On consistency of constrained spectral clustering under representation-aware stochastic block model

Shubham Gupta and Ambedkar Dukkipati
Department of Computer Science and Automation, Indian Institute of Science, Bangalore-560012, India.
[shubhamg, ambedkar]@iisc.ac.in

Abstract
Spectral clustering is widely used in practice due to its flexibility, computational efficiency, and well-understood theoretical performance guarantees. Recently, spectral clustering has been studied to find balanced clusters under population-level constraints. These constraints are specified by additional information available in the form of auxiliary categorical node attributes. In this paper, we consider a scenario where these attributes may not be observable, but manifest as latent features of an auxiliary graph. Motivated by this, we study constrained spectral clustering with the aim of finding balanced clusters in a given similarity graph $G$, such that each individual is adequately represented with respect to an auxiliary graph $\mathcal{R}$ (we refer to this as representation graph). We propose an individual-level balancing constraint that formalizes this idea. Our work leads to an interesting stochastic block model that not only plants the given partitions in $G$ but also plants the auxiliary information encoded in the representation graph $\mathcal{R}$. We develop unnormalized and normalized variants of spectral clustering in this setting. These algorithms use $\mathcal{R}$ to find clusters in $G$ that approximately satisfy the proposed constraint. We also establish the first statistical consistency result for constrained spectral clustering under individual-level constraints for graphs sampled from the above-mentioned variant of the stochastic block model. Our experimental results corroborate our theoretical findings.

1 Introduction

Theoretical performance guarantees and adaptability to real-world constraints determine the deployability, and hence the practical utility, of statistical methods and machine learning algorithms. For example, spectral clustering [Ng et al., 2001, von Luxburg, 2007], one of the most sought after algorithms for clustering and community detection, has been studied under various constraints such as must-link and cannot-link constraints [Kamvar et al., 2003, Wang and Davidson, 2010b], size-balanced clusters [Banerjee and Ghosh, 2006], and statistical fairness [Kleindessner et al., 2019]. Commonly used constraints in practice can be categorized as population-level (also known as statistical-level) or individual-level constraints. However, the only known consistency guarantees for constrained spectral clustering were established in Kleindessner et al. [2019] for the first case, where the goal is to find balanced clusters with respect to an auxiliary categorical attribute. In this paper, we
study a problem setting where the auxiliary information is encoded in a graph \( \mathcal{R} \), which we refer to as a *representation graph*. We study spectral clustering in a given *similarity graph* \( \mathcal{G} \) under an individual-level balancing constraint that is specified using the representation graph \( \mathcal{R} \). There are two-fold advantages of this setting: (i) our constraint selects clusters that are balanced from each individual’s perspective, and (ii) it enables us to define a new variant of the stochastic block model (SBM) [Holland et al., 1983] that plants the properties of \( \mathcal{R} \) into the sampled graphs, making this variant of SBM *representation-aware* (aware of the representation graph \( \mathcal{R} \)).

### 1.1 Problem setting and applications

Let \( \mathcal{G} \) denote a similarity graph based on which clusters have to be discovered. We consider a setting where each node in \( \mathcal{G} \) specifies a list of its representatives (other nodes in \( \mathcal{G} \)) using an auxiliary representation graph \( \mathcal{R} \). The graph \( \mathcal{R} \) is defined on the same set of nodes as \( \mathcal{G} \) and its edges specify the “is representative of” relationship. For example, \( \mathcal{R} \) may be a result of the interactions between individuals that result from values of certain latent node attributes like age and gender. This motivates a *representation constraint* that requires the clusters in \( \mathcal{G} \) to be such that each node has a sufficient number of representatives (as per \( \mathcal{R} \)) in all the clusters. Our goal is to develop and analyze variants of the spectral clustering algorithm with respect to this constraint.

We begin by briefly describing two applications that motivate the above discussed problem. The first application is concerned with the “fairness” of clusters. Informally, a node or individual finds the clusters fair if it has a sufficient representation in all the clusters. The work of Chierichetti et al. [2017] requires the clusters to be balanced with respect to various *protected groups* (like gender or race). For example, if 50% of the population is female then the same proportion should be respected in all clusters. This idea of proportional representation has been extended in various forms [Rösner and Schmidt, 2018, Bercea et al., 2019, Bera et al., 2019] and several efficient algorithms for discovering fair clusters under this notion have been proposed [Schmidt et al., 2018, Ahmadian et al., 2019, Kleindessner et al., 2019]. While the fairness notion mentioned above is a *statistical* fairness notion (i.e., constraints are applied on protected groups as a whole), Chen et al. [2019] and Mahabadi and Vakilian [2020] develop *individual* fairness notions that require examples to be “sufficiently close” to their cluster centroids. Anderson et al. [2020] pursue a different direction and adapt the fairness notion proposed by Dwork et al. [2012] to the problem of clustering. Only Kleindessner et al. [2019] study spectral clustering in the context of (statistical) fairness. In contrast, we show in Section 3.1 that our proposed constraint interpolates between statistical and individual fairness based on the structure of \( \mathcal{R} \).

**Example 1.1.** To understand the need for individual fairness notions, consider the representation graph \( \mathcal{R} \) specified in Figure 1a. All the nodes have a self-loop associated with them that has not been shown for clarity. In this example, \( N = 24, \ K = 2 \), and every node is connected to \( d = 6 \) nodes (including the self-loop). To use a statistical fairness notion [Chierichetti et al., 2017], one would begin by clustering the nodes in \( \mathcal{R} \) to approximate the protected groups, as the members of these protected groups will be each other’s representatives to the first order of approximation. A natural choice is to have two protected groups, as shown in Figure 1a using different colors. However, clustering nodes based on
these protected groups can produce the green and yellow clusters shown in Figure 1b. It is easy to verify that these clusters satisfy the statistical fairness criterion as they have an equal number of members from both protected groups. However, these clusters are very “unfair” from the perspective of each individual. For example, node $v_1$ does not have enough representation in the yellow cluster as only one of its six representatives are in this cluster, despite the equal size of both the clusters. A similar argument can be made for every other node in this graph. This example highlights an extreme case where a statistically fair clustering is highly unfair from the perspective of each individual. Figure 1c shows another clustering assignment, and it is easy to verify that each node in this assignment has the same representation in both red and blue clusters, making it individually fair with respect to $\mathcal{R}$. Our goal is to develop algorithms that prefer the clusters in Figure 1c over the clusters in Figure 1b.

Another possible application could be in balancing the load on computing resources in a cloud platform. Here, nodes in $\mathcal{G}$ correspond to processes and edges encode similarity among these processes in terms of shareable resources such as a read-only file. The edges in $\mathcal{R}$ on the other hand connect processes that share resources that can only be accessed by one process at a time (say a network channel). The goal is to cluster similar processes in $\mathcal{G}$ while ensuring that neighbors in $\mathcal{R}$ are spread out across clusters to avoid a collision.

### 1.2 Contributions and results

Our primary goal is to establish the statistical consistency of constrained spectral clustering, where the constraints are specified from the perspective of each individual node in the graph. Towards this end, we make four contributions.

First, in Section 3.1, we introduce the notion of balance from the perspective of each individual node in the graph and formally specify our representation constraint as a simple linear expression. While we focus on spectral clustering in this paper, the proposed constraint may be of independent interest for other clustering techniques as well.

Second, in Sections 3.2 and 3.3, we develop representation-aware variants of the un-normalized and normalized spectral clustering. The proposed algorithms incorporate our
representation constraint as a linear constraint in spectral clustering’s optimization objective. The resulting problem can be easily solved using eigen-decomposition and the returned clusters approximately satisfy the constraint.

Third, in Section 4.1, we propose a variant of SBM called representation-aware stochastic block model (R-SBM). R-SBM encodes a probability distribution over similarity graphs $G$ conditioned on a given representation graph $R$. It can be viewed as a model that plants the properties of $R$ into $G$. We show that R-SBM generates similarity graphs that present a “hard” problem instance to the spectral algorithms in a constrained setting. In Section 4.2, we consider the class of $d$-regular representation graphs and establish the weak-consistency of our algorithms (Theorems 4.1 and 4.2) for graphs sampled from R-SBM. To the best of our knowledge, these are the first consistency results for constrained spectral clustering under individual-level constraints.

Fourth, in Section 5, we present empirical studies on both simulated and real-world data to verify our theoretical guarantees. A comparison between the performance of the proposed algorithms and their closest counterparts in the literature demonstrates their practical utility. In particular, our experiments show that the $d$-regularity assumption on the representation graph is not necessary in practice.

We conclude the paper in Section 6 with a few remarks on promising directions for future work. The proofs of all technical lemmas are presented in the supplementary material [Gupta and Dukkipati, 2021b].

1.3 Related results

Several algorithms for unconstrained clustering such as $k$-means [Hofmann and Buhmann, 1997, Wagstaff et al., 2001], expectation-maximization based clustering [Shental et al., 2003], and spectral clustering [Kamvar et al., 2003] have been modified to satisfy the given must-link (ML) and cannot-link (CL) constraints [Basu et al., 2008] that specify pairs of nodes that should or should not belong to the same cluster. In this paper, we restrict our focus to spectral clustering as it provides a deterministic solution to the clustering problem in polynomial time and can detect arbitrarily shaped clusters [von Luxburg, 2007]. Existing approaches modify spectral clustering by preprocessing the input similarity graph [Kamvar et al., 2003, Lu and Carreira-Perpinán, 2008], post-processing the eigenvectors of the Laplacian matrix [Li et al., 2009], or modifying the optimization problem solved by spectral clustering [Yu and Shi, 2001, 2004, Bie et al., 2004, Wang and Davidson, 2010b, Eriksson et al., 2011, Kawale and Boley, 2013, Wang et al., 2014]. Researchers have also studied spectral approaches that, for example, handle inconsistent [Coleman et al., 2008] or sparse [Zhu et al., 2013] constraints, actively solicit constraints [Wang and Davidson, 2010a], or modify variants of spectral clustering [Rangapuram and Hein, 2012, Wang et al., 2009], to name but a few. Other types of constraints such as those on cluster sizes [Banerjee and Ghosh, 2006, Demiriz et al., 2008] and those that can be expressed as linear expressions [Xu et al., 2009] have also been explored. While this has been an active area of research, theoretical consistency guarantees on the performance of these constrained spectral clustering algorithms are largely missing from the literature.

(Unconstrained) spectral clustering is backed by strong statistical guarantees that usually take the form “the algorithm makes $o(N)$ mistakes with probability $1-o(1)$.” Here, $N$
is the number of nodes in the similarity graph. These results consider a random model that generates problem instances for the algorithm (similarity graph in this case) with known ground-truth clusters. The high probability bound is with respect to this random model and the mistakes are computed against the ground-truth clusters. An algorithm that satisfies the condition above is called weakly consistent [Abbe, 2018]. A common choice for the random model is the Stochastic Block Model (SBM) [Holland et al., 1983]. In this model, nodes have predefined community memberships that are used to sample edges with different probabilities. Rohe et al. [2011] established the weak consistency of spectral clustering under the SBM. Lei and Rinaldo [2015] instead used a variant of SBM to sample networks with a more realistic degree distribution [Karrer and Newman, 2011]. Several other variants of SBM have also been used to provide appropriate problem instances such as graphs with overlapping clusters [Zhang et al., 2014], observable node covariates [Binkiewicz et al., 2017], or two alternative set of clusters, one unfair and one fair [Kleindessner et al., 2019]. von Luxburg et al. [2008] use a different random model where the similarity graph encodes pairwise cosine similarity between input feature vectors that follow a particular probability distribution. Tremblay et al. [2016] study a variant of spectral clustering that is faster than the traditional algorithm. Strong consistency results are also known in some cases [Gao et al., 2017, Lei and Zhu, 2017, Vu, 2018]. Finally, Ghoshdastidar and Dukkipati [2017b,a] propose a variant of SBM for hypergraphs and establish the weak consistency of spectral algorithms in this setting.

Theoretical results for constrained clustering are primarily concerned with the computational complexity of the problem. It is known that the problem is NP-hard if one hopes for an exact solution that satisfies all the CL constraints [Davidson and Ravi, 2005] or the statistical fairness constraint [Chierichetti et al., 2017]. Hence, most existing methods only satisfy the constraints approximately [Cucuringu et al., 2016]. Results related to the cluster quality in the constrained setting only consider the algorithm’s convergence to the global optima of a relaxed optimization problem [Xu et al., 2009]. Kleindessner et al. [2019] is a notable exception, however, even they consider constraints that apply at the level of protected groups as explained above, and not at the level of individuals. They follow a similar strategy as ours and modify spectral clustering to add a statistical fairness constraint. In Section 3.1, we argue that a particular configuration of the representation graph \( R \) reduces our constraint to the statistical fairness criterion. Thus, the algorithms proposed in Kleindessner et al. [2019] are strictly special cases of the algorithms presented in this paper. To the best of our knowledge, we are the first to establish statistical consistency results for constrained spectral clustering for individual-level constraints. A subset of results presented in this paper are available online as a preliminary draft [Gupta and Dukkipati, 2021a].

2 Notation and preliminaries

Let \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) denote a similarity graph, where \( \mathcal{V} = \{v_1, v_2, \ldots, v_N\} \) is the set of \( N \) nodes, and \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) is the set of edges. The aim of clustering is to partition the nodes into \( K \geq 2 \) clusters \( \mathcal{C}_1, \ldots, \mathcal{C}_K \subseteq \mathcal{V} \) such that each node in \( \mathcal{V} \) belongs to exactly one cluster, i.e., \( \mathcal{C}_i \cap \mathcal{C}_j = \phi \) if \( i \neq j \) and \( \bigcup_{k=1}^{K} \mathcal{C}_k = \mathcal{V} \). We further assume the availability of a representation graph, denoted by \( \mathcal{R} = (\mathcal{V}, \hat{\mathcal{E}}) \), that encodes auxiliary information. Notice that \( \mathcal{R} \) is defined on the same set of vertices as the similarity graph \( \mathcal{G} \), but has a different
set of edges $\tilde{E} \subseteq V \times V$. The discovered clusters $C_1, \ldots, C_K$ are required to satisfy the constraint encoded by $R$, as described in Section 3.1. $A \in \{0, 1\}^{N \times N}$ and $R \in \{0, 1\}^{N \times N}$ denote the adjacency matrices of graphs $G$ and $R$, respectively. We assume that $G$ and $R$ are undirected. Further, $G$ has no self-loops. Thus, $A$ and $R$ are symmetric and $A_{ii} = 0$ for all $i \in [N]$, where $[n] := \{1, 2, \ldots, n\}$ for any integer $n$.

We propose modified variants of spectral clustering in Section 3. Before describing the proposed algorithms, we begin with a brief review of the standard spectral clustering algorithm.

### 2.1 Unnormalized spectral clustering

Given a similarity graph $G$, unnormalized spectral clustering finds clusters by approximately optimizing a quality metric known as the ratio-cut defined as [von Luxburg, 2007]

$$ RCut(C_1, \ldots, C_K) = \sum_{i=1}^{K} \frac{\text{Cut}(C_i, V \setminus C_i)}{|C_i|}. $$

Here, $V \setminus C_i$ denotes the set difference between sets $V$ and $C_i$. For any two subsets $X, Y \subseteq V$, $\text{Cut}(X, Y)$ is defined as $\text{Cut}(X, Y) = \frac{1}{2} \sum_{v_i \in X, v_j \in Y} A_{ij}$. That is, $\text{Cut}(X, Y)$ counts the number of edges that have one endpoint in $X$ and another endpoint in $Y$. The Laplacian matrix $L$ of the similarity graph $G$ is defined as

$$ L = D - A. \quad (1) $$

Here, $D \in \mathbb{R}^{N \times N}$ is the degree matrix, which is a diagonal matrix such that $D_{ii} = \sum_{j=1}^{N} A_{ij}$, for all $i \in [N]$. Further, define $H \in \mathbb{R}^{N \times K}$ as

$$ H_{ij} = \begin{cases} \frac{1}{\sqrt{|C_j|}} & \text{if } v_i \in C_j \\ 0 & \text{otherwise}. \end{cases} \quad (2) $$

One can easily verify that $RCut(C_1, \ldots, C_K) = \text{trace}\{H^\top LH\}$, where $H$ corresponds to clusters $C_1, \ldots, C_K$. Thus, to find good clusters, one can solve:

$$ \min_{H \in \mathbb{R}^{N \times K}} \text{trace}\{H^\top LH\} \quad \text{s.t. } H \text{ is of the form } (2). $$

It is computationally hard to solve this optimization problem due to the combinatorial nature of the constraint [Wagner and Wagner, 1993]. Unnormalized spectral clustering instead solves the following relaxed optimization problem:

$$ \min_{H \in \mathbb{R}^{N \times K}} \text{trace}\{H^\top LH\} \quad \text{s.t. } H^\top H = I. \quad (3) $$

The above relaxation is often referred to as the spectral relaxation. By Rayleigh-Ritz theorem [Lütkepohl, 1996, Section 5.2.2], the optimal matrix $H^*$ is such that it has $u_1, u_2, \ldots, u_K \in \mathbb{R}^N$ as its columns, where $u_i$ is the eigenvector corresponding to the $i^{th}$ smallest eigenvalue of $L$ for all $i \in [K]$. The algorithm clusters the rows of $H^*$ into $K$ clusters using $k$-means clustering [Lloyd, 1982] to return $\hat{C}_1, \ldots, \hat{C}_K$. Algorithm 1 summarizes this procedure.
Algorithm 1 Unnormalized spectral clustering

1: **Input:** Adjacency matrix \( A \), number of clusters \( K \geq 2 \)
2: Compute the Laplacian matrix \( L = D - A \).
3: Compute the first \( K \) eigenvectors \( u_1, \ldots, u_K \) of \( L \). Let \( H^* \in \mathbb{R}^{N \times K} \) be a matrix that has \( u_1, \ldots, u_K \) as its columns.
4: Let \( h^*_i \) denote the \( i^{th} \) row of \( H^* \). Cluster \( h^*_1, \ldots, h^*_N \) into \( K \) clusters using \( k \)-means clustering.
5: **Output:** Clusters \( \hat{C}_1, \ldots, \hat{C}_K \), s.t. \( \hat{C}_i = \{ v_j \in \mathcal{V} : h^*_j \text{ was assigned to the } i^{th} \text{ cluster} \} \).

