Diversification and Endogenous Financial Networks

Jean-Cyprien Héam*  Erwan Koch†

Abstract

We propose to test the assumption that interconnections across financial institutions can be explained by a diversification motive. This idea stems from the empirical evidence of the existence of long-term exposures that cannot be explained by a liquidity motive (maturity or currency mismatch). We model endogenous interconnections of heterogeneous financial institutions facing regulatory constraints using a maximization of their expected utility. Both theoretical and simulation-based results are compared to a stylized genuine financial network. The diversification motive appears to plausibly explain interconnections among key players. Using our model, the impact of regulation on interconnections between major banks—currently discussed at the Basel Committee on Banking Supervision—is analyzed.

Key words: Diversification; Financial networks; Regulation; Solvency; Systemic risk.
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*Autorité de Contrôle Prudentiel et de Résolution (ACPR) and CREST. jean-cyprien.heam@acpr.banque-france.fr
†ISFA, CREST and ETH Zürich (Department of Mathematics, RiskLab). erwan.koch@math.ethz.ch
1 Introduction

The behaviors of financial institutions, namely banks and insurance companies, constitute a paradox. On the one hand, they oppose one another in a competition to collect deposits as one may expect for firms in a common sector. In this perspective, the distress of one institution seems good news for the others since there is room for increasing market shares. However, on the other hand, financial institutions need to cooperate. For instance, the withdrawal of a bank from the short term interbank market means that a source of liquidity vanishes, triggering setbacks for others banks. In this case, one financial institution’s distress is definitely bad news for the other ones. Thus, even if they are in competition, banks cooperate, insurance companies cooperate and last but not least, banks cooperate with insurance companies. The last point has been ever more significant during the recent years. A support of this cooperation is the interconnections they develop between each other.

In a short-term view, interconnections mirror the resolution of the liquidity needs. As any other firms, banks and insurance companies face asynchronous in-flows and out-flows of cash. One solution is that every institution has its own cash buffer. Alternatively, institutions can create a liquidity pool by sharing their cash to mutualize the liquidity risk [Holmstrom and Tirole 1996; Tirole 2010; Rochet 2004]. Allen and Gale (2000) explicitly link the interconnectedness of banks to liquidity shocks. Besides the asynchronism of in-flows and out-flows, the liquidity risk is particularly salient since banks are exposed to runs [Diamond and Dybvig 1983] and operate large gross transactions in payment systems [Rochet and Tirole 1996]. Indeed flows between institutions are not netted.

However, one may argue that this analysis is not specific to banks and insurance companies since every firm actually faces asynchronous in-flows and out-flows. Liquidity concerns are not the only cause of interconnections between financial institutions. Moreover, there is evidence in the literature that banks are interconnected not only in the short term but also in the long run. For instance, according to Upper and Worms (2004), half of German interbank lending is composed of loans whose maturity is over 4 years (see Figure 1). According to Table 1 in Alves et al. (2013), interbank assets with residual maturity larger than one year account for about 50% of total interbank assets at the European level. These long term exposures cannot be explained by a liquidity motive since liquidity is a short term phenomenon. Other possible reasons are horizontal integration (share of a pool of customers via joint products), vertical integration (e.g. risk transfer between insurance and reinsurance companies), ego of top managers aiming at increasing their control of the market and last but not least diversification. Of course in practice the network formation stems from a combination of all these motives. However, for reasons explained further, diversification appears as a very important motive. Therefore, in this paper, we consider that these long-term exposures are accounted for by a diversification principle, in a sense that will be defined in the following.

Considering the diversification principle is supported by the existence of various business models for banks and insurance companies. The diversity of institutions leads to a diversity of debts and shares available for the other institutions as assets. In the case of insurance companies, there is a clear-cut distinction between mutual funds and profit-

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1The existence of long-term interconnections, through loans or shares, is also reported for other countries such as Canada (see Table 3 in Gauthier et al. 2012) or France (Fourel et al. 2013).
oriented insurance companies. The banking sector regroups heterogenous institutions from mutual saving banks to commercial banks. Moreover, the "bankinsurance" business model weakens the limit between banking and insurance activities. This variety can be explained by the different preferences of stakeholders or by historic patterns. Investors who have the same risk aversion gather and form an institution. This heterogeneity can also be linked to a specialization process. For instance, a mutual saving bank funded by farmers is very efficient in granting loans to farmers who in turn favor this bank since it provides the fairest interest rate. This auto-selection mechanism leads to a situation close to a local monopoly. We then understand that for a specific bank, getting interconnected to other institutions is a way to get access to their specific markets. Considering specific markets implicitly assumes that these are not perfectly correlated: for example retail differs from trading. Similarly insurance companies also specialize in specific risk classes. Thus being interconnected to other insurance companies allows diversifying one’s risk portfolio. All this goes to consider that the diversification principle may explain long-term interconnections among banks and insurance companies.

In order to properly model banking and insurance activities, one has to keep in mind that the banking and insurance sectors are characterized by a very specific production process and a heavy regulation. The core activity of a bank dwells in the selection of profitable loans to grant and in maturity transformation. Banks screen potential entrepreneurs for reliable projects and fairly price the interest rate. At the same time they manage the maturity gap: loans to entrepreneurs are long-term assets whereas deposits and issued bonds constitute short-term debt on the liability side. Information is also key to the core activity of insurance (e.g. damage insurance): the insurer has to efficiently assess the riskiness of the potential policyholder and to deduce the corresponding premium. Strictly speaking the insurance company does not provide maturity transformation. However, its production cycle is reversed: it first collects premia and cushions losses when claims occur. The regulation of the banking and insurance sectors is crucial to maintain people’s confidence in the system. In order to avoid bank runs, it is necessary that depositors consider their deposit as risk-free. Likewise if policy-holders are not confident in the capacity of the insurer to honor its commitments, they will make other insurance choices. A solvency ratio is imposed to banks and insurance companies: in the case of banks, the ratio compares the riskiness of granted loans with the own funds.

Table 1: Extract of Table 1 in [Upper and Worms (2004)].
while for insurance companies, the ratio balances the riskiness of insured risks and the collected premia.

Our paper has two main objectives. The first objective is to test whether a diversification motive is a plausible cause for interconnectedness across financial institutions. To do so, we build a model where interconnections are endogenous choices of financial institutions resulting from the maximization of their expected utility. After deriving some theoretical and simulation-based features of the resulting network, we compare these features to those of a stylized financial network (benchmark) based on empirical evidence. The second objective is to fairly assess the impact of regulation on interconnections using our model.

The cornerstone of this paper is the modeling of the endogenous balance sheet of a financial institutions, especially interconnections. Endogenous networks have been intensively analyzed in sociology or socio-economics (for a survey, see Goyal (2012) or Jackson and Zenou (2013)). However, finance is a new field for application. Usually, interconnections among financial institutions are considered as given, especially in applied papers (see among others Cifuentes et al. (2005), Arinaminpathy et al. (2012), or Anand et al. (2013)). Endogenous financial networks stem from the seminal paper by Allen and Gale (2000). For instance, Babus (2013) models interconnections across banks as the result of an insurance motive: interconnections represent a protection means against contagion. More recently, Acemoglu et al. (2013) focus on the short-term interbank market and model the network formation in the case of risk-neutral banks being able to renegotiate their claims in a case of distress. Elliott et al. (2012) make a case of showing the incentives that may drive financial network formations. Important insights are brought by this strand of literature inspired by microeconomics and game theory analysis. Nevertheless, in this field, financial institutions only compute their interconnections: the remaining elements of their balance sheet are completely exogenous. This assumption seems suitable in a short-term perspective but not anymore when considering long-term interconnections. Therefore, by including more balance sheet items than the sole interconnections, we distance ourselves from this strand. To the best of our knowledge, the unique paper that considers a complete balance sheet optimization (apart from the debt) is Bluhm et al. (2013). They propose a dynamic network formation with risk-neutral banks. Using a specific "trial and error" process, the authors first compute the volume of interbank assets (that corresponds to the network’s importance) and second its allocation (that corresponds to the network’s shape). The allocation is carried out using a matching algorithm based on the strict indifference of banks. In contrast, our paper considers heterogeneously risk-averse banks which explicitly get interconnected to specific counterparts. Last but not least, almost all papers consider lending (or debt securities) only whereas, inheriting from Gouriéroux et al. (2012), our paper also considers shares. This feature cannot be neglected in a long run perspective since financial institutions can take cross-share holdings.

The paper is organized as follows. Section 2 falls into two parts. First, the production process of banks and insurance companies and the regulatory constraints are described. Secondly, the financial network benchmark is described. Section 3 presents the theoretical results. After describing the optimization program of financial institutions, we show the existence of an equilibrium and discuss the conditions for its uniqueness. We show that interconnections are usually optimal for financial institutions. These theoretical properties

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3See among others Cohen-Cole et al. (2011), Gofman (2012), Farboodi (2014) or Georg (2014).
allow to characterize the shape of the network stemming from a diversification motive. Therefore we compare the shape of a genuine interbank network to a diversification-based one. Section 4 gathers simulation results. We first present the computational methodology and the calibration choices. Then we analyze the sensitivity of interconnections. These results lead us to assess the proximity of the obtained network to the benchmark network both in terms of balance sheet volume and support of interconnections (debt securities or cross-share holdings). Section 5 provides an analysis of financial interconnections with respect to the financial regulation. Elaborating on Repullo and Suarez (2013), we first show how to fairly analyze interconnectedness and then compare different regulations. Section 6 concludes. All proofs are gathered in Appendix.

2 Balance sheet structure and network benchmark

In this section, we first describe the economic setup which corresponds to the technology of financial institutions. We introduce the different elements of their balance sheet as well as the regulatory constraints. Then we present the stylized network later used as a benchmark.

2.1 Bank and insurance business

Each bank has access to a specific class of external illiquid assets and each insurance company specializes in one specific class of risk. These classes can be interpreted as main banking (respectively insurance) activities such as trading, commercial loans, mortgage loans, sovereign loans ...(resp. property insurance, liability insurance...).

The tight relationship between a specific class of asset (resp. risk) and a specific institution has to be interpreted as a consequence of costly portfolio management by investors followed by a specialization process. By portfolio management we mean the screening process. For banks that means selecting promising entrepreneurs to finance and offering a fair interest rate. In the case of insurance companies, it means organizing the mutualisation of risks, i.e. finding the adequate premium with respect to the policyholder’s risk profile. The specialization process strengthens the efficiency of managing a specific portfolio. Due to auto-selection of customers, specialization triggers further specialization.

2.1.1 Asset side

Bank $i$’s specific asset book value is labeled $A_{x_i}$. This asset is some illiquid loan and therefore cannot be exchanged on a market. Thus, no market value can be defined and only their book value is considered in the following. We denote by $r_i$ the return of $A_{x_i}$. The returns are jointly distributed. Their cumulative distribution function (c.d.f.) is denoted $F_S$. Banks have access to another external asset, similar to cash. This asset is denoted by $A_{\ell,i}$ and its return is $r_{\ell,f}$. We assume that insurance companies’ external assets are only $A_{\ell,i}$. Here $A_{\ell,i}$ is a very liquid and low-risk asset, the management of which does not require high technical skills (for instance AAA bonds or S&P 500 shares). Cash does not require any screening. Insurance companies are indeed assumed not to have the same capacity of selecting promising innovators as banks. Besides, institution $i$ can buy shares or debt securities issued by institution $j$ respectively in proportions $\pi_{i,j}$ and $\gamma_{i,j}$.
2.1.2 Liability side

The liability side is composed of equity (that is brought by investors) and nominal debt, whose book values are respectively denoted by $K_i$ and $Lx_i^*$ for institution $i$. Since equity and debt securities will be traded on the secondary market, it is necessary to introduce their market values, respectively denoted by $\tilde{K}_i$ and $\tilde{L}_x i$.

In the case of banks, $Lx_i^*$ includes different types of debts (deposits and bonds of various maturities) considered as homogeneous in terms of seniority. Banks issue debt along a common yield curve. In other words, bank debt securities are considered risky (the interest rate curve is above the risk free yield curve) but have a common degree of risk (the same rating, say). Despite this common feature, bank $i$ chooses its own degree of maturity transformation $\omega_i \in [0; 1]$. $\omega_i$ is the average of maturities of all types of debts. For instance deposits can be seen as every day re-funded overnight loans by households to banks and therefore correspond to $\omega_i = 1$. On the opposite, a debt whose maturity equals the asset maturity corresponds to $\omega_i = 0$. Banks usually assume that their short-term debt will be rolled over. However, it is not always the case, especially during crises. If a bank is only funded by deposits ($\omega_i = 1$), it may happen that all depositors suddenly quit, causing a funding liquidity shock. The same can happen in the case of debt issued with bonds if investors decide not to roll over. In the extreme opposite case ($\omega_i = 0$), there is no possible liquidity shock (but there is no maturity transformation). Banking activity is precisely profitable due to maturity transformation since the interest rate corresponding to long term lending (asset side) is superior to the one corresponding to short term borrowing. In our model $\omega_i$ directly affects the bank interest rate on its debt through a deterministic rule: $r_D(\omega_i)$.

In the case of insurance companies, the nominal debt $Lx_i^*$ mostly corresponds to technical provisions relative to the underwritten risks. Therefore $\omega_i$ can no longer be interpreted as an average maturity but as the mean severity of claims. Contrary to banks, the liability side of an insurer is stochastic. For instance, in line with standard ruin models (Lundberg, 1903 and Cramer, 1930), $\omega_i$ could be the parameter of the Pareto distribution followed by the claims. Of course, the collected premia directly reflect the risk profile of the insurance contracts.

The balance sheet of bank $i$ is represented at the initial date and the end date respectively in Tables 2 and 3.

| Asset            | Liability        |
|------------------|------------------|
| $\pi_{i,1}K_1^{(0)}$ | $Lx_i^*$         |
| ...              | $\vdots$         |
| $\pi_{i,n}K_n^{(0)}$ | $K_i^{(0)}$     |
| $\gamma_{i,1}Lx_1^{(0)}$ | $\leftrightarrow$ |
| ...              | $\vdots$         |
| $\gamma_{i,n}Lx_n^{(0)}$ | $Ax_i^{(0)}$    |
| $Ax_i^{(0)}$     | $\leftrightarrow$ |
| $A\ell_i^{(0)}$  |                  |

Table 2: Balance sheet of institution $i$ at the initial date $t = 0$. 

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It is important to note that the equity and the debt of the other institutions (on the asset side) must be priced at the market value at $t = 0$. At time $t = 1$, the book value can be considered.

In line with the model of Value-of-the-Firm (Merton, 1974), the value of the debt $Lx_i$ and the equity $K_i$ are given by the following equilibrium conditions

\[
K_i = \left( \sum_{j=1}^{n} \pi_{i,j}K_j + \sum_{j=1}^{n} \gamma_{i,j}Lx_j + A\ell_i + Ax_i - Lx_i^* \right)^+, \tag{1}
\]

\[
Lx_i = \min \left( \sum_{j=1}^{n} \pi_{i,j}K_j + \sum_{j=1}^{n} \gamma_{i,j}Lx_j + A\ell_i + Ax_i; Lx_i^* \right). \tag{2}
\]

These $2n$ equations define a liquidation equilibrium. Equation (1) corresponds to the simple accounting definition of equity as the net value of assets over debts. Equation (2) is very similar to (1) and directly inherited from Merton’s model: the debt value is the minimum between the asset value and the nominal debt. Note that $Lx_i^*$ gathers all types of debts (especially deposits and issued bonds). These different debt categories are considered as homogeneous in terms of seniority.

Previous Equations (1)–(2) provide consistent values for financial instruments across all institutions. The same contract (for e.g. a loan) is a liability for one institution (the borrower) and an asset for another institution (the lender). Proposition 2 in Gouriéroux et al. (2012) states that these equations define a suitable liquidation equilibrium (see Proposition 8 in Appendix B.1). The cornerstone of our strategy will consist in optimizing the balance sheet items of the financial institutions (apart from the equity which is exogenous). Proposition 8 states that whatever the balance sheet composition of each institution (whatever the values of $Ax_i$, $A\ell_i$, $\pi_{ij}$, $\gamma_{ij}$ and $Lx_i^*$ verifying Assumptions (A1), (A2) and (A3)), the obtained network can theoretically exist (under suitable unique values for $K_i$ and $Lx_i$, $i = 1, \ldots, n$). In particular our optimized network exists and thus our approach can be carried out.

Note that although Gouriéroux et al. (2012) do not consider any maturity, Proposition 1 still holds true in presence of $\omega_i$. It is sufficient to replace $Lx_i^*$ by $Lx_i^*(1 + r_D(\omega_i))$ in their proof.

\[\text{Table 3: Balance sheet of institution } i \text{ at the end date } t = 1.\]
2.2 Regulatory constraints

In line with the usual Basel regulation, the solvency constraint for institution $i$ is written

$$K_i \geq k^A_i A x_i + k^\pi \sum_{j=1}^{n} \pi_{i,j} K_j + k^\gamma \sum_{j=1}^{n} \gamma_{i,j} L x_j,$$

(3)

where $k^A_i$ and $k^\pi$ are regulatory parameters (risk weight) for external assets and inter-financial shareholdings and debtholdings, verifying: $0 < k^A_i < 1$, $0 < k^\pi < 1$ and $0 < k^\gamma < 1$. The parameter relative to the external assets is specific to each institution whereas those relative to inter-financial assets are common within a specific sector. This constraint means that the equity must be higher than the risk-weighted assets and aims at ensuring the existence of a sufficient capital buffer to avoid losses for creditors in most cases.

Note that in the case of insurance companies Equation (3) corresponds to Solvency I regulatory framework (apart from the term corresponding to interconnections). Since Solvency II is not implemented so far, we choose not to consider it in our modeling. Moreover, let us emphasize that the weights of banks differ from those of insurance companies. In case of an insurer, the constraints on $k^\pi$ and $k^\gamma$ can be relaxed: $0 \leq k^\pi_i < 1$ and $0 \leq k^\gamma_i < 1$.

