SOME NECESSARY AND SOME SUFFICIENT CONDITIONS FOR THE COMPACTNESS OF THE EMBEDDING OF WEIGHTED SOBOLEV SPACES

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Abstract. We give some necessary conditions and sufficient conditions for the compactness of the embedding of Sobolev spaces

\[ W^{1,p}(\Omega, w) \rightarrow L^p(\Omega, w), \]

where \( w \) is some weight on a domain \( \Omega \subset \mathbb{R}^n \).

1. Introduction

In the present paper, we give some necessary and sufficient conditions for the compactness of the embedding of Sobolev spaces

\[ (1.1) \quad W^{1,p}(\Omega, w) \rightarrow L^p(\Omega, w). \]

Our investigations were originated by a recent paper appeared in the Annals of Statistics (L7), in which a new definition of Nonlinear Principal Components is introduced as follows: if \( X \) is an absolutely continuous random vector on an open connected set \( \Omega \subseteq \mathbb{R}^n \), with density function \( f_X \), zero expectation and finite variance, the \( j^{th} \) Nonlinear Principal Component of \( X \) is defined as a solution \( \varphi_j \) of the maximization problem

\[ (1.2) \quad \max_{\psi(X) \in W^{1,2}(\Omega, f_X)} \frac{\mathbf{E}(\psi(X)^2)}{\mathbf{E}(|\nabla \psi(X)|^2)} \]

subject to the conditions

\[ \mathbf{E}(\psi(X), \varphi_s(X)) = 0 \quad \text{for } s = 1, 2, \ldots, j - 1, j > 1. \]

Moreover, it is required that \( \varphi_j \) has zero expectation.

Here \( \mathbf{E}(\psi, \varphi) \) denotes the usual scalar product in the Hilbert space \( L^2(\Omega, f_X) \) of square integrable functions on \( \Omega \) with respect to the measure \( f_X \, dx \), and \( W^{1,2}(\Omega, f_X) \) is the weighted Sobolev space

\[ W^{1,2}(\Omega, f_X) = \{ u \in L^2(\Omega, f_X) \mid \forall i = 1, \ldots, n, \; \partial_i u \in L^2(\Omega, f_X) \}. \]

It is well-known that the compactness of the embedding

\[ (1.3) \quad W^{1,2}(\Omega, f_X) \rightarrow L^2(\Omega, f_X) \]

turns out to be essential in order to prove the existence of an orthonormal set of Nonlinear Principal Components.

I would like to thank Prof. E. Salinelli who suggested me the problem.
The problem of the compactness of the embedding (1.1) for weighted Sobolev spaces has been studied by many authors (see e.g. [9], [10], [11]). For a rich bibliography on this kind of problems we refer also to [5], [14], [16], [19].

Nevertheless, the attention has mainly focused on those classes of weights which arise in the study of partial differential equations, such as polynomial weights in unbounded domains, or, in bounded domains, weights depending on the distance from the boundary and, as concerns the Poincaré-Wirtinger inequality, weights in $A_p$ classes and derivatives of quasi-conformal mappings. Under this point of view an enormous amount of work has been done in the last years and a complete list of contributions is not possible here. Just to have an idea, see for example [6], [7], [8], [13], [18] and the references therein.

On the contrary, in the maximization problem (1.2) we have to deal with general density functions $f_X$.

Moreover, most of the criteria for the compactness of the embedding (1.3) which can be found in mathematical literature are not simple to be handled. In view of possible applications of Nonlinear Principal Components, it is important, instead, to have simple, easily applicable necessary and sufficient conditions on the density function $f_X$ and on the set $\Omega$ for the compactness of (1.3).

In the present paper, we give some necessary conditions and sufficient conditions for the compactness of the embedding (1.1). In section 2, we recall some basic facts about weighted Sobolev spaces. In section 3, we prove the weighted versions of some necessary conditions for compactness due to Adams ([1]). In section 4, we prove a sufficient condition for the compactness of (1.1), showing that, under suitable hypothesis on the weight $w$, if the subgraph

\[ \Omega_w := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} | x \in \Omega, 0 < y < w(x)\} \]

of the weight is such that the embedding $W^{1,p}(\Omega_w) \hookrightarrow L^p(\Omega_w)$ is compact, then the embedding (1.1) is compact. In section 5, applying a sufficient condition for compactness in the non-weighted case due to Adams ([1]), we show some examples to which our sufficient condition is applicable.

