STOCHASTIC REVERSE ISOPERIMETRIC INEQUALITIES IN THE PLANE

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Abstract. In recent years, it has been shown that some classical inequalities follow from a local stochastic dominance for naturally associated random polytopes. We strengthen planar isoperimetric inequalities by attaching a stochastic model to some classical inequalities, such as Mahler’s Theorem, and a reverse Lutwak-Zhang inequality, the polar for $L_p$ centroid bodies. In particular, we obtain the dual counterpart to a result of Bisztriczky-Böröczky.

1. Introduction

In this paper we study functionals of convex bodies invariant under the general linear group, $GL(n)$. To pave the way, we begin with a fundamental example. Let $K$ be a convex body in $\mathbb{R}^n$, i.e., a compact convex set with non-empty interior. Whenever $K$ is symmetric with respect to the origin the volume product is defined by

$$P(K) := |K||K^o|,$$

where $|\cdot|$ denotes the $n$-dimensional Lebesgue measure and $K^o$ is the polar body of $K$ (see §2 for precise definitions). Since $P$ is continuous and $GL(n)$ invariant, a compactness argument shows that it attains a maximum and a minimum. The upper estimate is known as the Blaschke-Santaló inequality

$$P(K) \leq P(B^n_2),$$

where $B_2^n$ denotes the $n$-dimensional Euclidean unit ball centered at the origin. In addition, Alexander, Fradelizi, and Zvavitch [11] recently showed that among polytopes simplicial ones are maximizers; see [32] for earlier work in the plane. On the other hand, the lower estimate is known as the Mahler’s Conjecture, confirmed by Mahler for dimension 2 [27], and it is still one of the main open problems in convex geometry:

$$P(K) \geq P(L),$$
where $L$ denotes the $n$-parallelootope for the symmetric case, or a simplex for the non-symmetric one. A recent breakthrough by Iriyeh and Shibata [19] solves it in dimension 3 (see Fradelizi, Hubard, Meyer, Roldán-Pensado, and Zvavitch [15] for a shorter proof). The $n$-dimensional statement is known in special cases. Saint Raymond established it for unconditional bodies [42] (see Meyer [29] for a simpler proof). Reisner proved the result for zonoids [39]. For local versions and other known special cases see [2, 20, 21, 23, 34, 48, 40]. Moreover, the celebrated result by Bourgain and Milman [6] established (1.3) up to a constant. See [16, 33, 38, 44] for other proofs and related results. The sharpest known constant is due to Kuperberg [22].

On the topic of functional inequalities related to convex bodies, Paouris and Pivovarov studied stochastic forms of isoperimetric inequalities [35]. Subsequently, Cordero-Erausquin, Fradelizi, Paouris, and Pivovarov [13] showed randomized inequalities for polar bodies. A typical example of such random sets is given by the convex hull of the columns of a random matrix, for which the expectation of the volume is maximized by $N$ independent random vectors uniformly distributed in the Euclidean ball of volume one. In particular, motivated by the work in Stochastic Geometry [41, 18, 9, 12, 11], they gave a stochastic Blaschke-Santaló inequality. Let $K$ be a symmetric convex body in $\mathbb{R}^n$, $\{X_i\}_{i=1}^N$ random vectors sampled uniformly in $K$, and $\{Y_i\}_{i=1}^N$ sampled uniformly in $K^*$, the Euclidean ball of the same volume as $K$. Then

$$\mathbb{E} |(\{X_i\}_N)^\circ| \leq \mathbb{E} |(\{Y_i\}_N)^\circ|,$$

where $[K]_N$ stands for $\text{conv}\{\pm X_1, \ldots, \pm X_N\}$ and similarly for $[K^*]_N$. One may find the origin of this in 1864, when J.J. Sylvester [49] posed his four points problem, which ultimately led to the study of