Even if we do not focus on liquidity shocks, we introduce a liquidity constraint:

$$A l_i \geq k^L l(\omega_i, L x^*_i),$$

(4)

with $l$ being some increasing function (with respect to both variables) which will be characterized further and $k_L$ verifying $0 < k^L < 1$. This constraint aims at ensuring a sufficient liquid assets buffer to face exposure to liquidity risk (maturity transformation in the case of banks and claims in the case of insurance companies) stylized by $\omega_i$ and $L x_i$. Note that this constraint is similar to the Basel III Liquidity Coverage Ratio.

2.3 Summary of the optimization framework

In short, both banks and insurance companies select their balance sheet items under restriction (class of assets for banks and class of risks for insurance companies) and regulatory constraints. Their business model is reflected through a size variable and an intensity variable: the size is the total credit granted for a bank and the total of individual risks covered for an insurance company, while the intensity is the degree of maturity transformation for a bank and the sinistrality for an insurance company.

Let us emphasize that our modeling allows to take the specifics of banks and insurance companies into account in a unified way. The same parameters allow interpretation in terms of banks as well of insurance companies. However, as we mentioned, the nature of the debt $L x^*_i$ and that of the maturity $\omega_i$ are different when considering a bank or an insurance company. For the sake of simplicity we will therefore mainly focus on banks.

2.4 Network Benchmark

Our testing principle is to compare the network obtained through our modeling and a stylized network corresponding to a genuine situation. In this part, we describe this
stylized network through three dimensions. First, we provide the main aggregate items of bank balance sheet. Thus we will be able to check if apart from interbank assets the obtained balance sheet composition is close to real ones. Second, we focus on the network shape. This level provides a qualitative assessment of interconnections. Last, the size of interconnections along instruments in a typical banking network is described. This last level provides a quantitative assessment of interconnections. We restrain ourselves to interbank networks in industrial countries, typically the United-States, Canada or Europe. We identify four stylized facts that characterize an interbank network.

2.4.1 Main aggregate items of bank balance sheet

We consider Bank Holding Company Performance Report Peer Group Data (BHCPR Peer Group), published by the Federal Financial Institutions Examination Council (www.ffiec.gov), that provided the structure of asset and liability sides for banks above $10 Billion (from 69 banks in 12/2008 to 90 in 12/2012). Figure 1 provides the composition of the asset side and the leverage. Corresponding informations are summarized in the following stylized fact:

**Stylized fact 1:** For a typical bank, the external assets ($Ax_i$) represent about 95% of its total assets while its equity ($K_i$) represents about 5% of its total assets.

![Figure 1: excerpt of BHCPR Peer Group N1 between 12/2008 and 12/2012. Source: www.ffiec.gov.](image)

2.4.2 Network shape

National interbank networks are usually characterized by a core-periphery structure (Craig and Von Peter, 2010). The core is composed of large banks highly interconnected. The periphery is composed of smaller banks which are connected to core banks only. Figure 2 represents a typical national interbank network. Note that at the international

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5See Furfine (2003) for USA, Wells (2002) for UK, Upper and Worms (2004) for Germany, Lublóy (2005) for Hungary, van Lelyveld and Liedorp (2006) for the Netherlands, Degryse and Nguyen (2007) for Belgium, Toivanen (2009) for Finland, Gauthier et al. (2012) for Canada, Mistrulli (2011) for Italy, Fourel et al. (2013) for France...
level, the core-periphery structure is much less clear among major banks (Alves et al., 2013). A complete structure seems more representative of the reality. These observations are summarized in the two following stylized facts:

**Stylized fact 2:** For a network composed of banks heterogeneous in size, a core-periphery structure is ideally expected. In other words, the matrices containing the $\pi_{ij}$ and the $\gamma_{ij}$, $\Pi$ and $\Gamma$ present a block structure with a majority of zeros.

**Stylized fact 3:** For a network composed of large banks homogeneous in size, a complete structure is ideally expected. In other words, $\Pi$ and $\Gamma$ have few zero coefficients.

Comment: The core is composed of banks A to C while the periphery is composed of banks D to H.

Figure 2: Core-Periphery structure. Source: Figure 1 in Craig and Von Peter (2010).

### 2.4.3 Interconnections size and support

As mentioned above, total interbank assets account for about 5% of total asset. However, data concerning the relative importance of the different instruments are scarce. At the European level (at the end of 2011), according to Table 1 in Alves et al. (2013), credit claims (direct credit from one bank to another) and debt securities represent 90% of exposures. The remainder is composed of "other assets". For the 6 largest Canadian banks (at May 2008), there is a factor 4 between exposure through traditional lending and exposure through cross-share holding as reported in Table 3 in Gauthier et al. (2012).

**Stylized fact 4:** In the case of large banks, lending exposures represent a major part of exposures (between 80% and 90%). In other words, $\Gamma L \approx \alpha (\Pi K + \Gamma L)$ with $\alpha \in [80\%; 90\%]$. However, cross-share holdings can not be neglected.\(^{6}\)

\(^{6}\)Aside the relative weight of share securities, it is paramount to take them into account since they are more risky than debt/lending: shareholders loss as soon as the financial institution has losses while a debt holders is only affected if the losses of the financial institution are above its equity. For contagion analysis, cross-share holding cannot be neglected.
3 Model, theoretical properties and network shape

We model the network formation in two steps. The first one -dealt with in this section- concerns the modeling of the behavior of one institution, the state of the others being given. The aim is to determine how a financial institution defines its balance sheet and especially the interconnections knowing the main balance sheet elements of the other ones. For instance, how does a new bank get interconnected to previously existing ones? Or how does a bank adapt its balance sheet to modifications of the structure of others? The second step concerns the whole network formation using the modeling of individual behaviors and will be considered in Section 4.

Based on the framework introduced in the previous section, a one-period model is built. Financial institutions are risk-averse agents optimizing their balance sheet structure for the shareholder’s interest at the initial date $t = 0$ (represented by an upper-scripted index in parenthesis). The horizon is the final date $t = 1$.

The assumption that interconnections represent a long-term choice is a cornerstone of our analysis. Interconnections are not motivated by any liquidity features: they correspond to optimal choices on the long-run. Including liquidity-motivated interconnections that stem from daily work of Asset Liability Managers, as well as the interactions between short-term and long-term interconnections, is currently under research.

A very important concern is the reflective problem: how to technically manage the fact that the choices of financial institutions are interdependent? The main issue is that a complete Nash equilibrium modeling of the whole balance sheet structure -interconnections, external assets and debt- is clearly wishful thinking. It triggers difficulties with respect to privately available information, anticipation formation... Note that in models with Nash equilibrium such as in Babus (2013) or Acemoglu et al. (2013), choices are only interconnections: all the others components of the balance sheet are exogenous. This scope is arguably adapted in a short-term framework but is clearly unsuitable in a long-term perspective. In order to circumvent a complete game theory model we adopt simplifying but backed up assumptions.

3.1 Modeling strategy

We choose an efficient, albeit simple strategy: each financial institution is assuming that the asset side of the other financial institutions is only composed of their external assets. This implies that the institution optimizing its balance sheet is not taking into account the future reactions of the other financial institutions. In this perspective, the optimization program is not strategic: the institution plays fairly. Apart from simplifying the optimization program, this strategy corresponds to sound assumptions for each financial institution for several reasons.

First, the information set used in the optimization program is very close to the genuinely available one. Actually, bilateral exposures are private information. Publicly available information for any major financial institution is detailed income statement and balance sheet: return-on-asset, return-on-equity, cash, total interbank assets, loans on the asset side, debt and equity on the liability side... are easily extracted from public financial communication of firms or published reports (See Appendix A for an excerpt of the Consolidated Financial Statements for BHCs of Bank of America publicly published by the Federal Financial Institutions Examination Council.) Secondly, note that a large
part of debt securities and shares are traded on the secondary market. Therefore institution $i$ cannot know exactly who its creditors and shareholders are: institution $i$ knows its asset side but not the repartition of its liability side. The part of tradable shares is called the floating equity. By analogy, we called the floating debt the part of the debt traded on the secondary market.

Lastly, the absence of anticipation of reaction constitutes an approximation. As previously mentioned, there is no information on bilateral exposures. However, the total interbank assets represent about 5% or 10%. Each bilateral exposure should be much smaller: 0.5% of total assets seems a reasonable upper bound. Therefore when a new financial institution gets interconnected, the new interconnections do not significantly modify its balance sheet. It may trigger a reaction from its own counterparts but the effects can be neglected by comparison to the risk borne in the external assets for instance. As we will see in the simulation results, the reaction of counterparts only has a light influence on each institution, leading to a rapid stabilization of the network. This provides an indication that this assumption of absence of anticipation can be suited.

These assumptions -on the information set and on the horizon of optimization- allow us to derive some strong and tractable theoretical results. This is done immediately hereafter.

3.2 Optimization program

Financial institution $i$ is managed for the benefits of its investors (i.e. shareholders) who are risk-averse and endowed with an initial capital $K^{(0)}_i$. The risk-aversion of the investors of financial institution $i$ is represented by a utility function $u_i$ (with usual properties). We denote $1 - c^u_i$ (respectively $1 - c^d_i$) the floating equity (resp. debt) of institution $j$.

In line with our modeling strategy, we re-scale the total assets of institution $j$ by $\kappa_j = \frac{Lx^{(0)}_j + K^{(0)}_j}{Ax^{(0)}_j + A\ell^{(0)}_j}$. These scaling factors compensate for the lack of information and the absence of reactions. Thus, we get

$$K^{(1)}_i \equiv \left[ Ax^{(1)}_i + A\ell^{(1)}_i + \sum_{j=1}^n \pi_{i,j} \left( \kappa_j (Ax^{(1)}_j + A\ell^{(1)}_j) - Lx^{*(1)}_j \right) \right]^+$$

$$+ \sum_{j=1}^n \gamma_{i,j} \min \left( \kappa_j (Ax^{(1)}_j + A\ell^{(1)}_j); Lx^{*(1)}_j \right) - \left[ 1 + r_D(\omega_i) \right] Lx^{(0)}_i \right]^+.$$  \(\text{(5)}\)

\[For instance, at June 30, 2013, the proportion of interbank assets in the total asset is 3.4% for Bank of America, 13% for JPM, 8.40% for Citigroup 8.3% for Wells Fargo... according to Consolidated Financial Statements for BHCs.\]
Finally, the optimization program $P_i$ of financial institution $i$ is

$$
\mathcal{P}_i = \begin{cases}
\max & \mathbb{E}_0[u_i(K_i^{(1)})] \\
Ax_i^{(0)}, A\ell_i^{(0)} & \\
Lx_i^{(0)}, \omega_i & \\
\pi_{i,1}, \ldots, \pi_{i,n} & \\
\gamma_{i,1}, \ldots, \gamma_{i,n} & \\
\end{cases}
$$

subject to

$$Ax_i^{(0)} + A\ell_i^{(0)} + \sum_{j=1}^n \pi_{i,j} K_j^{(0)} + \sum_{j=1}^n \gamma_{i,j} Lx_j^{(0)} = K_i^{(0)} + Lx_i^{(0)} \quad (NOD)$$

$$K_i^{(0)} \geq k_i^A \ Ax_i^{(0)} + k_i^\pi \sum_{j=1}^n \pi_{i,j} K_j^{(0)} + k_i^\gamma \sum_{j=1}^n \gamma_{i,j} Lx_j^{(0)} \quad (RSC)$$

$$A\ell_i^{(0)} \geq k_L \ l(\omega_i, Lx_i^{(0)}) \quad (RLC)$$

$$Ax_i^{(0)} \geq 0, A\ell_i^{(0)} \geq 0, Lx_i^{(0)} \geq 0$$

$$\omega_i \in [0; 1]$$

$$\forall j \in [1; n], 0 \leq \pi_{i,j} \leq 1 - c_i^\pi, 0 \leq \gamma_{i,j} \leq 1 - c_i^\gamma.$$  \hspace{1cm} (6)

Constraint $(NOD)$ ensures the balance sheet equilibrium at the initial date. Note that this constraint allows the network resulting from our formation process (see Section 4) to verify Equation (1) for each institution. Inequalities $(RSC)$ and $(RLC)$ are respectively the regulatory solvency and liquidity constraints presented in Section 2.2.

### 3.3 Solution analysis

We define the position of a financial institution as the difference between its total assets (denoted by $A_i^{(1)}$) and its nominal debt: $P_i^{(1)} = A_i^{(1)} - Lx_i^{(1)}$. If this difference is positive, the position is simply the equity; if the difference is negative, the position is the loss for creditors (while the equity is equal to zero in this situation). $P$ can be interpreted as the profit-and-loss.

The uniqueness of the solution usually requires the strict concavity of the objective function. The concavity of function $u_i(K_i^{(1)})$ is not a necessary condition since we could expect that the integration operation makes the expectation strictly concave even if $u_i(K_i^{(1)})$ is not strictly concave everywhere (see Appendix for more details). Moreover, it would impose conditions on the joint distribution of the returns of the specific asset. Thus we look for conditions on $u_i(K_i^{(1)})$. Due to their limited liability, shareholders aim at maximizing the expected utility of the equity. The latter is defined as $K_i^{(1)} = \max(P_i^{(1)}; 0)$, making function $u_i \circ K_i^{(1)}$ non-differentiable and introducing a level shape. An unfortunate consequence is that for standard utility functions $u_i$, $u_i(K_i^{(1)})$ is not strictly concave and not even concave. Then our strategy is to approximate the real equity by a function $v(P_i^{(1)})$ to obtain the concavity. From an economic perspective, it is satisfactory to consider a transformation of the equity, as we will see in the following.

Therefore, we decompose the analysis of optimization program $\mathcal{P}_i$ into two steps. First we show that under mild assumptions there exists a solution (Proposition 1). Second we transform the optimization program $\mathcal{P}_i$ into a close one ($\mathcal{P}_i'$) for which existence and uniqueness are ensured (Proposition 2). Corollaries 1 and 2 provide examples of specifications compatible with the assumptions ensuring the existence and uniqueness.
3.3.1 Analysis of the exact optimization program

Contrary to usual optimization programs where the total wealth is exogenous, increasing wealth by issuing debt is allowed in optimization program $P_i$. Therefore, intuitively, the main difficulty in showing the existence is to show that the financial institution $i$ has no gain in issuing an infinite amount of debt. The argument is as follows. The equity is exogenously fixed. Therefore the regulatory solvency constraint ($RSC$) implies that the total value of risky assets is bounded. Thus, starting to a certain point, the funding obtained by issuing more debt is necessarily invested in the risk free liquid asset. But since the interest rate charged on the debt is higher than the risk free rate, it is not profitable to issue debt to invest in liquid assets. In other words, financial institutions are expected to invest in risky assets: granting credit is the core activity of banks.

All this goes to state the following proposition:

**Proposition 1** (Existence of solvency optimization program solution). *Considering class of shareholders $i$ endowed with a capital $K_i^{(0)}$, the optimal structure of the institution they found is given by solving $P_i$. If*

- (A1) The investors neglect interconnections among their counterparts;
- (A2) The utility function $u_i$ is continuous and strictly increasing;
- (A3) The joint c.d.f. of the $r_i$, denoted by $F_S$, is continuous. Moreover, the density $f_S$ is strictly positive on $[a; +\infty)^n$, where $a$ is a constant belonging to $\mathbb{R}$;
- (A4) The yield curve, $r_D$, is continuous and strictly higher than the risk free rate;

*there exists a solution of the solvency optimization program defined by (6).*

Assumption (A1) is both a technical assumption and a way to reflect the restricted information available for each agent. Assumptions (A2) and (A3) are very common in the literature and not restrictive. Assumption (A4) means that all institutions have access to the same risky debt market where only maturity matters. With this assumption, our analysis is restricted to institutions with overall similar risk profile.

3.3.2 Analysis of the approximated optimization program

As underlined before, it appears impossible to establish the uniqueness for $P_i$ except in particular cases of simple c.d.f. $F_S$ of shocks. We therefore consider an optimization problem $P'_i$ where the sole difference with $P_i$ is that the objective function is the expected utility of a transformation (denoted by $v$) of the position of financial institution $i$, $F_i^{(1)}$. Considering the position directly makes things easier. However, it means not taking into account the limited liability which has some important implications. Indeed, it plays the role of an insurance against extreme events for the managers. Therefore they are responsible for regular shocks but not for really extreme ones. Some phenomenon cannot be explained by macro-economic model ignoring this characteristic. The optimization
program $\mathcal{P}'_i$ is

$$
\mathcal{P}'_i = \left\{ \begin{array} {l}
\max \mathbb{E}_0 \left\{ u_i \left[ v(P_i) \right] \right\} \\
A x_i^{(0)}, A \ell_i^{(0)} \\
L x_i^{(0)}, \omega_i \\
\pi_{i1}, \ldots, \pi_{in} \\
\gamma_{i1}, \ldots, \gamma_{in}
\end{array} \right. \\
\text{s.t.} \quad A x_i^{(0)} + \sum_{j=1}^n \pi_{i,j} K_j^{(0)} + \sum_{j=1}^n \gamma_{i,j} L x_j^{(0)} = K_i^{(0)} + L x_i^{(0)} \quad \text{(NOD)} \\
K_i^{(0)} \geq k^A x_i^{(0)} + k^\pi \sum_{j=1}^n \pi_{i,j} K_j^{(0)} + k^\gamma \sum_{j=1}^n \gamma_{i,j} L x_j^{(0)} \\
A \ell_i^{(0)} \geq k^L l(\omega_i, L x_i^{(0)}) \quad \text{(RLC)} \\
A x_i^{(0)} \geq 0, A \ell_i^{(0)} \geq 0, L x_i^{(0)} \geq 0 \\
\omega_i \in [0; 1] \\
\forall j \in [1; n], 0 \leq \pi_{i,j} \leq 1 - c^\pi_j, 0 \leq \gamma_{i,j} \leq 1 - c^\gamma_j
\right. 
$$

(7)

With this specification, the level aspect of the limited liability is dodged and transformation $v$ ensures flexibility. For instance, with $v = \text{Id}$, one considers the usual maximization of the expected utility of profits. Alternatively, $v$ can be chosen to closely fit the design of the limited liability of shareholders while relaxing their complete indifference for loss amplitude. In this last case, optimization program $\mathcal{P}'_i$ is very close to optimization program $\mathcal{P}_i$.

