Entropy bounds for charged and rotating systems

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Abstract

It was shown in a previous work that, for systems in which the entropy is an extensive function of the energy and volume, the Bekenstein and the holographic entropy bounds predict new results. In this paper, we go further and derive improved upper bounds to the entropy of extensive charged and rotating systems. Furthermore, it is shown that for charged and rotating systems (including non-extensive ones), the total energy that appear in both the Bekenstein entropy bound (BEB) and the causal entropy bound (CEB) can be replaced by the internal energy of the system. In addition, we propose possible corrections to the BEB and the CEB.

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I. INTRODUCTION

According to classical general relativity, the second law of thermodynamics is violated when a system crosses the event horizon of a black hole. This had led Bekenstein [1] to conjecture that the area of a black hole (in suitable units) may be regarded as the black hole entropy. In addition, Bekenstein proposed to replace the ordinary second law of thermodynamics by the Generalized Second Law (GSL): The generalized entropy, \( S_g \equiv S + S_{BH} \), of a system consisting of a black hole (with entropy \( S_{BH} \)) and ordinary matter (with entropy \( S \)) never decreases with time (for an excellent review on the thermodynamics of black holes and the validity of the GSL, see Wald [2]).

It is not very clear, however, if the validity of the GSL depends on the existence of a universal upper bound to the entropy of a bounded system: Consider a gedanken-experiment such that one lowers adiabatically a spherical box of radius \( R \) toward a black hole (Geroch process). The box is lowered from infinity where the total energy of the box plus matter contents is \( E \). It was shown [3] that the entropy \( S \) of the box must obey (throughout the paper \( k_B = 1 \))

\[
S \leq \frac{2\pi RE}{\hbar c},
\]

in order to preserve the GSL. Recently, universal entropy bounds for charged and rotating systems have been proposed by several authors [4–8].

However, the naive derivation of Eq. (1) in [3] was criticized by Unruh and Wald [9,10] who have argued that, since the process of lowering the box is a quasi-static one (and therefore can be considered as a sequence of static-accelerating boxes), the box should experience a buoyant force due to the Unruh radiation [11]. Describing the acceleration radiation as a fluid, they have shown that this buoyant force alters the work done by the box such that no entropy bound in the form of Eq. (1) is necessary for the validity of the GSL. A few years ago, Pelath and Wald [12] gave further arguments in favor of this result.

Bekenstein [13,14], on the other hand, argued that, only for very flat systems, the Unruh-Wald effect may be important. Later on, he has shown [15] that, if the box is not almost at the horizon, the typical wavelengths in the radiation are larger than the size of the box and, as a result, the derivation of the buoyant force from a fluid picture is incorrect. The question of whether the Bekenstein bound follows from the GSL via the Geroch process remains controversial (see [16,2,17,18]). However, as it was shown by Bousso [19] (see the following paragraphs), there is another link connecting the GSL with the Bekenstein bound.

Susskind [20] has shown, by considering the conversion of a system to a black hole, that the GSL implies a spherical entropy bound (SEB)

\[
S \leq \frac{1}{4l_p^2} A,
\]

where \( S \) is the entropy of a system that can be enclosed by a sphere with area \( A \). A few years later, Bousso [21,22] had found an elegant way to generalize Eq. (2) and write it in a covariant form. He proposed the covariant entropy bound: “the entropy on any light-sheet \( L(B) \) of a surface \( B \) will not exceed the area of \( B \)”. That is,
\[ S[L(B)] \leq \frac{A(B)}{4l_p^2}, \quad (3) \]

where the light-sheet \( L[B] \) is constructed by the light rays that emanate from the surface \( B \) and are not expanding (for an excellent review see Bousso [22]).

Recently, Flanagan, Marolf and Wald [23] have generalized Eq. (3) into the following form:

\[ S[L(B, B')] \leq \frac{A(B) - A(B')}{4l_p^2}, \quad (4) \]

where \( L(B, B') \) is a light-sheet which starts at the cross-section \( B \) and cuts off at the cross-section \( B' \) before it reaches a caustic. They were motivated by the argument that when a matter system with initial entropy \( S \) falls into a black hole, the horizon surface area increases at least by \( 4l_p^2 S \) due to the GSL.

Unlike the controversial issues regarding the relationship between the GSL and Eq. (1), the entropy bounds in Eqs. (2, 3, 4) are closely related to the GSL. However, very recently, Bousso [19] has shown that the BEB follows from Eq. (4) for any isolated, weakly gravitating system. Hence, even though it is not clear whether quantum effects should be taken into consideration in the derivation of Eq. (1) (via the Geroch process), there is a strong link between the GSL and the Bekenstein bound.

