GENUS ONE HALF STACKY CURVES VIOLATING THE LOCAL-GLOBAL PRINCIPLE

HAN WU AND CHANG LV

Abstract. For any number field, we prove that there exists a stacky curve of genus 1/2 defined over the ring of its integers violating the local-global principle for integral points.

1. Introduction

1.1. Background. Given a number field $K$, let $\mathcal{O}_K$ be the ring of its integers, and let $\Omega_K$ be the set of all its nontrivial places. Let $K_v$ be the completion of $K$ at $v \in \Omega_K$. For $v$ is a finite place, let $\mathcal{O}_v$ be the valuation ring of $K_v$. For $v$ is an archimedean place, let $\mathcal{O}_v = K_v$. Let $X$ be a finite type algebraic stack over $\mathcal{O}_K$. If the set $X(\mathcal{O}_K) \neq \emptyset$, then the set $X(\mathcal{O}_v) \neq \emptyset$ for all $v \in \Omega_K$. The converse does not always hold. We say that $X$ violates the local-global principle for integral points if $X(\mathcal{O}_v) \neq \emptyset$ for all $v \in \Omega_K$, whereas $X(\mathcal{O}_K) = \emptyset$. For $K = \mathbb{Q}$, Darmo and Granville [DG95] implicitly gave an example of a stacky curve violating the local-global principle for integral points. In the paper [BP20], Bhargava and Poonen proved that for any stacky curve over $\mathcal{O}_K$ of genus less than 1/2, it satisfies local-global principle for integral points. For $K = \mathbb{Q}$, they gave an example of a genus-1/2 stacky curve violating the local-global principle for integral points in loc. cit.

Our goal is to generalize their counterexample to any number field. We will prove the following theorem.

Theorem 1.1.0.1 (Theorem 5.1.1). For any number field $K$, there exists a stacky curve of genus-1/2 over $\mathcal{O}_K$ violating the local-global principle for integral points.

The way to prove this theorem is to give an explicit construction of a genus-1/2 stacky curve violating the local-global principle for integral points. The paper is organised as follows. In Section 2, we set up the background by recalling some facts on stacky curves. Then we introduce a class of genus-1/2 stacky curves in Section 3. In Section 4, we prove that the stacky curves given in Section 3 have local integral points. Finally, in Section 5, we put some restrictions on the stacky curves given in Section 3 so that they do not have integral points, then Theorem 5.1.1 holds.

2. Notation and preliminaries

2.1. Notation. Given a number field $K$, let $\mathcal{O}_K$ be the ring of its integers, and let $\Omega_K$ be the set of all its nontrivial places. Let $K_\infty \subset \Omega_K$ be the subset of all real places. Let $K_v$ be the completion of $K$ at $v \in \Omega_K$. For $v$ is a finite place, let $\mathcal{O}_v$ be the valuation ring of $K_v$, and let $\mathbb{F}_v$ be the residue field. For $v$ is an archimedean place, let $\mathcal{O}_v = K_v$. We say that an element is a prime element, if the ideal generated by this element is a prime ideal. If an element $p \in \mathcal{O}_K$ is a prime element, we denote its associated valuation by $v_p$, and its associated valuation ring (field) by $\mathcal{O}_p$ (respectively $K_p$). Let $\overline{K}$ be an algebraic closure of $K$.

2020 Mathematics Subject Classification. Primary 11G30; Secondary 14A20, 14G25, 14H25.

Key words and phrases. stacky curves, local points, integral points, local-global principle for integral points.

H.W. was partially supported by NSFC Grant No. 12071448. C.L. was partially supported by NSFC Grant No. 11701362.
2.2. Stacky curves. In this subsection, we briefly recall some facts on stacky curves. We refer to [Ol91], [VZB19] and [BP20] for more details.

