Refined blow-up asymptotics for a perturbed nonlinear heat equation with a gradient and a non-local term

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Abstract
We consider in this paper a perturbation of the standard semilinear heat equation by a term involving the space derivative and a non-local term. In some earlier works [1, 2], we constructed a solution $u$ for that equation such that $u$ and $\nabla u$ both blow up at the origin and only there. We also gave the final blow-up profile. In this paper, we refine our construction method in order to get a sharper estimate on the gradient at blow-up.

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1 Introduction
We consider in this paper the following nonlinear parabolic equation

$$
\begin{align*}
\begin{cases}
 u_t &= \Delta u + |u|^{p-1}u + \mu|\nabla u| \int_{B(0,|x|)} |u|^{q-1}, \\
 u(0) &= u_0 \in W^{1,\infty}(\mathbb{R}^N),
\end{cases}
\end{align*}
$$

(1.1)
where \( u = u(x,t) \in \mathbb{R}, x \in \mathbb{R}^N \) and the parameters \( p, q \) and \( \mu \) are such that
\[
p > 3, \quad \frac{N}{2}(p - 1) + 1 < q < \frac{N}{2}(p - 1) + \frac{p + 1}{2}, \quad \mu \in \mathbb{R}.
\] (1.2)

Equation (1.1) is wellposed in the weighted functional space \( W^{1,\infty}_\beta(\mathbb{R}^N) \) defined as follows:
\[
W^{1,\infty}_\beta(\mathbb{R}^N) = \{ g, \ (1 + |y|^\beta)g \in L^\infty, \ (1 + |y|^\beta)\nabla g \in L^\infty \},
\] (1.3)
where
\[
0 \leq \beta < \frac{2}{p - 1}, \text{ if } \mu = 0 \quad \text{and} \quad \frac{N}{q - 1} < \beta < \frac{2}{p - 1}, \text{ if } \mu \neq 0,
\] (1.4)
as one may see from Appendix C in [1]. From the standard dichotomy, either the maximal solution is global in time, or it exists up to some maximal time \( T < +\infty \) with
\[
\|u(t)\|_{W^{1,\infty}_\beta} \to \infty \text{ as } t \to +\infty.
\]
In that case, we say that \( u(x,t) \) blows up in finite time, and we call \( T \) the blow-up time of the solution.

When \( \mu = 0 \), equation (1.1) becomes the standard semilinear heat equation with power nonlinearity:
\[
u_t = \Delta u + |u|^{p-1}u.
\] (1.5)
The existence of blow-up solutions for equation (1.5) has been extensively studied, see Fujita [13], Ball [3], Berger and Kohn [4], Herrero and Velázquez [21], Bricmont and Kupiainen [5], Merle and Zaag [25] and the references therein.

In particular, the authors in [5] and [25] constructed a solution \( u \) which approaches an explicit universal profile \( f \) depending only on \( p \) and independent from initial data as follows:
\[
\left\| (T-t)^{\frac{1}{p-1}}u(x,t) - f\left(\frac{x}{\sqrt{(T-t)|\log(T-t)|}}\right)\right\|_{L^\infty} \to 0,
\] (1.6)
as \( t \to T \), where \( f \) is the profile defined by
\[
f(z) = \left(p - 1 + \frac{(p-1)^2}{4p}|z|^2\right)^{-\frac{1}{p-1}}.
\] (1.7)

The proof relies on 2 parts:
- A formal part, where one finds an approximate solution, which will be considered as a profile for the exact solution to be constructed.
- A rigorous part, where one linearizes the equation around the approximate solution (i.e. the profile), and shows that the linearized solution has a solution which
converges to 0. This part relies itself on 2 steps: the first, where we reduce the control of the solution (which is infinite dimensional) to a finite dimensional problem. Then, the finite dimensional problem is solved thanks to index theory.

This method has proved to be efficient for different PDEs from different types, and no list can be exhaustive (see del Pino, Musso and Wei [7], Nouaili and Zaag [32], Tayachi and Zaag [34], Duong and Zaag [8], Mahmoudi, Nouaili and Zaag [23], Merle, Raphael, Rodnianski and Szeftel [24], Collot, Ghoul, Nguyen and Masmoudi [6], etc...).

In [1] and [2], we considered equation (1.1) as a challenge for the construction of blow-up solutions, since it presents a double difficulty: the gradient term and the nonlocal term, and we were successful in proving the following:

- In [1], we constructed a solution \( u(x,t) \) for equation (1.1) which blows up in finite time \( T \) at \( a = 0 \), and we proved that the solution approaches the profile \( f(1.7) \) in the sense that for all \((x,t) \in \mathbb{R}^N \times (0,T)\):

\[
|u(x,t) - (T-t)^{-\frac{1}{p-1}} f\left(\frac{x}{\sqrt{(T-t)\log(T-t)}}\right)| \leq \frac{C}{1 + (\frac{|x|^2}{T-t})^\frac{p-1}{2}} \frac{(T-t)^{-\frac{1}{p-1}}}{\log(T-t)^{\frac{1}{p-2}}},
\]

and

\[
|\nabla u(x,t) - (T-t)^{-\frac{1}{2}\frac{1}{p-1}} \nabla f\left(\frac{x}{\sqrt{(T-t)\log(T-t)}}\right)| \leq \frac{C}{1 + (\frac{|x|^2}{T-t})^\frac{p-1}{2}} \frac{(T-t)^{-\frac{1}{2}\frac{1}{p-1}}}{\log(T-t)^{\frac{1}{p-2}}},
\]

(note that \( 0 \leq \beta < 1 \) from (1.4) and (1.2)), and that \( f \) is called the “intermediate” profile).

- In [2], adapting the technique developed for equation (1.5) by Giga and Kohn [16], we prove that neither \( u \) nor \( \nabla u \) blow up outside the origin. Since \( u \) blows up at the origin by (1.8), this implies the single point blow-up property for \( u \). Using the mean value theorem, this yields that \( \nabla u \) blows up at the origin, which proves the single point blow-up property for \( \nabla u \) too. We also prove the existence of a blow-up final profile \( u^* \) such that \( u(x,t) \to u^*(x) \) as \( t \to T \) in \( C^1 \) of every compact of \( \mathbb{R}^N \setminus \{0\} \). Next, we find an equivalent of \( u^* \) and an upper bound on \( \nabla u^* \) near the blow-up point:

\[
u^*(x) \sim \left[ \frac{8p \log |x|}{(p-1)^2|x|^2} \right]^\frac{1}{p-1}, \text{ as } x \to 0,
\]

and for \( |x| \) small,

\[
|\nabla u^*(x)| \leq C|x|^{-\frac{p+1}{p-1}} \log |x|^{\frac{1}{p-1}}.
\]

Our goal in this paper is to give a refined asymptotic description of the blow-up solution. In fact, we prove more refined versions of (1.8), (1.9) and (1.11). This will be done through the introduction of a sharper shrinking set. More precisely, we prove the following theorem:
THEOREM 1.1 (A sharper description of blow-up). Let \( \mu \in \mathbb{R} \), \( p > 3 \), and \( q \in \mathbb{R} \) such that \( \frac{N}{2} (p-1) + 1 < q < \frac{N}{2} (p-1) + \frac{p+1}{2} \).

Consider an arbitrary \( \beta \) such that

\[
0 \leq \beta < \frac{2}{p-1}, \text{ if } \mu = 0 \quad \text{and} \quad \frac{N}{q-1} < \beta < \frac{2}{p-1}, \text{ if } \mu \neq 0.
\]

(1.12)

Consider also some arbitrary \( \varepsilon_1 \in (0, \frac{1}{2}] \) and \( \alpha \in (0, \frac{1}{2}) \).

Then, there exists \( T > 0 \) such that equation (1.1) has a solution \( u(x, t) \) such that \( u \) and \( \nabla u \) simultaneously blow up at time \( T \) at the point \( a = 0 \). Moreover,

1. For all \( t \in [0, T) \), for all \( x \in \mathbb{R}^N \),

\[
|u(x, t) - (T-t)^{-\frac{1}{p-1}} f\left( \frac{x}{\sqrt{(T-t) \log(T-t)}} \right)| \leq \frac{C}{1 + (\frac{|x|^2}{T-t})^\frac{\alpha}{2}} \left( T-t \right)^{-\frac{1}{p-1}} (\log(T-t))^{\frac{1}{2} - \frac{1}{2} - \varepsilon_1},
\]

(1.13)

and

\[
|\nabla u(x, t) - (T-t)^{-\frac{1}{2}} \frac{1}{\sqrt{\log(T-t)}} \nabla f\left( \frac{x}{\sqrt{(T-t) \log(T-t)}} \right)| \leq \frac{C}{1 + (\frac{|x|^2}{T-t})^\frac{\alpha}{2}} \left( T-t \right)^{-\frac{1}{2} - \frac{1}{p-1}} (\log(T-t))^{\frac{1}{2} - \frac{1}{2} - \varepsilon_1},
\]

(1.14)

where \( f \) is defined in (1.7).

