NON-EXTENSIVE STATISTICS AND SOLAR NEUTRINOS

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In this paper we will show that, because of the long-range microscopic memory of the random force, acting in the solar core, mainly on the electrons and the protons than on the light and heavy ions (or, equally, because of anomalous diffusion of solar core constituents of light mass and of normal diffusion of heavy ions), the equilibrium statistical distribution that these particles must obey, is that of generalized Boltzmann-Gibbs statistics (or the Tsallis non-extensive statistics), the distribution differing very slightly from the usual Maxwellian distribution. Due to the high-energy depleted tail of the distribution, the nuclear rates are reduced and, using earlier results on the standard solar model neutrino fluxes, calculated by Clayton and collaborators, we can evaluate fluxes in good agreement with the experimental data. While proton distribution is only very slightly different from Maxwellian there is a little more difference with electron distribution. We can define one central electron temperature as a few percent higher than the ion central temperature nearly equal to the standard solar model temperature. The difference is related to the different reductions with respect to the standard solar model values needed for B and CNO neutrinos and for Be neutrinos.

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I. INTRODUCTION

After many years of experimental and theoretical work in different laboratories, beyond the demonstration that the sun produces energy via nuclear fusion reactions (solar neutrinos have been detected with fluxes and energies approximately in the ranges predicted by the known standard solar models (SSM), see Ref.s [1–5] for the up to date state of the art in this field), the accepted conclusions for standard solar neutrinos are the following.

The solar luminosity is well known and the experimental results (Gallex, Sage, Chlorine, Kamiokande) seem to be mutually inconsistent; even if we discharge one of the four experiments.

Even neglecting these inconsistencies, the measured experimental fluxes of Be and CNO neutrinos are significantly smaller than the SSM predictions (we need a reduction factor of about 8 for $\Phi_{Be}$, about 2.4 for $\Phi_{CNO}$ and of 2.28 for $\Phi_{B}$, see Table 1).

The different reductions of the $^7$Be and $^8$B neutrino fluxes with respect to the SSM predictions are essentially in contradiction with the fact that both $^7$Be and $^8$B originate from the same parent $^7$Be nuclei. Nevertheless, we must remember, in relation to what will be stated in the following paragraphs, that in the reaction $^7$Be $+ e^- \rightarrow ^7$Li $+ \nu_e$ beryllium nuclei react with the electrons; while in the reaction $^7$Be $+ p \rightarrow ^8$B $+ \gamma, ^8$B $\rightarrow 2\alpha + e^+ + \nu_e, ^7$Be nuclei react with the protons.

The above results could suggest that the hypothesis that nothing happens to the neutrinos after they are created in the interior of sun is incorrect. The alternative is that some of the experiments are wrong, this must be checked.

There are accepted strategies to construct non-standard solar models in order to calculate reduced $^7$Be and $^8$B neutrino fluxes. They are as follows:
a) To produce models with a central temperature $T_c$ smaller than $T_{cSSM}$.
b) To play with the nuclear cross sections determining the branches of the fusion chain. However, we must say that the resonance $^3$He $\rightarrow ^3$He has not been found in the very recent measurements of the cross section by the LUNA group at Gran Sasso Laboratory [6] and, as it has been recently shown by Oberhümer [7], we are incapable of solving, in the pure nuclear physics framework, the solar neutrino problem.
c) To take into account correctly the effects of screening the nuclear charges by the stellar plasma [8]. In order to understand its contribution to the fluxes and make a comparison with the results from the laboratory experiments a new theory of energy loss at very low energies is needed. Very recently Brown and Sawyer have calculated the screening effects in stellar plasma and have found that the enhancement factor to the fusion cross sections is very close to one [9].
d) To calculate from the beginning the opacity, taking into account plasma collective effects, as done recently by Tsitovitch and collaborators [10] and by Ricci [11]. However, we must remember that, in this case, the reaction rates
must be recalculated if the electron distribution deviates from the Maxwellian one due to the presence of particular collective effects, external forces or other factors.

Generally it is accepted that for standard neutrinos the actual experimental results seem to be in disagreement with SSM and with all the solar models we could build even considering neutrino oscillations [12,13].

In spite of these arguments, we wish to discuss, within the standard solar model description, the thermostatistics to which ions and electrons of the solar core should adhere. In fact, we want to show that the Maxwell-Boltzmann statistics must be substituted with the non-extensive statistics, suitable for particles subjected to long-range microscopic memory. The resulting statistical distribution is, for the solar core, a very modest modification of the standard Maxwell-Boltzmann distribution [14,15].

The memory acts on the dynamics of the particles of the solar core because the electron mass is much smaller than the mass of the other constituents of the core plasma and because the proton mass is smaller than that of light and heavier ions. The normal memory is related to a time correlation of the random force which is a delta function. In this case the diffusion is normal, the equilibrium distribution is Maxwellian. The long-range memory implies a random force with time correlation different from a delta function as suggested by the results of measurements of neutrino fluxes.

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The diffusion coefficient of a system of particles, subjected to these forces, is anomalous (non-Brownian) and the equilibrium distribution function is a non-extensive Tsallis distribution [16,17].

We describe, in the following sections, the main features of the non-extensive Tsallis statistics, the normal and anomalous diffusion coefficient, a few applications of astrophysical interest, the application to the solar neutrino production using the results from the SSM of Clayton and collaborators [18,19] and give the physical motivations of the validity using, in this case, the non-extensive statistics.

