Raney numbers, threshold sequences and Motzkin-like paths

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Abstract
We provide new interpretations for a subset of Raney numbers, involving threshold sequences and Motzkin-like paths with long up and down steps.

Given three integers \( n, k, l \) such that \( n \geq 1, k \geq 2 \) and \( 0 \leq l \leq k - 2 \), a \((k, l)\)-threshold sequence of length \( n \) is any strictly increasing sequence \( S = (s_1 s_2 \ldots s_n) \) of integers such that \( ki \leq s_i \leq kn + l \). These sequences are in bijection with ordered \((l + 1)\)-tuples of \( k \)-ary trees. We prove this result and identify the Raney numbers that count the \((k, l)\)-threshold sequences. As a consequence, when \( k = 2 \) and \( k = 3 \), we deduce combinatorial identities involving Catalan numbers and powers of 2, and respectively Fuss-Catalan and Raney numbers. Finally, we show how to represent threshold sequences as Motzkin-like paths with long up and down steps, and deduce that these paths are enumerated by the same Raney numbers.

1 Introduction

In this paper, a \( k \)-ary tree \((k \geq 2)\) is any tree whose internal nodes, the root included, have exactly \( k \) children. It follows that such a tree either is a unique node (the tree is then called trivial), or is made of an internal node (the root) and \( k \) smaller \( k \)-ary subtrees. We assume that the \( k \) subtrees are ordered from left to right, meaning that two trees are equal if and only if either both of them are trivial, or both of them are non-trivial and they have the same \( k \)-trees identically ordered.

With the notation \( C_n^{(k)} \) for the number of \( k \)-ary trees with \( n \) internal nodes, we then have \( C_0^{(k)} = 1 \) and:

\[
C_n^{(k)} = \sum_{j_1 + j_2 + \ldots + j_k = n-1 \atop \forall h: 0 \leq j_h \leq n-1} C_{j_1}^{(k)} C_{j_2}^{(k)} \ldots C_{j_k}^{(k)} \tag{1}
\]

This recurrence defines the so-called Fuss-Catalan numbers, for which the following closed form is known [4]:

\[
C_n^{(k)} = \frac{1}{(k-1)n+1} \binom{kn}{n}. \tag{2}
\]

The Fuss-Catalan numbers are the particular case with \( r = 1 \) of the Raney numbers, defined as follows [10]:

\[
R_n^{(k,r)} = \frac{r}{kn+r} \binom{kn+r}{n} = \frac{r}{(k-1)n+r} \binom{kn+r-1}{n} \tag{3}
\]

As proved in [3], the Raney numbers are also related to the Fuss-Catalan numbers by the following relation:

\[
R_n^{(k,r)} = \sum_{i_1 + i_2 + \ldots + i_r = n \atop \forall h: 0 \leq i_h \leq n} C_{i_1}^{(k)} C_{i_2}^{(k)} \ldots C_{i_r}^{(k)} \tag{4}
\]
Using the interpretation of each \( C_{i_k}^{(k)} \), which represents the number of \( k \)-ary trees with \( i_k \) internal nodes, the previous recurrence relation implies that:

**Remark 1.1.** The Raney number \( R_n^{(k,r)} \) counts the number of ordered \( r \)-tuples of \( k \)-ary trees with a total number of \( n \) internal nodes.

The cases where \( k = 2 \) and \( k = 3 \) are particularly useful. Binary trees with \( n \) internal nodes, obtained when \( k = 2 \), are counted by \( C_n^{(2)} = \frac{1}{n+1} \binom{2n}{n} \), for \( n \geq 0 \), also called Catalan number and denoted \( C_n \).

Ternary trees with \( n \) internal nodes, obtained when \( k = 3 \), are counted by the Fuss-Catalan numbers \( C_n^{(3)} = \frac{1}{2n+1} \binom{3n}{n} = R_n^{(3,1)} \), and the ordered pairs of ternary trees with total number of \( n \) internal nodes are counted by the Raney number \( R_n^{(3,2)} = \frac{2}{2n+2} \binom{3n+1}{n} = \frac{1}{n+1} \binom{3n+1}{n} \). To simplify the notations, we denote by \( T_n = C_n^{(3)} = \frac{1}{2n+1} \binom{3n}{n} \) the number of ternary trees and by \( U_n = R_n^{(3,2)} = \frac{1}{n+1} \binom{3n+1}{n} \) the number of ordered pairs of ternary trees, both with \( n \) internal nodes. In the Online Encyclopedia of Integer Sequences [13], the sequences \( T_n \) and \( U_n \) are respectively OEIS A001764 and OEIS A006013, and enumerate many other combinatorial objects.

A \((k,l)\)-threshold sequence of length \( n \), for \( n \geq 1 \), \( k \geq 2 \) and \( 0 \leq l \leq k-2 \), is any strictly increasing sequence \( S = (s_1 s_2 \ldots s_n) \) of integers such that \( ki \leq s_i \leq kn + l \). See Figure [1] (Note that the case \( l = k - 1 \) is not significant, since the resulting sequences are easily identified as the \( n \)-length prefixes of the \((k,0)\)-threshold sequences of length \( n+1 \).) The contiguous subsequence \((s_p s_{p+1} \ldots s_q)\) of \( S \) is denoted by \( S[p,q] \). A proper \((k,l)\)-threshold sequence is a \((k,l)\)-threshold sequence such that \( s_n = kn + l \), i.e. either a \((k,0)\)-sequence, or a \((k,l)\)-threshold sequence with \( l \geq 1 \) which is not a \((k,l-1)\)-threshold sequence. Threshold-like sequences appear for instance in the characterization of maximally admissible pinnacle sets [12] (with \( k = 3 \)).

**Example 1.2.** Let \( k = 3 \) and \( n = 6 \). Then \( S_1 = (3 \ 6 \ 14 \ 15 \ 17 \ 18) \) is a proper \((3,0)\)-threshold sequence, whereas \( S_2 = (3 \ 6 \ 14 \ 15 \ 17 \ 19) \) is a proper \((3,1)\)-threshold sequence of length 6. Moreover, \( S_1 \) is also a \((3,1)\)-threshold sequence, but not a proper one. The sequence \( S_3 = (3 \ 4 \ 14 \ 15 \ 17 \ 18) \) is not a \((3,0)\)-threshold sequence since \( s_2 < 6 \).

**Remark 1.3.** The sequences \( S' = (s'_1 s'_2 \ldots s'_n) \) of integers such that \( ki + d \leq s'_i \leq kn + l + d \), for a fixed integer \( d \), are in bijection with the \((k,l)\)-threshold sequences, and are thus counted by the same formulas. We call them \((k,l)\)-sequences with offset \( d \).

