A note on anomalous Jacobians in 2 + 1 dimensions

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Abstract

There exist local infinitesimal redefinitions of the fermionic fields, which may be used to modify the strength of the coupling for the interaction term in massless $QED_3$. Under those (formally unitary) transformations, the functional integration measure changes by an anomalous Jacobian, which (after regularization) yields a term with the same structure as the quadratic parity-conserving term in the effective action. Besides, the Dirac operator is affected by the introduction of new terms, apart from the modification in the minimal coupling term. We show that the result coming from the Jacobian, plus the effect of those new terms, add up to reproduce the exact quadratic term in the effective action. Finally, we also write down the form a finite decoupling transformation would have, and comment on the likelihood of that transformation to yield a helpful answer to the non-perturbative evaluation of the fermionic determinant.

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1 Introduction

It has been known for quite a long time that some (non trivial) \(1+1\) dimensional partition functions may be exactly evaluated by performing a suitably chosen ‘decoupling’ change of variables [1]. That change of variables is such that, when the action is written in terms of the new fields, there is no interaction term. The infinitesimal version of this transformation, on the other hand, makes it possible to reduce the interaction term by a constant factor. Of course, that means that the coupled and decoupled theories are classically equivalent. Quantum mechanically, however, the need for a regularization of the path integral measure induces (when there are anomalies) a non trivial anomalous Jacobian [2]. This Jacobian, a purely quantum artifact, takes into account effects that would otherwise come from loop corrections in a standard perturbative calculation, and should therefore be included into the effective action. The infinitesimal form of the Jacobian is completely determined by the anomaly, which in turn may be exactly calculated, and it is not affected by radiative corrections [3]. This technique may be applied to solve some models exactly, like QCD in \(1+1\) dimensions [4].

Of course, the applicability of this procedure relies on the property that there exist local changes of variables whose net effect is tantamount to a modification of the coupling constant, something which is far from true for an arbitrary theory. We shall deal here with the particular case of fermionic determinants in the presence of background gauge fields.

Although it is always possible (both for the Abelian and non Abelian cases) to construct non local field redefinitions, that decouple the fields order by order in perturbation theory [3], local, formally unitary, transformations have not yet been used to achieve the same goal. Indeed, extensions of this powerful method to higher dimensional systems are lacking, except in very particular cases and restricted field configurations. In reference [6], local field redefinitions have been used to obtain the parity violating part of the effective action in three dimensional \(\text{QED}\), a result that can also be obtained by generalized parity transformations of the fermions [7]. The parity conserving part is much harder to find by an application of this method, since it does not seem to be related to an anomalous Jacobian.

It is our purpose in this note to show that there are, indeed, transformations in \(2+1\) dimensions that share some properties of their \(1+1\) dimensional counterparts, like locality and formal unitarity, allowing one to derive also the quadratic parity conserving part of the effective action. The re-
sult comes from the evaluation of two functional determinants, both of them naively equal to one, but nevertheless conspiring to produce the correct answer when properly regulated. The main difference with the 1+1 dimensional case shall be that the infinitesimal transformations also generate new, unwanted interactions. Their effect can be accounted for infinitesimally, but we will see that, for a finite transformation, the effect is so complicated that the method becomes not applicable. It will turn out that the main difference between the 2+1 and 1+1 dimensional cases shall be that in the former the decoupling transformation is not a neat redefinition of the coupling, but it modifies the Dirac operator as well. There are, however, some striking similarities, like the fact that at least part of the effective action can be obtained from anomalous Jacobians.

To proceed, we begin by giving a lighting review of the procedure for the case of a massless fermion field in the presence of an Abelian external field, in 1+1 dimensions. There, one wants to evaluate the Euclidean functional integral \( Z[A] \), corresponding to massless fermions in the presence of an external gauge field \( A_\mu \),

\[
Z[A] = \int D\bar{\psi} D\psi e^{-S[\bar{\psi}, \psi; A]},
\]

where

\[
S[\bar{\psi}, \psi; A] = \int d^2x \bar{\psi} D_A \psi,
\]

and \( D_A \) is the Dirac operator in the presence of an \( A_\mu \) background field. Namely, for massless fermions in a flat spacetime, \( D_A = \partial A_\mu \), where \( D_A \) is the gauge covariant derivative: \( D_A = \partial + iA \).

