SPHERICAL STEIN SPACES

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Abstract

Let $X$ be an irreducible reduced complex space on which a connected compact Lie group $K$ acts by holomorphic automorphisms. Let $G$ be the complexification of $K$ and $\mathfrak{g}$ the Lie algebra of $G$. Following the theory of algebraic transformation groups, we call the complex space $X$ spherical if $X$ is normal and its tangent space at some point is generated by the vector fields from a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$. We give several characterizations of spherical Stein spaces. In particular, we prove that a connected Stein manifold $X$ is spherical if and only if the algebra of $K$-invariant differential operators on $X$ is commutative.

1. Introduction

Let $X = (X, \mathcal{O})$ be a complex space, $T_x(X)$ the tangent space at $x \in X$, and $\mathcal{T} = \mathcal{T}_X$ the tangent sheaf of $X$. We will consider an action of a connected compact Lie group $K$ on $X$. We tacitly assume that our action is continuous and that each element of $K$ acts as a holomorphic transformation of $X$. The complexification of $K$ will be denoted by $G$. By definition, $G$ is a reductive algebraic group over $\mathbb{C}$ containing $K$ as a maximal compact subgroup. The action of $K$ gives rise to a local holomorphic action of $G$ on $X$, and so we have the associated Lie homomorphism $\mathfrak{g} \rightarrow \mathcal{T}(X)$. Though this homomorphism need not be injective even if $K$ acts effectively, we sometimes regard the elements of $\mathfrak{g}$ as vector fields on $X$.

Suppose now that $Y$ is an irreducible algebraic variety with an algebraic $G$-action and $L$ an algebraic line bundle on $Y$ with $G$-linearization. Assume that a Borel subgroup $B \subset G$ has an open orbit on $Y$. Then it is known that $\Gamma(Y, L)$ is a multiplicity free $G$-module, i.e., each irreducible $G$-module occurs in $\Gamma(Y, L)$ with multiplicity 0 or 1. This was first shown in [Se], and the argument is as follows. If $B$ has two weight vectors $s_1, s_2 \in \Gamma(Y, L)$ of the same weight then $f = s_1/s_2$ is a $B$-invariant rational function which should be constant on the open $B$-orbit and hence everywhere on $Y$. Thus a highest weight vector in $\Gamma(Y, L)$ is uniquely, up to a scalar multiple, determined by its weight.

In particular, the algebra of regular functions $\mathbb{C}[Y]$ is a multiplicity free $G$-module. Moreover, for $Y$ affine the converse is also true, see [VK]. Our first goal is a generalization of these results to complex spaces.

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We recall that an irreducible algebraic $G$-variety $Y$ is called spherical if $Y$ is normal and $B$ acts on $Y$ with an open orbit. Similarly, we say that an irreducible reduced complex space $X$ is spherical (under a $K$-action) if $X$ is normal and

(a) there exists a point $x \in X$ such that $T_x(X)$ is generated by the elements of $b$, where $b$ is a Borel subalgebra of $g$. Since any two Borel subalgebras are conjugated by an element of $K$, (a) is independent of the choice of $b$.

**Theorem 1** Let $X$ be an irreducible reduced complex space, $K$ a connected compact Lie group acting on $X$, and $L$ a holomorphic line bundle on $X$ with $K$-linearization. Assume that $X$ satisfies (a). Then $\Gamma(X, L)$ is a multiplicity free $K$-module. In particular,

(b) $\mathcal{O}(X)$ is a multiplicity free $K$-module.

We are mainly interested in $K$-actions on Stein spaces. For a reduced Stein space $X$ one has the complexification theorem, see [He]. Namely, there exists another reduced Stein space $X^C$, on which $G = K^C$ acts holomorphically, and a $K$-equivariant open embedding $i : X \hookrightarrow X^C$, such that $i(X)$ is a Runge domain in $X^C$ and $G \cdot i(X) = X^C$.

