A REPRESENTATION STABILITY THEOREM FOR VI-MODULES

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Abstract. Let VI be the category whose objects are the finite dimensional vector spaces over a finite field of order \( q \) and whose morphisms are the injective linear maps. A VI-module over a ring is a functor from the category VI to the category of modules over the ring. A VI-module gives rise to a sequence of representations of the finite general linear groups. We prove that the sequence obtained from any finitely generated VI-module over an algebraically closed field of characteristic zero is representation stable - in particular, the multiplicities which appear in the irreducible decompositions eventually stabilize. We deduce as a consequence that the dimension of the representations in the sequence \( \{ V_n \} \) obtained from a finitely generated VI-module \( V \) over a field of characteristic zero is eventually a polynomial in \( q^n \). Our results are analogs of corresponding results on representation stability and polynomial growth of dimension for FI-modules (which give rise to sequences of representations of the symmetric groups) proved by Church, Ellenberg, and Farb.

1. Introduction

The theory of representation stability was initiated by Church and Farb in their paper [3]. One of the main themes in this theory is to study, for an increasing chain of groups \( G_0 \subset G_1 \subset G_2 \subset \cdots \), the asymptotic behavior of certain sequences

\[
V_0 \xrightarrow{\phi_0} V_1 \xrightarrow{\phi_1} V_2 \xrightarrow{\phi_2} \cdots
\]

where each \( V_n \) is a representation of \( G_n \), and each \( \phi_n \) is a linear map. The sequence (1.1) is called a consistent sequence if, for every non-negative integer \( n \) and for every \( g \in G_n \), the

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following diagram commutes:

\[
\begin{array}{ccc}
V_n & \xrightarrow{\phi_n} & V_{n+1} \\
 g \downarrow & & \downarrow g \\
V_n & \xrightarrow{\phi_n} & V_{n+1}
\end{array}
\]

(where \( g \) acts on \( V_{n+1} \) by considering it as an element of \( G_{n+1} \)).

For the family of symmetric groups \( S_n \), it was discovered by Church, Ellenberg and Farb in [1] that many interesting consistent sequences of representations of \( S_n \) can be packaged into an FI-module, where FI is the category of finite sets and injective maps. An FI-module over a commutative ring \( k \) is, by definition, a functor from FI to the category of \( k \)-modules; thus, an FI-module \( V \) gives rise to a consistent sequence (1.1) where \( V_n = V(\{1, \ldots, n\}) \) and \( \phi_n \) is induced by the standard inclusion \( \{1, \ldots, n\} \hookrightarrow \{1, \ldots, n+1\} \). One of the main results of [1] is that the consistent sequence obtained from a finitely generated FI-module \( V \) over a field of characteristic zero is representation stable in the sense of [3].

Fix a finite field \( \mathbb{F}_q \) of order \( q \). The purpose of our present paper is to prove an analogous result for the family of finite general linear groups \( GL_n(\mathbb{F}_q) \). The role of the category FI will be played, in our paper, by the category VI whose objects are the finite dimensional vector spaces over \( \mathbb{F}_q \) and whose morphisms are the injective linear maps.

**Definition 1.1.**
(i) A VI-module over a commutative ring \( k \) is a functor from the category VI to the category of \( k \)-modules.

(ii) A homomorphism \( F : U \to V \) of VI-modules is a natural transformation from the functor \( U \) to the functor \( V \).

(iii) Suppose \( U \) and \( V \) are VI-modules such that \( U(X) \) is a \( k \)-submodule of \( V(X) \) for every object \( X \) of VI. We call \( U \) a VI-submodule of \( V \) if the collection of inclusion maps \( U(X) \hookrightarrow V(X) \) defines a homomorphism \( U \to V \) of VI-modules.

The category of VI-modules over a commutative ring \( k \) is an abelian category.

**Notation 1.2.**
Let \( \mathbb{Z}_+ \) be the set of non-negative integers. For each \( n \in \mathbb{Z}_+ \), we denote by \( n \) the object \( \mathbb{F}_q^n \) of VI.

The full subcategory of VI generated by the objects \( n \) for all \( n \in \mathbb{Z}_+ \) is a skeleton of VI. One has \( \text{End}_{VI}(n) = GL_n(\mathbb{F}_q) \).

**Notation 1.3.**
Suppose \( V \) is a VI-module. For each \( n \in \mathbb{Z}_+ \), set \( V_n = V(n) \) and denote by \( \phi_n : V_n \to V_{n+1} \) the map assigned by \( V \) to the standard inclusion \( n \hookrightarrow n+1 \).

The sequence (1.1) obtained from a VI-module \( V \) is a consistent sequence of representations of the groups \( GL_n(\mathbb{F}_q) \).

**Definition 1.4.**
A VI-module \( V \) is generated by a subset \( S \subset \bigcup_{n \in \mathbb{Z}_+} V_n \) if the only VI-submodule of \( V \) containing \( S \) is \( V \); we say that \( V \) is finitely generated if it is generated by a finite subset \( S \).