In short, the argument for the existence of a solution of $\mathcal{P}'_i$ is similar to the argument for the existence of a solution of $\mathcal{P}_i$. The uniqueness mainly stems from the strict concavity of the objective function we obtain by adjusting $v$. However the strict convexity of the constraints is necessary, imposing restrictions on the function form of the regulatory liquidity constraint (RLC) (see proof for details). The following Proposition provides the result concerning uniqueness:

**Proposition 2** (Existence and uniqueness of solvency optimization program solution). Considering class of shareholders $i$ endowed with a capital $K_i^{(0)}$, the optimal structure of the institution they found is unique and given by solving $\mathcal{P}'_i$.

Under Assumptions (A1), (A2), (A3), (A4) and Assumptions:

- (A5) The composition of the transformation function $v$ and the utility function $u_i$ is strictly concave: $\forall P, (u_i \circ v)^\prime(P) < 0$;
- (A6) The interest rate on debt is strictly concave: $\forall \omega_i \in [0, 1], r^\prime_D(\omega_i) < 0$;
- (A7) The interest rate on debt verifies $\forall \omega_i \in [0, 1], r''_D(\omega_i) \neq 0$;
- (A8) The function $l$ in the constraint (RLC) verifies

$$
\frac{\partial^2 l}{\partial \omega_i^2} \geq 0 \quad \text{and} \quad \frac{\partial^2 l}{\partial \omega_i^2} \times \frac{\partial^2 l}{\partial L x_i^{(0)} \partial L x_i^{(0)}} \geq \left( \frac{\partial^2 l}{\partial \omega \partial L x_i^{(0)}} \right)^2;
$$

there exists a unique solution of the solvency optimization program defined by (7) in the following sense. If all control variables appearing on the asset side of institution $i$ are fixed
apart from one variable, denoted by $Ac_i^{(0)}$, then there is unicity of the triplet $(Ac_i^{(0)}, Lx_i^{(0)}, \omega_i)$.

Note that the result of Proposition 2 is equivalent to saying that the main balance sheet items are unique. Indeed, the value of total assets $A_i^{(0)}$, the degree of maturity transformation $\omega_i$ and the debt $Lx_i^{(0)}$ are unique. Due to the high number of control variables in the asset side and the complexity of the problem, it seems impossible to demonstrate the uniqueness of all control variables. Indeed it is impossible to carry out any computation when considering the expectation operator (see Appendix). But it is precisely this expectation operator that can make the solution unique for all control variables. The unicity for all control variables will be verified on simulations.

3.3.3 Approximation properties

As mentioned before, the transformation function $v$ gives room to flexibility. Corollary 1 provides two specifications satisfying assumptions of Proposition 2, corresponding respectively to the position and a very good approximation of the equity.

**Corollary 1** (Some specifications of functions $v$ and $u$).

- **i)** If $v(P) = P$, Assumption (A5) reduces to $u'' < 0$.
- **ii)** If $v(P) = \ln(\exp(P) + 1)$, then Assumption (A5) is verified for the utility function $u = \ln$.

The approximation corresponding to function $v(P) = \ln(\exp(P) + 1)$ is shown on Figure 3. As we can see, the approximation error is very low. In the perspective of maximizing the utility, this function is probably even more satisfactory than the real equity. Indeed the utility of the equity is equal to zero whatever the position if the position is negative. In reality, one may think that the institution’s managers prefer a light insolvency situation to a large one, for example for the sake of reputation. It will be difficult to find funding to build a new project after letting an institution in a state of large insolvency. Our approximation function is strictly increasing and therefore takes this aspect into account. This is especially true for position values not too far away from the insolvency point.

![Figure 3: Approximation of the equity: ii) of Corollary 1](image)
Corollaries 2 and 3 provide a specification for the interest rate curve \( r_D \) and the function \( l \) (appearing in the regulatory liquidity constraint), respectively satisfying Assumptions (A6) and (A8).

**Corollary 2** (Specification of function \( l \)). \( l(\omega, Lx) = \exp(\omega) \cdot \exp(Lx) \) satisfies Assumption (A8).

**Corollary 3** (Specification of function \( r_D \)). An interest rate curve of the form:
\[
r_D(\omega) = \alpha - \beta \exp(\omega) \quad \text{for } \omega \in [0, 1],
\]
(8) satisfies Assumption (A6) in Proposition 3.

### 3.3.4 Choice

Previous theoretical results provide different suitable specifications (especially of function \( v \)) leading to a unique solution of the optimization program. In order to clarify the presentation, let us make a clear recommendation of choice. The following result is directly derived from Proposition 2 and Corollaries 1, 2 and 3.

**Proposition 3** (Existence and uniqueness of a specific solvency optimization program solution).

Additionally to Assumptions (A1) to (A4), let us consider:

- a logarithmic utility function
  \[
  u(x) = \ln(x);
  \]
- the following approximation of the limited liability of shareholders:
  \[
  v(P) = \exp(\ln(P) + 1);
  \]
- an exponential liquidity constraint:
  \[
  l(\omega, Lx) = \exp(\omega) \times \exp(Lx);
  \]
- the following interest rate curve:
  \[
  r_D(\omega) = \alpha - \beta \exp(\omega) \quad \text{for } \omega \in [0, 1].
  \]

Then, the associated optimization program \( \mathcal{P}' \) has a unique solution.

As mentioned above, all parameters or values used in our modeling can be calibrated on publicly available data. We can use Proposition 3 to compute the optimal choices of a new incoming financial institution in a preexisting network.

### 3.4 Optimal interconnections

Previous theoretical results ensure that the financial institution’ maximization program has a (unique) solution. However, we did not characterize the solution, in particular the interconnections. In this part, we show that it is optimal for a financial institution to get interconnected. In this section, in order to simplify the presentation and to explain the main features, we do not take into account the control variables \( A\ell_i, \omega_i \), as well as the liquidity constraint \( (RLC) \).

In order to start, let us consider a simplified case of a portfolio composed of a quantity \( Ax \) and a quantity \( \pi \) of assets having respectively random variables \( r \) and \( \pi^r \) as gross returns, under a solvency constraint\(^8\). The penalization weights are respectively \( k^A \) and \( k^\pi \).

\(^8\)For simplicity, the product \( \pi K \) of the complete program has been simplified into \( \pi \).
The corresponding optimization program is

\[ P_{RA} \equiv \begin{cases} \max_{Ax, \pi} & E[u(Ax r + \pi r^\pi)] \\ s.t. & k^A Ax + k^\pi \pi \leq 1 \\ & Ax \geq 0 \\ & 0 \leq \pi \leq 1 \end{cases} \]  

(9)

The Theorem by Karush, Kuhn and Tucker (KKT) allows to derive the following proposition:

**Proposition 4.** For the sake of simplicity, let us first denote \( f = E[u(Ax r_1 + \pi r_2)] \). Under the condition

\[ \frac{\partial f}{\partial Ax} k^A < \frac{\partial f}{\partial \pi} k^\pi, \]

the optimal \( \pi^* \) is necessarily different from 0.

This shows that under the condition that the derivative of the expected utility with respect to \( \pi \) (reported to its corresponding weight) is higher than the one with respect to \( Ax \), the optimal \( \pi^* \) will be strictly positive. Proposition 4 does not provide the solution but gives an indication that interconnections can be strictly positive under some conditions. This result can be generalized to a higher number of assets. Note that this illustrative program does not contain any equality constraint. However, such a constraint can be trimmed by replacing one control variable in function of the others. That reduces the problem’s dimension. This point will be further detailed in the following.

Due to the high complexity of our optimization problem (high dimension and high number of constraints), the KKT conditions are too numerous and therefore it seems impossible to derive the solution in a closed form. Therefore we decompose the study and the interpretations in different steps. We first consider a risk-neutral agent maximizing the value of its portfolio without limited liability. Second we consider the case of risk-averse agents and finally limited liability is taken into account.

**Risk-neutral agent without limited liability:**

In this case, it is sufficient to consider that \( r \) is deterministic. We then can consider the program

\[ \mathcal{P}_{RN} \equiv \begin{cases} \max_{Ax, \pi} & Ax r + \pi K^{(1)} \\ s.t. & k^A Ax + k^\pi \pi K^{(0)} \leq 1 \\ & Ax \geq 0 \\ & \pi \geq 0 \end{cases} \]  

(10)

where \( K^{(1)} \) indicates the equity value (book value) of another institution at time \( t = 1 \) and \( K^{(0)} \) the equity value (market value) of this institution at time \( t = 0 \). Similarly to \( r \), it is sufficient to consider that both \( K \) and \( K \) are deterministic.

By using the same type of argument as in Proposition 4, it is easy to show that if \( r > 0 \) or \( \frac{K^{(1)}}{K^{(0)}} > 0 \), then

- if \( \frac{r}{k^A} > \frac{K^{(1)}}{k^\pi K^{(0)}} \), the solution is: \( (Ax^* = \frac{1}{k^A}, \pi^* = 0) \).
• if \( \frac{r}{k^A} < \frac{K^{(1)}}{k^\pi K^{(0)}} \), the solution is: \( (Ax^* = 0, \pi^* = \frac{1}{k^\pi K^{(0)}}) \);

• if \( \frac{r}{k^A} = \frac{K^{(1)}}{k^\pi K^{(0)}} \), the solution is not unique.

Therefore, due to the solvency constraint, a risk-neutral agent only invests in the asset having the highest return \( \frac{K^{(1)}}{K^{(0)}} \) with respect to its specific penalty coefficient in the solvability constraint.

Let us now consider the case where a limit in the availability is introduced: the constraint \( \pi \geq 0 \) is replaced by \( 0 \leq \pi \leq c \). In this case, if \( \frac{r}{k^A} < \frac{K^{(1)}}{k^\pi K^{(0)}} \), \( \pi^* = \min \left( c; \frac{1}{k^\pi K^{(0)}} \right) \). Therefore, if \( c < \frac{1}{k^\pi K^{(0)}} \), investing all in \( K^{(0)} \) does not bind the solvency constraint. In this case (and if \( r > 0 \)), an investment in \( Ax \) completes the portfolio. This result can be easily generalized to the case of \( n \) institutions and where it is possible to invest in the debt \( Lx_j \) of other institutions. This is done in Proposition 5.

**Proposition 5.** Let us consider the following optimization program:

\[
\mathcal{P}_{RG} = \left\{ \begin{array}{l}
\max \ Ax_i \ r_i + \sum_{j=1}^{n} \pi_{ij} K_j^{(1)} + \sum_{j=1}^{n} \gamma_{ij} Lx_j^{(1)} \\
\text{s.t.} \ k^A Ax_i + k^\pi \sum_{j=1}^{n} \pi_{ij} K_j^{(0)} + k^\gamma \sum_{j=1}^{n} \gamma_{ij} Lx_j^{(0)} \leq 1 \\
0 \leq \pi_{ij} \leq c^\pi \\
0 \leq \gamma_{ij} \leq c^\gamma
\end{array} \right. 
\]

To find this problem’s solution, let us sort in decreasing order the following returns (relative to their penalty weight): \( \frac{r_i}{k^A} \), \( \frac{K_j^{(1)}}{k^\pi K_j^{(0)}} \) \( (\forall j \in [1, n]) \), \( \frac{Lx_j^{(1)}}{k^\gamma Lx_j^{(0)}} \) \( (\forall j \in [1, n]) \). The optimal solution consists in investing as much as possible in the asset having the highest return with respect to its penalty. When this asset is not available anymore, it is better to invest as much as possible in the second one, and so on... This is repeated until the solvency constrained is binding.

**Risk-averse agent without limited liability**
A risk-averse agent aims at decreasing the variance of its portfolio. To this purpose, it is necessary to diversify. Therefore, in this case, we can expect an investment in different assets, contrary to the "binary" investment described previously. This is confirmed by numerical experiments.

**Agent with limited liability**
In the previous considerations, we did not take into account the limited liability as well as the very different natures of equity and debt. Therefore we could not see the implications of the fact that the \( \pi_{ij} \) and the \( \gamma_{ij} \) are related to very different instruments. To pinpoint these implications, let us consider a stylized set-up with two financial institutions. One can identify four situations in which institution 1 (or 2) is either alive or in default. Table 4 reports these 4 states. Let us focus on the impact of limited liability for institution 1.
The expected utility of institution 1 is written as follows

$$EU_1 = P(e_{11}) \cdot PO(e_{11}) + P(e_{12}) \cdot PO(e_{12}) + P(e_{21}) \cdot PO(e_{21}) + P(e_{22}) \cdot PO(e_{22}),$$

where $P(e)$ is the probability of being in state $e$ and $PO(e)$ the associated payoff for institution 1. Due to limited liability, $PO(e_{11}) = PO(e_{12}) = 0$. Thus

$$EU_1 = P(e_{21}) \cdot PO(e_{21}) + P(e_{22}) \cdot PO(e_{22}).$$

In the state $e_{21}$, institution 2 defaults, meaning that its equity is equal to zero. It is therefore more interesting to invest in its debt. In the state $e_{22}$, institution 2 is alive. So if the equity of institution 2 has a higher return than its debt (after taking into account the regulatory penalization), institution 1 prefers investing in the share securities of institution 2, thus increasing the $\pi_{12}$. If the correlation $\rho$ between the external assets of both institutions is highly positive, both banks are likely to be alive and to default simultaneously. That means that $P(e_{21})$ is very low, giving: $EU_1 \approx P(e_{22}) \cdot PO(e_{22})$. In this situation, institution 1 prefers investing in share securities. On the contrary, if the correlation $\rho$ between the external assets of both institutions is highly negative, institution 2 is likely to default when institution 1 is alive. In this case $EU_1 \approx P(e_{21}) \cdot PO(e_{21})$ and institution 1 prefers investing in debt securities.

It is important to understand that the asymmetry between the cases $\rho > 0$ and $\rho < 0$ is due to the limited liability feature. Indeed, let us assume that bank 1 has no limited liability and thus is not indifferent to losses. If $\rho$ highly positive, $EU_1 \approx P(e_{11}) \cdot PO(e_{11}) + P(e_{22}) \cdot PO(e_{22})$. In state $e_{11}$, 2 defaults and it is better to invest in its debt whereas in state $e_{12}$, it is better to invest in its shares. Therefore, it can be appropriate to invest in both instruments and thus the asymmetry disappears. The same happens for a highly negative $\rho$.

### 3.5 Cost of funding

In the considerations of Section 3.4, we assumed that the agent owns a sufficient amount of wealth to invest until the solvency constrained is binding. However, the capital $K^0_i$ is very low compared to the total assets to invest (due to the regulatory weights values). Thus, once the total capital has been used, the institution must raise debt in order to continue to invest. Returns of shares and debt securities must be netted by the cost of funding. To make the investment attractive (in terms of net returns), the cost of raising debt should be lower than the returns of shares and debt securities.

Let us now state some results about the returns of investments in shares and debt securities of other institutions, compared to their funding cost. For the sake of simplicity of the interpretation, before stating the result for general functions $u$ and $v$, we propose a result in the case where $u$ and $v$ are the identity functions. It corresponds to the case of a risk-neutral institution maximizing its position $P^{(1)}_i$.

**Proposition 6** (Returns against opportunity cost).

*Case of a risk-neutral institution maximizing the expectation of its position*
• The expected return of share issued by \( j \) is larger than its opportunity cost if and only if
\[
\int_{-\infty}^{+\infty} \left( a_j + b_j r_j \right) f_{S_j}(r_j) \, dr_j > \left[ 1 + r_D(\omega_i) \right] K_j^{(0)},
\]
where \( a_j = \kappa_j A x_j^{(0)}, \ k_j = \kappa_j \left( A r_j^{(0)} + A l_j^{(0)} (1 + r_j) \right) - L x_j^{(0)} \left[ 1 + r_D(\omega_i) \right] \) and \( f_{S_j} \) being the marginal density of the return of institution \( j \).

• The expected return on debt issued by institution \( j \) is higher than its opportunity cost if
\[
\int_{-\infty}^{+\infty} \left( a_j r_j + b_j \right) f_{S_j}(r_j) \, dr_j + \mathcal{L} x_j^{(0)} \left[ c_j \left( 1 + r_D(\omega_i) \right) \right] > \mathcal{L} x_j^{(0)} \left[ 1 + r_D(\omega_i) \right],
\]
where \( c_j = P \left( r_j > \frac{L x_j^{(1)} - b_j}{a_j} \right) \).

**General case of an institution maximizing the expectation of the utility of its equity** In this case, Assumption 12 is replaced by
\[
\int_{-\infty}^{+\infty} \left( a_j r_j + b_j \right) w(r_j) \, dr_j > \left[ 1 + r_D(\omega_i) \right] K_j^{(0)} \int_{-\infty}^{+\infty} w(r_j) \, dr_j,
\]
by denoting
\[
w(r_j) = \int_{r_1} \ldots \int_{r_{j-1}} \int_{r_{j+1}} \ldots \int_{r_n} h_{11}(r_1, \ldots, r_{j-1}, \ldots, r_n) \, f(r_1, \ldots, r_n) \, dr_n \ldots dr_{j-1} \, dr_j \ldots dr_1,
\]
where
\[
h_{11}(r_1, \ldots, r_{j-1}, \ldots, r_n) = \frac{\partial (u \circ v)}{\partial P^{(1)}_i}.
\]
The same kind of expanding happens in the case of the debt.

