Fluctuations of the positions of nucleons within nuclei result in event-by-event fluctuations of the density of matter produced in ultrarelativistic nucleus-nucleus collisions \(1, 2\). These event-by-event fluctuations have been observed through the centrality and system-size dependence of elliptic flow \(4\), which is the second Fourier harmonic of the azimuthal distribution of outgoing particles. They may generate elliptic flow even in high-multiplicity proton-proton collisions at the LHC \(5, 6\), which might be responsible \(4, 8\) for the near-side ridge observed by CMS \(9\). Furthermore, fluctuations result in new, odd harmonics of the azimuthal distribution of outgoing particles, which are clearly seen experimentally through two-particle correlations \(10, 11\): The third harmonic, triangular flow \(12\), is largely responsible for the ridge and away-side structure seen in two-particle correlations and has recently been analyzed at RHIC \(13\) and LHC \(14\). A smaller first harmonic, directed flow \(15, 16\), is also predicted. Finally, there are non-trivial correlations between these three harmonics \(15, 17\), and they can be directly measured experimentally \(15\).

Flow fluctuations are typically studied within Monte-Carlo models of nucleus-nucleus collisions including initial-state fluctuations, such as NeXSPheRIO \(19\), UrQMD coupled to hydrodynamics \(20\), EPOS coupled to hydrodynamics \(21\), AMPT \(22, 23\), and Glauber models coupled to hydrodynamics \(24, 25\). These complex Monte-Carlo approaches do not allow for a simple understanding of the key quantities driving flow fluctuations.

In this paper, we propose a simple model of initial-state fluctuations and, by comparing with full Monte-Carlo calculations, show that it captures the essential physics contained in common models. We assume that the initial energy density profile in the transverse plane is the superposition of \(N\) random independent \(26\) and identical sources, namely:

\[
\epsilon(x) = \sum_{j=1}^{N} \rho(|x - x_j|),
\]

where \(x_j\) are \(N\) independent random variables with a smooth probability distribution \(p(x_j)\), and \(\rho(r)\) is the profile of a single source. The coordinate system is chosen such that centers of colliding nuclei are on the \(x\) axis, and the origin lies halfway between the two centers. For a collision of identical nuclei, the system then has the symmetries \(p(x, -y) = p(x, y)\) and \(p(-x, -y) = p(x, y)\).

A standard choice for \(\rho(r)\) is a Gaussian profile \(24, 25\):

\[
\rho(r) = \rho_0 e^{-r^2/\sigma^2},
\]

where \(\sigma\) controls the width. The parameters in our model are \(N\), the number of sources, the source distribution \(p(x)\) and the source profile \(\rho(r)\).

In Sec. \(\text{III}\) we introduce the initial anisotropies \(\epsilon_n\) and their moments, which are the relevant quantities. We discuss to what extent \(\epsilon_n\) depends on the source profile \(\rho(r)\). In Sec. \(\text{IV}\) we derive analytic expressions for the rms \(\epsilon_n\) and compare our results with two Monte-Carlo models. In Sec. \(\text{LV}\) our results are extended to higher moments of the distribution of \(\epsilon_n\) and mixed correlations between harmonics \(15, 17, 18\).

II. QUANTIFYING INITIAL FLUCTUATIONS

A. Azimuthal asymmetries of the initial distribution

In a given event, the azimuthal distribution of outgoing particles is driven by the azimuthal distribution
of the initial density \[27\,29\]. We define the participant eccentricity \[14\,\varepsilon_2\], the triangularity \(\varepsilon_3\), and the dipole asymmetry \[13\,\varepsilon_1\], and the corresponding orientations \(\Phi_n\), by:

\[
\varepsilon_2 e^{2i\Phi_2} = -\frac{\{r^2e^{2i\phi}\}}{\{r^2\}},
\]

\[
\varepsilon_3 e^{3i\Phi_3} = -\frac{\{r^3e^{3i\phi}\}}{\{r^3\}},
\]

\[
\varepsilon_1 e^{i\Phi_1} = -\frac{\{r^3e^{i\phi}\}}{\{r^3\}},
\]

where \(\cdots\) denotes an average over the transverse plane in a single event, weighted with the energy density:

\[
\{f(x, y)\} = \frac{\int f(x, y)e(x, y)dxdy}{\int e(x, y)dxdy},
\]

and \((r, \phi)\) are polar coordinates in which the pole is the center of the distribution, that is, \(\{re^{i\phi}\} = 0\).

The numerators in the right-hand side of Eq. (3) are obtained by expanding \(W\) in a single event, weighted with the energy density:

\[
\varepsilon\{n_1, \ldots, n_k\} \equiv \langle \varepsilon_{n_1} e^{in_1\Phi_{n_1}} \cdots \varepsilon_{n_k} e^{in_k\Phi_{n_k}} \rangle,
\]

where angular brackets denote an average over events in a centrality class. With this notation, the rms average of \(\varepsilon_n\) is

\[
\varepsilon_n \{2\} \equiv \langle \varepsilon_n^2 \rangle^{1/2} = \varepsilon\{n_1, -n_1\}^{1/2},
\]

where we have used the notation \(\varepsilon_n \{2\}\) for the rms value of \(\varepsilon_n\) [3].