In a previous work [24], it was shown that there is another link connecting the bound (1) with the entropy of thermal radiation and the Stephan-Boltzmann law. In our derivation, we have considered systems in which the entropy is an extensive function of the energy. We have also showed that for such systems, the SEB (2) yields the causal entropy bound (CEB) proposed by Brustein and Veneziano [25] and by Sasakura [26]. In the present paper, we generalize our results for charged rotating systems. The importance for such generalizations stems from the following.

The entropies of closed systems in flat spacetime are usually much smaller than the entropy bound (1) or its generalizations for charged and rotating systems [4–8] (see also section III and IV). Furthermore, since these bounds are not extensive functions of the energy, one can expect that there are much tighter bounds that are applicable only for extensive systems. As we shall see in this paper (see also our previous work [24]), such extensive bounds follow from the BEB and also from the SEB. Thus, in order to test the Bekenstein, as well as the spherical, entropy bounds, it is enough to test whether their predictions on extensive systems are valid.

For example, in our previous work, it was shown that the BEB (1) implies that the entropy of extensive spherical systems can not exceed \( (ER/\hbar c)^{3/4} \) (up to numerical factor). Since the entropy of thermal radiation saturates this extensive bound, it is a good guess that no other system has more entropy \(^1\). However, if one can find an extensive system which exceeds this bound, it will be a counter example to the BEB. As we shall see here,

\(^1\)The rest mass of ordinary particles only enhances gravitational instability without contributing to the entropy
the extensive entropy bound of charged rotating systems is tighter than \((ER/\hbar c)^{3/4}\) and therefore it provides a new challenge on the non-extensive entropy bounds.

This paper is organized as follows: In section II we discuss the validity of the BEB and we propose to include a logarithmic correction term. In sections III and IV, we derive new entropy bounds for extensive charged and rotating systems (i.e., systems with relatively small charge and small angular momentum). This bounds turn out to be much smaller than the non-extensive ones. In section V, the SEB is applied to extensive charged-rotating systems and new extensive bounds are proposed. In section VI we summarize our results and consider non-extensive systems. We propose improved entropy bounds for charged and rotating systems that generalize both the BEB and the CEB in a natural way. We also obtain a correction term to the CEB.

II. THE VALIDITY OF THE BEB

Beside the question whether the bound (1) is needed for the validity of the GSL, one may ask under what conditions it does hold. For composites of non-relativistic particles the bound is trivially satisfied since the entropy is of the same order as the number of particles involved. As we shall see now, there is one exception to this argument, but also then the bound is satisfied due to some other physical arguments. First, we would like to confirm that a system of \(N\) non-relativistic particles, each with mass \(m\), cannot have entropy greater than the BEB.

Let us denote the kinetic energy of the system by \(E_k\). Since the system is composed of non-relativistic particles we assume that the entropy of the system is a function of \(\hbar, m, E_k, R\) and \(N\). Since the entropy can be written as a function of dimensionless quantities, we infer that it is a function of only two independent parameters: \(N\) and \(w \equiv mER^2/\hbar^2\). Next, we assume that the chemical potential of the system vanishes. In that case the entropy is a function of \(w\) only, and since it is also an extensive function of \(E\) and \(V = \frac{4}{3} \pi R^3\) it must be proportional to \(w^{3/5}\). That is, the entropy is proportional to \(E_k^{3/5}R^{6/5}\).

The kinetic energy, \(E_k\), is smaller than the total energy, \(E\), that appears in the BEB; approximately, \(E_k = E - Nm\). However, at first sight, it looks that by taking \(R\) to be large enough such that \(R^{6/5} \gg R\) (ignoring the dimensions), one would be able to exceed the BEB.

The system described above might represent, for example, a degenerate Bose gas. Below the critical temperature \(T_0 \sim (N/V)^{2/3}\) the entropy is indeed proportional to \(E_k^{3/5}R^{6/5}\). However, this is true only below the critical temperature. Thus, for \(T_0 > T\), \(R\) is bounded from above\(^2\) and the entropy of the system can not exceed the BEB. This example illustrates that without the considerations of the actual physical system that exist in nature, mathematically, it is very easy to find counter examples for the BEB.