2. Preliminary facts

Let $\Omega$ be an open domain in $\mathbb{R}^n$. We will denote by $W(\Omega)$ the set of all real-valued, measurable, a.e. in $\Omega$ positive and finite functions $w(x)$. Elements in $W(\Omega)$ are called weight functions. For any Lebesgue-measurable set $U \subset \mathbb{R}^n$ and for $w \in W(\Omega)$, we will denote by $\mu_w(U)$ the Borel measure defined by

\[ \mu_w(U) = \int_U w(x) \, dx. \]

As usual, we will denote by $C^\infty_c(\Omega)$ the set of all the smooth, compactly supported functions in $\Omega$. Moreover, we will denote by $L^p(\Omega, w)$, for $1 \leq
\( p < \infty \), the set of measurable functions \( u = u(x) \) such that

\[
\| u \|_{L^p(\Omega, w)} = \left( \int_{\Omega} |u(x)|^p w(x) \, dx \right)^{1/p} < +\infty.
\]

It is a well-known fact that the space \( L^p(\Omega, w) \), endowed with the norm (2.1) is a Banach space. The following Proposition holds

**Proposition 2.1.** Let \( 1 \leq p < \infty \). If the weight \( w(x) \) is such that

\[
w(x)^{-\frac{1}{p-1}} \in L^1_{\text{loc}}(\Omega) \quad \text{(in the case } p > 1)
\]

\[
\text{ess sup}_{x \in B} \frac{1}{w(x)} < +\infty \quad \text{(in the case } p=1),
\]

for every ball \( B \subset \Omega \), then

\[
L^p(\Omega, w) \subseteq L^1_{\text{loc}}(\Omega).
\]

For a proof, see [15], [19].

As a consequence, under condition (2.2) (2.3), convergence in \( L^p(\Omega, w) \) implies convergence in \( L^1_{\text{loc}}(\Omega) \). Moreover, every function in \( L^p(\Omega, w) \) has distributional derivatives.

**Definition 2.2.** Let \( \Omega \) be an open set in \( \mathbb{R}^n \), let \( 1 \leq p < \infty \), and let \( w(x) \) be a weight function satisfying condition (2.2) (resp. (2.3)). We define the Sobolev space \( W^{m,p}(\Omega, w) \) as the set of those functions \( u \in L^p(\Omega, w) \) such that their distributional derivatives \( D^\alpha u \), for \( |\alpha| \leq m \), belong to \( L^p(\Omega, w) \).

It is a well-known fact that

**Theorem 2.3.** If \( w(x) \) fulfills condition (2.2) (resp. (2.3)), \( W^{m,p}(\Omega, w) \) is a Banach space.

For a proof, we refer to [15].

Throughout the paper, we will always assume that \( w \) satisfy condition (2.2) (resp. (2.3)).

### 3. Necessary conditions for compactness

In this section we derive some necessary conditions for the compactness of the embedding

\[
W^{1,p}(\Omega, w) \rightarrow L^p(\Omega, w),
\]

which are generalizations to the weighted case of analogous results obtained by Adams ([1], [3]) for non-weighted Sobolev spaces.

Let \( w(x) \in W(\Omega) \) be a weight on an open set \( \Omega \). Let \( T \) be a tessellation of \( \mathbb{R}^n \) with \( n \)-cubes of edge \( h \). If \( H \in T \), we will denote by \( N(H) \), as in [1],
the cube of side $3h$ concentric with $H$ and having faces parallel to those of $H$, and by $F(H)$ the fringe of $H$, defined by
$$F(H) := N(H) \setminus H.$$ 

The natural extension to the weighted case of the concept of $\lambda$-fatness employed by Adams is given by the following

**Definition 3.1.** Let $\lambda > 0$. A cube $H \in T$ is called $(\lambda, w)$-fat (with respect to $\Omega$) if
$$\mu_w(H \cap \Omega) > \lambda \mu_w(F(H) \cap \Omega).$$
If $H$ is not $(\lambda, w)$-fat, it is called $(\lambda, w)$-thin.

As in the non-weighted case, the following property holds:

**Theorem 3.2.** Let $1 \leq p < \infty$. If the embedding (3.1) is compact, then for every $\lambda > 0$ and for every tesselation $T$ of fixed edge $h$, $T$ has only finitely many $(\lambda, w)$-fat cubes.

**Proof.** The thesis follows from an easy extension of the proof of Theorem 6.33 in [1], where the Lebesgue measure $\mu$ is replaced by $\mu_w$ and $\lambda$-fat cubes are replaced by $(\lambda, w)$-fat cubes.

**Remark** Theorem 3.2 implies that if $w$ is a weight on $\mathbb{R}^n$ which has the doubling property, the embedding (3.1) cannot be compact. In particular, the embedding (3.1) for $A_p$ weights on $\mathbb{R}^n$ cannot be compact.

