SOME GENERALIZATIONS OF GOOD INTEGERS AND THEIR APPLICATIONS IN THE STUDY OF SELF-DUAL NEGACYCLIC CODES

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Abstract. Good integers introduced in 1997 form an interesting family of integers that has been continuously studied due to their rich number theoretical properties and wide applications. In this paper, we have focused on classes of $2^\beta$-good integers, $2^\beta$-oddly-good integers, and $2^\beta$-evenly-good integers which are generalizations of good integers. Properties of such integers have been given as well as their applications in characterizing and enumerating self-dual negacyclic codes over finite fields. An alternative proof for the characterization of the existence of a self-dual negacyclic code over finite fields has been given in terms of such generalized good integers. A general enumeration formula for the number of self-dual negacyclic codes of length $n$ over finite fields has been established. For some specific lengths, explicit formulas have been provided as well. Some known results on self-dual negacyclic codes over finite fields can be formalized and viewed as special cases of this work.

1. Introduction

For fixed coprime nonzero integers $a$ and $b$, a given positive integer $d$ is called a good integer (with respect to $a$ and $b$) (see [14]) if there exists a positive integer $k$ such that $d| (a^k + b^k)$. Otherwise, $d$ is called a bad integer. Some properties of the set $G_{a,b}$ of good integers defined with respect to $a$ and $b$ has been investigated in [8] and [14]. For a prime power $q$, the set $G_{(q,1)}$ has been applied in constructing BCH codes in [11]. In [7] and [9], $G_{(2^l,1)}$ has been applied in enumerating self-dual cyclic and abelian codes over finite fields.

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In [8], two subclasses of good integers defined with respect to coprime integers \( a \) and \( b \) have been introduced. A positive integer \( d \) is said to be *oddly-good* (with respect to \( a \) and \( b \)) if \( d|(a^k + b^k) \) for some odd integer \( k \geq 1 \), and *evenly-good* (with respect to \( a \) and \( b \)) if \( d|(a^k + b^k) \) for some even integer \( k \geq 2 \). Therefore, \( d \) is odd if it is oddly-good or evenly-good. Denote by \( OG_{(a,b)} \) (resp., \( EG_{(a,b)} \)) the set of oddly-good (resp., evenly-good) integers defined with respect to \( a \) and \( b \). In [10], some basic properties of \( OG_{(2',1)} \) and \( EG_{(2',1)} \) have been studied and applied in enumerating Hermitian self-dual abelian codes over finite fields. The characterizations of \( OG_{(a,b)} \) and \( EG_{(a,b)} \) was discussed in [8] and was applied to the study of the hull of abelian codes. However, there are errors in [8, Proposition 2.1] and [8, Proposition 2.3] and in some subsequent results using these.

In this paper, we first give necessary and sufficient conditions for \( 2^\beta d \) to be a good integer, while correcting the errors of [8]. Then we investigate the extended classes of \( G_{(a,b)} \) defined as follows. For a non-negative integer \( \beta \), a positive integer \( d \) is said to be \( 2^\beta \)-good (with respect to \( a \) and \( b \)) if \( 2^\beta d \in G_{(a,b)} \). Otherwise, \( d \) is said to be \( 2^\beta \)-bad. In the same fashion, a positive integer \( d \) is said to be \( 2^\beta \)-oddly-good (with respect to \( a \) and \( b \)) if \( 2^\beta d \in OG_{(a,b)} \), and \( 2^\beta \)-evenly-good (with respect to \( a \) and \( b \)) if \( 2^\beta d \in EG_{(a,b)} \). Therefore, \( d \) is \( 2^\beta \)-good if and only if it is \( 2^\beta \)-oddly-good or \( 2^\beta \)-evenly-good. For an integer \( \beta \geq 0 \), denote by \( G_{(a,b)}(\beta) \), \( OG_{(a,b)}(\beta) \), and \( EG_{(a,b)}(\beta) \) the sets of \( 2^\beta \)-good, \( 2^\beta \)-oddly-good, and \( 2^\beta \)-evenly-good integers, respectively. Here, we focus on these 3 types of generalized good integers. Applications of such generalized good integers in characterizing and enumerating self-dual negacyclic codes are discussed. Brief history, properties, and applications of (self-dual) negacyclic codes will be recalled in Section 3. For more details, the readers may refer to [1], [2], [3], [5], [6], [12], and references therein.

The paper is organized as follows. In Section 2, the family of \( 2^\beta \)-good integers are studied together with their subclasses of \( 2^\beta \)-evenly-good integers and \( 2^\beta \)-oddly-good integers. Corrections to [8, Propositions 2.1 and 2.3] and its consequence are also given in this section. Applications of such \( 2^\beta \)-good integers in characterizing and enumerating self-dual negacyclic codes are provided in Section 3.

### 2. \( 2^\beta \)-Good Integers

For pairwise coprime nonzero integers \( a, b \) and \( n > 0 \), let \( \text{ord}_n(a) \) denote the multiplicative order of \( a \) modulo \( n \). In this case, the multiplicative inverse \( b^{-1} \) of \( b \) exists in the multiplicative group \( \mathbb{Z}_n^\times \). Let \( \text{ord}_n(a) \) denote the multiplicative order of \( ab^{-1} \) modulo \( n \). Denote by \( 2^\gamma|n \) if \( \gamma \geq 0 \) is the largest integer such that \( 2^\gamma|n \), i.e., \( 2^\gamma|n \) but \( 2^{\gamma+1} \not| n \).

From the definition of a \( 2^\beta \)-good integer, a positive integer \( d \) is \( 2^\beta \)-good if and only if there exists a positive integer \( k \) such that \( 2^\beta d|(a^k + b^k) \). Hence, for each \( \beta \geq 1 \), \( d \) is \( 2^\beta \)-good whenever it is \( 2^\beta \)-good. It follows that \( G_{(a,b)}(\beta - 1) \supseteq G_{(a,b)}(\beta) \) for all \( \beta \geq 1 \).

We note that if \( ab \) is even, then \( d \not| (a^k + b^k) \) for all positive integers \( k \) and all even positive integers \( d \). Hence, \( d \not\in G_{(a,b)} \) for all even integers \( d \). Consequently, \( G_{(a,b)}(\beta) = \emptyset \) for all \( \beta \geq 1 \).

In the following subsections, we assume that \( a \) and \( b \) are coprime odd integers. In Subsection 2.1, we rectify the errors of [8]. Properties of odd \( 2^\beta \)-good integers are discussed in Subsection 2.2 and arbitrary \( 2^\beta \)-good integers are studied in Subsection 2.3. The subclasses of \( 2^\beta \)-evenly-good integers and \( 2^\beta \)-oddly-good integers are investigated in Subsection 2.4.
2.1. Good Integers: Correction of Results of [8]. The errors in [8] were caused because of the following false statements

$$ \text{ord}_b\left(\frac{a}{b}\right) = 2 \Rightarrow ab^{-1} \equiv -1 \mod 2^\beta \text{ i.e., } 2^\beta \mid a+b $$

and

$$ \text{ord}_d\left(\frac{a}{b}\right) = 2k \Rightarrow (ab^{-1})^k \equiv -1 \mod d, $$

used in the proofs of [8, Proposition 2.1] and [8, Proposition 2.3], respectively, where \(a, b\) and \(d \geq 1\) are pairwise coprime odd integers and \(\beta\) is a positive integer. It is not difficult to see that \(\text{ord}_8(11) = 2\) but \(11 \not\equiv -1 \mod 8\), and \(\text{ord}_{15}(11) = 2\) but \(11 \not\equiv -1 \mod 15\).

First we have a general lemma:

**Lemma 2.1.** Let \(x\) and \(d > 1\) be coprime odd integers. If \(k\) is the smallest positive integer such that \(x^k \equiv -1 \mod d\), then \(\text{ord}_d(x) = 2k\).

**Proof.** Let \(k\) be the smallest positive integer such that \(x^k \equiv -1 \mod d\). Write \(k = 2^\lambda k'\), where \(\lambda \geq 0\) and \(k'\) is an odd integer. Since \(x^k \equiv -1 \mod d\), we have \(x^{2k} \equiv 1 \mod d\). Therefore, \(\text{ord}_d(x)|2k\). Let \(\text{ord}_d(x) = 2^\mu r\), where \(0 \leq \mu \leq \lambda + 1\) and \(r\) is odd. Then \(r|k'\), i.e., \(k' = rr'\) for some positive integer \(r'\).

Suppose that \(\mu \leq \lambda\). Then \(x^{2^\mu r} \equiv 1 \mod d\). It gives that \(x^{2^\mu r}k' \equiv x^{2^\mu rr'} \equiv (x^{2^\mu r})^{2^\mu - rr'} \equiv 1 \mod d\), but \(x^{2^\mu k'} \equiv x \equiv -1 \mod d\), a contradiction, as \(d\) is odd. Therefore, we must have \(\mu = \lambda + 1\).

Write \(d = p_1^{e_1}p_2^{e_2}\cdots p_i^{e_i}\), where \(p_1, p_2, \ldots, p_i\) are distinct odd primes and \(e_1, e_2, \ldots, e_i\) are some positive integers. Since \(1 \equiv x^{\text{ord}_d(x)} = x^{2^\mu r} = x^{2^{\lambda + 1} r} \mod p_i^{e_i}\), we have \(p_i^{e_i} | (x^{2^\mu r} - 1)(x^{2^\mu r} + 1)\) for all \(i \in \{1, 2, \ldots, t\}\). Hence, for each \(i \in \{1, 2, \ldots, t\}\), either \(p_i^{e_i} | (x^{2^\mu r} - 1)\) or \(p_i^{e_i} | (x^{2^\mu r} + 1)\) but not both. If \(p_i^{e_i} | (x^{2^\mu r} - 1)\) for some \(i \in \{1, 2, \ldots, t\}\), then \(-1 \equiv x^k \equiv x^{2^\mu k'} \equiv x^{2^\mu r} \equiv (x^\lambda r)^{r'} \equiv 1 \mod p_i^{e_i}\), a contradiction. Hence, \(p_i^{e_i} | (x^{2^\mu r} + 1)\) for all \(i \in \{1, 2, \ldots, t\}\). Consequently, we have \(d | (x^{2^\mu r} + 1)\), i.e., \(x^{\frac{\text{ord}_d(x)}{2}} \equiv x^{2^\mu r} \equiv -1 \mod d\). By the minimality of \(k\), it can be deduced that \(k \leq \frac{\text{ord}_d(x)}{2}\). Since \(\text{ord}_d(x)|2k\), we have \(\text{ord}_d(x) = 2k\) as desired. \(\square\)

**Remark 2.2.** The converse of Lemma 2.1 is not always true. The converse holds only if \(d\) is an odd prime power or \(d = 2\). This is so because if \(\text{ord}_{p^r}(x) = 2\), where \(p\) is an odd prime, we have \(p^r|\left(x^s-1\right)(x^s+1)\). It cannot happen that \(p^r|\left(x^s-1\right)\) and \(p^r|\left(x^s+1\right)\) with \(i+j = r\), \(i \geq 1\), \(j \geq 1\). Because then \(p|\left(x^s-1\right)\) and \(p|\left(x^s+1\right)\) which gives \(p|2\); not possible. Hence, either \(p^r|\left(x^s-1\right)\) or \(p^r|\left(x^s+1\right)\) but not both. However, if \(p|\left(x^s+1\right)\), we get \(\text{ord}_{p^r}(x) \geq s\), not possible. Therefore, \(p^r \) must divide \(\left(x^s+1\right)\).

For each odd integer \(x\) and positive integer \(\beta\), we note that

\[
\text{ord}_{2^\beta}(x) = \begin{cases} 
1 & \text{if } \beta = 1, \\
2 & \text{if } \beta \geq 2 \text{ and } x \equiv -1 \mod 2^\beta. 
\end{cases}
\]

A correction of [8, Proposition 2.1] is now given as follows.

**Proposition 2.3.** Let \(a\) and \(b\) be coprime odd integers and let \(\beta \geq 1\) be an integer. Then \(2^\beta \in G(a,b)\) if and only if \(2^\beta|(a+b)\), i.e., \(1 \in G(a,b)(\beta)\) if and only if \(2^\beta|(a+b)\).

**Proof.** Suppose \(2^\beta \in G(a,b)\). If \(\beta = 1\), then clearly \(2^\beta|(a+b)\) since \(a+b\) is even. Let \(\beta > 1\). Then \(2^\beta|(a^k+b^k)\) for some integer \(k \geq 1\) and so \(4|(a^k+b^k)\). If \(k\) is even, then \(a^k \equiv 1 \mod 4\) and \(b^k \equiv 1 \mod 4\) which implies that \((a^k+b^k) \equiv 2 \mod 4\), a
contradiction. It follows that $k$ is odd. Since $a^k + b^k = (a + b) \left( \sum_{i=0}^{k-1} (-1)^i a^{k-1-i} b^i \right)$ and $\sum_{i=0}^{k-1} (-1)^i a^{k-1-i} b^i$ is odd (it being a sum of odd terms taken odd number of times), we have that $2^\beta | (a + b)$. The converse is obvious. \hfill \Box

The following results are needed in the proof of the correct version of \cite[Proposition 2.3]{j}.  

**Proposition 2.4** (\cite[Theorem 1]{p}). Let $d > 1$ be an odd integer. Then $d \in G_{(a,b)} = G_{(a,b)}(0)$ if and only if there exists an integer $s \geq 1$ such that $2^s | \text{ord}_{p^s}(\frac{a}{b})$ for every prime $p$ dividing $d$.

**Proposition 2.5** (\cite[Proposition 2.2]{j}). Let $a, b, d > 1$ be pairwise coprime odd integers. Then $d \in G_{(a,b)}$ if and only if $2d \in G_{(a,b)}$.

**Lemma 2.6** (\cite[Proposition 2]{p}). For an odd prime $p$, $\text{ord}_{p^s}(\frac{a}{b}) = \text{ord}_{p^s}(\frac{a}{b}) p^n$ for some $\alpha \geq 0$.

The next proposition is a correction of \cite[Proposition 2.3]{j}.

**Proposition 2.7.** Let $a, b$ and $d > 1$ be pairwise coprime odd positive integers and let $\beta \geq 2$ be an integer. Then $2^\beta d \in G_{(a,b)}$ if and only if $2^\beta | (a + b)$ and $2 | \text{ord}_{p^1}(\frac{a}{b})$ for every prime $p$ dividing $d$. In this case, $\text{ord}_{2^\beta}(\frac{a}{b}) = 2$ and $2 | \text{ord}_{2^\beta d}(\frac{a}{b})$.

Proof. Suppose $2^\beta d \in G_{(a,b)}$. Let $k$ be the smallest positive integer such that $2^\beta d | (a^k + b^k)$. Then $d | (a^k + b^k)$ and $2^\beta | (a^k + b^k)$ which implies that $(ab^{-1})^2k \equiv 1 \mod d$. Moreover, $2^\beta | (a + b)$ and $k$ must be odd by Proposition 2.3 and its proof. Let $k'$ be the smallest positive integer such that $d | (a^{k'} + b^{k'})$. Then $\text{ord}_{d}(\frac{a}{b}) = 2k'$ by Lemma 2.1. Since $(ab^{-1})^2k \equiv 1 \mod d$, we have $k'|k$. Consequently, $k'$ is odd and $(a + b)|(a^{k'} + b^{k'})$. Hence, $2^\beta d | (a^{k'} + b^{k'})$. By the minimality of $k$, we have $k = k'$ and $\text{ord}_{d}(\frac{a}{b}) = 2k' = 2k$, where $k$ is odd. Let $d = p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$ where $p_i$ are odd primes and $e_i \geq 1$. Then, using Lemma 2.6,

$$2k = \text{ord}_{d}(\frac{a}{b}) = \text{lcm}(\text{ord}_{p_1}(\frac{a}{b}) p_1^{\alpha_1}, \text{ord}_{p_2}(\frac{a}{b}) p_2^{\alpha_2}, \cdots, \text{ord}_{p_t}(\frac{a}{b}) p_t^{\alpha_t})$$

where $\alpha_i$ are some non-negative integers. Also $(ab^{-1})^{2} \equiv -1 \mod p_i$ for all $i$, $1 \leq i \leq t$. Therefore, $\text{ord}_{p_i}(\frac{a}{b})$ is even and so $2 | \text{ord}_{p_i}(\frac{a}{b})$ for each $i$.

Conversely let $2 | \text{ord}_{p_i}(\frac{a}{b})$ for each $p_i | d$. This gives $2 | \text{ord}_{p_i}(\frac{a}{b})$ for each $i$ by Lemma 2.6. Let $\text{ord}_{p_i}(\frac{a}{b}) = 2r_i$, where $r_i$ is odd. Therefore, by Remark 2.2, $(ab^{-1})^{r_i} \equiv -1 \mod p_i^{r_i}$ for all $i$, $1 \leq i \leq t$. Let $k = \text{lcm}(r_1, r_2, \cdots, r_t)$, $k$ is odd and let $k = r_1 r_i$. Each of $r_i$ is also odd. Then $(ab^{-1})^k \equiv (ab^{-1})^{r_1 r_i} \equiv (-1)^{r_i} \equiv -1 \mod p_i^{r_i}$ for each $i$. Therefore, $(ab^{-1})^k \equiv -1 \mod d$ which implies $\text{ord}_{d}(\frac{a}{b}) = 2k$ by Lemma 2.1. Now $2^\beta |(a + b)$ implies $2^\beta |a^k + b^k$ as $k$ is odd. Hence $(ab^{-1})^k \equiv -1 \mod 2^\beta d$, i.e., $2^\beta d \in G_{(a,b)}$.

In this case, we have $2^\beta |(a + b)$ which implies that $\text{ord}_{2^\beta}(\frac{a}{b}) = 2$ by (2.1). Moreover, $\text{ord}_{2^\beta d}(\frac{a}{b}) = \text{lcm}(\text{ord}_{2^\beta}(\frac{a}{b}), \text{ord}_{d}(\frac{a}{b})) = 2k$ and $k$ is odd. Hence, $2 | \text{ord}_{2^\beta d}(\frac{a}{b})$ as desired. \hfill \Box

**Remark 2.8.** As a consequence of the above corrections, the bullets (c) and (d) of [8, Theorem 2.1 and Theorem 3.1] should be rewritten as follows.

(c) $\beta \geq 2$, $d = 1$ and $2^\beta |(a + b)$. 

\vspace{1cm}
(d) $\beta \geq 2$, $d \geq 3$, $2^\beta(a + b)$ and $2|\text{ord}_p(\frac{a}{b})$ for every prime $p$ dividing $d$.

Note that the above corrections do not affect any other result given in [8].

2.2. ODD $2^\beta$-GOOD INTEGERS. The characterization of odd $2^\beta$-good integers greater than 1 is summarized in the next proposition.

**Proposition 2.9.** Let $a$, $b$ and $d > 1$ be pairwise coprime odd integers and let $\beta$ be a non-negative integer. Then the following statements hold.

1) ([14, Theorem 1]) If $\beta = 0$, then $d \in G_{(a,b)} = G_{(a,b)}(\beta)$ if and only if there exists $s \geq 1$ such that $2^s|\text{ord}_p(\frac{a}{b})$ for every prime $p$ dividing $d$.

2) ([8, Proposition 2.2]) If $\beta = 1$, then $d \in G_{(a,b)}(\beta)$ if and only if $d \in G_{(a,b)}$.

3) (Proposition 2.7) If $\beta \geq 2$, then $d \in G_{(a,b)}(\beta)$ if and only if $2^\beta(a + b)$ and $2|\text{ord}_p(\frac{a}{b})$ for every prime $p$ dividing $d$.

For coprime odd integers $a$ and $b$, it is obvious that 1 is an element in both $G_{(a,b)}(0)$ and $G_{(a,b)}(1)$. For $\beta \geq 2$, we have the following characterization.

**Corollary 2.10.** Let $a$ and $b$ be coprime odd integers and let $\beta \geq 2$ be an integer. Then the following statements are equivalent.

1) $1 \notin G_{(a,b)}(\beta)$.

2) $2^\beta \nmid (a + b)$.

3) $n \notin G_{(a,b)}(\beta)$ for all odd natural numbers $n$.

**Proof.** From Proposition 2.3, we have that 1) and 2) are equivalent.

To prove 2) implies 3), assume that $2^\beta \nmid (a + b)$. By Proposition 2.9, we therefore have $n \notin G_{(a,b)}(\beta)$ for all odd natural numbers $n$. That 3) implies 1) is clear.

Next, we consider a product of two odd $\beta$-good integers and the divisors of an odd $\beta$-good integer.

**Proposition 2.11.** Let $a$ and $b$ be coprime odd integers. Let $\beta$ be a non-negative integer and let $c$ and $d$ be odd positive integers.

1) If $cd \in G_{(a,b)}(\beta)$, then $c \in G_{(a,b)}(\beta)$ and $d \in G_{(a,b)}(\beta)$.

2) If $\beta \geq 2$, then $cd \in G_{(a,b)}(\beta)$ if and only if $c \in G_{(a,b)}(\beta)$ and $d \in G_{(a,b)}(\beta)$.

**Proof.** 1) and the sufficient part of 2) follow directly from the definition of $G_{(a,b)}(\beta)$. The necessary part of 2) follows from Proposition 2.9.

**Remark 2.12.** We note that the converse of 1) in Proposition 2.11 does not need to be true for $\beta \in \{0, 1\}$. Using Proposition 2.9 and a direct calculation, we have the following examples.

1. $5 \in G_{(3,1)}(0)$ and $7 \in G_{(3,1)}(0)$ but $5 \times 7 \notin G_{(3,1)}(0)$.

2. $5 \in G_{(3,1)}(1)$ and $7 \in G_{(3,1)}(1)$ but $5 \times 7 \notin G_{(3,1)}(1)$.

The following corollary can be obtained immediately from Proposition 2.11.

**Corollary 2.13.** Let $a$, $b$ and $d > 1$ be pairwise coprime odd integers and let $\beta$ be a non-negative integer. Then the following statements hold.

1. If $d \in G_{(a,b)}(\beta)$, then $c \in G_{(a,b)}(\beta)$ for all proper/prime divisors $c$ of $d$.

2. If $\beta \geq 2$, then $d \in G_{(a,b)}(\beta)$ if and only if $c \in G_{(a,b)}(\beta)$ for all proper/prime divisors $c$ of $d$.
2.3. Arbitrary $2^\beta$-Good Integers. Here, we focus on arbitrary $2^\beta$-good integers and derive the following results.

**Lemma 2.14.** Let $a$ and $b$ be coprime odd integers and let $\beta$ be a positive integer such that $2^\beta||(a+b)$. Then the following statements hold.
1) $G_{(a,b)}(\beta) \neq \emptyset$ and every element in $G_{(a,b)}(\beta)$ is odd.
2) $G_{(a,b)}(\beta + 1) = \emptyset$.

**Proof.** Clearly, $1 \in G_{(a,b)}(\beta)$. Next, suppose that $G_{(a,b)}(\beta)$ contains an even integer, denoted it by $2^i d$ for some positive integer $i$ and odd positive integer $d$. Then $d \in G_{(a,b)}(\beta + i)$. Since $\beta + i \geq 2$, we have $2^{\beta+i}|(a+b)$ by 3) of Proposition 2.9. This is a contradiction. Hence, 1) is proved.

To prove 2), suppose that $G_{(a,b)}(\beta + 1) \neq \emptyset$. Let $d \in G_{(a,b)}(\beta + 1)$ so $2^{\beta+1}d \in G_{(a,b)}$. Since $\beta + 1 \geq 2$, we have $2^{\beta+1}|(a+b)$ by 3) of Proposition 2.9. This is a contradiction. $\square$

The characterization of the set $G_{(a,b)}(\beta)$ is given in the next theorem.

**Theorem 2.15.** Let $a$ and $b$ be coprime odd integers and let $\beta \geq 0$ be an integer. Let $\gamma$ be a positive integer such that $2^\gamma||(a+b)$. Then the following statements hold.
1) If $\gamma < \beta$, then $G_{(a,b)}(\beta) = \emptyset$.
2) If $2 \leq \beta \leq \gamma$, then

\[ G_{(a,b)}(\gamma) = \{d \in \mathbb{N} \mid d = 1 \text{ or } d \text{ is odd such that } 2|\text{ord}_p\left(\frac{a}{b}\right) \text{ for every } p \text{ dividing } d\} \]

and for $2 \leq \beta < \gamma$

\[ G_{(a,b)}(\beta) = \bigcup_{i=0}^{\gamma-\beta} \{d2^i \mid d \in G_{(a,b)}(\gamma)\} \]

\[ = G_{(a,b)}(\beta+1) \cup \{d2^{\gamma-\beta} \mid d \in G_{(a,b)}(\gamma)\}. \]

3) If $\beta \in \{0,1\}$, then

\[ G_{(a,b)}(1) = \{2d \mid d \in G_{(a,b)}(2)\} \cup \{d \in \mathbb{N} \mid d = 1 \text{ or } d \text{ is odd and there exists } s \geq 1 \text{ such that } 2^s|\text{ord}_p\left(\frac{a}{b}\right) \text{ for every } p \text{ dividing } d\} \]

and

\[ G_{(a,b)}(0) = G_{(a,b)}(1) \cup \{2d \mid d \in G_{(a,b)}(1)\}. \]

**Proof.** From Lemma 2.14, 1) is clear.

To prove 2), assume that $2 \leq \beta \leq \gamma$. Since $2^\gamma||(a+b)$, we have $2^\beta|(a+b)$. By Lemma 2.14, $G_{(a,b)}(\gamma)$ is the set of odd $2^\gamma$-good integers and the result follows from Proposition 2.7. Let now $\beta < \gamma$. Let $c \in G_{(a,b)}(\beta)$. Let $c = 2^i d$ for some integer $i \geq 0$ and some odd integer $d$. It follows that $2^{\beta+i}d = 2^\beta c \in G_{(a,b)}(0)$. Hence, $2^{\beta+i} \in G_{(a,b)}(0)$ and $0 \leq i \leq \gamma - \beta$ because $2^\gamma||(a+b)$. Since $2 \leq \beta$, we have $d \in G_{(a,b)}(2)$ which implies that $d \in G_{(a,b)}(\gamma)$ by Proposition 2.7. Therefore, $c = 2^id \in \{d2^i \mid d \in G_{(a,b)}(\gamma)\}$ for some $0 \leq i \leq \gamma - \beta$. The reverse inclusion is clear. Therefore, $G_{(a,b)}(\beta) = \bigcup_{i=0}^{\gamma-\beta} \{d2^i \mid d \in G_{(a,b)}(\gamma)\}$. 

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Now
\[
G_{a,b}(\beta) = \bigcup_{i=0}^{\gamma-\beta-1} \{d2^i : d \in G_{a,b}(\gamma)\} \cup \{d2^{\gamma-\beta} : d \in G_{a,b}(\gamma)\}
\]

\[
= G_{a,b}(\beta+1) \cup \{d2^{\gamma-\beta} : d \in G_{a,b}(\gamma)\}.
\]

Next, we prove 3). If \( \gamma = 1 \), then \( G_{a,b}(2) = \emptyset \) by Lemma 2.14. The results follow from 1)-2) of Proposition 2.9. Assume that \( \gamma \geq 2 \).

**Case I:** \( \beta = 1 \). Let \( c \in G_{a,b}(1) \). If \( c \) is odd, then \( c \in G_{a,b}(0) = \{d \in \mathbb{N} | d = 1 \text{ or } d \text{ is odd and there exists } s \geq 1 \text{ such that } 2^s||\text{ord}_p(\frac{c}{2}) \text{ for every prime } p \text{ dividing } d\} \) by 1)-2) of Proposition 2.9. Suppose that \( c = 2d \) is even for some positive integer \( d \). Since \( 2d = c \in G_{a,b}(1) \), we have \( d \in G_{a,b}(2) \) and hence \( c = 2d \in \{2d : d \in G_{a,b}(2)\} \). The reverse inclusion is clear.

**Case II:** \( \beta = 0 \). Let \( c \in G_{a,b}(0) \). If \( c \) is odd, then \( c \in G_{a,b}(1) \) by 2) of Proposition 2.9. Suppose that \( c = 2d \) is even for some positive integer \( d \). If \( 2d = c \in G_{a,b}(0) \), we have \( d \in G_{a,b}(1) \) and hence \( c = 2d \in \{2d : d \in G_{a,b}(1)\} \). The reverse inclusion is clear. \( \square \)

**Corollary 2.16.** Let \( a \) and \( b \) be coprime odd integers and let \( \gamma \) be a positive integer such that \( 2^n || (a+b) \). Then
\[
G_{a,b} = G_{a,b}(0) \supseteq G_{a,b}(1) \supseteq G_{a,b}(2) \supseteq \cdots \supseteq G_{a,b}(\gamma) \supseteq G_{a,b}(\gamma+1) = \emptyset.
\]

**Proof.** From the definition, it is not difficult to see that \( G_{a,b}(i) \supseteq G_{a,b}(i+1) \) for all \( 0 \leq i \leq \gamma \). From Theorem 2.15, we have \( G_{a,b}(\gamma+1) = \emptyset \). Again, by Theorem 2.15, it can be seen that \( 2^{\gamma-i} \in G_{a,b}(i) \setminus G_{a,b}(i+1) \) for all \( 0 \leq i \leq \gamma - 1 \) and \( 1 \in G_{a,b}(\gamma) \setminus G_{a,b}(\gamma+1) \). \( \square \)

### 2.4. \( 2^\beta \)-Oddly-Good and \( 2^\beta \)-Evenly-Good Integers

In this subsection, we focus on families of \( 2^\beta \)-oddly-good and \( 2^\beta \)-evenly-good integers.

First, recall that a positive integer \( d \) is said to be \( 2^\beta \)-oddly-good (with respect to \( a \) and \( b \)) if \( 2^\beta d \in OG_{a,b} \), and \( 2^\beta \)-evenly-good (with respect to \( a \) and \( b \)) if \( 2^\beta d \in EG_{a,b} \). The useful characterization of \( OG_{a,b}(0) = OG_{a,b} \) and of \( EG_{a,b}(0) = EG_{a,b} \) from [8] are recalled as follows (These results are not affected by the errors of [8] discussed in Section 2.1).

**Proposition 2.17 ([8, Proposition 3.2]).** Let \( a \) and \( b \) be coprime non-zero integers and let \( d > 1 \) be an odd integer. Then \( d \in OG_{a,b}(0) = OG_{a,b} \) if and only if \( 2||\text{ord}_p(\frac{c}{2}) \) for every prime \( p \) dividing \( d \).

**Corollary 2.18 ([8, Corollary 3.2]).** Let \( a \) and \( b \) be coprime non-zero integers and let \( d \) be an odd positive integer. Then the following statements hold.

1) \( d \in OG_{a,b}(0) = OG_{a,b} \) if and only if \( 2d \in OG_{a,b}(0) \) if and only if \( d \in OG_{a,b}(1) \).

2) For each \( \beta \geq 2 \), \( d \in OG_{a,b}(\beta) \) if and only if \( d \in G_{a,b}(\beta) \).

**Proposition 2.19 ([8, Proposition 3.3]).** Let \( a \) and \( b \) be coprime nonzero integers and let \( d > 1 \) be an odd integer. Then \( d \in EG_{a,b}(0) = EG_{a,b} \) if and only if there exists \( s \geq 2 \) such that \( 2^s||\text{ord}_p(\frac{c}{2}) \) for every prime \( p \) dividing \( d \).
From the definitions, we have $G_{(a,b)}(\beta) = OG_{(a,b)}(\beta) \cup EG_{(a,b)}(\beta)$ for all non-negative integers $\beta$. In many cases, the following theorem shows that $EG_{(a,b)}(\beta) = \emptyset$.

**Theorem 2.20.** Let $a$ and $b$ be coprime odd integers and let $\beta$ be a non-negative integer. Then $EG_{(a,b)}(\beta) \neq \emptyset$ if and only if $\beta \in \{0,1\}$, i.e. $EG_{(a,b)}(\beta) = \emptyset$ if and only if $\beta \geq 2$.

**Proof.** Assume that $EG_{(a,b)}(\beta) \neq \emptyset$. Let $c \in EG_{(a,b)}(\beta)$. Then $2^\beta c \in EG_{(a,b)}$ which implies that $2^\beta \in EG_{(a,b)}$. It follows that $2^\beta \in G_{(a,b)}$ and $2^\beta|(a+b)$ by Proposition 2.3. Then $2^\beta \in OG_{(a,b)}$. By [8, Proposition 3.1], any positive integer greater than 2 can be either oddly-good or evenly-good but not both. Hence, $\beta \in \{0,1\}$ as desired.

For the converse, it is clear that $1 \in EG_{(a,b)}(0)$ and $1 \in EG_{(a,b)}(1)$.

From Theorem 2.20, we have that $OG_{(a,b)}(\beta) = G_{(a,b)}(\beta)$ for all $\beta \geq 2$ and they are determined in Theorem 2.15. Next we investigate $OG_{(a,b)}(\beta)$ and $EG_{(a,b)}(\beta)$ for $\beta \in \{0,1\}$.

**Theorem 2.21.** Let $a$ and $b$ be coprime odd integers an let $\gamma$ be a positive integer such that $2^0||(a+b)$. Then the following statements hold.

1) $OG_{(a,b)}(1) = \gamma^{-1} \bigcup \{d2^i \mid d \in OG_{(a,b)}(\gamma)\}$ for $\gamma \geq 2$, where $OG_{(a,b)}(\gamma) = \{d \in \mathbb{N} \mid d = 1$ or $d$ is odd such that $2|[\text{ord}_p(\frac{\gamma}{d})$ for every prime $p$ dividing $d\}$.

2) $EG_{(a,b)}(1) = \{d \in \mathbb{N} \mid d = 1$ or $d$ is odd and there exists $s \geq 2$ such that $2^s|[\text{ord}_p(\frac{\gamma}{d})$ for every prime $p$ dividing $d\}$.

3) $OG_{(a,b)}(0) = OG_{(a,b)}(1) \cup \{2d \mid d \in OG_{(a,b)}(1)\}$.

4) $EG_{(a,b)}(0) = EG_{(a,b)}(1) \cup \{2d \mid d \in EG_{(a,b)}(1)\}$.

**Proof.** If $\gamma = 1$, every element in $OG_{(a,b)}(1)$ is odd by Lemma 2.14 as $OG_{(a,b)}(1) \subseteq G_{(a,b)}(1)$. Hence, the characterization of $OG_{(a,b)}(1)$ follows from Corollary 2.18 and Proposition 2.17. Next, assume that $\gamma \geq 2$. Then we have $OG_{(a,b)}(\gamma) = G_{(a,b)}(\gamma)$ by Theorem 2.15. Let $c \in OG_{(a,b)}(1)$. Let $c = 2^d$ for some integer $i \geq 0$ and some odd integer $d$. It follows that $2^{i+1}d = 2c \in OG_{(a,b)}(0)$. Hence, $2^{i+1} \in OG_{(a,b)}(0)$ and $0 \leq i \leq \gamma - 1$ because $2^0||(a+b)$. Subsequently, we have $d \in OG_{(a,b)}(0)$. Since $d$ is odd, it can be concluded that $d \in \{d \in \mathbb{N} \mid d = 1$ or $d$ is odd such that $2|[\text{ord}_p(\frac{\gamma}{d})$ for every prime $p$ dividing $d\}$ = $OG_{(a,b)}(\gamma)$ by Proposition 2.17. Therefore, $c = 2d \in \{d2^i \mid d \in OG_{(a,b)}(\gamma)\}$ for some $0 \leq i \leq \gamma - 1$. The reverse inclusion is clear. Hence, the proof of 1) is completed.

Every element in $EG_{(a,b)}(1)$ is odd. For if $x = 2y \in EG_{(a,b)}(1)$, then $2^2y \in EG_{(a,b)}(0)$ and $y \in EG_{(a,b)}(2)$. But from Theorem 2.20, $EG_{(a,b)}(2) = \emptyset$. Now 2) follows immediately from Proposition 2.19.

Using arguments similar to those in the proof of 3) in Theorem 2.15, 3) and 4) follow.

From Corollary 2.16 and Theorem 2.20, the results can be summarized in the following diagram.
a subspace of the finite field of defined to be 3.1. Negacyclic codes.

Negacyclic codes in Subsection 3.2 (resp., Subsection 3.3). 3.1, brief history and basic properties of negacyclic codes are recalled. It is followed in the characterization and enumeration of self-dual negacyclic codes. In Subsection v, brief history and basic properties of negacyclic codes are recalled. It is followed in the characterization and enumeration of self-dual negacyclic codes. In Subsection v.

Unlike the cyclic case in [7], self-dual negacyclic code of length n exists only if \( l \mid n \) and \( l \) is an odd prime different from \( p \). The characterization for the existence of a Euclidean self-dual negacyclic code of length \( n \) has been given in [3]. In [5], algebraic structure of repeated root Euclidean self-dual negacyclic codes of length \( n = 2p^r \) over \( \mathbb{F}_p \) has been studied. In [1] and [2], all simple root self-dual negacyclic codes of lengths \( 2^r \) have been determined in terms of their generator polynomials, where \( q \) is an odd prime different from \( p \). In the said papers, the enumeration of such self-dual negacyclic codes has been given as well. All Euclidean self-orthogonal negacyclic codes of length \( q^t \) and \( 2q^t \) over \( \mathbb{F}_p \) have been determined in [2] and [16].

Due to their rich algebraic structures and wide applications, negacyclic codes with self-duality have been of interest and extensively studied (see [1], [2], [3], [5], [6], and [12]). Unlike the cyclic case in [7], self-dual negacyclic code of length \( n \) over \( \mathbb{F}_{p^l} \) exists only if \( p \) is odd. The characterization for the existence of a Euclidean self-dual negacyclic code over \( \mathbb{F}_p \) has been given in [3]. In [5], algebraic structure of repeated root Euclidean self-dual negacyclic codes of length \( n = 2p^r \) over \( \mathbb{F}_p \) has been studied. In [1] and [2], all simple root self-dual negacyclic codes of lengths \( 2^r \) have been determined in terms of their generator polynomials, where \( q \) is an odd prime different from \( p \). In the said papers, the enumeration of such self-dual negacyclic codes has been given as well. All Euclidean self-orthogonal negacyclic codes of length \( q^t \) and \( 2q^t \) over \( \mathbb{F}_p \) have been determined in [2] and [16].

Euclidean self-dual negacyclic codes of oddly-even length have been studied in [6].

3. SELF-DUAL NEGACYCLIC CODES

In this section, we focus on applications of 2\(^\beta\)-good and 2\(^\beta\)-oddly-good integers in the characterization and enumeration of self-dual negacyclic codes. In Subsection 3.1, brief history and basic properties of negacyclic codes are recalled. It is followed in the characterization and enumeration of Euclidean (resp., Hermitian) self-dual negacyclic codes in Subsection 3.2 (resp., Subsection 3.3).

3.1. NEGACYCLIC CODES. For a prime \( p \) and a positive integer \( l \), denote by \( \mathbb{F}_{p^l} \) the finite field of \( p^l \) elements. A linear code \( C \) of length \( n \) over \( \mathbb{F}_{p^l} \) is defined to be a subspace of the \( \mathbb{F}_{p^l} \)-vector space \( \mathbb{F}_{p^l}^n \). The Euclidean dual of a linear code \( C \) is defined to be

\[
C^\perp_E = \{ \mathbf{v} \in \mathbb{F}_{p^l}^n \mid \langle \mathbf{v}, \mathbf{c} \rangle_E = 0 \text{ for all } \mathbf{c} \in C \},
\]

where \( \langle \mathbf{v}, \mathbf{u} \rangle_E := \sum_{i=1}^{n} v_i u_i \) is the Euclidean inner product between \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) and \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) in \( \mathbb{F}_{p^l}^n \). Over \( \mathbb{F}_{p^{2l}} \), the Hermitian dual of a linear code \( C \) can be defined as well and it is defined to be

\[
C^\perp_H = \{ \mathbf{v} \in \mathbb{F}_{p^{2l}}^n \mid \langle \mathbf{v}, \mathbf{c} \rangle_H = 0 \text{ for all } \mathbf{c} \in C \},
\]

where \( \langle \mathbf{v}, \mathbf{u} \rangle_H := \sum_{i=1}^{n} v_i u_i^* \) is the Hermitian inner product between \( \mathbf{v} \) and \( \mathbf{u} \) in \( \mathbb{F}_{p^{2l}}^n \). A code \( C \) is said to be Euclidean self-dual (resp. Hermitian self-dual) if \( C = C^\perp_E \) (resp., \( C = C^\perp_H \)).

A linear code \( C \) is said to be negacyclic if it is invariant under the right negacyclic shift. Precisely, a linear code \( C \) is negacyclic if and only if

\[
(-c_{n-1}, c_0, c_1, \ldots, c_{n-2}) \in C \quad \text{whenever} \quad (c_0, c_1, c_2, \ldots, c_{n-1}) \in C.
\]

Due to their rich algebraic structures and wide applications, negacyclic codes with self-duality have been of interest and extensively studied (see [1], [2], [3], [5], [6], and [12]). Unlike the cyclic case in [7], self-dual negacyclic code of length \( n \) over \( \mathbb{F}_{p^l} \) exists only if \( p \) is odd. The characterization for the existence of a Euclidean self-dual negacyclic code over \( \mathbb{F}_p \) has been given in [3]. In [5], algebraic structure of repeated root Euclidean self-dual negacyclic codes of length \( n = 2p^r \) over \( \mathbb{F}_p \) has been studied. In [1] and [2], all simple root self-dual negacyclic codes of lengths \( 2^r \) have been determined in terms of their generator polynomials, where \( q \) is an odd prime different from \( p \). In the said papers, the enumeration of such self-dual negacyclic codes has been given as well. All Euclidean self-orthogonal negacyclic codes of length \( q^r \) and \( 2q^r \) over \( \mathbb{F}_p \) have been determined in [2] and [16].

Euclidean self-dual negacyclic codes of oddly-even length have been studied in [6].
In [12], construction and enumeration for Euclidean self-dual negacyclic code of length $2^vp^r$ have been provided together with a general concept for enumeration. However, there are no explicit formulas.

It is well known that every negacyclic code $C$ of length $n$ over $\mathbb{F}_{p^l}$ can be viewed as an (isomorphic) ideal in the principal ideal ring $\mathbb{F}_{p^l}[x]/(x^n+1)$ uniquely generated by a monic divisor $g(x)$ of $x^n+1$. Such polynomial is called the generator polynomial of $C$. For a monic polynomial $f(x) = \sum_{i=0}^{k} f_ix^i$ of degree $k$ in $\mathbb{F}_{p^l}[x]$ with $f_0 \neq 0$, the reciprocal polynomial of $f(x)$ is defined to be $f^*(x) := f_0^{-1}x^k \sum_{i=0}^{k} f_i(1/x)^i$. In $\mathbb{F}_{p^l}[x]$, the conjugate reciprocal polynomial of $f(x)$ is defined to be $f^l(x) := f_0^{-1}x^k \sum_{i=0}^{k} f_i^l(1/x)^i$. A polynomial $f(x)$ is called self-reciprocal (resp. self-conjugate-reciprocal) if $f(x) = f^*(x)$ (resp., $f(x) = f^l(x)$). Otherwise, $f(x)$ and $f^*(x)$ form a reciprocal polynomial pair (resp, $f(x)$ and $f^l(x)$ form a conjugate-reciprocal polynomial pair). In [4, Proposition 2.4] and [18, Proposition 2.3], it has been shown that the Euclidean and Hermitian duals of a negacyclic code $C$ over finite fields are again negacyclic codes. Moreover, if $C$ is a negacyclic code with the generator polynomial $g(x)$, then it is Euclidean self-dual (resp., Hermitian self-dual) if and only if $g(x) = h^*(x)$ (resp., $g(x) = h^l(x)$), where $h(x) = x^{\frac{n+1}{2}}$.

Consider an odd prime $p$ and $n = 2^vp^n$, where $\nu \geq 0$ and $r \geq 0$ are integers and $n'$ is an odd positive integer such that $p \mid n'$. We have

$$x^n + 1 = (x^{2^\nu n'} + 1)^{p^r} = \left(\frac{x^{2^{\nu+1}n'} - 1}{x^{2^n} - 1}\right)^{p^r} = \left(\prod_{d|2^{\nu+1}n'} Q_d(x)\right)^{p^r} = \left(\prod_{d|2^n} Q_{d^{2^\nu+1}}(x)\right)^{p^r},$$

where $Q_{d^{2^\nu+1}}(x) := \prod_{1 \leq i \leq d^{2^\nu+1}, \gcd(i,d^{2^\nu+1}) = 1} (x - \omega^i)$ is the $d^{2^\nu+1}$th cyclotomic polynomial and $\omega$ is a primitive $d^{2^\nu+1}$th root of unity.

### 3.2. Euclidean self-dual negacyclic codes

In this subsection, we focus on the characterization and enumeration of Euclidean self-dual negacyclic codes over $\mathbb{F}_{p^l}$. General properties of Euclidean self-dual negacyclic codes of any lengths are given in 3.2.1. Euclidean self-dual negacyclic codes of some specific lengths are discussed in 3.2.2.

#### 3.2.1. Euclidean Self-Dual Negacyclic Codes

From [13, Theorem 2.47], it has been shown that the $d^{2^\nu+1}$th cyclotomic polynomial $Q_{d^{2^\nu+1}}(x)$ in (3.1) can be factorized into a product of $\phi(d^{2^\nu+1})/\phi(d^{2^\nu+1}|p^r)$ distinct monic irreducible polynomials of the same degree in $\mathbb{F}_{p^l}[x]$, where $\phi$ is the Euler’s totient function.

The results in [7, Lemma 1] over a finite field of characteristic 2 can be straightforwardly generalized to the case of finite fields of odd characteristic as follows.

**Lemma 3.1.** Let $\nu \geq 0$ be an integer and let $d$ be an integer given in (3.1). Then the following statements hold.

1. $d \in G_{(p^l,1)}(\nu + 1)$ if and only if every irreducible factor of $Q_{d^{2^\nu+1}}(x)$ is self-reciprocal.

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2) If \( d \not\in G(p^r,1)(\nu+1) \) if and only if the irreducible factors of \( Q_{d2^{\nu+1}}(x) \) form reciprocal polynomial pairs.

Applying Lemma 3.1 and Equation (3.1) (cf. [17, Equation (29)]), it can be concluded that

\[
x^n + 1 = \prod_{d \mid n'} Q_{d2^{\nu+1}}(x) \prod_{d \not\mid n'} Q_{d2^{\nu+1}}(x) \]

(3.2)

where \( f_{d_i}(x) \) is a self-reciprocal irreducible polynomial for all \( d \) and \( i \), \( g_{d_j}(x) \) and \( g^*_j(x) \) are a reciprocal irreducible polynomial pair for all \( d \) and \( j \), \( \rho(d2^{\nu+1}, p^r) = \frac{\phi(d2^{\nu+1})}{\text{ord}_{d2^{\nu+1}}(p^r)} \) and \( \sigma(d2^{\nu+1}, p^r) = \frac{\phi(d2^{\nu+1})}{2\text{ord}_{d2^{\nu+1}}(p^r)} \).

The existence of a Euclidean self-dual negacyclic code of length \( n \) over \( \mathbb{F}_{p^r} \) can be determined using (3.2) as follows.

**Proposition 3.2.** Let \( p \) be an odd prime and let \( n = 2^r p^n \), where \( \nu \geq 0 \) and \( r \geq 0 \) are integers and \( n' \) is an odd positive integer such that \( p \not\mid n' \). Let \( l \) be a positive integer. Then there exists a Euclidean self-dual negacyclic code of length \( n \) over \( \mathbb{F}_{p^r} \) if and only if \( \nu > 0 \) and \( d \not\in G(p^r,1)(\nu+1) \) for all \( d \mid n' \).

**Proof.** Assume that there exists a Euclidean self-dual negacyclic code \( C \) of length \( n \) over \( \mathbb{F}_{p^r} \). Then \( n = 2^r p^n n' \) must be even which implies that \( \nu > 0 \). Let \( g(x) \) be the generator polynomial for \( C \) and let \( h(x) := \frac{x^{n+1}}{x^r} \). Since \( C \) is Euclidean self-dual, we have \( g(x) = h^*(x) \). Suppose that there exists a positive integer \( d \) such that \( d \mid n' \) and \( d \in G(p^r,1)(\nu+1) \). Then \( f_{d_i}(x) \) has the same multiplicity \( m \) in \( g(x) \) and in \( g^*(x) = h(x) \). It follows that the multiplicity of \( f_{d_i}(x) \) in \( x^{n+1} \) is \( 2m = p^r \), a contradiction. Hence, \( d \not\in G(p^r,1)(\nu+1) \) for all divisors \( d \) of \( n' \).

Conversely, assume that \( \nu > 0 \) and \( d \not\in G(p^r,1)(\nu+1) \) for all \( d \mid n' \). From (3.2), we have

\[
x^n + 1 = \prod_{d \mid n'} \prod_{j=1}^{\sigma(d2^{\nu+1}, p^r)} \left( g_{d_j}(x) g^*_j(x) \right)^{p^r}.
\]

It is not difficult to see that the negacyclic code generated by

\[
g(x) = \prod_{d \mid n'} \prod_{j=1}^{\sigma(d2^{\nu+1}, p^r)} \left( g_{d_j}(x) \right)^{p^r}
\]

is Euclidean self-dual. \( \square \)

The result of Blackford [3, Theorem 3] can be obtained as a corollary to Proposition 3.2 as follows.

**Corollary 3.3.** Let \( p \) be an odd prime and let \( n = 2^r p^n n' \), where \( \nu > 0 \) and \( r \geq 0 \) are integers and \( n' \) is an odd positive integer such that \( p \not\mid n' \). Let \( l \) be a positive
integer. Then there exists a Euclidean self-dual negacyclic code of length \( n \) over \( \mathbb{F}_{p^\nu} \) if and only if \( 1 \notin G(p^\nu,1)(\nu+1) \) i.e. if and only if \( 2^{\nu+1} \nmid (p^\nu + 1) \).

