LOW REGULARITY WELL-POSEDNESS FOR GROSS-NEVEU EQUATIONS

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Abstract. We address the problem of local and global well-posedness of Gross-Neveu (GN) equations for low regularity initial data. Combined with the standard machinery of $X_R^s$, $Y_R^s$ and $X^{s,b}$ spaces, we obtain local-wellposedness of (GN) for initial data $u, v \in H^s$ with $s \geq 0$. To prove the existence of global solution for the critical space $L^2$, we show non concentration of $L^2$ norm.

1. Introduction. We are interested in the nonlinear Dirac equations

$$
\begin{align*}
  i(\partial_t + \partial_x)U + mV &= \partial_x W(U,V), \\
  i(\partial_t - \partial_x)V + mU &= \partial_x W(U,V),
\end{align*}
$$

where $U, V$ are complex valued functions on $\mathbb{R}^{1+1}$ and $m(\geq 0)$ is a constant representing mass. We consider the potential $W$ of the form

$$W = a_1 |U|^2 |V|^2 + a_2 (\overline{UV} + VU)^2,$$

where $a_1$, $a_2$ are real constants and $\overline{U}$ is a complex conjugate of $U$.

When $W = 4|U|^2 |V|^2$, the system (1) is called Thirring equations [4]. The initial value problem of it has been studied by several authors [1, 2, 5, 8]. The global existence of solutions to the Thirring equations was studied in [2] in terms of Sobolev space $H^s$ ($s \geq 1$). Low regularity well-posedness was discussed in [1, 5, 8] showing that there exist a time $T > 0$ and solutions $U, V \in C([0,T], H^s(\mathbb{R}))$ ($s \geq 0$). Especially global existence of solution for $L^2$ initial data has recently been proved in [1].

When $W = \frac{1}{4}(\overline{UV} + VU)^2$, the system (1) is called Gross-Neveu equations [3] and takes the form

$$
\begin{align*}
  i(\partial_t + \partial_x)U + mV &= \text{Re}(\overline{UV})V, \\
  i(\partial_t - \partial_x)V + mU &= \text{Re}(\overline{UV})U.
\end{align*}
$$

Here the Cauchy problem of (GN) is studied with initial data

$$(U, V)(0, x) = (u, v)(x) \in H^s(\mathbb{R}) \times H^s(\mathbb{R}) \text{ for } s \geq 0.$$

The system (GN) has the charge conservation

$$\int_\mathbb{R} (|U|^2 + |V|^2) (t, x) \, dx = \int_\mathbb{R} (|u|^2 + |v|^2) (x) \, dx.$$

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Another important property of the system (GN) with zero mass $m = 0$ is invariance under the scaling

$$U^\lambda(t, x) = \lambda U(\lambda^2 t, \lambda^2 x), \quad V^\lambda(t, x) = \lambda V(\lambda^2 t, \lambda^2 x)$$

from which we deduce a scale invariant Sobolev space $L^2(\mathbb{R})$. The global existence of the strong solution for the massless (GN) with initial data having small and bounded charge was established in [10]. The global well-posedness of solutions to the massive (GN) was studied in [6] for initial data $H^s$ with $s > \frac{1}{2}$. We are interested in low regularity well-posedness of (GN). Our first result is concerned with the global solution in critical space $L^2$.

**Theorem 1.1.** For the initial data $(u, v) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$, there exists a global solution $(U, V) \in C([0, \infty); L^2(\mathbb{R})) \times C([0, \infty); L^2(\mathbb{R}))$ to (GN). Moreover, the solution is unique in a certain subspace of $C([0, \infty); L^2(\mathbb{R}))$ and has continuous dependence on initial data.

For this result, we follow the idea of Candy suggested in [1]. To obtain the $L^2$ local solution, we use null coordinates where the time of existence depends on the profile of the initial data. Then we show, using $L^\infty$ bound presented in [6], no concentration of charge of the $L^2$ local solution to extend it to a global solution. Our second result is concerned with the local solution in the space $H^s$ for $s > 0$. We adapt the idea of [8] where $X^{s,b}$ space is used together with a type of null form estimate.

**Theorem 1.2.** For the initial data $(u, v) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > 0$, there exists a time $T > 0$ such that we have a solution $(U, V) \in C([0, T]; H^s(\mathbb{R})) \times C([0, T]; H^s(\mathbb{R}))$ to (GN) which has continuous dependence on initial data. The solution is unique in a certain subspace of $C([0, T]; H^s(\mathbb{R}))$. Furthermore, if $s > \frac{1}{4}$, the solution is unique in $C([0, T]; H^s(\mathbb{R}))$.

We prove Theorem 1.1 and Theorem 1.2 in section 2 and 3 respectively. We conclude this section by giving a few notations. We use the standard Sobolev spaces $H^s(\mathbb{R})$ with the norm $\|f\|_{H^s} = \|(1 - \Delta)^{s/2}f\|_{L^2}$. We will use $c, C$ to denote various constants. When we are interested in local solutions, we may assume that $T \leq 1$. Thus we shall replace smooth function of $T, C(T)$ by $C$. We use $A \lesssim B$ to denote an estimate of the form $A \leq CB$.

