A seven-point algorithm for piecewise smooth univariate minimization

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Abstract In this paper, we construct an algorithm for minimising piecewise smooth functions for which derivative information is not available. The algorithm constructs a pair of quadratic functions, one on each side of the point with smallest known function value, and selects the intersection of these quadratics as the next test point. This algorithm relies on the quadratic function underestimating the true function within a specific range, which is accomplished using a adjustment term that is modified as the algorithm progresses.

Keywords Nonsmooth optimisation · nondifferentiable programming · univariate minimisation · quadratic approximation

1 Introduction

When solving the problem of minimizing a function of many variables, many existing solutions belong to a class of algorithms called line-search algorithms [15, 3, 5] which reduce the problem by iteratively restricting the search to 1-dimensional subspaces determined by a search direction to determine a step size. This 1-dimensional problem may be solved either exactly or inexactly. Most methods rely on the latter approach, as it is computationally cheaper while no less effective.

While exact line search is rarely used, due to its cost, there exist certain special cases where it is more effective such as Linear Programming. Moreover,
it has been observed by Yu et al. [17] that in the context of non-smooth optimisation, using exact line-search may result in stepping to a location from where a better subdifferential approximation may be constructed, thus resulting in selecting better search directions at future iterations.

In this paper, motivated by designing an exact line search algorithm for non-smooth optimisation, we examine the effectiveness of existing univariate optimisation algorithms on a class of non-smooth objective functions and introduce a new algorithm which is specifically designed for this class. The particular problem for which we develop this method is a black box, piecewise smooth objective function for which derivatives are not available.

When conducting an exact line-search, there are two distinct steps. First of all we must construct a bracket, which means finding a closed interval which is guaranteed to contain a local minimum of the univariate objective function. The second is find the local minimum within this interval. In this paper, we focus mainly on the second step which we rigorously state below.

**Problem 1.1** Given is a function \( f : \mathbb{R} \to \mathbb{R} \) of the form \( f = \max_{i=1,...,k} f_i(x) \) where \( f_i : \mathbb{R} \to \mathbb{R} \) are smooth functions, an oracle for calling function values of \( f(x) \), and an interval \([a, b]\). Design an optimisation method for finding a local minimiser of \( f \) in \([a, b]\) which makes use of only oracle calls for \( f \).

One class of algorithms which solves Problem 1.1 is that of 1d global optimisers. This class includes using interval analysis based methods such as the Moore-Skelboe algorithm [13]. However, these methods require that the objective function is known, and are not applicable to black box functions.

Another category of solutions is that of global Lipschitzian methods such as those summarised by Hansen et al [7]. While these might be feasible if we could guess a suitable Lipschitz Constant, in practice there is little gained from them. Even if more than one local optimum existed, we have no reason to expect that finding one minimum as opposed to another would make an exact line search based method more effective. Therefore, while we will make use of some of the ideas behind such algorithms, we reject them as they involve needless extra work.

When choosing a local univariate solver, one faces a trade off between speed and robustness. The basic methods are Golden Section [10,4] which converges Q-linearly for any continuous function, and interpolating methods [4,9] which, when successful, converge super linearly. The latter’s stability depends on their ability to construct a polynomial which approximates the objective function well locally. If the approximation does not fit the objective function well, then these algorithms may fail.

In practice, the most effective methods are bracketing interpolation hybrid methods such as Brent’s Method [21] and Hager’s cubic method [6] which combine the speed of interpolation methods with the stability of Golden Section by making use of a fall back option when the interpolation is not working. While convergence is guaranteed for such hybrid methods, they will still need their interpolations to match the objective function well if super linear conver-
gence is to be obtained. Therefore, we can’t expect them to converge quickly
for an objective function of the form stated in Problem [1.1].

There exist other bracketing-interpolation hybrid methods for non differ-
entiable functions of the form \( f(x) = \max_i f_i(x) \) [14,17]. However, these
algorithms assume that we can compute each \( f_i \) (from which \( f \) is computed)
separately. Therefore, these are unsuitable for Problem [1.1] given that the oracle
is defined to only return the value of \( f(x) \).

The only algorithm which seems to be optimised for our setting is the
five point method of Mifflin and Strodiot [12], to which we will from now
on refer to as “the Mifflin-Strodiot method”. This algorithm is designed to
converge rapidly to the local minimum \( x^* \) even when the objective function is
non-differentiable at \( x^* \).

The remainder of this paper is structured as follows: In Section 2, we define
the notion of a bracket rigorously and describe in brief the existing methods
which are currently most applicable to our problem. Sections 3 to 5 fo-
cus on the derivation of a new univariate optimiser, for which we present three variations.
We compare our new methods against relevant competitors in Section 6.

2 Bracketing Methods

We begin by rigorously defining a bracket:

**Definition 2.1 (Bracket)** Let \( f: \mathbb{R} \to \mathbb{R} \) be a function, and \( x^L, x^M, x^R \in \mathbb{R} \). We call the trio of points \( x^L, x^M, x^R \) a **bracket** of \( f \) if they satisfy the following:

1. \( x^L < x^M < x^R \),
2. \( f(x^L) \geq f(x^M) \leq f(x^R) \).

The significance of a bracket is that Definition 2.1 guarantees that there exists
a local minimum of \( f \) in \((x^L, x^R)\).

**Definition 2.2 (Set of Brackets)** We define \( B_f \) to be the set of all brackets
for the function \( f \).

In this paper, we refer to the elements of a bracket \( X \) as \( x^L, x^M, x^R \). If a bracket \( X_i \) is associated with a particular iteration of an algorithm, then
we write its elements as \( x^L_i, x^M_i, x^R_i \). Given this notation, we define the
function \( b(X) = x^R - x^L \).

Bracketing methods are the set of algorithms which are vaguely in the form
of Algorithm 1 given the inputs of a step function \( \sigma \) and an update function
\( U \). The step function should select a new point within \((x^L, x^M) \cup (x^M, x^R)\),
while the update function should return a bracket. When discussing existing
bracketing methods and constructing our new one, we use the structure of
Algorithm 1 where one algorithm is distinguished from another based on how
the step function \( \sigma \) and update function \( U \) are defined.

The purpose of this paper is to introduce a new bracketing method called
the Underestimating Polynomial Method (UPM) for Problem [1.1] for which, to
Algorithm 1 General Bracketing Method

Require: \( f \) Objective function, \( \sigma \) Step Function, \( U \) Update Function, \( X_0 \) Initial Bracket, \( \epsilon > 0 \) Tolerance.

\( i = 0; \)
while \( b(X_i) > 2 \epsilon \) do
\( \tilde{x}_i = \sigma(X_i). \)
\( X_{i+1} = U(\tilde{x}_i, X_i). \)
\( i = i + 1; \)
end while
return \( X_i. \)

our knowledge, there does not currently exist a robust and fast solution. When assessing the effectiveness of this algorithm, we will compare it to a small selection of existing bracketing methods including Golden Section, Brent’s Method [2,1], and the Mifflin-Strodiot method [11].

Of these algorithms, Golden Section may be considered to most robust as it is guaranteed to converge Q-linearly with a rate of approximately 0.618. Meanwhile Brent’s method is the most effective for smooth functions converging super-linearly, while sometimes performing surprisingly well for non-smooth functions. Finally, Mifflin’s method is theoretically the most comparable to ours, in that it is equipped for non-smooth functions. However, we will see that it lacks the robustness of the previous two algorithms.

The idea of bracketing methods such as Brent’s method [12], and Hager’s Cubic method [6] is that the polynomial which interpolates the bracket points approximates the objective function well locally. Therefore the local minimum of the quadratic is a sensible location to evaluate next. The error of this approximation can be quantified and bounded, yielding super-linear convergence guarantees [1, Theorem 4.1]. While this analysis works well for smooth functions, our problem is a piecewise smooth, black box function.

Definition 2.3 A function \( f : \mathbb{R}^n \supseteq \Omega \rightarrow \mathbb{R} \) is called piecewise smooth if \( f \) is continuous on \( \Omega \) and there exists a disjoint finite family of sets \( \{ \Omega_i; i = 1, 2, \ldots, k \} \) such that \( f \) is smooth on \( \Omega_i \) for all \( i \in I \) and \( \bigcup_{i=1}^{k} \Omega_i \) is dense in \( \Omega \). We call any point \( x \) such that \( x \in \overline{\Omega_i} \cap \overline{\Omega_j} \) for some \( i, j \in I \) a kink.

The theory behind algorithms such as Brent’s method collapses due to the existence of kinks, because interpolating across a kink has no meaning and yields no meaningful error bound. Therefore, if the local minimum of a function \( f \) is a kink, we expect algorithms like Brent’s method to converge to it slowly.

3 Static Underestimating Polynomial Method

The premise of the UPM is inspired by the approach used in the Mifflin-Strodiot method [11]. We approximate the objective function with two polynomials, one on each side of the bracket. If the local minimum is a kink, then the combination of these polynomials may be valid approximations and useful for locating the minimum. To construct these polynomials, we need more than
the three points contained in a bracket and therefore extend the definition of a bracket to include seven points:

**Definition 3.1 (Extended Bracket)** Given the function $f$, we call $X \in \mathbb{R}^7$ an extended bracket written in the following form

$$X = (x_3^L, x_2^L, x_1^L, x^M, x_2^R, x_1^R, x_3^R)^T,$$  \hspace{1cm} (3.1)

if the following conditions apply:

$$x_3^L < x_2^L < x_1^L < x^M < x_2^R < x_1^R < x_3^R$$ \hspace{1cm} (3.2a)

$$f(x_3^L) \geq f(x^M) \leq f(x_1^R).$$ \hspace{1cm} (3.2b)

**Remark 3.1** If $X$ is an extended bracket, then $(x_1^L, x^M, x_1^R)$ form a bracket.

For expressing an extended bracket’s size, we define the following functions.

**Definition 3.2 (Extended Bracket Length)** Let $X$ be an extended bracket of $f$. We define:

$$\text{diam}(X) = \text{diam(Conv}(X))),$$

$$b(X) = x_1^R - x_1^L.$$  \hspace{1cm} (3.3)

**Remark 3.2** Definition 3.2 is the natural extension of the function $b(X)$ from Section 2 where $X$ is merely a bracket as opposed to an extended bracket.

The UPM mostly conforms to the form of Algorithm 1. The main difference, apart from replacing the bracket with an extended bracket, is the fact that the UPM also depends on an input parameter $\alpha$. This parameter is used by the step function $\sigma$ when constructing the next point to evaluate. The three variations of the UPM presented in this paper differ in how they tread $\alpha$.

The Static Underestimating Polynomial Method (SUPM) requires an initial value of $\alpha$ to be supplied by the user, which is then kept constant for the duration of the algorithm. For the remainder of this section, we denote the step function $\sigma$ and update function $U$ which apply to the SUPM by $\sigma_S$ and $U_S$ respectively.

