Completeness of photon-added squeezed vacuum and one-photon states and of photon-added coherent states on a circle

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Abstract

A discrete completeness relation and a continuous one with a positive measure are found for the photon-added squeezed vacuum states. Extension to the photon-added squeezed one-photon states is considered. Photon-added coherent states on a circle are introduced. Their normalization and unity resolution relation are given.

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1 Introduction

Coherent states of the harmonic oscillator are known to have properties similar to those of the classical radiation field [1, 2, 3]. There also exist states of the electromagnetic field whose properties, such as squeezing [4], higher-order squeezing [3], antibunching [3], and sub-Poissonian statistics [3], are strictly quantum mechanical in nature. These states are called nonclassical. Since the introduction of squeezed coherent states in the early seventies [8], many nonclassical states of the radiation field have been constructed.

Among the latter, a class of states has attracted an ever increasing attention: the so-called photon-added (or excited) states, which are obtained by repeated application of photon creation operators on a given state and are distinct from displaced or squeezed number states (for a review of the latter see e.g. [3]). The earliest example was the photon-added coherent states (PACS), introduced by Agarwal and Tara [10]. They were soon followed by the photon-added squeezed vacuum state (PASVS), constructed by Zhang and Fan [4]. Since then photon-added squeezed coherent states [2], photon-added thermal states [3], photon-added even (PAECS) and odd (PAOCS) coherent states [14], for instance, have been considered. Various methods of generating such states have also been proposed [10, 15].

Some photon-added states have been interpreted in the context of nonlinear coherent states related to a deformed oscillator [16] or their generalizations [18]. Such is the case for the PACS and the PASVS, which were shown to be nonlinear coherent states [18] and even nonlinear coherent states [19], respectively. Similarly, the photon-added squeezed one-photon states (PASOPS) may be considered as odd nonlinear coherent states [19].

Most of the theoretical studies of photon-added states have concentrated so far on displaying their nonclassical properties, while leaving aside more fundamental questions, such as their completeness. It should be stressed however that proving their completeness is important both from theoretical and applied viewpoints. This property, together with normalizability and continuity in the label, indeed makes them qualify as generalized coherent states according to Klauder’s prescription [2] on one hand, and allows one to use them as (nonorthogonal) bases in many applications, on the other hand.
The completeness of PACS has recently been proved by Sixdeniers and Penson [20]. In the present letter, we consider the case of the PASVS and PASOPS, as well as that of the photon-added coherent states on a circle (PACSC), generalizing the PAECS and PAOCS of Ref. [14].

2 Definition and properties of photon-added squeezed vacuum states

The PASVS are defined by [11]

$$|\zeta, m\rangle = [N_m(|\zeta|)]^{-1/2} (a^\dagger)^m |\zeta\rangle,$$

where $m = 0, 1, 2, \ldots$, $a^\dagger$, $a$ are photon creation and annihilation operators satisfying the relation $[a, a^\dagger] = I$, $N_m(|\zeta|)$ is some normalization coefficient, and

$$|\zeta\rangle = S(|z\rangle)|0\rangle,$$

with $|0\rangle$ the vacuum state (i.e., $a|0\rangle = 0$). In (2), $S(z)$ is the squeezing operator

$$S(z) = e^{\frac{1}{4} [z(a^\dagger)^2 - z^2 a^2]} = e^{\frac{1}{4} \zeta^2 (a^\dagger)^2} \left(1 - |\zeta|^2\right)^{\frac{1}{2}} e^{-\frac{1}{2} \zeta a^2},$$

where $N = a^\dagger a$ is the number operator and $\zeta$ is related to $z$ through the relations

$$z = re^{i\phi}, \quad \zeta = \tanh re^{i\phi}.$$

Hence, $\zeta$ is restricted to the unit disc ($|\zeta| < 1$) when $z$ runs over the complex plane. In explicit form, the squeezed vacuum state (2) can be rewritten as

$$|\zeta\rangle = \left(1 - |\zeta|^2\right)^{1/4} e^{\frac{1}{2} \zeta (a^\dagger)^2} |0\rangle.$$

In the limit $\zeta \to 0$ (resp. $m \to 0$), the state $|\zeta, m\rangle$ reduces to the number state $|m\rangle = (m!)^{-1/2} (a^\dagger)^m |0\rangle$ (resp. the squeezed vacuum state $|\zeta\rangle$).

