Kinetic theory and Brazilian income distribution

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Abstract

We investigate the Brazilian personal income distribution using data from National Household Sample Survey (PNAD), an annual research available by the Brazilian Institute of Geography and Statistics (IBGE). It provides general characteristics of the country’s population. Using PNAD data background we also confirm the effectiveness of a semi-empirical model that reconciles Pareto power-law for high-income people and Boltzmann-Gibbs distribution for the rest of population. We use three measures of income inequality: the Pareto index, the average income and the crossover income. In order to cope with many dimensions of the income inequality, we calculate these three indices and also the Gini coefficient for the general population as well as for two kinds of population dichotomies: black/indigenous/mixed race versus white/yellow; and men versus women. We also followed the time series of these indices for the period 2001-2014. The results suggest a decreasing of Brazilian income inequality over the selected period. Another important result is that historically-disadvantaged subgroups (Women and black/indigenous/mixed race), that are the majority of the population, have a more equalitarian income distribution. These groups have also a smaller monthly income than the others and this social structure remained virtually unchanged in the period of time.

1 A simple model of income distribution

The first power-law has discovered about a century ago by economist Vilfredo Pareto long before the personal income distribution analysis to become an important meaning in the econophysics. Pareto stated there is a simple law which governs all distribution of income (at least for the asymptotic high-income regions) [11]:

\[ P(m) = \frac{A}{m^{(1+\alpha)}} \]

Then, applying log on both sides of the expression, \( \log(P(m)) = \log(A) - (1 + \alpha) \log(m) \), where \( A \) is a constant, \( m \) represents the personal income and \( \alpha \) is known as Pareto Index. It means that log-log plotted of Pareto power-law obey a linear behavior [7]. As a consequence, the higher is the value of \( \alpha \), the lower will be the income inequality between the wealthiest people.

After the Pareto power-law discovery it has been exhaustively confirmed by researches conducted in different countries, types of society at various periods of history over the following years.

Other extremely useful model to study of income distribution and wealth inequality in a determinate population of agents with an arbitrary individual income \( m_i \) is possible by analogy with the Boltzmann-Gibbs distribution (BGD). In classical kinetic theory
the BGD is the most probable distribution \( f(\varepsilon) \) for an equilibrium gas of \( N \) elements enclosed in a box. This consideration is independent of particle shape or details about interactions that occurs during elastic collisions (energy conservation), in this sense BGD is universal.

The transposition of this physical knowledge to the economy world happens when energy \( \varepsilon \) is replaced by money and for this reason the cash is also conserved. Therefore an initial arbitrary money distribution function must meet the conditions:

\[
\sum_{i=1}^{N} \phi_i = \Phi \tag{2}
\]
\[
\sum_{i=1}^{N} m_i \phi_i = M \tag{3}
\]

where we defined \( \phi_i \) as the number of agents and \( M \) as total income.

In order to satisfy that two specific needs, the stationary distribution of income \( P(m) \) should be exponential as equilibrium BGD. Mathematically after a few steps:

\[
P(m) = C \exp(-\lambda m_i) \tag{4}
\]

where \( C \) is a constant and \( \lambda \) is a Lagrange multiplier. In this way, normalizing \( P(m) \) to unity: \( \int_{0}^{\infty} P(m)dm = 1 \) and then calculating the average income \( <m> = \int_{0}^{\infty} mP(m)dm \). Finally, we can rewrite the distribution function of personal income as,

\[
P(m) = \frac{1}{T} \exp(-m/T) \tag{5}
\]

note that the “temperature” \( T \) is equal to the average income \( <m> \) \[3\].

An overview of some empirical studies of wealth distribution indicates that the exponential or log-normal behavior are observed for 90-95\% of the population and Pareto power-law for the wealthiest rest of population \[1,11\].

