Generalized Ponce’s inequality

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Abstract
We provide a generalization of a remarkable inequality by A. C. Ponce whose consequences are essential in several fields, such as a characterization of Sobolev spaces or nonlocal modelization.

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1 Definitions and preliminaries
Let \( \Omega \) be an open bounded set in \( \mathbb{R}^N \). We define the family of kernels \( (k_\delta)_{\delta>0} \) as a set of radial positive functions fulfilling the following properties:

1. \[
\frac{1}{C_N} \int_{B(0,\delta)} k_\delta (|s|) \, ds = 1,
\]

where

\[
C_N = \frac{1}{\text{meas}(S^{N-1})} \int_{S^{N-1}} |\sigma \cdot e|^p \, dH^{N-1}(\sigma),
\]

\( H^{N-1} \) stands for the \((N-1)\)-dimensional Hausdorff measure on the unit sphere \( S^{N-1} \), \( e \) is any unit vector in \( \mathbb{R}^N \), \( p > 1 \), and \( B(x, \delta) \) is the ball with center \( x \) and radius \( \delta \).

2. \( \text{supp} \, k_\delta \subset B(0,\delta) \).

We define the nonlocal operator \( B_h \) in \( L^p(\Omega) \times L^p(\Omega) \) by

\[
B_h(u, u) = \int_{\Omega} \int_{\Omega} H(x', x) \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u(x') - u(x)|^p \, dx' \, dx,
\]

where \( H(x', x) = \frac{h(x') + h(x)}{2}, \, h \in \mathcal{H} \),

\[ \mathcal{H} = \left\{ h : \Omega \rightarrow \mathbb{R} \mid h(x) \in [h_{\min}, h_{\max}] \text{ a.e. } x \in \Omega, h = 0 \text{ in } \mathbb{R}^N \setminus \Omega \right\}, \]

and \( 0 < h_{\min} < h_{\max} \) are given constants.
For $h = 1$, the following compactness result is well known (see, e.g., [4] and [9, proof of Theorem 1.2, p. 12]).

**Theorem 1** Let $(u_\delta)_\delta$ be a sequence uniformly bounded in $L^p(\Omega)$, and let $C$ be a positive constant such that

$$
\int_\Omega \int_\Omega \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \leq C
$$

for any $\delta$. Then from $(u_\delta)_\delta$ we can extract a subsequence, still denoted by $(u_\delta)_\delta$, and we can find $u \in W^{1,p}(\Omega)$ such that $u_\delta \rightharpoonup u$ strongly in $L^p(\Omega)$ as $\delta \to 0$ and

$$
\lim_{\delta \to 0} \int_\Omega \int_\Omega \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \geq \int_\Omega |\nabla u(x)|^p \, dx.
$$

Even though several authors are involved in the proof, we refer to estimate (1.2) as Ponce’s inequality.

**1.1 The objective**

Our goal is to prove the following extension of (1.2):

$$
\lim_{\delta \to 0} \int \int H(x', x) \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \geq \int \int |\nabla u(x)|^p \, dx,
$$

where $\Omega$ is an open bounded set, $H(x', x) = \frac{k_\delta(|x' - x|)}{|x' - x|^p} + h$, and $h \in H$.

As we will see, inequality (1.3) is equivalent to (1.2) for measurable sets, that is,

$$
\lim_{\delta \to 0} \int \int_{E \times E} \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \geq \int \int_{E} |\nabla u(x)|^p \, dx
$$

for all measurable sets $E$ in $\Omega$.

**1.2 Motivation and organization of the paper**

The context in which we locate the present paper is the study of the nonlocal $p$-Laplacian problem. Before proceeding, we make precise some notation. We define the spaces

$$L^p_0(\Omega) = \{ u \in L^p(\Omega) : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \}$$

and

$$X = \{ u \in L^p_0(\Omega) : \mathcal{B}(u, u) < \infty \},$$

where

$$\Omega_\delta = \Omega \cup \left( \bigcup_{x \in \partial \Omega} B(x, \delta) \right),$$

$\mathcal{B} = \mathcal{B}_1$, and $\mathcal{B}_h$ is the operator defined in $X \times X$ by

$$\mathcal{B}_h(u, v) = \int_{\Omega_\delta} \int_{\Omega_\delta} H(x', x) \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u(x') - u(x)|^{p-2} (u(x') - u(x)) (v(x') - v(x)) \, dx' \, dx,$$
We define now the following nonlocal variational problem: given $f \in L^p'(\Omega)$, where $p' = \frac{p}{p-1}$ and $p > 1$, find $u \in X$ such that

$$\mathcal{B}_h(u, w) = (f, w)_{L^{p'}(\Omega) \times L^p(\Omega)} \quad \text{in } X. \quad (1.5)$$