We say that $X$ is a stacky curve over $K$, if $X$ is a smooth, proper and geometrically connected 1-dimensional Deligne-Mumford stack over $K$ that contains a nonempty open substack isomorphic to a scheme, cf. [VZB19] Definition 5.2.1. Given a stacky curve $X$ over a number field $K$, by [KM97] Theorem 1.1, let $X_{\text{coarse}}$ be its coarse moduli space, which is a smooth, projective and geometrically connected curve over $K$. Let $\pi: X \to X_{\text{coarse}}$ be the coarse space morphism. For any finite extension $L/K$ and any closed point $P \in X_{\text{coarse}}(L)$, let $G_P$ be the stabilizer of $X$ above $P$, which is a finite group scheme over $K$. Let $P \subset X_{\text{coarse}}$ be the reduced finite subshebe above which the stabilizer is nontrivial. And $\pi$ is an isomorphism over the open subscheme $X_{\text{coarse}} \setminus P$. Motivated by the Riemann-Hurwitz formula, the genus of $X$ is defined by

\[
g(X) := g(X_{\text{coarse}}) + \frac{1}{2} \sum_{P \in P} \left(1 - \frac{1}{\deg G_P}\right) \deg P.
\]

This formula is stable under base field change. It can be defined using the geometrically closed points of $P$ by

\[
g(X) := g(X_{\text{coarse}}) + \frac{1}{2} \sum_{P \in P(K)} \left(1 - \frac{1}{\deg G_P}\right).
\]

In particular, the genus is a nonnegative rational number. From this formula, the following lemma follows.

**Lemma 2.2.1.** ([PV10] Lemma 6 and Proposition 8) Let $X$ be a stacky curve over a number field $K$, then $g(X) \geq 0$. If $g(X) < 1$, then $g(X_{\text{coarse}}) = 0$ and $X$ is geometrically isomorphic to $\mathbb{P}^1$.

It follows by the Hasse-Minkowski theorem that for a stacky curve of genus less than one over a number field, the local-global principle for rational points always holds. Bhargava and Poonen [PV10] Theorem 5] proved that the local-global principle for integral points always holds for a stacky curve of genus less than 1/2 over a number field. Furthermore, Christensen [Chris20] Theorem 13.0.6] proved that it satisfies strong approximation. Because of these, we consider the local-global principle for integral points of genus-1/2 stacky curves.

We say that $\mathcal{X}$ is a stacky curve over $O_K$, if $X$ is a proper algebraic stack over $O_K$ such that it is a stacky curve over $K$, (i.e. $X_K$ its base change to $K$, is a stacky curve). For any $O_K$-algebra $R$, let $\mathcal{X}(R)$ be the set of isomorphism classes of $O_K$-morphisms $\text{Spec } R \to \mathcal{X}$.

3. A class of genus-1/2 stacky curves

Let $K$ be a number field. Let $\mu_2 := \text{Spec } O_K[\lambda]/(\lambda^2 - 1) \subset \mathbb{G}_m := \text{Spec } O_K[\lambda, 1/\lambda]$ be the closed subgroup scheme. Let $\mathbb{Z}/2\mathbb{Z} := \text{Spec } O_K[\lambda]/(\lambda - 1) \bigcup \text{Spec } O_K[\lambda]/(\lambda + 1)$. The following lemma states that these two finite group schemes are isomorphic over $O_K[1/2]$.

**Lemma 3.0.1.** Given a number field field $K$, the natural morphism $\mathbb{Z}/2\mathbb{Z} \to \mu_2$ given by

\[O_K[\lambda]/(\lambda^2 - 1) \to O_K[\lambda]/(\lambda - 1) \times O_K[\lambda]/(\lambda + 1)\]

is a group homomorphism. And it is an isomorphism over $O_K[1/2]$.

**Proof.** By a direct check of group operators of these two group schemes, this is a group homomorphism. And the ring homomorphism base change to $O_K[1/2]$, is an isomorphism.

Let $p, q$ be two coprime integers in $K$. Let $z^2 - px^2 - qy^2$ be a homogeneous polynomial in $O_K[x, y, z]$ with homogeneous coordinates $(x : y : z)$. Let $Y_{(p,q)} := \text{Proj } O_K[x, y, z]/(z^2 - px^2 - qy^2)$, and let $Y_{(p,q)}$ be its base change to $K$. We define a $\mu_2$-action on $Y_{(p,q)}$ by letting $\lambda \in \mu_2$ act as $(x : y : z) \mapsto (x : y : \lambda z)$. Let $[Y_{(p,q)}/\mu_2]$ and $[Y_{(p,q)}/\mu_2]$ be the quotient stacks over $O_K$ and $K$ respectively.
Proposition 3.0.2. The quotient stack \( \mathcal{Y}_{(p,q)/\mu_2} \) is a Deligne-Mumford stack over \( \mathcal{O}_K[1/2] \). The quotient stack \( \mathcal{Y}_{(p,q)/\mu_2} \) is a genus-1/2 stacky curve.

