An improvement of the product integration method for a weakly singular Hammerstein equation
Laurence Grammont, Hanane Kaboul

To cite this version:
Laurence Grammont, Hanane Kaboul. An improvement of the product integration method for a weakly singular Hammerstein equation. 2016. hal-01295843
An improvement of the product integration method for a weakly singular Hammerstein equation

Laurence Grammont∗, Hanane Kaboul †

Abstract

We present a new method to solve nonlinear Hammerstein equations with weakly singular kernels. The process to approximate the solution, followed usually, consists in adapting the discretization scheme from the linear case in order to obtain a nonlinear system in a finite dimensional space and solve it by any linearization method. In this paper, we propose to first linearize, via Newton method, the nonlinear operator equation and only then to discretize the obtained linear equations by the product integration method. We prove that the iterates, issued from our method, tends to the exact solution of the nonlinear Hammerstein equation when the number of Newton iterations tends to infinity, whatever the discretization parameter can be. This is not the case when the discretization is done first: in this case, the accuracy of the approximation is limited by the mesh size discretization. A Numerical example is given to confirm the theoretical result.

Keywords: Nonlinear equations, Newton-like methods, Product integration method, Integral equations.

AMS Classification: 65J15, 45G10, 35P05

1 Introduction

The general framework of this paper is the following. Let $\mathcal{X}$ be a complex Banach space and $K: O \subseteq \mathcal{X} \to \mathcal{X}$ a nonlinear Fréchet differentiable integral operator of the Hammerstein type defined on a nonempty open set $O$ of $\mathcal{X}$:

$$K(x)(s) := \int_{a}^{b} H(s,t)L(s,t)F(t,x(t)) \, dt, \quad \text{for all } x \in \Omega,$$

(1)

∗Université de Lyon, Institut Camille Jordan, UMR 5208, 23 rue du Dr Paul Michelon, 42023 Saint-Étienne Cedex 2, France. laurence.grammont@univ-st-etienne.fr

†Université de Lyon, Institut Camille Jordan, UMR 5208, 23 rue du Dr Paul Michelon, 42023 Saint-Étienne Cedex 2, France. hanane.kaboul@univ-st-etienne.fr
where $H$ is the singular part of the kernel, $L$ is the regular part of the kernel and $F$, the nonlinear part of the operator, is a real-valued function of two variables:

$$(t, u) \in [a, b] \times \mathbb{R} \mapsto F(t, u) \in \mathbb{R},$$

with enough regularity so that $K$ is twice Fréchet-differentiable on $\mathcal{O}$.

The problem is a nonlinear Fredholm integral equation of the second kind:

(2) \quad \text{Find } \varphi \in \mathcal{O} : \quad \varphi - K(\varphi) = y,

for a given function $y \in \mathcal{X}$.

Let $T := K'$ denote the Fréchet derivative of $K$, i.e., for all $x \in \mathcal{O}$,

(3) \quad T(x)h(s) = \int_{a}^{b} H(s, t)L(s, t)\frac{\partial F}{\partial u}(t, x(t))h(t)\,dt, \quad h \in \mathcal{X}, s \in [a, b].

In the following, $\mathcal{X}$ will be the space of the real valued continuous functions over a real interval $[a, b], C^0([a, b], \mathbb{R})$, equiped with the supremum norm $\|\cdot\|$. If we consider a singular kernel such as $H(s, t) := \log(|s-t|)$ or $|s-t|^{\alpha}$, $0 < \alpha < 1$, an approximation based on standard numerical integrations is a poor idea. The main idea of the product integration method is introduced by Atkinson for linear integral equations ([7],[8] and [9]) and is motivated by Young [28]. The product integration method consists in performing a piecewise polynomial linear interpolation of the smooth part of the kernel times the function involving the unknown. This method is called product trapezoidal rule when the interpolation is linear. The solution of a second kind Fredholm integral equation with weakly singular kernel is typically nonsmooth near the boundary of the domain of integration. In order to obtain a high order of convergence, taking into account the singular behaviour of the exact solution, polynomial spline on graded mesh are developed (among other authors, Brunner, Pedas, Vainikko, Schneider [13], [24] and [26]). In [19], Kaneko, Noren and Xu discuss a standard product integration method with a general piecewise polynomial interpolation for weakly singular Hammerstein equation and indicate its superconvergence properties. Under particular assumptions on the right hand side $y$, on the function $F$ defining the nonlinearity and on the regularity of the exact solution of (2), they give an error estimation involving $n$ the discretization parameter and the degree $m$ of the piecewise polynomial interpolation. Hence the error depends on $n$ and $m$.

