Droplet Excitations for the Spin-1/2 XXZ Chain with Kink Boundary Conditions

B. Nachtergaele\textsuperscript{1}, W. Spitzer\textsuperscript{2} and S. Starr\textsuperscript{3}

\textsuperscript{1}Department of Mathematics
University of California, Davis
One Shields Avenue
Davis, CA 95616-8366, USA
bxn@math.ucdavis.edu

\textsuperscript{2}Department of Mathematics
University of British Columbia
Room 121, 1984 Mathematics Road
Vancouver, B.C., Canada V6T 1Z2
spitzer@math.ubc.ca

\textsuperscript{3}UCLA Mathematics Department
Box 951555
Los Angeles, CA 90095-1555, USA
sstarr@math.ucla.edu

August 24, 2005

Abstract

We give a precise definition for excitations consisting of a droplet of size \( n \) in the XXZ chain with various choices of boundary conditions, including kink boundary conditions and prove that, for each \( n \), the droplet energies converge to a boundary condition independent value in the thermodynamic limit. We rigorously compute an explicit formula for this limiting value using the Bethe Ansatz.

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\textsuperscript{1}Work partially supported by U.S. National Science Foundation grant under Grant # DMS-0303316.

\textsuperscript{2}Work partially supported by the Natural Sciences and Engineering Research Council of Canada.
1 Introduction

In this paper we study the low-energy spectrum of the one-dimensional spin-1/2 ferromagnetic XXZ Heisenberg Hamiltonian in the thermodynamic limit. The specific questions we are interested in concern the excitations that describe droplets, i.e., finite domains of reversed magnetization. The simplest case is where the infinite chain is in one of its two translation-invariant ground states with all spins parallel or antiparallel to the $z$-axis. A droplet excitation is then a state with $n$, $n \geq 1$, opposite spins that form, up to quantum fluctuations, a compact cluster which moves through the system as a unit. Since the model also has kink and antikink ground states \[11\] \[9\] in which two halves of the chain have opposite magnetization with a transition region in between, it is interesting to ask about droplet excitations with respect to such a ground state. This raises some interesting questions about how to define a droplet excitation in this case and how to approximate them by excited states in finite volume obtained by imposing boundary conditions or a constraint such as a particular value of the total magnetization.
We will consider the spin-1/2 XXZ chain of length $L$ with Hamiltonian

$$H_{[1,L]} = -\sum_{x=1}^{L-1} \frac{1}{\Delta} \left[ S^1_x S^1_{x+1} + S^2_x S^2_{x+1} \right] + S^3_x S^3_{x+1},$$

and study $H_{[1,L]} + h_{1,L}$, where $h_{1,L}$ is one of the following three choices of boundary term:

- periodic b.c.: $h_{1,L} = -\frac{1}{\Delta} \left[ S^1_1 S^1_L + S^2_1 S^2_L \right] - S^3_1 S^3_L$ (1)
- droplet b.c.: $h_{1,L} = -\delta \left( S^3_1 + S^3_L \right)$ (2)
- kink b.c.: $h_{1,L} = -\alpha \left( S^3_1 - S^3_L \right)$ (3)

See Section 2 for suitable choices of the constants $\alpha$ and $\delta$ as well as other definitions. In the first two cases one can define the droplet energy by restricting the Hamiltonian to invariant subspaces of fixed total third component of the spin. Let $\lambda(n)$ denote the smallest eigenvalue of the Hamiltonian under consideration restricted to the subspace of states with $n$ down spins and $L - n$ up spins, $0 \leq n \leq L$, which is called the space of $n$-magnon states. For the case of periodic or droplet boundary conditions, the energy of a droplet of size $n$ is then defined to be $\lambda(n) - \lambda(0)$. For the model with kink boundary conditions this strategy does not work, since $\lambda(n)$ is attained in a kink ground state and is independent of $n$. The kink ground states form a multiplet of maximal $SU_q(2)$ spin: $S_{\text{max}} = L/2$. It turns out that the correct subspace to define droplet excitations is the subspace of fixed total spin $S = S_{\text{max}} - n$. The mathematical explanation for this definition lies in the existence of a linear isomorphism between the space of $n$-magnon states and the “highest weight” vectors of “weight” $S_{\text{max}} - n$. The quotation marks are necessary here, since the isomorphism only exists for the infinite chain and the weights are not well-defined (infinite). See Section 5.3 for the definition of this isomorphism, which we will denote by $R$. $R$ is a bounded invertible operator that intertwines the Hamiltonians with kink and droplet boundary conditions on the infinite chain. Another, more physical interpretation, is that the total spin quantum number associated with a droplet of size $n$ is $n/2$, in agreement with the case $n = 1$, more commonly known as spin waves [3].

Our main results can be summarized in words as follows: the droplet energies defined with the different boundary conditions above all converge to the same value in thermodynamic limit and that value can be computed exactly by the Bethe Ansatz. The result is given in Theorem 2.1. In the proof of this theorem we use Perron-Frobenius type arguments to turn the Bethe Ansatz calculation into rigorous mathematics. For the precise definitions and mathematical statements we refer the reader to Section 2.

The main motivation for this study is to complete our understanding of the low-lying spectrum of the XXZ chain, which is important for a variety of problems involving the dynamics. As a by-product we have also come a step closer to a
complete proof of the completeness of the Bethe Ansatz in the thermodynamic limit.

2 Set-up and main results

2.1 The kink Hamiltonian

For \( L \in \mathbb{N}_+ \), consider a spin chain on the sites of \([1, L] \subset \mathbb{Z}\). The Hilbert space is \( \mathcal{H} = \mathcal{H}_{[1,L]} = \bigotimes_{x \in [1,L]} \mathcal{H}_x \), where \( \mathcal{H}_x \) is a two-dimensional Hilbert space for each \( x \in [1,L] \). We take an orthonormal basis of \( \mathcal{H}_x \) to be the Ising basis \( \{|\uparrow\rangle, |\downarrow\rangle\} \). The spin-1/2 representation of SU(2) is defined on \( \mathbb{C}^2 \) through the matrices

\[
S^1 = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}, \quad S^2 = \begin{bmatrix} 0 & -i/2 \\ i/2 & 0 \end{bmatrix}, \quad S^3 = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \end{bmatrix},
\]

in the \( \{|\uparrow\rangle, |\downarrow\rangle\} \) basis. For each \( x \in [1,L] \) and \( i \in \{1, 2, 3\} \) we have the operators \( S^i_x \) on \( \mathcal{H} \) where \( S^i \) acts on \( \mathcal{H}_x \) and is tensored with 1 on \( \mathcal{H}_y \) for all \( y \neq x \).

The XXZ model is the Hamiltonian

\[
H = \sum_{x=1}^{L-1} h_{x,x+1}
\]

\[
h_{x,x+1} = \frac{1}{4} \left[ S^3_x S^3_{x+1} - \frac{\Delta}{S^3_x S^3_{x+1} + S^2_x S^2_{x+1}} \right],
\]

with \( \Delta \geq 1 \). Since we consider \( 1/\Delta \), it is allowable that \( \Delta = +\infty \). Let \( q \) be the number in \([0, 1]\) such that \( \Delta = (q + q^{-1})/2 \). Then a modification of this Hamiltonian is the so-called kink Hamiltonian

\[
H^k = H - \frac{\alpha}{2} S^3_1 + \frac{\alpha}{2} S^3_L,
\]

where \( \alpha \) is the constant

\[
\alpha = \frac{1 - q^2}{1 + q^2} .
\]

When \( q = 1 \), this gives the isotropic Heisenberg model without boundary fields. It is useful to incorporate the alternating boundary fields into the nearest-neighbor interactions,

\[
h^k_{x,x+1} = \frac{1}{4} - S^3_x S^3_{x+1} - \frac{1}{\Delta} \left( S^1_x S^1_{x+1} + S^2_x S^2_{x+1} \right) - \frac{\alpha}{2} \left( S^3_x - S^3_{x+1} \right).
\]

Then the kink Hamiltonian can be written as

\[
H^k = \sum_{x=1}^{L-1} h^k_{x,x+1}.
\]
When we want to emphasize the chain for the Hamiltonian, we will write \( H_{[1,L]} \) for \( H \) and \( H^k_{[1,L]} \) for \( H^k \).

There are three important operators commuting with each nearest-neighbor kink interaction, separately. The first is

\[
S^3_{[1,L]} = \sum_{x=1}^{L} S^3_x. \tag{10}
\]

This is the usual total-magnetization operator for representations of SU(2). The other two operators are \( q \)-versions of the total raising and lowering operators

\[
S^+_x = \sum_{x=1}^{L} q^{-(S^3_x^2 + \cdots + S^3_{x-1})} S^+_x, \tag{11}
\]

\[
S^-_x = \sum_{x=1}^{L} S^-_x q^{2(S^3_{x+1}^2 + \cdots + S^3_L^2)}. \tag{12}
\]

Note that these operators are only well-defined when \( 0 < q \leq 1 \). These three operators together give a representation of the quantum group \( SU_q(2) \). (For readers unfamiliar with quantum groups, we will present all the details necessary for our results.)

The total magnetization eigenvalues are \( \{L/2 - n : n = 0, 1, \ldots, L\} \). The eigenspace for the eigenvalue \( L/2 - n \) will be denoted as \( \mathcal{H}(n) \). It is an invariant subspace for \( H^k \). For \( n < L/2 \), \( S^3_{[1,L]} \) maps \( \mathcal{H}(n) \) isomorphically onto its image in \( \mathcal{H}(n+1) \). (For a proof of this and other facts about the representations of \( SU_q(2) \) c.f. \[5\].) Moreover, this image is an invariant subspace of \( H^k \). Therefore, so is its orthogonal complement. For \( 1 \leq n \leq \lfloor L/2 \rfloor \), we may define \( \mathcal{H}^{bw}(n) \) as the subspace of \( \mathcal{H}(n) \) such that

\[
\mathcal{H}(n) = \mathcal{H}^{bw}(n) \oplus S^3_{[1,L]} \mathcal{H}(n-1).
\]

Define \( \mathcal{H}^{bw}(0) = \mathcal{H}(0) \). Then, \( \mathcal{H}^{bw}(n) \) consists of vectors in \( \mathcal{H} \) which have total \( SU_q(2) \) spin equal to \( L/2 - n \), and which are highest-weight vectors in the sense that \( S^3_{[1,L]} \) annihilates each such vector. (C.f., \[5\] for more information.) As noted above, \( \mathcal{H}^{bw}(n) \) is an invariant subspace for each \( n = 0, \ldots, \lfloor L/2 \rfloor \).

One can define a subspace \( \mathcal{H}^{sd}(n) \) to be the set of all vectors whose total \( SU_q(2) \) spin is \( L/2 - n \). We would call this the \( \text{"n-spin deviate"} \) subspace because the total spin deviates from the maximum possible value of \( L/2 \) by \( n \). (Total spin is a function of the Casimir operator, which generates the center of the algebra of \( SU_q(2) \), which matches the usual notion for \( SU(2) \) total spin when \( q = 1 \).) Since the total spin operator commutes with \( S^3_{[1,L]} \), one can define subspaces \( \mathcal{H}^{sd}(n,k) \) which are subspaces of \( \mathcal{H}^{sd}(n) \) with \( S^3_{[1,L]} \) eigenvalue equal to \( L/2 - k \). These
subspaces are trivial unless \( n \leq k \leq L - n \). Therefore
\[
\mathcal{H}^{sd}(n) = \bigoplus_{k=0}^{L-2n} \mathcal{H}^{sd}(n, n + k).
\]
Also, \( \mathcal{H}^{sd}(n, n) = \mathcal{H}^{hw}(n) \). For \( k = 1, \ldots, L - 2n \) one has \( \mathcal{H}^{sd}(n, n + k) = (S_{[1,L]}^{-1})^k \mathcal{H}^{hw}(n) \), and this is an isomorphic image. Since \( S_{[1,L]} \) commutes with \( H^k \), each subspace is an invariant subspace for \( H^k \).

It is natural to define
\[
E(L, n) = \inf \text{spec} \left( H^k \mid \mathcal{H}^{sd}(n) \right),
\]
which is the minimum energy of \( H^k \) ranging over all vectors in the \( n \)-spin deviate subspace. We make explicit reference to the length of the chain \([1,L]\) in this notation. On the other hand, \( H^k \) commutes with \( S_{[1,L]} \) and one can generate all of \( \mathcal{H}^{sd}(n) \) by acting on \( \mathcal{H}^{hw}(n) \) by \( S_{[1,L]} \) some number of times. Therefore, it is clear that
\[
\inf \text{spec} \left( H^k \mid \mathcal{H}^{sd}(n) \right) = \inf \text{spec} \left( H^k \mid \mathcal{H}^{hw}(n) \right).
\]
This is the definition we will use henceforth. We now define \( H_n^k = H^k \mid \mathcal{H}^{hw}(n) \). So, \( E(L, n) = \inf \text{spec}(H_n^k) \).

