On the approximation for singularly perturbed
stochastic wave equations

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Abstract

We explore the relation between fast waves, damping and imposed
noise for different scalings by considering the singularly perturbed
stochastic nonlinear wave equations

\[ \nu u_{tt} + u_t = \Delta u + f(u) + \nu^\alpha \dot{W} \]

on a bounded spatial domain. An asymptotic approximation to the
stochastic wave equation is constructed by a special transformation and
splitting of \( \nu u_t \). This splitting gives a clear description of the structure
of \( u \). The approximating model, for small \( \nu > 0 \), is a stochastic non-
linear heat equation for exponent \( 0 \leq \alpha < 1 \), and is a deterministic
nonlinear wave equation for exponent \( \alpha > 1 \).

Keywords Singular perturbation, stochastic wave equations, asymptotic
approximation.

1 Introduction

Our stochastic model is motivated by some material continuum in some
domain \( D \subset \mathbb{R}^n \), \( 1 \leq n \leq 3 \). The continuum is made of ‘particles’ with
‘displacement’ field \( u(t,x) \) and ‘velocity’ field \( v(t,x) \). The motion of the
particles in the continuum in a stochastic force field \( \sigma \dot{W} \), motivated by
Newton’s law, is assumed to be described by the following stochastic partial

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differential equations [1]

\[ u^{\nu}(t, x) = v^{\nu}(t, x), \quad (1) \]
\[ \nu u^{\nu}_t(t, x) = -kv^{\nu}(t, x) + \Delta u^{\nu}(t, x) + f(u^{\nu}(t, x)) + \sigma \dot{W}(t, x), \quad (2) \]
\[ u^{\nu}(0, x) = u_0, \quad v^{\nu}(0, x) = u_1, \quad (3) \]

for times \( t \geq 0 \), and locations \( x \in D \), with zero Dirichlet boundary condition on \( \partial D \). Here small \( \nu \) is the ‘density’ of the particles: we explore the singular limit as \( \nu \to 0 \) and so label the dependent fields with superscript \( \nu \). Damping is proportional to the velocity \( v^{\nu} \) with constant \( k \). The Laplacian governs near neighbour, particle-particle, quasi-elastic interaction forces, and a nonlinear reaction is characterised by \( f(u) \). The externally imposed stochastic force field is \( \sigma \dot{W}(t, x) \) where \( W(t, x) \) is an \( L^2(D) \) valued Wiener process defined on some complete probability space \( (\Omega, \mathcal{F}, P) \), and is assumed to be of strength that scales according to \( \sigma = \nu^\alpha \). For exponent \( \alpha = 0 \), the approximation of displacements \( u^{\nu} \), as \( \nu \to 0 \), is called the infinite dimensional Smolukowski–Kramers approximation which has been proved valid in the limit by estimating the remainder term [1, 2]. Our recent work [5] applied an averaging method to approximate the displacement field \( u^{\nu} \) for the case \( \sigma = \nu^\alpha \) with exponent \( 0 \leq \alpha \leq 1/2 \). Both of these methods are significantly complicated due to the coupling of displacement \( u \) and velocity \( v \) in the remainder term.

Here we apply a relatively simple method to derive suitable approximations for equations (1)–(3) with \( \sigma = \nu^\alpha \), for exponent \( \alpha \in [0, 1) \cup (1, \infty) \). We apply the following useful splitting of the velocity \( u^{\nu}_t \),

\[ u^{\nu}_t(t) = \nu \bar{v}^{\nu}_1(t) + v^{\nu}_2(t) + \nu^{\alpha-1/2} \bar{v}^{\nu}_3(t), \quad (4) \]

to avoid directly estimating the remainder term. The three parts of the above splitting are the initial value part, the mean value part, and the diffusion part, respectively. This decomposition gives a clear structure for the displacement \( u^{\nu} \) with

\[ u^{\nu}(t) - u_0 = \frac{1}{\nu} \int_0^t \bar{v}^{\nu}_1(s) \, ds + \int_0^t \bar{v}^{\nu}_2(s) \, ds + \nu^{\alpha-1/2} \int_0^t \bar{v}^{\nu}_3(s) \, ds. \]

