Z-scaling and space-time structural relativity

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Assuming fractality of hadronic constituents, we introduce elements of special realization of the relativity principle applied to physical quantities expressed with respect to various fractal structures. The construction is inspired by the premisses of the z-scaling observed in the inclusive reactions at high energies. The scheme concerns parton structure of hadrons and nuclei at small scales.

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I. INTRODUCTION

The inclusive spectra of secondaries produced with large transverse momenta in high energy collisions of hadrons and nuclei provide unique information about the properties of quark and gluon interactions. As follows from numerous studies in relativistic physics, common feature of these processes is local character of the hadron interactions. This leads to a conclusion about dimensionless constituents participating in the collisions. Fact that the interaction is local manifests naturally in a scale-invariance of the interaction cross sections. The invariance is a special case of the self-similarity which enables to predict and study various phenomenological regularities reflecting the point-like nature of the underlying interactions.

The particle spectra are often presented as a scaling function depending on the transverse mass \( m_\perp = \sqrt{p_\perp^2 + m_2^2} \). Such regularity in the behaviour of the differential cross sections concerns the central interaction region and holds in the limited range of the transverse momenta \( p_\perp \lesssim 3 \text{ GeV/c} \). For larger momenta \( p_\perp \lesssim 6 \text{ GeV} \), the scaling can be preserved in the variable \( m_\perp/K \), when introducing an energy dependent scale \( K(s) \). In analogy with the Koba-Nielsen-Olesen scaling of the multiplicity distributions, the energy dependence of the scale \( K(s) \) for the inclusive reactions was identified with the energy dependence of the average multiplicity density \( dN(0)/d\eta \) of particles produced in the central region of the interaction. The concept of self-similarity of hadron interactions at constituent level was complemented by considerations about fractal character of the objects undergoing the high energy collisions. This lead to introduction of the scaling variable

\[
z = z_0 \Omega^{-1},
\]

where

\[
\Omega(x_1, x_2) = (1 - x_1)^{\delta_1}(1 - x_2)^{\delta_2}.
\]

The variable \( z \) has character of a fractal measure. For a given production process, its finite part \( z_0 \) is ratio of the transverse energy released in the underlying collision of constituents and the average multiplicity density \( dN(0)/d\eta \). The divergent part \( \Omega^{-1} \) describes resolution at which the collision of the constituents can be singled out of this process. The \( \Omega(x_1, x_2) \) represents relative number of all initial configurations containing the constituents which carry the fractions \( x_1 \) and \( x_2 \) of the incoming momenta. The \( \delta_1 \) and \( \delta_2 \) are anomalous fractal dimensions of the colliding objects (hadrons and nuclei). Divergent character of the variable \( z \) secures that there is no \( z_{\text{max}} \) limit for any energy where the scaling has to be a priori violated. The scaling function was found to be independent of the center-of-mass energy and the angle of produced particles over a wide kinematic range. The energy and angular independence of the \( z \) scaling was shown for the production of high \( p_\perp \) jets. A-universality of the scaling was demonstrated for \( pA \) collisions for various nuclei.

The goal of the paper is to focus on general premisses of the \( z \) scaling which concern fractality of the constituent interactions. Fractals are mathematical concepts expressing the self-similarity and inexhaustible structure at small scales. The geometrical objects model the internal parton structure of hadrons and nuclei revealed in their interactions still more and more with increasing energies. This property encountered in modern physics is connected with scale dependence of physical laws gradually emerging in various experimental and theoretical investigations. Such extension of physics is intrinsically linked to the evolution of the concept of space-time.
Its structure is characterized by explicitly scale dependent metric potentials. Asking questions about the metrics leads one to question the relativity. The relativistic principle besides motion, applies also to the laws of scale \[14\]. The scale changes are expressed in terms of the "scale velocity" defined in a special way. In this paper we attempt to extend the realization of the relativity principle to various space-time structures emerging at small scales. Change of the structures is characterized by a "structural velocity" expressed in ratios of their anomalous fractal dimensions. Similarly as the scale velocity, the structural velocity does not represent any real motion. While first characterizes change of the state of scale, later expresses change of the structures at a given scale.

The paper is organized as follows. The description of the parton interactions as used by the construction of the $z$ scaling is recapitulated in Sec. II. Elements of the structural relativity for isolated fractal reference systems are introduced in Sec. III. In Sec. IV., we consider structural anisotropy of space-time induced by the interaction of the fractal objects possessing mutually different anomalous fractal dimensions. Relativistic mechanics in anisotropic space-time is discussed in Sec. V. Relations between the variables used in the kinematical and mechanical sector are presented in Sec. VI. Klein-Gordon equation and the conclusions are discussed in Sec. VII., VIII, and in the Appendix.

II. CONSTITUENT INTERACTIONS

At sufficient high energies, the interactions of hadrons and nuclei can be considered as an ensemble of individual interactions of their constituents. The constituents are partons in the parton model or quarks and gluons which are the building blocks in the theory of QCD. Production of particles with large transverse momenta from such reactions has relevance to physics at small interaction distances. In this region, the interactions of hadronic constituents are local relative to the resolution which depends on the kinematical characteristics of particles produced in the collisions. In accordance with the property of locality it has been suggested \[17\] that gross features of the single-inclusive particle distributions for the reaction

$$m_A + m_B \rightarrow m_c + X$$

(3)

can be described in terms of the corresponding kinematical characteristics of the sub-process

$$(x_1 m_A) + (x_2 m_B) \rightarrow m_c + (x_1 m_A + x_2 m_B + \bar{m}_c)$$

(4)

which is subject to the condition

$$(x_1 p_A + x_2 p_B - p_c)^2 = (x_1 m_A + x_2 m_B + \bar{m}_c)^2.$$ 

(5)

