Congestion Due to Random Walk Routing

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In this paper we derive an analytical expression for the mean load at each node of an arbitrary undirected graph for the uniform multicommodity flow problem under random walk routing. We show the mean load is linearly dependent on the nodal degree with a common multiplier equal to the sum of the inverses of the non-zero eigenvalue of the graph Laplacian. Even though some aspects of the mean load value, such as linear dependence on the nodal degree, are intuitive and may be derived from the equilibrium distribution of the random walk on the undirected graph, the exact expression for the mean load in terms of the full spectrum of the graph has not been known before. Using the explicit expression for the mean load, we give asymptotic estimates for the load on a variety of graphs whose spectral density are well known. We conclude with numerical computation of the mean load for other well-known graphs without known spectral densities.

Key words: multicommodity flow, network congestion, steady state, Laplacian of a graph, spectrum of a graph, random walk

1. Introduction

The study of network capacity, sometimes referred to as load or congestion, is over half a century old, and goes back to the pioneering work of Ford and Fulkerson [1] and Shannon [2] for the single commodity and to early attempts [3, 4, 5, 6, 7] for the multicommodity flow solutions of the problem. This rather large literature provides a characterization of the load or, more specifically, the minimal capacity required, in terms of sum of link capacities needed based on cut values, which in case of the single commodity model are both necessary and sufficient and for the multicommodity case generally provide necessary conditions.

Single commodity or multicommodity network flow models in communication, transportation and numerous other settings typically assume shortest path routing. There are natural settings in which alternative routing not involving shortest paths may be required. For example, it may happen that longer routes are used for load balancing or, in the case of capacitated networks, to avoid network expansion [8, 9]. Or the inverse problem may be posed: to determine weights so that shortest path routes determined from these weights result in smallest load across the network [10]. Given the universality of the network flow model, there are a vast number of applications of the model, and the list is too large to enumerate here.

There are few analytical results concerning the multicommodity flow problem with shortest path routing, in the sense of having a closed form solution as a function of a small number of parameters characterizing the network and the commodities. These include characterization of the maximal load for hyperbolic graphs [11, 12]. In this setting, for a network of \( N \) nodes one assumes 1 unit of (directed) flow between all \( N(N - 1) \) node pairs, and then asks how the load scales due to
shortest path routing as a function of \( N \). This measure is sometimes referred to as the betweenness centrality, see [13].

In this paper, we study the near opposite of shortest path routing: when flows are routed in a uniformly random manner, each flow starting from its source and moving at each step randomly to a neighboring node and only stopping when the destination of the flow is reached. More specifically, we consider the case when one unit of traffic, or a single packet, is injected into the network at every time step at each node \( i \) for each possible destination node \( j \neq i \). Thus there are \( N(N-1) \) units of traffic (or packets) injected into the \( N \)-node network at every time step. The network is assumed to be connected, i.e. have a single component. We first demonstrate that a steady-state distribution is achieved and then derive an expression for the expected flow, or the average number of packets passing through each node, in terms of the eigenvalues of the graph Laplacian. To illustrate the results more concretely, we estimate the largest mean loads for a set of networks whose distribution of Laplacian eigenvalues are known. We note that similar but not identical measures to the expected load at each node have been investigated numerically in the context of node ranking, see [14].

The organization of this paper is as follows. In Section 2, we obtain an expression for the steady state load at each node in the network: it is determined by the degree of each node, being equal to (up to an additive constant) \( N d \sum_{1}^{N-1} \lambda_n^{-1} \), where \( d \) is the degree of the node and \( \{\lambda_0, \lambda_1, \ldots, \lambda_{N-1}\} \) are the eigenvalues of the discrete Laplacian operator on the graph in increasing order (with \( \lambda_0 = 0 \)). This is in marked contrast to the result for geodesic routing, where congestion peaks at the network core. (For scale-free networks, there is a very good correlation between the degree of a node and its betweenness with geodesic routing, and so the distribution of congestion with geodesic and random routing is related, but this is not true in general.) Moreover, unlike the case for geodesic routing, there is no qualitative difference between the congestion behavior of hyperbolic and non-hyperbolic graphs. The approach used in this paper is similar to those in Refs. [15] and [16], but the questions we ask are slightly different. Section 3 has observations and analytical results for a few graph models using the expression derived in Section 2, and Section 3.3 has numerical results for several other graph models.

2. General results

2.1. Time evolution equations As described in the previous section, we consider an undirected connected graph \( G(N,E) \) with \( N \) nodes, in which packets of traffic are injected at various nodes in a deterministic manner and move towards specified destinations. The dynamics are discrete time, i.e. packets of traffic move from node to node at time \( t = 0,1,2,3,\ldots \). At each time step, exactly \( N-1 \) packets of unit size are injected into the graph at each node \( k \), with one packet heading towards each other node in the graph \( l \neq k \). Thus there are precisely \( N(N-1) \) packets injected into the graph at each time step. Any packet that is present at node \( i \) at time step \( t \) and whose destination is not \( i \) moves to one of the nodes adjacent to \( i \) at time \( t + 1 \). For this, one of the \( d_i \) nodes adjacent to \( i \) is chosen randomly, with probability equal to \( 1/d_i \). However, any packet of traffic that is at its destination at time \( t \) is removed from the network, and is no longer present at time \( t + 1 \). Note that a packet that returns to its source as it moves around randomly continues as it would from any other node. The congestion or load at any node at any time step is a random variable equal to the number of packets that are being processed at that node. We are interested in the expected value of the number of packets at each node. We expect that in steady state, if and when it exists, packets are (injected and) removed from each node at the same rate, i.e. \( N-1 \) packets per time step. We seek to find the steady state load, i.e. the average number of packets, at all the nodes of the network.