Given the objective function $f$, our strategy is to use the points $x_3^L, x_2^L, x_1^L$ and $x_3^R, x_2^R, x_1^R$ to construct the model functions $q^L(x; X, \alpha)$, and $q^R(x; X, \alpha)$ which approximate $f_L$ and $f_R$ respectively. We will employ Newton’s Divided Difference notation which we summarize below.

**Definition 3.3** Given $a, b, c \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$, the 1st and 2nd divided differences are defined by:

$$f[a, b] = \frac{f(a) - f(b)}{a - b},$$

$$f[a, b, c] = \frac{f[a, b] - f[a, c]}{b - c}.$$  \hspace{1cm} (3.3)

The reason we use this notation is that the 1st and 2nd divided differences are natural approximations for the 1st and 2nd derivatives of $f$ respectively. We will state the result in a later section when more rigour is needed.

Now we define our model functions $q^L$ and $q^R$ as the following:

$$q^L(x; X, \alpha) = f(x_3^L) + f[x_3^L, x_2^L](x-x_3^L) + (f[x_3^L, x_2^L, x_1^L] - \alpha h(X))(x-x_1^L)(x-x_2^L),$$

$$q^R(x; X, \alpha) = f(x_3^R) + f[x_3^R, x_2^R](x-x_3^R) + (f[x_3^R, x_2^R, x_1^R] - \alpha h(X))(x-x_1^R)(x-x_2^R).$$ (3.3)
where \( k \in \{L, R\} \), \( \alpha > 0 \) is a constant and \( h(\mathcal{X}) \) is a scaling function with the property that \( h(\mathcal{X}) \rightarrow 0 \) as \( \text{diam}(\mathcal{X}) \rightarrow 0 \). Note that \( q^k \) has the form of the 2\(^\text{nd}\) order Newton Interpolating Polynomial combined with the adjustment term \( \alpha h(\mathcal{X}) \). Finally we express \( \sigma_S \) in terms of \( q^L \) and \( q^R \):

\[
\sigma_S(\mathcal{X}; \alpha) = \arg\min_{x \in [x^L, x^R]} \max(q^L(x; \mathcal{X}, \alpha), q^R(x; \mathcal{X}, \alpha)).
\]  

(3.4)

**Remark 3.3** The model functions \( q^L \) and \( q^R \) are designed to underestimate \( f_L \) and \( f_R \) within \( [x^L, x^R] \). We will show how we achieve this in Lemma 3.2.

**Example 3.1** Consider the function \( f(x) = \max(\sin(\frac{1}{2}x\pi), 1 - \cos(\frac{1}{2}x\pi)) \). This function is piecewise smooth and unimodal in the interval \([-1, 1] \) with a local minimum at 0. At this local minimum, \( f \) is not differentiable.

Now suppose that \( \{x^L_i\}_{i=1}^3 = \{-0.75, -0.9, -1\} \) and \( \{x^R_i\}_{i=1}^3 = \{0.6, 0.8, 0.95\} \). We construct \( q^L \) and \( q^R \) by interpolating \( \{(x^L_i, f(x^L_i))\}_{i=1}^3 \) for \( k = L, R \). These two quadratics intersect at \( x \approx 0.005 \) as shown in Figure 3.1. This value is taken to be the next point at which we evaluate \( f \).

All that remains is for us to define the function \( U_S \).

\[
U_S(\tilde{x}; \mathcal{X}, \alpha) := \begin{cases} 
U_1(\tilde{x}; \mathcal{X}) & \text{if } \tilde{x} < x^M \text{ and } f(\tilde{x}) < f(x^M) \\
U_2(\tilde{x}; \mathcal{X}) & \text{if } \tilde{x} > x^M \text{ and } f(\tilde{x}) < f(x^M) \\
U_3(\tilde{x}; \mathcal{X}) & \text{if } \tilde{x} > x^M \text{ and } f(\tilde{x}) > f(x^M) \\
U_4(\tilde{x}; \mathcal{X}) & \text{if } \tilde{x} < x^M \text{ and } f(\tilde{x}) > f(x^M) 
\end{cases}
\]  

(3.5)

where

\[
U_1(\tilde{x}; \mathcal{X}) := (x^L_1 \ x^L_2 \ x^L_{\tilde{x}} \ x^M \ x^R_1 \ x^R_2)^T, \quad (3.6a)
\]

\[
U_2(\tilde{x}; \mathcal{X}) := (x^L_2 \ x^L_1 \ x^M \ x^R_1 \ x^R_2 \ x^R_3)^T, \quad (3.6b)
\]

\[
U_3(\tilde{x}; \mathcal{X}) := (x^L_3 \ x^L_2 \ x^L_1 \ x^M \ x^R_1 \ x^R_2)^T, \quad (3.6c)
\]

\[
U_4(\tilde{x}; \mathcal{X}) := (x^L_2 \ x^L_1 \ x^M \ x^R_2 \ x^R_3)^T. \quad (3.6d)
\]
Remark 3.4 The update function $\sigma_S(\mathcal{X})$ is the natural extension of the one used by the Mifflin-Strodiot method \cite{12} to 7 points.

Lemma 3.1 Let $\mathcal{X}_1$ be an extended bracket. If $\mathcal{x} = \sigma_S(\mathcal{X}_1) \in (x^1_r, x^M) \cup (x^M, x^R)$, then $\mathcal{X}_2 = U_S(\mathcal{x}; \mathcal{X}_1, \alpha)$ is an extended bracket and $b(\mathcal{X}_2) < b(\mathcal{X}_1)$.

Definition 3.4 A function $f: [a, b] \rightarrow \mathbb{R}$ is called unimodal if there exists $\zeta \in (a, b)$ such that $f$ is monotonically decreasing on $(a, \zeta)$ and monotonically increasing on $(\zeta, b)$. If a function is not unimodal, then it is multi-modal.

Definition 3.5 Let $f$ be a piece-wise smooth function. We call $f$ locally unimodal if for any local minimum $x^*$ of $f$, there exists an open set $(a, b) \ni x^*$ such that $f$ is unimodal on $(a, b)$.

When constructing convergence results, there are three levels of assumptions we might make. First is the case where $f$ is a piece-wise smooth function, $\mathcal{X}$ is a bracket, and nothing more is assumed. In this general context, we can do nothing beyond defining a minimum distance between points $\delta$. If $|\sigma_S(\mathcal{X}) - x^M| < \delta$, we replace $\sigma_S(\mathcal{X})$ with $\argmin_{x} |x - \sigma_S(\mathcal{X})|$ such that $|x - x^M| \geq \delta, x^R - x \geq \delta$ and $x - x^L \geq \delta$. If $\sigma_S(\mathcal{X}) = x^M$, then there exists two equally valid solutions: $x^M + \delta$ and $x^M - \delta$. When this applies, we arbitrarily set $\sigma_S(\mathcal{X}) = x^M - \delta$. By doing this, we are sure to converge eventually, even if slowly.

For the next level of assumptions, we additionally require $f$ to be in the form $f(x) = \max(f_L(x), f_R(x))$, and $\mathcal{X}$ satisfy $f(x^L_j) = f_L(x^L_j), j = 1, 2, 3$ and $f(x^R_j) = f_L(x^R_j), j = 1, 2, 3$. Given these assumptions, we show in Lemma 3.2 and Corollary 3.1 that $\sigma_S(\mathcal{X}; \alpha) \in (x^L_j, x^R_j)$ given a sufficiently large choice of $\alpha$. In short, when there are not too many kinks in one place, then the UPM selects sensible points to evaluate $f$ at.

Finally, we add the assumptions that $f$ is unimodal and $\text{diam}(\mathcal{X})$ is sufficiently small. Under these circumstances, we show in Lemma 3.3 and Theorem 3.2 that the distance between $\sigma_S(\mathcal{X})$ and the true solution $x^*$ can be bounded.

Taken in the context of locally unimodal functions, these results will imply that the \text{UPM} is stable when the function is not unimodal, and fast when it is. This is sufficient for our purposes given that a stable algorithm applied to a locally unimodal function will eventually converge to a bracket in which the function is unimodal.

For the analysis that follows, we require the following external result:

Theorem 3.1 (\cite{11}) Suppose that $k, n \geq 0$; $f \in C^{n+k}[a, b]$; $\zeta \in [a, b]$; and $x_0, \ldots, x_n$ are distinct points in $[a, b]$. Then

$$f[x_0, \ldots, x_n] = \frac{f^{(n)}(\zeta)}{n!} + \sum_{0 \leq r_1 \leq n} (x_{r_1} - \zeta) \frac{f^{(n+1)}(\zeta)}{(n + 1)!}$$

$$+ \ldots + \left( \sum_{0 \leq r_1 \leq \ldots \leq r_k \leq n} \prod_{j=1}^{k} (x_{r_j} - \zeta) \right) \frac{f^{(n+k)}(\zeta)}{(n + k)!} + E$$  \hspace{1cm} (3.7)
where
\[
E = \frac{1}{(n+k)!} \left( \sum_{0 \leq r_1 \leq \ldots \leq r_k \leq n} \left( \prod_{j=1}^{k} (x_{r_j} - \xi_j) \right) \right) f^{(n+k)}(\xi)
\]
and \(x_{r_j}, \ldots, x_{r_k}\) are points in the interval spanned by \(x_1, \ldots, x_n\) and \(\xi\).

**Lemma 3.2** Let \(f\) be a function of the form \(f = \max(f_L, f_R)\) such that \(f''_L\) and \(f''_R\) are both Lipschitz continuous with Lipschitz constants \(M_L\) and \(M_R\) respectively, and \(X\) be an extended bracket of \(f\) where \(f(x_j) = f_k(x_j)\) for \(j = 1, 2, 3, \ldots, k \in \{L, R\}\). If \(\alpha \geq \frac{1}{2} \max(M_L, M_R)\) and \(h(X) = \max(x_3^R - x_3^L, x_1^R - x_1^L)\), then \(q^R(x; X, \alpha) < f_L(x) \forall x \in [x_1^R, x_1^L]\) and \(q^L(x; X, \alpha) < f_R(x) \forall x \in [x_1^R, x_1^L]\).