\(^2\)Here we assume that \(T\) remains constant. Otherwise, by increasing \(R\) and decreasing \(T\) (such that \(T < T_0\)), the kinetic energy is decreased as well. Thus, it can be shown that the decrease in \(E_k\) is enough to protect the BEB.
For free massless quantum fields enclosed in volumes of various shapes the bound’s validity has been checked directly (see review by Bekenstein and Schiffer [27]). Nevertheless, in [24], it was shown that the bound impose a restriction on the number, \( n \), of massless fields to be no more then \( \sim 10^4 \).\(^3\)

In the derivation of the bound by Bousso [19], three assumptions have been made: weak gravity, the system is enclosed in a spatially compact region and the null energy condition. It is the second assumption that we would like to emphasize here. This assumption implies that \( E \) includes the entire system. As shown by Page [28], if this requirement is not satisfied one can find counter examples to the BEB.

According to quantum mechanics, the Compton wave length of the system is given by \( \lambda = \frac{hc}{E} \). Thus, the system will satisfy Bousso’s second assumption if \( ER \gg \bar{h}c \). Let us define the validity domain of the BEB as follows:

\[
\frac{ER}{\bar{h}c} \geq \gamma, \tag{5}
\]

where \( \gamma \) is a dimensionless constant of order unity. In order to determine \( \gamma \) one has to know how much energy is allowed to leak out of the box. Furthermore, \( \gamma \) determines the minimum possible value of the BEB. That is, the BEB implies that \( S \leq 2\pi \gamma \) for systems with radius \( R = \gamma hc/E \). Hence, a system that saturates the bound at this limit will provide information on \( \gamma \) (and vice versa).

In curved (non-spherical) spacetime it is not very clear how to define “E” and “R”. However, for spherical self-gravitating systems, Sorkin, Wald and Jiu [29] provided an indication that probably some version of the BEB may hold. If that is correct, one would expect logarithmic corrections to the BEB.

The leading order corrections to the Bekenstein-Hawking entropy are presumably logarithmic (see [30] and references therein). That is,

\[
S_{BH} = \frac{A}{4l_p^2} - k \log \left( \frac{A}{l_p^2} \right) + ... \tag{6}
\]

where there are few indications that \( k = 3/2 \). Therefore, the SEB may have the same corrections. As a result, the BEB (1) will exceed \( A/4l_p^2 - k \log(A/4l_p^2) \) in the limit when \( E \to c^2 R/2G \). This motivates us to introduce a logarithmic correction to the BEB:

\[
S \leq \frac{2\pi ER}{\bar{h}c} - k \log \left( \frac{ER}{\bar{h}c} \right). \tag{7}
\]

Since \( ER/\bar{h}c \) is usually much smaller than \( A/l_p^2 \), the corrections to the Bekenstein bound are relatively more important than the corrections to the entropy of a black hole.

\(^3\)The limitation on the number of massless species in nature is a consequence of any sort of entropy bound.
III. AN EXTENSIVE ENTROPY BOUND FOR CHARGED SYSTEMS

About ten years ago Zaslavskii [31] has suggested how to tighten Bekenstein’s bound on entropy when the object is electrically charged. Recently, Bekenstein and Mayo [5] have derived the bound by considering the accretion of an ordinary charged object by a black hole. They have found that the upper bound to the entropy $S$ of an arbitrary system of proper energy $E$, proper radius $R$, and charge $Q$:

$$S \leq \frac{2\pi ER}{hc} - \frac{\pi Q^2}{hc}.$$  \hspace{1cm} (8)

The entropy bound (8) is understandable from a dimensional point of view. That is, an entropy $S$ is a dimensionless function. For (charged) systems in flat space time, $S$ does not depend (explicitly) on the gravitational constant $G$. Furthermore, if the entropy bound does not depend on the mass of some particular species, it can be written in the form $S = f(x, y)$ where $x \equiv 2\pi ER/hc$, $y \equiv \pi Q^2/hc$, and $f$ is some arbitrary function of $x$ and $y$ (any other dimensionless quantity of $E, R, Q, h, c$ must be a function of $x$ and $y$). Now, the spherical entropy bound implies that

$$f(x, y) \leq \frac{\pi}{l_p^2} R^2 \equiv z,$$  \hspace{1cm} (9)

where $x$, $y$ and $z$ are all independent dimensionless parameters. Therefore, one can fix $x$ and $y$ and take $z$ to its minimal value. The minimal value of $z = z_{\text{min}}(x, y)$ is obtained when the system becomes a charged black hole with definite $x$ and $y$. Therefore, the expression for the radius of a Reissner-Nordstrom black hole implies that $z \geq z_{\text{min}} = x - y$. By substituting the minimum value of $z$ in Eq. (9), one obtains the bound (8).

