Noncommutative Quantum Mechanics and rotating frames

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Abstract

We study the effect of noncommutativity of space on the physics of a quantum interferometer located in a rotating disk in a gauge field background. To this end, we develop a path-integral approach which allows defining an effective action from which relevant physical quantities can be computed as in the usual commutative case. For the specific case of a constant magnetic field, we are able to compute, exactly, the noncommutative Lagrangian and the associated shift on the interference pattern for any value of \( \theta \).

1 Introduction and Results

The interest in noncommutative space, recently aroused in connection with developments in string theory [1]-[3], rapidly spread on other domains going from Quantum field theories and Quantum mechanics to Condensed matter physics [4]-[20] (See [21] for a complete list of references). Concerning quantum mechanical problems, since noncommutative physics can be connected with the dynamics of charged particles in a magnetic field (the Landau problem), many interesting results have been presented, going from the Aharonov-Bohm effect to the Quantum Hall effect [7]-[20].

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The purpose of the present work is two fold. On the one hand, we want to discuss
the specific quantum mechanical problem of a charged particle in a rotating disk, in the
presence of an electromagnetic field, when space is the anticommutative plane. This is
an interesting problem related to the Aharonov-Bohm effect, relevant to the physics of
superconducting interferometers.

On the other hand, we want to develop a simple procedure to handle, within the
path-integral approach, noncommutative quantum mechanical problems. The idea is to
provide the Feynman path-integral alternative to the wave equation approach developed
in [7]-[9]. This last approach is based in taking into account noncommutativity of the
base space by using the so called * product when the potential in the Hamiltonian acts
on the wave function. Now, at the Hamiltonian level, this amounts to an appropriate
shift in the coordinate dependence of the potential (and no change in momenta) so that,
finally, noncommutativity is encoded in the shifted potential through the noncommutative
parameter $\theta_{ij}$. Our approach starts precisely at this point and makes use of the Feynman
recipe for constructing, in phase space, the transition amplitude $Z$ for a quantum system
in noncommutative space as an integral over trajectories. Now, for simple (quadratic
both in $p$ and $x$) potentials, one can integrate over momenta ending with $Z$ written as
a path-integral over $x$, with an effective action where noncommutativity manifests just
through the parameter $\theta_{ij}$ appearing in the effective action.

Let us summarize the main results of our investigation. Concerning the path-integral
treatment of a general noncommutative planar system described by a quantum Hamiltonian $H$, we construct the transition amplitude between given initial and final states in the form

$$Z = \int \mathcal{D}\vec{p}\mathcal{D}\vec{x} \exp\left(\int_{t_i}^{t_f} dt \left(i(\vec{p}\dot{\vec{x}} - H_{\text{eff}}^\theta(\vec{p}, \vec{x}))\right)\right)$$

with the effective Hamiltonian $H_{\text{eff}}^\theta$ given by

$$H_{\text{eff}}^\theta = \frac{1}{2m}\vec{p}^2 + V(\vec{x} - \vec{\tilde{p}})$$

and

$$\vec{\tilde{p}}_i = \frac{1}{2} \theta \varepsilon_{ij} p_j$$

with $\theta$ the parameter characterizing noncommutativity. The only change with respect to
the usual Feynman formula in ordinary space is that the (classical) potential $V$ appears
with its argument shifted due to the presence of $\theta$.

Depending on the precise form of the potential, one should be able to integrate over
$\vec{p}$ in (1) ending with the Lagrangian version of the transition amplitude,

$$Z = \int \mathcal{D}\vec{x} \exp\left(i \int_{t_i}^{t_f} dt L_{\text{eff}}\right)$$

where $L_{\text{eff}}$ can be computed in close form or after some approximation depending of the
type of potential.
Formulæ (1)-(4) can be easily applied to the planar system we are interested in discussing, namely that of charged particles in a disk rotating with angular velocity $\omega$, subject to a constant magnetic field $2B$. In this case $L_{\text{eff}}$ can be computed closely, taking the form

$$L_{\text{eff}} = \frac{1}{2m} \left( m v_i - (qB(1 + qB\theta/2) + m\omega)\varepsilon_{ij}x_j \right)^2 - \frac{1}{2m}(qBx_i)^2. \quad (5)$$

This is an exact result to all orders in $\theta$ which reduces to the classical Lagrangian in the $\theta \to 0$ limit which corresponds to ordinary space. Also, it reproduces to first order in $\theta$ the approximate result presented in [12].

