ALGEBRAIC GROUPS OVER A 2-DIMENSIONAL LOCAL FIELD:
SOME FURTHER CONSTRUCTIONS

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Dedicated to A. Joseph on his 60th birthday

INTRODUCTION

0.1. Let $K$ be a local field, $G$ a split reductive group over $K$, and $G((t))$ the corresponding loop group, regarded as a group-indscheme. In [GK] we suggested a categorical framework in which one can study representations of the group $G((t))(K) = G(K((t)))$.

The main point is that $G := G((t))(K)$ admits no interesting representations on vector spaces, and we have to consider pro-vector spaces instead. In more detail, we regard $G$ as a group-like object in the category $\text{Set} := \text{Ind}(\text{Pro}(\text{Ind}(\text{Pro}(\text{Set}_0))))$, where $\text{Set}_0$ denotes the category of finite sets. We observe that $\text{Set}$ has a natural pseudo-action on the category $\text{Vect} = \text{Pro}(\text{Vect})$ of pro-vector spaces, and we define the category $\text{Rep}(G)$ to consist of pairs $(V, \rho)$, where $V \in \text{Vect}$, and $\rho$ is an action map $G \times V \to V$ in the sense of the above pseudo-action, satisfying the usual properties.

In [GK] several examples of objects of $\text{Rep}(G)$ were considered. One such example is the principal series representation $\Pi$, considered by M. Kapranov in [Ka]. Combining the results of loc. cit. and the formalism of adjoint functors developed in [GK] we showed that the endomorphism algebra of $\Pi$ could be identified with the Cherednik double affine Hecke algebra.

Another example is the "left regular" representation, corresponding to functions on $G$, with respect to the action of $G$ on itself by left translations, denoted $M(G)$. The main feature of $M(G)$ is that the right action develops an anomaly: instead of the action of $G$ we obtain an action of the Kac-Moody central extension $\widehat{G}_0$ of $G$ by means of the multiplicative group $G_m$, induced by the adjoint action of $G$ on its Lie algebra.

0.2. In the present paper we continue the study of the category $\text{Rep}(G)$. It is natural to subdivide the contents into three parts:

In the first part, which consists of Sections 1 and 2, we prove some general results about representability of various covariant functors on the category $\text{Rep}(G)$. These results are valid when $G$ is replaced by an arbitrary group-like object on $\text{Set}$. We also introduce the pro-vector space of distributions on an object of $\text{Set}$ with values in a pro-vector space; this notion is used in order to construct actions on invariants and coinvariants of representations of $G$.

The second part occupies Sections 3, 4, and Sect. 5. We study representations of a central extension $\widehat{G}$ of $G$ by means of $G_m$ with a fixed central character $c : G_m \to \mathbb{C}^*$; the corresponding category is denoted $\text{Rep}_c(\widehat{G})$, and $(\widehat{G}', c')$ denotes the opposite extension with its central character, cf. [GK], Sect. 5.9.

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Our goal here is to study the functor of semi-invariants
\[ \hat{\Box}_G : \text{Rep}_c(\hat{\mathbb{G}}) \times \text{Rep}_c'(\hat{\mathbb{G}}') \to \text{Vect}, \]
which couples the categories of representations at opposite levels. The motivation for the existence of such functor is provided by the semi-infinite cohomology functor on the category of representations of a Kac-Moody Lie algebra.

The construction of \( \hat{\Box}_G \) presented here follows the categorical interpretation of semi-infinite cohomology, developed by L. Positselsky (unpublished).

We use the functor of semi-invariants to prove the main result of the present paper, Theorem 3.3. This theorem describes for any quasi pro-unipotent subgroup \( H \) of \( G \) (cf. Sect. 2.6) the ring of endomorphisms of the functor \( \text{Coinv}_H : \text{Rep}(G) \to \text{Vect} \), as the algebra of endomorphisms of a certain object in the category of representations of \( G_0 \).

In particular, we obtain a functorial interpretation of the double affine (Cherednik) algebra in terms of the category \( \text{Rep}(G) \), as the algebra of endomorphisms of the functor of coinvariants with respect to the maximal quasi pro-unipotent subgroup of \( G \).

The third part consists of Sections 7 and 8, preceded by some preliminaries in Sect. 6. We construct some more examples of objects of \( \text{Rep}(G) \), this time using the moduli stack of bundles on an algebraic curve \( X \) over \( K \), when we think of the variable \( t \) as a local coordinate near some point \( x \in X \).

In particular, we show in Theorem 7.9 that in this way one naturally produces a pro-vector space, endowed with an action of \( G \times G \), such the space of bi-coinvariants with respect to the maximal quasi pro-unipotent subgroup \( I_0 \) of \( G \) is a bi-module over Cherednik’s algebra, isomorphic to the regular representation of this algebra.

0.3. Notation. We keep the notations introduced in [GK]. In particular, for a category \( \mathcal{C} \) we denote by \( \text{Ind}(\mathcal{C}) \) (resp., \( \text{Pro}(\mathcal{C}) \)) its ind- (resp., pro-) completion.

For a filtering set \( I \) and a collection \( A_i \) of objects of \( \mathcal{C} \) indexed by \( I \), we will denote by \( \lim_{\leftarrow} A_i \) the resulting object of \( \text{Ind}(\mathcal{C}) \) and by \( \lim_{\leftarrow} A_i := \text{limInd}(\lim_{\leftarrow} A_i) \in \mathcal{C} \) the inductive limit of the latter, if it exists. The notation for inverse families is similar.

As was mentioned above \( \text{Set}_0 \) denotes the category of finite sets. We use the short-hand notation \( \text{Set} = \text{Ind}(\text{Pro}(\text{Set}_0)) \) and \( \text{Set} = \text{Ind}(\text{Pro}(\text{Set})) \). We denote by \( \text{Vect}_0 \) the category of finite-dimensional vector spaces, \( \text{Vect} \simeq \text{Ind}(\text{Vect}_0) \) is the category of vector spaces, and \( \text{Vect} := \text{Pro}(\text{Vect}) \) is the category of pro-vector spaces.

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0.5. A correction to [GK]. As was pointed out by A. Shapira, Lemma 2.13 of [GK] is wrong. Namely, he explained to us a counter-example of a pro-vector space \( V \), acted on by a discrete set \( X \) (thought of as an object of \( \text{Set} \)), such that the action of every element of \( X \) on \( V \) is trivial, whereas the action of \( X \) on \( V \) in the sense of the pseudo-action of \( \text{Set} \subset \text{Set} \) on \( \text{Vect} \) is non-trivial. Namely, \( V = \lim_{\leftarrow} \text{Funct}_c(\mathbb{Z}^{\geq n}) \) and \( X = \mathbb{N} \), such that \( i \in \mathbb{N} \) acts on each
Funct_c(Z^{\geq n}) by
\[
\begin{align*}
  f(x_n, x_{n+1}, \ldots) &\mapsto f(x_n, x_{n+1}, \ldots, x_i + 1, \ldots) - f(x_n, x_{n+1}, \ldots, x_i, \ldots), \ i \geq n \\
  f(x_n, x_{n+1}, \ldots) &\mapsto 0, \ i < n.
\end{align*}
\]

However, we have the following assertion. Let $G$ be as in [GK], Sect. 1.12 let and $\Pi_1 = (\mathcal{V}_1, \rho_1)$, $\Pi_2 = (\mathcal{V}_2, \rho_2)$ be two objects of $\text{Rep}(G, \text{Vect})$. Assume that $\mathcal{V}_1$ is strict as a pro-vector space, i.e., that it can be represented as "$\lim\limits_{\longrightarrow} V_1'$", where the maps in the inverse system $V_1' \to V_1'$ are surjective. Let $\phi : \mathcal{V}_1 \to \mathcal{V}_2$ be a map in $\text{Vect}$, which intertwines the actions of the set $G(\mathbf{F}) = G_{\text{top}}$ on $\mathcal{V}_1$ and $\mathcal{V}_2$.

**Lemma 0.6.** Under the above circumstances, the map $\phi$ is a map in $\text{Rep}(G, \text{Vect})$.

**Proof.** We will prove a more general assertion, when we do not require $\mathcal{V}_1$ and $\mathcal{V}_2$ to be representations of $G$ on $\text{Vect}$, but just objects of endowed with an action of $G$, regarded as an object of $\text{Set}$. We claim that a map $\mathcal{V}_1 \to \mathcal{V}_2$ compatible with a point-wise action of $G_{\text{top}}$ is compatible with an action of $G$ as an object of $\text{Set}$, under the assumption that $\mathcal{V}_1$ is strict.

We represent $G$ as "$\lim\limits_{\longrightarrow} \mathbb{X}_k$, $\mathbb{X}_k \in \text{Pro(\text{Set})}$, and for each $k$, $\mathbb{X}_k \simeq \lim\limits_{\longrightarrow} \mathbf{X}_k^i$, such the maps $(\mathbf{X}_k^i)^{\text{top}} \to (\mathbf{X}_k^i)^{\text{top}}$ are surjective. The assertion of the lemma reduces immediately to the case when $\mathcal{V}_2 = \mathcal{W} \in \text{Vect}$, and $G$ is replaced by $\mathbb{X}_k$. In this case

\[
\text{Hom}(\mathbb{X}_k \otimes \mathcal{V}_2, \mathcal{W}) \simeq \lim\limits_{\longrightarrow} \text{Hom}(\mathbb{X}_k \otimes V_1^i, \mathcal{W}).
\]

However, by the assumption on the inverse system $\{V_1^i\}$, for every $i$ the map

\[
\text{Hom}((\mathbb{X}_k)^{\text{top}} \times V_1^i, \mathcal{W}) \to \text{Hom}((\mathbb{X}_k)^{\text{top}} \times V_1, \mathcal{W})
\]

is injective. This reduces us to the case when $\mathcal{V}_1 = \mathcal{V}$ is an object of $\text{Vect}$. The rest of the proof proceeds as in Lemma 2.13 of [GK].

\[\square\]

1. THE PRO-VESOR SPACE OF DISTRIBUTIONS

1.1. Let $\mathbb{X}$ be an object of $\text{Set}$ and $\mathcal{V} \in \text{Vect}$. Consider the covariant functor on $\text{Vect}$ that assigns to $\mathcal{W}$ the set of actions $\mathbb{X} \times V \to \mathcal{W}$. We claim that this functor is representable. We will denote the representing object by $\text{Dist}_c(\mathbb{X}, \mathcal{V}) \in \text{Vect}$; its explicit construction is given below. It is clear from the definition that covariant functor $V \to \text{Dist}_c(\mathbb{X}, \mathcal{V})$ is right exact.

We begin with some preliminaries of categorical nature:

**Lemma 1.2.** The category $\text{Vect}$ is closed under inductive limits.

**Proof.** Since $\text{Vect}$ is abelian, it is enough to show that it is closed under direct sums.

Let $\mathcal{V}^\kappa$ be a collection of pro-vector spaces, $\mathcal{V}^\kappa \simeq \lim\limits_{\longrightarrow} \mathcal{V}^\kappa_i$ with $i^\kappa$ running over a filtering set $I^\kappa$. Consider the set $\Pi_{\kappa} I^\kappa$, whose elements can be thought of as families $\{\varphi(\kappa) \in I^\kappa, \forall \kappa\}$. This set is naturally filtering, and

\[
\bigoplus_{\kappa} \mathcal{V}^\kappa \simeq \lim\limits_{\longrightarrow} \left( \bigoplus_{\kappa} \mathcal{V}_{\varphi(\kappa)} \right),
\]

where the inverse system is taken with respect to $\Pi_{\kappa} I^\kappa$.

\[\square\]
1.3. Let us now describe explicitly the pro-vector space \( \mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) \).

If \( \mathcal{X} \) is a finite set and \( \mathcal{V} \) is a finite-dimensional vector space, let \( \mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) \) be the set of \( \mathcal{V} \)-valued functions on \( \mathcal{X} \), thought of as distributions. If \( \mathcal{X}^0 \in \text{Pro}(\text{Set}_0) \) equals \( \lim \) \( \mathcal{X} \), with \( \mathcal{X}_i \in \text{Set}_0 \) and \( \mathcal{V} \) is as above, set

\[
\mathcal{Distr}_c(\mathcal{X}^0, \mathcal{V}) = \lim_{\leftarrow} \mathcal{Distr}_c(\mathcal{X}_i, \mathcal{V}) \in \text{Vect}.
\]

Set also \( \mathcal{Distr}(\mathcal{X}^0, \mathcal{V}) = \lim_{\leftarrow} \mathcal{Distr}_c(\mathcal{X}_i, \mathcal{V}) \in \text{Vect} \), i.e.

\[
\mathcal{Distr}(\mathcal{X}^0, \mathcal{V}) = \lim_{\leftarrow} \mathcal{Distr}_c(\mathcal{X}_i, \mathcal{V}) \in \text{Vect}.
\]

If \( \mathcal{X} \) is an object of \( \text{Set} \) equal to \( \lim \) \( \mathcal{X} \), \( \mathcal{X}^j \in \text{Pro}(\text{Set}_0) \) and \( \mathcal{V} \in \text{Vect} \) is \( \lim \) \( \mathcal{V}_m \) with \( \mathcal{V}_m \in \text{Vect}_0 \), set

\[
\mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) = \lim_{j,m} \mathcal{Distr}_c(\mathcal{X}^j, \mathcal{V}_m) \in \text{Vect},
\]

where the inductive limit is taken in \( \text{Vect} \). Set also

\[
\mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) = \lim_{j,m} \mathcal{Distr}_c(\mathcal{X}^j, \mathcal{V}_m) \in \text{Vect}.
\]

When \( \mathcal{V} \) is finite-dimensional, the latter is the vector space, which is the topological dual of the topological vector space \( \text{Funct}^c((\mathcal{X}, \mathcal{V}^\ast)) \) of locally constant functions on \( \mathcal{X} \) with values in \( \mathcal{V}^\ast \). Note that \( \mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) \) is not isomorphic to \( \lim \) \( \mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) \) even if \( \mathcal{V} \) is finite-dimensional.