As developed in the proof, in the case of a risk-neutral institution maximizing its position, the differentiation provides
\[
\frac{\partial E(P^{(1)}_i)}{\partial \pi_{ij}} = E \left[ K_j^{(1)} - \left[ 1 + r_D(\omega_i) \right] K_j^{(0)} \right] = E \left[ K_j^{(1)} \right] - \left[ 1 + r_D(\omega_i) \right] K_j^{(0)}.
\]
Equation (12) corresponds to the fact that \( E \left[ K_j^{(1)} \right] - \left[ 1 + r_D(\omega_i) \right] K_j^{(0)} > 0 \). It can be beneficial for institution \( i \) to increase its participation in \( j \) if the return on equity of \( j \) is higher than the interest rate that \( i \) must paid for its debt. Indeed, for fixed \( K_i^{(0)} \) as well as fixed \( A x_i^{(0)} \) and \( A l_i^{(0)} \), in order to increase its participation \( \pi_{i,j} \), institution \( i \) must raise debt.

In the general case, the same type of equation is obtained. However it takes the marginal \( u \circ v \) into account in \( w(r_j) \). For interpretation purpose, let us assume that \( v = Id \). For a given value of \( r_j \), the algebraic gain in increasing the participation \( \pi_{i,j} \) must be weighted by the marginal utility, which depends on the returns of all institutions. Integrating this marginal utility with respect to all returns \( r_1, \ldots, r_n \) apart from \( r_j \) yields the term \( w(r_j) \). The risk aversion of institution \( i \) is embedded in the term \( w(r_j) \). The same type of argument applies in the case of the debt.
Testing of the diversification motive: the network shape

Let us now compare the consequences of Proposition 6 and Stylized Facts 2 and 3 about the network shape, and discuss the impact of risk-aversion and limited liability features.

In the case of risk-neutral agents with unlimited liability, an institution gets interconnected to others by strict mechanical behaviors: it seeks sequentially for highest returns until binding the solvency constraint. Consequently, the network shape is very structured and directive since everyone gets interconnected in the same direction. Thus, in such a case, there is usually no general shape\(^9\). In other words, with risk-neutral agent and unlimited liability, diversification motive cannot provide interesting results.

In the case of risk-averse agents, the interconnections tend to shape a complete network. Institutions carry out a diversification to decrease the variance, in addition to their aim of obtaining higher returns. Note that even if all institutions have similar returns, a diversified portfolio has a lower variance than a concentrated one\(^{10}\). To significantly benefit from the diversification, the variance reduction must be high enough: situations where specific assets are not almost non-risky and/or when correlation is negative are prone to show complete network structure. These findings will be confirmed numerically in the next section. The limited liability feature modifies the balance between share securities and debt securities. When the correlation \(\rho\) is positive (respectively negative), share securities (respectively debt securities) will be dominant. When considering risk-averse agents, diversification motive generates complete financial networks which are usually observed among major institutions. Therefore, we cannot rule out diversification as explaining interconnections between key financial players\(^{11}\).

Network formation and simulation results

In this section, we derive simulations results in order to assess the diversification motive for the financial network formation. First, we present the specification we use and our calibration strategy. Second, we develop a network formation process taking advantage of the strong and tractable theoretical results obtained in the previous section. Then optimal choices for one financial institution and concerning the whole network are analyzed.

Specifications

For the sake of simplicity, two financial institutions are considered, \(n = 2\). Each institution is endowed with a capital amount of 1, \(K_1^{(0)} = K_2^{(0)} = 1\). Both institutions have \(x \mapsto \ln(x)\) as utility function. An initial capital of 1 implies that the equity value \(K^{(1)}\) at the optimization horizon is about 1. Therefore the objective function is close to be linear on the most likely area, meaning that financial institutions are only slightly risk-averse.

In order to properly understand the solvency feature of our model, we exclude \(A\ell\) and \(\omega\) from control variables. The interest rates paid by the two financial institutions, denoted by \(r_{D,1}\) and \(r_{D,2}\) are therefore fixed. To avoid drift effects, the risk-free interest rate is set to zero: \(r_{rf} = 0\).

\(^{9}\)Nevertheless, with particular set of returns, a star network occurs.

\(^{10}\)If \(X\) and \(Y\) are two random variables with mean \(\mu\), variance \(\sigma^2\) and correlation \(\rho < 1\), then: \(E(X + Y) = 2\mu = E(2X)\) whereas \(\text{Var}(X + Y) = 2(1 + \rho)\sigma^2 < 4\sigma^2 = \text{Var}(2X)\).

\(^{11}\)Note that our approach has no clue on the relevance of other motives (horizontal integration...). We simply show that diversification provides consistent results with empirical observations.
Finally, note that the expectations are computed using Monte-Carlo techniques; 100,000 simulation draws ensures a good precision.

4.2 Calibration strategy

The gross returns on external assets follow a bivariate log-normal distribution:

\[
\begin{pmatrix}
\ln \left( \frac{Ax_1^{(1)}}{Ax_1^{(0)}} \right) \\
\ln \left( \frac{Ax_2^{(1)}}{Ax_2^{(0)}} \right)
\end{pmatrix} 
\sim \mathcal{N}\left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}; \begin{pmatrix} \sigma_1^2 & \rho_{1,2}\sigma_1\sigma_2 \\ \rho_{1,2}\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right).
\] (17)

In order to calibrate the mean parameter, we consider the income statement in Consolidated Financial Statements for BHCs (FR Y-9C) for banks over USD 10B. Between 12/31/2010 and 12/31/2012, the (annual) net income varies from 0.51% to 0.71% of assets. We round this value, considering that on average the net income of our banks is equal to 1%. Over the same period, the interest expenses represent between 0.74% and 1.07% of earning asset. We basically consider that the cost of debt (\(r_{D,1}\) and \(r_{D,2}\)) varies between 0% and 1%, considering that the risk-free rate is set to zero. Finally, the expected return of the external assets for bank \(i\) is equal to 1% + \(r_{D,i}\ell_i\), where \(\ell_i\) is the bank \(i\)’s ratio of debt over total assets. For the variance parameter, a probability of default of 0.1% is in line with the current rating of major banks. We combine the informations relative to the net income and the probability of default to compute parameters \(\mu_1, \mu_2, \sigma_1\) and \(\sigma_2\) (see Appendix E for details). Parameter \(\rho\) lies between -0.9 to 0.9. A negative \(\rho\) can be interpreted as a sign of competition between the two banks or as the fact that banks operate in different markets (or geographical areas). Meanwhile, a positive \(\rho\) could be interpreted as an underlying common factor affecting both banks.

We consider the Basel 2 regulation. This regulation does not provide a unique set of values for the risk weights \(k^A_i, k^\pi\) and \(k^\gamma\). If the external assets correspond to a retail activity, i.e. loans to households, the required capital is 6% of the total exposure. If the external assets correspond to loans to unrated firms, i.e. small firms, the required capital is 8% of the total exposure. For quoted shares, the required capital is 23.2% of the total exposure. For debt securities issued by banks, the required capital can be 1.6% (when AAA or AA rated) or 4% (when A rated). Lastly, as discussed in Repullo and Suarez (2013), there is a factor between the regulatory capital and the (accounting) equity, that varies from 1 to 2. For simplicity, we consider that the regulatory capital is either equal to the equity or to a half of the equity. Bottom line, we have 8 possible sets of risk weights.

4.3 Discussion about the pricing of shares and debt securities

Recall that

\[
P_1^{(1)} = Ax_1(1 + r_1) + \pi_{12}[Ax_2(1 + r_2) - Lx_2^*(1 + r_{D2})] + \gamma_{12}Lx_2^*(1 + r_{D2}) \\
- \left( Ax_1 + \pi_{12}K_2^{(0)} + \gamma_{12}Lx_2^{(0)} \right)(1 + r_{D1}) \\
= Ax_1(r_1 - r_{D1}) + \pi_{12} \left[ Ax_2(1 + r_2) - Lx_2^*(1 + r_{D2}) - K_2^{(0)}(1 + r_{D1}) \right] \\
+ \gamma_{12} \left[ Lx_2^*(1 + r_{D2}) - Lx_2^{(0)}(1 + r_{D1}) \right].
\] (18)
Omitting the indices, Equation (18), $K$ and $Lx$ are respectively the market values of the share securities and the debt securities. In a complete market and with the usual assumptions, of course the price of an asset would be the discounted expected payoff under the risk-neutral probability:

$$K^{(0)}_2 = \frac{E_{RN}[K^{(1)}_2|\mathcal{F}_0]}{1 + r_{rf}},$$

where $\mathcal{F}_0$ denotes the available information at time $t = 0$. Since $K^{(t)}_2 = \max \left[ \kappa_2 Ax_2 (1 + r_2) - Lx_2^*(1 + r_{D2}); 0 \right]$, $K_2$ appears as a call whose underlying is $Ax_2$ and whose strike is $Lx_2^*(1 + r_{D2})$. However, since $Ax$ is the price of an illiquid asset, it is difficult to argue that there exists a unique probability (the risk-neutral probability) that makes $Ax$ a martingale. Therefore, we choose to consider that the price is the discounted expected payoff under the physical probability. The corresponding prices $K^{(0)}_2$ and $Lx^{(0)}_2$ are given in the following proposition.

**Proposition 7.** Considering $\ln \left( Ax^{(1)}/Ax^{(0)} \right) \sim \mathcal{N}(\mu; \sigma^2)$, the expected equity and debt values are

$$E_0 (K^{(1)}) = Ax^{(0)} e^{\mu + \frac{1}{2} \sigma^2} \left[ 1 - \Phi(\tilde{u} - \sigma) \right] - L^* (1 - \Phi(\tilde{u})),
$$

$$E_0 (L^{(1)}) = Ax^{(0)} e^{\mu + \frac{1}{2} \sigma^2} \Phi (\tilde{u} - \sigma) + (1 - \Phi(\tilde{u})),
$$

where $\tilde{u} = \frac{1}{\sigma} \left( \ln \left( \frac{L^*}{Ax^{(1)}} \right) - \mu \right)$ and $L^* = Lx^{(0)}(1 + r_{D}) = Lx^{(1)}$.

In order to understand some implications of our pricing choice, consider a situation where all returns are deterministic and $r_2 > r_{D2}$. In such a framework, we have

$$K^{(0)}_2 = \frac{\kappa_2 Ax_2 (1 + r_2) - Lx_2^*(1 + r_{D2})}{1 + r_{rf}} \quad \text{and} \quad Lx^{(0)}_2 = \frac{Lx_2^*(1 + r_{D2})}{1 + r_{rf}}.
$$

Therefore, injecting these prices in Equation (18), we obtain

$$P_1 = Ax_1 (r_1 - r_{D1}) + \pi_{12} \left( \kappa_2 Ax_2 (1 + r_2) - Lx_2^*(1 + r_{D2}) \right) \times \left( 1 - \frac{1 + r_{D1}}{1 + r_{rf}} \right),$$

$$+ \gamma_{12} \left[ Lx_2^*(1 + r_{D2}) \times \left( 1 - \frac{1 + r_{D1}}{1 + r_{rf}} \right) \right].$$

Generally, we have $r_{D1} > r_{rf}$, meaning that the factors of $\pi_{12}$ and $\gamma_{12}$ are negative and thus that the net yields on shares and debt securities are negative. Therefore, for a risk-neutral agent (i.e. not interested in variance reduction), it would not be optimal to invest in shares and debt securities. That stems partly from the fact that we have priced these instruments using the physical probability. Under the latter probability, the shares and debt securities yield in average the risk-free rate. This feature could of course be challenged. However, we should pay attention to the interpretations based on Equation (19) since it only gives the expression of the position in a very simplified case. Equation (19) must only be considered as an indication.

Contrary to the share and debt security prices, the initial value of $Ax_1$ does not take the future returns into account. As we already mentioned, $Ax_1$ is an illiquid asset that cannot be exchanged on the market. Therefore the assumption of absence of arbitrage is not necessarily verified and we price $Ax_1$ using its book value. Since generally $r_1 > r_{D1},$
the specific asset $Ax_1$ provides a positive return. This result is logical since getting positive
returns via maturity transformation constitutes the core business of banks. However, in
the pricing of $K_2^{(0)}$, we consider the future returns of $Ax_2$. This asymmetry can be
discussed but it is difficult to find an ideal solution given the narrow link between a
market asset ($K_2^{(0)}$) and an illiquid asset ($Ax_2$) in our model.

4.4 Methodology for network formation

The optimization program presented in Section 3 allows computing the balance sheet
of an institution knowing the state of the others. Here the aim is to build a complete
network using this individual optimization program. To this purpose, we operate in a
sequential way until an equilibrium in the network is reached.

We propose to use an iterative game. At each step, one institution optimizes its
balance sheet taking into account the state of the network obtained at the previous step.
Thanks to Proposition 3, there exists only one the network at each step. The procedure
is as follows:\footnote{Note that this formation process can be applied in the general framework of Section 3 but is here
presented using the previously mentioned specification.}

1. Bank 1 optimizes its balance sheet on $Ax_1$ and $Lx_1^*$. $\pi_{1,2}$ and $\gamma_{1,2}$ are forced to be
zero since at the initialization step, bank 2’s balance sheet is totally unknown.

2. Bank 2 optimizes its balance sheet on $Ax_2$, $Lx_2^*$, $\pi_{2,1}$ and $\gamma_{2,1}$ given bank 1’s balance
sheet from step 1.

3. Bank 1 optimizes its balance sheet on $Ax_1$, $Lx_1^*$, $\pi_{1,2}$ and $\gamma_{1,2}$ given bank 2’s balance
sheet from step 2. $\pi_{1,2}$ and $\gamma_{1,2}$ are optimized for the first time.

4. Bank 2 optimizes its balance sheet on $Ax_2$, $Lx_2^*$, $\pi_{2,1}$ and $\gamma_{2,1}$ given bank 1’s balance
sheet from step 3.

5. Bank 1 optimizes its balance sheet on $Ax_1$, $Lx_1^*$, $\pi_{1,2}$ and $\gamma_{1,2}$ given bank 2’s balance
sheet from previous step.

6. and so on...

Strictly speaking on a theoretical level, this procedure may be endless. However, variations
of the external assets and debt lower than 1% are observed in less than 10 steps.
We consider that the final situation constitutes an equilibrium. Moreover, if we accept
the numeric argument for the existence of the limit-network, we can affirm its uniqueness.
Indeed, if at each step, the network is unique then its final state is necessarily unique. It
is interesting to note that this method is inspired by the classical methodology used to
determine a Nash equilibrium (in the sense that no institution has any interest in devi-
ating from its current state). However, further investigations would be required to know
if the network obtained by our method effectively corresponds to a Nash equilibrium.

Last but not least, it is important to verify that the obtained network is consistent in
the sense that it verifies Equations (1) and (2). Firstly, at time $t = 0$, all banks considered
in the network are of course alive (otherwise they would disappear from the network).
That means that the initial debt equals the contractual one: $\forall i \in 1, \ldots, n \ Lx_i^{(0)} = Lx_i^*$. Therefore Equation (2) is automatically verified for each institution. Moreover, at each
step, being a constraint of the optimization program, Equation (1) is verified for the bank optimizing its balance sheet. If preliminary, this step has impacts on the other banks’ balance sheets and Equation (1) is not exactly verified anymore for them. Nevertheless, after some iterations the network does not evolve from one step to the next (due to the convergence), implying that Equation (1) is verified for all institutions. These two points show that the obtained network is actually consistent.

This sequential algorithm could appear a little artificial but it is actually close to what happens in reality. An example of a real formation process of a network is as follows:

1. Imagine an initial situation where there is no financial institution.

2. A first institution, denoted by $I_1$, is created during year $t = 0$. Since there are no other financial institutions, there are no possible interconnections. Thus $I_1$ optimizes $Ax_1$ and $Lx_1$. On Jan. 1st of year $t = 1$, $I_1$ publishes its balance sheet.

3. Imagine that on Jan. 3rd, a second institution $I_2$ is created. $I_2$ knows $Ax_1$ and $Lx_1$ and then can solve the optimization program to determine $Ax_2$, $Lx_2$, $\pi_{2,1}$ and $\gamma_{2,1}$. Once proportions $\pi_{2,1}$ and $\gamma_{2,1}$ have been determined, $I_2$ can buy on the secondary market shares and bonds issued by $I_1$ in these proportions.

4. On June 1st, $I_1$ and $I_2$ publish their balance sheets (apart from interconnections). Since the balance sheet of $I_1$ did not evolve since Jan. 1st, $I_2$ has no new optimization to carry out. On the other hand, $I_1$ discovers for the first time informations relative to $I_2$: $Ax_2$ and $Lx_2$. Then $I_1$ optimizes its balance sheet and thus obtains $Ax_1$, $Lx_1$, $\pi_{12}$ and $\gamma_{12}$. $I_1$ can buy on the secondary market shares and bonds issued by $I_2$.

5. On Jan. 1st of year $t = 2$, balance sheets of $I_1$ and $I_2$ are published. The balance sheet of $I_2$ did not change and thus $I_1$ has no optimization to do. On the other hand, $I_2$ must adapt to the new balance sheet of $I_1$.

6. ...

After such iterations, one may think that there is convergence to an equilibrium in the network. Balance sheets of institutions $I_1$ and $I_2$ do not evolve a lot from one step to the next.