Finally, using $L_{\text{eff}}$ we shall be able to analyse the interference pattern of charged particles when a two-slit device is put on a rotating disk in a gauge field background. We thus obtain a close expression to all orders in $\theta$ for the phase shift of the particle wavefunctions. Particularly interesting is the result that we obtain taking the accelerated interferometer as a rotating SQUID (superconducting quantum interference device). As an example, for a magnetic field confined to a thin center hole in the SQUID, the phase shift between two charged particles takes to first order in $\theta$ the form

$$\Delta \Phi^{0,1} = \left( -\frac{2\pi \Delta d}{\lambda} + \frac{\lambda \Delta d^3}{4\pi} m^2 \omega^2 + 2m\omega S_B \right) \left( 1 - m\omega \theta \right). \quad (6)$$

where $d$ is the distance from the source to the detector, $\Delta(d)$ the difference between the two paths, $S_B$ the area where the magnetic flux is different from zero and $S_\omega$ the corresponding one for an effective flux related to the rotation effect. We discuss the difficulties concerning the experimental settings in view of the present bounds on $\theta$.

The plan of the paper is the following. After discussing the classical system in Section 2, we present the path-integral treatment of the quantum problem in Section 3 and discuss the quantum interference device in section 4. Finally we present in Section 5 a summary of our results.

## 2 The classical system

Let us start by defining the Moyal $\ast$-product of functions on the noncommutative plane,

$$(f \ast g)(x) = \exp \left( \frac{i}{2} \theta_{ij} \partial_x \partial_y \right) f(x)g(y) \bigg|_{y=x} \quad (7)$$

Here $\theta_{ij} = \theta \delta_{ij}$ ($i,j = 1,2$), with $\theta$ a real parameter with dimensions of $(\text{length})^2$. The Moyal bracket is then defined as

$$\{f(x), g(x)\} = (f \ast g)(x) - (g \ast f)(x) \quad (8)$$

Now, for $f = x^1$ and $g = x^2$, eq.(8) takes the form

$$\{x^1, x^2\} = i\theta \quad (9)$$
which can be connected, using the Moyal-Weyl correspondence, with the operator algebra approach to noncommutative quantum mechanics where one starts from the commutation relation

\[ [\hat{x}^1, \hat{x}^2] = i\theta \]  

Let us consider a particle with mass \( m \) and charge \( q \) located in a disc rotating with constant angular velocity \( \omega \), in the presence of a gauge field background \( A_i \). It is an interesting system since, as we shall see, rotational effects are connected to magnetic ones, and the rotating disk introduces topological features equivalent to that resulting from a confined magnetic flux. In a region of a rotating frame that is not simply connected, the inertial forces can be cancelled without completely cancelling the inertial vector potential, and its presence can be detected in a quantum interference experiment as with the Aharonov-Bohm effect \[22\]. We shall construct here the Hamiltonian of such a system and then analyze the associated quantum problem in noncommutative space. Dynamics of such a classical system is governed by the Lagrangian

\[ L = -m\sqrt{g_{ij}\dot{x}^i\dot{x}^j} - qA_i\dot{x}^i - V \]  

where \( V \) is some additional potential. For nonrelativistic velocities in an inertial frame, we can write (we take for the moment \( c = 1 \))

\[ L = \frac{1}{2}mv^2 + q\vec{v} \cdot \vec{A} - V \]  

We want to discuss the case of a constant magnetic field. In the ordinary (commutative) case, this can be very simply achieved by considering a gauge field of the form

\[ A_i = \varepsilon_{ijk}B_jx_k \]  

and identifying \( 2B \) with the constant magnetic field (say in the \( z \) direction) computed from \( F_{12} \). Interestingly enough, one can see that already at the classical level coordinate noncommutativity can be established, this showing its link with the presence of a magnetic field. Indeed, consider the large magnetic field limit (equivalent to small \( m \)) in which the kinematical momentum \( \vec{p} = m\vec{v} \) vanishes so that \( \vec{p} = 0 \) should be imposed as a constraint. One has then to introduce Dirac brackets ending with the result (see \[20\], \[23\] for a detailed discussion)

\[ \{x^i, x^j\}_{\text{Dirac}} = \frac{1}{2qB}\varepsilon^{ij} \]  

We see in this very simple classical system how noncommutativity arises because of the presence of the constant magnetic field background, with \( \theta \) and \( B \) related according to \( \theta = 1/(2qB) \).

