A novel symmetry constraint of the super cKdV system

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Abstract
A new (1+1)-dimensional integrable system, i.e. the super coupled Korteweg–de Vries (cKdV) system, has been constructed by a super extension of the well-known (1+1)-dimensional cKdV system. For this new system, a novel symmetry constraint between the potential and the eigenfunction can be obtained by means of the binary nonlinearization of its Lax pairs. The constraints for even variables are explicit and the constraints for odd variables are implicit. Under the symmetry constraint, the spacial part and the temporal parts of the equations associated with the Lax pairs for the super cKdV system can be decomposed into the super finite-dimensional integrable Hamiltonian systems on the supersymmetry manifold \(R^{4N^2+2}\), whose integrals of motion are explicitly given.

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1. Introduction
Super extensions of classical integrable systems lead to super integrable systems and these have undergone extensive development in the past few years. There are many super integrable systems in the literature, such as the super AKNS system [1–3], the super KdV equation [4–7], the super KP hierarchy [8–11], etc. Super systems contain odd variables which provide more prolific fields for mathematical and physical researchers. The Darboux transformation [12–14], bi-Hamiltonian structure [15–17], Painlevé analysis [18] and so on have been widely studied. Very recently, nonlinearization of the super AKNS system and the super Dirac system has been investigated in [19–21].

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The mono-nonlinearization technique was first proposed by Cao in [22], and the binary-nonlinearization technique was proposed by Ma in [23]. Both mono- and binary-nonlinearization have the following characteristics. Firstly, the advantage of the nonlinearization method is to decompose infinite-dimensional systems into finite ones. Secondly, one of the essential steps of the nonlinearization method is to calculate the variational derivative. Finally, the key of the nonlinearization method is to find symmetry constraints between the potential and the eigenfunction by means of the variational derivative. On the one hand, nonlinearization of Lax pairs is valid for many classical integrable systems [24–27]. On the other hand, binary nonlinearization has been applied to the super AKNS system and the super Dirac system in [19–21]. However, is the nonlinearization method valid for other super integrable systems? For the cKdV system, the answer is affirmative in this paper. The cKdV system firstly proposed by Hirota and Satsuma in [28] is very important in classical integrable systems. Its mono-nonlinearization and Darboux transformation were studied in [29, 30].

The paper is organized as follows. In the next section, the cKdV system will be extended into the super one, and the super Hamiltonian structure will be obtained for the new system by means of the super trace identity. In section 3, the variational derivative of the spectral parameter with respect to the potential is calculated by lemma 2.1 in [21], and a symmetry constraint between the potential and the eigenfunction can be found. The symmetry constraint is an interesting constraint, and it is explicit for even elements, but implicit for odd elements. Then in section 4, after the introduction of two new odd variables, the novel symmetry constraint is substituted into the Lax pairs and the adjoint Lax pairs of the super cKdV system while considering the two new variables. We find that the constrained Lax pairs and the adjoint Lax pairs of the super cKdV system are super Hamiltonian systems, and are completely integrable systems in the Liouville sense. Integrals of motion with odd eigenfunctions are given explicitly. The conclusions and discussions are given in section 5.

2. The super cKdV soliton hierarchy

Let us begin with the following spectral problem:

\[ \phi_x = U(u, \lambda) \phi, \quad U(u, \lambda) = \begin{pmatrix} -\frac{1}{2} \lambda + \frac{1}{2} q & -r & \alpha \\ \frac{1}{2} \lambda - \frac{1}{2} q & 1 & \beta \\ -\alpha & -\frac{1}{2} q & 0 \end{pmatrix}, \quad u = \begin{pmatrix} q \\ r \\ \alpha \\ \beta \end{pmatrix}, \quad \phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \]

(1)

where \( u \) is a potential, and \( \lambda \) is a spectral parameter. Set \( p(q) = p(r) = p(\lambda) = 0, \) and \( p(\alpha) = p(\beta) = 1. \) Here \( p(f) \) means the parity of the arbitrary function \( f. \) Note that \( U \in B(0, 1), \) where \( B(0, 1) \) is a Lie super algebra.

Set

\[ V = \begin{pmatrix} A & B & \rho \\ C & -A & \delta \\ \delta & -\rho & 0 \end{pmatrix} \]

where \( p(A) = p(B) = p(C) = 0, p(\rho) = p(\delta) = 1. \) Noting that

\[ UV - VU = \begin{pmatrix} -B - rC + a\delta + \beta \rho & -\lambda B + 2rA + qB - 2\alpha \rho & -\frac{1}{2} \lambda \rho - aA - \beta B + \frac{1}{2} q \rho - r \delta \\ -\lambda C + 2A - qC + 2\beta \delta & -\frac{1}{2} \lambda \rho - aA - \beta B + \frac{1}{2} q \rho - r \delta \\ -\lambda C + 2A - qC + 2\beta \delta & -\frac{1}{2} \lambda \rho - aA - \beta B + \frac{1}{2} q \rho - r \delta \end{pmatrix}. \]

there then follows the co-adjoint representation equation

\[ V_t = [U, V] = UV - VU, \]