On the basis of this information we propose a general stationary income distribution function for a population:

\[
P(m) = \frac{1}{T} \theta(m_l - m) \exp(-m/T) + \frac{A}{m^{(1+\alpha)}} \tag{6}
\]

in the equation above \( \theta \) is the Heaviside function, \( m_l \) is the crossover income of personal income which the BGD is observed and \( A \) is a experimental constant. The effectiveness of that model became clear from the situations discussed in this text. Next for convenience, from equation (6) we can derive the cumulative probability distribution \( P(m' \geq m) \):

\[
P(m' \geq m) = \int_{m}^{\infty} P(m)dm \tag{7}
\]

hence, \( P(m' \geq m) = \exp(-m/T) - \exp(-m_l/T) + \frac{Am_l^{-\alpha}}{m^{(1+\alpha)}} \) if \( m \leq m_l \) and \( P(m' \geq m) = \frac{Am_l^{-\alpha}}{m^{(1+\alpha)}} \) if \( m \geq m_l \). Recognizing that in the vast majority of cases \( \alpha \sim 1.5 \) and which obviously \( m_l T > 1 \), we estimate (7) as

\[
P(m' \geq m) \approx \begin{cases} 
\exp(-m/T) & m \leq m_l \\
Bm_l^{-\alpha} & m \geq m_l 
\end{cases} \tag{8}
\]

to wit, the probability of a person having an individual income equal or greater than to \( m \). Thus, we can recognize that equation (8) is very closely linked to the econophysics “two-class” theory of Yakovenko \[9\].

\[2\]
2 Income inequality measures

Through previous semi-empirical model we can take three useful parameters to measure income inequality between subgroups of a same population or even to compare different populations, the Pareto index, the “temperature” and the crossover income \( m_l \). The Pareto index to be directly associated with wealthiest people and so it is limited by this range. But \( \alpha \) is still very convenient to make comparisons and even analyzing its temporal evolution as can be seen in some studies \([2,10,12]\).

The “temperature” and the crossover income can provide us a good perspective about income inequality between subgroups belonging to a given population at a same time; however, due to exchange rates between the base currency and foreign currencies, inflation and other complex aspects the average income and crossover income without adequate treatment can not contribute much more than that.

Beyond the proposed model the Gini coefficient \((G)\) has been the most popular instrument used to measure income inequality in the literature \([6,13]\).

Lorenz curve

![Lorenz curve](image)

In Figure 1 the simple function \( Y = X \) represents the situation where the income is perfectly distributed, i.e., the “poorest” 20% of population would earn the same percentage of total income and so on \([6]\). The other function commonly known as Lorenz curve shows the observed behavior in real societies. So, the Gini coefficient can be obtained through calculating the ratio of the area between the diagonal line and Lorenz curve \((OPQ)\) by area of the triangle beneath diagonal \((OPR)\). In other words, \( G = \frac{\text{Area}\,OPQ}{\text{Area}\,OPR} \). As a result, the key problem becomes estimating the value of the area below the Lorenz curve. Numerically this result can be achieved by successive approximation of trapezoids \( S_j \). Note that for \( N \) pairs \((X,Y)\) under the Lorenz curve we can build \( N-1 \) trapezoids \([5]\). The \( j \)th trapezoid base is \( Y_j \) and \( Y_{j-1} \) whose height is \((X_j - X_j-1)\). Thus,

\[
S_j = \frac{(Y_j + Y_{j-1})(X_j - X_{j-1})}{2}
\]

in view of previous equation we can define graphically the Gini coefficient by Brown’s formula:
while there are many positive contributions to the Gini coefficient in the evaluation of income inequality, it also has a few disadvantages. The most evident disadvantage is concerned with its highly sensitive to transfer especially of the middle classes [14].

3 Income distribution in Brazil

In this section, we investigate the Brazilian personal income distribution using microdata from National Household Sample Survey (PNAD) available by the Brazilian Institute of Geography and Statistics (IBGE). The PNAD is an annual research that study general aspects of Brazilian society, regarding labor, income, education and others [8]. Under the circumstances, we extracted from PNAD data the value of monthly income from all sources for people aged over 10 years (m) and also for four subsets: black / indigenous / mixed race, white / yellow, man and woman. After we got the income variable m and we neglected the persons without income as well as missing values in the new subset and once we begin making the analysis and fitting functions to the distributions. In this way, a part of the results of the paper is about economically active population.

