Singular value decomposition of large random matrices
(for two-way classification of microarrays)

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Abstract

Asymptotic behavior of the singular value decomposition (SVD) of blown up matrices and normalized blown up contingency tables exposed to Wigner-noise is investigated. It is proved that such an $m \times n$ matrix almost surely has a constant number of large singular values (of order $\sqrt{mn}$), while the rest of the singular values are of order $\sqrt{m+n}$ as $m, n \to \infty$. Concentration results of Alon at al. for the eigenvalues of large symmetric random matrices are adapted to the rectangular case, and on this basis, almost sure results for the singular values as well as for the corresponding isotropic subspaces are proved. An algorithm, applicable to two-way classification of microarrays, is also given that finds the underlying block structure.

Key words: Concentration of singular values, Two-way classification of microarrays, Perturbation of correspondence matrices, Almost sure convergence by large deviations

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1 Introduction

A general problem of multivariate statistics is to find linear structures in large real-world data sets like internet or microarray measurements. In [5], large

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symmetric blown up matrices burdened with a so-called symmetric Wigner-noise were investigated. It was proved that such an $n \times n$ matrix has some protruding eigenvalues (of order $n$), while the majority of the eigenvalues is at most of order $\sqrt{n}$ with probability tending to 1 as $n \to \infty$. Our goal is to generalize these results for the stability of SVD of large rectangular random matrices and to apply them for the contingency table matrix formed by categorical variables in order to perform two-way clustering of these variables.

First we introduce some notation.

**Definition 1** The $m \times n$ real matrix $W$ is a Wigner-noise if its entries $w_{ij}$ ($1 \leq i \leq m, 1 \leq j \leq n$) are independent random variables, $\mathbb{E}(w_{ij}) = 0$, and the $w_{ij}$’s are uniformly bounded (i.e., there is a constant $K > 0$, independently of $m$ and $n$, such that $|w_{ij}| \leq K, \forall i, j$).

Though, the main results of this paper can be extended to $w_{ij}$’s with any light-tail distribution (especially to Gaussian distributed $w_{ij}$’s), our almost sure results will be based on the assumptions of Definition 1.

**Definition 2** The $m \times n$ real matrix $B$ is a blown up matrix, if there is an $a \times b$ so-called pattern matrix $P$ with entries $0 \leq p_{ij} \leq 1$, and there are positive integers $m_1, \ldots, m_a$ with $\sum_{i=1}^a m_i = m$ and $n_1, \ldots, n_b$ with $\sum_{i=1}^b n_i = n$, such that the matrix $B$ can be divided into $a \times b$ blocks, where block $(i, j)$ is an $m_i \times n_j$ matrix with entries equal to $p_{ij}$ ($1 \leq i \leq a, 1 \leq j \leq b$).

Such schemes are sought for in microarray analysis and they are called chessboard patterns, cf. [9]. Let us fix the matrix $P$, blow it up to obtain matrix $B$, and let $A = B + W$, where $W$ is a Wigner-noise of appropriate size. We are interested in the properties of $A$ when $m_1, \ldots, m_a \to \infty$ and $n_1, \ldots, n_b \to \infty$, roughly speaking, at the same rate. More precisely, we make two different constraints on the growth of the sizes $m, n$, and the growth rate of their components. The first one is needed for all our reasonings, while the second one will be used in the case of noisy correspondence matrices, only.

**Definition 3**

**GC1** (Growth Condition 1)
There exists a constant $0 < c < 1$ such that $m_i/m \geq c$ ($i = 1, \ldots, a$) and there exists a constant $0 < d < 1$ such that $n_i/n \geq d$ ($i = 1, \ldots, b$).

**GC2** (Growth Condition 2)
There exist constants $C \geq 1, D \geq 1$, and $C_0 > 0, D_0 > 0$ such that $m \leq C_0 \cdot n^C$ and $n \leq D_0 \cdot m^D$ hold for sufficiently large $m$ and $n$.

**Remark 4** GC1 implies that

$$c \leq \frac{m_k}{m_i} \leq \frac{1}{c} \quad \text{and} \quad d \leq \frac{n_k}{n_j} \leq \frac{1}{d}$$  \hspace{1cm} (1)
hold for any pair of indices \( k, i \in \{1, \ldots, a\} \) and \( \ell, j \in \{1, \ldots, b\} \).

We want to establish some property \( P_{m,n} \) that holds for the \( m \times n \) random matrix \( A = B + W \) (briefly, \( A_{m \times n} \)) with \( m \) and \( n \) large enough. In this paper \( P_{m,n} \) is mostly related to the SVD of \( A_{m \times n} \).

**Definition 5** Property \( P_{m,n} \) holds for \( A_{m \times n} \) almost surely (with probability \( 1 \)) if \( P(\exists \ m_0, n_0 \in \mathbb{N} \text{ such that for } m \geq m_0 \text{ and } n \geq n_0 \ A_{m \times n} \text{ has } P_{m,n}) = 1 \). Here we may assume GC1 or GC2 for the growth of \( m \) and \( n \), while \( K \) is kept fixed.

In combinatorics literature convergence in probability, that is

\[
\lim_{m,n \to \infty} P(A_{m \times n} \text{ has } P_{m,n}) = 1
\]

is frequently considered, and – by the Borel–Cantelli Lemma – it implies almost sure convergence, if in addition \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} p_{mn} < \infty \) also holds, where

\[ p_{mn} = P(A_{m \times n} \text{ does not have } P_{m,n}). \]

According to a generalization of a theorem of Füredi and Komlós [7] to rectangular matrices, the spectral norm of an \( m \times n \) Wigner-noise is \( \sqrt{m+n} \) in probability. More precisely, it was shown (see [1]) that with probability tending to \( 1 \), \( \|W\| \leq \frac{2}{3}\sigma \sqrt{m+n} \), where \( \sigma \) is the common bound for the variances of the entries. Trivially, \( \sigma \leq K \) that does not depend on \( m \) and \( n \), hence \( \|W\| = \mathcal{O}(\sqrt{m+n}) \) in probability. Bounding the variances from below, authors also proved that \( \|W\| = \Theta(\sqrt{m+n}) \) with high probability for large \( m, n \).

To prove almost sure convergence, a sharp concentration theorem of N. Alon at al. plays a crucial role (cf. [2]). For completeness we formulate this result.

**Lemma 6** Let \( \bar{W} \) be a \( q \times q \) real symmetric matrix whose entries in and above the main diagonal are independent random variables with absolute value at most 1. Let \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_q \) be the eigenvalues of \( \bar{W} \). The following estimate holds for the deviation of the \( i \)th largest eigenvalue from its expectation with any positive real number \( t \):

\[
\mathbb{P}(|\lambda_i - \mathbb{E}(\lambda_i)| \geq t) \leq \exp\left(-\frac{(1-o(1))t^2}{32i^2}\right) \quad \text{when} \quad i \leq \frac{q}{2},
\]

and the same estimate holds for the probability \( \mathbb{P}(|\lambda_{q-i+1} - \mathbb{E}(\lambda_{q-i+1})| > t) \).
Now let $\mathbf{W}$ be a Wigner-noise with entries uniformly bounded by $K$. The $(m+n) \times (m+n)$ symmetric matrix

$$
\widetilde{\mathbf{W}} = \frac{1}{K} \cdot \begin{pmatrix} 0 & \mathbf{W} \\ \mathbf{W}^T & 0 \end{pmatrix}
$$

satisfies the conditions of Lemma 6, its largest and smallest eigenvalues are

$$
\lambda_i(\widetilde{\mathbf{W}}) = -\lambda_{n+m-i+1}(\widetilde{\mathbf{W}}) = \frac{1}{K} \cdot s_i(\mathbf{W}), \quad i = 1, \ldots, \min\{m, n\},
$$

the others are zeros, where $\lambda_i(.)$ and $s_i(.)$ denote the $i$th largest eigenvalue and singular value of the matrix in the argument, respectively (cf. [3]). Therefore

$$
\mathbb{P}\left( |s_1(\mathbf{W}) - \mathbb{E}(s_1(\mathbf{W}))| > t \right) \leq \exp\left( -\frac{(1-o(1))t^2}{32K^2} \right). \quad (2)
$$

The fact that $\|\mathbf{W}\| = \mathcal{O}(\sqrt{m+n})$ in probability and inequality (2) together ensure that $\mathbb{E}(\|\mathbf{W}\|) = \mathcal{O}(\sqrt{m+n})$. Hence, no matter how $\mathbb{E}(\|\mathbf{W}\|)$ behaves when $m \to \infty$ and $n \to \infty$, the following rough estimate holds.

