Percolation phenomena for Brownian motion from a
generic viewpoint

J O’Donovan
Department of Mathematics, University College Cork, Cork, Ireland
E-mail: j.odonovan@ucc.ie

Abstract. In this short review paper we describe the connection between harmonic measure
and Brownian motion. We also examine recent work done in the area of Brownian motion
avoiding configurations of obstacles in the unit disk and in d-dimensional space.

1. Introduction
Although potential theory is a rich area of pure mathematics, it is no stranger to the worlds of
applied mathematics and physics. Potential theory, the study of harmonic functions, solutions
of Laplace’s equation, arises in areas such as electromagnetism and fluid dynamics. In this
paper we are going to look at a very specific aspect of potential theory, the concept of harmonic
measure, and discuss the connection between harmonic measure and Brownian motion.

Brownian motion in the presence of obstacles, traps or absorbing spheres describes many
physical phenomena, for example bacterial migration in porus media, [1], and has been studied
for a number of decades; see [2]. In more recent years, Brownian motion in the presence of
random obstacles is also popular, [3], where the randomness is not only associated with the
fluid/gas but also with the medium. Other work on traps include [4], where in the discrete
setting of \( \mathbb{Z}^d \), random walks with traps are discussed and an integral condition for a collection
of traps to be avoidable, which they call non-massive, is given.

In this paper we are interested in when it is possible for Brownian motion to avoid a collection
of obstacles and escape from a domain. We will describe recent results in this area, namely those
in [5] and [6], which give necessary and sufficient integral conditions for a collection of obstacles
to be avoidable. In Sections 3 and 4, the setting is the unit disk in the complex plane and
\( \mathbb{R}^d \), d-dimensional space, respectively. We determine under which conditions there is positive
probability that Brownian motion will avoid a collection of obstacles. In each case a necessary
and sufficient integral condition for a configuration of obstacles to be avoidable is given.

This is a short review paper on a talk given at the International Workshop on Multi-Rate
Processes and Hysteresis in 2008 entitled “A fly on Murphys”. ¹

¹ Murphys is a popular stout in Ireland. The author finds it helpful to think of the paths of Brownian motion as
the possible paths of a drunk fly.
2. Potential Theory and Brownian Motion

2.1. Harmonic Measure
Let $\Omega$ be a bounded open connected subset of $\mathbb{R}^n$. Let $x \in \Omega$, $E$ a Borel subset of the boundary of $\Omega$, $\partial \Omega$ and $f$ be the characteristic function of $E$ defined on $\partial \Omega$, that is $f$ takes the value 1 on $E$ and 0 on $\partial \Omega \setminus E$. Then, harmonic measure at $x$ is the measure on $\partial \Omega$ for which

$$\int_{\partial \Omega} f(\zeta) d\omega(x, \zeta; \Omega)$$

returns the value at $x$ of the harmonic extension of $f$ into $\Omega$. Then $\omega(x, E; \Omega)$ is the harmonic measure at $x$ of the set $E$. For fixed $E$, $\omega(x, E; \Omega)$ is a harmonic function with values in $[0, 1]$ and can be thought of as the generalized solution of the Dirichlet problem with boundary value function equal to the characteristic function of the set $E$. For a fixed $x$, it is a probability measure on the subsets of $\partial \Omega$. For more on harmonic measure see for example the standard text [7] or the review paper [8].

2.2. Brownian Motion and Harmonic Measure
Brownian motion is a stochastic process which depends only on its current value, has independent increments and changes of its value over any finite time interval are normally distributed. It is named after Brown, who first observed the random movement of a particle on a liquid due to the molecules of the liquid, but the first mathematical study of Brownian motion was by Bachelier in the context of modelling stock market fluctuations. It can be thought of as a limit of random walks, a motion in which there is no preferred direction.