$$(1.4)\quad M(K) = \frac{1}{|K|} \mathbb{E} |[K]_N|,$$

for $N \geq n+1$. Thus, Sylvester’s problem is equivalent to study the extremals of $M(K)$. Blaschke [11] showed that in two dimensions the minimum is attained if and only if $K$ is an ellipse, and triangles are the only maximizers of (1.4) for $N = 4$ (see [11] for the extension to $N$ points on the plane). Indeed, since $M(K)$ is not increasing under Steiner symmetrization [17], ellipsoids are the only minimizers for the $n$-dimensional Sylvester’s problem. The maximum problem is still open for $n > 3$ and, as mentioned before, it is conjectured that the simplices are the only maximizers. Moreover, Campi, Colesanti, and Gronchi used RS-Movements (see §2) to determine maximizers of the $r$-th order moment of (1.4) in the plane for $N > n$. See Meckes [28] for the symmetric case.
We are interested in the polar version of Sylvester’s functional. Let $K$ be a centrally symmetric convex body in $\mathbb{R}^n$, $\{X_i\}_{i=1}^n$ independent random vectors sampled uniformly in $K$. We study a generalization of the functional

$$W(K) = \frac{1}{|K|} \mathbb{E} |([K]_N)^\circ|^{-1},$$

where $N > n$. In particular, we are interested in the functional of the normalized higher order moments of the volume of the polar of a random polytope in a centrally symmetric convex body in $\mathbb{R}^n$. Such functional, as Sylvester’s, is continuous with respect to the Hausdorff metric and invariant under invertible linear transformations.

The convexity of $W(K)$ under RS-Movements, Theorem 3.4, together with the setting beyond convex hull from Paouris and Pivovarov [35], provides a path to prove stochastic inequalities on the plane that strengthen planar isoperimetric inequalities. In particular, for centrally symmetric convex bodies $K$ in $\mathbb{R}^2$ with the origin as an interior point and $N \geq 2$, we are able to find a stochastic Mahler’s inequality.

**Theorem 1.1.** Let $N \geq 2$, $r \geq 1$, and $K$ a centrally symmetric convex body in $\mathbb{R}^2$. Then

$$\mathbb{E} |([K]_N)^\circ|^{-r} \leq \mathbb{E} |([Q]_N)^\circ|^{-r},$$

where $Q$ is a square with $|K| = |Q|$.

We will also be able to derive a stochastic reverse Lutwak-Zhang’s inequality on the plane. The deterministic inequality on the plane was shown by Campi and Gronchi [11]. Let $K$ be a star body about the origin in $\mathbb{R}^n$ and $1 \leq p \leq \infty$. The $p$-centroid body of $K$, $Z_p K$, is defined to be the convex body given by the function

$$h(Z_p K, x) = \left( \frac{1}{|K|} \int_K |\langle x, z \rangle|^p \, dz \right)^{1/p}.$$

In 1997 Lutwak and Zhang [26] showed, with a different normalization, that

$$|(Z_p K)^\circ| \leq |(Z_p K^*)^\circ|.$$  

Thus, one can think of the latter as a generalization of the Blaschke-Santaló inequality [12], since it can be obtained as the limit case $p = \infty$ for (1.7). For more recent developments see [24, 25].
In order to give a stochastic form of (1.7) on the plane, let $K$ be a centrally symmetric convex body in $\mathbb{R}^2$, $N \geq 2$, $\{X_i\}_{i=1}^N$ independent random vectors uniformly distributed in $K$, $r \geq 1$, and $1 \leq p \leq \infty$. We define the empirical $p$-centroid body of $K$, $Z_{p,N}(K)$, by its (random) support function

(1.8) \[ h^p(Z_{p,N}(K), z) = \frac{1}{N} \sum_{i=1}^N |\langle X_i, z \rangle|^p. \]

We denote $Z_{p,N}^o(K) = (Z_{p,N}(K))^o$, and $Z_N K$ the empirical centroid body of $K$, case $p = 1$.

