation using the hydrodynamical approach and the experimental data for high temperatures by Vincent Liu (1997 Phys. Rev. Lett. 79 4056) is not valid. For 2D systems, we show that the Beliaev damping rate is proportional to $k^3$ and the Landau damping rate is proportional to $T^2$ for low temperatures and to $T$ for high temperatures. We also show that in 2D the hydrodynamical approach gives the same result for zero temperature and for low temperature as HFB, but overestimates the Landau damping for high temperatures.
1. Introduction

The experimental realization of Bose–Einstein condensation (BEC) in magnetically trapped alkali atoms [1]–[3] provides a good tool to study the properties of the three-dimensional (3D) dilute Bose gas. Furthermore, with anisotropic traps [4, 5], one can further confine condensate atoms in a quasi-2D regime [5, 6]. Most theoretical work (for 3D and 2D systems, see reviews [7, 8], respectively) has focused on the dynamics of condensates and the zero-temperature behavior, which can be obtained by solving the nonlinear Gross–Pitaevskii (GP) equation. However, the finite-temperature behavior has still remained difficult to study, where experiments have shown damping of the condensate modes in the presence of a significant non-condensate component [9, 10]. The damping mechanism associated with collective excitations of Bose condensed atoms interacting with a non-condensed, thermal component is not well understood and remains a challenging problem in theoretical physics. The damping of collective modes can have various origins. There are two distinct contributions to the total decay rate $\gamma = \gamma_B + \gamma_L$: one arises at $T = 0$ from the process of decay of a quantum of excitation into two or more excitations with lower energy. This mechanism was first studied by Beliaev [11] in 3D uniform Bose superfluids and is known as Beliaev damping $\gamma_B$. At finite temperatures, a different mechanism of damping (known as Landau damping, $\gamma_L$) comes from the process of one quantum of excitation decaying due to coupling with transitions associated with other elementary excitations and occurring at the same frequency. Landau damping is not associated with the thermalization process and can be well described in the framework of mean field theory [12]–[14]. The subject of Landau damping in dilute BECs has been explored by several authors. Landau damping in a uniform Bose gas at low temperatures was first investigated by Popov [15] and Hohenberg and Martin [16], while at high temperatures it was first investigated by Szépfalusy and Kondor [17]. The relevance of Landau damping to explain experimental data of a trapped Bose gas was proposed by Liu and Schieve [18] and developed by Liu [19] using Popov’s hydrodynamical approach [15]. On the other hand, Pitaevskii and Stringari [12] investigated Landau damping in a weakly interacting uniform as well as a non-uniform Bose gas by means of semi-classical theory. They showed that for the uniform Bose gas, it reproduces known results for the low-temperature asymptotic behavior of the phonon coupling. However, for high temperatures, Liu
obtained a higher Landau damping rate than those obtained by Szépfalusy and Kondor, while the high-temperature behavior was reproduced by Pitaevskii and Stringari.

However, all investigations of the damping rate have been done for a 3D Bose gas. Two-dimensional Bose gases are interesting as their low temperatures physics is governed by strong long-range fluctuations. These fluctuations inhibit the formation of true long-range order, which is a key concept of phase transition theory in 3D. Thus a 2D uniform interacting Bose gas does not undergo BEC at finite temperature. However, this system turns superfluid below the BKT (Berezinski, Kosterlitz and Thouless) temperature $T_{KT}$ [20, 21]. The experimental indication of the BKT transition in a weakly interacting Bose system has been shown in [6]. Damping in a 2D Bose gas is an open question which was recently addressed by the authors for a uniform Bose gas [23] using the hydrodynamical theory of Popov [15]. In this work, we show that the hydrodynamical approach actually overestimates the damping rate at high temperatures, both for 3D and 2D systems, and we calculate the Beliaev and Landau damping rates for a 2D uniform Bose gas using the semi-classical Hartree–Fock–Bogoliubov (HFB) approach. In the limit of low temperatures, the results of this approach are in good agreement with that found from the hydrodynamical approach. In contrast to earlier work [19], we show that for the 3D case in the high temperature limit, the hydrodynamical approach cannot be used to explain the experimental data.

This paper is organized as follows. In section 2, we discuss the relation between the atom–atom interaction and the scattering length for 2D and 3D dilute gases. In section 3, we first introduce the hydrodynamical approach developed by Popov [15], and then calculate the Beliaev damping and Landau damping for 3D and 2D gases. The drawback of this approach for high temperatures is also discussed. In section 4, the HFB approximation is developed to calculate 3D and 2D Beliaev and Landau damping.

2. Atom–atom interaction and scattering length

The standard Hamiltonian of an interacting Bose gas is

$$H = \int d^3r \left[ -\frac{1}{2} \nabla \psi^\dagger(\mathbf{r}) \nabla \psi(\mathbf{r}) + V_{\text{ext}}(\mathbf{r}) \psi^\dagger(\mathbf{r}) \psi(\mathbf{r}) \right] + \frac{1}{2} \int d^3r d^3r' \psi^\dagger(\mathbf{r}) \psi^\dagger(\mathbf{r}') U(\mathbf{r} - \mathbf{r}') \psi(\mathbf{r}) \psi(\mathbf{r}') , \quad (1)$$

where $U$ is the atom–atom interaction and $V_{\text{ext}}$ is the external potential. For a uniform Bose gas, $V_{\text{ext}} = 0$. The true interaction between atoms is very complicated where one has to consider the fine structure of atoms. However, the scattering process can offer an effective potential to simplify the interaction. In order to do that, one has to introduce Green functions for bosonic systems with condensate. The difficulty of doing so arises from the fact that the terms containing an odd number of annihilation operators do not vanish for a Bose gas after averaging the ground state due to the existence of condensate, which unfortunately destroys the hope of applying the normal technique of Feynman diagrams to the system. This difficulty was successfully resolved by Beliaev [11, 24]. He separated the operators with zero momentum, which semi-classically can be regarded as a c-number, and the other operators with nonzero momenta. In this way, Feynman diagrams can be used for the Bose gas.

Beliaev considered a 3D system with short-range, central interaction potential with radius $1/a_3$ and then calculated the renormalized atom–atom interaction in the presence of
a condensate between two particles with nonzero momenta, which one should sum over all ladder diagrams. In this way, one can obtain the renormalized interaction in terms of the s-wave scattering amplitude according to elementary scattering theory [7, 24, 25]. Therefore, the effective potential can be written as

\[ U(r) = \frac{4\pi a_3}{m} \delta(r), \]  

(2)

with atom mass \( m \), and the momentum dependence of the scattering amplitude can be ignored in the low-temperature limit (note that \( \hbar = 1 \) throughout the paper). In the rest of the paper we define the atom–atom interaction strength \( g_3 \) as

\[ g_3 = \frac{4\pi a_3}{m}. \]  

(3)

For 2D, Schick followed the methods developed by Beliaev and examined a 2D system of hard-core bosons with a diameter \( a_2 \) at low density and zero temperature. In 3D systems, where ladder diagrams are independent of the dimensionless parameter \( na_3^2 \), it is natural to take it as a small perturbation to expand the quantities. For 2D systems, contributions from the ladder diagrams depend logarithmically on \( na_2^2 \), the dimensionless parameter for 2D systems, but not directly on \( na_2^2 \) itself. In particular, the renormalized interaction is proportional to \( 1/\ln(1/na_2^2) \):

\[ g_2 = \frac{4\pi}{m \ln(1/na_2^2)}. \]  

(4)

Schick concluded that the \( 1/\ln(1/na_2^2) \) plays a role of a small parameter in 2D dilute systems, and other quantities, like damping rate in this paper, can be expanded in terms of it.

