A REMARK ON THE FIRST EIGENVALUE OF THE LAPLACE OPERATOR ON 1-FORMS OR FUNCTIONS FOR COMPACT INNER SYMMETRIC SPACES

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Abstract. We remark that on a compact inner symmetric space $G/K$, endowed with the Riemannian metric given by the Killing form of $G$ signed-changed, the first (non-zero) eigenvalue of the Laplace operator on 1-forms or functions, is the Casimir eigenvalue of the highest either long or short root of $G$, according as the highest weight of the isotropy representation is long or short.

1. Introduction

It is well-known that symmetric spaces provide examples where the spectrum of Laplace or Dirac operators can be (theoretically) explicitly computed. However this explicit computation is far from being simple in general and only a few examples are known. On the other hand, several classical results in geometry involve the first (non-zero) eigenvalue of those spectra, so it seems interesting to get this eigenvalue without computing all the spectrum. The present paper is a proof of the following remark:

Proposition 1.1. Let $G/K$ be a compact inner symmetric space of “type I”, endowed with the Riemannian metric given by the Killing form of $G$ signed-changed. The first eigenvalue\(^1\) of the Laplace operator acting on 1-forms is given by the Casimir eigenvalue of the highest either long or short root of $G$ (relative to the choice of a common maximal torus $T$ in $G$ and $K$), according as the highest weight of the isotropy representation is long or short.

As a corollary, it is also obtained that

Corollary 1.2. The same result is valid for the first non-zero eigenvalue of the Laplace operator on functions.

Note that, although the result involves the choice of a basis of roots, it does not depend on this particular choice, by the transitivity of the Weyl group $W_G$ of $G$ on root bases. Indeed, by the Freudenthal formula, the Casimir eigenvalue of the highest (long or short) root $\beta$ is given by

$$\langle \beta + 2\delta_G, \beta \rangle,$$

where $\delta_G$ is the half-sum of the positive roots of $G$, and $\langle , \rangle$ the scalar product on the set of weights induced by the Killing form of $G$ signed-changed. Hence, by the $W_G$-invariance of the scalar product, two choices of a basis of roots (relative to the

\(^1\)by Bochner’s vanishing theorem, there are no harmonic 1-forms on the symmetric spaces considered here, since their Ricci curvature is positive.
choice of a common maximal torus \( T \) in \( G \) and \( K \) will lead to the same Casimir eigenvalue. On the other hand, recall that for the symmetric spaces considered here, the group \( G \) is simple, hence at most two lengths occur in the sets of roots (cf. for instance [Hum72]). So, if only one length occurs, the Casimir eigenvalue of the highest root has only to be considered.

The study of subgroups of maximal rank in a compact Lie group was initiated by A. Borel and J. De Siebenthal in [BDS49], with an explicit description for compact simple groups, resulting in the following complete list of irreducible compact simply-connected Riemannian inner symmetric spaces \( G/K \) of type I (cf. J. A. Wolf’s book [Wol72]), where the first eigenvalue is given.

| \( G/K \) | Number of \( G \)-root lengths | Lenght of the highest weight of the isotropy representation | First non-zero eigenvalue of the Laplacian on 1-forms or functions |
|------------|-----------------------------|-------------------------------------------------|----------------------------------|
| \( \text{SU}(p+q)/\text{SU}(p) \times \text{U}(q) \), \( 1 \leq p \leq q \) | 1 | long | 1 |
| \( \text{SO}(2p)/\text{SO}(2q+1) \), \( p \geq 1, q \geq 0 \) | 2 | short if \( q = 0 \) long if \( q \geq 1 \) | \( \frac{p}{2p+1} \) (\( G/K = S^{2p} \)) |
| \( \text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q) \), \( 1 \leq p \leq q \) | 2 | short | \( \frac{p+q}{p+q+1} \) |
| \( \text{SU}(2n)/\text{U}(n) \), \( n > 2 \) | 1 | long | 1 |
| \( \text{SO}(2n)/\text{SO}(2p) \), \( 1 \leq p \leq q \) | 1 | long | 1 |
| \( \text{Sp}(3)/\text{Sp}(1) \) | 2 | long | 1 |
| \( \text{Spin}(9) \) | 2 | short | 2/3 |
| \( \text{SO}(10)/\text{SO}(2) \) | 1 | long | 1 |
| \( \text{SU}(6)/\text{SU}(2) \) | 1 | long | 1 |
| \( \text{ES} \) | 1 | long | 1 |
| \( \text{SO}(16)/\text{SU}(2) \) | 1 | long | 1 |
| \( \text{ES} \) | 1 | long | 1 |

(By convention, if all roots have same length, they are called long. The notation \( \text{SO}'(n) \) is used to mention that \( \text{SO}(n) \) acts by means of a spin representation). The (rather puzzling) fact that the eigenvalue is equal to 1 in most of the cases is explained below.

Note that if a function \( f \) verifies \( \Delta f = \lambda f \), where \( \lambda \) is the first (nonzero) eigenvalue, then \( \Delta df = \lambda df \), hence \( \lambda \geq \mu \), where \( \mu \) is the first eigenvalue on 1-forms. We remark in the following that this inequality is an equality in all the cases considered here, hence the above table, obtained by considering the first eigenvalue of the Laplace operator on 1-forms, is also valid for functions.

It can be checked that the values given in the above table agree for Compact Rank One Symmetric Spaces: \( \mathbb{R}P^n, \mathbb{C}P^n, \mathbb{H}P^n, \mathbb{O}P^2 = \text{F}_4/\text{Spin}(9) \), with the explicit computations of the whole spectrum given in [CW76] and [Bes78]. The values may be also compared with the partial results (not always very explicit) obtained for
the spectrum of Grassmannians ([IT78], [Str80], [Tsu81], [TK03a], [Cha12]) and the spectrum of Sp(n)/U(n), ([TK03b], [HCT1]) \(^2\).