**Theorem 1.2.** Let $K$ be a centrally symmetric convex body in $\mathbb{R}^2$. Then

(1.9) \[ \mathbb{E} |Z_{p,N}^o(K)|^{-r} \leq \mathbb{E} |Z_{p,N}^o(Q)|^{-r}, \]

where $|K| = |Q|$. In particular, when $N \to \infty$

(1.10) \[ |(Z_p K)^o| \geq |(Z_p Q)^o|. \]

Bisztriczky and Böröczky [3] provided a planar converse of the Busemann-Petty centroid inequality [7, 8, 36, 37]. We recall the latter states that given an origin symmetric convex body $K \subset \mathbb{R}^n$ then

\[ |ZK| \geq |ZK^*|. \]

where $ZK$ denotes the centroid body, i.e., the case $p = 1$. Bisztriczky and Böröczky showed that given a centrally symmetric convex body $K \subset \mathbb{R}^2$, with $|K| = |Q|$, then

\[ |ZK| \leq |ZQ|. \]

Theorem 1.2 for $p = 1$ gives a stochastic polar version of such inequality, that is

(1.11) \[ \mathbb{E} |Z_N^o(K)|^{-r} \leq |Z_N^o(Q)|^{-r}. \]

When $N \to \infty$ and $r = 1$, one obtains the mentioned deterministic inequality

(1.12) \[ |(ZK)^o| \geq |(ZQ)^o|. \]
Furthermore, we will be able to generalize Theorem 3.4 for non-symmetric convex bodies $K$ in $\mathbb{R}^n$ considering a generalization of the functional

$$W^{s_0}(K) = \frac{1}{|K|} \mathbb{E} |([K]_N)^{s_0}|^{-1},$$

where $N \geq n + 1$, $[K]_N = \text{conv}\{X_1, \ldots, X_N\}$, and $K^{s_0}$ denotes the polar body of $K$ with respect to its Santaló point. This will allow us to prove a stochastic non-symmetric Mahler’s inequality.

**Theorem 1.3.** Let $K$ be a convex body in $\mathbb{R}^2$, $r \geq 1$, and $N \geq 3$. Then

$$\mathbb{E} |([K]_N)^{s_0}|^{-r} \leq \mathbb{E} |([T]_N)^{s_0}|^{-r},$$

where $T$ denotes a triangle with centroid at the origin and $|T| = |K|$.

We make special use of the work by Rogers and Shephard [41, 47] and developed by Campi, Colesanti, and Gronchi [9, 10, 11, 12]. For more about shadow systems see Saroglou [45].

## 2. Preliminaries

A set is (centrally) symmetric if $K = -K$. Let $K$, $L$ be two sets in $\mathbb{R}^n$, their Minkowski sum is given by

$$K + L = \{k + l : k \in K, l \in L\},$$

and the Hausdorff distance between $K$ and $L$ by

$$\delta^H(K, L) = \inf\{\epsilon > 0 : K \subset L + \epsilon B_2^n, L \subset K + \epsilon B_2^n\}.$$
Whenever $K$ and $\theta$ are clear by the context we will just write $P$, $u$, and $\ell$.

Also notice that, for $K$ convex, $u$ and $\ell$ are concave and convex, respectively.

Let $\langle \cdot, \cdot \rangle$ be the usual inner product, the support function of a convex set is defined as

$$h_K(w) = \max_{x \in K} \langle w, x \rangle, \forall w \in \mathbb{R}^n.$$ 

One can define the $p$-centroid body of $K$ by its support function as

$$(2.1) \quad h_{Z_pK}(z) = \left( \frac{1}{|K|} \int_K |\langle x, z \rangle|^p \, dx \right)^{1/p}.$$ 

Thus, the volume of $Z_pK$ and $Z_p\Phi K$ is the same whenever $\Phi$ is a linear transformation with determinant one. Indeed, the centroid body itself is an affine equivariant, see [24].

The polar set, $K^\circ$, of $K$ is given by

$$(2.2) \quad K^\circ = \{ \omega \in \mathbb{R}^n : \langle \omega, x \rangle \leq 1, \forall x \in K \}.$$ 

