Minimal Superspace Vector Fields for 5D Minimal Supersymmetry

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ABSTRACT

We investigate a minimal superspace description for 5D superconformal Killing vectors. The vielbein appropriate for AdS symmetry is discussed within the confines of this minimal supergeometry.

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(I.) Introduction

With the proposal of the existence of the AdS/CFT correspondence [1], the importance of understanding supersymmetry within the confines of a five dimensional spacetime became of major importance. Additionally, the introduction of five dimensional “semi-phenomenological” models highlighted [2, 3] the importance of this class of models. Most of these descriptions utilize a 5D superspace relying upon on Majorana-symplectic spinors to provide its fermionic coordinates. If this were only description possible, then there could be no question of uniqueness.

In this letter, we wish to initiate study of an aspect of 5D models with simple supersymmetry that has been largely overlooked. To our knowledge, all previous discussions of simple supersymmetry in 5D are based on the use of Majorana-symplectic spinors. However, the use of Majorana-symplectic spinors is not dictated by any fundamental principle. It is instead the legacy of the historical development of the field. So we wish to begin to investigate the alternative choice where the 5D superspace does not rely upon Majorana-symplectic spinors to provide its fermionic coordinates.

(II.) The Spinor Metric and Gamma Matrices in Five Dimensions

In a five-dimensional Lorentzian spacetime, the minimal spinors are four component complex objects that we can denote by $\theta^\alpha$. We may choose a representation of the Dirac gamma matrices according to the conventions below.

\begin{equation}
(\gamma^a)_{\alpha \beta} \equiv (\sigma^2 \otimes I_2, i\sigma^3 \otimes I_2, i\sigma^1 \otimes \sigma^1, i\sigma^1 \otimes \sigma^2, i\sigma^1 \otimes \sigma^3)
\end{equation}

These satisfy the equation

\begin{equation}
\gamma^a \gamma^b + \gamma^b \gamma^a = -2 \eta^{ab} I_4
\end{equation}

where the Minkowski space metric has the form,

\begin{equation}
\text{diag}(\eta_{ab}) \equiv (-1, 1, 1, 1, 1, 1)
\end{equation}

One feature that we have traditionally used to simplify the task of constructing spinor representations in various dimensions is the introduction of a “spinor metric.” The usual gamma matrices that appear in (1) and (2) may always be interpreted as type (1,1) bispinors, i.e. one spinor index is up and one is down. Matrix multiplication of such objects is always well defined. Interpreting the gamma matrices as elements of a Clifford algebra, together with the identity matrix and all possible higher order
anti-symmetrization of their spacetime vector indices, it is always possible to find a complete set \( \Gamma \) that provide a basis for all \((1,1)\) bispinors. Here we find
\[
\{ \Gamma \}_{(1,1)} \equiv \{ (\mathbb{I})_\alpha^\beta, (\gamma^\mathcal{A})_\alpha^\beta, (\sigma^{ab})_\alpha^\beta \} , \tag{4}
\]
constitutes such a set. The set \( \Gamma_{(1,1)} \) is obviously a sum of the \{1\} \( \oplus \) \{5\} \( \oplus \) \{10\} irreducible representations of the spacetime Lorentz group. The numbers of these matrices is given by \( 1 + 5 + 10 = 16 \) and since our matrices are \( 4 \times 4 \) we require 16 independent matrices for a complete basis. Next we introduce constant type \((0,2)\) and \((2,0)\) bispinors denoted by \( \eta^{\alpha \beta} \) and \( \eta_{\alpha \beta} \) that may be used to raise and lower spinor indices and such that \( \eta_{\gamma \beta} \eta^{\alpha \beta} = \delta_{\gamma}^{\alpha} \). Now the defining properties of \( \eta_{\alpha \gamma} \), the “spinor metric” and \( \eta^{\alpha \gamma} \), the “inverse spinor metric” are that when they are used to form \( \{ \Gamma \}_{(2,0)} \) and \( \{ \Gamma \}_{(0,2)} \), all elements of within each irreducible Lorentz representation are simultaneously members of the same of representation of \( S_2 \) on the exchange of spinor indices.

