On Gauge Invariance and Covariant Derivatives in Metric Spaces

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Abstract

In this manuscript, we will discuss the construction of covariant derivative operator in quantum gravity. We will find it is appropriate to use affine connections more general than metric compatible connections in quantum gravity. We will demonstrate this using the canonical quantization procedure. This is valid irrespective of the presence and nature of sources. The standard Palatini formalism, where metric and affine connections are the independent variables, is not sufficient to construct a source-free theory of gravity with affine connections more general than the metric compatible Levi-Civita connections. This is also valid for minimally coupled interacting theories where sources only couple with metric by using the metric compatible Levi-Civita connections exclusively. We will discuss a potential formalism and possible extensions of the action to introduce nonmetricity in these cases. This is also required to construct a general interacting quantum theory with dynamical general affine connections. We will have to use a modified Ricci tensor to state Einstein’s equation in the Palatini formalism. General affine connections can be described by a third rank tensor with one contravariant and two covariant indices. Antisymmetric part of this tensor in the lower indices gives torsion with a half factor. In the Palatini formalism or its generalizations considered here, symmetric part of this tensor in the lower indices is finite when torsion is finite. This part can give a massless scalar field in a potential formalism. We will have to extend the local conservation laws when we use general affine connections. General affine connections can become significant to solve cosmological problems.

Key-words: connections, nonmetricity, Palatini action, $f(P)$ gravity, scalar field

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I. Introduction

In this manuscript, we will discuss a few aspects of spacetime geometry relevant to quantum gravity. In Sec.II, we will discuss the construction of covariant derivative operator in quantum gravity. We can introduce a metric in a spacetime manifold provided it satisfies a few very general topological conditions [1]. To construct covariant differential equations for different tensor fields, we have to introduce additional structures in spacetime. These structures are known as connections. They can be introduced in a differential manifold independent of metric [2,3]. We can construct covariant derivatives in a manifold in terms of ordinary partial derivatives and connections. Covariant derivatives have a few general properties which are related to the corresponding properties of ordinary derivatives [2,3,4]. We will mention these properties in Sec.II. The most general connections for which covariant derivatives have these properties are known as affine connections or connection coefficients [2]. In the rest of this article, we will deal with only affine connections and denote general affine connections by affine connections or connection coefficients. The well-known Levi-Civita connections are a special set of affine connections which obey additional conditions. In section:II, we will find it is appropriate to use affine connections more general than metric compatible connections in quantum gravity. We will use the canonical quantization procedure to show this. This is valid irrespective of the presence and nature of sources. We have not considered the dynamics of quantum gravity. This is an involved problem due to presence of nontrivial constraints [4,5,6,7,8,9]. We have only considered a general mathematical issue which will be there, in a theory of quantum gravity which is not a quantum field theory in a fixed background, provided at least one component of metric can be taken as an independent variable in a neighborhood of the spacetime manifold. This can be done around any non-singular point of the manifold [7,8,9]. We have discussed these aspects in Sec.II. Affine connections may be asymmetric in the lower indices and need not to be compatible with metric or any symmetric second rank covariant tensor [2]. Affine connections give an additional field besides metric. This field is described by a third rank tensor with one contravariant index and two covariant indices. In this article, we denote this field by tensor \( C_{\alpha \mu \nu} \). Antisymmetric part of this tensor in the covariant indices is half of torsion [2], and give antisymmetric fluctuations in affine connections away from the Levi-Civita connections. Symmetric fluctuations of affine connections off the Levi-Civita connections are described by the symmetric part of \( C_{\alpha \mu \nu} \) in the lower indices and this field is not considered in conventional theories of quantum gravity even with sources. In the Palatini formalism or its generalizations considered in this article, this field is coupled with torsion and is usually finite when torsion is so. This is important since coupling of gravity with fermions introduces torsion in spacetime through local Lorentz invariance in the semiclassical limit of quantum gravity which presently is locally Lorentz invariant quantum field theory in curved spaces [10]. Metric is regarded as a classical variable in this theory and is given by Einstein’s equation. General affine connections can also get us beyond a locally Lorentz invariant theory of quantum fields in curved spaces. The symmetric part of \( C_{\alpha \mu \nu} \) in the lower indices can be described by a symmetric second rank covariant tensor when the corresponding Ricci tensor is symmetric. In section:III, we will discuss the suitability of the standard Palatini formalism, where metric and affine connections are the independent variables, to construct a quantum theory of gravity. We will consider two classes of theories. The first class consists of source free theory and also minimally coupled interacting theories where sources are only coupled with metric by using the Levi-Civita connections exclusively. Here, sources do not couple with \( C_{\alpha \mu \nu} \) explicitly. We call this class as generalized free theory. The second class consists of sources that explicitly couple with both metric and \( C_{\alpha \mu \nu} \). We will call this class as generalized interacting theory. The Palatini action leads to metric compatible Levi-Civita connections in generalized free theory and algebraic equations for \( C_{\alpha \mu \nu} \) in generalized interacting theory. We will have to extend the formalism to introduce nonmetricity in generalized free theory. We will also have to extend the formalism if we want to have dynamical equations for \( C_{\alpha \mu \nu} \) in generalized interacting theory. We can extend the formalism either by using a potential description of affine connections or by replacing the action by more general actions as can be done in \( f(P) \) gravity, where \( P \) is scalar curvature with connections more general than the Levi-Civita connections. We can also include higher order curvature scalars with finite \( C_{\alpha \mu \nu} \). We can use a potential description with more general actions. We have discussed a particular model in section:IV. General affine connections can give spin 3, 2, 1 and spin zero bosons in a quantum theory. We also have a massless scalar field in the potential formalism with torsion as one of its sources. We will later discuss if we can have massive scalar particles using \( f(P) \) gravity or other generalizations of the Palatini formalism. In the Palatini formalism, Einstein’s equation is expressed in terms of a modified Ricci tensor which gives a modified Einstein tensor. The antisymmetric parts of both the modified Ricci tensor and Ricci tensor are given by affine connections that are uniquely determined by the sources. We will have to impose additional constraints on the sources to make these tensors purely symmetric. We also note that the energy-momentum conservation law no longer remains exactly valid when we use affine connections more general than metric compatible affine connections. The above observations can become significant to
explain cosmological problems like inflation and dark energy. General affine connections can be important for potential description of electrodynamics, gauge invariance and the local charge conservation law. Gauge invariance does not necessarily imply the local charge conservation law when we use a general set of affine connections. We will discuss these aspects briefly in section IV.

II. Quantum Gravity and Covariant Derivatives

In this section, we will address the issue of construction of a covariant derivative operator in quantum gravity. We first consider the classical case. Four properties of ordinary derivatives and transformation rules of tensors under the change of coordinate systems are used to construct covariant derivatives in a general manifold [2,4]. Consequently, covariant derivatives in a curved spacetime inherit these properties. We state them in the following:

(I) The linearity property: \( \nabla_{\mu}(ap_{\alpha_1\beta_1} + bq_{\alpha_1\beta_1}) = a\nabla_{\mu}p_{\alpha_1\beta_1} + b\nabla_{\mu}q_{\alpha_1\beta_1} \). Where, \( a, b \) are two constants and \( p_{\alpha_1\beta_1}, q_{\alpha_1\beta_1} \) are two well-behaved tensor fields.

(II) For a well-behaved scalar field \( f \), and a vector field \( t^\mu \), \( t(f) = t^\mu \nabla_{\mu}(f) \). Here, \( t(f) \) denotes directional derivative of the scalar field in the direction of the vector field.

(III) The Leibnitz rule:

\[
\nabla_{\mu}(p_{\alpha_1\beta_1}q_{\alpha_2\beta_2}) = \nabla_{\mu}p_{\alpha_1\beta_1}q_{\alpha_2\beta_2} + p_{\alpha_1\beta_1}\nabla_{\mu}q_{\alpha_2\beta_2}
\]

Here, \( p_{\alpha_1\beta_1}, q_{\alpha_2\beta_2} \) are two arbitrary well-behaved tensor fields.

