RESEARCH ARTICLE

Clustering and percolation on superpositions of Bernoulli random graphs

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Abstract
A simple but powerful network model with \( n \) nodes and \( m \) partly overlapping layers is generated as an overlay of independent random graphs \( G_1, \ldots, G_m \) with variable sizes and densities. The model is parameterized by a joint distribution \( P_n \) of layer sizes and densities. When \( m \) grows linearly and \( P_n \to P \) as \( n \to \infty \), the model generates sparse random graphs with a rich statistical structure, admitting a nonvanishing clustering coefficient together with a limiting degree distribution and clustering spectrum with tunable power-law exponents. Remarkably, the model admits parameter regimes in which bond percolation exhibits two phase transitions: the first related to the emergence of a giant connected component, and the second to the appearance of gigantic single-layer components.

KEYWORDS
Overlapping communities, power law, clustering coefficient, random graph, intersection graph, complex network, bond percolation, site percolation, giant component

1 | INTRODUCTION

Applications in natural sciences, social sciences, and technology often deal with large networks of nodes linked by pairwise interactions which involve uncertainty due to noisy observations and missing data. Such uncertainties have been investigated using statistical models ranging from classical Bernoulli random graphs and uniform random graphs with given degree distributions to stochastic block models and more complex generative models involving various preferential attachment and rewiring mechanisms [1, 25, 30, 38, 46]. While succeeding to obtain a good fit for degree distributions

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and tractable percolation analysis, most earlier models fail to capture second-order effects related to clustering and transitivity. Random intersection graphs [5, 11, 18, 33, 41], spatial preferential attachment models [27–29], and hyperbolic random geometric graphs [13, 24, 34, 35] have been introduced to conduct percolation analysis on networks with nonvanishing transitivity and clustering properties.

Despite remarkable methodological advances, most sparse network models still appear somewhat rigid in what comes to modeling finer clustering properties, such as the clustering spectrum (degree-dependent local clustering coefficient) [3, 43, 47], which may significantly impact the percolation properties of the network [4, 19]. A decreasing clustering spectrum manifests the fact that high-degree nodes tend to have sparser local neighborhoods than low-degree nodes. Motivated by analyzing this phenomenon in a tractable quantitative framework, this article discusses a statistical network model generated as an overlay of mutually independent Bernoulli random graphs \( G_1, \ldots, G_m \) which can be interpreted as layers or communities. The layers have a variable size (number of nodes) and strength (link probability), and they may overlap each other. A key feature of the model is that the layer sizes and layer strengths are assumed to be correlated, which allows to model and analyze a rich class of networks with a tunable frequency of strong small communities and weak large communities.

1.1 Main contributions

This article presents a rigorous mathematical analysis of clustering and percolation of the overlay graph model in the natural sparse limiting regime where the number of nodes \( n \) tends to infinity, the number of layers \( m \) is linear in the number of nodes, and the joint distribution \( P_n \) of layer sizes and layer strengths converges to a limiting distribution \( P \). We derive exact formulas for the limiting degree distribution, clustering coefficient, clustering spectrum, and the largest component size in terms of cross-factorial moments and functional transforms of \( P \). We also investigate the model under bond and site percolation, and characterize critical parameter values of the associated phase transitions.

The descriptive power of the model is illustrated by a detailed investigation of an instance where the layer size follows a power law, and the layer strength is a deterministic function of the layer size following another power law. This setting leads to a power-law degree distribution and a power-law clustering spectrum with tunable exponents in ranges \((1, \infty)\) and \([0,2]\), respectively. A special case in which layer strengths are inversely proportional to their sizes corresponds to layers of bounded average degree. In this natural parameter regime we discover a remarkable double phase transition phenomenon with two critical values: the first characterizing the emergence of a giant component in the overlay graph, and the second characterizing the emergence of gigantic components in layers covering a typical node.

Finally, we highlight that the modelling framework in this article covers both deterministic and random layer types. Our approach of characterizing the regularity of layer types using averaged empirical distributions allows both cases to be treated in a uniform manner.

1.2 Related work

The overlay network model discussed in this article is naturally motivated and implicitly described by classical works in social networks [17, 22]. The explanatory power and wide applicability of the model in the context of social, collaboration, and information networks has been demonstrated in [48, 49] by experimental studies of a community-affiliation graph, which represents an instance of the present model where the node sets of layers are nonrandom or otherwise known to the observer. The superposition of Bernoulli random graphs considered here serves as a null model for sparse community-affiliation graphs.
The mathematical analysis in this article builds on earlier works on component evolution and clustering in inhomogeneous random graphs [14] and random intersection graphs [8, 9]. The special model instance with unit layer strengths reduces to the so-called passive random intersection graph [26], and as a byproduct, the present article also provides the first rigorous analysis of giant components in general passive random intersection graphs, extending [16, 37]. When layer strengths are constant but not necessarily one, clustering properties and subgraph densities of the model have been analyzed in [31, 32, 40], and the recovery of the layers in [21]. Another related work [45] (also part of [44]) on percolation in overlapping community networks assumes that layers are sampled from an arbitrary distribution on the space of finite connected graphs, and the layers are assigned to nodes via a bipartite configuration model. The restriction to connected layers and the use of a configuration model makes the model in [45] and its analysis fundamentally different from the present one, and limits its applicability by ruling out networks composed of weak communities.

Clustering spectra with power-law exponent 1 have been shown for random intersection graph models [7, 9] and spatial preferential attachment models [27, 36], and with a tunable power-law exponent in [0, 1] for random intersection graphs [10, 12] and recently also for a hyperbolic random geometric graph model [24]. Furthermore, [43] discusses an inhomogeneous Bernoulli graph model where the clustering spectrum vanishes, but its normalized version displays evidence of a power-law behavior with exponent in range (0,2).

To the best of our knowledge, the present work is the first of its kind where a degree dependent clustering with a tunable power-law exponent in the extended range [0,2] is rigorously analyzed in terms of a simple statistical network model. This model admits a clear explanation of the values of power-law exponents, and introduces a new analytical framework for studying ordinary and double phase transitions in bond and site percolation on sparse networks of overlapping communities of variable size and strength.

1.3 Outline

In the rest of the article, Section 2 presents model details and notations, and Section 3 the main results. Section 4 illustrates the main results in a power-law setting, and confirms the existence of double phase transition. The remaining Sections 5–10 are devoted to proofs, with technical details postponed to Appendix 11.

2 MODEL DESCRIPTION

2.1 Multilayer network

A multilayer network model with $n$ nodes and $m$ layers is defined by a list $(G_1, X_1, Q_1), \ldots , (G_m, X_m, Q_m)$ of mutually independent random variables with values in $G_n \times \{0, \ldots , n\} \times [0,1]$, where $G_n$ is the set of undirected graphs with node set contained in $\{1, \ldots , n\}$. We assume that conditionally on $(X_k, Q_k)$, the probability distribution of the vertex set $V(G_k)$ of $G_k$ is uniform on the subsets of $\{1, \ldots , n\}$ of size $X_k$, and conditionally on $(V(G_k), X_k, Q_k)$, each node pair of $V(G_k)$ is linked with probability $Q_k$, independently of other node pairs. Thus, $G_k$ is a Bernoulli random graph on node set $V(G_k)$, with edge set denoted $E(G_k)$. The variables $X_k$, $Q_k$, and $(X_k, Q_k)$ are called the size, strength, and type of layer $k$, respectively. Aggregation of layers produces an overlay random graph $G$ defined by

$$V(G) = \{1, \ldots , n\} \quad \text{and} \quad E(G) = \bigcup_{k=1}^{m} E(G_k).$$
This setting includes as special cases: (i) models with deterministic layer types, and (ii) models where the layer types are independent and identically distributed random variables.

### 2.2 | Large networks

A large network is analyzed by considering a sequence of network models \((G_{n,1}, X_{n,1}, Q_{n,1}), \ldots, (G_{n,m}, X_{n,m}, Q_{n,m})\) indexed by the number of nodes \(n = 1, 2, \ldots\) so that the number of layers \(m = m_n\) tends to infinity as \(n \to \infty\). The sequence of corresponding overlay random graphs is denoted \((G_n)\). We shall focus on a sparse parameter regime where there exists a probability measure \(P\) on (Borel’s \(\sigma\)-algebra \(B\) of \([0, 1, \ldots] \times [0, 1]\) which approximates in sufficiently strong sense the averaged layer type distribution

\[
P_n(B) = \frac{1}{m} \sum_{k=1}^{m} \mathbb{P}((X_{n,k}, Q_{n,k}) \in B), \quad B \in B.
\]

In this fundamental regime, the network features are described by limiting formulas with rich expressive power captured by cross moments and tail characteristics of \(P\).

### 2.3 | Notations

We denote \(\mathbb{Z}_+ = \{0, 1, \ldots\}, (a)_+ = \max\{0, a\}\), and \((x)_s = x(x - 1) \cdots (x - s + 1)\). The indicator function of a condition \(A\) is denoted by \(\mathbb{1}(A)\) or \(\mathbb{1}_A\), whichever is more convenient. Sets of size \(x\) are called \(x\)-sets. Unordered pairs and triples are abbreviated as \(ij\) and \(ijk\). We write \(\sum_{ij}\) and \(\sum_{ijk}\) to indicate sums over ordered pairs and ordered triples with distinct elements. We write \(a_n \ll b_n\) and \(a_n = o(b_n)\) when \(a_n/b_n \to 0\), \(a_n \preceq b_n\) and \(a_n = \Omega(b_n)\) when \(\limsup|a_n/b_n| < \infty\), and \(a_n \sim b_n\) when \(a_n/b_n \to 1\). For a sequence of bivariate random variables \((a_n, \beta_n)\), we write \(a_n = o_p(\beta_n)\) whenever \(\lim_{n \to \infty} \mathbb{P}(|a_n| < \epsilon|\beta_n|) = 1\) for any \(\epsilon > 0\); and \(a_n = O_p(\beta_n)\) if for every \(\epsilon > 0\) there exists a constant \(c_\epsilon > 0\) such that \(\lim_{n \to \infty} \mathbb{P}(|a_n| < c_\epsilon|\beta_n|) > 1 - \epsilon\). Notation \(a_n = o_p(\beta_n)\), \(a_n = O_p(\beta_n)\) extends to the case where the sequence \(\beta_n\) is deterministic (nonrandom).

A graph is a pair \(G = (V, E)\) where \(E\) is a set of unordered pairs of elements of \(V\). The degree and component of node \(i\) in graph \(G\) are denoted by \(\deg_G(i)\) and \(C_G(i)\), respectively. The transitive closure of graph \(G\) is defined as the graph \(\bar{G}\) with \(V(\bar{G}) = V(G)\) and \(E(\bar{G}) = \{ij : i \in C_G(j), j \in V(G)\}\) consisting of unordered node pairs connected by a path in \(G\).

The probability distribution of a random variable \(X\) is denoted by \(\mathcal{L}(X)\). For probability measures, \(d_{tv}(f, g)\) denotes the total variation distance, \(f \ast g\) the convolution, and \(f_n \xrightarrow{w} f\) refers to weak convergence. Convergence in probability is denoted \(\xrightarrow{p}\). On countable spaces, the same letter is used for both a probability measure \(\mathbb{1}(A)\) and its density \(f(t)\) with respect to the counting measure. The Dirac measure at \(x\) is denoted by \(\delta_x\). The densities of the binomial distribution \(\text{Bin}(x, q)\), with \(x \in \mathbb{Z}_+\) and \(q \in [0, 1]\), and the Poisson distribution \(\text{Poi}(\lambda)\) with \(\lambda > 0\), are denoted by

\[
\text{Bin}(x, q)(t) = \binom{x}{t} (1 - q)^{x - t} q^t, \quad \text{Poi}(\lambda)(t) = e^{-\lambda} \frac{\lambda^t}{t!},
\]

with the convention that the densities are zero for \(t\) outside \(\{0, \ldots, x\}\) and \(\mathbb{Z}_+\), respectively. The Bernoulli distribution is denoted \(\text{Ber}(q)(t) = \text{Bin}(1, q)(t)\). We also denote by

\[
\text{Bin}^\tau(x, q)(t) = \mathbb{P}\left(\deg_{H^\tau_{x+1, q}}(1) = t\right)
\]
the degree distribution of any particular node in the transitive closure $H_{x+1,q}$ of a Bernoulli random graph $H_{x+1,q}$ on node set $\{1, \ldots, x+1\}$, where each node pair is linked with probability $q$, independently of other node pairs. Alternatively, $\text{Bin}^+(x,q)(t)$ equals the probability that the connected component of any particular node in $H_{x+1,q}$ has size $t+1$. Both distributions have the same support $\{0, \ldots, x\}$, and $\text{Bin}(x,q) \leq_{st} \text{Bin}^+(x,q)$ in the strong stochastic order. No simple closed form expression is known for $\text{Bin}^+(x,q)(t)$, but its values can be efficiently computed with the help of Gontcharoff polynomials [2, 5]. The compound Poisson distribution with rate parameter $\lambda$ and increment distribution $g$ is denoted $\text{CPoi}(\lambda,g)$; recall that this is the law of a random variable $\sum_{k=1}^\Lambda X_k$ where $\Lambda,X_1,X_2, \ldots$ are mutually independent and such that $\mathcal{L}(\Lambda) = \text{Poi}(\lambda)$ and $\mathcal{L}(X_k) = g$.

For any probability measure $P$ on $\mathbb{Z}_+ \times [0, 1]$, any $P$-distributed random variable $(X, Q)$, and integers $r, s \geq 0$, we denote

$$
(P)_{rs} = \mathbb{E}(X, Q) = \int (x, q) \cdot P(dx, dq),
$$

and when this quantity is finite and nonzero, we define mixed probability distributions $\text{Bin}_{rs}(P)$ and $\text{Bin}^+_{rs}(P)$ on $\mathbb{Z}_+$ with probability mass functions

$$
\text{Bin}_{rs}(P)(t) = \mathbb{E} \left( \text{Bin}(X - r, Q)(t) \frac{(X, Q)_{rs}}{(P)_{rs}} \right),
$$

$$
\text{Bin}^+_{rs}(P)(t) = \mathbb{E} \left( \text{Bin}^+(X - r, Q)(t) \frac{(X, Q)_{rs}}{(P)_{rs}} \right),
$$

where $\text{Bin}(0, q)(t) = \text{Bin}^+(0, q)(t) = \mathbb{I}_{\{t=0\}}$ and $\text{Bin}(x, q) \equiv 0$, $\text{Bin}^+(x, q) \equiv 0$ for $x < 0$.

### 3 | MAIN RESULTS

#### 3.1 | Degree distribution

The model degree distribution is defined by

$$
f^{(n)}(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{P} \left( \deg_{G_{(n)}}(i) = t \right),
$$

and represents the probability distribution of the number of neighbors of a randomly chosen node. Because $G_{(n)}$ is an exchangeable random graph, we see that $f^{(n)} = \mathcal{L} \left( \deg_{G_{(n)}}(1) \right)$.

**Theorem 3.1.** Assume that $\frac{m}{n} \to \mu \in (0, \infty)$ and $P_n \to P$ weakly together with $(P_n)_{10} \to (P)_{10} \in (0, \infty)$ for some probability measure $P$ on $\mathbb{Z}_+ \times [0, 1]$. Then the model degree distribution $f^{(n)}$ converges weakly to the compound Poisson distribution $f = CPoi(\mu(P)_{10}, \text{Bin}_{10}(P))$.

The limiting degree distribution $f$ in Theorem 3.1 can be represented as the law of $\zeta = \sum_{k=1}^\Lambda \zeta_k$ where $\Lambda$ is Poisson distributed with mean $\mu(P)_{10}$, $\zeta_1, \zeta_2, \ldots$ follow a mixed binomial distribution $\text{Bin}_{10}(P)$, and the random variables in the sum are mutually independent. Here, $\Lambda$ represents the number of layers covering a particular node, and $\zeta_k$ the number of neighbors in a typical layer covering the node. The mean equals $\mathbb{E} \zeta = \mu(P)_{21} \leq \infty$, and the variance equals $\text{Var}(\zeta) = \mu((P)_{21} + (P)_{32})$ for $(P_{21}), (P_{32}) < \infty$. Moreover, $\mathbb{E}(\zeta^r) < \infty$ if and only if $(P)_{r+1,r} < \infty$. The generating function is
given by $\mathbb{E}(z^x) = e^{\hat{\beta}_0 g(z^{x-1})}$, where $\hat{g}_{10}(z) = \int (1 - q + qz)^{x-1} \frac{P(d_1, d_2)}{(P)_{10}}$. The structure of $P$ determines whether or not the limiting degree distribution is light-tailed or heavy-tailed. Section 4 illustrates both cases and provides examples of power laws with a tunable exponent. In Theorem 3.1 we have assumed that $(P)_{10} > 0$. One can show that for $(P)_{10} = 0$ the asymptotic degree distribution is degenerate at zero.

### 3.2 Clustering

Given a finite (nonrandom) graph $G = (V, E)$, the global clustering coefficient $\tau_G$ and the degree dependent local clustering coefficient (also called clustering spectrum) $\sigma_G(k)$ are defined as follows. Let $N_{\Delta}$ and $N_v$ denote the number of triangles and cherries (paths of length 2) of $G$, respectively. Let $N_{\Delta}(v)$ be the number of triangles containing vertex $v$. Then

$$\tau_G = \frac{3N_{\Delta}}{N_v}, \quad \sigma_G(k) = \frac{\sum_{v \in V} N_{\Delta}(v) I_{[\deg_G(v) = k]}}{\sum_{v \in V} I_{[\deg_G(v) = k]}}. \quad (8)$$

These network characteristics represent conditional probabilities of a link between two neighbors of a randomly selected vertex. Let $(v_1^*, v_2^*, v_3^*)$ be an ordered triple of vertices sampled uniformly at random. Let $\Delta_{v_1^*, v_2^*, v_3^*}$ denote the event that $v_1^*, v_2^*, v_3^*$ induce the triangle. Similarly, let $\lor_{v_1^*, v_2^*, v_3^*}$ denote the event that $v_3^*$ is adjacent to $v_1^*$ and $v_2^*$. A straightforward calculation shows that the ratios (8) can be written in the form

$$\tau_G = \mathbb{P}^* \left( \Delta_{v_1^*, v_2^*, v_3^*} | \lor_{v_1^*, v_2^*, v_3^*} \right), \quad \sigma_G(k) = \mathbb{P}^* \left( \Delta_{v_1^*, v_2^*, v_3^*} | \lor_{v_1^*, v_2^*, v_3^*}, \deg_G(v_3^*) = k \right).$$

Here, the probability $\mathbb{P}^*$ refers to random sampling of vertices $v_1^*, v_2^*, v_3^*$. Below we consider similar conditional probabilities, but defined for the random graph $G$ (instead of $G$)

$$\tau(G) = \mathbb{P} \left( \Delta_{v_1^*, v_2^*, v_3^*} | \lor_{v_1^*, v_2^*, v_3^*} \right), \quad \sigma(G)(k) = \mathbb{P} \left( \Delta_{v_1^*, v_2^*, v_3^*} | \lor_{v_1^*, v_2^*, v_3^*}, \deg_G(v_3^*) = k \right). \quad (9)$$

We call $\tau(G)$ the model (global) clustering coefficient and $\sigma(G)(k)$ the model clustering spectrum of the random overlay graph $G$. Note that conditional probabilities (9) refer to two sources of randomness, namely, the random graph $G$ and the random sampling of vertices $v_1^*, v_2^*, v_3^*$ (which is assumed to be independent of $G$). It is interesting to compare the probabilities (9) with the respective ratios $\tau_G$ and $\sigma_G(k)$ of (8), where $G$ is replaced by an instance of the random graph $G$. An argument bearing on the law of large numbers (applied to the sums of random variables in the numerators and denominators of ratios (8)) suggests that $\tau_G - \tau(G) = o_P(1)$ and $\sigma_G(k) - \sigma(G)(k) = o_P(1)$ as the number of vertices $n \to +\infty$. Therefore, the model characteristics $\tau(G)$ and $\sigma(G)(k)$ can be viewed as approximations to the clustering coefficients $\tau_G$ and $\sigma_G(k)$ and our asymptotic results for $\tau(G)$ and $\sigma(G)(k)$ shown below can likely be extended to $\tau_G$ and $\sigma_G(k)$.

Now we focus on the model clustering characteristics $\tau(G)$ and $\sigma(G)(k)$. We observe that since the distribution of $G$ is invariant under permutation of its vertices, we have that

$$\tau(G) = \frac{\sum_{ijk} P(G(ij), G(ik), G(ik))}{\sum_{ijk} P(G(ij), G(ik))}, \quad \sigma(G)(k) = \frac{\sum_{ijk} P(\deg_G(i) = k, G(ij), G(ik))}{\sum_{ijk} P(\deg_G(i) = k, G(ij), G(ik))},$$

where $G(ij)$ represents the event that nodes $i$ and $j$ are adjacent, and the sums are taken over ordered triples of distinct nodes. We denote $\tau^{(n)} = \tau(G_{(n)})$ and $\sigma^{(n)}(k) = \sigma(G_{(n)})(k)$. 


Theorem 3.2. Assume that \((P_n)_{rs} \to (P)_{rs} < \infty\) for \(rs = 21, 32, 33,\) and \((P)_{21} > 0\). Then
\[
\lim_{n \to \infty} \tau^{(n)} = \tau,
\]
where
\[
\tau = \begin{cases} 
(P)_{33} & \text{when } m \ll n \text{ and } (P)_{32} > 0, \\
\frac{(P)_{32}}{(P)_{32} + \mu(P)_{21}^2} & \text{when } \frac{m}{n} \to \mu \in (0, \infty), \\
0 & \text{when } n \ll m \ll n^2.
\end{cases}
\]

Remark (constant layer strengths). When \(Q_k = q\) is constant for all \(k\), we see that \((P)_{rs} = (p)_{rqs}\) where \((p)_{r}\) equals the \(r\)th factorial moment of the limiting layer size distribution. In this case the limiting model clustering equals \(\frac{q(p)_{3}}{(p)_{3} + \mu(p)_{21}}\) and agrees with [9, 32].

Theorem 3.3. Assume that \(\frac{m}{n} \to \mu \in (0, \infty)\), and \(P_n \to P\) weakly together with \((P_n)_{10} \to (P)_{10} \in (0, \infty)\) for \(rs = 10, 21, 32\). Then \(\sigma^{(n)} \to \sigma\) pointwise to the limit
\[
\sigma(t) = \frac{(P)_{33} (f * g_{33})(t - 2)}{(P)_{32}(f * g_{32})(t - 2) + \mu(P)_{21}^2(f * g_{21} * g_{21})(t - 2)},
\]
where \(f = \text{CPoi}(\mu(P)_{10}, \text{Bin}_{10}(P)) \) is the limiting degree distribution in Theorem 3.1, and the distributions \(g_{rs} = \text{Bin}_{rs}(P)\) are defined by (5).

Section 4 illustrates examples where the limiting clustering spectrum \(\sigma(t)\) follows a power law.

3.3 | Connected components

We denote by \(N_1(G_{(n)}) \geq N_2(G_{(n)})\) the two largest component sizes in \(G_{(n)}\). For a probability distribution \(h\) on \(\mathbb{Z}_+\), we denote by
\[
\rho_{(h)} = 1 - \min \left\{ s \geq 0 : \sum_{x \geq 0} s^x h(x) = s \right\}
\]
the probability of eternal survival of a Galton–Watson branching process with offspring distribution \(h\).

Theorem 3.4. Assume that \(\frac{m}{n} \to \mu \in (0, \infty)\) and \(P_n \to P\) weakly together with \((P_n)_{10} \to (P)_{10} \in (0, \infty)\). Then
\[
\frac{N_1(G_{(n)})}{n} \xrightarrow{P} \rho_{(f^+)} \quad \text{and} \quad \frac{N_2(G_{(n)})}{n} \xrightarrow{P} 0,
\]
where \(f^+ = \text{CPoi}(\mu(P)_{10}, \text{Bin}_{10}^+(P))\) is a compound Poisson distribution with rate parameter \(\mu(P)_{10}\) and increment distribution \(\text{Bin}_{10}^+(P)\) defined by (6).

In Theorem 3.4 we have assumed that \((P)_{10} > 0\). One can show that for \((P)_{10} = 0\) we have
\[
\frac{N_1(G_{(n)})}{n} \xrightarrow{P} 0.
\]
3.4 | Site percolation

We may analyze how a subset of nodes $S \subset \{1, \ldots, n\}$ is connected by considering a site-percolated graph defined as the subgraph

$$\tilde{G} = G[S]$$

(11)
of $G$ induced by $S$. The site-percolated overlay graph is an instance of the overlay graph model (1) on the vertex set $S$ with layers $(\tilde{G}_1, \tilde{X}_1, \tilde{Q}_1), \ldots, (\tilde{G}_m, \tilde{X}_m, \tilde{Q}_m)$ such that $\tilde{G}_k$ has vertex set $V(\tilde{G}_k) = S \cap V(G_k)$ of size $\tilde{X}_k := |V(\tilde{G}_k)|$ and $\tilde{Q}_k = Q_k$. Note that the conditional distribution of $\tilde{X}_k$ given $X_k = |V(G_k)|$ is hypergeometric. An approximation of the hypergeometric distribution by a binomial distribution where

$$H \approx \text{Bin}(\theta)$$

is hypergeometric. An approximation of the hypergeometric distribution by a binomial distribution $\text{Bin}(\theta)$ with $\frac{|S|}{n} \approx \theta$ suggests replacing the limiting layer type distribution $P$ by

$$\tilde{P}(A) = \int \left( \text{Bin}(x, \theta) \times \delta_q \right) (A) P(dx, dq).$$

The following result confirms that this modification is well justified, and summarizes the results of Theorems 3.1–3.4 adjusted to site percolation.

**Theorem 3.5.** Assume that $\frac{m}{n} \to \mu \in (0, \infty)$, $P_n \to P$ weakly together with $(P_n)_0 \to (P)_0 \in (0, \infty)$, and $S_n \subset \{1, \ldots, n\}$ satisfies $\frac{|S_n|}{n} \to \theta \in (0, 1]$. Then the following approximations are valid for the site-percolated graph $\tilde{G}_0 = \tilde{G}(S_n)$:

(i) The degree distribution converges weakly to $\tilde{\tau} = \text{CPoi} \left( \mu(\tilde{P})_0, \text{Bin}_1(\tilde{P}) \right)$.  

(ii) $|S_n|^{-1}N_1(\tilde{G}_0) \overset{P}{\to} \rho_{\tilde{\tau}}$, and $|S_n|^{-1}N_2(\tilde{G}_0) \overset{P}{\to} 0$ with $\tilde{\tau}^+ = \text{CPoi} \left( \mu(\tilde{P})_0, \text{Bin}_1(\tilde{P}) \right)$.

If we also assume that $(P_n)_r \to (P)_r \in (0, \infty)$ for $r = 21, 32, 33$, then

(iii) The model clustering coefficient $\tau(\tilde{G}_0)$ converges to $\tau$ where $\tau$ is the corresponding limit of the nonpercolated graph $G_0$.

(iv) The model clustering spectrum $\sigma(\tilde{G}_0)$ converges pointwise to $\tilde{\sigma}$ defined by replacing $f$ and $g_{rs}$ in (10) by $\tilde{f}$ and $\tilde{g}_{rs} = \text{Bin}_m(\tilde{P})$.

3.5 | Bond percolation

Bond percolation studies how well the nodes of a graph are connected along a subset of links obtained by random sampling. In a multilayer network, we may either sample (i) a subset of links of the overlay graph, or (ii) independent subsets of links for each layer separately. To analyze these cases for the overlay graph model $G = G_0$ in (1), we define an overlay bond-percolated graph by

$$\tilde{G} = G \cap H,$$

(12)

and a layerwise bond-percolated graph $\tilde{G}$ by

$$V(\tilde{G}) = \{1, \ldots, n\} \quad \text{and} \quad E(\tilde{G}) = \cup_{k=1}^m E(G_k \cap H_k),$$

(13)

where $H, H_1, \ldots, H_m$ are mutually independent random graphs on $\{1, \ldots, n\}$ in which each node pair is linked with probability $\theta$, independently of other node pairs, and independently of the layers $(G_k, X_k, Q_k)$. To emphasize the dependence on $\theta$ we sometimes write $\tilde{G} = \tilde{G}(\theta)$ and $\tilde{G} = \tilde{G}(\theta)$. 
In an epidemic modeling context, the standard SIR epidemic model is used to model individuals who infect their neighbors with probability \( \theta \), independently of each other [2]. The links of a graph \( G \) represent social contacts, and the bond-percolated component of node \( i \) corresponds to the set of eventually infected individuals in a population where node \( i \) is initially infectious and the other nodes susceptible. Bond percolation on the overlay graph can be used to develop finer models to model contacts of individuals generated by social communities (households, workplaces, schools) of variable size and strength. Layerwise percolation \( \hat{G} \) then models the case where infections occur independently inside the communities, and the overlay bond-percolation \( \hat{G} \) models the case where infections occur between individuals regardless of the underlying community structure.

The layerwise bond-percolated graph is an instance of the overlay model (1) with layer types \((X_k, \theta Q_k)\). This suggests considering a modified limiting layer type distribution

\[
\hat{P}(A) = \int (\delta_x \times \delta_{\theta q})(A) \, P(dx, dq).
\]

We expect the overlay bond-percolated model to behave similarly to the layerwise bond-percolated model in sparse regimes where the layers do not overlap much. The following result confirms this, and summarizes the results of Theorems 3.1–3.4 adjusted to bond percolation.

**Theorem 3.6.** Assume that \( \frac{m}{n} \to \mu \in (0, \infty) \), and \( P_n \to P \) together with \((P_n)_{10} \to (P)_{10} \in (0, \infty)\), and \( \theta \in (0, 1] \). Then the following approximations are valid for both the overlay bond-percolated graph \( \hat{G}(\theta) = \hat{G}(\theta) \) and layerwise bond-percolated graph \( \tilde{G}(\theta) = \tilde{G}(\theta) \):

(i) The degree distribution converges weakly to \( f = CPoi(\mu(\hat{P})_{10}, Bin_{10}(\hat{P})) \).
(ii) For \( N_1 \) and \( N_2 \) denoting the largest and the second largest component sizes we have \( n^{-1}N_1 \xrightarrow{\text{P}} \rho_{\hat{f}^+} \) and \( n^{-1}N_2 \xrightarrow{\text{P}} 0 \) with \( \hat{f}^+ = CPoi(\mu(\hat{P})_{10}, Bin_{10}^+(\hat{P})) \).

If we also assume that \((P_n)_{rs} \to (P)_{rs} \in (0, \infty) \) for \( rs = 21, 32, 33 \), then:

(iii) The model clustering coefficient converges to \( \theta \tau \) where \( \tau \) is the corresponding limit of the nonpercolated graph \( G(\theta) \).
(iv) The model clustering spectrum converges pointwise to \( \hat{\sigma} \) defined by replacing \( P, f, \) and \( g_{rs} \) in (10) by \( \hat{P}, \hat{f}, \) and \( \hat{g}_{rs} = Bin_{rs}(\hat{P}) \).

### 3.6 | Double phase transition

Theorem 3.6 shows that the largest relative component size in the bond-percolated graph is approximated by the survival probability \( \rho_{\hat{f}^+} \) of a Galton–Watson process with compound Poisson offspring distribution \( \hat{f}^+ = CPoi(\mu(\hat{P})_{10}, Bin_{10}^+(\hat{P})) \). The mean of the offspring distribution can be written as

\[
R_0(\theta) = \mu \int R(x - 1, \theta q) \, xP(dx, dq),
\]

where \( R(x, q) = \sum_{t \geq 0} t \, Bin^+(x, q)(t) \) defined using (3) represents the expected degree of a node in the transitive closure of Bernoulli random graph with \( x + 1 \) nodes and link probability \( q \). Classical branching process theory tells that \( \rho_{\hat{f}^+} > 0 \) if and only if \( R_0(\theta) > 1 \). Hence the largest component in
the bond-percolated graph is sublinear for $\theta < \theta_1$, and linear for $\theta > \theta_1$, where the critical threshold is defined by

$$\theta_1 = \sup\{\theta \in [0, 1] : R_0(\theta) < 1\}.$$  

The overlay graph model studied in this article involves another nontrivial phase transition associated with a critical threshold value $\theta_2 = \sup\{\theta \in [0, 1] : R_0(\theta) < \infty\}$.

