Discrete Affine Surfaces based on Quadrangular Meshes

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Abstract. In this paper we are interested in defining affine structures on discrete quadrangular surfaces of the affine three-space. We introduce, in a constructive way, two classes of such surfaces, called respectively indefinite and definite surfaces. The underlying meshes for indefinite surfaces are asymptotic nets satisfying a non-degeneracy condition, while the underlying meshes for definite surfaces are non-degenerate conjugate nets satisfying a certain natural condition. In both cases we associate to any of these nets several discrete affine invariant quantities: a metric, a normal and a co-normal vector fields, and a mean curvature. Moreover, we derive structural and compatibility equations which are shown to be necessary and sufficient conditions for the existence of a discrete quadrangular surface with a given affine structure.

Keywords: Discrete differential geometry, Discrete affine surfaces, Asymptotic nets, Discrete conjugate nets, Quad meshes.

1 Introduction

Discrete differential geometry has attracted much attention recently, mainly due to the growth of computer graphics. One of the main issues in discrete differential geometry is to define suitable discrete analogous of the concepts of smooth differential geometry. As far as surfaces are concerned, it has been observed that interesting classes of discrete surfaces are not merely discrete equivalent of smooth \textit{surfaces}, i.e. two-dimensional differential sets of points, but rather analogous to some \textit{parameterizations} of (smooth) surfaces, i.e. surfaces coming with a given coordinates system, such as asymptotic or isothermal coordinates.

Our purpose here is to continue the study toward a hopefully consistent theory of discrete surfaces in the affine 3-space $\mathbb{R}^3$. Let us mention some work that have been done on the subject: in \cite{2} a definition of discrete affine spheres is proposed, and the case of improper affine spheres were considered in \cite{5} and \cite{6}. In \cite{4}, we gave a constructive definition of discrete affine minimal surfaces with indefinite Berwald-Blaschke metric. In all these works, surfaces with indefinite metric were modeled by asymptotic net while surfaces with definite metric were modeled by discrete conjugate nets. Asymptotic, respectively discrete conjugate, nets are characterized by the fact that their crosses, respectively faces, are planar, and may be viewed as the discrete equivalents of parameterizations in asymptotic, respectively isothermal, coordinates. In this paper we introduce two general classes of discrete affine surfaces, inspired by the theory of smooth affine surfaces, and we try to get as many properties or structures analogous to the smooth case as possible. It turns out that our two classes include as particular cases the constructions of \cite{5,4,6} and \cite{2}.

In the indefinite case, we construct an asymptotic net satisfying a natural assumption of non-degeneracy which implies in particular that all the vertices of the surface are saddles. On such a discrete surface, we define the so-called \textit{co-normal vector field} that satisfies, as in the smooth case, Lelievre’s equations. In the definite case, we consider discrete conjugate nets satisfying a non-degeneracy condition insuring that the surface is locally convex at the vertices and, in addition, a simple equation which turns out to be equivalent to the existence of a co-normal vector field defined on the planar faces, orthogonal to them, and again satisfying Lelievre’s equations. In both cases, starting from the co-normal vector field, it is possible to obtain, by numerical integration, explicit examples of immersions.

Moreover, we are able to define several affine invariant structures on the surfaces we have defined, such as the affine metric and the normal field, and to get structural equations relating them. These equations feature several dimensionless parameters, called \textit{discrete parameters}, that are close to 1 when the discrete surface is close to a smooth surface. Simpler versions of these equations are found in \cite{2} and \cite{4}. We also discuss here how to define a discrete affine mean curvature $H$. 

Structural equations describe the discrete immersion in terms of the affine metric and discrete parameters, and they must satisfy compatibility equations that generalize the ones described in [2]. When the affine metric and discrete parameters satisfy the compatibility equations, one can define an asymptotic net, unique up to equi-affine transformations of $\mathbb{R}^3$, that satisfies the structural equations.

We conclude by emphasizing the similarity between the set of discrete equations we introduce and their smooth equivalent. The simplicity of these equations is somewhat surprising, since the construction is very general and we compute differences up to third order. We remark that the indefinite and definite constructions are independent, but it is quite interesting to compare both formalisms.

The paper is organized as follows: in Section 2 we review the basic equations of smooth surfaces in affine geometry, treating firstly the indefinite case with asymptotic parameters, and secondly the definite case with isothermal parameters. Section 3 is devoted to the description of discrete surfaces with indefinite metric as asymptotic nets. The affine metric, the normal and co-normal vector fields and the discrete parameter are introduced, and some examples of asymptotic nets are provided. Then we discuss a definition of discrete affine mean curvature, derive the compatibility equations and show that they are satisfied up to third order. We remark that the indefinite and definite constructions are independent, but it is quite interesting to compare both formalisms.

Notation. Given two vectors $V_1, V_2 \in \mathbb{R}^3$, we denote by $V_1 \times V_2$ the cross product and by $V_1 \cdot V_2$ the dot product between them. Given three vectors $V_1, V_2, V_3 \in \mathbb{R}^3$, we denote by $[V_1, V_2, V_3] = (V_1 \times V_2) \cdot V_3$ their determinant. For a discrete real or vector function $f$ defined on a domain $D \subset \mathbb{Z}^2$, we denote the discrete partial derivatives with respect to $u$ or $v$ by

$$f_1(u, v) = f(u + 1, v) - f(u, v)$$
$$f_2(u, v) = f(u, v + 1) - f(u, v).$$

The second order partial derivatives are defined by

$$f_{11}(u, v) = f(u + 1, v) - 2f(u, v) + f(u - 1, v)$$
$$f_{22}(u, v) = f(u, v + 1) - 2f(u, v) + f(u, v - 1)$$
$$f_{12}(u, v) = f(u + 1, v + 1) + f(u, v) - f(u + 1, v) - f(u, v + 1).$$

Acknowledgements. The first and third authors want to thank CNPq for financial support during the preparation of this paper.

2 Review of equations for smooth affine surfaces

In this section we review the concepts and equations of smooth affine surfaces. Although we shall not use these equations explicitly, they are important for comparing with the concepts and equations for discrete surfaces defined in this paper. For details and proofs of this section, see [3] and [7].

Consider a parameterized smooth surface $q : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, where $U$ is an open subset of the plane and denote

$$L(u, v) = [q_u, q_v, q_{uu}],$$
$$M(u, v) = [q_u, q_v, q_{uv}],$$
$$N(u, v) = [q_u, q_v, q_{vv}].$$

Here and in the remainder of this section, the partial derivatives are denoted by subscripts and $[A, B, C]$ denotes the determinant of the $3 \times 3$ matrix whose columns are $A, B$ and $C$. The surface is non-degenerate if $LN - M^2 \neq 0$, and, in this case, the Berwald-Blaschke metric is defined by

$$ds^2 = \frac{1}{|LN - M^2|^{1/4}} \left(L du^2 + 2M dudv + N dv^2\right).$$

If $LN - M^2 > 0$, the metric is definite while if $LN - M^2 < 0$, the metric is indefinite. The Berwald-Blaschke metric is conformal to the second fundamental form. In the definite case, the surface is locally convex, while in the indefinite case, the surface is locally hyperbolic, i.e., the tangent plane crosses the surface.
2.1 Surfaces with indefinite metric and asymptotic parameters

Assume that the affine surface has indefinite metric. We can make a change of coordinates such that $L = N = 0$. Such coordinates are called asymptotic. We may assume that $M > 0$, and the metric takes the form $ds^2 = 2\Omega du dv$, where $\Omega^2 = M$.

The vector field $\nu = \frac{\nu_u \times \nu_v}{\Omega}$ is called the co-normal vector field. It satisfies Lelieuvre’s equations

$$
\nu_u = \nu \times \nu_u,
$$
$$
\nu_v = -\nu \times \nu_v.
$$

The co-normal vector field satisfies $\nu_{uv} = H \Omega \nu$, where $H$ is a scalar function called the affine mean curvature. The affine normal vector field is defined as $\xi = \frac{\nu_u \times \nu_v}{\Omega}$ and satisfies $\nu \cdot \xi = 1$. It is easy to verify that

$$
\nu_u = \xi \times \nu_u,
$$
$$
\nu_v = -\xi \times \nu_v.
$$

We can write Gauss equations as

$$
q_{uu} = \frac{\Omega}{\Omega^2} q_u + A \Omega q_v,
$$
$$
q_{vv} = \frac{\Omega}{\Omega^2} q_u + B \Omega q_v,
$$
$$
q_{uv} = \Omega \xi,
$$

where

$$
A = [q_u, q_{uu}, \xi],
$$
$$
B = [q_v, q_{vv}, \xi].
$$

are the coefficients of the affine cubic form. The shape operator in basis $\{q_u, q_v\}$ is given by

$$
\xi_u = -H q_u + A \Omega q_v,
$$
$$
\xi_v = B q_u - H q_v.
$$

We remark that the trace of the shape operator is given by $-2H$.

The parameters $\Omega, A, B$ and $H$ must satisfy some compatibility equations: If we compare mixed third derivatives of $q$, we obtain

$$
H = \frac{\Omega u \Omega u - \Omega u v - AB}{\Omega^3}
$$

And if we compare mixed second derivatives of $\xi$, we obtain two other compatibility equations,

$$
H_u = \frac{AB u}{\Omega^3} - \frac{A}{\Omega} \frac{A_v}{\Omega} v
$$
$$
H_v = \frac{BA v}{\Omega^3} - \frac{B}{\Omega} \frac{B_u}{\Omega} u.
$$

Conversely, given $\Omega, A, B$ and $H$ satisfying the compatibility equations, there exists a surface $q(u, v)$ satisfying the above equations.

2.2 Surfaces with definite metric and isothermal parameters

Assume that the affine surface has definite metric. We can make a change of coordinates such that $L - N = M = 0$. Such coordinates are called isothermal. Moreover, we may assume that $L = N > 0$, and we define $\Omega$ by $\Omega^2 = L = N$. In this case, the metric takes the form $ds^2 = \Omega du^2 + dv^2$. 
As above, the vector field \( \nu = \frac{q_u \times q_v}{\Omega} \) is called the co-normal vector field and satisfies Lelieuvre’s equations

\[
q_u = \nu \times \nu_u, \\
q_v = -\nu \times \nu_v.
\]

The co-normal vector field satisfies \( \nu_{uu} + \nu_{vv} = H\Omega\nu \), where \( H \) is a scalar function called the affine mean curvature. The affine normal vector field is defined as \( \xi = \frac{q_{uu} + q_{vv}}{2\Omega} \) and satisfies \( \nu \cdot \xi = 1 \). It is easy to verify that

\[
\nu_u = \xi \times q_v, \\
\nu_v = -\xi \times q_u.
\]

We can write Gauss equations as

\[
q_{uv} = \left( \frac{\Omega_u + B}{2\Omega} \right) q_u + \left( \frac{\Omega_v + A}{2\Omega} \right) q_v \\
q_{uu} = \left( \frac{\Omega_u - A}{2\Omega} \right) q_u + \left( -\frac{\Omega_v + B}{2\Omega} \right) q_v + \Omega \xi \\
q_{vv} = -\left( \frac{\Omega_u - A}{2\Omega} \right) q_u - \left( -\frac{\Omega_v + B}{2\Omega} \right) q_v + \Omega \xi,
\]

where

\[
2A = \langle q_u, q_{uw}, \xi \rangle + \langle q_v, q_{uw}, \xi \rangle, \\
2B = \langle q_u, q_{uw}, \xi \rangle - \langle q_v, q_{uw}, \xi \rangle.
\]

are coefficients of the cubic form. One can also verify that the trace of the shape operator is \( H \).

As in the indefinite case, the parameters \( \Omega, A, B \) and \( H \) must satisfy some compatibility equations: If we compare mixed third derivatives of \( q \), we obtain

\[
H\Omega = \left( \frac{\Omega_u}{\Omega} \right)_u + \left( \frac{\Omega_v}{\Omega} \right)_v + \left( \frac{2B}{\Omega} \right)^2 + \left( \frac{2A}{\Omega} \right)^2.
\]

And if we compare mixed second derivatives of \( \xi \), we obtain two other compatibility equations. Conversely, given \( \Omega, A, B \) and \( H \) satisfying the compatibility equations, there exists a surface \( q(u, v) \) satisfying the above equations.