As a byproduct, we obtain the average time \( \tau \) (or number of steps in its path) that a packet takes to go from a randomly chosen source node to a randomly chosen destination. A packet that
hops from source to destination in \( t \) steps is in the network for \( t \) time steps. (We have assigned one time step each to the source and destination nodes.) The average number of packets at each node, summed over all the nodes in the graph, is therefore the product of the total injection rate \( N(N-1) \) and \( \tau \).

**Remark 1.** We shall use \( N \) to represent both the set of nodes in the graph as well as their count \(|N|\) without danger of confusion. Also, we write \( k \sim j \) to mean that node \( k \) is a neighbor of node \( j \), i.e., \( i \) and \( j \) are adjacent, and \( k \not\sim j \) when they are not; and refer to the adjacency matrix \( (A_{ij}) \) the Laplacian \( (L_{ij}) \) and the normalized Laplacian \( (L_{ij}) \) (for \( 0 \leq i, j \leq N \)) of the undirected graph \( G(N, E) \), with their standard definitions:

\[
A_{ij} = \begin{cases} 
0, & i = j \\
1, & i \sim j \\
0, & i \not\sim j 
\end{cases}, \quad L_{ij} = \begin{cases} 
d_i, & i = j \\
-1, & i \sim j \\
0, & i \not\sim j
\end{cases}, \quad L_{ij} = \begin{cases} 
1, & i = j \\
-(d_id_j)^{-\frac{1}{2}}, & i \sim j \\
0, & i \not\sim j
\end{cases} \tag{1}
\]

**Theorem 1.** For a connected graph \( G(N, E) \) with deterministic injection rate of one packet at each node destined for each other node, where each packet is routed uniformly randomly from its current node to its neighbors until it reaches its destination, there exists a unique steady state number of packets at each node.

**Proof.** We first consider the case of the traffic flowing from a single source node \( k \) to a single destination node \( l \). Let \( X_{kl}^i(t) \) be the random variable representing the number of packets at node \( i \) at time \( t \) and \( Z_{li}^k(t+1) \) be the random variable representing the number of packets sent out of node \( j \), a neighbor of \( i \), to \( i \) at time \( t \). This assumes tacitly that an outgoing packet from node \( j \) that leaves \( j \) at time \( t \) reaches a neighboring node \( i \) at time \( t+1 \); an incoming packet to node \( i \) from node \( j \) that reaches \( i \) at time \( t \) must leave node \( j \) at time \( t-1 \). Then the boundary condition Eq. (2), and the no-escape condition from destination \( l \) Eq. (3), both hold:

\[
X_{kl}^i(0) = \delta_{ik}, \quad 0 \leq i \leq N \tag{2}
\]

\[
Z_{li}^k(t) = 0, \quad \forall t \geq 0. \tag{3}
\]

Flow balance for outgoing packets implies that for all neighbors \( i \) of a node \( j \neq l \),

\[
X_{kl}^i(t) = \sum_{i \sim j \neq l} Z_{ji}^k(t+1), \quad 1 \leq i, j \leq N, 0 \leq t. \tag{4}
\]

which simply states that packets at node \( j \) at time \( t \) move out to its incident links at time \( t+1 \). These same packets arrive at time \( t+1 \) at adjacent nodes

\[
X_{kl}^i(t+1) = \delta_{ik} + \sum_{l \neq j \sim i} Z_{ji}^l(t+1), \quad 1 \leq i, j \leq N, 0 \leq t, \tag{5}
\]

Notice that the first term on the right hand side of Eq.(5) accounts for the fact that one packet is injected at node \( k \) for destination \( l \) at each time step. The second term represents the packets that move to node \( i \) at time \( t+1 \) from adjacent nodes at time \( t \). The sum in this term excludes the node \( l \) because any packet that was at the node \( l \) (the destination) at time \( t \) is removed from the network and is no longer present at time \( t+1 \).

Further, our assumption of uniformly random routing of packets from each node to its neighbors implies that for any neighbor \( i \) of a node \( j \neq l \),

\[
\mathbb{P}(Z_{ji}^k(t+1) = z) = \left( \frac{X_j(t)}{z} \right) \left( \frac{1}{d_j} \right)^{z} (1 - \frac{1}{d_j})^{X_j(t) - z}, \quad 0 \leq z \leq X_j(t), 0 \leq t. \tag{6}
\]
Taking ensemble expectation of Eqs. (6) and (5) and using the standard expression for the mean of the binomial distribution for Eq. (6), we get that for all $0 \leq i, j \leq N$

$$
\mathbb{E}[Z_{ji}^{kl}(t+1)] = \frac{1}{d_j} \mathbb{E}[X_j^{kl}(t)], \quad l \neq j \sim i \tag{7}
$$

$$
\mathbb{E}[X_i^{kl}(t+1)] = \delta_{ki} + \sum_{l \neq j \sim i} \mathbb{E}[Z_{ji}^{kl}(t + 1)] \tag{8}
$$

and substituting from Eq. (7) into (8), we get

$$
\mathbb{E}[X_i^{kl}(t+1)] = \delta_{ik} + \sum_{j \neq i} A_{ij} \frac{\mathbb{E}[X_j^{kl}(t)]}{d_j} \tag{9}
$$

or alternatively stated in terms of the adjacency matrix $A_{ij}$ of the graph,

$$
\mathbb{E}[X_i^{kl}(t+1)] = \delta_{ik} + \sum_{j \neq i} A_{ij} \frac{\mathbb{E}[X_j^{kl}(t)]}{d_j}. \tag{10}
$$

