Generating functions attached to some infinite matrices

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Abstract

Let $V$ be an infinite matrix with rows and columns indexed by the positive integers, and entries in a field $F$. Suppose that $v_{i,j}$ only depends on $i-j$ and is 0 for $|i-j|$ large. Then $V^n$ is defined for all $n$, and one has a “generating function” $G = \sum a_{1,1}(V^n)z^n$. Ira Gessel has shown that $G$ is algebraic over $F[z]$. We extend his result, allowing $v_{i,j}$ for fixed $i-j$ to be eventually periodic in $i$ rather than constant. This result and some variants of it that we prove will have applications to Hilbert-Kunz theory.

1 Introduction

Throughout, $\Lambda$ is a ring with identity element 1. Suppose that $w_{i,j}$, $i$ and $j$ ranging over the positive integers, are in $\Lambda$ and that $w_{i,j} = 0$ whenever $i-j$ lies outside a fixed finite set. Then $W^n$ is defined for all $n$, and one gets a generating function $G(W) = \sum a_n z^n$ in $\Lambda[[z]]$, where $a_n$ is the $(1,1)$ entry in the matrix $W^n$. We shall prove:

**Theorem I** Suppose that $w_{i,j} = 0$ if $i-j \notin \{-1, 0, 1\}$, and that $w_{i+1,j+1} = w_{i,j}$ unless $i = j = 1$. Suppose further that $\Lambda = M_s(F)$, $F$ a field, so that $G(W)$ may be viewed as an $s \times s$ matrix with entries in $F[[z]]$. Then these matrix entries are algebraic over $F(z)$.

**Corollary** Let $F$ be a field and $v_{i,j}$, $i$ and $j$ ranging over the positive integers, be in $F$. Suppose:

(a) $v_{i,j} = 0$ whenever $i-j$ lies outside a fixed finite set.
(b) For fixed $r$ in $Z$, $v_{i+r}$ is an eventually periodic function of $i$.

Then if $V$ is the matrix $[v_{i,j}]$, the generating function $G(V)$ is algebraic over $F(z)$.

**Proof** To derive the corollary we choose $s$ so that:
(1) \( v_{i,j} = 0 \) whenever \( i \leq s \) and \( j > 2s \) or \( j \leq s \) and \( i > 2s \).
(2) \( v_{i+s,j+s} = v_{i,j} \) whenever \( i + j \geq s + 2 \).

We then write the initial \( 2s \) by \( 2s \) block in \( V \) as \( |D_{\Lambda}| \) with \( A, B, C, D \) in \( M_s(F) \). Our choice of \( s \) tells us that \( V \) is built out of \( s \) by \( s \) blocks, where the blocks along the diagonal are a single \( D \), followed by \( B \)'s, those just below a diagonal block are \( A \)'s, those just above a diagonal block are \( C \)'s, and all other entries are 0. Now let \( \Lambda = M_s(F) \) and \( W = [w_{i,j}] \) where \( w_{i+1,i} = A \), \( w_{i,i+1} = C \), \( w_{1,1} = D \), \( w_{i,j} = B \) for \( i > 1 \), and all other \( w_{i,j} \) are 0. View \( G(W) \) as an \( s \) by \( s \) matrix with entries in \( F[[z]] \). One sees easily that \( G(V) \) is the (1,1) entry in this matrix, and Theorem I applied to \( W \) gives the corollary. \( \square \)

**Remark** When \( v_{i,j} \) only depends on \( i-j \), the above corollary is due to Gessel. (When the matrix entries of \( V \) are all 0's and 1's the result is contained in Corollary 5.4 of [1]. The restriction on the matrix entries isn't essential in Gessel's proof, as one can use a generating function for walks with weights.)

Our proof of Theorem I is easier than Gessel's proof of his special case of the corollary. The reason for this is that by working over \( \Lambda \) rather than over \( F \) we are able to restrict our study to walks with step-sizes in \( \{-1,0,1\} \). (A complication, fortunately minor, is that the weights must be taken in the non-commutative ring \( \Lambda \).) Our proof is well-adapted to finding an explicit polynomial relation between \( G(V) \) and \( z \); we'll work out a few examples. This paper would not have been possible without Ira Gessel's input. I thank him for showing me tools of the combinatorial trade.

## 2 Walks and generating functions

**Definition 2.1** If \( l \geq 0 \), an ordered \( l+1 \)-tuple \( \alpha = (\alpha_0, \ldots, \alpha_l) \) of integers is a (Motzkin) walk of length \( l = l(\alpha) \) if each of \( \alpha_1 - \alpha_0, \ldots, \alpha_l - \alpha_{l-1} \) is in \( \{-1,0,1\} \).

We say that the start of the walk is \( \alpha_0 \), the finish is \( \alpha_l \), and that \( \alpha \) is a walk from \( \alpha_0 \) to \( \alpha_l \).

