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Author list: \(^1\)Sadeq Hamdallah Naji and \(^2\)Emad Bakr Al-Zangana

Affiliations: \(^1,^2\) University of Mustansiriyah, College of Science, Department of Mathematics, Baghdad, Iraq

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In the Reference section, the following sources appear:

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Cubic Curves Over the Finite Field of Order Twenty Seven

Sadeq Hamdallah Naji and Emad Bakr Abdulkareem
University of Mustansiriyah, Baghdad, Iraq

Abstract

A cubic curve is a non-singular projective plane cubic curve. An \((k;3)\)-arc is a set of points no four are collinear but some three are linear. Most of the cubic curve can be regarded as an arc of degree three. In this paper, the projectively inequivalent cubic curves have been classified over the finite field of order twenty-seven with respect to its inflexions points and determined if they are complete or incomplete as arcs of degree three. Also the size of the largest arc of degree three that can be constructed form each incomplete cubic curve are given. The main conclusion that can be drawn is that, over \(\mathbb{F}_{27}\), the largest an arc of degree three can be constructed depending on the cubic curve is 38; that is, \(38 \leq m_3(2,27) \leq 55\).

Key words: Finite projective geometry, Non-singular cubic curve, Arc, Rational inflexion.

1. Introduction

A Galois field \(GF(\mathbb{F}) = \mathbb{F}_q\) of \(q\) elements, \(q = p^h\) where \(p\) is prime number and \(h\) is natural number is a finite field of \(q\) elements. Here \(p\) is called the characteristic of this field. Let the set \(V(3, q) = \{x = (x_1, x_2, x_3) \mid x_i \in \mathbb{F}_q\}\) be three dimensional vector space with entries in \(\mathbb{F}_q\), and \(PG(2, q)\) be the corresponding projective plane. The points \(P(x) = [x_1, x_2, x_3]\) of \(PG(2, q)\) are the one dimensional subspace of \(V(3, q)\), and the lines of \(PG(2, q)\) are the two dimensional subspace of \(V(3, q)\).

A \((k;r)\)-arc (arc of degree three) in \(PG(2, q)\) is a set of \(k\) points with property that, no \(r + 1\) of which are collinear and there is \(r\) of points in some lines. When \(k\) achieves its maximum value, this value will denoted by \(m_r(2, q)\). An \((n;r)\)-arc is complete if it is not contained in an \((n + 1;r)\)-arc.

Let \(F(X_0, X_1, X_2) = \sum_{0 \leq i \leq 2} a_i X_i X_0^2\) be a form of degree three. In \(PG(2, q)\), a projective plane cubic curve \(\mathcal{F}\) for the homogeneous polynomial \(F\), \(v(F)\) is the variety of \(F\); that is, is the zero set of the homogeneous cubic equation \(F\) in three variables over \(\mathbb{F}_q\).

\[\mathcal{F} = v(F) = \{P(X) \in PG(2, q) | F(X) = 0\}.\]
A point $P(X)$ of $F$ is a rational point of $\mathcal{F}$.

A non-singular (simple) point $P$ on $\mathcal{F}$ is a rational point with minimum intersection multiplicity one (there is a unique tangent through $P$). If all the rational points of $\mathcal{F}$ are non-singular, then the curve $\mathcal{F}$ is called non-singular projective plane cubic curve. This curve will call a cubic curve for shot.

A point $P$ on a cubic curve $\mathcal{F}$ is called inflexion if the unique tangent at $P$ of $\mathcal{F}$ has three points contact. cubic curve $\mathcal{F}$ is called harmonic or equianharmonic if the four tangents through a point form a harmonic or equianharmonic set. It is called general (denoted by $\mathcal{G}$) if it is not harmonic and equianharmonic and superharmonic if it is harmonic and equianharmonic (denoted by $S$).

gave by in the finite projective plane of cubic curves al details a rich gen [1] gave in Hirshfeld the maximum $N_q$ and minimum $L_q$ bounds of the number of rational points on cubic curves, the number of inflexion points on a cubic curve which are 0, 1, 3 or 9, and by total number of projective, distinct cubic curves $P_q$, the number of inflexions according to, and general form of cubic curves with certain inflexion points. In [2][3], a further studied of cubic curves focusing on the number of rational points on the curve in the plane. Recently, many authors continuing care about the bounds of rational points of algebraic curve in finite projective space of dimension greater than 2, as in [4][5][6][7][8].