2. Proof of Theorem 1.2. In this section, we will prove global well-posedness of (GN) with $L^2$ initial data. To prove the local existence of solution with critical regularity, we follow the similar way introduced in [1], where $L^2$ global solution for Thirring equations was proved. Since time of existence depends on the profile of initial data, the conservation of charge is not enough to extend the local solution globally. We need to rule out concentration of $L^2$ norm.

2.1. Preliminaries. Here we introduce null coordinates to prove global existence in $L^2$, similar to the method used in the work of Machihara, Nakanishi, and Tsugawa in [7]. We use the function spaces $X_R, Y_R$ and some related estimates. Such kind of spaces were introduced in [1] and applied in the study of Thirring equations.

We define the characteristic function on the interval $I_R(x_0) = \{x \in \mathbb{R} : |x - x_0| \leq R\}$ for $x_0 \in \mathbb{R}$ and $R > 0$,

$$\chi_{I_R(x_0)}(x) = \begin{cases} 1, & x \in I_R(x_0), \\ 0, & x \notin I_R(x_0). \end{cases}$$
We also define the characteristic function on the triangle area,
\[ \chi_{\Omega_R(x_0)}(t, x) = \begin{cases} 1, & (t, x) \in \Omega_R(x_0), \\ 0, & (t, x) \notin \Omega_R(x_0), \end{cases} \]
where \( \Omega_R(x_0) = \{(t, x) \in \mathbb{R}^{1+1} : |x - x_0| \leq R - t, 0 \leq t \leq R\} \). For the simplicity, we abbreviate \( I_R = I_R(0) \) and \( \Omega_R = \Omega_R(0) \). We define the spaces \( X_R \) and \( Y_R \) to be the completion of \( C^\infty(\mathbb{R}) \) with the norms,
\[
\|f\|_{X_R} = \|f^*\|_{L^\infty_x L^2_t(\Omega_R)} + \|\partial_\alpha f^*\|_{L^1_x L^2_t(\Omega_R)},
\]
\[
\|f\|_{Y_R} = \|f^*\|_{L^\infty_x L^2_t(\Omega_R)} + \|\partial_\beta f^*\|_{L^1_x L^2_t(\Omega_R)},
\]
where \( f(t, x) : \mathbb{R}^{1+1} \to \mathbb{C} \) and \( f^*(\alpha, \beta) = f(\frac{\alpha - \beta}{2}, \frac{\alpha + \beta}{2}) \). We refer to the coordinates \( (\alpha, \beta) = (x + t, x - t) \) as null coordinates. We will use \( X_R(x_0) \), \( Y_R(x_0) \) to indicate the \( X_R \), \( Y_R \) spaces centered at \( x_0 \). The first result we will require is the following linear inequalities.

**Lemma 2.1.** Assume \((U, V)\) is a solution to the Cauchy problem
\[
\begin{cases}
  i(\partial_t + \partial_x)U = F(x, t), \\
  i(\partial_t - \partial_x)V = G(x, t), \\
  (U, V)(0, x) = (u, v)(x),
\end{cases}
\]
where \((F, G) \in C^\infty(\Omega_R) \times C^\infty(\Omega_R)\) and \((u, v) \in C^\infty(I_R) \times C^\infty(I_R)\). Then,
\[
\|U\|_{X_R} \leq \|u\|_{L^2(I_R)} + \|F^*\|_{L^1_x L^2_t(\Omega_R)},
\]
\[
\|V\|_{Y_R} \leq \|v\|_{L^2(I_R)} + \|G^*\|_{L^1_x L^2_t(\Omega_R)}.
\]

We note that the solution to the nonhomogeneous transport equations (2) is given as
\[
\begin{cases}
  U(t, x) = u(x - t) - i \int_0^t F(s, x - t + s)ds, \\
  V(t, x) = v(x + t) - i \int_0^t G(s, x + t - s)ds.
\end{cases}
\]
Using change of variables and Minkowski’s inequality, we can derive desired results in Lemma 2.1. The following estimates and embeddings have been proved in [1].

**Lemma 2.2.** For any \( R > 0 \) we have
\[
\|f^*\|_{L^1_x L^\infty_t(\Omega_R)} \leq \|f\|_{X_R},
\]
\[
\|g^*\|_{L^1_x L^\infty_t(\Omega_R)} \leq \|g\|_{Y_R}.
\]

**Lemma 2.3.** Let \( 0 < T < R \). Then we have the continuous embeddings
\[
X_R, Y_R \hookrightarrow C([0, T]; L^2(I_{R-T})).
\]
The following inequalities are used for the control of nonlinear terms.