**Lemma 7** There exist positive constants $C_{K1}$ and $C_{K2}$, depending on the common bound on the entries of $\mathbf{W}$, such that

$$
\mathbb{P}\left( \|\mathbf{W}\| > C_{K1} \cdot \sqrt{m+n} \right) \leq \exp[-C_{K2} \cdot (m+n)]. \quad (3)
$$

The exponential decay of the right hand side of (3) implies that the spectral norm of a Wigner-noise $\mathbf{W}_{m \times n}$ is of order $\sqrt{m+n}$, almost surely. This observation will provide the base of almost sure results of Sections 2 and 3.

In Section 2 we shall prove that the $m \times n$ noisy matrix $\mathbf{A} = \mathbf{B} + \mathbf{W}$ almost surely has $r = \text{rank } (\mathbf{P})$ protruding singular values of order $\sqrt{mn}$. In Section 3 the distances of the corresponding isotropic subspaces are estimated and this gives rise to a two-way classification of the row and column items of $\mathbf{A}$ with sum of inner variances $\mathcal{O}(\frac{m+n}{mn})$, almost surely.

In Definition 2 we required that the entries of the pattern matrix $\mathbf{P}$ be in the $[0,1]$ interval. We made this restriction only for the sake of the generalized Erdős–Rényi hypergraph model with the entries of $\mathbf{P}$ as probabilities, see [6]. In fact, our results are valid for any pattern matrix with fixed sizes and with non-negative entries. For example, in microarray measurements the rows correspond to different genes, the columns correspond to different conditions, and the entries are the expression levels of a specific gene under a specific condition.
Sometimes the pattern matrix $P$ is an $a \times b$ contingency table with entries that are nonnegative integers. Then the blown up matrix $B$ can be regarded as a larger $(m \times n)$ contingency table that contains e.g., counts for two categorical variables with $m$ and $n$ different categories, respectively. As the categories may be measured in different units, a normalization is necessary. This normalization is made by dividing the entries of $B$ by the square roots of the corresponding row and column sums (cf. [9]). This transformation is identical to that of the correspondence analysis [8], and the transformed matrix remains the same when we multiply the initial matrix by a positive constant. The transformed matrix $B_{\text{corr}}$, which belongs to $B$, has entries in $[0,1]$ and maximum singular value 1. It is proved that there is a remarkable gap between the rank $(B) = \text{rank } (P)$ largest and the other singular values of $A_{\text{corr}}$, the matrix obtained from the noisy matrix $A = B + W$ by the correspondence transformation. This implies well two-way classification properties of the row and column categories (genes and expression levels) in Section 4.

In Section 5 a construction is given how a blown up structure behind a real-life matrix with a few protruding singular values and ‘well classifiable’ corresponding singular vector pairs can be found.

2 Singular values of a noisy matrix

**Proposition 8** If GC1 holds, then all the non-zero singular values of the $m \times n$ blown-up matrix $B$ are of order $\sqrt{mn}$.

**PROOF.** As there are at most $a$ and $b$ linearly independent rows and linearly independent columns in $B$, respectively, the rank $r$ of the matrix $B$ cannot exceed $\min\{a, b\}$. Let $s_1 \geq s_2 \geq \cdots \geq s_r > 0$ be the positive singular values of $B$. Let $v_k \in \mathbb{R}^m$, $u_k \in \mathbb{R}^n$ be a singular vector pair corresponding to $s_k$, $k = 1, \ldots, r$. Without loss of generality, $v_1, \ldots, v_r$ and $u_1, \ldots, u_r$ can be unit-norm, pairwise orthogonal vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

For the subsequent calculations we drop the subscript $k$, and $v, u$ denotes a singular vector pair corresponding to the singular value $s > 0$ of the blown-up matrix $B$, $\|v\| = \|u\| = 1$. It is easy to see that they have piecewise constant structures: $v$ has $m_i$ coordinates equal to $v(i)$ ($i = 1, \ldots, a$) and $u$ has $n_j$ coordinates equal to $u(j)$ ($j = 1, \ldots, b$). Then, with these coordinates the singular value–singular vector equation

$$Bu = s \cdot v$$  \hspace{1cm} (4)
has the form
\[ \sum_{j=1}^{b} n_j p_{ij} u(j) = s \cdot v(i) \quad (i = 1, \ldots, a). \] (5)

With the notations
\[ \tilde{u} = (u(1), \ldots, u(a))^T, \quad \tilde{v} = (v(1), \ldots, v(b))^T, \]
\[ D_m = \text{diag} (m_1, \ldots, m_a), \quad D_n = \text{diag} (n_1, \ldots, n_b) \]
the equations in (5) can be written as
\[ PD_n \tilde{u} = s \cdot \tilde{v}. \]

Introducing the following transformations of \( \tilde{u} \) and \( \tilde{v} \)
\[ w = D_n^{1/2} \tilde{u}, \quad z = D_m^{1/2} \tilde{v}, \] (6)
the equation is equivalent to
\[ D_m^{1/2} PD_n^{1/2} w = s \cdot z. \] (7)

Applying the transformation (6) for the \( \tilde{u}_k, \tilde{v}_k \) pairs obtained from the \( u_k, v_k \) pairs \( (k = 1, \ldots, r) \), orthogonormal systems in \( \mathbb{R}^a \) and \( \mathbb{R}^b \) are obtained:
\[ w_k^T \cdot w_\ell = \sum_{j=1}^{b} n_j u_k(j) u_\ell(j) = \delta_{k\ell} \quad \text{and} \quad z_k^T \cdot z_\ell = \sum_{i=1}^{a} m_i v_k(i) v_\ell(i) = \delta_{k\ell}. \]

Consequently, \( z_k, w_k \) is a singular vector pair corresponding to singular value \( s_k \) of the \( a \times b \) matrix \( D_m^{1/2} PD_n^{1/2} \) \( (k = 1, \ldots, r) \). With the shrinking
\[ \tilde{D}_m = \frac{1}{m} D_m, \quad \tilde{D}_n = \frac{1}{n} D_n \]
an equivalent form of (7) is
\[ \tilde{D}_m^{1/2} \tilde{D}_n^{1/2} w = \frac{s}{\sqrt{mn}} \cdot z, \]
that is the \( a \times b \) matrix \( \tilde{D}_m^{1/2} \tilde{D}_n^{1/2} \) has non-zero singular values \( \frac{s_k}{\sqrt{mn}} \) with the same singular vector pairs \( z_k, w_k \) \( (k = 1, \ldots, r) \). If the \( s_k \)'s are not distinct numbers, the singular vector pairs corresponding to a multiple singular value are not unique, but still they can be obtained from the SVD of the shrunken matrix \( \tilde{D}_m^{1/2} \tilde{D}_n^{1/2} \).