A spinor metric may be introduced according to
\[
\eta_{\alpha \beta} \equiv (\sigma^{3} \otimes \sigma^{2})_{\alpha \beta} , \quad \eta^{\alpha \beta} \equiv - (\sigma^{3} \otimes \sigma^{2})_{\alpha \beta} , \quad \eta_{\alpha \beta} = - \eta_{\beta \alpha} , \quad \eta^{\alpha \beta} = - \eta^{\beta \alpha} . \tag{5}
\]
This spinor metric may used to raise or lower one spinorial index on the gamma matrices
\[
(\gamma^\mathcal{A})_\alpha^\beta \equiv (\gamma^\mathcal{A})_\alpha^\gamma \eta_{\gamma \beta} , \quad (\gamma^\mathcal{A})^{\alpha \beta} \equiv \eta^{\alpha \gamma} (\gamma^\mathcal{A})_{\gamma \beta} , \tag{6}
\]
whereupon we find,
\[
(\gamma^\mathcal{A})_\alpha^\beta = - (\gamma^\mathcal{A})_{\beta \alpha} , \quad (\gamma^\mathcal{A})^{\alpha \beta} = - (\gamma^\mathcal{A})^{\beta \alpha} . \tag{7}
\]
The 5D spinor matrices whose commutator algebra is isomorphic to the Lorentz algebra is proportional to
\[
(\sigma^{ab})_\alpha^\beta \equiv i \frac{1}{2} (\gamma^{a} \gamma^{b} - \gamma^{b} \gamma^{a})_\alpha^\beta . \tag{8}
\]
The spinor metric may also used to raise or lower one spinorial index on this quantity
\[
(\sigma^{ab})_\alpha^\beta \equiv (\sigma^{ab})_\alpha^\gamma \eta_{\gamma \beta} , \quad (\sigma^{ab})^{\alpha \beta} \equiv \eta^{\alpha \gamma} (\sigma^{ab})_{\gamma \beta} , \tag{9}
\]
whereupon we find,
\[
(\sigma^{ab})_\alpha^\beta = (\sigma^{ab})_{\beta \alpha} , \quad (\sigma^{ab})^{\alpha \beta} = (\sigma^{ab})^{\beta \alpha} . \tag{10}
\]
So we have the following decomposition of \( \Gamma_{(2,0)} \)
\[
\Gamma_{(2,0)} = \Gamma_{(2,0)S} \oplus \Gamma_{(2,0)A} , \tag{11}
\]
\[
\Gamma_{(2,0)S} = \{ (\sigma^{ab})_\alpha^\beta \} , \quad \text{dim}(\Gamma_{(2,0)S}) = \frac{4 \cdot 5}{1 \cdot 2} = 10 ,
\]
\[
\Gamma_{(2,0)A} = \{ \eta_{\alpha \beta}, (\gamma^\mathcal{A})_\alpha^\beta \} , \quad \text{dim}(\Gamma_{(2,0)A}) = \frac{4 \cdot 3}{1 \cdot 2} = 6 = 1 + 5 ,
\]
where to calculate the dimensionality of the symmetric and antisymmetric subspaces of $\Gamma_{(2,0)}$ we simply have to take into account the symmetry or antisymmetry of the subspace and the dimensionality over which the spinor indices range. Similar features are observed for $\Gamma_{(0,2)}$.

Another useful object to introduce is a spinorial Levi-Civita tensor $\epsilon^{\alpha\beta\gamma\delta}$ with $\epsilon^{1234} \equiv +1$. Again the spinor metric can lower indices

$$
\epsilon^{\alpha\beta\gamma\delta} \equiv \epsilon^{\kappa\lambda\phi\varepsilon} \eta_{\kappa\alpha} \eta_{\lambda\beta} \eta_{\phi\gamma} \eta_{\varepsilon\delta} ,
$$

so that

$$
\epsilon^{\alpha\beta\gamma\delta} \epsilon^{\kappa\lambda\phi\varepsilon} = \delta^{[\alpha}_{[\kappa} \delta^{\beta]}_{\beta]} \delta^{\gamma]}_{\gamma]} \delta_{\phi]}_{\phi]}. \tag{13}
$$

The spinorial $\epsilon$-tensor also satisfies

$$
\frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} \eta_{\gamma\delta} = \eta^{\alpha\beta}, \quad \frac{1}{2} \epsilon^{\alpha\beta\gamma\delta} (\gamma^a)_{\gamma\delta} = - (\gamma^a)^{\alpha\beta}. \tag{14}
$$

