Large-\(c\) superconformal torus blocks

Konstantin Alkalaev\(^a,b\) Vladimir Belavin\(^{a,c,d}\)

\(^a\)I.E. Tamm Department of Theoretical Physics,
P.N. Lebedev Physical Institute,
Leninsky ave. 53, 119991 Moscow, Russia

\(^b\)Department of General and Applied Physics,
Moscow Institute of Physics and Technology,
7 Institutskiy per., Dolgoprudny, 141700 Moscow region, Russia

\(^c\)Department of Quantum Physics,
Institute for Information Transmission Problems,
Bolshoy Karetny per. 19, 127994 Moscow, Russia

\(^d\)Department of Particle Physics and Astrophysics, Weizmann Institute of Science,
Rehovot 7610001, Israel

E-mail: alkalaev@lpi.ru, belavin@lpi.ru

ABSTRACT: We study large-\(c\) SCFT\(_2\) on a torus specializing to one-point superblocks in the \(\mathcal{N} = 1\) Neveu-Schwarz sector. Considering different contractions of the Neveu-Schwarz superalgebra related to the large central charge limit we explicitly calculate three superblocks, \(osp(1|2)\) global, light, and heavy-light superblocks, and show that they are related to each other. We formulate the \(osp(1|2)\) superCasimir eigenvalue equations and identify their particular solutions as the global superblocks. It is shown that the resulting differential equations are the Heun equations. We study exponentiated global superblocks arising at large conformal dimensions and demonstrate that in the leading approximation the \(osp(1|2)\) superblocks are equal to the non-supersymmetric \(sl(2)\) block.

\(\dagger\)Weston Visiting Professorship at Weizmann Institute.
1 Introduction

The study of the large central charge regime in CFT\(_2\) allowed to identify conformal blocks as lengths of geodesic networks in the dual gravity theory with matter in asymptotically \(AdS_3\) space \([1–9]\) (for further development see \([10–22]\))

The aim of this paper is to study SCFT\(_2\) on a torus in the large-\(c\) regime. As a first step in this direction we consider the \(\mathcal{N} = 1\) superconformal Neveu-Schwarz (NS) algebra and 1-point superconformal blocks. We focus on global superblocks that are associated to the \(osp(1|2)\) subalgebra of the NS superalgebra. We develop the superCasimir approach where \(osp(1|2)\) global superblocks are realized as eigenfunctions of the super-Casimir differential operators.

The idea is that global blocks in CFT\(_2\) on any topology play the central role in studying the large-\(c\) regime because all other semiclassical blocks including light and heavy-light blocks are related to the global one \([6, 20, 23–25]\). We show that similar relations are valid in the supercase. Also, we consider the regime of large conformal dimensions and study the
exponentiated global superblocks. It is interesting that the supersymmetry turns out to be
degenerate in the leading order because the NS supermultiplet conformal dimensions are
indistinguishable in this regime, $\Delta + \frac{1}{2} \approx \Delta$.

The paper is organized as follows. Using $osp(1|2)$ representation theory described in
Section 2 we formulate two global superconformal torus blocks corresponding to different
structure constants in Section 3. The superCasimir approach for torus superblocks is elabo-
rated in Section 4. Here we derive two second order differential equations for the superblocks
and analyze their solutions. These equations are in fact the Heun equations and we describe
their local and global properties. In Section 4.3 the global superblocks are studied in the
regime of large conformal dimensions. We show that the resulting superblock block functions
are exponentiated and find explicit expressions. Remarkably, they all are expressed in terms
of the exponentiated non-supersymmetric block function. In Section 5 we consider supercon-
fomal blocks of the NS superalgebra and show that global, light, and heavy-light superblocks
can be obtained via particular contractions of the NS superalgebra when $1/c \to 0$. We close
with some concluding remarks in Section 6.

2 Representation theory of $osp(1|2)$ superalgebra

In this section we shortly review the $osp(1|2)$ superalgebra and Verma supermodules. It
basically serves to set our notation and conventions. For detailed reviews see, e.g., [26, 27].

The superalgebra $osp(1|2)$ is spanned by three even generators $L_{\pm 1,0}$ and two odd gener-
ators $G_{\pm \frac{1}{2}}$ with the graded commutation relations

\[ [L_m, L_n] = (m - n)L_{m+n} \quad [L_n, G_{\pm \frac{1}{2}}] = \left(\frac{n}{2} \mp \frac{1}{2}\right) G_{n\pm \frac{1}{2}}, \quad \{G_r, G_s\} = 2L_{r+s}, \quad (2.1) \]

where $m, n = 0, \pm 1, r, s = \pm \frac{1}{2}$. Obviously, $sl(2) \subset osp(1|2)$.

Verma supermodule. Let $V_{\Delta}$ denote $osp(1|2)$ supermodule of highest weight $\Delta$. A highest
weight (primary) state is defined as

\[ L_0|\Delta\rangle = \Delta|\Delta\rangle, \quad L_1|\Delta\rangle = 0, \quad G_{\frac{1}{2}}|\Delta\rangle = 0. \quad (2.2) \]

Integer powers of the other basis elements $L_{-1}$ and $G_{-\frac{1}{2}}$ act as raising operators generating
in this way the supermodule $V_{\Delta}$. From the graded commutation relations (2.1) it follows that
\((G_{-\frac{1}{2}})^2 = L_{-1}\) and, thus, the supermodule is spanned by

\[ |M, \Delta\rangle = (L_{-1})^m (G_{-\frac{1}{2}})^k |\Delta\rangle, \quad M = (m, k) : \quad m \in \mathbb{N}_0, \quad k = 0, 1. \quad (2.3) \]

A number $M = m + k/2$ defines a level, while $k$ defines $\mathbb{Z}_2$ grading of the supermodule.
The standard conjugation rules $L_m = (L_m)^\dagger$ and $G_s = (G_s)^\dagger$ are assumed. The $osp(1|2)$
supermodule $V_{\Delta}$ has a supermultiplet structure

\[ V_{\Delta} = V_{\Delta} \oplus V_{\Delta+\frac{1}{2}}, \quad (2.4) \]
where the factors are $sl(2)$ Verma modules of weights $\Delta$ and $\Delta + \frac{1}{2}$. These are generated from primary states $|\Delta\rangle$ and $|\Delta + \frac{1}{2}\rangle$ which are related by a supersymmetry transformation as $|\Delta + \frac{1}{2}\rangle = G_{-\frac{1}{2}} |\Delta\rangle$.

**Invariant operators.** The $osp(1|2)$ superCasimir operator reads

$$S_2 = -L_0^2 + \frac{1}{2}(L_{-1}L_1 + L_1L_{-1}) + \frac{1}{4}(G_{-\frac{1}{2}}G_{\frac{1}{2}} - G_{\frac{1}{2}}G_{-\frac{1}{2}}), \quad (2.5)$$

with eigenvalues $S_2F = -\Delta(\Delta - \frac{1}{4})F$, where $F$ is a supermultiplet (2.4). It is remarkable that there is the so-called Scasimir operator (see, e.g., [28–30])

$$\Upsilon_2 = \frac{1}{4}(G_{-\frac{1}{2}}G_{\frac{1}{2}} - G_{\frac{1}{2}}G_{-\frac{1}{2}}) - \frac{1}{8}, \quad (2.6)$$

that has the property of graded invariance: $[\Upsilon_2, L_{0,\pm 1}] = 0$ and $\{\Upsilon_2, G_{\pm \frac{1}{2}}\} = 0$ and satisfies the quadratic relation $\Upsilon_2^2 = S_2 + \frac{1}{64} (2.5)$. Then, the Scasimir eigenvalue equation says that $\Upsilon_2F = -\frac{1}{2}(\Delta + \frac{1}{4})F$. Introducing the $sl(2)$ Casimir $C_2$ we can represent the $osp(1|2)$ superCasimir as

$$S_2 = C_2 + \Upsilon_2 + \frac{1}{8}. \quad (2.7)$$

**Superconformal fields.** Using the superconformal version of the operator-state correspondence, the supermodule (2.4) can be realized as a supermultiplet of two conformal operators $(\phi_\Delta(z), \psi_{\Delta + \frac{1}{2}}(z))$ with conformal dimensions $\Delta$ and $\Delta + \frac{1}{2}$. Their $osp(1|2)$ superconformal transformations are given by [31, 32]

$$[L_m, \phi_\Delta(z)] = z^m(z\partial_z + (m + 1)\Delta)\phi_\Delta(z),$$
$$[G_r, \phi_\Delta(z)] = z^{r+\frac{1}{2}}\psi_{\Delta + \frac{1}{2}}(z),$$
$$[L_m, \psi_{\Delta + \frac{1}{2}}(z)] = z^m(z\partial_z + (m + 1)(\Delta + \frac{1}{2}))\psi_{\Delta + \frac{1}{2}}(z),$$
$$[G_r, \psi_{\Delta + \frac{1}{2}}(z)] = z^{r-\frac{1}{2}}(z\partial_z + (2r + 1)\Delta)\phi_\Delta(z), \quad (2.8)$$

where $m, n = 0, \pm 1, r, s = \pm \frac{1}{2}$. It follows that the supermultiplet components are related by a supersymmetry transformation $\psi_{\Delta + \frac{1}{2}}(z) = [G_{-\frac{1}{2}}, \phi_\Delta(z)]$.