In the 1950’s, the major theme in the domain of theoretical numerical analysis was the development of general frameworks in the domain of functional analysis to build and analyze numerical methods. A particular important contribution in this context is the paper of Kantorovich [20] and later [21]. It proposes a generalization of Newton’s method for solving nonlinear operator equations on Banach spaces. This idea is used everywhere when dealing with integral equations or partial differential equations. In [2], chapter 6, Anselone studies the Newton method to approximate solutions of nonlinear equations $P(x) = 0$ where $P$ is a nonlinear differentiable operator from a Banach space into itself. When dealing with the convergence of approximate solutions, these are defined as the solution of $P_n(x) = 0$ where $P_n$ is an approximation of $P$. The Newton method is then applied to the functional equation $P_n(x) = 0$. The
philosophy of most of the papers dealing with the numerical approximation of nonlinear integral operator equation consists in defining an approximate operator $P_n$ to $P$ and then apply Newton method ([4],[10], [11], [12], [13],[14],[16], [17],[19],[22],[23],[24] and [27]).

We propose to apply Newton’s method directly to the operator equation $P(x) = 0$ and then to discretize the linear operator equations, issued from the Newton’s iterations, by a product integration method. We will prove that the approximate iterates tend to the exact solution of the operator equation as the number of iterations tends to infinity. The important fact is that the convergence holds whatever the discretization parameter, defining the size of the linear system to be solved, can be. As we do not need the solution of each Newton iteration to be particularly accurate, we chose to apply the classical trapezoidal product integration method.

Section 2 is devoted to the description of our method (linearization via Newton’s method followed by discretization by the product integration method). In Section 3, the convergence result is proved. In the last section, the classical method (discretization followed by linearization) is recalled and we compare it with our method through a numerical example.

2 Description of the new method

To solve the problem (2) for a given function $y$ in $C^0([a,b],\mathbb{R})$, we propose to first apply the Newton method to the equation $\varphi - K(\varphi) = y$. It leads to the sequence $(\varphi^{(k)})_{k\geq 0} \in \mathcal{O}$:

$$\varphi^{(0)} \in \mathcal{O}, \quad (I - T(\varphi^{(k)}))(\varphi^{(k+1)} - \varphi^{(k)}) = -\varphi^{(k)} + K(\varphi^{(k)}) + y, \quad k \geq 0.$$  

Then we discretize this equation with the product integration method associated to the piecewise linear interpolation.

Let $\Delta_n$, defined by

$$a =: t_{n,0} < t_{n,1} < \cdots < t_{n,n} := b,$$

be the uniform grid of $[a,b]$ with mesh $h_n := \frac{b-a}{n}$.

Setting

$$f_x(t) := \frac{\partial F}{\partial u}(t, x(t)),$$

$[L(s, t)f_x(t)h(t)]_n$ denotes the piecewise linear interpolation of $L(s, t)f_x(t)h(t)$:

$\forall s \in [a,b], \forall i = 1, \ldots, n$:

$$[L(s, t)f_x(t)h(t)]_n := \frac{1}{h_n}(t_{n,i} - t)L(s, t_{n,i-1})f_x(t_{n,i-1})h(t_{n,i-1})$$

$$+ \frac{1}{h_n}(t - t_{n,i-1})L(s, t_{n,i})f_x(t_{n,i})h(t_{n,i})$$

3
for $t \in [t_{n,i-1}, t_{n,i}]$.

We define the approximate operator $T_n$ by

$$T_n(x)(h)(s) := \int_a^b H(s,t)[L(s,t)f_x(t)h(t)]_n \, dt, \quad x \in \mathcal{O}, h \in \mathcal{X}, s \in [a,b].$$

Then the approximate problem is:

$$\text{(7)} \quad \text{Find } \varphi_{n}^{(k+1)} \in X : (I - T_n(\varphi_{n}^{(k)}))(\varphi_{n}^{(k+1)} - \varphi_{n}^{(k)}) = -\varphi_{n}^{(k)} + K(\varphi_{n}^{(k)}) + y.$$