(IV) Commutativity between contraction and covariant derivative:

\[
\nabla_{\mu}[C(p_{\alpha_1\beta_1})] = C(\nabla_{\mu}[p_{\alpha_1\beta_1}])
\]

Here, \( C(\ ) \) denotes contraction operation between upper and lower indices. These properties and tensorial character of covariant derivatives lead to the following property:

(V) We define torsion tensor through the following relation:

\[
[\nabla_{\mu}\nabla_{\nu} - \nabla_{\nu}\nabla_{\mu}]f = -T^\alpha_{\mu\nu}\nabla_{\alpha}f
\]

where, \( f \) is a well-behaved scalar field. \( T^\alpha_{\mu\nu} \) is torsion tensor.

We can construct a covariant derivative operator having properties (I - IV):

\[
\nabla_{\mu}A_{\nu} = \nabla_{\nu}A_{\mu} - \Theta^\alpha_{\mu\nu}A_{\alpha}
\]

Here, \( \Theta^\alpha_{\mu\nu} \) are connections. In general relativity, we choose \( \nabla_{\mu} = \partial_{\mu} \), where \( \partial_{\mu} \) is ordinary partial derivative and we have: \( t(f) = t^\mu \partial_{\mu}(f) \). Corresponding connections for which \( \nabla_{\mu} \) is a covariant derivative operator and satisfy the above conditions are known as affine connections or connection coefficients [2]. In this article, we have used the terms affine connections or connection coefficients to denote the most general affine connections. The well known Levi-Civita connections are a special set of affine connections that obey additional conditions to be mentioned below. Tensorial character of covariant derivatives impose the following transformation rule on affine connections:

\[
\Theta^\alpha_{\mu\nu} = \frac{\partial\bar{\xi}^\alpha}{\partial x^\lambda} \frac{\partial x^\mu}{\partial \bar{\xi}^\kappa} \frac{\partial x^\tau}{\partial \bar{\xi}^\kappa} \Theta^\lambda_{\kappa\tau} + \frac{\partial^2\bar{\xi}^\alpha}{\partial^2\bar{\xi}^\mu \partial \bar{\xi}^\nu} \Theta^\lambda_{\mu\nu}
\]

We can make \( \Theta^\alpha_{\mu\nu} \) symmetric in the lower indices. Corresponding connections are known as the Christoffel symbols [4]. We can introduce additional conditions on the Christoffel symbols. In general relativity, we introduce the Levi-Civita connections through the following metric compatibility conditions [4]:

\[
\nabla_{\mu}[g_{\alpha\beta}] = 0
\]

We have the following expressions for them:
\[ \Theta^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} = \frac{1}{2} \left[ \partial_\mu (g_{\nu\kappa}) + \partial_\nu (g_{\mu\kappa}) - \partial_\kappa (g_{\mu\nu}) \right] g^{\alpha\kappa} \]  

The above expression shows that the Levi-Civita connections are dependent on partial derivatives of metric and are symmetric in the lower indices. Note that, we can have other solutions of Eq.(6). We will later use such connections. Torsion is zero when connections are symmetric in the lower indices. Thus, the right hand side of Eq.(3) is zero when we use the Levi-Civita connections. The familiar solutions of Einstein’s equation in general relativity satisfy the torsion-free condition [4]. In this article, we will show that we have to use affine connections more general than metric compatible connections in quantum gravity.

We next consider the Palatini action to describe gravity. The action is given by [4]:

\[ S = \int \sqrt{-g} R e \]  

Where, \( g \) is the determinant of metric, \( \sqrt{-g} \) is the natural volume element associated with metric and \( R \) is the scalar curvature. In the Palatini formalism, both metric and affine connections are the independent variables. We have the following expression for covariant derivative operator [4]:

\[ \nabla_\mu A_\nu = \nabla'_\mu A_\nu - C^\alpha_{\mu\nu} A_\alpha \]  

Here, \( C^\alpha_{\mu\nu} \) is an arbitrary well-behaved field. It can be symmetric or asymmetric in the lower indices. \( \nabla'_\mu \) is a given covariant derivative which have properties (I - IV) and obey the torsion-free condition. We choose \( \nabla'_\mu \) to be given by the following expression:

\[ \nabla'_\mu A_\nu = \partial_\mu A_\nu - \Gamma^\alpha_{\mu\nu} A_\alpha \]  

Here, \( \Gamma^\alpha_{\mu\nu} \) are the Levi-Civita connections associated with metric \( g_{\mu\nu} \). With this choice of \( \nabla'_\mu \), \( C^\alpha_{\mu\nu} \) is a third rank tensor. This follows from the transformation properties of affine connections given by Eq.(5) and the definition of Levi-Civita connections given by Eq.(7). In many cases, the symmetric part of affine connections in Eq.(9) can be given by the following expression [2]:

\[ S^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \tilde{C}^\alpha_{\mu\nu} \]  

\[ = \frac{1}{2} [\partial_\mu (b_{\nu\kappa}) + \partial_\nu (b_{\mu\kappa}) - \partial_\kappa (b_{\mu\nu})] \tilde{b}^{\alpha\kappa} \]  

Where, \( \Gamma^\alpha_{\mu\nu} \) are the Levi-Civita connections and given by Eq.(7). \( \tilde{C}^\alpha_{\mu\nu} \) is the symmetric part of \( C^\alpha_{\mu\nu} \) in the lower indices. \( b_{\mu\nu} \) is a non-symmetric symmetric covariant tensor and can be expressed as:

\[ b_{\mu\nu} = g_{\mu\nu} + a_{\mu\nu} \]  

The inverse of \( b_{\mu\nu} \), \( \tilde{b}^{\mu\nu} \), is a contravariant tensor [2] and can be expressed as \( \tilde{b}^{\mu\nu} = g^{\mu\nu} + d^{\mu\nu} \). Note that \( \tilde{b}^{\mu\nu} \) is different from \( b^{\mu\nu} \) and \( S^\alpha_{\mu\nu} \) satisfy the compatibility conditions: \( b_{\mu\nu|\alpha} = 0 \), where the bar denotes covariant derivative with connections \( S^\alpha_{\mu\nu} \). We will later find that the Ricci tensor is symmetric when affine connections are exclusively given by Eq.(11). However, Eq.(11) is not valid when the Ricci tensor is not symmetric. The antisymmetric part of \( C^\alpha_{\mu\nu} \) in the lower indices gives half of torsion tensor [2]. In the Palatini action principle, metric and \( C^\alpha_{\mu\nu} \) are taken to be independent variables. The Riemann curvature tensor is now defined by the following expressions, [2,4,11]:

\[ (\nabla_{\mu} \nabla_{\nu} - \nabla_{\nu} \nabla_{\mu}) A^\alpha_{\beta} = -R_{\mu\nu\kappa\beta} A^\kappa_{\beta} + R_{\mu\nu\beta\kappa} A^\alpha_{\kappa} - T^\kappa_{\mu\nu} \nabla_\kappa A^\alpha_{\beta} \]  

\[ R_{\mu\nu\kappa\beta} = R'_{\mu\nu\kappa\beta} + 2 \nabla_\nu [C^\kappa_{\mu\alpha}] |_{\alpha} + 2 [C^\lambda_{\mu\alpha} |_{\alpha} (C^\kappa_{\nu\lambda})] \]  

\[ T^\kappa_{\mu\nu} = C^\kappa_{\mu\nu} - C^{\kappa}_{\nu\mu} \]  

Here, \( R'_{\mu\nu\kappa\beta} \) is the Riemann curvature tensor associated with the derivative \( \nabla'_\mu \) in Eq.(10), and is given by the familiar expression in terms of ordinary partial derivatives of \( \Gamma^\alpha_{\mu\nu} \) [4,11]. \( T^\kappa_{\mu\nu} \) is torsion tensor. The
second equation is always valid when $\Gamma_{\mu\nu}^\alpha$ is symmetric in the lower indices. The curvature scalar is obtained by usual contractions and is given by Eq.(19).