The parts \( \bar{v}^{\nu}_1 \) and \( \bar{v}^{\nu}_3 \) satisfy linear equations, and section 3 establishes

\[ \frac{1}{\nu} \int_0^t \bar{v}^{\nu}_1(s) \, ds = O(\nu) \quad \text{and} \quad \nu^{\alpha-1/2} \int_0^t \bar{v}^{\nu}_3(s) \, ds = O(\nu^\alpha) \quad \text{as} \ \nu \to 0. \]

The mean part \( \bar{v}^{\nu}_2(t) \) of the velocity is \( O(1) \) as \( \nu \to 0 \) for \( t \in [0, T] \) with any fixed time \( T \). Then for small \( \nu \), section 3 determines which term is a high order term and gives an asymptotic approximation of the displacement \( u^{\nu} \).

Here one interesting case is when the exponent \( \alpha = 1 \). In this case there are two terms with the same order \( O(\nu) \) as \( \nu \to 0 \). Then if we keep all the
terms, the approximation to the displacement $u^\nu$ is just itself which is no modelling simplification. This case will be discussed further research.

Because of its motivation by physical continuum problems of wave motion in some random media [3], the system (1)–(3) is called a stochastic wave equation. For small $\nu$ and the particular case of $\sigma = \nu^{1/2}$, Lv and Wang [6, 8] studied the limit behaviour as $\nu \to 0$: in this case the random dynamics of (1)–(3) was proved to be described by that of the nonlinear heat equation

$$u_t(t,x) = \Delta u(t,x) + f(u(t,x)), \quad u(0,x) = u_0. \quad (5)$$

This paper extends this earlier research by approximating the behaviour of the solution on finite time interval $[0,T]$, $T > 0$, for the more general case of $\sigma = \nu^\alpha$ with any $\alpha \in [0,1) \cup (1,\infty)$. The tightness in the space $C(0,T;L^2(D))$, compact in sense of probability, has been proved in previous work [2, 8]. Consequently, here we just need to approximate the displacement $u^\nu$ in a weak sense; that is, we consider the approximation of the inner product $\langle u^\nu, \varphi \rangle$ in the space $C(0,T)$ for testing function $\varphi \in C^2(D \times [0,T])$ with $\varphi$ vanishing on the boundary $\partial D$.

Section 2 first gives some preliminaries and the main result, Theorem 4. Then section 3 details the proof.

2 Preliminary

Let $D \subset \mathbb{R}^n$, $1 \leq n \leq 3$, be a regular domain with boundary $\partial D$. Denote by $L^2(D)$ the Lebesgue space of square integrable real valued functions on $D$, which is a Hilbert space with inner product

$$\langle u,v \rangle = \int_D u(x)v(x) \, dx, \quad u, v \in L^2(D).$$

Write the norm on $L^2(D)$ by $\|u\|_0 = \langle u, u \rangle^{1/2}$. Define the following abstract operator

$$Au = -\Delta u, \quad u \in \text{Dom}(A) = \{u \in L^2(D) : \Delta u \in L^2(D), \ u|_{\partial D} = 0\}.$$  

Denoted by $\{\lambda_k\}$, assume the eigenvalues of operator $A$ satisfy $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$, and $\lambda_k \to \infty$ as $k \to \infty$. For any $s \geq 0$, denote by $H^s(D)$ the usual Sobolev space $W^{s,2}(D)$ and by $H^s_0(D)$ the closure of $C^\infty_0(D)$ in $H^s(D)$. In the space $H^s_0(D)$ we use the equivalent norm

$$\|u\|_s = \|A^{s/2}u\|_0, \quad u \in H^s_0(D).$$

We also denote the dual space of $H^s_0$ by $H^{-s}$. Here specify that the noise magnitude scales as $\sigma = \nu^\alpha$, $0 < \nu \leq 1$, for exponent $\alpha \geq 0$, in equation (2);
that is, we consider the following stochastic equations

\[ u_\nu' = v_\nu', \quad u_\nu'(0) = u_0, \]  
\[ v_\nu' = \frac{1}{\nu} [ -v_\nu - Au_\nu + f(u_\nu) ] + \nu^{\alpha - 1} \dot{W}, \quad u_\nu'(0) = u_1. \]