In order to state our main results, let us briefly recall the parametrization of irreducible representations of the groups \( GL_n(\mathbb{F}_q) \) over an algebraically closed field of characteristic zero, a more detailed discussion of which will be given below in Section 2.
Let $\mathcal{C}_n$ be the set of cuspidal irreducible representations of $\text{GL}_n(\mathbb{F}_q)$ (up to isomorphism), and let $\mathcal{C} = \bigsqcup_{n \geq 1} \mathcal{C}_n$. If $\rho \in \mathcal{C}_n$, we set $d(\rho) = n$. By a partition, we mean a non-increasing sequence of non-negative integers $(\lambda_1, \lambda_2, \ldots)$ where only finitely many terms are non-zero. If $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition, we set $|\lambda| = \lambda_1 + \lambda_2 + \cdots$. Let $\mathcal{P}$ be the set of partitions. For any function $\mu : \mathcal{C} \rightarrow \mathcal{P}$, let
\[\|\mu\| = \sum_{\rho \in \mathcal{C}} d(\rho) |\mu(\rho)|.\]

Then, following Zelevinsky [12], one has a natural parametrization of the isomorphism classes of irreducible representations of $\text{GL}_n(\mathbb{F}_q)$ by functions $\mu : \mathcal{C} \rightarrow \mathcal{P}$ such that $\|\mu\| = n$; we shall denote by $\varphi(\mu)$ the irreducible representation of $\text{GL}_n(\mathbb{F}_q)$ parametrized by $\mu$.

Let $\iota$ be the trivial representation of $\text{GL}_1(\mathbb{F}_q)$. Then $\iota \in \mathcal{C}_1$. Suppose $\lambda : \mathcal{C} \rightarrow \mathcal{P}$ is a function and $\lambda(\iota) = (\lambda_1, \lambda_2, \ldots)$. If $n$ is an integer $\geq |\lambda| + 1$, we define the function $\lambda[n] : \mathcal{C} \rightarrow \mathcal{P}$ with $\|\lambda[n]\| = n$ by

$$\lambda[n](\rho) = \begin{cases} (n - |\lambda|, \lambda_1, \lambda_2, \ldots) & \text{if } \rho = \iota, \\ \lambda(\rho) & \text{if } \rho \neq \iota. \end{cases}$$

Clearly, for each function $\mu : \mathcal{C} \rightarrow \mathcal{P}$ with $\|\mu\| < \infty$, there exists a unique function $\lambda : \mathcal{C} \rightarrow \mathcal{P}$ such that $\mu = \lambda[n]$, where $n = \|\mu\|$. 

**Definition 1.5.** A consistent sequence (1.1) of representations of the groups $\text{GL}_n(\mathbb{F}_q)$ over an algebraically closed field of characteristic zero is **representation stable** if there exists an integer $N$ such that for each $n \geq N$, the following three conditions hold:

- **(RS1) Injectivity:** The map $\phi_n : V_n \rightarrow V_{n+1}$ is injective.
- **(RS2) Surjectivity:** The span of the $\text{GL}_{n+1}(\mathbb{F}_q)$-orbit of $\phi_n(V_n)$ is all of $V_{n+1}$.
- **(RS3) Multiplicities:** There is a decomposition

$V_n = \bigoplus_{\lambda} \varphi(\lambda[n])^{c(\lambda)}$

where the multiplicities $0 \leq c(\lambda) \leq \infty$ do not depend on $n$; in particular, for any $\lambda$ such that $\lambda[N]$ is not defined, one has $c(\lambda) = 0$.

The naming of the notion defined above is consistent with [4, Definition 3.1]; in [11, Definition 3.3.2] and [3, Definition 2.6], this is called **uniformly** representation stable. Our main result reads:

**Theorem 1.6.** Let $V$ be a VI-module over an algebraically closed field of characteristic zero. Then $V$ is finitely generated if and only if the consistent sequence (1.1) obtained from $V$ is representation stable and $\dim(V_n) < \infty$ for each $n$.

After some preparation in Section 3 we give the proof of Theorem 1.6 in Section 4. In Section 5, we deduce from property (RS3) (which holds by Theorem 1.6) the following result.

**Theorem 1.7.** Let $V$ be a finitely generated VI-module over a field of characteristic zero. Then there exists a polynomial $P \in \mathbb{Q}[T]$ and an integer $N$ such that

$$\dim(V_n) = P(q^n) \quad \text{for all } n \geq N.$$
The various steps involved in the proof of Theorem 1.6 is summarized in the following diagram.

\[(1.2)\]

\[
\begin{align*}
V \text{ is noetherian} & \quad \text{(ii)} \quad \text{(RS1)} \\
V \text{ is finitely generated} & \quad \text{(i)} \\
\Rightarrow & \\
\text{dim}(V_n) < \infty \text{ for each } n & \quad \text{(RS2)} \\
V \text{ is weakly stable and weight bounded} & \quad \text{(v)} \quad \text{(RS3)}
\end{align*}
\]

Implication (i) in (1.2) was proved by the first author and Li in [5, Theorem 3.7 and Example 3.10]. (More generally, Putman and Sam [9, Theorem A], and Sam and Snowden [10, Corollary 8.3.3], proved implication (i) over any noetherian ring.) Implications (ii) and (iii) are straightforward; see [5, Proposition 5.1 and Proposition 5.2] for their proofs. Implications (iv) and (v) will be proved in the present paper by adapting the proofs for FI-modules due to Church, Ellenberg and Farb in [1]. A difference between our proof and theirs is that we do not try to minimize the \(N\) in Definition 1.5; we hope this streamlines the proof and makes it easier for the reader to grasp the essence of their argument, which is really short and nice.

We deduce Theorem 1.7 from Theorem 1.6 using the hook-length formula for dimensions of irreducible representations of \(\text{GL}_n(\mathbb{F}_q)\). The corresponding result for FI-modules was proved in [1, Theorem 1.5] via character polynomials. (More generally, it was proved for FI-modules over a field of any characteristic in [2, Theorem B].)