In the Appendix A we discuss the anomalous diffusion coefficient in the framework of the Tsallis statistics because this quantity represents the link between the distribution adopted and the motivation of its use (i.e. the long-range time correlation of the random force).

II. THE USE OF NON-MAXWELLIAN DISTRIBUTIONS IN THE RECENT PAST

A customary mode of procedure to change the SSM nuclear rates can be based on the use of the appropriate statistics (the generalization of the Boltzmann-Gibbs statistics or the Tsallis non-extensive thermostatistics [16]) which describes the behavior of the different particles composing the solar core. In this work we introduce a modified Maxwellian function which describes an equilibrium distribution and is the first order approximation of the Tsallis distribution function. The physical motivations of this choice are of a many-body nature as it will be explained later. This choice allows a reduction of the Maxwellian rates and a progression towards calculations of the neutrino fluxes not in conflict with experimental results.

To change the statistics and, of course, the distribution function means also to change the energy spectra of the neutrinos produced. Possibly, in future experiments at Gran Sasso Laboratory [20] or, for instance, in Hellaz experiment [21], spectra could be measured with enough precision to allow the verification of the validity of the generalized, non-extensive statistics in the solar plasma.

In the past, after the suggestion of Kocharov [22], Clayton and collaborators [18,19] calculated the evolutionary sequences for the sun under the assumption that the relative energies of the ions are not exactly Maxwellian. Depletion of the high energy tail of the distribution function of relative energies in excess of 10 $kT$ produces a marked reduction in the counting rate of the $Cl$ experiment, while producing minimal changes in the usual solar models. Those changes that do occur can be compensated for by small changes in the initial helium concentration. The mechanism leaves its own signature in the nuclear rates and in the energy spectra of solar neutrinos.

Clayton could not calculate a physical cause of the deficiency in the number of energetic pairs and suspected that the problem lies in many-body physics: long range Coulomb interaction conspires in some unknown way to quench relative energies above $\approx 15 \times kT$.

Correction to the Maxwellian distribution function is given by the factor $\exp[-\delta(E/kT)^2]$. Clayton found for the solar neutrino problem $\delta = 0.01 (E$ is the c.m. energy).

Departure of the two-particle distribution from the Maxwellian distribution function in the solar core is not unphysical. It should be possible and, as we will see in this work, the reason is long-range microscopic memory [23,24], more than the long-range interactions (Coulomb and gravitational).
Haubold and Mathai [25,26], avoiding mathematical approximations, derived closed form representations of nuclear reaction rates including resonant, non resonant, screened cases in terms of special functions. All these calculations have been accomplished without attributing an exact and precise physical meaning of the depletion of the Maxwellian distribution function used, except the following, that many-body effects in the solar core should be taken into account. We also performed in the recent past calculations of nuclear rates and recombination cross sections for opacity using non-Maxwellian distributions [27,28], before our first application of non-extensive statistics to solar neutrino problem [14].

The inclusion of the helium diffusion in solar models has been widely studied by Bahcall and collaborators in the frame of Maxwell-Boltzmann statistics [29,30].

In this paper, because of physical motivations, we want to introduce, for the description of the solar plasma, a generalization of the Boltzmann-Gibbs statistics and by means of a reevaluation of the nuclear rates of interest to calculate the neutrino fluxes.

### III. NON-EXTENSIVE STATISTICS

Recently, non-extensive statistics (generalization of Boltzmann-Gibbs statistics), based on a new generalized definition of entropy, becoming the well known standard definition when the characteristic parameter $q \rightarrow 1$, is widely considered in many different physical phenomena.

The distribution function of these statistics has not only a depleted high energy tail, but in reality vanishes when:

$$E = \frac{kT}{1-q} \quad \text{if} \quad q < 1 \ .$$

Tsallis [16] introduced in 1988 this generalization of thermodynamics and statistical physics to describe systems with long-range interaction (gravitational) and long-range memory [31–33].

Let us describe briefly Tsallis statistics.

For a system with $W$ microscopic state probabilities $p_i \geq 0$, normalized as $\sum_{i=1}^W p_i = 1$, we have 2 axioms:

1) the entropy is

$$S_q = \sum_{i=1}^W S_q^i \quad S_q^i = \frac{k}{q-1} p_i (1 - p_i^{q-1}) \ ,$$

for $q \rightarrow 1$, $S_1 = -k \sum_i p_i \log p_i$.

2) given an observable $O$ with $o_i$ eigenvalues, the mean value is

$$O_q = \sum_{i=1}^W p_i^q o_i, \quad q \rightarrow 1 : \quad O_1 = \sum_{i=1}^W p_i o_i.$$  

The validity of the two axioms lies in the comparison with experiments.

Properties are the following: systems with $q < 1$ give more weight to rare events, systems with $q > 1$ give more weight to frequent events.

The generalized entropy $S_q$ is positive, the microcanonical ensemble has equiprobability, entropy is concave ($q > 0$), convex ($q < 0$); Legendre transformation structure of thermodynamics is invariant for all $q$.

