Generalized non-associative structures on the 7-sphere

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Abstract. In this paper we provide a more general class of non-associative products using
the exterior and Clifford bundles on the 7-sphere \( S^7 \). Some additional properties encompass
the previous formalisms presented in [1, 2] in the Clifford algebra context, and wider classes
of non-associative structures on \( S^7 \) are investigated, evinced by the directional non-associative
products and the mixed composition of generalized non-associative products between Clifford
algebra multivectors. These non-associative products are further generalized by considering
the non-associative shear of arbitrary Clifford bundle \( C_\ell_0 \cdot 7 \) elements into octonions. We assert
new properties inherited from the non-associative structure introduced in the whole Clifford
bundle on \( S^7 \), which naturally induce involutions on the Clifford bundle and provide immediate
generalizations concerning well-established formal results in, e.g., [3, 4, 5, 6, 7] and potential
applications in physics as pioneered by, e. g., [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

1. Introduction

This paper aims to provide a comprehensive investigation concerning a more general class of
non-associative structures on \( S^7 \), in the Clifford algebra formulation of octonionic generalized
products. As the octonionic product can be defined from the Clifford algebra structure, such
formalism is closely related to the algebraic and geometric structures associated with the sphere
\( S^7 \) [15, 16, 17]. Generalized octonionic algebras and Moufang-like identities can be accomplished
in this formalism, addressing their possible generalizations and additional properties, as well as
prominent applications in physics.

The triality principle introduced by Cartan [18, 19] asserts that in an 8-dimensional vector
space \( V \) there exists an order three automorphism, which cyclically permutes vectors and
semispinors — also seen as minimal left ideals of the Clifford algebra \( C_\ell_8 \cdot 0 \), or \( C_\ell_0 \cdot 8 \), or \( C_\ell_4 \cdot 4 \),
or their equivalent \( C_\ell_8 (\mathbb{C}) \) complex case — that carry non-equivalent representations of the
Spin(8) group. In [1] it was proved that new deformed octonionic units with respect to the non-
associative product between Clifford bundle sections and octonionic fields can be introduced,
in order to better investigate the generalization of Moufang identities in this context. Using
the formalism presented and its generalization [20, 21], the Poincaré superalgebra is obtained
from the Clifford orthosymplectic algebra [21]. It was also shown in [20] — following [21] —
that the Chevalley product, an order three automorphism on the vector space constructed as
the direct sum of maximal index vector spaces and their semispinor associated spaces, induces
trality like morphisms on some subspaces of the associated complexified exterior algebra. See [22] for details including historical notes. Some interesting applications may be found in [8, 9, 10, 11, 12, 23, 14, 24, 25, 26, 27, 28].

A similar question can be formulated, and it motivates the main aim of this paper: to investigate in what extent the well known results, concerning the octonionic product deformations on the tangent bundle on $S^7$, can be completely generalized to the whole exterior and Clifford bundles on $S^7$. Furthermore, although there is a plethora of new products that can be defined in such scenario, we restrict our formalism to the prominent products that are immediate generalization of the results in, i. e., Cederwall, Bengtsson, Rooman, Preitschopf, Brink [2, 3, 4, 5, 6, 7], and some potential applications regarding the same references directly – as well as other ones as [8, 9, 10, 11, 12, 13].

First, the original non-associative deformed products between octonions are reviewed [1, 2, 3, 4, 5], together with the extended octonionic products between octonions and Clifford multivectors, and also the extended generalized non-associative products between Clifford multivectors, in the light of [1, 2]. Our results are immediately led to the formalism in [2, 24] in the very particular case where the paravector component of an arbitrary multivector in $\mathcal{Cl}_{0,7}$ is taken into account.

The results in [1] are generalized and more possibilities are considered, when the so called directional non-associative products are explicitly taken into account in the light of the former formalism [1]. Also, more non-equivalent non-associative products are introduced in order to encompass the exterior and Clifford bundles on $S^7$, in a more general context than the cases presented in [1].

Here we try to get the nature of the non-associative structures that can be defined on the exterior and Clifford bundles on $S^7$, and it is verified that all the additional unexpected properties concerning the non-associative structure in the Clifford bundle on $S^7$ can not be probed when only the underlying structure of the tangent bundle on $S^7$ is considered, like the results in [4, 2, 6, 5]. The formalism here presented and studied probes additional properties that cannot be realized only in the tangent bundle on $S^7$. Naturally, all the well known results concerning the deformed formalism on the tangent bundle on $S^7$ are re-obtained, when we only consider the particular case where the paravector subspace $\Lambda^0(\mathbb{R}^{0,7}) \oplus \Lambda^1(\mathbb{R}^{0,7}) \simeq \mathbb{R} \oplus \mathbb{R}^{0,7}$, associated to the standard octonionic product, is taken into account. To emphasize, this formalism encompass the most general underlying subspace $\Lambda(\mathbb{R}^{0,7})$ associated to the Clifford algebra constructed on the tangent space at an arbitrary point $X \in S^7$. We want here to stress that the exterior algebra $\Lambda(\mathbb{R}^7)$ is a structure that does not depend on a metric structure, but our notation $\Lambda(\mathbb{R}^{0,7})$ is opted in order to emphasize the underlying vector space $\mathbb{R}^7$ endowed with a metric diag($-1, -1, -1, -1, -1, -1, -1$). In this space the octonions are naturally described.

Concerning the X-product and its equivalent matrix representation in the Appendix, it was shown in [2, 5] that it introduces the Hopf fibration $S^3 \ldots S^7 \rightarrow S^4$, and we can also search for some extended correspondence between the products non-associative products, that define a paralellizable torsion on $S^7$, and some kind of generalized geometric structure that can be led to the Hopf fibration $S^3 \ldots S^7 \rightarrow S^4$, in the very particular case when $u \in \Lambda^0(\mathbb{R}^{0,7}) \oplus \Lambda^1(\mathbb{R}^{0,7}) \simeq \mathbb{R} \oplus \mathbb{R}^{0,7}$. In this specific case the product $e_a \circ_a e_b$ is made identical to the $X$-product between $e_a$ and $e_b$.

We can still ask which properties hold, when we consider the exterior and Clifford bundle instead of the tangent bundle on $S^7$ only. The formalism presented in [2, 3, 6, 7] shows that the octonionic product can be deformed, in order to encompass the parallelizable torsion on $S^7$ [13, 5]. The $X$-product as presented is exactly twice the torsion components, and we prove that, instead of considering the underlying vector space $\Lambda^0(\mathbb{R}^{0,7}) \oplus \Lambda^1(\mathbb{R}^{0,7})$ associated with octonions algebra, it is possible to consider the whole Clifford algebra at an arbitrary point on $S^7$, with the underlying vector space associated with $\mathcal{Cl}_{0,7}$. Possible ramifications of this formalism into
its applications may provide manageable models that extends, e.g., [1, 2, 6].

This paper is organized as follows: Section II reviews some mathematical tools and techniques related to the octonionic algebra in the Clifford algebra arena, and Section III concentrates on the fundamental properties already introduced in [1], and also additional properties on these topics are provided. The new definitions reveal a wealth of unexpected results and the subtle difference arising in the generalization of the $u$-product and the directional $u$-product. These products are introduced with the purpose to get more general non-associative structures on $S^7$. In Section IV we summarize some properties in [1] and present some examples elucidating the motivation around the formalism. In Section V, new classes of non-associative products are introduced in the Clifford bundle on $S^7$, together with the directional non-associative products and some new examples concerning counter-examples on the Moufang identities, that do not hold in our extended formalism. In Section VI, following [5], the scalar product between octonions is defined, together with two respective extensions, introduced in [2] is extended with respect to the $\circ$-product defined in [1]. In Section VII, four Propositions are presented and demonstrated, together with two respective extensions, introducing new octonionic involutions induced by and arbitrary multivector $u \in C_{0,7}$. The developments here can settle some open questions addressed in, e.g., [3, 6]. In Appendices A-F the respective demonstrations of Propositions 1-4 are provided. In Appendix G the construction of the tangent bundle on $S^7$ is considered and in Appendix H the matrix representation associated to the $\circ$-product and also to the whole Clifford algebra $C_{0,7}$ basis, acting on an arbitrary point at $S^7$, is presented following previous considerations in [2].

2. Preliminaries

Let $V$ be a finite $n$-dimensional real vector space and $V^*$ denotes its dual. We consider the tensor algebra $\bigoplus_{i=0}^{\infty} T^i(V)$ from which we restrict our attention to the space $\Lambda(V) = \bigoplus_{k=0}^{n} \Lambda^k(V)$ of multivectors over $V$. $\Lambda^k(V)$ denotes the space of the antisymmetric $k$-tensors, isomorphic to the $k$-forms vector space. Given $\psi \in \Lambda(V)$, $\hat{\psi}$ denotes the reversal, an algebra anti-automorphism given by $\hat{\psi} = (-1)^{|k|/2} \psi$ ($|k|$ denotes the integer part of $k$). $\hat{\psi}$ denotes the main automorphism or graded involution, given by $\hat{\psi} = (-1)^k \psi$. The conjugation is defined as the reversal followed by the main automorphism. If $V$ is endowed with a non-degenerate, symmetric, bilinear map $g : V^* \times V^* \to \mathbb{R}$, it is possible to extend $g$ to $\Lambda(V)$. Given $\psi = u_1 \wedge \cdots \wedge u_k$ and $\phi = v_1 \wedge \cdots \wedge v_l$, for $u_i, v_j \in V^*$, one defines $g(\psi, \phi) = \det(g(u_i, v_j))$ if $k = l$ and $g(\psi, \phi) = 0$ if $k \neq l$. The projection of a multivector $\psi = \psi_0 + \psi_1 + \cdots + \psi_n, \psi_k \in \Lambda^k(V)$, on its $p$-vector part is given by $\langle \psi \rangle_p = \psi_p$. Given $\psi, \phi, \xi \in \Lambda(V)$, the left contraction is defined implicitly by $g(\psi \lhd \phi, \xi) = g(\phi, \psi \wedge \xi)$. For $a \in \mathbb{R}$, it follows that $v \lhd a = 0$. Given $v \in V$, the Leibniz rule $v \lhd (\psi \wedge \phi) = (v \lhd \psi) \wedge \phi + \psi \wedge (v \lhd \phi)$ holds. The right contraction is analogously defined $g(\psi \rhd \phi, \xi) = g(\phi, \psi \wedge \xi)$ and its associated Leibniz rule $(\psi \wedge \phi) \lhd v = \psi \wedge (\phi \lhd v) + (\psi \rhd v) \wedge \phi$ holds. Both contractions are related by $v \lhd \psi = -\psi \lhd v$. The Clifford product between $\psi, \phi \in V$ is given by $\psi \phi = \psi \wedge \phi + \psi \lhd \phi$. The Grassmann algebra $(\Lambda(V), g)$ endowed with the Clifford product is denoted by $Cl(V, g)$ or $Cl_{p,q}$, the Clifford algebra associated with $V \cong \mathbb{R}^{p,q}$, $p + q = n$.

3. Octonions

The octonionic algebra $\mathbb{O}$ can be considered as the paravector space $\mathbb{R} \oplus \mathbb{R}^{0,7}$ [30] endowed with the product $\circ : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7}$, the so called octonionic standard product. The identity $e_0 = 1$ and an orthonormal basis $(e_a)_{a=1}^{7}$, in the underlying paravector space $\mathbb{R} \oplus \mathbb{R}^{0,7} \to Cl_{0,7}$ associated with $\mathbb{O}$, generate the octonion algebra [22, 31, 32]. The octonionic product can be constructed using the Clifford algebra $Cl_{0,7}$ as

$$A \circ B = \langle AB(1 - \psi) \rangle_{0 \oplus 1}, \quad A, B \in \mathbb{R} \oplus \mathbb{R}^{0,7},$$

(1)
where \( \psi = e_1e_2e_6 + e_2e_3e_7 + e_3e_4e_1 + e_4e_5e_2 + e_5e_6e_3 + e_6e_7e_4 + e_7e_1e_5 \in \Lambda^3(\mathbb{R}^{0,7}) \subset C_{l0,7,}
\) and the juxtaposition denotes the Clifford product [29]. The contrivance of introducing the octonionic product from the Clifford product in this context is to present hereon our formalism using Clifford algebras and the subsequent generalizations to the whole exterior and Clifford bundles on \( S^7 \). Indeed, as \( \mathcal{O} \) is isomorphic to \( \mathbb{R} \oplus \mathbb{R}^{0,7} \) as a vector space, the octonionic product takes two arbitrary elements of the paravecotor space \( \mathbb{R} \oplus \mathbb{R}^{0,7} \) — which is itself endowed with the octonionic product — resulting in another element of the paravecotor space. But looking at the octonions in the Clifford algebra arena it is possible to go beyond the paravector space and explore the whole exterior algebra underlying the Clifford algebra, which is one possibility we use to generalize the \( X \) - and \( XY \)-products, extending also the results in [1].

The exterior algebra \( \Lambda(\mathbb{R}^7) \) is denoted by \( \Lambda(\mathbb{R}^{0,7}) \), as in [1], to emphasize the underlying octonionic formalism character. It is well known that the exterior algebra is constructed on a vector space, without mention to any metric structure, and the notation \( \Lambda(\mathbb{R}^{0,7}) \) seems \textit{a priori} out of context, but we want to emphasize the fact that the underlying vector space is related to \( \mathbb{R}^7 \) endowed with the metric \( g \) of signature \((0,7)\).

In a close analogy, the octonionic product can be also expressed in terms of the Clifford algebra on the Euclidean space \( \mathbb{R}^{8,0} \) according to [29], in terms of a basis \( \{e_1, \ldots, e_8\} \) of \( \mathbb{R}^{8,0} \). The octonionic product is given by
\[
A \circ B = \langle A e_8 B (1 + \phi) - (1 - e_{12.8}) \rangle_1, \quad A, B \in \mathbb{R}^{8,0},
\]
where \( \phi = e_1e_2e_6 + e_2e_3e_7 + e_3e_4e_1 + e_4e_5e_2 + e_5e_6e_3 + e_6e_7e_4 + e_7e_1e_5 \in \Lambda^2(\mathbb{R}^{8,0}) \) and \( \frac{1}{2}(1 + \phi) \) is an idempotent. Both the approaches are equivalent: bivectors in \( C_{l8,0} \) correspond to the paravecctors of \( C_{l0,7} \) when the isomorphism \( e_\sigma \mapsto e_\sigma e_8 := e_\sigma, \ \sigma = 1,2,\ldots,7 \) is considered and \( e_8e_8 = 1 = e_0 \) denotes the octonionic unit in \( \mathbb{R} \mapsto \mathbb{R} \oplus \mathbb{R}^{0,7} \). In fact, \( e_8^2 = (e_\sigma e_8)^2 = -e_\sigma e_\sigma e_8 e_8 = -1 \). The octonion unit in \( \mathbb{R} \oplus \mathbb{R}^{0,7} \) corresponds to the vector \( e_8 \in \mathbb{R}^8 \). More details can be seen, e.g., in [29].