A consequence of Theorem 3.2 is the following result, which is the extension to the weighted case of Theorem 6.37 in [1]. The boundedness of the weight plays a crucial role and cannot be removed; a counterexample is given by the weight $w(x) = x^\alpha$ on $I = (0, 1) \subset \mathbb{R}$ for $\alpha \leq -1$, for which the embedding is compact (see [10]) even if $\int_I w(t) \, dt = +\infty$.

**Theorem 3.3.** If $w(x)$ is bounded and the embedding (3.1) is compact, then necessarily
$$\int_{\Omega} w(x) \, dx < +\infty.$$ 

**Proof.** The proof essentially follows the argument in [1]; we will give some details in order to show where the boundedness of the weight is essential. Let $T$ be a tesselation of $\mathbb{R}^n$ by cubes of unitary edge, and let $\lambda = (2(3^n - 1))^{-1}$. If $P$ is the union of the finitely many $(\lambda, w)$-fat cubes of $T$, then $\mu_w(P \cap \Omega) \leq \mu_w(P) < +\infty$.

If $H$ is a $(\lambda, w)$-thin cube of $T$, thanks to the choice of $\lambda$ we can choose $H_1 \in F(H)$ such that $\mu_w(H \cap \Omega) \leq \frac{1}{2} \mu_w(H_1 \cap \Omega)$. Analogously, if also $H_1$ is $(\lambda, w)$-thin, we can choose $H_2 \in F(H_1)$ such that $\mu_w(H_1 \cap \Omega) \leq \frac{1}{2} \mu_w(H_2 \cap \Omega)$.
If going on in this way we can construct an infinite chain \( \{H, H_1, H_2, \ldots\} \) of \((\lambda, w)\)-thin cubes, then for every \( j \in \mathbb{N} \)
\[
\mu_w(H \cap \Omega) \leq \frac{1}{2^j} \mu_w(H_j \cap \Omega),
\]
whence, thanks to the boundedness of \( w \),
\[
\mu_w(H \cap \Omega) \leq \frac{C}{2^j}
\]
for some positive constant \( C \) for every \( j \in \mathbb{N} \); thus, \( \mu_w(H \cap \Omega) = 0 \). As a consequence, if we denote by \( P_\infty \) the union of all the \((\lambda, w)\)-thin cubes in \( T \) for which it is possible to construct such an infinite chain, \( \mu_w(P_\infty \cap \Omega) = 0 \).

Let \( P_j \) denote the union of all \((\lambda, w)\)-thin cubes \( H \in T \) such that any chain of this type stops at the \( j \)-th step (that is, such that \( H_j \) is \((\lambda, w)\)-fat). Following the proof of [1], we get
\[
\mu_w(P_j \cap \Omega) \leq (2j + 1)^n 2^{-j} \mu_w(P \cap \Omega),
\]
whence
\[
\sum_{j=1}^{+\infty} \mu_w(P_j \cap \Omega) \leq \mu_w(P \cap \Omega) \sum_{j=1}^{+\infty} (2j + 1)^n 2^{-j} < +\infty.
\]
Since \( \mathbb{R}^n = P \cup P_\infty \cup P_1 \cup P_2 \cup \ldots \), the thesis follows.

Some stronger necessary conditions for compactness are given in the following Theorem, which is a generalization to the weighted case of Theorem 6.40 in [1]:

**Theorem 3.4.** Let \( w(x) \) be a continuous, bounded weight. For every \( r > 0 \), let \( \Omega_r, S_r \) be defined as
\[
\Omega_r := \{ x \in \Omega \mid |x| > r \},
\]
\[
S_r := \{ x \in \Omega \mid |x| = r \}.
\]
Moreover, let us denote by \( A_r \) the surface area, with respect to the weight \( w \), of \( S_r \). Then if the embedding (3.1) is compact,

1. for every \( \epsilon > 0, \delta > 0 \), there exists \( R > 0 \) such that if \( r \geq R \)
\[
\mu_w(\Omega_r) \leq \delta \mu_w(\{ x \in \Omega \mid r - \epsilon \leq |x| \leq r \});
\]
2. if \( A_r \) is positive and ultimately decreasing as \( r \to +\infty \) then for every \( \epsilon > 0 \)
\[
\lim_{r \to +\infty} \frac{A_{r+\epsilon}}{A_r} = 0.
\]

**Proof.** The thesis follows from an easy extension of the argument in [1], where the Lebesgue measure \( \mu \) is replaced by \( \mu_w \), \( \lambda \)-fat cubes are replaced by \((\lambda, w)\)-fat cubes and \( A_r \) is computed with respect to the weight \( w \). □
A consequence of Theorem 3.4 is the following Corollary, whose proof is a simple generalization to the weighted case of the proof of Corollary 6.41 in [1], and is therefore omitted.