Additive rule (non-extensivity) is :

$$S_q(A \cup B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B) \ .$$

The Tsallis distribution function is given by the expression

$$p_i = \frac{1}{Z_q} \left[ 1 - (1-q) \frac{E_i}{kT} \right]^{1/(1-q)} ,$$

and

$$Z_q = \sum_{i=1}^W \left[ 1 - (1-q) \frac{E_i}{kT} \right]^{1/(1-q)} .$$

Application of generalized non-extensive statistics are the following subjects: matter distribution of self gravitating systems, turbulence, anomalous diffusion, cosmological models, big bang background radiation [31–40]. We describe here only those applications related to astrophysical problems (see Sections 5, 6 and 7).

In Fig.1 the behavior of the function $p_i$ is shown for values of $q$ greater and smaller than one.
IV. NORMAL AND ANOMALOUS DIFFUSION COEFFICIENTS

It is well known that neutrinos of the four solar fluxes $\Phi_p$, $\Phi_{Be}$, $\Phi_{CNO}$ and $\Phi_B$ are produced in an equilibrium plasma in the solar core. In fact, the average time $\tau_{\text{nucl}}$ between two fusion reactions is much greater than the average time between two Coulomb collisions: $\tau_{\text{nucl}} \gg \tau_{\text{Coulomb}}$. We can define

$$\tau_{\text{nucl}} = \frac{1}{n_n <\sigma v>_{\text{nucl}}},$$

(6)

where $n_n$ is the nuclear density and $<\sigma v>_{\text{nucl}}$ is the nuclear fusion rate averaged over a Maxwellian distribution.

The quantity $\tau_{\text{Coulomb}}$ is related to the Brownian diffusion coefficient [41]

$$D_{\text{Br}} = \frac{kT}{m \tau_{\text{Coulomb}}},$$

(7)

A factor of $10^{20}$ is in favor of equilibrium and Maxwell-Boltzmann distribution function is used to describe both electrons and ions dynamics and reaction rates. In support of the validity of the Maxwell-Boltzmann distribution, explicit calculations in which the Fokker-Planck and Boltzmann equations were solved show that the Maxwellian tail is filled, after few Coulomb collisions, when the different colliding particles (both electrons and ions) are all considered Brownian.

It is known that neutrino production nuclear rates, reduced in respect of their SSM values, might lead to an evaluation of the solar neutrino fluxes in agreement with the solar luminosity and not too far from the results of measurements of fluxes accomplished in underground experiments.

Distribution functions of equilibrium are not only Maxwellian distributions. If the diffusion coefficient is not Brownian (or normal) because the medium is dense enough (like the solar core) and the random force has a long-time correlation, then the equilibrium steady state is described by a distribution which is not Maxwellian, although it may be very close to a Maxwellian.

Recently, we have shown that, in the energy space, we must introduce an anomalous diffusion coefficient to solve a generalized Fokker-Planck equation, given by the expression [42]:

$$D' = D_{\text{Br}} \left[ 1 - (1 - q) \frac{E}{kT} \right],$$

(8)

valid up to the energy $E = kT/(1 - q)$; to this coefficient we can relate the anomalous collision time

$$\tau' = \frac{1}{\rho <\sigma v>_{\text{Coul}}} \left[ 1 - (1 - q) \frac{E}{kT} \right],$$

(9)

where $\rho$ is the average density.

We are still in equilibrium conditions, but the distribution is a depleted Maxwellian and of non-extensive Tsallis type.

If the ions (proton, helium, light and heavy-ions) and the electrons can be considered Brownian particles, their diffusion coefficients are normal and their stationary distributions are Maxwellian. This description is correct also in the presence, in the dynamical equations, of a random force with a time correlation equal to a delta function [43–46]. The use of the distribution appropriate to the non-extensive ($q \neq 1$) thermostatistics has the physical meaning that the description of the dynamics of the solar core constituents contains anomalous, rather than normal, diffusion coefficients.

In the solar core we must add, in the dynamical equations of the different constituents, a random force $F(t)$ with time correlation different from the delta function (see the Appendix A for the needed details):

$$< F(0) F(t) > = F_0(\beta)t^{-\beta},$$

(10)

where $\beta$ is a parameter, in this case, smaller than one and related to $q$ by

$$\beta = \frac{1}{2 - q} = \frac{1}{1 + 2\delta},$$

(11)

(for $\beta = 1$, or $q = 1$ and $\delta = 0$ the delta function time correlation is recovered).

The choice of a non-delta time correlation is suggested by the comparison of calculations with the experimental results,
i.e. by the values of the parameter $q$ required by the results of the measurements and, consequently, by the values of the parameter $\beta$.  

In the solar core, the anomaly consists in a subdiffusion, because the value of the parameter $q$ to be used lies in the range: $q = 0.952 \div 0.994$ (or $\delta = 0.003 \div 0.024$), as explained in the following Section 7. The subdiffusion correction to the normal behavior is very small but sufficient to calculate neutrino fluxes in agreement with the measured fluxes.

In the solar core, the anomaly is not due to the long range gravitational interaction, rather it is due to the long-range memory of the random force.

In the Appendix A we discuss the meaning of a non-linear or non-Brownian diffusion coefficient and the solutions of the Fokker-Planck equation appropriate to the non-extensive thermostatistics.

V. STELLAR POLYTROPES

This is the first example of application of non-extensive statistics. The equation of state appropriate to the stellar polytropes is

$$\mathcal{P} = K \rho^\gamma ,$$

where $\mathcal{P}$ is the pressure, $\rho$ the density, $\gamma$ a constant related to specific heats defined by $\gamma = 1 + 1/n$ ($n$ is the polytropic index).

