MINIMUM POSITIVE ENTROPY OF COMPLEX ENRIQUES SURFACE AUTOMORPHISMS

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Abstract. We determine the minimum positive entropy of complex Enriques surface automorphisms. This together with McMullen’s work completes the determination of the minimum positive entropy of complex surface automorphisms in each class of Enriques-Kodaira classification of complex surfaces.

Contents

1. Introduction 1
2. Lattices 5
3. Twists 6
4. Glue 6
5. A new positivity criterion 11
6. Enriques surfaces and K3 surfaces 15
7. Enriques quadruple and realization conditions 17
8. Minimum entropy of Enriques surface automorphism 21
9. Ruling out Salem numbers 26
  9.1. Ruling out $\tau_3$ 26
  9.2. Ruling out $\tau_1$ 27
  9.3. Ruling out $\tau_6$ 29
  9.4. Ruling out $\tau_2$ 29
  9.5. Ruling out $\tau_7$ 31
  9.6. Ruling out $\tau_4$ 32
  9.7. Ruling out $\tau_5$ 34
Appendix A. Tables 39
References 39

1. Introduction

Throughout this paper, we work over the complex number field $\mathbb{C}$. The aim of this paper is to determine the minimum positive entropy of automorphisms of Enriques surfaces. This together with McMullen’s work [Mc07], [Mc11a], [Mc16] completes the problem to
determine the minimum positive entropy of compact Kähler surface automorphisms in each class of Enriques-Kodaira classification of complex surfaces (see [BHPV04] for basics on complex surfaces, and [Mc02a], [DS05] for basics on complex dynamics we shall use).

Let $X$ be a smooth compact Kähler surface and $f \in \text{Aut}(X)$ an automorphism of $X$. By the fundamental theorem of Gromov-Yomdin, the entropy $h(f)$ of $f$ is given by

$$h(f) = \log d_1(f) \geq 0.$$  

Here $d_1(f)(\geq 1)$ is the first dynamical degree of $f$, that is, the spectral radius of $f^*|H^2(X, \mathbb{C})$ when $f \in \text{Aut}(X)$, which coincides with the spectral radius of $f^*|\text{NS}(X)$ when $X$ is projective (see [ES13]). We call $f$ of positive entropy if $h(f) > 0$, i.e., if $d_1(f) > 1$.

If $X$ admits an automorphism $f$ of positive entropy, then $X$ is either a rational surface or bimeromorphic to one of the following surfaces: a K3 surface, a complex torus of dimension 2 or an Enriques surface. This important observation is due to Cantat ([Ca99]), which relates complex dynamics with algebraic geometry. In the last three cases, we may and will assume that the surface is minimal. This is because any bimeromorphic selfmap of a minimal surface with non-negative Kodaira dimension is a birational automorphism (see [BHPV04]) and the first dynamical degree is a birational invariant ([DS05]).

We call a real algebraic integer $\tau$ a Salem number if $\tau > 1$, conjugate to $1/\tau$ and all other conjugates lie on the unit circle $S^1$. The Salem polynomial of $\tau$ is the minimal monic polynomial $S(x) \in \mathbb{Z}[x]$ of $\tau$. The degree of $S(x)$, which we often call the degree of $\tau$, is an even integer. McMullen ([Mc02a] observed that $d_1(f)$ is a Salem number if $d_1(f) > 1$, i.e., if the entropy is positive. Furthermore, in [Mc07], McMullen also proved the following remarkable fact: if $f$ is of positive entropy, then

$$d_1(f) \geq \lambda_{10} \approx 1.17628,$$

where $\lambda_{10}$ is the Salem number whose Salem polynomial is

$$1 + x - x^3 - x^4 - x^5 - x^6 - x^7 + x^9 + x^{10}.$$

$\lambda_{10}$ is the smallest known Salem number called the Lehmer number. These two observations give unexpected relations between complex dynamics of surface automorphisms and number theory. Since then, relations between surface automorphisms and Salem numbers, such as realizability of Salem numbers as the first dynamical degree of surface automorphisms and the determination of the minimum Salem number obtained in this way in each class of Enriques-Kodaira classification, and so on, have been caught much attentions by many authors from various view points ([Mc07], [BK09], [Og10], [Mc11a], [Re11], [Re12], [Xie15], [Mc16], [BC16], [Ue16], [BG16], [EOY16], [Do17], [Sh17], [MOR17], [Yu18] and so on). Among many works, McMullen has also shown that there are a rational surface, a non-projective K3 surface and a projective K3 surface, with automorphism $f$ such that $d_1(f) = \lambda_{10}$ ([Mc07], [Mc11a], [Mc16]). For complex torus case, because of the degree reason ($10 > 6 = b_2(X)$, and also $10 > 4 \geq \text{rank NS}(X)$ when it is projective), there is a priori no automorphism such that $d_1(f) = \lambda_{10}$, while the minimum is determined for both projective and non-projective complex torus of dimension 2. They are the minimum Salem number $\lambda_4$ of degree 4 and the minimum Salem number $\lambda_6$ of degree 6 respectively ([Mc11a]). See also [Re11], [Re12] for more precise informations and Table 3 in Appendix for the list of the minimum Salem number $\lambda_{2d}$ in each degree $2d \leq 10$. 

Recall that a complex Enriques surface $S$ is a smooth compact complex surface whose universal cover, which is of degree 2, is a K3 surface. All Enriques surfaces are projective and they form ten dimensional moduli. Any Enriques surface admits a genus one fibration and its Jacobian fibration is a rational elliptic surface. So, Enriques surfaces are close to both K3 surfaces and rational surfaces. Also $b_2(S) = \rho(S) = 10$ for any Enriques surface. In spite of these facts, it has been shown that there is no Enriques surface automorphism $f$ such that $d_1(f) = \lambda_{10}$ (Og10). Since then, there are several works toward determination of the minimum Salem number realized as $d_1(f)$ of an Enriques surface automorphism $f$ ([Do17, Sh17, MOR17]). The current best record is due to Dolgachev [Do17], which is $d_1(f) = \lambda_{\text{Dol}} \approx 2.08101,$ where $\lambda_{\text{Dol}}$ is the Salem number whose Salem polynomial is $1 - x - 2x^2 - x^3 + x^4.$

Our main result is to show the following

**Theorem 1.1.** Let $\tau_8 \approx 1.58234$ be the Salem number whose Salem polynomial is

$$1 - x^2 - 2x^3 - x^4 + x^6.$$  

Then $\tau_8$ is the minimum Salem number which is realized as the first dynamical degree of an Enriques surface automorphism. That is,

$$d_1(f) \geq \tau_8 \approx 1.58234$$

for any Enriques surface automorphism $f$, and there are an Enriques surface $S$ and an automorphism $f \in \text{Aut} (S)$ such that $d_1(f) = \tau_8.$

**Remark 1.2.** The Salem number $\tau_8$ in Theorem 1.1 is the 4th smallest Salem number in degree 6. (See [Mos] for the list of small Salem numbers of small degrees.)

There are two issues to prove: (i) realizability of $\tau_8$ and (ii) unrealizability of the Salem numbers $\tau < \tau_8.$ Once we establish (i), it follows from a work of Matsumoto-Ohashi-Rams [MOR17] that $\tau$ in (ii) has to be one of seven Salem numbers $\tau_i \ (1 \leq i \leq 7)$ listed in Table 1 in Appendix.

As in [Mc16], our method for both (i) and (ii) is a lattice theoretic one being based on the Torelli theorem for the covering K3 surfaces and the automorphism (lifted to the covering K3), and twists and glues of lattices arising from the Salem polynomials and cyclotomic polynomials. In this approach, among other things, our new results, Theorem 5.6 and Theorem 7.3, are particularly important for us. They are also crucial to reduce the problem effectively to a computer algebra problem. We believe that these two theorems have their own interest and will be applicable for other problems.

Let us explain a bit more about these two theorems. As in the case of K3 surface automorphisms ([Mc16, BG16]), one of the essential points in geometric realization of an automorphism from an Hodge isometry of the K3 lattice is the preservation of the ample or Kähler cone. In lattice theoretic terms, this is the notion of positivity introduced by McMullen [Mc16] (see also Definition 5.1 for the precise definition). In general, it is very hard to check positivity. Theorem 5.6 is a new positivity criterion. Our statement of Theorem 5.6 is given in an equivalent form, so that it can be smoothly applied for both
realizability and unrealizability. Our proof of Theorem 5.6 is entirely free from computer algebra. However, our resulting formulation is the one which fits well with computer algebra (see Remark 5.8). Another new issue of realizability and unrealizability by an Enriques automorphism is to descend a candidate K3 surface automorphism to the original Enriques surface, i.e., commutativity of the covering involution and a candidate automorphism of the covering K3 surface. This makes our problem much more complicated than realizability or unrealizability by a K3 surface automorphism. To make this process clear and effective, we introduce a new notion, Enriques quadruple (Definition 7.1). This notion is described purely in terms of lattices and their isometries in which the information of a given Salem number is encoded. Theorem 7.4 shows that the realizability of a prescribed Salem number \( \tau \) as the first dynamical degree of an Enriques automorphism is equivalent to the existence of an Enriques quadruple with the same \( \tau \). Our proof of this theorem is again entirely free from computer algebra. However, again, our resulting formulation is the one which fits well with computer algebra.

We then use computer algebra to check the existence of Enriques quadruple with eight Salem numbers \( \tau_i \) (\( 1 \leq i \leq 8 \), see Table 1 in Appendix). This will be done in Sections 8 and 9. It turns out that \( \tau_i \) (\( 1 \leq i \leq 7 \)) are unrealizable (Section 9), while \( \tau_8 \) is realizable (Section 8). In this way, we complete the proof of Theorem 1.1. All computer algebra programs, which are based on \([MC11b]\), and the outputs needed in our proof are available from the second named author's home page \([Yu]\).

We conclude Introduction by posing some open questions closely related to our main result.

**Question 1.3.** Let \( S \) be an Enriques surface with automorphism of the minimal positive entropy \( \log \tau_8 \). Can one describe nicely a projective model of (some nice) \( S \) or a projective model of its covering K3 surface \( \tilde{S} \)?

In our construction, the transcendental lattice \( T_{\tilde{S}} \) of the covering K3 surface \( \tilde{S} \) is \( I_{2,2}(4) \) (Theorem 8.1). See \([MO15]\) for some explicit projective models of the covering K3 surfaces of Enriques surfaces with automorphisms of positive entropy.

**Question 1.4.** How about in positive characteristic \( p > 0 \)?

As our method is based on the Torelli theorem for complex K3 surfaces, it can not be applied to consider this question. See eg. \([EST13b, EO15, Xie15, BC16, BG16, Yu18]\) for some work related to Salem numbers and surface automorphisms in positive characteristics.

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2. Lattices

In this section, we recall some basics on lattices which we will use in our paper. Lemma 2.2 will be frequently used in the sequel.

A lattice $(L, \langle \cdot, \cdot \rangle)$ is a finite generated free $\mathbb{Z}$-module $L$, endowed with a $\mathbb{Z}$-valued symmetric bilinear form $(\langle \cdot, \cdot \rangle) = (\langle \cdot, \cdot \rangle)_L$. For brevity, we often denote $(x, x)$ by $x^2$. We call $L$ an even (resp. odd) lattice if $x^2 \in 2\mathbb{Z}$ for any $x \in L$ (resp. $x^2 \notin 2\mathbb{Z}$ for some $x \in L$). Let $(e_1, ..., e_n)$ be a $\mathbb{Z}$-basis of $L$. We call $\langle (e_i, e_j) \rangle_{1 \leq i, j \leq n}$ the Gram matrix of $L$ with respect to $(e_1, ..., e_n)$. The determinant $\det(L)$ of $L$ is defined to be the determinant of any Gram matrix of $L$. The lattice $L$ is non-degenerate if the symmetric bilinear form on $L$ is non-degenerate (equivalently, $\det(L) \neq 0$). For a sublattice $L' \subset L$, we say $L'$ is a primitive sublattice of $L$ if $(L' \otimes \mathbb{Q}) \cap L = L'$. If the signature of $L$, which we denote by $\text{sig} L$, is $(1, n-1)$ and $n > 1$, then $L$ is called a hyperbolic lattice. For a field $k$, we sometimes denote the $k$-linear space $L \otimes k$ by $L_k$. For a sublattice $M \subset L$ (resp. an element $x \in L$), we use $M_L^\perp$ (resp. $x_L^\perp$) to denote the orthogonal complement of $M$ (resp. $x$) in $L$ (we sometimes omit the subscript $L$ if there is no confusion).

For a nonzero $a \in \mathbb{Q}$, if $a(x, y)_L \in \mathbb{Z}$ for any $x, y \in L$, then the lattice $L(a)$ is defined to be the same $\mathbb{Z}$-module as $L$ with the form given by

$$(x, y)_{L(a)} := a(x, y)_L.$$ 

An element $x \in L$ is called a root if $x^2 = -2$. A lattice is called a root lattice if it is generated by roots. We use $A_k$ ($k \geq 1$), $D_l$ ($l \geq 4$), $E_m$ ($m = 6, 7, 8$) to denote the negative definite root lattice whose basis is given by the corresponding Dynkin diagram. We use $U$ (resp. $E_{10}$) to denote the unique even unimodular hyperbolic lattice of rank 2 (resp. rank 10). Let $r$ and $s$ be positive integers. We denote by $I_{r,s}$ (resp. $II_{r,s}$) the unique odd (resp. even) unimodular lattice of signature $(r,s)$ (See [Se73, Chapter V, Part I]).

For any isometry $f \in \text{O}(L)$, we denote the characteristic polynomial $\det(xI - f)$ by $\chi_f(x)$.

For any positive integer $k$, we denote the $k$-th cyclotomic polynomial by $\Phi_k(x)$.

**Definition 2.1.** Let $G$ be a finite abelian group. A quadratic form on $G$ is a map

$$q : G \to \mathbb{Q}/2\mathbb{Z}$$

together with a symmetric bilinear form

$$b : G \times G \to \mathbb{Q}/\mathbb{Z}$$

such that:

1) $q(nx) = n^2 q(x)$ for all $n \in \mathbb{Z}$ and $x \in G$, and

2) $q(x + x') - q(x) - q(x') \equiv 2b(x, x') \mod 2\mathbb{Z}$ for all $x, x' \in G.$

Note that, a quadratic form $q$ on $G$ is uniquely determined by its restriction to the Sylow subgroups $G_p$ of $G$ (see [Ni80, Proposition 1.2.2]).

The length of $G$, denoted by $l(G)$, is the minimum number of generators of $G$.

Let $L$ be a non-degenerate even lattice. The bilinear form of $L$ determines a canonical embedding $L \to L^* = \text{Hom}(L, \mathbb{Z})$, and we may view $L^*$ as a subset of $L \otimes \mathbb{Q}$. The quotient group $G(L) := L^*/L$ is finite abelian, and we call $G(L)$ the glue group of $L$, following
For any \( x \in L^* \), we use \( \pi \) to denote the image of \( x \) in \( L^*/L \) under the natural projection. The \((\mathbb{Q}\text{-valued})\) bilinear form on \( L^* \) induced by \((\ast,\ast)_L\) gives a bilinear form \( b_L \) and a quadratic form \( q_L \) on \( G(L) \) as follows

\[
b_L : G(L) \times G(L) \rightarrow \mathbb{Q}/\mathbb{Z}, \quad b_L(\pi,\pi) \equiv (x,y) \mod \mathbb{Z},
\]

and

\[
q_L : G(L) \rightarrow \mathbb{Q}/2\mathbb{Z}, \quad q_L(\pi) \equiv (x,x) \mod 2\mathbb{Z}.
\]

We call \( q_L \) the discriminant form of \( L \). For any prime \( p \), we use \( q_{L,p} \) to denote the restriction of \( q_L \) to the Sylow \( p \)-subgroup of \( G(L)_p \). Existence of an even lattice with given discriminant form and signature is characterized by [Ni80, Theorem 1.10.1].

If the glue group \( G(L) \) is a \( p \)-elementary abelian group for some prime \( p \), then we say \( L \) is a \( p \)-elementary lattice (See [RS89] for classification.)

The following lemma tells us that if the Sylow \( p \)-subgroup of the glue group of an even lattice is \( p \)-elementary of maximal length, then the lattice comes from a “simpler” even lattice.

**Lemma 2.2.** Let \( L \) be a non-degenerate even lattice of rank \( n \), and let \( p \) be a prime number. Suppose \( G(L)_p \cong \mathbb{F}_p^n \) (if \( p = 2 \), we require \( b_L(x,x) = 0 \in \mathbb{Q}/\mathbb{Z} \) for any \( x \in G(L)_2 \cong \mathbb{F}_2^n \)). Then

1) \( \frac{1}{p}L \subset L^* \);
2) \( L(\frac{1}{p}) \) is a well-defined non-degenerate even lattice.

**Proof.** 1) Since \( G(L)_p \cong \mathbb{F}_p^n \), then there exists a subgroup \( M \subset L^* \) such that \( M/L = G(L)_p \). Since \( pm \in L \) for any \( m \in M \), \( M \subset \frac{1}{p}L \). On the other hand, \((\frac{1}{p}L)/L \cong \mathbb{F}_p^n \). Thus, \( M = \frac{1}{p}L \).

2) Since \( \frac{1}{p}L \subset L^* \), it follows that \( \frac{1}{p}(x,y) \in \mathbb{Z} \) for any \( x,y \in L \), and hence \( L(\frac{1}{p}) \) is a well-defined lattice. Since \( L \) is non-degenerate, \( L(\frac{1}{p}) \) is also non-degenerate.

If \( p \geq 3 \). For any \( x \in L(\frac{1}{p}) \), since \( (x,x)_L = p(x,x)_{L(\frac{1}{p})} \) and \( (x,x)_L \) is even, it follows that \( (x,x)_{L(\frac{1}{p})} \) is also even. Thus, \( L(1/p) \) is an even lattice.

If \( p = 2 \) and \( b_L(x,x) = 0 \in \mathbb{Q}/\mathbb{Z} \) for any \( x \in G(L)_2 \cong \mathbb{F}_2^n \). For any \( y \in L(\frac{1}{2}) \),

\[
(y,y)_{L(\frac{1}{2})} = \frac{1}{2}(y,y)_L = 2\left(\frac{y}{2}\right)^2 \equiv \left(\frac{y}{2}\right)^2 \mod \mathbb{Z}.
\]

Since \( b_L(\frac{y}{2},\frac{y}{2}) = 0 \in \mathbb{Q}/\mathbb{Z} \), it follows that \( \left(\frac{y}{2}\right)^2 \) is an integer. Thus, \( (y,y)_{L(\frac{1}{2})} = 2\left(\frac{y}{2}\right)^2 \) is an even integer. Thus, \( L(\frac{1}{2}) \) is an even lattice. \( \square \)

### 3. Twists

In this section, following [Mc16], we discuss lattice automorphisms canonically associated to irreducible reciprocal polynomials. Theorem 3.2 below is a generalization of [Mc16, Theorem 5.2]. This generalization will be used to show unrealizability of \( \tau_d \) (which is pseudo-simple but not simple) in Section 9.

Let \( P(x) \in \mathbb{Z}[x] \) be a monic irreducible reciprocal polynomial of even degree \( d = 2m \). A \( P(x) \)-lattice is a pair \((L,f)\) of a non-degenerate lattice \( L \) and an isometry \( f \in O(L) \) such that the characteristic polynomial \( \chi_f(x) \) of \( f \) is equal to \( P(x) \).
Let \((L, f)\) be a \(P(x)\)-lattice. For any nonzero \(a \in \mathbb{Z}[f + f^{-1}]\), the new bilinear form
\[(v_1, v_2)_a = (av_1, v_2)\]
defines the new \(P(x)\)-lattice \((L(a), f)\), and we call \((L(a), f)\) the twist of \((L, f)\) by \(a\).

