SUBSPACE ARRANGEMENTS, CONFIGURATIONS OF LINEAR SPACES AND THE QUADRICS CONTAINING THEM

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ABSTRACT. A subspace arrangement in a vector space is a finite collection of vector subspaces. Similarly, a configuration of linear spaces in a projective space is a finite collection of linear subspaces. In this paper we study the degree 2 part of the ideal of such objects. More precisely, for a generic configuration of linear spaces \( \Lambda \) we determine \( HF(\Lambda, 2) \), i.e. the Hilbert function of \( \Lambda \) in degree 2.

1. Introduction

If \( V \) is an \( n+1 \) dimensional vector space then a subspace arrangement is a finite collection of vector subspaces of \( V \). This algebraic notion, and many properties of these objects, have been investigated from an algebraic point of view, see [Sid04, Sid07, BPS05]. Their geometric counterparts are also of interest and are obtained by projectivizing all the vector spaces involved. Doing this one obtains, in \( \mathbb{P}^n = \mathbb{P}(V) \), a finite collection of linear subspaces. Such a collection is referred to as a configuration of linear spaces.

Subspace arrangements arise in many contexts and in many applications and hence the same holds true for their alter ego, i.e. configurations of linear spaces. Derksen [Der07] showed applications to Statistics via Generalized Components Analysis. Also, Ma et al. gave applications to data modeling and segmentation in [MYDF08]. Moreover, in [CGG05] configurations of linear spaces and their Hilbert functions were shown to be related to the study of Segre-Veronese varieties and their higher secant varieties. Another application, this time to incidence properties of rational normal curves and linear spaces, is studied in [CC07, CC09]. In this paper we exhibit yet another application, relating configurations of linear spaces to the study of a special kind of polynomial decomposition (for more on polynomial decompositions see [Car06] and [Car05]).

As the Hilbert function of a configuration of linear spaces is of interest, we begin by recalling what is known about it. Derksen and Sidman
have discovered many interesting results about the Castelnuovo-Mumford regularity (CM-regularity) of the intersection of ideals generated by generic linear forms. In our context these results give bounds on the CM-regularity for the ideal of a generic configuration of linear spaces $\Lambda$. Hence, one knows an integer $d_0$ such that for $d \geq d_0$ the Hilbert function and the Hilbert polynomial for the ideal of $\Lambda$ agree, i.e. $HF(\Lambda, d) = hp(\Lambda, d)$. Moreover Derksen, in [Der07], gives an explicit formula for $hp(\Lambda, d)$ when the configuration of linear spaces is generic.

Thus, for a generic configuration of linear spaces $\Lambda$, we need only determine a finite number of values of $HF(\Lambda, d)$ in order to have complete knowledge of the Hilbert function of $\Lambda$. But, in general, this knowledge is available in only a few basic situations, i.e. when $\dim \Lambda = 0, 1$. When $\dim \Lambda = 0$ we are dealing with a generic set of points and for these the Hilbert function is known to be $HF(\Lambda, d) = \min\left\{\binom{n+d}{d}, hp(\Lambda, d)\right\}$ for all $d$. When $\dim \Lambda = 1$ (where it is enough to consider a generic union only of lines i.e. we need not consider lines and points) the problem is considerably harder. The first complete answer was given by Hartshorne and Hirschowitz in [HH82]. There it is shown that if $\Lambda$ is a generic collection of lines in $\mathbb{P}^n$ ($n > 2$) then

$$HF(\Lambda, d) = \min\left\{\binom{n+d}{d}, hp(\Lambda, d)\right\}.$$  

The proof they present is long and non-trivial and makes use of Castelnuovo’s sequence (la methode d’Horace) and degeneration techniques. When $\dim \Lambda > 1$ we are not aware of any general results in the literature.

Inasmuch as the general problem of describing the Hilbert function of any generic configuration of linear spaces seems extremely difficult, one may consider some distinct subproblems. One is to consider a generic configuration of linear spaces (with no restriction on the dimension or on the number of components) and determine its Hilbert function in the first unknown degree, i.e. in degree 2. Another is to consider families of generic configurations of a special kind, e.g. impose bounds on the dimension or on the number of components. The first subproblem is the subject of this paper. We postpone a discussion of the second subproblem to another paper (see [CCG09]) as it uses completely different techniques which are considerably inspired by [HH82] and involve multiple inductions coupled with Castelnuovo’s sequence.

In this paper we follow an approach which uses a fiber argument that reduces the problem to a chain of numerical inequalities. Using these ideas we are able to completely determine the Hilbert function,
in degree 2, for a generic configuration of linear spaces (see Theorem 4.3).

The paper is organized in the following way. Section 2 lays out some easy observations about certain Fano varieties. Sections 3 and 4 contain the technical heart of the paper. Section 3 deals with “small” generic linear configurations, i.e. generic configurations for which the dimension of the components are such that no two components intersect. A case by case argument shows that the number of quadrics containing such a generic configuration is exactly what one would expect.

Section 4 deals with “large” generic configurations, i.e. generic configurations for which the dimensions of the components force (some) of the components to intersect. There is again an “expected” behavior, but it is more complicated to express. We were able to calculate the Hilbert function in this case as well and thus show that the expression we had for the expected behavior was correct.

Finally, in Section 5, we give an application of our results to the problem of writing homogeneous polynomials in \( \mathbb{C}[x_0, \ldots, x_n] \) as a sum of polynomials in fewer variables.

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2. Notation and preliminary results

We say that \( \Lambda \subset \mathbb{P}^n \) is a configuration of linear spaces of \( \mathbb{P}^n \) if \( \Lambda \) is a finite union of linear spaces. We write

\[
\Lambda = \Lambda_1 + \cdots + \Lambda_s \subset \mathbb{P}^n,
\]

where \( \Lambda_i \simeq \mathbb{P}^{m_i} \) is a linear space of dimension \( m_i \) (\( 0 \leq m_i < n \)) and \( \Lambda_i \neq \Lambda_j \) for \( i \neq j \). We set \( L(\Lambda) = (m_1, \ldots, m_s) \) and call this the weight vector of \( \Lambda \).

Given a weight vector \( L \), the configurations of linear spaces having weight vector \( L \) are parameterized by

\[
\mathcal{D}_L = \mathbb{G}(m_1, \mathbb{P}^n) \times \ldots \times \mathbb{G}(m_s, \mathbb{P}^n).
\]

We notice that

\[
\dim \mathcal{D}_L = \sum_{i=1}^{s} (m_i + 1)(n - m_i).
\]
A configuration of linear spaces of \( \mathbb{P}^n \) is \textit{generic} if its components \( \Lambda_i \) are generic linear spaces of \( \mathbb{P}^n \), i.e., if \( \Lambda = \Lambda_1 + \cdots + \Lambda_s \) belongs to a specified open not empty subset of \( \mathcal{D}_L \).

Let \( S = \mathbb{C}[x_0, \ldots, x_n] = \bigoplus_{i \geq 0} S_i \) denote the coordinate ring of \( \mathbb{P}^n \). For any weight vector \( L \) we consider the following incidence correspondence

\[
\Sigma_L = \left\{ (Q, \Lambda) : Q \supset \Lambda \right\} \subset \mathbb{P}(S_2) \times \mathcal{D}_L
\]

and the projection maps

\[
\phi_L : \Sigma_L \rightarrow \mathbb{P}(S_2), \psi_L : \Sigma_L \rightarrow \mathcal{D}_L.
\]

(Whenever \( L \) is clear from the context we will drop the subscript \( L \).)