**Lemma 2.4.** For any \( R > 0 \) and \( f, g, h \in C^\infty(\Omega_R) \), we have
\[
\|\text{Re}(f^*g^*)h^*\|_{L^1_x L^2_t(\Omega_R)} \leq \|f\|_{X_R}\|g\|_{Y_R}\|h\|_{Y_R},
\]
\[
\|\text{Re}(f^*g^*)h^*\|_{L^1_x L^2_t(\Omega_R)} \leq \|f\|_{X_R}\|g\|_{Y_R}\|h\|_{X_R}.
\]
Applying the Hölder’s inequality and Lemma 2.2, we have
\[ \| \operatorname{Re}(\tilde{f}^* g^*) \|_{L^2_x L^2_t(\Omega_R)} \leq \| f \|_{X_R} \| g \|_{Y_R}^2, \]
\[ \| \operatorname{Re}(\tilde{f}^* g^*) f^* \|_{L^2_x L^2_t(\Omega_R)} \leq \| f \|_{X_R}^2 \| g \|_{Y_R}. \]

**Proof.** We only prove the first inequality as the remainders are almost identical. Applying the Hölder’s inequality and Lemma 2.2, we have
\[
\| \operatorname{Re}(\tilde{f}^* g^*)h^* \|_{L^2_x L^2_t(\Omega_R)} = \int_{I_R} \left( \int_{I_R} |\operatorname{Re}(\tilde{f}^* g^*)|^2 |h^*|^2 \, d\beta \right)^{1/2} \, dx d\alpha \\
\leq \| f \|_{L^2_x L^2_t(\Omega_R)} \| g^* \|_{L^2_x L^2_t(\Omega_R)} \| h^* \|_{L^2_x L^2_t(\Omega_R)} \\
\leq \| f \|_{X_R} \| g \|_{Y_R} \| h \|_{Y_R}.
\]

We can obtain the same results as Lemma 2.1 - 2.4 for \( X_R(x_0), \ Y_R(x_0) \) by considering translation.

### 2.2. Local and global solution in critical space

First of all, we construct a local solution on \( \Omega_R \) from the given initial data.

**Lemma 2.5.** For \( 0 < R < \frac{1}{16m} \), there exists a constant \( \epsilon > 0 \) such that if \( (u, v) \in L^2(I_R) \times L^2(I_R) \) satisfies
\[ \| u \|_{L^2(t_R)} + \| v \|_{L^2(t_R)} < \epsilon, \]
then there exists the unique solution \( (U, V) \in X_R \times Y_R \) to (GN) such that
\[ \| U \|_{X_R} + \| V \|_{Y_R} < 2\epsilon. \]

Moreover, the solution map from the initial data \( (u, v) \) satisfying the condition (4) to the solution \( (U, V) \) is Lipschitz continuous.

**Proof.** We define the function space
\[ Z_R = \{ (U, V) \in X_R \times Y_R : \| U \|_{X_R} + \| V \|_{Y_R} < 2\epsilon \}, \]
where \( \epsilon > 0 \) is a small constant to be chosen later. We consider the sequence of solutions \( \{ (U^{(n)}, V^{(n)}) \} \) defined inductively for \( n \geq 0 \),
\[
\begin{aligned}
& i(\partial_t + \partial_x) U^{(n+1)} = -m V^{(n)} + \text{Re}(\overline{U^{(n)}} V^{(n)}), \\
& i(\partial_t - \partial_x) V^{(n+1)} = -m U^{(n)} + \text{Re}(\overline{U^{(n)}} V^{(n)}) U^{(n)}, \\
& (U^{(n)}, V^{(n)})(0, x) = (u, v)(x),
\end{aligned}
\]
where the first step is \( (U^{(0)}, V^{(0)})(t, x) = (0, 0) \). We assume that \( \| U^{(n)} \|_{X_R} + \| V^{(n)} \|_{Y_R} < 2\epsilon \). Applying Lemma 2.1 and 2.4 to (5), we have
\[
\begin{aligned}
\| U^{(n+1)} \|_{X_R} + \| V^{(n+1)} \|_{Y_R} &
\leq \| u \|_{L^2(t_R)} + \| v \|_{L^2(t_R)} + m(\| V^{(n)} \|^* \|_{L^2_x L^2_t(\Omega_R)} + \| U^{(n)} \|^* \|_{L^2_x L^2_t(\Omega_R)}) \\
& + \| \text{Re}(\overline{U^{(n)}} V^{(n)}) V^{(n)} \|_{L^2_x L^2_t(\Omega_R)} + \| \text{Re}(\overline{U^{(n)}} V^{(n)}) U^{(n)} \|_{L^2_x L^2_t(\Omega_R)} \\
& \leq \epsilon + \epsilon(4Rm + 16\epsilon^2) < 2\epsilon,
\end{aligned}
\]
provided \( 0 < R < \frac{1}{16m} \) and \( \epsilon < \frac{1}{8} \). Then the sequence of solutions \( \{ (U^{(n)}, V^{(n)}) \} \) is well-defined on \( Z_R \). Next, we estimate the difference \( U^{(n+1)} - U^{(n)} \) and \( V^{(n+1)} - V^{(n)} \).
Denoting $U_n = U^{(n)} - U^{(n-1)}$ and $V_n = V^{(n)} - V^{(n-1)}$, we decompose the nonlinear term as

$$\text{Re}(\overline{U^{(n)}} V^{(n)}) - \text{Re}(\overline{U^{(n)}} V^{(n-1)}) V^{(n-1)}$$

$$= \text{Re}(\overline{U_n} V^{(n)}) + \text{Re}(\overline{U^{(n-1)}} V_n) V^{(n)} + \text{Re}(\overline{U^{(n-1)}} V^{(n-1)}) V_n.$$