(2)
which becomes
\[
\begin{align*}
Ax &= -B - rC + \alpha \delta + \beta \rho, \\
Bx &= -\lambda B + 2rA + qB - 2\alpha \rho, \\
Cx &= \lambda C + 2A - qC + 2\beta \delta, \\
\rho_x &= -\frac{1}{2}\lambda \rho - \alpha A - \beta B + \frac{1}{2}q \rho - r\delta, \\
\delta_x &= \frac{1}{2}\delta + \beta A - \alpha C + \rho - \frac{1}{2}q \delta.
\end{align*}
\]
(3)

Setting \( A = \sum_{j \geq 0} A_j \lambda^{-j}, \ B = \sum_{j \geq 0} B_j \lambda^{-j}, \ C = \sum_{j \geq 0} C_j \lambda^{-j}, \ \rho = \sum_{j \geq 0} \rho_j \lambda^{-j}, \ \delta = \sum_{j \geq 0} \delta_j \lambda^{-j}, \) equation (3) becomes equivalent to
\[
\begin{align*}
B_0 &= C_0 = \rho_0 = \delta_0 = 0, \\
A_{j,x} &= -B_j - rC_j + \beta \rho_j + \alpha \delta_j, \ j \geq 0, \\
B_{j,x} &= -B_{j+1} + 2rA_j + qB_j - 2\alpha \rho_j, \ j \geq 0, \\
C_{j,x} &= C_{j+1} + 2A_j - qC_j + 2\beta \delta_j, \ j \geq 0, \\
\rho_{j,x} &= -\frac{1}{2}\rho_{j+1} - \alpha A_j - \beta B_j + \frac{1}{2}q \rho_j - r\delta_j, \ j \geq 0, \\
\delta_{j,x} &= \frac{1}{2}\delta_{j+1} + \beta A_j - \alpha C_j + \rho_j - \frac{1}{2}q \delta_j, \ j \geq 0.
\end{align*}
\]
(4)

It can be written as the following recurrence relation:
\[
\begin{pmatrix}
A_{n+1} \\
C_{n+1} \\
2\delta_{n+1} \\
-2\rho_{n+1}
\end{pmatrix} = \mathcal{L}
\begin{pmatrix}
A_n \\
C_n \\
2\delta_n \\
-2\rho_n
\end{pmatrix},
\]
(5)

where the recursive operator is given by
\[
\mathcal{L} = \begin{pmatrix}
-\partial + \partial^{-1}q \partial & r + \partial^{-1}r \partial & \frac{1}{2}\alpha + \partial^{-1}\alpha \partial & -\frac{1}{2}\beta + \partial^{-1}\beta \partial \\
2 & -4\beta & -4\alpha & 2\beta + q & 2 \\
-4\beta \partial + 4\alpha & 4r \beta & 2r - 2\alpha \beta & -2\partial + q
\end{pmatrix}.
\]
with \( \partial = d/dx \) and \( \partial \partial^{-1} = \partial^{-1} \partial = 1. \)

Owing to \( B_0 = C_0 = \rho_0 = \delta_0 = 0, \) we find that \( A_{0,x} = 0. \) So we choose the initial value \( A_0 = -\frac{1}{2}. \) If we set all constants of integration to be zero, all \( A_j, \ B_j, \ C_j, \ \rho_j, \ \delta_j (j > 0) \) are uniquely given by (5). For instance
\[
\begin{align*}
A_1 &= 0, \quad B_1 = -r, \quad C_1 = 1, \quad \rho_1 = \alpha, \quad \delta_1 = \beta, \\
A_2 &= -r + 2\alpha \beta, \quad B_2 = r_x - qr, \quad C_2 = q, \quad \rho_2 = -2\alpha_x + q\alpha, \quad \delta_2 = 2\beta_x + q\beta.
\end{align*}
\]

Then, consider the auxiliary spectral problem associated with the spectral problem (1)
\[
\phi_{tn} = V^{(n)} \phi,
\]
(6)

where
\[
V^{(n)} = (\lambda^n V)_+ + \Delta_n = \sum_{j=0}^{n} \begin{pmatrix} A_j & B_j & \rho_j \\ C_j & -A_j & \delta_j \\ \delta_j & -\rho_j & 0 \end{pmatrix} \lambda^{-j} + \begin{pmatrix} 1 \ C_{n+1} & 0 & 0 \\ 0 & -\frac{1}{2}C_{n+1} & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
and \( (\lambda^n V)_+ \) denotes the non-negative power of \( \lambda \) in \( V. \)

The compatibility conditions of Lax pairs
\[
\phi_t = U \phi, \quad \phi_{tn} = V^{(n)} \phi,
\]
(7)
determine a hierarchy of the super cKdV system

\[
\begin{align*}
q_{tn} &= C_{n+1,x} + r C_{n+1}, \\
r_{tn} &= B_{n+1} + r C_{n+1}, \\
\alpha_{tn} &= \frac{1}{2} \alpha C_{n+1} - \frac{1}{2} \rho_{n+1}, \\
\beta_{tn} &= \frac{1}{2} \delta_{n+1} - \frac{1}{2} \beta C_{n+1}.
\end{align*}
\]  

(8)

The first nonlinear cKdV system in hierarchy (8) reads

\[
\begin{align*}
q_{t} &= q_{xx} + 2q q_{x} + 2r x - 4\alpha x\beta - 4\alpha \beta x - 4\beta \beta x, \\
r_{t} &= -r_{xx} + 2q r + 2q r_{x} + 4\alpha x\beta - 4\rho \beta x, \\
\alpha_{t} &= -2\alpha_{xx} + \frac{1}{2} q_{x} + 2q x_{x} + \beta + 2r x_{x} - 2\alpha \beta x, \\
\beta_{t} &= 2\beta_{xx} + \frac{1}{2} q_{x} + 2q \beta_{x} + 2\alpha x.
\end{align*}
\]

