On the sign problem in 2D lattice super Yang–Mills

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ABSTRACT: In recent years a new class of supersymmetric lattice theories have been proposed which retain one or more exact supersymmetries for non-zero lattice spacing. Recently there has been some controversy in the literature concerning whether these theories suffer from a sign problem. In this paper we address this issue by conducting simulations of the $\mathcal{N} = (2,2)$ and $\mathcal{N} = (8,8)$ supersymmetric Yang–Mills theories in two dimensions for the $U(N)$ theories with $N = 2, 3, 4$, using the new twisted lattice formulations. Our results provide evidence that these theories do not suffer from a sign problem in the continuum limit. These results thus boost confidence that the new lattice formulations can be used successfully to explore non-perturbative aspects of four-dimensional $\mathcal{N} = 4$ supersymmetric Yang–Mills theory.

KEYWORDS: Lattice Field Theory, Supersymmetric Gauge Theory, Topological Field Theories, Extended Supersymmetry, AdS/CFT.
1. Introduction

Supersymmetric Yang-Mills (SYM) theories are interesting from a variety of perspectives; as toy models for understanding theories such as QCD, as potential theories of BSM physics and via the AdS/CFT correspondence because of a possible connection to quantum gravity. Many features of these theories, for example, dynamical supersymmetry breaking, are inherently non-perturbative in nature and this serves as motivation to study such theories on the lattice.

Unfortunately, historically it has proven difficult to discretize supersymmetric theories using traditional methods. This stems from the fact that the supersymmetry algebra is an extension of the usual Poincaré algebra and hence is broken completely by naïve discretization on a space-time lattice. However, recently the development of a series of new theoretical tools have enabled us to construct certain supersymmetric theories on the lattice while preserving a subset of the continuum supersymmetries - see the reviews [1, 2, 3, 4] and references therein. Other recent complementary approaches to the problem of exact lattice supersymmetry can be found in [5, 6, 7, 8, 9, 10, 11, 12, 13].

One way to understand the new constructions is to realize that they correspond to discretizations of topologically twisted forms of the target continuum theories. Currently, lattice constructions exist for a set of SYM theories, including the four-dimensional $\mathcal{N} = 4$ SYM theory.

Lattice theories constructed this way are free of doublers, respect gauge-invariance, preserve a subset of the original supersymmetries and target the usual continuum theories...
in the naïve continuum limit. These constructions are possible only if the continuum SYM theories possess sufficient extended supersymmetry; the precise requirement is that the number of supercharges must be an integer multiple of $2^D$ where $D$ is the space-time dimension. This includes the $\mathcal{N} = (2, 2)$ SYM theory in two dimensions and $\mathcal{N} = 4$ SYM in four dimensions. In this paper we study both theories in two dimensions - the $\mathcal{N} = 4$ model yielding the $\mathcal{N} = (8,8)$ theory after dimensional reduction from four to two dimensions.

However, even when a supersymmetric lattice construction exists, it is still possible to encounter an additional difficulty that renders the use of numerical simulation problematic – the fermionic sign problem. To understand the nature of this problem consider a generic lattice theory with a set of bosonic $\phi$ and fermionic $\psi$ degrees of freedom. The partition function of the theory is

$$Z = \int [d\phi][d\psi] \exp \left( -S_B[\phi] - \psi^T M[\phi] \psi \right),$$

$$= \int [d\phi] \text{Pf}(M) \exp \left( -S_B[\phi] \right),$$

where $M$ is antisymmetric fermion matrix and $\text{Pf}(M)$ the corresponding Pfaffian. For a $2n \times 2n$ matrix $M$, the Pfaffian is explicitly given as $\text{Pf}(M)^2 = \text{Det}(M)$. In the supersymmetric lattice constructions we will consider in this paper, $M$ at non zero lattice spacing is a complex operator and one might worry that the resulting Pfaffian could exhibit a fluctuating phase depending on the background boson fields $\phi$. Since Monte Carlo simulations must be performed with a positive definite measure, the only way to incorporate this phase is through a reweighting procedure, which folds this phase in with the observables of the theory. Expectation values of observables derived from such simulations can then suffer drastic statistical errors which overwhelm the signal – the famous fermionic sign problem. Thus, if such a complex phase is present, the Monte Carlo technique is rendered effectively useless. Lattice theories such as QCD with finite chemical potential are known to suffer from a severe sign problem, which makes it very difficult to extract physical observables from simulations using conventional methods. The lattice sign problem exists not only in relativistic field theories but also in a variety of condensed matter systems [14].