3 Discrete affine surfaces with indefinite metric

3.1 Discrete affine concepts and examples

In this section, we define the affine metric, normal and co-normal vector fields associated with a non-degenerate asymptotic net. We also provide some concrete examples.

3.1.1 Affine metric, co-normal and normal vector fields

Non-degenerate asymptotic nets. An asymptotic net is defined to be a \( \mathbb{R}^3 \)-valued function defined on a subset \( D \) of \( \mathbb{Z}^2 \) such that “crosses are planar”, i.e., \( q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v), q_2(u, v + \frac{1}{2}) \) and \( q_2(u, v - \frac{1}{2}) \) are co-planar (see [1] Definition 2.15). For each quadrangle, let

\[
M(u + \frac{1}{2}, v + \frac{1}{2}) = \left[ q_1(u + \frac{1}{2}, v), q_2(u, v + \frac{1}{2}), q_2(u + 1, v + \frac{1}{2}) \right].
\]

We say that the asymptotic net is non-degenerate if \( M \) does not change sign. We shall assume throughout the paper that \( M(u + \frac{1}{2}, v + \frac{1}{2}) \) is non-degenerate for any \( (u, v) \in D \).

We remark that this condition implies that all vertices of the asymptotic net are saddle points, i.e., the plane spanned by \( q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v), q_2(u, v + \frac{1}{2}) \) and \( q_2(u, v - \frac{1}{2}) \) crosses the surface.
Affine metric. The affine metric $\Omega$ at a quadrangle $(u + \frac{1}{2}, v + \frac{1}{2})$ is defined as

$$\Omega(u + \frac{1}{2}, v + \frac{1}{2}) = \sqrt{M(u + \frac{1}{2}, v + \frac{1}{2})}.$$

Co-normal vector field and the discrete parameter. A co-normal vector field $\nu$ with respect to an asymptotic net $q$ is a vector-valued map defined at the vertices $(u, v)$, orthogonal to the planar crosses and satisfying the discrete Leibniz's equations

$$\nu(u, v) \times \nu(u + 1, v) = q_1(u + \frac{1}{2}, v),$$
$$\nu(u, v) \times \nu(u, v + 1) = -q_2(u, v + \frac{1}{2}).$$

The existence of the co-normal vector-field is guaranteed by the following proposition:

**Proposition 1** A non-degenerate asymptotic net $q$ admits a one-parameter family of co-normal fields. Any such co-normal field is determined by a positive map $\lambda(u + \frac{1}{2}, v + \frac{1}{2})$ defined on the faces of the net, which satisfies the following coincidence relations:

$$\nu(u, v) = \frac{\lambda^{-1}(u + \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})}(q_1(u + \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2})), $$
$$\nu(u, v) = \frac{\lambda(u - \frac{1}{2}, v + \frac{1}{2})}{\Omega(u - \frac{1}{2}, v + \frac{1}{2})}(q_1(u - \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2})), $$
$$\nu(u, v) = \frac{\lambda^{-1}(u - \frac{1}{2}, v - \frac{1}{2})}{\Omega(u - \frac{1}{2}, v - \frac{1}{2})}(q_1(u - \frac{1}{2}, v) \times q_2(u, v - \frac{1}{2})), $$
$$\nu(u, v) = \frac{\lambda(u + \frac{1}{2}, v - \frac{1}{2})}{\Omega(u + \frac{1}{2}, v - \frac{1}{2})}(q_1(u + \frac{1}{2}, v) \times q_2(u, v - \frac{1}{2})).$$

Finally, we can get the other co-normal fields of the family by

$$\nu_\rho(u, v) = \begin{cases} 
\rho \nu(u, v) & \text{if } u + v \text{ is even}, \\
\rho^{-1} \nu(u, v) & \text{if } u + v \text{ is odd}, 
\end{cases}$$

where $\rho$ is any positive constant.

**Proof.** We begin by fixing a positive initial value $\lambda(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})$ at some face of the net and define $\nu(u_0, v_0)$, $\nu(u_0 + 1, v_0)$, $\nu(u_0, v_0 + 1)$ and $\nu(u_0 + 1, v_0 + 1)$ by

$$\nu(u_0, v_0) = \frac{\lambda^{-1}(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}{\Omega(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}(q_1(u_0 + \frac{1}{2}, v_0) \times q_2(u_0, v_0 + \frac{1}{2})), $$
$$\nu(u_0 + 1, v_0) = \frac{\lambda(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}{\Omega(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}(q_1(u_0 + \frac{1}{2}, v_0) \times q_2(u_0, v_0 + \frac{1}{2})), $$
$$\nu(u_0 + 1, v_0 + 1) = \frac{\lambda^{-1}(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}{\Omega(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}(q_1(u_0 + \frac{1}{2}, v_0) \times q_2(u_0, v_0 + \frac{1}{2})), $$
$$\nu(u_0, v_0 + 1) = \frac{\lambda(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}{\Omega(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}(q_1(u_0 + \frac{1}{2}, v_0 + 1) \times q_2(u_0, v_0 + \frac{1}{2})).$$

We observe that

$$\nu(u_0, v_0) \times \nu(u_0 + 1, v_0) = \frac{(q_1(u_0 + \frac{1}{2}, v_0) \times q_2(u_0, v_0 + \frac{1}{2})) \times (q_1(u_0 + \frac{1}{2}, v_0) \times q_2(u_0 + 1, v_0 + \frac{1}{2}))}{\Omega^2(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})}$$
$$= q_1(u_0 + \frac{1}{2}, v_0),$$
so that Lelieuvre’s equation holds at \((u_0 + \frac{1}{2}, v_0)\). Analogous computations show that Lelieuvre’s equation holds as well at \((u_0, v_0 + \frac{1}{2})\), \((u_0 + \frac{1}{2}, v_0)\) and \((u_0 + 1, v_0 + \frac{1}{2})\).

Next, as the crosses are planar and using the coincidences relations, we can uniquely determine the values of \(\lambda(u_0 - \frac{1}{2}, v_0 + \frac{1}{2})\), \(\lambda(u_0 + \frac{1}{2}, v_0 - \frac{1}{2})\) and \(\lambda(u_0 - \frac{1}{2}, v_0 - 1)\) which in turn allows to define \(\nu\) at the five adjacent vertices \((u_0 - 1, v_0)\), \((u_0 - 1, v_0 - 1)\), \((u_0, v_0 - 1)\) and \((u_0 + 1, v_0 - 1)\).

By repeating this procedure, we can define \(\lambda\) in all the domain \(D\) and the coincidences relations prove that the scalar \(\lambda\) is uniquely defined at any face \((u + \frac{1}{2}, v + \frac{1}{2})\), independently from the path linking it to the initial face \((u_0 + \frac{1}{2}, v_0 + \frac{1}{2})\). Moreover at each step we can check that Lelieuvre’s equations are satisfied. Finally, if we multiply \(\lambda(u_0 + \frac{1}{2}, v_0 + \frac{1}{2})\) by a positive constant \(\rho\), the construction above gives a vector field \(\nu_\rho\) that satisfies the claimed relation with \(\nu\).

To construct examples of asymptotic nets, it is easier to begin with the co-normal vector field. It turns out that a given vector field is the co-normal vector field of an asymptotic net if and only if it is a Moutard net (see [1, Definition 2.12,Theorem 2.16]):

**Proposition 2** A vector field \(\nu(u, v)\) is the co-normal vector field of an asymptotic net if and only if it satisfies the equation

\[
H^*(u + \frac{1}{2}, v + \frac{1}{2})(\nu(u, v) + \nu(u + 1, v + 1)) = \nu(u, v + 1) + \nu(u + 1, v).
\]

for some map \(H^*(u + \frac{1}{2}, v + \frac{1}{2})\). One can also verify that necessarily \(H^*(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda^2(u + \frac{1}{2}, v + \frac{1}{2})\).

**Proof.** From Lelieuvre’s equations, we have

\[
q_1(u + \frac{1}{2}, v) + q_2(u + 1, v + \frac{1}{2}) = (\nu(u, v) + \nu(u + 1, v + 1)) \times \nu(u + 1, v)
\]

and

\[
q_1(u + \frac{1}{2}, v + 1) + q_2(u, v + \frac{1}{2}) = \nu(u, v + 1) \times (\nu(u, v) + \nu(u + 1, v + 1)).
\]

Thus

\[
q_1(u + \frac{1}{2}, v) + q_2(u + 1, v + \frac{1}{2}) - q_1(u + \frac{1}{2}, v + 1) - q_2(u, v + \frac{1}{2}) = (\nu(u, v) + \nu(u + 1, v + 1)) \times (\nu(u + 1, v) + \nu(u, v + 1))
\]

and the proposition is proved.

It is interesting to observe that, in terms of the co-normals, the affine metric is given as

\[
\Omega(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda^{-1}(u + \frac{1}{2}, v + \frac{1}{2}) [\nu(u, v), \nu(u, v + 1), \nu(u + 1, v)].
\]

The parameter \(\lambda\) will be called the discrete parameter.

**The normal vector field.** The affine normal vector field \(\xi\) is defined at each quadrangle \((u + \frac{1}{2}, v + \frac{1}{2})\) by

\[
\xi(u + \frac{1}{2}, v + \frac{1}{2}) = \frac{q_12(u + \frac{1}{2}, v + \frac{1}{2})}{\Omega(u + \frac{1}{2}, v + \frac{1}{2})}.
\]

It satisfies the following equations:

\[
\nu(u, v) \cdot \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda^{-1}(u + \frac{1}{2}, v + \frac{1}{2}),
\]

\[
\nu(u + 1, v) \cdot \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda(u + \frac{1}{2}, v + \frac{1}{2}),
\]

\[
\nu(u, v + 1) \cdot \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda(u + \frac{1}{2}, v + \frac{1}{2}),
\]

\[
\nu(u + 1, v + 1) \cdot \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda^{-1}(u + \frac{1}{2}, v + \frac{1}{2}).
\]

In Figure [1] we see an illustration of the co-normal and normal vector fields.
3.1.2 Examples

Example 3 Examples of asymptotic nets with $\lambda = 1$ can be obtained starting with a smooth 3-dimensional vector field $\nu(u, v)$ satisfying $\nu_{uv} = 0$. We consider a sampling $\nu(i\Delta u, j\Delta v)$ and the discrete immersion is obtained by integrating the discrete Lelieuvre’s equations. This nets were studied in [4], where they were called discrete affine minimal surfaces.

In figure 2 one can see such a surface obtained from $\nu(u, v) = (\frac{v^2 - u^2}{4}, \frac{u - v}{2}, -1)$, $2 \leq u \leq 4$, $0 \leq v \leq 2$, $\Delta u = 1/10$, $\Delta v = 2/21$.

Example 4 We begin with an asymptotic parameterization of the one-sheet smooth hyperboloid $y^2 + z^2 - x^2 = c^2$,

$$q(u, v) = \frac{c}{\sinh(u + v)}(- \cosh(u - v), - \sinh(u - v), \cosh(u + v)).$$

Taking derivatives

$$q_u = \frac{c}{\sinh^2(u + v)}(\cosh(2v), - \sinh(2v), -1),$$

$$q_v = \frac{c}{\sinh^2(u + v)}(\cosh(2u), \sinh(2u), -1).$$

Straightforward calculations show that

$$\Omega = \frac{2c^{3/2}}{\sinh^2(u + v)},$$

$$\nu(u, v) = c^{1/2}\frac{1}{\sinh(u + v)}(\cosh(u - v), \sinh(v - u), \cosh(u + v))$$

and

$$\xi(u, v) = c^{-1/2}\frac{1}{\sinh(u + v)}(- \cosh(u - v), \sinh(v - u), \cosh(u + v)).$$
One can also verify that \( A(u, v) = B(u, v) = 0 \) and that \( H = e^{-3/2} \) is the affine mean curvature. Since the asymptotic curves of the smooth hyperboloid are straight lines, sampling it in the domain of asymptotic parameters generate an asymptotic net. Denote by \( \Delta u \) and \( \Delta v \) the distance between samples in \( u \) and \( v \) directions, respectively.

