Gravitating monopoles in SU(3) gauge theory

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March 27, 2022

Abstract

We consider the Einstein-Yang-Mills-Higgs equations for an SU(3) gauge group in a spherically symmetric ansatz. Several properties of the gravitating monopole solutions are obtained and compared with their SU(2) counterpart.

1 Introduction

During the last few years a lot of attention was devoted to classical solutions of conventional field theories coupled to gravity, this after the remarkable work of Bartnik-McKinnon [1]. One of the most fashionable field theories in which static, regular, finite energy solution exist is the Georgi-Glashow model: an SU(2)-Yang-Mills gauge theory coupled to a triplet of scalar fields and with a Higgs potential breaking the symmetry. This model admits topological solitons among its classical solutions: the celebrated SU(2)-t’Hooft-Polyakov monopole [2, 3] and the multi-monopoles. Soon after the construction of the first monopole solution in SU(2), the classification of its counterparts in larger groups has been investigated, e.g. [4, 5, 6, 7]; with an Higgs-field multiplet in the adjoint representation of the gauge group.

Several years ago, it was shown that SU(2)-gravitating magnetic monopoles, as well as non-abelian black holes, exist in the Georgi-Glashow model coupled to gravity [8, 9, 10]. For instance, for a fixed value of the Higgs boson mass, the gravitating monopoles exist up to a critical value \(\alpha_c\) of the parameter \(\alpha\) (the ratio of the vector meson mass to the Planck mass). At the critical value \(\alpha_c\), a critical solution with a degenerate horizon is reached. In particular, for small values of the Higgs boson mass, the critical solution where an horizon first appears corresponds to an extremal Reissner-Nordstrøm (RN) solution outside the horizon while it is non-singular inside.

Recently, Lue and Weinberg reconsidered the SU(2) equations of the self-gravitating magnetic monopoles and discovered an insofar unsuspected phenomenon [11]. Indeed, for large enough values of the Higgs boson mass, the critical solution is an extremal black hole with non-abelian hair and a mass less than the extremal RN value. This new bifurcation pattern was checked to be also present for black-holes solutions [12] and it is natural to ask if it is also present for gauge groups larger than SU(2).

In this paper, we consider the SU(3)-generalisation of the Georgi-Glashow model that we couple to gravity. We establish the equations of motion of this theory in the spherically symmetric ansatz of the fields and we construct numerically the corresponding magnetic monopole solutions. We find that the main features of the SU(2)-gravitating monopoles [8, 11] also hold for the gauge group SU(3). The paper is organised as follows: in Sect. 2 we describe the Lagrangian, the spherically symmetric ansatz and the corresponding field equations. The main properties of the SU(2)-gravitating monopole are summarised in Sect. 3. Our results concerning the solutions with SU(3) as gauge group are discussed in Sect. 4 and illustrated by three figures.
The SU(3) Einstein-Yang-Mills-Higgs action can be constructed in analogy with the corresponding SU(2) one \[8, 9, 10\]; it is described by the action

\[
S = \int d^4x \sqrt{-g}(\mathcal{L}_E + \mathcal{L}_{YMH})
\]

with \(\mathcal{L}_E = 16\pi G \mathcal{R}\) (\(\mathcal{R}\) is the scalar curvature, \(G\) is Newton’s constant) and, for the matter fields

\[
\mathcal{L}_{YMH} = -\frac{1}{2} \text{Tr} F^{\mu\nu} F_{\mu\nu} + \frac{1}{2} \text{Tr} D_\mu \phi D^\mu \phi - V(\phi)
\]

The usual definitions of the covariant derivative and field strengths are used in the above formula:

\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]
\]

\[
D_\mu \phi = \partial_\mu \phi - i [A_\mu, \phi]
\]

with

\[
A_\mu = \frac{1}{2} A^a_\mu \lambda^a, \quad \phi = \phi^a \lambda^a
\]

\((\lambda^a, a = 1, \ldots, 8)\), are the Gell-Mann matrices normalised according to \(\text{tr}(\lambda_a \lambda_b) = 2\delta_{ab}\). The Higgs potential \(V(\phi)\) is the most general renormalisable potential which breaks the SU(3) symmetry \[13\]

\[
V(\phi) = p(v^2 - \frac{1}{2} \text{Tr} \phi^2)^2 + q(v^2 \frac{1}{2} \text{Tr} \phi^2 + v \sqrt{3} \text{Tr} \phi^3 - \frac{1}{2}(\text{Tr} \phi^2)^2 + \frac{3}{2} \text{Tr} \phi^4)
\]

\(p, q\) are positive constants. For generic values of the constants \(p, q\), the breakdown of the SU(3)-symmetry is such that the expectation value \(\langle \phi \rangle\) possesses two equal eigenvalues (i.e. the eigenvalues of this matrix are proportional to those of \(\lambda_8\)).

