Scattering in a Simple 2-d Lattice Model

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Lüscher has suggested a method to determine phase shifts from the finite volume dependence of the two-particle energy spectrum. We apply this to two models in $d=2$: (a) the Ising model, (b) a system of two Ising fields with different mass and coupled through a 3-point term, both considered in the symmetric phase. The Monte Carlo simulation makes use of the cluster updating and reduced variance operator techniques. For the Ising system we study in particular $O(a^2)$ effects in the phase shift of the 2-particle scattering process.

1. INTRODUCTION

Most of the particle entries in the annual review of particle properties of the particle data group refer to resonances. Usually they are created in two-particle collisions and observed in corresponding cross sections and scattering phase shifts. However, in quantum field theory unstable states still are one of the tantalizing problems. In the lattice approach, up to now efforts have concentrated on the identification of stable particle states. The determination of phase shifts and resonance parameters has been reserved for a future of more powerful computers and there are only very few (courageous) contributions.

Recently, Lüscher has refined earlier ideas on how to determine phase shifts in an elastic two-particle system from the energy spectrum in a finite volume and together with Wolff the power of the method was demonstrated in the $d=2$ $O(3)$ model. Here we discuss some of our results where we use this method to determine the phase shifts in a simple $d=2$ model with two particle types and resonating phase shifts. Other contributions to this conference present first results for $d=3$ and $d=4$ (cf. also) where life is harder and statistics scarcer. The case $d=2$ provides an excellent testbed to study some of the juicy details of the approach.

1.1. Method

Let us briefly review the idea. Consider the scattering of two identical particles of mass $m$ in a box of finite spatial extension $L$. The time extension is assumed to be sufficiently large, not to contribute to finite size effects. The size of the spatial volume $L$ and the periodic b.c., however, are responsible for the quantization of the momenta. In the elastic regime $2m \leq W < 4m$ (or $3m$, depending on the theory) the allowed momentum values $k$ are, in the two-dimensional case, related to the scattering phase shift via the quantization condition

$$2\delta(k_n) + k_n L = 2n\pi, \ n \in \mathbb{N}. \quad (1)$$

Assuming vanishing total momentum (CMS) the total energy of the 2-particle state is just twice the energy of the back-to-back single particle states

$$W_n = 2\sqrt{m^2 + k_n^2}. \quad (2)$$

Thus given $m$ and a couple of low lying energy levels $W_n$ one may obtain values of the (infinite volume) phase shift at the corresponding values $k_n$. Varying $L$ one may cover a whole range of momentum values. Relation (1) holds in that simple form for $d=2$ but can be generalized to higher dimensions.
One has to take care of the following restrictions.

- The interaction region and the single particle correlation length ought to be smaller than the spatial volume.
- Polarization effects due to virtual particles running around the torus should be under control.
- Lattice artifacts will turn up in $O(a^2)$ corrections.
- For the determination of the energy spectrum one should consider correlation functions of sufficiently many observables.

In $d=2$ all these can be controlled.

### 1.2. Model and Simulation

We choose a model, where two light particles $\varphi$ couple to a heavier particle $\eta$ giving rise to resonating behaviour. The action is given by

$$S = -\kappa_\varphi \sum_{x \in \Lambda, \mu=1,2} \varphi_x \varphi_x + \hat{\mu}$$

$$-\kappa_\eta \sum_{x \in \Lambda, \mu=1,2} \eta_x \eta_x + \hat{\mu}$$

$$+ \frac{g}{2} \sum_{x \in \Lambda, \mu=1,2} \eta_x \varphi_x (\varphi_{x-\hat{\mu}} + \varphi_{x+\hat{\mu}}). \quad (3)$$

The values of the fields are restricted to $\{+1, -1\}$. The sums run over all sites $(x_0, x_1)$ of the euclidean $L \times T$ lattice $\Lambda$ with periodic boundary conditions. The 3-point term was introduced in a nonlocal but symmetric way, because $\varphi^2 = 1$.

For $g = 0$ this is just a system of two independent Ising models, each describing in the scaling region interacting bosonic fields with mass $m(\kappa)$. The corresponding masses have been adjusted to $m_\varphi \simeq 0.19$ and $m_\eta \simeq 0.5$ ($m_\eta$ defined by the resonance peak position) by calibrating the couplings $\kappa_\varphi$ and $\kappa_\eta$.

When kinematically allowed, the term proportional to $g$ gives rise to transitions like $\eta \rightarrow \varphi \varphi$ rendering $\eta$ a resonance in the $\varphi \varphi$ channel. We study the model at $g = 0$, 0.02 and 0.04. Throughout this work we use $T = 100$; the spatial extension $L$ varies between 12 and 60. For each set of couplings and lattice size we performed typically $2 \times 10^5$ measurements. Our Monte Carlo simulation utilizes the cluster updating method introduced for the Ising model in [1]. The statistical errors are estimated with the Jackknife method. Details of the simulation technique and the phase diagram can be found in [2].

### 2. Observables

#### 2.1. Single particle state

The operator of a $\varphi$ state with momentum $p_1, \nu = 2\pi \nu / L, \nu = -L/2 + 1, \ldots, L/2$, is given through

$$\frac{1}{L} \sum_{x_1 \in \Lambda_{x_0}} \varphi_{x_0, x_1} \exp (ix_1 p_1, \nu), \quad (4)$$

where $\Lambda_{x_0}$ denotes a timeslice of $\Lambda$. Its connected correlation function over temporal distance $t$ decays exponentially $\propto \exp (-p_0, \nu t)$ defining $p_0, \nu$; in particular $p_{0, \nu=0} = m_\varphi$.