The set of matrices given by

$$
(I_4)_{\alpha\beta} , \quad (\gamma^a)_{\alpha\beta} , \quad (\sigma^{a\beta})_{\alpha\beta} , \tag{15}
$$

constitute a complete set of matrices over which type $(1,1)$ bi-spinors can be expanded. Similarly, the following two sets are such bases for type $(2,0)$ and $(0,2)$ bi-spinors, respectively.

$$
(I_4)_{\alpha\beta} , \quad (\gamma^a)_{\alpha\beta} , \quad (\sigma^{a\beta})_{\alpha\beta} , \tag{16}
$$

$$
(I_4)^{\alpha\beta} , \quad (\gamma^a)^{\alpha\beta} , \quad (\sigma^{a\beta})^{\alpha\beta} . \tag{17}
$$

There are also Fierz identities,

$$
\epsilon_{\alpha\beta\gamma\delta} = \frac{1}{2} \eta_{\alpha\beta} \eta_{\gamma\delta} + \frac{1}{2} (\gamma^a)_{\alpha\beta} (\gamma^a)_{\gamma\delta} ,
$$

$$
\delta^{[\alpha}_{[\beta} \delta^{\gamma]}_{\gamma]} = \frac{1}{2} \eta_{\alpha\beta} \eta_{\gamma\delta} - \frac{1}{2} (\gamma^a)_{\alpha\beta} (\gamma^a)_{\gamma\delta} ,
$$

$$
\delta_{[\alpha} \delta^{\beta]} \delta^{\gamma]}_{\gamma]} = - \frac{1}{4} (\sigma^{a\beta})_{\alpha\beta} (\sigma^{a\beta})_{\gamma\delta} ,
$$

$$
(\gamma^a)_{\alpha} \gamma (\gamma^a)_{\beta} \gamma = - \frac{1}{4} \left[ 5 \eta_{\alpha\beta} \eta_{\gamma\delta} + 3 (\gamma^a)_{\alpha\beta} (\gamma^a)_{\gamma\delta} - \frac{1}{2} (\sigma^{a\beta})_{\alpha\beta} (\sigma^{a\beta})_{\gamma\delta} \right] . \tag{18}
$$

The representation of the 5D gamma-matrices that we have given also satisfies,

$$
[(\gamma^a)_{\alpha}^{\beta}]^* = + (C \gamma^a C^{-1})_{\alpha}^{\beta} , \quad C_{\alpha}^{\beta} \equiv \sigma^1 \otimes \sigma^2 ,
$$

$$
\rightarrow C = C^{-1} \ & \ C_{\alpha}^{\gamma} \eta_{\gamma\beta} = - C_{\beta}^{\gamma} \eta_{\gamma\alpha} , \tag{19}
$$

$$
[(\sigma^{a\beta})_{\alpha}^{\beta}]^* = - (C \sigma^{a\beta} C^{-1})_{\alpha}^{\beta} = - (C \sigma^{a\beta} C)_{\alpha}^{\beta} .
$$
The multiplication table for the complete set of $4 \times 4$ matrices is given by

\[ \gamma^a \gamma^b = -\eta^{ab} I_4 - i \sigma^{ab} , \]  

(20)

\[ \gamma^a \sigma^{bc} = -i \eta^{[a} \gamma^{bc]} - \frac{1}{2} \epsilon^{a [bc} \sigma_{de]} \]  

(21)

\[ \sigma^{ab} \gamma^c = i \eta^{c [a} \gamma^{b]} - \frac{1}{2} \epsilon^{ac} \sigma_{bd} \gamma^d \]  

(22)

\[ \sigma^{ab} \sigma^{cd} = \eta^{ac} \gamma^d - \eta^{ad} \gamma^c - \epsilon^{abcde} \gamma^e \]  

(23)

The spinorial supercovariant derivatives for the superspace associated with these conventions can be defined by

\[ D_\alpha \equiv \partial_\alpha + \frac{1}{2} (\gamma^a)_{\alpha \beta} C_\gamma^\beta \bar{\theta}^\gamma \partial_\bar{a} , \quad \bar{D}_\beta \equiv C_\beta^\gamma \bar{\gamma} \partial_\gamma - \frac{1}{2} \eta^{ab} \gamma^\gamma \partial_\bar{a} . \]  

(24)