We have

$$T_n(x)(h)(s) := \sum_{j=0}^{n} w_{n,j}(s)L(s, t_{n,j})f_k(t_{n,j})h(t_{n,j}),$$

$$w_{n,0}(s) := \frac{1}{h_n} \int_{t_{n,0}}^{t_{n,1}} H(s, t)(t_{n,1} - t) \, dt,$$

$$w_{n,n}(s) := \frac{1}{h_n} \int_{t_{n,n-1}}^{t_{n,n}} H(s, t)(t - t_{n,n-1}) \, dt,$$

and for $1 \leq j \leq n - 1$,

$$w_{n,j}(s) := \frac{1}{h_n} \int_{t_{n,j-1}}^{t_{n,j}} H(s, t)(t - t_{n,j-1}) \, dt + \frac{1}{h_n} \int_{t_{n,j}}^{t_{n,j+1}} H(s, t)(t_{n,j+1} - t) \, dt.$$

Hence (7) can be rewritten as

$$\varphi_{n}^{(k+1)}(s) - \sum_{j=0}^{n} w_{n,j}(s)L(s, t_{n,j})f_k(t_{n,j})\varphi_{n}^{(k+1)}(t_{n,j})$$

$$= K(\varphi_{n}^{(k)})(s) + y(s) - \sum_{j=0}^{n} w_{n,j}(s)L(s, t_{n,j})f_k(t_{n,j})\varphi_{n}^{(k)}(t_{n,j}),$$

where

$$f_k(t_{n,j}) := \frac{\partial F}{\partial u}(t_{n,j}, \varphi_{n}^{(k)}(t_{n,j})).$$

Setting

$$x_{n}^{(k+1)}(j) := \varphi_{n}^{(k+1)}(t_{n,j}),$$

From the evaluations of equation (26) at the nodes of the grid, it is straightforward that the vector $x_{n}^{(k+1)}$ is the solution of the linear system

$$\text{(10)} \quad (I - A_{n}^{(k)})x_{n}^{(k+1)} = b_{n}^{(k)},$$

4
where
\[
A_{n}^{(k)}(i, j) := w_{n,j}(t_{n,i})L(t_{n,i}, t_{n,j})f_{k}(t_{n,j}),
\]
\[
b_{n}^{(k)} := K(\varphi_{n}^{(k)})(t_{n,i}) + y(t_{n,i}) - A_{n}^{(k)} x_{n}^{(k)}.
\]
\[
\varphi_{n}^{(k+1)}(s) = \sum_{j=1}^{n} w_{n,j}(s)L(s, t_{n,j})f_{k}(t_{n,j})\left(x_{n}^{(k+1)}(t_{n,j}) - x_{n}^{(k)}(t_{n,j})\right)
\]
\[
+ K(\varphi_{n}^{(k)})(s) + y(s).
\]

3 Convergence property of the new method

Existence, uniqueness and regularity properties of the solution of equation (2) have been already considered (for example by Kaneko, Noren and Xu [18] or Pedas and Vainikko [25]). In this section, we are only interested by the proof of the convergence of \( \varphi_{n}^{(k)} \) towards \( \varphi \) when \( k \to \infty \).

Hypotheses:

(H0) \( F \), defined in (1), is twice continuously differentiable on \([a, b] \times \mathbb{R}\).

(H1) \( L \in C^{0}([a, b] \times [a, b], \mathbb{R}) \).

(H2) \( H \) verifies:

(H2.1) \( c_{H} := \sup_{s \in [a, b]} \int_{a}^{b} |H(s, t)| dt < +\infty \).

(H2.2) \( \lim_{h \to 0} \omega_{H}(h) = 0 \), where

\[
\omega_{H}(h) := \sup_{|s-\tau| \leq h, \sigma, \tau \in [a, b]} \int_{a}^{b} |H(s, t) - H(\tau, t)| dt.
\]

(H3) \( \varphi \in \mathcal{O} \) is an isolated solution of \( \varphi - K(\varphi) = f \).

(H4) \( I - T(\varphi) \) is invertible.

These assumptions ensure that equations (2), (7) and (26) for \( n \) large enough, are uniquely solvable (see [9]).

Let \( a > 0 \) such that \( B(\varphi, a) \subset \mathcal{O} \), where \( B(\varphi, a) \) denotes the open ball in \( C^{0}([a, b], \mathbb{R}) \) centered in \( \varphi \) and of radius \( a \). As \( \varphi \in \subset C^{0}([a, b], \mathbb{R}) \), \( \varphi \) is bounded. As \( F \) is twice continuously differentiable, the following constant exists:

\[
M_{2}(a) := \sup_{t \in [a, b], |u| \leq a + \| \varphi \|} \left| \frac{\partial^{2} F}{\partial u^{2}}(t, u) \right|.
\]
The proof of convergence relies on the successive approximations convergence result (see [23] Theorem 2.3. pp 21). Let us recall this result (in a form slightly different from [23]).