**Corollary 3.5.** Let \( w(x) \) be a continuous, upper bounded weight. If the embedding (3.1) is compact, then for every \( k \in \mathbb{Z} \)
\[
\lim_{r \to +\infty} e^{kr} \mu_w(\Omega_r) = 0.
\]

4. A sufficient condition

Let \( w \in W(\Omega) \) be a lower semicontinuous weight defined on an open set \( \Omega \subseteq \mathbb{R}^n \). Let us suppose that \( w \) vanishes only on a closed subset \( \Omega_0 \subset \Omega \).

Moreover, let \( \Omega_\infty := \{ x \in \Omega \mid w(x) = +\infty \} \), and suppose that \( \Omega_\infty \) is closed. Both \( \Omega_0 \) and \( \Omega_\infty \) have (Lebesgue) measure equal to zero.

Moreover, we suppose that \( w \) is bounded from above and from below by positive constants on any compact set \( K \subseteq \Omega \setminus (\Omega_0 \cup \Omega_\infty) \).

Let us denote by \( \Omega_w \) the subgraph of the weight \( w(x) \), that is, the open set
\[
\Omega_w := \{ (x, y) \in \mathbb{R}^n \times \mathbb{R} \mid x \in \Omega, 0 < y < w(x) \}
\]
and consider the map
\[
J : W^{1,p}(\Omega, w) \rightarrow W^{1,p}(\Omega_w)
\]
defined by
\[
(Ju)(x, y) = u(x) \quad \text{a.e.}
\]
\( J \) is well-defined. It is not difficult to see that if \( u \in J(W^{1,p}(\Omega, w)) \), then the distributional derivative in the \( y \)-direction \( \nabla_y u \) is equal to zero. \( J \) is an isometry of \( W^{1,p}(\Omega, w) \) onto \( J(W^{1,p}(\Omega, w)) \) since for every \( u \in W^{1,p}(\Omega, w) \)
\[
||Ju||_{W^{1,p}(\Omega_w)}^p = \int_{\Omega_w} |Ju(x, y)|^p \, dx \, dy + \int_{\Omega_w} |\nabla_x (Ju)(x, y)|^p \, dx \, dy = \int_{\Omega} \left( \int_0^{w(x)} |u(x)|^p \, dy \right) \, dx + \int_{\Omega} \left( \int_0^{w(x)} |\nabla_x u(x)|^p \, dy \right) \, dx = \int_{\Omega} |u(x)|^p \, w(x) \, dx + \int_{\Omega} |\nabla_x u(x)|^p \, w(x) \, dx.
\]
We will denote by \( W^{1,p}_y(\Omega_w) \) the set \( J(W^{1,p}(\Omega, w)) \). Moreover, we will denote by \( L^p_y(\Omega_w) \) the completion of \( W^{1,p}_y(\Omega_w) \) with respect to the norm of \( L^p(\Omega_w) \).

**Lemma 4.1.** If \( w(x) \) satisfies the conditions above, then \( C^\infty_c(\Omega \setminus (\Omega_0 \cup \Omega_\infty)) \) is dense in \( L^p(\Omega, w) \), for \( 1 \leq p < +\infty \).
Proof. It suffices to show that, given $f \in L^p(\Omega, w)$, for every $\epsilon > 0$ there exists $g \in C^\infty_c(\Omega \setminus (\Omega_0 \cup \Omega_\infty))$ such that $\|f - g\|_{L^p(\Omega, w)} < \epsilon$. Let $\{\Omega_n\}$, $n \in \mathbb{N}$, be an exhaustion of $\Omega$, defined by

$$\Omega_n := \left\{ x \in \Omega \mid \min\{d(x, \Omega_0), d(x, \Omega_\infty), d(x, \partial \Omega)\} > \frac{1}{n} \right\}.$$ 

Let $f \in L^p(\Omega, w)$; for every $\epsilon > 0$, there exists $n$ such that

$$\left( \int_{\Omega \setminus \Omega_n} |f(x)|^p w(x) \, dx \right)^{\frac{1}{p}} < \frac{\epsilon}{2}.$$ 

Since $w$ is bounded from above and from below by positive constants on $\Omega_\infty$, $u \in L^p(\Omega_\infty, w)$ if and only if $u \in L^p(\Omega)$ and there exist $C_1, C_2 > 0$ such that for every $u \in L^p(\Omega)$

$$C_2 \|u\|_{L^p(\Omega_\infty)} \leq \|u\|_{L^p(\Omega)} \leq C_1 \|u\|_{L^p(\Omega_\infty)}.$$ 