The usual rules between basis elements under the octonionic product are verified when both the Eqs.(1) and (2) are used, and the definition in Eq.(1) is regarded hereupon, in this case \( \mathbb{R}^7 \) is considered instead of the usual \( \mathbb{R}^8 \) underlying vector space, concerning the definition in Eq.(2). In particular in the context of Eq.(1) the octonion multiplication table is constructed by
\[
e_a \circ e_b = e^c_{ab} e_c - \delta_{ab} \quad (a, b, c = 1, \ldots, 7),
\]
where \( e^c_{ab} = 1 \) for the cyclic permutations \( (abc) = (126), (237), (341), (452), (563), (674), (715) \). Explicitly, the multiplication is given by Table 1, wherein all the relations can be expressed as \( e_a \circ e_{a+1} = e_{a+5} \mod 7 \). Some useful identities follow from Eq.(3):
\[
\epsilon_{abc} \epsilon_{def} + \epsilon_{bde} \epsilon_{acf} = \delta_{ad} \delta_{bf} - \delta_{af} \delta_{db} - 2 \delta_{ab} \delta_{bf},
\]
and when an analogous of the Jacobi formula is computed, it reads
\[
[e_1, [e_j, e_k]] + [e_k, [e_i, e_j]] + [e_j, [e_k, e_i]] = 4(e_{ikm} \epsilon_{jmn} + e_{jim} \epsilon_{kmn} + e_{jim} \epsilon_{jmn}) e_n = 3 \epsilon_{ijkl} e_l \quad (4)
\]
where \( \epsilon_{ijkl} = -\epsilon_{mjkl} \epsilon_{nml} - \delta_{il} \delta_{jk} + \delta_{ik} \delta_{jl} \) [11]. Since the underlying vector space of \( \mathcal{O} \) can be considered as being \( \mathbb{R} \oplus \mathbb{R}^{0,7} \mapsto C_{l0,7} \), the Clifford conjugation of \( X = X^0 + X^a e_a \in \mathcal{O} \) is given by \( \bar{X} = X^0 - X^a e_a \), where \( X^0 \) and \( X^a \) are real coefficients. The Einstein's summation convention — asserting that when two equal indices appear, it indicates a sum over this indices, i.e., \( X^a e_a \) denotes \( X^1 e_1 + X^2 e_2 + \cdots + X^7 e_7 = \sum_{a=1}^7 X^a e_a \) — is used hereon. The underlying structure of the vector space is unable to assert whether the \( \mathcal{O} \)-conjugation is equivalent to the graded involution of the tensorial algebra, extended to the exterior and Clifford algebras, since the octonionic conjugation \( \bar{X} \) can be written either as \( \bar{X} \) or \( \bar{X} \), in terms of the Clifford algebra morphisms. But it is well known that the octonionic conjugation \( \bar{X} \) is involutive and an anti-automorphism, what immediately excludes the graded involution.
Table 1. The octonionic product between units in the \( \mathbb{O}_{4,5} \) convention.

|    | e₁ | e₂ | e₃ | e₄ | e₅ | e₆ | e₇ |
|----|----|----|----|----|----|----|----|
| e₁ | -1 | e₆ | e₂ | -e₃| e₇ | -e₂| -e₅|
| e₂ | -e₆| 1  | e₄ | -1 | e₇ | e₄ | e₁ |
| e₃ | -e₄| -e₇| 1  | e₁ | e₆ | -e₅| e₂ |
| e₄ | e₃ | -e₅| -e₁| -1 | e₂ | e₇ | -e₆|
| e₅ | -e₇| e₄ | -e₆| -1 | e₃ | e₁ |    |
| e₆ | e₂ | -e₁| e₅ | -e₇| -e₃| -1 | e₄ |
| e₇ | e₅ | e₃ | -e₂| e₆ | -e₁| -e₄| -1 |

4. The \( \bullet \)-product and the \( \circ \)-product on \( S^7 \)

Given \( X, Y \in \mathbb{R} \oplus \mathbb{R}^{0,7} \) fixed but arbitrary such that \( XX = X = YY = YY \) \( (X, Y \in S^7) \), the \( X \)-product is defined by [6, 3, 2]

\[
A \circ_X B := (A \circ X) \circ (\bar{X} \circ B).
\]  

(5)

For a particular value when \( X = B \in S^7 \), the usual octonionic product is obtained \( (A \circ B) \circ (B \circ B) = A \circ B \). The expressions below are shown in, e.g., [2, 6]

\[
(A \circ X) \circ (\bar{X} \circ B) = X \circ ((\bar{X} \circ A) \circ B) = (A \circ (B \circ X)) \circ X.
\]  

(6)

The \( XY \)-product is defined as

\[
A \circ_{X,Y} B := (A \circ X) \circ (\bar{Y} \circ B),
\]  

(7)

and in particular the \( (1, X) \)-product is given by

\[
A \circ_{1,X} B := A \circ (\bar{X} \circ B),
\]  

(8)

where \( X \) is the unit of the \( (1, X) \)-product above, since \( A \circ_{1,X} X = X \circ_{1,X} A = A \) [2, 6].

A non-associative product called the \( u \)-product was introduced in [1] as a natural generalization for the \( X \)-product. We briefly review the respective products obtained in [1] for completeness, and subsequently present a more general class of non-associative products on \( S^7 \). For homogeneous multivectors \( u = u_1 \ldots u_k \in \Lambda^k(\mathbb{R}^{0,7}) \hookrightarrow \mathcal{C} \ell_{0,7} \), where \( \{u_p\}_{p=1}^k \subset \mathbb{R}^{0,7} \) and \( A \in \mathbb{R} \oplus \mathbb{R}^{0,7} \), the products \( \bullet \), and \( \cdot \), are defined by [1]

\[
\bullet : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times \Lambda^k(\mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7}
\]

\[
(A, u) \rightarrow A \bullet u = ((u_2 \circ (u_1 \circ u_2) \circ \cdots) \circ u_k) \circ A
\]  

(9)

\[
\cdot : \Lambda^k(\mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \rightarrow \mathbb{R} \oplus \mathbb{R}^{0,7}
\]

\[
(u, A) \rightarrow u \circ A = \circ (u_2 \circ (\cdots (u_k \circ (u_k \circ A)))
\]  

(10)

The symbol \( \bullet \) uniquely denotes both the products \( \bullet \), and \( \cdot \), in Eqs.(9) and (10), since for this product it is implicit where the octonion enters in. The products (9, 10) are extended to the whole \( \Lambda(\mathbb{R}^{0,7}) \) by linearity.
Remark 1: The expression $XX = X \bar{X} = 1$ defines $S^7$, for $X \in \mathcal{O}$. The product in (5) $A \circ X B = (A \circ X) \circ (X \circ B)$ was motivated in the sense that it can be written as $A \circ X B = (A \circ X) \circ (X^{-1} \circ B)$, since for all $X \in S^7$ it follows that $X^{-1} = X$. Now, for a multivector $u \in \mathcal{C}_{0,7}$, we generalize this product in [1], and we must emphasize that in order that $(A \bullet u) \circ (u^{-1} \bullet B)$ be well defined, there must exist an inverse $u^{-1}$ associated to $u \in \mathcal{C}_{0,7}$.

As $u\overline{u}$ has not only a scalar component in general — indeed $u\overline{u} \neq \overline{u}u$ in general — the existence of an invertible element $u \in \mathcal{C}_{0,7}$ is a necessity to define a generalization of the $X$-product in order that the term $(A \bullet u) \circ (u^{-1} \bullet B)$ is proportional to $(A \bullet u) \circ (\overline{u} \bullet B)$.

The main aim of this work is to consider how the generalizations of $XX = X \bar{X} = 1$ defining $S^7$ in the Clifford algebra context for an arbitrary element $u \in \mathcal{C}_{0,7}$ affect the subsequent deformations on the octonionic algebra. In general $u\overline{u}$ and $\overline{u}u$ are not scalars, and the only case we can guarantee that $u^{-1} = \pm u$ — or equivalently $\overline{u}u = \pm 1$ is regarded — in general is to consider the homogeneous and simple elements $u \in \Lambda^k(\mathbb{O}^7)$, wherein it can be surely asserted that $u^{-1} = \pm u$.

Given an element $u \in \Lambda(\mathbb{O}^7)$, the $u$-product is defined as

$$o_u: (\mathbb{R} \oplus \mathbb{R}^0) \times (\mathbb{R} \oplus \mathbb{R}^0) \rightarrow \mathbb{R} \oplus \mathbb{R}^0, \quad (A,B) \mapsto A \circ_u B := (A \bullet u) \circ (\overline{u} \bullet B). \quad (11)$$

The authors in [1] ask whether the relations $A \circ_u B := (A \bullet u) \circ (\overline{u} \bullet B) = (A \circ (B \bullet u)) \bullet \overline{u} = u \bullet ((\overline{u} \bullet J) \circ B)$ hold, in a context where any similar generalization related to Eq.(5) can be constructed in the non-associative formalism induced by the $u$-product, where $u$ can also more generally denote a form field on the exterior bundle on $S^7$.

Example 1: Taking $u = e_1 - e_2 e_3 \in \mathcal{C}_{0,7}$, $A = A^2 e_2 + A^4 e_4 \in \mathcal{O}$, and $B = B^1 e_1 + B^5 e_5 \in \mathcal{O}$, it follows that

$$(A \bullet u) \circ (\overline{u} \bullet B) = [(A^2 e_2 + A^4 e_4) \bullet (e_1 - e_2 e_3)] \circ [(\overline{e}_1 - e_2 e_3) \bullet J (B^1 e_1 + B^5 e_5)] = (-A^2 e_6 + A^2 e_3 + A^4 e_3 - A^4 e_6) \circ (B^1 - B^5 e_7 + B^1 e_5 - B^5 e_1) = 2 (A^2 B^5 e_4 + A^2 B^1 e_3 + A^4 B^1 e_3 + A^4 B^5 e_4)$$

while

$$(A \circ (B \bullet u)) \bullet \overline{u} = [(A^2 e_2 + A^4 e_4) \circ [(B^1 e_1 + B^5 e_5) \bullet (e_1 - e_2 e_3)]] \bullet (\overline{e}_1 - e_2 e_3) = [(A^2 e_2 + A^4 e_4) \circ (-B^1 - B^5 e_5 - B^5 e_7 + B^5 e_1)] \bullet (-e_1 + e_3 e_2) = 2 (-A^2 B^1 e_6 - A^2 B^1 e_3 - A^4 B^5 e_2 + A^4 B^5 e_4).$$

An open question about the validity of the expression $(A \circ X) \circ (\overline{X} \circ B) = (A \circ (B \circ X)) \circ \overline{X}$ for a more general setting concerns how the use of the $\bullet$-product instead of the standard octonionic product affects Eq.(6), and it was presented in [1] in a particular context. Specifically, it has been argued whether the introduction of the $u$-product could allow us to generalize such expression immediately, in order to consider all the exterior algebra constructed on the tangent space on an arbitrary point on $S^7$. Instead of $X \in \mathbb{R} \oplus \mathbb{R}^0$, an analogous expression for $u \in \mathcal{C}_{0,7}$ that expresses the non-associative algebraic structure related to the exterior bundle on $S^7$ can be obtained.

In general Eqs.(6) are not generalizable in terms of a naive immediate substitution, as it can be verified that

$$(A \bullet u) \circ (\overline{u} \bullet B) \neq (A \circ (B \bullet u)) \bullet \overline{u}. \quad (12)$$
in the Example 1 above. When \( u \) is a paravector — an element of \( \mathbb{R} \oplus \mathbb{R}^{0,7} \) — it is clear that the \( u \)-product is equivalent to the \( X \)-product.

In a close analogy to the \( XY \)-product and the \( (1,X) \)-product, respectively defined by Eqs.\((7)\) and \((8)\), it is also possible to define another product, the \( (1,u) \)-product, as

\[
\circ_{1,u} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7} \quad (A,B) \mapsto A \circ_{1,u} B := (A \circ 1) \circ (\bar{u} \bullet_j B) = A \circ (\bar{u} \bullet_j B). \tag{13}
\]

Finally, Eq.\((7)\) can be generalized for \( u,v \in C\ell_{0,7} \) fixed, as follows:

\[
\circ_{u,v} : (\mathbb{R} \oplus \mathbb{R}^{0,7}) \times (\mathbb{R} \oplus \mathbb{R}^{0,7}) \to \mathbb{R} \oplus \mathbb{R}^{0,7} \quad (A,B) \mapsto A \circ_{u,v} B := (A \bullet u) \circ (\bar{v} \bullet_j B). \tag{14}
\]

Given \( u = u_1 \ldots u_k \) and \( v = v_1 \ldots v_l \in C\ell_{0,7} \), the non-associative product between Clifford algebra elements was defined in \([1]\) as

\[
\circ_{\u} : C\ell_{0,7} \times C\ell_{0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7} \quad (u,v) \mapsto u \circ_{\u} v := u_1 \circ (u_2 \circ (\cdots \circ (u_k \circ (u_{k-1} \circ (u_{k-2} \cdots u_2 \circ (u_1 \circ u)))) \cdots)). \tag{15}
\]

\[
\circ_{\v} : C\ell_{0,7} \times C\ell_{0,7} \to \mathbb{R} \oplus \mathbb{R}^{0,7} \quad (u,v) \mapsto u \circ_{\v} v := ((\cdots \circ ((u \bullet u_1) \circ u_2) \circ \cdots) \circ u_{k-1}) \circ u_k. \tag{16}
\]

The symbol \( \circ \) denotes both the products \( \circ_{\u} \) and \( \circ_{\v} \). It is easy to see that, when elements of \( C\ell_{0,7} \) are restricted to the paravector space \( \mathbb{R} \oplus \mathbb{R}^{0,7} \), then \( A \bullet B \equiv A \circ B \) and \( A \circ B \equiv A \circ B \), where \( A,B \in \mathbb{R} \oplus \mathbb{R}^{0,7} \).

**Example 2:** Let us calculate the product \((2e_1e_2 - 7e_5e_6) \circ_{\u} e_3e_4:\)

\[
(2e_1e_2 - 7e_5e_6) \circ_{\u} e_3e_4 = 2e_1e_2 \circ_{\u} e_3e_4 - 7e_5e_6 \circ_{\u} e_3e_4 = 2e_1 \circ e_4 - 7e_5 \circ (e_5 \circ e_4) = 2e_1 \circ e_6 - 7e_5 \circ (-e_2) = -2e_2 + 7e_4; \tag{17}
\]

while

\[
(2e_1e_2 - 7e_5e_6) \circ_{\v} e_3e_4 = 2e_1e_2 \circ_{\v} e_3e_4 - 7e_5e_6 \circ_{\v} e_3e_4 = 2(e_1 \circ e_7) \circ e_4 - 7(e_5 \circ e_5) \circ e_4 = 2(-e_5) \circ e_4 + 7e_4 = +2e_2 + 7e_4, \tag{18}
\]

so Eqs.\((17,18)\) can not be mapped into another through automorphisms or antiautomorphisms of \( C\ell_{0,7} \). After we shall see that it is possible to induce an involution on a deformed octonionic algebra, induced by an arbitrary Clifford element, that can make some extension of the Moufang identities possible.

In what follows it is implicit that \( u \in C\ell_{0,7} \) is not a scalar, since in this case would be nothing new to prove, since \( A = 1 \circ A = 1 \bullet A = 1 \circ A \), \( \forall A \in \mathbb{O} \).