**Theorem 2** Let $X$ be an irreducible reduced Stein space. Then (a) and (b) are equivalent and each of these conditions is equivalent to

(c) $X^C$ is an affine algebraic variety and $B$ has an open orbit on $X^C$.

If $X$ is normal then (a), (b) and (c) are equivalent to

(d) $X$ is a $K$-invariant domain in a spherical affine $G$-variety.

The proof of (b)$\Rightarrow$(c) for Stein manifolds is sketched in [HW], see p.266 - 267. We refer the reader to [Ma] for the definition (due to A.Grothendieck) of a linear differential operator with holomorphic coefficients on a complex space $X$. We recall that for local models of complex spaces that definition is equivalent to the following one. Let $(X, \mathcal{O}_X)$ be a model space in a Stein domain $U \subset \mathbb{C}^n$, defined by a coherent ideal sheaf $\mathcal{J} \subset \mathcal{O}_U$. Then linear differential operators with holomorphic coefficients on $X$ are such operators (in the usual sense) on $U$, which preserve $\mathcal{J}$, modulo those which map $\mathcal{O}_U$ in $\mathcal{J}$.

Let $\mathcal{D}_k = \mathcal{D}_{k,X}$ and $\mathcal{D} = \mathcal{D}_X$ be the sheaves of germs of such differential operators of order not exceeding $k$ and, respectively, of any order. For a $K$-sheaf $\mathcal{F}$ on $X$ and for a $K$-stable subset $U \subset X$ we denote by $\mathcal{F}(U)^K$ the set of all $K$-invariant sections of $\mathcal{F}$ on $U$. The sheaves $\mathcal{D}_k$ and $\mathcal{D}$ are $K$-sheaves. Moreover, $\mathcal{D}(X)$ is a $\mathbb{C}$-algebra on which $K$ acts as a group of automorphisms. In what follows, we consider $\mathcal{T}(X)$ as a $K$-submodule of $\mathcal{D}(X)$. Our second goal is a characterization of spherical Stein manifolds in terms of invariant differential operators. Again, such a characterization is known for non-singular affine algebraic varieties, see [HU],[Ag].

**Theorem 3** (i) If $X$ is a (not necessarily reduced) Stein $K$-space satisfying (b) then the algebra of invariant differential operators $\mathcal{D}(X)^K$ is commutative. In particular, $\mathcal{D}(X)^K$ is commutative for spherical Stein spaces.

(ii) If $X$ is a connected Stein manifold and $\mathcal{D}(X)^K$ is commutative then $X$ is spherical.
In the smooth case, (i) is not knew. Moreover, instead of compactness of $K$, one may assume that the union of irreducible invariant Hilbert subspaces in $\mathcal{O}(X)$ is total, see [FT], p.390. Assertion (ii) answers a question raised by J.-J. Loeb in a conversation with the first author several years ago.

Let $\mathcal{D}_{alg}(Y)$ be the algebra of algebraic linear differential operators on an algebraic variety $Y$. Suppose that $Y$ is acted on by a reductive group $G$ and consider the subalgebra of $G$-invariant elements $\mathcal{D}_{alg}(Y)^G \subset \mathcal{D}_{alg}(Y)$. For $Y$ spherical, not necessarily affine, $\mathcal{D}_{alg}(Y)^G$ is a polynomial algebra in $r$ generators, where $r$ is the rank of $Y$, see [Kn]. It is therefore interesting to ask whether $\mathcal{D}(X)^K$ is commutative for all spherical complex $K$-spaces. On the other hand, even if $Y$ is affine, the commutativity of $\mathcal{D}_{alg}(Y)^G$ does not imply sphericity of $Y$ for varieties with (normal) singularities. Moreover, it can happen that all $G$-orbits on $Y$ have positive codimension. The example is actually contained in [BGG] and is explained below, see Sect.4, example 3. This shows that (ii) in Theorem 3 is no longer true for Stein spaces with singularities.

The authors would like to thank Friedrich Knop for calling their attention to the paper [Ag] and for useful remarks.