For \( \mathcal{X}^0 \in \text{Pro}(\text{Set}) \) equal to \( \lim \) \( \mathcal{X} \) with \( \mathcal{X}_i \in \text{Set} \) and \( \mathcal{V} \) is a pro-vector space equal to \( \lim \) \( \mathcal{V} \), set

\[
\mathcal{Distr}_c(\mathcal{X}^0, \mathcal{V}) = \lim_{i,n} \mathcal{Distr}_c(\mathcal{X}_i, \mathcal{V}_n) \in \text{Vect}.
\]

Finally, for \( \mathcal{X} \in \text{Set} \) equal to \( \lim \) \( \mathcal{X} \) and \( \mathcal{V} \in \text{Vect} \), set

\[
\mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) = \lim_{k} \mathcal{Distr}_c(\mathcal{X}^k, \mathcal{V}).
\]

**Lemma-Construction 1.4.** For \( \mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) \in \text{Vect} \) constructed above, there exists a natural isomorphism

\[
\text{Hom}_{\text{Vect}}(\mathcal{Distr}_c(\mathcal{X}, \mathcal{V}), \mathcal{W}) \simeq \text{Hom}(\mathcal{X} \otimes \mathcal{V}, \mathcal{W}).
\]

**Proof.** By the definition of both sides, we can assume that \( \mathcal{X} \in \text{Pro}(\text{Set}) \) and \( \mathcal{W} = \mathcal{W} \in \text{Vect} \).

We have the following (evident) sublemma:

**Sublemma 1.5.** If \( \mathcal{U} = \lim \mathcal{U}_m \), where the projective limit is taken in the category \( \text{Vect} \), then for any \( \mathcal{X} \in \text{Pro}(\text{Set}) \) and \( \mathcal{W} \in \text{Vect} \),

\[
\text{Hom}(\mathcal{X} \otimes \mathcal{U}, \mathcal{W}) \simeq \lim \text{Hom}(\mathcal{X} \otimes \mathcal{U}_m, \mathcal{W}).
\]

The sublemma implies that we can assume that \( \mathcal{V} = \mathcal{V} \in \text{Vect} \). By applying again the construction of \( \mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) \), we reduce the assertion of the lemma further to the case when \( \mathcal{X} = \mathcal{X} \in \text{Set} \), i.e., we have to show that

\[
\text{Hom}_{\text{Vect}}(\mathcal{Distr}_c(\mathcal{X}, \mathcal{V}), \mathcal{W}) \simeq \text{Hom}(\mathcal{X} \otimes \mathcal{V}, \mathcal{W}).
\]

By the construction of \( \mathcal{Distr}_c(\mathcal{X}, \mathcal{V}) \) and the definition of the action, we can assume that \( \mathcal{X} \in \text{Pro}(\text{Set}_0) \) and \( \mathcal{V} \) is finite-dimensional. In this case the assertion is evident.

\( \square \)
1.6. Let now $X, Y$ be two objects of $\text{Set}$. The associativity constraint of the pseudo-action of $\text{Set}$ and $\text{Vect}$ gives rise to a map

$$\text{Distr}_c(X \times Y, V) \to \text{Distr}_c(X, \text{Distr}_c(Y, V)).$$

Let us now recall the following definition from [GK], Sect. 2.10:

An object $X \in \text{Set}$ is said to satisfy condition (***) if it can be represented as "$\lim$" $X_k$ with each $X_k \in \text{Pro}(\text{Set})$ being weakly strict. We remind (cf. [GK], Sect. 1.10) that an object $X' \in \text{Pro}(\text{Set})$ is said to be weakly strict if it can be represented as "$\lim$" $X'_i$, $X'_i \in \text{Set}$, such that the transition maps $X'_i \to X'_j$ are weakly surjective; in the case of interest when all $X'_i$’s are locally compact, the latter condition means that the map of topological spaces $X'_i^{\text{top}} \to X'_j^{\text{top}}$ has dense image.

As was shown in [GK], Sect. 2.12, if $G$ is an algebraic group over $K$, then the corresponding object $G \in \text{Set}$ satisfies condition (**).

**Proposition 1.7.** If $X \in \text{Set}$ satisfies condition (**), then the map in $\text{(1)}$ is surjective.  

This map is not in general an isomorphism. To construct a counter-example, it suffices to take $V = \mathbb{C}$ the 1-dimensional vector space, and $Y$ a discrete set $Y \in \text{Set} \simeq \text{Ind}(\text{Set}_0)$, regarded as an object of $\text{Set}$ by means of $\text{Set}_0 \to \text{Pro}(\text{Set})$.

**Proof.** We need to show that for a pro-vector space $W$, the map

$$\text{Hom}(X \otimes \text{Distr}_c(Y, V), W) \to \text{Hom}((X \otimes Y) \otimes V, W)$$

is injective. We will repeatedly use the facts that the functor $\lim \text{Ind}: \text{Ind}(\text{Vect}) \to \text{Vect}$ is exact and the functor $\lim \text{Proj}: \text{Pro}(\text{Vect}) \to \text{Vect}$ is left-exact.

By assumption, $X$ can be written as "$\lim$" $X_k$ with $X_k \in \text{Pro}(\text{Set})$ being weakly strict. Set also $W = "\lim" W_j$, $W_j \in \text{Vect}$. Both sides of $\text{(2)}$ are projective limits over $k$ and $j$ of the corresponding objects with $X$ replaced by $X_k$ and $W$ replaced by $W_j$. So, we can assume that $X$ is a weakly strict object of $\text{Pro}(\text{Set})$ and $W = W \in \text{Vect}$.

Let us write now $Y = "\lim" Y_{k'}$, $Y_{k'} \in \text{Pro}(\text{Set})$, in which case $\text{Distr}_c(Y, V) \simeq \lim \text{Distr}_c(Y_{k'}, V)$, and

$$\text{Hom}((X \otimes Y) \otimes V, W) \simeq \lim \text{Hom}((X \otimes Y_{k'}) \otimes V, W).$$

**Lemma 1.8.** If $U = \lim U_m$, the inductive limit taking place in $\text{Vect}$, then for an object $X \in \text{Set}$, satisfying condition (**), and $W \in \text{Vect}$, the natural map

$$\text{Hom}(X \otimes U, W) \to \lim \text{Hom}(X \otimes U_m, W)$$

is injective. If $X \in \text{Set}$, then this map is an isomorphism.

\footnote{We are grateful to Alon Shapira who discovered an error in the previous version of the paper, where the (**) assumption on $X$ was omitted.}
Proof. As above, we can assume that \( \mathcal{W} = \mathcal{W} \in \text{Vect} \), and \( \mathcal{X} \) is a weakly strict object of \( \text{Pro}(\text{Set}) \). Assume first that \( \mathcal{X} = \mathcal{X} \in \text{Set} \). In this case the assertion of the lemma follows from the description of inductive limits in \( \text{Vect} \) given in Lemma 1.2.

Thus, let \( \mathcal{X} \) be represented as \( "\lim\) \( \mathcal{X}_i \), \( \mathcal{X}_i \in \text{Set} \), with the transition maps \( \mathcal{X}_\nu \to \mathcal{X}_i \) being weakly surjective. Then

\[
\text{Hom}(\mathcal{X} \otimes \mathcal{U}, \mathcal{W}) \cong \lim_{I} \text{Hom}(\mathcal{X}_I \otimes \mathcal{U}, \mathcal{W}) \cong \lim_{m} \lim_{I} \text{Hom}(\mathcal{X}_I \otimes \mathcal{U}_m, \mathcal{W}),
\]

and

\[
\lim_{m} \text{Hom}(\mathcal{X} \otimes \mathcal{U}_m, \mathcal{W}) \cong \lim_{m} \lim_{I} \text{Hom}(\mathcal{X}_I \otimes \mathcal{U}_m, \mathcal{W}).
\]

However, by the assumption, the transition maps \( \text{Hom}(\mathcal{X}_I \otimes \mathcal{U}_m, \mathcal{W}) \to \text{Hom}(\mathcal{X}_\nu \otimes \mathcal{U}_m, \mathcal{W}) \) are injective. Therefore, the natural map

\[
\lim_{I} \lim_{m} \text{Hom}(\mathcal{X}_I \otimes \mathcal{U}_m, \mathcal{W}) \to \lim_{m} \lim_{I} \text{Hom}(\mathcal{X}_I \otimes \mathcal{U}_m, \mathcal{W})
\]

is injective.

Hence, we are reduced to the case when \( \mathcal{Y} \) is also an object of \( \text{Pro}(\text{Set}) \). Using Sublemma 1.5, we reduce the assertion further to the case when \( \mathcal{V} = \mathcal{V} \in \text{Vect} \) and \( \mathcal{Y} = \mathcal{Y} \in \text{Set} \).

If \( \mathcal{X} = "\lim\) \( \mathcal{X}_I \) then both sides of (2) are inductive limits over \( I \) of the corresponding objects with \( \mathcal{X} \) replaced by \( \mathcal{X}_I \). Thus, from now on we will assume that \( \mathcal{X} = \mathcal{X} \in \text{Set} \), and we have to show that the map

\[
\text{Hom}(\mathcal{X} \otimes \text{Distr}_c(\mathcal{Y}, \mathcal{V}), \mathcal{W}) \to \text{Hom}((\mathcal{X} \times \mathcal{Y}) \otimes \mathcal{V}, \mathcal{W})
\]

is injective, where on the left-hand side \( \text{Hom} \) is understood in the sense of the pseudo-action of \( \text{Set} \subseteq \text{Set} \) on \( \text{Vect} \).

By applying Lemma 1.8, we reduce the assertion to the case when \( \mathcal{Y} \in \text{Pro}(\text{Set}_0) \) and \( \mathcal{V} \) is finite-dimensional. It is clear that when \( \mathcal{Y} \) belongs to \( \text{Set}_0 \), the map in (3) is an isomorphism. Consider now the case when \( \mathcal{Y} = "\lim\) \( Y_i \) with \( Y_i \in \text{Set}_0 \) and \( \mathcal{X} = "\lim\) \( X_n \) with \( X_n \in \text{Pro}(\text{Set}_0) \). Then, by Sublemma 1.5,

\[
\text{Hom}(\mathcal{X} \otimes \text{Distr}_c(\mathcal{Y}, \mathcal{V}), \mathcal{W}) \cong \lim_{i} \text{Hom}(\mathcal{X} \otimes \text{Distr}_c(Y_i, \mathcal{V}), \mathcal{W}) \cong \lim_{i} \lim_{n} \text{Hom}(\mathcal{X}_n \otimes \text{Distr}_c(Y_i, \mathcal{V}), \mathcal{W}) \cong \lim_{n} \lim_{i} \text{Hom}((X_n \times Y_i) \otimes \mathcal{V}, \mathcal{W}).
\]

We also have and identification

\[
\text{Hom}((\mathcal{X} \times \mathcal{Y}) \otimes \mathcal{V}, \mathcal{W}) \cong \lim_{n} \text{Hom}((\mathcal{X}_n \times \mathcal{Y}) \otimes \mathcal{V}, \mathcal{W}) \cong \lim_{n} \lim_{i} \text{Hom}((X_n \times Y_i) \otimes \mathcal{V}, \mathcal{W}).
\]

Since \( Y_i \) are finite sets, we can assume that the transition maps \( Y_\nu \to Y_i \) are surjective. Therefore, the map

\[
\lim_{n} \lim_{i} \text{Hom}((X_n \times Y_i) \otimes \mathcal{V}, \mathcal{W}) \to \lim_{n} \lim_{i} \text{Hom}((X_n \times Y_i) \otimes \mathcal{V}, \mathcal{W})
\]

is injective.  

\[\square\]
1.9. As an application of Proposition 1.7, we will prove the following result.

Let $\rho : X \times V \to W$ be an action map. We can consider $\text{ker}(\rho)$ and $\text{coker}(\rho)$ as functors on $\text{Vect}$:

$$\text{ker}(\rho)(U) = \{ \phi : U \to V \mid \rho \circ \phi = 0 \} \text{ and } \text{coker}(\rho)(U) = \{ \psi : W \to U \mid \psi \circ \rho = 0 \}.$$

As in [GK], Proposition 2.8, one shows that $\text{coker}(\rho)$ is always representable, and $\text{ker}(\rho)$ is representable if condition (**) is satisfied.

**Corollary 1.10.** Let $Y \times V \to V$ and $Y \times W \to W$ be actions commuting in the natural sense with $\rho$. Then, if $Y$ satisfies (**), we have an action of $Y$ on $\text{coker}(\rho)$, and if $X$ satisfies condition (**), we have an action of $Y$ on $\text{ker}(\rho)$.

This corollary will be used when $V = W$, and both $X = G$ and $Y = H$ are group-like objects in $\text{Set}$, whose actions on $V$ commute. In this case we obtain that $G$ acts on both invariants and coinvariants of $H$ on $V$.

**Proof.** Let us first prove the assertion about the cokernel. Note that $\text{coker}(\rho)$ is isomorphic to the cokernel of the map $\text{Distr}_c(X, V) \to W$ obtained from $\rho$. We need to show that the composition

$$\text{Distr}_c(Y, W) \to W \to \text{coker}(\rho)$$

factors through $\text{Distr}_c(Y, \text{coker}(\rho))$. By the right-exactness of the functor $\text{Distr}_c(Y, \cdot)$,

$$\text{Distr}_c(Y, \text{coker}(\rho)) \simeq \text{coker}(\text{Distr}_c(Y, \text{Distr}_c(X, V)) \to \text{Distr}_c(Y, W)),$$

and it is enough to show that the composition

$$\text{Distr}_c(Y, \text{Distr}_c(X, V)) \to W \to \text{coker}(\rho)$$

vanishes.