Symmetry with respect to the reaction plane \((y \rightarrow -y)\) implies that all moments are real, therefore \(\varepsilon\{-n_1, \ldots, -n_k\} = \varepsilon\{n_1, \ldots, n_k\}\). In addition, \((x, y) \rightarrow (-x, -y)\) symmetry implies that the only nonvanishing moments are those such that \(n_1 + \cdots + n_k\) is even. Since the azimuthal orientation of each collision is uncontrolled experimentally, one can only measure rotationally symmetric quantities. Thus, the only relevant moments are those satisfying \(n_1 + \cdots + n_k = 0\). (Some of our intermediate results will involve other moments, such as \(\varepsilon\{1, 1\}\) or \(\varepsilon\{3, -1\}\).)

Within our independent-source model, all moments can be calculated analytically as a systematic expansion in powers of \(1/N\), where \(N\) is the number of sources.

### III. RMS ASYMMETRIES

The rms value of \(\varepsilon_n\) is of direct relevance to analyses of anisotropic flow. Indeed, the simplest measurement of anisotropic flow is a pair correlation, \((\cos n\Delta\phi)\), which is the average value over events of \(v_n^2\). Assuming that \(v_n \propto \varepsilon_n\) on an event-by-event basis, this scales like \(\langle \varepsilon_n^2 \rangle = \varepsilon_n \{2\}^2\). In this Section, we derive analytic results for \(\varepsilon_n \{2\}\) within our independent-source model, and compare with numerical results.

**C. Moments**

In practice, anisotropic flow is not analyzed in a single event, but in a centrality class. It is inferred from multiparticle azimuthal correlations [31]. To the extent that \(v_n\) is driven by \(\varepsilon_n\), the measured azimuthal correlations scale like the corresponding moments of the distribution of \(\varepsilon_n\) and their joint correlations. We introduce the notation [18]:

\[
\varepsilon\{n_1, \ldots, n_k\} \equiv \langle \varepsilon_{n_1} e^{in_1\Phi_{n_1}} \cdots \varepsilon_{n_k} e^{in_k\Phi_{n_k}} \rangle,
\]

where angular brackets denote an average over events in a centrality class. With this notation, the rms average of \(\varepsilon_n\) is

**B. Dependence of \(\varepsilon_n\) on source size**

Within our simple model, one can easily study the dependence of \(\varepsilon_n\) on the profile of a single source \(\rho(r)\). This is relevant to the early-time dynamics, which has a smearing effect on each source [28] and widens the source profile.

The generating function of cumulants is \(W(\tilde{K})\), where \[13\]

\[
e^{W(\tilde{K})} = \left\{e^{i\tilde{k}\cdot x}\right\}.
\]

Teaney and Yan’s cumulants are obtained by expressing \(W(k_x, k_y)\) in terms of \(K \equiv k_x + ik_y\) and \(\tilde{K} \equiv k_x - ik_y\), and expanding in power series of \(K^p \tilde{K}^q\). Specifically, the numerators of \(\varepsilon_1\), \(\varepsilon_2\) and \(\varepsilon_3\) in Eq. (3) are obtained by expanding \(W(K, \tilde{K})\) to order \(K^2\tilde{K}^2\), to \(K^2\), and \(K^3\), respectively, and \(\{r^2\}\) is obtained by expanding \(W(K, \tilde{K})\) to order \(K\tilde{K}\).

In our model, Eq. (1), the density profile is the convolution of the profile of a single source with the distribution of sources \(\sum_j \delta(x - \vec{x}_j)\). Hence

\[
W(\tilde{K}) = \ln \left(\frac{\tilde{\rho}(\tilde{K})}{\hat{\rho}(0)}\right) + \ln \left(\frac{1}{N} \sum_{j=1}^N e^{i\tilde{k}\cdot \vec{x}_j}\right),
\]

where \(\tilde{\rho}(\tilde{K})\) is the Fourier transform of the source profile \(\rho(r)\). By symmetry, \(\tilde{\rho}(\tilde{K})\) depends only on \(|\tilde{K}|^2 = K\tilde{K}\). Therefore the numerators of \(\varepsilon_n\) in Eq. (3) are strictly insensitive to the profile \(\rho(r)\). The only dependence of \(\varepsilon_n\) on the source profile is contained in the denominator, \(\{r^n\}\). If the source has a finite size, \(\{r^n\}\) increases, which results in a smearing of anisotropies. One expects that \(\{r^n\}\) scales approximately like \(|\tilde{K}|^2\) \(\propto \varepsilon_n \{2\}^2\), resulting in a stronger smearing for \(\varepsilon_2\) than for \(\varepsilon_3\), as observed in numerical calculations [28]. One also expects that \(\varepsilon_1/\varepsilon_3\) and \(\varepsilon_3/\varepsilon_2^{3/2}\) are largely independent of the source profile.
A. Analytic results