Proof. Since \( \mathcal{Y}_{(p,q)/(\mathbb{Z}/2\mathbb{Z})} \) is a Deligne-Mumford stack over \( \mathcal{O}_K \), the first argument follows from Lemma 3.0.1. In particular, the quotient stack \( \mathcal{Y}_{(p,q)/\mu_2} \) is a Deligne-Mumford stack. For a Deligne-Mumford stack, the properties of being smooth, proper and geometrically connected of dimension one follow from these properties of \( \mathcal{Y}_{(p,q)} \). Let \( \mathcal{P}_{z=0} \subset \mathcal{Y}_{(p,q)} \) be the finite \( K \)-subscheme defined by \( z = 0 \). The group \( \mu_2 \) acts freely on \( \text{Proj} \, K[x,y,z]/(z^2 - px^2 - qy^2) \mathcal{P}_{z=0} \), so the stack \( \text{Proj} \, K[x,y,z]/(z^2 - px^2 - qy^2) \mathcal{P}_{z=0} / \mu_2 \) is representable by a scheme, which is an open substack of \( \mathcal{Y}_{(p,q)/\mu_2} \). For \( \text{Proj} \, K[x,y,z]/(z^2 - px^2 - qy^2) \mathcal{P}_{z=0} / \mu_2 \) is geometrically isomorphic to \( \mathbb{G}_m \), geometrically this action over it can be viewed as the action from the Kummer sequence \( 1 \rightarrow \mu_2 \rightarrow \mathbb{G}_m \rightarrow \mathbb{G}_m \rightarrow 1 \), hence the stack \( \text{Proj} \, K[x,y,z]/(z^2 - px^2 - qy^2) \mathcal{P}_{z=0} / \mu_2 \) is geometrically isomorphism to \( \mathbb{G}_m \). So \( \mathcal{Y}_{(p,q)/\mu_2} \) is a stacky curve and \( g(\mathcal{Y}_{(p,q)/\mu_2}) = 0 \). For \( \mu_2 \) acts trivially on \( \mathcal{P}_{z=0} \) containing two geometrically point, by the genus formula \( \frac{1}{2} \), we have \( g(\mathcal{Y}_{(p,q)/\mu_2}) = 1/2 \). \( \square \)

The stacky curves that we consider in this paper, are the quotient stacks of form \( \mathcal{Y}_{(p,q)/\mu_2} \). And we denote \( \mathcal{Y}_{(p,q)/\mu_2} \) by \( \mathcal{X}_{(p,q)} \).

4. Existence of local points

In this section, we prove that the stacky curve \( \mathcal{X}_{(p,q)} \) has local integral points, i.e. the set \( \mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset \) for all \( v \in \Omega_K \).

Lemma 4.0.1. Given a number field \( K \), let \( p, q \) be two coprime integers in \( K \). Let \( S = K \cup \{ v \in \Omega_K | 2pq \neq 0 \} \) be a finite set. Then the set \( \mathcal{Y}_{(p,q)}(\mathcal{O}_v) \neq \emptyset \) for all \( v \in \Omega_K \setminus S \).

Proof. For any finite place \( v \in \Omega_K \), by Chevalley-Warning theorem (cf. [Ser73, Chapter I §2, Corollary 2]), the set \( \mathcal{Y}_{(p,q)}(\mathcal{O}_v) \neq \emptyset \). For any \( v \in \Omega_K \setminus S \), the scheme \( \mathcal{Y}_{(p,q)} \) is smooth over \( \mathcal{O}_v \). By the smooth lifting theorem, the set \( \mathcal{Y}_{(p,q)}(\mathcal{O}_v) \neq \emptyset \) for all \( v \in \Omega_K \setminus S \). \( \square \)

Remark 4.0.2. Consider the quotient morphism: \( \mathcal{Y}_{(p,q)} \rightarrow \mathcal{X}_{(p,q)} \). Then this lemma implies that the set \( \mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset \) for all \( v \in \Omega_K \setminus S \).

In order to prove that the stacky curves \( \mathcal{X}_{(p,q)} \) has local integral points. We need to check that the set \( \mathcal{X}_{(p,q)}(\mathcal{O}_v) \neq \emptyset \) for all \( v \in S \).

Let \( \mathcal{O}_K \)-algebra \( R \) be a principal ideal domain. We analysis the set \( \mathcal{X}_{(p,q)}(R) \) first.

