SELECTIVE SEPARABILITY ON SPACES WITH AN ANALYTIC TOPOLOGY

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Abstract. We study two forms of selective selective separability, $SS$ and $SS^+$, on countable spaces with an analytic topology. We show several Ramsey type properties which imply $SS$. For analytic spaces $X$, $SS^+$ is equivalent to have that the collection of dense sets is a $G_δ$ subset of $2^X$, and also equivalent to the existence of a weak base which is an $F_σ$-subset of $2^X$. We study several examples of analytic spaces.

1. Introduction

In this paper we study some combinatorial properties of countable topological spaces. We will focus on spaces with a definable topology ([17] [18]), that is to say, the topology of the space, viewed as a subset of $2^X$, has to be a definable set. Typically, the topology will be assumed to be an analytic subset of $2^X$. The interplay between combinatorial properties of a space and the descriptive complexity of the topology itself has shown to be quite fruitful [6, 15, 17, 18, 19].

A topological space $X$ is selectively separable, denoted $SS$, if for any sequence $(D_n)$ of dense subsets of $X$ there is a finite $a_n \subseteq D_n$ for all $n \in \mathbb{N}$ such that $\bigcup_n a_n$ is dense in $X$. This notion was introduced by Scheeper [14] and has received a lot of attention ever since (see for instance [1] [2] [3] [4] [5] [6] [9]). Bella et al. [4] showed that every separable space with countable fan tightness is $SS$. On the other hand, Barman and Dow [1] showed that every separable Fréchet space is also $SS$. In section 3 we present several combinatorial properties which implies $SS$.

Shibakov [15] showed a stronger result when the topology of the space is analytic. He showed that any Fréchet countable space with an analytic topology has a countable $\pi$-base (and thus it is $SS$). The existence of a countable $\pi$-base provides a characterization of a property quite similar to $SS$, it is a property related to a game naturally associated to a selection principle. Let $G_1$ be the two player game defined as follows. Player I plays dense subsets of $X$ and Player II picks a point from the set played by I. So a run of the game consists of a sequence of pairs $(D_n, x_n)$ where $D_n$ is a dense set played by I and $x_n$ is the response of II such that $x_n \in D_n$. We say that II wins if $\{x_n : n \in \mathbb{N}\}$ is dense. Scheeper [14] showed that $X$ has a countable $\pi$-base if, and only if, II has a winning strategy for $G_1$. For the property $SS$ a similar game, denoted $G_{fin}$, is defined as before but now player II picks a finite subset of the dense set played by I. A space $X$ has the property $SS^+$, if II has a winning strategy for $G_{fin}$. We show that, for $X$ with an analytic topology, the game $G_{fin}$ is determined. We also show that $SS^+$ is characterized by the existence of a $F_σ$ weak $\pi$-base (that is, a $F_σ$ subset $P$ of $2^X$ such that every set in $P$ has non empty interior and every non empty open set contains a set from $P$), which turns out to be also equivalent to having that the collection of dense subsets of $X$ is a $G_δ$ subset of $2^X$. We compare this notion of a weak base with that of a $\sigma$-compactlike family introduced in [2]. Our characterization of $SS^+$ allows to show very easily that the product of two $SS^+$ spaces with analytic topology is also $SS^+$. A result that

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holds in general as shown by Barman-Dow [2]. However our proof is different. We analyze a space constructed by Barman-Dow [2] which is SS and not $SS^+$ and show it has an analytic topology and has countable fan tightness. Finally, in the last section of the paper we present several examples of countable spaces.

2. Preliminaries

An ideal on a set $X$ is a collection $\mathcal{I}$ of subsets of $X$ satisfying: (i) $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$. (ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. (iii) $X \notin \mathcal{I}$ and $\emptyset \in \mathcal{I}$. We will always assume that an ideal contains all finite subsets of $X$. If $\mathcal{I}$ is an ideal on $X$, then $\mathcal{I}^+ = \{A \subseteq X : A \notin \mathcal{I}\}$. Fin denotes the ideal of finite subsets of the non negative integers $\mathbb{N}$. We denote by $A^{<\omega}$ the collection of finite sequences of elements of $A$. If $s$ is a finite sequence on $A$ and $i \in A$, $|s|$ denotes its length and $s^\frown i$ the sequence obtained concatenating $s$ with $i$. For $s \in 2^{<\omega}$ and $\alpha \in 2^{\mathbb{N}}$, let $s \prec \alpha$ if $\alpha(i) = s(i)$ for all $i < |s|$ and $|s| = \{\alpha \in 2^{\mathbb{N}} : s \prec \alpha\}$. The collection of all $[s]$ with $s \in 2^{<\omega}$ is a basis of clopen sets for $2^{\mathbb{N}}$. Let $a \in \text{Fin}$ and $A \subseteq \mathbb{N}$, we denote by $a \subseteq A$ if $a$ is an initial segment of $A$, i.e., $a = A \cap \{0, \ldots, n\}$, where $n = \max a$. For $A \subseteq \mathbb{N}$ and $m \in \mathbb{N}$, we denote by $A/m$ the set $\{n \in A : m < n\}$ and by $A \upharpoonright m$ the set $A \cap \{0, \ldots, m - 1\}$.

Let $X$ be a topological space and $x \in X$. All spaces are assumed to be regular and $T_1$. A space is crowded if does not have isolated points. A collection $\mathcal{B}$ of non empty open sets is a $\pi$-base, if every non empty open set contains an element of $\mathcal{B}$. For every non isolated point $x$, we use the following ideal

$$\mathcal{I}_x = \{A \subseteq X : x \notin A \setminus \{x\}\}.$$ 

The ideal of nowhere dense subsets of $X$ is denoted by $\text{nwd}(X)$.

Now we recall some combinatorial properties of ideals. We put $A^{<\ast} B$ if $A \setminus B$ is finite.

$(\mathbf{p}^+)$ $\mathcal{I}$ is $\mathbf{p}^+$, if for every decreasing sequence $(A_n)_n$ of sets in $\mathcal{I}^+$, there is $A \in \mathcal{I}^+$ such that $A \subseteq^* A_n$ for all $n \in \mathbb{N}$. Following [11], we say that $\mathcal{I}$ is $\mathbf{p}^-$, if for every decreasing sequence $(A_n)_n$ of sets in $\mathcal{I}^+$ such that $A_n \setminus A_{n+1} \in \mathcal{I}$, there is $B \in \mathcal{I}^+$ such that $B \subseteq^* A_n$ for all $n$.

The following notion was suggested by some results in [7 8]. By a scheme we understand a collection $(A_s)_{s \in 2^{<\omega}}$ such that $A_s = A_s^0 \cup A_s^1$ and $A_n^0 \cap A_n^1 = \emptyset$ for all $s \in 2^{<\omega}$. An ideal is $\mathbf{wp}^+$, if for every scheme $(A_s)_{s \in 2^{<\omega}}$ with $A_\emptyset \in \mathcal{I}^+$, there is $B \in \mathcal{I}^+$ and $\alpha \in 2^{\mathbb{N}}$ such that $B \subseteq^* A_{\alpha,n}$ for all $n$.

$(\mathbf{q}^+)$ $\mathcal{I}$ is $\mathbf{q}^+$, if for every $A \in \mathcal{I}^+$ and every partition $(F_n)_n$ of $A$ into finite sets, there is $S \in \mathcal{I}^+$ such that $S \subseteq A$ and $S \cap F_n$ has at most one element for each $n$. Such sets $S$ are called selectors for the partition. If we allow partitions with pieces in $\mathcal{I}$, we say that the ideal is weakly selective $\mathbf{ws}$ [11] (also called weakly Ramsey in [13]). Another natural variation is as follows: For every partition $(F_n)_n$ of a set $A \in \mathcal{I}^+$ with each piece $F_n$ in $\mathcal{I}$, there is $S \in \mathcal{I}^+$ such that $S \subseteq A$ and $S \cap F_n$ is finite for all $n$. It is known that the last property is equivalent to $\mathbf{p}^-$ (see Theorem 3.1).

If $\mathcal{F}$ is a filter over $\mathbb{N}$, recall that $A \in \mathcal{F}^+$ if $A \cap V \neq \emptyset$ for all $V \in \mathcal{F}$. If $\mathcal{F}^*$ is the dual ideal of $\mathcal{F}$, then $\mathcal{F}^* = (\mathcal{F}^+)^\perp$. Thus we say that a filter $\mathcal{F}$ is $\mathbf{q}^+$, if its dual ideal $\mathcal{F}^*$ is $\mathbf{q}^+$. We say that an ideal $\mathcal{I}$ is Ramsey, usually denoted by $\mathcal{I}^+ \to (\mathcal{I}^+)^2_2$, if for every $A \in \mathcal{I}^+$ and every coloring $c : A^{[2]} \to 2$, there is $B \subseteq A$ in $\mathcal{I}^+$ homogeneous for $c$, i.e. $c$ restricted to $B^{[2]}$ is constant.

A point $x$ of a topological space $X$ is called a Fréchet point, if for every $A \in \mathcal{I}^+_x$ there is a sequence $(x_n)_n$ in $A$ converging to $x$. We will say that $x$ is a $\mathbf{q}^+$-point, if $\mathcal{I}^+_x$ is $\mathbf{q}^+$. We say that space is a $\mathbf{q}^+$-space, if every point is $\mathbf{q}^+$. For each of the combinatorial notion about ideal introduced above, we define analogously the corresponding notion for points on a space.

Now we recall some other combinatorial properties of a topological space.