The hydrostatic equilibrium equation has spherical symmetric solutions corresponding to compact configurations of self-gravitating mass (stellar polytropes).

Relation for stellar polytropes between $n$ and the Tsallis parameter $q$ is

$$n = \frac{3}{2} + \frac{q}{1 - q} ,$$

The index $n$ must exceed $1/2$ to avoid singularity in the gamma function; $n > 5$ gives rise to unnormalizable mass distribution (unphysical). The Tsallis distribution function has a spatial cutoff of the mass distribution (compact nature of the stellar polytrope).

In the solar core we have a constraint which fixes differently the relation between $n$ and $q$

$$q = 1 - \frac{\tau}{n + 1}, \quad \tau = \frac{kT \rho}{\mathcal{P}}, \quad q = 1 - 2\delta ,$$

the quantity $\tau$ is very close to one if an ideal gas of particles having a certain average molecular weight is considered; in fact, the equation of state of an ideal gas, within the Tsallis statistics, differs from the classical one for the addition of terms whose contribution depends on powers of $\delta$ (in the limit $q \to 1$, this contribution can safely be disregarded).

VI. APPLICATION TO GALAXY CLUSTER VELOCITY

We have shown that the observational data recently provided by Giovannelli et al. [48] (COBE) and discussed by Neta Bahcall and Peng Oh [49] and by Moscardini et al. [50] concerning the velocity distribution of clusters of galaxies can be naturally fitted by a statistical distribution which generalizes the Maxwell-Boltzmann one.

In this generalization of Boltzmann Gibbs statistics the probability function of having a cluster with velocity greater than $v$ can be written as

$$P(> v) = \int_v^{v_{\text{max}}} \frac{dv}{v_{\text{max}}} \left[ 1 - (1 - q)(v/v_0)^2 \right]^{q/(1-q)} ,$$

$$v_{\text{max}} = \begin{cases} v_0(1 - q)^{-1/2} & \text{if } q < 1 \\ \infty & \text{if } q \geq 1 \end{cases} ,$$

A remarkably good fitting with the data is obtained for $q = 0.23 \pm 0.07$ and $v_0 = 490 \pm 5$ km/s.
VII. APPLICATION TO SOLAR NEUTRINO PROBLEM

Taking advantage of the calculations accomplished by Clayton and collaborators, based on their solar model (standard) which uses a distribution function of Maxwellian type with a depleted tail (this is a first and suitable approximation of the Tsallis distribution), we discuss the application of the non-extensive statistics to the solar neutrino problem. In Refs [14,15] we report how the constraints imposed by the solar core compel us to derive the relation (14) instead of Eq.(13) suitable for stellar polytropes; results of the application of the Tsallis statistics to solar neutrino problem are shown without mentioning the physical motivations to use this new statistics (mainly due to the anomalous diffusion of light particles rather than normal diffusion appropriate to heavier ions).

A value of the Clayton parameter $\delta$ different from zero (we remember that $q = 1 - 2\delta$) makes the star more luminous and reduces the rate of energy production at a given temperature. The solar core contracts to higher temperature. However, the increase of temperature at given solar luminosity does not increase the neutrino fluxes that decrease with $\delta$ (except $\Phi_B$, but its growth is very slow and its value is almost constant) because of the depleted Maxwellian tail and the consequent rearrangement of the particle distribution.

We introduce the reduction factor $f_{\text{red}}$ of the $B,\ CNO$ and $Be$ fluxes as a function of $\delta$ (or $q$)

$$\Phi_{B,CNO,Be} = \frac{1}{f_{\text{red}}} \Phi_{B,CNO,Be}^{\text{SSM}}. \quad (15)$$

and the enhancement factor $f_{\text{incr}}$ of the proton flux $\Phi_p$ as a function of $\delta$ (or $q$)

$$\Phi_p = f_{\text{incr}} \Phi_p^{\text{SSM}}. \quad (16)$$

In Fig. 2 we show both the reduction factor $f_{\text{red}}$ and the increasing factor $f_{\text{incr}}$ as function of $\delta$.

These two quantities are derived from the calculation of the neutrino fluxes as function of $\delta$ by Clayton and collaborators. The SSM fluxes are all normalized to one at $\delta = 0$.

We have obtained an expression for the factors valid up to $\delta = 0.025$ (an extrapolation of the results reported by Clayton and collaborators, in the range between $\delta = 0.020$ and $\delta = 0.025$, has been done):

$$f_{\text{red}}(B) = 1 + 92581.75\delta^{1.92},$$
$$f_{\text{red}}(CNO) = 1 + 23.66^{0.49} + 30\delta,$$
$$f_{\text{red}}(Be) = 1 + 851.71\delta^{1.29},$$
$$f_{\text{incr}}(p) = 1 + 0.89\delta^{0.62}. \quad (17)$$

We can clearly see that the selection of $\delta = 0.003$ ($q = 0.994$) produces the correct reduction and enhancement factors (see Section 1) of the SSM values of $\Phi_B, \Phi_{\text{CNO}}$ and $\Phi_p$ and then neutrino fluxes in agreement with the measured values and with the solar luminosity; the choice $\delta = 0.024$ ($q = 0.952$) is the right selection for $\Phi_{Be}$.

The reduction factors $f_{\text{red}}$ have been derived in such a way that the four fluxes satisfy the constraints imposed by the solar luminosity and by the results of measurements of the Gallex, Sage, Chlorine and Kamiokande experiments, within their precision [3].

