Riemann-Hilbert method and N-soliton solutions for the mixed Chen-Lee-Liu derivative nonlinear Schrödinger equation

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Abstract

In this paper, we aim to investigate the mixed Chen-Lee-Liu derivative nonlinear Schrödinger (CLL-NLS) equation via the Riemann-Hilbert (RH) method. We construct a RH problem based on the Jost solution of the Lax pair. By solving this RH problem corresponding to the non-reflection case, the N-soliton solution of PLL-NLS equation is obtained, which expression is the ratio of $(2N+1) \times (2N+1)$ determinant and $2N \times 2N$ determinant.

Keywords:
Riemann-Hilbert method; Chen-Lee-Liu derivative nonlinear Schrödinger equation; soliton solutions; boundary conditions

1. Introduction

Soliton theory is an important branch of nonlinear science. In physics, solitons are used to describe solitary waves with elastic scattering characteristics. In mathematics, soliton theory provides a series of methods for solving nonlinear partial differential equations, which attracts the attention of mathematical physicists. With the development of soliton theory, many methods to solve soliton equations with important physical background have been proposed, for example, inverse scattering transform (IST) [1, 2, 3], Darboux transformation (DT) [4], Bäcklund transformation [5], Hirota bilinear method [6, 7, 8], Wronskian technique [9, 10, 11] and so on.

The derivative nonlinear Schrödinger (DNLS) equation describing Alfven waves in magnetic field is as follows [12]:

\[ iu_t = u_{xx} + i|u|^2u_x = 0, \]

which is one of the most significant equations in physics. Kaup and Newell obtained the solution of Eq. (1.1) by using IST method [13]. The IST method has obvious advantages in solving the initial value of the soliton equation, but its calculation is large. Fortunately, on the basis of IST method, Riemann-Hilbert (RH) method, a relatively simple and direct method for solving soliton equation, was proposed by Novikov et al. [14]. This method is similar to the IST method, the RH method first considers the direct scattering problem, that is, the RH problem is constructed, from the initial data of the soliton equation, the scattering data at the initial time is obtained, and the scattering data at any time is obtained by using its time evolution law. Finally, the exact solution is established by using the IST method [15]. Since the RH method was proposed, many solutions of soliton equations have been discussed, such as,

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the coupled DNLS equation [16], the coupled higher-order NLS equation [17], the short pulse (SP) equation [18],
the coupled modified Korteweg-de Vries (mKdV) equation [19], the generalized Sasa-Satsuma equation [20], the
two-component Gerdjikov-Ivanov (GI) equation [21], the modified SP equation [22], and so on [23, 24, 25]. In particular,
RH method is an effective way to working the initial-boundary value problems (IBVPs) of the integrable systems
[26, 27, 28, 29, 30, 31].

Recently, Chan et al. [32] reported the the mixed Chen-Lee-Liu derivative nonlinear Schrödinger (CLL-NLS) equa-
tion as follows
\[ \begin{align*}
ir_t + r_{xx} + |r|^2 r - i|r|^2 r_x = 0,
\end{align*} \]
which is a completely integrable model, and a large number of solutions of Eq. (1.2) are discussed, such as, the
soliton solution by Hirota bilinear method [33], the higher-order soliton, breathers, and rogue wave solutions by DT
method [34]. In particular, the IBVPs of Eq. (1.2) to be investigated by Fokas method [35].

The design structure of this paper is as follows. Section 2, we will construct a basic RH problem based on the Jost
solution of Lax pair. Section 3, we will give the reconstruction of potential function and the law of scattering data
evolution with time. Section 4, the formula of N-soliton solution expressed by determinant ratio is proposed. Section
5 is the conclusions.