Integrating \( q_u \) and \( q_v \) one can show that

\[
q_1(u + \frac{\Delta u}{2}, v) = \frac{c \sinh(\Delta u)}{\sinh(u + v) \sinh(u + v + \Delta u)} (\cosh(2v) - \sinh(2v), -1),
\]

\[
q_2(u, v + \frac{\Delta v}{2}) = \frac{c \sinh(\Delta v)}{\sinh(u + v) \sinh(u + v + \Delta v)} (\cosh(2u), \sinh(2u), -1).
\]

Denoting by \( \nu(u, v) \) the co-normal vector of the smooth hyperboloid, one can verify that

\[
\nu(u, v) \times \nu(u + \Delta u, v) = q_1(u + \frac{\Delta u}{2}, v),
\]

\[
\nu(u, v) \times \nu(u + \Delta v, v) = q_2(u, v + \frac{\Delta v}{2}).
\]

Thus, since Lelievre’s formulas hold, we can consider \( \nu(u, v) \) as the co-normal of the discrete surface as well. Straightforward calculations show that

\[
\Omega(u + \frac{\Delta u}{2}, v + \frac{\Delta v}{2}) = \frac{2c^{3/2} \sinh(\Delta u) \sinh(\Delta v)}{\sqrt{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u) \sinh(u + v + \Delta v) \sinh(u + v)}}
\]

and

\[
\lambda(u + \frac{\Delta u}{2}, v + \frac{\Delta v}{2}) = \sqrt{\frac{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v)}{\sinh(u + v + \Delta u) \sinh(u + v + \Delta v)}}.
\]

The affine normal \( \xi(u + \frac{\Delta u}{2}, v + \frac{\Delta v}{2}) = (\xi_x, \xi_y, \xi_z) \) is given by

\[
2c^{1/2} \xi_x = \frac{-\cosh(\Delta u) \sinh(2v + \Delta v) - \cosh(\Delta v) \sinh(2u + \Delta u)}{\sqrt{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u) \sinh(u + v + \Delta v) \sinh(u + v)}}.
\]

\[
2c^{1/2} \xi_y = \frac{\cosh(\Delta u) \cosh(2v + \Delta v) - \cosh(\Delta v) \cosh(2u + \Delta u)}{\sqrt{\sinh(u + v + \Delta u) \sinh(u + v + \Delta u) \sinh(u + v + \Delta v) \sinh(u + v + \Delta v)}}.
\]

\[
2c^{1/2} \xi_z = \frac{\sinh(2u + 2v + \Delta u + \Delta v)}{\sqrt{\sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u + \Delta v) \sinh(u + v + \Delta u) \sinh(u + v + \Delta v)}}.
\]

In figure 3, we see the discrete hyperboloid with \( 1 \leq u \leq 3, 1 \leq v \leq 3, \Delta u = 2/50 \) and \( \Delta v = 2/51 \).

Figure 3: Discrete hyperboloid of example 4

Example 5 We first define \( \nu(u, v) \) for two consecutive values of \( c = \frac{u}{\Delta(u)} + \frac{v}{\Delta(v)} \) and then extend it to other values of \( c \) by (1). More explicitly, we define \( \nu(u, v) \), for \( c = 0 \) and \( c = 1 \), \( u_0 \leq u \leq u_1, -u_1 \leq v \leq u_0 \), and then extend to the triangle \( u_0 \leq u \leq u_1, -u_1 \leq v \leq u_0 \) using (1).

In figure 3, we see the surface obtained from \( \nu(u, v) = (u, v, u^2 + v^2) \), \( \Delta u = 0.14, \Delta v = 0.12, 0.3 \leq v \leq 1, u_0 = 0.3, \lambda^2(u, v) = 0.9, \) if \( u + v \) is even and \( \lambda^2(u, v) = 1.2, \) if \( u + v \) is odd.

3.2 Structural equations, mean curvature and compatibility

3.2.1 Discrete Gauss Equations I

Define

\[
A(u, v) = \begin{bmatrix} q_1(u - \frac{1}{2}, v), q_1(u + \frac{1}{2}, v), \lambda(u + \frac{1}{2}, v + \frac{1}{2}) \xi(u + \frac{1}{2}, v + \frac{1}{2}) \\ q_1(u + \frac{1}{2}, v), q_1(u + \frac{1}{2}, v), \lambda^{-1}(u, v) \xi(u + \frac{1}{2}, v - \frac{1}{2}) \\ q_1(u + \frac{1}{2}, v), q_1(u + \frac{1}{2}, v), \lambda^{-1}(u + \frac{1}{2}, v + \frac{1}{2}) \xi(u - \frac{1}{2}, v + \frac{1}{2}) \\ q_1(u - \frac{1}{2}, v), q_1(u - \frac{1}{2}, v), \lambda(u - \frac{1}{2}, v - \frac{1}{2}) \xi(u - \frac{1}{2}, v - \frac{1}{2}) \end{bmatrix}
\]
The functions $A(u,v)$ and $B(u,v)$ can be considered as the discrete equivalent of the coefficients of the smooth cubic form, but we shall not discuss this interpretation in the present paper.

**Notation.** In order to get more concise equations, from now on we shall use the following notation: $\lambda(u - \frac{1}{2}, v - \frac{1}{2}) = \lambda_0$, $\lambda(u + \frac{1}{2}, v - \frac{1}{2}) = \lambda_1$, $\lambda(u - \frac{1}{2}, v + \frac{1}{2}) = \lambda_2$ and $\lambda(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda_{12}$, and a similar notation for $\Omega$. For $A$, we use $A(u,v) = A_0$, $A(u+1,v) = A_1$, $A(u,v+1) = A_2$, $A(u-1,v) = A_T$ and $A(u,v-1) = A_{\overline{T}}$, with a similar notation for $B$.

**Proposition 6** The Gauss equations of the affine immersion are given by

\[
q_{11}(u,v) = \frac{\Omega_{12} - \lambda_{12}^{-1}\lambda_0^{-1}\Omega_2}{\Omega_{12}} q_1(u + \frac{1}{2}, v) + \frac{\lambda_1^{-1} A_0}{\Omega_{12}} q_2(u, v + \frac{1}{2}),
\]

\[
q_{22}(u,v) = \frac{\lambda_1^{-1} B_0}{\Omega_{12}} q_1(u + \frac{1}{2}, v) + \frac{\Omega_{12} - \lambda_{12}^{-1}\lambda_0^{-1}\Omega_1}{\Omega_{12}} q_2(u, v + \frac{1}{2}).
\]

We can also write

\[
q_{11}(u,v) = \frac{\Omega_1 - \lambda_0 \lambda_1 \Omega_0}{\Omega_1} q_1(u + \frac{1}{2}, v) + \frac{\lambda_1 A_0}{\Omega_1} q_2(u, v - \frac{1}{2})
\]

\[
= \frac{\lambda_2 \lambda_1 \Omega_2 - \Omega_2}{\Omega_2} q_1(u - \frac{1}{2}, v) + \frac{\lambda_2 A_0}{\Omega_2} q_2(u, v + \frac{1}{2})
\]

\[
= \frac{\lambda_0^{-1} \lambda_1^{-1} \lambda_2^{-1} \Omega_1 - \Omega_0}{\Omega_0} q_1(u - \frac{1}{2}, v) + \frac{\lambda_0^{-1} A_0}{\Omega_0} q_2(u, v - \frac{1}{2})
\]

and

\[
q_{22}(u,v) = \frac{\lambda_1 B_0}{\Omega_1} q_1(u + \frac{1}{2}, v) + \frac{\lambda_0 \lambda_2 \Omega_2 - \Omega_0}{\Omega_1} q_2(u, v - \frac{1}{2})
\]

\[
= \frac{\lambda_2 B_0}{\Omega_2} q_1(u - \frac{1}{2}, v) + \frac{\Omega_2 - \lambda_1 \lambda_2 \Omega_1}{\Omega_2} q_2(u, v + \frac{1}{2})
\]

\[
= \frac{\lambda_0^{-1} B_0}{\Omega_0} q_1(u - \frac{1}{2}, v) + \frac{\lambda_0^{-1} \lambda_2^{-1} \Omega_2 - \Omega_0}{\Omega_0} q_2(u, v - \frac{1}{2}).
\]

**Proof.** We prove the first formula, the others being similar. Since crosses are planar, $q_{11}(u,v)$ is a linear combination of $q_1(u + \frac{1}{2}, v)$ and $q_2(u, v + \frac{1}{2})$. The coefficient of $q_2(u, v + \frac{1}{2})$ is

\[
\frac{[q_1(u - \frac{1}{2}, v), q_1(u + \frac{1}{2}, v), \xi(u + \frac{1}{2}, v + \frac{1}{2})]}{\Omega_{12}} = \frac{\lambda_{12}^{-1} A_0}{\Omega_{12}},
\]
while the coefficient \( a \) of \( q_2(u, v + \frac{1}{2}) \) satisfies

\[
(q_1(u + \frac{1}{2}, v) - q_1(u - \frac{1}{2}, v)) \times q_2(u, v + \frac{1}{2}) = a \lambda_2 \Omega_1 \nu(u, v).
\]

Thus \( a = \frac{\Omega_2 - \lambda_2^{-1} \lambda_2^{-1} \lambda_2 \Omega_1}{\Omega_1} \).

### 3.2.2 Discrete Gaussian derivatives II

We define the weighted derivatives of \( \xi \) in the \( u \)-direction as

\[
D_1 \xi^- (u, v + \frac{1}{2}) = \lambda_{12} \xi (u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_{2}^{-1} \xi (u - \frac{1}{2}, v + \frac{1}{2}),
\]

\[
D_1 \xi^+ (u, v + \frac{1}{2}) = \lambda_{12}^{-1} \xi (u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_{2} \xi (u - \frac{1}{2}, v + \frac{1}{2}).
\]

Observe that these vectors are orthogonal to \( \nu(u, v) \) and \( \nu(u, v + 1) \), respectively. Similarly, define

\[
D_2 \xi^- (u + \frac{1}{2}, v) = \lambda_{12} \xi (u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_{2}^{-1} \xi (u + \frac{1}{2}, v - \frac{1}{2}),
\]

\[
D_2 \xi^+ (u + \frac{1}{2}, v) = \lambda_{12}^{-1} \xi (u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_{2} \xi (u + \frac{1}{2}, v - \frac{1}{2}),
\]

and observe that they are orthogonal to \( \nu(u, v) \) and \( \nu(u + 1, v) \), respectively.