In this paper, we show that \((k,l)\)-threshold sequences are related to \( k \)-ary trees in the sense that \((k,l)\)-threshold sequences of length \( n \) are in bijection with ordered \((l + 1)\)-tuples of \( k \)-ary trees with a total of \( n \) internal nodes, and are thus counted by the Raney numbers \( R_n^{(k,l+1)} \). We further deduce combinatorial identities involving Catalan, Fuss-Catalan and Raney numbers as well as bijections between \((k,l)\)-threshold sequences and related combinatorial objects.

To this end, in Section 2 we first consider the particular case \( k = 3 \), and obtain an implicit bijection by showing that the number of \((3,l)\)-threshold sequences of length \( n \) equals that of \((l+1)\)-tuples of ternary trees with \( n \) internal nodes, for \( l = 0, 1 \). Then, in Section 3, we propose, for arbitrary values \( k \) and \( l \), an explicit bijection between \((k,l)\)-threshold sequences and \((l+1)\)-tuples of \( k \)-ary trees. In Section 4, we present several combinatorial identities resulting from the recurrence relations proved in Sections 2 and 3. Finally, in Section 5, we represent threshold sequences as Motzkin-like paths with up steps \((1,u)\), \( u \geq 1 \), horizontal steps \((1,0)\) and down steps \((1,-d)\), \( 1 \leq d \leq k - 1 \). This representation allows us to deduce counting formulas for these paths too. Section 6 is the conclusion.
two depending whether the threshold sequence is simple or double. See Figure 1, with k = 3 values larger than 3. Sequences are in bijection with ternary trees, and the second result ensures the bijection between double ordered pairs of simple b-threshold sequences and ordered pairs of ternary trees.

In this section, k = 3 and the (3, l)-threshold sequences are called simple 3-threshold sequences when l = 0, and double 3-threshold sequences when l = 1. Simple 3-threshold sequences are thus also double 3-threshold sequences.

Let an (bn) be the number of simple (double) 3-threshold sequences of length n. Note that a1 = 1, a2 = 3, whereas b1 = 2, b2 = 7. By convention, we define a0 = bn = 1. We use various ways to count an and bn, then combine them to show that an satisfies the recurrence relation in Equation (1) and that bn counts ordered pairs of simple 3-threshold sequences. The first result allows us to deduce that simple 3-threshold sequences are in bijection with ternary trees, and the second result ensures the bijection between double 3-threshold sequences and ordered pairs of ternary trees.

For each i with 1 ≤ i ≤ n − 1, let the values 3i, 3i + 1, 3i + 2 be called standard values for si, and the values larger than 3i + 2 be called high values for si. For sn, there are only standard values, either one or two depending whether the threshold sequence is simple or double. See Figure 1 with k = 3.

### 2.1 First count: from left to right

A simple 3-threshold sequence of length n can be of two types with respect to s1.

- When s1 is a standard value, (s2 . . . sn) is any simple 3-threshold sequence of length n − 1 and offset 3. Using Remark 1.3, the number of such sequences is an−1, thus the number of simple 3-threshold sequence of length n with these properties is 3an−1, since there are three standard values available for s1.

- When s1 is a high value, there exists a largest value h such that s1, s2, . . . , sh are high values. Then sh+1 is a standard value, and 1 ≤ h ≤ n − 2, since sn = kn and thus no high value is available for sn−1. The simple 3-threshold sequence is thus made of a subsequence (s1 . . . sh+1) of length h + 1 containing only high values except for the last one, which is a standard value, followed by a subsequence (sh+2 . . . sn) of length n − h − 1 which may be any simple 3-threshold sequence of this length with offset 3(h + 1). Using again Remark 1.3, the number of such 3-threshold sequences for a fixed h is thus ch+1an−h−1, where ch+1 is the number of sequences (s1 . . . sh+1) described above. Each such sequence satisfies 3(i + 1) ≤ si ≤ 3(h + 1) + 2 for 1 ≤ i ≤ h and 3(h + 1) ≤ sh+1 ≤ 3(h + 1) + 2. Thus its h − 1 first elements are constrained as in a 3-threshold sequence with offset 3, whereas sh, sh+1 ∈ {3(h + 1), 3(h + 1) + 1, 3(h + 1) + 2}. Then we cannot have sh+1 = 3(h + 1),
since no value would be available for \( s_h \). Moreover, when \( s_{h+1} = 3(h+1) + 1 \) we have \( s_h = 3(h+1) \) and \((s_1 \ldots s_{h-1} s_h)\) is any simple 3-threshold sequence with offset 3 of length \( h \); there are \( a_h \) such sequences. And when \( s_{h+1} = 3(h+1) + 2 \) we have \( s_h \in \{3(h+1), 3(h+1) + 1\} \) and \((s_1 \ldots s_{h-1} s_h)\) is any double 3-threshold sequence with offset 3 of length \( h \); there are \( b_h \) such sequences. We deduce that \( c_{h+1} = a_h + b_h \).

In conclusion, the number of simple 3-threshold sequences of length \( n \) is given by:

\[
a_n = 3a_{n-1} + \sum_{h=1}^{n-2} (a_h + b_h)a_{n-h-1} \quad (5)
\]

For the double 3-threshold sequences, the enumeration is similar, except that we may have \( h = n - 1 \) when \( s_1 \) is a high value. This can only happen for sequences with \( s_{n-1} = 3n \) and \( s_n = 3n + 1 \). For these sequences, \((s_1 \ldots s_{n-1})\) is any simple 3-threshold sequence, so an amount of \( a_{n-1} \) must be added to the final count:

\[
b_n = 3b_{n-1} + \sum_{h=1}^{n-2} (a_h + b_h)b_{n-h-1} + a_{n-1} \quad (6)
\]

### 2.2 Second count: from right to left

Consider a simple 3-threshold sequence. Recall that \( s_n = 3n \) and make the following observation.

**Remark 2.1.** For every \( j \leq n - 1 \), the sequences \((s_1 s_2 \ldots s_j)\) satisfying for each \( i \) with \( 1 \leq i \leq j \) the constraint \( 3i \leq s_i \leq 3j + 2 \) (i.e. such that \( s_j \) is a standard value) are exactly the \( j \)-length prefixes of the simple 3-threshold sequences of length \( j + 1 \). Thus, there exist \( a_j+1 \) such sequences.

Again, a simple 3-threshold sequence may be of two types.

- When \( s_{n-1} \in \{3(n-1), 3(n-1) + 1\} \), the sequence \((s_1 \ldots s_{n-1})\) is any double 3-threshold sequence of length \( n - 1 \). There are \( b_{n-1} \) such simple 3-threshold sequences of length \( n - 1 \).