**Definition 2.2** If \( \alpha \) and \( \beta \) are walks of lengths \( l \) and \( m \), the concatenation \( \alpha \beta \) of \( \alpha \) and \( \beta \) is the walk \( (\alpha_0, \ldots, \alpha_l, \alpha_l + (\beta_1 - \beta_0), \ldots, \alpha_l + (\beta_m - \beta_0)) \) of length \( l + m \).

Now let \( \Lambda \) be a ring with identity element 1, and \( A, B, C, D \) lie in \( \Lambda \). To each walk \( \alpha \) we attach weights \( w(\alpha) \) and \( w^*(\alpha) \) in \( \Lambda \):

**Definition 2.3** If \( l(\alpha) = 0 \), \( w(\alpha) = w^*(\alpha) = 1 \). If \( l(\alpha) > 0 \), \( w(\alpha) = U_1 \cdots U_l \) where \( U_i = A, B \) or \( C \) according as \( \alpha_i - \alpha_{i-1} \) is \(-1,0,1 \). The definition of
$w^*(\alpha)$ is the same with one change: if $\alpha_i = \alpha_{i-1} = 0$ then $U_i = D$ rather than $B$.

Evidently $w(\alpha\beta) = w(\alpha)w(\beta)$. Furthermore $w^*(\alpha\beta) = w^*(\alpha)w^*(\beta)$ whenever $\alpha$ and $\beta$ are walks from 0 to 0.

**Definition 2.4** $\alpha$ is “standard” if each $\alpha_i \geq \alpha_l$. Note that a walk from 0 to 0 is standard if and only if each $\alpha_i \geq 0$.

**Definition 2.5** $\alpha$ is “primitive” if $l(\alpha) > 0$, $\alpha_0 = \alpha_l$ and no $\alpha_i$ with $0 < i < l$ is $\alpha_0$. Note that a standard walk from 0 to 0 is primitive if and only if $l(\alpha) > 0$ and each $\alpha_i$, $0 < i < l$, is $> 0$.

**Definition 2.6**

(1) $G(w) = \sum w(\alpha)z^{l(\alpha)}$, the sum extending over all standard walks from 0 to 0. $H(w)$ is the sum extending over all primitive standard walks from 0 to 0.

(2) $G(w^*)$ and $H(w^*)$ are defined similarly, using $w^*(\alpha)$ in place of $w(\alpha)$.

**Lemma 2.7** Let $G = G(w)$, $H = H(w)$. Then, in $\Lambda[[z]]$:

(1) $G = 1 + H + H^2 + \cdots$

(2) $H = Bz + CGAz^2$

**Proof** Every standard walk from 0 to 0 of length $> 0$ is either primitive or uniquely a concatenation of two or more primitive standard walks from 0 to 0. The multiplicative property of $w$ now gives (1). To prove (2) note that the primitive standard walk $(0,0)$ has $w = B$. And a primitive standard walk from 0 to 0 of length $l > 1$ is a concatenation of $(0,1)$, a standard walk, $\beta$, from 0 to 0 of length $l - 2$ and $(0,-1)$. Then $w(\alpha) = Cw(\beta)A$. Since $\alpha \rightarrow \beta$ gives a 1–1 correspondence between primitive standard walks of length $l$ from 0 to 0 and standard walks of length $l - 2$ from 0 to 0, we get the result. □

**Corollary 2.8** If $G = G(w)$, then $G - 1 - (BG)z - (CGAG)z^2 = 0$ in $\Lambda[[z]]$.

**Proof** By (1) of Lemma 2.7, $(1 - H) \cdot G = 1$. Substituting $H = Bz + CGAz^2$ gives the result. □

**Theorem 2.9** Suppose that $\Lambda = M_s(F)$, $F$ a field, so that $G(w)$ may be viewed as an $s$ by $s$ matrix with entries in $F[[z]]$. Then these matrix entries, $u_{i,j}$, are algebraic over $F(z)$.

**Proof** Let $U = |U_{i,j}|$ be an $s$ by $s$ matrix of indeterminates over $F$, and $p_{i,j}$ be the $(i,j)$ entry in $U - I_s - (BU)z - (CUA)z^2$. The $p_{i,j}$ are degree 2 polynomials in $U_{1,1}, \ldots, U_{s,s}$ with co-efficients in $F[z]$. By Corollary 2.8, $p_{i,j}(u_{1,1}, \ldots, u_{s,s}) = 0$. Now $p_{i,j} = U_{i,j} - \delta_{i,j} - zf_{i,j}(U_{1,1}, \ldots, U_{s,s}, z)$ where the
$f_{i,j}$ are polynomials with co-efficients in $F$. It follows that the Jacobian matrix of the $p_{i,j}$ with respect to the $U_{i,j}$, evaluated at $(u_{1,1}, \ldots, u_{s,s})$, is congruent to $I_{s^2}$ mod $z$ in the $s^2$ by $s^2$ matrix ring over $F[[z]]$, and so is invertible. Thus $(u_{1,1}, \ldots, u_{s,s})$ is an isolated component of the intersection of the hypersurfaces $p_{i,j}(U_{1,1}, \ldots, U_{s,s}) = 0$, and so its co-ordinates, $u_{1,1}, \ldots, u_{s,s}$, are algebraic over $F(z)$. $\square$