To study the Hilbert function, in degree 2, of a generic configuration of linear spaces having weight vector \( L \), we first notice that

\[
\dim(I_\Lambda)_2 = \dim \psi_L^{-1}(\Lambda) + 1
\]

where \( \Lambda \in \mathcal{D}_L \) is a generic point and \((I_\Lambda)_2\) is the ideal of \( \Lambda \) in degree 2. So, if we know \( \dim(I_\Lambda)_2 \) we can easily determine the Hilbert function of \( \Lambda \) in degree 2 since \( HF(\Lambda, 2) = \binom{n+2}{2} - \dim(I_\Lambda)_2 \).

**Note:** We will often prefer the ideal notation to the Hilbert function notation, as the first will be easier to use in our situation.

Recall the following theorem (see [Har92, Theorem 22.13]), that gives the dimension of the Fano variety of \( m \)-planes on a smooth quadric hypersurface.

**Theorem 2.1.** The variety \( F_{m,n-1} \subset G(m, \mathbb{P}^n) \) of \( m \)-planes on a smooth \((n-1)\)-dimensional quadric hypersurface is smooth and if \( m < \frac{n-1}{2} \) it is irreducible.

When \( m \leq \frac{n-1}{2} \) it has dimension

\[
\dim F_{m,n-1} = (m + 1) \left( n - 1 - \frac{3m}{2} \right),
\]

and it is empty otherwise.

\[\square\]

**Corollary 2.2.** The variety of \( m \)-planes on a quadric \( Q \subset \mathbb{P}^n \) of rank \( r > 2n - 2m \) is empty.

**Proof.** For \( r = n+1 \) the quadric \( Q \) is smooth and the conclusion follows from Theorem 2.1.

Let \( r < n + 1 \). Recall that \( Q \) is the cone over a smooth quadric \( \tilde{Q} \subset \mathbb{P}^{n'} \), \( n' = r - 1 \) having vertex \( V \simeq \mathbb{P}^{n-r} \). Hence, each \( \Lambda \simeq \mathbb{P}^m \subset Q \).
is projected from the vertex \( V \) into an \( m' \)-dimensional linear subspace of \( \tilde{Q} \), where
\[
m' = m - \dim(\Lambda \cap V) - 1 \geq m - (n - r) - 1.
\]
Since
\[
m' - \frac{n' - 1}{2} \geq m - (n - r) - 1 - \frac{(r - 1) - 1}{2} = \frac{2m - 2n + r}{2} > 0,
\]
by Theorem 2.1 there are no \( m' \)-planes on \( \tilde{Q} \), and the conclusion follows.

\[\square\]

**Lemma 2.3.** Let \( m_1 \geq \cdots \geq m_s \geq 0 \), and \( m_1 + m_2 < n \). Let
\[
\Lambda = \Lambda_1 + \cdots + \Lambda_s \subset \mathbb{P}^n
\]
be a generic configuration of linear spaces with weight vector \( L = (m_1, \ldots, m_s) \) and let \( r \leq 2n - 2m_1 \).

If \( r \) is even (say \( r = 2p \)) and one has:
\[
\binom{n + 2}{2} - \binom{n + 2 - r}{2} \leq \sum_{m_i \leq p - 1} \binom{m_i + 2}{2} + \frac{p}{2} \sum_{m_i \geq p} (2m_i + 3 - p),
\]
or if \( r \) is odd (say \( r = 2p + 1 \)) and one has
\[
\sum_{m_i \leq p - 1} \binom{m_i + 2}{2} + \frac{1}{2} \sum_{m_i \geq p} (2m_i + 2 + p(2m_i + 1 - p)),
\]
then \((I_\Lambda)_2\) does not contain any rank \( r \) quadric.

**Proof.** We let \( Q \) be a rank \( r \) quadric in \( I_\Lambda \). We want to compute \( \phi^{-1}_L(Q) \). To do this we have to describe all the \( \mathbb{P}^{m_i} \subset Q \) for \( i = 1, \ldots, s \). Since \( Q \) is a cone over a smooth quadric \( \tilde{Q} \subset \mathbb{P}^{r-1} \) having vertex a \( \mathbb{P}^{n-r} \), the generic \( \mathbb{P}^{m_i} \subset Q \) is projected from the vertex onto an \( m'_i \)-dimensional linear subspace of \( \tilde{Q} \).

**Claim:** If \( m_i \leq \frac{(r-1)-1}{2} \) then \( m_i = m'_i \).

**Proof of Claim:** Notice that \( \tilde{Q} \) is smooth in a \( \mathbb{P}^{r-1} \). By Theorem 2.1 this implies that \( m'_i \leq \frac{(r-1)-1}{2} \). Hence, using the genericity of \( \mathbb{P}^{m_i} \) and the formula for the intersection of linear spaces of \( \mathbb{P}^n \) we get that \( \mathbb{P}^{m_i} \cap V \) has dimension
\[
\leq \max \{m_i + (n - r) - n; 0\} = \max \{m_i - r; 0\}
\]
\begin{align*}
\leq \max\{\frac{(r - 1) - 1}{2} - r; 0\} = 0
\end{align*}

Thus the projection of $\mathbb{P}^m_i$ from $V$ maintains its dimension, i.e. $m'_i = m_i$.

It follows from Theorem 2.1 that the family of $m_i$-planes in $\tilde{Q}$ has dimension
\begin{align*}
(m_i + 1) \left( r - 2 - \frac{3}{2}m_i \right).
\end{align*}

Moreover, the $\mathbb{P}^m_i \subset Q$ projected onto the same $m_i$-plane form a family of dimension
\begin{align*}
(m_i + 1)(n - r + 1).
\end{align*}

(To see why this is so, note that these $m_i$-dimensional linear spaces all lie inside the linear space spanned by $V$ and $\mathbb{P}^m_i$. That span has dimension $n - r + m'_i + 1 = n - r + m_i + 1$ and since $\dim G(m_i, \mathbb{P}^{n-r+m_i+1}) = (m_i + 1)(n - r + 1)$ the statement follows.)

For $m_i > \frac{r - 2}{2}$, by Theorem 2.1 $\tilde{Q}$ does not contain any $\mathbb{P}^m_i$. Hence $m'_i$ is the biggest natural number $m'$ such that there exists a $\mathbb{P}^m' \subset \tilde{Q}$. Clearly we have
\begin{align*}
m' = \left\lfloor \frac{r - 2}{2} \right\rfloor.
\end{align*}

In this case the $\mathbb{P}^m' \subset \tilde{Q}$ form a family of dimension
\begin{align*}
(m'_i + 1) \left( r - 2 - \frac{3}{2}m'_i \right)
\end{align*}

and for each of these $\mathbb{P}^m'$, the family of $\mathbb{P}^m_i \subset Q$ projecting onto it has dimension
\begin{align*}
(m_i + 1)(n - r + m' - m_i + 1).
\end{align*}

Thus we get
\begin{align*}
\dim \phi^{-1}_L(Q) &= \sum_{m_i \leq \frac{r - 2}{2}} \left( (m_i + 1) \left( r - 2 - \frac{3}{2}m_i \right) + (m_i + 1)(n - r + 1) \right) \\
+ &\sum_{m_i > \frac{r - 2}{2}} \left( (m'_i + 1) \left( r - 2 - \frac{3}{2}m'_i \right) + (m_i + 1)(n - r + m' - m_i + 1) \right) \\
= &\sum_{m_i \leq \frac{r - 2}{2}} (m_i + 1) \left( n - \frac{3}{2}m_i - 1 \right) \\
+ &\sum_{m_i > \frac{r - 2}{2}} \left( (m'_i + 1) \left( r - 1 - \frac{3}{2}m' + m_i \right) + (m_i + 1)(n - r - m_i) \right).
Denoting by $Q_r$ the family of rank $r$ quadrics we get
\[
\dim \phi_L^{-1}(Q_r) = \binom{n+2}{2} - \binom{n-r+2}{2} - 1 + \dim \phi_L^{-1}(Q).
\]