Notice that $K^\circ$ depends on the location of the origin and it follows from the definition that if $K$ contains the origin $(K^\circ)^\circ = K$. In addition, a convex set is said to be a convex body if it is also compact with non-empty interior. In such case, the volume of the polar body of $K$ can be determined using the support function by

$$(2.3) \quad |K^\circ| = \frac{1}{n} \int_{\mathbb{S}^{n-1}} h_K^{-n}(x) \, dx,$$

where $\mathbb{S}^{n-1}$ denotes the unit sphere in $\mathbb{R}^n$, and $\cdot |$ the Lebesgue measure on $\mathbb{R}^n$. For the case of a non-symmetric convex body we define

$$K^{s\circ} = (K - s)^\circ = \{ \omega \in \mathbb{R}^n : \langle \omega, x - s \rangle \leq 1, \forall x \in K \},$$

where $s$ denotes the Santaló point of $K$, i.e. the unique point in the interior of $K$ such that

$$|K^{s\circ}| = \min_{x \in \text{int}(K)} \{|(K - x)^\circ|\}.$$ 

For $\{x_1, \ldots, x_N\} \subset \mathbb{R}^n$ we denote by $[x] := [x_1 \cdots x_N]$ the linear operator from $\mathbb{R}^N$ to $\mathbb{R}^n$, and $[x]C$ the set

$$[x]C = \left\{ \sum_{i=1}^N c_i x_i : c = (c_i) \in C \right\} \subset \mathbb{R}^n.$$
An example of this is when we consider $C = B_1^N$, then

$$\{x\}B_1^N = \text{conv}\{\pm x_1, \ldots, \pm x_N\},$$

where conv stands for convex hull.

3. Dual Version of Sylvester’s Functionals

We begin by recalling the notion of shadow system introduced by Rogers and Shephard [41].

**Definition 3.1.** A shadow system along a direction $\theta \in S^{n-1}$ is a family of convex sets $K_t \subset \mathbb{R}^n$ defined by

$$K_t = \text{conv}\{x + t\alpha(x)\theta : x \in A \subset \mathbb{R}^n\},$$

where $A$ is an arbitrary bounded set of points, $\alpha$ is a bounded function on $A$, and $t$ belongs to an interval of the real axis.

Rogers and Shephard [41, 47] proved that the volume of $K_t$ is a convex function of $t$, and many isoperimetric type inequalities have been shown using this technique. In the dual case, Campi and Gronchi showed the following fundamental theorem [11]:

**Theorem 3.2.** If $K_t$, $t \in [0,1]$, is a shadow system of origin symmetric convex bodies in $\mathbb{R}^n$, then $|K_t|^{-1}$ is a convex function of $t$.

We are interested in a particular case of shadow systems studied by Campi, Colesanti, and Gronchi [9], where the bounded function $\alpha$ is constant on each chord of $K$ parallel to $\theta$.

**Definition 3.3.** Let $K \subset \mathbb{R}^n$. A shadow system is called an RS-movement of $K$ if

$$K_t = \text{conv}\{x + t\beta(Px)\theta : x \in K\},$$

where $t \in [a,b]$, $0 \in [a,b]$, $P$ is the orthogonal projection defined as in § 2, and $\beta$ is a real valued function on $PK$.

Now, we introduce a functional motivated by Sylvester’s. It expresses the normalized higher negative moments of the volume of the polar of a random polytope in $K$. Denote by $K^n$ the class of all convex bodies in $\mathbb{R}^n$ and $K^N_0$ the class of all convex bodies with the origin as an interior point. Let $K \in K^n_0$ and $C \in K^N_0$ be centrally symmetric, $r \geq 1$, $N \geq n$, and $X_1, \ldots, X_N$ independent random vectors sampled uniformly in $K$. We
define the functional
\[ W_r(K; N; C) = \frac{1}{|K|^{N+r}} \int_{K^N} |([x_1 \cdots x_N]C)^{\circ} - r dx_1 \cdots dx_N \]
(3.1) \[ = \frac{1}{|K|^r} \mathbb{E} |([X_1 \cdots X_N]C)^{\circ} - r. \]

