FILTER QUOTIENTS AND NON-PRESENTABLE ($\infty, 1$)-TOPOSES

NIMA RASEKH

Abstract. We define filter quotients of ($\infty, 1$)-categories and prove that filter quotients preserve the structure of an elementary ($\infty, 1$)-topos and in particular lift the filter quotient of the underlying elementary topos. We then specialize to the case of filter products of ($\infty, 1$)-categories and prove a characterization theorem for equivalences in a filter product.

Then we use filter products to construct a large class of elementary ($\infty, 1$)-toposes that are not Grothendieck ($\infty, 1$)-toposes. Moreover, we give one detailed example for the interested reader who would like to see how we can construct such an ($\infty, 1$)-category, but would prefer to avoid the technicalities regarding filters.

Contents

0. Introduction 1
1. Review of Concepts 5
2. The Filter Quotient of ($\infty, 1$)-Categories 11
3. Filter Products 24
4. Future Directions 28
References 29

Introduction

Topos theory has two main branches: Grothendieck topos theory, introduced by Bourbaki in their study of algebraic geometry, and elementary topos theory, which generalizes the definition of a Grothendieck topos and is used in the study of logic [MM92].

The theory of ($\infty, 1$)-categories is a generalization of classical category theory allowing us to talk about a weak notion of composition and associativity and in particular a notion of equivalent morphisms, which gives us helpful categorical tools to do homotopy theory.

Date: January 2020.
There is a broad effort to generalize concepts from classical category theory to the \((\infty, 1)\)-categorical setting. One important aspect of this effort is to generalize the two branches of topos theory from the 1-categorical setting to \((\infty, 1)\)-categories.

Grothendieck \((\infty, 1)\)-topos theory has been introduced and studied quite extensively by Lurie, where it is called an \(\infty\)-topos \cite{Lu09}. It has been used effectively in the study of derived algebraic geometry, analogous to how a Grothendieck topos helped us advance algebraic geometry. On the other hand, the study of a generalization to an elementary \((\infty, 1)\)-topos is still in an early stage \cite{Ra18a}.

The goal of this work is to further our understanding and the applicability of elementary \((\infty, 1)\)-topos theory by introducing a new method for constructing elementary \((\infty, 1)\)-toposes: the filter construction. How could the filter construction be applied to topos theory and homotopy theory?

One noticeable problem in elementary \((\infty, 1)\)-topos theory is the lack of examples of elementary \((\infty, 1)\)-toposes without infinite colimits. Without actual examples we cannot reap the benefits of developing elementary \((\infty, 1)\)-topos theory, such as the elementary study of truncations \cite{Ra18c}.

A first step towards constructing non-presentable examples has been taken by Lo Monaco \cite{Lo19} using an appropriate cardinality argument on the \((\infty, 1)\)-category of spaces (analogous to how the subcategory of finite sets is a non-presentable elementary topos). But the resulting elementary \((\infty, 1)\)-topos is a subcategory of spaces and thus has countable colimits. On the other side, using the filter construction we can give examples of non-presentable elementary \((\infty, 1)\)-toposes that do not have infinite colimits and in particular have non-standard natural number objects (Subsection 3.2).

One longstanding hope (in particular of this author) has been to use higher topos theory to construct “non-standard models for spaces”, which are \((\infty, 1)\)-categories that share many similarities with spaces, but differ in some non-elementary ways (such as the existence of infinite colimits). In the 1-categorical case, it is known that the ultraproduct, \(\prod_{\mathcal{F}} \text{Set}\), of the category of sets on a non-principal ultrafilter exhibits this desired behavior \cite[Examples A2.1.13]{Jo03}. Having a filter construction for \((\infty, 1)\)-categories, which is just a lift of the 1-categorical one, we hope to show that ultraproducts on a non-principal ultrafilter for the \((\infty, 1)\)-category spaces give us examples of such non-standard models.

Filter constructions have already been studied in homotopy theory, concretely chromatic homotopy theory. In chromatic homotopy theory we usually break down spectra in two consecutive steps. First, we choose a prime \(p\) and then we choose a height \(n\) (corresponding to the various Morava \(K\)-theories). As the heights form a natural filtration we can then study the limiting behavior as the height increases, as long as we fix a prime.

On the other hand, the various prime numbers do not give us a filtration and thus there is no immediate way to study the limiting behavior as we increase the prime numbers. The insight of Barthel, Schlank and Stapleton \cite{BSS20} was to instead study the ultraproduct of the various \(p\)-local subcategories and show that the ultraproduct is equivalent to a category that comes from algebraic geometry, at the same time giving meaning and proving the statement “chromatic homotopy theory is asymptotically algebraic at a fixed height”.

These ultraproduct ($\infty, 1$)-categories are not well behaved (as observed by the authors [BSS20, Example 3.19]) and so we cannot study them using standard methods in stable homotopy theory as it has been primarily developed for presentable stable ($\infty, 1$)-categories. From this perspective, this paper can also be seen as a first step towards studying the homotopy theory of ultraproduct ($\infty, 1$)-categories that arise in chromatic homotopy theory.

0.1 Main Results. The main result can be summarized as following theorems:

First, we introduce a general method for constructing a new ($\infty, 1$)-category, namely the filter quotient:

**Theorem.** Let $\mathcal{C}$ be a finitely complete ($\infty, 1$)-category, which can be:

1. complete Segal space
2. quasi-category
3. Kan enriched category

and $\Phi$ a filter (Definition 1.29) of subobjects (Example 1.35).

Then there exists a finitely complete ($\infty, 1$)-category $\mathcal{C}_\Phi$ (Proposition 2.2, Definition 2.10, Theorem 2.13) along with a functor $P_\Phi : \mathcal{C} \to \mathcal{C}_\Phi$ (Definition 2.11, Definition 2.5).

Then we show the construction is well behaved from the perspective of elementary topos theory:

**Theorem.** Let $\mathcal{E}$ be an elementary ($\infty, 1$)-topos and $\Phi$ a filter of subobjects.

1. (Theorem 2.24) The functor $P_\Phi$ preserves all defining conditions of an elementary ($\infty, 1$)-topos:
   a. finite (co)limits
   b. subobject classifiers
   c. complete Segal Universes
   and so $\mathcal{E}_\Phi$ is an elementary ($\infty, 1$)-topos as well.
2. (Theorem 2.27) We have an equivalence of underlying elementary toposes $\tau_0(\mathcal{E}_\Phi) \simeq \tau_0(\mathcal{E})_\Phi$.
3. (Theorem 2.28) So, $P_\Phi$ also preserves natural number objects in $\mathcal{E}_\Phi$.
4. (Corollary 2.29) $\mathcal{E}_\Phi$ is not a Grothendieck ($\infty, 1$)-topos (Definition 1.13) if $\tau_0(\mathcal{E})_\Phi$ is not a Grothendieck topos.

This gives us a very effective recipe to construct non-presentable Grothendieck ($\infty, 1$)-toposes, namely constructing filter quotients $\mathcal{E}_\Phi$ such that the underlying elementary topos $\tau_0(\mathcal{E})_\Phi$ is not presentable.

In order to find such examples we next focus on a specific class of filter quotients, namely the filter product. We first show that in some ways they behave similar to the classical analogue:

**Theorem.** Let $\mathcal{C}$ be a finitely complete ($\infty, 1$)-category such that the final object has only two subobjects, $I$ is a set and $\Phi$ a filter on $P(I)$. Then we can construct the filter product $\prod_\Phi \mathcal{C}$ and we have following Los type results:
(1) (Theorem 3.4, Corollary 3.5) Two maps \((f_i)_{i \in I}, (g_i)_{i \in I}\) are equivalent if and only if
\[ \{ i \in I : f_i \simeq g_i \} \in \Phi \]

(2) (Theorem 3.6, Corollary 3.7) A map \((f_i)_{i \in I}\) is an equivalence if and only if
\[ \{ i \in I : f_i \text{ is an equivalence} \} \in \Phi \]

Finally, we use the filter product to give a large of examples of elementary \((\infty, 1)\)-toposes that are not Grothendieck \((\infty, 1)\)-toposes:

**Example.** (Example 3.11) There is a class of sets \(I\) and filter \(\Phi\) such that the filter product \(\prod_{\Phi} \text{Set}\) is not a Grothendieck topos and thus \(\prod_{\Phi} \text{Kan}\) is an elementary \((\infty, 1)\)-topos that is not a Grothendieck \((\infty, 1)\)-topos.

**0.2 Outline.** Section 1 gives a review of some of the concepts we need later on (see also Subsection 0.3 for an overview of the necessary background and Subsection 0.4 for notation). In particular, we review several important features of complete Segal spaces (Subsection 1.1) that play a crucial role in Section 2.

Section 2 breaks down into two subsections. In the first subsection (Subsection 2.1) we define the filter quotient. We give two separate definitions for Kan enriched categories and for complete Segal spaces and then prove these are equivalent. In the second subsection (Subsection 2.2) we prove that the filter quotient construction preserves all the properties of an elementary \((\infty, 1)\)-topos.

Section 3 focuses on a special filter quotient: filter products. In Subsection 3.1 we study general features of filter products and prove some results along the lines of Los’s theorem. In Subsection 3.2 we finally use filter products to give examples of elementary \((\infty, 1)\)-toposes that are not Grothendieck \((\infty, 1)\)-toposes. We will present one example in great detail, for the benefit of any reader who is only interested in understanding one example and wants to skip the technicalities about filters quotients.

Finally, Section 4 mentions some possible future directions.

**0.3 Background.** We will assume the reader has already some familiarity with the following concepts:

(1) The theory of \((\infty, 1)\)-categories, in particular the different models and how they are related.
(2) The theory of elementary \((\infty, 1)\)-toposes.

We only give a quick overview in Section 1 and refer the reader to the appropriate sources.

On the other hand, the following concepts require no familiarity and everything we need is covered in Section 1:

(1) Grothendieck \((\infty, 1)\)-topos theory
(2) Filters
0.4 Notation. We are using various models of $(\infty, 1)$-categories with various levels of strictness. In order to avoid any confusion we will use following conventions:

(1) Set, sSet and ssSet are the 1-categories of sets, simplicial sets and bisimplicial sets, respectively.
(2) The notation colim refers to a colim in a strict 1-category.
(3) The notation $\cong$ refers to an isomorphism of objects in a 1-category.
(4) We will use Kan complexes as our preferred model for the “homotopy theory of spaces” and therefore will avoid using the vague term space throughout.
(5) As we use various models of $(\infty, 1)$-categories we use a (rather arbitrary) convention to help the reader distinguish the various notations.
   (I) $C$: Arbitrary $(\infty, 1)$-Category
   (II) $E$: Arbitrary elementary $(\infty, 1)$-topos
   (III) $G$: Arbitrary Grothendieck $(\infty, 1)$-topos
   (IV) $Q$: Quasi-Category
   (V) $W$: Complete Segal space
   (VI) $K$: Kan enriched category

0.5 Acknowledgements. I want to thank Asaf Horev for pointing to the connection with the work by Barthel, Schlank and Stapleton [BSS20]. I also want to thank Peter Lumsdaine for making me aware of a more general form of Łoś theorem that applies to non-ultrafilter, which resulted in the material in Subsection 3.1.

Review of Concepts

We will give a minimal overview over the specific concepts required later on and refer the reader to the appropriate sources for more details.

