Quantization of Poisson families and of twisted Poisson structures

Pavol Ševera
Dept. of Theoretical Physics
Comenius University
Bratislava, Slovakia

Introduction

If $\phi$ is a closed 3-form on a smooth manifold $M$, a $\phi$-Poisson structure is, according to [5], a bivector field $\pi$ satisfying the equation
\[
[\pi, \pi] = 2 \wedge^3 \tilde{\pi}(\phi),
\]
where $\tilde{\pi} : T^*M \to TM$ is the linear map given by $\pi$. More conceptually, it is a Dirac structure in an exact Courant algebroid (see the appendix for the definitions). In this paper we try to quantize this structure if $\pi$ depends formally on an indeterminate $\hbar$ and $\pi = O(\hbar)$; $\phi$ is also allowed to depend on $\hbar$. Under the condition that the periods of $\phi$ are in $2\pi \sqrt{-1} \mathbb{Z}$ (here $\pi$ is 3.14... not a bivector) we construct a stack of algebras on $M$; the stack depends on a choice of an element $\Phi \in H^3(M, 2\pi \sqrt{-1} \mathbb{Z})$ and of a Dirac structure in an exact Courant algebroid $E$, such that the image of $\Phi$ in $H^3(M, \mathbb{C})$ is the characteristic class of $E$. The stack can reasonably be called a deformation of the “gerbe” corresponding to $\Phi$.

The basic idea is as follows: since locally the cohomology class of $\phi$ vanishes, $\pi$ is locally equivalent to a Poisson structure. This local Poisson structure is not unique, but any two are connected by a diffeomorphism formally depending on $\hbar$, and hence they give rise to isomorphic $*$-product. To keep track of these diffeomorphisms and isomorphisms we introduce a kind of families of Poisson structures, called tight families here; they are given by a Maurer-Cartan equation, and hence their quantization follows immediately from Kontsevich’s Formality theorem [2]. The idea and the interpretation of these families is again due to Kontsevich [3].

Acknowledgement

I am particularly grateful to P. Bressler and P. Xu who both suggested me that Dirac structures in exact Courant algebroids should be quantized to stacks, and to B. Jurčo for showing me his independent and very similar results. I would also like to thank to J. Stasheff for interesting comments.

1 Tight families of associative algebras

This section follows closely the appendix A.2 of [3], it is here just for reader’s convenience. Let $A$ be a vector space with a marked vector $1_A \in A$, $B$ a manifold and $\chi$ a closed 3-form on $B$; a $\chi$-tight family of algebras indexed by $B$ is an element $\gamma \in \Gamma(\wedge T^*B) \otimes$
\( C^* (A, \mathcal{A}) \) of total degree 2 (where \( C^k (A, \mathcal{A}) \) is the space of \( k \)-linear maps from \( A \) to \( \mathcal{A} \)), satisfying a modified Maurer–Cartan equation

\[
d\gamma + \frac{1}{2} [\gamma, \gamma] = \chi \otimes 1_{\mathcal{A}}
\]

and some additional constraints involving \( 1_{\mathcal{A}} \) (see below). In terms of bihomogeneous components (\( \gamma = \gamma_0 + \gamma_1 + \gamma_2 \), the bidegree of \( \gamma_k \) is \((k, 2-k)\))

1. \( [\gamma_0, \gamma_0] = 0 \), i.e. \( \gamma_0 \) is a family of associative algebras (on the fixed vector space \( \mathcal{A} \)) indexed by \( B \),
2. \( [\gamma_1, \gamma_0] + d\gamma_0 = 0 \), i.e. if \( \gamma_1 \) is interpreted as a connection on the vector bundle \( \mathcal{A} \times B \rightarrow B \), the algebra structure on the fibres is parallel,
3. \( [\gamma_2, \gamma_0] + ([\gamma_1, \gamma_1]/2 + d\gamma_1) = 0 \), i.e. the connection \( \gamma_1 \) need not be flat, but the result of a parallel transport along an infinitesimal closed curve is an inner automorphism given by \( \gamma_2 \),
4. \( [\gamma_2, \gamma_1] + d\gamma_2 = \chi \otimes 1_{\mathcal{A}} \);

the additional constraints are

5. \( 1_{\mathcal{A}} \) is the unit for every algebra,
6. \( 1_{\mathcal{A}} \) is parallel.

