JOINT AND CONDITIONAL LOCAL LIMIT THEOREMS FOR LATTICE RANDOM WALKS AND THEIR OCCUPATION MEASURES

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Abstract. Let $S_n$ be a lattice random walk with mean zero and finite variance, and let $A_n^a$ be its occupation measure at level $a$. We prove local limit theorems for the probabilities $P[S_n = x, A_n^a = \ell]$ and $P[S_n = x | A_n^a = \ell]$ in the cases where $a, |x - a|$ and $\ell$ are either zero or at least of order $\sqrt{n}$. The asymptotic description of these quantities matches the corresponding probabilities for Brownian motion and its local time process. Our method of proof relies on path decompositions that reduce the problem to the study of random walks with independent increments.

1. Introduction

Let $X$ be a random variable supported on the lattice $c + Z$ ($|c| < 1$) with mean zero and variance $\nu < \infty$. Let $X_1, X_2, X_3, \ldots$ be i.i.d. copies of $X$, and for each $n \in \mathbb{N}$, let $S_n := X_1 + \cdots + X_n$. The central limit theorem states that $S_n/\sqrt{n}$ converges in distribution as $n \to \infty$ to a Gaussian law. A natural refinement of this result consists of the local limit theorem, which describes the asymptotic behavior of $S_n$ on a finer scale.

Theorem 1.1 (Gnedenko’s Local Limit Theorem [GK68 §49]). Suppose that the lattice $c + Z$ is maximal for $X$. It holds that

$$\sqrt{n}P[S_n = nc + x] = e^{-(nc+x)^2/2\nu \cdot n} \cdot \frac{\sqrt{2\pi\nu}}{\sqrt{2\pi\nu} + o(1)}$$

as $n \to \infty$, where the error term $o(1)$ is uniform in $x \in \mathbb{Z}$.

Starting from the mid 1970s, there has been a growing interest in local limit theorems involving random walks that are conditioned on some event of interest, or for other closely related processes. Notable examples include local limit theorems for random walks conditioned to stay positive [BJD06, Car05, VW09], as well as lattice random walks conditioned to avoid returning to the origin [Kai73, Kai75]. Our aim in this paper is to contribute to this line of research by proving local limit theorems for lattice random walks considered jointly with their occupation...
measures

\[ \Lambda_n^a := \sum_{k=1}^{n} 1_{\{S_k = a\}}, \quad a \in \mathbb{R}, \ n \in \mathbb{N}. \]  

(1.1)

It is well known that, with the appropriate normalization, the couple \((S_n, \Lambda_n^a)\) converges in joint distribution to \((B^\nu_1, L^a)_{a \in \mathbb{R}}\), where \((B^\nu_t)_{t \geq 0}\) is a Brownian motion with variance \(\nu\) and \((L^a)_{a \in \mathbb{R}}\) is its local time process on the time interval \([0, 1]\) (see, for instance, [BK93]). It is thus natural to expect that the large \(n\) limits of the probabilities

\[ P[S_n = x, \Lambda_n^a = \ell], \quad x, a \in \mathbb{R}, \ \ell \in \mathbb{N} \]

should be described by the joint distribution of \((B^\nu_1, L^a)_{a \in \mathbb{R}}\). As we will see in Section 2, where we state our main results, this is indeed the case.

As one might expect, conditioning a random walk to avoid or visit a fixed level before some time \(n\) breaks the independence of increments. Thus, the methods used in the classical local limit theorems, such as Theorem 1.1, cannot be applied directly to joint or conditioned local limit theorems. A typical workaround for this problem (used, for example, in [Car05, Kai75, VW09]) is to use path transformations or decompositions to translate the problem at hand into one about random walks with independent increments. Our methods of proof rely on such techniques (see Lemmas 4.5 and 5.1).

The organization of this paper is as follows. In Section 2, we state our main results for random walks supported on \(\mathbb{Z}\). In Section 3, we prove that the results stated in Section 2 can be extended to non-centered lattices. In Sections 4 and 5, we prove our main theorems. In Section 6, we discuss avenues for future research related to our work.

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2. Main Results

For the sake of simplicity, we first state our results in the case where \(X\) is supported on the lattice \(\mathbb{Z}\).

Assumption 2.1. \((X_k)_{k \in \mathbb{N}}\) are i.i.d. copies of \(X\), which is supported on \(\mathbb{Z}\) and such that \(\mathbb{E}[X] = 0\) and \(\mathbb{E}[X^2] = \nu < \infty\). \(\mathbb{Z}\) is maximal for \(X\), that is, there is no \(c, h\) with \(h > 1\) such that \(X\) is supported on \(c + h\mathbb{Z}\). The random walk \(S_n = X_1 + \cdots + X_n\ (n \in \mathbb{N})\) is recurrent (i.e., \(S_n = 0\) occurs for infinitely many \(n\) almost surely), and aperiodic (i.e., for every \(x \in \mathbb{Z}\), there exists \(n \in \mathbb{N}\) such that \(P[S_m = x] > 0\) for every \(m \geq n\)).

We also use the following notation throughout the paper.

Notation 2.2. We use \(C\) to denote constants independent of \(n\) whose values may change from line to line. We denote the error functions

\[ \text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} \, dy \quad \text{and} \quad \text{erfc}(x) := 1 - \text{erf}(x), \quad x \in \mathbb{R}; \]

the Gaussian density

\[ \gamma_\nu(x) := \frac{e^{-x^2/2\nu}}{\sqrt{2\pi\nu}}, \quad x \in \mathbb{R}; \]
as well as

\[ \varphi_\nu(x,a,\ell) := \frac{(a + (x - a) + \nu \ell)e^{-(|a| + |x - a| + \nu \ell)^2/2\nu}}{\sqrt{2\pi \nu}}, \quad x, a \in \mathbb{R}, \ \ell > 0, \ \text{and} \]

\[ \psi_\nu(a,\ell) := 2\nu e^{-(|a|+\nu \ell)^2/2\nu}/\sqrt{2\pi \nu}, \quad a \in \mathbb{R}, \ \ell > 0. \]

Given two sequences \((\alpha_n)_{n \in \mathbb{N}}\) and \((\beta_n)_{n \in \mathbb{N}}\), we use \(\alpha_n \sim \beta_n\) as \(n \to \infty\) to denote that \(\lim_{n \to \infty} \alpha_n/\beta_n = 1\).

Our first result is the following.

**Theorem 2.3.** Under Assumption \(2.1\) for every \(\kappa > 0\), it holds that

\( (2.1) \)

\[ \sqrt{n}P[S_n = x, \Lambda_n^a = 0] = \gamma_\nu \left( \frac{x}{\sqrt{n}} \right) - \gamma_\nu \left( \frac{2a - x}{\sqrt{n}} \right) + o(1) \]

as \(n \to \infty\), where the error term \(o(1)\) is uniform in \(x, a \in \mathbb{Z}\) such that

1. \(a \geq \kappa \sqrt{n}\) and \(x \leq a\); or
2. \(a \leq -\kappa \sqrt{n}\) and \(x \geq a\).

Moreover,

\( (2.2) \)

\[ P[\Lambda_n^a = 0] = \text{erf} \left( \frac{|a|}{\sqrt{2\nu n}} \right) + o(1) \]

as \(n \to \infty\) with \(o(1)\) uniform in \(a \in \mathbb{Z}\) such that \(|a| \geq \kappa \sqrt{n}\).

**Remark 2.4.** Let us define the first hitting times

\[ T_a := \inf\{t \geq 0 : B^a_t = a\} \quad \text{and} \quad \tau_a := \min\{n \geq 1 : S_n = a\}, \quad a \in \mathbb{R} \]

so that \(\{L^a = 0\} = \{T_a > 1\}\) and \(\{\Lambda_n^a = 0\} = \{\tau_n > n\}\). \(2.1\) is consistent with

\[ P[B^a_1 \in dx, T_a > 1] = \gamma_\nu(x) - \gamma_\nu(2a - x) \ dx, \]

which holds for \(a > 0\) and \(x \leq a\), or \(a < 0\) and \(x \geq a\). (This is easily computed by using the joint distribution of \(B^a_t\) and \(\{B^a_t : t \in [0,1]\}\).) As for \(2.2\), the reflection principle for Brownian motion stipulates that

\[ P[T_a > 1] = \text{erf} \left( \frac{|a|}{\sqrt{2\nu}} \right), \quad a \in \mathbb{R}. \]

To the best of our knowledge, the local limit theorem most similar to Theorem \(2.3\) that is available in the literature is [Kai75 Theorem 1 and Equation (9)]. The latter concerns probabilities of the form \(nP[S_n = x, \tau_0 > n]\) and \(\sqrt{n}P[S_n = x|\tau_0 > n]\), that is, a lattice random walk that avoids returning to zero. As shown in [Kai75], in the regime \(x \sim \sqrt{n}\), the leading order contribution to these probabilities is given by the two-sided Rayleigh density, which is the endpoint distribution of a two-sided Brownian meander:

\[ P[B^\nu_1 \in dx|T_0 > 1] = \frac{|x|e^{-x^2/2\nu}}{2\nu} \ dx, \quad x \in \mathbb{R}. \]

Thus, \(2.1\) is a natural extension of this result for random walks that avoid visiting a level away from zero (i.e., \(|a| \geq \kappa \sqrt{n}\) for some \(\kappa > 0\)). Our method of proof for this result (i.e., Lemma \(4.5\)) uses a path decomposition similar to the one used in [Kai75].
As we will see in Section 4.1, (2.2) is a modest generalization of a similar result in [Kai75, Corollary 2]. Our motivation for including this result is that, by combining it with (2.1), we obtain a local limit theorem for $S_n$ conditioned on not visiting $a$.

Our second and main result is the following.

