Squares and Cubes Modulo \( n \)

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March 25, 2016

Abstract. We study the asymptotics of the average number of squares (or quadratic residues) in \( \mathbb{Z}_n \) and \( \mathbb{Z}_n^* \). Similar analyses are performed for cubes, square roots of 0 and 1, and cube roots of 0 and 1.

Let \( \mathbb{Z}_n \) denote the ring of integers modulo \( n \), and let \( \mathbb{Z}_n^* \) denote the group (under multiplication) of integers relatively prime to \( n \). The number of elements in \( \mathbb{Z}_n^* \) is \( \varphi(n) \), where \( \varphi \) is Euler’s totient function. What is the average number of elements in \( \mathbb{Z}_n^* \), given an arbitrary \( n \)? One way to answer this question is to apply the Selberg-Delange method [1, 2, 3] to the Dirichlet series

\[
\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{s+1}} = \prod_p \left( 1 + \sum_{r=1}^{\infty} \frac{\varphi(p^r)}{p^{r(s+1)}} \right)
\]

\[= \prod_p \left( 1 + \sum_{r=1}^{\infty} \frac{p-1}{p^{r(s+1)}} \right)
\]

\[= \prod_p \left( 1 + \frac{p-1}{p} \sum_{r=1}^{\infty} \frac{1}{p^{rs}} \right)
\]

\[= \prod_p \left( 1 + \frac{p-1}{p(p^s-1)} \right) = G(s) \cdot \zeta(s)
\]

where \( G(s) \) is bounded in a half plane \( \text{Re}(s) > c \) for some \( c < 1 \). In fact, \( G(s) = 1/\zeta(s+1) \) in this case, and hence

\[
\sum_{n \leq N} \frac{\varphi(n)}{n} \sim \frac{G(1)}{\Gamma(1)} N = \frac{1}{\zeta(2)} N
\]

as \( N \to \infty \). It follows by partial summation that

\[
\sum_{n \leq N} \varphi(n) \sim \frac{1}{2\zeta(2)} N^2 = \frac{3}{\pi^2} N^2.
\]

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A more elementary proof of this asymptotic formula appears in [4]. Since the Selberg-Delange method will be used throughout this paper, we choose to illustrate its application in this simple setting.

Many other questions can be asked for arbitrary \( n \):

- What is the average number of solutions of \( x^2 = 1 \) in \( \mathbb{Z}_n^* \)?
- What is the average number of solutions of \( x^3 = 1 \) in \( \mathbb{Z}_n^* \)?
- What is the average number of solutions of \( x^2 = 0 \) in \( \mathbb{Z}_n \)?
- What is the average number of solutions of \( x^3 = 0 \) in \( \mathbb{Z}_n \)?
- What is the average number of images of the map \( y \mapsto y^2 \) in either \( \mathbb{Z}_n \) or \( \mathbb{Z}_n^* \)?
- What is the average number of images of the map \( y \mapsto y^3 \) in either \( \mathbb{Z}_n \) or \( \mathbb{Z}_n^* \)?

Although the answers require only straightforward use of standard techniques, they do not seem to be explicitly given in the literature. We make no claim of originality: Our purpose is only to collect results in one place and to document relevant numerical techniques.

1. **Number Theory**

1.1. **Selberg-Delange Method.** Let \( F(s) = \sum_{n=1}^{\infty} a(n)n^{-s} \) be a Dirichlet series with positive coefficients and with the property that \( G(s) = F(s) \cdot \zeta(s)^{-z} \) can be analytically continued and is bounded over \( \text{Re}(s) > c \), for some \( c < 1 \) and some \( z \in \mathbb{C} \). Then

\[
\sum_{n \leq N} a(n) \sim \frac{G(1)}{\Gamma(z)} N \cdot (\ln N)^{z-1}
\]

as \( N \to \infty \). More terms of the asymptotic expansion are possible, as is an accurate estimate of the error, but we omit these details for brevity’s sake.

A generalization of this method is required for our work involving averages over arithmetic progressions. Let \( \chi \) denote the principal character modulo \( k = q^m \), where \( q \) is a prime and \( m \geq 1 \). Here we examine

\[
F_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)a(n)}{n^s} = G_\chi(s) \cdot L_\chi(s)^z
\]

where

\[
L_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \left(1 - \frac{1}{k^s}\right) \zeta(s)
\]
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is the L-series corresponding to \( \chi \). Assuming \( a(n) \) is a multiplicative function and \( q \nmid \ell \), it follows that

\[
\sum_{\substack{n \leq N, \\ n \equiv \ell \mod k}} a(n) \sim \frac{G_{\chi}(1)}{\Gamma(z)} N \cdot (\ln N)^{z-1}
\]

as \( N \to \infty \) and, further, that

\[
G_{\chi}(1) = \frac{1}{\varphi(k)} \left( 1 + \sum_{r=1}^{\infty} \frac{a(q^r)}{q^r} \right)^{-1} G(1).
\]

In our examples, \( q \) will be either 2 or 3 and the bracketed infinite series will always collapse to a closed-form expression. Rather than directly employing the formula for \( G_{\chi}(1) \), however, we prefer instead to deduce \( F_{\chi}(s) \) (and hence \( G_{\chi}(s) \)) from \( F(s) \) on basic principles.

1.2. Square Roots of Unity. The number \( a(n) \) of solutions of \( x^2 = 1 \) in \( \mathbb{Z}_n^* \) is

\[
a(n) = \begin{cases} 
2^\omega(n)-1 & \text{if } n \equiv 2, 6 \mod 8, \\
2^\omega(n) & \text{if } n \equiv 1, 3, 4, 5, 7 \mod 8, \\
2^\omega(n)+1 & \text{if } n \equiv 0 \mod 8
\end{cases}
\]

where \( \omega(n) \) denotes the number of distinct prime factors of \( n \). It is well-known that

\[
\sum_{n=1}^{\infty} \frac{2^\omega(n)}{n^s} = \prod_p \left( 1 + 2 \sum_{r=1}^{\infty} \frac{1}{p^{rs}} \right)
= \prod_p \left( 1 + \frac{2}{p^s - 1} \right) = \frac{\zeta(s)^2}{\zeta(2s)} = G(s) \cdot \zeta(s)^2
\]

and hence

\[
\sum_{n \leq N} 2^\omega(n) \sim \frac{1}{\zeta(2)} N \cdot \ln N = \frac{6}{\pi^2} N \cdot \ln N.
\]