**Proof** Suppose \(y \in (x_1^L, x_1^R)\). Since \(f(x_j) = f_L(x_j^L)\) for \(j = 1, 2, 3\), we have:
\[
\begin{align*}
& f_L(y) > q^L(y; X, \alpha), \\
& \Leftrightarrow f_L(y) > f_L(x_1^L) + f_L|x_1^L, x_2^L|(y - x_1^L) + (f_L|x_1^L, x_2^L| - \alpha h(X))(y - x_1^L), \\
& \Leftrightarrow f_L[y, x_1^L, x_2^L] > f_L|x_1^L, x_2^L| - \alpha h(X), \\
& \Leftrightarrow \alpha h(X) > f_L[x_1^L, x_2^L, x_3^L] - f_L[y, x_1^L, x_2^L].
\end{align*}
\]

Analogously, for \(y \in (x_1^R, x_1^R)\) we have:
\[
\alpha h(X) > f_R[x_1^R, x_2^R, x_3^R] - f_R[y, x_1^R, x_2^R].
\]

From Theorem 3.1, we know that there exists \(\xi^L \in [x_1^L, x_1^L]\) such that \(\frac{1}{2} f''(\xi^L) = f_R[x_1^L, x_2^L, x_3^L]\) and \(\xi^R \in [y, x_1^R]\) such that \(\frac{1}{2} f''(\xi^R) = f_R[y, x_1^R, x_2^R]\). Equivalently, the points \(\xi^L\) and \(\xi^R\) exist for \(f_L|x_1^L, x_2^L, x_3^L|\) and \(f_L[y, x_1^L, x_2^L]\) respectively. Therefore:
\[
\begin{align*}
f_L|x_1^L, x_2^L, x_3^L| - f_L[y, x_1^L, x_2^L] &= \left( \frac{1}{2} f''(\xi^L) - f''(\xi^R) \right), \\
&\leq \frac{1}{2} M_L |\xi^L - \xi^R|, \\
&\leq \frac{1}{2} M_L |\alpha h(X) - x_1^L|.
\end{align*}
\]

The bound on the RHS depends on the value of \(y\), and is itself bounded by \(\frac{1}{2} M_L |x_1^L - x_1^L|\) when considering \(y \in [x_1^L, x_1^L]\). Therefore, a sufficient condition for \(f_L(y) > q^L(y; X, \alpha) \forall y \in (x_1^L, x_1^R)\) is that \(\alpha h(X) > \frac{1}{2} M_L |x_1^L - x_1^L|\).

Similarly, a sufficient condition for \(f_R(y) > q^R(y; X, \alpha) \forall y \in (x_1^L, x_1^R)\) is that \(\alpha h(X) > \frac{1}{2} M_L |x_1^L - x_1^L|\).

Choosing \(h(X) = \max(x_3^R - x_3^L, x_1^R - x_1^L)\), and \(\alpha > \max(M_L, M_R)\) ensures that both of these are satisfied.

**Corollary 3.1** Let \(f\) be a function of the form \(f = \max(f_L, f_R)\) such that \(f''_L\) and \(f''_R\) are both Lipschitz continuous with Lipschitz constants \(M_L\) and \(M_R\) respectively, and \(X\) be an extended bracket of \(f\) where \(f(x_j) = f_k(x_j)\) for \(j = 1, 2, 3, \ldots, k \in \{L, R\}\). If \(\alpha \geq \frac{1}{2} \max(M_L, M_R)\) and \(h(X) = \max(x_1^R - x_1^L, x_3^R - x_3^L)\), then \(\alpha h(X) \in (x_1^L, x_1^R)\).
Proof As Lemma 3.2 is applicable, we know that \( q^R(x) < f_R(x) \forall x \in [x_1^L, x_1^R] \) and \( q^L(x) < f_L(x) \forall x \in [x_1^L, x_1^R] \). Since \( f = \max(f_L, f_R) \), it follows that \( q^L(x^M) < f(x^M) \) and \( q_R(x^M) < f(x^M) \). For convenience, write \( q(x) = \max(q^L(x; X, \alpha), q^R(x; X, \alpha)) \). Starting from Equation (3.4), we obtain:

\[
q(\sigma_S(X; \alpha)) \leq q(x^M) < f(x^M) \leq \min(f(x_1^L), f(x_1^R)) = \min(q^L(x_1^L), q^R(x_1^R)).
\]

But \( q(x_1^L) = q^L(x_1^L) \) and \( q(x_1^R) = q^R(x_1^R) \). Therefore \( q(\sigma_S(X; \alpha)) < \min(q(x_1^L), q(x_1^R)) \), which implies that \( \sigma_S(X; \alpha) \neq x_1^L \) or \( x_1^R \). Since \( \sigma_S(X; \alpha) \in [x_1^L, x_1^R] \) by definition (see Equation (3.4)), it follows that \( \sigma_S(X; \alpha) \in (x_1^L, x_1^R) \).

Remark 3.5 Between Corollary 3.1 and the requirement that the minimum distance between any two points in \( X \) is \( \delta \), we are assured that the conditions for Lemma 3.1 will be satisfied.

For the remainder of this section, we work towards bounding the convergence of the SUPM when \( f \) is piecewise-smooth and unimodal. We define \( x^* \) to be the unique local minimum of \( f \) in \( (x_1^L, x_1^R) \). When assessing the SUPM’s convergence rate, we consider two cases based on the values of \( f'_L(x^*) \) and \( f'_R(x^*) \).

If \( f'_L(x^*) = 0 \) or \( f'_R(x^*) = 0 \) holds, then we expect the SUPM to behave similarly to Brent’s Method. For example, if \( f'_L(x^*) = 0 \) and \( f'_R(x^*) < 0 \), then we expect \( q^R \) to behave similarly to the interpolated polynomial from Brent’s Method. The SUPM will be slower however for two reasons.

1. When the SUPM computes a new point, that value may be stored in \( x^M \) which does not impact the next computation but only the one after that.
2. As only \( q^R \) is converging, any point which is used to construct \( q^L \) will play little to no role in determining \( \sigma_S(X) \), and is therefore useless.

We focus our attention primarily on the case where \( f'_L(x^*) < 0 \) and \( f'_R(x^*) > 0 \). In order to bound \( |\sigma_S(X)| \), we need to understand what type of point \( \sigma_S(X) \) returns. We address this in Lemma 3.3. First however, we introduce a shorthand notation for Divided Differences which we use for the remainder of this section.

\[
\begin{align*}
    f^L_k &= f(x_1^L), \ k \in \{L, R\}, \\
    f^L_k &= f(x_1^L, x_2^L), \ k \in \{L, R\}, \\
    f^L_k &= f(x_1^L, x_2^L, x_3^L), \ k \in \{L, R\}.
\end{align*}
\]

Lemma 3.3 Let \( f \) be a piecewise smooth unimodal function of the form \( f = \max(f_L, f_R) \) such that \( f'_L \) and \( f'_R \) are both Lipschitz continuous with Lipschitz constants \( M_L \) and \( M_R \) respectively, and \( X \) be an extended bracket of \( f \). Further let \( \alpha \) be given such that \( 2\alpha \geq \max(M_L, M_R) \). If \( f'_L(0) < 0 \) and \( f'_R(0) > 0 \), then there exists \( \epsilon > 0 \) such that if \( X \in (-\epsilon, \epsilon)^L \), then the model functions \( q^L \) and \( q^R \) intersect exactly once within \( (x_1^L, x_1^R) \) and \( \sigma_S(X) \) returns this unique intersection point.
Proof Since \( f = \max(f_L, f_R) \) is unimodal, and \( X \) is an extended bracket, it follows that \( f(x_i^k) = f_k(x_i^k) \) for \( j = 1, 2, 3 \) and \( k \in \{L, R\} \). We use this along with Lemma 3.2 to show: \( q^L(x_i^1) = f(x_i^1) = f_L(x_i^1) \geq f_R(x_i^1) > q^R(x_i^1) \), and similarly \( q^R(x_i^1) > q^L(x_i^1) \). By the Intermediate Value Theorem [16, Theorem 4.23], it follows that \( q^L \) intersects with \( q^R \) at least once in \((x_i^1, x_i^3)\). Moreover, since \( q^L \) and \( q^R \) are both quadratic functions, they must intersect exactly once.

A sufficient but not necessary condition for the result to follow would be \( \exists \varepsilon > 0 \) such that \( q^L \) and \( q^R \) are monotonically decreasing and increasing respectively. Once this holds, \( \sigma_S(X) = \arg\min_x \max(q^L(x), q^R(x)) \) must refer to the intersection of \( q^L \) and \( q^R \).

To show that \( q^L \) is monotonically decreasing, it is sufficient to prove that \( (q^L)'(x_i^1) \leq 0 \) and \( (q^R)'(x_i^1) \leq 0 \). Once this is shown, we are done since \((q^L)'(x_i^1) \leq 0 \) is a linear function of \( x \). We find:

\[
(q^L)'(x_i^1) = f_L^2 = \frac{f(x_i^1) - f(x_i^2)}{x_i^1 - x_i^2} \leq 0,
\]
since \( f \) is unimodal and \( X \) is an extended bracket of \( f \) (recall Definition 3.1).

For \((q^L)'(x_i^1) \) we begin with

\[
(q^L)'(x_i^1) = f_L^4 + (f_L^3 - \alpha h(X))(2x_i^1 - x_i^1 - 3x_i^2) = (3.9)
\]

Next we apply Theorem 4.4 to conclude:

\[
\exists \xi_1^L, \xi_2^L \in [x_i^2, x_i^1] \text{ such that } f_L^4 = f_L^3 + (x_i^1 f_L''(\xi_1^L) + x_i^2 f_L''(\xi_2^L)),
\]
\[
\exists \xi_3^L \in [x_i^1, x_i^2] \text{ such that } f_L^4 = \frac{1}{2} f_L''(\xi_3^L).
\]

From this and Equation \((3.9)\) it follows that

\[
(q^L)'(x_i^1) = f_L^4(0) + \frac{1}{2} \left( x_i^1 f_L''(\xi_1^L) + x_i^2 f_L''(\xi_2^L) \right) + \frac{1}{2} (2x_i^1 - x_i^1 - x_i^2) \left( f_L''(\xi_3^L) - 2\alpha h(X) \right).
\]

Since \( f_L'' \) and \( f_R'' \) are Lipschitz continuous, we know that they are bounded on \([x_i^3, x_i^1]\). Let \( M \) be constant which bounds \( f_L'' \) and \( f_R'' \). We use this along with the fact that \( X \in (-\varepsilon, \varepsilon) \) to conclude:

\[
(q^L)'(x_i^1) \leq f_L^4(0) + \frac{1}{2} \left( -x_i^1 M - x_i^1 M + (2x_i^1 - x_i^1 - x_i^2)(M - 2\alpha h(X)) \right),
\]
\[
(q^L)'(x_i^1) \leq f_L^4(0) + \frac{1}{2} (2x_i^1 M + 4\alpha h(X)),
\]
\[
\leq f_L^4(0) + (3M - 4\alpha h(X)) \leq f_L^4(0) + 3\varepsilon M,
\]

since \( 0 < h(X) < 2\varepsilon \) and \( \alpha \) is constant. Therefore a sufficient condition for \((q^L)'(x_i^1) < 0 \) to hold is that \( \varepsilon < |f_L^4(0)|/3M \). That \( q^R \) is monotonically increasing given sufficiently small \( \varepsilon \) follows from the equivalent argument.