(9)

whose Lax pairs are \(U\) and

\[
V^{(2)} = \begin{pmatrix}
-\frac{1}{2} \lambda^2 + \frac{1}{2} q_{x} + \frac{1}{2} q^2 - 2\beta \beta - r \lambda + r x - q r - \alpha \lambda - 2\alpha x + q\alpha \\
\lambda + q \\
\frac{1}{2} \lambda^2 - \frac{1}{2} q_{x} - \frac{1}{2} q^2 + 2\beta \beta - \beta \lambda + 2\beta x + q\beta \\
\beta \lambda + 2\beta_{x} + q\beta \\
-\alpha \lambda + 2\alpha x - q\alpha \\
0
\end{pmatrix}.
\]

In what follows, the super Hamiltonian structures of the super cKdV system (8) can be achieved. Using the super trace identity \[31, 32\]

\[
\delta \int Str \left( V \frac{\partial U}{\partial \lambda} \right) dx = \left( \gamma - n \right) \left( \frac{A_{n}}{2\delta_{n+1}} \right),
\]

(10)

where \( Str \) means the super trace, we have

\[
\begin{pmatrix}
\frac{\delta}{\delta q} \\
\frac{\delta}{\delta r} \\
\frac{\delta}{\delta \alpha} \\
\frac{\delta}{\delta \beta}
\end{pmatrix}
\int -A_{n+1} \ dx = \left( \gamma - n \right) \left( \begin{array}{c}
A_{n} \\
-C_{n} \\
2\delta_{n+1} \\
-2\rho_{n+1}
\end{array} \right),
\]

where \( \gamma \) is an arbitrary constant. Letting \( n = 1 \) in the above equality, we obtain \( \gamma = 0 \). Therefore, we get the following identity:

\[
\begin{pmatrix}
A_{n+1} \\
-C_{n+1} \\
2\delta_{n+1} \\
-2\rho_{n+1}
\end{pmatrix}
= \delta \frac{H_{n}}{\delta u}, \quad H_{n} = \int \frac{1}{n+1} A_{n+2} dx.
\]

Thus, the super cKdV hierarchy can be written in the following super Hamiltonian form:

\[
\dot{u}_{n} = \begin{pmatrix}
q \\
\alpha \\
\beta
\end{pmatrix}
= K_{n} = J \begin{pmatrix}
A_{n+1} \\
-C_{n+1} \\
2\delta_{n+1} \\
-2\rho_{n+1}
\end{pmatrix}
= J \frac{\delta H_{n}}{\delta u},
\]

(11)
where the super symplectic operator is given by
\[
J = \begin{pmatrix}
0 & -\beta & 0 & 0 \\
-\beta & 0 & \frac{1}{2}\alpha & -\frac{1}{2}\beta \\
0 & \frac{1}{2}\alpha & 0 & \frac{1}{2} \\
0 & \frac{1}{2}\beta & \frac{1}{2} & 0
\end{pmatrix}.
\]

3. A novel symmetry constraint

In this section, a symmetry constraint between the potential and the eigenfunction can be obtained. To this end, consider the adjoint spectral problem associated with the spectral problem (1)
\[
\psi_x = -(U(u, \lambda))St \psi = \begin{pmatrix}
\frac{1}{2}\lambda - \frac{1}{2}q & -1 & \beta \\
r & -\frac{1}{2}\lambda + \frac{1}{2}q & -\alpha \\
-\alpha & -\beta & 0
\end{pmatrix} \psi, \quad \psi = \begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3
\end{pmatrix}, \quad (12)
\]
where St means super transposition.

Using lemma 2.1 in [21], we can easily get the variational derivative of the spectral parameter \(\lambda\) with respect to the potential \(u\):
\[
\frac{\delta \lambda}{\delta u} = \frac{1}{E} \begin{pmatrix}
\frac{1}{2}(\psi_1\phi_1 - \psi_2\phi_2) \\
-\psi_1\phi_2 \\
\psi_3\phi_2 - \psi_2\phi_3
\end{pmatrix}, \quad (13)
\]
where \(E = \int \frac{1}{2}(\psi_1\phi_1 - \psi_2\phi_2) \, dx\). When zero boundary conditions \(\lim_{|x| \to \infty} \phi = \lim_{|x| \to \infty} \psi = 0\) are imposed, it satisfies the following equation:
\[
\mathcal{L} \frac{\delta \lambda}{\delta u} = \lambda \frac{\delta \lambda}{\delta u}, \quad (14)
\]
where \(\mathcal{L}\) is defined as in (5). The above variational derivative will serve as a conserved covariant yielding a specific symmetry used in symmetry constraints.