We need to generalize this asymptotic formula to arithmetic progressions \( n \equiv \ell \mod k \), where \( k = 2^m \) for \( m \geq 1 \) and \( 2 \nmid \ell \). It can be shown that

\[
\sum_{n \equiv \ell \mod k} 2^\omega(n) \sim \frac{1}{\varphi(k)} \prod_{p^2 \geq k} \left( 1 + \frac{2}{p^s - 1} \right)
= \frac{2}{k} \left( 1 + \frac{2}{2^s - 1} \right)^{-1} \frac{\zeta(s)^2}{\zeta(2s)} = G_{\chi}(s) \cdot \zeta(s)^2
\]
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as $s \to 1$, and thus
\[
\sum_{n \leq N, \text{mod } k, n \equiv \ell} 2^{\omega(n)} \sim \frac{G_X(1)}{\Gamma(2)} N \cdot \ln N = \frac{4}{k\pi^2} N \cdot \ln N.
\]

The cases $(k, \ell) = (8, 1), (8, 3), (8, 5)$ and $(8, 7)$ follow immediately. The case $(k, \ell) = (8, 4)$ proceeds from the case $(k, \ell) = (2, 1)$:
\[
\frac{1}{N \cdot \ln N} \sum_{n \leq N, \text{mod } 8, n \equiv 4} 2^{\omega(n)} = \frac{1}{N \cdot \ln N} \sum_{4n \leq N, \text{mod } 2} 2^{\omega(n)+1} \longrightarrow \frac{1}{4} \cdot 2 \cdot \frac{2}{\pi^2} = \frac{1}{\pi^2}.
\]

The case $(k, \ell) = (8, 2)$ proceeds from the case $(k, \ell) = (4, 1)$:
\[
\frac{1}{N \cdot \ln N} \sum_{n \leq N, \text{mod } 8, n \equiv 2} 2^{\omega(n)} = \frac{1}{N \cdot \ln N} \sum_{2n \leq N, \text{mod } 4} 2^{\omega(n)+1} \longrightarrow \frac{1}{2} \cdot 2 \cdot \frac{1}{\pi^2} = \frac{1}{\pi^2}
\]
and $(8, 6)$ likewise proceeds from $(4, 1)$. By everything proved thus far, we have
\[
\frac{1}{N \cdot \ln N} \sum_{n \leq N, \text{mod } 8} 2^{\omega(n)} \longrightarrow \frac{6}{\pi^2} - 4 \cdot \frac{1}{2\pi^2} - 3 \cdot \frac{1}{\pi^2} = \frac{1}{\pi^2}.
\]

Therefore
\[
\sum_{n \leq N} a(n) \sim \left( \frac{1}{2} \cdot 2 \cdot \frac{1}{\pi^2} + \left( 4 \cdot \frac{1}{2\pi^2} + \frac{1}{\pi^2} \right) + 2 \cdot \frac{1}{\pi^2} \right) N \cdot \ln N = \frac{6}{\pi^2} N \cdot \ln N.
\]

It is interesting that $(8, 2)$ and $(8, 6)$ balance perfectly against $(8, 0)$ so that the mean value of $a(n)$ is asymptotically equivalent to the mean value of $2^{\omega(n)}$.

1.3. Cube Roots of Unity. The number $a(n)$ of solutions of $x^3 = 1$ in $\mathbb{Z}_n^*$ is \[7\]
\[
a(n) = \begin{cases} 
3\tilde{\omega}(n) & \text{if } n \equiv 1, 2, 3, 4, 5, 6, 7, 8 \text{ mod } 9, \\
3\tilde{\omega}(n)+1 & \text{if } n \equiv 0 \text{ mod } 9
\end{cases}
\]

where $\tilde{\omega}(n)$ denotes the number of distinct primes of the form $3k + 1$ dividing $n$:
\[
\tilde{\omega}(p^r) = \begin{cases} 
0 & \text{if } p = 3 \text{ or } p \equiv 2 \text{ mod } 3, \\
1 & \text{if } p \equiv 1 \text{ mod } 3.
\end{cases}
\]
First, note that

\[
\sum_{n=1}^{\infty} \frac{3\tilde{\omega}(n)}{n^s} = \prod_{p \equiv 3 \text{ or } p \equiv 2 \mod 3} \left(1 + \frac{1}{p^s} - 1\right) \cdot \prod_{p \equiv 1 \mod 3} \left(1 - \frac{1}{p^s}\right)^{-1} \left(1 - \frac{1}{p^{2s}}\right) \left(1 - \frac{2}{p^{s(p^2+1)}}\right)
\]

\[
= \zeta(s) \cdot \prod_{p \equiv 1 \mod 3} \left(1 - \frac{1}{p^s}\right)^{-2} \left(1 - \frac{1}{p^{2s}}\right) \left(1 - \frac{2}{p^{s(p^2+1)}}\right) = G(s) \cdot \zeta(s)^2
\]

and [8]

\[
\lim_{s \to 1} \prod_{p \equiv 1 \mod 3} \left(1 - \frac{1}{p^s}\right)^{-2} \cdot (s - 1) = \frac{\sqrt{3}}{2\pi} \prod_{p \equiv 1 \mod 3} \left(1 - \frac{1}{p^2}\right)^{-1};
\]

hence

\[
\sum_{n \leq N} 3\tilde{\omega}(n) \sim \frac{G(1)}{\Gamma(2)} N \cdot \ln N = C \cdot N \cdot \ln N
\]

where

\[
C = \frac{\sqrt{3}}{2\pi} \prod_{p \equiv 1 \mod 3} \left(1 - \frac{2}{p(p+1)}\right) = \frac{\sqrt{3}}{2\pi} (0.9410349413195354517900322...).
\]

We need to generalize this asymptotic formula to arithmetic progressions \(n \equiv \ell \mod k\), where \(k = 3^m\) for \(m \geq 1\) and \(3 \nmid \ell\). It can be shown that

\[
\sum_{n \equiv \ell \mod k} \frac{3\tilde{\omega}(n)}{n^s} \sim \frac{1}{\varphi(k)} \prod_{p \equiv 2 \mod 3} \left(1 + \frac{1}{p^s} - 1\right) \cdot \prod_{p \equiv 1 \mod 3} \left(1 + \frac{3}{p^s - 1}\right)
\]

\[
= \frac{3}{2k} \left(1 + \frac{1}{3^s - 1}\right)^{-1} G(s) \cdot \zeta(s)^2 = G_\chi(s) \cdot \zeta(s)^2
\]

as \(s \to 1\), and thus

\[
\sum_{n \leq N, \ n \equiv \ell \mod k} 3\tilde{\omega}(n) \sim \frac{G_\chi(1)}{\Gamma(2)} N \cdot \ln N = \frac{C}{k} N \cdot \ln N.
\]
The cases \((k, \ell) = (9, 1), (9, 2), (9, 4), (9, 5), (9, 7)\) and \((9, 8)\) follow immediately. The case \((9, 3)\) proceeds from the case \((3, 1)\):