**Lemma 2.10** $G(w^*)^{-1} - G(w)^{-1} = (B - D)z$. 

**Proof** The proof of Lemma 2.7 (1) shows that $G(w^*)^{-1} = 1 - H(w^*)$ with $H(w^*)$ as in Definition 2.6. So it suffices to show that $H(w) - H(w^*) = (B - D)z$. Now for a primitive walk $\alpha$ of length $> 1$ from 0 to 0 one cannot have $\alpha_{i-1} = \alpha_i = 0$, and so $w(\alpha) = w^*(\alpha)$. On the other hand, for the primitive walk $(0,0)$, $w = B$ and $w^* = D$. This gives the lemma. $\square$

Combining Lemma 2.10 with Theorem 2.9 we get:

**Theorem 2.11** If $\Lambda = M_s(F)$ the matrix entries of the $s$ by $s$ matrix $G(w^*)$ are algebraic over $F(z)$.

Now let $W = |w_{i,j}|$ where $w_{i+1,i} = A$, $w_{i,i+1} = C$, $w_{1,1} = D$, $w_{i,i} = B$ for $i > 1$, and all the other $w_{i,j} = 0$. In view of Theorem 2.11 the proof of Theorem I will be complete once we show that $G(W) = G(w^*)$ where $w^*$ is the weight function of Definition 2.3. The key to this is:

**Lemma 2.12** For $k \geq 1$ let $u_k^{(n)}$ be $\sum w^*(\alpha)$, the sum extending over all standard walks of length $n$ from $k - 1$ to 0. Then:

1. $u_k^{(0)} = 1$ or 0 according as $k = 1$ or $k > 1$.
2. $u_k^{(n+1)} = Du_k^{(n)} + Cu_k^{(n)}$.
3. If $k > 1$, $u_k^{(n+1)} = Au_{k-1}^{(n)} + Bu_k^{(n)} + Cu_{k+1}^{(n)}$.

Lemma 2.12 has the following immediate corollaries, with the first proved by induction on $n$.

**Corollary 2.13** The first column vector in $W^n$ is $(u_1^{(n)}, u_2^{(n)}, \ldots)$.

**Corollary 2.14** The $(1,1)$ co-efficient of $W^n$ is $\sum w^*(\alpha)$, the sum extending over all standard walks of length $n$ from 0 to 0. So $G(W) = G(w^*)$.

It remains to prove Lemma 2.12. (1) is evident. Let $\alpha$ be a standard walk of length $n$ from 0 or 1 to 0. Then $\beta = (0, \alpha_0, \ldots, \alpha_n)$ is a standard walk of length $n + 1$ from 0 to 0, and $w^*(\beta)$ is $Dw^*(\alpha)$ in the first case and $Cw^*(\alpha)$ in the second. Also each standard walk $\beta$ of length $n + 1$ from 0 to 0 arises in this way from some $\alpha$; explicitly $\alpha = (\beta_1, \ldots, \beta_n)$. Summing over $\beta$ we get (2). Similarly, suppose that $k > 1$ and that $\alpha$ is a standard walk of length $n$
from $k - 2$, $k - 1$ or $k$ to 0. Then $\beta = (k - 1, \alpha_0, \ldots, \alpha_n)$ is a standard walk of length $n + 1$ from $k - 1$ to 0 and $w^*(\beta) = Aw^*(\alpha)$ in the first case, $Bw^*(\alpha)$ in the second, and $Cw^*(\alpha)$ in the third. Also, each standard walk $\beta$ of length $n + 1$ arises from such an $\alpha$; explicitly $\alpha = (\beta_1, \ldots, \beta_n)$. Summing over $\beta$ we get (3), completing the proof. □

Remark 2.15 To calculate the matrix entries of $G(W)$ explicitly as algebraic functions of $z$ by the method of Theorem 2.9 involves solving a system of $s^2$ quadratic equations in $s^2$ variables. This isn’t practical when $s > 2$; in the next section we give a different proof of Theorem 2.9 that is often better adapted to explicit calculations.

3 A partial fraction proof of Theorem 2.9

Theorem 3.1 $\sum w(\alpha)x^{\alpha_0}$, the sum extending over all length $n$ walks (not necessarily standard) with finish 0, is the element $(Ax + B + Cx^{-1})^n$ of $\Lambda[x, x^{-1}]$.