The related theorems about these details which are given by Hirschfeld [1, Chapter 11] and relevant to our research are summarized in the next theorem.

**Theorem 1.1.** Over $F_q$, the followings are holds:

(i) The bounds $N_q$ and $L_q$ are as follows:

(1) $N_q = \begin{cases} 
q + [2 \sqrt{q}] & \text{if } q \text{ is exceptional} \\
q + 1 + [2 \sqrt{q}] & \text{if } q \text{ is non-exceptional}
\end{cases}$

(2) $L_q = \begin{cases} 
q + 2 - [2 \sqrt{q}] & \text{if } q \text{ is exceptional} \\
q + 1 - [2 \sqrt{q}] & \text{if } q \text{ is non-exceptional}
\end{cases}$

where the prime power $q = p^h$ is exceptional if $h$ is odd, $h \geq 3$, and $p$ divides $[2 \sqrt{q}]$.

(ii) The number of rational inflexions of cubic curves is

$0, 1, 3 \quad \text{if } q \equiv 0 \text{ or } 2 \pmod{3}$

$0, 1, 3, 9 \quad \text{if } q \equiv 1 \pmod{3}.$

(iii) Let $n_i$ for $= 0, 1, 3, 9$ be the number of inequivalent cubic curves with exactly $i$ rational inflexions. Then
\[ P_q = 2q + 2 + \left(\frac{-4}{q}\right) + \left(\frac{-3}{q}\right)^2 + 3\left(\frac{-3}{q}\right). \]

Hence \( P_q = n_0 + n_1 + n_2 + n_9 \), where \( \left(\frac{a}{q}\right) \) is Legendre and Legendre–Jacobi formula.

The theorems that give the general form of the cubic curves with certain inflexion points are given in the next sections.

The problem of maximum value of \( k, m_2(2, q) \) for a \((k;2)\)-arc was determined by Bose [9] which is

\[
m_2(2, q) = \begin{cases} 
q + 1 & \text{if } q \text{ odd} \\
q + 2 & \text{if } q \text{ even}
\end{cases}
\]

where the \((q + 1; 2)\)-arc is just the conic(projective plane curve of degree two) for \( q \) odd and \((q + 2; 2)\)-arc is conic plus its nucleus for \( q \) even. This problem is difficult to be determined in the case of arc of degree three. There are many mathematician worked in this problem especially when the linked between coding theory cryptography with finite projective space found during the last century[10][11]. The studied of an \((k; r)\)-arc was begun in the \( 50^{th} \) of the last century especially Barlotti [12] whose gave an upper bound by proving that, if \((r, q) \neq (2^t, 2^h)\) and \( 2 < r < q \), then

\[ k \leq (r - 1)q + r - 2. \]

This bound has been improved later on and reformulated by putting conditions on \( q \) and \( r \) also, in terms of linear codes. Ball and Hirschfeld [10] discussed the bound of \((k; r)\)-arc in the plane and gave latest bound and as special case at \( r = 3 \), showed that

\[
q + 1 + \left\lfloor \frac{2\sqrt{q}}{q} \right\rfloor \quad \text{if } q \text{ is non-exceptional;}
\]

\[
q + \left\lfloor 2\sqrt{q} \right\rfloor \quad \text{if } q \text{ is exceptional;}
\]

where the prime power \( q = p^h \) is exceptional if \( h \) is odd, \( h \geq 3 \), and \( p \) divides \( 2\sqrt{q} \).

Since a cubic curve with \( k \) rational points can be regarded as \( a(k; 3)\)-arc when \( q \) is large, so it is possible to use the rational points of a cubic curve as an initial arc of degree three and then extension it to large arc of degree three.

This technique is of central interest as much recent researches for example, Daskalov and Jiménez Contreras [13], constructed an \((k; r)\)-arc in over \( F_{13} \), Giulietti et al. [14] construct complete arcs on the curve. Also, Al-zangana[15][16] gave a classification of the cubic curves by its types and the exact length of complete arcs of degree three on the cubic curves over \( F_q, q = 11, 13, 19 \), and for \( F_{17} \), Al-Seraji [17] has gave all the distinct cubic curves. The recent work related to classification of all \((k; 3)\)-arcs with need to the cubic curve was by Cook[18] over \( F_{11} \).