We denote by $\langle f(x, y) \rangle$ the average value of $f(x, y)$ with the source probability density $p(\vec{x})$, and we introduce the notation $\delta f = \langle f \rangle - \langle f \rangle$ for the event-by-event fluctuations. We use the complex coordinate $z = x + iy$. The asymmetry $\varepsilon_n$ is given by Eq. (3), where we replace $re^{i\delta}$ by $z - \delta_z$ to take into account the recentering correction. To leading order in fluctuations, one obtains

$$\varepsilon_3 e^{3\Phi_3} = \frac{\{ (z - \delta_z)^3 \}^2}{\{ r^3 \}^2} \simeq \frac{\delta z - 3(z^2)\delta z}{\{ r^3 \}^2},$$

$$\varepsilon_1 e^{\Phi_1} = \frac{\{ (z - \delta_z)^2(\bar{z} - \delta_z) \}^2}{\{ r^3 \}^2} \simeq \frac{\delta z \bar{z} - 2(z \bar{z})\delta z - (z^2)\delta z}{\{ r^3 \}^2},$$

(9)

where $\bar{z} = x - iy$. The rms value of $\varepsilon_n$ involves an average over events of products of $\delta$'s. Two-point averages are computed using the following identity, which holds for independent sources [26]:

$$\langle \delta f \delta g \rangle = \frac{\langle fg \rangle - \langle f \rangle \langle g \rangle}{N}. \quad (10)$$

We thus obtain, using the identities $\langle z^n \bar{z}^m \rangle = 0$ for odd $n - m$ and $\langle z^n \bar{z}^m \rangle = \langle r^n + m \cos(n - m)\phi \rangle$ for even $n - m$:

$$\varepsilon_3 \{2\} = \frac{\langle r^6 \rangle + 6\varepsilon_3 \langle r^4 \cos 2\phi \rangle + 9\varepsilon_3^2 \langle r^2 \rangle^3}{N\langle r^3 \rangle^2},$$

$$\varepsilon_1 \{2\} = \frac{1}{N\langle r^3 \rangle^2} \left[ \langle r^6 \rangle - 4\langle r^4 \rangle^2 \right] + \frac{2\varepsilon_1 \langle r^4 \rangle^2 \cos 2\phi + 5\varepsilon_1^2 \langle r^2 \rangle^3}{N\langle r^3 \rangle^2}. \quad (11)$$

In these equations, $\varepsilon_n \equiv -\langle r^2 \cos 2\phi \rangle / \langle r^2 \rangle$ denotes the standard eccentricity. We also recall the result for $\varepsilon_2 \{2\}$ which has been derived earlier [26]:

$$\varepsilon_2 \{2\} = \varepsilon_2^2 + \frac{\langle r^4 \rangle (1 + 3\varepsilon_2^2) + 4\varepsilon_2 \langle r^4 \cos 2\phi \rangle}{N\langle r^3 \rangle^2}. \quad (12)$$

The first term in the right-hand side is the standard eccentricity, and the second term is the contribution of eccentricity fluctuations to order $1/N$. For a large number of sources, $N \gg 1$, the participant eccentricity $\varepsilon_2$ reduces to the standard eccentricity, while odd harmonics $\varepsilon_1$ and $\varepsilon_3$ vanish.

B. Comparison with Monte-Carlo results

We now compare analytic results derived from our independent-source model with results obtained using the mckt-v1.00 Monte-Carlo [32]. With this Monte-Carlo one can calculate results from both a Color-Glass-Condensate (CGC) inspired model — the MC-KLN [33] improved with running-coupling BK unintegrated gluon densities [34], as well as a standard Monte-Carlo Glauber [35]. We present only results for Pb-Pb collisions at 2.76 TeV per nucleon-nucleon collision, though results for 200 GeV Au-Au collisions agree equally well. In the Glauber model, each participant nucleon is given a weight $w = 1 - x + x N_{\text{coll}}$, where $N_{\text{coll}}$ is the number of binary collisions of that nucleon, and $x = 0.18$ [37].

One input of our model is the probability distribution of sources in the transverse plane, $p(\vec{x})$. For the sake of consistency, we assume that sources are distributed according to the average density profile: $p(\vec{x}) \equiv \langle e(\vec{x}) \rangle$, where $\langle \cdots \rangle$ denotes an average over many events in a centrality class. We assume pointlike sources for simplicity: $p(\vec{x}) = \delta(\vec{x})$. The last free parameter in our model is the number of independent sources $N$. One expects that this number scales typically like the number of participant nucleons in a collision. However, participants are not independent, but strongly correlated: for each participant of the projectile, there is by definition at least one participant from the target which is close enough in the transverse plane for a collision to occur. Nevertheless, it is plausible that the system behaves like a set of $N$ independent clusters of two or more nucleons. Again for simplicity, we use $N = 0.45 N_{\text{part}}$ for all centralities, though one could make the agreement with Monte-Carlo even better by fitting $N$ for each centrality class. Increasing $N$ by 20% typically decreases $\varepsilon_3 \{2\}$ and $\varepsilon_1 \{2\}$ by 10%, and $\varepsilon_2 \{2\}$ by less than 4%. If one uses a larger value of $N$ for peripheral collisions (say, $N = 0.6 N_{\text{part}}$ instead of $N = 0.45 N_{\text{part}}$), agreement with Monte-Carlo is significantly better for $\varepsilon_1 \{2\}$ and $\varepsilon_3 \{2\}$ but slightly worse for $\varepsilon_2 \{2\}$.