**Superfield description.** We use the superfield formalism (see, e.g., [33] and references therein), where $y = (z, \theta)$ are even and odd holomorphic coordinates in 2|2 dimensional superplane. Let us consider the (holomorphic) superfield

$$\Phi_\Delta(y) = \phi_\Delta(z) + \theta \psi_{\Delta + \frac{1}{2}}(z), \quad \text{where} \quad \psi_{\Delta + \frac{1}{2}}(z) = [G_{-\frac{1}{2}}, \phi_\Delta(z)]. \quad (2.9)$$

It is assumed that there is a common $Z_2$ grading $\pi = 0, 1$ for coordinates, fields, supermodule (2.4) and superalgebra elements:

$$\pi(\phi_\Delta) = 0, \quad \pi(\psi_{\Delta + \frac{1}{2}}) = 1; \quad \pi(z) = 0, \quad \pi(\theta) = 1; \quad \pi(V_\Delta) = 0, \quad \pi(V_{\Delta + \frac{1}{2}}) = 1; \quad \pi(L_n) = 0, \quad \pi(G_r) = 1. \quad (2.10)$$

Thus, the superfield is even, $\pi(\Phi_\Delta(y)) = 0$. 

---

3
3 Global torus superblocks

Let us consider $osp(1|2)$ superconformal theory on a two-dimensional torus. A one-point function of the primary superfield $\Phi_{\Delta, \bar{\Delta}}(x, \bar{x})$ with conformal dimensions $\Delta, \bar{\Delta}$ is given by

$$\langle \Phi_{\Delta, \bar{\Delta}}(x, \bar{x}) \rangle = \text{str} \left[ q^{L_0} \bar{q}^{\bar{L}_0} \Phi_{\Delta, \bar{\Delta}}(x, \bar{x}) \right],$$

(3.1)

where $\text{str}$ is the supertrace on the (super)space of states, and $(x, \bar{x}) = (w, \bar{w}, \eta, \bar{\eta})$ are the supercylindrical coordinates, see Appendix A. The modular parameter $q = e^{2\pi i \tau}$, where $\tau \in \mathbb{C}$ is the torus modulus.

From now on we consider holomorphic sector only. Assuming that the space of states can be decomposed into supermodules of various dimensions $\tilde{\Delta}$ we can project onto a particular supermodule. It is convenient to introduce the supertrace function

$$B(\Delta, \tilde{\Delta}, q|x) = \text{str}_{\tilde{\Delta}} \left[ q^{L_0} \Phi_{\Delta}(x) \right],$$

(3.2)

with the supertrace evaluated on the Verma supermodule of weight $\tilde{\Delta}$. Then, lower and upper superconformal blocks $B_0$ and $B_1$ can be defined by means of the decomposition\(^1\)

$$B(\Delta, \tilde{\Delta}, q|x) = C_{\Delta\Delta\tilde{\Delta}} B_0(\Delta, \tilde{\Delta}, q|w) + C_{\Delta\Delta+\frac{1}{2}\tilde{\Delta}} \eta B_1(\Delta, \tilde{\Delta}, q|w),$$

(3.3)

where $C_{\Delta\Delta\tilde{\Delta}} = \langle \tilde{\Delta} | \phi_{\Delta}(0) | \Delta \rangle$ and $C_{\Delta\Delta+\frac{1}{2}\tilde{\Delta}} = \langle \tilde{\Delta} | \psi_{\Delta+\frac{1}{2}}(0) | \Delta \rangle$ are two independent structure constant, see relations (3.6) below. Note that $B_{0,1}$ are even functions.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{One-point superconformal blocks of the supermultiplet $(\phi_{\Delta}, \psi_{\Delta+\frac{1}{2}})$. The solid (dashed) lines stand for Grassmann even (odd) operators. The left and right tadpole graphs correspond to the lower block (3.4) and upper block (3.5) and solid-dashed loop represents taking the supertrace.}
\end{figure}

\(^1\)Using slightly different conventions superconformal one-point torus blocks can be defined as e.g. in [34]. Related considerations of bosonic one-point blocks can be found in [35–38].
The lower and upper torus superblocks read off from relations (3.2) and (3.3) are given by

\[
B_0(\Delta, \bar{\Delta}|q) = \frac{q^{\bar{\Delta}}}{\langle \bar{\Delta}|\phi_\Delta(w)|\bar{\Delta} \rangle} \left[ \sum_{m=0}^{\infty} q^m \frac{\langle \bar{\Delta}, m|\phi_\Delta(w)|m, \bar{\Delta} \rangle}{\langle \bar{\Delta}, m|m, \bar{\Delta} \rangle} - \sum_{m=0}^{\infty} q^{m+\frac{1}{2}} \frac{\langle \bar{\Delta} + \frac{1}{2}, m|\phi_\Delta(w)|m, \bar{\Delta} + \frac{1}{2} \rangle}{\langle \bar{\Delta} + \frac{1}{2}, m|m, \bar{\Delta} + \frac{1}{2} \rangle} \right],
\]

(3.4)

\[
B_1(\Delta, \bar{\Delta}|q) = \frac{q^{\bar{\Delta}}}{\langle \bar{\Delta}|\psi_{\Delta+\frac{1}{2}}(w)|\bar{\Delta} \rangle} \left[ \sum_{m=0}^{\infty} q^m \frac{\langle \bar{\Delta}, m|\phi_{\Delta+\frac{1}{2}}(w)|m, \bar{\Delta} \rangle}{\langle \bar{\Delta}, m|m, \bar{\Delta} \rangle} + \sum_{m=0}^{\infty} q^{m+\frac{1}{2}} \frac{\langle \bar{\Delta} + \frac{1}{2}, m|\phi_{\Delta+\frac{1}{2}}(w)|m, \bar{\Delta} + \frac{1}{2} \rangle}{\langle \bar{\Delta} + \frac{1}{2}, m|m, \bar{\Delta} + \frac{1}{2} \rangle} \right].
\]

(3.5)

The superblocks (3.4) and (3.5) are decomposed according to parity of the exchanged channel that is manifested by respectively first and second sums of each expression. Taking into account the form of the matrix elements in the even/odd sectors,

\[
\begin{align*}
\langle \bar{\Delta} + \frac{1}{2}|\bar{\Delta} + \frac{1}{2} \rangle &= 2\bar{\Delta} \langle \bar{\Delta} |\bar{\Delta} \rangle = 2\bar{\Delta}, \\
\langle \bar{\Delta} + \frac{1}{2}|\phi_\Delta(z)|\bar{\Delta} + \frac{1}{2} \rangle &= (2\bar{\Delta} - \Delta)\langle \bar{\Delta} |\phi_\Delta(z)|\bar{\Delta} \rangle, \\
\langle \bar{\Delta} |\phi_\Delta(z)|\bar{\Delta} + \frac{1}{2} \rangle &= -\langle \bar{\Delta} |\phi_{\Delta+\frac{1}{2}}(z)|\bar{\Delta} \rangle, \\
\langle \bar{\Delta} + \frac{1}{2}|\phi_{\Delta+\frac{1}{2}}(z)|\bar{\Delta} + \frac{1}{2} \rangle &= -\langle \bar{\Delta} + \Delta - \frac{1}{2} \rangle \langle \bar{\Delta} |\phi_{\Delta+\frac{1}{2}}(z)|\bar{\Delta} \rangle,
\end{align*}
\]

(3.6)

we find the closed expressions for the lower superblock function,

\[
q^{-\Delta}B_0(\Delta, \bar{\Delta}|q) = \frac{1}{(1-q)\Delta} \, 2F_1(2\bar{\Delta} - \Delta, 1 - \Delta, 2\bar{\Delta}|q) - \frac{2\bar{\Delta} - \Delta}{2\bar{\Delta}} \frac{q^{1/2}}{(1-q)\Delta^2} \, 2F_1(2\bar{\Delta} - \Delta + 1, 1 - \Delta, 2\bar{\Delta} + 1|q),
\]

(3.7)

and for the upper superblock function,

\[
q^{-\Delta}B_1(\Delta, \bar{\Delta}|q) = \frac{1}{(1-q)\Delta + \frac{1}{2}} \, 2F_1(2\bar{\Delta} - \Delta - \frac{1}{2}, -\Delta + \frac{1}{2}, 2\bar{\Delta}|q) - \frac{2\bar{\Delta} + \Delta - \frac{1}{2}}{2\bar{\Delta}} \frac{q^{1/2}}{(1-q)\Delta + \frac{1}{2}} \, 2F_1(2\bar{\Delta} - \Delta + \frac{1}{2}, -\Delta + \frac{1}{2}, 2\bar{\Delta} + 1|q).
\]

(3.8)

Sending \( \Delta \to 0 \) in (3.7) we find the \( osp(1|2) \) (graded) character

\[
q^{-\Delta}B_0(0, \bar{\Delta}|q) = \frac{1}{1 + q^{1/2}} = 1 - q^{1/2} + q - q^{3/2} + q^2 - q^{5/2} + q^3 + \ldots.
\]

(3.9)
Indeed, substituting $\Phi_{\Delta} = 1$ into (3.2) yields the supertrace of the identity operator. In accordance with (2.3) it shows that there is one state on each level of the supermodule $V_{\tilde{\Delta}}$ (2.4). Note that $B_0(0, \tilde{\Delta}|q) = B_1(1/2, \tilde{\Delta}|q)$ and $B_0(1/2, \tilde{\Delta}|q) = B_1(0, \tilde{\Delta}|q)$.