However, using Proposition 1.7 we can replace $\text{Distr}_c(Y, \text{Distr}_c(X, V))$ by $\text{Distr}_c(Y \times X, V)$, and the required assertion follows from the commutative diagram:

$$\begin{array}{ccc}
\text{Distr}_c(X, V) & \xrightarrow{\rho} & W \\
\uparrow & & \uparrow \\
\text{Distr}_c(Y \times X, V) & \xrightarrow{\rho} & \text{Distr}_c(Y, W).
\end{array}$$

The proof for $\text{ker}(\rho)$ is similar. We have to show that the composition

$$\text{Distr}_c(X, \text{Distr}_c(Y, \text{ker}(\rho))) \to \text{Distr}_c(X, V) \to W$$

vanishes. Using Proposition 1.7 it is sufficient to show that the composition

$$\text{Distr}_c(X \times Y, \text{ker}(\rho)) \to \text{Distr}_c(X, V) \to W$$

vanishes, which follows from the assumption.

\[\square\]

## 2. Existence of Certain Left Adjoint Functors

2.1. In what follows $G$ will be group-like object in $\text{Set}$ satisfying assumption (**). Following [GK], we will denote by $\text{Rep}(G)$ the category of representations of $G$ on $\text{Vect}$.

**Proposition 2.2.** The forgetful functor $\text{Rep}(G) \to \text{Vect}$ admits a left adjoint.
Proof. We have to prove for any $\mathcal{W} \in \mathbf{Vect}$ the representability of the functor on $\mathbf{Rep}(G)$ given by $\Pi = (\mathcal{V}, \rho) \mapsto \text{Hom}_{\mathbf{Vect}}(\mathcal{W}, \mathcal{V})$. This functor obviously commutes with projective limits in $\mathbf{Rep}(G)$; so, by Proposition 1.2 of [GK] (with Ind replaced by Pro), it is enough to show that it is pro-representable.

Consider the category of pairs $(\Pi, \alpha)$, where $\Pi = (\mathcal{V}, \rho)$ is an object in $\mathbf{Rep}(G)$ and $\alpha : \mathcal{W} \to \mathcal{V}$ is a map in $\mathbf{Vect}$. For any such pair we obtain an action map $G \times \mathcal{W} \to \mathcal{V}$, and hence a map $\text{Distr}_c(G, \mathcal{W}) \to \mathcal{V}$. Since for an object of $\mathbf{Vect}$ the class of its quotient objects is clearly a set, the sub-class of those $(\Pi, \alpha)$, for which the above map $\text{Distr}_c(G, \mathcal{W}) \to \mathcal{V}$ is surjective, is also a set. This set is naturally filtered, and let us denote it by $A(\mathcal{W})$; it is endowed with a functor to $\mathbf{Rep}(G)$ given by $(\Pi, \alpha) \mapsto \Pi$.

We claim that $\lim_{(\Pi, \alpha) \in A(\mathcal{W})} \Pi$ is the object on $\text{Pro}(\mathbf{Rep})$, which pro-represents our functor.

Indeed, for $\Pi' = (\mathcal{V}', \rho') \in \mathbf{Rep}(G)$, the map

$$\text{Hom}_{\text{Pro}(\mathbf{Rep})}(\lim_{(\Pi, \alpha) \in A(\mathcal{W})} \Pi, \Pi') = \lim_{(\Pi, \alpha) \in A(\mathcal{W})} \text{Hom}_{\mathbf{Rep}}(\Pi, \Pi') \to \text{Hom}_{\mathbf{Vect}}(\mathcal{W}, \mathcal{V}')$$

is evident. Vice versa, given a map $\mathcal{W} \to \mathcal{V}'$ consider the induced map $\text{Distr}_c(G, \mathcal{W}) \to \mathcal{V}'$, and let $U$ be its image. We claim that the action map $G \times U \to \mathcal{V}'$ factors through $U$; this would mean that $\Pi := (U, \rho'|_U)$ is a sub-object of $\Pi'$, and we obtain a morphism from $\lim_{(\Pi, \alpha) \in A(\mathcal{W})} \Pi$ to $\Pi'$.

Consider the commutative diagram:

$$\text{Distr}_c(G \times G, \mathcal{W}) \xrightarrow{\text{mult}} \text{Distr}_c(G, \text{Distr}_c(G, \mathcal{W}))$$

$$\xrightarrow{\text{mult}} \text{Distr}_c(G, \mathcal{W}) \xrightarrow{\text{Distr}_c(G, \alpha)} \text{Distr}_c(G, U)$$

$$\xrightarrow{\alpha} U \xrightarrow{\text{mult}} \mathcal{V}'$$

We need to show that the image of the vertical map $\text{Distr}_c(G, U) \to \mathcal{V}'$ is contained in $U$. Since, by construction, the morphism $\text{Distr}_c(G, \mathcal{W}) \to U$ is surjective, and the functor $\text{Distr}_c(G, -)$ is right-exact, it suffices to show that the image of the composed vertical map is contained in $U$.

However, by Proposition 1.7 it is sufficient to check that the composed map

$$\text{Distr}_c(G \times G, \mathcal{W}) \to \mathcal{V}'$$

has its image contained in $U$, but this follows from the above diagram. \hfill \qed

2.3. Let us now derive some corollaries of Proposition 2.2. We will denote the left adjoint constructed above by $\mathcal{V} \mapsto \text{Free}(\mathcal{V}, G)$.

**Corollary 2.4.** Let $G_1 \to G_2$ be a homomorphism of group-objects of $\mathbf{Set}$. Then the natural forgetful functor $\text{Rep}(G_2) \to \text{Rep}(G_1)$ admits a left adjoint.

**Proof.** Let $\Pi_1$ be an object of $\text{Rep}(G_1)$. The functor on $\text{Rep}(G_2)$ given by $\Pi \mapsto \text{Hom}_{G_1}(\Pi_1, \Pi)$ commutes with projective limits. Therefore, by Lemma 1.2 of [GK] it suffices to show that it is pro-representable.

Let $V_1$ be the pro-vector space underlying $\Pi_1$. We have an injection $\text{Hom}_{G_1}(\Pi_1, \Pi) \to \text{Hom}_{\mathbf{Vect}}(V_1, \mathcal{V})$, where $\mathcal{V}$ is the pro-vector space underlying $\Pi$. \hfill \qed
By Proposition 2.2 we know that the functor $\Pi \mapsto \text{Hom}_{\text{Vect}}(V_1, V)$ is representable. Therefore, the assertion of the proposition follows from Proposition 1.4 of [GK].

We will denote the resulting functor $\text{Rep}(G_1) \to \text{Rep}(G_2)$ by $\Pi \mapsto \text{Coind}_{G_2}^{G_1}(\Pi)$ and call it the coinduction functor.

**Corollary 2.5.** The category $\text{Rep}(G)$ is closed under inductive limits.

**Remark.** Note that if $G = G$ is a group-object in $\text{Set}$, then the proof of Lemma 1.2 shows that the category $\text{Rep}(G, \text{Vect})$ is closed under inductive limits. Moreover, the forgetful functor $\text{Rep}(G, \text{Vect}) \to \text{Vect}$ commutes with inductive limits.

For an arbitrary $G \in \text{Set}$, the latter fact is not true, and we need to resort to Proposition 2.2 even to show the existence of inductive limits. We will always have a surjection from the inductive limit of underlying pro-vector spaces to the pro-vector space, underlying the inductive limit.

**Proof.** Let $\Pi_i = (V_i, \rho_i)$ be a filtering family of objects of $\text{Rep}(G)$. Consider the covariant functor $F$ on $\text{Rep}(G)$ given by $\Pi \mapsto \lim_{\leftarrow} \text{Hom}_{\text{Rep}(G)}(\Pi_i, \Pi)$.

Consider also the functor $F'$ that sends $\Pi = (V, \rho)$ to $\lim_{\leftarrow} \text{Hom}_{\text{Vect}}(V_i, V)$.

By Proposition 2.2 and Lemma 1.2, the functor $F'$ is representable. Hence, by Proposition 1.4 of [GK], we conclude that $F$ is pro-representable. Since $F$ obviously commutes with projective limits in $\text{Rep}(G)$, it is representable by Lemma 1.2 of [GK].

**2.6. Inflation.** Let us call a group-object $H$ of $\text{Set}$ quasi-unipotent if it can be presented as $\lim_{\rightarrow} H_i$, where $H_i$ are group-objects of $\text{Pro}(\text{Set}_0)$ and transition maps being homomorphisms, cf. [GK].

Let us call a group-object $\mathbb{H} \in \text{Pro}(\text{Set})$ quasi pro-unipotent if it can be presented as $\lim_{\leftarrow} H^i$, where $H^i$ are quasi-unipotent group-objects of $\text{Set}$, and the transition maps $H^i \to H^j$ being weakly surjective homomorphisms, cf. [GK], Sect. 1.10.

According to Lemma 2.7 of [GK], if $\mathbb{H}$ is quasi pro-unipotent, the functor of $\mathbb{H}$-coinvariants $\text{Coinv}_{\mathbb{H}} : \text{Rep}(\mathbb{H}, \text{Vect}) \to \text{Vect}$ is exact.

**Proposition 2.7.** If $\mathbb{H}$ is quasi pro-unipotent, the functor $\text{Coinv}_{\mathbb{H}}$ admits a left adjoint.

We will refer to the resulting adjoint functor as "inflation", and denote it by $V \mapsto \text{Inf}_{\mathbb{H}}(V)$.

**2.8. Proof of Proposition 2.7.** Let us first take $H$ to be a quasi-unipotent group-object of $\text{Set}$, isomorphic to $\lim_{\rightarrow} H_i$, where $H_i$ are group-objects in $\text{Pro}(\text{Set}_0)$.

Let us show that for a vector space $V$, the functor $\text{Rep}(H, \text{Vect}) \to \text{Vect}$ given by $\Pi \mapsto \text{Hom}(V, \Pi_H)$ is pro-representable.

For an index $i$, consider the object $\text{Coind}_{H}^{H_i}(V) \in \text{Rep}(H, \text{Vect})$, where $V$ is regarded as a trivial representation of $H_i$, and $\text{Coind}$ is as in Corollary 2.1. Using Proposition 2.4 of [GK], we obtain that $\text{Coind}_{H}^{H_i}(V)$ is a well-defined object of $\text{Pro}(\text{Rep}(H, \text{Vect}))$, which pro-represents the functor $\Pi \mapsto \Pi_{H_i}$. 

Note that if $H$ is locally compact, and $H_i \subset H$ is open, then $\text{Coind}^H_{H_i}(V)$ belongs in fact to $\text{Rep}(H, Vect)$, and is isomorphic to the space of compactly supported $V$-valued distributions on $H/H_i$, i.e., to the ordinary compact induction.

Since $H_i$ is compact, we have $\prod H_i \simeq \prod H_i$. Therefore, for $j > i$ we have natural maps

$$\text{Coind}^H_{H_j}(V) \to \text{Coind}^H_{H_i}(V).$$

Therefore, we can consider the object

$$\text{lim}^{-} \text{Coind}^H_{H_j}(V) \in \text{Pro}(\text{Rep}(H, Vect)),$$

where the projective limit is taken in the category $\text{Pro}(\text{Rep}(H, Vect))$.

For $\Pi \in \text{Rep}(H, Vect)$ we have:

$$\text{Hom}(\text{lim}^{-} \text{Coind}^H_{H_j}(V), \Pi) \simeq \text{lim}^{+} \text{Hom}(V, \Pi_H).$$

Since $\prod H \simeq \text{lim}^+ \Pi_H$, the RHS of the above expression is not in general isomorphic to $\text{Hom}(V, \Pi_H)$, except when $V$ is finite-dimensional. In the latter case we set $\text{Inf}^H(V) := \text{lim}^{-} \text{Coind}^H_{H_j}(V)$.

For general $V$, isomorphic to $\text{lim}^{-} V_k$ with $V_k \in Vect_0$, we set

$$\text{Inf}^H(V) = \text{lim}^{-} \text{Inf}^H(V_k),$$

where the inductive limit is taken in the category $\text{Rep}(H, Vect) \simeq \text{Pro}(\text{Rep}(H, Vect))$, cf. Lemma 1.2.

3. The functor of coinvariants

3.1. From now on we will assume that the group-like object $G$ is obtained from a split reductive group $G$ over $K$, as in [GK], Sect. 2.12. More generally, we will consider a central extension $\hat{G}$ of $G(t)$ as in Sect. 2.14 of [GK], and denote by $\text{Rep}_c(\hat{G})$ the category of representations of $\hat{G}$ at level $c$.

Let $\mathbb{H}$ be a quasi pro-unipotent group-object in $\text{Pro}(\text{Set})$. Let $\mathbb{H} \to G$ be a homomorphism, and we will assume that we are given a splitting of the induced extension $\hat{G}$ of $G(t)$ as in Sect. 2.14 of [GK], and denote by $\text{Rep}_c(\hat{G})$ the category of representations of $\hat{G}$ at level $c$. In particular, we have the forgetful functor $\text{Rep}_c(\hat{G}) \to \text{Rep}(\mathbb{H}, Vect)$.

Consider the functor

$$\text{Rep}_c(\hat{G}) \to Vect,$$

given by $\Pi \mapsto \text{Coinv}_{\mathbb{H}}(\Pi)$. Let $E(\mathbb{G}, \mathbb{H})_c$ denote the algebra of endomorphisms of this functor.