4.5 Simulation results about the optimal choice for one institution

Let us here focus on the second step of the iterative game where bank 2 optimizes its whole balance sheet (knowing the choice of bank 1 at step 1). For simplicity, we assume that bank 1’s external assets are equal to 10. We present the sensitivity of the optimal choices of external assets $Ax_2$, nominal debt $Lx_2$ and interconnections $\pi_{2,1}$ and $\gamma_{2,1}$. To ensure robustness, our results were found with various debt-issuing conditions (no costly with $r_{D,1} = r_{D,2} = r_{rf} = 0$, both costly with $r_{D,1} = r_{D,2} = 1% > r_{rf} = 0$ and only one costly with $r_{D,1} = 1% > r_{D,2} = r_{rf} = 0$). In each set-up, we consider the 8 sets of risk-weights and we let the correlation parameter vary between $-0.9$ and $+0.9$.

The corresponding results are summarized in Table 5. First, we observe that interconnections based on debt securities are never used. A direct consequence is that the
risk-weight on debt $k_\gamma$ has no impact on the balance sheet and thus does not appear in Table 5. Second, interconnections based on share securities are used only when the correlation is lower than -0.3 (independently of the interest rates) and when the associated risk-weight is equal to 23.2%. They linearly decrease from about 45% to 0% between $\rho - 0.9$ and $\rho = -0.3$. Third, the solvency constraint is binding. The optimal external assets represent about $1/k^A$. The last row-block displays the ratio of interbank assets over the total assets: when interconnections are present, their proportion in the total assets is in line with the stylized facts.

These results could be interpreted as follows. First, the bank plays its core business: it invests as much as it can in its external assets. Then, if the regulation is not too strict and if the competitor’s results are sufficiently anti-correlated, the bank opts for diversification: it slightly lowers its external assets to buy share securities issued by the competitor. Debt securities are not used since their net returns are negative (as a consequence of the pricing specification described in Section 4.3) and "nearly" deterministic (due to the low probability of default).

| $k^\pi$ | $k^A$ | $\rho = -0.9$ | $\rho = -0.6$ | $\rho = -0.3$ |
|---------|-------|----------------|----------------|----------------|
| $Ax$    | 23.2% | 6%             | 14             | 15             | 16             |
|         | 23.2% | 8%             | 11             | 12             | 11             |
|         | 46.4% | 12%            | 8              | 8              | 8              |
|         | 46.4% | 16%            | 6              | 6              | 6              |
| $\pi$   | 23.2% | 6%             | 45             | 25             | 0              |
| (%)     | 23.2% | 8%             | 45             | 25             | 0              |
|         | 46.4% | 12%            | 0              | 0              | 0              |
|         | 46.4% | 16%            | 0              | 0              | 0              |
| $\gamma$| 23.2% | 6%             | 0              | 0              | 0              |
| (%)     | 23.2% | 8%             | 0              | 0              | 0              |
|         | 46.4% | 12%            | 0              | 0              | 0              |
|         | 46.4% | 16%            | 0              | 0              | 0              |
| $IBA/TA$| 23.2% | 6%             | 3.1            | 1.6            | 0              |
| (%)     | 23.2% | 8%             | 3.9            | 2.0            | 0              |
|         | 46.4% | 12%            | 0              | 0              | 0              |
|         | 46.4% | 16%            | 0              | 0              | 0              |

Table 5: Stylized results for the optimal choice of one institution.

4.6 Iterative game results

The iterative game reaches an equilibrium in less than 5 steps and features pictured in the analysis of the behavior of one institution are still present. Especially, results are robust to the debt-issuing conditions.

Both institutions have the same balance sheet, whose composition is given in Table 6. Results are very similar to those for one institution only (Table 5). In particular, the proportion of interbank assets in the total asset is in agreement with the stylized facts. Note that for $\rho = -0.9$ and $\rho = -0.6$, the values of $\gamma_{12}$ and $\gamma_{21}$ are close to $10^{-4}$. However, we have reported 0 since such low values do not have any economic meaning.

Let us precise that results have been obtained using $\kappa_1 = \kappa_2 = 1$ in order to avoid numerical instability. Indeed, if the values of $\kappa_i$ become too large, the approximation of
our model is not verified anymore.

|       | $k^\pi$ | $k^A$ | $\rho = -0.9$ | $\rho = -0.6$ | $\rho = -0.3$ |
|-------|---------|-------|---------------|---------------|---------------|
| Ax    | 23.2%   | 6%    | 15            | 15            | 16            |
|       | 23.2%   | 8%    | 11            | 12            | 12            |
|       | 46.4%   | 12%   | 8             | 8             | 8             |
|       | 46.4%   | 16%   | 6             | 6             | 6             |
| $\pi$ | 23.2%   | 6%    | 70            | 45            | 16            |
| (%)   | 23.2%   | 8%    | 60            | 35            | 6             |
|       | 46.4%   | 12%   | 0             | 0             | 0             |
|       | 46.4%   | 16%   | 0             | 0             | 0             |
| $\gamma$ | 23.2%   | 6%    | 0             | 0             | 0             |
| (%)   | 23.2%   | 8%    | 0             | 0             | 0             |
|       | 46.4%   | 12%   | 0             | 0             | 0             |
|       | 46.4%   | 16%   | 0             | 0             | 0             |
| $IBA/TA$ | 23.2%   | 6%    | 3.2           | 2.3           | 1             |
| (%)   | 23.2%   | 8%    | 3.6           | 2.4           | 0.5           |
|       | 46.4%   | 12%   | 0             | 0             | 0             |
|       | 46.4%   | 16%   | 0             | 0             | 0             |

Table 6: Stylized results for the iterative game.

### 4.7 Testing the diversification motive

Regarding the capacity of the diversification motive to account for interconnections, previous results provide a quantitative assessment completing the qualitative arguments developed in Section 3. The key results is that when returns on specific assets are anti-correlated, diversification leads to interconnections with reasonable size in terms of proportion of the total assets. However, debt securities are never used, meaning that interconnections are only supported by share securities. This portfolio composition contrasts with empirical findings.

However, it is important to emphasize that in our simulation study, the choice of pricing shares and debt securities under the physical probability has large impacts. As explained in Section 4.3, it implies that the net yields of shares and bonds are negative. Therefore, in this framework, interconnections only allow for variance reduction but not for gain opportunity. We can expect this feature to be modified if the pricing is done under the risk-neutral probability. Interconnections in both shares and debt securities could then be observed, even for values of $\rho$ larger than $-0.3$. The study of the risk-neutral specification constitutes an ongoing work. In some sense, these two types of specification for the pricing allow disentangling the two aims of the diversification: opportunity and variance reduction.

The latter discussion shows that our model seems promising but that results are very sensitive to the different possible specifications. Moreover, two features that are not included in our model for the sake of simplicity may explain this discrepancy concerning debt securities. First, there are additional constraints -apart form the required capital-imposed to large shareholders such as mandatory public communication. Second, in our model, the debt is only composed of securities whereas one could distinguish deposit (with no paid interest) and bonds (with interest). Keeping similar cost of funding, 1% of
the total debt for example, would increase the coupons leading to better opportunities for banks. However, including one of these two features is clearly out of the scope of this paper.

5 Application: impact of interconnectedness regulation

In the previous section, we have developed and qualitatively tested an hypothesis to explain the drivers of interconnections. Note that all the relative results concern the initial network resulting from financial institutions’ choices based on their expectations. The diversification motive has proven an interesting explanation concerning bank size (Stylized Fact 1), the network shape (Stylized Facts 2 and 3) and the composition of interconnections (Stylized Fact 4). Spotting this motive is a necessary step to analyze the impact of regulation, since it allows building some plausible counterfactual.

5.1 Assessing interconnections

The interconnectedness across financial institutions has become a key concern of supervisors and regulatory authorities. Currently, interbank exposures, namely long-term, are covered by two main requirements. The first one concerns the solvency required capital for the interconnections, as for any other assets. It imposes a constraint on the total interbank exposure. The second one concerns "large" single exposures and imposes the risk-weighted exposure to be lower than a fraction of the capital. Currently, the Basel Committee proposes to consider that an exposure is large if above 5% (instead of 10%) of own funds and to impose that the risk-weighted exposure \( (\kappa_{ij} \pi_{ij} K_j + \kappa_{ij} \gamma_{ij} Lx_j) \) has to be lower than 25% of the capital (see BSBC 2013, Section II and Section IV.B). These requirements are valid for any type of exposure (corporate, sovereign...) but the weights can vary with respect to the type. For instance the Basel Committee proposes to introduce tighter rules about interbank exposures for the G-SIBs (Global Systemically Important Banks). An upper bound between 10% and 15% instead of 25% is in discussion (see BSBC 2013, Section V). These tighter rules about interbank exposures aim at reducing the risk of contagion.

These different aspects show that interconnectedness is generally assessed in a negative perspective. Actually, supervisors are primarily concerned with excessive risks and therefore either analyze the effects of interconnections under depressed scenarios (stress-test approach) or build indicators in order to monitor the current fragility of the financial sectors. In both approaches, interconnectedness usually means contagion only. For instance, seminal papers about network stress-tests -such as Furfine (2003) on US data or Upper and Worms (2004) on German data- sequentially consider the effects in their national banking sectors of the default of each bank. From their point of view, interconnected banks are likely to trigger defaults or to go bankrupt due to contagion.

However, although these analyses correspond to regulatory stress-test exercises or monitoring processes, they are not built on counterfactuals. They give informative insights about what could happen within the current network in cases of defaults of some institutions or difficult macroeconomic conditions. However, since the network reaction is not taken into account, such studies do not really provide any clue on the way to obtain a
more resilient network structure. Moreover, note that the question of regulation impact has hardly been addressed quantitatively, even in the case of a crystallized network.

The endogenous nature of interconnections in our model precisely allows us to assess the impact of regulation on interbank exposures, for instance of the one in discussion at the Basel Committee. To do so, we consider our 8 sets of regulatory weights associated to interbank exposure ($k^A$, $k^π$ and $k^γ$)$^{13}$ For each specific set, the initial network is derived using our formation process. This step accounts for the diversification motive. Then we simulate shocks and examine the network after the shocks. Let us emphasize that the shocks are properly propagated through the real interconnections. The unique set of values $K_i$ and $Lx_i$ (see Proposition 1) is determined using the algorithm described in Appendix F. This allows us to carry out a fair assessment of contagion. To do so, we build a welfare indicator that includes an explicit concern for the real economy and examine its sensitivity to the regulatory set of weights.

5.2 Welfare analysis

We adapt the welfare analysis by [Repullo and Suarez (2013)] to assess the impact of the regulatory parameters to the real economy.

The contribution of one bank is either negative or positive. When a bank defaults, its contribution is negative and proportional to the loss on its debt. This feature encompasses the cost of deposit insurance. When a bank is alive, its contribution is the volume of external assets, i.e. the lendings provided to the real economy. This component captures the capacity to finance the real economy. We have

$$w_i = -c \left( Lx_i^{*(1)} - Lx_i^{(1)} \right) + Ax_i^{(1)}, \quad (20)$$

where $c$ is the social cost for deposit insurance (in [Repullo and Suarez (2013)], $c$ varies in $[0\%; 60\%]$).

Our welfare indicator is the ratio the contribution of all banks over the initial lending to the real economy

$$W = \frac{\sum_{i=1}^{n} w_i}{\sum_{j=1}^{n} Ax_j^{(0)}}.$$

For $c = 0$, the welfare is given in Table 7. When there are interconnections, the welfare is higher than 1, indicating an increase of the banking capacity to lend to the real economy. In contrast, when there is no interconnection, the value of the external assets decreases. A complete analysis of the impact of interconnections would require further studies. However, these results suggest that interconnections stemming from diversification are beneficial for the real economy.

$^{13}$In reality only 4 since with the specification chosen, $k^γ$ has no impact.

$^{14}$Contrary to the assumption used in the individual optimization program: institutions do not consider interconnections of their counterparts.
\[ k^\pi \quad k^A \quad \rho = -0.9 \quad \rho = -0.6 \quad \rho = -0.3 \]

| Sum of contributions | 23.2% | 6% | 29.9 | 30.9 | 32.4 |
|-----------------------|-------|----|------|------|------|
| 23.2%                 | 8%    | 22.8 | 23.6 | 24.8 |
| 46.4%                 | 12%   | 15.6 | 15.6 | 15.6 |
| 46.4%                 | 16%   | 11.9 | 11.9 | 11.9 |

| Welfare (%) | 23.2% | 6% | 101.0 | 101.0 | 101.0 |
|-------------|-------|----|-------|-------|-------|
| 23.2%       | 8%    | 101.0 | 101.0 | 100.8 |
| 46.4%       | 12%   | 93.4 | 93.4 | 93.4 |
| 46.4%       | 16%   | 95.6 | 95.6 | 95.5 |

Table 7: Welfare.

6 Concluding remarks

A diversification motive appears as a sound candidate to account for long-term exposures across financial institutions. The first aim of this paper is to test this assumption.

To this purpose we build a model of financial network in which the balance sheets of all institutions (including interconnections) are totally endogenous apart from the equity. The network formation process involves two components. The first one explains how a bank optimizes its balance sheet knowing the state of the other banks in the network. We prove the existence and partial uniqueness of the solution of this optimization. The second part shows how to form the network using the individual optimization program. The existence and unicity of this network are shown by numerical arguments. An important feature of our model is its ability to account for the main features of the banking and the insurance business with the same set of parameters. Nevertheless we focus in this paper on the banking business.

Secondly, the characteristics of the resulting network are compared to features usually observed. As to the shape of the network, we theoretically find that the diversification motive leads to a network close to those observed across big banks. Concerning the size and support of the interconnections, we show that a correct magnitude is reached under standard calibration. Moreover, the results are sensitive to some specifications, concerning for example the pricing of shares and debt securities.

The fact that our network is totally endogenous allows studying how it adapts to regulatory changes. Thus the second aim is to apply our model to fairly assess the impact of regulation on interbank exposures. To this purpose we study the evolution of the welfare with respect to the regulatory weight relative to debt interconnections. A clear knife-edge effect appears.

Ongoing work includes the complete study in the case of insurance companies and the extension to short-term interconnections. An exhaustive sensitivity analysis of the obtained network with respect to macroeconomic parameters like the returns means as well as other specifications - e.g. concerning the pricing of shares and debt securities - are also under study. Finally, a simulation exercise in the case of 3 or 4 banks would also be of great interest.
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A  Example of public information on banks’ balance sheets

Figure 4: Excerpt of Consolidated Financial Statements for BHCs of Bank of America at 06/30/2013. Source: www.ffiec.gov.

B  The model by Gouriéroux et al. (2012)

In this part, we expose the model of Gouriéroux et al. (2012), that provides the conditions defining an equilibrium between $n$ financial institutions intertwined through stocks and
loans interconnections.

B.1 Existence and unicity of the equilibrium

**Proposition 8.** There exists a unique liquidation equilibrium, that is a unique set of values for $K$ and $Lx$ for any given values of $Lx^*$, $Ax$, $Aℓ$ if for all $(i,j) \in [1,n]^2$:

- $(A1') \quad \pi_{i,j} \geq 0, \gamma_{i,j} \geq 0$,
- $(A2') \quad Ax_i \geq 0, Aℓ_i \geq 0, Lx_i^* \geq 0$,
- $(A3') \quad \sum_{i=1}^{n} \pi_{i,j} < 1, \sum_{i=1}^{n} \gamma_{i,j} < 1$.

**Proof.** See Gouriéroux et al. (2012).

Assumptions $(A1')$ and $(A2')$ define a proper space for the parameters: all elements composing the balance sheet must obviously be positive.

Assumption $(A3')$ means that some shareholders and creditors do not belong to the perimeter of selected financial institutions. In practice, the first part of Assumption $(A3')$ is generally verified providing that we consider consolidated groups. Usual accounting rules indeed state that a subsidiary belongs to the consolidated perimeter if more than 20% of its capital is held by the parent company. Thus each $\pi_{ij}$ is necessarily lower than 20%. Furthermore studies by Gauthier et al. (2012) and Alves et al. (2013) clearly show that $\sum_{i=1}^{n} \pi_{i,j} < 1$. The constraint on the $\gamma_{ij}$ is largely verified since core deposits (deposits from external agents) represent approximately 55% of a bank’s debt.

B.2 Case of two financial institutions

For illustrative purposes, let us consider a network of two institutions whose balance sheets are shown in Table 8. In such a case the equilibrium conditions (1)-(2) are

$$
\begin{cases}
K_1 = \left( \pi_{1,1} K_1 + \pi_{1,2} K_2 + \gamma_{1,2} Lx_2 + Aℓ_1 + Ax_1 - Lx_1^* \right)^+,
Lx_1 = \min \left( \pi_{1,1} K_1 + \pi_{1,2} K_2 + \gamma_{1,2} Lx_2 + Aℓ_1 + Ax_1; Lx_1^* \right),
K_2 = \left( \pi_{2,1} K_1 + \pi_{2,2} K_2 + \gamma_{2,1} Lx_1 + Aℓ_2 + Ax_2 - Lx_2^* \right)^+,
Lx_2 = \min \left( \pi_{2,1} K_1 + \pi_{2,2} K_2 + \gamma_{2,1} Lx_1 + Aℓ_2 + Ax_2; Lx_2^* \right).
\end{cases}
$$

One can identify 4 regimes depending on the situations of institution 1 and institution 2. These regimes, represented in Figure 5, are:

- Regime 1: both institution 1 and institution 2 are alive,
- Regime 2: both institution 1 and institution 2 default,
- Regime 3: institution 1 defaults while institution 2 is alive,
- Regime 4: institution 1 is alive while institution 2 defaults.
| Asset | Liability | Asset | Liability |
|-------|-----------|-------|-----------|
| $\pi_{1,1}K_1$ | $Lx_1$ | $\pi_{2,1}K_1$ | $Lx_2$ |
| $\pi_{1,2}K_2$ | $K_1$ | $\pi_{2,2}K_2$ | $K_2$ |
| $\gamma_{1,2}L_2$ | | $\gamma_{2,1}L_1$ | |
| $A\ell_1$ | | $A\ell_2$ | |
| $Ax_1$ | | $Ax_2$ | |

Table 8: Balance Sheets of bank 1 and 2.