Let us now consider the same classical problem but in noncommutative plane. In this case, the appropriate field strength, which in fact changes covariantly under noncommutative \( U(1) \) gauge transformations, should be defined as

\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - iq\{A_\mu, A_\nu\} \]
so that the gauge potential (13) yields a field strength with the $F_{12}$ component in the form

$$F_{12} = 2B (1 + q\theta B/2) \quad (16)$$

As expected, $F_{12}$ coincides, to zeroth order in $\theta$, with $2B$, the value of the magnetic field associated with the gauge field (13) in the commutative plane. The term linear in $\theta$ modifies the commutative result giving, however, a field strength that is still constant.

It is worthwhile to mention here that, in general, local quantities in noncommutative electrodynamics are not gauge invariant and only integrated expressions can be given an invariant measurable meaning. Of course, a constant noncommutative field strength, is still gauge invariant since for constant $F_{12}$ one has $g^{-1} * F_{12} * g = F_{12}$. Furthermore, for general field strengths which are just gauge covariant, a possibility to work with gauge invariant objects is provided by Seiberg-Witten mapping [1]. Indeed, this mapping connects a noncommutative gauge theory with an ordinary one formulated in terms of ordinary (not star) products of gauge fields and an action having an explicit dependence on $\theta^{ij}$ which acts as a constant background field. Once the gauge theory is expressed in terms of ordinary gauge fields and of the background $\theta^{ij}$, it becomes a theory which is gauge invariant in the conventional sense with an action from which gauge invariant electric and magnetic fields can be defined. This is the strategy adopted in [20], [23] which also applies to define covariant coordinate transformations [24]. In general, the mapping can be determined by solving the Seiberg-Witten differential equation order by order in $\theta$. To order $\theta$, the transformation relating the noncommutative field strength $F_{ij}$ and the corresponding one, which in the commutative equivalent theory will be denoted as $F_{ij}^C$, is

$$F_{ij} = F_{ij}^C + \theta_{kl} \left( F_{ki}^C F_{lj}^C - A_{kl}^C \partial_l F_{ij}^C \right) + \ldots \quad (17)$$

Gauge invariant electric and magnetic fields can then be computed from $F^C$.

Since in the present quantum mechanical context one is in general interested in small $\theta$ effects, Eq. (17) is enough to establish a relationship between the noncommutative field strength and its commutative counterpart. Moreover, for an Abelian constant $F_{ij}$, the differential equation can be solved explicitly. The solution (with boundary condition $F_{ij}(\theta = 0) = F_{ij}^C$) written in order to have $F_{ij}^C$ in terms of $F_{ij}$ is [3]

$$F^C = F \frac{1}{1 - q\theta F} \quad (18)$$

where for notation simplicity we have suppressed indices. Formula (18) then gives an explicit way to connect magnetic and electric fields in the noncommutative and its equivalent commutative theories. As signaled above, gauge-covariant rules for the transformation of gauge fields under transformations of noncommutative coordinates can be defined using the Seiberg-Witten mapping.

We now come back to Lagrangian (12) and write it in the rotating frame. Using (13) we write

$$L = \frac{1}{2} m \ddot{v}^2 + q \vec{v} \cdot \vec{B} \times \vec{r} - V \quad (19)$$
(we ignore the additional potential \( V \) in what follows). In order to write the Lagrangian in the rotating frame (with constant angular velocity \( \omega \) which we take parallel to \( B \)) one has just to change \( \vec{v} \rightarrow \vec{v} + \vec{\omega} \times \vec{r} \) getting,

\[
L = \frac{1}{2} m \vec{v}^2 + m \vec{v} \cdot (\vec{\omega} \times \vec{r}) + \frac{1}{2} m (\vec{\omega} \times \vec{r})^2 + q \vec{r} \cdot (\vec{v} \times \vec{B}) + q \vec{B} \cdot (\vec{r} \times (\vec{\omega} \times \vec{r})).
\] (20)