Equivalently, the $osp(1|2)$ character can be obtained by expanding both superblock functions near $\tilde{\Delta} = \infty$,

$$B_0(\Delta, \infty|q) = B_1(\Delta, \infty|q) = B_0(0, \tilde{\Delta}|q). \tag{3.10}$$

### 4 SuperCasimir eigenvalue equations

It is known that $CFT_2$ global blocks can be described as solutions to the second-order differential equations interpreted as the $sl(2)$ Casimir operator eigenvalue conditions imposed on exchanged channels [39]. The original construction for 4-point blocks on the sphere can be extended to higher-point conformal blocks on the sphere and torus [19, 23]. The super-Casimir equations for 4-point sphere blocks were previously discussed in [40]. In $d$ dimensions (super)conformal Casimir equations were discussed in [40–43].

In what follows we elaborate the superCasimir approach for torus superblocks. To this end, we note that acting with the superCasimir operator (2.5) inside the supertrace operation we get the eigenvalue equation for the exchanged channel

$$\text{str}_\Delta \left[ S_2 q^{L_0} \Phi_{\Delta}(x) \right] = -\tilde{\Delta}(\tilde{\Delta} - \frac{1}{2}) B(\Delta, \tilde{\Delta}, q|x), \tag{4.1}$$

where the factor on the right-hand side is the eigenvalue of the superCasimir on the irreducible Verma supermodule $V_{\tilde{\Delta}}$. On the other hand, the external superfield $\Phi_{\Delta}$ satisfies the other eigenvalue equation

$$\tilde{S}_2 \Phi_{\Delta}(x) = -\Delta(\Delta - \frac{1}{2}) \Phi_{\Delta}(x), \tag{4.2}$$

with the superCasimir operator given by

$$\tilde{S}_2 = -L_0^2 + \frac{1}{2}(L_{-1}L_1 + L_1L_{-1}) + \frac{1}{4}(G_{-\frac{1}{2}}G_{\frac{1}{2}} - G_{\frac{1}{2}}G_{-\frac{1}{2}}), \tag{4.3}$$

where the $osp(1|2)$ generators are realized as differential operators in the supercylindrical coordinates, see Appendix A. Using the operator-state correspondence we can show that the eigenvalue superCasimir equation (4.2) is identically satisfied. Combining two eigenvalue conditions (4.1) and (4.2) we will obtain two equations for two superblocks. We will see that it is the supertrace function embracing two superblocks (3.3) provides a natural way to impose the superconformal invariance conditions.

\footnote{We note that $o(d - 1, 2)$ algebra in $CFT_d$ describes global conformal symmetries. In $CFT_2$ on general Riemannian surfaces Virasoro algebra always contains $sl(2)$ subalgebra that describes global symmetry only on the sphere.}
4.1 Derivation of the eigenvalue equation

Using the approach of [19] we find the following identities with the $osp(1|2)$ basis elements inserted into the supertrace

$$\frac{2nq^n}{1-q^nq\frac{\partial}{\partial q}} + \frac{1}{(1-q^n)(1-q^{-n})} \mathcal{L}_n \mathcal{L}_n \left[ q^{L_0} \Phi_\Delta(x) \right]$$

$$\frac{2q^k}{1-q^kq\frac{\partial}{\partial q}} + \frac{1}{(1-q^k)(1-q^{-k})} \mathcal{G}_k \mathcal{G}_k \left[ q^{L_0} \Phi_\Delta(x) \right]$$

where $n = \pm 1$ and $k = \pm \frac{1}{2}$. Here, in particular, we used the commutator $[L_0, T_s] = -sT_s$ with $T_s = (L_0, \pm 1, G_{\pm \frac{1}{2}})$ being $osp(1|2)$ basis elements that yields the identity $T_s q^{L_0} = q^{L_0+s} T_s$.

Also, it is easy to derive the polynomial homogeneity identity

$$\frac{(q\partial_q)^m}{(1-q^n)(1-q^{-n})} \mathcal{L}_n \mathcal{L}_n \left[ q^{L_0} \Phi_\Delta(x) \right] = 0$$

(4.6)

Loosely speaking, the identities (4.4) and (4.5) convert polynomial combinations of basis elements $L_n$ and $G_k$ acting on states of $\mathcal{V}_\Delta$ to differential operators $\mathcal{L}_n$ and $\mathcal{G}_k$ acting on the superfield along with $q$-differential operators. It is crucial here that the superblocks are combined into the supertrace function that allows using the graded cyclic property and commuting even/odd operators via (graded) cyclic permutations.

Recalling the $U(1) \times U(1)$ global symmetry of two-dimensional torus and relation (A.4) we find that $\mathcal{L}_0$ acts trivially, i.e.,

$$\mathcal{L}_0 \mathcal{L}_0 \left[ q^{L_0} \Phi_\Delta(x) \right] = 0 .$$

(4.7)

Then, substituting the identities (4.4) and (4.5) along with the relation (4.6) into the super-Casimir equation (4.1) we get

$$\left[ (q\partial_q)^2 + \frac{1}{2} - \frac{1}{1+q^{1/2}} \right] (q\partial_q) + \frac{q}{(1-q^2)(1-q^{1/2})^2} \left( \tilde{C}_2 + \frac{q^{1/2}}{1-q^{1/2}} \tilde{\Upsilon}_2 + \frac{1}{8} \right) -$$

$$\tilde{\Delta} \left( \tilde{\Delta} - \frac{1}{2} \right) \mathcal{L}_n \mathcal{L}_n \left[ q^{L_0} \Phi_\Delta(x) \right] = 0 ,$$

(4.7)

where operators $\tilde{C}_2$ and $\tilde{\Upsilon}_2$ are directly read off from (2.7) and (4.3). Now, we have to know how $\tilde{C}_2$ and $\tilde{\Upsilon}_2$ act on the superfield $\Phi_\Delta(x)$. To this end, we use the super-Casimir equation (4.2) to express

$$\left( \tilde{\Upsilon}_2 + \frac{1}{8} \right) \Phi_\Delta = -\Delta(\Delta - \frac{1}{2}) \Phi_\Delta - \tilde{C}_2 \Phi_\Delta ,$$

(4.8)
and substitute this relation into the equation (4.7)

\[
\left[ (q \partial_q)^2 + \left( \frac{1}{2} - \frac{1}{1 + q^{1/2}} \right) (q \partial_q) + \left( \frac{q}{(1 - q)^2} - \frac{q^{1/2}}{(1 - q^{1/2})^2} \right) \right] \tilde{C}_2 - \Delta(\Delta - \frac{1}{2})^2 \left( \frac{q^{1/2}}{(1 - q^{1/2})^2} - \Delta \left( \frac{1}{2} \right) \right) \Delta(\Delta - \frac{1}{2}) \Delta(\Delta - \frac{1}{2}) \text{str}_{\Delta} \left[ q^{L_0} \Phi_{\Delta}(x) \right] = 0 .
\]

Now, we observe that the \( sl(2) \) Casimir operator acts non-diagonally on the supermodule

\[
\tilde{C}_2 \Phi_{\Delta}(x) = -\Delta(\Delta - 1) \phi_{\Delta}(w) - (\Delta + \frac{1}{2})(\Delta - \frac{1}{2}) \eta \psi_{\Delta}(w) .
\]