Remark. One can regard $E(\mathbb{G}, \mathbb{H})_c$ as an analogue of the Hecke algebra of a locally compact subgroup with respect to an open compact subgroup. Indeed, if $G$ is a locally compact group-like object in $\text{Set}$ and $H \subset G$ is open and compact, the corresponding Hecke algebra, which by definition is the algebra of $H$-bi-invariant compactly supported functions on $G$, can be interpreted both, as the algebra of endomorphisms of the representation $\text{Coind}^G_H(C)$, where $C$ is...
the trivial representation, and as the algebra of endomorphisms of the functor $\Pi \mapsto \text{Coinv}_H(\Pi) : \text{Rep}(G, Vect) \to Vect.$

3.2. Recall now the representation $M_c(G)$, introduced in Sect. 5.6 of [GK]. According to the main theorem of \textit{loc.cit.}, the structure of $\hat{G}$-representation on $M_c(G)$ extends naturally to a structure of $\hat{G} \times \hat{G}'$-representation, where $\hat{G}'$ is the group-object of Set corresponding to the central extension $\hat{G}'$ of $G((t))$, the latter being the Baer sum of $\hat{G}$ and the canonical extension $\hat{G}_0$, corresponding to the adjoint action of $G$ on its Lie algebra. The action of $\hat{G}'$ of $M_c(G)$ has central character $c'$, given by the formula in Sect. 5.9 of [GK].

In what follows we will call objects of $\text{Rep}_{c'}(\hat{G}')$ "representations at the opposite level" to that of $\text{Rep}_c(\hat{G})$. We will refer to the $\hat{G}'$-action on $M_c(G)$ as the "right action".

Using Corollary 1.10, by taking $\mathbb{H}$-coinvariants with respect to $\mathbb{H}$ mapping to $\hat{G}'$, we obtain an object of $\text{Rep}_{c'}(\hat{G})$ which we will denote by $M_c(G, \mathbb{H})$. By construction, we have a natural map

$$E(G, \mathbb{H})_{c'} \to \text{End}_{Vect}(M_c(G, \mathbb{H})).$$

However, since the $\hat{G}$ and $\hat{G}'$ actions on $M_c(G)$ commute, from Lemma 0.6 we obtain that endomorphisms of $M_c(G, \mathbb{H})$, resulting from the above map, commute with the $\hat{G}$-action.

Hence, we obtain a map

$$(4) \quad E(G, \mathbb{H})_{c'} \to \text{End}_{\text{Rep}_c(\hat{G})}(M_c(G, \mathbb{H})).$$

We will prove the following theorem:

\textbf{Theorem 3.3.} The map in (4) is an isomorphism.

3.4. Let us consider a few examples. Suppose first that the group $\mathbb{H}$ is trivial. As a corollary of Theorem 3.3 we obtain:

\textbf{Theorem 3.5.} The algebra $E(G, c)$ of endomorphisms of the forgetful functor $\text{Rep}_c(\hat{G}) \to Vect$ is isomorphic to the algebra of endomorphisms of the object $M_c(G) \in \text{Rep}_{c'}(\hat{G}')$.

Let now $\mathbb{H}$ be a \textit{thick} subgroup of $G[[t]]$ (see [GK], Sect. 2.12). Note that in this case, the object $M_c(G, \mathbb{H})$ is isomorphic to the induced representation $i_{\mathbb{H}}^G(\mathbb{C})$ of [GK], Sect. 3.3, where $\mathbb{C}$ is the trivial 1-dimensional representation of $\mathbb{H}$.

In particular, let us take $\mathbb{H}$ to be $T^\circ$, the subgroup of $T$ equal to the kernel of the natural map $I \to T \to \Lambda$, where $I \subset G[[t]]$ is the Iwahori subgroup and $\Lambda$ is the lattice of cocharacters of $T$, regarded as a quotient of $T$ by its maximal compact subgroup.

The corresponding induced representation $i_{\mathbb{H}}^G(\mathbb{C})$ is isomorphic to Kapranov’s representation, denoted in Sect. 4 of [GK] by $V_c$. Assume now that $G$ is semi-simple and simply-connected. In this case it follows from Corollary 4.4 of [GK] that the algebra $\text{End}(V_c)$ is isomorphic to the Cherednik algebra $\mathcal{H}_{q,c'}$. From Theorem 3.3 we obtain:

\textbf{Corollary 3.6.} The Cherednik algebra $\mathcal{H}_{q,c'}$ is isomorphic to the algebra of endomorphisms of the functor $\Pi \mapsto \text{Coinv}_{\mathbb{H}}(\Pi) : \text{Rep}_c(\hat{G}) \to Vect$.

3.7. Note that by combining Proposition 2.7 and Corollary 2.4 we obtain that the above functor $\text{Coinv}_{\mathbb{H}} : \text{Rep}_c(\hat{G}) \to Vect$ admits a left adjoint:

$$V \mapsto \text{Coinv}_{\mathbb{H}}^G(\text{Inf}_{\mathbb{H}}(V)).$$

Of course, the algebra of endomorphisms of this functor is isomorphic to $E(G, \mathbb{H})_{c'}$. 


Consider now the functor $\text{Rep}_c(\hat{G}) \to \text{Vect}$ obtained by composing $\text{Coinv}_H$ with the functor $\lim \text{Proj} : \text{Vect} \to \text{Vect}$. Let $E(G, H)_c$ be the algebra of endomorphisms of this latter functor. We have a natural map $E(G, H)_c \to \overline{E}(G, H)_c$.

**Proposition 3.8.** (a) The map $E(G, H)_c \to \overline{E}(G, H)_c$ is injective.
(b) The algebra $\overline{E}(G, H)_c^c$ is isomorphic to $\text{End}_{\text{Rep}_c(\hat{G})} \left( \text{Coind}_H^G(\text{Inf}^G_H(\mathcal{C})) \right)$.

We do not know under what conditions on $H$ one might expect that the above map $E(G, H)_c \to \overline{E}(G, H)_c$ is an isomorphism.

**Proof.** To prove the first assertion of the proposition, note that by Theorem 3.8 the evaluation map $E(G, H)_c \to \text{End}_{\text{Vect}}(\text{Coinv}_H(M_c(G)))$ is injective.

By construction, the pro-vector space $M_c(G)$ can be represented as a countable inverse limit with surjective restriction maps. Hence, by Proposition 2.5 of [GK], $\text{Coinv}_H(M_c(G)) \in \text{Vect}$ will also have this property. We have:

**Lemma 3.9.** For any pro-vector space, which can be represented as a countable inverse limit with surjective restriction maps, the morphism $\lim \text{Proj}(\mathcal{V}) \to \mathcal{V}$ is surjective.

This lemma implies that the map $\text{End}_{\text{Vect}}(\mathcal{V}) \to \text{End}_{\text{Vect}}(\lim \text{Proj}(\mathcal{V}))$ is injective.

To prove the second assertion, we must analyze the endomorphism algebra of the functor $\text{Vect} \to \text{Rep}_c(\hat{G})$ given by

$$V \mapsto \text{Coind}_H^G(\text{Inf}^H_H(V)).$$

However, as every left adjoint, this functor commutes with inductive limits. Therefore, its enough to consider its restriction to the subcategory $\text{Vect}_0$. This implies the proposition. □

4. THE Functor OF SEMI-INVARIANTS

4.1. Our method of proof of Theorem 3.3 is based on considering the functor of $G$-semi-invariants

$$\overline{\otimes}_G : \text{Rep}_{c'}(\hat{G}') \times \text{Rep}_c(\hat{G}) \to \text{Vect},$$

where $c$ and $c'$ are opposite levels. The construction of this functor mimics the construction of the semi-infinite cohomology functor for associative algebras by L. Positselsky, [Pos].

For $\Pi_c \in \text{Rep}_c(\hat{G})$, $\Pi_{c'} \in \text{Rep}_{c'}(\hat{G}')$ consider the pro-vector spaces

$$\Pi_{c'} \otimes \Pi_c$$

and $\Pi_{c'} \otimes M_c(G) \otimes \Pi_c$.

We consider the former as acted on by the diagonal copy of $G[[t]]$, and the latter by two mutually commuting copies of $G[[t]]$: one acts diagonally on $\Pi_{c'} \otimes M_c(G)$ via the left $\hat{G}$-action on $M_c(G)$; the other copy acts diagonally on $M_c(G) \otimes \Pi_c$ via the right action. Consider the object

$$(\Pi_{c'} \otimes M_c(G) \otimes \Pi_c)_{G[[t]] \times G[[t]]}.$$ We will construct two natural maps

$$\text{(5)} \quad (\Pi_{c'} \otimes \Pi_c)_{G[[t]]} \Rightarrow (\Pi_{c'} \otimes M_c(G) \otimes \Pi_c)_{G[[t]] \times G[[t]]}.$$ To construct the first map recall from Lemma 5.8 of [GK] that

$$\text{(6)} \quad (M_c(G) \otimes \Pi_c)_{G[[t]]} \simeq _G^G \left( _G^{c_{G[[t]]}}(\Pi_c) \right).$$
Since \( G / G[[t]] \) is ind-compact, the functor \( r^G_{G[[t]]} \) is isomorphic to the induction functor, \( \widetilde{r}^G_{G[[t]]} \).

Therefore, we obtain a morphism of \( \hat{G} \)-representations

\[
\Pi_c \to \hat{r}^G_{G[[t]]}(\Pi_c) \simeq (M_c(G) \otimes \Pi_c)_{G[[t]]}
\]

by adjunction from the identity map \( r^G_{G[[t]]}(\Pi_c) \to r^G_{G[[t]]}(\Pi_c) \).

The first map in (5) comes from (7) by tensoring with \( \Pi_c' \) and taking \( G[[t]] \)-coinvariants.

To construct the second map in (5) we will use the following observation. Let \( \widetilde{M}_c(G) \) be a representation of \( \hat{G} \times \hat{G}' \), obtained from the representation \( M_c'(G) \) of \( \hat{G}' \times \hat{G} \), by flipping the roles of \( \hat{G} \) and \( \hat{G}' \). We have:

**Proposition 4.2.**

1. We have a natural isomorphism of \( \hat{G} \times \hat{G}' \)-representations \( \widetilde{M}_c(G) \simeq M_c'(G) \).
2. The resulting two morphisms

\[
M_c(G) \Rightarrow (M_c(G) \otimes M_c'(G))_{G[[t]]}
\]

one, coming from (4), and the other from interchanging the roles of \( c \) and \( c' \), coincide.

**Remark.** It will follow from the proof, that statement (2) of the proposition fixes the isomorphism of statement (1) uniquely.

The proof will be given in Sect. 5. Using this proposition we construct the second map in (5) by simply interchanging the roles of \( c \) and \( c' \).

4.3. For \( \Pi_c, \Pi_c' \) as above, we set \( \Pi_c' \overset{\oplus}{\otimes} \Pi_c \) to be the equalizer (i.e., the kernel of the difference) of the two maps in (5). Note that since the functor of \( G[[t]] \)-coinvariants is only right-exact, the resulting functor \( \overset{\oplus}{\otimes} \) is a priori neither right nor left exact.

Suppose now that \( \Pi_c \) is not only a representation of \( \hat{G} \), but carries an additional commuting action of some group-object \( H \in \text{Set} \), which satisfies condition (**). In this case it follows from Corollary 1.10 that \( \Pi_c' \overset{\oplus}{\otimes} \Pi_c \) is an object of \( \text{Rep}(H) \).

The key assertion describing the behavior of the functor of semi-invariants is the following:

**Proposition 4.4.** For \( M_c(G) \), regarded as an object of \( \text{Rep}_c(\hat{G}) \), we have a natural isomorphism \( \Pi_c' \overset{\oplus}{\otimes} M_c(G) \simeq \Pi_c' \). Moreover, this isomorphism is compatible with the \( \hat{G} \)-actions.

**Proof.** Consider the following general set-up. Let \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) be two abelian categories, \( G : \mathcal{C}_1 \to \mathcal{C}_2 \) be a functor, and \( F : \mathcal{C}_2 \to \mathcal{C}_1 \) its right adjoint. By composing with \( F \circ G \) on the left and on the right, the adjunction map \( \text{Id}_{\mathcal{C}_1} \to F \circ G \) gives rise to two maps

\[
F \circ G \Rightarrow F \circ G \circ F \circ G,
\]

such that \( \text{Id}_{\mathcal{C}_1} \) maps to the equalizer.

**Lemma 4.5.** Assume that the functor \( G \) is exact and faithful. Then the map

\[
\text{Id}_{\mathcal{C}_1} \to \text{Equalizer}(F \circ G \Rightarrow F \circ G \circ F \circ G)
\]

is an isomorphism.
Proof. By assumption o G, it is enough to show that

\[ G \to \text{Equalizer}(G \circ F \circ G) \cong G \circ F \circ G \circ F \circ G \]

is an isomorphism, but this happens for any pair of adjoint functors. \qed

We apply this lemma to \( c_1 = \text{Rep}_{c'}(\hat{G}') \), \( c_2 = \text{Rep}(G[[t]], \text{Vect}) \) with \( F = i^G_G, G = r^G_G[[t]] \).

To prove the Proposition it is sufficient to show that for \( \Pi_{c'} \in \text{Rep}_{c'}(\hat{G}') \) the terms and maps in \( c \) are equal to the corresponding ones in \( b \).

First, by (10) and Proposition 4.2(1), for \( \Pi_{c'} \) as above, \( F \circ G(\Pi_{c'}) \) is indeed isomorphic to \( (\Pi_{c'} \otimes M_c(G))_{G[[t]]} \). Furthermore, by applying the functor \( F \circ G \) to the adjunction map \( \Pi_{c'} \to F \circ G(\Pi_{c'}) \) we obtain the second of the two maps from \( b \).

Let us now calculate the adjunction map \( \text{Id}_{\text{Rep}_{c'}(\hat{G}')} \to (\hat{G}') \otimes r^G_G[G[[t]]] \) applied to

\[ F \circ G(\Pi_{c'}) \simeq (\Pi_{c'} \otimes M_c(G))_{G[[t]]}. \]

By construction, it is obtained from the adjunction map

\[ M_c(G) \to F \circ G(M_c(G)) \simeq (M_c(G) \otimes M_c(G))_{G[[t]]} \]

by tensoring with \( \Pi_{c'} \) and taking \( G[[t]] \)-coinvariants. Therefore, by Proposition 4.2(2), it coincides with the first map from \( b \). \qed

Remark. Note that by Proposition 4.2(2), the two identifications \( M_c(G) \hat{\otimes} M_c(G) \simeq M_c(G) \), one coming from Proposition 4.2(1) applied to \( \Pi_{c'} = M_c(G) \), and the other from interchanging the roles of \( c \) and \( c' \) as in Proposition 4.2(1), coincide.

