Abstract

Starting from a Born–Oppenheimer decomposition of the Wheeler-DeWitt equation for the quantum cosmology of the matter–gravity system, we have performed a Wigner–Weyl transformation and obtained equations involving a Wigner function for the scale factor and its conjugate momentum. This has allowed us to study in more detail than previously the approach to the classical limit of gravitation and the way time emerges in such a limit. To lowest order we reproduce the Friedmann equation and the previously obtained equation for the evolution of matter. We also obtain expressions for higher order corrections to the semi-classical limit.

Keywords: quantum cosmology, inflation, classical limit

1. Introduction

One of the main achievements of inflationary cosmology is the explanation of the origin of the large scale structure of the present Universe arising due to the stretching of the quantum fluctuations in the very early Universe [1]. This success [2] requires an investigation of the physics of the early Universe and leads one to study quantum cosmology, that is the treatment of the Universe as a unique quantum object governed by the laws of general relativity and quantum theory. The mathematical structure of quantum gravity and cosmology is that of a theory with first class constraints, having re-parametrisation invariance or invariance with respect to space-time diffeomorphisms. The main goal of quantum cosmology is the description of the quantum state of the Universe. Such a state should satisfy the Wheeler–DeWitt (WDW) equation [3] which
results from the application of the Dirac quantisation procedure to the Universe. The solutions of the WDW equation involving a limited number of homogeneous degrees of freedom are called mini-superspace models. One may then consider inhomogeneous degrees of freedom on such a homogeneous background. A further consequence of the reparametrization invariance is that in the resulting WDW equation, time (or time derivatives) does not appear. Clearly if one considers as a starting point the Universe as a quantum system it must be possible to re-obtain the usual classical Einstein and quantum matter (Schwinger–Tomonaga) equations with, as a boon, quantum gravitational corrections leading to observable (?) effects for sufficiently strong gravitational fields as occur at the beginning of inflation or near black hole horizons. This last step, however, does not appear to be immediate and is closely related to the way the classical limit arises.

Depending on the system being considered one has diverse approaches to the classical limit, see e.g. [4] and references therein. One may consider the particle (geometrical optics limit for light) or eikonal limit (WKB) in which, for a path integral representation of the wave function, neighbouring paths will tend to yield cancelling contributions on account of the rapid variation of the phase associated with the exponential of the (effective) action. An exception to this rule occurs at stationary points of the exponent and the associated paths are related to classical trajectories. In another approach the classical limit emerges through the use of the correspondence principle in which one considers large quantum numbers (including the use of coherent states). An example of this is the quantum harmonic oscillator with a large value of the quantum number \( n \) (level of the one-dimensional oscillator).

A further complication in inflationary cosmology arises because of the presence of matter. As before we shall apply the Born–Oppenheimer approach [5] to the composite quantum matter-gravity system [6, 7]. Such an approach is plausible since gravity is characterized by the Planck mass which is much greater than the usual matter mass. This allows one to suitable factorize the wave function of the Universe (composite system) into a purely gravitational part in which, to lowest order, gravity is driven by average matter Hamiltonian and a matter part wherein, again to lowest order, matter follows gravity adiabatically and time is seen to emerge through a semiclassical approximation to the gravitational wave function. We wish to again study, in such a context (however see also refs. [8, 9]), the semiclassical limit in order to examine the emergence of time, however, the method we shall use, in the present paper, is associated with the use of the Wigner function [10]. Such an approach was introduced in order to study quantum corrections to classical statistical mechanics and its goal was to link the wave function to a probability distribution in phase space [11]. Our approach will consist of studying the Wigner–Weyl transformations [12] for the gravitational part of the equations obtained in our BO approach with the aim of studying the semiclassical limit and the associated quantum corrections.