We now consider quantization of gravity by using the canonical quantization procedure. Canonical quantization is important to find the particle spectrum when we quantize a classical theory. In the canonical quantization of gravity, metric becomes operator on a Hilbert space. We represent such operators by carets. Affine connections present in the covariant derivatives act on the tensor operators and we represent them also by the symbols: $\Theta^\nu_{\mu\nu}$. Affine connections will contain components of metric and their spacetime derivatives and also other fields as evident from the previous discussions. In a Hamiltonian formulation, induced metric on a set of constant time surfaces is used as dynamical variable. We will discuss this construction later. The corresponding conjugate momenta are given in [4,9]. We use the symbols $\hat{h},\hat{\pi}$ to denote the corresponding collection of operators. In general, the Levi-Civita connections contain metric and time derivative of metric components and hence, will depend on the canonical conjugate variables ($\hat{h},\hat{\pi}$). We now express covariant derivative operator in the following form:

$$\nabla'_\mu \hat{A}_\nu = [\partial_\mu - \hat{\Gamma}^\nu_{\mu\nu}(\partial \hat{g}, \hat{g})] \hat{A}_\alpha = [\partial_\mu - \hat{\Gamma}^\alpha_{\mu\nu}(\hat{\pi}, \hat{h})] \hat{A}_\alpha$$  \hspace{1cm} (14)

Here, $\hat{\Gamma}_{\mu\nu}^\alpha$, are operator version of the Levi-Civita connections. We adopt the following operator ordering in connection coefficients. Whenever there appears a product between partial derivatives of metric and metric itself, the partial derivative is kept as the first term and metric is kept as the second term. The ordering of the operators ($\hat{h},\hat{\pi}$) in $\hat{\Gamma}_{\mu\nu}^\alpha$ is given to be the same as that written in the above equation, i.e., $\hat{h}$ is kept as the successor of $\hat{\pi}$. We now consider the operator: $q^\mu \nabla'_\mu q^\nu$, where $q^\mu$ is a vector field acting as $q^\mu \hat{I}$ on the Hilbert space. This operator contains canonical conjugate pairs of variables when we choose affine connections to be given by the Levi-Civita connections. In this case, we will have the following expression:

$$[q^\mu \nabla'_\mu q^\nu] |\Psi\rangle \neq 0$$  \hspace{1cm} (15)

remaining valid in a given state $|\Psi\rangle$ with an arbitrary well-behaved vector field $q^\mu$. We will not have a complete set of states for which the expectation value of the operator in the l.h.s is zero with negligible fluctuations for all well-behaved vector fields. This will be valid only in the classical limit, and is a subject similar to the familiar Ehrenfest’s theorems in non-relativistic quantum mechanics. Similar discussions will remain valid even if we choose affine connections to be given by the operator versions of Eqs.(9,10). In general, affine connections will contain canonical pairs of variables from metric sector to have proper classical limit of the Levi-Civita connections, and the concept of geodesics will not remain exact for all vector fields in a quantum state. This will also remain valid for parallel transport and the notion of parallel transport is not exact in a quantum theory of gravity. This is expected and indicates that we can use affine connections more general than the metric compatible connections in free quantum gravity. We now consider the metric compatibility conditions. The metric compatibility conditions given by Eq.(6) are to be replaced by the operator identity: $\nabla'_\mu [\hat{g}_{\alpha\beta}] \equiv 0$. The action of $\nabla'_\mu [\hat{g}_{\alpha\beta}]$ on any state is zero if connection operators are given by the Levi-Civita connection operators and we choose the operator ordering same as that mentioned below Eq.(14). Here, we always keep metric operators as the successors of the partial derivatives of themselves. Thus, the ordering of the different operators in the quantum version of the Levi-Civita connections will be the same as that given in Eq.(7). The same will also remain valid for $\nabla'_\mu [\hat{g}_{\alpha\beta}]$. Here, $\hat{g}_{\alpha\beta}$ will be kept at the right of the Levi-Civita connections. We also define the contravariant components of metric as: $\hat{g}^{\alpha\beta}$, $\hat{g}_{\alpha\beta} = \delta_{\beta}^\alpha$. This ordering leads to the operator identity: $\nabla'_\mu [\hat{g}_{\alpha\beta}] \equiv 0$ irrespective of the ordering of ($\hat{h},\hat{\pi}$) chosen in the partial derivatives of metric components. However, the operator version of metric compatibility conditions will not be consistent with a canonical quantization condition. We will demonstrate this in the following. As mentioned above, in a Hamiltonian formulation we use induced metric on a set of constant time surfaces as dynamical variable. Thus, $g_{\mu\nu}$ is replaced by: $\hat{g}_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu$. Here, $n_\mu$ is given by $(-N,0,0,0)$, $N$ being the lapse function. The contravariant $n^\mu$ is given by: $\frac{1}{N}(1,-N^1,-N^2,-N^3)$; where $N^i$ are the shift functions. We have: $g_{00} = N_k N^k - N^2$, $g_{0i} = N_i$ and $N_i = g_{ij} N^j$, [9]. The induced metric on the constant time surfaces coincide with the spatial part of $g_{\mu\nu}$ which are expressed as $\hat{g}_{ij}$. These fields are taken to dynamical variables in general relativity and we have the following Poisson brackets:

$$\{g_{ij}(t,\vec{x}), \pi^{kl}(t,\vec{y})\} = \delta_{ij} \delta_{kl} \delta(\vec{x}, \vec{y})$$  \hspace{1cm} (16)

Where, $\vec{x}$ refers to the spatial coordinates and the Poisson bracket is evaluated at equal time. The conjugate momentum is a spatial tensor density [4,9], and the delta function is defined without recourse to metric.
In a Hamiltonian formulation with the Einstein-Hilbert action, we have constraints and we can not naively replace the Poisson bracket by commutator when we try to quantize the theory [4]. There are two principal approaches to quantize the theory [8,9]. The first approach is similar to the Gupta-Bleuler method used to quantize electrodynamics and was initiated by Dirac [8,12]. In this approach all the classical variables are treated as independent variables and the constraints are imposed on the quantum states. Thus, we can replace the Poisson bracket in Eq.(16) by commutators when we quantize the theory. In the second approach, gauge fixing conditions are introduced to render the complete set of constraints second class. These conditions also determine the lapse and shift functions. We then pick two components of $g_{ij}$ as independent variables and quantize these components using standard commutation relations. We can solve the constraints to evaluate other commutators. We will demonstrate that it is not possible to use the Levi-Civita connections and impose the metric compatibility conditions, as long as we can regard at least one component of spatial metric on the constant time surfaces as an independent physical variable subjected to canonical quantization. We will also find that we can not have a Hilbert space on which we can impose the metric compatibility conditions when such quantization condition remains valid. These facts indicate that we can not use the Levi-Civita connections to quantize gravity. In the following, we will restrict our attention to a neighborhood around a regular point $'x'$. We can extend the neighborhood to the complete spacetime manifold leaving away singularities and other possible irregular points associated with the constraints [8,9].

