Blow Up for Some Semilinear Wave Equations in Multi-space Dimensions

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In this paper, we discuss a new nonlinear phenomenon. We find that in \( n \geq 2 \) space dimensions, there exists two indexes \( p \) and \( q \) such that the Cauchy problems for the nonlinear wave equations

\[
\square u(t, x) = |u(t, x)|^p, \quad x \in \mathbb{R}^n, \tag{0.1}
\]
and

\[
\square u(t, x) = |u_t(t, x)|^p, \quad x \in \mathbb{R}^n \tag{0.2}
\]
both have global existence for small initial data, while for the combined nonlinearity, the solutions to the Cauchy problem for the nonlinear wave equation

\[
\square u(t, x) = |u_t(t, x)|^p + |u(t, x)|^q, \quad x \in \mathbb{R}^n, \tag{0.3}
\]
with small initial data will blow up in finite time. In the two dimensional case, we also find that if \( q = 4 \), the Cauchy problem for the equation (0.1) has global existence, and the Cauchy problem for the equation

\[
\square u(t, x) = u(t, x) u_t(t, x)^2, \quad x \in \mathbb{R}^2 \tag{0.4}
\]
has almost global existence, that is, the life span is at least \( \exp(c \varepsilon^{-2}) \) for initial data of size \( \varepsilon \). However, in the combined nonlinearity case, the Cauchy problem for the equation

\[
\square u(t, x) = u(t, x) u_t(t, x)^2 + u(t, x)^4, \quad x \in \mathbb{R}^2 \tag{0.5}
\]
has a life span which is of the order of \( \varepsilon^{-18} \) for the initial data of size \( \varepsilon \), this is considerably shorter in magnitude than that of the first two equations. This solves the final open optimality problem for general theory of fully nonlinear wave equations (see [7]). Furthermore, we consider the finite time blow-up of solutions to another natural generalization problem of (0.5).

Keywords Blow up; Cauchy problem; Fully nonlinear wave equations; Life-Span.

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1. Introduction and Main Results

First we shall outline the general theory on the Cauchy problem for the following $n$-dimensional fully nonlinear wave equations:

\[
\begin{aligned}
\left\{
\begin{array}{l}
    u_{tt} - \Delta u = F(u, Du, D_t Du), \ (\Delta = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}), \ x \in \mathbb{R}^n, \ t > 0, \\
    t = 0 : \ u = \varepsilon f(x), \ u_t = \varepsilon g(x), \ (x = (x_1, \cdots, x_n)),
\end{array}
\right.
\end{aligned}
\]  

(1.1)

where

\[
Du = (u_{x_1}, u_{x_2}, \cdots, u_{x_n}), \ x_0 = t,
\]

\[
D_t Du = (u_{x_1}, u_{x_2}, \cdots, u_{x_n}), \ i, j = 0, 1, \cdots, n, i + j \geq 1,
\]

\[
f(x), g(x) \in C_0^\infty(\mathbb{R}^n) \text{ and } \varepsilon > 0 \text{ is a small parameter. Here, for simplicity of notations we write } x_0 = t.
\]

Let

\[
\hat{\lambda} = (\lambda; (\lambda_i), i = 0, 1, \cdots, n; (\lambda_{ij}), i, j = 0, 1, \cdots, n, i + j \geq 1).
\]

Suppose that in a neighborhood of $\hat{\lambda} = 0$, say, for $|\hat{\lambda}| \leq 1$, the nonlinear term $F = F(\hat{\lambda})$ in equation (1.1) is a sufficiently smooth function with

\[
F(\hat{\lambda}) = O(|\hat{\lambda}|^{1+\varepsilon}),
\]

where $\varepsilon$ is an integer and $\varepsilon \geq 1$.

We define a lifespan $T(\varepsilon)$ of solutions to problem (1.1) to be the supremum of all $\tau > 0$ such that there exists a classical solution to (1.1) for $x \in \mathbb{R}^n$ on $0 \leq t < \tau$. When $T(\varepsilon) = +\infty$, we mean that the problem (1.1) has global existence.

In chapter 2 of Li and Chen [8], we have long histories on the estimate for $T(\varepsilon)$. The lower bounds of $T(\varepsilon)$ are summarized in Table 1. Let $a = a(\varepsilon)$ satisfy

\[
a^2 \varepsilon^2 \log(a + 1) = 1
\]

and $c$ stand for a positive constant independent of $\varepsilon$. We have (see also a table in Li [9])

We note that all these lower bounds are known to be sharp except for the case $(n, \varepsilon) = (2, 2)$ and $\partial^l_u F(0) = 0$. The aim of this paper is to show that in this case the lower bound obtained by [7] is indeed sharp. Therefore, we solve the final open optimality problem for general theory of fully nonlinear wave equations. We remark that the sharpness for $(n, \varepsilon) = (4, 1)$ was only recently proved by Takamura and Wakasa [11], see also Zhou and Han [18]. For the case $(n, \varepsilon) = (2, 2)$ and $\partial^l_u F(0) = 0(l = 3, 4)$, the sharpness is due to Zhou and Han [17], in this case, it was believed that $l = 4$ is a technical condition which may be removed, however, we show in this paper that it is not the case, we show that if we drop this condition, the lifespan will be much shorter. Therefore our result shows that the condition $l = 4$ is necessary.