5. Generalized non-associative products on \( S^7 \)

All the possible products obtained from the combinations between \( \circ, \bullet, \circ_{\u}, \circ_{\v} \), and \( \circ_{\u} \) are listed below. Some examples are given to illustrate the different results obtained if only one of those products are interchanged. Moreover, the generalizations for the \( u-, (1,u)-, \) and \( (u,v)- \) products are provided by means of the directional octonionic products.
The definitions above allow us to see that the \((1, u)\)-product can be generalized to encompass and include multivectors of \(\mathcal{C}l_{0,7}\) in the first or second entry, so the \((1, u)\)-product for Clifford multivectors in the first entry is given by
\[
\circ_{1, u} : \mathcal{C}l_{0,7} \times \mathbb{O} \to \mathbb{O}
\]
\[
(\psi, A) \mapsto \psi \circ_{1, u} A := (\psi \bullet, 1) \circ (\bar{u} \circ_\phi A),
\]
which is the immediate generalization for the \((1, u)\)-product defined in Eq.(13), since \(A \circ 1 = A\), \(\forall A \in \mathbb{O}\) and \(\psi \bullet, 1 \neq \psi\), \(\forall \psi \in \mathcal{C}l_{0,7} \setminus (\mathbb{R} \oplus \mathbb{R}^{0,7})\). Therefore, the following product can be also defined
\[
\nabla_{1, u} : \mathcal{C}l_{0,7} \times \mathbb{O} \to \mathbb{O}
\]
\[
(\psi, A) \mapsto \psi \nabla_{1, u} A := \psi \bullet_j (\bar{u} \circ_\phi A).
\]
Such a product is a generalization that is not analogous to the \((1, u)\)-product defined in Eq.(13), but it is exactly the immediate generalization of the standard octonionic formalism, given by Eq.(8).

For a Clifford multivector in the right, two non-equivalent possibilities can be introduced by defining them in terms of the \(\circ_\phi\) or the \(\circ_{\bar{u}}\)-product as well:
\[
\circ_{1, u} : \mathbb{O} \times \mathcal{C}l_{0,7} \to \mathbb{O}
\]
\[
(A, \psi) \mapsto A \circ_{1, u} \psi := (A \circ 1) \circ (\bar{u} \circ_{\bar{u}} \psi) = A \circ (\bar{u} \circ_\phi \psi),
\]
while for the Clifford multivector in the right and the \(\circ_\phi\)-product in the left it reads
\[
\circ_{1, u} : \mathbb{O} \times \mathcal{C}l_{0,7} \to \mathbb{O}
\]
\[
(A, \psi) \mapsto A \circ_{1, u} \psi := (A \circ 1) \circ (\bar{u} \circ_\phi \psi) = A \circ (\bar{u} \circ_\phi \psi).
\]
The last extension of the \((1, u)\)-product for given \(\psi, \phi \in \mathcal{C}l_{0,7}\), fixed but arbitrary, is defined by
\[
\circ_{1, u} : \mathcal{C}l_{0,7} \times \mathcal{C}l_{0,7} \to \mathbb{O}
\]
\[
(\psi, \phi) \mapsto \psi \circ_{1, u} \phi := (\psi \bullet, 1) \circ (\bar{u} \circ_\phi \phi),
\]
\[
\circ_{1, u} : \mathcal{C}l_{0,7} \times \mathcal{C}l_{0,7} \to \mathbb{O}
\]
\[
(\psi, \phi) \mapsto \psi \circ_{1, u} \phi := (\psi \bullet, 1) \circ (\bar{u} \circ_\phi \phi).
\]
And, for the non-similar generalization it follows
\[
\nabla_{1, u} : \mathcal{C}l_{0,7} \times \mathcal{C}l_{0,7} \to \mathbb{O}
\]
\[
(\psi, \phi) \mapsto \psi \nabla_{1, u} \phi := \psi \bullet_j (\bar{u} \circ_\phi \phi),
\]
\[
\nabla_{1, u} : \mathcal{C}l_{0,7} \times \mathcal{C}l_{0,7} \to \mathbb{O}
\]
\[
(\psi, \phi) \mapsto \psi \nabla_{1, u} \phi := \psi \bullet_j (\bar{u} \circ_\phi \phi).
\]
**Remark 2:** Note that, \(A \circ_{1, u} \psi = A \nabla_{1, u} \psi\) and \(A \circ_{1, u} \psi = A \nabla_{1, u} \psi\), while \(\psi \circ_{1, u} \phi \neq \psi \nabla_{1, u} \phi\), i.e., where an octonion appear in the first entry the equality between the \(\circ_{1, u}\)- and \(\nabla_{1, u}\)-products is satisfied. Indeed,
\[
A \circ_{1, u} \psi = (A \circ 1) \circ (\bar{u} \circ_\phi \psi) = A \circ (\bar{u} \circ_\phi \psi)
\]
\[
\psi \circ_{1, u} \phi = (\psi \bullet, 1) \circ (\bar{u} \circ_\phi \phi) \neq \psi \bullet_j (\bar{u} \circ_\phi \phi) = \psi \nabla_{1, u} \phi
\]
since \(\psi \bullet, 1\) is in general an octonion and \(\psi\) is a Clifford multivector, the result of \(A \circ_{1, u} \psi = A \nabla_{1, u} \psi\) and \(\psi \circ_{1, u} \phi \neq \psi \nabla_{1, u} \phi\) follows analogously.
In order to generalize the \( u \)-product for a Clifford multivector, an octonion is replaced by a Clifford multivector in one of the entries. Firstly a Clifford multivector is placed in the first entry and using the \( \circ \)-product in the right and in the left, and subsequently for a Clifford multivector in the second entry:

\[
\circ_u^\bullet : C\ell_{0,7} \times \mathbb{O} \rightarrow \mathbb{O}
\]
\[
(\psi, A) \mapsto \psi \circ_u^\bullet A := (\psi \circ_{\downarrow} u) \circ (\bar{u} \bullet \downarrow A), \quad (27)
\]

\[
\circ_u^\downarrow : C\ell_{0,7} \times \mathbb{O} \rightarrow \mathbb{O}
\]
\[
(\psi, A) \mapsto \psi \circ_u^\downarrow A := (\psi \circ_{\downarrow} u) \circ (\bar{u} \bullet \downarrow A), \quad (28)
\]

\[
\circ_u^\circ : \mathbb{O} \times C\ell_{0,7} \rightarrow \mathbb{O}
\]
\[
(A, \psi) \mapsto A \circ_u^\circ \psi := (A \bullet u) \circ (\bar{u} \circ \downarrow \psi), \quad (29)
\]

\[
\circ_u^\circ : \mathbb{O} \times C\ell_{0,7} \rightarrow \mathbb{O}
\]
\[
(A, \psi) \mapsto A \circ_u^\circ \psi := (A \bullet u) \circ (\bar{u} \circ \downarrow \psi). \quad (30)
\]

Now an arbitrary multivector \( \psi \in C\ell_{0,7} \) is taken into account in both entries, and a choice needs to be made for the \( \circ \)-product direction, introducing the directional non-associative products. For Eq. (31) in Example 3 there are explicit computations illustrating how to perform the products \( \circ_u^\uparrow \) and \( \circ_u^\downarrow \).

\[
\circ_u^\uparrow : C\ell_{0,7} \times C\ell_{0,7} \rightarrow \mathbb{O}
\]
\[
(\psi, \phi) \mapsto \psi \circ_u^\uparrow \phi := (\psi \circ_{\downarrow} u) \circ (\bar{u} \circ \downarrow \phi), \quad (31)
\]

\[
\circ_u^\downarrow : C\ell_{0,7} \times C\ell_{0,7} \rightarrow \mathbb{O}
\]
\[
(\psi, \phi) \mapsto \psi \circ_u^\downarrow \phi := (\psi \circ_{\downarrow} u) \circ (\bar{u} \circ \downarrow \phi), \quad (32)
\]

\[
\circ_u^\circ : C\ell_{0,7} \times C\ell_{0,7} \rightarrow \mathbb{O}
\]
\[
(\psi, \phi) \mapsto \psi \circ_u^\circ \phi := (\psi \circ_{\downarrow} u) \circ (\bar{u} \circ \downarrow \phi), \quad (33)
\]

\[
\circ_u^\circ : C\ell_{0,7} \times C\ell_{0,7} \rightarrow \mathbb{O}
\]
\[
(\psi, \phi) \mapsto \psi \circ_u^\circ \phi := (\psi \circ_{\downarrow} u) \circ (\bar{u} \circ \downarrow \phi). \quad (34)
\]

The \((u, v)\)-product is now generalized [1], hence the product is considered with entries in \( \mathbb{O} \) and \( C\ell_{0,7} \) where in both the cases the \( \circ \)-product is performed in two directions.

\[
\circ_{u,v}^\bullet : C\ell_{0,7} \times \mathbb{O} \rightarrow \mathbb{O}
\]
\[
(\psi, A) \mapsto \psi \circ_{u,v}^\bullet A := (\psi \circ_{\downarrow} u) \circ (\bar{v} \bullet \downarrow A), \quad (35)
\]

\[
\circ_{u,v}^\downarrow : C\ell_{0,7} \times \mathbb{O} \rightarrow \mathbb{O}
\]
\[
(\psi, A) \mapsto \psi \circ_{u,v}^\downarrow A := (\psi \circ_{\downarrow} u) \circ (\bar{v} \bullet \downarrow A), \quad (36)
\]

\[
\circ_{u,v}^\circ : \mathbb{O} \times C\ell_{0,7} \rightarrow \mathbb{O}
\]
\[
(A, \psi) \mapsto A \circ_{u,v}^\circ \psi := (A \bullet u) \circ (\bar{v} \circ \downarrow \psi), \quad (37)
\]

\[
\circ_{u,v}^\circ : \mathbb{O} \times C\ell_{0,7} \rightarrow \mathbb{O}
\]
\[
(A, \psi) \mapsto A \circ_{u,v}^\circ \psi := (A \bullet u) \circ (\bar{v} \circ \downarrow \psi). \quad (38)
\]

Finally, the \((u, v)\)-product can be generalized for Clifford multivectors in both entries, as in [1],
but now explicitly considering both the directions related to the $\odot$-product:

\[
\begin{align*}
\odot_{\alpha,\beta} : \mathcal{C}_{0,7} \times \mathcal{C}_{0,7} & \rightarrow \mathcal{C}_1 \quad (\psi, \phi) \mapsto \psi \odot_{\alpha,\beta} \phi := (\psi \odot_\alpha u) \odot (\bar{\phi} \odot_\beta \phi), \\
\odot_{\alpha,\beta} : \mathcal{C}_{0,7} \times \mathcal{C}_{0,7} & \rightarrow \mathcal{C}_1 \quad (\psi, \phi) \mapsto \psi \odot_{\alpha,\beta} \phi := (\psi \odot_\alpha u) \odot (\bar{\phi} \odot_\beta \phi), \\
\odot_{\alpha,\beta} : \mathcal{C}_{0,7} \times \mathcal{C}_{0,7} & \rightarrow \mathcal{C}_1 \quad (\psi, \phi) \mapsto \psi \odot_{\alpha,\beta} \phi := (\psi \odot_\alpha u) \odot (\bar{\phi} \odot_\beta \phi), \\
\odot_{\alpha,\beta} : \mathcal{C}_{0,7} \times \mathcal{C}_{0,7} & \rightarrow \mathcal{C}_1 \quad (\psi, \phi) \mapsto \psi \odot_{\alpha,\beta} \phi := (\psi \odot_\alpha u) \odot (\bar{\phi} \odot_\beta \phi).
\end{align*}
\]

(39)

In addition, the non-associative products between Clifford algebra arbitrary elements in [1] are now completely constructed in the present context.

**Example 3:** Let us calculate $\psi \odot_{\alpha,\beta} \phi$ and $\psi \odot_{\alpha,\beta} \phi$ for $u = e_6 e_7 e_1 e_3 e_2 + 2 e_4 e_5$, $\bar{u} = -e_2 e_3 e_1 e_6 - 2 e_4 e_5$, $\psi = e_7 e_3$, and $\phi = e_3 e_4$:

\[
(\psi \odot_{\alpha,\beta} u) \odot (\bar{u} \odot_{\gamma,\delta} \phi) = \begin{cases} 
[e_7 e_3 \odot_{\alpha,\beta} (e_6 e_7 e_1 e_3 e_2 + 2 e_4 e_5)] \odot \left[(-e_2 e_3 e_1 e_7 e_6 - 2 e_5 e_4) \odot_{\gamma,\delta} e_5 e_4\right] \\
[e_7 \odot (e_3 \odot (e_6 e_7 e_1 e_3 e_2 + 2 e_4 e_5) + 2 e_7 \odot (e_3 \odot e_5 e_4))] \odot \left[(-e_2 \odot (e_3 \odot (e_1 \odot e_5)) - 2 e_5 \odot (e_1 \odot e_5)\right] \\
[e_7 \odot \left(\left(\left((-e_5 \odot e_7) \odot e_1\right) \odot e_3\right) \odot e_2\right) + 2 e_7 \odot e_7\right] \odot \left[(-e_2 \odot (e_3 \odot (e_1 \odot e_7 \odot (e_1))) - 2 e_5 \odot e_5\right] \\
\end{cases}
\]

(43)

On the another hand,

\[
(\psi \odot_{\gamma,\delta} u) \odot (\bar{u} \odot_{\alpha,\beta} \phi) = \begin{cases} 
[e_7 e_3 \odot_{\gamma,\delta} (e_6 e_7 e_1 e_3 e_2 + 2 e_4 e_5)] \odot \left[(-e_2 e_3 e_1 e_7 e_6 - 2 e_5 e_4) \odot_{\alpha,\beta} e_5 e_4\right] \\
[-1] \odot \left[(-e_2 \odot (e_3 \odot e_1 \odot e_5)) - 2 e_5 \odot e_5\right] \odot \left[(-e_2 \odot (e_3 \odot (e_1 \odot e_7 \odot (e_1))) - 2 e_5 \odot e_5\right] \\
\end{cases}
\]

(44)

This example shows the importance of the direction in the $\odot$-product, because in this case just by changing one direction different results are achieved for the same initial data.