2. Preliminary results

Let $z_1, \ldots, z_n$ be the coordinates in $\mathbb{C}^n$. We write $\partial_j$ instead of $\frac{\partial}{\partial z_j}$ and set $|\alpha| = \alpha_1 + \ldots + \alpha_n$, $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ for any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}_+^n$.

**Lemma 1** Let $\Omega \subset \mathbb{C}^n$ be a domain containing the origin and $E_1, \ldots, E_n$ holomorphic vector fields in $\Omega$. Assume that

$$ E_i = \partial_i + F_i, \quad F_i(0) = 0 \quad (i = 1, \ldots, n). $$

Then for any $k \geq 1$ and for any sequence $j = \{j_i\}_{i=1}^k$ of length $k$ with $j_i \in \{1, \ldots, n\}$ one has

$$ E_{j_1} \cdot \ldots \cdot E_{j_k} = \partial_{j_1} \cdot \ldots \cdot \partial_{j_k} + \sum_{|\alpha|=k} c_{j,\alpha} \partial^\alpha + G_j, \quad (\ast) $$

where $c_{j,\alpha} \in \mathcal{O}(\Omega)$, $c_{j,\alpha}(0) = 0$ and $G_j \in \mathcal{D}_{k-1}(\Omega)$.

**Proof** Multiplying (\ast) by $E_i$ on the left we get

$$ E_i \cdot E_{j_1} \cdot \ldots \cdot E_{j_k} = \partial_i \cdot \partial_{j_1} \cdot \ldots \cdot \partial_{j_k} + F_i \cdot \partial_{j_1} \cdot \ldots \cdot \partial_{j_k} + \sum_{|\alpha|=k} c_{j,\alpha} E_i \cdot \partial^\alpha + $$

$$ + \{ \sum_{|\alpha|=k} (E_i c_{j,\alpha}) \partial^\alpha + E_i \cdot G_j \}, $$

where the operator in brackets has order $\leq k$. Since $F_i(0) = 0$ and $c_{j,\alpha}(0) = 0$, our assertion follows by induction. \[\square\]
Lemma 2 With the notation above, let \( f \in \mathcal{O}(\Omega) \) be such a function that \( E_if \in \mathcal{O}(\Omega) \) for all \( i, \quad i = 1, \ldots, n \). If, in addition, \( f(0) = 0 \) then \( f = 0 \).

Proof It is easily seen by induction that

\[(E_{j_1} \cdots E_{j_k})f \in \mathcal{O}(\Omega)f\]

for all sequences \( j_1, \ldots, j_k \). We will show that all derivatives of \( f \) vanish at the origin. For \( k \geq 1 \) assume by induction that \( \partial^\alpha f(0) = 0 \) for all \( \alpha \) with \(|\alpha| < k\). Then, by Lemma 1,

\[(\partial_{j_1} \cdots \partial_{j_k})f(0) = (E_{j_1} \cdots E_{j_k})f(0) = 0,\]

and so we obtain

\[\partial^\alpha f(0) = 0\]

for all \( \alpha \) with \(|\alpha| = k\). \( \square \)

Lemma 3 Let \( X \) be a connected Stein manifold, \( K \) a compact connected Lie group acting on \( X \), and \( f \in \mathcal{O}(X)^K \) a non-constant function. Then there exists \( D \in \mathcal{T}(X)^K \), such that \( Df \neq 0 \).

Proof We identify \( X \) with its image \( i(X) \) in \( X^C \). Note that \( X^C \) is also non-singular. Furthermore, any \( K \)-invariant holomorphic function on \( X \) extends to a \( G \)-invariant holomorphic function on \( X^C \), see [He]. For this reason, we may assume that \( X = X^C \).