3. Hydrodynamical approach

In the low-temperature and low-energy limit, Popov [15] developed a hydrodynamical approach to find an effective Hamiltonian for a nonideal Bose gas. In order to do that, one has to separate the order parameter over rapidly and slowly oscillating field, and the hydrodynamical Hamiltonian can be obtained by integrating the functional over a rapidly oscillating field. The theory describes then the hydrodynamical Hamiltonian in terms of two slowly varying fields: phase \( \phi(x) \) and density fluctuation \( \pi(x) = n(x) - n_0 \) with \( n_0 \); the density of the ground state. Here the four-dimension Euclidean space \( x = (x, \tau) \) is used with the imaginary time \( \tau \). The hydrodynamical action for a \( d \)-dimensional nonideal Bose gas, according to Popov, can be written in the form

\[ S[\phi, \pi] = \int_0^\beta d\tau \int d^d x \left\{ \frac{i}{\hbar} \frac{\partial^2 p}{\partial \mu \partial n} \frac{1}{2m} \frac{\partial p}{\partial \mu} (\nabla \phi)^2 - \frac{1}{2} \frac{\partial^2 p}{\partial \mu^2} (\nabla \phi)^2 - \frac{1}{2} \frac{\partial^2 p}{\partial n^2} \pi^2 + \frac{1}{2} \frac{\partial^2 p}{\partial n^2} \pi^2 - \frac{(\nabla \pi)^2}{8mn_0} - \frac{\pi (\nabla \phi)^2}{2m} \right\} \]  

(5)

with atom mass \( m \), pressure of a homogeneous system \( p \), chemical potential \( \mu \) and atom density \( n \). The fields \( \phi \) and \( \pi \) are periodic in imaginary time \( \tau \) with period \( \beta = 1/(k_B T) \). For very low temperatures, as long as the non-condensate part can be neglected compared to the condensate, the pressure \( p(\mu, n) \) at zero temperature is a good approximation. Therefore, we can use the
expression of a weakly interacting dilute gas as \( p = \mu n - \frac{g_d}{2} n_0^2 \), where \( g_d \) is the atom–atom interaction related to the scattering length. It follows that

\[
\frac{\partial^2 p}{\partial \mu \partial n} = 1; \quad \frac{\partial p}{\partial \mu} = n \approx n_0; \quad \frac{\partial^2 p}{\partial \mu^2} = 0; \quad \frac{\partial^2 p}{\partial n^2} = -g_d,
\]

and the action (5) takes the form

\[
\int \mathrm{d} \tau \, \mathrm{d}^d x \left( i \pi \partial_\tau \phi - \frac{(n_0 + \pi)}{2m} (\nabla \phi)^2 - \frac{g_d}{2} \pi^2 - \frac{(\nabla \pi)^2}{8mn_0} \right).
\]

The action contains all quadratic functions except the term \( 1/2m \pi (\nabla \phi)^2 \), considered as an interacting potential. The Hamiltonian can be derived from the effective action (7) as

\[
\int \mathrm{d}^d x \left( \frac{m}{2} \nabla^2 + \frac{g_d}{2} (n - n_0)^2 + \frac{(\nabla n)^2}{8m n_0} \right),
\]

where the velocity field is defined as \( v = 1/m \nabla \phi \). This Hamiltonian is consistent with a particular realization of the Landau hydrodynamical Hamiltonian.

Fourier transforming the fields \( \phi \) and \( \pi \), the effective action (8) can be written in the form

\[
-\frac{1}{2} \sum \int \mathrm{d}^d k \, \frac{n_0}{m} k^2 \phi(k) \phi(-k) + 2 \omega_v \phi(k) \pi(-k) + \left( g_d + \frac{k^2}{4mn_0} \right) \pi(k) \pi(-k)
\]

\[
-\frac{1}{\sqrt{\beta}} \sum_{k_1+k_2+k_3=0} \frac{k_1 \cdot k_2}{2m} \phi(k_1) \phi(k_2) \pi(k_3),
\]

where \( k \) is the vector \((k, i \omega_v)\) and the Matsubara frequencies \( \omega_v = 2 \pi v / \beta \) with integers \( v \). From the action of the Fourier transformation (9) one can extract the important information needed for the perturbation calculations using a diagrammatic technique. First of all, the free Green’s function is defined as follows:

\[
G_0(k) = \begin{pmatrix}
\langle \phi(k) \phi(-k) \rangle_0 & \langle \phi(k) \pi(-k) \rangle_0 \\
\langle \pi(k) \phi(-k) \rangle_0 & \langle \pi(k) \pi(-k) \rangle_0
\end{pmatrix},
\]

where \( \langle \cdots \rangle_0 \) denotes the expectation value of fields calculated only with the quadratic action. From the action (9), the inverse of the free Green’s function can be found as

\[
G_0^{-1}(k) = \begin{pmatrix}
\frac{n_0}{m} k^2 & \omega_v \\
-\omega_v & g_d + \frac{k^2}{4mn_0}
\end{pmatrix}.
\]

Therefore,

\[
G_0(k) = \begin{pmatrix}
g_d + \frac{k^2}{4mn_0} & -\omega_v \\
\omega_v & \omega_v^2 + \epsilon^2(k)
\end{pmatrix}\begin{pmatrix}
\omega_v & \omega_v^2 + \epsilon^2(k) \\
\omega_v^2 + \epsilon^2(k) & (n_0/m) k^2
\end{pmatrix},
\]

where

\[
\epsilon(k) = \sqrt{\frac{k^2}{2m}} + c^2 k^2,
\]

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with \( c \equiv \sqrt{g_d n_0 / m} \). We represent the relation between the free Green’s functions and the Feynman diagrams in figure 1. The cubic term of the action (9) is known as phonon–phonon interaction in the low temperatures region, giving rise to a vertex of \( \delta^d(k_1 + k_2 + k_3)\delta_{\nu_1+\nu_2+\nu_3,0}(\zeta_1\cdot\zeta_2) \) represented by the last diagram of figure 1.

The exact Green’s function has to involve the phonon–phonon interaction, given by the Dyson equation \( G(k) = G_0(k) + G_0(k)\Sigma(k)G(k) \), where \( \Sigma(k) \) represents the self-energy matrix. The low-frequency spectrum of collective modes can be obtained by the poles of the exact Green’s function as

\[
\det G^{-1}(k) = \det[G_0^{-1}(k) - \Sigma(k)] = 0, \tag{14}
\]

through the analytical continuation \( i\omega_n = \omega + i\eta \) (\( \eta = 0^+ \)) after the Matsubara frequency sum. The complex frequency \( \omega = E - i\gamma(k) \) represents the energy spectrum \( E \) and the damping rate \( \gamma \). Neglecting the matrix \( \Sigma \), the zero-order approximation for equations (14) gives the square of the Bogoliubov energy spectrum:

\[
E^2 = \left( \frac{k^2}{2m} \right)^2 + c^2 k^2 = \epsilon(k)^2. \tag{15}
\]