The $x_1$ and $x_2$ are fractions of the incoming four-momenta $p_A$ and $p_B$ of the colliding objects with the masses $m_A$ and $m_B$. The $p_c$ is four-momentum of the inclusive particle with the mass $m_c$. The parameter $\bar{m}_c$ is minimal mass used in connection with the internal conservation laws (for isospin, baryon number, and strangeness). The relationship \[5\] can be conveniently written in the form

$$x_1 x_2 - x_1 \lambda_2 - x_2 \lambda_1 - \lambda_0 = 0,$$ 

(6)

where

$$\lambda_1 = \frac{(p_B p_c) + m_B \bar{m}_c}{(p_A p_B) - m_A m_B}, \quad \lambda_2 = \frac{(p_A p_c) + m_A \bar{m}_c}{(p_A p_B) - m_A m_B}, \quad \lambda_0 = \frac{0.5(\bar{m}_c^2 - m_c^2)}{(p_A p_B) - m_A m_B}.$$ 

(7)

We have determined \[8\] the momentum fractions $x_1$ and $x_2$ in the way to minimize the resolution $\Omega^{-1}$ of the fractal measure $z$ with respect to all possible sub-processes \[4\] which can lead to production of the inclusive particle with the four-momentum $p_c$. This corresponds to maximum of the functional

$$F(x_1, x_2) = \Omega(x_1, x_2) + \beta(x_1 x_2 - x_1 \lambda_2 - x_2 \lambda_1 - \lambda_0)$$

(8)

with a Lagrange multiplier $\beta$. The momentum fractions resulting from this requirement have the form

$$x_1 = \lambda_1 + \chi_1, \quad x_2 = \lambda_2 + \chi_2$$

(9)

where

$$\chi_1 = \sqrt{\mu_1^2 + \omega_1^2} - \omega_1, \quad \chi_2 = \sqrt{\mu_2^2 + \omega_2^2} + \omega_2.$$ 

(10)
Here we have used the notations
\[ \mu_1^2 = \lambda_2 \alpha \frac{(1-\lambda_1)}{(1-\lambda_2)}, \quad \mu_2^2 = \lambda_2 \frac{1}{\alpha} \frac{(1-\lambda_2)}{(1-\lambda_1)} \] (11)

\[ \omega_1 = \lambda_2 \frac{(\alpha - 1)}{2(1-\lambda_2)}, \quad \omega_2 = \lambda_2 \frac{(\alpha - 1)}{2\alpha(1-\lambda_1)}, \] (12)

\[ \lambda = \sqrt{\lambda_1 \lambda_2 + \lambda_0}. \] (13)

The structural parameter \( \alpha = \delta_2 / \delta_1 \) is ratio of the anomalous fractal dimensions of the colliding objects. Procedure of minimizing the fractal resolution \( \Omega^{-1} \) leads to the following consequences. The non-trivial result is that \( \omega_i \) and \( \mu_i \) are both related by the formulae
\[ \omega_1 = \mu_1 U, \quad \omega_2 = \mu_2 U \] (14) through the same value
\[ U = \frac{\alpha - 1}{2\sqrt{\alpha} \xi}. \] (15)

The quantity \( U \) consists of \( \alpha \) dependent structural part and a kinematical factor
\[ \xi = \frac{\lambda}{\sqrt{(1-\lambda_1)(1-\lambda_2)}}. \] (16)

The factor \( \xi \leq 1 \) is function of the center-of-mass energy and momenta of the observed secondaries. For the inclusive reactions, it characterizes the scale resolution. When approaching the phase-space limit of the reaction \( \mathbb{B} \), the \( \xi \) tends to unity. Along the whole phase-space limit \( x_1 = x_2 = 1 \) and \( \xi = 1 \). The phase-space boundary corresponds to the fractal limit with the infinite resolution \( \Omega^{-1} \). The fractal limit is thus equivalent to infinite value of the fractal measure \( z \). This extreme reflects situation when the whole reaction \( \mathbb{B} \) degenerates to the single sub-process \( \mathbb{D} \). Though kinematically accessible at any centre-of-mass energy, its probability is null. What is of physical meaning is the way the probability approaches this limit. The \( z \)-presentation of experimental data reveals power dependence of the scaling behaviour \( \mathbb{E} \) in the range of large \( z \) suggesting specific values of the anomalous fractal dimensions \( \delta_1 \).

The expressions (10) and (14) imply \( \chi_1 \chi_2 = \mu_1 \mu_2 \). Moreover, while \( \chi_i \) and \( \mu_i \) obtained by the minimalization procedure of \( \Omega^{-1} \) are non-trivial functions of structural parameter \( \alpha \), the combination
\[ \chi_1 \chi_2 = \mu_1 \mu_2 = \lambda^2 \] (17)
does not depend on \( \alpha \). This allows to write the sub-process \( \mathbb{D} \) in the symbolic form
\[ (\lambda_1 + \chi_1) + (\lambda_2 + \chi_2) \rightarrow (\lambda_1 + \lambda_2) + (\chi_1 + \chi_2). \] (18)

Equation (17) reflects the transverse momentum balance of this sub-process. The last relation should be understood that \( \lambda_i \) parts of the interacting constituents underly the production of the inclusive particle, while the \( \chi_i \) parts are responsible for the creation of its recoil. Other formal consequences resulting from minimal resolution in the fractal measure \( z \) will be discussed bellow.