\[
\frac{1}{N \cdot \ln N} \sum_{n \leq N, \ n \equiv 3 \pmod{9}} 3^{\tilde{\omega}(n)} = \frac{1}{N \cdot \ln N} \sum_{3n \leq N, \ n \equiv 1 \pmod{3}} 3^{\tilde{\omega}(n)} \rightarrow \frac{1}{3} \cdot \frac{C}{3} = \frac{C}{9}
\]

and \((9, 6), (9, 0)\) likewise proceed from \((3, 2), (3, 0)\). Therefore

\[
\sum_{n \leq N} a(n) \sim \left(8 \cdot \frac{C}{9} + 3 \cdot \frac{C}{9}\right) N \cdot \ln N = \frac{11}{9} C \cdot N \cdot \ln N = (0.317...) N \cdot \ln N.
\]

Unlike earlier, the mean value of \(a(n)\) is asymptotically greater than the mean value of \(3^{\tilde{\omega}(n)}\). Our estimate improves upon Cloitre [7], who gave \((0.4...) N \cdot \ln(N)\) on empirical grounds.

**1.4. Squares in \(\mathbb{Z}_{n}^*\).** Let \(a(n)\) be as defined in section [1.2]. The number of squares, that is, the cardinality of images under the map \(y \mapsto y^2\) in \(\mathbb{Z}_{n}^*\), is [9]

\[
b(n) = \frac{\varphi(n)}{a(n)} = \begin{cases} 
\frac{\varphi(n)}{2^{\omega(n)} - 1} & \text{if } n \equiv 2, 6 \pmod{8}, \\
\frac{\varphi(n)}{2^{\omega(n)}} & \text{if } n \equiv 1, 3, 4, 5, 7 \pmod{8}, \\
\frac{\varphi(n)}{2^{\omega(n) + 1}} & \text{if } n \equiv 0 \pmod{8}.
\end{cases}
\]

First, note that

\[
\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{s+1} 2^{\omega(n)}} = \prod_{p} \left(1 + \frac{1}{2} \sum_{r=1}^{\infty} \frac{p-1}{p^r s+1}\right) = \prod_{p} \left(1 + \frac{p-1}{2(p^s - 1)}\right) = G(s) \cdot \zeta(s)^{1/2},
\]

hence

\[
\sum_{n \leq N} \frac{\varphi(n)}{n^{2^{\omega(n)}}} \sim \frac{G(1)}{\Gamma(1/2)} N \cdot (\ln N)^{-1/2} = C \cdot N \cdot (\ln N)^{-1/2}
\]

where

\[
C = \frac{1}{\sqrt{\pi}} \prod_{p} \left(1 + \frac{1}{2p}\right) \left(1 - \frac{1}{p}\right)^{1/2} = \frac{1}{\sqrt{\pi}} (0.8121057111631225117062509...).
\]
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It follows by partial summation that

\[
\sum_{n \leq N} \frac{\varphi(n)}{2^{\omega(n)}} \sim \frac{C}{2} \cdot N^2 \cdot (\ln N)^{-1/2}.
\]

We need to generalize this asymptotic formula to arithmetic progressions \( n \equiv \ell \mod k \), where \( k = 2^m \) for \( m \geq 1 \) and \( 2 \nmid \ell \). It can be shown that

\[
\sum_{n \leq N, \ n \equiv \ell \mod k} \frac{\varphi(n)}{n^{s+1}2^{\omega(n)}} \sim \frac{1}{\varphi(k)} \prod_{p > 2} \left( 1 + \frac{p - 1}{2p(p^s - 1)} \right) = \frac{2}{k} \left( 1 + \frac{1}{4(2^s - 1)} \right)^{-1} G(s) \cdot \zeta(s)^{1/2} = G_\chi(s) \cdot \zeta(s)^{1/2}
\]
as \( s \to 1 \), and thus

\[
\sum_{n \leq N, \ n \equiv \ell \mod k} \frac{\varphi(n)}{n^{2^{\omega(n)}}} \sim \frac{G_\chi(1)}{\Gamma(1/2)} N \cdot (\ln N)^{-1/2} = \frac{8}{5k} C \cdot N \cdot (\ln N)^{-1/2}
\]
or

\[
\sum_{n \leq N, \ n \equiv \ell \mod k} \frac{\varphi(n)}{2^{\omega(n)}} \sim \frac{4}{5k} C \cdot N^2 \cdot (\ln N)^{-1/2}.
\]

The cases \((k, \ell) = (8, 1), (8, 3), (8, 5) \) and \((8, 7) \) follow immediately. The case \((k, \ell) = (8, 4)\) proceeds from the case \((k, \ell) = (2, 1)\):

\[
\frac{(\ln N)^{1/2}}{N^2} \sum_{n \leq N, \ n \equiv 4 \mod 8} \frac{\varphi(n)}{2^{\omega(n)}} = \frac{(\ln N)^{1/2}}{N^2} \sum_{4n \leq N, \ n \equiv 1 \mod 2} \frac{2\varphi(n)}{2^{\omega(n)+1}} \rightarrow \frac{1}{16} \cdot \frac{4}{10} C = \frac{C}{40}.
\]

The case \((k, \ell) = (8, 2)\) proceeds from the case \((k, \ell) = (4, 1)\):

\[
\frac{(\ln N)^{1/2}}{N^2} \sum_{n \leq N, \ n \equiv 2 \mod 8} \frac{\varphi(n)}{2^{\omega(n)}} = \frac{(\ln N)^{1/2}}{N^2} \sum_{2n \leq N, \ n \equiv 1 \mod 4} \frac{\varphi(n)}{2^{\omega(n)+1}} \rightarrow \frac{1}{4} \cdot \frac{1}{2} \cdot \frac{4}{20} C = \frac{C}{40}
\]
and \((8, 6)\) likewise proceeds from \((4, 1)\). By everything proved thus far, we have

\[
\frac{(\ln N)^{1/2}}{N^2} \sum_{n \leq N, \ n \equiv 0 \mod 8} \frac{\varphi(n)}{2^{\omega(n)}} \rightarrow \frac{C}{2} - 4 \cdot \frac{C}{10} - \frac{C}{40} - 2 \cdot \frac{C}{40} = \frac{C}{40}.
\]