Theorem 2.5. Under Assumption 2.1, for every $\kappa > 0$, it holds that

\[
\begin{align*}
nP[S_n = x, \Lambda_n^a = \ell] & = \varphi_\nu \left( \frac{a}{\sqrt{n}}, \frac{x-a}{\sqrt{n}}, \frac{\ell}{\sqrt{n}} \right) + o(1) \\
\sqrt{n}P[\Lambda_n^a = \ell] & = \psi_\nu \left( \frac{a}{\sqrt{n}}, \frac{\ell}{\sqrt{n}} \right) + o(1)
\end{align*}
\]

as $n \to \infty$, where the error terms $o(1)$ are uniform in $x,a \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ such that

1. $\ell \geq \kappa \sqrt{n}$;
2. $a = 0$ or $|a| \geq \kappa \sqrt{n}$; and
3. $|x-a| = 0$ or $|x-a| \geq \kappa \sqrt{n}$.

Remark 2.6. (2.3) is consistent with the joint density

\[
P[B_\nu^a dx, L_a^\nu d\ell] = \varphi_\nu(a, x-a, \ell) \, dx \, d\ell, \quad x,a \in \mathbb{R}, \ell > 0
\]

(see [Bor89 Page 6] and [Pit99]), valid for $x,a \in \mathbb{R}$ and $\ell > 0$. As for (2.4), by integrating the above with respect to $x$, one easily infers that

\[
P[L_a^\nu d\ell] = \psi_\nu(a, \ell) \, d\ell, \quad a \in \mathbb{R}, \ell > 0.
\]

Any statement of the form (2.3) appears to be completely new. As for (2.4), our statement can essentially be obtained from [Bor87, Theorem 1.1], though there are notable differences in the assumptions we make in this paper; for example, in [Bor87], $X$ is assumed to have finite moments of all orders, whereas we assume only a finite second moment.

As in Theorem 2.3, our main motivation for including (2.4) is that, when it is combined with (2.3), we obtain a conditioned local limit theorem. Moreover, as explained in Section 5.2, (2.4) is proved by using virtually the same method used for (2.3), up to a few details.

Remark 2.7. If we compare Theorems 2.3 and 2.5 with the more classical local limit theorems (such as Theorem 1.1), we note the unusual requirements that the parameters $a, |x-a|$, and $\ell$ be at least of order $\sqrt{n}$. As it turns out, these conditions are all the consequence of the same problem, which is that some of the asymptotic estimates on which our results depend are not optimal in those regimes. We address this question in detail in Section 6.1.

3. Non-Centered Lattices

Before embarking on the proof of Theorems 2.3 and 2.5, we show in this section that these results can be extended to random walks that satisfy the following assumption.

Assumption 3.1. $(X_k)_{k \in \mathbb{N}}$ are i.i.d. copies of $X$, which is supported on $c + \mathbb{Z}$ ($|c| < 1$) and such that $\mathbf{E}[X] = 0$, $\mathbf{E}[X^2] = \nu$, and $\mathbf{E}|X|^{2+\delta} < \infty$ for some $\delta > 0$. The lattice $c + \mathbb{Z}$ is maximal for $X$, and the quantity

\[
\rho := \min\{m \in \mathbb{N} : mc + \mathbb{Z} = \mathbb{Z}\}
\]
is finite and such that the random variable $Y := X_1 + \cdots + X_\rho$ gives rise to a random walk that satisfies Assumption 2.1.

**Remark 3.2.** We note in passing that the simple symmetric random walk normalized by 1/2 (i.e., $P[X = \pm 1/2] = 1/2$) satisfies Assumption 2.1 but not Assumption 2.4, as it is supported on the maximal lattice 1/2 + $\mathbb{Z}$ and is periodic.

**Remark 3.3.** The assumption that $E[|X^{2+\delta}|]$ is finite for some $\delta > 0$ is due to the condition that $|a|, |x - a| \geq \kappa \sqrt{n}$ in Theorems 2.1 and 2.5. Indeed, as will become clear in the proof of Theorem 3.4 below, if Theorems 2.1 and 2.5 hold uniformly in $|a|$ and $|x - a|$, then there is no need for this additional moment assumption.

For every $n \in \mathbb{N}$, $S_n$ takes values in $nc + \mathbb{Z}$. Let us define $c_n$ as the number among $\{c, 2c, \ldots, \rho c\}$ that satisfies $c_n + Z = cn + Z$, and let $
abla := \bigcup_{n=1}^{\rho} (c_n + Z)$ denote the state space of $S_n$. Assuming that Theorems 2.3 and 2.5 hold under Assumption 2.1, we have the following extension.

**Theorem 3.4.** Under Assumption 3.1, for every $\kappa > 0$, (2.1) and (2.2) hold as $n \to \infty$ uniformly in $x \in c_n + \mathbb{Z}$ and $a \in \Sigma$ such that

1. $a \geq \kappa \sqrt{n}$ and $x \leq a$; or
2. $a \leq -\kappa \sqrt{n}$ and $x \geq a$.

Moreover,

$$nP[S_n = x, \Lambda_n^a = \ell] = \rho \cdot \varphi_\nu \left( \frac{a}{\sqrt{n}}, \frac{x - a}{\sqrt{n}} + \rho \ell \right) + o(1) \quad \text{and}$$

$$\sqrt{n}p[\Lambda_n^a = \ell] = \rho \cdot \varphi_\nu \left( \frac{a}{\sqrt{n}}, \frac{\rho \ell}{\sqrt{n}} \right) + o(1)$$

as $n \to \infty$ uniformly in $x \in c_n + \mathbb{Z}$, $a \in \Sigma$, and $\ell \in \mathbb{N}$ such that

1. $\ell \geq \kappa \sqrt{n}$;
2. $a = 0$ or $|a| \geq \kappa \sqrt{n}$; and
3. $|x - a| = 0$ or $|x - a| \geq \kappa \sqrt{n}$.

**Proof.** We begin by proving that, as $n \to \infty$,

\begin{equation}
(3.1) \sup_{x \in c_n + \mathbb{Z}, \ a \in \Sigma, \ \ell \in \mathbb{N}} \left| nP[S_n = x, \Lambda_n^a = \ell] - \rho \cdot \varphi_\nu \left( \frac{a}{\sqrt{n}}, \frac{x - a}{\sqrt{n}} + \rho \ell \right) \right| = o(1).
\end{equation}

Let $m_1, m_2 \in \{0, 1, \ldots, \rho\}$ be such that $a \in m_1c + \mathbb{Z}$ and $x \in m_2c + \mathbb{Z}$. We can fragment $S_n$ into the summands

$$S_n = X_1 + \cdots + X_{m_1} + X_{m_1+1} + \cdots + X_{m_1+\rho} + X_{(m_1+\rho)+1} + \cdots + X_{m_1+2\rho}$$

$$+ \cdots + X_{m_2-m_1-\rho+1} + \cdots + X_{m_2} + X_{m_2+1} + \cdots + X_n.$$
For any \( m \in \mathbb{N} \), let \( \tilde{S}_m := Y_1 + \cdots + Y_m \) and

\[
\bar{\Lambda}_m^a := \sum_{k=1}^m 1\{\tilde{S}_k = b\}, \quad b \in \mathbb{Z},
\]

and for any \( n \in \mathbb{N} \), let \( \tilde{\nu} := (n - m_1 - m_2)/\rho \). We can then write

\[
P[\tilde{S}_m = x, \Lambda_m^a = \ell] = P[\tilde{S}_\tilde{\nu} = x - Z_1 - Z_2, \bar{\Lambda}_{\tilde{\nu}}^a - Z_1 = \ell].
\]

Since \( \tilde{S}_m \) satisfies Assumption 2.1 (with a variance of \( \rho \tilde{\nu} \)), our aim is now to apply Theorem 2.5 to the right-hand side of (3.2). The integers \( n \) and \( \tilde{\nu} \) are related to each other by \( n = \rho \tilde{\nu} + O(1) \). As for \( Z_1 \) and \( Z_2 \), the main concern is to preserve the requirement that \( |a - Z_1| \) and \( |x - a - Z_2| \) are at least of order \( \sqrt{n} \). Since \( Z_1 \) and \( Z_2 \) both have a finite moment of order \( 2 + \delta \), it follows from Markov’s inequality that \( P[|Z_1| > (\kappa/2)\sqrt{n}] \) or \( |Z_2| > (\kappa/2)\sqrt{n}] = O(n^{-1-\delta/2}) \). Thus, by Theorem 2.5,

\[
nP[\tilde{S}_m = x - Z_1 - Z_2, \bar{\Lambda}_{\tilde{\nu}}^a - Z_1 = \ell] = \sum_{|z_1|,|z_2| \leq (\kappa/2)\sqrt{n}} \rho \tilde{\nu} P[\tilde{S}_\tilde{\nu} = x - z_1 - z_2, \bar{\Lambda}_{\tilde{\nu}}^a = z_1 = \ell] P[Z_1 = z_1, Z_2 = z_2] + o(1)
\]

but reduced to showing that, as \( n \to \infty \),

\[
\sup_{x \in \mathbb{R}, a, \alpha \in \Sigma, \ell \in \mathbb{N}} \left| \mathbb{E}_{Z_1, Z_2} \phi_{\nu} \left( \frac{a - Z_1}{\sqrt{\rho / \tilde{\nu}}}, \frac{x - a - Z_2}{\sqrt{\rho / \tilde{\nu}}}, \frac{\rho / \tilde{\nu}}{\rho / \tilde{\nu}} \right) - \phi_{\nu} \left( \frac{a - Z_1}{\sqrt{\rho / \tilde{\nu}}}, \frac{x - a}{\sqrt{\rho / \tilde{\nu}}}, \frac{\rho / \tilde{\nu}}{\rho / \tilde{\nu}} \right) \right| = o(1).
\]

Note that the function \( y \mapsto ye^{-y^2/2\nu}/\sqrt{2\pi \nu} \) is Lipschitz with a constant depending on \( \nu \). Therefore, given that \( \sqrt{n/\rho \tilde{\nu}} = 1 + O(n^{-1}) \) uniformly in \( x, a \), and \( \ell \), it is easy to see that there is some \( C > 0 \) that depends only on \( \nu \) such that (3.3) is bounded above by \( C(\mathbb{E}[|Z_1|] + \mathbb{E}[|Z_2|])/\sqrt{n} \). This concludes the proof of (3.1).