Proof Denote the sum by $f_n$. Since $f_0 = 1$ it’s enough to show that $f_{n+1} = (Ax + B + Cx^{-1})f_n$. Let $v_k^{(n)}$ be the co-efficient of $x^k$ in $f_n$. Then $v_k^{(n)} = \sum w(\alpha)$, the sum extending over all length $n$ walks from $k$ to 0. The proof of (3) of Lemma 2.12, using all walks rather than all standard walks, shows that $v_k^{(n+1)} = Av_k^{(n)} + Bv_k^{(n)} + Cv_k^{(n)}$ for all $k$ in $Z$, giving the result. □

Definition 3.2

$M_0(w) = \sum w(\alpha)z^{l(\alpha)}$, the sum extending over all 0 to 0 walks.

$M_{-1}(w)$ is the sum extending over all $-1$ to 0 (or 0 to 1) walks.

$M_1$ is the sum extending over all 1 to 0 (or 0 to $-1$) walks.

We’ll generally omit the $w$ and just write $M_0$, $M_{-1}$ or $M_1$.

Corollary 3.3 Suppose that $i = 0$, $-1$ or 1. Then $M_i$ is the co-efficient of $x^i$ in the element $\sum_0^\infty (Ax + B + Cx^{-1})^n z^n$ of $\Lambda[x, x^{-1}][[z]]$.

Definition 3.4 $J_0 = J_0(w)$ is $\sum w(\alpha)z^{l(\alpha)}$, the sum extending over all primitive 0 to 0 walks.

Theorem 3.5

(1) $M_0 = 1 + J_0 + J_0^2 + \cdots$.

(2) $G(w) = M_0 - M_1M_0^{-1}M_{-1}$.

Proof (1) follows from the multiplicative property of $w$, as in the proof of Lemma 2.7. So $M_0^{-1} = 1 - J_0$, and (2) asserts that $G(w) = M_0 + M_1J_0M_{-1} - M_1M_{-1}$. If $\alpha$ is a walk from 0 to 0 let $r(\alpha)$ be the number of ways of writing
Suppose now that $\Lambda = M_s(F)$, $F$ a field, so that $M_0$, $M_1$ and $M_{-1}$ may be viewed as $s$ by $s$ matrices with entries in $F[[z]]$. Theorem 3.5, (2), will give a new proof of Theorem 2.9 once we show that these matrix entries are algebraic over $F(z)$. The facts about the matrix entries of $M_0$, $M_1$ and $M_{-1}$ follow from a standard partial fraction decomposition argument—we’ll give our own version.

The algebraic closure of the field of fractions of $F[[z]]$ is a valued field with value group $Q$, let $\Omega$ be the completion of this field and $\text{ord} : \Omega \to Q \cup \{\infty\}$ be the ord function in $\Omega$. Let $\Omega'$ consist of formal power series $\sum_{i=0}^{\infty} a_i x^i$ with $a_i \in \Omega$ and ord $a_i \to \infty$ as $|i| \to \infty$. $\Omega'$ has an obvious multiplication and is an overring of $F[x, x^{-1}][[z]]$. $l_0$, $l_1$ and $l_{-1}$ are the $\Omega$-linear maps $\Omega' \to \Omega$ taking $\sum a_i x^i$ to $a_0$, $a_1$ and $a_{-1}$. Note that $\overline{F(z)}$, the algebraic closure of $F(z)$, imbeds in $\Omega$.

**Lemma 3.6** Suppose $\lambda \in \overline{F(z)}$ with ord $\lambda \neq 0$. Then the element $x - \lambda$ of $\Omega'$ is invertible, and for all $k \geq 1$, $(x - \lambda)^{-k} = \sum_{i=0}^{\infty} a_i x^i$ in $\Omega'$ with the $a_i$ in $\overline{F(z)}$. In particular, $l_0$, $l_1$ and $l_{-1}$ take each $(x - \lambda)^{-k}$ to an element of $\overline{F(z)}$.

**Proof** If ord $\lambda > 0$, $x - \lambda = x(1 - \lambda x^{-1})$ has inverse $x^{-1}(1 + \lambda x^{-1} + \lambda^2 x^{-2} + \cdots)$, while if ord $\lambda < 0$, $x - \lambda = -\lambda(1 - \lambda^{-1} x)$ has inverse $-\lambda^{-1}(1 + \lambda^{-1} x + \lambda^{-2} x^2 + \cdots)$. □

**Lemma 3.7** Let $U_1$ and $U_2$ be elements of $F[z, x]$. Suppose that $U_2 \equiv x^s \bmod z$ for some $s$. Then $U_2$ has an inverse in $F[x, x^{-1}][[z]]$ and the co-efficients of $x^0$, $x^1$ and $x^{-1}$ in the element $U_1 U_2^{-1}$ of $F[x, x^{-1}][[z]]$ all lie in $\overline{F(z)}$.

