Regression for partially observed variables and nonparametric quantiles of conditional probabilities

Odile Pons
INRA, Mathématiques,
78352 Jouy en Josas cedex, France
e-mail: Odile.Pons@jouy.inra.fr

Abstract: Efficient estimation under bias sampling, censoring or truncation is a difficult question which has been partially answered and the usual estimators are not always consistent. Several biased designs are considered for models with variables \((X, Y)\) where \(Y\) is an indicator and \(X\) an explanatory variable, or for continuous variables \((X, Y)\). The identifiability of the models are discussed. New nonparametric estimators of the regression functions and conditional quantiles are proposed.

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Contents

1. Introduction .................................................. 1
2. Bias depending on the value of \(Y\) .................. 2
3. Bias due to truncation on \(X\) .................. 4
4. Truncation of a response variable in a nonparametric regression model 5
5. Truncation and censoring of \(Y\) in a nonparametric model 8
6. Observation by interval .................................. 11
References .................................................. 12

1. Introduction

Let \((X_i, Y_i)_{i \leq n}\) be a sample of the variable set \((X, Y)\) where \(Y\) is an indicator variable and \(X\) is an explanatory variable. Conditionally on \(X\), \(Y\) follows a Bernoulli distribution with parameter \(p(x) = \Pr(Y = 1|X = x)\). Usual examples are response variables \(Y\) to a dose \(X\) or to an expository time \(X\), economic indicators. The variable \(X\) may be observed at fixed values \(x_i, i \in \{1, \ldots, m\}\) on a regular grid \(\{1/m, \ldots, 1\}\) or at irregular fixed or random times \(t_j, j \leq n\), for a continuous process \((X_t)_{t \leq T}\).

Exponential linear models with known link functions are often used, especially the logistic regression model defined by \(p(x) = e^{\psi(x)}\{1 + e^{\psi(x)}\}^{-1}\) with a
parametric function $\psi$. The inverse function of $p$ is easily estimated using maximum likelihood estimators of the parameters and many authors have studied confidence sets for the parameters and the quantiles of the model.

In a nonparametric setting and for discrete sampling design with several independent observations for each value $x_j$ of $X$, the likelihood is written

\[
L_n = \prod_{i=1}^{n} p(X_i)^{Y_i} \{1 - p(X_i)\}^{1 - Y_i} = \prod_{j=1}^{m} \prod_{i=1}^{n} \{(p(x_j))^{Y_i} \{1 - p(x_j)\}^{1 - Y_i}\}^{1\{x_i = x_j\}}.
\]

The maximum likelihood estimator of $p(x_j)$ is the proportion of individuals with $Y_i = 1$ as $X_i = x_j$,

\[
\hat{p}_{1n}(x_j) = \frac{1}{\sum_{i=1}^{n} 1\{x_i = x_j\}} \sum_{i=1}^{n} Y_i 1\{x_i = x_j\}, j = 1, \ldots, m.
\]

Regular versions of this estimator are obtained by kernel smoothing or by projections on a regular basis of functions, especially if the variable $X$ is continuous. Let $K$ denote a symmetric positive kernel with integral 1, $h = h_n$ a bandwidth and $K_h(x) = h^{-1}K(h^{-1}x)$, with $h_n \to 0$ as $n \to \infty$. A local maximum likelihood estimator of $p$ is defined as

\[
\hat{p}_{2n}(x) = \frac{1}{\sum_{i=1}^{n} K_h(x - X_i)} \sum_{i=1}^{n} Y_i K_h(x - X_i)
\]

or by higher order polynomial approximations [2].

Under regularity conditions of $p$ and $K$ and ergodicity of the process $(X_t, Y_t)_{t \geq 0}$, the estimator $\hat{p}_{2n}$ is $P$-uniformly consistent and asymptotically Gaussian. When $p$ is monotone, the estimators are asymptotically monotone in probability. For large $n$, the inverse function $q$ is then estimated by $\hat{q}_n(u) = \sup \{x : \hat{p}_n(x) \leq u\}$ if $p$ is decreasing or by $\hat{q}_n(u) = \inf \{x : \hat{p}_n(x) \geq u\}$ if $p$ is increasing. The estimator $\hat{q}_n$ is also $P$-uniformly consistent and asymptotically Gaussian [7]. For small samples, a monotone version of $\hat{p}_n$ using the greatest convex minorant or the smallest concave majorant algorithm may be used before defining a direct inverse. Other nonparametric inverse functions have been defined [4].

Under bias sampling, censoring or truncation, the distribution function of $Y$ conditionally on $X$ is not always identifiable. The paper studies several cases and defines new estimators of conditional and marginal distributions, for a continuous bivariate set $(X, Y)$ and for a conditional Bernoulli variable $Y$.