4.6. Proof of Theorem 5.3. Let \( \Pi_{c'} \) be an object of \( \text{Rep}_{c'}(\hat{G}') \), and let \( \Pi_c \) be an object of \( \text{Rep}_c(\hat{G}) \), carrying an additional commuting action of a group-object \( H \in \text{Set} \), which is quasi pro-unipotent. Then, using Corollary 14.10 and the fact that the functor \( \text{Coinv}_H \) is exact (Lemma 2.7 of [GN]), we obtain an isomorphism:

\[ \Pi_{c'} \hat{\otimes} \Pi_c \simeq (\Pi_{c'} \otimes \Pi_c)_H. \]

Applying this for \( \Pi_c = M_c(G) \), we obtain a functorial isomorphism:

\[ (\Pi_{c'}) \hat{\otimes} M_c(G, H) \simeq (\Pi_{c'})_H. \]

Therefore, we obtain a map

\[ \text{End}_{\text{Rep}_c(\hat{G})}(M_c(G, H)) \to E(G, H)_{c'}. \]

The fact that the composition

\[ \text{End}_{\text{Rep}_c(\hat{G})}(M_c(G, H)) \to E(G, H)_{c'} \to \text{End}_{\text{Rep}_c(\hat{G})}(M_c(G, H)) \]

is the identity map follows from the remark following the proof of Proposition 14.2.

Therefore, to finish the proof of the theorem it suffices to show that the map of \( b \) is injective. For that note, that for any \( \Pi_{c'} \in \text{Rep}_{c'}(\hat{G}') \) we have an injection \( \Pi_{c'} \to (\Pi_{c'} \otimes M_c(G))_{G[[t]]} \) (coming from the above adjunction \( \text{Id}_{\text{Rep}_{c'}(\hat{G}')} \to i^G_G[[t]] \otimes r^G_G[[t]] \)) and a surjection \( \Pi_{c'} \otimes M_c(G) \to (\Pi_{c'} \otimes M_c(G))_{G[[t]]} \) of objects of \( \text{Rep}_{c'}(\hat{G}') \).
Lemma 4.7. Suppose an element $\alpha \in E(G, \mathbb{H})_{c}$ annihilates $(\Pi_{c})_{\mathbb{H}}$ for some $\Pi_{c} \in \text{Rep}_{c}(\hat{G})$. Then $\alpha$ annihilates all objects of the form $(V \otimes \Pi_{c})_{\mathbb{H}}$ for $V \in \text{Vect}$.

Proof. Suppose that $V = \text{"lim" } V_{i}$, $V_{i} \in \text{Vect}$. Then $\nabla \otimes \Pi_{c} \simeq \text{lim } (V_{i} \otimes \Pi_{c})$, where the projective limit is taken in the category $\text{Vect}$.

Using Corollary 2.6 of [GK], we have: $(V \otimes \Pi_{c})_{\mathbb{H}} \simeq \text{lim } (V_{i} \otimes \Pi_{c})_{\mathbb{H}}$. This shows that we can assume that $V$ is a vector space, which we will denote by $V$.

Let us write $V = \text{lim } V_{i}$, where $V_{i} \in \text{Vect}$.

Sublemma 4.8. For $V = \text{lim } V_{i}$ and $W \in \text{Vect}$ the natural map

$$\text{lim } (V_{i} \otimes W) \rightarrow (\text{lim } V_{i}) \otimes W$$

is surjective.

Therefore, we have a surjection

$$\text{lim } (V_{i} \otimes \Pi_{c}) \rightarrow V \otimes \Pi_{c},$$

and, hence, a surjection on the level of coinvariants.

Since by assumption, $\alpha$ annihilates every $(V_{i} \otimes \Pi_{c})_{\mathbb{H}}$, and the functor $\text{Coinv}_{\mathbb{H}}$ commutes with inductive limits (cf. Corollary 1.10), we obtain that $\alpha$ annihilates also $\left(\text{lim } (V_{i} \otimes \Pi_{c})\right)_{\mathbb{H}}$. Hence, by the above, it annihilates also $(V \otimes \Pi_{c})_{\mathbb{H}}$.

□

Using this lemma and the exactness of the functor of $\mathbb{H}$-coinvariants, we obtain that any $\alpha \in \ker(E(G, \mathbb{H})_{c} \rightarrow \text{End}(M_{c}(G, \mathbb{H}))$ annihilates all $(\Pi_{c} \otimes M_{c}(G))_{\mathbb{H}}$, and hence $(\Pi_{c})_{\mathbb{H}}$ for any $\Pi_{c}$.

Remark. Note that the same argument proves the following more general assertion. Let $\mathbb{H}_{1}, \mathbb{H}_{2}$ be two quasi pro-unipotent groups endowed with homomorphisms to $\hat{G}$. Then the space of natural transformations between the functors $\text{Coinv}_{\mathbb{H}_{1}}, \text{Coinv}_{\mathbb{H}_{2}} : \text{Rep}_{c}(\hat{G}) \rightarrow \text{Vect}$ is isomorphic to $\text{Hom}_{\text{Rep}_{c}(\hat{G})}(M_{c}(G, \mathbb{H}_{1}), M_{c}(G, \mathbb{H}_{2}))$.

5. Proof of Proposition 4.2

5.1. We will repeatedly use the following construction:

Let $Z_{1} \rightarrow Z_{2}$ be a map of schemes of finite type over $K$, such that $Z_{1}$ is principal bundle with respect to a smooth unipotent group-scheme $H$ on $Z_{2}$. Let $E$ be the line bundle on $Z_{2}$, given by $z \mapsto \det(h_{z})$, where $h_{z}$ is the fiber at $z \in Z_{2}$ of the sheaf of Lie algebras corresponding to $H$. Let $\tilde{Z}_{1}$ be the total space of the pull-back of the resulting $G_{m}$-torsor to $Z_{1}$.

Lemma 5.2. Under these circumstances we have a natural map

$$\left(\text{Funct}^{lc}_{\mathbb{C}}(\tilde{Z}_{1}) \otimes \mathbb{C}\right)_{G_{m}} \rightarrow \text{Funct}^{lc}_{\mathbb{C}}(Z_{2}),$$

where $G_{m}$ acts on $\mathbb{C}$ via the standard character $G_{m} \rightarrow \mathbb{Z} \xrightarrow{1-\gamma} \mathbb{C}^{\times}$.
5.3. Let us recall the construction of $M_c(\mathbb{G})$, following [GK], Sect. 5. To simplify the exposition, we will first assume that $c = 1$, in which case we will sometimes write $M(\mathbb{G})$ instead of $M_c(\mathbb{G})$.

Consider the set of pairs $(i, Y)$, where $Y$ is a sub-scheme of $G((t))$, stable under the right action of the congruence subgroup $G^i$. Note that in this case the quotient $Y/G^i$ is a scheme of finite type over $K$.

The above set is naturally filtered: $(i, Y) < (i', Y')$ if $i' \geq i$ and $Y \subset Y'$. Note also that $Y/G^i \to Y/G^i$ is a principal bundle with respect to the group $G^i/G^i$.

Let $Y/G^i$ denote the object of $\text{Set}$, corresponding to the scheme $Y/G^i$. Consider the vector space $V(i, Y) := \text{Funct}_{c,l}(Y/G^i) \otimes \mu(G[[t]]/G^i)$, cf. [GK], Sect. 3.2, where for a locally compact group $H$, we denote by $\mu(H)$ the space of left-invariant Haar measures on it.

Whenever $(i, Y) < (i', Y')$, we have a natural map $V(i', Y') \to V(i, Y)$. It is defined as the composition of the restriction map $\text{Funct}_{c,l}(Y'/G^{i'}) \to \text{Funct}_{c,l}(Y/G^i)$, followed by the map

$$\text{Funct}_{c,l}(Y/G^i) \otimes \mu(G^i/G^{i'}) \to \text{Funct}_{c,l}(Y/G^i),$$

coming from Lemma 5.2 using $\mu(G[[t]]/G^i) \simeq \mu(G[[t]]/G^i) \otimes \mu(G^i/G^{i'})$.

We have:

$$M_c(\mathbb{G}) = \lim_{(i, Y)} V(i, Y),$$
as a pro-vector space.

Let us now describe the action of $G \times \hat{G}_0$ on $M_c(\mathbb{G})$. For our purposes it would suffice to do so on the level of groups of $K$-valued points of the corresponding group-indschemes.

For $g \in G((t))(K)$ acting on $M_c(\mathbb{G})$ on the left, we define $V(i, Y) \to V(i, g \cdot Y)$ to be the natural map. In this way we obtain an action of $g$ on the entire inverse system.

To define the right action, for $(i, Y)$ as above, let $j$ be a large enough integer, so that $Ad_g^{-1}(G^j) \subset G^i$. Then the right multiplication by $g$ defines a map of schemes,

$$Y/G^j \to Y \cdot g/G^i,$$
such that the former is a principal $G^j/G^i$-bundle over the latter.

A lift of $g$ to a point $\hat{g}$ of the central extension $\hat{G}_0$ defines an identification $\mu(G[[t]]/G^j) \simeq \mu(G[[t]]/ Ad_g^{-1}(G^j))$. Hence, by Lemma 5.2 we obtain a map

$$V(j, Y) \to V(i, Y \cdot g),$$

and, hence, an action of $\hat{g}$ on the inverse system.

5.4. Let now $\hat{G}$ and $c$ be general. We modify the above construction as follows. For each $Y \subset G((t))$ as above, let $\hat{Y}$ be its pre-image in $\hat{G}$. Set

$$V_c(j, Y) := \left(\text{Funct}_{c,l}(G^j/\hat{Y}) \otimes \mathbb{C}\right)_{G_m} \otimes \mu(G[[t]]/G^j),$$

where $G_m$ acts naturally on $\hat{Y}$ and by the character $c$ on $\mathbb{C}$. We have:

$$M_c(\mathbb{G}) = \lim_{(j, Y)} V_c(j, Y),$$

and the action of $\hat{G} \times \hat{G}'$ is described in the same way as above.

By definition, the representation $\tilde{M}_c(\mathbb{G})$ is the same as $M_c(\mathbb{G})$, viewed as a representation of $\hat{G} \times \hat{G}' \simeq \hat{G}' \times \hat{G}$. Explicitly it can be written down as follows. Consider the set of pairs
on the group. Ad

\[ (12) \]

where the transition maps are given by fiber-wise integration. This space carries an action of \( \hat{M}_c(G) \), where the identity maps. 5.5. We shall now construct the sought-for map \( \hat{M}_c(G) \to M_c(G) \). Let us mention that when \( G \) is the multiplicative group \( G_m \) the sought-for isomorphism amounts to simply to the inversion on the group.

For a pair \((i, Y)\) as in the definition of \( M_c(G) \), there exists an integer \( j \) large enough so that \( \text{Ad}_{y^{-1}}(G^j) \subset G^i \) for \( y \in Y(K) \). In particular, over \( Y/G^j \) we obtain a group-scheme, denoted \( G^j_{Y} \), whose fiber over \( y \in Y \) is \( G^i / \text{Ad}_{y^{-1}}(G^j) \), and we have a map

\[ (11) \]

such that the former scheme is a principal \( G^j_{Y} \)-bundle over the latter.

Note that the fiber of \( \hat{Y} \) over a given point \( y \in Y \) identifies with \( \text{det}(\text{Ad}_y(g[[t]]), g[[t]]) \), where \( g \) is the Lie algebra of \( G \). Hence, we obtain a natural map

\[ \tilde{V}_{c'}(j, Y) \to V_c(i, Y) \]

from Lemma \[ \text{Lemma } 5.2 \]

Thus, we obtain a map \( \tilde{M}_c(G) \to M_c(G) \), and from the construction, it is clear that this map respects the action of \( \hat{G}(K) \times \hat{G}'(K) \). Now Lemma \[ \text{Lemma } 5.4 \] implies that the constructed map is a morphism of \( \hat{G} \times \hat{G}' \)-representations.

The map in the opposite direction: \( M_c(G) \to \tilde{M}_c(G) \) is constructed similarly, and by the definition of the transition maps giving rise to the inverse systems \( M_c(G) \) and \( \tilde{M}_c(G) \), it is clear that both compositions \( M_c(G) \to \tilde{M}_c(G) \to M_c(G) \) and \( \tilde{M}_c(G) \to M_c(G) \to M_c(G) \) are the identity maps.

This proves point (1) of Proposition \[ \text{Proposition } 5.6 \]

5.6. Following \[ \text{[GK]} \], let us denote by \( M(G[[t]]) \) the pro-vector space

\[ "\text{lim}^\Leftarrow \text{Funct}^G(G[[t]]/G^i)/G^i) \otimes \mu(G[[t]]/G^i), \]

where the transition maps are given by fiber-wise integration. This space carries an action of the group \( G[[t]] \times G[[t]] \). The convolution product defines an isomorphism

\[ (12) \]

where \( G[[t]] \) acts diagonally.

By construction, as a representation of \( \hat{G} \) under the left action, \( M_c(G) \) identifies with \( \hat{G}^G_{G[[t]]}(M(G[[t]])) \). Therefore,

\[ (13) \]

\[ \text{Hom}_{\text{Rep}}(\hat{G}, (\tilde{M}_c(G), M_c(G)) \simeq \text{Hom}_{G[[t]]}(\tilde{M}_c(G), M(G[[t]])). \]
The map $\tilde{M}_c(G) \to M_c(G)$ constructed above corresponds to the natural restriction morphism $\tilde{M}_c(G) \to M(G[[t]])$.