**Remark 3:** The Moufang identities [33, 2] for the octonionic algebra are listed below for all $A, B, C \in \mathcal{O}$:

\[
(A \circ B \circ A) \circ C = A \circ (B \circ (A \circ C)),
\]

(45)

\[
C \circ (A \circ B \circ A) = ((C \circ A) \circ B) \circ A,
\]

(46)

\[
(A \circ B) \circ (C \circ A) = A \circ (B \circ C) \circ A.
\]

(47)

In the case of the products $\circ : \mathcal{O} \times \Lambda^k(\mathbb{R}^{0,7}) \rightarrow \mathcal{O}$ and $\cdot : \Lambda^k(\mathbb{R}^{0,7}) \times \mathcal{O} \rightarrow \mathcal{O}$, the Moufang identities cannot be generalized only using conjugation and graded involution. Two counter-examples are presented below.
Example 4: One of the Moufang identities for octonions is expressed as

\[(A \circ B \circ A) \circ C = A \circ (B \circ (A \circ C)), \quad A, B, C \in \mathbb{O}.\]  

(48)

Suppose that an immediate generalization could be accomplished, by naively replacing the standard octonion product by the \(\cdot\)-product, as

\[(u \cdot_j B \cdot_j u) \cdot C = u \cdot_j (B \cdot (u \cdot_j C)), \quad u \in \mathcal{C}_{0,7},\]

(49)
or even as \((u \cdot B \cdot u) \cdot C = \tilde{u} \cdot (B \cdot (u \cdot C))\), \((u \cdot B \cdot u) \cdot C = \tilde{u} \cdot (B \cdot (u \cdot C))\), or the product above with any composition of graded involution and/or the Clifford conjugation over \(u\). For an easy understanding of the expressions, Eq.(49) is written down as

\[(u \cdot_j B \cdot_j u) \circ C = u \cdot_j (B \circ (u \cdot_j C)), \quad u \in \mathcal{C}_{0,7},\]

(50)
since the \(\cdot\)-product between octonions is identical to the \(\circ\)-product. Take \(u = e_2e_7e_4, B = e_1\) and \(C = e_3\). First,

\[(e_2e_7e_4 \cdot_j e_1 \cdot_j e_2e_7e_4) \circ e_3 = ((e_2 \circ (e_7 \circ e_4)) \cdot_j e_2e_7e_4) \circ e_3 = ((e_2 \circ (-e_2)) \cdot_j e_2e_7e_4) \circ e_3 = (-e_3 \circ e_4) \circ e_3 = -e_1 \circ e_3 = -e_4,\]

(51)

while

\[e_2e_7e_4 \cdot_j (e_1 \circ (e_2e_7e_4 \cdot_j e_3)) = e_2e_7e_4 \cdot_j (e_1 \circ (e_2 \circ (e_7 \circ (-e_1)))) = e_2e_7e_4 \cdot_j (e_1 \circ (e_2 \circ (-e_3))) = e_2e_7e_4 \cdot_j (-e_3) = e_2 \circ (e_7 \circ e_1) = e_2 \circ e_5 = -e_4.\]

(52)

On the other hand, by taking \(u = e_3e_5e_7, B = e_2\) and \(C = e_1\), it follows that:

\[(e_3e_5e_7 \cdot_j e_2 \cdot_j e_3e_5e_7) \circ e_1 = ((e_3 \circ (e_5 \circ e_3)) \cdot_j e_3e_5e_7) \circ e_1 = (e_5 \cdot_j e_3e_5e_7) \circ e_1 = (e_3 \circ e_7) \circ e_1 = e_2 \circ e_1 = -e_6,\]

(53)

while

\[e_3e_5e_7 \cdot_j (e_2 \circ (e_3e_5e_7 \cdot_j e_1)) = e_3e_5e_7 \cdot_j (e_2 \circ (e_3 \circ (e_5 \circ e_3))) = e_3e_5e_7 \cdot_j (-e_7) = e_3 \circ e_5e_6.\]

(54)

It can be seen that for distinct elements representing \(u \in A^3(\mathbb{R}^{0,7})\), both the relations \((u \cdot_j B \cdot_j u) \circ C = u \cdot_j (B \circ (u \cdot_j C))\) and \((u \cdot_j B \cdot_j u) \circ C = -u \cdot_j (B \circ (u \cdot_j C))\) are obtained. These last two relations cannot be mutually satisfied for elements with the same degree in \(\mathcal{C}_{0,7}\), the same for the product given by Eq.(49) with any composition of the graded involution and/or the Clifford conjugation on \(u\). Using the same counter-example above it can be shown that the other Moufang identities can not be generalized using Clifford conjugation and graded involution only.
For $A, B \in O$ defined in Eqs.(15,16) respectively. Indeed, computing the product $(u \odot B \odot u) \odot C$ and $u \odot (B \odot (u \odot C))$ for $u = e_1e_2e_7$, $B = e_2e_5e_6$ and $C = e_3e_5e_7$ it follows that:

\[
\begin{align*}
(e_1e_3e_7 \odot (e_2e_5e_6 \odot (e_1e_3e_7 \odot e_3e_5e_7))) &= (e_1 \circ (e_3 \circ ((e_7 \\bullet, e_2e_5e_6))) \odot (e_1e_3e_7) \odot e_3e_5e_7) \\
&= (e_1 \circ (e_3 \circ (e_6 \circ e_6)) \odot (e_1e_3e_7) \odot e_3e_5e_7) \\
&= (e_1 \circ (-e_3) \odot e_1e_3e_7) \odot e_3e_5e_7) \\
&= (-e_2 \circ e_5) \odot e_7 = e_4 \circ e_7 \\
&= -e_6
\end{align*}
\]

On the another hand we have:

\[
\begin{align*}
e_1e_3e_7 \odot (e_2e_5e_6 \odot & (e_1e_3e_7 \odot e_3e_5e_7)) \\
&= e_1e_3e_7 \odot (e_2e_5e_6 \odot (e_1 \circ (e_3 \circ ((e_7 \\bullet, e_3e_5e_7))))) \\
&= e_1e_3e_7 \odot (e_2e_5e_6 \odot (e_1 \circ (e_3 \circ (e_4 \circ e_7)))) \\
&= e_1e_3e_7 \odot (e_2e_5e_6 \odot (e_1 \circ (e_3 \circ (e_6)))) \\
&= e_1e_3e_7 \odot (e_2 \circ (e_5 \circ (e_6 \circ e_7))) \\
&= e_1e_3e_7 \odot (e_2 \circ (-e_2)) = e_1e_3e_7 \odot 1 \\
&= e_1e_3e_7 \circ_j 1 \\
&= e_1 \circ (e_3 \circ (e_7 \circ 1)) = e_1 \circ (e_3 \circ e_7) = e_1 \circ e_2 \\
&= e_6
\end{align*}
\]

Analogously, another counter-example with the product $\odot_j$ can be presented to show that the Moufang identities given by Eqs.(45, 46, 47) are not generalized in this context.

6. $\odot$-Scalar Product and $S^7$ Tangent Bundle Basis

For $A, B \in \odot$ the scalar product between two octonions is defined as $[5, 2, 29, 22]$: \[
\langle A, B \rangle = \frac{1}{2}(\bar{A} \circ B + B \circ A) = \frac{1}{2}(A \circ B + B \circ \bar{A}).
\]

Considering an arbitrary unit octonion $X = X^0 + X^a e_a$, its squared norm is given by $\|X\|^2 = \langle X, X \rangle = (X^0)^2 + (X^1)^2 + (X^2)^2 + (X^3)^2 + (X^4)^2 + (X^5)^2 + (X^6)^2 + (X^7)^2 = 1$, and the elements $\{e_a \circ X\}$ constitute the frame bundle on $S^7$, obtained by the left multiplication of $X \in S^7$ by an octonion basis $\{e_a\}$. They satisfy the relations:

\[
\begin{align*}
\langle e_a \circ X, e_b \circ X \rangle &= \delta_{ab}, & \langle e_a \circ X, X \rangle &= 0, \quad (55) \\
\langle e_a e_b \circ_j X, X \rangle &= 0. \quad (56)
\end{align*}
\]

Concerning the octonions $e_a \circ X$, Eqs.(55) tell us that these vectors are orthogonal to each other and more, these vectors lie in the tangent space at the arbitrary point $X \in S^7$, where they form an orthonormal basis. In the Appendix H the $\{e_a \circ X\}$ representations are explicitly constructed. For more details see [5, 2].
Note that the product $e_a e_b \bullet X$ also presents an associated matrix representation — in the Appendix II all are constructed, and some of them were first listed in [2] — it is possible to prove that $\forall a, b = 1, \ldots, 7, a \neq b$,

$$
\langle e_a e_b \bullet X, X \rangle = \langle e_a \circ (e_b \circ X), X \rangle = \langle e_b \circ X, e_a \circ X \rangle = - \langle e_b \circ X, e_a \circ X \rangle = 0,
$$

(57)

where Eq.(55) is used in the last one, i.e., $e_a \circ X$ is a basis for the tangent space in an arbitrary point on $S^7$. It is clear to see such property, in particular when the properties in the Appendix Appendix G are taken into account.

Eqs.(55) and (56) can be also easily demonstrated by computing all the cases for all $a, b = 1, \ldots, 7, a \neq b$, which shows that Eq.(55) relates a set of elements orthogonal to $X \in S^7$ for $a \neq b$, and then a basis for the tangent space is obtained.

In order to illustrate, in Appendix G particular cases of Eqs.(55) are explicitly demonstrated. Also in general for $a \neq b \neq c$,

$$
\langle e_a e_b e_c \bullet X, X \rangle \neq 0.
$$

(58)

Indeed, concerning the element $e_1 e_2 e_3$, it reads

$$
\langle e_1 e_2 e_3 \bullet X, X \rangle = 1/2\left[ \langle e_1 e_2 e_3 \bullet X, X \rangle \circ X + X \circ \langle e_1 e_2 e_3 \bullet X, X \rangle \right]
= 1/2 \left[ (+X^0 e_5 + X^1 e_7 - X^2 e_4 + X^3 e_6 - X^4 e_2 - X^5 + X^6 e_3 + X^7 e_1) \circ
( +X^0 + X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7) +
(+X^0 - X^1 e_1 - X^2 e_2 - X^3 e_3 - X^4 e_4 - X^5 e_5 - X^6 e_6 - X^7 e_7) \circ
(-X^0 e_5 - X^1 e_7 + X^2 e_4 - X^3 e_6 + X^4 e_2 - X^5 - X^6 e_3 - X^7 e_1) \right]
= 2(-X^0 X^5 - X^1 X^7 + X^2 X^4 - X^3 X^6).
$$

7. Properties of the $\bullet$-product

Some additional properties concerning the $\bullet$-product are asserted and proved for $u \in Cl_{0,7}$ homogeneous and simple, the only cases we can guarantee that there exists an inverse $u^{-1}$, in the light of the points presented in Remark 1. Such properties are helpful in case we want to obtain the most general expression that generalizes Eq.(6) and emulates it when $u \in Cl_{0,7}$ is considered instead of $X \in R \oplus R^{0,7}$.

Proposition 1: Given $e_a \in \bigcirc$ and $u = e_{i_1} e_{i_2} \ldots e_{i_k} \in \Lambda^k(R^{0,7})$, $k = 1, \ldots, 6$ with either $e_a \notin \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$ or $\{e_a, e_{i_1}, e_{i_2}\}$ not a $\mathbb{H}$-triple, for $\{i_{j_1}, i_{j_2}\} = \{i_1, \ldots, i_6\}$, then

$$
(\langle e_a \bullet u \rangle) \circ (\langle e_a \bullet u \rangle) = (-1)^{(\mid u \mid - 1)(\mid u \mid - 2)}/2 e_a
$$

(59)

Proposition 2: Given $e_a \in \bigcirc$ and $u = e_{i_1} e_{i_2} \ldots e_{i_k} \in \Lambda^k(R^{0,7})$, $k = 1, \ldots, 6$ then

$$
(\langle \bar{u} \bullet e_a \rangle) \circ (\langle 1 \bullet \bar{u} \rangle) = (-1)^{\mid u \mid (\mid u \mid + 1)}/2 e_a
$$

(60)

Observation 1: Propositions 1 and 2 when $e_a = e_0 = 1$ can be asserted as

Proposition 1': Given $u = e_{i_1} e_{i_2} \ldots e_{i_k} \in \Lambda^k(R^{0,7})$, $k = 1, \ldots, 6$ then

$$
(\langle 1 \bullet u \rangle) \circ (\langle 1 \bullet \bar{u} \rangle) = (-1)^{\mid u \mid (r_{\mid u \mid - r_{\mid u \mid - 1}) + (8 - [\mid u \mid (7 - [\mid u \mid)])/2
$$

(61)

where $r_j$ are the Radon-Hurwitz numbers defined by the following table and the recurrence relation $r_{j+8} = r_j + 4$. 

13
Appendices A, B, C, D, E, and F, for the Propositions 1, 1′ Propositions are given by making computations for all cases that can be completely checked in In all the cases above

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In this Section a more general class of non-associative products is introduced, presenting

8. Generalized Non-Associative Products

In this Section a more general class of non-associative products is introduced, presenting similarity to the -product between Clifford multivectors, but constructed without an additional element. The difference is that the product is computed without the insertion of any additional element , but now a vector is taken out the Clifford multivector itself, instead. This -valued map is called the non-associative shear of a Clifford multivector, and it cogently describes how to split an arbitrary multivector in a non-associative manner into an octonion.

Table 2. Radon-Hurwitz numbers.

| j  | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|----|---|---|---|---|---|---|---|---|
| r_j| 0 | 1 | 2 | 2 | 3 | 3 | 3 | 3 |

Proposition 2′: Given \( u = e_{i_1} e_{i_2} \ldots e_{i_k} \in \Lambda^k(\mathbb{R}^{0,7}) \), \( k = 1, \ldots, 6 \) then

\[
(\bar{u} \ast 1) \circ (1 \ast \bar{u}) = (-1)^{|u|(|r|_u - |r|_{u-1}) + (8 - |u|)(7 - |u|)/2}
\] (62)

Proposition 3: Given \( e_a \in \mathbb{O} \) and \( u = e_{i_1} e_{i_2} \ldots e_{i_k} \in \Lambda^k(\mathbb{R}^{0,7}), \ k = 1, \ldots, 6 \) with either \( e_a \notin \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\} \) or \( \{e_a, e_{j_1}, e_{j_2}\} \) not a \( \mathbb{H} \)-triple, for \( \{j_1, j_2\} = \{i_1, \ldots, i_k\} \), then

\[
e_a \ast u = (-1)^{|u|+1} e_a \circ (1 \ast u)
\] (63)

Proposition 4: Given \( \psi \in \mathcal{C}l_{0,7} \) and \( u \in \Lambda^k(\mathbb{R}^{0,7}), \ k = 1, \ldots, 6 \) then

\[
\psi \circ u = (-1)^{|u|+1} \psi \ast (1 \ast u)
\] (64)

In all the cases above \( |u| \) denotes the degree of \( u \): if \( u \in \Lambda^k(\mathbb{R}^{0,7}) \), then \( |u| = k \).

Note that the Proposition 4 is the generalization of the Proposition 3. The proofs for all the Propositions are given by making computations for all cases that can be completely checked in Appendices A, B, C, D, E, and F, for the Propositions 1, 1′, 2, 2′, 3, and 4, respectively.

Observation 2: Proposition 1 elicits an \( u \)-induced involution on \( \mathbb{O} \). By denoting it by

\[
\downarrow_u (e_a) := (e_a \ast u) \circ (1 \ast \bar{u})
\] (65)

it is immediate to see that \( \downarrow_u \downarrow_u (e_a) = (-1)^{|u|(|u|-1)(|u|-2)} e_a = e_a \).

Also, Proposition 2 provides another \( u \)-induced involution on \( \mathbb{O} \). By denoting it by

\[
\downarrow_u (e_a) := (\bar{u} \ast_u e_a) \circ (1 \ast \bar{u})
\] (66)

it is immediate to see that \( \downarrow_u \downarrow_u (e_a) = (-1)^{|u|(|u|+1)} e_a = e_a \).