The Laplacian given in (1) is more specifically known as unnormalized Laplacian. The next subsection describes a variant of spectral clustering, known as normalized spectral clustering [Shi and Malik, 2000, Ng et al., 2001], that uses the normalized Laplacian. Unless stated otherwise, we will use spectral clustering (without any qualification) to refer to unnormalized spectral clustering.

2.2 Normalized spectral clustering

The ratio-cut objective divides \( \text{Cut}(\mathcal{C}_i, \mathcal{V}\setminus\mathcal{C}_i) \) by the number of nodes in \( \mathcal{C}_i \) to balance the size of the clusters. The volume of a cluster \( \mathcal{C} \subseteq \mathcal{V} \), defined as \( \text{Vol}(\mathcal{C}) = \sum_{v_i \in \mathcal{C}} D_{ii} \), is another popular notion of its size. The normalized cut or NCut objective divides \( \text{Cut}(\mathcal{C}_i, \mathcal{V}\setminus\mathcal{C}_i) \) by \( \text{Vol}(\mathcal{C}_i) \), and is defined as,

\[
\text{NCut}(\mathcal{C}_1, \ldots, \mathcal{C}_K) = \sum_{i=1}^{K} \frac{\text{Cut}(\mathcal{C}_i, \mathcal{V}\setminus\mathcal{C}_i)}{\text{Vol}(\mathcal{C}_i)}.
\]

As before, one can show that \( \text{NCut}(\mathcal{C}_1, \ldots, \mathcal{C}_K) = \text{trace}\{T^\top L T\} \) [von Luxburg, 2007], where \( T \in \mathbb{R}^{N \times K} \) is specified below.

\[
T_{ij} = \begin{cases} 
\frac{1}{\sqrt{\text{Vol}(\mathcal{C}_j)}} & \text{if } v_i \in \mathcal{C}_j \\
0 & \text{otherwise.}
\end{cases}
\]

Note that \( T^\top D T = I \). Thus, the optimization problem for minimizing the NCut objective is

\[
\min_{T \in \mathbb{R}^{N \times K}} \text{trace}\{T^\top L T\} \text{ s.t. } T^\top D T = I \text{ and } T \text{ is of the form (4).}
\]

As before, this optimization problem is hard to solve, and normalized spectral clustering solves a relaxed variant of this problem. Let \( \mathbf{H} = D^{1/2} T \) and define the normalized graph Laplacian as \( \mathbf{L}_{\text{norm}} = I - D^{-1/2} \mathbf{A} D^{-1/2} \). Normalized spectral clustering solves the following relaxed problem:

\[
\min_{\mathbf{H} \in \mathbb{R}^{N \times K}} \text{trace}\{\mathbf{H}^\top \mathbf{L}_{\text{norm}} \mathbf{H}\} \text{ s.t. } \mathbf{H}^\top \mathbf{H} = I.
\]

Note that \( \mathbf{H}^\top \mathbf{H} = I \Leftrightarrow T^\top D T = I \). This is again the standard form of the trace minimization problem that can be solved using the Rayleigh-Ritz theorem. Algorithm 2 summarizes the normalized spectral clustering algorithm.
Algorithm 2 Normalized spectral clustering

1: Input: Adjacency matrix $A$, number of clusters $K \geq 2$
2: Compute the normalized Laplacian matrix $L_{\text{norm}} = I - D^{-1/2}AD^{-1/2}$.
3: Compute the first $K$ eigenvectors $u_1, \ldots, u_K$ of $L_{\text{norm}}$. Let $H^* \in \mathbb{R}^{N \times K}$ be a matrix that has $u_1, \ldots, u_K$ as its columns.
4: Let $h_i^*$ denote the $i^{th}$ row of $H^*$. Compute $\tilde{h}_i^* = \frac{h_i^*}{\|h_i^*\|_2}$ for all $i = 1, 2, \ldots, N$.
5: Cluster $\tilde{h}_1^*, \ldots, \tilde{h}_N^*$ into $K$ clusters using $k$-means clustering.
6: Output: Clusters $\hat{C}_1, \ldots, \hat{C}_K$, s.t. $\hat{C}_i = \{v_j \in V : \tilde{h}_j^* \text{ was assigned to the } i^{th} \text{ cluster}\}$.

3 Representation constraint and representation-aware spectral clustering

3.1 Representation constraint

Here, an individual-level constraint for clustering the graph $G = (V, E)$ is specified using a representation graph $R = (V, \hat{E})$. One intuitive explanation for $R$ is that nodes connected in this graph represent each other in some form (say based on opinions if nodes correspond to people). Let $\mathcal{N}_R(i) = \{v_j : R_{ij} = 1\}$ denote the set of neighbors of node $v_i$ in $R$. The size of $\mathcal{N}_R(i) \cap \hat{C}_k$ denotes node $v_i$’s representation in cluster $\hat{C}_k$. The goal of this constraint is to ensure that each node has an adequate amount of representation in all clusters.

Definition 3.1. The balance of given clusters $C_1, \ldots, C_K$ with respect to a node $v_i \in V$ is defined as

$$\rho_i = \min_{k, \ell \in [K]} \frac{|C_k \cap \mathcal{N}_R(i)|}{|C_{\ell} \cap \mathcal{N}_R(i)|}. \quad (7)$$

It is easy to see that $0 \leq \rho_i \leq 1$ and higher values of $\rho_i$ indicate that node $v_i$ has an adequate representation in all clusters. Thus, one objective could be to find clusters $C_1, \ldots, C_K$ that solve the following constrained optimization problem.

$$\min_{C_1, \ldots, C_K} f(C_1, \ldots, C_K) \quad \text{s.t.} \quad \rho_i \geq \alpha, \forall i \in [N]. \quad (8)$$

Here, $f(\cdot)$ is a function that is inversely proportional to the quality of clusters (such as RCut or NCut), and $\alpha \in [0, 1]$ is a user specified threshold. However, it not clear as to how this approach can be combined with spectral clustering to develop a consistent algorithm. Therefore, we take a slightly different approach, as described below.

First, note that $\rho_i \leq \min_{k, \ell \in [K]} \frac{|C_{\ell}|}{|C_k|}$ for all $i \in [N]$. Therefore, the balance $\rho_i$ is maximized when the representatives $\mathcal{N}_R(i)$ of node $v_i$ are split across clusters $C_1, \ldots, C_K$ in proportion of the sizes of these clusters. Our constraint, formally defined below, requires this proportionality condition to be satisfied for each node $v_i \in V$.

Definition 3.2 (Representation constraint). Given a representation graph $R$, clusters $C_1, \ldots, C_K$ in $G$ satisfy the representation constraint if $|C_k \cap \mathcal{N}_R(i)| \propto |C_k|$, for all $i \in [N]$ and $k \in [K]$, or equivalently,

$$\frac{|C_k \cap \mathcal{N}_R(i)|}{|C_k|} = \frac{|\mathcal{N}_R(i)|}{N}, \quad \forall k \in [K], \forall i \in [N]. \quad (9)$$
In other words, the representation constraint requires the representatives of any given node $v_i$ to have a proportional membership in all clusters. For example, if a node $v_i$ is connected to 30% of all the nodes in $\mathcal{R}$, then the clusters discovered in $G$ must be such that this node has 30% representation in all clusters. We wish to re-emphasize that the representation constraint applies at the level of individual nodes unlike the constraint in [Chierichetti et al., 2017] that applies at the level of protected groups. In Sections 3.2 and 3.3, we show that (9) can be integrated with the optimization problem solved by spectral clustering.

Appendix A presents additional remarks about the properties of this constraint, in particular, its relation to the statistical-level constraint for categorical attributes [Chierichetti et al., 2017, Kleindessner et al., 2019]. It shows that (9) recovers the statistical-level constraint for a particular configuration of the representation graph, and generalizes it to individual-level constraint for other configurations. Next, we turn to the feasibility of the constraint.

**Feasibility** The optimization problem in (8) can always be solved for a small enough value of $\alpha$ (with the convention that 0/0 = 1). On the other hand, the constraint in Definition 3.2 may not always be feasible. For example, if a node has only two representatives, i.e., $|\mathcal{N}_R(i)| = 2$, and there are $K > 2$ clusters, then (9) can never be satisfied as there will always be at least one cluster $C_k$ for which $|C_k \cap \mathcal{N}_R(i)| = 0$. We argue in the subsequent subsections that spectral relaxation ensures the approximate satisfiability of the constraint when it is added to spectral clustering’s optimization problem. We also describe a necessary assumption that is needed to ensure feasibility in practice and two additional assumptions that are required for our theoretical analysis. Thus, the way we use the proposed constraint makes it widely applicable, even though it is very strong. The general form of the constraint in (8) may be of independent interest in the context of other clustering algorithms, but we do not pursue this direction here.

### 3.2 Unnormalized representation-aware spectral clustering (URepSC)

Recall from Section 2.1 that unnormalized spectral clustering approximately minimizes the ratio-cut objective by relaxing an NP-hard optimization problem to solve (3). The lemma below specifies a sufficient condition that implies the constraint in (9).

**Lemma 3.1.** Let $H \in \mathbb{R}^{N \times K}$ have the form specified in (2). The condition

$$R \left( I - \frac{1}{N} 11^T \right) H = 0$$

(10)

implies that the corresponding clusters $C_1, \ldots, C_K$ satisfy the constraint in (9). Here, $I$ is the $N \times N$ identity matrix and $1$ is a $N$-dimensional all-ones vector.

Ideally, we would like to solve the following optimization problem to get the clusters that satisfy the representation constraint,

$$\min_H \quad \text{trace}\{H^T L H\} \quad \text{s.t.} \quad H \text{ is of the form (2);} \quad R \left( I - \frac{1}{N} 11^T \right) H = 0,$$  

(11)
Algorithm 3 URepSC

1: **Input:** Adjacency matrix \( A \), representation graph \( R \), number of clusters \( K \geq 2 \)
2: Compute \( Y \) containing orthonormal basis vectors of \( \text{null}\{R(I - \frac{1}{N}11^T)\} \)
3: Compute Laplacian \( L = D - A \)
4: Compute leading \( K \) eigenvectors of \( Y^TLY \). Let \( Z \) contain these vectors as its columns.
5: Apply \( k \)-means clustering to rows of \( H = YZ \) to get clusters \( \hat{C}_1, \hat{C}_2, \ldots, \hat{C}_K \)
6: **Return:** Clusters \( \hat{C}_1, \hat{C}_2, \ldots, \hat{C}_K \)

where \( L \) is the unnormalized graph Laplacian defined in (1). However, as noted in Section 2.1, the first constraint on \( H \) makes this problem NP-hard. Thus, we solve the following relaxed problem,

\[
\min_{H} \text{trace}\{H^T LH\} \quad \text{s.t.} \quad H^T H = I; \quad R \left(I - \frac{1}{N}11^T\right) H = 0. \tag{12}
\]

Clearly, the columns of any feasible \( H \) must belong to the null space of \( R(I - 1 1^T/N) \). Thus, any feasible \( H \) can be expressed as \( H = YZ \) for some matrix \( Z \in \mathbb{R}^{N-r \times K} \), where \( Y \in \mathbb{R}^{N \times N-r} \) is an orthonormal matrix containing the basis vectors of the null space of \( R(I - 11^T/N) \) as its columns. Here, \( r \) is the rank of \( R(I - 11^T/N) \). Because \( Y^TY = I \), \( H^TH = Z^TY^TYZ = Z^TZ \). Thus, \( H^TH = I \Leftrightarrow Z^TZ = I \). The following optimization problem is equivalent to (12) by setting \( H = YZ \).

\[
\min_{Z} \text{trace}\{Z^TY^TYZ\} \quad \text{s.t.} \quad Z^TZ = I. \tag{13}
\]

As in standard spectral clustering, the solution to (13) is given by the \( K \) leading eigenvectors of \( Y^TY \). Of course, for \( K \) eigenvectors to exist, \( N - r \) must be at least \( K \), as \( Y^TY \) has dimensions \( N-r \times N-r \). The clusters can then be recovered by using \( k \)-means clustering to cluster the rows of \( H = YZ \), as in Algorithm 1. Algorithm 3 summarizes this procedure. We refer to this algorithm as unnormalized representation-aware spectral clustering (URepSC).

### 3.3 Normalized representation-aware spectral clustering (NRepSC)

We use a similar strategy as the previous section to develop the normalized variant of RepSC. Recall from Section 2.2 that normalized spectral clustering approximately minimizes the NCut objective. The lemma below is a counterpart of Lemma 3.1. It formulates a sufficient condition that implies our constraint in (9), but this time in terms of the matrix \( T \) defined in (4).

**Lemma 3.2.** Let \( T \in \mathbb{R}^{N \times K} \) have the form specified in (4). The condition

\[
R \left(I - \frac{1}{N}11^T\right) T = 0 \tag{14}
\]

implies that the corresponding clusters \( C_1, \ldots, C_K \) satisfy (9). Here, \( I \) is the \( N \times N \) identity matrix and \( 1 \) is a \( N \) dimensional all-ones vector.
Algorithm 4 NRepSC

1: **Input:** Adjacency matrix $A$, representation graph $R$, number of clusters $K \geq 2$
2: Compute $Y$ containing orthonormal basis vectors of $\text{null}\{R(I - \frac{1}{N}11^\top)\}$
3: Compute Laplacian $L = D - A$
4: Compute $Q = \sqrt{Y^\top DY}$ using the matrix square root
5: Compute leading $K$ eigenvectors of $Q^{-1}Y^\top LYQ^{-1}$. Set them as columns of $V \in \mathbb{R}^{N-r \times K}$
6: Apply $k$-means clustering to the rows of $T = YQ^{-1}V$ to get clusters $\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_K$
7: **Return:** Clusters $\hat{C}_1, \hat{C}_2, \ldots, \hat{C}_K$

For NRepSC, we assume that the similarity graph $G$ is connected so that the diagonal entries of $D$ are strictly positive. We proceed as before to incorporate constraint (14) in optimization problem (5). After applying the spectral relaxation, we get

$$\min_{T} \text{trace}\{T^\top LT\} \quad \text{s.t.} \quad T^\top DT = I; \quad R(I - 11^\top/N)T = 0. \quad (15)$$

As before, $T = YZ$ for some $Z \in \mathbb{R}^{N-r \times K}$, where recall that columns of $Y$ contain orthonormal basis for $\text{null}\{R(I - 11^\top/N)\}$. This reparameterization yields

$$\min_{Z} \text{trace}\{Z^\top Y^\top LYZ\} \quad \text{s.t.} \quad Z^\top Y^\top DYZ = I.$$ 

Define $Q \in \mathbb{R}^{N-r \times N-r}$ such that $Q^2 = Y^\top DY$. Note that $Q$ exists as the entries of $D$ are non-negative. Let $V = QZ$. Then, $Z = Q^{-1}V$ and $Z^\top Q^2Z = V^\top V$ as $Q$ is symmetric. Reparameterizing again, we get:

$$\min_{V} \text{trace}\{V^\top Q^{-1}Y^\top LYQ^{-1}V\} \quad \text{s.t.} \quad V^\top V = I.$$ 

This again is the standard form of the trace minimization problem and the optimal solution is given by the leading $K$ eigenvectors of $Q^{-1}Y^\top LYQ^{-1}$. Algorithm 4 summarizes the normalized representation-aware spectral clustering algorithm, which we denote by NRepSC. Note that the algorithm assumes that $Q$ is invertible, which requires the absence of isolated nodes in the similarity graph $G$.