Kakutani, [9] discovered a remarkable connection between Brownian motion and harmonic measure. He discovered that harmonic measure at $x$ in $\Omega$ is the exit distribution from $\Omega$ of Brownian motion with initial point $x$, that is the harmonic measure of $E$ in $\Omega$ with respect to $x$, $\omega(x, E; \Omega)$, is the probability that Brownian motion started at $x$ first exits $\Omega$ in $E$. See for example [10] for a proof of this result. We give a simple example to illustrate this result. Consider the annulus in Figure 1. We want to compute the probability that Brownian motion starting at an interior point, $z$ of the annulus, will hit the inner circle before hitting the outer circle. Thus, we need a function which is harmonic (i.e. its Laplacian is zero) in the interior of the annulus, equal to 1 on the inner circle and equal to 0 on the outer circle. The following function satisfies these requirements,

$$u(z) = \log(|z|/R) - \log(r/R),$$

where $r \leq |z| \leq R$. Thus, $u(z)$ evaluated at $z$ in the annulus, gives the required probability.

3. Brownian Motion and Obstacles

3.1. Brownian motion and obstacles in the disk
In this section, we describe Ortega-Cerdà and Seip’s Theorem 1 in [5]. The setting is the unit disk which contains obstacles which are small disks, $D(\lambda, r) = \{ z \in \mathbb{D} : \rho(z, \lambda) \leq r \}$ where $\rho(z, \lambda)$ is the pseudo-hyperbolic distance between $z$ and $\lambda$ given by

$$\rho(z, \lambda) = \frac{|z - \lambda|}{|1 - \lambda \overline{z}|}$$

and $0 < r < 1$. Let $\Lambda$ be a sequence of points in $\mathbb{D}$. Following Akeroyd in [11], we call the unit disk less the circular obstacles a champagne subdomain, $\Omega$,

$$\Omega = \mathbb{D} \setminus \bigcup_{\lambda \in \Lambda} D(\lambda, r)$$.
We want to know when there is positive probability that Brownian motion starting at the origin can escape to the outer boundary of the unit disk without first hitting any of the circular obstacles along the way. We require the circular obstacles to be what Ortega-Cerdà and Carroll call regularly located in [6]. We say a configuration of disjoint disks is regularly located if the centres are separated and uniformly dense and the radius of any disk depends only on the distance from the disk’s centre to the origin.

**Definition.** We call a collection of disks **avoidable** if there is positive probability that Brownian motion starting from the origin hits the boundary of $D$ before hitting any of the disks in $D$.

We now give a technical definition of a regularly located configuration of disks.

**Definition.** We say that a configuration of disks is **regularly located** if (i) $\inf_{\lambda \neq \lambda'} \rho(\lambda, \lambda') > 0$ for $\lambda, \lambda' \in \Lambda$, (ii) there exists $R < 1$ such that $D = \bigcup_{\lambda \in \Lambda} D(\lambda, R)$ and (iii) for every $\lambda \in \Lambda$ the radius of the ball centred at $\lambda$ is given by $\phi(|\lambda|)$, where $\phi : [0, 1) \to [0, 1]$ is a decreasing function.

Finding when there is positive probability that Brownian motion starting at the origin can escape to the outer boundary of the region without first hitting any of the circular obstacles along the way is equivalent to the problem of determining when the harmonic measure at 0 of the boundary with respect to the region less the circular obstacles, $\omega(0, S^1; \Omega)$, where $S^1$ denotes the boundary of the unit disk, is positive.

**Theorem A.** The unit circle, $S^1$, has positive harmonic measure in $\Omega$ if and only if

$$\int_0^1 \frac{dt}{(1 - t) \log(1/\phi(t))} < \infty.$$ 

In their proof, Ortega-Cerdà and Seip use the conformal invariance of harmonic measure and properties of harmonic functions, for example the maximum principle. They also use the result that you may safely ignore finitely many balls and that the integral in Theorem A is convergent if and only if the sum

$$\sum_{j=0}^{\infty} \frac{1}{\phi(1 - K^{-j})}$$

is finite, where $K > 1$ is a constant.