We pick a particular component of spatial metric, say $g_{pl}$, as an independent physical variable. We then have the following equal time commutator:

$$[\hat{g}_{pl}(t, \vec{x}), \hat{\pi}^{\mu}(t, \vec{y})] = i\delta^{\mu}_{(\vec{y})}\delta^{\nu}_{(\vec{x})}[\delta(\vec{x}, \vec{y})]$$  
(17)

Where, the point $'y'$ belongs to the above mentioned neighborhood of $'x'$. The r.h.s of the commutator is taken to be a distribution that is a spatial tensor density in the spatial coordinates of $\hat{\pi}^{\mu}(t, \vec{y})$. The r.h.s will be supplemented by additional terms for dependent variables. We now consider the first order changes in both sides of the above equation when the point with coordinates $(t, \vec{x})$ is moved to a neighbouring point on the same spatial section. The new point obtained also lies within the neighborhood. This will lead to covariant derivatives $\nabla'_{xk}$ acting on variables with arguments $'x'$ in both sides of the above equation, where $k$ refers to the spatial coordinates $(x^1, x^2, x^3)$. The action of $\nabla'_{xk}$ on the r.h.s is same as that on a tensor and is given by partial derivatives of the delta function. This is not vanishing for all values of $\vec{x}$. The left hand side vanishes as can be found from the following expression:

$$\nabla'_{xk} \{\hat{g}_{pl}(t, \vec{x})\} \hat{\pi}^{\mu}(t, \vec{y}) - \hat{\pi}^{\mu}(t, \vec{y})\nabla'_{xk} \{\hat{g}_{pl}(t, \vec{x})\} = 0$$  
(18)

This follows since we are imposing the operator versions of the metric compatibility conditions. This can also be verified by applying the l.h.s of the above equation to any state, introducing a sum over a complete set of states between the products of the operators and using the fact that the action of $\nabla'_{\mu}[\hat{g}_{\alpha\beta}]$ on any state is zero if connection operators are given by the Levi-Civita connection operators with operator ordering chosen below Eq.(15). Thus, we obtain a contradiction when we have to consider application of $\nabla'_{xk}$ to operators with argument $\vec{x}$ in both sides of Eq.(17). We will obtain similar contradictions when we consider higher order changes in both sides of Eq.(17) for changes in $\vec{x}$. We will consider other operator ordering in the Levi-Civita symbols in Appendix:A and we will find that similar contradiction arise in these cases also. In the following, we will follow the ordering discussed below Eq.(15). Discussions below Eq.(18) indicate that the quantization condition given by Eq.(17) will not remain valid at all values of $\vec{x}$ (for all $\vec{y}$) within the neighborhood if it remains valid for a particular value of $\vec{x}$ (for all $\vec{y}$) within the neighborhood as long as we use the Levi-Civita connections. Similar situation will remain valid for any point in the manifold where we can introduce constant time surfaces and assume the existence of a metric component as an independent field in a neighborhood around that point. The above contradiction will also arise with a different choice of constant time surfaces. Thus, there will be a multitude of coordinate systems where we can not use the Levi-Civita connections as connection coefficients if we impose the quantization condition given by Eq.(17). This also indicates that we can not use the Levi-Civita connections as connection coefficients in all coordinate systems that are diffeomorphic to these coordinate systems. In Dirac’s approach to quantize gravity, we can not have a Hilbert space for which: $\langle \nabla'_{\mu}[\hat{g}_{\alpha\beta}] \rangle = 0$. This is easy to see if we take the expectation value of both sides of Eq.(17) on a given state of this space and apply arguments similar to those given below Eq.(18). Note that, we also have: $\langle \Phi|\nabla'_{\mu}[\hat{g}_{\alpha\beta}]|\Psi \rangle = 0$, in such a Hilbert space. Thus, with Eq.(17) assumed to remain valid, we can not impose the metric compatibility conditions as constraints on a Hilbert space which represents physical states. Lastly, problems similar to those mentioned below Eq.(18) will arise if we use
any set of metric compatible connections with covariantly nonconstant r.h.s of Eq.(17). Also, it is expected that $\{\hat{g}_{ij}(x^\mu), \hat{g}_{ij}(y^\nu)\}$ will depend on $(x^\mu, y^\nu)$ nontrivially [4], and it is appropriate to use connections more general than metric compatible connections in quantum gravity. The above discussions are valid irrespective of the presence and nature of sources. We can analyze this issue further in the following way. The Levi-Civita connections and metric compatibility conditions are taken as basic assumptions to calculate the scalar curvature when we use the Einstein-Hilbert action to describe classical and quantum gravity. It is better to discuss quantization and nonmetricity using the Palatini action where $C^\alpha_{\mu\nu}$ in Eq.(9) is an independent field. We will discuss the corresponding variational problem and nonmetricity in the next section.

### III. Affine Connections and the Palatini Action

In this section we will discuss the kinematics of the Palatini formalism. This will help us to identify the degrees of freedom when we use affine connections and the Palatini action. This is important in the quantum theory to construct consistent quantization conditions which require nonmetricity. We first discuss solutions of the variational problem for generalized free theory when we use affine connections and the Palatini action. Important examples are source free theory and minimally coupled scalar and electromagnetic field in the semiclassical limit of quantum gravity which presently is a locally Lorentz invariant theory of quantum fields in curved spaces. The Ricci tensor is given by: $R_{\mu\nu} = R'_{\mu\nu\kappa}$, where $R'_{\mu\nu\kappa}$ can be obtained from Eq.(13). The Ricci tensor, scalar curvature and Palatini action are given by the following expressions when we use affine connections given by Eqs.(9,10):

$$R_{\mu\alpha} = R'_{\mu\alpha\kappa} + \frac{2}{\sqrt{-g}} \nabla_{[\kappa}C_{\mu]}^\kappa\alpha + 2[C_{\kappa\lambda\mu\alpha}^\lambda + 2[C_{\mu\alpha\kappa\lambda}^\kappa]]$$

$$R (= \mathcal{P}) = R' + 2g^{\mu\alpha} \left\{ \nabla_{[\kappa}C_{\mu]}^\kappa\alpha + [C_{\mu\alpha\kappa\lambda}^\lambda + 2[C_{\kappa\lambda\mu\alpha}^\lambda]] \right\}$$

$$S = \int \sqrt{-g} R \mathcal{P}$$

Where, $R'_{\mu\alpha}$ is the Ricci tensor evaluated using the Levi-Civita connections and $R'$ is the corresponding scalar curvature. In this article, we have also used the symbol $\mathcal{P}$ to denote scalar curvature with finite $C_{\mu\nu}$. We now extremize the Palatini action given by the last equation w.r.t $C^\alpha_{\mu\nu}$. Covariant derivatives of the scalar density $\sqrt{-g}$ are zero when we use the Levi-Civita connections. The second term of the scalar curvature gives a boundary term by the Levi-Civita connections of $\nabla'_{\mu}$ and Gauss’s law form of Stoke’s theorem. This boundary term vanishes when $C^\alpha_{\mu\nu}$ is held fixed at the boundary. We then have the following equation as the solution of variational problem when $C^\alpha_{\mu\nu}$ is held fixed at the boundary:

$$C_{\kappa\lambda}b^{\mu\alpha} + C_{\kappa\lambda}b^{\nu\kappa} - C_{\mu\alpha}^\kappa - C_{\lambda}^\mu = 0 \quad (20)$$

There is no contribution from the source fields in the present case. This is an algebraic equation giving constraints on $C^\alpha_{\mu\nu}$. We obtain the following equation when we extremize the action w.r.t $g_{\mu\nu}$:

$$\mathcal{G}_{(\mu\alpha)} = \mathcal{R}_{(\mu\alpha)} - \frac{1}{2} \mathcal{R} g_{\mu\alpha} = 8\pi P_{\mu\alpha}$$

$$\mathcal{R}_{\mu\alpha} = R'_{\mu\alpha\kappa} + 2[C_{\kappa\mu\alpha\lambda}^\lambda + 2[C_{\kappa\lambda\mu\alpha}^\lambda]]$$

$$\mathcal{R} = g^{\mu\alpha} \mathcal{R}_{\mu\alpha}$$

Where, $\mathcal{R}_{\mu\nu}$ and $\mathcal{G}_{\mu\nu}$ are the modified Ricci tensor and modified Einstein tensor respectively. They coincide with the Ricci tensor and Einstein tensor when $C^\alpha_{\mu\nu}$ is absent. $P_{\mu\alpha}$ is matter field stress tensor which is related to the variational derivative of matter field action w.r.t $g^{\mu\alpha}$ by: $8\pi P_{\mu\alpha} = -\frac{\delta \mathcal{S}_{M}}{\delta g^{\mu\alpha}}$, where $S_{M}$ is the matter field action and $\kappa_{M}$ is a constant depending on the nature of source [4]. $P_{\mu\alpha}$ is symmetric. Again, the second term of the curvature scalar does not contribute to Einstein’s equation by the Levi-Civita connections of $\nabla'_{\mu}$ and the Gauss’s law form of Stoke’s theorem when metric is held fixed at the boundary. The first order change in $R'_{\mu\alpha}$ due to change in metric is given by: $\nabla'_{[\kappa}\delta \Gamma_{\mu\lambda]}^{\kappa}$, where $\delta \Gamma^{\kappa}_{\mu\lambda}$ is the change in Levi-Civita connection due to change in metric. This term does not contribute to the equation of motion by the Stoke’s theorem. Here, we have assumed that metric and its first order derivatives are held fixed at the boundary [4]. We now construct the solutions of Eq.(20). A contraction over $(\lambda, \mu)$ leads to the following equation:
\[ H^\kappa_{\alpha} + \frac{3}{2} C^{\alpha \kappa}_{\kappa} = 0 \] (22)

