The phase diagram of the one-dimensional quantum sine-Gordon system \( \beta^2 = 4\pi \) with a linear spatial modulation

Zhiguo Wang, Yumei Zhang

Department of Physics, Tongji University, Shanghai, 200092, China

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Abstract

The one-dimensional quantum sine-Gordon system with a linear spatial modulation is investigated in a special case, $\beta^2 = 4\pi$. The model is transformed into a massive Thirring model and then is exactly diagonalized, the energy spectrum of the model is obtained. Our result clearly demonstrates that cancelling the cosine term without any considering is unadvisable.

The one-dimensional (1D) quantum sine-Gordon model may probably be the most useful quantum model since it can be used to describe the most of the one- and two-dimensional (2D) models of either fermi or bose system, this fact attaches particular importance to the quantum sine-Gordon model. Many works have been done about this model both in field theory and in condensed matter physics. It is exactly solvable by quantum inverse scattering method. By a variational method Coleman first discovered that the energy density of the system is unbounded below when the coupling constant $\beta^2$ exceeds $8\pi$, so there is a phase transition as the coupling constant varies. This corresponds to the Kosterlitz-Thouless (K-T) phase transition by the equivalence of the 2D Coulomb gas and sine-Gordon model. The soliton mode of the sine-Gordon model corresponds to a one-fermion excitation in the fermi picture, which was clarified soon later by Mandelstam by introducing a Fermi-Bose relation.

As there are other spatial variations in the cosine potential, the low energy properties of the 1D quantum sine-Gordon system are more difficult to be analyzed. Here we shall discuss a simple case that there are a linear spatial modulation in the cosine potential. The Hamiltonian reads as

$$H = \int \left\{ \frac{1}{2} \left[ \Pi^2(x) + \left( \frac{\partial \phi(x)}{\partial x} \right)^2 \right] - \frac{\alpha}{\beta^2} \cos (\beta \phi(x) + \lambda x) \right\} dx , \tag{1}$$

here $\phi(x)$ is a bose field operator, $\Pi(x) = -i \frac{\delta}{\delta \phi(x)}$ is its conjugate momentum, they satisfy the commutation relation

$$[\phi(x), \Pi(y)] = i \delta(x - y) . \tag{2}$$
\( \lambda \) is a spatial modulated parameter. In condensed matter physics \( \lambda \) represents the fermi surface shifting from half filling in the fermi picture.

In the presence of finite \( \lambda \), most previous works suggested that the above Hamiltonian (1) describes a massless free field and cancelled the cosine potential directly. Has the cosine potential really no effect on the system in this case?

Schulz discussed a one-dimensional quantum sine-Gordon system with an additional gradient term. If we take a shift, \( \phi + \frac{4m\pi x}{\beta a} \to \phi \), the above Hamiltonian (1) is same as that of Schulz. But if one pays attention to the boundary condition of bose operator \( \phi \), he will find both model are different since the boundary conditions of system will be altered under this shift, namely, \( \int_{0}^{L} \nabla \phi(x) dx = 0 \to \int_{0}^{L} \nabla \phi(x) dx = \frac{\lambda L}{\beta} \), therefore the eigenstates will be also changed. Schultz pointed out that a commensurate-incommensurate transition happens in case of finite coefficient of the gradient term, this result implies that directly cancelling the cosine potential is unsuited.

In order to find an unquestionable answer for this, we investigate this Hamiltonian in a special case, \( \beta^2 = 4\pi \). First we transform the model into the massive Thirring model by the operator identities between fermions and bosons, and then exactly diagonalize the model using the bogliubov transformation. When the cosine potential may be omitted is obvious in our results.

For the case of a finite \( \lambda \), the above system has rarely been discussed before. After we use the bose-fermi relations

\[
\frac{1}{2} \int : \Pi^2(x) + \left( \frac{\partial \phi(x)}{\partial x} \right)^2 : dx =
\]

\[
- i \int \left[ \psi_1^\dagger(x) \frac{\partial}{\partial x} \psi_1(x) - \psi_2^\dagger(x) \frac{\partial}{\partial x} \psi_2(x) \right] \, dx ,
\]

\[
\cos \left( 2\sqrt{\pi} \phi(x) \right) = \pi \epsilon \left[ \psi_1^\dagger(x) \psi_2(x) + \psi_2^\dagger(x) \psi_1(x) \right] ,
\]

the Bose Hamiltonian (1) in the case \( \beta^2 = 4\pi \) can be transformed into a modified massive Thirring model.
\[ H = \int \left\{ -i \left[ \psi_1^\dagger(x) \frac{\partial}{\partial x} \psi_1(x) - \psi_2^\dagger(x) \frac{\partial}{\partial x} \psi_2(x) \right] \right. \\
- \left. \frac{\alpha \epsilon}{4} \left[ e^{i\lambda x} \psi_1^\dagger(x) \psi_2(x) + e^{-i\lambda x} \psi_2^\dagger(x) \psi_1(x) \right] \right\} dx \, . \] (5)

Here \( \epsilon \) is an infinitesimal positive parameter, it is of the order of the lattice constant in condensed matter physics. With following Fourier transformations
\[ c_{1k} = \frac{1}{\sqrt{L}} \int \psi_1(x) e^{i(k-\frac{\lambda}{2})x} dx \, , \]
\[ c_{2k} = \frac{1}{\sqrt{L}} \int \psi_2(x) e^{i(k+\frac{\lambda}{2})x} dx \, , \] (6)
the Hamiltonian (5) is rewritten as
\[ H = \sum_k \left[ (k - \frac{\lambda}{2}) c_{1k}^\dagger c_{1k} - (k + \frac{\lambda}{2}) c_{2k}^\dagger c_{2k} \right] - \frac{\alpha \epsilon}{4} \sum_k \left[ c_{1k}^\dagger c_{2k} + c_{2k}^\dagger c_{1k} \right] \, , \] (7)
where \( \sum \) denotes the summation over momentum which is cut off at \( \Lambda(\sim 1/\epsilon) \).

In order to diagonalize one particle terms we apply a Bogolubov transformation to the Hamiltonian (7)
\[ c_{1k} = u_k \beta_k + v_k \alpha_k^\dagger \, , \]
\[ c_{2k} = u_k \alpha_k^\dagger - v_k \beta_k \, . \] (8)
The standard technique gives us a quasi-particle Hamiltonian
\[ H = \sum_k \left[ \left( \sqrt{k^2 + \left( \frac{\alpha \epsilon}{4} \right)^2} - \frac{\lambda}{2} \right) \beta_k^\dagger \beta_k + \left( \sqrt{k^2 + \left( \frac{\alpha \epsilon}{4} \right)^2} + \frac{\lambda}{2} \right) \alpha_k^\dagger \alpha_k \right] \, . \] (9)

The quasi-particle creation operator \( \beta_k^\dagger \) and \( \alpha_k^\dagger \) correspond to two different quasi-particles such as the hole or particle. But now the excitations of them is not same. When \( \alpha \epsilon > 2\lambda \), both have excitation gap, while the excitation of \( \beta_k^\dagger \) is gapless in the case \( \alpha \epsilon < 2\lambda \). This shows that the original model can be treated as massless free system with spatial modulation when the parameters satisfy
\[ \alpha \epsilon < 2\lambda \, . \] (10)
Otherwise, the original model can not be treated as a massless system. In this case, one should be always aware of that the micro-structure of the ground state is essentially different with that of $\lambda = 0$ case.

In summary, we have concluded that the cosine potential term can be cancelled, i.e., the model can be considered as a massless free system, when the parameters satisfy equation (10). Although we only discussed in a special case that $\beta^2 = 4\pi$, it is sufficient for showing that cancelling the cosine potential term is unadvisable without considering the varying of parameters.
REFERENCES

[1] E.K. Sklyanin, L.A. Takhtadzhyan and L.D. Faddeev, Theor.Math.Phys., 40, 688(1980).

[2] P.Minnhagen, Phys.Rev.Lett., 54, 2351(1985); Phys.Rev., B32, 3088(1985); Rev.Mod.Phys., 59, 1001(1987).

[3] E.Fradkin, Field Theories of Condensed Matter Systems, Addison-Wesley, 1991.

[4] A.O. Gogolin, A.A. Nersesyan and A.M.Tsvelik, Bosonization and Strongly Correlated Systems, Cambridge University Press, 1998.

[5] S. Coleman, Phys.Rev. D11, 2088(1975).

[6] R.Ingermanson, Nucl.Phys., B266, 620(1986).

[7] S. Mandelstam, Phys.Rev., D11, 3026(1975).

[8] P.M. Stevenson, Phys.Rev., D32, 1389(1985).

[9] P. Sun and D. Schmeltzer, Phys.Rev., B61, 349(2000).

[10] H.J. Schulz, Phys. Rev., B 22, 5274 (1980).

[11] S. Takada and S. Misawa, Pro.Theor.Phys. 66, 101(1981).