An interval version of separation by semispaces in max–min convexity

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\begin{abstract}
In this paper, we study separation of a closed box from a max–min convex set by max–min semispaces. This can be regarded as an interval extension of the known separation results. We give a constructive proof of the separation in the case when the box satisfies a certain condition, and we show that the separation is never possible when the condition is not satisfied. We also study the separation of two max–min convex sets by a box and by a box and a semispace.
\end{abstract}

\section{Introduction}
Consider the closed interval $B = [0, 1]$ endowed with the operations $\vee = \max$, $\wedge = \min$. This is a well-known distributive lattice, and like any distributive lattice it can be considered as a semiring.
equipped with addition \( \lor \) and multiplication \( \land \). Importantly, both operations are idempotent, \( a \lor a = a \) and \( a \land a = a \), and closely related to the order:

\[
a \lor b = b \iff a \leq b \iff a \land b = a.
\]

For standard literature on lattices and semirings see e.g. [2, 11].

We consider \( B^n \), the cartesian product of \( n \) copies of \( B \), and equip this cartesian product with the operations of taking componentwise addition \( \lor \): \( (x \lor y)_i := x_i \lor y_i \) for \( x, y \in B^n \) and \( i = 1, \ldots, n \), and scalar \( \land \)-multiplication: \( (a \land x)_i := a \land x_i \) for \( a \in B, x \in B^n \) and \( i = 1, \ldots, n \). Thus \( B^n \) is considered as a semimodule over \( B \) [11]. Alternatively, one may think in terms of vector lattices [2].

A subset \( C \) of \( B^n \) is said to be max–min convex if the relations

\[
x, y \in C, \quad \alpha, \beta \in B, \quad \alpha \lor \beta = 1
\]

imply

\[
(\alpha \land x) \lor (\beta \land y) \in C.
\]

The interest in max–min convexity is motivated by the study of tropically convex sets, analogously defined over the semiring \( \mathbb{R}_{\text{max}} \), which is the completed set of real numbers \( \mathbb{R} \cup \{-\infty\} \) endowed with operations of idempotent addition \( a \oplus b := \max\{a, b\} \) and multiplication \( a \otimes b := a + b \). Constructed in [24, 25], tropical convexity and its lattice-theoretic generalizations received much attention and rapidly developed over the last decades; see [1, 5, 6, 8, 12, 16–18] among many others.

The matrix algebra developed over the max–min semiring, see [4, 10, 21] and references therein, is another related area. Max–min semimodules in \( B^n \), like max–min eigenspaces of matrices, are specific max–min convex sets. However, there is no immediate relation between max–min matrix algebra and the present article.

In this article, we continue the study of max–min convex structures started in [19, 20, 14, 15]. We are interested in separation of max–min convex sets by semispaces.

The set

\[
[x, y]_M = \{(\alpha \land x) \lor (\beta \land y) \in B^n | \alpha, \beta \in B, \alpha \lor \beta = 1\}
\]

\[
= \{(\max(\min(\alpha, x), \min(\beta, y)) \in B^n | \alpha, \beta \in B, \max(\alpha, \beta) = 1\}
\]

is fundamental for max–min convexity, it is called the max–min segment (or briefly, the segment) joining \( x \) and \( y \). As in the ordinary convexity in the real linear space, a set is max–min convex if and only if any two points are contained in it together with the max–min segment joining them. The max–min segments have been described in [19, 22].

Other relevant types of convex sets studied in the literature (see [20, 14, 15]) are max–min semispaces, hemispaces, halfspaces and hyperplanes. We recall below the definitions and the relationships between these notions.

For \( z \in B^n \), we call a subset \( S \) of \( B^n \) a max–min semispace (or, briefly, a semispace) at \( z \), if it is a maximal (with respect to set-inclusion) max–min convex set avoiding \( z \). Semispaces come from the abstract convexity, see e.g. [23]. One of their main application is in separation results: the family of semispaces is the smallest intersectional basis for the family of all convex sets. We recall (see [20]) that in \( B^n \) there exist at least one and most \( n + 1 \) semispaces at each point \( z \in B^n \), and exactly \( n + 1 \) at each finite point. Moreover, each convex set avoiding \( z \) is contained in at least one of those semispaces [20].

Another object from abstract convexity, which can also be straightforwardly introduced in max–min convexity, is the hemispace: this is any (max–min) convex set whose complement is also (max–min) convex. Hemispaces are used in separation results.