In this pattern, we are left with four massive vector bosons corresponding to \(A^a_\mu, a = 4, 5, 6, 7\) with equal masses \(M_W\) and four scalar fields. The masses of these fields are given by

\[
M_W^2 = \frac{3}{4} e^2 v^2, \quad M_1^2 = M_2^2 = M_3^2 = 18q v^2, \quad M_8^2 = 2(4p + q)v^2
\]

In contrast, in an SU(2) gauge theory with potential \(V(\phi) = p(v^2 - (1/2)\text{Tr} \phi^2)^2\), there are two vector bosons of equal masses \(M_W\) and one scalar of mass \(M_H\), with

\[
M_W^2 = e^2 v^2, \quad M_H^2 = 8pv^2
\]

For later use we define the following combinations of the coupling constants

\[
b \equiv \left( \frac{e M_W}{2 M_H} \right)^2 = 2p \quad \text{for SU(2)}
\]

\[
b \equiv \left( \frac{e M_W}{2 M_H} \right)^2 = \frac{2(4p + q)}{3}, \quad R \equiv \left( \frac{M_H}{M_W} \right)^2 = \frac{9q}{4p + q} \quad \text{for SU(3)}
\]

(remark that \(0 \leq R \leq 9\)). In relation with Newton’s constant \(G\), we define

\[
\alpha^2 = 4\pi G v^2, \quad a = 8\pi \frac{M_W^2}{e^2 M_{pl}^2}
\]

so that \(a = 2\alpha^2\) and \(a = (3/2)\alpha^2\) respectively for SU(2) and SU(3).

In order to obtain static, spherically symmetric and globally regular solutions, we use Schwarzschild like coordinates for the metric

\[
ds^2 = -A^2 N dt^2 + N^{-1} dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)
\]
The crucial step to obtain the flat monopole is the embedding of the \( \text{su}(2) \) algebra (whose generators will be labelled by \( T_1, T_2, T_3 \)) into the \( \text{su}(3) \) algebra \([4]\). Here we will use the embedding
\[
(T_1, T_2, T_3) = \left( \frac{1}{2} \lambda_1, \frac{1}{2} \lambda_2, \frac{1}{2} \lambda_3 \right) \equiv \frac{1}{2} \vec{r}
\]for which the monopole with the lowest classical energy can be constructed. Then, we use the spherically symmetric ansatz for the spatial components of the gauge field and for the Higgs fields (see e.g. \([3]\))
\[
\vec{A}_{\theta} = -\vec{e}_\phi \frac{1 - K(r)}{er} , \quad \vec{A}_{\phi} = \vec{e}_\theta \frac{1 - K(r)}{er} \sin \theta , \quad \phi = v(H_1(r) \vec{e}_r + H_2(r) \lambda_8) ,
\]
with the standard unit vectors \( \vec{e}_r, \vec{e}_\theta \) and \( \vec{e}_\phi \). It is convenient to use the dimensionless variable \( x = evr \) and to define the mass function \( \mu(x) \) by means of \( N(x) = 1 - 2\mu(x)/x \).

With these ansatz and definitions, the classical equations of the Lagrangian \([1]\) reduce to a system of five differential equations. For the functions parametrising the metric we have
\[
\mu' = \alpha^2 \left( NK' + \frac{1}{2} Nx^2(H_1'^2 + H_2'^2) \right.
\]
\[
+ \left. \frac{(K^2 - 1)^2}{2x^2} + H_1'^2 K^2 + px^2(H_1^2 + H_2^2 - 1)^2 + qx^2(H_1^2(1 + 2H_2)^2 + (H_2 + H_1^2 - H_2^2)^2) \right) ,
\]
and
\[
A' = \alpha^2 x \left( \frac{2K'^2}{x^2} + H_1'^2 + H_2'^2 \right) A ,
\]
(the prime indicates the derivative with respect to \( x \)). For the functions related to the matter fields, we obtain
\[
(ANK')' = AK \left( \frac{K^2 - 1}{x^2} + H_1^2 \right) ,
\]
\[
(x^2 ANH_1')' = A \left( 2H_1 K^2 + x^2 \frac{\partial V}{\partial H_1} \right) ,
\]
and
\[
(x^2 ANH_2')' = A \left( x^2 \frac{\partial V}{\partial H_2} \right) .
\]
The solutions of these equations can be studied for different values of the physical parameters \( \alpha^2, p, q \). They are characterised, namely, by the mass \( \mu_\infty \) (the asymptotic value of the function \( \mu(x) \)) and by their inertial mass \( M = \mu_\infty/\alpha^2 \), which, after multiplication by \( 4\pi v/e \) gives the ADM mass.

