The Features Radiation of a Charged Particle Moving Along an Arc of a Circle

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Abstract

A detailed analysis is given of the angular distribution of an charged particle moving along the arc of a circle. The areas of different angular distribution behavior are highlighted and studied.

Keywords: synchrotron radiation, angular distribution of power radiation, arc of a circle.

Recently discovered new properties of synchrotron radiation (SR) (see [1] and subsequent works [2-5]) gave a new impulse to the development of the SR theory. One of the questions of the SR theory is the study of the radiation characteristics of an electron moving along an arc of a circle. Such motion is considered as an integral part of more complex motion trajectories, available, for example, in synchrotrons, undulators or wigglers.

The main theoretical results in this area were obtained by using classical methods of electrodynamics in the works of Bagrov, Ternov, Fedosov [6-7].

A numerical analysis of the radiation structure was carried out and asymptotic formulas were in the articles [8-10] and in monograph [11]. The analysis was carried out for the particle velocity $v = \beta c$, where $c$ is the speed of light and $\beta > \sqrt{2/5}$.

In 1980s the results obtained for this speed range were considered sufficient (the a non relativistic and ultra relativistic cases). At the same time, a full study required significant computing power. Modern computer tools have solved this problem.

In this paper, a detailed analysis of the angular distribution of radiation was provided for charged particles moving in a plane with a constant velocity $v = \beta c$. Particle moves under the arc of a circle of opening angle $2\gamma$ in the plane of the orbit (Fig. 1.). The beta value takes values from 0 to 1.

We used the expressions for the angular distribution of radiation obtained in [6,7] by the classical electrodynamics methods. For numerical analysis we use exact expressions distribution of the average power of radiation integrated over the frequency for the $\sigma$– and $\pi$– components of linear polarization in the form:

$$d\varepsilon = W_0 TF(\beta, \gamma; \Theta, \varphi) \, d \Omega, \quad F = F_\sigma + F_\pi,$$

$$F_{\sigma,\pi}(\beta, \gamma; \Theta, \varphi) = \Phi_{\sigma,\pi}^0 + \Phi_{\sigma,\pi}^0(\varphi + \gamma) + \Phi_{\sigma,\pi}^1(\varphi - \gamma),$$

where $W_0$ — is total power synchrotron radiation of an electron moving in a circle of radius $R$, $T$ — is time of movement on an arc of circle, $\Theta, \varphi$ — are the polar and azimuthal angles of spherical coordinates, the direction of observation of radiation characterizing.
The functions $\Phi^0_{\sigma, \pi}(\beta, \Theta)$ describe the angular distribution of the average per revolution of synchrotron radiation:

$$\Phi^0_{\sigma}(\beta, \Theta) = \frac{3 (1 - \beta^2)^2 (4 + 3\mu^2)}{64\pi (1 - \mu^2)^{5/2}}, \quad \Phi^0_{\pi}(\beta, \Theta) = \frac{3 (1 - \beta^2)^2 (4 + \mu^2) \cos^2 \Theta}{64\pi (1 - \mu^2)^{7/2}},$$

where $\mu = \beta \sin \Theta$, $0 \leq \beta \leq 1$.

$$\Phi^1_{\sigma}(x) = Q(1 - \mu^2) \left\{ \frac{6\mu p^4 (4 + 3\mu^2)}{(1 - \mu^2)^{1/2}} \arctg \left( \frac{\mu \sin x}{p + (1 - \mu^2)^{1/2}} \right) + \left[ (23\mu^2 - 2) p^3 + (9\mu^2 - 2) (1 - \mu^2) p^2 - 2 (1 - \mu^2)^2 p + 6 (1 - \mu^2)^3 \right] \sin x \right\},$$

$$\Phi^1_{\pi}(x) = Q \cos^2 \Theta \left\{ \frac{6\mu p^4 (4 + \mu^2)}{(1 - \mu^2)^{1/2}} \arctg \left( \frac{\mu \sin x}{p + (1 - \mu^2)^{1/2}} \right) + \left[ (2 + 13\mu^2) p^3 + (3\mu^2 + 2) (1 - \mu^2) p^2 + 2 (1 - \mu^2)^2 p - 6 (1 - \mu^2)^3 \right] \sin x \right\},$$

where $Q = (1 - \beta^2) \left[ 128\pi \gamma \mu (1 - \mu^2)^3 p^1 \right]^{-1}, \quad p = 1 - \mu \cos x$.