We first recall that, in any number of spacetime dimensions, we can get rid of the longitudinal part of the gauge field, by a proper local transformation of the fermions. Indeed, decomposing \( A \) into transverse and longitudinal parts,

\[
A_\mu = A^\perp_\mu + \partial_\mu \varphi
\]

where \( \partial_\mu A^\perp_\mu = 0 \), the change of fermionic variables:

\[
\psi(x) \rightarrow \exp[-i\varphi(x)]\psi(x), \quad \bar{\psi}(x) = \bar{\psi}(x) \exp[i\varphi(x)]
\]

(which is non anomalous) leads to the identity:

\[
Z[A] = Z[A^\perp].
\]
In what follows, we shall assume that this has already been done, so that our starting point shall be $Z[A]$ with a transverse $A$. We shall omit the explicit $\perp$ notation, since transversality of $A$ will always be assumed.

Then, the infinitesimal change of variables
\[
\psi(x) \rightarrow \psi(x) + \xi \gamma_5 \alpha(x) \psi(x)
\]
\[
\bar{\psi}(x) \rightarrow \bar{\psi}(x) + \xi \bar{\psi}(x) \alpha(x) \gamma_5
\]
with the condition $\epsilon_{\mu\nu} \partial_\nu \alpha = A_\mu$, changes the action $S$ as follows:
\[
S[\bar{\psi}, \psi; A] \rightarrow S[\bar{\psi}, \psi; (1 - \xi) A] = \int d^2 x \bar{\psi} [\emptyset + i(1 - \xi) A] \psi .
\]

On the other hand, the measure acquires a Jacobian:
\[
J = \exp[-\frac{\xi}{\pi} \int d^2 x \alpha(x) \epsilon_{\mu\nu} \partial_\mu A_\nu] = \exp[+\frac{\xi}{\pi} \int d^2 x A_\mu A_\mu] .
\]
\[
= \exp[\frac{\xi}{2\pi} \int d^2 x F_{\mu\nu} \frac{1}{\partial^2} F_{\mu\nu}] .
\]
When properly iterated, this procedure may be used to obtain the correct answer for a finite transformation [1]. In short, there appears an extra $\frac{1}{2}$ factor before the non local Maxwell action (we are of course regarding $A$ as purely transverse).

In the present letter, we shall be concerned with a functional integral like (1), but with $S$ now denoting the $2 + 1$ dimensional Euclidean action:
\[
S[\bar{\psi}, \psi; A] = \int d^3 x \bar{\psi} D_A \psi , \quad D_A = \emptyset + i A ,
\]
the Dirac’s matrices being in a representation such that $\gamma_0 = \sigma_3$, $\gamma_1 = \sigma_1$, and $\gamma_2 = \sigma_2$, with $\sigma_j$ the standard Pauli matrices.

Regarding the actual form of the decoupling transformations, there is of course no equivalent to $\gamma_5$ in $2 + 1$ dimensions. Nevertheless, it is evident that to modify $A_\mu$, the transformation must involve the $\gamma$ matrices, indeed, the same transformation we used in [3] does the job. The main difference is that now we will pick up the parity conserving part of the corresponding Jacobian, by using a parity-conserving regulator. We shall see that the evaluation of that Jacobian involves a subtle point.

Thus, taking into account the constraints of unitarity and locality, we are led to consider the infinitesimal transformations:
\[
\delta \psi(x) = -\xi \emptyset(y(x)) \psi(x) , \quad \delta \bar{\psi}(x) = -\xi \bar{\psi}(y(x)) ,
\]
where $\xi$ is infinitesimal, and $b_\mu(x)$ is a vector field. $Z(A)$ cannot change under a change of variables like (10), however, it yields an equivalent way of writing it:

$$Z[A] = J_\xi[b] \times \int \mathcal{D}\bar{\psi}\mathcal{D}\psi e^{-S_\xi[\bar{\psi},\psi;A]} ,$$  \hspace{1cm} (11)

where $J_\xi$ is the Jacobian for the transformation (10), and

$$S_\xi[\bar{\psi},\psi;A] = \int d^3x \bar{\psi} \left[ \partial + i\gamma_\mu (A_\mu - \xi \tilde{f}_\mu(b)) - \xi (b \cdot D + D \cdot b) \right] \psi$$ \hspace{1cm} (12)