Suppose that \( \mathcal{A} \) is finite-dimensional, so that parallel transport makes sense. Then for any curve in \( B \) the parallel transport gives us an isomorphism between the algebras at the endpoints; moreover, for a closed curve spanned by a disk \( D \) this automorphism is inner, given by an element \( a_{D,x} \in \mathcal{A} \) (to be constructed below), where \( x \) is the starting point of the curve. If moreover \( D' \) is another disk homotopic to \( D \) rel boundary then \( a_{D',x} = a_{D,x} \exp \int \chi \), where the integral is over the volume swept by the homotopy (this is the meaning of 4.). These \( a \)’s behave in the obvious nice way under composition of curves and disks:

For the left picture, if we add to our closed curve a path from \( x_1 \) to \( x_2 \) and back again (with no other changes to the disk) then \( a_{D,x} \) doesn’t change for any \( x \) lying on the original closed curve; on the other hand, \( a_{D,x_2} = T(a_{D,x_1}) \), where \( T \) is the parallel transport from \( x_1 \) to \( x_2 \). For the right picture, if the disk is decomposed to \( D_1 \) and \( D_2 \) then \( a_{D,x} = a_{D_1,x} a_{D_2,x} \).

\(^1\)if we augment \( C^* (A, \mathcal{A}) \) by setting \( C^{-1} (A, \mathcal{A}) = \mathbb{R} \) (or \( \mathbb{C} \)) and \( d 1 = 1_{\mathcal{A}} \), this becomes the ordinary Maurer-Cartan equation.
These two composition properties also make it clear how to actually construct $a_{D,x}$:

\[
\begin{array}{cccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 \\
\end{array}
\]

For example, decompose $D$ as the square on the picture and number the little squares as indicated, then connect each of them with $x$ by a line that is first horizontal and then vertical. In this way we express $a_{D,x}$ as a product of $a$’s for the little squares. Hence it is enough to know $a_D$’s for infinitesimal $D$’s and these are (by 3.) $\exp(\int_D \gamma_2)$.

Finally, there are two natural group actions on tight families. If $\beta$ is a 2-form on $B$, we can add $\beta \otimes 1_A$ to $\gamma$, getting a $\chi + d\beta$-tight family; we shall call it outer transformation. Notice that only $\gamma_2$ gets changed; the connection remains the same and $a_{D,x}$’s are multiplied by $\exp(\int_D \beta)$. Secondly, the Lie algebra of elements $\alpha \in \Gamma(\wedge T^*B \otimes C^*(A,A))$ of total degree 1 acts on the space of $\chi$-tight families by $\alpha \cdot \gamma = d\alpha + [\alpha, \gamma]$, we shall call this inner transformations. In particular, the outer transformations by exact 2-forms are also inner.

(Let us finish with an unimportant remark that in a “global version” of tight families one should have a line bundle over the loop space of $B$, coming from a gerbe on $B$; $a_{D,x}$’s should be replaced by $a_p$’s, where $p$’s are elements of the associated principal bundle.)

2 Tight families of $\ast$-products

A $\chi$-tight family of $\ast$-products on $M$ indexed by $B$ appears when we take $A = C^\infty(M)[[h]]$ and instead of $C^*(A,A)$ we take $PDiff(M)[[h]]$ ($PDiff(M)$ are the polydifferential operators on $M$). $\chi$ itself is allowed to depend formally on $h$; moreover, we require that $\gamma_0$ is the ordinary product on $M$ plus $O(h)$ (i.e. a family of $\ast$-products on $M$) and that $\gamma_1$ is $O(h)$, so that parallel transport (and also $a_{D,x}$’s giving the inner automorphisms) make sense. Notice that the parallel transport is of the form $f \mapsto f + hD_1 f + h^2D_2 f + \ldots$ where $D_k$’s are differential operators and that $a_{D,x} = \exp(\int_D \gamma_2) + O(h)$. Just as above, for any point $x \in B$ we have a $\ast$-product on $M$, parallel transport gives isomorphisms between them and for closed contractible curves these automorphisms are inner, given by $a_{D,x}$’s. We also have to restrict the Lie algebra acting by inner transformations to those elements of total degree 1 whose $(0,1)$-bihomogeneous part is $O(h)$.