Those two involutions bring some new light on the generalization of Eq.(6) in the context of the \( \ast \)-product.
Given \( \psi = u_1u_2\ldots u_k \in \Lambda^k(\mathbb{R}^{0,7}) \) and \( \phi = v_1v_2\ldots v_j \in \Lambda^j(\mathbb{R}^{0,7}) \), where \( u_i \in \mathbb{R}^{0,7} \) and \( v_k \in \mathbb{R}^{0,7} \), we define the \( \triangleright \)- and \( \triangleleft \)-product that merge the \( u \)- and the \( \bullet \)-product, as:

\[
\triangleright : \mathcal{C}l_{0,7} \times \mathcal{C}l_{0,7} \to \mathbb{O} \\
(\psi, \phi) \mapsto \psi \triangleright \phi := ((u_1u_2\ldots u_{k-1}) \bullet u_k) \circ (\bar{u}_k \bullet (v_1v_2\ldots v_j)) \tag{67}
\]

\[
\triangleleft : \mathcal{C}l_{0,7} \times \mathcal{C}l_{0,7} \to \mathbb{O} \\
(\psi, \phi) \mapsto \psi \triangleleft \phi := ((u_1u_2\ldots u_k) \bullet v_1) \circ (\bar{v}_1 \bullet (v_2v_3\ldots v_j)) \tag{68}
\]

**Example 6:** Let \( \psi = e_1e_2e_3 \) and \( \phi = e_4e_5 \), and calculate the above products for this case. On the one hand,

\[
e_1e_2e_3 \triangleright e_4e_5 = (e_1e_2 \bullet e_3) \circ (e_3 \bullet e_4e_5) = (e_1e_2 \bullet e_3) \circ (-e_3 \bullet e_4e_5) = (e_1 \circ e_7) \circ (-e_1 \circ e_5) = -e_5 \circ (-e_7) = e_1.
\]

In the other hand,

\[
e_1e_2e_3 \triangleleft e_4e_5 = (e_1e_2e_3 \bullet e_4) \circ (e_4 \circ e_5) = (e_1e_2e_3 \bullet e_4) \circ (-e_4 \circ e_5) = (e_1 \circ (e_2 \circ e_1)) \circ (-e_2) = (e_1 \circ (-e_0)) \circ (-e_2) = e_2 \circ (-e_2) = 1.
\]

Those products can be expressed using the \( \circ_u^\infty \) product as

\[
\psi \triangleright \phi = (\psi u_k^{-1}) \circ_u^\infty \phi, \quad \psi \triangleleft \phi = \psi \circ_e^\infty (\psi_1^{-1}_1) \phi.
\tag{69}
\]

9. **Concluding remarks and outlook**

As Hopf fibrations can be accomplished by the use of the \( X \)-product [2], a natural task should be to ascertain about the geometric meaning — in the context of Hopf fibrations — of particular products of the type \( e_a \circ_u e_b \). The formalism presented here may make — for instance — the Hopf fibration \( S^3 \ldots S^7 \to S^4 \) and the parallelizable torsion on \( S^7 \) to arise from the immediate deformation of the octonionic product. The parameter \( A \circ_X B = (A \circ X) \circ (X \circ B) = X \circ ((X \circ A) \circ B) \) is twice the parallelizing torsion [34, 5] which components are given by \( T_{ijk}(X) = (\mathfrak{e}_i \circ X \circ (X \circ \mathfrak{e}_j)) \circ e_k \), which is exactly the \( X \)-product between \( \mathfrak{e}_i \) and \( \mathfrak{e}_j \), and the \( S^7 \) algebra can be written as \( [\delta_i, \delta_j] = 2T_{ijk}(X)\delta_k \), where \( \delta_A X = X \circ A \), and the variation \( \delta \) is indeed the parallelizing covariant derivative [6, 2, 4, 7, 5]. The question whether the analogous of the above structure of the form \( (A \circ (B \bullet u)) \bullet \bar{u} \), for \( u \in \mathcal{C}l_{0,7}, A, B \in \mathbb{O} \) also may allow for non-trivial central extensions [35] remains open. Also, it is well known the classical version of the \( S^7 \) Kac-Moody algebra, and the question concerning whether a generalization of the \( X \)-product as accomplished along this paper could ascertain the validity of immediate generalizations of some expressions in [6]. We can ask what is the meaning of the \( u \)-commutator \( [A, B]_u = A \circ_u B - B \circ_u A \) instead of the usual term \( [A, B]_X = A \circ_X B - B \circ_X A \).

The innate difficulties in computing non-associative products are circumvented, when these products can be incorporated in the multivector Clifford algebra structure. This is a strong property reminiscent of the assumption that there is defined an extension of the octonionic product in order to encompass also non-associative products between octonions and Clifford multivectors, and between Clifford multivectors themselves. By means of the \( \circ \)-product, all the arbitrary number of octonionic subsequent products are regarded as the \( \circ \)-product involving the Clifford multivector associated with the subsequent octonionic product as defined in Eqs. (9,10,15). The arbitrary number of octonionic products can be encoded in an unique product — the \( \circ \)-product — and their associated Clifford multivector structure.

When we deal with the homogeneous simple multivectors in \( \mathcal{C}l_{0,7} \), the non-associative structure related to the subsequent \( \circ \)-products among the \( \circ \)-units can be regarded in the anti-commutative structure in the underlying exterior algebra \( \Lambda(\mathbb{R}^{0,7}) \to \mathcal{C}l_{0,7} \). It is not a quite
straightforward task to consider the reversed non-associative products. For instance, given
\[ \alpha_0 = \frac{1}{3}(-3 + e_4 e_7 e_6 + e_5 e_1 e_7 + e_6 e_2 e_1 + e_7 e_3 e_2 + e_1 e_4 e_3 + e_2 e_5 e_4 + e_3 e_6 e_5) \] [2], it is possible to show that
\[-(e_a \circ e_b) \circ X = -(\alpha_0 \bullet_j (X \circ e_b)) \circ e_a + (\alpha_0 \bullet_j (X \circ e_a)) \circ e_b - (X \circ e_b) \circ e_a, \forall X \in \mathbb{O}. \quad (70)\]

The left actions introduced, in e.g. [2], can be completely described in the \( \bullet \)-product formalism, since it explicitly shows the manifest exterior algebra character of the subsequent left action. We completely formalize the framework introduced in [2, 6] in a robust platform provided by the Clifford bundle on \( S^7 \). The possibility of performing non-associative products between arbitrary multivectors of \( \mathcal{C} \ell_{0,7} \) naturally arises in our formalism that completes [1], and it also generalizes the formalism introduced in [2], concerning the original X-product.

The authors in [1] introduced the octonionic algebra in the Clifford algebra arena and dealt with non-associative products, firstly introduced by Dixon [2] with the X- and XY-products between octonions and also in [1] the \( u \)-, \( \bullet \)-, and \( \circ \)-products not only but also with an octonion and a Clifford multivector, and between Clifford multivectors as well. In this paper those products are generalized with respect to the direction, where more possibilities to make the product appear and a full list of them is presented. Moreover, four important Propositions with respect to the \( \bullet \)-product are shown, and a generalization for the non-associative products is provided, using both the \( u \)- and \( \bullet \)-product. Such propositions may be germane for the extension of Eq.(6) in order to encompass an arbitrary \( X \in \mathcal{C} \ell_{0,7} \) instead of \( X \in \mathbb{R} \oplus \mathbb{R}^{0,7} \). Although the Example 1 asserts that such a naive substitution of \( X \in \mathbb{O} \) by \( u \in \mathcal{C} \ell_{0,7} \) does not hold in general, we conjecture that the most general expression holding in this case must be of the form
\[ (A \bullet u) \circ (\bar{u} \bullet B) = [x^1_u(A) \circ (x^2_u(B) \bullet u)] \circ (1 \bullet \bar{u}), \]
where \( x^1_u \) and \( x^2_u \) are \( u \)-induced involutions on \( \mathbb{O} \), distinct from the \( \mathbb{O} \)-conjugation. In addition, \( f(u) \) is some \( \mathbb{O} \)-valued function involving \( u \in \mathcal{C} \ell_{0,7} \) and eventually some of the generalized non-associative products defined heretofore.

Finally, the non-associative shear was introduced as a map that takes into account the splitting of a multivector into an octonion, subsequently choosing its components to perform non-associative products among themselves, following a variety of possibilities as illustrated the tables throughout the text.

We emphasize that conformal field theory and the Kac-Moody algebra, e.g. in [6] are not concerned and are very far beyond the scope of the present paper, since the main aim here was to introduce and investigate a more general class of non-associative products on \( S^7 \). Also, the framework here introduced is a promising tool for considering its interplay with some applications, as introduced in [29, 36, 37, 38, 39, 40, 41, 42].

One more comment is worthwhile. As triality stabilizes the Lie algebra \( g_2 \) — the derivations of the octonions algebra — pointwise, the formalism presented in this paper can bring some new light on some deformations related to the exceptional group \( G_2 \). Some topological consequences can be addressed as, for example the cubic roots of the unit which describe two latitudes of the sphere \( S^6 \). In addition, the \( \bullet \)-product introduces deformations in \( g_2 \), and an open and fundamental question remains, regarding the construction of the derivation algebra of the octonions, but now the octonionic products are the extended non-associative \( u \)-, \( (1, u) \)-, and \( (u, v) \)-products. For the X-product particular case, such a question was completely posed and solved in [2], for the deformations induced by the paravectors in \( \Lambda^0(\mathbb{R}^{0,7}) \oplus \Lambda^1(\mathbb{R}^{0,7}) \), but the general case still claims for a solution. In addition, the (Leech) lattices leaving the X-product equal to the original octonionic product, and their interrelations to the exceptional Lie groups can be investigated using this formalism. Finally, the formalism presented here can reveal and bring some new light on the \( S^7 \) Hopf bundle of Yang-Mills fields over compactified \( \mathbb{R}^4 \). As the
exotic versions are no longer principal SU(2) bundles, but rather associated bundles with group 
SO(4), the formalism of the \( \bullet \) - and \( \circ \)-products introduced may provide some understanding of 
the differential geometric restrictions inherent to exotic structures, and how to possibly extended 
it to a more general formalism. The formalism introduced here originates more general non-
associative structures on \( S^7 \), which arise from the products already introduced in [1]. Further 
applications may concern some aspects of [9, 34, 36, 37, 39, 43, 44, 45, 46, 47, 48, 49], which are 
are beyond the scope of this paper.

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Appendix A. Proposition 1
In the next six Appendices the Propositions aforementioned in Section 7, and their respective 
demonstrations, are provided in details. Special attention for the case where \( u \in \Lambda^7(\mathbb{R}^{0,7}) \) is 
the volume element, which is not taken into account, since the elements not having common 

factors with \( e_u \) must be taken, and in this case \( u \in \Lambda^7(\mathbb{R}^{0,7}) \) is led back to the case where 
\( u \in \Lambda^6(\mathbb{R}^{0,7}) \). The volume element must have an \( e_u \) term that obviously commute with the \( e_u \) 
in the Proposition statement.

Proposition 1: Given \( e_u \in \Omega \) and \( u = e_{i_1} e_{i_2} \cdots e_{i_k} \in \Lambda^k(\mathbb{R}^{0,7}) \), \( k = 1, \ldots, 6 \) with either 
\( e_u \notin \{ e_{i_1}, e_{i_2}, \ldots, e_{i_k} \} \) or \( \{ e_u, e_{i_1}, e_{i_2} \} \) not a \( \mathbb{H} \)-triple, for \( \{ i_{j_1}, i_{j_2} \} = \{ i_1, \ldots, i_6 \} \), then

\[
(e_u \bullet u) \circ (1 \bullet \tilde{u}) = (-1)^{|u-(u-2)} e_u
\]  
(A.1)

Proof: Hereon the acronym MI denotes the Moufang identity given by Eq.(47).

0) For \( u \in \Lambda^0(\mathbb{R}^{0,7}) \):

\[
(e_u \bullet u) \circ (1 \bullet \tilde{u}) = (e_u \bullet e_0) \circ (1 \bullet \tilde{e}_0) = (e_u \circ e_0) \circ (1 \circ e_0) = (e_u \circ e_0) \circ e_0 = e_u
\]

1) For \( u \in \Lambda^1(\mathbb{R}^{0,7}) \):

(a) \( a \neq b \):

\[
(e_u \bullet u) \circ (1 \bullet \tilde{u}) = (e_u \bullet e_b) \circ (1 \bullet \tilde{e}_b) = (e_u \circ e_b) \circ (1 \circ e_b) = (e_u \circ e_b) \circ e_b = e_u
\]

(b) \( a = b \):

\[
(e_u \bullet u) \circ (1 \bullet \tilde{u}) = (e_u \bullet e_a) \circ (1 \bullet \tilde{e}_a) = (e_u \circ e_a) \circ (1 \circ e_a) = (e_u \circ e_a) \circ e_a = e_u
\]

2) For \( u \in \Lambda^2(\mathbb{R}^{0,7}) \):

(a) \( a \notin \{ b, c \} \), and \( (abc) \) is not a \( \mathbb{H} \)-triple:

\[
(e_u \bullet u) \circ (1 \bullet \tilde{u}) = (e_u \bullet e_{bc}) \circ (1 \bullet \tilde{e}_{bc}) = (e_u \circ e_{bc}) \circ (1 \circ (-e_{bc})) = (e_u \circ e_{bc}) \circ (1 \circ e_{bc}) = (e_u \circ e_{bc}) \circ e_{bc} = e_u
\]
(b) \( a = b \), or \( a = c \), or \((abc)\) is a \( \mathbb{H} \)-triple. Without loss of generality, consider \( a = b \):
\[
(e_a \cdot u) \circ (1 \cdot \tilde{u}) = (e_a \cdot e_{bc}) \circ (1 \cdot \tilde{e}_{bc}) = (e_a \cdot e_{ac}) \circ (1 \cdot \tilde{e}_{ac})
\]
\[
= (e_a \cdot e_{ac}) \circ (1 \cdot (\tilde{e}_{ac} - e_{ac})) = (e_a \cdot e_{ac}) \circ (1 \cdot e_{ac})
\]
\[
= ((e_a \cdot e_a) \circ e_c) \circ (e_a \cdot e_c) = -(e_c \circ (e_a \cdot e_a)) \circ (e_a \cdot e_c)
\]
\[
MI \quad -(e_a \cdot e_a) \circ e_a = e_a
\]

3) For \( u \in \Lambda^3(\mathbb{R}^7) \):
(a) \( a \notin \{b, c\} \), and \((abc)\) is not a \( \mathbb{H} \)-triple:
\[
(e_a \cdot u) \circ (1 \cdot \tilde{u}) = (e_a \cdot e_{bcd}) \circ (1 \cdot \tilde{e}_{bcd})
\]
\[
= (e_a \cdot e_{bcd}) \circ (1 \cdot (\tilde{e}_{bcd} - e_{bcd}))
\]
\[
= (e_a \cdot e_{bcd}) \circ (1 \cdot e_{bcd})
\]
\[
= (((e_a \cdot e_b) \circ e_c) \circ e_d) \circ (e_b \circ e_c) \circ e_d)
\]
\[
= -(e_d \circ ((e_a \cdot e_b) \circ e_c) \circ e_d) \circ ((e_b \circ e_c) \circ e_d)
\]
\[
MI \quad -(e_a \cdot e_b) \circ e_c) \circ (e_b \circ e_c) = -e_a
\]