Having established the conditions under which we can express \( \sigma_S(X) \) analytically, we now bound the distance between this point and the true minimiser.
Theorem 3.2 Let \( f : [a, b] \to \mathbb{R} \) be a piecewise smooth unimodal function of the form \( f = \max(f_L, f_R) \) such that \( x^* \in [a, b] \) is a minimiser of \( f \), \( f_x'(x^*) < 0 \), \( f_R'(x^*) > 0 \), and \( \mathcal{X} \) be an extended bracket of \( f \). If the functions \( f_R'' \) and \( f_L'' \) are both Lipschitz continuous (with constant \( M_L \) and \( M_R \)) and \( \alpha > \frac{1}{2} \max(M_L, M_R) \), then there exists \( \epsilon > 0 \) such that

\[
|\sigma_2(\mathcal{X})| \leq \frac{\sqrt{2}}{f_R''(x^*) - f_L''(x^*)} (|x^L - x^*||x_R - x^*|(|x^L - x^*| + |x_R^L - x^*|)M_L
+ |x^L - x^*||x_R - x^*|(|x^L - x^*| + |x_R^L - x^*|)M_R),
\]

when \( \mathcal{X} \in (-\epsilon, \epsilon)^2 \).

Proof Since Lemma \( \text{Lemma 3.3} \) applies, we know that \( \sigma_2(\mathcal{X}) \) returns the unique point of intersection between \( q^L \) and \( q^R \) in \( (x^L, x^R) \). The intersection points of these quadratics are the roots to the polynomial \( ax^2 + bx + c = 0 \), where:

\[
a = f_R^L - f_L^L,
b = f_R^L - f_L^L - (x^R + x^L)(f_R^L - \alpha h) + (x^L + x^L)(f_L^L - \alpha h),
c = f_R^L - f_L^L f_R^L + x^L f_L^L + x^R f_R^L (f_R^L - \alpha h) - x^L x^L (f_L^L - \alpha h).
\]

Assuming without loss of generality \( x^* = 0 \), we invoke Theorem 3.1 to make the following substitutions for \( \ell \in \{L, R\} \).

\[
f_1^\ell = f_\ell(0) + x^\ell f_\ell'(0) + \frac{1}{2} (x^\ell)^2 f_\ell''(\xi_1^\ell), \quad \xi_1^\ell \in [0, x^\ell],
f_2^\ell = f_\ell(0) + \frac{1}{2} (x^\ell f_\ell''(\xi_2^\ell) + x^\ell f_\ell''(\xi_2^\ell))), \quad \xi_2^\ell \in [0, x^\ell], \quad \xi_3^\ell \in [0, x^\ell],
f_3^\ell = \frac{1}{2} f_\ell''(\xi_4^\ell), \quad \xi_4^\ell \in [0, x^\ell].
\]

By inserting the substitutions above into the definitions of \( a, b \) and \( c \), we get:

\[
a = \frac{1}{2} (f_R''(0) - f_L''(0)) + K_a^L - K_a^R,
b = f_R^L (0) - f_L^L (0) + K_a^R - K_a^L,
c = K_a^L - K_a^R,
\]

where \( K_a^L, K_a^R \) and \( K_a^L \) are in turn defined for \( \ell \in \{L, R\} \) by:

\[
K_a^L = \frac{1}{2} (f_R''(\xi_1^L) - f_L''(\xi_1^L)),
K_a^R = \frac{1}{2} (f_R''(\xi_1^R) - f_L''(\xi_1^R) + x^L f_R''(\xi_2^L) - f_R''(\xi_2^L)) + (x^L + x^L) a h(\mathcal{X}),
K_a^L = \frac{1}{2} x^L (x^L f_R''(\xi_1^L) - f_R''(\xi_2^L)) + x^L (f_R''(\xi_2^L) - f_R''(\xi_2^L)) - x^L x^L a h(\mathcal{X}).
\]

Writing \( a, b \) and \( c \) in terms of these \( K \) values allows us to give bounds using the assumption that \( f_R'' \) and \( f_L'' \) are Lipschitz Continuous.

\[
|K_a^L| \leq \frac{1}{2} M x^L \leq \frac{1}{2} M \epsilon, \quad (3.10a)
|K_a^R| \leq \frac{1}{2} (|x^L| + |x^L|) (M x^L + a h(\mathcal{X})) \leq e^2 (M \epsilon + 2 \alpha), \quad (3.10b)
|K_a^L| \leq \frac{1}{2} x^L |x^L| M \epsilon (|x^L| + |x^L|) \leq M \epsilon e^3. \quad (3.10c)
\]
Choose an $\epsilon_1$ small enough to ensure \( \frac{1}{4}(f''_R(0) - f''_L(0)) > K^L_b - K^R_b \) (note that \( b \to f''_R(0) - f''_L(0) \) as \( \epsilon \to 0 \)). If \( \epsilon \leq \epsilon_1 \), then it follows that \( b > 0 \) which in turn implies:
\[
\sigma_S(X) = \frac{-b + \sqrt{b^2 - 4ac}}{2a},
\]
because the other root lies outside \([x^L, x^R]\) as \( \epsilon \to 0 \). We apply Taylors Theorem to conclude \( \exists \xi \in [-4ac, 4ac] \) such that \( \sqrt{b^2 - 4ac} = b + 2ac/\sqrt{b^2 - \xi} \). Since \( ac \to 0 \) as \( \epsilon \to 0 \), there exists \( \epsilon_2 \) such that \( |4ac| \leq \frac{1}{2}b^2 \) when \( \epsilon < \epsilon_2 \). Then if \( \epsilon < \min(\epsilon_0, \epsilon_1, \epsilon_2) \), we have:
\[
|\sigma_S(X)| = \left| \frac{-b + \sqrt{b^2 - 4ac}}{2a} \right| = \frac{1}{2|a|} \left| \frac{2ac}{\sqrt{b^2 - \xi}} \right|, \\
\leq \frac{|c|}{\sqrt{b^2 - \xi}} \leq \frac{|c|}{\sqrt{b^2 - 4|ac|}} \leq \frac{\sqrt{2}|c|}{b}, \\
= \sqrt{2} \frac{K^R_R - K^L_L}{f''_R(0) - f''_L(0)} \leq 2 \sqrt{2} \frac{|K^R_L| + |K^L_R|}{f''_R(0) - f''_L(0)}.
\]
Finally we insert Equation (3.10c) and the result follows.

Theorem 3.2 is remarkably similar to \([11, \text{Theorem 4.1}]\), the equivalent result which bounds the convergence of the Mifflin-Strodiot method. Using our own notation for their result, Mifflin and Strodiot showed \([x^L, x^R] \) as \( \epsilon \to 0 \). We apply Taylors Theorem to conclude \( \exists \xi \in [-4ac, 4ac] \) such that \( \sqrt{b^2 - 4ac} = b + 2ac/\sqrt{b^2 - \xi} \). Since \( ac \to 0 \) as \( \epsilon \to 0 \), there exists \( \epsilon_2 \) such that \( |4ac| \leq \frac{1}{2}b^2 \) when \( \epsilon < \epsilon_2 \). Then if \( \epsilon < \min(\epsilon_0, \epsilon_1, \epsilon_2) \), we have:

Finally, we insert Equation (3.10c) and the result follows.
4 Extremal Underestimating Polynomial Method

In Section 3, we showed that if $\alpha$ was sufficiently large, then the $\sigma_S$ is guaranteed to select a point within $(x^L, x^R)$. Coupled with a restriction on the minimum distance between points, this is enough to ensure convergence eventually. However, for an arbitrary piecewise smooth objective function, we have no idea what a suitable $\alpha$ might be. As there is no finite value of $\alpha$ which satisfies Lemma 3.2 for all functions, we ask what happens if we let $\alpha \to \infty$ and name the corresponding variation of the SUPM the Extremal Underestimating Polynomial Method (EUPM), denoting its step function and update function by $\sigma_E$ and $U_E$ respectively.

As $\alpha \to \infty$ we are assured by Lemma 3.2 that $q^L$ and $q^R$ will underestimate $f_L$ and $f_R$ respectively within $[x^L, x^R]$. Furthermore, if $h(x) > 0$, then we can pick a sufficiently large $\alpha$ such that the coefficient of $x^2$ in $q^L(x; X, \alpha)$ (see Equation (3.3)) will be negative. Once $\alpha$ satisfies both of these, then $q^L$ and $q^R$ will intersect at exactly one point, which is the point that $\sigma_S(X)$ returns.

We define $\sigma_E(X)$ to be this point which we calculate by identifying the root of $q^R(x; X, \alpha) - q^L(x; X, \alpha) = 0$, which remains bounded as $\alpha \to \infty$. By starting from Equation (3.11) and noting the Taylor Series as $\alpha \to \infty$, we find:

$$\sigma_E(X) = \frac{x^R_x - x^L_x}{x^R_x - x^L_x}.$$  \hspace{0.5cm} (4.1)

Note that the step function $\sigma_E$ does not depend on either $x^L_3$ or $x^R_3$. Therefore, within the context of this section, our extended bracket $X$ is of the form:

$$X = (x^L_2 \ x^M \ x^R_2 \ x^L_1 \ x^R_1)\text{T},$$ \hspace{0.5cm} (4.2)

In order to construct $U_E$, we first define $\tilde{U}_S$ to be the update function $U_S$, were we remove the entries corresponding to $x^R_3$ and $x^L_3$. Using this notation, we define:

$$U_E(X) := \tilde{U}_S(\sigma_E(X), X).$$ \hspace{0.5cm} (4.3)

Given $\sigma_E$ and $U_E$, the EUPM is precisely in the form of Algorithm 1.

Remark 4.1 Equation (4.3) is of the same form as the update function used by the Mifflin-Strodiot method [12].

Remark 4.2 We see the EUPM is qualitatively different from the SUPM in that $\sigma_E$ depends only on the values of $X$, and not on the function $f$ at all. Therefore, like Golden Section, the EUPM is equally suited to smooth, non-smooth, unimodal and multi-modal functions and is scale invariant.

4.1 Theoretical Convergence of the EUPM

In this section we derive a bound for the convergence rate of the EUPM. In particular, we find an integer $k$ and constant $C$ such that $b(X_{i+k})/b(X_i) \leq C$.
holds for any objective function. While there exist infinitely many possible objective functions, there are only finitely possible values for $X_{i+k}$ given $X_i$. This is because $\sigma_E$ does not depend on the objective function $f$ and thus the only effect that $f$ has on the EUPM is in the function $U_E$ (See Equation (3.3)) by determining which update function: $U_1, U_2, U_3$ or $U_4$ to apply, none of which depend on $f$.

Therefore, there are only 4 possible values of $X_{i+1}$ given $X_i$: $U_jX$ for $j = 1, 2, 3, 4$. As a consequence of this, there are at most $4^k$ possible values of $X_{i+k}$ given $X_i$. Each possible value of $X_{i+k}$ depends on the sequence of update functions which generated it.

**Remark 4.3** In this section, we will refer to the update functions $U_1, U_2, U_3$ and $U_4$ extensively. To simplify notation, we replace $U_j$ with $\tilde{U}_j$, where $\tilde{U}_j(X) := U_j(\sigma_E(X), X)$. For the remainder of Section 4, $U_j$ should be read as $\tilde{U}_j$.

**Definition 4.1** Let $\{X_i\}_{i=0}^n$ be a sequence of extended brackets generated by the EUPM such that $X_{i+1} = U_E(X_i)$ for each $i$. Let $I = \{i_j\}_{j=1}^{n-1} \in \{1, 2, 3, 4\}^n$ be the sequence of numbers such that $X_{j+1} = U_{i_j}(X_j)$, $\forall j \in \{0, 1, \ldots, n-1\}$. Then we call $I$ the sequence of EUPM iterations which generate $\{X_i\}_{i=0}^n$.

Moreover we write:

$$U_I := U_{i_0} \circ \ldots \circ U_{i_{n-1}}$$

where $I = \{i_j\}_{j=1}^n$.

**Definition 4.2** Define $\mathcal{I}_n$ to be the set of all length $n$ sequences of EUPM iterations.

**Definition 4.3** Define the $Q_k(f, X_0)$ to be the sequence of the first $k$ iterations generated by the EUPM when applied to the function $f$ with the starting bracket $X_0$.

**Lemma 4.1** Let $f$ be a function, $X \in B_f$ and $I \in \mathcal{I}_n$ such that $I = \{I_1, I_2\} = Q_n(f, X)$, where $I_1 \in \mathcal{I}_{n-k}$ and $I_2 \in \mathcal{I}_k$. Then it holds that: $Q_k(f, U_{I_2}X) = I_2$.

Using the notation from Definitions 4.1 and 4.3 it follows $X_{n+k} = U_{Q_n(f, X_0)}(X_n)$. Since $Q_k(f, X_0) \in \mathcal{I}_k$ for every function $f$, it follows that:

$$\frac{b(X_{n+k})}{b(X_n)} \leq C \text{ for any function } f \iff \frac{b(U_{I_2}X_n)}{b(X_n)} \text{, } \forall I \in \mathcal{I}_k.$$

**Definition 4.4** Given $I \in \mathcal{I}_n$, we define:

$$\mathcal{B}_I := \{X \in \mathbb{R}^5 | \exists f \text{ such that } X \in B_f, Q_n(f, X) = I\}.$$

**Lemma 4.2** Let $I \in \mathcal{I}_n$ be of the form $I = \{I_1, I_2\}$, where $I_1 \in \mathcal{I}_{n-k}$ and $I_2 \in \mathcal{I}_k$. Then it holds that: $\mathcal{B}_I \subseteq \mathcal{B}_{I_1}$ and $U_{I_2}(\mathcal{B}_{I_1}) \subseteq \mathcal{B}_{I_2}$.

**Proof** If $X \in \mathcal{B}_{I_1}$, then there exists a function $f$ such that $I = Q_n(f, X)$.

Since, $I = \{I_1, I_2\}$, then $I_1 = Q_{n-k}(f, X)$ by definition of $Q$. Hence $X \in \mathcal{B}_{I_1}$.

Moreover, $I_2 = Q_k(f, U_{I_1}X)$ by Lemma 4.1 which implies $U_{I_2}X \in \mathcal{B}_{I_2}$. 


We finally state our main result.

**Theorem 4.1** The EUPM converges $R$-linearly. In particular:

$$\max_{i \in \mathcal{I}_5} \max_{\mathbf{X} \in \tilde{B}_i} \frac{b(U_i \mathbf{X})}{b(\mathbf{X})} \leq \frac{1}{2},$$

for any bracket.

Given how long the proof of Theorem 4.1 is, we reserve an entire section for it. The proof may be found in Section 4.2, although it may be skipped on first reading.

Theorem 4.1 provides a bound on the linear convergence rate for the EUPM albeit a weak one. In practice, we observe far faster convergence as will be demonstrated in Section 6 and appendix B. However, the main justification for the EUPM is its robustness. In Section 3, we assumed that the SUPM started with a bracket that was already sufficiently close to the solution. Once equipped with this, we might expect the SUPM to converge quickly. The EUPM is useful for producing such a bracket in the first place. In Section 5, we present the final variation of the UPM in which we combine the initial robustness of the EUPM with the desirable properties of the SUPM.

4.2 Proof of Theorem 4.1

We divide this proof into two parts, which are defined by Lemmas 4.3 and 4.4.

**Lemma 4.3** It holds that

$$\max_{I \in \mathcal{A}} \max_{\mathbf{X} \in \tilde{B}_I} \frac{b(U_I \mathbf{X})}{b(\mathbf{X})} \leq \frac{1}{2} \quad \forall I \in \mathcal{A} = \{44, 111, 143, 422, 414, 434, 1411, 1141, 1423, 4322, 4314, 4114\}.$$

**Lemma 4.4** If

$$\max_{I \in \mathcal{A}} \max_{\mathbf{X} \in \tilde{B}_I} \frac{b(U_I \mathbf{X})}{b(\mathbf{X})} \leq \frac{1}{2} \quad \forall I \in \mathcal{I}_5,$$

where $A$ is as defined in Lemma 4.3.

Given that the proof of Lemma 4.3 is entirely algebraic calculations, we leave it for Appendix A. In this section, we work towards proving Lemma 4.4. Ultimately, Theorem 4.1 follows from these two results.

**Definition 4.5** Let $I = \{i_j\}_{j=1}^n \in \mathcal{I}_n$ be a length $n$ sequence of iterations of the EUPM. A sub-string of the sequence $I$ is a subsequence of the form $J = \{i_{j_1}\}_{j_1=k_1}^{j_2}$ such that $1 \leq k_1 < k_2 \leq n$. Denote by $S(I)$ the set of sub-strings of the sequence $I$.

The first step towards proving Lemma 4.4 is to show that if $I \in \mathcal{I}_5$ is a length 5 sequence of iterations and $J \in S(I)$ is a sub-string of $I$, then $b(U_J \mathbf{X})/b(\mathbf{X}) \leq \frac{1}{2}$ implies $b(U_I \mathbf{X})/b(\mathbf{X}) \leq \frac{1}{2}$.

**Lemma 4.5** Let $\mathbf{X}$ be an extended bracket of $f$. Given $I \in \mathcal{I}_n$, and $J \in S(I)$, it holds that:

$$\max_{\mathbf{X} \in B_J} \frac{b(U_I \mathbf{X})}{b(\mathbf{X})} \leq \max_{\mathbf{X} \in B_I} \frac{b(U_I \mathbf{X})}{b(\mathbf{X})}.$$
In the next few lemmas, we show that not all sequences of \{1, 2, 3, 4\} are possible sequences of EUPM iterations given a particular starting bracket \(X\).

**Lemma 4.6** Let \(I = \{i_1\}_{j=0}^{n-1} \in \mathcal{I}_n\) be a sequence of EUPM iterations. If \(i_j = 1\) then \(i_{j+1} \in \{1, 4\}\). Similarly, if \(i_j = 2\) then \(i_{j+1} \in \{2, 3\}\).

**Proof** If \(i_j = 1\), then by definition \(X_{j+1} = U_1X_j\). From this, we calculate:

\[
\sigma_E(X_{j+1}) - x_j^M = \frac{x_{1,j+1}^L x_{2,j+1}^L - x_{1,j+1}^L x_{2,j+1}^L}{x_{1,j+1}^L + x_{2,j+1}^L - x_{1,j+1}^L - x_{2,j+1}^L} - x_j^M,
\]

\[
= \frac{x_{1,j}^L x_j^M - x_{1,j}^L x_{2,j}^L}{x_{1,j}^L + x_j^M - x_{1,j}^L - x_{2,j}^L} - \frac{x_{1,j}^L x_{2,j}^L - x_{1,j}^L x_{2,j}^L}{x_{1,j}^L + x_{2,j}^L - x_{1,j}^L - x_{2,j}^L},
\]

\[
= \frac{x_{1,j}^L + x_j^M - x_{1,j}^L - x_{2,j}^L}{x_{1,j}^L + x_j^M - x_{1,j}^L - x_{2,j}^L} \left( -x_{2,j}^L x_{1,j}^L x_{2,j}^L \right) < 0,
\]

since each \(X_j\) is an extended bracket. Given that \(\sigma_E(X_{j+1}) - x_j^M < 0\), it follows from Equation (3.5) that \(i_{j+1} \in \{1, 4\}\). The equivalent argument applies for when \(i_j = 2\).

In the statement of Lemma 4.6, one may notice a symmetry between the functions \(U_1, U_4\) and \(U_2, U_3\) respectively. In the lemmas which follow, we derive a relation between \(U_1, U_4\) and \(U_2, U_3\) respectively such that if \(b(U_jX)/b(X) \leq \frac{1}{2}\) holds for some \(j\), it will still hold after replacing \(U_1\) and \(U_4\) with \(U_2\) and \(U_3\) and vice versa.

**Definition 4.6** Let \(I \in \mathcal{I}_n\) be a sequence of EUPM iterations. We define \(X : \mathbb{R}^5 \to \mathbb{R}^5\) to be the function which satisfies \(X(a, b, c, d, e) = -(e, d, c, b, a)\).

We make use of the function \(X\) because of the following useful properties.

**Lemma 4.7** It holds that \(XXX = X, \sigma_E(XX) = -\sigma_E(X), b(XX) = b(X), U_1X = XU_2XX, U_2X = XU_1XX, U_3X = XU_4XX\) and \(U_4X = XU_3XX\).

Having shown that the function \(X\) relates \(U_1\) to \(U_4\) and \(U_2\) to \(U_3\), we now show that this relation also holds for sequences of iterations.

**Definition 4.7** Let \(I = \{i_j\}_{j=1}^n \in \mathcal{I}_n\). Define \(W : \mathcal{I}_n \to \mathcal{I}_n\) to be the function: \(W(I) = (w(i_j))_{j=1}^n\), where \(w(1) = 2, w(2) = 1, w(3) = 4\) and \(w(4) = 3\).

**Lemma 4.8** If \(f\) and \(g\) are functions such that \(g(x) = f(-x)\) and \(X_0 \in \mathcal{B}_f\), then \(X^X \in \mathcal{B}_g\). Moreover, if \(\{X_j^f\}_{j=0}^n\) and \(\{X_j^g\}_{j=0}^n\) are the sequences of extended brackets generated by the EUPM when applied to \(f\) and \(g\) starting from \(X_0\) and \(XX_0\) respectively, then it holds that \(X_j^f = XX_j^g\) for every \(j = 0, 1, \ldots, n\) and \(Q_n(g, X, XX) = W(Q_n(f, XX))\).

