Computing the Extremal Possible Ranks with Incomplete Preferences

Aviram Imber
Technion, Israel
aviram.imber@cs.technion.ac.il

Benny Kimelfeld
Technion, Israel
bennyk@cs.technion.ac.il

Abstract

In an election via a positional scoring rule, each candidate receives from each voter a score that is determined only by the position of the candidate in the voter’s total ordering of the candidates. A winner (respectively, unique winner) is a candidate who receives a score not smaller than (respectively, strictly greater than) the remaining candidates. When voter preferences are known in an incomplete manner as partial orders, a candidate can be a possible/necessary (unique) winner based on the possibilities of completing the partial votes. The computational problems of determining the possible and necessary winners and unique winners have been studied in depth, culminating in a full classification of the class of “pure” positional scoring rules into tractable and intractable ones for each problem.

The above problems are all special cases of reasoning about the range of possible positions of a candidate under different tie breakers. Determining this range, and particularly the extremal positions, arises in every situation where the ranking plays an important role in the outcome of an election, such as in committee selection, primaries of political parties, and staff recruiting. Our main result establishes that the minimal and maximal positions are hard to compute (DP-complete) for every positional scoring rule, pure or not. Hence, none of the tractable variants of necessary/possible winner determination remain tractable for extremal position determination. We do show, however, that tractability can be retained when reasoning about the top-k and the bottom-k positions for a fixed k.

1 Introduction

A central task in social choice is that of winner determination—how to aggregate the candidate preferences of voters to select the winner. Relevant scenarios may be political elections, document rankings in search engines, hiring dynamics in the job market, decision making in multiagent systems, determination of outcomes in sports tournaments, and so on [5]. Different voting rules can be adopted for this task. The computational social-choice community has studied in depth the family of the positional scoring rules, where each voter assigns to each candidate a score based on the candidate’s position in the voter’s ranking, and then a winner is a candidate who receives the maximal sum of scores. Famous instantiations of the positional scoring rules include the plurality rule (where a winner is most frequently ranked first), the veto rule (where a winner is least frequently ranked last), their generalizations to $t$-approval and $t$-veto, respectively, and the Borda rule (where the score is the actual position in the reverse order).

The seminal work of Konczak and Lang [14] has addressed the situation where voter
preferences are expressed or known in just a partial manner. More precisely, a partial voting profile consists of a partial order for each voter, and a completion consists of a linear extension for each of the partial orders. The framework gives rise to the computational problems of determining the necessary winners who win in every completion, and the possible winners who win in at least one completion. In fact, each of these problems has two variants that correspond to two forms of winning: having a score not smaller than any other candidate (i.e., being a co-winner) and a having a score strictly greater than all other candidates (i.e., being the unique winner). These computational problems are challenging since, conceptually, they involve reasoning about the entire (exponential-size) space of such completions. The complexity of these problems has been thoroughly studied in a series of publications that established the tractability of the necessary winners [24], and a full classification of a general class of positional scoring rules (the “pure” scoring rules) into tractable and intractable for the problem of the possible winners [3,4,24].

Yet, the outcome of an election often goes beyond just reasoning about the maximal score. For example, the ranking among the other candidates might determine who will be the elected parliament members, the entries of the first page of the search engine, the job candidates to recruit, and the finalists of a sports competition. In the case of a positional scoring rule, the ranking order is determined by sum of score from voters under some tie-breaking mechanism [16]. When voter preferences are partial, a candidate can be ranked in a different positions for every completion, and it is then natural to reason about the range of these positions. In fact, the aforementioned computational problems can all be phrased as reasoning about the minimal and maximal ranks under different tie breakers. A candidate $c$ is a:

- possible co-winner if the minimal rank is one when the tie breaker favours $c$ most;
- possible unique winner if the minimal rank is one when the tie breaker favours $c$ least;
- necessary co-winner if the maximal rank is one when the tie breaker favours $c$ most;
- necessary unique winner if the minimal rank is one when the tie breaker favours $c$ least.

We study the computational problems $\text{Min} \{\theta\}$ and $\text{Max} \{\theta\}$, where $\theta$ is one of the comparisons $<$, $>$ and $=$. The input consists of a partial profile, a candidate, a tie breaking (total) order and a number $k$, and the goal is to determine whether $x \ast k$ where $x$ is the minimal rank and the maximal rank, respectively, of the candidate. Our results are summarized in Table 1 (Recall that “DP" is the class of problems that can be described as the intersection of a problem in NP and a problem in coNP.)

As the table shows, determining the extremal ranks of a candidate is fundamentally harder than their $k = 1$ counterparts (necessary and possible winners). For example, it is known that detecting the possible winners is NP-hard for every pure rule, with the exception of plurality and veto where the problem is solvable in polynomial time [3,4,24]. In contrast, we show that determining each of the minimum and maximum ranks is DP-complete for every positional scoring rule, pure or not, including plurality and veto. In particular, the tractability of the necessary winners does not extend to reasoning about the maximal rank.

We also study the impact of fixing $k$ and consider the problems $\text{Min} \{\theta k\}$ and $\text{Max} \{\theta k\}$ where the goal is to determine whether $x \ast k$ where, again, $x$ is the minimal rank and the maximal rank, respectively, of the candidate. We establish a more positive picture: tractability for the maximum (assuming that the scores are polynomial in the number of candidates), and tractability of the minimum under plurality and veto. The degree of the polynomials depend on $k$, and we show that this is necessary (under standard assumptions of parameterized complexity) at least for the case of maximum, where the problem is W[1]-hard for every positional scoring rule.
Table 1 Overview of the results. The symbol $\theta$ stands for each of $<$, $>$ and $=$, and $k$ stands for $m - k + 1$ where $m$ is the number of candidates.

| Problem | plurality, veto | pure – $\{\text{pl, veto}\}$ | non-pure | comment |
|---------|----------------|-------------------------------|----------|---------|
| $\text{Min}<$ | NP-c | NP-c | NP-c | [Thm. 3] |
| $\text{Min}>$ | coNP-c | coNP-c | coNP-c | [Thm. 4] |
| $\text{Min}=|$ | DP-c | DP-c | DP-c | [Thm. 5] |
| $\text{Max}<$ | coNP-c | coNP-c | coNP-c | [Thm. 6] |
| $\text{Max}>$ | NP-c | NP-c | NP-c | [Thm. 7] |
| $\text{Max}=|$ | DP-c | DP-c | DP-c | [Thm. 8] |
| $\text{Min}\{<k\}$ | P | W[2]-hard for pl. | ? | W[2]: [Thm. 9] |
| $\text{Min}\{>k\}$ | P | coNP-c for strongly pure | ? | P: [Thm. 10] |
| $\text{Min}\{=k\}$ | P | NP-hard for strongly pure | ? | NP, coNP: [Thm. 11] |
| $\text{Max}\{\theta k\}$ | P | P for poly. scores | P for poly. scores | [Thm. 12] |
| $\text{Min}\{\theta k\}$ | P | W[1]-hard | P for poly. scores | [Thm. 13] |
| $\text{Max}\{<\overline{k}\}$ | P | P for poly. scores | P for poly. scores | [Thm. 14] |
| $\text{Max}\{>\overline{k}\}$ | P | coNP-c for strongly pure | ? | P: [Cor. 15] |
| $\text{Max}\{=\overline{k}\}$ | P | NP-hard for strongly pure | ? | hardness: [Thm. 16] |

The study of the range of possible ranks, beyond the very top, is related to the problem of multiwinner election that has been studied mostly in the context of committee selection. Various utilities have been studied for qualifying selected committee, such as maximizing the number of voters with approved candidates [1] and, in that spirit, the Condorcet committees [9,11], aiming at proportional representation via frameworks such as Chamberlin and Courant’s [8] and Monroe’s [17], and the satisfaction of fairness and diversity constraints [6, 7]. In the case of incomplete voter preferences, the relevant problems are those of detecting the necessary and possible committee members. Note, however, that the problem of determining the elected committee can be intractable even if the preferences are complete [9,20,21,23], in contrast to rank determination (which is always in polynomial time in the framework we adopt). The problem of multiwinner determination for incomplete votes has been studied by Lu and Boutilier [15] in a perspective different from pure ranking: find a committee that minimizes the maximum objection (or “regret”) over all possible completions.

