Fourier Expansion, Integral Representation and Explicit Formula at Rational Arguments of the Tangent Polynomials of Higher-Order

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Abstract. In this paper, Fourier series expansion of Tangent polynomials are derived and the integral representation and explicit formula at rational arguments of these polynomials are established.

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1. Introduction

For \( r \in \mathbb{N} \), the higher-order tangent polynomials, \( T_r^n(x) (n \geq 0) \), are defined by the following generating function (see [1])

\[
\left( \frac{2}{e^{2t} + 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} T_r^n(x) \frac{t^n}{n!}, \quad |2t| < \pi.
\]

When \( r = 1 \), the above equation gives the generating function for the classical tangent polynomials (see [2]).

The study of tangent polynomials has become an interesting area for many mathematicians for they possess significant properties that can be found in the field of mathematics and physics (see [3], [4]). Analogues, explicit identities and symmetric properties for tangent polynomials are derived in (see [5], [6], [7]).

In this paper, the researchers derive the Fourier expansion and integral representation of the tangent polynomials of order \( r, r \in \mathbb{Z}^+ \) and present an explicit formula of these polynomials at rational arguments using the method of Luo [8].

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2. Fourier Expansions for Tangent polynomials of Higher-order

In this section, we give Fourier expansion for tangent polynomials of higher order.

**Theorem 2.1.** For \(0 \leq x \leq 1\),

\[
T^r_n(x) = 2 \cdot n! \left(\frac{2}{\pi}\right)^{r+n} \sum_{k=0}^{\infty} \sum_{j=0}^{r-1} (-1)^j \binom{r + n - j - 1}{r - j - 1} \frac{j!}{n!} \cos [(2k + 1)\pi x/2 - (r + n - j)\pi/2] \frac{B^r_j(x)}{(2k + 1)^{r+n-j}},
\]

where \(B^r_j\) denotes the Bernoulli polynomials of order \(r\) defined by

\[
\left(\frac{w}{e^w - 1}\right)^r e^{tw} = \sum_{j=0}^{\infty} B^r_j(x) \frac{w^n}{n!}.
\]

**Proof.** For \(r \geq 2\),

\[
\text{Res } (f(t), t = t_k) = \frac{1}{(r-1)!} \lim_{t \to t_k} \frac{d^{r-1}}{dt^{r-1}} (t - t_k)^r \left(\frac{2}{e^{2t} + 1}\right)^r e^{xt} \frac{t^{n+1}}{n!}.
\]

Consider the function

\[
(t - t_k)^r \left(\frac{2}{e^{2t} + 1}\right)^r e^{xt} \frac{t^{n+1}}{n!} = 2^r (t - t_k)^r e^{xt} \frac{t^{n+1}}{n!}.
\]

Writing \((e^{2t} + 1)^r\) as

\[
(e^{2t} + 1)^r = (-1)^r (e^{2t}(-1) - 1)^r = (-1)^r (e^{2t} - e^{-2t})^r = (-1)^r (e^{2(t-t_k)} - 1)^r,
\]

and since \(e^{-2t} = e^{-2(2k+1)\pi i} = e^{-(2k+1)\pi i} = -1\), we have

\[
(t - t_k)^r \left(\frac{2}{e^{2t} + 1}\right)^r e^{xt} \frac{t^{n+1}}{n!} = \frac{2^r(t - t_k)^r}{n!} \frac{e^{xt}}{t^{n+1}} = (-1)^r \left(\frac{2(t - t_k)}{e^{2(t-t_k)} - 1}\right)^r \frac{e^{xt}}{t^{n+1}} = (-1)^r \left(\frac{\sum_{n=0}^{\infty} B_n^r (2(t - t_k))^n}{n!}\right) e^{xt} \frac{1}{t^{-(n+1)}}.
\]

where \(B_n^r\) denotes the Bernoulli numbers of order \(r\) defined by the generating function.
To get the derivative, applying the Leibniz Rule yields