The Wigner function was previously used in a quantum cosmological context in [13]. Since the Universe is a closed system these papers studied the proposal that correlations of coordinates and their conjugate momenta could be predicted from peaks of the associated Wigner function. This is of interest since the Wigner function itself is a joint quasi-probability distribution of them. One may then recover classical behaviour from the prediction of a classical correlation in the quantum system. Classical general relativity can then be recovered from an (semiclassical) approximation to the Wigner function. If one considers an alternative approach to the semiclassical limit to the quantum Universe, such as a WKB approximation, one does not have any correlation between variables and their conjugate momenta. Even if the phase of such an approximation does lead to a classical Hamilton–Jacobi equation of motion it is clear that the sum of two such solutions will exhibit interference, a typical quantum effect. It is for the above motives that we decided to examine a Wigner function approach to the matter–gravity
system which we have been studying using a Born–Oppenheimer approach. To the best of our knowledge introduction of matter was not examined in previous approaches. Only one example has been previously given for the gravitational Wigner function from the WDW equation which consists of a simple model having the square of the scale factor \( a \) as dynamical variable and a cosmological constant. Such a choice of dynamical variable does not distinguish between a positive and negative (unphysical). On the other hand we shall use \( a \) as dynamical variable and obtain general coupled equations for the matter wave function and gravitational Wigner function. These equations can be solved to any order in Planck’s constant and we shall verify that to lowest order in Planck’s constant our previous results employing the simple WKB approximation are reproduced. Finally the general gravitational Wigner function is also obtained to next to leading order (NLO) and the procedure can be utilised to obtain even higher orders. In principle, for a particular matter–gravity system, on using the results obtained for the Wigner function one may determine when the quantum to classical transition occurs for gravitation during the expansion of the Universe. Or in more simple terms see when classical time evolution emerges [6]. The procedure we shall illustrate can also be generalised (as previously done for the WKB case) to inflationary perturbations. Indeed, concerning these perturbations, recently a detailed analysis of the anisotropies of the cosmic microwave background radiation from the point of view their ‘quantumness’ was done in the paper [14]. The new notion of quantum discord [15, 16] in quantum information theory was used. At the same time, the traditional tools of the Wigner function and the Weyl transformation were also employed in [14]. Yet another, more direct, study of the transition of the fluctuations of the early Universe from quantum to classical employing the Wigner formalism has been recently performed [17]. Thus, we see that the application of the Wigner function is still one of the best tools to analyse the quantum–classical relations and our approach and results will also effect these latter applications. Finally let us add that the Wigner function approach has also been recently employed in the context of loop quantum cosmology. Here the situation is more complicated since loop gravity is not quantised canonically but is quantised through a so-called polymer representation. This essentially means that rather than postulating canonical commutation relations between a coordinate and its conjugate momentum, thus leading to a representation of the momentum as a derivative with respect to the coordinate, one replaces the conjugate momentum by translations (or a limit thereof) with respect to the coordinate. Nonetheless several approaches have been suggested [18].

To end this section we observe that perhaps our discussion of the value of the Wigner function in the context of the classical limit in quantum cosmology has been (necessarily) brief, but on the other hand the approach has been studied in depth in the previous papers which provided us with the motivation for the present manuscript.

### 2. Born–Oppenheimer formalism

We shall consider the inflaton–gravity system which is described by the following action (see for example [19])

\[
S = \int d\eta \left[ -\frac{M_p^2}{2} a^2 + \frac{a^2}{2} \left( \phi^2 - 2V(\phi)a^2 \right) \right] + S_{MS},
\]

where \( M_p \) is the Planck mass, \( a \) is the scale factor, \( \eta \) is the conformal time, \( \phi \) is the homogenous part of the inflaton, \( S_{MS} \) is the action of the Mukhanov–Sasaki field [20] (which we shall not use in the present manuscript) and we consider a flat Robertson–Walker metric. The Hamiltonian
for the system then is

\[ H = -\frac{\pi_a^2}{2M_P^2} + \frac{\pi_\phi^2}{2a^2} + a^4V(\phi) + H_{MS}. \]  

(2)

where \( \pi_a = -M_P^2a' \) and \( \pi_\phi = a^2\phi' \) and \( H_{MS} \) is the Mukhanov–Sasaki contribution which we shall henceforth neglect. The canonical quantisation of the Hamiltonian constraints leads to the following WDW equation for the wave function of the Universe (gravity plus matter)

\[ \left[ \frac{\hbar^2}{2M_P^2} \frac{\partial^2}{\partial a^2} - \frac{\hbar^2}{2a^2} \frac{\partial^2}{\partial \phi^2} + a^4V(\phi) \right] \Psi(a, \phi) \equiv \left[ \frac{\hbar^2}{2M_P^2} \frac{\partial^2}{\partial a^2} + H_\phi \right] \Psi(a, \phi) = 0 \]  

(3)

which constitutes the starting point for our considerations.

Finding the general solution to the WDW equation (3) even in the absence of perturbations is a very complicated task due to the interaction between matter and gravity. A set of approximate solutions can be found in the BO approach [8]. The BO approach was originally introduced in order to simplify the Schrödinger equation of complex atoms and molecules and has been applied successfully to the inflaton gravity system [7]. It consist of factorizing the wave function \( \Psi(a, \phi) \) of the Universe into a product

\[ \Psi(a, \phi) = \psi(a)\chi(a, \phi), \]  

(4)

where \( \chi \) is normalized and not further separable. Coupled equations of motion for \( \psi \) and \( \chi \) may then be obtained

\[ \left( \frac{\hbar^2}{2M_P^2}\partial_a^2 + \langle H_\phi \rangle \right) \tilde{\psi} = -\frac{\hbar^2}{2M_P^2} (\partial_a^2 \tilde{\psi}), \]  