2 Preliminaries

We begin with some notation and terminology.

2.1 Voting Profiles and Positional Scoring Rules

Let $C = \{c_1, \ldots, c_m\}$ be the set of candidates (or alternatives) and let $V = \{v_1, \ldots, v_n\}$ be the set of voters. A voting profile $T = (T_1, \ldots, T_n)$ consists of $n$ linear orders on $C$, where each $T_i$ represents the ranking of $C$ by $v_i$. 
A positional scoring rule \( r \) is a series \( \{\vec{s}_m\}_{m\in\mathbb{N}^+} \) of \( m \)-dimensional score vectors \( \vec{s}_m = (\vec{s}_m(1), \ldots, \vec{s}_m(m)) \) of natural numbers where \( \vec{s}_m(1) \geq \cdots \geq \vec{s}_m(m) \) and \( \vec{s}_m(1) > \vec{s}_m(m) \). We denote \( \vec{s}_m(j) \) by \( r(m, j) \). Some examples of positional scoring rules include the plurality rule \((1, 0, \ldots, 0)\), the \( t \)-approval rule \((1, \ldots, 1, 0, \ldots, 0)\) that begins with \( t \) ones, the veto rule \((1, \ldots, 1, 0)\), the \( t \)-veto rule that ends with \( t \) zeros, and the Borda rule \((m - 1, m - 2, \ldots, 0)\).

Given a voting profile \( \mathbf{T} = (T_1, \ldots, T_n) \), the score \( s(T_i, c, r) \) that the voter \( v_i \) contributes to the candidate \( c \) is \( r(m, j) \) where \( j \) is the position of \( c \) in \( T_i \). The score of \( c \) in \( \mathbf{T} \) is \( s(\mathbf{T}, c, r) = \sum_{i=1}^{n} s(T_i, c, r) \) or simply \( s(\mathbf{T}, c) \) if \( r \) is clear from context. A candidate \( c \) is a winner (or co-winner) if \( s(\mathbf{T}, c) \geq s(\mathbf{T}, c') \) for all candidates \( c' \), and a unique winner if \( s(\mathbf{T}, c) > s(\mathbf{T}, c') \) for all candidates \( c' \neq c \).

We make standard assumptions about the positional scoring rule \( r \). We assume that \( r(m, i) \) is computable in polynomial time in \( m \). We also assume that the numbers in each \( \vec{s}_m \) are co-prime (i.e., their greatest common divisor is one).

A positional scoring rule is pure if \( \vec{s}_{m+1} \) is obtained from \( \vec{s}_m \) by inserting a score at some position, for all \( m > 1 \).

### 2.2 Partial Profiles

A partial voting profile \( \mathbf{P} = (P_1, \ldots, P_n) \) consists of \( n \) partial orders on set \( C \) of candidates, where each \( P_i \) represents the incomplete preference of the voter \( v_i \). A completion of \( \mathbf{P} = (P_1, \ldots, P_n) \) is a complete voting profile \( \mathbf{T} = (T_1, \ldots, T_n) \) where each \( T_i \) is a completion (i.e., linear extension) of the partial order \( P_i \). The problems of necessary winners and possible winners for partial voting preferences were introduced by Konczak and Lang [14].

Given a partial voting profile \( \mathbf{P} \), a candidate \( c \in C \) is a necessary winner if \( c \) is a winner in every completion \( \mathbf{T} \) of \( \mathbf{P} \), and \( c \) is a possible winner if there exists a completion \( \mathbf{T} \) of \( \mathbf{P} \) where \( c \) is a winner. Similarly, \( c \) is a necessary unique winner if \( c \) is a unique winner in every completion \( \mathbf{T} \) of \( \mathbf{P} \), and \( c \) is a possible unique winner if there exists a completion \( \mathbf{T} \) of \( \mathbf{P} \) where \( c \) is a unique winner.

The decision problems associated to a positional scoring rule \( r \) are those of determining, given a partial profile \( \mathbf{P} \) and a candidate \( c \), whether \( c \) is a necessary winner, a necessary unique winner, a possible winner, and a possible unique winner. We denote these problems by NW, NU, PW, and PU, respectively. A classification of the complexity of these problems has been established in a sequence of publications.

▶ **Theorem 1** (Classification Theorem [3424]). Each of NW and NU can be solved in polynomial time for every positional scoring rule. Each of PW and PU is solvable in polynomial time for plurality and veto; for all other pure scoring rules, PW and PU are NP-complete.

In this paper, we aim towards generalizing the Classification Theorem to determine the minimal and maximal ranks, as we formalize next.

### 2.3 Minimal and Maximal Ranks

The rank of a candidate is its position in the list of candidates, sorted by the sum of scores from the voters. However, for a precise definition, we need to resolve potential ties. Formally, let \( r \) be a positional scoring rule, \( C \) be a set of candidates, \( \mathbf{T} \) a voting profile, and \( \tau \) a tie breaker, which is simply a linear order over \( C \). Let \( R_{\mathbf{T}} \) be the linear order on \( C \) that sorts the candidates by their scores and then by \( \tau \); that is,

\[
R_{\mathbf{T}} := \{ c \gg c' : s(\mathbf{T}, c) > s(\mathbf{T}, c') \} \cup \{ c \gg c' : s(\mathbf{T}, c) = s(\mathbf{T}, c') \land c \tau c' \}.
\]
The rank of $c$ is the position of $c$ in $R_T$, and we denote it by $\text{rank}(c \mid T, \tau)$. If $T$ is replaced with a partial voting profile $P$, then we define $\text{ranks}(c \mid P, \tau)$ as the set of ranks that $c$ gets in the different completions of $P$:

$$\text{ranks}(c \mid P, \tau) := \{\text{rank}(c \mid T, \tau) \mid T \text{ extends } P\}$$

The minimal and maximal positions in $\text{ranks}(c \mid P, \tau)$ are denoted by $\min(c \mid P, \tau)$ and $\max(c \mid P, \tau)$, respectively.

Observe the following for a partial profile $P$ and a candidate $c$:

- $c$ is a possible winner if and only if $\min(c \mid P, \tau) = 1$ for any tie breaker $\tau$ that positions $c$ first.
- $c$ is a possible unique winner if and only if $\min(c \mid P, \tau) = 1$ for any tie breaker $\tau$ that positions $c$ last.
- $c$ is a necessary winner if and only if $\max(c \mid P, \tau) = 1$ for any tie breaker $\tau$ that positions $c$ first.
- $c$ is a necessary unique winner if and only if $\max(c \mid P, \tau) = 1$ for any tie breaker $\tau$ that positions $c$ last.

To investigate the computational complexity of calculating the minimal and maximal ranks for a scoring rule $r$, we will consider the decision problems of determining, given $P$, $c$, $\tau$ and a position $k$, whether $X(c \mid P, \tau) \theta k$ where $X$ is one of $\text{min}$ and $\text{max}$ and $\theta$ is one of $<, >, =$. We denote these problems by $\text{Min}_r\{\theta\}$ and $\text{Max}_r\{\theta\}$. Moreover, we will omit the rule $r$ when it is clear from the context. For example, $\text{Min}_r\langle \rangle$ (or just $\text{Min}\langle \rangle$) is the decision problem of determining whether $\min(c \mid P, \tau) < k$, and $\text{Max}_r\{\} = \{\} = \text{Max}\{\} = \text{Max}\{\}$ decides whether $\max(c \mid P, \tau) = k$.