Now define $p_i^{kl}(t) = (1 - \delta_{il})\mathbb{E}[X_j^{kl}(t)]$. In other words, $p_i^{kl}(t) = \mathbb{E}[X_j^{kl}(t)]$ except for the destination node, $i = l$, where $p_i^{kl} = 0$. The sum in Eq.(10) can now be unrestricted for $i \neq l$. The rate equation for the $p_i$'s is

$$
p_i^{kl}(t + 1) = \delta_{ik} + \sum_{j} A_{ij} \frac{p_j^{kl}(t)}{d_j} \tag{11}
$$

for $i \neq l$, with the boundary condition $p_i^{kl}(t + 1) = 0$. The restricted sum in Eq.(9) has been replaced by an unrestricted sum in Eq.(11), but the $l$'th node is now outside the domain of the equation. The boundary condition is an example of a Dirichlet boundary condition, where a function is defined in a region and is specified to be zero on the boundary of the region; in this case, the boundary is the node $l$ and the region is all the other nodes in the graph.

We now show that, under the time evolution of Eq.(11), the function $p_i^{kl}(t)$ reaches a $t$-independent unique steady state. Let $p_i^{kl(1)}(0)$ and $p_i^{kl(2)}(0)$ be two initial configurations at $t = 0$, that evolve according to Eq.(11). Define $q_i^{kl}(t)$ to be equal to $[p_i^{kl(1)}(t) - p_i^{kl(2)}(t)]/\sqrt{\delta_i}$. Then $q_i^{kl}$ satisfies

$$
q_i^{kl}(t+1) = \sum_j A_{ij} \frac{q_j^{kl}(t)}{\sqrt{d_j d_i}} \tag{12}
$$

with the Dirichlet boundary condition at $i = l$. This is equivalent to $q_i^{kl}(t+1) = \sum_j (\delta_{ij} - \mathcal{L}_{ij})q_j^{kl}(t)$, where $\mathcal{L}$ is the normalized Laplacian. Since $\mathcal{L}$ is a real symmetric matrix, it has a complete set of eigenfunctions. The eigenvalues are all in the interval $0 \leq \lambda \leq 2$, with an eigenvalue at $\lambda = 0$ iff one can construct a function $f$ on the graph for which $f_i = f_j$ for all nodes $(i, j)$, and an eigenvalue at $\lambda = 2$ iff one can construct $f$ such that $f_i = -f_j$ whenever $j \sim i$ [17]. With Dirichlet boundary conditions, since $f = 0$ on the boundary nodes, both of these are impossible, and therefore $0 < \lambda < 2$. Thus the operator $I - \mathcal{L}$ (with Dirichlet boundary conditions) is a contraction. Therefore $q_i^{kl}(t \to \infty) \to 0$, and as $t \to \infty$ all initial configurations tend to the same $t$-independent steady state configuration. Q.E.D.
2.2. Steady state solution  In this section, we solve the fixed point of the time evolution equation (11) with Dirichlet boundary condition as introduced in the proof of Theorem (1). As before, \( \{ \lambda_\alpha, \alpha < N \} \) represent the eigenvalues of the graph Laplacian.

**Theorem 2.** For a connected graph \( G(N,E) \) with deterministic injection rate of \( (N-1) \) packets at each node destined for every other node, where each packet is routed uniformly randomly from its current node to its neighbors until it reaches its destination, the unique steady state number of packets at each node \( j \) is given by \( \Lambda_j \) where

\[
\Lambda_j = (N - 1) + Nd_j \sum_{\alpha \neq 0} \frac{1}{\lambda_\alpha}.
\]

**Proof.** In steady state, we know that the load flowing into the node \( l \) at any time step must be equal to the load injected into the node \( k \), i.e. unity. Therefore \( \sum A_{ij}p_{j}^{kl}/d_j = 1 \), and we can extend Eq.(11) as

\[
p_i^{kl} = \delta_{ik} - \delta_{il} + \sum_j A_{ij} \frac{p_j^{kl}}{d_j}
\]

for all \( i \), with the additional condition \( p_l^{kl} = 0 \). It may seem that we have gained nothing by restricting our analysis to the steady state configuration, since we still have to impose Dirichlet boundary conditions at the \( l \)th node. However, as we shall see immediately, the solution to Eq.(14) can easily be found in terms of the eigenvectors of the Laplacian without the Dirichlet boundary condition, i.e. independent of \( k \) and \( l \).

In order to convert Eq.(14) to a Hermitean eigenvalue problem, we define \( p_j^{kl} = d_j r_j^{kl} \) and \( L_{ij} = d_j \delta_{ij} - A_{ij} \). Then

\[
\sum_j L_{ij} r_j^{kl} = \delta_{ik} - \delta_{il}
\]

with \( r_l^{kl} = 0 \). Here \( (L_{ij}) \) is the Laplacian for the graph. Since \( (L_{ij}) \) is a real symmetric matrix, it has a complete set of real eigenvalues \( \lambda_\alpha \) and real orthonormal eigenvectors \( \xi^\alpha \) for \( \alpha = 0, 1, 2 \ldots N - 1 \). Using the standard properties of the Laplacian, all the eigenvalues are non-negative, and since the graph has been assumed to have one component, there is only one zero eigenvalue \( \lambda_0 \) with eigenvector \( \xi^0 = (1, 1, 1, \ldots 1)/\sqrt{N} \). The denominator ensures that the normalization condition \( \sum_i \xi_i^0 \xi_i^0 = 1 \) is satisfied.