Let us also mention that if \(V\) is a finitely presented VI-module over a commutative ring, then it is known (independently by [6] and [9]) that the representations \(V_n\) have an inductive description called central stability; this inductive description does not say anything about the irreducible decomposition of \(V_n\) as a representation of \(\text{GL}_n(\mathbb{F}_q)\). In fact, the notion of central stability can be formulated in a very general setting. On the other hand, the notion of representation stability (more precisely property (RS3)) depends crucially on the sequence of groups involved. Theorems 1.6 and 1.7 do not follow from the results of [6] or [9].

2. Representations of finite general linear groups

Irreducible representations of \(\text{GL}_n(\mathbb{F}_q)\) over an algebraically closed field of characteristic zero were first classified by Green [7]. We collect in this section the basic facts we need, following [11] and [12].

2.1. Notations. From now on, we let \(k\) be an algebraically closed field of characteristic zero; by a representation or a VI-module, we mean a representation or a VI-module over \(k\).

For any finite group \(G\), we write \(R(G)\) for the Grothendieck group of the category of finite dimensional representations of \(G\). If \(\pi\) is a representation of \(G\), we write \(\pi^G\) for the subspace of \(G\)-invariants of \(\pi\), and \(\pi_G\) for the quotient space of \(G\)-coinvariants of \(\pi\).
For each \( n \in \mathbb{Z}_+ \), we set \( G_n = \text{GL}_n(\mathbb{F}_q) \). Let

\[ R = \bigoplus_{n \in \mathbb{Z}_+} R(G_n). \]

If \( m, r \in \mathbb{Z}_+ \) and \( n = m + r \), let \( P_{m,r} \subset G_n \) be the subgroup of matrices of the form

\[ p = \begin{pmatrix} g_{11} & g_{12} \\ 0 & g_{22} \end{pmatrix}, \quad \text{where} \ g_{11} \in G_m, \ g_{22} \in G_r. \]  

Define the subgroups \( G_{m,r}, H_{m,r}, U_{m,r} \) of \( P_{m,r} \) by the conditions that, for any element \( p \) of the form (2.1), one has:

\[ p \in G_{m,r} \iff g_{12} = 0, \]
\[ p \in H_{m,r} \iff g_{11} = 1_m, \]
\[ p \in U_{m,r} \iff g_{11} = 1_m \text{ and } g_{22} = 1_r, \]

where, for any \( m \in \mathbb{Z}_+ \), we write \( 1_m \) for the identity element of \( G_m \).

The composition \( G_{m,r} \hookrightarrow P_{m,r} \rightarrow P_{m,r}/U_{m,r} \) is an isomorphism. If \( \pi_1 \) is a representation of \( G_m \) and \( \pi_2 \) is a representation of \( G_r \), we denote by \( \pi_1 \times \pi_2 \) the representation of \( G_{m+r} \) obtained from the external tensor product \( \pi_1 \boxtimes \pi_2 \) by parabolic induction via \( P_{m,r} \), that is, we regard the representation \( \pi_1 \boxtimes \pi_2 \) of \( G_{m,r} \) as a representation of \( P_{m,r}/U_{m,r} \) via the isomorphism \( G_{m,r} \cong P_{m,r}/U_{m,r} \), then pull it back to a representation of \( P_{m,r} \) in which \( U_{m,r} \) acts trivially, and let \( \pi_1 \times \pi_2 \) be the induced representation of \( \pi_1 \boxtimes \pi_2 \) from \( P_{m,r} \) to \( P_{m+r} \).

This defines a multiplication on \( R \). It is a well-known result of Green [7, Lemma 2.5] that \( R \) is a commutative graded ring.

### 2.2. Decomposition into a tensor product.

By definition, an irreducible representation \( \rho \) of \( G_n \) is cuspidal if

\[ \rho^{U_{m,n-m}} = 0 \quad \text{for} \ m = 1, \ldots, n - 1. \]

Recall that we denote by \( \mathcal{C}_n \) the set of cuspidal irreducible representations of \( G_n \) (up to isomorphism), and write \( \mathcal{C} \) for \( \bigcup_{n \geq 1} \mathcal{C}_n \). For each \( \rho \in \mathcal{C} \), let \( R(\rho) \) be the additive subgroup of \( R \) generated by all \( \pi \in R \) such that \( \pi \) is a subrepresentation of \( \rho^{\times r} \) for some \( r \in \mathbb{Z}_+ \).

**Fact 2.1** ([12, §9]). For each \( \rho \in \mathcal{C} \), the additive subgroup \( R(\rho) \) of \( R \) is a subring of \( R \). Moreover, the multiplication map

\[ \bigotimes_{\rho \in \mathcal{C}} R(\rho) \rightarrow R \]

is a ring isomorphism.

The tensor product in (2.2) is defined as the inductive limit

\[ \lim_{\mathcal{J}} \bigotimes_{\rho \in \mathcal{J}} R(\rho) \]

where \( \mathcal{J} \) runs over the finite subsets of \( \mathcal{C} \) partially ordered by inclusion.
2.3. Ring of symmetric functions. Let

\[ \Lambda = \bigoplus_{r \in \mathbb{Z}_+} \Lambda_r \]

be the graded ring of symmetric functions in an infinite countable set of variables with coefficients in \( \mathbb{Z} \) (see [3, Chapter 1] or [12, §5]). For each partition \( \lambda \in \mathcal{P} \), we write \( s_\lambda \) for the Schur function corresponding to \( \lambda \). It is well-known that, for each \( r \in \mathbb{Z}_+ \), the Schur functions \( s_\lambda \) with \( |\lambda| = r \) form a \( \mathbb{Z} \)-basis for \( \Lambda_r \). (Recall that for a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \), we write \( |\lambda| = \lambda_1 + \lambda_2 + \cdots \).)