An alternate contraction over \((\alpha, \mu)\) leads to the following equation:

\[ 4H^\kappa_{\kappa \lambda} + 2\tilde{C}^\kappa_{\kappa \lambda} + \tilde{C}_\lambda^\kappa = 0 \] (23)

Here, \(\tilde{C}_{\mu \nu}^\alpha\) is the symmetric part of \(C^\alpha_{\mu \nu}\) and \(H^\alpha_{\mu \nu} = \frac{1}{2}(C^\alpha_{\mu \nu} - C^\alpha_{\nu \mu}) = \frac{1}{2}T^\alpha_{\mu \nu}\), is half of torsion tensor. These two equations give the following equations from Eq.(20):

\[ H^\kappa_{\rho \lambda}g^{\mu \rho} + H^\kappa_{\rho} (\alpha \delta^{\mu}_{\lambda}) + 3\tilde{C}^{(\mu \rho \lambda)}_{\lambda} = 0 \] (24)

and

\[ H^\kappa_{\rho}[\alpha \delta^{\rho}_{\lambda}] + 3H^{[\mu \rho \lambda]}_{\lambda} = 0 \] (25)

Eqs.(24,25) give 64 homogeneous constraints for the 64 components of \(C^\alpha_{\mu \nu}\). The solutions exist and are vanishing at all regular points and Eq.(21) reduces to familiar Einstein’s equation. We now consider the case when \(C^\alpha_{\mu \nu}\) is purely antisymmetric in the lower indices. In this case, we have the following equations:

\[ H^{[\mu \rho \lambda]}_{\rho} = 0 \; ; \] (26)

This gives vanishing solutions. With purely symmetric connections in the free theory, Eq.(24) also gives vanishing solutions. Thus, the problem mentioned at the end of the last section remains, and the standard Palatini formalism considered above is not suitable to describe generalized free quantum gravity which requires nonmetricity. The above equations will give nonvanishing solutions in presence of sources that couple with \(C^\alpha_{\mu \nu}\). This is the case with fermions in the locally Lorentz invariant theory of quantum fields in curved spaces. However, the inhomogeneous version of Eq.(20) remains an algebraic constraint and it is appropriate to extend the Palatini formalism in generalized interacting quantum gravity to have nontrivial dynamics for \(C^\alpha_{\mu \nu}\) given by differential equations with time derivatives. In the next section, we will discuss a few theories as possible candidates for a quantum theory of gravity. We note that we have the scope to have finite \(\tilde{C}_{\mu \nu}^\alpha\) when torsion is finite unless we have a special source such that: \(H^\kappa_{\kappa \lambda} = 0\). The antisymmetric part of the Ricci tensor is given by the following expression:

\[ R_{[\mu \rho \lambda]} = \nabla'^\kappa H^\kappa_{\rho \lambda} - \nabla'^\kappa [\rho C^\kappa_{|\kappa |\alpha}] + H^\lambda_{\mu \rho \lambda}C^\kappa_{\kappa \lambda} + \left( \tilde{C}^\lambda_{\kappa \alpha} H^\kappa_{\rho \lambda} - \tilde{C}^\kappa_{\lambda \mu} H^\lambda_{\kappa \alpha} \right) \] (27)

Where, \(\nabla'^\kappa\) is evaluated using the Levi-Civita part of complete connections. Note that, \(R'_{[\mu \rho \lambda]}\) is symmetric because: \(\Gamma^\kappa_{\kappa \alpha} = \partial^\kappa (ln \sqrt{|g|})\), where \(|g|\) is the absolute value of the metric determinant. A purely symmetric Ricci tensor will impose 6 additional constraints on the sources. Thus, in the Palatini formalism, the Ricci tensor is not symmetric in general in presence of sources. The same is valid for the modified Ricci tensor. In this case, the derivative terms will be absent. This is also important to construct a semiclassical theory of fermions in a curved spacetime that can be derived from a variational principle.

In the standard Palatini formalism discussed above, we can quantize the theory by considering \((g_{\mu \nu}, C^\alpha_{\mu \nu})\) as the complete set of variables. Finite conjugate momenta of \(C^\alpha_{\mu \nu}\) are given by: \(\pi^\mu_{\alpha \nu} = \sqrt{-g} \left[ g^{\mu \sigma} \delta_{\alpha}^\sigma - g^{\alpha \nu} \delta_{\kappa}^\nu \right] \); the rest are vanishing. Affine connections do not spoil the diffeomorphism invariance of the theory and the constraints will include three sets: (i) four secondary constraints coming from the non-dynamical nature of the lapse and shift functions and four gauge fixing conditions; (ii) 64 primary constraints: \(\pi^\mu_{\nu} = \sqrt{-g} \left[ g^{\mu \sigma} \delta_{\nu}^\sigma - g^{\nu \sigma} \delta_{\nu}^\sigma \right] \); (iii) 64 secondary constraints given by Eq.(20) which are equivalent to: \(C^\alpha_{\mu \nu} = 0\). The complete set of constraints are second class with suitable gauge fixing conditions and we can try to impose quantization conditions similar to Eq.(17). We will again have contradiction similar to that discussed in the previous section if we use the Levi-Civita connections. With \(C^\alpha_{\mu \nu} = 0\) being the unique solutions of Eq.(20), we will have to extend the standard Palatini formalism to quantize generalized free gravity. This will also lead to general affine connections. These aspects are also consistent with the discussions given below Eq.(26).
We now make a few comments on commutators. The following discussions are valid in general. We replace Eq.(17) by the following condition on the independent field operators:

\[ [\hat{\phi}_M(t, \vec{x}), \hat{\pi}^M(t, \vec{y})] = iX^M_\alpha(x^\mu, y^\nu, \hat{\phi}_P, \hat{\pi}^P) \]  

(28)

Where, \( \hat{\phi}_M \) is an independent field operator from the complete set of fields that include metric and \( C^\alpha_{\mu\nu} \). \( \hat{\pi}^M \) is the corresponding conjugate momentum. We will use the mixed tensor \( C^\alpha_{\mu\nu} \) to find the commutators since it can give covariant derivatives of mixed tensors. The commutators have nonvanishing covariant derivatives. \( X^M_\alpha \) will depend on the type of field and should be consistent with connection coefficients of the theory. For metric, we can not take the tensorial character of \( X^M_\alpha \) given only by a combination of Kronecker deltas and a kernel which is a scalar density in the spatial coordinates of the conjugate momentum. This makes \( X^M_\alpha \) similar to the \( r.h.s \) of Eq.(17). This is in order to avoid contradictions similar to those discussed before. To illustrate, let us consider Eq.(28) with: \( \phi_M = g_{pl} \), \( X^M_\alpha = \delta^\mu_\alpha \delta^M_1 [X(\vec{x}, \vec{y})] \) and \( X \) is a nonconstant kernel similar to \( \delta(\vec{x}, \vec{y}) \) of Eq.(17). Consider the action of a spatial covariant derivative \( \nabla_{xk} \) on both sides with connection coefficients given by the operator versions of Eqs.(9,10). This will act as a partial derivative on the \( r.h.s \) and we will obtain the condition: \( \{ \partial_k g_{pl}(t, \vec{x}) + \bar{C}^\alpha_{kp}(t, \vec{x}) \bar{g}_{al}(t, \vec{x}) + \bar{C}^\alpha_{k\lambda}(t, \vec{x}) \bar{g}_{pl}(t, \vec{x}) \} \), \( \bar{\pi}^p(t, \vec{y}) = 0 \). The above condition can hold for all \( (\vec{x}, \vec{y}) \) if the functional in the second parentheses is independent of \( \bar{g}_{pl}(t, \vec{x}) \).