More general, one can introduce segments, and consequently define convex sets, in any semimodule \( X \) over a semiring with multiplicative unity. Zimmermann [25] showed that under some assumptions, the segments of \( X \) satisfy Pasch-Peano axiom:

\[
\forall x, y_1, y_2, z_1, z_2 \in X, \quad z_i \in [x, y_i], \quad i = 1, 2 \Rightarrow [y_2, z_1] \cap [y_1, z_2] \neq \emptyset.
\]
When the segments in a semimodule satisfy Pasch-Peano axiom, which is the case for max–min convexity [25], a classical theorem of Kakutani (see [3] for a proof) tells us that for any two non-intersecting convex sets \( C_1 \) and \( C_2 \) there exists a hemispace \( H \) containing \( C_1 \) such that the complement of \( H \) contains \( C_2 \). The proof of Kakutani theorem is non-constructive and uses Zorn’s Lemma. A constructive proof of this theorem in the special case of max–min convexity is, to the authors’ knowledge, an open problem.

It is shown in [20] that any max–min semispace is a max–min hemispace.

A max–min hyperplane is the set of points in \( B^n \) that satisfies a max–min linear equation:

\[
(a_1 \land x_1) \lor \cdots \lor (a_n \land x_n) \lor a_{n+1} = (b_1 \land x_1) \lor \cdots \lor (b_n \land x_n) \lor b_{n+1},
\]

with \( a_i, b_i \in B, i = 1, \ldots, n+1 \). Similarly, a max–min halfspace is the set of points in \( B^n \) that satisfy the definition above with equality replaced by \( \leq \). In contrast to the case of the usual linear space, here one needs an affine function on each side of the equality/inequality sign. Indeed, if we regard the operation \( a \lor b \) as an addition, it does not have an inverse and one cannot move terms from one side of the equality/inequality sign to the other.

The structure of max–min hyperplanes is described in [14]. In particular, there are examples of hyperplanes that are not halfspaces. It follows from (1) that any max–min halfspace is a max–min hyperplane. The relationship between max–min hyperplanes and max–min semispaces is described in [15]: a semispace in \( B^n \) is a hyperplane if and only if it is a semispace at a point belonging to the main diagonal of \( B^n \). It follows from [14] that, in general, hyperplanes are not hemispaces, and hence not semispaces either. See Fig. 2.3 in [14], showing an example of hyperplane that does not have a connected complement, thus cannot be a hemispace. However, it is easy to show that a max–min halfspace is a hemispace.

In the max–min case, the hyperplanes cannot be used to separate a point from a max–min convex set. An example to this was first published in [14], followed by a simpler one in [15]. This is in contrast with very optimistic results in the tropical convexity and its lattice-theoretic generalizations [5, 6, 8, 9, 24], which behave like the ordinary convexity in linear spaces in this respect.

In this paper, we study the following interval version of the semispace separation: given a box \( B \), i.e. a Cartesian product of closed intervals, and a max–min convex set \( C \), decide whether it is possible to construct a semispace which contains \( C \) and avoids \( B \). In Section 2 we give our main result, Theorem 1, which shows that such separation is indeed possible when \( B \) satisfies a certain condition. This condition holds true in particular when \( B \) does not contain points with coordinates equal to 1, or when \( B \) reduces to a point. When the condition is not satisfied, we show that the separation by semispaces is never possible. However, separation can be saved if we also allow hemispaces of a certain kind. As a corollary of Theorem 1, we also recover the description of semispaces due to Nitica and Singer [20]. In Section 3 we study the separation of two convex sets by a box and by a box and a semispace. We show that this separation is always possible in \( B^2 \), and we provide a counterexample in \( B^3 \).

Fig. 1 summarizes the types of separation considered in this paper. The convex sets that need to be separated are colored in black, and the separating boxes or semispaces are colored in gray. The sets \( C_1, C_2 \) and \( C \) are convex and \( B \) is a box.

In view of the recent development of tropical interval linear algebra in [13] and [7, Chapter 6], the present paper may be seen as related to yet undeveloped area of the interval tropical convexity.

![Fig. 1. Separation types, \( n = 2 \).](image-url)
2. Separation of boxes from max–min convex sets

For any point $x^0 = (x^0_1, \ldots, x^0_n) \in \mathbb{B}^n$ we define a family of subsets $S_0(x^0), \ldots, S_n(x^0)$ in $\mathbb{B}^n$. The sets are introduced in [20, Proposition 4.1]. Recall that $x^0$ is called finite if it has all coordinates different from zeros and ones. Without loss of generality we may assume that

$$x^0_1 \geq \cdots \geq x^0_n. \quad (3)$$

The set $\{x^0_1, \ldots, x^0_n\}$ admits a natural subdivision into ordered subsets such that the elements of each subset are either equal to each other or are in strictly decreasing order, say,

$$x^0_1 = \ldots = x^0_{k_1} > \ldots > x^0_{k_1+l_1+1} = \ldots = x^0_{k_1+l_1+k_2} > \ldots$$