We note that this result can be translated into Euclidean terms but that the calculations are more complicated.
3.2. Brownian motion and obstacles in $\mathbb{R}^d$

In this section, our setting is $\mathbb{R}^d$, $d$-dimensional space with a Brownian motion starting at the origin. Given a configuration of spherical obstacles, we want to know if the Brownian motion can escape to infinity without hitting any of the spherical obstacles, i.e. is there positive probability that the configuration is avoidable? Brownian motion in $\mathbb{R}^2$ is recurrent, but Brownian motion in $\mathbb{R}^d$, $d \geq 3$ is transient. Thus, since in $\mathbb{R}^2$ any obstacle has probability 1 of being hit eventually, in this section we consider $\mathbb{R}^d$ for $d > 2$.

Let $\Lambda$ be a sequence of points in $\mathbb{R}^d$. We “fatten” the points by placing a ball of radius $\phi(|\lambda|)$, where $\phi$ is a decreasing function, at each $\lambda \in \Lambda$. We consider the collection of balls or spherical obstacles in $\mathbb{R}^d$, $B = \bigcup_{\lambda \in \Lambda} B(\lambda, \phi(|\lambda|))$.

We also suppose that the obstacles are regularly spaced, that is, $\inf_{\lambda \neq \lambda'} |\lambda - \lambda'| > 0$ for $\lambda, \lambda' \in \Lambda$ and there exists $R > 0$ such that $B(\lambda, R)$ contains at least one centre, $\lambda$. We ask the same question as in the last section, when are the obstacles avoidable? Under what condition on the radius function, $\phi$, is there positive probability that Brownian motion starting at the origin escapes to infinity before hitting any of the obstacles in $\mathbb{R}^d$. Carroll and Ortega-Cerdà answered this question in [6], proving the following theorem.

**Theorem B.** The collection of regularly spaced balls $B$ is avoidable in $\mathbb{R}^d$ if and only if

$$\int_0^\infty r\phi(r)^{d-2}dr < \infty.$$ 

In their proof, they use the result that the harmonic measure of a single ball in $\mathbb{R}^d$, $\omega(x, B(\lambda, r); \mathbb{R}^d \setminus B(\lambda, r))$ is $\left(\frac{r}{|x - \lambda|}\right)^{d-2}$. They also use properties of harmonic measure and harmonic functions.

**Acknowledgments**

The author would like to thank the organizers of the International Workshop on Multi-Rate Processes and Hysteresis, A. Pokrovskii and D. Rachinskii, for being allowed to speak at the conference, the participants for their questions and comments following the talk, particularly K. Dahmen for her useful references, and also T. Carroll for introducing the author to this topic.

**References**

[1] Duffy K J, Cummings P T and Ford R M 1995 Random Walk Calculations for Bacterial Migration in Porous Media Biophysical Journal 68 800–6

[2] Burdzy K, Holyst R and Sallisbury T 1992 2D Brownian motion in a system of traps: application of conformal transformations J. Phys. A 25 2463–71

[3] Lundh T 2001 Percolation Diffusion Stochastic Process. Appl. 95 235–44.

[4] den Hollander F, Menshikov M V and Volkov S E 1995 Two problems about random walks in a random field of traps Markov Process. Related Fields 1 185–202

[5] Ortega-Cerdà J and Seip K 2004 Harmonic measure and uniform densities Indiana Univ. Math. J. 53 905–23

[6] Carroll T and Ortega-Cerdà J 2007 Configurations of balls in Euclidean Space that Brownian motion cannot avoid Ann. Acad. Sci. Fenn. Math. 32 223–34

[7] Garnett J and Marshall D 2005 Harmonic Measure (Cambridge University Press)

[8] Carroll T 2004 Brownian motion and harmonic measure in conic sections Potential Theory in Matsue Adv. Stud. Pure Math. 44 25–41

[9] Kakutani S 1944 Two dimensional Brownian motion and harmonic functions Proc. Imp. Acad. Tokyo 20 706–14.

[10] Doob J 1984 Classical Potential Theory and its Probabilistic Counterpart, (Springer)

[11] Akeroyd J R 2002 Champagne subregions of the unit disk whose bubbles carry harmonic measure Math. Ann. 323 267–79

2 A drunk fly in $\mathbb{R}^2$ will always find its way home (eventually). A drunk fly in $\mathbb{R}^3$ may never find its way home.