**Proposition 7** The Gaussian equations for the derivatives of the normal vector field are given by

\[
\begin{align*}
D_1 \xi^- (u, v + \frac{1}{2}) & = \frac{-\lambda_2^{-2} - \lambda_2^2}{\Omega_{12} \lambda_{12}} q_1(u + \frac{1}{2}, v) + \frac{\lambda_{12} A_2 - \lambda_2^{-1} A_0}{\Omega_{12} \Omega_2 \lambda_2} q_2(u, v + \frac{1}{2}), \\
D_1 \xi^- (u, v + \frac{1}{2}) & = \frac{-\lambda_2^{-2} - \lambda_2^2}{\Omega_{12} \lambda_{12}^{-1}} q_1(u - \frac{1}{2}, v) + \frac{\lambda_2^{-1} A_2 - \lambda_{12} A_0}{\Omega_{12} \Omega_2 \lambda_2} q_2(u, v + \frac{1}{2}), \\
D_1 \xi^+ (u, v + \frac{1}{2}) & = \frac{-\lambda_2^{-2} - \lambda_2^2}{\Omega_{12} \lambda_2} q_1(u + \frac{1}{2}, v) + \frac{\lambda_{12} A_2 - \lambda_2^{-1} A_0}{\Omega_{12} \Omega_2 \lambda_{12}} q_2(u, v + \frac{1}{2}), \\
D_1 \xi^+ (u, v + \frac{1}{2}) & = \frac{-\lambda_2^{-2} - \lambda_2^2}{\Omega_{12} \lambda_2} q_1(u - \frac{1}{2}, v) + \frac{\lambda_{12}^{-1} A_2 - \lambda_{12} A_0}{\Omega_{12} \Omega_2 \lambda_{12}} q_2(u, v + \frac{1}{2}).
\end{align*}
\]

and

\[
\begin{align*}
D_2 \xi^- (u + \frac{1}{2}, v) & = \frac{-\lambda_2 B_1 - \lambda_{12}^{-1} B_0}{\Omega_{12} \Omega_1 \lambda_1} q_1(u + \frac{1}{2}, v) + \frac{-\lambda_2^{-2} - \lambda_2^2}{\Omega_{12} \lambda_{12}^{-1}} q_2(u, v + \frac{1}{2}), \\
D_2 \xi^- (u + \frac{1}{2}, v) & = \frac{-\lambda_2^{-1} B_1 - \lambda_{12}^2 B_0}{\Omega_{12} \Omega_1 \lambda_{12}^{-1}} q_1(u + \frac{1}{2}, v) + \frac{-\lambda_2^{-1} B_2 - \lambda_2^2 B_0}{\Omega_{12} \lambda_2 \lambda_{12}^{-1}} q_2(u, v + \frac{1}{2}), \\
D_2 \xi^+ (u + \frac{1}{2}, v) & = \frac{-\lambda_2 B_1 - \lambda_{12}^{-1} B_0}{\Omega_{12} \Omega_1 \lambda_1} q_1(u + \frac{1}{2}, v) + \frac{-\lambda_2^{-2} - \lambda_2^2}{\Omega_{12} \lambda_2 \lambda_{12}^-1} q_2(u, v + \frac{1}{2}), \\
D_2 \xi^+ (u + \frac{1}{2}, v) & = \frac{-\lambda_2^{-1} B_1 - \lambda_{12}^2 B_0}{\Omega_{12} \Omega_1 \lambda_{12}^{-1}} q_1(u + \frac{1}{2}, v) + \frac{-\lambda_2^{-1} B_2 - \lambda_2^2 B_0}{\Omega_{12} \lambda_2 \lambda_{12}^-1} q_2(u + 1, v + \frac{1}{2}).
\end{align*}
\]

**Proof.** Since the proofs of all formulas are similar, we only prove the first one. We can write

\[
D_1 \xi^- (u, v + \frac{1}{2}) = a q_1(u + \frac{1}{2}, v) + b q_2(u, v + \frac{1}{2}).
\]

If we multiply by \( q_2(u, v + \frac{1}{2}) \times \xi (u + \frac{1}{2}, v + \frac{1}{2}) \) we obtain

\[
a \Omega_{12} \lambda_2 = -[\xi (u - \frac{1}{2}, v + \frac{1}{2}) q_2(u, v + \frac{1}{2}), \xi (u + \frac{1}{2}, v + \frac{1}{2})].
\]

Since \( \Omega_2 \xi (u - \frac{1}{2}, v + \frac{1}{2}) = q_1(u - \frac{1}{2}, v + 1) - q_1(u - \frac{1}{2}, v) \), we conclude that

\[
a \Omega_{12} \lambda_2 = -\lambda_2 \lambda_{12} + \lambda_2^{-1} \lambda_{12}^{-1}.
\]
If we multiply by \( q_1(u + \frac{1}{2}, v) \times \xi(u - \frac{1}{2}, v + \frac{1}{2}) \) we obtain

\[
b\Omega_{12} \lambda_2 = [q_1(u + \frac{1}{2}, v), \xi(u + \frac{1}{2}, v + \frac{1}{2}), \xi(u - \frac{1}{2}, v + \frac{1}{2})].
\]

Since \( \Omega_2 \xi(u - \frac{1}{2}, v + \frac{1}{2}) = q_1(u - \frac{1}{2}, v + 1) - q_1(u - \frac{1}{2}, v) \), we conclude that

\[
b\Omega_{12} \lambda_2 \Omega_2 = \lambda_{12} A_2 - \lambda_{12}^{-1} A_0.
\]

### 3.2.3 Mean curvature

In the smooth case, the mean curvature is the trace of the shape operator. When we are using asymptotic parameters, we can calculate it directly from \( \tilde{H} \Omega = -\xi_u \cdot \nu_v = -\xi_v \cdot \nu_u \). Based on this idea, we discuss now a possible definition of discrete affine mean curvature.

We calculate

\[
D_1 \xi^-(u, v + \frac{1}{2}) \cdot \nu_2(u, v + \frac{1}{2}) = \lambda_{12}^2 - \lambda_2^{-2}, \\
D_1 \xi^+(u, v + \frac{1}{2}) \cdot \nu_2(u, v + \frac{1}{2}) = \lambda_2^2 - \lambda_{12}^{-2}.
\]

Also

\[
D_2 \xi^-(u + \frac{1}{2}, v) \cdot \nu_1(u + \frac{1}{2}, v) = \lambda_{12}^2 - \lambda_1^{-2}, \\
D_2 \xi^+(u + \frac{1}{2}, v) \cdot \nu_1(u + \frac{1}{2}, v) = \lambda_1^2 - \lambda_{12}^{-2}.
\]

We conclude that

\[
D_1 \xi^-(u, v + \frac{1}{2}) \cdot \nu_2(u, v + \frac{1}{2}) + D_1 \xi^+(u, v - \frac{1}{2}) \cdot \nu_2(u, v - \frac{1}{2}) = \lambda_{12}^2 - \lambda_2^{-2} + \lambda_0^2 - \lambda_1^{-2}
\]

\[
D_2 \xi^-(u + \frac{1}{2}, v) \cdot \nu_1(u + \frac{1}{2}, v) + D_2 \xi^+(u - \frac{1}{2}, v) \cdot \nu_1(u - \frac{1}{2}, v) = \lambda_{12}^2 - \lambda_1^{-2} + \lambda_0^2 - \lambda_2^{-2}.
\]

Our proposal is to define the affine mean curvature \( H \) at \((u, v)\) by

\[
H(u, v) = \lambda_{12}^2 - \lambda_1^{-2} - \lambda_2^{-2} + \lambda_0^2.
\]

It is interesting to observe that if the co-normal vector field is discrete harmonic, i.e., \( \lambda = 1 \), then \( H = 0 \). Thus the class of asymptotic nets with vanishing mean curvature contains the class of discrete minimal affine surfaces introduced in [4].

### 3.2.4 Affine Spheres

In [2], a definition of discrete affine spheres preserving the duality between \( q \) and \( \nu \) is proposed. But this class does not include the hyperboloid of Example 4 with a general sampling pair \((\Delta u, \Delta v)\). Actually, this hyperboloid satisfies the definition of [2] if and only if \( \Delta u = \Delta v \). We propose here a new definition of discrete affine sphere that includes the definition of [2] as a particular case. With this new definition, the hyperboloid of Example 4 with any sampling pair \((\Delta u, \Delta v)\) becomes an affine sphere.

We say that the asymptotic net is an affine sphere if

\[
\begin{align*}
\lambda_{12} A_2 &= \lambda_2 A_0, \\
\lambda_{13} B_1 &= \lambda_1 B_0.
\end{align*}
\]

These conditions imply that

\[
\begin{align*}
D_1 \xi^-(u, v + \frac{1}{2}) &= (\lambda_2^{-1} - \lambda_{12}) \left( \frac{q_1(u - \frac{1}{2}, v)}{\Omega_2} + \frac{q_1(u + \frac{1}{2}, v)}{\Omega_{12}} \right), \\
D_1 \xi^+(u, v + \frac{1}{2}) &= (\lambda_1^{-1} - \lambda_{12}) \left( \frac{q_1(u + \frac{1}{2}, v + 1)}{\Omega_2} + \frac{q_1(u - \frac{1}{2}, v + 1)}{\Omega_{12}} \right), \\
D_2 \xi^-(u + \frac{1}{2}, v) &= (\lambda_1^{-1} - \lambda_{12}) \left( \frac{q_2(u, v - \frac{1}{2})}{\Omega_1} + \frac{q_2(u, v + \frac{1}{2})}{\Omega_{12}} \right), \\
D_2 \xi^+(u + \frac{1}{2}, v) &= (\lambda_2^{-1} - \lambda_1) \left( \frac{q_2(u + 1, v - \frac{1}{2})}{\Omega_1} + \frac{q_2(u + 1, v + \frac{1}{2})}{\Omega_{12}} \right).
\end{align*}
\]
We remark that an affine sphere as defined above is also an affine sphere as defined in [2] if and only if
\[ \epsilon \Omega_{12} = \lambda_{12}^{-1} - \lambda_{12}, \]
for some constant \( \epsilon \).

### 3.2.5 Compatibility Equations

The first compatibility equation is obtained by comparing the expansions of \( q_{112} \) and \( q_{121} \):
\[
\frac{\Omega_2 \lambda_2^{-1} \lambda_{12}^{-1}}{\Omega_{12}} = \frac{\lambda_0 \lambda_1 \Omega_0}{\Omega_1} + \frac{A_0 B_0 \lambda_1 \lambda_{12}^{-1}}{\Omega_{12} \Omega_1}. \tag{2}
\]

There are two other compatibility equations, obtained by comparing the mixed second derivatives of \( \xi \):
\[
\frac{\lambda_0^2 - \lambda_1^2}{\lambda_1^{-1} \Omega_1} - \frac{\lambda_1^2 \lambda_2 - \lambda_2^2}{\lambda_1 \lambda_2 \Omega_1} = \frac{\lambda_1 B_1 - \lambda_{12}^{-1} B_0}{\Omega_{12} \Omega_1 \lambda_1} - \frac{\lambda_2 B_0 - \lambda_{12}^{-1} B_0}{\Omega_{12} \Omega_0 \lambda_0}. \tag{3}
\]
and
\[
\frac{\lambda_0^2 - \lambda_2^2}{\lambda_2^{-1} \Omega_2} - \frac{\lambda_1^2 \lambda_2 - \lambda_2^2}{\lambda_1 \lambda_2 \Omega_2} = \frac{\lambda_1 A_2 - \lambda_{12}^{-1} A_0}{\Omega_{12} \Omega_2 \lambda_2} - \frac{\lambda_2 A_0 - \lambda_{12}^{-1} A_0}{\Omega_{12} \Omega_0 \lambda_0}. \tag{4}
\]

The proofs of these equations are left to the appendix of section 3.

**Theorem 8** Given functions \( \Omega, \lambda, A \) and \( B \) satisfying the compatibility equations \( (2), (3) \) and \( (4) \), there exists an asymptotic net \( q \) satisfying Gauss equations. Moreover, two asymptotic nets with the same \( \Omega, \lambda, A \) and \( B \) are affine equivalent.

**Proof.** We begin by choosing four points \( q(0,0), q(1,0), q(0,1) \) and \( q(1,1) \) satisfying
\[ [q(1,0) - q(0,0), q(0,1) - q(0,0), q(1,1) - q(0,0)] = \Omega^2 \left( \frac{1}{2}, \frac{1}{2} \right). \]

Those four points are determined up to an affine transformation of \( \mathbb{R}^3 \).

From a quadrangle \( (u - \frac{1}{2}, v - \frac{1}{2}) \), one can extend the definition of \( q \) to the quadrangles \( (u + \frac{1}{2}, v - \frac{1}{2}) \) and \( (u - \frac{1}{2}, v + \frac{1}{2}) \) by the formulas of subsection 3.2.1. With these extensions, we can calculate \( \xi(u + \frac{1}{2}, v - \frac{1}{2}) \) and \( \xi(u - \frac{1}{2}, v + \frac{1}{2}) \). It is clear that \( \xi_1(u, v - \frac{1}{2}) \) and \( \xi_2(u - \frac{1}{2}, v) \) satisfy equations of subsection 3.2.2. The coherence of these extensions follows from the first compatibility equation.