**Proposition 1** Consider a nonlinear operator $A$ from a Banach space $X$ into itself, defined on an open set $V$. Let $x^* \in V$ be a fixed point of $A$. Let the operator $A$ be Fréchet differentiable at the point $x^*$. Let us assume that the following condition is fulfilled

$$\rho_0 := \rho(A'(x^*)) < 1,$$

where $\rho$ denotes the spectral radius and $A'$ denotes the Fréchet derivative of $A$.

Then, for all $\varepsilon > 0$ such that $\rho_0 + \varepsilon < 1$, there exist $r_\varepsilon > 0$ and $r'_\varepsilon > 0$ such that $B(x^*, r_\varepsilon) \subset V$ and $B(x^*, r'_\varepsilon) \subset V$ and such that for $x^{(0)} \in B(x^*, r'_\varepsilon)$, the successive approximations $x^{(k)}$ defined by

$$x^{(k+1)} = A(x^{(k)}), \quad k \in \mathbb{N},$$

remain in $B(x^*, r_\varepsilon)$ for all $k \in \mathbb{N}$, and the sequence $(x^{(k)})_{k \geq 0}$ converges to $x^*$. Moreover,

$$\|x^{(k)} - x^*\| \leq r_\varepsilon(\rho_0 + \varepsilon)^k.$$

The following four lemmas are needed to prove our main result.

**Lemma 1** For all $a > 0$ such that $B(\varphi, a) \subset O$, for all $x \in B(\varphi, a),$

$$\|T(x) - T(\varphi)\| \leq c_H c_L M_2(a)\|x - \varphi\|,$$

where

$$c_L := \max_{s,t \in [a,b]} |L(s,t)|.$$

**Proof:** From the mean value theorem applied to $\frac{\partial F}{\partial u}$ for all $t$, there exist a real number $c(t)$ between $\varphi(t)$ and $x(t)$ such that

$$\frac{\partial F}{\partial u}(t, x(t)) - \frac{\partial F}{\partial u}(t, \varphi(t)) = \frac{\partial^2 F}{\partial u^2}(t, c(t))(x(t) - \varphi(t)).$$

As $\forall t \in [a,b], |c(t)| \leq r + \|\varphi\|,$

$$\|T(x) - T(\varphi)\| \leq \|x - \varphi\| \sup_{s \in [a,b]} \left| \int_a^b |H(s, t)||L(s, t)| \right| \frac{\partial^2 F}{\partial u^2}(t, c(t)) |dt$$

(13)

$$\leq c_H c_L M_2(a)\|x - \varphi\|.$$

Hence (13) is deduced.

Now $a$ is fixed such that $B(\varphi, a) \subset O$. 

---

6
Lemma 2  There is a positive number \( r < a \) such that for all \( x \in B(\varphi, r) \), \( I - T(x) \) is invertible and

\[
\|(I - T(x))^{-1}\| \leq 2\mu,
\]
where \( \mu := \|(I - T(\varphi))^{-1}\| \).

Moreover, for all \( x \in B(\varphi, r) \), for \( n \) large enough, \( I - T_n(x) \) is invertible and there exists a constant \( c_x > 0 \), independent of \( n \), such that

\[
\|(I - T_n(x))^{-1}\| \leq c_x.
\]

Proof:

Let \( 0 < r < a \) be such that \( r \leq \frac{1}{2\mu HcLM_2(a)} \).

For all \( x \in B(\varphi, r) \),

\[
I - T(x) = I - T(\varphi) + T(\varphi) - T(x) = (I - T(\varphi))[I + (I - T(\varphi))^{-1}(T(\varphi) - T(x))].
\]

Since

\[
\|(I - T(\varphi))^{-1}(T(\varphi) - T(x))\| \leq \mu\|T(\varphi) - T(x)\| \leq \mu HcLM_2(a)r \leq \frac{1}{2},
\]

we conclude that \( I - T(x) \) is invertible and that its inverse is uniformly bounded on \( B(\varphi, r) \). In fact

\[
(I - T(x))^{-1} = [I + (I - T(\varphi))^{-1}(T(\varphi) - T(x))]^{-1}(I - T(\varphi))^{-1},
\]

so

\[
\|(I - T(x))^{-1}\| \leq \mu \sum_{k=0}^{\infty} \|(I - T(\varphi))^{-1}(T(\varphi) - T(x))\|^k \leq 2\mu.
\]

The function \((s, t) \rightarrow L(s, t) \frac{\partial F}{\partial u}(t, x(t))\) is in \( C^0([a, b] \times [a, b], \mathbb{R}) \). Hence, according to [9] or [2], \( T_n(x) \rightarrow p \rightarrow T(x) \), where \( \rightarrow p \rightarrow \) denotes the pointwise convergence, and the sequence \((T_n(x))_{n \geq 0}\) is collectively compact.