2. The Riemann-Hilbert problem

In this section, we shall construct a RH problem for the CLL-NLS equation, by using IST method. First of all, we
introduce the coupled CLL-NLS equation as following:
\[ \begin{align*}
\begin{cases}
    r_t - ir_{xx} + i r^2 q + q r_x = 0, \\
    q_t + i q_{xx} - i r^2 r + q r q_x = 0.
\end{cases}
\end{align*} \]
which reduces to the CLL-NLS equation while \( q = -r^* \) and the \( * \) denotes complex conjugation. These two equations
in (2.1) are the compatibility condition of the following Lax pair [34, 35, 36, 37]
\[ \begin{align*}
\Phi_x &= U \Phi = \left( i \left( \lambda^2 - \frac{1}{2} \sigma \right) + \lambda Q + \frac{1}{4} i Q^2 \sigma \right) \Phi, \\
\Phi_t &= V \Phi = \left[ -2 i \lambda^2 - \frac{1}{2} \lambda^2 \sigma - 2 i \lambda Q - i \lambda^2 Q^2 \sigma + \lambda \left( Q + i \sigma Q_x - \frac{1}{2} Q^2 \right) \right. \\
&\quad \left. - \frac{1}{8} i Q^4 \sigma + \frac{1}{4} (QQ_x - Q, Q) \right] \Phi,
\end{align*} \]
with
\[ \sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & r \\ -r^* & 0 \end{pmatrix}. \]
where \( \Phi = \Phi(x, t; \lambda) \) is a matrix function of the complex spectral parameter, \( \lambda \) is the spectral parameter. \( Q \) is named
potential function.

For the sake of convenience, we introduce a new matrix function \( J = J(x, t; \lambda) \) defined by
\[ \Phi = J e^{i(x^2 + 2 i x r - 2 i \lambda^2 - \frac{1}{4} \lambda^2 r^2 t)} \]

(2.4)
Obviously, we can check that \( J e^{i(k^2-\frac{1}{4})x^2-2ikx^2+\frac{1}{2}x^4} \) satisfies Eqs. (2.2a), (2.2b). Inserting (2.4) into (2.2a)-(2.2b), the form of the Lax pair \((2.2a)-(2.2b)\) becomes

\[
J_x = i(\lambda^2 - \frac{1}{2})[\sigma, J] + U_1 J, \tag{2.5a}
\]

\[
J_t = -2i(\lambda^2 - \frac{1}{2})[\sigma, J] + V_1 J. \tag{2.5b}
\]

where \([\sigma, J] = \sigma J - J \sigma \) is the commutator.

\[
U_1 = \lambda Q + \frac{1}{4} i \lambda^2 \sigma \quad V_1 = -2 \lambda^3 \sigma - i \lambda^2 \lambda^2 \sigma + \lambda \left( Q + i \lambda Q - \frac{1}{2} \tilde{Q}^2 \right) - \frac{1}{8} i \lambda^4 \sigma + \frac{1}{4} (Q^2 - Q_2, Q).
\]

As usual, in the following scattering process, we only concentrate on the \( x \)-part of the Lax pair \((2.2a)\). Indeed the \( x \)-part of the Lax pair \((2.2a)\) allows us to take use of the existing symmetry relations of the potential \( Q \). Consequently, we shall treat the time \( t \) as a dummy variable and omit it. Now, we calculate two Jost solutions \( J_+ = J_+ (x, \lambda) \) of Eq. (2.5a) for \( \lambda \in \mathbb{R} \). Based on the properties of the Jost solutions \( J_+ = J_+ (x, \lambda) \)

\[
J_+ = ([J_+], [J_+]_2), \quad \tag{2.7}
\]

\[
J_- = ([J_-], [J_-]_2), \quad \tag{2.8}
\]

with the boundary conditions

\[
J_+ \rightarrow I, \quad x \rightarrow -\infty, \quad \tag{2.9a}
\]

\[
J_- \rightarrow I, \quad x \rightarrow +\infty. \quad \tag{2.9b}
\]

the subscripts in \( J(x, \lambda) \) represent which end of the \( x \)-axis the boundary conditions are set. Where \( [J_+]_n (n = 1, 2) \) denote the \( n \)-th column vector of \( J_+ \), \( I = \text{diag}(1, 1) \) is the \( 2 \times 2 \) unit matrix. According to the method of variation of parameters as well as the boundary conditions, we can turn Eq. (2.5a) for \( \lambda \in \mathbb{R} \) into the Volterra integral equations.