We shall need a slight generalization: we choose an open subset $U \subset M \times B$ and suppose that $\gamma$ is defined just on $U$ (one could also consider a submersion $E \to B$ with a transversal foliation). The difference is that the $\ast$-products, parallel transport and $a_{D,x}$’s are defined only locally now: For any $x \in B$ let $U_x = U \cap (M \times \{x\})$; it is an open subset of $M$. Then $\gamma_0$ is a family of $\ast$-products on $U_x$’s. If $c$ is a curve in $B$ connecting 2 points $x_1, x_2$
and $V \subset M$ an open subset contained in $U_x$ for every $x \in c$, then $\gamma_1$ gives an isomorphism of the $*$-products on $V$ over $x_1$ and $x_2$. Finally, if $x_1 = x_2$ and $D$ is a disk in $B$ with boundary $c$, and if moreover $V \subset U_x$ for every $x \in D$, then we get a function $a_{D,x_1}$ on $V$ (formally depending on $h$) such that the automorphism given by $c$ is the inner automorphism given by $a_{D,x_1}$. Moreover, if we choose a different $D'$ homotopic to $D$ rel boundary, such that $V \subset U_x$ for every point swept by the homotopy, then $a_{D'} = a_D \exp \int \phi$, where the integral is over the volume swept by the homotopy.

3 Tight families of Poisson structures and their quantization

Let again $M$ and $B$ be manifolds and $\chi$ a closed 3-form on $B$. Let $p_{M,B}$ be the projections from $M \times B$ to $M$ and $B$. By a $p_B^*\chi$-tight family of Poisson structures on $M$ indexed by $B$ we shall mean an element $\sigma \in \Gamma(\wedge T^*B) \otimes \Gamma(\wedge TM)$ of total degree 2, satisfying the modified Maurer–Cartan equation (that again becomes ordinary Maurer-Cartan equation if we augment $\Gamma(\wedge TM)$)

$$d\sigma + \frac{1}{2}[\sigma, \sigma] = \chi,$$

where $d$ is the differential on $B$ and $[, ,]$ is the Schouten bracket on $M$. To be precise about the tensor product: $\sigma$ is a section of $\wedge^2(p_B^*T^*B \oplus p_M^*TM)$ where $p_{M,B}$ are the projections from $M \times B$ to $M$ and $B$. More generally, we shall consider an open subset $U \subset M \times B$ and suppose $\sigma$ is defined only there.

Equivalently, $\sigma$ is a $p_B^*\chi$-Dirac structure (see [4, 5] or the appendix) on $M \times B$ transversal to $TM \oplus T^*B$ (this works also for arbitrary submersions $E \rightarrow B$, not just for $M \times B \rightarrow B$; notice also that we have a pair of transversal Dirac structures, i.e. a Lie bialgebroid).

When $\sigma$ is decomposed to its bihomogeneous components $\sigma = \sigma_0 + \sigma_1 + \sigma_2$ (where $\sigma_i$ is an $i$-form on $B$ with values in $2-i$-vectors on $M$) then we get the same equations as for $\gamma$ above, with a similar interpretation: just replace isomorphisms of algebras by Poisson diffeomorphisms and inner automorphisms by Hamiltonian diffeomorphisms.

We again have an action of 2-forms on $B$ by $\sigma \mapsto \sigma + \beta$, making $\sigma$ to a $\chi + d\beta$-tight family (the outer transformations), and of the Lie algebra of the elements of total degree 1 (the inner transformations) by $\alpha \cdot \sigma = d\alpha + [\alpha, \sigma]$ (these $\alpha$’s are sections of the complementary Dirac structure).

For the purpose of this note, let a formal $p_B^*\chi$-tight family of Poisson structures be a $\sigma$ as above which is a formal power series in $h$ and moreover $\sigma_0$ and $\sigma_1$ are $O(h)$ (again, $\chi$ may also be allowed to depend on $h$). For the inner transformations we need a little restriction, the $(0,1)$-bihomogeneous component of $\alpha$ must be $O(h)$.

As an easy application of the Formality theorem, such a $\sigma$ can always be quantized to a $\chi$-tight family of $*$-products (see the last section). Moreover, the Formality theorem gives a bijection between their equivalence classes with respect to the inner transformations, and the quantization is equivariant w.r.t. the outer transformations.