The main goal is to construct large \((k; 3)\)-arcs in \( PG(2, 27) \) with \( k \) nearest integer to the upper bound to \( k \) through rational points of cubic curves by adding some extra points, and to do that the following steps have been taken:
1- Find the projectively distinct cubic curves.

2- Compute the rational inflexion points of each curve and then checked if they form an arc of degree three or not.

3- Give characterization of the arc by determined if they formed a complete or incomplete arc and give its stabilizer group type.

4- The incomplete arc extended until get the complete one with maximum length.

A mathematical programming language GAP [19] has been used to execute the programs that made to do the calculations.

2. Projectively distinct Cubic Curves

This section is devoted to classify the cubic curves over $\mathbb{F}_{27}$, and divided into three sections according to the number of inflexion points which are 0, 1 or 3.

The projective plane of order 27 has 757 points $P(X)$ and lines $\ell(X)$ such that the number of points on each line is 28 and the number of lines passing through each point is 28. The primitive polynomial $f(x) = x^3 - \beta x - \beta^2$ over $\mathbb{F}_{27}$ ($f$ is irreducible over $\mathbb{F}_{27}$ and reducible over $\mathbb{F}_{27^3}$ with $\beta$ as the primitive element of $\mathbb{F}_{27}$) has been used to setting up the companion matrix of $f$, $C(f)$ which is given by the $3 \times 3$ matrix

$$
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
\beta^{15} & \beta^{14} & 0
\end{pmatrix}.
$$

The matrix $C(f)$ is cyclic of order 757 (for general case see [1]), so the points and lines of $PG(2, 27)$ are generated as follows:

$$
\begin{align*}
\mathcal{P}_i &= P(1,0,0)C(f)^i, \\
\ell_i &= \ell(0,0,1)C(f)^i, i = 0, 1, \ldots, 756.
\end{align*}
$$

Here, $\ell_0$ is just the line passing through a point whose third coordinate is zero, which formed a difference set in $PG(2, 27)$. The numerical form of $\ell_0$ is:

\{ 1, 2, 4, 10, 28, 82, 149, 168, 212, 244, 309, 356, 386, 399, 438, 445, 461, 502, 506, 555, 576, 624, 634, 659, 674, 725, 730, 747 \}.

The corollaries are deducted from Theorem 1.

**Corollary 2.1:** Over $\mathbb{F}_{27},$

(i) the total number of projectively inequivalent cubic curves is 82; that is, $P_{27} = 82$;

(ii) the number of rational inflexions of non-singular cubic curves is zero, one and three;

(iii) since $27 \equiv 0 \pmod{3}$, the number of projectively inequivalent cubic curves with exactly nine rational inflexions is zero; that is, $n_9 = 0$;

(iv) $L_{27} = 18$ and $N_{27} = 38$. 


2.1. Projectively Distinct Cubic Curves with Three Rational Inflexions

The general formed of a projective cubic curve with three rational inflexions over $F_{27}$ is given in the next theorem. For more rich details see [1].

**Theorem 2.2:**

A cubic curve with three collinear rational inflexions and non- concurrent inflexional tangents has three rational inflexions and canonical form

$$\mathcal{F} = v(X_0X_1X_2 + e(X_0 + X_1 + X_2)^3)$$

for $e \neq 0, \frac{1}{27}$. When $q \equiv 0(\text{mod } 3)$, the curve $\mathcal{F}$ is non- singular for all $e$ in $F_q \setminus \{0\}$.

**Corollary 2.3:**

For all non-zero elements of $F_{27}$, the curve $\mathcal{F}$ is non- singular; that is, $\mathcal{F}$ is cubic curve.

**Theorem 2.4:**

Over $F_{27}$, the number of distinct cubic curves with three rational inflexions is 26, all of them are general.

