Stability of Schwarzschild black hole in f(R) gravity with the dynamical Chern-Simons term

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Abstract

We perform the stability analysis of the Schwarzschild black hole in $f(R)$ gravity with the parity-violating Chern-Simons (CS) term coupled to a dynamical scalar field $\theta$. For this purpose, we transform the $f(R)$ gravity into the scalar-tensor theory by introducing a scalaron $\phi$, providing the dynamical Chern-Simons modified gravity with two scalars. The perturbation equation for the scalar $\theta$ is coupled to the odd-parity metric perturbation equation, providing a system of two coupled second order equations, while the scalaron is coupled to the even-parity perturbation equation. This implies that the CS coupling affects the Regge-Wheeler equation, while $f(R)$ gravity does not affect the Zerilli equation. It turns out that the Schwarzschild black hole is stable against the external perturbations if the scalaron is free from the tachyon.

PACS numbers:


1 Introduction

$f(R)$ gravities [1, 2, 3] have much attention as one of strong candidates for explaining the current accelerating universe [4]. $f(R)$ gravities can be considered as Einstein gravity with an additional scalar (scalaron). For example, it was shown that the metric-$f(R)$ gravity is equivalent to the $\omega_{BD} = 0$ Brans-Dicke theory with a certain potential [5].

On the other hand, the Chern-Simons (CS) modified gravity was obtained by adding a parity-violating CS term to the Einstein-Hilbert action, where the CS term couples to gravity via a CS scalar field $\theta$ [6]. Originally the coupled scalar field $\theta$ was considered as a prescribed function, but on later this choice was not regarded as the well-motivated one. Indeed, the dynamical Chern-Simons (DCS) modified gravity has been formulated by treating the scalar field $\theta$ as a dynamical field [7]. For a review on the CS modified gravity, its astrophysical consequences, see [8] and for its critical gravity on the AdS$_4$ spaceimes, see [9, 10].

It is very interesting to investigate the Schwarzschild black hole obtained from a modified gravity of the $f(R)$ gravity with the dynamical CS term because astrophysical black holes are the most promising objects to probe the strongly gravitational field region of a modified gravity. The first study of the $f(R)$-black hole stability has very recently been performed in the $f(R, G)$ gravity [11]. In its scalar-tensor theory [12], the even-parity perturbations were affected by the scalaron and thus the black hole was stable against the whole perturbations if the scalaron did not have a tachyonic mass. In the context of the DCS modified gravity, the black hole perturbation has been carried out in [13], which indicates that if the background CS scalar $\bar{\theta}$ is a non-trivial, there was a serious mixing between odd- and even-parity metric perturbations. On the other hand, if $\bar{\theta} = 0$ or const., odd- and even-perturbations were decoupled as in Einstein gravity and odd-perturbations are affected only by the CS scalar field [14]. The odd-parity and CS scalar perturbations were described by a coupled system of two second-order equations, which has shown that the black hole is stable in the DCS modified gravity [15].

Very recently, there was a perturbation study on the black hole in the context of $f(R, C)$ modified gravity with $C$ the CS term [16]. The black hole is unstable because the perturbed Hamiltonian is not bounded from below, due to the CS term. In order to avoid the instability, either $\bar{R} = \text{const.}$ or \( \frac{\partial^2 f}{\partial R \partial C} = 0 \) is required. In this case, number of physically propagating degrees of freedom are three, one from odd-parity and two from even-parity and
scalaron because the $f(R, C)$ modified gravity belongs to the non-dynamical CS modified gravity. Those modes are too strongly coupled to decouple three independent modes, which shows a distinctive feature of a parity-violating theory. However, the no-ghost condition of $\frac{\partial f(R, C)}{\partial R} > 0$ and no-tachyon condition of $\frac{\partial^2 f(R, C)}{\partial R^2} > 0$ survive as in $f(R)$ gravities.