It follows that
\[
\dim \phi_L^{-1}(Q_r) - \dim \mathcal{D}_L
\]
\[
= \left[ \binom{n+2}{2} - \binom{n-r+2}{2} - 1 + \dim \phi_L^{-1}(Q) \right] - \sum_{i=1}^{s} (m_i + 1)(n - m_i).
\]

If $r$ is even (say $r = 2p$ and hence $m' = p - 1$) we obtain
\[
\dim \phi_L^{-1}(Q_r) - \dim \mathcal{D}_L
\]
\[
= \binom{n+2}{2} - \binom{n-2p+2}{2} - 1 + \sum_{m_i \leq p-1} (m_i + 1) \left( n - \frac{3}{2}m_i - 1 \right)
\]
\[
+ \sum_{m_i \geq p} \left( p \left( 2p + m_i - \frac{3}{2}p + \frac{1}{2} \right) + (m_i + 1)(n - 2p - m_i) \right)
\]
\[
- \sum_{i=1}^{s} (m_i + 1)(n - m_i)
\]
\[
= \binom{n+2}{2} - \binom{n-2p+2}{2} - 1 - \sum_{m_i \leq p-1} \binom{m_i + 2}{2} - \frac{p}{2} \sum_{m_i \geq p} (2m_i + 3 - p).
\]

On the other hand, for $r$ odd (say $r = 2p + 1$ and so $m' = p - 1$) we obtain
\[
\dim \phi_L^{-1}(Q_r) - \dim \mathcal{D}_L
\]
\[
= \binom{n+2}{2} - \binom{n-(2p+1)+2}{2} - 1 + \sum_{m_i \leq p-1} (m_i + 1) \left( n - \frac{3}{2}m_i - 1 \right)
\]
\[
+ \sum_{m_i \geq p} \left( p \left( 2p + 1 + m_i - \frac{3}{2}p + \frac{1}{2} \right) + (m_i + 1)(n - 2p - 1 - m_i) \right)
\]
\[
- \sum_{i=1}^{s} (m_i + 1)(n - m_i)
\]
\[
= \binom{n+2}{2} - \binom{n-2p+1}{2} - 1
\]
\[- \sum_{m_i \leq p-1} \binom{m_i + 2}{2} - \frac{1}{2} \sum_{m_i \geq p} (2m_i + 2 + p(2m_i + 1 - p)).\]

Hence
\[
\dim \phi_L^{-1}(Q_r) - \dim D_L < 0,
\]
and the conclusion follows.

\[\square\]

3. DISJOINT SPACES

We begin by considering configurations of linear spaces with non intersecting components, i.e.
\[
\Lambda = \Lambda_1 + \cdots + \Lambda_s, \text{ and } \Lambda_i \cap \Lambda_j = \emptyset \text{ for } i \neq j.
\]
In this situation it is easy to guess the expected behavior: if too many components are involved, then no quadric is expected to contain the configuration. We now give a precise statement and proof of this fact.

**Proposition 3.1.** Let \(s \geq 2, m_1 \geq \cdots \geq m_s \geq 0,\) and \(m_1 + m_2 < n\) (hence \(\Lambda_i \cap \Lambda_j = \emptyset \text{ for } i \neq j\)).

Let
\[
\Lambda = \Lambda_1 + \cdots + \Lambda_s \subset \mathbb{P}^n
\]
be a generic configuration of linear spaces having weight vector \((m_1, \ldots, m_s)\).

If
\[
\sum_{i=1}^{s} \binom{m_i + 2}{2} \geq \binom{n + 2}{2},
\]
then \(\dim(I_{\Lambda})_2 = 0\).

**Proof.** We will show that there are no quadrics of rank \(r\) in \((I_{\Lambda})_2\) for \(1 \leq r \leq n + 1\) and that will certainly prove the proposition.

First observe that for \(r > 2n - 2m_1\), the conclusion follows immediately from Corollary 2.2. So, we are reduced to considering the case where \(r \leq 2n - 2m_1\).

For \(r\) in this range and \(r\) even (say \(r = 2p\)) set
\[
(1) \quad f(p) = \sum_{m_i \leq p-1} \binom{m_i + 2}{2} + \frac{p}{2} \sum_{m_i \geq p} (2m_i + 3 - p).
\]

For \(r\) in this range and \(r\) odd (say \(r = 2p + 1\)) set
\[
(2) \quad f(p) = \sum_{m_i \leq p-1} \binom{m_i + 2}{2} + \frac{1}{2} \sum_{m_i \geq p} (2m_i + 2 + p(2m_i + 1 - p)).
\]
Notice that these expressions for $f(p)$ are precisely the expressions that appeared in the inequalities of Lemma 2.3. Exactly for that reason, if we can show that for $1 \leq r \leq \min\{2n - 2m_1; n + 1\}$ we have

$$
\left(\frac{n+2}{2}\right) - \left(\frac{n+2-r}{2}\right) \leq f(p),
$$

the conclusion will follow immediately from Lemma 2.3.

First notice that an easy computation gives us that

$$
f(p + 1) - f(p) = \sum_{m_i \geq p} (m_i - p + 1) \quad \text{when } r = 2p,
$$

and

$$
f(p + 1) - f(p) = \sum_{m_i \geq p} (m_i - p) \quad \text{when } r = 2p + 1.
$$

**Case 1:** $r = 2p$ and $n \leq 2m_1$.

In this case $2n - 2m_1 < n + 1$ and hence $\min\{2n - 2m_1; n + 1\} = 2n - 2m_1$. So, it will be enough to prove (3) for $1 \leq r \leq 2n - 2m_1$, that is for $1 \leq p \leq n - m_1$.

Notice that we have $m_1 - p \geq m_1 - n + m_1 \geq 0$, and hence $m_1 \geq p$.

We proceed by induction on $n - m_1 - p$.

If $n - m_1 - p = 0$ then $p = n - m_1$. Since $m_1 \geq p$ we have $m_2 \leq n - m_1 - 1 = p - 1$.

Hence, recalling that $\sum_{i=1}^{s} \left(\frac{m_i + 2}{2}\right) \geq \left(\frac{n+2}{2}\right)$, we have

$$
f(n - m_1) - \left(\frac{n+2}{2}\right) + \left(\frac{n+2-2p}{2}\right)
= \sum_{i \geq 2} \left(\frac{m_i + 2}{2}\right) + \frac{n-m_1}{2}(2m_1 + 3 - n + m_1)
- \left(\frac{n+2}{2}\right) + \left(\frac{2m_1 - n + 2}{2}\right)
\geq -\left(\frac{m_1 + 2}{2}\right) + \frac{n-m_1}{2}(3m_1 + 3 - n) + \left(\frac{2m_1 - n + 2}{2}\right) = 0.
$$

This finishes the case $n - m_1 - p = 0$.