This will not happen in the special case when all components of \( C^\alpha_{\mu\nu} \) are independent. In general, the definition of \( C^\alpha_{\mu\nu} \) tensor given by Eqs.(9,10) indicates that \( (C_{plk} + C_{lkp}) \) can contain linear terms in partial derivatives of metric by containing \( \nabla_{k} g_{pl} \) which is zero. Thus, \( (C_{pkl} + C_{klp}) \) in the above commutator can not cancel the noncovariant partial derivative of \( g_{pl}(t, \vec{x}) \) and we will have to generalize the form of \( X^M_\alpha \).

For metric, \( X^M_\alpha \) will explicitly depend on the fields to have nontrivial tensorial character. \( X^M_\alpha \) also depends on the coordinates of the two points and implicitly on metric. Similar situation can remain valid in the nonlinear field theories of QCD as follows from microscopic causality in a fixed background which quantize a theory by stating the field-field commutator at two spacetime points. In the case of gravity, \( [\hat{g}_{ij}(x^\mu), \hat{g}_{ij}(y^\nu)] \) will depend on \( (x^\mu, y^\nu) \) nontrivially [4], and it is appropriate to use connections more general than metric compatible connections in the quantum theory. Quantization in Dirac’s approach is more complicated and will be discussed later.

IV. Extensions of the Palatini Action and Quantum Gravity

The variational problem with Palatini action gives two sectors of equations. The first sector is obtained by varying the action \( w.r.t \) \( C^\alpha_{\mu\nu} \), and are given by Eq.(20). Eq.(20) is homogeneous and contains no time derivative. It acts like constraint. In generalized interacting theory, the \( r.h.s \) will be replaced by source terms. We found that the Palatini action is not suitable to describe quantum gravity. It gives vanishing \( C^\alpha_{\mu\nu} \) in generalized free theory and nondynamic algebraic equations for the same in generalized interacting theory. Matter and gauge field Lagrangians are usually polynomials in first order covariant derivatives and can not give derivatives of connection coefficients when we extremize the action \( w.r.t \) connections. We can try to extend the Palatini formalism to construct a quantum theory with finite and dynamic \( C^\alpha_{\mu\nu} \) field having propagators and loops. Here, we extend the Palatini formalism in two alternate ways. We can construct a theory by using potentials to describe connections [13]. We will then have a set of differential equations including time derivatives in place of Eq.(20) and we can introduce nonmetricity even in the source free theory by using nontrivial solutions of the homogeneous differential equations. We first discuss a particular model with zero torsion. We consider the special case when \( C^\alpha_{\mu\nu} \) is given by: \( C^\alpha_{\mu\nu} = \nabla^\alpha q_{\mu\nu} \), where \( q_{\mu\nu} \) is a symmetric tensor and \( \nabla^\alpha \) uses the Levi-Civita connections as given by Eqs.(9,10). We put this expression in the Palatini action given by Eqs.(8,19). We then take variational derivatives \( w.r.t \) \( q_{\mu\nu} \). In this case, the action contains a second order derivative which does not contribute to the variational derivative by the Levi-Civita connections of \( \nabla^\alpha \) and the Gauss’s law form of Stoke’s theorem. We have the following equation:

\[ 2\nabla^\alpha \nabla^\mu (q^\rho_{\alpha\mu}) - g^{\mu\alpha} \nabla^\lambda q^\nu_{\alpha\rho} - \nabla^\nu q^\rho_{\alpha\mu} = 0 \]  

(29)

Where, \( g_{\mu\nu} \) is a solution of the metric sector Einstein’s equation to be mentioned below, \( \nabla^\alpha \) is the corresponding metric compatible covariant derivative that uses the Levi-Civita connections and \( q \) is the trace of \( q_{\mu\nu} \). All the above equations are dynamical and all conjugate momenta are finite. Moreover, time derivative of all components of \( q^\lambda_{\nu} \) are present as a complete set of equations. In this case, the modified Ricci tensor is symmetric and the Ricci tensor can be made symmetric by imposing the constraint: \( \nabla^{\beta} [C^\alpha_{\nu\lambda}] = 0 \). This
leads us to introduce a scalar field: $\nabla'^{\kappa} q_{\kappa \alpha} = -\nabla'_\alpha \Phi$. Eq.(29) gives us the following relation between the trace of $q_{\mu \nu}$ and $\Phi$:

$$\nabla'^{\mu} \nabla'^{\mu} (q) = 2 \nabla'_\mu \nabla'^{\mu} (\Phi)$$  \hspace{1cm} (30)

There will be a nonholonomic constraint to ensure nonmetricity: $\nabla'^{(\mu q^{\kappa} \kappa) \neq 0}$. The trace of $\nabla'^{(\mu q^{\alpha})}$ is $-\nabla'_\kappa \Phi$. When we use Eqs.(11,12) to define symmetric connections, corresponding nonholonomic constraint is given by: $a_{\mu \nu} \neq n q_{\mu \nu}$, where $n$ is a constant which can be zero. This is a mild constraint, the otherwise of which gives the Levi-civita connections. The Ricci tensor is symmetric when we use Eqs.(11,12) to define symmetric connections since: $S^{\kappa \alpha \gamma} = \partial_{\alpha} (ln \sqrt{b})$, where $b$ is the determinant of $b_{\mu \nu}$ given in Eqs.(11,12).

When the Ricci tensor is symmetric, symmetric connections will be always given by: $(\Gamma_{\mu \alpha}^{\beta} + \nabla'^{\kappa} q_{\kappa \alpha})$, with $\nabla'^{\mu} q_{\mu \kappa} = -\nabla'_\kappa \Phi$, and $q_{\mu \nu}$ can be related to Eqs.(11,12) through a suitable $a_{\mu \nu}$. We can also construct a special model with: $H_{\mu \nu} = \nabla'^{\nu} f_{\mu \nu}$, where $f_{\mu \nu}$ is an antisymmetric tensor, and include both $q_{\mu \nu}$ and $f_{\mu \nu}$.

We have the following set of equations for $q_{\mu \nu}$ and $f_{\mu \nu}$:

$$2 \nabla'^{\kappa} \nabla'^{(\mu q^{\kappa})} - g^{\mu \alpha} \nabla'_{\kappa} q_{\alpha} = - \nabla'^{(\nu q^{\alpha})} q + g^{\mu \alpha} \nabla'^{\lambda} q^{\kappa} f_{\lambda \kappa} = 0$$

$$\nabla'_{\lambda} \{ q^{(\nu f^{\alpha})} - \nabla'^{(\nu q^{\alpha})} q = 0$$  \hspace{1cm} (31)

This is a set of coupled differential equations which are dynamical in nature. The last term in both the equations give coupling between $q_{\mu \nu}$ and $f_{\mu \nu}$ and vanishes when torsion is vanishing. We will later give an explicit expression for the last term of the first equation. We can have a quantum theory provided the above equations have nontrivial solutions. Otherwise, we have to use different potentials or generalize the action. We also find that we will have finite $q_{\mu \nu}$ when $f_{\mu \nu}$ is finite. General particle spectrum associated with $f_{\mu \nu}$ and the trace free part of $q_{\mu \nu}$ contains spin 2, spin 1 and spin 0 particles. We also have a scalar field and the corresponding scalar boson from the trace of $q_{\mu \nu}$. These quanta, although massless, can be important in inflation driven by scalar as well as higher spin fields [14,15,16,17]. We will later discuss the dynamics of this model. Alternatively, we can use different actions to construct the dynamics of $C_{\mu \nu}^{\alpha}$ field itself which will produce bosons including spin zero particles in the quantum theory. We can use extended theories of gravity like $f(\mathcal{P})$ gravity with $\mathcal{P}$ being the scalar curvature given by Eq.(19) to introduce dynamical $C_{\mu \nu}^{\alpha}$. A review on $f(R)$ gravity can be found in [18]. It is expected that in this case Eq.(20) will be replaced by dynamical equations. We can also include higher order curvature scalars like $R^\mu \nu R_{\mu \nu}$ with finite $C_{\mu \nu}^{\alpha}$. We will have to ensure nonmetricity and other secondary constraints associated with the vanishing components of $\pi_{\mu \nu}^{\alpha}$. We will again have finite $C_{\mu \nu}^{\alpha}$ in general when $H_{\mu \nu}$ is finite. We will later discuss if we can have massive spin zero particles from $C_{\mu \nu}^{\alpha}$ field in $f(\mathcal{P})$ gravity or other generalizations of the Palatini formalism.