In this paper, we firstly consider the following Cauchy problem with small initial data in two space dimensions

\[
\begin{aligned}
\left\{
\begin{array}{l}
    \Box u(t, x) = u(t, x)u_t(t, x)^2 + u^4(t, x), \ x \in \mathbb{R}^2, \ t > 0, \\
    t = 0 : \ u = 0, \ u_t = \varepsilon g(x), \ x \in \mathbb{R}^2,
\end{array}
\right.
\end{aligned}
\]  

(1.2)
Table 1

General Theory for the sharp lower bound of the lifespan for fully nonlinear wave equations

| $T(\varepsilon) \geq$ | $x = 1$ | $x = 2$ | $x \geq 3$ |
|-----------------------|---------|---------|----------|
| $ca(\varepsilon)$, in general case, | $ce^{-6}$, in general case, | | |
| $n = 2$ | $ce^{-1}$, if $\int g(x)dx = 0$, | $ce^{-18}$, if $\partial_t^l F(0) = 0$ | $\infty$ |
| | $ce^{-2}$, if $\partial_u^2 F(0) = 0$, | $exp(ce^{-2})$, if $\partial_u^3 F(0) = 0 (l = 3, 4)$ |
| $n = 3$ | $ce^{-2}$, in general case, | $\infty$ | $\infty$ |
| | $exp(ce^{-1})$, if $\partial_u^2 F(0) = 0$ | | |
| $n = 4$ | $exp(ce^{-2})$, in general case, | $\infty$ | $\infty$ |
| | $\infty$, if $\partial_u^3 F(0) = 0$ | | |
| $n \geq 5$ | $\infty$ | $\infty$ | $\infty$ |

where $\Box = \partial_t^2 - \sum_{i=1}^{n} \partial_{x_i}^2$ is the wave operator, and $g \in C_0^\infty (R^n)$, $\varepsilon > 0$ is a small parameter.

For problem (1.2), what interesting about this problem is that the Cauchy problem for the equation

$$\Box u(t, x) = u^4(t, x), \quad x \in R^2, \quad t > 0$$

has global existence (see [3]), and the Cauchy problem for the equation

$$\Box u(t, x) = u(t, x)u_t^2(t, x), \quad x \in R^2, \quad t > 0$$

has almost global existence (see [8]). However, in the combined nonlinearity case, the Cauchy problem for the equation

$$\Box u = uu_t^2 + u^4, \quad x \in R^2, \quad t > 0$$

has a life span which is of the order $\varepsilon^{-18}$, this is considerably shorter in magnitude than that of

$$\Box u = u(t, x)u_t(t, x)^2$$

and

$$\Box u = u^4(t, x).$$

For problem (1.2), we consider compactly supported, radial, nonnegative data $g \in C_0^\infty (R^2)$, and satisfy

$$g(x) = g(|x|) > 0, \quad \text{for } |x| < 1 \quad g(x) = 0, \quad \text{for } |x| > 1. \quad (1.3)$$

We establish the following theorem for (1.2):
**Theorem 1.1.** Let $g$ be a smooth function with compact support $g \in C_0^\infty(R^2)$ and satisfies (1.3), space dimensions $n = 2$. Suppose that problem (1.2) has a solution $(u, u_t) \in C([0, T), H^1(R^2) \times L^3(R^2))$ such that

$$\text{supp}(u, u_t) \subset \{(x, t) : |x| \leq t + 1\}.$$ 

Then the solution $u = u(t, x)$ will blow up in finite time, that is $T < \infty$. Moreover, we have the following estimates for the lifespan $T(\varepsilon)$ of solutions of (1.10) there exists a positive constant $A$ which is independent of $\varepsilon$ such that

$$T(\varepsilon) \leq A\varepsilon^{-18}. \quad (1.4)$$

**Remark 1.1.** Theorem 1.1 shows the sharpness of the lower bound of the lifespan obtained by Katayama [7], and it is the final optimality of the general theory for the initial value problem of nonlinear wave equations.

Secondly, we consider the following natural generalization Cauchy problem of (1.2) with small initial data in two space dimensions

$$\begin{cases} 
\Box u(t, x) = u(t, x)u_t(t, x)|^{p-1} + u(t, x)u(t, x)|^{q-1}, & x \in R^2, \ t > 0, \\
 t = 0: \ u = 0, \ u_t = \varepsilon g(x), \ x \in R^2, 
\end{cases} \quad (1.5)$$

where $\Box = \partial_t^2 - \sum_{i=1}^n \partial_{x_i}^2$ is the wave operator, and $g \in C_0^\infty(R^2)$, $\varepsilon > 0$ is a small parameter, $1 < p \leq 4$, $q > 1$.

For problem (1.5), we consider compactly supported, radial, nonnegative data $g \in C_0^\infty(R^2)$, and satisfy the condition (1.3).