Now suppose that $n - m_1 - p > 0$, i.e. $p < n - m_1$. By (4) we have

$$
f(p) - \left(\frac{n+2}{2}\right) + \left(\frac{n+2-r}{2}\right) = \quad \text{when } r = 2p + 1.
$$
\[ f(p+1) - \sum_{m_i \geq p} (m_i - p + 1) - \binom{n+2}{2} + \binom{n+2-2p}{2} \]  

(*)

By induction

\[ f(p+1) \geq \binom{n+2}{2} - \binom{n+2-2p+1}{2}. \]

Hence, induction gives us that

\[ (*) \geq \binom{n+2}{2} - \binom{n-2p}{2} - \sum_{m_i \geq p} (m_i - p + 1) - \binom{n+2}{2} + \binom{n+2-2p}{2} \]

\[ = (2n - 4p + 1) - \sum_{m_i \geq p} (m_i - p + 1). \]

We have thus proved

\[ f(p) - \binom{n+2}{2} + \binom{n+2-r}{2} \geq (2n - 4p + 1) - \sum_{m_i \geq p} (m_i - p + 1). \]

Clearly if \( \sum_{m_i \geq p} (m_i - p + 1) \leq 2n - 4p + 1 \), we get

\[ f(p) - \binom{n+2}{2} + \binom{n+2-r}{2} \geq 0, \]

and we are done.

It remains to see when this does not happen. That is the content of the following Claim.

**Claim:** Continuing with the hypothesis of Case 1 as well as the induction hypothesis that \( n - m_1 - p > 0 \), if

\[ (2n - 4p + 1) - \sum_{m_i \geq p} (m_i - p + 1) < 0 \]

then \( m_4 \geq p \).

**Note:** Once this claim is proved we need only show that (3) holds if \( m_4 \geq p \).

**Proof of Claim:** Since \( m_1 \geq p \) we first consider the possibility that \( m_2 < p \). In this case,

\[ (2n-4p+1)-\sum_{m_i \geq p} (m_i-p+1) = (2n-4p+1) - (m_1 - p + 1) = 2n - m_1 - 3p. \]

Since \( p \leq n - m_1 - 1 \) we have

\[ 2n - m_1 - 3p \geq 3 - n + 2m_1 > 0 \]

which is a contradiction.
Now suppose \( m_1 \geq m_2 \geq p \) and \( m_3 < p \). In like manner we get that

\[
(2n - 4p + 1) - \sum_{m_i \geq p} (m_i - p + 1) = 2n - 2p - m_1 - m_2 - 1.
\]

As above

\[
2n - 2p - m_1 - m_2 - 1 \geq m_1 - m_2 + 1 \geq 1 > 0
\]

again giving a contradiction.

Finally, suppose that \( m_1 \geq m_2 \geq m_3 \geq p \) and \( m_4 < p \). Then

\[
(2n - 4p + 1) - \sum_{m_i \geq p} (m_i - p + 1) = 2n - p - 2 - m_1 - m_2 - m_3.
\]

As before, with \( p \leq n - m_1 - 1 \) and, in addition, using \( n \geq m_1 + m_2 + 1 \), we get that

\[
(2n - 4p + 1) - \sum_{m_i \geq p} (m_i - p + 1) \geq m_1 - m_3 \geq 0
\]

which is again a contradiction.

Thus, \( m_4 \geq p \) as we wanted to show.

We return to the induction proof. However, now we can also assume that

\[
(6) \quad 2n - 4p + 1 - \sum_{m_i \geq p} (m_i - p + 1) < 0
\]

and hence that \( m_4 \geq p \).

It follows from (1) and (6) that

\[
f(p) - \binom{n+2}{2} - \binom{n+2-2p}{2}
\]

\[
= \sum_{m_i \leq p-1} \binom{m_i + 2}{2} + \frac{p}{2} \sum_{m_i \geq p} (2m_i + 3 - p) + p(2p - 2n - 3)
\]

\[
\geq \frac{p}{2} \sum_{m_i \geq p} (2m_i + 3 - p) + p(2p - 2n - 3)
\]

\[
= \frac{p}{2} \sum_{m_i \geq p} (2(m_i - p) + (p + 1)) + p(2p - 2n - 3)
\]

\[
\geq p(2n - 4p + 2) + \frac{p}{2} \cdot 4(p + 1) + p(2p - 2n - 3) = p \geq 0.
\]

**Case 2:** \( r = 2p \) and \( n \geq 2m_1 + 1 \). (This is very similar to Case 1 and so we will omit many details.)

In this case \( 2n - 2m_1 \geq n + 1 \), so we will prove (3) for \( 1 \leq r \leq n + 1 \), that is for \( 1 \leq p \leq \left\lfloor \frac{n+1}{2} \right\rfloor \).
We proceed by induction on $\left\lfloor \frac{n+1}{2} \right\rfloor - p$.

If $p = \left\lfloor \frac{n+1}{2} \right\rfloor$, since $n \geq 2m_1 + 1$, we have $m_1 \leq p - 1$, hence $f \left( \left\lfloor \frac{n+1}{2} \right\rfloor \right) = \sum_{i=1}^{n/2} \binom{m_i + 2}{2}$ and since $\sum_{i=1}^{n/2} \binom{m_i + 2}{2} \geq \binom{n+2}{2}$, then (3) holds for $p = \left\lfloor \frac{n+1}{2} \right\rfloor$.

Now assume $p < \left\lfloor \frac{n+1}{2} \right\rfloor$. As in the previous case, if

$$\sum m_i \geq p \left( m_i - p + 1 \right) \leq 2n - 4p + 1,$$

then (3) holds, hence let

$$(7) \quad \sum_{m_i \geq p} (m_i - p + 1) > 2n - 4p + 1,$$

and note that, since $p < \left\lfloor \frac{n+1}{2} \right\rfloor$, then $2n - 4p + 1 \geq 0$, so at least $m_1 \geq p$. If $m_i \geq p$ we have

$$(8) \quad (2m_i + 3 - p) \geq (m_i - p + 1) \cdot \frac{2m_1 + 3 - p}{m_1 - p + 1},$$

in fact

$$(2m_i + 3 - p)(m_1 - p + 1) - (m_i - p + 1)(2m_1 + 3 - p) = (m_1 - m_i)(p + 1) \geq 0.$$ Thus by (4) and (8) we have

$$f(p) = \sum_{m_i \leq p-1} \binom{m_i + 2}{2} + \frac{p}{2} \sum_{m_i \geq p} (2m_i + 3 - p)$$

$$\geq \frac{p}{2} \sum_{m_i \geq p} (m_i - p + 1) \cdot \frac{2m_1 + 3 - p}{m_1 - p + 1}$$

$$\geq \frac{p}{2} \left( 2n - 4p + 2 \right) \cdot \frac{2m_1 + 3 - p}{m_1 - p + 1}.$$ Since $\binom{n+2}{2} - \binom{n+2-2p}{2} = p(2n + 3 - 2p)$ if we prove that

$$(9) \quad p(n - 2p + 1) \cdot \frac{2m_1 + 3 - p}{m_1 - p + 1} \geq p(2n + 3 - 2p),$$

then (3) holds. By easy computation we have that (9) holds if and only if

$$p(n - 2m_1 - 1) - p + n - m_1 \geq 0,$$

so, since $p \leq n - m_1$ and $n \geq 2m_1 + 1$, the conclusion follows.

Case 3: $r = 2p + 1$ and $n \leq 2m_1 + 1$.

In this case $2n - 2m_1 \leq n + 1$, thus we will prove (3) for $1 \leq r \leq 2n - 2m_1$, that is for $0 \leq r \leq n - n_1 - 1$.

We have $m_1 - p \geq m_1 - n + m_1 + 1 \geq 0$, hence $m_1 \geq p$. 