Hence $f|_{\Omega_\infty} \in L^p(\Omega_\infty)$. As a consequence, there exists a function $g \in C^\infty_c(\Omega \setminus (\Omega_0 \cup \Omega_\infty))$ such that $\|g - f|_{\Omega_\infty}\|_{L^p(\Omega_\infty)} < (2C_1)^{-1} \epsilon$. Hence, $\|g - f\|_{L^p(\Omega_\infty)} < \frac{\epsilon}{2}$. This implies

$$\|g - f\|_{L^p(\Omega)} = \|g - f\|_{L^p(\Omega_\infty)} + \|g\|_{L^p(\Omega \setminus \Omega_n, w)} < \epsilon.$$ 

Hence, since $C^\infty_c(\Omega \setminus (\Omega_0 \cup \Omega_\infty)) \subset W^{1,p}(\Omega, w)$, $W^{1,p}(\Omega, w)$ is dense in $L^p(\Omega, w)$.

Since for every $u \in W^{1,p}(\Omega, w)$

$$\int_{\Omega} |u(x)|^p w(x) \, dx = \int_{\Omega_\infty} |Ju(x, y)|^p \, dxdy,$$

we get

**Lemma 4.2.** If $w(x)$ satisfies the above conditions, $J$ can be extended to an isometry

$$\overline{J} : L^p(\Omega, w) \longrightarrow L^p_y(\Omega_\infty).$$

As a consequence:

**Theorem 4.3.** Let $w(x)$ be a weight, satisfying the above conditions. If the subgraph $\Omega_w$ of $w(x)$ is such that the embedding

$$I_{\Omega_w} : W^{1,p}(\Omega_w) \longrightarrow L^p(\Omega_w)$$

is compact, then the embedding (3.1) is compact.

**Proof.** The embedding (3.1) is compact if and only if the embedding

$$I_y : W^{1,p}_y(\Omega_w) \longrightarrow L^p_y(\Omega_w)$$

is compact. But $I_y$ coincides with $P_y \circ I_{\Omega_w} \circ I$, where $I$ is the immersion

$$I : W^{1,p}(\Omega_w) \longrightarrow W^{1,p}(\Omega_w)$$
and \( P_y \) denotes the “projection”

\[
P_y : L^p(\Omega_w) \to L^p_y(\Omega_w),
\]

defined for a.e. \( x \in \Omega \) by

\[
(P_y u)(x) := \frac{1}{w(x)} \int_0^{w(x)} u(x, y) \, dy.
\]

Since \( P_y \) and \( I \) are continuous, the thesis follows. \( \square \)

We remark that the sufficient condition of Theorem 4.3 is not a necessary condition; as a matter of fact, whilst the embedding \( I_{\Omega_w} \) can be compact only if we assume a certain regularity for the weight \( w \), the embedding (3.1) can be compact even if the weight \( w \) is extremely irregular, once provided that it is controlled from above and below by a “regular” weight \( \Phi \). Indeed, the following Proposition holds

**Proposition 4.4.** Let \( \Phi \) be a weight for which the embedding

\[
W^{1,p}(\Omega, \Phi) \to L^p(\Omega, \Phi)
\]

is compact. Let \( w(x) \) be a weight such that there exist \( \alpha, \beta > 0 \) such that a.e. in \( \Omega \)

\[
\alpha \Phi(x) \leq w(x) \leq \beta \Phi(x).
\]

Then the embedding

\[
W^{1,p}(\Omega, w) \to L^p(\Omega, w)
\]

is compact.

**Proof.** It is immediate that \( u \in L^p(\Omega, \Phi) \) if and only if \( u \in L^p(\Omega, w) \) and

\[
\alpha \|u\|_{L^p(\Omega, \Phi)} \leq \|u\|_{L^p(\Omega, w)} \leq \beta \|u\|_{L^p(\Omega, \Phi)}.
\]

Analogously, \( u \in W^{1,p}(\Omega, \Phi) \) if and only if \( u \in W^{1,p}(\Omega, w) \) and

\[
\alpha \|u\|_{W^{1,p}(\Omega, \Phi)} \leq \|u\|_{W^{1,p}(\Omega, w)} \leq \beta \|u\|_{W^{1,p}(\Omega, \Phi)}.
\]

Hence, if \( \{u_n\} \) is a bounded sequence in \( W^{1,p}(\Omega, w) \), it is bounded also in \( W^{1,p}(\Omega, \Phi) \). Due to the compactness of the embedding (4.2), there exists a subsequence \( \{u_{n_k}\} \) such that \( u_{n_k} \) converges in \( L^p(\Omega, \Phi) \). Hence, \( u_{n_k} \) converges also in \( L^p(\Omega, w) \). \( \square \)