3.4 Comments on the proposed algorithms

Before proceeding with the theoretical analysis in Section 4, we first make two remarks about the proposed algorithms.

**Spectral relaxation** Note that the constraints $R(I - 11^\top/N)H = 0$ and $R(I - 11^\top/N)T = 0$ imply the satisfaction of our representation constraint only when $H$ and $T$ have the form given in (2) and (4), respectively. Thus, a feasible solution to the relaxed optimization problem in (12) or (15) may not necessarily result in representation-aware clusters. In fact, even in the unconstrained case, there are no general guarantees that bound the difference between the optimal solution of (3) or (6) and the respective optimal solutions of the original NP-hard ratio-cut/normalized-cut problems [Kleindessner et al., 2019]. Thus,
the representation-aware nature of the clusters discovered by solving (13) or (15) cannot
be guaranteed in the general case. Nonetheless, we show in Section 4 that the discovered
clusters indeed satisfy the representation constraint under certain additional assumptions.

**Computational complexity** Algorithms 3 and 4 have a time complexity of $O(N^3)$ and
space complexity of $O(N^2)$. Finding the null space of $\mathbf{R}(\mathbf{I} - \mathbf{1}\mathbf{1}^\top/N)$ to calculate $\mathbf{Y}$ and
computing the eigenvectors of appropriate matrices are the computationally dominant steps
in both cases. This matches the worst-case complexity of the standard spectral clustering
algorithm. For small $K$, several approximations can reduce this complexity, but most such
techniques require $K = 2$ [Yu and Shi, 2004, Xu et al., 2009].

4 Analysis

In this section, we show that Algorithms 3 and 4 recover the ground truth clusters with
a high probability under certain assumptions on the representation graph. As we will see
in Section 4.2, the ground truth clusters satisfy (9) by construction for similarity graphs
$\mathcal{G}$ sampled from a modified variant of the Stochastic Block Model (SBM) [Holland et al.,
1983], described in Section 4.1. Section 4.2 presents our main results that establish a high
probability upper bound on the number of mistakes made by the proposed algorithms.
Corollaries 4.1 and 4.2 then establish their weak-consistency.

4.1 $\mathcal{R}$-SBM

The well known Stochastic Block Model (SBM) [Holland et al., 1983] allows one to sample
random graphs with known clusters. It takes a function $\pi : \mathcal{V} \to [K]$ as input. This function
assigns each node $v_i \in \mathcal{V}$ to one of the $K$ clusters. Then, independently for all node pairs
$(v_i, v_j)$ such that $i > j$, $P(A_{ij} = 1) = B_{\pi(v_i)\pi(v_j)}$, where $\mathbf{B} \in [0, 1]^{K \times K}$ is a symmetric
matrix. The $(k, \ell)^{th}$ entry of $\mathbf{B}$ specifies the probability of a connection between two nodes
that belong to clusters $C_k$ and $C_\ell$, respectively. A commonly used variant of SBM assumes
$B_{kk} = \alpha$ and $B_{k\ell} = \beta$ for all $k, \ell \in [K]$ such that $k \neq \ell$. We define a variant of SBM with
respect to a representation graph $\mathcal{R}$ below and refer to it as Representation-Aware SBM or
$\mathcal{R}$-SBM.

**Definition 4.1 ($\mathcal{R}$-SBM).** A $\mathcal{R}$-SBM is defined by the tuple $(\pi, \mathcal{R}, p, q, r, s)$, where $\pi : \mathcal{V} \to [K]$ maps nodes in $\mathcal{V}$ to clusters, $\mathcal{R}$ is a representation graph, and $1 \geq p \geq q \geq r \geq s \geq 0$
are probabilities used for sampling edges. Under this model, for all $i > j$,

$$P(A_{ij} = 1) = \begin{cases} p & \text{if } \pi(v_i) = \pi(v_j) \text{ and } R_{ij} = 1, \\ q & \text{if } \pi(v_i) = \pi(v_j) \text{ and } R_{ij} = 1, \\ r & \text{if } \pi(v_i) = \pi(v_j) \text{ and } R_{ij} = 0, \\ s & \text{if } \pi(v_i) = \pi(v_j) \text{ and } R_{ij} = 0. \end{cases}$$  

(16)

The similarity graphs sampled from a $\mathcal{R}$-SBM have two interesting properties. First,
everything else being equal, nodes have a higher tendency to connect with other nodes in
the same cluster as $p \geq q$ and $r \geq s$. Thus, $\mathcal{R}$-SBM plants the clusters specified by $\pi$ in the
sampled graph $G$. Second, and more importantly, $\mathcal{R}$-SBM also plants the properties of the given representation graph $\mathcal{R}$ in the sampled graphs $G$. To see this, note that nodes that are connected in $\mathcal{R}$ have a higher probability of being connected in $G$ as well ($p \geq r$ and $q \geq s$).

Recall that our algorithms must discover clusters in $G$ in which the connected nodes in $\mathcal{R}$ are proportionally distributed. However, $\mathcal{R}$-SBM makes two nodes connected in $\mathcal{R}$ more likely to connect in $G$, even if they do not belong to the same cluster ($q \geq r$). In this sense, graphs sampled from $\mathcal{R}$-SBM are “hard” instances from the perspective of our algorithms.

When $\mathcal{R}$ itself has a community structure, there are two natural ways to cluster the nodes:

(i) based on the ground-truth clusters $C_1, C_2, \ldots, C_K$ specified by $\pi$; and

(ii) based on the communities in $\mathcal{R}$. The clusters based on communities in $\mathcal{R}$ are likely to not satisfy the representation constraint in Definition 3.2 as tightly connected nodes in $\mathcal{R}$ will be assigned to the same cluster in this case rather than being distributed across clusters.

We show in Section 4.2 that, under certain assumptions on $\mathcal{R}$, the ground-truth clusters can be constructed so that they satisfy the representation constraint (9). Assuming that the ground-truth clusters indeed satisfy (9), the goal is to show that Algorithms 3 and 4 recover the ground-truth clusters with high probability rather than returning any other natural but “representation-unaware” clusters.

4.2 Consistency results

As noted in Section 3.1, some representation graphs lead to constraints that cannot be satisfied. For our theoretical analysis, we restrict our focus to cases where the constraint in (9) is feasible. Towards this end, an additional assumption on $\mathcal{R}$ is required.

Assumption 4.1. $\mathcal{R}$ is a $d$-regular graph for some $K \leq d \leq N$. Moreover, $R_{ii} = 1$ for all $i \in [N]$, and each node in $\mathcal{R}$ is connected to $d/K$ nodes from cluster $C_i$, for all $i \in [K]$ (including the self-loop).

Assumption 4.1 ensures the existence of a $\pi$ for which the corresponding ground-truth clusters satisfy the representation constraint in (9). Namely, assuming equal-sized clusters, let $\pi(v_i) = k$, if $(k - 1)N/K \leq i \leq kN/K$ for all $i \in [N]$, and $k \in [K]$. It can be easily verified that the resulting clusters $C_k = \{v_i : \pi(v_i) = k\}$, $k \in [K]$ satisfy (9).

Before presenting our main results, we need to set up additional notation. Let $\Theta \in \{0,1\}^{N \times K}$ indicate the ground-truth cluster memberships, i.e., $\Theta_{ij} = 1 \iff v_i \in C_j$. Similarly, $\hat{\Theta} \in \{0,1\}^{N \times K}$ indicates the clusters returned by the algorithm, i.e., $\hat{\Theta}_{ij} = 1 \iff v_i \in \hat{C}_j$.

Further, let $J$ be the set of all $K \times K$ permutation matrices. The fraction of misclustered nodes [Lei and Rinaldo, 2015] is defined as

$$M(\Theta, \hat{\Theta}) = \min_{J \in J} \frac{1}{N} \|\Theta - \hat{\Theta}J\|_0.$$  

As the ground truth clusters $C_1, \ldots, C_K$ satisfy (9) by construction, a low $M(\Theta, \hat{\Theta})$ indicates that the clusters returned by the algorithm approximately satisfy (9). Theorems 4.1 and 4.2 also use the eigenvalues of the Laplacian matrix in the expected case. We use $L$ to denote this matrix, and define it as $L = D - A$, where $A = E[A]$ is the expected adjacency matrix of a graph sampled from $\mathcal{R}$-SBM and $D \in \mathbb{R}^{N \times N}$ is a diagonal matrix such that
Definition $D_{ii} = \sum_{j=1}^{N} A_{ij}$, for all $i \in [N]$. The next two results establish high-probability upper bounds on the fraction of misclustered nodes for URepSC and NRepSC for similarity graphs $G$ sampled from $\mathcal{R}$-SBM.

**Theorem 4.1** (Error bound for URepSC). Let $\text{rank}\{R\} \leq N - K$ and assume that all clusters have equal sizes. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-r}$ denote the eigenvalues of $Y^T L Y$, where $Y$ was defined in Section 3.2. Define $\gamma = \mu_{K+1} - \mu_K$. Under Assumption 4.1, there exists a universal constant $\text{const}(C, \alpha)$, such that if $\gamma$ satisfies

$$\gamma^2 \geq \text{const}(C, \alpha)(2 + \epsilon)pN \ln N,$$

and $p \geq C \ln N/N$ for some $C > 0$, then,

$$M(\Theta, \hat{\Theta}) \leq \text{const}(C, \alpha) \frac{(2 + \epsilon)}{\gamma^2} pN \ln N,$$

for every $\epsilon > 0$ with probability at least $1 - 2N^{-\alpha}$ when a $(1 + \epsilon)$-approximate algorithm for $k$-means clustering is used in Step 5 of Algorithm 3.

**Theorem 4.2** (Error bound for NRepSC). Let $\text{rank}\{R\} \leq N - K$ and assume that all clusters have equal sizes. Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-r}$ denote the eigenvalues of $Q^{-1} Y^T L Y Q^{-1}$, where $Q = \sqrt{Y^T D Y}$ and $Y$ was defined in Section 3.2. Define $\gamma = \mu_{K+1} - \mu_K$ and $\lambda_1 = q d + s(N - d) + (p - q) \frac{d}{K} + (r - s) \frac{N - d}{K}$. Under Assumption 4.1, there are universal constants $\text{const}_1(C, \alpha)$, $\text{const}_4(C, \alpha)$, and $\text{const}_5(C, \alpha)$ such that if:

1. $(\frac{\sqrt{pN \ln N}}{\lambda_1 - p}) \left( \frac{\sqrt{pN \ln N}}{\lambda_1 - p} + \frac{1}{6\sqrt{C}} \right) \leq \frac{1}{16(\alpha + 1)},$
2. $\frac{\sqrt{pN \ln N}}{\lambda_1 - p} \leq \text{const}_4(C, \alpha)$, and
3. $16(2 + \epsilon) \left[ \frac{8\text{const}_5(C, \alpha) \sqrt{K}}{\gamma} + \text{const}_1(C, \alpha) \right]^2 \frac{pN^2 \ln N}{(\lambda_1 - p)^2} < \frac{N}{K},$

and $p \geq C \ln N/N$ for some $C > 0$, then,

$$M(\Theta, \hat{\Theta}) \leq 32(2 + \epsilon) \left[ \frac{8\text{const}_5(C, \alpha) \sqrt{K}}{\gamma} + \text{const}_1(C, \alpha) \right]^2 \frac{pN \ln N}{(\lambda_1 - p)^2},$$

for every $\epsilon > 0$ with probability at least $1 - 2N^{-\alpha}$ when a $(1 + \epsilon)$-approximate algorithm for $k$-means clustering is used in Step 6 of Algorithm 4.

Next, we discuss our assumptions and use the error bounds above to establish the weak consistency of our algorithms.

**4.3 Discussion**

Note that $I - 11^T/N$ is a projection matrix and $1$ is its eigenvector with eigenvalue $0$. Any vector orthogonal to $1$ is also an eigenvector with eigenvalue $1$. Thus, $\text{rank}\{I - 11^T/N\} = N - 1$. Because $\text{rank}\{R(I - 11^T/N)\} \leq \min(\text{rank}\{R\}, \text{rank}\{I - 11^T/N\})$, requiring $\text{rank}\{R\} \leq N - K$ ensures that $\text{rank}\{R(I - 11^T/N)\} \leq N - K$, which is necessary for (13) and (15) to have a solution.
The assumption on the size of the clusters, together with the $d$-regularity assumption on $\mathcal{R}$, allows us to compute the smallest $K$ eigenvalues of the Laplacian matrix in the expected case. This is a crucial step in the proof of our main consistency results. The additional assumptions in Theorem 4.2 are easy to satisfy as $\lambda_1$ scales linearly with $N$ for appropriate values of $p, q, r,$ and $s$. Similar assumptions were also used in Kleindessner et al. [2019].

Remark. In practice, Algorithms 3 and 4 only require the rank assumption on $\mathcal{R}$ to ensure the existence of solutions to the corresponding optimization problems. The assumptions on the size of clusters and $d$-regularity of $\mathcal{R}$ are only needed for our theoretical analysis.

The next two corollaries are direct consequences of Theorems 4.1 and 4.2, respectively.

**Corollary 4.1** (Weak consistency of $\text{URepSC}$). Under the same setup as Theorem 4.1, for $\text{URepSC}$, $M(\Theta, \hat{\Theta}) = o(1)$ with probability $1 - o(1)$ if $\gamma = \omega(\sqrt{pNK \ln N})$.

**Corollary 4.2** (Weak consistency of $\text{NRepSC}$). Under the same setup as Theorem 4.2, for $\text{NRepSC}$, $M(\Theta, \hat{\Theta}) = o(1)$ with probability $1 - o(1)$ if $\gamma = \omega(\sqrt{pNK \ln N}/(\lambda_1 - p))$.

Thus, under the assumptions in the corollaries above, Algorithms 3 and 4 are weakly consistent [Abbe, 2018]. The conditions on $\gamma$ are satisfied in many interesting cases. For example, when there are $P$ protected groups, as is the case for statistical-level constraints, the equivalent representation graph has $P$ cliques that are not connected to each other (see Appendix A). Kleindessner et al. [2019] show that $\gamma = \theta(N/K)$ in this case (for the unnormalized variant), which satisfies the criterion given above if $K$ is not too large.

Finally, Theorems 4.1 and 4.2 require a $(1 + \epsilon)$-approximate solution to $k$-means clustering. Several efficient algorithms have been proposed in the literature for this task [Kumar et al., 2004, Arthur and Vassilvitskii, 2007, Ahmadian et al., 2017]. Such algorithms are also available in commonly used software packages like MATLAB and scikit-learn. The assumption that $p \geq C \ln N/N$ controls the sparsity of the graph, and is required in the the consistency proofs for standard spectral clustering as well [Lei and Rinaldo, 2015].

4.4 Proof of Theorems 4.1 and 4.2

The proof of Theorems 4.1 and 4.2 follow the commonly used template for such results [Rohe et al., 2011, Lei and Rinaldo, 2015]. In the context of $\text{URepSC}$ (similar arguments work for $\text{NRepSC}$ as well), we

1. Compute the expected Laplacian matrix $\mathcal{L}$ under $\mathcal{R}$-SBM and show that its top $K$ eigenvectors can be used to recover the ground-truth clusters (Lemmas 4.1–4.3).

2. Show that these top $K$ eigenvectors lie in the null space of $R(I - 11^T/N)$, and hence are also the top $K$ eigenvectors of $Y^T \mathcal{L} Y$ (Lemma 4.4). This implies that Algorithm 3 returns the ground truth clusters in the expected case.

3. Use matrix perturbation arguments to establish a high probability mistake bound in the general case when the graph $G$ is sampled from a $\mathcal{R}$-SBM (Lemmas 4.5–4.8).

We begin with a series of lemmas that highlight certain useful properties of eigenvalues and eigenvectors of the expected Laplacian $\mathcal{L}$. These lemmas will be used in Sections 4.4.1
and 4.4.2 to prove Theorem 4.1 and 4.2, respectively. See the supplementary material [Gupta and Dukkipati, 2021b] for the proofs of all technical lemmas. For the remainder of this section, we assume that all appropriate assumptions made in Theorems 4.1 and 4.2 are satisfied.