Let \(K\) be the number field \(\mathbb{Q}[x]/(P(x))\), and let \(R(x)\) be the trace polynomial of \(P(x)\), i.e., \(R(x) \in \mathbb{Z}[x]\) is the monic polynomial such that \(P(x) = x^mR(x + x^{-1})\). We define \(k = \mathbb{Q}[y]/(R(y))\). Then \(k\) is a subfield of \(K\) such that \([K : k] = 2\) under the natural inclusion \(k \subset K\) given by \(y = x + x^{-1}\). In particular, the extension \(k \subset K\) is Galois under \(\iota: x \mapsto x^{-1}\).

The principal \(P(x)\)-lattice \((L_0, f_0)\) is defined by
\[L_0 = \mathbb{Z}[x]/(P(x)) \subset K = \mathbb{Q}[x]/(P(x))\]
with the bilinear form
\[(g_1, g_2)_{L_0} = \sum_{i=1}^{d} \frac{g_1(x_i)g_2(x_i^{-1})}{R'(x_i + x_i^{-1})},\]
where \((x_i)_1^d\) are the roots of \(P(x)\) and \(R'\) denotes the formal derivative of \(R(x)\). The action \(f_0 \in O(L_0)\) is defined by multiplication by \(x\). Then \(L_0\) is an even lattice with \(\det(L_0) = |P(-1)P(1)|\).

As in \([Mc16]\), we say \(P(x)\) is simple if the class number of the number field \(K\) is 1, \(O_K = \mathbb{Z}[x]/P(x)\), and \(|P(-1)P(1)|\) is square free. All small Salem numbers in \([Mc16, Table 1]\) are simple. In the way of determining minimum positive entropy of automorphisms of Enriques surfaces, it turns out that we need to consider Salem numbers which are not simple. We say \(P(x)\) is pseudo-simple if the class number of the number field \(K\) is 1, \(O_K = \mathbb{Z}[x]/P(x)\), and there exists a \(P(x)\)-lattice \((L', f')\) such that \(|\det(L')|\) is square free. Thus, \(P(x)\) is pseudo-simple if it is simple.

**Example 3.1.** The polynomial \(x^2 + 1\) is pseudo-simple but not simple, and every \((x^2 + 1)\)-lattice is isomorphic to a twist of \((L, f)\) where
\[L = (\mathbb{Z}e_1 \oplus \mathbb{Z}e_2, ((e_i, e_j)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}), \quad f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.\]
This example tells us that \(L\) can be an odd lattice.

Our main result of this section is the following

**Theorem 3.2.** Let \(P(x)\) be a pseudo-simple monic irreducible reciprocal polynomial. Let \((L, f)\) be a \(P(x)\)-lattice such that \(|\det(L)|\) is square free. Then every \(P(x)\)-lattice is isomorphic to a twist \((L(a), f)\) of \((L, f)\), where \(a \in \mathbb{Z}[f + f^{-1}]\).

**Proof.** The inner product on the \(P(x)\)-lattice \((L, f)\) determines an isomorphism
\[L \cong bL^* \subset L^*\]
for some \(b \in O_K\) satisfying
\[N^K_Q(b) = \det(L).\]
By the assumption on \(L\), this norm is square-free.
Let \((L', f')\) be another \(P(x)\)-lattice. Since \(O_K \cong \mathbb{Z}[x]/P(x)\) is a PID, \((L', f') \cong (L, f)\) as \(O_K\)-modules. Then the inner product of \(L'\) is of the form
\[
(g_1, g_2)_{L'} = (ag_1, g_2)_L
\]
for some element \(a \in k\) (see [GMc02, Page 276, Remark]). Since \(a \in L^* \cong \frac{1}{2}L\), it follows that \(a \in b^{-1}O_K\).

We claim a \(\in O_k\), the ring of algebraic integers in \(k\). In fact, we may write \(aO_k = IJ^{-1}\), where \(I\) and \(J\) are relatively prime ideals in \(O_k\). We can also write \(a = c/d\), where \(c\) and \(d\) are relatively prime elements of \(O_K\). Then \(dO_K = JO_K\), and hence
\[
|N^K_C(d)| = |N^K_C(J)^2|.
\]
Since \(d|b\) and \(N^K_C(J)^2\) is square free, it follows that \(d \in O^*_K\) and hence \(a \in k \cap O_K = O_k\).

Note that \(O_K = \mathbb{Z}[f]\) and therefore
\[
O_k = \mathbb{Z}[f^i] = \mathbb{Z}[f + f^{-1}]
\]
by definition of \(i\). Thus, \(a \in \mathbb{Z}[f + f^{-1}]\). \(\square\)

**Remark 3.3.** Let \(\tau_1 < \cdots < \tau_8\) be the eight Salem numbers \(#1 - #8\) in [MOR17, Appendix] (See also Table 1 in Appendix). Then it turns out that \(\tau_i\) is simple (resp. pseudo-simple but not simple) for \(i = 1, 2, 3, 5, 6, 7\) (resp. \(i = 4, 8\)), as one can verify using computer algebra.

We close this section by recalling the notions of feasible prime and Salem factor from [Mc16] and their relations with isometries of the lattice \(E_{10}\) (instead of the K3 lattice \(\Pi_{3,19}\)).

Let \(\tau\) be a Salem number with Salem polynomial \(S(x)\) of degree 2d. We are interested in the conditions for realizability of \(\tau\) by isometries of \(E_{10}\), which is important in our study of Enriques quadruple (See Section 7).

Let \(p \in \mathbb{Z}\) be a prime. We say \(p\) is a **feasible prime** for \(S(x)\) if
\[
p|N = \prod_{\phi(k) \leq 10 - 2d} \text{res}(S(x), \Phi_k(x)).
\]

The difference between this definition and that in [Mc16] comes from the fact \(\text{rk}(E_{10}) = 10\), while \(\text{rk}(\Pi_{3,19}) = 22\).

For any positive integer \(n\), we use \(D(n)\) to denote the minimum \(D \geq 0\) such that \(\mathbb{Z}^D\) admits an automorphism of order \(n\). It satisfies \(D(1) = 0\), \(D(2) = 1\), and \(D(n) = D(n/2)\) if \(n > 2\) is even but \(n/2\) is odd.

Let \(g \in O(E_{10})\) such that the spectral radius of \(g\) is \(\tau\). Then \(\chi_g(x) = S(x)C(x)\) for some product \(C(x)\) of cyclotomic polynomials (see for instance [EOY16, Proposition 3.1]). Let
\[
L := \text{Ker}(S(g)) \subset E_{10}
\]
We call \(g|L\) the **Salem factor** of \(g\).

**Theorem 3.4.** ([Mc16, Theorem 6.2]) Let \(f : L \rightarrow L\) be the Salem factor for an isometry of \(E_{10}\) such that \(\chi_f(x) = S(x)\). Then:

1) The integer \(|G(L)|\) is divisible only by the feasible primes for \(S(x)\);
2) The order \(n\) of the natural map \(f|G(L)\) induced by \(f\) satisfies
\[
D(n) \leq 10 - \deg(S(x))
\]
3) There exists a product of distinct cyclotomic polynomials $C(x)$ such that

$$C(\bar{f}|G(L)) = 0, \quad \deg(C(x)) \leq 10 - \deg(S(x)).$$

4. Glue

In this section, we discuss gluing of lattices and isometries, and controlling of glue groups via resultants. We refer to [Mc16, Section 4] for more details. Our main result of this section is Theorem 4.6.

Let $L_i$ ($i = 1, 2$) be non-degenerate lattices. Let $H_i$ be a subgroup of $G(L_i)$. We say a map $\phi : H_1 \rightarrow H_2$ is a gluing map if 1) $\phi$ is an isomorphism of abelian groups, 2) $b_{L_1}(x, y) = -b_{L_2}(\phi(x), \phi(y))$ for any $x, y \in H_1$.

For any gluing map $\phi : H_1 \rightarrow H_2$, we define the lattice $L_1 \oplus_\phi L_2$ by

$$L_1 \oplus_\phi L_2 := \{(x, y) \in L_1^* \oplus L_2^* \mid \exists \overline{\tau} \in H_1, \overline{\sigma} \in H_2, \text{ and } \phi(\overline{\tau}) = \overline{\sigma}\} \subset (L_1 \oplus L_2) \otimes \mathbb{Q}.$$}

Clearly $L_1 \oplus L_2$ is a sublattice of $L_1 \oplus_\phi L_2$, and $L_i$ is a primitive sublattice of $L_1 \oplus_\phi L_2$, $i = 1, 2$. Moreover,

$$\det(L_1 \det(L_2) = \det(L_1 \oplus_\phi L_2)|H_1|^2.$$}

For a lattice $L \supset L_1 \oplus L_2$ of rank $\rk(L_1) + \rk(L_2)$, we say $L$ is a primitive extension of $L_1$ and $L_2$ if both $L_1$ and $L_2$ are primitive sublattices of $L$. Any primitive extension of $L_1$ and $L_2$ appears as $L_1 \oplus_\phi L_2$, in other words, any primitive extension of $L_1$ and $L_2$ can be obtained by gluing $L_1$ and $L_2$ via a gluing map.

Any isometry $f : L_1 \rightarrow L_2$ of lattices naturally induces an isomorphism of glue groups:

$$\overline{f} : G(L_1) \rightarrow G(L_2).$$

Moreover, $b_{L_1}(x, y) = b_{L_2}(\overline{f}(x), \overline{f}(y))$ for any $x, y \in G(L_1)$.

Let $f_i \in O(L_i)$, $i = 1, 2$. If a gluing map $\phi : H_1 \rightarrow H_2$ satisfies

$$\overline{f}_i(H_i) = H_i \quad (i = 1, 2) \quad \text{and} \quad \phi \circ \overline{f}_1 = \overline{f}_2 \circ \phi,$$

then $f_1 \oplus f_2$ can be naturally extended to an isometry $f_1 \oplus_\phi f_2 \in O(L_1 \oplus_\phi L_2)$. Conversely, for any primitive extension $L$ of $L_1$ and $L_2$, if an isometry $f \in O(L)$ satisfies $f(L_i) = L_i$, $i = 1, 2$, then $f$ must appear as $f_1 \oplus_\phi f_2$.

The following lemma characterizes the Sylow $p$-subgroup (for certain primes $p$) of the glue group and the discriminant-form of the lattice obtained by gluing two isometries.

**Lemma 4.1.** Let $p$ be a prime number. Let $f \in O(L)$ be an isometry of a non-degenerate lattice $L$. Let $L_1$ be a primitive $f$-stable non-degenerate sublattice. We set

$$L_2 := L_1^1 \subset L, \quad f_i := f|L_i \quad (i = 1, 2).$$

Suppose that

$$p \nmid \res(\chi_{f_1}(x), \chi_{f_2}(x)),$$

i.e., the resultant of the two polynomials $\chi_{f_1}(x)$ and $\chi_{f_2}(x)$ is not divided by $p$. Then there exists an isomorphism of abelian groups

$$\phi : G(L_1)_p \oplus G(L_2)_p \rightarrow G(L)_p$$
such that
\[ \tilde{f} \circ \phi = \phi \circ (f_1 \oplus f_2) \]
and \( \phi \) is an isomorphism between the quadratic forms \( q_{L_1,p} \oplus q_{L_2,p} \) and \( q_{L,p} \).

**Proof.** By the assumptions, \( L \) is a primitive extension of \( L_1 \) and \( L_2 \), and \( f(L_i) = L_i, i = 1, 2 \). Thus, there exists a gluing map \( \psi : H_1 \to H_2 \), for some \( H_i \subset G(L_i), i = 1, 2 \), such that \( L = L_1 \oplus \psi L_2 \) and \( f = f_1 \oplus \psi f_2 \). Since \( p \nmid |H_1| \), by \([Mc16\text{ Proposition 4.2}]\), \( p \nmid |H_1| \). Then the map
\[ \phi : G(L_1)_p \oplus G(L_2)_p \to G(L)_p \]
given by \( \phi(x_1, x_2) = x_1 + x_2 \) is well-defined, where \( x_i \in L_i \) satisfying \( x_i \in G(L_i)_p, i = 1, 2 \). Then \( \tilde{f} \circ \phi = \phi \circ (f_1 \oplus f_2) \), and
\[ q_{L_1,p}(x_1) + q_{L_2,p}(x_2) = q_{L,p}(\phi(x_1, x_2)) \]
for any \( \phi \in G(L_i)_p \). Since \( p \nmid |H_1| \), it follows that \( |G(L)_p| = |G(L_1)_p| \cdot |G(L_2)_p| \) and \( \phi \) is an isomorphism. \( \square \)

In the process of ruling out Salem numbers in Section 4.2, we often face the following problem: for an isometry \( f \in \text{O}(U \oplus E_{10}(2)) \) of finite order with characteristic polynomial \( C_1(x)C_2(x) \), where \( C_1(x) \) and \( C_2 \) are coprime polynomials in \( \mathbb{Z}[x] \), what can we say about invariants (e.g., glue group, signature) of the sublattice \( \text{Ker}(C_i(f)) \), \( i = 1, 2 \)? Motivated by this, we consider the following

**Set-up 4.2.** Let \( L_i \) (\( i = 1, 2 \)) be a non-degenerate lattice of rank \( r_i \), and let \( f_i \in \text{O}(L_i) \) (\( i = 1, 2 \)) be an isometry of finite order \( n_i \) such that \( n_1 \) and \( n_2 \) are coprime. Suppose \( H_i \subset G(L_i) \) is a subgroup satisfying \( f_i(H_i) = H_i \), and suppose there is a gluing map \( \phi : H_1 \to H_2 \) such that the isometry \( f_1 \oplus f_2 \) extends to \( f_1 \oplus \phi f_2 \in \text{O}(L_1 \oplus \phi L_2) \).

**Lemma 4.3.** Under Set-up 4.2, \( \tilde{f}_i|H_i = \text{id}_{H_i}, (i = 1, 2) \).

**Proof.** Since \( \phi \) is an isomorphism and \( \tilde{f}_2 \circ \phi = \phi \circ \tilde{f}_1 \), it follows that \( \text{Ord}(\tilde{f}_1|H_1) = \text{Ord}(\tilde{f}_2|H_2) \). On the other hand, \( \text{Ord}(f_i|H_i) \) divide \( n_i \). Since \( n_1 \) and \( n_2 \) are coprime, \( f_i|H_i = \text{id}_{H_i} \). \( \square \)

**Lemma 4.4.** Under Set-up 4.2, we suppose \( \chi_{f_1}(x) = (\Phi_{p^m}(x))^k \), where \( p \) is a prime and \( m, k > 0 \). Then \( H_i, i = 1, 2 \) is a \( p \)-elementary abelian group.

**Proof.** First we consider the case \( m = 1 \). Since \( \text{Ord}(f_1) < \infty \) and \( \chi_{f_1}(x) = (\Phi_p(x))^k \), it follows that
\[ f_1^{p-1} + ... + f_1 + \text{id}_{L_1} = 0. \]
Then
\[ f_1^{p-1}|H_1 + ... + f_1|H_1 + \text{id}_{H_1} = 0. \]
By Lemma 4.3 \( \tilde{f}_1|H_1 = \text{id}_{H_1} \). Thus, \( \text{pid}_{H_1} = 0 \), and \( H_1 (\cong H_2) \) is a \( p \)-elementary abelian group.

If \( m > 1 \), replacing \( f_i \) by \( f_i^{p^{m-1}} \), one reduces to the case \( m = 1 \). \( \square \)

**Lemma 4.5.** Under Set-up 4.2, we suppose \( n_1 \in \{5, 7, 9\} \). Assume that \( \chi_{f_1}(x) = \Phi_{n_1}(x) \). Then \( |H_1| = |H_2| = 1 \) or \( p \), where \( p \) is the unique prime factor of \( n_1 \), i.e., \( p \) is 5, 7, 3 for \( n_1 = 5, 7, 9 \) respectively.
In this section, we give a new criterion for positivity (Theorem 5.6). As mentioned in
Introduction, Theorem 5.6 and Remark 5.8 are crucial in our proof of Theorem 1.1.

Let $p$ be the unique prime factor of $n_1$. Then, by an easy computation, one obtain that
$$H = \left(\frac{c}{5} + \frac{2f_1'(e)}{5} + \frac{3f_1''(e)}{5} + \frac{4f_1'''(e)}{5}\right) \cong \mathbb{F}_5.$$ 

On the other hand, by Lemmas 4.3, 4.4 and by our assumption, $H_1$ is isomorphic to a
subgroup of $H$. Thus, $|H_2| = |H_1| = 1$ or 5. This completes the proof for the case
$n_1 = 5$. $\square$

The following theorem will play an important role in ruling out Salem numbers in Section 9. We will use this theorem to control the transcendental lattice of the covering K3 surface of an Enriques surface with an automorphism of a given entropy.

**Theorem 4.6.** Under Set-up 4.2, we suppose $n_1 \in \{5, 7, 9\}$,
$$\chi_{f_1}(x) = \Phi_{n_1}(x), \quad L_1 \oplus \phi \mathcal{L}_2 \cong U \oplus E_{10}(2).$$

Let $p$ be the unique prime factor of $n_1$. Then
$$G(L_1) = \mathbb{F}_2^k \oplus \mathbb{F}_p, \quad G(L_2) = \mathbb{F}_2^{10-k} \oplus \mathbb{F}_p$$
for some $k \leq r_1$. Moreover, if $k = r_1$ and the signature of $L_1$ is $(2, r_1 - 2)$, then $L_2$ has
roots.

**Proof.** Suppose $n_1 = 5$. Since $\Phi_5(x)$ is simple, by [Mc16, Theorem 5.2], $L_1$ is a twist of the
principal $\Phi_5(x)$-lattice whose glue group is of order $|\Phi_5(1)\Phi_5(-1)| = 5$. Thus, $\det(L_1)$ is
divided by 5. Since $L_1$ and $L_2$ glue to $U \oplus E_{10}(2)$ via $\phi : H_1 \rightarrow H_2$, it follows that
$$\det(L_1) \det(L_2) = \det(U \oplus E_{10}(2)) \cdot |H_1|^2 = 2^{10} |H_1|^2.$$ 

Thus, by Lemma 4.5, $|H_1| = |H_2| = 5$. Then by Lemma 4.4, $G(L_1) \cong \mathbb{F}_2^k \oplus \mathbb{F}_5$ for some
$k \leq r_1 = 4$, $G(L_2) \cong \mathbb{F}_2^{10-k} \oplus \mathbb{F}_5$. If $k = 4$ and the signature of $L_1$ is $(2, 2)$, by Lemma 2.2, $L_1(1/2)$ is a well-defined even lattice of determinant 5 and signature $(2, 2)$. Thus, by
classification, $L_1(1/2)$ (and hence $L_1$) is uniquely determined (see [RS89], [CS99]). Then the
discriminant-form $q_{L_2}$ of $L_2$ is uniquely determined (cf. [Ni80, Theorem 1.11.3]). By
computer algebra (Magma), there are exactly two non-isometric even lattices $S_i, i = 1, 2$
of signature $(0, 8)$, discriminant-form $q_{S_i} \cong q_{L_2}$, and both of $S_1$ and $S_2$ have roots. This
completes the proof for the case $n_1 = 5$.

Suppose $n_1 = 7$ or 9. Then, similar to the case $n_1 = 5$, we can prove that $G(L_1) = \mathbb{F}_2^k \oplus \mathbb{F}_p$,$G(L_2) = \mathbb{F}_2^{10-k} \oplus \mathbb{F}_p$, for some $k \leq r_1 = 6$. If $k = 6$ and the signature of $L_1$ is $(2, 4)$, then $L_2$
is a negative definite even lattice of determinant $2^4p$ and rank 6, where $p = 7$ or 3. Thus,$L_2$ has roots by [Mc16, Page 3] (and no need of computer algebra in these two cases). $\square$

5. A NEW POSITIVITY CRITERION

In this section, we give a new criterion for positivity (Theorem 5.6). As mentioned in
Introduction, Theorems 5.6 and Remark 5.8 are crucial in our proof of Theorem 1.1.
Let \((L, (*, **))\) be an even hyperbolic lattice of signature \((1, n)\) (note that a hyperbolic lattice in \([Mc16]\) is of signature \((n, 1)\)). The positive cone \(\mathcal{P}\) of \(L\) is defined to be one of the two connected components of
\[
\{ x \in L \otimes \mathbb{R} \mid x^2 > 0 \}.
\]
Let \(O^+(L) \subset O(L)\) be the subgroup consisting of isometries which preserve \(\mathcal{P}\). The positive cone \(\mathcal{P}\) is cut into chambers by the set of all root hyperplanes
\[
r^\perp := \{ x \in L \otimes \mathbb{R} \mid \langle x, r \rangle = 0 \},
\]
where \(r \in L\) and \(r^2 = -2\). Each chamber is a fundamental domain of the Weyl group \(W(L) \subset O^+(L)\) which is generated by the reflections
\[
s_r : x \mapsto x + \langle x, r \rangle r
\]
corresponding to the roots \(r \in L\).