In this work, we wish to perform the stability analysis of the Schwarzschild black hole in $f(R)$ gravity with the parity-violating CS term coupled to a dynamical scalar field $\theta$. In order to avoid the difficulty with fourth-order derivative terms, we first transform the $f(R)$ gravity into the scalar-tensor theory by introducing a scalaron $\phi$. This will provide the DCS modified gravity with two scalars, which means that four modes are physically propagating degrees of freedom. Interestingly, the perturbation equation for the CS scalar $\theta$ is coupled to the odd-parity metric perturbation equation, providing a system of two coupled second-order equations, while the scalaron $\phi$ is coupled to the even-parity perturbation equation. This enables us to perform the stability analysis of the Schwarzschild black hole obtained from $f(R)$+DCS modified gravity theory completely. To make all things clear, we mention our notations. The metric signature is $(-, +, +, +)$. The Riemann, Ricci tensor and Levi-Civita tensor are defined by

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}, \quad R_{\mu\nu} = R^\rho_{\mu\rho\nu}, \quad \epsilon^{\rho\nu\lambda\sigma} = \frac{1}{\sqrt{-g}}.$$  

2 $f(R)$ gravity with the DCS term

Let us consider $f(R)$ gravity with the dynamical Chern-Simons term in four dimensions which is given by

$$S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ f(R) + \frac{\theta}{4} \ast RR - \alpha \nabla_\mu \theta \nabla_\mu \theta \right]$$  

where $\kappa^2 = 8\pi G$, $\alpha$ is a dimensional constant, and $\ast RR = \ast R_{\xi}^{\eta \mu \nu} R_{\eta \mu \nu}$ is the Pontryagin density with

$$\ast R_{\xi}^{\eta \mu \nu} = \frac{1}{2} \epsilon^{\mu \nu \rho \sigma} R_{\xi \rho \sigma}^\eta.$$  

Here $\epsilon^{\mu \nu \rho \sigma}$ denotes the four-dimensional Levi-Civita tensor. It is well known that the action can be rewritten by introducing a scalaron field $\phi$ as follows [17]:

$$S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-g} \left[ \phi R - V(\phi) + \frac{\theta}{4} \ast RR - \alpha \nabla_\mu \theta \nabla_\mu \theta \right]$$  

3
with the potential \( V(\phi) = \phi A(\phi) - f(A(\phi)) \). Note that the mass dimensions of \( \phi, \theta \), and \( \alpha \) are given by \( [\phi] = 0 \), \( [\theta] = -2 \), \( [\alpha] = 4 \), respectively. Varying for the fields \( g_{\mu\nu}, \phi, \) and \( \theta \) lead to the following equations:

\[
\phi \left( R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) + \frac{1}{2} g_{\mu\nu} V(\phi) + \left( g_{\mu\nu} \nabla^2 \phi - \nabla_\mu \nabla_\nu \phi \right) = -C_{\mu\nu} + \alpha \left( \nabla_\mu \theta \nabla_\nu \theta - \frac{1}{2} g_{\mu\nu} \nabla_\rho \theta \nabla^\rho \theta \right),
\]

(2.4)

\[
R = V'(\phi),
\]

(2.5)

\[
\nabla^2 \theta = -\frac{1}{8\alpha} R R
\]

(2.6)

where \( ' \) denotes differentiation with respect to \( \phi \), and \( C_{\mu\nu} \) takes the form

\[
C_{\mu\nu} = \nabla_\rho \theta \epsilon^{\rho\sigma\gamma\mu} \nabla_\gamma R_{\nu}\sigma + \frac{1}{2} \nabla_\rho \nabla_\sigma \theta \epsilon^{\rho\gamma\delta} R^\sigma_{\mu\gamma\delta}.
\]

(2.7)

We take the trace of (2.4) to rewrite (2.5) as the scalaron equation

\[
3\nabla^2 \phi + 2V(\phi) - \phi V'(\phi) = -2\alpha \nabla_\mu \theta \nabla^\mu \theta.
\]

(2.8)

Also we can express Eq. (2.4) to be

\[
\phi R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} V(\phi) - \frac{1}{2} g_{\mu\nu} \nabla^2 \phi - \nabla_\mu \nabla_\nu \phi = -C_{\mu\nu} + \alpha \nabla_\mu \theta \nabla_\nu \theta.
\]

(2.9)

Taking the restricted background values\(^1\) as

\[
\bar{\theta} = \text{const}., \quad \phi = \bar{\phi} = \text{const}., \quad V(\bar{\phi}) = V'(\bar{\phi}) = 0, \quad V''(\bar{\phi}) \neq 0,
\]