(SS) $X$ is selectively separable [14] (see also [3]), if for any sequence $(D_n)_n$ of dense subsets of $X$ there is $a_n \subseteq D_n$ finite for all $n \in \mathbb{N}$ such that $\bigcup_n a_n$ is dense in $X$.
(SS+) Consider the following game $G_{\text{fin}}$. Player I picks a dense set $D_n$; player II picks a finite $a_n \subseteq D_n$. Player II wins if $\bigcup_n a_n$ is dense. $X$ is $SS^+$ if player II has a winning strategy for $G_{\text{fin}}$ (\cite{11}, see also \cite{14}). Notice that if $X$ is not $SS$, then player I has an obvious winning strategy. In Example 4.9 we show that the converse does not hold.

(RS) $X$ is $R$-separable \cite{3}, if for any sequence $(D_n)_n$ of dense subsets of $X$ there is $x_n \in D_n$ for each $n \in \mathbb{N}$ such that $\{x_n : n \in \mathbb{N}\}$ is dense in $X$.

A space $X$ has the countable fan tightness at $x$, if for every sequence $(A_n)_n$ with $x \in \overline{A_n}$ for all $n$, there are finite sets $K_n \subseteq A_n$ for all $n$ such that $x \in \bigcup_n K_n$. This last property is known to be equivalent to $p^+$ (see for instance \cite{13, Theorem 3.6}).

A subset $A$ of a Polish space is called analytic, if it is a continuous image of a Polish space. Equivalently, if there is a continuous function $f : \mathbb{N}^\mathbb{N} \to X$ with range $A$, where $\mathbb{N}^\mathbb{N}$ is the space of irrationals. For instance, every Borel subset of a Polish space is analytic. A general reference for all descriptive set theoretic notions used in this paper is \cite{12}. We say that a topology $\tau$ over a countable set $X$ is analytic, if $\tau$ is analytic as a subset of the cantor cube $2^X$ (identifying subsets of $X$ with characteristic functions) \cite{15, 17, 18, 19}, in this case we will say that $X$ is an analytic space. A regular countable space is analytic if, and only if, it is homeomorphic to a subspace of $C_p(\mathbb{N}^\mathbb{N})$ (see \cite{17}). If there is a base $\mathcal{B}$ of $X$ such that $\mathcal{B}$ is a $F_\sigma$ (Borel) subset of $2^X$, then we say that $X$ has a $F_\sigma$ (Borel) base. In general, if $X$ has a Borel base, then the topology of $X$ is analytic.

3. $p^+$, $q^+$ and $SS$.

In this section we show the following implications:

$$
\begin{array}{c}
\text{Fréchet} & \xrightarrow{ws} & \text{Ramsey} & \xleftarrow{wp} & F_\sigma\text{-base} \\
\text{Sequential} & \xleftarrow{q^+} & \text{Ramsey} & \xrightarrow{wp+} & p^+ \\
 & \text{Ramsey} & \xrightarrow{p^-} & SS \\
\end{array}
$$

We start with the ideal theoretic notions. Some of the implications above are known, however for the sake of completeness we include some of the proofs.

**Theorem 3.1.** The following hold for ideals on a countable set.

(i) $p^+$ implies $wp^+$.

(ii) $q^+$ and $wp^+$ together is equivalent to Ramsey.

(iii) Ramsey implies $ws$.

(iv) $wp$ implies $p^-$.

(v) $p^-$ is equivalent to saying that for every partition $(F_n)_n$ of a set $A \in \mathcal{I}^+$ with each piece $F_n$ in $\mathcal{I}$, there is $S \in \mathcal{I}^+$ such that $S \subseteq A$ and $S \cap F_n$ is finite for all $n$.

(vi) $ws$ is equivalent to $p^-$ together with $q^+$.

**Proof.** Let $\mathcal{I}$ be an ideal on a countable set $X$.

(i) Suppose $\mathcal{I}$ is $p^+$. Let $(A_s)_{s \in 2^{<\omega}}$ be a scheme with $A_s \in \mathcal{I}^+$. Let $\alpha \in 2^{\mathbb{N}}$ such that $A_{\alpha|n} \in \mathcal{I}^+$. Now we can applied $p^+$ to $(A_{\alpha|n})_n$ and finish the proof.

(ii) See \cite{8, Theorem 3.16}.

(iii) Let $(A_k)_k$ be a partition of a set $A \in \mathcal{I}^+$ such that each $A_k \in \mathcal{I}$. Consider the following coloring: $c(\{x, y\}) = 0$, if $x, y \in A_k$ for some $k$. Let $H$ be a $c$-homogeneous set in $\mathcal{I}^+$. Then $H$ cannot be a subset of any $A_k$, thus it has to be $1$-homogeneous and thus a selector.
(iv) Suppose $\mathcal{I}$ is $\text{wp}^+$. To see that $\mathcal{I}$ is $\text{p}^-$, let $(A_n)_n$ be a decreasing sequence of sets in $\mathcal{I}^+$ such that $A_n \setminus A_{n+1} \in \mathcal{I}$. Consider the following scheme:

$$B_0 = A_0,$$

$$B_{(0)} = A_0 \setminus A_1 \text{ and } B_{(1)} = A_1,$$

$$B_{(10)} = A_1 \setminus A_2 \text{ and } B_{(11)} = A_2,$$

$$B_{(11 \cdots 10)} = A_k \setminus A_{k+1} \text{ and } B_{(11 \cdots 11)} = A_{k+1}.$$ 

and $B_0$ can be chosen arbitrarily, if $s$ is not of the form $f \hat{t}$ where $t$ is a constant sequence with value 1.

By $\text{wp}^+$, there is $\alpha \in 2^\mathbb{N}$ and $C \in \mathcal{I}^+$ such that $C \subseteq B_{\alpha|m}$ for all $m$ (in fact, it is clear that the only possibility is that $\alpha$ is the constant sequence equal to 1). Hence $C \cap A_n$ is finite for all $n$.

(v) This result is well known but for the sake of completeness we include a proof. Suppose $\mathcal{I}$ satisfies the second property and we show it is $\text{p}^-$. Let $(A_n)_n$ be a decreasing sequence of sets in $\mathcal{I}^+$ such that $A_n \setminus A_{n+1} \in \mathcal{I}$ for all $n$. Let $F_n = A_n \setminus A_{n+1}$. Then $(F_n)_n$ is a partition of $A_0$ and each $F_n \in \mathcal{I}$. By hypothesis, there is $S \in \mathcal{I}^+$ such that $S \cap F_n$ is finite. Then clearly $S \subseteq B_{\alpha}$ for all $n$. Thus $\mathcal{I}$ is $\text{p}^-$. The other implication is shown analogously.

(vi) This is Theorem 3.2 of [11]. We include a proof for the sake of completeness. We use the equivalent version of $\text{p}^-$ given in (v). Clearly $\text{ws}$ implies $\text{p}^-$ and $\text{q}^+$. To check the converse, let $A \in \mathcal{I}^+$ and $(A_n)_n$ be a partition of $A$ such that $A_n \not\in \mathcal{I}^+$ for all $n$. By $\text{p}^-$ there are finite sets $F_n \subseteq A_n$ such that $B = \bigcup F_n$ belongs to $\mathcal{I}^+$. From $\text{q}^+$, applied to $B$ and the partition $(F_n)_n$, we get the required selector. \qed

The following result is well known (see, for instance, Theorem 3.3. of [11]).

**Theorem 3.2.** Every non trivial $F_\sigma$ ideal is $\text{p}^+$.

**Corollary 3.3.** Let $X$ be a countable space with a $F_\sigma$ basis. Then every point of $X$ is $\text{p}^+$ and thus $X$ is $SS$.

**Proof.** Suppose that $X$ has an $F_\sigma$ base, then it is easy to check that $\mathcal{I}_x$ is $\text{p}^+$ for all $x$. \qed

We recall that any countable subspace of $C_p(2^\mathbb{N})$ admits a $F_\sigma$ basis (see [17]).

**Theorem 3.4.** ([18]) Every point of a countable sequential space is $\text{q}^+$.

**Theorem 3.5.** Suppose $X$ is a countable sequential space. The following are equivalent:

(i) $X$ is Fréchet.

(ii) Every point is $\text{ws}$.

(iii) Every point is $\text{p}^-$.

**Proof.** (i) $\Rightarrow$ (ii). Suppose $X$ is Fréchet. Let $x \in X$, $A \in \mathcal{I}_x^+$ and $(F_n)_n$ be a partition of $A$ such that each $F_n \in \mathcal{I}_x$. We will show that there is a selector in $\mathcal{I}_x^+$. Since $X$ is Fréchet, then there is a sequence $(x_n)_n$ in $A$ converging to $x$. Since each $F_n \in \mathcal{I}_x$, then there are only finitely many elements of the sequence in each $F_n$. Thus we can find a subsequence of $(x_n)_n$ which contains at most one point of each piece $F_n$.

(ii) $\Rightarrow$ (iii). Straightforward.

(iii) $\Rightarrow$ (i). Suppose every point is $\text{p}^-$. Since $X$ is sequential, regular and countable, it suffices to show that $X$ does not contain a copy of the Arens space. In fact, it is easy to check that the property of being $\text{p}^-$ is hereditary and the Arens space is not $\text{p}^-$ (see [5,2]). \qed

**Theorem 3.6.** Let $X$ be a countable crowded space.

(i) If every point is $\text{p}^-$, then $X$ is $SS$.