2. Bias depending on the value of $Y$

In case-control studies, individuals are not uniformly sampled in the population: for rare events, they are sampled so that the cases of interest (individuals with $Y_i = 1$) are sufficiently represented in the sample but the proportion of cases in
the sample differs from its proportion in the general population \[6\]. Let \(S_i\) be the sampling indicator of individual \(i\) in the global population and

\[
Pr(S_i = 1|Y_i = 1) = \lambda_1, \quad Pr(S_i = 1|Y_i = 0) = \lambda_0.
\]

The distribution function of \((S_i, Y_i)\) conditionally on \(X_i = x\) is given by

\[
\begin{align*}
Pr(S_i = 1, Y_i = 1|x) &= Pr(S_i = 1|Y_i = 1) Pr(Y_i = 1|x) = \lambda_1 p(x), \\
Pr(S_i = 1, Y_i = 0|x) &= Pr(S_i = 1|Y_i = 0) Pr(Y_i = 0|x) = \lambda_0 \{1 - p(x)\}, \\
Pr(S_i = 1|x) &= Pr(S_i = 1, Y_i = 1|x) + Pr(S_i = 1, Y_i = 0|x) = \lambda_1 p(x) + \lambda_0 \{1 - p(x)\}.
\end{align*}
\]

Let

\[
\theta = \frac{\lambda_0}{\lambda_1}, \quad \alpha(x) = \theta \frac{1 - p(x)}{p(x)}.
\]

For individual \(i\), \((X_i, Y_i)\) is observed conditionally on \(S_i = 1\) and the conditional distribution function of \(Y_i\) is defined by

\[
\pi(x) = Pr(Y_i = 1|S_i = 1, X = x) = \frac{\lambda_1 p(x)}{\lambda_1 p(x) + \lambda_0 \{1 - p(x)\}}
\]

\[
= \frac{p(x)}{p(x) + \theta \{1 - p(x)\}} = \frac{1}{1 + \alpha(x)}.
\]

The probability \(p(x)\) is deduced from \(\theta\) and \(\pi(x)\) by the relation

\[
p(x) = \frac{\theta \pi(x)}{1 + (\theta - 1)\pi(x)}
\]

and the bias sampling is

\[
\pi(x) - p(x) = \frac{(1 - \theta)\pi(x)(1 - \pi(x))}{1 + (\theta - 1)\pi(x)}.
\]

The model defined by \((\lambda_0, \lambda_1, p(x))\) is over-parameterized and only the function \(\alpha\) is identifiable. The proportion \(\theta\) must therefore be known or estimated from a preliminary study before an estimation of the probability function \(p\).

In the logistic regression model, \(\psi(x) = \log[p(x)\{1 - p(x)\}^{-1}]\) is replaced by \(\log \alpha(x) = \log[\pi(x)\{1 - \pi(x)\}^{-1}] = \psi(x) - \log \theta\). Obviously, the bias sampling modifies the parameters of the model but not this model and the only stable parametric model is the logistic regression.

Let \(\gamma\) be the inverse of the proportion of cases in the population,

\[
\gamma = Pr(Y = 0)/Pr(Y = 1) = E(1 - Y)/EY = \frac{1 - \int p(x) \, dF_X(x)}{\int p(x) \, dF_X(x)}.
\]

Under the bias sampling,

\[
Pr(Y_i = 1|S_i = 1) = \frac{\lambda_1 \int p \, dF_X}{\lambda_0 (1 - \int p \, dF_X) + \lambda_1 \int p \, dF_X} = \frac{1}{1 + \theta \gamma},
\]

\[
Pr(Y_i = 0|S_i = 1) = \frac{\lambda_0 (1 - \int p \, dF_X)}{\lambda_0 (1 - \int p \, dF_X) + \lambda_1 \int p \, dF_X} = \frac{\theta \gamma}{1 + \theta \gamma}.
\]
\( \gamma \) is modified by the scale parameter \( \eta \): it becomes \( \Pr(Y = 0|S = 1)/\Pr(Y = 1|S = 1) = \theta \gamma \).

The product \( \theta \gamma \) may be directly estimated by maximization of the likelihood and

\[
\hat{\theta} \hat{\gamma}_n = 1 - \frac{\sum_i Y_i 1\{S_i = 1\}}{\sum_i 1\{S_i = 1\}}.
\]

In a discrete sampling design with several independent observations for fixed values \( x_j \) of the variable \( X \), the likelihood is

\[
\prod_{i=1}^{n} \pi(x_i)^{Y_i} (1 - \pi(x_i))^{1 - Y_i} = \prod_{j=1}^{m} \prod_{i=1}^{n} \pi(x_i)^{Y_i} (1 - \pi(x_i))^{1 - Y_i} \quad (x_i = x_j)
\]

and \( \alpha_j = \alpha(x_j) \) is estimated by

\[
\hat{\alpha}_j = \frac{\sum_i (1 - Y_i) 1\{S_i = 1\} 1\{x_i = x_j\}}{\sum_i Y_i 1\{x_i = x_j\} 1\{S_i = 1\}}.
\]