![Fig. 1](image_url) (Color online) $\varepsilon_n \{2\}$, with $n = 1, 2, 3$, versus centrality. Symbols are Monte-Carlo results, lines are our analytic results for independent sources. For $\varepsilon_2 \{2\}$, the solid line is the full result to order $1/N$ (Eq. (12)), while the solid line is the standard eccentricity (first term in the right-hand side of Eq. (12)).

Fig. 1 displays a comparison between our result (11)
and Monte-Carlo calculations. Our analytic formulae reproduce the centrality dependence of $\varepsilon_n$ computed using Monte-Carlo KLN or Monte-Carlo Glauber. Odd harmonics, $\varepsilon_1$ and $\varepsilon_3$, have a mild centrality dependence and essentially scale like $1/\sqrt{N_{\text{part}}}$. On the other hand, $\varepsilon_2$ is much larger for semi-central collisions: this increase is driven by the almond shape of the overlap area between the two nuclei. For $\varepsilon_2$, we show both the standard eccentricity, which is the leading-order term in a $1/N$ expansion, and the full expression including eccentricity fluctuations to order $1/N$ (Eq. (12)). The standard eccentricity is significantly larger for KLN than for Glauber, so that eccentricity fluctuations (which are roughly the same in both models) are a smaller relative correction.

Our model explains why $\varepsilon_1 < \varepsilon_3$, a feature which was observed in Monte-Carlo calculations but yet unexplained. This may be readily understood from Eq. (11) for central collisions, where $\varepsilon_3 = 0$: only the $\langle r^6 \rangle$ remains for $\varepsilon_3 \{2 \}^2$, while $\varepsilon_1 \{2 \}^2$ has an additional contribution proportional to $\langle r^2 \rangle^2 - \langle r^4 \rangle$, which is negative. For sake of completeness, we have also carried out two other sets of Glauber calculations with only wounded nucleon scaling ($x = 0$) or binary collision scaling ($x = 1$). Values of $\varepsilon_1 / \varepsilon_3$ turn out to be significantly smaller for wounded nucleons (between 0.3 and 0.4 for 0-50% centralities) than for binary collisions (between 0.6 and 0.7). This strong ordering is not reproduced by our analytic result, which gives intermediate results (between 0.5 and 0.6 in both cases). We do not have a simple explanation for this discrepancy.

IV. HIGHER-ORDER MOMENTS

A. Fluctuations of $\varepsilon_n$

The ALICE collaboration has recently measured $v_2 \{4 \}$ and $v_3 \{4 \}$. The relative magnitude of $v_n \{4 \}$ and $v_n \{2 \}$ depends on event-by-event fluctuations of $v_n$, if nonflow effects are small. Assuming that $v_n$ is proportional to $\varepsilon_n$ on an event-by-event basis, fluctuations of $v_n$ are due to fluctuations of $\varepsilon_n$:

$$\left( \frac{v_n \{4 \}}{v_n \{2 \}} \right)^4 = \left( \frac{\varepsilon_n \{4 \}}{\varepsilon_n \{2 \}} \right)^4 = 2 - \frac{\langle \varepsilon_n^4 \rangle}{\langle \varepsilon_n^2 \rangle^2}, \quad (13)$$

where we have introduced the 4-cumulant $\varepsilon_n \{4 \}$. $v_n \{4 \}$ is thus related to $\langle \varepsilon_n^4 \rangle$, which is $\varepsilon \{n, n, -n, -n\}$ in the notation of Eq. (7). Fig. 2 displays the ratio $\langle \varepsilon_n^4 \rangle / \langle \varepsilon_n^2 \rangle^2$ for $n = 1, 2, 3$. The larger the ratio, the larger the fluctuations of $\varepsilon_n$. Note that the order of the curves in Fig. 2 is reversed compared to Fig. 1: smaller values of $\varepsilon_n$ go with larger fluctuations. For the most central collisions, ratios are close to 2 for all $n$. This value corresponds to Gaussian fluctuations \[42\].

For $n = 2$, the ratio quickly decreases with centrality. This is due to the standard eccentricity $\varepsilon_2$, which is zero for central collisions but dominates over eccentricity fluctuations above 10% centrality. The smaller eccentricity fluctuations, the closer the ratio to 1. The decrease is stronger for the KLN model than for the Glauber model because of the larger $\varepsilon_2$. Monte-Carlo results are compared with an analytic result using the expression of $\varepsilon_2 \{4 \}$ to order $1/N$ derived in \[26\]. In practice, $\varepsilon_2 \{4 \} \approx \varepsilon_2$ for all centralities.