Let \( X//G \) be the categorical quotient and \( \pi : X \to X//G \) the quotient map. Take a closed orbit \( G \cdot x \subset X \), put \( p = \pi(x) \) and write \( H \) for the isotropy subgroup \( G_x \). Then \( H \) is reductive by Matsushima-Onishchik theorem and the action of \( G \) is locally described by the holomorphic version of Luna’s slice theorem, see [Sn] or [He]. Namely, there is a locally closed \( H \)-stable Stein subspace \( S \subset X \) containing \( x \), such that the natural map of the twisted product \( G \times_H S \to X \) is biholomorphic onto a \( \pi \)-saturated open Stein subset \( U \subset X \). Since \( X \) is non-singular, there is an \( H \)-module \( V \) and an \( H \)-equivariant biholomorphic map \( \varphi : S \to \varphi(S) \subset V \) of \( S \) onto an \( H \)-stable domain \( \varphi(S) \) containing the origin. We will identify \( S \) with \( \varphi(S) \) and \( U \) with \( G \times_H S \).

Let \( z_1, \ldots, z_n \) be the linear coordinate functions on \( V \) and \( E = \sum_{j=1}^n z_j \frac{\partial}{\partial z_j} \) the Euler vector field. If \( q_d \in C[V]^H \) is homogeneous of degree \( d > 0 \) then \( Ef = dq_d \) by Euler’s theorem. But \( E \) gives rise to a \( G \)-invariant vector field on \( U \) tangent to the fibers of the twisted product \( G \times_H S \). Call this vector field again \( E \). Then it follows that \( Ef \neq 0 \) for any non-constant \( G \)-invariant holomorphic function on \( U \). This proves our result for \( U \) in place of \( X \).

For any coherent \( G \)-sheaf \( \mathcal{F} \) on \( X \) the sheaf \((\pi_*\mathcal{F})^G\) is coherent on \( X//G \), see [HH]. In particular, this applies to \((\pi_*\mathcal{T})^G\) and (by the definition of the complex structure on \( X//G \)) to \((\pi_*\mathcal{O})^G\). Now, \( \pi(U) \) is an open neighborhood of \( p \in X//G \) which can be taken arbitrarily small. In particular, we may assume that \( \pi(U) \) has Runge property for any coherent sheaf on the Stein space \( X//G \). The assertion of the lemma follows from Runge approximation theorem for coherent sheaves, see [GR], Kap. V, § 6. Indeed, the subspaces \( \mathcal{O}(X)^G = (\pi_*\mathcal{O})^G(\pi(X)) \subset (\pi_*\mathcal{O})^G(\pi(U)) = \mathcal{O}(U)^G \) and \( \mathcal{T}(X)^G = (\pi_*\mathcal{T})^G(\pi(X)) \subset (\pi_*\mathcal{T})^G(\pi(U)) = \mathcal{T}(U)^G \) are dense and the Lie derivative \((D, f) \to Df\) is continuous in both arguments in the canonical Fréchet topology. \( \square \)
3. Proofs of the theorems

Proof of Theorem 1 Take a point \( x \in X \), such that \( T_x(X) \) is generated by global vector fields from \( \mathfrak{b} \). In a neighborhood of \( x \) we have a non-singular analytic set through \( x \) whose tangent space coincides with \( T_x(X) \). It follows that \( x \) is a non-singular point. Choose a coordinate neighborhood \( \Omega \) of \( x \), denote by \( z_1, \ldots, z_n \) the local coordinates satisfying \( z_i(x) = 0 \), and write \( \partial_i \) for \( \frac{\partial}{\partial z_i} \) in \( \Omega \). Then there exist \( E_1, \ldots, E_n \in \mathfrak{b} \) such that \( E_i(x) = \partial_i \) for all \( i \).

Let \( \mathcal{L} \) denote the sheaf of germs of holomorphic sections of \( L \). Without loss of generality assume that \( \mathcal{L}|\Omega \cong \mathcal{O} \). Let \( s_0 \in \mathcal{L}(\Omega) \) be a non-vanishing section.