By definition of the quotient stack, a morphism \( \text{Spec} \, R \rightarrow \mathcal{X}_{(p,q)} \) is given by a \( \mu_2 \)-equivariant morphism \( T \rightarrow \mathcal{Y}_{(p,q)} \). The torsors are classified by \( H^1_{\text{fppf}}(R, \mu_2) \), which is isomorphic to \( R^\times / R^{\times 2} \), since \( H^1_{\text{fppf}}(R, \mathbb{G}_m) = \text{Pic} \, R = 0 \). Explicitly, if \( t \in R^\times \), the corresponding \( \mu_2 \)-torsor is \( T_t := \text{Spec} \, R[u]/(u^2 - t) \) and the \( \mu_2 \)-action on \( T_t \) is given by letting \( \lambda \in \mu_2 \) act as \( u \mapsto \lambda u \). Let \( \mathcal{Y}_{(p,q)t} := \text{Proj} \, R[x,y,z]/(tz^2 - px^2 - qy^2) \) be the twist of \( \mathcal{Y}_{(p,q)} \) by \( t \). Consider the \( \mu_2 \)-torsor \( \mathcal{Y}_{(p,q)t} \times T_t \) over \( \mathcal{Y}_{(p,q)} \). Define a morphism \( \mathcal{Y}_{(p,q)t} \times T_t \rightarrow \mathcal{Y}_{(p,q)} \) given by

\[
\begin{align*}
&\mathcal{O}_K[x,y,z]/(z^2 - px^2 - qy^2) \rightarrow R[x,y,z]/(tz^2 - px^2 - qy^2, u^2 - t) \\
&\quad (x,y,z) \mapsto (x,y,u^2).
\end{align*}
\]

It is a \( \mu_2 \)-equivariant morphism. This gives a morphism \( \pi_1: \mathcal{Y}_{(p,q)t} \rightarrow \mathcal{X}_{(p,q)} \). To give a \( \mu_2 \)-equivariant morphism \( T_t \rightarrow \mathcal{Y}_{(p,q)} \) is the same as giving a triple \( (a_1, a_2, a_3) \in R^3 \), and the \( \mu_2 \)-equivariant morphism is given by

\[
\begin{align*}
&\mathcal{O}_K[x,y,z]/(z^2 - px^2 - qy^2) \rightarrow R[u]/(u^2 - t) \\
&\quad (x,y,z) \mapsto (a_1, a_2, a_3u).
\end{align*}
\]
And the triple \((a_1, a_2, a_3)\) gives a morphism \(\text{Spec } R \to \mathcal{Y}_{(p, q)}\) defined by
\[
R[x, y, z']/(tx^2 - px^2 - qy^2) \to R
\]
\[(x, y, z') \mapsto (a_1, a_2, a_3).
\]
Hence, to give a \(\mu_2\)-equivariant morphism \(T_t \to \mathcal{Y}_{(p, q)}\) is the same as giving a morphism \(\text{Spec } R \to \mathcal{Y}_{(p, q)}\). Thus we obtain
\[
\mathcal{X}_{(p, q)}(R) = \prod_{t \in \mathbb{R}^*/\mathbb{R}^2} \pi_t(\mathcal{Y}_{(p, q)}(R)).
\]
With this preparation, we have the following proposition.

**Proposition 4.0.3.** Given a number field \(K\), let \(p, q\) be two coprime integers in \(K\). Then the set \(\mathcal{X}_{(p, q)}(\mathcal{O}_v) \neq \emptyset\) for all \(v \in \Omega_K\).

**Proof.** By Lemma 4.0.1, we need to check that the set \(\mathcal{X}_{(p, q)}(\mathcal{O}_v) \neq \emptyset\) for all \(v \in S\).
Suppose that \(v \in \infty^*_K\) or \(v \nmid q\). Then \(q \in \mathcal{O}_v\). Since \(qx^2 - px^2 - qy^2 = 0\) has a nontrivial solution \((x : y : z) = (0 : 1 : 1)\), we have \(\mathcal{Y}_{(p, q)}(\mathcal{O}_v) \neq \emptyset\). Hence, the set \(\mathcal{X}_{(p, q)}(\mathcal{O}_v) \neq \emptyset\).