By Theorem 1.4 in [14],
\[
E(L, 0) \leq E(L, 1) \leq \ldots \leq E(L, [L/2]),
\]
for each finite \( L \). By Proposition 4.1 in that same paper we know that all the inequalities are strict, at least as long as \( 0 < q \leq 1 \). For \( q = 0 \) one cannot define \( E(L, n) \) because there is no quantum group representation, but taking the limit as \( q \to 0^+ \) gives \( E(L, 1) = \cdots = E(L, [L/2]) = 1 \) which satisfy the inequalities, but not strictly.

Moreover, by Proposition 7.1 in that paper, the sequence \( (E(L, n) : L \geq 2n) \) is decreasing in \( L \). Therefore, the limit \( \lim_{L \to \infty} E(L, n) \) necessarily exists. Obviously one has inequalities
\[
\lim_{L \to \infty} E(L, n + 1) \geq \lim_{L \to \infty} E(L, n),
\]
which are derived from the fact that \( E(L, n + 1) > E(L, n) \) for every finite \( L \). But note that one cannot automatically conclude that the inequality is strict in the limit. Whether or not this is so is a natural question. One might hope to resolve this question by finding an explicit formula for the limits. This is the first main result of the paper.

**Theorem 2.1.** For all \( n \in \mathbb{N} \), and \( 0 < q < 1 \)
\[
\lim_{L \to \infty} E(L, n) = \frac{(1-q^2)(1-q^n)}{(1+q^2)(1+q^n)}.
\]

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For two values of $n$ a formula for $\lim_{L \to \infty} E(L, n)$ was previously known. For $n = 0$, one obtains the ground state energy. It is well-known that the ground state energy is $E(L, 0) = 0$ for all finite $L$, in fact the Hamiltonian was constructed to satisfy this condition. For $n = 1$, $E(L, n)$ measures the spectral gap. The formula for this was calculated for all finite $L$ in [8]. The value of the limit is $\lim_{L \to \infty} E(L, 1) = 1 - \Delta^{-1}$. Since $\Delta = \frac{1}{2}(q + q^{-1})$, this is easily seen to agree with the result of the theorem.

**Remark 2.2.** A different method for bounding $E(1) = \lim_{L \to \infty} E(L, 1)$ was given in [12]. That method is based on the martingale method, which proved useful in interacting particle systems [10]. Some inequalities of [12] were made sharper in [17]. In particular, this led to an independent derivation of $E(1)$.

**Remark 2.3.** Let $E_q(L, n)$ and $E_q(n) = \lim_{L \to \infty} E_q(L, n)$ be the relevant $q$-dependent quantities for $q \in [0, 1]$. Using properties of the functions $E_q(L, n)$ (in particular monotonicity in $q$, c.f., Remark 5.3) one finds

$$\lim_{r \to q^+} \lim_{L \to \infty} E_r(L, n) = \lim_{L \to \infty} E_q(L, n) \leq \lim_{r \to q^-} \lim_{L \to \infty} E_r(L, n),$$

whenever the relevant limits exist. This can be used to recover the (obvious) fact that $E_q(n) = 1$ for $q = 0$ and $n \geq 1$. Also it can be used to obtain the upper bound $E_q(n) \leq 0$ for $q = 1$, which matches the obvious lower bound. For $q = 1$ direct spin-wave trial functions can also be used to verify $\lim_{L \to \infty} E_q(L, n) = 0$ directly. (In fact, in [8] the finite-size scaling was calculated for $E_q(L, n)$ when $q = 1$ and $n = 1$.) In our proof, we use the fact that $0 < q < 1$. Therefore, we will make this assumption, henceforth.

### 2.2 Droplet Hamiltonians

In [15], two of the authors investigated low-energy vectors for three different droplet-type Hamiltonians based on the XXZ model. For $\delta \in \mathbb{R}$ one may define

$$H_\delta^{[1,L]} = H_{[1,L]} + \frac{\delta}{2} (1 - S_1^3 - S_L^3),$$

on $\mathcal{H}_{[1,L]}$. We call this the “droplet” Hamiltonian. Also on $\mathcal{H}_{[1,L]}$ the spin chain with periodic boundary conditions (spin ring) is defined as

$$H_{[1,L]}^{cyc} = H_{[1,L]} + h_{1,L}.$$ 

We call this the “cyclic” Hamiltonian. Neither of these Hamiltonians has the full $SU_q(2)$ symmetry. But they both have the symmetry of $S_{[1,L]}^3$. We define

$$E_\delta(L, n) := \inf \text{spec} \left( H_\delta^{[1,L]} \upharpoonright \mathcal{H}_{[1,L]}(n) \right) \quad \text{and}$$

$$E^{cyc}(L, n) := \inf \text{spec} \left( H_{[1,L]}^{cyc} \upharpoonright \mathcal{H}_{[1,L]}(n) \right),$$

where $\mathcal{H}_{[1,L]}(n)$ is the subspace of $\mathcal{H}_{[1,L]}$ corresponding to the state with $n$ particles in the first spin.

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for each $L \ge n$.

Finally, let us recall that one of the ground-states of the infinite-volume Hamiltonian is the all-up-spin state. The GNS representation for this state will be constructed in Section 3. For now, let us write $(\mathcal{H}_\mathbb{Z}, \omega_\mathbb{Z}, H_\mathbb{Z})$ for the GNS representation. There are subspaces $\mathcal{H}_\mathbb{Z}(n)$ which are invariant for the Hamiltonian, and such that $\mathcal{H}_\mathbb{Z} = \bigoplus_{n=0}^\infty \mathcal{H}_\mathbb{Z}(n)$. These are the $n$-magnon subspaces. We define

$$E_\mathbb{Z}(n) := \inf_{\text{spec } (H_\mathbb{Z} \upharpoonright \mathcal{H}_\mathbb{Z}(n))},$$

for each $n$.

The second main result of the paper is the following.

**Theorem 2.4.** For $\delta \ge 1$,

$$\lim_{L \to \infty} E_\delta(L, n) = \lim_{L \to \infty} E_{\text{cyc}}(L, n) = E_\mathbb{Z}(n) = \lim_{L \to \infty} E(L, n).$$

3 Droplet energies in the infinite chain

In this section we will give a precise definition for $E_\mathbb{Z}(n)$, and calculate it for all $n \in \mathbb{N}$.

### 3.1 Set-up

We will start by constructing the GNS Hilbert space for the all-up-spin ground state of the infinite XXZ chain. Instead of following the usual GNS construction, for this special case one can define the representation directly. We define the GNS Hilbert space, $\mathcal{H}_\mathbb{Z}$ as the direct sum of Hilbert spaces $\mathcal{H}_\mathbb{Z}(n)$ for $n \in \mathbb{N} = \{0, 1, 2, \ldots \}$. Each of these subspaces are $\ell^2$-spaces on countable sets. Let $\mathcal{X}_0 = \{\emptyset\}$. Define $\mathcal{H}_\mathbb{Z}(0) = \ell^2(\mathcal{X}_0)$, which is a 1-dimensional space. For $n \in \mathbb{N}_+$, let $\mathcal{X}_n = \{x \in \mathbb{Z}^n : x_1 < \cdots < x_n\}$, and define $\mathcal{H}_\mathbb{Z}(n) = \ell^2(\mathcal{X}_n)$. This defines $\mathcal{H}_\mathbb{Z}$. Given $n \in \mathbb{N}$ and $x \in \mathcal{X}_n$, define $\delta_x \in \ell^2(\mathcal{X}_n)$ so that $\delta_x(y) = \delta_{x,y}$. These define the natural orthonormal basis. Note that the basis for $\mathcal{H}_\mathbb{Z}(0)$ is denoted $\delta_\emptyset$.

Physically, the $x_i$ are just the positions of the down spins.

We should next define operators on $\mathcal{H}_\mathbb{Z}$ satisfying the same commutation relations as the spin-matrices $S_i^\pm$ for $i = 1, 2, 3$ from the last section, except now for all $x \in \mathbb{Z}$, not just a finite set $x \in \{1, \ldots, L\}$. (These generate the $C^*$-algebra on which the infinite XXZ Hamiltonian operates by the Heisenberg dynamics.) For each $x \in \mathbb{Z}$, there is a representation of SU(2) on $\mathcal{H}_X$, given as follows. For each $n \in \mathbb{N}$, if $x \in \mathcal{X}_n$, then:

- $S^3_x \delta_x = m(x, x)\delta_x$, where $m(x, x)$ equals $+\frac{1}{2}$ if $x \in \{x_1, \ldots, x_n\}$ and $-\frac{1}{2}$ otherwise;

- $S^-_x \delta_x$ equals 0 if $x \in \{x_1, \ldots, x_n\}$, and otherwise it equals $\delta_y(x, x)$ where $y(x, x) \in \mathcal{X}_{n+1}$ is $y(x, x) = (x_1, \ldots, x_k, x, x_{k+1}, \ldots, x_n)$,
for that \( k \in \{1, \ldots, n\} \) such that \( x_k < x < x_{k+1} \) (considering \( x_0 = -\infty \) and \( x_{n+1} = +\infty \));

- \( S^+_x \delta_x \) equals 0 unless \( x \in \{x_1, \ldots, x_n\} \), and in that case it equals \( \delta_{z(x,x)} \) where \( z(x,x) \in \mathcal{X}_{n-1} \) is

\[
z(x,x) = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n),
\]

for that \( k \in \{1, \ldots, n\} \) such that \( x = x_k \).

This is similar to the Fock space representation of the CCR algebra, except that there is a restriction to have at most one particle per site. Therefore, this is sometimes called the hard-core Bose gas. (It is also related to a Fock space representation of the CAR algebra using the Jordan-Wigner transformation.)

The cyclic GNS vector is the vacuum vector, \( \delta_0 \in \mathcal{H}_Z(0) \). Then we may define the GNS Hamiltonian as

\[
H_Z = \sum_{x \in \mathbb{Z}} h_{x, x+1},
\]

where the interactions have the same formula as in \( (5) \), but relative to the present representation. As in the finite case, each \( n \)-magnon subspace is an invariant subspace for the Hamiltonian. It will be convenient to adopt a notation for the restriction to the \( n \)-magnon subspace

\[
H_Z(n) := H_Z \upharpoonright \mathcal{H}_Z(n).
\]

We define the droplet energies

\[
E_Z(n) = \inf \text{spec } H_Z(n),
\]

for each \( n \in \mathbb{N} \). The main purpose of this section is to prove the following result.

**PROPOSITION 3.1.** For each \( n \in \mathbb{N} \),

\[
E_Z(n) = \frac{(1 - q^2)(1 - q^n)}{(1 + q^2)(1 + q^n)}.
\]

The result is trivial for \( n = 0 \). Henceforth, we will consider \( n > 0 \). Let \( e_1, \ldots, e_n \) be the coordinate unit vectors in \( \mathbb{Z}^n \). Then \( H_Z(n) \) can be represented as a discrete integral operator by a kernel

\[
H_Z(n)f(x) = \sum_{y \in \mathcal{X}_n} K_n(x, y) f(y),
\]

for all \( x \in \mathcal{X}_n \). The kernel is given by

\[
K_n(x, y) = \sum_{k=0}^n K_n^{(k)}(x, y),
\]
where:

\[ K_n^{(0)}(x, y) = \frac{1}{2} \delta_{y, x} - \frac{1}{2\Delta} \delta_{y, x-e_1}; \]

for \( k = 1, \ldots, n - 1, \)

\[ K_n^{(k)}(x, y) = \left(1 - \delta_{x_{k+1}, x_{k+1}}\right) \delta_{y, x} - \frac{1}{2\Delta} \left(\delta_{y, x-e_{k+1}} + \delta_{y, x+e_k}\right); \]

and

\[ K_n^{(n)}(x, y) = \frac{1}{2} \delta_{y, x} - \frac{1}{2\Delta} \delta_{y, x+e_n}, \]

This kernel is symmetric, corresponding to the fact that \( H_Z(n) \) is self-adjoint. The kernel can also be used to define operators on \( \ell^p(\mathcal{X}_n) \) for \( p \) other than 2. One important preliminary step is to observe that \( H_{Z, n} \) is bounded.

**Lemma 3.2.** \( \|H_Z(n)\| \leq n(1 + \Delta^{-1}) \).

**Proof.** Let \( H_Z(n; p) \) be the operator on \( \ell^p(\mathcal{X}_n) \) with the kernel \( K_n \), as above. So \( H_Z(n) = H_Z(n; 2) \). One knows that

\[ \|H_Z(n; \infty)\| = \max_{x \in \mathcal{X}_n} \sum_{y \in \mathcal{X}_n} |K_n(x, y)|, \]

and

\[ \|H_Z(n; 1)\| \leq \max_{y \in \mathcal{X}_n} \sum_{x \in \mathcal{X}_n} |K_n(x, y)|. \]

Since the kernel is symmetric, these two numbers – the maximum column sum and maximum row sum – are equal. Both are bounded by \( n(1 + \Delta^{-1}) \). (They are actually equal to it.) This follows by considering the maximum number of off-diagonal entries in any row or column, which is \( 2n \), as well as the maximum diagonal entry, \( n \). Both occur when \( x_{i+1} > x_i + 1 \) for all \( i = 1, \ldots, n - 1 \). By the Riesz convexity theorem, (c.f., [18], Section 5.1), this gives the stated upper bound for \( \|H_Z(n; p)\| \) for all \( p \in [1, \infty] \), in particular \( p = 2 \).