1.1 The Theory of $(\infty, 1)$-Categories. An $(\infty, 1)$-category is a general idea of a weak category or homotopical category i.e. a category with a weak notion of composition and associativity. There are now various ways of making this notion precise, which are generally known as “models of an $(\infty, 1)$-category”. Throughout this work we will use three models of $(\infty, 1)$-categories: Kan enriched categories, quasi-categories, complete Segal spaces. All three have their own model structures, namely, Bergner, Joyal and Rezk model structure respectively and all are Quillen equivalent via various Quillen equivalences. Concretely we have following diagram of Quillen equivalences

\[
\begin{array}{cccc}
\text{sSet}^{\text{Joyal}} & \xrightarrow{p_1^*} & \text{ssSet}^{\text{Rezk}} & \xrightarrow{\pi_1} \\
\text{sSet}^{\text{Joyal}} & \xleftarrow{\sim} & \text{ssSet}^{\text{Rezk}} & \xleftarrow{\sim} \\
\text{sSet}^{\text{Joyal}} & \xrightarrow{\sim} & \text{ssSet}^{\text{Joyal}} & \xleftarrow{\sim} \\
\text{sSet}^{\text{Joyal}} & \xleftarrow{(\text{Cat}_\Delta)^{\text{Bergner}}} & \text{ssSet}^{\text{Rezk}} & \xrightarrow{\sim} \\
\text{sSet}^{\text{Joyal}} & \xrightarrow{\sim} & \text{ssSet}^{\text{Rezk}} & \xrightarrow{\sim} \\
\text{sSet}^{\text{Joyal}} & \xleftarrow{(\text{Cat}_\Delta)^{\text{Bergner}}} & \text{ssSet}^{\text{Rezk}} & \xleftarrow{\sim} \\
\text{ssSet}^{\text{Rezk}} & \xrightarrow{\sim} & \text{sSet}^{\text{Joyal}} & \xleftarrow{(\text{Cat}_\Delta)^{\text{Bergner}}} \\
\end{array}
\]

Remark 1.1. The functors $N_\Delta$ and $t'$ are Quillen right adjoints and thus for any Kan enriched category $\mathcal{K}$, $N_\Delta(\mathcal{K})$ is a quasi-category and $t'N_\Delta(\mathcal{K})$ is a complete Segal space.

Remark 1.2. On the other hand for a given quasi-category $Q$, $p_1^*(Q)$ is not a complete Segal space (unless $Q$ has no non-trivial automorphisms). So we need an alternative construction. Thus we introduce $\Gamma(Q)$, defined as the simplicial space (here $(-)^\text{core}$ denotes the underlying $(\infty, 1)$-groupoid)

\[\Gamma(Q)_n = (Q^{\Delta^n})^{\text{core}}\]
and observe that it is a complete Segal space with the property that \( \Gamma(Q)_n = \Gamma(Q)_{n0} \) (for more details see [JT07]).

**Remark 1.3.** Notice in particular if we have a Kan enriched category \( \mathcal{K} \) then the complete Segal space \( t^! N_\Delta(\mathcal{K}) \) has the same objects as \( \mathcal{K} \) and we have an equivalence of mapping Kan complexes

\[
Map_{\mathcal{K}}(X, Y) \simeq Map_{t^! N_\Delta(\mathcal{K})}(X, Y)
\]

In particular, we will need following notation.

**Notation 1.4.** We denote by \( \mathcal{K}an \) the Kan-enriched category of Kan complexes. On the other hand, we use the notation \( S \) for the quasi-category and the complete Segal space of Kan complexes. Note, in particular, \( S = N_\Delta(\mathcal{K}an) \).

For an overview about \((\infty, 1)\)-categories see [Be10], for quasi-categories [Lu09], for complete Segal spaces [Re01] and Kan enriched categories [Be07]. For the equivalence \((\mathcal{C}[\cdot], N_\Delta)\) see [Lu09] and for the other two equivalences see [JT07].

We will also need several basic facts about the category theory of complete Segal spaces, that we will review here. For more details see [Re01] and [Ra17].

**Notation 1.5.** In the context of complete Segal spaces we denote the free arrow as \( F(1) \), following notation in [Re01]. Similarly, \( F(0) \) denotes the complete Segal space with one object.

**Definition 1.6.** Let \( W \) be a complete Segal space. Then we call \( W_0 \) the *Kan complex of objects*. In particular we have \( W_{\text{core}} = W_0 \), where \( W_{\text{core}} \) denotes the underlying \((\infty, 1)\)-groupoid of \( W \).

**Definition 1.7.** For two objects \( x, y \) we define the *mapping Kan complex* as the pullback

\[
\begin{array}{ccc}
Map(W(x, y)) & \longrightarrow & W_1 \\
\downarrow \gamma & \downarrow \ (s, t) \\
\ast & \longrightarrow & W_0 \times W_0
\end{array}
\]

We also need some understanding of over-categories and limits in the context of complete Segal spaces. This material can be found in full detail in [Ra17].

**Definition 1.8.** Let \( W \) be a complete Segal space and \( x \) an object. Then we can define the slice CSS \( W_{/x} \) as the pullback

\[
\begin{array}{ccc}
W_{/x} & \longrightarrow & W^{F(1)} \\
\downarrow \gamma & \downarrow \ i \\
F(0) & \longrightarrow & W
\end{array}
\]

\[
\{x\}
\]
Definition 1.9. More generally, let $I$ be a simplicial space and $F : I \to \mathcal{W}$ be a fixed diagram in the CSS $\mathcal{W}$. Then we define the complete Segal space of cones as

$$\mathcal{W}_F = F(0) \times_{\mathcal{W}^I} \mathcal{W}^I \times_{\mathcal{W}^I} \mathcal{W}$$

Moreover, we have following results about the cocones.

Lemma 1.10. The map $F : I \to \mathcal{W}$ has a limit if and only if $\mathcal{W}_F$ has a final object.

Definition 1.11. If $\mathcal{W}$ is a complete Segal space we denote the complete Segal space of arrows as $\mathcal{W}^{F(1)}$ (using Notation 1.5). It comes with a target map

$$t : \mathcal{W}^{F(1)} \to \mathcal{W}$$

which is a Cartesian fibration if and only if $\mathcal{E}$ has finite limits. Concretely, this Cartesian fibration models the functor out of $\mathcal{W}^{op}$ that takes an object $x$ to the slice $\mathcal{W}/x$ and a morphism $f : x \to y$ to the pullback map $f^* : \mathcal{W}_{/y} \to \mathcal{W}/x$.

1.2 Grothendieck $(\infty,1)$-Topos. Grothendieck topos theory is very ubiquitous in algebraic geometry: Grothendieck 1-toposes in classical algebraic geometry and Grothendieck $(\infty,1)$-topos theory in derived algebraic geometry. However, we only focus on the fact they they are special cases of their elementary counterparts and thus we only give a minimal review.

Definition 1.12 ([Re05, Proposition 2.2]). A Grothendieck topos is a locally presentable 1-category that satisfies weak descent.

Definition 1.13 ([Re05, Theorem 6.9]). A Grothendieck $(\infty,1)$-topos is a presentable $(\infty,1)$-category that satisfies descent.

Example 1.14. The most simple example is $\mathcal{S}$, the $(\infty,1)$-category of Kan complexes.

We only need following important observation relating Grothendieck 1-toposes and $(\infty,1)$-toposes.

Corollary 1.15 ([Re05, Proposition 11.2]). Let $\mathcal{G}$ be a Grothendieck $(\infty,1)$-topos. Then the subcategory of 0-truncated objects, denoted $\tau_0 \mathcal{G}$, is a Grothendieck 1-topos.

We will not require (and mention) any further details about Grothendieck topos theory. We refer the interested reader to [MM92] for Grothendieck 1-topos theory and [Lu09] (using quasi-categories) or [Re05] (using model categories) for Grothendieck $(\infty,1)$-topos theory.

1.3 Elementary $(\infty,1)$-Topos. We will assume familiarity with elementary $(\infty,1)$-topos theory later on and only review basic definitions. We give one detailed example (Example 1.28) with the hope of giving the interested reader an intuition. The main source for elementary $(\infty,1)$-topos theory is [Ra18a].

Definition 1.16 ([Ra18a]). Let $\mathcal{W}$ be a complete Segal space with finite limits. A simplicial map $p : (U_\bullet) \to U_\bullet$ is a complete Segal universe if the induced map of Cartesian fibrations

$$\mathcal{W}/U_\bullet \to \mathcal{W}^{F(1)}$$

is an embedding. Here $\mathcal{W}^{F(1)}$ denotes the target Cartesian fibration as discussed in Definition 1.11.
Definition 1.17 ([Ra18a]). Let $\mathcal{W}$ be a complete Segal space with finite limits. We say $\mathcal{W}$ has **sufficient complete Segal universes** if there exists a collection of complete Segal universes $\{U_\bullet\}$ such that the embeddings $\mathcal{W}/U_\bullet \to \mathcal{E}^{F(1)}$ are jointly surjective.

Remark 1.18. Intuitively this is telling us that for any morphism $f : A \to B$ in $\mathcal{W}$ there exists a universe $U$ such that $f$ is a pullback of $p_{U_0} : (U_0)_* \to U_0$.

\[
\begin{array}{ccc}
A & \xrightarrow{\Gamma} & (U_0)_* \\
\downarrow f & & \downarrow p_{U_0} \\
B & \to & U_0
\end{array}
\]

Definition 1.19 ([Ra18a]). Let $\mathcal{W}$ be a complete Segal space. We say $\mathcal{W}$ is an **elementary complete Segal topos** if the following hold:

1. It has finite limits and colimits.
2. It has subobject classifier
3. It has sufficient complete Segal universes.

The definition we gave only applies to complete Segal spaces, however we will need a definition for other models of $(\infty, 1)$-categories as well.

Definition 1.20. A quasi-category $\mathcal{Q}$ is an **elementary quasi topos** if the complete Segal space $\Gamma(\mathcal{Q})$ (or equivalently $t'(\mathcal{Q})$) is an elementary complete Segal topos. Moreover, a Kan enriched category $\mathcal{K}$ is an **elementary Kan enriched topos** if the quasi-category $N_\Delta(\mathcal{K})$ is an elementary quasi topos.

Remark 1.21. We say $\mathcal{E}$ is an elementary $(\infty, 1)$-topos when we want to refer to the definition in any of those three models, without specifying which model.

This definition can be seen as a generalization of a Grothendieck $(\infty, 1)$-topos as well as elementary topos.

Definition 1.22 ([MM92]). An **elementary topos** is a locally Cartesian closed category with subobject classifier.

Proposition 1.23 ([Ra18a]). *Every Grothendieck $(\infty, 1)$-topos is an elementary $(\infty, 1)$-topos.*

Proposition 1.24 ([Ra18a]). *Let $\mathcal{E}$ be an elementary $(\infty, 1)$-topos. Then the subcategory of 0-truncated objects, denoted $\tau_0 \mathcal{E}$ is an elementary topos. We call it the underlying elementary topos.*

A general elementary $(\infty, 1)$-topos does not have infinite colimits, as it is not an elementary condition. We thus need an elementary alternative to infinite colimits that allow us to still recover some infinite constructions. This is achieved via the natural number object.

Definition 1.25 ([Ra18b]). A **natural number object** in an $(\infty, 1)$-category $\mathcal{E}$ is an object $\mathbb{N}$ along with two morphisms $s : \mathbb{N} \to \mathbb{N}$ and $o : 1 \to \mathbb{N}$, such that the triple $(\mathbb{N}, s, o)$ is initial.

Notice a natural number object is only useful in the context where we don’t have infinite colimits, as the next example shows.
Example 1.26 ([Ra18b]). If $E$ is an elementary $(\infty, 1)$-topos with countable colimits then the infinite coproduct $\coprod_{n \in \mathbb{N}} 1$ is the natural number object. This in particular applies to every Grothendieck $(\infty, 1)$-topos.

This example gives us following valuable recognition principle for elementary $(\infty, 1)$-toposes that are not Grothendieck $(\infty, 1)$-toposes.

Corollary 1.27. Let $E$ be an elementary $(\infty, 1)$-topos such that its natural number object $\mathbb{N}$ is not the infinite colimit $\coprod_{n \in \mathbb{N}} 1$. Then $E$ is not a Grothendieck $(\infty, 1)$-topos.

For more details on natural number objects for elementary $(\infty, 1)$-topos see [Ra18b].

In order to give a better understanding of the axioms of an elementary $(\infty, 1)$-topos we give one detailed example.