\[
J_+ (x, \lambda) = I - \int_{-\infty}^{\infty} e^{(\lambda^2-\frac{1}{4})\xi(x-\xi)} U_1 J_+(\xi, \lambda) d\xi, \quad \tag{2.10}
\]

\[
J_- (x, \lambda) = I + \int_{-\infty}^{\infty} e^{(\lambda^2-\frac{1}{4})(x-\xi)} U_1 J_- (\xi, \lambda) d\xi, \quad \tag{2.11}
\]

where \( \hat{\sigma} \) represents a matrix operator acting on \( 2 \times 2 \) matrix \( X \) by \( \hat{\sigma} X = [\sigma, X] \) and by \( e^{i\theta} X = e^{i\theta} X e^{-i\theta} \). Moreover, after simple analysis, we find that \( [J_+], [J_-] \) are analytic for \( \lambda \in D_+ \) and continuous for \( \lambda \in D_+ \cup \mathbb{R} \cup i\mathbb{R} \). Similarly \( [J_-], [J_+] \) are analytic for \( \lambda \in D_- \) and continuous for \( \lambda \in D_- \cup \mathbb{R} \cup i\mathbb{R} \), here

\[
D_+ = \left\{ \lambda : \arg \lambda \in \left( 0, \frac{\pi}{2} \right] \cup \left( \frac{3\pi}{2}, \pi \right) \right\}
\]

\[
D_- = \left\{ \lambda : \arg \lambda \in \left( \frac{\pi}{2}, \pi \right] \cup \left( \frac{3\pi}{2}, 2\pi \right) \right\}.
\]

Secondly, let us investigate the properties of \( J_+ \). We deduce the determinants of \( J_+ \) are constants for all \( x \) on the basis of the Abel’s identity and \( \text{Tr}(U_1) = 0. \) Furthermore, due to the boundary conditions Eq. (2.9a), (2.9b), we have

\[
\det J_+ = 1, \quad \lambda \in \mathbb{R} \cup i\mathbb{R}. \tag{2.13}
\]
By introducing a new function $E(x, \lambda) = e^{i(\lambda^2 - \frac{1}{4})x}$, we find that spectral problem Eq. (2.15) exists two fundamental matrix solutions $J_+E$ and $J_-E$, which are linearly related by a $2 \times 2$ scattering matrix $S(\lambda)$

$$J_-E = J_+E \cdot S(\lambda), \quad \lambda \in \mathbb{R} \bigcup i\mathbb{R}. \quad (2.14)$$

From Eq. (2.13) and (2.14), we know that

$$\det S(\lambda) = 1. \quad (2.15)$$

Furthermore, let $x$ go to $+\infty$, the $2 \times 2$ scattering matrix $S(\lambda)$ is given as

$$S(\lambda) = \lim_{x \to +\infty} E^{-1}J_-E = I + \int_{-\infty}^{+\infty} E^{-1}U_1J_+Ed\xi, \quad \lambda \in \mathbb{R} \bigcup i\mathbb{R}. \quad (2.16)$$

From the analytic property of $J_-$, we find that $s_{22}$ can be analytically extended to $D_+$, $s_{11}$ allows analytic extensions to $D_-$. Generally speaking, $s_{12}, s_{21}$ can only be defined in the $\mathbb{R} \bigcup i\mathbb{R}$.

In what follows, we shall construct a RH problem for the CLL-NLS equation. Firstly, using the analytic properties of $J_\pm$, we define a new Jost solution $P_1 = P_1(x, \lambda)$ as

$$P_1 = ([J_+], [J_-]). \quad (2.17)$$

which is obviously analytic for $\lambda \in D_+$. In addition, from Eqs. (2.10) and (2.11), we have

$$P_1 \to I, \quad \lambda \to +\infty, \quad \lambda \in D_+. \quad (2.18)$$

Next, we introduce the limit of $P_1$

$$P^+ = \lim_{\lambda \to i} P_1 \quad \Gamma = \mathbb{R} \bigcup i\mathbb{R}. \quad (2.19)$$

From (2.14) and (2.16), we can get

$$P^+ = J_+ \begin{pmatrix} 1 & e^{i(2\lambda^2 - 1)x}s_{12} \\ 0 & s_{22} \end{pmatrix}. \quad (2.20a)$$

To obtain the analytic counterpart of $P^+$ in $D_-$, denoted by $P_2$, we consider the inverse matrices $J_+^{-1}$ defined as

$$J_+^{-1} = \begin{pmatrix} [J_+^{-1}]_1 \\ [J_+^{-1}]_2 \end{pmatrix}, \quad J_-^{-1} = \begin{pmatrix} [J_-^{-1}]_1 \\ [J_-^{-1}]_2 \end{pmatrix}. \quad (2.20b)$$

here $[J_n^{-1}]_m(n = 1, 2)$ denote the $n$-th row vector of $J_+^{-1}$. Then we can see that $[J_+^{-1}]_1, [J_+^{-1}]_2$ are analytic for $\lambda \in D_-$ and continuous for $\lambda \in D_\pm \bigcup \mathbb{R} \bigcup i\mathbb{R}$, whereas $[J_-^{-1}]_1, [J_-^{-1}]_2$ are analytic for $\lambda \in D_+$ and continuous for $\lambda \in D_\pm \bigcup \mathbb{R} \bigcup i\mathbb{R}$.