\[
\frac{d^{r-1}}{dt^{r-1}} \left( (t - t_k)^r \left( \frac{2}{e^{2t} + 1} \right)^r e^{xt} \right) = \frac{d^{r-1}}{dr^{r-1}} \left\{ (-1)^r \left( \sum_{n=0}^{\infty} B_n^r \frac{(2(t - t_k))^n}{n!} \right) e^{xt} t^{-(n+1)} \right\} \\
= (-1)^r \frac{d^{r-1}}{dr^{r-1}} \left\{ \left( e^{xt} \sum_{n=0}^{\infty} B_n^r \frac{(2(t - t_k))^n}{n!} \right) t^{-(n+1)} \right\} \\
= (-1)^r \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{d^{r-1-j}}{dt^{r-1-j}} t^{-(n+1)} \cdot \frac{dj}{dt^j} \left( e^{xt} \sum_{n=0}^{\infty} B_n^r \frac{(2(t - t_k))^n}{n!} \right),
\]

\[
\frac{d^j}{dt^j} \left( e^{xt} \sum_{n=0}^{\infty} B_n^r \frac{2^n (t - t_k)^n}{n!} \right) \\
= \sum_{l=0}^{j} \binom{j}{l} x^j e^{xt} \sum_{n=l}^{\infty} B_n^r \frac{2^n n!}{n!} (n)_l (t - t_k)^{n-l} \\
= e^{xt} \sum_{l=0}^{j} \binom{j}{l} x^j \sum_{n=l}^{\infty} 2^n B_n^r \frac{(t - t_k)^{n-l}}{(n - l)!},
\]

\[
\frac{d^{r-1}}{dt^{r-1}} \left( (t - t_k)^r \left( \frac{2}{e^{2t} + 1} \right)^r e^{xt} \right) \\
= (-1)^r \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{d^{r-1-j}}{dt^{r-1-j}} t^{-(n+1)} \\
\times e^{xt} \sum_{l=0}^{j} \binom{j}{l} x^j \sum_{n=l}^{\infty} 2^n B_n^r \frac{(t - t_k)^{n-l}}{(n - l)!}.
\]

Thus,

\[
Res \left( f(t), t = t_k \right) = \frac{1}{(r-1)!} \lim_{t \to t_k} (-1)^r \sum_{j=0}^{r-1} \binom{r-1}{j} \frac{d^{r-1-j}}{dt^{r-1-j}} t^{-(n+1)} \\
\times \lim_{t \to t_k} e^{xt} \sum_{l=0}^{j} \binom{j}{l} x^j \sum_{n=l}^{\infty} 2^n B_n^r \frac{(t - t_k)^{n-l}}{(n - l)!}.
\]
Note that $B_n^r \frac{(t-t_k)^{n-r}}{(n-j)!} \to 0$ as $t \to t_k$ except when $n = l$. This gives

$$
\text{Res} \left( f(t), t = t_k \right) = \frac{1}{(r-1)!} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-j}(n+r-1-j)_{r-1-j} t_k^{-(n+r-j)}
$$

$$
\times e^{xt_k} \sum_{l=0}^{j} \binom{j}{l} x^{j-l} B_l^r
$$

$$
= \frac{(-1)^r}{(r-1)!} \sum_{j=0}^{r-1} \binom{r-1}{j} (-1)^{r-j}(n+r-1-j)_{r-1-j} t_k^{-(n+r-j)}
$$

$$
\times e^{xt_k} \sum_{l=0}^{j} \binom{j}{l} x^{j-l} B_l^r
$$

$$
= \sum_{j=0}^{r-1} (-1)^{j} 2^{j} \binom{n+r-1-j}{r-1-j} t_k^{-n-r} \sum_{l=0}^{j} \binom{j}{l} x^{j-l} / 2^{j-l} B_l^r.
$$

Recall that $B_j^r \left( \frac{x}{2} \right) = \sum_{l=0}^{j} \binom{j}{l} B_l^r \left( \frac{x}{2} \right)^{j-l}$. Thus,

$$
\text{Res} \left( f(t), t = t_k \right) = \sum_{j=0}^{r-1} (-1)^{j} 2^{j} \binom{n+r-1-j}{r-1-j} \frac{t_k^{-n-r}}{j!} e^{xt_k} B_j^r \left( \frac{x}{2} \right)
$$

$$
= \sum_{j=0}^{r-1} (-1)^{j} 2^{j} \binom{n+r-1-j}{r-1-j} \frac{B_j^r \left( \frac{x}{2} \right)}{j!} t_k^{-n-r-j}
$$

$$
= \sum_{j=0}^{r-1} (-1)^{j} 2^{j} \binom{n+r-j-1}{r-j-1} \frac{B_j^r \left( \frac{x}{2} \right)}{j!} t_k^{-n-r-j}.
$$