### 5. Some Applications

Let us now state some simple applications of Theorem 4.3. In the non-weighted case the following sufficient condition for compactness holds:

**Theorem 5.1.** (II) Let \( \Omega \) be an open set in \( \mathbb{R}^n \). If

1. there exists a sequence \( \{\Omega^*_N\}_{N=1}^{\infty} \) of open subsets of \( \Omega \) such that \( \Omega^*_N \subseteq \Omega^*_N+1 \) and for every \( N \) the embedding

\[
W^{1,p}(\Omega^*_N) \to L^p(\Omega^*_N)
\]

is compact;
(2) there exist a flow \( \Phi : U \to \Omega \) and a constant \( c > 0 \) such that if \( \Omega_N = \Omega \setminus \Omega_N^\ast \) then
(a) \( \Omega_N \times [0, c] \subset U \) for every \( N \);
(b) \( \Phi_t \) is one-to-one, for every \( t \);
(c) there exists \( M > 0 \) such that for every \( (x, t) \in U \)
\[ |\partial_t \Phi(x, t)| \leq M; \]

(3) the functions \( d_N(t) = \sup_{x \in \Omega_N} |\det J\Phi_t(x)|^{-1} \) satisfy
(a) \( \lim_{N \to \infty} d_N(c) = 0 \);
(b) \( \lim_{N \to \infty} \int_0^c d_N(t) \, dt = 0 \),
then the embedding
\[ W^{1,p}(\Omega) \to L^p(\Omega) \]
is compact.

We recall that a flow on \( \Omega \) is a continuously differentiable map \( \Phi : U \to \Omega \), where \( U \) is an open set in \( \Omega \times \mathbb{R} \) containing \( \Omega \times \{0\} \), with \( \Phi(x, 0) = x \) for every \( x \in \Omega \). Moreover, we denote by \( \Phi_t \) the map
\[ \Phi_t : x \mapsto \Phi(x, t), \]
and by \( J\Phi_t \) the Jacobian matrix of \( \Phi_t \).

Theorem 5.1 together with Theorem 4.3 can be used to get compactness results for weighted Sobolev spaces. Our first result is connected with Example 6.49 in [1]:

**Lemma 5.2.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), with \( C^\infty \) boundary, and let \( w(x) \in C^1(\Omega) \) be a weight on \( \Omega \), positive in every compact set \( K \subset \Omega \), such that, if we denote by \( r(x) \) the distance
\[ r(x) = \text{dist}(x, \partial \Omega), \]
near to the boundary \( w(x) \) can be expressed as
\[ w(x) = f(r), \]
where \( f \in C^1 \) is positive, nondecreasing, has bounded derivative \( f' \) and satisfies \( \lim_{r \to 0^+} f(r) = 0 \); then the embedding \( W^{1,p}(\Omega) \to L^p(\Omega) \) is compact.

**Proof.** Since the boundary is regular, there exist an open neighbourhood \( V \) of \( \partial \Omega \) in \( \Omega \), a constant \( a > 0 \) and a diffeomorphism
\[ \partial \Omega \times [0, a) \to V, \]
\[ (\xi, r) \to x(\xi, r) \]
such that \( x(\xi, r) \in \partial \Omega \) if and only if \( r = 0 \). We can suppose that \( r \) is equal to the distance of \( x(\xi, r) \) from the boundary.
Since \( w \) is strictly positive in \( \Omega \setminus V \), the embedding \((3.1)\) is compact if and only if \( W^{1,p}(V, w) \) is compactly embedded in \( L^p(V, w) \). Let us consider the subgraph \( V_w \) of \( w|_V \),

\[
V_w = \{ (\xi, r, y) \in \partial\Omega \times [0, a) \times \mathbb{R} \mid r > 0, 0 < y < f(r) \},
\]

and, for \( N \in \mathbb{N} \), the sets

\[
(V_w)_N := \left\{ (\xi, r, y) \in V_w \mid 0 < r \leq \frac{1}{N} \right\}.
\]

For \( N \in \mathbb{N} \), the open sets \((V_w)^*_N := V_w \setminus (V_w)_N\) are such that

\[
(V_w)^*_N \subseteq (V_w)^*_{N+1};
\]

moreover, they satisfy the cone property, hence the embedding

\[
W^{1,p}((V_w)^*_N) \longrightarrow L^p((V_w)^*_N)
\]

is compact for every \( N \in \mathbb{N} \). Let us consider the flow

\[
\Phi(\xi, r, y, t) := \left( \xi, r + t, \frac{f(r + t)}{f(r)} y \right)
\]

defined on the set

\[
U = \{ (\xi, r, y, t) \in \partial\Omega \times [0, a) \times \mathbb{R} \times \mathbb{R} \mid (\xi, r, y) \in V_w, -r < t < a - r \}.
\]