To derive this relation we used formulas from the Appendix A and the fact that even/odd components of the \( osp(1|2) \) superfield are themselves \( sl(2) \) conformal fields. Substituting this relation into (4.9) and using (3.3) we finally find two equations

\[
\left[ (q \partial_q)^2 + \left( \frac{1}{2} - \frac{1}{1 + q^{1/2}} \right) (q \partial_q) - \Delta(\Delta - 1) \left( \frac{q}{(1 - q)^2} - \frac{q^{1/2}}{(1 - q^{1/2})^2} \right) - \Delta(\Delta - \frac{1}{2}) \frac{q^{1/2}}{(1 - q^{1/2})^2} - \Delta(\Delta - \frac{1}{2}) \Delta(\Delta - \frac{1}{2}) \right] B_0(\Delta, \tilde{\Delta}|q) = 0 ,
\]

\[
\left[ (q \partial_q)^2 + \left( \frac{1}{2} - \frac{1}{1 + q^{1/2}} \right) (q \partial_q) - \left( \Delta + \frac{1}{2} \right)(\Delta - \frac{1}{2}) \left( \frac{q}{(1 - q)^2} - \frac{q^{1/2}}{(1 - q^{1/2})^2} \right) - \Delta(\Delta - \frac{1}{2}) \frac{q^{1/2}}{(1 - q^{1/2})^2} - \Delta(\Delta - \frac{1}{2}) \Delta(\Delta - \frac{1}{2}) \right] B_1(\Delta, \tilde{\Delta}|q) = 0 .
\]

These are the second order ODEs that differ only in the coefficient of the third terms. Since each of the equations has two independent solutions we fix the asymptotics

\[
B_{0,1}(\Delta, \tilde{\Delta}|q) \rightarrow q^{\tilde{\Delta}} \quad \text{as} \quad q \rightarrow 0 ,
\]

and find that the corresponding solutions to the superCasimir equations (4.11) and (4.12) are given by functions (3.7) and (3.8).

### 4.2 Properties of the eigenvalue equation

One might wonder whether the local OPE data and the modular properties of the torus correlation functions fix the form of the differential equations (4.11) and (4.12). To clarify this issue, we change variables as \( x = q^{1/2} \), for example, in the first equation (4.11) to obtain the second order ODE

\[
B_0''(x) + \frac{2}{x + 1} B_0'(x) - \left[ \frac{2 \left( \Delta x (2\Delta x + x^2 + 1) + 2\tilde{\Delta}^2 (x^2 - 1)^2 - \Delta (x^2 - 1)^2 \right)}{x^2 (x^2 - 1)^2} \right] B_0(x) = 0 ,
\]

(4.14)
(the prime denotes $x$-derivative) with the (regular) singular points $-1, 0, 1$ and $\infty$, and the Riemann P-symbol
\begin{align*}
\begin{cases}
x = -1 & x = 0 & x = 1 & x = \infty \\
\Delta - 1 & 1 - 2\widetilde{\Delta} & \Delta + 1 & 1 - 2\widetilde{\Delta}
\end{cases}.
\end{align*}
\tag{4.15}

From the entities $\alpha_{i,j}$ of the P-symbol (which are two characteristic exponents, $j = 1, 2$ at singular points $x_i$, where index $i = -1, 0, 1, \infty$ labels singular points) one can recognize the superCasimir eigenvalues in the exchanged channel (4.1). The equation (4.14) is Fuchsian that can be seen, in particular, by checking the Fuchs identity
\[
\sum_{i,j} \alpha_{i,j} = (p - 2)n(n - 1)/2,
\]
where $p = 4$ is the number of singular points and $n = 2$ is the order of the ODE.

An important characteristic of the Fuchsian ODEs is the so-called rigidity index
\[
\mathcal{I} = n^2(2 - p) + \sum_{i,j} m_{i,j}^2,
\tag{4.16}
\]
where $m_{i,j}$ are multiplicities of the characteristic exponents $\alpha_{i,j}$ (4.15). In our case the spectral type $\{m_{i,j}\} = (1, 1; 1, 1; 1, 1; 1, 1)$. If the rigidity index $\mathcal{I} = 2$, then the local data completely defines a differential equation, in particular, its explicit form and the monodromy group, etc. (for details see, e.g., [44]). In our case, however, the rigidity index $\mathcal{I} = 0$ implying that the ODE is not rigid and contains one accessory parameter which form cannot be determined only from the local data (4.15).

Doing the transformation $B_0(x) = (1 - x)^{\alpha_{1,2}}(1 + x)^{\alpha_{-1,1}}x^{\alpha_{0,2}}y(x)$ we get the differential equation of the form
\[
y''(x) + \left(\frac{a + b - c - d + 1}{x - t} + \frac{c}{x} + \frac{d}{x - 1}\right)y'(x) + \frac{(abx - s)}{x(x - 1)(x - t)} y(x) = 0.
\tag{4.17}
\]
This is the Heun equation with singular point $0, 1, t, \infty$, and the parameters $a, b, c, d$, along with the accessory parameter $s$ expressed in terms of conformal dimensions. The solution to the equation (4.17) is the Heun function $H_n(t, s; a, b, c, d; x)$.\footnote{In CFT$_2$ with Virasoro symmetry the Heun equation arises as the $c \rightarrow \infty$ limit of the BPZ equation for the 5-point conformal blocks with one degenerate light operator [45] (see also recent discussion in [46, 47]).} In our case, $t = -1$ and $s = 2\Delta(4\widetilde{\Delta} - 1)$. Matching other parameters one arrives at the differential equation of the form
\[
y''(x) + \frac{4(\Delta x^2 - \Delta x - \widetilde{\Delta})}{x^3 - x}y'(x) + \frac{\Delta(2 - 8\widetilde{\Delta})}{x^3 - x} y(x) = 0.
\tag{4.18}
\]
Hence, the lower superblock can be expressed in terms of the Heun function
\[
B_0(\Delta, \widetilde{\Delta}|q) = (1 - q^{1/2})^{-\Delta}(1 + q^{1/2})^{\Delta - 1}q^{\widetilde{\Delta} \times}
\times H_n(-1, 2\Delta(4\widetilde{\Delta} - 1); 0, 4\Delta - 1, 4\widetilde{\Delta}, -2\Delta; q^{1/2}).
\tag{4.19}
\]
The superCasimir equation for the upper superblock (4.12) can be considered along the same lines. The resulting Heun’s representation is given by

\[
B_1(\Delta, \tilde{\Delta}|q) = \left(1 - q^{1/2}\right)^{-\frac{\Delta}{2}} \left(1 + q^{1/2}\right)^{-\frac{\tilde{\Delta}}{2}} q^{\tilde{\Delta}} \times \text{Hn}(-1, (1 - 2\Delta)(4\tilde{\Delta} - 1); 0, 4\tilde{\Delta} - 1, 4\tilde{\Delta}, 2\Delta - 1; q^{1/2}).
\] (4.20)

### 4.3 Exponentiated global superblocks

Let us consider large conformal dimensions. In this regime \(\Delta\) and \(\tilde{\Delta}\) tend to infinity uniformly

\[
\Delta = \kappa \sigma , \quad \tilde{\Delta} = \kappa \tilde{\sigma},
\] (4.21)

where the scale parameter \(\kappa \gg 1\), and \(\sigma\) and \(\tilde{\sigma}\) are finite rescaled conformal dimensions. Remarkably, the large rescaled dimensions \(\Delta + \frac{1}{2}\) and \(\tilde{\Delta} + \frac{1}{2}\) are also given by \(\sigma\) and \(\tilde{\sigma}\).

Given that \(\sigma\) and \(\tilde{\sigma}\) are fixed, the large-\(\kappa\) asymptotics of the superblocks (3.7) and (3.8) can be conveniently defined using the superCasimir equations (4.11), (4.12). Indeed, in this case the solutions are exponentiated,

\[
q^{-\tilde{\Delta}} B_\alpha(\Delta, \tilde{\Delta}, q) \sim \exp \left[ \sum_{n=0}^{\infty} \kappa^{-n+1} b_{\alpha|n}(\sigma, \tilde{\sigma}, q) \right], \quad \alpha = 0, 1,
\] (4.22)

where \(b_{\alpha|0}(\sigma, \tilde{\sigma}, q)\) are the classical global blocks, while \(b_{\alpha|n}(\sigma, \tilde{\sigma}, q)\) at \(n = 1, 2, ...\) are \(1/\kappa^n\) corrections. Below we show that the leading asymptotics coincide \(b_{0|0}(\sigma, \tilde{\sigma}, q) = b_{1|0}(\sigma, \tilde{\sigma}, q)\) and the difference arises in higher orders. We note that exponential factors can be represented as follows

\[
b_{\alpha|n}(\sigma, \tilde{\sigma}, q) = \tilde{\sigma}^{1-n} b_{\alpha|n}(\delta, q), \quad \delta = \frac{\sigma}{\tilde{\sigma}},
\] (4.23)

where the lightness parameter \(\delta\) is dimensionless. This formula means that the scale factor \(\kappa\) in (4.22) is dummy because \(\kappa^{-n+1} b_{\alpha|n}(\sigma, \tilde{\sigma}, q) = \tilde{\Delta}^{-n+1} b_{\alpha|n}(\Delta/\tilde{\Delta}, q)\). However, working with dimensionless parameters \(\kappa\) and \(\delta\) turns out to be technically more convenient.