Example 1.28. Let $S_\bullet$ be the complete Segal space of Kan complexes (we denote it $S_\bullet$ rather than $S$ in this example to make it clear it is a simplicial object). We already know that it is an elementary $(\infty, 1)$-topos, as it is a Grothendieck $(\infty, 1)$-topos (Example 1.14). However, we want to use the fact that Kan complexes are well-understood to explain and give a better understanding of the axioms of an elementary $(\infty, 1)$-topos.

The existence of finite limits and colimits is a very standard condition and will not be discussed further. Thus we move on to the existence of a subobject classifier. For that we first need to better understand mono maps in the complete Segal space of Kan complexes. By definition a map of Kan complexes $f : X \to Y$ is mono if and only if the square

\[
\begin{array}{ccc}
X & \xrightarrow{id_X} & X \\
\downarrow^r & & \downarrow^f \\
X & \xrightarrow{f} & Y
\end{array}
\]

is pullback square. This only holds if the map $f : X \to Y$ is a local equivalence i.e. a map $f$ that restricted to each path component in $X$ becomes an equivalence.

Equivalence classes of such maps are completely determined by a choice of path components in $Y$ (namely we choose the ones we want to be in $X$) and such a choice is determined by a map $Y \to \{0, 1\}$ (where we map the desired path components to 1 and the rest to 0). Thus $\{0, 1\}$ is a subobject classifier in $S$.

We now want to gain a better understanding of universes in $S_\bullet$. Fix a large enough cardinal $\kappa$. Then we say a map of Kan complexes is $\kappa$-small if each fiber is $\kappa$-small. The sub-complete Segal space $S_\kappa^\bullet$ is itself a simplicial object in $S_\bullet$. Thus for any Kan complex $K$ we can define the complete Segal space $Maps(S(K, S_\kappa^\bullet))$. What are the objects and morphisms in this complete Segal space?

An object is a 0-cell in the Kan complex $Maps(S(K, S_\kappa^\bullet))$, which is a map of Kan complexes $f : K \to S_\kappa^\bullet$. Every Kan complex $K$ is also a simplicial set and thus

\[
K = \colim_{\Delta^n \to K} \Delta^n
\]
which means the map \( f : K \to S_0^\kappa \) is just a \( K \)-indexed family of maps \( \Delta^n \to S_0^\kappa = S_0^\kappa_n \). But \( n \)-cells in \( S_0^\kappa \) are just a choice of \( n \) homotopic Kan complexes. In particular, a map \( \Delta^0 \to S_0^\kappa \) is just a choice of \( \kappa \)-small Kan complex. Thus a map \( K \to S_0^\kappa \) is a \( K \)-indexed diagram of Kan complexes, which we can denote by a Kan fibration \( \int_K f \to K \).

We can recover this map as the pullback of Kan complexes

\[
\begin{array}{ccc}
\int_K f & \xrightarrow{\tau} & (S_\ast)_0^\kappa \\
\downarrow & & \downarrow \\
K & \xrightarrow{} & S_0^\kappa
\end{array}
\]

where \((S_\ast)_0^\kappa\) is the 0-level of the complete Segal space of pointed Kan complexes, which comes with a forgetful map to \( S_0^\kappa \).

We can make a similar argument to show that a map \( K \to S_1^\kappa \) corresponds to a choice of map of Kan complexes

\[
\int_K f \to \int_K g
\]

over \( K \). Thus the complete Segal space \( \mathbf{Map}(S, S_\ast^\kappa) \) is equivalent to the over category \((S/K)^\kappa\) of \( \kappa \)-small maps over \( K \).

This shows that \( S_\ast^\kappa \) is a complete Segal universe in \( S_\ast \). Using the fact that every map of Kan complexes is \( \kappa \)-small for a large enough cardinal \( \kappa \) proves that \( S_\ast \) has sufficient universes.

1.4 Filters. As part of our construction we will need the set-theoretical notion of a filter. The notion of a filter is very standard and can be found in any textbook such as [CK90]. We will only need the definition of a filter and so our review here will cover everything we need about a filter later on.

**Definition 1.29.** Let \((P, \leq)\) be a partially ordered set. A filter \( F \) is a subset of \( P \) that satisfies following conditions.

1. \( F \neq \emptyset \).
2. \( F \) is downward directed, meaning that for any two object \( x, y \in F \) there exists \( z \in F \) such that \( z \leq x \) and \( z \leq y \).
3. \( F \) is upward closed, meaning that if \( x \leq y \) and \( x \in F \), then \( y \in F \).

**Remark 1.30.** Notice if \( P \) has a maximum element, then every filter in \( P \) necessarily includes that maximum element. Thus we could replace condition (1) with the condition that the maximum is in \( F \).

We have following basic but crucial observation about filters that we will need in the next section.

**Remark 1.31.** If \( F \) is a filter (and thus also a category), then then the opposite category \( F^{op} \) is a filtered category and thus in particular any colimit with diagram \( F^{op} \) will commute with finite limits.
Definition 1.32. An ultrafilter $U$ of a poset $P$ is a filter of $P$ that is maximal, meaning there does not exist a filter $F$ such that $U \not\subset F \not\subset P$.

Example 1.33. Let $P$ be a poset and $x \in P$. Then the subset $\{y \in P : x \leq y\}$ is a filter. Any filter of this form is called an principal filter.

Example 1.34. Let $P$ be a poset with finite meets. Then any upward closed non-empty subset closed under finite meets is a filter.

There is only one kind of filter we care about in the next section and so will discuss here in more detail.

Example 1.35. Let $E$ be a finitely complete $(\infty,1)$-category (for example an elementary $(\infty,1)$-topos). Then $\text{Sub}(1)$ has the structure of a poset with finite meets. Here the meet of two subobjects $U, V$ of $1$ is given by their product $U \times V \to 1$. Given the explanation above, any subset of $\text{Sub}(1)$ that includes $1$, is closed under finite products and is upward closed is a filter in $\text{Sub}(1)$. We will call any such filter a product closed filter.

Example 1.36. There is one interesting example of a filter we will use in Section 3.2. Let $\mathbb{N}$ be the set of natural numbers and $P(\mathbb{N})$ be the power set, which is a poset with the inclusion relation. Then cofinite sets (subsets with finite complement) form a product closed filter (commonly called the Fréchet filter). Indeed, if two subsets of $\mathbb{N}$ have finite complement, then their intersection also has a finite complement.

The Filter Quotient of $(\infty,1)$-Categories

The goal of this section is to construct a new elementary $(\infty,1)$-topos using a filter of $\text{Sub}(1)$ (Example 1.35).

In the first subsection we start more generally with a finitely complete $(\infty,1)$-category $\mathcal{C}$ and use that to construct a new $(\infty,1)$-category $\mathcal{C}_\Phi$ along with a quotient functor from the original $(\infty,1)$-category $P_\Phi : \mathcal{C} \to \mathcal{C}_\Phi$. We will give two main constructions depending on the model of $(\infty,1)$-category:

(1) One for Kan enriched categories, which leaves the collection of objects untouched and thus resembles the definition of the filter quotient for elementary 1-toposes [Jo03, Example A2.1.13].

(2) Another for complete Segal spaces and quasi-categories.

We will then show these definitions agree with each other.

In the second section we prove that if $\mathcal{E}$ has finite limits, finite colimits, subobject classifiers, or universes, then $P_\Phi$ will preserve them. This then implies that if $\mathcal{E}$ is an elementary $(\infty,1)$-topos then $\mathcal{C}_\Phi$ is one as well.

Remark 2.1. Throughout this section the model of $(\infty,1)$-categories for $\mathcal{E}$ will change. The reader is advised to follow the convention introduced in Subsection 0.4. On other hand $\Phi$ will constantly denote a product closed filter in $\text{Sub}(1)$.
2.1 The Filter Quotient Construction. In this subsection we present two methods for constructing the filter quotient, one that applies to Kan enriched categories and one for complete Segal spaces and quasi-categories.

First we will describe the filter quotient construction for Kan-enriched categories. Thus, let $K$ be a finitely complete Kan enriched category. We will construct a new Kan enriched category which we denote by $K_{\Phi}$.

The objects of $K_{\Phi}$ are just the objects of $K$. For the morphisms we first need some preliminary observations:

1. $\Phi$ is a full subcategory of $K$ and in particular there is a canonical functor $\mathcal{I}: \Phi \hookrightarrow K$.
2. Let $- \times -: \mathcal{K} \times \mathcal{K} \to \mathcal{K}$ be the product functor. We can restrict it via $\mathcal{I}$ to a map $\mathcal{I}(-) \times -: \Phi \times \mathcal{K} \to \mathcal{K}$
3. We can apply opposite categories $\mathcal{I}^{op}(-) \times -: \Phi^{op} \times \mathcal{K}^{op} \to \mathcal{K}^{op}$
4. We can product this functor with the identity functor $(\mathcal{I}^{op}(-) \times -) \times id_{\mathcal{K}}: \Phi^{op} \times \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}^{op} \times \mathcal{K}$
5. Using the fact that $\mathcal{K}$ is a Kan-enriched category, we can post-compose the functor above $Map_{\mathcal{K}}(-,-) \circ ((\mathcal{I}^{op}(-) \times -) \times id_{\mathcal{K}}): \Phi^{op} \times \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}^{op} \times \mathcal{K} \to \mathcal{K}^{an}$
6. We can construct the adjoint to the functor above $\mathcal{F}: \mathcal{K}^{op} \times \mathcal{K} \to \text{Fun}(\Phi^{op}, \mathcal{K}^{an})$

defined as $\mathcal{F}(X,Y)(U) = Map_{\mathcal{K}}(X \times U, Y)$.
7. Now we observe that the diagram $\Phi^{op}$ is a filtered diagram in Kan complexes and we have Kan fibrant replacements that commute with filtered colimits (for example $Ex_{\infty}$) thus the colimit of this diagram in simplicial sets will again give us a Kan complex. Thus we get a functor $\text{colim} \circ \mathcal{F}: \mathcal{K}^{op} \times \mathcal{K} \to \text{Fun}(\Phi^{op}, \mathcal{K}^{an}) \to \mathcal{K}^{an}$
8. We define $Map_{\mathcal{K}_{\Phi}}(-,-) = \text{colim} \circ \mathcal{F}$

and so in particular $Map_{\mathcal{K}_{\Phi}}(X,Y) = \text{colim}(Map(- \times X,Y): \Phi^{op} \to \mathcal{K}^{an})$

First we prove that the Kan complexes $Map_{\mathcal{K}_{\Phi}}(X,Y)$ give us indeed a new Kan enriched category.

**Proposition 2.2.** $\mathcal{K}_{\Phi}$ as defined above is a Kan enriched category.

**Remark 2.3.** Part of the proof here is a generalization of the argument given in [Jo03, Example A2.1.13].

**Proof.** We use the observation that a simplicially enriched category is just a simplicial object in categories where each level has the same set of objects. Thus it suffices to prove that for every natural number $k$, we get a category with object the objects $\mathcal{K}$ and morphisms sets $Map_{\mathcal{K}_{\Phi}}(X,Y)_k$ and that composition respects simplicial boundary maps.
First, recall that colimits in the category of simplicial sets are computed level-wise, thus we have

\[ \text{Map}_{\mathcal{K}_\Phi}(X, Y)_k = \text{colim}(\text{Map}(- \times X, Y)_k : \Phi^{\text{op}} \to \text{Set}) \]

Let \( X, Y, Z \) be three objects, we need a composition map

\[ \text{Comp}(X, Y, Z)_k : \text{Map}_{\mathcal{K}_\Phi}(X, Y)_k \times \text{Map}_{\mathcal{K}_\Phi}(Y, Z)_k \to \text{Map}_{\mathcal{K}_\Phi}(X, Z)_k \]

Let \([f]\) in \( \text{Map}_{\mathcal{K}_\Phi}(X, Y)_k \) and \([g]\) in \( \text{Map}_{\mathcal{K}_\Phi}(Y, Z)_k \) be two morphisms. Then we can represent them as morphisms \( f : X \times U \to Y \) and \( g : Y \times V \to Z \). We now define the composition as

\[ [g] \circ [f] = [g(f \times id_V)] \]

The colimit construction implies that the composition is well-defined. Moreover, as composition in \( \mathcal{K} \) is associative this new composition is associative as well.