**Definition 5.1.** Let \(f \in O^+(L)\). We say \(f\) is positive if there exists a chamber \(\mathcal{M} \subset \mathcal{P}\) such that \(f(\mathcal{M}) = \mathcal{M}\).

**Example 5.2.** For any automorphism of a complex projective K3 surface, the induced isometry of the Picard lattice is positive since it preserves the ample cone of the surface.

In this geometric setting, we define the positive cone to be the connected component containing the ample cone.

Positivity of isometries of hyperbolic lattices is a subtle condition (see \([Mc16, BG16]\)).

**Definition 5.3.** A root \(r \in L\) is called an obstructing root if there is no \(\phi \in \text{Hom}(L \otimes \mathbb{R}, \mathbb{R})\) such that \(\text{Ker}(\phi)\) is negative definite and \(\phi(f^i(r)) > 0\) for all \(i \in \mathbb{Z}\).

The following result is a characterization of positivity in terms of obstructing roots.

**Theorem 5.4.** ([Mc16, Theorem 2.2]) An isometry \(f \in O^+(L)\) is positive if and only if \(f\) has no obstructing roots.

As a consequence of Theorem 5.4, the following lemma is useful for ruling out Salem numbers in Section 9.

**Lemma 5.5.** Let \(f \in O^+(L)\) be positive. Suppose \(L'\) is a hyperbolic sublattice of \(L\) such that \(f(L') = L'\). Then \(f|L' \in O^+(L')\) is positive.

**Proof.** Since \(f \in O^+(L)\) and \(L'\) is hyperbolic, it follows that \(f|L' \in O^+(L')\). Suppose \(f|L'\) is not positive. By Theorem 5.4 there exists an obstructing root \(r \in L'\) for \(f|L'\). Since \(f \in O^+(L)\) is positive, \(r\) is not an obstructing root of \(f\). Thus, there is \(\phi \in \text{Hom}(L \otimes \mathbb{R}, \mathbb{R})\) such that \(\text{ker}(\phi)\) is negative definite and \(\phi(f^i(r)) > 0\) for all \(i \in \mathbb{Z}\). Then \(\text{ker}(\phi|L'_R) = \text{ker}(\phi) \cap L'_R\) is negative definite, and \((\phi|L'_R)((f|L')^i(r))) > 0\) for all \(i \in \mathbb{Z}\), which means that \(r\) is not an obstructing root of \(f|L'\), a contradiction. Thus, \(f|L'\) is positive. \(\square\)

Let \(f \in O^+(L)\) be of spectral radius \(\tau > 1\). Then the characteristic polynomial of \(f\) can be written as \(\chi_f(x) = C(x)S(x)\), where \(C(x) \in \mathbb{Z}[x]\) is a product of cyclotomic polynomials and \(S(x) \in \mathbb{Z}[x]\) is a Salem polynomial (see for instance [EOY16, Section 3]). In particular, \(\tau\) must be a Salem number, and both \(\tau\) and \(\tau^{-1}\) are eigenvalues of \(f\) with multiplicity one. Let
\[
v, w \in L \otimes \mathbb{R}
\]
such that
\[ f(v) = \tau v, \quad f(w) = \tau^{-1}w. \]
Replacing \( v \) by \( -v \) if necessary, we may and will assume that \((v, w) > 0\).

Let \( h \in L \) such that \( h^2 > 0 \). We set
\[ C := \{ r \in L \mid r^2 = -2, \text{ and } r + f(r) + \ldots + f^i(r) = 0 \text{ for some } i \geq 1 \}; \]
\[ R_h := \{ r \in L \mid r^2 = -2 \text{ and } (r, h) = 0 \}, \]
\[ S_h := \{ r \in L \mid r^2 = -2 \text{ and } (r, h)(r, f(h)) < 0 \}. \]
A root in \( C \) is called a cyclic root. Cyclic roots are obstructing roots.

**Theorem 5.6.** Let \( f \in \mathcal{O}^+(L) \) be of spectral radius \( \tau > 1 \). Let \( h \in L \) such that \( h^2 > 0 \).

Then
1) The three sets \( C, R_h, S_h \) are finite sets.
2) \( f \) is positive if and only if both of the following two conditions are satisfied:
   i) \( C \) is empty,
   ii) \((r, v)(r, w) \geq 0\), for all \( r \in R_h \cup S_h \).

**Proof.** 1) As pointed above, we can write \( \chi(x) = C(x)S(x) \), where \( S(x) \) is the minimal polynomial of the Salem number \( \tau \). We can write \( C(x) = (x - 1)^kC_0(x) \) for some \( k \geq 0 \) such that \( C_0(x) \in \mathbb{Z}[x] \) is not divided by \( x - 1 \). Then \( C \) consists exactly of roots in \( \ker(C_0(f)) \). Since \( \ker(S(f)) \) is hyperbolic, it follows that \( \ker(C_0(f)) \) is negative definite. Thus, \( C \) is finite.

Since \( L \) is hyperbolic and \( h^2 > 0 \), it follows that the orthogonal complement \( h^\perp \subset L \) is negative definite. Thus, \( R_h \subset h^\perp \) is finite.

Next we show finiteness of \( S_h \). Let
\[ A := \{(a, b) \mid a = (r, h) \text{ and } b = (r, f(h)) \text{ for some } r \in S_h \}. \]

**Claim 5.7.** \( A \) is a finite set.

**Proof.** Let \( r \in S_h \). To simplify notation, we let \( x = (h, h), y = (h, f(h)), a = (h, r) \), and \( b = (f(h), r) \). We set \( B = \begin{pmatrix} x & y & a \\ y & x & b \\ a & b & -2 \end{pmatrix} \). Then the determinant of \( B \) is
\[ -2x^2 + 2y^2 + 2aby - xa^2 - xb^2. \]
Since \( L \) is hyperbolic and the three elements \( h, f(h), r \) generate a sublattice of \( L \), it follows that this determinant is greater than or equal to 0. Thus,
\[ (5.1) \quad -2x^2 + 2y^2 + 2aby \geq x(a^2 + b^2). \]
Note that \( x > 0, y > 0 \) (since \( f \in \mathcal{O}^+(L) \)), \( ab < 0 \) (since \( r \in S_h \)). Then \( 2aby < 0 \), and the inequality \((5.1)\) implies that both \( a \) and \( b \) are bounded (for fixed \( x \) and \( y \)). Thus, \( A \) is finite. This completes the proof of the claim. \( \square \)

For any \((a, b) \in A\), we set
\[ S_h^{(a, b)} := \{ r \in S_h \mid (r, h) = a, (r, f(h)) = b \}. \]
Then \( S_h^{(a,b)} \subset (−bh + af(h))^⊥ \). Since \((−bh + af(h))^2 > 0 \) by \( ab < 0 \), it follows that \((−bh + af(h))^⊥ \) is negative definite. Thus \( S_h^{(a,b)} \) is finite. Then \( S_h = \cup_{(a,b) \in A} S_h^{(a,b)} \) is also finite.

2) Suppose \( f \) is positive. Then \( f \) cannot have cyclic roots, i.e., \( C \) is empty. Let \( M \) be the \( f \)-invariant chamber. Since \((v, w) > 0\), we may assume both of \( v \) and \( w \) are contained in the closure of \( M \) by the Birkhoff-Perron-Frobenius theorem [Bi67]. Then \((r, v)(r, w) ≥ 0\) for any root \( r \in L \). Thus, the condition ii) is true.

Suppose both of the two conditions i) and ii) are satisfied. Let

\[ T_h = \{ r ∈ L | r^2 = −2, \text{ and } r ≠ f^k(r') \text{ for any } k ∈ Z, r' ∈ R_h ∪ S_h \}, \]

i.e., \( T_h \) consists of the roots which does not belong to any \( f \)-orbit of roots in \( R_h ∪ S_h \).

Let \( r ∈ T_h \). Then \( f^k(r) \) is not in \( R_h ∪ S_h \) for any \( k ∈ Z \). Thus, \((f^k(r), h)(f^{k−1}(r), h) > 0\) for any \( k \). Then either \((h, f^k(r)) > 0\) for all \( k \), or \((−h, f^k(r)) > 0\) for all \( k \). Thus, \( r \) is not an obstructing root.

Let \( r ∈ R_h ∪ S_h \). There are two possibilities: a) at least one of \((r, v)\) and \((r, w)\) is nonzero, b) both \((r, v)\) and \((r, w)\) are zero. In case a), by condition ii), interchanging \( v \) and \( w \) if necessary, we may assume \((r, v) > 0\) and \((r, w) ≥ 0\), then \((v + w, f^k(r)) > 0\) for all \( k ∈ Z \). Since \((v + w, v + w) > 0\), it follows that \((v + w)^⊥ ⊂ L ⊗ R \) is negative definite. Thus, \( r \) is not an obstructing root. In case b), \((v + w, f^k(r)) = 0\) for all \( k ∈ Z \). Since \((v + w)^⊥ \) is negative definite, it follows that \( f \)-orbit of \( r \) is a finite set. Then there exists \( m > 0 \) such that \( f^m(r) = r \). Let

\[ α = r + f(r) + ... + f^{m−1}(r). \]

Then \( f(α) = α, α ≠ 0 \) (since, by condition i), \( r \) is not a cyclic root), and \((α, α) < 0\). Since \((f^k(r), α) = (r, α)\) for any \( k \), it follows that \((r, α) = \frac{(α, α)}{m} < 0\). By \((v + w, α) = 0\), it follows that

\[ (N(v + w) − α, N(v + w) − α) > 0 \]

for sufficiently large \( N > 0 \). Then \( r \) is not an obstructing root since \((N(v +w) −α, f^k(r)) > 0\) for all \( k ∈ Z \).

Note that a root is an obstructing root if and only if some member of its \( f \)-orbit is an obstructing root. Therefore, we have proved that if the two conditions i) and ii) are satisfied, then \( f \) has no obstructing roots and, by Theorem 5.4, \( f \) is positive. This completes the proof of the theorem.

\[ \square \]

Remark 5.8. i) For practical purposes, Theorem 5.6 is easy to apply. In fact, firstly, it is often easy to find a vector \( h \) with positive self-intersection number in a given hyperbolic lattice \( L \). Then, using computer algebra, one can easily compute the finite sets \( C, R_h, S_h \) for any explicitly given isometry \( f ∈ O^+(L) \) with spectral radius greater than 1. The two sets \( C \) and \( R_h \) are easy to find. In order to find \( S_h \) one can first compute

\[ \mathcal{A}' := \{(a, b) ∈ Z × Z| −2x^2 + 2y^2 + 2aby ≥ x(a^2 + b^2), a > 0, b < 0\} \]

where \( x = (h, h), y = (h, f(h)) \). As explained in the proof of Claim 5.7, \( \mathcal{A}' \) is a finite set (note that, for any \((a, b) ∈ A\), either \((a, b) ∈ \mathcal{A}'\) or \((−a, −b) ∈ \mathcal{A}'\). The crucial point is the following: the elements of \( \mathcal{A}' \) and \( A \) can be easily found out by computer algebra. Then

\[ S_h = (\cup_{(a,b) ∈ \mathcal{A}'} \{r ∈ L | r^2 = −2, (r, −bh + af(h)) = 0\}) \setminus R_h. \]
After the three sets \( \mathcal{C}, \mathcal{R}_h \) and \( \mathcal{S}_h \) are computed, the two conditions i) and ii) in Theorem 5.6 is easy to check again by computer algebra.

ii) Let \( L \) be a hyperbolic lattice and let \( f \in \mathcal{O}^+(L) \) (here no need to assume \( f \) of spectral radius \( > 1 \)). Let \( h \in L \) such that \( h^2 > 0 \). Clearly, if both \( \mathcal{R}_h \) and \( \mathcal{S}_h \) are empty, then the chamber containing \( h \) is \( f \)-stable and \( f \) is positive.

6. Enriques Surfaces and K3 Surfaces

In this section, based on close relation between Enriques surfaces and K3 surfaces, we establish two constraints for automorphisms of Enriques surfaces (Lemmas 6.1 and 6.3).

Let \( Y \) be an Enriques surface and let \( X \) be the universal cover of \( Y \). Then there exists a fixed point free involution \( \sigma : X \to X \) such that \( X/\sigma = Y \). Let \( \pi : X \to Y \) denote the natural quotient map. To simplify notation, we use \( L \) to denote \( H^2(X,\mathbb{Z}) \). The isometry \( \sigma^* \in \mathcal{O}(L) \) induced by \( \sigma \) is of order 2, and we set

\[
L^+ := \{ \alpha \in L | \sigma^*(\alpha) = \alpha \}, \quad L^- := \{ \alpha \in L | \sigma^*(\alpha) = -\alpha \}.
\]

Then the lattice \( L \) is a primitive extension of \( L^+ \) and \( L^- \), and

\[
L^+ \cong E_{10}(2), \quad L^- \cong U \oplus E_{10}(2),
\]

cf. [BP83]. Let \( H^2(Y,\mathbb{Z})_f \) denote the free part of \( H^2(Y,\mathbb{Z}) \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2\mathbb{Z} \). Then, \( H^2(Y,\mathbb{Z})_f \cong U \oplus E_8 \), and \( \pi^*(H^2(Y,\mathbb{Z})_f) = L^+ \).

Lemma 6.1. Let \( x \in \text{NS}(X) \). If \( x \in (L^+)\perp \), then \( (x,x) \neq -2 \).

Proof. Since \( Y \) is projective, \( H^2(Y,\mathbb{Z})_f \) contains an ample class, say \( h \). Then \( \pi^*(h) \in L^+ \) is also ample since \( \pi \) is a finite map.

If \( x \in \text{NS}(X) \) and \( x^2 = -2 \), then by Riemann-Roch theorem, either \( x \) or \( -x \) is effective. Then either \( (x,\pi^*h) > 0 \) or \( (-x,\pi^*h) > 0 \). Thus, \( (x,\pi^*h) \neq 0 \), a contradiction to \( x \in (L^+)\perp \).

Any automorphism \( g \in \text{Aut}(Y) \) lifts (in two ways) to an automorphism \( \hat{g} \in \text{Aut}(X) \) commuting with \( \sigma \). Thus, if we set

\[
\text{Aut}(X,\sigma) := \{ f \in \text{Aut}(X) | f \circ \sigma = \sigma \circ f \},
\]
then \( \text{Aut}(Y) = \text{Aut}(X,\sigma)/\{\text{id},\sigma\} \). Since \( \hat{g}^*\sigma^* = \sigma^*\hat{g}^* \), both \( L^+ \) and \( L^- \) are \( \hat{g}^* \)-stable. We want to understand the relation between the characteristic polynomials of \( \hat{g}^*|L^+ \) and \( \hat{g}^*|L^- \).

Lemma 6.2. Let \( f \in \text{O}(U \oplus E_{10}(2)) \). Then

\[
\chi_f(x) \equiv (1 + x)^2 \chi_f(x) \mod 2.
\]

Proof. Let \( \alpha_1, \alpha_2 \) be a basis of \( U \) such that \( \alpha_1^2 = \alpha_2^2 = 0, (\alpha_1, \alpha_2) = 1 \). Let \( e_1, \ldots, e_{10} \) be a basis of \( E_{10} \) (hence also a basis of \( E_{10}(2) \)). Then \( (\alpha_1, \alpha_2, e_1, \ldots, e_{10}) \) is a basis of \( U \oplus E_{10}(2) \) and \( (\frac{\alpha_1}{2}, \frac{\alpha_2}{2}, \frac{e_1}{2}, \ldots, \frac{e_{10}}{2}) \) is a basis of \( G(U \oplus E_{10}(2)) \). We may write

\[
f(\alpha_1, \alpha_2, e_1, \ldots, e_{10}) = (\alpha_1, \alpha_2, e_1, \ldots, e_{10})A
\]

where

\[
A = \left( \begin{array}{cc} B & H \\ K & P \end{array} \right), B = \left( \begin{array}{cc} a & b \\ c & d \end{array} \right),
\]
By [BP83, Proposition (3.2)], \( \hat{\alpha} \).

By Lemma 6.2, \( \alpha, \beta \).

The isometry \( \hat{\alpha} \) is even for all \( i, j \). Then \( h_{ij} \) even for all \( 1 \leq i, j \leq 10 \).

Let \( p(x) = \det(xI - P) \), \( b(x) = \det(xI - B) \). Then

\[ (f(e_i), \alpha_j) = (e_i, f^{-1}(\alpha_j)) \]

is even for all \( i, j \). Then \( h_{ij} \) even for all \( 1 \leq i, j \leq 10 \).

Let \( p(x) = \det(xI - P) \), \( b(x) = \det(xI - B) \). Then

\[ \chi_f(x) \equiv b(x)p(x) \mod 2. \]

Note that

\[ \bar{f} = (\frac{e_1}{2}, ..., \frac{e_{10}}{2}) = (\frac{e_1}{2}, ..., \frac{e_{10}}{2}) \]

where \( \bar{P} \) denote mod 2 reduction of \( P \). Then

\[ \chi_{\bar{f}}(x) \equiv \det(xI - \bar{P}) \equiv p(x) \mod 2. \]

Thus,

\[ \chi_f(x) \equiv b(x)\chi_{\bar{f}}(x) \mod 2. \]

Note that

\[ 1 = (\alpha_1, \alpha_2) = (f(\alpha_1), f(\alpha_2)) = (a\alpha_1 + c\alpha_2 + \alpha, b\alpha_1 + d\alpha_2 + \beta) = ad + bc + (\alpha, \beta) \]

for some \( \alpha, \beta \in E_{10}(2) \). Since \( (\alpha, \beta) \) is even, \( ad + bc \equiv 1 \mod 2 \). Note that

\[ 0 = (f(\alpha_1), f(\alpha_1)) = 2ac + (\alpha, \alpha) \]

and \( (\alpha, \alpha) \in 4Z \). Thus, \( ac \) is even. Similarly, \( bd \) is even too.

Note that \( b(x) = x^2 - (a + d)x + ad - bc \). Since \( ad + bc \equiv 1 \mod 2 \) and both \( ac \) and \( bd \) are even, it follows that \( a + d \) is even. Thus,

\[ b(x) \equiv (1 + x)^2 \mod 2. \]

Then

\[ \chi_f(x) \equiv (1 + x)^2 \chi_{\bar{f}}(x) \mod 2. \]

This completes the proof of the lemma.

The relationship between \( \chi_{g^*|L^-}(x) \) and \( \chi_{g^*|L^+}(x) \) in the following lemma is important for us since it reduces the number of isometries of \( L^\pm \) which we need to consider to determine whether a given Salem number can be realized by automorphisms of Enriques surfaces.

**Lemma 6.3.** The isometry \( g^*|L^-(x) \in O(L^-) \) is of finite order, and

\[ \chi_{g^*|L^-}(x) \equiv (1 + x)^2 \chi_{g^*|L^+}(x) \equiv (1 + x)^2 \chi_{g^*}(x) \mod 2. \]

**Proof.** By [BP83, Proposition (3.2)], \( g^*|L^- \) is of finite order. Since \( L^- \) and \( L^+ \) are orthogonal to each other in the unimodular lattice \( L \), it follows that

\[ \chi_{g^*|L^-}(x) = \chi_{g^*|L^+}(x). \]

By Lemma 6.2,

\[ \chi_{g^*|L^-}(x) \equiv (1 + x)^2 \chi_{g^*|L^-}(x) \mod 2 \]

Note that

\[ \chi_{g^*|L^+}(x) \equiv \chi_{g^*|L^+}(x) \equiv \chi_{g^*}(x) \mod 2. \]
Thus,
\[\chi_{g^*L^+}(x) \equiv (1 + x)^2 \chi_{g^*L^+}(x) \mod 2.\]
This completes the proof of the Lemma. \(\square\)

7. **Enriques quadruple and realization conditions**

In this section, we introduce the notion of Enriques quadruple (Definition 7.1), and reduce realization problem to purely lattice theoretical problem in term of this notion (Theorem 7.4). This reduction is crucial in our proof of the main theorem.