We simply have to check that the sets \((V_w)^*_N\) and the flow \( \Phi \) satisfy the conditions of Theorem 5.1. It is easy to see that \((V_w)_N \times [0, \frac{a}{2}] \subset U\) for every \( N \in \mathbb{N} \). \( \Phi_t \) is one-to-one for every \( t \), since

\[
\det(J\Phi_t) = \frac{f(r + t)}{f(r)} > 0.
\]

Moreover,

\[
|\partial_t \Phi(\xi, r, y, t)| = |(0, 1, \frac{f'(r + t)}{f(r)} y)| \leq M
\]

for some \( M > 0 \) since \( f' \) is bounded and \( |\frac{y}{f(r)}| < 1 \) on \( V_w \).

Further,

\[
d_N(t) := \sup_{(V_w)_N} |\det(J\Phi_t)|^{-1} = \sup_{(V_w)_N} \left| \frac{f(r)}{f(r + t)} \right|
\]

satisfies

\[
\lim_{N \to +\infty} d_N \left( \frac{a}{2} \right) = \lim_{r \to 0^+} \frac{f(r)}{f(r + \frac{a}{2})} = 0.
\]

Analogously, for every \( t > 0 \)

\[
\lim_{N \to +\infty} d_N(t) = 0,
\]

and by dominated convergence

\[
\lim_{N \to +\infty} \int_0^t d_N(t) \, dt = 0.
\]
Hence, by Theorem 5.1, the embedding $I_{V_w}$ is compact, and Theorem 4.3 implies that the embedding (3.1) is compact.

**Remark** In particular, for $f(r) = r^\alpha$, $\alpha \geq 1$, we find compact embedding as in [10]. This holds also for $\alpha = p - 1$.

In the case of a radial weight on $\mathbb{R}^n$, combining Theorems 5.1, 4.3 and 3.4 we can even get a necessary and sufficient condition. The following result is a sort of “radial” version of Example 6.48 in [1]:

**Lemma 5.3.** Let $\Omega = \mathbb{R}^n$, and $w(x)$ be a radial function $w(x) = g(r)$ where $r = |x|$ and $g \in C^1([0, +\infty))$ is positive, nonincreasing, with bounded derivative $g'$; then the embedding (3.1) is compact if and only if

$$\lim_{s \to +\infty} \frac{g(s + \epsilon)}{g(s)} = 0$$

for every $\epsilon > 0$.

**Proof.** Suppose, first, that (5.1) holds for every $\epsilon > 0$. Let us consider, on $\mathbb{R}^n$, polar coordinates $(r, \theta)$. Then the subgraph of $w$ can be described by

$$\Omega_w = \{ (r, \theta, y) \mid 0 < y < g(r) \}.$$ 

For every $N \in \mathbb{N}$, let us consider the set

$$(\Omega_w)_N := \{ (r, \theta, y) \in \Omega_w \mid r \geq N \}.$$ 

Then $(\Omega_w)_N := \Omega_w \setminus (\Omega_w)_N$ is bounded and has the cone property; hence, the embedding

$$W^{1,p}(\Omega_w)_N^* \to L^p((\Omega_w)_N^*)$$

is compact for every $N \in \mathbb{N}$. Moreover $(\Omega_w)_N^* \subset (\Omega_w)_{N+1}^*$ for every $N \in \mathbb{N}$.

An easy computation shows that the flow

$$\Phi(r, \theta, y, t) := \left( r - t, \theta, \frac{g(r - t)}{g(r)} y \right)$$

defined on the set

$$U := \{ (r, \theta, y, t) \mid 0 < t < r \},$$

satisfies the conditions of Theorem 5.1 (with $c = 1$). As a consequence, the embedding $I_{\Omega_w}$ is compact, and Theorem 3.4 yields the thesis.

Conversely, suppose that the embedding (3.1) is compact. Then by Theorem 3.4

$$A_r = \int_{|x|=r} g(r)r^{n-1} drd\theta = C(n)r^{n-1}g(r)$$

must fulfill the condition

$$\lim_{r \to +\infty} \frac{A_{r+\epsilon}}{A_r} = 0.$$

As a consequence,

$$\lim_{r \to +\infty} \frac{g(r + \epsilon)}{g(r)} = \lim_{r \to +\infty} \frac{C(n)(r + \epsilon)^{n-1}g(r + \epsilon)}{C(n)r^{n-1}g(r)} = 0.$$
**Remark** In particular, for \( g(r) = e^{\alpha r} \) and for \( g(r) \sim r^\alpha \) as \( r \to +\infty \) (\( \alpha < 0 \)) we get that there is no compactness, as stated in [11].