$$> x^0_{k_1+l_1+k_2+l_2+1} = \ldots = x^0_{k_1+l_1+k_2+k_3} > \ldots$$

$$> x^0_{k_1+l_1+\cdots+k_{p-1}+l_{p-1}+1} = \ldots = x^0_{k_1+l_1+\cdots+k_{p-1}+l_{p-1}+k_p}$$

$$> \cdots > x^0_{k_1+l_1+\cdots+k_{p}+l_p} (= x^0_n). \quad (4)$$

Let us introduce the following notations:

$$L_0 = 0, \quad K_1 = k_1, \quad L_1 = K_1 + l_1 = k_1 + l_1, \quad (5)$$

$$K_j = L_{j-1} + k_j = k_1 + l_1 + \cdots + k_{j-1} + l_{j-1} + k_j \quad (j = 2, \ldots, p), \quad (6)$$

$$L_j = K_j + l_j = k_1 + l_1 + \cdots + k_j + l_j \quad (j = 2, \ldots, p); \quad (7)$$

we observe that $l_j = 0$ if and only if $K_j = L_j$.

We are ready to define the sets $S_i(x^0)$. We need to distinguish the cases when some of the coordinates of the point $x^0$ are zeros or ones, since some of the sets $S_i(x^0)$ become empty in that case.

**Definition 1.** (a) If $x^0$ is finite, then:

$$S_0(x^0) = \{x \in \mathbb{B}^n | x_i > x^0_i \text{ for some } i \text{ in } 1 \leq i \leq n\}, \quad (8)$$

$$S_{K_j+q}(x^0) = \{x \in \mathbb{B}^n | x_{K_j+q} < x^0_{K_j+q}, \text{ or } x_i > x^0_i \text{ for some } i \text{ in } K_j + q + 1 \leq i \leq n\}$$

$$(q = 1, \ldots, l_j; j = 1, \ldots, p) \text{ if } l_j \neq 0, \quad (9)$$

$$S_{l_{j-1}+q}(x^0) = \{x \in \mathbb{B}^n | x_{l_{j-1}+q} < x^0_{l_{j-1}+q}, \text{ or } x_i > x^0_i \text{ for some } i \text{ in } K_j + 1 \leq i \leq n\}$$

$$(q = 1, \ldots, k_j; j = 1, \ldots, p \text{ if } k_j \neq 0, \text{ or } j = 2, \ldots, p \text{ if } k_j = 0). \quad (10)$$

(b) If there exists an index $i \in \{1, \ldots, n\}$ such that $x^0_i = 1$, but no index $j$ such that $x^0_j = 0$, then the sets are $S_1(x^0), \ldots, S_n(x^0)$ of part (a).

(c) If there exists an index $j \in \{1, \ldots, n\}$ such that $x^0_j = 0$, but no index $i$ such that $x^0_i = 1$, then the sets are $S_0(x^0), S_1(x^0), \ldots, S_{\beta-1}(x^0)$ of part (a), where

$$\beta := \min\{1 \leq j \leq n | x^0_j = 0\}. \quad (11)$$

(d) If there exist an index $i \in \{1, \ldots, n\}$ such that $x^0_i = 1$, and an index $j$ such that $x^0_j = 0$, then the sets are $S_1(x^0), \ldots, S_{\beta-1}(x^0)$ of part (a), where $\beta$ is given by (11).

For future reference, we call the sets $S_i(x^0), 0 \leq i \leq n, x^0 \in \mathbb{B}^n$, admissible.
Proposition 1 [20]. For any \( x^0 \in B^n \) the sets \( S_i(x^0) \), \( 0 \leq i \leq n \), are max–min convex.

In the following \([a, c]\) denotes the ordinary interval on the real line \( \{b: a \leq b \leq c\} \), provided \( a \leq c \) (and possibly \( a = c \)).

We investigate the separation of a box \( B = [x_1, x_1] \times \cdots \times [x_n, x_n] \) from a max–min convex set \( C \subseteq B^n \), by which we mean that there exists a set \( S \) described in Definition 1, which contains \( C \) and avoids \( B \).

Assume that \( \bar{x}_1 \geq \cdots \geq \bar{x}_n \) and suppose that \( t(B) \) is the greatest integer such that \( \bar{x}_{t(B)} \geq x_i \) for all \( 1 \leq i \leq t(B) \). We will need the following condition:

\[
\text{If } (\bar{x}_1 = 1) \& (y_1 \geq x_i, 1 \leq l \leq n) \& (\bar{x}_l < y_l \text{ for some } l \leq t(B)), \text{ then } y \notin C. \tag{12}
\]

Note that if the box is reduced to a point and if \( \bar{x}_1 = 1 \), then \( t(B) = 1 \). Hence \( \bar{x}_l = 1 \) for all \( l \leq t(B) \) and \( \bar{x}_l < y_l \) is impossible. It follows that (12) always holds true in the case of a point.