In the specific case of the year 2014 we had 362,625 cases of which following the criteria mentioned earlier we worked with 219,288 of them. Initially to provide evidences of the effectiveness of model described here we calculated the cumulative probability distribution of income for all Brazilian population as you can see on the Figure 2.

PNAD of 2014

\[
G = 1 - 2 \sum_{j}^{N-1} S_j = 1 - \sum_{j}^{N-1} (Y_j + Y_{j-1})(X_j - X_{j-1})
\]
To complete this inequality parameters set of the Brazilian population we calculate the Gini coefficient associating the Brown’s formula and the method of convergent extrapolation oscillation \[4\]. This methodology allowed us to verify that the measure of the degree of uncertainty associated with this estimate is virtually nil due to extreme convergence. For this reason, we just estimate the accuracy of measure in the first divergent decimal case. Hence, \(G = 0.504\) for total population through 2014 PNAD data.

Gini coefficient from 2014 PNAD

![Figure 3: (a) Empirical Brazilian population Lorenz curve. (b) Method of convergent extrapolation oscillation.](image)

We made the same analysis for two different dichotomies of the population. The first is the division by gender (man and woman) and the second is the division by color/race (black / indigenous / mixed race and white / yellow). Then, we compare the inequality parameters for each of this cases.

| PNAD of 2014 | Group | Sex/Gender | Race/Color |
|--------------|-------|------------|------------|
| Subgroup     | Man   | Woman      | WY         | BIM        |
| Gini coefficient | 0.497 | 0.497      | 0.519      | 0.457      |
| Temperature (R$) | 1940 ± 9 | 1359 ± 6  | 2144 ± 11  | 1278 ± 5   |
| Pareto index | 2.172 ± 0.012 | 2.247 ± 0.020 | 2.306 ± 0.017 | 2.187 ± 0.012 |
| crossover income (R$) | 6000 | 4010       | 7000       | 3510       |

Table 1: Inequality parameters for each subgroup of 2014 PNAD, where BIM means black, indigenous, mixed race and WY means white, yellow. To construct the table we clean data using the same criteria applied to total population. Likewise, we fixed with good precision the crossover income \(m_l\) as lower limit to income of the 5% wealthiest population of each subgroup like we made for total population.

As can be seen from Table 1, for the gender group there is no difference between the Gini coefficient measured to men and women. For the 5% richest people we see a small improvement in economically active woman’s income distribution if we compare it with the man’s income distribution. This modest improvement is statistical significantly if we consider the 95% confidence interval of 0.024 around both man’s and woman’s Pareto index. However, the differences between men and women are more expressively if we focus on the monthly income values, since both the crossover income and the “temperature” are much higher for men. This contrast between the income parameters of
men and women can be better understood if we note that in the year of 2014 through the PNAD data, about 51.5% of the respondents were women, but only 40% of the total monthly income belonged to them. When we analysis the population separation by color, we note that the inequality in general is lower for the BIM subgroup. But joining to this result the differences between BIM and WY temperatures that it represents almost 68% of the BIM average monthly income, and the crossover income contrast at the same subgroups, we can recognize that the lower Gini coefficient and higher Pareto index calculated to BIM subgroup reflects the fact that the people belonging to this subset are concentrate at the poorer classes be either BG region or Pareto region. For the group separated by color 57.5% of the PNAD respondents are self-identified as black, indigenous or mixed-race while about 44% of the total monthly income is earned by BIM members.

4 Time series of inequality coefficients

To broaden our understanding of Brazilian income distribution, in this section we investigate the dynamics of that inequality parameters over the years 2001-2014. We use the same PNAD data source as before and still we maintain the criteria of data cleaning, statistical procedure and formation of groups previously adopted for the 2014 PNAD alone.