We define

\[
\pi_{kl}^\alpha = \sum_i \xi_i^\alpha (\delta_{ik} - \delta_{il}) = \xi_k^\alpha - \xi_l^\alpha
\]

which is the projection of the right hand side of Eq.(15) on to the \( \alpha \)’th eigenvector. Note that \( \pi_{kl}^0 = 0 \). With this definition,

\[
r_j^{kl} = \sum_{\alpha=1}^{N-1} \frac{\pi_{kl}^\alpha}{\lambda_\alpha} \xi_j^\alpha + c^{kl} \xi_j^0,
\]

where \( c^{kl} \) has to be chosen to make \( r_l^{kl} \) equal to zero. Since \( \xi_j^0 \) is independent of \( j \), the condition \( r_l^{kl} = 0 \) yields

\[
r_j^{kl} = \sum_{\alpha=1}^{N-1} \frac{\xi_k^\alpha - \xi_l^\alpha}{\lambda_\alpha} \xi_j^\alpha - \sum_{\alpha=1}^{N-1} \frac{1}{\lambda_\alpha} [\xi_k^\alpha \xi_l^\alpha - (\xi_j^\alpha)^2].
\]

Averaging over all the random paths taken by the traffic packets, the steady state load at any node \( j \neq l \) is \( p_{jl}^{kl} = r_j^{kl} d_j \). For the \( l \)th node, the load is \( \mathbb{E}[X_l^{kl}] \neq p_l^{kl} \), since we defined \( p_l^{kl} \) to be zero. However, in steady state we know that the traffic flowing out of node \( l \) at any time step is unity, and this is equal to the entire load \( \mathbb{E}[X_l^{kl}] \) at that time step. Therefore, in steady state, the load
at the \(j\)th node is equal to \(\Lambda_j^{kl} = d_j r_j^{kl} + \delta_j\). Note that a unit of load from \(k\) to \(l\) is counted at all the nodes it passes through, as well as the source and destination nodes. Depending on how traffic is actually processed by the network, it may be appropriate to change the weightage given to the nodes it passes through, as well as the source and destination nodes.

Summing over all source destination pairs, the total steady state load at the \(j\)th node is

\[
\Lambda_j = d_j \sum_l \sum_{k \neq l} r_j^{kl} + N - 1.
\]  

(19)

Since the first term on the right hand side of Eq.(18) is antisymmetric in \(k\) and \(l\), only the second term contributes to \(\sum_l \sum_{k \neq l} r_j^{kl}\). In the second term, we can replace the sum \(\sum_{k \neq l}\) with an unrestricted sum over \(k\), so that

\[
\Lambda_j = (N - 1) + d_j \sum_{\alpha=1}^{N-1} \frac{1}{\lambda_\alpha} \left[ N \sum_l (\xi_\alpha^l)^2 - (\sum_l \xi_\alpha^l)^2 \right]
\]

\[
= (N - 1) + N d_j \sum_{\alpha \neq 0} \frac{1}{\lambda_\alpha}.
\]

(20)

The load \(\Lambda_j\) at any node \(j\) is linearly dependent on the degree \(d_j\) of the node. Unlike the case when traffic between any source and destination flows along the geodesic path connecting them, there is no concept of a network core. Q.E.D.

**Remark 2.** The result \(\Lambda_j - (N - 1) \propto d_j\) can be obtained directly. An outline of the proof is as follows. The traffic from node \(k\) to node \(l\) can be represented as a stream of random walkers that diffuse through the network at discrete time steps. At every time step in addition to the diffusive dynamics, a walker is introduced at node \(k\), and all the walkers at node \(l\) are removed. Comparing with Eq.(11), the expected number of random walkers at node \(j\) at time \(t\) is equal to \(p_j^{kl}(t)\). If the random walks corresponding to all source destination pairs take place simultaneously, with each walker labelled with an index corresponding to its destination, we have random walkers with \(N\) different labels moving through the network. In addition to the random walk dynamics, walkers are created and destroyed at their sources and destinations respectively. In steady state, the number of walkers created and destroyed at any time step are equal to \(N - 1\) at each node, but they have different labels. If we ignore the labels on the random walkers, the creation and destruction of random walkers can be ignored. The steady state solution for \(\sum_k \sum_l p_j^{kl}(t)\) is proportional to the steady state solution for a diffusion process on the graph with no sources or sinks. It is easy to verify that, in this steady state, the number of random walkers at any node is proportional to the degree of the node. Although this tells us that \([\Lambda_j - (N - 1)]/d_j\) is a constant, independent of \(j\), it does not tell us that this constant is equal to \(N \sum_{\alpha \neq 0} 1/\lambda_\alpha\).

**Remark 3.** If instead of using the Laplacian, \(L\), of the graph, we had used the normalized Laplacian, \(\mathcal{L}\), the entire proof would have proceeded as presented except that equation (20) would have read as follows

\[
\Lambda_j = (N - 1) + d_j \sum_{\alpha=1}^{N-1} \frac{1}{\nu_\alpha} \left[ N \sum_l \left( \frac{\zeta_\alpha^l}{\sqrt{d_l}} \right)^2 - (\sum_l \frac{\zeta_\alpha^l}{\sqrt{d_l}})^2 \right]
\]

\[
= (N - 1) + N d_j \sum_{\alpha \neq 0} \frac{1}{\nu_\alpha} Var \left( \frac{\zeta_\alpha^l}{\sqrt{d_l}} \right).
\]

(21)

where \(0 \leq \nu_0, \ldots, \nu_{N-1} \leq 2\) are the eigenvalues and \(\{\zeta_\alpha\}\) are the corresponding orthonormal eigenvectors of \(\mathcal{L}\) and \(0 \leq \lambda_0, \ldots, \lambda_{N-1} \leq 2\) and \(\{\xi_\alpha\}\) are the eigenvalues and eigenvectors of \(L\). We note that expressions involving terms similar to the right-hand side of equation (21) were obtained in [18] in the context of hitting times of Markov chains, and it may be possible to obtain simpler
expressions there by using the Laplacian, as we did above. Equations (20) and (21) give an interesting relationship between the spectra of the Laplacian and those of the normalized Laplacian for an arbitrary graph which we had not come across before.