- When \( s_{n-1} = 3(n-1) + 2 \), two cases are possible. When \( s_{n-2} \) is a standard value, by Remark 2.1 we deduce that there are \( a_{n-1} \) 3-threshold sequences of length \( n \) with this property. When \( s_{n-2} \) is a high value, we have \( s_{n-2} \in \{3(n-1), 3(n-1) + 1\} \) and there exists a lowest index \( h \), with \( 0 \leq h \leq n - 3 \), such that \( s_{h+1} \), \ldots, \( s_{n-2} \) are high values. Thus \( s_h \), when \( h \geq 1 \), is a standard value. In this case, the simple 3-threshold sequence of length \( n \) is made of a subsequence \((s_1 s_2 \ldots s_h)\) satisfying Remark 2.1 for \( j = h \), followed by a double 3-threshold sequence \((s_{h+1} s_{h+2} \ldots s_{n-2})\) with offset 3 and length \( n - 2 - h \), which is on its turn followed by \( 3(n-1) + 2 \) and \( 3n \). The number of such sequences, for a fixed \( h \), is \( a_{h+1} b_{n-2-h} \).

We deduce that:

\[
a_n = b_{n-1} + a_{n-1} + \sum_{h=0}^{n-3} a_{h+1} b_{n-2-h} = b_{n-1} + a_{n-1} + \sum_{h=1}^{n-2} a_h b_{n-1-h} = \sum_{h=0}^{n-1} a_h b_{n-1-h} \quad (7)
\]

A double 3-threshold sequence is in one of the two following cases:

- When \( s_n \in \{3n, 3n + 1\} \) and \( s_{n-1} \) is a standard value, the double 3-threshold sequence of length \( n \) is made of a subsequence \((s_1 s_2 \ldots s_{n-1})\) satisfying Remark 2.1 followed either by \( 3n \) or by \( 3n + 1 \). There are \( 2a_n \) such sequences.
• When \( s_n = 3n + 1 \) and \( s_{n-1} = 3n \), two situations are possible for the sequence \((s_1 s_2 \ldots s_{n-2})\). If \( s_{n-2} \) is a standard value, by Remark \( 2.1 \) there are \( a_{n-1} \) simple 3-threshold sequences \((s_1 s_2 \ldots s_{n-2})\), and thus the same number of double 3-threshold sequences with the desired properties. If \( s_{n-2} \) is a high value, then we consider as before the minimum value of \( h \), \( 0 \leq h \leq n - 3 \), such that \( s_{h+1}, \ldots, s_{n-2} \) are high values. The double 3-threshold sequence of length \( n \) is made of a subsequence \((s_1 s_2 \ldots s_h)\) satisfying Remark \( 2.1 \) followed by a simple 3-threshold sequence \((s_{h+1} s_{h+2} \ldots s_{n-1})\) with offset 3, and further followed by \( 3n + 1 \). For a fixed \( h \), the number of such sequences is \( a_{h+1}a_{n-1-h} \).

In conclusion, we have:

\[
b_n = 2a_n + a_{n-1} + \sum_{h=0}^{n-3} a_{h+1}a_{n-1-h} = 2a_n + \sum_{h=1}^{n-1} a_{h}a_{n-h} = \sum_{h=0}^{n} a_{h}a_{n-h} \tag{8}
\]

### 2.3 Implicit bijections

**Proposition 2.2.** Simple 3-threshold sequences of length \( n \) are in bijection with ternary trees with \( n \) internal nodes.

**Proof.** We use Equations \((5)\) and \((8)\) to deduce that \( a_n \) satisfies the same recurrence relation as \( T_n \) (see Equation \((1)\), where \( C^{(3)}_n = T_n \)).

\[
a_n = 3a_{n-1} + \sum_{h=1}^{n-2} (a_h + b_h)a_{n-h-1}
\]

\[
= 3a_{n-1} + \sum_{h=1}^{n-2} a_ha_{n-h-1} + \sum_{h=1}^{n-2} b_ha_{n-h-1}
\]

\[
= 3a_{n-1} + \sum_{h=1}^{n-2} a_ha_{n-h-1} + \sum_{h=1}^{n-2} \left( \sum_{j=1}^{h} a_ja_{h-j} \right)a_{n-h-1}
\]

\[
= a_{n-1} + \sum_{j=0}^{n-1} a_ja_{n-j-1}a_{n-(n-1)-1} + \sum_{h=1}^{n-2} \sum_{j=0}^{h} a_ja_{h-j}a_{n-h-1}
\]

\[
= a_{n-1} + \sum_{h=1}^{n-1} \sum_{j=0}^{h} a_ja_{h-j}a_{n-h-1} = \sum_{h=0}^{n-1} \sum_{j=0}^{h} a_ja_{h-j}a_{n-h-1}
\]

Since \( a_0 = T_0 = 1 \) and \( a_n, T_n \) satisfy the same recurrence relation, we deduce that \( a_n = T_n \) for \( n \geq 1 \). The conclusion follows.

**Proposition 2.3.** Double 3-threshold sequences of length \( n \) are in bijection with ordered pairs of ternary trees whose total number of internal nodes is \( n \).

**Proof.** By Equation \((8)\), we have that \( b_n = \sum_{h=0}^{n} a_{h}a_{n-h} \). For each \( h \), the term \( a_{h}a_{n-h} \) counts the ordered pairs of simple 3-threshold sequences where the first (second) sequence in the pair is any simple 3-threshold...
sequence of length \( h \) \((n - h)\). But, by Proposition 2.2, simple 3-threshold sequences of length \( w \) are in a 1-to-1 correspondence with ternary trees with \( w \) internal nodes, so the conclusion follows.

The two previous propositions imply that:

**Corollary 2.4.** For each \( n \geq 1 \), \( a_n = T_n = \frac{1}{2n+1} \binom{3n}{n} \) and \( b_n = U_n = \frac{1}{n+1} \binom{3n+1}{n} \).

**Proposition 2.5.** Proper double 3-threshold sequences are in bijection with ordered 4-tuples of ternary trees with total number of internal nodes equal to \( n - 1 \). Moreover, we have

\[
b_n - a_n = \sum_{h=0}^{n-1} a_h a_{n-h} = \sum_{h=0}^{n-1} b_h b_{n-h-1} = U_n - T_n = \frac{2}{n+1} \binom{3n}{n-1}.
\]

**Proof.** Using Equations (6) and (7), we obtain:

\[
b_n = 3b_{n-1} + \sum_{h=1}^{n-2} a_h b_{n-1-h} + \sum_{h=1}^{n-2} b_h b_{n-1-h} + a_{n-1}
\]

\[
= 3b_{n-1} + (a_n - b_{n-1} - a_{n-1}) + \sum_{h=1}^{n-2} b_h b_{n-1-h} + a_{n-1}
\]

\[
= a_n + 2b_{n-1} + \sum_{h=1}^{n-2} b_h b_{n-1-h}
\]

\[
= a_n + \sum_{h=0}^{n-1} b_h b_{n-1-h}
\]

which implies together with Equation (8) that:

\[
b_n - a_n = \sum_{h=0}^{n-1} b_h b_{n-h-1} = \sum_{h=0}^{n-1} a_h a_{n-h}
\]

Now, \( \sum_{h=0}^{n-1} b_h b_{n-h-1} = \sum_{h=0}^{n-1} U_h U_{n-h-1} \) and thus counts, for \( n \geq 1 \), the number of ordered 4-tuples of ternary trees with total number of internal nodes equal to \( n - 1 \). This follows from the interpretation of \( U_k \), \( k \geq 0 \), which counts the pairs of ternary trees with \( k \) internal nodes. A simple computation further shows, using Corollary 2.4, that \( b_n - a_n = U_n - T_n = \frac{2}{n+1} \binom{3n}{n-1} \). The conclusion follows.