(5)

\[ \frac{\hbar^2}{M_P^2} \partial_a \tilde{\psi} \partial_a \tilde{\chi} + \tilde{\psi} \left( H_\phi - \langle H_\phi \rangle \right) \tilde{\chi} = \frac{\hbar^2}{2M_P^2} \tilde{\psi} \left[ (\partial_a^2) - \partial_a^2 \right] \tilde{\chi}, \]  

(6)

where \( \langle \hat{O} \rangle = \langle \hat{\chi}| \hat{O}| \hat{\chi} \rangle \), with \( \langle \hat{\chi}| \hat{\chi} \rangle = 1 \) and

\[ \psi = \tilde{\psi} e^{-i\mathcal{A} a}, \quad \chi = \tilde{\chi} e^{i\mathcal{A} a}, \quad \mathcal{A} = -i\langle \hat{\chi}| \partial_a |\hat{\chi} \rangle. \]  

(7)

The right hand sides (rhs) of equations (5) and (6) describe non-adiabatic transitions and are generally associated with quantum gravitational effects.

In order to discuss the quantum to classical transition for the gravitational sector and correspondingly define the (classical) time evolution for matter, one must consider the solution of the gravity equation and evaluate the probability current associated with it [8]. This approach has been discussed in several papers [7, 9]. Indeed when the gravity equation cannot be solved exactly, which is often the case, the WKB approach can be used. On considering the \( \hbar \to 0 \) limit one then obtains the semiclassical gravitational wave-function (in particular see [8]) and the Friedmann equation from the gravitational equation (5). On then substituting into the matter equation and keeping the contributions to order \( \hbar^2 \) one is finally led to a matter equation with quantum gravitational corrections included. Such corrections include the non-adiabatic effects and the quantum fluctuations arising from the introduction of time. However we shall here consider a different approach which consists of introducing the Wigner function for the gravitational sector in order to study how time then emerges.
2.1. Wigner function for gravity

The equation for the gravitational wave function is (5). One can define the Wigner function associated with it as

\[
W(a, p) \equiv \int_{-\infty}^{+\infty} d\xi e^{ip\xi} \tilde{\psi}_+^* \tilde{\psi}_+ ,
\]

where \( \tilde{\psi}_+ = \tilde{\psi}(a_+) \) and \( a_+ = a + s/2 \). Let us note that the gravitational wave function has support only when its argument is positive and therefore for \( a \) fixed \( s \) only varies in the interval \([-2a, 2a]\). One can, in principle, solve the equation for the gravitational wave function and then calculate the integral (8) to obtain the corresponding Wigner function. Apart from few particular cases, this procedure requires a series of approximations to first solve (5) and then calculate the integral (8). Let us note that, in order to evaluate the integral (8) one needs an accurate approximation to the gravitational wave function over the whole interval of integration, i.e., for values of its argument in \([0, 2a]\). In the large or small \( a \) limits the gravitational equation often has a simple form and an approximate solution may be easily guessed but this is seldom the case over the whole interval \([0, 2a]\) with \( a \) large. We shall briefly return to this point in the conclusions.

It then appears more convenient to obtain the exact equations satisfied by (8) and afterwards use some approximation scheme to obtain the Wigner function of the system directly from them.

On substituting \( a \to a_+ \) into (5), multiplying by \( \exp (ip\xi/\hbar) \tilde{\psi}_+^* \) and integrating over \( s \) one has

\[
\int_{-\infty}^{+\infty} d\xi e^{ip\xi} \tilde{\psi}_+^* \left[ \frac{\hbar^2}{2Mp^2} \partial_2^2 + \langle \hat{H}_\phi \rangle_+ + \frac{\hbar^2}{2Mp^2} (\partial_2^2)_+^* \right] \tilde{\psi}_+ = 0 ,
\]

where

\[
\langle \hat{O} \rangle_\pm \equiv \langle \chi(a_\pm, \phi)|\hat{O}|\chi(a_\pm, \phi)\rangle
\]

and \( \partial_\pm \equiv \partial_{a_\pm} \).