Section 4 describes an example where $0 < \theta_1 < \theta_2 < 1$.

The first phase transition at $\theta_1$ characterizes the emergence of a giant component in a bond-percolated overlay graph. To understand the second phase transition, note that $R_0(\theta)$ is proportional to the expected number of nodes which can be reached by paths within a typical bond-percolated layer covering a particular node. The second phase transition at $\theta_2$ hence amounts to the emergence of gigantic components inside bond-percolated layers covering a typical node.

In the epidemic context discussed in Section 3.5, we note that the critical quantity $R_0(\theta)$ does not refer to the number of individuals directly infected by a reference individual in an otherwise susceptible population, unlike in classical SIR models. Rather, $R_0(\theta)$ also counts the number of individuals indirectly infected by the reference individual via single-layer infection paths.

4 | POWER-LAW MODELS

This section illustrates the rich statistical features of the overlay model by discussing the results of Section 3 in a setting where the limiting layer strength is a deterministic function of layer size according to $Q = g(X)$ for some $g : \mathbb{Z}_+ \to [0, 1]$, and the limiting layer type distribution factorizes according to

$$P(dx, dq) = p(dx)\delta_{g(x)}(dq),$$

(15)

where the layer size distribution $p$ is a probability on $\mathbb{Z}_+$. We are especially interested in the case where the probability mass function $p(x) = \mathbb{P}(X = x)$ of the layer size distribution and $g(x)$ follow power laws

$$p(x) = L(x)x^{-\alpha} \quad \text{and} \quad g(x) = bx^{-\beta}$$

(16)

with exponents $\alpha \geq 2$ and $\beta \geq 0$. Here $L(x)$ is a slowly varying function at $+\infty$, $b > 0$ is a constant, and we choose $b \leq 1$ for $\beta = 0$. We will assume that (16) holds for large $x$. Note that for $r, s$ satisfying $\alpha + s\beta > r + 1$ the cross moment

$$(P)_{rs} = \sum_{x \geq 0} (x)^r g(x)^s p(x)$$

is finite. It is also finite in the case where $\alpha + s\beta = r + 1$ and $x^{-1}L(x)$ is integrable at $+\infty$.

4.1 | Degree distribution and clustering spectrum

Theorems 4.1 and 4.2 below establish power laws for the limiting degree distribution and clustering spectrum. Figures 1 and 2 illustrate how the associated power-law exponents relate to the
The assumptions of Theorem 4.2 do not hold in the grey areas where \((P)_{32} = \infty\).

corresponding exponents of layer sizes and layer strengths. Remarkably, the power law of the clustering spectrum admits a tunable exponent in \([0, 2]\). A similar power law with exponent 1 has earlier been established for a random intersection graph [9] and for a spatial preferential attachment random graph [27], and with exponent restricted to \([0, 1]\) for inhomogeneous random intersection graphs [7, 10, 12] and a hyperbolic random geometric graph model [24].

**Theorem 4.1.** Let \(\alpha \geq 2\), \(\beta \geq 0\), and \(b > 0\). Assume (16) and \((P)_{10} < \infty\).

(i) If \(\beta \in (0, 1)\), then the limiting degree distribution \(f\) satisfies as \(t \to +\infty\)

\[
f(t) \sim cL \left( t^{1/(1-\beta)} \right) t^{-\delta} \quad (17)
\]

for \(\delta = 1 + \frac{\alpha - 2}{1 - \beta}\) and \(c = \mu(1 - \beta)^{-1}b^{\delta - 1}\).

(ii) Relation (17) holds also for \(\beta = 0\). Note that in this case we have \(b \leq 1\).

(iii) If \(\beta \geq 1\), then the limiting degree distribution is light-tailed with the moment generating function bounded by \(\sum_{t \geq 0} e^{\alpha f(t)} \leq \exp \left\{ (e^{B(e^{\alpha - 1})} - 1)\mu(P)_{10} \right\} \forall s\). Here \(B = \max_{x} xg(x)\).

**Theorem 4.2.** Let \(\alpha > 2\). Let \(\beta \in [0, 1)\) be such that \(\alpha + 2\beta > 4\). Assume that for some \(a, b > 0\) (16) holds with \(L(x) = a + o(1)\) as \(x \to +\infty\). For \(\beta = 0\) we assume in addition that \(b < 1\). Then the model clustering spectrum defined by (10) follows a power law according to

\[
\sigma(t) \sim \begin{cases} 
    c_1 t^{-\beta/(1-\beta)}, & \beta < 2/3, \\
    c_2 t^{-2}, & \beta = 2/3, \\
    c_3 t^{-2}, & \beta > 2/3,
\end{cases}
\]

where \(c_1 = b^{1/(1-\beta)}\), \(c_3 = \mu(P)_{33}\), and \(c_2 = c_1 + c_3\). In particular, \(\sigma(t) \sim b\) for \(\beta = 0\).
We remark that the inequality $\alpha + 2\beta > 4$ implies $(P)_{10}, (P)_{21}, (P)_{32} < \infty$; in particular the asymptotic degree distribution has a finite second moment/variance.

Networks with $\sigma(t) \ll t^{-1}$ are sometimes called weakly clustered, and those with $\sigma(t) \gg t^{-1}$ strongly clustered [4]. According to Theorem 4.2, the overlay graph model produces weakly clustered networks for $\beta > \frac{1}{2}$, and strongly clustered networks for $\beta < \frac{1}{2}$. We believe Theorem 4.2 can be extended to more general subexponential distributions $p(x)$. We do not pursue this line here to avoid unnecessary technicalities.

4.2 Existence of double phase transition

For the power-law model (16), the function in (14) can be computed as $R_0(\theta) = \mu \sum_x R(x - 1, \theta g(x))x_p(x)$. By applying a classical giant component result for Bernoulli random graphs [30, Theorem 5.4], one may verify that

\[
\begin{align*}
\limsup_{x \to \infty} \theta x g(x) &\leq 1 - \epsilon \\
\liminf_{x \to \infty} \theta x g(x) &\geq 1 + \epsilon
\end{align*}
\]

implies that $\limsup_{x \to \infty} R(x - 1, \theta g(x)) \leq 2\epsilon^{-2}$, and $\liminf_{x \to \infty} x^{-1} R(x - 1, \theta g(x)) > 0$. \hspace{1cm} (18)

If $\alpha > 3$, then the limiting layer size distribution $p$ has a finite second moment and $R(x - 1, q) \leq x - 1$ implies that $R_0(1) < \infty$. Hence $\theta_2 = 1$, and the second phase transition cannot occur. On the other hand, when $\alpha \in (2, 3)$, the limiting layer size distribution has infinite second moment. In this case (18) yields the following conclusions:

1. $\beta = 1$ with $b > 1$. Then $R_0(\theta) < \infty$ for $\theta < b^{-1}$, and $R_0(\theta) = \infty$ for $\theta > b^{-1}$. Hence $\theta_2 = b^{-1} \in (0, 1)$. Assume in addition that the constant $a$ in (16) is large enough so that $\mu \theta(P)_{21} \geq 1$ for $\theta = \frac{1}{2}\theta_2$. Then $f^+ \geq a \tilde{f}$ implies that $R_0(\theta) = \sum_i f^+(i) \geq \sum_i \tilde{f}(i) = \mu \theta(P)_{21} \geq 1$ for $\theta = \frac{1}{2}\theta_2$, and the continuity of $R_0(\theta)$ on $(0, \theta_2)$ implies that $\theta_1 \in (0, \frac{1}{2} \theta_2)$. There are hence two critical values $0 < \theta_1 < \theta_2 < 1$ in which the model displays two distinct phase transitions.

2. $\beta \in (1, \infty)$, or $\beta = 1$ with $b < 1$. Then $R_0(\theta) < \infty$ for all $\theta \in [0, 1]$, so that $\theta_2 = 1$, and the second-type phase transition cannot occur.

3. $\beta \in [0, 1)$. Then one can show that $R_0(\theta) = \infty$ for all $\theta \in (0, 1)$, and hence $\theta_1 = \theta_2 = 0$, and there is no phase transitions of either type.

The above observations confirm the existence of a double phase transition in bond percolation, as postulated in [19], for a natural network model admitting tunable power-law exponents for both the degree distribution and the clustering spectrum. Together with Theorems 4.1 and 4.2, this opens up a flexible framework for studying the significance and interrelations of these power laws to bond and site percolation properties in clustered complex networks. The investigation of how these phase transitions are reflected in the core-periphery organization of the network [4, 19] remains an important topic for future research.

5 NOTATION USED IN PROOFS

Let $(X, Q)$ and $(X_{n,\pi}, Q_{n,\pi})$ be bivariate random variables with the distributions $P$ and $P_n$ respectively (one may interpret $\pi$ as a integer selected uniformly at random from $\{1, \ldots , m\}$). For a random variable

\[\sum_{i \geq 1} P(|C_{H_n}(1)| \geq t) \leq \sum_{i \geq 1} e^{-\frac{1}{2}(1-q)^2 t} \leq \int_0^\infty e^{-\frac{1}{2}(1-q)^2 t} dt \leq 2(1-q)^{-2},\]

so that $R(x - 1, q) = E[C_{H_n}(1)] - 1 \leq 2(1-q)^{-2}$.\hspace{1cm} (18)
ξ we denote by \( P_\xi \) the probability distribution of \( \xi \) and write \( \xi \sim P_\xi \) (equivalently, \( P_\xi = \mathcal{L}(\xi) \)). For a bivariate random variable \((\xi, \eta)\) we denote by \( P_{\xi \eta}^\eta \) the conditional probability distribution of \( \xi \) given \( \eta \). In particular, \( P_{\xi \eta}^\eta(B) = \mathbb{E}(\mathbb{1}_{\xi \in B} | \eta) \), for a Borel set \( B \). In the proof of Theorem 3.1 below we use the inequality

\[
d_{n}(P_\xi, P_\zeta) \leq \mathbb{E}(d_n(P_\xi, P_\zeta)) \tag{19}
\]

For vectors \( \vec{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m \) and \( \vec{q} = (q_1, \ldots, q_m) \in [0, 1]^m \) we denote \((\vec{x}, \vec{q}) = ((x_1, q_1), \ldots, (x_m, q_m))\). With a little abuse of notation we write \((\vec{x}_n, \vec{q}_n) = ((x_{n,1}, q_{n,1}), \ldots, (x_{n,m}, q_{n,m}))\) and \((\vec{X}_n, \vec{Q}_n) = ((X_{n,1}, Q_{n,1}), \ldots, (X_{n,m}, Q_{n,m}))\). To stress the dependence of the overlay graph on the vertex sets of the layers \( G_{n,j} \) we denote \( G_n = G(\vec{x}_n, \vec{Q}_n)\). By \( V = \{v_1, \ldots, v_n\} \) we denote the vertex set of \( G_n \). The vertex sets of the layers \( G_{n,j} \) are denoted \( D_j \), \( 1 \leq j \leq m \).

Next we introduce an \( \varepsilon \)-discretization of the space \([0, 1]\) of admissible layer strengths (edge densities) which helps to reduce the analysis of the general model with potentially uncountably many layer types into a one with finitely many layer types. For any \( \varepsilon \in (0, 1) \) we fix numbers \( 0 = s_0 < s_1 < \cdots < s_r = 1 \) (with \( r \leq 2/\varepsilon \)) such that \( \mathbb{P}(Q = s_i) = 0 \) for \( 0 < s_i < 1 \) and \( |s_i - s_{i-1}| < \varepsilon \) for \( 1 \leq i \leq r \). Relative to this mesh, we define down-rounding and up-rounding operations \( q \to q^- \) and \( q \to q^+ \) on \([0, 1]\) using the formulas

\[
q^+ = s_1 \mathbb{1}_{\{q = 0\}} + \sum_{i=1}^r s_i \mathbb{1}_{\{s_{i-1} < q \leq s_i\}}, \quad q^- = \sum_{i=1}^r s_{i-1} \mathbb{1}_{\{s_{i-1} < q \leq s_i\}}. \tag{20}
\]

For \( \vec{q} \in [0, 1]^m \) we denote \( \vec{q}^- = (q_1^-, \ldots, q_m^-), \vec{q}^+ = (q_1^+, \ldots, q_m^+) \) and write

\[
(\vec{x}, \vec{q}^-) = ((x_1, q_1^-), \ldots, (x_m, q_m^-)) \quad \text{and} \quad (\vec{x}, \vec{q}^+) = ((x_1, q_1^+), \ldots, (x_m, q_m^+)). \tag{21}
\]

Furthermore, we denote for short \( G^+ = G(\vec{x}, \vec{q}^+) \) and \( G^- = G(\vec{x}, \vec{q}^-) \). In particular, \( G^+ \) is the superposition of the layers \( \{G_{n,j}^+, X_{n,1}, Q_{n,1}^+, \ldots, G_{n,m}^+, X_{n,m}, Q_{n,m}^+\} \). In view of the coordinate-wise inequalities \( Q_{n,i} \leq Q_{n,j} \leq Q_{n,i}^+, 1 \leq i \leq m \), there is a natural coupling of random graphs \( G^-, G^+, \) and \( G_n \) such that \( \mathbb{P}(G^- \subset G_n \subset G^+) = 1 \). By \( d, d^+ \) and \( d^- \) we denote the degree of vertex \( v_1 \) in \( G_n \), \( G^+ \) and \( G^- \) respectively. The discretization is used in the proofs of Theorems 3.1 and 3.4. Given \( \varepsilon \in (0, 1) \) we first establish respective results for \( G^- \) and \( G^+ \) and then letting \( \varepsilon \downarrow 0 \) we carry them over to \( G_n \) using the coupling \( G^- \subset G_n \subset G^+ \).

By \( \mathbb{E}_{(\vec{x}, \vec{Q})} \) and \( \mathbb{P}_{(\vec{x}, \vec{Q})} \) we denote the conditional expectation and probability given the random vector \((\vec{X}, \vec{Q})\).

### 6 Analysis of Degree Distributions

Here we prove Theorem 3.1. Before the proof we collect some useful facts about the compound Poisson distribution and the total variation distance. Let \( \lambda_1, \lambda_2 > 0 \). Let \( \xi, \eta \) be random variables and let

\( Z_\xi \sim \text{CPoi}(\lambda_1, \mathcal{L}(\xi)) \) and \( Z_\eta \sim \text{CPoi}(\lambda_2, \mathcal{L}(\eta)) \) be independent compound Poisson random variables. Let \( I \) be Bernoulli random variable independent of \((\xi, \eta)\) having the success probability \( \mathbb{P}(I = 1) = \lambda_1/(\lambda_1 + \lambda_2) \). Then

\[
Z_\xi + Z_\eta \sim \text{CPoi}(\lambda_1 + \lambda_2, \mathcal{L}(\xi)), \quad \text{where} \quad \zeta = I\xi + (1 - I)\eta. \tag{22}
\]
In the particular case where $\xi$ and $\eta$ take values in $\mathbb{Z}_+$ we have

$$P(\xi = k) = \frac{\lambda_1}{\lambda_1 + \lambda_2} P(\xi = k) + \frac{\lambda_2}{\lambda_1 + \lambda_2} P(\eta = k), \quad k \in \mathbb{Z}_+. \quad (23)$$

If, in addition, $L(\xi) = L(\eta)$ then we have $L(\xi) = L(\xi) = L(\eta)$.

**Lemma 6.1.** Let $\gamma, \eta$ be non-negative integer valued random variables. Let $\{\xi_t, t \geq 0\}$ and $\{\zeta_t, t \geq 0\}$ be sequences of independent random variables. Assume that $\{\xi_t, t \geq 0\}$ is independent of $\gamma$, and $\{\zeta_t, t \geq 0\}$ is independent of $\eta$. Then

$$d_{iv}(L(\xi), L(\zeta)) \leq 2 d_{iv}(L(\gamma), L(\eta)) + \sum_{t \geq 0} d_{iv}(L(\xi_t), L(\zeta_t))P(\eta = t). \quad (24)$$

**Proof.** We can assume that $\{\xi_t, t \geq 0\}$ and $\eta$ are independent. By the triangle inequality

$$d_{iv}(L(\xi), L(\zeta)) \leq d_{iv}(L(\xi), L(\zeta)) + d_{iv}(L(\xi), L(\zeta)).$$

Furthermore, the identities that hold for any Borel set $B \subset \mathbb{R}$

$$P(\xi \in B) - P(\eta \in B) = \sum_{i \geq 0} (P(\gamma = i) - P(\eta = i)) P(\xi_t \in B),$$

$$P(\xi \in B) - P(\eta \in B) = \sum_{i \geq 0} (P(\xi_t \in B) - P(\eta_t \in B)) P(\eta = t)$$

imply

$$d_{iv}(L(\xi), L(\zeta)) \leq 2 d_{iv}(L(\gamma), L(\eta)).$$

$$d_{iv}(L(\xi), L(\zeta)) \leq \sum_{i \geq 0} d_{iv}(L(\xi_t), L(\zeta_t))P(\eta = t).$$

We apply Lemma 6.1 to compound Poisson random variables. Let $a, b > 0$. Let $\xi, \zeta$ be random variables. It follows from the lemma that

$$d_{iv}(CPoi(a, L(\xi)), CPoi(b, L(\zeta))) \leq 2|a - b| + bd_{iv}(L(\xi), L(\zeta)) \quad (25)$$

To see how (25) follows from (24) put $\gamma \sim \text{Poi}(a)$, $\eta \sim \text{Poi}(b)$ and $\xi_t := \xi_1 + \ldots + \xi_i$, $\zeta_t := \zeta_1 + \ldots + \zeta_j$, where $\xi_i, i \geq 1$, and $\zeta_j, j \geq 1$, are independent copies of $\xi$ and $\zeta$ respectively. Then use the triangle inequality $d_{iv}(L(\xi), L(\zeta)) \leq d_{iv}(L(\xi), L(\zeta))$ and the fact that $\mathbb{E}\eta = b$.

**Proof of Theorem 3.1.** In the proof we drop the subscript $n$ when it does not cause an ambiguity. Thus we write $G = G(n)$, $G_j = G_{n,j}$, $G_j^+ = G_j^+$ and $X_j = X_{n,j}$, $Q_j = Q_{n,j}$, $(X, \mathcal{G}) = (X_n, \mathcal{G}_n)$, $X_n = X_{n,n}$. We begin by outlining the idea of the proof. The degree $d$ of vertex $v_1$ in the overlay random graph $G$ is approximated by the sum of degrees of $v_1$ in the layers $G_j$ containing this vertex. We denote this sum

$$L_A = \sum_{1 \leq j \leq m} \mathbb{I}_{v_1 \in E_j} H(j). \quad (26)$$
Here, $\mathbb{I}_{\{\nu_j \in D_j\}}$ is the indicator of the event $\{\nu_j \in D_j\}$ and $H(j)$ stands for the degree of $\nu_j$ in $G_j$. In the sparse regime considered it is rather unlikely that two layers intersect in more than one point. Hence we approximate

$$d = L_A + o_P(1).$$  

(27)

Note that only the layers of size at least 2 may contribute to the sum $L_A$. We denote the respective number of layers $S_A = \sum_{1 \leq j \leq m} \mathbb{I}_{\{\nu_j \in D_j, X_j \geq 2\}}$. In order to analyze the distributions of $L_A$ and $S_A$ it is convenient to condition on $(X_j, Q_j)$, $1 \leq j \leq m$. Then $S_A$ becomes the sum of independent Bernoulli random variables and its distribution is approximately $Poi\left(\frac{m}{m} \bar{x}_s\right)$, where $\bar{x}_s = \frac{1}{m} \sum_{1 \leq j \leq m} X_j^{i}(X_j \geq 2)$. Furthermore, each $H(j)$ has binomial distribution $Bin(X_j - 1, Q_j)$ and the probability that it will contribute to the sum $L_A$ is proportional to $X_j$. This in turn yields that the typical contribution to the sum (26) by a layer of size at least 2 has the size biased mixed binomial distribution

$$h(t) = h(X, Q)(t) = \frac{1}{x_s} \sum_{1 \leq j \leq m} \text{Bin}(X_j - 1, Q_j)(t)X_j^{i}(X_j \geq 2), \quad t = 0, 1, \ldots ,$$  

(28)

while the conditional distribution of $L_A$ is approximately $CPoi\left(\frac{m}{m} \bar{x}_s, h\right)$.

Finally, letting $n, m \to +\infty$ we approximate $\frac{m}{m} \bar{x}_s \to \mu x_s$, where $x_s = \mathbb{E} X^{i}(X \geq 2)$. Furthermore, we approximate $h$ by the distribution $h_s$, where

$$h_s(t) = \frac{1}{x_s} \mathbb{E} \text{Bin}(X - 1, Q)(t)X^{i}(X \geq 2), \quad t = 0, 1, 2 \ldots .$$  

(29)

In this way we establish the approximation $CPoi\left(\frac{m}{m} \bar{x}_s, h\right) \xrightarrow{w} CPoi(\mu x_s, h_s)$. We conclude the outline with the observation that

$$CPoi(\mu x_s, h_s) = CPoi(\mu(P)_{10}, Bin_{10}(P)).$$  

(30)

Indeed, a simple calculation shows that

$$(P)_{10} = \mathbb{E} X = P(X = 1) + x_s,$$  

$$(P)_{10} Bin_{10}(P)(0) = x_s (P(X = 1) + h_s(0)),$$  

$$(P)_{10} Bin_{10}(P)(t) = x_s h_s(t), \quad t \geq 1.$$  

These identities imply the relation $(P)_{10}(\hat{g} - 1) = x_s (\hat{h}_s - 1)$ between the Fourier transforms $\hat{g}$ and $\hat{h}_s$ of the increment distributions $Bin_{10}(P)$ and $h_s$. The latter relation establishes the correspondence between the Fourier transforms of respective compound distributions (30).

Finally, we mention that the rigorous proof of Theorem 3.1 is a bit more complex and the proof idea is somewhat hidden by technicalities including the truncation and discretization.

Now we give a rigorous proof. We first consider the special case where there exists $M > 0$ such that $X_{n,j} \leq M$ almost surely for each $n$ and $1 \leq j \leq m$.

Given $0 < \varepsilon < 1$, consider $\varepsilon$-discretized random variables $Q^+$ and $Q^-$ and the overlay graphs $G^+$ and $G^-$ defined by the vectors $(\overline{X}, \overline{Q}_n)$ and $(\overline{X}, \overline{Q}_n)$, see (20), (21). Let $h^+_\varepsilon$ and $h^-\varepsilon$ denote the distributions defined by (29), where $Q$ is replaced by $Q^+$ and $Q^-$ respectively. Let $d^+_\varepsilon = d^+_\varepsilon(\varepsilon)$ and $d^-\varepsilon = d^-\varepsilon(\varepsilon)$.
be compound Poisson random variables with the distributions \( \text{CPoi}(\mu x, h^+ \varepsilon) \) and \( \text{CPoi}(\mu x, h^- \varepsilon) \). Note that as \( \varepsilon \to 0 \)

\[
\mathcal{L}(d^+ \varepsilon) \stackrel{w}{\to} \text{CPoi}(\mu x, h^+), \quad \text{and} \quad \mathcal{L}(d^- \varepsilon) \stackrel{w}{\to} \text{CPoi}(\mu x, h^-).
\]

(31)

Now we turn to the analysis of the degrees \( d, d^+, \) and \( d^- \). We will show below that

\[
\mathcal{L}(d^-) \stackrel{w}{\to} \mathcal{L}(d^+) \quad \text{and} \quad \mathcal{L}(d^+) \stackrel{w}{\to} \mathcal{L}(d^+).
\]

(32)

Since the coupling \( G^- \subset G \subset G^+ \) implies the coupling \( d^- \leq d \leq d^+ \) we have

\[
\mathbb{P}(d^+ \leq t) = \lim_{n} \mathbb{P}(d^+ \leq t) \leq \lim_{n} \mathbb{P}(d \leq t) \leq \mathbb{P}(d^- \leq t), \quad \forall \ t \geq 0.
\]

(33)

Combining (32) and (33) we obtain

\[
\mathbb{P}(d^+ \leq t) = \lim_{n} \mathbb{P}(d^+ \leq t) = \lim_{n} \mathbb{P}(d \leq t) \leq \lim_{n} \mathbb{P}(d^- \leq t) \leq \mathbb{P}(d^- \leq t).
\]

(34)

Finally, letting \( \varepsilon \downarrow 0 \) we obtain from (31), (34) that \( \mathcal{L}(d) \overset{w}{\to} \text{CPoi}(\mu x, h^+) \).

It remains to prove (32). We only show the second relation. The proof of the first one is the same. At this point we need some more notation. Recall \( \varepsilon \)-discretization (20). Let \( \eta_{k,i}^{(j)}, \xi_{k,i}^{(j)}, H_{k,i}^{(j)}, H^{(j)}_{k,i}, k \geq 2, 1 \leq i \leq r \) and \( j, l \geq 1 \) be independent random variables. We assume that \( \eta_{k,i}^{(j)} \) and \( \xi_{k,i}^{(j)} \) have Poisson and Bernoulli distributions with mean values \( \mathbb{E} \eta_{k,i}^{(j)} = \mathbb{E} \xi_{k,i}^{(j)} = k/n \). Furthermore, \( H_{k,i}^{(j)}, H_{k,i}^{(j)}, H^{(j)}_{k,i} \) have binomial distribution \( \text{Bin}(k - 1, s_i) \), where \( s_i, 1 \leq i \leq r \) are the same as in (20). Let

\[
L_1 = \sum_{1 \leq j \leq M} \|v_1\|_{H^+}(j) = \sum_{1 \leq j \leq M} \|v_1\|_{H^+}(j)_{X_{j} \geq 2},
\]

(35)

where \( H^+(j) \) stands for the number of neighbors of \( v_1 \) created by the layer \( G^+ \). In particular, given \( (X, Q) \), the random variable \( H^+(j) \) has binomial distribution \( \text{Bin}(X_j - 1, Q^+) \). Introduce random sets \( \mathcal{M}_{k,i} = \{ j : (X_j, Q^+) = (k, s_i) \} \) and denote their sizes \( m_{k,i} = |\mathcal{M}_{k,i}| \). Furthermore, put

\[
S = \sum_{1 \leq j \leq M} X_j I_{[X_j \geq 2]}, \quad \hat{\lambda} = \frac{S}{n} = \frac{m}{n}, \quad \hat{p}_{k,i} = \frac{k m_{k,i}}{S} I_{[S > 0]}, \quad 1 \leq i \leq r, \ k \geq 2.
\]

Next we condition on \( (X, Q) \). Given \( (X, Q) \) such that \( S > 0 \) define random variables

\[
L_2 = L_2(X, Q) = \sum_{2 \leq k \leq M} \sum_{i=1}^{r} \sum_{j=1}^{m_{k,i}} \xi_{k,i}^{(j)} H_{k,i}^{(j)}, \quad L_3 = L_3(X, Q) = \sum_{2 \leq k \leq M} \sum_{i=1}^{r} \sum_{j=1}^{m_{k,i}} H_{k,i}^{(j)}.
\]

For \( S = 0 \) we put \( L_2 \equiv 0 \) and \( L_3 \equiv 0 \).

We observe that the conditional distributions \( P_{L_2}^{(X, Q)} \) and \( P_{L_1}^{(X, Q)} \) do coincide. To see this we partition the index set \( \{ j : X_j \geq 2 \} = \bigcup_{k,i} \mathcal{M}_{k,i} \) of the second sum of (35) and note that the conditional distributions of \( \sum_{j=1}^{m_{k,i}} \xi_{k,i}^{(j)} H_{k,i}^{(j)} \) and \( \sum_{j \in \mathcal{M}_{k,i}} \|v_1\|_{H^+}(j) \) do coincide. Another useful fact is that for \( S > 0 \)
the conditional distribution $P^{(X, \overline{Q})}_{L_3}$ is a compound Poisson distribution. Moreover, using the property (22), (23) one can easily show that $P^{(X, \overline{Q})}_{L_3} = \text{CPoi}(\hat{\lambda}, h^+)$, where

$$h^+(t) = h^+(X, \overline{Q})(t) = \sum_{2 \leq k \leq M} \sum_{1 \leq i \leq r} \mathbb{P}(H_{k,i} = t) \hat{p}_{k,i}, \quad t = 0, 1, 2, \ldots .$$

Now we are ready for the proof of (32). We observe that $d^+ \neq L_1$ implies that some $v_i \neq v_1$ is a neighbor of $v_1$ in at least two different layers. We have, by the union bound and symmetry, that

$$\mathbb{P}(d^+ \neq L_1) \leq (n - 1) \sum_{i<j} \mathbb{P}(v_1, v_2 \in D_i \cap D_j) = (n - 1) \sum_{i<j} \left( \frac{\mathbb{P}(X_i) \mathbb{P}(X_j)}{(n)^2} \right) = \leq (n - 1) \frac{m^2}{n^4} \langle \mathbb{E}X^2 \rangle ^2.$$

Invoking $\mathbb{E}X^2 \leq M^2$ and using the fact that $\mathcal{L}(L_2) = \mathcal{L}(L_1)$ we obtain

$$d_{tv}(\mathcal{L}(d^+), \mathcal{L}(L_2)) = d_{tv}(\mathcal{L}(d^+), \mathcal{L}(L_1)) \leq \mathbb{P}(d^+ \neq L_1) = O(n^{-1}). \quad (36)$$

Next we evaluate $d_{tv}(\mathcal{L}(L_2), \mathcal{L}(L_3))$. To this aim we consider an array of random variables starting with $L_2$ and ending at $L_3$ where each subsequent element of the sequence is obtained from the previous one by replacing $\xi_{k,i}^{(j)} H_{k,i}^{(j)} = \sum_{1 \leq i \leq n} \sum_{1 \leq j \leq n} H_{k,i}^{(j)}$. We proceed until all the products $\xi_{k,i}^{(j)} H_{k,i}^{(j)}$ are replaced so that at the array ends with $L_3$. By the triangle inequality, the total variation distance between the conditional distributions

$$d_{tv}(P^{(X, \overline{Q})}_{L_2}, P^{(X, \overline{Q})}_{L_3}) \leq \sum_{2 \leq k \leq M} \sum_{1 \leq i \leq r} \sum_{1 \leq j \leq m_{x,i}} d_{tv}(\mathcal{L}(\xi_{k,i}^{(j)}), \mathcal{L}(\eta_{k,i}^{(j)})).$$

Invoking the bound $d_{tv}(\mathcal{L}(\xi_{k,i}^{(j)}), \mathcal{L}(\eta_{k,i}^{(j)})) \leq 2k^2/n^2$, which follows by Le Cam’s inequality [42], we obtain

$$d_{tv}(P^{(X, \overline{Q})}_{L_2}, P^{(X, \overline{Q})}_{L_3}) \leq 2 \sum_{1 \leq j \leq m} X^2 n^{-2} \leq \frac{2m}{n^2} \mathbb{E}X^2 X^2.$$

Now an application of (19) yields

$$d_{tv}(\mathcal{L}(L_2), \mathcal{L}(L_3)) \leq \frac{2m}{n^2} \mathbb{E}X^2 \leq \frac{2m}{n^2} M^2 = O(n^{-1}). \quad (37)$$

Finally, we evaluate the distance $d_{tv}(\mathcal{L}(L_3), \mathcal{L}(d^+))$, where $\mathcal{L}(d^+) = \text{CPoi}(\mu_x, h^+)$. For this purpose it is convenient to write $h^+_x$ in the form

$$h^+_x(t) = \sum_{2 \leq k \leq M} \sum_{1 \leq i \leq r} \mathbb{P}(H_{k,i} = t) p^+_x, \quad p^+_x := \frac{k}{x} X^+ \mathbb{P}(X = k, Q^+ = s_i).$$

Recall that given $(\overline{X}, \overline{Q})$ the conditional distribution of $L_3$ is $\text{CPoi}(\hat{\lambda}, h^+)$). It follows from (25) that

$$d_{tv}(\text{CPoi}(\hat{\lambda}, h^+), \text{CPoi}(\mu_x, h^+_x)) \leq 2|\hat{\lambda} - \mu_x| + \mu_x \cdot d_{tv}(h^+, h^+_x). \quad (38)$$
Furthermore, Lemma 6.1 implies
\[
d_{tv}(h^+, h^+_{\ast}) \leq 2 \sum_{2 \leq k \leq s} \sum_{1 \leq i \leq r} |\hat{p}_{k,i} - p^+_i|.
\] (39)

(Here we apply Lemma 6.1 to \( \xi_{k,i} = \zeta_{k,i} = H_{k,i} \) in the case where \( \gamma \) and \( \eta \) are bivariate random variables with the distributions \( \mathbb{P}(\gamma = (k, i)) = \hat{p}_{k,i} \) and \( \mathbb{P}(\eta = (k, i)) = p^+_i \)). We observe that both terms on the right of (38) vanish in probability. Indeed, by the weak law of large numbers (we use Chebyshev’s inequality), our assumption \( P_n \xrightarrow{w} P \) implies \( \hat{\lambda} - \lambda \xrightarrow{P} 0 \) and \( \hat{p}_{k,i} - p^+_i \xrightarrow{P} 0 \) for each \( k \) and \( i \). Since the number of different pairs \( (k, i) \) is finite, we conclude that \( \tau := d_{tv}(\text{CPoi}(\hat{\lambda}, h^+), \text{CPoi}(\mu_{x_{\ast}}, h^+_{\ast})) \xrightarrow{P} 0 \).