For Lax pairs (7), we choose the following symmetry constraint:
\[
\begin{pmatrix}
-r + 2\alpha \beta \\
-q \\
4\beta \lambda + 2q \beta \\
4\alpha \lambda - 2q \alpha
\end{pmatrix} = \frac{1}{2} \begin{pmatrix}
\langle \psi_1, \phi_1 \rangle - \langle \psi_2, \phi_2 \rangle \\
\langle \psi_1, \phi_2 \rangle \\
\langle \psi_1, \phi_3 \rangle + \langle \psi_3, \phi_2 \rangle \\
\langle \psi_2, \phi_1 \rangle - \langle \psi_3, \phi_1 \rangle
\end{pmatrix}, \quad (15)
\]
where \(\Phi_i = (\phi_{i1}, \ldots, \phi_{iN})^T, \Psi_i = (\psi_{i1}, \ldots, \psi_{iN})^T (i = 1, 2, 3)\), and \(\langle ., . \rangle\) denotes the standard inner product in \(\mathbb{R}^N\). We find that the odd potentials \(\alpha\) and \(\beta\) cannot be explicitly expressed by eigenfunctions, but the even potentials \(q\) and \(r\) can. Therefore, symmetry constraint (15) is a novel constraint.

Remark 1. In classical integrable systems, the symmetry constraint between the potential and the eigenfunction is either explicit or implicit. To date, we have no example where the symmetry constraint could combine the explicit and implicit constraint. Even in super integrable systems there is no example. Therefore, equation (15) is absolutely a novel symmetry constraint.
We then denote the expression of $P(u)$ under the symmetry constraint (15) by $\tilde{P}$. From the property (14) and the recurrence relation (5), we obtain

\[
\begin{align*}
\dot{A}_{n+1} &= \frac{1}{4}(\Lambda^{-1}\psi_1, \Phi_1) - (\Lambda^{-1}\psi_2, \Phi_2), \quad n \geq 1, \\
\dot{B}_{n+1} &= (\Lambda^{-1}\psi_2, \Phi_1), \quad n \geq 1, \\
\dot{C}_{n+1} &= (\Lambda^{-1}\psi_1, \Phi_2), \quad n \geq 1, \\
\dot{\rho}_{n+1} &= -\frac{1}{2}(\Lambda^{-1}\psi_2, \Phi_3) - (\Lambda^{-1}\psi_3, \Phi_1), \quad n \geq 1, \\
\delta_{n+1} &= \frac{1}{4}(\Lambda^{-1}\psi_1, \Phi_3) + (\Lambda^{-1}\psi_3, \Phi_2)), \quad n \geq 1, 
\end{align*}
\]

where $\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_N)$.

4. Binary nonlinearization

In the last section, we found a novel symmetry constraint (15). Because the odd potentials $\alpha$ and $\beta$ cannot be explicitly expressed by eigenfunctions, we introduce the following new independent variables:

\[
\phi_{n+1} = \alpha, \quad \psi_{n+1} = 4\beta.
\]

Choosing $N$ distinct parameters $\lambda_1, \ldots, \lambda_N$, we obtain the following two spatial and temporal systems:

\[
\begin{cases}
\begin{align*}
\left(\begin{array}{c}
\phi_{1,j} \\
\phi_{2,j} \\
\phi_{3,j}
\end{array}\right)_x &= U(u, \lambda_j) \left(\begin{array}{c}
\phi_{1,j} \\
\phi_{2,j} \\
\phi_{3,j}
\end{array}\right), \\
\left(\begin{array}{c}
\psi_{1,j} \\
\psi_{2,j} \\
\psi_{3,j}
\end{array}\right)_x &= -U^{(1)}(u, \lambda_j) \left(\begin{array}{c}
\psi_{1,j} \\
\psi_{2,j} \\
\psi_{3,j}
\end{array}\right), \\
\left(\begin{array}{c}
\phi_{1,j} \\
\phi_{2,j} \\
\phi_{3,j}
\end{array}\right)_{t_n} &= V^{(n)}(u, \lambda_j) \left(\begin{array}{c}
\phi_{1,j} \\
\phi_{2,j} \\
\phi_{3,j}
\end{array}\right), \\
\left(\begin{array}{c}
\psi_{1,j} \\
\psi_{2,j} \\
\psi_{3,j}
\end{array}\right)_{t_n} &= -(V^{(n)})^{(1)}(u, \lambda_j) \left(\begin{array}{c}
\psi_{1,j} \\
\psi_{2,j} \\
\psi_{3,j}
\end{array}\right),
\end{align*}
\end{cases}
\]

(18) \hspace{1cm} (19)

It is easy to verify that the compatibility condition of (18) and (19) is still the $n$th super cKdV system $u_n = K_n$. When the symmetry constraint (15) and new independent variables (17) are considered, the systems (18) and (19) become the following finite-dimensional system:

\[
\begin{align*}
\phi_{1,j,x} &= \frac{1}{2}(-\lambda_j + \langle \psi_1, \Phi_2 \rangle)\phi_{1,j} + \frac{1}{2}(\langle \psi_1, \Phi_1 \rangle - \langle \psi_2, \Phi_2 \rangle - \phi_{N+1}\psi_{N+1})\phi_{2,j} \\
&\quad + \phi_{N+1}\phi_{3,j}, \\
\phi_{2,j,x} &= \phi_{1,j} + \frac{1}{2}(\lambda_j - (\langle \psi_1, \Phi_2 \rangle)\phi_2_j + \frac{1}{2}\psi_{N+1}\phi_{3,j}, \\
\phi_{3,j,x} &= \frac{1}{2}(\langle \psi_2, \Phi_1 \rangle - \langle \psi_3, \Phi_1 \rangle) + \frac{1}{2}(\langle \psi_1, \Phi_2 \rangle)\phi_{N+1}, \\
\phi_{N+1,x} &= \frac{1}{2}(\langle \psi_2, \Phi_1 \rangle - \langle \psi_3, \Phi_1 \rangle) + \frac{1}{2}(\langle \psi_1, \Phi_2 \rangle)\phi_{N+1}, \\
\psi_{1,j,x} &= \frac{1}{2}(\lambda_j - (\langle \psi_1, \Phi_2 \rangle)\psi_{1,j} - \psi_{2,j} + \frac{1}{2}\psi_{N+1}\phi_{3,j}, \\
\psi_{2,j,x} &=\frac{1}{2}(-\langle \psi_1, \Phi_1 \rangle + \langle \psi_2, \Phi_2 \rangle + \phi_{N+1}\psi_{N+1})\psi_{1,j} + \frac{1}{2}(-\lambda_j + (\langle \psi_1, \Phi_2 \rangle)\psi_{2,j} - \phi_{N+1}\psi_{3,j}, \\
\psi_{3,j,x} &= -\phi_{N+1}\psi_{1,j} - \frac{1}{2}\psi_{N+1}\psi_{2,j}, \\
\psi_{N+1,x} &= (\langle \psi_1, \Phi_3 \rangle + \langle \psi_3, \Phi_2 \rangle - \frac{1}{2}(\langle \psi_1, \Phi_2 \rangle)\psi_{N+1}, \\
\end{align*}
\]

(20)
where $1 \leq j \leq N$. The system (20) can then be written as follows:

$$
\begin{align*}
\Phi_{1,x} &= \frac{1}{2} (-\Lambda + \langle \Phi_1, \Phi_2 \rangle) \Phi_1 + \frac{1}{2} \langle \langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle - \phi_{N+1} \psi_{N+1} \rangle \Phi_2 \\
&\quad + \phi_{N+1} \Phi_3 = \frac{\partial H_1}{\partial \phi_{N+1}}, \\
\Phi_{2,x} &= \Phi_1 + \frac{1}{2} (-\Lambda - \langle \Phi_1, \Phi_2 \rangle) \Phi_2 + \frac{1}{2} \psi_{N+1} \Phi_3 = \frac{\partial H_1}{\partial \psi_{N+1}}, \\
\Phi_{3,x} &= \frac{1}{2} \psi_{N+1} \Phi_1 - \phi_{N+1} \Phi_2 = \frac{\partial H_1}{\partial \phi_{N+1}}, \\
\phi_{N+1,x} &= \frac{1}{2} \langle \langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_1 \rangle \rangle + \frac{1}{2} \langle \langle \Psi_1, \Phi_2 \rangle \rangle \phi_{N+1} = \frac{\partial H_1}{\partial \psi_{N+1}}, \\
\Psi_{1,x} &= \frac{1}{2} (-\Lambda - \langle \Phi_1, \Phi_2 \rangle) \Psi_1 - \psi_2 + \frac{1}{2} \psi_{N+1} \Psi_1 = -\frac{\partial H_1}{\partial \psi_{N+1}}, \\
\Psi_{2,x} &= \frac{1}{2} (-\langle \Phi_1, \Phi_1 \rangle + \langle \Psi_1, \Phi_2 \rangle + \phi_{N+1} \psi_{N+1}) \Psi_1 + \frac{1}{2} (-\Lambda + \langle \Phi_1, \Phi_2 \rangle) \Psi_2 \\
&\quad - \phi_{N+1} \Psi_3 = -\frac{\partial H_1}{\partial \phi_{N+1}}, \\
\Psi_{3,x} &= -\phi_{N+1} \Psi_1 + \frac{1}{2} \psi_{N+1} \Psi_2 = \frac{\partial H_1}{\partial \phi_{N+1}}, \\
\psi_{N+1,x} &= \langle \Psi_1, \Phi_3 \rangle + \langle \Psi_3, \Phi_2 \rangle - \frac{1}{2} \langle \langle \Psi_1, \Phi_2 \rangle \rangle \psi_{N+1} = \frac{\partial H_1}{\partial \psi_{N+1}},
\end{align*}
$$

(21)

where the Hamiltonian function

$$
H_1 = -\frac{1}{2} (\Lambda \Psi_1, \Phi_1) + \frac{1}{2} (\Lambda \Psi_2, \Phi_2) + \frac{1}{2} (\Psi_1, \Phi_2) (\langle \Psi_1, \Phi_1 \rangle - \langle \Psi_2, \Phi_2 \rangle) \\
+ \langle \Psi_2, \Phi_1 \rangle - \frac{1}{2} \phi_{N+1} \psi_{N+1} (\langle \Psi_1, \Phi_2 \rangle + \phi_{N+1} (\langle \Psi_1, \Phi_3 \rangle + \langle \Psi_3, \Phi_2 \rangle) \\
+ \frac{1}{2} \psi_{N+1} (\langle \Psi_2, \Phi_3 \rangle - \langle \Psi_3, \Phi_1 \rangle).
$$