Then one can extend the definition of \( q \) to \( (u + \frac{1}{2}, v + \frac{1}{2}) \) in two different ways: from the quadrangle \( (u + \frac{1}{2}, v - \frac{1}{2}) \) and from \( (u - \frac{1}{2}, v + \frac{1}{2}) \). To see that both extensions lead to the same result, one has to verify that both affine normals \( \xi(u + \frac{1}{2}, v + \frac{1}{2}) \) are the same, which in fact reduces to verify that \( \xi_{12} = \xi_{21} \). But this last equation comes from the second and third compatibility equations, which completes the proof of the theorem. \( \blacksquare \)

### Appendix of section 3: Proofs of Compatibility Equations

To prove the first compatibility equation, we calculate \( q_{112}(u, v + \frac{1}{2}) \) in two different ways. Since \( q_{12}(u + \frac{1}{2}, v + \frac{1}{2}) = \Omega_{12} \xi(u + \frac{1}{2}, v + \frac{1}{2}) \), we write
\[
q_{112}(u, v + \frac{1}{2}) = (\Omega_{12} - \lambda_2 A_2 \Omega_2) \xi(u + \frac{1}{2}, v + \frac{1}{2}) + \Omega_{12} \lambda_2 D_1 \xi^{-1}(u, v + \frac{1}{2}).
\]

From equations of subsection 3.2.2, we conclude that the coefficient of \( q_1(u + \frac{1}{2}, v) \) of the expansion of \( \Omega_{12} \xi(u + \frac{1}{2}, v + \frac{1}{2}) \) is
\[
(\lambda_2^{-1} - \lambda_2 \lambda_{12}) \frac{\Omega_2}{\Omega_{12}}. \tag{5}
\]

On the other hand, \( q_{112}(u, v + \frac{1}{2}) = q_{11}(u, v + 1) - q_{11}(u, v) \), with
\[
q_{11}(u, v + 1) = \left(1 - \frac{\lambda_2 \lambda_{12} \Omega_2}{\Omega_{12}}\right) q_1(u + \frac{1}{2}, v + 1) + \frac{\lambda_2 A_2}{\Omega_{12}} q_2(u, v + \frac{1}{2})
\]
\[
q_{11}(u, v) = \left(1 - \frac{\lambda_0 \lambda_1 \Omega_0}{\Omega_1}\right) q_1(u + \frac{1}{2}, v) + \frac{\lambda_1 A_0}{\Omega_1} q_2(u, v - \frac{1}{2}).
\]
Since
\[ q_1(u + \frac{1}{2}, v) + \Omega_{12} \xi(u + \frac{1}{2}, v + \frac{1}{2}) \]
and
\[ q_2(u, v - \frac{1}{2}) = -\frac{\lambda_1^{-1}B_0}{\Omega_{12}} q_1(u + \frac{1}{2}, v) + \frac{\lambda_1^{-1} \lambda_2 \Omega_{2}}{\Omega_{12}} q_2(u, v + \frac{1}{2}), \]
the coefficient of \( q_1(u + \frac{1}{2}, v) \) of the expansion of \( q_{112}(u, v + \frac{1}{2}) \) in the basis \( \{ q_1(u + \frac{1}{2}, v), q_2(u, v + \frac{1}{2}), \xi(u + \frac{1}{2}, v + \frac{1}{2}) \} \) is
\[
-\frac{\lambda_2 \lambda_1 \Omega_2}{\Omega_{12}} + \frac{\lambda_2 \lambda_1 \Omega_2}{\Omega_1} + \frac{\lambda_1^{-1} \lambda_1 A_0 B_0}{\Omega_1 \Omega_{12}}.
\]
(6)
The first compatibility equation follows directly from (5) and (6).

To prove the second and third compatibility equation, we observe that
\[
D_1 \xi^-(u, v + \frac{1}{2}) - D_1 \xi^+(u, v - \frac{1}{2}) = D_2 \xi^-(u + \frac{1}{2}, v) - D_2 \xi^+(u - \frac{1}{2}, v).
\]
Using the formulas of subsection 3.2.2 and comparing the coefficients of \( q_1(u + \frac{1}{2}, v) \), we obtain the second compatibility equation. The third compatibility equation is obtained by comparing the coefficients of \( q_2(u, v + \frac{1}{2}) \).

4 Discrete affine surfaces with definite metric

4.1 Discrete affine concepts and examples

4.1.1 Affine metric, co-normal and normal vector fields.

Special non-degenerate discrete conjugate nets. A discrete conjugate net is defined to be a \( \mathbb{R}^3 \)-valued function defined on a subset \( D \) of \( \mathbb{Z}^2 \), such that “faces are planar”, i.e., \( q(u, v), q(u + 1, v), q(u, v + 1) \) and \( q(u + 1, v) \) are co-planar (see [1] Definition 2.1).

We say that the discrete conjugate net is non-degenerate if the sign of the following four quantities is the same and does not depend on \( (u, v) \):
\[
\Delta_1(u, v) := [q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v), q_2(u, v + \frac{1}{2})],
\]
\[
\Delta_2(u, v) := [q_1(u - \frac{1}{2}, v), q_2(u, v - \frac{1}{2}), q_2(u, v + \frac{1}{2})],
\]
\[
\Delta_3(u, v) := [q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v), q_2(u, v - \frac{1}{2})],
\]
\[
\Delta_4(u, v) := [q_1(u + \frac{1}{2}, v), q_2(u, v - \frac{1}{2}), q_2(u, v + \frac{1}{2})].
\]
In the following, we shall assume that \( \Delta_i(u, v) > 0, \forall (u, v) \in D, 1 \leq i \leq 4 \).

Unlike the indefinite case where we were able to describe an affine discrete structure on any non-degenerate asymptotic net, we need to consider here non-degenerate discrete conjugate nets with an extra assumption. We shall say that a non-degenerate asymptotic net is special if it satisfies the following compatibility equation
\[
\frac{\Delta_1(u, v) \Delta_3(u, v)}{\Delta_2(u, v) \Delta_4(u, v)} = 1, \quad \forall (u, v) \in D.
\]
(7)

Affine metric. At \( (u, v) \), we denote
\[
q_1 \times q_2 = \frac{1}{4} [q_1(u + \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2}) + q_1(u - \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2}) + q_1(u - \frac{1}{2}, v) \times q_2(u, v - \frac{1}{2}) + q_1(u + \frac{1}{2}, v) \times q_2(u, v - \frac{1}{2})],
\]
and define the metric by
\[
2 \Omega^2(u, v) = q_1 \times q_2(u, v) \cdot (q_{11}(u, v) + q_{22}(u, v)).
\]
Co-normal vector field and discrete parameters. A co-normal vector field $\nu$ with respect to a discrete conjugate net $q$ is a vector-valued map defined at any face $(u + \frac{1}{2}, v + \frac{1}{2})$ of the net, orthogonal to it, and satisfying the discrete Lelieuvre’s equations

$$
\begin{align*}
q_1(u + \frac{1}{2}, v) &= \nu(u + \frac{1}{2}, v - \frac{1}{2}) \times \nu(u + \frac{1}{2}, v + \frac{1}{2}) \\
q_2(u, v + \frac{1}{2}) &= -\nu(u - \frac{1}{2}, v + \frac{1}{2}) \times \nu(u + \frac{1}{2}, v + \frac{1}{2}).
\end{align*}
$$

It turns out that a necessary and sufficient condition for the existence of co-normal vector field with respect to a given non-degenerate discrete conjugate net is the special condition (7).

Proposition 9 A non-degenerate discrete conjugate net $q$ admits a one-parameter family of co-normal fields if and only if is satisfies the special condition (7). Any such co-normal field is determined by four positive parameters $\alpha(u, v), \beta(u, v), \gamma(u, v)$ and $\delta(u, v)$ that satisfy the following coincidence relations:

$$
\begin{align*}
\nu(u + \frac{1}{2}, v + \frac{1}{2}) &= \alpha(u, v)\Omega(u, v)q_1(u + \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2}) \\
\nu(u - \frac{1}{2}, v + \frac{1}{2}) &= \beta(u, v)\Omega(u, v)q_1(u - \frac{1}{2}, v) \times q_2(u, v + \frac{1}{2}) \\
\nu(u - \frac{1}{2}, v - \frac{1}{2}) &= \gamma(u, v)\Omega(u, v)q_1(u - \frac{1}{2}, v) \times q_2(u, v - \frac{1}{2}) \\
\nu(u + \frac{1}{2}, v - \frac{1}{2}) &= \delta(u, v)\Omega(u, v)q_1(u + \frac{1}{2}, v) \times q_2(u, v - \frac{1}{2}).
\end{align*}
$$

Finally, we can get the other co-normal fields of the family by

$$
\nu_p(u, v) = \begin{cases} 
\rho\nu(u, v) & \text{if } u + v \text{ is even,} \\
\rho^{-1}\nu(u, v) & \text{if } u + v \text{ is odd,}
\end{cases}
$$

where $\rho$ is any positive constant.

Proof. We first assume that there exists such a co-normal vector field. It follows that there exists maps $\alpha(u, v), \beta(u, v), \gamma(u, v)$ and $\delta(u, v)$ satisfying the coincidence relations. Making the vector product of the first two coincidence relations, using the identity $(A \times B) \times (A \times C) = [A, B, C]A$, and Lelieuvre’s equation we obtain

$$
-q_2(u, v + \frac{1}{2}) = \alpha(u, v)\beta(u, v)\Omega^2(u, v)q_2(u, v + \frac{1}{2})[q_2(u, v + \frac{1}{2}), q_1(u + \frac{1}{2}, v), q_1(u - \frac{1}{2}, v)]
$$

which simplifies to

$$
\alpha(u, v)\beta(u, v)\Omega^2(u, v)\Delta_1(u, v) = 1.
$$

Analogously, we get $\beta(u, v)\gamma(u, v)\Omega^2(u, v)\Delta_2(u, v) = 1$, $\gamma(u, v)\delta(u, v)\Omega^2(u, v)\Delta_3(u, v) = 1$ and $\delta(u, v)\alpha(u, v)\Omega^2(u, v)\Delta_4(u, v) = 1$. From these, Equation (7) follows easily, so we see that a discrete conjugate net must be special in order to admit a co-normal vector field. Moreover, we see that $\beta(u, v), \gamma(u, v)$ and $\delta(u, v)$ are uniquely determined by $\alpha(u, v)$.

The next step consists in fixing an initial value $\alpha(u_0, v_0)$ and define $\nu$ at the faces $(u_0 \pm \frac{1}{2}, v_0 \pm \frac{1}{2})$ by

$$
\begin{align*}
\nu(u_0 + \frac{1}{2}, v_0 + \frac{1}{2}) &= \alpha(u_0, v_0)\Omega(u_0, v_0)q_1(u_0 + \frac{1}{2}, v_0) \times q_2(u_0, v_0 + \frac{1}{2}) \\
\nu(u_0 - \frac{1}{2}, v_0 + \frac{1}{2}) &= \beta(u_0, v_0)\Omega(u_0, v_0)q_1(u_0 - \frac{1}{2}, v_0) \times q_2(u_0, v_0 + \frac{1}{2}) \\
\nu(u_0 - \frac{1}{2}, v_0 - \frac{1}{2}) &= \gamma(u_0, v_0)\Omega(u_0, v_0)q_1(u_0 - \frac{1}{2}, v_0) \times q_2(u_0, v_0 - \frac{1}{2}) \\
\nu(u_0 + \frac{1}{2}, v_0 - \frac{1}{2}) &= \delta(u_0, v_0)\Omega(u_0, v_0)q_1(u_0 + \frac{1}{2}, v_0) \times q_2(u_0, v_0 - \frac{1}{2}).
\end{align*}
$$

This in turn determine the value of $\alpha(u - 1, v), \alpha(u - 1, v - 1)$ and $\alpha(u, v - 1)$ and thus the corresponding values for the maps $\beta, \gamma$ and $\delta$. We can therefore determine $\nu$ at five adjacent faces, and from now on the proof follows closely the lines of the proof of Proposition 1 and we are able to define in a unique way the co-normal field on all the domain $D$.

As in the indefinite case, it is easier to construct examples beginning from the co-normal vector field.