### III. SPACE-TIME STRUCTURAL RELATIVITY

This section is devoted to the essential points to be clarified in our approach. It is based on suggestion that besides the kinematical variables there exist structural degrees of freedom to be taken into account for description of the hadronic constituent sub-processes at small scales. The construction relies on the constituent fractal-like compositeness as an universal property of hadronic matter. In this view, we focus on specific properties of the kinematical and mechanical variables defined relative to coordinate systems connected with different fractal structures. It will be shown that transformations between these variables form a group with the corresponding composition rules. The properties will be studied with respect to the "structural velocity" defined as follows
\[ u = \frac{U}{\sqrt{1 + U^2}} \] (19)

The natural structural parameter \( \alpha = \frac{\delta_2}{\delta_1} \) is ratio of the anomalous fractal dimensions of the colliding objects. Procedure of minimizing the fractal resolution \( \Omega^{-1} \) leads to the following consequences. The non-trivial result is that \( \omega_i \) and \( \mu_i \) are both related by the formulae
\[ \omega_1 = \mu_1 U, \quad \omega_2 = \mu_2 U \] (14) through the same value
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\[ u = \frac{U}{\sqrt{1 + U^2}} \] (19)

Here we have used the notations
\[ \mu_1^2 = \lambda_2 \alpha \frac{(1-\lambda_1)}{(1-\lambda_2)}, \quad \mu_2^2 = \lambda_2 \frac{1}{\alpha} \frac{(1-\lambda_2)}{(1-\lambda_1)} \] (11)

\[ \omega_1 = \lambda_2 \frac{(\alpha - 1)}{2(1-\lambda_2)}, \quad \omega_2 = \lambda_2 \frac{(\alpha - 1)}{2\alpha(1-\lambda_1)}, \] (12)

\[ \lambda = \sqrt{\lambda_1 \lambda_2 + \lambda_0}. \] (13)
where \( \mathbf{U} = (0, 0, U) \) is given by Eq. (15). Here we have oriented third coordinate axis in the direction of the collision beams. The inverse relation reads

\[
\mathbf{U} = \frac{\mathbf{u}}{\sqrt{1 - u^2}}
\]

(20)

This enables us to consider the quantities \( U_i \) as space components of structural four-velocity. The structural velocity \( \mathbf{u} \) has its origin in the structural asymmetry of the interaction and vanishes in the collision of the fractal objects possessing equal anomalous fractal dimensions \( \delta_1 = \delta_2 \). The quantity does not represent real motion but characterizes structural polarization in the interaction region. As the formula (15) for \( U / \mathbf{u} \) is Lorenz invariant with respect to motion, the value of the structural velocity is the same when evaluated in whatever inertial reference frame. This applies equally to any expressions which otherwise can depend on \( \mathbf{u} \) but are Lorenz invariants relative to motion. Such expressions can be evaluated in arbitrary motion inertial reference frame using the standard methods.

### A. Structural relativistic transformations in 1+1 dimensions

Exploiting the definition (19), we can rewrite the expressions (10) as follows

\[
\begin{align*}
\mu_1 - \mu_2 &= \frac{1}{\sqrt{1 - u^2}} \left[ (\chi_1 - \chi_2) + u(\chi_1 + \chi_2) \right], \\
\mu_1 + \mu_2 &= \frac{1}{\sqrt{1 - u^2}} \left[ (\chi_1 + \chi_2) + u(\chi_1 - \chi_2) \right],
\end{align*}
\]

(21)

or equivalently

\[
\begin{align*}
\chi_1 - \chi_2 &= \frac{1}{\sqrt{1 - u^2}} \left[ (\mu_1 - \mu_2) - u(\mu_1 + \mu_2) \right], \\
\chi_1 + \chi_2 &= \frac{1}{\sqrt{1 - u^2}} \left[ (\mu_1 + \mu_2) - u(\mu_1 - \mu_2) \right].
\end{align*}
\]

(22)

These relations have form of Lorenz transformations along the third coordinate axis with respect to the structural velocity \( u \). According to Eq. (17), their invariant

\[
(\chi_1 + \chi_2)^2 - (\chi_1 - \chi_2)^2 = (\mu_1 + \mu_2)^2 - (\mu_1 - \mu_2)^2
\]

(23)

does not depend on the structural parameter \( \alpha = \delta_2 / \delta_1 \). Therefore, for a given kinematical factor \( \xi \), Eqs. 21 and 22 can be considered as relativistic transformations of \( (\chi_1 - \chi_2, \chi_1 + \chi_2) \) and \( (\mu_1 - \mu_2, \mu_1 + \mu_2) \) expressed relative to the reference systems connected with fractal structures of different anomalous dimensions. As the combinations of fractions are Lorenz invariants with respect to motion, we can evaluate both of them in the center-of-mass system of the reaction (8). This gives

\[
\begin{align*}
\mu_1 + \mu_2 &= \frac{2}{\sqrt{s}} E_1, \quad \mu_1 - \mu_2 = \frac{2}{\sqrt{s}} p_{1z}, \quad \sqrt{\mu_1 \mu_2} = \frac{1}{\sqrt{s}} \sqrt{p_{1 \perp}^2 + m_c^2}, \\
\chi_1 + \chi_2 &= \frac{2}{\sqrt{s}} E_2, \quad \chi_1 - \chi_2 = \frac{2}{\sqrt{s}} p_{2z}, \quad \sqrt{\chi_1 \chi_2} = \frac{1}{\sqrt{s}} \sqrt{p_{2 \perp}^2 + m_c^2}.
\end{align*}
\]

(24)

or

\[
\begin{align*}
\mu_1 + \mu_2 &= \frac{2}{\sqrt{s}} E_1, \quad \mu_1 - \mu_2 = \frac{2}{\sqrt{s}} p_{1z}, \quad \sqrt{\mu_1 \mu_2} = \frac{1}{\sqrt{s}} \sqrt{p_{1 \perp}^2 + m_c^2}, \\
\chi_1 + \chi_2 &= \frac{2}{\sqrt{s}} E_2, \quad \chi_1 - \chi_2 = \frac{2}{\sqrt{s}} p_{2z}, \quad \sqrt{\chi_1 \chi_2} = \frac{1}{\sqrt{s}} \sqrt{p_{2 \perp}^2 + m_c^2}.
\end{align*}
\]

(25)

Here we have denoted \( E_1, p_1 \) and \( E_2, p_2 \) the center-of-mass energy and momentum of the recoil object \( m_c \) expressed relative to the structural reference frames \( S_1 \) and \( S_2 \), respectively. The single structural frames are associated with isolated fractal structures characterized by the anomalous fractal dimensions \( \delta_1 \) and \( \delta_2 \). The reference system \( S_1 \) is connected with the fractal structure of the first fractal object (nucleus A) and the reference system \( S_2 \) with the second fractal object (nucleus B). If both fractal structures possess mutual different anomalous dimensions \( \delta_1 \neq \delta_2 \), the variables are linked by the relativistic transformations

\[
\begin{align*}
p_{2z} &= \frac{1}{\sqrt{1 - u^2}} (p_{1z} - uE_1), \\
E_2 &= \frac{1}{\sqrt{1 - u^2}} (E_1 - up_{1z})
\end{align*}
\]

(26)
depending on the structural velocity \( u \neq 0 \). The invariant of these transformations, \( E_2^2 - p_{2z}^2 = E_1^2 - p_{1z}^2 \), corresponds to the invariant form (23).