Next, we would like to consider applications of this bound to extensive systems (i.e. systems in which the energy and charge are distributed uniformly). Since the energy for highly charged systems can not be distributed uniformly, we assume that the Coulomb energy, $E_c$, is much smaller than the total energy, $E$. When the charge is uniformly spread on a ball of radius $R$, $E_c = 3Q^2/5R$. Thus, in this section $E \gg Q^2/R$.

The entropy is a function of the internal energy $E_{\text{in}} = E - 3Q^2/5R$. Therefore, the bound (8) implies

$$S(E_{\text{in}}) \leq \frac{2\pi RE_{\text{in}}}{hc} + \frac{\pi Q^2}{5hc}.$$  \hspace{1cm} (10)

where we have substituted $E = E_{\text{in}} + 3Q^2/5R$. However, since the entropy does not depend explicitly on $Q$, we infer that

$$S(E_{\text{in}}) \leq \frac{2\pi RE_{\text{in}}}{hc}.$$  \hspace{1cm} (11)

Now, for systems in which the entropy is an extensive function of $E_{\text{in}}$, one can write

$$S(E_{\text{in}}, ...) = V s(\varepsilon_{\text{in}}, ...),$$  \hspace{1cm} (12)

where $\varepsilon_{\text{in}} \equiv E_{\text{in}}/V$ is the internal energy density and $s$ is the entropy density (the dots in Eq. (12) indicate that the entropy might depend on other extensive parameters such as
the number of particles etc). The entropy bound in the right hand side of Eq. (8) does not satisfy the above condition. Hence, we are motivated to seek a tighter bound for extensive systems (i.e. systems in which the entropy is an extensive function of energy, charge etc). Indeed, as we shall see now, such a bound exist and has some interesting features.

Applying the entropy bound (11) on extensive systems leads to the following bound on the entropy density of the system:

$$s(\varepsilon_{in}, ...) \leq \frac{2\pi \varepsilon_{in}}{\hbar c} R.$$  \hspace{1cm} (13)

Since $\varepsilon_{in}$ and $R$ can be considered to be independent, one can fix $\varepsilon_{in}$ and change $R$. However, there are limitations on the range allowed to $R$. First, as we have mentioned earlier, the Coulomb energy $Q^2/2R$ is much smaller than the total energy $E$. By imposing the condition $E \gg Q^2/2R$ we find that the radius of the ball must be much smaller than $\sim \varepsilon^{1/2}/\rho$, where $\rho$ is the charge density. Second, the size of the system can not be smaller than its one Compton wavelength (see Eq. (5)). Thus, $R$ is confined to the interval

$$\gamma \frac{\hbar c}{E} \leq R \ll \frac{\varepsilon^{1/2}}{\rho}.$$  

In order to minimize the right hand side of Eq. (13) we replace $R$ by its minimal value

$$R_{\min} = \gamma \hbar / E = (3\gamma \hbar c/4\pi \varepsilon)^{1/4}.$$  \hspace{1cm} (14)

This implies a new bound on the entropy density:

$$s(\varepsilon_{in}, ...) \leq a \left( \frac{\varepsilon_{in}}{\hbar c} \right)^{3/4},$$  

where $a = (12\gamma \pi^{3})^{1/4}$ is a dimensionless constant. Multiplying both sides by the volume of the system we obtain a new extensive entropy bound for charged systems:

$$S \leq \frac{2\pi \gamma^{1/4}}{(\hbar c)^{3/4}} \left( (ER)^{3/4} - \frac{9Q^2}{20(ER)^{1/4}} \right) + O(Q^4).$$  \hspace{1cm} (15)

Note that in the limit $ER \rightarrow \gamma \hbar c$, our extensive bound is a bit more liberal than the original bound (8). Of course, for $ER \gg \gamma \hbar c$ the bound (15) is much tighter.