**Fact 2.2** ([12, §9]). For each \( \rho \in \mathcal{C} \), there is a natural isomorphism of rings

\[ \Lambda \xrightarrow{\sim} R(\rho), \]

denoted by

\[ f \mapsto f(\rho), \]

such that if \( r \in \mathbb{Z}_+ \) and \( n = r \cdot d(\rho) \), then the elements \( s_\lambda(\rho) \) with \( |\lambda| = r \) are irreducible representations of \( G_n \). (Recall that \( d(\rho) = m \) if \( \rho \in \mathcal{C}_m \).)

For each \( n \in \mathbb{Z}_+ \), we write \( h_n \) for the \( n \)-th complete symmetric function, and \( \iota_n \) for the trivial representation of \( G_n \). Recall that \( \iota = \iota_1 \) and \( \iota \in \mathcal{C}_1 \).

**Fact 2.3** ([12, §9]). For each \( n \in \mathbb{Z}_+ \), one has \( h_n(\iota) = \iota_n \).

2.4. Classification of irreducible representations. Recall that for any function \( \mu : \mathcal{C} \to \mathcal{P} \), we let \( \|\mu\| = \sum_{\rho \in \mathcal{C}} d(\rho) \cdot |\mu(\rho)| \).

From Fact 2.1 and Fact 2.2, one can deduce the following parametrization of the irreducible representations of the groups \( G_n \).

**Fact 2.4** ([12, §9]). For each \( n \in \mathbb{Z}_+ \), the irreducible representations of the group \( G_n \) are parametrized by the functions \( \mu : \mathcal{C} \to \mathcal{P} \) such that \( \|\mu\| = n \). Under this parametrization, the irreducible representation \( \varphi(\mu) \) corresponding to \( \mu \) is

\[ \varphi(\mu) = \prod_{\rho \in \mathcal{C}} s_{\mu(\rho)}(\rho). \]

For our purposes, we do not need an explicit parametrization of the set \( \mathcal{C} \).

2.5. Pieri’s formula. The Pieri’s formula plays a central role in the proof of Theorem 1.6. We find it convenient to use the following (non-standard) notation.

**Notation 2.5.** For any \( \lambda, \mu \in \mathcal{P} \) and \( r \in \mathbb{Z}_+ \), we write \( \mu \sim \lambda + r \) if the Young diagram of \( \mu \) can be obtained by adding \( r \) boxes to the Young diagram of \( \lambda \) with no two boxes added in the same column. Similarly, we write \( \lambda \sim \mu - r \) if the Young diagram of \( \lambda \) can be obtained by removing \( r \) boxes from the Young diagram of \( \mu \) with no two boxes removed from the same column. (Thus, one has \( \mu \sim \lambda + r \) if and only if \( \lambda \sim \mu - r \).)

**Fact 2.6** (Pieri’s formula [3, Chapter 1, (5.16)]). For each \( \lambda \in \mathcal{P} \) and \( r \in \mathbb{Z}_+ \), one has

\[ s_\lambda h_r = \sum_{\mu \sim \lambda + r} s_\mu. \]
To apply Pieri's formula to representations of the groups $G_n$, we shall use the following notation.

Notation 2.7. For any functions $\lambda, \mu : C \to P$ and $r \in \mathbb{Z}_+$, we write $\mu \sim \lambda + r$ if $\mu(\iota) \sim \lambda(\iota) + r$, and $\mu(\rho) = \lambda(\rho)$ for all $\rho \neq \iota$. Similarly, we write $\lambda \sim \mu - r$ if $\lambda(\iota) \sim \mu(\iota) - r$, and $\lambda(\rho) = \mu(\rho)$ for all $\rho \neq \iota$.

Lemma 2.8. Let $m, r \in \mathbb{Z}_+$. Suppose that $\lambda : C \to P$ is a function such that $\|\lambda\| = m$. Then

$$\varphi(\lambda) \times \iota_r = \bigoplus_{\mu \sim \lambda + r} \varphi(\mu).$$

Proof. One has:

$$\varphi(\lambda) \times \iota_r = \left( \prod_{\rho \in \mathcal{C}} s_{\lambda(\rho)}(\rho) \right) \times h_r(\iota) \quad \text{(by Facts 2.3 and 2.4)}$$

$$= \bigoplus_{\mu \sim \lambda(\iota) + r} \left( s_{\mu}(\iota) \times \prod_{\rho \neq \iota} s_{\lambda(\rho)}(\rho) \right) \quad \text{(by Facts 2.2 and 2.6)}$$

$$= \bigoplus_{\mu \sim \lambda + r} \varphi(\mu) \quad \text{(by Fact 2.4)}.$$

Suppose $m, r \in \mathbb{Z}_+$ and $n = m + r$. If $\pi$ is a representation of $G_n$, then $\pi^{U_{m,r}}$ is a representation of $G_{m,r}$, and $\pi^{H_{m,r}}$ is a representation of $G_m$.

Lemma 2.9. Let $m, r \in \mathbb{Z}_+$. Suppose that $\mu : C \to P$ is a function such that $\|\mu\| = m + r$. Then

$$\varphi(\mu)^{H_{m,r}} = \bigoplus_{\lambda \sim \mu - r} \varphi(\lambda).$$

Proof. Let $n = m + r$. For any function $\lambda : C \to P$ with $\|\lambda\| = m$, the multiplicity of $\varphi(\lambda) \boxtimes \iota_r$ in $\varphi(\mu)^{U_{m,r}}$ is:

$$\dim \text{Hom}_{G_{m,r}} \left( \varphi(\mu)^{U_{m,r}}, \varphi(\lambda) \boxtimes \iota_r \right) = \dim \text{Hom}_{G_n} \left( \varphi(\mu), \varphi(\lambda) \times \iota_r \right)$$

$$= \begin{cases} 1 & \text{if } \lambda \sim \mu - r, \\ 0 & \text{else}, \end{cases}$$

where the first equality follows from Frobenius reciprocity for parabolic induction [12 §8.1], and the second equality follows from Lemma 2.8.

