On the non-predicative judgments in analytical mechanics

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Abstract The non-predicativity of some fundamental concepts of classical mechanics is investigated. It is shown that some fundamental concepts of classical mechanics and Newton’s laws are non-predicative. The different forms of the proof for the equivalence of general equation of dynamics to the equations for mechanical system with constraints are investigated. It is shown that some previous proofs in open literature are false because of use of non-predicative concepts. The reason of such non-predicativity is the incompleteness of concept of constraint force.

1. Introduction

The non-predicativity of some statements of classical mechanics is investigated. Thus, this contribution is devoted to some logic aspects of a description of classical mechanics. They play important role in the proof of statements and theorems of mechanics.

Russell has defined the concept of non-predicativity when he was creating the new logic which is distinct from the logic of Aristotle. By Russell, definitions of two concepts A and B are non-predicative, if A is mentioned in the definition of B and on the contrary.

Pic.1 The non-predicativity of concepts A and B

The concept A is non-predicative in itself if definition A refers on A.

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Example of non-predicative definition is the set of all sets. Non-predicative definitions lead to vicious circles of reasonings.

The non-predicative judgments are the main reason of logical antinomies. As is known, the antinomy is the paradoxical reasoning containing the contradictory statements when each of them has the convincing proof of the validity. An example of logical antinomy is the antinomy of a hairdresser. Let us consider the hairdresser which shaves all those clients and only those clients who do not shave itself. Whether hairdresser shaves itself? The answer is paradoxical. Hairdresser shaves itself and he doesn’t shave itself.

Poincare was the first who has paid attention on the non-predicativity of some concepts in mathematics. He has read a lecture "On the transfinite numbers" on Hilbert's seminar at 1908 [1]. He has analyzed the proofs of some inconsistent judgments which took place in the theory of sets and have become one of the reasons of crisis in mathematics at the beginning of XX century. One of the reasons of crisis is the non-predicativity of some concepts of mathematics. Poincare assumed, that each mathematical concept must be predicative otherwise the concept is not strict.

Mechanics as well as mathematics is not free from the non-predicative statements leading to closed, vicious circles of reasonings. As consequence, in one case the non-predicative statements lead to nonstrict definitions or to false proofs, in other they lead to paradoxical conclusions, they can limit the applied region of mathematical methods also.

One of the reasons of the non-predicativity in mechanics is the incompleteness of some system of concepts.

2. Non-predicativity in d'Alembert’s Principle

For an example, we consider a problem of calculation of constraint force by means of d'Alembert’s Principle. Let

\[ f_\alpha (r_1, \ldots, r_N, t) = 0, \quad \alpha = 1, \ldots, r \]  
\[ \varphi_\beta \equiv \sum_{i=1}^{N} b_i^{(\beta)} \dot{r}_i + d_\beta = 0, \quad \beta = 1, \ldots, s \]

be equations of holonomic and non-holonomic constraints. Here \( \mathbf{r}_j \) are the radius vector of particles with mass \( m_j \) \( b_i^{(\beta)} = b_i^{(\beta)} (r_1, \ldots, r_N, t) \), \( d_\beta = d_\beta (r_1, \ldots, r_N, t) \) are the coefficients. The equations of motions for a set of \( N \) particles can be written in the form

\[ \mathbf{F}_i + \mathbf{R}_i - m_i \mathbf{a}_i = 0 \quad i = 1 \ldots N \]

Here \( \mathbf{R}_j \) is the constraint force. We know nothing about it besides that \( \mathbf{R}_j \) is the result of action of constraints which have rejected according to a principle of release from constraints. For calculation of constraint force, we usually use Principle of d'Alembert. It follows from this principle that constraint force is the following form.
The result is paradoxical. The constraint force is dependent on the acceleration!

The reason of such paradoxical conclusion is the non-predicativity of concepts for equations of motion and the constraint force. Indeed, the equations of motion use the concept of constraint force, and on the contrary the constraint force is defined from the equations of motion.

![Pic. 3 The non-predicativity in d'Alembert’s Principle](image)

To remove the non-predicativity, it is necessary to calculate the constraint force with the help of equations of constraints.

Let constraint forces $R_i$ be ideal.