Notice that this functional is finite. Indeed, for \([K]_N = [x_1 \cdots x_N]B_1^N\), \([K]_N \subset K\) implies \(|([K]_N)^{\circ}| \geq |K^{\circ}|\). Thus, \(|([K]_N)^{\circ} - r \leq |K^{\circ} - r| \) for all \(r > 0\). For general \(C\), we have \(C \subset RB_1^N\) for some \(R > 0\). Hence, \(W_r(K; N, C)\) is finite. In addition, it is easy to check that \(W_r(K; N; C)\) is continuous with respect to the Hausdorff metric and it is invariant under invertible linear transformations \(T\), i.e., \(W_r(TK; N, C) = W_r(K; N, C)\). This follows from the change of variables \(x_i = Ty_i\) in (3.1) and using \(([Ty_1 \cdots Ty_N]C)^{\circ} = T^{-t}([y_1 \cdots y_N]C)^{\circ}\). However, in general, the functional is not invariant under invertible affine transformations as we are considering the polar body with respect to the origin.

**Theorem 3.4.** Let \(C \in K_0^N\) be centrally symmetric, \(K_t\) an RS-Movement for \(t \in [-1, 1]\), \(r \geq 1\), and \(N \geq n\). Then
\[ t \mapsto W_r(K_t; N; C) \]
is a convex function of \(t\).

We will use an analogue of Theorem 3.2 for linear images of convex sets, see [13, Corollary 3.8]. Namely, given an origin-symmetric convex body \(C\) in \(\mathbb{R}^N\), \(\theta \in S^{n-1}\), and \(x_1, \ldots, x_N \in \theta^\perp\), the map
\[ (t_1, \ldots, t_N) \mapsto |([x_1 + t_1\theta \cdots x_N + t_N\theta]C)^{\circ} - 1 \]
is convex on \(\mathbb{R}^N\).

**Proof.** Suppose without loss of generality \(K_0 = K\), so \(PK = PK_t\) and \(|K| = |K_t|\) for all \(t \in [-1, 1]\). Let \(u\) and \(\ell\) be as in §2 then one has
\[ K = \{(x, y) \in PK \times \mathbb{R} : \ell(x) \leq y \leq u(x)\} \]
\[ K_t = \{(x, y) \in PK \times \mathbb{R} : (\ell + t\beta)(x) \leq y \leq (u + t\beta)(x)\}. \]
Therefore by Fubini’s Theorem

\[
W_r(K_t; N; C) = \frac{1}{|K|^{N+r}} \int_{(PK)^N} \left( \prod_{i=1}^N \int_{(\ell_i + t\beta(x_i))}^{(u_i + t\beta)(x_i)} |(M_1 C)^\circ|^{-r} \, d\vec{y} \right) \, d\vec{x}
\]

\[
= \frac{1}{|K|^{N+r}} \int_{(PK)^N} \left( \prod_{i=1}^N \int_{u(x_i)}^{\ell(x_i)} |(M_2 C)^\circ|^{-r} \, d\vec{y} \right) \, d\vec{x},
\]

where

\[
M_1 C := [x_1 + y_1 \theta \cdot \cdot \cdot x_N + y_N \theta] C
\]

\[
= \left\{ \sum_{i=1}^N c_i (x_i + y_i \theta) : c = (c_i) \in C \right\}
\]

and after considering \( y_i = \tilde{y}_i + t\beta(x_i) \)

\[
M_2 C := [x_1 + (\tilde{y}_1 + t\beta(x_1)) \theta \cdot \cdot \cdot x_N + (\tilde{y}_N + t\beta(x_N)) \theta] C
\]

\[
= \left\{ \sum_{i=1}^N c_i (x_i + (\tilde{y}_i + t\beta(x_i)) \theta) : c = (c_i) \in C \right\}.
\]

Notice that by the convexity of (3.2), the functional in (3.1) is the repeated integral of the \( r \)-th power of a convex functional. Hence, it is convex.

\[ \square \]

At this point we generalize the previous functional for non-symmetric convex bodies. Let \( N \geq n + 1, K \in \mathcal{K}^n, C \in \mathcal{K}^N, X_1, \ldots, X_N \) independent random vectors sampled uniformly in \( K \), and \( r \geq 1 \). The functional

\[
W_r^{so}(K; N; C) = \frac{1}{|K|^{N+r}} \int_{K^N} |[x_1 \cdot \cdot \cdot x_N]C^{so}|^{-r} \, dx_1 \cdots dx_N
\]

\[
= \frac{1}{|K|} \mathbb{E} |([X_1 \cdot \cdot \cdot X_N]C^{so})^{-r}|
\]

expresses the normalized higher negative moments of the volume, of the polar of a non-symmetric random polytope in \( K \) with respect to its Santaló point.
The following theorem by Meyer and Reisner [31], allows us to show for (3.4) an analogous result to Theorem 3.4. We recall a non-degenerate shadow system, $K_t$, is a shadow system with non-empty interior for all $t$ in the interval.