We now compute $\langle \varepsilon_3^4 \rangle$ within our independent-source model. To leading order in the fluctuations, $\varepsilon_3$ is given by Eq. (21), and $\langle \varepsilon_3^4 \rangle$ involves products of four $\delta$'s. To leading order in $1/N$, average values of such products can be expressed in terms of two-point averages (Eq. (10)) using Wick's theorem. For instance, the four-point function is \[13\]:

$$\langle \delta_f \delta_g \delta_h \delta_k \rangle = \langle \delta_f \delta_g \rangle \langle \delta_h \delta_k \rangle + \langle \delta_f \delta_h \rangle \langle \delta_g \delta_k \rangle + \langle \delta_f \delta_k \rangle \langle \delta_g \delta_h \rangle \tag{14}$$

This identity gives

$$\varepsilon \{3, 3, -3, -3\} = 2\varepsilon \{3, -3\}^2 + \varepsilon \{3, 3\} \varepsilon \{-3, -3\}, \quad (15)$$

or, equivalently,

$$\frac{\langle \varepsilon_3^4 \rangle}{\langle \varepsilon_3^2 \rangle^2} = 2 + \left( \frac{\varepsilon \{3, 3\}}{\varepsilon \{3, 2\}^2} \right)^2. \quad (16)$$

---

1 We have also done comparisons with the PHOBOS Monte-Carlo Glauber \[37\], and found that $\varepsilon_1$ from this model is significantly smaller than predicted by our analytic formula.
\( \varepsilon_{3,3} \) can easily be calculated to order \( 1/N \) in the same way as \( \langle \varepsilon^3 \rangle \):

\[
\varepsilon_{3,3} = \langle \varepsilon^3 \rangle = \frac{1}{N(r^2)^2} \left( r^6 \cos 6\phi + 6 \varepsilon_s (r^2) (r^4 \cos 4\phi) - 9 \varepsilon_s^3 (r^2)^3 \right).
\]  

Inserting Eqs. (11) and (17) into (16), we obtain a leading-order analytic result for the ratio. This analytic result is independent of \( N \), and very close to 2 in practice, as seen in Fig. 2. This means that the first term on the right-hand side is the dominant contribution. It is easy to understand why the second term is small: \( \varepsilon_{3,3} \) involves the 6th Fourier harmonic of the initial distribution, which is of order \( \varepsilon_s^3 \). Therefore, the last term in Eq. (16) is of order \( \varepsilon_s^6 \), which is very small in practice.

The Monte-Carlo results in Fig. 2 show that the ratio \( \langle \varepsilon_3 \rangle^2 \) is slightly smaller than 2 for both Monte-Carlo KLN and Glauber models (a fact also implied by experimental results [44]), which cannot be explained by our leading-order result Eq. (16). The next-to-leading correction (restricted to central collisions) is derived in the Appendix. It is negative and scales like \( 1/N \). As shown in Fig. 2, it improves agreement.

Finally, \( \langle \varepsilon_4 \rangle \) can be computed in the same way. The leading-order result is:

\[
\frac{\langle \varepsilon_4 \rangle}{\langle \varepsilon_1 \rangle^2} = 2 + \left( \frac{\varepsilon_{1,1}}{\langle \varepsilon_1 \rangle} \right)^2,
\]  

where

\[
\varepsilon_{1,1} = \langle \varepsilon^3 \rangle = \frac{1}{N(r^2)^2} \left( r^6 \cos 2\phi + 8 \varepsilon_s (r^2)^3 + 2 \varepsilon_s (r^2) (r^4) - 4 (r^2)^3 (r^2 \cos 2\phi) - \varepsilon_s^3 (r^2)^3 \right).
\]  

This quantity is negative, which means that the dipole asymmetry develops mostly out of the reaction plane for non-central collisions. The last term in Eq. (18) gives a positive contribution to the ratio \( \langle \varepsilon_4 \rangle^2 / \langle \varepsilon_1 \rangle^2 \), which increases up to 3 for peripheral collisions. Note that \( \varepsilon_{1,1} \) is defined by Eq. (18) and is independent of \( N \). It is in good agreement with Monte-Carlo results.

Note that within our independent-source model, the ratios plotted in Fig. 2 are strictly insensitive to the profile of a single source \( \rho(r) \).

### B. Mixed correlations

We now study the correlations between \( \Phi_1, \Phi_2 \) and \( \Phi_3 \). The lowest order non-trivial correlations are

\[ \varepsilon_{23} \equiv \varepsilon_{3,3} = \langle \varepsilon_{3,3} \rangle \]

\[ \varepsilon_{12} \equiv \varepsilon_{1,1} = \langle \varepsilon_{1,1} \rangle \]

\[ \varepsilon_{13} = \langle \varepsilon_{3,1} \rangle \]

All these correlations were studied numerically in Ref. [18], with the exception of \( \varepsilon_{123} \) which is new. We derive an analytic prediction for these quantities using our independent-source model. To leading order in \( 1/N \), one can replace each factor of \( \varepsilon_{2k} \varepsilon_{2k+2} \) in Eq. (7) by the standard eccentricity \( \varepsilon_s \). The moments in Eq. (20) can then be expressed in terms of two-point moments using Wick's theorem:

\[ \langle \varepsilon_{3,3} \rangle = \langle \varepsilon_{3,3} \rangle \]

\[ \langle \varepsilon_{1,1} \rangle = \langle \varepsilon_{1,1} \rangle \]

\[ \langle \varepsilon_{3,1} \rangle = \langle \varepsilon_{3,1} \rangle \]

To leading order in \( 1/N \), \( \varepsilon_{3,3} \) and \( \varepsilon_{1,1} \) are given by Eqs. (17) and (19). The remaining two-point functions \( \langle \varepsilon_{3,1} \rangle \) and \( \varepsilon_{3,1} \) can be computed similarly:

\[ \langle \varepsilon_{3,3} \rangle = \langle \varepsilon_{3,3} \rangle \]

\[ \langle \varepsilon_{1,1} \rangle = \langle \varepsilon_{1,1} \rangle \]

\[ \langle \varepsilon_{3,1} \rangle = \langle \varepsilon_{3,1} \rangle \]

These equations define our analytic results for the moments.