Proposition 10 A vector field $\nu(u + \frac{1}{2}, v + \frac{1}{2})$ is the co-normal vector field of a special discrete conjugate net if and only if it satisfies

$$
\nu_{11}(u + \frac{1}{2}, v + \frac{1}{2}) + \nu_{22}(u + \frac{1}{2}, v + \frac{1}{2}) = H^*(u + \frac{1}{2}, v + \frac{1}{2})\nu(u + \frac{1}{2}, v + \frac{1}{2}).
$$

for some map $H^*$. $H^*$ can be calculated in terms of the discrete parameters, but we postpone this calculation until Subsection 4.2.4.
We define another parameter
\[ q_{12}(u + \frac{1}{2}, v + \frac{1}{2}) = \nu(u + \frac{1}{2}, v + \frac{1}{2}) \times \nu(u + \frac{1}{2}, v + \frac{3}{2}) - \nu(u + \frac{1}{2}, v - \frac{1}{2}) \times \nu(u + \frac{1}{2}, v + \frac{1}{2}) \]
and
\[ q_{21}(u + \frac{1}{2}, v + \frac{1}{2}) = -\nu(u + \frac{1}{2}, v + \frac{1}{2}) \times \nu(u + \frac{1}{2}, v + \frac{3}{2}) + \nu(u - \frac{1}{2}, v + \frac{1}{2}) \times \nu(u + \frac{1}{2}, v + \frac{1}{2}) \]
So
\[ q_{12} - q_{21} = \nu(u + \frac{1}{2}, v + \frac{1}{2}) \times (\lambda_{11}(u + \frac{1}{2}, v + \frac{1}{2}) + \lambda_{22}(u + \frac{1}{2}, v + \frac{1}{2})) \]
thus proving the proposition.

In terms of the co-normal vector field, the affine metric is given by
\[ 4\Omega^2 = [\nu(u+\frac{1}{2},v+\frac{1}{2}) - \nu(u-\frac{1}{2},v-\frac{1}{2}), \nu(u-\frac{1}{2},v+\frac{1}{2}) - \nu(u+\frac{1}{2},v-\frac{1}{2}), \nu(u-\frac{1}{2},v+\frac{3}{2}) - \nu(u+\frac{1}{2},v-\frac{3}{2})] \]
and the parameters \( \alpha(u,v), \beta(u,v), \gamma(u,v), \delta(u,v) \) are given by
\[
\begin{align*}
\Omega(u,v)\alpha(u,v) &= [\nu(u+\frac{1}{2},v-\frac{1}{2}), \nu(u+\frac{1}{2},v+\frac{1}{2}), \nu(u-\frac{1}{2},v+\frac{1}{2})] \\
\Omega(u,v)\beta(u,v) &= [\nu(u+\frac{1}{2},v+\frac{1}{2}), \nu(u-\frac{1}{2},v+\frac{1}{2}), \nu(u-\frac{1}{2},v-\frac{1}{2})] \\
\Omega(u,v)\gamma(u,v) &= [\nu(u-\frac{1}{2},v+\frac{1}{2}), \nu(u-\frac{1}{2},v-\frac{1}{2}), \nu(u+\frac{1}{2},v-\frac{1}{2})] \\
\Omega(u,v)\delta(u,v) &= [\nu(u-\frac{1}{2},v-\frac{1}{2}), \nu(u+\frac{1}{2},v-\frac{1}{2}), \nu(u+\frac{1}{2},v+\frac{1}{2})].
\end{align*}
\]
Observe that \( (\alpha + \gamma) \cdot (\beta + \delta) = 4 \) and
\[ \beta(u,v)\nu(u+\frac{1}{2},v-\frac{1}{2}) + \delta(u,v)\nu(u-\frac{1}{2},v+\frac{1}{2}) = \alpha(u,v)\nu(u-\frac{1}{2},v-\frac{1}{2}) + \gamma(u,v)\nu(u+\frac{1}{2},v+\frac{1}{2}). \]
We define another parameter \( \lambda(u,v) \) by
\[ \lambda(u,v) = \frac{\beta(u,v) + \delta(u,v)}{2} = \frac{2}{\alpha(u,v) + \gamma(u,v)}. \]
The parameters \( \alpha, \beta, \gamma, \delta \) and \( \lambda \) are called discrete parameters.

**Normal vector field.** The discrete affine normal \( \xi \) is defined as
\[ \xi(u,v) = \frac{q_{11}(u,v) + q_{22}(u,v)}{2\Omega(u,v)}. \]
One can verify that
\[
\begin{align*}
\nu(u+\frac{1}{2},v+\frac{1}{2}) \cdot \xi(u,v) &= \lambda(u,v) \\
\nu(u-\frac{1}{2},v+\frac{1}{2}) \cdot \xi(u,v) &= \lambda^{-1}(u,v) \\
\nu(u-\frac{1}{2},v-\frac{1}{2}) \cdot \xi(u,v) &= \lambda(u,v) \\
\nu(u+\frac{1}{2},v-\frac{1}{2}) \cdot \xi(u,v) &= \lambda^{-1}(u,v).
\end{align*}
\]
We remark that the above formulas imply that discrete conjugate nets satisfying (7) are necessarily convex. In Figure 5, we see an illustration of the co-normal and normal vector fields.

**Figure 5:** The normal vector at a vertex and four co-normals at its adjacent faces.
4.1.2 Examples

**Example 11** We say that a discrete conjugate net as above is an affine sphere if \( \xi(u, v) = -q(u, v) \). In this case, it is easy to see that \( \lambda = 1 \), and the above definitions reduce to the definitions proposed in [2]. In fact, all the equations obtained below for definite discrete affine surfaces can be seen as generalizations of the corresponding equations in [2].

**Example 12** We can define discrete conjugate nets beginning with co-normals. We first define

\[
\alpha = \frac{1}{2}, v = \frac{1}{2} \text{ for two consecutive values of } u \text{ and any } v \text{ and then extend it to other values of } u \text{ by (3). More explicitly, we define } \nu(u + \frac{1}{2}, v + \frac{1}{2}) \text{ for } u = u_0 \text{ and } u = u_0 + \Delta u, v_0 \leq v \leq v_0 + N\Delta v. \text{ Then we extend } \nu(u + \frac{1}{2}, v + \frac{1}{2}) \text{ to } (u, v) = (u + i\Delta u, v + j\Delta v \text{ with } 0 \leq i \leq N/2, i \leq j \leq N-i, \text{ by using (3).}

In figure 6 we can see a surface where we have taken \( \nu(u + \frac{1}{2}, v + \frac{1}{2}) = (u, v, uv) \), \( N = 12, \Delta u = 1/12, \Delta v = 1/11 \) and \( H^* = 0 \). In figure 7 we can see another surface where \( \nu(u + \frac{1}{2}, v + \frac{1}{2}) = (u, v, uv) \), \( N = 12, \Delta u = 1/12, \Delta v = 1/11 \) and \( H^*(u + \frac{1}{2}, v + \frac{1}{2}) = 0 \text{, if } u + v \text{ is even, and } H^*(u + \frac{1}{2}, v + \frac{1}{2}) = 0.1 \text{, if } u + v \text{ is odd.}

4.2 Structural equations, mean curvature and compatibility

4.2.1 Discrete Gauss equations I

In this subsection we generalize Theorem 5.1 of [2]. From now on, we shall change slightly the notation in order to write smaller equations. We shall write \( \alpha(u, v) = \alpha_0, \alpha(u + 1, v) = \alpha_1, \alpha(u, v + 1) = \alpha_2, \alpha(u - 1, v) = \alpha_3, \alpha(u, v - 1) = \alpha_4 \), \( \alpha(u + 1, v + 1) = \alpha_{12} \) and \( \alpha(u - 1, v + 1) = \alpha_{T2} \), with a similar notation for the parameters \( \beta, \gamma, \delta, \lambda \) and \( \Omega \).

**Lemma 13** We can write

\[
q_1(u + \frac{1}{2}, v + \frac{1}{2}) = (P - 1)q_1(u + \frac{1}{2}, v) + (Q - 1)q_2(u, v + \frac{1}{2})
\]

\[
q_1(u + \frac{1}{2}, v) + q_2(u, v + \frac{1}{2}) + q_1(u - \frac{1}{2}, v) - q_2(u, v - \frac{1}{2}) = Aq_1(u + \frac{1}{2}, v) + Bq_2(u, v + \frac{1}{2}) + (A + B + C)\xi(u, v)
\]

\[
q_1(u + \frac{1}{2}, v) + q_2(u, v + \frac{1}{2}) - q_1(u - \frac{1}{2}, v) - q_2(u, v - \frac{1}{2}) = 2\Omega(u, v)\xi(u, v)
\]

where

\[
\begin{align*}
P\alpha_0\Omega_0 &= \delta_2\Omega_2 \\
Q\alpha_0\Omega_0 &= \beta_1\Omega_1 \\
A\alpha_0 &= 2\beta_0\lambda_0^{-2} \\
B\alpha_0 &= -2\delta_0\lambda_0^{-2} \\
(A + B + C)\lambda(u, v) &= (\delta_0 - \beta_0)\Omega_0.
\end{align*}
\]
We begin this subsection by generalizing the dual discrete Lelieuvre’s equations of [2].

4.2.2 Discrete Gauss equations II

Theorem 15

**Proof.**

To calculate \( P \), observe that
\[
q_{12}(u + \frac{1}{2}, v + \frac{1}{2}) = q_1(u + \frac{1}{2}, v + 1) - q_1(u + \frac{1}{2}, v)
\]
and then multiply by \( q_2(u, v + \frac{1}{2}) \). We obtain
\[
P\alpha_0\Omega_0 = \delta_2\Omega_2.
\]
The calculations for \( Q \) are similar. The third and fourth equations can be written as
\[
2(q_1(u + \frac{1}{2}, v) - q_2(u, v - \frac{1}{2})) = Aq_1(u + \frac{1}{2}, v) + Bq_2(u, v + \frac{1}{2}) + (A + B + C + 2\Omega)\xi(u, v)
\]
\[
2(q_1(u - \frac{1}{2}, v) - q_2(u, v + \frac{1}{2})) = Aq_1(u - \frac{1}{2}, v) + Bq_2(u, v - \frac{1}{2}) + (A + B + C - 2\Omega)\xi(u, v).
\]
Multiplying the second equation by \( (q_2(u, v + \frac{1}{2}), \xi(u, v)) \) we obtain
\[
2\beta_0\lambda_0^{-1} = A\alpha_0\lambda_0
\]
Thus proving the third equation of the lemma. The fourth equation is proved similarly. Finally, the fifth equation is obtained by multiplying by \( \nu(u + \frac{1}{2}, v + \frac{1}{2}) \).

We now re-write lemma [13] in a more usual way:

**Theorem 14** Gauss equations can be written in the form

\[
q_{11}(u, v) = \frac{\alpha_0 - \beta_0\lambda_0^{-2}}{\alpha_0} q_1(u + \frac{1}{2}, v) + \frac{\delta_0\lambda_0^{-2} - \alpha_0}{\alpha_0} q_2(u, v + \frac{1}{2}) + \beta_0\Omega_0\lambda_0^{-1}\xi(u, v)
\]
\[
q_{22}(u, v) = \frac{\beta_0\lambda_0^{-2} - \alpha_0}{\alpha_0} q_1(u + \frac{1}{2}, v) + \frac{\alpha_0 - \delta_0\lambda_0^{-2}}{\alpha_0} q_2(u, v + \frac{1}{2}) + \delta_0\Omega_0\lambda_0^{-1}\xi(u, v)
\]
\[
q_{12}(u + \frac{1}{2}, v + \frac{1}{2}) = \frac{\delta_2\Omega_2 - \alpha_0\Omega_0}{\alpha_0\Omega_0} q_1(u + \frac{1}{2}, v) + \frac{\beta_1\Omega_1 - \alpha_0\Omega_0}{\alpha_0\Omega_0} q_2(u, v + \frac{1}{2}).
\]

4.2.2 Discrete Gauss equations II

We begin this subsection by generalizing the dual discrete Lelieuvre’s equations of [2].