**Theorem 3.5.** Let $K_t$, $t \in [a, b]$, be a non-degenerated shadow system in $\mathbb{R}^n$. Then $|K_t^{r\circ}|^{-1}$ is a convex function of $t$.

As in the symmetric case, (3.4) is finite, continuous with respect to the Hausdorff metric, and invariant under invertible linear transformations and translations; the Santaló point optimizes the translation of all points in the interior of $K$. In order to use Theorem 3.5, we consider $C = \text{conv}\{e_1, \ldots, e_N\}$.

**Theorem 3.6.** Let $C = \text{conv}\{e_1, \ldots, e_N\}$, $K_t$ an RS-Movement for $t \in [-1, 1]$, $r \geq 1$, and $N \geq n + 1$. Then

$$t \mapsto W^{r\circ}(K_t; N; C)$$

is a convex function of $t$.

**Proof.** Suppose without loss of generality $K_0 = K$, so $PK = PK_t$ and $|K| = |K_t|$ for all $t \in [-1, 1]$. Let $u$ and $\ell$ be as in §2 Then, as in Theorem 3.5, the convexity of (3.4) follows by Fubini’s Theorem and Theorem 3.5. \qed

4. **Applications**

Here we present some applications of Theorem 3.4 and Theorem 3.6. In particular, we argue the maximizers for our functionals (3.1) and (3.4) are squares and triangles, respectively, and prove by a random approximation procedure important results from Convex Geometry in the plane where a local stochastic dominance holds. We start by showing the maximizers for (3.1).

**Lemma 4.1.** Let $N \geq 2$, $C \in \mathcal{K}_0^N$ origin symmetric, and $r \geq 1$. Then

$$W_r(K; N; C) \leq W_r(B_{\infty}^2; N; C)$$

for all centrally symmetric $K \in \mathcal{K}_0^2$.

**Proof.** Let $m \geq 3$, $K$ a polygon with $\{\pm a_1, \ldots, \pm a_m\}$ ordered clockwise vertices, $\theta$ a direction parallel to the line joining $a_1$ and $a_3$, and set

$$K_t := \text{conv}\{\pm a_1, a_2 + t\theta, \pm a_3, \ldots, \pm a_m\}.$$ 

Then, there exists $\delta_1, \delta_2 > 0$ such that for $t \in [-\delta_1, \delta_2]$, $K_t$ is an RS-movement where $K_{-\delta_1}$ and $K_{\delta_2}$ have $m - 2$ vertices. As $W_r(K; N; C)$ is
a convex functional, Theorem 3.4

\[ W_r(K; N; C) \leq \max\{W_r(K_{-\delta_1}; N; C), W_r(K_{\delta_2}; N; C)\} \]

and iterating the above procedure one has

\[ W_r(K; N; C) \leq W_r(P; N; C), \]

where \( P \) is a parallelogram. The result follows by the invariance under linear transformations and the continuity of the functional.

We recall now a convergence result that will allow us to recover the classical inequalities, for the proof we refer to [46].

**Lemma 4.2.** Let \( K, K_1, \ldots, K_N \) be convex bodies with the origin as an interior point with \( K_N \xrightarrow{\delta_H} K \) as \( N \to \infty \). Then

\[ K_N^\circ \xrightarrow{\delta_H} K^\circ \text{ as } N \to \infty. \]

Let \( T \) be a triangle of volume one and centroid the origin, and \( B^N_p \) denote the unit ball in \( \ell^N_p \), for \( 1 \leq p \leq \infty \). Let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space, and assume we have the following independent random vectors sampled uniformly according to normalized the Lebesgue measure on the given set:

- \( \{X_i\}_{i=1}^N \) sampled in \( K \);
- \( \{Y_i\}_{i=1}^N \) sampled in \( B_\infty^2 \);
- \( \{\tilde{Y}_i\}_{i=1}^N \) sampled in \( T \).