We have constructed a collection of categories with the same objects and morphisms the hom set \( \text{Hom}_{\mathcal{K}_\Phi}(X, Y)_k \). We need to show that this collection is a simplicial object in categories. However, this follows immediately from the fact that for every simplicial map \( \delta : [k] \to [l] \) the following diagram commutes:

\[
\begin{align*}
\text{Hom}_{\mathcal{K}_\Phi}(X, Y)_l \times \text{Hom}_{\mathcal{K}_\Phi}(Y, Z)_l & \xrightarrow{\text{Comp}(X, Y, Z)_l} \text{Hom}_{\mathcal{K}_\Phi}(X, Z)_l \\
\downarrow{\delta \times \delta} & \quad \downarrow{\delta} \\
\text{Hom}_{\mathcal{K}_\Phi}(X, Y)_k \times \text{Hom}_{\mathcal{K}_\Phi}(Y, Z)_k & \xrightarrow{\text{Comp}(X, Y, Z)_k} \text{Hom}_{\mathcal{K}_\Phi}(X, Z)_k
\end{align*}
\]

Thus we have proven that \( \mathcal{K}_\Phi \) is a Kan enriched category. \( \square \)

**Definition 2.4.** Let \( \mathcal{K}_\Phi \) be the category defined above. We call the Kan enriched category \( \mathcal{K}_\Phi \) the **filter quotient of the Kan enriched category** \( \mathcal{K} \) **with filter** \( \Phi \).

Notice the category \( \mathcal{K}_\Phi \) comes with a distinguished functor from \( \mathcal{K} \).

**Definition 2.5.** There is a functor \( P_\Phi : \mathcal{K} \to \mathcal{K}_\Phi \). It is the identity map on objects and it takes a morphism \( f : X \to Y \) to the class \([f]\) of the morphism \( f : X \times 1 \to Y \).

We now will give an analogous construction for finitely complete complete Segal spaces (and by extension, quasi-categories). The goal is to construct the filter quotient as a filtered colimit of complete Segal spaces. This means we have to define our diagram and then prove the resulting colimit is a complete Segal space. Let \( \mathcal{W} \) be a fixed complete Segal space with product closed filter \( \Phi \) of \( \text{Sub}(1) \).

**Definition 2.6.** Let

\[ T_\Phi : \Phi^{\text{op}} \to \text{CSS} \]

be the diagram of complete Segal spaces that corresponds to the Cartesian fibration induced by the pullback.
Remark 2.7. Concretely, the diagram $\Phi$ takes an object $V$ to the over-category $W/V$ and morphism $W \leq V$ to the product map $- \times W : W/V \to W/W$.

We want to take the colimit of this diagram, but first we have to confirm that the colimit is a complete Segal space.

**Lemma 2.8.** Let $\Psi$ be a filtered category and let $F : \Psi \to \text{ssSet}$ be a diagram in bisimplicial sets such that $F(X)$ is a complete Segal space for each object $X$ in $\Psi$. Then the filtered colimit in $\text{ssSet}$ is a complete Segal space.

**Proof.** We have to prove that it is Reedy fibrant, satisfies the Segal condition and satisfies the completeness condition. First, we have to show that any diagram of the form

$$F(n) \times \Lambda^l_k \bigsqcup_{\partial F(n) \times \Lambda^l_k} \partial F(n) \times \Delta^l \longrightarrow \colim_{X \in \Psi} F(X)$$

The object $F(n) \times \Lambda^l_k \bigsqcup_{\partial F(n) \times \Lambda^l_k} \partial F(n) \times \Delta^l$ has finitely many 0-cells. Thus, using the fact that the diagram is filtered, we can find an object $X$ in $\Psi$ such that the horizontal map factors through $F(X)$. Now, the fact that a lift exists follows immediately from the fact that $F(X)$ is Reedy fibrant.

Next we prove the Segal condition. We have following chain of equivalences

$$\colim_{X \in \Psi} F(X)_n \overset{\simeq}{\longrightarrow} \colim_{X \in \Psi} [F(X)_1 \times F(X)_0] \overset{\simeq}{\longrightarrow} \colim_{X \in \Psi} [\colim_{X \in \Psi} F(X)_1]$$

where we used the fact that for each $X$, $F(X)$ is a Segal space and filtered colimits commute with finite limits.

The proof for the completeness condition is completely analogous to the proof of the Segal condition. A Segal space $\colim_{X \in \Psi} F(X)$ is complete if the following map is an equivalence (see [Re10,
Proposition 10.1)
\[
\text{colim}_{X \in \Psi} F(X)_0 \to \text{colim}_{X \in \Psi} F(X)_3 \times \text{colim}_{X \in \Psi} F(X)_1 \times \text{colim}_{X \in \Psi} F(X)_1
\]
which follows immediately from the fact that the filtered colimits commute with finite limits and for each \( X \in \Psi \), \( F(X) \) is complete. □

Remark 2.9. An analogous (independent) proof for the special case of ultraproducts can be found in [BSS20, Lemma 3.13].

We now have all the necessary ingredients to define the filter quotient complete Segal space.

Definition 2.10. Let \( W \) be a complete Segal space with finite limits. We define the filter quotient \( W_{\Phi} \) as
\[
W_{\Phi} = \text{colim}_{\Phi^{op}} T_{\Phi}
\]
and observe by the previous lemma that this is indeed a complete Segal space.

The colimit construction also gives us the desired quotient functor.

Definition 2.11. Let \( P_{\Phi} : W \to W_{\Phi} \) be the inclusion map into the filtered colimit, using the fact that \( W_{/1} = W \) and \( 1 \in \Phi \).

Notation 2.12. In order to simplify notation we will usually denote the object \( P_{\Phi}(X) \) in \( W_{\Phi} \) as \( X \) again.

We can give an analogous construction for quasi-categories.

Theorem 2.13. Let \( Q \) be a quasi-category and \( \Phi \) be a product closed filter on \( \text{Sub}(1) \). Then we define the simplicial set \( Q_{\Phi} \) defined as the colimit
\[
Q_{\Phi} = \text{colim}_{\Phi^{op}} T_{\Phi}
\]
has following properties:

1. It is a quasi-category.
2. It is compatible with the definition for complete Segal spaces, in the sense that for a given complete Segal space \( W \) we have equivalence
\[
i^{*}_{1}(W_{\Phi}) \simeq (i^{*}_{1}W)_{\Phi}.
\]
We call it the filter quotient of the quasi-category \( Q \) with respect to \( \Phi \).

Proof. The key observation is that \( \Gamma(Q) \) is a complete Segal space with \( \Gamma(Q)_{n0} = Q_n \) (Remark 1.2). Moreover, any filter \( \Phi \) in \( Q \) is also a filter in \( \Gamma(Q) \). Thus, we can apply the filter construction to get a complete Segal space \( \Gamma(Q)_{\Phi} \). Restricting to the first row via \( i^{*}_{1}(\Gamma(Q)) \) gives us a quasi-category. But, in the construction of \( \Gamma(Q)_{\Phi} \) we constructed colimits level-wise, thus we have
\[
i^{*}_{1}(\Gamma(Q)) = \text{colim}_{\Phi^{op}} T_{\Phi} = Q_{\Phi}
\]
This proves that \( Q_{\Phi} \) is a quasi-category and that it is compatible with the definition for complete Segal spaces. □
The question that remains is whether the construction for Kan enriched categories is compatible with the construction for complete Segal spaces (the same way we just showed that the quasi-category and complete Segal space constructions are compatible). Concretely, let $\mathcal{K}$ be a Kan enriched category. Then can we compare $t^{!}N_{\Delta}(\mathcal{K}_{\Phi})$ and $(t^{!}N_{\Delta}(\mathcal{K}))_{\Phi}$, noticing the fact that in the former we are using the Kan enriched filter quotient construction, whereas in the latter we are using the complete Segal space filter quotient construction.

Thus the final goal of this subsection is to prove that these constructions are indeed equivalent. For that we need an alternative characterization of the filter construction for Kan enriched categories.

**Definition 2.14.** Define an equivalence relation $\sim$ on the set of objects of $\mathcal{K}$ as follows

$$X \sim Y \iff \text{there exists a } V \in \Phi \text{ such that } X \times V \simeq Y \times V$$

**Definition 2.15.** Let $\text{Ob}(\mathcal{K}_{\Phi}^{\text{quot}})$ be a set of representatives of the equivalence classes of the equivalence relation given in the previous definition. Moreover, define $\mathcal{K}_{\Phi}^{\text{quot}}$ as the full subcategory of $\mathcal{K}_{\Phi}$ with set of objects in $\text{Ob}(\mathcal{K}_{\Phi}^{\text{quot}})$.

**Lemma 2.16.** The inclusion functor $\text{Ext}: \mathcal{K}_{\Phi}^{\text{quot}} \hookrightarrow \mathcal{K}_{\Phi}$ is an equivalence.

**Proof.** The functor is fully faithful by definition, thus it suffices to prove that the inclusion functor is essentially surjective. Let $X$ be an arbitrary object in $\mathcal{K}_{\Phi}$. Then there exists an object $Y$ in the set $\text{Ob}(\mathcal{K}_{\Phi}^{\text{quot}})$ and $V \in \Phi$ such that $X \times V \simeq Y \times V$. However, by definition of $\mathcal{K}_{\Phi}$, the two maps $\pi_{1}: X \times V \to X$ and $id_{X}: X \to X$ are identified in $\text{Map}_{\mathcal{K}_{\Phi}}(X, X)$, which implies that $X \simeq X \times V$. Similarly $Y \simeq Y \times V$. Thus we get

$$X \simeq X \times V \simeq Y \times V \simeq Y$$

which proves that $\text{Ext}$ is essentially surjective. □

**Lemma 2.17.** Let $X, Y$ be two objects in $\mathcal{K}$. Then we have a bijection

$$\text{Map}_{\mathcal{K}_{\Phi}}(X, Y) \cong \colim_{V \in \Phi^{op}} \text{Map}_{\mathcal{K}}(X \times V, Y \times V)$$

**Proof.** It suffices to prove that for every $V$ in $\Phi$ we have a functorial bijection

$$\text{Map}_{\mathcal{K}}(X \times V, Y \times V) \cong \text{Map}_{\mathcal{K}}(X \times V, Y)$$

We have following bijections:

$$\text{Map}_{\mathcal{K}}(X \times V, Y \times V) \cong \text{Map}_{\mathcal{K}}(X \times V, Y) \times \text{Map}_{\mathcal{K}}(X \times V, Y) \cong \text{Map}_{\mathcal{K}}(X \times V, Y)$$

where the last bijection follows from the fact that $\text{Map}_{\mathcal{K}}(X \times V, Y)$ has either none or one element as $V$ is a subobject of the final object and it is clearly not empty (we have $\pi_{2}: X \times V \to V$). □

We now want to give a construction of $\mathcal{K}_{\Phi}$ as a colimit analogous to our definition for complete Segal spaces. The problem is that there is no straightforward way to construct over-categories for Kan enriched categories. Thus, we introduce a way to circumvent that problem. The key is following observation from complete Segal spaces.

**Remark 2.18.** If $V$ is $(-1)$-truncated then the over-category $W/V$ is just a full subcategory of $W$. Moreover the objects $X$ in $W/V$ can be characterized by one of the following equivalent conditions:
(1) There exists a map $X \to V$.
(2) The map $\pi_1 : X \times V \to X$ is an equivalence.
(3) There is an equivalence $X \simeq Y \times V$ for some object $Y$.