For random observations of the variable \( X \), or for fixed observations without replications, \( \alpha(x) \) is estimated by

\[
\hat{\alpha}_n(x) = \frac{\sum_i (1 - Y_i) 1\{S_i = 1\} K_h(x - X_i)}{\sum_i Y_i 1\{S_i = 1\} K_h(x - X_i)}.
\]

If \( \theta \) is known, nonparametric estimators of \( p \) are deduced as

\[
\hat{p}_n(x_j) = \frac{\theta \sum_i Y_i 1\{S_i = 1\} 1\{x_i = x_j\}}{\sum_i (1 - Y_i + \theta Y_i) 1\{S_i = 1\} 1\{x_i = x_j\}}, \quad \text{in the discrete case},
\]

\[
\hat{p}_n(x) = \frac{\theta \sum_i Y_i 1\{S_i = 1\} K_h(x - X_i)}{\sum_i (1 - Y_i + \theta Y_i) 1\{S_i = 1\} K_h(x - X_i)}, \quad \text{in the continuous case}.
\]

### 3. Bias due to truncation on \( X \)

Consider that \( Y \) is observed under a fixed truncation of \( X \): we assume that \((X, Y)\) is observed only if \( X \in [a, b] \), a sub-interval of the support \( I_X \) of the variable \( X \), and \( S = 1_{[a,b]}(X) \). Then

\[
\Pr(Y_i = 1) = \int_{I_X} p(x) \, dF_X(x), \quad \Pr(Y_i = 1, S_i = 1) = \int_{a}^{b} p(x) \, dF_X(x)
\]

and the conditional probabilities of sampling, given the status value, are

\[
\lambda_1 = \Pr(S_i = 1|Y_i = 1) = \frac{\int_{a}^{b} p(x) \, dF_X(x)}{\int_{I_X} p(x) \, dF_X(x)},
\]

\[
\lambda_0 = \Pr(S_i = 1|Y_i = 0) = \frac{\int_{a}^{b} \{1 - p(x)\} \, dF_X(x)}{1 - \int_{I_X} p(x) \, dF_X(x)}.
\]
If the ratio $\theta = \lambda_0 / \lambda_1$ is known or otherwise estimated, the previous estimators may be used for the estimation of $p(x)$ from the truncated sample with $S_i \equiv 1$.

For a random truncation interval $[A, B]$, the sampling indicator is $S = 1_{[A, B]}(X)$ and the integrals of $p$ are replaced by their expectation with respect to the distribution function of $A$ and $B$ and the estimation is similar.

4. Truncation of a response variable in a nonparametric regression model

Consider then $(X, Y)$ a two-dimensional variable in a left-truncated transformation model: Let $Y$ denote a response to a continuous expository variable $X$, up to a variable of individual variations $\varepsilon$ independent of $X$,

$$Y = m(X) + \varepsilon, \quad E\varepsilon = 0, \quad E\varepsilon^2 < \infty,$$

$(X, \varepsilon)$ with distribution function $(F_X, F_\varepsilon)$. The distribution function of $Y$ conditionally on $X$ is defined by

$$F_{Y|X}(y; x) = P(Y \leq y | X = x) = F_\varepsilon(y - m(x)), \quad (2)$$

and the function $m$ is continuous. The joint and marginal distribution functions of $X$ and $Y$ are denoted $F_{X,Y}$, with support $I_{Y,X}$, $F_X$, with bounded support $I_X$, and $F_Y$, such that $F_Y(y) = \int F_\varepsilon(y - m(s)) \, dF_X(s)$ and $F_{X,Y}(x, y) = \int 1_{(s \leq x)} F_\varepsilon(y - m(s)) \, dF_X(s)$.

The observation of $Y$ is supposed left-truncated by a variable $T$ independent of $(X, Y)$, with distribution function $F_T : Y$ and $T$ are observed conditionally on $Y \geq T$ and none of the variables is observed if $Y < T$. Denote $\bar{F} = 1 - F$ for any distribution function $F$ and, under left-truncation,

$$\alpha(x) = P(T \leq Y | X = x) = \int_{-\infty}^{\infty} \bar{F}_\varepsilon(y - m(x)) \, dF_T(y),$$

$$A(y; x) = P(Y \leq y | X = x, T \leq Y)$$
$$= \alpha^{-1}(x) \int_{-\infty}^{y} F_T(v) \, dF_\varepsilon(v - m(x)) \quad (3)$$

$$B(y; x) = P(T \leq y \leq Y | X = x, T \leq Y)$$
$$= \alpha^{-1}(x) F_T(y) \bar{F}_\varepsilon(y - m(x)), \quad (4)$$

$$m^*(x) = E(Y | X = x, T \leq Y) = \alpha^{-1}(x) \int y F_T(y) \, dF_{Y|X}(y; x).$$