We establish the following Theorem 1.2 and Corollary 1.3 for (1.5):

**Theorem 1.2.** Let $g$ be a smooth function with compact support $g \in C_0^\infty(R^2)$ and satisfies (1.3), space dimensions $n = 2$. Suppose that problem (1.5) has a solution $(u, u_t) \in C([0, T), H^1(R^2) \times L^3(R^2))$ such that

$$\text{supp}(u, u_t) \subset \{(x, t) : |x| \leq t + 1\},$$

and the index $p, q$ satisfies the following conditions:

$$1 < p \leq 4, \ q > 1, \ (q - 1)(6 - p) > 4(q - 2). \quad (1.6)$$

Then the solution $u = u(t, x)$ will blow up in finite time, that is $T < \infty$. Moreover, we have the following estimates for the lifespan $T(\varepsilon)$ of solutions of (1.5): there exists a positive constant $A$ which is independent of $\varepsilon$ such that

$$T(\varepsilon) \leq A\varepsilon^{\frac{2(p-1)}{2q-1+6-p-4q+2}}. \quad (1.7)$$

As a direct consequence of the Theorem 1.2 above, we have the following
Corollary 1.3. Let $g$ be a smooth function with compact support $g \in C^\infty_0(R^2)$ and satisfies (1.3), space dimensions $n = 2$. Suppose that problem (1.5) has a solution $(u, u_t) \in C([0, T), \ H^1(R^2) \times L^3(R^2))$ such that

$$\text{supp}(u, u_t) \subset \{(x, t) : |x| \leq t + 1\},$$

and the index $p, q$ satisfies the following conditions:

$$p = 3, \quad 1 < q < 5. \quad (1.8)$$

Then the solution $u = u(t, x)$ will blow up in finite time, that is $T < \infty$. Moreover, we have the following estimates for the lifespan $T(\varepsilon)$ of solutions of (1.5): there exists a positive constant $A$ which is independent of $\varepsilon$ such that

$$T(\varepsilon) \leq A \varepsilon^{-\frac{6(p-1)}{q} - \frac{2}{q}}. \quad (1.9)$$

Thirdly, we will consider the following Cauchy problem with small initial data in $n(n \geq 2)$ space dimensions

$$\begin{align*}
\Box u(t, x) &= |u_t(t, x)|^p + |u(t, x)|^q, \quad x \in R^n, \quad t > 0, \\
         t = 0: \quad u = \varepsilon f(x), \quad u_t = \varepsilon g(x), \quad x \in R^n,
\end{align*} \quad (1.10)$$

where $f, g \in C^\infty_0(R^n)$, $\varepsilon > 0$ is a small parameter, $p > 1, q > 1$.

For problem (1.10), we consider compactly supported nonnegative data $f, g \in C^\infty_0(R^n), n \geq 2$ and satisfy

$$f(x) \geq 0, \quad g(x) \geq 0, \quad f(x) = g(x) = 0, \quad \text{for } |x| > 1 \quad \text{and } g(x) \neq 0. \quad (1.11)$$

We establish the following theorem for (1.10):

Theorem 1.4. Let $f, g$ be smooth functions with compact support $f, g \in C^\infty_0(R^n)$ and satisfy (1.11), space dimensions $n \geq 2$. Suppose that problem (1.10) has a solution $(u, u_t) \in C([0, T), \ H^1(R^n) \times L'^1(R^n))$, where $r = \max(2, p)$ such that

$$\text{supp}(u, u_t) \subset \{(x, t) : |x| \leq t + 1\},$$

and the index $p, q$ satisfies the following conditions:

$$\max\left(1, \frac{2}{n-1}\right) < p \leq \frac{2n}{n-1}; \quad (1.12)$$

$$1 < q < \min\left(\frac{4}{(n-1)p-2} + 1, \frac{2n}{n-2}\right). \quad (1.13)$$

Then the solution $u = u(t, x)$ will blow up in finite time, that is $T < \infty$. Moreover, we have the following estimates for the lifespan $T(\varepsilon)$ of solutions of (1.10) there exists a positive constant $A$ which is independent of $\varepsilon$ such that

$$T(\varepsilon) \leq A \varepsilon^{-\frac{2(p-1)}{2q+2-(n-1)(q-1)}}. \quad (1.14)$$
Remark 1.2. In the Theorem 1.4, we restrict $q < 2^* = \frac{2n}{n-2}$ in the condition (1.13), just to make that $H^1(R^n) \hookrightarrow L^q(R^n)$, so the nonlinearity $|u|^q$ can be integrable in $R^n$.

Remark 1.3. If we take $n = 2$, $p = 3$ and $q = 4$ in Theorem 1.4, we can also obtain the upper bound of the life span $Ae^{-\varepsilon}$ in the two space dimensions.