This observation gives us following definition:

**Definition 2.19.** Let $\mathcal{K}$ be a Kan enriched category and $V$ be a $(-1)$-truncated object. Then we denote by $\mathcal{K}^S_V$ the full subcategory of $\mathcal{K}$ consisting of objects which satisfy $X \simeq X \times V$, where $X$ is an object in $\mathcal{K}$.

**Definition 2.20.** Let

$$\mathcal{T}_\Phi : \Phi^{op} \to \text{Cat}_\Delta$$

be the Kan enriched functor defined as

$$\mathcal{T}_\Phi(V) = \mathcal{K}^S_V$$

**Remark 2.21.** In Definition 1.7 we described the mapping space as a finite limit diagram. Thus using the argument of Lemma 2.8 we see that for any filtered diagram $F : \Psi \to \text{ssSet}$ valued in complete Segal spaces we have an isomorphism of mapping Kan complexes:

$$\colim_{V \in \Psi} \text{Map}_F(V)(X, Y) \cong \text{Map}_{\colim_F}(X, Y)$$

We can now compare the filter construction for complete Segal spaces and Kan enriched categories.

**Theorem 2.22.** Let $(t^! N_\Delta) \circ \mathcal{T}_\Phi$ be the composition functor

$$(t^! N_\Delta) \circ \mathcal{T}_\Phi : \Phi^{op} \to \text{Cat}_\Delta \to \text{ssSet}$$

Then there is an equivalence of complete Segal spaces

$$\colim_{V \in \Phi^{op}} (t^! N_\Delta) \circ \mathcal{T}_\Phi \simeq (t^! N_\Delta)(\mathcal{K}_\Phi)$$

**Proof.** We have already shown that there is an equivalence $\text{Ext} : \mathcal{K}_\Phi^{\text{quot}} \to \mathcal{K}_\Phi$. Thus it suffices to prove that there is an equivalence

$$\colim_{V \in \Phi^{op}} (t^! N_\Delta) \circ \mathcal{T}_\Phi \to t^! N_\Delta(\mathcal{K}_\Phi)$$

As a first step we want to define the map above by defining a cocone. This means we have to construct functors $\mathcal{K}^S_V \to \mathcal{K}_\Phi^{\text{quot}}$ that are consistent with the functors $- \times V$.

Let

$$\mathcal{F}_V : \mathcal{K}^S_V \to \mathcal{K}_\Phi^{\text{quot}}$$

be defined on objects by $\mathcal{F}_V(X \times V) = [X]$ and on mapping Kan complexes it is the inclusion map into the colimit

$$\iota_V : \text{Map}_{\mathcal{K}^S_V}(X \times V, Y \times V) \to \text{Map}_{\mathcal{K}_\Phi^{\text{quot}}}(\{X\}, \{Y\}) = \colim_{W \in \Phi^{op}} \text{Map}_{\mathcal{K}}(X \times W, Y \times W)$$

that takes a map $f : X \times V \to Y \times V$ to the class $[f]$ in the colimit. Here we used the alternative characterization of mapping Kan complexes in $\mathcal{K}_\Phi$ as proven in Lemma 2.17.
We need to show that the collection of functors $\mathcal{F}_V$ give us a cocone over $\mathcal{K}^{\text{quot}}_Φ$. However, this follows immediately from the fact that for any object $W ∈ Φ \left[(X × V) × W] = [X × W\right]$, which proves that $\mathcal{F}_V(− × W) = \mathcal{F}_W$. Thus the maps $\mathcal{F}_V$ give us a cocone.

Applying the map $t^!N_Δ$ we get a cocone of complete Segal spaces, which gives us a universal map out of the colimit

$$\text{colim}_{V ∈ Φ^{op}}(\mathcal{F}_V) : \text{colim}_{V ∈ Φ^{op}} t^!N_Δ(\mathcal{K}^{S_V}) \to t^!N_Δ(\mathcal{K}^{\text{quot}}_Φ)$$

We want to show that this map is an equivalence.

We know that $t^!N_Δ(\mathcal{K}^{\text{quot}}_Φ)$ and $t^!N_Δ(\mathcal{K}^{S_V})$ are complete Segal spaces (Remark 1.1). Moreover, we prove in Lemma 2.8 that a filtered colimit of complete Segal spaces is a complete Segal space. This proves that the left hand side is also a complete Segal space. Thus it suffices to prove the map is Dwyer-Kan equivalence [Re01, Theorem 7.7].

Clearly the map is a surjection on objects, thus we need to show we have an equivalence of mapping Kan complexes. Fix two objects $X, Y$ in $\mathcal{K}$. Then we have a diagram of Kan complexes

\[
\begin{array}{ccc}
\text{colim}_{V ∈ Φ^{op}} Map_{\mathcal{K}^{S_V}}(X × V, Y × V) & \longrightarrow & Map_{\mathcal{K}^{\text{quot}}_Φ}([X], [Y]) \\
\downarrow \cong & & \downarrow \cong \\
\text{colim}_{V ∈ Φ^{op}} Map_{\mathcal{K}^{S_V}}(X × V, Y × V) & \longrightarrow & Map_{\mathcal{K}^{\text{quot}}_Φ}([X], [Y]) \\
\downarrow \cong & & \downarrow \cong \\
\text{colim}_{V ∈ Φ^{op}} Map_{\mathcal{K}}(X × V, Y × V) & \simeq & Map_{\mathcal{K}}(X, Y)
\end{array}
\]

where the numbered morphisms are equivalences for the following reasons:

1. The map $t^!N_Δ$ takes mapping Kan complexes to equivalent mapping Kan complexes (Remark 1.3).
2. $\mathcal{K}^{S_V}$ is a full subcategory $\mathcal{K}$, which gives us a bijection of mapping spaces (Definition 2.19)
3. The functor $\mathcal{E}xt$ is an equivalence of Kan enriched categories (Lemma 2.16)
4. This is an alternative characterization of mapping Kan complexes in $\mathcal{K}_Φ$ (Lemma 2.17)

This implies that the top horizontal map is also an equivalence of Kan complexes, which proves that we have a Dwyer-Kan equivalence of complete Segal spaces.

2.2 The Filter Quotient is an Elementary $(∞, 1)$-Topos. The next step is to prove that if $\mathcal{E}$ is an elementary $(∞, 1)$-topos then $\mathcal{E}_Φ$ is one as well. The key step of the proof is to show that $\mathcal{E}_Φ$ also has sufficient universes. For that we need a better understanding of the filter quotient of the complete Segal space of cones.
Lemma 2.23. Let $I$ be a finite simplicial space (i.e. with finitely many non-degenerate cells) and $\mathcal{W}$ a finitely complete complete Segal space with filter $\Phi$. Let $\Phi^I$ be the induced filter on $\mathcal{W}^I$ consisting of constant functor $I \to \mathcal{W}$ with value $V \in \Phi$. Then, we have

1. an equivalence of functor complete Segal spaces
   $$(\mathcal{W}^I)_{\Phi} \simeq (\mathcal{W}_{\Phi})^I$$

2. an equivalence of cocones
   $$(\mathcal{W}/I)_{\Phi} \simeq (\mathcal{W}_{\Phi})/I$$

Proof. (1) First notice we have an equivalence
$$((\mathcal{W}^I)/V) \simeq (\mathcal{W}/V)^I$$
as they both are the full subcategory of $\mathcal{W}^I$ consisting of diagrams that take value in the full subcategory $\mathcal{W}/V$. Thus, we get the equivalence
$$((\mathcal{W}^I)_{\Phi} = \text{colim}_{V \in \Phi} ((\mathcal{W}^I)/V) \simeq \text{colim}_{V \in \Phi} (\mathcal{W}/V)^I \simeq (\text{colim}_{V \in \Phi} \mathcal{W}/V)^I = (\mathcal{W}_{\Phi})^I$$
where the last equivalence follows from the fact that filtered colimits commute with exponents by finite simplicial spaces.

(2) We will now use the previous part and our explicit description of cocones to get the desired result:
$$(\mathcal{W}/I)_{\Phi} \simeq (\mathcal{W}(1)^I \times \mathcal{W}^I)_{\Phi} \simeq (\mathcal{W}(1)^I)_{\Phi} \times (\mathcal{W}^I)_{\Phi} \simeq (\mathcal{W}_{\Phi})^{(1)^I} \times (\mathcal{W}_{\Phi})^I = (\mathcal{W}_{\Phi})/I$$

Theorem 2.24. Let $\mathcal{W}$ be a complete Segal space with finite limits. Then the functor
$$P_{\Phi} : \mathcal{W} \to \mathcal{W}_{\Phi}$$
preserves

1. finite limits and colimits
2. subobject classifiers
3. complete Segal universes

In particular if $\mathcal{E}$ is an elementary $\text{(\infty,1)-topos}$, then $\mathcal{E}_{\Phi}$ is one as well.

Proof. We need to prove that $\mathcal{W}_{\Phi}$ satisfies the three conditions given in Definition 1.19. We will confirm each separately.

Finite Limits and Colimits: It suffices to prove that $\mathcal{W}_{\Phi}$ preserves finite limits and the case for finite colimits is analogous. We want to prove that the final object 1 in $\mathcal{W}$ is also the final object in $\mathcal{W}_{\Phi}$. We have
$$\text{Map}_{\mathcal{W}_{\Phi}}(X, 1) = \text{colim}_{V \in \Phi} \text{Map}_{\mathcal{W}}(X \times V, 1) \overset{(1)}{\simeq} \text{colim}_{V \in \Phi} \Delta[0] \overset{(2)}{=} \Delta[0]$$
where are using following facts:
(1) 1 is final in \( \mathcal{W} \) and thus \( \text{Map}_{\mathcal{W}}(X \times U, 1) \) is contractible and the colimit construction is homotopy invariant.

(2) filtered colimits of the final object is again final.

We now want to prove that \( \mathcal{W}_\Phi \) has \( I \)-shaped limits. We have following diagram

\[
\begin{array}{ccc}
\mathcal{W}_I & \xrightarrow{(P_{/I})_\Phi} & (\mathcal{W}_I)_{\Phi} \\
\downarrow{(P_{/I})_I} & & \downarrow{\simeq} \\
(\mathcal{W}_\Phi)_{/I} & & 
\end{array}
\]

By the previous lemma the vertical map is an equivalence. By the previous paragraph \((P_{/I})_\Phi\) preserves final objects. which then implies that \((P_{/I})_I\) also preserves final objects. But a final object in cone category is just the limit and thus \(P_\Phi\) preserves all finite limits.

**Subobject Classifier:** Let \( \Omega \) be the subobject classifier in \( \mathcal{W} \). We want to prove that \( \Omega \) is a subobject classifier in \( \mathcal{W}_\Phi \). First, we have to determine the subobjects in \( \mathcal{W}_\Phi \). Let \([f]\) be a morphism in \( \text{Map}_{\mathcal{W}_\Phi}(X, Y) \). Then \([f]\) is mono if and only if the following is a pullback square in \( \mathcal{W}_\Phi \)

\[
\begin{array}{ccc}
X & \xrightarrow{id_X} & X \\
\downarrow{id_X} & & \downarrow{[f]} \\
X & \xrightarrow{[f]} & Y
\end{array}
\]

However, we just proved that a pullback in \( \mathcal{W}_\Phi \) is evaluated as a pullback of any representative. Thus this is equivalent to the following being a pullback square

\[
\begin{array}{ccc}
X \times U & \xrightarrow{id_{X \times U}} & X \times U \\
\downarrow{id_{X \times U}} & & \downarrow{f} \\
X \times U & \xrightarrow{f} & Y
\end{array}
\]

where \( f : X \times U \to Y \) is any representative of \([f]\). However, this is equivalent to \( f : X \times U \to Y \) being mono in \( \mathcal{W} \). Thus we just proved that \([f]\) in \( \mathcal{W}_\Phi \) is mono if and only if there exists a representative \( f : X \times U \to Y \) that is mono in \( \mathcal{W} \), which we can state as a bijection

\[
\text{Sub}_{\mathcal{W}_\Phi}(X) \cong \text{colim}_{V \in \Phi^{op}} \text{Sub}_{\mathcal{W}}(X \times V)
\]
We will use this to prove $\Omega$ is a subobject classifier. We have following bijections

$$\text{Map}_{\mathcal{W}_\Phi}(X, \Omega) = \text{colim}_{V \in \Phi^{op}} \text{Map}_{\mathcal{W}}(X \times V, \Omega \times V) \cong \text{colim}_{V \in \Phi^{op}} \text{Sub}_{\mathcal{W}}(X \times V) \cong \text{Sub}_{\mathcal{W}_\Phi}(X)$$

where the last step follows from the bijection given in the last paragraph. Thus $\Omega$ is also the subobject classifier in $\mathcal{W}_\Phi$.