The first lemma shows that certain vectors that can be used to recover the ground-truth clusters indeed satisfy the representation constraint in (10) and (14).

**Lemma 4.1.** The $N$-dimensional vector of all ones, denoted by $\mathbf{1}$, is an eigenvector of $\mathbf{R}$ with eigenvalue $d$. Define $u_k \in \mathbb{R}^N$ for $k \in [K-1]$ as,

$$u_{ki} = \begin{cases} 1 & \text{if } v_i \in C_k \\ -\frac{1}{K-1} & \text{otherwise,} \end{cases}$$

where $u_{ki}$ is the $i^{th}$ element of $u_k$. Then, $\mathbf{1}, u_1, \ldots, u_{K-1} \in \text{null}\{\mathbf{R}(\mathbf{I} - \frac{1}{N}\mathbf{1}\mathbf{1}^\top)\}$. Moreover, $\mathbf{1}, u_1, \ldots, u_{K-1}$ are linearly independent.

Recall that we use $\mathcal{A} \in \mathbb{R}^{N \times N}$ to denote the expected adjacency matrix of the similarity graph $\mathcal{G}$. Clearly, $\mathcal{A} = \tilde{\mathcal{A}} - p\mathbf{I}$, where $\tilde{\mathcal{A}}$ is such that $\tilde{A}_{ij} = P(A_{ij} = 1)$ if $i \neq j$ (see (16)) and $\tilde{A}_{ii} = p$ otherwise. Note that

$$\tilde{\mathcal{A}}\mathbf{x} = \lambda \mathbf{x} \iff \mathcal{A}\mathbf{x} = (\lambda - p)\mathbf{x}. \quad (17)$$

Simple algebra shows that $\tilde{\mathcal{A}}$ can be written as

$$\tilde{\mathcal{A}} = q\mathbf{R} + s(\mathbf{1}\mathbf{1}^\top - \mathbf{R}) + (p - q) \sum_{k=1}^{K} G_k \mathcal{R} G_k + (r - s) \sum_{k=1}^{K} G_k (\mathbf{1}\mathbf{1}^\top - \mathbf{R}) G_k, \quad (18)$$

where, for all $k \in [K]$, $G_k \in \mathbb{R}^{N \times N}$ is a diagonal matrix such that $(G_k)_{ii} = 1$ if $v_i \in C_k$ and 0 otherwise. The next lemma shows that $\mathbf{1}, u_1, \ldots, u_{K-1}$ defined in Lemma 4.1 are eigenvectors of $\tilde{\mathcal{A}}$.

**Lemma 4.2.** Let $\mathbf{1}, u_1, \ldots, u_{K-1}$ be as defined in Lemma 4.1. Then,

$$\tilde{\mathcal{A}}\mathbf{1} = \lambda_1 \mathbf{1} \quad \text{where} \quad \lambda_1 = qd + s(N - d) + (p - q) \frac{d}{K} + (r - s) \frac{N - d}{K},$$

$$\tilde{\mathcal{A}} u_k = \lambda_{1+k} u_k \quad \text{where} \quad \lambda_{1+k} = (p - q) \frac{d}{K} + (r - s) \frac{N - d}{K}.$$
to optimize over vectors that belong to this null space. By Lemma 4.1, \( u_1, \ldots, u_{K-1} \in \text{null}(R(I - 11^T/N)) \) and these vectors are linearly independent. However, we need an orthonormal basis to compute \( Y \). Let \( y_1 = 1/\sqrt{N} \) and \( y_2, \ldots, y_K \) be orthonormal vectors that span the same space as \( u_1, \ldots, u_{K-1} \). The next lemma computes such \( y_2, \ldots, y_K \). The matrix \( Y \in \mathbb{R}^{N \times N-K} \) contains these vectors \( y_1, \ldots, y_K \) as its first \( K \) columns.

**Lemma 4.3.** Define \( y_{1+k} \in \mathbb{R}^N \) for \( k \in [K-1] \) as

\[
y_{1+k,i} = \begin{cases} 
0 & \text{if } v_i \in \mathcal{C}_{k'} \text{ s.t. } k' < k \\
\frac{1}{\sqrt{N/(k-k)(k-k+1)}} & \text{if } v_i \in \mathcal{C}_k \\
-\sqrt{\frac{1}{N}} & \text{otherwise.}
\end{cases}
\]

Then, for all \( k \in [K-1] \), \( y_{1+k} \) are orthonormal vectors that span the same space as \( u_1, u_2, \ldots, u_{K-1} \) and \( y_i^\top y_{1+k} = 0 \). As before, \( y_{1+k,i} \) refers to the \( i \)-th element of \( y_{1+k} \).

Let \( X \in \mathbb{R}^{N \times K} \) be such that it has \( y_1, \ldots, y_K \) as its columns. If two nodes belong to the same cluster, the rows corresponding to these nodes in \( XU \) will be identical for any \( U \in \mathbb{R}^{K \times K} \) such that \( U^\top U = UU^\top = I \). Thus, any \( K \) orthonormal vectors belonging to the span of \( y_1, \ldots, y_K \) can be used to recover the ground truth clusters. With the general properties of the eigenvectors and eigenvalues established in the lemmas above, we next move on to the proof of Theorem 4.1 in the next section and Theorem 4.2 in Section 4.4.2.

### 4.4.1 Proof of Theorem 4.1

Let \( Z \in \mathbb{R}^{N-r \times K} \) be a solution to the optimization problem (13) in the expected case with \( A \) as input. The next lemma shows that columns of \( YZ \) indeed lie in the span of \( y_1, \ldots, y_K \). Thus, the \( k \)-means clustering step in Algorithm 3 will return the correct ground truth clusters when \( A \) is passed as input.

**Lemma 4.4.** Let \( y_1 = 1/\sqrt{N} \) and \( y_{1+k} \) be as defined in Lemma 4.3 for all \( k \in [K-1] \). Further, let \( Z \) be the optimal solution of the optimization problem in (13) with \( L \) set to \( \mathcal{L} \). Then, the columns of \( YZ \) lie in the span of \( y_1, y_2, \ldots, y_K \).

Next, we use arguments from matrix perturbation theory to show a high-probability bound on the number of mistakes made by the algorithm. In particular, we need an upper bound on \( \|Y^\top LY - Y^\top \mathcal{L}Y\| \), where \( L \) is the Laplacian matrix for a graph randomly sampled from \( \mathcal{R}\text{-SBM} \) and \( \|P\| = \sqrt{\lambda_{\text{max}}(P^\top P)} \) for any matrix \( P \). Note that \( \|Y\| = \|Y^\top\| = 1 \) as \( Y^\top Y = I \). Thus,

\[
\|Y^\top LY - Y^\top \mathcal{L}Y\| \leq \|Y^\top\| \|L - \mathcal{L}\| \|Y\| = \|L - \mathcal{L}\|.
\]

Moreover,

\[
\|L - \mathcal{L}\| = \|D - A - (\mathcal{D} - A)\| \leq \|D - \mathcal{D}\| + \|A - A\|.
\]

The next two lemmas bound the two terms on the right hand side of the inequality above, thus providing an upper bound on \( \|L - \mathcal{L}\| \), and hence on \( \|Y^\top LY - Y^\top \mathcal{L}Y\| \) by (20).
Lemma 4.5. Assume that \( p \geq C \ln^N \frac{N}{N} \) for some constant \( C > 0 \). Then, for every \( \alpha > 0 \), there exists a constant \( \text{const}_1(C, \alpha) \) that only depends on \( C \) and \( \alpha \) such that

\[
\|D - D\| \leq \text{const}_1(C, \alpha) \sqrt{pN \ln N}
\]

with probability at least \( 1 - N^{-\alpha} \).

Lemma 4.6. Assume that \( p \geq C \ln^N \frac{N}{N} \) for some constant \( C > 0 \). Then, for every \( \alpha > 0 \), there exists a constant \( \text{const}_2(C, \alpha) \) that only depends on \( C \) and \( \alpha \) such that

\[
\|A - A\| \leq \text{const}_2(C, \alpha) \sqrt{pN}
\]

with probability at least \( 1 - N^{-\alpha} \).

From Lemmas 4.5 and 4.6, we conclude that there is always a constant \( \text{const}_3(C, \alpha) = \max\{\text{const}_1(C, \alpha), \text{const}_2(C, \alpha)\} \) such that, for any \( \alpha > 0 \), with probability at least \( 1 - 2N^{-\alpha} \),

\[
\|Y^\top LY - Y^\top LY\| \leq \|L - L\| \leq \text{const}_3(C, \alpha) \sqrt{pN \ln N}. \tag{21}
\]

Let \( \mathcal{Z} \) and \( \mathcal{Z} \) denote the optimal solution of (13) in the expected (\( L \) replaced with \( \mathcal{L} \)) and observed case. We use (21) to show a bound on \( \|YZ - YZ\|_F \) in Lemma 4.7 and then use this bound to argue that Algorithm 3 makes a small number of mistakes when the graph is sampled from \( \mathcal{R} \)-SBM.

Lemma 4.7. Let \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-r} \) be eigenvalues of \( Y^\top LY \). Further, let the columns of \( \mathcal{Z} \in \mathbb{R}^{N-r \times K} \) and \( \mathcal{Z} \in \mathbb{R}^{N-r \times K} \) correspond to the leading \( K \) eigenvectors of \( Y^\top LY \) and \( Y^\top L \), respectively. Define \( \gamma = \mu_{K+1} - \mu_K \). Then, with probability at least \( 1 - 2N^{-\alpha} \),

\[
\inf_{U \in \mathbb{R}^{K \times K}: U^\top U = U^\top U = I} \|YZ - YZU\|_F \leq \text{const}_3(C, \alpha) \frac{4\sqrt{2K}}{\gamma} \sqrt{pN \ln N},
\]

where \( \text{const}_3(C, \alpha) \) is from (21).

Recall that \( X \in \mathbb{R}^{N \times K} \) is a matrix that contains \( y_1, \ldots, y_K \) as its columns. Let \( x_i \) denote the \( i \)-th row of \( X \). Simple calculation using Lemma 4.3 shows that,

\[
\|x_i - x_j\|_2 = \begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ belong to the same cluster} \\ \sqrt{\frac{2K}{N}} & \text{otherwise.} \end{cases}
\]

By Lemma 4.4, \( Z \) can be chosen such that \( YZ = X \). Let \( U \) be the matrix that solves \( \inf_{U \in \mathbb{R}^{K \times K}: U^\top U = U^\top U = I} \|YZ - YZU\|_F \). As \( U \) is orthogonal, \( \|x_i^\top U - x_j^\top U\|_2 = \|x_i - x_j\|_2 \). The following lemma is a direct consequence of Lemma 5.3 in [Lei and Rinaldo, 2015].

Lemma 4.8. Let \( X \) and \( U \) be as defined above. For any \( \epsilon > 0 \), let \( \Theta \in \mathbb{R}^{N \times K} \) be the assignment matrix returned by a \((1 + \epsilon)\)-approximate solution to the \( k \)-means clustering problem when rows of \( YZ \) are provided as input features. Further, let \( \hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_K \in \mathbb{R}^K \) be the estimated cluster centroids. Define \( \hat{X} = \Theta \hat{\mu} \) where \( \hat{\mu} \in \mathbb{R}^{K\times K} \) contains \( \hat{\mu}_1, \ldots, \hat{\mu}_K \) as its rows. Further, define \( \delta = \sqrt{\frac{2K}{N}} \), and \( S_k = \{v_i \in C_k: \|\hat{x}_i - x_i\| \geq \delta/2\} \). Then,

\[
\delta^2 \sum_{k=1}^K |S_k| \leq 8(2 + \epsilon)\|XU^\top - YZ\|_F^2. \tag{22}
\]
Moreover, if γ from Lemma 4.7 satisfies \( \gamma^2 > \text{const}(C, \alpha)(2 + \epsilon)pNK \ln N \) for a universal constant \( \text{const}(C, \alpha) \), there exists a permutation matrix \( J \in \mathbb{R}^{K \times K} \) such that

\[
\hat{\theta}_i^\top J = \theta_i^\top, \quad \forall \ i \in [N] \setminus (\bigcup_{k=1}^K S_k).
\]

Here, \( \hat{\theta}_i \) and \( \theta_i \) represent the \( i \)th row of matrix \( \hat{\Theta}J \) and \( \Theta \) respectively.

By the definition of \( M(\Theta, \hat{\Theta}) \), for the matrix \( J \) used in Lemma 4.8, \( M(\Theta, \hat{\Theta}) \leq \frac{1}{N}||\Theta - \hat{\Theta}J||_0 \). But, according to Lemma 4.8, \( ||\Theta - \hat{\Theta}J||_0 \leq 2\sum_{k=1}^K |S_k| \). Using Lemma 4.7 and 4.8, we get:

\[
M(\Theta, \hat{\Theta}) \leq \frac{1}{N}||\Theta - \hat{\Theta}J||_0 \leq \frac{2}{N} \sum_{k=1}^K |S_k| \leq \frac{16(2 + \epsilon)}{N\delta^2}||XU^\top - YZ||_F^2.
\]

\[
\leq \text{const}_3(C, \alpha)^2 \frac{512(2 + \epsilon)}{N\delta^2 \gamma^2} pNK \ln N.
\]

Noting that \( \delta = \sqrt{\frac{2K}{N}} \) and setting \( \text{const}(C, \alpha) = 256 \times \text{const}_3(C, \alpha)^2 \) finishes the proof.

### 4.4.2 Proof of Theorem 4.2

Recall that \( Q = \sqrt{Y^\top D^\top Y} \) and analogously define \( Q = \sqrt{Y^\top D^\top Y} \), where \( D \) is the expected degree matrix. It was shown after Lemma 4.2 that \( D = (\lambda_1 - p)I \). Thus, \( Q = \sqrt{\lambda_1 - p} I \) as \( Y^\top Y = I \). Hence \( Q^{-1}Y^\top L^\top YQ^{-1} = \frac{1}{\lambda_1 - p}Y^\top L^\top Y \). Therefore, \( Q^{-1}Y^\top L^\top YQ^{-1} = \frac{\lambda}{\lambda_1 - p}x \iff Y^\top L^\top Yx = \lambda x \). Let \( Z \in \mathbb{R}^{N-r \times K} \) contain the leading \( K \) eigenvectors of \( Q^{-1}Y^\top L^\top YQ^{-1} \) as its columns. Algorithm 4 will cluster the rows of \( YQ^{-1}Z \) to recover the clusters in the expected case. As \( Q^{-1} = \frac{1}{\sqrt{\lambda_1 - p}} I \), we have \( YQ^{-1}Z = \frac{1}{\sqrt{\lambda_1 - p}} YZ \). By Lemma 4.4, \( Z \) can always be chosen such that \( YZ = X \), where recall that \( X \in \mathbb{R}^{N \times K} \) has \( y_1, \ldots, y_K \) as its columns. Because the rows of \( X \) are identical for nodes that belong to the same cluster, Algorithm 4 returns the correct ground truth clusters in the expected case.