(2.10)

the solution to the Eqs. (2.4), (2.5) and (2.6) is given by the Schwarzschild spacetime

\[
ds_{\text{Sch}}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu
\]

\[
= -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\varphi_1^2 + \sin^2 \varphi_1 d\varphi_2^2)
\]

(2.11)

\(^1\) When taking these values, it gives the background spacetimes with the constant curvature scalar \( \bar{R} \) which provides an easy step to find the solution of \( f(R) \) gravity. In obtaining the constant curvature-black hole solutions (for example, Schwarzschild and Schwarzschild-(A)dS black holes), it seems that there is no difference between \( \bar{\theta} = 0 \) and \( \bar{\theta} = \text{const} \). Hence we choose \( \bar{\theta} = \text{const} \. \) Here. Note that \( \bar{\phi} \) corresponds to \( f'(\bar{R}) \) in the original \( f(R) \) gravity and \( \bar{R} = V'(\bar{\phi}) = 2V(\bar{\phi})/\bar{\phi} \) from Eqs. (2.5) and (2.8). In this work, we focus on the Schwarzschild black hole solution with \( \bar{R} = 0 \), which implies that \( V'(\bar{\phi}) = V(\bar{\phi}) = 0 \). We mention that this is possible to occur when choosing a limited form of \( f(R) \) gravity: \( f(R) = a_1 R + a_2 R^2 + \cdots \) [18] [19] [20]. In this case, one finds that \( \bar{\phi} = a_1 \).
with the metric function
\[ f(r) = 1 - \frac{2M}{r}. \]

(2.12)

Now we introduce the perturbation around the background metric as
\[ g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}. \]

(2.13)

The perturbations around the background solution \( \tilde{\phi} \) and \( \tilde{\theta} \) are given by
\[ \theta = \bar{\theta} + \delta \theta, \quad \phi = \tilde{\phi} + \delta \phi. \]

(2.14)

The linearized equation to (2.9) can be written by
\[ \bar{\nabla}^2 - \frac{1}{3} \bar{\phi} V''(\bar{\phi}) \delta \phi = -\delta C_{\mu\nu} \]

(2.15)

where the linearized quantities of \( \delta R_{\mu\nu}(h) \), \( \delta R(h) \), and \( \delta C_{\mu\nu}(h) \) take the forms
\[ \delta R_{\mu\nu}(h) = \frac{1}{2} \left( \bar{\nabla}^2 h_{\mu\nu} + \bar{\nabla}^\gamma \bar{\nabla}_\nu h_{\mu\gamma} - \bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu \bar{\nabla}_\nu h \right) \]
\[ \delta R(h) = \bar{\nabla}^\mu \bar{\nabla}_\nu h_{\mu\nu} - \bar{\nabla}^2 h \]
\[ \delta C_{\mu\nu}(h) = \frac{1}{2} \bar{\nabla}_\rho \bar{\nabla}_\sigma \delta \theta \epsilon_{\mu\nu\sigma\delta} \bar{R}^\eta_{\rho\sigma} \bar{\nabla}_\rho \bar{\nabla}_\eta h_{\delta}. \]

(2.16)

In these expressions, the “overbar” denotes the background quantities. From Eq.(2.8) and (2.6), we obtain the linearized-scalaron equation
\[ \left[ \bar{\nabla}^2 - \frac{1}{3} \bar{\phi} V''(\bar{\phi}) \right] \delta \phi = 0 \]

(2.17)

and the linearized-\( \theta \) equation
\[ \bar{\nabla}^2 \delta \theta = -\frac{1}{4\alpha} \epsilon^{\mu\nu\rho\sigma} \bar{R}^\eta_{\rho\sigma} \bar{\nabla}_\rho \bar{\nabla}_\eta h_{\delta}. \]

(2.18)

3 Perturbation analysis

The metric perturbations \( h_{\mu\nu} \) are classified according to the transformation properties under parity, namely odd sector \( (h_0, h_1) \) and even sector \( (H_0, H_1, H_2, K) \). However it is nontrivial task to show how the decoupling process goes with two scalar fields \( (\delta \theta, \delta \phi) \) well.\(^2\)