2. Preliminaries for the proof

We consider a compact simply connected irreducible symmetric space \(G/K\) of "type \(\Gamma\)", where \(G\) is a simple compact and simply-connected Lie group and \(K\) is the connected subgroup formed by the fixed elements of an involution \(\sigma\) of \(G\). This involution induces the Cartan decomposition of the Lie algebra \(g\) of \(G\) into
\[
g = \mathfrak{k} \oplus \mathfrak{p},
\]
where \(\mathfrak{k}\) is the Lie algebra of \(K\) and \(\mathfrak{p}\) is the vector space \(\{X \in g : \sigma \cdot X = -X\}\). This space \(\mathfrak{p}\) is canonically identified with the tangent space to \(G/K\) at the point \(o\), \(o\) being the class of the neutral element of \(G\). We consider here irreducible symmetric spaces, that is, the isotropy representation
\[
\rho : K \rightarrow \text{GL}(\mathfrak{p})
\]
\[
k \rightarrow \text{Ad}(k)|_{\mathfrak{p}}
\]
is irreducible. Hence all \(G\)-invariant scalar products on \(\mathfrak{p}\), and so all \(G\)-invariant Riemannian metrics on \(G/K\) are proportional. We consider the metric induced by the Killing form of \(G\) sign-changed. With this metric, \(G/K\) is an Einstein space with scalar curvature \(\text{Scal} = n/2\), (cf. for instance Theorem 7.73 in [Bes87].)

As \(G/K\) is an homogeneous space, the bundle of \(p\)-forms on \(G/K\) may be identified with the bundle \(G \times \wedge^p \mathfrak{p}\). Any \(p\)-form \(\omega\) is then identified with a \(K\)-invariant function \(G \rightarrow \wedge^p \mathfrak{p}\), that is, a function verifying
\[
\forall g \in G, \quad \forall k \in K, \quad \omega(gk) = \wedge^p \rho(k^{-1}) \omega(g).
\]

Let \(L^2_K(G, \wedge^p \mathfrak{p})\) be the Hilbert space of \(L^2 \) \(K\)-equivariant functions \(G \rightarrow \wedge^p \mathfrak{p}\). The Laplacian operator \(\Delta_p\) extends to a self-adjoint operator on \(L^2_K(G, \wedge^p \mathfrak{p})\). Since it is an elliptic operator, it has a (real) discrete spectrum. By the Peter-Weyl theorem, the natural unitary representation of \(G\) on the Hilbert space \(L^2_K(G, \wedge^p \mathfrak{p})\) decomposes into the Hilbert sum
\[
\bigoplus_{\gamma \in \hat{G}} V_\gamma \otimes \text{Hom}_K(V_\gamma, \wedge^p \mathfrak{p}),
\]
where \(\hat{G}\) is the set of equivalence classes of irreducible unitary complex representations of \(G\), \((\rho_\gamma, V_\gamma)\) represents an element \(\gamma \in \hat{G}\) and \(\text{Hom}_K(V_\gamma, \wedge^p \mathfrak{p})\) is the vector space of \(K\)-equivariant homomorphisms \(V_\gamma \rightarrow \wedge^p \mathfrak{p}\), i.e.
\[
\text{Hom}_K(V_\gamma, \wedge^p \mathfrak{p}) = \{A \in \text{Hom}(V_\gamma, \wedge^p \mathfrak{p}) \text{ s.t. } \forall k \in K, A \circ \rho_\gamma(k) = \wedge^p \rho(k) \circ A\}.
\]
The injection \(V_\gamma \otimes \text{Hom}_K(V_\gamma, \wedge^p \mathfrak{p}) \hookrightarrow L^2_K(G, \wedge^p \mathfrak{p})\) is given by
\[
v \otimes A \mapsto \left( g \mapsto (A \circ \rho_\gamma(g^{-1})) \cdot v \right).
\]
The Laplacian \(\Delta_p\) respects the above decomposition, and its restriction to the space \(V_\gamma \otimes \text{Hom}_K(V_\gamma, \wedge^p \mathfrak{p})\) is nothing else but the Casimir operator \(\mathcal{C}_\gamma\) of the representation \((\rho_\gamma, V_\gamma)\), (see [IT78]):
\[
\Delta(v \otimes A) = v \otimes (A \circ \mathcal{C}_\gamma).
\]

\(^2\)Many references for the explicit computations of spectra may be found in https://mathoverflow.net/questions/219109/explicit-eigenvalues-of-the-laplacian.
But since the representation is irreducible, the Casimir operator is a scalar multiple of identity, $C_\gamma = c_\gamma \text{id}$, where the eigenvalue $c_\gamma$ only depends on $\gamma \in \hat{G}$. Hence the spectrum of $\Delta_p$ is the set of the $c_\gamma$ for which $\text{Hom}_K(V_\gamma, \wedge^p p)$ is non trivial.

Denote by $\wedge^p p = \oplus_{j=1}^N \rho_j^p$, the decomposition of the representation $K \to \wedge^p p$ into irreducible components. Note that for $p = 0$ or 1, the decomposition has only one component, since the representations are respectively the trivial one, and the isotropy representation, which are both irreducible.

Now, by the Frobenius reciprocity theorem, one has
\[
\dim(\text{Hom}_K(V_\gamma, \wedge^p p)) = \sum_{j=1}^N \text{mult}(\rho_j^p, \text{Res}_K^G(\rho_\gamma))
\]
where $\text{Res}_K^G(\rho_\gamma)$ is the restriction to $K$ of the representation $\rho_\gamma$.

So, finally,
\[
\text{Spec}(\Delta_p) = \{c_\gamma ; \gamma \in \hat{G} \text{ s.t. } \exists j \text{ s.t. } \text{mult}(\rho_j^p, \text{Res}_K^G(\rho_\gamma)) \neq 0\}.
\]

In particular
\[
\text{Spec}(\Delta_0) = \{c_\gamma ; \gamma \in \hat{G} \text{ s.t. } \text{mult}(\text{triv.repr.}, \text{Res}_K^G(\rho_\gamma)) \neq 0\},
\]
and
(1) \[
\text{Spec}(\Delta_1) = \{c_\gamma ; \gamma \in \hat{G} \text{ s.t. } \text{mult}(\rho, \text{Res}_K^G(\rho_\gamma)) \neq 0\}.
\]

3. Proof of the result

We furthermore assume that $G$ and $K$ have same rank and consider a fixed common maximal torus $T$.

Let $\Phi$ be the set of non-zero roots of the group $G$ with respect to $T$. According to a classical terminology, a root $\theta$ is called compact if the corresponding root space is contained in $\mathfrak{t}_C$ (that is, $\theta$ is a root of $K$ with respect to $T$) and noncompact if the root space is contained in $\mathfrak{p}_C$. Let $\Phi^+_G$ be the set of positive roots of $G$, with respect to a choice of a basis of simple roots. The half-sum of the positive roots of $G$ is denoted by $\delta_G$. The space of weights is endowed with the $W_G$-invariant scalar product $\langle , \rangle$ induced by the Killing form of $G$ sign-changed.