2. Blow-up occurs only at the origin.

3. For all \( x \neq 0 \), \( u(x, t) \to u^*(x) \) as \( t \to T \) in \( C^1 \) of every compact of \( \mathbb{R}^N \setminus \{0\} \), with

\[
u^*(x) \sim \left[ \frac{8p|\log|x||}{(p-1)^2|x|^2} \right]^{\frac{1}{p-1}}, \text{ as } x \to 0,
\]

(1.15)

and for \( |x| \) small,

\[
|\nabla u^*(x)| \leq C|x|^{-\frac{p+1}{p-1}} \log|x|^{\frac{p+1}{2(p-1)} - \alpha}.
\]

(1.16)

REMARK 1.2. 1. With respect to our earlier work [1] and [2], the improvement lays in (1.13), (1.14) and (1.16), since we could obtain the result in those papers only with \( \varepsilon_1 = \frac{1}{2} \) and \( \alpha = \frac{1}{4} \), whereas we are able now to get it for smaller values of \( \varepsilon_1 \) and larger values of \( \alpha \), namely for \( \varepsilon_1 \) arbitrarily close to 0 and \( \alpha \) arbitrarily close to \( \frac{1}{2} \).

2. As we have already noted in our earlier paper [1], already when \( \mu = 0 \), our error estimate in (1.13) is better than in [2], thanks to the term \( 1 + (\frac{|x|^2}{T-t})^\frac{\alpha}{2} \) in the denominator. In fact, our estimate is even better than (1.8) proved in [1], since we can take \( \varepsilon_1 \) arbitrarily small and not just equal to \( \frac{1}{2} \) as in (1.8).
3. As we have already noted in our previous paper [2], the blow-up of the gradient at the origin is a consequence of the single-point blow-up of the solution and the mean value theorem. In this paper, we do better, by exhibiting some \( \xi(t) \to 0 \) as \( t \to T \), so that \( |\nabla(\xi(t), t)| \to \infty \) (see below in page 20). The improvement in (1.14) is crucial in obtaining that result, keeping in mind that the version with \( \varepsilon_1 = \frac{1}{2} \) given in [1] (see (1.9) above) was not enough to derive such a sequence \( \xi(t) \).

4. We wonder whether one may be able to get an equivalent for the gradient final blow-up profile. Such a question seems to be hard, since, up to our knowledge, no such equivalent was proved in the literature, except in the recent work on the standard semilinear heat equation \( (1.5) \) by Duong, Ghoul and Zaag [9].

The paper is organized as follows. In Section 2, we give a formulation of the problem. In Section 3, we prove the existence of a solution of equation (2.7). Finally, in Section 4, we prove Theorem 1.1.

2 Formulation of the problem

In this section, we consider \( T > 0 \) and \( \varepsilon_1 \in (0, \frac{1}{2}] \), then we recall the following similarity variables transformation introduced by Giga and Kohn [16], [17], [18]:

\[
y = \frac{x}{\sqrt{T-t}}, \quad s = -\log(T-t) \quad \text{and} \quad w(y, s) = (T-t)^{\frac{1}{p-1}}u(x, t), \quad (2.1)
\]

where \( T \) is the time where we want to make the solution blow up.

If \( u(x, t) \) satisfies (1.1) for all \( (x, t) \in \mathbb{R}^N \times [0, T) \), then \( w(y, s) \) satisfies the following equation for all \( (y, s) \in \mathbb{R}^N \times [-\log T, +\infty) \):

\[
\partial_s w = \Delta w - \frac{1}{2}y \cdot \nabla w - \frac{1}{p-1}w + |w|^{p-1}w + \mu e^{-\gamma s} |\nabla w| \int_{B(0, |y|)} |w|^{q-1}, \quad (2.2)
\]

where \( \gamma = \frac{p-q}{p-1} + \frac{N-1}{2} \).

Note from (1.2) that \( \gamma > 0 \), which explains the little effect of the perturbation term for large time \( s \). This allows us to adopt a perturbative approach with respect to earlier constructions given for the standard semilinear heat equation \( (1.5) \).

With the transformation (2.1), we reduce the construction of a solution \( u(x, t) \) for equation (1.1) that blows up at \( T < \infty \) to the construction of a global solution \( w(y, s) \) for equation (2.2) such that

\[
\forall s \geq s_0, \quad \varepsilon_0 \leq \|w(s)\|_{L^\infty} \leq \frac{1}{\varepsilon_0},
\]
for some $\varepsilon_0 > 0$. In fact, we will require more information on the solution, in the sense that we would like to find $s_0 > 0$ and initial data $w_0$ such that the solution $w$ of equation (2.2) with $w(s_0) = w_0$ satisfies
\[
\|w(y, s) - f\left(\frac{y}{\sqrt{s}}\right)\|_{W^{1,\infty}} \to_{s \to \infty} 0,
\]
where $f$ is the profile defined by
\[
f(z) = (p - 1 + \frac{(p - 1)^2}{4p}|z|^2)^{-\frac{1}{p - 1}}.
\]
As we have already pointed-out in [1], we don’t linearize (2.2) around $f\left(\frac{y}{\sqrt{s}}\right) + \frac{\kappa N}{2ps}$ as in the case of equation (1.5) treated in [25], since this function is not in the space $W^{1,\infty}_\beta$. In fact, we linearize equation (2.2) around a new profile
\[
\varphi(y, s) = f\left(\frac{y}{\sqrt{s}}\right) + \frac{\kappa N}{2ps}\chi_0\left(\frac{y}{g_\varepsilon(s)}\right),
\]
where $\kappa = (p - 1)^{-\frac{1}{p - 1}}$ is a stationary solution for equation (2.2), $\chi_0 \in C_0^\infty$ with $\text{supp}(\chi_0) \subset B(0, 2)$, $\chi_0 \equiv 1$ on $B(0, 1)$ and $g_\varepsilon(s) = s^{\frac{1}{2} + \varepsilon}$, where $\varepsilon$ is a fixed constant satisfying
\[
0 < \varepsilon < \min(1, \varepsilon_1\beta^{-1}),
\]
(we could have taken $\varepsilon = \frac{1}{2} \min(1, \varepsilon_1\beta^{-1})$).

Introducing
\[
v(y, s) = w(y, s) - \varphi(y, s),
\]
the problem is reduced to constructing a function $v$ such that
\[
\lim_{s \to +\infty} \|v(s)\|_{W^{1,\infty}} = 0.
\]
If $w$ satisfies equation (2.2), then $v$ satisfies the following equation
\[
\partial_s v = (\mathcal{L} + V)v + B(v) + R(y, s) + N(y, s),
\]
where

- the linear term $(\mathcal{L} + V)v$ is given by
  \[
  \mathcal{L}(v) = \Delta v - \frac{1}{2} y \cdot \nabla v + v \quad \text{with} \quad V(y, s) = p\varphi^{p - 1} - \frac{p}{p - 1},
  \]
- the nonlinear term is
  \[
  B(v) = |v + \varphi|^{p - 1}(v + \varphi) - \varphi^p - p\varphi^{p - 1}v,
  \]
the remainder term involving $\varphi$ is

$$R(y, s) = \Delta \varphi - \frac{1}{2} y \cdot \nabla \varphi - \frac{1}{p-1} \varphi^p - \partial_s \varphi,$$  \hspace{1cm} (2.10)$$

and the “new” term is

$$N(y, s) = \mu e^{-\gamma s} |\nabla v + \nabla \varphi| \int_{B(0,|y|)} |v + \varphi|^{q-1}.$$  \hspace{1cm} (2.11)$$

Let us introduce the following integral form of equation (2.7), for each $s \geq \sigma \geq s_0$:

$$v(s) = K(s, \sigma)v(\sigma) + \int_{\sigma}^{s} K(s, t)(B(v(t)) + R(t) + N(t))dt,$$  \hspace{1cm} (2.12)$$

where $K$ is the fundamental solution of the operator $L + V$.

Since the dynamics of (2.7) are influenced by the linear operator $L + V$, we first need to recall some of its properties (for more details, see [5]).

The operator $L$ is self-adjoint in $D(L) \subset L^2_{\mu} (\mathbb{R}^N)$, where

$$L^2_{\mu} (\mathbb{R}^N) = \{ v \in L^2_{loc}(\mathbb{R}^N); \int_{\mathbb{R}^N} (v(y))^2 \rho(y)dy < \infty \}, \quad \rho(y) = \frac{e^{-\frac{|y|^2}{4}}}{(4\pi)^{\frac{N}{2}}}.$$  

The spectrum of $L$ consists only in eigenvalues given by

$$\text{spec}(L) = \{1 - \frac{m^2}{2}; \quad m \in \mathbb{N}\},$$

and its eigenfunctions are rescaled Hermite polynomials.

If $N = 1$, all the eigenvalues are simple, and the eigenfunction corresponding to $1 - \frac{m^2}{2}$ is

$$h_m(y) = \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-2k)!} (-1)^k y^{m-2k}.$$  \hspace{1cm} (2.13)$$

In particular, $h_0(y) = 1$, $h_1(y) = y$ and $h_2(y) = y^2 - 2$ are the eigenfunctions corresponding to nonnegative eigenvalues $\lambda \in \{1, \frac{1}{2}, 0\}$. We remark that $h_m$ satisfies:

$$\int_{\mathbb{R}} h_n h_m \rho dx = 2^n n! \delta_{n,m}.$$  

We also introduce $k_m = \frac{h_m}{\|h_m\|_{L^2(\mathbb{R})}^2}$.

If $N \geq 2$, the eigenspace corresponding to $1 - \frac{m^2}{2}$ is given by

$$E_m = \{ h_{m_1}(y_1) \cdots h_{m_N}(y_N); \quad m_1 + \cdots + m_N = m \}.$$  

In particular, these are the eigenspaces corresponding to nonnegative eigenvalues:

$E_0 = \{1\}$, $E_1 = \{y_i; \quad i = 1, \cdots, N\}$ and $E_2 = \{h_2(y_i), y_i y_j; \quad i, j = 1, \cdots, N, \ i \neq j \}$.

The potential $V(y, s)$ has two fundamental properties:
• $V(\cdot, s) \to 0$ in $L^p_\rho$ as $s \to +\infty$. In particular, on bounded sets or in the "blow-up area" $\{|y| \leq K_0 \sqrt{s}\}$, where $K_0 > 0$ will be fixed large enough, the effect of $V$ is considered as a perturbation of the effect of $L$, except maybe on the null eigenspace of $L$.

• Outside the "blow-up area", we have the following property: for all $\eta > 0$ the exist $C_\eta > 0$ and $s_\eta$ such that

$$
\sup_{s \geq s_\eta, |y| \geq C_\varepsilon \sqrt{s}} |V(y, s) - (-\frac{p}{p-1})| \leq \eta.
$$

This means that $L + V$ behaves like $L - \frac{p}{p-1}$ in the region $\{|y| \geq K_0 \sqrt{s}\}$ for $K_0$ large enough. Since $-\frac{p}{p-1} < -1$ and 1 is the largest eigenvalue of $L$, we can consider the operator $L + V$ outside the blow-up area as an operator with a purely negative spectrum, which greatly simplifies the analysis.

Bearing in mind that the behavior of $V$ is not the same inside and outside the "blow-up" area, we decompose $v$ as follows. Let us first introduce the following cut-off function:

$$
\chi(y, s) = \chi_0\left(\frac{|y|}{K_0 \sqrt{s}}\right),
$$

where $K_0 > 0$ will be fixed large enough so that various technical estimates hold, and the cut-off function $\chi_0$ was already introduced after (2.4).

We write

$$
v(y, s) = v_b(y, s) + v_e(y, s),
$$

with

$$
v_b(y, s) = v(y, s) \chi(y, s), \quad v_e(y, s) = v(y, s)(1 - \chi(y, s)).
$$

We note that $\text{supp } v_b(s) \subset B(0, 2K_0 \sqrt{s})$ and $\text{supp } v_e(s) \subset \mathbb{R}^N \setminus B(0, K_0 \sqrt{s})$.

In order to control $v_b$, we decompose it according to the sign of the eigenvalues of $L$ as follows (for simplicity, we give the decomposition only for $N = 1$, bearing in mind that the situation for $N \geq 2$ is the same, except for some more complicated notations that can be found in Nguyen and Zaag [29], where the formalism in higher dimensions is extensively given):

$$
v(y, s) = v_b(y, s) + v_e(y, s) = \sum_{m=0}^{2} v_m(s) h_m(y) + v_-(y, s) + v_e(y, s),
$$

where for $0 \leq m \leq 2$, $v_m = P_m(v_b)$ and $v_-(s) = P_-(v_b)$, with $P_m$ being the $L^2_\rho$ projector on $h_m$, the eigenfunction corresponding to $\lambda = 1 - \frac{p}{2}$, and $P_-$ the projector on $\{h_i \mid i \geq 3\}$, the nonpositive subspace of the operator $L$. 

8
3 Existence

This section is devoted to the proof of the existence of a solution $v$ of (2.7) such that

$$
\lim_{s \to +\infty} \|v(s)\|_{W^{1,\infty}} = 0. \tag{3.1}
$$

We proceed exactly as in our previous paper [1], except for the definition of the shrinking set below in Definition 3.1 where we will introduce a smaller set, leading to a sharper estimate on the norm in (3.1), at the expense of more involved estimates. For simplicity in the notations, we give the proof only when $N = 1$.

As in our previous paper [1], our argument is a non trivial adaptation of the method performed of equation (1.5) by Bricmont and Kupiainen [5], Merle and Zaag [25], Nguyen and Zaag [31] and Tayachi and Zaag [34]. Accordingly, as in [1], we proceed in 4 steps:

- In the first step, we define a new shrinking set $V_{A,\varepsilon_1}(s)$ and translate our goal in (3.1) to the construction of a solution belonging to that set.

- In the second step, using the spectral properties, we reduce our goal from the control of $v(s)$ (an infinite dimensional variable) in $V_{A,\varepsilon_1}(s)$ to the control of its two first components $(v_0(s), v_1(s))$ in $[-\frac{A}{s^2}, \frac{A}{s^2}]^2$ (a two-dimensional variable).

- In the third step, we prove a parabolic regularity estimate, in order to obtain the convergence of the gradient term.

- In the last step, we solve the two-dimensional problem using index theory and we prove the existence of a solution $v$ of (2.7) belonging to the new shrinking set $V_{A,\varepsilon_1}(s)$.

In the following, we fix $\beta$ satisfying (1.4) and we denote by $C$ a generic positive constant, depending only on $p$, $\mu$, $\beta$ and $K_0$. Note that $C$ does not depend on $A$, $\varepsilon_1$ and $s_0$, the constants that will appear below.

3.1 A New shrinking set $V_{\beta,K_0,A,\varepsilon_1}(s)$

In this subsection, we introduce the new shrinking set, which is the main novelty with respect to our previous work [1]:

**DEFINITION 3.1** (A set shrinking to zero). For all $K_0 \geq 1$, $A \geq 1$, $0 < \varepsilon_1 \leq \frac{1}{2}$ and $s \geq 1$, we define $V_{\beta,K_0,A,\varepsilon_1}(s)$ (or $V_{A,\varepsilon_1}(s)$ for simplicity) as the set of all functions $g$ such that $(1 + |y|^3)g \in L^\infty(\mathbb{R}^N)$ and

$$
|g_k| \leq \frac{A}{s^2}, \ k = 0, 1, \ |g_2| \leq \frac{A^2 \log s}{s^2}, \ \|\frac{g}{1 + |y|^3}\|_{L^\infty} \leq \frac{A}{s^{1 - \varepsilon_1}}, \tag{3.2}
$$
\[ \|g_e\|_{L^\infty} \leq \frac{A^2}{s^{1-\varepsilon_1}}, \quad \|(1 + |y|^\beta)g_e\|_{L^\infty} \leq \frac{A^2}{s^{1-\frac{2}{2} - \varepsilon_1}}, \]  

(3.3)

where \(g_k, g_-, g_e\) are defined in (2.16) and (2.17).

**REMARK 3.2.**

1. Since \(p > 3\) and \(\beta\) satisfy (1.4), it follows that \(\beta < \frac{2}{p - 1} < 1\).

Therefore, since \(\varepsilon_1 \in (0, 1 - \frac{\beta}{2})\), it follows that the bounds on \(g_-\) and \(g_e\) go to zero as \(s \to \infty\).

2. When \(\varepsilon_1 = \frac{1}{2}\), we recover the shrinking set of our previous paper [1]. Since we will take \(\varepsilon_1\) arbitrarily small here, we clearly improve our estimates. This is the key novelty of our work.