Since $\varphi(\mu)^{H_{m,r}} = (\varphi(\mu)^{U_{m,r}})^{G_r}$, the result follows.

3. Weak stability and weight boundedness

In this section, we define the notions of weak stability and weight boundedness for a VI-module, and prove that every finitely generated VI-module is weakly stable and weight bounded.
3.1. The VI-module $M(m)$. For each $m \in \mathbb{Z}_+$, define a VI-module $M(m)$ by

$$M(m)(-) = k \text{Hom}_{VI}(m, -),$$

that is, $M(m)$ is the composition of the functor $\text{Hom}_{VI}(m, -)$ followed by the free $k$-module functor. It is plain (see, for example, [5, Lemma 2.14]) that a VI-module $V$ is finitely generated if and only if there exists a surjective homomorphism

$$M(m_1) \oplus \cdots \oplus M(m_d) \twoheadrightarrow V \quad \text{for some } m_1, \ldots, m_d \in \mathbb{Z}_+.$$

**Lemma 3.1.** Suppose $m, r \in \mathbb{Z}_+$ and $n = m + r$. Then

$$M(m)_n = \pi \times t_r,$$

where $\pi$ is the regular representation of $G_m$.

**Proof.** The group $G_n$ acts transitively on $\text{Hom}_{VI}(m, n)$ and the stabilizer of the standard inclusion $m \hookrightarrow n$ is $H_{m,r}$. The result follows from the isomorphism

$$k[G_n/H_{m,r}] = k[G_n] \otimes k[P_{m,r}] k[G_m].$$

3.2. Weak stability. Consider a consistent sequence (1.1) with $G_n = \text{GL}_n(\mathbb{F}_q)$. Suppose $m, r \in \mathbb{Z}_+$ and $n = m + r$. The map $\phi_n : V_n \to V_{n+1}$ descends to a map

$$\phi_{m,r} : (V_n)_{H_{m,r}} \to (V_{n+1})_{H_{m,r+1}}$$

which is a homomorphism of representations of the group $G_m$.

**Definition 3.2.** A consistent sequence $\{V_n, \phi_n\}$ with $G_n = \text{GL}_n(\mathbb{F}_q)$ is weakly stable if for each $m \in \mathbb{Z}_+$, there exists $s \in \mathbb{Z}_+$ such that for each $r \geq s$, the map $\phi_{m,r}$ of (3.1) is an isomorphism. A VI-module $V$ is weakly stable if the consistent sequence obtained from $V$ is weakly stable (see Notation 1.3).

One can also define a stronger notion by requiring that the integer $s$ in Definition 3.2 can be chosen independently of $m \in \mathbb{Z}_+$ (see [1] Definition 3.1.3). (Unlike [1], we will not need to use this stronger notion.)

**Remark 3.3.** Consider a consistent sequence $\{V_n, \phi_n\}$ with $G_n = \text{GL}_n(\mathbb{F}_q)$. Suppose that $\text{dim}(V_n) < \infty$ for every $n \in \mathbb{Z}_+$. Let $m \in \mathbb{Z}_+$, and consider the sequence of maps

$$(V_m)_{H_{m,0}} \xrightarrow{\phi_{m,0}} (V_{m+1})_{H_{m,1}} \xrightarrow{\phi_{m,1}} (V_{m+2})_{H_{m,2}} \xrightarrow{\phi_{m,2}} \cdots.$$ 

It is plain that if $\phi_{m,r}$ is surjective for all $r$ sufficiently large, then $\phi_{m,r}$ is bijective for all $r$ sufficiently large.

**Lemma 3.4.** For each $m \in \mathbb{Z}_+$, the VI-module $M(m)$ is weakly stable.

**Proof.** Let $\ell \in \mathbb{Z}_+$. We claim that for each $r \geq m$, the map

$$\phi_{\ell,r} : (M(m)_n)_{H_{\ell,r}} \to (M(m)_{n+1})_{H_{\ell,r+1}} \quad \text{(where } n = \ell + r)$$

is surjective. By Remark 3.3, this will imply that $M(m)$ is weakly stable.

Suppose $r \geq m$ and $n = \ell + r$. We write the elements of $M(m)_{n+1}$ as $(n+1) \times m$-matrices of rank $m$. For every such matrix $A$, we can multiply it on the left by an element $g$ of $H_{\ell,r+1}$ so
that the last row of $gA$ is zero. The element $gA$ lies in the image of $\phi_n : M(m)_n \to M(m)_{n+1}$. It follows that $\phi_{\ell,r}$ is surjective, as claimed.

**Proposition 3.5.** Let $V$ be a finitely generated VI-module. Then $V$ is weakly stable.

*Proof.* Since $V$ is finitely generated, there exists $m_1, \ldots, m_d \in \mathbb{Z}_+$ and a surjective homomorphism $M \to V$ where $M = M(m_1) \oplus \cdots \oplus M(m_d)$. Let $m \in \mathbb{Z}_+$. By Lemma 3.4, the VI-module $M$ is weakly stable. Thus, there exists $s \in \mathbb{Z}_+$ such that

$$\phi_{m,r} : (M_{m+r})_{H_m,r} \to (M_{m+r+1})_{H_m,r+1}$$

is an isomorphism for every $r \geq s$. In the following commuting diagram

$$\begin{array}{ccc}
(M_{m+r})_{H_m,r} & \xrightarrow{\phi_{m,r}} & (M_{m+r+1})_{H_m,r+1} \\
\downarrow & & \downarrow \\
(V_{m+r})_{H_m,r} & \xrightarrow{\phi_{m,r}} & (V_{m+r+1})_{H_m,r+1}
\end{array}$$

the two vertical maps are surjective, so if the top horizontal map is surjective, then so is the lower horizontal map. By Remark 3.3, the result follows. \qed

### 3.3. Weight boundedness

Recall that for every function $\mu : \mathcal{C} \to \mathcal{P}$ with $\|\mu\| < \infty$, there is a unique function $\lambda : \mathcal{C} \to \mathcal{P}$ such that $\mu = \lambda[n]$, where $n = \|\mu\|$. The following definition is analogous to [1, Definition 3.2.1].