In view of the obvious inequality \( \tau \leq 1 \) we have also that \( \mathbb{E}\tau = o(1) \). Now (19) implies \( d_{tv}(\mathcal{L}(L_3), \mathcal{L}(d^+_n)) = o(1) \).

Finally, from the latter bound combined with (36), (37) we derive (32), by the triangle inequality:
\[
d_{tv}(\mathcal{L}(d^+), \mathcal{L}(d^+_{\ast})) \leq d_{tv}(\mathcal{L}(d^+), \mathcal{L}(L_2)) + d_{tv}(\mathcal{L}(L_2), \mathcal{L}(L_3)) + d_{tv}(\mathcal{L}(L_3), \mathcal{L}(d^+_{\ast})) = o(1).
\]

Now we revoke the extra condition that \( X_{n,j} \leq M \) almost surely for each \( n \) and \( 1 \leq j \leq m \). From now on let \( \{(X_{n,j}, Q_{n,j}), n \geq 1, 1 \leq j \leq m_n\} \) be arbitrary bivariate random variables satisfying conditions of the theorem. Given \( M > 0 \), let \( G^{[M]}(n) \) be the random overlay graph defined by the sequence \( \left( X_{n,1}^{[M]}, Q_{n,1}, \ldots, (X_{n,m}, Q_{m}) \right) \), where \( X_{n,j}^{[M]} = X_{n,j} \mathbb{I}_{\{X_{n,j} \leq M\}} \). In the proof above we have shown that the degree \( d^{[M]} \) of vertex \( v_1 \) in \( G^{[M]}(n) \) has asymptotic compound Poisson distribution \( \text{CPoi}(\mu_{x}^{[M]}, h_{\ast}^{[M]}) \).

Here \( x_{\ast}^{[M]} \) and \( h_{\ast}^{[M]} \) are defined in the same way as \( x_{\ast} \) and \( h_{\ast} \) above, but with \( X \) replaced by \( X^{[M]} = X \mathbb{I}_{\{X \leq M\}} \).

Now we let \( M \to \infty \) and observe that \( \text{CPoi}(\mu_{x}^{[M]}, h_{\ast}^{[M]}) \xrightarrow{w} \text{CPoi}(\mu_{x_{\ast}}, h_{\ast}) \). Furthermore, the natural coupling \( G^{[M]}(n) \subset G(n) \) implies \( d^{[M]} \leq d \), where \( d \) stands for the degree of \( v_1 \) in \( G(n) \). Moreover, \( d \neq d^{[M]} \) implies that \( v_1 \) belongs to a layer of size greater than \( M \). Hence, by the union bound,
\[
\mathbb{P}(d \neq d^{[M]}) \leq \sum_{1 \leq j \leq m} \mathbb{P}(v_1 \in D_j, X_{n,j} > M) = \sum_{1 \leq j \leq m} \frac{\mathbb{E}X_{n,j} \mathbb{I}_{\{X_{n,j} > M\}}}{n} = \frac{m}{n} \mathbb{E}(X_{n,s} \mathbb{I}_{\{X_{n,s} > M\}}).
\]

Note that the quantity on the right is \( o(1) \) uniformly in \( n \) as \( M \to \infty \). Indeed our assumptions \( P_n \xrightarrow{w} P \) and \( (P)_{10} \xrightarrow{(P)_{10}} \) imply \( \lim_{M \to \infty} \sup_n \mathbb{E}X_{n,s} \mathbb{I}_{\{X_{n,s} > M\}} = 0 \). Therefore, \( \mathcal{L}(d) \xrightarrow{w} \text{CPoi}(\mu_{x_{\ast}}, h_{\ast}) \).

7 | ANALYSIS OF CLUSTERING

Here we prove Theorems 3.2 and 3.3. In the proof we drop the subscript \( n \) when it does not cause an ambiguity. Thus we write \( G = G(n) \), \( V = V(G(n)) = \{1, 2, \ldots, n\} \) and \( G_k = G_{n,k} \). Let \( K_{12} \) be the two-star on the vertex set \( \{1, 2, 3\} \) with links \( \{12, 13\} \). Let \( K_3 \) be the triangle on the vertex set \( \{1, 2, 3\} \). Denote events \( \mathcal{K}_3 = \{G \supseteq K_3\} \) and \( \mathcal{K}_{12} = \{G \supseteq K_{12}\} \). We also denote \( d = \text{deg}_G(1) \). Theorems 3.2 and 3.3 are derived from Theorems 7.3 and 7.4 below, where we evaluate the probabilities \( \mathbb{P}(\mathcal{K}_3), \mathbb{P}(\mathcal{K}_{12}) \),...
\( \mathbb{P}(d = t, K_3) \) and \( \mathbb{P}(d = t, K_{12}) \) defining the ratios \( \tau^{(n)} = \frac{\mathbb{P}(K_3)}{\mathbb{P}(K_{12})} \) and \( \sigma^{(n)}(t) = \frac{\mathbb{P}(d = t, K_3)}{\mathbb{P}(d = t, K_{12})} \). In the proof of Theorem 7.3 we approximate \( \mathbb{P}(K_3) \) by the probability that \( K_3 \) is produced by a single layer (with \( m \) different layers available). Similarly, in the proof of Theorem 7.4 \( \mathbb{P}(K_{12}) \) is approximated by the sum of two probabilities: the first one being the probability that \( K_{12} \) is produced by a single layer (with \( m \) different layers available) and the second one being the probability that the edges of \( K_{12} \) are produced by different layers (with \( m(m - 1) \) layer pairs available). We note the corresponding sum in the denominator of \( (10) \).

Before the proof we introduce some notation and collect auxiliary results. Subgraph frequencies in the overlay graph will be characterized using cross moments

\[
(P_n)_{rs} = \int (x)_r y^s dP_n, \quad (P_n)_{rs,ju} = \int (x)_r y^s (x)_t y^u dP_n
\]

(40) of the averaged layer type distribution \( P_n \) defined by (2), and normalized cross moments defined by

\[
\mu_{rs} = \sum_{k=1}^{m} p_{rs}(k), \quad \mu_{rs,ju} = \sum_{k=1}^{m} p_{rs}(k) p_{tu}(k),
\]

(41)

where \( p_{rs}(k) = (n)^{-1} E(X_{n,k}) Q_{n,k}^k \). Note that \( \mu_{rs} = m(n)^{-1} (P_n)_{rs} \).

**Lemma 7.1.** Recall the averaged empirical distribution \( P_n \) defined by (2). If \( P_n \xrightarrow{w} P \) and \( (P_n)_{rs} \to (P)_{rs} < \infty \), then the cross moments defined in (40)–(41) satisfy \( \mu_{10,rs} \ll m(n)^{-1} \) and \( (P_n)_{10,rs} \ll n \).

**Proof.** Denote \( A_k = X_k \) and \( B_k = (X_k)_r Q_{k}^k \). Observe that \( A_k \leq a + A_k \| (A_k > a) \| E_k \leq a E_k + b n \| (A_k > a) \| E_k \). A similar inequality holds for \( B_k \).

(42)

By taking expectations and averaging with respect to \( k \), we find that

\[
\frac{1}{m} \sum_{k=1}^{m} E A_k E B_k \leq a E E_k + b n \| (A_k > a) \| E_k \].
\]

(43)

where \( A_\pi = X_\pi, B_\pi = (X_\pi)_r Q_{\pi}^k \), and \( (X_\pi, Y_\pi) \) is a generic \( P_n \)-distributed random variable. Because the left side above equals \( m^{-1} n(n)_{rs} \mu_{10,rs} \), we conclude that

\[
m^{-1} n(n)_{rs} \mu_{10,rs} \leq a c + b \phi(a) + \psi(b),
\]

where \( c = \sup_{rs} (P_n)_{rs} \), \( \phi(t) = \sup_{rs} \int \| (x > i) \| dP_n \) and \( \psi(t) = \sup_{rs} \int (x)_r y^s \| (x)_t y^u \| i dP_n \). Then the tightness of \( P_n \) implies that \( \phi(a_n) \to 0 \) for \( a_n = n^{1/2} \). Hence also \( b_n \phi(a_n) \to 0 \) for \( b_n = \phi(a_n)^{-1/2} \to \infty \). The uniform \( (x)_r y^s \)-integrability of \( P_n \) further implies that \( \psi(b_n) \to 0 \). Hence the right side above vanishes and first claim follows. The uniform \( (x)_r y^s \)-integrability of \( P_n \) further implies that \( \psi(b_n) \to 0 \). Hence the right side above vanishes and first claim follows. The uniform \( (x)_r y^s \)-integrability of \( P_n \) further implies that \( \psi(b_n) \to 0 \). Hence the right side above vanishes and first claim follows. The uniform \( (x)_r y^s \)-integrability of \( P_n \) further implies that \( \psi(b_n) \to 0 \). Hence the right side above vanishes and first claim follows.

For the second claim, we may repeat the above reasoning to verify that (42) holds also with the \( E \)-symbol removed. Therefore, (43) also holds when the left side is replaced by \( (P_n)_{10,rs} = \frac{1}{m} \sum_{k=1}^{m} E A_k B_k \). Hence the second claim follows by the same argument. \( \blacksquare \)
Let $k^*$ be a random number uniformly distributed in $\{1, \ldots, m\}$ and independent of $G = G(n)$. In the following result $G_{k^*} = G(n,k^*)$ represents a randomly chosen layer.

**Lemma 7.2.** Let $F_{rs}$ be a graph with node set in $\{1, \ldots, n\}$ such that $|V(F_{rs})| = r$ and $|E(F_{rs})| = s$, and let $i$ be a node in $V(F_{rs})$ with $\deg_{F_{rs}}(i) = r - 1$. Select $k^* \in \{1, \ldots, m\}$ uniformly at random and independently of the layers. Then:

(i) $\mathbb{P}(G_{k^*} \supset F_{rs}) = m^{-1} \mu_{rs}$,

(ii) $\mathbb{P}(\deg_{G_{k^*}}(i) = t \mid G_{k^*} \supset F_{rs}) = \text{Bin}(P_n(t - r + 1))$ for all $t$.

**Proof.** (i) Because $\mathbb{P}(V(G_k) \supset V(F_{rs}) \mid X_k, Q_k) = (\frac{X_k}{n})^{t-r}$ for any $k$, we see that $\mathbb{P}(G_k \supset F_{rs}) = \mathbb{E}(\frac{X_k}{n})^{t-r}$.

The corresponding probability for a randomly selected $k^*$ equals $\mathbb{P}(G_{k^*} \supset F_{rs}) = \frac{1}{m} \sum_{k=1}^{m} \mathbb{P}(G_k \supset F_{rs}) = (n)^{-1}(P_n)^{-1}(r)$. Finally, recall that $(n)^{-1}(P_n)^{-1}(r) = m^{-1} \mu_{rs}$.

(ii) Denote $d_k = \deg_{G_k}(i)$. On the event that $G_k \supset F_{rs}$, we see that $d_k = \ell' + d'_k$ where $d'_k = |N_{G_k}(i) \setminus V(F_{rs})|$ and $\ell' = r - 1$. Conditionally on $(X_k, Q_k) = (x, q)$ and $G_k \supset F_{rs}$, the random integer $d'_k$ is Bin$(x-r, q)$-distributed. Hence

$$\mathbb{P}(d_k = t, G_k \supset F_{rs}) = \mathbb{E} \left( \text{Bin}(X_k - r, Q_k)(t - \ell') \frac{(X_k)^{t-r}}{(n)^{t-r}} Q_k \right).$$

The corresponding probability for a randomly chosen $k^*$ is

$$\mathbb{P}(d_{k^*} = t, G_{k^*} \supset F_{rs}) = \int \left( \text{Bin}(x-r, q)(t - \ell') \frac{(x)^{t-r}}{(n)^{t-r}} q \right) P_n(dx, dq),$$

so the claim follows by dividing both sides by $\mathbb{P}(G_{k^*} \supset F_{rs}) = (n)^{-1}(P_n)^{-1}(r)$. $\blacksquare$

Now we are ready to state and prove Theorems 7.3 and 7.4. We use the short hand notation $g_{rs}^{(n)} = \text{Bin}_{rs}(P_n)$, where the mixed binomial distribution $\text{Bin}_{rs}(P_n)$ is defined in (5).

**Theorem 7.3.** We have

(i) $|\mathbb{P}(G \supset K_3) - \mu_{33}| \leq 4 \mu_{21} \mu_{32} + \mu_{21}^3$.

(ii) $\mathbb{P}(\deg_G(1) = t, G \supset K_3) = \mu_{33} f^{(n)}(t) * g_{rs}^{(n)}(t - 2) + \varepsilon(t)$, where $f^{(n)}$ is the model degree distribution defined by (7) and the approximation error is bounded by

$$|\varepsilon(t)| \leq (4 + t)\mu_{21} \mu_{32} + \mu_{21}^3 + 2 \mu_{0,33}.$$

**Proof.** Denote by $A_k = \{G_k \supset K_3\}$ the event that all node pairs of the triangle are linked by the layer $k$. We also denote $d_k = \deg_{G_k}(1)$, and $d_{-k} = \deg_{G_{-k}}(1)$ with $G_{-k} = \cup_{k' \neq k} G_{k'}$.

Proof of (i). Denote

$$\varepsilon_1(t) = \mathbb{P}(d = t, K_3) - \mathbb{P}(d = t, \cup_k A_k),$$

and observe that $0 \leq \varepsilon_1(t) \leq \mathbb{P}(d = t, E_{12}) + \mathbb{P}(d = t, E_{111})$, where $E_{12}$ is the event that there exists one layer covering one link and a different layer covering the remaining two links of $K_3$, and $E_{111}$ is the event that three distinct layers cover distinct links of $K_3$. We write $p(abc) = \mathbb{P}(G^1_{a'}, G^1_{b'}, G^2_{c'})$, where
denotes the event that node pair $ij$ is linked in layer $a$. We note that $p(abc) = p_{21}(a)p_{21}(b)p_{21}(c)$, $p(aab) = p_{32}(a)p_{21}(b)$, and $p(aaa) = p_{33}(a)$ for distinct layers $a, b, c$. Hence

$$\mathbb{P}(E_{12}) \leq \sum_{t=0}^{t'} \sum_{a,b} (p(aab) + p(aba) + p(baa)) \leq 3\mu_{21}\mu_{32},$$

and $\mathbb{P}(E_{111}) \leq \sum_{t=0}^{t'} p(abc) \leq \mu_{21}^3$. Thus, $\sum_{t=0}^{t} |\epsilon_1(t)| \leq 3\mu_{21}\mu_{32} + \mu_{21}^3$.

Then denote

$$\epsilon_2(t) = \mathbb{P}(d = t, \cup_k \mathcal{A}_k) - \sum_k \mathbb{P}(d = t, \mathcal{A}_k).$$

Bonferroni’s inequalities imply that $0 \leq -\epsilon_2(t) \leq \sum_{k,k'} \mathbb{P}(d = t, \mathcal{A}_k, \mathcal{A}_{k'})$, and hence, noting that $\mu_{33} \leq \mu_{32} \leq \mu_{21}$,

$$\sum_{t=0}^{t} |\epsilon_2(t)| \leq \sum_{k,k'} \mathbb{P}(\mathcal{A}_k, \mathcal{A}_{k'}) = \sum_{k,k'} p_{33}(k)p_{33}(k') \leq \mu_{33}^2 \leq \mu_{21}\mu_{32}.$$ 

By combining this with the bound for $\epsilon_1(t)$, we conclude that

$$\mathbb{P}(d = t, \mathcal{K}_3) = \sum_k \mathbb{P}(d = t, \mathcal{A}_k) + \epsilon_1(t) + \epsilon_2(t),$$

where $\sum_{t=0}^{t} (|\epsilon_1(t)| + |\epsilon_2(t)|) \leq 4\mu_{21}\mu_{32} + \mu_{21}^3$. Hence claim (i) follows by summing the above equality over $t$, and noting that $\sum_k \mathbb{P}(\mathcal{A}_k) = \mu_{33}$.

Proof of (ii). We start with (45) and approximate

$$\sum_k \mathbb{P}(d = t, \mathcal{A}_k) \approx \sum_k \mathbb{P}(d_{-k} + d_k = t, \mathcal{A}_k)$$

$$\quad = \sum_k \sum_{r+s=t} \mathbb{P}(d_{-k} = r) \mathbb{P}(d_k = s, \mathcal{A}_k)$$

$$\quad \approx \sum_k \sum_{r+s=t} \mathbb{P}(d = r) \mathbb{P}(d_k = s, \mathcal{A}_k).$$

It follows from Lemma 7.2 that

$$\sum_k \mathbb{P}(d_k = s, \mathcal{A}_k) = m \mathbb{P}(d_k = s, G_{k^c} \supseteq \mathcal{K}_3)$$

$$\quad = m \mathbb{P}(d_k = s | G_{k^c} \supseteq \mathcal{K}_3) \mathbb{P}(G_{k^c} \supseteq \mathcal{K}_3) = \mu_{33} \text{Bin}_{33}(P_s)(s - 2).$$

Hence the last term above equals $\mu_{33} f^{(n)}_{33}(t - 2)$, and to prove the claim it suffices to analyze the approximation errors in (46)–(47).

The approximation error in (46) equals $\epsilon_3(t) = \sum_k \epsilon_{3k}(t)$, where

$$\epsilon_{3k}(t) = \mathbb{P}(d = t, \mathcal{A}_k) - \mathbb{P}(d_{-k} + d_k = t, \mathcal{A}_k).$$

By applying Lemma 11.2 with $A = \{k\}, B = [n] \setminus \{k\}$, $\mathcal{E}_A = \mathcal{A}_k$, and $\mathcal{E}_B$ being the sure event, we see that $|\epsilon_{3k}(t)| \leq c_B t \mathbb{P}(d_k \leq t, \mathcal{A}_k) \leq c_B \mathbb{P}(\mathcal{A}_k)$, where $c_B = \mathbb{P}(G_{-k} \supseteq \mathcal{K}_3) \leq 12 \leq \sum_{\ell \neq k} p_{21}(\ell) \leq \mu_{21}$. Hence

$$|\epsilon_3(t)| \leq t\mu_{21} \sum_k p_{33}(k) = t\mu_{21}\mu_{33} \leq t\mu_{21}\mu_{32}.$$
The approximation error in (47) equals $\varepsilon_4(t) = \sum_k \varepsilon_{4k}(t)$ where

$$\varepsilon_{4k}(t) = \sum_{r+s=t} (\mathbb{P}(d = r) - \mathbb{P}(d_{-k} = r)) \mathbb{P}(d_k = s, A_k).$$

By Lemma 11.1, $\sum_{i \geq 0} |\varepsilon_{4k}(t)| \leq 2\mathbb{P}(d_k > 0)\mathbb{P}(A_k).$ Because $\mathbb{P}(d_k > 0) \leq p_{10}(k)$ and $\mathbb{P}(A_k) = p_{33}(k),$ it follows that $\sum_{i \geq 0} |\varepsilon_4(t)| \leq 2\mu_{10,33}$. Claim (ii) follows by combining the above estimates for the total approximation error $\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t) + \varepsilon_4(t).$ $\blacksquare$

**Theorem 7.4.** We have

(i) $|\mathbb{P}(G \supset K_{12}) - (\mu_{32} + \mu_{21})| \leq 6\mu_{21}\mu_{32} + 6\mu_{21}^3 + \mu_{21,21}$.

(ii) $\mathbb{P}(\deg_G(1) = t, G \supset K_{12}) = \mu_{32} f^{(n)} * g^{(n)}(t-2) + \mu_{21}^2 f^{(n)} * g^{(n)} * g^{(n)}(t-2) + \varepsilon(t),$ where $f^{(n)}$ is the degree distribution of $G$, and the approximation error is bounded by

$$|\varepsilon(t)| \leq (6 + 3t)(\mu_{21}\mu_{32} + \mu_{21}^3) + \mu_{21}^4 + 4\mu_{10,32} + 4\mu_{21}\mu_{10,21} + \mu_{21,21}. \quad (49)$$

**Proof.** Recall that $K_{12}$ is the two-star with node set $\{1, 2, 3\}$ and link set $\{e_1, e_2\} := \{12, 13\}$. We denote by $G_k^\ell$ the event that $ij \in E(G_k)$ and we set $A_k^\ell = G_k^\ell \cap G_{\ell}^\ell$. We denote $G_{k^\ell} = G_k \cup G_{\ell}^\ell$ and $G_{-k^\ell} = \cup_{q \in \mathcal{E}, k, \ell} G_q$, and we set $d = \deg_G(1)$, $d_{k^\ell} = \deg_{G_{k^\ell}}(1)$ and $d_{-k^\ell} = \deg_{G_{-k^\ell}}(1)$. We also denote $h_{k^\ell}(s) = \mathbb{P}(d_{k^\ell} = s, A_{k^\ell}).$

We start with an outline of the proof. First we approximate

$$\mathbb{P}(d = t, K_{12}) \approx \sum_{k, \ell} \mathbb{P}(d = t, A_{k^\ell}) \quad (50)$$

$$\approx \sum_{k, \ell} \mathbb{P}(d_{k^\ell} + d_{-k^\ell} = t, A_{k^\ell}) \quad (51)$$

$$= \sum_{k, \ell} \sum_{r+s=t} \mathbb{P}(d_{-k^\ell} = r) h_{k^\ell}(s) \quad (52)$$

so that

$$\mathbb{P}(d = t, K_{12}) \approx \sum_{r+s=t} f^{(n)}(r) \sum_k h_{kk}(s) + \sum_{r+s=t} f^{(n)}(r) \sum_{k, \ell} h_{k^\ell}(s). \quad (53)$$

We note that

$$\sum_k h_{kk}(s) = m\mathbb{E}\left(\deg_{G_{k^\ell}}(1) = s, K_{12} \subset G_{k^\ell}\right) \quad (54)$$

$$= m\mathbb{E}\left(\deg_{G_{k^\ell}}(1) = s | K_{12} \subset G_{k^\ell}\right) \mathbb{P}(K_{12} \subset G_{k^\ell}) = \mu_{32} g^{(n)}(s - 2),$$

where the conditional probability is evaluated using Lemma 7.2. Hence the first term on the right of (53) equals

$$\sum_{r+s=t} f^{(n)}(r) \sum_k h_{kk}(s) = \mu_{32} f^{(n)} * g^{(n)}(t-2). \quad (55)$$
Next we approximate, denoting $h_k(s) = \mathbb{P}(d_k = s, G_{k}^{12})$,

$$\sum_{k, \ell} h_{k\ell}(s) = \sum_{k, \ell} \mathbb{P}(d_k = s, G_k^{12}, G_{\ell}^{13})$$

$$\approx \sum_{k, \ell} \mathbb{P}(d_k + d_{\ell} = s, G_k^{12}, G_{\ell}^{13})$$

$$= \sum_{k, \ell} \sum_{s_1 + s_2 = s} h_k(s_1) h_{\ell}(s_2)$$

$$\approx \sum_{k, \ell} \sum_{s_1 + s_2 = s} h_k(s_1) h_{\ell}(s_2). \quad (56)$$

After noting (see Lemma 7.2) that $\sum_k h_k(s) = \mu_{21} g_{21}^{(o)}(s) - 1$, we conclude that

$$\sum_{k, \ell} \sum_{s_1 + s_2 = s} h_k(s_1) h_{\ell}(s_2) = \mu_{21} g_{21}^{(o)} (s - 2),$$

and hence the second term on the right side of (53) is approximately

$$\sum_{r+s=t} f^{(o)}(r) \sum_{k, \ell} h_{k\ell}(s) \approx \mu_{21} g_{21}^{(o)} (t - 2). \quad (57)$$

By combining (53), (55) and (58), we conclude that

$$\mathbb{P}(d = t, K_{12}) \approx \mu_{32} f^{(o)} * g_{32}^{(o)} (t - 2) + \mu_{21} f^{(o)} * g_{21}^{(o)} (t - 2). \quad (59)$$

The total approximation error in (59) can be written as $\varepsilon(t) = \varepsilon_1(t) + \varepsilon_2(t) + \varepsilon_3(t) + \varepsilon_4(t)$, where $\varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t)$ are the approximation errors in (50), (51), (52), respectively, and the approximation error in (58) equals

$$\varepsilon_4(t) = \sum_{r+s=t} f^{(o)}(r) (\varepsilon_{41}(s) + \varepsilon_{42}(s)), \quad (60)$$

where $\varepsilon_{41}(s), \varepsilon_{42}(s)$ denote the errors made in (56), (57), respectively.

Now we give a rigorous proof of (i) and (ii), where we analyze the individual approximation errors one by one.

Proof of claim (i). The union bound shows that the approximation error $\varepsilon_1(t)$ in (50) is nonpositive for all $t$, and hence $\sum_{i \geq 0} |\varepsilon_1(t)| = \sum_{k, \ell} \mathbb{P}(A_{k\ell}) - \mathbb{P}(\bigcup_{k, \ell} A_{k\ell})$. Bonferroni’s inequalities imply that

$$\sum_{i \geq 0} |\varepsilon_1(t)| \leq \sum_{(k_1, k_2, \ell_1, \ell_2)} \mathbb{P}(A_{k_1 k_2, \ell_1, \ell_2}) =: \Delta.$$ 

We split the right side above by $\Delta = \Delta_2 + \Delta_3 + \Delta_4$, where $\Delta_i, i = 2, 3, 4$, is the sum over layer pairs $(k_1, k_2) \neq (\ell_1, \ell_2)$ such that the list $(k_1, k_2, \ell_1, \ell_2)$ contains precisely $i$ distinct elements. Denote

$$p(k_1 k_2 \ell_1 \ell_2) = \mathbb{P}(G_{k_1} \ni e_1, G_{k_2} \ni e_2, G_{\ell_1} \ni e_1, G_{\ell_2} \ni e_2).$$
Then
\[
\Delta_2 = \sum_{a,b} (p(aabb) + p(abba) + p(aaab) + p(aba) + p(baa) + p(baa)),
\]
\[
\Delta_3 = \sum_{a,b,c} (p(aabc) + p(abac) + p(baca) + p(baca) + p(baca) + p(baca)).
\]

In the sum of \(\Delta_2\), the terms \(p(aabb)\) and \(p(abba)\) equal \(p_{32}(a)p_{32}(b)\) and the other terms equal \(p_{32}(a)p_{21}(b)\). Because \(p_{32}(b) \leq p_{21}(b)\), it follows that \(\Delta_2 \leq 6\sum_{a,b} p_{32}(a)p_{32}(b) \leq 6\mu_{21}\mu_{32}\). In the sum of \(\Delta_3\), the terms \(p(abac)\) and \(p(baca)\) equal \(p_{21}(a)p_{21}(b)p_{21}(c)\) and the other terms equal \(p_{32}(a)p_{21}(b)p_{21}(c)\). Because \(p_{32}(a) \leq p_{21}(a)\), it follows that \(\Delta_3 \leq 6\mu_{21}^3\). Furthermore, \(\Delta_4 = \sum_{a,b,c,d} p(abcd) \leq \mu_{21}^4\). As a conclusion, it follows that
\[
\sum_{t \geq 0} |\varepsilon_1(t)| \leq 6\mu_{21}\mu_{32} + 6\mu_{21}^3 + \mu_{21}^4. \tag{61}
\]

Claim (i) now follows by combining the above bound with the equality
\[
\sum_{k,\ell} \mathbb{P}(A_{k\ell}) = \sum_k p_{32}(k) + \sum_{k,\ell} p_{21}(k)p_{21}(\ell) = \mu_{32} + \mu_{21}^2 - \mu_{21,21}.
\]

Proof of claim (ii). The approximation error in (51) equals \(\varepsilon_2(t) = \sum_{k,\ell} \varepsilon_{2k\ell}(t)\) where
\[
\varepsilon_{2k\ell}(t) = \mathbb{P}(d = t, A_{k\ell}) - \mathbb{P}(d_{k\ell} + d_{-k\ell} = t, A_{k\ell}).
\]

By applying Lemma 11.2 with \(A = \{k, \ell\}\), \(B = [m] \setminus \{k, \ell\}\), \(\mathcal{E}_A = A_{k\ell}\), and \(\mathcal{E}_B\) being the sure event, we see that
\[
|\varepsilon_{2k\ell}(t)| \leq t c_B \mathbb{P}(d_{k\ell} \leq t, A_{k\ell}) \leq t c_B \mathbb{P}(A_{k\ell}),
\]
where \(c_B \leq \mathbb{P}(G_{-k\ell} \ni 12) \leq \mathbb{P}(G \ni 12) \leq \mu_{21}\). Hence
\[
|\varepsilon_2(t)| \leq t \mu_{21} \sum_{k,\ell} \mathbb{P}(A_{k\ell}) \leq t(\mu_{21}\mu_{32} + \mu_{21}^3). \tag{62}
\]

The approximation error in (52) equals \(\varepsilon_3(t) = \sum_{k,\ell} \varepsilon_{3k\ell}(t)\) where
\[
\varepsilon_{3k\ell}(t) = \sum_{r+s=t} \left(\mathbb{P}(d = r) - \mathbb{P}(d_{-k\ell} = r)\right) h_{k\ell}(s).
\]

By applying Lemma 11.1 with \(g(s) = \frac{h_{k\ell}(s)}{\mathbb{P}(A_{k\ell})}\), it follows that \(\sum_{t \geq 0} |\varepsilon_{3k\ell}(t)| \leq 2 \mathbb{P}(A_{k\ell})\mathbb{P}(d_{k\ell} > 0)\). Observe now that \(\mathbb{P}(d_{k\ell} > 0) \leq p_{10}(k) + p_{10}(\ell)\). Hence,
\[
\sum_{t \geq 0} |\varepsilon_3(t)| \leq 2 \sum_{k,\ell} (p_{10}(k) + p_{10}(\ell)) \mathbb{P}(A_{k\ell})
\leq 4 \sum_k p_{10}(k)p_{32}(k) + 4 \sum_{k,\ell} p_{10}(k)p_{21}(k)p_{21}(\ell)
\leq 4\mu_{10,32} + 4\mu_{21}\mu_{10,21}. \tag{63}
\]
The approximation error in (56) equals $\varepsilon_{41}(s) = \sum_{k, \ell} \varepsilon_{4k\ell}(s)$ where

$$
\varepsilon_{4k\ell}(s) = \mathbb{P}(d_{k\ell} = s, A_{k\ell}) - \mathbb{P}(d_k + d_\ell = s, A_{k\ell}).
$$

By applying Lemma 11.2 with $A = \{k\}$ and $B = \{\ell\}$, together with $E_A = \{12 \in G_k\}$ and $E_B = \{13 \in G_\ell\}$, it follows that $|\varepsilon_{4k\ell}(s)| \leq sp_{21}(k)p_{32}(\ell) + sp_{21}(\ell)p_{32}(k)$. By summing the above inequality with respect to $k, \ell$, it follows that $|\varepsilon_{41}(s)| \leq 2s\mu_{21}\mu_{32}$.