From (5), it follows that the expansion of the states (2) in the number-state basis $|n\rangle$, $n = 0, 1, 2, \ldots$, is given by

$$|\zeta, m\rangle = [N_m(|\zeta|)]^{-1/2} \left(1 - |\zeta|^2\right)^{1/4} \sum_{k=0}^{\infty} \frac{\sqrt{(2k + m)!}}{k!} \left(\frac{1}{2} \zeta\right)^k |2k + m\rangle.$$
Hence, for a given $m$ value, the states $|\zeta, m\rangle$ belong to the subspace $\mathcal{F}_\mu^{(m)}$ of Fock space $\mathcal{F}$, spanned by the states $|2k + m\rangle$, $k = 0, 1, 2, \ldots$, with a photon number not less than $m$ and of the same parity as $m = \mu \mod 2$.

The overlap $\langle \xi, n|\zeta, m\rangle$ of two PASVS vanishes except if $|n - m|$ is an even integer. If $n - m$ is a nonnegative even integer, the overlap can be written in any one of the three following equivalent forms,

$$
\langle \xi, n|\zeta, m\rangle = [N_m(|\zeta|)N_n(|\xi|)]^{-1/2} \left[ (1 - |\zeta|^2) \left( 1 - |\xi|^2 \right) \right]^{1/4} \frac{n!}{(n-m/2)!} \left( \frac{1}{2} \zeta \right)^{(n-m)/2} \left( 1 - \zeta \right)^{-2(n-m)/2}
$$

$$
\times \binom{2F1}{n+1/2, n+2/2; n-m/2 + 1; \xi \zeta}
$$

$$
= [N_m(|\zeta|)N_n(|\xi|)]^{-1/2} \left( \zeta \zeta \right)^{-1/4} \left[ (1 - |\zeta|^2) \left( 1 - |\xi|^2 \right) \right]^{1/4} \left( 1 - \xi \zeta \right)^{-(n+m)/2}
$$

$$
\times \binom{2F1}{-n-1/2, -n-2/2; -n-m/2 + 1; \xi \zeta}
$$

$$
= [N_m(|\zeta|)N_n(|\xi|)]^{-1/2} \left( \zeta \zeta \right)^{-1/4} \left[ (1 - |\zeta|^2) \left( 1 - |\xi|^2 \right) \right]^{1/4} \left( 1 - \xi \zeta \right)^{(n-m)/2}
$$

$$
\times \binom{P}{(m-n)/2; (1 - \xi \zeta)^{-1/2}}
$$

(7)

where $\langle \xi|\zeta\rangle$ is the overlap of two squeezed vacuum states,

$$
\langle \xi|\zeta\rangle = \left[ (1 - |\zeta|^2) \left( 1 - |\xi|^2 \right) \right]^{1/4} \left( 1 - \xi \zeta \right)^{-1/2}.
$$

(8)

The first equality in (7) directly follows from (6) and the definition of the hypergeometric function $2F1(a, b; c; z)$, while the other two equalities result from well-known properties of the latter and of Legendre functions of the first kind $P^\mu_n(z)$ [21, 22]. If $n - m$ is a negative even integer, the corresponding overlap can be deduced from (6) by using the Hermiticity property $\langle \xi, n|\zeta, m\rangle = \langle \zeta, m|\xi, n\rangle$

As a special case of (7), we get back the overlap $\langle \zeta, n|\zeta, m\rangle$ determined in [11] by a different method. We also obtain the normalization coefficient of the PASVS,

$$
N_m(|\zeta|) = m! \left( 1 - |\zeta|^2 \right)^{-m/2} P_m \left( (1 - |\zeta|^2)^{-1/2} \right)
$$

(9)

in terms of Legendre polynomials.