This graph motivates the interconnections between institutions. In a situation without interconnections, the four regimes would be defined by rectangles. Here the bounds deviate due to the presence of interconnections. In the case where the external assets of institution 2, $Ax_2 + A\ell_2$, are just above the limit value $Ax_2^*$, if it is interconnected and if $Ax_1 + A\ell_1$ is low, then institution 2 can default ($\mathcal{R}_3$ is larger in presence of interconnections). In this case, interconnections have a negative effect since the predicament of institution 1 negatively impacts institution 2 by contagion. When $Ax_2 + A\ell_2$ is very low, institution 2 necessarily defaults if not linked to institution 1. However, if institution 2 owns shares of institution 1, institution 2 can survive if the external assets' value of institution 1 is sufficient ($\mathcal{R}_1$ is larger in presence of interconnections). In such a case institution 2 takes advantage of the high yield investments of institution 1. Thus we understand that the impacts of interconnections are not necessarily negative and must be fairly assessed.
C Proofs

For Proposition 1

Proof. Let us denote the vectors of all control variables by $\mathbf{X}$. We have

$$\mathbf{X} = \left( A\varepsilon_i^{(0)}, A\ell_i^{(0)}, L\varepsilon_i^{(0)}, \omega_i, \pi_{i,1}, \ldots, \pi_{i,n}, \gamma_{i,1}, \ldots, \gamma_{i,n}\right) \in \mathcal{X}_{ad},$$

where $\mathcal{X}_{ad}$ is the admissible space verifying all constraints of program $\mathcal{P}_i$. The proof relies on the Weierstrass theorem: a continuous function on a compact set reaches its bounds. Therefore we first show the continuity of the objective function and then the compactness of the admissible set $\mathcal{X}_{ad}$.

Continuity of the objective function

Under Assumptions $(A2)$ and $(A3)$, both the utility function and the c.d.f. of shocks $F_S$ are continuous. Therefore, the expectation is also continuous and the objective function is continuous.

Compactness of the admissible set $\mathcal{X}_{ad}$

To prove the compactness of $\mathcal{X}_{ad}$, we show that it is a closed and a bounded set. Before we prove that $\mathcal{X}_{ad}$ is not empty.

$\mathcal{X}_{ad}$ is non-empty:

Let us consider the vector of parameters $\mathbf{X}_0$ defined as

$$\mathbf{X}_0 = \left( K_i^{(0)} - kLl(0,0), kLl(0,0), 0, \ldots, 0\right)'.$$

All constraints apart from $Ax_i \geq 0$, $(NOD)$, $(RSC)$ and $(RLC)$ are obviously satisfied. $Ax_i \geq 0$ imposes that $K_i^{(0)} \geq kLl(0,0)$ which is not restrictive due to the low value of $kL$ and the fact that $l(0,0)$ can be taken equal to one. Constraint $(NOD)$ reduces to $K_i^{(0)} - kLl(0,0) + kLl(0,0) = K_i^{(0)}$ and is thus satisfied. Constraint $(RSC)$ is written

$$K_i^{(0)} \geq k_i^A Ax_i^{(0)} \iff K_i^{(0)} \geq k_i^A [K_i^{(0)} - kLl(0,0)] \iff K_i^{(0)}(1 - k_i^A) \geq -k_i^A kLl(0,0).$$

Due to the inequality $k_i^A < 1$ and the positivity of $k_i^A$, $kL$ and function $l$, the left hand term is positive whereas the right one is negative. Constraint $(RSC)$ is satisfied. Thus $\mathbf{X}_0$ belongs to the admissible set $\mathcal{X}_{ad}$ which is therefore not empty.

$\mathcal{X}_{ad}$ is a closed set:

In order to show that the admissible set $\mathcal{X}_{ad}$ is a closed set, we show that it is the intersection of closed sets.
i) Constraint (NOD) can be written

\[ Ax_i^{(0)} + A\ell_i^{(0)} + \sum_{j=1}^{n} \pi_{i,j} K_j^{(0)} + \sum_{j=1}^{n} \gamma_{i,j} Lx_j^{(0)} - K_i^{(0)} - Lx_i^{(0)} = 0. \]

The corresponding admissible space is the reciprocal image of the singleton \( \{0\} \), which is a closed set of \( \mathbb{R} \), by a continuous function. Therefore, constraint (NOD) defines a closed set.

ii) Constraint (RSC) is derived in

\[ k^A Ax_i^{(0)} + k^\pi \sum_{j=1}^{n} \pi_{i,j} K_j^{(0)} + k^\gamma \sum_{j=1}^{n} \gamma_{i,j} Lx_j^{(0)} - K_i^{(0)} \leq 0. \]

The corresponding admissible space is the reciprocal continuous image of \([−\infty; 0]\), which is a closed set of \( \mathbb{R} \). Therefore, constraint (RSC) defines a closed set.

iv) Constraint (RLC) is derived in

\[ k^L l(\omega_i, Lx_i^{(0)}) - A\ell_i^{(0)} \leq 0. \]

The corresponding admissible space is the reciprocal continuous image of \([−\infty; 0]\), which is a closed set of \( \mathbb{R} \). Therefore, constraint (RLC) defines a closed set.

v) The positivity constraints \( Ax_i^{(0)} \geq 0, A\ell_i^{(0)} \geq 0 \) and \( Lx_i^{(0)} \geq 0 \) also define closed sets, as the continuous reciprocal images of \([0; +\infty]\), which is a closed set of \( \mathbb{R} \).

vi) Constraints \( \omega_i \in [0; 1], 0 \leq \pi_{i,j} \leq 1 - c^\pi_j \) and \( 0 \leq \gamma_{i,j} \leq 1 - c^\gamma_j \) (\( \forall j \in [1; n] \)) define a closed admissible set as the reciprocal images of \([0; 1], [0; c^\pi_j] \) and \([0; c^\gamma_j] \), which are closed sets of \( \mathbb{R} \), by a continuous function.

The admissible set \( \mathcal{X}_{ad} \) is the intersection of the admissible sets defined by each constraint. Moreover, an intersection of closed sets is a closed set. Thus, combining points i) to vi), we obtain that \( \mathcal{X}_{ad} \) is a closed set.

\( \mathcal{X}_{ad} \) is a bounded set:

Let us show that the admissible set is bounded.

Conditions \( 0 \leq \pi_{i,j} \leq 1 - c^\pi_j \) and \( 0 \leq \gamma_{i,j} \leq 1 - c^\gamma_j \) (\( \forall j \in [1; n] \)) show that all the \( \pi_{i,j} \) and \( \gamma_{i,j} \) are bounded. The same is true for \( \omega_i \in [0; 1] \). Let us now prove that \( A\ell_i, Ax_i \) and \( Lx_i \) are bounded.

i) Bound for \( A\ell_i \)

The combination of constraint \( Lx_i^{(0)} \geq 0 \) and constraint (NOD) implies that the institution invests at least all its own capital.

If \( Lx_i^{(0)} = 0 \), \( K_i^{(0)} \) is an upper-bound for \( A\ell_i \).

Let us now consider the case \( Lx_i^{(0)} > 0 \). The constraint (NOD) can be used to express the debt as a function of other control variables,

\[ Lx_i^{(0)} = Ax_i^{(0)} + A\ell_i^{(0)} + \sum_{j=1}^{n} \pi_{i,j} K_j^{(0)} + \sum_{j=1}^{n} \gamma_{i,j} Lx_j^{(0)} - K_i^{(0)}. \]
Using this last equation, one can express the PnL, \( P_i^{(1)} \), as a function of the other control variables:

\[
P_i^{(1)} = A x_i^{(0)} (1 + r_i) + \ell_i^{(0)} (1 + r_{rf}) + \sum_{j=1}^{n} \pi_{i,j} \left( \kappa_j (A x_j^{(0)} (1 + r_j) + \ell_j^{(0)} (1 + r_{rf})) - L x_j^{* (1)} \right) + \sum_{j=1}^{n} \gamma_{i,j} \min \left( \kappa_j (A x_j^{(0)} (1 + r_j) + \ell_j^{(0)} (1 + r_{rf})); L x_j^{* (1)} \right) - \left[ 1 + r_D(\omega_i) \right] \left( A x_i^{(0)} + \ell_i^{(0)} + \sum_{j=1}^{n} \pi_{i,j} K_j^{(0)} + \sum_{j=1}^{n} \gamma_{i,j} L x_j^{(0)} - K_i^{(0)} \right) = \ell_i^{(0)} \left[ r_{rf} - r_D(w_i) \right] + d(X_{-A\ell}, R), \tag{22}
\]

where \( X_{-A\ell} = ( A x_i^{(0)}, L x_i^{(0)}, \omega_i, \pi_{i,1}, \ldots, \pi_{i,n}, \gamma_{i,1}, \ldots, \gamma_{i,n} )' \) is the vector of control variables apart from \( \ell_i^{(0)} \), \( R = (r_1, \ldots, r_n)' \) is the vector of the net returns of the external assets and \( d \) is a function. The PnL \( P_i^{(1)} \) is a function of \( \ell_i^{(0)}, X_{-A\ell} \) and \( R \), denoted by \( P_i^{(1)}(\ell_i^{(0)}, X_{-A\ell}, R) \). Assumption (A4) says that \( r_D(\omega_i) > r_{rf} \), giving that \( P_i^{(1)}(\cdot) \) is strictly decreasing with respect to \( \ell_i^{(0)} \).

Let us consider a value \( V_i > K_i^{(0)} \) for \( \ell_i^{(0)} \). From Equation \( \tag{22} \), we see that, for all \( X_{-A\ell} \), there exists a set \( \varepsilon_1, \ldots, \varepsilon_n \) of values such that, if \( \forall k \in [1, n], r_k \geq \varepsilon_k \), then \( P_i^{(1)}(V_1, X_{-A\ell}, R) > 0 \). For a second value \( V_2 \) such that \( K_i^{(0)} \leq V_2 < V_1 \), we have

\[
P_i^{(1)}(V_2, X_{-A\ell}, R) > P_i^{(1)}(V_1, X_{-A\ell}, R). \tag{23}
\]

Therefore,

\[
\text{if } \forall k \in [1, n], r_k \geq \varepsilon_k, P_i^{(1)}(V_2, X_{-A\ell}, R) > 0. \tag{24}
\]

Now, let us compare the expected utility at \( \ell_i^{(0)} = V_1 \) and \( \ell_i^{(0)} = V_2 \). We have

\[
E \left[ u \left( K_i^{(1)} \right) \right] (V_1, X_{-A\ell}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u \left( \max \left( P_i^{(1)}(V_1, X_{-A\ell}, R); 0 \right) \right) f_S(R) \, dR + \int_{\varepsilon_1}^{+\infty} \cdots \int_{\varepsilon_n}^{+\infty} u \left( P_i^{(1)}(V_1, X_{-A\ell}, R) \right) f_S(R) \, dR. \tag{25}
\]

By the same decomposition and using Equation \( \tag{24} \), we obtain, for \( \ell_i^{(0)} = V_2 > V_1 \),

\[
E \left[ u \left( K_i^{(1)} \right) \right] (V_2, X_{-A\ell}) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} u \left( \max \left( P_i^{(1)}(V_2, X_{-A\ell}, R); 0 \right) \right) f_S(R) \, dR + \int_{\varepsilon_1}^{+\infty} \cdots \int_{\varepsilon_n}^{+\infty} u \left( P_i^{(1)}(V_2, X_{-A\ell}, R) \right) f_S(R) \, dR. \tag{26}
\]

Using Equation \( \tag{23} \), we have, for \( R \in [-\infty, \varepsilon_1] \times \cdots \times [-\infty, \varepsilon_n] \),

\[
\max \left[ P_i^{(1)}(V_2, X_{-A\ell}, R); 0 \right] \geq \max \left[ P_i^{(1)}(V_1, X_{-A\ell}, R); 0 \right],
\]

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and, since $u$ is strictly increasing (Assumption (A2)),
\[
u \left( \max \left[ P_i(1)(V_2, X_{-Aℓ}, R); 0 \right] \right) \geq u \left( \max \left[ P_i(1)(V_1, X_{-Aℓ}, R); 0 \right] \right).
\]
Using Equation (23), we have, for all $R \in [\varepsilon_1; +\infty] \times \cdots \times [\varepsilon_n, +\infty]$,
\[
P_i(1)(V_2, X_{-Aℓ}, R) > P_i(1)(V_1, X_{-Aℓ}, R),
\]
and, since $u$ is strictly increasing,
\[
u \left[ P_i(1)(V_2, X_{-Aℓ}, R) \right] > u \left[ P_i(1)(V_1, X_{-Aℓ}, R) \right].
\]
Moreover, for all $R \in [a; +\infty]^n, f_S(R) > 0$ (Assumption (A3)). Therefore, combining Equations (25) and (26) yields
\[
\forall X_{-Aℓ}, \quad E \left[ u \left( K_{i}^{(1)} \right) \right] (V_1, X_{-Aℓ}) < E \left[ u \left( K_{i}^{(1)} \right) \right] (V_2, X_{-Aℓ}),
\]
meaning that, for $Aℓ_i^{(0)} \geq K_{i}^{(0)}$, the objective function is strictly decreasing with respect to $Aℓ_i^{(0)}$. Consequently, the optimization program $P_i$ is equivalent if we upper-bound the space of $Aℓ_i$. Moreover, since $Aℓ_i$ is lower-bounded by 0, $Aℓ_i$ is bounded.

ii) Bounds for $Ax_i$ and $Lx_i$

Let us recall constraint (RSC):
\[
K_i^{(0)} \geq k^A Ax_i^{(0)} + k^\pi \sum_{j=1}^{n} \pi_{i,j} K_j^{(0)} + k^\gamma \sum_{j=1}^{n} \gamma_{i,j} Lx_j^{(0)}.
\]
$K_i^0$ is fixed as an endowment. Moreover, in the right hand term of inequality (RSC), all components are positive. Thus it imposes that each term is bounded. Therefore, $k^A Ax_i$ is bounded and, since $k^A > 0$ by assumption, $Ax_i$ is upper-bounded. Moreover, $Ax_i \geq 0$ and thus $Ax_i$ is bounded.

Using the fact that $k^\pi > 0$ and $k^\gamma > 0$, we also obtain that both $\sum_{j=1}^{n} \pi_{i,j} K_j^{(0)}$ and $\sum_{j=1}^{n} \gamma_{i,j} Lx_j^{(0)}$ are upper-bounded. Let us recall that constraint (NOD) gives
\[
Lx_i^{(0)} = Ax_i^{(0)} + Aℓ_i^{(0)} + \sum_{j=1}^{n} \pi_{i,j} K_j^{(0)} + \sum_{j=1}^{n} \gamma_{i,j} Lx_j^{(0)} - K_i^{(0)},
\]
implying that $Lx_i^{(0)}$ is upper-bounded since all terms in the right part of the equation are bounded. Since moreover $Lx_i^{(0)} \geq 0$ by assumption, $Lx_i^{(0)}$ is bounded.

Existence

To summarize, the admissible set is not empty. It is also closed and bounded, and therefore compact. The objective function is continuous and Weierstrass’s theorem ensures the existence of a solution. □
For Proposition 2

Proof.

Existence

The existence can be shown exactly in the same way than for Proposition 1.

Uniqueness

The uniqueness is based on a fundamental theorem of optimization: a strictly concave function on a closed convex set admits a unique maximum. We first show that the admissible set is convex and then that the objective function is strictly concave.

Convexity of the admissible set

As before, we denote

\[ X = \left( Ax_i(0), A^i(0), Lx_i(0), \omega_i, \pi_{i,1}, \ldots, \pi_{i,n}, \gamma_{i,1}, \ldots, \gamma_{i,n} \right) \in \mathcal{X}_{ad}, \]

where \( \mathcal{X}_{ad} \) is the admissible space of program \( P_i' \).

Let us show that each constraint of \( P_i' \) defines a convex set. All constraints excluding constraint \( (RLC) \) involve linear functions of the control variables and thus each of these constraints obviously defines a convex set.

Constraint \( (RLC) \) requires more attention. For the sake of simplicity, let us denote by \( x = \omega_i, y = Lx_i(0), z = A^i \). Constraint \( (RLC) \) can therefore be re-written \( z > l(x,y) \). The corresponding set is the epigraph of the function \( l \). The epigraph is convex if and only if \( l \) is convex, i.e. if and only if the Hessian of \( l \), \( H_l \), is semi definite positive. We have

\[
H_l = \begin{pmatrix}
\frac{\partial^2 l}{\partial x^2} & \frac{\partial^2 l}{\partial x \partial y} \\
\frac{\partial^2 l}{\partial x \partial y} & \frac{\partial^2 l}{\partial y^2}
\end{pmatrix}.
\]

Sylvester’s criterion states that a matrix is semi definite positive if and only if all its leading principal minors are positive, i.e.

\[
\frac{\partial^2 l}{\partial x^2} \geq 0 \quad \text{and} \quad \frac{\partial^2 l}{\partial x^2} \times \frac{\partial^2 l}{\partial y^2} \geq \left( \frac{\partial^2 l}{\partial x \partial y} \right)^2.
\]

Thus, under Assumption \( (A8) \), Constraint \( (RLC) \) defines a convex set and finally all constraints define a convex set. Since the intersection of convex sets is a convex set, \( \mathcal{X}_{ad} \) is a convex set.