Since this shrinking set is different from all the previous studies, we need to introduce new estimates. Let us first note that the set \(V_{A,\varepsilon_1}(s)\) is increasing (for fixed \(s, \beta, K_0, \varepsilon_1\)) with respect to \(A\) in the sense of inclusion. Let us now give some properties of the shrinking set in the following:

**PROPOSITION 3.3.** For all \(K_0 \geq 1, A \geq 1\) and \(0 < \varepsilon_1 \leq \frac{1}{2}\), there exists \(s_1(K_0, A, \varepsilon_1)\) such that, for all \(s \geq s_1(K_0, A, \varepsilon_1)\) and \(g \in V_{A,\varepsilon_1}(s)\), we have

\[ \|g\|_{L^\infty(|y| \leq 2K_0\sqrt{s})} \leq \frac{CA}{s^{1-\varepsilon_1}} \quad \text{and} \quad \|g\|_{L^\infty(\mathbb{R})} \leq \frac{CA^2}{s^{1-\varepsilon_1}}. \]  

(3.4)

\[ \|(1 + |y|^\beta)g\|_{L^\infty(|y| \leq 2K_0\sqrt{s})} \leq \frac{CA}{s^{1-\frac{2}{2} - \varepsilon_1}} \quad \text{and} \quad \|(1 + |y|^\beta)g\|_{L^\infty(\mathbb{R})} \leq \frac{CA^2}{s^{1-\frac{2}{2} - \varepsilon_1}}. \]  

(3.5)

**REMARK 3.4.** Here and throughout the paper, the generic constant \(C\) depends on \(p\) and \(N\), and may depend on \(K_0\) too.

We omit the proof of this proposition, since it is obtained by a simple modification of the proof of Proposition 3.8 in [1] page 14.

The construction of a solution \(v\) of (2.7) is based on a careful choice of initial data at time \(s_0 = -\log T\), as we do in the following:

**DEFINITION 3.5.** (Choice of the initial data) For all \(K_0 \geq 1, A \geq 1, s_0 = -\log T > 1\) and \(d_0, d_1 \in \mathbb{R}\), we consider the following function as initial data for equation (2.7):

\[ \psi_{s_0, d_0, d_1}(y) = \frac{A}{s_0^2} (d_0 h_0(y) + d_1 h_1(y)) \chi(2y, s_0), \]  

(3.6)

where \(h_i, i = 0, 1\) are defined in (2.13) and \(\chi\) is defined in (2.14).
Reasonably, we choose the parameter \((d_0, d_1)\) such that the initial data \(\psi_{s_0, d_0, d_1} \in \mathcal{V}_{A, \varepsilon_1}(s_0)\). More precisely, we claim the following result:

**Proposition 3.6.** (Properties of initial data) For each \(0 < \varepsilon_1 \leq \frac{1}{2}\), \(K_0 \geq 1\), \(A \geq 1\), there exists \(s_{01}(K_0, A, \varepsilon_1) > 1\) such that for all \(s_0 \geq s_{01}(K_0, A, \varepsilon_1)\):

i) There exists a rectangle \(D_{s_0} \subset [-2, 2]^2\) such that the mapping

\[
\Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2, \quad (d_0, d_1) \mapsto (\psi_0, \psi_1),
\]

(where \(\psi := \psi_{s_0, d_0, d_1}\)) is linear, one to one from \(D_{s_0}\) onto \([-\frac{A}{s_0}, \frac{A}{s_0}]^2\) and maps \(\partial D_{s_0}\) into \(\partial([-\frac{A}{s_0}, \frac{A}{s_0}]^2)\). Moreover, it has degree one on the boundary.

ii) For all \((d_0, d_1) \in D_{s_0}\), \(\psi \in \mathcal{V}_{A, \varepsilon_1}(s_0)\) with strict inequalities except for \((\psi_0, \psi_1)\), in the sense that

\[
\psi_e \equiv 0, \quad |\psi_-(y)| < \frac{1}{s_0^{2-\varepsilon_1}}(1+|y|^3), \quad \forall y \in \mathbb{R}, \quad |\psi_k| \leq \frac{A}{s_0}, \quad k = 0, 1, \quad |\psi_2| < \frac{\log s_0}{s_0^2}.
\]

iii) Moreover, for all \((d_0, d_1) \in D_{s_0}\), we have

\[
\|(1 + |y|^\beta) \nabla \psi\|_{L^\infty} \leq \frac{CA}{s_0^{2-\frac{\beta}{2}}} \leq \frac{1}{s_0^{1-\frac{\beta}{2}-\varepsilon_1}}, \quad (3.9)
\]

\[
|\nabla \psi_-(y)| \leq \frac{1}{s_0^{\frac{3}{2}-\varepsilon_1}}(1 + |y|^3). \quad (3.10)
\]

We omit the proof, since it is obtained by a simple modification of the proof of Proposition 4.5 of [34]. For more details, we refer the interested reader to pages 5915 – 5918 of [34], and pages 14 – 15 of [1].

### 3.2 Parabolic Regularity

As stated above, our goal is to get the convergence in \(W^{1, \infty}_\beta\). Thus, we need to prove that the solution \(v\) of (2.7) satisfies

\[
\lim_{s \rightarrow +\infty} \|(1 + |y|^\beta) \nabla v(s)\|_{L^\infty} = 0.
\]

More precisely, we need the following parabolic regularity result for equation (2.7):
PROPOSITION 3.7. (Parabolic regularity in $V_{A,\varepsilon_1}(s)$)
For all $0 < \varepsilon_1 \leq \frac{1}{2}$, $K_0 \geq 1$ and $A \geq 1$, there exists $s_{02}(K_0, A, \varepsilon_1)$ such that for all $s_0 \geq s_{02}(K_0, A)$, if $v$ is the solution of (2.7) for all $s \in [s_0, s_1]$, $s_0 \leq s_1$, with initial data at $s_0$, given in (3.6) with $(d_0, d_1) \in D_{s_0}$ defined in Proposition 3.6 and $v(s) \in V_{A,\varepsilon_1}(s)$, then, for all $s \in [s_0, s_1]$, we have

$$\|\nabla v(s)\|_{L^\infty} \leq \frac{CA^2}{s^{1-\varepsilon_1}} \quad \text{and} \quad \|(1 + |y|^\beta)\nabla v(s)\|_{L^\infty} \leq \frac{CA^2}{s^{1-\frac{\beta}{2}-\varepsilon_1}}. \quad (3.11)$$

The proof of the previous proposition follows exactly as in [1]. For that reason, the proof is omitted and we refer the reader to subsection 3.2.3 page 21-24 in [1] for details.

Since $v(s) \in V_{A,\varepsilon_1}(s)$, we clearly see from (3.5) and (3.11) that

$$\|(1 + |y|^\beta)v(s)\|_{L^\infty} + \|(1 + |y|^\beta)\nabla v(s)\|_{L^\infty} \leq \frac{CA^2}{s^{1-\frac{\beta}{2}-\varepsilon_1}}. \quad (3.12)$$

3.3 Reduction to a finite dimensional problem

The following section is crucial in the proof of the existence of the blow-up solution. In fact, we reduce here the construction question to a finite dimensional problem. As in [27], [10], [34] and [1], we prove that it is enough to control the 2 components $(v_0(s), v_1(s)) \in [-\frac{A}{s^2}, \frac{A}{s^2}]^2$ in order to control the solution $v(s)$ in $V_{A,\varepsilon_1}(s)$, which is infinite dimensional. Precisely, we have the following proposition.

PROPOSITION 3.8. (Control of $v(s)$ by $(v_0, v_1)(s)$ in $V_{A,\varepsilon_1}(s)$ ) For all $0 < \varepsilon_1 \leq \frac{1}{2}$, there exists $K_3(\varepsilon_1) \geq 1$ such that for any $K_0 \geq K_3(\varepsilon_1)$, there exists $A_3(K_0, \varepsilon_1) \geq 1$ such that for each $A \geq A_3(K_0, \varepsilon_1)$, there exists $s_{03}(K_0, A, \varepsilon_1) \in \mathbb{R}$ such that for all $s_0 \geq s_{03}(K_0, A, \varepsilon_1)$, the following holds:

If $v$ is a solution of (2.7) with initial data at $s = s_0$ given by (3.6) with $(d_0, d_1) \in D_{s_0}$, and $v(s) \in V_{A,\varepsilon_1}(s)$ for all $s \in [s_0, s_1]$, with $v(s_1) \in \partial V_{A,\varepsilon_1}(s_1)$ for some $s_1 \geq s_0$, then:

i) (Reduction to a finite dimensional problem) We have:

$$(v_0(s_1), v_1(s_1)) \in \partial([-\frac{A}{s^2}, \frac{A}{s^2}]^2).$$

ii) (Transverse crossing) There exist $m \in \{0, 1\}$ and $\omega \in \{-1, 1\}$ such that

$$\omega v_m(s_1) = \frac{A}{s^2} \quad \text{and} \quad \omega v'_m(s_1) > 0.$$
The remainder of this section is devoted to the proof of Proposition 3.8. We proceed in 2 steps:

- In the first step, we prove that if \( v(s) \in V_{A,\epsilon_1}(s) \), then \( B(v), R(y,s) \) and \( N(y,s) \) given in (2.7) belong \( V_{C,\epsilon_1}(s) \) and the potential term \( Vv(s) \in V_{CA,\epsilon_1}(s) \), for some positive constant \( C \).