We will work by induction on \( n - m_1 - 1 - p \).
Let \( p = n - m_1 - 1 \). Since \( n \geq m_1 + m_2 + 1 \), we have \( m_2 \leq p \). Let \( \alpha \) denote the number of \( m_i \) equal to \( p \).

We have (recalling the definition (2) and that \( \sum_{i=1}^{s} \binom{m_i+2}{2} \geq \binom{n+2}{2} \))

\[
f(p) - \binom{n+2}{2} + \binom{n+1-2p}{2}
\]

\[
\geq - \sum_{m_i \geq p} \binom{m_i+2}{2} + \frac{1}{2} \sum_{m_i \geq p} (2m_i + 2 + p(2m_i + 1 - p)) + \binom{n+1-2p}{2}
\]

\[
= - \left( \frac{m_1 + 2}{2} \right) - \alpha \left( \frac{p + 2}{2} \right) + m_1 + 1 + \frac{p}{2} (2m_1 + 1 - p) + \alpha (p + 1 + \frac{p}{2} (2p + 1 - p))
\]

\[
+ \binom{n+1-2p}{2} = m_1 - p + 1.
\]

Hence, since \( p = n - m_1 - 1 \), and \( n \leq 2m_1 + 1 \), we get \( m_1 - p + 1 \geq 1 \), and (3) holds for \( p = n - m_1 - 1 \).

Now let \( p \leq n - m_1 - 2 \).

By (5) and by the inductive hypothesis we have

\[
f(p) - \binom{n+2}{2} + \binom{n+1-2p}{2}
\]

\[
= f(p + 1) - \sum_{m_i \geq p} (m_i - p) - \binom{n+2}{2} + \binom{n+1-2p}{2}
\]

\[
\geq \binom{n+2}{2} - \binom{n-2p-1}{2} - \sum_{m_i \geq p} (m_i - p) - \binom{n+2}{2} + \binom{n+1-2p}{2}
\]

\[
= 2n - 4p - 1 - \sum_{m_i \geq p} (m_i - p).
\]

Hence for \( \sum_{m_i \geq p} (m_i - p) \leq 2n - 4p - 1 \), we obtain the conclusion.

Let

\[(10) \quad \sum_{m_i \geq p} (m_i - p) \geq 2n - 4p,\]

and let \( \alpha \) denote the number of \( m_i \) equal or bigger than \( p \). Recalling that \( m_1 + m_2 + 1 \leq n \leq 2m_1 + 1 \) and \( p \leq n - m_1 - 2 \), it is easy to show that \( \alpha \geq 5 \). Hence by (2), (10) and by eliminating the \( m_i < p \), we have

\[
f(p) - \binom{n+2}{2} + \binom{n+1-2p}{2}
\]

\[
\geq \frac{1}{2} \sum_{m_i \geq p} (2m_i + 2 + p(2m_i + 1 - p)) + 2p^2 - 2np - p - n - 1
\]
\[ \sum_{m_i \geq p} \left( (m_i - p)(p + 1) + \frac{1}{2}(p^2 + 3p + 2) \right) + 2p^2 - 2np - p - n - 1 \]
\[ \geq (2n - 4p)(p + 1) + \frac{\alpha}{2}(p^2 + 3p + 2) + 2p^2 - 2np - p - n - 1 \]
\[ \geq n - 2p^2 - 5p - 1 + \frac{5}{2}(p^2 + 3p + 2) > 0. \]

**Case 4:** \( r = 2p + 1 \) and \( n \geq 2m_1 + 2. \)

In this case \( 2n - 2m_1 \geq n + 1, \) so we will prove (3) for \( 1 \leq r \leq n + 1, \) that is for \( 0 \leq p \leq \left\lfloor \frac{n}{2} \right\rfloor. \) As in the previous cases, we will work by induction. For \( p = \left\lfloor \frac{n}{2} \right\rfloor, \) since \( n \geq 2m_1 + 2, \) we have \( m_1 \leq p - 1, \) hence \( f \left( \left\lfloor \frac{n}{2} \right\rfloor \right) = \sum_{i=1}^{s} \binom{m_i + 2}{2} \) and since \( \sum_{i=1}^{s} \binom{m_i + 2}{2} \geq \binom{n+2}{2}, \) then (3) holds for \( p = \left\lfloor \frac{n}{2} \right\rfloor. \)

Let \( p < \left\lfloor \frac{n}{2} \right\rfloor. \)

As in Case 3, if \( \sum_{m_i \geq p} (m_i - p) \leq 2n - 4p - 1, \) then (3) holds. Let

\[ (11) \sum_{m_i \geq p} (m_i - p) \geq 2n - 4p, \]

and note that, since \( p \leq \left\lfloor \frac{n}{2} \right\rfloor - 1, \) then \( 2n - 4p > 0, \) so at least \( m_1 > p. \)

For \( m_i \geq p \) we have

\[ (12) 2m_i + 2 + p(2m_i + 1 - p) \geq (m_i - p) \cdot \frac{2m_1 + 2 + p(2m_1 + 1 - p)}{m_1 - p}, \]

in fact

\[ (2m_i + 2 + p(2m_i + 1 - p))(m_1 - p) - (m_i - p)(2m_1 + 2 + p(2m_1 + 1 - p)) \]
\[ = (m_1 - m_i)(p^2 + 3p + 2) \geq 0. \]

Omitting the \( m_i < p, \) by (11) and (12) we obtain

\[ f(p) \geq \frac{1}{2} \sum_{m_i \geq p} (2m_i + 2 + p(2m_i + 1 - p)) \]
\[ \geq \frac{1}{2} \sum_{m_i \geq p} (m_i - p) \cdot \frac{2m_1 + 2 + p(2m_1 + 1 - p)}{m_1 - p}, \]
\[ \geq \frac{1}{2} (2n - 4p) \frac{2m_1 + 2 + p(2m_1 + 1 - p)}{m_1 - p}. \]

Since \( \binom{n+2}{2} - \binom{n+1-2p}{2} = 2np - 2p^2 + n + p + 1 \) if we prove that

\[ (n - 2p) \frac{2m_1 + 2 + p(2m_1 + 1 - p)}{m_1 - p} \geq 2np - 2p^2 + n + p + 1, \]
we obtain the conclusion. Now, since $n \geq 2m_1 + 2$ and $m_1 \geq p$, we easily get
\[
(n - 2p)(2m_1 + 2 + p(2m_1 + 1 - p)) - (m_1 - p)(2np - 2p^2 + n + p + 1)
\]
\[
= n(m_1 + p^2 + 2p + 2) - 5m_1p - 3p - 2m_1p^2 - p^2 - m_1
\]
\[
\geq 2(m_1 + 1)(m_1 + 2) + (p + 1)(p - m_1)
\]
\[
\geq 2(p + 1)(m_1 + 2) + (p + 1)(p - m_1) = (p + 1)(m_1 + p + 4) > 0,
\]
and the conclusion follows.

\[\Box\]

The previous proposition describes the behavior of configurations of linear spaces with non intersecting components. More precisely, we know that if no quadric is expected then no quadric containing the configuration exists. Using this knowledge we now show that generic configurations having disjoint components always have the expected behavior.

**Theorem 3.2.** Let $s \geq 2$, $m_1 \geq \cdots \geq m_s \geq 0$, and $m_1 + m_2 < n$ and so $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$.