For all \( x \in B(\varphi, r) \), \( T_n(x) \) is a collectively compact approximation of \( T(x) \) (see [2]). Hence for \( n \) large enough, \( I - T_n(x) \) is invertible and \( (I - T_n(x))^{-1} \) is uniformly bounded in \( n \). This means that there is a constant \( c_x \) such that for \( n \) large enough,

\[
\|(I - T_n(x))^{-1}\| \leq c_x.
\]

This ends the proof.

Let \( A_n \) be the operator defined on \( B(\varphi, r) \) by

\[
A_n(x) := x + S_n(x)(K(x) + y - x),
\]

(17)
where

\[ S_n(x) := (I - T_n(x))^{-1}. \]  

(18)

Notice that we have

\[ A_n(\varphi) = \varphi. \]  

(19)

**Lemma 3** The operator \( S_n \) is Fréchet differentiable at \( \varphi \).

**Proof:** As for all \( h \in X \), \( T_n(x)(h)(s) = \sum_{j=0}^{n} w_{n,j}(s) L(s, t_{n,j}) \frac{\partial F}{\partial u}(t_{n,j}, x(t_{n,j})) h(t_{n,j}) \), the operator \( x \mapsto T_n(x) \) is differentiable at \( \varphi \), and we have for all \( h \in X \) and \( \delta \in X \),

\[ T_n'(x)(\delta, h)(s) = \sum_{j=0}^{n} w_{n,j}(s) L(s, t_{n,j}) \frac{\partial^2 F}{\partial^2 u}(t_{n,j}, x(t_{n,j})) \delta(t_{n,j}) h(t_{n,j}). \]

As, at the first order, \( (L + E)^{-1} - (L)^{-1} \preceq -(L)^{-1} E(L)^{-1} \), we have

\[ S_n(\varphi + \delta) - S_n(\varphi) = (I - T_n(\varphi + \delta))^{-1} - (I - T_n(\varphi))^{-1} \]

\[ = (I - T_n(\varphi) - T_n'(\varphi) \delta + O(\delta^2))^{-1} - (I - T_n(\varphi))^{-1} \]

\[ \preceq S_n(\varphi) T_n'(\varphi) \delta S_n(\varphi), \]

so that \( S_n \) is Fréchet differentiable at \( \varphi \) and

\[ S_n'(\varphi) \delta = S_n(\varphi) T_n'(\varphi) \delta S_n(\varphi). \]

**Lemma 4** The operator \( A_n \) is Fréchet differentiable at \( \varphi \). For \( n \) large enough,

\[ \rho(A_n'(\varphi)) < 1. \]  

(20)

**Proof:** Notice that \( A_n(\varphi) = \varphi \), hence

\[ A_n(\varphi + h) - A_n(\varphi) = \varphi + h + S_n(\varphi + h)(K(\varphi + h) + y - \varphi - h) - \varphi \]

\[ = h + S_n(\varphi + h)(K(\varphi) + T(\varphi) h + O(h^2) + y - \varphi - h) \]

\[ = h + S_n(\varphi + h) ( T(\varphi) - I ) h + O(h^2), \]

hence \( A_n \) is differentiable at \( \varphi \) and

\[ A_n'(\varphi) = I - S_n(\varphi)(I - T(\varphi)). \]  

(21)
We have

\begin{equation}
\rho(I - S_n(\varphi)(I - T(\varphi))) = \inf_n \|(I - S_n(\varphi)(I - T(\varphi)))^n\|^\frac{1}{n}
\end{equation}

(22)

\begin{equation}
\leq \|(I - S_n(\varphi)(I - T(\varphi)))^2\|^\frac{1}{2}.
\end{equation}

(23)

