Multipath Wave-Particle Duality in Classical Optics

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Compiled March 11, 2020

It is well known that in classical optics, the visibility of interference, in a two-beam light interference, is related to the optical coherence of the two beams. A wave-particle duality relation can be derived using this mutual coherence. The issue of wave-particle duality in classical optics is analyzed here, in the more general context of multipath interference. New definitions of interference visibility and path distinguishability have been introduced, which lead to a duality relation for multipath interference. The visibility is shown to be related to a new multi-point optical coherence function.

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http://dx.doi.org/10.1364/ao.XXXXXX

The dual nature of light has been a subject of debate for a long time. Corpuscular theory and the wave theory of light constituted two opposite viewpoints on the nature of light. With the advent of quantum mechanics, the concept of dual nature of light grew much deeper as quantum mechanics itself is a wave theory. Niels Bohr formalized the notion of wave-particle duality by his principle of complementarity \cite{1}. The idea behind it is that any experiment can fully reveal either the particle nature or the wave nature of light, or of any other quantum entity, but never both of them. A two-slit interference experiment, using photons or massive particles, is an ideal testbed to study the principle of complementarity. The interference is an obvious signature of the wave nature in such an experiment. Finding out which of the two paths the particle followed, amounts to revealing its particle nature. Wooters and Zurek \cite{2} tried to investigate Bohr’s principle by asking if one can \textit{partially} observe both wave and particle natures at the same time? The expectation was that if an experiment only partially reveals (say) the particle nature, it may also allow partial revelation of the wave nature. The sharpness of interference may constitute a measure of wave nature, and the amount of knowledge, regarding which slit the particle went through, may quantify the particle nature. Wooters and Zurek’s line of investigation was later extended by Englert who derived a wave-particle duality relation \cite{3}

\begin{equation}
\mathcal{P}^2 + \mathcal{V}^2 \leq 1, \tag{1}
\end{equation}

where \(\mathcal{P}\) is path distinguishability, a measure of the particle nature, and \(\mathcal{V}\) the visibility of interference, a measure of wave nature. The above study assumed that a measuring device is introduced which can tell which the two slits the particle went through.

However, there is another line of thought in which wave-particle duality can be studied, without introducing a which-way measuring, or path-detecting device. If the two beams emerging from the two slits, have different intensities, one could predict which of the two paths the particle might have taken, with a success which is more than a random 50-50 guess. Greenberger and Yasin \cite{4}, and later Jaeger, Shimoni and Vaidmann \cite{5}, followed this line of thought and derived different kind of duality relation

\begin{equation}
P^2 + V^2 \leq 1, \tag{2}\end{equation}

where \(P\) is a path-predictability, and \(V\) the visibility of interference. The ability to predict the path of the particle, allows one to argue that there is a certain degree of particle nature associated with the objects passing through the double-slit. The fringe visibility, which quantifies the wave nature, is taken to be just the Michelson’s fringe contrast \cite{6}

\begin{equation}V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}, \tag{3}\end{equation}

where the notations have the usual meaning. Although both inequalities (1) and (2) have been referred to as wave-particle duality relations in the literature, it is important to remember that the two are different, and pertain to different experimental
situations. However, the name distinguishability is being used interchangeably for both $D$ and $P$.

The issue of wave-particle duality, in the realm of classical optics, has come into a lot of recent attention [7–12]. It should be emphasized here that for classical light, since one does not talk at the level of single photons, it is meaningless to talk about duality relations of the type (1). One can only predict with a good success rate that light would pass through the more intense beam. Then it is meaningful to talk of duality relation of the kind (2).