When the phonon–phonon interaction is considered, the imaginary part appears in the spectrum. Figure 2 shows the one-loop diagrams for the self-energy \( \Sigma \). The contribution to the imaginary part of the spectrum is given by the last five figures (b)–(f). The damping rate can be obtained
from equations (11) and (14) [19] as
\[
\gamma(k) = \frac{1}{2E} \left[ g_d + \frac{k^2}{4m\gamma} \right] \text{Im} \Sigma_{\phi\phi}(k, \omega + i\eta) + \frac{n_0k^2}{m} \text{Im} \Sigma_{\pi\pi}(k, \omega + i\eta) - \text{Re} \Sigma_{\phi\pi}(k, \omega + i\eta). \tag{16}
\]

Inserting (15) into equation (16) and calculating the diagrams, we obtain \(\gamma(k) = \gamma_B(k) + \gamma_L(k)\), where \(\gamma_B\) is the Beliaev damping rate given as
\[
\gamma_B(k) = \frac{1}{2d+2\pi d-1} \int d^d k' \delta(\varepsilon(k) - \varepsilon(k') - \varepsilon(k - k')) \left[ f^0(\varepsilon(k')) - f^0(-\varepsilon(k - k')) \right] \\
\times \left\{ \frac{(k - k')^2(k \cdot k')^2\varepsilon(k')\varepsilon(k)}{2mn_0k^2k^2\varepsilon(k - k')} + \frac{(k \cdot k')(k \cdot (k - k'))\varepsilon(k)}{2mn_0k^2} \\
+ \frac{k^2(k' \cdot (k - k'))^2\varepsilon(k')\varepsilon(k - k')}{4mn_0k^2(k - k')^2\varepsilon(k)} + \frac{(k' \cdot (k - k'))(k' \cdot k')\varepsilon(k')}{mn_0k^2} \right\}, \tag{17}
\]
and \(\gamma_L\) is known as Landau damping and is given as
\[
\gamma_L(k) = \frac{1}{2d+2\pi d-1} \int d^d k' \delta(\varepsilon(k) + \varepsilon(k') - \varepsilon(k + k')) \left[ f^0(\varepsilon(k')) - f^0(\varepsilon(k + k')) \right] \\
\times \left\{ \frac{\varepsilon(k)}{2mn_0} \left[ \frac{k^2(k \cdot (k + k'))\varepsilon(k + k')}{k^2(k + k')^2\varepsilon(k')} + \frac{(k + k')^2(k' \cdot k')^2\varepsilon(k')}{k^2k^2\varepsilon(k)} \right] \\
+ \frac{(k \cdot k')(k \cdot (k + k'))\varepsilon(k)}{mn_0k^2} + \frac{k^2(k' \cdot (k + k'))^2\varepsilon(k')\varepsilon(k + k')}{2mn_0k^2(k + k')^2\varepsilon(k)} \\
+ \frac{(k' \cdot (k + k'))}{mn_0} \left[ \frac{(k \cdot k')\varepsilon(k')}{k^2} + \frac{(k \cdot (k + k'))\varepsilon(k + k')}{(k + k')^2} \right] \right\}, \tag{18}
\]
where the bosonic distribution function \(f^0(\varepsilon) = 1/\exp(\beta\varepsilon) - 1\) with \(\beta = 1/k_B T\).

### 3.1. Quantum regime \(ck \gg k_B T\)

At \(T = 0\), the Landau damping disappears and the Beliaev damping contributes to the damping rate. In the Beliaev damping mechanism the momenta of the three excitations are comparable, \(|k| \simeq |k'| \simeq |k - k'|\). Then equation (17) yields
\[
\gamma_B(k) = \frac{9c}{2d+2\pi d-1m\gamma} \int d^d k ||k'|||k - k'||\delta(\varepsilon(k) - \varepsilon(k') - \varepsilon(k - k')). \tag{19}
\]
In 3D the damping rate for small \(k (k \ll mc)\) is
\[
\gamma_B^{d=3}(k) = \frac{3k^5}{640\pi mn_0}, \tag{20}
\]
known as Beliaev’s result [11].

For 2D systems, equation (19) can be written as
\[
\gamma_B^{d=2}(k) = 2 \left[ \frac{9}{64\pi mn_0} \int d^d k' \frac{|k'|(|k - k'|)^2}{\sin \theta} \right], \tag{21}
\]

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where $\theta$ is the angle between $k$ and $k'$. The factor two in front of the bracket comes from the fact that there are two angles corresponding to the energy conservation for Beliaev damping ($\epsilon(k) - \epsilon(k') = \epsilon(k - k')$): $\sin \theta \simeq \pm \frac{\sqrt{3} |k - k'|}{2mc}$, and the Beliaev damping rate for a 2D Bose gas has the form

$$\gamma_B^{d=2} = \frac{\sqrt{3}c}{2\pi n_0} k^3. \quad (22)$$

This result corrects that previously given by Chung and Bhattacherjee [22] and the factor of two will appear naturally in the 2D Landau damping.

### 3.2. Thermal regime $ck \ll k_BT$

For finite temperatures and small momenta such that $ck \ll k_BT$ and $ck \ll n_0 g_d$, Beliaev damping is much smaller than Landau damping. In 3D, the damping rate to the lowest order in $k$ can be obtained from equation (18) as

$$\frac{\gamma_L^{d=3}(k)}{\epsilon(k)} = \frac{k_0^3}{16\pi mn_0 k_B T} I_3(\tau), \quad (23)$$

where

$$I_3(\tau) = \frac{1}{4} \int_0^{\infty} dz z^2 \sech^2 \left( \frac{z}{2\tau} \right) \left[ \frac{1}{2} + \frac{3}{2(z^2 + 1)} + \frac{2z^2}{(z^2 + 1)^2} - \frac{2}{(z^2 + 1)^2} \right], \quad (24)$$

with $k_0 = \sqrt{mn_0 g_3}$ and $\tau = k_B T/n_0 g_3$. This result was first obtained by Liu [19]. For $k_B T \ll mc^2$, equation (24) reduces to Hohenberg and Martin’s result [16]

$$\gamma_L^{d=3}(k) = \frac{3\pi^3 k (k_B T)^4}{40 mn_0 c^4}. \quad (25)$$

This low-temperature limit gives the same result as that using the HFB approach, which will be introduced in the next section. For high temperatures $k_B T \gg mc^2$, $I_3(\tau) \sim 38.735 \tau$, and the damping rate is approximated by

$$\frac{\gamma_L^{d=3}(k)}{\epsilon(k)} \simeq 9.648 \frac{k_B T a_3}{c}. \quad (26)$$

with the 3D scattering length $a_3 = mg_3/4\pi$. Unfortunately, this result is different from that investigated by Szépfalusy and Kondor [17], which reads

$$\frac{\gamma_L^{d=3}(k)}{\epsilon(k)} \simeq \frac{3\pi k_B T a_3}{8 \frac{c}{c}}. \quad (27)$$

We see that the Landau damping (26) at high $T$ obtained from equation (24) is 8.2 of magnitude larger than the high-temperature asymptotic value (27). Therefore, the hydrodynamical approach is no longer correct for high temperatures. The reason is that in the hydrodynamical Hamiltonian only the slowly oscillating fields are considered by integrating out the fast oscillating fields. For high temperatures the fast oscillating fields should also be considered to reduce the damping rate. We can conclude that the hydrodynamical approach is very good for low temperatures. However, for high temperatures, other methods should be developed, for example the HFB approach. We will see that HFB will give the same asymptotic behavior as equation (24). We will discuss that in the next section.