**Definition 7.1.** Let \(L^+\) and \(L^-\) be two lattices isometric to \(E_{10}(2)\) and \(U \oplus E_{10}(2)\) respectively. Let \(f^+ \in O(L^+), f^- \in O(L^-)\), let \(T \subset L^-\) be a primitive sublattice, and let \(\phi : G(L^-) \to G(L^+)\) be a gluing map.

We say the 4-tuple \((f^+, f^-, T, \phi)\) an **Enriques quadruple** if all of the following eight conditions are satisfied:

1) the spectral radius of \(f^+\) is a Salem number \(\tau\),
2) \(\chi_{f^{-}}(x) \equiv (1 + x)^2 \chi_{f^+}(x) \mod 2,\)
3) \(f^-\) is of finite order,
4) the signature of \(T\) is \((2, r)\), where \(r \geq 0,\)
5) \(f^-(T) = T\) and the minimal polynomial of \(f^-|T\) is irreducible,
6) \(T_{L^+}^\perp\) has no roots,
7) \(L^- \oplus_\phi L^+ \simeq \Pi_{3,19}\) and \(f^- \oplus f^+\) extends to \(f^- \oplus_\phi f^+ \in O(L^- \oplus_\phi L^+)\),
8) there exists \(h \in L^+\) such that:
   i) \((h, h) > 0,\)
   ii) \(h_{L^+}^\perp \oplus_\phi L^+\) has no roots, and
   iii) \(h\) and \((f^- \oplus_\phi f^+)(h)\) are in the same chamber of \(T_{L^+ \oplus_\phi L^+}^\perp.\)

The **entropy** of an Enriques quadruple is defined to be the entropy of \(f^+,\) i.e., \(\log \tau.\)

**Remark 7.2.** Condition 2) follows from condition 7) (cf. Lemma 6.3), and clearly condition 6) follows from condition 8). However, we include conditions 2) and 6) in Definition 7.1 as we will frequently use them later.

We need the following lemma in our proof of Theorem 7.4.

**Lemma 7.3.** Let \(T\) be a lattice of signature \((2, r),\) where \(0 \leq r \leq 10.\) Let \(f \in O(T)\) be an isometry of finite order such that the minimal polynomial of \(f\) is irreducible. Then \(T^\perp_\mathbb{R}\) contains an \(f\)-invariant plane \(P\) such that \(P\) has signature \((2, 0), f|P \in SO(P),\) and \(P_{T^\perp}^\perp \cap T = 0.\)

**Proof.** By the assumption on \(f,\) the characteristic polynomial \(\chi_f(x) = \Phi_k^\phi(x),\) where \(k = \text{ord}(f).\) The lemma is true when \(k = 1, 2\) (see [Og02, Lemma 2.13]). In fact, we may choose a \(\mathbb{Q}\)-basis \(e_1, e_2, \ldots, e_{2+r}\) of \(T^\perp_\mathbb{Q}\) such that \((e_i, e_i) > 0\) for \(i = 1, 2, (e_i, e_i) < 0\) for \(i = 3, \ldots, 2 + r,\) and \((e_i, e_j) = 0\) for \(i \neq j.\) Let \(t \in \mathbb{R}\) be a transcendental number such that \(t > 1.\) Let \(N > r\) be a sufficient large integer such that

\[v_1 := t^N e_1 + te_3 + \ldots + t^r e_{2+r}\]
satisfying \( v_1^2 > 0 \). Let \( P \subset T_{\mathbb{R}} \) be the plane generated by \( v_1, e_2 \). Then \( P \) is of signature \((2,0)\) and \( P_{\mathbb{R}} \cap T = 0 \).

From now on, we assume \( k > 2 \).

Suppose \( n = 1 \). Then \( \chi_f(x) = \Phi_k(x) \), and \( \mathbb{Z}[x]/\Phi_k(x) \) is a PID (here we use the fact \( \deg(\Phi_k(x)) = 2 + r \leq 12 \)). \( T \) is a free \( \mathbb{Z}[x]/\Phi_k(x) \)-module of rank 1. Note that \( \text{rk}(T) = 2 + r = 2m \) for some positive integer \( m \). Thus, the roots of \( \Phi_k(x) \) are of the form \( \xi_1, \xi_2, \ldots, \xi_m, \xi_m \). Then we have the following decomposition

\[
(7.1) \quad T_C = V(\xi_1) \oplus V(\xi_1) \oplus \cdots \oplus V(\xi_m) \oplus V(\xi_m) = (V(\xi_1) \oplus V(\xi_1)) \perp \cdots \perp (V(\xi_m) \oplus V(\xi_m)),
\]

where \( V(\xi_i) \) (resp. \( V(\xi_i) \)) denotes the one-dimensional eigen space of \( f|T \) with respect to \( \xi_i \) (resp. \( \xi_i \)). For any \( i \), choose a nonzero \( v_i \in V(\xi_i) \). Then \( \overline{v_i} \in V(\xi_i) \). We write \( v_i = x_i + \sqrt{-1}y_i \), where \( x_i, y_i \in T_{\mathbb{R}} \). By

\[
(v_i, v_i) = (f(v_i), f(v_i)) = (\xi_i v_i, \xi_i v_i) = \xi_i^2 (v_i, v_i)
\]

and \( \xi_i^2 \neq 1 \), we have \( (v_i, v_i) = 0 \). Similarly, \( (\overline{v_i}, \overline{v_i}) = 0 \). Thus, \( (x_i, x_i) = (y_i, y_i) \) and \( (x_i, y_i) = 0 \). Let \( a_i := (x_i, x_i) \). Then \( (v_i, \overline{v_i}) = 2a_i \). Thus, the intersection matrix on \( V(\xi_i) \oplus V(\xi_i) \) with respect to the basis \( x_i, y_i \) is \( \begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix} \). Since \( T \) is of signature \((2, r)\), it follows that there exists a unique \( i \), say 1, such that \( a_i > 0 \). Then we choose \( P := \mathbb{R}(x_1, y_1) \).

Note that \( P \otimes \mathbb{C} = V(\xi_1) \oplus V(\xi_1) \). By \( f(v_1) = \xi_1 v_1, f(\overline{v_1}) = \overline{\xi_1 \overline{v_1}} = 1 \), we have \( f(P) = P \) and \( f|P \in \text{SO}(P) \). For any \( x \in P_{\mathbb{R}} \cap T \), we have \( (v_1, x) = 0 \) and \( (\overline{v_1}, x) = 0 \).

Since the Galois group \( \text{Gal}(\overline{\mathbb{Q}}(\xi_1))/\overline{\mathbb{Q}} \) acts on \( \{\xi_1, \xi_2, \ldots, \xi_m, \xi_m\} \) transitively, it follows that \( (v_1, x) = (\overline{v_1}, x) = 0 \) for any \( i \). Thus, \( x = 0 \) since \( T \) is a non-degenerate lattice.

Suppose \( n > 1 \). Let \( s = \deg(\Phi_k(x)) \). We choose any \( v \in T \) of positive norm. Let \( L_1 \subset T \) be the sublattice generated by \( v, f(v), \ldots, f^{s-1}(v) \). By \( \text{Ord}(f|T) = k \) and the minimal polynomial of \( f \) is \( \Phi_k(x) \), it follows that \( f(L_1) = L_1 \) and \( \chi_f|L_1(x) = \Phi_k(x) \). Then by considering decomposition as in \((7.1)\), we deduce that \( L_1 \) is of signature \((2, s-2)\). Then \( (L_1)_{\mathbb{R}} \) is negative definite. We choose any nonzero \( v_2 \in (L_1)_{\mathbb{R}} \), and let \( L_2 \subset T \) be the sublattice generated by \( v_2, f(v_2), \ldots, f^{s-1}(v_2) \). Then \( L_2 \) is a negative definite lattice such that \( f(L_2) = L_2 \) and \( \chi_f|L_2(x) = \Phi_k(x) \). Repeating this process, we obtain sublattices \( L_i, i = 1, 2, \ldots, n \) such that the following conditions 1) - 4) are satisfied:

1) \( L_i \perp L_j \) for \( i \neq j \), 2) \( f(L_i) = L_i \) for any \( i \),
3) \( \text{sig}(L_1) = (2, s-2) \) and \( \text{sig}(L_i) = (0, s) \) for \( i > 1 \) and,
4) \( \chi_f|L_i(x) = \Phi_k(x) \) for any \( i \).

Then \( L_1 \oplus \cdots \oplus L_m \) is a sublattice of \( T \) of finite index. Then there exists a root, say \( \xi \), of \( \Phi_k(x) \) such that 1) \( f(w_1) = \xi w_1 \) for some nonzero \( w_1 \in (L_1)_{\mathbb{Q}(\xi)} \) and 2) \( \mathbb{R}(x_1, y_1) \) is of signature \((2,0)\), where \( w_1 = x_1 + \sqrt{-1}y_1 \). For any \( i \geq 2 \), we choose a nonzero \( w_i = x_i + \sqrt{-1}y_i \in (L_i)_{\mathbb{Q}(\xi)} \) such that \( f(w_i) = \xi w_i \). Let \( t \in \mathbb{R} \) be any transcendental number such that \( t > 1 \). For sufficiently large integer \( N > n \), the plane

\[
P := \mathbb{R}(t^N x_1 + t x_2 + \cdots + t^{n-1} x_n, t^N y_1 + t y_2 + \cdots + t^{n-1} y_n) \subset T_{\mathbb{R}}
\]

is of signature \((2,0)\). Then, one can verify that \( f(P) = P \), \( f|P \in \text{SO}(P) \), and \( P_{\mathbb{R}} \cap T = 0 \).

This completes the proof of the lemma. \( \square \)

The main result of this section is the following:
Theorem 7.4. A Salem number \( \tau \) can be realized by an automorphism of an Enriques surface if and only if there exists an Enriques quadruple of entropy \( \log \tau \).

Proof. Suppose \( \tau \) can be realized by an automorphism \( g : Y \rightarrow Y \) of an Enriques surface \( Y \). Let \( \sigma : X \rightarrow X \) be the fixed point free involution of the covering K3 surface \( X \) such that \( X/\sigma = Y \). Let \( \pi : X \rightarrow Y \) be the natural quotient map. Let \( T_X \) and \( \omega_X \in T_X \otimes \mathbb{C} \) denote the transcendental lattice and a nonzero holomorphic two form on \( X \) respectively. Let \( \hat{g} \in \text{Aut}(X) \) denote a lift of \( g \). Recall

\[
H^2(X,\mathbb{Z})^{\sigma^*} \cong E_{10}(2), \quad (H^2(X,\mathbb{Z})^{\sigma^*})^\perp \cong U \oplus E_{10}(2)
\]

see (6.1). Note that the even unimodular lattice \( H^2(X,\mathbb{Z}) \) is a primitive extension of \( H^2(X,\mathbb{Z})^{\sigma^*} \) and \( (H^2(X,\mathbb{Z})^{\sigma^*})^\perp \), and both \( H^2(X,\mathbb{Z})^{\sigma^*} \) and \( (H^2(X,\mathbb{Z})^{\sigma^*})^\perp \) are \( \hat{g}^* \)-stable. Thus, there exists a gluing map

\[
\phi : G((H^2(X,\mathbb{Z})^{\sigma^*})^\perp) \rightarrow G(H^2(X,\mathbb{Z})^{\sigma^*})
\]

such that

\[
(H^2(X,\mathbb{Z})^{\sigma^*})^\perp \oplus_\phi H^2(X,\mathbb{Z})^{\sigma^*} = H^2(X,\mathbb{Z}) \cong \Pi_{3,19}
\]

and \( \hat{g}^* = \hat{g}^*(H^2(X,\mathbb{Z})^{\sigma^*})^\perp \oplus_\phi \hat{g}^*(H^2(X,\mathbb{Z})^{\sigma^*})^\perp \). To simplify notations, we set

\[
f^+ := \hat{g}^*(H^2(X,\mathbb{Z})^{\sigma^*}), \quad f^- := \hat{g}^*(H^2(X,\mathbb{Z})^{\sigma^*})^\perp.
\]

Since the entropy of \( g \) is \( \log \tau \), the entropy of \( f^+ \) is also \( \log \tau \).

By Lemma (6.3) \( f^- \) is of finite order and

\[
\chi_{f^-}(x) \equiv (1 + x)^2 \chi_{f^+}(x) \mod 2.
\]

Since \( T_X \) is \( \hat{g}^* \)-stable, \( T_X \) is also \( f^- \)-stable. Let \( \omega_X \) be a nonzero holomorphic 2-form on \( X \). Since \( T_X \) is the unique minimal sublattice of \( H^2(X,\mathbb{Z}) \) such that

\[
\mathbb{C}[\omega_X] \subset T_X \otimes \mathbb{C},
\]

the minimal polynomial of \( f^-|T_X \) is irreducible. By Lemma (6.1) the orthogonal complement \( N \) to \( T_X \) in \( (H^2(X,\mathbb{Z})^{\sigma^*})^\perp \) has no roots. Note that \( f^- \oplus_\phi f^+ = \hat{g}^* \) preserves the ample cone, and \( H^2(X,\mathbb{Z})^{\sigma^*} \) contains an ample class, say \( h \). Thus, \( h \) and \( (f^- \oplus_\phi f^+)(h) \) are in the same chamber of \( \text{NS}(X) = (T_X)_{H^2(X,\mathbb{Z})} \). Then the 4-tuple \( (f^+, f^-, T_X, \phi) \) is an Enriques quadruple of entropy \( \log \tau \). This completes the proof of \( "\text{only if }" \) part of the theorem.

Suppose \( (f^+, f^-, T, \phi) \) is an Enriques quadruple of entropy \( \log \tau \). By the three conditions (3)-5), we can apply Lemma (7.3) to our \( T \). Hence, \( T_\mathbb{R} \) contains an \( f^- \)-invariant plane \( P \) such that \( P \) has signature \( (2,0) \), \( f^-|P \in \text{SO}(P) \), and \( P_{T_\mathbb{R}} \cap T = 0 \). Take an orthonormal basis \( u, v \) of \( P \). Let \( \omega = u + \sqrt{-1}v \). Then \( (\omega, \omega) = 0 \) and \( (\omega, \omega) > 0 \), and \( \omega \) is an eigenvector of \( f^- \). Note that \( \omega \in (L^- \oplus_\phi L^+) \otimes \mathbb{C} \) and \( L^- \oplus_\phi L^+ \cong \Pi_{3,19} \). Thus, by surjectivity of Period mapping for complex K3 surfaces, there exist a complex K3 surface \( X \), a nonzero holomorphic two form \( \omega_X \) on \( X \), and an isometry

\[
F : H^2(X,\mathbb{Z}) \rightarrow L^- \oplus_\phi L^+
\]

such that \( F(\omega_X) = \omega \). To simplify notations, we identify \( H^2(X,\mathbb{Z}) \) with \( L^- \oplus_\phi L^+ \) via \( F \). By the choice of \( P \), the sublattice \( T \) is the minimal primitive sublattice of \( L^- \oplus_\phi L^+ \) containing \( \omega \) after tensoring with \( \mathbb{C} \). Thus,

\[
T_X = T, \quad \text{NS}(X) = T_{L^- \oplus_\phi L^+}^\perp,
\]
where $T_X$ and $\text{NS}(X)$ denote the transcendental lattice and Néron-Severi lattice of $X$ respectively.

Choose $h \in L^+$ in the condition 8) of Definition 7.1. Then there exists $w \in W(T^\perp_{L^- \oplus \phi}L^+)$ such that $w(h)$ is an ample class of $X$, where $W(T^\perp_{L^- \oplus \phi}L^+)$ is the Weyl group of $T^\perp_{L^- \oplus \phi}L^+$. Here we use the fact that the ample cone of a projective K3 surface is the fundamental domain of the action on the positive cone by the Weyl group. Let

$$\hat{f} := w \circ (f^- \oplus \phi f^+) \circ w^{-1}, \quad \hat{\sigma} := w \circ (-\text{id}_{L^-} \oplus \phi \text{id}_{L^+}) \circ w^{-1}.$$

Then $\hat{f}(w(h)) = w((f^- \oplus \phi f^+)(h))$ and $\hat{\sigma}(w(h)) = w(h)$ are ample classes of $X$. Note that $\hat{f} \hat{\sigma} = \hat{\sigma} \hat{f}$. Then, by global Torelli Theorem, there exist automorphisms $f, \sigma \in \text{Aut}(X)$ such that

$$\hat{f} = f^*, \quad \hat{\sigma} = \sigma^*, \quad f \sigma = \sigma f.$$

Note that $(L^- \oplus \phi L^+)^\sigma = L^+$. Thus, $\text{rk}((L^- \oplus \phi L^+)^\sigma) = 10$ and $l(G((L^- \oplus \phi L^+)^\sigma)) = 10$. Then by [Ni83, Page 1425], the fixed point locus $X^\sigma = \emptyset$ or $X^\sigma = C$, where $C$ is an elliptic curve. If $X^\sigma = C$, for $f \sigma = \sigma f$, we have $f^*(C) = C$, hence $f$ has zero-entropy by [Og07, Theorem 1.4 (1)], a contradiction to the condition 1) of Definition 7.1. Then $\sigma$ is fixed point free, and $f$ descends to an automorphism of the Enriques surface $X/\sigma$ of entropy $\tau$. This completes the proof of the theorem. \hfill \Box

**Remark 7.5.** Any automorphism $\varphi$ of an Enriques surface $S$ admits exactly two liftings, say $\psi_1, \psi_2$, to the covering K3 surface $\tilde{S}$. Moreover, $\psi_1 = \psi_2 \sigma$, where $\sigma$ is the fixed-point involution of $\tilde{S}$ such that $S = \tilde{S}/\sigma$. In fact, from the proof of the theorem, clearly both $(f^+, f^-, T_X, \phi)$ and $(f^+, -f^-, T_X, \phi)$ are Enriques quadruples if one of them is an Enriques quadruple.

We conclude this section with the following two lemmas which will be used later.

**Lemma 7.6.** Let $Y$ be a K3 surface such that $T_Y \cong U \oplus U(2)$. Then any automorphism of $Y$ is of zero entropy.

**Proof.** Since $Y$ is a 2-elementary K3 surface, $Y$ has a unique automorphism $\theta$ such that $\theta^*|T_Y = -\text{id}_{T_Y}$ and $\theta^*|\text{NS}(Y) = \text{id}_{\text{NS}(Y)}$. By [Ni83, Theorem 4.2.2], the fixed locus of $\theta$ is disjoint union of a smooth elliptic curve $C$ and eight smooth rational curves. Let $\varphi$ be any automorphism of $Y$. Since $\theta$ is in the center of $\text{Aut}(Y)$, $\varphi(C) = C$. Then by [Og07, Theorem 1.4 (1)], $\varphi$ is of zero entropy. \hfill \Box

**Lemma 7.7.** Let $f \in O(L^-)$ be an isometry of finite order such that 1) there exists a $f$-stable primitive sublattice $N \subset L^-$ satisfying $N \cong E_8(2)$ and 2) $T$ is isomorphic to $U \oplus U(2)$, where $T := N^\perp \subset L^-$. Let $g \in O(L^+)$ be an isometry with spectral radius $>1$. Then there exists no gluing map $\phi : G(L^-) \to G(L^+)$ such that both of the following two statements are true

i) the map $f \oplus g$ extends to $L^- \oplus \phi L^+ \cong \mathbb{II}_{3,19}$, and

ii) the restriction of $f \oplus \phi g$ to $N \oplus \phi L^+ \subset L^- \oplus \phi L^+$ is positive.