The symmetric spaces considerer here being irreducible, the space $\mathfrak{p}_C$ is irreducible.

Let $\alpha$ be the highest weight of this representation. As the group $G$ is simple, there are at most two root lengths, and all roots of a given length are conjugate under the Weyl group $W_G$ of $G$ (see [Hum72], §10.4, lemma C). So $\alpha$ is conjugate to either the maximal root of $G$ or the highest short root. Denote by $\beta$ this root, and let $w$ be any element in $W_G$ such that $w \cdot \alpha = \beta$. We claim that

Lemma 3.1. The multiplicity $\text{mult}(\rho, \text{Res}_K^G(\rho_\beta))$ is $\neq 0$.

As the proof differs if either $\alpha$ is long or short, we first have a glance to symmetric spaces for which $\alpha$ is short.

3.1. Symmetric spaces for which the highest weight of the isotropy representation of $K$ is a short root. First note that if $G$ has only one root-length, then the highest weight $\alpha$ of the isotropy representation is necessarily a long root. So we only have to consider symmetric spaces $G/K$ for which $G$ has two root-lengths. Using for instance the table 2, p. 66 in [Hum72], we have to look to the following symmetric spaces.
(1) SO(2p+2q+1)/SO(2p) × SO(2q+1). We consider here G = Spin(2p+2q+1).
Identifying $\mathbb{R}^{2p}$ and $\mathbb{R}^{2q+1}$ with the subspaces spanned respectively by $e_1, \ldots, e_{2p}$ and $e_{2p+1}, \ldots, e_{2p+2q+1}$, where $(e_1, \ldots, e_{2p+2q+1})$ is the canonical basis of $\mathbb{R}^{2p+2q+1}$, $K$ is the subgroup of $G$ defined by

$$\text{Spin}(2p) \cdot \text{Spin}(2q+1) = \{ \psi \in \text{Spin}(2p + 2q + 1) : \psi = \varphi \phi, \varphi \in \text{Spin}(2p), \phi \in \text{Spin}(2q + 1) \} \,.$$

(Note that $K = \text{Spin}(2p)$, when $q = 0$).

We consider the common torus of $G$ and $K$ defined by

$$T = \left\{ \sum_{k=1}^{p+q} \left( \cos(\beta_k) e_{2k-1} \cdot e_{2k}^\ast : \beta_1, \ldots, \beta_{p+q} \in \mathbb{R} \right) \right\}.$$

The Lie algebra of $T$ is

$$t = \left\{ \sum_{k=1}^{p+q} \beta_k e_{2k-1} \cdot e_{2k}^\ast : \beta_1, \ldots, \beta_{p+q} \in \mathbb{R} \right\}.$$

We denote by $(x_1, \ldots, x_{p+q})$ the basis of $t^\ast$ given by

$$x_k \cdot \sum_{j=1}^{p+q} \beta_j e_{2j-1} \cdot e_{2j} = \beta_k.$$

We introduce the basis $(\tilde{x}_1, \ldots, \tilde{x}_{p+q})$ of $it^\ast$ defined by

$$\tilde{x}_k := 2i x_k, \quad k = 1, \ldots, p + q.$$

A vector $\mu \in it^\ast$ such that $\mu = \sum_{k=1}^{p+q} \mu_k \tilde{x}_k$, is denoted by

$$\mu = (\mu_1, \mu_2, \ldots, \mu_{p+q}).$$

The restriction to $t$ of the Killing form $B$ of $G$ is given by

$$B(e_{2k-1} \cdot e_{2k}, e_{2l-1} \cdot e_{2l}) = -8(2p + 2q - 1) \delta_{kl}.$$

It is easy to verify that the scalar product on $it^\ast$ induced by the Killing form sign changed is given by

$$\forall \mu = (\mu_1, \ldots, \mu_{p+q}) \in it^\ast, \forall \mu' = (\mu'_1, \ldots, \mu'_{p+q}) \in it^\ast,$$

$$< \mu, \mu' > = \frac{1}{2(2p + 2q - 1)} \sum_{k=1}^{p+q} \mu_k \mu'_k.$$

Considering the decomposition of the complexified Lie algebra of $G$ under the action of $T$, it is easy to verify that $T$ is a common maximal torus of $G$ and $K$, and that the respective roots are given by (see for instance chapter 12.4 in [BHM+15] for details),

$$\begin{cases}
\pm (\tilde{x}_i + \tilde{x}_j), \pm (\tilde{x}_i - \tilde{x}_j), & 1 \leq i < j \leq p + q, \\
\pm \tilde{x}_i, & 1 \leq i \leq p + q,
\end{cases} \text{ for } G,$$

$$\begin{cases}
\pm (\tilde{x}_i + \tilde{x}_j), \pm (\tilde{x}_i - \tilde{x}_j), & 1 \leq i < j \leq p, \quad p + 1 \leq i < j \leq p + q, \\
\pm \tilde{x}_i, & p + 1 \leq i \leq p + q,
\end{cases} \text{ for } K.$$

We consider as sets of positive roots

$$\Phi^\ast_G = \{ \tilde{x}_i + \tilde{x}_j, \tilde{x}_i - \tilde{x}_j, 1 \leq i < j \leq p + q, \tilde{x}_i, 1 \leq i \leq p + q \} \,.$$

$$\Phi^\ast_K = \{ \tilde{x}_i + \tilde{x}_j, \tilde{x}_i - \tilde{x}_j, 1 \leq i < j \leq p + q, \tilde{x}_i, 1 \leq i \leq p + q \} \,.$$
and
\[ \Phi^+_K = \left\{ \begin{array}{l} \hat{x}_i + \hat{x}_j, \quad 1 \leq i < j \leq p, \\
\hat{x}_i - \hat{x}_j, \quad p + 1 \leq i < j \leq p + q, \\
P_i, \quad p + 1 \leq i \leq p + q \end{array} \right\}. \]

so the set of positive non-compact roots is
\[ \Phi^+_p = \left\{ \begin{array}{l} \hat{x}_i + \hat{x}_j, \quad 1 \leq i \leq p, \quad p + 1 \leq j \leq p + q, \\
\hat{x}_i - \hat{x}_j, \quad 1 \leq i \leq p \end{array} \right\}. \]