\(^2\)It turns out that for \( \theta \neq \text{const.} \), there was mixing between odd and even modes in Chern-Simons modified gravity \[^{13}\]. Here we can avoid this difficulty by choosing \( \theta = \text{const.} \).
to see this explicitly, we must consider the full metric perturbation as

\[ h_{\mu\nu} = \begin{pmatrix}
H_0(r)Y & H_1(r)Y & \frac{-\partial_{\psi 1}Y}{\sin \phi_1} h_0(r) & \sin \phi_1 \partial_{\psi 1} Y h_0(r) \\
H_1(r)Y & H_2(r)Y & \frac{-\partial_{\psi 2}Y}{\sin \phi_1} h_1(r) & \sin \phi_1 \partial_{\psi 2} Y h_1(r) \\
\frac{-\partial_{\psi 1}Y}{\sin \phi_1} h_0(r) & \frac{-\partial_{\psi 2}Y}{\sin \phi_1} h_1(r) & r^2 Y K(r) & 0 \\
\sin \phi_1 \partial_{\psi 1} Y h_0(r) & \sin \phi_1 \partial_{\psi 2} Y h_1(r) & 0 & r^2 \sin^2 \phi_1 Y K(r)
\end{pmatrix} e^{-ikt} \] (3.1)

with \( Y \equiv Y^{LM}(\phi_1, \phi_2) \) spherical harmonics. The form of \( \delta \theta \) and \( \delta \phi \) are given by

\[ \delta \theta = \frac{\psi(r)}{r} Y e^{-ikt}, \quad \delta \phi = \frac{\Phi(r)}{r} Y e^{-ikt}. \] (3.2)

Substituting Eqs. (3.1) and (3.2) into Eq. (2.15) and after tedious manipulations, we find the perturbation equations for ten components as

\[
\begin{align*}
(t, t); & \quad e^{-ikt} E_1 Y = 0 \\
(t, r); & \quad e^{-ikt} E_2 Y = 0 \\
(t, \phi_1); & \quad e^{-ikt} \left( E_3 \partial_{\psi_1} Y + O_1 \partial_{\phi_2} Y \right) = 0 \\
(t, \phi_2); & \quad e^{-ikt} \left( E_3 \partial_{\psi_2} Y + O_2 \partial_{\phi_1} Y \right) = 0 \\
(r, r); & \quad e^{-ikt} E_4 Y = 0 \\
(r, \phi_1); & \quad e^{-ikt} \left( E_5 \partial_{\psi_1} Y + O_3 \partial_{\phi_2} Y \right) = 0 \\
(r, \phi_2); & \quad e^{-ikt} \left( E_5 \partial_{\psi_2} Y + O_3 \partial_{\phi_1} Y \right) = 0 \\
(\phi_1, \phi_1); & \quad e^{-ikt} \left( E_6 Y + E_7 \partial_{\psi_1}^2 Y + O_5 \partial_{\phi_2}Y + O_6 \partial_{\phi_1} \partial_{\phi_2} Y \right) = 0 \\
(\phi_1, \phi_2); & \quad e^{-ikt} \left( E_8 \partial_{\phi_1} Y + E_7 \partial_{\psi_1} \partial_{\phi_2} Y + O_7 Y + O_8 \partial_{\phi_1}^2 Y \right) = 0 \\
(\phi_2, \phi_2); & \quad e^{-ikt} \left( E_9 Y + E_7 \partial_{\psi_2}^2 Y + E_{10} \partial_{\phi_1} Y + O_9 \partial_{\phi_2} Y + O_{10} \partial_{\phi_1} \partial_{\phi_2} Y \right) = 0,
\end{align*}
\] (3.3)

where \( E_i \) with \( i = 1, \ldots, 10 \) are functions of \( (H_0, H_1, H_2, K, \Phi) \) and \( O_i \) with \( i = 1, \ldots, 10 \) are functions of \( (h_0, h_1, \psi) \) (see Appendix for the details). It is important to note that for \( L > 1 \), the perturbation equations \( \{E_i\} \) imply twenty conditions like

\[ E_i = 0, \quad O_i = 0, \quad \text{for } i = 1, \ldots, 10 \] (3.4)