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3 We thank Professor Semenov for pointing out the factor of two in the 2D result.
In figure 3, the 3D Landau damping per unit energy using the hydrodynamical approach (dashed line) and HFB (solid line) is plotted as a function of $\tau$. Also shown are the asymptotic behavior at high temperatures (the dashed-dot line) and the low-temperature limit (the dashed-dot-dot line). We can see that the hydrodynamical approach gives very good agreement with the low-temperature limit for $\tau \leq 0.5$. However, it becomes too large at high temperatures and it does not approach the asymptotic value given by Szépfalusy and Kondor. Therefore, the conclusion made by Liu [19] that the results of the hydrodynamical approach can fit the experimental data is not valid.

In 2D, the damping rate reads
\[
\frac{\gamma_d^{2D}(k)}{\epsilon(k)} = \frac{\sqrt{2} k_0^4}{16 m n_0 k_B T} I_2(\tau),
\]
where
\[
I_2(\tau) = \frac{1}{4} \int_0^\infty dz z^2 \frac{\text{sech}^2(\frac{z}{\tau})}{\sqrt{z^2 + 1} - 1 - \frac{z^2}{2(z^2 - 1)}} \left[ \frac{1}{2} + \frac{3}{2(z^2 + 1)} + \frac{2z^2}{(z^2 + 1)^2} - \frac{2}{(z^2 + 1)^2} \right].
\]

In the low-temperature limit: $k_B T \ll mc^2$, $I_2(\tau) \rightarrow \sqrt{6} \pi^2 \tau^2$; therefore the damping coefficient is given by
\[
\frac{\gamma_d^{2D}(k)}{\epsilon(k)} = \frac{\sqrt{3} \pi (k_B T)^2}{8 n_0 c^2}.
\]

In this low-temperature regime, the damping rate is proportional to $T^2$. As far as we know, this quadratic dependence of the temperature for the damping rate in the low-temperature regime has been found for the first time in this paper.

In the high-temperature regime, as in the 3D case, the hydrodynamic Hamiltonian overestimates the damping. In the next section, we will use the HFB approximation to obtain the 2D damping at high temperatures.

**Figure 3.** Landau damping rate per unit energy versus $\tau$ in 3D. The unit of $\gamma_L / \epsilon(k)$ is $\sqrt{a^2 n_o}$. The solid black line represents the result obtained by the HFB method and the dashed blue line represents the hydrodynamical approach. The high-temperature asymptotic behavior and low-temperature limit are also shown by the dashed-dot green line and dashed-dot-dot red line, respectively.
Figure 4. Landau damping rate per unit energy in 2D in units of $8\pi/|\ln n a^2|$. The black solid line represents the result from HFB and the blue dashed line is that from the hydrodynamical approach. The high-temperature asymptotic behavior and the low-temperature limit are also reported as the green dashed-dot line and the red dashed-dot-dot line, respectively.

Figure 4 shows the 2D Landau damping rate per unit energy using both hydrodynamical approach (dashed line) and HFB method (solid line). In this figure, the low-temperature limit (dashed-dot-dot line) and the asymptotic value at the high-temperature limit (dashed-dot line) are also shown. The hydrodynamical result is in agreement with the low-temperature limit for $\tau < 0.2$; however, it does not approach the asymptotic value for high temperatures, similar to the 3D case.

4. HFB approach

In this section, we represent a semi-classical method: HFB. We will see that in the low-temperature regime this approach is in good agreement with the hydrodynamical approach. For high temperatures, HFB offers a precise asymptotic behavior for the decay rate in contrast to the hydrodynamic approach.

We start with the method of Giorgini [13]. The grand-canonical Hamiltonian of a system with a non-uniform external field $V_{\text{ext}}(\mathbf{r})$ reads

$$K = H - \mu N = \int d^d r \psi^\dagger(\mathbf{r}, t) H_0 \psi(\mathbf{r}, t) + \frac{g_d}{2} \int d^d r \psi^\dagger(\mathbf{r}, t) \psi^\dagger(\mathbf{r}, t) \psi(\mathbf{r}, t) \psi(\mathbf{r}, t),$$

(31)

where

$$H_0 = -\frac{\nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) - \mu$$

(32)

and $\psi^\dagger(\mathbf{r}, t)$ and $\psi(\mathbf{r}, t)$ are the creation and annihilation field operators. Since the system is in the regime where the condensate exists, we define a time-dependent condensate wave function $\Phi(\mathbf{r}, t)$ [16]

$$\Phi(\mathbf{r}, t) = \langle \psi(\mathbf{r}, t) \rangle$$

(33)

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with the average \( \langle \cdots \rangle \) using the grand-canonical Hamiltonian (31). We note that the equation (33) can always be used if the condensate exists. However, for a homogeneous 2D system, i.e. \( V_{\text{ext}}(r) = 0 \), the condensate does not exist at finite temperatures. In the 2D case, the long-range order disappears, and a quasi-long-range order appears. That means that, although a macroscopic order number of a single state does not exist, there exists a small value of \( k_c \) in the momentum space where a macroscopic order number of the states \( k < k_c \) still forms a quasi-condensate. Therefore, the bracket in equation (33) should count all the states in the quasi-condensate. We can see that \( \Phi(r, t) \) allows us to describe the oscillating condensate away from equilibrium. To avoid confusion for the notation, we define here the stationary value of the condensate in its equilibrium state as \( \Phi_0(r) \)

\[
\Phi_0(r) = \langle \psi(r) \rangle_0, \tag{34}
\]

where \( \langle \cdots \rangle_0 \) denotes the time-independent average of the condensate in its equilibrium. The particle field can be decomposed into a condensate and a non-condensate component

\[
\psi(r, t) = \Phi(r, t) + \tilde{\psi}(r, t). \tag{35}
\]

By the definition of the condensate (33), the non-condensate component has to satisfy the condition:

\[
\langle \tilde{\psi}(r, t) \rangle = 0. \tag{36}
\]

By applying the decomposition (34) to the grand-canonical Hamiltonian, it can be separated to a quadratic and a quartic term: \( K = K_2 + K_4 \), where

\[
K_2 = \int d^d r \left( \tilde{\psi}^+(r, t) H_0 \tilde{\psi}(r, t) + 2 g_d \Phi(r, t) |2 \tilde{\psi}^+(r, t) \tilde{\psi}(r, t) + \frac{g_d}{2} \Phi^2 \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) \right) + \frac{g_d}{2} \Phi r^2 \tilde{\psi}(r, t) \tilde{\psi}(r, t) \tag{37}
\]

and

\[
K_4 = \frac{g_d}{2} \int d^d r \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}(r, t) \tilde{\psi}(r, t). \tag{38}
\]