Obviously, $J_+^{-1}$ satisfy the adjoint scattering equation of Eq. (2.15):

$$K_+ = i(\lambda^2 - \frac{1}{2})[x, K] - KU_1. \quad (2.21)$$

In addition, it is not difficult to find that the inverse matrices $J_+^{-1}$ and $J_-^{-1}$ satisfy the following boundary conditions.

$$J_+^{-1} \to I, \quad x \to -\infty, \quad \quad J_-^{-1} \to I, \quad x \to +\infty. \quad (2.22a)$$

$$J_+^{-1} \to I, \quad x \to -\infty, \quad \quad J_-^{-1} \to I, \quad x \to +\infty. \quad (2.22b)$$
Taking the similar procedure as above, a matrix function $P_2$ which is analytic in $D_-$

$$P_2 = \begin{pmatrix} [J_+^{-1}]_1 \\ [J_+^{-1}]_2 \end{pmatrix}.$$

(2.23)

and $P^-$

$$P^- = \lim_{i \to 1} P_2 \quad \Gamma = \mathbb{R} \bigcup i \mathbb{R}. \quad (2.24)$$

are expressed. Moreover, we can get that

$$P_2 \to I, \quad \lambda \to -\infty, \quad \lambda \in D_-.$$

(2.25)

and

$$P^- = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i(2\lambda - 1)x} r_{21} \quad r_{22} J_+^{-1}. \quad (2.26)$$

with $R(\lambda) \equiv (r_{ij})_{2 \times 2} = S^{-1}(\lambda)$. Similar to the scattering coefficients $s_{ij}$ above, it is easy to know that $r_{11}$ allows an analytic extension to $D_+$ and $r_{22}$ is analytically extendible to $D_-$. Generally speaking, $r_{12}$, $r_{21}$ can only be defined in $\mathbb{R} \bigcup i \mathbb{R}$. In addition, from (2.14), we get

$$E^{-1} J_-^{-1} = R(\lambda) \cdot E^{-1} J_+^{-1}, \quad \lambda \in \mathbb{R} \bigcup i \mathbb{R}. \quad (2.27)$$

Summarizing the above results, we find that two matrix functions $P^+$ and $P^-$ which are analytic in $D_+$ and $D_-$, respectively, are related by

$$P^-(x, \lambda)P^+(x, \lambda) = \begin{pmatrix} 1 \\ r_{21}e^{-i(2\lambda - 1)x} \end{pmatrix} s_{12} e^{i(2\lambda - 1)x} \quad 1, \quad \lambda \in \mathbb{R} \bigcup i \mathbb{R}. \quad (2.28)$$

Eq. (2.28) is just the RH problem for the CLL-NLS equation. In order to obtain solution for this RH problem, we assume that the RH problem is non-regular when $\det P_1$ and $\det P_2$ can be zero for $\lambda_k \in D_+$ and $\lambda_k \in D_-$, respectively. $1 \leq k \leq N$. where $N$ is the number of these zeros. Recalling the definitions of $P_1$ and $P_2$, we can see that

$$\det P_1(x, \lambda) = s_{22}(\lambda), \quad \lambda \in D_+, \quad (2.29)$$

$$\det P_2(x, \lambda) = r_{22}(\lambda), \quad \lambda \in D_-.$$

(2.30)

To specify these zeros, we first can take use of a symmetry relation for $U_1$

$$U_1^+ = \sigma U_1 \sigma,$$

where the superscript $\dagger$ means the Hermitian of a matrix. Therefore from Eq. (2.21), we arrive at

$$J_+^*(x, t, \lambda^*) = \sigma J_+^{-1}(x, t, \lambda) \sigma,$$

(2.31)

Then from (2.14), we also gain

$$S^\dagger(\lambda^*) = \sigma S^{-1}(\lambda) \sigma,$$

(2.32)

which implies the following relations

$$r_{11}(\lambda) = s_{11}^*(\lambda^*) \quad \lambda \in D_+, \quad (2.33)$$
Moreover from Eq. (2.32) and the definitions of $P_1, P_2,$ we point out that the analytic solutions $P_1, P_2$ satisfy the involution property

$$P_1(x, \lambda^\ast) = \sigma P_2(x, \lambda) \sigma, \quad \lambda \in D_-.$$  

(2.37)

The similar analysis shows that the potential matrix $Q$ also satisfies another symmetry relations

$$Q = -\sigma Q \sigma$$

It follows that

$$J_a(-\lambda) = \sigma J_a(\lambda) \sigma.$$  

(2.38)

and

$$P_1(-\lambda) = \sigma P_1(\lambda) \sigma.$$  

(2.39)

thus

$$s_{11}(-\lambda) = -s_{11}(\lambda), \quad \lambda \in D_-, \quad (2.40)$$

$$s_{22}(-\lambda) = -s_{22}(\lambda), \quad \lambda \in D_+, \quad (2.41)$$

$$s_{12}(-\lambda) = -s_{12}(\lambda), \quad \lambda \in \mathbb{R} \cup \mathbb{R}, \quad (2.42)$$

$$s_{21}(-\lambda) = -s_{21}(\lambda), \quad \lambda \in \mathbb{R} \cup \mathbb{R}. \quad (2.43)$$

Therefore, from (2.44) and view of (2.45), we know that if $\lambda_j$ is a zero of $\det P_1,$ then $\hat{\lambda}_j = \lambda_j^\ast$ is a zero of $\det P_2.$ Moreover, in view of (2.41), we know that $-\lambda$ is also a zero of $\det P_1.$ Hence we suppose that $\det P_1$ has $2N$ simple zeros $\{\lambda_j\}_1^{2N}$ satisfying $\lambda_{N+l} = -\lambda_l \quad (1 \leq l \leq N),$ which all lie in $D_+.$ From the zeros of $\det P_1,$ we see that $\det P_2$ possesses $2N$ simple zeros $\{\hat{\lambda}_j\}_1^{2N}$ satisfying $\hat{\lambda}_j = \lambda_j^\ast \quad (1 \leq j \leq 2N),$ which are all in $D_-.$ Obviously the zeros of $\det P_1$ and $\det P_2$ always appear in quadruples. To solve the RH problem, we need the scattering data including the continuous scattering data $\{s_{12}, s_{21}\}$ and the discrete scattering data $\{\lambda_j, \hat{\lambda}_j, v_j, \hat{v}_j\}$ which a single column vector $v_j$ and row vector $\hat{v}_j$ satisfying

$$P_1(\lambda_j) v_j = 0, \quad \hat{v}_j P_2(\hat{\lambda}_j) = 0. \quad (2.44)$$

On the one hand, by taking the Hermitian conjugate of $P_1(\lambda_j) v_j = 0,$ we can construct the relationship between each pair of $v_j$ and $\hat{v}_j.$

$$v_j = \sigma v_j - N, \quad N + 1 \leq j \leq 2N,$$

$$\hat{v}_j = v_j^\ast \sigma, \quad 1 \leq j \leq N. \quad (2.45)$$

On the other hand, in order to obtain the spatial evolutions for vectors $v_j(x),$ taking the $x$-derivative to equation $P_1 v_j = 0$ and using (2.56), we obtain

$$v_j = e^{i(\hat{\lambda}_j^\ast - \lambda_j) x} v_{\hat{h}_j}, \quad 1 \leq j \leq N. \quad (2.46)$$
where \( v_{1i} = v_{j1} = 0 \).

Using these vectors, the RH problem which corresponds to the reflection-less case, that is to say, we set the vanishing coefficient \( s_{12} = s_{21} = 0 \) in the RH problem (2.28), can be solved exactly, and the result is

\[
P_1(\lambda) = 1 - \sum_{k=1}^{2N} \sum_{j=1}^{2N} \frac{v_k(M^{-1})_{kj}}{\lambda - \lambda_j},
\]

\[
P_2(\lambda) = 1 + \sum_{k=1}^{2N} \sum_{j=1}^{2N} \frac{v_k(M^{-1})_{kj}}{\lambda - \lambda_k}.
\]

where \( M = (m_{kj})_{2N \times 2N} \) is a matrix whose entries are

\[
m_{kj} = \frac{e_{kij}}{\lambda_j - \lambda_k}, \quad 1 \leq k, j \leq 2N.
\]

3. Inverse scattering transform

In this section, with the help of \( P_1 \) in (2.47a), we can write out explicitly the potential \( Q \). Owing to \( P_1(\lambda) \) is the solution of spectral problem (2.5a), we assume that the asymptotic expansion of \( P_1(\lambda) \) at large \( \lambda \) as

\[
P_1 = 1 + \frac{P_1^{(1)}}{\lambda} + \frac{P_1^{(2)}}{\lambda^2} + O(\lambda^{-3}) \quad \lambda \to \infty,
\]

Then by substituting the above expansion into (2.5a) and comparing \( O(\lambda) \) terms, we obtains

\[
Q = -i[\sigma, P_1^{(1)}] = \begin{pmatrix} 0 & -2i(P_1^{(1)})_{12} \\ 2i(P_1^{(1)})_{21} & 0 \end{pmatrix},
\]

which implies that \( r \) can be reconstructed as

\[
r = -2i(P_1^{(1)})_{12}.
\]

where \( P_1^{(1)} = (P_1^{(1)})_{2 \times 2} \) and \((P_1^{(1)})_{ij}\) is the \((i; j)\)-entry of \( P_1^{(1)} \), \( i, j = 1, 2 \). Here, the matrix function \( P_1^{(1)} \) can be found from (2.47a)

\[
P_1^{(1)} = \sum_{k=1}^{2N} \sum_{j=1}^{2N} v_k(M^{-1})_{kj} \hat{v}_j.
\]

4. The soliton solutions

To derive the solutions for the CLL-NLS equation, we also need the scattering data at time \( t \), which need investigate the time evolution of scattering data. In fact, by using (2.35b) and (2.14), making the limit \( x \to +\infty \), and taking into account the boundary condition (2.2a) for \( J_t \) as well as \( V_1 \to 0 \) as \( x \to \pm\infty \), we arrive at

\[
s_{11} = s_{22} = 0, \quad s_{12} = -4i(\lambda^2 - \frac{1}{2})^2 s_{12}, \quad s_{21} = -4i(\lambda^2 - \frac{1}{2})^2 s_{21},
\]

\[
\frac{dA_j}{dt} = 0, \quad v_{j1} = -2i(\lambda^2 - \frac{1}{2})^2 v_{j2}.
\]

Combining (2.45) with (2.46), we can derive the column vectors \( v_j \) and the row vector \( \hat{v}_j \) explicitly,

\[
v_j = \begin{cases} e^{\theta_j} v_{j1}, & 1 \leq j \leq N \\ \sigma e^{\theta_j} v_{j-N,0}, & N + 1 \leq j \leq 2N \end{cases}
\]

where \( \theta_j = \theta_{j-N}\theta_j \pm \theta_{j-N} \pm \theta_j \pm \theta_{j-N} \pm \theta_j \).
and

\[
\tilde{v}_j = \begin{cases} 
    v_{j_0}^{1} e^{\theta_{j_0} \sigma}, & 1 \leq j \leq N \\
    v_{j-N0}^{1} e^{\theta_{j-N0} \sigma}, & N + 1 \leq j \leq 2N
\end{cases}
\]

(4.3)

where \(\theta_j = i(\lambda_j^2 - \frac{1}{2})x - 2i(\lambda_j^2 - \frac{1}{2})^2 t\) \((\lambda_j \in D^+), v_{j_0}\) is a constant vector.

by (4.2) it follows from that the N-soliton solutions for the CLL-NLS equation reads

\[
r = 2\sum_{k=1}^{N} \sum_{j=1}^{N} c_k e^{\theta_{k_j}} (M^{-1})_{kj} + 2i \sum_{k=N+1}^{2N} \sum_{j=1}^{N} c_k e^{\theta_{k_j}} (M^{-1})_{kj}
\]

\[
-2i \sum_{k=1}^{N} \sum_{j=N+1}^{2N} c_k e^{\theta_{k_j}} (M^{-1})_{kj} - 2i \sum_{k=N+1}^{2N} \sum_{j=N+1}^{2N} c_k e^{\theta_{k_j}} (M^{-1})_{kj}.
\]