which mean that ten perturbation equations can be decoupled into two classes: odd-parity (\( \{O_i\} \)) and even-parity (\( \{E_i\} \)).
For the even-parity case, we observe that the condition of $E_7 = 0$ yields

$$H_0(r) - f^2 H_2(r) - \frac{2f}{\phi r} \Phi(r) = 0. \quad (3.5)$$

By using the above condition together with $E_i = 0$ ($i = 1, \cdots, 6$), one finds the central constraint equation as

$$\left\{ \lambda f^{-1} - 2 + rf^{-1} f' \right\} H_0 + \left\{ 2k^2 r^2 f^{-1} + 2f + rf' + \frac{r^2}{2} f^{-1} (f')^2 - \lambda \right\} K$$

$$- \left\{ 2ikr + \frac{\lambda}{2ik} \right\} H_1 - \left\{ 2\lambda - 4f - 2k^2 r^2 f^{-1} - \frac{r^2}{2} f^{-1} (f')^2 \right\} \frac{\Phi}{\phi r} = 0,$$

where $\lambda = L(L + 1)$. Manipulating two equations of $E_2 = 0$ and $E_3 = 0$ by using the Eqs. (3.5) and (3.6) lead to

$$\frac{d}{dr} \left( K + \frac{\Phi}{\phi r} \right) = \frac{\lambda r f^{-1} f' - 4k^2 r^2 f^{-1} - 6r f'}{2r(\lambda - 2f + rf')} \left( K + \frac{\Phi}{\phi r} \right) + \frac{2\lambda f - 4k^2 r^2 - \lambda^2}{2ir^2(\lambda - 2f + rf')} \left( \frac{H_1}{k} \right),$$

$$\frac{d}{dr} \left( \frac{H_1}{k} \right) = \frac{2\lambda - 4f - 2k^2 r^2 f^{-1} - r^2 f^{-1} f'^2/2}{i(\lambda f - 2f^2 + rf f')} \left( K + \frac{\Phi}{\phi r} \right)$$

$$- \frac{3\lambda f^{-1} f'/2 - 2f' + rf f'^2 - 2k^2 rf^{-1}}{\lambda - 2f + rf'} \left( K + \frac{\Phi}{\phi r} \right) \left( \frac{H_1}{k} \right).$$

Now we introduce the tortoise coordinate ($r^* = \int \frac{dr}{f}$) and a new field defined by

$$\hat{\mathcal{M}} = \frac{1}{pq - h} \left\{ p \left( K + \frac{\Phi}{\phi r} \right) - \frac{H_1}{k} \right\},$$

where

$$q(r) = \frac{\lambda(\lambda + 1)r^2 + 3\lambda M r + 6M^2}{r^2(\lambda r + 3M)}, \quad h(r) = \frac{i(-\bar{\lambda} r^2 + 3\lambda M r + 3M^2)}{(r - 2M)(\lambda r + 3M)},$$

$$p(r) = -\frac{i r^2}{r - 2M}, \quad \bar{\lambda} = \frac{\lambda}{2} - 1.$$

As a result, from the Eqs. (3.7), (3.8) and (3.9) we arrive at the Zerilli equation

$$\frac{d^2 \hat{\mathcal{M}}}{dr^*^2} + \left[ k^2 - V_Z \right] \hat{\mathcal{M}} = 0,$$

where the Zerilli potential is given by

$$V_Z(r) = f \left[ \frac{2\lambda^2 (\bar{\lambda} + 1)r^3 + 6\bar{\lambda}^2 Mr^2 + 18\bar{\lambda} M^2 r + 18M^3}{r^3(\lambda r + 3M)^2} \right].$$

$$7$$
The potential $V_Z(r^*)$ is always positive for whole range of $-\infty \leq r^* \leq \infty$, which implies that the even-parity perturbation is stable, even though the scalaron $\Phi$ is coupled to making the even-perturbation [12]. In addition, using the tortoise coordinate $(r^*)$, the scalaron equation (2.17) becomes

$$\frac{d^2}{dr^{*2}}\Phi + \left[k^2 - V_\Phi\right]\Phi = 0,$$

(3.13)

where the scalaron potential $V_\Phi$ is given by

$$V_\Phi = f\left(\frac{\lambda}{r^2} + \frac{2M}{r^3} + m^2_\phi\right),$$

(3.14)

with $m^2_\phi = \bar{\phi}V''(\bar{\phi})/3$. The potential $V_\Phi$ is always positive exterior the event horizon if the mass squared $m^2_\Phi$ is positive [12].