These curves are given in Table (1) with exact canonical form $\mathcal{F}$, number of rational points $|\mathcal{F}|$, description (complete or incomplete as $(k; 3)$-arc), the maximum size of a complete arc contains each incomplete curve and the stabilizer group of each curve $G$:

| No. | Canonical form | $|\mathcal{F}|$ | Description | $M(\mathcal{F})$ | $G$ |
|-----|---------------|----------------|-------------|-----------------|----|
| 1.  | $X_0X_1X_2 + (X_0 + X_1 + X_2)^3$ | 18 | incomplete | 30 | $S_3$ |
| 2.  | $X_0X_1X_2 + \beta^{2.1}(X_0 + X_1 + X_2)^3$ | 21 | incomplete | 33 | $S_3$ |
| 3.  | $X_0X_1X_2 + \beta^{1.7}(X_0 + X_1 + X_2)^3$ | 21 | incomplete | 33 | $S_3$ |
| 4.  | $X_0X_1X_2 + \beta^{2.7}(X_0 + X_1 + X_2)^3$ | 21 | incomplete | 33 | $S_3$ |
| 5.  | $X_0X_1X_2 + \beta^{1.5}(X_0 + X_1 + X_2)^3$ | 24 | complete | - | $S_3$ |
| 6.  | $X_0X_1X_2 + \beta^{20}(X_0 + X_1 + X_2)^3$ | 24 | incomplete | 25 | $S_3$ |
| 7.  | $X_0X_1X_2 + \beta^{24}(X_0 + X_1 + X_2)^3$ | 24 | incomplete | 25 | $S_3$ |
| 8.  | $X_0X_1X_2 + \beta^{18}(X_0 + X_1 + X_2)^3$ | 24 | incomplete | 25 | $S_3$ |
| 9.  | $X_0X_1X_2 + \beta^{5}(X_0 + X_1 + X_2)^3$ | 24 | complete | - | $S_3$ |
| 10. | $X_0X_1X_2 + \beta^{19}(X_0 + X_1 + X_2)^3$ | 24 | complete | - | $S_3$ |
| 11. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 27 | complete | - | $S_3$ |
| 12. | $X_0X_1X_2 + \alpha^{7}(X_0 + X_1 + X_2)^3$ | 27 | complete | - | $S_3$ |
| 13. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 27 | complete | - | $S_3$ |
| 14. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 30 | complete | - | $S_3$ |
| 15. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 30 | complete | - | $S_3$ |
| 16. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 30 | complete | - | $S_3$ |
| 17. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 30 | complete | - | $S_3$ |
| 18. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 30 | complete | - | $S_3$ |
| 19. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 30 | complete | - | $S_3$ |
| 20. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 33 | complete | - | $S_3$ |
| 21. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 33 | complete | - | $S_3$ |
| 22. | $X_0X_1X_2 + \alpha^{9}(X_0 + X_1 + X_2)^3$ | 33 | complete | - | $S_3$ |
2.2. Projectively Distinct Cubic Curves with One Rational Inflexion

The cubic curves with one rational inflexion point has more characteristics cubic curves with three inflexions because of they have two parameters in the general form of cubics. The knowledge that related to the aim this section and a good characteristic of cubic curves with one inflexions is given below. e that theorems that related to the

Theorem 2.5 [2]:

If \( \beta = 3 \), a cubic curve defined over \( \mathbb{F}_q \), \( q = p^h \), \( h \) is the natural number with at least one rational inflexion, then the cubic form has one of two the following canonical form:

(i) \( F = X_2^2 X_1 + X_3 + b X_0^2 X_1 + X_1^3 \), where \( b d \neq 0 \) and is never harmonic;

(ii) \( F = \frac{X_2^2 X_1 + X_3}{X_0^2 X_1 + d X_1^3} \), where \( c \neq 0 \) and is always superharmonic.

In case of (i) write, \( F = \frac{X_2^2 X_1 + X_3}{X_0^2 X_1 + d X_1^3} \). The two cubic curves \( v(F) \) and \( v(\bar{F}) \) are equivalent if and only if \( b^3 / d = \bar{b}^3 / \bar{d} \) and \( b / \bar{b} \) is a square.

Corollary 2.6:

Over \( \mathbb{F}_{27} \), there are 692 order pairs \( (b, d) \) that satisfy \( b d \neq 0 \) of \( F \) and 702 order pairs \( (c, d) \) that satisfy \( c \neq 0 \) of \( \bar{F} \).

Using Theorem 2.5 and Corollary 2.6 the following theorem is deduced.