The approximation error in (57) equals

$$
|\varepsilon_{42}(s)| = \sum_k \sum_s \mathbb{P}(d_k = s_1, G_k^1)\mathbb{P}(d_k = s_2, G_k^2),
$$

where $\sum_{s \geq 0} |\varepsilon_{42}(s)| = \sum_k p_{21}(k)^2 = \mu_{21,21}$. Hence, by (60),

$$
|\varepsilon_{4}(t)| \leq \sum_{r+s=t} f^{(n)}(r) (|\varepsilon_{41}(s)| + |\varepsilon_{42}(s)|) \leq \max_{s \leq t} |\varepsilon_{41}(s)| + \max_{s \leq t} |\varepsilon_{42}(s)|
$$

$$
\leq 2t\mu_{21}\mu_{32} + \mu_{21,21}.
$$

(64)

Claim (ii) follows by collecting all the bounds in (61–64) together.

**Proof of Theorem 3.2.** We evaluate the ratio $\tau^{(n)} = \frac{\mathbb{P}(\mathcal{K}_3)}{\mathbb{P}(\mathcal{K}_{12})}$. By Theorems 7.3 and 7.4,

$$
\mathbb{P}(\mathcal{K}_3) = \mu_{33} + O \left( \mu_{21}\mu_{32} + \mu_{21}^2 \right),
$$

$$
\mathbb{P}(\mathcal{K}_{12}) = \mu_{32} + \mu_{21}^2 + O \left( \mu_{21}\mu_{32} + \mu_{21}^3 + \mu_{21} + \mu_{21,21} \right),
$$

where $\mu_{rs} = (m/n)_r^{-1}(P_n)_s$. Now relations $(P_n)_{21} \leq 1$ and $(P_n)_3 \leq 1$ imply $\mu_{21} \leq m/n^2$ and $\mu_{32} \leq m/n^3$. Hence $\mu_{21}\mu_{32} \leq m^2n^{-5}$, $\mu_{21}^3 \leq m^3n^{-6}$, and $\mu_{21}^4 \leq m^4n^{-8}$. Next, we note that $\mu_{21,21} \leq m(n)_2^{-2}(P_n)_21,21$ by Jensen’s inequality. Note also that $(x_2q)^2 \leq 2x(x_3q)^2$ for $x \geq 3$. Hence, $(x_2q)^3 \leq 4 + 2x(x_3q)^2$, and $(P_n)_21,21 \leq 4 + 2(P_n)_{10,32}$. Furthermore, Lemma 7.1 implies that $(P_n)_{10,32} \ll 1$. Hence $\mu_{21,21} \ll mn^{-3}$.

(i) Consider the case $m/n \to \mu \in [0, \infty)$. Then $\mu_{32} = ((P)_{32} + o(1)) mn^{-3}$ and

$$
\mu_{21}^2 = (\mu(P)_{21}^2 + o(1)) mn^{-3} \text{ imply that}
$$

$$
\mathbb{P}(\mathcal{K}_{12}) = (P_{32})_{32} mn^{-3} + \mu(P)_{21}^2 mn^{-3} + O(mn^{-3}).
$$

Similarly, $\mu_{33} = ((P)_{33} + o(1)) n^{-3}m$ implies

$$
\mathbb{P}(\mathcal{K}_3) = (P_{33})_{33} mn^{-3} + O(mn^{-3}),
$$

and hence the first two claims of Theorem 3.2 follow (the first claim corresponds to $\mu = 0$).

(ii) Assume now that $n \ll m \ll n^2$. Then $mn^{-3}, m^2n^{-5}, m^3n^{-6} \ll m^2n^{-4}$. Hence, $\mathbb{P}(\mathcal{K}_3) \ll m^2n^{-4}$.

Furthermore, $m^4n^{-8} \ll m^2n^{-4}$, and we conclude that $\mathbb{P}(\mathcal{K}_{12}) = (P_{21})_{21} m^2n^{-4} + O(mn^{-3})$. Hence, $\frac{\mathbb{P}(\mathcal{K}_3)}{\mathbb{P}(\mathcal{K}_{12})} \to 0$ implies the third claim of Theorem 3.2.

**Proof of Theorem 3.3.** We evaluate the ratio $\sigma^{(n)}(t) = \frac{\mathbb{P}(d=t, K_3)}{\mathbb{P}(d=t, K_{12})}$. By Theorems 7.3 and 7.4,

$$
\mathbb{P}(d = t, K_3) = \mu_{33} f^{(n)} \ast G_{33}^{(n)}(t-2) + \varepsilon_A(t),
$$

$$
\mathbb{P}(d = t, K_{12}) = \mu_{32} f^{(n)} \ast G_{32}^{(n)}(t-2) + (\mu_{21})^2 f^{(n)} \ast G_{21}^{(n)} \ast G_{21}^{(n)}(t-2) + \varepsilon_B(t),
$$

where $G_{rs}^{(n)} = \mathbb{P}(d = t, A_{rs})$.
where the remainder terms $\varepsilon_A(t)$ and $\varepsilon_B(t)$ are upper bounded in (44) and (49) respectively. The conditions $m/n \to \mu$ and $(P_n)_r \to (P)_r$ imply $\mu_{rs} = (\mu + o(1))(P)_r n^{1-r}$ for $rs = 21, 32, 33$. Invoking these relations in (44), (49) and using the bounds $\mu_{10,33} \ll n^{-2}, \mu_{10,32} \ll n^{-2}$ and $\mu_{21,21} \ll n^{-2}$ (the latter bound is shown in the proof of Theorem 3.2 above) we upper bound the remainders $\varepsilon_A(t) \ll n^{-2}$ and $\varepsilon_B(t) \ll n^{-2}$. We obtain

$$
\begin{align*}
\mu_{33} f^{(n)} * g_{33}^{(n)}(t - 2) &= (P)_{33} \mu n^{-2} f^{(n)} * g_{33}^{(n)}(t - 2) + o(n^{-2}), \\
\mu_{32} f^{(n)} * g_{32}^{(n)}(t - 2) &= (P)_{32} \mu n^{-2} f^{(n)} * g_{32}^{(n)}(t - 2) + o(n^{-2}), \\
\mu_{21} f^{(n)} * g_{21}^{(n)}(t - 2) &= (P)_{21} \mu n^{-2} f^{(n)} * g_{21}^{(n)}(t - 2) + o(n^{-2}).
\end{align*}
$$

Now the claim follows by the fact that $f^{(n)} \xrightarrow{w} f = \text{CPOi}(\mu(P)_{10}, g_{10})$ (see Theorem 3.1) and $g_{rs}^{(n)} \xrightarrow{w} g_{rs}$ for $rs = 21, 32, 33$.

8 | CLUSTERING AND DEGREE IN POWER-LAW MODELS

Here we prove Theorems 4.1 and 4.2. In the proof we use the fact that a compound Poisson distribution $\text{CPOi}(\lambda, h)$ is heavy-tailed whenever the increment distribution $h$ is [23, Theorem 4.30]. Namely, for a subexponential distribution $h$ we have as $t \to +\infty$

$$
\text{CPOi}(\lambda, h)(t) \sim \lambda h(t).
$$

(65)

Furthermore, we show below that for $P$ satisfying (15), (16) the mixed binomial distribution (5) follows a power law

$$
\text{Bin}_{rs}(P)(t) \sim d_{rs} L(t^{1/(1-\beta)}) t^{-\delta_{rs}},
$$

(66)

with parameters

$$
\delta_{rs} = 1 + \frac{\alpha + s \beta - r - 1}{1 - \beta} \quad \text{and} \quad d_{rs} = \frac{b^\delta_{rs} b^{\delta_{rs} - 1}}{P_{rs}} \frac{1 - \beta}{1 - \beta}.
$$

(67)

It is an immediate consequence of (65) and (66) that the limiting degree distribution $f = \text{CPOi}(\mu(P)_{10}, \text{Bin}_{10}(P))$ obeys a power law (17). We similarly establish respective power law asymptotics for the distributions $f * g_{33}, f * g_{32}$ and $f * g_{21} * g_{21}$ that appear in (10). Now a simple analysis of the ratio (10) as $t \to +\infty$ shows the asymptotics of Theorem 4.2.

Proof of Theorem 4.1. Statements (i) and (ii) are immediate consequences of (65), (66) as described above. To prove (iii) we note that $\beta \geq 1$ implies $XQ = Xg(X) \leq B$ almost surely. Denote for short $\lambda = \mu(P)_{10}$ and let $H$ be a random variable with the distribution $\text{Bin}_{10}(P)$ and $\Lambda$ be a Poisson random variable with parameter $\lambda$. We have

$$
\sum_{t \geq 0} e^{ht} f(t) = \mathbb{E} \left( e^{H} \right)^{\Lambda} = e^{(\mathbb{E} e^{H} - 1) \lambda},
$$

where

$$
\mathbb{E} e^{H} = \sum_{h \geq 0} e^{h} e^{X - 1 \choose h} Q^h (1 - Q)^{X - 1 - h} \frac{X}{(P)_{10}}.
$$
The almost sure inequality $QX \leq B$ together with the identity $(P)_{10} = \mathbb{E}X$ complete the proof. \hfill \Box

**Proof of Theorem 4.2.** Theory of discrete subexponential densities [23, Lemmas 4.9 and 4.14], implies that $(f_{1} * f_{2})(t) \sim f_{1}(t) + f_{2}(t)$ for all probability densities on the positive integers such that $f_{i}(t) \sim a_{i}t^{-\alpha_{i}}$ with $a_{i} > 0$ and $\alpha_{i} > 1$. By Theorem 4.1, we know that $f(t) \sim \mu(P)_{10}d_{10}t^{-\delta_{10}}$, and by (66), we find that $g_{rs}(t) = \text{Bin}_{rs}(P)(t) \sim d_{rs}t^{-\delta_{rs}}$ with parameters given by (67). Because $\delta_{32} < \delta_{21} < \delta_{10}$, it follows that

$$(f * g_{32})(t) \sim f(t) + g_{32}(t) \sim g_{32}(t)$$

and

$$(f * g_{21} * g_{21})(t) \sim f(t) + g_{21}(t) + g_{21}(t) \ll g_{32}(t).$$

Hence by formula (10),

$$\sigma(t) \sim (P)_{32} f(t) + g_{32}(t) \sim \frac{(P)_{33} \mu(P)_{10}d_{10}t^{-\delta_{10}} + d_{33}t^{-\delta_{33}}}{(P)_{32}}.$$

Because $\delta_{33} - \delta_{10} = \frac{3\beta - 2}{1 - \beta}$, we see that $\sigma(t)$ follows a power law with density exponent $\delta_{33} - \delta_{32} = \frac{\beta}{1 - \beta}$ for $\beta \leq \frac{2}{3}$, and density exponent $\delta_{10} - \delta_{32} = 2$ for $\beta \geq \frac{2}{3}$. The constant term of the power law is determined by (67). \hfill \Box

Let us prove (66). Given $r \in \mathbb{Z}_{+}$, let $H_{(r)}$ be a mixed binomial random variable with the distribution

$$\mathbb{P}(H_{(r)} = t) = \mathbb{E}(\text{Bin}(X - r, g(X))(t)), \quad t = 0, 1, 2, \ldots, \quad (68)$$

where the distribution $p(x) = \mathbb{P}(X = x)$ of $X$ and function $g$ are given by (16). In Lemma 8.1 below we show that the distribution of $H_{(r)}$ follows a power law

$$\mathbb{P}(H_{(r)} = t) \sim \frac{b^{\frac{z-1}{1-\beta}}}{1 - \beta} L\left(t^{1/(1-\beta)}\right) t^{-1 - \frac{z-1}{1-\beta}} \quad \text{as} \quad t \to +\infty. \quad (69)$$

Next we observe that the distribution $\text{Bin}_{rs}(P)$ can be written in the form

$$\text{Bin}_{rs}(P)(t) = \mathbb{E}\left(\text{Bin}(\bar{X} - r, g(\bar{X}))(t)\right) = \sum_{x=r}^{\infty} \text{Bin}(x - r, g(x))(t) \tilde{p}_{rs}(x),$$

where the random variable $\bar{X}$ has the power law distribution

$$\mathbb{P}(\bar{X} = x) = \tilde{p}_{rs}(x) = \frac{(x)_{r}g(x)_{r}p(x)}{(P)_{rs}} \sim \frac{b^{x}}{(P)_{rs}} L(x)x^{-(a+\beta-r)}.$$
Hence (69) yields (66). It remains to prove (69). In the proof we use the fact that binomial distribution is highly concentrated around its mean. We apply this fact to mixed binomial random variable \( H_{(r)} \) conditionally, given the mixing random variable \( X \).

**Lemma 8.1.** Let \( \alpha \geq 1, 0 \leq \beta < 1 \) and \( b > 0 \). Assume that (15), (16) hold. For \( \beta = 0 \) we assume, in addition, that \( b \leq 1 \). Then for each \( r \in \mathbb{Z}_+ \) (69) holds.

**Proof of Lemma 8.1.** For \( \beta = 0, b = 1 \) relation (69) follows from the identity \( \mathbb{P}(H_{(r)} = t) = \mathbb{P}(X = t + r) \), which holds for large \( t \). Indeed, for large \( t \) the second relation of (16) implies \( g(x) \equiv 1 \) for \( x \geq t \). Hence Bin\((x - r, g(x))(t) = \mathbb{I}_{[x-r=W]} \), and thus \( \mathbb{P}(H_{(r)} = t) = \mathbb{P}(X = t + r) \).

In what follows we assume that either \( 0 < \beta < 1 \) or \( \beta = 0 \) and \( b < 1 \). We only consider the case where \( r = 0 \). The proof for \( r \geq 1 \) is much the same.

In the proof limits are taken as \( t \to +\infty \). Let \( H_k \sim \text{Bin}(k, g(k)) \) be a binomial random variable. We use the shorthand notation

\[
\mu_k = \mathbb{E}H_k = kg(k), \quad \sigma_k^2 = \text{Var}H_k = kg(k)(1 - g(k)), \quad \tilde{p}_k = \mathbb{P}(X = k).
\]

Given \( t \), let \( \delta_t = t^{1/2}\ln^4 t \). We split the probability

\[
\mathbb{P}(H_{(0)} = t) = \sum_{k,t} \mathbb{P}(H_k = t) \tilde{p}_k = I_1 + I_2 + I_3, \quad I_j := \sum_{k \in A_j} \mathbb{P}(H_k = t) \tilde{p}_k,
\]

where

\[
A_1 = \{k \geq t : \mu_k < t - \delta_t\}, \quad A_2 = \{k : |t - \mu_k| \leq \delta_t\}, \quad A_3 = \{k : \mu_k > t + \delta_t\}.
\]

We assume that \( t \) is large enough so that \( g(k) = bk^{-\beta} \) for \( k > t \). Then \( \mu_k = bk^{1-\beta} \) and \( A_1 \neq \emptyset \). To prove (69) we show that

\[
I_2 = \frac{1}{1 - \beta} b^{\frac{\alpha - 1}{\beta}} L\left(t^{1/(1-\beta)} \right) t^{-\frac{\alpha - 1}{\beta}} \left(1 + o(1)\right) \quad \text{and} \quad I_1, I_3 = O(e^{-0.5\ln^4 t}). \tag{70}
\]

Let us evaluate \( I_2 \). By the local limit theorem [39], [50] we approximate uniformly in \( k \in A_2 \)

\[
\mathbb{P}(H_k = t) = \frac{1}{\sqrt{2\pi \sigma_k^2}} e^{-\frac{\sigma_k^2}{2\sigma_k^2}} (1 + o(1)). \tag{71}
\]

Furthermore, we have \( \tilde{p}_k = L\left(t^{1/(1-\beta)}\right) \left(b/t\right)^{\alpha/(1-\beta)} (1 + o(1)) \), uniformly in \( k \in A_2 \). In what follows we consider the cases \( 0 < \beta < 1 \) and \( \beta = 0 \) separately.

For \( 0 < \beta < 1 \) we have \( \sigma_k^2 = \mu_k(1 - g(k)) = t(1 + O(\delta_t/t) + O(\tau^{-\beta/(1-\beta)}) \) uniformly in \( k \in A_2 \). Now (71) implies

\[
I_2 = (1 + o(1)) \sum_{k \in A_2} \frac{1}{\sqrt{2\pi \sigma_k^2}} e^{-\frac{\sigma_k^2}{2\sigma_k^2}} \tilde{p}_k = (1 + o(1)) \frac{S}{\sqrt{2\pi t}} L\left(t^{1/(1-\beta)}\right) \left(b/t\right)^{\alpha/(1-\beta)},
\]
where \( S = \sum_{k \in \mathcal{A}_2} e^{-\mu_k \frac{x^2}{2}} \). Next we approximate

\[
S = I + O(1), \quad I = \int_{|t - by^{1-\beta}| \leq \delta} e^{-(t-by^{1-\beta})^2/(2t)} dy
\]

and use the substitution \( x = (by^{1-\beta} - t)/\sqrt{t} \) to write the integral in the form

\[
I = \frac{1}{1 - \beta} \int_{|x| < \ln t} \frac{1}{\sqrt{t}} \frac{1}{\beta} I_1, \quad I_1 := \int_{|x| < \ln t} e^{-x^2/2} (1 + x/\sqrt{t})^{1 - \beta/2} dx.
\]

Now it is easily seen that the integral \( I_1 \) converges to \( \sqrt{2\pi} \) as \( t \to +\infty \). Hence,

\[
S = (1 + o(1)) \frac{1}{1 - \beta} b^{-\frac{1}{1 - \beta}} \frac{1}{\beta} \sqrt{2\pi} \quad \text{as} \quad t \to +\infty.
\]

We have arrived to the first relation of (70).

For \( \beta = 0 \) and \( b < 1 \) we have \( \sigma_k^2 = t(1 - b)(1 + O(\delta t / t)) \). Now (71) implies

\[
I_2 = \frac{S'}{\sqrt{2\pi t(1 - b)}} L(t)(b/t)^\alpha(1 + o(1)), \quad \text{where} \quad S' = \sum_{k \in \mathcal{A}_2} e^{-(t-x^2)/2(1-b)}.
\]

Invoking the approximation \( S' = I' + O(1) \), where

\[
I' = \int_{|t - by| \leq \delta} e^{-(t-by)^2/(2t(1-b))} dy = \sqrt{2\pi t(1 - b)} b^{-1}(1 + o(1)),
\]

we obtain the first relation of (70).

We derive the bounds on the right of (70) from the upper bounds

\[
\mathbb{P}(H_k = t) \leq e^{-0.5n^t}(1 + o(1)). \tag{72}
\]

that hold uniformly in \( k \in \mathcal{A}_1 \cup \mathcal{A}_3 \). Indeed, (72) follows from the well-known exponential inequalities for Binomial probabilities see for example, Theorem 2.1 of [30],

\[
\mathbb{P}(H_k \geq \mu_k + s) \leq e^{-\frac{s^2}{2\mu_k + h(x)}}, \quad \mathbb{P}(H_k \leq \mu_k - s) \leq e^{-\frac{s^2}{2\mu_k}}, \quad s > 0. \tag{73}
\]

We only show (72) for \( k \in \mathcal{A}_3 \). Let \( \mu = \min\{\mu_k : k \in \mathcal{A}_3\} \). The function \( h(x) = (x - t)^2x^{-1} \) is increasing for \( x > t \). Hence \( h(\mu) \geq h(\mu) \) for \( k \in \mathcal{A}_3 \). The second inequality of (73) implies

\[
\mathbb{P}(H_k = t) \leq \mathbb{P}(H_k \leq \mu_k - (\mu_k - t)) \leq e^{-0.5h(\mu_k)} \leq e^{-0.5h(\mu)} = e^{-0.5\ln t}(1 + o(1)).
\]

9 | GIANT COMPONENT

Here we prove Theorem 3.4. We start with an outline of the proof. In the proof we apply the approach developed in [15]: we approximate the number \( N_1(G(n)) \) of vertices in the largest connected component of \( G(n) \) by the number of vertices \( u \) with the property that the breadth first search (BFS) tree rooted
at \( u \) contains at least \( \omega \) vertices, where \( \omega = \omega(n) \to +\infty \) as \( n \to +\infty \). Then the BFS exploration process rooted at \( u \) is approximated by respective branching process and the fraction of vertices \( u \) having large BFS trees (of size at least \( \omega \)) is approximated by the survival probability \( \rho = \rho(\omega) \) of the branching process. Hence \( N_1(G_n)/n \to \mathbb{P} \). We briefly comment on the branching process, which is much different from that of \([15]\). Let us consider the BFS exploration process at the moment when it enters a new layer, say, \( G_{n,i} \). The first vertex of \( G_{n,i} \) detected by BFS will be included into the BFS tree together with the component of \( G_{n,i} \) the vertex belongs to. The other members of the component are called children of the vertex. Since the vertex may belong to several layers, it can have children from the other layers as well. Consequently, the total number of children is approximated by the sum of degrees of the vertex in the transitive closures \( \overline{G}_{n,i} \) of layers \( G_{n,i} \) that cover this vertex. Accordingly, the offspring distribution is approximated by \( f^+ \). We note that layers do not need to be connected. Each layer \( G_{n,i} \) may split into components, but for moderately growing \( \omega = \omega(n) \) the BFS exploration process will not visit the same layer twice within the first \( O(\omega) \) steps with high probability. Therefore the offspring numbers are approximately independent.

The proof is organized as follows. We start with Lemma 9.1 which establishes the result in the special case where the layer types are deterministic, that is, \( \forall n \mathbb{P}(\mathcal{(X}_n, \overline{Q}_n) = (\mathcal{X}_n, \overline{Q}_n)) = 1 \) for some \( \mathcal{X}_n = (x_{n,1}, \ldots, x_{n,m}) \) and \( \overline{Q}_n = (q_{n,1}, \ldots, q_{n,m}) \). Furthermore, we assume that the distribution of \((X, Q)\) has a finite support, say, \( A \subset \{0, 1, 2, \ldots, M\} \times [0, 1] \), where \( M \geq 2 \) is an integer. Hence, \( \mathbb{P}(X = t, Q = q) > 0 \) whenever \( (t, q) \in A \). Moreover, we assume in Lemma 9.1 that the set

\[
A^0 = \{(t, q) : t \geq 2, q > 0, (t, q) \in A\}
\]

is nonempty and

\[
\forall \ n \geq 1 \quad \forall 1 \leq i \leq m_n \quad (x_{n,i}, q_{n,i}) \in A. \quad (74)
\]

Note that for each \((x_{n,i}, q_{n,i}) \in A \setminus A^0\) respective layer \( G_{n,i} \) has no edges. Next, in Lemma 9.2 we relax condition (74) by allowing a negligible fraction of layer types \((x_{n,i}, q_{n,i})\) to take their values outside \( A \). In the last step of the proof we reduce the general case to that considered in Lemma 9.2. To this aim we truncate the layer sizes \( X_{n,i} \) (at level \( M \)) and \( \varepsilon \)-discretize the edge densities \( Q_{n,i} \) as in (20). Then we apply Lemma 9.2 conditionally given the truncated and discretized layers \( (\mathcal{X}_n^M, \overline{Q}_n^-) \). Here we use notation (21) and for \( \mathcal{X}_n = (x_{n,1}, \ldots, x_{n,m}) \) we denote \( x_{n}^M = (x_{n,1}, \ldots, x_{n,m}^M) \), where \( x_{n}^M = x_{n}^\|_{1 \leq m} \). Finally, letting \( \varepsilon \downarrow 0 \) and \( M \uparrow \infty \) we approximate the distribution of \((X, Q)\) and survival probability \( \rho \) by respective characteristics of \((X^M, Q^\pm)\) thus completing the proof of Theorem 3.4.

**Notation.** Before the proof we introduce some notation. Given a Galton-Watson (G-W) branching process \( \mathcal{X} \) we denote by \( |\mathcal{X}| \) the total progeny of \( \mathcal{X} \), \( \rho^k(\mathcal{X}) = \mathbb{P}(|\mathcal{X}| \geq k) \) and \( \rho(\mathcal{X}) = \mathbb{P}(|\mathcal{X}| = \infty) \). Let \( T(t, q) = \text{deg}_{H_{t,q}}(\mathcal{X}) \) be the degree of a randomly selected vertex \( \mathcal{X} \) in the transitive closure \( H_{t,q} \) of Bernoulli random graph \( H_{t,q} \) on \( t \) vertices and with edge density \( q \). Let \( T = \{T_s(t, q), T_s^M(t, q) : q \in [0, 1], j, s, t \in \mathbb{N}\} \) be a collection of independent random variables such that \( T_s(t, q) \) and \( T_s^M(t, q) \) have the same distribution as \( T(t, q) \).

Let \( C \subset V \) be the vertex set of the connected component of \( G = G_n = (\mathcal{X}_n, \overline{Q}_n) \) containing vertex \( v \in V \). Let \( \omega : \mathbb{N} \to \mathbb{N} \) be a function satisfying \( \omega(n) \to \infty \), \( \omega(n) = o(n) \) as \( n \to +\infty \). Let \( B^{k} = \{u : |C_u| \geq k\} \subset V \) be the set of vertices belonging to connected components of \( G \) of size at least \( k \). We write \( B^\omega = B^{o(\omega)} \). Let \( C \subset V \) denote the vertex set of the largest connected component of \( G \). Note that for each integer \( k \geq 1 \)

\[
|C| \leq \max\{|k, |B^k|\}. \quad (75)
\]
In Lemma 9.1 below we assume that the distribution of \((X, Q)\) has a finite support \(A\) and denote

\[
q_0 = \min\{q : (t, q) \in A^0\} \quad \text{and} \quad h_{t,q} = \mathbb{P}(X = t, Q = q), \quad (t, q) \in A.
\]

Given \(0 < \delta < 1/4\), let \(Y_0^+, \ Y_0^-\), \(\mathcal{Y}\) be G-W processes with the offspring numbers

\[
Y_0^+ = \sum_{(t,q) \in A^0} \sum_{s=1}^{\Lambda_{t,q}} T_s(t,q), \quad Y_0^- = \sum_{(t,q) \in A^0} \sum_{s=1}^{\Lambda_{t,q}} T_s(t,q), \quad Y = \sum_{(t,q) \in A^0} \sum_{s=1}^{\Lambda_{t,q}} T_s(t,q).
\]

Here \(\Lambda_{t,q} \sim \text{Poi}(\lambda_{t,q})\) and \(\Lambda_{t,q}^+ \sim \text{Poi}(\lambda_{t,q}^+\mu)\) with \(\lambda_{t,q} = \lambda_{t,q}(1 \pm \delta)\). Furthermore, we assume that the collection of random variables \(\{\Lambda_{t,q}, \Lambda_{t,q}^+, \Lambda_{t,q}^-, (t, q) \in A^0\}\) and \(\mathcal{T}\) are independent. Note that offspring numbers (76) have compound Poisson distributions. In particular, \(Y\) has the probability distribution \(\text{CPoi}(\lambda, \mathcal{L}(T_*))\), where \(\lambda = \mu x_*\) with \(x_* = \mathbb{E}(X|X_{t} \geq 1)\) and the random variable \(T_*\) has the probability distribution

\[
\mathbb{P}(T_* = \ell) = x_*^{-1} \mathbb{E}\left(\mathbb{P}(T(X, Q) = \ell|X, Q)X_{t} \geq 1\right)
\]

(77)

By the same reasoning as in (30) above, we obtain the equality of distributions

\[
\text{CPoi}(\lambda, \mathcal{L}(T_*)) = \text{CPoi}(\mu(P)_{10}, \text{Bin}_{10}(P)),
\]

(78)

where the increment distribution \(\text{Bin}_{10}(P)\) is defined by (6).

We recall that \(D_i\) denotes the vertex set of layer \(G_{n,i}\), \(1 \leq i \leq m\). With a little abuse of notation we also refer to \(D_i\) as the layer \(G_{n,i}\). Furthermore, we assign label \(D_i\) to the edges of \(G_{n,i}\). In particular, an edge of \(G\) may receive several labels if it is present in several layers. Given \(n\) and \((t, q) \in A\), let \(D_{t,q}\) be the collection of layers \(D_i\) having size \(x_{n,i} = t\) and edge density \(q_{n,i} = q\). Put \(D_0 = \cup_{(t,q) \in A^0} D_{t,q}\). For every \(D_i \in D_0\) the probability that a randomly chosen vertex of \(D_i\) has a neighbor in \(D_i\) connected by an edge labeled \(D_i\) is

\[
1 - (1 - q_{n,i})^{x_{n,i} - 1} \geq \min\{1 - (1 - q)^{t-1} : (t, q) \in A^0\} =: \hat{q}.
\]

Note that \(\hat{q} \geq q_0\).

**Lemma 9.1.** Let \(\mu > 0\). Let \(M \geq 2\) be an integer. Let \(n, m \to +\infty\). Assume that \(m/n \to \mu\). Let \(\varepsilon_n \downarrow 0\) be a positive sequence. Assume that \(\mathbb{P}(X \leq M) = 1\) and \((X, Q)\) has a finite support denoted by \(A\). Assume that \(A^0 = \{(t, q) \in A : t \geq 2, q > 0\}\) is nonempty. Assume that (74) holds and the numbers

\[
m_{t,q} := \#\{i \in [1, \ldots, m] : (x_{n,i}, q_{n,i}) = (t, q)\}
\]

satisfy

\[
\forall \ n \quad \max_{(t,q) \in A^0} \left| h_{t,q} - m_{t,q}/m \right| \leq \varepsilon_n.
\]

(79)

Assume that \(o(n) \leq n\ln^{-2} n\) and

\[
\forall n \quad |m/n - \mu| < \varepsilon_n.
\]

(80)
There exists sequences $\epsilon'_n \downarrow 0, \epsilon''_n \downarrow 0$ (depending on $\epsilon_n$, $\omega$ and $\mu, A^0, \{h_{t,q}, (t,q) \in A^0\}$) such that for each $n$ we have

$$\max_{v \in V} |\mathbb{P}(|C_v| \geq \omega(n)) - \rho(\mathcal{Y})| \leq \epsilon'_n, \quad (81)$$

$$\mathbb{P}\left(\left|B^{(n)} - n \rho(\mathcal{Y})\right| > \epsilon''_n n\right) \leq \epsilon'_n. \quad (82)$$

There exists sequences $\epsilon'_n \downarrow 0, \epsilon''_n \downarrow 0$ (depending on $\epsilon_n$, $\mu, A^0, \{h_{t,q}, (t,q) \in A^0\}$) such that for each $n$ we have

$$\mathbb{P}\left(\left|C - n \rho(\mathcal{Y})\right| > \epsilon''_n n\right) \leq \epsilon'_n. \quad (83)$$

We note that $A^0 \neq \emptyset$ implies $q_0 > 0$ and $\hat{q} > 0$.

**Proof of Lemma 9.1.** We recall that the idea of the proof is outlined in the first paragraph of the section above. Proofs of (81), (82), (83) are given in separate steps.