Let us now consider a scenario where light passes through an array of equally spaced $n$ slits (see Fig. 1). The complex amplitude of the light field depends on its two-dimensional transverse coordinate $r$, time $t$, and intrinsic polarization $s$. The field at the location of the $i$th slit, at time $t$, has arbitrary amplitude and unit polarization: $E_i = \hat{s}_i E_i(r, t)$. The field received from the $i$th slit, at a point $r_x$ on the detection screen, is given by

$$E_{ix} = K_i \hat{s}_i E_i(r_x, t - t_i),$$

(4)

where $t_i$ is the time taken by light to travel from the $i$th slit to the point $r_x$ on the screen, and $K_i$ is the usual purely imaginary propagation factor [6]. The total intensity at the point $r_x$ on the screen is a result of fields received from all the $n$ slits:

$$I(r_x, t) = \left[ \sum_{i=1}^{n} K_i \hat{s}_i E_i(r_x, t - t_i) \right]^2$$

$$= \sum_{i=1}^{n} |K_i|^2 |E_i(r_x, t - t_i)|^2 + \sum_{i \neq j} K_i K_j^* \langle E_i(r_x, t - t_i) E_j^*(r_x, t - t_j) \rangle |\hat{s}_i \cdot \hat{s}_j|,$$

(5)

where the angular brackets denote statistical average. The field correlation functions can be translated in time. With this in mind, the above relation can be written as

$$I(r_x, t) = \sum_{i=1}^{n} I_i + \sum_{i \neq j} |K_i||K_j| |\hat{s}_i \cdot \hat{s}_j| \Gamma_{ij}(r_x, r_y, \tau_{ij}),$$

(6)

where $\tau_{ij} = t_i - t_j$, and $\Gamma_{ij}(r_x, r_y, \tau_{ij})$ is the vector-mode mutual coherence function between the $i$th and $j$th slits, given by

$$\Gamma_{ij}(r_x, r_y, \tau) = \langle \hat{s}_i \cdot \hat{s}_j \rangle \langle E_i(r_x, 0) E_j^*(r_y, \tau) \rangle.$$  

(7)

Here $I_i$ is to be interpreted as the intensity at the point $r_x$ on the screen, if only the $i$th slit were open. The mutual coherence function can be normalized as

$$\gamma_{ij}(\tau) = \frac{\Gamma_{ij}(r_x, r_y, \tau)}{\sqrt{\Gamma_{ii}(r_x, r_y, 0) \Gamma_{jj}(r_x, r_y, 0)}}$$

(8)

For monochromatic light $\gamma_{ij}(\tau)$ may be represented as [6]

$$\gamma_{ij}(\tau) = |\gamma_{ij}(\tau)| e^{i \omega \tau + \phi_{ij}},$$

where $\phi_{ij}$ are certain unspecified phases. Using this, eqn. (6) assumes the following form

$$I(r_x, t) = \sum_{i=1}^{n} I_i + \sum_{i \neq j} \sqrt{I_i I_j} |\gamma_{ij}(\tau_{ij})| \cos(\omega \tau_{ij} + \phi_{ij}),$$

(9)

where $\phi_{ij}$ are certain phase factors. The above relation describes the intensity at a point on the screen, in terms of a sum of normalized mutual coherences between all pairs of slits.

Now if one tries to evaluate the visibility of interference from (9) using (3), one does not get a compact and elegant answer. However, one can check that for $n = 2$, one does recover the known result $V = |\gamma_{12}(\tau_{12})|$, which indicates that we are on the right track. For $n > 2$ one has to look for another strategy. Looking at (9) one would notice the first term represent the incoherent contribution to the interference, basically the sum of intensities reaching a point on the screen from each slit separately, without any interference. The second term represents interference. If one subtracts out the incoherent contribution from the full interference pattern, what remains is just the interference term, i.e., the second term. Following this line of thought, a new definition of visibility has been introduced [13, 14]

$$V_C = \frac{1}{n-1} \frac{I_{\text{max}} - I_{\text{inc}}}{I_{\text{inc}}},$$

(10)