The first interesting observation is about the Pareto region that remained fairly constant over the selected years ranging from 4% to 6% of total population. For this reason, we fixed the crossover income as the lower monthly income of the 5% richest people that is represents asymptotic high-income region. The next result relates to the time series of the values obtained for temperature and crossover income of the total population, both measured in local currency (Real).

Time series of temperature and crossover income

![Time series of the average and crossover income](image)

In this context, if we analyze the Figure 4 we can observe that the percentage difference between the “temperature” and the crossover income in the year of 2001 corresponded to 241% of the average income. For the last year studied (2014) this percentage difference decreases to 202%.

Focusing on the time series of Gini coefficient, we estimated it for the same subsets as before: gender (man, woman) and color (BIM, WY).

For all the subgroups considered a significant improvement of the Gini coefficient in these 14 years is observed. In specific for the total population this inequality parameter
changed from 0.578 to 0.504 an development of approximately 13%. These findings suggest an decreasing of Brazilian income inequality. One of the possible reasons for this improvement is an increase in the number of people in the middle income class where the value of the Gini coefficient is more sensitive to changes. Moreover, as can be shown in Figure 5, historically-disadvantaged subgroups (woman and BIM) have a better Gini coefficient. In the gender case the difference between man’s and woman’s Gini coefficient over the years selected was decreasing and in 2014 it became zero. Therefore, another pertinent question has been raised, how is money distributed among these subgroups? To answer this question, we evaluate the percentage difference of the average income and the crossover income for each subgroup in comparison with the total population.

In Figure 6 it is indicated that the income inequality is more expressive when the population is divided by color and also that the percentage difference are larger for the crossover income than the average income. Although their similar behavior, average income and crossover income have no strong correlation. Focusing the gender dichotomy (blue lines), there is no substantial shift towards a better income equality for studied parameters if we take into account the standard errors. In other words, despite the reduction over time of the Gini coefficient, the fact that Brazilian women earn on average 40% less than men remained unchanged over the investigated years. When the Brazilian people is separated by color/race (red lines), the percentage differences for
Percentage difference relative to the total population

![Average income](image1)

![Crossover income](image2)

Figure 6: (a) Average income percentage difference from the total population. The standard error adopted for the graph is 2% and it was obtained by calculating the propagated error for each individual measure where we chose the highest uncertainty among all errors. (b) Crossover income percentage difference relative to the total population. The standard error is also about 2% due to ranging of 4% to 6% in high-income region.

Average income and crossover income with respects to the total population are higher. Whites and yellows earn about 65% to 50% more than blacks, mixed race and indigenous. Nevertheless, for BIM subgroup we could notice a decrease of 10% of its percentage difference in comparison with the total population for average income and crossover income over the selected years.

Another Gini coefficient limitation is that it does not contain information about absolute national or personal incomes [13]. Hence, even with a better Gini coefficient for historically-disadvantaged subsets through Figure 6 we can see that these disadvantaged subgroups of the Brazilian population have a much smaller monthly income than the others. For this reason, it is possible to observe that the social structure of men earning more than women and WY earning more than BIM remained over the times series.

We use the Pareto exponent to investigated the income distribution tail comprising roughly 5% of the population that earn a high monthly income. The insert of Figure 7 shows that for almost all subsets of total population the Pareto index fluctuates around its respectively means. This means are indicated for horizontal dashed lines. For the group divided by sex and for the WY colors the slope coefficient has magnitude $10^{-3}$. Otherwise, a slight increase is observed in the BIM group where the slope coefficient is

8
Time series of Pareto index

![Figure 7](image)

Figure 7: (a) Pareto index of total population divided by gender. (b) Pareto index for total population divided by color. The horizontals dashed lines represents the means which Pareto coefficients of each subgroups fluctuates around.

Thus, for the gender group thought unpaired T test with Welch’s correction the mean of Pareto coefficient over the selected years is significantly higher for women than for men. This represents a better distribution of income for the richer women. For the color group the situation was inverted: the Pareto index for WY is statistically higher.

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