Eckhoff’s problem on convex sets in the plane

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Abstract  
Eckhoff proposed a combinatorial version of the classical Hadwiger–Debrunner $(p,q)$-problems as follows. Let $\mathcal{F}$ be a finite family of convex sets in the plane and let $m \geq 1$ be an integer. If among every $\left( \binom{m+2}{2} \right)$ members of $\mathcal{F}$ all but at most $m-1$ members have a common point, then there is a common point for all but at most $m-1$ members of $\mathcal{F}$. The claim is an extension of Helly’s theorem ($m = 1$). The case $m = 2$ was verified by Nadler and by Perles. Here we show that Eckhoff’s conjecture follows from an old conjecture due to Szemerédi and Petruska concerning 3-uniform hypergraphs. This conjecture is still open in general; its solution for a few special cases answers Eckhoff’s problem for $m = 3, 4$. A new proof for the case $m = 2$ is also presented.

Mathematics Subject Classifications: 52A10, 52A35, 05C62, 05D05, 05D15, 05C65

1 Introduction

The subject of this note is a combinatorial version of the classical Hadwiger–Debrunner $(p,q)$-problems proposed by Eckhoff [2] (see also [1]). A family $\mathcal{F}$ of convex sets in the
plane has the $\Delta(m)$-property if $\mathcal{F}$ has at least $|\mathcal{F}| - m + 1$ sets with non-empty intersection. We restate Eckhoff’s conjecture using this notation.

**Problem 1. (Eckhoff [2, Problem 6])** Let $m \geq 1$, $k = \binom{m+2}{2}$ be integers, and let $\mathcal{F}$ be a family of at least $k$ convex sets in $\mathbb{R}^2$. If every $k$ members of $\mathcal{F}$ has the $\Delta(m)$-property, then $\mathcal{F}$ also has the $\Delta(m)$-property.

Due to Helly’s theorem [5], Problem 1 has a positive answer for $m = 1$. The claim was verified also for $m = 2$ by Nadler [8] and by Perles [9]. In this note we show that Eckhoff’s conjecture follows from an old conjecture due to Szemerédi and Petruska [10] on 3-uniform hypergraphs.

In Section 2, Problem 1 is restated first (Problem 2) in terms of 2-representable 3-uniform hypergraphs. The Szemerédi-Petruska conjecture, as reformulated by Lehel and Tuza [11, Problem 18.(a)] states that $\binom{m+2}{2}$ is the maximum order of a 3-uniform $\tau$-critical hypergraph with transversal number $m$. Thus Eckhoff’s conjecture becomes equivalent to a particular instance of a general extremal hypergraph problem (Theorem 6). The Szemerédi-Petruska conjecture is verified for $m = 2, 3, 4$ (see [7]) using the concept of 3-uniform $\tau$–critical hypergraphs, cross-intersecting set-pair systems, and $\tau$-critical graphs; this solves Eckhoff’s problem for $m = 3, 4$, with a new proof for $m = 2$ (Corollary 7).

Eckhoff made the remark that the value of $k$ in Problem 1 is not expected to be tight. Examples in Section 5 show that $k = \binom{m+2}{2}$ cannot be lowered for $m = 2, 3$, but it is not optimal for $m = 4$.

## 2 Convex hypergraphs

Given a family $\mathcal{F}$ of convex sets in $\mathbb{R}^2$, let $H$ be the 3-uniform intersection hypergraph defined by vertex set $V(H) = \{F : F \in \mathcal{F}\}$ and edge set $E(H) = \{\{A, B, C\} : A, B, C \in \mathcal{F} \text{ and } A \cap B \cap C \neq \emptyset\}$.

A 3-uniform hypergraph $H$, that is the intersection hypergraph of some family $\mathcal{F}$ of planar convex sets is called a 2-representable or convex hypergraph. Observe that a $k$-clique $N \subset V$ of the intersection hypergraph indicates that the $k$ convex sets of $\mathcal{F}$ corresponding to the vertices of $N$ have a common point in the plane, due to Helly’s theorem. Eckhoff’s problem is stated next in terms of convex hypergraphs.

**Problem 2.** Let $m \geq 1$ and $n \geq \binom{m+2}{2}$ be integers, and let $H$ be a 2-representable 3-uniform hypergraph of order $n$. If $\omega(H[X]) \geq \binom{m+2}{2} - m + 1$, for every $X \subseteq V$, $|X| = \binom{m+2}{2}$, then $\omega(H) \geq n - m + 1$.