In the construction of supersymmetric lattice gauge theories, there has been an ongoing debate on the existence of a sign problem in the two-dimensional $\mathcal{N} = (2, 2)$ supercharge lattice theory [15, 16, 17]. The resolution of this sign problem is crucial as the extraction of continuum physics from the lattice model depends very much on whether the results from phase quenched simulations can be trusted. Moreover, if a sign problem were to be found in this model it makes it more likely that the four-dimensional $\mathcal{N} = 4$ theory also suffers from a sign problem which would render practical simulation of this theory impossible. In [15], it was shown that there is a potential sign problem in the two-dimensional $\mathcal{N} = (2, 2)$ SYM lattice theory. Furthermore, in [16] numerical evidence was presented of a sign problem in a phase quenched dynamical simulation of the theory at non-zero lattice spacing. More recently Hanada et al. [17] have argued that there is no sign problem for this theory in the continuum limit. However, the models studied by these various groups differed in detail; Catterall et al. studied an $SU(2)$ model obtained by truncating the supersymmetric
$U(2)$ theory and utilized bosonic link fields valued in the group $SL(2,C)$, while Hanada et al. used a $U(2)$ model where the complexified bosonic variables take their values in the algebra of $U(2)$ together with the inclusion of supplementary mass terms to control scalar field fluctuations.

In this paper, we present results from simulations of the two dimensional $\mathcal{N} = (2,2)$ $U(N)$ SYM theory (which we will refer to from now on as the $Q = 4$ theory, with $Q$ the number of supercharges) and the maximally supersymmetric $\mathcal{N} = (8,8)$ $U(N)$ SYM theory (we refer to this theory as the $Q = 16$ theory). Our results provide strong evidence that there is no sign problem in the supersymmetric continuum limit for these theories. In the next four sections we summarize the details of the lattice constructions of both theories including a discussion of the possible parameterizations of the bosonic link fields. We then present our numerical results for $Q = 4$ and $Q = 16$ lattice SYM theories in two dimensions.

2. Supersymmetric Yang–Mills theories on the lattice

As discussed in the introduction it is possible to discretize a class of continuum SYM theories using ideas based on topological twisting\(^1\). Though the basic idea of twisting goes back to Witten in his seminal paper on topological field theory [21], it actually had been anticipated in earlier work on staggered fermions [22]. In our context, the idea of twisting is to decompose the fields of the Euclidean SYM theory in $D$ space-time dimensions in representations not in terms of the original (Euclidean) rotational symmetry $SO_{\text{rot}}(D)$, but a twisted rotational symmetry, which is the diagonal subgroup of this symmetry and an $SO_R(D)$ subgroup of the R-symmetry of the theory, that is,

$$SO(D)' = \text{diag}(SO_{\text{Lorentz}}(D) \times SO_R(D)) \, .$$

As an example, let us consider the case where the total number of supersymmetries is $Q = 2^D$. In this case we can treat the supercharges of the twisted theory as a $2^{D/2} \times 2^{D/2}$ matrix $q$. This matrix can be expanded on the Dirac–Kähler basis as

$$q = QI + Q_a \gamma_a + Q_{ab} \gamma_a \gamma_b + \ldots$$

The $2^D$ antisymmetric tensor components that arise in this basis are the twisted supercharges that satisfy the corresponding supersymmetry algebra inherited from the original algebra

$$Q^2 = 0 \quad \quad \quad \quad (2.3)$$
$$\{Q, Q_a\} = p_a \quad \quad \quad \quad (2.4)$$
$$\vdots \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (2.5)$$

The presence of the nilpotent scalar supercharge $Q$ is most important; it is the algebra of this charge that is compatible with discretization. The second piece of the algebra expresses the fact that the momentum is the $Q$-variation of something which makes the statement

\(^1\)Note that the lattice actions constructed using orbifold and twisted methods are equivalent [18, 19, 20].
plausible that the energy-momentum tensor and hence the entire action can be written in a \(Q\)-exact form\(^2\). Notice that an action written in such a \(Q\)-exact form is trivially invariant under the scalar supersymmetry \(Q\) provided the latter remains nilpotent under discretization.

The recasting of the supercharges in terms of twisted variables can be repeated for the fermions of the theory and yields a set of antisymmetric tensors \((\eta, \psi_a, \chi_{ab}, \ldots)\), which for the case of \(Q = 2^D\) matches the number of components of a real \(\text{Kähler–Dirac field}\). This repackaging of the fermions of the theory into a \(\text{Kähler–Dirac field}\) is at the heart of how the discrete theory avoids fermion doubling as was shown by Becher, Joos and Rabin in the early days of lattice gauge theory \([23, 24]\). It is important to recognize that the transformation to twisted variables corresponds to a simple change of variables in flat space – one more suitable for discretization.

### 2.1 Two-dimensional \(Q = 4\) SYM on the lattice

The two-dimensional \(Q = 4\) SYM theory is the simplest example of a gauge theory that permits topological twisting and thus satisfies our requirements for supersymmetric lattice constructions. Its R-symmetry possesses an \(SO(2)\) subgroup corresponding to rotations of the its two degenerate Majorana fermions into each other. After twisting the fields and supersymmetries of the target theory, the action takes the following form in the continuum

\[
S = \frac{1}{g^2} Q \int \text{Tr} \left( \chi_{ab} \mathcal{F}_{ab} + \eta [\mathcal{D}_a, \mathcal{D}_b] - \frac{1}{2} \eta d \right),
\]

(2.6)

where \(g\) is the coupling parameter. We use an anti-hermitian basis for the generators of the gauge group with \(\text{Tr}(T^a T^b) = -\delta^{ab}\).

The degrees of freedom appearing in the above action are just the twisted fermions \((\eta, \psi_a, \chi_{ab})\) and a complexified gauge field \(A_a\). The latter is built from the usual gauge field \(A_a\) and the two scalars \(B_a\) present in the untwisted theory: \(A_a = A_a + iB_a\). The twisted theory is naturally written in terms of the complexified covariant derivatives

\[
\mathcal{D}_a = \partial_a + A_a, \quad \overline{\mathcal{D}}_a = \partial_a + \overline{A}_a,
\]

(2.7)

and complexified field strengths

\[
\mathcal{F}_{ab} = [\mathcal{D}_a, \mathcal{D}_b], \quad \overline{\mathcal{F}}_{ab} = [\overline{\mathcal{D}}_a, \overline{\mathcal{D}}_b].
\]

(2.8)

Notice that the original scalar fields transform as vectors under the original R-symmetry and hence become vectors under the twisted rotation group while the gauge fields are singlets under the R-symmetry and so remain vectors under twisted rotations. This structure makes the appearance of a complex gauge field in the twisted theory possible. This action is invariant under the original \(U(N)\) gauge symmetry from the untwisted theory.

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\(^2\)In the case of four-dimensional \(\mathcal{N} = 4\) SYM there is an additional \(Q\)-closed term in the action.
The nilpotent transformations associated with the scalar supersymmetry $Q$ are given explicitly by

\[ Q A_a = \psi_a \]
\[ Q \psi_a = 0 \]
\[ Q \mathcal{A}_a = 0 \]
\[ Q \chi_{ab} = -F_{ab} \]
\[ Q \eta = d \]
\[ Q d = 0 \] \hspace{1cm} (2.9)

Performing the $Q$-variation on the action and integrating out the auxiliary field $d$ yields

\[ S = \frac{1}{g^2} \int \text{Tr} \left( -F_{ab}F_{ab} + \frac{1}{2} [D_a, D_a]^2 - \chi_{ab}D_{[a} \psi_{b]} - \eta \overline{D}_a \psi_a \right) . \] \hspace{1cm} (2.10)

The prescription for discretization is somewhat natural. The complexified gauge fields are represented as complexified Wilson gauge fields

\[ A_a(x) \to U_a(n) , \] \hspace{1cm} (2.11)

living on links of a lattice, which for the moment can be thought of as hypercubic, with integer-valued basis vectors

\[ \widehat{\mu}_1 = (1, 0), \quad \widehat{\mu}_2 = (0, 1) . \] \hspace{1cm} (2.12)

They transform in the usual way under $U(N)$ lattice gauge transformations

\[ U_a(n) \to G(n)U_a(n)G^\dagger(n + \widehat{\mu}_a) . \] \hspace{1cm} (2.13)

Supersymmetric invariance then implies that $\psi_a(n)$ live on the same links and transform identically. The scalar fermion $\eta(n)$ is clearly most naturally associated with a site and transforms accordingly

\[ \eta(n) \to G(n)\eta(n)G^\dagger(n) . \] \hspace{1cm} (2.14)

The field $\chi_{ab}(n)$ is slightly more difficult. Naturally as a 2-form it should be associated with a plaquette. In practice we introduce diagonal links running through the center of the plaquette and choose $\chi_{ab}(n)$ to lie with opposite orientation along those diagonal links. This choice of orientation will be necessary to ensure gauge invariance. Figure 1 shows the resultant lattice theory.

To complete the discretization we need to describe how continuum derivatives are to be replaced by difference operators. A natural technology for accomplishing this in the case of adjoint fields was developed many years ago and yields expressions for the derivative operator applied to arbitrary lattice p-forms \[25\]. In the case discussed here we need just two derivatives given by the expressions

\[ D_a^{(+)} f_b(n) = U_a(n)f_b(n + \widehat{\mu}_a) - f_b(n)U_a(n + \widehat{\mu}_b) , \] \hspace{1cm} (2.15)
\[ \overline{D}_a^{(-)} f_a(n) = f_a(n)\overline{U}_a(n) - \overline{U}_a(n - \widehat{\mu}_a)f_a(n - \widehat{\mu}_a) . \] \hspace{1cm} (2.16)
Figure 1: The 2d lattice for the four supercharge theory with field orientation assignments.

The lattice field strength is then given by the gauged forward difference acting on the link field: $F_{ab}(n) = D^+(a)U_b(n)$, and is automatically antisymmetric in its indices. Furthermore, it transforms like a lattice 2-form and yields a gauge invariant loop on the lattice when contracted with $\chi_{ab}(n)$. Similarly the covariant backward difference appearing in $\overline{D}(-)aU_a(n)$ transforms as a 0-form or site field and hence can be contracted with the site field $\eta(n)$ to yield a gauge invariant expression.

This use of forward and backward difference operators guarantees that the solutions of the lattice theory map one-to-one with the solutions of the continuum theory and hence fermion doubling problems are evaded \[23\]. Indeed, by introducing a lattice with half the lattice spacing one can map this Kähler–Dirac fermion action into the action for staggered fermions \[26\]. Notice that, unlike the case of QCD, there is no rooting problem in this supersymmetric construction since the additional fermion degeneracy is already required in the continuum theory.

As for the continuum theory the lattice action is again $Q$-exact:

$$S = \sum_n \text{Tr} \left( \chi_{ab}(n)D^+(a)U_b(n) + \eta(n)\overline{D}^(-)aU_a(n) - \frac{1}{2}\eta(n)d(n) \right).$$  \hspace{1cm} (2.17)

Acting with the $Q$ transformation on the lattice fields and integrating out the auxiliary field $d$, we obtain the gauge and $Q$-invariant lattice action:

$$S = \sum_n \text{Tr} \left( F^+_{ab}(n)F_{ab}(n) + \frac{1}{2}\left(\overline{D}(-)aU_a(n)\right)^2 - \chi_{ab}(n)D^+(a)\psi_{a}(n) - \eta(n)\overline{D}^(-)a\psi_{a}(n) \right).$$ \hspace{1cm} (2.18)

2.2 Four-dimensional $Q = 16$ SYM on the lattice

In four dimensions the constraint that the target theory possess sixteen supercharges singles out a unique theory for which this construction can be undertaken – the $\mathcal{N} = 4$ SYM theory.

The continuum twist of $\mathcal{N} = 4$ that is the starting point of the twisted lattice construction was first written down by Marcus in 1995 \[27\] although it now plays an important role
in the Geometric-Langlands program and is hence sometimes called the GL-twist \[28\]. This four-dimensional twisted theory is most compactly expressed as the dimensional reduction of a five-dimensional theory in which the ten (one gauge field and six scalars) bosonic fields are realized as the components of a complexified five-dimensional gauge field while the 16 twisted fermions naturally span one of the two Kähler–Dirac fields needed in five dimensions. Remarkably, the action of this theory contains a $\mathcal{Q}$-exact term of precisely the same form as the two-dimensional theory given in Eq. \((2.14)\) provided one extends the indices labeling the fields to run from one to five. In addition, the Marcus twist of $\mathcal{N} = 4$ YM requires a new $\mathcal{Q}$-closed term which was not possible in the two-dimensional theory

$$S_{\text{closed}} = -\frac{1}{8} \int \text{Tr} \; \epsilon_{mpqr} \chi_{qr} \mathcal{F}_p \chi_{mn} .$$ \hfill (2.19)$$

The supersymmetric invariance of this term then relies on the Bianchi identity

$$\epsilon_{mpqr} \mathcal{F}_p \mathcal{F}_{qr} = 0 .$$ \hfill (2.20)

The four-dimensional lattice that emerges from examining the moduli space of the resulting discrete theory is called the $A_4^\ast$-lattice and is constructed from the set of five basis vectors $\hat{e}_a$ pointing out from the center of a four-dimensional equilateral simplex out to its vertices together with their inverses $-\hat{e}_a$. It is the four-dimensional analog of the two-dimensional triangular lattice. Complexified Wilson gauge link variables $U_a$ are placed on these links together with their $\mathcal{Q}$-superpartners $\psi_a$. Another 10 fermions are associated with the diagonal links $\hat{e}_a + \hat{e}_b$ with $a > b$. Finally, the exact scalar supersymmetry implies the existence of a single fermion for every lattice site. The lattice action corresponds to a discretization of the Marcus twist on this $A_4^\ast$-lattice and can be represented as a set of traced closed bosonic and fermionic loops. It is invariant under the exact scalar supersymmetry $\mathcal{Q}$, lattice gauge transformations and a global permutation symmetry $S^5$ and can be proven free of fermion doubling problems as discussed above. The $\mathcal{Q}$-exact part of the lattice action is again given by Eq. \((2.18)\) where the indices $a, b$ now correspond to the indices labeling the five basis vectors of $A_4^\ast$.

While the supersymmetric invariance of this $\mathcal{Q}$-exact term is manifest in the lattice theory, it is not clear how to discretize the continuum $\mathcal{Q}$ closed term. Remarkably, it is possible to discretize Eq. \((2.14)\) in such a way that it is indeed exactly invariant under the twisted supersymmetry

$$S_{\text{closed}} = -\frac{1}{8} \sum_n \text{Tr} \; \epsilon_{mpqr} \chi_{qr} (n + \hat{\mu}_m + \hat{\mu}_n + \hat{\mu}_p) \mathcal{F}_p^{-} \chi_{mn} (n + \hat{\mu}_p)$$ \hfill (2.21)

and can be seen to be supersymmetric since the lattice field strength satisfies an exact Bianchi identity \[23\].

$$\epsilon_{mpqr} \mathcal{F}_p^{+} \mathcal{F}_{qr} = 0 .$$ \hfill (2.22)

The renormalization of this theory has been recently studied in perturbation theory with some remarkable conclusions \[29\]; namely that the classical moduli space is not lifted to all orders in the coupling, that the one loop lattice beta function vanishes and that no fine tuning of the bare lattice parameters with cut-off is required at one-loop for the theory to recover full supersymmetry as the lattice spacing is sent to zero.
3. Towards the continuum limit

3.1 Parametrizations of the gauge links

There exist two distinct parameterizations of the gauge fields on the lattice that have been proposed for these theories. The first one follows the standard Wilson prescription where the complexified gauge fields in the continuum are mapped to link fields \( U_a(n) \) living on the link between \( n \) and \( n + \hat{\mu}_a \) through the mapping

\[
U_a(n) = e^{A_a(n)},
\]  

(3.1)

where \( A_a(n) = \sum_{i=1}^{N_G} A^i_a T^i \) and \( T^i = 1, \ldots, N_G \) are the anti-hermitian generators of \( U(N) \). The resultant gauge links belong to \( GL(N, C) \). We call this realization of the bosonic links the exponential or group based parametrization\(^3\).

The other parametrization of the bosonic link fields that has been used, particularly in the orbifold literature, simply takes the complex gauge links as taking values in the algebra of the \( U(N) \) group

\[
U_a(n) = A_a(n).
\]  

(3.2)

In this case to obtain the correct continuum limit one must subsequently expand the fields around a particular point in the moduli space of the theory corresponding to giving an expectation value to a component of the link field proportional to the unit matrix. This field can be identified as the trace mode of the scalar field in the untwisted theory.

\[
U_a(n) = I_N + A_a(n).
\]  

(3.3)

Usually the use of such an algebra based or non compact parametrization would signal a breaking of lattice gauge invariance. It is only possible here because the bosonic fields take values in a complexified \( U(N) \) theory – so that the unit matrix appearing in Eq. (3.3) can be interpreted as the expectation value of a dynamical field - the trace mode of the scalars. We will refer to this parametrization as the linear or algebra based parametrization\(^4\).

Both parameterizations of the gauge links are equivalent at leading order in the lattice spacing, yield the same lattice action and can be considered as providing equally valid representations of the lattice theory at the classical level. The exponential parametrization was used in studies of both \( Q = 4 \) and \( Q = 16 \) theories in [16] while in [17] the linear parametrization was employed to perform simulations of the \( Q = 4 \) theory. In this work we have concentrated on the linear parametrization principally because it is naturally associated with a manifestly supersymmetric measure in the path integral - the flat measure. Explicit comparison with results from the exponential parametrization can be found in [34].

\(^3\)Notice that our lattice gauge fields are dimensionless and hence contain an implicit factor of the lattice spacing \( a \).

\(^4\)In fact, a non-compact parametrization of the gauge-fields has also been recently used to restore BRST symmetry on the lattice in Ref. [31], i.e., to evade the so-called Neuberger 0/0 problem [32] (see also Refs. [33] and [34] for the recent progress, and [34] for the relation between the Neuberger 0/0 problem and sign problem for the lattice SYM theories.).
3.2 Potential terms

As we have described in the previous section, the linear parameterization only yields the correct naïve continuum limit if the trace mode of the scalars develops a vacuum expectation value so that appropriate kinetic terms are generated in the tree level action. In addition, we require that the fluctuations of all dimensionless lattice fields vanish as the lattice spacing is sent to zero; a non-trivial issue in theories possessing flat directions associated with extended supersymmetry. Since no classical scalar potential is present in the lattice theory\(^5\) it is crucial to add by hand a suitable gauge invariant potential to ensure these features\(^6\). Specifically we add a potential term of the following form\(^\cite{17}\)

\[ S_M = \mu^2 \sum_n \left( \frac{1}{N} \text{Tr}(U_\alpha^\dagger(n)U_\alpha(n)) - 1 \right)^2, \]  

(3.4)

to the lattice action. Here \(\mu\) is a tunable mass parameter, which can be used to control the expectation values and fluctuations of the lattice fields. Notice that such a potential obviously breaks supersymmetry – however because of the exact supersymmetry at \(\mu = 0\) all supersymmetry breaking counterterms induced via quantum effects will possess couplings that vanish as \(\mu \to 0\) and so can be removed by sending \(\mu \to 0\) at the end of the calculation.

To understand the effect of this term let us consider the full set of vacuum equations for the lattice theory. These are given by setting the bosonic action to zero

\[ F_{ab}(n) = 0, \]  

(3.5)

\[ \overline{D}_a^\dagger U_a(n) = 0, \]  

(3.6)

\[ \frac{1}{N} \text{Tr}\left(U_\alpha^\dagger(n)U_\alpha(n)\right) - 1 = 0. \]  

(3.7)

The first two equations imply that the moduli space consists of constant complex matrices taking values in the \(N\)-dimensional Cartan subalgebra of \(U(N)\).

Assuming that the matrix valued complexified link fields \(U_\alpha(n)\) are nonsingular\(^7\), we can decompose them in the following way

\[ U_\alpha(n) = P_\alpha(n)U_\alpha(n), \]  

(3.8)

where \(P_\alpha(n)\) is a positive semidefinite hermitian matrix and \(U_\alpha(n)\) a unitary matrix. The form of the mass term clearly does not depend on the unitary piece and clearly is minimized by setting \(P_\alpha(n) = I_N\). Expanding about this configuration gives the following expression for the complex link matrices

\[ U_\alpha(n) = P_\alpha(n)U_\alpha(n) = \left(I_N + p_\alpha(n)\right)U_\alpha(n), \]  

(3.9)

\(^5\)Lattice theories based on supersymmetric mass deformations have also been proposed in two dimensions\(^\cite{17,13}\).

\(^6\)It was precisely this requirement that led to a truncation of the \(U(N)\) symmetry to \(SU(N)\) in the original simulations of these theories. One can think of this truncation as corresponding to the use of a delta function potential for the \(U(1)\) part of the field\(^\cite{17}\).

\(^7\)Having zero eigenvalues for the matrices \(U_\alpha(n)\) would not cause a problem for us, as we are interested in expanding these fields around the point \(I_N\) instead of the origin of the moduli space.
where \( p_a(n) \) is a hermitian matrix. Minimizing the mass term leads to

\[
0 = \frac{1}{N} \text{Tr} \left( U_a^\dagger(n) U_a(n) \right) - 1 ,
\]

\[
= \frac{1}{N} \text{Tr} \left[ U_a^\dagger(n) (I_N + p_a(n)) \right] \left[ (I_N + p_a(n)) U_a(n) \right] - 1 ,
\]

\[
= \frac{1}{N} \text{Tr} \left[ I_N + 2p_a(n) + p_a^2(n) \right] - 1 ,
\]

\[
= \frac{1}{N} \left[ \frac{2}{\sqrt{N}} p_a^0(n) + \sum_{A=1}^{N} (p^A_a(n))^2 \right] .
\]

where we have adopted a basis in which \( T^0 \) is proportional to the unit matrix and all other (Cartan) generators are traceless. Analyzing the gauge transformation properties of the complexified link fields,

\[
U_a(n) \to G(n) U_a(n) G^\dagger(n + \hat{\mu}_a) ,
\]

we see that the unitary piece \( U_a(n) \) transforms like a link field

\[
U_a(n) \to G(n) U_a(n) G^\dagger(n + \hat{\mu}_a)
\]

while the hermitian matrix \( p_a(n) \) transforms like a scalar field

\[
p_a(n) \to G(n) p_a(n) G^\dagger(n) .
\]

Thus in this language we can identify the \( p_a(n) \) with the scalar field fluctuations \( B_a(n) \). The mass term then becomes

\[
S_M = \mu^2 \sum_n \frac{1}{N^2} \left[ \frac{2}{\sqrt{N}} p_a^0(n) + \sum_{A=1}^{N} (p^A_a(n))^2 \right] .
\]

From this expression it is straightforward to see that the fluctuations of the scalar trace mode are governed by a quadratic potential while the traceless scalar field fluctuations feel only a quartic potential. Thus, if we keep \( \mu \equiv \mu a \) fixed as \( a \to 0 \) the trace mode will acquire an infinite mass in the continuum limit and hence fluctuations of the trace more around its vacuum expectation value will be completely suppressed in that limit. In the same limit the presence of the quartic potential for the traceless Cartan generators is sufficient to regulate possible infrared problems associated with the flat directions of the \( SU(N) \) sector. Finally, once the continuum limit is attained, we can restore supersymmetry by taking the final limit \( \mu \to 0 \).

Notice that the fact that this potential term selects out preferentially the trace mode of the scalars is trivially obvious if we adopt the exponential parametrization of the complexified gauge links since in that case we can identify \( I + p_a \) with \( e^{iB_a} \).

4. Simulation Results

As noted previously, we have rescaled all lattice fields by powers of the lattice spacing to make them dimensionless. This leads to an overall dimensionless coupling parameter of the
form \(N/(2\lambda a^2)\), where \(a = \beta/T\) is the lattice spacing, \(\beta\) is the physical extent of the lattice in the Euclidean time direction and \(T\) is the number of lattice sites in the time-direction. Thus, the lattice coupling is

\[
\kappa = \frac{NL^2}{2\lambda \beta^2},
\]

for the symmetric two-dimensional lattice where the spatial length \(L = T^8\). Note that \(\lambda \beta^2\) is the dimensionless physical \('t\) Hooft coupling in units of the area. In our simulations\(^9\), the continuum limit can be approached by fixing \(\lambda \beta^2\) and \(N\) and increasing the number of lattice points \(L \to \infty\). In practice we fix the value of \(\beta = 1\) and vary \(\lambda\). We have taken three different values for this coupling \(\lambda = 0.5, 1.0, 2.0\) and lattice sizes ranging from \(L = 2, \cdots, 16\). Systems with \(U(N)\) gauge groups with \(N = 2, 3, 4\) have been examined.

The simulations are performed using anti-periodic (thermal) boundary conditions for the fermions\(^{10}\). An RHMC algorithm was used for the simulations as described in \([30]\). The use of a GPU accelerated solver \([35]\) allowed us to reach larger lattices than have thus far been studied.

### 4.1 \(Q = 4\) Supersymmetries

In figure 2, we show results for the absolute value of the (sine of) the Pfaffian phase \(|\sin \alpha|\) as a function of lattice size \(L = 1/a\) for the \(Q = 4\) model with gauge group \(U(2)\). The data corresponds to \(\lambda = 1\) but similar results are obtained for \(\lambda = 0.5, 2.0\) and larger numbers of colors. Three values of \(\mu\) are shown corresponding to \(\mu = 0.1, \mu = 1.0\) and \(\mu = 10.0\). While modest phase fluctuations are seen for small lattices for the smallest value of \(\mu\), we see that they disappear as the continuum limit is taken. As a practical matter, these results make it clear that no re-weighting of observables is needed over much of the parameter space. This point is reinforced when we plot a histogram of the phase angle in figure 3. Clearly the angle fluctuations contract towards the origin as the continuum limit is approached.

To check for the restoration of supersymmetry in the continuum limit and as the scalar potential is sent to zero, we show in figure 4, a plot of the bosonic action density vs lattice size \(L\). While the curves plateau for large \(L\) indicating a well defined continuum limit it is clear that in general supersymmetry is broken there. Indeed, the exact value of the bosonic action which is shown by the dotted line in the plot can be computed using a simple \(Q\) Ward identity and yields \([16]\)

\[
< \frac{1}{L^2} \kappa S_B > = \frac{3}{2} N_G
\]

(4.2)

It should be clear from the plot that the measured action indeed approaches this supersymmetric value if the subsequent limit \(\mu \to 0\) is taken\(^{11}\). Thus the regulating procedure we have described does indeed provide a well defined procedure for studying the supersymmetric lattice theory.

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\(^8\) Notice that this coupling multiples all terms in the bosonic action including those associated with the scalar potential.

\(^9\) See \([30]\) for the details of the code we used to simulate these theories.

\(^{10}\) This forbids exact zero modes that are otherwise present in the fermionic sector.

\(^{11}\) Actually strictly we only expect this as \(\beta \equiv \lambda \to \infty\) and thermal effects are suppressed. These appear to be already small for \(\lambda = 1\) in this theory.
Finally, to reassure ourselves that $L \to \infty$ indeed corresponds to a continuum limit, figure 3 shows a plot of the expectation value of the maximal eigenvalue of the operator $(U_a^\dagger U_a - 1)$ averaged over the lattice as a function of $L$ for $\lambda = 1$. To leading order, this expression yields the largest scalar field eigenvalue in units of the lattice spacing. Reassuringly we see that the eigenvalue indeed approaches zero as $L \to \infty$ corresponding to a vanishing lattice spacing.

4.2 $Q = 16$ Supersymmetries

The results for the absolute value of the (sine of) the Pfaffian phase for the $Q = 16$ supercharge model with $U(2)$ gauge group in two dimensions are shown in figure 6. As for the $Q = 4$ case, we see that the average Pfaffian phase is small and decreases with $L$. Indeed, the magnitude of these angular fluctuations are $O(10^{-4})$ for all $L$ and $\mu$ - much smaller than that observed for $Q = 4$. Thus, even on the coarsest lattice and smallest $\mu$, there is clearly no practical sign problem and certainly no sign problem in the continuum limit. Again, this picture is reinforced by looking at a histogram of the phase angle $\alpha$ as seen in figure 7.

The corresponding plot of the expectation value of the bosonic action vs lattice size $L$ is shown in figure 8. In the case of the $Q = 16$ model the exact expression for the bosonic
The action is given by
\[ < \frac{1}{L^2} \kappa S_B > = \frac{9}{2} N_G \]  \hspace{1cm} (4.3)

The data shown in this plot allow us to conclude that a well defined continuum limit exists for non-zero \( \mu \) and furthermore, \( Q \)-supersymmetry can be restored by subsequently sending the parameter \( \mu \to 0 \). As a final cross check that the limit \( L \to \infty \) indeed corresponds to a true continuum limit, we have again examined the behavior of the maximal eigenvalue of \( U^\dagger U - I \) as \( L \to \infty \). The result is shown in figure 9 and is consistent with a vanishing lattice spacing in this limit.

These results generalize to large numbers of colors as can be seen in figure 10, where we plot the expectation value of the absolute value of the sine of the Pfaffian phase for the case of the \( U(4) \) group. Notice that the Pfaffian can be proven real in the limit that the \( Q = 16 \) theory is reduced to zero dimensions for two and three colors so that it is necessary to examine the \( U(4) \) case to be sure of seeing truly generic behavior.

Nevertheless we see that \( U(4) \) looks qualitatively the same as for \( U(2) \). In fact the fluctuations in the phase angle that we observe are even smaller than those seen for the \( U(2) \) theory. This again indicates that this theory exhibits no sign problem even on small lattices and certainly in the continuum limit.

The plot of the bosonic action for \( U(4) \) is shown in figure 11. While the largest lattice we have been able to simulate thus far is rather too small to get a good continuum limit the measured bosonic action is nevertheless within a percent or so of the exact value expected.
on the basis of $Q$-supersymmetry. The scalar field fluctuations also decrease toward zero as the number of lattice points increase as shown in figure 4.

It is at first sight rather remarkable that the observed Pfaffian phase fluctuations are small in the $Q = 16$ theory given that the Pfaffian is certainly complex when evaluated on a generic set of background scalar and gauge fields. It appears to be a consequence of very specific dynamics in the theory which ensure that only certain special regions of field space are important in the path integral. Of course the continuum theory does possess very special dynamics; for example the twisted supersymmetry ensures that the torus partition function $Z$ is a topological invariant. One immediate consequence of this is that $Z$ may be computed exactly at one loop where Marcus has argued that it simply reduces to an unsigned sum over isolated points in the moduli space of flat complexified connections up to complex gauge transformations [27]. Furthermore, much of this structure survives in the lattice theory; the full partition function including any Pfaffian phase may be calculated exactly at one loop. As in the continuum theory there is a perfect cancellation of contributions from fermions and bosons and the final result is real [29]. Of course this does not mean that simulations at finite gauge coupling should not suffer from sign problems but certainly makes it less likely. More prosaically, it is easy to see that the Pfaffian is real positive if the lattice scalar fields are set to zero - and this is what effectively happens in

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**Figure 4:** $\langle \kappa S_B \rangle$ for $Q = 4$, $U(2)$ and $\mu = 0.1, 1, 10$.
Figure 5: Ensemble average for the eigenvalues of the \((U_a^\dagger U_a - 1)\) operator for \(Q = 4\), U(2) and \(\mu = 0.1, 1, 10\).

the continuum limit as a result of the scalar potential that we use to control the vacuum expectation value and fluctuations of the trace mode.

5. Conclusions

We have performed numerical simulations of the four and sixteen supercharge lattice SYM theories in two dimensions to investigate the occurrence of a sign problem in these theories. In contrast to the usual situation in lattice gauge theory, we utilize a non compact parameterization of the gauge fields in which the lattice fields are expanded on the algebra of the group. While such a scheme would ordinarily break lattice gauge invariance we show that in the case of these twisted supersymmetric models this preserves gauge symmetry since the models in question are formulated in terms of a complexified gauge field valued in \(U(N)\). The correct continuum limit is then ensured by adding an appropriate gauge invariant potential term which picks out a non-zero vacuum expectation value for the trace mode of the scalar fields in the continuum limit. We argue that the effects of this potential on the remaining traceless modes can be subsequently removed by sending the potential to zero after the continuum limit is taken.
Figure 6: $\langle |\sin \alpha| \rangle$ for $Q = 16$, $U(2)$ and $\mu = 0.1, 1, 10$.

We have examined both supersymmetric theories for several values of the dimensionless 't Hooft coupling $\lambda \beta^2$ and for gauge groups $U(2)$, $U(3)$ and $U(4)$. We take a careful continuum limit by simulating the theories over a range of lattice size $L = 2 - 14$. In both cases we see that the average Pfaffian phase goes to zero for a fixed gauge invariant potential as the continuum limit is taken. We also examine the subsequent limit in which the potential is removed and show evidence that supersymmetry is restored. While the absence of a sign problem is not surprising in the $Q = 4$ case (where one can prove the Pfaffian reduces to a real positive definite determinant in the continuum limit) it is much more non trivial matter in the $Q = 16$ supercharge case. In that case the Pfaffian evaluated on a generic background is complex even in the continuum limit. Nevertheless, we observe that the Pfaffian phase is small and decreases to zero as the continuum limit is taken. Indeed, in practice it is sufficiently small even on coarse lattices that there is no need to use a reweighting procedure to compute expectation values of observables. The analysis of the $Q = 16$ model is complicated by the fact that the $U(2)$ and $U(3)$ theories exhibit some special properties since in the matrix model limit they are real positive definite and real respectively. Nevertheless, the pattern we observe for the $U(4)$ group is similar to that seen for the smaller groups and the trend supports the conjecture that the sign problem is
absent in the continuum limit.

These results thus help to strengthen the case that there may be no sign problem for the $Q = 16$ theory in four dimensions and hence no a priori barrier to numerical studies of this theory.

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