where $n$ is the number of slits, $I_{\text{max}}$ is the intensity at a primary maximum of the interference pattern, and $I_{\text{inc}}$ is the intensity at the position of a primary maximum if the light is made incoherent before entering the slits, and consequently the interference is destroyed (see Fig. 2). For example, in (9), the first term represents $I_{\text{inc}}$. For simplicity, let us assume that all $\phi_{ij}$ are zero. In (9), the maximum intensity will be when all cosine terms are equal to unity at the same time. These are the locations of primary maxima. Using (9) and (10) leads one to

$$V_C = \frac{1}{n-1} \sum_{i \neq j} \sqrt{I_i I_j} |\gamma_{ij}(\tau_{ij})|.$$  

(11)

First thing to notice is that for $n = 2$, $V_C = |\gamma_{12}(\tau_{12})|$, which shows that for $n = 2$, $V_C$ gives the same value as the traditional visibility (3). If intensities at all the slits are equal, then

$$V_C = \frac{1}{n(n-1)} \sum_{i \neq j} |\gamma_{ij}(\tau_{ij})|,$$

(12)

which is just the average of normalized mutual coherence over all pairs of slits. It is satisfying to see that, even for multislit interference, interference visibility is related to the degree of mutual coherence.
Next we turn our attention to the degree of distinguishability of paths, which is technically predictability, but in optics literature it is called distinguishability. We will also refer to it here as distinguishability, but keeping in mind that it only pertains to guessing the path when the paths are unequal. For the case of two slits, it is defined as

$$D = \frac{|I_1 - I_2|}{I_1 + I_2},$$

(13)

which means that if the intensities on the screen, coming from the two slits, are equal, there is no way one can distinguish between the two paths, and $D = 0$. On the other extreme, if the intensity (say) $I_2$ is negligibly small, one can be almost sure that any light reaching the screen came from slit 1, and $D \approx 1$. Now if one moves on to n-slit interference, there is no obvious way to generalize (13) to more than two slits. We start by rewriting (13) as

$$D = \sqrt{1 - \frac{4I_1I_2}{(I_1 + I_2)^2}}.$$

(14)

Note that the second term in the square root in the above equation can also be written as $(\sum_{j \neq i} \sqrt{I_j I_i} / (I_1 + I_2)^2$. This form can be intuitively generalized to n-slit case. We introduce the following path distinguishability for the case of n slits,

$$D \equiv \sqrt{1 - \left( \frac{1}{n-1} \sum_{i \neq j} \sqrt{\frac{I_i I_j}{\sum_{k=1}^{n} I_k}} \right)^2},$$

(15)

where the role of the factor $\frac{1}{n-1}$ is to normalize the second term inside the square root. For $n = 2$, (15) reduces to (14).

Before proceeding further, it may be worthwhile to test this new distinguishability in various limits. If the intensities coming from all the slits, except one, are zero, we should have complete knowledge about which slit the light comes from. As expected, in such a situation, (15) yields $D = 1$. If the intensities from all the slits are equal (i.e., all $I_j = I_0$), there is no way one can distinguish between the $n$ paths. For this case (15) yields $D = 0$.

Now that we have our n-path distinguishability and generalized visibility, defined, we proceed to finding out the constraints on their values together. Using (11) and (15), we can write

$$D^2 + V_C^2 = 1 - \left( \frac{1}{n-1} \sum_{i \neq j} \frac{\sqrt{I_i I_j}}{\sum_{k=1}^{n} I_k} \right)^2$$

$$+ \left( \frac{1}{n-1} \sum_{i \neq j} \frac{\sqrt{I_i I_j}}{\sum_{k=1}^{n} I_k} |\gamma_{ij}(\tau_{ij})| \right)^2$$

(16)

Since $|\gamma_{ij}| \leq 1$, we can write the inequality

$$D^2 + V_C^2 \leq 1.$$  

(17)

This is a new wave-particle duality relation involving the path distinguishability and interference visibility, and works for interference from any number of slit. The inequality saturates if all $|\gamma_{ij}| = 1$ (fully coherent light). For $n = 2$ the inequality reduces to the known duality relation for two-slit interference.