**Proof of (81).** Note that the distribution of $|C_v|$ is the same for each $v \in V$. Hence, it suffices to approximate $\mathbb{P}(|C_v| \geq \omega(n))$ for $v = v_1$.

Before the proof we introduce some notation. Given $0 < \delta < 4^{-1}$ we denote

$$m_{t,q}(\delta) = mh_{t,q}(1 - \delta) \quad \text{and} \quad p_{t,\delta} = t(1 - \delta)n^{-1}. \quad (84)$$

We assume that $n,m$ are large enough so that $m_{t,q} > m_{t,q}(\delta)$, for $(t,q) \in A_0$. Let $\omega' : \mathbb{N} \rightarrow \mathbb{N}$ be such that $\omega'(n) = o(\sqrt{n}), \omega'(n) \rightarrow +\infty, \omega' \leq \omega$. We write, for short, $\rho^{(a)} = \rho^{(\omega(a))(\mathcal{Y})}, \rho^{(a\omega)} = \rho^{(\omega(a))(\mathcal{Y})}$.

Let

$$\mathcal{N}^+ = \{N_{t,q}^+, 2 \leq t \leq M\}, \quad \mathcal{N}^- = \{N_{t,q}^-, 2 \leq t \leq M\},$$

$$\mathcal{N}^+ = \{\tilde{N}_{t,q}^+, 2 \leq t \leq M\}, \quad \mathcal{N}^- = \{\tilde{N}_{t,q}^-, 2 \leq t \leq M\}$$

be collections of independent random variables having binomial and Poisson distributions:

$$N_{t,q}^+ \sim \text{Bin}(m_{t,q}, t/(n - \omega'(n))), \quad N_{t,q}^- \sim \text{Bin}(m_{t,q}(2\delta), p_{t,\delta}),$$

$$\tilde{N}_{t,q}^+ \sim \text{Poi}(t_{t,q}/(n - \omega'(n))), \quad \tilde{N}_{t,q}^- \sim \text{Poi}(m_{t,q}(2\delta)p_{t,\delta}). \quad (85)$$

Note that $\mathbb{E}N_{t,q}^+ = \mathbb{E}\tilde{N}_{t,q}^+$ and $\mathbb{E}N_{t,q}^- = \mathbb{E}\tilde{N}_{t,q}^-$. We assume that $\mathcal{T}$ is independent of $\mathcal{N}^-, \mathcal{N}^+, \tilde{N}^-, \tilde{N}^+$.

Let $Z^\pm$ (respectively $\tilde{Z}^\pm$) be defined as $Y^\pm_{\delta}$ in (76), but with $\Lambda_{t,q}$ replaced by $N_{t,q}^\pm$ (respectively $\tilde{N}_{t,q}^\pm$). Let $\mathcal{X}^\pm$ and $\tilde{\mathcal{X}}^\pm$ be Galton-Watson processes with the offspring numbers $Z^\pm$ and $\tilde{Z}^\pm$ respectively.

Using the total variation distance bound $d_{\mathcal{T}}(\text{Bin}(n,p), \text{Poi}(np)) \leq p$, see (1.23) in [6], we show by coupling the offspring numbers of $\mathcal{X}^\pm$ and $\tilde{\mathcal{X}}^\pm$ that

$$\left|\rho^{(k)}(\mathcal{X}^-) - \rho^{(k)}(\tilde{\mathcal{X}}^-)\right| \leq Mkn^{-1}, \quad (87)$$

$$\left|\rho^{(k)}(\mathcal{X}^+) - \rho^{(k)}(\tilde{\mathcal{X}}^+)\right| \leq Mk(n - \omega(n))^{-1}. \quad (88)$$

From (79), (80), (87) we obtain for $k = k(n) = o(n)$ as $m, n \rightarrow +\infty$

$$\rho^{(k)}(\mathcal{Y}^-_{3\delta}) \leq \rho^{(k)}(\tilde{\mathcal{X}}^-) \leq \rho^{(k)}(\mathcal{X}^-) + Mkn^{-1} = \rho^{(k)}(\mathcal{X}^+) + o(1), \quad (88)$$
\[ \rho^{(k)}(Y_{26}^+) \geq \rho^{(k)}(\hat{\lambda}^+) \geq \rho^{(k)}(\lambda^+) - M(k(n - \omega(n))^{-1} \geq \rho^{(k)}(\lambda^+) - o(1). \]  

(89)

To show the first inequality of (88) we couple the offspring numbers \( Y_{36}^- \leq \hat{Z}^- \). The coupling is feasible whenever \( \lambda_{t,q}(1 - 3\delta) \leq m_{t,q}(2\delta)p_{t,q} \), \( \forall (t,q) \in A^0 \) or, equivalently, \( \mu(1 - 3\delta) \leq (m/n)(1 - 3\delta + 2\delta^2) \).

In view of (80) the latter inequality holds true for sufficiently large \( n \), say, \( n > n_\varepsilon \), where \( n_\varepsilon \) depends on \( \delta \) and \( \{\varepsilon_n\} \). Similarly, the first inequality of (89) holds whenever \( \lambda_{t,q}(1 + 2\delta) \geq m_{t,q}/(n - \omega(n)) \) \( \forall (t,q) \in A^0 \). In view of (79), (80) the latter inequality holds for sufficiently large \( n \), say, \( n > n_\varepsilon \), where \( n_\varepsilon \) depends on \( \delta \) and \( \{\varepsilon_n\} \).

Furthermore, we show in (101), (109) below that

\[ \rho^{(o)}(\lambda^+) - o(1) \leq \mathbb{P}(|C_\varepsilon| \geq \omega(n)) \leq \mathbb{P}(|C_\varepsilon| \geq \omega'(n)) \leq \rho^{(o)}(\lambda^+) + o(1) \]  

(90)

(the second inequality follows by \( \omega \geq \omega' \)). We note that \( o(1) \) on the left of (90) depends on \( \delta, \omega, M, \mu, |A^0| \), and \( \{\varepsilon_n\} \), while \( o(1) \) on the right depends on \( \omega' \), \( M \) and \( \hat{q} \). (88), (89), (90) imply

\[ \rho^{(o)}(Y_{36}^-) - o(1) \leq \mathbb{P}(|C_\varepsilon| \geq \omega(n)) \leq \rho^{(o)}(Y_{36}^+) + o(1). \]  

(91)

Finally, letting \( \delta \downarrow 0 \) we obtain as \( n \to +\infty \)

\[ \mathbb{P}(|C_\varepsilon| \geq \omega(n)) = \rho(Y) + o(1), \]  

(92)

where the remainder \( o(1) \) depends on \( M, A^0, \{h_{t,q}, (t,q) \in A^0\}, \mu, \{\varepsilon_n\} \) and function \( \omega \). Indeed the lower bound of (92) follows from \( \rho^{(o)}(Y_{36}^-) \geq \rho(Y) \) as \( \delta \downarrow 0 \). More precisely, given \( \tau > 0 \) we choose \( \delta_\varepsilon > 0 \) such that \( \delta \leq \delta_\varepsilon \) implies \( \rho(Y_{36}^-) \geq \rho(Y) - \tau \) and then letting \( n \to \infty \) we apply the left inequality of (91) to get

\[ \mathbb{P}(|C_\varepsilon| \geq \omega(n)) \geq \rho^{(o)}(Y_{36}^-) - \tau \geq \rho(Y) - 2\tau. \]

For the upper bound we push \( \rho^{(k)}(Y_{26}^+) \) arbitrarily close to \( \rho(Y) \) choosing large \( k \) and small \( \delta \). Indeed, given \( \tau > 0 \) we find large \( k_\tau > 0 \) such that \( \rho^{(k)}(Y) \leq \rho(Y) + \tau \). Next, we find small \( \delta_\tau \) such that \( \delta \leq \delta_\tau \) implies \( \rho^{(k)}(Y_{26}^+) \leq \rho^{(k)}(Y) + \tau \). For \( \delta \leq \delta_\tau \) and \( k > k_\tau \) we have

\[ \rho^{(k)}(Y_{26}^+) \leq \rho^{(k)}(\lambda^+) \leq \rho^{(k)}(Y) + \tau \leq \rho(Y) + 2\tau. \]  

(93)

Now letting \( n \to \infty \) we apply the right inequality of (91) to get

\[ \mathbb{P}(|C_\varepsilon| \geq \omega(n)) \leq \rho^{(o)}(Y_{26}^+) + \tau \leq \rho(Y) + 2\tau. \]

Finally, we note that (92) implies (81). It remains to prove the first and the last inequality of (90).

**Proof of (90).** We fix an order \( v_1 < v_2 < \cdots < v_n \) of elements of \( V \). Let \( m' = \sum_{(t,q) \in A^0} m_{t,q} \) be the number of sets in the collection \( \mathbb{D}_0 \). We can assume without loss of generality that \( \mathbb{D}_0 = \{D_1, \ldots, D_{m'} \} \) and \( |D_1| \leq |D_2| \leq \cdots \leq |D_{m'}| \).

**Upper bound (the last inequality of (90)).** Given \( v \in V \), define the list \( L_v \) of vertices using a BFS type exploration procedure. In the beginning all vertices are uncolored, all sets \( D_i \in \mathbb{D}_0 \) are unmarked, and \( L_v = \emptyset \). After a vertex is added to \( L_v \), the vertex is colored white. We add \( v \) to the list. Next we proceed recursively. We choose the oldest (with respect to inclusion to \( L_v \)) white vertex, say \( u \), from \( L_v \).
For \( i = 1, 2, \ldots, m' \) such that \( u \in D_i \) and \( D_i \) is not marked, we mark \( D_i \) (we say that \( D_i \) is marked by \( u \)) and add to \( L_v \) (in increasing order) all uncolored vertices of \( D_i \) that are connected to \( u \) by paths of edges labelled \( D_i \). We say that \( D_i \) brings these vertices to the list and attach label \( D_i \) to each of them. Afterwards we color \( u \) black. Vertices added to \( L_v \) in this step are called children of \( u \). We then chose the oldest white vertex from \( L_v \), add to \( L_v \) its children and color this vertex black etc. We stop when there are no more white vertices in \( L_v \) or there are no more unmarked sets \( D_i \) left. We denote \( L_v = \{ u_1, u_2, \ldots \} \), where elements are listed in the order of their inclusion to the list (\( u_i \) is older than \( u_j \) for \( i < j \) and \( u_1 = v \)). We denote \( L_{v,k} = \{ u_1, \ldots, u_k \} \) the set of \( k \) oldest vertices of \( L_v \). Note that \( L_v \) is a subset of \( C_v \). For any \( u_i \in L_v \) with \( i \geq 2 \) there is unique \( i^* \in [1, i) \) such that \( u_i \) is a child of \( u_{i^*} \) (equivalently, \( u_{i^*} \) is the parent of \( u_i \)). While constructing the list \( L_v \) we keep track of the sets \( D_{i_1}, D_{i_2}, \ldots \) that have been marked one after another (\( D_{i_s} \) was marked before \( D_{i_r} \) for \( s < r \)). For \( u_i \in L_v \) the number \( r = r(j) \) tells us that \( u_j \) was brought to the list by \( D_{i_r} \), the \( r \)th member of the sequence \( \{ D_{i_1}, D_{i_2}, \ldots \} =: D_v \). A set \( D_i \) marked by \( u \in L_v \) is called void if \( u \) has no neighbors in \( D_i \) linked to \( u \) by edges labeled \( D_i \) (in this case \( D_i \) brings no children to \( u \)). Note that any \( D_i \) is void with probability at most \( 1 - \hat{q} \). A set \( D_{i_1} \in D_v \) is called regular if \( U_{j=1}^{s-1} D_{i_1} \) and \( D_{i_1} \) intersect in a single point. Vertex \( v \) is called \( k \)-regular if \( |L_v| \geq k \) and each \( D_{i_1} \) is regular for \( j = 2, 3, \ldots, r(k) \). The set of \( k \)-regular vertices of \( G \) is denoted \( V_k = \{ v \in V : v \text{ is } k \text{-regular} \} \). Note that the events \( \{ |C_v| \geq k, v \in V_k \} \) and \( \{ |L_v| \geq k, v \in V_k \} \) are equivalent.

We observe that the number of vertices brought to the list \( L_v \) by a regular set \( D_{i_1} \in D_v \) has the same distribution as \( T(t, q) \). For a nonregular set this number may be smaller, since white vertices of a nonregular set \( D_{i_1} \) that have been colored in previous steps of the exploration cannot be brought to \( L_v \) by \( D_{i_1} \). Therefore as long as \( k \leq \omega'(n) \) a coupling of the exploration process with the branching process \( \mathcal{X}^+ \) shows that

\[
\mathbb{P}(|L_v| \geq k) \leq \mathbb{P}(|\mathcal{X}^+| \geq k).
\]

Next we show that

\[
\mathbb{P}(|L_v| \geq k, v \not\in V_k) \leq \hat{k}^2 M^2 n^{-1} + 2k^{-1},
\]

where \( \hat{k} := 2k/\hat{q} \). For \( v \) with \( |L_v| \geq k \) the event \( \{ v \not\in V_k \} \) implies that one or more nonregular sets have been marked during the exploration. Then either the first marked nonregular set \( D_{i_1} \) has index \( s \) satisfying \( s \leq \hat{k} \) (we denote this event \( A_k \)) or we have \( s > \hat{k} \). In the latter case there are at least \( \hat{k} - k + 2 \) void sets \( D_{i_1} \) with \( l \leq \hat{k} \) (this event we denote \( B_k \)). Indeed, on the event \( \hat{A}_k \cap \{ v \not\in V_k \} \cap \{ |L_v| \geq k \} \) we have that the index \( s \) of the first observed nonregular set \( D_{i_1} \) satisfies \( \hat{k} < s \leq r(k) \). But the inequality \( \hat{k} < r(k) \) implies that among the first \( \hat{k} \) sets from \( D_v \) there are less than \( k - 1 \) nonvoid ones as each nonvoid set contributes at least one new vertex to the list. Now (95) follows from the inequalities

\[
\mathbb{P}(B_k) \leq \mathbb{P}(Y < k - 1) \leq 2k^{-1},
\]

\[
\mathbb{P}(A_k) \leq \sum_{2 \leq s \leq \hat{k}} \mathbb{P}(D_{i_1} \text{ is non regular}) \leq (\hat{k} - 1)(\hat{k} - 1)M^2 n. \tag{97}
\]

Here, \( Y \sim \text{Bin}(\hat{k}, \hat{q}) \) and (96) follows by Chebyshev’s inequality. In (97) we estimated \( \mathbb{P}(D_{i_1} \text{ is non regular}) \leq (\hat{k} - 1)M^2/(n - (\hat{k} - 1)) \). Indeed, given \( H_{s-1} = \cup_{1 \leq l \leq s-1} D_{i_1} \), the size \( |D_{i_1}| = t \) and the event that \( D_{i_1} \) is marked by \( u_i \), the probability that \( D_{i_1} \) is non regular is the conditional probability
\[ p^* = \mathbb{P} \left( |H_{s-1} \cap D^*| \geq 2 | u_j \in D^* \right), \]  
where \( D^* \) is a random subset of size \( t \) of the set \( V \setminus \{ u_1, \ldots, u_{j-1} \} \). For \( |H_{s-1}| = h \) we have

\[ p^* = \frac{\mathbb{P}(|H_{s-1} \cap D^*| \geq 2, u_j \in D^*)}{\mathbb{P}(u_j \in D^*)} \leq \frac{(h-j)(t-1)}{n-j}. \]  
(98)

The last fraction upper bounds the probability that \( D^* \setminus \{ u_j \} \) of size \( t-1 \) intersects with \( H_{s-1} \setminus \{ u_1, \ldots, u_j \} \) of size \( h-j \). Note that \( (h-j)/(n-j) \leq h/n \) and \( h \leq (k-1)M \) and \( t \leq M \). Therefore the right side of (98) is at most \((k-1)M^2/n\). This shows (97) and we arrive to (95). It follows from (95) that for \( k = k_n \to +\infty \)

\[ k = o(\sqrt{n}) \Rightarrow \mathbb{P}(|L_v| \geq k, v \notin V_k) = o(1). \]  
(99)

We similarly show

\[ k = o(\sqrt{n}) \Rightarrow \mathbb{P}(|C_v| \geq k, v \notin V_k) = o(1). \]  
(100)

Namely, given \( v \) with \( |C_v| \geq k \), the event \( v \notin V_k \) implies that either \( |L_v| < k \) or \( |L_v| \geq k \) and \( D_j \) is nonregular for some \( 2 \leq j \leq r(k) \). The probability of the latter event is bounded by (99). Now we show that the remaining event \( C_k := \{ |C_v| \geq k, |L_v| < k \} \) has probability \( \mathbb{P}(C_k) = o(1) \). We observe 3 that event \( C_k \) implies that a nonregular set \( D_j \) has been marked by some \( u_r \in L_v \), where \( r < k \). Next we consider two alternatives: either the index \( s \) of the first marked nonregular set \( D_j \) satisfies \( s \leq \hat{k} \) (the probability of such event is upperbounded in (97) and is \( o(1) \)) or \( s > \hat{k} \). But the inequality \( s > \hat{k} \) implies that at most \( k-1 \) elements of the list \( L_v \) have marked at least \( \hat{k} - k + 2 \) void sets before a nonregular set was marked. The probability of such event is upperbounded by (96) and is \( o(1) \).

Finally, we observe that the events \( \{|C_v| \geq k, v \in V_k \} \) and \( \{|L_v| \geq k, v \in V_k \} \) are equal. Now (99), (100) combined with (94) imply

\[ \mathbb{P}(|C_v| \geq k) = \mathbb{P}(|L_v| \geq k) + o(1) \leq \mathbb{P}(|X^+| \geq k) + o(1). \]  
(101)

**Lower bound (the first inequality of (90)).** We modify a bit our exploration procedure. Given \( v \in V \), we construct the list \( L_v^* = \{ u_1, u_2, \ldots \} \) similarly as \( L_v \) above, but now each \( u_j \in L_v^* \) only accepts children brought by regular sets. Moreover, not every regular set is allowed to contribute to the list \( L_v^* \). Permission to contribute is granted at random. The construction of \( L_v^* \) is described in the algorithm \( \mathbb{A} \), which uses a slightly modified definition of regular set.

In the algorithm \( \mathbb{A} \) we use the following notation. \( D_1^*, D_2^*, \ldots \) denote the regular marked sets that were allowed to contribute to the list \( L_v^* \) one after another during the exploration; \( H_v^* = \{ u_1 \} \cup (\cup_{1 \leq i \leq s} D_i^* \), \( s \geq 1 \). We set \( H_0^* = \{ u_1 \} \). Furthermore, \( M^{(1)}, M^{(2)}, \ldots \) denote the numbers of sets marked by \( u_1, u_2, \ldots \in L_v^* \) respectively; \( M^{(0)}_{t,q} \) denotes the number of sets from \( D_{t,q} \) marked by \( u_j \) (so that \( M^0 = \sum_{(t,q)\in A^0} n_{t,q}^{(0)} \)). For each \( (t, q) \in A^0 \) we define the integer sequence \( m^{(0)}_{t,q} = m_{t,q}(6) - (j - 1)[3 \ln m], j \geq 1 \). For integers \( h, t \) we denote, see (84),

\[ p^*(h, t, j) = \frac{p_{t,\delta}^*}{p_t^*(h, t, j)}, \quad p_t^*(h, t, j) = \binom{n-h}{t-1} \binom{n-j+1}{t-1}^{-1}. \]  
(102)
Algorithm $\mathcal{A}$
1. for $(t, q) \in A^0$ set $\mathcal{D}_{t,q}^{(0)} := \mathcal{D}_{t,q}$;
2. $L^*_t \leftarrow v, \text{ color}[v] = \text{white}, j \leftarrow 0$;
3. while $L^*_t$ contains a white vertex:
4. $j \leftarrow j + 1, u_j \leftarrow$ the oldest white vertex of $L^*_t, \text{ color}[u_j] = \text{black},$
5. for $(t, q) \in A^0$:
6. select a subset $\mathcal{D}_{t,q}^{(j)} \subseteq \mathcal{D}_{t,q}^{(j-1)}$ of size $|\mathcal{D}_{t,q}^{(j)}| = n_{t,q}^{(j)}$,
7. examine each $D \in \mathcal{D}_{t,q}^{(j)}$; mark $D$ whenever $u_j \in D$, and if $D$ is regular then accept $D$ with probability $p^*(h, t, j)$, where $t = |D|$ and where $h = |H^*_t|$ refers to the union $H^*_t$ of regular sets accepted so far. Color white the children of $u_j$ brought by $D$ and insert them to the list $L^*_t$. Set $\mathcal{D}_{t,q}^{(j)}$ to be the family of unmarked sets from the collection $\mathcal{D}_{t,q}^{(j)}$. Stop if either (a) $|L^*_t| = k$ or (b) $M_{t,q}^{(j)} > [3 \ln m]$.

Let us show that for any sequence $k = k_n \rightarrow +\infty, k_n \leq n \ln^{-2}n$ we have
\[\mathbb{P}(|L^*_t| \geq k) \geq \mathbb{P}(|\mathcal{X}^-| \geq k) + o(1).\] More precisely, we show that there exist integer $n'_t$ depending on $\delta$ and $\{h_{t,q}, (t, q) \in A^0\}$ such that for $n > n'_t$ we have $\mathbb{P}(|L^*_t| \geq k) \geq \mathbb{P}(|\mathcal{X}^-| \geq k) + r_{st},$ where the sequence $r_{st} = o(1)$ may depend on $M,$ $\mu,$ $\{\varepsilon_n\}$ and $|A^0|$.

We firstly show that the acceptance probability $p^*(h, t, j)$ of step 7 is well defined, that is, for sufficiently large $n, m$ it is always less than 1. Indeed, while deciding whether to accept a regular set $D$ in step 7 we know that $|L^*_t| < k$. Hence, the number of sets marked so far is at most $\sum_{n \in L^*_t} M^{(j)} < k |A^0| 3 \ln m$ as each $M^{(j)}$ contribute at most $|A^0| 3 \ln m$. The number of marked regular accepted sets is even less. Hence $h \leq M k |A^0| 3 \ln m = O(n \ln^{-1}n)$. Here we used the fact that the layer sizes $|D^*_t|$ is finite and $m/n \rightarrow \mu \in (0, +\infty)$. Now for $j = O(n \ln^{-2}n)$ and $2 \leq t \leq M$ we have
\[p^*_1(h, t, j) = \frac{(n - O(n \ln^{-1}n))^{t-1}}{(n - O(n \ln^{-2}n))^t} = (1 + o(1)) \frac{t}{n} > (1 - \delta) \frac{t}{n} = p_{t, \delta}.\]

In the inequality above we use $n > n'_t$. A similar reasoning shows that each time we perform step 6 the collection $\mathcal{D}_{t,q}^{(j-1)}$ has more than $m_{t,q}(2\delta)$ members. Indeed, we have for $j < n \ln^{-2}n$
\[|\mathcal{D}_{t,q}^{(j)}| = |\mathcal{D}_{t,q}^{(j)}| - M_{t,q}^{(j)} \geq m_{t,q}^{(j)} - [3 \ln m] \geq m_{t,q}(\delta) - j[3 \ln m] \geq m_{t,q}(2\delta).\]

In the last inequality above we use $n > n'_t$.

We secondly show (see (105), (106) below) that the probability that algorithm $\mathcal{A}$ stops for the reason (b) is negligibly small. Introduce the event
\[E_r = \{M_{t,q}^{(j)} > 3 \ln m, \text{ for some } 1 \leq j \leq r \text{ and some } (t, q) \in A^0\}.\]

If the algorithm did not stop before $u_j$ started exploring its children in the layers of size $t$ and strength $q$, then $M_{t,q}^{(j)}$ has binomial distribution $\text{Bin} \left( m_{t,q}, t/(n - j + 1) \right)$. In this case we have for some constant $c$ depending on $\mu$ and $M$
\[\mathbb{P}(M_{t,q}^{(j)} > [3 \ln m]) \leq cm^{-2}.\]
To show (104) we couple $M_{t,q}^{(j)}$ with binomial random variable $M_j^* \sim \text{Bin}(m, t/(n - j + 1))$ so that 
$$
\mathbb{P}(M_{t,q}^{(j)} \leq M_j^*) = 1. \quad \text{Then we apply exponential Markov inequality } \mathbb{P}(M_j^* > x) \leq e^{-x} \mathbb{E}e^{M_j^*} \text{ with } x = [3 \ln m] \text{ and use the bound } 5 \mathbb{E}e^{M_j^*} \leq c, \text{ where } c \text{ depends on } \mu, M \text{ and the sequence } \{\varepsilon_n\}. \text{ If the algorithm stops before } u_j \text{ starts exploring its children in the layers of size } l \text{ and strength } q, \text{ then we set } M_{t,q}^{(j)} \equiv 0. \text{ In the latter case (104) is obvious. For } r \leq k \text{ we obtain from (104) by the union bound that } \mathbb{P}(E_r) \leq \mathbb{P}(E_k) \leq c|A^0|m^{-1} = O(m^{-1}).}
$$

(105)

By $i^*(k)$ and $r^*(k)$ we denote the positive integers such that $u_k \in L_v^*$ is a child of $u_{i^*(k)} \in L_v^*$ and $u_k$ is brought to the list by the set $D_{r^*(k)}^*$. Observe that $r^*(k) \leq M_1 + \cdots + M_{r^*(k)}$ and $i^*(k) < k$. Therefore (105) implies

$$
\mathbb{P}(\overline{E}_{r^*(k)}) = 1 - O(m^{-1}), \quad \mathbb{P}(|L_v^*| \geq k \mid \overline{E}_{r^*(k)}) = \mathbb{P}(|L_v^*| \geq k) + O(m^{-1}).
$$

(106)

Here, the second relation follows from the first one. The conditioning on $\overline{E}_{r^*(k)}$ means that the algorithm has not stopped for the reason (b).

We now are ready to prove (103). We consider the probability $\mathbb{P}(|L_v^*| \geq k \mid \overline{E}_{r^*(k)})$. We claim that as long as the algorithm does not stop for the reason (b), we have for each $1 \leq j \leq i^*(k)$, each $(t, q) \in A^0$ and each $D \in \mathbb{D}_{t,q}^{(j)}$ that the probability that $D$ is marked by $u_j$, $D$ is regular and it is allowed to contribute to $L_v^*$ is $p_{i,j}^\overline{\delta}$. To show this we examine the probability $p^*(h, t, j)$ of step 7. Note that $p^*_j(h, t, j)$ is the probability that given $H \subseteq W$ of size $|H| = h$ and $u_1, \ldots, u_j \in H$, a random subset $D \subseteq V \setminus \{u_1, \ldots, u_{j-1}\}$ of size $t$ intersect with $H$ and the intersection $D \cap H = \{u_j\}$ (i.e., $p^*_j(h, t, j)$ is the probability that $D$ is marked by $u_j$ and it is regular). The random acceptance of $D$ with probability $p^*(h, t, j)$ (in step 7) makes the final acceptance probability equal $p_{i,j}^\overline{\delta}$. Now we can write the total number of children of $u_j$, $1 \leq j \leq i^*(k)$, in the form

$$
\sum_{(t,q)\in A^0} \sum_{j=1}^{i^*(k)} \|_{t,q}^{(j)}(t,q)T_j^{(j)}(t,q).
$$

(107)

Here, $\|_{t,q}^{(j)}(t,q)$ is a Bernoulli random variable (independent of all the other random variables) with success probability

$$
p_{j,i}^\overline{\delta} := \mathbb{P}(D \text{ is marked by } u_j) = \frac{p_{i,j}^\overline{\delta}}{t/(n - j + 1)} = (1 - \delta)\frac{n - j + 1}{n}.
$$

(note that $p_{j,i}^\overline{\delta}$ is the conditional probability that $D \in \mathbb{D}_{t,q}^{(j)}$ is allowed to contribute to $L_v^*$ given that $D$ is marked by $u_i$). Furthermore, by $\eta_{t,q}^{(j)}$ we denote the random variable $M_{t,q}^{(j)}$ conditioned on the event $M_{t,q}^{(j)} \leq [3 \ln m]$. Let us compare the exploration process $L_v^*$ (conditioned on the event $\overline{E}_{r^*(k)}$) with the branching process $\mathcal{L}$, which produces an ordered list of particles $\{u_1, u_2, \ldots\}$ and where the offspring number

\[5\text{We have } \mathbb{E}e^{M_j^*} = \left(1 + (e - 1)\frac{t}{n-j+1}\right)^m \leq e^{\frac{m(e-1)}{n-j+1}} \leq e^{\frac{3\ln m}{n-j+1}} \leq e^{3\ln \frac{2}{\varepsilon}}, \text{ for } n \geq 5. \]
of \(\mu_j\) is defined by (107), but with \(P_{t,q}^{(i)}\) replaced by \(M_{t,q}^{(j)}\). Note that the total variation distance between their distributions \(\mathcal{L}(\eta_{t,q}^{(i)})\) and \(\mathcal{L}(M_{t,q}^{(j)})\)

\[
d_{tv}\left(\mathcal{L}(\eta_{t,q}^{(i)}), \mathcal{L}(M_{t,q}^{(j)})\right) \leq \mathbb{P}\left(M_{t,q}^{(j)} > 3 \ln m\right) \leq cm^2.
\]

Here, the second inequality is shown in (104) and the first one follows from the inequality\(^6\) \(|\mathbb{P}(A|B) - \mathbb{P}(A)| \leq \mathbb{P}(\overline{B})\), which holds for any events \(A, B\) with \(\mathbb{P}(B) > 0\). Hence, we have

\[
\mathbb{P}\left(|L_n^*| \geq k \mid \mathcal{E}_{P_t(k)}\right) = \mathbb{P}(|L| \geq k) + O(k/m^2).
\]  

(108)

Furthermore, we have \(\mathbb{P}(|L| \geq k) \geq \mathbb{P}(|\mathcal{Y}| \geq k\). Indeed, we can represent the offspring number of \(\mathcal{L}\) as

\[
\sum_{(t,q) \in A^2} \sum_{s=1}^{M_{t,q}^{(j)}} (t,q)T_s^{(j)}(t,q) = \sum_{(t,q) \in A^2} \sum_{s=1}^{M_{t,q}^{(j)}} T_s^{(j)}(t,q),
\]

where \(M_{t,q}^{(j)} \sim \text{Bin}(m_{t,q}^{(j)}, P_{t,q}^{(j)})\), and then couple \(M_{t,q}^{(j)}\) with \(N_{t,q}^{(j)}\) so that \(\mathbb{P}\left(M_{t,q}^{(j)} \geq N_{t,q}^{(j)}\right) = 1\). Now (106), (108) imply (103). Finally, (103) together with the simple inequality \(\mathbb{P}(|C_{i}| \geq k) \geq \mathbb{P}(L_n^* \geq k)\) shows

\[
\mathbb{P}(|C_{i}| \geq k) \geq \mathbb{P}(|\mathcal{Y}| \geq k) + o(1).
\]  

(109)

**Proof of (82).** We use the shorthand notation \(\iota_{y} := \iota_{|C_{i}| \geq \omega(n)}\). We have

\[
|B^\omega| = \sum_{y \in V} \iota_{y}, \quad \left(\frac{|B^\omega|}{2}\right) = \sum_{\{x,y\} \subset V} \iota_{x} \iota_{y}.
\]  

(110)

The first identity combined with (81) yield

\[
\mathbb{E}|B^\omega| = n\rho(\mathcal{Y}) + o(n).
\]  

(111)

For \(\rho(\mathcal{Y}) = 0\) this implies (82). For \(\rho(\mathcal{Y}) > 0\) we establish (82) by showing that \(|B^\omega|\) concentrates around its mean \(\mathbb{E}|B^\omega|\).