**Complete Segal Universes:** We will now prove that $\mathcal{W}_\Phi$ has sufficient complete Segal universes.

Let $\mathcal{U}^S$ be a complete Segal universe in $\mathcal{W}$ classifying the collection of maps $S$ in $\mathcal{W}$. We show that $\mathcal{U}^S$ is a complete Segal universe in $\mathcal{W}_\Phi$ by proving there is an equivalence of complete Segal spaces

$$(\mathcal{W}_\Phi)_{/X}^S \simeq \text{Map}_{\mathcal{W}_\Phi}(X, \mathcal{U}^S).$$

Let $X$ be a fixed object. Then we have following equivalences

$$\text{Map}_{\mathcal{W}_\Phi}(X, \mathcal{U}^S) = \text{colim}_{V \in \Phi^{op}} \text{Map}_{\mathcal{W}/V}(X \times V, \mathcal{U}^S \times V) \overset{(1)}{=} \text{colim}_{V \in \Phi^{op}} (\mathcal{W}_{/V}^F(1) \times F(0))^S \overset{(2)}{=} \text{colim}_{V \in \Phi^{op}} (\mathcal{W}_{/V}^F(1) \times F(0))^S \overset{(3)}{=} \text{colim}_{V \in \Phi^{op}} (\mathcal{W}_{/V}^F(1) \times F(0))^S \overset{(4)}{=} \text{colim}_{V \in \Phi^{op}} (\mathcal{W}_{/V}^F(1) \times F(0))^S$$

where the numbered equivalences hold for the following reasons:

1. This equivalence holds because $\mathcal{U}^S$ is a complete Segal universe in $\mathcal{W}$.
2. This follows from the observation that an object over a morphism $\pi_2 : X \times V \to V$ is determined by the map into the domain $X \times V$.
3. This is the definition of an over-category using complete Segal spaces (Definition 1.8).
4. This follows from the fact that filtered colimits commute with finite limits (as used in Lemma 2.8) and exponents by finite simplicial spaces (as used in Lemma 2.23).
5. This is just the definition of $\mathcal{W}_\Phi$.

This shows that $\mathcal{U}^S$ is a complete Segal object in $\mathcal{W}_\Phi$.

All that remains is to prove $\mathcal{W}_\Phi$ has sufficient universes. Let $[f] : X \to Y$ be an arbitrary map in $\mathcal{W}_\Phi$. Then it can be represented by a map $f : X \times U \to Y \times U$ in $\mathcal{W}$. As $\mathcal{W}$ has sufficient universes $f$ is classified by some universe $\mathcal{U}$. Given that pullbacks in $\mathcal{W}_\Phi$ are computed by pullbacks in $\mathcal{W}$, it follows that $[f]$ is classified by $\mathcal{U}$ in $\mathcal{W}_\Phi$ as well.

This proves that if $\mathcal{E}$ is an elementary complete Segal topos then $\mathcal{E}_\Phi$ is one as well. As we defined other models of $(\infty, 1)$-categories to be an elementary $(\infty, 1)$-topos if the corresponding complete Segal space is one, the same result holds for the other two models and hence we are done. □

**Remark 2.25.** Some parts (such as preservation of limits and colimits) restricted to the case of ultraproducts were already proven in [BSS20, Lemma 3.17].

Notice there are trivial examples of filter quotients.
Example 2.26. Let $\Phi$ be minimal filter (which only includes 1 itself). Then $\mathcal{E}_\Phi = \mathcal{E}$. On the other hand if $\Phi = \text{Sub}(1)$, then $\mathcal{E}_\Phi$ is the trivial category with one object and identity map. This immediately follows from the fact that in this case the poset $\Phi^{op}$ has a final object and thus all colimits are just evaluation at that final object. However, that final object is just 0 and $\mathcal{E}/_0$ is the trivial category.

We finish this section by observing that this construction lifts the construction of the underlying toposes.

Theorem 2.27. Let $\mathcal{E}$ be a locally Cartesian closed $(\infty,1)$-category with subobject classifier and $\tau_0\mathcal{E}$ the elementary topos of 0-truncated objects. Moreover, let $\Phi$ be a product closed filter of $\text{Sub}(1)$. Then the functor

$$\tau_0(\mathcal{E})\Phi \to \tau_0(\mathcal{E}_\Phi)$$

induced by $P_\Phi : \mathcal{E} \to \mathcal{E}_\Phi$ is an equivalence of 1-categories.

Proof. First we determine 0-truncated objects in $\mathcal{E}_\Phi$. Let $X$ be an object in $\mathcal{E}_\Phi$. Then $X$ is 0-truncated if and only if the following commutative square

$$
\begin{array}{ccc}
X & \xrightarrow{id_X} & X \\
\downarrow{id_X} & & \downarrow{\Delta_X} \\
X & \xrightarrow{\Delta_X} & X \times X
\end{array}
$$

is a pullback in $\mathcal{E}_\Phi$. This is equivalent to

$$
\begin{array}{ccc}
X \times U & \xrightarrow{id_{X \times U}} & X \times U \\
\downarrow{id_{X \times U}} & & \downarrow{\Delta_X \times id_U} \\
X \times U & \xrightarrow{\Delta_X \times id_U} & (X \times X) \times U
\end{array}
$$

being a pullback in $\mathcal{E}$ for some object $U$ in $\Phi$. But the map $\Delta_X \times id_U$ is isomorphic to $\Delta_X \times U$ (as we have $U \cong U \times U$). This means the square above being a pullback in $\mathcal{E}$ is equivalent to $X \times U$ being 0-truncated in $\mathcal{E}$. Thus we have proven that $X$ is 0-truncated in $\mathcal{E}_\Phi$ if and only if it $X \times U$ is 0-truncated in $\mathcal{E}$ for some $U$ in $\Phi$.

In particular, if $X$ is 0-truncated in $\mathcal{E}$ then $P_\Phi(X)$ is an object in the subcategory $\tau_0(\mathcal{E}_\Phi)$, which means we have following commutative diagram
FILTER QUOTIENTS AND NON-PRESENTABLE ($\infty$, 1)-TOPOSES

\[
\begin{array}{ccc}
\tau_0(E) & \xrightarrow{\tau_0(P_\Phi)} & \tau_0(E_\Phi) \\
\downarrow & & \downarrow \\
E & \xrightarrow{P_\Phi} & E_\Phi
\end{array}
\]

Notice $\Phi$ is also a filter in $\tau_0(E)$ and so we can apply the filter quotient construction to get a diagram

\[
\begin{array}{ccc}
\tau_0(E) & \xrightarrow{\tau_0(P_\Phi)} & \tau_0(E_\Phi) \\
\downarrow & & \downarrow \\
\tau_0(E)_\Phi \\
\end{array}
\]

We now want to show that we can lift the diagram to a functor $P_\Phi : \tau_0(E)_\Phi \to \tau_0(E_\Phi)$.

The map is already defined for objects, but we need to define it for $\text{Hom}$ sets, which means we need a map

\[
\text{Hom}_{\tau_0(E)_\Phi}(X,Y) \to \text{Hom}_{\tau_0(E_\Phi)}(X,Y)
\]

Using their definition as colimits this means we need a map

\[
\text{colim}_{V \in \Phi^{op}} \text{Hom}_{\tau_0(E)}(X \times V, Y) \to \text{colim}_{V \in \Phi^{op}} \text{Hom}_E(X \times V, Y)
\]

But $\tau_0E$ is a full subcategory of $E$ and so we can take this map to be the identity.

We now want to prove that this induced functor is an equivalence of categories. The functor is fully faithful by construction, so we only need to prove it is essentially surjective. Let $X$ be an object in $\tau_0(E_\Phi)$. Then there exists a $U$ such that $X \times U$ is 0-truncated. This means that $X \times U$ is an object in $\tau_0(E)$ and thus also an object in $(\tau_0(E))_\Phi$. Finally by construction of the filter quotient, the map $U \times X \to X$ is an equivalence. Thus we have proven that every object $X$ is equivalent to an object in the image, namely $X \times U$.

One interesting implication of this result is the preservation of natural number objects.

**Theorem 2.28.** Let $E$ be an elementary ($\infty$, 1)-topos and $\mathbb{N}$ be the natural number object. Moreover, let $\Phi$ be a product closed filter on $E$. Then $P_\Phi(\mathbb{N})$ is the natural number in $E_\Phi$.

**Proof.** A natural number object $\mathbb{N}$ is always 0-truncated and thus lives in $\tau_0(E)$. By [Jo03] $P_\Phi(\mathbb{N})$ is thus also a natural number object in the filter quotient elementary topos $\tau_0(E)_\Phi$. By the equivalence above, it is thus also a natural number object in $\tau_0(E_\Phi)$. But by [Ra18b] every natural number object in the underlying elementary topos is a natural number object in the whole elementary ($\infty$, 1)-topos. Thus $E_\Phi$ has a natural number object, namely $P_\Phi(\mathbb{N})$. \qed
We can now combine this theorem with Corollary 1.15 to get a powerful method to construct many elementary \((\infty,1)\)-toposes that are not Grothendieck \((\infty,1)\)-toposes.

**Corollary 2.29.** Let \(\mathcal{E}\) be an elementary \((\infty,1)\)-topos and \(\Phi\) be a product closed filter such that the filter quotient elementary topos \(\tau_0(\mathcal{E})_\Phi\) is not a Grothendieck topos. Then the filter quotient \(\mathcal{E}_\Phi\) is an elementary \((\infty,1)\)-topos that is not a Grothendieck \((\infty,1)\)-topos.

**Remark 2.30.** The conditions given in the previous corollary are sufficient conditions for \(\mathcal{E}\) to be elementary without being Grothendieck. It is not known whether the conditions are also necessary i.e. if there is an elementary \((\infty,1)\)-topos that is not a Grothendieck \((\infty,1)\)-topos, but for which the elementary topos is a Grothendieck topos.

In the next section we will focus on specific filter quotients and then use it, combined with the corollary above, to construct elementary \((\infty,1)\)-toposes are not Grothendieck \((\infty,1)\)-toposes

**Filter Products**

In this section we want to restrict our attention to specific filter quotients: *filter products*. In the first subsection we study general filter product. In the second subsection we focus on one specific example of a filter product.

### 3.1 Loš Theorem for Equivalences

Throughout this subsection \(\mathcal{C}\) is a fixed finitely complete \((\infty,1)\)-category such that \(1\) only has two subobjects, \(I\) is a set and \(\Phi\) is a product closed filter of \(P(I)\), the power set of \(I\). Notice \(\mathcal{C}^I\) is also finitely complete. We want to use the filter \(\Phi\) of \(P(I)\) to build a filter on \(\mathcal{C}^I\).

First, we observe by assumption there is a bijection

\[\{0,1\} \cong \text{Sub}_{\mathcal{C}}(1)\]

Moreover, we have

\[\text{Sub}_{\mathcal{C}^I}(1) = \text{Sub}_{\mathcal{C}}(1)^I = \{0,1\}^I = P(I)\]

and so a filter \(\Phi\) on \(P(I)\) is automatically a filter on \(\mathcal{C}^I\) and we can use it to define the filter quotient \((\mathcal{C}^I)_{\Phi}\).