Let us now examine fractal limit which is equivalent to the infinite resolution. As follows from the definition of the scaling variable \( z \), the fractal limit can be achieved kinematically at any energy. It corresponds to the phase-space limit with \( x_1 = x_2 = 1 \). In this case, the fractions \( \chi_i \) approach their limiting values \( \chi_1 = \cos^2(\theta_B/2) \) and \( \chi_2 = \sin^2(\theta_B/2) \) where \( \theta_B \) is the center-of-mass angle of the recoil particle expressed relative to the structural reference frame \( S_2 \). The corresponding kinematics is given by the sphere

\[
\chi_1 - \chi_2 = \cos \theta_B, \quad \chi_1 + \chi_2 = 1
\]  

in this frame. When transforming the sphere to the reference system associated with the fractal structure of the object \( A \), we get

\[
\mu_1 - \mu_2 = \sqrt{\alpha} \sin^2(\theta_B/2) - \frac{1}{\sqrt{\alpha}} \cos^2(\theta_B/2),
\]

\[
\mu_1 + \mu_2 = \sqrt{\alpha} \sin^2(\theta_B/2) + \frac{1}{\sqrt{\alpha}} \cos^2(\theta_B/2).
\]  

These equations represent angular parameterization of an ellipse with the focus in the origin of the reference system. Similarly, the spherical kinematics

\[
\mu_1 - \mu_2 = \cos \theta_A, \quad \mu_1 + \mu_2 = 1
\]  

expressed in terms of the center-of-mass angle \( \theta_A \) of the recoil particle \( \bar{m} \) in the reference frame \( S_1 \) transforms to the ellipse

\[
\chi_1 - \chi_2 = \frac{1}{\sqrt{\alpha}} \sin^2(\theta_A/2) - \sqrt{\alpha} \cos^2(\theta_A/2),
\]

\[
\chi_1 + \chi_2 = \frac{1}{\sqrt{\alpha}} \sin^2(\theta_A/2) + \sqrt{\alpha} \cos^2(\theta_A/2)
\]  

in the fractal reference system \( S_2 \). The transverse components \( \chi_\perp = \sin \theta_B = \sin \theta_A = \mu_\perp \) are conserved by these structural transformations.

To end up with infinite resolution, we mention one remarkable property concerning the composition of the structural velocities in 1+1 dimensions. As the kinematical scale factor (16) is unity in the fractal limit \( (\xi = 1) \), the relation (15) takes the simple form

\[
U = \frac{\alpha - 1}{2\sqrt{\alpha}} = \frac{u}{\sqrt{1 - u^2}}.
\]  

When solving this equation with respect to \( u \), we get the structural velocity

\[
u = \frac{\alpha - 1}{\alpha + 1}
\]  

as an exclusive function of the ratio \( \alpha = \delta_2/\delta_1 \) of the anomalous fractal dimensions of the colliding objects. This relation satisfies the standard relativistic composition rule

\[
u_a = \frac{u + u_b}{1 + uu_b},
\]

provided

\[
\alpha_a = \alpha a_b.
\]  

The conclusion one has to make for the fractal limit is following. While the composition of structural velocities in 1+1 dimensions is governed by Einstein-Lorenz law, the composition of the corresponding ratio of the anomalous fractal dimensions obeys the multiplicative group law. Such correspondence is specific expression of structural relativity in which single fractal structures play analogous roles as the inertial systems in the motion relativity.
In this section we discuss generalization of the structural relativistic transformations for 3 spatial dimensions. First we consider the transformations which left untouched the variables transverse to the relativistic boost. This is characteristic for the Lorenz transformations

\[
p_2 = p_1 + u \left( \frac{u p_1}{u^2} (\gamma - 1) - \gamma E_1 \right),
E_2 = \gamma (E_1 - u p_1).
\]

(35)

As the transformations concern the structural relativity, the relativistic factor

\[
\gamma = (1 - u^2)^{-1/2}
\]

(36)
is given in terms of the structural velocity \( u \). In order to preserve the standard relativistic relations in both fractal reference frames \( S_1 \) and \( S_2 \), we have to require the same transformations also for coordinates and time,

\[
r_2 = r_1 + u \left( \frac{u r_1}{u^2} (\gamma - 1) - \gamma t_1 \right),
t_2 = \gamma (t_1 - u r_1).
\]

(37)

Last equations entail composition of the structural velocity \( u \) with the motion velocity \( v_1 = dr_1/dt_1 \) in formally standard way

\[
v_2 = \gamma^{-1} v_1 + u \left[ (1-\gamma^{-1}) u \cdot v_1/u^2 - 1 \right] / (1 - u \cdot v_1).
\]

(38)

The result is the motion velocity \( v_2 = dr_2/dt_2 \) with respect to the fractal reference system \( S_2 \). The composition of the structural velocities alone is given by the group structure of the structural transformations. When using the four-dimensional notation \( r_2 = \Lambda(u)r_1 \) \((p_2 = \Lambda(u)p_1)\) with

\[
\Lambda(u) = \left( \begin{array}{cc}
\delta_{ij} + (\gamma - 1)u_i u_j / u^2 & -\gamma u_j \\
-\gamma u_j & \gamma
\end{array} \right),
\]