We are interested in the regime where the condensate slightly differs from the equilibrium state, that is,

\[
\Phi(r, t) = \Phi_0(r) + \delta \Phi(r, t) \tag{39}
\]

with a small fluctuation \( \delta \Phi(r, t) \). Expanding \( K_2 \) up to the linear term in \( \delta \Phi(r, t) \), we can rewrite it as \( K_2 = K_2^{(0)} + K_2^{(1)} \), where \( K_2^{(0)} \) is the zero-order term of the condensate

\[
K_2^{(0)} = \int d^d r \tilde{\psi}^+(r, t) (H_0 + 2 g_d n_0(r)) \tilde{\psi}(r, t) + \frac{g_d}{2} n_0(r) (\tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) + \tilde{\psi}(r, t) \tilde{\psi}(r, t)) \tag{40}
\]

with the condensate density \( n_0(r) = |\Phi_0(r)|^2 \), while \( K_2^{(1)} \) is the linear term in the fluctuation:

\[
K_2^{(1)} = \int d^d r 2 g_d \Phi_0(r) [\delta \Phi_0(r, t) + \delta \Phi_0^*(r, t)] \tilde{\psi}^+(r, t) \tilde{\psi}(r, t) + g_d \Phi_0(r) \times [\delta \Phi_0(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}(r, t) + \delta \Phi_0^*(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}(r, t)]. \tag{41}
\]

As for the quartic term \( K_4 \), the mean-field decomposition is first used

\[
\tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) = 4 \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) + \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t) \tilde{\psi}^+(r, t), \tag{42}
\]

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where
\begin{align}
\tilde{n}(\mathbf{r}, t) &= \langle \tilde{\psi}^\dagger(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) \rangle, \\
\tilde{m}(\mathbf{r}, t) &= \langle \tilde{\psi}(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) \rangle
\end{align}
are the normal and anomalous time-dependent densities, respectively. Under the linearization in the fluctuation (39), the normal and anomalous density are also displaced with a small fluctuation as
\begin{align}
\tilde{n}(\mathbf{r}, t) &= \tilde{n}^0(\mathbf{r}) + \delta \tilde{n}(\mathbf{r}, t), \\
\tilde{m}(\mathbf{r}, t) &= \tilde{m}^0(\mathbf{r}) + \delta \tilde{m}(\mathbf{r}, t),
\end{align}
and expanding around their stationary values \( \tilde{n}^0(\mathbf{r}) = \langle \tilde{\psi}^\dagger(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) \rangle_0 \) and \( \tilde{m}^0(\mathbf{r}) = \langle \tilde{\psi}(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) \rangle_0 \). In the literature, often referred to the so-called Popov approximation
\begin{equation}
\tilde{n}^0(\mathbf{r}) = 0
\end{equation}
has been used in the mean-field treatment. Actually, this approximation was never suggested by Popov, as indicated by Yukalov [26], but was first proposed by Shohno [27] and we will refer to it as the Shohno approximation or Shohno ansatz in the remainder of this paper. The Shohno approximation is necessary in the treatment because without this approximation the elementary excitation would have a gap, which disobeys the gapless spectrum of the Goldstone modes caused by the continuous gauge symmetry breaking in the ground state. However, use of the Shohno approximation is still under debate. Several attempts have been made to go beyond the Shohno approximation (for example, see [26, 28, 29]). The Shohno approximation is needed in the mean-field factorization (42). By avoiding factorization, for example, using the perturbation or random-phase approximation (RPA) to calculate the quartic terms, the gapless dispersion can be obtained even without the Popov approximation.

Inserting equations (39), (42), (44) and the Shohno ansatz (45) into (38), the quartic term \( K_4 \) can be expanded up to the first-order terms \( K_4 = K_4^{(0)} + K_4^{(1)} \) in fluctuations \( \delta \tilde{n} \) and \( \delta \tilde{m} \) as
\begin{equation}
K_4^{(0)} = 2g_d \int d^4r \tilde{n}^0(\mathbf{r}) \tilde{\psi}^\dagger(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t)
\end{equation}
is the zero-order term, which represents the coupling to the condensate from the quartic term, while the first-order term reads
\begin{equation}
K_4^{(1)} = \frac{g_d}{2} \int d^4r (4 \delta \tilde{n}(\mathbf{r}, t) \tilde{\psi}^\dagger(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) + \delta \tilde{m}(\mathbf{r}, t) \tilde{\psi}^\dagger(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) + \delta \tilde{m}^*(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t) \tilde{\psi}(\mathbf{r}, t)).
\end{equation}

Unlike the \( K_2^{(1)} \) term which represents the coupling between the fluctuations of the condensate and the non-condensate particles, the \( K_4^{(1)} \) term is related to the coupling of the non-condensate particles to the normal and anomalous densities. If the density of the non-condensate particles is much smaller than the density of the condensate, \( K_2^{(1)} \) is more important than \( K_4^{(1)} \), therefore \( K_4^{(1)} \) can be neglected and \( K = K_2^{(0)} + K_4^{(0)} + K_2^{(1)} \).

In the case of a large occupation number of particles in the condensate, \( K_2^{(1)} \) is much smaller than \( K_2^{(0)} \) and \( K_4^{(0)} \), and we can use \( K_2^{(0)} + K_4^{(0)} \) as basis to develop a perturbation expansion in terms of \( K_2^{(1)} \). To diagonalize \( K_2^{(0)} + K_4^{(0)} \), one can apply a Bogoliubov transformation
\begin{align}
\tilde{\psi}(\mathbf{r}, t) &= \sum_j u_j(\mathbf{r}) \alpha_j(t) + v_j^*(\mathbf{r}) \alpha_j^\dagger(t), \\
\tilde{\psi}^\dagger(\mathbf{r}, t) &= \sum_j u_j^*(\mathbf{r}) \alpha_j^\dagger(t) + v_j(\mathbf{r}) \alpha_j(t),
\end{align}

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with the quasi-particle creation and annihilation operators $\alpha^\dagger_i, \alpha_j$ obeying the Bose commutation relations $[\alpha_i, \alpha_j^\dagger] = \delta_{ij}$, which gives the normalization condition for the functions $u_j(r, t), v_j(r, t)$ as

$$\int d^d r \left[ u^*_j(r) u_j(r) - v^*_j(r) v_j(r) \right] = \delta_{ij}. \quad (49)$$

Therefore, the operator $K_2^{(0)} + K_4^{(0)}$ can be diagonalized if the Bogoliubov–de Gennes equations are satisfied:

$$\mathcal{L}u_j(r) + gn_0(r) v_j(r) = \epsilon_j u_j(r),$$
$$\mathcal{L}v_j(r) + gn_0(r) u_j(r) = -\epsilon_j v_j(r), \quad (50)$$

where a Hermitian operator is introduced as

$$\mathcal{L} = H_0 + 2g_d n(r) \quad (51)$$

with the total density $n(r)$ defined as the sum of the condensate density and normal density in the equilibrium: $n(r) = n_0(r) + n^0(r)$. As a result, the grand-canonical Hamiltonian (31) becomes

$$K = K_2 + K_4 = \sum_j \epsilon_j \alpha_j^\dagger \alpha + K_2^{(1)} \quad (52)$$

with the eigenvalues $\epsilon_j$ obtained from the Bogoliubov–de Gennes equations (50).