Now we want to establish relations between the singular values of \( P \) and \( \tilde{D}_m^{1/2} \tilde{D}_n^{1/2} \). Let \( s_k(Q) \) denote the \( k \)th largest singular value of a matrix \( Q \). By the Courant–Fischer–Weyl minimax principle (cf. [3, p.75])
\[ s_k(Q) = \max_{\dim H = k} \min_{x \in H} \frac{\|Qx\|}{\|x\|}. \]
Since we are interested only in the first $r$ singular values, where $r = \text{rank}(B) = \text{rank}(\tilde{D}_m^{1/2}PD_n^{1/2})$, it is sufficient to consider vectors $x$, for which $\tilde{D}_m^{1/2}PD_n^{1/2}x \neq 0$. Therefore with $k \in \{1, \ldots, r\}$ and an arbitrary $k$-dimensional subspace $H \subset \mathbb{R}^b$ one can write

$$
\min_{x \in H} \frac{\|\tilde{D}_m^{1/2}PD_n^{1/2}x\|}{\|x\|} = \min_{x \in H} \frac{\|\tilde{D}_m^{1/2}PD_n^{1/2}x\|}{\|PD_n^{1/2}x\|} \cdot \frac{\|PD_n^{1/2}x\|}{\|D_n^{1/2}x\|} \cdot \frac{\|D_n^{1/2}x\|}{\|x\|} 
\geq s_k(\tilde{D}_m^{1/2}) \cdot \min_{x \in H} \frac{\|PD_n^{1/2}x\|}{\|D_n^{1/2}x\|} \cdot s_k(\tilde{D}_n^{1/2}) \geq \sqrt{cd} \cdot \min_{x \in H} \frac{\|PD_n^{1/2}x\|}{\|D_n^{1/2}x\|},
$$

with $c,d$ of GC1. Now taking the maximum for all possible $k$-dimensional subspaces $H$ we obtain that $s_k(\tilde{D}_m^{1/2}PD_n^{1/2}) \geq \sqrt{cd} \cdot s_k(P) > 0$. On the other hand,

$$s_k(\tilde{D}_m^{1/2}PD_n^{1/2}) \leq \|\tilde{D}_m^{1/2}PD_n^{1/2}\| \leq \|\tilde{D}_m^{1/2}\| \cdot \|P\| \cdot \|D_n^{1/2}\| \leq \|P\| \leq \sqrt{ab}.
$$

These inequalities imply that $s_k(\tilde{D}_m^{1/2}PD_n^{1/2})$ is a nonzero constant, and because of $s_k(\tilde{D}_m^{1/2}PD_n^{1/2}) = \frac{s_k}{\sqrt{mn}}$ we obtain that $s_1, \ldots, s_r = \Theta(\sqrt{mn})$. □

**Theorem 9** Let $A = B + W$ be an $m \times n$ random matrix, where $B$ is a blown up matrix with positive singular values $s_1, \ldots, s_r$ and $W$ is a Wigner-noise. Then, under GC1, the matrix $A$ almost surely has $r$ singular values $z_1, \ldots, z_r$, such that

$$|z_i - s_i| = \mathcal{O}(\sqrt{m + n}), \quad i = 1, \ldots, r
$$

and for the other singular values almost surely

$$z_j = \mathcal{O}(\sqrt{m + n}), \quad j = r + 1, \ldots, \min\{m, n\}.
$$

**PROOF.** The statement follows from the analog of the Weyl's perturbation theorem for singular values of rectangular matrices (see [3, p.99]) and from Lemma 7. If $s_i(A)$ and $s_i(B)$ denote the $i$th largest singular values of the matrix in the argument then for the difference of the corresponding pairs

$$|s_i(A) - s_i(B)| \leq \max_i s_i(W) = \|W\|, \quad i = 1, \ldots, \min\{m, n\}.
$$

By Lemma 7, $\mathbb{P}\left(|s_i(A) - s_i(B)| > C_{K_1} \cdot \sqrt{m + n}\right) \leq \mathbb{P}\left(\|W\| > C_{K_1} \cdot \sqrt{m + n}\right) \leq \exp[-C_{K_2}(m + n)]$. The right hand side of the last inequality is the general term of a convergent series (defined as a double summation), thus the convergence in probability implies the almost sure statement of the theorem. □

**Corollary 10** With notations

$$\varepsilon := \|W\| = \mathcal{O}(\sqrt{m + n}) \quad \text{and} \quad \Delta := \min_{1 \leq i \leq r} s_i(B) = \min_{1 \leq i \leq r} s_i = \Theta(\sqrt{mn}) \quad (8)$$

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there is a spectral gap of size $\Delta - 2\varepsilon$ between the $r$ largest and the other singular values of the perturbed matrix $A$, and this gap is significantly larger than $\varepsilon$.

3 Classification via singular vector pairs

With the help of Theorem 9 we can estimate the distances between the corresponding right- and left-hand side eigenspaces (isotropic subspaces) of the matrices $B$ and $A = B + W$. Let $v_1, \ldots, v_m \in \mathbb{R}^m$ and $u_1, \ldots, u_n \in \mathbb{R}^n$ be orthonormal left- and right-hand side singular vectors of $B$,

$$Bu_i = s_i \cdot v_i \quad (i = 1, \ldots, r) \quad \text{and} \quad Bu_j = 0 \quad (j = r + 1, \ldots, n).$$

Let us also denote the unit-norm, pairwise orthogonal left- and right-hand side singular vectors corresponding to the $r$ protruding singular values $z_1, \ldots, z_r$ of $A$ by $y_1, \ldots, y_r \in \mathbb{R}^m$ and $x_1, \ldots, x_r \in \mathbb{R}^n$, respectively. Then $Ax_i = z_i \cdot y_i$ ($i = 1, \ldots, r$). Let

$$F := \text{Span} \{v_1, \ldots, v_r\} \quad \text{and} \quad G := \text{Span} \{u_1, \ldots, u_r\}$$

denote the spanned linear subspaces in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively; further, let $\text{dist}(y, F)$ denote the Euclidean distance between the vector $y$ and the subspace $F$.

**Proposition 11** With the above notation, under GC1, the following estimate holds almost surely:

$$\sum_{i=1}^r \text{dist}^2(y_i, F) \leq r \frac{\varepsilon^2}{(\Delta - \varepsilon)^2} = \mathcal{O} \left( \frac{m + n}{mn} \right)$$

and analogously,

$$\sum_{i=1}^r \text{dist}^2(x_i, G) \leq r \frac{\varepsilon^2}{(\Delta - \varepsilon)^2} = \mathcal{O} \left( \frac{m + n}{mn} \right).$$

**PROOF.** Let us choose one of the right-hand side singular vectors $x_1, \ldots, x_r$ of $A = B + W$ and denote it simply by $x$ with corresponding singular value $z$. We shall estimate the distance between $x$ and $G$, similarly between $y = Ax/z$ and $F$. For this purpose we expand $x$ and $y$ in the orthonormal bases $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$, respectively:

$$x = \sum_{i=1}^n t_i u_i \quad \text{and} \quad y = \sum_{i=1}^m l_i v_i.$$
Then
\[ \mathbf{A} \mathbf{x} = (\mathbf{B} + \mathbf{W}) \mathbf{x} = \sum_{i=1}^{r} t_i s_i \mathbf{v}_i + \mathbf{W} \mathbf{x}, \quad (11) \]
and, on the other hand,
\[ \mathbf{A} \mathbf{x} = \mathbf{z} \mathbf{y} = \sum_{i=1}^{m} z_i \mathbf{v}_i. \quad (12) \]
Equating the right-hand sides of (11) and (12) we obtain
\[ \sum_{i=1}^{r} (z_i - t_i s_i) \mathbf{v}_i + \sum_{i=r+1}^{m} z_i \mathbf{v}_i = \mathbf{W} \mathbf{x}. \]
Applying the Pythagorean Theorem
\[ \sum_{i=1}^{r} (z_i - t_i s_i)^2 + z^2 \sum_{i=r+1}^{m} l_i^2 = \|\mathbf{W} \mathbf{x}\|^2 \leq \varepsilon^2, \quad (13) \]
because \(\|\mathbf{x}\| = 1\) and \(\|\mathbf{W}\| = \varepsilon\).
As \(z \geq \Delta - \varepsilon\) holds almost surely by Theorem 9,
\[ \text{dist}^2(\mathbf{y}, \mathbf{F}) = \sum_{i=r+1}^{m} l_i^2 \leq \frac{\varepsilon^2}{z^2} \leq \frac{\varepsilon^2}{(\Delta - \varepsilon)^2}. \]
The order of the above estimate follows from the order of \(\varepsilon\) and \(\Delta\) of (8):
\[ \text{dist}^2(\mathbf{y}, \mathbf{F}) = \mathcal{O}\left(\frac{m + n}{mn}\right) \quad (14) \]
almost surely. Applying (14) for the left-hand side singular vectors \(\mathbf{y}_1, \ldots, \mathbf{y}_r\), by the Definition 5
\[
\mathbb{P} \{ \exists m_0, n_0 \in \mathbb{N} \text{ such that for } m \geq m_0 \text{ and } n \geq n_0 : \text{dist}^2(\mathbf{y}_i, \mathbf{F}) \leq \varepsilon^2/(\Delta - \varepsilon)^2 \} = 1
\]
for \(i = 1, \ldots, r\). Hence,
\[
\mathbb{P} \{ \exists m_0, n_0 \in \mathbb{N} \text{ such that for } m \geq m_0 \text{ and } n \geq n_0 : \text{dist}^2(\mathbf{y}_i, \mathbf{F}) \leq \varepsilon^2/(\Delta - \varepsilon)^2 , i = 1, \ldots, r \} = 1,
\]
consequently,
\[
\mathbb{P} \{ \exists m_0, n_0 \in \mathbb{N} \text{ such that for } m \geq m_0 \text{ and } n \geq n_0 : \sum_{i=1}^{r} \text{dist}^2(\mathbf{y}_i, \mathbf{F}) \leq r\varepsilon^2/(\Delta - \varepsilon)^2 \} = 1
\]
also holds, and this finishes the proof of the first statement.

The estimate for the squared distance between $G$ and a right-hand side singular vector $x$ of $A$ follows in the same way starting with $A^T y = z \cdot x$ and using the fact that $A^T$ has the same singular values as $A$. □

By Proposition 11, the individual distances between the original and the perturbed subspaces and also the sum of these distances tend to zero almost surely as $m, n \to \infty$.

Now let $A$ be a microarray on $m$ genes and $n$ conditions, with $a_{ij}$ denoting the expression level of gene $i$ under condition $j$. We suppose that $A$ is a noisy random matrix obtained by adding a Wigner-noise $W$ to the blown up matrix $B$. Let us denote by $A_1, \ldots, A_a$ the partition of the genes and by $B_1, \ldots, B_b$ the partition of the conditions with respect to the blow-up (they can also be thought of as clusters of genes and conditions).

Proposition 11 also implies the well-clustering property of the representatives of the genes and conditions in the following representation. Let $Y$ be the $m \times r$ matrix containing the left-hand side singular vectors $y_1, \ldots, y_r$ of $A$ in its columns. Similarly, let $X$ be the $n \times r$ matrix containing the right-hand side singular vectors $x_1, \ldots, x_r$ of $A$ in its columns. Let the $r$-dimensional representatives of the genes be the row vectors of $Y$: $y^1, \ldots, y^m \in \mathbb{R}^r$, while the $r$-dimensional representatives of the conditions be the row vectors of $X$: $x^1, \ldots, x^n \in \mathbb{R}^r$. Let $S^2_a(Y)$ denote the $a$-variance, introduced in [4], of the genes’ representatives

$$S^2_a(Y) = \min_{\{A'_1, \ldots, A'_a\}} \sum_{i=1}^{a} \sum_{j \in A'_i} \|y^j - \bar{y}^i\|^2,$$

where $\bar{y}^i = \frac{1}{m_i} \sum_{j \in A'_i} y^j$,

while $S^2_b(X)$ denotes the $b$-variance of the conditions’ representatives

$$S^2_b(X) = \min_{\{B'_1, \ldots, B'_b\}} \sum_{i=1}^{b} \sum_{j \in B'_i} \|x^j - \bar{x}^i\|^2,$$

where $\bar{x}^i = \frac{1}{n_i} \sum_{j \in B'_i} x^j$,

the partitions $\{A'_1, \ldots, A'_a\}$ and $\{B'_1, \ldots, B'_b\}$ varying over all $a$- and $b$-partitions of the genes and conditions, respectively.

**Theorem 12** With the above notation, under GC1, for the $a$- and $b$-variances of the representation of the microarray $A$ the relations

$$S^2_a(Y) = \mathcal{O}\left(\frac{m+n}{mn}\right) \quad \text{and} \quad S^2_b(X) = \mathcal{O}\left(\frac{m+n}{mn}\right)$$

hold almost surely.
**PROOF.** By the proof of Theorem 3 of [4] it can be easily seen that \( S_a^2(Y) \leq \sum_{j=1}^{a} \sum_{j \in A_i} \| y^j - \bar{y}^i \|^2 \) and \( S_b^2(X) \leq \sum_{i=1}^{b} \sum_{j \in B_i} \| x^j - \bar{x}^i \|^2 \), the right-hand sides being equal to the left-hand sides of (9) and (10), respectively, therefore they are also of order \( \frac{m+n}{mn} \). \( \square \)

Hence, the addition of any kind of a Wigner-noise to a rectangular matrix that has a blown up structure \( B \) will not change the order of the protruding singular values, and the block structure of \( B \) can be reconstructed from the representatives of the row and column items of the noisy matrix \( A \).

With an appropriate Wigner-noise, we can achieve that the matrix \( B + W \) in its \((i, j)\)-th block contains 1's with probability \( p_{ij} \), and 0's otherwise. That is, for \( i = 1, \ldots, a \), \( j = 1, \ldots, b \), \( l \in A_i \), \( k \in B_j \), let

\[
    w_{lk} := \begin{cases} 
        1 - p_{ij}, & \text{with probability } p_{ij} \\
        -p_{ij} & \text{with probability } 1 - p_{ij} 
    \end{cases}
\]

be independent random variables. This \( W \) satisfies the conditions of Definition 1 with entries uniformly bounded by 1, zero expectation and variance

\[
    \sigma^2 = \max_{1 \leq i \leq a; 1 \leq j \leq b} p_{ij}(1 - p_{ij}) \leq \frac{1}{4}.
\]

The noisy matrix \( A \) becomes a 0-1 matrix that can be regarded as the incidence matrix of a hypergraph on \( m \) vertices and \( n \) edges. (Vertices correspond to the genes and edges correspond to the conditions. The incidence relation depends on whether a specific gene is expressed or not under a specific condition).

By the choice (15) of \( W \), vertices of the vertex set \( A_i \) appear in edges of the edge set \( B_j \) with probability \( p_{ij} \) (set \( i \) of genes equally influences set \( j \) of conditions, like the chess-board pattern of [9]). It is a generalization of the classical Erdős–Rényi model for random hypergraphs and for several blocks, see [6]. The question, how such a chess-board pattern behind a random (especially 0-1) matrix can be found under specific conditions, is discussed in Section 5.