Remark. From the latter description it is not immediately clear why this map is compatible with the right $\hat{G}'$-action.

Note also that the map $\tilde{M}_c(G) \to M_c(G)$ can be described by a similar adjunction property with respect to the right $\hat{G}'$-action.

Let us prove now point (2) of Proposition 6.2. For any $\Pi$, which is a representation of $\hat{G} \times \hat{G}'$ at levels $(c, c')$ we have:

$$\text{Hom}_{\hat{G} \times \hat{G}'}\left(\Pi, (M_c(G) \otimes \tilde{M}_c(G))_{G[[t]]}\right) \simeq \text{Hom}_{G[[t]] \times G[[t]]}(\Pi, M(G[[t]])),$$

with the isomorphism being given by the restriction map

$$(M_c(G) \otimes \tilde{M}_c(G))_{G[[t]]} \to (M(G[[t]])) \otimes M(G[[t]]))_{G[[t]]},$$

followed by the map of (12).

Let us apply this to $\Pi = M_c(G)$. It is clear that both maps appearing in Proposition 6.2, correspond under the above isomorphism to the restriction map $M_c(G) \to M(G[[t]])$. Therefore, these two maps coincide.

6. Distributions on a stack

6.1. First, let $X$ be a locally compact object of $\text{Set}$. Recall that $\text{Funct}^lc(X)$ denotes the corresponding (strict) object in $\text{Vect}$ (cf. [GK], Sect. 3.2), and $\text{Funct}^lc(X) = \limProj \text{Funct}^lc(X)$. The vector space $\text{Distr}^c(X)$ introduced in Sect. 3.2 identifies with $\text{Hom}_{\text{Vect}}(\text{Funct}^lc(X), \mathbb{C})$, or, which is the same, with the space of linear functionals $\text{Funct}^lc(X) \to \mathbb{C}$, continuous in the topology of projective limit.

Suppose now that $X = X(K)$, where $X$ is a smooth algebraic variety over $K$. In this case we can introduce the subspace $\text{Distr}^lc(X)$ of locally constant distributions on $X$ (cf., e.g., [GK], Sect. 5.1).

Indeed, it is well-known that a choice of a top differential form $\omega$ on $X$ defines a measure $\mu(\omega)$ on $X$, i.e., a functional on the space $\text{Funct}^lc(X)$. For $\omega' = \omega \cdot f$, where $f$ is an invertible function on $X$, we have: $\mu(\omega') = \mu(\omega) \cdot |f|$. Hence, the subset of elements in $\text{Distr}^c(X)$, which can be (locally) written as $\mu(\omega) \cdot g$, where $g$ is a locally constant function on $X$ with compact support, is independent of the choice of $\omega$. This subset is by definition $\text{Distr}^lc(X)$.

Although the following is well-known, we give a proof for the sake of completeness:

Proposition 6.2. Let $f : X_1 \to X_2$ be a smooth map between smooth varieties over $K$. Then

1. The push-forward map $\text{Distr}^c(X_1) \to \text{Distr}^c(X_2)$ sends $\text{Distr}^lc(X_1)$ to $\text{Distr}^lc(X_2)$.

2. If $X_1(K) \to X_2(K)$ is surjective, then $f_1 : \text{Distr}^lc(X_1) \to \text{Distr}^lc(X_2)$ is also surjective.

Proof. Statement (1) is local in the analytic, and a fortiori in the Zariski topology on $X_1$. Therefore, we can assume that our morphism $f$ factors as $X_1 \overset{f'}{\to} X_2 \times Z \overset{f''}{\to} X_2$, where $Z$ is another smooth variety, with $f'$ being étale, and $f''$ being the projection on the first factor.

Since an étale map induces a local isomorphism in the analytic topology, it is clear that $f'_1$ maps $\text{Distr}^lc(X_1)$ to $\text{Distr}^lc(X_2 \times Z)$.
From the definition of \( \text{Distr}^{lc}(-) \), it is clear that
\[
\text{Distr}^{lc}(\mathbb{Z}_1) \otimes \text{Distr}^{lc}(\mathbb{Z}_2) \longrightarrow \text{Distr}^{lc}(\mathbb{Z}_1 \times \mathbb{Z}_2)
\]

(14)
\[
\text{Distr}_{c}(\mathbb{Z}_1) \otimes \text{Distr}_{c}(\mathbb{Z}_2) \longrightarrow \text{Distr}_{c}(\mathbb{Z}_1 \times \mathbb{Z}_2).
\]

So the map \( f_1^* : \text{Distr}^{lc}(\mathbb{X}_2 \times \mathbb{Z}) \to \text{Distr}_{c}(\mathbb{X}_2) \) can be identified with
\[
\text{Distr}^{lc}(\mathbb{Z}) \otimes \text{Distr}^{lc}(\mathbb{X}_2) \xrightarrow{f \times \text{id}} \text{Distr}^{lc}(\mathbb{X}_2),
\]
implying assertion (1) of the proposition.

We will prove a slight strengthening of assertion (2). Note that since \( f \) is smooth, the image of \( \mathbb{X}_1 \) in \( \mathbb{X}_2 \) is open, and hence, also closed in the analytic topology. We will show that \( f_1 \) maps \( \text{Distr}^{lc}(\mathbb{X}_1) \) surjectively onto the subspace of \( \text{Distr}^{lc}(\mathbb{X}_2) \), consisting of distributions, supported on the image.

The assertion is local in the analytic topology on \( \mathbb{X}_2 \). Let \( x_2 \in \mathbb{X}_2(\mathbb{K}) \) be a point, and let \( x_1 \in \mathbb{X}_1(\mathbb{K}) \) be some its pre-image. Then the local factorization of \( f \) as \( f'' \circ f' \) as above makes the assertion manifest.

\[ \square \]

6.3. In what follows we will need a relative version of the above notions. For a smooth morphism \( g : \mathbb{X} \to \mathbb{Z} \) let \( \omega_{rel} \) be a relative top differential form on \( \mathbb{X} \). It defines a relative measure \( \mu(\omega_{rel}) : \text{Funct}^{lc}(\mathbb{X}) \to \text{Funct}^{lc}(\mathbb{Z}) \). As in the absolute situation, by multiplying \( \mu(\omega_{rel}) \) by locally constant functions on \( \mathbb{X} \), whose support is proper over \( \mathbb{Z} \), we obtain a pro-vector sub-space inside \( \text{Hom}_{\text{Funct}^{lc}(\mathbb{Z})}(\text{Funct}^{lc}(\mathbb{X}), \text{Funct}^{lc}(\mathbb{Z})) \), which we will denote by \( \text{Distr}^{lc}(\mathbb{X}/\mathbb{Z}) \). Note that when \( X = X' \times \mathbb{Z} \), we have: \( \text{Distr}^{lc}(\mathbb{X}/\mathbb{Z}) \simeq \text{Distr}^{lc}(\mathbb{X}') \otimes \text{Funct}^{lc}(\mathbb{Z}) \) (the tensor product being taken in the sense of \( \text{Vect} \)). We will denote by \( \text{Distr}^{lc}(\mathbb{X}/\mathbb{Z}) \) the vector space \( \lim \text{Proj} \text{Distr}^{lc}(\mathbb{X}/\mathbb{Z}) \).

When \( f : \mathbb{X}_1 \to \mathbb{X}_2 \) is a smooth map of schemes smooth over \( \mathbb{Z} \), as in Proposition 6.2 we have a push-forward map \( f_1 : \text{Distr}^{lc}(\mathbb{X}_1/\mathbb{Z}) \to \text{Distr}^{lc}(\mathbb{X}_2/\mathbb{Z}) \), which is surjective if \( f : \mathbb{X}_1(\mathbb{K}) \to \mathbb{X}_2(\mathbb{K}) \) is; moreover, in this case the map \( f_1 : \text{Distr}^{lc}(\mathbb{X}_1/\mathbb{Z}) \to \text{Distr}^{lc}(\mathbb{X}_2/\mathbb{Z}) \) is also easily seen to be surjective. In the particular case when \( \mathbb{X}_2 = \mathbb{Z} \) we obtain a map \( f_1 : \text{Distr}^{lc}(\mathbb{X}/\mathbb{Z}) \to \text{Funct}^{lc}(\mathbb{Z}) \).

If \( \mathbb{Y} \) is another scheme over \( \mathbb{Z} \), consider the Cartesian diagram
\[
\begin{array}{ccc}
\mathbb{X} \times \mathbb{Y} & \xrightarrow{f} & \mathbb{X} \\
\mathbb{Y} \downarrow & & \downarrow g \\
\mathbb{Y} & \xrightarrow{f} & \mathbb{Z}.
\end{array}
\]

(15)

We have a pull-back map \( f^* : \text{Distr}^{lc}(\mathbb{X}/\mathbb{Z}) \to \text{Distr}^{lc}(\mathbb{X} \times \mathbb{Y}/\mathbb{Z}) \).

Suppose now that the scheme \( \mathbb{Z} \) is itself smooth, and \( \mathbb{X} \) is smooth over \( \mathbb{Z} \) as above. In this case the spaces \( \text{Distr}^{lc}(\mathbb{X}) \) and \( \text{Distr}^{lc}(\mathbb{Z}) \) are well-defined, and we have an isomorphism
\[
\text{Distr}^{lc}(\mathbb{X}) \simeq \text{Distr}^{lc}(\mathbb{X}/\mathbb{Z}) \otimes_{\text{Funct}^{lc}(\mathbb{Z})} \text{Distr}^{lc}(\mathbb{Z}).
\]
If in the situation of (15) $Y$ is also smooth over $\mathbb{Z}$, and $\xi_Y \in \text{Distr}^{lc}_{\mathbb{Z}}(Y)$, $\xi_{X/Z} \in \text{Distr}^{lc}_{\mathbb{Z}}(X/Z)$, consider the element $f^*(\xi_{X/Z}) \otimes \xi_Y \in \text{Distr}^{lc}_{\mathbb{Z}}(X \times Y)$. We have:

\begin{align}
(16) \quad f'_!(f^*(\xi_{X/Z}) \otimes \xi_Y) &= \xi_{X/Z} \otimes f_!(\xi_Y) \in \text{Distr}^{lc}_{\mathbb{Z}}(X), \quad \text{and} \\
(17) \quad g'_!(f^*(\xi_{X/Z}) \otimes \xi_Y) &= f^*(g_!(\xi_{X/Z})) \cdot \xi_Y \in \text{Distr}^{lc}_{\mathbb{Z}}(Y).
\end{align}

Finally, let us assume that both maps $f$ and $g$ induce surjections on the level of $K$-valued points.

**Lemma 6.4.** The maps $f_!, g_!$ induce an isomorphism

$$
\text{Distr}^{lc}_{\mathbb{Z}}(Z) \simeq \text{coker} \left( \text{Distr}^{lc}_{\mathbb{Z}}(X \times Y) \xrightarrow{(f'_!, -g_!)} \text{Distr}^{lc}_{\mathbb{Z}}(X) \oplus \text{Distr}^{lc}_{\mathbb{Z}}(Y) \right).
$$

**Proof.** Let $(\xi_X, \xi_Y) \in \text{Distr}^{lc}_{\mathbb{Z}}(X) \oplus \text{Distr}^{lc}_{\mathbb{Z}}(Y)$ be an element such that $f_!(\xi_X) = g_!(\xi_Y)$. We need to find an element $\xi' \in \text{Distr}^{lc}_{\mathbb{Z}}(X_1 \times Y)$, such that $f'_!(\xi') = \xi_X$ and $g'_!(\xi') = \xi_Y$. Using Lemma 6.2 we can assume that $\xi_X = 0$.

Let $\xi_{X/Z}$ be an element in $\text{Distr}^{lc}_{\mathbb{Z}}(X/Z)$, such that $f_! \xi = 1 \in \text{Funct}(Z)$. Then $\xi' := f^*(\xi_{X/Z}) \otimes \xi_Y$ satisfies our requirements, by (16).

Let now $X$ and $Y$ be smooth varieties, and $f : Z \times X \to Y$ a map, such that the corresponding map $f' : Z \times X \to Z \times Y$ is smooth.

**Lemma-Construction 6.5.** Under the above circumstances we have a natural action map

$$
Z \times \text{Distr}^{lc}_{\mathbb{Z}}(X) \to \text{Distr}^{lc}_{\mathbb{Z}}(Y).
$$

**Proof.** Consider the map

$$
f'_! : \text{Distr}^{lc}_{\mathbb{Z}}(Z \times X/Z) \to \text{Distr}^{lc}_{\mathbb{Z}}(Z \times Y/Z).
$$

By composing it with $\cdot \otimes 1 : \text{Distr}^{lc}_{\mathbb{Z}}(X) \to \text{Distr}^{lc}_{\mathbb{Z}}(Z \times X/Z)$ we obtain a map

$$
\text{Distr}^{lc}_{\mathbb{Z}}(X) \to \text{Distr}^{lc}_{\mathbb{Z}}(Y) \otimes \text{Funct}^{lc}(Z).
$$

The latter is, by definition, the same as an action map $Z \times \text{Distr}^{lc}_{\mathbb{Z}}(X) \to \text{Distr}^{lc}_{\mathbb{Z}}(Y)$.

6.6. Let $\mathcal{Y}$ be an algebraic stack. We will say that $\mathcal{Y}$ is $K$-admissible (or just admissible) if there exists a smooth covering $Z \to \mathcal{Y}$, such that for any map $X \to \mathcal{Y}$, the corresponding map of schemes

$$
X \times Z \to X
$$

is surjective on the level of $K$-points.

If $\mathcal{Y}$ is admissible, a covering $Z \to \mathcal{Y}$ having the above property will be called admissible. It is clear that the class of admissible coverings is closed under Cartesian products. It is also clear that if $\mathcal{Y}$ is admissible, and $\mathcal{Y}' \to \mathcal{Y}$ is a representable map, then $\mathcal{Y}'$ is also admissible.