Applying Lemma 2.1 and 2.4 on the difference $U^{(n+1)} - U^{(n)}$, we have

$$\|U^{(n+1)} - U^{(n)}\|_{X_R} \leq m(R_n^* \|L^1 X^2(\Omega_R)\| + \|\text{Re}(\overline{U^{(n)}} V^{(n)})\|_{X^{1/2}_R} + \|\text{Re}(\overline{U^{(n-1)}} V^{(n)})\|_{X^{1/2}_R} + \|\text{Re}(\overline{U^{(n-1)}} V^{(n-1)})\|_{X^{1/2}_R} \|V^{(n)}\|_{Y_R} \|V^{(n-1)}\|_{Y_R}$$

$$\leq m2R\|V_n\|_{Y_R} + \|U_n\|_{X_R} \|V^{(n)}\|_{X_R}^2 + \|U^{(n-1)}\|_{X_R} \|V^{(n)}\|_{Y_R} + \|U^{(n-1)}\|_{X_R} \|V^{(n-1)}\|_{Y_R} \|V_n\|_{Y_R}$$

Do the similar calculation to $V^{(n+1)} - V^{(n)}$ and combining the estimates, we obtain

$$\|U^{(n+1)} - U^{(n)}\|_{X_R} + \|V^{(n+1)} - V^{(n)}\|_{Y_R} \leq (2mR + 3\epsilon^2) \left( \|U^{(n)} - U^{(n-1)}\|_{X_R} + \|V^{(n)} - V^{(n-1)}\|_{Y_R} \right).$$

Taking into account the sufficient small assumptions on $R$, $\epsilon$ and the standard induction argument on $n$, we have

$$\|U^{(n+1)} - U^{(n)}\|_{X_R} + \|V^{(n+1)} - V^{(n)}\|_{Y_R} \leq \left( \frac{1}{2} \right)^n$$

which implies that $\{(U^{(n)}), V^{(n)}\}$ is Cauchy sequence in $Z_R$ and hence it converges to the unique solution in $Z_R$. Lipschitz continuity of the solution map follows from the similar estimates used to deduce that $\{(U^{(n)}), V^{(n)}\}$ is Cauchy sequence. \(\square\)

Now we are ready to construct a $L^2$ local solution in Theorem 1.1. For a given initial data $(u, v) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})$ and a constant $\epsilon > 0$ in Lemma 2.5, choose sufficiently small $0 < R < \frac{1}{16\epsilon}$ such that

$$\sup_{x_0 \in \mathbb{R}} \left( \|u\|_{L^2(I_R(x_0))} + \|v\|_{L^2(I_R(x_0))} \right) < \epsilon. \quad (6)$$

With this $R$, we consider $\mathbb{R}$ as the following two ways,

$$\mathbb{R} = \bigcup_{j \in \mathbb{Z}} I_R(x_{1,j}) = \bigcup_{j \in \mathbb{Z}} I_R(x_{2,j}),$$

where $x_{1,j} = (2j+1)R$, $x_{2,j} = (2j)R$. For each $k = 1, 2$ and $j \in \mathbb{Z}$, applying Lemma 2.5 with the initial data $(\chi_{I_R(x_{k,j})} u, \chi_{I_R(x_{k,j})} v)$, we can construct the corresponding solution $U_{k,j}$, $V_{k,j}$ on $\Omega_R(x_{k,j})$. We split the plane $[0, \frac{R}{2}] \times \mathbb{R}$ into disjoint squares $S_{k,j}$

$$S_{1,j} = [0, \frac{R}{2}] \times I_{\frac{R}{2}}(x_{1,j}), \quad S_{2,j} = [0, \frac{R}{2}] \times I_{\frac{R}{2}}(x_{2,j}).$$
We note that \( S_{k,j} \subset \Omega_R(x_{k,j}) \). We define functions on \([0, \frac{T}{2}] \times \mathbb{R}\),

\[
U(t,x) = \sum_{k=1}^\infty \sum_{j \in \mathbb{Z}} \chi_{S_{k,j}}(t,x)U_{k,j}(t,x),
\]

\[
V(t,x) = \sum_{k=1}^\infty \sum_{j \in \mathbb{Z}} \chi_{S_{k,j}}(t,x)V_{k,j}(t,x).
\]

We claim that \((U,V)\) defined in (7) is a local solution to (GN) in Theorem 1.1. Firstly we observe that the information of the initial data on the outside of \( I_R(x_{k,j}) \) does not influence the restricted solution \((U,V)\) on \( S_{k,j} \). Considering (3) and the influence region of the solution,

\[
\chi_{I_R(x_{k,j})}U(t,x) = \chi_{I_R(x_{k,j})}u(x-t) - i \int_0^t \chi_{I_R(x_{k,j})} (\text{Re}(UV)V - mV) (s,x-t+s)ds,
\]

\[
\chi_{I_R(x_{k,j})}V(t,x) = \chi_{I_R(x_{k,j})}v(x-t) - i \int_0^t \chi_{I_R(x_{k,j})} (\text{Re}(UV)U - mU) (s,x+t-s)ds.
\]

Since \( \chi_{I_R(x_{k,j})}(t,x) = 1 \), for \((t,x) \in S_{k,j} \subset \Omega_R(x_{k,j})\), the restricted solution \((\chi_{S_{k,j}}U, \chi_{S_{k,j}}V)\) is determined by the information of initial data on \( I_R(x_{k,j}) \).