Note that the sets
\[ \Delta_G = \{ \hat{x}_i - \hat{x}_{i+1}, \quad 1 \leq i \leq p + q - 1, \hat{x}_{p+q} \} \]
and
\[ \Delta_K = \left\{ \begin{array}{l} \hat{x}_i - \hat{x}_{i+1}, \quad 1 \leq i \leq p - 1, \quad \hat{x}_{p+1} + \hat{x}_p, \\
\hat{x}_i - \hat{x}_{i+1}, \quad p + 1 \leq i \leq p + q - 1, \quad \hat{x}_{p+q}, \end{array} \right\}, \]
are basis of G-roots and K-roots respectively. So, Any \( \mu = (\mu_1, \ldots, \mu_{p+q}) \in i \mathfrak{t}^\ast \)
• is a dominant G-weight if and only if
\[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{p+q} \geq 0, \]
and the \( \mu_i \) are all simultaneously integers or half-integers,
• is a dominant K-weight if and only if
\[ \begin{cases} \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{p-1} \geq |\mu_p|, \\
\mu_{p+1} \geq \mu_{p+2} \geq \cdots \geq \mu_{p+q} \geq 0, \end{cases} \]
and the \( \mu_i \), for \( 1 \leq i \leq p \) or \( p + 1 \leq i \leq p + q \), are all simultaneously integers or half-integers.

Hence
• If \( q = 0 \), the highest weight of \( \mathfrak{p}_C \) is the short root \( \alpha = \hat{x}_1 \), (which is also the highest shortest root \( \beta \) of \( G \)).
• If \( q > 0 \), the highest weight of \( \mathfrak{p}_C \) is the long root \( \alpha = \hat{x}_1 + \hat{x}_{p+1} \).

(2) \( \text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q) \). The space \( \mathbb{H}^{p+q} \) is viewed as a right vector space on \( \mathbb{H} \) in such a way that \( G \) may be identified with the group
\[ \{ A \in M_{p+q}(\mathbb{H}); \; ^tAA = I_{p+q} \}, \]
acting on the left on \( \mathbb{H}^{p+q} \) in the usual way. The group \( K \) is identified with the subgroup of \( G \) defined by
\[ \{ A \in M_{p+q}(\mathbb{H}); \; A = \begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}, \; ^tBB = I_p, \; ^tCC = I_q \}. \]
Let \( T \) be the common torus of \( G \) and \( K \)
\[ T := \left\{ \begin{pmatrix} e^{i\hat{\beta}_1} & & \\
& \ddots & \\
& & e^{i\hat{\beta}_{p+q}} \end{pmatrix}, \; \hat{\beta}_1, \ldots, \hat{\beta}_{p+q} \in \mathbb{R} \right\}, \]
where
\[ \forall \beta \in \mathbb{R}, \quad e^{i\beta} := \cos(\beta) + \sin(\beta) i, \]
\( (1, i, j, k) \) being the standard basis of \( \mathbb{H} \).
The Lie algebra of $T$ is

$$t = \left\{ \begin{pmatrix} i \beta_1 \\ \vdots \\ i \beta_{p+q} \end{pmatrix} ; \beta_1, \beta_2, \ldots, \beta_{p+q} \in \mathbb{R} \right\}.$$ 

We denote by $(x_1, \ldots, x_{p+q})$ the basis of $t^*$ given by

$$x_k \cdot \begin{pmatrix} i \beta_1 \\ \vdots \\ i \beta_{p+q} \end{pmatrix} = \beta_k.$$ 

A vector $\mu \in \mathfrak{g}^*$ such that $\mu = \sum_{k=1}^{p+q} \mu_k \tilde{\xi}_k$, in the basis $(\tilde{\xi}_k = i x_k)_{k=1,\ldots,p+q}$, is denoted by

$$\mu = (\mu_1, \mu_2, \ldots, \mu_{p+q}).$$

The restriction to $\mathfrak{t}$ of the Killing form $B$ of $G$ is given by

$$\forall X \in t, \forall Y \in t, \quad B(X, Y) = 4(p + q + 1) \Re \left( \text{tr}(X Y) \right).$$

It is easy to verify that the scalar product on $\mathfrak{g}^*$ induced by the Killing form sign changed is given by

$$\forall \mu = (\mu_1, \ldots, \mu_{p+q}) \in \mathfrak{g}^*, \forall \mu' = (\mu'_1, \ldots, \mu'_{p+q}) \in \mathfrak{g}^*,$$

$$< \mu, \mu' > = \frac{1}{4(p + q + 1)} \sum_{k=1}^{p+q} \mu_k \mu'_k. \quad (5)$$

Now, considering the decomposition of the complexified Lie algebra of $G$ under the action of $T$, it is easy to verify that $T$ is a common maximal torus of $G$ and $K$, and that the respective roots are given by

$$\begin{cases}
\pm (\tilde{\xi}_i + \tilde{\xi}_j), \\
\pm (\tilde{\xi}_i - \tilde{\xi}_j), \\
1 \leq i < j \leq p + q,
\end{cases}$$

$$\begin{cases}
\pm 2 \tilde{\xi}_i, 1 \leq i \leq p + q & \text{for } G,
\end{cases}$$

and

$$\begin{cases}
\pm (\tilde{\xi}_i + \tilde{\xi}_j), \\
\pm (\tilde{\xi}_i - \tilde{\xi}_j), \\
1 \leq i < j \leq p,
\end{cases}$$

$$\begin{cases}
\pm 2 \tilde{\xi}_i, 1 \leq i \leq p + q & \text{for } K,
\end{cases}$$

$$1 \leq i < j \leq p + q,$$

$$p + 1 \leq i < j \leq p + q.$$ 

We consider as sets of positive roots

$$\Phi_G^+ = \{ \tilde{\xi}_i + \tilde{\xi}_j, \tilde{\xi}_i - \tilde{\xi}_j, 1 \leq i < j \leq p + q, 2 \tilde{\xi}_i, 1 \leq i \leq p + q \}$$

and

$$\Phi_K^+ = \left\{ \begin{array}{l}
\tilde{\xi}_i + \tilde{\xi}_j, \\
\tilde{\xi}_i - \tilde{\xi}_j,
\end{array} \begin{array}{l}
1 \leq i < j \leq p \\
p + 1 \leq i < j \leq p + q,
\end{array} 2 \tilde{\xi}_i, 1 \leq i \leq p + q \right\},$$

so the set of positive non-compact roots is

$$\Phi_p^+ = \{ \tilde{\xi}_i + \tilde{\xi}_j, \tilde{\xi}_i - \tilde{\xi}_j, 1 \leq i \leq p, p + 1 \leq j \leq p + q \}.$$ 