**Proof.** Suppose otherwise, i.e., there exists a gluing map $\phi : G(L^-) \to G(L^+)$ satisfying both i) and ii).

Clearly, we can choose a sufficiently large $n$ such that both $f^n|L^-$ and $g^n|G(L^+)$ are identity maps. By ii), the restriction of $f^n \oplus \phi g^n$ to $N \oplus \phi L^+$ is positive. By Torelli
Theorem and subjectivity of Period mapping, there exist an automorphism $F : X \rightarrow X$ of a K3 surface $X$ and an isometry $\Phi : H^2(X, \mathbb{Z}) \rightarrow L^- \oplus_\phi L^+$ such that

a) $\Phi \circ F^* = (f^m \oplus_\phi g^n) \circ \Phi$, and

b) $\Phi(T_X) = T$, where $T_X$ denotes the transcendental lattice of $X$.

Thus, $F$ is of positive entropy, which contradicts to Lemma 7.6. This completes the proof of the lemma. \hfill \square

8. Minimum entropy of Enriques surface automorphism

In this section, we prove realizability of the Salem number $\tau_8$ in Theorem 1.1 as the first dynamical degree of an Enriques surface automorphism. Recall that the Salem polynomial of $\tau_8$ is

$$S_8(x) := 1 - x^2 - 2x^3 - x^4 + x^6.$$ \hfill \(8.1\)

**Theorem 8.1.** There exists an automorphism $g : S \rightarrow S$ of an Enriques surface $S$ such that:

i) The characteristic polynomial of $g^* : H^2(S, \mathbb{C}) \rightarrow H^2(S, \mathbb{C})$ is

$$(-1 + x)^3(1 + x)S_8(x);$$

ii) Let $\tilde{S}$ be the universal cover of $S$. Then there is a lifting, say $\tilde{g} : \tilde{S} \rightarrow \tilde{S}$, of $g$ such that the characteristic polynomial of $\tilde{g}^* : H^2(\tilde{S}, \mathbb{C}) \rightarrow H^2(\tilde{S}, \mathbb{C})$ is

$$(-1 + x)^5(1 + x)^3(1 + x^2)^2(1 + x^4)S_8(x);$$

and

iii) The transcendental lattice $T_{\tilde{S}}$ of $\tilde{S}$ is isometric to $\mathbb{Z}_{2,2}(4)$, and the action $\tilde{g}^*|T_{\tilde{S}}$ is of order 8.

In particular, the entropy of $g$ is $h(g) = \log \tau_8$, i.e., $d_1(g) = \tau_8$.

**The Salem factor and the isometry of $E_{10}$**. Let $(L_0, f_0)$ be the principal $S_8(x)$-lattice (see Section 3). Then $L_0$ is an even lattice of signature $(3,3)$ and $G(L_0) \cong \mathbb{F}_2^2$. Let $a = P(f_0 + f_0^{-1}) \in \mathbb{Z}[f_0 + f_0^{-1}]$, where $P(y) = 1 + y$. Note that $1 + y$ is a unit of the ring $\mathbb{Z}[y]/(R_8(y))$, where $R_8(y)$ is the trace polynomial of the Salem polynomial of $\tau_8$. Then the twist $L_0(a)$ is an even lattice of signature $(1,5)$ and $G(L_0(a)) \cong \mathbb{F}_2^2$. The order of $\overline{f_0}|G(L_0(a))$ is 2. The bilinear form $b_{L_0(a)}$ on $G(L_0(a))$ is isomorphic to the bilinear form $-b_{D_4}$ on $G(D_4)$. There exists, up to conjugation, a unique isometry $f_1 \in O(D_4)$ such that

$$\chi_{f_1}(x) = (-1 + x)^3(1 + x)$$

and the order of $\overline{f_1}|G(D_4)$ is 2. Then there exists a gluing map

$$\phi_1 : G(D_4) \rightarrow G(L_0(a)),$$

such that

$$D_4 \oplus_\phi L_0(a) \cong E_{10}, \quad \text{and} \quad f_1 \oplus f_0 \text{ extends to}$$

$$f_1 \oplus_\phi f_0 \in O(D_4 \oplus_\phi L_0(a)). \quad \text{\hfill (8.2)}$$
The transcendental factor and the isometry of \( U \oplus E_{10}(2) \). Let \((L_2, f_2)\) be a \((1+x^4)\)-lattice. Since \( \Phi_8(x) = 1 + x^4 \) and \( \mathbb{Z}[x]/(\Phi_8(x)) \) is a PID, there exists \( e \in L_2 \) such that \( \{e, f_2(e), f_2^2(e), f_2^3(e)\} \) is a basis of \( L_2 \). To simplify notations, let \( b := (e, e) \) and \( c := (e, f_2(e)) \). Then,

\[
(e, f_2^2(e)) = (f_2^2(e), f_2^4(e)) = (f_2^2(e), -e) = -(e, f_2^2(e)) = 0,
\]

and

\[
(e, f_2^3(e)) = (f_2(e), f_2^4(e)) = (f_2(e), -e) = -c.
\]

Thus, the Gram matrix \((f_2^2(e), f_2^4(e))_{0 \leq i,j \leq 3}\) is

\[
\begin{pmatrix}
  b & c & 0 & -c \\
  c & b & c & 0 \\
  0 & c & b & c \\
  -c & 0 & c & b
\end{pmatrix}.
\]

From now on, we suppose that \( b = -4 \) and \( c = -4 \). Then \( L_2 \cong I_{2,2}(4) \) and \( G(L_2) \cong (\mathbb{Z}/4)^4 \) (note that, for any monic irreducible polynomial \( P(x) \in \mathbb{Z}[x] \) such that \( \mathbb{Z}[x]/(P(x)) \) is a PID, in principal, one may determine any \( P(x) \)-lattice in this way). Let \((L_3, f_3)\) be a pair of a lattice \( L_3 \) of rank 8 and an isometry \( f_3 \) of \( L_3 \) such that, in terms of a basis of \( L_3 \), the Gram matrix of \( L_3 \) and the matrix of \( f_3 \) are given by

\[
\begin{pmatrix}
-4 & 0 & 2 & 0 & -1 & 2 & -2 & 0 \\
0 & -4 & 2 & 4 & -1 & 2 & -2 & 4 \\
2 & 2 & -4 & -2 & 2 & -2 & 4 & -4 \\
0 & 4 & -2 & -8 & 2 & 0 & 4 & -8 \\
-1 & -1 & 2 & 2 & -4 & 1 & -6 & 2 \\
2 & 2 & -2 & 0 & 1 & -4 & 2 & 0 \\
-2 & -2 & 4 & 4 & -6 & 2 & -12 & 4 \\
0 & 4 & -4 & -8 & 2 & 0 & 4 & -12
\end{pmatrix}
\]

and

\[
f_3 = \begin{pmatrix}
0 & -1 & 1 & 1 & 0 & 0 & 0 & 1 \\
0 & -2 & 2 & 2 & -1 & 0 & -1 & 3 \\
1 & -1 & 1 & 2 & -1 & -1 & -2 & 2 \\
0 & -2 & 2 & 3 & -1 & 0 & -2 & 4 \\
1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\
0 & 1 & -1 & -2 & 1 & 0 & 2 & -2
\end{pmatrix}.
\]

Then \( L_3 \) is an even lattice of signature \((0,8)\) and \( G(L_3) \cong (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^4 \) (one possible approach to find all rank 8 even negative definite lattices of glue group isomorphic to
(\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^4$ is to search for such lattices by considering sublattices in $E_8 \oplus E_8(-1)$ generated by eight randomly chosen elements in $E_8 \oplus E_8(-1)$, cf. [Ni80, Theorem 1.1.2]).

The lattice $L_3$ has no roots. Moreover,

$$\chi_{f_3}(x) = (-1 + x)^2(1 + x)^2(1 + x^2)^2.$$ 

Let $H_1 := \{2x|x \in G(L_2)\}$ and $H_2 := \{2x|x \in G(L_3)\}$. Clearly $H_1 \cong H_2 \cong \mathbb{F}_2^4$. Then there exists a gluing map

$$\phi_2 : H_1 \rightarrow H_2,$$

such that

$$L_2 \oplus_{\phi_2} L_3 \cong U \oplus E_{10}(2), \quad (8.3)$$

and $f_2 \oplus f_3$ extends to

$$f_2 \oplus_{\phi_2} f_3 \in O(L_2 \oplus_{\phi_2} L_3). \quad (8.4)$$

**Proof of Theorem 8.1.** Let $(L^-, f^-)$ be the pair $(L_2 \oplus_{\phi_2} L_3, f_2 \oplus_{\phi_2} f_3)$ (see (8.3), (8.4)). Then, for a suitable choice of $\phi_2$ satisfying (8.3) and (8.4), in terms of a basis $e_1, \ldots, e_{12}$, of $L^-$, the Gram matrix of $L^-$ and the matrix of $f^-$ are given by

$$((e_i, e_j)) = \begin{pmatrix}
-4 & 2 & -2 & -2 & -4 & 2 & -2 & 2 & -4 & 2 & -2 \\
2 & -4 & 0 & 0 & 1 & -1 & 0 & 2 & -2 & 1 & -3 \\
-2 & 0 & -4 & 0 & -3 & -3 & 2 & 0 & 2 & -1 & 1 \\
-2 & 0 & 0 & -4 & 2 & -6 & 2 & -2 & -2 & -6 & -2 \\
-2 & 1 & -3 & 2 & -6 & 0 & 1 & 1 & 3 & 1 & 0 \\
-4 & -1 & -3 & 2 & -6 & 0 & -10 & 5 & -3 & -1 & -7 \\
2 & 0 & 2 & 2 & 1 & 5 & -4 & 2 & 0 & 3 & 1 \\
-2 & 2 & 2 & -2 & 2 & -4 & 0 & -4 & 0 & -3 & 1 \\
-2 & 0 & -2 & 2 & -2 & 3 & -1 & 0 & 0 & -4 & 1 & -3 \\
-4 & 1 & -1 & -6 & 1 & -7 & 3 & -3 & -1 & -8 & -1 & -3 \\
-2 & -3 & 1 & -2 & 0 & -2 & 1 & 1 & -3 & -1 & -6 & 1 \\
-2 & 2 & 0 & -2 & 1 & -3 & 2 & -2 & 0 & -3 & 1 & -4
\end{pmatrix}$$
Note that the fact

\[ L^- \cong U \oplus E_{10}(2) \]

can be also verified by classification of 2-elementary lattices due to Nikulin ([Ni83, Theorem 4.3.1]), see below for \( G(L^-) \). Moreover, \( f^- \) is an isometry of order 8, and

\[
\chi_{f^-}(x) = (-1 + x)^2(1 + x)^2(1 + x^2)^2(1 + x^4).
\]

By computation, \( \mathbb{F}_2^{10} \cong G(L^-) \) is generated by

\[
\alpha_1 := \frac{e_1}{2}, \alpha_2 := \frac{e_2 + e_3}{2}, \alpha_3 := \frac{e_4}{2}, \alpha_4 := \frac{e_5 + e_6}{2}, \alpha_5 := \frac{e_7}{2},
\]

\[
\alpha_6 := \frac{e_2 + e_8}{2}, \alpha_7 := \frac{e_2 + e_9}{2}, \alpha_8 := \frac{e_2 + e_5 + e_{10}}{2}, \alpha_9 := \frac{e_5 + e_{11}}{2}, \alpha_{10} := \frac{e_2 + e_{12}}{2}.
\]

Let

\[ T := \text{Ker}((f^-)^4 + 1) \subset L^- \text{, } N := T_{L^-}^\perp. \]

Then, by computation, \( T \cong I_{2,2}(4) \), and \( N \) is a negative definite even lattice with glue group \( G(N) \cong (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/4)^4 \). Moreover, \( N \) has no roots. Note that

\[
\chi_{f^-|T}(x) = 1 + x^4, \quad \chi_{f^-|N}(x) = (-1 + x)^2(1 + x)^2(1 + x^2)^2.
\]

Let \((L^+, f^+)\) be the pair \(((D_4 \oplus \varphi_1 L_0(a))(2), f_1 \oplus \varphi_1 f_0)\) (see (8.1), (8.2)). Then, for a suitable choice of \( \varphi_1 \) satisfying (8.1) and (8.2), in terms of a basis \( \eta_1, ..., \eta_{10} \), of \( L^+ \), the Gram matrix of \( L^+ \) and the matrix of the isometry \( f^+ \) are given by

\[
\begin{pmatrix}
-4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -4
\end{pmatrix}
\]

\[
((\eta_i, \eta_j)) =
\]

\[
\begin{pmatrix}
-4 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -4 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & -4
\end{pmatrix}
\]
and
\[
\begin{pmatrix}
0 & -1 & -1 & 1 & 0 & 1 & 0 & 0 & -1 & 1 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & -2 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 1 \\
0 & -3 & -1 & 1 & 2 & -1 & 1 & -2 & 2 & 1 \\
0 & -3 & -1 & 1 & 2 & -1 & 1 & -2 & 1 & 1 \\
1 & -2 & -1 & 0 & 1 & 2 & -1 & 1 & -2 & 1 \\
1 & -1 & -1 & 0 & 1 & 1 & -1 & 1 & -1 & 0 \\
1 & -1 & -1 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}.
\]

Then
\[
L^+ \cong E_{10}(2)
\]
and
\[
\chi_{f^+}(x) = (-1 + x)^3(1 + x)(1 - x^2 - 2x^3 - x^4 + x^6).
\]
Note that the glue group \(G(L^+) \cong \mathbb{F}_2^{10}\) is generated by
\[
\eta_1, \ldots, \eta_{10}.
\]

We define a map
\[
\phi : G(L^-) \rightarrow G(L^+)
\]
by
\[
(\phi(\alpha_1), \ldots, \phi(\alpha_{10})) = (\frac{\eta_1}{2}, \ldots, \frac{\eta_{10}}{2})M_{\phi},
\]
where
\[
M_{\phi} =
\begin{pmatrix}
0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \end{pmatrix}
\]

Then \(\phi\) is a gluing map such that i) \(L^- \oplus_{\phi} L^+ \cong \Pi_{3,19}\), and ii) \(f^- \oplus f^+\) extend to \(f^- \oplus_{\phi} f^+ \in O(L^- \oplus_{\phi} L^+)\). Note that \(-id_{L^-} \oplus id_{L^+}\) extends to an involution
\[
\sigma := -id_{L^-} \oplus_{\phi} id_{L^+} \in O(L^- \oplus_{\phi} L^+),
\]
and
\[
\sigma \circ (f^- \oplus_{\phi} f^+) = (f^- \oplus_{\phi} f^+) \circ \sigma.
\]
Let
\[
h := 97\eta_1 + 75\eta_2 + 131\eta_3 + 194\eta_4 + 172\eta_5 + 149\eta_6 + 118\eta_7 + 92\eta_8 + 53\eta_9 + 15\eta_{10}.
\]
Then \( h \) is a vector in \( L^+ \) and \( h^2 = 496 \). Let \( L \) denote \( T_{L^-}^{\perp} \oplus qL^+ \) and let 
\[
f = (f^- \oplus f^+) \mid L.
\]
Then, following the method in Remark 5.8 one can compute the following two sets
\[
\mathcal{R}_h := \{ r \in L \mid r^2 = -2 \text{ and } (r, h) = 0 \},
\]
\[
\mathcal{S}_h := \{ r \in L \mid r^2 = -2 \text{ and } (r, h)(r, f(h)) < 0 \}.
\]
It turns out that both \( \mathcal{R}_h \) and \( \mathcal{S}_h \) are empty, which means i) none of roots in \( L \) is perpendicular to \( h \), and ii) \( h \) and \( f(h) \) are in the same chamber of \( L \).

Since all the conditions 1)-8) of Definitions 7.1 are satisfied, it follows that \((f^+, f^-, T, \phi)\) is an Enriques quadruple of entropy \( \log \tau_8 \). Thus, by Theorem 7.4, \( \tau_8 \) is realized by an automorphism \( g : S \rightarrow S \) of an Enriques surface \( S \). Moreover, by the proof of Theorem 7.4 and the three conditions i)-iii) in Theorem 8.1 are satisfied. This completes the proof. □

**Remark 8.2.** Roughly speaking, the Enriques quadruple \((f^+, f^-, T, \phi)\) in the proof are obtained in the process of trying to rule out \( \tau_8 \) like ruling out the other 7 Salem numbers in Section 9 based on Theorem 7.4 (recall that \( \tau_8 \) is pseudo-simple, see Remark 3.3). However, Theorem 7.4 is if and only if formulation. So, if one obtains a final output, then it is a realization. In this way, we obtained Theorem 8.1.

## 9. Ruling out Salem numbers

In this section, we prove unrealizability of Salem numbers \( \tau_i \) (1 \( \leq \) i \( \leq \) 7) in Table 1 as the first dynamical degree of an Enriques surface automorphism. We will rule out \( \tau_i \) (1 \( \leq \) i \( \leq \) 7) separately in the following seven subsections.

In this section, we use \( L^+ \) and \( L^- \) to denote \( E_{10}(2) \) and \( U \oplus E_{10}(2) \) respectively.

### 9.1. Ruling out \( \tau_3 \)

Recall that \( \tau_3 \) be the Salem number whose Salem polynomial is
\[
S_3(x) = 1 - x - x^3 + x^4 - x^5 - x^7 + x^8
\]
and \( \tau_3 \) is simple (Remark 3.3). Unlike the six cases below, the remaining arguments in this subsection is free from computer algebra.

**Lemma 9.1.** Let \( f \in \text{O}(E_{10}) \) be of entropy \( \log \tau_3 \). Then 
\[
\chi_f(x) \equiv (1 + x)^2(1 + x + x^4)(1 + x^3 + x^4) \mod 2.
\]

**Proof.** Since the entropy of \( f \) is \( \log \tau_3 \), \( \chi_f(x) = S_3(x)Q(x) \), where \( Q(x) \) is a product of cyclotomic polynomials, and \( \deg(Q(x)) = 2 \). There are exactly five cyclotomic polynomials (i.e., \( \Phi_k(x) \), \( k = 1, 2, 3, 4, 6 \)) with degree less than or equal to 2. By computation, \( 7 \mid \text{res}(\Phi_2(x), S_3(x)) \) and \( 7 \nmid \text{res}(\Phi_k(x), S_3(x)) \) for \( k = 1, 3, 4, 6 \). Since \( \tau_3 \) is simple and \( |\text{det}(L_0)| = 7 \), the determinant of any \( S_3(x) \)-lattice must be divided by 7. Since \( E_{10} \) is unimodular, by [Mc16] Theorem 4.3, \( \Phi_2(x) \) divides \( Q(x) \). Then 
\[
\chi_f(x) \equiv (1 + x)^2S_3(x) \equiv (1 + x)^2(1 + x + x^4)(1 + x^3 + x^4) \mod 2.
\]
□
Theorem 9.2. The Salem number $\tau_3$ can not be realized by an automorphism of any Enriques surface.