We first consider the special case of \(\omega = \omega_r\), where \(\omega_r(n) = \ln n\). Let \(\{x,y\} \subset V\) denote a pair of vertices selected uniformly at random. We show below that

\[
\mathbb{E}(\iota_{x} \iota_{y}) \leq \rho(\mathcal{Y}) \times \rho(\mathcal{Y}) + o(1),
\]  

(112)

where the remainder \(o(1)\) only depends on \(M, A^0\) and \(\{\epsilon_n\}\). (112) combined with (110), (111) imply \(\mathbb{E}|B^\omega|^2 \leq (\mathbb{E}|B^\omega|)^2 + o(n^2)\). From the latter inequality we conclude that \(\text{Var}|B^\omega| = o(n^2)\). Now Chebyshev’s inequality implies

\[
\forall \gamma > 0 \quad \mathbb{P}\left(\left|\mathbb{E}|B^\omega| - \mathbb{E}|B^\omega|\right| > \gamma n\right) \leq (\gamma n)^{-2}\text{Var}(|B^\omega|) = o(1).
\]  

(113)

Letting \(\gamma \downarrow 0\) we obtain (82). It remains to prove (112).

\(^6\)For \(\Delta := |\mathbb{P}(A|B) - \mathbb{P}(A)| \geq \mathbb{P}(A|B)\mathbb{P}(\overline{B}) \leq \mathbb{P}(\overline{B})\) for \(\Delta \leq 0\) we have \(|\Delta| = |\mathbb{P}(A) - \mathbb{P}(A|B)| \leq \mathbb{P}(A) - \mathbb{P}(A|B) \leq \mathbb{P}(\overline{B})\).
Proof of (112). We start with an outline. Denote for short $\bar{k} = \bar{o}(n)$. We select $y$ and perform exploration $L_y$ (as in the proof of the upper bound of (90)). We stop the exploration after the first $\bar{k}$ elements of the list $L_y$ are discovered. Let $D_y$ denote the collection of sets marked during this exploration and $H(y) = \cup_{D \in D_y} D$. We will show that $|H(y)| = O_P(\bar{k})$ (see (115) below). Next we select $x$. The event $L_{x|y} := \{ x \notin H(y) \}$ has probability $1 - O(\bar{k})/n = 1 - o(1)$ (see (116) below). We then consider the exploration $L_x$ (until the first $\bar{k}$ elements of the list $L_x$ are discovered) conditionally on the event $L_{x|y}$. Using the fact that $|H(y)| = O_P(\bar{k})$ we show that the sets marked during the exploration $L_x$ do not intersect with $H(y)$ with high probability (see (118)). Hence the event $\hat{L}_x := \{|L_x| \geq \bar{k}\}$ is asymptotically independent of $L_y := \{|L_y| \geq \bar{k}\}$. Finally, we establish (112) by approximating $P(\hat{L}_x)$ and $P(L_x|L_y) = P(L_x)(1 + o(1))$ by the survival probabilities of related branching processes. The rigorous argument below adds the details.

Recall that $V_k$ denotes the set of $k$-regular vertices. Denote the events

$$H_y = \{|H(y)| \leq M\bar{k}, \, |D_y| \leq \bar{k}\}, \quad \hat{L}_y = \hat{L}_x \cap \{ z \in V_{\bar{k}} \}, \quad z = x, y,$$

where $\bar{k} = 2\bar{k}/\dot{\gamma}$. (100) and the fact that events $\{|C_v| \geq \bar{k}, v \in V_{\bar{k}}\}$ and $\hat{L}_y$ are equal imply

$$E(\bar{L}_x) = E(\{L_x \cup \{x\} \}_{1 \leq |V_x|} \cup \{y \}) + o(1), \quad E(\{L_x \cup \{x\} \}_{1 \leq |V_x|} \cup \{y \}) = P(L_x \cap L_y). \quad (114)$$

Note that event $\hat{L}_y \cap \{|D_y| > \bar{k}\}$ implies that among the first $\bar{k}$ sets marked by $L_y$ less than $\bar{k} - 1$ are nonvoid. The probability of such event is $o(1)$, see (96). Since $|D_y| \leq \bar{k}$ implies $H_y$ we conclude that

$$P(L_x \cap H_y) = o(1). \quad (115)$$

Combining the latter bound with the obvious bound

$$P(L_x|H_y) \leq M\bar{k}/n = o(1) \quad (116)$$

we obtain

$$P(L_x \cap L_y) = P(L_x \cap L_y \cap H_y) + o(1) = P(L_x \cap L_y \cap H_y \cap \hat{L}_x) + o(1). \quad (117)$$

In the last step we used the inequalities

$$P(L_x \cap L_y \cap H_y \cap \hat{L}_x) \leq P(H_y \cap \hat{L}_x) \leq P(L_x|H_y) = o(1).$$

Let $D = \{D_{i_1}, D_{i_2}, \ldots \}$ denote the sets marked during the exploration $L_x$ ($D_{i_s}$ is marked before $D_{i_{s+1}}$). We call $D_{i_s}$ healthy whenever $D_{i_s} \cap H(\gamma) = \emptyset$. Exploration $L_x$ is called healthy if all marked sets $D_{i_s}$ are healthy (recall that we stop marking the sets after $L_x$ collects $\bar{k}$ elements). Introduce events

$$S_x = \{L_x \text{ is healthy} \}, \quad S_x^* = \{ \text{there is no nonhealthy } D_{i_s} \text{ with } s \leq \bar{k} \}.$$

Next we show that

$$P(L_x \cap L_y \cap H_y \cap \hat{L}_x) = P(L_x \cap L_y \cap H_y \cap \hat{L}_x \cap S_x) + o(1). \quad (118)$$
Given integer $0 < t \leq M$, let $D \subset V$ be a random set of size $|D| = t$. Assuming that $D$ and $H(y)$ are independent we estimate the conditional probability

$$
\mathbb{P}\left(D \cap H(y) \neq \emptyset | H(y), x \notin H(y), x \in D\right) \leq (t - 1)|H(y)|(n - 1)^{-1}.
$$

(119)

Now we consider the exploration $L_x$ conditionally, given the event $H_y \cap \mathcal{L}_{xly}$. The conditional probability that $D_{i_t}$ marked by $x$ is not healthy is at most $(M - 1)\hat{m}(n - 1)^{-1}$. Here, we applied (119) and used the fact that $x \notin H(y)$ implies $D_{i_t} \notin \mathcal{D}_y$ and therefore $D_{i_t}$ and $H(y)$ are (conditionally) independent. Furthermore, for $s = 1, 2, \ldots$, given the event that $D_{i_t}, \ldots, D_{i_s}$ are all healthy and that $D_{i_{s+1}}$ was marked by the $j$th element (where $j < \hat{k}$) of the list $L_s$, the probability that $D_{i_{s+1}}$ is not healthy is at most $(M - 1)\hat{m}(n - j)^{-1} \leq (M - 1)\hat{m}(n - \hat{k})^{-1}$. Here we used the fact that $u_j \notin H(y)$ implies $D_{i_{s+1}} \notin \mathcal{D}_y$.

By the union bound applied to $S_t^x = \bigcup_{1 \leq s \leq \hat{k}} \{D_{i_t}, \ldots, D_{i_{s+1}} \text{ are healthy and } D_{i_t} \text{ is not healthy}\}$, we have

$$
\mathbb{P}\{S_t^x | H_y, x \notin H(y)\} \leq \hat{k} \cdot (M - 1)\hat{m}(n - \hat{k})^{-1} = o(1).
$$

This bound implies

$$
\mathbb{P}(S_t^x \cap H_y \cap \mathcal{L}_{xly}) = \mathbb{P}(H_y \cap \mathcal{L}_{xly}) - o(1).
$$

(120)

Furthermore, on the event $S_t^x$, the exploration $L_x$ does not encounter $H(y)$ and therefore $L_x$ is determined solely by the sets $\mathcal{D}_x = \{D_{i_t}, D_{i_r}, \ldots\}$ (which are subsets of $V \setminus H(y)$). The same argument as that of (96), (97) above yields that the event $\mathcal{L}_t^x \cap (|\mathcal{D}_x| > \hat{k})$ has probability $o(1)$, that is, we have

$$
\mathbb{P}\{\mathcal{L}_t^x \cap (|\mathcal{D}_x| > \hat{k}) | S_t^x, \mathcal{L}_{xly}, H_y\} = o(1).
$$

Consequently,

$$
\mathbb{P}\{\mathcal{L}_t^x \cap (|\mathcal{D}_x| \leq \hat{k}) | S_t^x \cap \mathcal{L}_{xly} \cap H_y\} = \mathbb{P}\{\mathcal{L}_t^x \cap S_t^x \cap \mathcal{L}_{xly}, H_y\} - o(1).
$$

(121)

In view of the fact that the event $(|\mathcal{D}_x| \leq \hat{k}) \cap S_t^x$ implies $S_x$ and event $S_x$ implies $S_t^x$, we obtain from (121) that

$$
\mathbb{P}\{\mathcal{L}_t^x \cap S_x \cap \mathcal{L}_{xly} \cap H_y\} = \mathbb{P}\{\mathcal{L}_t^x \cap S_t^x \cap \mathcal{L}_{xly}, H_y\} - o(1).
$$

This relation together with (117), (120) imply (118).

In the last step of the proof of (112) we estimate

$$
\mathbb{P}\{\mathcal{L}_t^x \cap \mathcal{L}_t^y \cap H_y \cap \mathcal{L}_{xly} \cap S_x\} \leq \mathbb{P}\{\mathcal{L}_t^x \cap S_x \cap \mathcal{L}_y^y \cap H_y \cap \mathcal{L}_{xly}\} \mathbb{P}(\mathcal{L}_y^+)
$$

(122)

and

$$
\mathbb{P}(\mathcal{L}_y^+) \leq \rho(\mathcal{Y}) + o(1), \quad \mathbb{P}\{\mathcal{L}_t^x \cap S_x \cap \mathcal{L}_y^y \cap H_y \cap \mathcal{L}_{xly}\} \leq \rho(\mathcal{Y}) + o(1).
$$

(123)

The first bound of (123) follows from (89,93,94). The second one is obtained by a similar argument, but now we perform exploration $L_x$ in the subgraph of $G$ induced by the vertex set $V \setminus H(y)$. In particular, we set $N_{t,q}^+ \sim \text{Bin}(m_q, p_t^*)$ and $\hat{N}_{t,q}^+ \sim \text{Poi}(m_q p_t^*)$ in (85), (86). Here $p_t^* = t/(n - \hat{k} - M\hat{k})$ upperbounds the probability $\mathbb{P}(u_j \in D)$ that $u_j \in L_x = \{u_1, u_2, \ldots\}$ with $j < \hat{k}$ marks a “healthy” random set $D \subset V \setminus \{u_1, \ldots, u_{t-1}\} \cup H(y))$ of size $|D| = t$. Note that $2 < t \leq M$ implies $p_t^* = tn^{-1}(1 + o(1))$.

Relation (112) follows from (114), (117), (118), (122), and (123). We have shown (82) in the special case of $\omega = \overline{m}$, where $\overline{m}(n) = \ln n$. 
Next we prove (82) for general ω. To this aim we show that \(|B^ω| − |B^\overline{ω}|| = o_p(n). \) Let \(ω_1 = ω ∨ \overline{ω}\) and \(ω_2 = ω ∧ \overline{ω}\) so that \(|B^ω| − |B^\overline{ω}|| = |B^{ω_1}| − |B^{ω_2}|| ≥ 0. \) Now (111) implies

\[
E|B^ω| − |B^\overline{ω}|| = E(|B^{ω_1}| − |B^{ω_2}||) = o(n).
\]

Hence \(|B^ω| − |B^\overline{ω}|| = o_p(n).\)

**Proof of (83).** The upper bound \(P(\{|C| ≤ nρ(\mathcal{Y}) + \epsilon''_n n\}) ≥ 1 − o(1)\) follows from (82). Indeed, we can assume without loss of generality that \(\epsilon''_n\) from (82) satisfies \(\epsilon''_n n ≥ ln^2 n\). For \(o(n) = ln n\) we obtain from (75) for large \(n, m\) that

\[
P(\{|C| > nρ(\mathcal{Y}) + \epsilon''_n n\}) ≤ P\left(\max\{\omega(n), |B^ω|\} > nρ(\mathcal{Y}) + \epsilon''_n n\right)
\]

\[
= P(|B^ω| > nρ(\mathcal{Y}) + \epsilon''_n n) = o(1). \tag{124}
\]

For \(ρ(\mathcal{Y}) = 0\) relation (83) follows from (124). It remains to prove (83) for \(ρ(\mathcal{Y}) > 0\). To this aim we show the matching lower bound \(|C| ≥ ρ(\mathcal{Y}) n + o_p(n)\). Fix \((t, q) ∈ A^0\). Choose \(δ = δ_n > 0\) such that \(ρ(\mathcal{Y}_{δ^{-}}) > 0\). We select a subset \(D_{t, q}^{δ} \subseteq D_{t, q}\) of size \(|D_{t, q}^{δ}| = [δm_{t, q}]\) and color sets from \(D_{t, q}^{δ}\) blue. The collection \(D_{0}^{δ} = D_{0} \backslash D_{0}^{δ}_{t, q}\) is obtained from \(D_{0}\) by removal of the blue sets. Let \(G^{δ}\) (respectively \(G^*\)) be the overlay graph on the vertex set \(V\) defined by the collection of layers \(D_{t, q}^{δ}\) (respectively \(D_{0}^{δ}\)). We color edges of \(G^{δ}\) blue. We couple \(G^{δ}\) and \(G^*\) so that \(G = G^{δ} ∪ G^*\). Let \(ω(n) = n^{2/3}\) and let \(B^* \subseteq V\) be the set of vertices belonging to connected components of \(G^*\) having at least \(o(n)\) vertices. Clearly, there are at most \(n^{1/3}\) such components. Given a pair of such components \(C', C'' \subseteq V\), for any \(D ∈ D_{t, q}^{δ}\), the probability that \(C', C''\) are connected by a blue edge labeled \(D\) is at least

\[
p^* : = P\left(D \cap C', D \cap C'' \mid C', C''\right) . q ≥ 2n^{4/3} n^{-2}. \tag{125}
\]

Indeed, \(P\left(D \cap C', D \cap C'' \mid C', C''\right)\), the probability that a randomly selected pair of elements of \(D\) intersects with \(C'\) and \(C''\) simultaneously, is at least \(n^{2/3} n^{2/3} / \binom{n}{2}\). Furthermore, by Chernoff’s bound (see formula (2.6) of [30]), for any \(c > 0\) the probability that there are less than \(c \ln n\) blue edges between \(C'\) and \(C''\) is at most

\[
P\left\{X^* < c \ln n\right\} ≤ e^{-c \bar{c} n^{1/3}}. \tag{125}
\]

where the random variable \(X^*\) has binomial distribution \(Bin([δm_{t, q}], p^*).\) Furthermore, the constant \(c > 0\) in (125) depends on \(h_{t, q}\), constant \(c > 0\) and the sequence \(m_{t, q} / n → \mu.\) Next, by the union bound, the probability that there exists a pair of components connected by less than \(c \ln n\) blue edges is at most

\[
\left(\frac{ln^{1/3}}{2}\right) e^{-c \bar{c} n^{1/3}} = o(1), \tag{126}
\]

provided that \(\delta n^{1/3} ≥ ln^2 n.\)

We let \(δ = n^{-1/6}\) and apply (82) to the set \(B^*\) of vertices of \(G^*\). In view of (126) these vertices belong to the same connected component of \(G = G^{δ} ∪ G^*\) with high probability. Hence, \(|C| ≥ |B^*| ≥ nρ(\mathcal{Y}) + o_p(n).\)

In the next Lemma, we relax condition (74) by allowing a negligible fraction of layer types \((x_{n, j}, q_{n, i})\) to take their values outside \(A.\)
Lemma 9.2. Statements (82), (83) of Lemma 9.1 remain true if we replace condition (74) by the condition

$$\forall n \geq 1 \quad \forall 1 \leq i \leq m_n \quad x_{n,i} \leq M,$$

and condition (79) by the condition

$$\forall n \quad \max_{(i,q) \in A} \left| h_{i,q} - m_{i,q} / m \right| \leq \varepsilon_n.$$  \hfill (127)

Proof. Given \( n \) and \((\overline{x}_n, \overline{q}_n)\) we color a pair \((x_{n,i}, q_{n,i})\) red whenever \((x_{n,i}, q_{n,i}) \in A\). Otherwise we color \((x_{n,i}, q_{n,i})\) blue. The layers defined by red (blue) pairs are colored red (blue) as well. By (127), the number \( m_B \) of blue layers satisfies \( m_B = o(m) \). The overlay graphs defined by the families of red (blue) pairs are denoted by \( G_R (G_B) \). Then \( G = G_R \cup G_B \). Let \( C_R \) and \( C \) be the vertex sets of the largest components of \( G_R \) and \( G \).

We first show that (83) holds (under conditions of Lemma 9.2). We observe that results (81), (82), (83) of Lemma 9.1 apply to \( G_R \) because \( m_B = o(m) \). In particular, (83) remains true with \( C \) replaced by \( C_R \). This observation together with the simple inequality \( |C| \geq |C_R| \) yields the lower bound \( |C| \geq n \rho(\mathcal{Y}) + o_\rho(n) \). To prove the matching upper bound we apply the inequality

$$|B^k| \leq |B^k_R| + m_B M k,$$  \hfill (128)

where \( B^k_R \) is the set of vertices that belong to components of \( G_R \) of sizes at least \( k \). Let us show inequality (128). We observe that each \( v \in B^k \setminus B^k_R \) belongs to a component of \( G_R \) of size less than \( k \) and this component intersects with some blue layer. Furthermore, each blue layer (being of size at most \( M \)) may intersect with at most \( M \) distinct components. Hence each blue layer may contribute at most \( k M \) vertices to \( B^k \setminus B^k_R \). Consequently, blue layers altogether contribute at most \( m_B M k \) vertices. From (75) and (128) we obtain

$$|C| \leq \max \{ k, |B^k| \} \leq |B^k_R| + m_B M k + k.$$  \hfill (129)

Choosing \( k = k_n \rightarrow +\infty \) so that

$$m_B k_n = o(n)$$  \hfill (130)

we obtain \( |C| \leq |B^k_R| + o(n) \). Finally we apply (82) to \( |B^k_R| \) and obtain \( |C| \leq n \rho(\mathcal{Y}) + o_\rho(n) \).

Next we show that (82) holds (under conditions of Lemma 9.2). For \( k_n \) satisfying (130) the upper bound \( |B^k_n| \leq n \rho(\mathcal{Y}) + o_\rho(n) \) follows from (128) and relation (82) applied to \( B^k_R \). We extend this upper bound to arbitrary \( \omega \). Fix some \( \{ k_n \} \) satisfying (130). Then \( \omega_2(n) = o(n) \land k_n \) satisfies (130) and the upper bound applies to \( |B^\omega| \geq |B^\omega| \).

The lower bound \( |B^\omega| \geq n \rho(\mathcal{Y}) + o_\rho(n) \) makes sense when \( \rho(\mathcal{Y}) > 0 \). For \( \rho(\mathcal{Y}) > 0 \) the lower bound follows from the lower bound \( |C| \geq |C_R| \geq n \rho(\mathcal{Y}) + o_\rho(n) \) (see (83)) and the fact that \( |B^\omega| \geq |C| \) provided that \( o(n) = o(n) \) and \( |C| \geq 0.5 n \rho(\mathcal{Y}) \).

Proof of Theorem 3.4. The proof of Theorem 3.4 rests on Lemma 9.1. With the aid of truncation and discretization we reduce the general case to the case of a finite set of layer types considered in Lemma 9.1.

We start with some notation. Let \( M \geq 2 \) be a positive integer. Recall the notation \( x^{[M]}_n \) and \( \overline{x}^{[M]}_n \): given \( x_n = (x_{n,1}, \ldots, x_{n,m}) \) we denote \( x^{[M]}_n = (x^{[M]}_{n,1}, \ldots, x^{[M]}_{n,m}) \), where \( x^{[M]} = x^{[M]}_{\{1 \leq i \leq M\}} \). Furthermore, given \( 0 < \varepsilon < 1 \) and \( \overline{Q}_n = (Q_{n,1}, \ldots, Q_{n,m}) \) we denote \( \overline{Q}^\varepsilon_n = (Q^+_{n,1}, \ldots, Q^+_{n,m}) \) the \( \varepsilon \)-discretization as...
in (20). Let \( A^\pm_\epsilon \) denote the support of \((X^{[M]}_n, Q^\pm_n)\). That is, \( A^\pm_\epsilon \) is a subset of \([0, 1, \ldots, M] \times \{s_0, \ldots, s_r\}\), where \( s_0, \ldots, s_r \) are possible values of \( Q^\pm \), and \((t, q) \in A^\pm_\epsilon \) whenever \( h^\pm_{t,q,\epsilon} > 0 \). Here,

\[
h^\pm_{t,q,\epsilon} := \mathbb{P}\left( (X^{[M]}_n, Q^\pm_n) = (t, q) \right), \quad (t, q) \in [0, 1, \ldots, M] \times \{s_0, \ldots, s_r\}.
\]

We consider the overlay random graphs \( G^+_M = G(X^{[M]}_n, Q^+_n) \) and \( G^-_M = G(X^{[M]}_n, Q^-_n) \) defined by the sequences of layer types \((X^{[M]}_n, Q^+_n)\) and \((X^{[M]}_n, Q^-_n)\) respectively. We can couple \( G^+_M \) with \( G^-_M \), so that \( \mathbb{P}(G^+_M \subset G^-_M \subset G^+_M) = 1 \). Furthermore, we couple \( G^+_M \) and \( G = G(X^{[M]}_n, Q^-_n) \) so that \( \mathbb{P}(G^+_M \subset G) = 1 \). Let \( C^+_M \) and \( C^-_M \) denote the vertex sets of the largest components of \( G^+_M \) and \( G^-_M \) respectively. The couplings above imply the couplings

\[
\mathbb{P}\left(|C^+_M| \leq |C^-_M| \leq |C^+_M|\right) = 1 \quad \text{and} \quad \mathbb{P}\left(|C^-_M| \leq |C^+_M|\right) = 1. \tag{131}
\]

Denote \( \lambda_M = \mu x^{[M]} \), where \( x^{[M]} = \mathbb{E}(X|2 \leq X \leq M) \). Let \( T_{s}^{[M]} \) be a random variable with the distribution

\[
\mathbb{P}\left(T^{[M]} = \ell \right) = x^{[M]} \mathbb{E}\left(\text{Bin}(X-1, Q)(\ell)X|2 \leq X \leq M\right), \quad \ell = 0, 1, \ldots.
\]

Note that \( T^{[M]} \) has the same distribution as (77) above, but with \((X, Q)\) replaced by \((X^{[M]}_n, Q)\). Let \( Y_{M} \) be a G-W process with the offspring number \( Y_{M} \sim \text{CPoi}(\lambda_M, \mathcal{L}(T^{[M]})) \).

Now we are ready to prove the theorem. We will assume that \( \mathbb{P}(X \geq 2, Q > 0) > 0 \). (The case where \( \mathbb{P}(X \geq 2, Q > 0) = 0 \) is treated at the very end of the proof.) Let \( M \) be large enough so that \( \mathbb{P}(2 \leq X \leq M, Q > 0) > 0 \) and \( \epsilon > 0 \) is small enough so that \( A^+_\epsilon \) and \( A^-_\epsilon \) are both nonempty (this will ensure the condition \( A^0 \neq \emptyset \) of Lemma 9.1).

In the first step of the proof we show that as \( n \to +\infty \)

\[
|C^+_M| = n\rho(Y_{M}) + o_p(n). \tag{132}
\]

We note that the remainder term \( o_p(n) \) depends on \( M \). To prove (132) we apply Lemma 9.2 to overlay graphs \( G^+_M \) and \( G^-_M \) conditionally, given their layer types \((X^{[M]}_n, Q^+_n)\) and \((X^{[M]}_n, Q^-_n)\) respectively. We will show below that as \( n \to +\infty \)

\[
|C^+_M| = n\rho(Y_{M}^+) + o_p(n). \tag{133}
\]

where \( o_p(n) \) depends on \( M, \epsilon, A^\pm_\epsilon, \{h^\pm_{t,q,\epsilon}, (t, q) \in A^\pm_\epsilon\} \), the sequence \( m/n \to \mu \), and sequences \( \delta^\pm_n \downarrow 0 \) constructed below. Here \( Y^\pm_M \) is a G-W processes with offspring numbers \( Y^\pm_M \sim \text{CPoi}(\lambda_M, \mathcal{L}(T^{[M]})) \), where \( T^{[M]} \) is a random variable with the probability distribution

\[
\mathbb{P}\left(T^{[M]} = \ell \right) = x^{[M]} \mathbb{E}\left(\text{Bin}(X-1, Q)(\ell)X|2 \leq X \leq M\right), \quad \ell = 0, 1, \ldots.
\]

Letting \( \epsilon \downarrow 0 \) we obtain \( \rho(Y^+_M) \to \rho(Y_{M}) \). Now (133) together with the first identity of (131) yield (132). We explain the implications (131), (133) \(\Rightarrow (132) \) in more detail. Given \( \tau > 0 \) we choose small \( \epsilon \) so that \( |\rho(Y^+_M) - \rho(Y_{M})| < \tau \). Then we build sets \( A^\pm_\epsilon \), \( \{h^\pm_{t,q,\epsilon}, (t, q) \in A^\pm_\epsilon\} \) and sequences \( \delta^\pm_n \) (depending on \( \epsilon \) and \( M \), see the proof of (133) below) and apply (133). We obtain

\[
|C^+_M| - n\rho(Y_{M}) \leq \tau n + |C^+_M| - n\rho(Y^+_M) \leq \tau n + o_p(n).
\]

This yields (132): \( \forall \tau > 0 \) we have \( \mathbb{P}\left(|C^+_M| - n\rho(Y_{M}) > 2\tau n\right) = o(1) \) as \( n \to +\infty \).
Let us show (133). We only consider $C_{[M]}^+$. For $(t, q) \in A_t^+$ denote

$$m_{t,q} = \# \left\{ i : (X_{n,i}^{[M]}, Q_{n,i}^+) = (t, q) \right\}.$$ 

Let $A_n(\delta)$ denote the event that $\max_{(t, q) \in A_t^+} |h_{t,q,\epsilon} - m_{t,q}/m| \leq \delta$. We claim that there exists a positive sequence $\delta_n^+ \downarrow 0$ such that

$$\mathbb{P}(A_n(\delta_n^+)) = 1 - o(1) \quad \text{as} \quad n, m \to +\infty. \quad (134)$$

Indeed, for each $(t, q) \in A_t^+$ we have

$$\mathbb{E}(m_{t,q}/m) = \mathbb{P}\left((X_{n,\pi}^{[M]}, Q_{n,\pi}^+) = (t, q)\right) - \mathbb{P}\left((X_{n,\pi}^{[M]}, Q_{n,\pi}^+) = (t, q)\right) = o(1). \quad (135)$$

In the very last step we use the fact that our assumption $P_\mu \xrightarrow{w} P$ implies $\mathcal{L}\left((X_{n,\pi}^{[M]}, Q_{n,\pi}^+)\right) \xrightarrow{w} \mathcal{L}\left((X^{[M]}, Q^+)\right)$. Noting that $m_{t,q}$ is a sum of independent Bernoulli random variables (some of them may be degenerate) we have, by Chebyshev’s inequality, for any $\delta > 0$

$$\mathbb{P}\left((m_{t,q} - \mathbb{E}m_{t,q}) > m\delta \right) \leq \frac{\text{Var}(m_{t,q})}{m^2\delta^2} \leq \frac{1}{m\delta^2}. \quad (136)$$

Combining (135) and (136) we obtain (134). Now we are ready to derive (133) from Lemma 9.2. We apply (83) of Lemma 9.2 to $G_{[M]}^+$ conditionally, given the event $A_n(\delta_n^+)$. By (83) there exists a sequence $\epsilon_n'' \downarrow 0$ (depending on $M, A_t^+, h_{t,q,\epsilon}, (t, q) \in A_t^+$) and the sequences $\delta_n^+ \downarrow 0$ and $m/n \to \mu$ such that

$$\mathbb{P}\left(|C_{[M]}^+| - n\rho(\gamma_{[M]}^+)| > \epsilon_n'' n \mid A_n(\delta_n^+) \right) = o(1).$$

Combining this bound with (134) we obtain (133).

In the second step of the proof we let $M \to +\infty$. Denote $\lambda = \mu x_s$, where $x_s = \mathbb{E}(X\mathbb{1}_{X \geq 2})$, and let $T_s$ be a random variable with the distribution

$$\mathbb{P}(T_s = \ell) = x_s^{-1}\mathbb{E}\left(\text{Bin}^+(X - 1, Q)(\ell)X\mathbb{1}_{[2 \leq X]}\right), \quad \ell = 0, 1, \ldots.$$ 

Note that $T_s$ has the same distribution as (77) above, but now we drop the restriction on $(X, Q)$ of having a finite support. Let $Y$ be a G-W process with the offspring number $Y \sim \text{CPOi}(\lambda, \mathcal{L}(T_s))$. We mention that the equality of distributions (78) extends to the general setup (where we drop the restriction on $(X, Q)$ of having a finite support). Consequently, the respective survival probabilities $\rho(Y)$ and $\rho(Y_{[2, s]})$ are the same.

To prove the first relation of Theorem 3.4 we show that

$$|C| = n\rho(Y) + o_P(n).$$

We have, by continuity, that $\rho(Y_M) \to \rho(Y)$ as $M \to +\infty$. Now the second identity of (131) together with (132) yield the lower bound $|C| \geq n\rho(Y) + o_P(n)$. Indeed, given $\tau > 0$ we choose large $M$ such that $|\rho(Y_M) - \rho(Y)| < \tau$. Then we apply (131), (132) to get $|C| \geq |C_{[M]}| \geq (\rho(Y) - \tau)n + o_P(n)$.
To show the matching upper bound $|C| \leq np(\mathcal{Y}) + o_P(n)$ we use the inequalities

$$|C| \leq \max\{k, |B^k|\} \leq \left|B^k_{[M]}\right| + k \sum_{1 \leq i \leq m} X_{n,i} \mathbb{I}_{\{X_{n,i} > M\}} + k. \tag{137}$$

Here, $B^k$ (respectively $B^k_{[M]}$) is the set of vertices of $G$ (respectively $G_{[M]}$) that belong to components of size at least $k$. The first inequality of (137) is obvious. The second one is obtained in the same way as (128), (129) above.

Now we upper bound $|B^k_{[M]}|$. An inspection of the proof of (82) in Lemmas 9.1 and 9.2 shows that (82) remains true if we replace $B^m$ and $\rho(\mathcal{Y})$ by $B^k$ and $\rho^{(k)}(\mathcal{Y})$, respectively, where integer $k \geq 1$ is fixed. (Note that sequences $\epsilon_n \downarrow 0$, $\epsilon'_n \downarrow 0$ of (82) now depends on $k$ (instead of $\omega$). Using this fact and proceeding as in the proof of (132) above we show that as $n \to +\infty$

$$|B^k_{[M]}| = np^{(k)}(\mathcal{Y}_M) + o_P(n). \tag{138}$$

Note that the remainder term $o_P(n)$ depends on $k$ and $M$.

Next we upper bound the sum $\sum (\cdots)$ on the right of (137). We observe that as $M \to \infty$

$$Z_{M,n} := m^{-1} \sum_{1 \leq i \leq m} X_{n,i} \mathbb{I}_{\{X_{n,i} > M\}} \mathbb{P} \to 0 \tag{139}$$

uniformly in $n$. Indeed, our conditions $P_n \overset{w}{\to} P$ and $(P_n)_{10} \to (P)_{10}$ imply the uniform integrability of the sequence $\{X_{n,i}, n \geq 1\}.$ That is, we have

$$\varphi_M := \sup_n \mathbb{E} (X_{n,i} \mathbb{I}_{\{X_{n,i} > M\}}) \to 0 \quad \text{as} \quad M \to +\infty.$$ 

Then for any $\tau > 0$ and $n$ we have, by Markov’s inequality, that

$$\mathbb{P} \left( Z_{M,n} > \tau \right) \leq \tau^{-1} \mathbb{E} Z_{M,n} = \tau^{-1} \mathbb{E} \left( X_{n,i} \mathbb{I}_{\{X_{n,i} > M\}} \right) \leq \tau^{-1} \varphi_M.$$ 

Hence (139) holds uniformly in $n$.