**Remark 2.6.** The sequence corresponding to the number of ordered 4-tuples of ternary trees is known as OEIS A006629 [13].

### 3. Explicit bijections

We start this section by defining a labeling for the \( k \)-ary trees. Note that a \( k \)-ary tree with \( n \) internal nodes has \( nk + 1 \) nodes.

Let \( w \) be an integer. A \textit{k-ary w-tree} is a \( k \)-ary tree \( A \) whose nodes are labeled such that a breadth first traversal of \( A \) yields the list of nodes \( w, w - 1, \ldots, w - nk \), where \( n \) is the number of internal nodes of \( A \). Equivalently, the root of \( A \) is \( w \), and the nodes on each level are labeled in decreasing order from left to right, starting with the largest value not used on the previous level.
Figure 2: a) The quaternary tree $A^1_S$ computed for the $(4, 0)$-threshold sequence $S = (7, 12, 14, 16)$. b) The two quaternary trees $A^1_{V}$ (left), $A^2_{V}$ (right) computed for the $(4, 2)$-threshold sequence $V = (7, 9, 17, 18)$.

Example 3.1. The trees in Figure 2 are respectively a quaternary 16-tree, a quaternary 18-tree and a quaternary 9-tree.

Remark 3.2. Two $k$-ary $w$-trees are equal if and only if the sets of their internal nodes, seen as labeled nodes, are equal.

Remark 3.3. In each $k$-ary $w$-tree with $j$ internal nodes, the smallest label $x$ of a node satisfies $x = w - jk$.

Let $S = (s_1s_2 \ldots s_n)$ be a $(k, l)$-threshold sequence. The cut index of $S$ is the largest index $i < n$ such that

$$s_i < s_n - (n - i)k$$  \hspace{1cm} (11)

if such an element exists, and is equal to 0 otherwise. Intuitively, $s_i$ is the largest element in $S$ whose value is not large enough to be a label in the $k$-ary $s_n$-tree whose internal nodes have the labels $s_n, s_{n-1}, \ldots, s_{i+1}$ (see also Remark 3.3). For each $(k, l)$-threshold sequence $S$ of length $n$, let $\text{Forest}(S)$ be the set of trees, initially empty, defined as follows (see Figure 2) as well as Examples 3.4 and 3.5:

1. Let $A^1_S$ be the $k$-ary $s_n$-tree whose internal nodes ordered according to a breadth first traversal are the elements $s_n, s_{n-1}, \ldots, s_{i+1}$ of $S$. Then $A^1_S$ belongs to $\text{Forest}(S)$ (discard the node labels, they were needed only to define the tree).

2. If $i \neq 0$, let $Q = S[1, i]$ and add $\text{Forest}(Q)$ to $\text{Forest}(S)$.

Assume $\text{Forest}(S) = \{A^1_S, A^2_S, \ldots, A^t_S\}$ where $1, 2, \ldots, t$ indicate the order of computation of the trees. Let $Q_1 = S$ and let $Q_2, \ldots, Q_t$ be the successive sequences computed in step (2) above, respectively generating $A^2_S, A^3_S, \ldots, A^t_S$ using step (1). We prove below that the unique index $l_p$ such that $Q_p$ is a proper $(k, l_p)$-threshold sequence, $1 \leq p \leq t$, satisfies $l = l_1 > l_2 > \ldots > l_t$. Then define $\text{Tuple}(S)$ as the $(l+1)$-tuple containing $A^p_S$ in position $l_p + 1$, $1 \leq p \leq t$, and the trivial $k$-ary tree $\lambda$ on each of the remaining positions.

Example 3.4. For the proper $(4, 0)$-threshold sequence $S = (7, 12, 14, 16)$ of length 4, also denoted by $Q_1$, none of the indices satisfies the definition of a cut index, thus $\text{Forest}(S) = \{A^1_S\}$ and the quaternary tree $A^1_S$ is depicted in Figure 2 (a). The $(l + 1)$-tuple associated with $S$ is a 1-tuple, represented by $A^1_S$ alone. When $S$ is seen as a $(4, 2)$-threshold sequence, the $(l + 1)$-tuple associated with $S$ is the 3-tuple $(A^1_S, \lambda, \lambda)$. Indeed, $16 = 4 \cdot 4 + 0$ thus for the sequence $V (= Q_1)$ we have $l_1 = 0$.  


Example 3.5. For the proper (4, 2)-threshold sequence \( V = (7, 9, 17, 18), \) also denoted by \( Q_1, \) we have \( s_3 = 17. \) Then \( s_3 \) does not satisfy the condition \( \Box \), since \( 17 \not\in (4 - 3)4 = 14, \) so the index 3 is not the cut index of \( V. \) But \( s_2 = 9 \) and \( 9 < (4 - 2)4 = 10, \) so that 9 does not belong to the tree with internal nodes 18 and 17, denoted by \( A_1^l \) and depicted in Figure 3(b), left. Therefore \( i = 2 \) is the cut index of \( V, \) and 9 is the root of the next tree. The next tree, called \( A_2^l \) is thus computed in the same way using the (4, 2)-threshold sequence \( Q_2 = V[1, i] = (7, 9), \) which has no cut index. Then \( A_2^l \) is the tree in Figure 3(b), right. We deduce that \( Forest(V) = \{ A_1^l, A_2^l \}. \) The \( (l + 1) \)-tuple associated with \( V \) is the 3-tuple \( (\lambda, A_1^l, A_2^l) \) since \( 18 = 4 \cdot 4 + 2 \) (thus \( l_1 = 2 \) for \( Q_1 = V \)) and \( 9 = 2 \cdot 4 + 1 \) (thus \( l_2 = 1 \) for \( Q_2 = V[1, 2] = (7, 9) \)).

Theorem 3.6. Let \( n, k, l \) be three integers with \( n \geq 1, k \geq 2 \) and \( 0 \leq l \leq k - 2. \) The function \( Tuple(S) \) is well-defined for the pair \( (k, l), \) and is a bijection between \( (k, l) \)-threshold sequences of length \( n \) and \( (l + 1) \)-tuples of \( k \)-ary trees with total number of \( n \) internal vertices.