The need for two different choices of $\delta$ is due to the $Be$ flux and depends on the electron distribution, while the $B$ flux (and, of course, the proton and $CNO$ fluxes) depends on the proton distribution.

In fact:

$$\Phi_{Be} \propto N(e^-)N(7\text{Be}) < e^- 7\text{Be} > \text{ and } \Phi_B \propto N(p)N(7\text{Be}) < p 7\text{Be} >,$$

where $N(e^-)$ is the electron density, $N(p)$ the proton density, $< e^- 7\text{Be} >$ the nuclear capture rate electron-beryllium and $< p 7\text{Be} >$ the nuclear rate proton-beryllium (analogously we can say of the other two fluxes). See the Table where the results on the fluxes are reported.

The consequences of this derivation is that, in the solar core, electrons behave differently from protons and from the other light and heavy ions. The latter are nearly Maxwellian, the electron distribution must have a more depleted tail. In addition, the polytropic index $n$ of electrons differs from that of the other constituents. The solar core plasma model, consisting of two main components (electrons and ions), with different features, is under study. Here we still want to examine the effect of these physical facts on the central solar temperature by using the well known dependencies of the fluxes on central temperature as given in the literature [4,29,30].
VIII. TEMPERATURE OF THE CENTRAL CORE

We derive the dependence of the four fluxes upon central temperature \( T_c \) when all SSM calculations are executed at \( T_c^{SSM} \), taking advantage of the expressions reported in Ref. [430]. We have derived from Clayton and collaborators calculations the expression

\[
T_c = T_c^{SSM} (1 + 3.12\delta) ,
\]

valid for \( \delta < 0.1 \).

The dependence is as follows

\[
\begin{align*}
\Phi_p &\propto (1 - 0.884t^{11})(1 + 0.44t^{0.62}) , \\
\Phi_B &\propto T_c^{25}[1 + 10^4(t - 1)^{1.92}]^{-1} , \\
\Phi_{CNO} &\propto (xT_c^{20} + (1 - x)T_c^{23})[1 + 13.66(t - 1)^{0.43}]^{-1} , \\
\Phi_{Be} &\propto T_c^{11}[1 + 196(t - 1)^{1.29}]^{-1} ,
\end{align*}
\]

where \( t = T_c/T_c^{SSM} = 1 + 3.12\delta, x \) represents the percentage of the contribution to the CNO flux from the ions \(^{13}\)N and \(^{15}\)O and \( 1 - x \) the contribution from \(^{17}\)F.

We can see that \( T_c \) increases by 1 per cent in respect to \( T_c^{SSM} \) using the value of \( \delta \) appropriate to \( \Phi_{p,B,CNO} \) fluxes (this temperature is the ionic plasma temperature \( T_c^{(1)} = 1.01 T_c^{SSM} \), and increases by seven per cent using the value of \( \delta \) appropriate to \( \Phi_{Be} \) (this temperature is the electron plasma temperature \( T_c^{(e)} = 1.07 T_c^{SSM} \)).

The meaning of the results of the above relations is different from that of the results coming from the relations of other authors. While, according to these authors, the central temperature must decrease in respect to the SSM value to reproduce the behavior of the ratio of the fluxes, in our approach the central temperature \( T_c \) must slightly increase with \( \delta \) and the fluxes (except \( \Phi_p \), which is almost constant) decrease because of the depleted tail and the consequent rearrangement of the particles.

Helioseismology is probing the interior structure and dynamics of the sun with great precision [52–54]. Acoustic waves and internal gravity waves are strongly influenced by the central temperature and by the structure of central regions. The reliability of the non-extensive statistics applied to the solar core can be tested in the context of helioseismological studies. This is the topic of our investigation in the near future.

IX. CONCLUSION

The validity of the description of the solar core shown in this work, by means of a generalized Boltzmann-Gibbs statistics, could be verified in future planned experiments by measuring the electron central temperature \( T_e \), through the detection of the \( Be \) line and by measuring the neutrino energy spectra, looking for a shift of the Gamow peak and of the neutrino maximum energy due to non-extensive thermostatistics effects, if very high precision will be obtained. It could be of practical use, from a theoretical point of view, to evaluate the effect of the neutrino oscillations of the fluxes in the context of the non-extensive statistical description of the solar core.

We have shown that anomalous diffusion (different from Brownian) of electrons and of protons, caused by the long-range microscopic memory of the random force in the solar core plasma, is the cause of the stationary distributions of the non-extensive statistical type we must use. Light and heavy ions have a normal diffusive behavior. As a consequence, the values of the nuclear rates of the different reactions decrease in respect to Maxwellian evaluations (we have taken advantage of the SSM of Clayton and collaborators).

A particular choice of the value of the non-extensive Tsallis parameter \( q \) for the electrons \((q = 0.952)\) and for the protons \((q = 0.994)\) allows the calculation of neutrino fluxes. The values of \( q \) differ from the value \( q = 1 \) (Maxwellian distribution) for very small magnitudes. The value \( q = 1 \) is assumed for light and heavy ions.

This description implies that the electrons have a behavior which differs from the Maxwellian behavior more than that of the protons, while we can assume that light and heavy ions have a normal behavior, or normal diffusion coefficient, which implies Maxwellian distribution.