### 2.4 Additional Notation.

We use the following notation. For a set $A$ and a partition $A_1, \ldots, A_t$ of $A$:

- $P(A_1, \ldots, A_t)$ denotes the partitioned partial order $\{a_1 \succ \cdots \succ a_t : \forall i \in [t], a_i \in A_i\}$.
- $O(A_1, \ldots, A_t)$ denotes an arbitrary linear order on $A$ that completes $P(A_1, \ldots, A_t)$.

A linear order $a_1 \succ \cdots \succ a_t$ is also denoted as a vector $(a_1, \ldots, a_t)$. The concatenation $(a_1, \ldots, a_t) \circ (b_1, \ldots, b_t)$ is $(a_1, \ldots, a_t, b_1, \ldots, b_t)$.

### 3 Complexity of Minimum and Maximum Ranks

In this section, we show that the problems we study are computationally hard for all positional scoring rules. We begin with a lemma that is proved by combining some well known results.

#### Lemma 2.

The following problems are DP-complete: Given a cubic graph $G$ and $k$,

determine whether

- the largest independent set of $G$ has size exactly $k$;
- the smallest vertex cover of $G$ has size exactly $k$.

**Proof.** First, the problems of deciding whether a graph contains an independent set (resp. vertex cover) of size $k$ is known to be NP-complete for cubic graphs [19]. By a straightforward reduction, we get that deciding whether a graph contains a clique of size $k$ is NP-complete for graphs $G = (U, E)$ where $\deg(u) = |E| - 3$ for every $u \in U$. Determining whether the largest clique has size exactly $k$ in such graphs can be shown to be DP-complete using the same proof of Papadimitriou and Yannakakis [18] that exact clique is DP-complete. Then, using the standard reductions from cliques to independent sets and from cliques to vertex covers, we can deduce the DP-completeness for the two problems of the lemma. ▶
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The following theorems state the hardness of computing the minimal and maximal rank for all positional scoring rules. We begin with Min$_r$\{<\} and Min$_r$\{=\}. Note that the NP-completeness of Min$_r$\{<\} immediately implies the coNP-completeness of Min$_r$\{>\}.

**Theorem 3.** For every positional scoring rule $r$, Min$_r$\{<\} is NP-complete and Min$_r$\{=\} is DP-complete.

**Proof.** Memberships in the corresponding classes (NP and DP) are straightforward. We show hardness for Min$_r$\{=\} by a reduction from vertex cover in cubic graphs, which is DP-complete by Lemma 2. This reduction also shows the NP-hardness of Min$_r$\{<\}.

Let $r$ be a positional scoring rule, and denote $r$ by $\{s_m\}_{m \geq 1}$. Assume, w.l.o.g., that $s_m(m) = 0$ for every $m > 1$. (Otherwise, we can subtract $s_m(m)$ from all the entries in the vector without affecting the ranks in any profile.) Let $G = (U,E)$ be a cubic graph with $U = \{u_1,\ldots,u_n\}$. We construct an instance $(C,P,\tau)$ under $r$. The candidate set is $C = U \cup \{c^*,d\}$ and the tie breaker is $\tau = O(\{c^*,d\},U)$. The voting profile $P = P^1 \circ T^2$ is the concatenation of two parts $P^1$ and $T^2$ that we describe next.

Note that $|C| = n + 2$. Let $\ell < n + 2$ be an index such that $s_{n+2}(\ell) > s_{n+2}(\ell + 1) = 0$. We know that such an $\ell$ exists due to the definition of a scoring rule and our assumption that $s_m(m) = 0$ for every $m > 1$.

The first part of the profile contains a profile for every $P^1 = \{P^1_e\}_{e \in E}$. For every edge $e = \{u,w\} \in E$, the profile $P^1_e = (P^1_e(1),\ldots,P^1_e(n))$ consists of $n$ voters, as illustrated in Figure 1 for every $i \in [n]$, denote $M_i(C \setminus e) = (c_{i1},\ldots,c_{in})$ where $M_i$ is the $i$th circular vote as defined by Baumeister, Roos and Jörj 2:

$$M_i(a_1,\ldots,a_\ell) := (a_i,a_{i+1},\ldots,a_\ell,a_1,a_2,\ldots,a_{i-1}).$$

The $i$th voter in $P^1_e$ is

$$P^1_e(i) = (c_{i1},c_{i2},\ldots,c_{i\ell-1},\{u,w\},c_{i\ell},c_{i\ell+1},\ldots,c_{in}).$$

**Figure 1** The voters of the profile $P^1_e = (P^1_e(1),\ldots,P^1_e(n))$ for the edge $e = \{u,w\}$ used in the proof of Theorem 3. The other candidates are denoted as $C \setminus U = \{c_1,\ldots,c_n\}$.

The second part of the profile contains a profile for the edge $\{u,w\} \in E$.

$$P^2_e = (P^2_e(1),\ldots,P^2_e(n))$$

The voters of the profile $(T^2_1,\ldots,T^2_n)$ used in the proof of Theorem 3.
This means that in \( P^1 \), the candidates \( u \) and \( w \) can only be at positions \( \ell \) and \( \ell + 1 \), and the other candidates are circulating at all other positions.

The second part of the profile, \( T^2 \), consists of three copies of the profile \( (T^1_1, \ldots, T^1_n) \), as illustrated in Figure 2. For every \( i \in [n] \), denote \( M_i(U) = (c_{i_1}, \ldots, c_{i_n}) \) and define

\[
T^2_i = (c_{i_1}, c_{i_2}, \ldots, c_{i_{\ell-1}}, d, c_{i_1}, c_{i_{\ell+1}}, \ldots, c_{i_n}).
\]

This means that \( d \) and \( c^* \) are always at positions \( \ell \) and \( \ell + 1 \), respectively, and the candidates of \( U \) are circulating at all other positions.

This completes the construction of \((C, P, \tau)\). Next, we state some observations. Let \( T = T^1 \circ T^2 \) be a completion of \( P \) where \( T^1 = \{T^1_e\}_{e \in E} \).

The scores of \( c^* \) and \( d \) in \( T^1 \) are

\[
s(T^1, c^*) = s(T^1, d) = \sum_{e \in E} s(T^1_e, d) = |E| \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i).
\]

For every \( u \in U \), denote by \( E(u) \) the set of edges incident to \( u \), and denote \( \overline{E}(u) = E \setminus E(u) \).

Recall that \( |E(u)| = 3 \) since the graph is cubic. By definition, it holds that

\[
s(T^1, u) = \sum_{e \in E(u)} s(T^1_e, u) + \sum_{e \in \overline{E}(u)} s(T^1_e, u).
\]

Observe that

\[
\sum_{e \in E(u)} s(T^1_e, u) \leq 3n \cdot \bar{s}_{n+2}(\ell)
\]

and that

\[
\sum_{e \in \overline{E}(u)} s(T^1_e, u) = (|E| - 3) \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i).
\]

In \( T^2 \), the score of \( c^* \) is \( s(T^2, c^*) = 3n \cdot \bar{s}_{n+2}(\ell + 1) = 0 \) and the score of \( d \) is \( s(T^2, d) = 3n \cdot \bar{s}_{n+2}(\ell) \). For every \( u \in U \), \( s(T^2, u) = 3 \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i) \).

Overall,

\[
s(T, c^*) = s(T^1, c^*) + s(T^2, c^*) = |E| \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i) + 0 = |E| \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i) \tag{1}
\]

\[
s(T, d) = s(T^1, d) + s(T^2, d) = |E| \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i) + 3n \cdot \bar{s}_{n+2}(\ell) \tag{2}
\]

and for every \( u \in U \),

\[
s(T, u) = s(T^1, u) + s(T^2, u) = \sum_{e \in E(u)} s(T^1_e, u) + \sum_{e \in \overline{E}(u)} s(T^1_e, u) + s(T^2, u)
\]

\[
= \sum_{e \in E(u)} s(T^1_e, u) + (|E| - 3) \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i) + 3 \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i)
\]

\[
= \sum_{e \in E(u)} s(T^1_e, u) + |E| \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i) \tag{3}
\]

\[
\leq 3n \cdot \bar{s}_{n+2}(\ell) + |E| \sum_{i \neq \ell, \ell+1} \bar{s}_{n+2}(i) = s(T, d) \tag{4}
\]
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From Equations (2) and (4) and the definition of \( \tau \) we conclude that \( d \) always defeats all other candidates. From Equations (2) and (3) we conclude that \( c^* \) defeats \( u \) if and only if \( \sum_{e \in E(u)} s(T^1_e, u) = 0 \).