Therefore

\[
\sum_{n \leq N} b(n) \sim \left( 2 \cdot 2 \cdot \frac{C}{40} + \left( 4 \cdot \frac{C}{10} + \frac{C}{40} \right) + \frac{1}{2} \cdot \frac{C}{40} \right) N^2 \cdot (\ln N)^{-1/2}
\]

\[
= \frac{43}{80} C \cdot N^2 \cdot (\ln N)^{-1/2} = (0.246...) N^2 \cdot (\ln N)^{-1/2}.
\]
1.5. Cubes in $\mathbb{Z}_n^*$. Let $a(n)$ be as defined in section [1.3]. The number of cubes, that is, the cardinality of images under the map $y \mapsto y^3$ in $\mathbb{Z}_n^*$, is $[10]$ 

$$b(n) = \frac{\varphi(n)}{a(n)} = \begin{cases} \frac{\varphi(n)}{3^{\omega(n)}} & \text{if } n \equiv 1, 2, 3, 4, 5, 6, 7, 8 \mod 9, \\ \frac{\varphi(n)}{3^{\omega(n)}+1} & \text{if } n \equiv 0 \mod 9. \end{cases}$$

First, note that

$$\sum_{n=1}^{\infty} \frac{\varphi(n)}{n^{s+1}3^{\omega(n)}} = \prod_{p=3 \text{ or } p \equiv 2 \mod 3} \left(1 + \sum_{r=1}^{\infty} \frac{p-1}{p^r}ight) \cdot \prod_{p \equiv 1 \mod 3} \left(1 + \frac{1}{3} \sum_{r=1}^{\infty} \frac{p-1}{p^r}ight) = G(s) \cdot \zeta(s)^{2/3},$$

hence

$$\sum_{n \leq N} \frac{\varphi(n)}{n^{s+1}3^{\omega(n)}} \sim \frac{G(1)}{\Gamma(2/3)} N \cdot (\ln N)^{-1/3} = C \cdot N \cdot (\ln N)^{-1/3}$$

where

$$C = \frac{1}{\Gamma(2/3)} \prod_{p=3 \text{ or } p \equiv 2 \mod 3} \left(1 + \frac{1}{p}ight) \left(1 - \frac{1}{p}ight)^{2/3} \cdot \prod_{p \equiv 1 \mod 3} \left(1 + \frac{1}{3p}ight) \left(1 - \frac{1}{p}ight)^{2/3}$$

$$= \frac{1}{\Gamma(2/3)} (0.9477556177621765519078142...).$$

It follows by partial summation that

$$\sum_{n \leq N} \frac{\varphi(n)}{3^{\omega(n)}} \sim \frac{C}{2} \cdot N^2 \cdot (\ln N)^{-1/3}.$$ 

We need to generalize this asymptotic formula to arithmetic progressions $n \equiv \ell \mod k$, where $k = 3^m$ for $m \geq 1$ and $3 \nmid \ell$. It can be shown that

$$\sum_{n \equiv \ell \mod k} \frac{\varphi(n)}{n^{s+1}3^{\omega(n)}} \sim \frac{1}{\varphi(k)} \prod_{p=2 \mod 3} \left(1 + \frac{p-1}{p(p^s-1)}\right) \cdot \prod_{p \equiv 1 \mod 3} \left(1 + \frac{p-1}{3p(p^s-1)}\right)$$

$$= \frac{3}{2k} \left(1 + \frac{2}{3(3^s-1)}\right)^{-1} G(s) \cdot \zeta(s)^{2/3} = G_\chi(s) \cdot \zeta(s)^{2/3}$$
as \( s \to 1 \), and thus
\[
\sum_{\substack{n \leq N, \\ n \equiv \ell \mod k}} \frac{\varphi(n)}{n^{3\varphi(n)}} \sim \frac{G_\chi(1)}{\Gamma(2/3)} N \cdot (\ln N)^{-1/3} = \frac{9}{8k} C \cdot N \cdot (\ln N)^{-1/3}
\]
or
\[
\sum_{\substack{n \leq N, \\ n \equiv \ell \mod k}} \frac{\varphi(n)}{3^{\varphi(n)}} \sim \frac{9}{16k} C \cdot N^2 \cdot (\ln N)^{-1/3}.
\]
The cases \((k, \ell) = (9, 1), (9, 2), (9, 4), (9, 5), (9, 7)\) and \((9, 8)\) follow immediately. The case \((9, 3)\) proceeds from the case \((3, 1)\):
\[
\frac{\ln N}{N^2} \sum_{\substack{n \leq N, \\ n \equiv 3 \mod 9}} \frac{\varphi(n)}{3^{\varphi(n)}} = \frac{\ln N}{N^2} \sum_{\substack{3n \leq N, \\ n \equiv 1 \mod 3}} \frac{2\varphi(n)}{3^{\varphi(n)}} \to \frac{1}{2} \cdot C \cdot \frac{9}{48} C = \frac{C}{24}
\]
and \((9, 6)\) likewise proceeds from \((3, 2)\). By everything proved thus far, we have
\[
\frac{\ln N}{N^2} \sum_{\substack{n \leq N, \\ n \equiv 0 \mod 9}} \frac{\varphi(n)}{3^{\varphi(n)}} \to \frac{C}{2} - 6 \cdot \frac{C}{16} - 2 \cdot \frac{C}{24} = \frac{C}{24}.
\]
Therefore
\[
\sum_{n \leq N} b(n) \sim \left( 6 \cdot \frac{C}{16} + 2 \cdot \frac{C}{24} + \frac{1}{3} \cdot \frac{C}{24} \right) N^2 \cdot (\ln N)^{-1/3}
\]
\[
= \frac{17}{36} C \cdot N^2 \cdot (\ln N)^{-1/3} = (0.330...)N^2 \cdot (\ln N)^{-1/3}.
\]

1.6. Square Roots of Nullity. The number \( a(n) \) of solutions of \( x^2 = 0 \) in \( \mathbb{Z}_n \) is a multiplicative function of \( n \), with \( a(p^r) = p^{[r/2]} \), thus [11, 12]
\[
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} + \frac{p}{p^{2s}} + \frac{p^2}{p^{4s}} + \frac{p^3}{p^{5s}} + \frac{p^3}{p^{6s}} + \frac{p^3}{p^{7s}} + \cdots \right)
\]
\[
= \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \frac{1}{p^{4s}} + \frac{1}{p^{5s}} + \frac{1}{p^{6s}} + \frac{1}{p^{7s}} + \cdots \right)
\]
\[
= \prod_p \left( \frac{1}{1 - \frac{1}{p^{2s}}} \right) = \prod_p \left( 1 - \frac{1}{p^{2s-1}} \right)^{-1} \left( 1 + \frac{1}{p^s} \right)
\]
\[
= \frac{\zeta(2s - 1)\zeta(s)}{\zeta(2s)} = G(s) \cdot \zeta(s)^2
\]
and \( \lim_{s \to 1} \zeta(2s - 1) \cdot (s - 1) = 1/2 \), hence
\[
\sum_{n \leq N} a(n) \sim \frac{G(1)}{\Gamma(2)} N \cdot \ln N = \frac{3}{\pi^2} N \cdot \ln N.
\]