To bound the number of mistakes made by Algorithm 4, we show that \( YQ^{-1}Z \) is close to \( YQ^{-1}Z \). Here, \( Z \in \mathbb{R}^{N-r \times K} \) contains the top \( K \) eigenvectors of \( Q^{-1}Y^\top L^\top YQ^{-1} \). As in the proof of Lemma 4.7, we use Davis-Kahan theorem to bound this difference. This requires us to compute \( ||Q^{-1}Y^\top L^\top YQ^{-1} - Q^{-1}Y^\top L^\top YQ^{-1}|| \). Note that:

\[
||Q^{-1}Y^\top L^\top YQ^{-1} - Q^{-1}Y^\top L^\top YQ^{-1}|| = ||Q^{-1} - Q^{-1}|| \cdot ||Y^\top L^\top Y|| \cdot ||Q^{-1}|| + ||Q^{-1}|| \cdot ||Y^\top L^\top Y - Y^\top L^\top Y|| \cdot ||Q^{-1}|| + ||Q^{-1}|| \cdot ||Y^\top L^\top Y|| \cdot ||Q^{-1} - Q^{-1}||.
\]

We already have a bound on \( ||Y^\top L^\top Y - Y^\top L^\top Y|| \) in (21). Also, note that \( ||Q^{-1}|| = \frac{1}{\sqrt{\lambda_1 - p}} \) as \( Q^{-1} = \frac{1}{\sqrt{\lambda_1 - p}} I \). Similarly, as \( Y^\top Y = I \), \( ||Y^\top L^\top Y|| \leq ||L|| = \lambda_1 - \tilde{\lambda} \), where \( \tilde{\lambda} = \lambda_{\text{min}}(\hat{A}) \). Finally,

\[
||Q^{-1}|| \leq ||Q^{-1} - Q^{-1}|| + ||Q^{-1}|| = ||Q^{-1} - Q^{-1}|| + \frac{1}{\sqrt{\lambda_1 - p}}, \quad \text{and}
\]

\[
||Y^\top L^\top Y|| \leq ||Y^\top L^\top Y - Y^\top L^\top Y|| + ||Y^\top L^\top Y|| = ||Y^\top L^\top Y - Y^\top L^\top Y|| + \lambda_1 - \tilde{\lambda}.
\]

Thus, to compute a bound on \( ||Q^{-1}Y^\top L^\top YQ^{-1} - Q^{-1}Y^\top L^\top YQ^{-1}|| \), we only need a bound on \( ||Q^{-1} - Q^{-1}|| \). The next lemma provides this bound.
Lemma 4.9. Let $Q = \sqrt{Y^{\top}DY}$, $Q = \sqrt{Y^{\top}DY}$, and assume that
\[
\left(\frac{\sqrt{pN\ln N}}{\lambda_1 - p}\right)\left(\frac{\sqrt{pN\ln N}}{\lambda_1 - p} + \frac{1}{6\sqrt{C}}\right) \leq \frac{1}{16(\alpha + 1)},
\]
where $C$ and $\alpha$ are used in $\text{const}_1(C, \alpha)$ defined in Lemma 4.5. Then,
\[
||Q^{-1} - Q^{-1}|| \leq \sqrt{\frac{2}{(\lambda_1 - p)^3}}||D - D||.
\]

Using the lemma above with (21), we get
\[
||Q^{-1}Y^{\top}LYQ^{-1} - Q^{-1}Y^{\top}LYQ^{-1}|| \leq \frac{2(\lambda_1 - \lambda)}{(\lambda_1 - p)^2} \left[\sqrt{\frac{2}{(\lambda_1 - p)}} \left\|\frac{2\sqrt{2}||D - D||}{\lambda_1 - p} + \frac{2||D - D||^2}{(\lambda_1 - p)^2} + 1\right\|\right] \sqrt{pN\ln N}. \tag{24}
\]

The next lemma uses the bound above to show that $YQ^{-1}Z$ is close to $YQ^{-1}Z$.

**Lemma 4.10.** Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N - p}$ be eigenvalues of $Q^{-1}Y^{\top}LYQ^{-1}$. Further, let the columns of $Z \in \mathbb{R}^{N - r \times K}$ and $Z \in \mathbb{R}^{N - r \times K}$ correspond to the leading $K$ eigenvectors of $Q^{-1}Y^{\top}LYQ^{-1}$ and $Q^{-1}Y^{\top}LYQ^{-1}$, respectively. Define $\gamma = \mu_{K + 1} - \mu_K$ and let there be a constant $\text{const}_4(C, \alpha)$ such that $\frac{\sqrt{pN\ln N}}{\gamma(\lambda_1 - p)^{3/2}} \leq \text{const}_4(C, \alpha)$. Then, with probability at least $1 - 2N^{-\alpha}$, there exists a constant $\text{const}_5(C, \alpha)$ such that
\[
\inf_{U:U^{\top}U = I} \|YQ^{-1}Z - YQ^{-1}ZU\|_F \leq \frac{16K\text{const}_5(C, \alpha)}{\gamma(\lambda_1 - p)^{3/2}} + \frac{2\text{const}_1(C, \alpha)\sqrt{K}}{(\lambda_1 - p)^{3/2}} \sqrt{pN\ln N},
\]
where $\text{const}_1(C, \alpha)$ is defined in Lemma 4.5.

Recall that, by Lemma 4.4, $Z$ can always be chosen such that $YZ = X$, where $X$ contains $y_1, \ldots, y_K$ as its columns. As $Q^{-1} = \frac{1}{\sqrt{\lambda_1 - p}}I$, one can show that:
\[
||(Q^{-1}X)_i - (Q^{-1}X)_j||_2 = \begin{cases} 0 & \text{if } v_i \text{ and } v_j \text{ belong to the same cluster} \\ \sqrt{\frac{2K}{N(\lambda_1 - p)}} & \text{otherwise}. \end{cases}
\]

Here, $(Q^{-1}X)_i$ denotes the $i^{th}$ row of the matrix $YQ^{-1}Z$. Let $U$ be the matrix that solves $\inf_{U:U^{\top}U = I} \|YQ^{-1}Z - YQ^{-1}ZU\|_F$. As $U$ is orthogonal, $||(Q^{-1}X)_i - (Q^{-1}X)_j||_2 = ||(Q^{-1}X)_i - (Q^{-1}X)_j||_2$. As in the previous case, the following lemma is a direct consequence of Lemma 5.3 in [Lei and Rinaldo, 2015].

**Lemma 4.11.** Let $X$ and $U$ be as defined above. For any $\epsilon > 0$, let $\hat{\Theta} \in \mathbb{R}^{N \times K}$ be the assignment matrix returned by a $(1 + \epsilon)$-approximate solution to the $k$-means clustering problem when rows of $YQ^{-1}Z$ are provided as input features. Further, let $\hat{\mu}_1, \hat{\mu}_2, \ldots, \hat{\mu}_K \in \mathbb{R}^K$ be the estimated cluster centroids. Define $\hat{X} = \hat{\Theta}\hat{\mu}$ where $\hat{\mu} \in \mathbb{R}^{K \times K}$ contains
Figure 2: Comparing URepSC with other “unnormalized” algorithms using synthetically generated $d$-regular representation graphs.

\[ \hat{\mu}_1, \ldots, \hat{\mu}_K \] as its rows. Further, define \( \delta = \sqrt{\frac{2K}{N(\lambda_1 - p)}} \), and \( S_k = \{ v_i \in C_k : ||\hat{x}_i - x_i|| \geq \delta/2 \} \). Then,

\[ \delta^2 \sum_{k=1}^{K} |S_k| \leq 8(2 + \epsilon)||XU - YQ^{-1}Z||^2_F. \]

Moreover, if \( \gamma \) from Lemma 4.10 satisfies

\[ 16(2 + \epsilon) \left[ \frac{8\text{const}_5(C, \alpha)\sqrt{K}}{\gamma} + \text{const}_1(C, \alpha) \right]^2 \frac{pN^2\ln N}{(\lambda_1 - p)^2} < \frac{N}{K}, \]

then, there exists a permutation matrix \( J \in \mathbb{R}^{K \times K} \) such that

\[ \hat{\theta}_i^T J = \theta_i^T, \quad \forall i \in [N] \backslash (\bigcup_{k=1}^{K} S_k). \]

Here, \( \hat{\theta}_i \) and \( \theta_i \) represent the \( i \)th row of matrix \( \hat{\Theta} J \) and \( \Theta \) respectively.

The proof of Lemma 4.11 is similar to that of Lemma 4.8, and has been omitted. The result follows by using a similar calculation as was done after Lemma 4.8 in Section 4.4.1.

5 Numerical Results

We perform three types of experiments. In the first two cases, we use synthetically generated data to validate our theoretical results using $d$-regular representation graphs (Section 5.1) and non-$d$-regular representation graphs (Section 5.2). In the third case, we demonstrate the effectiveness of the proposed algorithms on a real-world dataset (Section 5.3). Notably, our experiments in Sections 5.2 and 5.3 demonstrate that the $d$-regularity assumption on $\mathcal{R}$ is not needed in practice. Before proceeding further, we mention two important details below: (i) How do we compare with the algorithms presented in Kleindessner et al. [2019]? and (ii) What do we do when the rank assumption on $\mathcal{R}$ is not satisfied?
Comparison with Kleindessner et al. [2019] We refer to the algorithms proposed in Kleindessner et al. [2019] as UFairSC and NFairSC, corresponding to the unnormalized and normalized fair spectral clustering, respectively. These algorithms assume that each node belongs to one of the $P$ protected groups $\mathcal{P}_1, \ldots, \mathcal{P}_P \subseteq V$ that are observed by the learner. Recall that these algorithms are special cases of our algorithms when $\mathcal{R}$ is block diagonal (Section 3.1). To demonstrate the generality of our algorithms, we only experiment with representation graphs that are not of the form specified above. Naturally, UFairSC and NFairSC are not directly applicable in this setting. Nonetheless, to compare with these algorithms, we approximate the protected groups by clustering the nodes in $\mathcal{R}$ using standard spectral clustering. Each discovered cluster is then treated as a protected group.

Approximate URepSC and NRepSC Recall that the rank assumption on $\mathcal{R}$ requires \( \text{rank}\{\mathcal{R}\} \leq N - K \). It is not possible to find $K$ orthonormal eigenvectors in Algorithms 3 and 4 if $\mathcal{R}$ violates this assumption. Unlike other assumptions in our theoretical analysis, this assumption is necessary in practice. If a graph $\mathcal{R}$ violates the rank assumption, we instead use the best rank $R$ approximation of its adjacency matrix $\mathcal{R}$ ($R \leq N - K$). This approximation does not have binary elements, but it works well in practice. Whenever this approximation is used, we refer to URepSC and NRepSC as URepSC (approx.) and NRepSC (approx.), respectively.

5.1 Experiments with $d$-regular representation graphs

For these experiments, we sampled $d$-regular representation graphs using $p = 0.4$, $q = 0.3$, $r = 0.2$, and $s = 0.1$, for various values of $d$, $N$, and $K$. We ensured that the sampled $\mathcal{R}$ satisfies Assumption 4.1 and \( \text{rank}\{\mathcal{R}\} \leq N - k \). Further, the ground-truth clusters have equal size and are representation-aware by construction as described in Section 4.2. Figure 2 compares the performance of URepSC with unnormalized spectral clustering (USC) (Algorithm 1) and UFairSC. Figure 2a shows the effect of varying $N$ for a fixed $d = 40$ and $K = 5$. Figure 2b varies $K$ and keeps $N = 1200$ and $d = 40$ fixed. Similarly, Figure 2c keeps $N = 1200$ and $K = 5$ fixed and varies $d$. In all cases, we use $R = P = N/10$,
where recall that $R$ is the rank used for approximation in URepSC (approx.) and $P$ is the number of protected groups discovered in $R$ for running UFairSC. The figures plot the accuracy on $y$-axis and report the mean and standard deviation across 10 independent executions of the algorithms in each case.

As the ground truth clusters satisfy Definition 3.2 by construction, a high accuracy of cluster recovery implies that the algorithm returns representation-aware clusters. Figure 3 shows the corresponding results for NRepSC, where we compare it with the normalized variants of other algorithms. In Figures 2a and 3a, it appears that even the standard spectral clustering algorithm will return representation-aware clusters for a large enough graph. However, Figures 2b and 3b show that this is not true if the number of clusters also increases with $N$, as is more common in practice.

It may also be tempting to think that UFairSC and NFairSC may perform well with a more carefully chosen value of $P$, the number of protected groups. However, Figures 4a and 4b show that this is not true. These figures plot the performance of UFairSC and NFairSC as a function of the number of protected groups $P$. Also shown in these plots is the performance of the approximate variants of our algorithms for various values of rank $R$. As expected, the accuracy increases with $R$ as the approximation of $R$ becomes better.

### 5.2 Experiments with representation graphs sampled from SBM

In this case, we divide the nodes into $P = 5$ protected groups and sample a representation graph $R$ using a Stochastic Block Model. Nodes in $R$ are connected with probability $p_{in} = 0.8$ (resp. $p_{out} = 0.2$) if they belong to the same (resp. different) protected group(s). Conditioned on $R$, we then sample an adjacency matrix from $R$-SBM as before. As an $R$ generated this way may violate the rank assumption, we only experiment with the approximate variants of URepSC and NRepSC in this case. Moreover, as such an $R$ may not be $d$-regular, the notion of accuracy no longer conveys information about the representation awareness of an algorithm. Thus, we instead compute the individual balance $\rho_i$ with respect
to each node, as defined in (7). Recall that \( 0 \leq \rho_i \leq 1 \) and higher values indicate that the representatives of node \( v_i \) are well spread out across clusters \( \hat{C}_1, \ldots, \hat{C}_K \). We use average balance \( \bar{\rho} = \frac{1}{N} \sum_{i=1}^{N} \rho_i \) to measure the representation-awareness of the clusters.

While average balance measures the representation awareness of the clusters, we also need to ensure that they have a high quality. Thus, we compute the ratio of the average balance to the ratio-cut objective. A high value indicates balanced clusters with a high quality (low ratio-cut score). Figure 5 fixes the value of \( P = 5 \) and shows the variation of the metric described above on \( y \)-axis as a function of the number of protected groups used by UFairSC and rank \( R \) used by URepSC (APPROX.), for various values of \( N \) and \( K \). We used the same values of parameters \( p, q, r, \) and \( s \), as in Section 5.1. The plots in Figure 5 show a trade-off between clustering accuracy and representation awareness. One can choose an appropriate value of \( R \) and use URepSC (APPROX.) to get good quality clusters with a high balance. Figure 6 presents analogous results for NRepSC (APPROX.).

5.3 Experiments with a real-world network

For the final set of experiments, we use the FAO trade network [Domenico et al., 2015], which is a multiplex network based on the data made available by the Food and Agriculture Organization (FAO) of the United Nations. It has 214 nodes representing countries and 364 layers corresponding to commodities like coffee, banana, barley, etc. An edge between two countries in a layer indicates the volume of the corresponding commodity traded between
these countries. We convert the weighted graph in each layer to an unweighted graph by connecting every node with its five nearest neighbors. We then make all the edges undirected and use the first 182 layers to construct the representation graph $R$. Nodes in $R$ are connected if they are linked in either of these layers. Similarly, the next 182 layers are used to construct the similarity graph $G$. Note that $R$ constructed this way is not $d$-regular. The goal is to find clusters in $G$ that satisfy Definition 3.2 with respect to $R$.

To motivate this further, note that clusters based only on $G$ only consider the trade of commodities 183–364. However, countries also have other trade relations in $R$, leading to shared economic interests. Assume that the members of each cluster would jointly formulate the economic policies for that cluster. However, the policies made in one cluster affect everyone, even if they are not part of the cluster, as they all share a global market. This incentivizes the countries to influence the economic policies of all the clusters. Being representation aware with respect to $R$ entails that each country has members in other clusters with shared interests. This enables a country to indirectly shape the policies of other clusters.

As before, we use the low-rank approximation for the representation graph in URepSC (APPROX.) and NRepSC (APPROX.). Figure 7 compares URepSC (APPROX.) with UFairSC, and has the same semantics as Figure 5. Different plots in Figure 7 correspond to different choices of $K$. URepSC (APPROX.) achieves a higher ratio of average balance to ratio-cut. In practice, a user would choose $R$ by assessing the relative importance of
Figure 7: Comparing URepSC (approx.) with UFairSC on FAO trade network.

a quality metric like ratio-cut and representation metric like average balance. Figure 8 presents analogous results for NRepSC (approx.).

6 Conclusion

The primary focus of this work has been on studying the consistency of constrained spectral clustering under an individual-level representation constraint. The proposed representation constraint naturally generalizes similar population-level constraints [Chierichetti et al., 2017] by using auxiliary information encoded in a representation graph $R$. We showed that the constraint can be expressed as a linear expression that when added to the optimization problem solved by spectral clustering results in the representation-aware variants of the algorithm. An interesting consequence of this problem is a variant of the stochastic block model that plants the properties of $R$ in a similarity graph in addition to the given clusters to provide a hard problem instance to our algorithms. Under this model, we derive a high-probability upper bound on the number of mistakes made by the algorithms and establish conditions under which they are weakly consistent. To the best of our knowledge, these are the first consistency results for constrained spectral clustering under individual-level constraints. Next, we make a few additional remarks.

The $d$-regularity assumption  The $d$-regularity assumption on $R$ ensures the representation-awareness of the ground-truth clusters in our analysis. Note that the representation graph
Figure 8: Comparing NRepSC (approx.) with NFairSC on FAO trade network.

that recovers the statistical-level constraint is also $d$-regular (see Appendix A), hence our analysis strictly generalizes the previously known results. It would be interesting to study the performance of our algorithms under weaker assumptions on $R$. One can also use a similar strategy as ours to modify more expressive variants of the stochastic block model, like the degree-corrected SBM [Karrer and Newman, 2011], to establish the consistency of the algorithms on more realistic similarity graphs.