Note that the sets

$$\Delta_G = \{ \tilde{\xi}_i - \tilde{\xi}_{i+1}, 1 \leq i \leq p + q - 1, 2 \tilde{\xi}_{p+q} \},$$
and

\[ \Delta_K = \left\{ \begin{array}{l} \hat{x}_i - \hat{x}_{i+1}, 1 \leq i \leq p - 1, 2\hat{x}_p, \\ \hat{x}_i - \hat{x}_{i+1}, p + 1 \leq i \leq p + q - 1, 2\hat{x}_{p+q} \end{array} \right\}, \]

are basis of \(G\)-roots and \(K\)-roots respectively. So, Any \(\mu = (\mu_1, \ldots, \mu_{p+q}) \in \mathfrak{t}^*\)
- is a dominant \(G\)-weight if and only if
  \[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_{p+q} \geq 0, \]
  and the \(\mu_i\) are all integers,
- is a dominant \(K\)-weight if and only if
  \[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_p \geq 0, \mu_{p+1} \geq \mu_{p+2} \geq \cdots \geq \mu_{p+q} \geq 0 \]
  and the \(\mu_i\) are all integers.

Hence the highest weight of \(\mathfrak{p}_C\) is the short root \(\alpha = \hat{x}_1 + \hat{x}_{p+1}\).

(3) \(\text{Sp}(n)/U(n)\). With the same notations as above, we consider the subgroup \(K\) of \(G = \text{Sp}(n)\):

\[ K = \{ A = (a_{ij}) \in \text{Sp}(n) ; a_{ij} \in \mathbb{R} + i\mathbb{R} \} \cong U(n). \]

Note that \(K\) is the set of fixed points of the inner involution:

\[ \sigma : \text{Sp}(n) \to \text{Sp}(n), \ A \mapsto IAI^{-1}, \]

where

\[ I = iI_\mathbb{R}. \]

The subspace \(\mathfrak{p} = \{ A \in \mathfrak{sp}(n) ; \sigma_*(A) = -A \}\) is then the set

\[ \mathfrak{p} = \{ A = (a_{ij}) \in \mathfrak{sp}(n) ; a_{ij} = j\mathbb{R} + k\mathbb{R} \}. \]

The torus \(T\) introduced above is a common torus of \(G\) and \(K\). Considering the decomposition of the complexified Lie algebra of \(G\) under the action of \(T\), it is easy to verify that \(T\) is a common maximal torus of \(G\) and \(K\), and that the respective roots are given by

\[ \{ \pm(\hat{x}_i - \hat{x}_j), 1 \leq i < j \leq n, \pm(\hat{x}_i + \hat{x}_j), 1 \leq i \leq j \leq n \} \quad \text{for } G, \]

\[ \{ \pm(\hat{x}_i - \hat{x}_j), 1 \leq i < j \leq n \} \quad \text{for } K, \]

We consider as sets of positive roots

\[ \Phi^+_G = \{ \hat{x}_i - \hat{x}_j, 1 \leq i < j \leq n, \hat{x}_i + \hat{x}_j, 1 \leq i \leq j \leq n \} \]

and

\[ \Phi^+_K = \{ \hat{x}_i - \hat{x}_j, 1 \leq i < j \leq n \}, \]

so the set of positive non-compact roots is

\[ \Phi^+_p = \{ \hat{x}_i + \hat{x}_j, 1 \leq i \leq j \leq n \}. \]

Note that the sets

\[ \Delta_G = \{ \hat{x}_i - \hat{x}_{i+1}, 1 \leq i \leq n - 1, 2\hat{x}_n \}, \]

and

\[ \Delta_K = \{ \hat{x}_i - \hat{x}_{i+1}, 1 \leq i \leq n - 1 \}, \]
are basis of $G$-roots and $K$-roots respectively (there are only $n - 1$ simple $K$-roots as $K$ is not semi-simple). So, any $\mu = (\mu_1, \ldots, \mu_n) \in i t^*$ is
- a dominant $G$-weight if and only if
\[ \mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0, \]
and the $\mu_i$ are all integers,
- a dominant $K$-weight if and only if
\[ \mu_i - \mu_{i+1} \in \mathbb{N}, \ 1 \leq i \leq n - 1. \]
Hence there are two dominant $K$-weights in the representation $p_C$: $2 \hat{x}_1$ and $\hat{x}_1 + \hat{x}_2$, but the highest weight is $2 \hat{x}_1$ since $\hat{x}_1 + \hat{x}_2 < 2 \hat{x}_1$. Hence the highest weight of $p_C$ is the long root $\alpha = 2 \hat{x}_1$.

(4) $G_2/\text{SO}(4)$. We use here the results given in [CG88] (see page 226), which follow from a general result of Borel-de Siebenthal, [BDS49]. A set of $G$-roots $\Phi_G$ is given by the elements $x \in \mathbb{R}^3$, whose coordinates are integers verifying
\[ \sum_{i=1}^{3} x_i = 0 \quad \text{and} \quad \|x\|^2 = 2 \text{ or } 6, \]
hence
\[ \Phi_G = \{ \pm (e_1 - e_2), \pm (e_2 - e_3), \pm (e_3 - e_1), \pm (2 e_1 - e_2 - e_3), \pm (2 e_2 - e_1 - e_3), \pm (2 e_3 - e_1 - e_2) \}. \]
The following system of positive $G$-roots is choosen:
\[ \Phi^+_G = \{ e_1 - e_2, e_3 - e_2, e_3 - e_1, -2 e_1 + e_2 + e_3, -2 e_2 + e_3 + e_1, 2 e_3 - e_1 - e_2 \}. \]
It can be checked that a basis of $G$-roots is given by
\[ \Delta_G = \{ e_1 - e_2, -2 e_1 + e_2 + e_3 \}. \]
A system of positive $K$-roots (which appears to be also a basis of $K$-roots) is then given by
\[ \Phi^+_K = \{ e_1 - e_2, -e_1 - e_2 + 2 e_3 \}. \]
The set of positive non-compact roots is
\[ \Phi^+_p = \{ e_3 - e_2, e_3 - e_1, -2 e_1 + e_2 + e_3, -2 e_2 + e_3 + e_1 \}. \]
There are two dominant weights in $p_C$: $-2 e_2 + e_3 + e_1$ and $e_3 - e_2$, but the highest weight is $-2 e_2 + e_3 + e_1$ since $e_3 - e_2 < -2 e_2 + e_3 + e_1$. Hence the highest weight of $p_C$ is the long root $\alpha = -2 e_2 + e_3 + e_1$.