The remaining claims in the statement of the present theorem use the same arguments, and are in fact easier than (3.1). Thus, their proofs will be omitted. In order to justify the appearance of the limiting formulas involved, we propose the following heuristics (which are guided by the facts that \( n \sim \rho \tilde{\nu} \) and that the effects of \( Z_1 \) and \( Z_2 \) are negligible). For (2.1) and (2.2), we have

\[
\sqrt{\rho} P[\tilde{S}_m = x, \bar{\Lambda}_m^a = 0] \approx \sqrt{\rho} P[\tilde{S}_m = x, \bar{\Lambda}_m^a = 0] \approx \sqrt{\rho} \left( \gamma_{\rho \tilde{\nu}} \left( \frac{x}{\sqrt{\rho / \tilde{\nu}}} \right) - \gamma_{\rho \tilde{\nu}} \left( \frac{2a - x}{\sqrt{\rho / \tilde{\nu}}} \right) \right) \approx \gamma_{\nu} \left( \frac{x}{\sqrt{\rho / \tilde{\nu}}} \right) - \gamma_{\nu} \left( \frac{2a - x}{\sqrt{\rho / \tilde{\nu}}} \right)
\]
and
\[ P[\Lambda_n = 0] = P[\bar{\Lambda}_n = 0] \approx \text{erf} \left( \frac{|a|}{\sqrt{2\rho n}} \right) \approx \text{erf} \left( \frac{|a|}{\sqrt{2m}} \right). \]

As for the occupation measure, we have that
\[ \sqrt{n}P[\Lambda_n^a = \ell] \approx \sqrt{m}P[\Lambda_n^a = \ell] \approx \sqrt{\rho \psi_n} \left( \frac{a}{\sqrt{n}}, \frac{\ell}{\sqrt{n}} \right) \approx \rho \psi_n \left( \frac{a}{\sqrt{n}}, \frac{\rho \ell}{\sqrt{n}} \right), \]
concluding the proof. \( \square \)

4. Proof of Theorem [2.3]

4.1. Proof of \((2.2)\). We begin the proof of our main theorems with the simplest of our results, namely, the limit \((2.2)\). We take this opportunity to introduce several of the ideas that will make repeated appearances in the proofs of our other results.

Given that \( A_0 \), for every \( \Delta \in \mathbb{Z} \), we can make \( \lim_{n \to \infty} |a|e^{-a^2/(2v_n)} \) to show \((2.2)\), we need only prove that
\[
\lim_{n \to \infty} \sup_{|a| \geq \sqrt{n}} \left| \sum_{k=1}^{n} P[\tau_a = k] - \int_{0}^{1} \left| \frac{|a|e^{-a^2/(2v_n)}}{\sqrt{2\pi v_n n^3}} \right| du \right| = 0. \tag{4.1}
\]
(Recall that \( \tau_a \) is a first hitting time, defined in Remark [2.4] and that \( P[\Lambda_n^a = 0] \) is equal to \( P[\tau_a > n] \).) Our main tool for studying the probabilities \( P[\tau_a = k] \) is the following local limit theorem due to Kaigh.

Lemma 4.1 ([Kai75] (9) and Section 4). For every \( a \in \mathbb{Z} \) and \( n \in \mathbb{N} \), it holds that \( P[\tau_n = n] = P[S_n = a, \tau_0 > n] \). Moreover,
\[
\lim_{n \to \infty} \sup_{a \in \mathbb{Z}} \left| nP[S_n = a, \tau_0 > n] - \frac{|a|e^{-a^2/(2v_n)}}{\sqrt{2\pi v_n n^3}} \right| = 0. \tag{4.2}
\]
For every \( \Delta \in (0, 1) \), \((4.1)\) is bounded by the sum of the limits
\[
A_1 := \lim_{n \to \infty} \sup_{a \in \mathbb{Z}} \left| \sum_{k=[n\Delta]+1}^{n} P[\tau_a = k] - \int_{\Delta}^{1} \frac{|a|e^{-a^2/(2v_n)}}{\sqrt{2\pi v_n n^3}} \right| du,
\]
\[
A_2 := \lim_{n \to \infty} \sup_{|a| \geq \sqrt{n}} \int_{0}^{\Delta} \frac{|a|e^{-a^2/(2v_n)}}{\sqrt{2\pi v_n n^3}} \right| du, \text{ and}
\]
\[
A_3 := \lim_{n \to \infty} \sup_{|a| \geq \sqrt{n}} P[\tau_a \leq n\Delta].
\]
Our strategy is to prove that \( A_1 + A_2 + A_3 \) can be made arbitrarily small by taking \( \Delta \to 0 \), which then implies that \((4.1)\) holds.

For \( A_1 \), according to \((4.2)\),
\[
\sum_{k=[n\Delta]+1}^{n} P[\tau_a = k] = \sum_{k=[n\Delta]+1}^{n} \left( \frac{|a|e^{-a^2/(2\nu k^3)}}{\sqrt{2\pi \nu k^3}} + o(1) \right), \tag{4.3}
\]

as \( n \to \infty \), with \( o(1) \) uniform in \( a \in \mathbb{Z} \) and \( |n\Delta| + 1 \leq k \leq n \). Thus,

\[
\sum_{k=|n\Delta|+1}^{n} \mathbb{P}[\tau_a = k] = \frac{1}{n} \sum_{k=|n\Delta|+1}^{n} \frac{|a|e^{-a^2/2\nu n(k/n)}}{\sqrt{2\pi \nu n(k/n)^3}} + o(1),
\]

with \( o(1) \) uniform in \( a \in \mathbb{Z} \). Next, in order to relate (4.4) with the integral in \( A_1 \), we need the following uniform Riemann sum approximation result.

**Lemma 4.2** \([\text{Kai73 Lemma 2.21}]\). Let \( f : [c_1, c_2] \times \mathbb{R}^d \to \mathbb{R} \ (d \in \mathbb{N}) \) be a continuous function, where \( c_1 < c_2 \in \mathbb{R} \). If

\[
\lim_{\|y\| \to \infty} \sup_{c_1 \leq u \leq c_2} |f(u, y)| = 0,
\]

then

\[
\lim_{n \to \infty} \sup_{y \in \mathbb{R}^d} \left| \frac{1}{n} \sum_{k=|nc_1|}^{[nc_2]-1} f(k/n, y) - \int_{c_1}^{c_2} f(u, y) \, du \right| = 0.
\]

If we apply Lemma 4.2 with the function

\[
f(u, y) := \frac{|y|e^{-y^2/2u}}{\sqrt{2\pi u^3}}, \quad (u, y) \in [\Delta, 1] \times \mathbb{R},
\]

then we get the desired result: For any \( \Delta \in (0, 1) \),

\[
A_1 = \lim_{n \to \infty} \sup_{a \in \mathbb{Z}} \left| \frac{1}{n} \sum_{k=|n\Delta|+1}^{n} \frac{|a|e^{-a^2/2\nu n(k/n)}}{\sqrt{2\pi \nu n(k/n)^3}} \right| = 0.
\]

For \( A_2 \), since \( \text{erfc} \) is strictly decreasing, a direct computation shows that

\[
\sup_{|a| \geq \kappa \sqrt{\nu}} \int_{0}^{\Delta} \frac{|a|e^{-a^2/2\nu nu}}{\sqrt{2\pi \nu n u^3}} \, du = \sup_{|a| \geq \kappa \sqrt{\nu}} \text{erfc} \left( \frac{|a|}{\sqrt{2\nu n}} \right) = \text{erfc} \left( \frac{\kappa}{\sqrt{2\nu \Delta}} \right).
\]

This vanishes as \( \Delta \to 0 \).

Finally, to control \( A_3 \), we use the following result.

**Lemma 4.3** \([\text{Kai75 Corollary 2}]\). For every fixed \( z \geq 0 \), it holds that

\[
\lim_{n \to \infty} \mathbb{P}[\tau_n \leq n^2z] = \text{erfc} \left( \frac{1}{\sqrt{2\nu z}} \right).
\]

For every \( |a| \geq \kappa \sqrt{n} \), let \( \kappa_a \geq \kappa \) be such that \( |a| = \kappa_a \sqrt{n} \). Then,

\[
\mathbb{P}[\tau_a \leq n\Delta] = \mathbb{P}[\tau_{\kappa_a \sqrt{\nu} \Delta} \leq \kappa_a^2 n(\Delta/\kappa_a^2)] \leq \mathbb{P}[\tau_{\kappa_a \sqrt{\nu} \Delta} \leq \kappa_a^2 n(\Delta/\kappa_a^2)].
\]

Thus, it follows from Lemma 4.3 that

\[
A_3 \leq \lim_{n \to \infty} \sup_{|a| \geq \kappa \sqrt{n}} \mathbb{P}[\tau_a \leq n^2(\Delta/\kappa^2)] = \text{erfc} \left( \frac{\kappa}{\sqrt{2\nu \Delta}} \right).
\]

**Remark 4.4.** For any \( \Delta > 0 \), the arguments in this section also imply that

\[
\lim_{n \to \infty} \sup_{|a| \geq \kappa \sqrt{n}} \left| \mathbb{P}[\tau_a > n\Delta] - \text{erf} \left( \frac{|a|}{\sqrt{2\nu n\Delta}} \right) \right| = 0.
\]

\( ^1 \)We note that [\text{Kai73 Lemma 2.21}] is stated only for \( f : [a, b] \times \mathbb{R}^d \to \mathbb{R} \) with \( d = 1 \), and without the supremum over \( \alpha \). However, a trivial modification of their argument gives the result stated here.
4.2. Proof of (2.1). We now prove (2.1). By symmetry, it is enough to show that

\[
\lim_{n \to \infty} \sup_{x, a \in \mathbb{Z}, a \geq \sqrt{n}, x \leq a} |\sqrt{n}P[S_n = x, \tau_a > n] - \gamma_\nu \left( \frac{x}{\sqrt{n}} \right) + \gamma_\nu \left( \frac{2a - x}{\sqrt{n}} \right)| = 0.
\]

Our main tool for deriving (4.5) is the following summation by parts identity, which reduces \(P[S_n = x, \tau_a > n]\) to quantities for which we already have local limit theorems.