Observe that by Helly’s theorem, a family $\mathcal{F}$ of $k$ convex sets in $\mathbb{R}^2$ has the $\Delta(m)$-property if and only if the 3-uniform intersection hypergraph $H$ defined by $\mathcal{F}$ has clique number $\omega(H) \geq k - m + 1$. This implies the equivalence of Problem 1 and Problem 2.
3 \( \tau \text{-critical 3-uniform hypergraphs} \)

Let \( H = (V, E) \) be an \( r \)-uniform hypergraph. For \( X \subseteq V \) define the subhypergraph \( H[X] \) on vertex set \( X \) with all those edges in \( E \) that are contained by \( X \). For \( e \in E \), denote \( H - e \) the partial hypergraph with vertex set \( V \) and edge set \( E \setminus \{e\} \). Let \( \hat{H} = (V, \hat{E}) \) be the \( r \)-uniform hypergraph obtained as the complement of \( H \) with \( \hat{E} \) containing all \( r \)-element subsets of \( V \) not in \( E \).

The transversal number of a hypergraph \( H \) is defined by

\[
\tau(H) = \min\{|T| : T \subseteq V, \ e \cap T \neq \emptyset, \ \text{for each } e \in E\}.
\]

A hypergraph \( H \) is \( \tau \)-critical if it has no isolated vertex \( (\bigcup_{e \in E} e = V) \) and \( \tau(H - e) = \tau(H) - 1 \) for every \( e \in E \). Let \( v_{\max}(r, t) \) be the maximum order of an \( r \)-uniform \( \tau \)-critical hypergraph \( H \) with \( \tau(H) = t \). This function was introduced and investigated by Gyárfás et al. \([4]\) and by Tuza \([11, \text{Section 4.2}]\).

Denote \( \omega(H) \) the clique number of \( H \) defined as the maximum cardinality of a subset \( N \subseteq V \) such that every \( r \)-element set of \( N \) belongs to \( E \).

**Lemma 3.**

(a) If \( \hat{H} \) is a \( \tau \)-critical \( r \)-uniform hypergraph, then the maximum cliques of \( H \) have no common vertex.

(b) If the maximum cliques of an \( r \)-uniform hypergraph \( H \) have no common vertex, then \( |V| \leq v_{\max}(r, t) \), where \( t = \tau(\hat{H}) \).

**Proof.** Notice that \( N \subseteq V \) is a minimum cardinality transversal of \( \hat{H} \) if and only if \( T = V \setminus N \) is the vertex set of a maximum cardinality clique of \( H \).

(a) By definition, \( \hat{H} \) has no isolated vertex. Furthermore, for every \( x \in V \) and \( e \in \hat{E} \), \( x \in e \), we have \( \tau(\hat{H}[V \setminus \{x\}]) \leq \tau(\hat{H} - e) = \tau(\hat{H}) - 1 \). Then the union of \( \{x\} \) with a \((\tau(\hat{H}) - 1)\)-element transversal of \( \hat{H}[V \setminus \{x\}] \) forms a minimum transversal of \( \hat{H} \). Therefore, every \( x \in V \) belongs to some minimum transversal of \( \hat{H} \). Equivalently, the complements of the minimum transversals of \( \hat{H} \), the maximum cliques of \( H \), have no common vertex.

(b) Because the maximum cliques in \( H \) have no common vertex, the union of their complement in \( V \), that is, the union of the \( t \)-element transversals of \( \hat{H} \), is equal to \( V \). Let \( H' \) be a \( \tau \)-critical partial hypergraph of \( \hat{H} \) with vertex \( V' \) and \( \tau(H') = t \). We claim that \( |V'| = |V| \).

Because every vertex \( x \in V \setminus V' \) belongs to some \( t \)-element transversal \( T \) of \( \hat{H} \), the set \( T \setminus \{x\} \) is a \((t - 1)\)-element transversal for all edges of \( \hat{H} \) not containing \( x \); hence \( \tau(H') < t \), a contradiction. Thus \( |V'| = |V| \) implies \( |V'| \leq v_{\max}(r, t) \) follows.