### 3.2 Direct integral, Fourier decomposition

Let us define \( d = e_1 + \cdots + e_n \in \mathbb{Z}^n \). Note that if the coordinates of \( x \in \mathbb{Z}^n \) are ordered as \( x_1 < \cdots < x_n \), then the same is true for \( x + d \). Therefore, the function \( \tau(x) = x + d \) defines a bijection on \( \mathcal{X}_n = \{x \in \mathbb{Z}^n : x_1 < \cdots < x_n\} \). It is trivial to check that \( K_n(x, y) = K_n(\tau(x), \tau(y)) \) for all \( x, y \in \mathcal{X}_n \). Therefore, defining \( T : H_Z(n) \rightarrow H_Z(n) \) such that \( T\delta_x = \delta_{\tau(x)} \), it follows that \( T \) and \( H_Z(n) \) commute. The shift operator \( T \) has absolutely continuous spectrum. The analogue of block-diagonalizing \( H_Z(n) \) according to the eigenspaces of \( T \) is to make a direct-integral decomposition of \( H_Z(n) \) using the usual Fourier transform with respect to \( T \). We describe this in some detail, next.
Let \( Y_n = \{ x \in X_n : 0 \leq x \cdot d \leq n - 1 \} \). There is a natural identification of this as the quotient space \( X_n/\tau \). Namely, for every \( x \in X_n \), there is a unique \( y \in Y_n \) and \( k \in \mathbb{Z} \) such that \( x = \tau^k(y) \). We define \([x]\) to be this \( y \). Let \( \ell(X_n) \) be the set of all sequences on \( X_n \). The operator \( T \) extends naturally to this vector space. We let \( \ell_0(X_n) \) be the set of \( T \)-invariant sequences \( f \in \ell(X_n) \). Moreover, we define \( \ell_0^2(X_n) \) to be the Hilbert space of functions in \( \ell_0(X_n) \) such that the following norm is finite:

\[
\| f \|^2 = \sum_{y \in Y_n} |f(y)|^2.
\]

Let \( S^1 = \mathbb{R}/2\pi \mathbb{Z} \) be the unit circle. Let \( L^2(S^1, \ell_0^2(X_n)) \) be the Hilbert space consisting of the set of functions \( \Phi : S^1 \to \ell_0^2(X_n) \) such that for each \( x \in X_n \) the function \( \Phi(\cdot)(x) \) is measurable on \( S^1 \). The norm is

\[
\| \Phi \|^2 = \int_0^{2\pi} \| \Phi(\theta) \|^2 \frac{d\theta}{2\pi},
\]

where \( \Phi(\theta) \in \ell_0^2(X_n) \) so \( \| \Phi(\theta) \| \) is the norm in \( \ell_0^2(X_n) \). There is an analogous Banach space \( L^p(S^1, \ell_0^2(X_n)) \) for each \( p \in [1, \infty] \) such that

\[
\| \Phi \|_p^p = \int_0^{2\pi} \| \Phi(\theta) \|^p \frac{d\theta}{2\pi},
\]

where \( \| \Phi(\theta) \| \) is still the norm in \( \ell_0^2(X_n) \). We continue to denote \( \| \Phi \|_2 \) by just \( \| \Phi \| \). Then \( L^\infty(S^1, \ell_0^2(X_n)) \) is a dense subspace of \( L^2(S^1, \ell_0^2(X_n)) \), as in the case of finite-dimensional vector-valued functions.

We define a map, \( \mathcal{G} \), which is an analogue of the Fourier series, by

\[
\mathcal{G} : L^\infty(S^1, \ell_0^2(X_n)) \to \ell^2(X_n) : \mathcal{G}(\Phi)(x) = \int_0^{2\pi} e^{i\theta x \cdot d} \Phi(\theta)(x) \frac{d\theta}{2\pi}.
\]

As in the case of the usual Fourier series, we define the map on a dense subspace first, and will eventually extend to the full Hilbert space using the isometry property. (We remind the reader to think of \( \ell_0^2(X_n) \) as functions on \( \ell^2(Y_n) \), where \( X_n \equiv Y_n \times \mathbb{Z} \) to see the analogy with the Fourier series.) One can determine that for these functions,

\[
\| \mathcal{G} \Phi \|^2 = \frac{1}{n} \sum_{k=0}^{n-1} \int_0^{2\pi} \sum_{y \in Y_n} \Phi(\theta)(y) \Phi(\theta + 2\pi k/n)(y) e^{-2\pi i k y \cdot d/n} \frac{d\theta}{2\pi}.
\]

This map has a nontrivial null-space (except in the case \( n = 1 \)) because of the sum over \( k \). But it is if we restrict to the closed subspace \( L_+^2(S^1, \ell_0^2(X_n)) \) of \( L^2(S^1, \ell_0^2(X_n)) \), defined as all those \( \Phi \) such that

\[
\Phi(\theta + 2\pi/n)(x) = e^{-2\pi i x \cdot d/n} \Phi(\theta)(x),
\]

(26)
then \( G \) is a partial isometry between \( L^2(S^1, \ell_0^2(\mathcal{X}_n)) \cap L^\infty(S^1, \ell_0^2(\mathcal{X}_n)) \) (with the \( L^2(S^1, \ell_0^2(\mathcal{X}_n)) \) norm) and its range in \( L^2(\mathcal{X}_n) \). One can then extend \( G \) to all of \( L^2(S^1, \ell_0^2(\mathcal{X}_n)) \), to obtain an isometry with \( \ell^2(\mathcal{X}_n) \).

This map is surjective. For example, given any \( \tilde{x} \in \mathcal{X} \), defining

\[
\Phi(\theta)(x) = \delta_{|x|,|\tilde{x}|} e^{-i\theta \tilde{x} \cdot d},
\]

one can easily check that \( G\Phi = \delta_{\tilde{x}} \). Recall that \( \ell^2(\mathcal{X}_n) \) is \( \mathcal{H}_Z(n) \).

Let us define the map \( F : \mathcal{H}_Z(n) \to L^2(S^1, \ell_0^2(\mathcal{X}_n)) \) as the inverse of \( G \).

In a situation such as this, it is usual to call the Fourier-type decomposition a direct-integral decomposition, and to write

\[
\mathcal{H}_Z(n) \overset{\mathcal{F}}{\cong} \int_{S^1} \mathcal{L}_0^2(\mathcal{X}_n) \frac{d\theta}{2\pi}.
\]

(This matches the notation of [16], section 13.16.) Actually, we have a slightly more involved situation because of the constraint (26). We could reduce to the usual situation by restricting attention to \( \theta \in [0, 2\pi/n] \), which is a fundamental domain for such \( \Phi \). However, for notational purposes which arise shortly, we prefer to keep the present convention, and merely remember that (26) must be satisfied.

We can define a family of bounded operators \( H_Z(n; \theta) \) on \( \ell_0^2(\mathcal{X}_n) \), such that

\[
\mathcal{F} H_Z(n) f(\theta) = H_Z(n; \theta) \mathcal{F} f(\theta),
\]

for all \( \theta \in S^1 \). This can be done precisely because \( H_Z(n) \) commutes with \( T \). One usually then says that \( H_Z(n) \) is a decomposable operator, and writes

\[
H_Z(n) \overset{\mathcal{F}}{\cong} \int_{S^1} H_Z(n; \theta) \frac{d\theta}{2\pi}.
\]

We have the same caveat about remembering (26) as before. Particularly, we should pay attention to the fact that the proposed \( H_Z(n; \theta) \) should preserve this property.

The operators \( H_Z(n; \theta) \) are easiest to express as discrete integral operators, as was the case for \( H_Z(n) \) itself. For each \( \theta \in S^1 \), we define

\[
K_{n, \theta}(x, y) = \sum_{k=0}^{n} K_{n, \theta}^{(k)}(x, y),
\]

where:

\[
K_{n, \theta}^{(0)}(x, y) = \frac{1}{2} \delta_{y, x} - \frac{1}{2\Delta} e^{i\theta} \delta_{y, x - e_1};
\]

for \( k = 1, \ldots, n - 1 \),

\[
K_{n, \theta}^{(k)}(x, y) = \left(1 - \delta_{x_{k+1}, x_{k+1}}\right) \delta_{y, x} - \frac{1}{2\Delta} \left( e^{i\theta} \delta_{y, x - e_{k+1}} + e^{-i\theta} \delta_{y, x+e_k}\right);
\]

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and
\[ K_{n,\theta}(x, y) = \frac{1}{2} \delta_{y,x} - \frac{1}{2\Delta} e^{-i\theta} \delta_{y,x+e_n}, \]

In addition to \( H_Z(n; \theta) \) it is useful to consider the operator on the vector space \( \ell(\mathcal{X}_n) \) defined through this same kernel. We denote this operator as \( K(\theta) \). It is easy to check that the kernel does map the vector space to itself (i.e., there are no divergences, because all sums are finite). It is also easy to see that \( \ell_0(\mathcal{X}_n) \) is an invariant subspace. The restriction of \( K(\theta) \) to \( \ell_0^2(\mathcal{X}_n) \) is \( H_Z(n; \theta) \).

From the decomposition, it is clear that for any \( f \in H_Z(n) \), we have
\[ (f, H_Z(n) f) = \int_0^{2\pi} (Ff(\theta), H_Z(n, \theta) Ff(\theta)) \frac{d\theta}{2\pi}. \]

Also, the map \( \theta \mapsto H_Z(n; \theta) \) is norm continuous. Therefore, by the Rayleigh-Ritz variational principle,
\[ \text{infspec } H_Z(n) = \min_{\theta \in \mathbb{S}^1} \text{infspec } H_Z(n; \theta). \]

(At this point, the reader may be concerned that because of (26) this might not be true. We leave it as an easy exercise to check that \( \text{infspec } H_Z(n; \theta) = \text{infspec } H_Z(n; \theta + 2\pi k/n) \) for all \( \theta \) and that moreover, the equation above is true.) We can see that the minimum is attained for \( \theta = 0 \) (as well as possibly other values as per the last comment). This follows from the observation that for any \( f \in \ell_0^2(\mathcal{X}_n) \), one has
\[ (|f|, H_Z(n, 0)|f|) \leq (f, H_Z(n, \theta)f), \]
for all \( \theta \in \mathbb{S}^1 \), where \( |f| \in \ell_0^2(\mathcal{X}_n) \) is the function \( |f|(x) = |f(x)| \). (This, in turn, follows because the kernel \( K_{n,0} \) has nonpositive signs for the off-diagonal entries.) From this, one knows that Proposition 3.1 will follow if we show that
\[ \text{infspec } H_{Z,n}(0) = \frac{(1 - q^2)(1 - q^n)}{(1 + q^2)(1 + q^n)}. \]

3.3 The Bethe ansatz

We will now develop the simplest possible application of the Bethe ansatz. The following is a well-known result, which we include for completeness.

**Lemma 3.3.** Let \( \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \). Suppose that \( \xi \in (\mathbb{C}^\times)^n \) satisfies, for \( k = 1, \ldots, n - 1 \),
\[ e^{i\theta} \xi_k + e^{-i\theta} \xi_{k+1} = 2\Delta. \]

Define the function, \( f_\xi \in \ell(\mathcal{X}_n) \) as
\[ f_\xi(x) = \prod_{k=1}^n \xi_k^{x_k}. \]
Then \( f \) is an eigenvector of \( K(\theta) \) with eigenvalue equal to

\[
E(\xi, \theta) = \sum_{k=1}^{n} \left( 1 - \frac{1}{2\Delta} \left[ e^{i\theta} \xi_{k} + e^{-i\theta} \xi_{k}^{-1} \right] \right).
\]  

(30)

Proof. The key is to consider an operator on \( \ell(\mathbb{Z}^n) \) which restricts to \( K(\theta) \) on \( \ell(\mathcal{X}_n) \). Define the following kernel on \( \mathbb{Z}^n \),

\[
L_{n,\theta}(x, y) = \sum_{k=1}^{n} \left( \delta_{y,x} - \frac{1}{2\Delta} \left( e^{i\theta} \delta_{y,x-e_k} + e^{-i\theta} \delta_{y,x+e_k} \right) \right).
\]

Let \( \mathcal{L}(\theta) \) be the operator with this kernel. This is easily related to the Laplacian on \( \mathbb{Z}^n \). In particular, if one defines \( F_\xi \in \ell(\mathbb{Z}^n) \) by the same formula as in (29) (except on all of \( \mathbb{Z}^n \)), one has

\[
\mathcal{L}(\theta) F_\xi = E(\xi, \theta) F_\xi.
\]

For \( k = 1, \ldots, n-1 \), define a kernel on \( \mathbb{Z}^n \) by

\[
M_{n,\theta}^{k,k+1}(x, y) = \delta_{y_{k+1},y_k+1} \left( \delta_{y,x} - \frac{1}{2\Delta} \left( e^{i\theta} \delta_{y,x-e_k} + e^{-i\theta} \delta_{y,x+e_k} \right) \right).
\]

Let \( M_{k,k+1}(\theta) \) be the operator with this kernel. Then we claim the following is true. First, for any \( x, y \in \mathcal{X}_n \),

\[
K_{n,\theta}(x, y) = L_{n,\theta}(x, y) - \sum_{k=1}^{n-1} M_{n,\theta}^{k,k+1}(x, y).
\]

Second, if \( y \in \mathcal{X}_n \) and \( x \in \mathbb{Z}^n \setminus \mathcal{X}_n \), then

\[
L_{n,\theta}(x, y) - \sum_{k=1}^{n-1} M_{n,\theta}^{k,k+1}(x, y) = 0.
\]

The reader can check both properties easily. Because of these two properties the following is a fact. Suppose that \( F \in \ell(\mathbb{Z}^n) \) is an eigenvector of \( \mathcal{L}(\theta) \) with eigenvalue \( E(\theta) \), and suppose that \( M_{k,k+1}(\theta) F = 0 \) for all \( k = 1, \ldots, n-1 \). Then defining \( f \in \ell(\mathcal{X}_n) \) to be the restriction of \( F \), one knows that \( f \) is an eigenvector of \( K(\theta) \) with the same eigenvalue \( E(\theta) \).