### 4 Perturbation results for correspondence matrices

Now the pattern matrix \( P \) contains arbitrary non-negative entries, so does the blown up matrix \( B \). Let us suppose that there are no identically zero rows or columns. We perform the correspondence transformation described below on \( B \). We are interested in the order of singular values of matrix \( A = B + W \) when the same correspondence transformation is applied to it. To this end, we introduce the following notations:
\[
D_{Brow} = \text{diag} \left( d_{Brow 1}, \ldots, d_{Brow m} \right) := \text{diag} \left( \sum_{j=1}^{n} b_{1j}, \ldots, \sum_{j=1}^{n} b_{mj} \right)
\]
\[
D_{Bcol} = \text{diag} \left( d_{Bcol 1}, \ldots, d_{Bcol n} \right) := \text{diag} \left( \sum_{i=1}^{m} b_{i1}, \ldots, \sum_{i=1}^{m} b_{in} \right)
\]
\[
D_{Arow} = \text{diag} \left( d_{Arow 1}, \ldots, d_{Arow m} \right) := \text{diag} \left( \sum_{j=1}^{n} a_{1j}, \ldots, \sum_{j=1}^{n} a_{mj} \right)
\]
\[
D_{Acol} = \text{diag} \left( d_{Acol 1}, \ldots, d_{Acol n} \right) := \text{diag} \left( \sum_{i=1}^{m} a_{i1}, \ldots, \sum_{i=1}^{m} a_{in} \right)
\]

Further, set
\[
B_{corr} := D_{Brow}^{-1/2} B D_{Bcol}^{-1/2} \quad \text{and} \quad A_{corr} := D_{Arow}^{-1/2} A D_{Acol}^{-1/2}
\]
for the transformed matrices obtained from B and A while carrying out correspondence analysis on B and the same correspondence transformation on A. It is well known [8] that the leading singular value of \(B_{corr}\) is equal to 1 and the multiplicity of 1 as a singular value coincides with the number of irreducible blocks in B. Let \(s_i\) denote a non-zero singular value of \(B_{corr}\) with unit-norm singular vector pair \(v_i, u_i\). With the transformations
\[
v_{corr i} := D_{Brow}^{-1/2} v_i \quad \text{and} \quad u_{corr i} := D_{Bcol}^{-1/2} u_i
\]
the so-called correspondence vector pairs are obtained. If the coordinates \(v_{corr i}(j), u_{corr i}(\ell)\) of such a pair are regarded as possible values of two discrete random variables \(\beta_i\) and \(\alpha_i\) (often called the \(i\)th correspondence factor pair) with the prescribed marginals, then, as in canonical analysis, their correlation is \(s_i\), and this is the largest possible correlation under the condition that they are uncorrelated with the previous random variables \(\beta_1, \ldots, \beta_{i-1}\) and \(\alpha_1, \ldots, \alpha_{i-1}\), respectively \((i > 1)\).

If \(s_1 = 1\) is a single singular value, then \(v_{corr 1}\) and \(u_{corr 1}\) are the all 1 vectors and the corresponding \(\beta_1, \alpha_1\) pair is regarded as a trivial correspondence factor pair. This corresponds to the general case. Keeping \(k \leq \text{rank}(B_{corr}) = \text{rank}(B) = \text{rank}(P)\) singular values with the coordinates of the corresponding \(k - 1\) non-trivial correspondence factor pairs, the following \((k - 1)\)-dimensional representation of the \(j\)th and \(\ell\)th categories of the underlying two discrete variables is obtained:
\[
v_{corr j} := (v_{corr 2}(j), \ldots, v_{corr k}(j)) \quad \text{and} \quad u_{corr \ell} := (u_{corr 2}(\ell), \ldots, u_{corr k}(\ell))
\]

This representation has the following optimality properties: the closeness of categories of the same variable reflects the similarity between them, while the closeness of categories of different variables reflects their frequent simultaneous occurrence. For example, B being a microarray, the representatives of similar
function genes, as well as representatives of similar conditions are close to each other; also, representatives of genes that are responsible for a given condition, are close to the representatives of those conditions. Now we prove the following.

**Proposition 13** Given the blown up matrix $B$, under GC1 there exists a constant $\delta \in (0, 1)$, independent of $m$ and $n$, such that all the $r$ non-zero singular values of $B_{corr}$ are in the interval $[\delta, 1]$, where $r = \text{rank}(B) = \text{rank}(P)$.

**PROOF.** It is easy to see that $B_{corr}$ is the blown up matrix of the $a \times b$ pattern matrix $\tilde{P}$ with entries

$$
\tilde{p}_{ij} = \frac{p_{ij}}{\sqrt{\left(\sum_{\ell=1}^{b} p_{i\ell n_{\ell}}\right)\left(\sum_{k=1}^{a} p_{kj m_{k}}\right)}}.
$$

Following the considerations of the proof of Proposition 8, the blown up matrix $B_{corr}$ has exactly $r = \text{rank}(P) = \text{rank}(\tilde{P})$ non-zero singular values that are the singular values of the $a \times b$ matrix $P' = D_{m}^{-1/2}PD_{n}^{-1/2}$ with entries

$$
p'_{ij} = \frac{p_{ij} \sqrt{m_{i} n_{j}}}{\sqrt{\left(\sum_{\ell=1}^{b} p_{i\ell n_{\ell}}\right)\left(\sum_{k=1}^{a} p_{kj m_{k}}\right)}} = \frac{p_{ij}}{\sqrt{\left(\sum_{\ell=1}^{b} p_{i\ell n_{\ell}}\right)\left(\sum_{k=1}^{a} p_{kj m_{k}}\right)}}.
$$

Since the matrix $P$ contains no identically zero rows or columns, the matrix $P'$ varies on a compact set of $a \times b$ matrices determined by the inequalities (1). The range of the non-zero singular values depends continuously on the matrix that does not depend on $m$ and $n$. Therefore, the minimum non-zero singular value does not depend on $m$ or $n$. Because the largest singular value is 1, this finishes the proof. \(\square\)

**Theorem 14** Under GC1 and GC2, there exists a positive number $\delta$ (independent of $m$ and $n$) such that for every $0 < \tau < 1/2$ the following statement holds almost surely: the $r$ largest singular values of $A_{corr}$ are in the interval $[\delta - \max\{n^{-\tau}, m^{-\tau}\}, 1 + \max\{n^{-\tau}, m^{-\tau}\}]$, while all the others are at most $\max\{n^{-\tau}, m^{-\tau}\}$.

**PROOF.** First notice that

$$
A_{corr} = D_{Arow}^{-1/2}A^{1/2}D_{Acol}^{-1/2} = D_{Arow}^{-1/2}B D_{Acol}^{-1/2} + D_{Arow}^{-1/2}W D_{Acol}^{-1/2}.
$$

The entries of $D_{Brow}$ and those of $D_{Bcol}$ are of order $\Theta(n)$ and $\Theta(m)$, respectively. Now we prove that for every $i = 1, \ldots, m$ and $j = 1, \ldots, n$ $|d_{Brow,i} - d_{Brow,i}| < n \cdot n^{-\tau}$ and $|d_{Acol,j} - d_{Bcol,j}| < m \cdot m^{-\tau}$ hold almost surely. To this
end, we use Chernoff’s inequality for large deviations (cf. [5], Lemma 4.2):

\[
\mathbb{P}\left(|d_{\text{arrow} \ i} - d_{\text{brow} \ i}| > n \cdot n^{-\tau}\right) = \mathbb{P}\left(\left|\sum_{j=1}^{n} w_{ij}\right| > n^{1-\tau}\right)
\]

\[
< \exp\left\{-\frac{n^{2-2\tau}}{2(\text{Var } (\sum_{j=1}^{n} w_{ij}) + K n^{1-\tau} / 3)}\right\} \leq \exp\left\{-\frac{n^{2-2\tau}}{2(n\sigma^2 + K n^{1-\tau} / 3)}\right\}
\]

\[
= \exp\left\{-\frac{n^{1-2\tau}}{2(\sigma^2 + K n^{-\tau} / 3)}\right\} \quad (i = 1, \ldots, m),
\]

where the constant \( K \) is the uniform bound for \( |w_{ij}| \)'s and \( \sigma^2 \) is the bound for their variances. In virtue of \( \text{GC2} \) the following estimate holds with some \( C_0 > 0 \) and \( C \geq 1 \) (constants of \( \text{GC2} \)) and large enough \( n \):

\[
\mathbb{P}\left(|d_{\text{arrow} \ i} - d_{\text{brow} \ i}| > n^{1-\tau} \text{ for all } \ i \in \{1, \ldots, m\}\right)
\]

\[
\leq m \cdot \exp\left\{-\frac{n^{1-2\tau}}{2(\sigma^2 + K n^{-\tau} / 3)}\right\} \leq C_0 \cdot n^C \cdot \exp\left\{-\frac{n^{1-2\tau}}{2(\sigma^2 + K n^{-\tau} / 3)}\right\}
\]

\[
= \exp\left\{\ln C_0 + C \ln n - \frac{n^{1-2\tau}}{2(\sigma^2 + K n^{-\tau} / 3)}\right\}.
\]