Finally, combining (137), (138) and invoking the simple inequality $\rho^{(k)}(\mathcal{Y}_M) \leq \rho^{(k)}(\mathcal{Y})$ we obtain for any $k, M$ as $n, m \to +\infty$

$$|C| \leq np^{(k)}(\mathcal{Y}) + kmZ_{M,n} + k + o_P(n),$$

where $o_P(n)$ depends on $k$ and $M$. Now, by choosing large $k$, we can make $\rho^{(k)}(\mathcal{Y})$ arbitrarily close to $\rho(\mathcal{Y})$. Then, by choosing large $M$, we can make $kZ_{M,n}$ arbitrarily close to zero whp. In this way we obtain the desired upper bound $|C| \leq np(\mathcal{Y}) + o_P(n)$. For readers convenience we explain these steps in more detail. From the fact that $m/n \to \mu$ we conclude that the sequence $\{m/n\}$ is bounded, that is, $m/n < C_\star$ for some $C_\star > 0$ and all $n$ (recall that $m = m_n$). Given $\tau > 0$ we choose (sufficiently large) $k$ such that $\rho^{(k)}(\mathcal{Y}) \leq \rho(\mathcal{Y}) + \tau$. Then we choose (sufficiently large) $M$ such that $\varphi_M < \tau^2/(C_\star k)$. Now (137), (138) yields

$$|C| \leq np(\mathcal{Y}) + \tau n + kmZ_{M,n} + k + nr_{k,M,n},$$
where \( n r_{k,M,n} \) stands for the remainder \( o_P(n) \) in (138). In particular, \( r_{k,M,n} = o_P(1) \) as \( n \to +\infty \). Consequently,

\[
\mathbb{P}(|C| > n\rho(Y) + 4\tau n) \leq \mathbb{P}(kmZ_{M,n} > \tau n) + \mathbb{P}(k > \tau n) + \mathbb{P}(r_{k,M,n} > \tau).
\]

The first probability on the right

\[
\mathbb{P}(kmZ_{M,n} > \tau n) \leq (km/\tau n)\mathbb{E}Z_{M,n} \leq (km/\tau n)\rho_M \leq \tau,
\]

by Markov’s inequality. Furthermore, \( \mathbb{P}(k > \tau n) + \mathbb{P}(r_{k,M,n} > \tau) = o(1) \) as \( n \to +\infty \). Hence \( \mathbb{P}(|C| > n\rho(Y) + 4\tau n) \leq \tau + o(1) \). The latter inequality yields the bound \( |C| \leq n\rho(Y) + o_P(n) \).

It remains to show that \( N_2(G_{(n)}) = o_P(n) \). We write for short \( N_1 = N_1(G_{(n)}) \). For \( \rho = 0 \) we have \( N_2 \leq N_1 = o_P(n) \). For \( \rho > 0 \) we use the simple inequality \( N_1 + N_2 \leq |B^k| + 2k \) \( \forall k = 2, 3, \ldots \). From (138) and the second inequality of (137) we obtain \( N_1 + N_2 \leq n\rho^{(k)}(Y) + kmZ_{M,n} + 2k + o_P(n) \). Hence \( N_1 + N_2 \leq n\rho + o_P(n) \). Now \( N_2 = o_P(n) \) follows from the relation \( N_1 \geq n\rho + o_P(n) \) shown above (recall that \( N_1 = |C| \)).

Now we consider the case where \( \mathbb{P}(X \geq 2, Q > 0) = 0 \). In this case the distribution \( f^+ = \text{CPoi}(\mu(P)_{10}, \text{Bin}^i_0(P)) \) is degenerate, \( f^+(0) = 1 \), and the theorem claims that \( N_1(G_{(n)}) = o_P(n) \), for \( i = 1, 2 \). To prove this claim we show that \( \mathbb{E}[N_1(G_{(n)})] = o(n) \). Note that the asymptotic degree distribution \( f = \text{CPoi}(\mu(P)_{10}, \text{Bin}_0(P)) \) defined by Theorem 3.1 is degenerate as well. In particular, we have \( f^{(n)}(0) \to f(0) = 1 \). Consequently, we have

\[
\mathbb{E}[N_1(G_{(n)})] \leq \mathbb{E}\left( \sum_{i \in V} \mathbb{1}_{[\deg_{G_{(n)}}(i) \geq 1]} \right) = \sum_{i \in V} \mathbb{P}(\deg_{G_{(n)}}(i) \geq 1) = n(1 - f^{(n)}(0)) = o(n).
\]

\[\blacksquare\]

10 | PERCOLATION MODELS

Proof of Theorem 3.5. The site-percolated graph \( \tilde{G} \) is an instance of the overlay graph model (1) with \( \tilde{n} = |S| \) nodes and \( m \) layers \( \tilde{G}_1, \ldots, \tilde{G}_m \) where \( \tilde{G}_k \) is the subgraph of \( G_k \) induced by vertices belonging to \( S_k \) and \( G_k = \tilde{G}_k \). The layer types \( (X_k, Q_k) \) in the site-percolated model are mutually independent, and \( \mathcal{L}(X_k | X_k = x_k) \) is hypergeometric with probability mass function

\[
\text{Hyp}(n, \tilde{n}, x_k | t) = \frac{\binom{\tilde{n}}{t} \binom{n-\tilde{n}}{x_k-t}}{\binom{n}{x_k}}.
\]

We consider the sequence of site-percolated graphs \( \tilde{G}_{(n)} \) defined by a sequence of overlay graphs \( G_{(n)} \) and sets \( (S_n) \). For each \( n \), the site-percolated graph \( \tilde{G}_{(n)} \) is an instance of the overlay model with \( |S_n| = \tilde{n} = \tilde{n}_n \) nodes, \( m = m_n \) layers, and averaged layer type distribution

\[
\tilde{P}_n(A) = \int (\text{Hyp}(n, \tilde{n}, x) \times \delta_q(A)) P_n(dx, dq).
\]
We claim that \( \hat{P}_n \xrightarrow{w} \hat{P} \) as \( n \to +\infty \). In the proof of the claim we use the following bounds

\[
    d_{tv}(\text{Hyp}(n, \bar{n}, x), \text{Bin} \left( x, \frac{n}{\bar{n}} \right)) \leq 4 \frac{x}{n}, \quad d_{tv}(\text{Bin} \left( x, \frac{n}{\bar{n}} \right), \text{Bin}(x, \theta)) \leq \frac{\bar{n}}{n} - \theta \mid x.
\]

The first bound is shown in [20, Theorem 4]. The second one is obtained by coupling of coin flips. Recall that \((X_{n,\pi}, Q_{n,\pi})\) is a bivariate random variable with the distribution \( P_n \). Let \( S^n_\pi \) be a random subset of \( \{1, \ldots, n\} = V \) of size \( \bar{n} \) independent of \((X_{n,\pi}, Q_{n,\pi})\). Then the random variable \( \hat{X}_{n,\pi} = |\{1, \ldots, X_{n,\pi}\} \cap S^n_\pi| \) has the mixed hypergeometric distribution \( \mathbb{P}(\hat{X}_{n,\pi} = t) = \mathbb{E}(\text{Hyp}(n, \bar{n}, X_{n,\pi})(t)) \) and the random vector \((\hat{X}_{n,\pi}, Q_{n,\pi})\) has the distribution \( \hat{P}_n \). Let \((\hat{X}, Q)\) be a random variable with the distribution \( \hat{P} \). To prove the weak convergence \( \hat{P}_n \xrightarrow{w} \hat{P} \) we show for every \((t, s) \in \mathbb{Z}_+ \times [0, 1]\) that

\[
    \Delta := \hat{P}_n(t \times [0, s]) - \hat{P}(t \times [0, s]) = o(1).
\]

We split

\[
    \Delta = \mathbb{E} \left( \text{Hyp}(n, \bar{n}, X_{n,\pi})(t) \mathbb{I}_{Q_{n,\pi} \leq s l} \right) - \mathbb{E} \left( \text{Bin}(X, \theta)(t) \mathbb{I}_{Q \leq s l} \right) = \Delta_1 + \Delta_2.
\]

where

\[
    \Delta_1 = \mathbb{E} \left( \text{Hyp}(n, \bar{n}, X_{n,\pi})(t) \mathbb{I}_{Q_{n,\pi} \leq s l} \right) - \mathbb{E} \left( \text{Bin}(X_{n,\pi}, \theta)(t) \mathbb{I}_{Q_{n,\pi} \leq s l} \right)
\]

and

\[
    \Delta_2 = \mathbb{E} \left( \text{Bin}(X_{n,\pi}, \theta)(t) \mathbb{I}_{Q_{n,\pi} \leq s l} \right) - \mathbb{E} \left( \text{Bin}(X, \theta)(t) \mathbb{I}_{Q \leq s l} \right).
\]

By (140), we have \( |\Delta_1| \leq \left( \frac{4}{n} + \theta \right) \mathbb{E} X_{n,\pi} \). Our assumption \((P_{n,10}) \xrightarrow{} (P_{10})\) implies \( \limsup_n \mathbb{E} X_{n,\pi} < \infty \). Hence \( \Delta_1 = o(1) \). Furthermore, the assumption \( P_n \xrightarrow{w} P \) implies \( \Delta_2 = o(1) \). We obtain \( \Delta = o(1) \) thus proving the claim.

Next we observe that \((\hat{P}_{n,10}) \xrightarrow{} (P_{10})\) and \( m \xrightarrow{} \bar{\mu} = \theta^{-1} \mu \). Theorem 3.5(i)–(ii) now follow by applying Theorems 3.1 and 3.4 to \( \hat{G}_{(n)} \) and noting that \( \bar{\mu}(\hat{P}_{10}) = \mu(P_{10}) \).

Now assume that \((P_{n,rs}) \xrightarrow{} (P_{rs}) \in (0, \infty) \) for \( rs = 21, 32, 33 \). A direct computation shows that \( \hat{P}_{rs} = \theta^r(P_{rs}) \). Theorem 3.5(iii) now follows by applying Theorem 3.2 to conclude that the clustering coefficient \( \tau(\hat{G}_{(n)}) \) converges to

\[
    \frac{(\hat{P}_{10})_{12}}{(\hat{P}_{10})_{21} + \bar{\mu}(P_{10})_{21}} = \frac{(P_{10})_{12}}{(P_{10})_{21} + \mu(P_{10})_{21}} = \tau.
\]

Theorem 3.5(iv) follows similarly from Theorem 3.3.

**Proof of Theorem 3.6 for the layerwise bond-percolated graph \( \tilde{G}_{(n)} \)** The graph \( \tilde{G}_{(n)} \) is an instance of the overlay model with \( n \) nodes and \( m \) layers \( \tilde{G}_{n,1}, \ldots, \tilde{G}_{n,m} \) where \( \tilde{G}_{n,k} \) has size \( X_{n,k} \) and strength \( \theta Q_{n,k} \). The layers \( \tilde{G}_{n,k}, X_{n,k}, \theta Q_{n,k} \) are mutually independent, with averaged layer type distribution

\[
    \tilde{P}_n(A) = \int (\delta_x \times \delta_{\theta q})(A) P_n(dx, dq)
\]

converging according to \( \tilde{P}_n \xrightarrow{w} \tilde{P} \) and \( \tilde{P}_{10} \to (\tilde{P})_{10} \). Furthermore, a direct computation shows that \( \tilde{P}_{rs} = \theta^r(P_{rs}) \). Statements (i)–(ii) of Theorem 3.6 now follow by Theorems 3.1 and 3.4, and noting that \( \tilde{P}_{10} = (P)_{10} \). Statements (iii)–(iv) follow analogously by Theorems 3.2 and 3.3.
Proof of Theorem 3.6 for overlay bond percolated graph $\hat{G}_{(n)}$. In the proof we use a coupling argument. We will utilize the fact that the overlay bond-percolated graph does not differ much from the layerwise bond-percolated graph $\tilde{G}_{(n)}$, for which the theorem has already been proved. The conditional distribution of $\tilde{G}_{(n)}$ given the layers $(G_{n,k}, X_{n,k}, Q_{n,k})$ is an inhomogeneous Bernoulli graph on $\{1, \ldots, n\}$ where each node pair $ij$ is linked with probability $\hat{p}_{ij} = \theta(M_{ij} \wedge 1)$ where $M_{ij} = \sum_k I(E(G_{n,k}) \ni ij)$ is the number of layers linking a node pair $ij$. The corresponding conditional distribution of $\tilde{G}_{(n)}$ is a similar inhomogeneous Bernoulli graph with link probabilities $\hat{p}_{ij} = 1 - (1 - \theta)^{M_{ij}}$. Because $\hat{p}_{ij} \leq \hat{p}_{ij}$, this suggests the following coupling construction:

(i) Sample the layers $(G_{n,k}, X_{n,k}, Q_{n,k}), k = 1, \ldots, m$.

(ii) Sample independent inhomogeneous Bernoulli graphs $\tilde{H}$ and $H^*$ with link probabilities $\tilde{p}_{ij}$ and $p^*_{ij} = \tilde{p}_{ij}$ with the convention $0/0 = 1$. Note that $p^*_{ij} \geq \theta$.

(iii) Define $\hat{G}_{(n)} = G_{(n)} \cap \tilde{H}$ and $\tilde{G}_{(n)} = G_{(n)} \cap \tilde{H}$ with $G_{(n)}$ defined by (1) and $\tilde{H} = \tilde{H} \cap H^*$. Note the identity $\hat{G}_{(n)} = \tilde{G}_{(n)} \cap H^*$.

Then $(\hat{G}_{(n)}, \tilde{G}_{(n)}, G_{(n)})$ constitutes a coupling of the overlay bond-percolated, layerwise bond-percolated, and nonpercolated graphs such that

$$\hat{G}_{(n)} \subset \tilde{G}_{(n)} \subset G_{(n)} \quad \text{almost surely.} \quad (142)$$

Proof of Theorem 3.6(i). Let us denote by $\hat{d}_n = \deg \hat{G}_{(n)}(i)$ and $\tilde{d}_n = \deg \tilde{G}_{(n)}(i)$ the degrees of node $i$ in the overlay bond-percolated and layerwise bond-percolated graph, respectively. By the coupling (142), we have $\hat{d}_n = \tilde{d}_n$ on the event $M_i \leq 1$, where $M_i = \max_{j \neq i} M_{ij}$. Hence $d_{nt}(\mathcal{L}(\tilde{d}_n), \mathcal{L}(\tilde{d}_n)) \leq \mathbb{P}(M_i > 1)$. The union bound implies that

$$\mathbb{P}(M_{ij} > 1) \leq \sum_{k \neq \ell} \mathbb{P}(E(G_{n,k}) \ni ij) \mathbb{P}(E(G_{n,\ell}) \ni ij) \leq \left( \sum_k \mathbb{P}(E(G_{n,k}) \ni ij) \right)^2.$$  

By noting that $\mathbb{P}(E(G_{n,k}) \ni ij) = \mathbb{P}(X_{n,k} > 0)_{ij} Q_{n,k}$, we conclude that

$$\mathbb{P}(M_{ij} > 1) \leq (m(n)^{-1}(P_{n,21})^2, \quad (143)$$

Another union bound shows that $\mathbb{P}(M_i > 1) \leq \sum_{j \neq i} \mathbb{P}(M_{ij} > 1)$ and hence

$$d_{nt}(\mathcal{L}(\tilde{d}_n), \mathcal{L}(\tilde{d}_n)) \leq (m/n)^2 (P_{n,21})^2 (n - 1)^{-1}. \quad (144)$$

In the particular case, where the layer sizes are bounded, that is, there exists constant $M > 0$ such that

$$\mathbb{P}(X_{n,k} \leq M) = 1 \quad \forall \ n = 1, 2, \ldots \quad \text{and} \quad k = 1, \ldots, m_n,$$  

the right side of (144) is $o(1)$ as $n \to +\infty$. Now (144) together with the weak convergence (shown above) $\mathcal{L}(\tilde{d}_n) \rightarrow \text{CPoi}(\mu(P)_{10}, \text{Bin}_{10}(\hat{P}))$ yields the weak convergence $\mathcal{L}(\tilde{d}_n) \rightarrow \text{CPoi}(\mu(P)_{10}, \text{Bin}_{10}(\tilde{P}))$.

Finally, to treated the general case we revoke the boundedness condition (145) in the same way as in the proof of Theorem 3.1. The proof of Theorem 3.6(i) is complete.
Proof of Theorem 3.6(iii). By the fact that random overlay graph $G_n$ is independent of the percolating graph $H$, see (12), for any distinct nodes $i, j, k$, we have
\[
\Pr\left(\hat{C}_{(n)}(ij), \hat{G}_{(n)}(ik), \hat{G}_{(n)}(jk)\right) = \theta^3 \Pr\left(G_{(n)}(ij), G_{(n)}(ik), G_{(n)}(jk)\right),
\]
\[
\Pr\left(\hat{C}_{(n)}(ij), \hat{G}_{(n)}(ik)\right) = \theta^2 \Pr\left( (G_{(n)}(ij), G_{(n)}(ik)) \right).
\]
Hence $\tau(\hat{G}_{(n)}) = \theta \tau(G_{(n)})$ and the claim follows by applying Theorem 3.2 to the nonpercolated model.

Proof of Theorem 3.6(iv). The proof is similar to that of Theorem 3.3, but it is a bit more technical. For reader’s convenience we give it in the Appendix (Section A) below.

Proof of Theorem 3.6(ii). Let $\tilde{N}_1$ and $\tilde{N}_2$ (respectively, $\tilde{N}_1$ and $\tilde{N}_2$) denote the number of vertices of the largest and second largest component of $\hat{G}_{(n)}$ (respectively $\hat{G}_{(n)}$). Let $\tilde{\rho} = \rho^{\hat{\lambda}}$ denote the survival probability of the Galton-Watson branching process with the offspring distribution $\hat{f}^+$.

We observe that coupling (142) yields a coupling of $\tilde{N}_1$ and $\tilde{N}_1$ such that $\Pr(\tilde{N}_1 \leq \tilde{N}_1) = 1$. Furthermore, an application of Theorem 3.4 to the sequence of overlay graphs $\hat{G}_{(n)}$ yields the approximation $\tilde{N}_1 = \tilde{\rho} n + o_P(n)$. These two facts taken together imply the upper bound $\tilde{N}_1 \leq \tilde{\rho} n + o_P(n)$. To show the matching lower bound $\tilde{N}_1 \geq n\tilde{\rho} + o_P(n)$ and the bound $\tilde{N}_2 = o_P(n)$ we use the same argument as that of the proof of Theorem 3.4. The only place where we need a minor modification of the argument is the proof of Lemma 9.1. In what follows we review the proof of Lemma 9.1 and pinpoint the changes needed to be made.

At this point we need some notation. Let $\hat{C}, \hat{C}_v, \hat{B}^k$ and $\hat{C}, \hat{C}_v, \hat{B}^k$ denote the largest component, the component containing vertex $v$ and the set of vertices belonging to components of size at least $k$ in $\hat{G}_{(n)}$ and $\hat{G}_{(n)}$ respectively. Recall the notation $\lambda = \mu x_*$ and $x_* = \mathbb{E}(X_{\|X\|\geq 2})$, and let $\tilde{T}_*$ be a random variable with the distribution
\[
\Pr(\tilde{T}_* = \ell) = \chi X^{-1} \mathbb{E}\left( \text{Bin}^+(X - 1, \theta Q)(\ell)X_{\|X\|\geq 2}\right), \quad \ell = 0, 1, \ldots ,
\]
which is obtained from (77) by attaching the factor $\theta$ to $Q$. We note the equality of distributions $\hat{f}^+ = \text{CPOi}(\hat{\lambda}, \mathcal{L}(\tilde{T}_*))$, which is shown by the same argument as (30) above. In particular, $\tilde{\rho}$ is the survival probability of the Galton-Watson branching process with the offspring distribution $\text{CPOi}(\hat{\lambda}, \mathcal{L}(\tilde{T}_*))$.

We claim that the results (81), (82), (83) of Lemma 9.1 hold true if we replace $C, C_v, B^o, \rho(\hat{\gamma})$ by $\hat{C}, \hat{C}_v, \hat{B}^o, \tilde{\rho}$, respectively. In the proof we use the fact that the coupling $\hat{G}_{(n)} \subset \hat{G}_{(n)}$ implies couplings $\hat{C} \subset \hat{C}, \hat{C}_v \subset \hat{C}_v, \hat{B}^o \subset \hat{B}^o$, and that the results of Lemma 9.1 hold true for $\hat{C}, \hat{C}_v, \hat{B}^o, \tilde{\rho}$.

Proof of (81). We apply Lemma 9.1 to $\hat{G}_{(n)}$ and obtain the upper bound of (81) via the coupling $\hat{C}_v \subset \hat{C}_v$:
\[
\Pr(\hat{C}_v \geq o(\nu)) \leq \tilde{\rho} + \tilde{\epsilon}_n.
\]
The corresponding lower bound
\[
\Pr(\hat{C}_v \geq o(\nu)) \geq \tilde{\rho} - \tilde{\epsilon}_n
\]
is obtained by the same argument as in the proof of respective result of Lemma 9.1 (note that regular exploration will not detect any difference between the coupled graphs $\hat{G}_{(n)} \subset \hat{G}_{(n)}$).

Proof of (82). In the proof we take a shortcut (compared to the original argument of Lemma 9.1) while establishing the analog of the main intermediate inequality (112). For the overlay graph $\hat{G}_{(n)}$ the inequality (112) holds and it reads as follows
\[
\mathbb{E}\left( \{\hat{G}_{(n)} \geq o(n)\} \mathbb{I}(\hat{G}_{(n)} \geq o(n)) \right) \leq \tilde{\rho} \times \tilde{\rho} + o(1).
\]
The coupling \( \tilde{C}_y \subset \tilde{G}_y, \tilde{C}_y \subset \tilde{G}_y \) implies
\[
\mathbb{E} \left( \mathbb{1}_{|\tilde{C}_y| \geq \omega o(n)} \mathbb{1}_{|\tilde{C}_y| \geq \omega o(n)} \right) \leq \mathbb{E} \left( \mathbb{1}_{|\tilde{G}_y| \geq \omega o(n)} \mathbb{1}_{|\tilde{G}_y| \geq \omega o(n)} \right).
\]

Hence
\[
\mathbb{E} \left( \mathbb{1}_{|\tilde{C}_y| \geq \omega o(n)} \mathbb{1}_{|\tilde{C}_y| \geq \omega o(n)} \right) \leq \bar{\rho} \times \bar{\rho} + o(1).
\]

This is the analogue of (112) that yields (82) for \( \tilde{G}_y \). The rest of the proof of (82) goes without changes.

Proof of (83). The upper bound
\[
\mathbb{P}(|\tilde{C}| \leq n \bar{\rho} + \epsilon_n'' n) \geq 1 - o(1)
\]
follows from the respective upper bound for \( |\tilde{C}| \) combined with the coupling \( \tilde{C} \subset \tilde{C} \). To show the matching lower bound
\[
\mathbb{P}(|\tilde{C}| \geq n \bar{\rho} - \epsilon_n'' n) \geq 1 - o(1)
\]
we use the coupling \( \tilde{G}_y \subset \tilde{G}_y \) and slightly modify the corresponding argument of the proof of Lemma 9.1. Recall that \( \tilde{G}_y \) is obtained from \( \tilde{G}_y \) by deleting certain edges at random and the probability of deletion is at most \( 1 - p \). We call this process thinning. Let \( \tilde{C}', \tilde{C}'' \) be connected components of \( \tilde{G}_y \) that have at least \( n^{2/3} \) vertices each. We show that for any pair of such components \( \tilde{C}', \tilde{C}'' \) there is at least one blue edge of \( \tilde{G}_y \) (blue edges are defined in the proof of Lemma 9.1 when applied to \( \tilde{G}_y \)) connecting \( \tilde{C}' \) and \( \tilde{C}'' \) that has not been removed when we intersect \( H^* \) with \( \tilde{G}_y \), to get \( \tilde{G}_y = \tilde{G}_y \cap H^* \) in our coupling construction (142). For this purpose we choose \( \bar{\epsilon} = 10\theta^{-1} \) in (125). Now the probability that all blue edges of \( \tilde{G}_y \) connecting a given pair \( \tilde{C}', \tilde{C}'' \) have been removed by the intersection of \( \tilde{G}_y \) with \( H^* \) is at most \( (1 - \theta)^{10n} \leq n^{-10} \). As the number of pairs does not exceed \( \binom{n^{1/3}}{2} \) the probability that at least one pair \( \tilde{C}', \tilde{C}'' \) is not connected by a blue edge is at most \( \binom{n^{1/3}}{2} n^{-10} = o(1) \), by the union bound. The rest of the proof of the lower bound of (83) goes without changes. Hence (83) holds.

\[ \Box \]

## 11 | SUPPLEMENTARY RESULTS

Recall the overlay graph \( G \) defined by (1). Given \( A \subset [m] \), we consider the subgraph \( G_A \subset G \) on the vertex set \( V(G_A) = V(G) = \{1, \ldots, n\} \) defined by the layers \( (G_a, X_a, Q_a), a \in A \). Thus, the edge set of \( G_A \) is \( E(G_A) = \bigcup_{a \in A} E(G_a) \). In the following two results, we denote by \( N_A \) the set of neighbors of node \( i \) in \( G_A \), and we set \( D_A = |N_A| \) to denote the degree of \( i \) in \( G_A \).

**Lemma 11.1.** Let \( g \) be a probability density on \( \mathbb{Z}_+ \). Let
\[
\epsilon(t) = \sum_{i+s=t} (\mathbb{P}(D_{A \cup B} = r) - \mathbb{P}(D_A = r)) g(s).
\]
Then \( \sum_{t \geq 0} |\epsilon(t)| \leq 2 \mathbb{P}(D_B > 0) \).
Proof. Denote the densities of the degrees by $f_{A \cup B} = \mathcal{L}(D_{A \cup B})$ and $f_A = \mathcal{L}(D_A)$. Then

$$\sum_{t \geq 0} |\varepsilon(t)| = \|f_{A \cup B} * g - f_A * g\|_1 = 2d_{iv}(f_{A \cup B} * g, f_A * g) \leq 2d_{iv}(f_{A \cup B}, f_A).$$

Further, $d_{iv}(f_{A \cup B}, f_A) \leq \mathbb{P}(D_{A \cup B} \neq D_A) \leq \mathbb{P}(D_B > 0)$.

Lemma 11.2. Assume that $G_1, \ldots, G_m$ are mutually independent, let $A, B \subset [m]$ be disjoint, and let $\mathcal{E}_A, \mathcal{E}_B$ be events determined by $(G_a)_{a \in A}$ and $(G_b)_{b \in B}$, respectively. Then

$$\mathbb{P}(D_{A \cup B} = t, \mathcal{E}_A, \mathcal{E}_B) = \mathbb{P}(D_A + D_B = t, \mathcal{E}_A, \mathcal{E}_B) + \varepsilon(t),$$

where the error term is bounded by $|\varepsilon(t)| \leq c_B \mathbb{P}(D_A \leq t, \mathcal{E}_A)$, and where $c_B = \max_{j \neq i} \mathbb{P}(ij \in E(G_B), \mathcal{E}_B)$. In the particular case where $i = 1, A = \{k\}, B = \{\ell\}$, $\mathcal{E}_A = \{12 \in G_k\}$, $\mathcal{E}_B = \{13 \in G_\ell\}$ we have $|\varepsilon(t)| \leq t \mathbb{P}(12, 13 \in G_k) \mathbb{P}(13 \in G_\ell) + t \mathbb{P}(12, 13 \in G_\ell) \mathbb{P}(12 \in G_k)$.

Proof. Because $D_{A \cup B} = D_A + D_B$ outside the event $\mathcal{F} = \{|N_A \cap N_B| > 0\}$, we see that

$$\varepsilon(t) = \mathbb{P}(D_{A \cup B} = t, \mathcal{E}_A, \mathcal{E}_B, \mathcal{F}) - \mathbb{P}(D_A + D_B = t, \mathcal{E}_A, \mathcal{E}_B, \mathcal{F}).$$

Hence, it follows that $|\varepsilon(t)| \leq \mathbb{P}(D_A \leq t, \mathcal{E}_A, \mathcal{E}_B, \mathcal{F})$, where the upper bound can be expressed as

$$\mathbb{P}(D_A \leq t, \mathcal{E}_A, \mathcal{E}_B, \mathcal{F}) = \sum_{U:|U| \leq t, j \notin U} \mathbb{P}(N_A = U, \mathcal{E}_A) \mathbb{P}(|U \cap N_B| > 0, \mathcal{E}_B).$$

Because $\mathbb{P}(|U \cap N_B| > 0, \mathcal{E}_B) \leq \sum_{j \notin U} \mathbb{P}(ij \in E(G_B), \mathcal{E}_B) \leq c_B t$ whenever $|U| \leq t$, the inequality $|\varepsilon(t)| \leq c_B t \mathbb{P}(D_A \leq t, \mathcal{E}_A)$ follows.

To show the remaining bound we note that $N_A \cap N_B \neq \emptyset$ implies that at least one of the events $\{3 \in N_A\}$ and $\{\exists j \in N_A \cap N_B, j \neq 3\}$ occurs. Hence

$$\mathbb{P}(D_A \leq t, \mathcal{E}_A, \mathcal{E}_B, \mathcal{F}) = \mathbb{P}(\mathcal{E}_A, \mathcal{E}_B, 13 \in G_k) + \sum_{U:|U| \leq t, j \notin U} \mathbb{P}(N_A = U, \mathcal{E}_A) \sum_{j \notin U, j \neq 3} \mathbb{P}(\mathcal{E}_B, 1j \in G_\ell).$$

By symmetry, the inner sum is at most $t \mathbb{P}(12, 13 \in G_\ell)$. Consequently, the second term on the right is at most $\mathbb{P}(\mathcal{E}_A) t \mathbb{P}(12, 13 \in G_\ell)$. The first term equals $t \mathbb{P}(12, 13 \in G_k) \mathbb{P}(13 \in G_\ell)$.