In order to obtain the decay rate, we have to find the time evolution of the fluctuation of the condensate: $\delta \Phi(r, t)$. The equation of motion:

$$i \frac{\partial}{\partial t} \psi(r, t) = [\psi(r, t), K] \quad (53)$$

leads to the result

$$i \frac{\partial}{\partial t} \psi(r, t) = H_0 \psi(r, t) + g_d \psi^\dagger(r, t) \psi(r, t) \psi(r, t). \quad (54)$$

Inserting the decomposition (35) into the equation of motion (54), it reads

$$i \frac{\partial}{\partial t} \Phi(r, t) = \left( H_0 + g_d |\Phi(r, t)|^2 \right) \Phi(r, t) + g_d \Phi^\dagger(r, t) \tilde{n}(r, t) + g_d \Phi(r, t) \tilde{n}(r, t). \quad (55)$$

Here, we assume that the cubic product of the non-condensate contributes very little to the dynamics of the condensate, therefore the average value is set equal to zero: $\langle \tilde{\psi}^\dagger \tilde{\psi} \tilde{\psi} \rangle = 0$. The wavefunction in the equilibrium can be obtained by setting $\partial \Phi_0/\partial t = 0$, which leads to the stationary equation

$$(H_0 + g_d [n_0(r) + 2\tilde{n}^0(r)]) \Phi_0(r) = 0. \quad (56)$$

By inserting equation (39) and the stationary equation (56) into the equation of motion for the condensate (55), the equation of motion for the small amplitude $\delta \Phi$ reads

$$i \frac{\partial}{\partial t} \delta \Phi(r, t) = (H_0 + 2g_d n(r)) \delta \Phi(r, t) + g_d n_0(r) \delta \Phi^\dagger(r, t) + 2g_d \Phi_0 \delta \tilde{n}(r, t) + g_d \Phi_0 \delta \tilde{m}(r, t). \quad (57)$$

Applying the Bogoliubov transformation (48) to equation (57), equation (57) gives the final form:

$$i \frac{\partial}{\partial t} \delta \Phi(r, t) = \left( H_0 + 2g_d n(r) \right) \delta \Phi(r, t) + g_d n_0(r) \delta \Phi^\dagger(r, t) + g_d \Phi_0(r) \sum_{ij} \left\{ 2[u^*_i u_j + v^*_i v_j + v^*_j u_j] \right\} \times f_{ij}(t) + [2v_i u_j + u_i u_j]g_{ij}(t) + [2v^*_i u^*_j + u^*_i u^*_j]g_{ij}^*(t), \quad (58)$$
where \( f_{ij}(t) \equiv \langle \alpha_i(t) \alpha_j(t) \rangle \) and \( g_{ij} \equiv \langle \alpha_i(t) \alpha_j(t) \rangle \) are normal and anomalous quasi-particle distribution functions.

To calculate the normal and anomalous quasi-particle distribution functions using the perturbation Hamiltonian (52), we have to use the equation of motion:

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} f_{ij}(t) = \left\{ [\alpha_i(t), K] \right\}.
\]

(59)

\[
\frac{i}{\hbar} \frac{\partial}{\partial t} g_{ij}(t) = \{ [\alpha_i(t), K] \}.
\]

(60)

To first order, the Fourier transform of \( f_{ij} \) and \( g_{ij} \) at the frequency \( \omega \) is given by

\[
f_{ij}(\omega) = 2 g_d \frac{f_{ij}^0 - f_{ij}^0}{\omega + (\epsilon_i - \epsilon_j) + i0^+} \int d^3r \Phi_0 \left[ \delta \Phi_1 \left( r, \omega \right) \right.
\]

\[
\times (u_i u_j^* + v_i v_j^*) \left. \right] ;
\]

(61)

\[
g_{ij}(\omega) = 2 g_d \frac{1 + f_{ij}^0 + f_{ij}^0}{\omega - (\epsilon_i + \epsilon_j) + i0^+} \int d^3r \Phi_0 \left[ \delta \Phi_1 \left( r, \omega \right) \right.
\]

\[
\times (u_i^* u_j^* + v_i^* v_j^*) \left. \right] ,
\]

(62)

where \( \delta \Phi_1 \left( r, \omega \right) \) and \( \delta \Phi_2 \left( r, \omega \right) \) are the Fourier transform of \( \delta \Phi \left( r, t \right) \) and \( \delta \Phi^* \left( r, t \right) \):

\[
\delta \Phi_1 \left( r, \omega \right) = \int dt e^{-i\omega t} \Phi \left( r, t \right),
\]

(63)

\[
\delta \Phi_2 \left( r, \omega \right) = \int dt e^{-i\omega t} \Phi^* \left( r, t \right),
\]

(64)

and \( f_{ij}^0 \) is the bosonic distribution function

\[
f_{ij}^0 = \frac{1}{e^{\hbar \epsilon_i} - 1}.
\]

(65)

Fourier transforming the equation of motion (58) and inserting equations (61) and (62) into it, we obtain the perturbed eigenfrequency:

\[
\omega = \omega_0 + 4g_d^2 \sum_{ij}(f_{ij}^0 - f_{ij}^0) \frac{|A_{ij}|^2}{\omega_0 + (\epsilon_i - \epsilon_j) + i0^+} + 2g_d^2 \sum_{ij} \left( 1 + f_{ij}^0 + f_{ij}^0 \right) \frac{|B_{ij}|^2}{\omega_0 - (\epsilon_i + \epsilon_j) + i0^+}
\]

\[- \frac{|\tilde{B}_{ij}|^2}{\omega_0 + (\epsilon_i + \epsilon_j) + i0^+} \right),
\]

(66)

where the unperturbed eigenfrequency \( \omega_0 \) is obtained from the unperturbed RPA equation [30]

\[
\left( \begin{array}{cc} L & g_d n_0 \\ -g_d n_0 & -L \end{array} \right) \left( \begin{array}{c} \delta \Phi_1^0 \\ \delta \Phi_2^0 \end{array} \right) = \omega_0 \left( \begin{array}{c} \delta \Phi_1^0 \\ \delta \Phi_2^0 \end{array} \right)
\]

(67)

with the normalization condition

\[
\int d^3r (|\delta \Phi_1^0|^2 - |\delta \Phi_2^0|^2) = 1.
\]

(68)
and \( A_{ij}, B_{ij} \) and \( \tilde{B}_{ij} \) are defined as

\[
A_{ij} = \int \! \! \! \! \! d^d \mathbf{r} \, \Phi_0 \left[ \delta \Phi_1^0 (u_i u_j^* + v_i v_j^* + v_i u_j^* + u_i v_j^*) + \delta \Phi_2^0 (u_i u_j^* + v_i v_j^* + u_i v_j^* + v_i u_j^*) \right],
\]

\[
B_{ij} = \int \! \! \! \! \! d^d \mathbf{r} \, \Phi_0 \left[ \delta \Phi_1^0 (u_i^* v_j^* + v_i^* u_j^* + u_i^* u_j^* + v_i^* v_j^*) + \delta \Phi_2^0 (u_i^* v_j^* + v_i^* u_j^* + v_i^* v_j^* + u_i^* u_j^*) \right],
\]

\[
\tilde{B}_{ij} = \int \! \! \! \! \! d^d \mathbf{r} \, \Phi_0 \left[ \delta \Phi_1^0 (u_i v_j + v_i u_j + u_i u_j) + \delta \Phi_2^0 (u_i v_j + v_i u_j + v_i u_j) \right].
\]  

The real part of the right-hand side (equation (66)) gives the eigenenergy of the system and the imaginary part tells us the damping coefficient \( \gamma \). Using the relation

\[
\frac{1}{x + i0^+} = P \frac{1}{x} - i\pi \delta(x),
\]

we can divide the damping rate into two different types: one comes from the process whereby one phonon with the frequency \( \omega_0 \) is absorbed by a thermal excitation \( \epsilon_i \) jumping to another thermal excitation with the energy \( \epsilon_j = \epsilon_i + \omega_0 \). This mechanism is so-called Landau damping given by the second term on the right-hand side of equation (66),

\[
\gamma_L = 4\pi g_0^2 \sum_{ij} |A_{ij}|^2 (f_i^0 - f_j^0) \delta(\omega_0 + \epsilon_i - \epsilon_j).
\]  

This process happens mostly at finite temperatures, and is therefore a thermal process. Another kind of decay arises from the process of a long wavelength phonon decaying into two phonons, as indicated by Beliaev, and it can be obtained by the imaginary part of the first term in brackets on the right-hand side of equation (66):

\[
\gamma_B = 2\pi g_0^2 \sum_{ij} |B_{ij}|^2 (1 + f_i^0 + f_j^0) \delta(\omega_0 - \epsilon_i - \epsilon_j).
\]

This process occurs mostly at zero temperatures, which is a pure quantum effect. The total damping rate is the sum of the two damping coefficients:

\[
\gamma = \gamma_B + \gamma_L.
\]

In this paper, we are interested in homogeneous Bose gases, i.e. \( V_{\text{ext}}(\mathbf{r}) = 0 \). For homogeneous systems the condensate density remains the same throughout the space: \( \Phi_0 = \sqrt{n_0} \), while the excitations and the fluctuations satisfying equation (50) and equation (67) can be described by the plane waves

\[
\begin{align*}
\left( \delta \Phi_1 (\mathbf{r}) \right) & = \frac{1}{\sqrt{V}} \int \! \! \! \! \! d^d \mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}} \begin{pmatrix} u_k \\ v_k \end{pmatrix}, \\
\left( \delta \Phi_2 (\mathbf{r}) \right) & = \frac{1}{\sqrt{V}} \int \! \! \! \! \! d^d \mathbf{k} e^{i\mathbf{k} \cdot \mathbf{r}} \begin{pmatrix} u_k^* \\ v_k^* \end{pmatrix},
\end{align*}
\]

where \( u_k \) and \( v_k \) satisfy the Bogoliubov relations:

\[
\begin{align*}
u_k^2 & = 1 + u_k^2 = \frac{(\epsilon^2(\mathbf{k}) + g_d^2 n_0^2)^{1/2} + \epsilon(\mathbf{k})}{2 \epsilon(\mathbf{k})}, \\
u_k u_k & = -\frac{g_d n_0}{2 \epsilon(\mathbf{k})},
\end{align*}
\]

and \( \epsilon(\mathbf{k}) \) is the Bogoliubov energy (13).
4.1. Quantum regime

As mentioned in the last section, the decay rate is mostly contributed by Beliaev damping, which can be obtained by setting $f^0_j = f^0_j = 0$ in equation (72). The matrix element $B_{k',-k,k'}$ reads

$$B_{k',-k,k'} = \sqrt{\frac{n_0}{V}} \left[(u_k (u_{k'} v_{k'-k} + v_{k'} u_{k'-k}) + v_k (u_{k'} v_{k'-k} + v_{k'} u_{k'-k} + v_{k'} v_{k'-k}))\right],$$

and the other elements are zero. At zero temperatures, only the momenta with long wavelength are involved in the Beliaev damping process, i.e. $k' \sim k \sim |k' - k| \ll mc$. Therefore, the long-wavelength approximation for the energy $\epsilon(k)$ and the wave functions $u_k$ and $v_k$ can be used:

$$\epsilon(k) \simeq ck + \frac{k^3}{8mc^2},$$

$$u_k \simeq \left(\frac{mc}{2k}\right)^{1/2} + \frac{1}{2} \left(\frac{k}{2mc}\right)^{1/2} - \frac{1}{8} \left(\frac{k}{2mc}\right)^{3/2} - \frac{1}{8} \left(\frac{k}{2mc}\right)^{5/2},$$

$$v_k \simeq - \left(\frac{mc}{2k}\right)^{1/2} + \frac{1}{2} \left(\frac{k}{2mc}\right)^{1/2} - \frac{1}{8} \left(\frac{k}{2mc}\right)^{3/2} - \frac{1}{8} \left(\frac{k}{2mc}\right)^{5/2}. \quad \text{(78)}$$

Substituting (77) and (78) into equation (76), we obtain the result

$$B_{k',-k,k'} = \sqrt{\frac{n_0}{V}} \frac{3}{4\sqrt{2}} \frac{|k||k'||k'-k|}{(mc)^{3/2}}. \quad \text{(79)}$$

Inserting equation (79) into equation (72) and summing all the momenta $k$ and $k'$, one obtains the same result (19) as that using the hydrodynamical approach. Therefore, we reproduce the results for 3D and 2D decay as equations (20) and (22).

4.2. Thermal regime

In the thermal regime where $\omega_0 \sim k_B T$, a long-wavelength Goldstone mode with eigenfrequency $(\omega_0 \simeq ck)$ describes the behavior of the condensate in the thermal clouds. In this limit, the $u$ and $v$ functions can be expanded as

$$u_k \simeq \left(\frac{mc}{2\epsilon(k)}\right)^{1/2} + \frac{1}{2} \left(\frac{\epsilon(k)}{2mc^2}\right)^{1/2},$$

$$v_k \simeq - \left(\frac{mc}{2\epsilon(k)}\right)^{1/2} + \frac{1}{2} \left(\frac{\epsilon(k)}{2mc^2}\right)^{1/2}. \quad \text{(80)}$$

Using this expansion, the long-wavelength behavior for the nonzero elements of the matrix $A$ can be expressed as

$$A_{k',k+k} = \sqrt{n_0} \sqrt{\frac{\epsilon(k)}{V}} \left(\frac{\epsilon(k)}{2mc^2}\right)^{1/2} \left(u_k^2 + v_k^2 + u_k v_k - \frac{v_k}{c} \cos \theta \frac{2u_k^2 v_k^2}{u_k^2 + v_k^2}\right), \quad \text{(81)}$$

with the angle $\theta$ between the vectors $k'$ and $k$, and the group velocity of the excitation defined as $v_g = \partial \epsilon(k) / \partial k$.

In 3D, the damping rate can be obtained by inserting the nonzero coefficients (81) into (71) and integrating out the angle $\theta$ as follows:

$$\frac{\gamma_{d=3}}{\epsilon(k)} \simeq \frac{\gamma_{d=3}}{\epsilon(k)} = (a^3 n_0)^{1/2} F(\tau). \quad \text{(82)}$$

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where $\tau = k_B T/mc^2$ is the dimensionless temperatures, $a_3$ is the 3D scattering length, and $F(\tau)$ is defined as

$$F(\tau) = \sqrt{\frac{\pi}{\tau}} \int dz \sech^2\left(\frac{z}{2\tau}\right) \left(1 - \frac{1}{2u} - \frac{1}{2u^2}\right)^2$$

with the definition $u = \sqrt{1 + z^2}$. This expression was first found by Pitaevskii and Stringari [12].

The HFB method and the results for Beliaev and Landau damping in 3D were first found by Stringari and Pitaevskii [12] and then generalized by Giorgini [13]. We present the details here first to offer the readers a general review, and, second, to apply this method to 2D systems and then to compare the results to the hydrodynamical approach.