Denote by \( \alpha(G) \) the minimal size of a vertex cover in \( G \). The following two claims show that for any \( k \), \( \min(c^* \mid P, \tau) \leq k + 2 \) if and only if \( \alpha(G) \leq k \), which implies NP-completeness for \( \text{Min}_r\{\prec\} \).

\[ \forall \text{ Claim. If } \alpha(G) \leq k \text{ then } \min(c^* \mid P, \tau) \leq k + 2. \]

\[ \text{Proof. Assume that } \alpha(G) \leq k \text{, and let } B \text{ be a vertex cover of size at most } k \text{ in } G. \text{ Consider} \]

the following completion \( T = \{T^1\}_{e \in E} \circ T^2 \) of \( P \). For every \( e = \{u, w\} \in E \), recall that only the positions of \( u \) and \( w \) are not determined in the voters of \( P^1_e \).

- If \( u \in B \), then in all voters of \( T^1_e \) the candidate \( u \) is placed at the \( \ell \)th position and \( w \) is placed at the \((\ell + 1)\)th position.

- Otherwise, \( w \in B \) (since \( B \) is a vertex cover), and then in all voters of \( T^1_e \) the candidate \( w \) is placed at the \( \ell \)th position and \( u \) is placed at the \((\ell + 1)\)th position.

So, for every \( u \notin B \) the candidate \( u \) is placed at the \((\ell + 1)\)th position in all voters of \( \{T^1_e\}_{e \in E(u)} \), hence \( \sum_{e \in E(u)} s(T^1_e, u) = 0 \) and \( c^* \) defeats \( u \). These are at least \( n - k \) candidates which \( c^* \) defeats, therefore \( \text{rank}(c^* \mid T, \tau) \leq k + 2 \) and \( \min(c^* \mid P, \tau) \leq k + 2. \]

\[ \forall \text{ Claim. If } \min(c^* \mid P, \tau) \leq k + 2 \text{ then } \alpha(G) \leq k. \]

\[ \text{Proof. Assume that } \min(c^* \mid P, \tau) \leq k + 2 \text{, and let } T = \{T^1_e\}_{e \in E} \circ T^2 \text{ be a completion of } P \]

where \( \text{rank}(c^* \mid T, \tau) \leq k + 2 \). Let \( B \subseteq U \) be the candidates of \( U \) that defeat \( c^* \) in \( T \), we know that \( |B| \leq k \) because \( d \) always defeats \( c^* \).

For every \( e = \{u, w\} \in E \), all voters of \( T^1_e \) placed a vertex from \( B \) at the \( \ell \)th position. (If a vertex from \( U \setminus B \) is placed at the \( \ell \)th position for some voter, then this vertex defeats \( c^* \), in contradiction to the definition of \( B \).) Since these voters can only place \( u \) and \( w \) at the \( \ell \)th position, we get that either \( u \in B \) or \( w \in B \). Hence \( B \) is a vertex cover, which implies \( \alpha(G) \leq k \).

Finally, observe that \( \min(c^* \mid P, \tau) = \alpha(G) + 2 \) (by choosing \( k = \alpha(G) \) and \( k = \min(c^* \mid P, \tau) - 2 \) in the two claims), which implies DP-completeness for \( \text{Min}_r\{\prec\} \).
Theorem 4. For every positional scoring rule \( r \), \( \text{Max}_r\{\geq\} \) is NP-complete and \( \text{Max}_r\{=\} \) is DP-complete.

Proof. Memberships in the corresponding classes (NP and DP) are straightforward. We show hardness for \( \text{Max}_r\{=\} \) by a reduction from independent set in cubic graphs, which is DP-complete by Lemma 2. This reduction also shows hardness for \( \text{Max}_r\{>\} \) and uses some parts from the reduction of the proof of Theorem 3.

Let \( r \) be a positional scoring rule, and denote \( r \) by \( \{s_m\}_{m>1} \). Assume, w.l.o.g., that \( s_m(m) = 0 \) for every \( m > 1 \). Let \( G = (U, E) \) be a cubic graph with \( U = \{u_1, \ldots, u_n\} \). We construct an instance \((C, P, \tau)\) under \( r \). The candidates set is \( C = U \cup \{c^*, d\} \) and the tie breaker is \( \tau = O(U, \{c^*, d\}) \). The voting profile is the concatenation \( P = P^1 \circ P^2 \circ P^3 \) of three parts described next.

Note that \( |C| = n + 2 \). Let \( \ell < n + 2 \) be an index such that \( s_{n+2}(\ell) > s_{n+2}(\ell + 1) = 0 \). The first two parts \( P^1 = \{P_i\}_{i \in E} \) and \( P^2 \) are the same as in the proof of Theorem 3. The third part, \( P^3 \), consists of \( 3n \) copies of the profile \((T^1_1, \ldots, T^1_n)\), as illustrated in Figure 3. We start with \( T^3_i = M_i(u_1, \ldots, u_n, d, c^*) \) for the circular votes as defined in the proof of Theorem 3 and then perform the following change. There exists some \( i \in [n + 2] \) such that \( d \) and \( c^* \) are placed at position \( \ell \) and \( \ell + 1 \), respectively, in \( T^3_i \). In this voter, switch the positions of \( d \) and \( c^* \). This means that in \((T^1_1, \ldots, T^3_n)\), the candidate \( c^* \) is placed at the \( \ell \)th position twice, and \( d \) is placed at the \((\ell + 1)\)th position twice.

Since \( s_{n+2}(\ell + 1) = 0 \), observe that \( s(T^3, c^*) = 3n \left( \sum_{i=1}^{n+2} s_{n+2}(i) + s_{n+2}(\ell) \right) \) and \( s(T^3, d) = 3n \left( \sum_{i=1}^{n+2} s_{n+2}(i) - s_{n+2}(\ell) \right) \). For every \( u \in U \) we have that \( s(T^3, u) = 3n \sum_{i=1}^{n+2} s_{n+2}(i) \). By combining this with the observations from the proof of Theorem 3 we get that for every completion \( T = \{T^1_{e_i}\}_{e \in E} \circ P^2 \circ P^3 \) of \( P \), the following holds. The score of \( c^* \) is given by

\[
s(T, c^*) = s(T^1, c^*) + s(T^2, c^*) + s(T^3, c^*)
\]

\[
= \left| E \right| \sum_{i \neq \ell, \ell+1} s_{n+2}(i) + 0 + \left( 3n \sum_{i=1}^{n+2} s_{n+2}(i) + 3n \cdot s_{n+2}(\ell) \right)
\]

\[
= \left| E \right| \sum_{i \neq \ell, \ell+1} s_{n+2}(i) + 3n \sum_{i=1}^{n+2} s_{n+2}(i) + 3n \cdot s_{n+2}(\ell).
\]

The score of \( d \) is given by

\[
s(T, d) = s(T^1, d) + s(T^2, d) + s(T^3, d)
\]

\[
= \left| E \right| \sum_{i \neq \ell, \ell+1} s_{n+2}(i) + (3n \cdot s_{n+2}(\ell)) + \left( 3n \sum_{i=1}^{n+2} s_{n+2}(i) - 3n \cdot s_{n+2}(\ell) \right)
\]

\[
= \left| E \right| \sum_{i \neq \ell, \ell+1} s_{n+2}(i) + 3n \sum_{i=1}^{n+2} s_{n+2}(i).
\]
The score of every $u \in U$ is given by

\[
s(T, u) = \sum_{e \in E(u)} s(T^1_e, u) + \left( \sum_{e \in E(u)} s(T^1_e, u) + s(T^2_e, u) \right) + s(T^3_e, u)
\]

\[
= \sum_{e \in E(u)} s(T^1_e, u) + |E| \sum_{i \neq \ell, \ell+1} s_{n+2}(i) + 3n \sum_{i=1}^{n+2} s_{n+2}(i)
\]

\[
= 3n \cdot s_{n+2}(\ell) + |E| \sum_{i \neq \ell, \ell+1} s_{n+2}(i) + 3n \sum_{i=1}^{n+2} s_{n+2}(i)
\]

\[
= s(T, c^*).
\]