Obviously, the mean of $Y$ is biased under the truncation and a direct estimation of the conditional distribution function $F_{Y|X}$ is of interest for the estimation of $m(x) = E(Y | X = x)$ instead of the apparent mean $m^*(x)$. The function $\bar{F}_\varepsilon$ is
also written \( \exp\{-\Lambda_{\varepsilon}\} \) with \( \Lambda_{\varepsilon}(y) = \int_{-\infty}^{y} \tilde{F}_{\varepsilon}^{-1}(s) \, ds \) and the expressions \( \Phi_{\varepsilon}^{-1} \) of \( A \) and \( B \) imply that

\[
\Lambda_{\varepsilon}(y - m(x)) = \int_{-\infty}^{y} B^{-1}(s; x) \, A(ds; x)
\]

and \( F_{Y|X}(y; x) = \exp\{-\Lambda_{\varepsilon}(y - m(x))\} \).

An estimator of \( F_{Y|X}(y; x) \) is obtained as the product-limit estimator \( \tilde{F}_{\varepsilon,n}(y - m(x)) \) of \( F_{\varepsilon}(y - m(x)) \) based on estimators of \( A \) and \( B \): For a sample \( (X_i, Y_i)_{1 \leq i \leq n} \), let \( x \) in \( I_{X,n,h} = [\min_i X_i + h, \max_i X_i - h] \) and

\[
\tilde{A}_n(y; x) = \frac{\sum_{i=1}^{n} K_h(x - X_i) I(T_i \leq Y_i)}{\sum_{i=1}^{n} K_h(x - X_i) I(T_i \leq Y_i)},
\]

\[
\tilde{B}_n(y; x) = \frac{\sum_{i=1}^{n} K_h(x - X_i) I(T_i \leq Y_i)}{\sum_{i=1}^{n} K_h(x - X_i) I(T_i \leq Y_i)};
\]

\[
\tilde{F}_{Y|X,n}(y; x) = 1 - \prod_{1 \leq i \leq n} \left\{ 1 - \frac{\tilde{A}_n(Y_i; x)}{\tilde{B}_n(Y_i; x)} \right\},
\]

with \( 0/0 = 0 \). That is a nonparametric maximum likelihood estimator of \( F_{Y|X} \), as is the Kaplan-Meier estimator for the distribution function of a right-censored variable. Then an estimator of \( m(x) \) may be defined as an estimator of \( \int y \, F_{Y|X}(dy; x) \),

\[
\tilde{m}_n(x) = \sum_{i=1}^{n} Y_i I(T_i \leq Y_i) \left\{ \tilde{F}_{Y|X,n}(Y_i; x) - \tilde{F}_{Y|X,n}(Y_i^-; x) \right\}
\]

\[
= \sum_{i=1}^{n} Y_i I(T_i \leq Y_i) K_h(x - X_i) \tilde{F}_{Y|X,n}(Y_i^-; x) \sum_{j=1}^{n} K_h(x - X_j) I(T_j \leq Y_j),
\]

(6)

By the same arguments, from means in \( \Phi_{\varepsilon}^{-1} \), \( \tilde{F}_{Y}(y) = E\tilde{F}_{\varepsilon}(y - m(X)) \) is estimated by

\[
\tilde{F}_{Y,n}(y) = \prod_{1 \leq i \leq n} \left\{ 1 - \frac{I(T_i \leq Y_i \leq y)}{\sum_{j=1}^{n} I(T_j \leq Y_j)} \right\},
\]

the distribution function \( F_T \) is simply estimated by the product-limit estimator for right-truncated variables \( \Phi_{\varepsilon}^{-1} \)

\[
\tilde{F}_{T,n}(t) = \prod_{1 \leq i \leq n} \left\{ 1 - \frac{I(t \leq T_i \leq Y_i)}{\sum_{j=1}^{n} I(T_j \leq T_i \leq Y_i)} \right\}
\]

and an estimator of \( \tilde{F}_{\varepsilon} \) is deduced from those of \( \tilde{F}_{Y|X}, F_X \) and \( m \) as

\[
\tilde{F}_{\varepsilon,n}(s) = n^{-1} \sum_{1 \leq i \leq n} \tilde{F}_{Y|X,n}(s + \tilde{m}_n(X_i); X_i).
\]
The estimators \( \hat{\mu}_{T,n} \) and \( \hat{\mu}_{Y,n} \) are estimated by

\[
\hat{\mu}_{T,n} = n^{-1} \sum_{i=1}^{n} \frac{T_i I_{\{T_i \leq Y_i\}}}{\sum_{j=1}^{n} I_{\{T_j \leq Y_j\}}} \hat{F}_{T,n}(T_i), \quad \hat{\mu}_{Y,n} = n^{-1} \sum_{i=1}^{n} \frac{Y_i I_{\{T_i \leq Y_i\}}}{\sum_{j=1}^{n} I_{\{T_j \leq Y_j\}}} \hat{F}_{Y,n}(Y_i).
\]