Let now $\psi$ be a closed 3-form on $M \times B$; a $\psi$-tight family of Poisson structures is a $\psi$-Dirac structure on $M \times B$ transversal to $TM \oplus T^*B$ (again it can be expressed as an element of $\Gamma(\wedge T^*B) \otimes \Gamma(\wedge TM)$ of total degree 2, satisfying some equation not written here). If $\beta$ is a 2-form on $M \times B$ and $\sigma$ a $\psi$-tight family, we can define a $\psi + d\beta$-Dirac structure $\tau_{\beta}\sigma$. It need not be transversal to $TM \oplus T^*B$; however, if $\sigma$ is a formal tight
family, \( \tau_\beta \sigma \) is also a formal tight family. If there is a 2-form \( \beta \) on \( M \times B \) such that \( \psi + d\beta = p_B^* \chi \), we can quantize \( \tau_\beta \sigma \) to a \( \chi \)-tight \( \ast \)-product family; this can be called a quantization of \( \sigma \) itself.

4 Quantization of twisted Poisson structures to tight \( \ast \)-product families

Let \( \phi \) be a closed 3-form on \( M \) (possibly depending formally on \( \hbar \)). We choose \( B = M \) (other choices would do in special cases). Let \( U \subset M \times B \) be an open subset containing the diagonal such that \( p_M^* \phi - p_B^* \phi \) is exact on \( U \) (e.g. \( U \) is a tubular neighborhood of the diagonal).

Let \( \pi \) be a formal \( \phi \)-Poisson structure on \( M \); we make it to a (constant) \( p_M^* \phi \)-tight Poisson family (by setting \( \sigma_0 = \pi, \sigma_1 = \sigma_2 = 0 \)). We restrict this family from \( M \times B \) to \( U \) and denote it \( \sigma \).

Now choose a 2-form \( \beta \) on \( U \) s.t. \( p_B^* \phi = p_M^* \phi + d\beta \). Then \( \tau_\beta \sigma \) is a \( p_B^* \phi \)-tight Poisson family, hence it can be quantized to a \( \phi \)-tight family of \( \ast \)-products.

5 Quantization of twisted Poisson structures to stacks

Suppose now that the periods of \( \phi \) are in \( 2\pi \sqrt{-1} \mathbb{Z} \) (hopefully this \( \pi = 3.14 \ldots \) will not be confused with the bivector field \( \pi \)). Let us also choose a cohomology class \( \Phi \in H^3(M, 2\pi \sqrt{-1} \mathbb{Z}) \) whose image in \( H^3(M, \mathbb{C}) \) is the class of \( \phi \).

Let us choose a triangulation of \( M \) which is fine enough (or \( U \) is large enough) for the following property to hold: if \( x \) is in a simplex \( s \) then \( S_s \subset U_x \), where \( S_s \) is the star of \( s \). Recall that the star is the union of the interiors of the simplices containing \( s \); \( S_s \) is the intersection of \( S_t \)'s, where \( i \)'s are the vertices of \( s \), and \( S_i \)'s, where \( i \)'s are all the vertices of the complex, form an open cover of \( M \):

Whenever two vertices \( i, j \) form an edge, we denote \( T_{ij} \) the parallel transport from \( i \) to \( j \) along the edge, whenever \( i, j \) and \( k \) form a triangle \( \triangle \), we denote \( a_{ijk} \) the function \( a_{\triangle, i} \), and whenever \( i, j, k \) and \( l \) form a tetrahedron, we denote \( c_{ijkl} = \exp(\int \phi) \), where we integrate over the tetrahedron. On each \( S_t \), we have a \( \ast \) product, \( T_{ij} \) gives isomorphisms on the overlaps \( S_i \cap S_j \) and on triple overlaps these isomorphisms are inner, given by \( a_{ijk} \)'s. We obviously have the identities

\[
T_{ij} = T_{ji}^{-1}, \quad a_{ijk} = a_{ikj}^{-1}, \quad a_{jki} = T_{ij}(a_{ijk}), \quad T_{ki}T_{jk}T_{ij} = Ad(a_{ijk})^{-1}
\]

and

\[
a_{ikl}a_{ijk} = c_{ijkl}a_{ijl}T_{ji}(a_{jkl})
\]
(the products between $a$’s are the $*$-products in $S_i$).