Remark 1.4. In problem (1.10), let $p_0 = p_0(n) = \frac{2}{n-1} + 1$, and $q_0 = q_0(n)$ is the positive root of the quadratic equation $\gamma(n, q) = (n - 1)q^2 - (n + 1)q - 2 = 0$, that is

$$q_0 = q_0(n) = \frac{n + 1 + \sqrt{n^2 + 10n - 7}}{2(n - 1)}.$$

Then we know that $p_0$ and $q_0$ are the critical index of the following semilinear wave equations respectively:

\[ \Box u(t, x) = |u_t(t, x)|^p, \quad (1.15) \]

and

\[ \Box u(t, x) = |u(t, x)|^q. \quad (1.16) \]

If $p > p_0$, then the solution of the initial problem for the above equation (1.15) will exists globally, see [5, 15]; and also if $q > q_0$, then the solution of the initial problem for the above equation (1.16) will exists globally, this problem has a long history, one can see [1–4, 10, 14]. But it can be showed that there exist $p > p_0$ and $q > q_0$ such that the conditions (1.12) and (1.13) can be still satisfied simultaneously, for that purpose, we take for example $p > p_0$ and sufficiently close to $p_0$, then (1.12) will be satisfied. Take $q$ satisfying (1.13) and sufficiently close to $\frac{4}{n-1} + 1$ then $q$ will be sufficiently close to $\frac{4}{n-1} + 1$ which is larger than $q_0$, this can be seen from the fact that

$$\gamma \left( n, \frac{4}{n-1} + 1 \right) = \frac{8}{n-1} > 0. \quad (1.17)$$

Consequently, from Theorem 1.4, the solutions of the Cauchy problem (1.10) will blow up in finite time, while the Cauchy problems for (1.15) and (1.16) have global existence.

The rest of the paper is arranged as follows. We state a preliminary Lemma in Section 2. In Section 3, we prove the sharpness of the lower bound obtained by [7], i.e., Theorem 1.1 and Theorem 1.2 for a natural generalization Cauchy problem. Section 4 is devoted to the proof for our Theorem 1.4, i.e., blow up and the upper bound estimate of lifespan of solutions to some semilinear wave equations with small initial data in $n(n \geq 2)$ space dimensions.

2. Preliminaries

To prove the main results in this paper, we will employ the following important ODE result:

Lemma 2.1. (see [10], also see [16]) Let $\beta > 1$, $a \geq 1$, and $(\beta - 1)a > \alpha - 2$. If $F \in C^2([0, T])$ satisfies

1. $F(t) \geq \delta(t + 1)^a$,
(2) \( \frac{d^2 F(t)}{dt^2} \geq k(t + 1)^{-3}[F(t)]^\delta, \)

with some positive constants \( \delta, k, \) then \( F(t) \) will blow up in finite time, \( T < \infty. \)

Furthermore, we have the following estimate for the life span \( T(\delta) \) of \( F(t) : \)

\[
T(\delta) \leq c\delta^{-\frac{(\delta-1)}{(1+n-2\delta)}} , \tag{2.1}
\]

where \( c \) is a positive constant depending on \( \delta \) but independent of \( \delta. \)

**Proof.** For the proof of blow up result part see Sideris [10]. For the estimate of the life span of \( F(t), \) one can see Lemma 2.1 in [16]. \( \square \)

To outline the method, following Yordanov and Zhang [12], we will introduce the following functions: Let

\[
\phi_1(x) = \int_{S^{n-1}} e^{x\omega} d\omega \geq 0.
\]

Obviously, \( \phi_1(x) \) satisfy: \( \Delta \phi_1 = \phi_1. \)

When space dimensions \( n \geq 2, \) by rotation invariance, we have

\[
\phi_1(x) = \int_{S^{n-1}} e^{x|\omega|} d\omega = \int_{S^{n-1}} e^{x|\omega|} d\omega_1 d\omega_2 = c \int_0^1 e^{x|\omega_1|} (\sqrt{1 - \omega_1^2})^{n-3} d\omega_1 + c \int_0^1 e^{-x|\omega_1|} (\sqrt{1 - \omega_1^2})^{n-3} d\omega_1 \leq c e^{|x|} \int_0^1 e^{x|\omega_1|} (1 - \omega_1^2)^{n-3} d\omega_1 + C \leq c e^{|x|} \int_0^1 e^{-x|\omega_1|} (1 - \omega_1^{n-1}) d\omega_1 + C (\text{we take } \lambda = |x|(1 - \omega_1)) = c e^{|x|} |x|^{-\frac{n-1}{2}} \int_0^{|x|} e^{-\lambda \frac{x-n}{2}} d\lambda + C \leq c e^{|x|} |x|^{-\frac{n-1}{2}} \int_0^\infty e^{-\lambda \frac{x-n}{2}} d\lambda + C = \tilde{C} e^{|x|} |x|^{-\frac{n-1}{2}}.
\]

Moreover, obviously we have

\[
|\phi_1(x)| \leq e^{|x|} \int_{S^{n-1}} d\omega = Ce^{|x|}.
\]

Thus we can conclude that

\[
|\phi_1(x)| \leq C e^{|x|} (1 + |x|)^{-\frac{n-1}{2}} \quad (n \geq 2). \tag{2.2}
\]

By the positivity of \( \phi_1(x), \) so we get that when \( n \geq 2, \)

\[
0 \leq \phi_1(x) \leq C e^{|x|} (1 + |x|)^{-\frac{n-1}{2}} , \quad (n \geq 2).
\]

In order to describe the following methods, we define the following test function

\[
\psi_1(x, t) = \phi_1(x)e^{-t}, \quad \forall x \in R^n, \quad t \geq 0. \tag{2.3}
\]
Then $\Delta \psi_1 = \psi_1$, and $\Box \psi_1 = 0$.

One can see [12], also see [13].