The estimators \( \hat{F}_{Y,n} \) and \( \hat{F}_{T,n} \) are known to be \( P \)-uniformly consistent and asymptotically Gaussian. For the further convergence restricted to the interval \( I_{n,h} = \{(y,x) \in I_{Y,X} : x \in I_{X,n,h}\} \), assume

**Condition 4.1**

C1. \( h = h_n \to 0 \) and \( nh^3 \to \infty \) as \( n \to \infty \), \( \int K = 1 \), \( \kappa_1 = \int x^2 K(x) \, dx \) and \( \kappa_2 = \int K^2 < \infty \).

C2. the conditional probability \( \alpha \) is strictly positive in the interior of \( I_X \).

C3. The distribution function \( F_{Y,X} \) is twice continuously differentiable with respect to \( x \) and differentiable with respect to \( y \).

C4. \( \varepsilon^{2+\delta} < \infty \) for a \( \delta \) in \((1/2,1)\).

Let us denote \( \hat{F}_{Y,X,2}(y,x) = \partial F_{Y|X}(y,x)/\partial x \), \( \hat{F}_{Y,X,2}(y,x) = \partial^2 F_{Y|X}(y,x)/\partial x^2 \), and \( \hat{F}_{Y,X,1}(y,x) = \partial F_{Y|X}(y,x)/\partial y \).

**Proposition 4.1** \( \sup_{I_{n,h}} |\hat{A}_n - A| \overset{P}{\to} 0 \) and \( \sup_{I_{n,h}} |\hat{B}_n - B| \overset{P}{\to} 0 \),

\[
b_{n,h}^A(y;x) = (E\hat{A}_n - A)(y;x) = \frac{h^2}{2\alpha(x)} \kappa_1 \left\{ \int_{-\infty}^{y} F_T(v) \hat{F}_{Y,X,2}(dv, dx) \right\} + o(h^2),
\]

\[
b_{n,h}^B(y;x) = (E\hat{B}_n - B)(y;x) = \frac{h^2}{2\alpha(x)} \kappa_1 \{F_T(y) \hat{F}_{Y,X,2}(dy, dx) \} + o(h^2),
\]

\[
v_{n,h}^A(y;x) = \text{var} \hat{A}_n(y;x) = (nh)^{-1} \kappa_2 A(1-A)(y;x)\alpha^{-1}(x) + o((nh)^{-1}),
\]

\[
v_{n,h}^B(y;x) = \text{var} \hat{B}_n(y;x) = (nh)^{-1} \kappa_2 B(1-B)(y;x)\alpha^{-1}(x) + o((nh)^{-1}).
\]

If \( nh^5 \to 0 \), \((nh)^{1/2}(\hat{A}_n - A)\) and \((nh)^{1/2}(\hat{B}_n - B)\) converge in distribution to Gaussian processes with mean zero, variances \( \kappa_2 A(1-A)(y;x)\alpha^{-1}(x) \) and \( \kappa_2 B(1-B)(y;x)\alpha^{-1}(x) \) respectively, and the covariances of the limiting processes are zero.

The proof relies on an expansion of the form

\[
(nh)^{1/2}(\hat{A}_n - A)(y;x) = (nh)^{1/2}c^{-1}(x)\{ (\hat{a}_n - a)(y;x) - A(\hat{c}_n - c)(x) \} + o_L(1)
\]

with \( \hat{A}_n = \hat{c}_n^{-1}\hat{a}_n \) and \( \hat{B}_n = \hat{c}_n^{-1}\hat{b}_n \), where

\[
\hat{c}_n(x) = n^{-1} \sum_{i=1}^{n} K_h(x - X_i)I_{\{T_i \leq Y_i\}}, \quad \hat{a}_n(y;x) = n^{-1} \sum_{i=1}^{n} K_h(x - X_i)I_{\{T_i \leq Y_i \leq y\}},
\]
A similar approximation holds for $\hat{b}_n$. The biases and variances are deduced from those of each term and the weak convergences are established as in [7].

From proposition 4.1 and applying the results of the nonparametric regression,

**Proposition 4.2** The estimators $\hat{F}_{Y|X,n}, \hat{m}_n, \hat{F}_\epsilon,n \text{ converge } P\text{-uniformly to } F_{Y|X}, m, F_\epsilon, \hat{\mu}_{Y,n} \text{ and } \hat{\mu}_{T,n} \text{ converge } P\text{-uniformly to } EY \text{ and } ET \text{ respectively.}

The weak convergence of the estimated distribution function of truncated survival data was proved in several papers [11, 12]. As in [3] and by proposition 4.1 their proof extends to their weak convergence on $(\min\{Y_i : T_i < Y_i\}, \max\{Y_i : T_i < Y_i\})$ under the conditions $\int F_T dF_{Y|X} < \infty$ and $\int \hat{F}_{Y|X}^{-1} dF_T < \infty$ on $I_{X,n,h}$, which are simply satisfied if for every $x$ in $I_{X,n,h}$, $\inf\{t : F_T(t) > 0\} < \inf\{t : F_{Y|X}(t ; x) > 0\}$ and $\sup\{t : F_{Y|X}(t ; x) > 0\} < \sup\{t : F_T(t) > 0\}$.