For low temperatures $k_B T \ll mc^2$, i.e. $\tau \ll 1$ and the function $F$ takes approaches the limit $F \approx 3 \pi^{3/2}\tau^4/5$ and one finds Hohenberg and Martin’s result (25). As mentioned in the last section, the hydrodynamical approach and HFB gives the same limit at low temperatures. However, at high temperatures, the hydrodynamic approach fails. For temperatures $\tau \gg 1.$, i.e. $k_B T \gg mc^2$, the function $F$ takes the asymptotic limit $F \rightarrow 3 \pi^{3/2}/2\tau$, and the damping rate approaches the Szépfalusy and Kondor result (27). Therefore, the HFB gives the correct asymptotic value at high temperatures.

In figure 3, the famous result for the 3D damping rate using the HFB method obtained by Stringari and Pitaevskii [12], and then recovered by Giorgini [13], is shown as the solid line. We can see that the 3D damping rate leaves the low-temperature limit very quickly, while it approaches the high-temperature linear law very slowly.

In 2D, HFB offers a good approximation for all regimes of temperature. After inserting the matrix elements (81) into (71) and then integrating out the angle $\theta$, one obtains the damping rate:

$$\gamma_{d=2} = 2 \frac{\epsilon(k)}{\epsilon(k)} = \frac{2m g_2 G(\tau)}{\gamma_d \approx 0.013}.$$  

where

$$G(\tau) = \frac{\sqrt{2}}{\pi \tau} \int_0^\infty dz \frac{\sech^2(z/\tau)}{\sqrt{z^2 + 1} - 1 - z^2/(2(2z^2-1))} \left(1 - \frac{1}{2u} - \frac{1}{2u^2}\right)^2.$$  

In the low-temperature limit $\tau \ll 1$, the function $G$ approaches the limit: $G(\tau) \rightarrow \sqrt{3}\pi \tau^2/16$, the damping rate converges to the result (30). As in 3D HFB gives the same result as the hydrodynamical approach at low temperatures in 2D.

For high temperatures $k_B T \gg mc^2$, the function $G$ takes its asymptotic value: $G(\tau) \rightarrow 0.013\tau$, and the damping rate is given by

$$\gamma_{d=2} \approx 0.026 \frac{m k_B T}{n_0}.$$  

Therefore the damping rate itself reads

$$\gamma_{d=2} \approx 0.013 \frac{8\pi}{\ln(1/na_1^2)} \frac{k_B T}{mc}.$$  

In figure 4, the 2D damping rate per unit energy using the HFB method is also shown as a function of $\tau$ (solid line). We can see that the 2D damping rate approaches the high-temperatures linear law much more quickly than the 3D case. That means the 2D systems go to the high-temperature asymptotic value at lower value of temperature compared to that in 3D. This
behavior has been found by Guilleumas and Pitaevskii studying a quasi-2D system [31]. In this figure, one can also see that the hydrodynamical approach can only give good results in the regime where \( \tau < 0.5 \).

5. Discussion and conclusion

The damping of Bogoliubov excitations in 2D and 3D optical lattices has been considered by Tsuchiya and Griffin [32]. In their consideration, a very deep sinusoidal potential is applied to a Bose gas so that the one band Bose–Hubbard model can be used in the system. The hopping constant \( J \) and on-site interaction strength \( U \) can both be adjusted by the laser intensity. Unlike Bose gases without an optical lattice, for which energy dispersion (13) cannot be changed, the dispersion for a Bose gas in an optical lattice can be easily adjusted by changing the laser intensity, which alters \( J \) and \( U \). Therefore in optical lattices of any dimension, the Bogoliubov excitation spectrum must exhibit anomalous dispersion for the damping process to occur, which means that the excitation energy has to bend upward at low momentum. When \( g_{d\ell n_0}(T) \lesssim 6dJ \), the damping disappears due to the fact that the energy spectrum bends downward for small momenta under this condition. On the other hand, dilute Bose gases without optical lattices cannot have this property because the energy dispersion is not tunable.

In this work, we have compared the hydrodynamical approach and the HFB approach to calculate the damping rate in 2D and 3D Bose gases. The hydrodynamical approach is a powerful tool due to the fact that one can use the Green’s function technique based on the effective action (5). This works very well at zero and low temperatures. However, this method truncates the rapid oscillating fields, which is not the case for the high-temperature regime, therefore it overestimates the Landau damping. On the other hand, the HFB approximation can explain either low-temperature or high-temperature damping. It seems that using the mean-field approach (HFB) one can obtain the high-temperature behavior for Bose gases. The HFB approach based on the mean field method factorizes the quartic terms and therefore the Shohno ansatz has to be used to avoid anomalous behavior. In the absence of the Shohno ansatz, there would exist a gap in the low-excitation spectrum. Therefore the mean-field approach cannot guarantee a zero energy gap. From a physical point of view, the existence of a gapless excitation is a general rule for Bose systems. In order to avoid errors in the higher order calculations, the mean-field approach should be used carefully. Therefore, we can see the benefit of the hydrodynamical approach for the low-temperature regime. The low-energy excitation is always gapless using the hydrodynamical approach. Another benefit of using the hydrodynamical approach, as indicated by Popov [15], is that it can avoid the strange divergence at high and low momenta, the so-called ultraviolet and infrared catastrophe, respectively, which can be caused by perturbation theory based on the mean-field approach.

The HFB approach can also be generalized to confined systems. Experimentally a true 2D homogeneous system cannot be produced, therefore a quasi-2D condensate trapped in a harmonic potential can verify the properties of the systems in 2D. For example, Chevy et al [33] investigated transverse collective oscillation in an elongate condensate confined in an axially symmetric harmonic trap. It turns out that at low temperatures, the frequencies of both monopole and quadrupole transverse modes are in good agreement with the predictions of the GP theory for a 2D condensate, thus they reveal the quasi-2D character. In this context, Guilleumas and Pitaevskii [31] theoretically studied a cylindrical condensate with a harmonic trap in the radial (transverse) direction and compared the damping rate with experiment [34]. They found that the
Landau damping rate of the transverse quadrupole mode is in agreement with the experimental result. Figure 1 in [31] shows similar curve as the solid line of figure 4 in our case. The linear behavior at high $T$ appears at relatively small temperature in comparison to 3D systems. Due to this behavior of Landau damping in 2D a linear approximation for damping can be used down to low temperatures, in contrast to that for a 3D condensate.

In contrast, the Landau damping for the transverse monopole mode is one order of magnitude smaller than the experimental decay. Thus, Landau damping cannot explain the experimental decay of the transverse breathing mode. Jackson and Zaremba [34] claimed that the effect of the collisions between elementary excitation has to be included to explain the decay of the transverse breathing mode. This effect is usually smaller in comparison with Landau damping, while for this particular mode the collision seems to dominate the damping process.

We have to address that in the two approaches discussed in this paper, the thermal cloud is assumed to be in a state of thermodynamic equilibrium and hence one can neglect its motion. Therefore this method cannot be applied to 2D breathing modes, because the thermal cloud has the same frequency as the condensate. Therefore the cloud could possibly oscillate in phase with the condensate, which will reduce the Landau damping.

For 2D Bose gases we have found for the first time that the Beliaev damping rate is proportional to $k^3$ at zero temperature and the Landau damping rate is proportional to $T^2$ for low temperatures and to $T$ for high temperatures. The behavior of the 2D damping is also totally different from the 3D damping. While the 3D Landau damping approaches the linear regime very slowly with increasing temperature, the 2D damping become linear very quickly. The linear regime symbolizes the classical high-temperature behavior; therefore the 2D system goes to the high-temperature asymptotic value at a lower value of temperature as compared with the 3D system.

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