- In the second step, we prove the result of the reduction to a finite dimensional problem.

3.3.1 Step 1: Preliminary estimates on various terms of equation (2.7)

In the following, we prove that the remainder term \( R(y,s) \) is in \( V_{C,\epsilon_1}(s) \) for \( s \) large enough and some \( C > 0 \). We also prove that if \( v(s) \in V_{A,\epsilon_1}(s) \), then the nonlinear term \( B(v) \) and the potential term \( Vv \) is trapped in \( V_{CA,\epsilon_1}(s) \). Furthermore, under some additional assumptions on \( v \), we prove that the new term \( N(y,s) \) is also trapped in \( V_{C,\epsilon_1}(s) \). More precisely, we prove the following result:

**Lemma 3.9.** 1. For all \( K_0 \geq 1 \) and \( 0 < \epsilon_1 \leq \frac{1}{2} \), there exists \( s_3(K_0, \epsilon_1) \) sufficiently large such that for all \( s \geq s_3(K_0, \epsilon_1) \), the remainder term \( R(s) \in V_{C,\epsilon_1}(s) \).

2. For all \( K_0 \geq 1 \), \( 0 < \epsilon_1 \leq \frac{1}{2} \) and \( A \geq 1 \), there exists \( s_4(K_0, A, \epsilon_1) \) sufficiently large such that for all \( s \geq s_4(K_0, A, \epsilon_1) \), if \( v(s) \in V_{A,\epsilon_1}(s) \), then the nonlinear term \( B(v) \) and the potential term \( Vv \) is trapped in \( V_{CA,\epsilon_1}(s) \).

We need the following technical lemma before proving Lemma 3.9.

**Lemma 3.10.** For all \( K_0 \geq 1 \), there exists \( s_2(K_0) \) such that for all \( s \geq s_2(K_0) \), the following holds:

1. if \( g(y) = 1 \) then \( \| g^{-}(y) \|_{L^\infty} \leq \frac{C}{K_0^2s^2} \),

2. if \( g(y) = y^2 \), then \( \| g^{-}(y) \|_{L^\infty} \leq \frac{C}{K_0\sqrt{s}} \).

**Proof.** The proof of this technical lemma follows exactly as in [34]. For that reason, the proof is omitted, and we refer the interested reader to Lemma 4.8 page 5916 – 5917 in [34] for a similar case.

Now, we are ready to prove Lemma 3.9.

**Proof of Lemma 3.9.** In comparison with [1], the estimates for \( B(v) \) and \( vV \) can be adapted smoothly (see subsection 3.2.2 page 14 – 17 in [1] and subsection 4.2.2 page 5918 – 5923 in [34]), unlike for \( R(y,s) \). For that reason, we only prove the estimate for the latter. Moreover, since the constraints on \( R_m(s) \) for \( m = 0, 1, 2 \) in our new shrinking set are the same as with the old shrinking set in [1] (see Remark 3.2, we...
refer the reader to Lemma 3.9 page 15 in that paper for the proof of the estimates on those components, and focus only on the proofs of the estimates related to \( R_-(y,s) \) and \( R_e(y,s) \).

Since \(-\frac{1}{2}zf'(z) - \frac{1}{p-1}f(z) + f(z)^p = 0\) by definition (2.3) of \( f(z) \), we write \( R(y,s) \) introduced in (2.10) as follows

\[
R(y,s) = \frac{1}{s} f''(z) + \frac{1}{2s} z f'(z) + \left( f(z) + \chi_0(Z) \frac{\kappa}{2ps} \right)^p - f(z)^p
\]

\[
+ \frac{\kappa}{2ps} \left[ \frac{1}{g_z} \chi''_0(Z) - \left( \frac{1}{2} - \frac{g'_z(s)}{g_z(s)} \right) \frac{Z \chi'_0(Z)}{s} + \left( \frac{1}{s} - \frac{1}{p-1} \right) \chi_0(Z) \right]
\]

\[= R_i + R_{ii}, \tag{3.13} \]

where \( z = \frac{y}{\sqrt{s}} \), \( Z = \frac{y}{g_z(s)} \), \( g_z(s) = s^{\frac{1}{2} + \varepsilon} \), and \( \varepsilon \in (0, \min(1, \varepsilon_1 \beta^{-1})) \) was already fixed in (2.5).

First, noting that \( f(z) \) is bounded by definition (2.3), then recalling that \( p > 3 \) (see (1.2)) and \( 0 \leq \chi_0(Z) \leq 1 \) (see right after (2.4)), we write

\[|f(z) + \chi_0(Z) \frac{\kappa}{2ps} - f(z)^p| \leq C \frac{s}{s}. \tag{3.14} \]

Noting that \( f''(z) \) and \( zf'(z) \) are also bounded, still by (2.3), then recalling that \( \chi_0 \) is smooth and compactly supported, we clearly see from (3.13) that

\[|R(y,s)| \leq C \frac{s}{s}. \tag{3.15} \]

In particular, by definition (2.16), we have for \( s \) large enough:

\[|R_e(y,s)| \leq C \frac{s}{s} \leq C \frac{s}{s^{1-\varepsilon_1}}. \]

Now, we estimate \( \|(1 + |y|^{\beta})R_e(y,s)\|_{L^\infty} \). Since we have the same profile as in [1], arguing as in pages 15-17 of [1], we easily prove the following estimate

\[\|(1 + |y|^{\beta})R_e(y,s)\|_{L^\infty} \leq C \frac{s}{s^{1-\frac{\beta}{2} - \varepsilon_1}}. \]

Since \( \varepsilon \beta \leq \varepsilon_1 \) by (2.3), we obtain for \( s \) large enough

\[\|(1 + |y|^{\beta})R_e(y,s)\|_{L^\infty} \leq C \frac{s}{s^{1-\frac{\beta - \varepsilon_1}{2}}}. \]

Finally, we bound \( R_-(y,s) \).

Refining the Taylor expansion (3.14) up to the second order, we write

\[
R_i = \frac{1}{s} \left( f''(z) + \frac{z}{2} f'(z) + \frac{\kappa}{2p} \chi_0(Z) f(z)^{p-1} \right) + \frac{p(p-1)}{2} \left( \frac{\kappa}{2ps} \right)^2 \chi_0^2(Z) f^{p-2}(z) + O \left( \frac{1}{s^3} \right) \tag{3.16} \]
Since $\chi_0$ is constant on the unit ball $B(0, 1)$ (see right after (2.4)), making a Taylor expansion of $R_i$ (3.16) in $z$ and in $Z$, we see that

$$R_i = \frac{1}{s} \left(a + bz^2 + O(z^3) \right) + \frac{c}{s^2} \left(1 + O(z^2) \right) + O \left(\frac{1}{s^3} \right) + O \left(\frac{|Z|^3}{s} \right)$$

for some real numbers $a$, $b$ and $c$. Similarly, we have from (3.13)

$$R_{ii} = \frac{\kappa}{2ps} \left(\frac{1}{s} - \frac{1}{p-1} \right) + O \left(\frac{|Z|^3}{s} \right).$$

Using Lemma 3.10 together with straightforward estimates, we see that

$$|R_i| \leq C s^5 \left(1 + |y|^3 \right) \leq C s^5 \left(1 + |y|^3 \right)$$

for $s$ large enough. This concludes the proof of Lemma 3.9.

We now estimate the new term defined in (2.11). For this end, we claim first, the following Lemma:

**Lemma 3.11.** For all $0 < \varepsilon_1 \leq \frac{1}{2}$, $K_0 \geq 1$ and $A \geq 1$, there exists $s_5(K_0, A, \varepsilon_1)$ sufficiently large, such that for all $s \geq s_5(K_0, A, \varepsilon_1)$, if $v \in V_{A, \varepsilon_1}(s)$ satisfies the following

$$\|\nabla v(s)\|_{L^\infty} \leq \frac{CA^2}{s^\tau}, \quad \|(1 + |y|^3)\nabla v(s)\|_{L^\infty} \leq \frac{CA^2}{s^{\tau'}},$$

for some positive constants $\tau$ and $\tau'$, then, the new term defined in (2.11) satisfies

1. $\|N(y, s)\|_{L^\infty} \leq Ce^{-\frac{2}{\tau} s}$,

2. $\| (1 + |y|^3) N(y, s) \|_{L^\infty} \leq Ce^{-\frac{2}{\tau} s}$,

where $C$ is a positive constant.