Fig. 2 shows an illustration of condition (12). One has \( t(B) = 3, \bar{x}_1 = 1 \), and the point \( y = (y_1, y_2, y_3, y_4) \) satisfies \( y_1 \geq \bar{x}_1, 1 \leq l \leq 4 \) and \( \bar{x}_3 < y_3 \) for \( 3 \leq t(B) = 3 \), hence \( y \in C \) is not allowed.

The formulation of our main result will also use an oracle answering the question, whether or not a given max–min convex set \( C \subseteq B^n \) lies in an admissible set \( S \) (see Definition 1). As in the conventional convex geometry or tropical convex geometry, this question can be answered in \( O(mn) \) time if \( C \) is a convex hull of \( m \) points. Indeed it suffices to answer whether any of the inequalities defining \( S \) is satisfied for each of the \( m \) points generating \( C \).

Theorem 1. Let \( B = [x_1, x_1] \times \cdots \times [x_n, x_n] \), and let \( C \subseteq B^n \) be a max–min convex set avoiding \( B \). Suppose that \( B \) and \( C \) satisfy (12). Then there is a set \( S \) described by Definition 1, which contains \( C \) and avoids \( B \). This set is constructed in at most \( n + 1 \) calls to the oracle.

Proof. If \( x_i < 1 \) for all \( i \), then we try to separate \( B \) from \( C \) by \( S_0(x_1, \ldots, x_n) \) given by (8). Suppose we fail. Then there exists \( y \in C \) such that \( y_i \leq x_i \) for all \( i \).

Otherwise, \( 1 = \bar{x}_1 \geq \cdots \geq \bar{x}_n \). Let \( \bar{x}_1 = \max_{k \leq t(B)} x_k \), and define \( u \in B^n \) by

\[
u_i = \begin{cases} x_i, & \text{if } i \leq t(B), \\ \bar{x}_i, & \text{if } i > t(B). \end{cases}
\tag{13}
\]

It follows from the definition of \( t(B) \) that \( x_i = \max_{1 \leq i \leq n} u_i \).

We try to separate \( B \) from \( C \) by \( S_i(u) \), which is given by (9) or (10). If we fail then there exists \( y \in C \) such that \( y_i \leq \bar{x}_i \) for all \( i > t(B) \) (and trivially \( y_i \leq \bar{x}_i \) for \( \bar{x}_1 = 1 \)).

Thus we either separate \( C \) from \( B \), or there is a point \( y \in C \) such that \( y_i \leq \bar{x}_i \) for all \( i > t(B) \). In the latter case, condition (12) and \( B \cap C = \emptyset \) assure that there is at least one \( i \) such that \( y_i < \bar{x}_i \). Indeed,
The segments $i$ for some otherwise if $x_i \leq y_i$ for all $i$ and $y_i \leq \overline{x}_i$ for all $i \leq t(B)$ then $y \in B$; and if $x_i \leq y_i$ for all $i$ and $\overline{x}_i < y_i$ for some $i \leq t(B)$ then $y \notin C$ by condition (12).

Now assume without loss of generality that $x_1 \geq \cdots \geq x_n$ (the order of $x_i$ is now arbitrary).

The set $\{1, \ldots, n\}$ is naturally partitioned by the following procedure. See Fig. 3 for an illustration.

The segments $[x_i, \overline{x}_i]$ are drawn vertically and counted from left to right.

Let $s_1$ be the smallest number such that $x_{s_1} \leq \overline{x}_i$ for all $i = s_1, \ldots, n$.

If $s_1 \neq 1$ then there exists $t_1 \in \{s_1, \ldots, n\}$ such that $x_{s_1-1} > \overline{x}_{t_1}$. In this case let $T_1$ be the set of such $t_1$. Otherwise if $s_1 = 1$ we take $T_1 = \{1, \ldots, n\}$. In Fig. 3 one has $T_1 = \{12\}$.

We define

$$a_1 = \min \{x_i : i \in T_1\}. \quad (14)$$

We have

$$x_i \leq a_1 \leq \overline{x}_i \quad \forall i = s_1, \ldots, n. \quad (15)$$

Thus $a_1$ is a common level in all intervals $[x_{s_1}, \overline{x}_{s_1}], \ldots, [x_n, \overline{x}_n]$, but not $[x_{s_1-1}, \overline{x}_{s_1-1}]$.