**Computational complexity** Our current approach involves finding the null space of a $N \times N$ matrix. This operation has $O(N^3)$ complexity. Existing methods for speeding up the standard spectral clustering algorithm focus on making the eigen-decomposition and/or the $k$-means step faster (see Tremblay et al. [2016] and the references within). However, even with these modifications, the null space computation would still be the computationally dominant step. Xu et al. [2009] proposed an efficient algorithm for solving the normalized cut problem under a linear constraint. However, their algorithm assumes that $K = 2$. Developing similar algorithms for general values of $K$ and exploring their theoretical guarantees will enable the application domains that involve very large graphs to utilise our ideas.

Other possible extensions of our work include similar algorithms for weighted similarity graphs, overlapping clusters, and other types of graphs such as hypergraphs. This paper provides the first step towards consistency analysis of spectral clustering under individual-level constraints.
A Representation constraint: Additional details

In this section, we make two additional remarks about the properties of the proposed constraint, both in the context of fairness.

Statistical fairness as a special case  Recall that our constraint specifies an individual fairness notion. Contrast this with several existing approaches that assign each node to one of the $P$ protected groups $P_1, \ldots, P_P \subseteq V$ [Chierichetti et al., 2017], and require these protected groups to have a proportional representation in all clusters, i.e.,

$$\frac{|P_i \cap C_j|}{|C_j|} = \frac{|P_i|}{N}, \quad \forall i \in [P], \ j \in [K].$$

This is an example of statistical fairness. In Example 1.1, we argued that statistical fairness may not be enough in some cases. We now show that the constraint in Definition 3.2 is equivalent to a statistical fairness notion for an appropriately constructed representation graph $R$ from the given protected groups $P_1, \ldots, P_P$. Namely, let $R$ be such that $R_{ij} = 1$ if and only if $v_i$ and $v_j$ belong to the same protected group. In this case, it is easy to verify that the constraint in Definition 3.2 reduces to the statistical fairness criterion given above. In general, for other configurations of the representation graph, we strictly generalize the statistical fairness notion. We also strictly generalize the approach presented in Kleindessner et al. [2019], where the authors use spectral clustering to produce statistically fair clusters. Also noteworthy is the assumption made by statistical fairness, namely that every pair of vertices in a protected group can represent each others’ interests ($R_{ij} = 1 \iff v_i$ and $v_j$ are in the same protected group), or they are very similar with respect to some sensitive attributes. This assumption becomes unreasonable as protected groups grow in size.

Sensitive attributes and protected groups  Viewed as a fairness notion, the proposed constraint only requires a representation graph $R$. It has two advantages over existing fairness criteria: (i) it does not require observable sensitive attributes (such as age, gender, and sexual orientation), and (ii) even if sensitive attributes are provided, one need not specify the number of protected groups or explicitly compute them. This ensures data privacy and helps against individual profiling. Our constraint only requires access to the representation graph $R$. This graph can either be directly elicited from the individuals or derived as a function of several sensitive attributes. In either case, once $R$ is available, we no longer need to expose any sensitive attributes to the clustering algorithm. For example, individuals in $R$ may be connected if their age difference is less than five years and if they went to the same school. Crucially, the sensitive attributes used to construct $R$ may be numerical, binary, categorical, etc.

References

Emmanuel Abbe. Community detection and stochastic block models: Recent developments. *Journal of Machine Learning Research*, 18(177):1–86, 2018.
Sara Ahmadian, Ashkan Norouzi-Fard, Ola Svensson, and Justin Ward. Better guarantees for $k$-means and euclidean $k$-median by primal-dual algorithms. *Symposium on Foundations of Computer Science*, 2017.

Sara Ahmadian, Alessandro Epasto, Ravi Kumar, and Mohammad Mahdian. Clustering without over-representation. *In Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*, pages 267–275, 2019.

Nihesh Anderson, Suman K. Bera, Syamantak Das, and Yang Liu. Distributional individual fairness in clustering. *arXiv*, 2006.12589, 2020.

David Arthur and Sergei Vassilvitskii. $k$-means++: The advantages of careful seeding. *Symposium on Discrete Algorithms*, 2007.

Arindam Banerjee and Joydeep Ghosh. Scalable clustering algorithms with balancing constraints. *Data Mining and Knowledge Discovery*, 13:365–395, 2006.

Sugato Basu, Ian Davidson, and Kiri Wagstaff. *Constrained clustering: Advances in algorithms, theory, and applications*. CRC Press, 2008.

Suman Bera, Deeparnab Chakrabarty, Nicolas Flores, and Maryam Negahbani. Fair algorithms for clustering. *Advances in Neural Information Processing Systems*, 32:4954–4965, 2019.

Ioana O. Bercea, Martin Gross, Samir Khuller, Aounon Kumar, Clemens Rösner, Daniel R. Schmidt, and Melanie Schmidt. On the cost of essentially fair clusterings. *APPROX-RANDOM*, pages 18:1–18:22, 2019.

Tijl De Bie, Johan Suykens, and Bart De Moor. Learning from general label constraints. *Structural, Syntactic, and Statistical Pattern Recognition. Lecture Notes in Computer Science*, 3138, 2004.

Norbert Binkiewicz, Joshua T. Vogelstein, and Karl Rohe. Covariate assisted spectral clustering. *Biometrika*, 104(2):361–377, 2017.

Xingyu Chen, Brandon Fain, Liang Lyu, and Kamesh Munagala. Proportionally fair clustering. *In Proceedings of the 36th International Conference on Machine Learning*, 97:1032–1041, 2019.

Flavio Chierichetti, Ravi Kumar, Silvio Lattanzi, and Sergei Vassilvitskii. Fair clustering through fairlets. *Advances in Neural Information Processing Systems*, 30:5029–5037, 2017.

Tom Coleman, James Saunderson, and Anthony Wirth. Spectral clustering with inconsistent advice. *In Proceedings of the 25th International Conference on Machine Learning*, pages 152–159, 2008.

Mihai Cucuringu, Ioannis Koutis, Sanjay Chawla, Gary Miller, and Richard Peng. Simple and scalable constrained clustering: A generalized spectral method. *Proceedings of the 19th International Conference on Artificial Intelligence and Statistics*, 41:445–454, 2016.
Ian Davidson and S. S. Ravi. Clustering with constraints: Feasibility issues and the k-means algorithm. *In Proceedings of the 5th SIAM Data Mining Conference*, pages 138–149, 2005.

Ayhan Demiriz, Kristin Bennett, and P.S. Bradley. Constrained Clustering: Advances in Algorithms, Theory, and Applications, chapter Using assignment constraints to avoid empty clusters in k-means clustering, pages 201–220. 2008.

Manlio De Domenico, Vincenzo Nicosia, Alexandre Arenas, and Vito Latora. Structural reducibility of multilayer networks. *Nature Communications*, 6(1), 2015.

Cynthia Dwork, Moritz Hardt, Toniann Pitassi, Omer Reingold, and Richard Zemel. Fairness through awareness. *ITCS ’12*, pages 214—226, 2012.

Anders Eriksson, Carl Olsson, and Fredrik Kahl. Normalized cuts revisited: A reformulation for segmentation with linear grouping constraints. *Journal of Mathematical Imaging and Vision*, 39:45–61, 2011.

Chao Gao, Zongming Ma, Anderson Y. Zhang, and Harrison H. Zhou. Achieving optimal misclassification proportion in stochastic block models. *Journal of Machine Learning Research*, 18:1–45, 2017.

Debarghya Ghoshdastidar and Ambedkar Dukkipati. Consistency of spectral hypergraph partitioning under planted partition model. *Annals of Statistics*, 45(1):289–315, 2017a.

Debarghya Ghoshdastidar and Ambedkar Dukkipati. Uniform hypergraph partitioning: Provable tensor methods and sampling techniques. *Journal of Machine Learning Research*, 18(1):1638–1678, 2017b.

Shubham Gupta and Ambedkar Dukkipati. Protecting individual interests across clusters: Spectral clustering with guarantees. *arXiv*, 2105.03714, 2021a.

Shubham Gupta and Ambedkar Dukkipati. Supplement to “on consistency of constrained spectral clustering under representation-aware stochastic block model”. 2021b.

Thomas Hofmann and Joachim M. Buhmann. Pairwise data clustering by deterministic annealing. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 19(1):1–14, 1997.

Paul W. Holland, Kathryn Blackmond Laskey, and Samuel Leinhardt. Stochastic blockmodels: First steps. *Social Networks*, 5(2):109–137, 1983.

Sepandar D. Kamvar, Dan Klein, and Christopher D. Manning. Spectral learning. *In Proceedings of the 18th International Joint Conference on Artificial Intelligence*, pages 561–566, 2003.

Brian Karrer and Mark Newman. Stochastic blockmodels and community structure in networks. *Physical Review E*, 83(1):016107, 2011.

Jaya Kawale and Daniel Boley. Constrained spectral clustering using l1 regularization. *In Proceedings of the International Conference on Data Mining*, pages 103–111, 2013.
Matthäus Kleindessner, Samira Samadi, Pranjal Awasthi, and Jamie Morgenstern. Guarantees for spectral clustering with fairness constraints. In Proceedings of the 36th International Conference on Machine Learning, 97:3458–3467, 2019.

Amit Kumar, Yogish Sabharwal, and Sandeep Sen. A simple linear time \((1 + \epsilon)\)-approximation algorithm for k-means clustering in any dimensions. Symposium on Foundations of Computer Science, 2004.

Jing Lei and Alessandro Rinaldo. Consistency of spectral clustering in stochastic block models. The Annals of Statistics, 43(1):215–237, 2015.

Jing Lei and Lingxue Zhu. A generic sample splitting approach for refined community recovery in stochastic block models. Statistica Sinica, 27(4):1639–1659, 2017.

Zhenguo Li, Jianzhuang Liu, and Xiaoou Tang. Constrained clustering via spectral regularization. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, pages 421–428, 2009.

Stuart P. Lloyd. Least squares quantisation in pcm. IEEE Transactions on Information Theory, 28(2):129–137, 1982.

Zhengdong Lu and Miguel Á. Carreira-Perpinán. Constrained spectral clustering through affinity propagation. In Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition, 2008.

Helmut Lütkepohl. Handbook of matrices. Wiley, 1996.

Sepideh Mahabadi and Ali Vakilian. Individual fairness for k-clustering. In Proceedings of the 37th International Conference on Machine Learning, 119:6586–6596, 2020.

Andrew Ng, Michael Jordan, and Yair Weiss. On spectral clustering: Analysis and an algorithm. Advances in Neural Information Processing Systems, 14, 2001.

Syama S. Rangapuram and Matthias Hein. Constrained 1-spectral clustering. In Proceedings of the 15th International Conference on Artificial Intelligence and Statistics, 22:1143–1151, 2012.

Karl Rohe, Sourav Chatterjee, and Bin Yu. Spectral clustering and the high-dimensional stochastic blockmodel. The Annals of Statistics, 39(4):1878–1915, 2011.

Clemens Rösner and Melanie Schmidt. Privacy preserving clustering with constraints. ICALP, 2018.

Melanie Schmidt, Chris Schwiegelshohn, and Christian Sohler. Fair coresets and streaming algorithms for fair k-means clustering. arXiv, 1812.10854, 2018.

Noam Shental, Aharon Bar-Hillel, Tomer Hertz, and Daphna Weinshall. Computing gaussian mixture models with em using equivalence constraints. In Proceedings of the 16th International Conference on Neural Information Processing Systems, pages 465–472, 2003.
Jianbo Shi and Jitendra Malik. Normalized cuts and image segmentation. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 22(8):888–905, 2000.

Nicolas Tremblay, Gilles Puy, Remi Gribonval, and Pierre Vandergheynst. Compressive spectral clustering. *In Proceedings of The 33rd International Conference on Machine Learning*, 48:1002–1011, 2016.

Ulrike von Luxburg. A tutorial on spectral clustering. *Statistics and Computing*, 17(4):395–416, 2007.

Ulrike von Luxburg, Mikhail Belkin, and Olivier Bousquet. Consistency of spectral clustering. *The Annals of Statistics*, 36(2):555–586, 2008.

Van Vu. A simple svd algorithm for finding hidden partitions. *Combinatorics, Probability and Computing*, 27(1):124–140, 2018.

Vincent Q. Vu and Jing Lei. Minimax sparse principal subspace estimation in high dimensions. *The Annals of Statistics*, 41(6):2905–2947, 2013.

Dorothea Wagner and Frank Wagner. Between min cut and graph bisection. *Mathematical Foundations of Computer Science*, 711:744–750, 1993.

Kiri Wagstaff, Claire Cardie, Seth Rogers, and Stefan Schrödl. Constrained k-means clustering with background knowledge. *In Proceedings of the 18th International Conference on Machine Learning*, pages 577–584, 2001.

Fei Wang, Chris Ding, and Tao Li. Integrated kl (k-means- laplacian) clustering: A new clustering approach by combining attribute data and pairwise relations. *In Proceedings of the SIAM International Conference on Data Mining*, pages 38–48, 2009.

Xiang Wang and Ian Davidson. Active spectral clustering. *In Proceedings of the IEEE International Conference on Data Mining*, pages 561–568, 2010a.

Xiang Wang and Ian Davidson. Flexible constrained spectral clustering. *In Proceedings of the 16th ACM SIGKDD International Conference on Knowledge Discovery and Data Mining*, pages 563–572, 2010b.

Xiang Wang, Buyue Qian, and Ian Davidson. On constrained spectral clustering and its applications. *Data Mining and Knowledge Discovery*, 28:1–30, 2014.

Linli Xu, Wenye Li, and Dale Schuurmans. Fast normalized cut with linear constraints. *IEEE Conference on Computer Vision and Pattern Recognition*, 2009.

Stella X. Yu and Jianbo Shi. Grouping with bias. *Advances in Neural Information Processing Systems*, 14:1327–1334, 2001.

Stella X. Yu and Jianbo Shi. Segmentation given partial grouping constraints. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 26(2):173–183, 2004.
Yuan Zhang, Elizaveta Levina, and Ji Zhu. Detecting overlapping communities in networks using spectral methods. *SIAM Journal on Mathematics of Data Science*, 2(2):265–283, 2014.

Xiatian Zhu, Chen Change Loy, and Shaogang Gong. Constrained clustering: Effective constraint propagation with imperfect oracles. *In Proceedings of the 13th International Conference on Data Mining*, pages 1307–1312, 2013.
Proof of technical lemmas from Section 3

Proof of Lemma 3.1
Fix an arbitrary node $v_i \in V$ and $k \in [K]$. Because $R(I - 11^\top/N)H = 0$,

$$\sum_{j=1}^{N} R_{ij} H_{jk} = \frac{1}{N} \left( \sum_{j=1}^{N} R_{ij} \right) \left( \sum_{j=1}^{N} H_{jk} \right)$$

$$\Rightarrow \frac{1}{\sqrt{|C_k|}} \left| \{ v_j \in V : R_{ij} = 1 \land v_j \in C_k \} \right| = \frac{1}{N} \left| \{ v_j \in V : R_{ij} = 1 \} \right| \frac{|C_k|}{\sqrt{|C_k|}}$$

$$\Rightarrow \frac{|\{ v_j \in V : R_{ij} = 1 \land v_j \in C_k \}|}{|C_k|} = \frac{|\{ v_j \in V : R_{ij} = 1 \}|}{N}.$$

Because this holds for an arbitrary $v_i \in V$ and $k \in [K]$, $R(I - 11^\top/N)H = 0$ implies the constraint in Definition 3.2.

Proof of Lemma 3.2
Fix an arbitrary node $v_i \in V$ and $k \in [K]$. Because $R(I - 11^\top/N)T = 0$,

$$\sum_{j=1}^{N} R_{ij} T_{jk} = \frac{1}{N} \left( \sum_{j=1}^{N} R_{ij} \right) \left( \sum_{j=1}^{N} T_{jk} \right)$$

$$\Rightarrow \frac{1}{\sqrt{\text{Vol}(C_k)}} |\{ v_j \in V : R_{ij} = 1 \land v_j \in C_k \}| = \frac{1}{N} |\{ v_j \in V : R_{ij} = 1 \}| \frac{|C_k|}{\sqrt{\text{Vol}(C_k)}}$$

$$\Rightarrow \frac{|\{ v_j \in V : R_{ij} = 1 \land v_j \in C_k \}|}{|C_k|} = \frac{|\{ v_j \in V : R_{ij} = 1 \}|}{N}.$$

Here, recall that $\text{Vol}(C_k) = \sum_{v_i \in C_k} D_{ii}$ is the volume of the cluster $C_k$, which is used in (4). Because this holds for an arbitrary $v_i \in V$ and $k \in [K]$, $R(I - 11^\top/N)T = 0$ implies the constraint in Definition 3.2.