**1.7. Cube Roots of Nullity.** The number \( a(n) \) of solutions of \( x^3 = 0 \) in \( \mathbb{Z}_n \) is a multiplicative function of \( n \), with \( a(p^r) = p^{r/3} \), thus
\[
\sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} \right) = \prod_p \left( 1 - \frac{1}{p^{3s}} \right)^{-1} \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s-1}} \right)
\]
\[
= \zeta(3s - 2) \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s-1}} \right) = G(s) \cdot \zeta(s)^3.
\]

We have
\[
\lim_{s \to 1} \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s-1}} \right) \cdot (s - 1)^2 = \lim_{s \to 1} \frac{1}{\zeta(s) \cdot 2\zeta(2s - 1)} \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s-1}} \right)
\]
\[
= \frac{1}{2} \lim_{s \to 1} \prod_p \left( 1 + \frac{1}{p^s} + \frac{1}{p^{2s-1}} \right) \left( 1 - \frac{1}{p^s} \right) \left( 1 - \frac{1}{p^{2s-1}} \right)
\]
\[
= \frac{1}{2} \prod_p \left( 1 + \frac{2}{p} \right) \left( 1 - \frac{1}{p} \right)^2
\]
\[
= \frac{1}{2} \prod_p \left( 1 + \frac{3}{p - 1} \right) \left( 1 - \frac{1}{p} \right)^3
\]
\[
= \frac{1}{2} \prod_p \left( 1 - \frac{1}{p^2} \right) \left( 1 - \frac{2}{p(p + 1)} \right)
\]
\[
= \frac{1}{2\zeta(2)} \prod_p \left( 1 - \frac{2}{p(p + 1)} \right)
\]

and \( \lim_{s \to 1} \zeta(3s - 2) \cdot (s - 1) = 1/3 \), hence
\[
\sum_{n \leq N} a(n) \sim \frac{G(1)}{\Gamma(3)} N \cdot (\ln N)^2 = C \cdot N \cdot (\ln N)^2
\]
where $\[6, 13\]

$$C = \frac{1}{2\pi^2} \prod_p \left(1 - \frac{2}{p(p+1)}\right) = \frac{1}{12}(0.2867474284344787341078927...)$$

1.8. Squares in $\mathbb{Z}_n$. The number $b(n)$ of images under the map $y \mapsto y^2$ in $\mathbb{Z}_n$ is a multiplicative function of $n$, with $\[14, 15, 16\]

$$b(p^r) = \begin{cases} 
\frac{1}{3}(2^{r-1} + 4) & \text{if } p = 2 \text{ and } r \equiv 0 \mod 2, \\
\frac{1}{3}(2^{r-1} + 5) & \text{if } p = 2 \text{ and } r \equiv 1 \mod 2, \\
\frac{1}{2(p+1)}(p^{r+1} + p + 2) & \text{if } p > 2 \text{ and } r \equiv 0 \mod 2, \\
\frac{1}{2(p+1)}(p^{r+1} + 2p + 1) & \text{if } p > 2 \text{ and } r \equiv 1 \mod 2
\end{cases}$$

and

$$F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s+1}} = \left(1 + \sum_{r=1}^{\infty} \frac{b(2^r)}{2^{r(s+1)}}\right) \cdot \prod_{p>2} \left(1 + \sum_{r=1}^{\infty} \frac{b(p^r)}{p^{r(s+1)}}\right).$$

The left-hand factor in $F(s)$ simplifies to

$$1 + \frac{1}{3} \sum_{i=1}^{\infty} \frac{2^{2i-1} + 4}{2^{(2i)(s+1)}} + \frac{1}{3} \sum_{j=1}^{\infty} \frac{2^{2j-1} - 1}{2^{(2j-1)(s+1)}}$$

$$= 1 + \frac{1}{2} \left(\frac{4^s - 3}{(4^s + 1)(4^s - 1)} + 2^s \frac{2 \cdot 4^s - 7}{(4^s + 1)(4^s - 1)}\right)$$

$$= \left(1 + \frac{2^{2s+1} - 2^{s+1} - 1}{2^{s+2}(2^{2s+1} - 2^{s+1} - 1)}\right) \left(1 - \frac{2^{s+1} + 2}{2(2^{s+1} + 1)(2^{s+1} - 1)}\right) \left(1 - \frac{1}{2^s}\right)^{-1}$$

and the $p$th right-hand factor simplifies to

$$1 + \frac{1}{2(p+1)} \sum_{i=1}^{\infty} \frac{p^{2i+1} + p + 2}{p^{(2i)(s+1)}} + \frac{1}{2(p+1)} \sum_{j=1}^{\infty} \frac{p^{(2j-1)+1} + 2p + 1}{p^{(2j-1)(s+1)}}$$

$$= 1 + \frac{1}{2(p+1)} \left(\frac{p^{2s+3} + p^{2s+1} + 2p^{2s} - 2p - 2}{(p^{2s} - 1)(p^{2s} - 1)} + p^{s+1} \frac{p^{2s+2} + 2p^{2s+1} + p^{2s} - 2p - 2}{(p^{2s} + 1)(p^{2s} - 1)}\right)$$

$$= \left(1 - \frac{(p^{s+1} + 2)(p - 1)}{2(p^{s+1} + 1)(p^{s+1} - 1)}\right) \left(1 - \frac{1}{p^s}\right)^{-1}.$$
We have

\[ F(s) = \zeta(s) \left( 1 + \frac{2^{2s+1} - 2^{s+1} - 1}{2^{s+2}(2^{s+1} - 2^{s-1} - 1)} \right) \prod_p \left( 1 - \frac{(p^{s+1} + 2)(p - 1)}{2(p^{s+1} + 1)(p^{s+1} - 1)} \right) = G(s) \cdot \zeta(s)^{1/2} \]

and hence

\[ \sum_{n \leq N} b(n) \sim \frac{G(1)}{\Gamma(1/2)} N \cdot (\ln N)^{-1/2} = C \cdot N \cdot (\ln N)^{-1/2} \]

where

\[ C = \frac{17}{16} \frac{1}{\sqrt{\pi}} \prod_p \left( 1 - \frac{p^2 + 2}{2(p^2 + 1)(p + 1)} \right) \left( 1 - \frac{1}{p} \right)^{-1/2} \]

\[ = \frac{17}{16} \frac{1}{\sqrt{\pi}} (1.2569136102101885959492115...) \]