Substituting the ansatz (4.22) into the superCasimir equations we find out that each leading asymptotic satisfies the same residual differential equation

\[
\left(\partial_q b_{\alpha|0}(\sigma, \tilde{\sigma}, q)\right)^2 + \frac{2\tilde{\sigma}}{q} \partial_q b_{\alpha|0}(\sigma, \tilde{\sigma}, q) - \frac{\sigma^2}{q(1-q)^2} = 0.
\] (4.24)

Therefore, classical lower/upper superblocks coincide with each other and equal to the known classical \(sl(2)\) global block [24]

\[
b_{\alpha|0}(\sigma, \tilde{\sigma}, q) = \tilde{\sigma} \int_0^q dx \left( -\frac{1}{x} + \sqrt{\frac{1}{x^2} + \frac{\delta^2}{x(x-1)^2}} \right), \quad \alpha = 1, 2.
\] (4.25)

The integral is divergent in \(x = 0\). However, the divergency can be bypassed if we expand in the lightness parameter (4.23) near \(\delta = 0\) meaning that the exchanged channel is much heavier than external primary operator (see [24] for more details).
All higher-order corrections are iteratively expressed in terms of the leading contribution (4.25). For example, the $O(k^0)$ corrections $b_{0|1}(\sigma, \tilde{\sigma}, q)$ are defined by inhomogeneous first order differential equations given in Appendix B. The solutions have the integral form

$$b_{0|1}(\sigma, \tilde{\sigma}, q) = \int_0^q dx \left[ \frac{(x+1)(x-1)^2 - 2Wx^{1/2}}{4W(1-x)x} + \frac{(x-1)^2 - (x+1)x^{1/2}\delta}{4W^{1/2}(x-1)x} \right],$$

$$b_{1|1}(\sigma, \tilde{\sigma}, q) = \int_0^q dx \left[ \frac{(x+1)(x-1)^2 - 2Wx^{1/2}}{4W(1-x)x} + \frac{(x^{1/2} + 1)^2 (x+1 + (\delta - 2)x^{1/2})}{4W^{1/2}(x-1)x} \right],$$

(4.26)

where $W = (1-x)^2 + x\delta^2$. The above integrals differ in the last two terms. Evaluating the integrals and further expanding in the smallness parameter $\delta$ we can obtain a first few terms

$$b_{0|1}(\sigma, \tilde{\sigma}, q) = -\log \left( 1 + q^{1/2} \right) + \frac{1}{2} \frac{q^{1/2}}{1-q} \delta - \frac{1}{8} \frac{q(q+1)}{(1-q)^2} \delta^2 + O(\delta^3),$$

$$b_{1|1}(\sigma, \tilde{\sigma}, q) = -\log \left( 1 + q^{1/2} \right) - \frac{1}{2} \frac{q^{1/2}}{1-q^{1/2}} \delta - \frac{1}{8} \frac{q(q+1)}{(1-q)^2} \delta^2 + O(\delta^3).$$

(4.27)

(4.28)

In particular, recalling that these corrections are dimensionless (4.23) we observe that osp(1|2) character (3.9) is reproduced by the logarithmic terms in the limit $\delta \to 0$. Indeed, the limit can be achieved by $\tilde{\sigma} \to \infty$ and therefore we can use (3.10).

5 Conformal blocks of contracted Neveu-Schwarz superalgebras

In this section we consider the Inonu-Wigner contractions of the NS superalgebra with respect to the inverse central charge $1/c$ in the limit $c \to \infty$. Analogously to the Virasoro algebra case [24] we compute associated torus superblocks and identify them with different types of semiclassical torus superblocks.\(^4\)

5.1 NS superalgebra and superblocks

Let us consider $\mathcal{N} = 1$ NS superalgebra which generators $L_m$ and $G_r$ satisfy the graded commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0},$$

$$[L_n, G_r] = \left( \frac{n}{2} - r \right) G_{n+r},$$

$$\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3} \left( r^2 - \frac{1}{4} \right) \delta_{r+s,0},$$

(5.1)

where $m, n \in \mathbb{Z}$ and $r, s \in \mathbb{Z} + 1/2$. A primary superfield $\Phi_{\Delta}(x)$ transformations on a torus are given in (A.4).

\(^4\)On the sphere, $\mathcal{N} = 1, 2$ vacuum 4-point heavy-light superblocks were considered in [48]; 4-point light superblocks were calculated in [49]; the global $\mathcal{N} = 1, 2$ superblocks were discussed in [40]; extended SCFT superblocks, including $\mathcal{N} = 4$ case can be found in [50]. The torus NS superblocks were studied in [34].
We introduce the supertrace function on the NS supermodule of the weight $\Delta$ and define (cf. Section 3) the NS superblocks as

$$\text{str}_\Delta \left[ q^{L_0 - \frac{c}{2}} \Phi_\Delta(x) \right] = \Upsilon(\Delta, \Delta, q|x) = C_{\Delta \Delta} \Upsilon_0(\Delta, \Delta, q|w) + C_{\Delta \Delta + \frac{1}{2}} \Upsilon_1(\Delta, \Delta, q|w),$$

(5.2)

where $C_{\Delta \Delta}$ and $C_{\Delta \Delta + \frac{1}{2}}$ are the structure constants, and the component form of the NS lower/upper superblocks reads

$$\Upsilon_0(\Delta, \Delta, c|q) = \sum_{n \in \frac{1}{2} \mathbb{Z}} (-)^{2n} q^{\Delta+n} \sum_{n=|M|=|N|} B_{M|N} \frac{\langle \Delta, M|\phi_\Delta(w)|N, \Delta \rangle}{\langle \Delta|\phi_\Delta(w)\Delta \rangle},$$

(5.3)

$$\Upsilon_1(\Delta, \Delta, c|q) = \sum_{n \in \frac{1}{2} \mathbb{Z}} q^{\Delta+n} \sum_{n=|M|=|N|} B_{M|N} \frac{\langle \Delta, M|\psi_{\Delta+\frac{1}{2}}(w)|N, \Delta \rangle}{\langle \Delta|\psi_{\Delta+\frac{1}{2}}(w)\Delta \rangle}. \tag{5.4}$$

Here, summation over superindices is not graded and basis monomials of the NS supermodule denoted by $\mathcal{M}_\Delta$ are given by

$$|N, \Delta \rangle = L_{i_1 m_1}^1 \cdots L_{i_k m_k}^k G_{j_1 s_1}^{j_1} \cdots G_{j_l s_l}^{j_l} \sigma \Delta \rangle,$$

(5.5)

where $N = (K, S)$ labels basis monomials, $|N| = |K| + |S|$, where $|K| = i_1 m_1 + \ldots + i_k m_k$ and $|S| = j_1 s_1 + \ldots + j_l s_l$ is the level number. The supermatrix $B_{M|N}$ is the inverse of the Gram supermatrix $\mathcal{B}_{M|N} = \langle \Delta, M|N, \Delta \rangle$.

In what follows we consider various semiclassical superblocks arising in the limit $c \to \infty$ of the original NS superblocks (5.3), (5.4). We distinguish between heavy and light conformal dimensions $\Delta = O(c^1)$ and $\Delta = O(c^0)$ so that there are three types of one-point superblocks with external light operators: global superblock, light superblock, and heavy-light superblock. It will be shown that these superblocks are associated to different contractions of the NS superalgebra and are related to each other.

5.2 Contracted NS superalgebras

Let us first consider contractions that leave a finite-dimensional subalgebra intact. Even and odd generators rescale as

$$L_{0,\pm 1} \to l_{0,\pm 1} = L_{0,\pm 1}, \quad L_m \to a_m = L_m/c^\gamma, \quad |m| \geq 2,$$

$$G_{\pm \frac{1}{2}} \to g_{\pm \frac{1}{2}} = G_{\pm \frac{1}{2}}, \quad G_r \to b_r = G_r/c^\gamma, \quad |r| \geq 3/2.$$

(5.6)

We consider two cases: type A contraction $\gamma = 1$ and type B contraction $\gamma = 1/2$. The rescaled transformations of the primary field $\Phi_\Delta(x)$ with respect to $l_n$ and $g_s$ take the form (2.8), while higher order NS generators act trivially,

$$[a_m, \Phi_\Delta] = 0, \quad [b_s, \Phi_\Delta] = 0.$$

(5.7)
Thus, $\Phi_\Delta$ is an $osp(1|2)$ superconformal quasi-primary field.

In general, the resulting contracted NS superalgebras are isomorphic to semi-direct sum

$$osp(1|2) \ltimes \mathcal{F},$$

where $\mathcal{F}$ is an infinite-dimensional superalgebra with two branches $\mathcal{F} = \mathcal{F}_- \oplus \mathcal{F}_+$ spanned by basis elements $a_m$ and $b_n$ with $n \in \frac{1}{2}\mathbb{Z}$. The two branches $\mathcal{F}_\pm$ are highest weight $osp(1|2)$ supermodules. The factor $\mathcal{F}$ is defined by a particular type A/B contraction.