By this analysis and the definition of $\tau$, $d$ is always defeated by all other candidates, and $u$ defeats $c^*$ if and only if $\sum_{e \in E(u)} s(T^1_e, u) = 3n \cdot s_{n+2}(\ell)$. Denote the maximal size of an independent set in $G$ by $\beta(G)$. The following two claims show that for any $k$, $\max(c^* | P, \tau) \geq k + 1$ if and only if $\beta(G) \geq k$, which implies NP-completeness for $\text{Max}_r \{\cdot\}$.

\textbf{Claim.} If $\beta(G) \geq k$ then $\max(c^* | P, \tau) \geq k + 1$.

\textbf{Proof.} Assume that $\beta(G) \geq k$, let $B$ be an independent set of size at least $k$ in $G$. Consider a completion $T = \{T^1_e\}_{e \in E} \circ T^2 \circ T^3$ of $P$ as follows. For every $e = \{u, w\} \in E$, recall that only the positions of $u$ and $w$ are not determined in the voters of $P^1_e$. If $u, w \notin B$ then complete all voters of $P^1_e$ arbitrarily. If $u \in B$ then in all voters of $T^1_e$, $u$ is placed at the $\ell$th position and $w$ is placed at the $(\ell + 1)$th position. Finally, if $w \in B$ then in all voters of $T^1_e$, $w$ is placed at the $\ell$th position and $u$ is placed at the $(\ell + 1)$th position. Note that we cannot have $u, w \in B$ because $B$ is an independent set.

For every $u \in B$, $u$ is placed at the $\ell$th position in all voters of $\{T^1_e\}_{e \in E(u)}$, hence $\sum_{e \in E(u)} s(T^1_e, u) = 3n \cdot s_{n+2}(\ell)$ and $u$ defeats $c^*$. These are at least $k$ candidates which defeat $c^*$, therefore $\text{rank}(c^* | T, \tau) \geq k + 1$ and $\max(c^* | P, \tau) \geq k + 1$. ◀

\textbf{Claim.} If $\max(c^* | P, \tau) \geq k + 1$ then $\beta(G) \geq k$.

\textbf{Proof.} Assume that $\max(c^* | P, \tau) \geq k + 1$, there exists a completion $T = \{T^1_e\}_{e \in E} \circ T^2 \circ T^3$ of $P$ where $\text{rank}(c^* | T, \tau) \geq k + 1$. Let $B$ be the candidates which defeat $c^*$ in $T$. $B \subseteq U$ because $c^*$ always defeats $d$, and $|B| \geq k$. For every $u \in B$ we get that $\sum_{e \in E(u)} s(T^1_e, u) = 3n \cdot s_{n+2}(\ell)$, otherwise $u$ does not defeat $c^*$.

Assume to the contrary that $e = \{u, w\} \in B$ for some pair $u, w \in B$. In the voters of $T^1_e$, both $u$ and $w$ should be placed at the $\ell$th position, that is a contradiction. Hence $B$ is an independent set, which implies $\beta(G) \geq k$. ◀

Finally, observe that $\max(c^* | P, \tau) = \beta(G) + 1$ (by choosing $k = \beta(G)$ and $k = \max(c^* | P, \tau) - 1$ in the two claims), which implies DP-completeness for $\text{Max}_r \{\cdot\}$. ◀

\section{Comparison to a Bounded Rank}

In the previous section, we established that the problems of computing the minimal and maximal ranks are very often intractable. We now investigate the complexity of comparing the minimal and maximal ranks to some fixed rank $k$. Hence, the input consists of only $P$, $c$ and $\tau$, but not $k$. We denote these problems by $\text{Min}_r \{\theta k\}$ and $\text{Max}_r \{\theta k\}$ where, as usual, $\theta$ is one of $<, >, =$. Again, we will omit the rule $r$ when it is clear from the context. For
example, Min\{< k\} is the decision problem of determining whether \( \min(c \mid P, \tau) < k \), and Max\{= k\} decides whether \( \max(c \mid P, \tau) = k \).

We will show that the complexity picture for Min\{= k\} and Max\{= k\} is way more positive, as we generalize the tractability of almost all of the tractable scoring rules for NW and PW. We will also generalize hardness results from PW to Min\{= k\}; interestingly, this generalization turns out to be quite nontrivial.

In addition to comparing to the fixed \( k \), we will consider the problem of comparing to \( k := m - k + 1 \) where \( m \) is, as usual, the number of candidates. Note that for \( k = \{1, \ldots, m\} \), the position \( k \) is the \( k \)th rank from the end (bottom). Hence, Max\{> k\} is the problem of deciding whether the candidate can end up in the bottom \( k - 1 \) positions.

### 4.1 Complexity of Min\{ θk\}

We begin with the problems of Min\{ θk\} and focus first on the plurality and veto rules.

#### 4.1.1 Plurality and Veto

We first show that the positional scoring rules that are tractable for PW, namely plurality and veto, are also tractable for Min\{ θk\}. This is done by a lemma which uses a reduction to the problem of polygamous matching [13]. Given a bipartite graph \( G = (U \cup W, E) \) and natural numbers \( \alpha_w \leq \beta_w \) for all \( w \in W \), determine whether there is a subset of \( E \) where each \( u \in U \) is incident to exactly one edge and every \( w \in W \) is incident to at least \( \alpha_w \) edges and at most \( \beta_w \) edges. This problem is known to be solvable in polynomial time [10,22].

> **Lemma 5.** The following decision problem can be solved in polynomial time for the plurality and veto rules: given a partial profile \( P \) over a set \( C \) of candidates and numbers \( \gamma_c \leq \delta_c \) for every candidate \( c \), is there a completion \( T \) such that \( \gamma_c \leq s(T, c) \leq \delta_c \) for every \( c \in C \)?

**Proof.** For both rules, we apply a reduction to polygamous matching, where \( U = V \) (the set of voters) and \( W = C \). For plurality, \( E \) connects \( v_i \in V \) and \( c \in C \) whenever \( c \) can be in the top position in one or more completions of \( P_i \), and the bounds are \( \alpha_c = \gamma_c \) and \( \beta_c = \delta_c \). For veto, receiving a score \( s \) is equivalent to being placed in the bottom position of \( n - s \) voters, so \( E \) connects \( v_i \in V \) and \( c \in C \) whenever \( c \) can be in the bottom position in one or more completions of \( P_i \). The bounds are \( \alpha_c = n - \delta_c \) and \( \beta_c = n - \gamma_c \).

To solve Min\{< k\} (Min\{> k\} and Min\{= k\} immediately follow) given \( C, P, \tau \) and \( c \), we search for a completion where \( c \) defeats more than \( m - k \) candidates. For this goal we consider every set \( D \subseteq C \setminus \{c\} \) of size \( m - k + 1 \) and search for a completion where \( c \) defeats all candidates of \( D \). For that, we iterate over every integer score \( 0 \leq s \leq n \) and use Lemma 5 to test whether there exists a completion \( T \) such that \( s(T, c) \geq s \), and for every \( d \in D \) we have \( s(T, d) \leq s \) if \( c \tau d \) or \( s(T, d) < s \) otherwise. Hence, we conclude that:

> **Theorem 6.** For every fixed \( k \geq 1 \), Min\{ θk\} is solvable in polynomial time under the plurality and veto rules, where \( \theta \) is one of \(<, >, =\).