Proof. We first prove two affirmations, named (A) and (B):

(A) For any \( (k, l) \)-threshold sequence \( S \) of length \( n, \) the number of trees in \( Forest(S) \) is upper bounded by \( l + 1. \)

When \( A_S^l \) is built, the value \( s_i \) given by the cut index \( i \) of \( S \) is not a node of \( A_S^l. \) Indeed, \( s_i \) is the \( (n - i + 1) \)-th element of \( S \) in decreasing order of the indeces, thus \( A_S^l \) is built on \( (n - i) \) internal nodes and by Remark 3.3, its smallest label is \( s_n - (n - i)k. \) Condition \( \Box \) then ensures that \( s_i \) is not a node label from \( A_S^l. \) Then, either \( A_S^l \) contains all the elements in \( S \) (this is the case \( i = 0), \) or \( S[1, i] \) must be used to complete the set \( Forest(S) \) (this is the case \( i \geq 1). \)

We use induction on \( l \) to show Affirmation (A). When \( l = 0, s_n = kn \) and for each \( r < n \) we have \( s_r \geq kr = kn - (n - r)k = s_n - (n - r)k, \) thus \( s_r \) belongs to the \( k \)-ary \( s_n \)-tree with internal nodes \( s_n, s_{n-1}, \ldots, s_{r+1}. \) We deduce that \( A_S^l \) contains all the elements in \( S \) and the conclusion follows.

Assume now that Affirmation (A) is true for all \( (k, l') \)-threshold sequences \( Q \) with \( l' < l. \) Assume moreover that \( S \) is a proper \( (k, l) \)-threshold sequence, i.e. \( s_n = kn + l, \) otherwise the conclusion follows by inductive hypothesis. Let \( i \) be the cut index of \( S, \) denote by \( Q = S[1, i] \) and note that \( q_i = s_i < s_n - (n - i)k = kn + l - nk + ik = ik + l, \) thus \( q_i \leq ik + l - 1. \) This means \( Q \) is a proper \( (k, l_Q) \)-threshold sequence of length \( i, \) for some \( l_Q \leq l - 1. \) Then, using the inductive hypothesis for \( Q, \) we deduce that the number of trees in \( Forest(Q) \) is upper bounded by \( l_Q + 1, \) thus by \( l, \) and therefore the number of trees in \( Forest(S) \) is upper bounded by \( l + 1. \)

(B) With the notation \( Q_1 = S, Q_2, \ldots, Q_l \) for the successive sequences respectively generating \( A_2^l, A_3^l, \) \ldots, \( A_l^l, \) and assuming each \( Q_p \) is a proper \( (k, l_p) \)-sequence, the sequence \( l_1, l_2, \ldots, l_i \) is a strictly decreasing sequence of integers.

As proved above for \( S \) and \( Q = S[1, i], \) where \( i \) is the cut index of \( S, \) we have \( q_i = s_i < ik + l \) meaning that \( q_i = ik + l' \) with \( l' < l. \) The same reasoning may be applied to each pair \( Q_i, Q_{i+1} \) and Affirmation (B) follows.

By Affirmations (A) and (B), the function \( Tuple(S) \) is well defined, since no pair of trees \( A_S^l, A_T^l \) is affected to the same position, and there are enough positions in the \( (l + 1) \)-tuple to contain all the trees in \( Forest(S). \)

We show that \( Tuple(S) \) is a bijection.

Injectivity. Assume that \( Tuple(S) = Tuple(V) \) for two \( (k, l) \)-threshold sequences \( S \) and \( V. \) Denote by \( y_1, \ldots, y_k \) from right to left the positions of the non-trivial trees, which are the same in both tuples. Then by the definition of the function \( Tuple \) and the equation \( Tuple(S) = Tuple(V), \) we have that \( A_S^l = A_T^l \)
Tuple(S)[wp] = Tuple(V)[wp] = A^p, for 1 ≤ p ≤ t. Recall that all the trees in the tuples are unlabeled k-ary trees.

Let \( r_{p-1} \) be the number of internal nodes in \( A^1_S, \ldots, A^{p-1}_S \). By the definition of the function Tuple, and using the same notations \( Q_i \) as above with respect to \( S \), during the construction of Tuple(S) the root of \( A^p_S \) has the label \( w_p = k(n-r_{p-1}) + l_p \). The same reasoning for \( V \), assuming the sequences used to build Tuple(V) are \( Y_1, \ldots, Y_t \), imply that the root \( A^p_V \) has the label \( k(n-r_{p-1}) + l'_p \), where \( Y_p \) is a proper \((k, l'_p)\)-sequence. But \( A^p_S \) and \( A^p_V \) are placed on the same position in Tuple(S) and Tuple(V) respectively, meaning that \( l_p = l'_p = y_p - 1 \), and thus the roots of \( A^p_S \) and \( A^p_V \) have the same label \( w_p \). Then \( A^p_S \) and \( A^p_V \) are not only identical when seen as k-ary trees, but also when seen as k-ary \( w \)-trees (see Remark 3.2).

Then let \( I_p \) be the increasing sequence of the internal nodes of \( A^p_S \), seen as a k-ary \( w_p \)-tree. Then \( S = I_1 I_{t-1} \ldots I_1 \). The sequence \( V \) is computed similarly using the same increasing sequences, since the labeled trees are identical. Thus \( S = V \).

**Surjectivity.** Let \( A = (A_1, \ldots, A_{l+1}) \) be a \((l+1)\)-tuple of k-ary trees, and assume that \( A_{y_1}, A_{y_2}, \ldots, A_{y_t} \) are the non-trivial k-ary trees, with \( y_1 > y_2 > \ldots > y_t \). Let \( r_p \) be the total number of internal nodes in \( A_{y_1}, \ldots, A_{y_t} \). We label \( A_{y_p} \) as a k-ary \( w_p \)-tree with \( w_p = k(n-r_{p-1}) + y_p - 1 \), and we denote by \( I_p \) the increasing sequence of its internal nodes. Then the length of \( I_p \) is \( |I_p| = r_p - r_{p-1} \).