Since \((I - S_n(\varphi)(I - T(\varphi))) = S_n(\varphi)(T(\varphi) - T_n(\varphi)),\)

\[
\|(I - S_n(\varphi)(I - T(\varphi)))^2\| = \|S_n(\varphi)(T(\varphi) - T_n(\varphi))S_n(\varphi)(T(\varphi) - T_n(\varphi))\|
\leq \|S_n(\varphi)\| \| (T(\varphi) - T_n(\varphi)) S_n(\varphi) (T(\varphi) - T_n(\varphi)) \|
\leq c_\varphi \| (T(\varphi) - T_n(\varphi)) S_n(\varphi) (T(\varphi) - T_n(\varphi)) \|.
\]

As \(S_n(\varphi)\) is uniformly bounded (see Lemma 2) and \((T(\varphi) - T_n(\varphi))\) is collectively compact, the closure of the set \(S := \cup_n \{S_n(\varphi)(T(\varphi) - T_n(\varphi))x, \|x\| \leq 1\}\) is compact so that

\[
\|(I - S_n(\varphi)(I - T(\varphi)))^2\| \leq c_\varphi \sup_{x \in S} \| (T(\varphi) - T_n(\varphi)) x \| \xrightarrow{n \to \infty} 0.
\]

Then

\[
\rho(A_n'(\varphi)) \xrightarrow{n \to \infty} 0.
\]

This ends the proof.

\begin{flushright}
\hfill \blacksquare
\end{flushright}

**Theorem 1** Under assumptions (H0) to (H4), there exists \(r > 0\) such that, for a fixed \(n\) large enough to have

\[
\rho_n := \rho(A_n'(\varphi)) < 1,
\]

and for any \(\varepsilon > 0\) such that \(\rho_n + \varepsilon < 1\), there exist \(B(\varphi, r'_n, \varepsilon) \subset B(\varphi, r)\) and \(B(\varphi, r_n, \varepsilon) \subset B(\varphi, r)\) such that, if \(\varphi_n^{(0)} \in B(\varphi, r'_n, \varepsilon)\), then the sequence \((\varphi_n^{(k)})_{k \geq 0}\) solution of

\[
(I - T_n(\varphi_n^{(k)}))(\varphi_n^{(k+1)} - \varphi_n^{(k)}) = -\varphi_n^{(k)} + K(\varphi_n^{(k)}) + f,
\]

is defined, belongs to \(B(\varphi, r_n, \varepsilon)\) and

\[
\varphi_n^{(k)} \xrightarrow{k \to \infty} \varphi.
\]

Moreover, the following estimation holds:

\begin{equation}
\|(\varphi_n^{(k)} - \varphi)\| \leq r_{n, \varepsilon} (\rho_n + \varepsilon)^k.
\end{equation}

(24)

**Proof:** The exact solution \(\varphi\) of the nonlinear problem (2) is a fixed point of \(A_n\) (see (19)). From Lemma 4, \(A_n\) is Fréchet differentiable at \(\varphi\) and for \(n\) large enough \(\rho(A_n'(\varphi)) < 1\). The conditions needed to apply Proposition 1 to the operator \(A_n\) are satisfied so that Proposition 1 gives the result.

\begin{flushright}
\hfill \blacksquare
\end{flushright}
4 Numerical evidence

4.1 The classical product integration method

Let us recall the classical product integration method applied to nonlinear operators (see [19]).

The classical product integration approximation \( \psi_n \) solves the nonlinear equation

\[
\psi_n(s) - \int_0^b H(s, t) [L(s, t) F(t, \psi_n(t))] dt = y(s),
\]

where \([L(s, t) F(t, \psi_n(t))]_n\) denotes the piecewise linear interpolant of \( t \mapsto L(s, t) F(t, \psi_n(t)) \).

Using the uniform grid \( \Delta_n \), we obtain

\[
\psi_n(s) - \sum_{j=0}^n w_{n,j}(s) L(s, t_{n,j}) F(t_{n,j}, \psi_n(t_{n,j})) = y(s),
\]

where

\[
w_{n,0}(s) := \frac{1}{h_n} \int_{t_{n,0}}^{t_{n,1}} H(s, t)(t_{n,1} - t) dt,
\]

\[
w_{n,j}(s) := \frac{1}{h_n} \int_{t_{n,j-1}}^{t_{n,j}} H(s, t)(t - t_{n,j-1}) dt + \frac{1}{h_n} \int_{t_{n,j}}^{t_{n,j+1}} H(s, t)(t_{n,j+1} - t) dt, \quad j = 1, \ldots, n - 1,
\]

\[
w_{n,n}(s) := \frac{1}{h_n} \int_{t_{n,n-1}}^{t_{n,n}} H(s, t)(t - t_{n,n-1}) dt.
\]