**Definition 3.1.** Let \(\mathcal{C}\) be a finitely complete \((\infty,1)\)-category such that \(1\) has only two subobjects, \(S\) a set and \(\Phi\) a product filter of \(P(I)\). Then the resulting filter quotient \((\mathcal{C}^I)_{\Phi}\) is called the *filter product* and denoted \(\prod_{\Phi} \mathcal{C}\).

**Remark 3.2.** If the filter \(\Phi\) is an ultrafilter, then \(\prod_{\Phi} \mathcal{C}\) is often called the *ultraproduct*.

Our goal is to prove analogues of Loš theorem for homotopies and equivalences in a filter product. We will actually prove a more general statement about filter quotients, from which the desired results for filter products will follow as an immediate corollary.

**Lemma 3.3.** Let \(\Phi^{op}\) be a filtered diagram and \(F : \Phi^{op} \to S\) be a diagram of spaces. Then

1. We have isomorphism of sets

\[\pi_0(\text{colim}F) \cong \text{colim}(\pi_0(F))\]
(2) Two points \( x, y \in \text{colim}(F) \) are homotopic if and only if there exists \( S \in \Phi^{op} \) and \( x', y' \in F(S) \) such that \( x' \simeq y' \) and \( \iota(x') = x, \iota(y') = y \), where \( \iota : F(S) \to \text{colim}(F) \) is the universal cocone map.

Proof. The first part follows immediately from the fact that \( \pi_0 \) commutes with filtered colimits. Given that the second part focuses on the existence of equivalences we can restrict our attention to \( \pi_0 \) of that diagram.

Thus we have to prove that \( x = y \) in \( \text{colim}(\pi_0(F)) \) if and only if there exists \( S \) and \( x', y' \in \pi_0(F(S)) \) such that \( \iota(x') = x, \iota(y') = y \) and \( x' = y' \). However, that is just the definition of the colimit. \( \square \)

Theorem 3.4. Let \( \mathcal{C} \) be a finitely complete \((\infty, 1)\)-category and \( \Phi \) a filter. Then for two morphisms \( f, g : X \to Y \) in \( \mathcal{E}_\Phi \)

\[
f \simeq g \text{ in } \mathcal{E}_\Phi \iff \exists U \in \Phi(f \times \text{id}_U \simeq g \times \text{id}_U \text{ in } \mathcal{C})
\]

Proof. Observe that

\[
\text{Map}_{\mathcal{E}_\Phi}(X, Y) = \text{colim}_{V \in \Phi^{op}} \text{Map}_{\mathcal{C}}(X \times V, Y \times V)
\]

So, the result follows immediately from Lemma 3.3 as two morphism are equivalent if and only if they are in the same path-component. \( \square \)

Using the result for filter products we immediately get following corollary.

Corollary 3.5. Let \( (f_i)_{i \in I}, (g_i)_{i \in I} : (X_i)_{i \in I} \to (Y_i)_{i \in I} \) be two maps. Then \( (f_i)_{i \in I} \simeq (g_i)_{i \in I} \) in \( \prod_{\Phi} \mathcal{C} \) if and only if

\[
\{ i \in I : f_i \simeq g_i \text{ in } \mathcal{C} \} \in \Phi
\]

Theorem 3.6. Let \( \mathcal{C} \) be a finitely complete \((\infty, 1)\)-category, \( \Phi \) a filter on \( \mathcal{C} \). Then a map \( f \) in \( \mathcal{C}_\Phi \) is an equivalence if and only if there exists \( U \in \Phi \) such that \( f \times \text{id}_U \) is an equivalence in \( \mathcal{C} \).

Proof. Let \( \mathcal{C}^{\text{core}} \) be the underlying \((\infty, 1)\)-groupoid of \( \mathcal{C} \). Then we have an equivalence

\[
(\mathcal{C}_\Phi)^{\text{core}} \simeq (\mathcal{C}^{\text{core}})_\Phi
\]

Indeed, this follows from the fact that in the complete Segal space model, we have \( \mathcal{C}^{\text{core}} = \mathcal{C}_0 \) (Definition 1.6) and the filter quotient is defined as a level-wise colimit (Definition 2.10).

A morphism \( f \) in \( \mathcal{C}_\Phi \) is an equivalence if and only if it is in \( (\mathcal{C}_\Phi)^{\text{core}} \), which is equivalent to \( (\mathcal{C}^{\text{core}})_\Phi \). But, by definition of filtered colimits, this is equivalent to \( f \times \text{id}_U \) being invertible for some \( U \in \Phi \). \( \square \)

Again, we can restrict our attention to filter products.

Corollary 3.7. Let \( \mathcal{C} \) be a finitely complete \((\infty, 1)\)-category such that 1 has two subobjects, \( I \) a set and \( \Phi \) a filter on \( P(I) \). Then a map \( (f_i)_{i \in I} \) in \( \prod_{\Phi} \mathcal{C} \) is an equivalence if and only if

\[
\{ i \in I : f_i \text{ is an equivalence} \} \in \Phi
\]

Remark 3.8. We can use these results to characterize truncated objects in a filter quotient. See [Ra18c, Subsection 6.2] for more details.
Remark 3.9. The restriction that 1 in \( C \) has only two subobject is such that \( \Phi \) is also a filter on \( C \). With enough care that condition could possibly be relaxed.

Remark 3.10. Notice in the actual Löś’s theorem, the filter needs to be an ultrafilter, which we did not assume. That is because Löś’s theorem (as used in model theory) holds for all formulas. In particular, it holds for formulas that include negations, which require ultrafilters, as they have a certain closure property under set complements.

On the other hand we prove very particular results, none of which include any negation and that is why these statements hold for any filter. I want to thank Peter Lumsdaine for making me aware of this fact.

We have now gathered enough background to give examples of non-presentable \((\infty, 1)\)-toposes.

3.2 Examples of Filter-Quotients that are not Grothendieck \((\infty, 1)\)-Toposes. In this subsection we finally give examples of elementary \((\infty, 1)\)-toposes that are not Grothendieck toposes. First we use our general knowledge to construct a whole class of examples. Then we focus on one specific example and make the construction as explicit as possible.

Example 3.11. As before, let \( \mathcal{K}an \) be the Kan enriched category of Kan complexes, let \( S \) be a set and \( \Phi \) a filter on \( P(S) \). Notice that we indeed have \( \text{Sub}(1) = \{0, 1\} \). Thus, we can define the filter product \( \prod_{\Phi} \mathcal{K}an \) and by Theorem 2.24 it is still an elementary \((\infty, 1)\)-topos. By Theorem 2.27 the underlying elementary topos is \( \prod_{\Phi} \text{Set} \). Based on Corollary 2.29 we only need to show that \( \prod_{\Phi} \text{Set} \) is not a Grothendieck topos.

For example let \( S \) be a set and \( \Phi \) be a non-principal filter. Then \( \prod_{\Phi} \text{Set} \) is not a Grothendieck topos [AJ82, Theorem 3.4]. Thus there are at least as many elementary \((\infty, 1)\)-toposes that are not Grothendieck \((\infty, 1)\)-toposes as there are non-principal filters of sets.

Here is one particular example that satisfies the conditions given in Example 3.11: Let \( \Phi \) be the filter of cofinite subsets of \( \mathbb{N} \), the set of natural numbers (Example 1.36). We can thus apply the filter quotient to the topos \( \mathcal{K}an^\mathbb{N} \). The resulting topos \( \prod_{\Phi} \mathcal{K}an \) is not a Grothendieck \((\infty, 1)\)-topos as its subcategory of 0-truncated objects is not a Grothendieck topos [Jo03, Example D5.1.7].

We end this section by giving a detailed description of this particular example without using the language of filter quotients. Thus a reader who only wants to see an example can avoid the technical details of the previous section. We will refrain from giving detailed proofs here and refer the interested reader to the proofs in the previous section.

Let \( \mathcal{K}an \) be the Kan enriched category of Kan complexes. Then \( \mathcal{K}an^\mathbb{N} \) is the Kan enriched category with

1. objects tuples \((X_n)_{n \in \mathbb{N}}\), where \( X_n \) is a Kan complex, and
2. morphisms level-wise morphisms

\[
\text{Map}_{\mathcal{K}an^\mathbb{N}}((X_n)_{n \in \mathbb{N}}, (Y_n)_{n \in \mathbb{N}}) = \prod_{n \in \mathbb{N}} \text{Map}_{\mathcal{K}an}(X_n, Y_n)
\]

We will now define a new Kan enriched category \( \prod_{\Phi} \mathcal{K}an \). However for that we need an equivalence relation on the mapping Kan complexes \( \text{Map}_{\mathcal{K}an}(X_n, Y_n) \). We will define an equivalence relation
for each level of the Kan complex and then show that the simplicial operator maps respect the equivalence relation, thus giving us a new simplicial set.

Denote the set of natural numbers bigger than \( m \) by \( \mathbb{N}_{> m} \). We define an equivalence relation on the set \( \prod_{m \in \mathbb{N}} (\text{Map}_{\mathcal{Kan}}((X_n)_{n \in \mathbb{N}_{\geq m}}, (Y_n)_{n \in \mathbb{N}_{\leq m}}))_k \) as follows. Let

\[
(f_n)_{n \in \mathbb{N}_{\geq m_1}} \sim (g_n)_{n \in \mathbb{N}_{\geq m_2}) \iff \text{there exists } N > m_1, m_2 \text{ such that for all } n > N : (f_n = g_n)
\]

Notice if we have two maps

\[
f, g : \Delta[k] \times X_n \to Y_n
\]

for \( n > m_1 \) and \( f \sim g \) then \( f \circ \delta \sim g \circ \delta \) for any simplicial map \( \delta : \Delta[l] \to \Delta[k] \), as \( f_n \circ \delta = g_n \circ \delta \) for \( n \) large enough. Thus imposing the equivalence relation level-wise still gives us a simplicial set, and in fact a Kan complex.

We can use that to define a new category \( \prod_{\Phi} \mathcal{Kan} \) with

1. objects tuples \( (X_n)_{n \in \mathbb{N}} \) (so the same objects as \( \mathcal{Kan}^{\mathbb{N}} \)) and
2. morphisms level-wise equivalence classes of morphisms

\[
\text{Map}_{\prod_{\Phi} \mathcal{Kan}}((X_n)_{n \in \mathbb{N}}, (Y_n)_{n \in \mathbb{N}}) = \left[ \prod_{n \in \mathbb{N}} \prod_{n \in \mathbb{N}_{\geq m}} \text{Map}_{\mathcal{Kan}}(X_n, Y_n) \right] / \sim
\]

Intuitively a \( k \)-cell in \( \text{Map}_{\prod_{\Phi} \mathcal{Kan}}((X_n)_{n \in \mathbb{N}}, (Y_n)_{n \in \mathbb{N}}) \) is a class

\[
[f_n] : (X_n)_{n \in \mathbb{N}_{\geq m_1}} \times \Delta[k] \to (Y_n)_{n \in \mathbb{N}_{\geq m_2}}
\]

where two morphisms \( f_n, g_n \) are in the same class if \( f_n = g_n \) for \( n \) large enough.

We need to confirm that we actually get a category. It suffices to check that we have a composition. For two morphisms classes \( [f_n] : (X_n)_{n \in \mathbb{N}_{\geq m_1}} \to (Y_n)_{n \in \mathbb{N}_{\geq m_2}} \) and \( [g_n] : (Y_n)_{n \in \mathbb{N}_{\geq m_2}} \to (Z_n)_{n \in \mathbb{N}_{\geq m_2}} \) we define the composition as

\[
[g_n \circ f_n] : (X_n)_{n \in \mathbb{N}_{\geq \max(m_1, m_2)}} \to (Z_n)_{n \in \mathbb{N}_{\geq \max(m_1, m_2)}}
\]

and notice this definition of composition is indeed well-defined.