The states $|\zeta, m\rangle$ are distinct from the squeezed number states [1]

$$
|m, \zeta\rangle = S(z)|m\rangle, \quad m = 0, 1, 2, \ldots.
$$

(10)
Contrary to the former, the latter are defined in a subspace of Fock space \( \mathcal{F} \) including photon numbers less than \( m \), namely the subspace \( \mathcal{F}_\mu \) of even or odd number states according to whether \( m \) is even (\( \mu = 0 \)) or \( m \) is odd (\( \mu = 1 \)). Since \( S(z) \) is a unitary operator, the set of squeezed number states, corresponding to a given \( z \) or \( \zeta \) value and \( m = 0, 1, 2, \ldots \), is an orthogonal basis of \( \mathcal{F} \):

\[
\langle n, \zeta | m, \zeta \rangle = \delta_{n,m},
\]

(11)

\[
\sum_{m=0}^{\infty} |m, \zeta \rangle \langle m, \zeta| = I.
\]

(12)

Any PASVS can be expressed as a linear combination of squeezed number states

\[
|m, \zeta\rangle = \left[ (1 - |\zeta|^2)^{m/2} P_m \left( (1 - |\zeta|^2)^{-1/2} \right)^{-1/2} \sqrt{m!} \right. \\
\left. \times \sum_{k=0}^{m} \frac{1}{2} \left[ 1 + (-1)^{m-k} \right] \frac{\bar{\zeta}^{(m-k)/2}}{(m-k)!! \sqrt{k!}} |k, \zeta\rangle, \right.
\]

(13)

and conversely

\[
|\zeta, m\rangle = \sqrt{m!} \sum_{k=0}^{m} \frac{1}{2} \left[ 1 + (-1)^{m-k} \right] \left[ (1 - |\zeta|^2)^{k/2} P_k \left( (1 - |\zeta|^2)^{-1/2} \right) \right]^{-1/2} \\
\left. \times \frac{(-\bar{\zeta})^{(m-k)/2}}{(m-k)!! \sqrt{k!}} |\zeta, k\rangle. \right.
\]

(14)

In proving (13), we used the property \( S^{-1}(z)a^\dagger S(z) = \left( a^\dagger + \bar{\zeta} a \right) / \sqrt{1 - |\zeta|^2} \), resulting from Baker-Campbell-Hausdorff formula, and equation (2.1) of [11]. We conclude that the set of PASVS corresponding to a given \( \zeta \) value and \( m = 0, 1, 2, \ldots \) forms a nonorthogonal basis of \( \mathcal{F} \).

### 3 Completeness of photon-added squeezed vacuum states

We may consider two different types of completeness or resolution of unity for the PASVS: one in \( \mathcal{F} \), obtained for a given \( \zeta \) by summing over the discrete label \( m \), and the other in \( \mathcal{F}_\mu^{(m)} \), obtained for a given \( m \) by integrating over the continuous label \( \zeta \).
The former directly follows from the unity resolution relation (12) for the squeezed number states and the relation (14) between the latter and the PASVS:

\[
(1 - |\zeta|^2)^{-1/2} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} (1 + \delta_{n,m})^{-1} \left[ (n!/m!) P_m \left( (1 - |\zeta|^2)^{-1/2} \right) \right] \left[ (1 - |\zeta|^2)^{-1/2} \right]^{1/2} \\
\times P_{(m+n)/2}^{(m-n)/2} \left( (1 - |\zeta|^2)^{-1/2} \right) \left[ (-e^{-i\phi})^{(m-n)/2} |\zeta, m\rangle \langle \zeta, n| + (-e^{i\phi})^{(n-m)/2} |\zeta, n\rangle \langle \zeta, m| \right] \\
= I. 
\] (15)

It has a nondiagonal form characteristic of a nonorthogonal basis, with coefficients given by the elements of the overlap matrix inverse.

The derivation of the latter is more involved. The problem amounts to determining a positive measure \( d\rho_m(\zeta, \bar{\zeta}) \) such that

\[
\int d\rho_m(\zeta, \bar{\zeta}) |\zeta, m\rangle \langle \zeta, m| = I^{(m)}, 
\] (16)

where the integration is carried out over the unit disc and \( I^{(m)} \equiv \sum_{k=0}^{\infty} |2k + m\rangle \langle 2k + m| \) denotes the unit operator in \( F^{(m)} \).