The condition \( M_{k,k+1}(\theta) F_\xi = 0 \), is called the “meeting condition” in the context of the Bethe ansatz. It is

\[
2\Delta F(y) = e^{i\theta} F(y + e_k) + e^{-i\theta} F(y - e_{k+1}),
\]

for every \( y \in \mathbb{Z}^n \) such that \( y_{k+1} = y_k + 1 \). For \( F = F_\xi \), this is equivalent to

\[
2\Delta \xi_k^{y_k} \xi_{k+1}^{y_{k+1}} = e^{i\theta} \xi_{k+1}^{y_k+1} \xi_k^{y_{k+1}} + e^{-i\theta} \xi_{k+1}^{y_k} \xi_k^{y_{k+1}}.
\]

Dividing by \( \xi_{k+1}^{y_k} \xi_k^{y_{k+1}} \), this is precisely the relation in (28). \( \square \)
The condition in (28) is the same as the linear fractional relation

\[ e^{i\theta} \xi_{k+1} = \frac{1}{2\Delta - e^{i\theta} \xi_k} . \]

Let us define the matrix

\[ A = \begin{bmatrix} 0 & 1 \\ -1 & q + q^{-1} \end{bmatrix} . \]

Then the linear fractional relation is expressed as

\[ A \begin{bmatrix} e^{i\theta} \xi_k \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ e^{-i\theta} \xi_{k+1}^{-1} \end{bmatrix} . \]

More generally, suppose \( v_k \in \mathbb{C}^2 \setminus \{0\} \) is a vector such that \( v_k^1/v_k^2 = e^{i\theta} \xi_k \). Then, defining \( v_{k+1} = Av_k \), we see that \( v_{k+1} \in \mathbb{C}^2 \setminus \{0\} \) and \( v_{k+1}^1/v_{k+1}^2 = e^{i\theta} \xi_{k+1} \). The eigenpairs of \( A \) are

\[ \lambda_+ = q, \quad \mu_+ = \begin{bmatrix} q^{-1/2} \\ q^{1/2} \end{bmatrix} \quad \text{and} \quad \lambda_- = q^{-1}, \quad \mu_- = \begin{bmatrix} q^{1/2} \\ q^{-1/2} \end{bmatrix} . \]

Therefore, the most general solution to the linear recurrence relation \( v_{k+1} = Av_k \) for \( k = 1, \ldots, n - 1 \) is

\[ v_k = \alpha q^k \mu_+ + \beta q^{-k} \mu_- . \]

Therefore, the most general solution to the linear fractional recurrence relation (28) is

\[ e^{i\theta} \xi_k = \frac{z^{1/2} q^{k-1/2} + z^{-1/2} q^{-k+1/2}}{z^{1/2} q^{k+1/2} + z^{-1/2} q^{-k-1/2}} . \]

(For this, we have taken \( \alpha = z^{1/2} q^{1/2} \) and \( \beta = z^{-1/2} q^{-1/2} \), which is allowed since the two variables \( \alpha \) and \( \beta \) only amount to one independent quantity in the ratio.)

Let us define

\[ \Xi_m(z) = \frac{z^{1/2} q^{m-1/2} + z^{-1/2} q^{-m+1/2}}{z^{1/2} q^{m+1/2} + z^{-1/2} q^{-m-1/2}} , \]

for all \( z \). Then another way to write the most general solution of (28) is \( \xi_k = \Xi_m(k)(z) \), where \( m(k) = k - (n + 1)/2 \) and \( z \in \mathbb{C} \). If we wish to have \( T \)-invariance, then we require \( \xi_1 \cdots \xi_n = 1 \). Since it is more convenient to work with \( e^{i\theta} \xi_k \), this is rewritten as \( (e^{i\theta} \xi_1) \cdots (e^{i\theta} \xi_n) = e^{im\theta} \). One easily sees that

\[ \Xi_{M}(z) \Xi_{M+1}(z) \cdots \Xi_M(z) = \frac{z^{1/2} q^{-M-1/2} + z^{-1/2} q^{M+1/2}}{z^{1/2} q^{M+1/2} + z^{-1/2} q^{-M-1/2}} . \]

So the condition for \( T \)-invariance is that \( z \) solves

\[ \frac{z^{1/2} q^{-M-1/2} + z^{-1/2} q^{M+1/2}}{z^{1/2} q^{M+1/2} + z^{-1/2} q^{-M-1/2}} = e^{im\theta} , \]

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where $M = (n - 1)/2$. One solution is $z = e^{i\Theta}$, where $\Theta = \Theta(q, n, \theta)$ is

$$\Theta = 2 \tan^{-1} \left( \frac{1 + q^n}{1 - q^n \tan(n\theta/2)} \right).$$

For each choice of $N_2, \ldots, N_n \in \mathbb{N} = \{1, 2, 3, \ldots\}$, there is a unique point $y \in \mathcal{Y}_n$ with $y_k - y_{k-1} = N_k$ for $k = 2, \ldots, N$, and this labels all possible points in $\mathcal{Y}_n$. Therefore, one has

$$\sum_{y \in \mathcal{Y}_n} |f_{\xi}(y)|^2 = \sum_{N_2, \ldots, N_n \in \mathbb{N}} \left( \prod_{k=2}^{n} |\xi_k| \right)^{N_k} = \prod_{k=2}^{n} \left( \frac{\prod_{j=k}^{n} |\xi_j|}{1 - \prod_{j=k}^{n} |\xi_j|} \right).$$

But, for $2 \leq k \leq n$, we have

$$\prod_{j=k}^{n} \xi_j = \Xi_m(z) \cdots \Xi_M(z)$$

$$= \frac{\cos(\Theta/2)[q^{m-1/2} + q^{-m+1/2}] + i \sin(\Theta/2)[q^{m-1/2} - q^{-m+1/2}]}{\cos(\Theta/2)[q^{M+1/2} + q^{-M-1/2}] + i \sin(\Theta/2)[q^{M+1/2} - q^{-M+1/2}]},$$

for $M = (n - 1)/2$ and $m = (2k - n - 1)/2$. This clearly has norm less than 1. So $||f_{\xi}||$, relative to $\ell^2(\mathcal{X}_n)$, is finite.

The eigenvalue is

$$E(\xi, \theta) = \sum_{m=-M}^{M} \left( 1 - \frac{1}{2\Delta} [\Xi_m(z) + \Xi_m(z)^{-1}] \right),$$

where $z = e^{i\Theta}$. But one can easily verify that

$$1 - \frac{1}{2\Delta} [\Xi_m(z) + \Xi_m(z)^{-1}] = \frac{1 - q^2}{1 + q^2} \left[ \frac{1}{1 + q^{2m+1}z} - \frac{1}{1 + q^{2m-1}z} \right].$$

Therefore, by a telescoping sum

$$E(\xi, \theta) = \frac{1 - q^2}{1 + q^2} \left[ \frac{1}{1 + q^{2M+1}z} - \frac{1}{1 + q^{2M-1}z} \right]$$

$$= \frac{1 - q^2}{1 + q^2} \left[ \frac{1}{1 + q^nz} - \frac{1}{1 + q^{-n}z} \right].$$

Putting this all together, we obtain the following.

**Lemma 3.4.** Given $n \in \mathbb{N}_+$, we set $M = (n - 1)/2$. For $\theta \in (-\pi/n, \pi/n)$, we define

$$\Theta = 2 \tan^{-1} \left( \frac{1 + q^n}{1 - q^n \tan(n\theta/2)} \right) \in (-\pi, \pi),$$
and for \( m = -M, -M + 1, \ldots, M \) we define

\[
\Xi_m = \frac{q^{m-1/2}e^{i\Theta/2} + q^{-m+1/2}e^{-i\Theta/2}}{q^{m+1/2}e^{i\Theta/2} + q^{-m-1/2}e^{-i\Theta/2}}.
\]

Then setting

\[
\xi_k = e^{-i\theta} \Xi_{k-(n+1)/2},
\]

for \( k = 1, \ldots, n \), we have that \( f_\theta(x) = \prod_{k=1}^{n} \xi_k^{x_k} \) defines a (normalizable) eigenvector of \( H_Z(n;\theta) \). Its energy eigenvalue is equal to

\[
E_n(\theta) = \frac{(1 - q^2)(1 - q^{2n})}{(1 + q^2)(1 + q^n e^{i\Theta})(1 + q^n e^{-i\Theta})}.
\]

In particular, when \( \theta = 0 \), this gives the formula from [21].

**Remark 3.5.** Parts of this lemma are standard. For example, the linear fractional transformation was solved by Babbitt and Gutkin in [1]. However, they did not consider the direct integral decomposition. Instead they considered “generalized eigenvectors”. Also, no reference is made to the exact formula for the energy. A more explicit formula is

\[
E_n(\theta) = \frac{1 - q^2}{(1 + q^2)(1 + q^n)} \left( 1 - q^n + \frac{2(1 - \cos \theta)}{1 - q^n} \right).
\]

From this we can derive the well-known dispersion relation for the isotropic model, \( \lim_{q \uparrow 1} E_n(\theta) = \frac{1}{n}(1 - \cos \theta) \). However, to prove that this is the minimum energy for \( H_Z(n;\theta) \) is beyond our calculations because we would have to obtain the full diagonalization of \( H_Z(n;\theta) \), which we have not done. For \( q = 1 \), this was done in the important work by Babbitt and Thomas [2]. (The generalization to other \( q \) was outlined in [1].) All that we need to calculate is \( E_n(\theta = 0) \), which we handle by a different technique, next.

**Remark 3.6.** In [21], Yang and Yang considered the ferromagnetic XXZ model to complement their famous and important work on the antiferromagnetic XXZ model [20]. They derived the linear fractional recurrence relation for \( \theta = 0 \) with respect to the problem of calculating \( E^{\text{cyc}}(n) \). However, they did not solve the linear fractional recurrence relation, although they did set up a graphical method of solution which allowed them to determine the important fact that \( \lim_{m \to +\infty} \Xi_m = q \) and \( \lim_{m \to -\infty} \Xi_m = 1/q \).

This would have given them the result that \( \lim_{n \to \infty} E^{\text{cyc}}(n) \) is finite. (In fact it is \( \alpha = (1 - q^2)/(1 + q^2) \).) But there is an unfortunate typographical error in their paper. They mistyped the formula for the energy in the equation just before equation (24) of [21] (compare to their definition in equation (11) of [20]). In our notation, their error is equivalent to saying \( E_n(\xi, \theta = 0) \) is equal to

\[
\sum_{k=1}^{n} \left[ 1 - (\xi_k + \xi_k^{-1}) \right].
\]
In other words, they left off an important factor \((2\Delta)^{-1}\). For this reason, they determined that the energy of an “edge spin” is asymptotically equal to \(-2\Delta + 1\) (with our notation) instead of the correct value, which is 0. As a consequence the droplet nature of these states was not recognized at the time.

### 3.4 Positive eigenvectors are ground states

In Lemma 3.4, setting \(\theta = 0\) gives \(\Theta = 0\) and therefore \(\xi_k > 0\) for all \(k\). Hence \(f_\xi\) is a strictly positive eigenvector of \(H_\mathcal{E}(n; 0)\). Moreover, we claim that \(\text{infspec } H_\mathcal{E}(n; 0)\) is the eigenvalue of \(f_\xi\). This would be enough to prove Proposition 3.1. We also know that \(c_1 - H_\mathcal{E}(n; 0)\) is positivity preserving (when considered as an operator on \(\ell^2(\mathcal{Y}_n)\)) and bounded. Therefore, the proof is completed by applying the following theorem.

**Theorem 3.7.** Let \(\mathcal{Y}\) be a countable set. Suppose that \(A\) is a positivity preserving, bounded, self-adjoint operator on \(\ell^2(\mathcal{Y})\). If \(A\) has a strictly positive eigenvector \(f\), then the eigenvalue of \(f\) equals the spectral radius.