The polynomial degree in Theorem 6 depends on \( k \). The following result shows that this is unavoidable, at least for the plurality rule, under conventional assumptions in parameterized complexity.

> **Theorem 7.** Under the plurality rule, Min\{<\} is W[2]-hard for the parameter \( k \).
Proof. We show an FPT reduction from the dominating set problem, which is the following: Given an undirected graph \( G = (U, E) \) and an integer \( k \), is there a set \( D \subseteq U \) of size \( k \) such that every vertex is either in \( D \) or adjacent to some vertex in \( D \)? This problem is known to be \( W[2] \)-hard for the parameter \( k \).

Given a graph \((U, E)\) with \( U = \{u_1, \ldots, u_n\} \), we construct an instance for Min\{\(<\)\} under plurality where the candidates are \( C = U \cup \{c^*\} \), the tie breaker is \( \tau = O(\{c^*\}, U) \), and the voting profile is \( P = (P_1, \ldots, P_n) \). Let \( N(u_i) \) be the set of neighbours of \( u_i \) in the graph, and let \( N[u_i] = N(u_i) \cup \{u_i\} \). For all \( i \in \{n\} \) we have \( P_i = P(N[u_i]) \cup N[u_i], \{c^*\} \). Hence, the \( i \)th voter can vote only for vertices that dominate \( u_i \). To complete, we show that the graph has a dominating set of size \( k \) if and only if \( \min(c^* | P, \tau) < k + 2 \).

Suppose there is a dominating set \( D \) of size \( k \), consider the profile \( T = (T_1, \ldots, T_n) \) where

\[
T_i := O(N[u_i] \cap D, N[u_i] \setminus D, U \setminus N[u_i], \{c^*\}).
\]

In this completion, for each \( u \notin D \) we get \( s(T, u) = 0 \). These are \( n - k \) candidates that \( c^* \) defeats, therefore rank\((c^* | T, \tau) < k + 2 \) (at most \( k \) candidates defeat \( c^* \)) and \( \min(c^* | P, \tau) < k + 2 \). Conversely, if \( \min(c^* | P, \tau) < k + 2 \) then in some completion \( T \) it defeats at least \( n - k \) candidates, and these candidates have a score 0 in \( T \). Let \( D \) be the set of candidates that \( c^* \) does not defeat in \( T \), all voters voted for candidates in \( D \) and \(|D| \leq k \). A voter \( P_i \) can only vote for vertices which dominate \( u_i \), hence \( D \) is a dominating set of size at most \( k \).

4.1.2 Beyond Plurality and Veto

The Classification Theorem (Theorem 1) states that PW is intractable for every pure scoring rule \( r \) other than plurality or veto. While this hardness easily generalizes to Min\{\(\leq k\)\} for \( k = 1 \), it is not at all clear how to generalize it to any \( k > 1 \). In particular, we cannot see how to reduce PW to Min\{\(\leq k\)\} while assuming only the purity of the rule. We can, however, show such a reduction under a stronger notion of purity, as long as the scores are bounded by a polynomial in the number \( m \) of candidates. In this case, we say that the rule has polynomial scores. Note that this assumption is in addition to our usual assumption that the scores can be computed in polynomial time.

A rule \( r \) is strongly pure if the score sequence for \( m + 1 \) candidates is obtained from the score sequence for \( m \) candidates by inserting a new score, either to beginning or the end of the sequence. More formally, \( r = \{s_m\}_{m \in \mathbb{N}^+} \) is strongly pure if for all \( m \geq 1 \), either \( s_{m+1} = s_m + 1 \) or \( s_{m+1} = s_m \). Note that \( t \)-approval, \( t \)-veto and Borda are all strongly pure.

\[ \textbf{Theorem 8.} \text{ Suppose that the positional scoring rule is strongly pure, has polynomial scores, and is neither plurality nor veto. Then Min\{<\} is NP-complete and Min\{\(= k\)\} is NP-hard for all fixed } k \geq 1. \]

Proof. Let \( r \) be a positional scoring rule that satisfies the conditions of the theorem, and let us denote \( r \) by \( \{\bar{s}_m\}_{m \geq 1} \). We use a reduction from PW under \( r \). Consider the input \( P = (P_1, \ldots, P_n) \) and \( c \) for PW over a set \( C \) of \( m \) candidates. Let \( m' = m + k - 1 \). Since \( r \) is strongly pure, there is an index \( t \leq k - 1 \) such that

\[ \bar{s}_{m'} = (\bar{s}_m(1), \ldots, \bar{s}_m(t)) \circ \bar{s}_m \circ (\bar{s}_m(t + m + 1), \ldots, \bar{s}_m(m')). \]

That is, \( \bar{s}_{m'} \) is obtained from \( \bar{s}_m \) by inserting \( t \) values at the top coordinates and \( k - 1 - t \) values at the bottom coordinates. We define \( C', P' \) and \( \tau' \) as follows. The candidates are
\[ C' = C \cup D_1 \cup D_2 \text{ where } D_1 = \{ d_1, \ldots, d_t \} \text{ and } D_2 = \{ d_{t+1}, \ldots, d_{k-1} \}, \text{ denote } D = D_1 \cup D_2. \]
The tie breaker is \( \tau' = O(D_1 \setminus \{ c \}, C \setminus \{ c \}) \). The profile \( P' \) is the concatenation \( Q \circ M \) of two partial profiles. The first is \( Q = \{ Q_1, \ldots, Q_n \} \), where \( Q_i \) is the same as \( P_i \), except that the candidates of \( D_1 \) are placed at the top positions and the candidates of \( D_2 \) are placed at the bottom positions. Formally, \( Q_i := P_i \cup P(D_1, C, D_2) \). The second, \( M \), consists of \( n \cdot \overset{\rightarrow}{s}_{m'}(1) \) copies of the profile \( \{ M_{i,j} \}_{i=1, \ldots, k-1, j=1, \ldots, m} \) where \( M_{i,j} \) is \( M_i(D) \circ M_j(C) \) for the circular votes \( M_i(D) \) and \( M_j(C) \) as defined in the proof of Theorem 8.

We show that the candidates of \( D \) always defeat all other candidates. For every \( d \in D \), the score of \( d \) in \( M \) is \( s(M, d) = n \cdot \overset{\rightarrow}{s}_{m'}(1) \cdot m \sum_{i=1}^{k-1} \overset{\rightarrow}{s}_{m'}(i) \), and for every \( c' \in C \) the score in \( M \) is
\[
s(M, c') = n \cdot \overset{\rightarrow}{s}_{m'}(1) \cdot (k-1) \sum_{i=k}^{m'} \overset{\rightarrow}{s}_{m'}(i) \leq n \cdot \overset{\rightarrow}{s}_{m'}(1) \cdot \left( m \sum_{i=1}^{k-1} \overset{\rightarrow}{s}_{m'}(i) - 1 \right)
\]
where the inequality is due to the assumption that \( \overset{\rightarrow}{s}_{m'}(1) > \overset{\rightarrow}{s}_{m'}(m') \). Let \( T' \) be a completion of \( P' \), the total score of \( c' \) is
\[
s(T', c') \leq n \cdot \overset{\rightarrow}{s}_{m'}(1) + s(M, c') \leq n \cdot \overset{\rightarrow}{s}_{m'}(1) + s(M, d) - n \cdot \overset{\rightarrow}{s}_{m'}(1) \leq s(T', d).
\]
Since the candidates of \( D \) are the first candidates in \( \tau' \), they always defeat the candidates of \( C \). We show that \( c \) is a possible winner for \( P \) if and only if \( \min(c | P', \tau') = k \). Since the candidates of \( D \) are always the first \( k-1 \) candidates, this is equivalent to \( \min(c | P', \tau') < k+1 \).