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Here, we prove Theorem 3.6(iv). The proof is similar to that of Theorem 3.3 above. In particular, we use notation of section 7. Furthermore, for a graph $F \subseteq G_{(n)}$ we denote by $\hat{F} = F \cap H$ the percolated graph, where each edge of $F$ is retained independently with probability $\theta$, see (12). We write $\hat{G} = \hat{G}_{(n)}$, $\hat{d} = \deg_{\hat{G}}(1)$. We also denote $\hat{\mathcal{G}}^{(n)} = \text{Bin}_{rs}(\hat{P}_{n})$, where $\hat{P}_{n}$ is defined in (141) and where the mixed binomial distribution $\text{Bin}_{rs}$ is defined in (5). The degree distribution $f^{(n)}$ of $\hat{G}$ is defined as in (7), but with $\deg_{\hat{G}^{(n)}}(i)$ replaced by $\deg_{\hat{G}_{(n)}}(i)$. Introduce events $\hat{\mathcal{G}}_{3} = \{ \hat{G} \supset K_{3} \}$ and $\hat{\mathcal{G}}_{12} = \{ \hat{G} \supset K_{12} \}$ and denote

$$
\Delta_{rs} = \frac{m^{3}}{(\langle n \rangle_{2})^{3}}(P_{n})_{rs}(P_{n})_{21}(P_{n})_{21}, \quad \hat{\Delta}_{rs} = \frac{m^{2}}{(\langle n \rangle_{3}(\langle n \rangle_{2})^{2}}(P_{n})_{rs}(P_{n})_{21}.
$$

We derive Theorem 3.6(iv) from the relations

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APPENDIX

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(i) $\mathbb{P}(\hat{d} = t, \hat{K}_3) = \theta^3 \mu_{33} f^{(n)}(t) * \hat{g}_{33}^{(n)}(t - 2) + \hat{\epsilon}(t)$, where

$$|\hat{\epsilon}(t)| \leq 4\mu_{21} \mu_{32} + \mu_0^3 + \mu_{10,33} + 2\Delta_{44} + 4\Delta_{33};$$

(ii) $\mathbb{P}(\hat{d} = t, \hat{K}_{12}) = \theta^2 \mu_{32} f^{(n)}(t) * \hat{g}_{32}^{(n)}(t - 2) + \theta^2 \mu_{21}^3 f^{(n)}(t) * \hat{g}_{21}^{(n)}(t - 2) + \hat{\epsilon}(t)$, where

$$|\hat{\epsilon}(t)| \leq 10\mu_{21} \mu_{32} + 6\mu_0^3 + \mu_2^4 + 4\mu_{10,32} + 4\mu_{21} \mu_{10,21} + 2(n - 3)\mu_{32}^2 + 2\Delta_{43} + 4(\Delta_{32} + \Delta_{32} + \Delta_{21}).$$

We first show (i) and (ii). Afterwards we prove Theorem 3.6 (iv). Introduce events $\hat{A}_k = \{ \hat{G}_k \supset K_3 \}$ and $\hat{A}_{k\ell} = \hat{G}_k^{12} \cap \hat{G}_\ell^{13}$, where $\hat{G}_j^{ij}$ is the event that $ij \in E(\hat{G}_j)$. Recall the overlay graphs $G_{-k} = \cup_{k' \neq k} G_{k'}$, $G_{k\ell} = G_k \cup G_\ell$ and $G_{-k\ell} = \cup_{q \not\in \{k, \ell\}} G_q$ and denote

$$\hat{d}_k = \deg_{\hat{G}_k}(1), \quad \hat{d}_{-k} = \deg_{\hat{G}_{-k}}(1), \quad \hat{d}_{k\ell} = \deg_{\hat{G}_{k\ell}}(1), \quad \hat{d}_{-k\ell} = \deg_{\hat{G}_{-k\ell}}(1),$$

$$\delta_k = \deg_{\hat{G}_0 \cap G_{-k}}(1), \quad \delta_{k\ell} = \deg_{\hat{G}_0 \cap G_{-k\ell}}(1), \quad \delta_{k\ell} = \deg_{G_0 \cap G_{-k\ell}}(1).$$

We also denote $\hat{h}_{k\ell}(s) = \mathbb{P}(\hat{d}_{k\ell} = s, \hat{A}_{k\ell})$, and $\hat{\gamma}_{k\ell}(s) = \mathbb{P}(\hat{d}_k = s, \hat{G}_k^{ij})$.

Proof of (i). The proof is similar to that of Theorem 7.3 (ii): we denote

$$\hat{\epsilon}_1(t) = \mathbb{P}(\hat{d} = t, \hat{K}_3) - \mathbb{P}(\hat{d} = t, \cup_k \hat{A}_k), \quad \hat{\epsilon}_2(t) = \mathbb{P}(\hat{d} = t, \cup_k \hat{A}_k) - \sum_k \mathbb{P}(\hat{d} = t, \hat{A}_k)$$

and estimate (cf. proof of Theorem 7.3 (ii))

$$0 \leq \hat{\epsilon}_1(t) \leq \mathbb{P}(E_{12}) + \mathbb{P}(E_{111}) \leq 3\mu_{21} \mu_{32} + \mu_0^3,$$

$$0 \leq -\hat{\epsilon}_2(t) \leq \sum_{k,k'} \mathbb{P}(\hat{d} = t, A_k, A_{k'}) \leq \sum_{k,k'} \mathbb{P}(A_k, A_{k'}) \leq \mu_{21} \mu_{32}.$$

Hence $|\mathbb{P}(\hat{d} = t, \hat{K}_3) - \sum_k \mathbb{P}(\hat{d} = t, \hat{A}_k)| \leq 4\mu_{21} \mu_{32} + \mu_0^3$. We next approximate

$$\sum_k \mathbb{P}(\hat{d} = t, \hat{A}_k) \approx \sum_k \sum_{r+s=t} \mathbb{P}(\hat{d}_{-k} = r) \mathbb{P}(\hat{d}_k = s, \hat{A}_k) \approx \sum_k \sum_{r+s=t} \mathbb{P}(\hat{d} = r) \mathbb{P}(\hat{d}_k = s, \hat{A}_k)$$

and observe that $\sum_k \mathbb{P}(\hat{d}_k = s, \hat{A}_k) = \theta^3 \mu_{33} \hat{g}_{33}^{(n)}(s - 2)$ (the identity follows from Lemma 7.2 applied to $\hat{G}_{k\ell} = G_{k\ell} \cap H$, cf. (48)). Hence the right side of (A3) equals $\theta^3 \mu_{33} f^{(n)}(t) * \hat{g}_{33}^{(n)}(t - 2)$, and to prove (i) it suffices to analyze the approximation errors in (A2), (A3).

To upper bound the approximation error in (A2), denoted $\hat{\epsilon}_3(t)$, we apply the first inequality of Lemma A.1 with $F = K_3$, $A = \{k\}$, $B = [m] \setminus \{k\} \forall k$. We obtain

$$|\hat{\epsilon}_3(t)| \leq \sum_k 2\mathbb{P}(A_k, \delta_k > 0) \leq 2\Delta_{44} + 4\Delta_{33},$$

where in the last step we apply inequality (A12) of Lemma A.2.
The approximation error in (A3) equals \( \bar{\varepsilon}_4(t) = \sum_k \hat{\varepsilon}_4k(t) \) where
\[
\hat{\varepsilon}_4k(t) = \sum_{r+k=s} (\mathbb{P}(\hat{d} = r) - \mathbb{P}(\hat{d}_{-k} = r)) \mathbb{P}(\hat{d}_k = s, \hat{A}_k).
\]

Noting that \( \deg_{G_k}(1) = 0 \) implies \( \hat{d} = \hat{d}_{-k} \) we estimate
\[
\left| \mathbb{P}(\hat{d} = r) - \mathbb{P}(\hat{d}_{-k} = r) \right| \leq \mathbb{P}(\deg_{G_k}(1) > 0) \leq \mathbb{P}(1 \in V(G_k)) = p_{10}(k)
\]
which follows by Lemma 7.2 applied to \( \hat{G}_k \), cf. (54)) we write the right side of (A8) in the form \( \theta^2 \mu_{32} \hat{g}_{32}(t - 2) \).

Next, we approximate the second inner sum on the right of (A8) in the form \( \theta^2 \mu_{32} \hat{g}_{32}(t - 2) \).

Invoking identities \( \sum_k \bar{h}_{k2}(s) = \sum_k \bar{h}_{r3}(s) = \theta \mu_{21} \bar{g}_{21}(s - 1) \) (the first one follows by symmetry; the second one follows by Lemma 7.2 applied to \( \hat{G}_{k^*} \), cf. (54)) we write the right side of (A10) in the form \( \theta^2 \mu_{21} \hat{g}_{32}(t - 2) \).

Hence, \( |\hat{\varepsilon}_4(t)| \leq \sum_k p_{10}(k)p_{33}(k) = \mu_{10,33} \). Claim (i) follows by combining the above estimates for the total approximation error \( \hat{\varepsilon}(t) = \sum_{i=1}^4 \hat{\varepsilon}_i(t) \).

Proof of (ii). The proof is similar to that of Theorem 7.4 (ii): we approximate
\[
\mathbb{P}(\hat{d} = t, \hat{K}_{12}) \approx \sum_{k, \ell} \mathbb{P}(\hat{d} = t, \hat{A}_{k\ell}),
\]
which follows by Lemma 7.2 applied to \( \hat{G}_{k^*} \), cf. (54)) we write the first term on the right of (A8) in the form \( \theta^2 \mu_{32} \hat{g}_{32}(t - 2) \).

Next, we approximate the second inner sum on the right of (A8) in the form \( \theta^2 \mu_{21} \hat{g}_{32}(t - 2) \).

Invoking identities \( \sum_k \bar{h}_{k2}(s) = \sum_k \bar{h}_{r3}(s) = \theta \mu_{21} \bar{g}_{21}(s - 1) \) (the first one follows by symmetry; the second one follows by Lemma 7.2 applied to \( \hat{G}_{k^*} \), cf. (54)) we write the right side of (A10) in the form \( \theta^2 \mu_{21} \hat{g}_{32}(t - 2) \).

Hence, the second term on the right side of (A8) is approximately \( \theta^2 \mu_{21} \hat{g}_{32}(t - 2) \). Finally, we conclude that
\[
\mathbb{P}(\hat{d} = t, \hat{K}_{12}) \approx \theta^2 \mu_{32} \hat{g}_{32}(t - 2) + \theta^2 \mu_{21} \hat{g}_{32}(t - 2),
\]
where the total approximation error can be written as \( \varepsilon(t) = \sum_{i=1}^{4} \varepsilon_i(t) \). Here \( \varepsilon_1(t), \varepsilon_2(t), \varepsilon_3(t) \) are the approximation errors in (A5), (A6), (A7), respectively, and

\[
\varepsilon_4(t) = \sum_{r+s=t} f^{(n)}(r) (\varepsilon_{41}(s) + \varepsilon_{42}(s)),
\]

where \( \varepsilon_{41}(s) \) and \( \varepsilon_{42}(s) \) denote the errors made in (A9) and (A10).

Now, we analyze individual approximation errors \( \varepsilon_i(t) \). The cases \( i = 1, 3 \) are treated similarly as in the corresponding proof of Theorem 7.4 (ii) Bonferroni’s inequalities imply

\[
|\varepsilon_1(t)| \leq \sum_{(k_1,k_2) \in \ell(1,1,2)} \mathbb{P}(A_{k_1,k_2}, \varepsilon_i, \varepsilon_2) \leq \sum_{(k_1,k_2) \in \ell(1,1,2)} \mathbb{P}(A_{k_1,k_2}, A_{\varepsilon_i, \varepsilon_2})
\]

and therefore (61) extends to \(|\varepsilon_1(t)|\). Furthermore, invoking inequalities (cf. (A4))

\[
|\mathbb{P}(\hat{d} = r) - \mathbb{P}(\hat{d}_{-k\ell} = r)| \leq \mathbb{P}(1 \in V(G_1)) + \mathbb{P}(1 \in V(G_{\ell})) = p_{10}(k) + p_{10}(\ell)
\]

and \( \sum_{r+s=t} \hat{h}_{k\ell}(s) \leq \mathbb{P}(\hat{A}_{k\ell}) \leq \mathbb{P}(A_{k\ell}) \) we obtain

\[
|\varepsilon_3(t)| \leq \sum_{k,\ell} (p_{10}(k) + p_{10}(\ell)) \mathbb{P}(A_{k\ell}) \leq \sum_{k,\ell} (p_{10}(k) + p_{10}(\ell)) \mathbb{P}(A_{k\ell}).
\]

Therefore (63) extends to \(|\varepsilon_3(t)|\).

To upper bound \( \varepsilon_2(t) \) we apply the first inequality of Lemma A.1 with \( F = K_{12} \), where we set
\( A = \{k, \ell\}, B = [m] \setminus \{k, \ell\} \) for \( k \neq \ell \), and we set \( A = \{k\}, B = [m] \setminus \{k\} \) for \( k = \ell \). We obtain

\[
|\varepsilon_2(t)| \leq 2 \sum_{k,\ell} \mathbb{P}(A_{k\ell}, \delta_{k\ell} > 0) \leq 2\hat{\Delta}_{43} + 4\hat{\Delta}_{32} + \Delta_{31} + \Delta_{21},
\]

where the last inequality follows by inequality (A13) of Lemma A.2.

To upper bound \( \varepsilon_{41}(s) \) we apply the second inequality of Lemma A.1 with \( A = \{k\}, B = \{\ell\} \) and \( F, F' \) being the edges 12, 13. We obtain

\[
|\varepsilon_{41}(s)| \leq \sum_{k,\ell} 2 \mathbb{P}(A_{k\ell}, \delta_{k\ell} > 0) \leq 2(n-3)\mu^2_{32} + 2\mu_{32}\mu_{21},
\]

where the last inequality follows by inequality (A11) of Lemma A.2.

Finally the approximation error in (A10)

\[
|\varepsilon_{42}(s)| = \sum_{k} \sum_{s_1,s_2=m} \mathbb{P}(d_k = s_1, \hat{G}^k_{12}) \mathbb{P}(d_k = s_2, \hat{G}^k_{12})
\]

satisfies
\[ \sum_{s \geq 0} |\varepsilon_{42}(s)| = \sum_{k} \left( \mathbb{P}(\hat{G}^k_{12}) \right)^2 = \theta^2 \sum_{k} \left( p_{21}(k) \right)^2 = \theta^2 \mu_{21,21}. \]

We conclude that \(|\varepsilon_4(t)| \leq 2(n-3)\mu^2_{32} + 2\mu_{32}\mu_{21} + \mu_{21,21} \).

Claim (ii) follows by collecting the bounds for \( \varepsilon_i, i = 1, 2, 3, 4 \) altogether.
Proof of Theorem 3.6(iv). We evaluate the ratio \( \sigma(\hat{G})(k) = \frac{\mathbb{P}(d = n, K_k)}{\mathbb{P}(d = n, \bar{K}_k)} \). To this aim we invoke approximations (i) and (ii) above, where respective leading terms

\[
\theta^3 \mu_3^3 f_{33}^2(\hat{\gamma}^{(n)}_{33}(t-2)) = (\hat{P})_{33} \mu n^{-2} f_{33}^2 \hat{g}_{33}(t-2) + o(n^{-2})
\]

and

\[
\theta^2 \mu_2 f_{32}^2(\hat{\gamma}^{(n)}_{32}(t-2)) = (\hat{P})_{32} \mu n^{-2} f_{32}^2 \hat{g}_{32}(t-2) + (\hat{P})_{21} \mu n^{-2} f_{21}^2 \hat{g}_{21}(t-2) + o(n^{-2})
\]

are of order \( \Theta(n^{-2}) \) as \( n \to +\infty \). It remains to show that the remainders \( \hat{\epsilon}(t) \) and \( \mathcal{E}(t) \) of (i) and (ii) are of order \( o(n^{-2}) \). In the proof of Theorem 3.3 all the terms contributing to \( \hat{\epsilon}(t) \) and \( \mathcal{E}(t) \) are shown to be \( o(n^{-2}) \), but \( \hat{\Delta}_{43}, \hat{\Delta}_{44}, \hat{\Delta}_{33}, \hat{\Delta}_{32}, \hat{\Delta}_{23}, \hat{\Delta}_{21}, \) and \( (n-3)\mu_{32}^2 \).

Here, we only show that \( \hat{\Delta}_{43} = o(n^{-2}) \). The proof of \( o(n^{-2}) \) bound for the remaining terms is easy. We have \( \hat{\Delta}_{4,3} \leq n^{-3}(P_{n,43} = n^{-3} \mathbb{E}(X_{n,\pi}^3 Q_{n,\pi}^3), \text{where } (X_{n,\pi}, Q_{n,\pi}) \) is a bivariate random variable with the distribution \( P_n \). Our conditions \( (P_n)_{3,3} \to (P)_{3,3} \) and \( P_n \to P \) implies that the sequence \( \{X_{n,\pi}^3 Q_{n,\pi}, n \geq 1\} \) is uniformly integrable. Note also that \( \mathbb{P}(X_{n,\pi} \leq n) = 1 \ \forall n \). Now fix \( 0 < \tau < 1 \) and split

\[
(P_{n,43} = \mathbb{E}(X_{n,\pi}^4 Q_{n,\pi}^3) = \mathbb{E}(X_{n,\pi}^4 Q_{n,\pi}^3 \mathbb{I}_{[X_{n,\pi} \geq \tau n]})) + \mathbb{E}(X_{n,\pi}^4 Q_{n,\pi}^3 \mathbb{I}_{[X_{n,\pi} < \tau n]}) \leq \tau n \mathbb{E}(X_{n,\pi}^3 Q_{n,\pi}^2) + n \mathbb{E}(X_{n,\pi}^4 Q_{n,\pi}^3 \mathbb{I}_{[X_{n,\pi} \geq \tau n]})
\]

The last term above is \( o(n) \), by the uniform integrability property. Hence \( (P_n)_{43} = o(n) \). Consequently, we have \( \hat{\Delta}_{43} = o(n^{-2}) \).

In the next lemma we use notation \( G_A = (V(G_A), E(G_A)), A \subseteq [m] \), introduced in section 11. We also denote \( N_A \) the set of neighbors of vertex 1 in \( G_A \) and write \( D_A = |N_A| \). For \( A, B \subseteq [m] \) we denote \( \delta_{A,B} = |N_A \cap N_B| \). We denote by \( \hat{D}_A \) the degree of vertex 1 in the percolated graph \( \hat{G}_A = G_A \cap H \). Given a graph \( F \) with the vertex set \( V(F) \subseteq V = \{1, \ldots, n\} \) we denote by \( A_{F,A} \) and \( \hat{A}_{F,A} \) the events that \( F \subseteq G_A \) and \( F \subseteq \hat{G}_A \) respectively. Lemma A.1 is used in the proof of claims (i), (ii) above where we take \( F = K_3, K_{12} \).

Lemma A.1. Let \( A, B \subseteq [m] \). Assume that \( A \cap B = \emptyset \). Assume that \( 1 \in V(F) \). Then

\[
\mathbb{P}(\hat{D}_{A \cup B} = t, \hat{A}_{F,A}) - \sum_{t+s=m} \mathbb{P}(\hat{D}_B = r) \mathbb{P}(\hat{D}_A = s, \hat{A}_{F,A}) \leq 2 \mathbb{P}(A_{F,A}, \delta_{A,B} > 0).
\]

Assume, in addition, that graph \( F' \) has vertex set \( V(F') \subseteq V \) and \( 1 = V(F) \cap V(F') \). Then

\[
\mathbb{P}(\hat{D}_{A \cup B} = t, \hat{A}_{F \cap F',A \cup B}, A_{F,A}, A_{F',B}) - \sum_{t+s=m} \mathbb{P}(\hat{D}_B = r, \hat{A}_{F',B}) \mathbb{P}(\hat{D}_A = s, \hat{A}_{F,A}) \leq 2 \mathbb{P}(A_{F,A}, A_{F',B}, \delta_{A,B} > 0).
\]

Proof. We only prove the first inequality. The proof of the second one is much the same.

In the proof we write, for short, \( A = A_{F,A} \) and \( \hat{A} = \hat{A}_{F,A} \). We note that random variables \( D_A \) and \( D_B \) are independent and \( \delta_{A,B} = 0 \) implies \( D_{A \cup B} = D_A + D_B \) and, consequently, \( \hat{D}_{A \cup B} = \hat{D}_A + \hat{D}_B \).
Introduce Bernoulli random graph \( \tilde{H} \) with \( V(H) = V \) and edge probability \( \theta \), which is independent of \( G_1, \ldots, G_n \) and \( H \). Let \( \tilde{D}_B \) denote the degree of vertex 1 in the percolated graph \( \tilde{G}_B = G_B \cap \tilde{H} \). Clearly, random variables \( \tilde{D}_B \) and \( D_B \) have the same probability distributions. Moreover, on the event \( \delta_{A,B} = 0 \) the pairs \((\tilde{D}_A, \tilde{A}); (\tilde{D}_B, \tilde{B})\) have the same distributions. Hence

\[
\mathbb{P}(\tilde{D}_{A,B} = t, \tilde{A}, \delta_{A,B} = 0) = \mathbb{P}(\tilde{D}_A + \tilde{D}_B = t, \tilde{A}, \delta_{A,B} = 0) = \mathbb{P}(\tilde{D}_A + \tilde{D}_B = t, \tilde{A}, \delta_{A,B} = 0).
\]

Consequently, we obtain

\[
\left| \mathbb{P}(\tilde{D}_{A,B} = t, \tilde{A}) - \mathbb{P}(\tilde{D}_A + \tilde{D}_B = t, \tilde{A}) \right| \leq 2 \mathbb{P}(A, \delta_{A,B} > 0).
\]

Next, by the independence of \( \tilde{D}_B \) and \( (\tilde{D}_A, \tilde{A}) \) we can factorize the probability \( \mathbb{P}(\tilde{D}_A + \tilde{D}_B = t, \tilde{A}) = \sum_{r+s=t} \mathbb{P}(\tilde{D}_A = s, \tilde{A}) \mathbb{P}(\tilde{D}_B = r) \). Now the identity \( L(\tilde{D}_B) = L(\tilde{D}_B) \) completes the proof.

To prove the second inequality of the lemma we note that \( V(F) \cap V(F') = 1 \) implies that \( F \) and \( F' \) have no common edges. Therefore, given the event \( F \subset G_A, F' \subset G_B \) and \( \delta_{A,B} = 0 \), the percolation process acts on \( F \) and \( F' \) independently. The rest of the proof is much the same as above.

**Lemma A.2.** For \( \Delta_{rs} \) and \( \Delta_{rs} \) defined in (A1), we have

\[
\sum_{k \in [m]} \mathbb{P}(A_k, \delta_k > 0) \leq (n - 3) \mu_{32}^2 + 2 \mu_{21} \mu_{32}, \tag{A11}
\]

\[
\sum_{k \in [m]} \mathbb{P}(A_k, \delta_k > 0) \leq \tilde{\Delta}_{44} + 2 \tilde{\Delta}_{33}, \tag{A12}
\]

\[
\sum_{k \in [m]} \mathbb{P}(A_k, \delta_k > 0) \leq \Delta_{43} + 2(\Delta_{32} + \Delta_{32} + \Delta_{21}). \tag{A13}
\]

**Proof.** By \( \mathbb{P} = \mathbb{P}\left(\bar{X}_n, \bar{Q}_n\right) \), we denote the conditional probability given \( (\bar{X}_n, \bar{Q}_n) \).

Proof of (A11). We write event \( \{A_{k, \ell}, \delta_{k, \ell} > 0\} \) in the form

\[
G_{k, \ell}^{12} \cap G_{\ell, \ell}^{13} \cap \left( G_{k, \ell}^{13} \cup G_{\ell, \ell}^{12} \cup \bigcup_{s=4}^{n} (G_{k, \ell}^{13} \cap G_{s, \ell}^{13}) \right).
\]

Invoking identities

\[
\mathbb{P}(G_{k, \ell}^{12}, G_{\ell, \ell}^{13}, G_{k, \ell}^{13}) = \frac{(X_k)_3 (X_{\ell})_2}{(n)_3} Q_{k}^2 (X_{\ell})_2 Q_{\ell},
\]

\[
\mathbb{P}(G_{k, \ell}^{12}, G_{\ell, \ell}^{13}, G_{k, \ell}^{13}) = \frac{(X_k)_2 (X_{\ell})_3}{(n)_2} Q_{k}^2 (X_{\ell})_2 Q_{\ell},
\]

\[
\mathbb{P}(G_{k, \ell}^{12}, G_{\ell, \ell}^{13}, G_{k, \ell}^{13}) = \frac{(X_k)_3 (X_{\ell})_3}{(n)_3} Q_{k}^2 (X_{\ell})_3 Q_{\ell}^2
\]

we obtain, by the union bound and symmetry,

\[
\mathbb{P}(A_{k, \ell}, \delta_{k, \ell} > 0) \leq \frac{(X_k)_3}{(n)_3} Q_{k}^2 (X_{\ell})_2 Q_{\ell} + \frac{(X_k)_2}{(n)_2} Q_{k} (X_{\ell})_3 Q_{\ell}^2.
\]

\[
+ \frac{(X_k)_3}{(n)_3} Q_{k}^2 (X_{\ell})_3 Q_{\ell}^2.
\]
Similarly, \[
P(A_{k'}, \bar{\delta}_{k'} > 0) = \mathbb{E}[P(A_{k'}, \bar{\delta}_{k'} > 0)] \leq p_{32}(k)p_{21}(\epsilon') + p_{21}(k)p_{32}(\epsilon') + (n-3)p_{32}(k)p_{32}(\epsilon').
\]

Now, summing over \(k \neq \ell\) we obtain (A11).

Proof of (A12). We write \(\delta_k = \delta_k' + \delta_k''\), where \(\delta_k'\) (respectively \(\delta_k''\)) is the number of vertices \(s \in V(G) \setminus \{1, 2, 3\}\) (respectively \(s \in \{2, 3\}\)) such that \(1s \in E(G_k) \cap E(G_{-k})\). Then \(P(A_k, \delta_k > 0) \leq P(A_k, \delta_k' > 0) + P(A_k, \delta_k'' > 0)\) and we estimate the probabilities on the right separately.

To evaluate \(P(A_k, \delta_k' > 0)\) we write event \(\{\delta_k' > 0\}\) in the form

\[
\bigcup_{s \in V(G_k) \setminus \{1,2,3\}} \left( \mathcal{G}_k^{1s} \cap \left( \bigcup_{j \in [m] \setminus \{k\}} \mathcal{G}_j^{i_s} \right) \right).
\]

(A14)

Then, using the union bound and symmetry, we upper bound the conditional probability

\[
P(A_k, \delta_k' > 0) \leq (X_k - 3)\frac{(X_k)_3}{(n)_3} \sum_{j \in [m] \setminus \{k\}} \frac{(X_j)_2}{(n)_2} Q_j.
\]

(A15)

Here, \(X_k - 3\) is the number of vertices \(s \in V(G_k) \setminus \{1, 2, 3\}\). \(\frac{(X_k)_3}{(n)_3}\) is the probability that \(\{1, 2, 3\} \subset V(G_k)\).

Similarly, \(\frac{(X_j)_2}{(n)_2}\) is the probability that \(\{1, s\} \subset V(G_j)\). Next, we take the expectation \(\mathbb{E}[P(A_k, \delta_k' > 0)] = P(A_k, \delta_k' > 0)\) and sum over \(k\) to obtain

\[
\sum_k P(A_k, \delta_k' > 0) \leq \sum_k \frac{\mathbb{E}((X_k)_3 Q_k^3)}{(n)_3} \frac{\mathbb{E}((X_j)_2 Q_j)}{(n)_2}.
\]

(A16)

Note that, the right side is upper bounded by

\[
\sum_k \frac{\mathbb{E}((X_k)_3 Q_k^3)}{(n)_3} \times \sum_j \frac{\mathbb{E}((X_j)_2 Q_j)}{(n)_2} = \Delta_{44}.
\]

(A17)

Now, we evaluate \(P(A_k, \delta_k'' > 0)\). We write event \(\{A_k, \delta_k'' > 0\}\) in the form

\[
A_k \cap \bigcup_{s \in \{2,3\}} \left( \bigcup_{j \in [m] \setminus \{k\}} \mathcal{G}_j^{i_s} \right)
\]

and proceeding similarly as in (A15) above we obtain that

\[
P(A_k, \delta_k'' > 0) \leq 2 \sum_{j \in [m] \setminus \{k\}} \frac{\mathbb{E}((X_k)_3 Q_k^3)}{(n)_3} \frac{\mathbb{E}((X_j)_2 Q_j)}{(n)_2} \leq 2 \frac{\mathbb{E}((X_k)_3 Q_k^3)}{(n)_3} m \frac{P_{21}(n)}{(n)_2}.
\]

(A18)

Consequently, the sum \(\sum_k P(A_k, \delta_k'' > 0) \leq 2\Delta_{33}\). The latter bound together with (A17) implies (A12).

Proof of (A13). We write \(\delta_{k'} = \delta_{k'}' + \delta_{k'}''\), where \(\delta_{k'}'\) (respectively \(\delta_{k'}''\)) is the number of vertices \(s \in V(G) \setminus \{1, 2, 3\}\) (respectively \(s \in \{2, 3\}\)) such that \(1s \in E(G_{k'}) \cap E(G_{-k'})\).
We split
\[ \sum_{k,\ell} \mathbb{P}(A_{k\ell}, \delta_{k\ell} > 0) = \sum_{k\ell} \mathbb{P}(A_{k\ell}, \delta_{k\ell} > 0) + \sum_{k} \mathbb{P}(A_{kk}, \delta_{kk} > 0) =: S_1 + S_2 \]
and estimate
\[ S_1 \leq \sum_{k\ell} \mathbb{P}(A_{k\ell}, \delta'_{k\ell} > 0) + \sum_{k\ell} \mathbb{P}(A_{k\ell}, \delta''_{k\ell} > 0) =: S'_1 + S''_1. \]

In what follows we upper bound \( S'_1, S''_1 \) and \( S_2 \). To estimate \( S'_1 \) we write event \( \{ \delta'_{k\ell} > 0 \} \) in the form
\[ \bigcup_{s \in V(G_{k\ell}) \setminus \{1,2,3\}} \left( G^1_{k\ell} \cup G^2_{k\ell} \cap \left( \bigcup_{j \in [m] \setminus [k,\ell]} G^1_{j} \right) \right) \]
and upper bound the conditional probability \( \mathbb{P}(A_{k\ell}, \delta'_{k\ell} > 0) \) using the union bound and symmetry,
\[ \mathbb{P}(A_{k\ell}, \delta'_{k\ell} > 0) \leq \frac{(X_{k})_2 Q_k (X_{\ell})_2 Q_{\ell}}{(n)^2} ((X_k - 2)Q_k + (X_{\ell} - 2)Q_{\ell}) \] \[ \times \sum_{j \in [m] \setminus [k,\ell]} \frac{(X_j)_2 Q_j}{(n)^2}. \] (A19)
Here, \( \frac{(X_{k})_2 Q_k (X_{\ell})_2 Q_{\ell}}{(n)^2} \) is the (conditional) probability of event \( A_{k\ell} \); \( (X_k - 2)Q_k \) counts \( s \in V(G_k) \setminus \{1,2\} \) linked to \( 1 \) by the layer \( G_k \); \( (X_{\ell} - 2)Q_{\ell} \) counts \( s \in V(G_{\ell}) \setminus \{1,3\} \) linked to \( 1 \) by the layer \( G_{\ell} \).

Furthermore, each such \( s \) is linked to \( 1 \) by \( G_{-k\ell} \) with probability \( \mathbb{P}(\cup_{j \in [m] \setminus [k,\ell]} G^1_j) \leq \sum_{j \in [m] \setminus [k,\ell]} (X_j)_2 Q_j / (n)^2. \)

It follows from (A19) that
\[ S'_1 = \sum_{k,\ell} \mathbb{E}\mathbb{P}(A_{k\ell}, \delta'_{k\ell} > 0) \leq \frac{2}{(n)^2} m^3 (P_n)_{32} (P_n)_{21}^2 = 2\Delta_{32}. \]

We similarly estimate \( S''_1 \). The event \( \delta''_{k\ell} > 0 \) means that at least one link 12 or/and 13 is present in \( G_{-k\ell} \). Hence
\[ \mathbb{P}(A_{k\ell}, \delta''_{k\ell} > 0) \leq \frac{2}{(n)^2} m^3 (P_n)_{32} (P_n)_{21}^2 \times \sum_{j \in [m] \setminus [k,\ell]} (X_j)_2 Q_j / (n)^2. \]

Consequently, \( S''_1 \leq \sum_{k,\ell} \mathbb{E}\mathbb{P}(A_{k\ell}, \delta_{k\ell} - \delta''_{k\ell} > 0) \leq \frac{2}{(n)^2} m^3 (P_n)_{21}^2 = 2\Delta_{21}. \)

Finally, we show \( S_2 \leq \hat{A}_{43} + 2\Delta_{32} \) in much the same way as (A12) above. Collecting the bounds for \( S'_1, S''_1 \), and \( S_2 \) we obtain (A13).