Define \( S = I_1 \ldots I_t \) and notice that the number \( n \) of elements in \( S \) is the total number \( n \) of internal vertices in all the trees of \( A \). Also denote by \( Q_p = I_1 \ldots I_p \) for \( 1 ≤ p ≤ t \). Then the rightmost element of \( Q_p \) is the root of \( A_{y_p} \), that is \( w_p \), and its length is:

\[
|Q_p| = n - (|I_1| + \ldots + |I_{p-1}|) = n - (r_1 - r_0 + r_2 - r_1 + \ldots + r_{p-1} - r_{p-2}) = n - r_{p-1}.
\]

Then we also have \( w_p = s_{|Q_p|} \). We note that:

\[
w_p = k(n - r_{p-1}) + y_p - 1 = k(n - r_{p-2}) - (r_{p-1} - r_{p-2})k + y_p - 1 + y_p - y_{p-1}
\]

\[
= w_{p-1} - (r_{p-1} - r_{p-2})k + y_p - y_{p-1}
\]

\[
< w_{p-1} - (r_{p-1} - r_{p-2})k = w_{p-1} - (|Q_{p-1}| - |Q_p|)k.
\]

Now, \( w_p < w_{p-1} - (|Q_{p-1}| - |Q_p|)k \) is condition (11) applied to \( Q_{p-1} \) and the index \( |Q_p| \) of \( Q_{p-1} \), given that \( w_p = s_{|Q_p|} \). Moreover, by the definition of a k-ary \( w_{p-1} \)-tree, for any element \( s_j \) situated between \( w_{p-1} \) and \( w_p \) Equation (11) cannot hold. Thus \( |Q_p| \) is the cut index of \( Q_{p-1} \), and thus \( A = Tuple(S) \).

**Corollary 3.7.** The number of \((k, l)\)-threshold sequences of length \( n \) is equal to the Raney number:

\[
R^{(k,l+1)}_n = \frac{l + 1}{kn + l + 1} \binom{kn + l + 1}{n}.
\]

**Proof.** The result follows from Theorem 3.6, Remark 3.5 and Equation (3). \( \square \)

**Corollary 3.8.** For each \( l ≥ 1 \), the number of proper \((k, l)\)-threshold sequences of length \( n \) is equal to the Raney number:

\[
P_{n-1}^{(k,k+l)} = \frac{k + l}{(k-1)(n-1) + k + l} \binom{kn + l - 1}{n-1}.
\]

**Proof.** By Corollary 3.7, the number of proper \((k, l)\)-threshold sequences of length \( n \) is equal to \( R^{(k,l+1)}_n - R^{(k,l)}_n \), and a well known recurrence relation or a simple verification indicates that this value is \( R^{(k,k+l)}_{n-1} \). \( \square \)
4 Combinatorial identities

In this section, we deduce from our previous results three combinatorial identities, obtained when \( k = 2 \) and \( k = 3 \).

4.1 Case \( k = 2 \): Catalan numbers

Proposition 4.1. Catalan numbers \( C_n \) satisfy for all \( n \geq 1 \) the recurrence relation:

\[
C_n = \sum_{r+s+t=n-1 \atop r,s \geq 1; t \geq 0} C_r C_s 2^t + 2^{n-1}
\]  

(12)

Proof. By Corollary 3.7 for \( k = 2, l = 0 \) and recalling that \( C_n = C_n(2) = R_n^{(2,1)} \), we deduce that \( C_n \) is the number of \((2,0)\)-threshold sequences of length \( n \).

As in Section 2, we propose a count of the \((2,0)\)-threshold sequences \( S \) of length \( n \) that allows us to obtain Equation (12). The last element of these sequences is always \( s_n = 2n \). Similarly to the case \( k = 3 \), for each \( i \) with \( 1 \leq i \leq n - 1 \), the values \( 2i, 2i + 1 \) are called standard values for \( s_i \), and the values larger than \( 2i + 1 \) are called high values for \( s_i \). See Figure 1. We assume that \( s_0 = 0 \), and this value is neither high nor standard.

For each \((2,0)\)-threshold sequence \( S \) of length \( n \), we introduce the following notation:

\[
a = \max \{ j \mid s_{j-1} \text{ is not a standard value and } s_j \text{ is a standard value} \}
\]

\[
b = \max \{ j \mid s_j \text{ is not a high value value and } s_{j+1} \text{ is a high value} \}
\]

Then \( a = 1 \) if and only if the sequence \( S \) contains only standard values. There are \( 2^{n-1} \) such \((2,0)\)-threshold sequences.

Otherwise, \( 2 \leq a \leq n - 1 \) since there are no high values for \( s_{n-1} \). In this case, \( S \) is made of 1) a \((2,0)\)-threshold subsequence of length \( a \) such that \( s_{a-1} = 2a \) and \( s_a = 2a + 1 \) (these are the only possible values), concatenated with 2) any subsequence \((s_{a+1} \ldots s_{n-1})\) of length \( n - 1 - a \) made of standard elements, and followed by \( 2n \).

The subsequences described in 2) are counted by \( 2^{n-1-a} \) since each \( s_i \) may take one of two precise values, for each \( i \) with \( a + 1 \leq i \leq n - 1 \). We need \( b \) to count the subsequences described in 1). We have \( 0 \leq b \leq a - 2 \). Each of the subsequences described in 1) are made of 1') an arbitrary \((2,0)\)-subsequence \((s_1 \ldots s_b)\) with \( s_b \in \{2b, 2b + 1\} \), followed by 2') an arbitrary \((2,0)\)-subsequence \( s_{b+1}, \ldots, s_{a-1} \) with \( s_i \geq 2i + 2 \) for each \( i \) with \( b + 1 \leq i \leq a \) and \( s_{a-1} = 2a \) as explained above; the sequence ends with \( s_a = 2a + 1 \). By Remark 2.1, the subsequences in 1') are in bijection with the \((2,0)\)-threshold sequences of length \( b+1 \). Then these sequences are enumerated by \( C_{b+1} \). The subsequences in 2') are the \((2,0)\)-threshold subsequences of length \( a - 1 - b \) and offset 2.

Thus the total number of \((2,0)\)-threshold sequences of length \( n \) is

\[
C_n = \sum_{a=2}^{n-1} \sum_{b=0}^{a-2} C_{b+1} C_{a-1-b} 2^{n-1-a} + 2^{n-1}
\]  

(13)

Let \( r = b + 1, s = a - r \) and \( t = n - 1 - (r + s) \). Then we successively have:
\[ C_n = \sum_{a=2}^{n-1} \sum_{r=1}^{n-1-a} C_r C_{a-r} 2^{n-1-a} + 2^{n-1} = \sum_{r=1}^{n-1} \sum_{a=r+1}^{n-1} C_r C_{a-r} 2^{n-1-a} + 2^{n-1} \]
\[ = \sum_{r=1}^{n-1} \sum_{s=1}^{n-1-r} C_r C_s 2^{n-1-(s+r)} + 2^{n-1} = \sum_{r,s \geq 1; t \geq 0} C_r C_s 2^t + 2^{n-1} \]

The conclusion is proved. \( \square \)

### 4.2 Case \( k = 3 \): Relations between \( T_n \) and \( U_n \)

According to the introduction and to Corollary 2.4, we have:

1. \( T_n = a_n = C_n^{(3)} \), and this value counts the number of ternary trees with \( n \) internal nodes, as well as the simple 3-threshold sequences of length \( n \).

2. \( U_n = b_n = R_n^{(3,2)} \), and this value counts the number of ordered pairs of ternary trees with a total of \( n \) internal nodes, as well as the double 3-threshold sequences of length \( n \).