Now we would like to multiply each $a_{ijk}$ by a non-zero number $b_{ijk}$ so as to remove the factors $c_{ijkl}$ from the last formula; this is indeed possible, since the periods of $\phi$ are in $2\pi\sqrt{-1}\mathbb{Z}$, hence $c_{ijkl}$ is a coboundary. The choice of $b$’s is fixed modulo coboundaries by the cohomology class $\Phi$.

According to Kashiwara [1], the $*$-products on $S_i$’s, the isomorphisms $T_{ij}$’s and the modified $a_{ijk}$’s define a stack of algebras on $M$.

6 The proof involving Formality theorem

In this section we will show how Formality theorem may be used to quantize tight Poisson families to tight $*$-product families. We will use the language of graded $Q$-manifolds, as in [1]. Recall that Kontsevich constructed there an equivariant map between the graded $Q$-manifolds $\mathcal{U} : \Gamma(\bigwedge TM)[2] \to PDiff(M)[2]$, sending 0 to the ordinary product $m_0$ (the map is given by a divergent Taylor series, but that is enough for deformation quantization; $PDiff$ denotes the space of polydifferential operators); moreover, the graded $Q$-manifold $PDiff(M)[2]$ (more precisely, the formal neighborhood of $m_0$) becomes isomorphic to the direct product of $\Gamma(\bigwedge TM)[2]$ with a contractible graded $Q$-manifold.

For our purpose it is important that $\mathcal{U}$ is also equivariant w.r.t. translations by constants, and hence it can be extended to the augmented spaces. A grading-and-$Q$-equivariant map from $T[1]B$ to the augmented $\Gamma(\bigwedge TM)[2]$ is the same as a choice of a closed 3-form $\chi$ on $B$ and of a $p_B^*\chi$-tight Poisson family $\sigma$. We will allow this map to depend formally on $h$ in such a way that setting $h = 0$ yields a map constant to second order along $B \subset T[1]B$; this is a formal tight Poisson family. Now the composition with the augmented $\mathcal{U}$ is a formal $\chi$-tight $*$-product family.

A Exact Courant algebroids and $\phi$-Dirac structures

A Courant algebroid over a manifold $M$ is a vector bundle $E \to M$ equipped with a field of nondegenerate symmetric bilinear forms $(\cdot, \cdot)$ on the fibres, an $\mathbb{R}$-bilinear bracket $[\cdot, \cdot] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ on the space of sections of $E$, and a bundle map $\rho : E \to TM$ (the anchor), such that the following properties are satisfied:

1. for any $e_1, e_2, e_3 \in \Gamma(E)$, $[e_1, [e_2, e_3]] = [[e_1, e_2], e_3] + [e_2, [e_1, e_3]]$;
2. for any $e_1, e_2 \in \Gamma(E)$, $\rho(e_1, e_2) = [\rho e_1, \rho e_2]$;
3. for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$, $[e_1, fe_2] = f[e_1, e_2] + (\rho(e_1)f)e_2$;
4. for any $e, h_1, h_2 \in \Gamma(E)$, $\rho(e)(h_1, h_2) = ([e, h_1], h_2) + (h_1, [e, h_2])$;
5. for any $e, h \in \Gamma(E)$, $(e, [h, h]) = ([e, h], h)$.

Equivalently, instead of the bracket $[\cdot, \cdot]$, we can use a linear map $e \mapsto Z_e$ which maps sections of $E$ to vector fields on the total space of $E$. The vector field $Z_e$ is a lift of $\rho(e)$ from $M$ to $E$, and the first four axioms just say that the flows of $Z_e$’s preserve the structure of $E$. The bracket $[e_1, e_2]$ is the Lie derivative of $e_2$ by $Z_{e_1}$.
A Dirac structure in $E$, also called an $E$-Dirac structure on $M$, is a maximal isotropic subbundle $L$ of $E$ whose sections are closed under the bracket, i.e. which is preserved by the flow of $Z_{\psi}$ for any $\psi \in \Gamma(L)$. The restriction of the bracket and anchor to any Dirac structure $L$ form a Lie algebroid structure on $L$.