Let \([X], [Y], \text{ and } [\tilde{Y}] \) as in §2. We now recall that under almost sure convergence, e.g. [13], integration and limit operation can be interchanged, so by Lemma 4.2 one has the following almost surely convergence in \( \delta_H^p \):

- \([X]B_1^N \to K.\]
- \( \frac{1}{N^{1/p}}[X]B_q^N \to Z_p K.\]
- \( \frac{1}{N}[X]B_\infty^N \to ZK.\]

Now, using Theorem 3.4 and Lemma 4.1, we deduce the proofs of the stochastic forms on the plane for Mahler’s Theorem [27], Theorem 1.1 and a reverse Lutwak-Zhang’s inequality [26], Theorem 1.2.
Proof of Theorem 1.1. Consider without loss of generality $K$ such that $|K| = |B_2|$, and notice we are able to write

$$[K]_N = [X]B_1^N \quad \text{and} \quad [B_2]_N = [Y]B_1^N.$$  

By Lemma 4.1 with $C = B_1^N$ we have

$$\frac{1}{|K|^r} \mathbb{E} \left( |(X(B_1^N)|^{r} \right) \leq \frac{1}{|B_2|^r} \mathbb{E} \left( |(Y(B_1^N)|^{r} \right)$$

and inequality (1.5) follows. \qed

Remark 4.3. From (1.5) one has

$$\lim_{N \to \infty} \mathbb{E} \left( |(X(B_1^N)|^{r} \right) \leq \lim_{N \to \infty} \mathbb{E} \left( |(Y(B_1^N)|^{r} \right).$$

Thus, by the continuity of the Lebesgue measure and Dominated Convergence Theorem

$$\mathbb{E} \left( \lim_{N \to \infty} |(X(B_1^N)|^{r} \right) \leq \mathbb{E} \left( \lim_{N \to \infty} |(Y(B_1^N)|^{r} \right).$$

Therefore by Lemma 4.2 one has the almost sure convergence mentioned above. It follows that

$$|K^{o}|^r \geq |(B_2^2)|^r = |B_1^2|^r.$$  

In particular, we recover (1.3) on the plane, i.e.,

$$|K| |K^{o}| \geq |B_2^2| |B_1^2|.$$  

Proof of Theorem 1.2. First consider without loss of generality $K$ such that $|K| = |B_2^2|$. Notice for $1/p + 1/q = 1$, (1.8) can be compare in matrix form with

$$Z_{p,N}(K) = \frac{1}{N^{1/p}} [X]B_q^N \quad \text{and} \quad Z_{p,N}(B_2^2) = \frac{1}{N^{1/p}} [Y]B_q^N.$$  

Then, by Lemma 4.1 with $C = B_q^N$ one has

$$\frac{1}{|K|^r} \mathbb{E} \left( |(X(B_q^N)|^{r} \right) \leq \frac{1}{|B_2^2|^r} \mathbb{E} \left( |(Y(B_q^N)|^{r} \right),$$

and inequality (1.5) follows.
so it follows that

$$E \left| \left( \frac{1}{N^{1/p}} [X] B^N_q \right)^{\circ -r} \right| \leq E \left| \left( \frac{1}{N^{1/p}} [Y] B^N_q \right)^{\circ -r} \right|$$

which is (1.9). Moreover, taking limits on both sides of the expression above

$$\lim_{N \to \infty} E \left| \left( \frac{1}{N^{1/p}} [X] B^N_q \right)^{\circ -r} \right| \leq \lim_{N \to \infty} E \left| \left( \frac{1}{N^{1/p}} [Y] B^N_q \right)^{\circ -r} \right|.$$

Inequality (1.10) follows from the continuity of the Lebesgue measure, the double application of the Dominated Convergence Theorem, and the almost sure convergence to the $p$-centroid body from the beginning of this section.

\[\square\]