**Proposition 4.2.** Assuming \( T_0 = U_0 = 1 \), the following relations hold for \( n \geq 1 \):

\[ 2 \sum_{h=0}^{n-1} \frac{1}{h+1} T_h T_{n-1-h} = 3U_{n-1} - T_n \]
\[ 2 \sum_{h=0}^{n-1} \frac{1}{3h+1} U_h U_{n-1-h} = 4T_n - U_n \]

**Proof.** We use the equations proved in Section 2 and thus also the notations \( a_n \) and \( b_n \), rather than \( T_n \) and \( U_n \), during the proof.

For each \( n \geq 1 \), the following equations may be obtained by basic computations using the close forms for \( a_n \) and \( b_n \):

\[ a_n + b_n = \frac{2}{n+1} \binom{3n}{n} = \frac{2}{n+1} (2n+1) a_n = 2 \left( 2 - \frac{1}{n+1} \right) a_n \]
\[ \frac{1}{n+1} a_n = \frac{1}{3n+1} b_n \]
\[ 3a_n - b_n = \frac{1}{n} (b_n - a_n) = \frac{2}{3n+1} b_n \]

They are used below without recalling it. Then Equation (5) and Equation (8) imply:

\[ a_n - 3a_{n-1} = \sum_{h=1}^{n-2} (a_h + b_h) a_{n-h-1} = 2 \sum_{h=1}^{n-2} \left( 2 - \frac{1}{h+1} \right) a_h a_{n-h-1} \]
\[ = 4 \sum_{h=1}^{n-2} a_h a_{n-h-1} - 2 \sum_{h=1}^{n-2} \frac{1}{h+1} a_h a_{n-h-1} \]
\[ = 4 (b_{n-1} - 2a_{n-1}) - 2 \sum_{h=1}^{n-2} \frac{1}{h+1} a_h a_{n-h-1} \]
We deduce:

\[ 2 \sum_{h=1}^{n-2} \frac{1}{h+1} a_h a_{n-h-1} = 4b_{n-1} - a_n - 5a_{n-1} \]

\[ 2 \sum_{h=0}^{n-2} \frac{1}{h+1} a_h a_{n-h-1} = 4b_{n-1} - a_n - 3a_{n-1} \]

\[ 2 \sum_{h=0}^{n-1} \frac{1}{h+1} a_h a_{n-h-1} = 3b_{n-1} - a_n \]

The first equation in the proposition is proved. The approach is similar for the second one. Using Equation (6) and Equation (7) we deduce:

\[ b_n - 3b_{n-1} - a_{n-1} = \sum_{h=1}^{n-2} (a_h + b_h)b_{n-h-1} = 2 \sum_{h=1}^{n-2} \left( 2 - \frac{1}{h+1} \right) a_h b_{n-h-1} \]

\[ = 4 \sum_{h=1}^{n-2} a_h b_{n-h-1} - 2 \sum_{h=1}^{n-2} \frac{1}{h+1} a_h b_{n-h-1} \]

\[ = 4(a_n - b_{n-1} - a_{n-1}) - 2 \sum_{h=1}^{n-2} \frac{1}{h+1} a_h b_{n-h-1} \]

This implies:

\[ 2 \sum_{h=1}^{n-2} \frac{1}{h+1} a_h b_{n-h-1} = 4a_n - b_n - b_{n-1} - 3a_{n-1} \]

\[ 2 \sum_{h=1}^{n-2} \frac{1}{3h+1} b_h b_{n-h-1} = 4a_n - b_n - b_{n-1} - 3a_{n-1} \]

\[ 2 \sum_{h=1}^{n-1} \frac{1}{3h+1} b_h b_{n-h-1} = 4a_n - b_n - b_{n-1} - 3a_{n-1} + \frac{2}{3n-2} b_{n-1} \]

\[ 2 \sum_{h=1}^{n-1} \frac{1}{3h+1} b_h b_{n-h-1} = 4a_n - b_n - b_{n-1} - 3a_{n-1} + 3a_{n-1} - b_{n-1} \]

\[ 2 \sum_{h=1}^{n-1} \frac{1}{3h+1} b_h b_{n-h-1} = 4a_n - b_n - 2b_{n-1} \]

\[ 2 \sum_{h=0}^{n-1} \frac{1}{3h+1} b_h b_{n-h-1} = 4a_n - b_n \]

The second equation in the conclusion is now proved.
5 Threshold sequences and Motzkin-like paths

We now show that threshold sequences may be seen as variants of Motzkin paths.

Defined in [3] in relation with the Motzkin numbers used in [3], a Motzkin path of length \( n \) is a path on the integral lattice \( \mathbb{Z} \times \mathbb{Z} \) never going below the \( x \)-axis, starting in position \((0, 0)\) and whose steps are of three types: up steps \((1, 1)\), horizontal steps \((1, 0)\) and down steps \((1, -1)\). Here, step \((a, b)\) indicates that the next position in the path is reached by moving \( a \) units to right and \( b \) units to top with respect to the current position. Motzkin paths enumerate various combinatorial objects, as shown in [3, 14] but also more recently in [3, 15, 2, 9]. Some generalizations of Motzkin paths have also been investigated [7], among which those with long horizontal steps [1].

Motzkin \( n \)-paths are Motzkin paths of length \( n \) ending in \((n, 0)\). They are counted by the formula below [3]:

\[
M_n = \sum_{k=0}^{\left\lfloor n/2 \right\rfloor} \binom{n}{2k} C_k
\]

(14)

where \( C_k \) is the Catalan number.

Let us call a \((k, l)\)-extended Motzkin path of length \( n \) every path on the integral lattice \( \mathbb{Z} \times \mathbb{Z} \) never going below the \( x \)-axis, starting in position \((0, 0)\) and ending in position \((n, h)\), with \( 0 \leq h \leq l \), whose \( n \) steps are of three types: up steps \((1, u)\) with \( u \geq 1 \), horizontal steps \((1, 0)\) and down steps \((1, -t)\) with \( 1 \leq t \leq k-1 \). Similarly to Motzkin \( n \)-paths, we call a \((k, l)\)-extended Motzkin \( n \)-path every \((k, l)\)-extended Motzkin path with \( n \) steps that starts in \((0, 0)\) and ends in \((n, 0)\).

**Proposition 5.1.** Let \( k, l, n \) be three integers such that \( k \geq 2 \), \( 0 \leq l \leq k-2 \) and \( n \geq 1 \). The \((k, l)\)-extended Motzkin paths of length \( n \) and endpoint \((n, l)\) are in bijection with the proper \((k, l)\)-threshold sequences of length \( n \).