(17)

The estimation of probability

\[
\mathbb{P}\left(|d_{\text{col} \ j} - d_{\text{bcol} \ j}| > m^{1-\tau} \text{ for all } \ j \in \{1, \ldots, n\}\right)
\]

can be treated analogously (with \( D_0 > 0 \) and \( D \geq 1 \) of \( \text{GC2} \)). The right-hand side of (17) forms a convergent series, therefore

\[
\min_{i \in \{1, \ldots, m\}} |d_{\text{arrow} \ i}| = \Theta(n), \quad \min_{j \in \{1, \ldots, n\}} |d_{\text{col} \ j}| = \Theta(m)
\]

(18)

hold almost surely.

Now it is straightforward to bound the norm of the second term of (16) by

\[
\|D_{\text{arrow}}^{-1/2}\| \cdot \|W\| \cdot \|D_{\text{bcol}}^{-1/2}\|.
\]

(19)

As by Lemma 7, \( \|W\| = \mathcal{O}(\sqrt{m + n}) \) holds almost surely, the quantity (19) is at most of order \( \sqrt{\frac{m+n}{mn}} \) almost surely. Hence, it is almost surely less than \( \max\{n^{-\tau}, m^{-\tau}\} \).

To estimate the norm of the first term of (16) let us write it in the form

\[
D_{\text{arrow}}^{-1/2}BD_{\text{bcol}}^{-1/2} = D_{\text{arrow}}^{-1/2}BD_{\text{bcol}}^{-1/2} + \left[D_{\text{arrow}}^{-1/2} - D_{\text{brow}}^{-1/2}\right]BD_{\text{bcol}}^{-1/2}
\]

\[
+ D_{\text{arrow}}^{-1/2}B\left[D_{\text{bcol}}^{-1/2} - D_{\text{bcol}}^{-1/2}\right].
\]

(20)
The first term is just \( B_{\text{corr}} \), so due to Proposition 13, we should prove only that the norms of both remainder terms are almost surely less than \( \max\{n^{-\tau}, m^{-\tau}\} \). These two terms have a similar appearance, therefore it is enough to estimate one of them. For example, the second term can be bounded by

\[
\|D_{\text{Arow}}^{-1/2} - D_{\text{Brow}}^{-1/2}\| \cdot \|B\| \cdot \|D_{\text{Bcol}}^{-1/2}\|. \tag{21}
\]

The estimation of the first factor in (21) is as follows:

\[
\|D_{\text{Arow}}^{-1/2} - D_{\text{Brow}}^{-1/2}\| = \max_{i \in \{1, \ldots, m\}} \left( \frac{1}{\sqrt{d_{\text{Arow}i}}} - \frac{1}{\sqrt{d_{\text{Brow}i}}} \right)
\]

\[
= \max_{i \in \{1, \ldots, m\}} \frac{|d_{\text{Arow}i} - d_{\text{Brow}i}|}{\sqrt{d_{\text{Arow}i} \cdot d_{\text{Brow}i}} \cdot (\sqrt{d_{\text{Arow}i}} + \sqrt{d_{\text{Brow}i}})}
\]

\[
\leq \max_{i \in \{1, \ldots, m\}} \frac{|d_{\text{Arow}i} - d_{\text{Brow}i}|}{\sqrt{d_{\text{Arow}i} \cdot d_{\text{Brow}i}}} \cdot \max_{i \in \{1, \ldots, m\}} \left( \frac{1}{\sqrt{d_{\text{Arow}i}} + \sqrt{d_{\text{Brow}i}}} \right).
\]  \tag{22}

By relations (18), \( \sqrt{d_{\text{Arow}i} \cdot d_{\text{Brow}i}} = \Theta(n) \) for any \( i = 1, \ldots, m \), and hence,

\[
\frac{|d_{\text{Arow}i} - d_{\text{Brow}i}|}{\sqrt{d_{\text{Arow}i} \cdot d_{\text{Brow}i}}} \leq n^{-\tau}
\]

almost surely, further \( \max_{i \in \{1, \ldots, m\}} \frac{1}{\sqrt{d_{\text{Arow}i} + \sqrt{d_{\text{Brow}i}}}} = \Theta(\frac{1}{\sqrt{n}}) \) almost surely.

Therefore the left hand side of (22) can be estimated by \( n^{-\tau-1/2} \) from above almost surely. For the further factors in (21) we obtain \( \|B\| = \Theta(\sqrt{mn}) \) (see Proposition 8), while \( \|D_{\text{Bcol}}^{-1/2}\| = \Theta(\frac{1}{\sqrt{m}}) \) almost surely. These together imply that

\[
n^{-\tau-1/2} \cdot n^{1/2}m^{1/2} \cdot m^{-1/2} \leq n^{-\tau} \leq \max\{n^{-\tau}, m^{-\tau}\}.
\]

This finishes the estimation of the first term in (16), and by he Weyl’s perturbation theorem the proof, too. \( \square \)

**Remark 15** In the Gaussian case the large deviation principle can be replaced by the simple estimation of the Gaussian probabilities with any \( \kappa > 0 \):

\[
P\left( \left| \frac{1}{n} \sum_{j=1}^{n} w_{ij} \right| > \kappa \right) < \min \left( 1, \frac{4\sigma}{\kappa \sqrt{2\pi n}} \exp \left\{ -\frac{n}{2\sigma^2 \kappa^2} \right\} \right).
\]

Setting \( \kappa = n^{-\tau} \) we get an estimate, analogous to (17).

Suppose that the blown up matrix \( B \) is irreducible and its non-negative entries sum up to 1. This restriction does not effect the result of the correspondence analysis, that is the SVD of the matrix \( B_{\text{corr}} \). Remember that the non-zero singular values of \( B_{\text{corr}} \) are the numbers \( 1 = s_1 > s_2 \geq \cdots \geq s_r > 0 \) with unit-norm singular vector pairs \( v_i, u_i \) having piecewise constant structure.
(i = 1, \ldots, r). Set 
\[ F := \text{Span} \{ v_1, \ldots, v_r \} \quad \text{and} \quad G := \text{Span} \{ u_1, \ldots, u_r \}. \]

Let \( 0 < \tau < 1/2 \) be arbitrary and \( \epsilon := \max\{ n^{-\tau}, m^{-\tau} \} \). Let us also denote the unit-norm, pairwise orthogonal left- and right-hand side singular vectors corresponding to the \( r \) singular values \( z_1, \ldots, z_r \in [\delta - \epsilon, 1 + \epsilon] \) of \( A_{\text{corr}} \) – guaranteed by Theorem 14 under \( GC2 \) – by \( y_1, \ldots, y_r \in \mathbb{R}^m \) and \( x_1, \ldots, x_r \in \mathbb{R}^n \), respectively.