Let \( T = (T_1, \ldots, T_n) \) be a completion of \( P \) where \( c \) is a winner. Consider the completion \( T' = (T'_{1}, \ldots, T'_{n}) \circ M \) of \( P' \) where \( T'_{i} = O(D_1) \circ T_i \circ O(D_2) \). For every \( c' \in C \), we know that

![Figure 4](image-url) The voters \( M_{i,j} \) used in the proof of Theorem 8.
We prove that for any fixed $q$, when $q = k$, hence $\pi$ is a winner in $T$.

4.2 Complexity of $\text{Max}\{\theta k\}$

We prove that for any fixed $k \geq 1$ and positional scoring rules $r$ with polynomial scores, $\text{Max}\{\theta k\}$ is solvable in polynomial time when $\theta$ is one of $<, >, =$.

Note that all of the specific rules mentioned so far (i.e., $t$-approval, $t$-veto, Borda and so on) have polynomial scores, and hence, are covered by Theorem 9. An example of a rule that is not covered is the rule defined by $r(m, j) = 2^{m-j}$.

The polynomial degree in Theorem 9 depends on $k$. This is unavoidable under conventional assumptions in parameterized complexity. This is shown by the proof of Theorem 4 that gives an FPT reduction from the problem of independent set in cubic graphs problem, which is $W[1]$-hard for the parameter $k$, to $\text{Max}\{>\}$. Therefore:

Theorem 10. For every positional scoring rule, $\text{Max}\{>\}$ is $W[1]$-hard for the parameter $k$.

In the remainder of this section, we prove Theorem 9. We show that $\text{Max}\{>\}$ is in polynomial time, which immediately implies that $\text{Max}\{=k\}$ and $\text{Max}\{<k\}$ are in polynomial time. To determine whether $\max(c \mid \rho, \tau) > k$, we search for $k$ candidates that defeat $c$ in some completion $T$, since $\max(c \mid \rho, \tau) > k$ if and only if at least $k$ candidates defeat $c$ in $T$. For that, we iterate over every subset $\{c_1, \ldots, c_k\} \subseteq C \setminus \{c\}$ and determine whether these $k+1$ candidates can get a combination of scores where $c_1, \ldots, c_k$ all defeat $c$.

More formally, let $C$ be a set of candidates and $r$ a positional scoring rule. For a partial profile $P = (P_1, \ldots, P_n)$ and a sequence $S = (c_1, \ldots, c_q)$ of candidates from $C$, we denote by $\pi(P, S)$ the set of all possible scores that the candidates in $S$ can obtain jointly in a completion:

$$\pi(P, S) := \{s(T, c_1), \ldots, s(T, c_q) \mid T \text{ completes } P\}.$$

Note that $\pi(P, S) \subseteq \{0, \ldots, n \cdot \bar{s}_m(1)\}^q$. When $P$ consists of a single voter $P$, we write $\pi(P, S)$ instead of $\pi(P, S)$. To show that $\max(c \mid P, \tau) > k$ we need a sequence $S = (c_1, \ldots, c_q)$ where $q = k + 1$ and $c_q = c$, and a sequence $(s_1, \ldots, s_q) \in \pi(P, S)$ such that each $c_i$ beats $c$ when for $i = 1, \ldots, k$ the score of $c_i$ is $s(c_i) = s_i$ and $s(c) = s_q$. The following two lemmas show that, indeed, if such sequences exist, we can find them in polynomial time.

Lemma 11. Let $q$ be a fixed number and $r$ a positional scoring rule. Whether $(s_1, \ldots, s_q) \in \pi(P, S)$ can be determined in polynomial time, given a partial order $P$ over a set of candidates, a sequence $S$ of $q$ candidates, and scores $s_1, \ldots, s_q$. 

$s(T', d) \geq s(T', c')$ for every $d \in D$, and from the property of $\bar{s}_m$, we get that

$$s(T', c') = s(T, c') + n \cdot \bar{s}_m(1) \cdot (k - 1) \sum_{i=k} s_m(i).$$

From the choice of $\tau'$, $c$ defeats all candidates of $C \setminus \{c\}$ in $T'$, hence $\max(c \mid T', \tau') = k$.

Conversely, let $T' = (T'_1, \ldots, T'_n) \circ M$ be a completion of $P'$ where $\max(c \mid T', \tau') = k$, define a completion $T$ of $P$ by removing $D$ from all orders in $(T'_1, \ldots, T'_n)$. For every $c' \in C$ we have

$$s(T, c') = s(T', c') - n \cdot \bar{s}_m(1) \cdot (k - 1) \sum_{i=k} s_m(i)$$

hence $c$ is a winner in $T$. □
Proof. We use a reduction to a scheduling problem where tasks have execution times, release times, deadlines, and precedence constraints (i.e., task \( x \) should be completed before starting task \( y \)). This scheduling problem can be solved in polynomial time \([12]\). In the reduction, each candidate \( c \) is a task with a unit execution time. For every \( c_i \) in \( S \), the release time is \( \min \{ j \in [n] : r(m, j) = s_i \} \), and the deadline is \( 1 + \max \{ j \in [n] : r(m, j) = s_i \} \). For the rest of the candidates, the release time is 1 and the deadline is \( m + 1 \). The precedence constraints are \( P \). It holds that \( (s_1, \ldots, s_q) \in \pi(P, S) \) if and only if the tasks can be scheduled according to all the requirements.

From Lemma \([11]\) we can conclude that when \( q \) is fixed and \( r \) has polynomial scores, we can construct \( \pi(P, S) \) in polynomial time, via straightforward dynamic programming.

\[\text{Lemma 12.}\] Let \( q \) be a fixed natural number and \( r \) a positional scoring rule with polynomial scores. The set \( \pi(P, S) \) can be constructed in polynomial time, given a partial profile \( P \) and a sequence \( S \) of \( q \) candidates.

**Proof.** First, for every \( i \in [n] \), construct \( \pi(P_i, S) \) by applying Lemma \([11]\). Then, given \( \pi((P_1, \ldots, P_n), S) \), observe that

\[
\pi((P_1, \ldots, P_{n+1}), S) = \{ \tilde{u} + \tilde{w} : \tilde{u} \in \pi((P_1, \ldots, P_n), S), \tilde{w} \in \pi(P_{n+1}, S) \}
\]

where \( \tilde{u} + \tilde{w} \) is a point-wise sum of the two vectors \( (\tilde{u} + \tilde{w})(j) = \tilde{u}(j) + \tilde{w}(j) \). Hence, \( \pi(P, S) \) can be constructed in polynomial time.

### 4.3 Complexity of \( \text{Min}\{\theta k\} \)

Recall that \( k := m - k + 1 \). We now show that the problem of \( \text{Min}\{\theta k\} \) is tractable for every positional scoring rule with polynomial scores. This is surprising because \( \text{Min}\{1\} \) is NP-complete for every pure positional scoring rule other than plurality and veto, by a reduction from \( P \).

Given a binary positional scoring rule \( r \) and functions \( a, b : \mathbb{N}_+ \to \mathbb{N}_+ \), we define the \((a, b)\)-reversed scoring rule, denoted \( r^{a,b} \), to be the one given by \( r^{a,b}(m, i) = a(m) - b(m) \cdot r(m, m + 1 - i) \). For example, the \((1, 1)\)-reversed rule of plurality is veto, and more generally, the \((1, 1)\)-reversed rule of \( t \)-approval is \( t \)-veto. Also, the \((m, 1)\)-reversed rule of Borda is Borda. In the following lemma, we use a generalized notation for our decision problems where instead of fixed \( k \) or \( \theta k \) we use a fixed function \( f \), which is applied to the number of candidates to produce a number \( f(m) \).