**Remark 4.** So far we have dealt with connected undirected graphs. We point out that when the graph is directed, then assuming that steady state distribution is achieved, Remark 2 implies that the expected load \( \Lambda_j = N - 1 + C \pi_j \) where \( C \) is some constant independent of the node and \((\pi_j)\) is the principal eigenvector of the random walk matrix for the directed graph, which for undirected graphs is equal to \((d_j)\).

**Remark 5.** We observe that the proofs of both theorems carry through essentially unchanged if we replace the deterministic arrival of one packet at each source node for each destination node at each time step with a Poisson arrival process with a mean of one packet arrival per node per unit time for each destination node. The same is true if we replace the uniform random routing from each node to its neighbors with a more general value \( w_{jk} / w_j \) with \( w_j = \sum_{l \sim j} w_{jl} \) for the probability of moving from a node \( j \) to any of its neighbors \( k \), so long as \( w_{jk} = w_{kj} \neq 0 \). However, the normalized Laplacian \( (L_{jk}) \) and its eigenvalues \( \{\lambda_n, \alpha < N\} \) in Theorem (2) are now replaced by \( (L^w_{jk}) \) and its eigenvalues \( \{\lambda^w_n, \alpha < N\} \) where \( (L^w_{jk}) \) is now the weighted normalized Laplacian [17], defined analogously as \( L^w_{jk} = \delta_{jk} - (1 - \delta_{jk})w_{jk}/\sqrt{w_jw_k} \) instead of \( L_{jk} = \delta_{jk} - (1 - \delta_{jk})/\sqrt{d_jd_k} \), see (1) in Remark 1.

### 2.3. Discussion

In the large-\( N \) limit, the spectral density of the Laplacian \( \sum_{\alpha} \delta(\lambda - \lambda_\alpha) \) tends to \( N \rho(\lambda) \) where \( \rho(\lambda) \) is smooth. If \( \rho(\lambda \to 0) = 0 \), we have

\[
N \sum_{\alpha \neq 0} \frac{1}{\lambda_\alpha} \to N^2 \int \frac{\rho(\lambda)}{\lambda} d\lambda \sim N^2
\]  

for large \( N \). The simplest example of this is when the graph Laplacian has a spectral gap in the large \( N \) limit. A more subtle case is the Erdős-Rényi model [19], where the spectral density is empirically found [20] to be close to that of a infinite regular tree whose nodes all have the same degree as the average degree of the Erdős-Rényi graph. Even though the infinite tree has a spectral gap, the corresponding Erdős-Rényi spectral density has a narrow tail extending down to \( \lambda = 0 \), so that there is no spectral gap [16]. However, in the next section of this paper, we find numerically that \( N \sum_{\alpha} \lambda^{-1}_\alpha \sim N^2 \) for Erdős-Rényi graphs, presumably because the density in the tail as \( \lambda \to 0 \) is \( \rho(\lambda \to 0) = 0 \). The same result is also shown numerically for scale-free graphs.

If \( \rho(\lambda \to 0) \) is not zero, \( N \sum_{\alpha \neq 0} 1/\lambda_\alpha \) diverges faster than \( \sim N^2 \) for large \( N \). If \( \rho(\lambda \to 0) \) is finite, the spectral gap for large but finite \( N \) is proportional to \( 1/N \). Then \( N^2 \int \rho(\lambda)/\lambda d\lambda \) diverges as \(-N^2 \ln \lambda_{\min} \sim N^2 \ln N \). This is the case for the square lattice and, as is shown in the next section of the paper, a finite regular tree. For hyperbolic grids \( \mathbb{H}_{p,q} \) (where \( p \) and \( q \) are positive integers satisfying \((p-2)(q-2) > 4\)), which are infinite regular planar graphs with constant degree \( q \) and \( p \)-sided polygons as faces, we show numerically in the next section of this paper that \( N \sum \lambda^{-1}_\alpha \sim N^2 \ln N \).

The average congestion in the network is, up to an additive constant, equal to the product of \( N \sum 1/\lambda_\alpha \) and \( d_{\max} \). The large-\( N \) dependence of the latter depends on the degree distribution, e.g growing as \( \sim \ln N \) for Erdős-Rényi graphs and as a power of \( N \) for scale-free networks.

The average time \( \tau \) that a packet spends in the network is obtained from the equation \( \tau = \sum_j \Lambda_j \), from which

\[
\tau = \frac{\sum_j d_j}{N-1} \sum_{\alpha = 1}^{N-1} \frac{1}{\lambda_\alpha} + 1 \to \bar{d} \sum_{\alpha = 1}^{N-1} \frac{1}{\lambda_\alpha}
\]  

in the large \( N \) limit, where \( \bar{d} \) is the average degree of nodes in the graph. If \( \sum \lambda^{-1}_\alpha \sim N \), the average sojourn time in the graph is \( O(N) \). To express this in terms of the diameter of the graph instead
of the number of nodes, we have to know how the diameter grows as \( N \) is increased; for small world graphs, \( \tau \) grows exponentially as the diameter of the graph is increased. Exponential growth implies that a shortest-path walk starting at a site \( k \) and aimed at site \( l \) can reach destination \( l \) exponentially faster on average than the random walk.