**Proof.** Since $V_{A, \varepsilon_1}$ is increasing (in the sense of inclusion) with respect to $\varepsilon_1$, recalling that $0 < \varepsilon_1 \leq \frac{1}{2}$ and that the case $\varepsilon_1 = \frac{1}{2}$ was treated in Abdelhedi and Zaag [1], we refer the reader to Lemma 3.11 page 18 of [1].

Thanks to Lemma 3.11, we obtain the following results:

**Proposition 3.12.** For all $0 < \varepsilon_1 \leq \frac{1}{2}$, $K_0 \geq 1$ and $A \geq 1$, there exists $s_6(K_0, A, \varepsilon_1)$ sufficiently large, such that for all $s \geq s_6(K_0, A, \varepsilon_1)$, if $v \in V_{A, \varepsilon_1}(s)$ and $\nabla v$ satisfies the estimate stated in Proposition 3.7, then, the new term $N$ defined in (2.11) satisfies

$$|N_m(s)| \leq Ce^{-\frac{2}{\tau} s}, \quad 0 \leq m \leq 2, \quad \|\frac{N_1(y, s)}{1 + |y|^3}\|_{L^\infty} \leq Ce^{-\frac{2}{\tau} s}$$

and $N \in V_{C, \varepsilon_1}(s)$, for some positive constant $C$.

**Proof.** The proof follows by standard estimates exactly as in the case $\varepsilon_1 = \frac{1}{2}$ treated in [1]. See Proposition 3.10 page 18 in that paper.
3.3.2  Step 2: Reduction to a finite dimensional problem

In this subsection, we reduce the problem to a finite dimensional one. We prove through a priori estimates that the control of $v(s)$ in $\mathcal{V}_{A,\varepsilon_1}(s)$ can be reduced to the control of $(v_0, v_1)(s)$ in $[-\frac{A}{2^{\varepsilon_1}}, \frac{A}{2^{\varepsilon_1}}]$. For this end, we recall the integral equation (2.12), for each $s \geq \sigma \geq s_0$:

$$
v(s) = K(s, \sigma)v(\sigma) + \int_{\sigma}^{s} K(s, t)(B(v(t)) + R(t) + N(t)) dt
= A(s) + B(s) + C(s) + D(s),
$$

(3.18)

and we estimate the different components of this Duhamel formulation.

In the first step, we recall from [1] this fundamental property concerning the kernel $K(s, \sigma)$:

**Lemma 3.13.** There exists $K_0 \geq 1$ such that for all $K_0 \geq K_5$, for all $\rho > 0$, there exists $\sigma_0 = \sigma_0(K_0, \rho)$ such that if $\sigma \geq \sigma_0 \geq 1$ and $g(\sigma)$ satisfies

$$
\sum_{m=0}^{2} |g_m(\sigma)| + \frac{g_r(y, \sigma)}{1 + |y|^3} \|L^\infty\| + \|(1 + |y|^2)g(\sigma)\|_{L^\infty} < +\infty,
$$

(3.19)

then $\theta(s) = K(s, \sigma)g(\sigma)$ satisfies for all $s \in [\sigma, \sigma + \rho]$,

1. $\|\theta(y, s)\|_{L^\infty} \leq C e^{s-\sigma}((s-\sigma)^2 + 1)\|g_0(\sigma)\| + |g_1(\sigma)| + \sqrt{s}|g_2(\sigma)|$

2. $|\theta(\sigma)| \leq C e^{s-\sigma}(\sum_{l=0}^{2} s^{\frac{l}{2}}|g_1(\sigma)|) + s^{\frac{3}{2}}\|g_r(y, \sigma)\|_{L^\infty} + C e^{-\frac{s}{2}} \|g(\sigma)\|_{L^\infty}$

3. $\|(1 + |y|^2)\theta(y, s)\|_{L^\infty} \leq C e^{-\frac{s-\sigma}{\rho+\frac{1}{2}} - \frac{\sigma}{2}} \|(1 + |y|^2)g(\sigma)\|_{L^\infty}$

$$
+ C e^{\frac{\sigma}{2}(s-\sigma)} s^{\frac{3}{2}}(\sum_{l=0}^{2} s^{\frac{l}{2}}|g_1(\sigma)|) + s^{\frac{3}{2}}\|g_r(y, \sigma)\|_{L^\infty}.
$$

**Proof.** The original version of this lemma is due to Bricmont and Kupiainen in [3], where they gave the proof in the case where some particular bounds hold on each component appearing in (3.19) only for $\beta = 0$. Later in [31], Nguyen and Zaag simply restated the lemma of Bricmont and Kupiainen and checked that the proof of [5] works without those particular bounds, still for $\beta = 0$. In our earlier paper [1], we handled the case $0 \leq \beta < \frac{2}{p-1}$. Accordingly, we refer the reader to Lemma 3.15 page 26 in [1] for the detailed justification. □
Applying the above Lemma, we get a new bound on all terms in the decomposition (3.18). More precisely, we have the following results:

**Lemma 3.14.** For all $0 < \varepsilon_1 \leq \frac{1}{2}$, there exists $K_0(\varepsilon_1)$ such that for any $K_0 \geq K_0(\varepsilon_1)$, there exists $A_0(K_0, \varepsilon_1) > 0$ such that for all $A \geq A_0(K_0, \varepsilon_1)$ and $\rho > 0$, there exists $s_0(0, A, \rho, \varepsilon_1) > 0$ with the following property: for all $s_0 \geq s_0(0, A, \rho, \varepsilon_1)$ assume that for all $s \in [\sigma, \sigma + \rho]$, $v(s)$ satisfies (2.7), $v(s) \in V_{A, \varepsilon_1}(s)$ and $\nabla v$ satisfies the estimates stated in Proposition 3.7. Then, we have:

1. **Linear term:**
   
   $$
   \frac{A - (y, s)}{1 + |y|^3} \leq C e^{-\frac{v_-(s, \sigma)}{1 + |y|^3}} L_{\infty} + C e^{-\frac{v_-(s, \sigma)}{s^2}} \|v(\sigma)\|_{L_{\infty}} + \frac{C}{s^2 - \varepsilon_1}.
   $$

2. **Nonlinear term:**
   
   $$
   \frac{B_-(y, s)}{1 + |y|^3} \leq C \frac{e^\frac{\beta}{2}(s, \sigma)}{s^2 - \varepsilon_1}(s - \sigma), \quad \frac{B_-(s)}{1 + |y|^3} \leq C \frac{e^\frac{\beta}{2}(s, \sigma)}{s^2 - \varepsilon_1}(s - \sigma) e^{s - \sigma}.
   $$

3. **Remainder term:**
   
   $$
   \frac{C_-(y, s)}{1 + |y|^3} \leq C \frac{e^\frac{\beta}{2}(s, \sigma)}{s^2 - \varepsilon_1}(s - \sigma), \quad \frac{C_-(s)}{1 + |y|^3} \leq C \frac{e^\frac{\beta}{2}(s, \sigma)}{s^2 - \varepsilon_1}(s - \sigma) e^{s - \sigma}.
   $$

4. **New term:**
   
   $$
   \frac{D_-(y, s)}{1 + |y|^3} \leq C(s - \sigma) e^{s - \sigma} e^{-\frac{\beta}{2}s}, \quad \|D_-(s)\|_{L_{\infty}} \leq C(s - \sigma) e^{s - \sigma} e^{-\frac{\beta}{2}s}.
   $$

   $$
   (1 + |y|^3) D_-(s) \|_{L_{\infty}} \leq C(s - \sigma) e^{-\frac{\beta}{2}s} (1 + s^2 e^{\frac{\beta}{2}(s - \sigma)}).
   $$
Proof. Although we have a different shrinking set here, in comparison with our earlier work [1], the proof follows exactly in the same way. See Lemma 3.16 page 30 in that paper for details.

In the following statement, we add the bounds of Lemma 3.14 in order to obtain bounds on \( v(y, s) \), thanks to the expression (3.18). We also project equation (2.7) in order to write differential inequalities satisfies by \( v_m \) for \( m = 0, 1 \) and 2:

**Proposition 3.15.** For all \( 0 < \varepsilon_1 \leq \frac{1}{2} \), there exists \( K_7(\varepsilon_1) \) such that for any \( K_0 \geq K_7(\varepsilon_1) \), there exists \( A_7(K_0, \varepsilon_1) \geq 1 \) such that for all \( A \geq A_7(K_0, \varepsilon_1) \), there exists \( s_0(A, K_0, A, \varepsilon_1) \) large enough such that the following holds for all \( s \geq s_0(A, K_0, A, \varepsilon_1) \):

Assume that for some \( s_1 \geq \sigma \geq s_0 \), we have

\[
v(s) \in V_{A, \varepsilon_1}(s), \quad \text{for all } s \in [\sigma, s_1],
\]

and that \( \nabla v \) satisfies the estimates stated in Proposition 3.7. Then, the following holds for all \( s \in [\sigma, s_1] \):

i) (ODE satisfied by the positive modes) : For \( m \in \{0, 1\} \), we have

\[
|v'_m(s) - (1 - \frac{m}{2})v_m(s)| \leq C \frac{s^2}{s^2}.
\]

ii) (ODE satisfied by the null mode) : We have

\[
|v'_2(s) + \frac{2}{s^2} v_2(s)| \leq C \frac{s^3}{s^3}.
\]

iii) (Control of the negative and outer modes): We have

\[
\begin{align*}
\|v_-(s)\|_{L^\infty} &\leq Ce^{-\frac{s \sigma}{2}} \|v_-(\sigma)\|_{L^\infty} + Ce^{\frac{(s - \sigma)^2}{s^2}} \|v_-(\sigma)\|_{L^\infty} + C \frac{1 + s - \sigma}{s^{2 - \varepsilon_1}}, \\
\|v_+(s)\|_{L^\infty} &\leq Ce^{-\frac{s \sigma}{2}} \|v_+(\sigma)\|_{L^\infty} + Ce^{\frac{(s - \sigma)}{s^2}} \|v_+(\sigma)\|_{L^\infty} + C \frac{1 + (s - \sigma)e^{s - \sigma}}{s^{1 - \varepsilon_1}}.
\end{align*}
\]

iv) (Weighted estimate of the outer mode): We have

\[
\begin{align*}
\|v_+(s)\|_{L^\infty} &\leq C e^{-\frac{s \sigma}{2}(\frac{1}{p^2 - 1} - \frac{2}{p})} \|v_+(\sigma)\|_{L^\infty} + Ce^{\frac{(s - \sigma)}{s^2}} \|v_+(\sigma)\|_{L^\infty} + C \frac{1 + (s - \sigma)e^{s - \sigma}}{s^{1 - \varepsilon_1}}.
\end{align*}
\]

Proof. The proof of items iii) and iv) is straightforward from Lemma 3.14 and the decomposition (3.18).

As for items i) and ii), they follow exactly in the same way as in any paper about the standard heat equation (\( \mu = 0 \)), since the effect of the “new term” is exponentially small, thanks to Lemma 3.11. See for example Lemma 3.10 page 1293 of Nguyen and Zaag [29].
Now, with Proposition 3.15, which gives estimates on the different components of the decomposition (3.18), the proof of Proposition 3.8 follows exactly as in our earlier work [1]. For that reason, we omit the proof and refer the reader to page 31 of that paper, where the analogous statement (numbered Proposition 3.5) is proved.

3.4 Proof of the existence of a solution in $V_{A,\varepsilon_1}(s)$

In this subsection, we solve the two-dimensional problem using index theory and we prove the existence of a solution $v$ of (2.7) in $V_{A,\varepsilon_1}(s)$. More precisely, we have the following statement:

**PROPOSITION 3.16** (Existence of a solution in the shrinking set). For all $0 < \varepsilon_1 \leq \frac{1}{2}$, there exists $K_4(\varepsilon_1) \geq 1$ such that for all $K_0 \geq K_4(\varepsilon_1)$, there exists $A_4(K_0, \varepsilon_1) \geq 1$ such that for all $A \geq A_4(K_0, \varepsilon_1)$ there exists $s_0(K_0, A, \varepsilon_1)$ such that for all $s \geq s_0(K_0, A, \varepsilon_1)$, there exists $(d_0, d_1)$ such that if $v$ is the solution of (2.7) with initial data at $s_0$, given in (3.6), then $v(s) \in V_{A,\varepsilon_1}(s)$, for all $s \geq s_0$.

**Proof.** The derivation of this proposition from the reduction to a finite dimensional problem and the transverse crossing (both given in Proposition 3.8) is classical in the literature concerning construction of solutions with prescribed behavior. For that reason, we omit it and refer the reader to page 13 of [1] where an analogous statement is given (namely, Proposition 3.7). The proof in that paper holds here with natural modifications. \qed

4 Proof of the main result

This section is dedicated to the proof of Theorem 1.1.

**Proof of Theorem 1.1.** Consider some $\varepsilon_1 \in (0, \frac{1}{2})$ and $\alpha \in (0, \frac{1}{2})$. Using Proposition 3.16, we derive the existence of a solution $v$ of equation (2.7) defined for all $y \in \mathbb{R}$ and $s \geq s_0$, such that $v(s) \in V_{A,\varepsilon_1}(s)$ for some $s_0 \geq 1$ and $A > 0$. Moreover, using (3.12), we have for all $s \geq s_0$,

$$
\|(1 + |y|^\beta)v(s)\|_{L^\infty} + \|(1 + |y|^\beta)\nabla v(s)\|_{L^\infty} \leq \frac{C(A)}{s^{1 - \frac{\beta}{2} - \varepsilon_1}},
$$

where $C(A) = CA^2$. For simplicity, we omit the dependence on $A$ in the following.

By definition (2.6) of $v$, we see that for all $s \geq s_0$

$$
\|(1 + |y|^\beta)(w(y, s) - \varphi(y, s))\|_{L^\infty} + \|(1 + |y|^\beta)\nabla_y (w(y, s) - \varphi(y, s))\|_{L^\infty} \leq \frac{C}{s^{1 - \frac{\beta}{2} - \varepsilon_1}},
$$

where $\varphi$ is the profile introduced in (2.4).

First, we estimate this profile. We remark that our "intermediate" profile satisfies:

$$
|f\left(\frac{y}{\sqrt{s}}\right)| \sim \left(\frac{s}{b|y|^2}\right)^{\frac{1}{\beta+1}} \text{ as } \frac{y}{\sqrt{s}} \to +\infty,
$$
and for all $K_0 \geq 1$ and $|y| \geq 2K_0\sqrt{s}$, we have
\[
\left| \frac{1}{\sqrt{s}} \nabla f \left( \frac{y}{\sqrt{s}} \right) \right| \leq \frac{C}{(1 + |y|^\beta) s^{\frac{1}{2} - \frac{\beta}{2}}},
\]
on the one hand (remember that $\beta < 2/(p - 1)$). On the other hand, the cut-off function $\chi_0$ satisfies for all $y \in \mathbb{R}$,
\[
\left| \frac{\kappa N}{2 ps} \chi_0 \left( \frac{y}{g_\varepsilon(s)} \right) \right| \leq \frac{C}{(1 + |y|^\beta) s^{1 - \frac{\beta}{2} - \varepsilon}}, \quad \left| \frac{\kappa N}{2 ps} \nabla \chi_0 \left( \frac{y}{g_\varepsilon(s)} \right) \right| \leq \frac{C}{(1 + |y|^\beta) s^{\frac{1}{2} - \frac{\beta}{2} - \varepsilon}}.
\]
With the choice of $\varepsilon \in (0, \min(1, \varepsilon_1 \beta^{-1}))$ (see (2.5)), we obtain
\[
\left| \frac{\kappa N}{2 ps} \chi_0 \left( \frac{y}{g_\varepsilon(s)} \right) \right| \leq \frac{C}{(1 + |y|^\beta) s^{1 - \frac{\beta}{2} - \varepsilon_1}}, \quad \left| \frac{\kappa N}{2 ps} \nabla \chi_0 \left( \frac{y}{g_\varepsilon(s)} \right) \right| \leq \frac{C}{(1 + |y|^\beta) s^{\frac{1}{2} - \frac{\beta}{2} - \varepsilon_1}}.
\]
Using the above estimates, we deduce that
\[
\| (1 + |y|^\beta)(w(y, s) - f(\frac{y}{\sqrt{s}})) \|_{L^\infty} + \| (1 + |y|^\beta) \nabla_y (w(y, s) - f(\frac{y}{\sqrt{s}})) \|_{L^\infty} \leq \frac{C}{s^{\frac{1}{2} - \frac{\beta}{2} - \varepsilon_1}}.
\]
Therefore, thanks to the similarity variables transformation (2.1), the solution $u$ of equation (1.1) defined for all $x \in \mathbb{R}$ and $t \in [0, T)$ satisfies
\[
|(T - t)^{\frac{1}{p-1}} u(x, t) - f(\frac{x}{\sqrt{(T - t)|\log(T - t)|}}) \right| \leq \frac{C}{(1 + (\frac{|x|^2}{T - t})^{\frac{\beta}{2}})|\log(T - t)|^{1 - \frac{\beta}{2} - \varepsilon_1}},
\]
and
\[
|(T - t)^{\frac{1}{2} - \frac{\beta}{2}} \nabla u(x, t) - \frac{1}{\sqrt{|\log(T - t)|}} \nabla f \left( \frac{x}{\sqrt{(T - t)|\log(T - t)|}} \right) \right| \leq \frac{C}{(1 + (\frac{|x|^2}{T - t})^{\frac{\beta}{2}})|\log(T - t)|^{1 - \frac{\beta}{2} - \varepsilon_1}}.
\]
and identities (1.13) and (1.14) follow. In particular, $\lim_{t \to T} (T - t)^{\frac{1}{p-1}} u(0, t) = (p - 1)^{\frac{1}{p-1}}$ and $u$ blows up at time $t = T$ at the origin.