For the determination of the energy spectrum a precise knowledge of the single particle mass and related finite size effects is important. We find [3], that our results for $p_0, \nu$ follow with high precision the spectral relation for the lattice propagator of a Gaussian particle with mass $m$,

$$p_{0, \nu} = \cosh^{-1}(1 - \cos p_{1, \nu} + \cosh m). \quad (5)$$

This expression deviates from the continuum dispersion relation (d.r.) $p_{0, \nu} = (m^2 + p_1^2, \nu)^{1/2}$ by a leading correction $O((ap)^2)$.

The observed mass, as compared to the “real” mass at vanishing lattice spacing and infinite volume, is also affected by polarization due to self interaction around the torus. We confirm this behaviour and find good agreement with the expected exponential decrease [4]. We also determined the wave function renormalization constants for the fields [3].

#### 2.2. Scattering sector

We consider operators with total zero momentum and quantum numbers of the $\eta$,

$$N_1(x_0) = \frac{1}{L} \sum_{x_1 \in \Lambda_{x_0}} \eta_{x_1, x_0}, \quad (6)$$
Figure 1. Phase shifts for the Ising model for two values of the mass, determined with the continuum d.r. (2); full line: $\delta = -\pi/2$, dashed line: 2-particle threshold.

$$N_j(x_0) = \frac{1}{L^2} \sum_{x_1,y_1 \in \Lambda_{x_0}} e^{ip_j(x_1-y_1)} \varphi_{x_1,y_1} \varphi_{y_1,x_0},$$

with $p_j = \frac{2\pi(j-2)}{L}$, $j = 2,3\ldots (7)$

and measure all (connected) cross-correlations $M_{nm}(t) = \langle N_n(t) N_m(0) \rangle_c$ ($t$ denotes the separation of the time slices). The operators $N_{j>1}$ describe two $\varphi$-particles in the CM system with relative momentum $2p_j$. Because of the interaction they do not correspond to eigenstates of our model. Indeed they are eigenstates of the Gaussian model of free bosons. However, if the set is complete, a diagonalization of the correlation matrix provides the necessary information on the energy spectrum of this channel. The transfer matrix formalism yields the spectral decomposition

$$M_{nm}(t) = \sum_{l=1}^{\infty} v_{n}^{(l)*} v_{m}^{(l)} e^{-tW_l},$$

where $v_{n}^{(l)} = \langle l | N_n \rangle_0$ are the projections of the states $|N_n\rangle_0$ (generated by the operators $N_n$ out of the vacuum) on the energy eigenstates $|l\rangle$ of the scattering problem.

The number of operators considered should be chosen larger than the number of states in the elastic regime $2m_\varphi \leq W < 4m_\varphi$ and not larger than $L/2$ to be linearly independent. A larger set provides a better representation of the eigenstates but enhances the numerical noise.

We work with between 4 and 6 operators depending on $L$ and distances $t = 1\ldots8$. Details on the determination of the eigenspectrum in the scattering sector and on the representation of the eigenstates are discussed in [6].

Figure 2. Like fig. 1, but now using the lattice d.r. (9)
3. RESULTS FOR THE PHASE SHIFTS

3.1. Resonance

In [6], we present our results for the observed energy levels and the resulting phase shifts. Since the pure Ising model has an S-matrix equal to $-1$, the phase shift starts with $-\pi/2$ and then shows, for non-vanishing $g$, a clear resonance behaviour manifesting itself in a fast increase by $\pi$. It can be nicely approximated by a standard effective range resonance formula [2,6].

3.2. $O(a^2)$ corrections

Let us consider the Ising case $(g = 0)$ in more detail. Here we increased the statistics significantly (1.5 million measurements) and repeated the calculation for $m = 0.19$ and a higher value of the mass $m = 0.5$ (fig.1). As mentioned we expect a phase shift of $-\pi/2$ and the results are in agreement with this. At small $k$ the energy is close to the 2-particle threshold and the statistical error of the energy transforms via (2) into a relatively larger error of $k$ and thus $\delta(k)$. Higher $k$, on the other hand, stem from large values of the energy with intrinsically larger statistical fluctuations. However, due to the enhanced statistics we do identify a systematic deviation from $-\pi/2$ increasing with $k$ and $m$. We attribute this behaviour to $O(a^2)$ corrections.

As mentioned earlier, the d.r. (2) gives the total energy of the asymptotic 2-particle state which (under the assumption of localized interaction region) is just twice the energy of the outgoing particles. Now the $O(a^2)$ corrections of the single particle d.r. have been nicely described by replacing the continuum d.r. by the lattice relation (8). We therefore replace (8) by the corresponding lattice expression,

$$W_n = 2 \cosh^{-1}(1 - \cos k_n + \cosh m).$$

(9)

Our data for $W_n$ now produce slightly different values of $k_n$ and $\delta$ exhibited in fig.2 in better agreement with a constant value of $-\pi/2$. We conclude that the leading $O(a^2)$ corrections can be expressed by replacing the continuum d.r. by the lattice relation, at least in the 2 dimensional Ising Model. In general we suspect that lattice artifacts in the phase shift can be diminished by studying carefully the dispersion relation of single particle states.

4. CONCLUSION

We have determined phase shifts in the Ising model and resonating phase shifts in a model with two types of particles and a three-point coupling. We find that Lüscher’s suggestion for determining these phase shifts is indeed a very reliable method, at least in $d=2$. The leading $O(a^2)$ effects in the 1- and 2-particle channel may be explained by the differences between lattice vs. continuum dispersion relations.

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