(5) $F_4/\text{Sp}(3) \cdot \text{Sp}(1)$. We use here also results given in [CG88] (see page 227).

A set $\Phi_G$ of $G$-roots is given by the elements $x \in \mathbb{R}^4$ whose coordinates are integers or half-integers satisfying $\|x\|^2 = 1$ or 2, [Hum72]. hence
\[ \Phi_G = \left\{ \pm e_i, 1 \leq i \leq 4, \pm e_i \pm e_j, 1 \leq i < j \leq 4, \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \right\}. \]
We consider the system of positive roots
\[ \Phi^+_G = \left\{ e_i, 1 \leq i \leq 4, e_i \pm e_j, 1 \leq i < j \leq 4, \frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4) \right\}. \]
It can be check that a basis of $G$-roots is given by
\[ \Delta_G = \{ \alpha_1 := e_2 - e_3, \alpha_2 := e_3 - e_4, \alpha_3 := e_4, \alpha_4 := \frac{1}{2}(e_1 - e_2 - e_3 - e_4) \}. \]
A system of positive $K$-roots is then given by
\[
\Phi_K^+ = \left\{ e_3, e_4, e_1 + e_2, e_1 - e_2, e_3 + e_4, e_3 - e_4, \frac{1}{2} (e_1 - e_2 - e_3 - e_4), \frac{1}{2} (e_1 - e_2 - e_3 + e_4), \frac{1}{2} (e_1 - e_2 + e_3 - e_4), \frac{1}{2} (e_1 - e_2 + e_3 + e_4) \right\}.
\]

It can be check that a basis of $K$-roots is given by
\[
\Delta_K = \left\{ e_1 + e_2, e_4, e_3 - e_4, \frac{1}{2} (e_1 - e_2 - e_3 - e_4) \right\}.
\]

The set of positive non-compact roots is
\[
\Phi_p^+ = \left\{ e_1, e_2, e_1 + e_3, e_1 + e_4, e_2 + e_3, e_2 + e_4, e_1 - e_3, e_1 - e_4, e_2 - e_3, e_2 - e_4, \frac{1}{2} (e_1 + e_2 + e_3 + e_4), \frac{1}{2} (e_1 + e_2 + e_3 - e_4), \frac{1}{2} (e_1 + e_2 - e_3 + e_4), \frac{1}{2} (e_1 + e_2 - e_3 - e_4) \right\}.
\]

There are two dominant weights in $p_C$: $e_1$ and $e_1 + e_3$, but the highest weight is $e_1 + e_3$ since $e_1 \prec e_1 + e_3$. Hence the highest weight of $p_C$ is the long root $\alpha = e_1 + e_3$.

(6) $F_4/\text{Spin}(9)$. In that case, a system of positive $K$-roots is
\[
\Phi_K^+ = \{ e_i, 1 \leq i \leq 4, e_i \pm e_j, 1 \leq i < j \leq 4 \}.
\]

A basis of simple $K$-roots is given by
\[
\Delta_K = \{ e_1 - e_2, e_2 - e_3, e_3 - e_4, e_4 \}.
\]

The set of positive non-compact roots is
\[
\Phi_p^+ = \left\{ \frac{1}{2} (e_1 \pm e_2 \pm e_3 \pm e_4) \right\},
\]

hence the highest weight of the representation $p_C$ is the only dominant weight: the short root $\alpha = \frac{1}{2} (e_1 + e_2 + e_3 + e_4)$.

3.2. Proof of the lemma [3.1]

Proof. The proof is very simple if there is only one root length, or if $\alpha$ is a long root. In that case, $\beta$ is necessarily the maximal root, hence the highest weight of the adjoint representation of the simple group $G$ in its complexified Lie algebra $g_C$.

But the decomposition of $g_C = \mathfrak{t}_C \oplus p_C$ into $K$-invariant subspaces implies at once that $\rho$ is contained in the restriction of $\rho_\beta$ to $K$.

The proof is a little more involved when two roots lengths occur and $\alpha$ is a short root. As we saw it just before, there are only three cases to be considered here: $\text{SO}(2p+1)/\text{SO}(2p)$, $\text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q)$, and $F_4/\text{Spin}(9)$.
Let \( v_\beta \) be the maximal vector (unique up to a scalar multiple) of the representation \( V_\beta \) and let \( g \in G \) be some representative of \( w^{-1} \in W_G \). First \( \rho_\beta(g) \cdot v_\beta \) is a weight-vector for the weight \( \alpha \) since for all \( X \in \mathfrak{t} \),

\[
\rho_\beta(X) \cdot (\rho_\beta(g) \cdot v_\beta) = \frac{d}{dt}_{t=0} (\rho_\beta(\exp(tX)g) \cdot v_\beta) = \frac{d}{dt}_{t=0} \left( \rho_\beta(gg^{-1}\exp(tX)g) \cdot v_\beta \right) = \rho_\beta(g) \cdot (\rho_\beta(A) \cdot X) \cdot v_\beta = \rho_\beta(g) \cdot \beta(w \cdot X) v_\beta = \beta^{-1}(\beta)(X) \rho_\beta(g) \cdot v_\beta = \alpha(X) \rho_\beta(g) \cdot v_\beta.
\]

Now, we claim that there exists \( w \in W_G \) such that \( \beta = w \cdot \alpha \), and the weight-vector \( \rho_\beta(g) \cdot v_\beta \), where \( g \) is a representative of \( w^{-1} \), is a maximal vector for the action of \( K \). We may consider only the action of a basis of simple \( K \)-roots \( \{ \theta_1, \ldots, \theta_s \} \).