Taking $t_k = \frac{1}{2} (2k+1) \pi i$, we get

$$
\text{Res} \left( f(t), t = t_k \right) = \sum_{j=0}^{r-1} (-1)^{j} 2^{j} \binom{n+r-j-1}{r-j-1} B_j^r \left( \frac{x}{2} \right) \frac{e^{\frac{1}{2} (2k+1) \pi i x}}{(2k+1) \pi i} t_k^{-n-r-j}
$$

$$
= \frac{1}{(\frac{1}{2} \pi i)^{r+n}} \sum_{j=0}^{r-1} (-1)^{j} 2^{j} \binom{n+r-j-1}{r-j-1} B_j^r \left( \frac{x}{2} \right) \frac{e^{\frac{1}{2} (2k+1) \pi i x}}{(2k+1)^{r+n-j}}.
$$

This gives

$$
T_n^r(x) = n! \left( \frac{2}{\pi i} \right)^{r+n} \sum_{k \in \mathbb{Z}} \left( \sum_{j=0}^{r-1} (-1)^j \left( \frac{\pi i}{j!} B_j^r \left( \frac{x}{2} \right) \right) \frac{e^{\frac{1}{2} (2k+1) \pi i x}}{(2k+1)^{r+n-j}} \right).
$$
Now, from (3), we look at
\[ i^{-(r+n-j)} \sum_{k \in \mathbb{Z}} e^{(2k+1) \frac{n}{2} x} = i^{-(r+n-j)} \sum_{k=0}^{\infty} e^{(2k+1) \frac{n}{2} x} + (-1)^{r+n-j} \sum_{k=0}^{\infty} e^{-(2k+1) \frac{n}{2} x} \]
\[ = \sum_{k=0}^{\infty} e^{((2k+1)x/2-(r+n-j)/2)\pi i} + e^{-(2k+1)x/2-(r+n-j)/2)\pi i} \]
\[ = \sum_{k=0}^{\infty} 2 \cos [(2k+1)\pi x/2-(r+n-j)/2] \]
\[ = 2 \sum_{k=0}^{\infty} \cos [(2k+1)\pi x/2-(r+n-j)/2] \]

Replacing (4) with (5) in (3), we get the desired formula (2).

### 3. Integral representation for Tangent polynomials of Higher-order

In this section, we establish an integral representation for tangent polynomials of higher order.

**Theorem 3.1.** For \( n \in \mathbb{N} \), \( r \geq 2 \), and \( 0 \leq \Re(x) \leq 1 \),
\[
T^{r}_{n}(x) = 2^{r+n} \sum_{j=0}^{r-1} \frac{(-1)^j}{j!} \cdot \frac{B^{r}_{j}(\frac{x}{2})}{(r-j-1)!} \left\{ \int_{0}^{\infty} \frac{e^{\pi t} \cos \left(\frac{\pi x}{2} - \frac{(r+n-j)\pi}{2}\right)}{\cosh(2\pi t) - \cos(\pi x)} \right. \]
\[ \left. - \int_{0}^{\infty} \frac{e^{-\pi t} \cos \left(\frac{\pi x}{2} + \frac{(r+n-j)\pi}{2}\right)}{\cosh(2\pi t) - \cos(\pi x)} \right \} \]
\[ = 2^{r+n+1} \sum_{j=0}^{r-1} \frac{(-1)^j B^{r}_{j}(\frac{x}{2})}{(r-j-1)!} \cdot \frac{(r+n-j-1)!}{j!} \cdot \frac{\pi^{j} B^{r}_{j}(\frac{x}{2})}{(2k+1)^{r+n-j}} \]
\[ \times \cos \left[\left(2k+1\right)\frac{\pi x}{2} - \left(r+n-j\right)\frac{\pi}{2}\right] \]
\[ = 2^{r+n+1} \sum_{j=0}^{r-1} \frac{(-1)^j B^{r}_{j}(\frac{x}{2})}{(r-j-1)!} \cdot \frac{(r+n-j-1)!}{j!} \cdot \frac{\cos \left[\left(2k+1\right)\frac{\pi x}{2} - \left(r+n-j\right)\frac{\pi}{2}\right] \sum_{k=0}^{\infty} \cos \left(2k+1\right)\frac{\pi x}{2} - \left(r+n-j\right)\frac{\pi}{2}\right]}{(2k+1)^{r+n-j}} \]
\[ = \sum_{j=0}^{r-1} \frac{(-1)^j B^{r}_{j}(\frac{x}{2})}{(r-j-1)!} \cdot \frac{(r+n-j-1)!}{j!} \cdot \frac{\cos \left(\frac{\pi x}{2} - \frac{(r+n-j)\pi}{2}\right)}{\cosh(2\pi t) - \cos(\pi x)} \]
\[ \times \cos \left[\left(2k+1\right)\frac{\pi x}{2} - \left(r+n-j\right)\frac{\pi}{2}\right] \]
\[ = \sum_{j=0}^{r-1} \frac{(-1)^j B^{r}_{j}(\frac{x}{2})}{(r-j-1)!} \cdot \frac{(r+n-j-1)!}{j!} \cdot \frac{\cos \left[\left(2k+1\right)\frac{\pi x}{2} - \left(r+n-j\right)\frac{\pi}{2}\right]}{(2k+1)^{r+n-j}} \]
We look at
\[
\frac{(r + n - j - 1)!}{\pi^{r+n-j}} \sum_{k=0}^{\infty} \cos \left(\frac{(2k+1)\pi x}{2} - (r + n - j)\pi/2 \right) \frac{(r + n - j - 1)!}{(2k+1)^{r+n-j}}.
\]