3. The Proof of Theorem 1.1 and Theorem 1.2

In order to prove Theorem 1.1, we need to consider the following Cauchy problem

$$\begin{cases}
\Box u(t, x) = u(t, x)u_t(t, x)^2 + u^4(t, x), & x \in \mathbb{R}^2, \quad t > 0, \\
t = 0 : u = 0, \quad u_t = \varepsilon g(|x|), & x \in \mathbb{R}^2,
\end{cases}$$

where

$$g(x) = g(|x|) > 0, \quad \text{for} \ |x| < 1 \quad \text{and} \ g(x) = 0, \quad \text{for} \ |x| > 1.$$ 

By the Picard iteration technique, the positivity of the fundamental solution of the wave operator in two space dimensions and Mathematical induction, we can prove that $u$ is nonnegative, the proof is similar as the corresponding part of the Proof of Theorem 1.2 (see below for detail). And then Theorem 1.1 is a direct consequence of Theorem 1.2 with $p = 3, q = 4$. So we only need to prove Theorem 1.2.

**Proof of Theorem 1.2.** To prove Theorem 1.2, we will consider the following Cauchy problem

$$\begin{cases}
\Box u(t, x) = u(t, x)|u_t(t, x)|^{p-1} + u(t, x)|u(t, x)|^{q-1}, & x \in \mathbb{R}^2, \quad t > 0, \\
t = 0 : u = 0, \quad u_t = \varepsilon g(|x|), & x \in \mathbb{R}^2,
\end{cases}$$

where

$$g(x) = g(|x|) > 0, \quad \text{for} \ |x| < 1 \quad \text{and} \ g(x) = 0, \quad \text{for} \ |x| > 1.$$ 

We first prove that $u$ is nonnegative.

By the local existence of classical solutions, the solution to Cauchy problem (3.1) can be approximated by Picard iteration. Let

$$u^{(0)} \equiv 0$$

and

$$\begin{cases}
\Box u^{(m)}(t, x) = u^{(m-1)}(t, x)|u_t^{(m-1)}(t, x)|^{p-1} + u^{(m-1)}(t, x)|u^{(m-1)}(t, x)|^{q-1}, & x \in \mathbb{R}^2, \quad t > 0, \\
t = 0 : u^{(m)} = 0, \quad u_t^{(m)} = \varepsilon g(|x|), & x \in \mathbb{R}^2,
\end{cases}$$

where $u^{(m)}(t, x)$ is a series of approximate solutions to (3.1).

Since $u^{(0)} \equiv 0$, we have $u_t^{(0)} \equiv 0$, and $g(x)$ is non-negative. So by the positivity of the fundamental solution of the wave operator in two space dimensions, we can prove that all $u^{(m)}$ are nonnegative by induction. Let $m \to \infty$, we can conclude that $u$ is nonnegative.

Let

$$r = |x|, \quad x \in \mathbb{R}^2.$$
The radial symmetric form of problem (3.1) can be written as

\[
\begin{cases}
  u_t - u_{rr} - \frac{u}{r} = u|u|^p - 1 + u|u|^{q-1}, & t > 0, \\
  t = 0 : \quad u = 0, \quad u_t = \varepsilon g(r).
\end{cases}
\]  

(3.4)

It follows from (3.4) that

\[
\begin{cases}
  \left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r^2}\right) \left(\frac{1}{r} u\right) = \frac{1}{4} r^{-\frac{3}{2}} u + r^\frac{3}{2} u|u|^{p-1} + r^\frac{3}{2} u|u|^{q-1}, & t > 0, \\
  t = 0 : \quad r^\frac{3}{2} u = 0, \quad r^\frac{3}{2} u_t = \varepsilon r^\frac{3}{2} g(r).
\end{cases}
\]  

(3.5)

Let \( G(r) = \frac{1}{2} r^\frac{3}{2} g(r) \), by D’Alembert’s formula (see [4]), in the domain \( r > t \), we have

\[
r^\frac{3}{2} u(t, r) = \varepsilon G(t + r) + \varepsilon G(r - t) + \frac{1}{8} \int_0^t \int_{r-(t-\tau)}^{t+\tau} \frac{u(\tau, \lambda)}{\lambda^\frac{3}{2}} d\lambda d\tau \\
+ \frac{1}{2} \int_0^t \left[ \frac{\lambda^\frac{3}{2}}{\lambda^\frac{3}{2}} (u|u|^{p-1} + u|u|^{q-1})(\tau, \lambda) \right]_{\lambda=r-(t-\tau)}^{\lambda=r+(t-\tau)} d\tau.
\]  

(3.6)

Differentiate with respect to \( t \) yields:

\[
r^\frac{3}{2} u_t(t, r) = \varepsilon G(t + r) + \varepsilon G(r - t) + \frac{1}{8} \int_0^t \left[ \left. \frac{u(\tau, \lambda)}{\lambda^\frac{3}{2}} \right|_{\lambda=r-(t-\tau)} \left. \frac{u(\tau, \lambda)}{\lambda^\frac{3}{2}} \right|_{\lambda=r-(t-\tau)} \right] d\tau \\
+ \frac{1}{2} \int_0^t \left( \left. \frac{\lambda^\frac{3}{2}}{\lambda^\frac{3}{2}} (u|u|^{p-1} + u|u|^{q-1})(\tau, \lambda) \right|_{\lambda=r-(t-\tau)} \left. \frac{\lambda^\frac{3}{2}}{\lambda^\frac{3}{2}} (u|u|^{p-1} + u|u|^{q-1})(\tau, \lambda) \right|_{\lambda=r-(t-\tau)} \right) d\tau.
\]  