For the $t_2$-part, we have the following spectral problem:

$$
\phi_{t_2} = V^{(2)} \phi = \begin{pmatrix}
-\frac{1}{2} \lambda^2 + \frac{1}{2} q_s - \frac{1}{2} q^2 - 2 \beta \beta_s & -r \lambda + r_s - q r & \alpha \lambda - 2 \alpha s - q a \\
\lambda + q & \frac{1}{2} \lambda^2 - \frac{1}{2} q_s - \frac{1}{2} q^2 + 2 \beta \beta_s & \beta \lambda + 2 \beta = q \beta \\
\beta \lambda + 2 \beta s + q \beta & -\alpha \lambda + 2 \alpha s - q a & 0
\end{pmatrix} \phi,
$$

(22)

and its adjoint spectral problem

$$
\psi_{t_2} = -(V^{(2)})^T \psi = \begin{pmatrix}
\frac{1}{2} \lambda^2 - \frac{1}{2} q_s - \frac{1}{2} q^2 + 2 \beta \beta_s & -r \lambda - r_s + q r & -\lambda - q \\
-r \lambda - r_s + q r & -\frac{1}{2} \lambda^2 + \frac{1}{2} q_s + \frac{1}{2} q^2 - 2 \beta \beta_s & -\alpha \lambda + 2 \alpha s - q a \\
-\alpha \lambda + 2 \alpha s - q a & -\beta \lambda - 2 \beta s - q \beta & 0
\end{pmatrix} \psi.
$$

(23)

Considering $N$ copies of (22) and (23) under the symmetry constraint (15), we obtain the following finite-dimensional system:

$$
\begin{align*}
\phi_{1,t_2} &= \left( -\frac{1}{2} \lambda_j^2 + \frac{1}{2} q_s + \frac{1}{2} q^2 - 2 \beta \beta_s \right) \phi_{1,j} + (-r \lambda_j + r_s - q r) \phi_{2,j} + (\tilde{\alpha} \lambda_j - 2 \tilde{\alpha} s + \tilde{q} \tilde{a}) \phi_{3,j}, \\
\phi_{2,t_2} &= (\lambda_j + q) \phi_{1,j} + \left( \frac{1}{2} \lambda_j^2 - \frac{1}{2} q_s - \frac{1}{2} q^2 + 2 \beta \beta_s \right) \phi_{2,j} + (\tilde{\beta} \lambda_j + 2 \tilde{\beta} s + \tilde{q} \tilde{b}) \phi_{3,j}, \\
\phi_{3,t_2} &= (\tilde{\beta} \lambda_j + 2 \tilde{\beta} s + \tilde{q} \tilde{b}) \phi_{1,j} + (-\tilde{\alpha} \lambda_j + 2 \tilde{\alpha} s - \tilde{q} \tilde{a}) \phi_{2,j}, \\
\psi_{1,t_2} &= \left( \frac{1}{2} \lambda_j^2 - \frac{1}{2} q_s - \frac{1}{2} q^2 + 2 \beta \beta_s \right) \psi_{1,j} - (\lambda_j + q) \psi_{2,j} + (\tilde{\beta} \lambda_j + 2 \tilde{\beta} s + \tilde{q} \tilde{b}) \psi_{3,j}, \\
\psi_{2,t_2} &= (\tilde{\beta} \lambda_j - r_s + \tilde{q} \tilde{b}) \psi_{1,j} + (\tilde{\alpha} \lambda_j + 2 \tilde{\alpha} s - \tilde{q} \tilde{a}) \psi_{2,j} + (-\tilde{\alpha} \lambda_j + 2 \tilde{\alpha} s - \tilde{q} \tilde{a}) \psi_{3,j}, \\
\psi_{3,t_2} &= (-\tilde{\alpha} \lambda_j + 2 \tilde{\alpha} s - \tilde{q} \tilde{a}) \psi_{1,j} - (\tilde{\beta} \lambda_j + 2 \tilde{\beta} s + \tilde{q} \tilde{b}) \psi_{2,j},
\end{align*}
$$

(24)
where \( 1 \leq j \leq N \), \( \tilde{q}_j \), \( \tilde{r}_j\), \( \tilde{\alpha}_j\), \( \tilde{\beta}_j\) respectively denote \( q, r, \alpha, \beta \) under the symmetry constraint (15), and \( \tilde{q}_j, \tilde{r}_j, \tilde{\alpha}_j, \tilde{\beta}_j \) are given by the following identities:

\[
\tilde{q}_j = \frac{1}{2} \left( \gamma \psi_j \phi_j - (\psi_j, \phi_j) \right),
\]

\[
\tilde{r}_j = \frac{1}{2} \left( \gamma \phi_j \psi_j - (\phi_j, \psi_j) \right),
\]

\[
\tilde{\alpha}_j = \frac{1}{2} \left( \gamma \delta \phi_j - (\delta, \phi_j) \right),
\]

\[
\tilde{\beta}_j = \frac{1}{2} \left( \gamma \delta \psi_j - (\delta, \psi_j) \right).
\]

Thus, the constrained system (24) becomes

\[
\left\{ \begin{array}{l}
\Phi_{1,t} = \frac{1}{2} (-\gamma + \gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Phi_1 + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \\
- \phi_j \phi_j \phi_j + 2 (\Phi_j, \Phi_j) \Omega + \frac{1}{2} (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \Phi_1 - (\gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Omega \Phi_1 + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \Phi_1
\end{array} \right.
\]

\[
\Phi_{2,t} = \frac{1}{2} (-\gamma + \gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Phi_2 + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \Phi_2 - (\gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Omega \Phi_2 + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \Phi_2
\]

\[
\Phi_{3,t} = \frac{1}{2} (-\gamma + \gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Phi_3 + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \Phi_3 - (\gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Omega \Phi_3 + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \Phi_3
\]

\[
\psi_{N+1,t} = \frac{1}{2} \phi_j \psi_j \phi_j + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1} - (\gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Omega \psi_{N+1} + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1}
\]

\[
\psi_{N+1,t} = \frac{1}{2} \phi_j \psi_j \phi_j + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1} - (\gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Omega \psi_{N+1} + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1}
\]