These satisfy the relations

\[ [D_\alpha , \bar{D}_\beta] = (\gamma^a)_{\alpha \beta} C_\gamma^\beta \bar{\theta}^\gamma \partial_\bar{a} , \quad [D_\alpha , D_\beta] = [\bar{D}_\alpha , \bar{D}_\beta] = 0 , \]  

(25)

\[ (D_\alpha)^* = -C_\alpha^\beta \bar{D}_\beta , \quad (\bar{D}_\alpha)^* = -C_\alpha^\beta D_\beta . \]

Note that the last equations imply

\[ ((D_\alpha)^*)^* = -D_\alpha , \quad ((\bar{D}_\alpha)^*)^* = -\bar{D}_\alpha , \]  

(26)

which is characteristic of a spacetime in which it is not possible to define Majorana spinors.

At this point, it has long been the custom to append an SU(2) index to the spinor coordinates of the superspace ($\theta^\alpha \rightarrow \theta^{\alpha i}$) and to further impose a “Majorana” condition on the enlarged spinorial coordinate,

\[ (\theta^{\alpha i})^* \equiv \bar{\theta}^a_i = C_{ij} \theta^{\alpha j} , \]  

(27)

which may be consistently imposed on the simplectic spinor $\theta^{\alpha i}$. So the net effect of all of this is to first double the number of independent spinorial coordinates ($\theta^\alpha \rightarrow \theta^{\alpha i}$) and then cut this number in half (back to four independent spinorial coordinates) by the imposition of the condition in (27). This naturally raises the question of what supergeometrical structures occur in the absence of this process?

(III.) Simple Poincaré Supervector Fields in Minimal 5D Superspace
The supersymmetry algebra associated with this minimal superspace can be read off from the results in (24) and (26). The generators of the minimal 5D superspace include a complex 4-component spinor $Q_\alpha$ (and $\overline{Q}_\alpha$), the Lorentz 5-vector translation operator $P_\underline{a}$ and the second-rank Lorentz rotation generator $L_{\underline{ab}}$.

\[
P_\underline{a} \equiv i \partial_\underline{a} \quad , \quad L_{\underline{ab}} \equiv i x_4 \partial_\underline{a} - \frac{1}{2} \theta^\alpha (\sigma_{\underline{ab}})^\alpha_\beta \partial_\beta - \frac{1}{2} \bar{\theta}^\alpha (\sigma_{\underline{ab}})_{\alpha}^\beta \bar{\partial}_\beta \quad ,
\]

\[
Q_\alpha \equiv i \left[ \partial_\alpha - \frac{1}{2} (\gamma^\underline{a})_{\alpha}^\beta C_{\gamma} \bar{\partial}_\gamma \partial_\underline{a} \right] \quad , \quad \overline{Q}_\beta \equiv i \left[ C_\beta \gamma \bar{\partial}_\gamma + \frac{1}{2} (\gamma^\underline{a})_{\beta}^\gamma \theta^\gamma \partial_\underline{a} \right] .
\]

The only non-vanishing commutators are given by

\[
\begin{align*}
[ Q_\alpha , \overline{Q}_\beta ] & = i (\gamma^\underline{a})_{\alpha}^\beta P_\underline{a} \quad , \quad [ L_{\underline{ab}}, Q_\beta ] = \frac{1}{2} (\sigma_{\underline{ab}})^\alpha_\beta Q_\beta \quad , \\
[ L_{\underline{ab}}, \overline{Q}_\beta ] & = \frac{1}{4} (\sigma_{\underline{ab}})^\alpha_\beta \overline{Q}_\beta \quad , \quad [ L_{\underline{ab}}, P_\underline{a} ] = -i \eta_{\underline{ab}} L_{\underline{cd}} + i \eta_{\underline{cd}} L_{\underline{ab}} , \quad (30)
\end{align*}
\]

as expected. So the super Poincaré group has a representation in terms of the super-vector fields constructed from the derivatives with respect coordinates of the minimal 5D superspace.