**Proof** Write $U_2$ as $x^s(1 - zp)$ with $p \in F[x, x^{-1}, z]$. Then $x^{-s}(1 + zp + z^2p^2 + \cdots)$ is the desired inverse of $U_2$. If $\lambda$ in $\Omega$ has ord 0 then $1 - zp(\lambda, \lambda^{-1}, z)$ has ord 0 and cannot be 0. So when we factor $U_2$ in $\overline{F(z)}[x]$ as $q \cdot \Pi(x - \lambda_i)_{i\in I}$ with $q$ in $F[z]$ and $\lambda_i$ in $\overline{F(z)}$, no ord $(\lambda_i)$ can be 0. View $U_1 U_2^{-1}$ as an element of $\overline{F(z)}(x)$. As such it is an $\overline{F(z)}$ linear combination of powers of $x$ and powers of the $(x - \lambda_i)^{-1}$. Since $l_0$, $l_1$ and $l_{-1}$ are $\Omega$-linear they are $\overline{F(z)}$-linear. Lemma 3.6 then tells us that $U_1 U_2^{-1}$, viewed as an element of $\Omega'$, is mapped by each of $l_0$, $l_1$ and $l_{-1}$ to an element of $\overline{F(z)}$. This completes the proof. □
Lemma 3.8 Let $A$, $B$ and $C$ be in $M_s(F)$ and $u \in F[x, x^{-1}][[z]]$ be an entry in the matrix $(I_s - z(Ax + B +Cx^{-1}))^{-1}$. Then the co-efficients of $x^0$, $x^1$ and $x^{-1}$ in $u$ all lie in $F(z)$.

Proof $u$ may be written as $U_1/U_2$ where $U_1$ and $U_2$ are in $F[z, x]$ and $U_2 = \det (xI_s - z(Ax^2 + Bx + C))$. Then $U_2 \equiv x^s \mod z$, and we apply Lemma 3.7.

Corollary 3.9 If $\Lambda = M_s(F)$, $F$ a field, then the matrix entries of $M_0$, $M_1$ and $M_{-1}$ are algebraic over $F(z)$. (So by Theorem 3.5 the same is true of the matrix entries of $G(w)$.)

Proof $(I_s - z(Ax + B +Cx^{-1}))^{-1} = \sum_0^\infty (Ax + B +Cx^{-1})^n z^n$, and we combine Lemma 3.8 with Corollary 3.3.

4 Examples

Example 4.1 For $i$, $j$ positive integers define $v_{i,j}$ by:

1. $v_{i,j} = 1$ if $i - j \in \{-1, 0, 1\}$.
2. $v_{i,j} = 1$ if $j = i + 3$ and $i$ is odd.
3. All other $v_{i,j}$ are 0.

We calculate $G(V)$ where $V = |v_{i,j}|$. If we take $s = 2$, (1) and (2) in the corollary to Theorem I are satisfied, and $D = B = (\frac{1}{1} \frac{1}{1}), A = (\frac{0}{0} \frac{1}{0}), C = (\frac{1}{0} \frac{1}{0})$. Let $G = G(w) = G(w^*)$. $G$ is a 2 by 2 matrix $(\frac{g_1}{g_2} \frac{g_3}{g_4})$ with entries in $F[[z]]$, and $g_1 = G(V)$. By Corollary 2.8, $CGAGz^2 + BGz - G + I_2 = 0$. Two of the four equations this gives are:

\begin{align*}
z^2g_1g_3 + z(g_1 + g_3) - g_3 &= 0 \\
z^2g_3^2 + z(g_1 + g_3) - g_1 + 1 &= 0
\end{align*}

Solving the first equation for $g_3$ and substituting in the second we find that $G(V) = g_1$ is a root of:

$(z^5 - z^4)x^3 + (3z^4 - 4z^3 + 2z^2)x^2 + (2z^3 - 4z^2 + 3z - 1)x + (z^2 - 2z + 1) = 0$.

Example 4.2 For $i$, $j$ positive integers define $v_{i,j}$ by:

1. $v_{i,j} = 1$ if $i - j \in \{-1, 0, 1\}$.
2. $v_{i,j} = 1$ if $j = i + 3$ and $i$ is even.
3. All other $v_{i,j}$ are 0.
We calculate $G(V^*)$ where $V^* = |v_{i,j}|$. Since $v_{2,5} = 1$, condition (1) of the corollary to Theorem I is not met when $s = 2$, and we instead take $s = 4$.