(25)

where the Hamiltonian function is as follows:

\[
H_j = -\frac{1}{2} \left( (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1} - (\gamma \phi_j \psi_j - (\phi_j, \psi_j)) \Omega \psi_{N+1} + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1}
\]

\[
+ \phi_j \phi_j \phi_j \phi_j \phi_j \phi_j + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1}
\]

\[
+ \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1}
\]

\[
+ \phi_j \phi_j \phi_j \phi_j \phi_j \phi_j + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1}
\]

\[
+ \phi_j \phi_j \phi_j \phi_j \phi_j \phi_j + \frac{1}{2} \zeta (\gamma \psi_j \phi_j - (\psi_j, \phi_j)) \Lambda \psi_{N+1}
\]

Let us construct integrals of motion for (21). An obvious equality \((\vec{V})^2 = [\vec{U}, \vec{V}]^2\) leads to

\[
F_x = \frac{1}{2} \text{Str} \left( \vec{V}^2 \right) = \frac{d}{dx} \left( \lambda^2 + \hat{B} \hat{C} + 2 \hat{p} \hat{d} \right) = 0,
\]

(26)

that is to say, \( F \) is a generating function of integrals of motion for the constrained spatial system (21). Since \( F = \sum_{n \geq 0} F_n \lambda^{-n} \), we obtain the following expressions:

\[
F_n = \sum_{i=0}^{n} (\hat{A}_i \hat{A}_{n-i} + \hat{B}_i \hat{B}_{n-i} + 2 \hat{p} \hat{d} \hat{n}_{n-i}).
\]

Using (16), we get

\[
F_0 = \frac{1}{4}, \quad F_1 = F_2 = 0,
\]

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Here $F_i(n \geq 0)$ are all polynomials of $6N+2$ dependent variables $\phi_{ij}$, $\psi_{ij}$, $\phi_{N+1}$ and $\psi_{N+1}$, with $i = 1, 2, 3$ and $j = 1, \ldots, N$. Note that for the temporal part, $V_v = [V^{(v)}, V]$ is true. With the similar discussion, we found that $F = \frac{1}{2} \text{Str} \ V^2$ is also a generating function of integrals of motion for (19). Moreover, when the symmetry constraint (15) and new independent variables (17) are considered, the system (19) is constrained as follows:

\[
\begin{align*}
\phi_{1j,tn} &= \left( \sum_{m=0}^{n} \tilde{A}_m \lambda_j^{m-n} + \frac{1}{2} \tilde{C}_{n+1} \right) \phi_{1j} + \sum_{m=0}^{n} \tilde{B}_m \lambda_j^{m-n-1} \phi_{2j} + \sum_{m=0}^{n} \tilde{\rho}_m \lambda_j^{m-n-1} \phi_{3j}, \quad 1 \leq j \leq N, \\
\phi_{2j,tn} &= \sum_{m=0}^{n} \tilde{C}_m \lambda_j^{m-n} \phi_{1j} - \left( \sum_{m=0}^{n} \tilde{A}_m \lambda_j^{m-n} + \frac{1}{2} \tilde{C}_{n+1} \right) \phi_{2j} + \sum_{m=0}^{n} \tilde{\delta}_m \lambda_j^{m-n} \phi_{3j}, \quad 1 \leq j \leq N, \\
\phi_{3j,tn} &= \sum_{m=0}^{n} \tilde{\delta}_m \lambda_j^{m-n} \phi_{1j} - \sum_{m=0}^{n} \tilde{\rho}_m \lambda_j^{m-n} \phi_{2j}, \quad 1 \leq j \leq N, \\
\phi_{N+1,tn} &= \frac{1}{x} \phi_{N+1} (\Lambda^{n-1} \Psi_1, \Phi_2) + \frac{1}{x} ((\Lambda^{n-1} \Psi_2, \Phi_3) - (\Lambda^{n-1} \Psi_3, \Phi_2)), \\
\psi_{1j,tn} &= -\left( \sum_{m=0}^{n} \tilde{A}_m \lambda_j^{m-n} + \frac{1}{2} \tilde{C}_{n+1} \right) \psi_{1j} - \sum_{m=0}^{n} \tilde{C}_m \lambda_j^{m-n} \psi_{2j} + \sum_{m=0}^{n} \tilde{\delta}_m \lambda_j^{m-n} \psi_{3j}, \quad 1 \leq j \leq N, \\
\psi_{2j,tn} &= -\sum_{m=0}^{n} \tilde{B}_m \lambda_j^{m-n} \psi_{1j} + \left( \sum_{m=0}^{n} \tilde{A}_m \lambda_j^{m-n} + \frac{1}{2} \tilde{C}_{n+1} \right) \psi_{2j} - \sum_{m=0}^{n} \tilde{\rho}_m \lambda_j^{m-n} \psi_{3j}, \quad 1 \leq j \leq N, \\
\psi_{3j,tn} &= -\sum_{m=0}^{n} \tilde{\rho}_m \lambda_j^{m-n} \psi_{1j} - \sum_{m=0}^{n} \tilde{\delta}_m \lambda_j^{m-n} \psi_{2j}, \quad 1 \leq j \leq N, \\
\psi_{N+1,tn} &= (\Lambda^{n-1} \Psi_1, \Phi_3) + (\Lambda^{n-1} \Psi_3, \Phi_2) - \frac{1}{x} \psi_{N+1} (\Lambda^{n-1} \Psi_1, \Phi_2).
\end{align*}
\]
After a direct calculation, we have
\[
\begin{align*}
\Phi_{1,t_0} &= \frac{\partial F_{1,2}}{\partial \Phi_{1}}, \\
\Phi_{2,t_0} &= \frac{\partial F_{1,2}}{\partial \Phi_{2}}, \\
\Phi_{3,t_0} &= \frac{\partial F_{1,2}}{\partial \Phi_{3}}, \\
\Phi_{N+1,t_0} &= \frac{\partial F_{1,2}}{\partial \Phi_{N+1}}, \\
\Psi_{1,t_0} &= -\frac{\partial F_{1,2}}{\partial \Psi_{1}}, \\
\Psi_{2,t_0} &= -\frac{\partial F_{1,2}}{\partial \Psi_{2}}, \\
\Psi_{3,t_0} &= \frac{\partial F_{1,2}}{\partial \Psi_{3}}, \\
\Psi_{N+1,t_0} &= \frac{\partial F_{1,2}}{\partial \Psi_{N+1}},
\end{align*}
\]
which shows that the constrained system (28) is a super Hamiltonian system.