**Definition 3.6.** A consistent sequence $\{V_n, \phi_n\}$ with $G_n = \text{GL}_n(\mathbb{F}_q)$ is *weight bounded* if there exists $\xi \in \mathbb{Z}_+$ such that for every $n \in \mathbb{Z}_+$ and every irreducible subrepresentation $\varphi(\lambda[n])$ of $V_n$, one has $\|\lambda\| \leq \xi$; we call the minimal such $\xi$ the *weight* of the consistent sequence. A VI-module $V$ is *weight bounded* if the consistent sequence obtained from $V$ is weight bounded, and we define the *weight* of $V$ to be the weight of its consistent sequence.

**Proposition 3.7.** Let $V$ be a finitely generated VI-module. Then $V$ is weight bounded.

*Proof.* Since every quotient of a weight bounded VI-module is also weight bounded, it suffices to show that for each $m \in \mathbb{Z}_+$, the VI-module $M(m)$ is weight bounded.

Let $m, r \in \mathbb{Z}_+$ and $n = m + r$. Let $\mu : \mathcal{C} \to \mathcal{P}$ be a function with $\|\mu\| = n$. Suppose that the irreducible representation $\varphi(\mu)$ of $G_n$ is a subrepresentation of $M(m)_n$. By Lemma 2.8 and Lemma 3.3, there exists a function $\nu : \mathcal{C} \to \mathcal{P}$ such that $\|\nu\| = m$ and $\mu \sim \nu + r$. In particular, the number of columns in the Young diagram of $\mu(i)$ is at least $r$. If $\mu = \lambda[n]$, then the number of columns in the Young diagram of $\mu(i)$ is $n - \|\lambda\|$, so

$$\|\lambda\| \leq n - r = m.$$  

\qed

**Remark 3.8.** The *generating degree* of a nonzero VI-module $V$ is the smallest $m \in \mathbb{Z}_+ \cup \{\infty\}$ such that $V$ is generated by $\bigcup_{n=0}^{m} V_n$. The proof of the above proposition shows that if $V$ is nonzero and has generating degree $m$, then the weight of $V$ is at most $m$. 
4. Multiplicity stability

In this section, we complete the proof of Theorem 1.6 by showing that every VI-module which is weakly stable and weight bounded is multiplicity stable.

4.1. Proof of multiplicity stability. The proof of the following key proposition is an adaptation of the arguments in the proof of [1 Proposition 3.3.3].

Proposition 4.1. Let \( \{V_n, \phi_n\} \) be a consistent sequence with \( G_n = \text{GL}_n(\mathbb{F}_q) \). Suppose that \( \{V_n, \phi_n\} \) is weakly stable and weight bounded. Then there exists an integer \( N \) such that for each \( n \geq N \), the consistent sequence \( \{V_n, \phi_n\} \) satisfies condition (RS3) in Definition 1.5.

Proof. Let \( a \) be the weight of the consistent sequence \( \{V_n, \phi_n\} \). By weak stability, we can choose \( s \in \mathbb{Z}_+ \) such that the map \( \phi_{m,r} \) of (3.1) is an isomorphism whenever \( m \leq a \) and \( r \geq s \). Let \( N = \max\{a + s, 2a\} \).

For each \( n \in \mathbb{Z}_+ \), let

\[
(4.1) \quad V_n = \bigoplus_{||\lambda|| \leq a} \varphi(\lambda[n])^{\oplus c(\lambda,n)} \quad \text{(where } 0 \leq c(\lambda,n) \leq \infty)\]

be a decomposition of \( V_n \) into a direct sum of irreducible representations of \( G_n \). We claim that if \( ||\lambda|| \leq a \) and \( n \geq N \), then one has \( c(\lambda,n) = c(\lambda,N) \). We shall prove the claim by induction on \( ||\lambda|| \).

Let \( m \in \mathbb{Z}_+ \) such that \( m \leq a \). Assume that \( c(\lambda,n) = c(\lambda,N) \) whenever \( ||\lambda|| < m \) and \( n \geq N \). (This assumption is vacuously true for \( m = 0 \).)

Suppose \( n \geq N \), and set \( r = n - m \). Taking \( H_{m,r} \)-invariants on both sides of (4.1), and applying Lemma 2.9, we obtain

\[
(V_n)^{H_{m,r}} = \bigoplus_{||\lambda|| \leq a} \left( \varphi(\lambda[n])^{H_{m,r}} \right)^{\oplus c(\lambda,n)} = \bigoplus_{||\lambda|| \leq a} \left( \bigoplus_{\mu \sim \lambda[n] - r} \varphi(\mu)^{\oplus c(\lambda,n)} \right).
\]

Keeping in mind that the number of columns in the Young diagram of \( \lambda[n](\iota) \) is \( n - ||\lambda|| \), we make the following observations:

- If \( ||\lambda|| > m \), then \( n - ||\lambda|| < n - m = r \). In this case, there is no function \( \mu : \mathcal{C} \to \mathcal{P} \) satisfying \( \mu \sim \lambda[n] - r \).
- If \( ||\lambda|| = m \), then \( n - ||\lambda|| = n - m = r \). In this case, the only function \( \mu : \mathcal{C} \to \mathcal{P} \) satisfying \( \mu \sim \lambda[n] - r \) is \( \mu = \lambda \).

Hence, we obtain

\[
(4.2) \quad (V_n)^{H_{m,r}} = \left( \bigoplus_{||\lambda|| < m} \left( \bigoplus_{\mu \sim \lambda[n] - r} \varphi(\mu)^{\oplus c(\lambda,N)} \right) \right) \oplus \left( \bigoplus_{||\lambda|| = m} \varphi(\lambda)^{\oplus c(\lambda,n)} \right).
\]

Since \( r = n - m \geq N - a \geq s \), the map \( \phi_{m,r} \) of (3.1) is an isomorphism, and so we have isomorphisms of \( G_m \)-representations

\[
(4.3) \quad (V_n)^{H_{m,r}} \cong (V_n)_{H_{m,r}} \cong (V_{n+1})_{H_{m,r+1}} \cong (V_{n+1})^{H_{m,r+1}}.
\]
where the first isomorphism is the composition of the inclusion map \((V_n)_{H_{m,r}} \to V_n\) and the quotient map \(V_n \to (V_n)_{H_{m,r}}\) (and similarly for the third isomorphism).

We claim that if \(\|\lambda\| \leq a\), then
\[
\{ \mu | \mu \sim \lambda[n] - r \} = \{ \mu | \mu \sim \lambda[n+1] - (r+1) \}.
\]

It is clear that the left hand side is a subset of the right hand side. To see that the right hand side is contained in the left hand side, we note that
\[r + 1 > n - m \geq N - a \geq a \geq \|\lambda\| \geq \|\lambda(\iota)\|,\]
so if the Young diagram of \(\mu(\iota)\) is obtained from the Young diagram of \(\lambda[n+1](\iota)\) by the removal of \(r + 1\) boxes, then one of the \(r + 1\) boxes removed must be from the first row.

It follows from (4.2), (4.3), and (4.4) that we have an isomorphism of \(G_m\)-representations
\[
\bigoplus_{|\lambda| = m} \varphi(\lambda)^{\oplus c(\lambda,n)} \cong \bigoplus_{|\lambda| = m} \varphi(\lambda)^{\oplus c(\lambda,n+1)}.
\]

Hence, if \(\|\lambda\| = m\), then \(c(\lambda,N) = c(\lambda,N + 1) = c(\lambda,N + 2) = \cdots\). This completes the proof of the inductive step. \(\Box\)

4.2. Proof of Theorem 1.6. As explained in Section 1, we only have to prove implications (iv) and (v) in (1.2).

Implication (iv) is the combined statements of Proposition 3.5 and Proposition 3.7. Implication (v) is immediate from Proposition 4.1. This completes the proof of Theorem 1.6.

Remark 4.2. We would like to point out that the noetherian property of VI is not used in the proofs of implications (iv) and (v). (In [1], although the noetherian property of FI is not used in the proof of [1] Proposition 3.3.3, it is used in the first paragraph in the proof of [1] Theorem 1.13 to show that [1] Proposition 3.3.3 can be applied.)

One can also deduce condition (RS1) from weak stability and weight boundedness. We give the proof below, which is adapted from the proof of [1] Proposition 3.3.3. It follows that a proof of Theorem 1.6 can be given without using the noetherian property at all.

Proposition 4.3. Let \(\{V_n, \phi_n\}\) be a consistent sequence with \(G_n = GL_n(\mathbb{F}_q)\). Suppose that \(\{V_n, \phi_n\}\) is weakly stable and weight bounded. Then there exists an integer \(N\) such that for each \(n \geq N\), the consistent sequence \(\{V_n, \phi_n\}\) satisfies condition (RS1) in Definition 1.5.

Proof. Let \(a\) be the weight of the consistent sequence \(\{V_n, \phi_n\}\) and choose \(s \in \mathbb{Z}_+\) such that the maps \(\phi_{a,r}\) are isomorphisms for every \(r \geq s\). Let \(N = a + s\) and suppose that \(n \geq N\). Set \(r = n - a\). Let \(K_n\) be the kernel of \(\phi_n : V_n \to V_{n+1}\). Then \((K_n)_{H_{a,r}} \) is contained in the kernel of \(\phi_{a,r}\). Since \(r \geq s\), the map \(\phi_{a,r}\) is injective, so \((K_n)_{H_{a,r}} = 0\); equivalently, one has \((K_n)_{H_{a,r}} = 0\). If \(K_n \neq 0\), then it contains an irreducible subrepresentation \(\varphi(\lambda[n])\) for some \(\lambda : \mathcal{C} \to \mathcal{P}\). The number of columns in the Young diagram of \(\lambda[n](\iota)\) is \(n - \|\lambda\|\). But \(n - \|\lambda\| \geq r\), so there exists \(\mu : \mathcal{C} \to \mathcal{P}\) such that \(\mu \sim \lambda[n] - r\). It follows by Lemma 2.9 that \(\varphi(\lambda[n])_{H_{a,r}} \neq 0\). Thus \((K_n)_{H_{a,r}} \neq 0\), a contradiction. Therefore we must have \(K_n = 0\) when \(n \geq N\). \(\Box\)

5. Dimension growth

In this section, we prove Theorem 1.7 using Theorem 1.6 and the hook-length formula.
5.1. **Hook-length formula.** For each \( n \geq 1 \), let
\[
\Phi_n(q) = \prod_{i=1}^{n} (q^i - 1).
\]
Suppose \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a partition. We set
\[
\varepsilon(\lambda) = \sum_{i \geq 1} (i - 1)\lambda_i.
\]
We denote by \( h(x) \) the hook-length at the box \( x \in \lambda \) of the Young diagram of \( \lambda \). Let
\[
\Psi_\lambda(q) = q^{\varepsilon(\lambda)} \cdot \prod_{x \in \lambda} \left(q^{h(x)} - 1\right)^{-1}.
\]

Let us recall the hook-length formula for the dimension of an irreducible representation of the group \( G_n \).