**Theorem 4.1** $(nh)^{1/2}(\hat{F}_{Y|X,n} - F_{Y|X})1_{I_{n,h}} \text{ converges weakly to a centered Gaussian process } W \text{ on } I_{Y,X}$. The variables $(nh)^{1/2}(\hat{m}_n - m)(x)$, for every $x \in I_{X,n,h}$, and $(nh)^{1/2}(\hat{\mu}_{Y,n} - EY)$ converge weakly to $EW(Y; x)$ and $E \int W(Y; x) dF_X(x)$.

If $m$ is supposed monotone with inverse function $r$, $X$ is written $X = r(Y - \varepsilon)$ and the quantiles of $X$ are defined by the inverse functions $q_1$ and $q_2$ of $F_{Y|X}$ at fixed $y$ and $x$, respectively, are defined by the equivalence between

$$F_{Y|X}(y; x) = u \quad \text{and} \quad \begin{cases} x = r(y - Q_\varepsilon(u)) \equiv q_1(u; y) \\ y = m(x) + Q_\varepsilon(u) \equiv q_2(u; x), \end{cases}$$

where $Q_\varepsilon(u)$ is the inverse of $F_\varepsilon$ at $u$. Finally, if $m$ is increasing, $F_{Y|X}(y; x)$ is decreasing in $x$ and increasing in $y$, and it is the same for its estimator $\hat{F}_{Y|X,n}$, up to a random set of small probability. The thresholds $q_1$ and $q_2$ are estimated by

$$\hat{q}_{1,n,h}(u; y) = \sup\{x : \hat{F}_{Y|X,n}(y; x) \leq u\},$$

$$\hat{q}_{2,n,h}(u; x) = \inf\{y : \hat{F}_{Y|X,n}(u; x) \geq u\}.$$

As a consequence of theorem 4.1 and generalizing known results on quantiles

**Theorem 4.2** For $k = 1, 2$, $\hat{q}_{k,n,h} \text{ converges } P\text{-uniformly to } q_k \text{ on } \hat{F}_{Y,X,n}(I_{n,h})$. For every $y$ and (respect.) $x$, $(nh)^{1/2}(\hat{q}_{1,n,h} - q_1)(y; :)$ and $(nh)^{1/2}(\hat{q}_{2,n,h} - q_2)(: x)$ converge weakly to the centered Gaussian process $W \circ q_1[\hat{F}_{Y|X,n}(y; q_1(y))]^{-1}$ and, respect., $W \circ q_2[\hat{F}_{Y|X,n}(q_2(y); :)]^{-1}$.

5. Truncation and censoring of $Y$ in a nonparametric model

The variable $Y$ is supposed left-truncated by $T$ and right-censored by a variable $C$ independent of $(X, Y, T)$. The notations $\alpha$ and those of the joint and marginal
distribution function of $X$, $Y$ and $T$ are in section 4, and $F_C$ is the distribution function of $C$. The observations are $\delta = 1_{\{Y \leq C\}}$, and $(Y \wedge C, T)$, conditionally on $Y \wedge C \geq T$. Let

$$
A(y; x) = P(Y \leq y \wedge C|X = x, T \leq Y) = \alpha^{-1}(x) \int_{-\infty}^{y} F_C(v) F_{Y|X}(dv; x)
$$

$$
B(y; x) = P(T \leq y \leq Y \wedge C|X = x, T \leq Y) = \alpha^{-1}(x) F_C(y) \tilde{F}_{Y|X}(y; x),
$$

$$
\tilde{F}_{Y|X}(y; x) = \exp\{-\int_{-\infty}^{y} B^{-1}(v; x) A(dv; x)\}.
$$

The estimators are now written

$$
\hat{\tilde{F}}_{Y|X,n}(y; x) = \prod_{1 \leq i \leq n} \left\{ 1 - \frac{K_h(x - X_i) I\{T_i \leq Y \leq y \wedge C_i\}}{\sum_{j=1}^{n} K_h(x - X_j) I\{T_j \leq Y \leq y \wedge C_j\}} \right\},
$$

$$
\hat{m}_n(x) = \frac{\sum_{i=1}^{n} Y_i I\{T_i \leq Y \leq C_i\} K_h(x - X_i) \tilde{F}_{Y|X,n}(Y_i; x)}{\sum_{j=1}^{n} K_h(x - X_j) I\{T_j \leq Y \leq y \wedge C_j\}}.
$$

$$
\hat{\tilde{F}}_{Y|n}(y) = \prod_{1 \leq i \leq n} \left\{ 1 - \frac{I\{T_i \leq Y \leq y \wedge C_i\}}{\sum_{j=1}^{n} I\{T_j \leq Y \leq y \wedge C_j\}} \right\}.
$$