Hereafter we non-dimensionalise the time scale with the drag rate so that, in effect, the drag coefficient is one. We assume \( \{W(t,x)\}_{t \in \mathbb{R}} \) is an \( L^2(D) \)-valued, two sided, Wiener process, defined on a complete probability space \( (\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P}) \) with covariance operator \( Q \) such that

\[ Q e_k = b_k e_k, \quad k = 1, 2, \ldots, \]

where \( \{e_k\} \) is a complete orthonormal system in \( H \) and \( \{b_k\} \) is a bounded sequence of non-negative real numbers. Then the noise process \( W(t,x) \) has the spectral expansion

\[ W(t,x) = \sum_{k=1}^{\infty} \sqrt{b_k} e_k w_k(t), \]

where \( w_k \) are real, mutually independent, standard scalar Brownian motions \([7]\). Further, we assume boundedness of the sums

\[ \text{tr} \, Q = \sum_{k=1}^{\infty} b_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \lambda_k b_k < \infty. \]  

**Assumption 1.** For the nonlinearity \( f \) we assume

1. \(|f(s)| \leq C_1(1 + |s|^3), \quad |f'(s)| \leq C_2(1 + |s|^2); \]
2. \( F(s) \leq -C_3(|s|^4 - 1), \quad sf(s) \leq -C_4(F(s) - 1); \)

for some positive constants \( C_i, \, i = 1, 2, 3, 4 \), and where \( F(s) = \int_0^s f(r) \, dr \).

One simple example satisfying these assumptions is \( f(u) = u - u^3 \).

Then we have the following theorem.

**Theorem 2.** Assume that the boundedness \([5]\) and Assumption \([2]\) hold. For any \((u_0, u_1) \in H_0^1(D) \times L^2(D)\), there is a unique solution \((u_\nu', v_\nu')\) to \((6)-(7)\), with

\[ u_\nu' \in L^2(\Omega, C(0,T;H_0^1(D))), \quad v_\nu' \in L^2(\Omega, C(0,T;L^2(D))), \]

for any \( T > 0 \). Moreover, for any \( T > 0 \) there is a positive constant \( C_T \) which is independent of \( \nu \) such that the expectation

\[ \mathbb{E} \sup_{0 \leq t \leq T} ||u_\nu'(t)||_1 \leq C_T (||u_0||^2_1 + ||u_1||^2_0), \]

and \( \{u_\nu'\}_{0 < \nu \leq 1} \) is tight in the space \( C(0,T;L^2(D)) \).
Proof. To prove the existence of the solution we define 
\[ A = \begin{bmatrix} 0, & \text{id}_{L^2(D)} \\ \frac{1}{nu}, & -\frac{1}{nu} \end{bmatrix}, \quad F(u^\nu, v^\nu) = \begin{bmatrix} 0 \\ \frac{1}{nu}f(u^\nu) \end{bmatrix} \quad \text{and} \quad W(t) = \begin{bmatrix} 0 \\ \sqrt{\nu}W(t) \end{bmatrix}. \]

Let \( \Phi = (u^\nu, v^\nu) \), so equation (6)–(7) can be rewritten in the following abstract stochastic evolutionary form
\[
\dot{\Phi} = A\Phi + F(\Phi) + \dot{W}, \quad \Phi(0) = (u_0, u_1).
\]
Notice that operator \( A \) generates a strong continuous semigroup and the nonlinearity \( F \) is locally Lipschitz continuous, then by a standard method for stochastic evolutionary equations [7] we have the first part of the theorem.

For \( 0 \leq \alpha < 1/2 \) the energy estimate for \((u^\nu, v^\nu)\) and tightness result can be obtained via a similar argument to that of Cerrai and Freidlin [2], and for \( \alpha \geq 1/2 \) the energy estimate and tightness were obtained by Lv and Wang [6]. The proof is complete.

In the following approach we need the following lemma on weak convergence of a sequence of functions due to Lions [4].

Lemma 3. For any given functions \( h^\nu \) and \( h \in L^p([0, T] \times D) \) \((1 < p < \infty)\), if \( \|h^\nu\|_{L^p([0, T] \times D)} \leq C \) for some positive constant \( C \), and \( h^\nu \rightarrow h \) on \([0, T] \times D\) almost everywhere as \( \nu \rightarrow 0 \), then \( h^\nu \rightarrow h \) weakly in \( L^p([0, T] \times D) \).

Now we give the main theorem on the approximation of the displacement \( u^\nu \) in our stochastic wave equation.

Theorem 4. Assume that the boundedness (8) and Assumption 1 hold, and \((u_0, u_1) \in H^1_0(D) \times L^2(D)\). If exponent \( 0 \leq \alpha < 1 \), for any \( T > 0 \), and for small \( \nu > 0 \), then with probability one
\[
\nu^{-\alpha}\|u^\nu - \bar{u}^\nu\|_{C([0, T]; L^2(D))} \rightarrow 0, \quad \text{as} \ \nu \rightarrow 0,
\]
with the approximation \( \bar{u}^\nu \) solving the stochastic nonlinear heat equation
\[
\bar{u}^\nu_t = \Delta \bar{u}^\nu + f(\bar{u}^\nu) + \nu^\alpha \hat{W}, \quad \bar{u}^\nu(0) = u_0.
\]

Conversely, if \( \alpha > 1 \),
\[
\nu^{-1}\|u^\nu - \bar{u}^\nu\|_{C([0, T]; L^2(D))} \rightarrow 0, \quad \text{as} \ \nu \rightarrow 0,
\]
with the approximation \( \bar{u}^\nu \) solving the deterministic nonlinear wave equation
\[
\nu \bar{u}_{tt} + \bar{u}_t = \Delta \bar{u}^\nu + f(\bar{u}^\nu), \quad \bar{u}^\nu(0) = u_0, \quad \bar{u}^\nu_t(0) = u_1.
\]

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3 Approximation: proof of Theorem 4

By Theorem 2, \( \{u^\nu(t)\}_{0<\nu\leq 1} \) is tight in the space \( C(0,T;L^2(D)) \), so we approximate the displacement \( u^\nu \) in a weak sense: we approximate \( \langle u^\nu, \varphi \rangle \) for any \( \varphi \in C^2([0,T] \times D) \) with \( \varphi|_{\partial D} = 0 \).

In order to avoid the coupling between the displacement \( u^\nu \) and the velocity \( v^\nu \), we scale the velocity field as

\[
\tilde{v}^\nu = \nu v^\nu. \tag{13}
\]

Then

\[
\begin{align*}
\tilde{u}^\nu_t &= \frac{1}{\nu} \tilde{v}^\nu, \quad \tilde{u}^\nu(0) = u_0, \\
\tilde{v}^\nu_t &= -\frac{1}{\nu} \tilde{v} + \Delta u^\nu + f(u^\nu) + \nu^a \dot{W}, \quad \tilde{v}^\nu(0) = \nu u_1.
\end{align*}
\]

Further, we make the decomposition

\[
\begin{align*}
\tilde{v}^\nu &= \tilde{v}^\nu_1 + \nu \tilde{v}^\nu_2 + \nu^{a+1/2} \tilde{v}^\nu_3, \tag{14} \\
\text{where} \quad \tilde{v}^\nu_{1,t} &= -\frac{1}{\nu} \tilde{v}^\nu_1, \quad \tilde{v}^\nu_1(0) = \nu u_1, \\
\tilde{v}^\nu_{2,t} &= -\frac{1}{\nu} [\tilde{v}^\nu_2 - \Delta u^\nu - f(u^\nu)], \quad \tilde{v}^\nu_2(0) = 0, \\
\tilde{v}^\nu_{3,t} &= -\frac{1}{\nu} \tilde{v}^\nu_3 + \frac{1}{\sqrt{\nu}} \dot{W}, \quad \tilde{v}^\nu_3(0) = 0. \tag{15}
\end{align*}
\]

Then

\[
\begin{align*}
\tilde{u}^\nu &= \frac{1}{\nu} \tilde{v}^\nu + \tilde{v}^\nu_2 + \nu^{a-1/2} \tilde{v}^\nu_3, \quad \tilde{u}^\nu(0) = u_0. \tag{18}
\end{align*}
\]

The decomposition of \( \tilde{v}^\nu \) makes the problem easier. The two sdes \([14]\) and \([17]\) for the two components \( \tilde{v}^\nu_1 \) and \( \tilde{v}^\nu_3 \) are just linear sdes whose properties are well known. The properties of \( \tilde{v}^\nu_2 \) can be derived straightforwardly from the pde \([16]\) by the estimates in Theorem 2. We state the following results.

Lemma 5. Assume that the boundedness \([3]\) and Assumption \([1]\) hold. Let \( u_1 \in L^2(D) \), then for any \( \varphi \in C^2([0,T] \times D) \) with \( \varphi|_{\partial D} = 0 \),

\[
\begin{align*}
&\frac{1}{\nu} \int_0^t \langle \tilde{v}^\nu_1(s), \varphi(s) \rangle ds \to 0, \quad 0 \leq t \leq T, \tag{19} \\
\text{and} \quad &\nu^{-1/2} \int_0^t \langle \tilde{v}^\nu_3(s), \varphi(s) \rangle ds \to \int_0^t \langle \varphi(s), dW(s) \rangle, \quad 0 \leq t \leq T, \tag{20}
\end{align*}
\]

in \( L^2(\Omega) \), as \( \nu \to 0 \).

Proof. The proof is direct. First,

\[
\tilde{v}^\nu_1(t) = \nu u_1 e^{-t/\nu}.
\]
Then for $\varphi \in C^2([0,T] \times D)$

$$\frac{1}{\nu} \int_0^t \langle \bar{v}_1^\nu(s), \varphi(s) \rangle \, ds = \int_0^t \langle u_1, \varphi(s) \rangle e^{-s/\nu} \, ds$$

$$= \nu \int_0^t \langle u_1, \varphi(\nu \tau) \rangle e^{-\tau} \, d\tau \to 0, \quad \text{as } \nu \to 0,$$

uniformly on $[0,T]$, which yields the first convergence. Second, for any $\varphi \in C^2([0,T] \times D)$ with $\varphi|_{\partial D} = 0$ by equation (17),

$$\nu^{-1/2} \int_0^t \langle \bar{v}_3^\nu(s), \varphi(s) \rangle \, ds = -\sqrt{\nu} \int_0^t \langle \bar{v}_3^\nu(s), \varphi(s) \rangle \, ds + \int_0^t \langle \varphi(s), dW(s) \rangle$$

$$= -\sqrt{\nu} \langle \bar{v}_3^\nu(t), \varphi(t) \rangle + \sqrt{\nu} \int_0^t \langle \bar{v}_3^\nu(s), \varphi(s) \rangle \, ds$$

$$+ \int_0^t \langle \varphi(s), dW(s) \rangle. \quad (21)$$

So it remains to show that $\bar{v}_3^\nu(t)$ is uniformly bounded in the space $L^2(\Omega, L^2(D))$. By equation (17), applying Itô formula to $\|\bar{v}_3^\nu\|^2_0$ gives

$$\frac{1}{2} \frac{d}{dt} \|\bar{v}_3^\nu(t)\|^2_0 = -\frac{1}{\nu} \|\bar{v}_3^\nu\|^2_0 + \frac{1}{2\nu} \text{tr } Q + \frac{1}{\sqrt{\nu}} \langle \bar{v}_3^\nu, \dot{W} \rangle.$$

Then by the Gronwall lemma

$$\mathbb{E} \|\bar{v}_3^\nu(t)\|^2_0 \leq \text{tr } Q, \quad t \geq 0.$$

The proof is complete. \qed

**Lemma 6.** Assume the conditions in Theorem 2 holds, then there is a parameter $\nu$ and independent positive constant $C_T$ such that

$$\mathbb{E} \|\bar{v}_3^\nu(t)\|_{-1} \leq C_T, \quad 0 \leq t \leq T.$$

**Proof.** For any $\psi \in H^1_0(D)$, from equation (16)

$$\frac{d}{dt} \langle \bar{v}_2^\nu(t), \psi \rangle = -\frac{1}{\nu} \langle \bar{v}_2^\nu, \psi \rangle - \frac{1}{\nu} \langle \nabla u^\nu, \nabla \psi \rangle + \frac{1}{\nu} \langle f(u^\nu), \psi \rangle.$$

Then

$$\langle \bar{v}_2^\nu(t), \psi \rangle = \frac{1}{\nu} e^{-t/\nu} \int_0^t e^{s/\nu} \left[ -\langle \nabla u^\nu(s), \nabla \psi \rangle + \langle f(u^\nu(s)), \psi \rangle \right] \, ds.$$

By the estimates in Theorem 2 and the embedding $H^1_0(D) \subset L^6(D)$ for $1 \leq n \leq 3,

$$\mathbb{E} |\langle \bar{v}_2^\nu(t), \psi \rangle| \leq C_T \|\psi\|_1, \quad 0 \leq t \leq T.$$

The proof is complete. \qed
From the above lemma we prove the main Theorem. First, for any \( \kappa > 0 \), by the tightness of displacement \( u^\nu \) in the space \( C(0, T; L^2(D)) \), there is a compact set \( B_\kappa \subset C(0, T; L^2(D)) \) such that

\[
\mathbb{P}\{u^\nu \in B_\kappa\} \geq 1 - \kappa/2. \tag{22}
\]

By the Markov inequality and the estimate in Lemma for any \( \kappa > 0 \) there is a positive constant \( C_T^\kappa \) such that

\[
\mathbb{P}\{\|\bar{u}^\nu_2(t)\|_{-1} \leq C_T^\kappa\} \geq 1 - \kappa/2. \tag{23}
\]

Then for any \( \kappa > 0 \), define a probability space \( (\Omega_\kappa, \mathcal{F}_\kappa, \mathbb{P}_\kappa) \)

\[\Omega_\kappa = \{\omega \in \Omega : \text{events (22) and (23) hold}\}, \quad \mathcal{F}_\kappa = \{F \cap \Omega_\kappa : F \in \mathcal{F}\}, \]

and for any \( F \in \mathcal{F}_\kappa \)

\[
\mathbb{P}_\kappa(F) = \frac{\mathbb{P}(F \cap \Omega_\kappa)}{\mathbb{P}(\Omega_\kappa)}. \]

In the following we restrict our problem to the above new probability space. For any \( \omega \in \Omega_\kappa \), the convergence \( (19) \) still holds. The convergence \( (20) \) is in the \( L^2(\Omega) \) sense which yields the convergence for \( \mathbb{P} \) almost all \( \omega \in \Omega \), then we also have the convergence \( (20) \) for \( \mathbb{P}_\kappa \) almost all \( \omega \in \Omega_\kappa \). So we can assume that for all \( \omega \in \Omega_\kappa \), the convergence \( (20) \) holds.

Furthermore, we establish the limit

\[
f(u^n) \to f(u) \quad \text{weakly in } L^2(0, T; L^2(D)) \tag{24}
\]

for any \( u^n \to u \) in \( C(0, T; L^2(D)) \). By the embedding \( H^6_0(D) \subset L^6(D) \) we have \( \|f(u^n)\|_{L^6(0, T; L^2(D))} \leq C_T \) for some positive constant \( C_T \), and by the strong convergence of \( u^n \to u \), \( f(u^n(t, x)) \to f(u(t, x)) \) on \([0, T] \times D\) almost everywhere. Then Lemma gives the limit.

Next we give an asymptotic approximation to the displacement \( u^\nu \). For this we consider \( \langle u^\nu(t), \varphi(t) \rangle \) with \( \varphi \in C^2([0, T] \times D) \) and \( \varphi|_{\partial D} = 0 \). From equation

\[
\langle u^\nu(t), \varphi(t) \rangle - \langle u_0, \varphi(0) \rangle - \int_0^t \langle u^\nu(s), \varphi_t(s) \rangle \, ds
\]

\[
= \frac{1}{\nu} \int_0^t \langle \bar{u}^\nu_2(s), \varphi(s) \rangle \, ds + \int_0^t \langle \bar{v}^\nu_2(s), \varphi(s) \rangle \, ds + \nu^{\alpha-1/2} \int_0^t \langle \bar{v}^\nu_3(s), \varphi(s) \rangle \, ds.
\]

From equation

\[
\int_0^t \langle \bar{v}^\nu_2(s), \varphi(s) \rangle \, ds = \int_0^t \langle u^\nu(s), \Delta \varphi(s) \rangle \, ds + \int_0^t \langle f(u^\nu(s)), \varphi(s) \rangle \, ds
\]

\[
- \nu \langle \bar{v}^\nu_2(t), \varphi(t) \rangle + \nu \int_0^t \langle \bar{v}^\nu_2(s), \varphi_t(s) \rangle \, ds.
\]
By the definition of $\Omega_\kappa$, and Lemmas 5 and 6
\begin{align*}
\frac{1}{\nu} \int_0^t \langle \bar{v}'_1(s), \varphi(s) \rangle \, ds = O(\nu), \quad \nu \langle \bar{v}'_2(t), \varphi(t) \rangle = O(\nu),
\end{align*}
and
\begin{align*}
\nu \int_0^t \langle \bar{v}'_2(s), \varphi_t(s) \rangle \, ds = O(\nu).
\end{align*}
Further, by (21)
\begin{align*}
\nu^{\alpha-1/2} \int_0^t \langle \bar{v}'_3(s), \varphi(s) \rangle \, ds = \nu^\alpha \int_0^t \langle \varphi(s), dW(s) \rangle + O(\nu^{\alpha+1/2}).
\end{align*}
Then
\begin{align*}
\langle u' (t) , \varphi(t) \rangle - \langle u_0 , \varphi(0) \rangle &- \int_0^t \langle u'(s) , \varphi_t(s) \rangle \, ds - \int_0^t \langle u'(s) , \Delta \varphi(s) \rangle \, ds \\
&- \int_0^t \langle f (u'(s)) , \varphi(s) \rangle \, ds \\
&= \nu^{\alpha-1/2} \int_0^t \langle \bar{v}'_3(s) , \varphi(s) \rangle \, ds + \frac{1}{\nu} \int_0^t \langle \bar{v}'_1(s), \varphi(s) \rangle \, ds \\
&- \nu \langle \bar{v}'_2(t) , \varphi(t) \rangle + \nu \int_0^t \langle \bar{v}'_2(s) , \varphi_t(s) \rangle \, ds \\
&= \nu^\alpha \int_0^t \langle \varphi(s) , dW(s) \rangle + O(\nu^{\alpha+1/2}) + O(\nu).
\end{align*}
(25)

Now for exponent $0 \leq \alpha < 1$, noticing the convergence (24) and neglecting the $o(\nu^\alpha)$ terms we have the following equation:
\begin{align*}
\langle \bar{u}'(t) , \varphi(t) \rangle - \langle u_0 , \varphi(0) \rangle &- \int_0^t \langle \bar{u}'(s) , \varphi_t(s) \rangle \, ds - \int_0^t \langle \bar{u}'(s) , \Delta \varphi(s) \rangle \, ds \\
&- \int_0^t \langle f (\bar{u}'(s)) , \varphi(s) \rangle \, ds = \nu^\alpha \int_0^t \langle \varphi(s) , dW(s) \rangle.
\end{align*}
Then we deduce the following approximation equation holds:
\begin{align*}
\bar{u}' = \Delta \bar{u}' + f (\bar{u}') + \nu^\alpha \dot{W}, \quad \bar{u}'(0) = u_0.
\end{align*}
(26)

For exponent $\alpha > 1$ we need to show the rate of decay in $\nu$ of $v'_1$ and $v'_2$ as $\nu \to 0$. First, as $\{u'(\nu)\}_\nu \subset B_\kappa$ which is compact in $C(0,T; L^2(D))$, for any sequence, there are a subsequence, say $u'(\nu_n)$, with $\nu_n \to 0$ as $n \to \infty$, and $\bar{u} \in C(0,T; L^2(D))$ with $\bar{u}(0) = u_0$ such that
\begin{align*}
\nu_n &\to 0, \quad n \to \infty.
\end{align*}

First we assume
\begin{align*}
\Delta \bar{u} + f(\bar{u}) \neq 0 \quad \text{in } H^{-1} \quad \text{for all } t \in [0,T].
\end{align*}
By the estimates in Lemma 6, and by the convergence (24),
\[ \alpha > 0. \]
Then for exponent \( \nu \) in (14),
\[ \nu \Phi(\nu) \]
we have the following approximation equation
\[ \nu \Phi(\nu) = O(\nu) \]
instead of \( o(\nu) \) as \( \nu \to 0. \)

Further from equation (15), for any \( \varphi \in C^2([0,T] \times D) \) with \( \varphi_{|\partial D} = 0, \)
\[ \frac{1}{\nu} \langle \bar{v}^\nu_1(t), \varphi(t) \rangle - \langle u_1, \varphi(0) \rangle - \frac{1}{\nu} \int_0^t \langle \bar{v}^\nu_1(s), \varphi(t) \rangle ds = -\frac{1}{\nu^2} \int_0^t \langle \bar{v}^\nu_1(s), \varphi(s) \rangle ds. \]
Then
\[ \frac{1}{\nu^2} \int_0^t \langle \bar{v}^\nu_1(s), \varphi(s) \rangle ds \to \langle u_1, \varphi(0) \rangle, \quad \nu \to 0. \quad (28) \]

Then for exponent \( \alpha > 1, \) in the asymptotic expansion (25) for small \( \nu, \)
eglecting the \( o(\nu) \) term consisting of \( \bar{v}^\nu_2, \) and by the transformation (13) and decomposition (14),
\[ \langle u^\nu(t), \varphi(t) \rangle - \langle u_0, \varphi(0) \rangle - \int_0^t \langle u^\nu(s), \varphi(t) \rangle ds - \int_0^t \langle u^\nu(s), \Delta \varphi(s) \rangle ds \]
\[ - \int_0^t \langle f(u^\nu(s)), \varphi(s) \rangle ds \]
\[ = \frac{1}{\nu} \int_0^t \langle \bar{v}^\nu_1(s), \varphi(s) \rangle ds - \nu \langle \bar{v}^\nu_2(t), \varphi(t) \rangle + \nu \int_0^t \langle \bar{v}^\nu_2(s), \varphi(t) \rangle ds \]
\[ = -\nu \langle v^\nu(t), \varphi(t) \rangle + \nu \int_0^t \langle v^\nu(s), \varphi(t) \rangle ds + \nu \langle u_1, \varphi(0) \rangle \]
\[ + \nu^{\alpha+1/2} \langle \bar{v}^\nu_3(t), \varphi(t) \rangle - \nu^{\alpha+1/2} \int_0^t \langle \bar{v}^\nu_3(s), \varphi(t) \rangle ds. \]
Then noticing that velocity \( v^\nu = u^\nu_1, \) and neglecting the \( O(\nu^{\alpha+1/2}) \) terms, we have the following approximation equation
\[ \nu \bar{u}^\nu + \bar{u}^\nu = \Delta \bar{u}^\nu + f(\bar{u}^\nu). \quad (29) \]
Second if for $t \in [0, T]$

$$\Delta \bar{u} + f(\bar{u}) = 0, \quad \bar{u}(0) = u_0$$

in $H^{-1}$. Then $\bar{u}$ is a stationary solution of (29).

The above approximation is in the sense of $\mathbb{P}_\kappa$ almost surely. Then by the arbitrary choice of $\kappa$, and the well-posedness of (26) and (29), this establishes the approximation with $\mathbb{P}$ probability one.

This completes our proof of the main Theorem 4 on the approximations of the stochastic wave equation (1)–(3) for different scaling of the noise process.

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