Set

\[
Y_n(i) := y(t_{n,i}), \quad A_n(i, j) := w_{n,j}(t_{n,i}) L(t_{n,i}, t_{n,j}),
\]

\[
X_n := \begin{bmatrix}
\psi_n(t_{n,0}) \\
\vdots \\
\psi_n(t_{n,n})
\end{bmatrix}, \quad F(X_n) := \begin{bmatrix}
F(t_{n,0}, \psi_n(t_{n,0})) \\
\vdots \\
F(t_{n,n}, \psi_n(t_{n,n}))
\end{bmatrix}.
\]

Evaluating the equation (25) at the nodes \( t_{n,i}, \quad i = 0, \ldots, n \), the following non linear system is obtained:

\[
X_n - A_n F(X_n) = Y_n,
\]

which we can solve by the classical finite dimensional Newton method ([5] and [6]). Notice that the Newton’s iterates tends to \( \psi_n \) when \( k \to \infty \) and not to \( \varphi \), the exact solution. It means that the accuracy of the approximate solution is limited by \( n \).
4.2 Numerical Illustration

Numerical experiments are now carried out to illustrate the accuracy of our method. Let us consider in $\mathcal{X} := C^0([0,1], \mathbb{R})$, the operator

$$ K(\varphi)(s) := \int_0^1 \kappa(s, t, \varphi(t)) \, dt, \quad \varphi \in \mathcal{O} \subseteq \mathcal{X}, \; s \in [0,1], $$

with the real valued kernel function $\kappa$ :

$$(s, t, u) \in [0,1] \times [0,1] \times \mathbb{R} \mapsto \kappa(s, t, u) := \log(|s - t|) \sin(\pi u).$$

The exact solution of $\varphi(s) - K(\varphi)(s) = y(s)$ is

$$\varphi(s) := 1$$

for

$$y(s) := 1.$$ 

Implementation remark : To solve the linear system at each Newton iteration, the integral

$$\int_0^b \log(|s - t|) \sin(\varphi^{(k)}_n(t)) \, dt$$

needs to be evaluated. To evaluate it, we use the singularity subtraction technique ([3] and also [1]).

In the following, our method is called the linearization-discretization method and the classical one is called the discretization-linearization method. We compare them.

![Figure 1: Comparison of the errors for $n = 50$](image)
The classical discretization-linearization method can not be accurate if it is performed on a coarse grid \((n = 50)\). We obtain the approximate solution \(\psi_n\) of \(\varphi\) and the error is constant with the number of iterations. On the other hand, our linearization-discretization method approaches the exact solution \(\varphi\) even if the discretization is done with a coarse grid. It confirms the theoretical result.

![Figure 2: errors of the linearization-discretization method for increasing values of \(n\)](image)

When \(n\) increases, the number of Newton’s iterations needed to reach a fixed accuracy decreases. It can be useful to apply higher order product integration methods (see the idea of Diogo, Franco and Lima in [15]), instead of the product trapezoidal rule, to reduce the number of Newton’s iterations and therefore the computational cost of the method. An high order product integration approximation, adapted to a Fredholm equation, can also be a good starting point for the Newton’s iterations of our method.

**Acknowledgements**

The first author is partially supported by the Indo-French Centre for Applied Mathematics (IFCAM).

**References**

[1] M. Ahues, A. Largillier and B. V. Limaye, *Spectral Computations for Bounded Operators*, Applied Mathematics 18, Chapman and Hall/CRC, 2001.
[2] P.M Anselone, *Collectively Compact Operator Approximation Theory and Application to Integral Equations*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1971.

[3] P.M Anselone, Singularity Subtraction in the Numerical Solution of Integral Equations, *J. Austral. Math. Soc. (Series B)*, 22, pp 408-418, 1981.

[4] R. Ansorge, Convergence of Discretizations of Nonlinear Problems. A General Approach, *Z. angew. Math. Mech.* 73, 10, pp 239-253, 1993.

[5] I. K. Argyros, *Convergence and Applications of Newton-type Iterations*, Springer Science+Business Media, LLC, 2008.

[6] I. K. Argyros, Some methods for finding errors bounds for Newton-like methods under mild differentiability conditions, *Acta Math. Hung.*, 61 (3-4): pp 183-194, 1993.