with $\tilde{f}_\mu(b) = \epsilon_{\mu\nu\lambda} \partial_\nu b_\lambda$. It is clear that, if we choose $b_\mu$ such that $\tilde{f}_\mu(b) = A_\mu$, then the minimal coupling term will be suppressed (if $\xi > 0$) by a $(1 - \xi)$ factor, although there have appeared two unwelcome extra terms, which were not present in the original action. Since $\xi$ is infinitesimal, we may also write (11) as:

$$Z(A) = J_\xi \times K_\xi \times Z[(1 - \xi)A] ,$$ \hspace{1cm} (13)

where

$$K_\xi(A) = \det[1 - \xi \mathcal{D}^{-1}(b \cdot D + D \cdot b)]$$ \hspace{1cm} (14)

is a factor that takes care of the extra term in the action. Thus, to first order in $\xi$, the change in the functional integral is determined by the two factors $J_\xi$ and $K_\xi$, of which $J_\xi$ is a Jacobian. Note that, to first order in $\xi$, both are ill-defined, since they involve a $0 \times \infty$.

Let us consider now each factor separately. The Jacobian $J_\xi(b)$, is

$$J_\xi(b) = [\det(1 - \xi \mathcal{D})]^{-2}$$ \hspace{1cm} (15)

with a negative power because of the Grassmann character of the fields, and a 2 because both $\psi$ and $\bar{\psi}$ contribute by the same amount. Thus for an infinitesimal $\xi$,

$$J_\xi = \exp[2\xi \mathcal{A}]$$ \hspace{1cm} (16)

with

$$\mathcal{A} = \text{Tr}[\mathcal{D}] .$$ \hspace{1cm} (17)

The (functional and Dirac) trace of $\mathcal{D}$ suffers from the same kind of ill behaviour the trace of $\gamma_5$ times a scalar field does in $1 + 1$ dimensions. Hence the need for a regularization. However, a delicate question arises here, since a naive calculation easily misleads one to a wrong answer: let $\mathcal{A}_\Lambda$, the regularized version of (17) be

$$\mathcal{A}_\Lambda \equiv \text{Tr} \left[ \psi f(-\frac{D^2}{\Lambda^2}) \right]$$ \hspace{1cm} (18)
where Λ is an UV cutoff, and f a regularizing function, verifying:
\[ f(0) = 1, \quad f(\pm \infty) = f'(\pm \infty) = \ldots f^{(k)}(\pm \infty) = \ldots = 0. \quad (19) \]

Taking, for example, \( f(x) = \frac{1}{1+x} \), we see after some standard manipulations that the only term that does not vanish when \( \Lambda \to \infty \) is given (in Fourier space) by:
\[
A_\Lambda = i \int \frac{d^3k}{(2\pi)^3} \tilde{b}_\mu(-k) I(k) \epsilon_{\mu\nu\lambda} k_\nu \tilde{A}_\lambda(k)
\]
where the tilde denotes the Fourier transformed of the corresponding object, while \( I(k) \) is defined by the expression:
\[
I(k) = -2\Lambda^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{(p^2 + \Lambda^2)[(p + k)^2 + \Lambda^2]}.
\]

Being the momentum integral convergent, we evaluate it and consider its \( \Lambda \to \infty \) limit. This leads to a result diverging linearly with \( \Lambda \). Using afterwards the relation between \( A_\mu \) and \( b_\mu \), the upshot of this procedure is a *divergent* answer for \( A_\Lambda \):
\[
A_\Lambda \sim \Lambda \int d^3x A_\mu A_\mu, \quad \Lambda \sim \infty,
\]
which is dimensionally correct: in our conventions, the mass dimension of \( A \) is 1. Note that no perturbative approximation in \( A \) has been used, the \( \Lambda \to \infty \) limit selects that contribution, and higher order terms simply fade away, as negative powers of \( \Lambda \).