Note that \( \mathcal{L}(X) \) and \( \mathcal{O}(X) \) are \( (K, \mathfrak{g}) \)-modules. Furthermore, the action of \( \mathfrak{g} \) is local in the sense that for any open set \( U \subset X \) the algebra \( \mathfrak{g} \) acts on \( \mathcal{O}(U) \) and \( \mathcal{L}(U) \) commuting with restriction maps. For the multiplication map \( \mathcal{O}(U) \times \mathcal{L}(U) \to \mathcal{L}(U) \) and any element of \( \mathfrak{g} \) we have the Leibniz rule.

Assume that \( \mathcal{L}(X) \) contains two isomorphic irreducible \( K \)-submodules which do not coincide as subspaces. Then these modules have the same highest weight with respect to \( \mathfrak{b} \). Let \( s_1, s_2 \in \mathcal{L}(X) \) be the corresponding weight vectors. The sections \( s_1, s_2 \) are linearly independent, and one has \( Es_k = \lambda(E)s_k \quad (k = 1, 2) \)

for all \( E \in \mathfrak{b} \), where \( \lambda : \mathfrak{b} \to \mathbb{C} \) is a Lie algebra character. Take a non-trivial linear combination \( s = c_1s_1 + c_2s_2 \), such that \( s(x) = 0 \). Restricting \( s \) to \( \Omega \), we can write \( s|\Omega = fs_0 \), where \( f \in \mathcal{O}(\Omega) \). Since \( Es_0 = \varphi_Es_0 \), where \( \varphi_E \in \mathcal{O}(\Omega) \), we get \( Ef = (\lambda(E) - \varphi_E)f \in \mathcal{O}(\Omega)f \)

by Leibniz rule. But then Lemma 2 shows that \( f = 0 \), contradictory to the fact that \( s_1, s_2 \) are linearly independent. \( \square \)

Remark Here is another proof of Theorem 1 for Stein spaces. Condition (a) implies in this case that \( B \) has an open orbit on \( X^C \). In particular, \( \mathcal{O}(X^C)^G = \mathbb{C} \). Since any two closed \( G \)-orbits are separated by invariant functions, \( G \) has exactly one closed orbit on \( X^C \). It follows that \( X^C \) is \( (G\text{-equivariantly biholomorphic to}) \) an affine algebraic variety on which \( G \) acts algebraically, see [Sn], Cor. 5.6. On the other hand, the line bundle \( L \) extends to a holomorphic line bundle \( \tilde{L} \) on \( X^C \) with \( G \)-linearization and, moreover, the multiplicities of irreducible \( K \)-modules in \( \Gamma(X, L) \) and \( \Gamma(X^C, \tilde{L}) \) are the same, see [HH], Cor. 3 and Identity Principle. The assertion follows now from the corresponding result for affine algebraic varieties. \( \square \)

Proof of Theorem 2 (a) \( \Rightarrow \) (b) is already proved.

(b) \( \Rightarrow \) (c). Since \( \mathcal{O}(X^C) \) is a (dense) \( K \)-invariant subspace in \( \mathcal{O}(X) \), it is a multiplicity free \( K \)-module and, equivalently, a multiplicity free \( G \)-module. In particular, \( \mathcal{O}(X^C)^G = \mathbb{C} \). As in the above remark, we see that \( X^C \) is an affine algebraic variety with algebraic action of \( G \). The algebra of regular functions on this variety is a multiplicity free \( G \)-module. Therefore \( B \) acts on \( X^C \) with an open orbit, see [VK].
(c) ⇒ (a) and (d) ⇒ (a) are obvious.

(c) ⇒ (d) follows from the fact each point of $X^C$ is contained in the $G$-orbit of some point of $i(X)$. Therefore, if $X$ is normal then $X^C$ is also normal. \hfill \Box

Proof of Theorem 3 (i) Let $\hat{K}$ denote the set of equivalence classes of irreducible linear representations of $K$. For each $\delta \in \hat{K}$ we have the isotypic component $O_\delta(X) \subset O(X)$ of type $\delta$. By a theorem of Harish-Chandra, $O_\delta(X)$ is closed in the canonical Fréchet topology of $O(X)$ and the sum of all subspaces $O_\delta(X)$ is dense, see [Ha-Ch] or [Akh], §5.1.