(3.7)

Therefore, we obtain, in the domain \( r > t \),

\[
r^\frac{3}{2} u(t, r) \geq \varepsilon \int_{t-r}^{t+r} G(\lambda) d\lambda,
\]  

(3.8)

\[
r^\frac{3}{2} u_t(t, r) \geq \varepsilon G(r - t).
\]  

(3.9)

It then easily follows that in the domain \( t \geq \frac{1}{2} \) and \( \frac{1}{4} \leq r - t \leq \frac{3}{4} \), we have

\[
u(t, r), u_t(t, r) \geq c_0 \varepsilon r^{-\frac{1}{2}}.
\]  

(3.10)

Let

\[
F(t) = \int_{R^2} u(t, x) dx,
\]  

(3.11)

then by integrating the equation in (3.1) with respect to \( x \), we obtain

\[
F''(t) = \int_{R^2} (u|u|^{p-1})(t, x) dx + \int_{R^2} (u|u|^{q-1})(t, x) dx.
\]

Thus, by the positivity of the solution \( u \), it follows

\[
F''(t) \geq \int_{R^2} (u|u|^{p-1})(t, x) dx,
\]  

(3.12)

and

\[
F''(t) \geq \int_{R^2} (u|u|^{q-1})(t, x) dx = \int_{R^2} u^q(t, x) dx.
\]  

(3.13)
Noting (3.10), we obtain
\[
\int_{\mathbb{R}^2} (u|u|^{p-1})(t, x) dx = c \int_{0}^{t+1} (u|u|^{p-1})(t, r) dr \\
\geq c \int_{t+\frac{1}{4}}^{t+\frac{3}{4}} (u|u|^{p-1})(t, r) dr \\
\geq ce^p \int_{t+\frac{1}{4}}^{t+\frac{3}{4}} r^{-\frac{q}{2}} \cdot r dr = ce^p \int_{t+\frac{1}{4}}^{t+\frac{3}{4}} r^{-\frac{q+2}{2}} dr \\
\geq ce^p (1 + t)^{-\frac{q+2}{2}}.
\]

It then follows from (3.12) that
\[
F(t) \geq ce^p (1 + t)^{\frac{6-p}{2}}, \quad t \geq 1. \tag{3.14}
\]

On the other hand, by the positivity of \( u(u|u|^{q-1} = |u|^q = u^q) \), it follows from H"older’s inequality that
\[
F(t) = \int_{|x| \leq t+1} u(t, x) dx \leq \left( \int_{\mathbb{R}^2} u^q(t, x) dx \right)^{\frac{1}{q}} \left( \int_{|x| \leq t+1} dx \right)^{\frac{q-1}{q}} \\
\leq C_1 (1 + t)^{\frac{2(q-1)}{q}} \left( \int_{\mathbb{R}^2} u^q(t, x) dx \right)^{\frac{1}{q}},
\]

where \( C_1 = \pi^{\frac{q-1}{2}} \).

Noting (3.13), we obtain
\[
F''(t) \geq C_2 \frac{F^q(t)}{(1 + t)^{2(q-1)}}, \tag{3.15}
\]

where \( C_2 = 1/C_1^q = 1/\pi^{q-1} > 0 \).

Noting (3.14) and (3.15), we may apply Lemma 2.1, we take \( \delta = e^p, \beta = q, a = \frac{6-p}{2}, x = 2(q-1) \), from the conditions (1.6) in Theorem 1.2, we have \( \beta > 1, a \geq 1 \), and \( (\beta - 1)a > x - 2 \) all can be deduced from (1.6). So by the Lemma 2.1, \( F(t) \) will blow up in finite time and thus the solutions to problem (1.5) will blow up in finite time, and also we have the following desired estimate of the life span of solutions to problem (1.5):
\[
T(\varepsilon) \leq A\delta^{-\frac{(q-1)}{(\beta-1)a-\frac{a}{2}}} \frac{(q-1)}{(q-1)\times \frac{a}{2}-2(q-1)+2} \\
= A\varepsilon^{-\frac{2(q-1)}{(q-1)(6-p)(6-p-2q+2)}} \\
= A\varepsilon^{-\frac{2(q-1)}{(q-1)(6-p)(6-p-2q+2)}}, \tag{3.16}
\]

where \( A \) is a positive constant which is independent of \( \varepsilon \). The proof of Theorem 1.2 is complete. \( \square \)

4. The Proof of Theorem 1.4

Theorem 1.4 is a consequence of the blowup result and the upper bound estimate about nonlinear differential inequalities in Lemma 2.1.