We want to show that there is unicity of the solution of the optimization of the triple \( (Ac_i(0), Lx_i(0), \omega_i) \), where \( Ac_i(0) \) is one of the variables appearing on the asset side, i.e. among \( Ax_i(0), A^i(0), \pi_{i,1}, \ldots, \pi_{i,n}, \gamma_{i,1}, \ldots, \gamma_{i,n} \). Let us denote

\[ X_r = \left( Ac_i(0), Lx_i(0), \omega_i \right) \in \mathcal{X}_{r_{ad}}, \]

where \( \mathcal{X}_{r_{ad}} \) is the admissible set of the three-dimensional optimization program. By using the same arguments as for \( \mathcal{X}_{ad} \), \( \mathcal{X}_{r_{ad}} \) defines a convex set, whatever the control variable \( Ac_i(0) \) that is chosen. Moreover, note that one can show that \( \mathcal{X}_{r_{ad}} \) is a closed set, as for Proposition 1.
Expectation and underlying objective function

In the following, we generally denote the PnL by \( P_i^{(1)}(X_r, R) \) but sometimes we omit the arguments \( X_r \) and \( R \) for simplicity. The strict concavity of \( u_i \left[ v \left( P_i^{(1)} \right) \right] \) is a sufficient condition to obtain the strict concavity of \( \mathbb{E} \left\{ u_i \left[ v \left( P_i^{(1)} \right) \right] \right\} \) with respect to \( X \). Indeed, let us assume that \( u_i \left[ v \left( P_i^{(1)} \right) \right] \) is strictly concave. Combining the latter assumption with the fact that \( f_i \) is strictly positive on \([-0; +\infty]^n \) (Assumption \((A3)\)), we get, for all \((X_{r1}, X_{r2}) \in X \), and for all \( \lambda \in [0; 1] \),

\[
\mathbb{E} \left\{ u_i \left[ v \left( P_i^{(1)} \right) \right] \right\} (\lambda X_{r1} + (1 - \lambda) X_{r2}) = \int_{[-\infty; +\infty]^n} u_i \left( v \left[ P_i^{(1)} \left( \lambda X_{r1} + (1 - \lambda) X_{r2}, R \right) \right] \right) \ f_S(R) \ dR
\]

\[
> \int_{[-\infty; +\infty]^n} \left[ \lambda u_i \left( v \left[ P_i^{(1)}(X_{r1}, R) \right] \right) + (1 - \lambda) u_i \left( v \left[ P_i^{(1)}(X_{r2}, R) \right] \right) \right] \ f_S(R) \ dR
\]

\[
= \lambda \mathbb{E} \left\{ u_i \left[ v \left( P_i^{(1)} \right) \right] \right\} (X_{r1}) + (1 - \lambda) \mathbb{E} \left\{ u_i \left[ v \left( P_i^{(1)} \right) \right] \right\} (X_{r2}),
\]

showing the strict concavity of the expected utility.

Strict concavity of the underlying objective function

Let us then now focus on function \( u_i \left[ v \left( P_i^{(1)} \right) \right] \). We consider that only one control variable is free in the asset side. For the sake of simplicity, let us denote by: \( x_1 = Ac_i^{(0)} \), \( x_2 = \omega_i \) and \( x_3 = Lx_i^{(0)} \). Let us introduce the function \( f \) defined by

\[
f : \mathbb{R}^+ \times [0, 1] \times \mathbb{R}^+ \to \mathbb{R}
\]

\[
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto t(x_1) - [1 + r_D(x_2)]x_3,
\]

where \( t(.) \) is a linear transformation. Function \( t \) maps the chosen control variable into the value of the total assets \( Ax_i^{(0)}(1 + r_i) + A_i^{(0)}(1 + r_{rf}) + \sum_{j=1}^n \pi_{ij} K_j^{(0)} + \sum_{j=1}^n \gamma_{ij} L x_j^{(0)} \). Therefore \( u_i \left[ v \left( P_i^{(1)} \right) \right] \) is equivalent to \( g = u \circ v \circ f \). Let us now study the strict concavity of function \( g \). For simplicity, we denote by \( m = u \circ v \), yielding \( g = m \circ f \). Function \( g \) is strictly concave if and only if its Hessian matrix \( H_g \) is definite negative. We have

\[
H_g = m^v \begin{pmatrix}
1 & -x_3 r_D(x_2) & -[1 + r_D(x_2)] \\
-x_3 r_D(x_2) & -x_3 \left[ \frac{m'}{m} r_N(x_2) + r_D^2(x_2) x_3 \right] & r_D'(x_2) \left[ -\frac{m'}{m} + x_3[1 + r_D(x_2)] \right] \\
-[1 + r_D(x_2)] & r_D'(x_2) \left[ -\frac{m'}{m} + x_3[1 + r_D(x_2)] \right] & \left[ 1 + r_D(x_2) \right]^2
\end{pmatrix}.
\]

As before, Sylvester’s criterion states that \( H_g \) is definite negative if and only if all its leading principal minors are strictly negative. Let us now study the three corresponding minors.
i) First minor
The first minor is
\[ \text{Det}_1 = |m''|. \]
According to Sylvester’s criterion, \( m'' < 0 \) is imposed.

ii) Second minor
The second minor is
\[ \text{Det}_2 = m'^2 \times \left[ -x_3 \left[ \frac{m'}{m''} r''_D(x_2) + r'^2_D(x_2) x_3 \right] + x_3^2 r'^2_D(x_2) \right]. \]
Thus Sylvester’s condition imposes, \( \forall x_2 \in [0, 1], x_3 \in \mathbb{R}^+ \),
\[ x_3 \left[ m'.m''.r''_D(x_2) + m'^2.r'^2_D(x_2) x_3 \right] > x_3^2 r'^2_D(x_2).m'^2 \]
\[ \iff m'.m''.r''_D(x_2) + m'^2.r'^2_D(x_2) x_3 > x_3^2 r'^2_D(x_2).m'^2 \]
\[ \iff m'.m''.r''_D(x_2) > 0 \]
\[ \iff r''_D(x_2) < 0, \]
since \( m' > 0 \) by assumption and previous condition (see i) imposes \( m'' < 0 \).

iii) Third minor
The third minor is computed using Sarrus’ rule. We obtain
\[ \text{Det}_3 = m'^3 \left\{ -x_3 \left[ \frac{m'}{m''} r''_D(x_2) + r'^2_D(x_2) x_3 \right] + x_3^2 r'^2_D(x_2) \right\}^2 \]
\[ + 2(-x_3 r'_D(x_2)) r'_D(x_2) \left[ -\frac{m'}{m''} + x_3[1 + r_D(x_2)] \right] \times (-[1 + r_D(x_2)]) \]
\[ - \left[ 1 + r_D(x_2) \right] \left[ -x_3 \left[ \frac{m'}{m''} r''_D(x_2) + r'^2_D(x_2) x_3 \right] \right] + r'_D(x_2) \left[ -\frac{m'}{m''} + x_3[1 + r_D(x_2)] \right] \]
\[ + x_3^2 r'^2_D(x_2) \left[ 1 + r_D(x_2) \right] \left[ -x_3 \left[ \frac{m'}{m''} r''_D(x_2) + r'^2_D(x_2) x_3 \right] \right] + r'_D(x_2) \left[ -\frac{m'}{m''} + x_3[1 + r_D(x_2)] \right] \]
\[ = m'^3 \left\{ 2x_3 r'^2_D(x_2) + r'^2_D(x_2) \left( -\frac{m'}{m''} + x_3[1 + r_D(x_2)] \right) \right\} \left( -\frac{m'}{m''} + x_3[1 + r_D(x_2)] \right) \]
\[ + x_3^2 r'^2_D(x_2) \left[ 1 + r_D(x_2) \right] \left[ -x_3 \left[ \frac{m'}{m''} r''_D(x_2) + r'^2_D(x_2) x_3 \right] \right] + r'_D(x_2) \left[ -\frac{m'}{m''} + x_3[1 + r_D(x_2)] \right]. \]

Considering \( m'' < 0 \) (see ii) and \( m' > 0 \) (by assumption), we have \( \frac{m''}{m'} < 0 \). Thus, assuming \( \forall x_2 \in [0, 1], r'_D(x_2) \neq 0 \), all terms in the brackets are strictly positive. Moreover \( m'^3 < 0 \)
and thus the condition \( \text{Det}_3 < 0 \) is satisfied.

Summary
The following assumptions

- \( m''(x) < 0 \) (Assumption (A5)),

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• $r_D^* < 0$ (Assumption (A6)),
• $r_D' \neq 0$ (Assumption (A7))

are sufficient to ensure that the Hessian matrix of $g$ is definite negative and therefore that $g$ is strictly concave with respect to the control variable $Ac_i^{(0)}$, the debt $Lx_i^{(0)}$ and the maturity transformation $\omega_i$.

Finally, under Assumption (A5), (A6), (A7) and (A8), the objective function $E \{ u_i \left[ v \left( P_i^{(1)} \right) \right] \}$ is strictly concave on a closed convex set, showing the unicity.

**Remark 1.** In the proof, we consider the total asset. However, since the total asset is the simple sum all assets, the proof can be written to state that the function $u \circ v$ is strictly concave with respect to one control variable in the asset side, setting the others constant.

**Remark 2.** Let us now then come back to the choice of working directly on the integrand. As we have shown, studying the concavity of a multivariate function involves studying its hessian. This is already quite complicated in case of the integrand. But the Hessian of the expectation matrix implies very complicated expressions, especially products of integral, apart from the first leading minor. The condition on this first leading minor is expressed as follows:

$$\int_{\mathbb{R}} (u \circ v \circ P)^{\prime\prime}(X, R) f_S(R) \, dR > 0.$$  \hfill (27)

Thus even in the case of the first leading minor, it seems difficult to obtain results except in particular cases of very simple density functions $f_S$. Moreover, if function $u \circ v$ is not strictly concave everywhere, one may expect the strict concavity coming from the integrating operation with respect to returns $R$. But this imposes to consider the complete expression of $P$ (with respect to $R$ and thus with respect to most of parameters) and thus prevents from using the dimension reduction operated by function $h$. That means that a hessian in high dimension must be considered.

**For Corollary [1]**

**Proof.** i) With $v = Id$, we have $v'(P) = 1$ and $v''(P) = 0$.

Therefore Assumption (A5) imposes $w''(P) < 0 \forall P$.

ii) $v : P \mapsto \ln(\exp(P) + 1)$. We have

$$v'(P) = \frac{e^P}{e^P + 1} \text{ and } v''(P) = \frac{e^P(e^P + 1) - e^P e^P}{(e^P + 1)^2} = \frac{e^P}{(e^P + 1)^2}.$$

Let us study the function $g = (u \circ v) = \ln(v)$. We have

$$g'(P) = \frac{v'(P)}{v(P)}.$$
and thus
\[ g''(P) = \frac{v''(P) v(P) - v^2(P)}{v^2(P)} = \frac{e^P}{(e^P + 1)^2} \times \frac{1}{\ln(e^P + 1)} - \frac{e^P}{(e^P + 1)^2} \times \frac{1}{(\ln(e^P + 1))^2} \]
\[ = \frac{e^P}{(e^P + 1)^2} \times \frac{1}{\ln(e^P + 1)} \times \left(1 - \frac{e^P}{\ln(e^P + 1)}\right).\]
The two first factors are positive whereas the third one is negative (since \(\forall x \in R^*_+, \ln(1 + x) < x\)). Consequently, \(\forall P, g''(P) < 0\). Hence the result.

\[\square\]

**For Corollary 2**

*Proof.* We have \(l(\omega, Lx) = \exp(\omega) \exp(Lx)\). For simplicity, we denote \(x = \omega\) and \(y = Lx\). We then have \(l(x, y) = \exp(x) \exp(y)\), giving
\[
\forall x \in [0, 1] \text{ and } \forall y \in \mathbb{R}^+, \frac{\partial^2 l}{\partial x^2} = \exp(x) \exp(y) > 0 \quad \text{and} \quad \frac{\partial^2 l}{\partial y^2} = \left(\frac{\partial^2 l}{\partial x \partial y}\right)^2.
\]
That shows that Assumption \((A8)\) is verified.

\[\square\]

**For Proposition 4**

*Proof.* The proof hinges on the Karuch, Kuhn, Tucker (KKT) Theorem. KKT Theorem provides necessary conditions on a local optimum of an optimization problem under equality and inequality constraints. We show that assuming \(\pi = 0\) is inconsistent.

KKT Theorem states that there exist coefficients \(\mu_i \geq 0\) such that a local maximum is a local maximum of the objective function \(\mathcal{L}\)
\[
\mathcal{L} = f - \mu_1(k^A Ax + k^\pi \pi - 1) + \mu_2 Ax + \mu_3 \pi - \mu_4(\pi - 1),
\]
where \(f\) is the initial objective function, i.e. the expected utility function in our case. Moreover, the \(\mu_i\) coefficients verify
\[
\forall i, \mu_i \times C_i = 0,
\]
where \(C_i\) is the \(i\)-th constraint.

The KKT conditions (first-order conditions) are
\[
\begin{cases}
\frac{\partial f}{\partial Ax} - \mu_1 k^A + \mu_2 = 0 \\
\frac{\partial f}{\partial \pi} + \mu_3 - \mu_1 k^\pi - \mu_4 = 0 \\
\mu_1(k^A Ax + k^\pi \pi - 1) = 0 \\
\mu_2 Ax = 0 \\
\mu_3 \pi = 0 \\
\mu_4(\pi - 1) = 0
\end{cases}
\]
Let us assume $\pi = 0$ and let us show that there is a contradiction. The last equation directly provides $\mu_4 = 0$. Since $f$ is strictly increasing, $Ax$ is necessarily strictly positive. Therefore, we have $\mu_2 = 0$. Thus, the first equation provides

$$
\mu_1 = \frac{\partial f}{\partial Ax} \times \frac{1}{k^A}.
$$

Injecting this result into the second equation gives

$$
\mu_3 = \frac{\partial f}{\partial Ax} \times \frac{k^\pi}{k^A} - \frac{\partial f}{\partial \pi} < 0.
$$

Equation (28) is in contradiction with the KKT theorem, stating that $\forall i, \mu_i \geq 0$. Therefore $\pi \neq 0$.

**For Proposition 6**

**Proof.** First, we need to recall that

$$
P_i^{(1)} = Ax_i^{(1)} + \ell_i^{(1)} + \sum_{j=1}^n \pi_{i,j} K_j^{(1)} + \sum_{j=1}^n \gamma_{i,j} L x_j^{(1)} - [1 + r_D(\omega_i)] \left( Ax_i^{(0)} + \ell_i^{(0)} + \sum_{j=1}^n \pi_{i,j} K_j^{(0)} + \sum_{j=1}^n \gamma_{i,j} L x_j^{(0)} - K_i^{(0)} \right)
$$

$$
= Ax_i^{(1)} + \ell_i^{(1)} - [1 + r_D(\omega_i)] \left( Ax_i^{(0)} + \ell_i^{(0)} - K_i^{(0)} \right) + \sum_{j=1}^n \pi_{i,j} \left( K_j^{(1)} - [1 + r_D(\omega_i)] K_j^{(0)} \right)
$$

$$
+ \sum_{j=1}^n \gamma_{i,j} \left( L x_j^{(1)} - [1 + r_D(\omega_i)] L x_j^{(0)} \right).
$$

**Case of a risk-neutral institution maximizing its position**

In this particular case, $u o v = Id$.

Then the derivative of the objective function with respect to $\pi_{i,j}$ is written

$$
\frac{\partial E(P_i^{(1)})}{\partial \pi_{ij}} = E \left[ K_j^{(1)} - [1 + r_D(\omega_i)] K_j^{(0)} \right] = E \left[ K_j^{(1)} - [1 + r_D(\omega_i)] K_j^{(0)} \right],
$$

where

$$
K_j^{(1)} = \max \left( \kappa_j \left( Ax_j^{(1)} + \ell_j^{(1)} \right) - L x_j^{(1)}, 0 \right).
$$

Let us now explicit the latter expression:

$$
\kappa_j \left( Ax_j^{(1)} + \ell_j^{(1)} \right) - L x_j^{(1)} = \kappa_j \left( Ax_j^{(0)} (1 + r_j) + \ell_j^{(0)} (1 + r_j) \right) - L x_j^{(0)} [1 + r_D(\omega_i)]
$$

$$
= a_j r_j + b_j
$$

by denoting $a_j = \kappa_j Ax_j^{(0)}$ and $b_j = \kappa_j \left( Ax_j^{(0)} + \ell_j^{(0)} (1 + r_j) \right) - L x_j^{(0)} [1 + r_D(\omega_i)]

Then,

$$
E \left[ K_j^{(1)} \right] = E \left[ \max \left( a_j r_j + b_j; 0 \right) \right] = \int_{-\infty}^{+\infty} (a_j r_j + b_j) f_{\phi_j}(r_j) \, dr_j
$$

$$
> [1 + r_D(\omega_i)] K_j^{(0)} \text{ by Assumption (A9).}
$$
Therefore $\frac{\partial E(P^{(1)}_i)}{\partial \pi_{ij}} > 0$.

The derivative with respect to $\gamma_{ij}$ is written

$$
\frac{\partial E(P^{(1)}_i)}{\partial \gamma_{ij}} = \mathbb{E}[L x^{*(1)}_j] - [1 + r_D(\omega_i)] L x^{(0)}_j
$$

$$
= \int_{-\infty}^{L x^{*(1)}_j - b_j} (a_j r_j + b_j) f_{S_j}(r_j) \, dr_j + \mathcal{L} x^{(1)}_j P \left( r_j > \frac{L x^{*(1)}_j - b_j}{a_j} \right) - [1 + r_D(\omega_i)] L x^{(0)}_j
$$

$$
= \int_{-\infty}^{L x^{*(1)}_j - b_j} (a_j r_j + b_j) f_{S_j}(r_j) \, dr_j + \mathcal{L} x^{(0)}_j [c_j (1 + r_D(\omega_j)) - (1 + r_D(\omega_i))]
$$

> 0 \text{ by Assumption (A10)}.