(ii) If every point is $\text{ws}$, then $X$ is $RS$. 


Proof. (i) Let \((D_n)_n\) be a sequence of dense subsets of \(X\). Let \((L_k)_k\) be a partition of \(\mathbb{N}\) into infinite subsets. Let \((x_k)_k\) be an enumeration of \(X\) and \((O_n^k)_{n \in L_k}\) be a maximal family of open disjoint subsets of \(X\) such that \(x_k \notin O_n^k\) for all \(n \in L_k\). Since each \(D_n\) is dense, from the maximality we get that, for every \(k\), \(\bigcup_{n \in L_k} O_n^k \cap D_n\) accumulates to \(x_k\). Since each \(O_n^k \cap D_n \in \mathcal{I}_{x_k}\) and \(x_k\) is a \(p\)-point, then for every \(k\) and \(n \in L_k\) there is \(E_n^k \subseteq O_n^k \cap D_n\) finite such that \(\bigcup_{n \in L_k} E_n^k \in \mathcal{T}_{x_k}^+\) for all \(k\). Then \(\bigcup_{k \in \mathbb{N}} \bigcap_{n \in L_k} E_n^k\) is the required dense set.

(ii) As before, but using \(\text{ws}\) we can assure that \(|E_n^k| = 1\) for all \(k\) and all \(n \in L_k\).

\(\square\)

The following result is an immediate corollary of Theorems 3.5 and 3.6.

**Corollary 3.7.** (Barman-Dow [1]) Every countable Fréchet space is RS.

Our next example shows that \(p^+\) does not imply RS.

**Example 3.8.** Let \(X = CL(2^\mathbb{N})\) be the collection of all clopen subsets of \(2^\mathbb{N}\) as a subspace of \(2^{2^\mathbb{N}}\). Then \(X\) has a \(F_\sigma\) base (see [17]) and thus it is \(p^+\) and thus SS (see Corollary 3.3). It is also known that \(X\) is not RS (see [2]). We include a proof for the sake of completeness. Let \(s, t \in 2^\mathbb{N}\), we say that \(s\) and \(t\) are linked if there is a sequence \(u \in 2^{n-1}\) such that \(s = u^{-i}\) and \(t = u^{-j}\) with \(i + j = 1\). Each \(x \in X\) is a finite union of sets of the form \([s]\) for \(s \in 2^{<\omega}\) (see [2]). For \(k\) a positive integer, we say that a \(x \in X\) is \(k\)-adequated, if \(x\) can be written as \([s_1] \cup \cdots \cup [s_m]\) with each \(s_i \in 2^k\) and any pair of them are not linked. Let \(A_k = \{x \in X : x\text{ is }k\text{-adequated}\}\). Notice that \(\bigcup_{k \in \mathbb{N}} A_k\) is dense in \(X\) for any infinite set \(B \subseteq \mathbb{N}\). Let \((B_i)_i\) be a partition of \(\mathbb{N}\) into infinite sets and let \(D_i = \bigcap_{k \in B_i} A_k\). Then \((D_i)_i\) witnesses that RS fails. In fact, for any selection \(x_i \in D_i\), \(i \in \mathbb{N}\) it is easy to construct an \(\alpha \in 2^\mathbb{N}\) such that \(\alpha \notin x_i\) for all \(i\). Since \(\{x \in X : \alpha \in x\}\) is an open set, then \(\{x_i : i \in \mathbb{N}\}\) is not dense in \(X\).

### 4. The property \(SS^+\)

As we said in the introduction, Scheepers [14] showed that the existence of a countable \(\pi\)-base is equivalent to required that player II has a winning strategy in the game \(G_1\). In this section we are going to show an analogous result for the game \(G_{fin}\).

Let us say that a collection \(\mathcal{P}\) of subsets of a space \(X\) is a weak base (respectively weak \(\pi\)-base) if every set in \(\mathcal{P}\) has non empty interior and for every non empty open set \(W\) and every \(x \in W\), there is \(V \in \mathcal{P}\) such that \(x \in \text{int}(V) \subseteq V \subseteq W\) (respectively, for every non empty open set \(W\) there is \(V \in \mathcal{P}\) such that \(V \subseteq W\)). Observe that if \(X\) has a countable weak \(\pi\)-base, then it obviously has a countable \(\pi\)-base. The relevance of this notion is when we impose a complexity restriction on \(\mathcal{P}\) (for instance, to require \(\mathcal{P}\) to be \(F_\sigma\)).

We denote by \(\mathfrak{D}(X)\) the collection of all dense subsets of \(X\). Our first observation is the following

**Proposition 4.1.** Let \(X\) be a countable analytic space. Then \(\mathfrak{D}(X)\) is co-analytic.

**Proof.** Let \(\tau\) be the topology of \(X\). Then \(D \notin \mathfrak{D}(X)\) if, and only if, there is \(V \in \tau\) such that \(V \cap E = \emptyset\). Hence the complement of \(\mathfrak{D}(X)\) is a projection of an analytic set and thus it is analytic.

\(\square\)

We present in [5,4] an example of an analytic space \(X\) such that \(\mathfrak{D}(X)\) is not Borel.

**Proposition 4.2.** Let \(Y\) be a countable set. Suppose \(\mathfrak{D} \subseteq 2^Y\) is closed upwards, i.e. whenever \(D \subseteq E\) and \(D \in \mathfrak{D}\), then \(E \in \mathfrak{D}\). The following are equivalent:

1. \(\mathfrak{D}\) is \(G_\delta\).
2. There are closed sets \(F_n \subseteq 2^Y\), for \(n \in \mathbb{N}\), such that \(2^Y \setminus \mathfrak{D} = \bigcup_n \{E : \exists V \in F_n, E \cap V = \emptyset\}\).
Proof. It is clear that (ii) implies that the complement of $\mathcal{D}$ is $F_\sigma$. Suppose now that $\mathcal{D}$ is $G_\delta$. Let $(K_n)_n$ be closed sets in $2^Y$ such that
\[ 2^X \setminus \mathcal{D} = \bigcup_n K_n. \]
Consider $F_n = \{ Y : Y \setminus K_n \subseteq 2Y \}$. We claim that $(F_n)_n$ satisfies (ii). In fact, suppose $E \notin \mathcal{D}$. Then there is $n$ such that $E \in K_n$. Hence $V = Y \setminus E \notin F_n$ and $V \cap E = \emptyset$. Conversely, suppose $V \in F_n$ and $E \cap V = \emptyset$. Since $Y \setminus V \notin \mathcal{D}$ and $E \subseteq Y \setminus V$, then $E \notin \mathcal{D}$. To see that each $F_n$ is closed in $2^Y$ recall that the map $A \mapsto Y \setminus A$ is an homeomorphism of $2^Y$ to itself.

\[ \square \]

**Proposition 4.3.** Let $X$ be a countable crowded space.

(i) If $X$ has a $F_\sigma$ base, then $\mathcal{I}_x$ is $F_\sigma$ for all $x \in X$.

(ii) If $\mathcal{I}_x$ is $F_\sigma$ for all $x \in X$, then $X$ has a $F_\sigma$ weak base.

(iii) Suppose $\mathcal{D}(X)$ is $G_\delta$ and let $(F_n)_n$ be a sequence of closed sets satisfying (ii) in Proposition 4.2. We claim that $\bigcap_n F_n$ is a weak $\pi$-base for $X$. First we show that if $V \in F_n$, then $V$ has no empty interior. Since $(X \setminus V) \cap V = \emptyset$, then by (ii), $X \setminus V \notin \mathcal{D}(X)$. Thus $V$ has non empty interior. Now let $W$ be a nonempty open set. Then $X \setminus W \notin \mathcal{D}(X)$. Thus there is $n$ and $V \in F_n$ such that $(X \setminus W) \cap V = \emptyset$. Thus $V \subseteq W$. Conversely, suppose $\bigcap_n F_n$ is a $F_\sigma$ weak $\pi$-base for $X$. Then it is easy to verify that $(F_n)_n$ satisfies (ii) in Proposition 4.2.

As we saw in Example 3.8, $CL(2^N)$ is a countable space with a $F_\sigma$ base. Since $CL(2^N)$ does not satisfy $RS$, it cannot have a countable $\pi$-base.

**Theorem 4.4.** Let $X$ be a countable crowded analytic space. Then the game $G_{fin}$ is determined. More precisely:

(i) If $\mathcal{D}(X)$ is $G_\delta$, then II has a winning strategy for $G_{fin}$.

(ii) If $\mathcal{D}(X)$ is not $G_\delta$, then I has a winning strategy for $G_{fin}$.

Proof. We assume w.l.o.g. that $X = \mathbb{N}$ and, for simplicity, we will write $\mathcal{D}$ in place of $\mathcal{D}(X)$. For each $a \in F_{\mathbb{N}}$, let $[[a]] = \{ A \in 2^\mathbb{N} : a \subseteq A \}$. Notice that if $O \subseteq 2^\mathbb{N}$ is open and $A \in O$ with $A$ finite, then there is $a \in F_{\mathbb{N}}$ such that $A \in [[a]] \subseteq O$.