Secondly, let \((u_n, v_n)\) be the sequence of functions converging to \((u,v) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})\). We choose sufficiently large \( N \) such that for any \( n \geq N\),

\[
\sup_{x_0 \in \mathbb{R}} \left( \|u_n\|_{L^2(I_R(x_0))} + \|v_n\|_{L^2(I_R(x_0))} \right) < \epsilon.
\]

Then we get a local solution \((U_n, V_n)\) on \([0, \frac{T}{2}] \times \mathbb{R}\) by applying Lemma 2.5 and repeating the above process. By the embedding presented in Lemma 2.3 and Lipschitz continuity proved in Lemma 2.5, for each \( k = 1, 2 \) and \( j \in \mathbb{Z}, \)

\[
\|U - U_n\|_{L^\infty_t L^2_x(\mathbb{R})} + \|V - V_n\|_{L^\infty_t L^2_x(\mathbb{R})} \lesssim \|U - U_n\|_{L^\infty_t L^2_x(S_{k,j})} + \|V - V_n\|_{L^\infty_t L^2_x(S_{k,j})} \lesssim \|u - u_n\|_{L^2(I_R(x_{k,j}))} + \|v - v_n\|_{L^2(I_R(x_{k,j}))}.
\]

We sum these inequalities over \( k = 1, 2 \) and \( j \in \mathbb{Z}\) to get

\[
\|U - U_n\|_{L^\infty_t L^2_x([0,\frac{T}{2}] \times \mathbb{R})} + \|V - V_n\|_{L^\infty_t L^2_x([0,\frac{T}{2}] \times \mathbb{R})} \lesssim \|u - u_n\|_{L^2(\mathbb{R})} + \|v - v_n\|_{L^2(\mathbb{R})}
\]

and the continuous dependence on initial data. It is easy to prove \((U,V) \in C([0, \frac{T}{2}); L^2(\mathbb{R})) \times C([0, \frac{T}{2}); L^2(\mathbb{R}))\).

Now we will show global extension of \( L^2 \) local solution. Note that the time of existence of the local solution above depends on the size of \( R \) which satisfies (6). That is, it depends on the profile of initial data. Therefore charge conservation is not sufficient to extend our solution globally. We need the following lemma saying that non concentration of \( L^2 \) norm of the solution to (GN) implies global existence. As a result, we only need to show \( L^2 \) norm of our solution doesn’t concentrate at a point. The following type lemma has been shown in [1] for studying Thirring model and the proof can be adapted easily for our equation (GN).

**Lemma 2.6.** Suppose \((u,v) \in L^2(\mathbb{R}) \times L^2(\mathbb{R})\). Let \( T \) be the maximal time such that the corresponding solution \((U,V) \in C([0,T); L^2(\mathbb{R})) \times C([0,T); L^2(\mathbb{R}))\) to (GN)
exists. If
\[
\limsup_{t \to T} \sup_{x \in \mathbb{R}} \int_{|y-x|<4(T-t)} |U(t, y)|^2 + |V(t, y)|^2 \, dy = 0, \tag{8}
\]
then \((U, V)\) can be extended beyond \(0, T)\).

In point of Lemma 2.6, it only remains to show (8) to conclude global existence of (GN). To show (8) we make use of the idea in [6] where the following local version of charge conservation has been shown
\[
2 \int_0^{t_0} |U|^2(s, x_0 + t_0 - s) \, ds + 2 \int_0^{t_0} |V|^2(s, x_0 - t_0 + s) \, ds
= \int_{x_0-t_0}^{x_0+t_0} (|u(s)|^2 + |v(s)|^2) \, ds \leq M, \tag{9}
\]
where we denote \(M = \int_\mathbb{R} (|u(y)|^2 + |v(y)|^2) \, dy\). Multiplying (GN) by \(U\) and \(\overline{V}\) respectively, we have
\[
\partial_t |U|^2 + \partial_x |U|^2 + 2m \text{Im}(\overline{U}V) = 2 \text{Re}(|U|^2 \text{Im}(U \overline{V})), \tag{10}
\]
\[
\partial_t |V|^2 - \partial_x |V|^2 + 2m \text{Im}(\overline{V}U) = 2 \text{Re}(|U|^2 \text{Im}(U \overline{V})). \tag{11}
\]
Integrating (10) along characteristic and cancelling \(|U|\) both sides, we have
\[
\frac{d}{dt} |U(t, x + t)| \leq |U(t, x + t)||V(t, x + t)|^2 + m|V(t, x + t)|
\]
which implies
\[
|U(t, x + t)| \leq \exp \left( \int_0^t |V(s, x + s)|^2 \, ds \right) \left( |u(x)| + m \int_0^t |V(s, x + s)| \, ds \right)
\leq e^{\frac{Mt}{2}} \left| u(x) \right| + m \sqrt{\frac{M}{2}} t^{\frac{3}{2}}, \tag{12}
\]
where the estimate (9) is used. The similar argument applied to (11) leads us to
\[
|V(t, x - t)| \leq e^{\frac{Mt}{2}} \left| v(x) \right| + m \sqrt{\frac{M}{2}} t^{\frac{3}{2}}. \tag{13}
\]
We can rewrite (12) and (13) as follows.
\[
|U(t, y)| \leq e^{\frac{Mt}{2}} \left| u(y - t) \right| + m \sqrt{\frac{M}{2}} t^{\frac{3}{2}},
\]
\[
|V(t, y)| \leq e^{\frac{Mt}{2}} \left| v(y + t) \right| + m \sqrt{\frac{M}{2}} t^{\frac{3}{2}}. \tag{14}
\]
Taking (14) into account, we obtain
\[
\sup_{x \in \mathbb{R}} \int_{|y-x|<4(T-t)} |U(t, y)|^2 + |V(t, y)|^2 \, dy
\leq 2e^{\frac{Mt}{2}} \sup_{x \in \mathbb{R}} \int_{|y-x|<4(T-t)} |u(y-t)|^2 + |v(y+t)|^2 + m^2 Mt \, dy
\leq 2e^{\frac{Mt}{2}} \left( \int_{|y-x+T|<6(T-t)} |u(y)|^2 \, dy + \int_{|y-x-T|<6(T-t)} |v(y)|^2 \, dy \right)
+ 8m^2 M(T-t) t.
Letting $t \to T$, we get (8) and so we have shown non concentration of $L^2$ norm to prove global solution in time.

3. Proof of Theorem 1.2. In this section we prove the local existence of solution with initial data $(u, v) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > 0$ by using the $X^{s, b}$ space. It is used by Selberg and Tefahun in [8] to prove the local existence of solution for Thirring equations with initial data in $H^s(\mathbb{R})$, $s > 0$. Since the contraction argument only gives uniqueness in the iteration space $X^{s, b}$, we also prove unconditional uniqueness for $s > 1/4$.

First of all, we give some definitions and related estimates. Especially we refer to [8]. The Fourier transform in space-time is defined by

$$\hat{f}(\tau, \xi) = \int \int_{\mathbb{R}^{1+1}} e^{-i(\tau + x \xi)} f(t, x) \, dt \, dx.$$  

We use the function space $X^{s, b}_\pm$ and $H^{s, b}$ to be the completion of $S(\mathbb{R}^{1+1})$ with respect to the norms

$$\|f\|_{X^{s, b}_\pm} = \|\langle \xi \rangle^s (\tau \pm \xi)^b \hat{f}(\tau, \xi)\|_{L_x^2 L_t^2},$$

$$\|f\|_{H^{s, b}} = \|\langle \xi \rangle^s (\tau - \xi)^b \hat{f}(\tau, \xi)\|_{L_x^2 L_t^2},$$

where $\langle \cdot \rangle = \sqrt{1 + (\cdot)^2}$ and $s, b \in \mathbb{R}$. Let $S_T = (0, T) \times \mathbb{R}$. The restriction space $X^{s, b}_\pm(S_T)$ is a Banach space with norm

$$\|f\|_{X^{s, b}_\pm(S_T)} = \inf_{g \mid [S_T] = f} \|g\|_{X^{s, b}_\pm}.$$ 

For $b > 1/2$, we have

$$\|f(t, \cdot)\|_{H^s} \leq C \|f\|_{X^{s, b}_\pm(S_T)} \quad \text{for} \quad 0 \leq t \leq T,$$

where $C$ depends only on $b$. The following is the linear estimate associated with $X^{s, b}_\pm$ space.

Lemma 3.1. Let $s > 0$, $\frac{1}{2} < b \leq 1$, $0 \leq \delta \leq 1 - b$ and $0 < T \leq 1$. Then, for all $(F, G) \in X^{s, b-1+\delta}_+(S_T) \times X^{s, b-1+\delta}_-(S_T)$ and $(f, g) \in H^s \times H^s$, the Cauchy problem

$$\begin{cases}
    i(\partial_t + \partial_x) U = F(x, t), \\
    i(\partial_t - \partial_x) V = G(x, t), \\
    (U, V)(0, x) = (u, v)(x)
\end{cases}$$

has a unique solution in $(U, V) \in X^{s, b}_+(S_T) \times X^{s, b}_-(S_T)$ such that

$$\|U\|_{X^{s, b}_+(S_T)} \leq C \left( \|u\|_{H^s} + T^\delta \|F\|_{X^{s, b-1+\delta}_+(S_T)} \right),$$

$$\|V\|_{X^{s, b}_-(S_T)} \leq C \left( \|v\|_{H^s} + T^\delta \|G\|_{X^{s, b-1+\delta}_-(S_T)} \right),$$

where $C$ depends only on $b$.

Lemma 3.1 has been proven in [8, 9]. Note that $T^\delta$ factor is useful in a contraction argument. We need the following nonlinear estimate.