### IV. AN EXTENSIVE ENTROPY BOUND FOR ROTATING SYSTEMS

Using arguments similar to those that motivated Bekenstein to propose the universal upper entropy bound (1), Hod [4] was able to infer a tighter entropy bound for rotating systems:

$$S \leq \frac{2\pi E R}{\hbar c} \left( 1 - \frac{J^2}{E^2 R^2} \right)^{1/2},$$  \hspace{1cm} (16)

where $J$ is the total angular momentum (spin) of the system. Similarly to the arguments following Eq. (8), one can derive this bound from dimensional considerations assuming the spherical entropy bound (2). Let us now consider the applications of this bound to extensive systems.
It is well known that in a state of thermal equilibrium, only a uniform rotation of a body as a whole is possible [32]. The total energy of a rotating (in our case spherical) body may be written as the sum of its internal energy, $E_{in}$, and its kinetic energy of rotation: $E = E_{in} + J^2/2I$, where $I$ is the moment of inertia of a spherical ball. In general, rotation changes the distribution of mass in the system, and therefore both $I$ and $E_{in}$ are functions of $J$. However, in this section, we will consider only a sufficiently slow rotation ($E \gg J^2/2I$), so that $I$ and $E_{in}$ may be regarded as constants independent of $J$. In this case, Eq. (16) implies (see [5])

$$S \leq 2\pi R \left( E - \frac{J^2}{2\mu R^2} \right) + O(J^4),$$

(17)

where $E$ has been replaced with the rest mass, $\mu$, in the denominator.

Now, since the entropy is not a function of the total energy $E$, but a function of the internal energy, $E_{in}$, both the Bekenstein bound (1), as well as Eq. (17), imply Eq. (11), where in this section $E_{in}$ is the internal energy of a (spherical) rotating body. Thus, the bound given in Eq. (14) is valid also for rotating systems, assuming the entropy is an extensive function of $E_{in}$.

For a spherical ball with rest mass, $\mu$, the moment of inertia has the value, $\frac{2}{5}\mu R^2$, so that the internal energy for given $E$ and $R$ is: $E_{in} = E - 2J^2/5\mu R^2$. Substituting the value of the internal energy density $\varepsilon_{in} = E_{in}/V$ in Eq. (14), and then multiplying both sides by the volume, $V$, we obtain a new extensive bound for rotating systems:

$$S \leq \frac{2\pi \gamma^{1/4}}{(\hbar c)^{3/4}} \left[ (ER)^{3/4} - \frac{3}{10} \frac{J^2}{(ER)^{5/4}} \right] + O(J^4),$$

(18)

where we have replaced $\mu \to E$ in the denominator. Again, in the limit $ER \to \gamma \hbar c$, the extensive bound is a bit more liberal than the original bound (17).

**V. APPLICATIONS OF THE SEB ON EXTENSIVE CHARGED-ROTATING SYSTEMS**

Another interesting question that we would like to address here concerns the application of the SEB on extensive systems (not necessarily in flat spacetime). In our previous work it was shown that the Holographic principle predicts the CEB for extensive systems. However, in our derivation we did not include the net charge (or angular momentum) of the system. As it will be shown in this section (without pretending to any rigor), for charged rotating systems the causal bound takes a different form.

Applying the SEB (2) on extensive systems leads to the following bound on the entropy density of the system:

$$s(\varepsilon_{in}, ...) \leq \frac{1}{R},$$

(19)

where in this section we set $c = G = \hbar = 1$ and we will stress only functional dependence, while ignoring numerical factors. Now, since $\varepsilon_{in}$ and $R$ can be considered to be independent, one can fix $\varepsilon_{in}$ and increase $R$. However, by increasing $R$ and at the same time keeping $\varepsilon_{in}$
constant, one reduces the ratio $R/E$. Therefore, the maximum possible value of $R$ occurs when the system becomes a (charged-rotating) black hole.

For the Kerr-Newman black hole, the horizon is located at

$$r_+ = E + (E^2 - Q^2 - J^2/E^2)^{1/2},$$

(20)

where here $E$ represents the black hole’s mass. The horizon area of such a black hole is:

$$A = 4\pi (r_+^2 + J^2/E^2).$$

Thus, we conclude that $R \geq \sqrt{A/4\pi}$. In terms of $\varepsilon_{in}$ this condition can be written in the form

$$R < \frac{1}{\varepsilon_{in}^{1/2}} + O(Q^2, J^2).$$

(21)

Next, substituting the maximum value of $R$ in Eq. (19) we infer the entropy density bound

$$s(\varepsilon_{in}, ...) \leq \sqrt{\varepsilon_{in}} + O(Q^2, J^2).$$

(22)

Now, since $s$ depends on $Q$ and $J$ only through $\varepsilon_{in}$, we infer

$$s(\varepsilon_{in}, ...) \leq \sqrt{\varepsilon_{in}}.$$

(23)

Finally, multiply both sides of the equation above by the volume of the system, we find that for extensive, charged and rotating systems, the SEB implies:

$$S(E, V, Q, J, ...) < \sqrt{EV} \left[ 1 - \frac{3Q^2}{10ER} - \frac{J^2}{5E^2R^2} \right].$$

(24)

where we have substitute $E_{in} = E - 2J^2/5\mu R^2 - 3Q^2/5R$. This result will be generalized in the next section for the case of non-extensive systems.