Now
\[
D = B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.
\]

Let the entries in the first column of the 4 by 4 matrix $G = G(w)$ be $a$, $b$, $c$ and $d$. Examining the entries in the first column of the matrix equation $G = BGz + CGAGz^2 + I_4$ we see:

\[
\begin{align*}
a &= (a + b)z + 1 \\
b &= (a + b + c)z + bdz^2 \\
c &= (b + c + d)z \\
d &= (c + d)z + d(a + c)z^2
\end{align*}
\]

Using Maple to eliminate $b$, $c$, and $d$ from this system we find that $a = G(V^*)$ is a root of:

\[
(z^2) \cdot (z - 1)^3 \cdot (3z^2 + 3z - 2) \cdot x^3 \\
+ (z - 1)^2 \cdot (9z^4 + 6z^3 - 11z^2 + 5z - 1) \cdot x^2 \\
+ (2z - 1) \cdot (5z^4 - 13z^2 + 9z - 2) \cdot x \\
+ (2z - 1)^2 \cdot (z^2 + 2z - 1) = 0.
\]

**Example 4.3** For $i$, $j$ positive integers define $v_{i,j}$ by:

1. $v_{i,j} = 1$ if $i - j \in \{-1, 1\}$.
2. $v_{i,j} = 1$ if $i - j \in \{-3, 3\}$ and $i \equiv 2 \pmod{3}$.
3. All other $v_{i,j}$ are 0.

We calculate $G(V)$ where $V = |v_{i,j}|$. Take $s = 3$. Then:

\[
A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad B = D = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.
\]

The determinant of the matrix $xI_3 - z(Ax^2 + Bx + C)$ is $-x^2(zx^2 + (3z^2 - 1)x + z)$. The splitting field of this polynomial over $F(z)$ is the extension of $F(z)$ generated by $\sqrt{1 - 10z^2 + 9z^4}$. The arguments of section 3 show that $M_0$, $M_1$ and $M_{-1}$ have entries in this extension field. It’s not hard to write down these matrices explicitly using the partial-fraction decomposition argument. Theorem 3.5 and a Maple calculation then show that the $(1, 1)$ entry in $G(w)$ is $4/(3 + z^2 + \sqrt{1 - 10z^2 + 9z^4})$. Since $D = B$, $G(w^*) = G(w)$, and this $(1, 1)$ entry is the desired $G(V)$. 

8
5 More algebraic generating functions

Definition 5.1 Suppose that $\Lambda = M_s(F)$, $F$ a field, and that $A$, $B$, $C$, $D$ are in $\Lambda$. Then $L \subset \mathbb{Q}(z)$ is the extension field of $F(z)$ generated by the matrix entries of the $M_0$, $M_1$ and $M_{-1}$ of Definition 3.2.

Remark 5.2 As we’ve seen $L$ contains the matrix entries of $G(w)$ and $G(w^*)$ and is finite over $F(z)$. Indeed the proofs of Lemmas 3.7, 3.8 and Corollary 3.9 show that $L$ is a finite extension field over $F(z)$. One can see a bit more. The above polynomial splits into linear factors in $\Omega[z]$, and one may view its splitting field as a subfield of the valued field $\Omega$. By examining the partial-fraction decomposition one finds that $L$ is fixed elementwise by each automorphism of the splitting field that is the identity on $F(z)$ and permutes the roots that have positive ord among themselves.

The goal of this section is to show that some generating functions related to $G(w)$ also have their matrix entries in $L$. These results will be used in a sequel to show the algebraicity (under a conjecture) of certain Hilbert-Kunz series and Hilbert-Kunz multiplicities.

Now let $u_k^{(n)}$ be as in Lemma 2.12 where $k$ is a positive integer. By definition, $G^*(w) = \sum u_1^{(n)}z^n$.

Lemma 5.3 $\sum_n u_{k+1}^{(n)}z^n = G(w)(Az)\sum_n u_k^{(n)}z^n$.

Proof A standard walk from $k$ to 0 can be written in just one way as the concatenation of a standard walk from $k$ to $k$, the walk $(k, k-1)$ and a standard walk from $k-1$ to 0.

Corollary 5.4 Fix $k \geq 1$. The generating function arising from the $(k, 1)$ entries of the matrices $W^n$ has its matrix entries in $L$.

Proof Corollary 2.13 shows that this generating function is $\sum u_k^{(n)}z^n$, and we use Lemma 5.3 and induction.

Definition 5.5 $G^*_r = \sum \binom{\alpha}{r}w^*(\alpha)z^{(\alpha)}$, the sum extending over all standard walks finishing at 0.

Evidently $G^*_0 = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} u_{k+1}^{(n)}z^n$. By Lemma 5.3, this is

$$(1 + G(w)Az + (G(w)Az)^2 + \cdots)G(w^*).$$

So:

Lemma 5.6 $(1 - G(w)Az)G^*_0 = G(w^*)$. 