Let
\[
\Lambda = \Lambda_1 + \cdots + \Lambda_s \subset \mathbb{P}^n
\]
be a generic configuration of linear spaces having weight vector $(m_1, \ldots, m_s)$. Then
\[
\dim(I_\Lambda)_2 = \max \left\{ \binom{n + 2}{2} - \sum_{i=1}^{s} \binom{m_i + 2}{2}; 0 \right\}
\]

**Proof.** For $\sum_{i=1}^{s} \binom{m_i + 2}{2} \geq \binom{n + 2}{2}$ the conclusion follows from Proposition 3.1.

If $\sum_{i=1}^{s} \binom{m_i + 2}{2} < \binom{n + 2}{2}$, let $\tilde{\Lambda}$ be the configuration of linear spaces obtained by adding $x = \binom{n + 2}{2} - \sum_{i=1}^{s} \binom{m_i + 2}{2}$ generic points to $\Lambda$, that is,
\[
\tilde{\Lambda} = \Lambda_1 + \cdots + \Lambda_s + \Lambda_{s+1} + \cdots + \Lambda_{s+x},
\]
where $\Lambda_{s+1}, \ldots, \Lambda_{s+x}$ are $x$ generic points. By applying Proposition 3.1 to the configuration $\tilde{\Lambda}$, of weight vector $(m_1, \ldots, m_s, 0, \ldots, 0)$, we have $\dim(I_{\tilde{\Lambda}})_2 = 0$, hence, since the $x$ points are generic, we obtain $\dim(I_{\Lambda})_2 = x$.

\[\Box\]

From Theorem 3.2, we obtain the following easy remark, which we will use in Section 4.:
Remark 3.3. If $\Lambda \subset \mathbb{P}^n$ is a generic configuration of four linear spaces with the same dimension $m$, and $n = 2m + 1$, then $\dim(I_{\Lambda})_2 = 0$. In fact
\[
\left(\begin{array}{c} n + 2 \\ 2 \end{array}\right) - \sum_{i=1}^{4} \left(\begin{array}{c} m + 2 \\ 2 \end{array}\right) = -m - 1 < 0.
\]
Moreover, if $\Lambda \subset \tilde{\Lambda} \subset \mathbb{P}^n$, then obviously $\dim(I_{\tilde{\Lambda}})_2 = 0$.

4. INTERSECTING SPACES

In this section we deal with the case of generic configurations of linear spaces for which some of the components intersect. The configurations are still generic but now intersections arise because of dimension reasons. In this situation it is harder to express the notion of "expected behavior" in simple terms.

Before stating the Main Theorem of this section, we want to make an easy, but useful, observation about projections of families of quadrics which contain a common linear space in their vertex.

Remark 4.1. Let
\[
\Lambda = \Lambda_1 + \cdots + \Lambda_s \subset \mathbb{P}^n
\]
be a generic configuration of linear spaces having weight vector
\[
L = (m_1, \ldots, m_s).
\]
Assume that the forms in $(I_{\Lambda})_2$ define cones, each of which has vertex containing a fixed linear space, $V$, of dimension $d$. Consider the projection from $V$ onto a generic linear space $H \subset \mathbb{P}^n$ of complementary dimension $n' = n - d - 1$. Then each $\Lambda_i$ is projected from $V$ onto a linear space $\Lambda'_i \subset H$ where
\[
\dim\Lambda'_i := m'_i = m_i - \dim(V \cap \Lambda_i) - 1
\]
(if $V \cap \Lambda_i = \emptyset$, we use the convention that $\dim(V \cap \Lambda_i) = -1$).

We have
\[
\dim(I_{\Lambda})_2 = \dim(I_{\Lambda'})_2
\]
where
\[
\Lambda' = \Lambda'_1 + \cdots + \Lambda'_s \subset H \simeq \mathbb{P}^{n'} = \mathbb{P}^{n-d-1}
\]
is a generic configuration of linear spaces and $I_{\Lambda'}$ is the ideal of $\Lambda'$ in the homogeneous coordinate ring of $\mathbb{P}^{n'}$.

We now state and prove the Main Theorem of this section.
Theorem 4.2. Let \( m_1 \geq \cdots \geq m_s \geq 0 \), and \( m_1 + m_2 \geq n \). Let
\[
\Lambda = \Lambda_1 + \cdots + \Lambda_s \subset \mathbb{P}^n
\]
be a generic configuration of linear spaces having weight vector \((m_1, \ldots, m_s)\), and let
\[
\tau = \max \{ i \in \mathbb{N} \mid m_1 + m_i \geq n \},
\]
and
\[
v = \sum_{i=2}^{\tau} (m_1 + m_i - n + 1).
\]
Then the following statements hold.

(i) If \( v \geq m_1 + 1 \), then \( \dim (I_\Lambda)^2 = 0 \).

(ii) If \( v \) is such that \( 2m_1 - n + 2 \leq v \leq m_1 \) and
(a) if \( \tau \geq 4 \), then \( \dim (I_\Lambda)^2 = 0 \);
(b) if \( \tau = 3 \), \( s \geq 4 \), \( 2n \leq \sum_{i=1}^{4} m_i + 2 \), then \( \dim (I_\Lambda)^2 = 0 \);
(c) if \( \tau = 3 \), \( s \geq 4 \) and \( 2n \geq \sum_{i=1}^{4} m_i + 2 \), or \( \tau = s = 3 \), then
\[
\dim (I_\Lambda)^2 = \max \left\{ \left( \frac{n+2}{2} \right) - \sum_{i=1}^{s} \left( \frac{m_i + 2}{2} \right) + \sum_{i,j=1,2,3} \left( \frac{m_i + m_j - n + 2}{2} \right) ; 0 \right\}.
\]

(iii) If \( v \leq 2m_1 - n + 1 \), then
\[
\dim (I_\Lambda)^2 = \max \left\{ \left( \frac{n+2}{2} \right) - \sum_{i=1}^{s} \left( \frac{m_i + 2}{2} \right) + \sum_{i=2}^{\tau} \left( \frac{m_1 + m_i - n + 2}{2} \right) ; 0 \right\}.
\]

Proof. We fix the following notation:
\[
\Lambda_{i,j} = \Lambda_i \cap \Lambda_j \quad \text{and} \quad V = \langle \Lambda_{1,2}, \Lambda_{1,3}, \ldots, \Lambda_{1,\tau} \rangle \subset \Lambda_1
\]
Since \( v = \sum_{i=2}^{\tau} (\dim \Lambda_{1,i} + 1) \) and the \( \Lambda_i \) are generic linear spaces, then the linear span \( V \) has dimension
\[
(13) \quad \dim V = \min \{ v - 1 ; m_1 \},
\]
and the points of \( V \) are singular points for the quadrics defined by the forms of \( I_\Lambda \). Hence the quadrics through \( \Lambda \) are cones whose vertex contains \( V \).