The reason for the (wrong) result (22), is that it is obtained by regarding \( b_\mu \) as a local field, while it is related non locally to \( A_\mu \). Indeed, the relation between \( A \) and \( b \) is momentum dependent, since
\[
b_\mu = -\frac{1}{\partial^2} \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda,
\]
and this relation must be used before doing the momentum integration, since we really want to know the Jacobian as a function of \( A_\mu \), which is a true local field \[^1\]. Using this relation before the integration, there appears a quite different momentum space integral, since now we may write \( A_\Lambda \) as follows:
\[
A_\Lambda = \int \frac{d^3k}{(2\pi)^3} \tilde{F}_{\mu\nu}(-k) L(k) \tilde{F}_{\mu\nu}(k)
\]

\[^1\]Note that the locality of the fermionic field transformation is beyond any question: the new field value at point \( x \) does not involve the values of the fermionic field at different points.
where $\tilde{F}_{\mu\nu}(k) = i(k_{\mu}\tilde{A}_{\nu} - k_{\nu}\tilde{A}_{\mu})$, $L(k)$ is defined by:

$$L(k) = -\Lambda^2 \int \frac{d^3p}{(2\pi)^3} \frac{1}{p^2 (p^2 + \Lambda^2)((p + k)^2 + \Lambda^2)},$$

and we have only kept terms that do not vanish when $\Lambda \to \infty$.

Surprisingly enough, $L$ is *convergent* when $\Lambda \to \infty$:

$$L(k) \to -\frac{1}{|k|}, \quad \Lambda \to \infty.$$  \hfill (26)

In coordinate space, the answer for $J_\xi$ is then:

$$J_\xi = \exp \left[ -\frac{\xi}{4} \int d^2 F_{\mu\nu}(A) \frac{1}{\sqrt{-\partial^2}} F_{\mu\nu}(A) \right],$$

which, except for the factor, has the same structure of the exact quadratic parity conserving term, known to be:

$$(Z[A])_{\text{quad.}} = \exp \left[ -\frac{1}{4} \int d^3 x F_{\mu\nu} \frac{1}{16 \sqrt{-\partial^2}} F_{\mu\nu} \right].$$ \hfill (28)

The $K_\xi$ factor may be evaluated in a very similar way, namely, by using the regulating function $f(x) = \frac{1}{1+x}$ and keeping only the non-vanishing terms when $\Lambda \to \infty$. The result has of course the same structure, and differs from the contribution of $J_\xi$ only by a factor:

$$K_\xi = \exp \left[ \frac{\xi}{4} \int d^3 x F_{\mu\nu} \frac{7}{8\sqrt{-\partial^2}} F_{\mu\nu} \right].$$ \hfill (29)

Putting together both $J_\xi$ and $K_\xi$, we see that:

$$\ln Z[A] - \ln Z[(1 - \xi)A] = \int d^3 x \left[ -\frac{\xi}{4} \int d^3 x F_{\mu\nu} \frac{1}{8\sqrt{-\partial^2}} F_{\mu\nu} + O(\xi^2) \right],$$ \hfill (30)

which is the exact result we where looking for, since it leads to (28) by a simple integration.

We end up by considering the form a finite version of the decoupling transformation might have. We apply it to the free Dirac operator, $\partial$, to see whether the interacting operator may be reproduced or not by a change of variables, as it happens in 1+1 dimensions. We expect ‘spurious’ terms, of
course, but it is interesting to see their explicit form. They will be, in fact, the real obstruction for a local, finite decoupling transformation to exist.

To be unitary, and to reduce to the infinitesimal transformations we have considered, we must have:

\[
\psi(x) \rightarrow [a(x) - \Psi(x)]\psi(x) \\
\bar{\psi}(x) \rightarrow \bar{\psi}(x)[a(x) - \Psi(x)]
\]

(31)

where \(a^2 - b_\mu b^\mu = 1\). Under this transformation, the free action written in the new variables may be written in the following form:

\[
S = \int d^3x \bar{\psi}(x) \left\{ \phi + [\partial \cdot bb_\mu - ia\bar{f}_\mu + b \cdot (\partial b_\mu) - i\epsilon_{\mu\nu\lambda}b_\nu \partial_\lambda a]\gamma_\mu \\
+ [i\epsilon_{\mu\nu\lambda}b_\mu \partial_\nu b_\lambda - (a\partial \cdot b + (b \cdot \partial a)) \right\}.
\]

(32)

One can immediately see that it is not possible to fulfill at the same time the conditions required to have the proper coupling term and no extra term at the same time. The conditions to have the coupling term with an external field \(A_\mu\) cannot me met, since the corresponding equations are not compatible. This negative result is to be expected, since we knew that the infinitesimal version already produces new terms in the action.

We may then conclude by saying that the main obstruction to the application of the local decoupling transformations in higher dimensions is ‘kinematical’, since there are no (local) transformations which amount to just a redefinition of the coupling. The fact that part of the effective action may indeed be obtained from an anomalous change of variables is, surprisingly, true.

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