In what follows, for (6N+2)-dimensional super Hamiltonian systems (21) and (29), we find 3N+1 integrals of motion. It is natural to find that
\[
f_k = \psi_{1k} \phi_{1k} + \psi_{2k} \phi_{2k} + \psi_{3k} \phi_{3k}, \quad 1 \leq k \leq N,
\]
are integrals of motion for constrained systems (21) and (29). Therefore, for constrained systems (21) and (29), we choose 3N+1 integrals of motion
\[
f_1, \ldots, f_N, F_3, F_4, \ldots, F_{2N+3}.
\]

After a simple calculation, we get
\[
\{ F_m, F_{n+2} \} = \frac{\partial}{\partial t_n} F_m = 0,
\]
where the Poisson bracket is defined by
\[
\{ F, G \} = \sum_{j=1}^{N} \sum_{i=1}^{N} \left( \frac{\partial F}{\partial \psi_{ij}} \frac{\partial G}{\partial \phi_{ij}} - (-1)^{p(\phi_{ij})p(\psi_{ij})} \frac{\partial G}{\partial \psi_{ij}} \frac{\partial F}{\partial \phi_{ij}} \right) + \frac{\partial F}{\partial \phi_{N+1}} \frac{\partial G}{\partial \psi_{N+1}} + \frac{\partial F}{\partial \psi_{N+1}} \frac{\partial G}{\partial \phi_{N+1}}.
\]
The identity (32) means that \( \{ F_m \}_{m \geq 0} \) are in involution. The property of involution among \( \{ f_k \}_{k=1}^{N} \) is obvious. About the independence of \( \{ f_k \}_{k=1}^{N} \) and \( \{ F_m \}_{m=3}^{2N+3} \), we can refer to the proof of proposition 1 in [19]. Thus, we obtain the following theorem.

**Theorem 1.** The constrained systems (21) and (29) are Liouville integrable super Hamiltonian systems, whose integrals of motion are given by (31).

5. Conclusions and discussions

In this paper, the cKdV system is successfully extended to the super cKdV. For the new system, its super Hamiltonian structure is expressed in the form of (11). In our previous papers [19–21], the binary nonlinearization has been applied to the super AKNS system and the super Dirac system. For the super AKNS system, two kinds of nonlinearization of Lax pairs, including nonlinearization under an explicit symmetry constraint [19] and nonlinearization under an implicit symmetry constraint [20], have been considered respectively. For the super Dirac system, we only consider binary nonlinearization under an explicit symmetry constraint [21]. From these three kinds of nonlinearization of Lax pairs, the symmetry constraint is either implicit or explicit. The novelty of constraint (15) for the super cKdV system is due to the combination of the explicit constraint for even potentials \((q, r)\) and the implicit constraint for odd potentials \((\alpha, \beta)\). Such a combination will make the process of binary nonlinearization complex. It is highly non-trivial to solve \((\alpha, \beta)\) from implicit constraints (15) because it is related to a coupled differential equation with variable coefficients. We introduce two new odd variables (17) following the technique of the implicit constraint [33]. Thus, the spatial and temporal parts of the super cKdV system are nonlinearized respectively to the constrained spatial system (21) and to the constrained temporal system (29). Then, we see that the systems (21) and (29) are super Hamiltonian systems. Furthermore, constrained systems (21) and (29) are integrable in the Liouville sense.
However, we are not able to do this for the supersymmetric cKdV system because the spectral matrix of the supersymmetric cKdV system cannot be described by a certain Lie super algebra. In a word, how to make nonlinearization of the supersymmetric cKdV system is an interesting problem. Furthermore, it is also interesting to find an explicit solution of the super finite-dimensional integrable system. We shall consider these problems in the future.

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