**Theorem** [6] For restrictions (1), (2) to be ideal it is necessary and sufficient to have

$$
R_i = \sum_{\alpha=1}^{r} \lambda_{\alpha} \text{grad} f_{\alpha} + \sum_{\beta=1}^{s} \mu_{\beta} b_{i}^{(\beta)}, \quad i = 1, \ldots, N
$$

(3)

where $\lambda_{\alpha}$, $\mu_{\beta}$ are the uncertain multipliers.

Let us differentiate equations (1), (2) with respect to $t$. We then have

$$
\sum_{k=1}^{N} \text{grad} f_{\alpha} a_k + D_{\alpha} (r_k, v_k, t) = 0, \quad \alpha = 1, \ldots, r
$$

$$
\sum_{k=1}^{N} b_k^{(\beta)} a_k + \sum_{k=1}^{N} \dot{b}_k^{(\beta)} v_k + \dot{d}_\beta = 0, \quad \beta = 1, \ldots, s
$$

The equations in Lagrange’s multipliers take the form:

$$
\sum_{k=1}^{N} \text{grad} f_{\alpha} \frac{1}{m_k} \left( F_k + \sum_{n=1}^{r} \lambda_n \text{grad} f_n + \sum_{m=1}^{s} \mu_m b_{k}^{(m)} \right) + D_{\alpha} = 0, \quad \alpha = 1, \ldots, r
$$

(4)

$$
\sum_{k=1}^{N} b_k^{(\beta)} \frac{1}{m_k} \left( F_k + \sum_{n=1}^{r} \lambda_n \text{grad} f_n + \sum_{m=1}^{s} \mu_m b_{k}^{(m)} \right) + \sum_{k=1}^{N} \dot{b}_k^{(\beta)} v_k + \dot{d}_\beta = 0, \quad \beta = 1, \ldots, s
$$

(5)

From (3)-(5), we find the constraint forces as the functions of coordinates and speeds:

$$
R_i = R_i(r_j, v_j, t)
$$

So, the reason of non-predicativity of judgments is the absence of necessary information on constraint forces.

3. Non-predicativity of base concepts of classical mechanics
The non-predicativity of some concepts of classical mechanics is caused by uncertainty of concept for active forces also. What is the force? There are two definitions of active forces. Lagrange and Kirchhoff are the authors of them.

Lagrange simulates the force by the vector. Therefore we should define the equality of two forces. Let's consider Poincare's reasonings [2].

Let's assume that two forces $F$ and $F'$ are directed vertically upwards and they are enclosed to two identical bodies. If we have equilibrium in both cases, then forces $F$ and $F'$ are equal one another.

However these reasonings are not free from some lacks. It is impossible to say, that the weight $P$ of a body $B$ put in equilibrium the force $F$. Really, force $F$ is enclosed to elementary volume $dB'$ of body B. Similarly, force $F'$ is enclosed to elementary volume $d'B$ of same body. The elementary volume $dB$ is under action of force $F$ and constraint force $R$. Therefore $F$ puts in equilibrium $R$. The body $B$ is under action of reaction $R'$ and force $P$. Obviously

$$R = -R'$$

From (6), it follows that $F = -P$. We obviously get $F' = -P$ therefore $F' = F$. So, the concept of force by Lagrange refers to Third Law of Newton which uses the concept of force. Thus, the concepts of force by Lagrange and Third Law of Newton are non-predicative therefore Lagrange’s definition of force is not strict.

Let's consider the definition of force by Kirchhoff. The total applied force $F$ is equal to the mass multiplying by the acceleration, that is

$$F = ma$$

The substantiation of such definition is well-known. If we consider free isolated particle of mass $m$, moving with a speed $v$, thus we have

$$mv = c$$

The derivative $d mv / dt$ is not equal to zero when some body acts to particle. In this case, the derivative $d mv / dt$ will be some measure of such interaction. Taking into account that interaction is the force, we then have

$$F = ma$$

So, these reasonings use Newton’s Law of inertia. On the contrary, Newton’s Law of inertia is refers to concept of force. Thus, the definition of force by Kirchhoff and Newton’s Law of inertia are non-predicative therefore Kirchhoff’s definition of force is not strict.