If $s_1 = 1$ then we stop. Otherwise we proceed by induction. Let $s_k$ be the smallest number such that $x_{s_k} \leq \overline{x}_i$ for all $i \in \{s_k, \ldots, n\} \setminus T_1 \cup \cdots \cup T_{k-1}$ and let $t_k \in \{s_k, \ldots, n\}$ such that $x_{s_k-1} > \overline{x}_{t_k}$. In this case let $T_k$ be the set of such $t_k$. Otherwise if $s_k = 1$, then define $T_k := \{1, \ldots, n\} \setminus T_1 \cup \cdots \cup T_{k-1}$.

We take

$$a_k = \min \{x_i : i \in T_k\}. \quad (16)$$

We have

$$x_i \leq a_k \leq \overline{x}_i \quad \forall i \in \{s_k, \ldots, n\} \setminus T_1 \cup \cdots \cup T_{k-1}. \quad (17)$$

Thus $a_k$ is a common level in all intervals $[x_{s_k}, \overline{x}_{s_k}], \ldots, [x_n, \overline{x}_n]$ excluding the intervals with indices in $T_1 \cup \cdots \cup T_{k-1}$ which are below that level. The interval $[x_{s_k-1}, \overline{x}_{s_k-1}]$ is above $a_k$ (and possibly several other such levels going into $T_k$).

In Fig. 3, the sets $T_i$ are $T_1 = \{12\}$, $T_2 = \{10\}$, $T_3 = \{8\}$, $T_4 = \{7, 9, 11\}$, $T_5 = \{4\}$, $T_6 = \{1, 2, 3, 5, 6\}$.

Next we recall our point $y \in C$. It has $y_i < x_i$ for some $i$. Denote $K = \{i : y_i > \overline{x}_i\}$. Pick the greatest $i$ such that $y_i < x_i$ (note that for such $i$ we necessarily have $x_i > 0$), and let $s_k \leq i < s_{k-1}$, which

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**Fig. 3.** Construction of $a_i$ and $s_i$ in the proof of Theorem 1.
implies $x_j \leq y_j \leq \overline{x}_j$ for all $j \in \{s_{k-1}, \ldots, n\}\setminus K$. We try to separate $B$ from $C$ by the sets

$$S_i(u^i) = \{x \in B^+: x_i < x_j \text{ or } x_j > \overline{x}_j \text{ for some } j \in T_1 \cup \ldots \cup T_{k-1}\},$$

where $u^i$ can be defined by

$$u^i = \begin{cases} x_j, & l < i, \\ x_i, & l \geq i \text{ and } l \notin T_1 \cup \ldots \cup T_{k-1}, \\ \overline{x}_i, & l \in T_1 \cup \ldots \cup T_{k-1}, \end{cases}$$

for all $i$ with $y_i = x_i$ and $s_k \leq i < s_{k-1}$. Indeed, (18) is of the form (9) or (10), where $u^i$ is substituted for $x^0$. (In particular, it can be checked that $u^i_1 \geq \cdots \geq u^i_n$.)

Suppose the separation always fails. Then it gives us points $x^i \in C$ such that

$$x^i_j \geq x_i \quad \text{and} \quad x^i_j \leq \overline{x}_j \quad \forall j \in T_1 \cup \ldots \cup T_{k-1}.$$  

(20)

Then (17) implies that

$$x_i \leq a_k \wedge x^i_j \leq \overline{x}_i \quad \text{and} \quad a_k \wedge x^i_j \leq \overline{x}_j \quad \forall j = s_k, \ldots, n,$$

(21)

since $a_k \in \{x_i, x_j\}$ for $j \in \{s_k, \ldots, n\}\setminus T_1 \cup \ldots \cup T_{k-1}$ by (17), and we use (20) for $j \in T_1 \cup \ldots \cup T_{k-1}$. The point

$$z = \bigvee_i (a_k \wedge x^i) \lor y \in C$$

(22)

will be in some sense better than $y$. Indeed, (21) implies that $x^i_i \leq z_i \leq \overline{x}_i$ for all $i \in \{s_k, \ldots, n\}\setminus K$, versus $x_i \leq y_i \leq \overline{x}_i$ for all $i = \{s_{k-1}, \ldots, n\}\setminus K$. As $z \geq y$ we have $z_i > \overline{x}_i$ for all $i \in K$.

Proceeding with this improvement we obtain a point $z$ in $C$ which satisfies $x_i \leq z_i$ for all $i$ and $z_i \leq \overline{x}_i$ for all $i \in \{1, \ldots, n\}\setminus K$. This contradicts either $B \cap C = \emptyset$, or condition (12). This contradiction shows that we should succeed with separation at some stage. Clearly, the number of calls to the oracle does not exceed $n + 1$. \qed

We note that Theorem 1 also yields a method which verifies condition (12) in no more than $n + 1$ calls to the oracle.

The box $B$ can be a point and in this case condition (12) always holds true. Therefore, some known results on max–min semispaces [20] can be deduced from Theorem 1. The following statement is an immediate corollary of Theorem 1 and Proposition 1.