Applying the integral formula
\[
\int_0^\infty t^n e^{-at} \, dt = \frac{n!}{a^{n+1}},
\]
for \( n \geq 0 \) and \( \Re(a) > 0 \), then (8) becomes
\[
\frac{(r + n - j - 1)!}{\pi^{r+n-j}} \sum_{k=0}^{\infty} \cos \left(\frac{(2k+1)\pi x}{2} - (r + n - j)\pi/2 \right) \frac{(r + n - j - 1)!}{(2k+1)^{r+n-j}}
\]
\[
= \frac{1}{\pi^{r+n-j}} \sum_{k=0}^{\infty} \cos \left(\frac{(2k+1)\pi x}{2} - (r + n - j)\pi/2 \right) \left( \int_0^\infty t^{r+n-j-1} e^{-(2k+1)t} \, dt \right)
\]
\[
= \frac{1}{\pi^{r+n-j}} \int_0^\infty t^{r+n-j-1} \sum_{k=0}^{\infty} e^{-(2k+1)t} \cos \left(\frac{(2k+1)\pi x}{2} - (r + n - j)\pi/2 \right) \, dt
\]
\[
= \frac{1}{\pi^{r+n-j}} \int_0^\infty t^{r+n-j-1} \left\{ \cos \left(\frac{(r + n - j)\pi}{2} \right) \sum_{k=0}^{\infty} e^{-(2k+1)t} \cos \left(\frac{(2k+1)\pi x}{2} \right) + \sin \left(\frac{(2k+1)\pi x}{2} \right) \sum_{k=0}^{\infty} e^{-(2k+1)t} \sin \left(\frac{(2k+1)\pi x}{2} \right) \right\} \, dt
\]
\[
= \frac{1}{\pi^{r+n-j}} \left\{ \cos \left(\frac{(r + n - j)\pi}{2} \right) \sum_{k=0}^{\infty} e^{-(2k+1)t} \cos \left(\frac{(2k+1)\pi x}{2} \right) + \sin \left(\frac{(r + n - j)\pi}{2} \right) \sum_{k=0}^{\infty} e^{-(2k+1)t} \sin \left(\frac{(2k+1)\pi x}{2} \right) \right\} \int_0^\infty t^{r+n-j-1} \, dt.
\]