**Lemma 4.5.** For any \(2 \leq m \leq n\) and \(x \in \mathbb{Z}\) and \(a \in \mathbb{N}\) such that \(x \leq a\), one has

\[
P[S_n = x, \tau_a > n] = P[S_n = x] - P[S_{n-1} = x - a]
\]

\[
+ \sum_{k=1}^{m-1} P[\tau_a > k] \left( P[S_{n-k} = x - a] - P[S_{n-k-1} = x - a] \right)
\]

\[
+ P[S_{n-m} = x - a] P[\tau_a > m]
\]

\[
- \sum_{k=m+1}^{n} P[S_{n-k} = x - a] P[\tau_a = k].
\]

**Proof.** According to the strong Markov property,

\[
P[S_n = x, \tau_a > n] = P[S_n = x] - \sum_{k=1}^{n} P[S_n = x, \tau_a = k]
\]

\[
= P[S_n = x] - \sum_{k=1}^{n} P[S_{n-k} = x - a] P[\tau_a = k].
\]

For each \(1 \leq k \leq n\), one has \(-P[\tau_a = k] = P[\tau_a > k] - P[\tau_a > k - 1]\). Therefore, a simple rearrangement yields that

\[
- \sum_{k=1}^{m} P[S_{n-k} = x - a] P[\tau_a = k]
\]

\[
= \sum_{k=1}^{m} P[S_{n-k} = x - a] P[\tau_a > k] - \sum_{k=1}^{m} P[S_{n-k} = x - a] P[\tau_a > k - 1]
\]

\[
= P[S_{n-m} = x - a] P[\tau_a > m] - P[S_{n-1} = x - a]
\]

\[
+ \sum_{k=1}^{m-1} P[S_{n-k} = x - a] P[\tau_a > k] - \sum_{k=2}^{m} P[S_{n-k} = x - a] P[\tau_a > k - 1]
\]

\[
= P[S_{n-m} = x - a] P[\tau_a > m] - P[S_{n-1} = x - a]
\]

\[
+ \sum_{k=1}^{m-1} P[\tau_a > k] \left( P[S_{n-k} = x - a] - P[S_{n-k-1} = x - a] \right),
\]

from which the result follows. \(\square\)

We note that Lemma 4.5 also features prominently in [Bel70, Kai73, Kai75], though in a slightly different form.

Let us define

\[
J_\Delta(y, z) := \int_\Delta e^{-y^2/2t} e^{-z^2/2(1-t)} dt, \quad y > 0, \ z \geq 0, \ \Delta \in [0, 1/2).
\]
According to Lemma \[\text{Lemma A.2}\], \(J_0(y, z) = e^{-(y+z)^2/2\sqrt{2\pi}}\), and thus we may write
\[
\gamma_{\nu} \left( \frac{2a-x}{\sqrt{n}} \right) = J_0 \left( \frac{a-x}{\sqrt{\nu n}} \right).
\]

With this in mind, for every \(0 < \Delta < 1/2\), it then follows from Lemma \[\text{Lemma 4.5}\] with \(m = \lfloor n\Delta \rfloor\) that \(A_{1\Delta} \) is bounded above by the sum of the limits
\[
B_1 := \limsup_{n \to \infty} \sup_{x \in \mathbb{Z}} \sqrt{n} P[S_n = x] - \gamma_{\nu} \left( \frac{x}{\sqrt{n}} \right),
\]
\[
B_2 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}} \left| \sqrt{n} P[S_{n-1} = x-a] - \gamma_{\nu} \left( \frac{x-a}{\sqrt{n}} \right) \right|,
\]
\[
B_3 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}} \sqrt{n} \sum_{k=1}^{\lfloor n\Delta \rfloor-1} P[\tau_a > k] \left( P[S_{n-k} = x-a] - P[S_{n-k-1} = x-a] \right),
\]
\[
B_4 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}} \left| \sqrt{n} P[S_{n-\lfloor n\Delta \rfloor} = x-a] P[\tau_a > \lfloor n\Delta \rfloor] - \gamma_{\nu} \left( \frac{x-a}{\sqrt{n}} \right) \right|,
\]
\[
B_5 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}} \left( \sum_{k=1}^{\lfloor n(1-\Delta) \rfloor} \sqrt{n} P[S_{n-k} = x-a] P[\tau_a = k] \right)
\]
\[
\quad - \frac{1}{\sqrt{\nu}} J_\Delta \left( \frac{a}{\sqrt{\nu n}} \frac{a-x}{\sqrt{\nu n}} \right),
\]
\[
B_6 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}} \sum_{a \geq n\Delta \sqrt{\nu}} \sqrt{n} P[S_{n-k} = x-a] P[\tau_a = k],
\]
\[
B_7 := \sup_{n \in \mathbb{N}} \sup_{x, a \in \mathbb{Z}} \left| J_\Delta \left( \frac{a}{\sqrt{\nu n}} \frac{a-x}{\sqrt{\nu n}} \right) - J_0 \left( \frac{a}{\sqrt{\nu n}} \frac{a-x}{\sqrt{\nu n}} \right) \right|.
\]

We prove that \(B_1 + \cdots + B_7\) can be made arbitrarily small by taking \(\Delta \to 0\).

According to Theorem \[\text{Theorem 1.1}\], \(B_1 + B_2 = 0\).

For \(B_3\), we need the following result.

Lemma 4.6 \((\text{Kai75} \text{ Lemma in Section 2})\). It holds that
\[
\limsup_{n \to \infty} \sup_{x \in \mathbb{Z}} \left| n^{3/2} \left( P[S_n = x] - P[S_{n-1} = x] \right) - \left( \frac{x^2}{\nu n} - 1 \right) \frac{e^{-x^2/2\nu n}}{2\sqrt{2\pi \nu}} \right| = 0.
\]

Given that \(y \mapsto (y^2 - 1) e^{-y^2/2\sqrt{2\pi}}\) is uniformly bounded in \(y\), it follows from the above lemma and the trivial bound \(P[\tau_a > k] \leq 1\) that we can find \(C > 0\) independent of all parameters except \(\nu\) such that
\[
B_3 \leq \lim_{n \to \infty} C \sum_{k=1}^{\lfloor n\Delta \rfloor-1} \frac{\sqrt{n}}{(n-k)^{3/2}} = \lim_{n \to \infty} C \frac{n^{3/2}}{n} \sum_{k=1}^{\lfloor n\Delta \rfloor-1} \frac{1}{(1-k/n)^{3/2}}.
\]

We conclude with the remark that
\[
\lim \lim_{\Delta \to 0} \frac{1}{n} \sum_{k=1}^{\lfloor n\Delta \rfloor-1} \frac{1}{(1-k/n)^{3/2}} = \lim_{\Delta \to 0} \int_{0}^{\Delta} \frac{du}{(1-u)^{3/2}} = 0.
\]
For $B_4$, Theorem 1.1 and Remark 4.4 imply that
\[ \sqrt{n}P[S_{n-\lfloor n\Delta \rfloor} = x-a]P[\tau_a > \lfloor n\Delta \rfloor] \]
\[ = \left( \frac{1}{\sqrt{1-\Delta}} + o(1) \right) \left( \gamma_\nu \left( \frac{x-a}{n(1-\Delta)} \right) + o(1) \right) \left( \text{erf} \left( \frac{a}{\sqrt{2\nu n\Delta}} \right) + o(1) \right), \]
as $n \to \infty$, where the $o(1)$ are uniform in $a \geq \kappa \sqrt{n}$ and $x \in \mathbb{Z}$. Since the exponential and error functions above are uniformly bounded, this yields
\[ \sqrt{n}P[S_{n-\lfloor n\Delta \rfloor} = x-a]P[\tau_a > \lfloor n\Delta \rfloor] \]
\[ = \left( \frac{1}{\sqrt{1-\Delta}} \gamma_\nu \left( \frac{x-a}{n(1-\Delta)} \right) \right) \left( \text{erf} \left( \frac{a}{\sqrt{2\nu n\Delta}} \right) + o(1) \right), \]
with $o(1)$ uniform in $a \geq \kappa \sqrt{n}$ and $x \in \mathbb{Z}$. On the one hand,
\[ \lim_{\Delta \to 0} \sup_{a \geq \kappa \sqrt{n}} \left| \text{erf} \left( \frac{a}{\sqrt{2\nu n\Delta}} \right) - 1 \right| = \lim_{\Delta \to 0} \text{erfc} \left( \frac{\kappa}{\sqrt{2\nu \Delta}} \right) = 0. \]
On the other hand, an elementary calculus computation shows that for any $C > 0$, one has
\[(4.6) \quad \sup_{y \in \mathbb{R}} \left| \frac{e^{-y^2/2\nu} - e^{-(Cy)^2/2\nu}}{\sqrt{2\pi \nu}} \right| = \frac{1}{\sqrt{2\pi \nu}} \left| C^{2/(1-C^2)} - C^{2C^2/(1-C^2)} \right|.
\]
Given that
\[ \lim_{C \to 1} \left| C^{2/(1-C^2)} - C^{2C^2/(1-C^2)} \right| = 0, \]
if we apply (4.6) with $C = (1-\Delta)^{-1/2}$, then we see that
\[ \lim_{\Delta \to 0} \sup_{x,a \in \mathbb{Z}} \left| \gamma_\nu \left( \frac{x-a}{n(1-\Delta)} \right) - \gamma_\nu \left( \frac{x-a}{n} \right) \right| = 0. \]
We conclude that $B_4$ can be made arbitrarily small by taking $\Delta \to 0$.