**Type A contraction.** In this case the contracted superalgebra is

$$NS_A = osp(1|2) \ltimes \mathcal{S}A,$$

where $\mathcal{S}A$ is an infinite-dimensional Abelian superalgebra. The graded commutation relations are given by

\[
[l_m, l_n] = (m - n)l_{m+n}, \quad \{g_r, g_s\} = 2l_{r+s}, \quad [l_n, g_r] = \left(\frac{n}{2} - r\right)g_{n+r},
\]

\[
[a_m, a_n] = 0, \quad \{b_r, b_s\} = 0, \quad [a_n, b_r] = 0,
\]

\[
[l_m, a_n] = (m - n)a_{m+n}, \quad |m + n| \geq 2; \quad [l_m, a_n] = 0, \quad |m + n| \leq 1,
\]

\[
[l_n, b_r] = \left(\frac{n}{2} - r\right)b_{n+r}, \quad |n + r| \geq 3/2; \quad [l_n, b_r] = 0, \quad |n + r| = \frac{1}{2},
\]

\[
[g_r, a_n] = -(\frac{n}{2} - r)b_{n+r}, \quad \{g_r, b_s\} = 2a_{r+s}, \quad |r + s| \geq 2; \quad \{g_r, b_s\} = 0, \quad |r + s| \leq 1.
\]

Here, the first group of relations defines $osp(1|2)$ superalgebra, the second group defines Abelian superalgebra, the third group defines the highest weight supermodule structure.

**Type B contraction.** Here, the contracted superalgebra is

$$NS_B = osp(1|2) \ltimes \mathcal{H}C,$$

where $\mathcal{H}C$ is the infinite-dimensional Heisenberg-Clifford superalgebra. The respective graded commutation relations are different from those of $NS_A$ (5.10)–(5.12) only in the part defining the infinite-dimensional factor. Namely,

\[
[a_m, a_n] = \frac{m(m^2 - 1)}{12} \delta_{m+n,0}, \quad \{b_r, b_s\} = \frac{1}{3} \left(r^2 - \frac{1}{4}\right) \delta_{r+s,0}, \quad [a_n, b_r] = 0.
\]

**Type C contraction.** Also, there is a third type of contraction when all NS generators are rescaled so that the $osp(1|2)$ subalgebra is also contracted,

$$L_0 \rightarrow l_0 = L_0 / c, \quad L_m \rightarrow l_m = L_m / \sqrt{2}c, \quad G_r \rightarrow g_r = G_r / \sqrt{2}c, \quad m \neq 0, \quad \forall r.$$

Then, the resulting contraction is given by the Heisenberg-Clifford superalgebra

$$NS_C = \mathcal{H}C,$$
with non-zero graded commutation relations
\[
[l_m, l_{-m}] = ml_0 + \frac{m(m^2 - 1)}{24}, \quad m \in \mathbb{Z},
\]
\[
\{g_r, g_{-r}\} = l_0 + \frac{1}{6} \left( r^2 - \frac{1}{4} \right), \quad r \in \mathbb{Z} + \frac{1}{2}.
\] (5.17)

The superfield \( \Phi_\Delta \) is light and, therefore, \( \mathcal{HC} \)-invariant,
\[
[l_m, \Phi_\Delta] = 0, \quad [g_r, \Phi_\Delta] = 0, \quad m \in \mathbb{Z}, \quad r \in \mathbb{Z} + \frac{1}{2}.
\] (5.18)

### 5.3 Associated superblocks

In what follows we argue that the superblock functions associated to the three types of contracted NS superalgebra correspond to \( osp(1|2) \) global, light, and heavy-light one-point superblocks, respectively.

Consider first the representation theory of the contracted superalgebras (5.8). A supermodule \( \mathcal{M}_\tilde{\Delta} \) is spanned by basis monomials
\[
|N, \tilde{\Delta}\rangle = a^Rb^S l^1_{-1} g^k_{-\frac{1}{2}} |\tilde{\Delta}\rangle \equiv a^Rb^S |N, \tilde{\Delta}\rangle,
\] (5.19)
where we split into \( osp(1|2) \) and \( F \) subalgebra generators, and denoted \( a^R = a^i_{-m_1} \ldots a^i_{-m_k} \), with a level \( |R| = i_1 m_1 + \cdots + i_k m_k \), and \( b^S = b^j_{-s_1} \ldots b^j_{-s_k} \), with a level \( |S| = j_1 s_1 + \cdots + j_k s_k \). The total level number is given by \( |N| = |N| + |R| + |S| \in \frac{1}{2} \mathbb{Z} \), where \( N \) labels \( osp(1|2) \) indices and \( |N| = s + k \) and \( s = 1, 2, \ldots \) and \( k = 0, 1 \), cf. (2.3) and (5.5).

Let \( \mathcal{V}_\tilde{\Delta} \) be an \( osp(1|2) \) supermodule of weight \( \tilde{\Delta} \), and \( F \) be the Fock supermodule of the factor \( F \) in the truncated superalgebra (5.8). Then, considering \( \mathcal{M}_\tilde{\Delta}, \mathcal{V}_\tilde{\Delta}, \) and \( F \) on their own as linear spaces we conclude from (5.19) that \( \mathcal{M}_\Delta \) is the (graded) tensor product of vector spaces \( \mathcal{M}_\tilde{\Delta} = F \otimes \mathcal{V}_\tilde{\Delta} \). On the other hand, the primary superfield \( \Phi_\Delta \) is \( F \)-invariant (5.7) and, therefore, can be represented as \( 1_F \otimes \hat{\Phi}_\Delta \), where \( \hat{\Phi}_\Delta \) is \( osp(1|2) \) quasi-primary superfield. Using that a supertrace trace on the tensor product is a product of supertraces one can explicitly show that the supertrace function (5.2) associated to the contracted type A/B superalgebras reads
\[
\Upsilon(\Delta, \tilde{\Delta}, q|\eta) = \chi_F(q) \cdot str_{\mathcal{V}_\tilde{\Delta}} \left[ q^{L_0} \hat{\Phi}_\Delta(x) \right],
\] (5.20)
where the first factor here is the \( F \) character on the Fock module, while the second factor is the \( osp(1|2) \) supertrace function (3.3).

For the type A superalgebra (5.9) the \( F \) character is trivial \( \chi_F(q) = 1 \) so that the resulting supertrace function is \( \Upsilon(\Delta, \tilde{\Delta}, q|\eta) = B(\Delta, \tilde{\Delta}, q|\eta) \) given in (3.2). Thus, we conclude that global superblocks correspond to the type A contracted NS superalgebra. In other words, a truncation of the NS superalgebra to \( osp(1|2) \) subalgebra is equivalent to a particular contraction.
For the type B superalgebra (5.13) the \( \mathcal{F} \) character is a truncated Heisenberg-Clifford (graded) character. Indeed, the Heisenberg-Clifford character is known to be (see, e.g., [51])

\[
\chi_{HC}(q) = \prod_{n=0}^{\infty} \frac{1 + q^{n+\frac{1}{2}}}{1 - q^{n+\frac{1}{2}}}.
\]

(5.21)

On the other hand, basis elements \( a_m \) and \( b_r \) of the \( \mathcal{F} \) factor are labeled by (half-)integers \( |m| \geq 2 \) and \( |r| \geq 3/2 \). It means that states generated by basis elements with \( m = 0, \pm 1 \) and \( r = \pm 1/2 \) do not contribute because they belong to the \( osp(1|2) \) module. The \( \mathcal{F} \) character, if compared to the Heisenberg-Clifford character, does not take into account those lower label states so that the truncated character \( \tilde{\chi}_{HC}(q) \) is given by

\[
\tilde{\chi}_{HC}(q) = \frac{\chi_{HC}(q)}{\chi_{osp}(q)} = \prod_{n=1}^{\infty} \frac{1 - q^{n+\frac{1}{2}}}{1 - q^{n+\frac{1}{2}}} ,
\]

(5.22)

where the \( osp(1|2) \) character given by (3.9). Finally, the resulting block is

\[
\mathcal{L}(\Delta, \bar{\Delta}, q|\eta) = \tilde{\chi}_{HC}(q) B(\Delta, \bar{\Delta}, q|\eta) ,
\]

(5.23)

where the prefactor is given by (5.22). The block on the left-hand side is the light NS superblock that equivalently can be obtained as the \( c \to \infty \) limit of the original NS super-conformal block (5.2) at fixed conformal dimensions \( \Delta, \bar{\Delta} \).