First note that if \( \theta \) is a positive \( K \)-root, and \( E_\theta \) a root-vector associated to it, then \( \text{Ad}(g^{-1}) \cdot E_\theta \) is a root-vector for the root \( w \cdot \theta \), since for all \( X \in \mathfrak{t} \),

\[
[X, \text{Ad}(g^{-1}) \cdot E_\theta] = \text{Ad}(g^{-1}) \cdot [\text{Ad}(g) \cdot X, E_\theta] = \text{Ad}(g^{-1}) \cdot [w^{-1} \cdot X, E_\theta] = \theta(w^{-1} \cdot X) \text{Ad}(g^{-1}) \cdot E_\theta = (w \cdot \theta)(X) \text{Ad}(g^{-1}) \cdot E_\theta.
\]

Now, if \( w \cdot \theta \) is a positive root, then, since \( v_\beta \) is a maximal vector killed by any root-vector associated to a positive root,

\[
\rho_\beta(E_\theta) \cdot (\rho_\beta(g) \cdot v_\beta) = \frac{d}{dt}_{t=0} (\rho_\beta(\exp(tE_\theta)g) \cdot v_\beta) = \rho_\beta(g) \cdot (\rho_\beta(A) \cdot E_\theta) \cdot v_\beta = \rho_\beta(g) \cdot 0 = 0.
\]

So \( \rho_\beta(g) \cdot v_\beta \) is killed by the action of any positive \( K \)-root \( \theta \) such that \( w \cdot \theta \) is a positive root. Hence we may conclude by proving that, for each symmetric space under consideration, there exists \( w \in W_G \) such that \( w \cdot \alpha = \beta \), and \( w \cdot \theta_i \) is a positive \( G \)-root, for any simple \( K \)-root \( \theta_i \).

1. \( \text{SO}(2p+1)/\text{SO}(2p) \). We saw above that the highest weight of the representation \( p_C \) is the short root \( \alpha = \hat{x}_1 \), which is also the highest shortest root \( \beta \) of \( G \). We may choose \( w = \text{id} \), and as all the \( K \)-simple roots in \( \Delta_K \) are \( G \)-positive roots, the claim is verified in that case.

2. \( \text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q) \). We saw above that the highest weight of the representation \( p_C \) is the short root \( \alpha = \hat{x}_1 + \hat{x}_{p+1} \). Now the highest short \( G \)-root is \( \beta = \hat{x}_1 + \hat{x}_2 \). Let \( w \) be the element of the Weyl group \( W_G \) given by the \( p \)-cycle permutation \( (2 \cdots p+1) \). One has \( w \cdot \alpha = \beta \), and it is easily verified that \( w \cdot \theta_i \) is a positive \( G \)-root for any simple \( K \)-root \( \theta_i \), hence the claim is also proved in that case.

3. \( \text{F}_4/\text{Spin}(9) \). We saw above that the highest weight of the representation \( p_C \) is the short root \( \alpha = \frac{1}{2}(e_1 + e_2 + e_3 + e_4) \). Now, the highest short \( G \)-root is \( \beta = e_1 \), and the reflection \( \sigma_{\alpha_4} \) across the hyperplane \( \alpha_4^\perp \) verifies \( \sigma_{\alpha_4} \cdot \alpha = \beta \).
It is easily verified that $\sigma_{\alpha_4} \cdot \theta_i$ is a positive $G$-root for any simple $K$-root $\theta_i$ since
\[
\begin{align*}
\sigma_{\alpha_4} : & e_1 - e_2 \mapsto e_3 + e_4, \\
& e_2 - e_3 \mapsto e_2 - e_3, \\
& e_3 - e_4 \mapsto e_3 - e_4, \\
& e_4 \mapsto \frac{1}{2} (e_1 - e_2 - e_3 + e_4).
\end{align*}
\]
Hence the claim is also proved in that case.

3.3. First eigenvalue of the Laplace operator acting on 1-forms. In order to conclude, we have to verify that the Casimir eigenvalue $c_\beta$ is the lowest eigenvalue of the Laplacian.

**Lemma 3.2.** Let $(\rho_\gamma, V_\gamma)$ be an irreducible $G$ representation such that the multiplicity $\text{mult}(\rho, \text{Res}_K^G(\rho_\gamma)) \neq 0$. Then $c_\beta \leq c_\gamma$.

**Proof.** Recall that the highest weight of $\rho$ is $\alpha$, hence if $\text{mult}(\rho, \text{Res}_K^G(\rho_\gamma)) \neq 0$, then $\alpha$ and $\beta = w \cdot \alpha$ are actually weights of the representation $\rho_\gamma$. But then, as $\gamma$ is the highest weight,
\[
\langle \beta + \delta_G, \beta + \delta_G \rangle \leq \langle \gamma + \delta_G, \gamma + \delta_G \rangle,
\]
cf. Lemma C, p.71 in [Hum72], hence
\[
c_\beta = \|\beta + \delta_G\|^2 - \|\delta_G\|^2 \leq \|\gamma + \delta_G\|^2 - \|\delta_G\|^2 = c_\gamma.
\]

**Remark 3.3.** The fact that $c_\beta = 1$ when $\beta$ is the highest long root may seem to be rather puzzling. This is indeed a consequence of Freudenthal’s formula (cf. 48.2 in [FdV69] or p. 123 in [Hum72]): for any $G$-weight $\mu$ of the representation $(\rho_\beta, V_\beta) = (\text{Ad}, g_C)$, the multiplicity $\text{mult}(\mu)$ of $\mu$ is given recursively by the formula
\[
(\langle \beta + \delta_G, \beta + \delta_G \rangle - \langle \mu + \delta_G, \mu + \delta_G \rangle) \text{mult}(\mu) = 2 \sum_{\theta > 0} \sum_{i=1}^\infty \text{mult}(\mu + i\theta) (\mu + i\theta, \theta).
\]

Applying this formula to the weight $\mu = 0$, one obtains since
- $\text{mult}(0) = \dim(t)$, as $T$ is a maximal common torus,
- for any integer $i \geq 1$, $\text{mult}(i\theta) \neq 0 \Leftrightarrow i = 1$, and $\text{mult}(\theta) = 1$, by properties of roots, since the only multiple of a root $\theta$ which is itself a root is $\pm \theta$,

\[
(\|\beta + \delta_G\|^2 - \|\delta_G\|^2) \dim(t) = 2 \sum_{\theta > 0} \|\theta\|^2.
\]

But Gordon Brown’s formula ([Bro64] or 21.5 in [FdV69]) states that
\[
2 \sum_{\theta > 0} \|\theta\|^2 = \dim(t),
\]
hence
\[
c_\beta = 1.
\]
According to (1), we only have to verify that \mult(\text{triv.repr.}, \Res^G_K(\rho_\beta)) \neq 0, where \beta is the highest long or short root. Indeed, if that is verified, then the first non-zero eigenvalue \lambda of the Laplacian on functions verifies \lambda \geq c_\beta, as \lambda belongs to the spectrum. But, as it was remarked in the introduction, \lambda \geq c_\beta, since \lambda has to be greater or equal to the first eigenvalue of Laplace operator on 1-forms.