We now compare our analytic results with Monte-Carlo results. We first scale \( \varepsilon_{n,1} \) by \( \varepsilon_{n,1} \) to single out the angular correlation as in Ref. [18]. We compute these ratios both with the Monte-Carlo and with our analytic formulas. To leading order in \( 1/N \), we use the approximation \( \varepsilon_{2k} \approx \varepsilon_s \). One easily shows, using Eq. (21), that the resulting leading-order prediction for the ratios is independent of \( N \).

Fig. 5 displays a comparison between Monte-Carlo results and analytic results. Our leading-order results explain the sign, magnitude, and centrality dependence of all ratios.

The scaled \( \varepsilon_{23} \) is much smaller than unity, in agreement with earlier observations that triangularity and eccentricity are uncorrelated except for peripheral collisions [45]. This means that there is a strong positive correlation between \( \Phi_1 \) and \( \Phi_3 \) [17]. On the other hand, \( \varepsilon_{12} \) is negative, which means that
FIG. 3. (Color online) Mixed correlations versus centrality. From top to bottom: $\varepsilon_{13}/(\varepsilon_1\varepsilon_3)$ (labeled 13), $\varepsilon_{123}/(\varepsilon_1\varepsilon_2\varepsilon_3)$ (labeled 123), $\varepsilon_{23}/(\varepsilon_2\varepsilon_3)$ (labeled 23), $\varepsilon_{123}/(\varepsilon_1\varepsilon_2\varepsilon_3)$ (labeled 123) and $\varepsilon_{12}/(\varepsilon_1\varepsilon_2)$ (labeled 12). Symbols are Monte-Carlo results. Full lines are analytic results to leading-order in $1/N$. Dashed lines for $\varepsilon_{12}$ and $\varepsilon_{123}$ include next-to-leading order corrections derived in Appendix. As in Fig. 2 we display NLO results up to 40% centrality, although they are only valid for central collisions.

$\Phi_1$ is more likely to be perpendicular to the participant plane. Because of the strong correlation between $3\Phi_1$ and $3\Phi_3$, this also explains why the mixed correlation $\varepsilon_{123}$ is also negative.\cite{12,17}.

A closer look at Fig. 3 reveals that our leading-order results for $\varepsilon_{12}$ and $\varepsilon_{123}$ do not agree well with Monte-Carlo for central collisions, where both ratios are small but not zero, while our leading-order results go smoothly to zero. Agreement is improved by including the next-to-leading term, which is of order $1/N^2$, negative, and does not vanish for central collisions. This term is calculated in the Appendix for central collisions.

V. CONCLUSIONS

We have shown that the magnitude of initial-state anisotropies $\varepsilon_n$, their fluctuations and mutual correlations can be understood within a simple model where fluctuations stem from identical, independent sources, which can be viewed as “hot spots” scattered across the interaction region.\cite{46,47} The independent-source model reproduces results from both CGC-inspired and and Glauber Monte-Carlo models.

In our model, all information about the initial state is encoded in a few parameters: the number of sources $N$, the profile of a single source, and the distribution of sources across the transverse plane. We have considered pointlike sources for simplicity. The magnitudes of $\varepsilon_1$ and $\varepsilon_3$ show that $N$ is roughly half the number of participant nucleons, which means that participant nucleons are correlated pairwise. The angular correlations between $\Phi_1$, $\Phi_2$ and $\Phi_3$ are independent of $N$ to leading order in $1/N$. They only depend on the distribution of sources, and are mostly driven by the almond shape of the overlap area between the two nuclei, which is responsible for the large elliptic flow.\cite{48}

There is an ambiguity in the definition of the “initial state”. What is computed in Monte-Carlo models (KLN or Glauber) is typically the distribution of energy right after the nuclei have passed through each other. On the other hand, the relevant anisotropies for collective flow are the anisotropies at a somewhat later time, when the system reaches local equilibrium. The value of $\varepsilon_n$ decreases during this thermalization time.\cite{28} We have shown that this decrease is solely due to the increase in the system size, due to the smearing of each source, $\varepsilon_3$ depends more strongly on the system size than $\varepsilon_2$, which explains why it decreases more strongly during the thermalization phase. (This effect was interpreted by the authors of Ref. \cite{28} as an interference between $\varepsilon_2$ and $\varepsilon_3$.)