**Corollary 1** [20]. Let $x \in B^0$ and $C \subseteq B^0$ be a max–min convex set avoiding $x$. Then $C$ is contained in one of the admissible $S_i(x)$, $1 \leq i \leq n$, introduced in Definition 1. Consequently admissible sets are the family of semispaces at $x$.

**Proof.** The proof of Theorem 1 applied to $B = \{x\}$ shows that any max–min convex set avoiding $x$ is contained in one of the sets $S_i(x)$. Proposition 1 implies that these sets are max–min convex and do not contain $x$. Obviously, they are not included in each other. If $S_i(x)$ is not maximal, let $S$ be a max–min convex set strictly containing $S_i(x)$. Then Theorem 1 implies that there exists other $S_j(x)$, $i \neq j$, such that $S \subset S_j(x)$. But this implies $S_j(x) \subset S_i(x)$, a contradiction. Hence $S_i(x)$ are all maximal and $\{S_i(x)\}_i$ is the family of semispaces at $x$. \qed

Thus we recover a result of [20] that Definition 1 actually yields all semispaces at a given point. We now show that separation by semispaces is impossible when $B$ and $C$ do not satisfy (12).
We assume that

\[ x \in \mathbb{R}^n \quad \text{where} \quad 0 \leq a \leq b \leq 1 \text{.} \]

A simple example of interval non-separation is shown in Fig. 4. The box is \( B = [0, 1] \times [a, b] \) where \( 0 \leq a \leq b < 1 \) and the convex set is \( C = \{ z \} \) where \( z = (z_1, z_2) \) with \( z_2 > b \). Note that \( B \) and \( C \) do not satisfy condition \( (12) \).

**Remark 1.** A simple example of interval non-separation is shown in Fig. 4. The box is \( B = [0, 1] \times [a, b] \) where \( 0 \leq a \leq b < 1 \) and the convex set is \( C = \{ z \} \) where \( z = (z_1, z_2) \) with \( z_2 > b \). Note that \( B \) and \( C \) do not satisfy condition \( (12) \).

**Remark 2.** Theorem 1 can be easily modified to allow any case, if in addition to the admissible sets from Definition 1 we also allow the sets

\[ T_0^M(x^0) = \{ x : x_i > x_i^0 \text{ for some } i \in M \}, \]

where \( M \) is a proper subset of \( \{1, 2, \ldots, n\} \).

By Corollary 1, these sets cannot be semispaces. Nevertheless, they are *hemispaces*. Indeed, a point \( x \in T_0^M(x^0) \) is characterized by \( x_i > x_i^0 \) for some \( i \in M \). So if \( x, y \in T_0^M(x^0) \), then one has \( x_i > x_i^0 \) and \( y_i > y_i^0 \) for some \( i \in M \) with \( i \neq j \).
3. Separation of two max–min convex sets

Throughout this section, we assume that $\mathbb{B}^n = [0, 1]^n$ is endowed with the induced topology coming from the standard Euclidean topology of $\mathbb{R}^n$. All topological notions used in the sequel: boundary, closure etc. refer to this topology.

We will investigate the separation of two disjoint closed max–min convex sets by a box, property called in the introduction box separation, and by a box and a semispace, property called in the introduction box-semispace separation. Both properties are illustrated in Fig. 1.

We recall the structure of 2-dimensional max–min segments as presented in [19]. Pictures of all types of max–min segments are shown in Fig. 5, taken from [19].

**Theorem 3.** Let $C_1, C_2 \in \mathbb{B}^2, C_1 \cap C_2 = \emptyset$, be two closed max–min convex sets. Then there exist a permutation $i : \{1, 2\} \rightarrow \{1, 2\}$ and a box $B \subset \mathbb{B}^2$ such that $C_{i(1)} \subset B$ and $B \cap C_{i(2)} = \emptyset$.

**Proof.** Let

$$x_c := \max\{x|(x, y) \in C_1 \text{ for some } y\},$$

$$y_c := \max\{y|(x, y) \in C_1 \text{ for some } x\}. \tag{27}$$

As $C_1$ is compact, there exist $(x_c, y), (x, y_c) \in C_1$. Moreover, the convexity of $C_1$ implies that

$$c := (x_c, y_c) = (x_c, y) \lor (x, y_c) \in C_1. \tag{28}$$

Let

$$x_a := \min\{x|(x, y) \in C_1 \text{ for some } y\},$$

$$y_b := \min\{y|(x, y) \in C_1 \text{ for some } x\}. \tag{29}$$

Consider the points in $C_1$, guaranteed again by compactness:

$$a := (x_a, y_a),$$

$$b := (x_b, y_b). \tag{30}$$

The values $y_a$ and $x_b$ are chosen arbitrarily.