Making the polar decomposition \( \zeta = |\zeta|e^{i\phi} \), given in (10), and the ansatz

\[
d\rho_m(\zeta, \bar{\zeta}) = m! (1 - y)^{-(m+1)/2} P_m \left( (1 - y)^{-1/2} \right) h_m(y) d^2\zeta, \\
y = |\zeta|^2, \\
d^2\zeta \equiv |\zeta|d|\zeta|d\phi, 
\] (17)

and using the expansion (10), we find after integrating over \( \phi \) that equation (16) is equivalent to the set of conditions

\[
\int_0^1 dy y^k h_m(y) = \frac{[(2k)!!]^2}{\pi (m+k)!}, \quad k = 0, 1, 2, \ldots. 
\] (18)

Consequently, the requirement that for a given \( |m\), \( |\zeta, m\rangle \) form a complete (in fact, overcomplete) set in \( F^{(m)}_\mu \) is equivalent to the resolution of a power-moment problem [23].

As is usual in such a problem (see e.g. [20, 24]), it is convenient to set for complex \( s \) and \( \text{Re} s > 0, k \to s - 1, \) to define

\[
g_m(y) = \begin{cases} 
  h_m(y) & \text{if } 0 < y < 1 \\
  0 & \text{if } 1 < y < \infty 
\end{cases}, 
\] (19)
and to interpret (18) as the Mellin transform $g_m^*(s)$ of $g_m(y)$ \cite{25},
\[ \int_0^\infty dy y^{s-1} g_m(y) = g_m^*(s) \equiv \begin{cases} \frac{1}{2\pi} B\left(s, \frac{1}{2}\right) & \text{if } m = 1 \\ \frac{1}{4\pi(m-2)!} B\left(s, \frac{m}{2}\right) B\left(s, \frac{m-1}{2}\right) & \text{if } m = 2, 3, \ldots \end{cases} \]  
(20)

In (20), $B(z, w)$ denotes the beta function, i.e., $B(z, w) = \Gamma(z)\Gamma(w)/\Gamma(z + w)$. To find $g_m(y)$, we must perform the inverse Mellin transform on $g_m^*(s)$.

For $m = 1$, $g_1(y)$ is given in tables of Mellin transforms \cite{22}, leading to the following result for $h_1(y)$:
\[ h_1(y) = \frac{1}{2\pi} (1 - y)^{-1/2}. \]  
(21)

This is a positive function on $(0, 1)$, increasing from $1/(2\pi)$ to $+\infty$ when $y$ goes from 0 to 1.

For higher $m$ values, it is convenient to use the Mellin convolution property of inverse Mellin transforms, which states that if $g_m^*(s) = g_{m1}^*(s)g_{m2}^*(s)$ and $g_{m1}(y), g_{m2}(y)$ exist, then the inverse Mellin transform of $g_m^*(s)$ is
\[ g_m(y) = \int_0^\infty \frac{dt}{t} g_{m1}\left(\frac{y}{t}\right) g_{m2}(t). \]  
(22)

In applying (22) to (20) for $m \geq 2$, we choose $g_{m1}^*(s) = [4\pi(m-2)!]^{-1}B(s, \frac{m}{2})$ and $g_{m2}^*(s) = B(s, \frac{m-1}{2})$, which identifies $g_{m1}(y)$ and $g_{m2}(y)$ as
\[ g_{m1}(y) = \begin{cases} [4\pi(m-2)!]^{-1}(1 - y)^{(m-2)/2} & \text{if } 0 < y < 1 \\ 0 & \text{if } 1 < y < \infty \end{cases}, \]  
(23)

and
\[ g_{m2}(y) = \begin{cases} (1 - y)^{(m-3)/2} & \text{if } 0 < y < 1 \\ 0 & \text{if } 1 < y < \infty \end{cases}, \]  
(24)

respectively \cite{22}. Hence we obtain
\[ g_m(y) = [4\pi(m-2)!]^{-1} \int_0^1 dt t^{-m/2}(t - y)^{(m-2)/2}(1 - t)^{(m-3)/2} \]  
(25)

for $0 < y < 1$, and $g_m(y) = 0$ for $1 < y < \infty$. It is obvious that the right-hand side of (25) is a positive function, thus providing a solution for $h_m(y)$ for $m \geq 2$. 
To express the latter in terms of known functions, we introduce a new variable $u = (1 - t)/(1 - y)$, thereby obtaining

$$h_m(y) = [4\pi(m - 2)!]^{-1} (1 - y)^{m - \frac{3}{2}} \int_0^1 du \, (1 - u) \left(1 - (1 - y)u\right)^{-m/2},$$