We multiply the equation in (1.10) by the test function \( \psi_1(x, t) \in C^2(\mathbb{R}^n \times \mathbb{R}) \) and integrate over \( \mathbb{R}^n \), then we use integration by parts.
First,

\[
\int_{\mathbb{R}^n} \psi_1 (u_t - \Delta u) dx = \int_{\mathbb{R}^n} \psi_1 |u_t|^p dx + \int_{\mathbb{R}^n} \psi_1 |u|^q dx.
\]

By the expression \(\psi_1(x,t) = \phi_1(x) e^{-t}\), we have

\[
\int_{\mathbb{R}^n} \psi_1 \Delta u dx = \int_{\mathbb{R}^n} \Delta \psi_1 u dx = \int_{\mathbb{R}^n} \psi_1 u dx.
\]

So we have

\[
\int_{\mathbb{R}^n} (\psi_1 u_t - \psi_1 u) dx = \int_{\mathbb{R}^n} \psi_1 (|u_t|^p + |u|^q) dx.
\] (4.1)

Notice that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \psi_1 u_t dx = \int_{\mathbb{R}^n} (\psi_1 \cdot u_t - u_t \psi_1) dx,
\] (4.2)

\[
\frac{d}{dt} \int_{\mathbb{R}^n} (\psi_1 u) dx = \int_{\mathbb{R}^n} [(\psi_1) \cdot (u + u_t \cdot \psi_1)] dx = \int_{\mathbb{R}^n} [\psi_1 \cdot u_t - u \psi_1] dx.
\] (4.3)

Adding up the above two expressions, we obtain the following

\[
\frac{d}{dt} \int_{\mathbb{R}^n} (\psi_1 u_t + \psi_1 u) dx = \int_{\mathbb{R}^n} (\psi_1 \cdot u_t - u \cdot \psi_1) dx = \int_{\mathbb{R}^n} (\psi_1 \cdot |u_t|^p + \psi_1 \cdot |u|^q) dx.
\] (4.4)

So we have

\[
\int_{\mathbb{R}^n} (\psi_1 u_t + \psi_1 u) dx \bigg|_{t=0} + \int_0^t \int_{\mathbb{R}^n} \psi_1 \cdot (|u_t|^p + |u|^q) dx d\tau
\]

\[
= \varepsilon \int_{\mathbb{R}^n} \phi_1(x) [f(x) + g(x)] dx + \int_0^t \int_{\mathbb{R}^n} \psi_1 \cdot (|u_t|^p + |u|^q) dx d\tau.
\] (4.5)

Adding the expressions (4.1) and (4.5), we have

\[
\int_{\mathbb{R}^n} (\psi_1 u_t + \psi_1 u) dx
\]

\[
= \varepsilon \int_{\mathbb{R}^n} \phi_1(x) [f(x) + g(x)] dx + \int_{\mathbb{R}^n} \psi_1 (|u_t|^p + |u|^q) dx
\]

\[
+ \int_0^t \int_{\mathbb{R}^n} \psi_1 \cdot (|u_t|^p + |u|^q) dx d\tau
\]

\[
\geq \varepsilon \int_{\mathbb{R}^n} \phi_1(x) [f(x) + g(x)] dx = C_0 \varepsilon.
\] (4.6)

Also, we know that

\[
\frac{d}{dt} \int_{\mathbb{R}^n} \psi_1 u_t dx + 2 \int_{\mathbb{R}^n} \psi_1 \cdot u_t dx = \int_{\mathbb{R}^n} [\psi_1 u_t + u_t (\psi_1)_t + 2\psi_1 u_t] dx
\]

\[
= \int_{\mathbb{R}^n} (\psi_1 u_t + \psi_1 u_t) dx
\]

\[
\geq C_0 \varepsilon.
\] (4.7)
Multiplying the above differential inequality by $e^{2t}$, we get the following expression
\[
\frac{d}{dt} \left( e^{2t} \int_{\mathbb{R}^n} \psi_1 u_t \, dx \right) \geq C_0 e^{2t}.
\] (4.8)

So we have
\[
e^{2t} \int_{\mathbb{R}^n} \psi_1 u_t \, dx \geq C_0 (e^{2t} - 1) e + e \int_{\mathbb{R}^n} \phi_1 g \, dx.
\] (4.9)

Therefore, noting the positivity of $\phi_1$ and $g$, we have
\[
\int_{\mathbb{R}^n} \psi_1 u_t \, dx \geq \tilde{C}_0 e,
\] (4.10)

where $\tilde{C}_0$ is a positive constant.

Let $F(t) = \int_{\mathbb{R}^n} u \, dx$, we integrate the equation

\[
\Box u = |u_t|^p + |u|^q,
\]

we have
\[
\frac{d}{dt} \left( \int_{\mathbb{R}^n} u \, dx \right) \geq \int_{\mathbb{R}^n} |u_t|^p \, dx + \int_{\mathbb{R}^n} |u|^q \, dx.
\] (4.11)

That is
\[
F''(t) \geq \int_{\mathbb{R}^n} |u_t|^p \, dx + \int_{\mathbb{R}^n} |u|^q \, dx.
\] (4.12)