**Proof** Note that \(XX \in \mathcal{B}_g\) follows from Definitions 3.1 and 4.6. Let \(\{X_j^f\}_{j=0}^n\) and \(\{X_j^g\}_{j=0}^n\) be the sequences of extend brackets generated by applying the EUPM with the starting brackets \(X_0\) and \(XX_0\) to \(f\) and \(g\) respectively. Define:

\[
\varphi(X) = \sigma_E(X) - x^M.
\]
Since $\sigma_E(\mathcal{X}) = -\sigma_E(X^\mathcal{X})$, it follows that $f(\sigma_E(X_0)) = g(\sigma_E(X_0))$ and $\varphi(X_0) < 0 \Leftrightarrow \varphi(X_0) > 0$. Given that $\text{sign}(\varphi(X))$ is precisely the condition in Equation (3.36) which determines whether to apply update functions $U_1, U_4$ or $U_2, U_3$, it follows that $X_j^q = U_jX_0 \Leftrightarrow X_j^q = U_W(j)X_0 = U_{W(j)}X_0$.

Applying Lemma 4.7, we note that $U_{W(j)}X_0 = XX_0$ which implies that $X_j^q = X_j^q$. From this it follows inductively that $X_j^q = X_j^q$ for all $j = 1, 2, \ldots, n$. Moreover, if we write $Q_n(f, g)$ and $Q_n(g, f, g)$ in the forms $\{i_j\}_{j=0}^{n-1}$ and $\{j_j\}_{j=0}^{n-1}$ respectively, it also follows inductively that $i_j^{j} = W(i_j^q)$ for each $j = 0, 1, \ldots, n - 1$; in other words $Q_n(g, f, g) = W(Q_n(f, g))$.

**Corollary 4.1** If $I \in I_n$, then: $X(\tilde{B}) = \tilde{B}(W(I))$ (recall Definition 4.4).

**Corollary 4.2** It holds that $U_I \mathcal{X} = XU_{W(I)}X \mathcal{X}, \forall I \in I_n$.

**Proof** Let $f$ be a function such that $I = Q(f, \mathcal{X})$. By Lemma 4.3 the function $g(x) = f(-x)$ has the property that $Q_n(g, \mathcal{X}) = W(Q_n(f, \mathcal{X}))$ and $U_I = U_Q(f, \mathcal{X}) = XU_Q(g, \mathcal{X}) = XU_{W(I)}(X) = XU_{W(I)}(X).$

The map $W$ serves as a natural bijective between sequences of EUPM iterations which start with 1 or 4 and those which start with 2 or 3. Dividing $\mathcal{I}_5$ is useful due to the following result.

**Corollary 4.3** Given $I \in I_n$ a sequence of EUPM iterations, it holds that:

$$\max_{\mathcal{X} \in \mathcal{B}} \frac{b(U_I \mathcal{X})}{b(\mathcal{X})} = \max_{\mathcal{X} \in \mathcal{B}_W(3)} \frac{b(U_{W(I)} \mathcal{X})}{b(\mathcal{X})}$$

**Proof** Using Lemma 4.7 and Corollaries 4.1 and 4.2 it follows that:

$$\max_{\mathcal{X} \in \mathcal{B}} \frac{b(U_I \mathcal{X})}{b(\mathcal{X})} = \max_{\mathcal{X} \in \mathcal{B}} \frac{b(XU_{W(I)} \mathcal{X})}{b(\mathcal{X})} = \max_{\mathcal{X} \in \mathcal{B}} \frac{b(U_{W(I)} \mathcal{X})}{b(\mathcal{X})} = \max_{\mathcal{X} \in \mathcal{B}} \frac{b(U_{W(I)} \mathcal{X})}{b(\mathcal{X})}$$

We are now equipped to prove Lemma 4.4. For this proof, we write particular sequences of iterations such as $\{4, 1, 3\}$ simply as 413.

**Proof** (Lemma 4.4) To prove this lemma, we list out elements of $\mathcal{I}_3, \mathcal{I}_4$ and $\mathcal{I}_5$ which begin with either 1 or 4 and mark them in the following way. If a sequence is contained in $\mathcal{A}$, we overline it. If a sequence contains a sub-string which is contained in $\mathcal{A}$, we underline the relevant sub-string. Finally, if a sequence contains a sub-string $J$ such that $W(J) \in \mathcal{A}$, then we place square brackets around $J$. We will show that all elements of $\mathcal{I}_3$ can be marked in one of these three ways. First we list elements of $\mathcal{I}_3$ which start with either 1 or 4. There is no need to list the sequences starting with 2 or 3 because each of those sequences is merely $W(J)$ for some $J$ listed below.

| 114 | 132 | 133 | 141 | 142 | 143 | 144 | 145 |
| 114 | 132 | 133 | 141 | 142 | 143 | 144 | 145 |
We see that only 7 (or 14 if you count the equivalent sequences starting with 2 or 3) of the sequences above are unmarked. We take these seven sequences, and list all elements of \( I^4 \) which begin with any of these 7 sub-strings. We neglect to list out elements of \( I^4 \) which begin with a marked sub-string because these elements are themselves guaranteed to be marked.

\[
\begin{align*}
1141 & \quad 1143 & \quad 1411 & \quad 4311 & \quad 42[33] & \quad 4[311] & \quad 4322 \\
1142 & \quad 1144 & \quad 1414 & \quad 1423 & \quad 4114 & \quad 4232 & \quad 4[234] & \quad 4[323]
\end{align*}
\]

Using the same procedure as earlier, we see only 3 sequences are left unmarked. We list out all elements of \( I^5 \) which begin with one of these three sequences and note that all of them are marked.

\[
\begin{align*}
11422 & \quad 42311 & \quad 4[2322] & \quad 11423 & \quad 4[2314] & \quad 42[323]
\end{align*}
\]

If a sequence \( J \) is marked in any of these ways, then it follows from Lemma 4.5 and Corollary 4.3 that \( b(U_I X)/b(X) \leq \frac{1}{2} \forall I \in A \) implies \( b(U_J X)/b(X) \leq \frac{1}{2} \).

\section{5 Dynamic Underestimating Polynomial Method}

In Section 3 we identified two main weaknesses of the SUPM: that of choosing a suitable \( \alpha \), and that of ensuring that both \( x^R_{1,i} \) and \( x^L_{1,i} \) converge to \( x^* \) fast enough. In this section, we introduce a heuristic which makes use of the features of the SUPM, while being equipped with a safeguarding option designed to avoid the situations where the SUPM fails. We call this heuristic the Dynamic UPM (DUPM), and denote the step and update functions for the DUPM by \( \sigma_D \) and \( U_D \) respectively.

The effect that \( \alpha \) has on \( \sigma_S \) lies in how the model functions (see Equation (3.3)) are constructed. We see that in this definition, \( \alpha \) is multiplied to \( h(X) \). Since \( h(X) \to 0 \) should occur as the algorithm converges, a finite but excessively large value of \( \alpha \) should not a problem ultimately. However, the DUPM may stall temporarily when \( \alpha \) is too small. Therefore, we construct the DUPM such that it will increase \( \alpha \) when necessary, but never decrease it.

In order for Theorem 3.2 to apply, we need \( \alpha \) to satisfy Lemma 3.2. While there is no way to check this, a necessary condition for Lemma 3.2 to apply is that \( f(x^M) \geq q^L(x^M) \) and \( f(x^M) \geq q^R(x^M) \). Using a method similar to the proof of Lemma 3.2, we find that this is equivalent to:

\[
\alpha \geq \frac{1}{h(X)} \max_{k \in \{L, R\}} \left( f(x^k_1, x^k_2, x^k_3) - f(x^M, x^k_1, x^k_2) \right). \tag{5.1}
\]

Therefore, at each iteration of the DUPM we set \( \alpha_{i+1} \geq \max(\alpha_i, \alpha^*) \) where \( \alpha^* \) is the smallest value \( \alpha \) which satisfies Equation (5.1).

Next we wish to force the DUPM to update both \( x^L_{1,i} \) and \( x^R_{1,i} \) regularly. In order to do this, we must first equip the DUPM to recognize when this occurs. We append the three variables \( u_1, u_2, u_3 \), whose purpose is record whether the update function \( U_D \) updated \( x^L \) or \( x^R \) during each of the last three iterations,
to the extended bracket $\mathcal{X}$. The update function $U_D$ is defined to be the natural extension of $U_S$ (see Equation (3.5)) which includes $u_1, u_2$ and $u_3$:

$$U_D(\bar{x}; \mathcal{X}, \alpha) = \begin{cases} 
(U_1(\bar{x}; \mathcal{X}), R, u_1, u_2) & \text{if } \bar{x} < x^M \text{ and } f(\bar{x}) < f(x^M) \\
(U_2(\bar{x}; \mathcal{X}), L, u_1, u_2) & \text{if } \bar{x} > x^M \text{ and } f(\bar{x}) < f(x^M) \\
(U_3(\bar{x}; \mathcal{X}), R, u_1, u_2) & \text{if } \bar{x} > x^M \text{ and } f(\bar{x}) > f(x^M) \\
(U_4(\bar{x}; \mathcal{X}), L, u_1, u_2) & \text{if } \bar{x} < x^M \text{ and } f(\bar{x}) > f(x^M) 
\end{cases} \quad (5.2)$$

where the functions $U_1, \ldots, U_4$ are as defined in Equation (3.6).

If $u_1 = u_2 = u_3$, then this means that the DUPM has updated either $x^L_{i+1}$, or $x^R_{i+1}$, for each of the last three iterations. When this occurs, we force the DUPM to take an EUPM step instead, $\sigma_D(\mathcal{X}; \alpha) = \sigma_E(\mathcal{X})$. This interference is similar to the fall back option used by Brent’s method [21]. The choice to interfere after three iterations is based on practical experience, and not on any theoretical insight.

Finally, we define one more condition under which we interfere with the SUPM. From practical experience, we observe situations where either $q^L$ or $q^R$ is a convex function whose local minimum is returned by $\sigma_S$, which result in slow convergence. As a response to this, we ensure that $\alpha$ is sufficiently big that $\sigma_D$ returns a point of intersection between $q^L$ and $q^R$. More rigorously, we require $\alpha$ such that

$$q^L(\sigma_S(\mathcal{X}; \alpha); \mathcal{X}, \alpha) = q^R(\sigma_S(\mathcal{X}; \alpha); \mathcal{X}, \alpha) \quad (5.3)$$

holds, where $\sigma_S$ is the step function for the SUPM (see Equation (5.4)).

Define $\chi$ to be the the smallest value of $\alpha$ such that Equation (5.3) holds for all $\alpha \geq \chi(\mathcal{X})$. Then at each iteration of the DUPM, we require $\alpha_{i+1} \geq \max(\alpha_i, \chi(\mathcal{X}))$.

In order to compute $\chi(\mathcal{X})$, note Equation (5.3) is satisfied whenever $q^L$ and $q^R$ are concave functions, provided that Equation (5.1) holds. Define $\alpha^+$ to be $\max_k f(x_{i,k}^L, x_{i,k}^R, x_{i,k}^M) \in \{L, R\}$. Then it follows that for all $\alpha > \alpha^+$, Equation (5.3) is satisfied, implying that $\chi(\mathcal{X}) \leq \alpha^+$.