Similarly, starting from the equation for \( \tilde{\psi}_+^* \) with \( a \to a_- \), multiplying by \( \exp (ip\xi/\hbar) \tilde{\psi}_+ \) and finally integrating over \( s \) one has

\[
\int_{-\infty}^{+\infty} d\xi e^{ip\xi} \tilde{\psi}_+ \left[ \frac{\hbar^2}{2Mp^2} \partial_2^2 + \langle \hat{H}_\phi \rangle_- + \frac{\hbar^2}{2Mp^2} (\partial_2^2)_-^* \right] \tilde{\psi}_+^* = 0 .
\]

The resulting equations (9) and (11) can now be summed and subtracted obtaining

\[
\int_{-\infty}^{+\infty} d\xi e^{ip\xi} \left[ \frac{\hbar^2}{2Mp^2} (\partial_2^2_+ \pm \partial_2^2_-) + \langle \hat{H}_\phi \rangle_+ \pm \langle \hat{H}_\phi \rangle_- \right. \\
+ \frac{\hbar^2}{2Mp^2} (\langle \partial_2^2 \rangle_+ \pm \langle \partial_2^2 \rangle_-^*) \left. \right] \tilde{\psi}_+^* \tilde{\psi}_+ = 0
\]

and we used the fact that \( a_+ \) and \( a_- \) are independent variables. The integrand of (12) can be rewritten in terms of \( a \) and \( s \). Let us note that \( \partial_2^2_+ + \partial_2^2_- = \langle \partial_2^2 \rangle + 4\langle \hat{O} \rangle /2 \) and \( \partial_2^1_+ - \partial_2^1_- = 2\partial_\phi \partial_\sigma \).

On remembering that the gravitational wave functions are zero on the boundaries, integration by parts leads to

\[
\int_{-\infty}^{+\infty} d\xi e^{ip\xi} \partial_2^0 \tilde{\psi}_+^* \tilde{\psi}_+ = \left( \frac{i}{\hbar} p \right)^a W(a, p)
\]
whereas
\[ \int_{-\infty}^{+\infty} d\sigma \hat{\Psi}_{\psi}^* \hat{\Psi}_{-} = (-i\hbar)^p \partial_p^p W(a, p). \] (14)

Thus on Taylor expanding the expectation values which appear in (12) for \( s \to 0 \) and using (14) one finally obtains
\[
\frac{\hbar^2}{4M_p^2} \partial_p^2 W = \frac{p^2}{M_p^2} W + \sum_{n=0}^{\infty} \left[ (-1)^n + 1 \right] \frac{(i\hbar)^n}{2^n n!} \frac{d^n}{da^n} (\partial_p^n W)
\]
\[ + \frac{\hbar^2}{2M_p^2} \sum_{n=0}^{\infty} \left[ (-i\hbar)^n \frac{d^n}{da^n} (\partial_a^n) + \text{c.c.} \right] \partial_p^n W = 0 \] (15)

and
\[
-\frac{i\hbar}{M_p^2} \partial_a W + \sum_{n=0}^{\infty} \left[ (-1)^n - 1 \right] \frac{(i\hbar)^n}{2^n n!} \frac{d^n}{da^n} (\partial_p^n W)
\]
\[ + \frac{\hbar^2}{2M_p^2} \sum_{n=0}^{\infty} \left[ (-i\hbar)^n \frac{d^n}{da^n} (\partial_a^n) - \text{c.c.} \right] \partial_p^n W = 0. \] (16)

the latter being a quantum Liouville equation. The Wigner function must satisfy the above equations simultaneously. Let us note that each equation may contain infinite derivatives w.r.t \( p \) unless the Taylor expansion of \( \langle H_p \rangle \) and \( \langle \partial_a^p \rangle \) contains a finite number of contributions. Since each contribution is associated with a different power of \( \hbar \) and, if \( \langle H_p \rangle \) and \( \langle \partial_a^p \rangle \) can be written as a finite series of powers of \( a \), then higher derivatives with respect to \( p \) are multiplied by decreasing coefficients for \( \hbar \to 0 \) and \( a \to \infty \).

In the classical limit, one must keep the leading contribution for each equation. Equation (15) then becomes
\[
\left[ -\frac{p^2}{M_p^2} + 2 \langle H_p \rangle \right] W = 0
\] (17)

which is satisfied by
\[
W_c(a, p) = \delta \left( -\frac{p^2}{M_p^2} + 2 \langle H_p \rangle \right),
\] (18)
i.e. when \( p^2 \) is equal to its classical value \( p_c^2 \equiv M_p^4 \lambda^2 = 2M_p^2 \langle H_p \rangle \) and this latter relation is the classical Friedmann equation. Equation (16), to LO, is
\[
-\frac{i\hbar}{M_p^2} \partial_a W - i\hbar \frac{d}{da} \partial_p W = 0
\] (19)

and is satisfied by any function \( W = W(-\frac{p^2}{M_p^2} + 2 \langle H_p \rangle) \), in agreement with (18).

2.2. The matter equation

Given the classical limit for the gravitational wave function one may now study the matter equation (6). In order to introduce the classical time one must replace \( a \to a_+ \), multiply by \( \hat{\psi}_+^* \) and integrate over \( s \). Finally one must integrate over \( p \) in an arbitrary integral \( \mathcal{I}_+ \) domain.
around one the two peaks of the Dirac delta (18) thus selecting either the contracting or the expanding phase of the Universe.