Indeed, by Assumption (A10), $w_j < w_i$ and $r_D(.)$ is a strictly decreasing function.

**General case**

Note that in case where $u \circ v = \text{Id}$, the returns of other interconnection assets than $j$ are eliminated. It is different otherwise. Let us at first consider the derivative with respect to $\pi_{ij}$. We have

$$
\frac{\partial u \left[ v \left( P^{(1)}_i \right) \right]}{\partial \pi_{ij}} = \frac{\partial (u \circ v)}{\partial P^{(1)}_i} \frac{\partial P^{(1)}_i}{\partial \pi_{ij}}.
$$

The first term $\frac{\partial (u \circ v)}{\partial P^{(1)}_i}$ can be interpreted as some kind of marginal utility (with function...). It depends on returns of all banks connected to $i$ and not only on the return of bank $j$. Let us denote

$$
h_{i1}(r_1, \ldots, r_j, \ldots, r_n) = \frac{\partial (u \circ v)}{\partial P^{(1)}_i}.
$$

Moreover, we have

$$
\frac{\partial P^{(1)}_i}{\partial \pi_{ij}} = K^{(1)}_j - [1 + r_D(\omega_i)] K^{(0)}_j = (a_j r_j + b_j; 0) - [1 + r_D(\omega_i)].
$$

Let us introduce

$$
h_{i2}(r_j) = \frac{\partial P^{(1)}_i}{\partial \pi_{ij}}.
$$
Thus,

\[
\frac{\partial E \left\{ u \left[ v \left( P_i^{(1)} \right) \right] \right\}}{\partial \pi_{ij}} = E \left[ \frac{\partial u(v(P_i^{(1)}))}{\partial \pi_{ij}} \right]
\]

\[
= \int_{r_1} \cdots \int_{r_j} \cdots \int_{r_n} h_{i1}(r_1, \ldots, r_j, \ldots, r_n) h_{i2}(r_j) f_S(r_1, \ldots, r_n) \, dr_n \cdots dr_j \cdots dr_1
\]

\[
= \int_{r_j} \left[ \int_{r_1} \cdots \int_{r_n} h_{i1}(r_1, \ldots, r_j, \ldots, r_n) h_{i2}(r_j) f_S(r_1, \ldots, r_n) \, dr_n \cdots dr_1 \right] \, dr_j
\]

\[
= \int_{r_j} h_{i2}(r_j) \left[ \int_{r_1} \cdots \int_{r_n} h_{i1}(r_1, \ldots, r_j, \ldots, r_n) f_S(r_1, \ldots, r_n) \, dr_n \cdots dr_1 \right] \, dr_j
\]

\[
= \int_{r_j} h_{i2}(r_j) \, w(r_j) \, dr_j
\]

\[
= \int_{-b_j \varepsilon_{ij}}^{+\infty} (a_j r_j + b_j) \, w(r_j) \, dr_j - \int_{-\infty}^{+\infty} \left[ 1 + r_D(\omega_i) \right] K^{(0)}_j w(r_j) \, dr_j
\]

\[
= \int_{-b_j \varepsilon_{ij}}^{+\infty} (a_j r_j + b_j) \, w(r_j) \, dr_j - \left[ 1 + r_D(\omega_i) \right] K^{(0)}_j \int_{-\infty}^{+\infty} w(r_j) \, dr_j,
\]

where

\[
w(r_j) = \int_{r_1} \cdots \int_{r_{j-1}} \int_{r_{j+1}} \cdots \int_{r_n} h_{i1}(r_1, \ldots, r_j, \ldots, r_n) f(r_1, \ldots, r_n) \, dr_n \cdots dr_{j+1} \cdots dr_{j-1} \cdots dr_1.
\]

Thus we have under Assumption (A11)

\[
\frac{\partial E \left\{ u \left[ v \left( P_i^{(1)} \right) \right] \right\}}{\partial \pi_{ij}} > 0.
\]

Let us now consider the case of \( \gamma_{ij} \). As in the previous case, the corresponding derivative is written

\[
\frac{\partial u \left[ v \left( P_i^{(1)} \right) \right]}{\partial \gamma_{ij}} = \frac{\partial (u \circ v)}{\partial P_i^{(1)}} \frac{\partial P_i^{(1)}}{\partial \gamma_{ij}}.
\]

The first term is equal to \( h_{i1}(r_j) \) and the second is \( \frac{\partial P_i^{(1)}}{\partial \gamma_{ij}} = h_{i3}(r_j) \). The same computation as in the case of \( \pi_{ij} \) yields

\[
\frac{\partial E \left\{ u \left[ v \left( P_i^{(1)} \right) \right] \right\}}{\partial \gamma_{ij}} = \int_{r_j} h_{i3}(r_j) \, w(r_j) \, dr_j.
\]

\( \square \)
For Proposition 7

Proof. Recall that we consider the following dynamic for $Ax$:

$$
\ln \left( \frac{Ax^{(1)}}{Ax^{(0)}} \right) \sim \mathcal{N}(\mu; \sigma) \quad \text{i.e.} \quad Ax^{(1)} = Ax^{(0)} e^{\mu + \sigma u} \text{ where } u \sim \mathcal{N}(0; 1).
$$

We have

$$
K^{(1)} = \max(Ax^{(1)} - L^*; 0) \quad \text{and} \quad Lx^{(1)} = \min(Ax^{(1)}; L^*).
$$

We define $\hat{u}$ such that $Ax^{(0)} e^{\mu + \sigma \hat{u}} = L^*$, i.e. $\hat{u} = \frac{1}{\sigma} \left( \ln \left( \frac{L^*}{Ax^{(0)}} \right) - \mu \right)$. We have

$$
\mathbb{E}_t(K) = \mathbb{E}_t(\max(Ax^{(1)} - L^*; 0))
$$

$$
= \left( Ax^{(0)} e^{\mu} \right) \int_{\hat{u}}^{\infty} e^{\sigma u} \varphi(u) \, du - L^* \int_{\hat{u}}^{\infty} \varphi(u) \, du
$$

$$
= \left( Ax^{(0)} e^{\mu} \right) \int_{\hat{u}}^{\infty} e^{\frac{1}{2}\sigma^2} e^{-\frac{1}{2}u^2} \, du - L^* (1 - \Phi(\hat{u}))
$$

$$
= \left( Ax^{(0)} e^{\mu} \right) \int_{\hat{u}+\sigma}^{\infty} \varphi(v) \, dv - L^* (1 - \Phi(\hat{u}))
$$

$$
= Ax^{(0)} e^{\mu + \frac{1}{2}\sigma^2} (1 - \Phi(\hat{u} - \sigma)) - L^* (1 - \Phi(\hat{u})).
$$

In the same way,

$$
\mathbb{E}_t(L) = \mathbb{E}_t(\min(Ax^{(1)}; L^*))
$$

$$
= \int - \infty \min(Ax^{(0)} e^{\mu + \sigma u}; L^*) \varphi(u) \, du
$$

$$
= \int_{-\infty}^{\hat{u}} Ax^{(0)} e^{\mu + \sigma u} \varphi(u) \, du + \int_{\hat{u}}^{\infty} L^* \varphi(u) \, du
$$

$$
= Ax^{(0)} e^{\mu + \frac{1}{2}\sigma^2} \int_{-\infty}^{\hat{u}+\sigma} \varphi(v) \, dv + L^* (1 - \Phi(\hat{u}))
$$

$$
= Ax^{(0)} e^{\mu + \frac{1}{2}\sigma^2} \Phi(\hat{u} - \sigma) + L^* (1 - \Phi(\hat{u})).
$$

We can verify that $\mathbb{E}_t(K + L) = \mathbb{E}_t(Ax^{(1)})$. Indeed,

$$
\mathbb{E}_t(K + L) = \mathbb{E}_t(K) + \mathbb{E}_t(L)
$$

$$
= Ax^{(0)} e^{\mu + \frac{1}{2}\sigma^2} (1 - \Phi(\hat{u} - \sigma) + \Phi(\hat{u} - \sigma)) - L^* (1 - \Phi(\hat{u}) - \Phi(\hat{u}) - 1)
$$

$$
= Ax^{(0)} e^{\mu + \frac{1}{2}\sigma^2}
$$

$$
= \mathbb{E}_t(Ax^{(1)}).
$$

\square
D Algorithm of network formation

In the case of two institutions ($n = 2$), the algorithm of network formation is the following:

1. **Optimization for institution 1 without interconnections.** Indeed in this first step, $K_2^{(0)}$ and $L_2^{(0)}$ are not known.
   We then have to optimize $E \left\{ u \left( P_1^{(1)}(Ax_1^{(0)}, A\ell_1^{(0)}, \omega_1) \right) \right\}$, where
   
   $$P_1^{(1)} = Ax_1^{(0)} (1 + r_1) + A\ell_1^{(0)} (1 + r_f) - Lx_1^{(0)} (1 + r_D(w_1)).$$

   This step provides: $Ax_1^{(0)}, A\ell_1^{(0)}, Lx_1^{(0)}, \omega_1$.

2. **Optimization for institution 2 with interconnections.** We have
   
   $$P_2^{(1)} \equiv \left( Ax_2^{(1)} + A\ell_2^{(1)} + \pi_2,1 \left( \kappa_1(Ax_1^{(1)} + A\ell_1^{(1)}) - Lx_1^{(0)} (1 + r_D(w_1)) \right) \right) +
   + \gamma_{2,1} \min \left( \kappa_1(Ax_1^{(1)} + A\ell_1^{(1)}) ; Lx_1^{(0)} (1 + r_D(w_1)) \right) - (1 + r_D(w_2)) Lx_2^{(0)},$$

   where $\kappa_1 = \frac{Lx_1^{(0)} + K_1^{(0)}}{Ax_1^{(0)} + A\ell_1^{(0)}}$ is the scaling factor compensating the absence of interconnections (it keeps the balance sheet of institution 1 balanced). Since $K_1^{(0)}$ has been obtained at step 1 under the assumption that institution 1 is not interconnected, here $\kappa_1 = 1$. But this will be corrected in further iterations.

   This step gives: $Ax_2^{(0)}, A\ell_2^{(0)}, Lx_2^{(0)}, \omega_2, \pi_2,1, \gamma_{2,1}$.

3. **Optimization for institution 1 with interconnections.** We have
   
   $$P_1^{(2)} \equiv \left( Ax_1^{(2)} + A\ell_1^{(2)} + \pi_1,2 \left( \kappa_2(Ax_2^{(2)} + A\ell_2^{(2)}) - Lx_2^{(0)} (1 + r_D(w_2)) \right) \right) +
   + \gamma_{1,2} \min \left( \kappa_2(Ax_2^{(2)} + A\ell_2^{(2)}) ; Lx_2^{(0)} (1 + r_D(w_2)) \right) - (1 + r_D(w_1)) Lx_1^{(0)},$$

   where $\kappa_2 = \frac{Lx_2^{(0)} + K_2^{(0)}}{Ax_2^{(0)} + A\ell_2^{(0)}}$. This step gives: $Ax_1^{(0)}, A\ell_1^{(0)}, Lx_1^{(0)}, \omega_1, \pi_1,2, \gamma_{1,2}$.

4. **New optimization for institution 2 with interconnections.**
   
   Note that at this step, $\kappa_1 = \frac{Lx_1^{(0)} + K_1^{(0)}}{Ax_1^{(0)} + A\ell_1^{(0)}} > 1$, since at the previous step the optimization has been done for institution 1 being interconnected.

5. **New optimization for institution 1 with interconnections.**

   Further iterations can be carried out if the variation in the estimates from one step to the next is higher than a predefined threshold.

E Calibration of external assets returns

Given the values of the mean net returns and the probability of default, let us derive the corresponding values of $\mu_1$ and $\mu_2$, as well as $\sigma_1$ and $\sigma_2$. We respectively denote
by \( GR \) and \( NR \) the gross and the net return. They of course verify the relationship \( NR = GR - 1 \). Thus
\[
E(NR) = E(GR) - 1 = \exp \left( \mu + \frac{\sigma^2}{2} \right) - 1.
\]

If we denote by \( m \) the empirical mean of the net return, we then have
\[
m = \exp \left( \mu + \frac{\sigma^2}{2} \right) - 1,
\]
that gives
\[
\mu = \ln(1 + m) - \frac{\sigma^2}{2}.
\] (29)

We need a second equation to find \( \mu \) and \( \sigma \). Of course we could use the expression
\[
\text{Var}(NR) = \text{Var}(GR) = (\exp(\sigma^2) - 1) \exp(2\mu + \sigma^2)
\]
i.e., by denoting by \( v \) the empirical variance of \( RN \),
\[
v = (\exp(\sigma^2) - 1) \exp(2\mu + \sigma^2).
\]
However, it is difficult to find reliable values for \( v \). If we consider banks data, only one return is available per year and thus the estimation of the variance is inaccurate. Another possibility is to compute the variance of the net returns of an index like the CAC 40. However, such an index is not representative of the external assets of a financial institution since it only contains shares. Moreover it does not take the hedging strategy of the institution into account.

Therefore we choose to derive the needed equation from the probability of default. This quantity is indeed easier to obtain. Actually, the usual rating for large banks corresponds to a probability of default about 0.1%. Considering an autarkic stylized bank with debt \( L \) and a total asset of \( A \) whose gross returns are log-normal of parameter \((\mu, \sigma)\), the probability of default is
\[
PD(\mu, \sigma) = \Phi \left( \frac{\ln \left( \frac{L}{A} \right) - \mu}{\sigma} \right).
\] (30)

Using in Equation (30) the expression of \( \mu \) in Equation (29), we obtain
\[
PD(\sigma) = \Phi \left( \frac{\ln \left( \frac{L}{A} \right) - \ln(1 + m) + \frac{\sigma^2}{2}}{\sigma} \right) = \Phi \left( \frac{\ln \left( \frac{L}{A(1 + m)} \right) + \frac{\sigma^2}{2}}{\sigma} \right).
\]

If we denote by \( p \) the empirical probability of default, the equation to solve is
\[
p = \Phi \left( \frac{\ln \left( \frac{L}{A(1 + m)} \right) + \frac{\sigma^2}{2}}{\sigma} \right) \iff \frac{\sigma^2}{2} - \sigma \Phi^{-1}(p) + \ln \left( \frac{L}{A(1 + m)} \right) = 0.
\]
This is a quadratic equation whose discriminant \( \Delta = (\Phi^{-1}(p))^2 - 2\ln \left( \frac{L}{A(1 + m)} \right) \). With chosen values of \( A, L \) and \( m \), \( \Delta > 0 \) and thus \( \sigma = \Phi^{-1}(p) + \sqrt{\Delta} \), since the other solution is strictly negative and thus unsuitable for a volatility. Finally the implied volatility is written

\[
\sigma(p) = \Phi^{-1}(p) + \sqrt{(\Phi^{-1}(p))^2 - 2\ln \left( \frac{L}{A(1 + m)} \right)}.
\]

We then obtain \( \mu \) using Equation (29).

\[ (31) \]

\section{Algorithm of equilibrium computation}

The computation of the equilibrium involving \( n \) financial institutions requires to solve up to \( 2^n \) linear systems with a brutal force approach (see Gouriéroux et al. (2012) for details) implying a total complexity in \( O(n^3 \times 2^n) \). The square term stems from the resolution of a linear system that requires to invert a \( n \times n \) matrix. Only a little gain can be obtained on this term. The exponential term comes from testing each possible situation: each institution is either alive or in default.

Instead, we adopt an heuristic algorithm. The key idea is to test the \( 2^n \) potential regimes in a "proper" order and to use the existence property to stop the algorithm as soon as one feasible solution is computed. The way of sorting the regimes relies on the fact that interconnections are small.

To do so, let us define regime \( r \) by \( d^r = (d^r_1, \ldots, d^r_n)' \), where \( d^r_i = -1 \) if institution \( i \) is in default and 1 otherwise (for \( i = 1, \ldots, n \)). We define a weight vector \( w = (w_1, \ldots, w_n) \), where \( w_i = (Ax_i + \ell_i - Lx_i^*)/Lx_i^* \) (for \( i = 1, \ldots, n \)). When \( w_i \) is positive, the external assets of financial institution \( i \) are higher than its nominal debt. Therefore, whatever the situations of other financial institutions, financial institution \( i \) is always alive at equilibrium. On the opposite, when \( w_i \) is negative, the financial institution needs a sufficient amount of inter-financial assets to be alive. In that case, since interconnections are assumed to be small, the (absolute) value of \( w_i \) indicates the likelihood (in a non-statistical sense) of default of institution \( i \). One can associate to regime \( r \) a score given by \( w.d^r \). The latter measures a distance from the situation without interconnections.

The idea is then to construct the list of all potential regimes. Actually, the regime with the lowest score can easily be derived from \( w \). This regime, labeled \( \bar{r} \), is defined by \( d^\bar{r}_i = I\{w_i > 0\} - I\{w_i \leq 0\} \) (for \( i = 1, \ldots, n \)). Then, one can consider deviations from regime \( \bar{r} \). Keeping in mind that assuming the default of an institution with positive weight is dead-end, one can switch the components of \( d^\bar{r} \) one by one to get new regimes with low scores. This mechanism of building new regimes from the previous one can be carried on until having sorted all the potential regimes (i.e. excluding ones were there exists \( i \) such that \( w_i > 0 \) and \( d_i = -1 \)).

Strictly speaking, the complexity of this algorithm is still in \( O(n^3 \times 2^n) \). However, the algorithm performs well in practice. For example, with 10 financial institutions having log-normal returns with random interconnections, the equilibrium lies in the 10 first tested regimes in most cases.

NB: If one remains concerned by exploring all the regimes (implying keeping the exponential term in the complexity), one solution is to stop the exploring after an arbi-
arbitrary (for instance $n$) number of regimes. When the exploration approach is stopped, a
pure numerical approach can be carried out (for instance routines for finding zeros or for
minimizing program).
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