By Holder’s inequality, we can obtain
\[
\int_{\mathbb{R}^n} u_t \psi_1 \, dx \leq \left( \int_{|x| \leq t+1} |u_t|^p \, dx \right)^{\frac{1}{p}} \left( \int_{|x| \leq t+1} |\psi_1|^{p'} \, dx \right)^{\frac{1}{p'}} ,
\] (4.13)

where $p$ and $p'$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

Noting (2.2), it turns out that
\[
\left( \int_{|x| \leq t+1} |\psi_1|^{p'} \, dx \right)^{\frac{1}{p'}} = e^{-t} \left( \int_{|x| \leq t+1} |\phi_1(x)|^{p'} \, dx \right)^{\frac{1}{p'}} \leq C e^{-t} \left( \int_0^{t+1} e^{p' r} (1 + r)^{-\frac{(n-1)p'}{2}} r^{n-1} \, dr \right)^{\frac{1}{p'}} \leq C e^{-t} \left( \int_0^{t+1} e^{p' r} \, dr \right)^{\frac{1}{p'}} \leq C e^{-t} \left( t + 1 \right)^{\frac{n-1}{p'} - \frac{n-1}{2} \left[ \frac{1}{p'} \left( \psi'_t (t+1) \right) - \frac{1}{p'} \right]^{\frac{1}{p'}}}
\] (4.14)

\[
\leq C e^{-t} \left( t + 1 \right)^{\frac{n-1}{p'} - \frac{n-1}{2} \left( \frac{1}{p'} \right)^{\frac{1}{p'}} e^{t+1}} \leq C (1 + t)^{\frac{n-1}{p'} - \frac{n-1}{2}}.
\]
Noting (4.10) and (4.14), it follows from (4.13) that

\[
\int_{\mathbb{R}^n} |u_t|^p dx \geq C \frac{\tilde{C}_0^p}{(1 + t)^{(n-1)p - \frac{n}{2}}}. \tag{4.15}
\]

Since \( p' = \frac{p}{p-1} \), so we have

\[
\left[ \frac{n-1}{p'} - \frac{n-1}{2} \right] p = \frac{(n-1)p}{p'} - \frac{(n-1)p}{2} = \frac{(n-1)(p-2)}{2},
\]

thus the expression (4.15) leads to the following

\[
\int_{\mathbb{R}^n} |u_t|^p dx \geq C \frac{\tilde{C}_0^p}{(1 + t)^{(n-1)p - \frac{n}{2} - 1}}. \tag{4.16}
\]

By Holder’s inequality, we obtain

\[
F(t) = \int_{\mathbb{R}^n} u dx \leq \left( \int_{\mathbb{R}^n} |u|^q dx \right)^{\frac{1}{q}} \left( \int_{|x| \leq t+1} dx \right)^{\frac{1}{q'}} \leq C \left( \int_{\mathbb{R}^n} |u|^q dx \right)^{\frac{1}{q}} (1 + t)^{\frac{n}{q}}, \tag{4.17}
\]

where \( q \) and \( q' \) are conjugate numbers, they satisfy \( \frac{1}{q} + \frac{1}{q'} = 1 \). Therefore, we have

\[
\int_{\mathbb{R}^n} |u|^q dx \geq C \frac{F(t)^q}{(1 + t)^{\frac{n}{q}}}. \tag{4.18}
\]

Noticing that \( q' = \frac{q}{q-1} \), so \( \frac{nq}{q} = n(q-1) \). So the expression (4.18) leads to the following

\[
\int_{\mathbb{R}^n} |u|^q dx \geq C \frac{F(t)^q}{(1 + t)^{n(q-1)}}. \tag{4.19}
\]

Hence, \( F(t) \) satisfies the following inequality

\[
F''(t) \geq C \frac{F(t)^q}{(1 + t)^{n(q-1)}}, \tag{4.20}
\]

On the other hand, by (4.16), we get:

\[
F''(t) \geq C \frac{\epsilon^p}{(1 + t)^{\frac{n-1(p-2)}{2}}}. \tag{4.21}
\]

So integrating the above expression twice, we have the following

\[
F(t) \geq \epsilon^p (1 + t)^{-\frac{(n-1)(p-2)}{2}}. \tag{4.22}
\]

We take \( a = 2 - \frac{(n-1)(p-2)}{2} \), \( \alpha = n(q-1) \), \( \beta = q > 1 \) in Lemma 2.1, from the conditions (1.12) and (1.13) in Theorem 1.4, we have \( \beta > 1, a \geq 1 \) can be deduced
from (1.12) and \((\beta - 1)\alpha > \alpha - 2\) can be deduced from (1.13). So by the Lemma 2.1, \(F(t)\) will blow up in finite time and thus the solutions to problem (1.10) will blow up in finite time, and also we have the following

\[
T(\delta) \leq c\delta^{-\frac{\beta - 1}{\alpha + \beta - 1}} \quad (4.23)
\]

\[
= c\delta^{-\frac{\alpha - 1}{\alpha + \beta - 1}} \quad (4.24)
\]

here we take \(\delta = \varepsilon^p\), then we have the estimate for the lifespan of the solution to the problem (1.10):

\[
T(\varepsilon) \leq C\varepsilon^{-\frac{p(\alpha - 1)}{\alpha + 2(\alpha - 1)p - 1}} = C\varepsilon^{-\frac{2\alpha(\alpha - 1)}{2\alpha - 2(n - 1)p - 1}} \quad (4.25)
\]

where \(C\) is a positive constant which is independent of \(\varepsilon\). The proof of Theorem 1.4 is complete.

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