**Proof.** Suppose there exists a Euclidean self-dual negacyclic code of length \( n \) over \( \mathbb{F}_{p^\nu} \). Then, by Proposition 3.2, we have that \( d \notin G(p^\nu,1)(\nu+1) \) for all \( d \mid n' \), in particular \( 1 \notin G(p^\nu,1)(\nu+1) \). Conversely if \( 1 \notin G(p^\nu,1)(\nu+1) \), then \( d \notin G(p^\nu,1)(\nu+1) \) for all odd numbers \( d \), in particular for all divisors \( d \) of \( n' \), by Corollary 2.10. The result therefore follows from Proposition 3.2.

From Corollary 2.10, the last two statements are equivalent.

**Corollary 3.4.** Let \( p \) be an odd prime and let \( \nu > 0 \) and \( r \geq 0 \) be integers. Let \( n' \) be an odd positive integer such that \( p \nmid n' \). Let \( l \) be a positive integer. If there exists a Euclidean self-dual negacyclic code of length \( 2^{\nu} p^r n' \) over \( \mathbb{F}_{p^\nu} \), then there exist a Euclidean self-dual negacyclic code of length \( 2^{\nu} p^r n' \) over \( \mathbb{F}_{p^\nu} \) for all positive integers \( \mu \geq \nu \).

Using the above results, we can prove two corollaries which are due to [12, Theorem 3.3].

**Corollary 3.5.** Let \( p \) be an odd prime and \( l \) be a positive integer. Assume that \( 2^\nu \mid (p^\nu - 1) \). Then there exists a Euclidean self-dual negacyclic code of length \( 2^{\nu} p^r \) over \( \mathbb{F}_{p^\nu} \) if and only if \( \nu \geq r \).

**Proof.** As \( 2^\nu \mid (p^\nu - 1) \), let \( p^\nu - 1 = 2^\gamma c \) for some odd integer \( c \) and integer \( \gamma \geq 2 \). Then \( p^\nu + 1 = 2^\gamma c + 2 = 2(2^{\gamma-1} c + 1) \), where \( 2^{\gamma-1} c + 1 \) is odd. We see that \( 2^{\nu+1} \nmid (p^\nu + 1) \) if and only if \( \nu \geq r \). Then, the result follows from Corollary 3.3.

**Corollary 3.6.** Let \( p \) be an odd prime and \( l \) be a positive integer. Let \( \gamma \) be a positive integer such that \( \gamma \geq 2 \) and \( 2^\gamma \mid (p^\nu + 1) \). Then there exists a Euclidean self-dual negacyclic code of length \( 2^{\nu} p^r \) over \( \mathbb{F}_{p^\nu} \) if and only if \( \nu \geq \gamma \).

**Proof.** Since \( 2^\gamma \mid (p^\nu + 1) \), we see that \( 2^{\nu+1} \nmid (p^\nu + 1) \) if and only if \( \nu \geq \gamma \). The desired result therefore follows by Corollary 3.3.

The explicit number of Euclidean self-dual negacyclic codes of specific lengths \( n = 2^\nu, \nu \geq 1 \) and \( n = 2^\nu q^{\nu}, \nu = 1, n' = q, q \) an odd prime) was obtained in [1, Theorems 3.4], [2, Theorems 3.4] respectively. The idea for a general formula for the number of Euclidean self-dual negacyclic codes of length \( n \) over \( \mathbb{F}_{p^\nu} \) was given in [12, Corollary 2.6]. However, there are no explicit formulas for general \( n \). Using \( 2^\beta \)-good integers discussed in Section 2, a general formula for the number of Euclidean self-dual negacyclic codes of length \( n \) over \( \mathbb{F}_{p^\nu} \) can be deduced.

**Theorem 3.7.** Let \( p \) be an odd prime and let \( n = 2^\nu p^r n' \), where \( \nu > 0 \) and \( r \geq 0 \) are integers and \( n' \) is an odd positive integer such that \( p \nmid n' \). Let \( l \) be a positive integer. The number of Euclidean self-dual negacyclic codes of length \( n \) over \( \mathbb{F}_{p^\nu} \) is

\[
NE(p^\nu, n) := \begin{cases} 
(p^\nu + 1) \frac{1}{2} \sum_{d \mid n'} \frac{g(d^{2^{\nu+1}})}{\phi(d^{2^{\nu+1}})} & \text{if } 2^{\nu+1} \nmid (p^\nu + 1), \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** If \( 2^{\nu+1} \nmid (p^\nu + 1) \), there are no Euclidean self-dual negacyclic codes of length \( n \) over \( \mathbb{F}_{p^\nu} \) by Corollary 3.3. Hence, \( NE(p^\nu, n) = 0 \).
Assume that $2^{\nu+1} \nmid (p^l + 1)$. From (3.2), we have

$$x^n + 1 = \prod_{d|n'} \prod_{j=1}^{\sigma(d2^{\nu+1},p^l)} (g_{d_j}(x)g_{d_j}(x)^{b_{d_j}})^{r^j}.$$ 

Let

$$g(x) = \prod_{d|n'} \prod_{j=1}^{\sigma(d2^{\nu+1},p^l)} g_{d_j}(x)^{a_{d_j}r^j}g_{d_j}^*(x)^{b_{d_j}}$$

be the generator polynomial of a Euclidean self-dual cyclic code of length $n$ over $\mathbb{F}_{p^t}$, where $0 \leq a_{d_j} \leq p^r$ and $0 \leq b_{d_j} \leq p^r$. Then

$$g(x) = h^*(x) = \prod_{d|n'} \prod_{j=1}^{\sigma(d2^{\nu+1},p^l)} g_{d_j}(x)^{r^j-b_{d_j}}g_{d_j}^*(x)^{r^j-a_{d_j}}.$$ 

This implies that $a_{d_j} + b_{d_j} = p^r$, and hence, the number of choices for $(a_{d_j}, b_{d_j})$ is $p^r + 1$ for all $d \mid n'$ and $1 \leq j \leq \sigma(d2^{\nu+1},p^l)$. Therefore, the formula is proved. \(\square\)

From the above theorem, the remaining difficult part is to compute

$$t(n'2^{\nu+1},p^l) := \frac{1}{2} \sum_{d|n'} \frac{\phi(d2^{\nu+1})}{\text{ord}_{d2^{\nu+1}}(p^l)}$$

which is independent of a factor $p^r$ of $n$. Some results on a specific $n'$ are given in 3.2.2.

3.2.2. Euclidean Self-Dual Negacyclic Codes of Lengths $2^\nu$ and $2^\nu p^r$. In this part, we give explicit formulae of Euclidean self-dual negacyclic codes of lengths $2^\nu$ and $2^\nu p^r$ over $\mathbb{F}_{p^t}$, where $\nu$ is a positive integer. First, we compute $\text{ord}_{2^{\nu+1}}(p^l)$ which is a key to determine $t(2^{\nu+1},p^l)$.

First, useful number theoretical results are recalled.

**Theorem 3.8** ([15, Theorem 3.10]). If $k \geq 3$, then 5 has order $2^{k-2}$ modulo $2^k$. If $a \equiv 1 \pmod{4}$ then there exists a unique integer $i \in \{0, 1, \ldots, 2^{k-2}-1\}$ such that $a \equiv 5^i \pmod{2^k}$. If $a \equiv 3 \pmod{4}$ then there exists a unique integer $i \in \{0, 1, \ldots, 2^{k-2}-1\}$ such that $a \equiv -5^i \pmod{2^k}$.

For convenience, for each $\nu \geq 1$, let $\alpha_p$ denote the unique integer in the set $\{0, 1, \ldots, 2^{\nu-1}-1\}$ such that

$$p \equiv 5^{\alpha_p} \pmod{2^{\nu+1}} \quad \text{if } p \equiv 1 \pmod{4},$$

$$p \equiv -5^{\alpha_p} \pmod{2^{\nu+1}} \quad \text{if } p \equiv 3 \pmod{4}.$$ 

Note that $\alpha_p = 0$ if $\nu = 1$ and the existence of $\alpha_p$ is guaranteed by Theorem 3.8 for all $\nu \geq 2$.