**Proof.** Without loss of generality, assume \(\mathcal{Y} = \mathbb{N}_+\). Let us denote \(\mathcal{H} = \ell^2(\mathbb{N}_+)\). For each \(N \in \mathbb{N}_+\), let \(P_N\) be the orthogonal projection from \(\mathcal{H}\) onto \(\mathcal{H}_N = \ell^2(\{1, \ldots, N\})\). Let us define

\[
A_N := P_N A P_N.
\]

Obviously this is positivity preserving. Let us also define

\[
f_N := P_N f,
\]

which is a strictly positive vector in \(\mathcal{H}_N\). Let \(E\) be the eigenvalue of \(f\). Then

\[
P_N A f = E f_N. \tag{31}
\]

Let us define \(P'_N = I - P_N\), and let us define another nonnegative vector in \(\mathcal{H}_N\),

\[
\tilde{f}_N := P_N A P'_N f.
\]

Since \(P'_N f = f - f_N\), an obvious bound is \(\|\tilde{f}_N\| \leq \|A\| \cdot \|f - f_N\|\). Using \(\|f\|\), we have

\[
A_N f_N + \tilde{f}_N = E f_N. \tag{32}
\]

Let \(t_N = (f_N, \tilde{f}_N) \geq 0\) and \(r_N = \|f_N\|^2 > 0\). We know that \(r_N\) is an increasing sequence with limit equal to \(\|f\|^2\). Consider the operator \(\tilde{A}_N\), which is self-adjoint on \(\mathcal{H}_N\), defined by

\[
\tilde{A}_N g = A_N g + r_N^{-1} \left[ f_N (f_N, g) + f_N (\tilde{f}_N, g) \right].
\]
Note that each of the three summands is positivity preserving, since \( f_N \) and \( \tilde{f}_N \) are nonnegative vectors. On the other hand,

\[
\tilde{A}_N f_N = A_N f_N + \tilde{f}_N + \frac{t_N}{r_N} f_N .
\]

Therefore, by equation (32), we have that

\[
\tilde{A}_N f_N = \left( E + \frac{t_N}{r_N} \right) f_N .
\]

In other words, \( f_N \) is an eigenvector of \( \tilde{A}_N \). Since \( f_N \) has strictly positive components in \( \mathcal{H}_N \), the Perron-Frobenius theorem guarantees that the spectral radius of \( \tilde{A}_N \) equals \( E + (t_N/r_N) \). From this we determine that

\[
\lim_{N \to \infty} \max \text{spec}(\tilde{A}_N) = E ,
\]

because \( r_N \uparrow \|f\|^2 \), while

\[
0 \leq t_N \leq \|f_N\| \cdot \|\tilde{f}_N\| \leq \|A\| \cdot \|f\| \cdot \|f - f_N\| ,
\]

and \( \|f - f_N\| \downarrow 0 \) as \( N \to \infty \).

We claim that \( \rho(A) \leq \lim_{N \to \infty} \rho(\tilde{A}_N) \) by the variational principle. This would imply that \( \rho(A) \leq E \). We already know that \( E \) is in the spectrum of \( A \), so that if \( \rho(A) \leq E \) we have \( \rho(A) = E \). So we just need to prove \( \rho(A) \leq \lim_{N \to \infty} \rho(\tilde{A}_N) \).

To begin with, note that by the (Rayleigh-Ritz) variational principle for self-adjoint operators, we have

\[
\rho(A) = \sup \{(g, Ag) : \|g\| = 1\} .
\]

Therefore, it suffices to observe that for any \( g \in \mathcal{H} \), we have that \( g = \lim_{N \to \infty} P_N g \) and

\[
Ag = \lim_{N \to \infty} \tilde{A}_N P_N g .
\]

The fact that \( g = \lim_{N \to \infty} P_N g \) is the usual density result (which we have already implicitly used). Since \( \lim_{N \to \infty} \|g - P_N g\|^2 = 0 \), one has

\[
\lim_{N \to \infty} \|A g - A_N P_N g\|^2 = \lim_{N \to \infty} \left( g - P_N g, A^2 (g - P_N g) \right) 
\]

\[
\leq \|A\|^2 \lim_{N \to \infty} \|g - P_N g\|^2 = 0 ,
\]

and the perturbation

\[
[\tilde{A}_N - A_N] g = r_N^{-1} \left( f_N \left( \tilde{f}_N, g \right) + \tilde{f}_N \left( f_N, g \right) \right) ,
\]

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is bounded, in norm, by $\|\tilde{A}_N - A_N\| \leq 2 \left\| \tilde{f}_N \right\| / \|f_N\|$. This converges to zero for reasons we have explained before. Therefore, for any $g$ satisfying $\|g\| = 1$,

$$(g, A g) = \lim_{N \to \infty} \left( P_N g, \tilde{A}_N P_N g \right) \leq \lim_{N \to \infty} \rho(A_N),$$

as was claimed. $\square$

4 Droplet energies in the droplet Hamiltonian and cyclic chain

Our next goal is to compare the energies (recall their definitions in (15–18)) $E(L, n)$, $E^{\text{cyc}}(L, n)$ and $E^\delta(L, n)$, for $\delta \geq 1$, to $E_Z(n)$. The simplest case is $E^\delta(L, n)$ for $\delta \geq 1$. The desired result follows by two applications of the Rayleigh-Ritz variational principle.

**LEMMA 4.1.** For any $\delta \in \mathbb{R}$,

$$E_Z(n) \geq \limsup_{L \to \infty} E^\delta(L, n).$$

**Proof.** Given $a \leq b$, both in $\mathbb{Z}$, define $X_n([a, b]) = \{ x \in \mathbb{Z}^n : a \leq x_1 < \cdots < x_n \leq b \}$. By definition,

$$E_Z(n) = \inf\{(f, H_Z f) : f \in H_Z(n), \|f\| = 1\}.$$

Since the functions with finite support are dense in $H_Z(n)$, and since $H_Z$ is bounded on $H_Z(n)$, one can replace this by

$$E_Z(n) = \inf_{L \in \mathbb{N}^+} \inf \left\{ (f, H_Z f) : f \in L^2(X_n([-L, L])), \|f\| = 1 \right\}.$$

Moreover, by translation-invariance one can shift a function on $[-L, L]$ to a function on $[2, 2L + 2]$. Therefore,

$$E_Z(n) = \lim_{L \geq 3} \inf \left\{ (f, H_Z f) : f \in L^2(X_n([2, L - 1])), \|f\| = 1 \right\}.$$

Suppose $f \in L^2(X_{[2, L-1],n}$ and $\|f\| = 1$. Define $\psi \in H_{[1,L]}(n)$ by

$$\psi = \sum_{1 \leq x_1 < \cdots < x_n \leq L} f(x_1, \ldots, x_n) S_{x_1}^- \cdots S_{x_n}^- |\uparrow\rangle_{[1,L]}.$$

Then it will be apparent that $(\psi, h_{x,x+1}\psi) = (f, h_{x,x+1}f)$ for $\{x, x+1\} \subset [1, L]$. Moreover, it is apparent that $(\psi, S_1^+\psi) = 1/2$ and that $(\psi, S_L^+\psi) = 1/2$ because $f$ vanishes if there is any down-spin at sites 1 or $L$. Therefore, $(\psi, H^\delta_{[1,L]}\psi) = (\psi, H_{[1,L]}\psi)$ and this is equal to $(f, H_Z f)$. So

$$(f, H_Z f) = (\psi, H^\delta_{[1,L]}\psi) \geq E^\delta(L, n).$$
In fact, by exactly the same argument,

\[ (f, H_Z f) \geq E^\delta_L(n), \]

for any \( L' \geq L \). From this, one sees that

\[
\inf \{ (f, H_Z f) : f \in \ell^2(\mathcal{X}_{[2,L-1],n}), \| f \| = 1 \} \geq \sup_{L' \geq L} E^\delta(L', n).
\]

But taking the limit as \( L \to \infty \), we obtain the result.

**Lemma 4.2.** For any \( \delta \geq 1 \), one has

\[ E_Z(n) \leq \liminf_{n \to \infty} E^\delta(L, n). \]

**Proof.** Suppose that \( \psi \in \mathcal{H}(L, n) \) is any normalized vector, so that \( \| \psi \| = 1 \). We can write this vector as

\[
\psi = \sum_{1 \leq x_1 < \cdots < x_n \leq L} F(x_1, \ldots, x_n) S^-_{x_1} \cdots S^-_{x_n} |\uparrow\rangle_{[1, L]},
\]

where \( F : \mathcal{X}_{[1, L], n} \to \mathbb{C} \) is such that

\[
\| F \|^2 := \sum_{x \in \mathcal{X}_{[1, L], n}} |F(x)|^2 = 1.
\]

Then we can extend \( F \) to a function on \( \mathcal{X}_n \) by defining

\[
f(x) = \begin{cases} F(x) & \text{if } x \in \mathcal{X}_{[1, L], n}, \\ 0 & \text{if } x \in \mathcal{X}_n \setminus \mathcal{X}_{[1, L], n}. \end{cases}
\]

It is easy to see, from the construction of the single site representations of \( \text{SU}(2) \) on \( \mathcal{H}_Z \), that \((f, h_{x,x+1} f) = (\psi, h_{x,x+1} \psi)\) for \( \{x, x+1\} \subset [1, L] \). It is also easy to see that \((f, h_{x,x+1} f) = 0\) for \( \{x, x+1\} \subset \mathbb{Z} \setminus [1, L] \), because, in this case, \( f \) has no down-spin at either \( x \) or \( x + 1 \). Furthermore, one can easily see that

\[
(f, h_{0,1} f) = (\psi, \tilde{h}^+_x \psi) \quad \text{and} \quad (f, h_{L,L+1} f) = (\psi, \tilde{h}^+_L \psi),
\]

where \( h^+_x = \text{Tr}_y(h_{x,y} P^+_y) \) is the partial trace of the XXZ interaction with \( P^+_y = \frac{1}{2} 1 + S^3_y \) the projection onto the up-spin vector. But one easily calculates

\[
h^+_x = \frac{1}{4} 1 - \frac{1}{2} S^3_x = \frac{1}{2}(1 - P^+_{x}).
\]

Therefore, one sees that the energy of \( f \) relative to \( H_Z \) is exactly equal to

\[
(f, H_Z f) = (\psi, H^\delta_{[1, L]} \psi),
\]

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with $\delta = 1$. Since the boundary field operator is positive semi-definite, one obtains

$$ (f, H_Z f) \leq \left( \psi, H_{[1,L]}^\delta \psi \right), $$

as long as $\delta \geq 1$. From this, it follows that

$$ E_Z(n) \leq \left( \psi, H_{[1,L]}^\delta \psi \right). $$

By minimizing over $\psi$, and using the Rayleigh-Ritz variational principle, one obtains the result. \hfill \Box

Combining these two lemmas proves the following.

**Proposition 4.3.** For $\delta \geq 1$, $\lim_{L \to \infty} E^\delta(L, n)$ exists and equals $E_Z(n)$. \hfill \Box

Let us now consider $E^\text{cyc}(L, n)$. There is a proof completely analogous to Lemma 4.1 for the following.

**Lemma 4.4.** $E_Z(n) \geq \limsup_{L \to \infty} E^\text{cyc}(L, n)$.

To prove the analogue of Lemma 4.2 requires a different argument. Given any $x \in [1, L]$, we define the projections $P_x^+$ and $P_x^-$ on $H_{[1,L]}$ by

$$ P_x^+ = \frac{1}{2} (1 - S_x) \quad \text{and} \quad P_x^- = 1 - P_x^+. $$

Suppose that $[a, b] \subset [1, L]$. Then we define

$$ P_{[a,b]}^\pm = \prod_{x=a}^b P_x^\pm. $$

Let us also define $P_{[a,b]} = P_{[a,b]}^+ + P_{[a,b]}^-$. The following result follows from Corollary 4.3 in [13].

**Lemma 4.5.** Let $K_L$ be a self-adjoint operator on $H_{[1,L]}$ and $M = \|K_L - H_L\|$. Given $E < \infty$, suppose $\psi \in H_{[1,L]}$ is any vector with $\|\psi\| = 1$ and $(\psi, K_L \psi) \leq E$. Given any interval $[a, a + \ell - 1] \subset [1, L]$ and any $\ell' < \ell$, there is a subinterval $[b, b + \ell' - 1] \subset [a, a + \ell - 1]$ such that

$$ \|P_{[b,b+\ell'-1]} \psi\| \geq 1 - \epsilon, $$

where

$$ \epsilon = \frac{2(E + M)}{\gamma \lfloor \ell/\ell' \rfloor}. $$

Moreover, when $\epsilon < 1$, defining $I = [b, b + \ell' - 1]$,

$$ (P_I \psi, K_L P_I \psi) \leq (\psi, K_L \psi) + \left( M \epsilon + 2(\Delta^{-1} + 2M) \sqrt{\epsilon(1 - \epsilon)} \right). $$

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We can now prove the following.