$m = 2, 3, \ldots$. (26)

Formula 3.197.3 of [26] then leads to

$$h_m(y) = \frac{1}{2\pi(2m - 3)!!} (1 - y)^{m - \frac{3}{2}} {\binom{m}{2}}_0 \binom{m - 1}{2} \left(1 - y\right)^{-1/2}, \quad m = 2, 3, \ldots. \quad (27)$$

By using formula 3.2.5 in volume 1 of [21], we can rewrite $h_m(y)$, $m \geq 2$, in terms of a Legendre function of the second kind $Q^\mu_\nu(z)$, for which $\mu = 0$ and $\nu$ is a nonnegative integer,

$$h_m(y) = \frac{1}{2\pi(2m - 3)!!} (1 - y)^{(m - 2)/2} Q_{m - 2} \left(1 - y\right)^{-1/2}, \quad m = 2, 3, \ldots. \quad (28)$$

Such a function can be expressed in terms of Legendre polynomials combined with a logarithmic function [21]:

$$Q_0 \left(1 - y\right)^{-1/2} = \frac{1}{2} \ln \frac{1 + \sqrt{1 - y}}{1 - \sqrt{1 - y}}, \quad (29)$$

$$Q_{m - 2} \left(1 - y\right)^{-1/2} = \frac{1}{2} P_{m - 2} \left(1 - y\right)^{-1/2} \ln \frac{1 + \sqrt{1 - y}}{1 - \sqrt{1 - y}}$$

$$- \sum_{k=0}^{[m-3/2]} \frac{2m - 4k - 5}{(m - k - 2)(2k + 1)} P_{m - 2k - 3} \left(1 - y\right)^{-1/2}, \quad m = 3, 4, \ldots. \quad (30)$$

Here $[x]$ denotes the largest integer contained in $x$.

For the first few values of $m$, we find

$$h_2(y) = \frac{1}{4\pi} \ln \frac{1 + \sqrt{1 - y}}{1 - \sqrt{1 - y}}, \quad (31)$$

$$h_3(y) = \frac{1}{4\pi} \left( \ln \frac{1 + \sqrt{1 - y}}{1 - \sqrt{1 - y}} - 2\sqrt{1 - y} \right), \quad (32)$$

$$h_4(y) = \frac{1}{16\pi} \left[ (2 + y) \ln \frac{1 + \sqrt{1 - y}}{1 - \sqrt{1 - y}} - 6\sqrt{1 - y} \right], \quad (33)$$

$$h_5(y) = \frac{1}{144\pi} \left[ 3(2 + 3y) \ln \frac{1 + \sqrt{1 - y}}{1 - \sqrt{1 - y}} - 2(11 + 4y)\sqrt{1 - y} \right]. \quad (34)$$
From (28), (29), and (30), it can be shown that $h_m(y) \to +\infty$ or 0 according to whether $y \to 0$ or 1. This is confirmed by Fig. 1, which displays $h_m(y)$ for several $m$ values.

Having found a solution for the problem stated in (18), we may now ask whether this solution is unique. An answer is provided by the (sufficient) condition of Carleman [23]: if

$$S \equiv \sum_{k=1}^{\infty} a_k, \quad a_k \equiv \left( \frac{[(2k)!]^2}{\pi(m+2k)!} \right)^{-1/(2k)},$$

(35)
diverges, then the solution is unique. The convergence of $S$ can be tested by applying the logarithmic test [22]: if $\lim_{k \to \infty} [\ln(a_k)/\ln(k)] > -1$, then $S$ diverges. By using Stirling formula for the asymptotic form of $\Gamma(z)$ [21], we obtain $\lim_{k \to \infty} [\ln(a_k)/\ln(k)] = 0$. We conclude that $h_m(y)$ given in (21) and (28) is the unique solution to the problem.