Our results, based on the standard solar model developed by Clayton and collaborators, imply a weak increase of the central temperatures (both electronic and ionic), as they must slightly increase because of the depleted tail of the distribution.

Finally, we must remember that the validity of the non-extensive statistics in the solar core plasma is also based on the polytropic nature of the sun, whose polytropic index \( n \) has a finite value with the consequence that the distribution
must generally be non-Maxwellian.
The approach described in this work will be refined in order to take into account the information given by the helioseismological studies and measurements.

Our approach is based on a standard solar model with diffusion (anomalous and normal) of the constituents of the solar core plasma, therefore it is highly possible that the test with the helioseismological data be positive, but it must be checked; this will be accomplished soon.

We realize that the results reported in this paper are not definitive. In fact we do not indicate errors and precisions of the figures we have derived and shown. This is due to the reason that we do not have used a solar model code and we have not accomplished calculations abinitio. However, we have taken advantage of a standard solar model previsions and our results are an indication of a possible solution of the solar neutrino problem (anomalous diffusion for the lighter constituents of the core). We hope that in the next future the non-extensive Tsallis statistics will be tested within one of the standard solar models actually in use.

X. APPENDIX A

In the solar core we can distinguish the following constituents: protons; helium ions; other light ions and heavy ions and electrons (with \( m_{\text{ion}} \gg m_{\text{He}} \gg m_p \gg m_e \)).

Due to the magnitude of \( m_e \), the electrons diffusive behavior is a little less normal than that of light ions, because their dynamics, described for instance by a generalized Langevin equation, require the inclusion of a particular random force. For the same reason, the magnitude of their mass, heavy ions have a normal diffusive behavior. We assume that also helium has a normal diffusion.

More explicitly, the deterministic equation \( m \dot{v} + \alpha v = 0 \), whose solution is \( v(t) = v(0) \exp(-\gamma t) \) (\( \gamma = \alpha/m = \tau^{-1} \)), is not sufficient to describe the diffusion of particles with a mass smaller than the masses of the particles composing the rest of the plasma (their velocity due to thermal fluctuations is considerable).

If the time correlation of the random force \( F(t) \), which we must add, is a delta function, the equilibrium distribution is also a Maxwellian distribution.

From another aspect, a long-range microscopic memory, i.e. a time correlation of the random force different from a delta function, produces an equilibrium distribution of a non-extensive statistical category.

Subdiffusion is responsible for the depletion of the Maxwellian tail. The equilibrium distribution of electrons and of protons and helium ions can be derived from dynamical equations containing a colored noise, more coloured for electrons than for proton and for helium ions.

Anomalous diffusion, like subdiffusion, is related to the normalized equilibrium velocity autocorrelation function \( C_v(t) \) \[13\]

\[
C_v(t) = \frac{1}{\langle v^2 \rangle} \langle v(0)v(t) \rangle ,
\]

where \( v(t) \) is the velocity of the particle at time \( t \) and \( C_v(t) \) does non depend on the time origin.

Mean square displacement is

\[
\langle x(t)^2 \rangle = t \int_0^t C_v(\tau)d\tau - \int_0^t \tau C_v(\tau)d\tau .
\]

If \( C_v(u) \) decays faster than \( u^{-2} \) after a long period of time, one observes normal (or Brownian) diffusion.

We are interested here in subdiffusion because our analysis of the measured neutrino fluxes suggests that this is the type of diffusion of the solar core particles. This regime is reached when \( C_v(u) \approx Au^{-(2-\beta)} \) as \( u \to \infty \) (\( 0 < \beta < 1 \), \( A < 0 \) and \( \int_0^\infty C_v(u)du = 0 \)) then \( \langle x(t)^2 \rangle \approx Bt^\beta \) as \( t \to \infty \).

If the particle is moving in the positive \( x \) direction, it is more likely to move in the negative \( x \) direction in the next instant. The fluctuating velocity reverses its direction very often.

The time correlation of the random force can be written, for instance, as

\[
\langle F(0)F(t) \rangle = \frac{D}{\vartheta} \exp(-t/\vartheta) .
\]

If \( \vartheta \to 0 \) the correlation is given by \( 2D\delta(t) \) which refers to a normal diffusion (therefore \( q \) is linked to \( \vartheta \)).

Following Wang \[44\], we can write generally
As here we want to preserve the norm of the distribution, we must pose $q = 2 - \nu = 1 - 2\delta$ and then $\nu = 1 + 2\delta$ ($\delta$ is the Clayton parameter). By comparison, we have the relation

$$\beta = \frac{1}{2 - q} = \frac{1}{1 + 2\delta} = \frac{1}{\nu} .$$

(25)

The normal diffusion is recovered when $\beta = 1$ ($\nu = 1$), i.e., as we know, when $q = 1$ ($\delta = 0$).