We now want to prove that \( \prod_{\Phi} \mathcal{Kan} \) is an elementary \((\infty,1)\)-topos. We need to show that it has finite limits and colimits, subobject classifier and complete Segal universes. We will treat each separately.

First, \( \prod_{\Phi} \mathcal{Kan} \) has a final object, namely \((\Delta[0])_{n \in \mathbb{N}} \). Now, for two morphisms \( (f_n)_{n \in \mathbb{N}_{\geq m_1}} : (X_n)_{n \in \mathbb{N}_{\geq m_1}} \to (Z_n)_{n \in \mathbb{N}_{\geq m_1}} \) and \( (g_n)_{n \in \mathbb{N}_{\geq m_2}} : (Y_n)_{n \in \mathbb{N}_{\geq m_2}} \to (Z_n)_{n \in \mathbb{N}_{\geq m_2}} \) a routine calculation shows that the pullback is just the level-wise pullback \((X_n \times_{Z_n} Y_n)_{n \in \mathbb{N}_{\geq \max(m_1, m_2)}}\). This proves that \( \prod_{\Phi} \mathcal{Kan} \) has a finite limits. The argument for finite colimits is analogous.

Next we want to find the subobject classifier. We showed in Example 1.28 that the subobject classifier in \( \mathcal{Kan} \) is \( \{0,1\} \). We will now show that the constant sequence \((\{0,1\})_{n \in \mathbb{N}}\) is the subobject classifier in \( \prod_{\Phi} \mathcal{Kan} \). First, notice that a map \((f_n)_{n \in \mathbb{N}_{\geq m}} : (X_n)_{n \in \mathbb{N}_{\geq m}} \to (Y_n)_{n \in \mathbb{N}_{\geq m}}\) is mono if and only if \( f_n \) is mono for \( n > N \) for some \( N \). But every mono \( f_n : X_n \to Y_n \) is uniquely determined by a map \( Y_n \to \{0,1\} \) as it is a subobject classifier in \( \mathcal{Kan} \). This means we get maps \( Y_n \to \{0,1\} \) for \( n > N \), which is exactly the data of a map \((Y_n)_{n \in \mathbb{N}} \to (\{0,1\})_{n \in \mathbb{N}}\). So, every mono map in \( \prod_{\Phi} \mathcal{Kan} \).
is determined by a map into \(\{0, 1\}\) and we can show this assignment is unique, proving it is a subobject classifier.

Finally, we need to show \(\prod_{\Phi} \mathcal{K}an\) has universes. Recall that the universes in \(\mathcal{K}an\) are the simplicial objects \(S^\kappa_n\), where \(\kappa\) is a cardinal (see Example 1.28 for more details). We want to show that the simplicial objects \(S^\kappa_n\) give us universes in \(\prod_{\Phi} \mathcal{K}an\). Let \((f_n)_{n \in \mathbb{N}_{\geq m}} : (X_n)_{n \in \mathbb{N}_{\geq m}} \to (Y_n)_{n \in \mathbb{N}_{\geq m}}\) be an arbitrary map. Choose a cardinal \(\kappa\), such that every morphism \(f_n\) is \(\kappa\)-small. Then every map \(f_n\) is a pullback of a map \(Y_n \to S^\kappa_0\). However, we just showed that pullbacks in the category \(\prod_{\Phi} \mathcal{K}an\) are evaluated level-wise. Thus \((f_n)_{n \in \mathbb{N}_{\geq m}}\) is the pullback of \((Y_n)_{n \in \mathbb{N}_{\geq m}} \to (S^\kappa)_{n \in \mathbb{N}_{\geq m}}\), which shows that every map is classified by a universe. We can show that this assignment gives us an equivalence.

In order to finish this example we need to show that \(\prod_{\Phi} \mathcal{K}an\) is not a Grothendieck \((\infty, 1)\)-topos. Following Corollary 1.27 it suffices to prove that its natural number object is not equivalent to the countable colimit.

The argument we give here is analogous to [Jo03, Example D5.1.7]. The natural number object in \(\prod_{\Phi} \mathcal{K}an\) is the constant sequence \((\mathbb{N})_n\). Let \(\Delta : (1)_{n \in \mathbb{N}} \to (\mathbb{N})_n\) be the map that at level \(n\) is just the map \(\{n\} : 1 \to \mathbb{N}\). We can think of this map as a “diagonal map”. Let \((P^m_n)_{n \in \mathbb{N}}\) be the following pullbacks

\[
\begin{array}{ccc}
(P^m_n)_{n \in \mathbb{N}} & \xrightarrow{\varphi^m} & (1)_{n \in \mathbb{N}} \\
\downarrow & & \downarrow \Delta \\
(1)_{n \in \mathbb{N}} & \xrightarrow{m} & (\mathbb{N})_{n \in \mathbb{N}}
\end{array}
\]

By descent if the cocone formed by the maps \(\{m\} : (1)_{n \in \mathbb{N}} \to (\mathbb{N})_n\) is a colimiting cocone, then the cocone formed by \(\{\varphi^m : (P^m_n)_{n \in \mathbb{N}} \to (1)_{n \in \mathbb{N}}\}_{m \in \mathbb{N}}\) is also a colimiting cocone. However, the maps \(\Delta, \{m\} : (1)_{n \in \mathbb{N}} \to (\mathbb{N})_n\) only coincide when \(n = m\) and disagree otherwise. Thus for \(n > m\), we have \(P^m_n = \emptyset\), which implies that \((1)_{n \in \mathbb{N}}\) is isomorphic to the colimit \(\coprod_{n \in \mathbb{N}} \emptyset \cong \emptyset\), which only happens in the topos with one element.

Notice, by the explanation above, that \(\prod_{\Phi} \mathcal{K}an\) also does not have infinite colimits. Thus we cannot use traditional methods to study its homotopy theory (such as define truncations and prove Blakers-Massey theorem). This necessitates developing elementary \((\infty, 1)\)-topos theory and proving those results in the elementary context. For such an elementary approach and a careful analysis of truncations in \(\prod_{\Phi} \mathcal{K}an\) see [Ra18c].

Remark 3.12. For an alternative argument why \(\prod_{\mathbb{N}} \mathcal{K}an\) does not have infinite limits see [BSS20, Example 3.19].

**Future Directions**

Up to here we answered the important question on whether there is an elementary \((\infty, 1)\)-topos that is not a Grothendieck \((\infty, 1)\)-topos. In this section we want to pose some new questions about elementary \((\infty, 1)\)-topos theory motivated by the response.
**Homotopy Filter Quotient:** Although the filter quotient construction enables us to construct new elementary \((\infty, 1)\)-toposes, it does have certain limitations. In particular, a filter on \(E\) is completely determined by a filter on the underlying elementary topos \(\tau_0 E\) as we are just using \(\text{Sub}(1)\), which is a subcategory of both categories. This is particularly problematic when we have interesting categorical data as we will illustrate in the following example.

**Example 4.1.** Let \(\mathcal{K}an\) be the Kan enriched category of Kan complexes and let \(K\) be a connected Kan complex. Then the slice category \(\mathcal{K}an/_{K}\) is also an elementary \((\infty, 1)\)-topos with final object \(id_K : K \to K\), which has only two subobjects, namely \(\emptyset \to K\) and \(id_K : K \to K\). Thus \(\mathcal{K}an/_{K}\) only has trivial filter quotients (see Example 2.26).

Ideally we would like to have a notion of filter quotient which allows us to take the higher homotopical data of \(K\) into account and use that to form a quotient, which should lead us to a notion of homotopy filter quotient.

**Filter Quotients of Sheaves:** We give explicit examples of filter quotients for the simplest of all Grothendieck \((\infty, 1)\)-toposes, namely \(\mathcal{K}an^S\), where \(S\) is a set (Example 3.11). However, given that the filter construction works for all Grothendieck \((\infty, 1)\)-toposes the next step is to construct the filter quotient on the category of sheaves.

**Non-standard Models of Spaces:** One long standing goal of elementary \((\infty, 1)\)-topos theory is to study models for spaces (similar to how elementary toposes are used to study models of set theory) and the filter quotient might be a step in developing such models. Concretely, let \(S\) be an infinite set and \(\Phi\) a non-principal ultrafilter (Definition 1.32) on the power set \(P(S)\). Then we can construct the ultraproduct \(\prod_{\Phi} \mathcal{K}an\). The underlying elementary topos is the ultraproduct \(\prod_{\Phi} \text{Set}\), which has many interesting properties [Jo03, A2.2]:

1. It is Boolean
2. It is generated by the final object
3. It doesn’t have infinite colimits

Thus \(\prod_{\Phi} \text{Set}\) shares many properties with the category of sets. This suggests that \(\prod_{\Phi} \mathcal{K}an\) should behave similar to the \((\infty, 1)\)-category of Kan complexes. An important first step would be to prove that \(\prod_{\Phi} \mathcal{K}an\) is generated by the final object.

**Models of Homotopy Type Theory:** One important goal of elementary \((\infty, 1)\)-topos theory is to construct models of homotopy type theory. We already know that every Grothendieck \((\infty, 1)\)-topos is a model of homotopy type theory [Sh19]. However, as of now there are no other known models. Given that we have an explicit construction of the filter quotient the hope is that we can show the filter quotient of a Grothendieck \((\infty, 1)\)-topos is also a model of homotopy type theory.

References

[AJ82] M. Adelman, P. T. Johnstone. *Serre classes for toposes*. Bulletin of the Australian Mathematical Society 25.1 (1982): 103-115.

[Be07] J.E. Bergner, *A model category structure on the category of simplicial categories*. Transactions of the American Mathematical Society 359.5 (2007): 2043-2058.

[Be10] J.E. Bergner, *A survey of \((\infty, 1)\)-categories*, The IMA Volumes in Mathematics and its Applications. Springer, 2010. pp. 69-83
Chromatic homotopy theory is asymptotically algebraic.

Inventiones mathematicae (2020): 1-109.

C. Chang, J. Keisler. Model theory. Amsterdam: North-Holland, 1990.

P. Johnstone, Sketches of an elephant: A topos theory compendium-2 volume set. Oxford University Press, Jul 2003. ISBN-10: ISBN-13: 9780198524960 (2003): 1288.

A. Joyal, M. Tierney, Quasi-categories vs Segal spaces, Contemp. Math 431 (2007): pp. 277-326

G. L. Monaco. Dependent products and 1-inaccessible universes. arXiv preprint arXiv:1905.10220 (2019).

J. Lurie, Higher Topos Theory, Annals of Mathematics Studies 170, Princeton University Press, Princeton, N.J, 2009, xviii+925 pp. A

S. Mac Lane, I. Moerdijk. Sheaves in Geometry and Logic, Springer-Verlag, New York (1992).

N. Rasekh. Yoneda lemma for simplicial spaces. arXiv preprint arXiv:1711.03160 (2017).

N. Rasekh, A Theory of Elementary Higher Toposes. arXiv preprint arXiv:1805.03805 (2018)

N. Rasekh, Every Elementary Higher Topos has a Natural Number Object. arXiv preprint arXiv:1809.01734 (2018)

N. Rasekh, An Elementary Approach to Truncations. arXiv preprint arXiv:1812.10527 (2018).

C. Rezk. A model for the homotopy theory of homotopy theory, Trans. Amer. Math.Soc., 353(2001), no. 3, 973-1007.

C. Rezk Toposes and Homotopy Toposes https://faculty.math.illinois.edu/~rezk/homotopy-topos-sketch.pdf

C. Rezk, A Cartesian presentation of weak n-categories. Geometry & Topology 14.1 (2010): 521-571.

M. Shulman, All (∞, 1)-toposes have strict univalent universes. arXiv preprint arXiv:1904.07004 (2019).

École Polytechnique Fédérale de Lausanne, SV BMI UPHESS, Station 8, CH-1015 Lausanne, Switzerland

E-mail address: nima.rasekh@epfl.ch