It follows by partial summation that

\[ \sum_{n \leq N} b(n) \sim \frac{C}{2} \cdot N^2 \cdot (\ln N)^{-1/2} = (0.376...)N^2 \cdot (\ln N)^{-1/2}. \]

1.9. Cubes in \( \mathbb{Z}_n \). The number \( b(n) \) of images under the map \( y \mapsto y^3 \) in \( \mathbb{Z}_n \) is a multiplicative function of \( n \), with

\[
\begin{align*}
    b(p^r) &= \begin{cases} 
        \frac{1}{13} (3^{r+1} + 10) & \text{if } p = 3 \text{ and } r \equiv 0 \text{ mod } 3, \\
        \frac{1}{13} (3^{r+1} + 30) & \text{if } p = 3 \text{ and } r \equiv 1 \text{ mod } 3, \\
        \frac{1}{13} (3^{r+1} + 12) & \text{if } p = 3 \text{ and } r \equiv 2 \text{ mod } 3, \\
        \frac{1}{p^2 + p + 1} (p^{r+2} + p + 1) & \text{if } p \equiv 2 \text{ mod } 3 \text{ and } r \equiv 0 \text{ mod } 3, \\
        \frac{1}{p^2 + p + 1} (p^{r+2} + p^2 + p) & \text{if } p \equiv 2 \text{ mod } 3 \text{ and } r \equiv 1 \text{ mod } 3, \\
        \frac{1}{p^2 + p + 1} (p^{r+2} + p^2 + 1) & \text{if } p \equiv 2 \text{ mod } 3 \text{ and } r \equiv 2 \text{ mod } 3, \\
        \frac{1}{3(p^2 + p + 1)} (p^{r+2} + 2p^2 + 3p + 3) & \text{if } p \equiv 1 \text{ mod } 3 \text{ and } r \equiv 0 \text{ mod } 3, \\
        \frac{1}{3(p^2 + p + 1)} (p^{r+2} + 3p^2 + 3p + 2) & \text{if } p \equiv 1 \text{ mod } 3 \text{ and } r \equiv 1 \text{ mod } 3, \\
        \frac{1}{3(p^2 + p + 1)} (p^{r+2} + 3p^2 + 2p + 3) & \text{if } p \equiv 1 \text{ mod } 3 \text{ and } r \equiv 2 \text{ mod } 3
    \end{cases}
\end{align*}
\]
and

\[ F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^{s+1}} = \left( 1 + \sum_{r=1}^{\infty} \frac{b(3^r)}{3^r(s+1)} \right) \cdot \prod_{p \equiv 2 \pmod{3}} \left( 1 + \sum_{r=1}^{\infty} \frac{b(p^r)}{p^r(s+1)} \right) \cdot \prod_{p \equiv 1 \pmod{3}} \left( 1 + \sum_{r=1}^{\infty} \frac{b(p^r)}{p^r(s+1)} \right). \]

The expressions for \( b(p^r) \) follow from a conjecture by Wilson [17]; a proof for the case \( p = 2, r \equiv 0 \pmod{3} \) was given by Wilmer & Schirokauer [18]. The left-hand factor in \( F(s) \) simplifies to

\[
1 + \frac{1}{13} \left( \sum_{i=1}^{\infty} \frac{3^{3i+1} + 10}{3^{(3i)(s+1)}} + \sum_{j=1}^{\infty} \frac{3^{3j-2}+30}{3^{(3j-2)(s+1)}} + \sum_{k=1}^{\infty} \frac{3^{3k-1}+12}{3^{(3k-1)(s+1)}} \right)
= 1 + \left( \frac{7 \cdot 27^s - 1}{(27^s+1-1)(27^s-1)} + \frac{3^{2s+1} \cdot 9 \cdot 27^s - 7}{(27^s+1-1)(27^s-1)} + \frac{3^{s+1} \cdot 3 \cdot 27^s - 1}{(27^s+1-1)(27^s-1)} \right)
= \left( 1 - \frac{2(3s+2+1)}{(3s+1+3(s+1)/2+1)(3s+1-3(s+1)/2+1)(3s+1-1)} \right) \left( 1 - \frac{1}{3s} \right)^{-1}.
\]

The \( p \text{th} \) right-hand factor simplifies to

\[
1 + \frac{1}{p^2 + p + 1} \left( \sum_{i=1}^{\infty} \frac{p^{3i+2} + p + 1}{p^{(3i)(s+1)}} + \sum_{j=1}^{\infty} \frac{p^{3j-2}+p^2+p}{p^{(3j-2)(s+1)}} + \sum_{k=1}^{\infty} \frac{p^{3k-1}+p^2+1}{p^{(3k-1)(s+1)}} \right)
= 1 + \frac{1}{p^2 + p + 1} \left( \frac{p^{3s+5} + p^{3s+1} + p^{3s} - p^2 - p - 1}{(p^{3s+3} - 1)(p^{3s} - 1)} \right)
+ \frac{p^{2s+2}p^{3s+3} + p^{3s+2} + p^{3s+1} - p^2 - p - 1}{(p^{3s+3} - 1)(p^{3s} - 1)}
+ \frac{p^{s+1}p^{3s+4} + p^{3s+2} + p^{3s} - p^2 - p - 1}{(p^{3s+3} - 1)(p^{3s} - 1)}
= \left( 1 - \frac{(p^{s+1}+1)(p-1)}{(p^{s+1} + p(s+1)/2 + 1)(p^{s+1} - p(s+1)/2 + 1)(p^{s+1} - 1)} \right) \left( 1 - \frac{1}{p^s} \right)^{-1}.
\]
when \( p \equiv 2 \mod 3 \) and

\[
1 + \frac{1}{3(p^2 + p + 1)} \left( \sum_{i=1}^{\infty} \frac{p^{3i+2} + 2p^2 + 3p^3}{p^{3i+1}(p+1)} + \sum_{j=1}^{\infty} \frac{p^{3j+2} + 2p^2 + 3p^3}{p^{3j+2}(p+1)} + \sum_{k=1}^{\infty} \frac{p^{3k+2} + 2p^2 + 3p^3}{p^{3k+1}(p+1)} \right)
\]