For $B_5$, we can write
\[(4.7) \quad \sum_{k=\lfloor n(1-\Delta) \rfloor + 1}^{\lfloor n(1-\Delta) \rfloor} \sqrt{n}P[S_{n-k} = x-a]P[\tau_a = k] \]
\[ = \frac{1}{n} \sum_{k=\lfloor n\Delta \rfloor + 1}^{\lfloor n(1-\Delta) \rfloor} \sqrt{n} \sqrt{n-k}P[S_{n-k} = x-a] \frac{n}{k}P[\tau_a = k]. \]
It follows from Theorem 1.1 and (4.2) that
\[ \sqrt{n-k}P[S_{n-k} = x-a] \frac{n}{k}P[\tau_a = k] \]
\[ = \frac{1}{\sqrt{1-k/n}} \left( \gamma_\nu \left( \frac{x-a}{\sqrt{n-k}} \right) + o(1) \right) \frac{1}{k/n} \left( \frac{|a|e^{-a^2/(2\nu(n-k))}}{\sqrt{2\pi n-k}} + o(1) \right), \]
where the error terms $o(1)$ are uniform in $x, a \in \mathbb{Z}$ and $\lfloor n\Delta \rfloor + 1 \leq k \leq \lfloor n(1-\Delta) \rfloor$. Since all above terms are uniformly bounded for such values of $k$, (4.7) is equal to
\[ \left( \frac{1}{n} \sum_{k=\lfloor n\Delta \rfloor + 1}^{\lfloor n(1-\Delta) \rfloor} \frac{1}{\sqrt{2\pi(1-k/n)}} \frac{|a|e^{-(x-a)^2/(2\nu n(1-k/n))}}{\sqrt{2\pi \nu(n-k)^3}} \right) + o(1), \]
with $o(1)$ uniform in $x, a \in \mathbb{Z}$. Thus, if we apply Lemma 1.2 with

$$f(u; y, z) := \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} e^{-z^2/2(1-u)}, \quad (u; y, z) \in [\Delta, 1 - \Delta] \times \mathbb{R}^2,$$

then we conclude that $B_6 = 0$ for every fixed $\Delta$.

For $B_6$, since $y \mapsto e^{-y^2/2\sqrt{2\pi}}$ and $y \mapsto |y|e^{-y^2/2\pi}$ are uniformly bounded in $y$, it follows from Theorem 1.1 and 4.2 that

$$\sum_{k=\lfloor n(1-\Delta) \rfloor + 1}^{n} \sqrt{n} P[S_{n-k} = x-a] P[r_a = k] \leq C \sum_{k=\lfloor n(1-\Delta) \rfloor + 1}^{n-1} \frac{\sqrt{n}}{n-k}\frac{\sqrt{n}}{k}$$

for some $C > 0$ independent of all parameters. If $k \geq \lfloor n(1-\Delta) \rfloor + 1$, then

$$\frac{\sqrt{n}}{k} \leq \frac{\sqrt{n}}{\lfloor n(1-\Delta) \rfloor + 1} \sim \frac{1}{\sqrt{n(1-\Delta)}} \quad \text{as } n \to \infty.$$ 

Moreover,

$$\sum_{k=\lfloor n(1-\Delta) \rfloor + 1}^{n} \frac{1}{\sqrt{n-k}} \sim 2\sqrt{n\Delta} \quad \text{as } n \to \infty,$$

and thus $B_6 \leq C\sqrt{\Delta}/(1-\Delta)$, which vanishes as $\Delta \to 0$.

Finally for $B_7$, let us define for each $y > 0$ and $z \geq 0$

$$J_1^\Delta(y, z) := \int_0^\Delta \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} e^{-z^2/2(1-u)} \, du \quad \text{and}$$

$$J_2^\Delta(y, z) := \int_{1-\Delta}^1 \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} e^{-z^2/2(1-u)} \, du.$$ 

It is enough to prove that

$$\lim_{\Delta \to 0} \sup_{y \geq \kappa, z \geq 0} J_i^\Delta(y, z) = 0, \quad i = 1, 2.$$ 

For every $0 \leq u < 1$, the function $z \mapsto e^{-z^2/2(1-u)}/\sqrt{2\pi(1-u)}$ attains its maximum at $z = 0$, in which case the maximal value is $(2\pi(1-u))^{-1/2}$. If $u \leq \Delta$, this itself is bounded above by $(2\pi(1-\Delta))^{-1/2}$. Therefore,

$$J_1^\Delta(y, z) \leq \frac{1}{\sqrt{2\pi(1-\Delta)}} \int_0^\Delta \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} \, du = \frac{1}{\sqrt{2\pi(1-\Delta)}} \text{erfc} \left( \frac{y}{\sqrt{2\Delta}} \right), \quad z \geq 0.$$ 

If we take a supremum over $y \geq \kappa$ of the above, we get an upper bound of

$$\frac{1}{\sqrt{2\pi(1-\Delta)}} \text{erfc} \left( \frac{\kappa}{\sqrt{2\Delta}} \right),$$

which converges to zero as $\Delta \to 0$. For every $0 < u \leq 1$, the function $y \mapsto ye^{-y^2/2u}/\sqrt{2\pi u^3}$ attains its maximum at $y = \sqrt{u}$, in which case the maximal value is $(2\pi u^2)^{-1/2}$. If $u \geq 1 - \Delta$, then $(2\pi u^2)^{-1/2} \leq (2\pi(1-\Delta)^2)^{-1/2}$. Therefore,

$$J_2^\Delta(z, y) \leq \frac{1}{\sqrt{2\pi(1-\Delta)}} \int_{1-\Delta}^1 \frac{1}{\sqrt{2\pi(1-u)}} \, du = \frac{1}{\sqrt{2\pi(1-\Delta)}} \sqrt{\frac{2\Delta}{\pi}}$$

for any $y > 0$ and $z \geq 0$, and this vanishes as $\Delta \to 0$. 

(4.8)
5. Proof of Theorem 2.5

We focus our proof on the case \( x, |x - a| \geq \kappa \sqrt{n} \). As \( n \to \infty \),

\[
\sup_{x, a, \ell \in \mathbb{Z}} |a|, |x - a|, \ell \geq \kappa \sqrt{n} \left[ n \mathbb{P}[S_n = x, \Lambda_n^a = \ell] - \varphi_\nu \left( \frac{a}{\sqrt{n}}, \frac{x - a}{\sqrt{n}}, \frac{\ell}{\sqrt{n}} \right) \right] = o(1) \quad \text{and}
\]

\[
\sup_{a, \ell \in \mathbb{N}} |a|, \ell \geq \kappa \sqrt{n} \left[ \sqrt{n} \mathbb{P}[\Lambda_n^a = \ell] - \psi_\nu \left( \frac{a}{\sqrt{n}}, \frac{\ell}{\sqrt{n}} \right) \right] = o(1).
\]

Indeed, as explained below in Section 5.3, the proof for the cases where \( a = 0 \) or \( |x - a| = 0 \) uses essentially the same arguments (and is in fact easier) and will thus be omitted.

We begin with the path decomposition that serves as the basis of our proof. The idea, illustrated in Figure 1, is to break down a random walk path that visits some level \( a \) a total of \( \ell \) times before time \( n \) into three parts. The first part consists of the steps before the first visit to level \( a \). These steps are counted by the first hitting time \( \tau_a \). The second part consists of the \( \ell - 1 \) excursions of the random walk between consecutive visits to level \( a \). The number of such steps is equal in distribution to a sum of \( \ell - 1 \) i.i.d. copies of \( \tau_0 \). The third part consists of the steps after the last visit to level \( a \). This part behaves like a random walk that avoids returning to zero. More precisely, we have the following.

**Lemma 5.1.** Let \((\omega_k)_{k \in \mathbb{N}}\) be i.i.d. copies of the hitting time \( \tau_0 \), and for all \( \ell \in \mathbb{N} \), let \( \Omega_\ell := \omega_1 + \cdots + \omega_\ell \). For every \( a, x \in \mathbb{Z} \setminus \{0\} \) and \( \ell \in \mathbb{N} \), it holds that

\[
P[S_n = x, \Lambda_n^a = \ell] = \sum_{k=1}^{n} P[\tau_a = k] \sum_{m=1}^{n-k} P[\Omega_{\ell-1} = m] P[\tau_{x-a} = n-k-m],
\]

(5.3)

\[
P[\Lambda_n^a = \ell] = \sum_{k=1}^{n} P[\tau_a = k] \sum_{m=1}^{n-k} P[\Omega_{\ell-1} = m] P[\tau_0 > n-k-m].
\]

(5.4)
Proof. As the proof of (5.4) is easier, we show only (5.3). According to the strong Markov property,
\[
P[S_n = x, \Lambda^a_n = \ell] = \sum_{k=1}^{n} P[\tau_a = k, S_n = x, \Lambda^a_n = \ell]
\]
\[
= \sum_{k=1}^{n} P[\tau_a = k] P[S_{n-k} = x-a, \Lambda^0_{n-k} = \ell-1].
\]
By repeating the same procedure with \(\ell-1\) successive return times to zero, we obtain
\[
P[S_{n-k} = x-a, \Lambda^0_{n-k} = \ell-1] = \sum_{m=1}^{n-k} P[\Omega_{\ell-1} = m] P[S_{n-k-m} = x-a, \tau_0 > n-k-m].
\]
The result then follows from the fact that
\[
P[S_u = y, \tau_u > u] = P[\tau_y = u], \quad y \in \mathbb{Z}, \ u \in \mathbb{N}
\]
(see [Kai75, Section 4]). □

The proof of Theorem 2.5 is thus reduced to an analysis of the probabilities of the events \(\{\tau_a = k\}\) and \(\{\Omega_{\ell-1} = m\}\). The former is taken care of by the local limit theorem (4.2). For the latter, we have the following lemma.

Lemma 5.2. It holds that
\[
\lim_{n \to \infty} \sup_{m, \ell \in \mathbb{N}} \left| \frac{n P[\Omega_{\ell-1} = m] - (\nu \ell e^{-(\nu \ell)^2/2\nu n(m/n)}}{2\pi \nu n(m/n)^3} \right| = 0.
\]

Proof. According to [Kes63, Equation (10.6)],
\[
1 - P[\tau_0 \leq n] \sim \frac{n^{-1/2} \sqrt{2\nu}}{\Gamma(1/2)} = \frac{\sqrt{2\nu}}{\sqrt{\pi n}} \quad \text{as } n \to \infty.
\]
Therefore, it follows from [Fel71, Section XIII.6] that \(\tau_0\) is in the domain of attraction of the distribution with Laplace transform \(e^{-\sqrt{t}}\) \((t \geq 0)\) and with norming sequence \(2\nu n^2\) \((n \in \mathbb{N})\). It is easy to see that the Lévy distribution satisfies this condition:
\[
\int_0^{\infty} e^{-1/4y} \cdot e^{-ty} \, dy = e^{-\sqrt{t}}, \quad t \geq 0,
\]
and thus a straightforward change of variables implies that
\[
\lim_{u \to \infty} P \left[ \frac{\Omega_u}{u^2} \in dy \right] = \frac{\sqrt{\nu e^{-\nu u^2/2m}}}{\sqrt{2\pi y^3}} \, dy.
\]

Clearly, \(\tau_0\) is supported on the lattice \(\mathbb{Z}\). In fact, \(\mathbb{Z}\) is maximal for \(\tau_0\): If \(\tau_0\) is supported on \(m\mathbb{Z}\) for some \(m > 1\), then \(P[S_n = 0] > 0\) only if \(n\) is a multiple of \(m\), which contradicts that \(S_n\) is aperiodic.