On the other hand, for odd-parity perturbation ($\{O_i = 0\}$), the first five equations provide three:

$$O_1 = 0 \text{ or } O_2 = 0$$

$$r^3(-4M + \lambda r)h_0 - rf\left(2kir^4 - \frac{12}{\bar{\phi}}Mr\psi + kir^5h_1 + \frac{6}{\bar{\phi}}Mr\psi' + r^5h''_0\right) = 0,$$

(3.15)

$$O_3 = 0 \text{ or } O_4 = 0$$

$$-ikr^3\left(2h_0 - ikr h_1 - rh_0'\right) + r^2f(\lambda - 2)h_1 + \frac{6}{\bar{\phi}}ikMr\psi = 0,$$

(3.16)

$$O_5 = 0$$

$$ikr^3h_0 - (2M - r)\left\{2Mh_1 - (2M - r)rh_1'\right\} = 0,$$

(3.17)

and all remaining equations $O_i$ with $i = 6, \cdots, 10$ are redundant. Introducing the tortoise coordinate and a new field $Q$ defined by $Q = fh_1/r$, the above three equations (3.15)$\sim$(3.17) become one coupled second-order equation

$$\frac{d^2}{dr^{*2}}\tilde{Q} + \left\{k^2 - f\left(\frac{\lambda}{r^2} - \frac{6M}{r^3}\right)\right\}\tilde{Q} = \frac{6ikMf}{r^5}\psi,$$

(3.18)

where $\tilde{Q} = \tilde{\phi}Q$. Also, the perturbation equation (2.18) for the dynamical scalar $\theta$ becomes a coupled second-order equation

$$\frac{d^2}{dr^{*2}}\psi + \left[k^2 - f\left(\frac{18M^2}{r^2} + \frac{2M}{r^3}\right)\right]\psi = -\frac{3\lambda(\lambda - 2)iMf}{kr^5\bar{\alpha}}\tilde{Q},$$

(3.19)

In the original $f(R)$ gravity, the quantity of $\tilde{\phi}V''(\tilde{\phi})/3$ corresponds to $f'(0)/3f''(0) |\tilde{\phi} \equiv f'(0), V'' \equiv 1/f''(0)|$. Therefore, the condition of $m^2_\Phi > 0$ implies no-tachyon ($f''(0) > 0$) if $f'(0) > 0$ (no-ghost) in $f(R)$ gravity.
where $\tilde{\alpha} = \tilde{\phi} \alpha$. This is an important feature of CS coupling to $f(R)$ gravity. Actually these coupled equation are the same found in [15]. Hence, it is clear that the black hole is stable against the perturbations of $\tilde{Q}$ and $\psi$ when using two independent numerical approaches of time evolution and a formation of frequency domain employed in Ref.[15].

Finally, we wish to mention the $f(R)$-form dependence on the stability of the Schwarzschild black hole. In writing down two Eqs.(3.18) and (3.19), we introduce two new variables $\tilde{Q} = \tilde{\phi} Q$ and $\tilde{\alpha} = \tilde{\phi} \alpha$ which show the connection to the original $f(R)$ gravity because $\tilde{\phi} = f'(0)$. As was mentioned in footnote 1, our analysis is valid for a limited form of $f(R) = a_1 R + a_2 R^2 + \cdots$. In this limit from, we have $f'(0) = a_1 = \tilde{\phi}$, which is fixed by choosing the limited $f(R)$ gravity. In general, we can say that different $f(R)$ theories with different corresponding $\tilde{\phi}$ have different model parameters. However, as far as the constant curvature-black hole stability is concerned, we expect that Eqs.(3.18) and (3.19) remain unchanged except $\tilde{\phi}$, leading to the stable Schwarzschild black hole.