3. Analytical results for models

3.1. Hypercubic lattices

Suppose the graph is a square lattice, with nodes labeled as \( \{x_1, x_2\} \) with \( 1 \leq x_1, x_2 \leq w \). Two nodes \( x \) and \( y \) are connected if \( \sum_{\alpha} |x_\alpha - y_\alpha| = 1 \). All nodes in the interior of the square have four nearest neighbors, but the nodes on the perimeter have three and the nodes at the corners have two. The action of the Laplacian on a node is \( Lr_x = d_x r_x - \sum_y r_y \), where \( d_x \) is the degree of the node \( x \) and the sum over \( y \) is restricted to the nearest neighbors of \( x \). As shown in Figure 1, if one adds a layer of new nodes around the perimeter of the square, with \( r_y \) at a perimeter node \( y \) equal to \( r_x \) at its adjacent new node, the action of the Laplacian is the same at all the nodes in the lattice. This corresponds to imposing discrete Neumann boundary conditions on the lattice. (As compared to Dirichlet boundary conditions that have been discussed earlier, Neumann boundary conditions for a function defined in a region require that the normal component of the gradient of the function should be zero everywhere on the boundary.)

The eigenfunctions of the (discrete) Laplacian are then of the form

\[
\xi_{m,n}(x) \propto \cos \left[ m \pi (x_1 - \frac{1}{2}) / w \right] \cos \left[ n \pi (x_2 - \frac{1}{2}) / w \right]
\] (24)

with eigenvalues

\[
\lambda_{m,n} = 4 - 2 \cos (m \pi / w) - 2 \cos (n \pi / w)
\] (25)

where \( m, n = 0, 1, 2 \ldots (w-1) \).

When \( w \) is large, the summation in \( \sum 1/\lambda_\alpha \) can be divided into three parts. In the middle part, either \( m \) or \( n \) is greater than \( N_1 \), where \( N_1 \) is sufficiently large that the summation in \( \sum 1/\lambda_\alpha \) can be replaced by an integral. At the same time, \( k_x = m/w \) and \( k_y = n/w \) are less than \( \delta \), where \( \delta \) is sufficiently small that \( \lambda_{m,n} \) is approximately \( (m^2 + n^2) \pi^2 / w^2 \). The error in these approximations can be made as small as desired by decreasing \( \delta \) and increasing \( N_1 \). For any choice of \( (\delta, N_1) \) it is possible to make \( w \) sufficiently large that \( w \delta \gg N_1 \). As shown in Figure 2, for convenience we further reduce the middle region to be a circular annulus in the \( m - n \) plane. In terms of \( (k_x, k_y) \) the inner and outer radii of the annulus are \( N_1 \sqrt{2}/w \) and \( \delta \). Then
Figure 2. Division of the summation in $\sum 1/\lambda_{mn}$ into three parts. (The point $m = n = 0$ is excluded from the sum.) Outside the inner dashed line, the sum can be replaced by an integral; inside the outer dashed line, the $\lambda_{mn}$ is approximately $(m^2 + n^2)\pi^2/w^2$. The middle part of the summation is all the $(m,n)$ values that are inside the annulus quadrant.

$$\sum_{\alpha \neq 0} \frac{1}{\lambda_{\alpha}} \to w^2 \int_{|k| > \delta} dk_x dk_y \frac{4 - 2 \cos \pi k_x - 2 \cos \pi k_y}{\pi^2(k_x + k_y)^2} + \sum_{|m^2 + n^2| \leq 2N^2} \frac{w^2}{(m^2 + n^2)\pi^2}.$$  \hfill (26)

The first and the third terms on the right hand side are clearly proportional to $w^2$. The second term is equal to $w^2 \int (2\pi kd\delta)/k^2 = 2\pi w^2 \ln w\delta/N_1 \sqrt{2}$. Thus for large $w$, the leading term in the summation is $2\pi w^2 \ln w = \pi N \ln N$, since the number of nodes in the graph is $N = w^2$.

It is straightforward to generalize this approach to a $d$-dimensional hypercubic lattice. For $d > 2$, the first and second parts of the summation are proportional to $w^d = N$, while the third part is proportional to $w^2$, which can be neglected in comparison. For $d = 1$, the second and third parts are proportional to $w^2 = N^2$ while the first part is proportional to $w = N$ which can be neglected in comparison. Thus

$$\sum_{\alpha \neq 0} \frac{1}{\lambda_{\alpha}} \sim \begin{cases} N, & d > 2 \\ N \ln N, & d = 2 \\ N^{2/d}, & d < 2. \end{cases}$$

### 3.2. Regular trees

In this subsection, we calculate $\sum_{\alpha \neq 0} \lambda_{\alpha}^{-1}$ for a regular tree. We show that the spectrum of a finite tree near $\lambda = 0$ has a large-$N$ form that results in $\sum_{\alpha \neq 0} \lambda_{\alpha}^{-1} \sim N \ln N$ for large $N$. Thus as a function of the number of levels $h$ in the tree, the average sojourn time of a random walker in the graph is $\sim h \exp[ah]$ for large $h$.

The graph is generated from a tree that extends $h$ levels from the root node, with each node except the leaf nodes at level $h$ having degree $q$. The eigenvalues of the Laplacian fall in two categories.