Clearly, each $O_\delta(X)$ is stable under $D(X)^K$. Condition (b) means that $O_\delta(X)$ is irreducible. Therefore any $D \in D(X)^K$ acts on $O_\delta(X)$ as a scalar operator by Schur’s lemma. It follows that if $D$ is of the form $D = [D_1,D_2]$, where $D_1,D_2 \in D(X)^K$, then $Df = 0$ for all $f \in O(X)$. For a Stein space, this implies $D = 0$. Indeed, let $x \in X$ be any point, $D_x$ the germ of $D$ at $x$ and $g_x \in O_x$ any element of the local ring. In some neighborhood $U$ of $x$ one can represent $g_x$ by a function $g \in O(U)$. Choose $U$ to be a Runge domain in $X$ so that the image of the restriction map $O(X) \rightarrow O(U)$ is dense in the canonical Fréchet topology. Since $D$ is continuous in this topology we get $Dg = 0$ showing that $D_x = 0$.

(ii) Suppose $X$ is a connected Stein manifold with commutative algebra $D(X)^K$. In order to prove that $X$ is spherical it suffices to show that $X^C$ is an affine algebraic $G$-variety. Indeed, since $X^C$ is obviously non-singular, the commutativity of $D_{alg}(X^C)$ implies then that $X^C$ is spherical, see [Ag], Satz 2.5. On the other hand, we know that $O(X^C)^G = \mathbb{C}$ implies that $X^C$ is an affine algebraic $G$-variety (see the above remark). Therefore, since $O(X^C)^G = O(X)^K$, we only have to check that $O(X)^K = \mathbb{C}$.

Assume the contrary. Let $f \in O(X)^K$ be a non-constant function. By Lemma 3 there is an invariant holomorphic vector field $D$ on $X$, such that $Df \neq 0$. Denote by $M_f$ the operator (of order 0) of multiplication by $f$. Then $M_f$ is also invariant and $[D,M_f] = Md_f \neq 0$. This contradicts the commutativity of $D(X)^K$. \hfill \Box

4. Some examples

1) Let $X$ be a compact complex manifold without non-constant meromorphic functions, $\dim X \geq 2$, and let $K = \{\text{id}\}$ be the trivial transformation group of $X$. For any holomorphic line bundle $L$ the space $\Gamma(X,L)$ has dimension 0 or 1. Thus $\Gamma(X,L)$ is a multiplicity free $K$-module. However, $X$ is obviously non-spherical, showing that the converse statement to Theorem 1 is false. Due to Rosenlicht’s theorem, this phenomenon does not occur in the algebraic setting. More precisely, suppose that a non-singular algebraic variety $X$ is non-spherical with respect to an algebraic action of a reductive group $G$. Then there exist a line bundle $L$ on $X$ and a $G$-linearization of $L$, such that some irreducible $G$-module occurs in $\Gamma(X,L)$ with multiplicity greater than 1.

2) Let $\Gamma$ be a discrete Zariski dense subgroup in $G = K^C$ and $X = G/\Gamma$. Then it is known that $O(X) = \mathbb{C}$, see e.g. [Akh], §5.3. In this situation, one has a natural isomorphism between $D(X)$ and the universal enveloping algebra $U(\mathfrak{g})$. Indeed, any $D \in$
\[ \mathcal{D}(X) \text{ can be viewed as a holomorphic differential operator on } G \text{ commuting with right translations } R_g \text{ for } g \in \Gamma. \] For any \( f \in \mathcal{O}(G) \) define a function \( \varphi = \varphi_f \in \mathcal{O}(G \times G) \) by
\[
\varphi(g, x) = (DR_{g^{-1}}f)(x) - (Df)(xg^{-1}).
\]
Then for \( \gamma \in \Gamma \) we have
\[
\varphi(g\gamma, x) = (DR_{\gamma^{-1}}R_{g^{-1}}f)(x) - (Df)(x\gamma^{-1}g^{-1}) =
\]
\[
= R_{\gamma^{-1}}(DR_{g^{-1}}f - R_{g^{-1}}Df)(x) = \varphi(g, x\gamma^{-1}).
\]
Since \( \mathcal{O}(X) = \mathbb{C} \), it follows that
\[
\varphi(g, x) = \varphi(e, xg^{-1}) = 0,
\]
showing that \( D \) commutes with all right translations.