(b) \( a = b \) or \( a = c \) or \((abc)\) is a \( \mathbb{H} \)-triple. Without loss of generality, consider \( a = b \):
\[
(e_a \cdot u) \circ (1 \cdot \tilde{u}) = (e_a \cdot e_{bdf}) \circ (1 \cdot \tilde{e}_{bdf})
\]
\[
= (e_a \cdot e_{bdf}) \circ (1 \cdot e_{bdf})
\]
\[
= (e_f \circ (e_a \cdot e_b) \circ e_c) \circ e_d)
\]
\[
= -(e_d \circ ((e_a \cdot e_b) \circ e_c) \circ e_d) \circ ((e_b \circ e_c) \circ e_d)
\]
\[
MI \quad -(e_a \cdot e_b) \circ e_c) \circ (e_b \circ e_c) \circ e_d) = -e_a
\]

4) For \( u \in \Lambda^4(\mathbb{R}^7) \):
(a) \( a \notin \{b, c\} \), and \((abc)\) is not a \( \mathbb{H} \)-triple:
\[
(e_a \cdot u) \circ (1 \cdot \tilde{u}) = (e_a \cdot e_{bdf}) \circ (1 \cdot \tilde{e}_{bdf})
\]
\[
= (e_a \cdot e_{bdf}) \circ (1 \cdot e_{bdf})
\]
\[
= (e_f \circ (e_a \cdot e_b) \circ e_c) \circ e_d)
\]
\[
= -(e_d \circ ((e_a \cdot e_b) \circ e_c) \circ e_d) \circ ((e_b \circ e_c) \circ e_d)
\]
\[
MI \quad -(e_a \cdot e_b) \circ e_c) \circ (e_b \circ e_c) \circ e_d) = -e_a
\]

(b) \( a = b \) or \( a = c \) or \((abc)\) is a \( \mathbb{H} \)-triple. Without loss of generality, consider \( a = b \):
\[
(e_a \cdot u) \circ (1 \cdot \tilde{u}) = (e_a \cdot e_{bdf}) \circ (1 \cdot \tilde{e}_{bdf})
\]
\[
= (e_a \cdot e_{bdf}) \circ (1 \cdot e_{bdf})
\]
\[
= (e_f \circ (e_a \cdot e_b) \circ e_c) \circ e_d)
\]
\[
= -(e_d \circ ((e_a \cdot e_b) \circ e_c) \circ e_d) \circ ((e_b \circ e_c) \circ e_d)
\]
\[
MI \quad -(e_a \cdot e_b) \circ e_c) \circ (e_b \circ e_c) \circ e_d) = -e_a
\]
5) For \( u \in \Lambda^5(\mathbb{R}^{0,7}) \):
   (a) \( a \notin \{ b, c \} \), and \((abc)\) is not a \( \mathbb{H} \)-triple:
   \[
   (e_a \bullet u) \circ (1 \bullet \bar{u}) = (e_a \bullet e_{bcdfg}) \circ (1 \bullet \bar{e}_{bcdfg}) = (e_a \bullet e_{bcdfg}) \circ (1 \bullet e_{bcdfg}) = -(e_a \bullet e_{bcdfg}) \circ (1 \bullet e_{bcdfg}) \]
   \[
   -(((e_a \circ e_b \circ e_c) \circ e_d \circ e_f) \circ e_g) \circ (((e_b \circ e_c \circ e_d) \circ e_f) \circ e_g) = (e_g \circ (((e_a \circ e_b \circ e_c) \circ e_d) \circ e_f)) \circ (((e_b \circ e_c \circ e_d) \circ e_f) \circ e_g)
   \]
   \[
   \overset{MI}{=} (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f \circ (((e_b \circ e_c) \circ e_d) \circ e_f)
   \]
   \[
   \overset{(4)(a)}{=} e_a
   \]

(b) \( a = b \) or \( a = c \) or \((abc)\) is a \( \mathbb{H} \)-triple. Without loss of generality, consider \( a = b \):
   \[
   (e_a \bullet u) \circ (1 \bullet \bar{u}) = (e_a \bullet e_{bcdfg}) \circ (1 \bullet \bar{e}_{bcdfg}) = (e_a \bullet e_{acdfg}) \circ (1 \bullet e_{acdfg}) = -(e_a \bullet e_{acdfg}) \circ (1 \bullet e_{acdfg}) = -(((e_a \circ e_b \circ e_c) \circ e_d \circ e_f) \circ e_g) \circ (((e_b \circ e_c \circ e_d) \circ e_f) \circ e_g) = (e_g \circ (((e_a \circ e_b \circ e_c) \circ e_d) \circ e_f)) \circ (((e_b \circ e_c \circ e_d) \circ e_f) \circ e_g)
   \]
   \[
   \overset{MI}{=} (((e_a \circ e_a) \circ e_c) \circ e_d) \circ e_f \circ (((e_b \circ e_c) \circ e_d) \circ e_f)
   \]
   \[
   \overset{(4)(b)}{=} e_a
   \]

6) For \( u \in \Lambda^6(\mathbb{R}^{0,7}) \):
   (a) \( a \notin \{ b, c \} \), and \((abc)\) is not a \( \mathbb{H} \)-triple:
   \[
   (e_a \bullet u) \circ (1 \bullet \bar{u}) = (e_a \bullet e_{bcdfgh}) \circ (1 \bullet \bar{e}_{bcdfgh}) = (e_a \bullet e_{bcdfgh}) \circ (1 \bullet e_{bcdfgh}) = ((e_a \bullet e_{bcdfgh}) \circ (1 \bullet e_{bcdfgh}) \overset{MI}{=} (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f \circ (((e_b \circ e_c) \circ e_d) \circ e_f)
   \]
   \[
   \overset{(5)(a)}{=} e_a
   \]

(b) \( a = b \) or \( a = c \) or \((abc)\) is a \( \mathbb{H} \)-triple. Without loss of generality, consider \( a = b \):
   \[
   (e_a \bullet u) \circ (1 \bullet \bar{u}) = (e_a \bullet e_{bcdfgh}) \circ (1 \bullet \bar{e}_{bcdfgh}) = (e_a \bullet e_{acdfgh}) \circ (1 \bullet e_{acdfgh}) = (e_a \bullet e_{acdfgh}) \circ (1 \bullet e_{acdfgh}) = (e_a \bullet e_{acdfgh}) \circ (1 \bullet e_{acdfgh}) = (e_a \bullet e_{acdfgh}) \circ (1 \bullet e_{acdfgh})
   \]
   \[
   \overset{MI}{=} -(((e_a \circ e_a) \circ e_c) \circ e_d) \circ e_f \circ e_g \circ (((e_b \circ e_c) \circ e_d) \circ e_f) \circ e_g
   \]
   \[
   \overset{(5)(b)}{=} e_a
   \]
Table B1. Radon-Hurwitz numbers.

| j | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|---|
| r_j | 0 | 1 | 2 | 3 | 3 | 3 | 3 | 3 |

Appendix B. Proposition 1′

Proposition 1′: Given $u = e_{i_1}e_{i_2} \ldots e_{i_k} \in \Lambda^k(\mathbb{R}^{0,7})$, $k = 1, \ldots, 6$ then

$$
(1 \bullet u) \circ (1 \bullet \bar{u}) = (1 \bullet e_0) \circ (1 \bullet \bar{e}_0) = e_0 \circ e_0 = 1
$$

where $r_j$ are the Radon-Hurwitz numbers defined by the following table and the recurrence relation $r_{j+8} = r_{j+4}$.

Proof: Hereon the acronym MI denotes the Moufang identity given by Eq.(47).

0) For $u \in \Lambda^0(\mathbb{R}^{0,7})$:

$$
(1 \bullet u) \circ (1 \bullet \bar{u}) = (1 \bullet e_0) \circ (1 \bullet \bar{e}_0) = e_0 \circ e_0 = 1
$$

1) For $u \in \Lambda^1(\mathbb{R}^{0,7})$:

$$
(1 \bullet u) \circ (1 \bullet \bar{u}) = (1 \bullet e_a) \circ (1 \bullet \bar{e}_a) = -e_a \circ e_a = 1
$$

2) For $u \in \Lambda^2(\mathbb{R}^{0,7})$:

$$
(1 \bullet u) \circ (1 \bullet \bar{u}) = (1 \bullet e_a e_b) \circ (1 \bullet \bar{e}_a \bar{e}_b) = (e_a \circ e_b) \circ (1 \bullet (-e_a e_b)) = (e_a \circ e_b) \circ (e_a \circ e_b) = - (e_b \circ e_a) \circ (e_a \circ e_b) = -1
$$

3) For $u \in \Lambda^3(\mathbb{R}^{0,7})$:

(a) $(abc)$ is not a $\mathbb{H}$-triple:

$$
(e_a \bullet u) \circ (1 \bullet \bar{u}) = (e_a \bullet e_b e_c \bar{e}_c) \circ (1 \bullet \bar{e}_a \bar{e}_b \bar{e}_c) = ((e_a \circ e_b) \circ e_c) \circ (1 \bullet (-e_a e_b e_c)) = ((e_a \circ e_b) \circ e_c) \circ ((e_a \circ e_b) \circ e_c) = -(e_c \circ (e_a \circ e_b)) \circ ((e_a \circ e_b) \circ e_c) \overset{MI}{=} -(e_a \circ e_b) \circ (e_a \circ e_b) = 1
$$

(b) $(abc)$ is a $\mathbb{H}$-triple:

$$
(e_a \bullet u) \circ (1 \bullet \bar{u}) = (e_a \bullet e_b e_c) \circ (1 \bullet \bar{e}_a \bar{e}_b \bar{e}_c) = ((e_a \circ e_b) \circ e_c) \circ (1 \bullet (-e_a e_b e_c)) = ((e_a \circ e_b) \circ e_c) \circ ((e_a \circ e_b) \circ e_c) = (e_c \circ (e_a \circ e_b)) \circ ((e_a \circ e_b) \circ e_c) \overset{MI}{=} (e_a \circ e_b) \circ (e_a \circ e_b) = 1
$$
4) For $u \in \Lambda^4(\mathbb{R}^{0,7})$:
   (a) $(abc)$ is not a $\mathbb{H}$-triple:
   \[
   (1 \circ u) \circ (1 \circ u) = (1 \circ_u e_a e_b e_c e_d) \circ (1 \circ_u e_a e_b e_c e_d)
   = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ (1 \circ_u e_a e_b e_c e_d)
   = -((((e_a \circ e_b) \circ e_c) \circ e_d) \circ (e_a \circ e_b) \circ e_c \circ e_d)
   = (e_f \circ (((e_a \circ e_b) \circ e_c) \circ e_d)) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f)
   \]
   \[
   MI = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \quad (3)(a) = -1
   \]

   (b) $(abc)$ is a $\mathbb{H}$-triple:
   \[
   (1 \circ u) \circ (1 \circ u) = (1 \circ_u e_a e_b e_c e_d) \circ (1 \circ_u e_a e_b e_c e_d)
   = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ (1 \circ_u e_a e_b e_c e_d)
   = -((((e_a \circ e_b) \circ e_c) \circ e_d) \circ (e_a \circ e_b) \circ e_c \circ e_d)
   = (e_f \circ (((e_a \circ e_b) \circ e_c) \circ e_d)) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f)
   \]
   \[
   MI = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \quad (3)(b) = 1
   \]

5) For $u \in \Lambda^5(\mathbb{R}^{0,7})$:
   (a) $(abc)$ is not a $\mathbb{H}$-triple:
   \[
   (1 \circ u) \circ (1 \circ u) = (1 \circ_u e_a e_b e_c e_d e_f) \circ (1 \circ_u e_a e_b e_c e_d e_f)
   = (((((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ (1 \circ_u e_a e_b e_c e_d e_f)
   = -(((e_a \circ e_b) \circ e_c) \circ e_d) \circ (e_a \circ e_b) \circ e_c \circ e_d) \circ e_f)
   = (e_f \circ (((e_a \circ e_b) \circ e_c) \circ e_d)) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f)
   \]
   \[
   MI = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \quad (4)(a) = -1
   \]

   (b) $(abc)$ is a $\mathbb{H}$-triple:
   \[
   (1 \circ u) \circ (1 \circ u) = (1 \circ_u e_a e_b e_c e_d e_f) \circ (1 \circ_u e_a e_b e_c e_d e_f)
   = (((((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ (1 \circ_u e_a e_b e_c e_d e_f)
   = -(((e_a \circ e_b) \circ e_c) \circ e_d) \circ (e_a \circ e_b) \circ e_c \circ e_d) \circ e_f)
   = (e_f \circ (((e_a \circ e_b) \circ e_c) \circ e_d)) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f)
   \]
   \[
   MI = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \quad (4)(b) = 1
   \]

6) For $u \in \Lambda^6(\mathbb{R}^{0,7})$:
   (a) $(abc)$ is not a $\mathbb{H}$-triple:
   \[
   (1 \circ u) \circ (1 \circ u) = (1 \circ_u e_a e_b e_c e_d e_f e_g) \circ (1 \circ_u e_a e_b e_c e_d e_f e_g)
   = -(((e_a \circ e_b) \circ e_c) \circ e_d) \circ (e_f) \circ (e_f) \circ (1 \circ u e_a e_b e_c e_d e_f e_g)
   \]
   \[
   MI = -(((e_a \circ e_b) \circ e_c) \circ e_d) \circ (e_f) \circ (e_f) \circ (1 \circ u e_a e_b e_c e_d e_f e_g)
   \]
   \[
   (5)(a) = -1
   \]
(b) \((abc)\) is a \(\mathbb{H}\)-triple:

\[
(1 \bullet u) \circ (1 \bullet u) = (1 \bullet e_a e_b e_c e_d e_f e_g) \circ (1 \bullet e_a e_b e_c e_d e_f e_g)
\]

\[
= -(((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ (1 \bullet e_a e_b e_c e_d e_f e_g)
\]

\[
\overset{\text{MI}}{=} -(((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f)
\]

\[
\overset{(5)(b)}{=} 1
\]

7) For \(u \in \Lambda^7(\mathbb{R}^{0,7})\):

(a) \((abc)\) is not a \(\mathbb{H}\)-triple:

\[
(1 \bullet u) \circ (1 \bullet u) \overset{(6)(a)}{=} 1
\]

(b) \((abc)\) is a \(\mathbb{H}\)-triple:

\[
(1 \bullet u) \circ (1 \bullet u) \overset{(6)(b)}{=} -1
\]

Appendix C. Proposition 2

**Proposition 2:** Given \(e_a \in \mathcal{O}\) and \(u = e_{i_1} e_{i_2} \ldots e_{i_k} \in \Lambda^k(\mathbb{R}^{0,7})\), \(k = 1, \ldots, 6\) then

\[
(\bar{u} \bullet_j e_a) \circ (1 \bullet \bar{u}) = (-1)^{|u|(|u|+1)/2} e_a \tag{C.1}
\]

**Proof:** \((\bar{u} \bullet_j e_a) \circ (1 \bullet \bar{u})\) is computed explicitly for all the homogeneous multivectors that do not have common factors with \(e_a\).