It follows from the definition that the sequence $0 \rightarrow T^*M \rightarrow E \rightarrow TM \rightarrow 0$ is a complex (the second arrow is $\rho^*$ composed with the isomorphism $E^* \simeq E$ given by $(\cdot, \cdot)$, the third arrow is $\rho$). If it is an exact sequence, $E$ is an exact Courant algebroid (ECA). They are classified by the $3$rd de Rham cohomology $[\mathbb{R}]$ (see also $[\mathbb{R}]$): If we choose an isotropic splitting of the exact sequence, so that $E$ becomes $TM \oplus T^*M$ with the bilinear form $((X_1, \xi_1), (X_2, \xi_2)) = \xi_1(X_2) + \xi_2(X_1)$, the bracket is

$$[(X_1, \xi_1), (X_2, \xi_2)] = ([X_1, X_2], \mathcal{L}_{X_1} \xi_2 - i_{X_2} d\xi_1 + \phi(X_1, X_2, \cdot))$$

for some closed $3$-form $\phi$; this ECA will be denoted $(TM \oplus T^*M)_\phi$. The space of isotropic splitting is an affine space over the space of $2$-forms, and adding a $2$-form $\beta$ changes $\phi$ to $\phi + d\beta$. In other words, there is an action of the additive group of $2$-forms on the bundle $TM \oplus T^*M$ given by $\tau_\beta(X, \xi) = (X, \xi + \beta(X, \cdot))$; $\tau_\beta$ is an isomorphism from $(TM \oplus T^*M)_\phi$ to $(TM \oplus T^*M)_{\phi + d\beta}$. The cohomology class of $\phi$ is the characteristic class of the ECA.

A $(TM \oplus T^*M)_\phi$-Dirac structure will be called simply a $\phi$-Dirac structure. If $\pi$ is a bivector field on $M$, the graph of $\tilde{\pi} : T^*M \rightarrow TM$ is a $\phi$-Dirac structure if and only if $\pi$ is a $\phi$-Poisson structure, i.e. if $[\pi, \pi] = 2 \wedge^3 \tilde{\pi}(\phi)$.

Although it is not strictly needed for the paper, I include here graded $Q$-manifold version of some of the notions defined above. Let $\mathcal{Y}$ be a principal $\mathbb{R}[2]$-bundle over $T[1]M$ in the category of graded $Q$-manifold. Such $\mathcal{Y}$’s are classified by the $3$rd de Rham cohomology of $M$. Indeed: there is always a grading-preserving splitting of $\mathcal{Y}$ to $T[1]M \times \mathbb{R}[2], Q = d + \phi \partial/\partial t$, where $d$ is the de Rham differential on $M$, $t$ the coordinate on $\mathbb{R}[2]$ and $\phi$ a closed $3$-form on $M$. To pass to another splitting we have to choose a $2$-form $\beta$ on $M$ and after a simple computation we find that $\phi$ gets changed to $\phi + d\beta$. There is an ECA corresponding to such a $\mathcal{Y}$: its sections are the vector fields on $\mathcal{Y}$ of degree $-1$, the bracket is $[[Q, v_1], v_2]$ and the bilinear form is $[v_1, v_2]$.

Let now $j^1\mathcal{Y}$ be the space of $1$-jets of sections of $\mathcal{Y} \rightarrow T[1]M$. The Dirac structures in the corresponding ECA can be encoded as Legendrian graded $Q$-submanifolds of $j^1\mathcal{Y}$. Their formal neighborhoods in the infinite-dimensional graded $Q$-manifold of all Legendrian submanifolds of $j^1\mathcal{Y}$ give rise to $L_\infty$-algebras; the graded Lie algebra $\Gamma(\Lambda TM)$ appears in the case $\pi = 0, \phi = 0$.

References

[1] M. Kashiwara, Quantization of contact manifolds, Publ. RIMS 32 no.1 (1996).
[2] M. Kontsevich, Deformation quantization of Poisson manifolds I, q-alg/9709040
[3] M. Kontsevich, Deformation quantization of algebraic varieties, math.AG/0106006
[4] P. Ševera, an obscure unpublished letter to A. Weinstein, 1998.
[5] P. Ševera, A. Weinstein, Poisson geometry with a $3$-form background, Prog. Theor. Phys. Suppl. 144 (2001), math.SG/0107133.