Operators on Partial Inner Product Spaces: Towards a Spectral Analysis

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Abstract. Given a Lattice of Hilbert spaces $V_J$ and a symmetric operator $A$ in $V_J$, in the sense of partial inner product spaces, we define a generalized resolvent for $A$ and study the corresponding spectral properties. In particular, we examine, with help of the KLMN theorem, the question of generalized eigenvalues associated to points of the continuous (Hilbertian) spectrum. We give some examples, including so-called frame multipliers.

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1. Introduction

In physics, rigged Hilbert spaces (RHS) are standard tools in Quantum Mechanics, in particular for reconciling the convenient bra–ket formalism of Dirac with the mathematically rigorous approach of von Neumann [5, Chap. 7]. For instance, the question of generalized eigenvalues of observables, associated to points of the continuous spectrum, is solved with help of the celebrated Maurin–Gel’fand theorem.

In a recent paper, Bellomonte et al. [12] have attacked this problem by considering observables as operators in $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$, for a suitable RHS $\mathcal{D} \subset \mathcal{H} \subset \mathcal{D}^\times$, where $\mathcal{L}(\mathcal{D}, \mathcal{D}^\times)$ is the space of all continuous linear maps from $\mathcal{D}$ into $\mathcal{D}^\times$. However, the framework they use in a large part of their paper is in fact a partial inner product space (PIP-space), more precisely a Lattice of Hilbert spaces (LHS).

Indeed, the basic ingredient in [12] is that of a family $\mathcal{F}$ of interspaces between $\mathcal{D}$ and $\mathcal{D}^\times$ [5, Sec. 5.4.1]. By interspace, one means a locally convex space $\mathcal{E}[\tau(\mathcal{E}, \mathcal{E}^\times)]$, equipped with the Mackey topology from its conjugate dual, and such that $\mathcal{D} \subset \mathcal{E} \subset \mathcal{D}^\times$, where both embeddings are continuous and have dense range. In addition, one requires that the family $\mathcal{F}$ of interspaces
be a multiplication framework, that is, (i) $D \in \mathcal{G}$; (ii) for every $E \in \mathcal{G}$, the conjugate dual $E^\times$ also belongs to $\mathcal{G}$; and (iii) for every pair $E, F \in \mathcal{G}$, $E \cap F \in \mathcal{G}$. Then, if every interspace $E \in \mathcal{G}$ (except $D$ and $D^\times$) is a Hilbert space, as assumed in most of [12], the resulting structure is a LHS $V_J$ in the sense of [5] and $\mathcal{L}(D, D^\times) \equiv \text{Op}(V_J)$.

In view of this fact, we feel that the analysis becomes simpler if one uses the language of pip-spaces from the beginning. Thus, we will make a few steps towards a spectral theory of symmetric operators in a LHS, following in part [12]. Our framework will be a LHS $V_J$ and we adopt the definitions and notations of our monograph [5]. For the convenience of the reader, we summarize in the Appendix the salient features of pip-spaces and operators on them.

The paper is organized as follows. Section 2 is devoted to the notion of inverse operator in the pip-space context, with application to resolvents, and in particular, their analyticity properties. In Sect. 3, we discuss the various aspects of spectral analysis of Hilbert space operators, including the generalized eigenvalues and eigenvectors, in the light of the well-known KLMN theorem. In particular we revisit the notion of tight rigging. Section 4, finally, is devoted to several examples of spectral analysis of rather singular operators. As for notations, the domain of a Hilbert space operator $A$ is denoted $D(A)$ and its range by $\text{Ran}(A)$.

## 2. Inverses and Resolvents

### 2.1. Invertible Operators

The key ingredient of the spectral theory of operators is the notion of resolvent. For fixing ideas, given a closed operator $A$ in a Hilbert space $\mathcal{H}$, consider $A - \lambda I : D(A) \to \mathcal{H}$. Then the resolvent of $A$ is $R_\lambda(A) := (A - \lambda I)^{-1}$, for those $\lambda \in \mathbb{C}$ for which this inverse exists as an everywhere defined bounded operator in $\mathcal{H}$, that is, $\lambda \in \rho(A) \subset \mathbb{C}$, the resolvent set of $A$. To extend this notion to a pip-space, we have first to define an appropriate concept of inverse of an operator, and this is nontrivial.

Let $V_J$ be a LBS/LHS and $A \in \text{Op}(V_J)$. According to [5, Sec. 3.3.2], we shall say that a representative $A_{pq}$ is invertible if it is bijective, hence it has a continuous inverse $B_{qp} := (A_{pq})^{-1} : V_p \to V_q$. Any successor $A_{p'q'}$, $q' \leq q, p' \geq p$, of an invertible representative $A_{pq}$ is injective and has dense range. An invertible representative has in general no predecessors, that is, a representative $A_{p'q'}$ with $q' \geq q, p' \leq p$. This does not exclude the possibility for an invertible operator $A$ to have a nontrivial null-space. Indeed, $A$ may have a noninjective representative $A_{sr}$, where $r$ is not comparable to $q$, i.e., there may exist a $g \in V_r$ such that $Ag = 0$, provided $g \notin V_q$. Note that, if $A_{pq}$ is invertible, $A_{pq}^\times$ is also invertible and $(A_{pq}^\times)^{-1} = (A_{pq}^{-1})^\times : V_q^\times \to V_p^\times$.

Given an operator $A \in \text{Op}(V_J)$, we recall that $(q, p) \in \mathcal{j}(A)$ means that $A$ has a continuous representative $A_{pq} : V_q \to V_p$.

**Lemma 2.1.** Let $V_J$ be a LBS/LHS and $A \in \text{Op}(V_J)$. Then the following conditions are equivalent: