Legendre Spectral Method for the 1-D Maxwell Equation

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Abstract. A time and space Legendre spectral method is established for the 1-D Maxwell equation, that is, both the time and the space are discretized by the Legendre-tau method. The specific scheme of the method and the implementation process of the algorithm are given. Also, the time multi-interval method is considered and the specific scheme is given. By taking appropriate basis functions, the unknown functions can be decoupled in computation. Numerical results show that the two methods are efficient.

Keywords: Maxwell equation, Legendre spectral method, error estimation

1. Introduction
The spectral method has high-order accuracy for smooth problems, and it has attracted much attention in the field of scientific and engineering computing. In recent years, the high-order discrete scheme in time direction has been widely concerned for solving evolution differential equations. The discontinuous Galerkin method in time is constantly developing, and a higher-order discrete scheme in time is better established [1, 2]. The spectral method in time is applied for convection-diffusion equations and parabolic equations, respectively in [3, 4]. In [5, 6], the time spectral method and the time multi-interval method are also proposed.

The Maxwell equation is a partial differential equation that reveal the basic properties of electromagnetic fields. Some effective numerical methods have been applied for the Maxwell equation by many scholars [7, 8]. For the work related to the spectral method, the difference method is mainly adopted for the time approximation. A multidomain Legendre- Galerken method is considered for the 1-D Maxwell equation with discontinuous solutions in [9]. In [10], the multidomain Legendre-tau method is established for the 1-D Maxwell equation of nonhomogeneous media with discontinuous solutions.

Let \( I_x = (-1,1) \), \( I_t = (0,T] \) and \( W = I_x \times I_t \). Consider the following 1-D Maxwell equation [11].

\[
\begin{align*}
\varepsilon \frac{\partial}{\partial t} E_z &= \frac{\partial}{\partial x} H_y, & (x,t) \in W, \\
\frac{\partial}{\partial t} H_y &= \frac{\partial}{\partial x} E_z, & (x,t) \in W, \\
E_z(x,1,t) &= E_z(x,T) = 0, & t \in I_t, \\
E_z(x,0) &= E_{z0}(x), & H_y(x,0) = H_{y0}(x), & x \in I_x,
\end{align*}
\]

where \( E_z \) and \( H_y \) are the unknown functions, and \( \varepsilon \) and \( m \) are the positive constants.

In the paper, we present a time and space spectral method for (1) and its time multi-interval method.
is developed. The polynomials of different degrees are used to approximate the unknown functions, so that they can be decoupled in computation. Numerical results show that the two methods are efficient.

2. Legendre Spectral Method

In the section, we consider a time and space Legendre spectral scheme for (1) and give the algorithm implementation of the method.

2.1. Preliminaries

Let \((\times, \times)_{Q}\) be the inner product of \(L^2(Q)\), and \(\|\cdot\|_{Q}\) be the norm of \(L^2(Q)\), where \(Q\) stands for \(I_x\) or \(I_t\), respectively. For a integer \(m > 0\), let \(H^m(I)\) the standard Sobolev space, and \(\|\cdot\|_{m}\) and \(|\cdot|_{m}\) be the norm and semi-norm, respectively, where \(I\) stands for \(I_x\) or \(I_t\), respectively. Define

\[ H^0_0(I_x) = \{ v \in H^1(I_x) : v(-1) = v(1) = 0 \} \]

For a pair of integers \(N > 0\) and \(M > 0\), let \(P_N(I_x)\) and \(P_M(I_t)\) be the polynomials of \(x\) and to \(t\) degrees \(N\) and \(M\), respectively. Let \(t_j^C\) and \(w_j^C\) \((0 \leq j \leq M)\) be the Chebyshev-Gauss-Lobatto (CGL) points and the weights on \(I_t\). Let \(x_j^C\) and \(w_j^C\) \((0 \leq j \leq N)\) be the CGL points and the weights on \(I_x\). We define the CGL interpolation operator \(I_N^C v \in V_N\) which satisfies

\[ I_N^C v(x_j^C) = v(x_j^C), \quad 0 \leq j \leq N. \]

**Lemma 2.1** [12] If \(u \in H^r(I_x)\), then

\[ \| P_N u - u \|_{L^r} \leq C N^{-r} \| u \|_{L^r}, \quad r \geq 0, \]

\[ \| I_N^L u - u \|_{L^r} \leq C N^{l-r} \| u \|_{L^r}, \quad r \geq 0, \quad l = 0, 1, \]