**Lemma 6.7.** Suppose that $\mathcal{Y}$ is a stack, which is locally in the Zariski topology has the form $Z/G$, where $Z$ is a scheme, and $G$ is an affine algebraic group. Then $\mathcal{Y}$ is admissible.

**Proof.** First, we can assume that $G = GL_n$. Indeed, by assumption, there is an embedding $G \to GL_n$, and consider the scheme $Z' := Z \times GL_n$. Then $\mathcal{Y} = Z'/GL_n$.

Now the assertion follows from Hilbert’s 90: for $y \in \mathcal{Y}(K)$ its pre-image in $Z$ is a $GL_n$-torsor, which is necessarily trivial.
From now on, we will assume that $Y$ is admissible. Assume in addition that $Y$ is smooth. We will now define the space, denoted, $\text{Distr}_{lc}^c(Y)$, of locally constant compactly supported distributions on $Y$.

Namely, given two admissible coverings $Z_1, Z_2 \to Y$ we define

$$\text{Distr}_{lc}^c(Y) := \text{coker} \left( \text{Distr}_{lc}^c(Z_1 \times_Y Z_2) \to \text{Distr}_{lc}^c(Z_1) \oplus \text{Distr}_{lc}^c(Z_2) \right).$$

Lemma 6.4 combined with Proposition 6.2(2), implies that $\text{Distr}_{lc}^c(Y)$ is well-defined, i.e., independent of the choice of $Z_1, Z_2$.

If $f : Y_1 \to Y_2$ is a smooth representable map of (smooth admissible) stacks, from Proposition 6.2(1) we obtain that there exists a well-defined map $f_1 : \text{Distr}_{lc}^c(Y_1) \to \text{Distr}_{lc}^c(Y_2)$.

Assume now that $Y = Z/G$, where $G$ is an algebraic group acting on $Z$. By Lemma 6.3 we have an action of $G$ on the vector space $\text{Distr}_{lc}^c(Z)$. From Lemma 6.4 we obtain:

**Corollary 6.8.** For $Y$ as above,

$$\text{Distr}_{lc}^c(Y) \simeq \text{Coinv}_G(\text{Distr}_{lc}^c(Z)).$$

6.9. **Relative version.** Assume now that $Y$ is a stack, endowed with a smooth map to a scheme $Z$. For a pair of admissible coverings $X_1, X_2 \to Y$, we define the pro-vector space $\text{Distr}_{lc}^c(Y/Z)$ as

$$\text{coker} \left( \text{Distr}_{lc}^c(X_1 \times_Y X_2/Z) \to \text{Distr}_{lc}^c(X_1/Z) \oplus \text{Distr}_{lc}^c(X_2/Z) \right).$$

A relative version of Lemma 6.4 shows that this is well-defined, i.e., independent of the choice of $X_1$ and $X_2$.

Finally, the assertion of Lemma-Construction 6.5 remains valid, where $Z$ is a scheme, $Y, Y'$ are smooth stacks, and the map $f : Z \times Y \to Y'$ is such that the corresponding map $f' : Z \times Y \to Z \times Y'$ is smooth and representable.

## 7. Induction via the moduli stack of bundles

7.1. Let $X$ be a (smooth complete) algebraic curve over $K$, $x \in X$ a rational point, and let $t$ be a coordinate near $x$.

If $G$ be a split reductive group, let $\text{Bun}_G$ denote the moduli stack of principal $G$-bundles on $X$. For $i \in Z$, let $\text{Bun}_{G,x}^i$ denote the stack classifying bundles equipped with a trivialization on the $i$-th infinitesimal neighbourhood of $x$. By construction, $\text{Bun}_{G,x}^i$ is a principal $G[[t]]/G$-bundle over $\text{Bun}_G$.

If $Y \subset \text{Bun}_G$ is an open sub-stack of finite type, we let $Y^{i,x}$ denote its pre-image in $\text{Bun}_{G,x}^i$. The following is well-known:

**Lemma 7.2.** For any $Y \subset \text{Bun}_G$ of finite type and $i$ large enough, the stack $Y^{i,x}$ is a scheme of finite type.

For $Y$ as above, we let $Y^{i,x}$ denote the object of $\text{Pro}(Sch^{ft})$ equal to "$\lim_{\rightarrow Y} Y^{i,x}".$ We let $\text{Bun}_{G,x}^{\infty,x}$ denote the object

$$\text{"lim}_{\rightarrow Y} Y^{i,x} \in \text{Ind(Pro}(Sch^{ft})).$$

Another basic fact is that $G((t))$, viewed as a group-object of $\text{Ind}(\text{Pro}(Sch^{ft}))$, acts on $\text{Bun}_{G,x}^{\infty,x}$ in the sense of the tensor structure on $\text{Ind}(\text{Pro}(Sch^{ft})).$
7.3. By Lemma 6.7, the stacks $Y^{i,x}$ are admissible. Set $W^i_{Y^1} = \text{Distr}_{\text{lc}}(Y^{i,x})$. For $Y_1 \hookrightarrow Y_2$ we have a natural push-forward map on the level of distributions $W^i_{Y_1} \to W^i_{Y_2}$. Set

$$W^i := \lim_{\leftarrow y} W^i_{Y^1} \in \text{Vect}.$$  

For a fixed $Y$ and $j > i$ we have a smooth representable map of stacks $Y^{j,x} \to Y^{i,x}$; hence we obtain a map $W^i_{Y^1} \to W^i_{Y^2}$ and, finally, a map $W^j \to W^i$.

We define the pro-vector space

$$W_{X,x} := \text{"lim"}_i W^i.$$  

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We define the pro-vector space

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Now we are ready to state:

**Theorem 7.4.** The pro-vector space $W_{X,x}$ carries a natural action of the group $G$, such that $\text{Coinv}_{G^i}(W_{X,x}) \simeq W^i$.

Note that by construction we have:

**Corollary 7.5.** The $G$-representation $W_{X,x}$ is admissible.

Indeed, the coinvariants $\text{Coinv}_{G^i}(W_{X,x}) \simeq W^i$ all belong to $\text{Vect}$.

7.6. **Proof of Theorem 7.4.** Let $G((t)) = \text{"lim"}_k Z_k$ with $Z_k = \text{"lim"}_l Z^j_k$, where $Z^j_k$ are schemes of finite type.

To define an action

$$G \times W_{X,x} \to W_{X,x}$$

we need to give for every $k$ and $i$ a map

$$Z^j_k \times W^j \to W^i$$

defined for $j$ and $l$ sufficiently large.

For $k$ and $i$ as above let $j$ be such that $\text{Ad}_{Z^j_k}(G^j) \subset G^i$. The action of $G((t))$ on $\text{Bun}_{G^j}^{\infty}$ yields a map of stacks $Z^j_k \times \text{Bun}_{G^j}^{\infty} \to \text{Bun}_{G^i}^{\infty}$, which factors through $Z^j_k$ for some $l$. Moreover, for every sub-stack $Y,\text{Bun}_{G}$ of finite type, there exists another sub-stack $Y'$ of finite type, such that we have a map

$$Z^j_k \times Y^{j,x} \to Y'^{j,x}.$$  

We claim that for $i, j, k, l, Y, Y'$ as above, we have a map

$$Z^j_k \times W^j \to W^i.$$  

This follows from the stack-theoretic version of Lemma-Construction 6.5 cf. Sect. 6.9. The fact that the resulting action map $G \times W_{X,x} \to W_{X,x}$ respects the group law on $G$ is a straightforward verification.

To compute $\text{Coinv}_{G^i}(W_{X,x})$ note that $G[[t]]$, and hence all $G^i$, act on each $\text{Bun}_{G^j}^{\infty}$ individually.

Hence,

$$\text{Coinv}_{G^i}(W_{X,x}) \simeq \text{"lim"}_{j \geq i} \text{Coinv}_{G^j/G^i}(W^j).$$  

We claim that for $j \geq i$, $\text{Coinv}_{G^j/G^i}(W^j) \simeq W^i$. Indeed, since each $Y^{j,x}$ is stable under $G^j/G^i$, we have:

$$\text{Coinv}_{G^j/G^i}(W^j) \simeq \lim_{\leftarrow Y} \text{Coinv}_{G^j/G^i}(W^j_Y) \simeq \lim_{\leftarrow Y} W^j_Y \simeq W^i.$$
where the middle isomorphism follows from Corollary 5.8.

7.7. Variants and Generalizations. Recall that the stack $\mathrm{Bun}_G$ is endowed with a canonical line bundle $\mathcal{L}_{\mathrm{Bun}_G}$, with the basic property that the $G$-action on $\mathrm{Bun}_G^{\infty}$ extends to an action of a central extension $\hat{G}$ on the pull-back of $\mathcal{L}_{\mathrm{Bun}_G}$ to $\mathrm{Bun}_G^{\infty}$.

By the same token, we consider now a representation $\hat{W}_{X,c}$ of $\hat{G}$, and for every $c : G_m \to \mathbb{C}^*$ the object
\[ \hat{W}_{X,c} := (\hat{W}_X \otimes \mathbb{C})G_m \in \text{Rep}_c(\hat{G}). \]

Note that instead of a single point $x$ we could have considered any finite collection $X = x_1, \ldots, x_n$ of rational points. By repeating the construction we obtain a pro-vector space $\hat{W}_{X,x}$ acted on by the product $\Pi \hat{G}_{x_k}$, where each $\hat{G}_{x_k}$ identifies with $\hat{G}$ once we identify the local ring of $X$ at $x_k$ with $F$.

Again, for a choice of a character $c : G_m \to \mathbb{C}^*$, we obtain a representation of $\Pi \hat{G}_{x_k}$, denoted $\hat{W}_{X,x,c}$, such that the center $G_m^c$ acts via the multiplication map $G_m^c \to G_m$.

7.8. From now on we will suppose that $X$ is isomorphic to the projective line $P^1$, and the number of points is two, which we will denote by $x_1$, and $x_2$, respectively. Assume also that $G$ is semi-simple and simply connected.

Consider the representation $\hat{W}_{P^1,x_1,x_2,c}$ of $\hat{G}_{x_1} \times \hat{G}_{x_2}$. Let us take its coinvariants with respect to $I_{x_1}^0 \subset \hat{G}_{x_1}$. By Corollary 1.10 on the resulting pro-vector space we will have an action of $\hat{G}_{x_2}$; we will denote this representation by $\Pi_{c}^{\text{thick}}$, i.e.,
\[ \Pi_{c}^{\text{thick}} = \text{Coinv}_{I_{x_1}^0}(\hat{W}_{P^1,x_1,x_2,c}). \]

By Theorem 3.3 the algebra $H_{q,c'}$ acts on $\Pi_{c}^{\text{thick}}$ by endomorphisms. Consider now
\[ U_c := \text{Coinv}_{I_{x_1}^0 \times I_{x_2}^0}(\hat{W}_{P^1,x_1,x_2,c}) \cong \text{Coinv}_{I_{x_1}^0}(\Pi_{c}^{\text{thick}}). \]

By Corollary 6.5 this is a vector space, endowed with two commuting actions of $H_{q,c'}$. We have:

**Theorem 7.9.** There exists a canonically defined vector $1_{U_c} \in U_c$, which freely generates $U_c$ under each of the two $H_{q,c'}$-actions.

8. Proof of Theorem 7.9

8.1. Let $W_{aff}$ be the affine Weyl group corresponding to $G$. Since $G$ was assumed simply connected, $W_{aff}$ is a Coxeter group.

If $\alpha$ is a simple affine root, let $I_{\alpha} \subset \hat{G}$ denote the corresponding sub-minimal parahoric; let $N(I_{\alpha})$ denote the (pro)-unipotent radical of $I_{\alpha}$, and $M_{\alpha} := I_{\alpha}/N(I_{\alpha})$ the Levi quotient.

By definition, $M_{\alpha}$ is a reductive group of semi-simple rank $1$, with a distinguished copy of $G_m$ in its center; we will denote by $M'_{\alpha}$ the quotient $M_{\alpha}/G_m$. Let $B_{\alpha}$ denote the Borel subgroup of $M_{\alpha}$, and $B_{\alpha}^0$ the kernel of
\[ B_{\alpha} \to T \to \Lambda. \]

Let $\Pi^0_{\alpha}$ be the quotient of the principal series representation of $M_{\alpha}$, given by the condition that $G_m \subset M_{\alpha}$ acts by the character $c$, i.e.,
\[ \Pi^0_{\alpha} = \left( \text{Funct}_{c}^!(M_{\alpha}/B_{\alpha}^0) \otimes \mathbb{C} \right)_{G_m}. \]
Let us denote by $\mathcal{H}_{q,c}^\alpha$ the corresponding affine Hecke algebra of $\mathcal{M}_\alpha$, i.e., the algebra of endomorphisms of the functor $\text{Coinv}_{B^\alpha} : \text{Rep}(\mathcal{M}_\alpha, \text{Vect})_c \to \text{Vect}$, or which is the same, the algebra of endomorphisms of $\Pi^\alpha_{c}$ as a $\mathcal{M}_c$-representation. It is well-known that $U_\alpha := \text{Coinv}_{B^\alpha} (\Pi^\alpha_c)$, as a bi-module over $\mathcal{H}_{q,c}^\alpha$, is isomorphic to the regular representation. In a sense, Theorem 7.9 generalizes this result to the affine case.