4 Extension to the photon-added squeezed one-photon states

The PASOPS are defined by [19]

$$|1, \zeta, m\rangle = [N_{1m}(|\zeta|)]^{-1/2} (a^\dagger)^m |1, \zeta\rangle,$$

(36)
where $m = 0, 1, 2, \ldots,$

$$|1, \zeta\rangle = S(z)|1\rangle = \left(1 - |\zeta|^2\right)^{3/4} e^{\frac{1}{2} \zeta (a^\dagger)^2} |1\rangle,$$

(37)
and $|1\rangle = a^\dagger |0\rangle$. In the limit $\zeta \to 0$ (resp. $m \to 0$), they reduce to the number state $|m+1\rangle$ (resp. the squeezed one-photon state $|1, \zeta\rangle$).

Their expansion in the number-state basis is given by

$$|1, \zeta, m\rangle = [N_{1m}(|\zeta|)]^{-1/2} \left(1 - |\zeta|^2\right)^{3/4} \sum_{k=0}^{\infty} \frac{\sqrt{(2k + m + 1)!}}{k!} \left(\frac{1}{2} \zeta\right)^k |2k + m + 1\rangle,$$

(38)
showing that for a given $m$ value, they belong to the same subspace $F_{\mu'}^{(m+1)} (\mu' \equiv (m + 1) \mod 2)$ of $F$ as the PASVS $|\zeta, m + 1\rangle$. We actually obtain

$$[N_{1m}(|\zeta|)]^{1/2} \left(1 - |\zeta|^2\right)^{-1/2} |1, \zeta, m\rangle = [N_m(|\zeta|)]^{1/2} |\zeta, m + 1\rangle,$$

(39)
which enables us to easily extend some of the results of the two previous sections to the PASOPS.

For instance, their overlap for \( n - m \) an even nonnegative integer and their normalization coefficient are given by

\[
\langle 1, \xi, n | 1, \zeta, m \rangle = [N_{1m}(|\zeta|)N_{1n}(|\xi|)]^{-1/2} \langle 1, \xi | 1, \zeta \rangle (n + 1)! \bar{\xi}^{(m-n)/4} \xi^{(n-m)/4} \times (1 - \bar{\xi} \zeta)^{-m-n-2/4} P_{(m+n+2)/2}^{(m-n)/2} (1 - \bar{\xi} \zeta)^{-1/2},
\]

and

\[
N_{1m}(|\zeta|) = (m + 1)! (1 - |\zeta|^2)^{-m-1/2} P_{m+1} ((1 - |\zeta|^2)^{-1/2}),
\]

respectively. In (40), \( \langle 1, \xi | 1, \zeta \rangle \) is the overlap of two squeezed one-photon states,

\[
\langle 1, \xi | 1, \zeta \rangle = [(1 - |\xi|^2)(1 - |\zeta|^2)]^{3/4} (1 - \bar{\xi} \zeta)^{-3/2}.
\]

Similarly, it can be shown that they form a nonorthogonal basis of \( F^{(1)} \) (i.e., the Fock space from which the one-dimensional subspace spanned by the vacuum state has been removed) and an (over)complete set in \( F_{\mu}^{(m+1)} \) with a positive measure given by

\[
d\rho_{1m}(\zeta, \bar{\zeta}) = (m + 1)! (1 - y)^{-m-2/2} P_{m+1} ((1 - y)^{-1/2}) h_{1m}(y) d^2\zeta, \quad y \equiv |\zeta|^2,
\]

\[
h_{1m}(y) = \frac{1}{2\pi(m - 1)!} (1 - y)^{(m-1)/2} Q_{m-1} ((1 - y)^{-1/2}), \quad m = 1, 2, \ldots.
\]

### 5 Definition and completeness of photon-added coherent states on a circle

A special class of multiphoton coherent states is provided by the eigenstates of a power \( a^\lambda \) \((\lambda = 2, 3, 4, \ldots)\) of the photon annihilation operator [27, 28, 29], satisfying the relation

\[
a^\lambda |z, \mu\rangle = z|z, \mu\rangle, \quad \mu = 0, 1, \ldots, \lambda - 1.
\]