Within our independent-source model, ratios such as $\langle \varepsilon^4_1 \rangle/\langle \varepsilon^2_1^2 \rangle$, as well as the various angular correlations between event planes considered in \cite{18}, are independent of the number of sources to leading order in $1/N$ (with the exception of $\langle \varepsilon^4_2 \rangle/\langle \varepsilon^2_2^2 \rangle$). Remarkably, these ratios are also unchanged through the early-time dynamics, and are therefore well-defined observables for initial-state fluctuations.

ACKNOWLEDGMENTS

We would like to thank Clément Gombeaud for help with the Monte-Carlo models, and in particular for providing the code for centrality determination. This work is funded by “Agence Nationale de la Recherche” under grant ANR-08-BLAN-0093-01 and by CEFIPRA under project 4404-2. RSB acknowledges the hospitality of IPhT Saclay where part of this work was done.

Appendix A: Higher-order calculations

In this Appendix, we present next-to-leading order calculations in $1/N$ for a few moments. Since next-to-leading order calculations are considerably more involved than leading-order calculations, we assume azimuthal symmetry for sake of simplicity. We compute three quantities for which our leading-order results vanish in the limit of azimuthal symmetry, namely, $\varepsilon_3\{4\}$, $\varepsilon_{12}$ and $\varepsilon_{123}$.

We first compute $\varepsilon_3\{4\}$. Taking the recentering cor-
After some algebra, we get
\[ \varepsilon^2 = \frac{(z - \delta^3)}{|z - \delta^3|^2}. \] (A1)

Treating \( \delta \) as the expansion parameter, we can expand this expression to any desired order in \( \delta \). For instance, \( \{z - \delta^3\} = \delta^3 - 3z^2\delta^2 - 3z\delta^2 + 2(\delta^3) \). We organize the calculation as follows. Let
\[ \varepsilon^2 = B + C + D + \cdots, \] (A2)
where \( B \sim O(\delta^2), C \sim O(\delta^3), \) etc. (There are no terms independent of \( \delta \) or of order \( \delta \).) Hence
\[ \varepsilon^2 = \varepsilon^2 + \frac{1}{N} \epsilon_\varepsilon \left[ 2\varepsilon^2 \right] \] and
\[ \varepsilon^2 = \varepsilon^2 + \frac{1}{N} \epsilon_\varepsilon \left[ 2\varepsilon^2 \right] \] (A3)

To proceed further, we make use of the expressions for \( \langle \delta f \delta \rangle, \langle \delta f \delta \delta \rangle \) and \( \langle \delta f \delta \delta \delta \rangle \) given in \[43\]. (These expressions ignore correlations between participant positions.) Dominant terms in these five quantities are of \( O(1/N) \), \( O(1/N^2) \), \( O(1/N^3) \) and \( O(1/N^4) \), respectively, where \( N \) is the number of participants. For central collisions, azimuthal symmetry allows to simplify the previous expression:
\[ \varepsilon^2 = \frac{2}{N} \left( \varepsilon^2 \right)^2 - \left( \varepsilon^4 \right) \] (A5)

After some algebra, we get
\[ \varepsilon^2 = \frac{1}{N^3} \left[ 2(\varepsilon^4)^2 - (\varepsilon^6) + 8(\varepsilon^2)(\varepsilon^2)^3 - 8(\varepsilon^6) \right]. \] (A6)

A similar result has been derived previously for \( \varepsilon_2 \), which is also of order \( 1/N^3 \) for independent sources and central collisions \[43\]. This scaling in \( 1/N^3 \) is natural for the four-particle cumulant \[31\].

Other moments can be computed in a similar way. Our results for \( \varepsilon_{123} \) and \( \varepsilon_{12} \) are given in Appendix \[B\].

### Appendix B: Summary of results

The expressions of \( \varepsilon_{12} \) to order \( 1/N \) are given by Eqs. (11) and (12). The higher-order cumulant \( \varepsilon_{123} \) has been derived to order \( 1/N^2 \):
\[ \varepsilon_{123} = \frac{1}{N^2} \left[ 2\varepsilon(\varepsilon^2)^2 - \varepsilon^2 \right] \] (A7)

The expression of \( \varepsilon_{123} \) was derived in \[43\] up to order \( 1/N^2 \) for central collisions:
\[ \varepsilon_{123} = 2\varepsilon^2 - 2\varepsilon \]
\[ = \varepsilon^4 + \frac{1}{N} \epsilon_\varepsilon \left[ 2\varepsilon^4 \right] \] plus \[A2\], \[A3\], \[A4\]... (B2)

Our result for \( \varepsilon_{123} \) to order \( 1/N^3 \) is
\[ \varepsilon_{123} = 2\varepsilon^2 - 2\varepsilon \]
\[ = \varepsilon^4 + \frac{1}{N} \epsilon_\varepsilon \left[ 2\varepsilon^4 \right] \] plus \[A2\], \[A3\], \[A4\]...

The 3-point mixed correlations are to order \( 1/N^2 \),
\[ \varepsilon_{12} = \frac{\varepsilon_{12}}{N} \left[ (\varepsilon^2)^2 - (\varepsilon^4) \right] \] plus \[A2\], \[A3\], \[A4\]...

and
\[ \varepsilon_{123} = \frac{\varepsilon_{12}^3}{N} \left[ (\varepsilon^2)^2 - (\varepsilon^4) \right] \] plus \[A2\], \[A3\], \[A4\]...