4 Discussions

In this work, we have performed the stability analysis of the Schwarzschild black hole in $f(R)$ gravity with the parity-violating CS term coupled to a dynamical scalar field $\theta$. In order to avoid the difficulty with fourth-order derivative terms appeared in $f(R)$ gravity, we first transformed the $f(R)$ gravity into the scalar-tensor theory by introducing a scalaron $\phi$. This will provide the DCS modified gravity with two scalars, which provides four physically propagating degrees of freedom.

Interestingly, the perturbation equation for the CS scalar $\theta$ is coupled to the odd-parity metric perturbation equation, providing a system of two coupled second-order equations, while the scalaron $\phi$ is coupled to the even-parity perturbation equation. This enables us to perform the stability analysis of the Schwarzschild black hole obtained from $f(R) + \text{DCS}$ modified gravity theory completely. It was shown that the CS coupling affects the Regge-Wheeler equation significantly, while $f(R)$ gravity does not affect the Zerilli equation. It turns out that the Schwarzschild black hole is stable against four external perturbations of $\{\mathcal{M}, \Phi, \tilde{Q}, \psi\}$ if the scalaron is free from the tachyon.

However, the role of DCS term is limited here because its perturbation $\delta C_{\mu \nu}$ in (2.16) does not involve third-order derivative terms. This higher derivative may appear when
the background solution contains a spherically symmetric CS scalar \[13\]. In this case, one could not decouple five massive gravitons successfully because there exists a mixing between odd- and even-parity modes, and third-order derivative terms are present. Even in the Minkowski background, it is not clear which modes are propagating with their own masses. It has been argued that a spacelike vector of \( v^\mu = \partial^\mu \bar{\theta} = (0, \vec{\mu}) \) renders the theory free from ghosts and tachyons, while a timelike vector of \( v^\mu = (\mu, \vec{0}) \) yields an inconsistent quantum theory \[23, 24\]. On the contrary, the opposite case is true: the only tachyon- and ghost-free model is the one with a timelike vector \[25\]. Hence, it seems to be a formidable task to perform the stability of the Schwarzschild black hole when including the third-order derivative terms.

**Acknowledgments**

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) through the Center for Quantum Spacetime (CQUeST) of Sogang University with grant number 2005-0049409. Y. Myung was partly supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (No.2011-0027293).
Appendix: The explicit forms of twenty perturbation equations of $E_i = 0$ and $O_i = 0$ where

\[
E_1 = \frac{1}{2r^2} \left[ 2r^5 k^2 K(r) + r(-2M^2 f^{-1} + r^2 \lambda) H_0(r) + 2i(3M - 2r)r^3 k H_1(r) \\
-r f(2M^2 - r^4 k^2) H_2(r) + 2Mr^3 f K'(r) + r^3 (5M - 2r) H'_0(r) \\
-2ir^5 f k H'_1(r) - Mr^3 f^2 H_2'(r) - r^5 f H''_0(r) + \frac{1}{3} r^4 f \tilde{V}''(\Phi(r)) \\
+ \frac{2}{\phi} \left\{ (2M - Mr + k^2 r^4) \Phi(r) + Mr^2 f \Phi'(r) \right\} \right],
\]

\[
E_2 = \frac{i}{2r^2} \left[ 2(r - 3M) k f^{-1} K(r) - 2kr f H_2(r) + 4kr^2 K'(r) - i\lambda H_1(r) \\
- \frac{2k}{\phi} \left\{ \left( 1 - \frac{M}{r} \right) f^{-1} \Phi(r) - r \Phi'(r) \right\} \right],
\]

\[
E_3 = \frac{i}{2r^2} \left[ kr^2 K(r) - 2iMH_1(r) + kr^2 f H_2(r) - ir^2 f H'_1 + \frac{2kr}{\phi} \Phi(r) \right],
\]

\[
E_4 = \frac{1}{2r^3 f^2} \left[ 2M(2r - 3M) f^{-1} H_0(r) + 2iMrkr^2 H_1(r) - f \left( 6M^2 - 4Mr \\
+ k^2 r^4 - r^2 f \lambda \right) H_2(r) - 2r(6M^2 - 7Mr + 2r^2) K'(r) - Mr^2 H'_0 \\
+ 2irkr^4 f H'_1 + rf(6M^2 - 7Mr + 2r^2) H'_2 - 2r^4 f^2 K''(r) + r^4 f H''_0 \\
- \frac{r^3 f}{3} \tilde{V}''(\Phi(r)) - \frac{2f}{\phi} \left\{ (-5M + 2r) \Phi(r) + r(5M - 2r) \Phi'(r) + r^3 f \Phi''(r) \right\} \right],
\]