We can also use the potential formalism in $f(\mathcal{P})$ gravity. A relevant formulation is the dynamical theory of torsion considered by a few authors with different actions [19,20]. These papers introduce torsion in an effective action of quantum gravity obtained from the Einstein-Hilbert action and discuss propagators for the corresponding particles. Corresponding potential theory of torsion is ghost free in this formalism [19]. For our purpose, it is appropriate to extend the Einstein-Hilbert-Palatini action at the classical level. We can also use theories like string theory.

Source free Einstein’s equation is given by the following expression when we use the Palatini formalism with a potential description of $C_{\mu \nu}^{\alpha}$:

$$G_{\mu \nu}^{\prime} = \Xi_{(\mu \nu)}$$  \hspace{1cm} (32)

Where, $G_{\mu \nu}^{\prime}$ is the Einstein tensor evaluated using the Levi-Civita part of complete connections. $\Xi_{(\mu \nu)}$ includes the symmetric part of the contribution of the connection potentials to the modified Einstein tensor and also possible contributions from the variational derivatives w.r.t to metric of the Levi-Civita connections present in the second term in the second parentheses of the scalar curvature given by Eq.(19). Thus, we can have: $R_{\mu \nu}^{\prime} \neq 0$, even in source free theory due to nonmetricity. $\Xi_{(\mu \nu)}$ can be regarded as the stress-tensor of an artificial fluid and it will be interesting to find if it can generate the cosmological constant. Similar situation will remain valid when we extend the action. Affine connections introduce additional particles in quantum gravity when we extend the theory as mentioned before. These observations can be useful in quantum cosmology. We can consider a semiclassical locally Lorentz invariant generalized free theory without fermionic sources where covariant derivatives acting on minimally coupled source fields use only the Levi-civita connections although $C_{\mu \nu}^{\alpha}$ can be finite and gives a source term in Einstein’s equation. This can be useful to explain dark energy.
We can also consider a generalized interacting quantum theory with dynamic $C_{\mu\nu}$ where the classical cosmological models with the Levi-Civita connections can be taken as a subset of the probable vacuum configurations of $C_{\mu\nu}$ field. It is expected that quantum fluctuations are important in the very early universe. Quantum fluctuations in $C_{\mu\nu}$ will contribute to nonmetricity in the quantum theory. The inflation problem also indicates that the present universe need not to be in the vacuum of $C_{\mu\nu}$ throughout its evolution. It requires a consistent theory of quantum gravity to elucidate these aspects. Extended theories mentioned in this section will be useful in this respect. As a first approximation of the general quantum theory, we can study the effects of $C_{\mu\nu}$ by considering semiclassical theories like quantum fields in curved spaces with covariant derivatives given by general affine connections that are general solutions of equations similar to Eq.(31) with sources. Coupling of gravity with fermions through local Lorentz invariance introduces torsion with metricity in spacetime [10,21,22]. General affine connections and nonmetricity can get us beyond a strict local Lorentz invariant theory. Alternatively, we can consider a minimally coupled locally Lorentz invariant semiclassical theory where only the part of $C_{\mu\nu}$ required by local Lorentz invariance is coupled with source fields while the complete $C_{\mu\nu}$ field including finite homogeneous solutions like those of Eq.(31) influences metric. Both formalisms can be useful to explain dark energy. We can use the Palatini action with a potential formalism or $f(P)$ gravity to construct these semiclassical theories. Torsion had been used previously to formulate an alternate theory to cosmic inflation [23]. The present article may be significant in this respect. Theoretical and observational aspects of torsion is discussed in [13,24].

We now discuss a few consequences of using general affine connections. We note that, we no longer have: $\nabla^\alpha G_{\mu\nu} = 0$, when we do not use metric compatible connections. We also have: $\nabla^\alpha P_{\mu\nu} \neq 0$, where $P_{\mu\nu}$ is matter field stress tensor. Thus, the energy-momentum conservation principle no longer remains exactly valid when we use affine connections. We next consider electrodynamics in a curved space with general affine connections. This is important when fluctuations in affine connections away from the Levi-Civita connections become significant. We only consider a minimally coupled theory. We assume the electromagnetic field tensor, $F_{\mu\nu}$, to be antisymmetric and is described by a set of potentials in a gauge invariant way, [22]:

$$F^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu} F_{\mu\nu} = g^{\alpha\mu} g^{\beta\nu} [\nabla_\mu A_\nu - \nabla_\nu A_\mu + T^{\alpha\beta}_{\mu\nu} A_\alpha] = g^{\alpha\mu} g^{\beta\nu} [\nabla_\mu A_\nu - \nabla_\nu A_\mu]$$

(33)

This is consistent in the Palatini formalism, since the potentials are analogous to connections in the geometric theory of gauge fields [25]. $\nabla_\mu$ is a covariant derivative with symmetric connections. The electromagnetic field is invariant under gauge transformations: $A_\mu \rightarrow A_\mu + \partial_\mu \chi$. We now consider the Maxwell’s equations and electric charge conservation law when the electromagnetic Lagrangian density is given by the Palatini-Maxwell expression:

$$\mathcal{L} = -\frac{1}{4} \sqrt{-g} F^{\mu\nu} F_{\mu\nu}$$

(34)

Where, $g$ is the metric determinant. The variational derivative of the action w.r.t $A_\mu$ is determined by: $\frac{\delta \mathcal{L}}{\delta (\nabla_\mu A_\nu)}$, where $\nabla_\mu$ is the torsionless covariant derivative in Eq.(33), and this term vanish when variation in $A_\nu$ is of pure gauge. This ensures the gauge invariance of the theory. The most general choice of connections in $\nabla_\mu$ is: $\Gamma^{\alpha}_{\mu\nu} + C_{\mu\nu}$. $\mathcal{L}$ is a scalar density. Hence, $\frac{\delta \mathcal{L}}{\delta (\nabla_\mu A_\nu)} = \left( \frac{\delta \mathcal{L}}{\delta (\nabla_\mu A_\nu)} \right)$ is a contravariant vector density (relative contravariant vector of weight one). We can replace covariant derivatives in the four-divergence part of: $\nabla_\mu \left[ \frac{\delta \mathcal{L}}{\delta (\nabla_\mu A_\nu)} \right]$ by partial derivatives and apply general Gauss’s law form of Stoke’s theorem to eliminate it from the action as a vanishing boundary term [2]. We then have the following inhomogeneous Maxwell equations:

$$\nabla_\nu \left( \sqrt{-g} F^{\nu\mu} \right) = -j^\mu$$

(35)

Here, $j^\mu$ is the current density which is a contravariant vector density. We also have the following generalization of the local charge conservation law, when we use Eq.(13):

$$\nabla_\mu j^\mu = -\sqrt{-g} \tilde{R}_{\mu\nu} F^{\mu\nu} + (\nabla_\mu \nabla_\nu \sqrt{-g}) F^{\mu\nu}$$

(36)