0) For \(u \in \Lambda^0(\mathbb{R}^{0,7})\):

\[
(\bar{u} \bullet_j e_a) \circ (1 \bullet \bar{u}) = (\bar{e}_0 \bullet_j e_a) \circ (1 \bullet \bar{e}_0) = (e_0 \circ e_a) \circ (1 \circ e_0)
\]

\[
= (e_0 \circ e_a) \circ e_0 = e_a
\]

1) For \(u \in \Lambda^1(\mathbb{R}^{0,7})\):

\[
(\bar{u} \bullet_j e_a) \circ (1 \bullet \bar{u}) = (\bar{e}_b \bullet_j e_a) \circ (1 \bullet \bar{e}_b) = (-e_b \circ e_a) \circ (1 \circ e_b)
\]

\[
= -(e_b \circ e_a) \circ e_b = -e_a
\]

2) For \(u \in \Lambda^2(\mathbb{R}^{0,7})\):

\[
(\bar{u} \bullet_j e_a) \circ (1 \bullet \bar{u}) = (\bar{e}_{bc} \bullet_j e_a) \circ (1 \bullet \bar{e}_{bc}) = (e_{cb} \bullet_j e_a) \circ (1 \bullet (-e_{bc}))
\]

\[
= -(e_c \circ (e_b \circ e_a)) \circ (e_b \circ e_c) \overset{\text{MI}}{=} -(e_b \circ e_a) \circ e_b = -e_a
\]

3) For \(u \in \Lambda^3(\mathbb{R}^{0,7})\):

\[
(\bar{u} \bullet_j e_a) \circ (1 \bullet \bar{u}) = (\bar{e}_{bcd} \bullet_j e_a) \circ (1 \bullet \bar{e}_{bcd}) = (-e_{deb} \bullet_j e_a) \circ (1 \bullet (-e_{bcd}))
\]

\[
= (e_{deb} \bullet_j e_a) \circ (1 \bullet e_{bcd})
\]

\[
= (e_d \circ (e_c \circ (e_b \circ e_a)) \circ ((e_b \circ e_c) \circ e_d) \overset{\text{MI}}{=} (e_c \circ (e_b \circ e_a)) \circ (e_b \circ e_c) \overset{(2)}{=} e_a
\]
Proposition 2′

Appendix D. Proposition 2′

Proposition 2′: Given $u = e_{i_1} e_{i_2} \ldots e_{i_k} \in \Lambda^k(\mathbb{R}^{0,7})$, $k = 1, \ldots, 6$ then

$$ (\bar{u} \circ (1 \cdot \bar{u}) = (-1)^{|u|(|r|_u - |r|_{\bar{u}} - 1) + (8 - |u|)(7 - |u|)/2} \quad (D.1) $$

Proof:

0) For $u \in \Lambda^0(\mathbb{R}^{0,7})$:

$$(\bar{u} \circ (1 \cdot \bar{u}) = (e_0 \circ (1 \cdot e_0) = e_0 \circ e_0 = 1$$

1) For $u \in \Lambda^1(\mathbb{R}^{0,7})$:

$$(\bar{u} \circ (1 \cdot \bar{u}) = (e_a \circ (1 \cdot e_a) = -e_a \circ 1 \circ e_a = -e_a \circ e_a = 1$$

2) For $u \in \Lambda^2(\mathbb{R}^{0,7})$:

$$(\bar{u} \circ (1 \cdot \bar{u}) = (e_b e_c \circ (1 \cdot e_b e_c) = (e_b e_c \circ (1 \cdot (e_a e_b)) = -e_b \circ e_a \circ e_b = -1$$

3) For $u \in \Lambda^3(\mathbb{R}^{0,7})$:

$$(\bar{u} \circ (1 \cdot \bar{u}) = (e_a e_b e_c \circ (1 \cdot e_a e_b e_c) = (e_a e_b e_c \circ (1 \cdot (e_a e_b e_c)) = (e_a \circ (e_b \circ e_a)) \circ ((e_a \circ e_b) \circ e_c) = e_b \circ e_a \circ e_b = 1$$
4) For $u \in \Lambda^4(\mathbb{R}^{0,7})$:

\[
(\bar{u} \bullet 1) \circ (1 \bullet \bar{u}) = (e_a e_b e_c e_d \bullet 1) \circ (1 \bullet e_a e_b e_c e_d)
\]

\[
= (e_a e_b e_c e_d \bullet 1) \circ (1 \bullet e_a e_b e_c e_d)
\]

\[
= (e_d \circ (e_c \circ (e_b \circ e_a))) \circ (((e_a \circ e_b) \circ e_c) \circ e_d)
\]

\[
\overset{MI}{=} (e_c \circ (e_b \circ e_a)) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \overset{(3)}{=} 1
\]

5) For $u \in \Lambda^5(\mathbb{R}^{0,7})$:

\[
(\bar{u} \bullet 1) \circ (1 \bullet \bar{u}) = (e_a e_b e_c e_d e_f \bullet 1) \circ (1 \bullet e_a e_b e_c e_d e_f)
\]

\[
= (e_f e_d e_c e_b e_a \bullet 1) \circ (1 \bullet e_a e_b e_c e_d e_f)
\]

\[
= (e_f \circ (e_d \circ (e_c \circ (e_b \circ e_a)))) \circ (((e_a \circ e_b) \circ e_c) \circ e_d)
\]

\[
\overset{MI}{=} (e_d \circ (e_c \circ (e_b \circ e_a))) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \overset{(4)}{=} -1
\]

6) For $u \in \Lambda^6(\mathbb{R}^{0,7})$:

\[
(\bar{u} \bullet 1) \circ (1 \bullet \bar{u}) = (e_a e_b e_c e_d e_f e_g \bullet 1) \circ (1 \bullet e_a e_b e_c e_d e_f e_g)
\]

\[
= (e_g e_f e_d e_c e_b e_a \bullet 1) \circ (1 \bullet (e_a e_b e_c e_d e_f e_g))
\]

\[
= (e_g \circ (e_f \circ (e_d \circ (e_c \circ (e_b \circ e_a)))))) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f)
\]

\[
\overset{MI}{=} (e_f \circ (e_d \circ (e_c \circ (e_b \circ e_a))) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \overset{(5)}{=} -1
\]

7) For $u \in \Lambda^7(\mathbb{R}^{0,7})$:

\[
(\bar{u} \bullet 1) \circ (1 \bullet \bar{u}) = (e_a e_b e_c e_d e_f e_g e_h \bullet 1) \circ (1 \bullet e_a e_b e_c e_d e_f e_g e_h)
\]

\[
= (-e_h e_g e_f e_d e_c e_b e_a \bullet 1) \circ (1 \bullet (e_a e_b e_c e_d e_f e_g e_h))
\]

\[
= (e_h \circ (e_g \circ (e_f \circ (e_d \circ (e_c \circ (e_b \circ e_a))))) \circ (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g)
\]

\[
\overset{(6)}{=} 1
\]

Appendix E. Proposition 3

**Proposition 3**: Given $e_a \in \mathbb{H}$ and $u = e_{i_1} e_{i_2} \ldots e_{i_k} \in \Lambda^k(\mathbb{R}^{0,7})$, $k = 1, \ldots, 6$ with either $e_a \notin \{e_{i_1}, e_{i_2}, \ldots, e_{i_k}\}$ or $\{e_a, f_{i_1}, f_{i_2}\}$ not a $\mathbb{H}$-triple, for $\{i_{j_1}, i_{j_2}\} = \{i_1, \ldots, i_6\}$, then

\[
\bullet u = (-1)^{|u|+1} e_a \circ (1 \bullet u)
\]

**Proof**:

0) When $u = e_0 \in \Lambda^0(\mathbb{R}^{0,7})$ it follows that:

\[
e_a \bullet e_0 = e_a \circ e_0 = e_a \circ (1 \circ e_0) = e_a \circ (1 \bullet u)
\]
1) When \( u = e_b \in \Lambda^1(\mathbb{R}^{0,7}) \) it follows that:
   (a) \( a \neq b \):
   \[
   e_a \cdot_u u = e_a \cdot_u e_b = e_a \circ (1 \circ e_b) = e_a \circ (1 \cdot_u u)
   \]
   (b) \( a = b \):
   \[
   e_a \cdot_u u = e_a \cdot_u e_a = e_a \circ (1 \circ e_a) = e_a \circ (1 \cdot_u u)
   \]

2) When \( u = e_{bc} \in \Lambda^2(\mathbb{R}^{0,7}) \) it follows that:
   (a) \( a \notin \{b, c\} \), and \((abc)\) is not a \( H \)-triple:
   \[
   e_a \cdot_u u = e_a \cdot_u e_{bc} = (e_a \circ e_b) \circ e_c
   = -e_a \circ (e_b \circ e_c) = -e_a \circ (1 \cdot_u e_{bc}) = -e_a \circ (1 \cdot_u u)
   \]
   (b) \( a \in \{b, c\} \) or \((abc)\) is a \( H \)-triple. Without loss of generality consider \( a = b \):
   \[
   e_a \cdot_u u = e_a \cdot_u e_{bc} = e_a \cdot_u e_{ac} = (e_a \circ e_a) \circ e_c
   = e_a \circ (e_a \circ e_c) = e_a \circ (1 \cdot_u e_{ac})
   = e_a \circ (1 \cdot_u u)
   \]

3) When \( u = e_{bcd} \in \Lambda^3(\mathbb{R}^{0,7}) \), follows that:
   (a) \( a \notin \{b, c, d\} \), and \((ijk)\) is not a \( H \)-triple where \( i, j, k \in \{a, b, c, d\} \):
   \[
   e_a \cdot_u u = e_a \cdot_u e_{bcd} = ((e_a \circ e_b) \circ e_c) \circ e_d
   = -(e_a \circ (e_b \circ e_c)) \circ e_d
   = e_a \circ ((e_b \circ e_c) \circ e_d) = e_a \circ (1 \cdot_u e_{bcd}) = e_a \circ (1 \cdot_u u)
   \]
   (b) \( a \in \{b, c, d\} \) or \((abc)\) is a \( H \)-triple. Without loss of generality consider \( a = b \):
   \[
   e_a \cdot_u u = e_a \cdot_u e_{bcd} = e_a \cdot_u e_{acd} = ((e_a \circ e_a) \circ e_c) \circ e_d
   = (e_a \circ (e_a \circ e_c)) \circ e_d = -(e_a \circ ((e_a \circ e_c) \circ e_d) = -e_a \circ (1 \cdot_u e_{acd})
   = -e_a \circ (1 \cdot_u u)
   \]
   (c) \((ijk)\) is a \( H \)-triple where \( i, j, k \in \{a, b, c, d\} \) and \((ijk) \neq (abc)\). Let us suppose that \((abd)\) is a \( H \)-triple:
   \[
   e_a \cdot_u u = e_a \cdot_u e_{bcd} = ((e_a \circ e_b) \circ e_c) \circ e_d
   = -(e_a \circ (e_b \circ e_c)) \circ e_d
   = e_a \circ ((e_b \circ e_c) \circ e_d) = e_a \circ (1 \cdot_u e_{bcd}) = e_a \circ (1 \cdot_u u)
   \]

4) When \( u = e_{bcdf} \in \Lambda^4(\mathbb{R}^{0,7}) \), follows that:
   (a) \( a \notin \{b, c, d, f\} \), and \((ijk)\) is not a \( H \)-triple where \( i, j, k \in \{a, b, c, d, f\} \):
   \[
   e_a \cdot_u u = e_a \cdot_u e_{bcdf} = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f
   = -(e_a \circ (e_b \circ e_c)) \circ e_d) \circ e_f
   = (e_a \circ ((e_b \circ e_c) \circ e_d)) \circ e_f = -e_a \circ ((e_b \circ e_c) \circ e_d) \circ e_f)
   = -e_a \circ (1 \cdot_u e_{bcdf})
   = -e_a \circ (1 \cdot_u u)
   \]

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(b) \( a \in \{b, c, d, f\} \) or \((abc)\) is a \( \mathbb{H}\)-triple. Without loss of generality consider \( a = b \):
\[
\begin{align*}
e_a \bullet u &= e_a \bullet e_{bcdf} = e_a \bullet e_{acdf} \\
&= (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f = ((e_a \circ (e_b \circ e_c)) \circ e_d) \circ e_f \\
&= -(e_a \circ ((e_b \circ e_c) \circ e_d)) \circ e_f \\
&= e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f) = e_a \circ (1 \bullet e_{acdf}) \\
&= e_a \circ (1 \bullet u)
\end{align*}
\]

(E.2)

(c) \((ijk)\) is a \( \mathbb{H}\)-triple where \( i, j, k \in \{a, b, c, d, f\} \) and \((ijk) \neq (abc)\). Let us suppose that \((abd)\) is a \( \mathbb{H}\)-triple:
\[
\begin{align*}
e_a \bullet u &= e_a \bullet e_{bcd} \\
&= (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f = -(e_a \circ (e_b \circ e_c)) \circ e_d) \circ e_f \\
&= (e_a \circ ((e_b \circ e_c) \circ e_d)) \circ e_f = -e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f) \\
&= -e_a \circ (1 \bullet e_{bcd}) \\
&= -e_a \circ (1 \bullet u)
\end{align*}
\]

5) When \( u = e_{bcdfg} \in \Lambda^5(\mathbb{R}^{0,7}) \), follows that:
(a) \( a \notin \{b, c, d, f, g\} \), and \((ijk)\) is not a \( \mathbb{H}\)-triple where \( i, j, k \in \{a, b, c, d, f, g\} \):
\[
\begin{align*}
e_a \bullet u &= e_a \bullet e_{bcdfg} = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g \\
&= -((e_a \circ (e_b \circ e_c)) \circ e_d) \circ e_f) \circ e_g \\
&= ((e_a \circ ((e_b \circ e_c) \circ e_d)) \circ e_f) \circ e_g \\
&= -(e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f)) \circ e_g \\
&= e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f) \circ e_g \\
&= e_a \circ (1 \bullet e_{bcdfg}) = e_a \circ (1 \bullet u)
\end{align*}
\]

(b) \( a \in \{b, c, d, f, g\} \) or \((abc)\) is a \( \mathbb{H}\)-triple. Without loss of generality, consider \( a = b \):
\[
\begin{align*}
e_a \bullet u &= e_a \bullet e_{bcdfgh} = e_a \bullet e_{acdfgh} = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g \\
&= -((e_a \circ (e_b \circ e_c)) \circ e_d) \circ e_f) \circ e_g \\
&= ((e_a \circ ((e_b \circ e_c) \circ e_d)) \circ e_f) \circ e_g \\
&= -(e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f)) \circ e_g \\
&= e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f) \circ e_g \\
&= -e_a \circ (1 \bullet e_{acdfgh}) = -e_a \circ (1 \bullet u)
\end{align*}
\]

(c) \((ijk)\) is a \( \mathbb{H}\)-triple where \( i, j, k \in \{a, b, c, d, f, g\} \) and \((ijk) \neq (abc)\). Let us suppose that \((abd)\) is a \( \mathbb{H}\)-triple:
\[
\begin{align*}
e_a \bullet u &= e_a \bullet e_{bcdfg} = (((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g \\
&= -((e_a \circ (e_b \circ e_c)) \circ e_d) \circ e_f) \circ e_g \\
&= ((e_a \circ ((e_b \circ e_c) \circ e_d)) \circ e_f) \circ e_g \\
&= -(e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f)) \circ e_g \\
&= e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f) \circ e_g \\
&= e_a \circ (1 \bullet e_{bcdfg}) = e_a \circ (1 \bullet u)
\end{align*}
\]
(d) $(ijk)$ and $(lmn)$ are $\mathbb{H}$-triples where $i, j, k, l, m, n \in \{a, b, c, d, f, g, h\}$, $\{i, j, k\} \neq \{l, m, n\}$ and $(ijk) \neq (abc)$. Let us suppose that $(abd)$ and $(cfg)$ are $\mathbb{H}$-triples:

\[
e_a \bullet u = e_a \bullet e_{bcdfg} = (((((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g) = e_a \circ (e_b \bullet e_{bcdfg}) = e_a \circ (1 \bullet u)
\]