Finally, let us shortly consider heavy-light superblocks \( \mathcal{H}_{0,1}(\Delta, \bar{\Delta}, q) \) which have one heavy dimension \( \bar{\Delta} = \mathcal{O}(c^1) \) and light dimension \( \Delta = \mathcal{O}(c^0) \) at \( c \to \infty \). A contraction of the NS superalgebra underlying the heavy-light superblocks is given by the type C superalgebra (5.16). Applying the definitions (5.3) and (5.4) to the type C we can explicitly calculate the associated superblocks that are given by the Heisenberg-Clifford character (5.21), i.e.

\[
\mathcal{H}_{0,1}(\Delta, \bar{\Delta}, q) = \chi_{HC}(q) .
\]

(5.24)

Also, using relations (3.10), (5.22), and (5.23) we see that the heavy-light superblocks can be seen as the limiting case of the light superblocks

\[
\mathcal{H}(\Delta, \bar{\Delta}, q|\eta) = \mathcal{L}(\Delta, \infty, q|\eta) .
\]

(5.25)

The above relations between various superblocks (5.23), (5.24), and (5.25) are a supersymmetric version of the analogous relations between semiclassical Virasoro torus block and \( sl(2) \) global torus blocks [24].

\[\text{For analogous relations between higher-point global and light non-supersymmetric torus blocks see also [25].}\]

On the sphere, it is shown that 4-point heavy-light \( N = 1 \) vacuum superblocks with pairwise equal dimensions coincide with non-supersymmetric heavy-light blocks in the leading order of the large-\( c \) approximation [48].
6 Concluding remarks

In this paper we have developed a framework to study large-$c$ behavior of torus SCFT$_2$ superblocks. We have explicitly calculated various types of semiclassical one-point superblocks and established relations between them.

We have seen that two exponentiated osp(1|2) global superblocks are equal to the single non-supersymmetric exponentiated sl(2) global block. On the other hand, there is a lot of evidence that exponentiated global blocks in the leading approximation are related to the linearized classical conformal blocks [6, 20, 23, 24]. We expect that in SCFT$_2$ these two types of superblocks are similarly related and, therefore, the linearized classical superblocks are equal to the linearized non-supersymmetric classical blocks. Indeed, here we have the same phenomenon that classical conformal dimensions of the supermultiplet operators $\epsilon_b = \frac{\Delta}{c}$ and $\epsilon_f = \frac{\Delta + \frac{1}{2}}{c}$ coincide in the large-$c$ regime: $\epsilon_b = \epsilon_f$. It is similar to the argument of [52] that classical $\mathcal{N} = 1$ superblocks on the sphere should have the large-$c$ asymptotic given by the purely bosonic Zamolodchikov’s classical block.

From the bulk perspective, both classical global and linearized classical torus blocks are realized as lengths of geodesic tadpole-type networks stretched in the thermal AdS$_3$ space [13, 19, 20]. One might expect that classical superblocks could be realized in terms of superparticles propagating on the particular background that solves 3d supergravity equations. However, presently, we may conclude only that in the leading order of the large-$c$ approximation the superblocks are realized by duplicated system of bosonic geodesic networks.

The other interesting question is to understand the bulk dual realization of higher-order corrections to the classical global (super)blocks of Section 4.3. It appears that using the worldline approach they can be calculated by accounting for the backreaction of particles in the bulk (see [21] for recent discussion of the worldline formalism in the context of the semiclassical AdS$_3$/CFT$_2$ correspondence).

Acknowledgements. We thank R. Metsaev for useful discussions. The work of K.A. was supported by the Russian Science Foundation grant 14-42-00047. The work of V.B. was supported by the Foundation for the advancement of theoretical physics BASIS.

A Supercylindrical coordinates

The superfield transformations on the superplane are known to be [31, 32]

\[
\begin{align*}
[L_n, \Phi_\Delta(y)] &= \left(z^{n+1} \partial_z + \frac{1}{2} (n + 1) z^n \theta \partial_\theta + \Delta (n + 1) z^n \right) \Phi_\Delta(y), \\
[G_r, \Phi_\Delta(y)] &= \left(z^{r+\frac{1}{2}} \left(\partial_\theta - \theta \partial_z\right) - 2\Delta (r + \frac{1}{2}) \theta z^r z^{-\frac{1}{2}} \right) \Phi_\Delta(y),
\end{align*}
\]

where $n \in \mathbb{Z}$ and $r \in \mathbb{Z} - \frac{1}{2}$. To find the superconformal transformations on the torus we change from the superplane coordinates $y = (z, \theta)$ to the supercylindrical coordinates.

\footnote{In that context, it would be instructive to examine geodesic Witten diagrams with fermions [53].}
x = (w, η) as
\[ w = i \log z , \quad η = (-iz)^{-1/2} \theta . \tag{A.2} \]
This is the superconformal map so that the superfield \( Φ_Δ(y) \) transforms as \(^8\)
\[ Φ_Δ(y) = (Dη)^2 Δ Φ_Δ(x) , \quad \text{where} \quad Dη = (-iz)^{-1/2} , \tag{A.3} \]
and \( D = ∂_θ + z∂_z \) is the supercovariant (left) derivative. In the supercylindrical coordinates we find
\[ \mathcal{L}_n Φ_Δ(x) \equiv [L_n, Φ_Δ(x)] = (-i)e^{-inw} \left[ ∂_w + in \left( \frac{1}{2} η ∂_η + Δ \right) \right] Φ_Δ(x) , \tag{A.4} \]
\[ \mathcal{G}_n Φ_Δ(x) \equiv [G_r, Φ_Δ(x)] = (-i)^{-\frac{1}{2}} e^{-irw} \left[ (∂_η + η ∂_w) + 2irΔη \right] Φ_Δ(x) . \]
As expected, the Hamiltonian on a cylinder is realized by \( w \)-translations, \( \mathcal{L}_0 = -i∂_ω \) contrary to dilatations \( \mathcal{L}_0 = z∂_z + Δ \) on the plane.

B Corrections to the classical global block

The first corrections \( b_{a|1} = b_{a|1}(σ, ˜σ, q) \) to the classical global block are defined by the following differential equations
\[ ∂_q b_{0|1} = \frac{A}{B} , \quad ∂_q b_{1|1} = -\frac{C}{D} , \tag{B.1} \]
where
\[ A = -2(q - 1)^2 q^{3/2} ∂_q b_{0|0} - \left( q^{1/2} - 1 \right)^2 \left( 3q^{3/2} + 4q + q^{1/2} \right) ∂_q b_{0|0} + 2 \left( -q^{3/2} + q + q^{1/2} - 1 \right) ˜σ + (q + 1)σ , \]
\[ B = 4(q - 1)^2 q^{1/2} \left( q∂_q b_{0|0} + ˜σ \right) , \]
\[ C = q^{1/2} (q^{1/2} - 1)^2 \left( 2 \left( q^{1/2} + 1 \right) q∂_q^2 b_{0|0} + \left( 3q^{1/2} + 1 \right) ∂_q b_{0|0} \right) + \left( q^{1/2} + 1 \right) σ + 2 \left( q^{1/2} - 1 \right)^2 ˜σ , \]
\[ D = 4(q^{1/2} - 1)^2 (q + q^{1/2}) \left( q∂_q b_{0|0} + ˜σ \right) , \tag{B.2} \]
and the classical global block \( b_{0|0} \) is given by (4.25). The solutions are given in (4.26)-(4.28).

Below we give the first correction to the \( sl(2) \) global block (4.25) included here for completeness,
\[ b_1(σ, ˜σ, q) = -\log(1 - q) + \frac{1}{2} \frac{q}{(q - 1)} σ - \frac{1}{4} \frac{q^2}{(q - 1)^2} δ^2 + \frac{1}{24} \frac{(q - 3)q^2}{(q - 1)^3} δ^3 + \mathcal{O}(δ^4) . \tag{B.3} \]
It can be read off from the Casimir equation in [19, 24]. Similar to the superblocks (4.28), the first term here yields the \( sl(2) \) character.

\(^8\)For more details on 2d superconformal geometry see, e.g., [54, 55].
References

[1] T. Hartman, Entanglement Entropy at Large Central Charge, 1303.6955.

[2] A. L. Fitzpatrick, J. Kaplan and M. T. Walters, Universality of Long-Distance AdS Physics from the CFT Bootstrap, JHEP 1408 (2014) 145, [1403.6829].

[3] P. Caputa, J. Simon, A. Stikonas and T. Takayanagi, Quantum Entanglement of Localized Excited States at Finite Temperature, JHEP 01 (2015) 102, [1410.2287].

[4] J. de Boer, A. Castro, E. Hijano, J. I. Jottar and P. Kraus, Higher spin entanglement and $W_N$ conformal blocks, JHEP 07 (2015) 168, [1412.7520].

[5] E. Hijano, P. Kraus and R. Snively, Worldline approach to semi-classical conformal blocks, JHEP 07 (2015) 131, [1501.02260].