Where, $\nabla_\mu$ is the covariant derivative with connections: $\Gamma^{\alpha}_{\mu\nu} + \tilde{C}^{\alpha}_{\mu\nu}$, $\tilde{R}_{\mu\nu}$ is the Ricci tensor associated with $\nabla_\mu$ which is not symmetric when we use general symmetric affine connections. It is an important problem to interpret this relation. This relation indicates that geometry can influence the measurement of electric
charge. The r.h.s of the above equation vanishes when we use $\nabla'_{\mu}$ for $\widetilde{\nabla}_{\mu}$. We note that gauge invariance does not necessarily indicate vanishing four-divergence of the electric current density. The exterior derivative of the electromagnetic field tensor vanishes although we have the following three form field:

$$\nabla_{[\alpha} F_{\beta\gamma]} = \frac{1}{3} [T^\beta_{\alpha\mu} F_{\nu\beta} + T^\beta_{\mu\nu} F_{\alpha\beta} + T^\beta_{\nu\alpha} F_{\mu\beta}]$$  \hspace{1cm} (37)

Note that, we have used the contravariant field tensor to write the inhomogeneous equations coupled with the sources while the covariant field tensor to introduce the potentials and calculate $\nabla_{[\alpha} F_{\beta\gamma]}$. This is important to use the definitions of Riemann curvature tensor and torsion tensor in absence of the metric compatibility conditions. $\nabla_{[\alpha} F_{\beta\gamma]}$ is finite and arises from geometric coupling between gravity and gauge fields in presence of torsion. This field vanish in a spacetime with the Levi-Civita connections as affine connections. The above discussions illustrate the nontrivial differential geometry of manifolds with general affine connections [2].

We conclude the present article with the following discussions within the context of quantum fields in curved spaces. If we accept the Big bang model of the universe, time reversal symmetry may not be much meaningful in the very early universe. To illustrate, in nonrelativistic quantum mechanics time reversal symmetry in a time independent potential means: $u^*(\vec{r},-t) = u(\vec{r},t)$, where $u(\vec{r},t)$ is the solution of the Schrodinger equation [26]. Thus, we have to be able to define the negative time states: $u(\vec{r},-t)$ which is not obvious if we take the Big bang singularity to be the beginning of time [4]. We can try to solve the problem by extending time through the initial singularity and by considering it to be the Big crunch singularity of another universe. On the other hand, we can consider the formation of a black hole through gravitational collapse and its subsequent evolution by Hawking radiation. We can easily consider a system in a pure state collapsed to a black hole and subsequently evolved into thermal radiation [27]. If we can assemble the thermal radiation to collapse to a black hole, the black hole will subsequently evolve into a mixed state and not into the initial pure state we began with. This is not time reversal symmetric. The presently observed anisotropy in the cosmological microwave radiation may indicate that parity was only an approximate symmetry in the early universe [28]. Thus, it may be so, that charge conjugation symmetry were broken in the early universe. This may be useful to explain the observed particle-antiparticle asymmetry. The affine connections discussed in this article can introduce discrete symmetry breaking in quantum gravity. There will be other issues like the spin-statistics theorem.

IV. Conclusion

To conclude, in this article we have considered the issue of construction of covariant derivative operator in quantum gravity. We have used the canonical quantization approach and Palatini action to illustrate this issue. We have found that all basic covariant structures of geometry will be required to formulate a consistent quantum theory of gravity. This is valid irrespective of the presence and nature of sources. These covariant structures of geometry include metric and a third rank tensor $C^\alpha_{\mu\nu}$. We call this field as supertorsion. The later field leads to affine connections more general than the metric compatible Levi-Civita connections. Symmetric part of $C^\alpha_{\mu\nu}$ in the lower indices can introduce a symmetric second rank covariant tensor. We have found that the standard Palatini formalism is not sufficient to construct a quantum theory of gravity. We have considered two possible extensions of the Palatini formalism to construct a quantum theory. We can extend the Palatini formalism by using potentials to express connections. Alternatively, we can use more general actions than the Palatini action to construct a quantum theory. We can use $f(\mathcal{P})$ gravity. Both formalisms can give finite and zero spin bosons in the quantum theory. Potential formalism can give a scalar field. General affine connections can get us beyond a locally Lorentz invariant theory of quantum fields in curved spaces. In this article, we have also considered the possibility of a semiclassical theory where covariant derivatives on minimally coupled sources use only a part of $C^\alpha_{\mu\nu}$ that is required by local Lorentz invariance while the complete $C^\alpha_{\mu\nu}$ couples with gravity. This can be useful to explain dark energy. Issues like unitarity, renormalizability and ghosts in quantum gravity with general affine connections need to be addressed from modern perspective [12,29,30]. In the Palatini formalism, we have to use a modified Ricci tensor to state Einstein’s equation. It is not possible to have a symmetric modified Ricci tensor with finite $C^\alpha_{\mu\nu}$ in all the formalisms mentioned above unless the sources satisfy additional conditions. Supertorsion brings anomalies to the local energy-momentum and charge conservation laws. These anomalies can not be removed in a general quantum theory of gravity. We found that gauge invariance does not necessarily imply the local charge conservation law in electrodynamics when we use a general set of affine connections. Supertorsion together with an appropriate action will become important to explain cosmological problems like cosmic epoch, dark energy, dark matter, and symmetry breaking effects like particle-antiparticle asymmetry. Supertorsion can provide possible source for the inflation field(s).
Here, we make a few comments on what will happen to Eq.(18) when we choose a different operator ordering in the Levi-Civita connections than that discussed below Eq.(15). We first discuss what happens to the discussions below Eq.(15) when we consider a symmetric ordering in the Levi-Civita symbol. The r.h.s of Eq.(18) is now non-vanishing and we get the following condition when we consider the changes in both sides of Eq.(17) due to a small spatial change in the point with coordinates $\vec{x}$:

$$
-\left[\hat{g}^{\alpha\tau}(t, \vec{x}), \hat{M}_{\tau k}(l, \vec{x})\hat{g}_{\alpha l}(t, \vec{x}), \hat{r}^p(t, \vec{y})\right] = i\delta^p_{\rho}\delta^l_{\beta}\left[\partial_{\mu k}\hat{\delta}(\vec{x}, \vec{y})\right]
$$

(38)

Where, $M_{\alpha\mu} = \frac{1}{2}\left(\partial_{\nu}(g_{\alpha\nu}) + \partial_{\nu}(g_{\mu\nu}) - 2\partial_{\nu}(g_{\mu\alpha})\right)$ and we have kept $\hat{g}_{al}$ and $\hat{g}_{ap}$ at the right of the connections. We have used the fact that $\hat{\nabla}_{\mu}[\hat{g}_{\alpha\beta}] = 0$, when the operator ordering in the Levi-Civita connections is taken to be the same as that discussed below Eq.(15) and given by Eq.(7). We also have, $\hat{g}^{\alpha\mu}\hat{g}_{\alpha\beta} = \delta_{\beta}^{\alpha}$. We need to solve the constraints to find the exact expression of the l.h.s which is not possible in the present case. We can construct a general form as follows. The commutator within the commutator in the l.h.s of the above equation will be independent of $\vec{y}$ and will in general contain singular quantities in coordinates $\vec{x}$ depending on gauge fixing conditions. In the most favorable situation, the l.h.s can be of the form: $is(\vec{x}, \vec{x})\delta^p_{\rho}\delta^l_{\beta}\left[\partial_{\mu k}\hat{\delta}(\vec{x}, \vec{y})\right]$, where $s(\vec{x}, \vec{x})$ is a singular quantity. This is not same as the r.h.s. Thus, we again obtain a contradiction similar to that discussed below Eq.(18). Lastly, any other ordering of $\partial_{\lambda}g_{\alpha\beta}$ and $g^{\mu\nu}$ in the Levi-Civita connections can be expressed as a linear combination of the ordering given by Eq.(7) and the symmetric ordering considered here. Corresponding covariant derivative is given as: $\hat{\nabla}_\mu = m\hat{\nabla}_{1\mu} + (1 - m)\hat{\nabla}_{2\mu}$, where the covariant derivatives in the r.h.s correspond to the two orderings mentioned before and $m$ can be negative. With $\hat{\nabla}_{1\mu}(\hat{g}_{\alpha\beta}) = 0$, we will again have contradictions similar to those discussed in this article.

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