The Fokker Plank equation for the particles of the solar core is ($\mu = 1, \nu = 1 - 2\delta$)

$$\frac{\partial}{\partial t}[p(x,t)]^\mu = -\frac{\partial}{\partial x}[F(x)p(x,t)] + D\frac{\partial^2}{\partial x^2}[p(x,t)]^{1+2\delta} ,$$

(27)

where $F(x) = -\partial V/\partial x$ is an external (drift) force associated with the potential $V(x)$ ($F(x) = k_1 - k_2 x$, $k_1, k_2 \geq 0$). The Tsallis parameter $q$ is related to $\mu$ and $\nu$

$$q = 1 + \mu - \nu .$$

(26)

The stationary solution of the above equation is the Tsallis distribution in the coordinate space ($x$ has the dimension of an energy square root)

$$p(x) = \begin{cases} N(q, kT) \left[1 - \left(\frac{1 - q}{kT} x^2\right)^{1/(1-q)}\right], & |x| < \sqrt{\frac{kT}{1 - q}} \\ 0 & \text{otherwise} \end{cases}$$

(28)

where $N$ is the following normalization constant

$$N(q, kT) = \pi^{-1/2} \left(\frac{kT}{1 - q}\right)^{-1/2} \frac{\Gamma\left(\frac{5 - 3q}{2} - \frac{q}{1 - q}\right)}{\Gamma\left(\frac{2 - q}{1 - q}\right)} .$$

(29)

The results obtained in the Appendix can be recovered using the approach described in Ref. [42] where a new definition of the diffusion coefficient in the momentum space, whose expression is reported in Eq.(8) of the text, was introduced.

In the momentum space the equilibrium distribution can be derived by means of the Fourier transformation of the distribution $p(x)$, following the procedure outlined by Tsallis et al. [54] (its analytic expression is in terms of the modified Bessel function $K$ [55]). In the situations where $q \rightarrow 1$ (as in the case of the solar neutrino problem) we can easily verify that the Tsallis distribution in the coordinate space, given by Eq.(28), corresponds to another Tsallis distribution in the momentum space. In fact, for $q \rightarrow 1$ we can write (here $x$ is adimensional):

$$p(x) \propto [1 - (1 - q)x^2]^{1/(1-q)} \approx \exp(-x^2 - \frac{1-q}{2} x^4) .$$

(30)

The Fourier transformation of $p(x)$ is the Clayton distribution

$$\mathcal{F}[p(x)] \approx \exp^{2 \pi^2} \left[1 + (1 - q)\frac{p^2}{4} - (1 - q)\frac{p^4}{32}\right] \approx \exp\left[-\frac{p^2}{4} - (1 - q)\frac{p^4}{32}\right] ,$$

(31)