The isomorphism \( \mathcal{D}(X) = U(\mathfrak{g}) \) is compatible with the \( G \)-action. Thus \( \mathcal{D}(X)^K \) is isomorphic to the center of \( U(\mathfrak{g}) \). Though \( \mathcal{D}(X)^K \) is commutative, \( X \) is spherical only if \( K \) is a torus. Thus, in Theorem 3 (ii) it is essential that \( X \) is Stein.

3) Following [BGG], consider the normal surface
\[
Y = \{ z = (z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1^3 + z_2^3 + z_3^3 = 0 \}
\]
with \( G = \mathbb{C}^* \) and \( K = \{ \zeta \in \mathbb{C}^* \mid |\zeta| = 1 \} \) acting on \( Y \) by \( z \to \zeta \cdot z \). The associated infinitesimal operator is the Euler vector field \( E = z_1 \partial_1 + z_2 \partial_2 + z_3 \partial_3 \in \mathcal{D}(Y) \). For any \( p \in \mathbb{Z}, \ p \geq 0 \), set
\[
\mathcal{O}^{(p)}(Y) = \{ f \in \mathcal{O}(Y) \mid Ef = pf \}, \quad \mathcal{D}^{(p)}(Y) = \{ D \in \mathcal{D}(Y) \mid [E, D] = pD \}.
\]
Note that \( \mathcal{O}^{(p)}(Y) \) is the space of all homogeneous functions of degree \( p \) on \( Y \), in particular, \( \mathcal{O}^{(p)}(Y) \subset \mathbb{C}[Y] \). On the other hand, it is easily seen that
\[
\mathcal{D}^{(p)}(Y) = \{ D \in \mathcal{D}(Y) \mid D \cdot \mathcal{O}^{(q)}(Y) \subset \mathcal{O}^{(p+q)}(Y) \text{ for all } q \geq 0 \},
\]
hence \( \mathcal{D}^{(p)}(Y) \subset \mathcal{D}_{alg}(Y) \). A differential operator \( D \in \mathcal{D}(Y) \) is \( G \)-invariant if and only if \( [E, D] = 0 \). Thus
\[
\mathcal{D}(Y)^K = \mathcal{D}_{alg}(Y)^G = \mathcal{D}^{(0)}(Y).
\]
One of the results in [BGG] yields an isomorphism of \( \mathbb{C} \)-algebras \( \mathcal{D}_{alg}(Y)^G \simeq \mathbb{C}[E] \). This shows that assertion (ii) in Theorem 3 is false for singular Stein spaces.

For the same reason, its algebraic counterpart is false if an affine variety has (normal) singularities. Moreover, one can say why the proof in the non-singular case does not go through for singular varieties (cf. [Bi], Lemma 4.2). That proof is based on the density property of \( \mathcal{D}_{alg}(Y) \). Namely, given \( l \) linearly independent functions \( f_1, \ldots, f_l \in \mathbb{C}[Y] \) and \( l \) arbitrary functions \( g_1, \ldots, g_l \in \mathbb{C}[Y] \), there exists an algebraic differential operator \( D \) such that \( Df_i = g_i \) for all \( i, \ i = 1, \ldots, l \). In general, this is no longer true for varieties with singularities, e.g., for the surface \( Y \) considered above. Indeed, it is shown in [BGG] that any \( D \in \mathcal{D}_{alg}(Y) \) can be written as a finite sum \( D = D_0 + D_1 + D_2 + \ldots \), where \( D_p \in \mathcal{D}^{(p)}(y) \), i.e., \( D_p \) raises the homogeneity degree of a function by \( p \). Thus density fails already for \( l = 1 \).
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