By making use of
\[
\sum_{k=0}^{\infty} e^{-(2k+1)t} \sin \left(\frac{(2k+1)x}{2} \right) = \frac{\sin x \cosh t}{\cosh (2t) - \cos (2x)},
\]
and
\[
\sum_{k=0}^{\infty} e^{-(2k+1)t} \cos \left(\frac{(2k+1)x}{2} \right) = \frac{\cos x \sinh t}{\cosh (2t) - \cos (2x)},
\]
which may be deduced from
\[
\sum_{k=0}^{\infty} e^{(x-i)(2k+1)} = \frac{\cos x \sinh t + i \sin x \cosh t}{\cosh (2t) - \cos (2x)},
\]
for $t > 0$, (9) then becomes

$$\frac{(r + n - j - 1)!}{\pi^{r + n - j}} \sum_{k=0}^{\infty} \frac{\cos [(2k + 1)\pi x/2 - (r + n - j)\pi/2]}{(2k + 1)^{r + n - j}}$$

$$= \frac{1}{\pi^{r + n - j}} \int_{0}^{\infty} \left\{ \cos [(r + n - j)\pi/2] \frac{\cos \frac{\pi x}{2} \sinh t}{\cosh (2t) - \cos (\pi x)} + \sin [(r + n - j)\pi/2] \frac{\sin \frac{\pi x}{2} \cosh t}{\cosh (2t) - \cos (\pi x)} \right\} t^{r + n - j - 1} dt. \quad (10)$$

Applying the transformation $t = \pi t$, (10) becomes

$$\frac{(r + n - j - 1)!}{\pi^{r + n - j}} \sum_{k=0}^{\infty} \frac{\cos [(2k + 1)\pi x/2 - (r + n - j)\pi/2]}{(2k + 1)^{r + n - j}}$$

$$= \frac{1}{\pi^{r + n - j}} \int_{0}^{\infty} \left\{ \cos [(r + n - j)\pi/2] \frac{\cos \frac{\pi x}{2} \sinh \pi t}{\cosh (2\pi t) - \cos (\pi x)} + \sin [(r + n - j)\pi/2] \frac{\sin \frac{\pi x}{2} \cosh \pi t}{\cosh (2\pi t) - \cos (\pi x)} \right\} \pi^{r + n - j} t^{r + n - j - 1} dt$$

$$= \int_{0}^{\infty} \left\{ \cos [(r + n - j)\pi/2] \frac{\cos \frac{\pi x}{2} (e^{\pi s} - e^{-\pi s})}{2 \cosh (2\pi t) - \cos (\pi x)} + \sin [(r + n - j)\pi/2] \frac{\sin \frac{\pi x}{2} (e^{\pi s} + e^{-\pi s})}{2 \cosh (2\pi t) - \cos (\pi x)} \right\} t^{r + n - j - 1} dt$$

$$= \frac{1}{2} \int_{0}^{\infty} \left\{ e^{\pi s} \cos [(r + n - j)\pi/2] \frac{\cosh (2\pi t) + \cos [(r + n - j)\pi/2] \sin \frac{\pi x}{2}}{\cosh (2\pi t) - \cos (\pi x)} - e^{-\pi s} \frac{\cosh (2\pi t) + \cos [(r + n - j)\pi/2] \sin \frac{\pi x}{2}}{\cosh (2\pi t) - \cos (\pi x)} \right\} t^{r + n - j - 1} dt$$

$$= \frac{1}{2} \int_{0}^{\infty} e^{\pi s} \cos \left[ \frac{\pi x}{2} - (r + n - j)\frac{\pi}{2} \right] - e^{-\pi s} \cos \left[ \frac{\pi x}{2} + (r + n - j)\frac{\pi}{2} \right] \cosh (2\pi t) - \cos (\pi x) t^{r + n - j - 1} dt. \quad (11)$$

Applying (11) to (7), we get the desired formula (6).

4. Explicit formula for Tangent polynomials of Higher-order at rational arguments

In this section, we obtain an explicit formula for tangent polynomials of higher order at rational arguments by applying the Fourier expansion (2). Here let $\mathbb{Z}_0 = \{0, -1, -2, \cdots \}$ denote the set of nonpositive integers.