Proof. Suppose $\tau_3$ can be realized. By Theorem 7.4 there exists an Enriques quadruple $(f^+, f^-, T, \phi)$ of entropy $\log \tau_3$ (see Definition 7.1). By condition 2) of Definition 7.1 and Lemma 9.1,

$$\chi_{f^-}(x) \equiv (1 + x)^2 \chi_{f^+}(x) \equiv (1 + x)^4(1 + x + x^4)(1 + x^3 + x^4) \bmod 2.$$  

By factorizations of such cyclotomic polynomials in $\mathbb{F}_2[x]$ (see Table 2), either $\Phi_{15}(x)$ or $\Phi_{30}(x)$ divides $\chi_{f^-}(x)$. Replacing $f^-$ by $-f^-$ if necessary (see Remark 7.3), we may and will assume that $\Phi_{15}(x)$ divides $\chi_{f^-}(x)$. Then $\chi_{f^-}(x) = \Phi_{15}(x)C(x)$, where $C(x)$ is a product of cyclotomic polynomials, and $C(x) \equiv (x + 1)^4 \bmod 2$. Let

$$L_1 := \text{Ker}(\Phi_{15}(f^-)) \subset L^- \text{, } L_2 := \text{Ker}(C(f^-)) \subset L^-.$$  

By computation, $|\text{res}(\Phi_{15}(x), \Phi_k(x))| = 1$, for $k = 1, 2, 4, 8$. Thus,

$$|\text{res}(\Phi_{15}(x), C(x))| = 1,$$

therefore

$$L^- = L_1 \oplus L_2$$

by [Mc16] Proposition 4.2. Recall that $L^- = U \oplus E_{10}(2)$. Thus, $L_1$ and $L_2$ are even 2-elementary lattices of rank 8 and 4 respectively. By [Ni83] Theorem 4.1, any negative definite even 2-elementary lattice of rank $\leq 8$ has roots with the only exception $E_8(2)$. If $L_2$ is negative definite, then $L_2$ has roots and $T = L_1$, a contradiction to condition 6) of Definition 7.1. If $L_2$ is not negative definite, then $L_2$ must be of signature $(2, 2)$, $T \subset L_2$, $L_1 \cong E_8(2)$, and $L_2 \cong U \oplus U(2)$. Let $R \subset L^- \oplus_\phi L^+$ be the smallest primitive sublattice containing both $L_1$ and $L^+$. By condition 8) of Definition 7.1 the restriction of $f^- \oplus_\phi f^+$ to $T_{L^- \oplus_\phi L^+}$ is positive. Note that $R \subset T_{L^- \oplus_\phi L^+}$. Thus, by Lemma 5.5 the restriction $(f^- \oplus_\phi f^+)|_R$ is positive. This contradicts to Lemma 7.1. Therefore, $\tau_3$ can not be realized by an automorphism of any Enriques surface. This completes the proof of the theorem. $\square$

9.2. Ruling out $\tau_1$. Recall that $\tau_1$ is the Salem number whose Salem polynomial is

$$S_1(x) = 1 - x - x^4 + x^5 - x^6 - x^9 + x^{10}.$$  

Let $R_1(x)$ be the trace polynomial of $S_1(x)$.

We set $K := \mathbb{Q}[x]/(S_1(x))$ and $k := \mathbb{Q}[x + x^{-1}] \subset K$ (See section 3).

First we make some preparation using computer algebra. We have $\mathcal{O}_K \cong \mathbb{Z}[x]/(S_1(x))$, $\mathcal{O}_k \cong \mathbb{Z}[x]/(R_1(x))$, and they are PIDs (see Remark 4.3).

Clearly, finding all possible $S_1(x)$-lattice isomorphic to $L^+$ is equivalent to finding all possible $S_1(x)$-lattices isomorphic to $E_{10}$.

Lemma 9.3. There exists a subset $\mathcal{R} \subset \mathcal{O}(E_{10})$ consisting of 4 elements such that any element in $\mathcal{O}(E_{10})$ whose characteristic polynomial is equal to $S_1(x)$ is conjugate to some element in $\mathcal{R}$. Moreover, $\mathcal{R}$ is explicitly given, and the mod 2 reduction of $S_1(x)$ is

$$(1 + x + x^2)^2(1 + x + x^3)(1 + x^2 + x^3).$$
Proof. Let \((L_0, f_0)\) denote the principal \(S_1(x)\)-lattice. Let \(\mathcal{U} \subset \mathcal{O}_k^\times\) denote a set of representatives for the units modulo squares. There are exactly four units \(u_i \in \mathcal{U}, i = 1, 2, 3, 4\) such that the twists \(L_0(u_i)\) are isomorphic to \(E_{10}\). Thus, by Theorem 3.2, up to conjugation, there are at most four isometries, say \(g_i \in O(L^+)\), \(i = 1, 2, 3, 4\) with characteristic polynomial \(S_1(x)\). Then we set \(R = \{g_1, g_2, g_3, g_4\}\). (See Section 3 for terminologies used here.)

Lemma 9.4. Let \(R\) be as in Lemma 9.3. Let \(g \in R \subset O(L^+)\), and let \(H \subset G(L^+)\) be a subgroup isomorphic to \(F_2^6\). Let \(f \in O(A_6(2))\) with characteristic polynomial \(\Phi_7(x)\). Then there exists no gluing map \(\psi : G(A_6(2))_2 \to H\) such that

i) the map \(f \oplus g\) extends to \(f \oplus \psi g \in O(A_6(2) \oplus \psi L^+)\), and

ii) \(f \oplus \psi g\) is positive.

Proof. This is proved by computer algebra, and here we explain how it works. First of all, one can check, up to conjugation in \(O(A_6(2))\), there is a unique isometry \(f \in O(A_6(2))\) with characteristic polynomial \(\Phi_7(x)\). Next, one can observe that, for each \(g \in R\), there are exactly seven pairs \((H, \psi)\), where \(H \subset G(L^+)\) is a subgroup of order \(2^6\) and \(\psi : G(A_6(2))_2 \to H\) is a gluing map, such that the map \(f \oplus g\) extends to \(f \oplus \psi g \in O(A_6(2) \oplus \psi L^+)\). Thus, totally, there are 28 candidates \(f \oplus \psi g\), but it turns out that none of them is positive by computer algebra (see Remark 5.8).

Now we prove the main result of this subsection.

Theorem 9.5. The Salem number \(\tau_1\) can not be realized by an automorphism of any Enriques surface.

Proof. Suppose \(\tau_1\) can be realized. Then, by Theorem 7.4, there exists an Enriques quadruple \((f^+, f^-, T, \phi)\) of entropy \(\log \tau_1\). Thus,

\[
\chi_{f^-}(x) \equiv (1 + x)^2 \chi_{f^+}(x) \equiv (1 + x)^2(1 + x + x^2)^2(1 + x + x^3)(1 + x^2 + x^3) \mod 2.
\]

Since \(f^-\) is of finite order, \(\chi_{f^-}(x)\) is a product of cyclotomic polynomials of degrees not greater than 12. By factorizations of such cyclotomic polynomials in \(\mathbb{F}_2[x]\) (see Table 2), either \(\Phi_7(x)\) or \(\Phi_{14}(x)\) divides \(\chi_{f^-}(x)\). Replacing \(f^-\) by \(f^-\) if necessary, we may and will assume that \(\Phi_7(x)\) divides \(\chi_{f^-}(x)\) (see Remark 7.5). Then

\[
\chi_{f^-}(x) = \Phi_7(x)C(x),
\]

where \(C(x)\) is a product of polynomials in \(\{\Phi_1(x), \Phi_2(x), \Phi_3(x), \Phi_4(x), \Phi_6(x), \Phi_{12}(x)\}\), and

\[
C(x) \equiv (1 + x)^2(1 + x + x^2)^2 \mod 2.
\]

We set

\[
L_1 := \text{Ker}(\Phi_7(f)) \subset L^-, L_2 := \text{Ker}(C(f)) \subset L^-.
\]

We use \(f_i, i = 1, 2\) to denote \(f|L_i\). Since both \(L_1\) and \(L_2\) are primitive sublattices of \(L^-\), and \([L^- : L_1 \oplus L_2] < \infty\), it follows that there exist subgroups \(H_i \subset G(L_i), i = 1, 2\) and a gluing map \(\psi : H_1 \to H_2\) such that \((L_1 \oplus \psi L_2, f_1 \oplus \psi f_2)\) is isomorphic to \((L^-, f)\). By Lemmas 4.5 and 4.6,

\[
|H_1| = |H_2| = 7, G(L_1) = \mathbb{F}_2^k \oplus \mathbb{F}_7, G(L_2) = \mathbb{F}_2^{10-k} \oplus \mathbb{F}_7.
\]
for some $k \leq 6$. Note that the characteristic polynomial of $f^- : G(L^-) \to G(L^-)$ is 
\[(1 + x + x^2)(1 + x + x^3)(1 + x^2 + x^3).\]
Thus, by Lemma 4.1, $k = 6$ and $b_{L_1}(v, v) = 0 \in \mathbb{Q}/\mathbb{Z}$ for all $v \in G(L_1)_2$.

If the signature of $L_1$ is $(2, 4)$, then, by condition 4) of Definition 7.1, $T = L_1$. Then, by Lemma 4.6, $L_2$ has roots, a contradiction to condition 6) of Definition 7.1. Thus, the signature of $L_1$ is not $(2, 4)$ and $T \neq L_1$. Then, by condition 5) of Definition 7.1, $T \subset L_2$. Thus, the signature of $L_1$ is $(0, 6)$.

Then by Lemma 2.2, $L_1(1/2)$ is a well-defined negative definite even lattice of determinant 7 and rank 6. By classification, $L_1(1/2) \cong A_6$ (see CSSS Table 1) and $L_1 \cong A_6(2)$.

Let $R \subset L^- \oplus \phi L^+$ be the smallest primitive sublattice containing both $L_1$ and $L^+$. By condition 8) of Definition 7.1, the restriction of $f^- \oplus \phi f^+$ to $T_{L^- \oplus \phi L^+}$ is positive. Note that $R \subset T_{L^- \oplus \phi L^+}$. Thus, by Lemma 5.5, the restriction $(f^- \oplus \phi f^+)|R$ is positive. This contradicts to Lemma 9.4. Thus, $\tau_1$ can not be realized. This completes the proof. □

9.3. Ruling out $\tau_6$. Recall that $\tau_6$ is the Salem number whose Salem polynomial

\[S_6(x) = 1 - x - x^2 + x^5 - x^8 - x^9 + x^{10}.\]

Let $R_6(x)$ be the trace polynomial of $S_6(x)$. We set $K := \mathbb{Q}[x]/(S_6(x))$, $k := \mathbb{Q}[x + x^{-1}] \subset K$. Both $\mathcal{O}_K \cong \mathbb{Z}[x]/(S_6(x))$ and $\mathcal{O}_k \cong \mathbb{Z}[x]/(R_6(x))$ are PIDs.

Exactly in the same way as in subsection 9.2, computer algebra verifies the following two lemmas.

**Lemma 9.6.** There exists a subset $\mathcal{R} \subset \mathcal{O}(E_{10})$ consisting of 4 elements such that any element in $\mathcal{O}(E_{10})$ whose characteristic polynomial is equal to $S_6(x)$ is conjugate to some element in $\mathcal{R}$. Moreover, $\mathcal{R}$ is explicitly given, and the mod 2 reduction of $S_6(x)$ is

\[(1 + x + x^2 + x^3 + x^4)(1 + x^3 + x^6).\]

**Lemma 9.7.** Let $g \in \mathcal{R}$, and let $H \subset G(L^+)$ be a subgroup isomorphic to $\mathbb{F}_2^9$. Let $f \in \mathcal{O}(E_{6}(2))$ with characteristic polynomial $\Phi_0(x)$. Then there exists no gluing map $\psi : G(E_{6}(2))_2 \to H$ such that both of the following two statements are true

1) the map $f \oplus g$ extends to $f \oplus \psi g \in \mathcal{O}(E_{6}(2) \oplus \psi L^+)$, and
2) $f \oplus \psi g$ is positive.

**Theorem 9.8.** The Salem number $\tau_6$ can not be realized by an automorphism of any Enriques surface.

**Proof.** One proves this theorem exactly in the same way as in Theorem 9.3 based on lemmas 9.6 and 9.7 (instead of lemmas 9.3 and 9.4). □

9.4. Ruling out $\tau_2$. Recall that $\tau_2$ is the Salem number whose Salem polynomial is

\[S_2(x) = 1 - x^2 - x^3 - x^4 + x^6.\]

Let $R_2(x)$ be the trace polynomial of $S_2(x)$.

We set $K := \mathbb{Q}[x]/(S_2(x))$ and $k := \mathbb{Q}[x + x^{-1}] \subset K$. As in subsection 9.2, $\mathcal{O}_K \cong \mathbb{Z}[x]/(S_2(x))$, $\mathcal{O}_k \cong \mathbb{Z}[x]/(R_2(x))$, and they are PIDs. Let $(L_0, g_0)$ be the principal $S_2(x)$-lattice.
Lemma 9.9. There exists a subset $\mathcal{R} \subset O(E_{10})$ consisting of $84$ elements such that any element in $O(E_{10})$ whose characteristic polynomial divided by $S_2(x)$ is conjugate to some element in $\mathcal{R}$. Moreover, $\mathcal{R}$ is explicitly given, and the set of characteristic polynomials of elements in $\mathcal{R}$ consists of $\Phi(3)\Phi_4^2(x)S_2(x)$, $\Phi(3)\Phi_4(x)\Phi_1(x)S_2(x)$, $\Phi_3^2(x)S_2(x)$, $\Phi_6(x)\Phi_2(x)\Phi_1(x)S_2(x)$, $\Phi_6(x)\Phi_2^3(x)S_2(x)$, $\Phi_6(x)\Phi_4(x)S_2(x)$, and $\Phi_{12}(x)S_2(x)$. The set of mod 2 reduction of these 7 polynomials consists of

$$(1 + x)^2(1 + x + x^2)^2(1 + x + x^2 + x^3 + x^4) \text{ and } (1 + x + x^2)^3(1 + x + x^2 + x^3 + x^4).$$

Proof. This is proved by computer algebra again, and here we explain how it works. The Salem number $\tau_2$ has two feasible primes $2, 7$ (see [31]), and $|\text{det}(L_0)| = 1$. The prime number 2 factors in $O_k$ consist of elements of primes of degree 1 and 2 respectively. Let $U \subset O_k$ denote a set of representatives for the units modulo squares.

Since $\tau_2$ is simple, the Salem factor of any isometry of $E_{10}$ with spectral radius $\tau_2$ must be isomorphic to a twist $f_0\gamma_{L_0}^a$ for some $a \in O_k$. By Theorem 3.4, it turns out that $a$ must be an associate of one of the four elements $a_1, a_2, b_1, a_3$, i.e., up to units in $O_k$, $a$ coincides one of the four elements.

Next we discuss case by case for the four cases: associates of $a_1, a_2, b_1, a_3$.

Case associates of $a_1$. There are exactly four units, say $u_{i,j}, i = 1, 2, 3, 4$, in $U$ such that the twists $L_0(u_{1,i},a_1)$ have signature $(1,5)$. Then $G(L_0(u_{1,i},a_1)) \cong F_2^2$ and the orders of $f_0|G(L_0(u_{1,i},a_1))$ are equal to 3. For any $i \in \{1, 2, 3, 4\}$, the discriminant form $q_{L_0(u_{1,i},a_1)}$ is isomorphic to $-q_{D_4}$, where $D_4$ is the root lattice of type $D_4$. Thus, if $f_0|G(L_0(u_{1,i},a_1))$ is the Salem factor of an isometry of $E_{10}$, the orthogonal complement $L_0(u_{1,i},a_1)^{\perp} \subset E_{10}$ must be isomorphic to $D_4$. By computing all possible glue between $f_0|L_0(u_{1,i},a_1)$ and isometries of $D_4$, we explicitly find a subset $\mathcal{R} \subset O(E_{10})$ such that i) any isometry of $E_{10}$ whose Salem factor is one of the four $f_0|L_0(u_{1,i},a_1)$ is conjugate to an element in $\mathcal{R}$, ii) the Salem factor of any element in $\mathcal{R}$ is one of the four $f_0|L_0(u_{1,i},a_1)$, iii) $\mathcal{R}$ has 84 elements.

Case associates of $a_2$. There are exactly four units, say $u_{2,i}, i = 1, 2, 3, 4$, in $U$ such that the twists $L_0(u_{2,i},a_2)$ have signature $(1,5)$. Then $G(L_0(u_{2,i},a_2)) \cong F_2^2$ and the orders of $f_0|G(L_0(u_{2,i},a_2))$ are equal to 5. By [31], there exists no isometry of $E_{10}$ with characteristic polynomial either $\Phi_5(x)S_2(x)$ or $\Phi_6(x)S_2(x)$. Thus, none of $L_0(u_{2,i},a_2)$, $i = 1, 2, 3, 4$ can be the Salem factor of an isometry of $E_{10}$.

Case associates of $b_1$. There are exactly four units, say $u_{3,i}, i = 1, 2, 3, 4$, in $U$ such that the twists $L_0(u_{3,i},b_1)$ have signature $(1,5)$. Then $G(L_0(u_{3,i},b_1)) \cong F_2^2$ and the orders of $f_0|G(L_0(u_{3,i},b_1))$ are equal to 8. By [31], again, none of $L_0(u_{3,i},b_1)$ $i = 1, 2, 3, 4$ can be the Salem factor of an isometry of $E_{10}$.

Case associates of $a_3$. There are exactly four units, say $u_{4,i}, i = 1, 2, 3, 4$, in $U$ such that the twists $L_0(u_{4,i},a_3)$ have signature $(1,5)$. Then $G(L_0(u_{4,i},a_3)) \cong (Z/8)^2$ and the orders of $f_0|G(L_0(u_{4,i},a_3))$ are equal to 6. It turns out that there exists no suitable isometry $g'$ of any negative definite rank 4 even lattice $g'$ such that $(g', g')$ and $(L_0(u_{4,i},a_3), f_0)$ can glue to an isometry of $E_{10}$ (for classification of even lattices of rank 4 and determinant 64, see [51]). Thus, none of $f_0|L_0(u_{4,i},a_3) i = 1, 2, 3, 4$ can be the Salem factor of an isometry of $E_{10}$.

As in subsection 9.2, we get the following lemma by computer algebra.
Lemma 9.10. Let $\mathcal{R}$ be as in Lemma 9.9. Let $g \in \mathcal{R} \subset \text{O}(L^+)$, and let $H \subset \text{G}(L^+)$ be any subgroup isomorphic to $\mathbb{F}_2^4$. Let $f \in \text{O}(A_4(2))$ with characteristic polynomial $\Phi_5(x)$ (up to conjugation, such $f$ is unique). Then there exists no gluing map $\psi : \text{G}(A_4(2))_2 \to H$ such that both of the following two statements are true:

i) the map $f \oplus g$ extends to $f \oplus \psi g \in \text{O}(A_4(2) \oplus \psi \, L^+)$, and

ii) $f \oplus \psi g$ is positive.

Now we can prove the main result of this subsection.

Theorem 9.11. The Salem number $\tau_2$ can not be realized by an automorphism of any Enriques surface.

Proof. One proves this theorem exactly in the same way as in Theorem 9.5 based on lemmas 9.9 and 9.10 (instead of lemmas 9.3 and 9.4).

\[\Box\]

9.5. Ruling out $\tau_7$. Let $\tau_7$ be the Salem number with the minimal polynomial

$$S_7(x) = 1 - x - x^2 + x^3 - x^4 - x^5 + x^6.$$ 

Let $R_7(x)$ be the trace polynomial of $S_7(x)$. Like before, we set $K := \mathbb{Q}[x]/(S_7(x))$ and $k := \mathbb{Q}[x + x^{-1}] \subset K$. It turns out that both $O_K \cong \mathbb{Z}[x]/(S_7(x))$ and $O_k \cong \mathbb{Z}[x]/(R_7(x))$ are PIDs. Let $(L_0, g_0)$ be the principal $S_7(x)$-lattice.

Lemma 9.12. There exists a subset $\mathcal{R} \subset \text{O}(E_{10})$ consisting of 120 elements such that any element in $\text{O}(E_{10})$ whose characteristic polynomial divided by $S_7(x)$ is conjugate to some element in $\mathcal{R}$. Moreover, $\mathcal{R}$ is explicitly given, and the set of mod 2 reduction of characteristic polynomials of elements in $\mathcal{R}$ consists of

$$\begin{align*}
(1 + x)^4(1 + x + x^3)(1 + x^2 + x^3), \\
(1 + x)^2(1 + x + x^2)(1 + x + x^3)(1 + x^2 + x^3), \\
(1 + x + x^2)^2(1 + x + x^3)(1 + x^2 + x^3).
\end{align*}$$

Proof. The proof proceeds along the same line as in the proof of Lemma 9.9. However, some differences appear as we will indicate below.