**Lemma 3.9.** Let $\nu$ and $l$ be positive integers and let $p$ be an odd prime. Then

$$\text{ord}_{2^{\nu+1}}(p^l) = \begin{cases} 
1 & \text{if } \nu = 1 \text{ and } p^l \equiv 1 \pmod{4}, \\
2 & \text{if } \nu = 1 \text{ and } p^l \equiv 3 \pmod{4}, \\
\frac{2^{\nu-1}}{\gcd(2^{\nu-1}, \alpha_p l)} & \text{if } \nu \geq 2.
\end{cases}$$
Lemma 3.12. Let polynomials over a finite field of odd characteristic as follows.

The result in [10, Lemma 3.5] over a finite field of characteristic 2 can be straightforwardly generalized to the case of Hermitian self-dual negacyclic codes.

3.3. General properties of Hermitian self-dual negacyclic codes of any lengths are discussed in 3.3.2. Otherwise, the characterization and enumeration of Hermitian self-dual negacyclic codes over \( F_{2^\nu} \) are given in 3.3.1. Hermitian self-dual negacyclic codes of some specific lengths are discussed in 3.3.2.

3.3.1. Hermitian self-dual negacyclic codes. The result in [10, Lemma 3.5] over a finite field of characteristic 2 can be straightforwardly generalized to the case of polynomials over a finite field of odd characteristic as follows.

Lemma 3.12. Let \( \nu \geq 0 \) be an integer and let \( d \) be an integer given in (3.1). Then the following statements hold.

1) \( d \in \text{OG}(p^{\nu + 1}) \) if and only if every irreducible factor of \( Q_{d\nu}^{\nu + 1}(x) \) is self-conjugate-reciprocal.

2) \( d \notin \text{OG}(p^{\nu + 1}) \) if and only if the irreducible factors of \( Q_{d\nu}^{\nu + 1}(x) \) form conjugate-reciprocal polynomial pairs.
Applying Lemma 3.12 and Equation (3.1) (cf. [17, Equation (29)]), it can be concluded that

\[ x^n + 1 = \left( \prod_{d \in OG(p^r, 1)} \mu(d^{n+1}p^{2^i}) \prod_{j=1}^\eta(d^{n+1}p^{2i}) \prod_{d \notin OG(p^r, 1)} g_{d_j}(x) \right)^{p^r}, \]

(3.6)

where \( f_{d_i}(x) \) is a self-conjugate-reciprocal irreducible polynomial for all \( d \) and \( i \), \( g_{d_j}(x) \) and \( g^\dagger_{d_j}(x) \) are a conjugate-reciprocal irreducible polynomial pair for all \( d \) and \( j \), \( \mu(d^{n+1}p^{2i}) = \frac{\phi(d^{n+1}p^{2i})}{\ord_{d^{n+1}p^{2i}}(p+1)} \) and \( \eta(d^{n+1}p^{2i}) = \frac{\phi_2(d^{n+1}p^{2i})}{\ord_{d^{n+1}p^{2i}}(p+1)} \).

The existence of a Hermitian self-dual negacyclic code of length \( n \) over \( \mathbb{F}_{p^2} \) can be determined using (3.6) as follows.

**Proposition 3.13.** Let \( p \) be an odd prime and let \( n = 2^\nu p^rn' \), where \( \nu \geq 0 \) and \( r \geq 0 \) are integers and \( n' \) is an odd positive integer such that \( p \nmid n' \). Let \( l \) be a positive integer. There exists a Hermitian self-dual negacyclic code of length \( n \) over \( \mathbb{F}_{p^2} \) if and only if \( \nu > 0 \) and \( d \notin OG(p^r, 1)(\nu + 1) \) for all \( d|n' \).

**Proof.** Assume that there exists a Hermitian self-dual negacyclic code \( C \) of length \( n \) over \( \mathbb{F}_{p^2} \). Then \( n = 2^{\nu}p^rn' \) must be even which implies that \( \nu > 0 \). Let \( g(x) \) be the generator polynomial for \( C \) and let \( h(x) := \frac{x^{n+1}}{p(x)} \). Since \( C \) is Hermitian self-dual, we have \( g(x) = h^\dagger(x) \). Suppose that there exists a positive integer \( d \) such that \( d|n' \) and \( d \notin OG(p^r, 1)(\nu + 1) \). Then \( f_{d_1}(x) \) has the same multiplicity \( m \) in \( g(x) \) and in \( g^\dagger(x) = h(x) \). It follows that the multiplicity of \( f_{d_1}(x) \) in \( x^n + 1 \) is \( 2m = p^r \), a contradiction.

Conversely, assume that \( \nu > 0 \) and \( d \notin OG(p^r, 1)(\nu + 1) \) for all \( d|n' \). From (3.6), we have

\[ x^n + 1 = \prod_{d|n'} \prod_{j=1}^{\eta(d^{n+1}p^{2i})} \left( g_{d_j}(x) g^\dagger_{d_j}(x) \right)^{p^r}. \]

It is not difficult to see that the negacyclic code of length \( n \) generated by

\[ g(x) = \prod_{d|n'} \prod_{j=1}^{\eta(d^{n+1}p^{2i})} (g_{d_j}(x))^{p^r} \]

is Hermitian self-dual.

\[ \square \]

For \( \nu > 0 \) and an odd positive integer \( d \), we have \( d \notin OG(p^{r+1}, 1)(\nu + 1) \) if and only if \( d \notin G(p^{r+1}, 1)(\nu + 1) \) by Corollary 2.18. Then the conditions in Proposition 3.13 can be simplified using the above discussion and Corollary 3.3 as follows.

**Corollary 3.14.** Let \( p \) be an odd prime and let \( n = 2^\nu p^rn' \), where \( \nu > 0 \) and \( r \geq 0 \) are integers and \( n' \) is an odd positive integer such that \( p \nmid n' \). Let \( l \) be a positive
Let Corollary 3.17. The result can be deduced.

**Proof.** By replacing (3.7) for some specific \(\nu > 0\) and \(r \geq 0\) are integers and \(\nu\) is an odd positive integer such that \(p \nmid \nu\). Let \(l\) be a positive integer. The number of Hermitian self-dual negacyclic codes of length \(n\) over \(\mathbb{F}_{p^{2l}}\) is

\[
NH(p^{2l}, n) := \begin{cases} 
(p^r + 1)^{\frac{1}{2}} \sum_{d|n'} \phi(d2^{\nu+1}) & \text{if } 2^{\nu+1} \nmid (p^l + 1), \\
0 & \text{otherwise.}
\end{cases}
\]

**Proof.** By replacing \(G_{(p', 1)}(\nu + 1)\) with \(OG_{(p', 1)}(\nu + 1)\) in the proof of Theorem 3.7, the result can be deduced. \(\square\)

Next, we focus on the determination of

\[
\tau(n'2^{\nu+1}, p^l) := \frac{1}{2} \sum_{d|n'} \frac{\phi(d2^{\nu+1})}{\operatorname{ord}_{d2^{\nu+1}}(p^l)}
\]

for some specific \(n'\). Note that \(\tau(n'2^{\nu+1}, p^l)\) is independent of the factor \(p^r\) of \(n\).

3.3.2. **Hermitian Self-Dual Negacyclic Codes of Lengths \(2^{\nu}\) and \(2^r p^r\).** From Proposition 3.13, we assume that \(\nu > 0\) is an integer and conclude the following results.

**Proposition 3.16.** Let \(p\) be an odd prime and let \(\nu\) and \(l\) be positive integers. If \(2^{\nu+1} \nmid (p^l + 1)\) then

\[
\tau(2^{\nu+1}, p^l) = \gcd(2^{\nu-1}, 2\alpha p^l).
\]

In particular, if \(p = 5\), then

\[
\tau(2^{\nu+1}, 5^l) = \gcd(2^{\nu-1}, 2l).
\]

**Proof.** The result follows from the fact that \(\operatorname{ord}_{2^{\nu+1}}(p^l) = \gcd(2^{\nu-1}, 2\alpha p^l)\). \(\square\)

Combining Theorem 3.15 and Proposition 3.16, the number of Hermitian self-dual negacyclic codes of lengths \(2^{\nu}\) and \(2^r p^r\) over \(\mathbb{F}_{p^{2l}}\) is determined as follow.

**Corollary 3.17.** Let \(\nu\) and \(l\) be positive integers and let \(p\) be an odd prime. Let \(r \geq 0\) be an integer. If \(2^{\nu+1} \nmid (p^l + 1)\), then

\[
NH(p^{2l}, 2^r p^r) = (p^r + 1)^{\frac{1}{2}} \sum_{d|n'} \phi(d2^{\nu+1}) = (p^r + 1)^{\gcd(2^{\nu-1}, 2\alpha p^l)}.
\]

Otherwise, \(NH(p^{2l}, 2^r p^r) = 0\).

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