First, we consider eigenvectors whose value is the same at all the nodes at the same level, i.e. they are azimuthally symmetric. The eigenvalue equation for $L_{ij}$ reduces to

$$q\xi_i - \xi_{i-1} - (q-1)\xi_{i+1} = \lambda \xi_i.$$  \hfill (27)

For the root node, which has $q$ daughters, the equation is $\xi_0 - \xi_1 = \lambda \xi_0$, while for the leaf nodes the equation is $\xi_h - \xi_{h-1} = \lambda \xi_h$. These nodes also satisfy Eq. (27) if we impose the boundary conditions
Figure 3. Regular tree with \( q = 3 \) nearest neighbors of each node (except the leaf nodes) and \( h = 3 \) levels below the root node.

\[ \xi_{-1} = \xi_1 \text{ and } \xi_{h+1} = \xi_h. \]  

The eigenvalues of the Laplacian are now those of a \( h + 1 \times h + 1 \) tridiagonal matrix acting on the column vector \([\xi_0, \xi_1, \ldots, \xi_h] \), instead of a \( N \times N \) matrix:

\[
\begin{pmatrix}
q & -q & 0 & 0 & 0 & \cdots \\
-1 & q & -(q-1) & 0 & 0 & \cdots \\
0 & -1 & q & -(q-1) & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\cdots & 0 & -1 & q & -(q-1) & \cdots \\
\cdots & 0 & 0 & -1 & 1 & \cdots
\end{pmatrix}.
\]  

(28)

There are \( h \) eigenvectors of this matrix excluding the trivial one where all the \( \xi_i \)'s are the same and \( \lambda = 0 \). They are obtained by solving the recursion relation Eq.(27). The general solution to this equation is \( \xi_i = A \rho_1^i + B \rho_2^i \) with \( \rho_{1,2} \) the roots of the equation \((q-1)\rho^2 + 1 + (\lambda - q)\rho = 0\). This has solutions \( \rho_{1,2} = e^{\pm i\alpha}/\sqrt{q-1} \), with

\[
\lambda = q - 2\sqrt{q-1}\cos\alpha.
\]  

(29)

Since \( \lambda \) is real, either the real or the imaginary part of \( \alpha \) must be zero. The boundary conditions are

\[
A(\rho_1 - 1/\rho_1) + B(\rho_2 - 1/\rho_2) = 0 \\
A\rho_1^h(\rho_1 - 1) + B\rho_2^h(\rho_2 - 1) = 0.
\]  

(30)

If \( \lambda \neq 0 \), \( \rho_{1,2} \neq 1 \) and \( \rho_1^{h+1}(1+\rho_2) = \rho_2^{h+1}(1+\rho_1) \), which is equivalent to \( \sqrt{q-1}\sin(h+1)\alpha = -\sin h\alpha \), i.e.

\[
\frac{\sqrt{q-1}\sin\alpha}{1 + \sqrt{q-1}\cos\alpha} = -\tan h\alpha.
\]  

(31)

There are no solutions to this equation with imaginary \( \alpha \). Therefore all the \( h \) non-trivial eigenvalues correspond to real \( \alpha \), i.e. from Eq.(29)

\[
q + 2\sqrt{q-1} > \lambda > q - 2\sqrt{q-1}.
\]  

(32)

Thus there is a spectral gap between the lowest non-zero eigenvalue and \( \lambda = 0 \). The contribution of these eigenvalues to \( \sum_{\alpha} 1/\lambda_\alpha \) is \( O(h) \), i.e. \( O(\ln N) \). (If the root node has \( q - 1 \) daughters instead of \( q \), the boundary condition at the root is \( \xi_0 = \xi_1 \), which results in the first equation in Eqs.(30) being replaced with \( A(\rho_1 - 1) + B(\rho_2 - 1) = 0 \). With the second equation, this immediately yields \( \rho_1^h = \rho_2^h \), i.e. \( \alpha = m\pi/h \) with integer \( m \), and hence a spectral gap.)

Second, we consider eigenvectors which are only non-zero at two daughters of a node at the \( k \)'th level and descendants thereof, with \( h > k \geq 0 \). The eigenvector must be equal and opposite at the two daughters at the \( k+1 \)'th level in order that the eigenvalue equation for their parent should be
satisfied. Thereafter, the eigenvector must be azimuthally symmetric within the sectors descending from either daughter. The recursion relation inside either of the two sectors is then the same as Eq.(27), but the first boundary condition is replaced by $\xi_0 = 0$. There are $h-k$ eigenvalues for any $k$, each with degeneracy $N_{k+1} - N_k$, where $N_k$ is the number of nodes at the $k$’th level. Adding up the eigenvectors of both types, we have

$$h + 1 + \sum_{k=0}^{h-1} [N_{k+1} - N_k](h - k) = h + 1 + \sum_{k=1}^{h} N_k - hN_0 = N$$

and thus we have accounted for all the eigenvectors.

For these second type of eigenvectors, the eigenvalues are those of the $h-k \times h-k$ matrix $S_k$ that, acting on the column vector $[x_{k+1} \ldots x_h]$ corresponding to a sector, codifies the recursion relation and boundary conditions:

$$S_k = \begin{pmatrix} q & -(q-1) & 0 & 0 & \ldots \\ -1 & q & -(q-1) & 0 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \\ \ldots & -1 & q & -(q-1) & \ldots \\ \ldots & 0 & -1 & 1 \end{pmatrix}. \quad (34)$$

Including degeneracies, the contribution of these eigenvectors to $\sum_\alpha 1/\lambda_\alpha$ is

$$\sum_{k=0}^{h-1} \{N_{k+1} - N_k\} \text{Tr}[S_k^{-1}]. \quad (35)$$

To find $\text{Tr}[S_k^{-1}]$, we calculate the cofactors and determinant of $S_k$ using the tridiagonal form of $S_k$. Let $r_n$ and $s_n$ be the determinants of of the top left and bottom right $n \times n$ submatrices of $S_k$ respectively; these are independent of $k$. Then

$$r_n = qr_{n-1} - (q-1)r_{n-2}; \quad r_1 = q, r_0 = 1$$
$$s_n = qs_{n-1} - (q-1)s_{n-2}; \quad s_1 = 1, s_0 = 1 \quad (36)$$

with the $i$’th cofactor of $S_k$ equal to $r_{i-1}s_{h-k-i}$ and $\text{det}[S_k] = s_{h-k}$. The solutions to the recursion equations for $r_n$ and $s_n$ are

$$r_n = \frac{(q-1)^{n+1} - 1}{q - 2}$$
$$s_n = 1. \quad (37)$$

From these, it is straightforward to obtain

$$\text{Tr}[S_k^{-1}] = \frac{1}{s_{h-k}} \sum_{n=0}^{h-k-1} r_n s_{h-k-1-n} = \frac{1}{(q - 2)^2} [(q-1)^{h-k+1} - (q-1) - (h-k)(q-2)]. \quad (38)$$