**Proof.** The result is very simple when \beta is the highest long root, hence the highest weight of the adjoint representation of the simple group G in its complexified Lie algebra \mathfrak{g}_C; as T is a common maximal torus of G and K, the group K acts trivially on the Lie algebra \mathfrak{g}_C, hence \mult(\text{triv.repr.}, \Res^G_K(\rho_\beta)) \neq 0.

So we only have to look after the three symmetric spaces where two root lengths occur and \beta is the highest short root: SO(2p+1)/SO(2p), Sp(p+q)/Sp(p) \times Sp(q), and and F_4/Spin(9). Going back to the case-by-case study of those three symmetric spaces above, one gets:

1. SO(2p+1)/SO(2p). The highest short root is \beta = \hat{x}_1, which is the highest weight of the fundamental standard representation of the group Spin(2p+1) (or SO(2p+1)) in the space \mathbb{C}^{2p+1}, see for instance chapter 12 in [BHM+15].

Now, the group K acts trivially on the last vector e_{2p+1} of the canonical basis \(e_1,\ldots,e_{2p+1}\) of \mathbb{C}^{2p+1} since the inclusion K \subset G is induced by the natural inclusion

\[
A \in \text{SO}(2p) \rightarrow \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \text{SO}(2p+1).
\]

Hence \mult(\text{triv.repr.}, \Res^G_K(\rho_\beta)) \neq 0.

2. Sp(p+q)/Sp(p) \times Sp(q). The highest short root is \beta = \hat{x}_1 + \hat{x}_2, which is the highest weight of the fundamental representation of the group Sp(p + q) in the space \wedge^2(\mathbb{C}^{2p+q}). To be more explicit, \mathbb{F}^{p+q} is identified with \mathbb{C}^{2(p+q)} here, such that Sp(p + q) \simeq SU(2(p + q)) \cap Sp(2(p + q), \mathbb{C}), and the representation of Sp(p + q) in the space \wedge^2(\mathbb{C}^{2(p+q)}) decomposes into two irreducible pieces:

\[
\wedge^2(\mathbb{C}^{2(p+q)}) = \wedge^2_0(\mathbb{C}^{2(p+q)}) \oplus \mathbb{C} \cdot \omega_{p+q},
\]

where the first piece is the fundamental representation with highest weight \(\hat{x}_1 + \hat{x}_2\), and the second one the trivial representation on the space generated by the symplectic 2-form:

\[
\omega_{p+q} := e_1 \wedge e_{-1} + e_2 \wedge e_{-2} + \cdots + e_{p+q} \wedge e_{-(p+q)},
\]

where \((e_1,e_2,\ldots,e_{p+q},e_{-1},e_{-2},\ldots,e_{-(p+q)})\) is the canonical basis of \wedge^2(\mathbb{C}^{2(p+q)}), see for instance chapter 12 in [BHM+15].

Note then that the 2-form

\[
e_1 \wedge e_{-1} + e_2 \wedge e_{-2} + \cdots + e_p \wedge e_{-p},
\]

is K = Sp(p) \times Sp(q)-invariant, so its (non-zero) component in \wedge^2_0(\mathbb{C}^{2(p+q)}) under the decomposition (6) is also an invariant of the group K, hence \mult(\text{triv.repr.}, \Res^G_K(\rho_\beta)) \neq 0.
(3) $F_4/\text{Spin}(9)$. The highest short root is $\beta = e_1$. The function “branch” in the LiE Program, \cite{LCL92},

$$\text{branch}([0,0,0,1], B4, \text{res\textunderscore} \text{mattr} (B4, (F4)), (F4)),$$

returns that the dominant weights in the decomposition of $\text{Res}^G_K(\rho_{\beta})$ (expressed in the basis of fundamental weights) are

$$[0,0,0,0], \ [0,0,0,1] \ \text{and} \ [1,0,0,0],$$

hence $\text{mult(\text{triv.\textunderscore} \text{repr.}, \text{Res}^G_K(\rho_{\beta}))} = 1 \neq 0.$ □

To conclude, we have to compute the Casimir eigenvalue $c_\beta$ for the three symmetric spaces considered above.

(1) $\text{SO}(2p+1)/\text{SO}(2p)$. Here $\beta = \widehat{x}_1$. The half-sum of the positive $G$-roots is

$$\delta_G = \left( p - \frac{1}{2}p - \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2} \right).$$

Using (3), one gets

$$c_\beta = \langle \beta + 2\delta_G, \beta \rangle = \frac{2p}{2(2p-1)} = \frac{p}{2p-1}.$$ (2) $\text{Sp}(p+q)/\text{Sp}(p) \times \text{Sp}(q)$. Here $\beta = \widehat{x}_1 + \widehat{x}_2$. The half-sum of the positive roots is

$$\delta_G = (p+q, p+q-1, \ldots, 2, 1).$$

Using (5), one gets

$$c_\beta = \langle \beta + 2\delta_G, \beta \rangle = \frac{4(p+q)}{4(p+q+1)} = \frac{p+q}{p+q+1}.$$ (3) $F_4/\text{Spin}(9)$. Here $\beta = e_1$. The half-sum of the positive roots is

$$\delta_G = \frac{1}{2} (11e_1 + 5e_2 + 3e_3 + e_4).$$

We considered above the scalar product on weights induced by the usual scalar product on $\mathbb{R}^4$. In order to compare it with the scalar product induced by the Killing form sign-changed, we use the “strange formula” of Freudenthal and de Vries (see 47-11 in \cite{FdV69}). For the scalar product $(\ , \ )$ induced by the usual scalar product on $\mathbb{R}^4$:

$$\langle \delta_G, \delta_G \rangle = 39,$$

whereas for the scalar product $\langle \ , \ \rangle$ induced by Killing form sign-changed:

$$\langle \delta_G, \delta_G \rangle = \frac{\dim(G)}{24} = \frac{13}{6}.$$ Hence, as the two $\text{Ad}_G$-invariant scalar products have to be proportional since $G$ is simple,

$$\langle \ , \ \rangle = \frac{1}{18} (\ , \ ),$$

hence

$$c_\beta = \langle \beta + 2\delta_G, \beta \rangle = \frac{12}{18} = \frac{2}{3}.$$
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