Theorem 2.7:

Over \( \mathbb{F}_{27} \), there are twenty seven cubic curves with exactly one inflexion of type \( \mathcal{G} \) are given Table (3), and three of type \( S \) are given in Table (3).

| No. | Canonical form | \( |\mathcal{F}| \) | Description | \( M(\mathcal{F}) \) | \( \mathcal{G} \) |
|-----|----------------|-------------|--------------|-----------------|-----------|
| 1   | \( X_2^2 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^2 X_1^3 \) | 20          | incomplete   | 32              | \( Z_2 \) |
| 2   | \( X_2^2 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{20} X_1^3 \) | 20          | incomplete   | 32              | \( Z_2 \) |
| 3   | \( X_2^2 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{22} X_1^3 \) | 20          | incomplete   | 32              | \( Z_2 \) |
| 4   | \( X_2^2 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{19} X_1^3 \) | 20          | incomplete   | 30              | \( Z_2 \) |
| 5   | \( X_2^2 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{19} X_1^3 \) | 23          | incomplete   | 29              | \( Z_2 \) |
| Row | Expression                                                                 | Status  | Symbol |
|-----|---------------------------------------------------------------------------|---------|--------|
| 6   | $X_0^2 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^3 X_1^3$                 | incomplete | $Z_2$ |
| 7   | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^5 X_1^3$                 | incomplete | $Z_2$ |
| 8   | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{10} X_1^3$              | incomplete | $Z_2$ |
| 9   | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{16} X_1^3$              | incomplete | $Z_2$ |
| 10  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{18} X_1^3$              | incomplete | $Z_2$ |
| 11  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{24} X_1^3$              | incomplete | $Z_2$ |
| 12  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{12} X_1^3$              | incomplete | $Z_2$ |
| 13  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{8} X_1^3$               | incomplete | $Z_2$ |
| 14  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{15} X_1^3$              | incomplete | $Z_2$ |
| 15  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{9} X_1^3$                | complete  | $Z_2$ |
| 16  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{7} X_1^3$                | complete  | $Z_2$ |
| 17  | $X_0^3 X_1 + X_0^3 + \beta^{10} X_0^2 X_1 + \beta^{13} X_1^3$           | complete  | $Z_2$ |
| 18  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{14} X_1^3$               | complete  | $Z_2$ |
| 19  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{4} X_1^3$                | complete  | $Z_2$ |
| 20  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + X_1^3$                          | complete  | $Z_2$ |
| 21  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{21} X_1^3$               | complete  | $Z_2$ |
| 22  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{25} X_1^3$               | complete  | $Z_2$ |
| 23  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{11} X_1^3$               | complete  | $Z_2$ |
| 24  | $X_0^3 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta X_1^3$                    | complete  | $Z_2$ |
| 25  | $X_0^2 X_1 + X_0 X_1^2 + \beta^2 X_0^2 X_1 + \beta^{17} X_1^3$          | complete  | $Z_2$ |
| 26  | $X_0^2 X_1 + X_0^3 + \beta^2 X_0^2 X_1 + \beta^{13} X_1^3$               | complete  | $Z_2$ |
Table (3): Cubic curves with exactly one inflexion of type S

| No. | Canonical form | $|\mathcal{F}|$ | Description | $M(\mathcal{F})$ | $G$ |
|-----|----------------|----------------|-------------|-----------------|-----|
| 1   | $X_0^2X_1 + X_0^3 + \beta X_0^2 X_1 + \beta^2 X_1^2$ | 19 | incomplete | 38 | $Z_6$ |
| 2   | $X_0^2X_1 + X_0^3 + \beta^2 X_0 X_1^2$ | 28 | incomplete | 38 | $Z_2$ |
| 3   | $X_0^3X_1 + X_0^3 + \beta^2 X_0 X_1^2 + \beta^2 X_1^3$ | 37 | complete | – | $Z_6$ |

2.3. Projectively Distinct Cubic Curves with No Rational Inflexions

In this section a non-singular cubic curves with no rational inflection are given.

**Theorem 2.8 [2]:**

Over $\mathbb{F}_q$, $q \equiv 0 \pmod{3}$, a cubic curves $\mathcal{F} = v(F)$ with no rational inflections has the canonical form

$F = X_0^3 + X_1^3 + cX_2^3 + dX_0^2X_2 + dX_0X_1^2 + d^2X_0X_2^2 + dX_1X_2^2$, where $c \neq 1$ and $X^3 + dX - 1$ is a fixed irreducible polynomial.