If $Y$ is only right-truncated by $C$ independent of $(X, Y)$, with observations $(X, Y)$ and $C$ conditionally on $Y \leq C$, the expressions $\alpha$, $A$ and $B$ are now written

$$
\alpha(x) = P(Y \leq C|X = x) = \int_{-\infty}^{\infty} \tilde{F}_C(y) F_{Y|X}(dy; x),
$$

$$
A(y; x) = P(Y \leq y|X = x, Y \leq C) = \alpha^{-1}(x) \int_{-\infty}^{y} \tilde{F}_C(v) F_{Y|X}(dv; x),
$$

$$
B(y; x) = P(Y \leq y \leq C|X = x, Y \leq C) = \alpha^{-1}(x) \tilde{F}_C(y) F_{Y|X}(y; x),
$$

$$
A'(y; x) = P(Y \leq C \leq y|X = x, Y \leq C) = \alpha^{-1}(x) \int_{-\infty}^{y} \tilde{F}_{Y|X}(v; x) dF_C(v).
$$

The distribution function $F_C$ and $F_{Y|X}$ are both identifiable and their expression differs from the previous ones,

$$
\tilde{F}_C = \exp\{-\int_{-\infty}^{\infty} E B^{-1}(v; X) E A'(dv; X)\},
$$

$$
F_{Y|X}(y; x) = \exp\{-\int_{-\infty}^{y} B^{-1}(v; x) A(dv; x)\}.
$$

The estimators are now

$$
\hat{\tilde{F}}_{Y|X,n}(y; x) = \prod_{1 \leq i \leq n} \left\{ 1 - \frac{K_h(x - X_i) I\{Y_i \leq y \wedge C_i\}}{\sum_{j=1}^{n} K_h(x - X_j) I\{Y_j \leq y \wedge C_j\}} \right\},
$$

\[
\hat{F}_{C,n}(y) = \prod_{1 \leq i \leq n} \left\{ 1 - \frac{I_{Y_i \leq C} \leq y}{\sum_{j=1}^{n} I_{Y_j \leq C_i}} \right\}, \\
\hat{m}_n(x) = \frac{\sum_{i=1}^{n} Y_i I_{Y_i \leq C} K_h(x - X_i) \hat{F}_{Y \mid X,n}(Y_i^-; x)}{\sum_{j=1}^{n} K_h(x - X_j) I_{Y_j \leq C_j}}.
\]

If \( Y \) is left and right-truncated by variables \( T \) and \( C \) independent and independent of \( (X,Y) \), the observations are \( (X,Y) \), \( C \) and \( T \), conditionally on \( T \leq Y \leq C \),

\[
\begin{align*}
\alpha(x) &= P(T \leq Y \leq C \mid X = x) = \int_{-\infty}^{\infty} F_T(y) \hat{F}_C(y) \; F_{Y \mid X}(dy; x), \\
A(y; x) &= P(Y \leq y \mid X = x, T \leq Y \leq C) \\
&= \alpha^{-1}(x) \int_{-\infty}^{y} F_T(v) \hat{F}_C(v) \; F_{Y \mid X}(dv; x), \\
B(y; x) &= P(T \leq y \leq Y \mid X = x, T \leq Y \leq C) \\
&= \alpha^{-1}(x) F_T(y) \int_{y}^{\infty} \hat{F}_C(v) \; F_{Y \mid X}(dv; x), \\
A'(y) &= P(y \leq T \mid T \leq Y \leq C) = \int_{-\infty}^{\infty} dF_T(t) \int_{t}^{\infty} \hat{F}_C dF_Y, \\
B'(y) &= P(Y \leq y \leq C \mid T \leq Y \leq C) = \hat{F}_C(y) \int_{-\infty}^{y} F_T dF_Y, \\
B''(y) &= P(C \leq y \mid T \leq Y \leq C) = \int_{-\infty}^{y} \left\{ \int_{-\infty}^{s} F_T(v) \; dF_Y(v) \right\} dF_C(s).
\end{align*}
\]

The distribution functions \( F_C, F_T \) and \( F_{Y \mid X} \) are identifiable, with \( F_{Y \mid X} \) defined by \( F_{Y \mid X}(y; x) = -\int_{-\infty}^{y} \hat{F}_C^{-1} dH(\cdot; x) \) and

\[
\begin{align*}
H(y; x) &= \int_{y}^{\infty} \hat{F}_C(v) dF_{Y \mid X}(dv; x) = \exp\left\{ -\int_{-\infty}^{y} B^{-1}(v; x) A(dv; x) \right\}, \\
\hat{F}_C(s) &= \exp\left\{ -\int_{-\infty}^{s} B^{-1} dB'' \right\}, \\
F_T(t) &= \exp\left\{ -\int_{t}^{\infty} (EB(\cdot; X))^{-1} dA' \right\}.
\end{align*}
\]