6) When $u = e_{bcdfg} \in \Lambda^6(\mathbb{R}^{0,7})$, follows that:

(a) $a \notin \{b, c, d, f, g, h\}$, and $(ijk)$ and $(lmn)$ are not $\mathbb{H}$-triples where $i, j, k, l, m, n \in \{a, b, c, d, f, g, h\}$ and $\{i, j, k\} \neq \{l, m, n\}$:

\[
e_a \bullet u = e_a \bullet e_{bcdfg} = (((((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g) \circ e_h = e_a \circ (e_b \bullet e_{bcdfgh}) = e_a \circ (1 \bullet u)
\]

(b) $a \in \{b, c, d, f, g, h\}$ or $(abc)$ is a $\mathbb{H}$-triple. Without loss of generality, consider $a = b$:

\[
e_a \bullet u = e_a \bullet e_{bcdfg} = e_a \bullet e_{acdfgh} = (((((e_a \circ e_a) \circ e_c) \circ e_d) \circ e_f) \circ e_g) \circ e_h = e_a \circ (e_a \bullet e_{acdfgh}) = e_a \circ (1 \bullet u)
\]

(c) $(ijk)$ is a $\mathbb{H}$-triple where $i, j, k \in \{a, b, c, d, f, g, h\}$ and $(ijk) \neq (abc)$. Let us suppose that $(abd)$ is a $\mathbb{H}$-triple:

\[
e_a \bullet u = e_a \bullet e_{bcdfg} = (((((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g) \circ e_h = e_a \circ (e_b \bullet e_{bcdfgh}) = e_a \circ (1 \bullet u)
\]

(d) $(ijk)$ and $(lmn)$ are $\mathbb{H}$-triples where $i, j, k, l, m, n \in \{a, b, c, d, f, g, h\}$, $\{i, j, k\} \neq \{l, m, n\}$ and $(ijk) \neq (abc)$, or $(lmn) \neq (abc)$. Let us suppose that $(abd)$ and $(cfg)$ are $\mathbb{H}$-triples:

\[
e_a \bullet u = e_a \bullet e_{bcdfg} = (((((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g) \circ e_h = e_a \circ (e_b \bullet e_{bcdfgh}) = e_a \circ (1 \bullet u)
\]
Let us suppose that \( (abd) \) and \( (cfg) \) are \( \mathbb{H} \)-triples:

\[
e_a \bullet u = e_a \bullet e_{bcdfgh} = (((((e_a \circ e_b) \circ e_c) \circ e_d) \circ e_f) \circ e_g) \circ e_h
\]

\[
= (((e_a \circ (e_b \circ e_c)) \circ e_d) \circ e_f) \circ e_g) \circ e_h
\]

\[
= ((e_a \circ ((e_b \circ e_c) \circ e_d) \circ e_f)) \circ e_g) \circ e_h
\]

\[
= (e_a \circ (((e_b \circ e_c) \circ e_d) \circ e_f) \circ e_g)) \circ e_h
\]

\[
e_a \circ ((((e_b \circ e_c) \circ e_d) \circ e_f) \circ e_g) \circ e_h
\]

\[
e_a \circ (1 \bullet e_{bcdfgh}) = e_a \circ (1 \bullet u)
\]

On the hypothesis above \( u \in \Lambda(\mathbb{R}^{0,7}) \) should not be homogeneous. For instance, by taking Eqs.(E.1) and (E.2). When \( u = e_b e_c + e_b e_d e_b \in \Lambda^2(\mathbb{R}^{0,7}) \oplus \Lambda^4(\mathbb{R}^{0,7}) \) is defined, the Propositions shown in cases 2(b) and 4(b) above imply that \( e_a \bullet u = e_a \circ (1 \bullet u) \), and therefore in many cases the referred Proposition holds for elements that are not homogeneous. Taking similar examples, it can shown that there exists some cases where \( u \in \Lambda^k(\mathbb{R}^{0,7}) \oplus \Lambda^{k+1}(\mathbb{R}^{0,7}) \) and it satisfies the Proposition 3.

Appendix F. Proposition 4

Here the result of the Proposition 3 for \( e_a \in \mathbb{O} \) and \( u \in \mathcal{C}_{0,7} \) is considered, \( e_a \bullet u = \pm e_a \circ (1 \bullet u) \). Hence the Proposition 4 is a generalization of the Proposition 3.

**Proposition 4:** Given \( \psi \in \mathcal{C}_{0,7} \) and \( u \in \Lambda^k(\mathbb{R}^{0,7}) \), \( k = 1, \ldots, 6 \) then

\[
\psi \circ_\psi u = (-1)^{|u|+1} \psi \bullet_\psi (1 \bullet u)
\]

**F.1**

In all the cases above \( |u| \) denotes the degree of \( u \): if \( u \in \Lambda^k(\mathbb{R}^{0,7}) \), then \( |u| = k \).

**Proof:** It must be evaluated the computation for all the cases, but considering Proposition 3 the work is reduced.

0) For \( \psi = e_0 \in \Lambda^0(\mathbb{R}^{0,7}) \):

\[
e_0 \circ_\psi u = e_0 \bullet_\psi u = e_0 \circ (1 \bullet u) = \psi \bullet_\psi (1 \bullet u)
\]

1) For \( \psi = e_a \in \Lambda^1(\mathbb{R}^{0,7}) \):

\[
e_a \circ_\psi u = e_a \bullet_\psi u = \pm e_a \circ (1 \bullet u) = \pm \psi \bullet_\psi (1 \bullet u)
\]

2) For \( \psi = e_{ab} \in \Lambda^2(\mathbb{R}^{0,7}) \):

\[
e_{ab} \circ_\psi u = e_{ab} \circ (e_b \bullet_\psi u) = \pm e_a \circ (e_b \circ (1 \bullet u)) = \pm e_{ab} \bullet_\psi (1 \bullet u) = \pm \psi \bullet_\psi (1 \bullet u)
\]

3) For \( \psi = e_{abc} \in \Lambda^3(\mathbb{R}^{0,7}) \):

\[
e_{abc} \circ_\psi u = e_{abc} \circ (e_{bc} \bullet_\psi u) = \pm e_{abc} \circ (e_c \circ (1 \bullet u)) = \pm e_{abc} \bullet_\psi (1 \bullet u) = \pm \psi \bullet_\psi (1 \bullet u)
\]

4) For \( \psi = e_{abcd} \in \Lambda^4(\mathbb{R}^{0,7}) \):

\[
e_{abcd} \circ_\psi u = e_{abcd} \circ (e_{bc} \circ (e_d \bullet_\psi u)) = \pm e_{abcd} \circ (e_d \circ (1 \bullet u)) = \pm e_{abcd} \bullet_\psi (1 \bullet u) = \pm \psi \bullet_\psi (1 \bullet u)
\]
Note that the indices of \( \psi \) and \( u \) must be different, otherwise the signal changes.

Appendix G. The frame \( \{ e_a \circ X \} \)

As \( X \in \mathcal{O} \) does not take any privileged unit, without loss of generality, by setting \( e_a = e_1 \), once the process is similar for all \( e_a, a = 1, \ldots, 7 \), it shall be checked that \( \langle e_a \circ X, X \rangle = 0 \).

\[
\langle e_1 \circ X, X \rangle = 1/2 [ e_1 \circ X ] \circ X + \bar{X} \circ ( e_1 \circ X )
= 1/2 [ ( X^0 e_1 - X^1 e_2 e_1 + X^3 e_3 - X^4 e_4 + X^5 e_5 + X^6 e_6 - X^7 e_7 ) \circ ( X^0 e_1 + X^1 e_2 e_1 + X^3 e_3 - X^4 e_4 + X^5 e_5 - X^6 e_6 + X^7 e_7 ) + ( X^0 - X^1 e_3 - X^2 e_3 - X^4 e_4 + X^5 - X^6 e_5 - X^7 e_7 ) \circ ( X^0 e_1 - X^1 e_2 e_1 + X^3 e_3 - X^4 e_4 + X^5 e_5 - X^6 e_6 - X^7 e_7 ) ]
= 0.
\]

Now, it will be shown that \( \langle e_a e_b \bullet_j X, X \rangle = 0 \). Again, without loss of generality, choosing \( e_1 e_2 \) comes:

\[
\langle e_1 e_2 \bullet_j X, X \rangle = 1/2 [ ( e_1 e_2 \bullet_j X ) \circ X + \bar{X} \circ ( e_1 e_2 \bullet_j X ) ]
= 1/2 [ ( X^0 e_6 - X^1 e_2 e_1 + X^3 e_3 - X^4 e_4 + X^5 e_5 - X^6 e_6 + X^7 e_7 ) \circ ( X^0 - X^1 e_3 - X^2 e_3 - X^4 e_4 + X^5 - X^6 e_5 + X^7 e_7 ) + ( X^0 e_1 - X^1 e_2 e_1 + X^3 e_3 - X^4 e_4 + X^5 e_5 - X^6 e_6 - X^7 e_7 ) \circ ( X^0 + X^1 e_2 e_1 - X^3 e_3 + X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7 ) + ( X^0 e_1 - X^1 e_2 e_1 + X^3 e_3 - X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7 ) \circ ( X^0 - X^1 e_3 - X^2 e_3 - X^4 e_4 + X^5 - X^6 e_5 + X^7 e_7 ) ]
= 0.
\]

Appendix H. The \( X \)-product representations

It is a well known assertion (see e.g. [2]) that the octonionic algebra is a non-associative algebra, not being possible to represent it on a matrix algebra. The adjoint algebras of the left and right actions on octonions itself are associative. In our case, the \( \bullet \)-product on the right and left is presented in Eqs.(9,10) respectively.

The main purpose of this Section is to evince the matrix representation for the left actions on the octonions in our formalism\(^1\) [2]. The following matrices generate \( \mathcal{M}(8, \mathbb{R}) \) and all \( u \bullet_j X \) representations for \( u \in \mathcal{O}_{0,7} \). Indeed, the Hodge dual in \( \mathcal{O}_{0,7} \) can be expressed as

\[
* u = \hat{u} e_1 e_2 e_3 e_4 e_5 e_6 e_7,
\]

\(^1\) In fact, Dixon exhibited the matrix representations for the left and right actions on the octonions by computing — in his notation — \( e_{Ra} e_{Lab} e_{Labc} \) and also \( e_{Ra} e_{Rab} e_{Rabc} \), for octonions provided from both the \( e_a e_{a+1} = e_{a+5 \, \text{mod} \, 7} \) and \( e_a e_{a+1} = e_{a+3 \, \text{mod} \, 7} \) rules.
therefore in order to generate the matrices associated to the set
\[ \{ e_a e_b e_c e_d, e_a e_b e_c e_f e_g, e_1 e_2 e_3 e_4 e_5 e_6 e_7 \} \]

that acts on \( X \in \mathcal{O} \) by \( \bullet \)-product — it must just to be considered respectively the correspondence to \( \{ * 1, * e_a, * e_b e_0, * e_a e_b e_c \} \) in this order. Moreover, the set \( \{ 1, e_a, e_b e_0, e_a e_b e_c \} \) has dimension equal to 64.

An octonion \( X = X^0 + X^a e_a \) can be written as \((X^0, X^1, \ldots, X^7)^T\) representing its vector space underlying structure. Below we present the \( \circ \)-product by the action of matrices:

\[
e_1 \circ (X) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ X^0 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \\ X^6 \\ X^7 \end{pmatrix}
\]

Analogously, each \( e_a \circ (\cdot) \) action corresponds respectively to the following matrices:

\[
e_2 \circ (\cdot) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{e}_3 \circ (\cdot) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
e_4 \circ (\cdot) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{e}_5 \circ (\cdot) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

\[
e_6 \circ (\cdot) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{e}_7 \circ (\cdot) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
\]

Also, the \( \bullet \)-product can be represented by their matrices left actions. Below we present explicitly all the \( \Lambda(V) \) basis vectors action

\[
(e_1 e_2) \bullet (X) = e_1 \circ (e_2 \circ (X^0 + X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7)) = e_1 \circ (X^0 e_2 - X^1 e_1 - X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7) = X^0 e_2 + X^1 e_1 - X^2 e_2 - X^3 e_3 + X^4 e_4 + X^5 e_5 - X^6 e_6 - X^7 e_7 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} X^0 \\ X^1 \\ X^2 \\ X^3 \\ X^4 \\ X^5 \\ X^6 \\ X^7 \end{pmatrix}
\]
Analogously, each \( e_i e_j \cdot (\cdot) \) action corresponds respectively to the following matrices:

\[
\begin{align*}
(e_1 e_3) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \\
(e_1 e_4) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \\
(e_1 e_5) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \\
(e_2 e_3) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
(e_2 e_4) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
(e_2 e_5) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
(e_3 e_3) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
(e_3 e_4) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \\
(e_3 e_5) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
(e_3 e_6) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \\
(e_4 e_6) \cdot (\cdot) & \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. 
\end{align*}
\]
Analogously, each \( e_i \) action corresponds respectively to the following matrices:

\[
(e_1 e_2 e_3) \cdot (X) = e_1 \circ (e_2 \circ (e_3 \circ (X^0 + X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5 + X^6 e_6 + X^7 e_7)))
\]

\[
= e_1 \circ (e_2 \circ (X^0 e_7 - X^1 e_7 + X^2 e_6 - X^3 e_5 + X^4 e_4 + X^5 e_3 - X^7 e_1))
\]

\[
= e_1 \circ (X^0 e_7 - X^1 e_7 + X^2 e_6 - X^3 e_5 + X^4 e_4 + X^5 e_3 - X^7 e_1)
\]

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
X^0 \\
X^1 \\
X^2 \\
X^3 \\
X^4 \\
X^5 \\
X^6 \\
X^7 \\
\end{pmatrix}
\]

Analogously, each \( e_i e_5 e_6 \cdot (\cdot) \) action corresponds respectively to the following matrices:

\[
(e_1 e_2 e_4) \cdot (\cdot) \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

\[
(e_1 e_2 e_5) \cdot (\cdot) \mapsto \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

\[
(e_1 e_2 e_6) \cdot (\cdot) \mapsto \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

\[
(e_1 e_3 e_4) \cdot (\cdot) \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
(e_1 e_3 e_5) \cdot (\cdot) \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
(e_1 e_3 e_6) \cdot (\cdot) \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
(e_1 e_3 e_7) \cdot (\cdot) \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\[
\begin{align*}
(e_1 e_4 e_5) \cdot (\cdot) & \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix} & (e_1 e_4 e_6) \cdot (\cdot) & \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} \\
(e_1 e_4 e_7) \cdot (\cdot) & \mapsto \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix} & (e_1 e_5 e_6) \cdot (\cdot) & \mapsto \begin{pmatrix}
0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\end{align*}
\]
\((e_3 e_4 e_7) \cdot_j (\cdot) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \)

\((e_3 e_5 e_6) \cdot_j (\cdot) \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \)

\[(e_3 e_5 e_7) \cdot_j (\cdot) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[(e_4 e_5 e_6) \cdot_j (\cdot) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[(e_4 e_5 e_7) \cdot_j (\cdot) \mapsto \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \]

\[(e_5 e_6 e_7) \cdot_j (\cdot) \mapsto \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \]

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