(39)

the group structure of the Lorenz transformations is expressed as follows

\[
R(\phi)\Lambda(u_a) = \Lambda(u_b)\Lambda(u).
\]

(40)

The matrix

\[
R(\phi) = \left( \begin{array}{cc}
r_{ij} & 0 \\
0 & 1
\end{array} \right)
\]

(41)
describes three dimensional Thomas precession \( \sim u_a \times u \) known in the theory of relativity. The corresponding composition of the structural velocities defined with respect to various isolated fractal reference frames reads

\[
u_b = \gamma^{-1} u_a + u \left[ (1-\gamma^{-1}) u \cdot u_a/u^2 - 1 \right] / (1 - u \cdot u_a).
\]

(42)

Though formally identical, this should be distinguished from the composition of the motion velocities, \( v'_i = v'_i \oplus v_i \), following from the relations \( r'_i = \Lambda(v_i)r'_i, i = 1, 2 \). For the situation considered above we conclude that the structural velocities and the motion velocities are composed separately and mutually in the same way.

There exists another generalization of Eqs. connected with mutual spinning of the interacting fractal structures around the collision axis. This transformation has the form

\[
p_2 = \gamma^{-1} p_1 - \sigma u \times p_1 - \gamma u (E_1 - u \cdot p_1),
E_2 = \gamma (E_1 - u \cdot p_1)
\]

(43)

where \( \sigma = \pm 1 \) corresponds to the right/left spinning (or torsion) of the colliding fractals. For similar reasons as above, the same should apply to the kinematical variables \( r \) and \( t \),

\[
r_2 = \gamma^{-1} r_1 - \sigma u \times r_1 - \gamma u (t_1 - u \cdot r_1),
t_2 = \gamma (t_1 - u \cdot r_1).
\]

(44)
The inverse transformations are obtained by the interchange $1 \leftrightarrow 2$ with replacing $u$ by $-u$. The above transformations preserve the invariant forms $E^2 - p^2 = m_0^2$ and $t^2 - r^2 = \tau^2$. The relations
\[ E_1 = \frac{m_0}{\sqrt{1 - v_1^2}}, \quad p_1 = E_1 v_1, \quad E_2 = \frac{m_0}{\sqrt{1 - v_2^2}}, \quad p_2 = E_2 v_2 \] (45)
are thus automatically fulfilled. This secures that the standard motion relativity remains valid in both fractal reference frames $S_1$ and $S_2$. The relativistic transformations with respect to motion, $r''_i = \Lambda(v_i) r'_i$ ($p''_i = \Lambda(v_i) p'_i$), $i=1,2$, imply the usual Lorenz composition of the motion velocities, $v''_i = v'_i \odot v_i$, also in this case. However, according to Eq. (44), the motion velocities $v_1$ and $v_2$ expressed relative to the structural reference systems $S_1$ and $S_2$ are linked by the expressions
\[ v_2 = \frac{v_1 - \sigma \gamma u \times v_1}{\gamma^2(1 - u \cdot v_1)} - u, \quad v_1 = \frac{v_2 + \sigma \gamma u \times v_2}{\gamma^2(1 + u \cdot v_2)} + u \] (46)
which differ from Eq. (45). Similarly, the composition of the structural velocities alone is different here. It is given by the group structure of the structural transformations (43) and (44). When using the four-dimensional notation $r_2 = \Pi(u) r_1$ ($p_2 = \Pi(u) p_1$) with
\[ \Pi(u) = \left( \begin{array}{c} \gamma^{-1} \delta_{ij} + \sigma \epsilon_{ijk} u_k + \gamma u_i u_j - \gamma u_i \\ -\gamma u_j \\ \gamma \end{array} \right), \] (47)
the group structure of these transformations is expressed by the relation
\[ R(\psi) \Pi(u_a) = \Pi(u_b) \Pi(u). \] (48)
The corresponding structural velocities are composed as follows
\[ u_b = \frac{u_a - \sigma \gamma u \times u_a}{\gamma^2(1 - u \cdot u_a)} - u. \] (49)
Really, one can compute the matrix product $\Pi(u_b) \Pi(u) \Pi(-u_a)$ and find that the result has the structure (41). One can convince itself also, that the corresponding spatial part $r_{ij}$ has the form
\[ r(\psi) = \left( \begin{array}{ccc} n_1^2 + (1 - n_1^2) \cos \psi & n_1 n_2 (1 - \cos \psi) - n_3 \sin \psi & n_1 n_3 (1 - \cos \psi) + n_2 \sin \psi \\ n_1 n_2 (1 - \cos \psi) + n_3 \sin \psi & n_2^2 + (1 - n_2^2) \cos \psi & n_2 n_3 (1 - \cos \psi) - n_1 \sin \psi \\ n_1 n_3 (1 - \cos \psi) - n_2 \sin \psi & n_2 n_3 (1 - \cos \psi) + n_1 \sin \psi & n_3^2 + (1 - n_3^2) \cos \psi \end{array} \right) \] (50)
representing general parametrization of 3D rotation around an unit vector $n = \psi/\psi$. Unlike the standard Thomas precession, the precession (50) is non-zero even for composition of the collinear velocities. This circumstance is connected with existence of the vector product in the transformation equations. Once convincing ourselves in the above expressions, we can state that the structural transformations (43) and (44) form a group. The corresponding composition of the structural velocities includes term characterising torsion. While the motion velocities alone are composed in the standard Lorenzian way, the composition including at least one structural velocity results in formula which contains the term with torsion.

The transformations (45) and (46), or (43) and (44) represent the mathematical expression of special realization of the space-time structural relativity. They concern the relativity with respect to the self-similar scale structures which are modeled by fractals of various anomalous dimensions. The corresponding structural transformations relate physical quantities given in one fractal reference frame with the quantities expressed relative to the other one. Single isolated reference frames associated with fractal structures of different anomalous fractal dimensions play analogous role as the inertial systems in the motion relativity. The above relativistic transformations suggest that there does not exist any absolute structural reference system connected either with a fractal object or with a particular structure of the (QCD) vacuum.

IV. INDUCED ANISOTROPY OF SPACE-TIME

In our construction we associate single fractal reference systems with the extended structural objects colliding at high energies. Their fractal structure models parton content of hadrons and nuclei at small scales. It concerns subtle net of quarks, anti-quarks and gluons revealed still more and more with increasing resolution. Pursuing the ideas of space-time structural relativity, we discuss consequences in the proposed thinking frame. The consequences lay...
beyond the relativistic transformations [35 and 37], or [13] and [14] which are the transformations connecting two isolated structural systems. Beside the fractal objects, one has to consider general frames related to any structure of space-time as well. According to the accepted notions of quantum field theories, the space-time vacuum is not an empty space. Its intimate structure is governed by the same processes which influence the very structure of hadrons and nuclei. The self-similarity and infinity of the elementary creation and annihilation processes allows us to consider the vacuum in the framework of fractal geometry as well. The way through which space-time properties are related to matter properties is instructive. It consists in attributing to space-time those properties of matter which are universal. It was suggested by many authors (see e.g. [13, 14, 15]), that one of such universal property is fractality, the never ending self-similar content of matter forming its intimate structure at small scales. In this view we go beyond the isolated fractal objects and generalize our working hypothesis as follows: Ones we adopt fractality of hadronic constituents and consider ultra-relativistic collisions of hadrons and nuclei as collisions of fractals, we can conjecture that interaction of these fractal objects induces deformation to the very structure of space-time. If the colliding objects possess mutually different anomalous fractal dimensions (δ1 ≠ δ2), it is natural to imagine that, due to fractality, vacuum structure acquires a polarization (or anisotropy) along the collision axis. Anisotropy of space-time induced by the interaction is visualized as a fractal background or sort of a fractal medium "moving" with the structural velocity \( \mathbf{u} \). The elementary constituent interactions take place on this background in disturbed space-time. As far as our working hypothesis.

One of the attributes of scale dependent fractal space-time is the fundamental consequence, namely breaking of the reflection invariance [15] with regard to real motion. If one reverses the sign of time in the proper time differential element, the velocity \( \mathbf{v} \), becomes \( -\mathbf{v} \), and there is no reason for these velocities to be equal, in contrast to what happens in the standard case. The quadratic relativistic invariant of the special relativity is thus not conserved. Let us consider violation of the reflection invariance caused by the structural disparity characterized with the structural velocity \( \mathbf{u} \). We insist simultaneously on the requirement, that breaking of the reflection invariance does not disturb spatial isotropy. This can be achieved by the transition to a new structural reference system \( S^\prime \) by the relation

\[
\mathbf{r}_2 = \mathbf{r}, \quad t_2 = t - \mathbf{u} \cdot \mathbf{r}.
\]

The reference system \( S \) is associated with the fractal structure of the object \( B \) which interacts with the fractal object \( A \). The interaction of both fractals makes substantial distinction between the systems \( S \) and \( S^\prime \). In the same way one can introduce the coordinate system connected with the fractal structure of the object \( A \) which interacts with the fractal \( B \). The only difference is the interchange \( \mathbf{u} \leftrightarrow -\mathbf{u} \). Let us examine the reference system \( S \) in more detail. Distortion of space-time in this system is given by the metrics

\[
\eta(\mathbf{u}) = \begin{pmatrix} -\delta_{ij} + u_i u_j & -u_i \\ -u_j & 1 \end{pmatrix}
\]

which corresponds to the invariant

\[
t^2 - r^2 - 2t \mathbf{u} \cdot \mathbf{r} + (\mathbf{u} \cdot \mathbf{r})^2 = \tau^2.
\]

Simple metrics of this type has been used in the 3+1 formalism of general relativity [21]. In this formalism, space-time is described as a foliation of space-like hyper-surfaces of constant time \( t \). The quantity \( \mathbf{u} \) has meaning of a vector relating the spatial coordinate systems on different hyper-surfaces. In difference from the structural velocity \( \mathbf{u} \), we denote the motion velocity as

\[
\mathbf{v} = \frac{d\mathbf{r}}{dt}.
\]

Using the four-dimensional notation \( r = (r, t) \), the relativistic transformations with respect to motion which preserve the invariant [35] can be expressed as follows

\[
r'' = \Delta_D(\mathbf{v}, \mathbf{u})r',
\]

where

\[
\Delta_D(\mathbf{v}, \mathbf{u}) = D^{-1}(\mathbf{u})\Lambda(\mathbf{v}_2)D(\mathbf{u}).
\]

The \( \Lambda(\mathbf{v}_2) \) is Lorenz transformation matrix of the form [36] which depends on the motion velocity vector

\[
\mathbf{v}_2 = \frac{\mathbf{v}}{1 - \mathbf{u} \cdot \mathbf{v}}.
\]
and
\[
D(u) = \begin{pmatrix}
\delta_{ij} & 0 \\
-u_j & 1
\end{pmatrix}.
\] (58)

The transformation matrix \( \Delta_D \) can be rewritten into the compact form
\[
\Delta_D(v, u) = \begin{pmatrix}
\delta_{ij} + G v_i v_j + \Gamma v_i u_j & -\Gamma v_i \\
-G_v v_j - \Gamma u_j & 1 + \Gamma
\end{pmatrix}.
\] (59)

Here we have used the notations
\[
\Gamma = \frac{1}{\sqrt{(1 - u \cdot v)^2 - v^2}},
\] (60)
\[
G = \frac{(1 - u \cdot v) \Gamma - 1}{v^2},
\] (61)

and
\[
\Gamma_{\pm} = G v^2 \pm \Gamma u \cdot v, \quad G_{\pm} = \Gamma \pm G u \cdot v.
\] (62)

The factor \( \Gamma \) is analogue of the Lorenz factor for non-zero space-time anisotropy \( u \) relating particle’s proper time \( \tau \) with the time \( t \) in the reference system \( S \),
\[
t = \tau \Gamma.
\] (63)