**Theorem 15** We have that

\[
\lambda_0^{-1}\nu(u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_0\nu(u - \frac{1}{2}, v + \frac{1}{2}) = -q_2(u, v + \frac{1}{2}) \times \xi(u, v)
\]
\[
\lambda_2\nu(u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_0^{-1}\nu(u - \frac{1}{2}, v + \frac{1}{2}) = -q_2(u, v + \frac{1}{2}) \times \xi(u, v + 1)
\]
\[
\lambda_0^{-1}\nu(u + \frac{1}{2}, v - \frac{1}{2}) - \lambda_0\nu(u + \frac{1}{2}, v - \frac{1}{2}) = q_1(u + \frac{1}{2}, v) \times \xi(u, v)
\]
\[
\lambda_1\nu(u + \frac{1}{2}, v - \frac{1}{2}) - \lambda_1^{-1}\nu(u + \frac{1}{2}, v - \frac{1}{2}) = q_1(u + \frac{1}{2}, v) \times \xi(u + 1, v).
\]

**Proof.** Observe first that \( \lambda_0^{-1}\nu(u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_0\nu(u - \frac{1}{2}, v + \frac{1}{2}) \) is orthogonal to \( q_2(u, v + \frac{1}{2}) \) and \( \xi(u, v) \). So
\[
\lambda_0^{-1}\nu(u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_0\nu(u - \frac{1}{2}, v + \frac{1}{2}) = mq_2(u, v + \frac{1}{2}) \times \xi(u, v).
\]
Multiplying by \( q_1(u + \frac{1}{2}, v) \) we obtain \( m = -1 \).

We define the weighted derivatives of the affine normal by the following formulas:

\[
D^+_1\xi(u + \frac{1}{2}, v) = \lambda_1\xi(u + 1, v) - \lambda_0^{-1}\xi(u, v)
\]
\[
D^+_1\xi(u + \frac{1}{2}, v) = \lambda_1^{-1}\xi(u + 1, v) - \lambda_0\xi(u, v)
\]
\[
D^+_2\xi(u + \frac{1}{2}, v) = \lambda_2\xi(u, v + 1) - \lambda_0^{-1}\xi(u, v)
\]
\[
D^+_2\xi(u + \frac{1}{2}, v) = \lambda_2^{-1}\xi(u, v + 1) - \lambda_0\xi(u, v)
\]

Observe that \( D^+_1\xi(u + \frac{1}{2}, v) \) and \( D^+_2\xi(u + \frac{1}{2}, v) \) are orthogonal to \( \nu(u + \frac{1}{2}, v + \frac{1}{2}) \). Similarly, \( D^-_1\xi(u + \frac{1}{2}, v) \) is orthogonal to \( \nu(u + \frac{1}{2}, v - \frac{1}{2}) \) and \( D^-_2\xi(u, v + \frac{1}{2}) \) is orthogonal to \( \nu(u - \frac{1}{2}, v + \frac{1}{2}) \). The coefficients of the expansion of these derivatives in the corresponding faces are given by next lemma.
Lemma 16

\[ D^+_{1} \xi(u + \frac{1}{2}, v) = \left( \lambda^2 E_0 + \frac{(\lambda_1^2 - \lambda_0^2)(\alpha_0 \lambda_0^2 - \beta_0)}{\alpha_0 \Omega_0 \delta_0} \right) q_1(u + \frac{1}{2}, v) + \frac{\lambda_1^2 - \lambda_0^2}{\alpha_0 \Omega_0} q_2(u, v + \frac{1}{2}) \]

\[ D^+_{2} \xi(u + \frac{1}{2}, v) = E_0 q_1(u + \frac{1}{2}, v) + \frac{\lambda_1^2 - \lambda_0^2}{\alpha_0 \Omega_0} q_2(u, v - \frac{1}{2}) \]

\[ D^+_{2} \xi(u, v + \frac{1}{2}) = \frac{\lambda_1^2 - \lambda_0^2}{\beta_0 \Omega_0} q_1(u - \frac{1}{2}, v) + \left( \lambda_2^2 F_0 + \frac{(\lambda_2^2 - \lambda_0^2)(\delta_0 - \alpha_0 \lambda_0^2)}{\alpha_0 \Omega_0 \beta_0} \right) q_2(u, v + \frac{1}{2}), \]

where \( E \) and \( F \) are parameters whose formulas will be given below.

**Proof.** Writing

\[ D^+_{2} \xi(u, v + \frac{1}{2}) = a q_1(u + \frac{1}{2}, v) + b q_2(u, v + \frac{1}{2}) \]

and making cross product with \( q_2(u, v + \frac{1}{2}) \), we obtain

\[ \lambda_1^2 - \lambda_0^2 = a \alpha_0 \Omega_0. \]

The same argument can be used to calculate the coefficient of \( q_1(u - \frac{1}{2}, v) \) in the fourth equation and the coefficients of \( q_2(u, v + \frac{1}{2}) \) and \( q_2(u, v - \frac{1}{2}) \) in the first and second equation, respectively.

Also, we can calculate the coefficient \( b \) of \( q_1(u + \frac{1}{2}, v) \) in the first formula from \( E \). In fact,

\[ b = \lambda_1^2 E_0 + \frac{(\lambda_1^2 - \lambda_0^2)(\alpha_0 \lambda_0^2 - \beta_0)}{\alpha_0 \Omega_0 \delta_0}. \]

The coefficient of \( q_2(u, v + \frac{1}{2}) \) in the fourth formula can be calculated from \( F \) in a similar way. ■

The calculations to obtain \( E \) and \( F \) are very laborious, and we postpone them to the appendix of this section. They are obtained by comparing \( q_{112}(u, v + \frac{1}{2}) \) with \( q_{121}(u, v + \frac{1}{2}) \) expanded in basis \( q_1(u + \frac{1}{2}, v), q_2(u, v + \frac{1}{2}), \xi(u, v) \).

\[ E_0 = \frac{\delta_2 \Omega_2 - \alpha_0 \Omega_0}{\alpha_0 \Omega_0 \beta_1 \Omega_1} + \lambda_1^{-2} \frac{\alpha_0 \gamma_1 \Omega_1 - \delta_0 \Omega_0}{\delta_0 \Omega_0 \gamma_1 \Omega_1} + \frac{2 \lambda_0 - \delta_0 - \alpha_0 \lambda_0^2}{\alpha_0 \Omega_0 \delta_0} + \frac{2 \lambda_2^{-1} - \gamma_1 - \beta_1 \lambda_1^{-2}}{\gamma_1 \beta_1 \Omega_1} \] (14)

\[ F_0 = \frac{\alpha \Omega_2 - \beta_0 \Omega_0}{\beta_0 \Omega_0 \gamma_2 \Omega_2} + \lambda_2^2 \frac{\beta_2 \Omega_2 - \beta_0 \Omega_0}{\alpha_0 \Omega_0 \beta_2 \Omega_2} + \frac{2 \lambda_2^{-1} - \alpha_0 - \beta_0 \lambda_0^{-2}}{\alpha_0 \beta_0 \Omega_0} + \frac{2 \lambda_2^{-1} - \alpha_0 - \beta_0 \lambda_0^{-2}}{\gamma_2 \beta_2 \Omega_2} \] (15)

Another important formula that we obtain in this expansion is

\[ \alpha_0 \Omega_0 + \gamma_1 \Omega_1 \beta_1 = \beta_1 \Omega_1 + \delta_2 \Omega_2. \] (16)

This formula can be seen as a first compatibility equation, or first discrete Gauss-Codazzi equation.

### 4.2.3 Mean curvature

We would like to define the affine mean curvature as the trace of the shape operator, as in the smooth case. At the center of a face, it is not clear which basis we should take. Our choice is to calculate the derivatives of the affine normal in the diagonal basis.

We denote the diagonal vectors at a face by

\[ d_1(u + \frac{1}{2}, v + \frac{1}{2}) = q(u + 1, v + 1) - q(u, v) \]

\[ d_2(u + \frac{1}{2}, v + \frac{1}{2}) = q(u, v + 1) - q(u + 1, v). \]

and define the derivatives of \( \xi \) in the diagonal directions by

\[ D_{d_1} \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda_2 \xi(u, v + 1) - \lambda_1 \xi(u + 1, v) \]

\[ D_{d_2} \xi(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda_1^{-1} \xi(u + 1, v + 1) - \lambda_0^{-1} \xi(u, v). \]
Theorem 17 Let
\[
H_1^*(u + \frac{1}{2}, v + \frac{1}{2}) = 4 - H^*(u + \frac{1}{2}, v + \frac{1}{2}) - 2\lambda_{12}^2 - 2\lambda_0^2
\]
\[
H_2^*(u + \frac{1}{2}, v + \frac{1}{2}) = 4 - H^*(u + \frac{1}{2}, v + \frac{1}{2}) - 2\lambda_1^2 - 2\lambda_2^2.
\]

Then
\[
H_2^*(u + \frac{1}{2}, v + \frac{1}{2}) \nu(u + \frac{1}{2}, v + \frac{1}{2}) = d_1(u + \frac{1}{2}, v + \frac{1}{2}) \times D_d\xi(u + \frac{1}{2}, v + \frac{1}{2})
\]
\[
H_1^*(u + \frac{1}{2}, v + \frac{1}{2}) \nu(u + \frac{1}{2}, v + \frac{1}{2}) = -d_2(u + \frac{1}{2}, v + \frac{1}{2}) \times D_d\xi(u + \frac{1}{2}, v + \frac{1}{2}).
\]

Proof. For the first equation observe that
\[
\lambda_1^{-1} \nu(u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_1 \nu(u + \frac{1}{2}, v + \frac{1}{2}) = -q_2(u + 1, v + \frac{1}{2}) \times \xi(u + 1, v)
\]
\[
\lambda_2^{-1} \nu(u + \frac{1}{2}, v + \frac{1}{2}) - \lambda_2 \nu(u + \frac{1}{2}, v + \frac{1}{2}) = q_2(u, v + \frac{1}{2}) \times \xi(u + 1, v + 1).
\]

So
\[
\nu(u + \frac{1}{2}, v + \frac{1}{2}) + \nu(u + \frac{1}{2}, v + \frac{1}{2}) - 2\lambda_1^2 \nu(u + \frac{1}{2}, v + \frac{1}{2}) = -\lambda_2(q_2(u + 1, v + \frac{1}{2}) + q_1(u + \frac{1}{2}, v)) \times \xi(u + 1, v)
\]
\[
\nu(u + \frac{1}{2}, v + \frac{1}{2}) + \nu(u + \frac{1}{2}, v + \frac{1}{2}) - 2\lambda_2^2 \nu(u + \frac{1}{2}, v + \frac{1}{2}) = \lambda_2(q_2(u, v + \frac{1}{2}) + q_1(u + \frac{1}{2}, v + 1)) \times \xi(u, v + 1),
\]
thus proving the first equation. The proof of the second equation is similar.

\[
\text{Corollary 18} \quad \text{The trace of the shape operator is given by}
\]
\[
H(u + \frac{1}{2}, v + \frac{1}{2}) = \frac{H^*(u + \frac{1}{2}, v + \frac{1}{2}) + 4 - \lambda_0^2 - \lambda_{12}^2 - \lambda_1^2 - \lambda_2^2}{\beta_1\Omega_1 + \delta_2\Omega_2}.
\]

We call this number the affine mean curvature.

4.2.4 Compatibility equations

We shall obtain compatibility equations by comparing the expansions of the weighted of \(\xi\) in basis \(\{q_1, q_2\}\) and \(\{d_1, d_2\}\).