The matter equation then takes the following form

\[
\int \frac{dp}{\mathcal{I}_+} \int dx e^{i \pi^\mu \chi^+} \left[ \frac{\hbar^2}{M_p^2} \left( \partial_+ \tilde{\chi}^+ \right) \left( \partial_+ \tilde{\chi}^+ \right) + \tilde{\psi}^+ \left( H_{\phi,0} - \langle H_{\phi} \rangle_+ \right) \tilde{\chi}^+ \right]
+ \frac{\hbar^2}{2M_p^2} \tilde{\psi}^+ \left( \left( \partial^2_+ \right)_+ - \partial^2_+ \right) \tilde{\chi}^+ = 0, \tag{20}
\]

where the expanding branch has been chosen. Let us consider the first contribution in (20). If one Taylor expands the wave function for matter for \( s \rightarrow 0 \), then its derivative is

\[
\partial_+ \tilde{\chi}^+ = \left( \frac{1}{2} \partial_+ + \partial_0 \right) \left( \tilde{\chi}^+ + \partial_0 \tilde{\chi}^+ + \partial^2_0 \tilde{\chi}^+ \right) = \partial_0 \tilde{\chi}^+ + \partial_0^2 \tilde{\chi}^+ + \cdots \tag{21}
\]

while the derivative of the product of the gravitational wave functions is

\[
\partial_+ \left( \tilde{\psi}^+ \tilde{\psi}^+ \right) = \frac{1}{2} \partial_0 \left( \tilde{\psi}^+ \tilde{\psi}^+ \right) + \partial_0 \tilde{\psi}^+ \tilde{\psi}^+. \tag{22}
\]

On using the relations (13), (14) one then finds

\[
\int \frac{dp}{\mathcal{I}_+} \int dx e^{i \pi^\mu \chi} (\partial_0 \tilde{\chi}) \left( \tilde{\psi}^+ \tilde{\psi}^+ \right) = -\frac{\mathcal{P}}{\hbar} (\partial_0 \tilde{\chi}) W \sim h^{-1}, \tag{23}
\]

\[
\int \frac{dp}{\mathcal{I}_+} \int dx e^{i \pi^\mu \chi} \frac{1}{2} \partial_0 (\tilde{\psi}^+ \tilde{\psi}^+) = \frac{1}{2} (\partial_0 \tilde{\chi}) (\partial_0 W) \sim h^0, \tag{24}
\]

\[
\int \frac{dp}{\mathcal{I}_+} \int dx e^{i \pi^\mu \chi} (\partial^2_0 \tilde{\chi}) \frac{1}{2} s \partial_0 (\tilde{\psi}^+ \tilde{\psi}^+) = -\frac{1}{2} (\partial^2_0 \tilde{\chi}) (W + p \partial_0 W) \sim h^0, \tag{25}
\]

\[
\int \frac{dp}{\mathcal{I}_+} \int dx e^{i \pi^\mu \chi} (\partial^2_0 \tilde{\chi}) \frac{1}{2} s \partial_0 (\tilde{\psi}^+ \tilde{\psi}^+) = \frac{1}{4} (\partial^2_0 \tilde{\chi}) \left( -\frac{d^2 W}{dp \, da} \right) \sim h^1, \tag{26}
\]

and higher powers of \( s \) in (21) contribute higher powers of \( h \) to the first contribution in (20) and can be neglected.

The remaining contributions to equation (20) have the form

\[
\int \frac{dp}{\mathcal{I}_+} e^{i \pi^\mu \chi} \left( D_+ \tilde{\chi}_+ \right) \left( \tilde{\psi}^+ \tilde{\psi}^+ \right) = \left( D_+ \tilde{\chi} \right) W - \partial_0 \left( D_+ \tilde{\chi} \right) i \hbar \partial_0 W + \cdots \tag{27}
\]

where \( D \) generically indicates an \( a \) dependent operator acting on \( \tilde{\chi}, \ (D_+ \tilde{\chi}_+) \) is expanded in a Taylor series for \( s \) small and the ellipsis denote higher orders in \( \hbar \). The integral is finally simplified using (14) leading to a power series of \( h \) and derivative w.r.t. \( p \) of the Wigner function.