The transformations inverse to (55) are obtained by the interchange \( r'' \leftrightarrow r' \) and replacing \( v \) by \( v_{\text{inv}} \), where
\[
v_{\text{inv}} = -\frac{v}{1 - 2u \cdot v}.
\] (64)

This formula connects the motion velocity \( v \) of a system \( S' \) in the reference system \( S \) with the motion velocity \( v_{\text{inv}} \) of the system \( S \) in the \( S' \) reference frame. Because of space-time structural anisotropy \( u \), the magnitudes of the two motion velocities are not equal. Exploiting the symmetry properties
\[
\Gamma(v_{\text{inv}}) = (1 - 2u \cdot v) \Gamma(v), \quad \Gamma_{\pm}(v_{\text{inv}}) = \Gamma_{\mp}(v),
\]
\[
G(v_{\text{inv}}) = (1 - 2u \cdot v)^2 G(v), \quad G_{\pm}(v_{\text{inv}}) = (1 - 2u \cdot v) G_{\pm}(v),
\] (65)

the transformation matrix of the inverse transformations reads
\[
\Delta_D^{-1}(v, u) = \begin{pmatrix}
\delta_{ij} + G v_i v_j - \Gamma v_i u_j + \Gamma v_i \\
+ G v_i v_j - \Gamma u_j & 1 + \Gamma
\end{pmatrix}.
\] (66)

As follows from the relation
\[
\Delta_D^{-1}(v, u) \eta(u) \Delta_D(v, u) = \eta(u),
\] (67)

the motion transformations (55) preserve the invariant (53). The transformations comply the principle of relativity expressed by their group properties. The composition of the transformations has the form
\[
\Omega_D(\phi, u) \Delta_D(v'', u) = \Delta_D(v', u) \Delta_D(v, u)
\] (68)

with \( \Omega_D(\phi, u) = D^{-1}(u) R(\phi) D(u) \), provided
\[
v'' = \frac{v' + v [\Gamma(1 - u \cdot v') + G v \cdot v']}{1 + \Gamma_{\pm}(1 - u \cdot v') + G_{\pm} v \cdot v'}
\] (69)

The inverse relation reads
\[
v' = \frac{v'' - v [\Gamma(1 - u \cdot v'') - G v \cdot v'']}{1 + \Gamma_{\mp}(1 - u \cdot v'') - G_{\mp} v \cdot v''}.
\] (70)
One can obtain the above relations from the standard composition of the Λ matrices. Range of the accessible values of the velocities \( v \) is given by the rotational ellipsoid

\[
(v_{\|} + e)^2 + \gamma^2 v_{\perp}^2 = \gamma^4.
\]

Here \( v_{\|} \) and \( v_{\perp} \) denote the velocity components which are parallel and perpendicular to the space-time structural anisotropy \( u \), respectively. The ellipsoid is given by the major semi-axis \( a = \gamma^2 \) and by the minor semi-axis \( b = \gamma \). Its eccentricity is \( e = \gamma \sqrt{\gamma^2 - 1} \). One focus of the ellipsoid corresponds to the point \( v = 0 \). The velocity ellipsoid is invariant with respect to the relations \( \Theta \) and \( \Theta' \).

In the case of \( v = (0, 0, v) \) and \( u = (0, 0, u) \), the motion relativistic transformations have the simple form

\[
\begin{align*}
\xi'' &= \Gamma[(1 - 2uv)v' + vt'], \\
\tau'' &= \Gamma[t' + vr'' \gamma^{-2}]
\end{align*}
\]

(72)

\[
\begin{align*}
\xi' &= \Gamma[(v' - vt')]
\end{align*}
\]

(73)

The transverse components are conserved, \( v''_i = v'_i, i = x, y \). Composition of the corresponding velocities reads

\[
\begin{align*}
v''_x &= \frac{v'_x + v - 2uvv'_t}{1 + uvv'_t \gamma^{-2}}, \\
v''_y &= \frac{v'_y}{1 + uvv'_t \gamma^{-2}}
\end{align*}
\]

(74)

\[
\begin{align*}
v'_x &= \frac{v''_x - v}{1 - 2uv - vv''_t \gamma^{-2}}, \\
v'_y &= \frac{v''_y}{1 - 2uv - vv''_t \gamma^{-2}}
\end{align*}
\]

(75)

Detailed classification of the linear transformations of the type \( \Theta \) and \( \Theta' \) was performed in 1+1 dimensions in Ref. [18].

V. RELATIVISTIC MECHANICS IN ANISOTROPIC SPACE-TIME

In standard relativistic mechanics, the position and momentum of an elementary particle is given by the four-vectors \( r^\mu = \{r, t\} \) and \( p^\mu = \{p, E\} \), respectively. We comprehend the notion of elementarity as a relative concept which relies on the scales and structures we are dealing with. For the infinite resolution the elementary particle should be a perfect point without any internal structure. For an arbitrary small but still finite resolution, the perfect point is approximated by a particle which we call "elementary" with respect to this resolution. It is therefore natural to assume that the concepts of the momentum, energy, mass and the velocity of the "elementary" particle have good physical meaning also in the region where the space-time isotropy is violated. Let us consider space-time structural anisotropy induced by collisions of two interacting fractals. Suppose the anisotropy is characterized by the structural velocity \( u \). In such case we have to impose general requirements on mechanical variables, which remain still valid. Based on our physical intuition, we formulate these requirements as follows:

1. Energy of a free particle cannot be pumped from the structure of space-time. This condition reads

\[
E = E_{\text{min}} = m_0 \quad \text{for} \quad v = 0
\]

(76)

where \( m_0 \) is the rest mass and \( v \) is the velocity of the free particle.