(4.4)

and \(M = (m_{kj})_{2N \times 2N}\) is given by

\[
m_{kj} = \begin{cases} 
    \frac{e^{i(\lambda_k^2 \lambda_j^2 - \alpha_k \alpha_j)}}{e^{i(\lambda_k^2 \lambda_j^2 + \alpha_k \alpha_j)}}, & 1 \leq k, j \leq N, \\
    \frac{e^{iX}}{e^{iX}} e^{i(\lambda_k^2 \lambda_j^2 + \alpha_k \alpha_j)} e^{i(\lambda_k^2 \lambda_j^2 + \alpha_k \alpha_j)}, & 1 \leq k \leq N, N + 1 \leq j \leq 2N, \\
    \frac{e^{iX}}{e^{iX}} e^{i(\lambda_k^2 \lambda_j^2 + \alpha_k \alpha_j)} e^{i(\lambda_k^2 \lambda_j^2 + \alpha_k \alpha_j)}, & 1 \leq j \leq N, N + 1 \leq k \leq 2N, \\
    \frac{e^{iX}}{e^{iX}} e^{i(\lambda_k^2 \lambda_j^2 + \alpha_k \alpha_j)} e^{i(\lambda_k^2 \lambda_j^2 + \alpha_k \alpha_j)}, & N + 1 \leq k, j \leq 2N.
\end{cases}
\]

(4.5)

with \(\theta_j = i(\lambda_j^2 - \frac{1}{2})x - 2i(\lambda_j^2 - \frac{1}{2})^2 t\) \((\lambda_j \in D^+)\).

The simplest situation occurs when \(N = 1\) in formula (4.4). the single-soliton solution is

\[
r = -\frac{2i\xi e^{\theta_{\lambda_1} - \xi_1} (m_{11} + m_{21} - m_{12} - m_{22})}{\det M}
\]

(4.6)

where \(M = (m_{kj})_{2 \times 2}\) is given by

\[
m_{11} = \left| c_1 \left( e^{\theta_{\lambda_1} + \xi_1} - e^{-\theta_{\lambda_1} - \xi_1} \right) \right|, m_{12} = \left| c_1 \left( e^{\theta_{\lambda_1} + \xi_1} + e^{-\theta_{\lambda_1} - \xi_1} \right) \right|, \\
m_{21} = \left| c_1 \left( e^{\theta_{\lambda_1} + \xi_1} + e^{-\theta_{\lambda_1} - \xi_1} \right) \right|, m_{22} = \left| c_1 \left( e^{\theta_{\lambda_1} + \xi_1} - e^{-\theta_{\lambda_1} - \xi_1} \right) \right|.
\]

(4.7)

Letting \(\lambda_1 = \lambda_{11} + i\lambda_{12}, |c_1| = e^{\xi_1}\), then the single-soliton solution can be written as

\[
r = \frac{4\lambda_{11}\lambda_{22} e^{\lambda_{11} \sinh X + i\lambda_{12} \cosh X}}{\lambda_{12}^2 \cosh^2 X + \lambda_{11}^2 \sinh^2 X}.
\]

(4.8)

with

\[
X = 8\lambda_{11}\lambda_{12}(\lambda_{11}^2 - \lambda_{12}^2)t - 2\lambda_{11}\lambda_{12}x - 4\lambda_{11}\lambda_{12}t + \ln|c_1|, \\
y = 2(\lambda_{11}^2 - \lambda_{12}^2)x - 4(\lambda_{11}^2 - 6\lambda_{11}^2 + \lambda_{12}^2 - \lambda_{11}^2 - \lambda_{12}^2)t.
\]
Now we investigate the case for $N = 2$ in formula (4.4), we arrive at the two-soliton solution as follows:

$$
\begin{align*}
& r = -2i \sum_{k=1}^{2} \sum_{j=1}^{2} c_k e^{\theta_j - \theta_l} (M^{-1})_{kj} - 2i \sum_{k=3}^{4} \sum_{j=1}^{2} c_k e^{\theta_j - \theta_l} (M^{-1})_{kj} \\
& + 2i \sum_{k=1}^{4} \sum_{j=3}^{4} c_k e^{\theta_j - \theta_l} (M^{-1})_{kj} + 2i \sum_{k=3}^{4} \sum_{j=3}^{4} c_k e^{\theta_j - \theta_l} (M^{-1})_{kj},
\end{align*}
$$