Now, if we define a vector field \( \vec{V} \) such that

\[
\vec{V} = \vec{\omega} \times \vec{r},
\] (21)

so that the canonical momentum reads

\[
\vec{P} = \frac{\partial L}{\partial \vec{v}} = m \vec{v} + m \vec{V} + q \vec{A}.
\] (22)

With this, the Hamiltonian in the rotating frame takes the form

\[
H = \frac{1}{2m} (\vec{P} - m \vec{V} - q \vec{B} \times \vec{r})^2 - \frac{m}{2} \vec{V}^2 - q \vec{V} \cdot (\vec{B} \times \vec{r}).
\] (23)

### 3 The quantum system in noncommutative space: the path-integral approach

In order to discuss the quantum mechanical problem, let us start by noting that in the noncommutative plane, the Heisenberg algebra takes the form (we put \( \hbar = 1 \))

\[
[\hat{x}^1, \hat{x}^2] = i \theta,
\]
\[
[\hat{p}^1, \hat{p}^2] = 0,
\]
\[
[\hat{x}^i, \hat{p}^j] = i \delta^{ij}.
\] (24)

Now, because of (24), it is not possible to construct eigenstates \( |x^1, x^2\rangle \) common to \( x^1 \) and \( x^2 \) and hence the definition of a probability density for a given state \( |\psi\rangle \) becomes problematic. However, as noted in [7]-[12], one can find a new coordinate system

\[
x^i = \hat{x}^i + \tilde{p}^i, \quad p^i = \hat{p}^i,
\] (25)

with

\[
\tilde{p}^i = \frac{1}{2} \theta \varepsilon^{ij} \hat{p}_j,
\] (26)

which satisfies the canonical commutation relations

\[
[x^1, x^2] = 0,
\]
\[
[p^1, p^2] = 0,
\]
\[
[x^i, p^j] = i \delta^{ij}.
\] (27)
It is then possible in this new system to define the probability density associated with a state $|\psi\rangle$ as $|\langle y^1 y^2 |\psi\rangle|^2$. One may then use realization (27) to solve specific quantum mechanical problems.

An alternative approach to investigate noncommutative quantum mechanical systems is related to the way in which noncommutative quantum field theories have been investigated. In this approach, one starts from the Schrödinger equation with ordinary products replaced by Moyal $\ast$-products as defined in (7) while coordinates are treated as in ordinary space [7],[9]. We shall follow this last approach and consider, for definiteness, the simple case in which the system corresponds to a particle of mass $m$, in a potential $V(\vec{x})$. The Schrödinger equation for such a system should then be written as

$$i \frac{\partial \psi(\vec{x},t)}{\partial t} = \frac{1}{2m} \vec{p}^2 \psi(\vec{x},t) + V(\vec{x}) \ast \psi(\vec{x},t).$$

(28)

Now, one can eliminate the $\ast$ product in the potential term by using [7]-[9]

$$V(\vec{x}) \ast \psi(\vec{x},t) = V(\vec{x} - \vec{p})\psi(\vec{x},t),$$

(29)

an identity that can be proven just by Fourier tranforming the l.h.s. Once this is done, the wave equation reads

$$i \frac{\partial \psi(\vec{x},t)}{\partial t} = \frac{1}{2m} \vec{p}^2 \psi(\vec{x},t) + V(\vec{x} - \vec{p})\psi(\vec{x},t) \equiv \hat{H}_{eff} \psi(\vec{x},t),$$

(30)

which is a “normal” (ordinary space) Schrödinger equation for a system with a modified Hamiltonian $\hat{H}_{eff}$. One can make contact between this approach and that referred at the beginning of this section by noting that a redefinition of coordinates according to eq.(25) turns eq.(30) into the Schrödinger equation for the original Hamiltonian but with coordinates obeying the algebra (24).