4. Investigation of different proofs for equivalence of general equation of dynamics to equations of motion for mechanical system with constraints.

Let’s consider the different proofs of the theorem of equivalence for the general equation of dynamics

$$\sum_{k=1}^{N} \left( F_k - m_k a_k \right) \delta r_k = 0$$

to equations

$$m_i a_i = F_i + R_i \quad i = 1 \ldots N$$

of motion for mechanical system with constraints (1), (2). Let the constraint force be ideal:

$$\sum_{k=1}^{N} R_k \delta r_k = 0$$

here $\delta r_k$ are the virtual movements.

There are three proofs of this theorem [3-6]. Two proofs [3, 4] use the concept of constraint force in indeterminate form. The constraint force is calculated either with the help of formal equality
\( \mathbf{R}_k = m_i \mathbf{a}_k - F_i \) or by means of principle of release from constraints. It leads to vicious circles of reasonings because of the existence of some non-predicative judgments.

Let’s consider the first proof [3] of equivalence mentioned above. It is well-known that equation (7) is the consequence of equation (8), (9). The converse statement is not obvious.

Let’s assume that equation (7) is given. We shall define constraint forces \( \mathbf{R}_j \) formally:

\[
\mathbf{R}_k = m_i \mathbf{a}_k - F_i
\]

Therefore we can write down the general equation (7) in the form of (8), (9). The definition of constraint forces is lawful if and only if the expression for \( \mathbf{R}_j \) is an actual forces. However, general equation (7) speaks nothing about it. Therefore, it is necessary to add to the definition of \( \mathbf{R}_j \) a hypothesis about a reality of forces. So, we assume that there exist a actually forces \( \mathbf{F}_k \) therefore considered movement corresponds to active forces \( \mathbf{F}_k \). However, this hypothesis is the latent form of equations (8). Thus, the first proof is false.

The non-predicativity of definition of equations (8) and hypothesis about a reality of force is the reason of falsity of proof.

\[
\sum_{k=1}^{N} (m_i \mathbf{a}_k - F_k - \mathbf{R}_k) \delta \mathbf{r}_k = 0
\]

(10)

Here \( \delta \mathbf{r}_k \) are the arbitrary independent virtual movements of free particles \( m_k \). Hence it follows equations (8), (9). This completes the second proof.

However, this proof has the non-predicative concepts. Indeed, equations (8) are deduced from equation (10), not from equation (7). Therefore, it is necessary to demand the equivalence of equation (10) to equation (7). However, this equivalence takes place if and only if mechanical system with constraints is equivalent to released system.

So, let’s consider the mechanical system with constraints and the released mechanical system. They have identical sets of particles \( \{ \mathbf{m}_k \} \), \( \{ m_k \} \) \( k = 1, ..., N \) but they have the different sets of virtual movements \( \{ \delta \mathbf{r}_k \} \), \( \{ \mathbf{r}_k \} \) because of conditions \( \{ \delta \mathbf{r}_k \} \subset \{ \delta \mathbf{r}_k \} \).
Definition. We say that system with constraints is dynamically equivalent to released system if both mechanical systems have identical kinematical states of actual movements provided that initial data are identical.

Hence it follows that definition refers to general equations (7) and (10) because these equations describe the kinematical states of mechanical systems. Thus, we have the non-predicativity between the concept of equivalence of equations (7), (10) and the concept of equivalence of two mechanical systems.

As a consequence, second proof is false.

The non-predicativity in such proofs is caused by incompleteness of concepts when artificial definition of reactions $\mathbf{R}_i$ does not use equations of constraints (1), (2) and, as a consequence, closes the system of the concepts on "itself".

The third proof [5, 6] is true because there is not the non-predicativity. The completeness of concepts is provided by representation of constraint forces $\mathbf{R}_i$ by means of Lagrange’s multipliers.

In the conclusion we note, that Russell [7] has attempted to construct the predicative analysis. It has been shown that this problem is impracticable. The matter is that the phenomenon of the non-predicativity is inherent in our consciousness and language because the language contains non-predicative words and concepts, as for example, a word “All”. The absolute sense of this word is non-predicative because it contains itself.

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