2 Proof of technical lemmas from Section 4

2.1 Proof of Lemma 4.1
Because $R$ is a $d$-regular graph, it is easy to see that $R1 = d1$. Recall from Section 4.2 that $I - \frac{1}{N} 11^\top$ is a projection matrix that removes the component of any vector $x \in \mathbb{R}^N$ along.
the all ones vector $1$. Thus, $(I - \frac{1}{N}11^\top)1 = 0$ and hence $1 \in \text{null}(R(I - \frac{1}{N}11^\top))$. Moreover, as all clusters have the same size,

$$1^\top u_k = \sum_{i=1}^N u_{ki} = \sum_{i: v_i \in C_k} u_{ki} + \sum_{i: v_i \notin C_k} u_{ki} = \frac{N}{K} - \frac{1}{K-1}(N - \frac{N}{K}) = 0.$$  

Thus, $R(I - \frac{1}{N}11^\top)u_k = Ru_k$. Let us compute the $i$th element of the vector $Ru_k$ for an arbitrary $i \in [N]$.

$$(Ru_k)_i = \sum_{j=1}^N R_{ij} u_{kj} = \sum_{j: R_{ij} = 1 \land v_j \in C_k} 1 - \sum_{j: R_{ij} = 1 \land v_j \not\in C_k} k - \frac{1}{K-1} = d - \frac{1}{K-1} = 0.$$  

Here, the second last equality follows from Assumption 4.1 and the assumption that all clusters have the same size. Thus, $Ru_k = 0$ and hence $u_k \in \text{null}(R(I - \frac{1}{N}11^\top))$.

Because $1^\top u_k = 0$ for all $k \in [K-1]$, to show that $1, u_1, \ldots, u_{K-1}$ are linearly independent, it is enough to show that $u_1, \ldots, u_{K-1}$ are linearly independent. Consider the $i$th component of $\sum_{k=1}^{K-1} \alpha_k u_k$ for arbitrary $\alpha_1, \ldots, \alpha_{K-1} \in \mathbb{R}$ and $i \in [N]$. If $v_i \in C_k$, then

$$\left(\sum_{k=1}^{K-1} \alpha_k u_k\right)_i = \frac{-1}{K-1} \sum_{k=1}^{K-1} \alpha_k.$$  

Similarly, when $v_i \in C_{k'}$ for some $k' \in [K-1]$, we have,

$$\left(\sum_{k=1}^{K-1} \alpha_k u_k\right)_i = \alpha_{k'} - \frac{1}{K-1} \sum_{k \neq k'}^{K-1} \alpha_k.$$  

Thus, $\sum_{k=1}^{K-1} \alpha_k u_k = 0$ implies that $-\frac{1}{K-1} \sum_{k=1}^{K-1} \alpha_k = 0$ and $\alpha_{k'} - \frac{1}{K-1} \sum_{k \neq k'}^{K-1} \alpha_k = 0$ for all $k' \in [K-1]$. Subtracting the first equation from the second gives $\alpha_{k'} + \frac{1}{K-1} \alpha_{k'} = 0$, which in turn implies that $\alpha_{k'} = 0$ for all $k' \in [K-1]$. Thus, $1, u_1, \ldots, u_{K-1}$ are linearly independent.

### 2.2 Proof of Lemma 4.2

Using the representation of $\tilde{A}$ from (18), Lemma 4.1, and the assumption on equal size of the clusters, we get,

$$\tilde{A}1 = qR1 + s(11^\top - R)1 + (p - q)\sum_{k=1}^K G_k RG_k1 + (r - s)\sum_{k=1}^K G_k (11^\top - R)G_k1$$

$$= qd1 + sN1 - sd1 + (r - s)\sum_{k=1}^K G_k 11^\top G_k1 + [(p - q) - (r - s)]\sum_{k=1}^K G_k RG_k1$$

$$= \left[qd + s(N - d) + (p - q)\frac{d}{K} + (r - s)\frac{N - d}{K}\right]1.$$
Similarly, for any $k' \in [K]$,
\[
\tilde{A}u_{k'} = qRu_{k'} + s(11^T - R)u_{k'} + (p - q) \sum_{k = 1}^{K} G_k RG_k u_{k'} + (r - s) \sum_{k = 1}^{K} G_k (11^T - R)G_k u_{k'}
\]
\[
= 0 + 0 + (r - s) \sum_{k = 1}^{K} G_k 11^T G_k u_{k'} + [(p - q) - (r - s)] \sum_{k = 1}^{K} G_k RG_k u_{k'}
\]
\[
= \left[(p - q) \frac{d}{K} + (r - s) \frac{N - d}{K}\right] u_{k'}.
\]

### 2.3 Proof of Lemma 4.3

It is easy to verify that vectors $y_2, \ldots, y_K$ are obtained by applying the Gram-Schmidt normalization process to the vectors $u_1, \ldots, u_{K-1}$. Thus, $y_2, \ldots, y_K$ span the same space as $u_1, \ldots, u_{K-1}$. Recall that $y_1 = 1/\sqrt{N}$. We start by showing that $y_1^Ty_{1+k} = 0$.

\[
y_1^Ty_{1+k} = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} y_{(1+k)i} = \frac{1}{\sqrt{N}} \left[ \sum_{i: v_i \in C_k} (K - k)q_k - \sum_{i: v_i \in C_{k'}, k' > k} q_k \right]
\]
\[
= \frac{1}{\sqrt{N}} \left[ \frac{N}{K} (K - k)q_k - \left( N - k \frac{N}{K} \right) q_k \right] = 0.
\]

Here, $q_k = \frac{1}{\sqrt{N}(K-k)(K-k+1)}$. Now consider $y_1^Ty_{1+k_1}y_{1+k_2}$ for $k_1, k_2 \in [K-1]$ such that $k_1 \neq k_2$. Assume without loss of generality that $k_1 < k_2$.

\[
y_1^Ty_{1+k_1}y_{1+k_2} = \sum_{v_i \in C_{k_2}} (-q_{k_1})(K - k_2)q_{k_2} + \sum_{v_i \in C_{k'}, k' > k_2} (-q_{k_1})(-q_{k_2})
\]
\[
= -q_{k_1}q_{k_2}(K - k_2)\frac{N}{K} + q_{k_1}q_{k_2}\left( N - k_2 \frac{N}{K} \right) = 0.
\]

Finally, for any $k \in [K-1]$,

\[
y_1^Ty_{1+k} = \sum_{i: v_i \in C_k} (K-k)^2q_k^2 + \sum_{i: v_i \in C_{k'}, k' > k} q_k^2 = q_k^2 \left[ \frac{N}{K} (K - k)^2 + N - k \frac{N}{K} \right] = 1,
\]

where the last equality follows from the definition of $q_k$.

### 2.4 Proof of Lemma 4.4

Note that the columns of $Z$ are also the dominant $K$ eigenvectors of $Y^T\tilde{A}Y$, as $Z$ is the solution to (13) with $L$ set to $L$. The calculations below show that for all $k \in [K]$, $e_k \in \mathbb{R}^{N-r}$, the $k^{th}$ standard basis vector, is an eigenvector of $Y^T\tilde{A}Y$ with eigenvalue $\lambda_k$, where $\lambda_1, \ldots, \lambda_K$ are defined in Lemma 4.2.

\[
Y^T\tilde{A}Ye_k = Y^T\tilde{A}y_k = \lambda_k Y^Ty_k = \lambda_k e_k.
\]

The second equality follows from Lemma 4.2 because $y_2, \ldots, y_K \in \text{span}\{u_1, \ldots, u_{K-1}\}$, and $u_1, \ldots, u_{K-1}$ are all eigenvectors of $\tilde{A}$ with the same eigenvalue. To show that the columns of $YZ$ lie in the span of $y_1, \ldots, y_K$, it is enough to show that $e_1, \ldots, e_K$ are the dominant $K$ eigenvectors of $Y^T\tilde{A}Y$. 

3
Let \( \alpha \in \mathbb{R}^{N-r} \) be an eigenvector of \( Y^T \hat{A} Y \) such that \( \alpha \notin \text{span}\{e_1, \ldots, e_K\} \) and \( \|\alpha\|^2 = 1 \). Then, because \( Y^T \hat{A} Y \) is symmetric, \( \alpha^T y_1 = 0 \), i.e. \( \alpha_i = 0 \), where \( \alpha_i \) denotes the \( i \)th element of \( \alpha \). The eigenvalue corresponding to \( \alpha \) is given by

\[
\lambda_\alpha = \alpha^T Y^T \hat{A} Y \alpha.
\]

Let \( x = Y \alpha = \sum_{i=1}^{N-r} \alpha_i y_i \), then \( \lambda_\alpha = x^T \hat{A} x \). Using the definition of \( \hat{A} \) from (18), we get,

\[
x^T \hat{A} x = (q-s)x^T Rx + s x^T 11^T x + [(p-q) - (r-s)] \sum_{k=1}^{K} x^T G_k R G_k x + (r-s) \sum_{k=1}^{K} x^T G_k 11^T G_k x. \tag{25}
\]

We will consider each term in (25) separately. Before that, note that \( y_2, \ldots, y_{N-r} \in \text{null}\{R\} \). This is because \( y_1 = 1/\sqrt{N} \) and \( y_2, \ldots, y_{N-r} \) are orthogonal to \( y_1 \). Thus,

\[
R(I - 11^T/N)y_1 = 0 \Rightarrow R(I - y_1 y_1^T)y_i = 0 \Rightarrow Ry_i = 0, \quad i = 2, 3, \ldots, N-r. \tag{26}
\]

Now consider the first term in (25).

\[
x^T Rx = \sum_{i=1}^{N-r} \sum_{j=1}^{N-r} \alpha_i \alpha_j y_i^T y_j \Rightarrow \alpha_i y_i^T Ry_i = 0.
\]

Here, the second equality follows from (26), and the third equality follows as \( \alpha_1 = 0 \). Similarly, for the second term in (25),

\[
x^T 11^T x = N x^T y_1 y_1^T x = N \sum_{i=1}^{N-r} \sum_{j=1}^{N-r} \alpha_i \alpha_j y_i y_j = N \alpha_1^2(\alpha_1 y_1)^2 = 0.
\]

Note that \( G_k = G_k G_k \) as \( G_k \) is a diagonal matrix with either 0 or 1 on its diagonal. For the third term in (25),

\[
x^T G_k R G_k x = x^T G_k R G_k G_k x = x^T R_k x_k \leq \frac{d}{K} \|x_k\|_2^2, \tag{27}
\]

where \( x_k \in \mathbb{R}^{N/K} \) contains elements of \( x \) corresponding to vertices in \( C_k \). Similarly, \( R_k \in \mathbb{R}^{N/K \times N/K} \) contains the submatrix of \( R \) restricted to rows and columns corresponding to vertices in \( C_k \). The last inequality holds because \( R_{[k]} \) is a \( d/K \)-regular graph by Assumption 4.1, hence its maximum eigenvalue is \( d/K \). Further,

\[
\sum_{k=1}^{K} x^T G_k R G_k x \leq \frac{d}{K} \sum_{k=1}^{K} \|x_k\|_2^2 = \frac{d}{K} \|x\|_2^2 = \frac{d}{K}.
\]

Similarly, for the fourth term in (25),

\[
x^T G_k 11^T G_k x = x^T G_k 11^T G_k G_k x = x^T 1_{N/K} 1_{N/K}^T x_k \leq \frac{N}{K} \|x_k\|_2^2.
\]

Here, \( 1_{N/K} \in \mathbb{R}^{N/K} \) is an all-ones vector and the last inequality holds because \( 1_{N/K} 1_{N/K}^T \) is a \( N/K \)-regular graph. Because \( x_k \notin \text{span}\{y_1, \ldots, y_K\} \), there is at least one \( k \in [K] \) for which \( x_k \) is not a constant vector (if this was not true, \( x_k \) will belong to span of \( y_1, \ldots, y_K \)). Thus, at least for one \( k \in [K] \), \( x^T G_k 11^T G_k x < \frac{N}{K} \|x_k\|_2^2 \). Summing over \( k \in [K] \), we get,

\[
\sum_{k=1}^{K} x^T G_k 11^T G_k x < \frac{N}{K} \sum_{k=1}^{K} \|x_k\|_2^2 = \frac{N}{K} \|x\|_2^2 = \frac{N}{K}.
\]
Adding the four terms we get the following bound. For eigenvector \( \alpha \) of \( Y^\top \tilde{A} Y \) such that \( \alpha \not\in \text{span}\{e_1, \ldots, e_N\} \) and \( ||\alpha||_2^2 = 1 \),

\[
\lambda_\alpha = x^\top \tilde{A} x < [(p - q) - (r - s)] \frac{d}{K} + (r - s) \frac{N}{K} = \lambda_K.
\]

Thus, \( \lambda_1, \ldots, \lambda_K \) are the highest \( K \) eigenvalues of \( Y^\top \tilde{A} Y \) and hence \( e_1, \ldots, e_K \) are the top \( K \) eigenvectors. Thus, the columns of \( YZ \) lie in the span of \( y_1, \ldots, y_K \).

2.5 Proof of Lemma 4.5

As \( D \) and \( \tilde{D} \) are diagonal matrices, \( ||D - \tilde{D}|| = \max_{i \in [N]} |D_{ii} - \tilde{D}_{ii}| \). Applying union bound, we get,

\[
P(\max_{i \in [N]} |D_{ii} - \tilde{D}_{ii}| \geq \epsilon) \leq N \sum_{i=1}^{N} P(|D_{ii} - \tilde{D}_{ii}| \geq \epsilon).
\]

We consider an arbitrary term in this summation. For any \( i \in [N] \), note that \( D_{ii} = \sum_{j \neq i} A_{ij} \) is a sum of independent Bernoulli random variables such that \( E[D_{ii}] = D_{ii} \).

Case 1: \( p > \frac{1}{2} \) By Hoeffding’s inequality,

\[
P(|D_{ii} - \tilde{D}_{ii}| \geq \epsilon) \leq 2 \exp\left(-\frac{2\epsilon^2}{N}\right).
\]

Setting \( \epsilon = \sqrt{2(\alpha + 1)} / \sqrt{pN \ln N} \), we get for any \( \alpha > 0 \),

\[
P(|D_{ii} - \tilde{D}_{ii}| \geq \sqrt{2(\alpha + 1)} / \sqrt{pN \ln N}) \leq 2 \exp\left(-\frac{4p(\alpha + 1)N \ln N}{N}\right) \leq N^{-(\alpha + 1)}.
\]

Case 2: \( p \leq \frac{1}{2} \) By Bernstein’s inequality, as \( |A_{ij} - \tilde{A}_{ij}| \leq 1 \) for all \( i, j \in [N] \),

\[
P(|D_{ii} - \tilde{D}_{ii}| \geq \epsilon) \leq 2 \exp\left(-\frac{\epsilon^2/2}{\sum_{j \neq i} E[(A_{ij} - \tilde{A}_{ij})^2] + \epsilon/3}\right).
\]

Also note that,

\[
E[(A_{ij} - \tilde{A}_{ij})^2] \leq A_{ij}(1 - A_{ij})^2 + (1 - A_{ij})(-A_{ij})^2 = A_{ij}(1 - A_{ij}) \leq A_{ij} \leq p.
\]

Thus,

\[
P(|D_{ii} - \tilde{D}_{ii}| \geq \epsilon) \leq 2 \exp\left(-\frac{\epsilon^2/2}{Np + \epsilon/3}\right).
\]

Let \( \epsilon = c \sqrt{pN \ln N} \) for some constant \( c > 0 \) and assume that \( p \geq C \frac{\ln N}{N} \) for some \( C > 0 \). We get,

\[
2 \exp\left(-\frac{\epsilon^2/2}{Np + \epsilon/3}\right) = 2 \exp\left(-\frac{c^2 pN \ln N}{2(Np + c \sqrt{pN \ln N}/3)\right)} = 2 \exp\left(-\frac{c^2 \ln N}{2(1 + \frac{3}{3\sqrt{c}})}\right) \leq 2 \exp\left(-\frac{c^2 \ln N}{2(1 + \frac{3}{3\sqrt{c}})}\right).
\]
Let \( c \) be such that \( \frac{c^2}{2(1+\varepsilon/3\sqrt{c})} \geq 2(\alpha + 1) \). Such a \( c \) can always be chosen as \( \lim_{c \to \infty} \frac{c^2}{2(1+\varepsilon/3\sqrt{c})} = \infty \). Then,

\[
P(|D_{ii} - D_{ii}| \geq \epsilon) \leq N^{-(\alpha+1)}.
\]

Thus, there always exists a constant \( \text{const}_1(C,\alpha) \) that depends only on \( C \) and \( \alpha \) such that for all \( \alpha > 0 \) and for all values of \( p \geq C\ln N/N \),

\[
P(|D_{ii} - D_{ii}| \geq \text{const}_1(C,\alpha)\sqrt{pN\ln N}) \leq N^{-(\alpha+1)}.
\]

Applying the union bound over all \( i \in [N] \) yields the desired result.