(i) Since \( v \geq m_1 + 1 \) we must have \( \dim V = m_1 \). Projecting from \( V \), and using the notation of Remark 4.1, we have \( n' = n - m_1 - 1 \) and
\[
m'_2 = m_2 - \dim \Lambda_{1,2} - 1 = m_2 - (m_1 + m_2 - n) - 1 = n - m_1 - 1.
\]
Hence \( H = \Lambda'_2 \) and so \( (I_{\Lambda'})_2 = (0) \) and, by Remark 4.1 so is \( (I_\Lambda)_2 \).
Hence contradiction: 
\[ V \cap h = A_1 \cap \Lambda_i\]

Hence by Remark 3.3 it follows that 
\[ \dim(\Lambda'') = 2, \quad m_2 = 1 - v; \quad m_3 = m_1 - v; \]
\[ m''_2 = n - m_1 - 1 - (v - 2m + n - 2 + 1) = m_1 - v; \]
\[ m''_3 = n - m_1 - 1 - \dim(A'_4 \cap A'_3) - 1 \geq m_1 - v. \]

Hence by Remark 3.3 it follows that 
\[ \dim(I_{A''}) = \dim(I_{\Lambda''}) = 0. \]

(b), (c) In these cases 
\[ v \geq 2m_1 - n + 2 \]

implies 
\[ 2m_1 + m_2 + m_3 - 2n + 2 - 2m_1 + n - 2 \geq 0, \]

that is 
\[ m_2 + m_3 \geq n. \] 

It follows that 
\[ \Lambda_2 \cap \Lambda_3 \neq \emptyset. \]

Let 
\[ W = \langle A_{1,2}, A_{1,3}, A_{2,3} \rangle. \]

\[ \dim < A_{1,2}, A_{1,3} >= \dim A_{1,2} + \dim A_{1,3} + 1 = 2m_1 + m_2 + m_3 - 2n + 1. \]

Since 
\[ v \leq m_1, \]
we have 
\[ 2m_1 + m_2 + m_3 - 2n + 2 - m_1 \leq 0, \]
and from this inequality it follows that 
\[ \Lambda_{2,3} \cap \Lambda_1 = \emptyset. \]

Thus 
\[ \dim W = \dim A_{1,2} + \dim A_{1,3} + \dim A_{2,3} + 2 \]
\[ = 2m_1 + 2m_2 + 2m_3 - 3n + 2. \]

By projecting \( \Lambda \) from \( W \) into a linear space 
\[ H \simeq \mathbb{P}^n \] 
(following Remark 4.1 using \( W \) instead of \( V \)), we get
\[ \Lambda' = \Lambda_1' + \cdots + \Lambda_s' \subset H \simeq \mathbb{P}^{n'}; \quad \Lambda'_i \simeq \mathbb{P}^{m'_i}; \]

\[ n' = n - \dim W - 1 = 4n - 2m_1 - 2m_2 - 2m_3 - 3; \]

\[ m'_1 = m'_2 = m'_3 = 2n - m_1 - m_2 - m_3 - 2 \]

and, for \( s \geq 4 \), we have also:

\[ m'_4 = m_4 - \dim \Lambda_4 \cap W - 1. \]

Note that for \( s = 3 \) we may apply Theorem \[3.2\].

Let \( s \geq 4 \). Now, \( \Lambda_4 \cap W \neq \emptyset \) if and only if

\[ m_4 + 2m_1 + 2m_2 + 2m_3 - 3n + 2 \geq n, \]

hence for \( m_4 \geq 4n - 2m_1 - 2m_2 - 2m_3 - 2 \) we have

\[ m'_4 = m_4 - (m_4 + 2m_1 + 2m_2 + 2m_3 - 3n + 2 - n) - 1 = n', \]

that is \( \Lambda'_4 \simeq \mathbb{P}^{n'} \), and \( \dim(I_{\Lambda})_2 = \dim(I_{\Lambda'})_2 = 0 \) immediately follows.

For \( m_4 < 4n - 2m_1 - 2m_2 - 2m_3 - 2 \) we have \( m'_4 = m_4 \). Then in case (ii)(b), where \( 2n \leq \sum_{i=1}^4 m_i + 2 \), we obtain

\[ m'_4 \geq m'_1 \]

Since \( n' = 2m'_1 + 1 \), by Remark \[3.3\] we have \( \dim(I_{\Lambda})_2 = \dim(I_{\Lambda'})_2 = 0 \), and this completes the proof of case (ii)(b).

If we are in case (ii)(c) we have \( 2n > \sum_{i=1}^4 m_i + 2 \) hence \( m_1 + m_2 + m_3 < 2n - 2 - m_4 \). It follows that \( m_4 - 4n + 2m_1 + 2m_2 + 2m_3 + 2 < m_4 - 4n + 2(2n - 2 - m_4) = -4 - m_4 < 0 \), so \( \Lambda_4 \cap W = \emptyset \) and \( m'_4 = m_4 \leq m'_i \). Since \( n' = 2m'_1 + 1 \), in order to compute the dimension of \( (I_{\Lambda'})_2 \), we apply Theorem \[3.2\] and we obtain:

\[ \dim(I_{\Lambda'})_2 = \max \left\{ \binom{n' + 2}{2} - \sum_{i=1}^s \binom{m'_i + 2}{2}; 0 \right\} \]

where \( m'_i = m_i \) for \( i > 3 \).

In case (ii)(c) with \( s = 3 \), as we noted above, by Theorem \[3.2\] we have:

\[ \dim(I_{\Lambda'})_2 = \max \left\{ \binom{n' + 2}{2} - \sum_{i=1}^3 \binom{m'_i + 2}{2}; 0 \right\} \]

If we prove that for \( s \geq 4 \)

\[ \binom{n + 2}{2} - \sum_{i=1}^s \binom{m_i + 2}{2} + \sum_{i,j=1,2,3} \binom{m_i + m_j - n + 2}{2} = \binom{n' + 2}{2} - \sum_{i=1}^s \binom{m'_i + 2}{2} \]

(14)
and for $s = 3$:

\[
\binom{n + 2}{2} - \sum_{i=1}^{3} \binom{m_i + 2}{2} + \sum_{i,j=1,2,3 \atop i \neq j} \binom{m_i + m_j - n + 2}{2} = \binom{n' + 2}{2} - \sum_{i=1}^{3} \binom{m_i' + 2}{2}
\]

we are done. Since for $i > 3$ we have $m_i' = m_i$, if the equality (15) holds, then (14) holds. Recall that $n' = 4n - 2m_1 - 2m_2 - 2m_3 - 3$ and $m_i' = m_i' = m_i' = 2n - m_1 - m_2 - m_3 - 2$. Now (15) holds if and only if

\[
(n + 2)(n + 1) - (m_1 + 2)(m_1 + 1) - (m_2 + 2)(m_2 + 1) - (m_3 + 2)(m_3 + 1) + (m_1 + m_2 - n + 2)(m_1 + m_2 - n + 1) + (m_1 + m_3 - n + 2)(m_1 + m_3 - n + 1) + (m_2 + m_3 - n + 2)(m_2 + m_3 - n + 1) - (2n - m_1 - m_2 - m_3) - 1)(2n - m_1 - m_2 - m_3) - 2) + 3(2n - m_1 - m_2 - m_3)(2n - m_1 - m_2 - m_3 - 1) = 0,
\]

and this equality is easy to check with a direct computation.

(iii) Since $v \leq 2m_1 - n + 1 \leq m_1$, we have $v - 1 \leq 2m_1 - n < m_1$ and so $\dim V = v - 1$. Projecting from $V$ onto $H$ (as in Remark 4.1) we obtain

\[
\Lambda' = \Lambda'_1 + \cdots + \Lambda'_s \subset H \simeq \mathbb{P}^{n'}; \quad \Lambda'_i \simeq \mathbb{P}^{m_i'}
\]

\[
n' = n - v; \quad m_i' = m_i - v;
\]

\[
m_i' = n - m_i - 1, \quad \text{for } 2 \leq i \leq \tau; \quad m_i' = m_i, \quad \text{for } i > \tau.
\]

(Recall that for $i > \tau$, we have $\Lambda_1 \cap \Lambda_i = \emptyset$).