We are now ready to investigate the path-integral approach to the quantum problem in noncommutative space. To this end, we shall proceed to the construction of the quantum transition amplitude using the Feynman integral over trajectories. As it is well known, this approach replaces the analysis of the wave equation for a system with quantum Hamiltonian $\hat{H}$ by the phase space path-integral $Z$ giving the transition amplitude between some given initial and final states

$$Z = \int \mathcal{D}\vec{p}\mathcal{D}\vec{x} \exp\left(\int_{t_i}^{t_f} dt \left(i(\vec{p}\cdot\dot{\vec{x}} - H(\vec{p},\vec{x}))\right)\right)$$

(31)

where $H(\vec{p},\vec{x}) = \langle \vec{p} \hat{H} |\vec{x}\rangle$.

Now, in view of eq.(30), one can apply the usual Feynman recipe to the system with Hamiltonian $\hat{H}_{eff}$, and write the transition amplitude in the form

$$Z = \int \mathcal{D}\vec{x}\mathcal{D}\vec{p} \exp\left(i(\vec{p}\cdot\dot{\vec{x}} - H_{eff}(\vec{p},\vec{x}))\right)$$

(32)
with \( H_{\text{eff}}(\vec{p}, \vec{x}) = \langle \vec{p} | \hat{H}_{\text{eff}} | \vec{x} \rangle \). It is just in the shifted potential term in \( H_{\text{eff}} \) where noncommutativity manifests. Depending on the form of the potential, which depends now on \( \vec{p} \) because of the shift (29), the integral over momenta could be done in close form, leading to a Lagrangian version of \( Z \).

In the case of Hamiltonian (23), the shift (29) amounts to

\[
A_i = -\varepsilon_{ij} B x_j \rightarrow -\varepsilon_{ij} B (x_j - \tilde{p}_j) \\
V_i = -\varepsilon_{ij} \omega x_j \rightarrow -\varepsilon_{ij} \omega (x_j - \tilde{p}_j)
\]  

(33)

As a result, Hamiltonian \( H_{\text{eff}} \) can be written in the form

\[
H_{\text{eff}} = \frac{1}{2m} \left( 1 + q B \theta / 2 \right)^2 \left( \vec{p} - \frac{q \vec{A}}{1 + q B \theta / 2} \right)^2 + \frac{1}{2} \omega p^2 \theta - \vec{p} \cdot \vec{\omega} \times \vec{r}.
\]  

(34)

In the present case, this expression can be used to define effective mass and charge resulting from deformation of space at the noncommutative scale [9],

\[
m_{\text{eff}} = \frac{m}{(1 + q B \theta / 2)^2} \\
q_{\text{eff}} = \frac{q}{1 + q B \theta / 2}.
\]  

(35)

Being \( H_{\text{eff}} \) quadratic in \( \vec{p} \), one can integrate out the momenta in (32), this yielding to the Lagrangian version of the path-integral \( Z \). The answer is

\[
Z = \int D\vec{x} \exp \left( i \int dt L_{\text{eff}} \right)
\]  

(36)

where the effective Lagrangian \( L_{\text{eff}} \) is given by

\[
L_{\text{eff}} = \frac{1}{2m} \left( m v_i + \frac{q B}{m} A_i \right) (v_i + V_i)^2 - \frac{1}{2m} (q B x_i)^2.
\]  

(37)

Note that this is the exact expression for the Lagrangian, to all orders in \( \theta \). As expected, it reduces to the classical one for \( \theta = 0 \). It is important to stress that applying the noncommutative transformation defined in eq.(33) to the classical Lagrangian (21) does not yield the effective Lagrangian eq.(37). This is due to the fact that the former is obtained after path-integrating the momenta (this implying that factors in the numerator of the shifted Hamiltonian appear as denominators in the Lagrangian) and not just by a simple shift in the \( \vec{x} \) variables.

Up to first order in \( \theta \), the effective Lagrangian can be written, in terms of vector fields, as

\[
L_{\theta,1} = \frac{1}{2} \left( m (v_i + V_i + \frac{2q}{m} A_i) (v_i + V_i) - \frac{1}{2} m \theta_{jk} \partial_j A_i (v_i + V_i + \frac{q}{m} A_i) (v_k + V_k) + \frac{1}{2} \theta_{jk} (v_i + V_i) (v_k + V_k + 2 \frac{q}{m} A_k) + \frac{q^2}{m^2} A_i A_k \right).
\]  

(38)
Written in this way it is instructive to show the structure of the approximate noncommutative Lagrangian for a generic case.