2.2. Scheme

The Legendre spectral scheme to (1) is: Find \(E_{zL} \in V_N^0 \quad V_M\) and \(H_{yL} \in V_{N+1} \quad V_M\) such that

\[
\begin{align*}
& \left( \frac{\partial}{\partial t} E_{zL}, v \right) + \left( H_{yL}, \frac{\partial}{\partial x} v \right) = 0, & v \in V_N^0 \otimes V_M, \\
& \left( \frac{\partial}{\partial t} H_{yL}, w \right) + \left( \frac{\partial}{\partial x} E_{zL}, w \right) = 0, & w \in V_{N+1} \otimes V_M, \\
& E_{zL}(x,0) = I_N^C e_{z0}, \quad H_{yL}(x,0) = P_{N+1} I_N^C H_{y0}, & x \in I_x.
\end{align*}
\]
We briefly describe the algorithm implementation of (4). For simplicity, taking 
\[ \begin{bmatrix} 1,1 \end{bmatrix} \times \begin{bmatrix} 1,1 \end{bmatrix}. \] The basis functions in space are to be chosen as 
\[ (x) = \begin{bmatrix} \frac{1}{2} x, \frac{1+x}{2}, x, \ldots, \lambda(x) \end{bmatrix}, \]
\[ 0(x) = \begin{bmatrix} \lambda(x), \ldots, \lambda^N(x) \end{bmatrix}, \]
\[ L(x) = \begin{bmatrix} L_0(x), L_1(x), \ldots, L_{N-1}(x) \end{bmatrix}, \]
\[ Y(x) = x \hat{L}(x, L_k(x, \ldots, L_{N-1}(x)), \]
\[ = L_k(x) \quad \text{and} \quad M = M_0 \times M_1 \times \cdots \times M_0. \]

The basis functions in time are to be chosen as 
\[ (t) = \begin{bmatrix} 1,1+t, 2(t), \ldots, \lambda(t) \end{bmatrix}, \]
\[ L(t) = \begin{bmatrix} L_0(t), L_1(t), \ldots, L_{M-1}(t) \end{bmatrix}, \]
\[ \text{where} \quad k(t) = L_k(t) \quad \text{and} \quad L_k(z(t)). \]

We denote the approximate solutions and test functions as follows 
\[ E_{\eta}(x,t) = (t) \hat{E} \quad T(x), \quad H_{\eta}(x,t) = (t) \hat{H} \quad T(x), \]
\[ v(x,t) = L(t) \hat{v} \quad T(x), \quad w(x,t) = L(t) \hat{w} \quad T(x). \]

By (4), we derive the following algebraic system 
\[ \begin{cases} (t, L) \hat{E}(0, \lambda) + (t, L) \hat{H}(x, 0, L) \lambda = 0, \\ (t, L) \hat{H}(L, L) \lambda + (t, L) \hat{E}(L, x) \lambda = 0, \end{cases} \]
and then the equation can be written in matrix form as 
\[ \begin{bmatrix} K^t \hat{E} M^t + M^t K^o \hat{E} = 0, \\ K^t \hat{H} D + M^t \hat{E} K^x T = 0, \end{bmatrix} \]
where \( \hat{E} \) and \( \hat{H} \) are matrices composed of coefficients that approximate solutions \( E_{\eta} \) and \( H_{\eta} \), respectively. In order to separate the initial conditions from the coefficient matrix, \( \hat{E} \), \( \hat{H} \) and \( M^t \) is divided into the following forms as 
\[ \hat{E} = \begin{bmatrix} \hat{E}_i \\ \hat{E}_0 \end{bmatrix}, \quad \hat{H} = \begin{bmatrix} \hat{H}_i \\ \hat{H}_0 \end{bmatrix}, \quad M^t = \begin{bmatrix} M^t_i \\ M^t_0 \end{bmatrix}, \]
where \( \hat{E}_i \) and \( \hat{H}_i \) are the first rows of the coefficient matrix \( \hat{E} \) and \( \hat{H} \) respectively, corresponding to the initial value, \( M^t_i \) is the first column of \( M^t \), which leads to 
\[ M^t_0 \hat{H} K^x = M^t_i \hat{H} K^x + M^t_0 \hat{H} K^x, \]
\[ M^t \hat{E} K^x T = M^t_i \hat{E} K^x T + M^t_0 \hat{E} K^x T. \]

By the properties of the basis function 
\[ \begin{bmatrix} L_n(t) & L_{n-2}(t) \end{bmatrix} = (2n \quad 1)^x L_{n-1}(t), \quad n \quad 2, \]
the orthogonality of Legendre polynomials shows that both $K^t$ and $D$ are diagonal matrices, and the elements on the diagonal of $K^t$ are $2$ except that the first element is zero. Thus, (6) can be expressed as

\[
\begin{align*}
2 \hat{E}_0 M' x + M_0' \hat{H}_0 K^s_0 &= M_i' \hat{H}_i K^s_0, \\
2 \hat{H}_0 D M_0' \hat{E}_0 K^{xt} &= M_i' \hat{E}_i K^{xt}.
\end{align*}
\] (8)

By the second equation of (8), we get

\[
2 \hat{H}_0 = M_0' \hat{E}_0 K^{xt} D^{-1} + M_i' \hat{E}_i K^x_0,
\] (9)

substituting (9) into the first equation of (8), let $K^{xx} = K^{xt} D^{-1} K^s_0$, we have

\[
4 \hat{E}_0 M' x + (M_0')^2 \hat{E}_0 K^{xx} = 2 M_i' \hat{H}_i K^x_0 M_0' M_i' \hat{E}_i K^{xx}.
\] (10)

Note that the first two columns of $\hat{E}_0$ correspond to the boundary values of $E_z$, respectively. It can be seen that (10) after decoupling is an equation only about $E_z$, and it can be solved by the method in [12].

3. Time Multi-Interval Method

In this section, a time multi-interval method is developed.

3.1. Preliminaries

For an integer $K>0$, we break $I_t$ into $K$ disjoint subintervals

\[
I_t = \bigcup_{k=1}^{K} I_k, \quad I_k = (a_k, a_{k+1}), \quad k = a_k, \quad a_{k+1}, \quad 1 \leq k \leq K,
\] (11)

where

\[
0 = a_0 < a_1 < \cdots < a_k < \cdots < a_K = T.
\]

Let $\mathcal{M} = (M_1, \cdots, M_K)$ and

\[
X_{\mathcal{M}} = W_{\mathcal{M}} \cap H^1(I_t), \quad W_{\mathcal{M}} = \{v: v|_{I_k} \in \mathbb{P}_{M_k}(I_k), 1 \leq k \leq K\},
\] (12)

where $\mathbb{P}_{M_k}(I_k)$ be the polynomials of $t$ of degree $M_k$. Let

\[
W_{M-1} = \{v: v|_{I_k} \in \mathbb{P}_{M_k-1}(I_k), 1 \leq k \leq K\},
\] (13)

where $\mathcal{M}-1 = (M_1-1, \cdots, M_K-1)$.

Let $\hat{I} = (1,1)$ be a reference interval, $\hat{t}^{k,c}_j$ and $\hat{w}^{k,c}_j (0 \leq j \leq M_k)$ be the CGL points and the weights on $\hat{I}$, let $\{t^{k,c}_j\}$ and $\{w^{k,c}_j\}$ be the CGL points and the weights on $I_k = (a_{k-1}, a_k), 1 \leq k \leq K$. Set
\begin{equation}
I_M^j = \{ t_j^{k,c} : t_j^{k,c} = \frac{\tau_{ij}^{k,c} + a_{k-1} + a_k}{2}, 0 \leq j \leq M_k, 1 \leq k \leq K \},
\end{equation}

where \( k = a_k - a_{k-1} \).

Let \( v^k \big|_{I_k} \), \( u, v \in C(\bar{T}) \) and \( k = \frac{1}{2} \), \( \gamma_k \), we define

\begin{equation}
(u, v)_{M,k} = \sum_{j=0}^{M_k} u^k(t_j^{k,c})v^k(t_j^{k,c})\omega_j, \quad (u, v)_{M} = \sum_{k=1}^{K}(u, v)_{M,k}.
\end{equation}

We define CGL interpolation operator \( I_M^c : C(\bar{T}) \rightarrow W_M \), which satisfies

\begin{equation}
I_M^c(u^{k,c}) = u(t_j^{k,c}), \quad 0 \leq M_k, \quad 1 \leq k \leq K.
\end{equation}

Set the relation as follow

\begin{equation}
v(t) = \hat{v}(\hat{t}), \quad t = \frac{1}{2}(\hat{t} + a_k + a_{k-1}), \quad a_{k-1} \leq t \leq a_k.
\end{equation}

We denote \( \hat{P}_{M_k-1} : L^2(I_k) \rightarrow \mathbb{P}_{M_k-1} \) the \( L^2 \)-Legendre spectral projection operator generated by \( P_{M_k-1} : L^2(I_k) \rightarrow W_{M_k-1} \) such that

\begin{equation}
(P_{M_k-1}v)\big|_{I_k}(t) = \hat{P} \big|_{I_k}(v)\big|_{I_k}(\hat{t}).
\end{equation}