**Lemma 4.6.** \( \liminf_{L \to \infty} E_{\text{cyc}}(L, n) \geq E_Z(n) \).

**Proof.** The result is trivial for \( n = 0 \). Let us assume \( n > 0 \).

Given \( L \in \mathbb{N} \), let \( \psi_L \) be a vector in \( \mathcal{H}_{[1, L]}(n) \) such that \( \|\psi\| = 1 \) and \( (\psi_L, H_{L}^{\text{cyc}} \psi_L) = E_{\text{cyc}}(L, n) \). We will now apply Lemma 4.5. We take \( \ell = L \) so that \( a = 1 \) and \( a + \ell - 1 = L \). We take \( \ell' = 2n \). We have \( M = 1 \) because \( H_L - H_{L}^{\text{cyc}} = h_{1, L} \) and \( \|h_{1, L}\| = 1 \). By Lemma 4.4 we can take \( E = 2E_Z(n) \) as long as \( L \) is large enough. Hereafter, we assume that \( L \) is large enough. Then we obtain

\[
\epsilon = \epsilon_L := \frac{4E_Z(n) + 2}{\gamma \lfloor L/2n \rfloor}.
\]

We obviously have \( \lim_{L \to \infty} \epsilon_L = 0 \) since \( \epsilon_L = O(1/L) \). On the other hand, given any interval \( I \subset [1, L] \) with \( |I| = 2n \), we have

\[
P_L^+ \psi_L = 0,
\]

since \( \psi_L \) only has \( n \) downspins, and \( P_L^+ \) projects onto vectors which have downspins at all \( 2n \) sites in \( I \). Therefore, we obtain

\[
\|P_L^+ \psi_L\|^2 \geq 1 - \epsilon.
\]

Since we have translation invariance, we can translate the interval \( I \) as long as we translate \( \psi_L \) by the same amount. Then, without loss of generality, we may assume that

\[
\|P_{[1, n]}^+ P_{[L-n+1, L]}^+ \psi_L\|^2 \geq 1 - \epsilon.
\]

Let \( \psi'_L = P_{[1, n]}^+ P_{[L-n+1, L]}^+ \psi_L \). This can be written as

\[
\psi'_L = \sum_{x \in \mathcal{X}_{[n+1, L-n], n}} f'_L(x) S_{x_1}^- \cdots S_{x_n}^- |\uparrow\rangle_{[1, L]},
\]

for some \( f'_L \in \ell^2(\mathcal{X}_{[n+1, L-n], n}) \). It is easily seen that \( \|f_L'\| = \|\psi'_L\| \) and that \( (f'_L, H_Z f'_L) = (\psi'_L, H_{Z}^{\text{cyc}} \psi'_L) \). Therefore, using the second half of Lemma 4.5 and the fact that \( \lim_{L \to \infty} \epsilon_L = 0 \), we conclude that

\[
\liminf_{L \to \infty} E_{\text{cyc}}(L, n) \geq E_Z(n),
\]
as desired.

Thus we have proved the following.

**Proposition 4.7.** For each \( n \in \mathbb{N} \), \( \lim_{L \to \infty} E_{\text{cyc}}(L, n) \) exists and equals \( E_Z(n) \).

**Remark 4.8.** Yang and Yang actually calculated \( \lim_{L \to \infty} E_{\text{cyc}}(L, n) \) using the Bethe ansatz in [21]. However, their proof was more involved than ours, because they used the Bethe ansatz for finite \( L \) and took the asymptotic limit. Also, as mentioned before, because of a typographical error, they miscalculated \( E_{\text{cyc}}(n) \).
5 Droplet energies in the kink Hamiltonian

highest weight vectors are vectors with total spin equal to the total $S^3$-component. We denoted the subspace of highest weight vectors on the chain $[1,L]$ with total spin $L/2 - n$ by $H_{\text{hw}}(n)$. We also defined the energy $E(L,n)$ to be the lowest energy of the kink Hamiltonian $H^k_n$, which is the restriction of $H^k_{[1,L]}$ to $H_{\text{hw}}(n)$. We are interested in the limit, $\lim_{L \to \infty} E(L,n)$. Throughout this section, we require $0 < q < 1$; for $q = 1$ we can use the SU(2) symmetry to show instantly that $\lim_{L \to \infty} E(L,n) = 0$. Here is the main result.

**PROPOSITION 5.1.** For each $n \in \mathbb{N}$ and $0 < q < 1$,

$$\lim_{L \to \infty} E(L,n) = E_Z(n) = \frac{(1-q^2)(1-q^n)}{(1+q^2)(1+q^n)}.$$  \hfill (34)

The essential part of this statement relates the ground state energy of the kink Hamiltonian for highest weight vectors with the ground state energy of the translation invariant Hamiltonian in the $n$-magnon sector. A detailed description of this relation is a bit lengthy but the first steps are well-known and standard.

First we recall the construction of a basis for the highest weight vectors for the finite chain. Then, we extend this basis to infinite volume and define the action of the infinite volume kink Hamiltonian, $H^k_{Z,n}$. We then construct a continuous bijection, $R$, between the highest weight vectors and the $n$-magnon vectors with the property that $H^k_{Z,n} = R^{-1}H^k_{Z,n}R$. The main proposition is proved in Section 5.5.

5.1 Basis for the highest weight vectors

We use the notation $[2]_q = q + q^{-1}$. Let us start by introducing a set of “valid brackets” from which we then define the “generalized Hulthén bracket vectors”. They were first introduced by Lieb and Temperley in [19]. Let $n \in \mathbb{N}$ be fixed throughout this section. A valid bracket $b = ([x_1,y_1], \ldots, [x_n,y_n])$ is a collection of $n$ brackets $[x_i,y_i]$ with $1 \leq x_i < y_i \leq L$ for all $i$ and $y_1 < \ldots < y_n$. In addition, a valid bracket satisfies the three conditions:

1. **(Exclusion)** For each $x \in [1,L]$, let $d_b(x) = \# \{ i : x_i = x \text{ or } y_i = x \}$. Then $d_b(x) \leq 1$ for all $x \in [1,L]$.

2. **(Non-crossing)** If $y_i > x_j$ for some $j > i$ then $x_i < x_j < y_j < y_i$.

3. **(Non-spanning)** For any $i$ if there is some $x$ such that $x_i < x < y_i$, then $d_b(x) = 1$.

We define $|b| = n$. The set of all valid brackets on the chain $[1,L]$ is denoted by $V([1,L],n)$. Now, given a valid bracket $b \in V([1,L],n)$, we define the “generalized
Hulthén bracket vector $\psi(b) \in \mathcal{H}_{[1,L]}$ as

$$\psi(b) := \prod_{i=1}^{|b|} \left( q^{-1/2} S_{x_i}^- - q^{1/2} S_{y_i}^- \right) |\uparrow\rangle_{[1,L]} \ . \quad (35)$$

$|\uparrow\rangle_{[1,L]}$ is, of course, the all up-spin vector in $\mathcal{H}_{[1,L]} = (\mathbb{C}^2)^{\otimes L}$. Lieb and Temperley proved that for any $n \in [0, [L/2]]$, the set $\mathcal{V}([1,L], n)$ forms a basis spanning $\mathcal{H}_{\text{hw}}([1,L], n)$. Therefore, to calculate $E(L, n)$, it suffices to find the minimum eigenvalue for the matrix of $H_x^k$ in the basis of $\psi(b)$. In addition, the action of $h_x^k$ upon these basis vectors is simple and has a very appealing graphical representation which we will see after the following Lemma.

**Lemma 5.2.** For $x \in [1, L-1]$ and $b \in \mathcal{V}([1,L], n)$, let

$$\phi = -[2]_q h_{x,x+1}^k \psi(b) \ .$$

Then $\phi$ has the following values, depending on the case.

- If $d_b(x) = d_b(x + 1) = 0$ then $\phi = 0$.
- If $d_b(x) = 1$ but $d_b(x + 1) = 0$ then there is a bracket in $b$, $[x_i, y_i]$, with $y_i = x$. Then $\phi = \psi(b')$ where $b'$ is defined relative to $b$ by the replacement $[x_i, y_i] \to [x_i, x + 1]$.
- If $d_b(x) = 0$ but $d_b(x + 1) = 1$ then there is a bracket in $b$, $[x_i, y_i]$, with $x_i = x + 1$. Then $\phi = \psi(b')$ where $b'$ is defined relative to $b$ by the replacement $[x_i, y_i] \to [x_i, x + 1]$.
- If $d_b(x) = d_b(x + 1) = 1$ then one of four possibilities occurs.
  - There are two brackets in $b$, $[x_i, y_i]$ and $[x_j, y_j]$, with $y_i = x$ and $x_j = x + 1$. Then $\phi = \psi(b')$, where $b'$ has the replacements $\{[x_i, y_i], [x_j, y_j]\} \to \{[x, x + 1], [x_i, y_j]\}$.
  - There are two brackets in $b$, $[x_i, y_i]$ and $[x_j, y_j]$, with $y_i = x$ and $y_j = x + 1$. Then $\phi = \psi(b')$, where $b'$ has the replacements $\{[x_i, y_i], [x_j, y_j]\} \to \{[x, x + 1], [x_j, x_i]\}$.
  - There are two brackets in $b$, $[x_i, y_i]$ and $[x_j, y_j]$, with $x_i = x$ and $x_j = x + 1$. Then $\phi = \psi(b')$, where $b'$ has the replacements $\{[x_i, y_i], [x_j, y_j]\} \to \{[x, x + 1], [y_j, y_i]\}$.
  - The bracket $[x, x + 1]$ is in $b$. Then $\phi = -[2]_q \psi(b)$.

The proof is left to the reader who should notice that the vector $(q^{-1/2} S_{x}^- - q^{1/2} S_{y}^-) |\uparrow\rangle_{[1,L]}$ is just a multiple of the singlet state between sites $x$ and $y$.

In the graphical representation proposed by Lieb and Temperley we consider the graph whose vertex set is the ordered chain $[1,L]$. The edges are the edges
specified by brackets. Namely, there is an edge between \( x \) and \( y \) if \([x, y] \in b\). Because of rules (1–3) these are precisely the set of partial matchings which can all be made above the line passing through \([1, L]\), and such that no two edges intersect, nor does any edge span an unpaired vertex. We call the edges “arcs”, and moreover since we specify that they should be above the line through \([1, L]\), we call them “upper arcs”. An example for \( L = 8 \) is

\[
\{[3, 4][2, 5], [7, 8]\} \rightarrow \\
\begin{array}{c}
\infty
\end{array}
\]

Then, if we define \( U_{x,x+1} = -[2]_q h^k_{x,x+1} \), we have that \( U_{x,x+1} \) acts as follows. First of all, associate to each \( U_{x,x+1} \) a diagram of a pair of arcs; e.g,

\[
U_{1,2} = \\
U_{2,3} = \\
\vdots
\]

\[
U_{L-1,L} = \\
\begin{array}{c}
\infty
\end{array}
\]

Imagine concatenating this graph below the arc system for \( b \). Then contract all loose ends down to vertices, and stretch the arcs to their normal shapes. Then one obtains the correct arc system corresponding to \( b' \). The one exception is if \( b' = b \) because \([x, x + 1] \in b\). But in this case one obtains \( b \) with one additional circle, or “bubble”. If one declares that the bubble takes a scalar value \(-[2]_q\) to remove, then one has the correct answer for \( \phi \) also in this case.

\[
\bullet - (q + q^{-1}).
\]

At this point let us mention that the matrices \( U_{x,x+1} = -[2]_q h^k_{x,x+1} \) satisfy the Temperley-Lieb relations and that \( \mathcal{H}_{[1,L]}(n) \) is an irreducible representation space of the Temperley-Lieb algebra.

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**REMARK** 5.3. In this matrix representation, the only dependence on \( q \) comes from the “bubble” terms, which contribute a factor \(-[2]_q\) to the diagonal elements. Since this is clearly decreasing in \( q \), it is trivial to see that the minimum energy of the kink Hamiltonian in any \( n \)-magnon subspace is decreasing in \( q \). This is pertinent to Remark 2.3.

### 5.2 The kink Hamiltonian on \( \mathbb{Z} \) in the Hulthen bracket basis

The purpose of this section is to define the kink Hamiltonian on the Hulthen bracket basis vectors for brackets on \( \mathbb{Z} \) instead of \([1, L] \). So let us first extend the notion of valid brackets and set \( \mathcal{V}([-L, L], n) \) to be the set of valid brackets \( b = ([x_1, y_1], \ldots, [x_n, y_n]) \) on \([-L, L] \) satisfying the same conditions (1–3) from above. Then we define the set of valid brackets on \( \mathbb{Z} \), \( \mathcal{V}(\mathbb{Z}, n) = \bigcup_{L \in \mathbb{N}} \mathcal{V}([-L, L], n) \).

For any \( b \in \mathcal{V}(\mathbb{Z}, n) \) we take some \( L \) so that \([-L, L] \) contains all points in \( b \) and then unambiguously define the Hulthen bracket basis vectors \( \psi(b) \) as in (35) and tensor with the all spin-up vector on \( \mathbb{Z} \setminus [-L, L] \).