**Lemma 13.** Let \( r \) be a positional scoring rule \( r \) and let \( f, a, b : \mathbb{N}_+ \to \mathbb{N}_+ \). Define \( f'(m) = m + 1 - f(m) \), there exists a reduction

1. from \( \text{Min}_r\{<f\} \) to \( \text{Max}_{r^{a,b}}\{>f\} \);
2. from \( \text{Max}_r\{>f\} \) to \( \text{Min}_{r^{a,b}}\{<f\} \);
3. from \( \text{Min}_r\{=f\} \) to \( \text{Max}_{r^{a,b}}\{=f\} \);
4. from \( \text{Max}_r\{=f\} \) to \( \text{Min}_{r^{a,b}}\{=f\} \).

**Proof.** For a partial order \( P \), the reversed order is defined by \( P^R := \{ x > y : (y > x) \in P \} \). Note that \( T \) extends \( P \) if and only if \( T^R \) extends \( P^R \). Given \( (C, \pi, \tau) \) as input under \( r \) with \( P = (P_1, \ldots, P_n) \), consider \( (C, P', \tau') \) under \( r^R \) where \( P' = (P'_1, \ldots, P'_n) \).

Let \( \mathbf{T} = (T_1, \ldots, T_n) \) be a completion of \( P \), observe the completion \( \mathbf{T}' = (T'_1, \ldots, T'_n) \) of \( P' \). For every candidate \( c \) and a voter \( v_i \) we get \( s(T_i^R, c, r^{a,b}) = a(m) - b(m) \cdot s(T_i, c, r) \) so overall \( s(T', c, r^{a,b}) = n \cdot (a(m) - b(m) \cdot s(T, c, r) \). Since the tie-breaking order is also reversed, the rank is \( \text{rank}(c \mid T', \tau^R) = m + 1 - \text{rank}(c \mid T, \tau) \). In same way, if \( T' \) is
a completion of $\mathbf{P}'$ then by reversing the orders we get a completion $\mathbf{T}$ of $\mathbf{P}$ such that
\[
\text{rank}(c | \mathbf{T}, \tau) = m + 1 - \text{rank}(c | \mathbf{T'}, \tau^R)
\]
for every $c \in \mathbf{C}$. We can deduce that
1. $\min(c | \mathbf{P}, \tau) < f(m)$ if and only if $\max(c | \mathbf{P}', \tau^R) > f'(m)$;
2. $\max(c | \mathbf{P}, \tau) > f(m)$ if and only if $\min(c | \mathbf{P}', \tau^R) < f'(m)$;
3. $\min(c | \mathbf{P}, \tau) = f(m)$ if and only if $\max(c | \mathbf{P}', \tau^R) = f'(m)$;
4. $\max(c | \mathbf{P}, \tau) = f(m)$ if and only if $\min(c | \mathbf{P}', \tau^R) = f'(m)$.
From the above points we conclude the parts of the lemma, respectively.

\[\blacktriangleright\textbf{Theorem 14.} \min\{\theta^k\} \text{ is solvable in polynomial time for every fixed } k \geq 1, \text{ positional scoring rules } r \text{ with polynomial scores, and } \theta \text{ one of } <, >, \text{ and } =.\]

\[\textbf{Proof.} \text{ Let } r \text{ be a positional scoring rule with polynomial scores, and denote } r \text{ by } \{\tilde{s}_m\}_{m>1}.
\]

Define the functions $a(m) = \tilde{s}_m(1), b(m) = 1$, and observe $r^{a,b}$. For any $m > 1$, the vector for $r^{a,b}$ is
\[
(\tilde{s}_m(1) - \tilde{s}_m(m), \tilde{s}_m(1) - \tilde{s}_m(m-1), \ldots, \tilde{s}_m(1) - \tilde{s}_m(2), \tilde{s}_m(1) - \tilde{s}_m(1))
\]
therefore $r^{a,b}$ is also with polynomial scores. For any fixed $k$, $\max_{r^{a,b}}\{\theta^k\}$ is solvable in polynomial time by Theorem 9. Then, by Lemma 13, $\min_{r}\{\theta^k\}$ is solvable in polynomial time.

\[\blacktriangleright\textbf{4.4 Complexity of } \max\{\theta^k\} \text{.}\]

First, for plurality and veto, by Theorem 6 and Lemma 13 we can deduce the following:

\[\blacktriangleright\textbf{Corollary 15.} \text{ For every fixed } k \geq 1, \max\{\theta^k\} \text{ is solvable in polynomial time under the plurality and veto rules, where } \theta \text{ is one of } <, >, \text{ and } =.\]

A positional scoring rule $r$ is $p$-valued, where $p$ is a positive integer greater than 1, if there exists a positive integer $m_0$ such that for all $m \geq m_0$, the scoring vector $\tilde{s}_m$ of $r$ contains exactly $p$ distinct values. A rule is \textit{bounded} if it is $p$-valued for some $p > 1$. Note that for a pure bounded rule there exists some constant $t$ such that for every $m$, the values in $\tilde{s}_m$ are at most $t$ because for every $m > m_0$, the vector $\tilde{s}_m$ contains values which do not appear in $\tilde{s}_m$.

\[\blacktriangleright\textbf{Theorem 16.} \text{ Suppose that a positional scoring rule } r \text{ is bounded, strongly pure, and is neither plurality nor veto. Then } \max_r\{\theta^k\} \text{ is NP-complete and } \max_r\{\leq\theta^k\} \text{ is NP-hard for all fixed } k \geq 1.\]

\[\textbf{Proof.} \text{ Let } r \text{ be a positional scoring rule that satisfies the conditions of the theorem, and let us denote } r \text{ by } \{\tilde{s}_m\}_{m>1}. \text{ Since } r \text{ is bounded and strongly pure, there exists some constant } t \text{ such that for every } m, \text{ the values in } \tilde{s}_m \text{ are at most } t. \text{ Observe the scoring rule } r' = r^{t,1}. \text{ For every } m \geq 1, \text{ if } \tilde{s}_{m+1} = \tilde{s}_{m+1}(1) \circ \tilde{s}_m \text{ then the vector of } r' \text{ for } m + 1 \text{ candidates is}
\]
\[
(t - \tilde{s}_{m+1}(1), t - \tilde{s}_{m+1}(m), \ldots, t - \tilde{s}_{m+1}(1))
\]
\[
= (t - \tilde{s}_m(m), t - \tilde{s}_m(m-1), \ldots, t - \tilde{s}_m(1)) \circ (t - \tilde{s}_{m+1}(1))
\]
Otherwise, $\tilde{s}_{m+1} = \tilde{s}_m \circ \tilde{s}_{m+1}(m + 1)$, and the vector of $r'$ for $m + 1$ candidates is
\[
(t - \tilde{s}_{m+1}(m + 1), t - \tilde{s}_{m+1}(m), \ldots, t - \tilde{s}_{m+1}(1))
\]
\[
= (t - \tilde{s}_m(m + 1)) \circ (t - \tilde{s}_m(m), t - \tilde{s}_m(m - 1), \ldots, t - \tilde{s}_m(1))
\]
Therefore $r'$ is strongly pure, has polynomial scores (the scores are bounded by $t$), and is neither plurality nor veto (because $r$ is neither plurality nor veto). By Theorem 8, $\min_{r}\{\leq k\}$ is NP-complete and $\min_{r}\{<k\}$ is NP-hard. Since $r = (r')^{t,1}$, by Lemma 13 we deduce that $\max_r\{\leq k\}$ is NP-complete and $\max_r\{= k\}$ is NP-hard.
5 Concluding Remarks

We studied the problems determining and minimal and maximal ranks of a candidate in a partial voting profile. We showed that these problems are fundamentally harder than the necessary and possible winners that reason about being top ranked: determining whether the maximal/minimal rank is equal to a given number is DP-complete for every positional scoring rule, pure or not, including plurality and veto. For the problems of comparison to a fixed $k$, we have generally recovered the tractable positional scoring rules of the necessary winners (for maximum rank) and possible winners (for minimum rank). Many problems are left for investigation in future research, including: (a) establishing useful tractability conditions for an input $k$; (b) completing our results towards full classifications (of the class of pure rules) for fixed $k$; and (c) further investigating the parameterized complexity of the problem when $k$ is the parameter.

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