*Proof.* We associate with the proper \((k, l)\)-threshold sequence \( S \) of length \( n \) the path \( \text{Path}(S) \) starting in position \((0, 0)\) and whose \( i \)-th step depends on the value of \( s_i - s_{i-1} - k \), for \( 1 \leq i \leq n \) (\( s_0 \) is supposed to be equal to \( 0 \)):

- if \( s_i - s_{i-1} > k \), then the \( i \)-th step is an up step \((1, s_i - s_{i-1} - k)\)
- if \( s_i - s_{i-1} = k \), then the \( i \)-th step is an horizontal step \((1, 0)\)
- if \( s_i - s_{i-1} < k \), then the \( i \)-th step is a down step \((1, s_i - s_{i-1} - k)\)

Let \((i, y_i)\), \( 1 \leq i \leq n \), be the point the path reaches after the \( i \)-th step. Then, according to the definition of \( \text{Path}(S) \):

\[
y_i = \sum_{p=1}^{i} (s_p - s_{p-1} - k) = s_i - ik.
\]

(15)

Since \( S \) is a proper \((k, l)\)-threshold sequence, we have that \( y_i \geq 0 \) and also that \( y_n = s_n - nk = l \). Thus the endpoint of \( \text{Path}(S) \) is \((n, l)\). In order to show that \( \text{Path}(S) \) is a \((k, l)\)-extended Motzkin path of length \( n \), it remains to verify that the down steps satisfy \( 1 \leq -(s_i - s_{i-1} - k) \leq k - 1 \). We have \( s_i - s_{i-1} < k \) by the definition of a down step, and thus the inequality on the left side is verified. Moreover, \(- (s_i - s_{i-1} - k) \leq k - 1 \) is equivalent with \( s_i - s_{i-1} \geq 1 \) and this is necessarily true since threshold sequences are strictly increasing sequences. Thus the right side inequality is verified too.

Equation (15) easily implies that the function \( \text{Path}(\cdot) \) is an injective function. To show it is a bijection, consider a \((k, l)\)-extended Motzkin path of length \( n \) and endpoint \((n, l)\), and let \((i, y_i)\) be the coordinates of
Figure 3: The $(5, 4)$-extended Motzkin path associated with the $(5, 4)$-threshold sequence $S = (7\ 15\ 16\ 21\ 28\ 30\ 38)$.

the point reached after $i$ steps. We have $y_i \geq 0$ by definition. We also have $y_n = l$ and, since the down steps $(1, -t)$ satisfy $1 \leq t \leq k - 1$, we deduce that $y_i \leq y_n + (n - i)(k - 1) \leq l + nk - ik - n + i$, thus $y_i + ik \leq nk + l + (i - n) \leq nk + l$. Then, defining $S$ as the sequence with $s_i = y_i + ik$ for all $i$, $1 \leq i \leq n$, we have $s_i \leq nk + l$ and $s_i - ik = y_i \geq 0$, thus $S$ is a $(k, l)$-threshold sequence. Moreover, $s_n - nk = y_n = l$ and the proof is finished.

Example 5.2. Consider the proper $(5, 3)$-threshold sequence $S$ of length 7 given by $S = (7\ 15\ 16\ 21\ 28\ 30\ 38)$. The $(5, 4)$-extended Motzkin path $Path(S)$ associated to it uses the following steps: $(1, 2)$, $(1, 3)$, $(1, -4)$, $(1, 0)$, $(1, 2)$, $(1, -3)$, $(1, 3)$. It starts in $(0, 0)$ and ends in $(7, 3)$. See Figure 3.

Now, Proposition 5.1 and Corollary 3.8 imply:

**Corollary 5.3.** For each $l \geq 1$, the number of $(k, l)$-extended Motzkin paths of length $n$ ending in $(n, l)$ is equal to the Raney number:

$$R_{n-1}^{(k,k+l)} = \frac{k + l}{(k - 1)(n - 1) + k + l} \binom{kn + l - 1}{n - 1}.$$

Using Proposition 5.1 for each $h = 0, 1, \ldots, l$ and Corollary 3.7, we deduce that:

**Corollary 5.4.** The $(k, l)$-extended Motzkin paths of length $n$ are in bijection with the $(k, l)$-threshold sequences of length $n$. Thus they are counted by the Raney number:

$$R_n^{(k,l+1)} = \frac{l + 1}{kn + l + 1} \binom{kn + l + 1}{n}.$$

When $l = 0$, the $(k, 0)$-extended Motzkin paths of length $n$ are exactly the $(k, 0)$-extended Motzkin $n$-paths. The case $k = 2$ is worth to be noticed, since the down steps are in this case $(1, -1)$ steps only, as in the Motzkin paths. We deduce from the previous corollary and the remark that $R_n^{(2,1)} = C_n$ that:

**Corollary 5.5.** Motzkin-like $n$-paths obtained by allowing arbitrarily long up steps in Motzkin $n$-paths are enumerated by the Catalan numbers:

$$C_n = \frac{1}{n + 1} \binom{2n}{n}.$$

Example 5.6. Figure 4 shows the five Motzkin-like $4$-paths with arbitrarily long up steps which are not Motzkin $4$-paths. The number of Motzkin $4$-paths, computed with Equation (14) is 9. The total number of Motzkin-like $4$-paths with arbitrarily long up steps is thus 14, which is the Catalan number $C_4$. 

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Conclusion

The bijections between \((k,l)\)-threshold sequences, ordered \((l+1)\)-tuples of \(k\)-ary trees and Motzkin-like paths with long up and down steps we presented in the paper provide new combinatorial interpretations for the Raney numbers, when the parameters \(r\) and \(k\) satisfy \(r \leq k - 1\). The case where \(k \leq r \leq 2k - 2\) also gets new interpretations, since in this case the Raney numbers count proper \((k,l)\)-threshold sequences and Motzkin-like paths with long up and down steps that have a fixed endpoint.

Threshold sequences may also be represented as particular cases of other combinatorial objects. For instance, \((k,l)\)-threshold sequences of length \(n\) are also in bijection with \(k\)-ballot sequences \([11]\) over the alphabet \(\{A,B\}\) with \(a = kn + l + 1\) letters \(A\), \(b = n\) letters \(B\), whose letters \(B\) are isolated and whose last letter is \(B\). This is done by associating with each \((k,l)\)-threshold sequence \(S\) of length \(n\) the \(k\)-ballot sequence \(W(S) = AW_1W_2\ldots W_n\), where \(W_i\) is a sequence of \(s_i - s_{i-1}\) letters \(A\), followed by a letter \(B\) \((s_0 = 0\) by convention). Then the number of \(k\)-ballot sequences with \(kb < a < kb + k\), whose letters \(B\) are isolated and whose last letter is \(B\), is equal to:

\[
R_b^{(k,a-kb)} = \frac{a - kb \binom{a}{b}}{a}
\]

To see this, it is sufficient to define \(l = a - kb - 1\) and to conclude using Corollary \([3,7]\) and Equation \([3]\).

In consequence, threshold sequences are closely related to existing, and useful, combinatorial objects, and show efficient in bringing a new point of view on these objects.

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