**Corollary 2.9:**

There are 26 ordered pairs that satisfy $c \neq 1$ and $X^3 + dX - 1$ is a fixed irreducible polynomial in $(2, 27)$.

Combining Theorem 2.8 and Corollary 2.9 with doing some calculation to find the distinct inequivalent cubic curve, stabilizer group and the complete arc the following theorem is founded.

**Theorem 2.10:**

Over $\mathbb{F}_{27}$, the number of projectively inequivalent cubic curves with no rational iniplexions is 26, all of them are of general type.

In Table (4), These curves are given with details of canonical form, number of rational numbers, description, size of complete curves and the type of the groups of stabilizers.

Table (4): Cubic curves with no rational inflexion

| No. | Canonical form | $|\mathcal{F}|$ | Description | $M(\mathcal{F})$ | $G$ |
|-----|----------------|----------------|-------------|-----------------|-----|
| 1   | $X_0^3X_1^3 + \beta^2X_2^3 + \beta^15X_0X_2 + \beta^15X_0X_1^2 +$ | 18 | incomplet | 35 | $Z_3$ |
| Equation | Description | Value | Status |
|----------|-------------|-------|--------|
| $\beta_1 X_0 X_2^2 + \beta_5 X_1 X_2^2$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_6 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 30 | Z_3 |
| $X_0^3 + X_1^3 + \beta_7 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_8 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_9 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_10 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_11 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_12 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_13 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_14 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_15 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_16 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_17 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_18 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_19 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_20 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_21 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_22 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_23 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_24 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_25 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_26 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_27 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_28 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_29 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_30 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_31 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
| $X_0^3 + X_1^3 + \beta_32 X_0 X_2 X_2 + \beta_5 X_0 X_2 X_1 + \beta_5 X_0 X_2 X_1$ | | 21 | incomple te |
3. Conclusion

Results from Theorem 2.4, 2.7 and 2.10 additionally with the Tables (1),(2),(3) and (4) f provide a basis to establish the following theorem.

**Theorem 3.1:**

In $PG(2,27)$,

(i) the 82 inequivalent cubic curves are divided into 49 complete and 33 incomplete arcs of degree three. Table (5) lists the number of each type of stabilizer group of complete and incomplete projectively distinct cubic curves.

| Table (5): Summary of groups and descriptions of cubic curves |
|---------------------------------------------------------------|
| Complete | $G$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_6$ | $S_3$ | |
| No. | 13 | 12 | 1 | 23 |
| Incomplete | $G$ | $\mathbb{Z}_2$ | $\mathbb{Z}_3$ | $\mathbb{Z}_6$ | $S_3$ | |
| No. | 15 | 10 | 1 | 7 |

(ii) Let cell $m : n$ means that $n$ is the number of points on the curve and $m$ is the number of such distinct curves. Then the cubic curves with certain inflexions with cell $m : n$ are deduced.

| 3 inflexions | 18:1 | 21:3 | 24:6 | 27:3 | 30:6 | 33:3 | 36:4 |
| 1 inflexion | 19:1 | 20:4 | 23:3 | 26:6 | 28:1 | 29:4 | 32:6 | 35:3 | 37:1 |
| 0 inflexion | 18:1 | 21:3 | 24:6 | 27:3 | 30:6 | 33:3 | 36:4 |

The data indicate in Theorem 3.1 gave the following statistic in $PG(2,27)$ about cubic curves.

**Corollary 3.2:** In $PG(2,27)$ ,

(i) $n_0 = 26$, $n_1 = 30$ and $n_3 = 26$. So, $P_{27} = 82$;
(ii) $L_{27}(1) = 18$, $N_{27}(1) = 38$ and the number of rational points on the cubic curves for $q = 27$ takes every value between 18 and 38, except the values 22, 25, 31, 34.

(iii) A cubic curve with $k$ points is a complete $(k; 3)$-arc when $k$ has one of the following values:

$$24, 25, 26, 27, 28, 29, 30, 32, 33, 35, 36, 37, 38.$$ 

Therefore, $38 \leq m_3(2, 27) \leq 55$.

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