The functor $\Pi \mapsto \text{Coinv}_{I^0} (\Pi)$ on $\text{Rep}_c (\hat{G})$ can be factored into two steps. We first apply the functor
\[
\text{Rep}_c (\hat{G}) \to \text{Rep}_c (\text{Vect}) \text{ Coinv}_{I^0} \to \text{Rep}_c (\mathcal{M}_\alpha, \text{Vect}),
\]
where the first arrow is the forgetful functor, and then apply
\[
\text{Coinv}_{B^\alpha} : \text{Rep}_c (\mathcal{M}_\alpha, \text{Vect}) \to \text{Vect}.
\]
In particular, endomorphisms of the latter functor map to endomorphisms of the composition. As a result, we obtain the canonical embedding $\mathcal{H}_{q,c}^\alpha \to \mathcal{H}_{q,c}$. Recall also that the group-algebra $\mathbb{C}[\Lambda]$ is canonically a subalgebra in $\mathcal{H}_{q,c}$, contained in each $\mathcal{H}_{q,c}^\alpha$.

8.2. The strategy of the proof of Theorem 7.9 will be as follows. We will endow the vector space $U_c$ with an increasing filtration
\[
U_c = \bigcup_{w \in W_{aff}} U_w
\]
with $U_{w_1} \subset U_{w_2}$ if and only if $w_1 \leq w_2$ in the Bruhat order. This filtration will be stable under the action of $\mathbb{C}[\Lambda] \subset \mathcal{H}_{q,c}$ with respect to both actions of the latter on $U_c$.

The subquotients
\[
U^w := U_w / \bigcup_{w' < w} U_{w'}
\]
will be free $\Lambda$-modules of rank 1 (with respect to each of the actions of $\mathcal{H}_{q,c}$). In particular, for $w = 1$, the space $U^1 \simeq U_1$ will contain a canonical element $1_{U_1} \in U^1$, which generates $U_1$ under each of the $\Lambda$-actions. This will be the element $1_{U_c}$ of Theorem 7.9.

Moreover, the following crucial property will be satisfied. Suppose that $w$ is an element of $W_{aff}$, and $s_\alpha$ is a simple affine reflection, such that $s_\alpha \cdot w > w$ (resp., $w \cdot s_\alpha > w$). Then the subquotient
(20) $U_{s_\alpha \cdot w} / \bigcup_{w' < s_\alpha \cdot w, w' \neq w} U_{w'}$, (resp., $U_{w \cdot s_\alpha} / \bigcup_{w' < w \cdot s_\alpha, w' \neq w} U_{w'}$)
is stable under the action of $\mathcal{H}_{q,c}^\alpha$, embedded into the first (resp., second) copy of $\mathcal{H}_{q,c}$, and as a $\mathcal{H}_{q,c}^\alpha$-module, it is isomorphic to $\mathcal{H}_{q,c}^\alpha \otimes \mathbb{C}[\Lambda]$.

The existence of a filtration with the above properties clearly implies the assertion of the theorem.

8.3. Let $I^0$ denote the (pro)-unipotent radical of $I$; we have $I^{00} / I^0 \simeq T^0$, where $T^0 \subset T$ is the maximal compact subgroup of $T$.

Consider the scheme $\mathcal{G} := \text{Bun}^{\infty}_{\mathcal{G}} / I_{x_2}$, called the thick Grassmannian of $G$. By definition, it classifies principal $G$-bundles on $X = P^1$, endowed with a trivialization at the formal neighbourhood of $x_1$ and a reduction to $B$ of their fiber at $x_2$. Consider also the base affine...
space \( \widetilde{S}_G := \text{Bun}_G^\infty / I_{x_1} \), which is a principal \( T \)-bundle over \( S_G \). The loop group \( G((t)) \), where \( t \) is the coordinate near \( x_1 \), acts naturally on both \( S_G \) and \( \widetilde{S}_G \).

It is well-known that \( S_G \) can be written as a union of open sub-schemes \( S_{G,w}, w \in W_{aff} \), each being stable under the action of \( I_{x_1} = I \subset G((t)) \), such that \( S_{G,w_1} \subset S_{G,w_2} \) if and only if \( w_1 < w_2 \) in the Bruhat order. Let us denote by \( \mathcal{G}_G^w \) the locally closed sub-scheme \( S_{G,w} - \bigcup_{w' < w} S_{G,w'} \), and by \( \widetilde{S}_G^w, \mathcal{G}_G^w \) the corresponding sub-schemes in \( \widetilde{S} \). It is well-known that the group \( I^0 \) (resp., \( I \)) acts transitively on each \( \mathcal{G}_G^w \) (resp., \( \widetilde{S}_G^w \)) with finite-dimensional unipotent stabilizers. Choosing a point in each \( \mathcal{G}_G^w \), we will denote by \( N_w \) its stabilizer in \( I \), or, which is the same, the stabilizer in \( I^0 \) of the projection of this point to \( S_{G,w} \).

Consider the stack
\[
\text{Bun}_G^{x_1,x_2} := \text{Bun}_G^\infty / (I_{x_1}^0 \times I_{x_2}^0) \simeq \widetilde{S}_G / I^0.
\]

By definition, it classifies \( G \)-bundles on \( X = \mathbb{P}^1 \) with a reduction to the maximal unipotent at \( x_1 \) and \( x_2 \), and it carries a natural action of the group \( T \times T \). From the above discussion, we obtain that \( \text{Bun}_G^{x_1,x_2} \) can be canonically written as a union of open sub-stacks of finite type
\[
\text{Bun}_G^{x_1,x_2} = \bigcup_{w \in W_{aff}} \mathcal{Y}_w
\]
with \( \mathcal{Y}_{w_1} \subset \mathcal{Y}_{w_2} \) if and only if \( w_1 \leq w_2 \).

Consider the locally-closed sub-stack \( \mathcal{Y}_w := \mathcal{Y}_w - \bigcup_{w' < w} \mathcal{Y}_{w'} \). We obtain that \( \mathcal{Y}_w \) is isomorphic to \( T \times (\text{pt} / N_w) \), where \( N_w \) as above. The first copy of \( T \) acts via multiplication on the first factor, and the action of the second copy is twisted by the projection of \( w \) to the finite Weyl group, acting by automorphisms on \( T \).

We will denote by \( \mathcal{Y}_{w_1}, \mathcal{Y}_{w_2} \) the pull-back of the total space of the \( G_m \)-torsor corresponding to \( \mathcal{L}_{\text{Bun}_G} \) to these sub-stacks.

8.4. We have:
\[
U_c \simeq \lim_{w} \left( \text{Distr}_c^I(\mathcal{Y}_w) \otimes \mathbb{C} \right)_{T^0 \times T^0 \times G_m}.
\]
Set
\[
U_w := (\text{Distr}_c^I(\mathcal{Y}_w) \otimes \mathbb{C})_{T^0 \times T^0 \times G_m}.
\]

We claim that each \( U_w \) maps injectively into \( U_c \); and the images of \( U_w \) define a filtration with the required properties. One thing is clear, however: by construction, \( U_w \) carries an action of \( \Lambda \times \Lambda \), and its map to \( U_c \) is compatible with this action.

8.5. To proceed we need to introduce some more notation. Let \( Z \) be a smooth scheme, and let \( \mathcal{L} \) be a line bundle on \( Z \). Let \( \mathcal{L} \) denote the total space of the corresponding \( G_m \)-torsor over \( Z \). We will denote by \( \text{Distr}_c^I(\mathcal{L})_{\mathcal{L}} \) the space
\[
\left( \text{Distr}_c^I(\mathcal{L}) \otimes \mathbb{C} \right)_{G_m},
\]
where \( G_m \) acts on \( \mathbb{C} \) via the standard character \( G_m \to \mathbb{Z} \xrightarrow{1 \mapsto i} \mathbb{C}^* \).

Let now \( Z_1 \subset Z \) be a smooth closed sub-scheme, and let \( Z_2 \) be its complement. We have:
Lemma 8.6. There exists a natural short exact sequence:

$$0 \to \text{Distr}^I_c(Z_2) \to \text{Distr}^I_c(Z) \to \text{Distr}^I_c(Z_1) \to 0,$$

where $\mathcal{L}_n$ is the top power of the normal bundle to $Z_1$ inside $Z$.

Proof. Note that by definition we have

$$\text{Distr}^I_c(Z)_{\mathcal{L}_0} \simeq \text{Funct}^I_c(Z),$$

where $\mathcal{L}_0$ is the inverse of the line bundle of top forms on $Z$. The assertion of the lemma follows now from the fact that for any $Z_1 \subset Z$ we have a short exact sequence for the corresponding spaces of locally constant functions with compact support:

$$0 \to \text{Funct}^I_c(Z_2) \to \text{Funct}^I_c(Z) \to \text{Funct}^I_c(Z_1) \to 0.$$

For each $w < w'$, the open embedding $Y_w \hookrightarrow Y_{w'}$ can be covered by an open embedding of schemes $Z_w \hookrightarrow Z_{w'}$, such that $Y_{w'} = Z_{w'}/N$, $Y_w = Z_w/N$ with $N$ being a unipotent algebraic group. Therefore, by Lemma 8.6 and the exactness of the functor Coinv, the map $U_w \to U_{w'}$ is an embedding. Hence, $U_w \to U_c$ is also an embedding.

Moreover, we claim that from Lemma 8.6 we obtain a (non-canonical) isomorphism

$$(22) \quad U^w \simeq (\text{Distr}^I_c(Y^w))_{T^0 \times T^0},$$

compatible with the $\Lambda \times \Lambda$-action.

Indeed, a priori, $U^w \simeq \left((\text{Distr}^I_c(Y^w))\right)_{T^0 \times T^0}$ for a certain $T \times T$-equivariant line bundle $\mathcal{L}$ on $Y^w$. However, from the description of $Y^w$ as $T \times (\text{pt}/N_w)$, this line bundle is (non-canonically) trivial. Note, however, that this line bundle is canonically trivial for $w = 1$.

Now, the same description of $Y^w$ implies that $\text{Distr}^I_c(Y^w) \simeq \text{Funct}^I_c(T)$, with the first action of $T$ being given by multiplication, and the second action is twisted by $w$. This implies that $U^w \simeq (\text{Funct}^I_c(T))_{T^0 \times T^0} \simeq \mathbb{C}[\Lambda]$.

8.7. We will now study the subquotient $U_{s_{\alpha} \cdot w}/\bigcup_{w' < s_{\alpha} \cdot w, w' \neq w} U_{w'}$, where $s_{\alpha}$ is a simple affine reflection such that $s_{\alpha} \cdot w > w$. (The case $w \cdot s_{\alpha} > w$ is analyzed similarly.)

Note first of all that for any $w' \in W_{aff}$, we have $I_{\alpha} \cdot G_{w'} \subseteq G_{w'} \cup G_{-w'}$. Hence, the open subset $G_{w', w'} = I_{\alpha}$-stable, and so is the union $\bigcup_{w' < s_{\alpha} \cdot w, w' \neq w} G_{w'}$. Therefore, the subquotient in

(20)

is indeed $H^x_{q,c}$-stable.

We will consider two additional stacks. One is $\acute{\gamma} := Bun_G^{\infty, x_1, x_2} / (N(I_{\alpha})_{x_1} \times I^0_{x_2})$, on which we have an action of $M'_{\alpha}$. We will denote by $pr$ the projection

$$\acute{\gamma} \overset{pr}{\to} \acute{\gamma}/N_{\alpha} \simeq Bun_{x_1, x_2}^G,$$

where $N_{\alpha} := B_{\alpha} \cap I^0$.

Another stack is the quotient

$$\acute{\gamma} := Bun_G^{\infty, x_1, x_2} / (M'_{\alpha} \times I^0).$$

The stack $\acute{\gamma}$ can be written as a union of open sub-stacks $\acute{\gamma}_w$ numbered by left cosets of $\{1, s_{\alpha}\} \backslash W_{aff}$; we will denote by $\acute{\gamma}_w$ the corresponding locally closed sub-stacks. Let also $\acute{\gamma}_w$ and $\acute{\gamma}_w$ denote the pre-images of the corresponding sub-stacks in $\acute{\gamma}$, and $\acute{\gamma}_w$, $\acute{\gamma}_w$ the total spaces of the $G_m$-torsors, corresponding to the pull-backs of the line bundle $\mathcal{L}_{Bun_G}$. 

If $w$ is an element of $W_{aff}$ we have:
\[(23)\]
\[\tilde{\gamma} w = pr^{-1}(\gamma w \cup \gamma x_\alpha w).\]

Using Lemma 8.6, the subquotient (20) is isomorphic to
\[
\left(\text{Distr}^{lc}(\tilde{\gamma} w) \otimes \mathbb{C}\right)_{(\mathcal{B}_0^w)_{x_1} \times (\mathbb{T}^e)_{x_2} \times G_m}.
\]
The vector space \[
\left(\text{Distr}^{lc}(\tilde{\gamma} w) \otimes \mathbb{C}\right)_{(\mathbb{T}^e)_{x_2} \times G_m}
\]
is naturally a representation of the group $\mathcal{M}_\alpha$.

We claim that as such,
\[(24)\]
\[
\left(\text{Distr}^{lc}(\tilde{\gamma} w) \otimes \mathbb{C}\right)_{(\mathbb{T}^e)_{x_2} \times G_m} \simeq \Pi^q.
\]

Clearly, the above isomorphism implies our assertion about the action of $\mathcal{H}_{q,c'}^{c'}$ on the subquotient in (20).

8.8 To prove (24) let us observe that $\tilde{\gamma} w \simeq \gamma / M'_\alpha$ and that $\tilde{\gamma} w \simeq pt / N_{w,\alpha}$, where $N_{w,\alpha}$ is a unipotent group, so that the Cartesian product
\[
pt \times \tilde{\gamma} w
\]
is isomorphic to $M'_\alpha$, and the action of $N_{w,\alpha}$ on $M'_\alpha$ comes from a surjective homomorphism $N_{w,\alpha} \to N_\alpha$ and the action of the latter on $M'_\alpha$ by right multiplication. Hence,
\[
\text{Distr}^{lc}(\tilde{\gamma} w) \simeq \left(\text{Distr}^{lc}(\mathcal{M}_\alpha)\right)_{N_\alpha} \simeq \left(\text{Distr}^{lc}(\mathcal{M}_\alpha / N_\alpha)\right),
\]
implying (24).

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