The formulas (1)-(5) describe a continuous transition from the instantaneous angular distribution of synchrotron radiation to the average per revolution when the opening angle $2\gamma$ changes from 0 to $\pi$.

Since the angular distribution is symmetric with respect to the plane $y = 0$, in numerical analysis we restrict ourselves to considering angles $0 \leq \varphi \leq \pi$.

A numerical analysis of the $\sigma$–component linear polarization of the radiation shows that with increasing particle velocity the radiation concentration occurs in the orbit plane. This is completely consistent with the known results from the SR theory. Therefore further we consider radiation in the orbit plane $\Theta = \pi/2$. In this case, the angular distribution is conveniently represented as a function

$$\chi(\varphi) = F_{\sigma} \left( \beta, \gamma, \frac{\pi}{2}, \varphi \right), \quad 0 \leq \gamma \leq \pi.$$

Fig. 1: The arc of a circle traced by a moving particle
The function $\chi(\varphi)$ always has extrema at the points $\varphi = 0, \varphi = \pi$ for any $\beta, \gamma$. Besides this function may have extrema at points $\varphi = \varphi_k$, $(k = 1, 2, 3, 4)$.

The functions $\varphi_k$ satisfy inequalities

$$0 \leq \varphi_1 < \varphi_2 < \gamma < \varphi_3 < \varphi_4 < \pi$$

(7)

Let’s consider these functions in more detail. The functions $\varphi_k$ can be obtained as solutions of the equation $\chi'(\varphi) = 0$. The solutions of this equation substantially depend on the particle velocity $\beta$. The threshold value here is $\beta = 2/5$. When passing through this value, significant changes are observed in the structure of the angular distribution associated with the appearance of two additional extrema. For $0 < \beta \leq 2/5$ there are only two internal extrema $\varphi_2(\beta, \gamma)$ and $\varphi_3(\beta, \gamma)$. For $2/5 < \beta < 1$ the radical expression in formula for $\chi'(\varphi)$ becomes positive and extrema appear $\varphi_1(\beta, \gamma)$ and $\varphi_4(\beta, \gamma)$. Thus we consider two cases: $0 < \beta \leq 2/5$ and $2/5 < \beta < 1$.

**Angular distribution at $0 < \beta \leq 2/5$**

Solving the equation $\chi'(\varphi) = 0$ for $0 < \beta \leq \frac{2}{5}$ and, discarding the roots $\varphi = 0, \varphi = \pi$, we obtain two solutions $\varphi_2(\beta, \gamma)$ and $\varphi_3(\beta, \gamma)$. These functions are considered as implicit functions for the fixed values of $\beta$. See the Fig. 2.

![Fig. 2: The structure of extrema $\varphi_2, \varphi_3$ of the function $\chi(\varphi)$ for $\beta < 2/5$](image)

Let us consider the function $\psi_1(\beta)$ at which $\varphi_2(\beta, \gamma)$ reaches a minimum. This function is exactly found and it is equal to $\psi_1(\beta) = \arccos(\beta))$. In this case, we have three areas at which the behavior of the function $\chi(\varphi)$ is uniquely determined. The distribution of extremes is shown on the Fig. 2. At low values of beta (less than 2/5) there are only three qualitatively different extremum locations (Fig. 3-5).
The structure of the angular distribution becomes more complex with increasing particle velocity. The new additional extrema $\varphi_1$ and $\varphi_4$ appear. This is due to the fact that the equation $\chi'(\varphi) = 0$ has new roots that do not exist for $\beta < 2/5$. We define the auxiliary function $\psi_2(\beta)$ as the maximum of $\psi_4$. This maximum value is reached at $\varphi = \pi$. Thus, we obtain the exact value