Using a three-dimensional notation, one can further rewrite the first order expression compactly, as

\[
L_{\theta} = -\frac{q m}{4 \hbar^2} \vec{\theta} \cdot \left( (\vec{v} + \vec{V}) \times \nabla A_i \right) \left( v_i + V_i + \frac{q}{m} A_i \right) \\
- \frac{m^2}{4 \hbar^2} \vec{\theta} \cdot \left( (\vec{v} + \vec{V} + 2 \frac{q}{m} \vec{A}) \times \nabla V_i \right) (v_i + V_i) \\
- \frac{q^2}{4 \hbar^2} \vec{\theta} \cdot \left( \vec{A} \times \nabla V_i \right) A_i. \tag{39}
\]

where we have defined \( \vec{\theta} = \epsilon_{ijk} \theta_{jk} \). One can easily see that eq.(39) coincides with the approximate (first order in \( \theta \)) result derived in [12] for the special case of \( V = 0 \). It should be stressed, however, that eq.(37) provides an exact form for the Lagrangian to be considered in the transition amplitude \( Z \) for the quantum noncommutative model.

4 The quantum interference device

We are now in conditions to discuss the quantum dynamics of charged particles in a rotating disk, in the presence of a gauge field background. In this way, by studying the interference pattern of the particles when a two slit device is put on a rotating disk, we shall be able to determine noncommutative effects in connection both with the gauge field and with the non-inertial frame. As we shall see, both effects interfere each other and provide a \( \theta \) shift which can be accurately calculated.

Let us start by observing that the phase shift \( \Delta \Phi \) between two electrons reaching a detector through different paths can be computed from the formula

\[
\Delta \Phi = \Delta \int_{t_i}^{t_f} dt \ L_{eff}
\]

where \( \Delta \) indicates subtraction between both integrals computed in the interval \((t_i, t_f)\) that the particle takes from the source to the detector. For the particular case of a constant \( F_{12} \), the full Lagrangian eq.(37) takes the simple form

\[
L_{eff} = \alpha v_i^2 + \beta x_i^2 + \gamma v_i \epsilon_{ij} x_j \tag{41}
\]

with

\[
\alpha = \frac{m}{2 f_\theta} \\
\beta = \frac{1}{2m} \left( \frac{g_\theta^2}{f_\theta} - q^2 B^2 \right) \\
\gamma = -\frac{g_\theta}{f_\theta}
\]
\[ f_\theta = 1 + \theta(m \omega + qB) + \theta^2 \frac{q^2 B^2}{4} \]
\[ g_\theta = m \omega + qB + \theta \frac{q^2 B^2}{2}. \] (42)

Now, the result of the integration in eq.(40) in terms of the de Broglie wavelength \( \lambda = 2\pi/p \) associated with the particle is

\[ \Delta \Phi = \frac{2\pi}{\lambda f_\theta} \Delta(d) + \frac{\lambda}{4\pi} \left( \frac{g_\theta^2}{f_\theta} - q^2 B^2 \right) \Delta(d^3) + \oint \vec{\gamma} \cdot d\vec{x} \] (43)

where \( d \) is the distance from the source to the detector (thus, \( \Delta(d) \) represents the difference between the two paths to the same point in the detector) and \( \gamma_i = \gamma \epsilon_{ij} x_j \). While the first two terms do not depend on the flux of the fields, the last one does. Indeed, the \( \gamma \) factor in the third term depends on the area where the flux of magnetic and \( \nu \) fields are non-zero. Suppose that one confines the \( B \) flux into a solenoid in the center of the rotating disk. Then the \( B \) part of the curl \( \vec{\gamma} \) flux would be multiplied by the area of the solenoid while the \( \omega \) part by the area defined by the path difference. In this way we would have an Aharonov-Bohm effect combined with a rotational effect. If the solenoid is very thin, and the magnetic field not too strong, then the relevant phase shift will depend on the angular velocity. Nevertheless, it must be noted that these effects are not only summed but also multiplied each other, (see for example eq.(39)).