Let us now integrate each contribution with respect to \( p \). Since

\[
W_{\chi} = \delta \left( -\frac{p^2}{M_p^2} + 2\langle H_{\phi} \rangle \right) \left( p + \sqrt{2M_p^2 \langle H_{\phi} \rangle} \right) + \delta \left( p + \sqrt{2M_p^2 \langle H_{\phi} \rangle} \right) \left( p - \sqrt{2M_p^2 \langle H_{\phi} \rangle} \right), \tag{28}
\]
Equation (23) becomes
\[
- \int dp \frac{p^2}{\hbar} (\partial_a \bar{\chi}) W = - \frac{i}{2\hbar} \partial_a \bar{\chi}. \tag{29}
\]

The contribution (24) can be simplified by using the classical ‘Liouville’ equation (19) as follows
\[
\int dp \frac{1}{2} (\partial_a \bar{\chi}) (\partial_p W_{cl}) = - \frac{1}{2\hbar} \int dp \frac{1}{2} (\partial_a \bar{\chi}) \left( \frac{d<H_o>/da}{p} \partial_p W_{cl} \right) = - \partial_a \bar{\chi} \frac{d<H_o>/da}{2p} \bigg|_{I_+} - \int dp \partial_a \bar{\chi} \frac{d<H_o>/da}{2p^2} W_{cl} = - \partial_a \bar{\chi} \frac{d}{da} \frac{M_p^2}{4p^3} \frac{1}{2\hbar} \bigg|_{I_+}.
\tag{30}
\]
where \( p_{cl} = \sqrt{2M_p^2(H_o)} \) and the boundary term vanishes. The third contribution is
\[
\int dp \frac{1}{2} (\partial_a \bar{\chi}) (W_{cl} + p\partial_p W_{cl}) = - \frac{1}{4p_{cl}} \partial_a \bar{\chi} \left( 1 + pW_{cl}|_{I_+} - 1 \right) = 0. \tag{31}
\]
The fourth contribution is next to next to leading for \( \hbar \rightarrow 0 \) and is then neglected. Therefore only two of the four terms survive the integration with respect to \( p \). The remaining contributions in (20) have the form
\[
\int dp [ (D\chi) W_{cl} - \partial_a (D\chi) i\hbar \partial_p W_{cl} ] = \frac{(D\chi)}{2p_{cl}} - \partial_a \frac{\partial_p W_{cl}}{2p_{cl}}|_{I_+}, \tag{32}
\]
where the last contribution is the boundary term which being proportional to the Dirac delta vanishes.

Finally on summing the diverse contributions up to order \( \hbar^2 \) and multiplying by \( 2p_{cl} \) one obtains
\[
- \frac{i\hbar}{M_p} p_{cl} \partial_a \bar{\chi} - \frac{\hbar^2}{M_p^2} \frac{d}{da} \frac{\ln<H_o>/a}{\partial_a \bar{\chi}} + (\dot{H}_o - <H_o>) \bar{\chi} = \frac{\hbar^2}{2M_p^2} \left[ (\partial_a^2 - \partial_a^2) \bar{\chi} \right] \tag{33}
\]
and
\[
- \frac{i\hbar}{M_p} p_{cl} \partial_a \bar{\chi} = - \frac{i\hbar}{M_p} M_p^2 a' \partial_a \bar{\chi} = - i\hbar \partial_p \chi, \quad \text{with} \quad \eta \quad \text{the conformal time. We end this section by noting that (33) is exactly the same as was obtained by the previous approach on using the WKB solution for the gravitational equation (see for example [6]).}

3. NLO corrections in \( \hbar \)

In the previous Section we found that, to LO, the correct classical limit (Friedmann) is reproduced for \( \hbar \rightarrow 0 \). In this Section we are interested in evaluating the NLO corrections to the classical limit for the gravitational part of the system which will then have consequences on all matter evolution.

To the NLO, the equation (15) is
\[
(p^2 - p_{cl}^2) W = \frac{\hbar^2}{4} \left[ \partial_a^2 W - M_p^2 \frac{d^2<H_o>/da^2}{\partial_p^2} \partial_p W + 2 (\langle \partial_a^2 \rangle + \text{c.c.}) W \right] + \mathcal{O}(\hbar^4) \tag{34}
\]
and we shall again just consider the positive branch (expanding Universe case). When the classical limit ($\hbar \to 0$) is studied one is interested in calculating the solution of (34) in a small interval around the classical solution $p/p_\text{cl} \in [1 - \epsilon, 1 + \epsilon]$ and

$$ \left( p^2 - p_\text{cl}^2 \right) W = (p + p_\text{cl}) (p - p_\text{cl}) W = 2p_\text{cl} (p - p_\text{cl}) W + \mathcal{O} \left( \epsilon^2 \right). $$  