\(\text{(IV.) Conformal Supervector Fields in Minimal 5D Superspace}\)

To describe the superconformal algebra requires that we introduce additional operators over and above those required to describe the super Poincaré algebra. In particular, we must introduce $S_\alpha$, $\overline{S}_\alpha$, $K_\underline{a}$, $\Delta$, $J^+$, $J^-$ and $J^0$ for the s-supersymmetry (and its conjugate), the special conformal boosts, dilatation generator and SU(2) charge operators, respectively. The most interesting of the generators is the s-supersymmetry generator (and its conjugate) defined by,

\[
S_\alpha \equiv \left[ -i x^\underline{a} (\gamma^\underline{a})_{\alpha}^\beta \overline{Q}_\beta + C_\rho \delta_{\beta} (\gamma^\underline{a})_{\alpha}^\rho \theta^\sigma \theta_\alpha \partial_\underline{a} + \frac{1}{4} (\gamma^\underline{a} C)_{\alpha}^\delta \bar{\theta}_\delta \theta^\rho \theta_\alpha \partial_\underline{a} \\
+ \frac{3}{4} C_\rho \delta_{\beta} (\gamma^\underline{a})_{\alpha}^\rho \theta^\sigma \theta_\alpha \partial_\underline{a} + \theta_\alpha \bar{\theta}^\rho \bar{\theta}_\rho - \frac{1}{2} C_\delta \sigma \bar{\theta}_\sigma \theta_\alpha \gamma \bar{\partial}_\gamma - 2 C_\alpha \delta_{\beta} \theta^\sigma C_\epsilon \gamma \bar{\partial}_\epsilon \gamma - \theta_\alpha \theta^\rho \partial_\rho + \frac{1}{2} \theta^\rho \theta_\rho \partial_\alpha \right] , \quad (31)
\]

\[
\overline{S}_\alpha \equiv s_1 \left[ -i x^\underline{a} (\gamma^\underline{a})_{\alpha}^\beta Q_\beta + \frac{1}{4} (\gamma^\underline{a})_{\alpha}^\beta \delta_{\beta} \bar{\theta}_\rho \partial_\underline{a} + \theta_\rho (\gamma^\underline{a} C)_{\alpha}^\rho \bar{\theta}^\sigma \bar{\theta}_\sigma C_\alpha \gamma \bar{\partial}_\gamma \\
+ \frac{3}{4} C_\rho \delta_{\beta} (\gamma^\underline{a} C)_{\alpha}^\rho \bar{\theta}^\sigma \bar{\theta}_\sigma - C_\alpha \gamma \bar{\partial}_\gamma \theta^\rho \partial_\rho + \frac{1}{2} \bar{\theta}^\rho \bar{\theta}_\rho C_\alpha \gamma \bar{\partial}_\gamma \right] , \quad (32)
\]

The coefficient $s_1 = \pm 1$ for the present. We also require additional bosonic genera-
The full commutator algebra with the choice $s_1 = -1$ is given below:

\[
\begin{align*}
\{ S_\alpha, \overline{S}_\beta \} &= + i (\gamma^2)_{\alpha\beta} K_\alpha, \quad \{ L_{ab}, K_c \} = - i \eta_{e[a} K_{b]} \ , \\
\{ L_{ab}, S_\alpha \} &= \frac{i}{2} (\sigma_{ab})^{\beta} S_\beta, \quad \{ L_{ab}, \overline{S}_\alpha \} = \frac{i}{2} (\sigma_{ab})^{\beta} \overline{S}_\beta \ , \\
\{ S_\alpha, P_\beta \} &= - (\gamma_\alpha)_\gamma \overline{Q}_\gamma, \quad \{ \overline{S}_\alpha, P_\beta \} = + (\gamma_\alpha)_\gamma Q_\gamma \ , \\
\{ Q_\alpha, K_\beta \} &= + (\gamma_\alpha)_\gamma S_\gamma, \quad \{ \overline{Q}_\alpha, K_\beta \} = - (\gamma_\alpha)_\gamma \overline{S}_\gamma \ , \\
\{ Q_\alpha, S_\beta \} &= - i \frac{1}{2} (\sigma_{ab})_{\alpha\beta} L_{ab} + \eta_{\alpha\beta} [ \Delta + 3 J^0 ] \ , \\
\{ \overline{Q}_\alpha, \overline{S}_\beta \} &= - i \frac{1}{2} (\sigma_{ab})_{\alpha\beta} L_{ab} - s_1 \eta_{\alpha\beta} [ \Delta + 3 J^0 ] \ , \\
\{ \Delta, Q_\alpha \} &= - i \frac{1}{2} Q_\alpha, \quad \{ \Delta, S_\alpha \} = i \frac{1}{2} S_\alpha \ , \\
\{ \Delta, \overline{Q}_\alpha \} &= - i \frac{1}{2} \overline{Q}_\alpha, \quad \{ \Delta, \overline{S}_\alpha \} = i \frac{1}{2} \overline{S}_\alpha \ , \\
\{ J^0, Q_\alpha \} &= - \frac{1}{2} Q_\alpha, \quad \{ J^0, S_\alpha \} = \frac{1}{2} S_\alpha \ , \\
\{ J^0, \overline{Q}_\alpha \} &= \frac{1}{2} \overline{Q}_\alpha, \quad \{ J^0, \overline{S}_\alpha \} = - \frac{1}{2} \overline{S}_\alpha \ , \\
\{ J^+, Q_\alpha \} &= 0, \quad \{ J^+, S_\alpha \} = + \frac{1}{\sqrt{2}} \overline{S}_\alpha \ , \\
\{ J^+, \overline{Q}_\alpha \} &= - \frac{1}{\sqrt{2}} Q_\alpha, \quad \{ J^+, \overline{S}_\alpha \} = 0 \ , \\
\{ J^-, Q_\alpha \} &= - \frac{1}{\sqrt{2}} Q_\alpha, \quad \{ J^-, S_\alpha \} = 0 \ , \\
\{ J^-, \overline{Q}_\alpha \} &= 0, \quad \{ J^-, \overline{S}_\alpha \} = + \frac{1}{\sqrt{2}} S_\alpha \ , \\
\{ \Delta, P_\alpha \} &= - i P_\alpha, \quad \{ \Delta, K_\alpha \} = i K_\alpha \ , \\
\{ P_\alpha, K_\beta \} &= + i 2 L_{ab} - i 2 \eta_{ab} [ \Delta + 3 J^0 ] \ , \\
\{ \overline{Q}_\alpha, S_\beta \} &= - i 3 \sqrt{2} \eta_{\alpha\beta} J^- \ , \quad \{ Q_\alpha, \overline{S}_\beta \} = + i 3 \sqrt{2} \eta_{\alpha\beta} J^+ \ , \\
\{ J^+, J^0 \} &= J^- \ , \quad \{ J^0, J^- \} = J^+ \ , \quad \{ J^-, J^+ \} = J^0 \ .
\end{align*}
\]

The major feature that emerges from the construction of the supervector fields that generate the superconformal group is the unavoidable appearance of a triplet of SU(2) generators ($J^+, J^0, J^-$).

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The terms in the ellipsis below can explicitly be determined from the first result in (34).

\(^4\)The terms in the ellipsis below can explicitly be determined from the first result in (34).
Simple AdS Supervector Fields in Minimal 5D Superspace

Once the supervectors fields for a conformal group are known for a superspace, there follows as an immediate consequence the possibility of realizing an anti-de-Sitter (AdS) supersymmetry. The key point is that the AdS supersymmetry generators (\( Q \)) and AdS translation operators (\( P \)) can be defined in terms of operators for the conformal group by the equations

\[
Q_\alpha = Q_\alpha + \lambda S_\alpha , \quad \overline{Q}_\alpha = Q_\alpha - s_1 \lambda S_\alpha , \quad P_\underline{a} = P_\underline{a} + |\lambda|^2 K_\underline{a} .
\]

All of these features were discussed many years ago in *Superspace*. The non-vanishing terms in the 5D AdS supersymmetry algebra take the form

\[
[ Q_\alpha , Q_\beta ] = \iota (\gamma^a)_{\alpha \beta} P_\underline{a} + \iota s_1 (\lambda + \bar{\lambda}) (\sigma^{ab})_{\alpha \beta} L_{\underline{a} \underline{b}}
\]

\[
+ s_1 (\lambda - \bar{\lambda}) \eta_{\alpha \beta} (\Delta + 3 \mathcal{J}^0 ) , \quad (36)
\]

The second of these has the characteristic form of the algebra of translation operators in a space of constant negative curvature. One of the uses of this information is that it allows the construction of the rigid AdS limit of five dimensional supergravity in minimal superspace. This is done first by noting that the supergravity vielbein superfield can be directly defined in terms of the AdS supervector fields defined immediately above.