[7] K. Atkinson, *Extensions of the Nyström Method for the Numerical Solution of Integral Equations of the Second Kind*, Ph.D. dissertation, Univ. of Wisconsin, Madison, 1966

[8] K. Atkinson, The Numerical Solution of Integral Equations of the Second Kind for nonlinear integral equation, *SIAM J. Num. Anal.*, 4: pp 337-348, 1967.

[9] K. E. Atkinson, *The Numerical Solution of Integral Equations of the Second Kind*, Cambridge University Press, 1997.

[10] K. E. Atkinson and F. A. Potra, Projection and iterated projection methods for nonlinear integral equations, *SIAM J. Numer. Anal.*, 24 (6): pp 1352-1373, 1987.

[11] K. E. Atkinson and J. Flores, The discrete collocation method for nonlinear integral equation, *IMA J. Numer. Anal.*, 13: pp 195-213, 1993.

[12] K.E. Atkinson, A Survey of Numerical Methods for Solving Nonlinear Integral Equation, *Journal of integral equations and applications*, Vol. 4, No. 1, pp 15-46, Winter 1992.

[13] H. Brunner, A. Pedas, G. Vainikko, The piecewise polynomial collocation method for nonlinear weakly singular Volterra Equations. *Helsinki University of Technology, institute of Mathematics*, Research Report A392, 1997.

[14] D. R. Dellew and M. B. Friedman, Accelerated projection and iterated projection methods with applications to nonlinear integral equations, *SIAM J. Numer. Anal.*, 28 (1): pp 236-250, 1991.

[15] T. Diogo, N. B. Franco and P. Lima, High Order Product Integration Method for a Volterra Integral Equation with Logarithmic singular Kernel, *Communications on Pure and Applied Analysis* Vo. 3, Issue 2, pp 217-235, June 2004

[16] L. Grammont, R.P. Kulkarni and P. Vasconcelos, Modified projection and the iterated modified projection methods for nonlinear integral equations, *Journal of Integral Equations and Applications*, vol 25, number 4, pp 481-516, 2013.

[17] L. Grammont, R.P. Kulkarni, T.J. Nidhin, Modified projection method for Urysohn integral equations with non smooth kernels, *Journal of Computational and Applied Mathematics* 294: pp 309-322, 2016.
[18] H. Kaneko, R.D Noren and Y. Xu, Regularity of the Solution of Hammerstein Equations With Weakly Singular Kernel, *Integral Equations and Operator Theory* Vol. 13, pp 660-670, 1990.

[19] H. Kaneko, R.D Noren and Y. Xu, Numerical solution for Weakly Singular Hammerstein Equations And Their Superconvergence, *J. Integral Equation Appl.* Volume 4, Number 3, pp 391-407, Summer 1992.

[20] L.V Kantorovich, Functional analysis and applied mathematics *Uspehi Mat. Nauk* 3: pp 89-185, 1948. Translated from the Russian by Curtis Benster, in *NSB report 1509* ed. by G. Forsythe, 1952.

[21] L.V Kantorovich and G. Akilov, *Functional analysis in normed spaces*, 2nd edition, Pergamon Press, 1982. Translated from the Russian by Curtis Benster.

[22] M. A. Krasnoselskii and P. P. Zabreiko, *Geometrical Methods of Nonlinear Analysis*, Springer Verlag, Berlin, Heidelberg, New York, Tokyo, 1984.

[23] M. A. Krasnoselskii, G. Vainikko, P. P. Zabreiko, Ya. B. Rutitskii and V. Ya. Stetsenko, *Approximate Solution of Operator Equations*, Noordhoff, Groningen, the Nederlands, 1972.

[24] A. Pedas and G. Vainikko, Superconvergence of piecewise polynomial collocations for nonlinear weakly singular integral equations, *J. Integral Equat. Appl.*, volume 9, number 4, pp 379-406, Fall 1997.

[25] A. Pedas and G. Vainikko, The smoothness of solutions to nonlinear weakly singular integral equations, *Journal for Analysis and its Applications*, volume 13, number 3, pp 463-476, 1997.

[26] C. Schneider, Product integration for weakly singular integral equations, *Math. Comp.* 36, pp 207-213, 1981.

[27] G. M. Vainikko, Galerkin’s perturbation method and general theory of approximate methods for nonlinear equations, *USSR Comput. Math. and Math. Phys.*, 7 (4): pp 1-58, 1967.

[28] A. Young, The application of approximate product integration to the numerical solution of integral equations, *Proc. Royal Soc.*, London A224 : pp 561-573, 1954.