The quantum Liouville equation (16) and its derivative with respect to the scale factor can then be used to simplify equation (34). On retaining contributions up to $\hbar^2$ one finally obtains

$$ \frac{\hbar^2}{4} \left[ \frac{1}{2} \frac{d^2 p_\text{cl}}{da^2} \left( \frac{\partial p_\text{cl}}{p_\text{cl}} + \partial_y W \right) + \left( \frac{dp_\text{cl}}{da} \right)^2 \left( \frac{\partial_y W}{p_\text{cl}} - \partial_y^2 W \right) \right] $$

$$ + \left[ 2p_\text{cl} (p - p_\text{cl}) - \frac{\hbar^2}{2} \left( \partial_y^2 c + \text{c.c.} \right) \right] W = 0, $$  

where we replaced $p \to p_\text{cl}$ in the $\mathcal{O} (\hbar^2)$ part. This last equation (36) can be solved exactly in momentum space. On setting

$$ W(a, p) = \int_{-\infty}^{+\infty} dy \, e^{iyp} \tilde{W}(a, y), $$

one finds the following equation for the transformed Wigner function $\tilde{W}$

$$ \frac{\hbar^2}{4} \left[ p_\text{cl}^2 \frac{d^2 p_\text{cl}}{da^2} + 2 \left( \frac{dp_\text{cl}}{da} \right)^2 \right] \frac{i\hbar}{2} \left( \frac{d\tilde{W}}{p_\text{cl}} - \frac{\hbar^2}{4p_\text{cl}} \frac{d^2 p_\text{cl}}{da^2} \frac{\partial \tilde{W}}{y} \right) $$

$$ + 2i p_\text{cl} \frac{\partial \tilde{W}}{\partial y} - \left[ 2p_\text{cl}^2 + \frac{\hbar^2}{2} \left( \partial_y^2 c + \text{c.c.} \right) \right] \tilde{W} = 0 $$

which can be easily solved obtaining

$$ \tilde{W} = \tilde{W}_0 \exp \left\{ -\frac{\hbar^2}{8} \frac{d^2 p_\text{cl}}{da^2} \frac{iy^3}{3} - \frac{\hbar^2}{16} \left[ \frac{d^2 p_\text{cl}}{da^2} + \frac{2}{p_\text{cl}} \left( \frac{dp_\text{cl}}{da} \right)^2 \right] \frac{y^2}{p_\text{cl}} \right\} $$

$$ - \left[ p_\text{cl} + \frac{\hbar^2}{4p_\text{cl}} \left( \partial_y^2 c + \text{c.c.} \right) \right] iy . $$

The above expression can be transformed back so as to obtain the Wigner function with the $\mathcal{O} (\hbar^2)$ corrections included. Let us note that the Airy function $\text{Ai}(x)$ admits an integral representation [21] which, on shifting the integration variable and rescaling it properly, leads to the following relation

$$ \int_{-\infty}^{+\infty} dt \, e^{i \left[ \frac{2}{3} \left( \frac{t}{b} \right)^{3/2} + b t^{1/2} \right]} = \frac{2\pi}{b^{1/3}} \exp \left[ i \frac{c}{b} \left( \frac{2c^2}{3b} - d \right) \right] \text{Ai} \left[ \frac{d - \frac{c^2}{b}}{b^{1/3}} \right], $$

which can be used in (37) and (39) setting

$$ b = -\frac{\hbar^2}{8} \frac{d^2 p_\text{cl}}{da^2}, $$

$$ c = \frac{i\hbar^2}{16p_\text{cl}} \left[ \frac{d^2 p_\text{cl}}{da^2} + \frac{2}{p_\text{cl}} \left( \frac{dp_\text{cl}}{da} \right)^2 \right]. $$
\[ d = p - p_{cl} - \left[ \frac{\hbar^2}{4p_{cl}} \left( \partial_x^2 + \text{c.c.} \right) \right]. \] (43)

Let us observe that [22]

\[ \frac{1}{|\alpha|} \text{Ai} \left( \frac{x}{\alpha} \right) \xrightarrow{\alpha \to 0} \delta(x) + \frac{\alpha^3}{3} \frac{d^3 \delta(x)}{dx^3}. \] (44)

and to the LO and the \( h \to 0 \) limit

\[
W(a, p) \simeq \frac{2\pi}{b^{1/3}} \text{Ai} \left[ \frac{p - p_{cl} + \mathcal{O}(\hbar^2)}{b^{1/3}} \right] \\
\times \exp \left[ \left( 1 + \frac{2}{p_{cl}^2} \frac{d^2 p_{cl}}{da^2} \right) \frac{(p_{cl} - p)}{2p_{cl}} + \mathcal{O}(\hbar^2) \right] \\
\xrightarrow{h \to 0} 2\pi \delta(p - p_{cl}) \] (45)

with \( b^{1/3} \sim h^{2/3} \) and thus the correct classical limit is recovered. Let us note that the NLO correction to the limit of the Airy function (44) is proportional to \( b \sim \hbar^2 \) and the other contributions, of the same order, must be added in order to properly calculate the NLO corrections to the limit of the expression (40). The above results appear rather unwieldy for an immediate use (for example in the general matter evolution equation) and we shall return to their application and significance in the conclusions.