VI. DISCUSSION: GENERALIZATION OF THE BEKENSTEIN AND CAUSAL ENTROPY BOUNDS FOR NON-EXTENSIVE CHARGED AND RotATING SYSTEMS

In this paper we have proposed two new entropy bounds for an isolated extensive, charged and rotating system. It was assumed that the entropy of the system is an extensive function of the internal energy. Furthermore, our results applicable only for $E \gg Q^2/R$ and $E \gg J^2/\mu R^2$; otherwise the particles will be concentrated on the edge of the system, and thus the entropy will not be an extensive function of $E_{in}$. However, if the gravitational force is taken into account, it is possible to imagine extensive systems with relatively high charge and angular momentum. For such systems our bounds in the previous sections can be applied.

Although we have used Eqs. (8,16) to derive the bounds given in Eqs. (15,18), it was not really mandatory. We could obtain the same extensive bounds assuming only the BEB (1). This is a consequence of the following “generalization” of the Bekenstein bound.

The entropy of charged rotating systems depends on the internal energy, $E_{in}$ [32]. Thus, the entropy can be written as a function $S(E_{in}, V, ...)$. The dots indicates possible dependence on other parameters such as the number of particles, but not on the charge or the
angular momentum of the system, because their dependence is included in $E_{in}$. Now, the total energy of the system, can be written in the form: $E = E_{in} + \Delta(Q, J, V)$, where $\Delta$ is the sum of the Coulomb and rotational energies. Hence, the BEB can be written in the form:

$$S(E_{in}, V, \ldots) \leq \frac{2\pi E_{in} R}{\hbar c} + \frac{2\pi R}{\hbar c} \Delta(Q, J, V).$$

(25)

However, the right hand side depends explicitly on $R, E_{in}, Q$ and $J$, whereas the left hand side depends only on $E_{in}$ and $R$ (and maybe on some other parameters that are not relevant for our argument). Thus, we infer that

$$S \leq \frac{2\pi R E_{in}}{\hbar c}.$$  

(26)

This is a generalized version of the BEB (1).

The bound (26) for lots of systems is tighter than Eq. (8) or Eq. (16). For charged non-rotating spherical systems it coincides with Eq. (8), when the charge is uniformly spread on the edge of the system (a spherical shell). Although the bound (26) is (usually) smaller, the bounds in Eq. (8) and Eq. (16) have the advantage that one does not have to know $E_{in}$, but only the total energy, charge and angular momentum.

The exact same arguments that motivated us to propose Eq. (26), lead us to generalize the CEB in the following form (we set $k_B = c = G = 1$ and ignore numerical factors):

$$S < \sqrt{E_{in} V}.$$  

(27)

This is reduced to Eq. (24) when the charge and angular momentum are relatively small. However, Eq. (27) can be applied also for highly charged-rotating systems.

In our previous work, we have obtained the CEB for extensive systems assuming the SEB (2). However, if one includes the logarithmic correction to the SEB (see Eq. (6)), than the same arguments that led to Eq. (23), implies the following correction term (for simplicity we assume $J = Q = 0$ so that $\varepsilon_{in} = \varepsilon$):

$$s(\varepsilon, \ldots) \leq \sqrt{\varepsilon} + k \varepsilon^{3/2} \ln \varepsilon.$$  

(28)

Thus, Eq. (27) is replaced by

$$S < \sqrt{E V} + k \frac{E^{3/2}}{V^{1/2}} \ln \left( \frac{E}{V} \right).$$  

(29)

The above correction term may be compared with first order correction to the original definition of the CEB [25,26]. We leave this comparison for a future work.

In conclusion, if the entropy of a black hole includes a logarithmic correction term (see Eq. (6)), then both the causal and the Bekenstein entropy bound have to be modified as given in Eq. (29) and Eq. (7), respectively. Furthermore, it has been shown that one can replace $E \rightarrow E_{in}$ in the formulas of these bounds in order to obtain tighter bounds for charged and rotating systems.
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