The Salem number $\tau_7$ has three feasible primes $3, 5, 7$, and $|\text{det}(L_0)| = 1$. The prime number 3 factors in $O_k$ as $3 = a_1 a_2$ of primes of degree 1 and 2 respectively. The prime number 5 factors in $O_k$ as $5 = b_1 b_2$ of primes of degree 1 and 2 respectively. The prime number 7 factors in $O_k$ as $7 = c_1 c_2$ of primes of degree 1 and 2 respectively. Let $\mathcal{U} \subset O_k^\times$ denote a set of representatives for the units modulo squares.

Note that $\tau_7$ is simple as before. Like in the proof of Lemma 9.9, it turns out that $a$ must be an associate of one of the four elements $a_1, a_2, b_1, c_1$.

Next we discuss case by case for the four cases: associates of $a_1, b_1, a_2, c_1$.

Case associates of $a_1$. There are exactly four units, say $u_{i,1}$, $i = 1, 2, 3, 4$, in $\mathcal{U}$ such that the twists $L_0(u_{i,1}a_1)$ have signature $(1, 5)$. Then $G(L_0(u_{i,1}a_1)) \cong \mathbb{F}_3^2$ and the orders of $g_0|G(L_0(u_{i,1}a_1))$ are equal to 4. The only rank 4 negative definite even lattice with glue group isomorphic to $\mathbb{F}_3^2$ is $A_2 \oplus A_2$. Thus, if $g_0|L_0(u_{i,1}a_1)$ is the Salem factor of an isometry of $E_{10}$, then the orthogonal complement $L_0(u_{i,1}a_1)^\perp \subset E_{10}$ must be isomorphic to $A_2 \oplus A_2$. By computing all possible glue between $g_0|L_0(u_{i,1}a_1)$ and isometries of $A_2 \oplus A_2$, we find a subset $\mathcal{R}_1 \subset \text{O}(E_{10})$ such that i) any isometry of $E_{10}$ whose Salem factor is one of the four
$g_0|L_0(u_1,a_1)$ is conjugate to an element in $\mathcal{R}_1$, ii) the Salem factor of any element in $\mathcal{R}_1$ is one of the four $g_0|L_0(u_1,a_1)$, iii) $\mathcal{R}_3$ has 48 elements.

**Case associates of $b_1$.** There are exactly four units, say $u_{3,i}$, $i = 1, 2, 3, 4$, in $\mathcal{U}$ such that the twists $L_0(u_{3,i}b_1)$ have signature $(1,5)$. Then $G(L_0(u_{3,i}b_1)) \cong \mathbb{F}_2^3$ and the orders of $g_0|G(L_0(u_{3,i}b_1))$ are equal to $3$. The only rank $4$ negative definite even lattice with glue group isomorphic to $\mathbb{F}_2^3$ is

$$
\begin{pmatrix}
-2 & -1 & -1 & -1 \\
-1 & -2 & 0 & -1 \\
-1 & 0 & -4 & -2 \\
-1 & -1 & -2 & -4 \\
\end{pmatrix},
$$

which we denote by $M$ (see [Nip91]). By considering all possible glue between $g_0|L_0(u_{3,i}b_1)$ and isometries of $M$, we find a subset $\mathcal{R}_2 \subset O(E_{10})$ such that i) any isometry of $E_{10}$ whose Salem factor is one of the four $g_0|L_0(u_{3,i}b_1)$ is conjugate to an element in $\mathcal{R}_2$, ii) the Salem factor of any element in $\mathcal{R}_2$ is one of the four $g_0|L_0(u_{3,i}b_1)$, iii) $\mathcal{R}_2$ has $72$ elements.

**Cases associates of $a_2$ and $c_1$.** One concludes impossibility as in the case associates of $a_2$ of the proof of Lemma 9.9. □

As in subsection 9.2, we get the following lemma by computer algebra.

**Lemma 9.13.** Let $g \in \mathcal{R} \subset O(L^+)$ (recall that $O(E_{10})$ can be naturally identified with $O(L^+)$ since $L^+ = E_{10}(2)$, and let $H \subset G(L^+)$ be any subgroup isomorphic to $\mathbb{F}_2^n$. Let $f \in O(A_6(2))$ with characteristic polynomial $\Phi_7(x)$. Then there exists no gluing map $\psi : G(A_6(2))_2 \longrightarrow H$ such that both of the following two statements are true

i) the map $f \oplus g$ extends to $f \oplus \psi g \in O(A_6(2) \oplus \psi L^+)$, and

ii) $f \oplus \psi g$ is positive.

**Theorem 9.14.** The Salem number $\tau_7$ can not be realized by an automorphism of any Enriques surface.

**Proof.** One proves this theorem exactly in the same way as in Theorem 9.14 based on lemmas 9.12 and 9.13 (instead of lemmas 9.3 and 9.4). □

9.6. **Ruling out $\tau_4$.** Let $\tau_4$ be the Salem number with the minimal polynomial

$$S_4(x) = 1 - x^2 - x^3 - x^5 - x^6 + x^8$$

Here $\tau_4$ is just pesduo-simple but not simple, and this is the main difference from other subsections.

Let $R_4(x)$ be the trace polynomial of $S_4(x)$. Like before, we set $K := \mathbb{Q}[x]/(S_4(x))$ and $k := \mathbb{Q}[x+x^{-1}] \subset K$. Let $(L_0,g_0)$ be the principal $S_4(x)$-lattice. Then $|\det(L_0)| = $
$|S_4(-1)S_4(1)| = 4$. Consider the pair $(L'_0, g'_0)$

$$L'_0 := \begin{pmatrix} 1 & 0 & 1 & 1 & 1 & 2 & 2 & 4 \\ 0 & 1 & 0 & 1 & 1 & 1 & 2 & 2 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 \\ 4 & 2 & 2 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}$$

and $g'_0 := \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$.

Then it can be easily verified that $(L'_0, g'_0)$ is $S_4(x)$-lattice and $\det(L'_0) = 1$ (hence square free). Then by Theorem 3.32 any $S_4(x)$-lattice is a twist of $(L'_0, g'_0)$.

**Lemma 9.15.** There exists a subset $\mathcal{R} \subset O(E_{10})$ consisting of 8 elements such that any element in $O(E_{10})$ whose characteristic polynomial divided by $S_4(x)$ is conjugate to some element in $\mathcal{R}$. Moreover, $\mathcal{R}$ is explicitly given, and the set of characteristic polynomials of elements in $\mathcal{R}$ consists of $\Phi_1(x)\Phi_2(x)S_4(x)$, $\Phi_4(x)S_4(x)$. The mod 2 reduction of these two polynomials is equal to

$$(1 + x)^4 (1 + x^3 + x^6).$$

**Proof.** $\tau_4$ has one feasible prime 2. The prime number 2 factors in $O_k$ as $2 = a_1a_2$ of degree one and three respectively. Let $\mathcal{U} \subset O_k^\times$ denote a set of representatives for the units modulo squares.

Since $\tau_4$ is pesudo-simple, the Salem factor of any isometry of $E_{10}$ with spectral radius $\tau_4$ must be isomorphic to a twist $g'_0|L'_0$ for some $a \in O_k$. By Theorem 3.41 it turns out that $a$ must be an associate of one of the two elements $a_1, a_2^2$.

Next we discuss case by case for the two cases: associates of $a_1, a_2^2$.

**Case associates of $a_1$.** There are exactly two units, say $u_{1,i}$, $i = 1, 2$, in $\mathcal{U}$ such that the twists $L'_0(u_{1,i}a_1)$ have signature $(1,7)$. Then $G(L'_0(u_{1,i}a_1)) \cong \mathbb{F}_2^2$ and the orders of $g'_0|G(L'_0(u_{1,i}a_1))$ are equal to 2. For any $i \in \{1,2\}$, the discriminant form $q_{L'_0(u_{1,i}a_1)}$ is isomorphic to $-q_{10,2(2)}$, where $\mathcal{I}_2$ is the unique negative definite odd unimodular lattice of rank 2. Thus, if $g'_0|L'_0(u_{1,i}a_1)$ is the Salem factor of an isometry of $E_{10}$, then the orthogonal complement $L'_0(u_{1,i}a_1)^\perp \subset E_{10}$ must be isomorphic to $\mathcal{I}_{10,2(2)}$. By considering all possible glue between $g'_0|L'_0(u_{1,i}a_1)$ and isometries of $\mathcal{I}_{10,2(2)}$, we find a subset $\mathcal{R} \subset O(E_{10})$ such that i) any isometry of $E_{10}$ whose Salem factor is one of the two $g'_0|L'_0(u_{1,i}a_1)$ is conjugate to an element in $\mathcal{R}$, ii) the Salem factor of any element in $\mathcal{R}$ is one of the two $g'_0|L'_0(u_{1,i}a_1)$, iii) $\mathcal{R}$ has 8 elements.

**Case associates of $a_2^2$.** There are exactly two units, say $u_{2,i}$, $i = 1, 2$, in $\mathcal{U}$ such that the twists $L'_0(u_{2,i}a_2^2)$ have signature $(1,7)$. Then $G(L'_0(u_{2,i}a_2^2)) \cong (\mathbb{Z}/4\mathbb{Z})^2$ and the orders of $g'_0|G(L'_0(u_{2,i}a_2^2))$ are equal to 4. Any rank 2 negative definite even lattice with glue group $(\mathbb{Z}/4\mathbb{Z})^2$ must be isometric to $\mathcal{I}_{10,2(4)}$. It turns out that the discriminant form $q_{L'_0(u_{2,i}a_2^2)}$ is not isomorphic to $-q_{10,2(4)}$. Thus, none of $L'_0(u_{2,i}a_2^2)$, $i = 1, 2$ can be the Salem factor of an isometry of $E_{10}$. 

Again as in subsection 9.2 we get the following lemma by computer algebra.
Lemma 9.16. Let $g \in \mathcal{R} \subset \text{O}(L^+)$ (recall that $\text{O}(E_{10})$ can be naturally identified with $\text{O}(L^+)$ since $L^+ = E_{10}(2)$), and let $H \subset G(L^+)$ be any subgroup isomorphic to $\mathbb{F}_2^5$. Let $f \in \text{O}(E_6(2))$ with characteristic polynomial $\Phi_0(x)$. Then there exists no gluing map $\psi : G(E_6(2))_2 \rightarrow H$ such that both of the following two statements are true

i) the map $f \oplus g$ extends to $f \oplus \psi g \in \text{O}(E_6(2) \oplus \psi L^+)$, and

ii) $f \oplus \psi g$ is positive.

Theorem 9.17. The Salem number $\tau_4$ cannot be realized by an automorphism of any Enriques surface.

Proof. One proves this theorem exactly in the same way again as in Theorem 9.5 based on lemmas 9.15 and 9.16 (instead of lemmas 9.3 and 9.4). □

9.7. Ruling out $\tau_5$. Let $\tau_5$ be the Salem number with the minimal polynomial

$$S_5(x) = 1 - x - x^3 - x^4 + x^6.$$ 

Let $R_5(x)$ be the trace polynomial of $S_5(x)$. Like before, we set $K := \mathbb{Q}[x]/(S_5(x))$ and $k := \mathbb{Q}[x + x^{-1}] \subset K$. It turns out $\tau_5$ is simple, $\mathcal{O}_K \cong \mathbb{Z}[x]/(S_5(x))$ and $\mathcal{O}_k \cong \mathbb{Z}[x]/(R_5(x))$ are PIDs. Let $(L_0, g_0)$ be the principal $S_5(x)$-lattice. Then $|\det(L_0)| = 5$.

However, this case is the most complicated case in our approach, and the main reason is mod 2 reduction of $S_5(x)$ is

$$(1 + x + x^2)^3$$

only one irreducible factor of small degree, so there are more possible candidates for cyclotomic factors.

The Salem number $\tau_5$ has two feasible primes 2, 5. The prime number 2 factors in $\mathcal{O}_k$ as $2 = a_1^4$, where $a_1$ is a prime of degree 1. The prime number 5 factors in $\mathcal{O}_k$ as $5 = b_1 b_2$ of primes of degree 1 and 2 respectively. Let $\mathcal{U} \subset \mathcal{O}_k^\times$ denote a set of representatives for the units modulo squares. The Salem factor of any isometry of $E_{10}$ with spectral radius $\tau_5$ must be isomorphic to a twist $g_0 \mathcal{L}_0(a)$ for some $a \in \mathcal{O}_k$. By Theorem 3.4 it turns out that $a$ must be either 1) a unit or 2) an associate of $a_1$.

Case 1). There are exactly two units, say $u_{1,i}, i = 1, 2$, in $\mathcal{U}$ such that the twists $L_0(u_{1,i})$ have signature (1, 5). Then $G(L_0(u_{1,i})) \cong \mathbb{F}_5$ and the orders of $g_0 G(L_0(u_{1,i}))$ are equal to 2. The only rank 4 negative definite even lattice with glue group $\mathcal{L}_0(u_{1,i})^\perp \subset \mathcal{L}_{10}$ is isomorphic to $A_4$. Thus, if $g_0 \mathcal{L}_0(u_{1,i})$ is the Salem factor of an isometry of $E_{10}$, then the orthogonal complement $L_0(u_{1,i})^\perp \subset \mathcal{L}_{10}$ must be isomorphic to $A_4$. By considering all possible glue between $g_0 \mathcal{L}_0(u_{1,i})$ and isometries of $A_4$, we find a subset $\mathcal{R}_1 \subset \text{O}(E_{10})$ such that i) any isometry of $E_{10}$ whose Salem factor is one of the two $g_0 \mathcal{L}_0(u_{1,i})$ is conjugate to an element in $\mathcal{R}_1$, ii) the Salem factor of any element in $\mathcal{R}_1$ is one of the two $g_0 \mathcal{L}_0(u_{1,i})$, iii) $\mathcal{R}_1$ has 28 elements.

Case 2). There are exactly two units, say $u_{2,i}, i = 1, 2$, in $\mathcal{U}$ such that the twists $L_0(u_{2,i})$ have signature (1, 5). Then $G(L_0(u_{2,i})) \cong \mathbb{F}_2^5 \oplus \mathbb{F}_2$ and the orders of $g_0 G(L_0(u_{2,i}))$ are equal to 6. Let $\mathcal{L}$ be the lattice with Gram matrix

$$
\begin{pmatrix}
-4 & 15 & -2 & -6 \\
15 & -58 & 9 & 23 \\
-2 & 9 & -4 & -2 \\
-6 & 23 & -2 & -12
\end{pmatrix}.
$$

Then $q_M = -q_{L_0(a_2,a_1)}$. Thus, if $g_0 | L_0(u_2,a_1)$ is the Salem factor of an isometry of $E_{10}$, then the orthogonal complement $L_0(u_1,a_1)^L \subset E_{10}$ must be isomorphic to $M$ (see [Nip91]). By considering all possible glue between $g_0 | L_0(u_2,a_1)$ and isometries of $M$, we find a subset $\mathcal{R}_2 \subset O(E_{10})$ such that i) any isometry of $E_{10}$ whose Salem factor is one of the four $g_0 | L_0(u_2,a_1)$ is conjugate to an element in $\mathcal{R}_2$, ii) the Salem factor of any element in $\mathcal{R}_2$ is one of the four $g_0 | L_0(u_2,a_1)$, iii) $\mathcal{R}_2$ has 24 elements.

In this way, we obtain the following lemma.

**Lemma 9.18.** Let $\mathcal{R}_1$, $\mathcal{R}_2$ be as above. There exists a subset $\mathcal{R} \subset O(E_{10})$ consisting of 52 elements such that any element in $O(E_{10})$ whose characteristic polynomial divided by $S_5(x)$ is conjugate to some element in $\mathcal{R}$. Moreover, $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ is explicitly given, and the set of mod 2 reduction of characteristic polynomials of elements in $\mathcal{R}$ consists of

\[(9.1)\quad (1 + x)^4(1 + x + x^2)^3,
\]

\[(9.2)\quad (1 + x + x^2)^3(1 + x + x^2 + x^3 + x^4),
\]

\[(9.3)\quad (1 + x)^2(1 + x + x^2)^4.
\]

**Lemma 9.19.** Let $f \in O(L^-)$ of finite order. Suppose $\chi_f(x) = C_1(x)C_2(x)$, where $C_1(x)$ is a product of cyclotomic polynomials in $\{\Phi_3(x), \Phi_6(x), \Phi_{12}(x)\}$, and $C_2(x)$ is a product of cyclotomic polynomials in $\{\Phi_1(x), \Phi_2(x), \Phi_4(x), \Phi_5(x), \Phi_8(x), \Phi_{10}(x)\}$. Suppose $\deg(C_1(x)) = 6$, and

$$
\chi_{f|G(L^-)}(x) = (1 + x)^4(1 + x + x^2)^3, \text{ or } (1 + x + x^2)^3(1 + x + x^2 + x^3 + x^4)
$$

i.e., the cases $9.1$ and $9.2$ in Lemma 9.18. Let

$$
L_1 := \text{Ker}(C_1(f)) \subset L^-, \quad L_2 := \text{Ker}(C_2(f)) \subset L^-.
$$

Then

i) If $\text{sig}(L_1) = (2,4)$, then $L_2$ has roots.

ii) If $\text{sig}(L_1) = (0,6)$, then $L_1$ is isometric to either $E_6(2)$ or $A_2(2)^{\oplus 3}$.

**Proof.** Let $f_i := f|L_i$, $i = 1,2$. By the assumption, there exist subgroups $H_i \subset G(L_i)$ and a gluing map $\phi : H_1 \rightarrow H_2$ such that $L^- = L_1 \oplus_\phi L_2$ and $f = f_1 \oplus_\phi f_2$. Then $f^n = f_1^n \oplus_\phi f_2^n$, for any $n > 0$. Then $f^{40} = f_1^{40} \oplus_\phi f_2^{40}$. Since $\chi_{f_1^{40}}(x) = \Phi_3(x)^3$ and $\chi_{f_2^{40}}(x) = \Phi_1(x)^6$, by Lemma 4.4, $H_1 \cong H_2 \cong F_k^3$, for some $k$. Here we have $k \leq 3$, because the dimension of the eigen space (of eigenvalue 1) of the action of $f_1^{40}$ on $F_1^{40}/L_1$ is exactly three. Thus, by Lemma 4.1 and assumption on $\chi_{f|G(L^-)}(x)$, $G(L_1) \cong F_2^6 \oplus F_3^k$ and $G(L_2) \cong F_2^4 \oplus F_3^k$. By Lemma 2.2, $L_1(1/2)$ is an even 3-elementary lattice of rank 6 and determinant $3^k$.

If $\text{sig}(L_1) = (2,4)$, then by classification of 3-elementary lattices (cf. [CS99] Chapter 15, Theorem 13), $k = 1$ or 3. If $k = 1$, then $L_2$ is a negative definite even lattice of determinant $2^4 \cdot 3$ and rank 6, and hence $L_2$ has roots (see [Mo44] Page3). If $k = 3$, then $L_2$ has roots
since $L_2$ must be isometric to the lattice with Gram matrix

$$
\begin{pmatrix}
-4 & 2 & 0 & 0 & 0 & 0 \\
2 & -4 & 0 & 0 & 0 & 0 \\
0 & 0 & -2 & 0 & -1 & -1 \\
0 & 0 & 0 & -2 & -1 & -1 \\
0 & 0 & -1 & -1 & -4 & -1 \\
0 & 0 & -1 & -1 & -1 & -4 \\
\end{pmatrix}.
$$

(In fact, any even negative definite lattice of rank 6 with the same discriminant form as this lattice must be isometric to this lattice, which can be verified by computer algebra (Magma).)