In order to clearly distinguish the noncommutative contributions from those already present in the ordinary case, let us analyse the complete first order approximation, given by

\[ \Delta \Phi^{0,1} = -\frac{2\pi \Delta d}{\lambda} + \frac{\lambda \Delta d^3}{4\pi} \left( m^2 \omega^2 + 2qBm\omega \right) + 2(m\omega S_w + qBS_B) + \theta \left( \frac{2\pi \Delta d}{\lambda} (m\omega + qB) - \frac{\lambda \Delta d^3}{4\pi} (m\omega + qB) (m^2 \omega^2 + 2qBm\omega) 
-2m^2 \omega^2 S_w - 2qBm\omega(S_w + S_B) - q^2 B^2 S_B \right), \] (44)

where \( S_B \) and \( S_w \) are respectively the areas where the magnetic flux and \( \omega \) flux are nonzero (it must be noted that since \( S_w \) is defined by the particle’s contour, then it is always \( S_B < S_w \)). The last term in the first line represents the usual Aharonov-Bohm contribution, while the second one is the rotational analog. In the third line, we find the corresponding noncommutative shifts, and their interference becomes apparent. The other terms do not depend on the topology of the device but we can also see both the ordinary and their noncommutative counterparts. Our result shows that the device can be used to exhibit the noncommutative shift in the Aharonov-Bohm effect, and introduces a new physical effect due to the interference of the two potential fields \( A \) and \( \nu \).

At this order, the \( \nu = 0 \) phase shift is simply

\[ \Delta \Phi^{0,1}_{\nu=0} = -\frac{2\pi \Delta d}{\lambda} (1 - qB\theta) - qBS_B (2 - qB\theta). \] (45)
The accelerated interferometer could be also realized as a rotating SQUID (superconducting quantum interference device). In this case, irrespective of the external field distribution, the particle in the SQUID would not see any $B$ field as a result of the Meissner effect, and thus no magnetic force would be measured in the rotating frame. Nevertheless, there is still a magnetic flux through the center of the SQUID, together with that related to $\omega$, this amounting to the effect just described.

If on the other hand one has not a strong magnetic field or it is confined to a thin center hole in the SQUID, then the $\omega$ field still affects the particle resulting in a phase shift of the same nature, also depending on $\theta$ as follows

$$\Delta \Phi^{0,1} = \left( -\frac{2\pi \Delta(d)}{\lambda} + \frac{\lambda \Delta(d^3)}{4\pi} m^2 \omega^2 + 2m\omega S_w \right) (1 - m\omega\theta). \quad (46)$$

In order to have a measurable noncommutative effect, the first order contribution should be a measurable fraction of the $\theta=0$ result, say a 1%. In this case, from eq.(45), one should have a strong magnetic field satisfying

$$qB \simeq 10^{-2} \theta^{-1}. \quad (47)$$

Similarly, in eq.(46) one needs the angular velocity as fast as

$$m\omega \simeq 10^{-2} \theta^{-1}. \quad (48)$$

Using the current bound for the noncommutative parameter, $\theta \leq (10 TeV)^{-2}$ [11,19], one finds that both requirements are experimentally hard to realize. One could perhaps think about an experimental setting involving astronomic velocities and field-strengths, but it seems to us that it is still beyond the current possibilities.

5 Summary

We have presented a path-integral approach to noncommutative quantum mechanics in the plane and discussed how the physics of a rotating interferometer is affected by the fact that spatial coordinates do not commute. One advantage of this approach is that all noncommutative effects are encoded in an effective Lagrangian which can be used to compute transition amplitudes within the usual framework provided by Feynman integral over trajectories. The transition amplitude, originally written as a path-integral over phase space, eq.(32), can be reduced to a path-integral over coordinates, eq.(36), with an effective Lagrangian which includes the effects of noncommutativity of space coordinates. Using this result, we have investigated a rotating interferometer device to see how noncommutativity will eventually modify the physics of a quantum particle in a gauge field background together with the effects of locating the system in a noninertial frame. From the associated quantum effective Lagrangian, which for the present gauge we calculated to all orders in $\theta$, we were able to analytically compute the phase shift for electrons reaching the detector through different paths in an exact way. Finally, we discussed the possible
implicancies of noncommutativity on the phenomenology of a rotating SQUID showing that the current bounds on $\theta$ require an experimental setting which is far beyond the present possibilities.

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