(4.9)

and $M = (m_{ij})_{4 \times 4}$ is given by

$$
\begin{align*}
& m_{11} = \frac{c_1 e^{\theta_1 + \theta_2} - e^{-\theta_i - \theta_j}}{\lambda_i^2 - \lambda_j^2}, m_{12} = \frac{c_1 c_2 e^{\theta_1 + \theta_2} - e^{-\theta_i - \theta_j}}{\lambda_i^2 - \lambda_2^2}, \\
& m_{13} = \frac{c_1 c_2 e^{\theta_1 + \theta_2} + e^{-\theta_i - \theta_j}}{\lambda_i^2 + \lambda_j^2}, m_{14} = \frac{c_1 e^{\theta_1 + \theta_2} + e^{-\theta_i - \theta_j}}{\lambda_i^2 + \lambda_2^2}, \\
& m_{21} = \frac{c_2 e^{\theta_1 + \theta_2} - e^{-\theta_i - \theta_j}}{\lambda_i^2 - \lambda_j^2}, m_{22} = \frac{c_2 e^{\theta_1 + \theta_2} - e^{-\theta_i - \theta_j}}{\lambda_i^2 - \lambda_2^2}, \\
& m_{23} = \frac{c_2 e^{\theta_1 + \theta_2} + e^{-\theta_i - \theta_j}}{\lambda_i^2 + \lambda_j^2}, m_{24} = \frac{c_2 e^{\theta_1 + \theta_2} + e^{-\theta_i - \theta_j}}{\lambda_i^2 + \lambda_2^2}, \\
& m_{31} = \frac{c_1 e^{\theta_1 + \theta_2} + e^{-\theta_i - \theta_j}}{-\lambda_i^2 - \lambda_j^2}, m_{32} = \frac{c_2 e^{\theta_1 + \theta_2} + e^{-\theta_i - \theta_j}}{-\lambda_i^2 - \lambda_2^2}, \\
& m_{33} = \frac{c_1 e^{\theta_1 + \theta_2} - e^{-\theta_i - \theta_j}}{-\lambda_i^2 + \lambda_j^2}, m_{34} = \frac{c_1 e^{\theta_1 + \theta_2} - e^{-\theta_i - \theta_j}}{-\lambda_i^2 + \lambda_2^2}, \\
& m_{41} = \frac{c_1 e^{\theta_1 + \theta_2} + e^{-\theta_i - \theta_j}}{-\lambda_i^2 - \lambda_j^2}, m_{42} = \frac{c_1 e^{\theta_1 + \theta_2} + e^{-\theta_i - \theta_j}}{-\lambda_i^2 - \lambda_2^2}, \\
& m_{43} = \frac{c_1 e^{\theta_1 + \theta_2} - e^{-\theta_i - \theta_j}}{-\lambda_i^2 + \lambda_j^2}, m_{44} = \frac{c_1 e^{\theta_1 + \theta_2} - e^{-\theta_i - \theta_j}}{-\lambda_i^2 + \lambda_2^2}.
\end{align*}
$$

(4.10)

Further, we can also write the obtained N-soliton solutions (4.4) into the form of the determinant ratio

$$
\begin{align*}
& r = 2i \text{det} \tilde{M}_{12} \text{det} M, \\
& \text{where } \tilde{M}_{12} \text{ is}
\end{align*}
$$

(4.11)

$$
\tilde{M}_{12} = \begin{pmatrix}
0 & f_1 \\
g_2 & M
\end{pmatrix}
$$

(4.12)

where the matrix $M$ is defined by Eq (4.5), and $f_1 = (v_{11}, v_{21}, \cdots v_{2N_1})$, $g_2 = (\hat{v}_{11}, \hat{v}_{21}, \cdots \hat{v}_{2N_1})^T$

5. Conclusions

In this work, by applying RH method, we establish the N-soliton solutions for the CLL-NLS equation. First of all, the Lax pair of the coupled CLL-NLS equation are transformed to obtain the corresponding Jost solution. Then we study spectrum analysis and construct the particular RH problem, which is non-regular case. Then we derive the scattering data including the continuous scattering data $\{v_{ij}, \hat{v}_{ij}\}$ and the discrete scattering data $\{\lambda_i, \hat{\lambda}_j, v_j, \hat{v}_j\}$. With aid of reconstructing the potential, we solve the N-soliton solutions for the CLL-NLS. Finally we obtain a simple and compact N-soliton solutions formula.
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