### 4. Conclusions

Following a Born–Oppenheimer decomposition of the wave function of the Universe, separate wave equations for the matter and gravity parts of it were obtained from the initial WDW equation. On then multiplying both the resulting equations with respect to the conjugate gravitational wave function and simultaneously performing a Wigner–Weyl transformation, one is led to a wave equations involving gravitational phase space (Wigner function). In particular since we limited ourselves to the homogeneous degrees of freedom of gravitation these are the scale factor and its conjugate momentum. The resulting wave equations will then involve, besides the matter wave function, the Wigner function for gravitation. This Wigner function is of particular interest since it is only in the classical limit for gravitation that time appears, otherwise it is absent in a quantum formulation. Matter was allowed to remain in a quantum state since the energies for quantum gravity are much higher than those for quantum matter and therefore gravity becomes classical first in an expanding Universe [7]. We then concentrated our attention on the resulting constraints on the gravitational Wigner function.

The equation for the gravitational Wigner function was further separated into a part relating the Wigner function to matter and a quantum Liouville equation for it. Both included an infinite expansion with respect to the Planck constant. On taking the classical limit for both the equations, the Friedmann equation is recovered and, on substitution of the semiclassical solution for the Wigner function into the equation for quantum matter, the previously obtained [7] evolution equation (Schrödinger or Schwinger–Tomonaga) to lowest order is reproduced. This of course, as mentioned in the introduction, puts on a firmer setting our previously obtained results [7, 9].

The equations for the Wigner function were then solved while retaining higher order terms in the Planck constant. In particular we kept terms to order \( \hbar^2 \) which are of the same order
as the non-adiabatic contributions to matter, which we previously considered perturbatively in the context of the Mukhanov equation and whose contribution to the power spectrum we evaluated previously [7, 9]. Our new corrections will modify to the same order the introduction of time in the Mukhanov equation and then one may consider such an additional contribution perturbatively and examine its effect on the power spectrum. In particular the gravitational Wigner function found was an Airy function type solution which, on taking the classical limit, reduces in lowest LO to the previous result together with corrections which can be expressed in terms of generalised functions.

Concerning this last point we observe that, since such an expansion requires that the Airy function be multiplied by a relatively slowly varying function of its argument [22], it may well be that this is not always possible, in particular for situations for which the classical limit may not exist (since it is associated with maxima of the Wigner function). Further we observe that the Airy function in equation (45) was obtained by studying our equations in the vicinity of the (classical) maximum of the Airy, indeed it falls exponentially for \( p > p_{cl} \) while it damps oscillatorily on the \( p < p_{cl} \) side. This may have consequences on the study of the approach to the classical limit and the introduction of time in our evolution equations. Our general corrections to the previous introduction of time appear rather unwieldy, thus it will be convenient to apply them to the evolution of matter after the choice of a suitable inflationary potential.

Let us note that in order to determine the Wigner function we have, in a sense, ‘inverted’ the usual procedure which consisted of solving the equation for the gravitational wave function and subsequently obtaining the Wigner function. Instead through a Wigner–Weyl transformation we directly obtained an equation for the Wigner function and its associated \( \hbar \) corrections. This latter approach has various advantages, since performing the Weyl transformation, and correctly reproducing the resulting quantum corrections, needs a careful treatment of the approximations and in the literature the transformation sometimes is not treated with due care. Therefore starting from the Wigner function and obtaining its quantum corrections appears easier and is physically justified, since it is exactly what one needs for the introduction of the classical time. Finally we stress that our approach is novel as it is applied to the matter(inflaton)–gravity system. For such a system, the recovery of the classical limit for gravity is not just important by itself, but is necessary in order to introduce time for matter.

Finally let us remember a point we mentioned in section 2.1, which is however generally glossed over, is that \( a > 0 \) and the gravitational wave function does not have support for \( a < 0 \). This can actually have strong consequences on the domain of integration in the Wigner–Weyl transformation (although for the case of suitably localised gravitational wave functions, for example in the presence of a bounce for \( a \) small, one need not be concerned). The detailed implementation of such a constraint is cumbersome and has been attempted [23] and studied [24]. We may now end by emphasising that our formalism can be generalised, and applied, to the case of cosmological fluctuations, since these contain parts of the metric, and the results presented here may significantly alter other approaches. It is our intention to return to this point in order to improve on our previously obtained results on the power spectrum.

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Data availability statement

No new data were created or analysed in this study.

ORCID iDs

Alexander Y Kamenshchik https://orcid.org/0000-0002-0575-486X
Alessandro Tronconi https://orcid.org/0000-0003-1913-9654

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