The smallest box in $\mathbb{B}^2$ containing the convex set $C_1$ is $B_0 := [x_a, x_c] \times [y_b, y_c]$. The point $c$ is the upper right corner of $B_0$.

We need the following Lemma, which can be proved by drawing all possible special cases and using the structure of max–min segments shown in Fig. 5. This proof is routine and will be omitted.

**Lemma 1.** The box $B_0$ can be partitioned as $B_0 = T_0 \cup T_1 \cup T_2 \cup T_3$, where

$$T_0 = \{(\alpha \land a) \lor (\beta \land b) \lor (\gamma \land c) : \alpha \lor \beta \lor \gamma = 1\},$$

$$T_1 = B_0 \cap \{(x, y) : x < x_b, y < y_a\},$$

$$T_2 = B_0 \cap \{(x, y) : y > y_a, x < x_c, y > x\},$$

$$T_3 = B_0 \cap \{(x, y) : x > x_b, y < y_c, y < x\}.$$ 

All regions $T_0, T_1, T_2, T_3$ are max–min convex (or possibly empty).

The regions $T_0, T_1, T_2, T_3$ are shown in Fig. 6.

Evidently $T_0 \subseteq C_1$ (note that $T_0$ is the max–min convex hull of $a, b, c$). In particular, the max–min segments $[a, b]_M, [a, c]_M, [b, c]_M$ are included in $C_1$ and any point from $C_2$ stays away from them. The other regions may contain points from both $C_1$ and $C_2$. 

$y_j > x^0_i$ for some $i, j \in M$. It follows now from (2) that any point $z \in [x, y]$ has either $z_1 > x^0_i$ or $z_j > x^0_j$. The complement of $T^M_0$ is max–min convex as the intersection of the max–min convex sets $\{x_i : x_i \leq x^0_i\}$, for $i \in M$.

The condition $C \subseteq T^M_0(x^0)$ can be verified by the same type of oracle as in Theorem 1.
We show that if the convex set $C_2$ intersects one of the regions $T_1$ and $T_2$, then there is a box $B_1 \subseteq B^2$ such that $C_2 \cap B_1 \cap C_1 = \emptyset$. Due to the symmetry about the main diagonal and the definitions of $T_2$ and $T_3$, there is no need to consider the case where $C_2$ intersects $T_3$.

**Case 1.** Assume $C_2$ intersects the region $T_1$.

The intersection $C_2'$ of $C_2$ with the region $T_1$ is max–min convex (as the intersection of two max–min convex sets). Thus there exists a point $(x_M, y_M) \in C_2'$, away from the boundary of $C_1$, and consequently away from the segment $[a, b]_M$, that has the maximum $x$-coordinate and maximum $y$-coordinate for $C_2'$. We show that $C_2$ is included in the box $B_1 = [0, x_M] \times [0, y_M]$.

Assume by contradiction that there exists $(x', y') \in C_2$ such that $x' > x_M$ or $y' > y_M$, then $(x'', y'') := (x_M, y_M) \lor (x', y')$ has either $x'' = x_M$ and $y'' = y' > y_M$, or $x'' = x' > x_M$ and $y'' = y_M$, or $(x'', y'') = (x', y')$. If $x'' < x_b$ and $y'' < y_a$ then $(x'', y'') \in C_2'$, which contradicts the maximality of $x_M$ and $y_M$. Otherwise, the segment $[(x_M, y_M), (x'', y'')]_M$ intersects $[a, b]_M$ and hence $C_1 \cap C_2 = \emptyset$, a contradiction.

We show that the box $B_1$ does not intersect with $C_1$. Assume that there exists $(x, y) \in B_1 \cap C_1$. Then $x \leq x_M$ and $y \leq y_M$. There exist $(x', y_M) \in [a, b]_M$ and $(x_M, y') \in [a, b]_M$ such that $x' > x_M$ and...
\[ y' > y_M. \] Using these points, we obtain that
\[ (x_M, y_M) = (x, y) \lor (x_M \land (x', y_M)), \quad \text{if } x_M \geq y_M, \]
\[ (x_M, y_M) = (x, y) \lor (y_M \land (x_M, y')), \quad \text{if } x_M \leq y_M. \] (31)

In both cases \((x_M, y_M) \in C_1\) and hence \(C_1 \cap C_2 \neq \emptyset\), a contradiction.

**Case 2.** Assume now that \(C_2\) intersects \(T_2\), and let \(C'_2 := C_2 \cap T_2\). Let \(x_M\) be the largest \(x\) coordinate of a point in \(C'_2\) and \(y_M\) the smallest \(y\) coordinate of a point in \(C'_2\). Let \((x_0, y_M), (x_M, y_0) \in C'_2\). From the definition of \(T_2\) we have \(x_0 \leq y_M\) and \(x_M \leq y_0\). Let \([t_1, t_2] := [x_a, x_c] \cap [y_a, y_c]\) (where all segments are ordinary on the real line).