$$\psi_2(\beta) = \arccos \frac{5\beta^2 - 2}{3\beta}. \quad (8)$$

It should be noted that a study of the radiation of a charge moving along an arc of a circle in the relativistic case was carried out in [8,9]. However, in [8,9] the threshold value
\( \beta = 2/5 \) was not found and the results were obtained under the assumption that \( \beta \) is greater than \( \sqrt{2/5} \). For the remaining values of \( \beta \) the angular distribution was not studied.

Numerical calculations made it possible to determine five areas related to the angle \( \gamma \), where the radiation structure changes significantly with increasing \( \beta \).

The key intervals here are: \( 2/5 < \beta < 1/2 \), \( 1/2 < \beta < \sqrt{2/5} \) and \( \beta > \sqrt{2/5} \). The angular structure of radiation for \( 2/5 < \beta < 1/2 \) is shown in Fig. 6.

![Fig. 6: The structure of extrema \( \varphi_k \) of the function \( \chi(\varphi) \) for \( 2/5 < \beta < 1/2 \).](image)

In the range \( 2/5 < \beta < 1/2 \) the condition \( \psi_1(\beta) > \psi_2(\beta) \) is satisfied. When the particle reaches the velocity \( \beta = 1/2 \), the equality holds \( \psi_1(\beta) = \psi_2(\beta) \). With a further increase of the particle velocity, it turns into an inequality \( \psi_1(\beta) < \psi_2(\beta) \). See Fig. 7. This fact explains the changes in angular distribution structure.

The next significant change in the structure of radiation occurs when the velocity of particle \( \beta > \sqrt{2/5} \). In this case zone (area) IV is located above zone III in contrast to Fig. 6, where the picture is opposite. See Fig. 8.

The behavior of the function \( \chi(\varphi) \) in this case (\( \beta > \sqrt{2/5} \)) was studied in [9]. The numerical analysis shows that for \( 0 \leq \gamma < \psi_1 \) (zone I in Fig. 8) the point \( \varphi = 0 \) is the global (absolute) maximum, the point \( \varphi = \varphi_3 \) — the global (absolute) minimum, the point \( \varphi = \varphi_4 \) — the local maximum and the point \( \varphi = \pi \) is the local minimum.

For \( \psi_1 \leq \gamma < \pi - \psi_2 \) (zone II in Fig. 8) the point \( \varphi = 0 \) is the local minimum, the point \( \varphi = \varphi_2 \) is the global (absolute) maximum, the point \( \varphi = \varphi_3 \) — the global (absolute) minimum, the point \( \varphi = \varphi_4 \) — the local maximum and the point \( \varphi = \pi \) is the local minimum.

For \( \pi - \psi_2 \leq \gamma < \psi_2 \) (zone III in Fig. 8) the function \( \chi(\varphi) \) has in the points \( \varphi = \varphi_1, \varphi_3, \pi \) the minimums and in the points \( \varphi = 0, \varphi_2, \varphi_4 \) — maximums.

For \( \psi_2 \leq \gamma < \pi - \psi_1 \) (zone IV in Fig. 8) the function \( \chi(\varphi) \) has the five extreme points. The point \( \varphi = 0 \) is the global (absolute) maximum, the point \( \varphi = \varphi_1 \) is the local minimum, the point \( \varphi = \varphi_2 \) — the local maximum, the point \( \varphi = \varphi_3 \) — global (absolute) minimum and the point \( \varphi = \pi \) is the local maximum.
For $\pi - \psi_1 \leq \gamma < \pi$ (zone V in Fig. 8) the point $\varphi = 0$ is the global (absolute) maximum, the point $\varphi = \varphi_1$ is the local minimum, the point $\varphi = \varphi_2$ — the local maximum and the point $\varphi = \pi$ is the global (absolute) minimum.

For velocities $\beta$ close to unity, asymptotic formulas for $\varphi_k$ were obtained in [9]

$$\varphi_{1,2}(\beta, \gamma) = \gamma + \sqrt{\varepsilon_0} \pm \varepsilon_0^2 (2\sqrt{2}\sin^3 \gamma)^{-1},$$

$$\varphi_{3,4}(\beta, \gamma) = \gamma - \sqrt{\varepsilon_0} \pm \varepsilon_0^2 (2\sqrt{2}\sin^3 \gamma)^{-1},$$

where $\varepsilon_0 = 1 - \beta^2$.

The numerical analysis made it possible to clarify the action areas of asymptotic formulas (10)–(11). We can conclude that asymptotic expressions have a large error for small values of the angle $\gamma$ and for $\gamma$ near to $\pi$. The formulas (10) and (11) give good results in the range of angles $\pi/4 < \gamma < 3\pi/4$ for $\beta > 0.9$. This can be seen in more detail from the tables 1-4 (Appendix). The error of $\delta_{\varphi_i}$ of asymptotic formulas was calculated using the formula

$$\delta_{\varphi_i} = \frac{|\varphi_{\text{real}} - \varphi_{\text{asympt}}|}{\varphi_{\text{real}}}, \ i = 1 \ldots 4,$$

where $\varphi_{\text{real}}$ is calculated as the solution of the equation $\chi'(\varphi) = 0$. The results are presented in table 5.

To summarize we can say that numerical analysis and computer simulations allowed us to investigate in sufficient detail the structure of the angular distribution of particle radiation at all kinds of velocity values $0 < \beta < 1$.

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## Appendix

### Table 1

| $\beta$ | $\gamma$ | \(\phi_{\text{real}}\) | \(\phi_{\text{asympt}}\) |
|---------|-----------|---------------------|---------------------|
| 0.7     | $\pi/16$  | 1.386               | 6.967               |
|         | $\pi/8$   | 1.763               | 1.810               |
|         | $\pi/4$   | 2.5979              | 1.515               |
|         | $\pi/2$   | undef.              | 2.2166              |
|         | $3\pi/4$  | undef.              | 3.0857              |
|         | $7\pi/8$  | undef.              | 4.166               |
|         | $15\pi/16$| undef.              | 9.7162              |
| 0.8     | $\pi/16$  | 1.087               | 0.765               |
|         | $\pi/8$   | 1.760               | 0.902               |
|         | $\pi/4$   | 2.377               | 1.259               |
|         | $\pi/2$   | 2.2903              | 2.0376              |
|         | $3\pi/4$  | undef.              | 3.910               |
|         | $7\pi/8$  | undef.              | 4.166               |
|         | $15\pi/16$| undef.              | 9.7162              |
| 0.9     | $\pi/16$  | 0.569               | 0.961               |
|         | $\pi/8$   | 1.109               | 1.107               |
|         | $\pi/4$   | 1.8921              | 1.8864              |
|         | $\pi/2$   | undef.              | 2.678               |
|         | $3\pi/4$  | undef.              | 3.1211              |
|         | $7\pi/8$  | undef.              | 3.7101              |
| 0.99    | $\pi/16$  | 0.3444              | 0.061               |
|         | $\pi/8$   | 1.2394              | 0.765               |
|         | $\pi/4$   | 2.12498             | 1.9952              |
|         | $\pi/2$   | undef.              | 2.5313              |
|         | $3\pi/4$  | undef.              | -2.6257             |
| 0.999   | $\pi/16$  | 0.24121             | 0.4375              |
|         | $\pi/8$   | 0.83013             | 0.83013             |
|         | $\pi/4$   | 2.40101             | 2.40101             |
|         | $\pi/2$   | undef.              | 2.7936              |
|         | $3\pi/4$  | undef.              | 2.9901              |