Lemma 3.2. For any $s > 0$, $\epsilon > 0$ and $f, g \in S(\mathbb{R}^{1+1})$, we have

$$\|\text{Re}(f g)\|_{X^{s, -\frac{1}{2}+\epsilon}_+} \lesssim \|f\|_{X^{s, \frac{1}{2}+\epsilon}_+} \|g\|^2_{X^{s, \frac{1}{2}+\epsilon}_-},$$  

$$\|\text{Re}(f g)\|_{X^{s, -\frac{1}{2}+\epsilon}_-} \lesssim \|g\|_{X^{s, \frac{1}{2}+\epsilon}_-} \|f\|^2_{X^{s, \frac{1}{2}+\epsilon}_+},$$  

(16) (17)
In the following proof, we follow almost same calculation as in [8] which is used to prove low regularity well-posedness for the Thirring equation. We remark that nonlinear terms of Gross-Neveu and Thirring equations are unlike, but we could have similar low regularity solutions due to Lemma 3.2.

**Proof.** We only prove (16). The estimate (17) is similar to (16). For any $w \in S(\mathbb{R}^{1+1})$, we have

\[
\left| \int_{\mathbb{R}^{1+1}} \text{Re}(f \bar{g}) g \bar{w} \, dt dx \right| \leq \int_{\mathbb{R}^{1+1}} |fgw| \, dt dx \\
\lesssim \|f\|_{H^{s,0}} \|g\|_{H^{-s,0}} \\
\lesssim \|f\|_{X^{s,0}_+} \|g\|_{X^{-s,0}_+} \|w\|_{X^{-s,0}_+}.
\]

The last inequality is due to the followig estimates which are (4.5) and (4.6) in [8],

\[
\|f\|_{H^{s,0}} \lesssim \|f\|_{X^{s,0}_+} \|g\|_{X^{-s,0}_+}, \\
\|g\|_{H^{-s,0}} \lesssim \|g\|_{X^{-s,0}_+} \|w\|_{X^{-s,0}_+}.
\]

Then we have (16) by duality. \qed

Finally, we shall use the standard product estimate for the Sobolev spaces $H^s$ to prove unconditional uniqueness.

**Lemma 3.3.** If $a, b, c \in \mathbb{R}$ satisfy

\[ a + b + c > 1/2, \quad a + b \geq 0, \quad b + c \geq 0, \quad c + a \geq 0, \]

or

\[ a + b + c = 1/2, \quad a < 1/2, \quad b < 1/2, \quad c < 1/2, \]

then we have

\[ \|fg\|_{H^{-c}} \lesssim \|f\|_{H^s} \|g\|_{H^b}. \]

3.1. **Local solution in $H^s$.** Now we are ready to prove Theorem 1.2. We set $(U^{(0)}, V^{(0)})(t, x) = (0, 0)$ and define inductively, for $n \geq 0$,

\[
\begin{cases}
  i(\partial_t + \partial_x) U^{(n+1)} = -m V^{(n)} + \text{Re}(\overline{U^{(n)}} V^{(n)}), \\
  i(\partial_t - \partial_x) V^{(n+1)} = -m U^{(n)} + \text{Re}(\overline{U^{(n)}} V^{(n)}) U^{(n)},
\end{cases}
\]

with initial data $(U^{(n+1)}, V^{(n+1)})(0, x) = (u, v)(x)$. Applying Lemma 3.1 and 3.2, we have

\[
\|U^{(n+1)}\|_{X^{s,0}_{+}^{1/2 + \epsilon}(S_T)} \\
\lesssim \left[ \|u\|_{H^s} + T^\epsilon \left( m \|V^{(n)}\|_{X^{-s,0}_{-}^{1/2 + \epsilon}(S_T)} + \|U^{(n)}\|_{X^{s,0}_{+}^{1/2 + \epsilon}(S_T)} \|V^{(n)}\|_{X^{-s,0}_{+}^{1/2 + \epsilon}(S_T)} \right) \right] \|V^{(n+1)}\|_{X^{s,0}_{+}^{1/2 + \epsilon}(S_T)},
\]

\[
\|V^{(n+1)}\|_{X^{s,0}_{-}^{1/2 + \epsilon}(S_T)} \\
\lesssim \left[ \|v\|_{H^s} + T^\epsilon \left( m \|U^{(n)}\|_{X^{s,0}_{+}^{1/2 + \epsilon}(S_T)} + \|V^{(n)}\|_{X^{-s,0}_{-}^{1/2 + \epsilon}(S_T)} \|U^{(n)}\|_{X^{s,0}_{-}^{1/2 + \epsilon}(S_T)} \right) \right].
\]