\[
E_5 = \frac{-1}{2r^3 f} \left[ (r - M) rf^{-1} H_0(r) - ikr^3 H_1(r) - (2M^2 - 3Mr + r^2) H_2(r) \\
+ r^3 f K'(r) - r^3 H'_0 + \frac{2rf}{\phi} \left\{ -2\Phi(r) + r \Phi'(r) \right\} \right],
\]

\[
E_6 = \frac{-1}{2r^2} \left[ 2Mr f^{-1} H_0(r) - 2ikr^3 H_1(r) + 2(2M^2 + Mr - r^2) H_2(r) \\
+ r^2 (k^2 r^2 f^{-1} + \lambda + 2) K(r) - 2r^2 (3M - 2r) K'(r) - r^3 H'_0(r) \\
- r^3 f^2 H_2'(r) + r^4 f K''(r) + 2rf \left\{ \frac{r^2}{6} f^{-1} \tilde{V}''(\Phi(r)) + \frac{1}{\phi} \right\} \right],
\]

\[
E_7 = \frac{1}{2r} \left[ rf^{-1} H_0(r) - rf H_2(r) - \frac{2}{\phi} \Phi(r) \right],
\]

\[
E_8 = -E_7 \cot \varphi_1,
\]

\[
E_9 = E_6 \sin^2 \varphi_1,
\]

\[
E_{10} = E_7 \cos \varphi_1 \sin \varphi_1.
\]
\[O_1 = \frac{csc \varphi_1}{2r^3} \left[ ir^2 f \left\{ 2kh_1(r) + krh_1'(r) - irh_0''(r) \right\} + (4M - \lambda r)h_0(r) \right. \\
\left. - \frac{6}{\phi r^2} M f \left\{ 2\psi(r) - r\psi'(r) \right\} \right],
\]
\[O_2 = \frac{-1}{2r^3} \left[ ir^2 f \sin \varphi_1 \left\{ 2kh_1(r) + krh_1'(r) - irh_0''(r) \right\} - (rf + 2M \cos 2\varphi_1) \csc \varphi_1 h_0(r) \right. \\
\left. + r(csc \varphi_1 - \lambda \sin \varphi_1)h_0(r) - \frac{6}{\phi r^2} M f \sin \varphi_1 \left\{ 2\psi(r) - r\psi'(r) \right\} \right],
\]
\[O_3 = \frac{csc \varphi_1}{2r^3 f} \left[ 2ikr^2 h_0(r) - ikr^3 h_0'(r) + (2rf + k^2 r^3 - \lambda rf)h_1(r) - \frac{6i}{\phi r} kM \psi(r) \right],
\]
\[O_4 = \frac{-1}{2r^3 f} \left[ ikr^2 \sin \varphi_1 \left\{ 2h_0(r) - rh_0' \right\} + \left\{ -rf \cos \varphi_1 \cot \varphi_1 + (rf + k^2 r^3) \sin \varphi_1 \right. \\
\left. + rf(csc \varphi_1 - \lambda \sin \varphi_1) \right\} h_1(r) - \frac{6i}{\phi r} kM \sin \varphi_1 \psi(r) \right],
\]
\[O_5 = \frac{csc \varphi_1 \cot \varphi_1}{r^3 f} \left[ ikr^3 h_0(r) + rf \left\{ 2Mh_1(r) + r^2 fh_1'(r) \right\} \right],
\]
\[O_6 = -O_5 \tan \varphi_1,
\]
\[O_7 = \frac{O_5 \lambda}{2} \left[ \sin^2 \varphi_1 \tan \varphi_1 \right],
\]
\[O_8 = O_5 \sin^2 \varphi_1 \tan \varphi_1,
\]
\[O_9 = -O_5 \sin^2 \varphi_1,
\]
\[O_{10} = O_8.\]
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