If \(y_M \leq x_M\), then due to convexity
\[ (x_0, x_0) = (x_0, y_M) \lor (x_0 \land (x_M, y_0)) \in C'_2, \]
\[ (y_M, y_M) = (x_0, y_M) \lor (y_M \land (x_M, y_0)) \in C'_2, \] (32)

and hence the whole diagonal (and max–min) segment \([x_0, x_0), (y_M, y_M)\] is included in \(C'_2\). It can be observed that any point in the closure of \(T_2\) that belongs to the main diagonal lies in \([a, c]\), which is in \(C_1\). Thus \(C_1 \cap C_2 \neq \emptyset\), a contradiction, hence we must have \(y_M > x_M\).

When \(y_M > x_M\), due to convexity we have
\[ (x_M, y_M) = (x_0, y_M) \lor (y_M \land (x_M, y_0)) \in C'_2. \] (33)

In this case we claim that \(C_2\) is contained in the box \(B_1 := [0, x_M] \times [y_M, 1]\), which avoids \(C_1\).

Assume by contradiction that there exists \((x', y') \in C_2\) which does not lie in \(B_1\). This implies that \(x' > x_M\) or \(y' < y_M\). We also have \(y_M > x'\) and \(y' > x_M\), otherwise the segment \([x', y'), (x_M, y_M)]\) has points on the main diagonal, in which case it intersects \([a, c]\). Consider the convex combinations
\[ (x', y_M) = (y_M \land (x', y')) \lor (x_M, y_M), \quad \text{if } x' > x_M, \]
\[ (x_M, y') = (x', y') \lor (y' \land (x_M, y_M)), \quad \text{if } y' < y_M \text{ and } x' \leq x_M. \] (34)

Thus we obtain either \((x', y_M) \in C_2\) with \(x' > x_M\), or \((x_M, y') \in C_2\) with \(y' < y_M\) and \(x' \leq x_M\), leading to a contradiction with the maximality of \(x_M\) or the minimality of \(y_M\).

To prove that \(B_1\) avoids \(C_1\), assume by contradiction that there exists \((x, y) \in C_1\) where \(x \leq x_M\) and \(y \geq y_M\). We observe that there is a point \((x_M, y') \in [a, c]\), where \(y' \leq y_M\). Using this point we obtain
\[ (x_M, y_M) = (y_M \land (x, y)) \lor (x_M, y'), \] (35)

which implies \((x_M, y_M) \in C_1\), hence \(C_1 \cap C_2 \neq \emptyset\), a contradiction. \(\square\)

**Theorem 4.** Let \(C_1, C_2 \in \mathbb{B}^2\), \(C_1 \cap C_2 = \emptyset\), be two closed max–min convex sets that do not intersect the boundary of \(\mathbb{B}^2\). Then there exist a permutation \(i : \{1, 2\} \rightarrow \{1, 2\}\), a box \(B \subset \mathbb{B}^2\) and a semispace \(S \subset \mathbb{B}^2\) such that \(C_i(1) \subset B\), \(C_i(2) \subset S\) and \(B \cap S = \emptyset\).

**Proof.** The statement follows from Theorem 1 and Theorem 3. Indeed, Theorem 3 implies that either the minimal containing box of \(C_1\) does not intersect with \(C_2\), or the minimal containing box of \(C_2\) does not intersect with \(C_1\). The condition (12) is satisfied due to the fact that the convex sets do not intersect the boundary of \(\mathbb{B}^0\) and hence so are the minimal containing boxes. Applying Theorem 1 we obtain the statement. \(\square\)

**Remark 3.** We observe that Theorem 3 and Theorem 4 are not valid in dimension 3 or higher.

Let \(C_1\) be the max–min segment \([a, a, a)\), \((b, b, b)]\) and \(C_2\) be the max–min segment \([b, a, a)\), \((a, b, a)]\), where \(0 \leq a < b \leq 1\). It follows from [19] that \(C_1\) is part of the main diagonal, and that \(C_2\) is the concatenation of two pieces with parametrizations \([t, b, a)\, | \, a \leq t \leq b\) and \([(b, t, a)\, | \, a \leq t \leq b]\). It follows from Fig. 7 that the smallest box containing \(C_1\) is \([a, b]^2\) and the smallest box containing
$C_2$ is $[a, b]^2 \times \{a\}$. Since one box is completely included in the other, neither box separation nor box-semispace separation of $C_1$ and $C_2$ is possible.

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