Now, for any \( x \in [-L, L - 1] \) and \( b \in \mathcal{V}([-L, L], n) \) we define the matrix representation,

\[
h_{x,x+1}^k \psi(b) = \sum_{b' \in \mathcal{V}([-L, L], n)} A_{x}^x(b', b) \psi(b') .
\]

Note that for any such lattice point \( x \) and bracket \( b \) we have the following fact: If \( L' \geq L \), then by Lemma 5.2

- If \( b' \in \mathcal{V}([-L, L], n) \) then \( A_{x}^{x'}(b', b) = A_{x}^{x'}(b', b) \).
- If \( b' \in \mathcal{V}([-L', L'], n) \setminus \mathcal{V}([-L, L], n) \) then \( A_{x}^{x'}(b', b) = 0 \).

Thus we can define for any \( x \in \mathbb{Z} \) the kernel \( A^{x} \) by

\[
A^{x}(b', b) = \lim_{L \to \infty} A_{x}^{x'}(b', b) .
\]

Moreover, by the same rules explained in Lemma 5.2 we have

- If \( b \in \mathcal{V}([-L, L], n) \) and \( x \in \mathbb{Z} \setminus [-L - 1, L] \), then \( A^{x}(b', b) \) is zero for every \( b' \in \mathcal{V}(\mathbb{Z}, n) \);
- If \( b' \in \mathcal{V}([-L, L], n) \) and \( x \in \mathbb{Z} \setminus [-L, L] \), then \( A^{x}(b', b) \) is zero for every \( b \in \mathcal{V}(\mathbb{Z}, n) \).

Therefore, we can define a valid, finite kernel, \( A : \mathcal{V}(\mathbb{Z}, n) \times \mathcal{V}(\mathbb{Z}, n) \to \mathbb{R} \) by

\[
A(b', b) = \sum_{x \in \mathbb{Z}} A^{x}(b', b) ,
\]

since all but a finite number of summands will be zero. Moreover, from this we conclude that if \( b \in \mathcal{V}([-L, L], n) \), then \( A(b', b) = 0 \) unless \( b' \in \mathcal{V}([-L - 1, L], n) \).
Therefore, defining \( \ell(\mathcal{V}(\mathbb{Z}, n)) \) to be the set of all sequences on the countable set \( \mathcal{V}(\mathbb{Z}, n) \), we may define the linear transformations \( \mathcal{A}_n \) and the infinite chain kink Hamiltonian \( H^k_{\mathbb{Z},n} \) by

\[
\mathcal{A}_n \delta_b = \sum_{b' \in \mathcal{V}(\mathbb{Z}, n)} A(b', b) \delta_{b'},
\]

\[
H^k_{\mathbb{Z},n} \psi(b) = \sum_{b' \in \mathcal{V}(\mathbb{Z}, n)} A(b', b) \psi(b').
\]

Again, all but finitely many terms in the sum are zero. Since \( \mathcal{A}_n \) and \( H^k_{\mathbb{Z},n} \) are defined by the same kernel they have the same spectrum.

**Lemma 5.4.** The linear operator \( \mathcal{A}_n \) restricted to \( \ell^2(\mathcal{V}(\mathbb{Z}, n)) \) is bounded.

**Proof.** For this we will use again the Riesz-convexity theorem. From Lemma 5.2, we observe the following facts:

- If \( b = ([x_1, y_1], \ldots, [x_n, y_n]) \in \mathcal{V}(\mathbb{Z}, n) \), then \( A^x(b', b) \) is zero unless \( |x - x_i| \leq 1 \) or \( |x - y_i| \leq 1 \) for some \( i \in [1,n] \). In any of these cases, there is exactly one \( b' \) such that \( A^x(b', b) \) is nonzero.

- If \( b' = ([x'_1, y'_1], \ldots, [x'_n, y'_n]) \in \mathcal{V}(\mathbb{Z}, n) \), then \( A^x(b', b) \) is zero unless \( x = x'_i \) and \( x + 1 = y'_i \) for some \( i \in [1,n] \). In this case, the only way for \( A^x(b', b) \) to be nonzero is if either: \( b = b' \); or if \( b \) differs from \( b' \) by the replacement of the arc \([x'_i, y'_i]\) by another arc with one endpoint in \( \{x, x+1\} \); the replacement of the arc \([x'_i, y'_i]\) and another arc, bracketing it, \([x'_j, y'_j]\) by the arcs \([x'_j, x'_i]\) and \([y'_j, y'_i]\). The total number of these possibilities is bounded by \( n \).

- Whenever \( A^x(b, b') \) is nonzero, the absolute value is bounded by 1.

Using these facts, one can conclude that \( \mathcal{A}_n \) is bounded both in \( \ell^1(\mathcal{V}(\mathbb{Z}, n)) \) and \( \ell^\infty(\mathcal{V}(\mathbb{Z}, n)) \) by bounding the maximum “row sum” and “column sum” of \( A \). Therefore, by the Riesz-convexity theorem, it happens that \( \mathcal{A}_n \) is bounded on every \( \ell^p(\mathcal{V}(\mathbb{Z}, n)) \) for \( 1 \leq p \leq \infty \). In particular, it works for \( p = 2 \).

5.3 Bounds for the change-of-basis transformation

We show now that the change from the Hulthén bracket basis to the Ising basis in the \( n \)-magnon sector \( \mathcal{H}_Z(n) \) is bijective and bounded. It is important that we do this on \( \mathbb{Z} \) and to remember that \( q < 1 \). We use the notation from Section 3.

**Lemma 5.5.** Define a transformation \( R : \ell^p(\mathcal{V}(\mathbb{Z}, n)) \to \ell^p(\mathcal{X}_n) \) such that

\[
\psi(b) = \sum_{x \in \mathcal{X}_n} \delta_x R(b, x).
\]
Then $R$ is invertible and bounded for $p = 1$ and $p = \infty$. More precisely,
\[
\|R\|_1 \leq q^{-1/2} \frac{(2n)!}{n!}, \quad \text{and} \quad \|R\|_\infty = \left(q^{-1/2} + q^{1/2}\right)^n.
\]
Moreover, $R$ intertwines between the operator $A_n$ and $H_{Z(n)}$, namely, $RA_n = H_{Z(n)}R$.

**Proof.** The intertwining property follows immediately since the kink boundary fields will telescope to 0 in the $L \to \infty$ limit of the all-up-spin GNS representation.

The columns of $R$ represent the coefficients of $\psi(b)$ in the expansion in the Ising basis. The map $R$ is clearly injective. On the other hand, given an Ising vector $\delta_x$, we can write this as a norm-convergent telescoping sum of Hulthé bracket vectors. This is not possible on a finite chain nor when $q = 1$.

Recall the following basic fact. If $B : \mathbb{C}^{n_1} \to \mathbb{C}^{n_2}$ is a linear transformation, and
\[
(Bu)_i = \sum_{j=1}^{n_1} B_{ij} u_j, \quad i = 1, \ldots, n_2.
\]
Then
\[
\|B\|_\infty = \max_{1 \leq i \leq n_2} \sum_{j=1}^{n_1} |B_{ij}|.
\]
Furthermore, $\|B\|_1 \leq \|B\|_\infty$. [Although $R$ itself is not a finite dimensional matrix, we could restrict the discussion of the change-of-basis transformation to finite chains where we would show that our bounds are uniform in $L$.]

If we expand $\psi(b)$ in the Ising basis $\delta_x$, then we see that the sum of the absolute values of the coefficients is $\left(q^{-1/2} + q^{1/2}\right)^n$, and we therefore have that $\|R\|_\infty = \left(q^{-1/2} + q^{1/2}\right)^n$.

The other bound is combinatorial and rests on the following claim.

**CLAIM:** Let $w = (w_1, \ldots, w_n)$ be a collection of points in $X_n$. There is an upper bound on the number of nonspanning systems of arcs
\[
b = ([x_1, y_1], \ldots, [x_n, y_n]),
\]
for which there exists a permutation $\pi \in S_n$ such that $w_{\pi(k)} \in \{x_k, y_k\}$ for each $k \in [1, n]$. The bound is $(2n)!/n!$.

Given this estimate we instantly get
\[
\|R\|_1 \leq \|R\|_\infty \leq q^{-1/2} \frac{(2n)!}{n!}.
\]

**Proof of CLAIM:** Suppose that $b = ([x_1, y_1], \ldots, [x_n, y_n])$ is a nonspanning system of arcs (i.e., satisfying conditions 1 and 3) but possibly with crossings, and such that there exists a $\pi \in S_n$, such that $w_k \in \{x_{\pi(k)}, y_{\pi(k)}\}$ for all $k \in [1, n]$. Let
$\tilde{v}_k$ be the complementary point so that $\{w_k, \tilde{v}_k\} = \{x_{\pi(k)}, y_{\pi(k)}\}$ for each $k$. Let $(v_1, \ldots, v_n)$ be the rearrangement of $\tilde{v}_1, \ldots, \tilde{v}_n$ in increasing order. Then we claim that the arc system $\mathcal{B}$, whose arcs are (the ordered rearrangement of) $\{w_k, v_k\}$ is also nonspanning. The reason is that one can transpose the $(\tilde{v}_k, \tilde{v}_{k+1})$ such that $\tilde{v}_k > \tilde{v}_{k+1}$, one-at-a-time to obtain the desired goal, and never span new sites. Particularly, the only new sites which could be spanned by transposing $\tilde{v}_k > \tilde{v}_{k+1}$ are the sites in $[\tilde{v}_k, \tilde{v}_{k+1}]$. But since $w_k < w_{k+1}$, these sites must have already been spanned by one (or both) of the two arcs whose endpoints are (the ordered rearrangements of) $\{\tilde{v}_k, w_k\}$ and $\{\tilde{v}_{k+1}, w_{k+1}\}$.

Now, we can determine a set of points $\{u_1^-, \ldots, u_n^-, u_1^+, \ldots, u_n^+\}$ such that $\{v_1, \ldots, v_n\}$ must be a subset of this one. This would obviously prove the claim, since there are at most $(2n)!/(n!)^2$ such subsets, and $\mathcal{B}$ is uniquely determined by $(\tilde{v}_1, \ldots, \tilde{v}_n)$ which is obtained from $(v_1, \ldots, v_n)$ by permuting by one of the $n!$ permutations in $\mathcal{S}_n$.

Let $\{u_1^-, \ldots, u_n^-\}$ be the points such that $u_n^-$ is the first point to the left of $w_n$ which is not among $\{w_1, \ldots, w_n\}$, and for each $k < n$, the point $u_k^-$ is the first point to the left of $w_k$ which is not among $\{w_1, \ldots, w_n\}$ or $\{u_{k+1}^-, \ldots, u_n^-\}$. We claim that if $v_k < w_k$ then it must be among $\{u_k^-, \ldots, u_n^-\}$. This is because, in this case, $\{v_k, w_k\}$ spans no sites other than $\{w_1, \ldots, w_n\}$ or $\{v_1, \ldots, v_n\}$. Since $v_j < v_k$ for $j < k$, in fact it spans no sites other than $\{w_1, \ldots, w_n\}$ and $\{v_k, \ldots, v_n\}$. A similar construction for $\{u_1^+, \ldots, u_n^+\}$ allows the conclusion that if $v_k > w_k$ then $v_k$ is in the set $\{u_1^+, \ldots, u_n^+\}$. 

\[\Box\]

**REMARKS 5.6.**

1. By the open mapping theorem, $R^{-1}$ is also bounded.

2. From the Riesz-convexity theorem we derive the bound

$$\|R\|_p \leq \|R\|_{1/p}^{1/p} \|R\|_{1/q}^{1/q}$$

for any $p \in [1, \infty]$ with $1/p + 1/q = 1$. In particular, the map

$$R : \ell^2(V(\mathbb{Z}, n)) \to \mathcal{H}_{\mathbb{Z}}(n)$$

and its inverse are bounded.

### 5.4 A Wielandt theorem

This section extends a Wielandt-type theorem \[13\] applicable to Banach spaces. We actually prove a stronger statement then needed. So let us consider a countable set $X$. Then, let $k : X \times X \to \mathbb{R}$ be a kernel with the following properties

1. There exists a uniform $k_0 < \infty$ so that $0 \leq k(x, y) \leq k_0$ for all $x, y \in X$;

2. There is an integer $N$ such that

$$\sup_{y \in X} \# \{x \in X : k(x, y) \neq 0\} \leq N, \text{ and } \sup_{x \in X} \# \{y \in X : k(x, y) \neq 0\} \leq N.$$
Interpolating between $\ell^1$ and $\ell^\infty$, we know from the Riesz-convexity theorem that this kernel defines a linear bounded operator, $K : \ell^2(X) \to \ell^2(X)$. The first partial result concerns the spectral radius of a restriction of $K$. So let $Y \subset X$. Then we define the operator $K | Y$ to be the operator on $\ell^2(Y)$ whose kernel is $k | Y \times Y$.