Recall that \( v_{\max}(3, m) \) is the maximum order of a 3-uniform \( \tau \)-critical hypergraph \( H \) with \( \tau(H) = m \). The conjecture that \( v_{\max}(3, m) = \binom{m + 2}{2} \) for every \( m \) \([11, \text{Problem 18.(a)}]\) was verified only for a few small values of \( m \):
Proposition 4 ([7]). Let \( m = 2, 3, \) or \( 4, \) and \( n > m. \) If \( H \) is a 3-uniform hypergraph of order \( n \) with clique number \( \omega(H) = n - m = k \geq 3 \) and the intersection of the \( k \)-cliques of \( H \) is empty, then \( n \leq \binom{m+2}{2}. \)

Corollary 5. \( v_{\text{max}}(3, m) = \binom{m+2}{2}, \) for \( m = 2, 3 \) and \( 4. \)

Proof. For every \( m \geq 1, \) a 3-uniform \( \tau \)-critical hypergraph of order \( n = \binom{m+1}{2} + m + 1 \) with transversal number \( m \) is obtained from the complete graph \( K_{m+1} \) by extending each edge with one vertex using additional distinct vertices. This construction implies \( v_{\text{max}}(3, m) \geq \binom{m+2}{2}. \)

Let \( \hat{H} \) be a \( \tau \)-critical 3-uniform hypergraph with \( \tau(\hat{H}) = m \) and \( |V| = v_{\text{max}}(3, m). \) By Lemma 3(a) and by applying Proposition 4, we obtain \( |V| = v_{\text{max}}(3, m) \leq \binom{m+2}{2}, \) \( m = 2, 3, 4. \) Thus \( v_{\text{max}}(3, m) = \binom{m+2}{2} \) follows for \( m = 2, 3 \) and \( 4. \)

4 Eckhoff’s problem and \( \tau \)-critical hypergraphs

Eckhoff’s problem relates to the hypergraph extremal problem of determining \( v_{\text{max}}(3, m) \) as is shown by the next theorem.

Theorem 6. For \( m \geq 1 \) and \( n \geq k \geq v_{\text{max}}(3, m), \) let \( \mathcal{F} \) be a family of \( n \) convex sets in \( \mathbb{R}^2. \) If every \( k \) members of \( \mathcal{F} \) have the \( \Delta(m) \)-property, then \( \mathcal{F} \) has the \( \Delta(m) \)-property.

Proof. Assume that the claim is not true. Let \( H_0 \) be a 3-uniform convex hypergraph of minimum order \( n_0 \) such that \( \omega(H_0) \leq n_0 - m, \) but \( \omega(H_0[X]) > k - m + 1, \) for every \( X \subseteq V_0, \) \( |X| = k. \) Notice that the definition of \( H_0 \) implies \( n_0 > k; \) furthermore, since \( n_0 \) is minimal, \( \omega(H_0) = n_0 - m. \)

We claim that the intersection of the maximum cliques of \( H_0 \) is empty. If \( x \in V_0 \) was a common vertex of all maximum cliques, then \( H' = H_0[V_0 \setminus \{x\}] \) has order \( n' = n_0 - 1, \) and for its clique number we have \( \omega(H') = \omega(H_0) - 1 = n_0 - m - 1 = n' - m. \) At the same time, \( \omega(H'[X]) \geq k - m + 1, \) for every \( k \)-element subset \( X \subseteq V_0 \setminus \{x\}. \) Hence \( H' \) is a counterexample of order \( n', \) contradicting the minimality of \( n_0. \) Therefore, the maximum cliques of \( H_0 \) have no common vertex, and because \( \tau(H_0) = n_0 - \omega(H_0) = m, \) Lemma 3 implies \( k < n_0 \leq v_{\text{max}}(3, m) \leq k, \) a contradiction.

As an immediate corollary of Theorem 6 and Proposition 5 we obtain an extensions of Helly’s theorem together with a combinatorial proof for the case \( m = 2 \) (verified earlier by Nadler [8] and by Perles [9]).

Corollary 7. Let \( 1 \leq m \leq 4, \) \( k = \binom{m+2}{2}, \) and let \( \mathcal{F} \) be a family of at least \( k \) convex sets in \( \mathbb{R}^2. \) If every \( k \) members of \( \mathcal{F} \) has the \( \Delta(m) \)-property, then \( \mathcal{F} \) also has the \( \Delta(m) \)-property.
5 Concluding remarks

5.1 The best known general bound $v_{\text{max}}(3, m) \leq \frac{3}{2}m^2 + m + 1$ is obtained by Tuza\(^1\) using the machinery of $\tau$-critical hypergraphs. This bound combined with Theorem 6 yields the following finiteness result on Eckhoff’s problem, for every $m$.