Since $N_k = q(q-1)^{k-1}$ for $k > 0$ and $N_0 = 1$, one can perform the summation in Eq.(35). For $h \to \infty$, the expression simplifies to leading order as

$$N \left[ h\{1 + O(1/N)\} + \left\{ \frac{2 - 2q^2 + 2q^3 - q^4}{2q(q-2)(q-1)^2} + O(1/N) \right\} \right] \to Nh = O(N \ln N). \quad (39)$$
shows the results for scale free networks. Following the extension of Ref. [4], we show the results for a few cases involving prototypical graphs including the Erdős–Rényi model of preferential attachment [22, 21], and hyperbolic grids.

Because of its zero eigenvalue, the matrix $L$ is not invertible. We define the matrix $M = L + P$, where $P_{ij} = 1/N$. Then $P$ is a projection operator: $P \sum_{\alpha} c_{\alpha} \xi^{\alpha} = c_{0} \xi^{0}$. Therefore

$$M \sum_{\alpha} c_{\alpha} \xi^{\alpha} = \sum_{\alpha} (\lambda_{\alpha} + \delta_{0}) c_{\alpha} \xi^{\alpha}. \quad (40)$$

Therefore $M$ is an invertible matrix, with $\text{Tr}[M^{-1}] = \sum_{\alpha} (\lambda_{\alpha} + \delta_{0})^{-1}$, which is equal to $1 + \sum_{\alpha \neq 0} \lambda_{\alpha}^{-1}$. We have to numerically evaluate $\text{Tr}[M^{-1}] - 1$.

Figure 4 shows the results for $N \sum \lambda_{\alpha}^{-1}$ for the Erdős–Rényi model as $N$ is increased. Two cases are considered: when the average nodal degree $d_{a}$ is 2 and 4. Since $d_{a} > 1$, there is a giant component in each graph, containing an $N$-independent fraction of the nodes in the large-$N$ limit. All the other nodes are in components whose size does not diverge as $N$ is increased. Since we are considering graphs with a single component in this paper, only the giant component of each graph is retained. This means that the actual number of nodes in the graph is a $d_{a}$-dependent fraction of the $N$ shown in Figure 4, but this does not affect the functional form of large-$N$ behavior. Each point shown in the figure comes from averaging over eighty random graphs. We see that $N \sum \lambda_{\alpha}^{-1} \sim N^{2}$.

Figure 4 also shows results for scale free networks. Following the extension of Ref. [21] of the original model of Ref. [22], nodes enter the network one by one, with each node born with $p$ edges that link it to pre-existing nodes; the probability of linking to any pre-existing node is proportional to its degree with an offset of $q$; the results for various values of $(p, q)$ are shown. The curves are flat for all the cases, demonstrating that $N \sum \lambda_{\alpha}^{-1} \sim N^{2}$.

In the Erdős–Rényi model, if $d_{a} = c \ln N$ instead of being independent of $N$, there is a phase transition in the behavior of the model when $c$ is increased to 1: the fraction of the nodes in the giant
Figure 5. Plot of $\frac{\sum_{\alpha \neq 0} \lambda_\alpha^{-1}}{N}$ versus $N$ for the hyperbolic grid, with seven triangles meeting at every node. All the nodes that are less than some distance $r$ from a center node are included; $N$ increases with $r$. With the $x$-axis on a logscale, the straight line fit demonstrates that $N \sum \lambda_\alpha^{-1} \sim N^{2} \ln N$.

Figure 6. Log-log plot of $\frac{\sum_{\alpha \neq 0} \lambda_\alpha^{-1}}{N}$ versus $N$ for the Erdős-Rényi model with the average degree of the nodes equal to $\ln N$. The straight line shown corresponds to $0.75 N^{-0.185}$. The component approaches 1. The behavior of graphs constructed using this model is very different in this regime. Figure 6 shows the results for $N \sum \lambda_\alpha^{-1}$ when $d_a = \ln N$. We see that $N \sum \lambda_\alpha^{-1}$ grows slower than $N^2$ for large $N$. Although the data are not conclusive, they suggest a $\sim N^{2-\alpha}$ form. As with the other random graph models, each point in the figure is obtained by averaging over eighty random graphs.

4. Conclusions We showed for the uniform multicommodity flow problem on an arbitrary connected graph under random routing, the mean load (or congestion) at each node of the graph exists, is unique and derived an explicit expression for it in terms of the spectrum of the graph Laplacian. Using this explicit expression, we obtained analytical estimates for the mean load for all the nodes in a graph consists of — apart from an additive term — the product of $N \sum \lambda_\alpha^{-1}$ and the highest nodal degree in the graph. For scale free graphs, if the probability of a node having a degree $d$ scales as $p(d) \sim d^{-\gamma}$ for large $d$, the highest nodal degree in a graph with $N$ nodes scales as $N^{1/(\gamma-1)}$ for large $N$. 
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hypercubic lattices and regular trees in the large-size regime using their known spectral densities and computed numerically the mean load for the Erdős-Rényi random graphs, the scale-free Barabási-Albert preferential attachment graphs and hyperbolic grids.

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