By combining the above convergence in distribution with the fact that \(\mathbb{Z}\) is maximal for \(\tau_0\), we conclude from the general local limit theorem for stable distributions [GK68, Theorem in §50] that
\[
u^3 P[\Omega_u = m] = \frac{u^3 \sqrt{\nu e^{-\nu u^2/2m}}}{\sqrt{2\pi m^3}} + o(1) \quad \text{as } u \to \infty,
\]
with $o(1)$ uniform in $m \in \mathbb{N}$. If we let $\kappa_{\ell,n} := (\ell - 1)/\sqrt{n} \geq \kappa + n^{-1/2}$, then it follows from (5.7) that
\[
\begin{align*}
nP[\Omega_{\ell-1} = m] &= (\kappa_{\ell,n} \sqrt{n})^2 P[\Omega_{\kappa_{\ell,n} \sqrt{n}} = m]/\kappa_{\ell,n}^2 \\
&= \frac{\kappa_{\ell,n} n^{3/2} \sqrt{\nu n^2/2m}}{2 \pi n^3} + o(1) \\
&= \frac{(\nu \ell) e^{-2(\nu \ell)^2/2nu(1-k/n)}}{2 \pi n(1-k/n)^3} + o(1),
\end{align*}
\]
with $o(1)$ uniform in $m \in \mathbb{N}$. \hfill \Box

5.1. Proof of (5.1).

5.1.1. Step 1. Our first step is to show that for every fixed $\Delta \in (0, 1)$, one has

\begin{equation}
\lim_{n \to \infty} \sup_{x, a \in \mathbb{Z}, \ell, t \in \mathbb{N}} \left| \sum_{m=1}^{n-k} P[\Omega_{\ell-1} = m]P[\tau_{x-a} = n - k - m] \right| = 0.
\end{equation}

Define
\[
I_{t,\tilde{\Delta}}(y, z) := \int_{\Delta}^{(1-t)(1-\tilde{\Delta})} \frac{ye^{-y^2/2u} - u e^{-u^2/2(1-t-u)}}{\sqrt{2\pi}u} \frac{z e^{-z^2/(2(1-t-u))}}{\sqrt{2\pi}(1-t-u)^3} \, du,
\]
y, $z > 0$, $t, \tilde{\Delta} \in [0, 1)$.

According to Lemma A.1, it holds that
\[
I_{t,0}(y, z) = \frac{(y+z)e^{-((y+z)^2/2(1-t))}}{\sqrt{2\pi}(1-t)^3}, \quad y, z > 0, \ t \in [0, 1).
\]

Thus, (5.8) is bounded above by the sum of the terms
\[
C_1 := \sup_{n \in \mathbb{N}, x, a \in \mathbb{Z}, \ell, t \in \mathbb{N}} \left| \sum_{m=1}^{n-k} P[\Omega_{\ell-1} = m]P[\tau_{x-a} = n - k - m] \right|,
\]
\[
C_2 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}, \ell, t \in \mathbb{N}} \left| \sum_{m=1}^{n-k} P[\Omega_{\ell-1} = m]P[\tau_{x-a} = n - k - m] \right|, \quad n \in \{n-k(1-\Delta)\} + 1,
\]
\[
C_3 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}, \ell, t \in \mathbb{N}} \left| \sum_{m=1}^{n-\tilde{\Delta}} P[\Omega_{\ell-1} = m]P[\tau_{x-a} = n - k - m] \right|, \quad \text{and}
\]
\[
C_4 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}, \ell \geq \kappa \sqrt{n}} \left| \sum_{m=1}^{\lfloor (n-k)(1-\Delta) \rfloor} P[\Omega_{\ell-1} = m]P[\tau_{x-a} = n - k - m] \right| - I_{k/n,\tilde{\Delta}} \left( \frac{\nu \ell}{\sqrt{nu}}, \frac{|x-a|}{\sqrt{nu}} \right),
\]
We prove that these terms vanish as $\tilde{\Delta} \to 0$.

For $\tilde{C}_1$, if we let

\[
I_{i,\tilde{\Delta}}^1(y, z) := \int_{(1-t)(1-\tilde{\Delta})}^{(1-t)} \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} \frac{z e^{-z^2/2(1-t-u)}}{\sqrt{2\pi(1-t-u)^3}} \, du \quad \text{and}
\]
\[
I_{i,\tilde{\Delta}}^2(y, z) := \int_0^{\tilde{\Delta}} \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} \frac{z e^{-z^2/2(1-t-u)}}{\sqrt{2\pi(1-t-u)^3}} \, du,
\]

then it is enough to prove that for any fixed $\Delta$, one has

\[
\lim_{\Delta \to 0} \sup_{y, z \geq \kappa} I_{i,\tilde{\Delta}}^1(y, z) = 0, \quad i = 1, 2.
\]

As shown for (18), it holds that

\[
ye^{-y^2/2u}/\sqrt{2\pi u^3} \leq (2\pi(1-t)^2(1-\tilde{\Delta})^2)^{-1/2} \leq (2\pi\Delta^2(1-\tilde{\Delta})^2)^{-1/2}
\]

for any $y > 0$, $(1-t)(1-\tilde{\Delta}) \leq u$, and $(1-t) \geq \Delta$. Moreover, for $z \geq \kappa$ and $t \geq 0$,

\[
\int_{(1-t)(1-\tilde{\Delta})}^{1-t} \frac{z e^{-z^2/2(1-t-u)}}{\sqrt{2\pi(1-t-u)^3}} \, du = \erfc\left(\frac{z}{\sqrt{2(1-t)\Delta}}\right) \leq \erfc\left(\frac{\kappa}{\sqrt{2\Delta}}\right).
\]

Combining these two bounds gives (5.10) for $i = 1$. Up to trivial modifications, the case $i = 2$ is handled in the same way.

For $\tilde{C}_2$, if we combine the fact that $y \mapsto \frac{y e^{-y^2/2u}/\sqrt{2\pi}}{1-u}$ is uniformly bounded in $y$ with $\ell \geq \kappa \sqrt{n}$, $m \geq (n-k)(1-\tilde{\Delta})$, and $k \leq (1-\Delta)n$, then (5.5) implies that

\[
P[\Omega_{\ell-1} = m] \leq \frac{1}{n} C \leq \frac{1}{n} \frac{C}{(m/n)(1-\tilde{\Delta})} \leq \frac{1}{n} \frac{C}{\Delta(1-\tilde{\Delta})}
\]

for $C > 0$ independent of all parameters. Moreover,

\[
\sum_{m=(n-k)(1-\tilde{\Delta})+1}^{n-k} \Pr[	au_{x-a} = n-k-m] \leq \Pr[	au_{x-a} \leq n\tilde{\Delta}].
\]

Since $|x-a| \geq \kappa \sqrt{n}$, combining these bounds with Remark 4.4 implies that $\tilde{C}_2 \to 0$ as $\tilde{\Delta} \to 0$.

For $\tilde{C}_3$, by combining (4.2), $m \leq \tilde{\Delta}n$, and $k \leq (1-\Delta)n$, we obtain the bound

\[
n \Pr[	au_{x-a} = n-k-m] \leq C \frac{1-\Delta}{n-k/m} \leq \frac{C}{\Delta - \Delta}
\]

for $C$ independent of all parameters. Moreover, it follows from (5.3) that

\[
\sum_{k=1}^{\tilde{\Delta}n} \Pr[\Omega_{\ell-1} = k] = \frac{1}{n} \sum_{k=1}^{\tilde{\Delta}n} \frac{\nu e^{-(\nu t)^2/2\nu n(k/n)}}{\sqrt{2\pi n(k/n)^3}} + o(1)
\]

with $o(1)$ uniform in $\ell \geq \kappa \sqrt{n}$. Given that

\[
f(u, y) := \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}}, \quad (u, y) \in [0, 1] \times [\kappa, \infty)
\]
is uniformly continuous on its specified domain, it follows from a uniform Riemann sum approximation that

$$
\sum_{k=1}^{[\Delta n]} P[\Omega_{\ell-1} = k] = \int_0^\Delta \frac{e^{-(\nu \ell)^2/2\nu n}}{\sqrt{2\pi \nu n}} \, du + o(1)
$$

with $o(1)$ uniform in $\ell \geq \kappa \sqrt{n}$. This can then be controlled as $\Delta \to 0$ in the same way as $A_2$ in Section 4.1

Finally, for $C_4$, if we write

$$
nP[\Omega_{\ell-1} = m]P[\tau_{x-a} = n - k - m] = \frac{1}{n} nP[\Omega_{\ell-1} = m] \frac{n - k - m}{1 - k/n - m/n} P[\tau_{x-a} = n - k - m],
$$

then we see that $\widetilde{C}_4 = 0$ for any fixed $\Delta$ by combining (5.12), (5.5), and Lemma 4.2 applied to the function

$$
f(u; y, z) := \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} z e^{-z^2/2(1-u)} \sqrt{2\pi (1-t-u)^3}, \quad (u; y, z) \in [\Delta, 1 - \Delta] \times \mathbb{R}^2.
$$