\[
= 1 + \frac{1}{3(p^2 + p + 1)} \left( \frac{p^{3s+5} + 2p^{3s+2} + 3p^{3s+1} + 3p^{3s} - 3p^2 - 3p - 3}{(p^{3s+3} - 1)(p^{3s} - 1)} + p^{2s+2} \frac{3p^{3s+2} + 3p^{3s+1} + 2p^{3s} - 3p^2 - 3p - 3}{(p^{3s+3} - 1)(p^{3s} - 1)} + p^{s+1} \frac{3p^{3s+4} + 3p^{3s+2} + 2p^{3s+1} + 3p^{3s} - 3p^2 - 3p - 3}{(p^{3s+3} - 1)(p^{3s} - 1)} \right)
\]

\[
= \left( 1 - \frac{(2p^{2s+2} + 3p^{s+1} + 3)(p - 1)}{3(p^{s+1} + p^{(s+1)/2} + 1)(p^{s+1} - p^{(s+1)/2} + 1)(p^{s+1} - 1)} \right) \left( 1 - \frac{1}{p^s} \right)^{-1}
\]

when \( p \equiv 1 \mod 3 \). We have

\[
F(s) = \zeta(s) \left( 1 - \frac{2(3^{s+2} + 1)}{(3^{s+1} + 3^{(s+1)/2} + 1)(3^{s+1} - 3^{(s+1)/2} + 1)(3^{s+1} - 1)} \right)
\]

\[
\cdot \prod_{p \equiv 2 \mod 3} \left( 1 - \frac{p^2 + 1}{(p^2 + p + 1)(p^2 - p + 1)(p + 1)} \right) \left( 1 - \frac{1}{p} \right)^{-1/3}
\]

\[
\cdot \prod_{p \equiv 1 \mod 3} \left( 1 - \frac{2p^4 + 3p^2 + 3}{3(p^2 + p + 1)(p^2 - p + 1)(p + 1)} \right) \left( 1 - \frac{1}{p} \right)^{-1/3}
\]

\[
= G(s) \cdot \zeta(s)^{2/3}
\]

and hence

\[
\sum_{n \leq N} \frac{b(n)}{n} \sim \frac{G(1)}{\Gamma(2/3)} N \cdot (\ln N)^{-1/3} = C \cdot N \cdot (\ln N)^{-1/3}
\]

where

\[
C = \frac{12}{13} \Gamma(2/3) \left( 1 - \frac{1}{3} \right)^{-1/3}
\]

\[
\cdot \prod_{p \equiv 2 \mod 3} \left( 1 - \frac{p^2 + 1}{(p^2 + p + 1)(p^2 - p + 1)(p + 1)} \right) \left( 1 - \frac{1}{p} \right)^{-1/3}
\]

\[
\cdot \prod_{p \equiv 1 \mod 3} \left( 1 - \frac{2p^4 + 3p^2 + 3}{3(p^2 + p + 1)(p^2 - p + 1)(p + 1)} \right) \left( 1 - \frac{1}{p} \right)^{-1/3}
\]

\[
= \frac{12}{13} \Gamma(2/3) (1.4225831466986636811460982...).
\]
It follows by partial summation that

\[ \sum_{n \leq N} b(n) \sim \frac{C'}{2} \cdot N^2 \cdot (\ln N)^{-1/3} = (0.484...) N^2 \cdot (\ln N)^{-1/3}. \]

We emphasize that this result is only conjectural.

1.10. Other Problems. The power of the Selberg-Delange method is evident (many deeper applications occur elsewhere in the literature). We merely mention that the number \( a(n) \) of solutions of \( x^2 = -1 \) in \( \mathbb{Z}_n^* \) satisfies

\[ \sum_{n \leq N} a(n) \sim \frac{3}{2\pi} N; \]

in particular, \( x^2 = -1 \) has asymptotically far fewer solutions than \( x^2 = 1 \). Such asymmetry does not occur for \( x^3 = \pm 1 \) (just replace \( x \) by \( -x \)). See other modular polynomial equations at [19] and the enumeration of weakly primitive Dirichlet characters at [20, 21].

A more difficult exercise concerns the number \( b(n) \) of elements of \( \mathbb{Z}_n \) that are both squares and cubes. If \( w = z^6 \), then clearly \( w = (z^3)^2 = (z^2)^3 \). Conversely, if \( w = u^2 = v^3 \), then \( (uw^{-1})^6 = (u^2v^3(v^{-1})^2) = w^3w^{-2} = w \). Hence \( b(n) \) is the same as the number of sixth-powers in \( \mathbb{Z}_n \). Wilson’s conjecture again provides expressions for \( b(p^r) \), which in turn give formulas for \( F(s) \) and \( G(s) \). The details of this and other higher-power problems are left to someone else [22].
2. Numerical Techniques

2.1. Prime Products. Here is a method for evaluating constants of the form

\[ C = \prod_{p \equiv \ell \mod k} f(p) \]

to high precision, where the product is taken over all primes of the form \( p = mk + \ell \). Suppose that the function \( \ln f \) has asymptotic expansion

\[ \ln f(p) = \frac{c_2}{p^{s_2}} + \frac{c_3}{p^{s_3}} + \cdots + \frac{c_n}{p^{s_n}} + \cdots \]

as \( p \to \infty \), where \( (c_n, s_n) \) are real numbers and \( 1 < s_2 < \ldots < s_n < \ldots \) (Often \( s_n = n \) occurs.) Define the \((k, \ell)\)th prime zeta function

\[ P_{k,\ell}(s) = \sum_{p \equiv \ell \mod k} \frac{1}{p^s} \]

for \( \Re(s) > 1 \); it follows that

\[ \ln C = \sum_{p \equiv \ell \mod k} \ln f(p) = \sum_{n \geq 2} c_n P_{k,\ell}(s_n). \]

Let \( p_{k,\ell} \) denote the smallest prime of the form \( mk + \ell \); clearly \( P_{k,\ell}(n) \sim 1/p_{k,\ell}^n \) as \( n \to \infty \). Consequently, if the coefficients \( c_n \) are uniformly bounded, the convergence of the sum is fast (geometric). It hence remains to accurately compute the values \( P_{k,\ell}(s_n) \).