### 3.2. Scheme

The space Legendre scheme and the time multi-interval scheme for (1) is: Find \( E_{zL}^k \in V_N^0 \otimes W_M \) and \( H_{yL}^k \in V_{N-1} \otimes W_M \) such that

\begin{equation}
\begin{cases}
(\varepsilon \partial_x E_{zL}^k, v)_\Omega + (H_{yL}^k, \partial_x v)_\Omega = 0, & \forall v \in V_N^0 \otimes W_{M-1}, \\
(\mu \partial_x H_{yL}^k, w)_\Omega - (\partial_x E_{zL}^k, w)_\Omega = 0, & \forall w \in V_{N-1} \otimes W_{M-1}, \\
E_{zL}^k(x,0) = I_N^c E_{z0}, \quad H_{yL}^k(x,0) = P_{N-1} I_N^c H_{y0}, & \forall x \in I_x.
\end{cases}
\end{equation}

In the computation and following analysis, we set

\begin{equation}
v^k(x,t) = v(x,t + a_k), \quad t + \hat{t_k} = (0, k), \quad 1 \leq k \leq K.
\end{equation}

Let \( k = I_k \times \hat{I}_k \), and then (14) can be expressed as: For \( 1 \leq k \leq K \), find \( E_{zL}^k \in V_N^0 \otimes \mathbb{P}_{M_k} \) and \( H_{yL}^k \in V_{N-1} \otimes \mathbb{P}_{M_k} \) such that
\[
\left\{
\begin{array}{l}
(\varepsilon \partial_t E_{zL}^k, v^k)_{\Omega_1} + (H_{yL}^k, \partial_x v^k)_{\Omega_1} = 0, \\
(\mu \partial_t H_{yL}^k, w^k)_{\Omega_1} - (\partial_t E_{zL}^k, w^k)_{\Omega_1} = 0, \\
E_{zL}^k(x, 0) = E_{zL}^{k-1}(x, \tau_{k-1}), \\
H_{yL}^k(x, 0) = H_{yL}^{k-1}(x, \tau_{k-1}), \\
x \in I_x,
\end{array}
\right.
\]

when \( k = 1 \), \( E_{zL}^0(x, 0) = I_N^L E_{z0}(x) \) and \( H_{yL}^0(x, 0) = P_N^L I_N^L H_{y0}(x) \).

4. Numerical Examples

In the section, the results of some numerical examples are given. We define

\[
E_\infty(E_z) = \max_{0 \leq j \leq N} \left| E_{zL}(x_j^c, t) - E_z(x_j^c, t) \right|,
\]

\[
E_\infty(H_y) = \max_{0 \leq j \leq N} \left| H_{yL}(x_j^c, t) - H_y(x_j^c, t) \right|.
\]

**Example 4.1** Consider (1) with \( I_x = (0,1), \ I_y = (0,1), \ I = I_x \times I_y, \ m = 1 \) and \( e = 1 \). The solution is as

\[
\begin{align*}
E_z(x,t) &= \cos(3t) \sin(3x), \quad (x,t) \in W, \\
H_y(x,t) &= \sin(3t) \cos(3x), \quad (x,t) \in W.
\end{align*}
\]

The example is computed by (4). From the table 1, we can see that the time direction has the same high precision as the space direction.

**Table 1.** \( L_\infty \)-error at \( t = 1 \) of the method (4).

| \( (N, M) \) | \( E_\infty(E_z) \) | \( E_\infty(H_y) \) | time |
|---------------|----------------|----------------|--------|
| (8,8)         | 4.04e-04       | 1.99e-02       | 0.07s  |
| (12,12)       | 9.38e-07       | 3.99e-05       | 0.08s  |
| (16,16)       | 6.57e-15       | 3.33e-08       | 0.10s  |
| (20,20)       | 1.38e-12       | 7.86e-12       | 0.10s  |
| (24,24)       | 1.69e-15       | 2.99e-15       | 0.11s  |
| (28,28)       | 1.94e-15       | 3.05e-15       | 0.12s  |

**Example 4.2** We use the time multi-interval method (14) to solve Example 4.1. Taking \( N = M = 28 \) and \( t = 5 \), table 2 shows the numerical results of (14).

**Table 2.** \( L_\infty \)-error of the method (14) \( (N = M = 28) \).

| \( t \) | \( E_\infty(E_z) \) | \( E_\infty(H_y) \) | time |
|--------|----------------|----------------|--------|
| 1.00   | 1.94e-15       | 3.05e-15       | 0.12s  |
| 2.00   | 2.44e-15       | 3.88e-15       | 0.13s  |
| 3.00   | 5.10e-15       | 3.77e-15       | 0.14s  |
| 4.00   | 6.63e-15       | 7.93e-15       | 0.17s  |
| 5.00   | 7.24e-15       | 7.43e-15       | 0.20s  |
5. Conclusion
In this paper, a time and space Legendre spectral method is investigated for the 1-D Maxwell equation and its time multi-interval method is developed. We will continue the above work and apply the method to high-dimensional Maxwell equations.

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