5.1.2. Step 2. With (5.8) proved, if we define

$$
I_\Delta(y, z) := \int_{-\Delta}^{1-\Delta} \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} z e^{-z^2/2(1-u)} \sqrt{2\pi (1-t-u)^3} \, du, \quad y, z > 0, \Delta \in [0, 1/2)
$$

so that $I_0(y, z) = (y + z)e^{-(y+z)^2/2\sqrt{\pi}}$ by Lemma 4.1 then we need only show that the following limits can be made arbitrarily small by taking $\Delta \to 0$.

$$
C_1 := \sup_{n \in \mathbb{N}} \sup_{|a|, |x-a|, \ell \in \mathbb{N}, \ell \geq \kappa \sqrt{n}} \left| I_\Delta \left( \frac{|a|}{\sqrt{\nu n}}, \frac{|x-a| + \nu \ell}{\sqrt{\nu n}} \right) - I_0 \left( \frac{|a|}{\sqrt{\nu n}}, \frac{|x-a| + \nu \ell}{\sqrt{\nu n}} \right) \right|
$$

$$
C_2 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}, \ell \in \mathbb{N}, |x-a| \geq \kappa \sqrt{n}} \sum_{k=(1-\Delta)n+1}^{n \wedge k} P[\tau_a = k]
$$

$$
C_3 := \lim_{n \to \infty} \sup_{x, a \in \mathbb{Z}, \ell \in \mathbb{N}, \ell \geq \kappa \sqrt{n}} \left| \sum_{k=(n\Delta)+1}^{(n+1)\Delta} P[\tau_a = k] \frac{(x-a) e^{-((x-a)+\nu \ell)^2/2\nu n(1-k/n)}}{\sqrt{2\pi \nu n(1-k/n)^3}} \right|

- I_\Delta \left( \frac{|a|}{\sqrt{\nu n}}, \frac{|x-a| + \nu \ell}{\sqrt{\nu n}} \right), \text{ and}
$$

$$
C_4 := \limsup_{n \to \infty} \sup_{x, a \in \mathbb{Z}, \ell \in \mathbb{N}, |a|, |x-a|, \ell \geq \kappa \sqrt{n}} \sum_{k=1}^{n \wedge k} P[\tau_a = k] \frac{(x-a) e^{-((x-a)+\nu \ell)^2/2\nu n(1-k/n)}}{\sqrt{2\pi \nu n(1-k/n)^3}}.
$$

For $C_1$, we proceed as in (5.10).
For $C_2$, $k \geq (1 - \Delta)n$ implies by (4.2) that $P[\tau_a = k] \leq C/(1 - \Delta)n$ for $C > 0$ independent of all parameters. Thus, it is enough to control

$$\sup_{|x-a|\leq n} \frac{P[\Omega_{\ell-1} = m]P[\tau_{x-a} = n - k - m]}{n-k}.$$

With the trivial bound $P[\Omega_{\ell-1} = m] \leq 1$, this is bounded by

$$\sum_{m=1}^{n-k} P[\tau_{x-a} = \Delta n - |(1 - \Delta)n - 1 - m| \leq P[\tau_{x-a} \leq \Delta n].$$

Taking a supremum over $|x - a| \geq \kappa \sqrt{n}$, we can prove that this vanishes as $\Delta \to 0$ by Remark 1.3.

For $C_3$, we need only combine (4.2) with Lemma 4.2. Finally, for $C_4$, if we use the bound

$$(|x-a| + \nu \ell) e^{-|x-a|+\nu \ell)^2/2\nu n(1-k/n)} \leq \frac{C}{(1-k/n)^3} \leq \frac{C}{1-\Delta},$$

valid for $k \leq \Delta n$ and any $x, a, \ell$, and $n$, then we need only control

$$\sum_{k=1}^{\lfloor n\Delta \rfloor} P[\tau_a = k] = P[\tau_a \leq \Delta n].$$

Taking a supremum over $|a| \geq \kappa \sqrt{n}$, this vanishes as $\Delta \to 0$.

5.2. **Proof of (5.2).** The technical details of the proof of (5.2) are virtually the same as in (5.1), the only difference being that the terms $P[\tau_{x-a} = n - k - m]$ are replaced by $P[\tau_0 > n - m - k]$. Consequently, we omit the proof. To illustrate the need for a different normalization (i.e., $\sqrt{n}$ instead of $n$) and the appearance of the function $\psi_j(a, \ell) := 2\nu e^{-|a|+\nu \ell)^2/2\nu n(1-k/n)}$, we offer the following heuristic.

By combining (4.2), (5.3), and (5.6) as well as Lemmas A.1 and A.2, we see that

$$\sqrt{n} \sum_{k=1}^{n} P[\tau_a = k] \sum_{m=1}^{n-k} P[\Omega_{\ell-1} = m]P[\tau_0 > n - k - m]$$

$$= \frac{1}{n} \sum_{k=1}^{n} \frac{k}{k/n} P[\tau_a = k] \frac{1}{n} \sum_{m=1}^{n-k} nP[\Omega_{\ell-1} = m] \frac{\sqrt{n-k-m}}{\sqrt{1-k/n-m/n}} P[\tau_0 > n - k - m]$$

$$\approx \int_{0}^{1} \int_{0}^{1} \frac{e^{-|a|+\nu \ell)^2/2\nu n}}{\sqrt{2\pi \nu n} t^3} \frac{e^{-|a|+\nu \ell)^2/2\nu n}}{\sqrt{2\pi (1-t-u)}} \frac{2\sqrt{\nu}}{\sqrt{2\pi (1-t-u)}} \frac{dudt}{\sqrt{2\pi (1-t-u)}}$$

$$= \frac{2\nu}{e^{-|a|+\nu \ell)^2/2\nu n}} \frac{e^{-|a|+\nu \ell)^2/2\nu n}}{\sqrt{2\pi (1-t-u)}}.$$

5.3. **Proof of Theorem 2.5 for Zero Endpoints.** In cases where $a = 0$ or $|x - a| = 0$, we have the following analog of Lemma 5.1.

**Lemma 5.3.** With the same notation as in Lemma 5.1,

$$P[S_n = 0, \Omega_n = \ell] = P[\Omega_n = n] \quad P[\Omega_n = \ell] = \sum_{m=1}^{n} P[\Omega_{\ell-1} = m]P[\tau_0 > n - m].$$
If $a, |x - a| \neq 0$, then

$$P[S_n = x, \Lambda_n^0 = \ell] = \sum_{m=1}^{n} P[\Omega_{\ell - m} = m]P[\tau_{x - a} = n - m]$$

and

$$P[S_n = a, \Lambda_n^a = \ell] = \sum_{k=1}^{n} P[\tau_a = k]P[\Omega_{\ell - k} = n - k].$$

With this result in hand and the full details to the proof of (5.1), it is immediately clear how to proceed.

6. Discussion

6.1. Conditions on the Sizes of $a, |x - a|, \text{ and } \ell$. In this section, we assess the suitability of the methods used in this paper to extend Theorems 2.3 and 2.5 to the cases where $|a|, |x - a|, \text{ and } \ell$ are of order $o(\sqrt{n})$. We will see that the local limit theorems cited in this paper are not powerful enough to undertake this, and we explore what improvements on those results might be enough.

6.1.1. $a = o(\sqrt{n})$ in Theorem 2.3. According to the classical strong ratio theorems for random walk hitting times [Spj76, Section 32], for every fixed $a \in \mathbb{Z} \setminus \{0\}$, it holds that

$$P[\tau_a > n] \sim \frac{P(a)}{\sqrt{n}} \left( \frac{2\nu}{\pi} \right)^{1/2} \text{ as } n \to \infty,$$

where

$$P(a) := \sum_{k=1}^{\infty} (P[S_k = 0] - P[S_k = -a]), \quad a \in \mathbb{Z}.$$

Given that $P(a) \sim |a|/\sqrt{n}^\nu$ as $a \to \infty$ (see, for instance, [Spj76, Page 354]) and that $\text{erf}(x) \sim 2x/\sqrt{\pi}$ as $x \to 0$, the following conjecture seems like a plausible interpolation between (6.1) and (2.2).

**Conjecture 6.1.** Suppose that $a = a_n \to \infty$ as $n \to \infty$ at a rate slower than $\sqrt{n}$. Under Assumption 2.7, it holds that

$$P[\tau_a > n] = \text{erf} \left( \frac{|a|}{\sqrt{2\nu n}} \right) (1 + o(1))$$

as $n \to \infty$.

In principle, our argument in Section 4.1 could be used to study this problem. Indeed, if we write

$$P[\tau_a \leq n] = P[\tau_a \leq a^2 \Delta] + \sum_{k=|a^2 \Delta| + 1}^{n} P[\tau_a = k],$$

then we can control $P[\tau_a \leq a^2 \Delta]$ when $a \to \infty$ and $\Delta \to 0$ by using Lemma 4.3 and we expect from the local limit theorem in Lemma 4.1 and a Riemann sum...
approximation that
\[
\sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{n} P[\tau_a = k] \approx \sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{n} \frac{|a|e^{-a^2/2\nu k}}{\sqrt{2\pi \nu k^3}} \leq \int_{a^2 \Delta/n}^{1} \frac{|a|e^{-a^2/2\nu nu}}{\sqrt{2\pi \nu nu^3}} \, du = \text{erfc} \left( \frac{|a|}{\sqrt{2\nu n}} \right).
\]

However, a precise implementation of this idea does not work because the local limit theorem fails to identify the leading order terms in \(P[S_n = a, \tau_0 > n]\) unless \(a \sim \sqrt{n}\): In its current form, Lemma 4.1 implies that
\[
P[S_n = a, \tau_0 > n] = \frac{|a|e^{-a^2/2\nu n}}{\sqrt{2\pi \nu n^3}} + o(1) \quad \text{as} \quad n \to \infty,
\]
and the rightmost term in (6.2) is of order \(o(1) \cdot \log(\sqrt{n}/a)\) as \(n \to \infty\), which is not guaranteed to vanish if \(a = o(\sqrt{n})\). In fact, even if we have a version of Lemma 4.1 that correctly identifies the leading order term, i.e.,
\[
P[S_n = a, \tau_0 > n] = \frac{|a|e^{-a^2/2\nu n}}{\sqrt{2\pi \nu n^3}} + o\left( \frac{|a|}{n^{3/2}} \right) \quad \text{as} \quad n \to \infty,
\]
then our method still falls short of Conjecture 6.1. Indeed, (6.3) would imply that
\[
\sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{n} P[\tau_a = k] = \sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{n} \frac{|a|e^{-a^2/2\nu k}}{\sqrt{2\pi \nu k^3}} + o(1) \sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{n} \frac{|a|}{k^{3/2}},
\]
and while the rightmost term is now of order \(o(1)\), this is still too weak to identify \(\text{erf}(a/\sqrt{n})\) as the leading order contribution.