The first step in this construction is to introduce a AdS background-fixed vielbein superfield \( \tilde{E}_{\underline{a} M}(\theta, \bar{\theta}, x) \). We next observe that there exist a canonical relation between the supervector fields above and the AdS background-fixed vielbein superfield. This relation is,

\[
Q_\alpha \equiv \iota \tilde{E}_\alpha^M(-\theta, -\bar{\theta}, x) D_M , \quad P_\underline{a} \equiv \iota \tilde{E}_\underline{a}^M(-\theta, -\bar{\theta}, x) D_M . \quad (37)
\]

Upon solving both of the above equation for \( \tilde{E}_{\underline{a} M}(\theta, \bar{\theta}, x) \), one obtains an explicit expression for the vielbein superfield. Since the superconformal vector fields are \( \lambda \)-dependent, the AdS background-fixed vielbein superfield \( \tilde{E}_{\underline{a} M} \) is also \( \lambda \)-dependent. The general superspace measure (“D-term”) measure for the five dimensional superspace of minimal supersymmetry is then

\[
\int d\mu_{AdS} = \int d^5 x d^4 \theta d^4 \bar{\theta} [s\text{det}(\tilde{E}_{\underline{a} M})]^{-1} . \quad (38)
\]

There remains the separate problem of how to construct the AdS superspace chiral measure. This is a problem that will have to be addressed in a future work.

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5The parameter \( s_1 \) may conveniently be set equal to minus one.
(VI.) Discussion and Future Prospectives

As we have seen, there appear no major impediments to a hitherto unconventional description of simple supersymmetry in a five dimensional superspace. The formalism that we have established is based on the fact that complex spinor 4-component spinors provide a perfectly adequate basis for describing the fermionic components of a 5D superspace. The super Poincaré group associated with this construction is consistent with either a U(1) or SU(2) holonomy. The superconformal group associated with this construction requires a triplet of SU(2) generators to close.

One of our purposes to studying the realization of superconformal vector fields in the present context was in preparation for an investigation on how superconformal symmetry is geometrically realizable (GR) in all superspaces. Between the experience gained in the present investigation and that of some previous works, one is led to expect that superconformal Killing vector can be realized in all dimensions! In terms of supergeometry, there is every reason to expect that 10D and 11D generalizations of the results in (29), (32) and (33) should exist.

This statement is at variance with long held beliefs about this topic. There are “no-go” theorems which purport to limit the spacetime dimensions in which superconformal symmetry can appear. In fact, an examination of the purported proofs reveals that none of them are based on the vector bundles naturally associated with the differential geometry of Salam-Strathdee superspace. As such these “proofs” have no a priori validity with regard to placing restrictions on the representations carried by superfields. As is often the case for no-go theorems in mathematical physics, their restrictions can often be avoided by relaxing one or more of the assumptions that go into their construction. The caveat that allows for the existence of geometrically realizable superconformal symmetries for all superspaces is likely the existence of new charges in the algebra that are multi-spinor representations of the Lorentz sub-algebra.

This solution has been hinted at by much of the recent literature on conformal symmetry especially in 11D supersymmetric models. The extra 2-form and 5-form charges to which appropriate branes are expected to couple play this role. In the model-independent description of centerless super-Virasoro algebras [1], in particular, new charges (called the “U” and “R” charges) must occur. Relating these 1D charges to the 11D 2-brane and 5-brane charges seems a worthy task. Upon reduction to 1D, all the generators of the higher D GR superconformal algebra may be expected
to become the complete set of generators for the 1D $N$-extended super $GR$ Virasoro algebras.

“The absurd has meaning only in so far as it is not agreed to” – Albert Camus.

References

[1] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231.

[2] P. Hořava and E. Witten, Nucl. Phys. B460 (1998) 506.

[3] E.A.Mirabelli and M.E.Peskin, Phys. Rev. D58 (1998) 065002.

[4] S. J. Gates, Jr. and L. Rana, Phys. Lett. 438B (1998) 80 [hep-th/9806038]; C. Curto, S. J. Gates, Jr. and V. G. J. Rodgers, Phys. Lett. 450B (2000) 337 [hep-th/0002010]; S. J. Gates, Jr. and V. G. J. Rodgers, Phys. Lett. B512 (2001) 189 [hep-th/0105161]; A. Boviea, S. J. Gates, Jr., D. M. Kimberly, B. A. Larson and V. G. J. Rodgers, Phys. Lett. 529B (2002) 222 [hep-th/0201094].