Suppose $\mathcal{D}$ is $G_\delta$ and let $(O_n)_n$ be a decreasing sequence of open dense sets such that $\bigcap_n O_n = \mathcal{D}$. We will describe a winning strategy of player II for the game $G_{fin}$. Let $X_0$ be the first move of player I. Since $X_0 \in O_0$, player II chooses $a_0 \in X_0$ such that $[[a_0]] \subseteq O_0$. Suppose $\langle X_0, a_0, \ldots, X_{n-1}, a_{n-1}, X_n \rangle$ has been played such that $X_m$ is dense for all $m \leq n$, $a_m$ is a finite subset of $X_m$ and $[[a_0 \cup \cdots \cup a_m]] \subseteq O_m$ for all $m < n$. Since $Y = (X_n/\max(a_0 \cup \cdots \cup a_{n-1})) \cup (a_0 \cup \cdots \cup a_{n-1})$ is dense in $X$, there is a finite set $a_n \subseteq Y$ such that $\max(a_0 \cup \cdots \cup a_{n-1}) < \min a_n$ and $[[a_0 \cup \cdots \cup a_n]] \subseteq O_n$. Clearly $\bigcap_n a_n \in \bigcap_n O_m$ and hence it is dense. This is a winning strategy for II.

Conversely, suppose $\mathcal{D}$ is not $G_\delta$. Since $\mathcal{D}$ is co-analytic, we use Hurewicz's theorem [12, Theorem 21.18] to find a countable set $Q \subseteq \mathcal{D}$ homeomorphic to the rationals such that $\overline{Q} \cap \mathcal{D} = Q$. We will describe a winning strategy for player I. Let $\{D_n : n \in \mathbb{N}\}$ be an enumeration of $Q$. Player I starts playing $X_0 = D_0$. Suppose player II plays a finite set $a_0 \subseteq X_0$. Since $Q$ has no isolated points, there is $k$ such that $a_0 \subseteq D_k$ and $D_k \neq D_0$; so let $l_1$ be the first such $k$. Let $m_1$ be such that $D_0 \upharpoonright m_1 \neq D_1 \upharpoonright m_1$ and $a_0 \subseteq D_1 \upharpoonright m_1$. Then I plays $D_{l_1}/m_1$. In general, assume the strategy has been defined up round $n$ such that if $\langle X_0, a_0, \ldots, X_{n-1} \rangle$ has been played such that player I followed this strategy, then there are natural numbers $l_j$ and $m_j$ for $j < n$ such that, for $i \leq n - 1$,
(i) $X_i = D_{l_i}/m_i$,
(ii) $a_0 \cup a_1 \cup \cdots \cup a_{i-1} \subset D_{l_i} \upharpoonright m_i$,
(iii) $D_{l_{i-1}} \upharpoonright m_{i-1} \subset D_{l_i}$,
(iv) $D_{l_i} \upharpoonright m_i \neq D_j \upharpoonright m_j$ for all $j \leq i$.

Then $I$ plays a finite set $a_{n-1} \subset D_{l_{n-1}}/m_{n-1} = X_{n-1}$. For the next move, as $Q$ has no isolated points, there are $l_n$ and $m_n$ such that $a_0 \cup a_1 \cup \cdots \cup a_{n-1} \subset D_{l_n} \upharpoonright m_n$, $D_{l_{n-1}} \upharpoonright m_{n-1} \subset D_{l_n}$ and $D_{l_i} \upharpoonright m_i \neq D_j \upharpoonright m_j$ for all $j \leq n$. Then $I$ plays $X_n = D_{l_n}/m_n$.

Finally, let $E = \bigcup_i a_i$ and $F = \bigcup_i (D_{l_i} \upharpoonright m_i)$. Then by (ii) $E \subseteq F$ and by (iii) $F \in \overline{Q}$. By (iv), $F \notin Q$ and therefore neither $F$ nor $E$ are dense. Hence this strategy is winning for player $I$. □

**Corollary 4.5.** Let $X$ be a countable crowded analytic space. Then $X$ is SS$^+$ if, and only if, $X$ has a $F_\sigma$ weak $\pi$-base.

Barman and Dow [2] introduced the notion of a compactlike family which was instrumental for their study of the property SS$^+$. A collection $\mathcal{C}$ of subset of a space $X$ is **compactlike**, if for all $D \in \mathcal{D}(X)$ there is $a \subseteq D$ finite such that $a \cap V \neq \emptyset$ for all $V \in \mathcal{C}$. A space $X$ is **$\sigma$-compactlike**, if there is a sequence $(\mathcal{C}_n)_n$ of compactlike families of open subsets of $X$ such that the topology of $X$ is equal to $\bigcup_n \mathcal{C}_n$.

**Proposition 4.6.** Let $X$ be a countable space. Let $\mathcal{C}$ be a family of subsets of $X$. Then
(i) $\mathcal{C}$ is compactlike if, and only if, every set in $\overline{\mathcal{C}}$ (closure in $2^X$) has non empty interior.
(ii) If $\mathcal{C}$ is compactlike, then $\overline{\mathcal{C}}$ is also compactlike.
(iii) If $X$ is $\sigma$-compactlike, then $X$ has a $F_\sigma$ weak base.

Proof. (i) Suppose $\mathcal{C}$ is compactlike. Let $V \in \mathcal{C}$. If $V$ has empty interior, then $D = X \setminus V$ is dense. But this contradicts the definition of a compactlike family. Conversely, suppose that every set in $\overline{\mathcal{C}}$ has non empty interior. Suppose $\mathcal{C}$ is not compactlike and let $D$ be a dense set such that for all finite subset $a$ of $D$ there is $V \in \mathcal{C}$ missing $a$. Let $(x_n)_n$ be an enumeration of $D$ and pick $V_n \in \mathcal{C}$ such that $\{x_0, \ldots, x_n\} \cap V_n = \emptyset$ for all $n$. Since $\overline{\mathcal{C}}$ is compact, then there is $(n_k)_k$ increasing such that $V_{n_k}$ converges to some $V \in \overline{\mathcal{C}}$. Let $W$ be the interior of $V$ and $m$ such that $x_m \in W \cap D$. Thus there is $k$ such that $n_k > m$ and $x_m \in V_{n_k}$ which is a contradiction.

(iii) Let $\mathcal{C}_n$ be compactlike families of open sets such that the topology of $X$ is equal to $\bigcup_n \mathcal{C}_n$. Then by (i) and (ii), $\bigcup_n \overline{\mathcal{C}}_n$ is a $F_\sigma$ weak base for $X$.

Barman and Dow [2] showed that a countable space satisfies SS$^+$ if, and only if, it is $\sigma$-compactlike. Now we can show some other equivalent characterization of SS$^+$

**Theorem 4.7.** Let $X$ be a countable space. The following are equivalent:
(i) $X$ is SS$^+$.
(ii) $X$ is $\sigma$-compactlike.
(iii) $X$ has a weak $F_\sigma$ base.
(iv) $X$ has a weak $F_\sigma$ $\pi$-base.
(v) $\mathcal{D}(X)$ is $G_\delta$.

Proof. (i) $\Rightarrow$ (ii) was shown in [2]. (ii) $\Rightarrow$ (iii) follows from Proposition 4.6. (iii) $\Rightarrow$ (iv) is obvious. (iv) $\Rightarrow$ (v) follows from Proposition 4.3. (v) $\Rightarrow$ (i) follows from the proof of Theorem 4.1(i), just observe that in that proof the hypothesis that the topology of $X$ is analytic is not necessary.

Barman and Dow [2] have shown that the product of two countable SS$^+$ spaces is SS$^+$. Their proof is based in the characterization of SS$^+$ in terms of compactlike families which makes an extensive use of ultrafilters. We present below a different proof for analytic spaces.
Theorem 4.8. Let X and Y be countable crowded analytic spaces. If X and Y are SS⁺, then so is $X \times Y$.

Proof. It is clear that $X \times Y$ has an analytic topology. By Corollary 4.5 it suffices to show that having a $F_\sigma$ weak $\pi$-base is a productive property. Let $(F_n)_n$ and $(G_n)_n$ be closed subsets of $2^X$ and $2^Y$, respectively, such that $\bigcup_n F_n$ and $\bigcup_n G_n$ are weak $\pi$-bases for X and Y respectively. Let $L_n^m = \{V \times W : V \in F_n \text{ and } W \in G_m\}$. Then each $L_n^m$ is a closed subset of $2^X \times 2^Y$ as the map $(A, B) \in 2^X \times 2^Y \mapsto A \times B \in 2^X \times 2^Y$ is continuous. To finish the argument we observe that $\bigcup_{n,m} L_n^m$ is clearly a weak $\pi$-base for $X \times Y$. □

Barman and Dow [2, Theorem 2.12] constructed a countable space X which is SS and not SS⁺. We recall its definition in order to verify it has an analytic topology. We also show that it is not SS⁺ using Theorem 4.4. They constructed that space to show that SS⁺ is not preserved by unions, in contrast to what happen with SS which is preserved by unions [9]. In fact, there are two disjoint subspaces A and B such that $X = A \cup B$, A and B are SS⁺, however X is not SS⁺ but it is (hereditarily) SS. Moreover, A and B both have a $F_\sigma$ base.

Example 4.9. We use $\alpha$, $\beta$, etc. to denote elements of the Cantor set $2^N$. For each $\alpha \in 2^N$, let $\alpha^* \in 2^N$ be the flipping of $\alpha$ in the first value, i.e. $\alpha^*(0) = 1 - \alpha(0)$ and $\alpha^*(n) = \alpha(n)$ for all $n \geq 1$. Consider the following subset of $2^{2^N}$

$$Z = \{z \in 2^{2^N} : z(\alpha) \cdot z(\alpha^*) = 0 \text{ for all } \alpha \in 2^N\}.$$ 

We need to modify the collection of clopen subsets of $2^{2^N}$. Let Q be the collection of eventually zero sequences. Then Q is a copy of the rationals inside $2^{2^N}$. For each $z \in 2^N$, let

$$z_* = z \setminus Q \cup \{\alpha^* : \alpha \in z \cap Q\}.$$ 

Consider the map f from $2^{2^N}$ to itself given by $f(z) = z_*$. Notice that $(z_*)_* = z$ for all $z \in 2^N$. Clearly f is an homeomorphism. Moreover $Z = f[Z]$.

Let $A = CL(2^N) \cap Z$, $B = \{z_* : z \in A\}$ and $X = A \cup B$. Notice that A and B are homeomorphic and $A \cap B = \emptyset$. Since $CL(2^N)$ has a $F_\sigma$ base (see [17]), then both A and B has a $F_\sigma$ base. We will show that X has analytic topology, is $p^+$ and not SS⁺. To see that the topology of X is analytic, we just need to check that the standard subbase of X is analytic. In fact, for each $\alpha \in 2^N$ consider the subbasic open sets of X

$$[\alpha; 1] = \{x \in X : \alpha \in x\} \text{ and } [\alpha; 0] = \{x \in X : \alpha \notin x\}.$$ 

We claim that the functions from $2^N$ to $2^X$, $\alpha \mapsto [\alpha; i]$ are Borel measurable for $i = 0, 1$. Let us check for $i = 1$. Fix $x_0 \in X$, it suf\-fles to verify that $N = \{\alpha \in 2^N : x_0 \in [\alpha; 1]\}$ is Borel. Obviously $N = x_0$. Notice that $x_0$ is either a clopen subset of $2^N$ or it is of the form $z_*$ for some $z \in A$ which is obviously a Borel set. Hence the collection of all subbasic open sets of the form $[\alpha; 0]$ or $[\alpha; 1]$ is the Borel image of the Cantor space and thus is a analytic subbase for X. Therefore the topology of X is also analytic (see [17]). Since both A and B are $p^+$, then it is easy to verify that X is also $p^+$.

Now we will show that X is not SS⁺ by showing that $\mathcal{D}(X)$ is not $G_\delta$. For that end we will find a copy of the rationals $Q_1 \subseteq 2^X$ such that $\overline{Q_1} \cap \mathcal{D}(X) = Q_1$. Consider

$$Q_1 = \{[q; 0] \cap A \cup [q^*; 0] \cap B : q \in Q\}.$$ 

We claim that $\overline{Q_1} = Q_1 \cup \{[\alpha, 0] : \alpha \in 2^N \setminus Q\}$. We state several facts in order to verify it. For notational simplicity, let

$$D_\beta = [\beta; 0] \cap A \cup [\beta^*; 0] \cap B$$

for each $\beta \in 2^N$. 8
(1) The map $\alpha \in 2^N \mapsto [\alpha; 0] \cap A$ is continuous: This follows immediately from the fact that every element of $A$ is a clopen subset of $2^N$.

(2) If $q_n \in Q$ and $q_n \to \alpha$ with $\alpha \notin Q$, then $[q_n; 0] \cap B \to [\alpha^*; 0] \cap B$: Let $z_\alpha \in [\alpha^*; 0] \cap B$ with $z \in A$, in particular, $z$ is clopen. Since $\alpha^* \notin z_\alpha$ and $\alpha \notin Q$, then $\alpha^* \notin z$. Clearly $q_n^\alpha \to \alpha^*$, thus $q_n^\alpha \notin z$ eventually. Therefore $q_n \notin z_n^\alpha$ eventually. That is to say $z_n \in [q_n; 0] \cap B$ eventually. Analogously, one shows that if $z_n \notin [\alpha^*; 0] \cap B$, then $z_n \notin [q_n; 0] \cap A$ eventually.

(3) If $q_n \in Q$ and $q_n \to \alpha$ with $\alpha \in Q$, then $[q_n; 0] \cap B \to [\alpha; 0] \cap B$: This is shown analogously as before.

(4) The map $q \in Q \mapsto D_q$ is continuous: Let $q_n \in Q$ converging to $q \in Q$. Then $D_{q_n} = [q_n; 0] \cap A \cup [q_n^\alpha; 0] \cap B$ converges to $[q; 0] \cap A \cup [q^\alpha; 0] \cap B$ by (1), (3) and the fact that $q_n^\alpha \to q^\alpha$.

(5) Suppose $D_{q_n} \to D$ for some sequence $(q_n)_n$ in $Q$ converging to some $\alpha \in 2^N$. If $\alpha \in Q$, then $D = D_\alpha$ (notice that $D_{q_n} = [q_n; 0] \cap A \cup [q_n^\alpha; 0] \cap B$ converges to $[\alpha; 0] \cap A \cup [\alpha^*; 0] \cap B = D_\alpha$).

On the other hand, if $\alpha \notin Q$, then $D = [\alpha; 0]$ (notice that $D_{q_n} = [q_n; 0] \cap A \cup [q_n^\alpha; 0] \cap B$ converges to $[\alpha; 0] \cap A \cup [\alpha^*; 0] \cap B = [\alpha; 0]$).

(6) $D_\alpha$ is dense for every $\beta$ and $[\alpha; 0]$ is not dense for every $\alpha$.

From (4), $Q_1$ has no isolated points and hence it is homeomorphic to the rationals. Finally $\overline{Q_1 \cap D(X)} = Q_1$ by (6).

This example also shows that having a winning strategy for player I in $G_{fin}$ does not imply the failure of $SS$. As this space is analytic, the game $G_{fin}$ is determined and, in fact, player I has a winning strategy because the space is not $SS^+$ (by Theorem [44]). However, $X$ is $SS$. On the other hand, notice that in general when a space is not $SS$, player I has an obvious winning strategy.

5. Some examples

Now we present examples to illustrate that some of implications shown in the previous section are strict. Sequentiality, $SS$ and $SS^+$ are the only notions we are considering which are strictly topological (i.e. they are no reduced to a combinatorial property of the ideal $\mathcal{I}_s$). We will present examples showing that $SS^+$, $p^-$ and $q^+$ are independent, i.e. all boolean combination of them are realizable by analytic spaces.

It is well known that filters (or dually, ideals) are viewed as spaces with only one non isolated point. We recall this basic construction. Suppose $Z = \mathbb{N} \cup \{\infty\}$ is a space such that $\infty$ is the only accumulation point. Then $\mathcal{F}_\infty = \{A \subseteq \mathbb{N} : \infty \in \text{int}_Z(A \cup \{\infty\})\}$ is the neighborhood filter of $\infty$. Conversely, given a filter $\mathcal{F}$ over $\mathbb{N}$, we define a topology on $\mathbb{N} \cup \{\infty\}$ by declaring that each $n \in \mathbb{N}$ is isolated and $\mathcal{F}$ is the neighborhood filter of $\infty$. We denote this space by $Z(\mathcal{F})$. It is clear that the combinatorial properties of $\mathcal{F}$ and $Z(\mathcal{F})$ are the same. Since in $Z(\mathcal{F})$ all points except $\infty$ are isolated, we use other methods to associate a crowded space to a filter. We give two different such constructions. The first one is defined on $\mathbb{N}^{<\omega}$ and is a generalization of Arkhangel’stvii-Franklin space. The second one uses inverse limits.

We need some general facts about analytic ideals and topologies.

Theorem 5.1. (Jalali-Naini, Talagrand [16] Theorem 1, p. 32) Let $\mathcal{I}$ be an ideal over a countable set $X$ containing all finite subsets of $X$. Suppose $\mathcal{I}$ has the Baire property as a subset of $2^X$. Then there is a partition $(K_n)_n$ of $X$ into finite sets such that $\bigcup_{i \in A} K_i \notin \mathcal{I}$ for all infinite $A \subseteq X$.

Proposition 5.2. Let $X$ be a countable analytic crowded space and $A \subseteq X$. For each partition $(F_n)_n$ of $A$ into finite sets there is a coarser partition $(E_n)_n$ such that $\bigcup_{n \in M} E_n$ is dense in $A$ for every infinite $M \subseteq \mathbb{N}$.

Proof. For each $x \in \overline{A}$, the ideal $\mathcal{I}_x$ (restricted to $A$) is analytic and thus it has the Baire property. By Theorem [5,1] there is a partition $(G^x_n)_n$ of $A \setminus \{x\}$ into finite sets such that $x \in \bigcup_{n \in M} G^x_n$ for
every infinite \( M \subseteq \mathbb{N} \). Fix a point \( x \in \overline{\mathbb{A}} \). Now we define by recursion a new partition \((E_n^x)\), with the property for any infinite \( M \subseteq \mathbb{N} \cup E_i \) accumulates to \( x \).

We will omit the superscript \( x \). Let \( a_0 \) be a finite set such that \( G_0 \subseteq \bigcup_{n \in a_0} F_n \). Pick \( n_1 \) such that \( G_{n_1} \cap (\bigcup_{n \in a_0} F_n) = \emptyset \) and let \( a_1 \) be a finite set disjoint from \( a_0 \) and such that \( G_{n_1} \subseteq \bigcup_{n \in a_1} F_n \). Then we construct a sequence of disjoint finite sets \((a_i)_i\) and a sequence of integers \((n_i)_i\), such that \( G_{n_i} \subseteq \bigcup_{n \in a_i} F_n \) and \( G_{n_i} \cap \bigcup_{n \in a_{i-1}} F_n = \emptyset \). Let \((k_i)_i\) be an enumeration of \( \mathbb{N} \setminus \bigcup_k a_k \). Let \( E_i = F_{k_i} \cup \bigcup_{n \in a_1} F_n \). Then \((E_i)_i\) is a partition with the property that for any infinite \( M \subseteq \mathbb{N} \), \( \bigcup_{E_i} \subseteq M \) accumulates to \( x \).