**Remark 4.4.** Using (2.1) for $p = 1$ and (2.3), one can determine the volume of the polar centroid body of $B^2_{\infty}$

$$|(ZB^2_{\infty})^\circ| = 8 \int_S \left( \int_{B_{\infty}^Z} |\langle x, z \rangle| dx \right)^{-2} dz$$

$$= 32 \int_0^{\pi/2} \left( \int_{-1}^{1} \int_{-1}^{1} |x \cos \theta + y \sin \theta| dxdy \right)^{-2} d\theta$$

\[(4.1)\]

$$= \frac{4\pi}{\sqrt{3}} + 6.$$

Therefore, using (4.1) and (1.12) we can give an estimate for the planar Centroid volume product:

$$|(ZK)^\circ| |K| \geq \frac{16\pi}{\sqrt{3}} + 24.$$

Lastly, to finish the section we first show in Lemma 4.5 that triangles are the maximizers for functional (3.4) which allows us to give a stochastic Mahler’s Theorem on the plane for non-symmetric convex bodies.

**Lemma 4.5.** Let $T$ be a triangle, $N \geq 3$, $C = \text{conv}\{e_1, \ldots, e_N\}$, and $r \geq 1$. Then

$$W^\circ_r(K; N; C) \leq W^\circ_r(T; N; C)$$

for all $K \in \mathcal{K}_0^N$. 
Proof. Let $m \geq 3$, $K$ a polygon with \{a_1, \ldots, a_m\} ordered clockwise vertices, $\theta$ a direction parallel to the line joining $a_1$ and $a_3$, and set

$$K_t := \text{conv}\{a_1, a_2 + t\theta, a_3, \ldots, a_m\}.$$

Then, there exists $\delta_1, \delta_2 > 0$ such that for $t \in [-\delta_1, \delta_2]$, $K_t$ is an RS-motion where $K_{-\delta_1}$ and $K_{\delta_2}$ have $m - 1$ vertices. As $W_{r, p, r}(K; N; C)$ is a convex functional, Theorem 3.6

$$W_{r, p, r}(K; N; C) \leq \max\{W_{r, p, r}(K_{-\delta_1}; N; C), W_{r, p, r}(K_{\delta_2}; N; C)\}$$

and iterating the above procedure one has

$$W_{r, p, r}(K; N; C) \leq W_{r, p, r}(T; N; C).$$

The result follows by the invariance under linear transformations and the continuity of the functional. \hfill \square

Proof of Theorem 1.3. Let $e_1, \ldots, e_N$ be the standard orthonormal basis in $\mathbb{R}^n$, and $\tilde{C} = \text{conv}\{e_1, \ldots, e_N\}$. Consider without loss of generality $K$ such that $|K| = |T| = 1$, then we are able to write

$$[K]_N = [X] \tilde{C} \quad \text{and} \quad [T]_N = [\tilde{Y}] \tilde{C}.$$

By Lemma 4.5 with $C = \tilde{C}$ we have

$$\frac{1}{|K|} \mathbb{E}\left(\left|[X] \tilde{C}\right|^{r}\right) \leq \frac{1}{|T|} \mathbb{E}\left(\left|[\tilde{Y}] \tilde{C}\right|^{r}\right)$$

and inequality (1.13) follows. \hfill \square

Remark 4.6. From (1.13) one has

$$\lim_{N \to \infty} \mathbb{E}\left(\left|[X] \tilde{C}\right|^{r}\right) \leq \lim_{N \to \infty} \mathbb{E}\left(\left|[\tilde{Y}] \tilde{C}\right|^{r}\right).$$

Thus, by the continuity of the Lebesgue measure and Dominated Convergence Theorem

$$\mathbb{E}\left(\lim_{N \to \infty} \left|[X] \tilde{C}\right|^{r}\right) \leq \mathbb{E}\left(\lim_{N \to \infty} \left|[\tilde{Y}] \tilde{C}\right|^{r}\right).$$
Therefore by Lemma 4.2 one has the almost sure convergence mentioned above. It follows that

\[ |K^{sg}| \geq |T^{sg}| = \frac{27}{4}, \]

which recovers (1.3).

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