**PROPOSITION 5.7 (Generalized Wielandt theorem).** We assume the same conditions on $K$ as above. Let $Y$ be a finite subset of $X$ and let $j \geq 0$ be a kernel on $Y \times Y$. Let $J$ be the operator on $\ell^2(Y)$ defined by this kernel. If $j(x,y) \leq k(x,y)$ for all $(x,y) \in Y \times Y$, then $\rho(J) \leq \rho(K)$.

**REMARK 5.8.** The norm for $\ell(Y)$ is immaterial since $Y$ is a finite set.

**Proof.** By the standard Perron-Frobenius theorem for matrices, there is a vector $\psi \in \ell(Y)$ with eigenvalue $\lambda = \rho(Y)$.

By extending it to be zero on $X \setminus Y$, we can also consider this as a vector in $\ell^2(X)$. Moreover, by the our hypotheses, we have that $J\psi \leq K\psi$. This implies that $(K - \lambda)\psi \geq 0$. Writing $(K^n - \lambda^n)\psi = K(K^{n-1} - \lambda^{n-1})\psi + \lambda^{n-1}(K - \lambda)\psi$ we conclude inductively that $K^n\psi \geq \lambda^n\psi$ for all $n \in \mathbb{N}$. Since the kernel has positive entries, we get that $\|K^n\psi\| \geq \lambda^n\|\psi\|$. Therefore, $\|K^n\| \geq \lambda^n$, and

$$\rho(K) = \lim_{n \to \infty} \|K^n\|^{1/n} \geq \lambda = \rho(J).$$

**REMARK 5.9.** This proposition and proof follow [13].

### 5.5 Proof of the main proposition

The last item of our business is to prove Proposition 5.1 about the ground state energies of the kink Hamiltonian. Since $H_n$ is self-adjoint, it follows that

$$\inf \text{spec}(H_{Z,n}) = \inf_{\psi \in \mathcal{H}_n} \frac{(\psi, H_{Z,n}\psi)}{\|\psi\|}.$$

On the other hand, using Lemma 5.5, we can write

$$\inf \text{spec}(H_{Z,n}) = \inf_{\phi \in \ell^2(\mathcal{V}(Z,n))} \frac{(\phi, A_n\phi)}{\|\phi\|}.$$

Moreover, with the natural identification of $\mathcal{V}([-L,L],n) \subset \mathcal{V}(Z,n)$, we have density

$$\ell^2(\mathcal{V}(Z,n)) = \text{cl} \left( \bigcup_{L \in \mathbb{N}} \ell^2(\mathcal{V}([-L,L],n)) \right).$$

Therefore,

$$\inf \text{spec}(H_{Z,n}) = \inf_{L \in \mathbb{N}} \min_{\phi \in \ell^2(\mathcal{V}([-L,L],n))} \frac{(\phi, A_n\phi)}{\|\phi\|}.$$
Now, let $\varepsilon > 0$. Since both $H_{Z,n}$ and $A_n$ are bounded below, there does exist an $L \in \mathbb{N}$ and $\phi \in \ell^2(V([-L, L], n))$ such that $\|R\phi\| = 1$ and $(\phi, A_n\phi) \leq \inf\text{spec}(A_n) + \varepsilon$. For such a vector $\phi \in \ell^2(V([-L, L], n))$ we have a unique vector $\psi \in H^L_{\text{bw}}([-L, L], n)$ with the property that $\|\psi\| = 1$ and $(\phi, A_n\phi) = (\psi, H^L_{[-L, L]}\psi)$.

By shifting the interval $[-L, L]$ to the right by $L + 1$ units, we conclude that

$$E(2L + 1, n) \leq \varepsilon + \inf\text{spec}(H_{Z,n}).$$

Since $\varepsilon$ was arbitrary, and using the monotonicity of $E(L, n)$ in $L$, we have

$$E(n) \leq \inf\text{spec}(H_{Z,n}).$$

For the opposite inequality we use Proposition 5.7. Let $X = V(Z, n).$ Let $L \in \mathbb{N}$, and $Y = V([-L, L], n)$. Then we consider the matrix $B : Y \times Y \to \mathbb{R}$ given by the kernel

$$B(b', b) = \sum_{x=-L}^{L} A_x^L(b', b).$$

The operators $J = n - B$ and $K = n - A_n$ satisfy the conditions of Proposition 5.7. Therefore, we conclude that

$$\rho(n - B) \leq \rho(n - A_n).$$

But using the fact that the spectra of $B$ and $K$ are both real subsets (because the associated operators are similar to self-adjoint operators), we conclude that

$$\rho(n - B) = n - \inf\text{spec}(B)$$

and

$$\rho(n - K) = n - \inf\text{spec}(K).$$

Therefore,

$$E(L, n) \geq \inf\text{spec}(A_n),$$

as desired.

A Results for small $q$

In this appendix, we collect some results for small $q$. The primary purpose of this is to verify the Bethe ansatz formulas for droplet eigenstates of the reduced Hamiltonian $H(n, \theta)$ for other values of $\theta \in S^1$ than $\theta = 0$. Using the methods of Section 4 we can also treat the cyclic Hamiltonian $H^\text{cyc}$ in the $e^{i\theta}$ eigenspaces of the translation operator. In particular, the latter is interesting because this is the framework analyzed by Yang and Yang in [21]. Previously, this regime of the XXZ model (small $q$ and cyclic boundary conditions) was rigorously analyzed by Kennedy in [14], using the methods developed in [7]. (The purpose of [9] was partly to give a pedagogic introduction to the methods of [7], but it also gave new and interesting results for the XXZ model, some of which we describe below.)
Before going further, we would like to mention that in the paper proper, none of the arguments were perturbative. All applied to the entire region \( q \in (0, 1) \), which is the maximal interval where the results are valid. This is important to keep in mind when one considers the relatively simple arguments to follow.

### A.1 Droplet energies in the infinite chain for small \( q \)

Let us fix \( n \in \mathbb{N}_+ \). Before stating the main result of this section, we recall the following. The Hamiltonians \( H_Z(n, \theta) \) are periodic of period \( 2\pi/n \) in the sense that there is a unitary phase multiplication, as in (26), such that after conjugating by that \( H_Z(n, \theta) \) and \( H(n, \theta + 2\pi/n) \) are equal. In particular, this means that the spectrum is \( 2\pi/n \) periodic. Moreover, if there is an eigenvector of \( H_Z(n, \theta) \), then multiplying this eigenvector by the necessary phase produces the relevant eigenvector for \( H(n, \theta + 2\pi/n) \).

**Proposition A.1.** There exists a constant \( q_0 = q_0(n) > 0 \) such that for \( 0 < q < q_0 \), the infspec of \( H_Z(n, \theta) \) is an eigenvalue for all \( \theta \), and for \( \theta \in (-\pi/N, \pi/N) \) the eigenvector is the one given in Lemma 3.4. The eigenvectors are norm continuous in \( \theta \), and are determined for all \( \theta \) using this and periodicity. Moreover, there is a constant \( \gamma(n, q) > 0 \) such that there is a spectral gap above the ground state of \( H_Z(n, \theta) \) of size at least \( \gamma(n, q) \), uniformly in \( \theta \).

**Proof.** Fixing \( n \) and \( \theta \), there is obviously a spectral gap above the bound state for the Ising model, \( q = 0 \). It is easy to see that the gap is 1 at \( q = 0 \). But the kernel \( K_{n,\theta} \), when thought of as a function of \( q \), varies in a way such that the associated operators are norm-continuous with respect to \( q \), on \( \ell^2_0(\mathcal{X}_n) \). (As used before in the paper, this can be proved by obtaining row and column sum bounds, which pertain to \( \ell^1 \) and \( \ell^\infty \), and then using Riesz convexity.) Therefore, there is some \( q_0 \) and some curve \( \gamma(n, q) \), positive for \( q < q_0 \), such that \( H_Z(n, \theta) \) has a unique ground state and a spectral gap of size at least \( \gamma(n, q) \) for all \( \theta \) as long as \( 0 < q < q_0 \). But the bound states found in Lemma 3.4 vary continuously in \( q \), therefore, they must be the actual eigenstates.

**Remark A.2.** The argument of the proof is, to some extent, an analogue of Yang and Yang’s argument from [20] but starting from the Ising model, not the XY model, and valid directly in the infinite volume limit. We would like to mention that more sophisticated and more powerful arguments of the Yang, Yang style were employed by Goldbaum in [4] to handle the more complex – but still Bethe ansatz solvable – Hubbard model.

**Remark A.3.** Note that in the proposition above, one cannot choose \( q_0 \) to be independent of \( n \). The reason is that in our \( \ell^1, \ell^\infty \) interpolation, the rowsums and columnsums do depend on \( n \) because of the number of matrix entries. On the other hand, in [15] two of the authors proved a positive spectral gap for all \( q \) and \( n \) with \( q^n \) small enough, which is uniform in this regime. Therefore, using that result and the present argument, one can obtain a single \( q_0 \) which works for all \( n > 0 \).
Note that not only are the energies for the bound states continuous in $\theta$ and $q$, they are easily seen to be analytic. This is simply because the kernel entries of the operator are analytic in $\theta$ and $q$, and using the properties of the kernel (that there are a finite number of nonzero entries in each row and column) we deduce analyticity of ($q$-dependent) Fourier-reduced Hamiltonian $H_q(n, \theta)$ in the weak-topology. Using the spectral gap this is sufficient to guarantee analyticity of the eigenvectors. Using analyticity in $\theta$, we can obtain the following result.

**COROLLARY A.4.** The spectrum of $H_Z(n)$ in the range $(0, \gamma_l(n, q))$ is absolutely continuous.

We will not give a detailed proof, but the reader is referred to Theorem XIII.86 of [16]. After conjugating by the spectral projection onto $(0, \gamma(n, q))$ the Hamiltonian satisfies the conditions of that theorem.

**REMARK A.5.** One probably expects that the entire spectrum of $H_Z(n)$ is absolutely continuous for all $0 < q \leq 1$. Using the results of [2] this is presumably provable at $q = 1$. But in general the translation-invariance suggests it is true.

### A.2 The Hamiltonian for the cyclic chain for small $q$

Note that for the finite cyclic chain, just as for the infinite chain, there is a well-defined translation operator, commuting with the Hamiltonian. The following result was proved by Kennedy in [7].

**PROPOSITION A.6.** There exists a $q_0$ such that for $0 \leq q \leq q_0$ the $L$ lowest energy levels of $H_{cyc}^{[1, L]}$ in the sector with $n$ downspins ($0 < n < L$) can be indexed by the translation eigenvalues $e^{i\theta}$, for $\theta = 2\pi k/L$ and $k \in \mathbb{Z}/L\mathbb{Z}$. For all $\theta$, there is an analytic expression for the energy eigenvalue $E_{cyc}^{[L,n]}(L, n, \theta)$ satisfying

$$\lim_{L \to \infty} E_{cyc}^{[L,n]}(L, n, \theta) = 1 + \sum_{s=-\infty}^{\infty} d_s e^{i\theta s}.$$  

The coefficients $d_s = d_s(n, q)$ are of order $O(q^n)$.

**REMARK A.7.** The arguments in [15] prove that there is also a gap for large enough $n$ and small enough $q$, and calculates the asymptotic form of the energy in the $n \to \infty$ limit, with $q$ fixed. As is easily seen from our present analysis, in that limit the energy converges to $\alpha = (1 - q^2)/(1 + q^2)$. A simpler argument, but which is not robust to changes in $L$, can follow the proof of Proposition A.1. Namely, one can construct a kernel in each subspace of $n$ downspins and translation eigenvalue $\theta$, and check that as a function of $q$ the kernel is continuous, and moreover it is uniformly continuous for $q \in [0, 1]$. If one considers the sequence of operators for all $L$ (acting on different Hilbert spaces depending on $L$) one can even deduce that they are in some sense equicontinuous, because the stronger fact is true that the operators are Lipschitz with Lipschitz constants which are uniformly bounded in $L \in \mathbb{N}_+$, and $q \in [0, 1]$.  

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REMARK A.8. The important technique of Kennedy, which follows the previous work [7], is to obtain a perturbation expansion which can be performed for all $L$ at once, therefore allowing comparison of different $L$.

COROLLARY A.9. The series expansion of Kennedy for the $L \to \infty$ limit matches the analytic expressions obtainable from Lemma 3.4.

Proof. One wants to show that

$$
\lim_{L \to \infty, \theta_n \to \theta} E^\text{cyc}(L, n, \theta_L) = E_{\text{Z}, n}(\theta).
$$

One knows the existence of a spectral gap in the $\theta_n$ subspaces for small enough $q$ uniform in $L$ for $H^\text{cyc}[[L]]$. In the last subsection, we established a similar spectral gap for $H_{\text{Z}}(n, \theta)$ on the infinite chain. Therefore, we can use exactly the same argument as in Section 4 to establish the same result for all $\theta \in S^1$ that we established for $\theta = 0$, there: namely Proposition 4.7. The reader will find that translation invariance played no special rôle in that argument.

Acknowledgement. B.N. acknowledges support and hospitality from the Erwin Schrödinger Institute for Mathematical Physics, Vienna, and the Centre de Physique Théorique, Luminy, where part of this work was carried out.

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