### Table 2

| $\beta$ | $\gamma$ | \(\phi_{\text{real}}\) | \(\phi_{\text{asympt}}\) |
|---------|-----------|---------------------|---------------------|
| 0.7     | $\pi/16$  | -11.4743            | -11.4743            |
|         | $\pi/8$   | -0.5340             | -0.5340             |
|         | $\pi/4$   | 1.2394              | 1.2394              |
|         | $\pi/2$   | 2.1930              | 2.1930              |
|         | $3\pi/4$  | 2.8102              | 2.8102              |
|         | $7\pi/8$  | 1.8222              | 1.8222              |
|         | $15\pi/16$| -8.7254             | -8.7254             |
| 0.8     | $\pi/16$  | 0.7375              | -5.3746             |
|         | $\pi/8$   | 1.1924              | 1.1924              |
|         | $\pi/4$   | 1.3689              | 1.3689              |
|         | $\pi/2$   | 2.1627              | 2.1627              |
|         | $3\pi/4$  | 2.8939              | 2.8939              |
|         | $7\pi/8$  | 2.5315              | 2.5315              |
|         | $15\pi/16$| -2.6257             | -2.6257             |
| 0.9     | $\pi/16$  | 0.5804              | -1.0867             |
|         | $\pi/8$   | 0.7998              | 0.6008              |
|         | $\pi/4$   | 1.2167              | 1.1852              |
|         | $\pi/2$   | 2.0080              | 1.9939              |
|         | $3\pi/4$  | 2.7618              | 2.7560              |
|         | $7\pi/8$  | 2.9570              | 2.9570              |
|         | $15\pi/16$| 1.6622              | 1.6622              |
| 0.95    | $\pi/16$  | 0.4739              | 0.4739              |
|         | $\pi/8$   | 0.6924              | 0.6924              |
|         | $\pi/4$   | 1.0969              | 1.0969              |
|         | $\pi/2$   | 1.8848              | 1.8848              |
|         | $3\pi/4$  | 2.6066              | 2.6066              |
|         | $7\pi/8$  | 3.0204              | 3.0204              |
|         | $15\pi/16$| undef.              | undef.              |
| 0.99    | $\pi/16$  | 0.3321              | 0.3321              |
|         | $\pi/8$   | 0.5328              | 0.5328              |
|         | $\pi/4$   | 0.9266              | 0.9266              |
|         | $\pi/2$   | 1.7125              | 1.7125              |
|         | $3\pi/4$  | 2.4973              | 2.4973              |
|         | $7\pi/8$  | 2.8945              | 2.8945              |
| 0.999   | $\pi/16$  | 0.2412              | 0.2412              |
|         | $\pi/8$   | 0.4375              | 0.4375              |
|         | $\pi/4$   | 0.83013             | 0.83013             |
|         | $\pi/2$   | 1.6155              | 1.6155              |
|         | $3\pi/4$  | 2.4009              | 2.4009              |
|         | $7\pi/8$  | 2.7937              | 2.7937              |
|         | $15\pi/16$| 2.9876              | 2.9876              |
### Table 3