**Proposition 16** With the above notation, under \( GC1 \) and \( GC2 \) the following estimate holds almost surely for the distance between \( y_i \) and \( F \):
\[
\text{dist}(y_i, F) \leq \frac{\epsilon}{(\delta - \epsilon)} = \frac{1}{(\frac{\delta}{\epsilon} - 1)} \quad (i = 1, \ldots, r) \tag{23}
\]
and analogously, for the distance between \( x_i \) and \( G \):
\[
\text{dist}(x_i, G) \leq \frac{\epsilon}{(\delta - \epsilon)} = \frac{1}{(\frac{\delta}{\epsilon} - 1)} \quad (i = 1, \ldots, r). \tag{24}
\]

**PROOF.** Follow the method of proving Proposition 11 – under \( GC1 \) – with \( \delta \) instead of \( \Delta \) and \( \epsilon \) instead of \( \epsilon \). Here \( GC2 \) is necessary only for \( A_{\text{corr}} \) to have \( r \) protruding singular values. □

**Remark 17** The left-hand sides of (23) and (24) are almost surely of order \( \max\{ n^{-\tau}, m^{-\tau} \} \) that tend to zero as \( m, n \to \infty \) under \( GC1 \) and \( GC2 \).

Proposition 16 implies the well-clustering property of the representatives of the two discrete variables by means of the noisy correspondence vector pairs
\[
y_{\text{corr}} i := D_{\text{Arow}}^{-1/2} y_i, \quad x_{\text{corr}} i := D_{\text{Acol}}^{-1/2} x_i \quad (i = 1, \ldots, r).
\]

Let \( Y_{\text{corr}} \) denote the \( m \times r \) matrix that contains the left-hand side vectors \( y_{\text{corr}} 1, \ldots, y_{\text{corr}} r \) in its columns. Similarly, let \( X_{\text{corr}} \) denote the \( n \times r \) matrix that contains the right-hand side vectors \( x_{\text{corr}} 1, \ldots, x_{\text{corr}} r \) in its columns. The \( r \)-dimensional representatives of \( \alpha \) are the row vectors of \( Y_{\text{corr}} \) denoted by \( y_{\text{corr}} 1, \ldots, y_{\text{corr}} m \in \mathbb{R}^r \), while the \( r \)-dimensional representatives of \( \beta \) are the row vectors of \( X_{\text{corr}} \) denoted by \( x_{\text{corr}} 1, \ldots, x_{\text{corr}} n \in \mathbb{R}^r \). With respect to the marginal distributions, let the \( a \)- and \( b \)-variances of these representatives be defined by
\[
S_a^2(Y_{\text{corr}}) = \min_{\{ \Lambda_1', \ldots, \Lambda_a' \}} \sum_{i = 1}^a \sum_{j \in \Lambda_i'} d_{\text{Arow}} j \| y_{\text{corr}} j - \hat{y}_{\text{corr}} i \|^2,
\]
\[ S_b^2(X_{\text{corr}}) = \min_{\{B'_1, \ldots, B'_b\}} \sum_{i=1}^b \sum_{j \in B'_i} d_{\text{Aclo}} j \| x^{(j)}_{\text{corr}} - x^i_{\text{corr}} \|^2, \]

where \( \{A'_1, \ldots, A'_a\} \) and \( \{B'_1, \ldots, B'_b\} \) are \( a \)- and \( b \)-partitions of the genes and conditions, respectively.

\[ y^i_{\text{corr}} = \sum_{j \in A'_i} d_{\text{Arow}} j y^j_{\text{corr}} \quad \text{and} \quad x^i_{\text{corr}} = \sum_{j \in B'_i} d_{\text{Aclo}} j x^j_{\text{corr}}. \]

**Theorem 18** With the above notation, under GC1 and GC2,

\[ S_a^2(Y_{\text{corr}}) \leq \frac{r}{(\frac{\epsilon}{\epsilon}-1)^2} \quad \text{and} \quad S_b^2(X_{\text{corr}}) \leq \frac{r}{(\frac{\epsilon}{\epsilon}-1)^2} \]

hold almost surely, where \( \epsilon = \max\{n^{-\tau}, m^{-\tau}\} \) with every \( 0 < \tau < 1/2 \).

**PROOF.** An easy calculation shows that

\[ S_a^2(Y_{\text{corr}}) \leq \sum_{i=1}^a \sum_{j \in A_i} d_{\text{Arow}} j \| y^j_{\text{corr}} - y^i_{\text{corr}} \|^2 = \sum_{i=1}^r \text{dist}^2(y_i, F), \]

\[ S_b^2(X_{\text{corr}}) \leq \sum_{i=1}^b \sum_{j \in B_i} d_{\text{Aclo}} j \| x^{(j)}_{\text{corr}} - x^i_{\text{corr}} \|^2 = \sum_{i=1}^r \text{dist}^2(x_i, G), \]

hence the result of Proposition 16 can be used. \( \square \)

Under GC1 and GC2 with \( m, n \) large enough, Theorem 18 implies that after performing correspondence analysis on the noisy matrix \( A \), the representation through the correspondence vectors belonging to \( A_{\text{corr}} \) will also reveal the block structure behind \( A \).

### 5 Recognizing the structure

One might wonder where the singular values of an \( m \times n \) matrix \( A = (a_{ij}) \) are located if \( a := \max_{i,j} |a_{ij}| \) is independent of \( m \) and \( n \). On one hand, the maximum singular value cannot exceed \( O(\sqrt{mn}) \), as it is at most \( \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2} \).

On the other hand, let \( Q \) be an \( m \times n \) random matrix with entries \( a \) or \(-a\) (indepedently of each other). Consider the spectral norm of all such matrices and take the minimum of them: \( \min_{Q \in \{-a, +a\}^{m \times n}} \|Q\| \). This quantity measures the minimum linear structure that a matrix of the same size and magnitude as \( A \) can possess. As the Frobenius norm of \( Q \) is \( a\sqrt{mn} \), in virtue of inequalities between spectral and Frobenius norms, the above minimum is at least \( \frac{a}{\sqrt{2}} \sqrt{m + n} \), which is exactly the order of the spectral norm of a Wigner-noise.
So an \( m \times n \) random matrix (whose entries are independent and uniformly bounded) under very general conditions has at least one singular value of order greater than \( \sqrt{m+n} \). Suppose there are \( k \) such singular values and the representatives by means of the corresponding singular vector pairs can be well classified in the sense of Theorem 12 (cf. the introduction to that theorem). Under these conditions we can reconstruct a blown up structure behind our matrix.

**Theorem 19** Let \( A_{m\times n} \) be a sequence of \( m \times n \) matrices, where \( m \) and \( n \) tend to infinity. Assume, that \( A_{m\times n} \) has exactly \( k \) singular values of order greater than \( \sqrt{m+n} \) (\( k \) is fixed). If there are integers \( a \geq k \) and \( b \geq k \) such that the \( a \)- and \( b \)-variances of the row- and column-representatives are \( O(\frac{m+n}{mn}) \), then there is a blown up matrix \( B_{m\times n} \) such that \( A_{m\times n} = B_{m\times n} + E_{m\times n} \), with \( \|E_{m\times n}\| = O(\sqrt{m+n}) \).

**PROOF.** The proof gives an explicit construction for \( B_{m\times n} \). In the sequel the subscripts \( m \) and \( n \) will be dropped. We shall speak in terms of microarrays (genes and conditions).