9
A variant of this is:

**Lemma 5.7** \((1 - G(w)Az)G^*_r = G(w)(Az)G^*_r.\)

**Proof** We introduce new weight functions \(w|t\) and \(w^*|t\) as follows. Replace \(\Lambda, A\) and \(C\) by \(\Lambda[[t]], A(1 + t)\) and \(C(1 + t)^{-1}\), and let \(w|t\) and \(w^*|t\) be the new \(w\) and \(w^*\) that arise. If \(\alpha = (\alpha_0, \ldots, \alpha_l)\) is a walk from \(k\) to \(0\) then there are \(k = \alpha_0\) more steps of size \(-1\) in the walk than there are steps of size \(1\). It follows that \(w|t(\alpha)\) and \(w^*|t(\alpha)\) are \((1 + t)^{\alpha_0}w(\alpha)\) and \((1 + t)^{\alpha_0}w^*(\alpha)\). In particular, \(G(w|t) = G(w)\) and \(G(w^*|t) = G(w^*)\). Applying Lemma 5.6 in this new situation we find:

\[
((1 - G(w)Az) - G(w)Azt) \left(\sum_{k=0}^{\infty} \sum_{n=0}^{\infty} (1 + t)^k u_{i+1}^{(n)} z^n\right) = G(w^*).
\]

In particular, the co-efficient of \(t^{r+1}\) in the left-hand side of the above equation is \(0\). Evaluating this co-efficient we get the lemma. \(\square\)

**Theorem 5.8** Let \(a_1, a_2, \ldots\) be elements of \(F\). Suppose there is a polynomial function whose value at \(j\) is \(a_j\) for sufficiently large \(j\). Let \(R_n = \sum_1^{\infty} a_k u_k^{(n)}\). Then all the matrix entries of \(\sum R_n z^n\) lie in \(L\).

**Proof** Corollary 5.4 shows that the generating function arising from any single \((j, 1)\) entry has matrix entries in \(L\). So we may assume that \(j \to a_j\) is a polynomial function. Since any polynomial function is an \(F\)-linear combination of the functions \(j \to \binom{j-1}{r}\), \(r = 0, 1, 2, \ldots\) we may assume \(a_j = \binom{j-1}{r}\). But then \(\sum R_n z^n\) is \(G^*_r\), and we use Lemmas 5.6, 5.7 and induction. \(\square\)

**Corollary 5.9** Suppose \(V = |v_{i,j}|, i, j \geq 1\) is a matrix with entries in \(F\) satisfying:

1. \(v_{i,j} = 0\) whenever \(i \leq s\) and \(j > 2s\) or \(j \leq s\) and \(i > 2s\).
2. \(v_{i+s,j+s} = v_{i,j}\) whenever \(i + j \geq s + 2\).
3. The initial 2s by 2s block in \(V\) is \((P_{AB})\).

Suppose further that \(a_1, a_2, \ldots\) are in \(F\) and that for each \(i, 1 \leq i \leq s\), there is a polynomial function agreeing with \(k \to a_{i+sk}\) for large \(k\). Let \(v_{i}^{(n)}\) be the \((i, 1)\) entry in \(V^n\). Then \(\sum v_{i}^{(n)} a_i z^n\) is in \(L\).

**Proof** Construct \(W\) as in the proof of the corollary to Theorem I. As the first column of \(W^n\) is \(u_1^{(n)}, u_2^{(n)}, \ldots\) it follows that \(v_{i+sk}^{(n)}\) is just the \((i, 1)\) entry in the \(s\) by \(s\) matrix \(u_{i+1}^{(n)}\). Theorem 5.8 shows that for each \(i\) with \(1 \leq i \leq s\), \(\sum_{k,n} v_{i+sk}^{(n)} a_{i+sk} z^n\) is in \(L\). Summing over \(i\) we get the result. \(\square\)

The following results may seem artificial but they’re convenient for our intended applications to Hilbert-Kunz theory.
Lemma 5.10 Let $Y$ be a finite dimensional vector space over $F$, $T : Y \rightarrow Y$ and $l : Y \rightarrow F$ linear maps and $y_1, y_2, \ldots$ a sequence in $Y$. Let $V$ and $s$ be as in Corollary 5.9. Suppose that for each $i, 1 \leq i \leq s$, each co-ordinate of $y_{i+s}$ with respect to a fixed basis of $Y$ is an eventually polynomial function of $k$. Define $y^{(n)}$ inductively by $y^{(0)} = 0$, $y^{(n+1)} = Ty^{(n)} + \sum v_i^{(n)} y_i$—see Corollary 5.9 for the definition of $v_i^{(n)}$. Then $\sum l(y^{(n)}) z^n$ is in $\mathcal{L}$.