| $\beta$  | $\gamma$ | $\pi/16$ | $\pi/8$ | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $7\pi/8$ | $15\pi/16$ |
|---------|---------|----------|---------|---------|---------|---------|---------|-----------|
| 0.7     | $\varphi_{\text{real}}$ | undef.   | undef.  | 0       | 0.8815  | 1.6727  | 2.0969  | 2.2741    |
|         | $\varphi_{\text{asympt}}$  | 11.867   | 1.3194  | 0.3314  | 0.9486  | 1.9022  | 3.6756  | 14.6159   |
| 0.8     | $\varphi_{\text{real}}$ | undef.   | undef.  | 0.2477  | 0.9789  | 1.7727  | 2.2014  | 2.2903    |
|         | $\varphi_{\text{asympt}}$  | 5.7673   | 0.6103  | 0.3150  | 1.0166  | 1.8858  | 2.9665  | 8.5162    |
| 0.9     | $\varphi_{\text{real}}$ | undef.   | undef.  | 0.3798  | 1.1336  | 1.9249  | 2.3417  | 2.5612    |
|         | $\varphi_{\text{asympt}}$  | 1.4794   | 0.1846  | 0.3856  | 1.1477  | 1.9564  | 2.5407  | 4.2283    |
| 0.95    | $\varphi_{\text{real}}$ | undef.   | 0.1212  | 0.4810  | 1.2568  | 2.0447  | 2.4492  | 2.6657    |
|         | $\varphi_{\text{asympt}}$  | 0.3367   | 0.1404  | 0.4827  | 1.2619  | 2.0535  | 2.4966  | 3.0856    |
| 0.99    | $\varphi_{\text{real}}$ | 0        | 0.2547  | 0.6443  | 1.4294  | 2.2150  | 2.6088  | 2.8095    |
|         | $\varphi_{\text{asympt}}$  | 0.0741   | 0.2541  | 0.6447  | 1.4299  | 2.2155  | 2.6103  | 2.8230    |
| 0.999   | $\varphi_{\text{real}}$ | 0.1518   | 0.34801 | 0.7407  | 1.5261  | 2.3115  | 2.7042  | 2.9006    |
|         | $\varphi_{\text{asympt}}$  | 0.1518   | 0.3480  | 0.7407  | 1.5261  | 2.3115  | 2.7042  | 2.9007    |