6.1.2. \(\ell = o(\sqrt{n})\) in Theorem 2.3. Assuming for simplicity that \(x = a = 0\), we recall from Lemma 5.3 that
\[
P[S_n = 0, A_n^0 = \ell] = P[\Omega_\ell = n], \quad P[A_n^0 = \ell] = \sum_{m=1}^{n} P[\Omega_{\ell-1} = m]P[\tau_0 > n - m].
\]
The leading order of \(P[\tau_0 > n]\) as \(n \to \infty\) is known (see (5.6)), and thus we need only identify the leading order contribution to \(P[\Omega_\ell = n]\). At this point, we run into the same problem discussed in the previous section: The general local limit theorem that we cite in the proof of Lemma 5.2 fails to identify the leading term when \(\ell = o(\sqrt{n})\). Indeed, it holds that
\[
P[\Omega_\ell = n] = \frac{\ell \sqrt{n} e^{-\ell^2/2n}}{2\pi n^3} + o(1) \quad \text{as} \quad \ell \to \infty \quad \text{with} \quad o(1) \quad \text{uniformly in} \quad n,
\]
and this ceases to be optimal when \(\ell = o(\sqrt{n})\).

6.1.3. \(a, |x - a| = o(\sqrt{n})\) in Theorem 2.3. If we assume that \(\ell \geq \kappa \sqrt{n}\) for some \(\kappa\), then the most natural generalization of Theorem 2.3 would be the following.

**Conjecture 6.2.** Under Assumption 2.1 for every \(\kappa > 0\), it holds that
\[
nP[S_n = x, A_n^0 = \ell] = \varphi_{\nu} \left( \frac{a}{\sqrt{n}}, \frac{x - a}{\sqrt{n}}, \frac{\ell}{\sqrt{n}} \right) + o(1) \quad \text{and}
\]
\[
\sqrt{n}P[A_n^0 = \ell] = \psi_{\nu} \left( \frac{a}{\sqrt{n}}, \frac{\ell}{\sqrt{n}} \right) + o(1)
\]
as $n \to \infty$, where $o(1)$ are uniform in $x, a \in \mathbb{Z}$ and $\ell \in \mathbb{N}$ such that $\ell \geq \kappa \sqrt{n}$.

If a result such as (6.3) is true, then there is hope that the methods in this paper could be used to study this question. Indeed, if we consider for simplicity the case where $x = a$, then we can write (recall Lemma 5.3)

$$n \mathbb{P}[S_n = a, \Lambda^a_n = \ell] = n \sum_{k=1}^{\lfloor n(1-\Delta) \rfloor} \mathbb{P}[\tau_a = k] \mathbb{P}[\Omega_{\ell-1} = n-k]$$

$$+ n \sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{\lfloor n(1-\Delta) \rfloor} \mathbb{P}[\tau_a = k] \mathbb{P}[\Omega_{\ell-1} = n-k] + n \sum_{k=n(1-\Delta) \lfloor 1 \rfloor + 1}^{n} \mathbb{P}[\tau_a = k] \mathbb{P}[\Omega_{\ell-1} = n-k].$$

Assuming that $a \to \infty$, the first and third sums on the right-hand side of the above equation could be controlled similarly to $\tilde{C}_2$ and $\tilde{C}_3$ in Section 5.1.1 by taking $\Delta \to 0$. As for the second sum, a combination of Lemma 5.2 and (6.3) would imply that

$$n \sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{\lfloor n(1-\Delta) \rfloor} \mathbb{P}[\tau_a = k] \mathbb{P}[\Omega_{\ell-1} = n-k]$$

$$= \sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{\lfloor n(1-\Delta) \rfloor} \left( \frac{|a|e^{-a^2/2nk^3}}{\sqrt{2\pi nk^3}} + \frac{|a|}{k^{3/2} o(1)} \right) \left( \frac{\nu e^{-(\nu \ell)^2/2\nu n(1-k/n)}}{\sqrt{2\pi n(1-k/n)^3}} + o(1) \right)$$

$$= \frac{1}{n} \sum_{k=\lfloor a^2 \Delta \rfloor + 1}^{\lfloor n(1-\Delta) \rfloor} \frac{|a|e^{-a^2/2nk(k/n)}}{\sqrt{2\pi nk(k/n)^3}} \frac{\nu e^{-(\nu \ell)^2/2\nu n(1-k/n)}}{\sqrt{2\pi n(1-k/n)^3}} + o(1),$$

and then we expect a Riemann sum approximation to yield the result.

6.2. Stable Laws and Infinite Variance. The most fundamental ingredients in the proof of our main results (i.e., Lemmas 4.3 and 5.1) do not require that $X$ has finite variance. Thus, in principle, it seems that the methods in this paper could be used to prove analogous statements in the case where $X$ has a lattice distribution in the domain of attraction of some general stable law (see, for instance, VW09 for conditioned local limit theorems that hold in this level of generality).

Much of the technical difficulties in treating this general case comes from the necessity of dealing with normalizations of the form $n^\alpha v(n)$, where $v$ is a slowly varying function, as well as limiting densities that have no closed-form formula. These problems are compounded by the fact that many of the results that are essential in our arguments, such as Kes63 Theorem 8 and every result in Kai75, have not been proved in such generality. Thus, using the methods in this paper to extend Theorems 2.3 and 2.5 to general stable laws appears to be a challenging undertaking.

6.3. Nonlattice Random Walks. Another interesting generalization of this paper would be to extend Theorems 2.3 and 2.5 for nonlattice random variables and hopefully prove a statement in the style of Car05 Theorem 1 or VW09 Theorem 3. Of course, if $X$ is nonlattice, then the definition of the occupation measure that we use must be different from (1.1).

The existing strong invariance principles involving nonlattice occupation measures (such as BK03 Section 4) suggest that we use an occupation measure of the
form
\[ \sum_{k=1}^{n} 1(a_k \in [a-\varepsilon, a+\varepsilon]), \quad a \in \mathbb{R}, \ n \in \mathbb{N}, \]
where \( \varepsilon > 0 \) is a fixed constant (see Figure 2 below). However, it is clear that a

path decomposition similar to the one described in Figure 1 is not as well behaved in this case. For example, given that successive visits of the walk to \([a-\varepsilon, a+\varepsilon)\) occur at different levels, the time increments between these visits are not i.i.d.

Consequently, it appears that a different path transformation or decomposition, a different occupation measure, or an altogether different approach may be required to tackle this problem.

**Appendix A. Integral Identities**

**Lemma A.1.** For every \( y, z, t > 0 \),
\[
\int_0^t \frac{ye^{-y^2/2u} z e^{-z^2/2(t-u)}}{\sqrt{2\pi u^3} \sqrt{2\pi(t-u)^3}} \, du = \frac{(y + z)e^{-(y+z)^2/2t}}{\sqrt{2\pi t^3}}.
\]

**Proof.** If \( T_y \) denotes the first hitting time of \( y > 0 \) by a standard Brownian motion, then it is well known that
\[
P[T_y \in dt] = \frac{ye^{-y^2/2t}}{\sqrt{2\pi t^3}} \, dt.
\]
According to the strong Markov property, if \( T_y \) and \( T_z \) are independent Brownian hitting times, then \( T_y + T_z \) is equal in distribution to \( T_{y+z} \). Therefore, the result follows directly by a convolution computation. \( \square \)

**Lemma A.2.** For every \( y, t > 0 \) and \( z \geq 0 \),
\[
\int_0^t \frac{ye^{-y^2/2u} e^{-z^2/2(t-u)}}{\sqrt{2\pi u^3} \sqrt{2\pi(t-u)^3}} \, du = \frac{e^{-(y+z)^2/2t}}{\sqrt{2\pi t}}.
\]
Proof. Note that
\[
\frac{\partial}{\partial z} \int_0^t \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} \frac{e^{-z^2/2(t-u)}}{\sqrt{2\pi(t-u)^3}} \, du = - \int_0^t \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} \frac{ze^{-z^2/2(t-u)}}{\sqrt{2\pi(t-u)^3}} \, du.
\]
Therefore, Lemma A.1 implies that
\[
\int_0^t \frac{ye^{-y^2/2u}}{\sqrt{2\pi u^3}} \frac{e^{-z^2/2(t-u)}}{\sqrt{2\pi(t-u)^3}} \, du = e^{-(y+z)^2/2t} + \Phi(y), \quad y, z, t > 0
\]
for some function \(\Phi\).

We note that for any \(y, t > 0\),
\[
\frac{d}{du} \left( -\frac{e^{-y^2/2t}}{\sqrt{2\pi t}} \text{erf} \left( \frac{y}{\sqrt{2t}} \frac{\sqrt{t-u}}{\sqrt{u}} \right) \right) = -\frac{e^{-y^2/2u}}{\sqrt{2\pi u^3}} \frac{1}{\sqrt{2\pi(t-u)}}.
\]
Moreover,
\[
\lim_{u \to t} \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} \text{erf} \left( \frac{y}{\sqrt{2t}} \frac{\sqrt{t-u}}{\sqrt{u}} \right) = 0 \quad \text{and} \quad \lim_{u \to 0} \frac{e^{-y^2/2t}}{\sqrt{2\pi t}} \text{erf} \left( \frac{y}{\sqrt{2t}} \frac{\sqrt{t-u}}{\sqrt{u}} \right) = e^{-y^2/2t}.
\]
This proves the lemma for \(z = 0\) by the fundamental theorem of calculus. If we then extend (A.1) to \(z = 0\) by continuity, this also implies that \(\Phi(y) = 0\) for all \(y\), concluding the proof of the lemma for \(z > 0\). \(\square\)

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