Since our results in item 1) of this Theorem are better that our earlier result in (1), the single point blow-up property for $u$ and $\nabla u$ follows exactly as in our paper. If for $u$, we clearly have from (1.13) that $u(0, t) \to \infty$ as $t \to T$, the question to find some $\xi(t) \to 0$ such that $|\nabla u(\xi(t), t)| \to 0$ was left open after our previous work. In this new work, we are able to prove it, as we have already stated in the third remark following the statement of Theorem (1.1) and as we show in the following: If we choose $\varepsilon_1$ such that $\varepsilon_1 \leq \frac{1}{2} - \frac{\beta}{2}$, then we have
\[
(T - t)^{\frac{1}{p-1}} \sqrt{(T - t)|\log(T - t)|} \nabla u(\sqrt{(T - t)|\log(T - t)|}, t) \sim \nabla f(K_0) \quad \text{as } t \to T.
\]
In particular,
\[ \| \nabla u(t) \|_{L^\infty} \geq \frac{C}{(T - t)^{\frac{1}{p-1}} \sqrt{(T - t) | \log(T - t) |}}. \]

Thus \( \nabla u \) blows up at time \( t = T \) at the origin. This concludes the proof of parts 1) and 2) of Theorem 1.1.

Finally, for the proof of part 3) of Theorem 1.1 the reader can adapt easily the proof in [2] (section 4 pages 2618-2622), where we replace the cut off function in [2] (step 1 page 2619), by \( \phi_r(\xi) = \phi(r \frac{\xi}{|\log(T - t_0)|^\alpha}) \), \( 0 < \alpha < \frac{1}{2} \) and the bounded in estimate (4.1) in [2] page 2618, by \( |\log(T - t_0)|^{1-\varepsilon_1} \), \( 0 < \varepsilon_1 \leq \frac{1}{2} \). \( \square \)

References

[1] B. Abdelhedi and H. Zaag, Construction of a blow-up solution for a perturbed nonlinear heat equation with a gradient term, J. Differential Equations, 272 (2021) 1-45.

[2] B. Abdelhedi and H. Zaag, Single point blow-up and final profile for a perturbed nonlinear heat equation with a gradient term, Discrete Contin. Dyn. Syst. Ser. S (2021), 14(8): 2607-2623.

[3] J. M. Ball, Remarks on blow-up and nonexistence theorems for nonlinear evolution equations. Quart. J. Math. Oxford Ser.(2), 28(112)4 :73-486, 1977.

[4] M. Berger and R. V. Kohn. A rescaling algorithm for the numerical calculation of blowing-up solutions. Comm. Pure Appl. Math., 41(6):841-863, 1988.

[5] J. Bricmont and A. Kupiainen, Universality in blow-up for nonlinear heat equations. Nonlinearity, 7 (1994), 539-575.

[6] C. Collot, T. Ghoul, N. Masmoudi and V. T. Nguyen, Refined description and Stability of singular solutions for the two dimensional Keller-Segel system, Comm. Pure Appl. Math., 2020 (to appear).

[7] M. del Pino, M. Musso, and J. Wei. Infinite-time blow-up for the 3-dimensional energy-critical heat equation. Anal. PDE, 13(1):215–274, 2020.

[8] G.K. Duong and H. Zaag, Profile of touch-down solution to a nonlocal MEMS model. Math. Models Methods Appl. Sci. 29 (2019), no. 7, 1279-1348.

[9] G.K. Duong, T.E. Ghoul and H. Zaag. Sharp equivalent for the blowup profile to the gradient of a solution to the semilinear heat equation. submitted. arXiv:2109.03497.
[10] M. A. Ebde and H. Zaag, Construction and stability of a blow up solution for a nonlinear heat equation with a gradient term, SEMA J., 55 (2011), 5-21.

[11] S. Filippas and R. V. Kohn, Refined asymptotics for the blow-up of $u_t - \Delta u = u^p$. Comm. Pure Appl. Math., 45(7): 821-869, 1992.

[12] S. Filippas and W. X. Liu. On the blowup of multidimensional semilinear heat equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 10(3) :313-344, 1993.

[13] H. Fujita, On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sect. I, 13 : 109-124, 1966.

[14] V. A. Galaktionov and J. L. Vazquez, Regional blow-up in a semilinear heat equation with convergence to a Hamilton-Jacobi equation. SIAM J. Math. Anal., 24 (1993), 1254-1276.

[15] V. A. Galaktionov and J. L. Vazquez, Blow-up for quasilinear heat equations described by means of nonlinear Hamilton-Jacobi equations. J. Differential Equations, 127 (1996), 1-40.

[16] Y. Giga and R. V. Kohn, Asymptotically self-similar blow-up of semilinear heat equations. Comm. Pure Appl. Math., 38(3): 297-319, 1985.

[17] Y. Giga and R. V. Kohn, Characterizing blowup using similarity variables. Indiana Univ. Math. J., 36(1): 1-40, 1987.

[18] Y. Giga and R. V. Kohn, Nondegeneracy of blowup for semilinear heat equations. Comm. Pure Appl. Math., 42(6): 845-884, 1989.

[19] M. A. Herrero and J. J. L. Velázquez, Flat blow-up in one-dimensional semilinear heat equations. Differential Integral Equations, 5(5):973-997, 1992.

[20] M. A. Herrero and J. J. L. Velázquez, Generic behaviour of one-dimensional blow up patterns. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 19(3):381-450, 1992.

[21] M. A. Herrero and J. J. L. Velázquez, Blow-up behaviour of one-dimensional semilinear parabolic equations. Ann. Inst. H. Poincaré Anal. Non Linéaire, 10(2):131-189, 1993.

[22] M. A. Herrero and J. J. L. Velázquez, Comportement générique au voisinage d’un point d’explosion pour des solutions d’équations paraboliques unidimensionnelles. C. R. Acad. Sci. Paris Sér. I Math., 314(3):201-203, 1992.

[23] F. Mahmoudi, N. Nouaili, and H. Zaag, Construction of a stable periodic solution to a semilinear heat equation with a prescribed profile, Nonlinear Anal. 131 (2016), 300-324.
[24] F. Merle, P. Raphael, I. Rodnianski, J. Szeftel, On blow up for the energy super critical defocusing non linear Schrödinger equations, 2019, arXiv:1912.11005.

[25] F. Merle and H. Zaag, Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. Duke Math. J., 86(1):143-195, 1997.

[26] F. Merle and H. Zaag, Optimal estimates for blowup rate and behavior for nonlinear heat equations. Comm. Pure Appl. Math., 51(2):139-196, 1998.

[27] F. Merle and H. Zaag, Refined uniform estimates at blow-up and applications for nonlinear heat equations. Geom. Funct. Anal., 8(6):1043-1085, 1998.

[28] V. T. Nguyen, Numerical analysis of the rescaling method for parabolic problems with blow-up in finite time. Physica D Nonlinear Phenomena 339 (2017), 49-65.

[29] V.T. Nguyen and H. Zaag, Finite degrees of freedom for the refined blow-up profile of the semilinear heat equation. Ann. Scient. Éc. Norm. Supér (4). 50:5 (2017), 1241-1282.

[30] V. T. Nguyen and H. Zaag, Blow-up results for a strongly perturbed semilinear heat equation: Theoretical analysis and numerical method. Anal. PDE 9 (2016), no. 1, 229-257.

[31] V. T. Nguyen and H. Zaag, Contruction of a stable blow-up solution for a class of strongly perturbed semilinear heat equation. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 14 (2016), no. 4, 1275-1314.

[32] N. Nouaili and H. Zaag, Construction of a blow-up solution for the complex Ginzburg-Landau equation in a critical case. Arch. Ration. Mech. Anal., 228(3):995-1058, 2018.

[33] P. Souplet, S. Tayachi, and F. B. Weissler, Exact self-similar blow-up of solutions of a semilinear parabolic equation with a nonlinear gradient term. Indiana Univ. Math. J., 45(3):655-682, 1996.

[34] S. Tayachi and H. Zaag, Existence of a stable blow-up profile for the nonlinear heat equation with a critical power nonlinear gradient term. Trans. Amer. Math. Soc., 371 (2019), 5899-5972.

[35] F. B. Weissler, Single point blow-up for a semilinear initial value problem. J. Differential Equations, 55(2): 204-224, 1984.