Here \( \mu \) distinguishes between the \( \lambda \) orthogonal solutions of (44), belonging to the subspaces \( F_{\mu} \) of Fock space \( F \) spanned by the number states \(|k\lambda + \mu\rangle, k = 0, 1, 2, \ldots: \)

\[
|z, \mu\rangle = [N_{\mu}(|z|)]^{-1/2} \sum_{k=0}^{\infty} \left( \frac{\mu!}{(k\lambda + \mu)!} \right)^{1/2} z^k |k\lambda + \mu\rangle,
\]

\[
N_{\mu}(|z|) = a F_{\lambda-1} \left( \frac{1}{\lambda} + 1, \frac{2}{\lambda} + 1, \ldots, \frac{\mu}{\lambda} + 1, \frac{\mu + 1}{\lambda}, \frac{\mu + 2}{\lambda}, \ldots, \frac{\lambda - 1}{\lambda}; y \right),
\]

\[
y \equiv |z|^2/\lambda^2.
\]
where \( pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) \) denotes a generalized hypergeometric function \[21\].

The states \[15\] may also be written as linear combinations of \((\text{standard})\) coherent states equidistantly separated from each other along a circle of radius \(|t| = |z|^{1/\lambda}\) \[28\],

\[
|z, \mu\rangle = [N_{\mu}(|z|)]^{-1/2} \frac{\sqrt{\mu!}}{\sqrt{\lambda}} e^{\frac{1}{2}\mu t^2} t^{-\mu} e^{-\mu|t|e^{\epsilon}} |t^\lambda, \mu\rangle, \quad t^\lambda = z, \tag{47}
\]

\[
|t^\nu\rangle = e^{-\frac{1}{2}|t|^2 + t^\nu a^\dagger} |0\rangle, \quad \epsilon \equiv e^{2\pi i/\lambda}, \tag{48}
\]

\[
N_{\mu}(|z|) = \mu! |t|^{-2\mu} h_{\mu+1}(|t|^2, \lambda), \tag{49}
\]

hence the name of coherent states on a circle that is often used for them \[30\]. In \[43\], \( h_i(x, n) \) denotes a hyperbolic function of order \( n \), i.e., a generalization of the hyperbolic cosine and sine functions to which it reduces for \( n = 2 \) and \( i = 1 \) or \( 2 \), respectively \[21\].

Let us define PACSC by the relation

\[
|z, \mu, m\rangle = [N_{\mu m}(|z|)]^{-1/2} (a^\dagger)^m |z, \mu\rangle, \tag{50}
\]

where \( m = 0, 1, 2, \ldots \), \( N_{\mu m}(|z|) \) is some normalization coefficient, and \(|z, \mu\rangle\) is given by \[14\] or \[17\]. For \( \lambda = 2 \) and \( \mu = 0 \) or \( 1 \), they reduce to the PAECS or PAOCS, respectively \[14\].

According to whether we use the expansions \[14\] or \[17\], we can express the PACSC either in the number-state basis,

\[
|z, \mu, m\rangle = [N_{\mu m}(|z|)] N_{\mu}(|z|)^{-1/2} \sum_{k=0}^{\infty} \frac{\mu!(k\lambda + m + \mu)!}{(k\lambda + \mu)!} z^k |k\lambda + m + \mu\rangle, \tag{51}
\]

or in terms of the PACS \(|t^\nu, m\rangle\) of \[21\],

\[
|z, \mu, m\rangle = [N_{\mu m}(|z|)] N_{\mu}(|z|)^{-1/2} [N_{\mu}(|t|)]^{1/2} \frac{\sqrt{\mu!}}{\sqrt{\lambda}} e^{\frac{1}{2}|t|^2} t^{-\mu} e^{-\mu|t|e^{\epsilon}} |t^\nu, m\rangle, \tag{52}
\]

where

\[
|t^\nu, m\rangle = [N_m(|t|)]^{-1/2} (a^\dagger)^m |t^\nu\rangle, \quad N_m(|t|) = m! L_m(-|t|^2), \tag{53}
\]

and \( L_m(x) \) denotes a Laguerre polynomial \[21\]. From \[51\], it is clear that for given \( \mu \) and \( m \) values, the states \(|z, \mu, m\rangle\) belong to the subspace \( F_{\mu m}^{(m+\mu)} \) of Fock space \( \mathcal{F} \) spanned by the states with photon number \( n \) not less than \( m + \mu \) and congruent with \( \mu m \), defined by \( m + \mu = \mu m \mod \lambda \).
By using methods similar to those employed in Secs. 2 and 3, it is straightforward to obtain the overlap of two PACSC and their normalization coefficient, as well as two different kinds of completeness relations. Here we only mention two of these results.