[6] A. L. Fitzpatrick, J. Kaplan and M. T. Walters, Virasoro Conformal Blocks and Thermality from Classical Background Fields, JHEP 11 (2015) 200, [1501.05315].

[7] K. B. Alkalaev and V. A. Belavin, Classical conformal blocks via AdS/CFT correspondence, JHEP 08 (2015) 049, [1504.05943].

[8] E. Hijano, P. Kraus, E. Perlmutter and R. Snively, Semiclassical Virasoro blocks from $AdS_3$ gravity, JHEP 12 (2015) 077, [1508.04987].

[9] K. B. Alkalaev and V. A. Belavin, Monodromic vs geodesic computation of Virasoro classical conformal blocks, Nucl. Phys. B904 (2016) 367–385, [1510.06685].

[10] M. Beccaria, A. Fachechi and G. Macorini, Virasoro vacuum block at next-to-leading order in the heavy-light limit, JHEP 02 (2016) 072, [1511.05452].

[11] A. L. Fitzpatrick and J. Kaplan, Conformal Blocks Beyond the Semi-Classical Limit, JHEP 05 (2016) 075, [1512.03052].

[12] P. Banerjee, S. Datta and R. Sinha, Higher-point conformal blocks and entanglement entropy in heavy states, JHEP 05 (2016) 127, [1601.06794].

[13] K. B. Alkalaev and V. A. Belavin, Holographic interpretation of 1-point toroidal block in the semiclassical limit, JHEP 06 (2016) 183, [1603.08440].

[14] B. Chen, J.-q. Wu and J.-j. Zhang, Holographic Description of 2D Conformal Block in Semi-classical Limit, JHEP 10 (2016) 110, [1609.00801].

[15] K. B. Alkalaev, Many-point classical conformal blocks and geodesic networks on the hyperbolic plane, JHEP 12 (2016) 070, [1610.06717].

[16] P. Kraus and A. Maloney, A Cardy formula for three-point coefficients or how the black hole got its spots, JHEP 05 (2017) 160, [1608.03284].

[17] O. Hulk, T. Prochzka and J. Raeymaekers, Multi-centered $AdS_3$ solutions from Virasoro conformal blocks, JHEP 03 (2017) 129, [1612.03879].

[18] A. L. Fitzpatrick, J. Kaplan, D. Li and J. Wang, Exact Virasoro Blocks from Wilson Lines and Background-Independent Operators, 1612.06385.

[19] P. Kraus, A. Maloney, H. Maxfield, G. S. Ng and J.-q. Wu, Witten Diagrams for Torus Conformal Blocks, JHEP 09 (2017) 149, [1706.00047].
[20] K. B. Alkalaev and V. A. Belavin, *Holographic duals of large-c torus conformal blocks*, JHEP 10 (2017) 140, [1707.09311].

[21] H. Maxfield, *A view of the bulk from the worldline*, 1712.00885.

[22] E. M. Brehm, D. Das and S. Datta, *Probing thermality beyond the diagonal*, 1804.07924.

[23] K. B. Alkalaev and V. A. Belavin, *From global to heavy-light: 5-point conformal blocks*, JHEP 03 (2016) 184, [1512.07627].

[24] K. B. Alkalaev, R. V. Geiko and V. A. Rappoport, *Various semiclassical limits of torus conformal blocks*, JHEP 04 (2017) 070, [1612.05891].

[25] M. Cho, S. Collier and X. Yin, *Recursive Representations of Arbitrary Virasoro Conformal Blocks*, 1703.09805.

[26] I. P. Ennes, A. V. Ramallo and J. M. Sanchez de Santos, *OSP(1|2) conformal field theory*, AIP Conf. Proc. 419 (1998) 138–150, [hep-th/9708094].

[27] G. Gotz, T. Quella and V. Schomerus, *Representation theory of sl(2|1)*, J. Algebra 312 (2007) 829–848, [hep-th/0504234].

[28] A. Lesniewski, *A remark on the Casimir elements of Lie superalgebras and quantized Lie superalgebras*, J. Math. Phys. 36 (1995) 1457.

[29] D. Arnaudon and M. Bauer, *Scasimir operator, scentre and representations of U-q(osp(1|2)), Lett. Math. Phys. 40* (1997) 307–320, [q-alg/9605020].

[30] P. K. Ghosh, *SuperCalogero model with OSp(2|2) supersymmetry: Is the construction unique?*, Nucl. Phys. B681 (2004) 359–373, [hep-th/0309183].

[31] D. Friedan, Z.-a. Qiu and S. H. Shenker, *Superconformal Invariance in Two-Dimensions and the Tricritical Ising Model*, Phys. Lett. 151B (1985) 37–43.

[32] M. A. Bershadsky, V. G. Knizhnik and M. G. Teitelman, *Superconformal Symmetry in Two-Dimensions*, Phys. Lett. 151B (1985) 31–36.

[33] L. Alvarez-Gaume and P. Zaugg, *Structure constants in the N=1 superoperator algebra*, Annals Phys. 215 (1992) 171–230, [hep-th/9109050].

[34] L. Hadasz, Z. Jaskolski and P. Suchanek, *Recurrence relations for toric N=1 superconformal blocks*, JHEP 09 (2012) 122, [1207.5740].

[35] R. Poghossian, *Recursion relations in CFT and N=2 SYM theory*, JHEP 12 (2009) 038, [0909.3412].

[36] L. Hadasz, Z. Jaskolski and P. Suchanek, *Recursive representation of the torus 1-point conformal block*, JHEP 01 (2010) 063, [0911.2353].

[37] M. Piatek, *Classical torus conformal block, N = 2* twisted superpotential and the accessory parameter of Lame equation*, JHEP 03 (2014) 124, [1309.7672].

[38] P. Menotti, *Torus classical conformal blocks*, 1805.07788.

[39] F. Dolan and H. Osborn, *Conformal Partial Waves: Further Mathematical Results*, 1108.6194.

[40] A. L. Fitzpatrick, J. Kaplan, Z. U. Khandker, D. Li, D. Poland and D. Simmons-Duffin, *Covariant Approaches to Superconformal Blocks*, JHEP 08 (2014) 129, [1402.1167].
[41] D. Simmons-Duffin, *Projectors, Shadows, and Conformal Blocks*, JHEP 04 (2014) 146, [1204.3894].

[42] Y. Gobeil, A. Maloney, G. S. Ng and J.-q. Wu, *Thermal Conformal Blocks*, 1802.10537.

[43] D. Poland, S. Rychkov and A. Vichi, *The Conformal Bootstrap: Numerical Techniques and Applications*, 1805.04405.

[44] T. Oshima, *Fractional Calculus of Weyl Algebra and Fuchsian Differential Equations*, The Mathematical Society of Japan 28 (2012) 1–203.

[45] A. Litvinov, S. Lukyanov, N. Nekrasov and A. Zamolodchikov, *Classical Conformal Blocks and Painlevé VI*, JHEP 1407 (2014) 144, [1309.4700].

[46] M. Pitek and A. R. Pietrykowski, *Solving Heun’s equation using conformal blocks*, 1708.06135.

[47] M. Leness and F. Novaes, *Classical Conformal Blocks and Accessory Parameters from Isomonodromic Deformations*, JHEP 04 (2018) 096, [1709.03476].

[48] H. Chen, A. L. Fitzpatrick, J. Kaplan, D. Li and J. Wang, *Degenerate Operators and the 1/c Expansion: Lorentzian Resummations, High Order Computations, and Super-Virasoro Blocks*, 1606.02659.

[49] H. Poghosyan, *The light asymptotic limit of conformal blocks in $\mathcal{N} = 1$ super Liouville field theory*, JHEP 09 (2017) 062, [1706.07474].

[50] Y.-H. Lin, S.-H. Shao, D. Simmons-Duffin, Y. Wang and X. Yin, *$\mathcal{N} = 4$ superconformal bootstrap of the K3 CFT*, JHEP 05 (2017) 126, [1511.04065].

[51] P. Goddard, A. Kent and D. I. Olive, *Unitary Representations of the Virasoro and Supervirasoro Algebras*, Commun. Math. Phys. 103 (1986) 105–119.

[52] L. Hadasz, Z. Jaskolski and P. Suchanek, *Elliptic recurrence representation of the $N = 1$ Neveu-Schwarz blocks*, Nucl. Phys. B798 (2008) 363–378, [0711.1619].

[53] M. Nishida and K. Tamaoka, *Fermions in Geodesic Witten Diagrams*, 1805.00217.

[54] D. Friedan, *Notes on string theory and two-dimensional conformal field theory*, preprint EFI 85-99 (1986).

[55] M. Dorzapr, *The Definition of Neveu-Schwarz superconformal fields and uncharged superconformal transformations*, Rev. Math. Phys. 11 (1999) 137–169, [hep-th/9712107].

– 20 –