If at iteration $i$, $\alpha_i$ satisfies Equation (5.3), then there is no need to compute $\chi(\mathcal{X})$. Otherwise, we know that $\chi(\mathcal{X})$ lies in the interval $(\alpha_i, \alpha^+]$. Therefore, we can apply bisection to compute $\chi(\mathcal{X})$ to any desired accuracy.

To define the DUPM rigorously, we must insert an additional line into Algorithm 1. Before computing $\sigma_D$, we set

$$\alpha_{i+1} = \max \left( \alpha_i, \chi(\mathcal{X}_i), \frac{1}{h(\mathcal{X})} \max_{k \in \{L, R\}} \left( f(x_{1,k}^L, x_{2,k}^R, x_{3}^M) - f(x_{1}^M, x_{1,k}^L, x_{2,k}^R) \right) \right).$$

Finally we define

$$\sigma_D(\mathcal{X}; \alpha) := \begin{cases} 
\sigma_E(\mathcal{X}) & \text{if } u_1 = u_2 = u_3 \\
\sigma_S(\mathcal{X}; \alpha) & \text{else} 
\end{cases} \quad (5.4)$$

Ultimately the purpose of these interferences is to ensure that Theorem 3.2 applies as soon as possible while forcing both $x^L_{i+1}$ and $x^R_{i+1}$ to be updated.
6 Numerical Results

Thus far, we have presented three algorithms: the SUPM, EUPM, and DUPM. Of these, we have only bounded the theoretical convergence rate for EUPM. While Theorem 3.2 is insightful, it does not yield any concrete bound. Moreover, while the DUPM has some nice properties which are designed to strengthen the SUPM, we have not offered any proof as to their effectiveness. Therefore, in order to justify these algorithms, we demonstrate their use on a variety of test functions, comparing them to Brent’s method [1,2] and the Mifflin-Strodiot method [11]. Also, recall Golden Section which converges Q-linearly with rate 0.618.

We consider three categories of test functions: smooth unimodal, non-smooth unimodal and smooth multimodal. To our knowledge there is no commonly used set of univariate test functions. For our experiments, we have taken some test functions from Jamil et al [8], and have added a few of our own, which are designed to be adversarial examples based on our observations. We scale each test function $f$ such that the minimum Lipschitz constant for $f''$ lies between 0.8 and 1. For the category of smooth unimodal, we use the following test functions:

$$f_{SU}^1 = -\frac{1}{\sqrt{e}}\exp\left(-\frac{1}{2}x^2\right), \quad -1 \leq x \leq 1, \quad (6.1)$$

$$f_{SU}^2 = \frac{1}{24}x^4, \quad -1 \leq x \leq 1, \quad (6.2)$$

$$f_{SU}^3 = \frac{1}{11}\left(-\sin(2x - \frac{1}{2}\pi) - 3\cos x - \frac{1}{2}x\right), \quad -2.5 \leq x \leq 3, \quad (6.3)$$

$$f_{SU}^4 = \frac{1}{2500}\left(\frac{x^2}{2} - \frac{\cos(5\pi x)}{25\pi^2} - \frac{x\sin(5\pi x)}{5\pi}\right), \quad -10 \leq x \leq 10, \quad (6.4)$$

$$f_{SU}^5 = -\frac{1}{250}\left(x^\frac{3}{2} + (1 - x^2)^\frac{3}{2}\right), \quad 0.1 \leq x \leq 0.9, \quad (6.5)$$

$$f_{SU}^6 = \frac{1}{6000}\left(e^x + \frac{1}{\sqrt{x}}\right), \quad 0.1 \leq x \leq 3, \quad (6.6)$$

$$f_{SU}^7 = -\frac{1}{13}(16x^2 - 24x + 5)e^{-x}, \quad 1.3 \leq x \leq 3.9. \quad (6.7)$$

For each test function, we have an interval $[a, b]$ on which the function is intended to be used. We sample 4 points uniformly from both $[a, a + \frac{1}{5}(b - a)]$ and $[b - \frac{1}{5}(b - a), b]$. After evaluating the function at these 8 points, we select the point where $f$ is minimised to be $x^M$; usually this will be one of the points closer to $\frac{1}{2}(b - a)$. Finally we order the remaining points, and select the 3 closest on the left and right to be $x_L^1, x_L^2, \ldots, x_R^3$. Selecting our initial points in this way yields an extended bracket.

Having generated the bracket, we run each algorithm and compute the average convergence rate over 1000 different random initialisations. If an algorithm converged for a particular starting bracket, then its average convergence rate is by definition between 0 and 1, the smaller the better. If for any random
A seven-point algorithm for piecewise smooth univariate minimization

Other methods

Functions | SUPM with parameter value | Other methods |
|-----------|---------------------------|---------------|
| $f_{SU}^1$ | $\alpha = 0$ 0.3268 0.4273 0.4912 0.5472 0.6192 0.48 0.2159 0.1477 | EUPM 0.3268 0.4273 0.4912 0.5472 0.6192 0.48 0.2159 0.1477 |
| $f_{SU}^2$ | $\alpha = 0.1$ 0.68 0.6164 0.6183 0.6187 0.6188 0.6182 0.3282 0.6614 | DUPM 0.6105 0.6104 0.6104 0.6107 0.6103 0.5626 $\infty$ |
| $f_{SU}^3$ | $\alpha = 1$ $\infty$ 0.4848 0.5438 0.5882 0.6349 0.5293 0.3329 $\infty$ | Brent 0.5871 0.615 0.6246 0.6249 0.5569 0.2812 $\infty$ |
| $f_{SU}^4$ | $\alpha = 10$ 0.5469 0.5925 0.6233 0.6323 0.4854 0.2756 0.2581 | Mifflin 0.5871 0.615 0.6246 0.6249 0.5569 0.2812 $\infty$ |
| $f_{SU}^5$ | $\alpha = 0$ 0.35 0.4966 0.5521 0.5944 0.6353 0.4885 0.2575 $\infty$ | $\infty$ |

Table 6.1 Average convergence rate of different algorithms on the set of smooth unimodal test functions.

initialisation, an algorithm did not converge, we assign $\infty$ instead. We present these results in Table 6.1.

First of all, we see that Brent’s method consistently converges the fastest. This is not surprising, given that Brent’s method is the only algorithm shown here which is specifically designed for smooth functions. While the Mifflin-Strodiot method is faster in select circumstances, it fails entirely on others. This is also what we observe from the SUPM with $\alpha = 0$.

Looking at the SUPM’s performance, we see that the larger $\alpha$ is, the slower the SUPM will converge (assuming it converges in the first place). The DUPM outperforms the SUPM with $\alpha = 1$, but is not always faster than the SUPM with $\alpha = 0.1$, which suggests that while Lemma 3.2 is required for our theoretical results, satisfying it is by no means necessary. Meanwhile, the EUPM is both consistent and slow, converging with an average rate comparable to Golden Section.

In short, the DUPM is not competitive with Brent’s method for this class of test functions. However, it converge with an average rate which is better than for Golden Section, implying that it is not prohibitively slow.

Next we define our set of non-smooth unimodal test functions:

\[
\begin{align*}
f_{1}^{NU} &= -60000 \exp \left( \frac{|x|}{50} \right), & -32 \leq x \leq 32, \\
f_{2}^{NU} &= \frac{1}{6} \max \left( \frac{1}{x+3}, \log(x) \right), & -2 \leq x \leq 10, \\
f_{3}^{NU} &= \frac{1}{24} \max \left( \frac{1}{x+3}, \frac{1}{(x-3)^2} \right), & -2 \leq x \leq 2, \\
f_{4}^{NU} &= \frac{1}{160} \max \left( \frac{1}{x+3}, \exp(x) \right), & -2 \leq x \leq 5, \\
f_{5}^{NU} &= \frac{1}{150} \max (\exp(-x), \exp(x)), & -5 \leq x \leq 5.
\end{align*}
\]

The results for these functions are shown in Table 6.2. These Results are particularly important as it was for this class of functions that the UPM was designed in the first place. As we’d hope, we see the SUPM with $\alpha = 1$ and the DUPM outperform Brent’s method for every function. Despite the fact
that the Mifflin-Strodiot method is also designed for this context, it does not always successfully converge. In particular, we observe that the Mifflin-Strodiot method struggles with non-convex functions.

Finally we define our smooth multimodal test functions.

\[
\begin{align*}
&f_1^{SM} = \frac{x^6}{300} \left( 2 + \sin \left( \frac{1}{x} \right) \right), \quad -1 \leq x \leq 1, \quad (6.13) \\
&f_2^{SM} = -\frac{1}{80000} \sin (5\pi x)^6, \quad -1 \leq x \leq 1, \quad (6.14) \\
&f_3^{SM} = -\frac{1}{250000} \sin \left( 5\pi (x^{\frac{1}{2}} - \frac{1}{20}) \right)^6, \quad 0.01 \leq x \leq 1, \quad (6.15) \\
&f_4^{SM} = \frac{1}{5} \left( \sin \left( \frac{16}{15} x - 1 \right) + \sin \left( \frac{16}{15} x - 1 \right) \right)^2, \quad -1 \leq x \leq 1, \quad (6.16) \\
&f_5^{SM} = \frac{x^2}{4000} - \cos x + 1, \quad 100 \leq x \leq 100, \quad (6.17) \\
&f_6^{SM} = \frac{1}{71} \left( (\log(x - 2))^2 + (\log(10 - x))^2 - x^{\frac{1}{2}} \right), \quad 2.5 \leq x \leq 9.5, \quad (6.18) \\
&f_7^{SM} = \frac{1}{40} \left( \sin(x) + \sin \left( \frac{10x}{3} \right) + \log(x) + \frac{21}{25} x \right), \quad 0.5 \leq x \leq 10. \quad (6.19)
\end{align*}
\]

It is important to note that theory for neither the UPM nor indeed Brent’s method is suitable for multi-modal functions. However, both algorithms are equipped with features ensuring sufficient robustness in the general case that they converge (possibly slowly) to a locality where the function is unimodal.

For these test-functions, Brent’s method is once again the best. While slower than Brent’s method, the DUPM consistently performs better than Golden Section, which is an ideal result considering that these are not ideal circumstances for the DUPM. Finally we see that the Mifflin-Strodiot method is entirely unsuitable for this type of problem, and as such consistently fails to converge in reasonable time.
Table 6.3  Average convergence rate of different algorithms on the set of smooth multimodal test functions.

| Functions | SUPM with parameter value | Other methods |
|-----------|---------------------------|---------------|
|           | α = 0                     | EUPM | DUPM | Brent | Mifflin |
| $f_{SM}$  | ∞                         | 0.6146 | 0.6144 | 0.6149 | 0.6224 | 0.4828 | ∞      |
| $f_{SM}$  | ∞                         | 0.5076 | 0.5579 | 0.5961 | 0.6183 | 0.486  | 0.3588 | ∞      |
| $f_{SM}$  | ∞                         | 0.5481 | 0.5918 | 0.6177 | 0.6234 | 0.5085 | 0.3372 | ∞      |
| $f_{SM}$  | ∞                         | 0.4536 | 0.5308 | 0.5789 | 0.6338 | 0.5282 | 0.2703 | ∞      |
| $f_{SM}$  | ∞                         | 0.4704 | 0.5188 | 0.5582 | 0.6165 | 0.5143 | 0.311  | ∞      |
| $f_{SM}$  | ∞                         | 0.5099 | 0.5962 | 0.6191 | 0.624  | 0.5353 | 0.2982 | ∞      |
| $f_{SM}$  | ∞                         | 0.4733 | 0.5353 | 0.5776 | 0.6224 | 0.4815 | 0.273  | ∞      |

7 Conclusion

In this paper, we have constructed a univariate optimization algorithm for black box, piece-wise smooth functions. As seen in Section 6, this new method, the DUPM, converges both more robustly and often faster than existing methods for such problems. Furthermore, while it is not the fastest algorithms for standard smooth test functions, it is not prohibitively slow either.