The regularity of the solution at the origin (including the function \( N(x) \)), the integrability of the mass function \( \mu(x) \) and the requirement that the metric \([12]\) asymptotically approaches the Minkowski metric lead to the following set of boundary conditions
\[
\mu(0) = 0 , \quad K(0) = 1 , \quad H_1(0) = 0 , \quad H_2'(0) = 0
\]
\[
A(\infty) = 1 , \quad K(\infty) = 0 , \quad H_1(\infty) = \frac{\sqrt{3}}{2} , \quad H_2(\infty) = -\frac{1}{2}
\]
which complete the mathematical problem posed by Eqs. \([13]-[19]\). In fact, alternative boundary conditions for the matter functions \( K, H_1, H_2 \) are also possible \([3]\) but we will restrict ourselves to the ones above which, in the flat limit, corresponds to the monopole with the lowest energy.
In the absence of gravity (i.e. $\alpha = 0, N = A = 1$), the first two equations are trivial and the SU(3)-monopole solutions are recovered. They were studied numerically in some details in [7].

Setting $H_2 = 0, q = 0$ in Eqs. (15)-(18) (Eq.(19) is then trivial) leads to the equations of the SU(2) gravitating monopole. These were studied in details in [9, 10]. In particular, it was shown that the non-Abelian gravitating dyon bifurcates at some critical value $\alpha = \alpha_c$ into an extremal Schwarzschild solution with

$$N(x) = \frac{(x - \alpha)^2}{x^2}, \quad A(x) = 1$$  \hspace{1cm} (22)

$$K(x) = 0, \quad H(x) = 1$$  \hspace{1cm} (23)

defined on $x \in [\alpha, \infty]$. The dependence of the critical value $\alpha_c$ (with $a$ defined in Eq. [11]) on the parameter $b$ is presented in Fig. 1 by the line with the circles.

![Figure 1: The critical value $\alpha_c$ of the SU(2)-monopoles (bullets line) and of the SU(3)-monopoles (squares and triangles lines) for the small values of the parameter $b$.](image)

### 3 SU(2)-Gravitating monopole

Since the purpose of this paper is to compare the properties of the gravitating monopoles available in SU(2) and in SU(3), we first briefly recall how the magnetic monopole solutions approach the critical solution when the vector boson mass, i.e. $a$, is varied while the ratio of the Higgs boson mass to the vector boson mass, i.e. $b$, is kept fixed.

Recently, Lue and Weinberg [11] realized that there are two regimes of $b$, each with its own type of critical solution.

In the first regime, $b$ is small ($b < 25$) and the metric function $N(r)$ of the monopole solutions possesses a single minimum. As the critical solution is approached, i.e. as $a \to a_{cr}$, the minimum of the function $N(r)$ decreases until it reaches zero at $r = r_0$. The limiting solution corresponds to an extremal RN black hole solution with horizon radius $r_h = r_0$ and unit magnetic charge for $r \geq r_0$. Consequently, also the mass of the limiting solution coincides with the mass of this extremal RN black hole. However, the limiting solution is not singular in the interior region $r < r_0$. We will call this type of limiting approach the RN-type.

In the second regime, $b$ is large and the metric function $N(r)$ of the monopole solutions develops a second minimum as the critical solution is approached. This second minimum arises for a value $r = r*$ which is smaller...
than the first one at \( r = r_0 \). Keeping \( b \) fixed and increasing \( a \) one observes that the internal minimum \( N(r_*) \) decreases faster than the external one \( N(r_0) \). Therefore, the critical solution is reached for \( a = a_{cr} \) when the inner minimum reaches zero. This solution thus possesses an extremal horizon at \( r_* < r_0 \), and corresponds to an extremal black hole with non-abelian hair and a mass less than the extremal RN value. Consequently, we call this second type of limiting approach the NA-type (non-abelian-type).

To summarise: the non-Abelian gravitating monopoles exist on a portion of the \((a, b)\) plane limited by a curve \( a_{cr}(b) \). At a particular point \((a_{tr}, b_{tr})\) on this curve, the separation between the RN-type and NA-type of ending occurs. The value \( a_{tr} = 3/2 \) is determined algebraically in \([11]\), a numerical analysis \([12]\) indicates \( b_{tr} \sim 26.7 \).

4 SU(3)-Gravitating monopole

The numerical solutions of eqs. \([15, 19]\) strongly indicates that the bifurcation pattern of the gravitating monopoles in SU(3) is very similar to the one in SU(2). The analysis is rendered more involved by the presence of an additional equation and of a second parameter in the Higgs potential.