Similarly, suppose that \(v \nmid p\), the sets \(\mathcal{Y}_{(p, q)}(\mathcal{O}_v) \neq \emptyset\) and \(\mathcal{X}_{(p, q)}(\mathcal{O}_v) \neq \emptyset\).

Since \(p, q\) are two coprime integers, the set \(\mathcal{X}_{(p, q)}(\mathcal{O}_v) \neq \emptyset\) for all \(v \in \Omega_K\). \(\square\)

5. Genus 1/2-stacky curves violating the local-global principle for integral points

Given a number field \(K\), we put some restrictions on the choice of integers \(p, q\) so that the stacky curve \(\mathcal{X}_{(p, q)}\) has no integral points, i.e. the set \(\mathcal{X}_{(p, q)}(\mathcal{O}_K) = \emptyset\). We choose \(p, q\) in the following way.

5.1. Choosing prime elements. Given a number field \(K\), since the ideal class group of \(K\) is finite, we take a positive integer \(N\) such that \(\mathcal{O}_K[1/N]\) is a principal ideal domain. By Dirichlet’s unit theorem, the group \(\mathcal{O}_K[1/N]^\times\) is a finitely generated abelian group. We assume that it is generated by \(\{a_i\}\) for \(i = 1, \cdots, n\). By Čebotarev’s density theorem and global class field theory applied to a ray class field, we can find a pair of two different odd prime elements \((p, q)\) such that

1. \(a_i \in K_p^{1/p}\) for all \(i = 1, \cdots, n\),
2. \(q \notin K_p^{1/p}\).

We refer to [Wu21] and [Wu22] for more details. Then we have the following theorem.

**Theorem 5.1.1.** Let \(K\) be a number field. Let a positive integer \(N\) and a pair of two different odd prime elements \((p, q)\) be chosen as in Subsection 5.1. Let \(\mathcal{X}_{(p, q)}\) be the stacky curve defined in Section 3. Then \(\mathcal{X}_{(p, q)}\) is a stacky curve of genus-1/2 over \(\mathcal{O}_K\) violating the local-global principle for integral points.

**Proof.** By Proposition 3.0.2, the genus of \(\mathcal{X}_{(p, q)}\) is 1/2. By Proposition 4.0.3, the set \(\mathcal{X}_{(p, q)}(\mathcal{O}_v) \neq \emptyset\) for any \(v \in \Omega_K\).

Next, we prove that the set \(\mathcal{X}_{(p, q)}(\mathcal{O}_K[1/N]) = \emptyset\). For the ring \(\mathcal{O}_K[1/N]\) is a principal ideal domain, in order to prove that \(\mathcal{X}_{(p, q)}(\mathcal{O}_K[1/N]) = \emptyset\), by the equality of sets (4), it will be sufficient to prove that for any \(t \in \mathcal{O}_K[1/N]^\times\), the set \(\mathcal{Y}_{(p, q)}(\mathcal{O}_K[1/N]) = \emptyset\). For \(\mathcal{O}_K[1/N]^\times\) is generated by \(\{a_i\}\) for \(i = 1, \cdots, n\), and by the chosen condition of Subsection 5.1 that \(a_i \in K_p^{1/p}\), we have \(\mathcal{Y}_{(p, q)}(\mathcal{O}_K[1/N]) = \emptyset\).

By the choice of elements \(q\), the set \(\mathcal{Y}_{(p, q)}(\mathcal{K}_p) = \emptyset\). So \(\mathcal{X}_{(p, q)}(\mathcal{O}_K[1/N]) = \emptyset\), which implies that \(\mathcal{X}_{(p, q)}(\mathcal{O}_K) = \emptyset\).

So the stacky curve \(\mathcal{X}_{(p, q)}\) is of genus-1/2 and violating the local-global principle for integral points. \(\square\)
Remark 5.1.2. This theorem implies that the chosen stacky curve $X_{(p,q)}$ violates strong approximation in the sense of [Chr20].

Acknowledgements. The authors would like to thank D.S. Wei and W.Z. Zheng for many fruitful discussions.

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University of Science and Technology of China, School of Mathematical Sciences, No.96, JinZhai Road, Baohe District, Hefei, Anhui, 230026. P.R.China.

Email address: wuhan90@mail.ustc.edu.cn

State Key Laboratory of Information Security, Institute of Information Engineering, Chinese Academy of Sciences, Beijing 100093, P.R. China

Email address: lvchang@amss.ac.cn