**Fact 5.1** ([12, Proposition 11.10]). Let \( n \geq 1 \). Let \( \mu : \mathcal{C} \to \mathcal{P} \) be a function such that \( \|\mu\| = n \). Then
\[
\dim(\varphi(\mu)) = \Phi_n(q) \cdot \prod_{\rho \in \mathcal{C}} \Psi_{\mu}(\rho) \left(q^{d(\rho)}\right).
\]

5.2. **Proof of Theorem 1.7.** We now prove Theorem 1.7. By extension of scalars, we may assume that \( V \) is a finitely generated VI-module over an algebraically closed field of characteristic zero. By Theorem 1.6, it suffices to prove the following proposition.

**Proposition 5.2.** Let \( m \in \mathbb{Z}_+ \). Let \( \lambda : \mathcal{C} \to \mathcal{P} \) be a function such that \( \|\lambda\| = m \). Then there exists \( N \in \mathbb{Z}_+ \) and a polynomial \( P \in \mathbb{Q}[T] \) such that
\[
\dim(\varphi(\lambda[n])) = P(q^n) \quad \text{for all } n \geq N.
\]

**Proof.** In the formula for \( \dim(\varphi(\lambda[n])) \) given by Fact 5.1, the only factors which depend on \( n \) are \( \Phi_n(q) \) and \( \Psi_{\lambda[n]}(\mu)(q) \).

Write \( \lambda(i) \) as \( (\lambda_1, \lambda_2, \ldots) \), and let \( N = m + \lambda_1 \). Suppose \( n \geq N \). It is clear that \( \varepsilon(\lambda[n](i)) \) is an integer independent of \( n \). Moreover, the number of boxes in the first row of the Young diagram of \( \lambda[n](i) \) is \( n - m \); the hook-lengths at these boxes are:
\[
n - r_1, \ldots, n - r_{\lambda_1}, n - N, \ldots, 2, 1.
\]
for some \( r_1 < \cdots < r_{\lambda_1} < N \). The integers \( r_1, \ldots, r_{\lambda_1} \) do not depend on \( n \). Let \( s_1 < \cdots < s_m \) be the \( m \) integers such that
\[
\{r_1, \ldots, r_{\lambda_1}\} \cup \{s_1, \ldots, s_m\} = \{0, 1, \ldots, N - 1\}.
\]
It follows from Fact 5.1 that one has
\[
\dim(\varphi(\lambda[n])) = c(q^{n-s_1} - 1) \cdots (q^{n-s_m} - 1)
\]
for some \( c \in \mathbb{Q} \) which does not depend on \( n \). Choosing \( P \in \mathbb{Q}[T] \) to be the polynomial
\[
P(T) = c(q^{-s_1}T - 1) \cdots (q^{-s_m}T - 1)
\]
of degree \( m \), we are done. \( \square \)

**Remark 5.3.** From the above proof, we see that the degree of the polynomial \( P \) in Theorem 1.7 is at most the weight of \( V \).
A REPRESENTATION STABILITY THEOREM FOR $\text{VI}$-MODULES

References

[1] T. Church, J. Ellenberg, B. Farb, FI-modules and stability for representations of symmetric groups, Duke Math. J. 164 (2015), no. 9, 1833-1910, [arXiv:1204.4533]
[2] T. Church, J. Ellenberg, B. Farb, R. Nagpal, FI-modules over Noetherian rings, Geom. Top. 18-5 (2014), 2951-2984, [arXiv:1210.1854]
[3] T. Church, B. Farb, Representation theory and homological stability, Adv. Math. 245 (2013), 250-314, [arXiv:1008.1368]
[4] B. Farb, Representation stability, Proc. International Congress of Mathematicians, Seoul 2014, Vol. II, 1173-1196, [arXiv:1404.4065]
[5] W.L. Gan, L. Li, Noetherian property of infinite EI categories, New York J. Math. 21 (2015), 369-382, [arXiv:1407.8235]
[6] W.L. Gan, L. Li, On central stability, [arXiv:1504.07675]
[7] J.A. Green, The characters of the finite general linear groups, Trans. Amer. Math. Soc. 80 (1955), 402-447.
[8] I.G. MacDonald, Symmetric functions and Hall polynomials, Second edition, Oxford Math. Monographs, Oxford Sci. Publ., The Clarendon Press, Oxford Univ. Press, New York, 1995.
[9] A. Putman, S. Sam, Representation stability and finite linear groups, [arXiv:1408.3694]
[10] S. Sam, A. Snowden, Gröbner methods for representations of combinatorial categories, to appear in J. Amer. Math. Soc., [arXiv:1409.1670]
[11] T. Springer, A. Zelevinsky, Characters of $\text{GL}(n, F_q)$ and Hopf algebras, J. London Math. Soc. (2) 30 (1984), no. 1, 27-43.
[12] A. Zelevinsky, Representations of finite classical groups. A Hopf algebra approach, Lecture Notes in Math., 869, Springer-Verlag, Berlin-New York, 1981.