**Corollary 8.** Let $\mathcal{F}$ be a family of at least $k \geq \frac{3}{2}m^2 + m + 1$ convex sets in $\mathbb{R}^2$. If every $k$ members of $\mathcal{F}$ has the $\Delta(m)$-property, then $\mathcal{F}$ also has the $\Delta(m)$-property. \(\square\)

5.2 In Corollary 7 the value of $k$ is optimal (the smallest possible) if there is a family of $n \geq k$ convex sets in $\mathbb{R}^2$ such that every $k - 1$ members of $\mathcal{F}$ satisfy the $\Delta(m)$-property, but $\mathcal{F}$ fails it. It was proved by Nadler [8] that $k = \binom{m+2}{2}$ is optimal for $m = 2$, but as noted by Eckhoff [1], it is ‘somewhat unlikely’ that it is optimal for every $m$. We address optimality for $m = 2, 3, 4$ by defining a family $\mathcal{F}_m$ of convex sets as follows.

$\mathcal{F}_2$: $m = 2$, $k = 6$. Let $\mathcal{F}_2$ be the family of $n = 6$ line segments, taken each side of the triangle $T = (p,q,r)$ twice. Then any vertex of $T$ covers only $4 = n - m$ members of $\mathcal{F}_2$; meanwhile, when removing a copy of one side, say $qr$, vertex $p$ covers $(k-1) - (m-1) = 4$ members.

$\mathcal{F}_3$: $m = 3$, $k = 10$. Let $p_0, p_1, p_2, p_3, p_4 \in \mathbb{R}^2$ be the vertices of a regular pentagon $P$, and let $\mathcal{F}_3$ be the family of $n = 10$ convex sets: the five triangles $T_i = (p_i, p_{i+1}, p_{i+2})$ plus the five quadrangles $Q_i = (p_i, p_{i+1}, p_{i+2}, p_{i+3})$, $0 \leq i \leq 4$, with (mod 5) index arithmetic. Notice that among eight members there are at least three triangles, and among three triangles the intersection of some two is a vertex of $P$, which covers only $7 = n - m$ members of $\mathcal{F}_3$. On the other hand when removing some member $C$ from $\mathcal{F}_3$, any vertex of $P$ not in $C$ covers $(k-1) - (m-1) = 7$ members.

$\mathcal{F}_4$: $m = 4$, $k = \binom{m+2}{2} - 1 = 14$. Let $S = \{p_0, p_1, \ldots, p_7\}$ be the set of vertices of a regular octagon, and let $\mathcal{F}_4$ be the family of $n = 14$ convex sets defined as follows. Take the eight hexagons determined by the vertex sets $S \setminus \{p_i, p_{i+1}\}$, $0 \leq i \leq 7$, and take the six quadrangles $Q_i = (p_i, p_{i+1}, p_{i+2}, p_{i+3})$, for $i \in \{1, 2, 3, 5, 6, 7\}$, with (mod 8) index arithmetic. Notice that the undefined $Q_0, Q_4$ do not belong to $\mathcal{F}_4$, furthermore, the six quadrangles defined in $\mathcal{F}_4$ form three disjoint pairs. Taking one quadrangle from each pair plus the eight hexagons form a subfamily of 11 convex sets with no common point, thus at most 10 = $n - m$ members of $\mathcal{F}_4$ can be covered by one point. On the other hand, three intersecting quadrangles plus seven more hexagons contained in every subfamily $\mathcal{F}_4 \setminus \{C\}$, that is, $(k-1) - (m-1) = 10$ members have a common point $q$ of the plane as it is seen in Fig.1.

Family $\mathcal{F}_m$ shows that $k = \binom{m+2}{2}$ is optimal in Corollary 7 for $m = 2, 3$. Each of $\mathcal{F}_2$ and $\mathcal{F}_3$ is derived from a 3-uniform hypergraph witnessing $v_{\text{max}}(3, m) = \binom{m+2}{2}$. For $m = 4$ the 3-uniform witness hypergraphs are not 2-representable. This fact was observed by Jobson et al. [6] when a similar method using convex hypergraphs was applied to

\(^1\)Personal communication
another geometry problem on convex sets in the plane [6]. Thus the optimum for \( m = 4 \)

is less than \( \binom{4+2}{2} = 15 \); and \( \mathcal{F}_4 \) shows that for \( m = 4 \) the optimum value in Corollary 7 is

actually \( k = \binom{4+2}{2} - 1 = 14 \).

5.3 In the light of the discussions above, Eckhoff’s problem takes the form of an extremal

problem asking for the smallest integer \( k(m) \leq \binom{m+2}{2} \) such that Theorem 6 remains true

when \( v_{\text{max}}(3, m) \) is replaced with \( k(m) \). The exact values, which we know are

\( k(1) = 3 \), \( k(2) = 6 \), \( k(3) = 10 \), \( k(4) = 14 \), and we ask the question whether \( k(m) = \Omega(m^2) \).

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