For each \( x \in \overline{\mathbb{A}} \), fix a partition \((E_n^x)\) (coarser than \((F_n)_n\)) as before. The required partition \( P = (E_n^x)_n \) can be constructed recursively so that for each \( x \in \overline{\mathbb{A}} \), \( P \) contains infinitely many of the blocks of the partition \((E_n^x)_n\).

\[ \square \]

5.1. **Ideals.** We present some example of ideals satisfying some of the properties we have considered. Probably they are known, we have included them to illustrate some of the combinatorial properties and also because we will use them later to construct crowded spaces with similar properties.

**Example 5.3.** We present an example of a \( \text{ws} \) and not Ramsey ideal. It is well known that \( nwd(\mathbb{Q})^+ \) is not Ramsey (see for instance [8]). We include the proof for the sake of completeness. Let \( (x_n)_n \) be an enumeration of \( \mathbb{Q} \) and define a coloring as follows: \( c : \mathbb{Q}^2 \rightarrow \{0,1\} \), for \( r < q \) in \( \mathbb{Q} \) put \( c(q,r) = 0 \) if \( q = x_n \), \( r = x_m \) and \( n < m \). Then any homogeneous set for \( c \) is the range of a strictly monotone subsequence of \((x_n)_n\) and hence is nowhere dense.

We show that \( nwd(\mathbb{Q})^+ \) is \( \text{ws} \). Let \( A \not\subseteq nwd(\mathbb{Q}) \) and \((F_n)_n\) be a partition of \( A \) with each \( F_n \) a nwd set. Let \((V_n)_n\) be an enumeration of an open basis for \( W = \text{int}(\overline{\mathbb{A}}) \). By recursion we define a sequence of points \((x_n)_n\) and a sequence of integers \((n_k)_k\). Pick \( x_0 \in V_0 \cap A \) and let \( n_0 \) be such that \( x_0 \in F_{n_0} \). Since \( E_0 = \bigcup_{k=0}^{n_0} F_k \) is nwd and \( V_1 \cap A \) is not nwd, then pick \( x_1 \in (V_1 \cap A) \setminus E_0 \). Let \( n_1 \) be such that \( x_1 \in F_{n_1} \). As \( E_1 = \bigcup_{k=0}^{n_1} F_k \) is nwd and \( V_2 \cap A \) is not nwd, then pick \( x_2 \in (V_2 \cap A) \setminus E_1 \) and \( n_2 \) such that \( x_2 \in F_{n_2} \) and so on. The set \( S = \{ x_n : n \in \mathbb{N} \} \) is a selector for \((F_n)_n \) which is dense in \( W \).

\[ \square \]

**Example 5.4.** The Fubini product \( \mathcal{I} \times \mathcal{J} \) of two ideals \( \mathcal{I} \) and \( \mathcal{J} \) over \( \mathbb{N} \) is the ideal over \( \mathbb{N} \times \mathbb{N} \) given by

\[ A \in \mathcal{I} \times \mathcal{J} \iff \{ n : (n,m) \in A \} \not\subseteq \mathcal{J} \in \mathcal{I}. \]

It is easy to verify that \( \mathcal{I} \times \mathcal{J} \) is not \( p^- \) for all ideals \( \mathcal{I} \) and \( \mathcal{J} \). In fact, consider the sets \( A_n = \{ n \} \times \mathbb{N} \) for \( n \in \mathbb{N} \). Then each \( A_n \in \mathcal{I} \times \mathcal{J} \), \( \mathbb{N} \times \mathbb{N} = \bigcup_n A_n \) and for any \( S \subseteq \mathbb{N} \times \mathbb{N} \) such that \( S \cap A_n \) is finite for all \( n \), we have that \( S \in \mathcal{I} \times \mathcal{J} \). Thus \( \mathcal{I} \times \mathcal{J} \) is not \( p^- \) (see Proposition 3.1).

If \( \mathcal{J} \) is not \( q^+ \), then \( \mathcal{I} \times \mathcal{J} \) is not \( q^+ \) for any ideal \( \mathcal{I} \). In fact, let \( A \not\in \mathcal{J} \) and \((L_m)_m\) be a partition of \( A \) into finite sets such that any selector for \((L_m)_m\) belongs to \( \mathcal{J} \). Consider the set \( B = \mathbb{N} \times A \). It is clear that \( B \not\in \mathcal{I} \times \mathcal{J} \) and \( B = \bigcup_{n,m} \{ n \} \times L_m \). Let \( S \subseteq B \) be such that \( S \cap \{ n \} \times L_m \) is finite for all \( n \) and \( m \). We claim that \( S \in \mathcal{I} \times \mathcal{J} \). In fact, otherwise there is \( n \) such that \( S' = \{ m : (n,m) \in S \} \not\subseteq \mathcal{J} \) and \( S' \) is a selector for \((L_m)_m\), which is a contradiction.

We show below in Theorem 5.10 that the converse is also true for analytic ideal, i.e. \( \mathcal{I} \times \mathcal{I} \) is \( q^+ \), when \( \mathcal{I} \) is an analytic \( q^+ \) ideal.

**Example 5.5.** We recall a well known example of a \( p^+ \) but not \( q^+ \) ideal. We include it for the sake of completeness. Let \((K_n)_n \) be a partition of \( \mathbb{N} \) into finite sets such that \( |K_n| < |K_{n+1}| \) for all \( n \). Let \( \mathcal{I} \) be the collection of all \( A \subseteq \mathbb{N} \) such that \( \sup_n |A \cap A_n| < \infty \). Then \( \mathcal{I} \) is an ideal and clearly the partition \((K_n)_n \) shows that \( \mathcal{I} \) is not \( q^+ \). It is easy to see that \( \mathcal{I} \) is an \( F_\sigma \) ideal and therefore it is \( p^+ \).

\[ \square \]
5.2. The space $\text{Seq}(\mathcal{F})$. In this section we present a method to construct non $SS$ spaces. We use a well-known construction of a family of topologies on $\mathbb{N}^{<\omega}$, following the presentation given in [IS]. These topologies are defined using filters $\mathcal{F}$ over $\mathbb{N}$. Define a topology $\tau_{\mathcal{F}}$ over $\mathbb{N}^{<\omega}$ by letting a subset $U$ of $\mathbb{N}^{<\omega}$ be open if and only if

$$\{n \in \mathbb{N} : s \in U\} \in \mathcal{F} \quad \text{for all} \quad s \in U.$$ 

Let $\text{Seq}(\mathcal{F})$ denote the space $(\mathbb{N}^{<\omega}, \tau_{\mathcal{F}})$. This space is $T_2$, zero dimensional and has no isolated points. Notice that when $\mathcal{F}$ is the filter of co-finite sets, then $\text{Seq}(\mathcal{F})$ is homeomorphic to the Arkhangel’ski˘ı-Franklin space $\text{Seq}$ and thus Arens space is homeomorphic to $\mathbb{N}^{<\omega}$ as a subspace of $\text{Seq}$. It is also clear that $\tau_{\mathcal{F}}$ is analytic if $\mathcal{F}$ is analytic (see [20] for more descriptive set theoretic properties of this space). Now we characterize the closure operator of $\text{Seq}(\mathcal{F})$. Let $A \subseteq \mathbb{N}^{<\omega}$ and put

$$c(A) = A \cup \{s \in \mathbb{N}^{<\omega} : \{n \in \mathbb{N} : s \in A\} \in \mathcal{F}^+\}.$$ 

We can iterate this operator and define $c_\alpha(A)$ for all $\alpha < \omega_1$ as follows: $c_{\alpha+1}(A) = c(c_\alpha(A))$ and $c_\lambda(A) = \bigcup_{\alpha < \lambda} c_\alpha(A)$ for $\lambda$ limit. Then

$$\overline{A} = \bigcup_{\alpha < \omega_1} c_\alpha(A).$$

For $s \in \overline{A}$, we define $rk(s, A)$ to be the smallest $\alpha$ such that $s \in c_\alpha(A)$.

**Lemma 5.6.** For each $A \subseteq \mathbb{N}^{<\omega}$ and every $s \in \overline{A}$, there is $D \subseteq A$ such that $s \in \overline{D}$ and $rk(t, E) \leq rk(s, A)$ for all $E \subseteq D$ and every $t \in \overline{E}$.

**Proof.** By induction on $rk(s, A)$. If $rk(s, A) = 0$, then let $D = \{s\}$. Suppose $\alpha = rk(s, A) > 0$ and the conclusion holds for sets with rank smaller than $\alpha$. By the definition of the rank, we know that

$$H = \{n \in \mathbb{N} : s \in c_\beta(A)\} \in \mathcal{F}^+.$$ 

Let $A_n = N_{c_\alpha} \cap A$ for $n \in H$. Notice that if $n \in H$, then $rk(s_n, A) = rk(s_n, A_n) < \alpha$. For each $n \in H$, by inductive hypothesis applied to $s_n$ and $A_n$, there is $D_n \subseteq A_n$ such that $s_n \in \overline{D_n}$ and $rk(t, E) \leq rk(s_n, A_n)$ for all $E \subseteq D_n$ and every $t \in \overline{E}$. Let $D$ be the union of the $D_n$’s. Then $D$ satisfies the conclusion.