We first discuss the case when \( b \) is small. Fixing values for \( b, R \) and increasing \( \alpha \), the function \( N(x) \) develops a minimum, say \( N_m \) at \( x = x_m \), which becomes deeper and deeper. At some critical value \( \alpha_c \) we have

\[ N_m = 0 \quad , \quad \alpha_c = \mu_\infty = x_m \]  

characterising a bifurcation into an extremal RN black hole with an horizon \( x_h \) coinciding with \( x_m \). The limiting solution is given again by \([2]\), \( K(x) = 0 \), \( H_1(x) = \sqrt{3}/2 \), \( H_2(x) = -1/2 \).

This approach of the RN black hole by the gravitating monopole is illustrated in Fig. 2 for \( b = R = 1 \); in this case we find \( \alpha_c \approx 1.2483 \). The figure further illustrates the evolutions of the mass \( \mu_\infty \) of the solution and of the inertial mass \( M = \mu(\infty)/\alpha^2 \). Pushing the numerical analysis up to \( N_m \approx 0.0001 \), we have not gotten any evidence of a second branch of solutions. Backbending branches, which occur in the SU(2) case (see \([9]\) pp.365, Fig. 3) likely exist too in SU(3) for lower values of the parameters \( R \) and \( b \); we have not attempted to construct them.

Figure 2: The \( \alpha \) dependance of the mass \( M \) and of the minimal value \( N_m \) of \( N(x) \); \( N_m = 0 \) at \( \alpha \approx 1.2483 \).

The critical value \( \alpha_c \) (and correspondingly \( a_c \)) of course depends on \( b \) and \( R \). The dependence of \( a_c \) on the other parameters is illustrated in Fig. 1 for \( R = 0 \) and \( R = 1 \). For all values covered by Fig. 1 the gravitating monopole bifurcates into a extremal RN black hole. As far of the horizon \( x_h = x_m \) of the limiting monopole
solution is concerned, we see that, for fixed $b$, the SU(3) solution having the largest horizon and the highest inertial mass are those corresponding to the masslessness of the three supplementary Higgs particles ($M_1 = M_2 = M_3$ in (7)), i.e., to $q = 0$.

Again keeping $b$ fixed but decreasing $R$ we observed that the value $a_c(R)$ decreases from the value $a_c(0)$ (symbolised by the curve with the squares in Fig. 1) and quickly reaches a minimum for $R \approx 4.5$ (virtually overlapping the curve for $R = 1$ in Fig. 1). For fixed $b$, the value $a_c(R)$ varies only a little for $1 \leq R \leq 9$; however for $R > 9$ we observed that $a_c(R)$ raises up sensibly, as illustrated in Fig. 1 for $R = 14$. Due to the numerical difficulties we have not attempted to obtain the limiting curve $a_c(R = \infty)$.

Next we considered the case when $b$ is large and attempted to reproduce the phenomenon observed by Lue-Weinberg [11] (and described in the previous section) for SU(3)-gravitating monopoles. Choosing the values $R = 1$, $b = 90$ we found that an inner minimum appears for $\alpha \approx 0.929$ (while the outer minimum has $N_m \approx 0.0007$). Rapidly the inner minimum becomes lower than the outer one and it is attained for $x \approx 0.78$ while $\alpha$ approaches the critical value $\alpha_c \approx 0.9331$. Several profiles illustrating the evolution of $N(x)$ are presented in Fig. 3. For $R = 1$ and lower values of $b$, the pattern is, qualitatively and quantitatively, very similar to the one obtained in SU(2) and discussed at length in [11, 12]. However, choosing small values of the mass ratio $R$ (typically $R \sim 0.1$), we got no evidence of this type of phenomenon.

The numerical analysis of the equation is very lengthy and our goal is just to demonstrate that the "NA-type" of bifurcation is also present for groups larger than SU(2), we therefore have not attempted to refine the evaluation of $b_{tr}$ as a function of $R$.

5 Conclusion

The Einstein-Georgi-Glashow model constitutes a good theoretical laboratory for testing the properties of gravitational solitons. The set of solutions is particularly rich and contains patterns of bifurcations of several types [10, 11, 12]. In this paper we have shown that many properties of these solutions are present with the larger group SU(3), in particular gravitating monopoles can bifurcate into both Abelian and non-Abelian (hairy) black holes, according to the values of the coupling constants. This result suggests that these two types of critical phenomenon are not specific to SU(2) [11, 12] and occur in SU(N) and, in particular, in the "grand-unifying models".
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