We set $E_0 = \|u\|_{H^s} + \|v\|_{H^s}$. If

\[
\|U^{(n)}\|_{X^{s,0}_{+}^{1/2 + \epsilon}(S_T)} + \|V^{(n)}\|_{X^{-s,0}_{-}^{1/2 + \epsilon}(S_T)} \leq 2E_0,
\]
then
\[ \|U^{(n+1)}\|_{X^\frac{s}{2}+\epsilon,(S_T)} + \|V^{(n+1)}\|_{X^\frac{s}{2}+\epsilon,(S_T)} \leq 2E_0 \]

provided that \( T \leq (2m + 16E_0^2)^{-\frac{1}{4}}. \) This shows that the sequence \((U^{(n+1)}, V^{(n+1)})\) is well-defined in a ball of \(X^\frac{s}{2}+\epsilon,(S_T) \times X^\frac{s}{2}+\epsilon,(S_T)\). Similar application of Lemma 3.1, 3.2 and the decomposition used in section 2.2, for sufficiently small \( T > 0 \), lead us to
\[ \|U^{(n+1)} - U^{(n)}\|_{X^\frac{s}{2}+\epsilon,(S_T)} + \|V^{(n+1)} - V^{(n)}\|_{X^\frac{s}{2}+\epsilon,(S_T)} \]
\[ \leq \frac{1}{2} \left( \|U^{(n)} - U^{(n-1)}\|_{X^\frac{s}{2}+\epsilon,(S_T)} + \|V^{(n)} - V^{(n-1)}\|_{X^\frac{s}{2}+\epsilon,(S_T)} \right), \]

which shows that \((U^{(n)}, V^{(n)})\) is Cauchy sequence in \(X^\frac{s}{2}+\epsilon,(S_T) \times X^\frac{s}{2}+\epsilon,(S_T)\).

Thus, there is a unique solution \((u, v) \in X^\frac{s}{2}+\epsilon,(S_T) \times X^\frac{s}{2}+\epsilon,(S_T)\) which is continuously embedded in \(C([0,T]; H^s(R)) \times C([0,T]; H^s(R))\) due to (15). A continuous dependence of solution on initial data and uniqueness in \(X^\frac{s}{2}+\epsilon,(S_T) \times X^\frac{s}{2}+\epsilon,(S_T)\)

follow from the standard argument.

### 3.2. Unconditional uniqueness when \( s > 1/4 \)

We prove uniqueness of the solution \((U, V) \in C([0,T]; H^s(R)) \times C([0,T]; H^s(R))\) for \( s > \frac{1}{4} \). Since we already have uniqueness in the iteration space \((U, V) \in X^\frac{4}{7} \times X^\frac{4}{7},(S_T) \times X^\frac{4}{7} \times X^\frac{4}{7},(S_T)\), it suffices to show that if
\[ (U, V) \in C([0,T]; H^\frac{1}{4}+4\epsilon(R)) \times C([0,T]; H^\frac{1}{4}+4\epsilon(R)) \]

is a solution of (GN) with initial data \((u, v) \in H^\frac{1}{4}+4\epsilon(R) \times H^\frac{1}{4}+4\epsilon(R)\), then
\[ (U, V) \in X^\frac{2}{7} \times X^\frac{2}{7},(S_T). \]

Note that (19) implies that
\[ (U, V) \in X^\frac{4}{7}+4\epsilon,0,(S_T) \times X^\frac{4}{7}+4\epsilon,0,(S_T). \]

On the other hand, we will show
\[ (U, V) \in X^\frac{4}{7}+4\epsilon,1,(S_T) \times X^\frac{4}{7}+4\epsilon,1,(S_T). \]

If (22) is true, then interpolation between (21) and (22) gives us
\[ (U, V) \in X^\frac{4}{7}+4\epsilon,0,\theta,(S_T) \times X^\frac{4}{7}+4\epsilon,0,\theta,(S_T) \]

for \( 0 \leq \theta \leq 1 \). In particular, taking \( \theta = \frac{1}{2} + \epsilon \), we obtain (20). Thus it only remains to prove (22). Indeed, applying Lemma 3.1 to (GN), we reduce it to
\[ (\text{Re}(UV), \text{Re}(UV)) \in X^\frac{4}{7}+4\epsilon,0,(S_T) \times X^\frac{4}{7}+4\epsilon,0,(S_T). \]

Applying Lemma 3.3 two times, respectively, we have
\[ \|\text{Re}(UV)\|_{X^\frac{4}{7}+4\epsilon,0,(S_T)} \lesssim \|\text{Re}(UV)\|_{L^\infty H^\frac{4}{7}+4\epsilon(R)} \|\text{Re}(UV)\|_{L^\infty H^\frac{4}{7}+4\epsilon(R)} \]
\[ \lesssim \|U\|_{L^\infty H^\frac{4}{7}+4\epsilon,(S_T)} \|V\|_{L^\infty H^\frac{4}{7}+4\epsilon,(S_T)}, \]
\[ \|\text{Re}(UV)\|_{X^\frac{4}{7}+4\epsilon,0,(S_T)} \lesssim \|\text{Re}(UV)\|_{L^\infty H^\frac{4}{7}+4\epsilon(R)} \|\text{Re}(UV)\|_{L^\infty H^\frac{4}{7}+4\epsilon(R)} \]
\[ \lesssim \|U\|_{L^\infty H^\frac{4}{7}+4\epsilon,(S_T)} \|V\|_{L^\infty H^\frac{4}{7}+4\epsilon,(S_T)}. \]
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