2.2. Prime Zeta Functions. Let \( \Re(s) > 1 \). The classical prime zeta function \( P(s) = P_{1,0}(s) \) can be related to the classical zeta function by Euler’s famous product:

\[ \ln \zeta(s) = -\sum_p \ln \left( 1 - \frac{1}{p^s} \right) = \sum_p \sum_{n \geq 1} \frac{1}{np^{ns}} = \sum_{n \geq 1} \frac{P(ns)}{n}. \]

Applying the Möbius inversion formula, we obtain \[23, 24\]

\[ P(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \ln \zeta(ns). \]

Since \( \ln \zeta(ns) \sim 2^{-ns} \) as \( n \to \infty \), only a few terms in this series are required to compute an accurate value of \( P(s) \). Also \( P(s) \sim -\ln(s - 1) \) as \( s \to 1^+ \). These facts are useful in computing constants of the form \( \prod_p f(p) \).
For constants of the form \( \prod_{p \equiv \ell \mod 3} f(p) \), we need \( P_{3,1}(s) \) and \( P_{3,2}(s) \). To achieve this, it is necessary to introduce the two characters modulo 3:

\[
\chi_0(n) = \begin{cases} 
1 & \text{if } n \equiv 1 \mod 3, \\
1 & \text{if } n \equiv 2 \mod 3, \\
0 & \text{if } n \equiv 0 \mod 3,
\end{cases} \quad \chi_1(n) = \begin{cases} 
1 & \text{if } n \equiv 1 \mod 3, \\
-1 & \text{if } n \equiv 2 \mod 3, \\
0 & \text{if } n \equiv 0 \mod 3,
\end{cases}
\]

and their associated Dirichlet L-series:

\[
L_j(s) = L_{\chi_j}(s) = \sum_{n=1}^{\infty} \frac{\chi_j(n)}{n^s} = \frac{1}{3^s} \left( \chi_j(1) \zeta(s, \frac{1}{3}) + \chi_j(2) \zeta(s, \frac{2}{3}) \right), \quad j = 0, 1
\]

where \( \zeta(s,a) \) is the Hurwitz zeta-function. By well-known acceleration procedures, series of this nature can be evaluated to many decimal places.

From the Euler product expressions

\[
L_0(s) = \prod_{p} \left( 1 - \frac{\chi_0(p)}{p^s} \right)^{-1} = \prod_{p \equiv 1 \mod 3} \left( 1 - \frac{1}{p^s} \right)^{-1} \prod_{p \equiv 2 \mod 3} \left( 1 - \frac{1}{p^s} \right)^{-1},
\]

\[
L_1(s) = \prod_{p} \left( 1 - \frac{\chi_1(p)}{p^s} \right)^{-1} = \prod_{p \equiv 1 \mod 3} \left( 1 - \frac{1}{p^s} \right)^{-1} \prod_{p \equiv 2 \mod 3} \left( 1 + \frac{1}{p^s} \right)^{-1},
\]

we obtain

\[
\frac{1}{2} \ln \left( \frac{L_0(s)}{L_1(s)} \right) = \frac{1}{2} \sum_{p \equiv 2 \mod 3} \ln \left( \frac{1 + p^{-s}}{1 - p^{-s}} \right) = \sum_{n=0}^{\infty} \frac{P_{3,2}((2n+1)s)}{2n+1},
\]

\[
\frac{1}{2} \ln \left( \frac{L_0(s)L_1(s)}{L_0(2s)} \right) = \frac{1}{2} \sum_{p \equiv 1 \mod 3} \ln \left( \frac{1 + p^{-s}}{1 - p^{-s}} \right) = \sum_{n=0}^{\infty} \frac{P_{3,1}((2n+1)s)}{2n+1}
\]

and, again, by Möbius inversion,

\[
P_{3,2}(s) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\mu(2n+1)}{2n+1} \ln \left( \frac{L_0((2n+1)s)}{L_1((2n+1)s)} \right),
\]

\[
P_{3,1}(s) = \frac{1}{2} \sum_{n=0}^{\infty} \frac{\mu(2n+1)}{2n+1} \ln \left( \frac{L_0((2n+1)s)L_1((2n+1)s)}{L_0((4n+2)s)} \right),
\]

Of course, \( L_0(s) = \zeta(s)(1 - 1/3^s) \) and \( L_1(s) = 1 - 1/2^s + 1/4^s - 1/5^s + \cdots \). Also, \( P_{3,1}(s) \sim -\frac{1}{2} \ln(s-1) \) and \( P_{3,2}(s) \sim -\frac{1}{2} \ln(s-1) \) as \( s \to 1^+ \).

Similar techniques involving characters modulo \( k \) can be used to compute constants of the form \( \prod_{p \equiv \ell \mod k} f(p) \), but for brevity’s sake we do not discuss these here.
2.3. A Simple Example. Let us compute the constant

\[ C = \prod_{p \equiv 1 \mod 3} \left( 1 - \frac{2}{p(p+1)} \right) = \prod_{p \equiv 1 \mod 3} \left( \frac{(p-1)(p+2)}{(p+1)p} \right) \]

that appears in section [1.3]. It is easy to establish that

\[ \ln C = \sum_{p \equiv 1 \mod 3} \left( \ln \left( \frac{p-1}{p+1} \right) + \ln \left( 1 + \frac{2}{p} \right) \right) = \sum_{n \geq 2} \frac{c_n}{n} P_{3,1}(n) \]

where \( c_n = 2^n - 2 \) when \( n \) is odd and \( c_n = -2^n \) when \( n \) is even. Since \( c_n = O(2^n) \) and \( P_{3,1}(n) = O(7^{-n}) \), it is more efficient to compute directly the product up to a certain cutoff \( p_c \). For example, if we take \( p_c = 31 \), we find

\[ C = \frac{3247695}{3430336} \prod_{p \equiv 1 \mod 3, p > 31} \left( 1 - \frac{2}{p(p+1)} \right) \]

and consequently

\[ \ln C = \ln \frac{3247695}{3430336} + \sum_{n = 2}^{\infty} \frac{c_n}{n} \left( P_{3,1}(n) - \frac{1}{7^n} - \frac{1}{13^n} - \frac{1}{19^n} - \frac{1}{31^n} \right) \]

enjoys much faster convergence because \( P_{3,1}(n) - \frac{1}{7^n} - \frac{1}{13^n} - \frac{1}{19^n} - \frac{1}{31^n} = O(37^{-n}) \). The first few terms of this series produce

| \( n \) | \( C \) |
|-------|-------|
| 2 | 0.94 (09438379523896292195206...) |
| 3 | 0.94103 (87732177050567463275...) |
| 4 | 0.941034 (8096648041499806620...) |
| 5 | 0.94103494 (70255355752383278...) |
| 10 | 0.94103494131953 (43277214763...) |
| 15 | 0.941034941319535451790 (3566...) |

and only 15 terms are necessary to obtain 20 correct decimal places.

3. Acknowledgements

We thank Gérald Tenenbaum & Jean-Marie De Koninck for their expertise in the Selberg-Delange method, Pieter Moree for his help in evaluating a complex residue, and David Wilson & Benoit Cloitre for their many contributions to Neil Sloane’s sequence database. Our work is extended in [25] and we gratefully acknowledge Greg Martin for his mastery of the subject.
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