Proof $(I - zT)\sum y^{(n)} z^n = \sum_i v_i^{(n)} y_i z^{n+1}$. By Corollary 5.9, all the co-ordinates of $(I - zT)\sum y^{(n)} z^n$ with respect to a fixed basis of $Y$ lie in $\mathcal{L}$. Since $\det (I - zT)$ is a non-zero element of $F(z) \subset \mathcal{L}$, the same is true of the co-ordinates of $\sum y^{(n)} z^n$, giving the lemma. □

Theorem 5.11 Suppose $X$ is a vector space over $F$, $Y$ is a finite dimensional subspace, $T : X \rightarrow X$ is linear with $T(Y) \subset Y$, and $E_1, E_2, \ldots$ lie in $X$. Suppose further that $T(E_i) = \sum v_{i,j} E_j + y_j$, where $V = \{v_{i,j}\}$ is as in Lemma 5.10 and $y_1, y_2, \ldots$ is a sequence in $Y$ satisfying the condition of Lemma 5.10. Then if $l : X \rightarrow F$ is linear with each $l(E_i) = 0$, the power series $\sum_{0}^{\infty} l(T^n(E_1)) z^n$ is in $\mathcal{L}$.

Proof Define $y^{(n)}$ as in Lemma 5.10. Using the identity $\sum_j v_{i,j} v_j^{(n)} = v_i^{(n+1)}$ and induction we find that $T^n(E_1) = \sum v_i^{(n)} E_i + y^{(n)}$. So $l(T^n(E_1)) = l(y^{(n)})$ and we apply Lemma 5.10. □

The following example is closely related to our calculations in [2]. We’ll explain how this and similar examples relate to Hilbert-Kunz theory in a sequel to this paper.

Example 5.12 Suppose $\delta_1$ and $\delta_2$ are a basis of $Y$, that $y_1 = 6\delta_1$ and that $y_k = (8k - 2)\delta_1 + \delta_2$, $k > 1$. Suppose further that $T(\delta_1) = 16\delta_1$, $T(\delta_2) = 4\delta_1 + 4\delta_2$, $T(E_1) = E_1 + E_2 + y_1$, and that $T(E_k) = E_{k-1} + E_{k+1} + y_k$ for $k > 1$. Suppose $l : X \rightarrow F$ takes $\delta_1$ to 1, and $\delta_2$ and each $E_k$ to 0. We shall calculate the power series $S = \sum l(T^n(E_1)) z^n$ explicitly.

In the above situation, $v_{1,1} = v_{i,i+1} = v_{i+1,i} = 1$ and all other $v_{i,j}$ are 0. So we can take $s = 1$, $A = C = D = 1$ and $B = 0$. Since $s = 1$, $v_{k}^{(n)} = u_{k}^{(n)}$. It follows from this and the definition of the $y_k$ that $\sum_{k,n} v_k^{(n)} y_k z^{n+1} = z(8G_1^* + 6G_0^*) \delta_1 + z(G_0^* - G(w^*)) \delta_2$.

Now the matrix of $T : Y \rightarrow Y$ on the basis $(\delta_1, \delta_2)$ is $\left[\begin{array}{cc} 16 & -4 \\ 0 & 1 \end{array}\right]$. It follows that the matrix of $I - zT$ is $\left[\begin{array}{cc} 1 - 16z & -4z \\ 0 & 1 - 4z \end{array}\right]$. Since $S$ is the co-efficient of $\delta_1$ in $\sum l(y^{(n)}) z^n = (I - zT)^{-1} \sum_{k,n} v_k^{(n)} y_k z^{n+1}$, the last paragraph shows that $(1 - 16z)(1 - 4z)S = (z - 4z^2)(8G_1^* + 6G_0^*) + 4z^2(G_0^* - G(w^*))$. It only remains to calculate $G(w^*)$, $G_0^*$ and $G_1^*$.

Lemma 2.7 and Corollary 2.8 show that $H(w) = z^2G(w)$, and $z^2G(w)^2 -$
\[ G(w) + 1 = 0. \] So \( G(w) \) and \( H(w) \) are \( \frac{1 - \sqrt{1 - 4z^2}}{2z^2} \) and \( \frac{1 - \sqrt{1 - 4z^2}}{2} \). Lemma 2.10 then shows \( G(w^*) = \frac{1}{2z(1 - 2z)}(-1 + 2z + \sqrt{1 - 4z^2}) \). Making use of Lemmas 5.6 and 5.7 we find that \( G_0^* \) and \( G_1^* \) are \( \frac{1}{1 - 2z} \) and \( \frac{1}{2(1 - 2z)^2}(-1 + 2z + \sqrt{1 - 4z^2}) \). A brief calculation then gives the explicit formula:

\[
(1 - 16z)(1 - 4z)(1 - 2z)^2 S = 4z(1 - 2z)^2 + (2z - 12z^2)\sqrt{1 - 4z^2}.
\]

References

[1] I. Gessel, A factorization for formal Laurent series and lattice path enumeration, J. Combinatorial Theory Ser A 28 (1980), 321–337.

[2] P. Monsky, Rationality of Hilbert-Kunz multiplicities: a likely counterexample, Michigan Math. J. 57 (2008), 605–613.