**Theorem 5.7.** $\text{Seq}(\mathcal{F})$ is not $SS$ for any filter $\mathcal{F}$. In particular, $\text{Seq}(\mathcal{F})$ is not $p^-$.

**Proof.** Consider $D_n = \{s \in \mathbb{N}^{<\omega} : |s| \geq n\}$. Then each $D_n$ is dense in $\text{Seq}(\mathcal{F})$ but for any $F_n \subseteq D_n$ finite, the set $\bigcup_n F_n$ is not dense in $\text{Seq}(\mathcal{F})$.

The previous result implies that, among the properties we are considering, the only ones that $\text{Seq}(\mathcal{F})$ could possibly satisfy are sequentiality and $q^+$. When this space is sequential have been already characterized as stated in the following theorem. Recall that a filter $\mathcal{F}$ is Fréchet if for all $A \in \mathcal{F}^+$ there is $B \subseteq A$ such that every infinite subset of $B$ belongs to $\mathcal{F}^+$.

**Theorem 5.8.** ([IS]) $\text{Seq}(\mathcal{F})$ is sequential if, and only if, $\mathcal{F}$ is Fréchet.

In [10] were constructed $\aleph_1$ non homeomorphic spaces of the form $\text{Seq}(\mathcal{F})$ with $\mathcal{F}$ a Fréchet analytic filter.

Now we address the question of when $\text{Seq}(\mathcal{F})$ is $q^+$.

**Theorem 5.9.** Let $\mathcal{F}$ be a analytic filter on $\mathbb{N}$. Then $\text{Seq}(\mathcal{F})$ is $q^+$ if, and only if, $\mathcal{F}$ is $q^+$. 


Proof. Suppose that \( \mathcal{F} \) is not \( \mathfrak{q}^+ \). We will show that \( \text{Seq}(\mathcal{F}) \) is not \( \mathfrak{q}^+ \) at \( \emptyset \). Let \( A \in \mathcal{F}^+ \) and \( (F_n)_n \) be a partition of \( A \) into finite sets such that every selector belong to \( \mathcal{F}^* \). Let \( B = \{ s^n : n \in A \} \). Clearly \( \emptyset \in \mathcal{B} \) and the corresponding partition \( G_m = \{ s^n : n \in F_m \} \) of \( B \) does not have a selector which accumulates to \( \emptyset \).

Conversely, suppose that \( \mathcal{F} \) is \( \mathfrak{q}^+ \). For each \( s \in \mathbb{N}^{<\omega} \), let

\[ \mathcal{I}_s = \{ A \subseteq \mathbb{N}^{<\omega} : s \notin A \}, \]

where the closure is in \( \text{Seq}(\mathcal{F}) \). We need to show that each \( \mathcal{I}_s \) is \( \mathfrak{q}^+ \). Let \( A \subseteq \mathbb{N}^{<\omega} \) with \( s \in \overline{A} \) and \( (F_n)_n \) be a partition of \( A \) into finite sets. Let \( \alpha \) be a countable ordinal such that \( s \in c_\alpha(A) \).

We will show, by induction on \( \alpha \), that there is a selector for \( (F_n)_n \) in \( \mathcal{I}_s^\alpha \). If \( s \in c(A) \), then \( B = \{ n \in \mathbb{N} : s^n \in A \} \in \mathcal{F}^+ \). Then \( (F_n)_n \) induces a partition of \( B \) and \( \mathcal{F} \) is \( \mathfrak{q}^+ \), there is a selector \( S \in \mathcal{F}^+ \) for the induced partition. Then \( \{ s^n : n \in S \} \) is the required selector for \( (F_n)_n \).

Suppose now that the result holds for every \( A \) and every \( s \in c_\beta(A) \) with \( \beta < \alpha \). Let \( s \in c_\alpha(A) \) and \( (F_n)_n \) a partition of \( A \). Since the topology of \( \text{Seq}(\mathcal{F}) \) is analytic, then by Proposition 5.2 we can assume that \( \bigcup_{k \in \mathbb{N}} F_k \) is dense in \( A \) for every infinite \( M \subseteq \mathbb{N} \). Let \( B = \{ n \in \mathbb{N} : s^n \in \bigcup_{\gamma < \alpha} c_\gamma(A) \} \), then \( B \in \mathcal{F}^+ \). Notice that \( s^n \in A \cap N_{s^n} \) and \( s^n \in \bigcup_{\gamma < \alpha} c_\gamma(A \cap N_{s^n}) \) for every \( n \in B \). By Lemma 5.6 by passing to a subset of \( A \cap N_{s^n} \), we can assume that \( s^n \in A \cap N_{s^n} \) and \( rk(t, E) \leq rk(s^n, A \cap N_{s^n}) \) for all \( E \subseteq A \cap N_{s^n} \) and every \( t \in E \).

Let \( (M_n)_{n \in B} \) be a partition of \( \mathbb{N} \) into infinite sets. Let \( A_n = \bigcup_{k \in M_n} F_k \), for \( n \in B \). Since each \( A_n \) is dense in \( A \), then \( s^n \in \overline{A}_n \) for \( n \in B \) and by construction, \( rk(s^n, A_n) < \alpha \). Let \( P_n \) be the partition that \( (F_m)_{m \in M_n} \) induces on \( A_n \cap N_{s^n} \). By the inductive hypothesis, for each \( n \in B \), there is a selector \( S_n \in \mathcal{I}_s^\alpha \) for \( P_n \). Then \( \bigcup_{n \in B} S_n \) is a selector of the original partition and it belongs to \( \mathcal{I}_s^\alpha \).

\[ \square \]

Corollary 5.10. Let \( \mathcal{I} \) be an analytic \( \mathfrak{q}^+ \) ideal over \( \mathbb{N} \). Then \( \mathcal{I} \times \mathcal{I} \) is \( \mathfrak{q}^+ \).

Proof. Let \( \mathcal{F} \) be the dual filter of \( \mathcal{I} \) and \( X = \mathbb{N}^2 \cup \{ \emptyset \} \) as a subspace of \( \text{Seq}(\mathcal{F}) \). Then \( \emptyset \) is an accumulation point of \( X \). It is easy to verify that

\[ \mathcal{I}_\emptyset = \{ A \subseteq X : \emptyset \notin \overline{A} \} = \mathcal{I} \times \mathcal{I}. \]

Since \( \text{Seq}(\mathcal{F}) \) is \( \mathfrak{q}^+ \), then so is \( X \) and thus \( \mathcal{I} \times \mathcal{I} \) is also \( \mathfrak{q}^+ \).

The argument above can be easily modified to show that \( \mathcal{I} \times \mathcal{J} \) is \( \mathfrak{q}^+ \) when both \( \mathcal{I} \) and \( \mathcal{J} \) are \( \mathfrak{q}^+ \). \[ \square \]

Example 5.11. \( \text{Seq} \) is \( \mathfrak{q}^+ \) (as \( \text{Fin} \) is \( \mathfrak{q}^+ \)) and it is not \( \mathfrak{SS}^+ \).

Example 5.12. Let \( \mathcal{F} \) be a non \( \mathfrak{q}^+ \) filter (for instance, see Example 5.3). Then by Theorems 5.9 and 5.7 \( \text{Seq}(\mathcal{F}) \) is neither \( \mathfrak{q}^+ \) nor \( \mathfrak{SS}^+ \).

5.3. Inverse limits. In this section we use inverse limits to construct, for each boolean combination of the properties \( \mathfrak{p}^- \) and \( \mathfrak{q}^+ \), a crowded \( \mathfrak{SS}^+ \) analytic space satisfying them. Notice first that the rationals are \( \mathfrak{p}^+ \), \( \mathfrak{q}^+ \) and \( \mathfrak{SS}^+ \) by being first countable.

We recall a construction and some examples of countable spaces presented in [9]. Let \( Z \) be a space and \( f : Z \to Z \) continuous. As usual, we denote by \( \lim\{Z, f\} \) the inverse limit of the constant sequence \( \{Z; f\} \) which is defined by

\[ X_\infty = \{(x_n)_n \in Z^\mathbb{N} : x_n = f(x_{n+1}) \text{ for all } n \in \mathbb{N}\}. \]

The projection functions \( \pi_n : X_\infty \to Z \) are defined by \( \pi_n(x_m)_m = x_n \).

Theorem 5.13. Let \( \mathcal{F} \) be a filter on \( \mathbb{N} \) and \( Z = Z(\mathcal{F}) \). Let \( f : Z \to Z \) be a continuous and closed surjection such that \( f(\infty) = \infty \). Let \( X_\infty = \lim\{Z, f\} \). Then

(i) Every subspace of \( X_\infty \) has a countable \( \pi \)-base. In particular, \( X_\infty \) is hereditarily \( \mathfrak{SS}^+ \).
Let \( Z \) be a crowded countable analytic space \( \text{Example 5.14.} \)
then the topology of \( \{ Z \} \times \text{Fin} \) is not \( q \).

**Proof.** (ii)-(iv) were proven in [6]. To show (i), let \( Y \subseteq X_\infty \).
Since each \( n \in \mathbb{N} \) is isolated in \( Z \),
then \( \{ \pi_1^{-1}(n) \cap Y : n, m \in \mathbb{N} \} \) is a \( \pi \)-base for \( Y \).

\( \Box \)

**Example 5.14.** There exists a crowded countable analytic space \( X \) which is \( q^+ \), \( SS^+ \) and not \( p^- \).