In the following we would like to propose another way of defining the degree of distinguishability of paths, which will lead to a simpler duality relation. Instead of (15) we propose the following distinguishability

$$D' \equiv 1 - \frac{1}{n-1} \sum_{i \neq j} \sqrt{\frac{I_i I_j}{\sum_{k=1}^{n} I_k}}.$$  

(18)

This definition is simpler in form, without involving the square of a sum. For $n = 2$, it reduces to

$$D' = 1 - \frac{2\sqrt{I_1 I_2}}{I_1 + I_2}$$

(19)

which can also be written as

$$D' = (\sqrt{I_1} - \sqrt{I_2})^2 / I_1 + I_2.$$  

(20)

As one can see, this is not any more complicated than (13), and follows the same physical idea. Now it is easy to see that (18), together with (11), yields

$$D' + V_C = 1 - \frac{1}{n-1} \sum_{i \neq j} \frac{\sqrt{I_i I_j}}{\sum_{k=1}^{n} I_k} (1 - |\gamma_{ij}(\tau_{ij})|).$$

(21)

Since $|\gamma_{ij}| \leq 1$, we can write a new duality relation

$$D' + V_C \leq 1.$$  

(22)

For the case of two slits, this reduces to the following duality relation for two-slit interference

$$D' + V \leq 1,$$  

(23)

where $V$ is the traditional visibility (3). As one can see, this is a simpler duality relation and using the squares of the two measures was an unnecessary baggage, being carried along just because of trying to mimic the form of the first duality relation (2) derived by Greenberger and Yasin [4].

Finally we take a closer look at (12), where the new visibility $V_C$ seems to suggest a n-point degree of coherence. We suggest a general n-point degree of mutual coherence as:

$$\gamma_n(\tau_1, \tau_2, \ldots, \tau_n, \tau_{12}, \tau_{23}, \ldots) = \frac{1}{n(n-1)} \sum_{i \neq j} |\gamma_{ij}(\tau_{ij})|,$$

(24)

which is a real quantity, with magnitude $0 \leq \gamma_n \leq 1$. It appears to be the classical optical counterpart of a recently introduced measure of quantum coherence $C$ [15, 16]:

$$C = \frac{1}{n(n-1)} \sum_{i \neq j} |\rho_{ij}|,$$

(25)

where $|\rho_{ij}|$ are the matrix elements of the density operator of the particle, in a particular basis. We claim so because in multipath quantum interference, the new way of measuring visibility, that is described here, yields the quantum coherence $C$ [13, 14]. Since this n-point degree of coherence is directly related to the visibility of interference, we believe it may turn out to be useful in some other situations too. The usefulness of $\gamma_n(\tau_1, \tau_2, \ldots, \tau_n, \tau_{12}, \tau_{23}, \ldots)$ needs to be explored further.

In summary, wave-particle duality in multipath interference is now well studied in the quantum domain [16–21], however, it has not been well studied in classical optics. We have analyzed multipath interference in classical optical domain, and introduced a new way of measuring interference visibility. This
visibility turns out to be related to a new multipoint degree of mutual coherence. We also introduced a quantitative measure of the degree of distinguishability of the paths in a \( n \)-beam interference. These two together, lead to a new wave-particle duality relation. In another extension of the analysis we define yet another path distinguishability, which leads to a simpler wave-particle duality relation. For two-slit interference, it leads to a new duality relation which is simpler than the one currently known. The new multipoint degree of mutual coherence appears to the classical optical analogue of a recently introduced measure of quantum coherence, in the context of quantum information theory. Further study is needed to explore the usefulness of the multipoint mutual coherence.

Acknowledgement. Bibhash Paul and Sammi Kamal thank the Centre for Theoretical Physics at Jamia Millia Islamia, for providing the facilities of the Centre during the course of this work.

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