For higher-order mixed correlations, we have only carried out leading-order calculations. The 4-point correlations considered in this paper are
\[ \varepsilon_{123} = \frac{\varepsilon_{12}^2}{N} \left[ (\varepsilon^2)^2 - (\varepsilon^4) \right] \] plus \[A2\], \[A3\], \[A4\]...

and
\[ \varepsilon_{123} = \frac{\varepsilon_{12}^3}{N} \left[ (\varepsilon^2)^2 - (\varepsilon^4) \right] \] plus \[A2\], \[A3\], \[A4\]...
and
\[
\varepsilon_{13} = \frac{3}{N^2(r^3)} \left[ (r^6 \cos 2\phi) - 8\varepsilon_s(r^2)^3 + 2\varepsilon_s(r^2) \langle r^4 \rangle \right. \\
\left. - 4\langle r^4 \rangle \langle r^2 \cos 2\phi \rangle - \varepsilon_s^3 \langle r^2 \rangle^3 \right] \\
\times \left[ (r^6 \cos 2\phi) + 3\varepsilon_s(r^2) \langle r^4 \rangle - 6\varepsilon_s(r^2)^3 \right. \\
\left. - 3\varepsilon_s(r^2) - 2\varepsilon_s \langle r^4 \rangle \langle r^2 \cos 2\phi \rangle + \varepsilon_s(r^2)^3 \right].
\] (B7)

Finally, we have computed the 5-point correlation
\[
\varepsilon_{23} = \frac{\varepsilon_s^3}{N(r^3)^2} \left[ (r^6 \cos 6\phi) + 6\varepsilon_s(r^2) \langle r^4 \rangle \cos 4\phi \right. \\
\left. - 9\varepsilon_s^3 \langle r^2 \rangle^3 \right].
\] (B8)

For sake of completeness, we list all other 4- and 5-point correlations which can be constructed out of the first three harmonics, and their expressions to leading order, obtained using Wick's theorem:
\[
\varepsilon \{2, -2, 3, -3\} = \varepsilon_2 \{2\}^2 \varepsilon_3 \{2\}^2 \\
\varepsilon \{1, -1, 2, -2\} = \varepsilon_1 \{2\} \varepsilon_2 \{2\}^2 \\
\varepsilon \{1, -1, 3, -3\} = \varepsilon_1 \{2\}^2 \varepsilon_3 \{2\}^2 + \varepsilon \{3\} \{1\}^2 + \varepsilon \{3\} \{1\} \varepsilon \{3\} \{1\} \\
\varepsilon \{1, 2, 3, -3\} = \varepsilon_2 \{2\} \varepsilon_3 \{2\} \varepsilon \{3\} \{1\} + \varepsilon \{1\} \{1\} \varepsilon \{3\} \{1\} \\
\varepsilon \{1, 1, -2, 3, -3\} = \varepsilon_2 \{2\} \varepsilon_3 \{2\} \varepsilon \{3\} \{1\} + \varepsilon \{2\} \{1\} \varepsilon \{3\} \{1\} \\
\varepsilon \{1, 1, 2, -2, -2\} = \varepsilon_2 \{2\} \varepsilon_3 \{2\} \\
\varepsilon \{1, 1, 1, -1, -2\} = 3\varepsilon_2 \{2\} \varepsilon_3 \{2\}.
\] (B9)

The first of these equations means that there are no correlations between the magnitudes of \(\varepsilon_2\) and \(\varepsilon_3\) to leading order. This could be tested experimentally by measuring the correlation between the magnitudes of \(v_2\) and \(v_3\).
[39] T. Hirano, U. W. Heinz, D. Kharzeev, R. Lacey, Y. Nara, Phys. Lett. B636, 299-304 (2006). [nucl-th/0511046].
[40] T. Lappi, R. Venugopalan, Phys. Rev. C74, 054905 (2006). [nucl-th/0609021].
[41] K. Aamodt et al. [ The ALICE Collaboration ], Phys. Rev. Lett. 105, 252302 (2010). [arXiv:1011.3914 [nucl-ex]].
[42] S. A. Voloshin, A. M. Poskanzer, A. Tang and G. Wang, Phys. Lett. B 659, 537 (2008) [arXiv:0708.0800 [nucl-th]].
[43] B. Alver et al., Phys. Rev. C 77, 014906 (2008) [arXiv:0711.3724 [nucl-ex]].
[44] R. S. Bhalerao, M. Luzum, J. -Y. Ollitrault, [arXiv:1106.4940 [nucl-ex]].
[45] J. L. Nagle and M. P.McCumber, Phys. Rev. C 83, 044908 (2011) [arXiv:1011.1853 [nucl-ex]].
[46] M. Gyulassy, D. H. Rischke, B. Zhang, Nucl. Phys. A613, 397-434 (1997) [nucl-th/9609030].
[47] G. -L. Ma, X. -N. Wang, Phys. Rev. Lett. 106, 162301 (2011). [arXiv:1011.5249 [nucl-th]].
[48] J. -Y. Ollitrault, Phys. Rev. D46, 229-245 (1992).