It must be acknowledged that the comparison between the UPM, the Mifflin-Strodiot method and Brent’s method is not entirely fair given that they require a different number of points to start. Therefore, in the context of a line-search, we expect to start with a method like Golden Section, and switch to the DUPM once we have accumulated enough points to form and extended bracket. Regardless, the DUPM offers a univariate solver which performs well in most contexts and is therefore suitable for non-smooth functions.

A Proof of Lemma 4.3

For the calculations in this section, we use the following change of variables: $t = (p \ q \ r \ s)^T = (x_1^L - x_2^L, x^M - x_1^L, x_1^R - x_2^R, x_2^R - x_1^R)^T$. Converting our calculations to being in terms of $t$ both reduces the number of variables, and simplifies the constraint $x \in \tilde{B}$ (recall Definition 4.3) into $t \in \mathbb{R}_+^4$. 
The algebra for the calculations which follow is tedious, and therefore we employed Mathematica to compute the values of $U_I$, $I \in A$ as well as changing the variables used.

\[
\frac{b(U_{444}X)}{b(X)} = \frac{(q + r)(p + q + r)}{2(q + r)(p + q + r) + (q + r)^2 + s^2 + s(p + 3q + 3r)} < \frac{1}{2}
\]

\[
\frac{b(U_{111}X)}{b(X)} = \frac{q}{p + 2q + r} < \frac{1}{2}
\]

\[
\frac{b(U_{111}X)}{b(X)} = \frac{q(p + q)(p + q + r)}{(p + 2q + r)(3q + r + r^2 + p(2q + r))} \frac{q(p + q)}{(p + q + r)} < \frac{1}{2}
\]

\[
\frac{b(U_{422}X)}{b(X)} = \frac{1}{2(q + r)(p + q + r) + (q + r)^2 + s^2 + s(p + 3q + 3r)} < \frac{1}{2}
\]

\[
\frac{b(U_{114}X)}{b(X)} = \frac{1}{2 + \left(\frac{4q^2 + p^2r + 7q^2r + 5qr^2 + r^3 + p(q + r)(3q + 2r)}{q(p + q)(p + q + r)}\right)} < \frac{1}{2}
\]

\[
\frac{b(U_{122}X)}{b(X)} = \frac{1}{2 + \left(\frac{4q^2 + p^2r + 7q^2r + 5qr^2 + r^3 + p(q + r)(3q + 2r)}{q(p + q)(p + q + r)}\right)} < \frac{1}{2}
\]

\[
\frac{b(U_{144}X)}{b(X)} = \frac{1}{2 + \left(\frac{4q^2 + p^2r + 7q^2r + 5qr^2 + r^3 + p(q + r)(3q + 2r)}{q(p + q)(p + q + r)}\right)} < \frac{1}{2}
\]

For the sequence 414, we need to employ another piece of information. From Equation (3.5), we see that $U_3$ is only applied to $X$ if $\varphi(X) < 0$ (see Equation (4.4)). This is equivalent to $q(p + q) - r(r + s) > 0$ when written in terms of our alternative variables. Proceeding with the calculation we find:

\[
\frac{b(U_{414}X)}{b(X)} = \frac{1}{2 + \left(\frac{4q^2 + p^2r + 7q^2r + 5qr^2 + r^3 + p(q + r)(3q + 2r)}{q(p + q)(p + q + r)}\right)} < \frac{1}{2}
\]

which is bounded above by $\frac{1}{2}$ precisely when $q(p + q) - r(r + s) > 0$. Therefore $b(U_{414}X)/b(X) < \frac{1}{2}$ as required.

There remain 4 elements of $A$ for which we must establish a bound: 434, 4322, 4314 and 4114. As the previous change of variables does little to simplify the calculations for these subsequences, we instead use the following alternative change of variables: $(a, b, c, d) = (x^2, x^3, x^4, x^5, \varphi(X))$. For this set of variables, $X \in B$ is equivalent to
a, b, c > 0, b > a and c > a. Using this transformation, we find:

\[
\frac{b(U_{434}, \mathcal{X})}{b(\mathcal{X})} = \frac{bc(b + c)(b + c - a)}{(ab + bc + c^2)(ac(b + c)^2 - d(ab + bc + c^2))} < \frac{bc(b + c)}{ab^2 + 2b^2c + 3bc^2 + c^3} < \frac{bc(b + c)}{b + c} < \frac{1}{2}
\]

\[
\frac{b(U_{4322}, \mathcal{X})}{b(\mathcal{X})} = \frac{b(b + c)}{bc(b + c)(b + c - a)} < \frac{b(b + c)}{ab^2 + 2b^2c + 3bc^2 + c^3} < \frac{b}{b + c} < \frac{1}{2}
\]

**Remark A.1** From the calculations above, we see that \(b(U_{434}, \mathcal{X})/b(\mathcal{X}) = b(U_{4322}, \mathcal{X})/b(\mathcal{X})\) and \(b(U_{44}, \mathcal{X})/b(\mathcal{X}) = b(U_{422}, \mathcal{X})/b(\mathcal{X})\). While this may imply another relation which we have not taken advantage of, we have not examined this further.

Finally we have the sequences 4314 and 4114. Similar to what we did with the sequence 414, we need to make use of additional information, specifically the fact that \(U_4\) is only applied if \(\varphi(\mathcal{X}) = d < 0\). In addition, \(U_1\) may only follow after \(U_4\) if \(\varphi(U_4, \mathcal{X}) < 0\). This is equivalent to: \(-d(ab + bc + c^2) > abc(b + c - a)\). From this it follows that:

\[
\frac{b(U_{4114}, \mathcal{X})}{b(\mathcal{X})} = \frac{abc(b + c)(b + c - a)}{(ab + bc + c^2)(ac(b + c)^2 - d(ab + bc + c^2))} < \frac{abc(b + c)}{ab^2 + 2b^2c + 3bc^2 + c^3} < \frac{abc(b + c)}{b + c} < \frac{1}{2}
\]

Similarly for the sequence 4314, \(U_4\) may only follow after \(U_4\) if \(\varphi(U_4, \mathcal{X}) > 0\). This is equivalent to: \(-d(ab + bc + c^2) < abc(b + c - a)\). First we compute

\[
\frac{b(U_{4314}, \mathcal{X})}{b(\mathcal{X})} = \frac{-d(ac - d)(ab + bc + c^2)}{a(b + c)(bc(b + c)^2 - d(ab + bc + c^2))}.
\]

Next we observe that \(\varphi(U_4, \mathcal{X}) > 0\) is equivalent to:

\[
-d > \frac{abc(b + c - a)}{ab + bc + c^2} \iff ac - d > \frac{ac(b + c)^2}{ab + bc + c^2}.
\]

This implies that:

\[
\frac{b(U_{4314}, \mathcal{X})}{b(\mathcal{X})} < \frac{-dc(b + c)}{ac(b + c)^2 - d(ab + bc + c^2)}.
\]

Finally we that a sufficient condition for \(b(U_{4314}, \mathcal{X})/b(\mathcal{X}) < \frac{1}{2}\) is:

\[
-2dc(b + c) < ac(b + c)^2 - d(ab + bc + c^2),
\]

\[
\iff -d(-ab + bc + c^2) < ac(b + c)^2.
\]

We see that this is true when we combine the requirement that \(\varphi(U_4, \mathcal{X}) > 0\) must hold for the sequence 4314 to occur with Equation (A.1). In particular:

\[
-d(-ab + bc + c^2) < -d(ab + bc + c^2) < (ac - d)(ab + bc + c^2) < ac(b + c)^2.
\]

Therefore, \(b(U_{4314}, \mathcal{X})/b(\mathcal{X}) < \frac{1}{2}\) holds and moreover \(b(U_1, \mathcal{X})/b(\mathcal{X}) < \frac{1}{2} \forall I \in \mathcal{I}_5\).
B Numerical Performance of the EUPM

In Section 4.1 we bounded the convergence rate over 5 iterations of the EUPM. In practice we observe far superior convergence rates. From our observation, this seems to be because the extremal values of \( t_0 \) which yield the worst convergence rate over 5 iterations tend to yield best case performance over 6 iterations and so on. In this section, we show what performance might realistically be expected from the EUPM.

In order to construct a suitable experiment, we first must ask what factors affect the convergence of the EUPM? The answer to this question is the function to minimise \( f \), along with the initial value of \( t_0 \). However, the only effect that the function \( f \) actually has is to determine whether \( f(\tilde{x}) < f(x^M) \). This along with the current value of \( t \) uniquely determines which update function will be used. Therefore, when testing the convergence rate of the EUPM, we do not test it on a range of functions, but rather for different binary sequences, where 1’s mean \( f(\tilde{x}) < f(x^M) \) and 0’s the opposite.

The main advantage of this is that while the set \( C([a, b]) \) has infinite cardinality, the set of binary sequences of length \( n \) is finite. Therefore, given an initial bracket \( t_0 \), it is possible to determine how the EUPM will perform on literally any function. Given \( t_0 \) and \( f \in \{0, 1\}^n \), we plot the average convergence rate of the EUPM.

![Fig. B.1 We compare the convergence rate of the EUPM with Golden Section over all possible sequences \( I \in \{0, 1\}^1 \). These sequences are ordered such that the slowest ones are plotted first. For each sequences, we sample a large number of values for \( t_0 \) and average over these.](image)

For each sequence \( f \in \{0, 1\} \), we sample a large number \( t_0 \) values, constrained such that \( |t_0| = 1 \), and average over these. This leaves us with an average convergence rate for every possible sequence in \( \{0, 1\} \), and by extension for every possible function. We choose \( n = 10 \) and plot the resulting data corresponding to both the EUPM and Golden Section in Figure B.1.

What we see is that Golden Section is the more conservative of the two. It performs notably better in the worst case scenario even if notably worse in the best case scenario. Even though we see some convergence rates from the EUPM which are significantly worse than those observed in Tables 6.1 to 6.3, they are far better than the upper bound from Theorem 4.1.

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