The space \( X \) will be a subspace of an inverse limit of the form \( \lim \{ Z, f \} \). We first define the space \( Z \).
Consider the following ideal on \( \mathbb{N} \times \mathbb{N} : \)
\[
\text{Fin} \times \text{Fin} = \{ A \subseteq \mathbb{N} \times \mathbb{N} : n \in \mathbb{N} : \{ m \in \mathbb{N} : (m, n) \in A \} \notin \text{Fin} \} \in \text{Fin}. 
\]
Let \( Z = Z(\mathcal{F}) \) where \( \mathcal{F} \) is the dual filter of \( \text{Fin} \times \text{Fin} \). Since \( \text{Fin} \times \text{Fin} \) is analytic, then it is clear that the topology of \( Z \) is analytic. \( Z \) is \( q^+ \), as \( \text{Fin} \times \text{Fin} \) is \( q^+ \) (see Corollary 5.10). \( Z \) is not \( p^- \), as \( \text{Fin} \times \text{Fin} \) is not \( p^- \) (see Example 5.1).

Let \( f : Z \rightarrow Z \) be any function such that:

(a) \( f(\infty) = \infty \);
(b) \( f^{-1}(\{ n \} \times \mathbb{N}) = \{ n \} \times \mathbb{N} \), for each \( n \), and \( f^{-1}(\{ n, m \}) \) has two points for each \( (n, m) \in \mathbb{N}^2 \).

It is not difficult to see that \( f \) is a continuous, closed and open surjection. Let \( X_\infty = \lim \{ Z, f \} \).

In [6] Example 1) it was shown that every countable subspace of \( X_\infty \) has an analytic topology, is \( q^+ \), and \( SS \). By Proposition 5.13 there is an embedding \( i : Z \rightarrow X_\infty \) such that \( |\pi_1^{-1}(n, m) \cap i(Z)| = 1 \), for each \( (n, m) \in \mathbb{N} \times \mathbb{N} \).

By (b), \( X_\infty \) has no isolated points, so let \( D_{n,m} \subseteq \pi_1^{-1}(n, m) \) be a countable set without isolated points such that \( |D_{n,m} \cap i(Z)| = 1 \), for each \( (n, m) \in \mathbb{N} \times \mathbb{N} \).

Let \( X = \bigcup_{(n,m)\in N} D_{(n,m)} \cup \{ p \} \). Since \( i(Z) \subseteq X \) and \( i(Z) \) is not \( p^- \), then \( X \) neither is \( p^- \).

**Example 5.15.** There exists a regular crowded space \( X \) with a \( F_\gamma \) basis (thus \( p^+ \) and \( SS^+ \)) which is not \( q^+ \).

Let \( \mathcal{I} \) be the ideal given in Example 5.3 associated to a partition \( (K_n)_n \) of \( \mathbb{N} \). Let \( Z = Z(\mathcal{I}) \). It is clear that the topology of \( Z \) is analytic. It is obvious that \( Z \) is not \( q^+ \). Notice that \( \mathcal{I} \) is a \( F_\gamma \) subset of \( 2^\mathbb{N} \), therefore \( Z \) is \( p^+ \) (see Theorem 5.2). Let \( f : Z \rightarrow Z \) be any surjective function such that:

(a) \( f(\infty) = \infty \);
(b) \( f^{-1}(K_n) = K_{n+1} \), for each \( n \), and \( f^{-1}(x) \) has two points for each \( x \in \mathbb{N} \).

It is not difficult to see that \( f \) is a continuous, closed and open map. Let \( X_\infty = \lim \{ Z, f \} \).
In [6] Example 2) it was shown that every countable subspace of \( X_\infty \) has a \( F_\gamma \) basis and thus it is \( SS \).

By (b), \( X_\infty \) has no isolated points. For each \( n \in \mathbb{N} \), let \( D_n \subseteq \pi_1^{-1}(n) \) be a countable set without isolated points. Let \( X = \bigcup_n D_n \cup \{ p \} \). By Proposition 5.13 there is an embedding from \( Z \) into \( X \), therefore \( X \) is not \( q^+ \).

**Example 5.16.** There exists a crowded countable analytic space \( X \) which is \( SS^+ \) and neither \( q^+ \) nor \( p^- \).

Let \( \mathcal{I} \) be the ideal given in Example 5.3 associated to a partition \( (K_n)_n \) of \( \mathbb{N} \). Then as we have seen \( \mathcal{I} \times \mathcal{I} \) is neither \( q^+ \) nor \( p^- \). Let \( f : \mathbb{N} \rightarrow \mathbb{N} \) be an onto map such that \( f^{-1}(K_n) = K_{n+1} \), for each \( n \), and \( f^{-1}(x) \) has two points for each \( x \in \mathbb{N} \).
Let \( Z = Z(\mathcal{I}) \). Consider \( g : Z \rightarrow Z \) given by \( g(n, m) = (n, f(m)) \) and \( g(\infty) = \infty \).
Let \( X_\infty = \lim \{ Z, g \} \). As in the previous example,
let $D_{(n,m)} \subseteq \pi_1^{-1}((n,m))$ be a countable set without isolated points, for each $(n,m) \in \mathbb{N} \times \mathbb{N}$. Let $X = \bigcup_{(n,m)\in\mathbb{N}\times\mathbb{N}} D_{(n,m)} \cup \{p\}$. By Proposition 5.13 there is an embedding from $Z$ into $X$. Therefore $X$ is the required space.

5.4. **Complexity of $\mathcal{D}(X)$.** We have seen that $\mathcal{D}(X)$ is co-analytic when the topology of $X$ is analytic. The aim of this section is to show that there are examples where the complexity of $\mathcal{D}(X)$ is co-analytic and not Borel, i.e. it is as high as possible.

**Theorem 5.17.** Let $\mathcal{F}$ be an analytic filter. Then $\mathcal{D}(\text{Seq}(\mathcal{F}))$ is a complete co-analytic set.

**Proof.** Suppose $\mathcal{F}$ is an analytic filter, then the topology $\tau_{\mathcal{F}}$ is analytic and hence $\mathcal{D}(\text{Seq}(\mathcal{F}))$ is co-analytic. For simplicity, put $\mathcal{D} = \mathcal{D}(\text{Seq}(\mathcal{F}))$. Let $\text{Tree}$ be the collection of trees on $\mathbb{N}$ and $\text{WFT}$ the collection of well founded trees on $\mathbb{N}$. It is a classical fact that $\text{Tree}$ is a Polish space (as a subset of $2^{\mathbb{N}^{<\omega}}$) and $\text{WFT}$ is a complete co-analytic set (see [12] Section 32B). To see that $\mathcal{D}$ is complete co-analytic we define a Borel reduction of $\text{WFT}$ into $\mathcal{D}$, that is to say, a Borel map $G : \text{Tree} \to 2^{\mathbb{N}^{<\omega}}$ such that $T$ is well founded iff $G(T) \in \mathcal{D}$. Consider $F : \text{Tree} \to 2^{\mathbb{N}^{<\omega}}$ given by

$$F(T) = \{s \in \mathbb{N}^{<\omega} : \exists t \in T \ | |t| = |s| \ \& \ \forall n < |t|, \ t(n) \leq s(n) \}.$$ 

Let $G : \text{Tree} \to 2^{\mathbb{N}^{<\omega}}$ be given by $G(T) = \mathbb{N}^{<\omega} \setminus F(T)$. We will show that $G$ is the required Borel reduction.

**Claim 1:** $T$ is a well founded tree iff $F(T)$ is a well founded tree.

**Proof:** Notice that $F(T)$ is also a tree and $T \subseteq F(T)$. So it remains to show that if $F(T)$ is not well founded, then so is $T$. Let $\alpha$ be an infinite branch of $F(T)$. Consider

$$S = \{t \in T : \ t(n) \leq \alpha(n) \text{ for all } n < \text{lh}(t)\}.$$ 

Then $S$ is a finitely branching subtree of $T$ and moreover $S$ is infinite (as for all $n$ there is $t_n \in T$ such that $t_n \leq \alpha(n)$). So $S$ is not well founded.

**Claim 2:** If $T$ is not well founded, then $\text{int}_{\tau_{\mathcal{F}}}(F(T)) \neq \emptyset$.

**Proof:** Let $\alpha$ be an infinite branch of $F(T)$ and consider

$$V = \{t \in \mathbb{N}^{<\omega} : \alpha(i) \leq t(i) \text{ for all } i < \text{lh}(t)\}.$$ 

Then $V \subseteq F(T)$ and $V \in \tau_{\mathcal{F}}$ (if $t \in V$ and $|t| = m$, then $\ell j \in V$ for all $j \geq \alpha(m)$).

**Claim 3:** If $S$ is a well founded tree, then $S \in \text{nwd}(\text{Seq}(\mathcal{F}))$.

**Proof:** Since $S$ is a tree, then it is closed in $\text{Seq}$ and therefore it is also $\tau_{\mathcal{F}}$-closed. Thus it suffices to check that $\text{int}_{\tau_{\mathcal{F}}}(S) = \emptyset$. But this is obvious, since any $\tau_{\mathcal{F}}$ basic nbhd inside $S$ will provide an infinite branch in $S$.

Finally, we check that $G$ is the required Borel reduction of $\text{WFT}$ into $\mathcal{D}$. Clearly $F$ is a Borel function and so is $G$. Now, if $T \in \text{WFT}$, then by Claim 3, $G(T)$ contains a open dense set. On the other hand, if $T$ is not well founded, by Claim 2, $G(T)$ is not dense.

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