Theorem 5.1, together with Theorem 4.3, can be used also to deal with weights which are not bounded from above.

**Lemma 5.4.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), with \( C^\infty \) boundary, and let \( w(x) \in C^1(\Omega) \) be a weight on \( \Omega \), positive in every compact set \( K \subset \Omega \); moreover, we suppose that, if we denote by \( r(x) \) the distance

\[
r(x) = \text{dist}(x, \partial \Omega),
\]

near to the boundary \( w(x) \) can be expressed as

\[
w(x) = f(r),
\]

where \( f(r) \to +\infty \) as \( r \to 0^+ \), \( f \) is strictly decreasing on \( 0 < r < \delta \) for some \( \delta > 0 \), \( |f'(r)| \geq \frac{1}{C} \) and

\[
\lim_{y \to +\infty} \frac{f^{-1}(y + \epsilon)}{f^{-1}(y)} = 0
\]

for every \( \epsilon > 0 \). Then the embedding (3.1) is compact.

**Proof.** As in Lemma 5.2, it suffices to show that \( W^{1,p}(V,w) \) is compactly embedded in \( L^p(V,w) \), where \( V \) is a tubular neighbourhood of \( \partial \Omega \) in \( \Omega \). To this end, consider the subgraph \( V_w \) of \( w|_V \), and, for \( N \in \mathbb{N} \), the sets

\[
(V_w)_N := \{(\xi,r,y) \in V_w | n \leq y < f(r)\}.
\]

The open sets \( (V_w)_N^* := V_w \setminus (V_w)_N \) and the flow

\[
\Phi(\xi,r,y,t) := \left( \xi, \frac{f^{-1}(y - t)}{f^{-1}(y)} r, y - t \right),
\]

defined for \( 0 < t < y < f(r) \), fulfill the conditions of Theorem 5.1. Hence, Theorem 4.3 implies that the embedding (3.1) is compact. \( \Box \)

**Remark** The previous Lemma does not apply to the weight \( w(x) = r^\alpha \) when \( \alpha < 0 \). In this case, the compactness of the embedding (3.1) has been proved in [10].

Via a similar proof it can be shown that

**Lemma 5.5.** Let \( \Omega \) be an open domain in \( \mathbb{R}^n \) such that \( 0 \in \Omega \), and let \( w \in C^1(\Omega \setminus \{0\}) \) be a weight of the type \( w(x) = f(|x|) \), where \( f(s) \to +\infty \) as \( s \to 0^+ \), \( f \) is strictly decreasing on \( 0 < s < \delta \) for some \( \delta > 0 \), \( |f'(s)| \geq \frac{1}{C} \) and

\[
\lim_{y \to +\infty} \frac{f^{-1}(y + \epsilon)}{f^{-1}(y)} = 0
\]

for every \( \epsilon > 0 \). Then the embedding (3.1) is compact.
Finally, we show an example of weight not belonging to the \( A_p \) class on a bounded domain:

**Lemma 5.6.** Let us consider the set \( \Omega := (-\frac{1}{2}, \frac{1}{2}) \) and the weight

\[
 w(x) := \begin{cases} 
 1 & \text{if } x \leq 0 \\
 (\log \frac{1}{x})^{\frac{1}{2}} & \text{if } x > 0.
\end{cases}
\]

Then the embedding (3.1) is compact.

**Proof.** Consider the subgraph \( \Omega_{w}^- \). Since the set

\[
 \Omega_{w}^- := \{(x, y) \in \mathbb{R}^2 \mid x \in \Omega, x < 0, 0 < y < w(x)\}
\]

has the cone property, in order to prove the compactness of the embedding (3.1), it suffices to show that the embedding \( I_{\Omega_{w}^+} \) is compact, where

\[
 \Omega_{w}^+ := \{(x, y) \in \mathbb{R}^2 \mid x \in \Omega, x > 0, 0 < y < w(x)\}.
\]

To this purpose, it is not difficult to check that the subsets \( (\Omega_{w}^+)_{N} := \Omega_{w}^+ \setminus (\Omega_{w}^+)_{N} \),

\[
 (\Omega_{w}^+)_{N} := \{(x, y) \in \Omega_{w}^+ \mid N < y < w(x)\},
\]

and the flow \( \Phi \), defined by

\[
 \Phi(x, y, t) := \left( \frac{e^{-(y-t)^2}}{e^{-y^2}} x, y - t \right)
\]

for \( 0 < t < y \), satisfy the conditions of Theorem 5.1. Via Theorem 4.3, the thesis follows.

Hence in this case a Poincaré-Wirtinger inequality holds even if the weight does not belong to the \( A_p \) class.

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