The normalization coefficient can be written either in closed form as

\[
N_{\mu m}(|z|) = [N_{\mu}(|z|)\mu!]^{-1}(m + \mu)! \lambda F_{2\lambda-1}\left(\frac{m + \mu + 1}{\lambda}, \frac{m + \mu + 2}{\lambda}, \ldots, \frac{m + \mu + \lambda}{\lambda}; \frac{1}{\lambda} + 1, \frac{2}{\lambda} + 1, \ldots, \frac{\mu + 1}{\lambda}, \frac{\mu + 2}{\lambda}, \ldots, \frac{\lambda - 1}{\lambda} + 1, \frac{2}{\lambda} + 1, \ldots, \frac{\lambda - 1}{\lambda} \right) y, \quad y \equiv |z|^2 / \lambda^2 \lambda,
\]

or as a linear combination of Laguerre polynomials,

\[
N_{\mu m}(|z|) = [\lambda N_{\mu}(|z|)]^{-1}\mu! |t|^{-2\mu} \sum_{\nu=0}^{\lambda-1} e^{-\nu y} \lambda^\nu L_m(-|t|^2) L_m(y), \quad y \equiv |z|^2 / \lambda^2 \lambda,
\]

The PACSC satisfy a unity resolution relation in \( F^{(m+\mu)}_{\mu m} \),

\[
\int d\rho_{\mu m}(z, \bar{z}) |z, \mu, m\rangle \langle z, \mu, m| = I^{(m+\mu)}_{\mu m},
\]

with a positive measure given by

\[
d\rho_{\mu m}(z, \bar{z}) = N_{\mu m}(|z|)N_{\mu}(|z|)\mu! h_{\mu m}(y) d^2 z, \quad y \equiv |z|^2 / \lambda^2 \lambda,
\]

\[
h_{\mu m}(y) = \frac{1}{\pi \lambda^{\mu+1-\mu}} y^{(\mu+1-\lambda)/\lambda} e^{-\lambda y^{1/\lambda}} U\left(m, 1, \lambda y^{1/\lambda}\right),
\]

where \( U(a, b; z) = \Psi(a, b; z) \) is Kummer’s confluent hypergeometric function \cite{21}.

6 Conclusion

In the present letter, we demonstrated that the PASVS satisfy two different types of unity resolution relations, a discrete one in \( F \) and a continuous one in \( F^{(m+\mu)}_{\mu m} \), and we extended such results to the PASOPS. In addition, we introduced the PACSC and obtained both their normalization and their continuous unity resolution relation in \( F^{(m+\mu)}_{\mu m} \).

Proving the completeness of photon-added squeezed coherent states along similar lines is a much more difficult problem since such states depend upon two continuous variables instead of one \cite{12}. We hope however to solve it in a near future.
Another interesting open question is whether completeness relations of the second type exist for photon-subtracted squeezed states. It is already clear that this is true neither for photon-subtracted squeezed vacuum states, nor for the even nonlinear coherent states proposed in [19] by extending the results for positive integer $m$ values to negative integer ones. In both cases, the states are indeed nonnormalizable in the limit $\zeta \to 0$. Photon-subtracted squeezed excited states might, on the contrary, be good candidates for the existence of completeness relations provided $m$ remains low enough.

As a final point, it is worth stressing that a central requirement of this work has been to find a continuous resolution of unity of the usual type with a positive measure. Relaxing this demand may lead to generalized unity resolution relations of the type considered in Ref. [31] for the nonclassical states studied in the present work, as well as for their extensions.
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Figure captions

Fig. 1. The weight function $h_m(y)$ as a function of $y$ for various $m$ values: (a) $m = 1$ (solid line), $m = 2$ (dashed line); (b) $m = 3$ (solid line), $m = 4$ (dashed line), $m = 5$ (dotted line).
Figure 1