It is not difficult to check that $n' > m_i' + m_j'$, for $i \neq j$ i.e. the generic linear spaces $\Lambda'_i$ do not intersect. So we may use Theorem 3.2 in order to compute $\dim(I_{\Lambda'})_2$ and we get

\[
\dim(I_{\Lambda'})_2 = \max\{\binom{n' + 2}{2} - \sum_{i=1}^{s} \binom{m_i' + 2}{2}; 0\}.
\]

If we show that

\[
\binom{n + 2}{2} - \sum_{i=1}^{s} \binom{m_i + 2}{2} + \sum_{i=2}^{7} \binom{m_1 + m_i - n + 2}{2} = \binom{n' + 2}{2} - \sum_{i=1}^{s} \binom{m_i' + 2}{2},
\]

we have achieved our goal.
then the proof of part (3) of the theorem is complete. A long, and
tedious, calculation leads us to prove that

\[(n + 2)(n + 1) - (m_1 + 2)(m_1 + 1)\]

\[+ \sum_{i=2}^{\tau} \left( -(m_i + 2)(m_i + 1) + (m_1 + m_i - n + 2)(m_1 + m_i - n + 1) \right)\]

\[= -(n - v + 2)(n - v + 1) + (m_1 - v + 2)(m_1 - v + 1) + \tau(n - m_1 + 1)(n - m_1) = 0,\]

and this can be checked by an easy direct computation. \qed

Finally we can summarize the results of the paper in a theorem giving
a complete description of \(I_2(\Lambda)\), and hence of \(HF(\Lambda, 2)\), for a generic
configuration of linear spaces \(\Lambda\).

**Theorem 4.3.** Let \(m_1 \geq \cdots \geq m_s \geq 0\). Consider a generic configura-
tion of linear spaces of weight vector \((m_1, \ldots, m_s)\)

\[\Lambda = \Lambda_1 + \cdots + \Lambda_s \subset \mathbb{P}^n,\]

and let \(I_\Lambda\) be its defining ideal.

If \(m_1 + m_2 < n\), then

\[\dim(I_\Lambda)_2 = \max \left\{ \binom{n + 2}{2} - \sum_{i=1}^{s} \binom{m_i + 2}{2}; 0 \right\}.\]

If \(m_1 + m_2 \geq n\), we let

\[\tau = \max\{i \in \mathbb{N} | m_1 + m_i \geq n\},\]

\[v = \sum_{i=2}^{\tau} (m_1 + m_i - n + 1),\]

then the following statements hold:

(i) if \(v \geq m_1 + 1\), then \(\dim(I_\Lambda)_2 = 0;\)

(ii) if \(v\) is such that \(2m_1 - n + 2 \leq v \leq m_1\), and

(a) if \(\tau \geq 4\), then \(\dim(I_\Lambda)_2 = 0;\)

(b) if \(\tau = 3, s \geq 4, 2n \leq \sum_{i=1}^{s} m_i + 2\), then \(\dim(I_\Lambda)_2 = 0;\)

(c) if \(\tau = 3, s \geq 4 \) and \(2n \geq \sum_{i=1}^{s} m_i + 2\), or \(\tau = s = 3\), then

\[\dim(I_\Lambda)_2 = \max \left\{ \binom{n + 2}{2} - \sum_{i=1}^{s} \binom{m_i + 2}{2} + \sum_{i,j=1,2,3} \binom{m_i + m_j - n + 2}{2}; 0 \right\}.\]
(iii) If $v \leq 2m_1 - n + 1$, then

$$\dim(I_\Lambda)_2 = \max \left\{ \binom{n+2}{2} - \sum_{i=1}^{s} \binom{m_i + 2}{2} + \sum_{i=2}^{\tau} \binom{m_1 + m_i - n + 2}{2}; 0 \right\}. \quad \Box$$

5. An application: decomposition of polynomials

In this section we will consider the problem of writing homogeneous polynomials in a special way (see Remark 5.1). A classical result in this direction says that a quadratic form in $n + 1$ variables can always be written as the sum of at most $n + 1$ squares of linear forms.

We consider the rings $S = \mathbb{C}[x_0, \ldots, x_n]$ and $T = \mathbb{C}[y_0, \ldots, y_n]$, and we denote by $S_d$ and $T_d$ their homogeneous pieces of degree $d$. We consider $T$ as an $S$-module by letting the action of $x_i$ on $T$ be that of partial differentiation with respect to $y_i$. We also use some basic notions about apolarity (for more on this see [Ger96, IK99]).

Let $I \subset S$ be a homogeneous ideal and denote by $I^\perp \subset T$ the submodule of $T$ annihilated by every element of $I$. We recall that $(I_d^\perp) = (I^\perp)_d$.

Given linear forms $l_{i,j} \in T_1, i = 1, \ldots, r, j = 0, \ldots, n_i$ we ask the question $(\star)$: does the following vector space equality hold

$$T_d = (\mathbb{C}[l_{i,0}, \ldots, l_{i,n_i}])_d + \ldots + (\mathbb{C}[l_{s,0}, \ldots, l_{s,n_s}])_d,$$

where $(\mathbb{C}[l_{i,0}, \ldots, l_{i,n_i}])_d$ is the degree $d$ part of the subring of $T$ generated by the $l_{i,j}$’s for a fixed $i$.

Remark 5.1. Notice that if $(\star)$ has an affirmative answer then any degree $d$ form $f(y_0, \ldots, y_n)$ can be written as

$$f_1(l_{1,0}, \ldots, l_{1,n_1}) + \ldots + f_s(l_{s,0}, \ldots, l_{s,n_s})$$

for suitable homogeneous polynomials $f_i$.

The connection with configurations of linear spaces is given by the following results.

Lemma 5.2. Let $\Lambda \subset \mathbb{P}^n$ be an $i$ dimensional linear space having defining ideal $I$. Then, for any $d$, we have the following:

$$(I^\perp)_d = (\mathbb{C}[l_0, \ldots, l_i])_d$$

where the linear forms $l_i \in T_1$ generate $(I^\perp)_1$. 

Proof. Obvious, since after a linear change of variables, we may assume 
\[ I = (x_0, \ldots, x_{n-i-1}). \]

\[ \square \]

**Proposition 5.3.** Let \( \Lambda = \Lambda_1 + \ldots + \Lambda_s \subset \mathbb{P}^n \) be a configuration of linear spaces having defining ideal \( I \) and such that \( \dim \Lambda_i = n_i \). Then, for any \( d \), the following holds:

\[ (I^\perp)_d = (C[l_{i,0}, \ldots, l_{i,n_i}])_d + \ldots + (C[l_{s,0}, \ldots, l_{s,n_s}])_d \]

where the linear forms \( l_{i,j} \in T_1 \) are such that the degree 1 piece of \( (l_{i,0}, \ldots, l_{i,n_i})^\perp \) generates the ideal of \( \Lambda_i \).

Proof. The proof follows readily from the previous lemma once we recall that \((I \cap J)^\perp = I^\perp + J^\perp\). \[ \square \]

Now we can make clear the connection with question (\( \star \)). Given linear forms \( l_{i,j} \in T_1, i = 1, \ldots, r, j = 0, \ldots, n_i \), consider the linear spaces \( \Lambda_i \subset \mathbb{P}^n \) having defining ideal generated by \( (l_{i,0}, \ldots, l_{i,n_i})^\perp \). Then any degree \( d \) element in \( T \) can be written as a form in the \( l_{i,j} \)'s if and only if \( I_d = 0 \) where \( I \) is the ideal of the configuration of linear spaces \( \Lambda = \Lambda_1 + \ldots + \Lambda_s \). Hence, Theorem 4.3 produces a complete answer to question (\( \star \)) for \( d = 2 \) and for a generic choice of linear forms \( l_{i,j} \)'s.

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