**Theorem 4.1.** For $n, q \in \mathbb{N}$ and $p \in \mathbb{Z}$,

$$T_n^p \left( \frac{2p}{q} \right) = \frac{2 \cdot n!}{(q\pi)^{r+n}} \sum_{j=0}^{r-1} (-1)^j \binom{r+n-j-1}{r-j-1} \frac{(2q\pi)^j}{j!} B_j^r \left( \frac{p}{q} \right)$$
× \sum_{l=1}^{q} \zeta \left( r + n - j, \frac{2l - 1}{2q} \right) \cos \left[ \frac{(2l - 1)p\pi}{q} - \frac{(r + n - j)\pi}{2} \right], \quad (12)

where

\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)^s}, \quad (13)

for \Re(s) > 1 and a \notin \mathbb{Z}_0^- is Hurwitz zeta function.

Proof. We look at

\sum_{k=0}^{\infty} \cos \left[ \frac{(2k + 1)p\pi}{q} - \frac{(r + n - j)\pi}{2} \right] \frac{(2k + 1)^{r+n-j}}{(2k + 1)^{r+n-j}}. \quad (14)

Replacing \( k \) with \( k - 1 \):

\sum_{k=0}^{\infty} \cos \left[ \frac{(2k + 1)p\pi}{q} - \frac{(r + n - j)\pi}{2} \right] = \sum_{k=1}^{\infty} \cos \left[ \frac{(r + n - j)\pi}{2} - \frac{(2k - 1)p\pi}{q} \right] \frac{(2k - 1)^{r+n-j}}{(2k - 1)^{r+n-j}}. \quad (15)

Applying the elementary series identity

\sum_{k=1}^{\infty} f(k) = \sum_{l=1}^{q} \sum_{k=0}^{\infty} f(qk + l), \quad q \in \mathbb{N},

(15) becomes

\sum_{k=0}^{\infty} \cos \left[ \frac{(2k + 1)p\pi}{q} - \frac{(r + n - j)\pi}{2} \right] \frac{(2k + 1)^{r+n-j}}{(2k + 1)^{r+n-j}}

= \sum_{l=1}^{q} \sum_{k=0}^{\infty} \cos \left[ \frac{(r + n - j)\pi}{2} - \frac{(2kq + 2l - 1)p\pi}{q} \right] \frac{(2kq + 2l - 1)^{r+n-j}}{(2kq + 2l - 1)^{r+n-j}}

= \sum_{l=1}^{q} \sum_{k=0}^{\infty} \cos \left[ \frac{(r + n - j)\pi}{2} - \frac{(2l - 1)p\pi}{q} - \frac{(r + n - j)\pi}{2} \right] \frac{1}{(2q)^{r+n-j}} \frac{1}{\left( k + \frac{2l-1}{2q} \right)^{r+n-j}}. \quad (16)

Setting \( x = 2p/q \), (16) becomes

\sum_{k=0}^{\infty} \cos \left[ \frac{(2k + 1)p\pi}{q} - \frac{(r + n - j)\pi}{2} \right] \frac{(2k + 1)^{r+n-j}}{(2k + 1)^{r+n-j}}

= \sum_{l=1}^{q} \sum_{k=0}^{\infty} \cos \left[ \frac{(r + n - j)\pi}{2} - \frac{(2l - 1)p\pi}{q} - 2\pi(pk) \right] \frac{1}{(2q)^{r+n-j}} \frac{1}{\left( k + \frac{2l-1}{2q} \right)^{r+n-j}}
\[
\begin{align*}
\sum_{k=0}^{\infty} \cos \left[ (2k+1)\frac{\pi x}{2} - (r + n - j)\frac{\pi}{2} \right] \\
= \frac{1}{(2q)^{r+n-j}} \sum_{l=1}^{q} \sum_{k=0}^{\infty} \cos \left[ (r + n - j)\frac{\pi}{2} - (2l-1)\frac{p\pi}{q} \right] \frac{1}{k + \frac{2l-1}{2q}}^{r+n-j}.
\end{align*}
\]

By (13), (17) becomes

\[
\begin{align*}
\sum_{k=0}^{\infty} \cos \left[ (2k+1)\frac{\pi x}{2} - (r + n - j)\frac{\pi}{2} \right] \\
= \frac{1}{(2q)^{r+n-j}} \sum_{l=1}^{q} \cos \left[ \frac{(2l-1)p\pi}{q} - \frac{(r + n - j)\pi}{2} \right] \zeta \left( r + n - j, \frac{2l-1}{2q} \right).
\end{align*}
\]

Replacing (14) with (18) in (2), we obtain the desired formula (12).

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