### Table 4

| $\beta$  | $\gamma$ | $\pi/16$ | $\pi/8$ | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $7\pi/8$ | $15\pi/16$ |
|---------|---------|----------|---------|---------|---------|---------|---------|-----------|
| 0.7     | $\varphi_{\text{real}}$ | undef.   | undef.  | undef.  | 0.5437  | 1.379   | 1.6677  | 1.7554    |
|         | $\varphi_{\text{asympt}}$  | 12.9026  | -1.9623 | -0.1888 | 0.7647  | 1.3820  | 0.3939  | -10.153   |
| 0.8     | $\varphi_{\text{real}}$ | undef.   | undef.  | undef.  | 0.8513  | 1.6312  | 1.9492  | 2.1627    |
|         | $\varphi_{\text{asympt}}$  | -6.5746  | -1.0249 | 0.0558  | 0.9250  | 1.6266  | 1.3313  | -3.8257   |
| 0.9     | $\varphi_{\text{real}}$ | undef.   | 0.2315  | 1.1040  | 1.8824  | 2.2395  | 2.3764  |           |
|         | $\varphi_{\text{asympt}}$  | -1.9585  | -0.2709 | 0.3134  | 1.1221  | 1.8842  | 2.0853  | 0.7904    |
| 0.95    | $\varphi_{\text{real}}$ | undef.   | 0.4512  | 1.2495  | 2.0322  | 2.41036 | 2.5730  |           |
|         | $\varphi_{\text{asympt}}$  | -0.5685  | 0.0205  | 0.4636  | 1.2552  | 2.0344  | 2.3767  | 2.1803    |
| 0.99    | $\varphi_{\text{real}}$ | undef.   | 0.2470  | 0.6434  | 1.4294  | 2.2143  | 2.6059  | 2.7972    |
|         | $\varphi_{\text{asympt}}$  | 0.0364   | 0.2491  | 0.6439  | 1.4296  | 2.2147  | 2.6053  | 2.7853    |
| 0.999   | $\varphi_{\text{real}}$ | undef.   | 0.3480  | 0.7407  | 1.5261  | 2.3115  | 2.7042  | 2.9005    |
|         | $\varphi_{\text{asympt}}$  | 0.1514   | 0.3479  | 0.7407  | 1.5261  | 2.3115  | 2.7042  | 2.9003    |
| $\beta$ | $\gamma$ | $\pi/16$ | $\pi/8$ | $\pi/4$ | $\pi/2$ | $3\pi/4$ | $7\pi/8$ | $15\pi/16$ |
|--------|----------|----------|----------|----------|----------|----------|----------|----------|
| 0.7    | $\delta_{\varphi_1}$ | 859%     | 86%      | 0.17%    | 8.5%     | -        | -        | -        |
|        | $\delta_{\varphi_2}$ | 1423%    | 151%     | 15.6%    | 3%       | -        | -        | -        |
|        | $\delta_{\varphi_3}$ | -        | -        | undef.   | 7.6%     | 13.7%    | 75%      | 543%     |
|        | $\delta_{\varphi_4}$ | -        | -        | -        | 41%      | 0.21%    | 76%      | 678%     |
| 0.8    | $\delta_{\varphi_1}$ | 541%     | 51%      | 0.33%    | 3.2%     | -        | -        | -        |
|        | $\delta_{\varphi_2}$ | 828%     | 85%      | 8.26%    | 1.7%     | 2.3%     | -        | -        |
|        | $\delta_{\varphi_3}$ | -        | -        | 27%      | 3.85%    | 6.4%     | 35%      | 272%     |
|        | $\delta_{\varphi_4}$ | -        | -        | 8.65%    | 0.28%    | 32%      | 32%      | 277%     |
| 0.9    | $\delta_{\varphi_1}$ | 207%     | 17%      | 0.16%    | 0.89%    | 2.8%     | -        | -        |
|        | $\delta_{\varphi_2}$ | 287%     | 24.9%    | 2.6%     | 0.7%     | 0.2%     | -        | -        |
|        | $\delta_{\varphi_3}$ | -        | -        | 1.5%     | 1.2%     | 1.6%     | 8.5%     | 65%      |
|        | $\delta_{\varphi_4}$ | -        | -        | 35%      | 1.6%     | 0.1%     | 6.9%     | 67%      |
| 0.95   | $\delta_{\varphi_1}$ | 69%      | 4.6%     | 0.16%    | 0.3%     | 0.4%     | -        | -        |
|        | $\delta_{\varphi_2}$ | 88%      | 6.8%     | 0.8%     | 0.27%    | 0.06%    | 0.64%    | -        |
|        | $\delta_{\varphi_3}$ | -        | 16%      | 0.34%    | 0.4%     | 0.4%     | 1.9%     | 16%      |
|        | $\delta_{\varphi_4}$ | -        | -        | 2.76%    | 0.46%    | 0.11%    | 1.4%     | 15%      |
| 0.99   | $\delta_{\varphi_1}$ | 3.4%     | 0.1%     | 0.05%    | 0.01%    | 0.02%    | -        | -        |
|        | $\delta_{\varphi_2}$ | 4%       | 0.29%    | 0.06%    | 0.045%   | 0.017%   | 0.2%     | -        |
|        | $\delta_{\varphi_3}$ | undef.   | 0.2%     | 0.06%    | 0.03%    | 0.02%    | 0.06%    | 0.48%    |
|        | $\delta_{\varphi_4}$ | -        | 0.86%    | 0.08%    | 0.01%    | 0.019%   | 0.02%    | 0.4%     |
| 0.999  | $\delta_{\varphi_1}$ | 0.02%    | 0.015%   | 0.002%   | 0.001%   | 0.004%   | 0.003%   | -        |
|        | $\delta_{\varphi_2}$ | 0.14%    | 0.026%   | 0.003%   | 0.0003%  | 0.0003%  | 0.004%   | -        |
|        | $\delta_{\varphi_3}$ | 0.02%    | 0.001%   | 0.001%   | 0.0008%  | 0.0005%  | 0.0003%  | 0.003%   |
|        | $\delta_{\varphi_4}$ | -        | 0.03%    | 0.002%   | 0.001%   | 0.0009%  | 0.002%   | 0.005%   |