which approximates, in the momentum space, the Tsallis distribution

$$\mathcal{F}[p(x)] \propto [1 - (1 - q)p^2/4]^{1/(1-q)} .$$

(32)
[1] J.Bahcall, M.Pinsonneault: Neutrino 96, ed.s K.Huitu, K.Enqvist and J.Maalampi, World Scientific, Singapore, 1996, in press.
[2] J.Bahcall, M.Pinsonneault, S.Basu and J.Christensen-Dalsgaard: Are standard solar models reliable?, astro-ph/9610250, submitted to Phys. Rev. Lett., 1996.
[3] E.Calabrese, N.Ferrari, G.Fiorentini, M.Lissia, Astropart. Phys. 4 (1995) 159.
[4] V.Castellani, S.Dei'Innocenti, G.Fiorentini, M.Lissia, B.Ricci, Solar Neutrinos: beyond standard solar models, astro-ph/9606186, Phys. Rep. (1996), in press.
[5] A.Dar: Standard physics solution to the solar neutrino problem?, astro-ph/9611014; A.Dar, G.Shaviv, Standard solar neutrinos astro-ph/9604009.
[6] C.Arpesella et al., Phys. Lett. B 389 (1996) 452; G.Gervino private communication, 1996.
[7] H.Oberhummer, Contribution to Astrophysics and Basic Space Science Int. Conf., Bonn, Sept. 1996.
[8] L.Bracci, G.Fiorentini, V.Meleshchik, G.Mezzorani and P.Quarati, Nucl. Phys. A513 (1990) 316.
[9] L.S.Brown and R.F.Sawyer, Nuclear reaction rates in a plasma, astro-ph/9610256.
[10] V.Tsytovich, R.Bingham, V.de Angelis, A.Forlani, Phys. Lett. A 205 (1995) 199.
[11] B.Ricci, preprint INFNFE09-96, submitted to Astropart. Phys., 1996.
[12] P.Krastev, S.Petcov, Phys. Rev. D 53 (1996) 1665.
[13] P.Krastev, S.Petcov, Constraints on energy independent solutions of the solar neutrino problem, hep-ph/9612243.
[14] G.Kianiadakis, A.Lavagno, P.Quarati, Phys. Lett. B369 (1996) 308.
[15] P.Quarati et al., Constraints for solar neutrino fluxes, Nucl. Phys. A (Suppl.) (1996), in press.
[16] C.Tsallis, J. Stat. Phys. 52 (1988) 479; Physica A 221 (1995) 277.
[17] C.Tsallis, J. Phys. A (Math Gen.) 24 (1991) L69; 24 (1991) 3187; 25 (1992) 1019.
[18] D.Clayton et al., Nature 249 (1974) 131.
[19] D.Clayton et al., Ap. J. 199 (1975) 494.
[20] E.Bellotti et al., Proposal for a permanent gallium neutrino observatory (GNO) at Lab. Naz. Gran Sasso, preprint 1996.
[21] T.Ypsilantis, Europhysics News 27 (1996) 97.
[22] G.Kocharov, Ioffe Institute Report n.298 (Leningrad) 1972.
[23] H.Mori, Prog. Theor. Phys. 33 (1965) 423, 43 (1965) 399.
[24] P.Quarati, Progr. Theor. Phys. 56 (1976) 599.
[25] H.Haubold, A.M.Mathai, Astroph. Space Sci. 228 (1995) 77; 228 (1995) 113.
[26] H.Haubold, A.M.Mathai, Contribution to Astrophysics and Basic Space Science Int. Conf., Bonn, Sept. 1996.
[27] G.Lapenta, P.Quarati, Nucl. Phys. B(Suppl) 14 (1992) 150; Zeit. Phys. A 346 (1993) 243.
[28] E.Erdas, P.Quarati, Zeit. Phys. D28 (1993) 185; D31 (1994) 161.
[29] J.Bahcall, M.Pinsonneault, Rev. Mod. Phys. 67 (1995) 781.
[30] J.Bahcall, M.Pinsonneault, G.Wasserburg, Rev. Mod. Phys. 67 (1995) 781.
[31] B.Boghosian, Phys. Rev. E 53 (1996) 4754.
[32] P.Jund, S.G.Kim, C.Tsallis, Phys. Rev. B 52 (1995) 50.
[33] C.Tsallis, P.Sá Barreto, E.Loh, Phys. Rev. B 52 (1995) 50.
[34] C.Tsallis et al., Phys. Rev. Lett. 75 (1996) 3589.
[35] X.-P.Huang, C.Driscoll, Phys. Rev. Lett. 72 (1994) 2187.
[36] C.Tsallis, D.Bukman, Phys. Rev. E 54 (1996) R2197.
[37] D.Zanette, P.Alemany, Phys. Rev. Lett. 75 (1995) 366.
[38] M.O.Cacerás, C.Budde, Phys. Rev. Lett. 77 (1996) 2589 (comment).
[39] D.Zanette, P.Alemany, Phys. Rev. Lett. 77 (1996) 2590 (comment).
[40] A.R.Plastino, A.Plastino, Phys. Lett. A 174 (1993) 384.
[41] H.Risken, The Fokker-Plank Equation, Springer-Verlag, Berlin, 1988.
[42] G.Kianiadakis, P.Quarati, Physica A 192 (1993) 667; Physica A (1996), in press.
[43] R.Muralidhar et al., Physica A 167 (1990) 539.
[44] K.G.Wang, Phys. Rev. A 45 (1992) 833.
[45] A.Compte, D.Jou, J. Phys. A (Math. Gen.) 29 (1996) 4321.
[46] A.Compte, D.Jou, Y.Katayama, Anomalous diffusion in linear shear flows, J. Phys. A (Math. Gen.), (1996), in press.
[47] A.R.Plastino, A.Plástino, C.Tsallis, J. Phys. A (Math. Gen.) 27 (1994) 5707.
[48] R.Giovannelli et al., preprints: astro-ph/9610117, astro-ph/9610118, submitted to Ap. J. (1996).
[49] N.Bahcall, S.P.Oh, Ap. J. Lett. 462 (1996) L49.
[50] L.Moscadini et al. M.N.R.A.S. 1996, in press.
[51] A.Lavagno, G.Kianiadakis, M.Rego-Monteiro, P.Quarati, C.Tsallis, Non-extensive thermostatistical approach to the peculiar velocity function of galaxy cluster, submitted to Ap. J. Lett., 1996.
Figure Caption

Fig.1
The Tsallis distribution function $p(E_i)$ for different values of $q$: $q = 0$, 0.5, 1 (MB), 1.5. The typical depletion of the high-energy tail of the Maxwellian ($q = 1$) is clearly evidentiated for $q < 1$.

Fig.2
The reduction factor $f_{\text{red}} (B, CNO, Be)$ and the increasing factor $f_{\text{incr}} (p)$ as functions of the parameter $\delta (q = 1 - 2\delta)$, up to $\delta = 0.025$. We call the attention to the fact that SSM values of the $B$, CNO and Be fluxes must be divided by $f_{\text{red}} (B, CNO, Be)$ to obtain the fluxes within the non-extensive statistics, while the SSM proton flux must be multiplied by $f_{\text{incr}} (p)$. All SSM fluxes are normalized to one at $\delta = 0$.

Table Caption

Table 1
The SSM fluxes ($\delta = 0$) from Ref. [30], the flux $\Phi_p = f_{\text{incr}} \Phi_p^{\text{SSM}}$ and the fluxes $\Phi_{B,CNO,Be} = \Phi_{B,CNO,Be}^{\text{SSM}} / f_{\text{red}}$ evaluated within the non-extensive statistics. All fluxes are in $10^9$ cm$^{-2}$ s$^{-1}$ but $\Phi_B$ in $10^6$ cm$^{-2}$ s$^{-1}$ (we recall that the Kamiokande experiment gives the result $\Phi_B = 2.9 \pm 0.4$).

| $\delta$ | $\Phi_p$ | $\Phi_B$ | $\Phi_{CNO}$ | $\Phi_{Be}$ |
|----------|-----------|----------|---------------|-------------|
| 0        | 59.24(1 ± 0.01) | 6.62(1^{+0.14}_{-0.17}) | 1.17(1±0.2) | 5.15(1±0.06) |
| 0.003    | 60.66 ($f_{\text{incr}} = 1.024$) | 2.85 ($f_{\text{red}} = 2.326$) | 0.48 ($f_{\text{red}} = 2.46$) | 0.65 ($f_{\text{red}} = 7.93$) |
| 0.024    |           |          |               |             |