Note that (1.5) is equivalent to

$$\int_{\Omega_\delta} \int_{\Omega_\delta} H(x', x) k_\delta(|x' - x|) \frac{|u(x') - u(x)|^{p-2}(u(x') - u(x))(w(x') - w(x))}{|x' - x|^p} \, dx' \, dx = \int_{\Omega_\delta} f w \, dx \quad (1.6)$$

for all $w \in X$. Since the existence and uniqueness of solution for this problem is a well-known fact, for $h$ fixed and any $\delta$, there exists a solution $u_\delta$. The aim is to check whether the sequence of solutions $(u_\delta)_\delta$ converges to the solution of the corresponding local $p$-Laplacian equation. This convergence (or $G$-convergence) clearly entails the study of the minimization principle

$$\min_{w \in X} \left\{ \frac{1}{p} \mathcal{B}_h(w, w) - \int_{\Omega} f(x) w(x) \, dx \right\},$$

and, consequently, this task inevitably leads us to the study of the problem posed above; [1–3, 5] are some references where this type of convergence is analyzed.

The paper is organized by means of three sections containing different proofs of (1.3) and (1.4).

2 First proof

Our essential tool in to generalize (1.3) is a convenient Vitali covering of the set $\Omega$ (see [11, Chap. 4, Sect. 3, p. 109] for details or [6, Chap. 2, Sect. 2, p. 26] for an elegant proof in the case of Lebesgue-measurable sets). Recall that the family $\{V_i\}_{i \in \mathbb{K}}$ is a Vitali covering for $\Omega \subset \mathbb{R}^N$ if with any $x \in \Omega$ we can associate a number $\alpha > 0$, a sequence of $V_i$, and a sequence of balls $B(x, \epsilon_i)$ such that $V_i \subset B(x, \epsilon_i)$ and $|V_i| \geq \alpha |B(x, \epsilon_i)|$, where $\epsilon_i \to 0$ as $i \to \infty$.

**Theorem 2** (Vitali covering theorem) Let $\mathcal{A} = \{V_i\}_{k \in \mathbb{K}}$ be a Vitali covering of closed subsets of $\mathbb{R}^N$ for $\Omega$. There is a sequence of $(i_j)_j \in \mathbb{K}$ such that $|\Omega \setminus \bigcup_j V_{i_j}| = 0$ and the sets $(V_{i_j})$ are pairwise disjoint.

A particular and useful version of this chief result is the following:

**Proposition 1** Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, let $K$ be a compact set included in $\Omega$, and let $\xi$ be a nonnegative function in $L^1(\Omega \times \Omega)$. Then there is a sequence of pairwise disjoint closed balls $(\overline{B}_i) \subset \Omega$ such that $|K \setminus \bigcup_{i=1}^\infty \overline{B}_i| = 0$ and

$$\iint_{K \times K} \xi(x', x) \, dx' \, dx \geq \sum_{i=1}^\infty \iint_{\overline{B}_i \times \overline{B}_i} \xi(x', x) \, dx' \, dx.$$
Proposition 2
Let \( K \) be a compact inside \( \Omega \) and \( \Omega \) is open, we have \( d \equiv \text{dist}(K, \mathbb{R}^N \setminus \Omega) > 0. \) In particular, any closed ball \( \overline{B}_r = \overline{B}(x, r) \subset \Omega \) for any \( r < d. \) Moreover, the family \( \mathcal{F} = \{ \overline{B}(x, s) : x \in K, s < r/2 \} \) is a Vitali covering of \( K, \) because every point of \( K \) is contained in an arbitrarily small ball belonging to \( \mathcal{F}. \) Consequently, there are disjoint balls \( \overline{B}_i \) such that \( |K \setminus \bigcup_{i=1}^{\infty} \overline{B}_i| = 0. \) This covering also serves to approximate \( K \times K \) because \(|(K \times K) \setminus (\bigcup_{i=1}^{\infty} (\overline{B}_i \times \overline{B}_i))| = 0, \) and therefore

\[
\int_{K \times K} \xi(x', x) \, dx' \, dx = \sum_{i,j=1}^{\infty} \int_{\overline{B}_i \times \overline{B}_j} \xi(x', x) \, dx' \, dx \geq \sum_{i=1}^{\infty} \int_{\overline{B}_i \times \overline{B}_i} \xi(x', x) \, dx' \, dx. \tag{1}\]

In a first step, we assume that \( h \) is continuous a.e. in \( \Omega. \) We adapt [7, Lemma 7.9, p. 129] to prove our key result.

Proposition 2
Let \( \Omega \subset \mathbb{R}^N \) be an open bounded set such that \( |\partial \Omega| = 0, \) and let \( f \) be a positive a.e. continuous function on \( \Omega. \) Let \( r_k : \Omega \setminus N \to \mathbb{R}^+ \) be a sequence of functions, where \( N \) is the set of discontinuity points of \( f. \) There exist a set of points \( \{\alpha_{k_i}\} \subset \Omega \setminus N \) and positive numbers \( \{\epsilon_{k_i}\}_{i}, \) such that for each \( k, \epsilon_{k_i} \leq r_k(a_{k_i}). \)

\[
\{\alpha_{k_i} + \epsilon_{k_i} \Omega\} \quad \text{are pairwise disjoint,}
\]

\[
\Omega = \bigcup_i \{\alpha_{k_i} + \epsilon_{k_i} \Omega\} \cup N_k, \quad \text{where } |N_k| = 0,
\]

and

\[
\int_{\Omega} f(x) \xi(x) \, dx = \sum_i f(a_{k_i}) \int_{a_{k_i} + \epsilon_{k_i} \Omega} \xi(x) \, dx + o(1) \quad \text{as } k \to +\infty \tag{2.1}
\]

for all \( \xi \in L^1(\Omega). \)

Proof
Let \( C = \Omega \setminus N \) be the set of points of continuity of \( f. \) We define the families

\[
\mathcal{F}_k = \left\{ a + \epsilon \Omega : \epsilon \in C, \epsilon \leq r_k(a), |f(x) - f(a)| \leq \frac{1}{k} \text{ for any } x \in a + \epsilon \Omega \right\}.
\]

For each fixed \( k > 0, \) the family \( \mathcal{F}_k \) covers \( C \) (and \( \Omega \)) in the sense of Vitali. Thus, Theorem 2 allows us to choose a numerable sequence of disjoint sets \( \{a_{k_i} + \epsilon_{k_i} \Omega\} \in \mathcal{F}_k \) such that \( |\Omega \setminus \bigcup \{a_{k_i} + \epsilon_{k_i} \Omega\}| = 0. \) Since \( f \) is continuous at \( a_{k_i}, \) the sequence \( \epsilon_{k_i} \) can be chosen so that

\[
|f(x) - f(a_{k_i})| \leq \frac{1}{k} \quad \text{for any } x \in a_{k_i} + \epsilon_{k_i} \Omega \text{ and any } j.
\]

Consequently,

\[
\left| \int_{\Omega} \xi(x)f(x) \, dx - \sum_j f(a_{k_i}) \int_{a_{k_i} + \epsilon_{k_i} \Omega} \xi(x) \, dx \right| = \left| \sum_j \int_{a_{k_i} + \epsilon_{k_i} \Omega} (f(x) - f(a_{k_i}))\xi(x) \, dx \right|
\]
\[ \leq \sum_{i} \int_{a_{i} + \epsilon_{i} \Omega} |f(x) - f(a_{i})| \xi(x) \, dx \]
\[ \leq \frac{1}{k} \sum_{i} \int_{a_{i} + \epsilon_{i} \Omega} \xi(x) \, dx \]
\[ = \frac{1}{k} \| \xi \|_{L^{1}(\Omega)}. \]

**2.1 Application**

We apply the previous analysis to the integral
\[ I = \int_{\Omega} \int_{\Omega} H(x',x) \xi_{\delta}(x',x) \, dx' \, dx, \]
where
\[ \xi_{\delta}(x',x) = \frac{k_{3}(|x' - x|)}{|x' - x|^{p}} |u_{\delta}(x') - u_{\delta}(x)|. \]

We consider \( \Omega \times \Omega \) instead of \( \Omega \), and now \( f(x',x) \) is the symmetric function \( H(x',x) = \frac{k_{3}(|x' - x|)}{|x' - x|^{p}} \) with \( h \in H \). We assume that \( h \) is continuous, and we take \( \bigcup_{i} (a_{i} + \epsilon_{i} \Omega) \times (a_{i} + \epsilon_{i} \Omega) \), the union of a family of pairwise of disjoint sets covering \( \Omega \times \Omega \). Then, according to the previous discussion, we trivially deduce

\[ I = \sum_{i} H(a_{i},a_{i}) \int_{a_{i} + \epsilon_{i} \Omega} \int_{a_{i} + \epsilon_{i} \Omega} \frac{k_{3}(|x' - x|)}{|x' - x|^{p}} |u_{\delta}(x') - u_{\delta}(x)| \, dx' \, dx + o(1) \]
\[ \geq \sum_{i} h(a_{i}) \int_{a_{i} + \epsilon_{i} \Omega} \int_{a_{i} + \epsilon_{i} \Omega} \frac{k_{3}(|x' - x|)}{|x' - x|^{p}} |u_{\delta}(x') - u_{\delta}(x)| \, dx' \, dx + o(1) \]
\[ = \sum_{i} h(a_{i}) \int_{a_{i} + \epsilon_{i} \Omega} \int_{a_{i} + \epsilon_{i} \Omega} \frac{k_{3}(|x' - x|)}{|x' - x|^{p}} |u_{\delta}(x') - u_{\delta}(x)| \, dx' \, dx + o(1). \]

We pass to the limit as \( \delta \to 0 \) in \( I \): we use (1.1), Fatou’s lemma and (1.2) for open sets to derive

\[ \liminf_{\delta \to 0} I \geq \liminf_{\delta \to 0} \sum_{i} h(a_{i}) \int_{a_{i} + \epsilon_{i} \Omega} \int_{a_{i} + \epsilon_{i} \Omega} \frac{k_{3}(|x' - x|)}{|x' - x|^{p}} |u_{\delta}(x') - u_{\delta}(x)| \, dx' \, dx + o(1) \]
\[ \geq \sum_{i} h(a_{i}) \left( \liminf_{\delta \to 0} \int_{a_{i} + \epsilon_{i} \Omega} \int_{a_{i} + \epsilon_{i} \Omega} \frac{k_{3}(|x' - x|)}{|x' - x|^{p}} |u_{\delta}(x') - u_{\delta}(x)| \, dx' \, dx \right) \]
\[ + o(1) \]
\[ \geq \sum_{i} h(a_{i}) \left( \int_{a_{i} + \epsilon_{i} \Omega} |\nabla u(x)|^{p} \, dx \right) + o(1). \]

If we take limits as \( k \to +\infty \), then this estimate gives

\[ \liminf_{\delta \to 0} I \geq \lim_{k \to +\infty} \sum_{i} h(a_{i}) \int_{a_{i} + \epsilon_{i} \Omega} |\nabla u(x)|^{p} \, dx. \]

By using again Proposition 2 the last inequality clearly provides inequality (1.3).
Remark 1 The analysis and conclusion we have just arrived at remain valid if we consider any open set $O \subset \Omega$ such that $|\partial O| = 0$. We can go a step further: we have

$$\liminf_{\delta \to 0} \int \int_{\Omega} F(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_5(x') - u_5(x)|^p \, dx' \, dx \geq \int \int_{\Omega} F(x, x) \left| \nabla u(x) \right|^p \, dx$$ (2.3)

for any symmetric nonnegative continuous function $F \in L^\infty(O \times O)$.

2.2 Extension to the case of measurable functions

Let now $h$ be just measurable; without loss of generality, supp $H \subset \Omega \times \Omega$ and $H = 0$ otherwise. By Luzin’s theorem (see [10, Theorem 2.24, p. 62]), given arbitrary $\delta > 0$, there exists a Vitali covering of $\Omega$ such that

$$\liminf_{\delta \to 0} \int \int_{\Omega} F(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_5(x') - u_5(x)|^p \, dx' \, dx \geq \int \int_{\Omega} F(x, x) \left| \nabla u(x) \right|^p \, dx$$ (2.3)

for any symmetric nonnegative continuous function $F \in L^\infty(O \times O)$.

Remark O

At this stage, we consider any compact set $K \subset \Omega \setminus E \subset \Omega$. Since $\Omega$ is open, we can use Proposition 1: there is a number $r > 0$ such that the family $\mathcal{F} = \{B(x, \delta) : x \in K, \delta < r/2\}$ is a Vitali covering of $K$, and therefore there exists a sequence of pairwise disjoint closed balls $(\overline{B_i})_{i=1}^\infty \subset \mathcal{F}$ such that $|K \setminus \bigcup_{i=1}^\infty \overline{B_i}| = 0$.

We take the limits as $\delta \to 0$ to get

$$\liminf_{\delta \to 0} \int \int_{\Omega} F(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_5(x') - u_5(x)|^p \, dx' \, dx \geq \liminf_{\delta \to 0} \sum_i \int \int_{\overline{B_i}} G(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_5(x') - u_5(x)|^p \, dx' \, dx$$

where the second inequality is true because of (2.3) and Fatou’s lemma. Then, since $K$ is any compact set in $\Omega \setminus E$, we obtain

$$\liminf_{\delta \to 0} \int \int_{\Omega} F(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_5(x') - u_5(x)|^p \, dx' \, dx = \int \int_{\Omega} \frac{k_3(|x' - x|)}{|x' - x|^p} |u_5(x') - u_5(x)|^p \, dx' \, dx$$.
\[
\int_{\Omega \setminus E} G(x, x) |\nabla u(x)|^p \, dx \\
= \int_{\Omega \setminus E} H(x, x) |\nabla u(x)|^p \, dx \\
= \int_{\Omega} h(x) |\nabla u(x)|^p \, dx - \int_{E} h(x) |\nabla u(x)|^p \, dx.
\]

By letting \( \epsilon \downarrow 0 \) and using \(|E| \leq \epsilon \), we obtain (1.3), that is,

\[
\liminf_{\delta \rightarrow 0} \int_{\Omega} \int_{\Omega} H(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_3(x') - u_3(x)|^p \, dx' \, dx \\
\geq \sum_{i} \int_{B_i} \int_{B_i} H(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_3(x') - u_3(x)|^p \, dx' \, dx.
\]

Finally, to circumvent the assumption \(|\partial \Omega| = 0\), the procedure we follow is identical to that just employed. Take any compact set \( K \) included in \( \Omega \). Since \( \Omega \) is assumed to be open, thanks to Proposition 1, \( K \) can be exhaustively covered by the union of a numerable sequence of pairwise disjoint closed balls \( B_i \in \mathcal{F} = \{B(x, s) : x \in K, s < r/2\} \subset \Omega, i = 1, 2, \ldots \). Then we realize that

\[
\liminf_{\delta \rightarrow 0} \int_{\Omega} \int_{\Omega} H(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_3(x') - u_3(x)|^p \, dx' \, dx \\
\geq \int_{\Omega} H(x, x) |\nabla u(x)|^p \, dx.
\]

By taking into account that \(|\partial B_i| = 0\) we can apply (2.3) and Fatou’s lemma in (2.5) to obtain

\[
\liminf_{\delta \rightarrow 0} \int_{\Omega} \int_{\Omega} H(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_3(x') - u_3(x)|^p \, dx' \, dx \\
\geq \int_{K} H(x, x) |\nabla u(x)|^p \, dx.
\]

Since \( K \subset \Omega \) is arbitrary, we arrive at (2.4) for any open set \( \Omega \),

\[
\liminf_{\delta \rightarrow 0} \int_{\Omega} \int_{\Omega} H(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_3(x') - u_3(x)|^p \, dx' \, dx \\
\geq \int_{\Omega} H(x, x) |\nabla u(x)|^p \, dx.
\]

**2.3 Corollary**

We apply (2.4) to the case \( F(x', x) = I_{G \setminus G}(x', x) \), where \( G \) is any measurable set included in \( \Omega \); on the one hand, (2.6) guarantees

\[
\liminf_{\delta \rightarrow 0} \int_{\Omega} \int_{\Omega} F(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_3(x') - u_3(x)|^p \, dx' \, dx \\
\geq \int_{\Omega} F(x, x) |\nabla u(x)|^p \, dx \\
= \int_{G} I_{G}(x) |\nabla u(x)|^p \, dx \geq \int_{G} |\nabla u(x)|^p \, dx,
\]

and, on the other hand, it is obvious that

\[
\int_{\Omega} \int_{\Omega} F(x', x) \frac{k_3(|x' - x|)}{|x' - x|^p} |u_3(x') - u_3(x)|^p \, dx' \, dx
\]
\[
\int_G \int_G \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx.
\]

Consequently, (1.4) is proved for any measurable set \( G \subset \Omega \).

### 3 A second proof

We firstly prove (1.4) and then (1.3). By having a look at the work done in the previous section we will be able to provide a straightforward proof of (1.4). Indeed, if \( E \) is a measurable set included in \( \Omega \), then we can find a compact set \( K \subset E \) such that \(|E \setminus K|\) is arbitrarily small. Proposition 1 ensures the existence of a numerable sequence of pairwise disjoint \( \bar{B}_i \in \mathcal{F} \) such that \(|K \setminus \bigcup_{i=1}^\infty \bar{B}_i| = 0 \), \( \bar{B}_i \subset \Omega \) for any \( i \) and

\[
\begin{align*}
\int_E \int_E \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \\
\geq \int_K \int_K \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \\
\geq \sum_i \int_{\bar{B}_i} \int_{\bar{B}_i} \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx.
\end{align*}
\]

We apply (1.2) for open sets and Fatou’s lemma in the last chain of inequalities to derive

\[
\liminf_{\delta \to 0} \int_E \int_E \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \geq \sum_i \int_{\bar{B}_i} |\nabla u(x)|^p \, dx = \int_K |\nabla u(x)|^p \, dx.
\]

Since \( K \subset E \) is arbitrary, we arrive at (1.4), that is,

\[
\liminf_{\delta \to 0} \int_E \int_E \frac{k_\delta(|x' - x|)}{|x' - x|^p} |u_\delta(x') - u_\delta(x)|^p \, dx' \, dx \geq \int_E |\nabla u(x)|^p \, dx. \quad (3.1)
\]

### 3.1 Corollary

We prove (1.3). Let \( h \) be a given simple function defined in \( \Omega \). Then \( h \) can be written as \( h(x) = \sum_{i=1}^m h_i \chi_{B_i}(x) \), where \( \{B_i\} \) is a finite covering of disjoint measurable subsets of \( \Omega \), and \( \{h_i\} \) is a set of numbers such that \( h_{\min} \leq h_i \leq h_{\max} \). Consequently, we can easily check that

\[
I = \int_{\Omega} \int_{\Omega} H(x', x)k_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|^p}{|x' - x|^p} \, dx' \, dx \\
\geq \sum_{i=1}^m h_i \int_{\bar{B}_i} \int_{\bar{B}_i} k_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|^p}{|x' - x|^p} \, dx' \, dx.
\]

Using inequality (1.4) for measurable sets that we have just proved, we straightforwardly infer

\[
\liminf_{\delta \to 0} I \geq \sum_{i=1}^m h_i \int_{\bar{B}_i} |\nabla u(x)|^p \, dx = \int_\Omega h(x) |\nabla u(x)|^p \, dx.
\]

Let \( h \) be a measurable function. By recalling that any measurable function \( h \) can be pointwise approximated by an increasing sequence \( (s_n)_n \) of simple functions we can write

\[
\liminf_{\delta \to 0} \int_\Omega \int_\Omega H(x', x)k_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|^p}{|x' - x|^p} \, dx' \, dx
\]
\[ = \liminf_{\delta \to 0} \int_{\Omega} h(x) \int_{\Omega} k_{3}(|x' - x|) \frac{|u_{\delta}(x') - u_{\delta}(x)|^p}{|x' - x|^p} \, dx' \, dx \]
\[ \geq \liminf_{\delta \to 0} \int_{\Omega} s_{\delta}(x) \int_{\Omega} k_{3}(|x' - x|) \frac{|u_{\delta}(x') - u_{\delta}(x)|^p}{|x' - x|^p} \, dx' \, dx \]
\[ \geq \int_{\Omega} s_{\delta}(x) \nabla u(x) \, dx. \]

It suffices to take the limits in \( n \) and apply the monotone convergence theorem to establish (1.3).

### 4 A third proof

The idea is reproducing the arguments from [9]. In a first step, we assume that \( h : \overline{\Omega} \to [h_{\min}, h_{\max}] \) is a continuous function. Moreover, without loss of generality, we suppose that \( h \) is a continuous function in the set \( \overline{\Omega} = \Omega \cup \bigcup_{p \in \Omega} B(p, s) \), where \( s \) is a fixed positive number.

Now, for the proof of (1.3), the key idea is extending the Stein inequality (see [8, Lemma 4, p. 245]) in the following sense: by using Jensen’s inequality and performing a change of variables we deduce the inequality

\[ \int_{\Omega} \int_{\Omega} H_{r}(x', x) k_{3}(|x' - x|) \frac{|u_{\delta}(x') - u_{\delta}(x)|^p}{|x' - x|^p} \, dx' \, dx \]
\[ \geq \int_{\Omega} \int_{\Omega} H(x', x) k_{3}(|x' - x|) \frac{|u_{r, \delta}(x') - u_{r, \delta}(x)|^p}{|x' - x|^p} \, dx' \, dx \]

for any \( \delta < r \), where \( u_{r, \delta} = \eta_{\delta} u_{r, \delta}, \eta_{r}(x) = \frac{1}{r} \eta\left( \frac{x}{r} \right), x \in \mathbb{R}^{N}, \eta \) is a nonnegative radial function from \( C^{\infty}_{c}(B(0, 1)) \) such that \( \int \eta(x) \, dx = 1 \),

\[ H_{r}(x', x) = \frac{(\eta_{r} \ast h)(x') + (\eta_{r} \ast \delta_{r})(x)}{2}, \]

and \( \Omega_{-r} = \{ x \in \Omega : \text{dist}(x, \delta \Omega) > r \} \). By the continuity of \( H \) in \( \Omega_{r} \times \Omega_{r} \), we know that \( H_{r}(x', x) \rightarrow H(x', x) \) uniformly on compact sets of \( \Omega_{r} \times \Omega_{r} \), whereby, for any \( \epsilon > 0 \), we can choose \( r_{0} > 0 \) such that

\[ \left| \int_{\Omega} \int_{\Omega} (H(x', x) - H_{r}(x', x)) k_{3}(|x' - x|) \frac{|u_{\delta}(x') - u_{\delta}(x)|^p}{|x' - x|^p} \, dx' \, dx \right| \leq \epsilon \]

for any \( r < r_{0} \) and uniformly in \( \delta > 0 \). Then

\[ \lim_{\delta \to 0} \int_{\Omega} \int_{\Omega} H(x', x) k_{3}(|x' - x|) \frac{|u_{\delta}(x') - u_{\delta}(x)|^p}{|x' - x|^p} \, dx' \, dx \]
\[ \geq \lim_{\delta \to 0} \int_{\Omega} \int_{\Omega} H(x', x) k_{3}(|x' - x|) \frac{|u_{r, \delta}(x') - u_{r, \delta}(x)|^p}{|x' - x|^p} \, dx' \, dx - \epsilon \]

for any \( r < r_{0} \). At this point, we notice that Proposition 1 from [8, p. 242] can be modified by including the term \( H(x', x) \) within the integrand; this is factually what Remark 1 establishes. Then passing to the limit as \( \delta \to 0 \) and using the convergence of \( \rho_{r} \ast u_{\delta} \to \rho_{r} \ast u \) in \( C^{2}(\overline{\Omega_{-r}}) \), we get

\[ \lim_{\delta \to 0} \int_{\Omega_{-r}} \int_{\Omega_{-r}} H(x', x) k_{3}(|x' - x|) \frac{|u_{r, \delta}(x') - u_{r, \delta}(x)|^p}{|x' - x|^p} \, dx' \, dx \]
\[
\geq \int_{\Omega \cap r} h(x) |\nabla (\rho_r \ast u)(x)|^p \, dx' \, dx.
\]

Consequently, letting \( r \to 0 \) in this inequality and taking into account that \( \nabla (\rho_r \ast u) \) strongly converges to \( \nabla u \) in \( L^p(\Omega) \), we derive

\[
\lim_{\delta \to 0} \int_{\Omega} \int_{\Omega} H(x', x) k_\delta(|x' - x|) \frac{|u_\delta(x') - u_\delta(x)|^p}{|x' - x|^p} \, dx' \, dx \geq \int_{\Omega} h(x) |\nabla u(x)|^p \, dx' \, dx - \epsilon C.
\]

Now, since \( \epsilon \) is arbitrarily small, the statement is proved under the assumption that \( h \) is continuous in \( \Omega_\ast \).

If \( h: \Omega \to [h_{\min}, h_{\max}] \) is a measurable function, then we extend it by zero to \( \Omega_\ast \), and then apply Luzin's theorem to this extended function. The remaining details follow along the lines of Sect. 2.2.

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No dataset has been used for the conclusions or contributions of this work.

Competing interests

The author declares that they have no competing interests.

Author's contributions

The contributions correspond to the only author of the manuscript. Author read and approved the final manuscript.

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