Remember that \(H_1^*\) is the coefficient of \(d_1\) in the expansion of \(D_d\xi\) in basis \(\{d_1, d_2\}\). On the other hand
\[
D_{d_1}\xi(u + \frac{1}{2}, v + \frac{1}{2}) = aq_1(u + \frac{1}{2}, v) + bq_2(u + 1, v + \frac{1}{2})
\]
\[
D_{d_2}\xi(u + \frac{1}{2}, v + \frac{1}{2}) = cq_1(u + \frac{1}{2}, v) + dq_2(u, v + \frac{1}{2}),
\]
where
\[
a = \left( \frac{\lambda_2^2 E_0 + (\lambda_2^2 - \lambda_0^2)(\alpha_0\Omega_0 - \beta_0)}{\alpha_0\Omega_0\delta_0} \right) + \frac{\lambda_2^2 - \lambda_{12}^2}{\beta_1\Omega_1} + \frac{\alpha_0\Omega_0 - \delta_2\Omega_2}{\beta_1\Omega_1} \left( \frac{\lambda_2^2 - \lambda_0^2}{\alpha_0\Omega_0} \right)
\]
\[
b = \left( \frac{\lambda_{12}^2 F_1 + (\lambda_{12}^2 - \lambda_1^2)(\delta_1 - \alpha_1\lambda_1^2)}{\alpha_1\Omega_1\beta_1} \right) + \frac{\alpha_0\Omega_0}{\beta_1\Omega_1} \left( \frac{\lambda_2^2 - \lambda_0^2}{\alpha_0\Omega_0} \right)
\]
\[
c = \frac{\lambda_2^2 - \lambda_0^2}{\alpha_0\Omega_0} - \left( \frac{\lambda_2^2 E_0 + (\lambda_2^2 - \lambda_0^2)(\alpha_0\Omega_0 - \beta_0)}{\alpha_0\Omega_0\delta_0} \right)
\]
\[
d = F_0 - \frac{\lambda_2^2 - \lambda_0^2}{\alpha_0\Omega_0}.
\]

Now, expanding \(q_1\) and \(q_2\) in basis \(\{d_1, d_2\}\) we obtain
\[
H_1^*(u + \frac{1}{2}, v + \frac{1}{2}) = \alpha_0\Omega_0a + \gamma_{12}\Omega_{12}b
\]
\[
H_2^*(u + \frac{1}{2}, v + \frac{1}{2}) = \beta_1\Omega_1c - \delta_2\Omega_2d.
\]

It is worthwhile to observe that, in case of affine spheres, \(a = 1, b = 1, c = 1, d = -1\) and the second and third compatibility equations reduce to \(\alpha_0\Omega_0 + \gamma_{12}\Omega_{12} = H_1^* = H^*\) and \(\beta_1\Omega_1 + \delta_2\Omega_2 = H_2^* = H^*\), respectively.
Theorem 19 Given functions $\Omega$, $\lambda$, $\alpha$ and $\beta$ satisfying the compatibility equations (13), (17) and (18), there exists a discrete conjugate net $q$ satisfying Gauss equations. Moreover, two discrete conjugate nets with the same $\Omega$, $\lambda$, $\alpha$ and $\beta$ are affine equivalent.

Proof. We begin by choosing a point $q(0, 0)$ and three co-normals $\nu(\frac{1}{2}, \frac{1}{2})$, $\nu(-\frac{1}{2}, \frac{1}{2})$ and $\nu(\frac{1}{2}, -\frac{1}{2})$ satisfying (9). These four vectors are determined up to an affine transformation of $\mathbb{R}^3$. Then define $\nu(-\frac{1}{2}, -\frac{1}{2})$ by (13). It is clear that equations (10), (11) and (12) hold at $(u, v) = (0, 0)$. Also, we can define $q(u, v)$ at the four vertices connected to $(0, 0)$ and calculate $\xi(0, 0)$.

Then one can extend the definition of $q$ to the vertices connected to $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$ by the formulas of subsection 13. With these extensions, we can calculate $\xi(1, 0)$, $\xi(0, 1)$, $\xi(-1, 0)$ and $\xi(0, -1)$. It is clear that $D_1\xi(\frac{1}{2}, 0)$, $D_2\xi(0, \frac{1}{2})$, $D_1\xi(-\frac{1}{2}, 0)$ and $D_2\xi(0, -\frac{1}{2})$ satisfy equations of subsection 4, 2, 2. The coherence of these extensions is assured by the first compatibility equation.

Then one can extend the definition of $q$ to the vertices connected to $(1, 1), (-1, 1), (1, -1)$ and $(-1, -1)$. The coherence of these extensions comes from the second and third compatibility equations. Following in this way, one can construct all the discrete conjugate net.

Appendix of section 4: Proof of formulas (14), (15) and (16)

We shall compare $q_{121}(u, v + \frac{1}{2})$ with $q_{121}(u, v + \frac{1}{2})$ expanded in basis $q_1(u + \frac{1}{2}, v), q_2(u, v + \frac{1}{2}), \xi(u, v)$.

We have

$$q_{12}(u + \frac{1}{2}, v + \frac{1}{2}) = \frac{\delta_2\Omega_2 - \alpha_0\Omega_0}{\alpha_0\Omega_0} q_1(u + \frac{1}{2}, v) + \frac{\beta_1\Omega_1 - \alpha_0\Omega_0}{\alpha_0\Omega_0} q_2(u, v + \frac{1}{2})$$

$$q_{12}(u - \frac{1}{2}, v + \frac{1}{2}) = \frac{\delta_2\Omega_2 - \alpha_0\Omega_0}{\beta_0\Omega_0} q_1(u - \frac{1}{2}, v) + \frac{\beta_1\Omega_1 - \alpha_0\Omega_0}{\beta_0\Omega_0} q_2(u, v + \frac{1}{2})$$

and

$$q_1(u - \frac{1}{2}, v) = \frac{\beta_0\lambda_0^{-2}}{\alpha_0} q_1(u + \frac{1}{2}, v) + \frac{\alpha_0 - \delta_0\lambda_0^{-2}}{\alpha_0} q_2(u, v + \frac{1}{2}) - (\beta_0\Omega_0\lambda_0^{-1})\xi(u, v)$$

Writing

$$q_{121}(u, v + \frac{1}{2}) = E^1 q_1(u + \frac{1}{2}, v) + E^2 q_2(u, v + \frac{1}{2}) + E^3 \xi(u, v)$$

we conclude that

$$E^1 = \frac{\delta_2\Omega_2 - (\delta_2\Omega_2 - \alpha_0\Omega_0)\lambda_0^{-2}}{\alpha_0\Omega_0} - 1$$

$$E^2 = \frac{\beta_1\Omega_1}{\alpha_0\Omega_0} + \frac{\alpha_1\Omega_1}{\beta_0\Omega_0} - 2 - \frac{\delta_2\Omega_2 - \alpha_0\Omega_0}{\alpha_0\beta_0\Omega_0}(\alpha_0 - \delta_0\lambda_0^{-2})$$

$$E^3 = (\delta_2\Omega_2 - \alpha_0\Omega_0)\lambda_0^{-1}$$

From

$$q_{11}(u, v + 1) = \frac{\alpha_2\lambda_0^2 - \beta_2}{\delta_2} q_1(u + \frac{1}{2}, v + 1) + \frac{\delta_2 - \alpha_2\lambda_0^2}{\delta_2} q_2(u, v + \frac{1}{2}) + \gamma_2\Omega_2\lambda_2\xi(u, v + 1)$$

$$q_{11}(u, v) = \frac{\alpha_0 - \beta_0\lambda_0^{-2}}{\alpha_0} q_1(u + \frac{1}{2}, v) + \frac{\delta_0\lambda_0^{-2} - \alpha_0}{\alpha_0} q_2(u, v + \frac{1}{2}) + \beta_0\Omega_0\lambda_0^{-1} \xi(u, v)$$

and

$$\lambda_2\xi(u, v + 1) = \lambda_0^{-1} \xi(u, v) + F_1 q_1(u + \frac{1}{2}, v) + F_2 q_2(u, v + \frac{1}{2})$$

$$q_1(u + \frac{1}{2}, v + 1) = \frac{\delta_2\Omega_2}{\alpha_0\Omega_0} q_1(u + \frac{1}{2}, v) + \frac{\beta_1\Omega_1 - \alpha_0\Omega_0}{\alpha_0\Omega_0} q_2(u, v + \frac{1}{2})$$
we conclude that

\[ E^1 = \frac{(\alpha_2 \lambda^2 - \beta_2)\Omega_2 + \beta_0 \Omega_0 \lambda_0^{-2}}{\alpha_0 \Omega_0} + \gamma_2 \Omega_2 F_1 - 1 \]

\[ E^2 = \frac{(\alpha_2 \lambda^2 - \beta_2)(\beta_1 \Omega_1 - \alpha_0 \Omega_0)}{\delta_2 \alpha_0 \Omega_0} - \frac{\alpha_2 \lambda^2}{\delta_2} - \frac{\delta_0 \lambda_0^{-2}}{\alpha_0} + \gamma_2 \Omega_2 F_2 + 2 \]

\[ E^3 = (\gamma_2 \Omega_2 - \beta_0 \Omega_0) \lambda_0^{-1} \]

From the equality of \( E^1 \) we obtain

\[ \delta_2 \Omega_2 - (\delta \Omega_2 \Omega_2 - \alpha_0 \Omega_1) \lambda_0^{-2} = \frac{(\alpha_2 \lambda^2 - \beta_2)\Omega_2 + \beta_0 \Omega_0 \lambda_0^{-2}}{\alpha_0 \Omega_0} + \gamma_2 \Omega_2 F_1. \]

Using now that

\[ F_1 \alpha_0 \Omega_0 = \lambda^2 - \lambda_0^{-2}, \]

we conclude that (16) holds.

From the equality of \( E^3 \) we obtain

\[ \delta \Omega_2 \Omega_2 - \alpha_0 \Omega_1 = \gamma_2 \Omega_2 - \beta_0 \Omega_0 \]

and we again conclude (16).

From the equality of \( E^2 \) we obtain

\[ \frac{\beta_1 \Omega_1}{\alpha_0 \Omega_0} + \frac{\alpha_1 \Omega_1}{\beta_0 \Omega_0} - 2 - \frac{\delta \Omega_2 \Omega_2 - \alpha_0 \Omega_1}{\delta_2 \alpha_0 \Omega_0} (\alpha_0 - \delta_0 \lambda_0^{-2}) \]

\[ = \frac{(\alpha_2 \lambda^2 - \beta_2)(\beta_1 \Omega_1 - \alpha_0 \Omega_0)}{\delta_2 \alpha_0 \Omega_0} - \frac{\alpha_2 \lambda^2}{\delta_2} - \frac{\delta_0 \lambda_0^{-2}}{\alpha_0} + \gamma_2 \Omega_2 F_2 + 2 \]

By using (16) and \( \alpha_2 \lambda^2 - \beta_2 = \delta_2 - \gamma_2 \lambda^2_3 \) we obtain

\[ \frac{\beta_1 \Omega_1}{\alpha_0 \Omega_0} + \frac{\alpha_1 \Omega_1}{\beta_0 \Omega_0} - 1 - \frac{\gamma_2 \Omega_2}{\alpha_0 \beta_0 \Omega_0} (\alpha_0 - \delta_0 \lambda_0^{-2}) \]

\[ = \frac{(\delta_2 - \gamma_2 \lambda^2_3)(\beta_1 \Omega_1 - \alpha_0 \Omega_0)}{\delta_2 \alpha_0 \Omega_0} - \frac{\alpha_2 \lambda^2}{\delta_2} + \gamma_2 \Omega_2 F_2 + 2 \]

So

\[ \frac{\alpha_0 \Omega_1 - \beta_0 \Omega_0}{\beta_0 \Omega_0 \gamma_2 \Omega_2} + \frac{\beta_1 \Omega_1 - \alpha_0 \Omega_0}{\alpha_0 \gamma_2 \Omega_2} - \frac{\alpha_0 - \delta_0 \lambda_0^{-2}}{\alpha_0 \beta_0 \Omega_0} \]

\[ = \frac{\lambda_2^2 \Omega_2 - \beta_2}{\gamma_2 \Omega_2 \alpha_0 \delta_2 \Omega_2} + \frac{\delta_2 - \alpha_2 \lambda^2}{\alpha_2 \gamma_2 \Omega_2} + F_2 \]

We conclude that

\[ F_0 = \frac{\alpha_0 \Omega_1 - \beta_0 \Omega_0}{\beta_0 \Omega_0 \gamma_2 \Omega_2} + \lambda_2^2 \frac{\beta_1 \Omega_1 - \beta_0 \Omega_0}{\alpha_0 \delta_2 \Omega_2} + \frac{\lambda_0 \Omega_0 \lambda_0^{-2}}{\alpha_0 \beta_0 \Omega_0} + \frac{2 \lambda_2^2 - \delta_2 - \gamma_2 \lambda^2_3}{\gamma_2 \delta_2 \Omega_2}, \]

thus proving (13). The proof of (14) is similar.

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