### 2.6 Proof of Lemma 4.6

Note that \( \max_{i,j} \) \( C,\alpha \) is \( p \). Define \( g = pN = N \max_{i,j} \) \( C,\alpha \). Note that, \( g \geq C\ln N \) as \( p \geq C\ln N/N \). By Theorem 5.2 from Le and Rinaldo [2015], for any \( \alpha > 0 \), there exists a constant \( \text{const}_2(C,\alpha) \) such that,

\[
\|A - A\| \leq \text{const}_2(C,\alpha)\sqrt{g} = \text{const}_2(C,\alpha)\sqrt{pN}
\]

with probability at least \( 1 - N^{-\alpha} \).

### 2.7 Proof of Lemma 4.7

Because \( Y^\top Y = I \), for any orthonormal matrix \( U \in \mathbb{R}^{K \times K} \) such that \( UU^\top = U^\top U = I \),

\[
\|YZ - YZU\|_F^2 = \|Y(Z - ZU)\|_F^2 = \text{trace}((Z - ZU)^\top Y(Z - ZU)) = \|Z - ZU\|_F^2.
\]

Thus, it is enough to show an upper bound on \( \|Z - ZU\|_F \), where recall that columns of \( Z \in \mathbb{R}^{N-r \times K} \) and \( Z \in \mathbb{R}^{N-r \times K} \) contain the leading \( K \) eigenvectors of \( Y^\top CY \) and \( Y^\top EN \) respectively. Thus,

\[
\inf_{U \in \mathbb{R}^{K \times K}, U^\top U = I} \|YZ - YZU\|_F = \inf_{U \in \mathbb{R}^{K \times K}, U^\top U = I} \|Z - ZU\|_F.
\]

By equation (2.6) and Proposition 2.2 in Vu and Le [2013],

\[
\inf_{U \in \mathbb{R}^{K \times K}, U^\top U = I} \|Z - ZU\|_F \leq \sqrt{2}\|ZZ^\top (I - ZZ^\top)\|_F.
\]

Moreover, \( \|ZZ^\top (I - ZZ^\top)\|_F \leq \sqrt{K}\|ZZ^\top (I - ZZ^\top)\|_F \) as \( \text{rank}\{ZZ^\top (I - ZZ^\top)\} \leq K \). Thus, we get,

\[
\inf_{U \in \mathbb{R}^{K \times K}, U^\top U = I} \|Z - ZU\|_F \leq \sqrt{2K}\|ZZ^\top (I - ZZ^\top)\|_F.
\]

Let \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-r} \) be eigenvalues of \( Y^\top CY \) and \( \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{N-r} \) be eigenvalues of \( Y^\top EN \). By Weyl’s perturbation theorem,

\[
|\mu_i - \alpha_i| \leq \|Y^\top EN - Y^\top CY\|, \quad \forall i \in [N-r].
\]

Define \( \gamma = \mu_{K+1} - \mu_K \) to be the eigen-gap between the \( K^{th} \) and \((K+1)^{th}\) eigenvalues of \( Y^\top CY \).

**Case 1:** \( \|Y^\top EN - Y^\top CY\| \leq \frac{\gamma}{2} \). If \( \|Y^\top EN - Y^\top CY\| \leq \frac{\gamma}{2} \), then \( |\mu_i - \alpha_i| \leq \frac{\gamma}{4} \) for all \( i \in [N-r] \) by the inequality given above. Thus, \( \alpha_1, \alpha_2, \ldots, \alpha_K \in [0, \mu_K + \frac{\gamma}{4}] \) and \( \alpha_{K+1}, \alpha_{K+2}, \ldots, \alpha_{N-r} \in [\mu_{K+1} - \frac{\gamma}{4}, \infty) \). Let \( S = [0, \mu_K + \frac{\gamma}{4}] \), then \( \mu_1, \ldots, \mu_K \in S \) and \( \alpha_1, \ldots, \alpha_{N-r} \notin S \). Define \( \delta \) as,

\[
\delta = \min\{\alpha_i - s, \alpha_i \notin S, s \in S\}.
\]

Then, \( \delta \geq [\mu_{K+1} - \gamma/4] - [\mu_K + \gamma/4] = \gamma/2 \). By Davis-Kahan sin \( \Theta \) theorem,

\[
\|ZZ^\top (I - ZS^\top)\| \leq \frac{1}{\delta} \|Y^\top EN - Y^\top CY\| = \frac{\gamma}{2} \|Y^\top EN - Y^\top CY\|.
\]

6
Case 2: $||Y^\top LY - Y^\top LY|| > \frac{\gamma}{4}$ Note that $||ZZ^\top(I - ZZ^\top)|| \leq 1$ as,

$$||ZZ^\top(I - ZZ^\top)|| \leq ||Z|| ||I - ZZ^\top|| = 1.1 = 1.$$ Thus, if $||Y^\top LY - Y^\top LY|| > \frac{\gamma}{4}$, then,

$$||ZZ^\top(I - ZZ^\top)|| \leq \frac{4}{\gamma}||Y^\top LY - Y^\top LY||.$$ In both cases, $||ZZ^\top(I - ZZ^\top)|| \leq \frac{4}{\gamma}||Y^\top LY - Y^\top LY||$. Using (21) and (29), we get with probability at least $1 - 2N^{-\alpha}$,

$$\inf_{U \in R^{K \times K}, UU^\top = U^\top U = I} ||Z - ZU||_F \leq \text{const}_3(C, \alpha) \frac{4\sqrt{2K}}{\gamma} \sqrt{pN \ln N}.$$

### 2.8 Proof of Lemma 4.8

Equation (22) directly follows from Lemma 5.3 in Lei and Rinaldo [2015]. We only need to show that

$$\frac{8(2 + \epsilon)}{\delta^2} ||XU - YZ||_F^2 < \frac{N}{K}.$$ Equation (23) then follows from Lemma 5.3 in Lei and Rinaldo [2015]. Recall that $\delta = \sqrt{\frac{2K}{N}}$. Using Lemma 4.7, we get

$$\frac{8(2 + \epsilon)}{\delta^2} ||XU^\top - YZ||_F^2 \leq \text{const}_3(C, \alpha)^2 \frac{128(2 + \epsilon)}{\gamma^2} pN^2 \ln N < \frac{N}{K}.$$ Here, the last inequality follows from the assumption that $\gamma^2 > \text{const}_3(C, \alpha)^2 1.128(2 + \epsilon) pNK \ln N$.

### 2.9 Proof of Lemma 4.9

We begin by showing a simple result. Let $a, b > 0$. Then,

$$|\sqrt{a} - \sqrt{b}| = \left\{\frac{(|\sqrt{a} - \sqrt{b}|)(|\sqrt{a} + \sqrt{b}|)}{\sqrt{a} + \sqrt{b}}\right\} = \frac{|a - b|}{\sqrt{a} + \sqrt{b}} \leq \frac{|a - b|}{\sqrt{b}}.$$ Further,

$$\left\{\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right\} = \left\{\frac{|\sqrt{a} - \sqrt{b}|}{\sqrt{a} \sqrt{b}}\right\} \leq \frac{|a - b|}{|\sqrt{a} \sqrt{b}|}.$$ Coming back to the bound on $||Q^{-1} - Q^{-1}||$, note that, as $Q^{-1} = (\lambda_1 - p)^{-1/2}I$, we have that

$$||Q^{-1} - Q^{-1}|| = \max \left\{\nu_i - \frac{1}{\sqrt{\lambda_1 - p}} : \nu_i \text{ is an eigenvalue of } Q^{-1}\right\}.$$ As $Q = Y^\top DY$, the eigenvalues of $Q^{-1}$ are given by $1/\mu_1', \ldots, 1/\mu_{N-r}'$, where $\mu_1', \ldots, \mu_{N-r}'$ are the eigenvalues of $Y^\top DY$. Moreover, by substituting $a = \mu_i'$ and $b = \lambda_1 - p$ in the inequality derived above, we get

$$\frac{1}{\mu_i'} - \frac{1}{\sqrt{\lambda_1 - p}} \leq \frac{\lambda_1 - p}{\mu_i'} \frac{1}{(\lambda_1 - p)\sqrt{\mu_i'}} \forall i \in [N - r].$$ (30)

By Weyl’s perturbation theorem, for any $i \in [N - r]$,

$$|\mu_i' - (\lambda_1 - p)| \leq ||Y^\top DY - Y^\top DY|| \leq ||D - D||,$$
where the last inequality follows as $\mathbf{Y}^\top \mathbf{Y} = \mathbf{I}$. Let us assume for now that $\|\mathbf{D} - \mathcal{D}\| \leq \frac{\lambda_1 - p}{2}$ (we prove this below). Then, $|\mu'_i - (\lambda_1 - p)| \leq \frac{\lambda_1 - p}{2}$ for all $i \in [N - r]$. Hence,

$$\mu'_i \geq \frac{\lambda_1 - p}{2}, \quad \forall \ i \in [N - r].$$

Using this in (30) results in

$$\frac{1}{\sqrt{\mu'_i}} - \frac{1}{\sqrt{\lambda_1 - p}} \leq \frac{\sqrt{2}}{\sqrt{(\lambda_1 - p)^3}} \|\mathbf{D} - \mathcal{D}\|, \quad \forall \ i \in [N - r].$$

Taking the maximum over all $i \in [N - r]$ yields the desired result. Next, we prove that $\|\mathbf{D} - \mathcal{D}\| \leq \frac{\lambda_1 - p}{2}$.

Recall from Lemma 4.5 that $\|\mathbf{D} - \mathcal{D}\| \leq \text{const}_1(C, \alpha)\sqrt{pN \ln N}$, where the proof of Lemma 4.5 requires that $\text{const}_1(C, \alpha)$ satisfies the following condition:

$$\frac{\text{const}_1(C, \alpha)^2}{2 \left( 1 + \frac{\text{const}_1(C, \alpha)}{3\sqrt{C}} \right)} \geq 2(\alpha + 1).$$

Additionally, to show that $\|\mathbf{D} - \mathcal{D}\| \leq \frac{\lambda_1 - p}{2}$, we need to ensure that $\text{const}_1(C, \alpha) \leq \frac{\lambda_1 - p}{2\sqrt{pN \ln N}}$. A constant $\text{const}_1(C, \alpha)$ that satisfies both these conditions exists if:

$$\frac{(\lambda_1 - p)^2/4pN \ln N}{2 \left( 1 + \frac{(\lambda_1 - p)/2\sqrt{pN \ln N}}{3\sqrt{C}} \right)} \geq 2(\alpha + 1).$$

Simplifying the expression above results in

$$\frac{1}{\sqrt{\frac{pN \ln N}{\lambda_1 - p}} \left( \frac{\sqrt{pN \ln N}}{\lambda_1 - p} + \frac{1}{6\sqrt{C}} \right)} \geq 16(\alpha + 1).$$

The assumption made in the lemma guarantees that such a condition is satisfied. Hence, $\text{const}_1(C, \alpha)$ can be set such that $\|\mathbf{D} - \mathcal{D}\| \leq \frac{\lambda_1 - p}{2}$.

### 2.10 Proof of Lemma 4.10

As in the proof of Lemma 4.7, because $\mathbf{Y}^\top \mathbf{Y} = \mathbf{I}$, for any orthonormal matrix $\mathbf{U} \in \mathbb{R}^{K \times K}$ such that $\mathbf{U}^\top \mathbf{U} = \mathbf{U} \mathbf{U}^\top = \mathbf{I}$,

$$\|\mathbf{Y} \mathbf{Q}^{-1} \mathbf{Z} - \mathbf{Y} \mathbf{Q}^{-1} \mathbf{Z} \mathbf{U}\|_F = \|\mathbf{Q}^{-1} \mathbf{Z} - \mathbf{Q}^{-1} \mathbf{Z} \mathbf{U}\|_F.$$ 

As $\mathbf{Q}, \mathbf{Q} \in \mathbb{R}^{N - r \times N - r}$ and $\mathbf{Z}, \mathbf{Z} \in \mathbb{R}^{N - r \times K}$, we have that $\text{rank} \{\mathbf{Q}^{-1} \mathbf{Z}\} \leq K$ and $\text{rank} \{\mathbf{Q}^{-1} \mathbf{Z} \mathbf{U}\} \leq K$, and hence $\text{rank} \{\mathbf{Q}^{-1} \mathbf{Z} - \mathbf{Q}^{-1} \mathbf{Z} \mathbf{U}\} \leq 2K$. Therefore,

$$\|\mathbf{Q}^{-1} \mathbf{Z} - \mathbf{Q}^{-1} \mathbf{Z} \mathbf{U}\|_F \leq \sqrt{2K} \|\mathbf{Q}^{-1} \mathbf{Z} - \mathbf{Q}^{-1} \mathbf{Z} \mathbf{U}\|.$$ 

Moreover, using $\mathbf{Q}^{-1} = (\sqrt{\lambda_1 - p})^{-1} \mathbf{I}$ and Lemma 4.9,

$$\|\mathbf{Q}^{-1} \mathbf{Z} - \mathbf{Q}^{-1} \mathbf{Z} \mathbf{U}\| \leq \|\mathbf{Q}^{-1}\| \cdot \|\mathbf{Z} - \mathbf{Z} \mathbf{U}\| + \|\mathbf{Q}^{-1} - \mathbf{Q}^{-1}\| \cdot \|\mathbf{Z} \mathbf{U}\|$$

$$\leq \frac{1}{\sqrt{\lambda_1 - p}} \|\mathbf{Z} - \mathbf{Z} \mathbf{U}\| + \sqrt{\frac{2}{(\lambda_1 - p)^3}} \|\mathbf{D} - \mathcal{D}\| \cdot \|\mathbf{Z} \mathbf{U}\|.$$
Note that $\|ZU\| = \sqrt{\lambda_{\max}(U^\top Z^\top ZU)} = \sqrt{\lambda_{\max}(U^\top U)} = \sqrt{\lambda_{\max}(|I|)} = 1$. Also note that $\|Z - ZU\| \leq \|Z - U\|_F$. Combining all of this information, we get,

$$\inf_{U: U^\top U = I} \|YQ^{-1}Z - Y^{-1}ZU\|_F \leq \sqrt{\frac{2K}{\lambda_1 - p}} \inf_{U: U^\top U = I} \|Z - ZU\|_F + \sqrt{\frac{4K}{(\lambda_1 - p)^3}} \|D - D\|.$$

Let $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_{N-r}$ be eigenvalues of $Q^{-1}Y^\top LyQ^{-1}$ and $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_{N-r}$ be eigenvalues of $Q^{-1}Y^\top LyQ^{-1}$. Define $\gamma = \mu_{K+1} - \mu_K$. Using a strategy similar to the one used in the proof of Lemma 4.7, we get:

$$\inf_{U: U^\top U = I} \|Z - ZU\|_F \leq \frac{4\sqrt{2K}}{\gamma} \|Q^{-1}Y^\top LyQ^{-1} - Q^{-1}Y^\top LyQ^{-1}\|.$$

Using (24) and Lemma 4.5 results in

$$\inf_{U: U^\top U = I} \|Z - ZU\|_F \leq \frac{8\sqrt{2K}}{\gamma(\lambda_1 - p)} \left[ \left( \frac{(\lambda_1 - \lambda)\gamma(\alpha, C, \alpha)\sqrt{2}}{\lambda_1 - p} + \frac{\gamma(\alpha, C, \alpha)^2}{2} \right) \sqrt{pN \ln N} + \frac{\gamma(\alpha, C, \alpha)}{pN \ln N} \left( \frac{\gamma(\alpha, C, \alpha)^2}{(\lambda_1 - p)^2} \right) \right] \sqrt{pN \ln N}.$$

Here, the second inequality follows from the assumption that there is a constant $\gamma(\alpha, C, \alpha)$ that satisfies $\gamma(\alpha, C, \alpha) \leq \gamma(\alpha, C, \alpha)$. The last inequality follows by choosing $\gamma(\alpha, C, \alpha)$ such that the expression between the square brackets in the second inequality is less than $\gamma(\alpha, C, \alpha)$. Using the expression above, we get:

$$\inf_{U: U^\top U = I} \|YQ^{-1}Z - YQ^{-1}ZU\|_F \leq \left[ \frac{16K \gamma(\alpha, C, \alpha)}{\gamma(\alpha, C, \alpha)^{3/2}} + \frac{2\gamma(\alpha, C, \alpha)^{3/2}}{(\lambda_1 - p)^{3/2}} \right] \sqrt{pN \ln N}.$$