Let \( y_1, \ldots, y_k \in \mathbb{R}^m \) and \( x_1, \ldots, x_k \in \mathbb{R}^n \) denote the left- and right-hand side unit-norm singular vectors corresponding to \( z_1, \ldots, z_k \), the singular values of \( A \) of order larger than \( \sqrt{m+n} \). The \( k \)-dimensional representatives of the genes and conditions – that are row vectors of the \( m \times k \) matrix \( Y = (y_1, \ldots, y_k) \) and those of the \( n \times k \) matrix \( X = (x_1, \ldots, x_k) \), respectively – by the condition of the theorem form \( a \) and \( b \) clusters in \( \mathbb{R}^k \), respectively with sum of inner variances \( O(\frac{m+n}{mn}) \). Reorder the rows and columns of \( A \) according to the clusters. Denote by \( y^1, \ldots, y^m \in \mathbb{R}^k \) and \( x^1, \ldots, x^n \in \mathbb{R}^k \) the Euclidean representatives of the genes and conditions (the rows of the reordered \( Y \) and \( X \)), and let \( \bar{y}^1, \ldots, \bar{y}^a \in \mathbb{R}^k \) and \( \bar{x}^1, \ldots, \bar{x}^b \in \mathbb{R}^k \) denote the cluster centers, respectively. Now let us choose the following new representation of the genes and conditions. The genes’ representatives be row vectors of the \( m \times k \) matrix \( \tilde{Y} \) such that the first \( m_1 \) rows of \( \tilde{Y} \) be equal to \( \tilde{y}^1 \), the next \( m_2 \) rows to \( \tilde{y}^2 \), and so on, the last \( m_a \) rows of \( \tilde{Y} \) be equal to \( \tilde{y}^a \); similarly, the conditions’ representatives be row vectors of the \( n \times k \) matrix \( \tilde{X} \) such that the first \( n_1 \) rows of \( \tilde{X} \) be equal to \( \tilde{x}^1 \), and so on, the last \( n_b \) rows of \( \tilde{X} \) be equal to \( \tilde{x}^b \).

By the considerations of Theorem 12 and the assumption for the clusters,

\[
\sum_{i=1}^{k} \text{dist}^2(y_i, F) = S^2_a(Y) = O\left(\frac{m+n}{mn}\right) \tag{25}
\]

and

\[
\sum_{i=1}^{k} \text{dist}^2(x_i, G) = S^2_b(X) = O\left(\frac{m+n}{mn}\right) \tag{26}
\]
hold respectively, where the $k$-dimensional subspace $F \subset \mathbb{R}^m$ is spanned by the column vectors of $\tilde{Y}$, while the $k$-dimensional subspace $G \subset \mathbb{R}^n$ is spanned by the column vectors of $\tilde{X}$. We follow the construction given in [4] (see Proposition 2) of a set $v_1, \ldots, v_k$ of orthonormal vectors within $F$ and another set $u_1, \ldots, u_k$ of orthonormal vectors within $G$ such that

$$\sum_{i=1}^{k} \|y_i - v_i\|^2 = \min_{v'_1, \ldots, v'_k} \sum_{i=1}^{k} \|y_i - v'_i\|^2 \leq 2 \sum_{i=1}^{k} \text{dist}^2(y_i, F)$$  \hspace{1cm} (27)$$

and

$$\sum_{i=1}^{k} \|x_i - u_i\|^2 = \min_{u'_1, \ldots, u'_k} \sum_{i=1}^{k} \|x_i - u'_i\|^2 \leq 2 \sum_{i=1}^{k} \text{dist}^2(x_i, G)$$  \hspace{1cm} (28)$$

hold, where the minimum is taken over orthonormal sets of vectors $v'_1, \ldots, v'_k \in F$ and $u'_1, \ldots, u'_k \in G$, respectively. The construction of the vectors $v_1, \ldots, v_k$ is as follows ($u_1, \ldots, u_k$ can be constructed in the same way). Let $v'_1, \ldots, v'_k \in F$ an arbitrary orthonormal system (obtained e.g., by the Schmidt orthogonalization method). Let $V' = (v'_1, \ldots, v'_k)$ be $m \times k$ matrix and

$$Y^T V' = QSZ^T$$

be SVD, where the matrix $S$ contains the singular values of the $k \times k$ matrix $Y^T V'$ in its main diagonal and zeros otherwise, while $Q$ and $Z$ are $k \times k$ orthogonal matrices (containing the corresponding unit norm singular vector pairs in their columns). The orthogonal matrix $R = ZQ^T$ will give the convenient orthogonal rotation of the vectors $v'_1, \ldots, v'_k$. That is, the column vectors of the matrix $V = V'R$ form also an orthonormal set that is the desired set $v_1, \ldots, v_k$.

Define the error terms $r_i$ and $q_i$, respectively:

$$r_i = y_i - v_i \quad \text{and} \quad q_i = x_i - u_i \quad (i = 1, \ldots, k).$$

In view of (25) – (28),

$$\sum_{i=1}^{k} \|r_i\|^2 = O\left(\frac{m+n}{mn}\right) \quad \text{and} \quad \sum_{i=1}^{k} \|q_i\|^2 = O\left(\frac{m+n}{mn}\right).$$  \hspace{1cm} (29)$$

Consider the following decomposition:

$$A = \sum_{i=1}^{k} z_i y_i x_i^T + \sum_{i=k+1}^{\min\{m,n\}} z_i y_i x_i^T.$$  \hspace{1cm} (27)$$

The spectral norm of the second term is at most of order $\sqrt{m+n}$. Now con-
Consider the first term,
\[
\sum_{i=1}^{k} z_i y_i x_i^T = \sum_{i=1}^{k} z_i (v_i + r_i)(u_i^T + q_i^T) =
\]
\[
= \sum_{i=1}^{k} z_i v_i u_i^T + \sum_{i=1}^{k} z_i v_i q_i^T + \sum_{i=1}^{k} z_i r_i u_i^T + \sum_{i=1}^{k} z_i r_i q_i^T.
\]
(30)

Since \(v_1, \ldots, v_k\) and \(u_1, \ldots, u_k\) are unit vectors, the last three terms in (30) can be estimated by means of the relations
\[
\|v_i u_i^T\| = \sqrt{\|v_i v_i^T u_i u_i^T\|} = 1 \quad (i = 1, \ldots, k),
\]
\[
\|v_i q_i^T\| = \sqrt{\|q_i v_i^T v_i q_i^T\|} = \|q_i\| \quad (i = 1, \ldots, k),
\]
\[
\|r_i u_i^T\| = \sqrt{\|r_i u_i^T u_i r_i^T\|} = \|r_i\| \quad (i = 1, \ldots, k),
\]
\[
\|r_i q_i^T\| = \sqrt{\|r_i q_i^T q_i r_i^T\|} = \|q_i\| \cdot \|r_i\| \quad (i = 1, \ldots, k).
\]

Taking into account that \(z_i\) cannot exceed \(\Theta(\sqrt{mn})\) and \(k\) is fixed, due to (29) we get that the spectral norms of the last three terms in (30) – for their finitely many subterms the triangle inequality is applicable – are at most of order \(\sqrt{m + n}\). Let \(B\) be the first term, i.e.,
\[
B = \sum_{i=1}^{k} z_i v_i u_i^T,
\]
then \(\|A - B\| = O(\sqrt{m + n})\).

By definition, the vectors \(v_1, \ldots, v_k\) and the vectors \(u_1, \ldots, u_k\) are in the subspaces \(F\) and \(G\), respectively. Both spaces consist of piecewise constant vectors, thus the matrix \(B\) is a blown up matrix containing \(a \times b\) blocks. The ‘noise’ matrix is
\[
E = \sum_{i=1}^{k} z_i v_i q_i^T + \sum_{i=1}^{k} z_i r_i u_i^T + \sum_{i=1}^{k} z_i r_i q_i^T + \sum_{i=k+1}^{\min\{m,n\}} z_i y_i x_i^T
\]
that finishes the proof. \(\square\)

Then, provided the conditions of Theorem 19 hold, by the construction given in the proof above, an algorithm can be written that uses several SVD’s and produces the blown up matrix \(B\). This \(B\) can be regarded as the best blown up approximation of the microarray \(A\). At the same time clusters of the genes and conditions are also obtained. More precisely, first we conclude the clusters from the SVD of \(A\), rearrange the rows and columns of \(A\) accordingly, and after we use the above construction. If we decide to perform correspondence
analysis on $A$ then by (16) and (20), $B_{corr}$ will give a good approximation to $A_{corr}$ and similarly, the correspondence vectors obtained by the SVD of $B_{corr}$ will give representatives of the genes and conditions.

To obtain SVD of large matrices, randomized algorithms are at our disposal, e.g., [1]. There is nothing to loose when applying these algorithms because they give the required results only if our matrix had a primary linear structure.

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