Their estimators are

\[
\begin{align*}
\hat{F}_{C,n}(s) &= \prod_{i=1}^{n} \left\{ 1 - \frac{I_{T_i \leq Y_i \leq C \leq s}}{\sum_{j=1}^{n} I_{T_j \leq Y_j \leq C_j}} \right\}, \\
\hat{F}_{T,n}(t) &= \prod_{i=1}^{n} \left\{ 1 - \frac{I_{T_i \leq Y_i \leq C \leq t}}{\sum_{j=1}^{n} I_{T_j \leq T_j \leq Y_j \leq C_j}} \right\}, \\
\hat{F}_{Y \mid X,n}(y; x) &= \frac{\sum_{i=1}^{n} \hat{F}_{C,n}(Y_i) I_{T_i \leq Y_i \leq C \leq y} K_h(x - X_i) \hat{F}_{Y \mid X,n}(Y_i^-; x)}{\sum_{j=1}^{n} K_h(x - X_j) I_{T_j \leq Y_j \leq C_j}}.
\end{align*}
\]
\[
\tilde{H}_{Y|X}(y; x) = \prod_{i=1}^{n} \left\{ 1 - \frac{K_h(x - X_i)I_{\{T_i \leq Y_i \leq C_i \land y\}}}{\sum_{j=1}^{n} K_h(x - X_j)I_{\{T_j \leq Y_j \leq C_j\}}} \right\}.
\]

The other nonparametric estimators of the introduction and the results of section \[4\] generalize to all the estimators of this section. Right and left-truncated distribution functions \(F_{Y|X}\) and the truncation distributions are estimated in a closed form by the solutions a self-consistency equation \[8, 9\]. The estimators still have asymptotically Gaussian limits even with dependent truncation distributions, when the martingale theory for point processes does not apply.

6. Observation by interval

Consider model (2) with an independent censoring variable \(C\) for \(Y\). For observations by intervals, only \(C\) and the indicators that \(Y\) belongs to the interval \([-\infty, C]\) or \([C, \infty]\) are observed. The function \(F_{Y|X}\) is not directly identifiable and efficient estimators for \(m\) and \(F_{Y|X}\) are maximum likelihood estimators. Let \(\delta = I_{\{Y \leq C\}}\) and assume that \(F_{\varepsilon}\) is \(C^2\). Conditionally on \(C\) and \(X = x\), the log-likelihood of \((\delta, C)\) is

\[
l(\delta, C) = \delta \log F_{\varepsilon}(C - m(x)) + (1 - \delta) \log F_{\varepsilon}(C - m(x))
\]

and its derivatives with respect to \(m(x)\) and \(F_{\varepsilon}\) are

\[
i_{m(x)}(\delta, C) = -\frac{f_{\varepsilon}}{F_{\varepsilon}}(C - m(x)) + (1 - \delta) \frac{f_{\varepsilon}}{F_{\varepsilon}}(C - m(x)),
\]

\[
i_{\varepsilon}a(\delta, C) = \delta \int_{-\infty}^{C - m(x)} a \, dF_{\varepsilon} + (1 - \delta) \int_{C - m(x)}^{\infty} a \, dF_{\varepsilon}
\]

for every \(a\) s.t. \(\int a \, dF_{\varepsilon} = 0\) and \(\int a^2 \, dF_{\varepsilon} < \infty\). With \(a_F = -f_{\varepsilon}f_{\varepsilon}^{-1}\), \(i_{\varepsilon}a_F = i_{m(x)}\) then \(i_{m(x)}\) belongs to the tangent space for \(F_{\varepsilon}\) and the estimator of \(m(x) = E(Y|X = x)\) must be determined from the estimator of \(F_{\varepsilon}\) through the conditional probability function of the observations

\[
B(t; x) = P(Y \leq C \leq t|X = x) = \int_{-\infty}^{t} F_{\varepsilon}(s - m(x)) \, dF_{C}(s).
\]

Let \(\hat{F}_{C,n}\) the empirical estimator of \(F_{C}\) and

\[
\hat{B}_n(t; x) = \frac{\sum_{i=1}^{n} K_h(x - X_i)I_{\{Y_i \leq C_i \leq t\}}}{\sum_{i=1}^{n} K_h(x - X_i)}.
\]

an estimator \(\hat{F}_{\varepsilon,n}(t - m(x))\) of \(F_{\varepsilon,n}(t - m(x))\) is deduced by deconvolution and

\[
\hat{m}_n(x) = \int t \, d\hat{F}_{\varepsilon,n}(t - m(x)).
\]
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