If $\text{sig}(L_1) = (0, 6)$, then by classification of 3-elementary lattices again, $k = 1$ or 3. Then $L_1(1/2)$ is isometric to either $E_6$ or $A_2^\oplus 3$ (by Magma again). Thus, $L_1$ is isometric to either $E_6(2)$ or $A_2(2)^\oplus 3$. This completes the proof of the Lemma. □

Lemma 9.20. Let $f \in O(L^-)$ of finite order. Suppose $\chi_f(x) = C_1(x)C_2(x)$, where $C_1(x)$ is a product of cyclotomic polynomials in $\{\Phi_3(x), \Phi_6(x), \Phi_{12}(x), \Phi_{24}(x)\}$, and $C_2(x)$ is a product of cyclotomic polynomials in $\{\Phi_1(x), \Phi_2(x), \Phi_4(x), \Phi_8(x)\}$. Suppose $\deg(C_2(x)) = 8$, and

$$
\chi_{f|G(L^-)}(x) = (1 + x)^2(1 + x + x^2)^4
$$

i.e., the case 9.3 in Lemma 9.19. Let

$$
L_1 := \text{Ker}(C_1(f)) \subset L^-; L_2 := \text{Ker}(C_2(f)) \subset L^-.
$$

Then

i) If $\text{sig}(L_1) = (2, 6)$, then $L_2$ has roots.

ii) If $\text{sig}(L_1) = (0, 8)$, then $L_1$ is isometric to one of the following four lattices: $E_8(2)$, $E_6(2) \oplus A_2(2)$, $A_2(2)^\oplus 4$, $M'(2)$, where $M'$ is the lattice with Gram matrix

$$
\begin{pmatrix}
-2 & -1 & 0 & -1 & 0 & 1 & -1 & 1 \\
-1 & -2 & -1 & -1 & 0 & 1 & -1 & 0 \\
0 & -1 & -2 & -1 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & -2 & 1 & 1 & -1 & 1 \\
0 & 0 & 0 & 1 & -4 & -2 & 2 & -2 \\
1 & 1 & 0 & 0 & 1 & -2 & -4 & 1 & -2 \\
-1 & -1 & 0 & -1 & 2 & 1 & -4 & 2 \\
1 & 0 & 0 & 1 & -2 & -2 & 2 & -4 \\
\end{pmatrix}.
$$

Proof. The proof is similar to that of Lemma 9.19.

Let $f_i := f|L_i$, $i = 1, 2$. By the assumption, there exist subgroups $H_i \subset G(L_i)$ and a gluing map $\phi : H_1 \rightarrow H_2$ such that $L^- = L_1 \oplus \phi L_2$ and $f = f_1 \oplus \phi f_2$. Then $f^n = f_1^n \oplus \phi f_2^n$, for any $n > 0$. Then $f^8 = f_1^8 \oplus \phi f_2^8$. Since $\chi_{f_1^8}(x) = \Phi_3(x)^4$ and $\chi_{f_2^8}(x) = \Phi_1(x)^4$, by Lemma 4.4 $H_1 \cong H_2 \cong F_3^k$ for some $k \leq 4$ (since $\text{rk}(L_2) = 4$). Thus, by Lemma 4.4 and assumption on $\chi_{f|G(L^-)}(x)$, $G(L_1) \cong F_3^8 \oplus F_3^k$ and $G(L_2) \cong F_2^8 \oplus F_3^k$. By Lemma 2.2 $L_1(1/2)$ is an even 3-elementary lattice of rank 8 and determinant $3^k$.

If $\text{sig}(L_1) = (2, 6)$, then by classification of 3-elementary lattices, $k = 0, 2$ or 4. If $k = 0$ or 2, then $L_2$ is a negative definite even lattice of determinant $2^2$ or $2^2 \cdot 3^2$ and rank 4, and...
hence $L_2$ has roots (see [Mo44, Page3]). If $k = 4$, then, by the classification of indefinite 3-elementary lattices, $L_1 = A_2(-2) \oplus A_2(2) \oplus 3$, $L_2(1/3)$ is a well-defined even 2-elementary lattice, and $L_2(1/3) \cong D_4$ (see [CSSS Table 1]). On the other hand, by computer algebra, there exists no gluing map between $G(A_2(-2) \oplus A_2(2) \oplus 3)$ and $G(D_4(3))_3$, a contradiction. Thus, $k = 4$ is impossible.

If $\text{sig}(L_1) = (0,8)$, then by classification of 3-elementary lattices again, $k = 0,2$ or 4. Then $L_1(1/2)$ is isometric to either $E_8$, $E_6 \oplus A_2$, $A_2(3)$, or $M'$ (again by Magma). Thus, $L_1$ is isometric to $E_8(2)$, $E_6(2) \oplus A_2(2)$, $A_2(2) \oplus 4$, or $M'(2)$ (note that $A_2(2) \oplus 4$ and $M'(2)$ are non-isometric but they have the same discriminant form). This completes the proof of the Lemma.

\begin{theorem}
The Salem number $\tau_5$ can not be realized by an automorphism of any Enriques surface.
\end{theorem}

\begin{proof}
Suppose $\tau_5$ can be realized. Then, by Theorem 7.4 there exists an Enriques quadruple $(f^+, f^-, T, \phi)$ of entropy $\log \tau_5$. Thus, we may and will assume $f^+$ is an element in $\mathcal{R}$ (cf. Lemma 9.18). Then

$$\chi_{f^+}(x) \equiv (1 + x)^2 \chi_{f^+}(x) \equiv (1 + x)^2 Q(x) \mod 2,$$

where $Q(x)$ is $(1 + x)^4(1 + x + x^2)^3$, $(1 + x + x^2)^3(1 + x + x^2 + x^3 + x^4)$, or $(1 + x)^2(1 + x + x^2)^4$. We set

$$\mathcal{R}_3 := \{g \in \mathcal{R}| \chi_{f^+}(x) \equiv \Phi_1(x) \Phi_3(x)^3, \text{ or } \bar{\Phi}_1(x) \Phi_3(x)^3 \mod 2\},$$

$$\mathcal{R}_4 := \{g \in \mathcal{R}| \chi_{f^+}(x) \equiv \Phi_1(x)^2 \Phi_3(x)^4 \mod 2\}.$$

One observes that $\mathcal{R} = \mathcal{R}_3 \cup \mathcal{R}_4$.

Then we may write $\chi_{f^+}(x) = C_1(x)C_2(x)$, where $C_1(x)$ is a product of cyclotomic polynomials in $\{\Phi_3(x), \Phi_6(x), \Phi_{12}(x), \Phi_{24}(x)\}$, and $C_2(x)$ is a product of cyclotomic polynomials in $\{\Phi_1(x), \Phi_2(x), \Phi_4(x), \Phi_5(x), \Phi_8(x), \Phi_{10}(x)\}$. Moreover, $\deg(C_1(x)) = 6$ or 8. Let

$L_1 := \ker(C_1(f^-)) \subset L^-$, $L_2 := \ker(C_2(f^-)) \subset L^-$.

Then by Lemmas 9.19 and 9.20, $T \subset L_2$ and $L_1$ is isometric to one of the following six lattices: i) $E_6(2)$, ii) $A_2(2) \oplus 3$, iii) $E_8(2)$, iv) $E_6(2) \oplus A_2(2)$, v) $A_2(2) \oplus 4$, vi) $M'(2)$.

We treat each of these six cases separately.

Case i). Then $\deg(C_1(x)) = 6$, $C_1(x) \equiv \Phi_3(x)^3 \mod 2$, and $f^+ \in \mathcal{R}_3$. Let $\mathcal{C}_{E_6(2)}$ denote a set of representatives $g$ of conjugacy classes $[g]$ in $O(E_6(2))$ such that $\chi_{g}(x) \equiv \Phi_3(x)^3 \mod 2$. Now we proceed the same method as in Lemma 9.4. First of all $\mathcal{C}_{E_6(2)}$ contains exactly 6 elements. One can verify that for any $g_1 \in \mathcal{C}_{E_6(2)}$, $g_2 \in \mathcal{R}_3$, and any subgroup $H \subset G(L^+)$ isomorphic to $\mathbb{F}_2^6$, there exists no gluing map $\psi : G(E_6(2))_2 \rightarrow H$ such that both of the following two statements are true: i) the map $g_1 \oplus g_2$ extends to $g_1 \oplus \psi g_2 \in O(E_6(2) \oplus \psi L^+)$, and ii) $g_1 \oplus \psi g_2$ is positive. Thus, $L_1 \cong E_6(2)$ is impossible.

Case ii). Then $\deg(C_1(x)) = 6$, $C_1(x) \equiv \Phi_3(x)^3 \mod 2$, and $f^+ \in \mathcal{R}_3$. Let $\mathcal{C}_{A_2(2) \oplus 3}$ denote a set of representatives $g$ of conjugacy classes $[g]$ in $O(A_2(2) \oplus 3)$ such that $\chi_{g}(x) \equiv \Phi_3(x)^3 \mod 2$. It turns out that $\mathcal{C}_{A_2(2) \oplus 3}$ contains exactly 8 elements. One can verify that for any $g_1 \in \mathcal{C}_{A_2(2) \oplus 3}$, $g_2 \in \mathcal{R}_3$, and any subgroup $H \subset G(L^+)$ isomorphic to $\mathbb{F}_2^6$, there exists no gluing map $\psi : G(A_2(2) \oplus 3)_2 \rightarrow H$ such that the map $g_1 \oplus g_2$ extends to $g_1 \oplus \psi g_2 \in O(A_2(2) \oplus \psi L^+)$. Thus, $L_1 \cong A_2(2) \oplus 3$ is impossible.
Case iii). This case is free from computer algebra. Then \( \deg(C_1(x)) = 8, C_1(x) \equiv \Phi_3(x)^4 \mod 2 \) and \( f^+ \in R_4 \). Since \( E_8(2) \) is 2-elementary, \( L_2 = U \oplus U(2) \) (cf. proof of Lemma 10.19). Then by Lemma 11.4 \((f^+, f^-, T, \phi)\) can not be an Enriques quadruple, a contradiction. Thus, \( L_1 \cong E_8(2) \) is impossible.

Case iv). Then \( \deg(C_1(x)) = 8, C_1(x) \equiv \Phi_3(x)^4 \mod 2 \) and \( f^+ \in R_4 \). Let \( C_{E_6(2) \oplus A_2(2)} \) denote a set of representatives \( g \) of conjugacy classes \([g] \) in \( O(E_6(2) \oplus A_2(2)) \) such that \( \chi_g(x) \equiv \Phi_3(x)^4 \mod 2 \). It turns out that \( C_{E_6(2) \oplus A_2(2)} \) contains exactly 12 elements. For the same reason as in case i), this case is impossible.

Case v). Then \( \deg(C_1(x)) = 8, C_1(x) \equiv \Phi_3(x)^4 \mod 2 \) and \( f^+ \in R_4 \). Let \( C_{A_2(2)^{\oplus 4}} \) denote a set of representatives \( g \) of conjugacy classes \([g] \) in \( O(A_2(2)^{\oplus 4}) \) such that \( \chi_g(x) \equiv \Phi_3(x)^4 \mod 2 \). It turns out that \( C_{A_2(2)^{\oplus 4}} \) contains exactly 16 elements. We define \( \mathcal{D} \) as the set of pairs

\[
(A_2(2)^{\oplus 4} \oplus_\psi L^+, g_1 \oplus_\psi g_2)
\]

such that 1) \( \psi : G(A_2(2)^{\oplus 4})_2 \to H \) is a gluing map, for some subgroup \( H \subset G(L^+) \) isomorphic to \( \mathbb{F}_2^8 \), and 2) \( g_1 \oplus_\psi g_2 \in O(A_2(2)^{\oplus 4} \oplus_\psi L^+) \) is positive.

We define \( \mathcal{D}_1 \) to be the set of pairs \((L', f') \in \mathcal{D}\) such that

\[
G(L') \cong \mathbb{F}_2^2 \oplus \mathbb{F}_3^4, \quad \text{Ord}(\overline{f'}|G(L')_2) = 1, \quad \text{Ord}(\overline{f'}|G(L')_3) = 4
\]

and

\[
\chi_{f'}(x) = (1 + x)^2(1 - x + x^2)^2(1 + x + x^2)(1 - x^2 + x^4)(1 - x - x^3 - x^5 + x^6).
\]

We define \( \mathcal{D}_2 \) to be the set of pairs \((L', f') \in \mathcal{D}\) such that

\[
G(L') \cong \mathbb{F}_2^2 \oplus \mathbb{F}_3^4, \quad \text{Ord}(\overline{f'}|G(L')_2) = 1, \quad \text{Ord}(\overline{f'}|G(L')_3) = 8
\]

and

\[
\chi_{f'}(x) = (1 + x)^2(1 - x + x^2)(1 - x - x^3 - x^5 + x^6)(1 - x^4 + x^8).
\]

It turns out that both \( \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are not empty, and \( \mathcal{D} = \mathcal{D}_1 \cup \mathcal{D}_2 \).

Now we are going to consider if there exists some \((L', f') \in \mathcal{D}_i, i = 1, 2\), which can be glued with a pair \((L, f)\) to obtain an Enriques quadruple. It turns out \((L, f)\) must satisfy the following conditions

1) there exists a gluing map \( \theta : G(L) \to G(L') \) such that \( L \oplus L' \cong \Pi_{3,19} \) and \( f \oplus f' \)

extends to \( f \oplus f' \in O(L \oplus L') \).

2) \( L \cong U(3) \oplus U(6), f \in O(L) \) (a direct consequence of 1)),

3) \( \chi_f(x) = (-1 + x)(1 + x)^2 \) when \( i = 1 \), and \( \chi_f(x) = \Phi_8(x) \) when \( i = 2 \),

4) \( \ker(f^2 + 1) \subset L \) is of signature \((2, 0)\) when \( i = 1 \).

However, it turns out that, for any \((L', f') \in \mathcal{D}_i, i = 1, 2\), there exists no \((L, f)\) satisfying these four conditions 1)-4). Thus, \( L_1 \cong A_2(2)^{\oplus 4} \) is impossible.

Case vi). Then \( \deg(C_1(x)) = 8, C_1(x) \equiv \Phi_3(x)^4 \mod 2 \) and \( f^+ \in R_4 \). Let \( C_{M'(2)} \) denote a set of representatives \( g \) of conjugacy classes \([g] \) in \( O(M'(2)) \) such that \( \chi_g(x) \equiv \Phi_3(x)^4 \mod 2 \). It turns out that \( C_{M'(2)} \) contains exactly 9 elements. For the same reason as in case i), this case is impossible.

This completes the proof of the theorem.
Table 1. The eight candidate small Salem numbers $\tau_i$

| $\tau_i$ | $S_i(x) := \text{minimal polynomial of } \tau_i$ | factorization of $S_i(x)$ in $\mathbb{F}_2[x]$ |
|-----------|---------------------------------|---------------------------------------------------|
| $\tau_1$ | $1 - x - x^2 + x^5 - x^6 - x^8$ | $(1 + x + x^5)(1 + x^2 + x^6)$ |
| $\tau_2$ | $1 - x^2 - x^3 + x^5$ | $(1 + x + x^4)(1 + x + x^2 + x^3)^2$ |
| $\tau_3$ | $1 - x - x^3 + x^5 - x^6 - x^8$ | $(1 + x + x^4)(1 + x^2 + x^3)^2$ |
| $\tau_4$ | $1 - x^2 - x^3 + x^5 - x^6 + x^8$ | $(1 + x)^2(1 + x^3 + x^6)$ |
| $\tau_5$ | $1 - x - x^3 - x^5 + x^6$ | $(1 + x + x^7)^2$ |
| $\tau_6$ | $1 - x^2 - x^3 - x^5 - x^6 + x^8$ | $(1 + x + x^4 + x^5 + x^6)(1 + x^3 + x^6)$ |
| $\tau_7$ | $1 - x - x^3 + x^5 - x^6 - x^8$ | $(1 + x + x^4)(1 + x^2 + x^3)^2$ |
| $\tau_8$ | $1 - x^2 - 2x^3 - x^5 + x^6$ | $(1 + x)^2$ |

Table 2. Cyclotomic polynomials of degree $\leq 12$ and their reductions mod 2

| $i$ | $\Phi_i(x)$ | factorization of $\Phi_i(x)$ in $\mathbb{F}_2[x]$ |
|-----|-------------|-----------------------------------------------|
| 1   | $1 + x$     | $1 + x$                                        |
| 2   | $1 + x$     | $1 + x$                                        |
| 3   | $1 + x + x^2$ | $1 + x + x^2$                                |
| 4   | $1 + x^2$   | $(1 + x)^2$                                   |
| 5   | $1 + x + x^2 + x^4 + x^6$ | $1 + x + x^3 + x^4$ |
| 6   | $1 - x + x^2$ | $1 - x + x^2$                                |
| 7   | $1 + x + x^2 + x^4 + x^5 + x^6$ | $(1 + x + x^3)(1 + x^2 + x^3)$ |
| 8   | $1 + x^5$   | $(1 + x)^4$                                   |
| 9   | $1 + x^2 + x^5$ | $1 + x^3 + x^5$                              |
| 10  | $1 - x + x^2 - x^3 + x^4$ | $1 + x + x^3 + x^5$ |
| 11  | $1 + x + x^2 + x^3 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11}$ | $1 + x + x^7$ |
| 12  | $1 - x + x^2$ | $(1 + x^3)^2$                                |
| 13  | $1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^8 + x^9 + x^{10} + x^{11} + x^{12}$ | $1 + x + x^3 + x^4 + x^7 + x^8 + x^9 + x^{10}$ |
| 14  | $1 - x + x^2 - x^3 + x^4 - x^5 + x^6$ | $(1 + x + x^3)(1 + x^2 + x^3)^2$ |
| 15  | $1 - x + x^2 - x^3 + x^5 - x^6$ | $(1 + x^3)(1 + x^3 + x^4)$ |
| 16  | $1 + x^5$   | $(1 + x)^5$                                   |
| 17  | $1 - x + x^2$ | $1 + x^3 + x^5$                              |
| 18  | $1 - x + x^2$ | $1 + x^3 + x^5$                              |
| 19  | $1 + x^3 - x^4 + x^6 + x^7$ | $(1 + x + x^3 + x^4)^2$ |
| 20  | $1 - x + x^2 + x^3 - x^4 + x^5 - x^6 + x^7 + x^8 - x^9 + x^{10} + x^{11}$ | $1 + x + x^3 + x^5 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12}$ |
| 21  | $1 - x + x^2 + x^3 - x^4 + x^5 - x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12}$ | $1 + x + x^7$ |
| 22  | $1 - x + x^2 + x^3 + x^4 - x^5 + x^6 - x^7 - x^9 + x^{10} + x^{11} + x^{12}$ | $1 + x + x^7$ |
| 23  | $1 - x + x^5$ | $(1 + x)^3$                                   |
| 24  | $1 - x^2 + x^3 - x^5 - x^6 + x^7 + x^8 - x^9 + x^{10} + x^{11} + x^{12}$ | $(1 + x^3)^2(1 + x^7 + x^8)$ |
| 25  | $1 - x + x^2 - x^3 + x^5 + x^7$ | $(1 + x + x^3)(1 + x^3 + x^4)$ |
| 26  | $1 - x + x^2 + x^3 + x^4 - x^5 + x^6 + x^7 + x^8 + x^9 + x^{10} + x^{11} + x^{12}$ | $(1 + x + x^3)^2(1 + x^8)^2$ |
| 27  | $1 + x - x^2 - x^3 - x^4 + x^5 + x^6$ | $(1 + x + x^3)(1 + x^3 + x^4)$ |
| 28  | $1 - x + x^2 + x^3 + x^4 - x^5 + x^6$ | $(1 + x + x^3)(1 + x^8)$ |
| 29  | $1 - x + x^2 + x^3 + x^4$ | $(1 + x^3)^2$                                |

Appendix A. Tables

References

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Table 3. The minimum Salem number $\lambda_{2d}$ in degree $2d$

| $\lambda_{2d}$ | Minimal polynomial of $\lambda_{2d}$ |
|----------------|-----------------------------------|
| $\lambda_0$    | $1 + x - x^3 - x^4 + x^5 + x^9 + x^{11}$ |
| $\lambda_1$    | $1 - x^3 - x^5 + x^{10}$ |
| $\lambda_2$    | $1 - x^2 - x^4 + x^8$ |
| $\lambda_3$    | $1 - x - x^2 + x^8$ |
| $\lambda_4$    | $1 - 3x + x^7$ |

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