2. Rate of clocks is slowest in the centre of gravity system. The only source of gravity is free particle itself and not the structure of space-time. This condition reads

\[
dt = dt_{\text{min}} \quad \text{for} \quad P = 0
\]

(77)

where the vector \( P \) defines effect of force on the particle. The physical requirements are obvious in the Minkovski space-time. In the case of the space-time anisotropy, they lead to specific constraints on the mechanical variables. The construction depends on the way the anisotropy is induced by the interaction. It will be instructive to discuss the situation corresponding to the transformations \( \Theta \) and \( \Theta' \) separately.

A. Fractal interaction with torsion

The transformations with structural torsion model situation when the interacting fractal structures are in mutual spinning position. Amount of the mutual torsion depends on the values of anomalous fractal dimensions of the
colliding fractal objects which can be functions of their spin states. Explicit spin dependencies of these parameters have to be determined from experiment and require further independent study. Without going to details, we show here that the phenomenological aspects of the requirements \((76)\) and \((77)\) can be fulfilled in the following way. Relying on the transformations \((43)\), we link the mechanical variables \(p_2\) and \(E_2\) defined in the isolated fractal reference frame \(S_2\) with their counterparts \(P\) and \(E\) in the reference system \(S\) by the expressions

\[
p_2 = \gamma^{-1}P - \sigma u \times P - u E, \quad E_2 = E.
\]  

(78)

The \(u\) is the structural velocity considered in the previous sections. Las t equation suggests that energy of a free particle is the same in the isolated fractal system \(S_2\) and the reference frame \(S\) which is essentially the same fractal system disturbed by the interaction with the fractal object \(A\). This is not longer true for the quantities \(p_2\) and \(P\). The relation between them which is inverse to Eq. \((78)\) reads

\[
P = \gamma^{-1}p_2 + \sigma u \times p_2 + \gamma u (E_2 + u \cdot p_2).
\]  

(79)

The standard relativistic invariant in the system \(S_2\) is replaced by the invariant expression

\[
\gamma^{-2} [E^2 - P^2 + 2EU \cdot P - (U \times P)^2] = m_0^2.
\]  

(80)

This corresponds to the the metrics

\[
\bar{\eta}(u) = (-\delta_{ij} + u_i u_j \gamma^{-1}) u_i \gamma^{-1} \gamma^{-2}
\]  

(81)

in the \((P,E)\) space. The relativistic transformations of the mechanical variables with respect to motion, which preserve the invariant \((80)\), can be written as follows

\[
P'' = \Delta_H(W,U)P'
\]  

(82)

where

\[
\Delta_H(W,U) = H^{-1}(u)\Lambda(v_2)H(u).
\]  

(83)

Here \(\Lambda(v_2)\) has the same form as in Eq. \((39)\) and

\[
H(u) = \left( \begin{array}{cc} \gamma^{-1} \delta_{ij} + \sigma \epsilon_{ijk} u_k & -u_i \\ 0 & 1 \end{array} \right).
\]  

(84)

When evaluating the right-hand side of the expression \((83)\) it can be shown that the transformation matrix \(\Delta_H\) is explicit function of the anisotropy \(U\) and the vector

\[
W = \gamma^{-1}v + \sigma u \times v.
\]  

(85)

It takes the form

\[
\Delta_H(W,U) = \left( \begin{array}{cc} \delta_{ij} + GW_i W_j - G_- U_i W_j & -G_+ W_i + GW^2 U_i \\ -GW_j & 1 + \Gamma_+ \end{array} \right)
\]  

(86)

where the symbols \(\Gamma, \Gamma_\pm, \text{ and } G_\pm\) are given by Eqs. \((60), (61), \text{ and } (62)\), respectively. They can be expressed in terms of \(U\) and \(W\) in the following way

\[
\Gamma = \frac{1}{\sqrt{1 - 2U \cdot W - W^2}}, \quad G = \frac{(1 - U \cdot W) \Gamma - 1}{W^2 + (U \cdot W)^2},
\]  

(87)

\[
\Gamma_\pm = GW^2 + G(U \cdot W)^2 \pm \Gamma U \cdot W, \quad G_\pm = \Gamma \pm GU \cdot W.
\]  

(88)

The transformations inverse to Eq. \((82)\) are obtained by the interchange \(P'' \leftrightarrow P'\) and replacing \(W\) by \(W_{\text{inv}}\), where

\[
W_{\text{inv}} = -\frac{W}{1 - 2U \cdot W}
\]  

(89)
Exploiting the symmetry properties (65), the inverse transformation matrix reads
\[
\Delta_{H}^{-1}(W, U) = \left( \begin{array}{ccc}
\delta_{ij} + GW_i W_j + G_+ U_i W_j & G_- W_i + GW^2 U_i \\
GW_j & 1 + \Gamma_-
\end{array} \right).
\]
(90)
As follows from the relation
\[
\Delta_{H}^{j}(W, U)\tilde{\eta}(u)\Delta_{H}(W, U) = \tilde{\eta}(u),
\]
(91)
the motion transformations of mechanical variables preserve the invariant (80). The composition of the transformations has the form
\[
\Omega_{H}(\phi, U)\Delta_{H}(W'', U) = \Delta_{H}(W', U)\Delta_{H}(W, U)
\]
with \( \Omega_{H}(\phi, U) = H^{-1}(u)R(\phi)H(u) \), provided
\[
W'' = \frac{W' + W \left[ \Gamma - G_- U \cdot W' + GW \cdot W' \right]}{1 + \Gamma_+ - GW^2 U \cdot W' + G_+ W \cdot W'},
\]
(93)
The inverse formula reads
\[
W' = \frac{W'' - W \left[ \Gamma - G_+ U \cdot W'' - GW \cdot W'' \right]}{1 + \Gamma_- - GW^2 U \cdot W'' - G_- W \cdot W''}.
\]
(94)
The above relations can be obtained from the standard composition of the \( \Lambda \) matrices. We end up this section by the following observation. White the transformations of the kinematical variables depend on the quantities \( u \) and \( v \), the transformations of the mechanical variables are explicit functions of \( U \) and \( W \). The variables \( r^\mu \) and \( P^\mu \) posses different transformation properties with respect to motion. The first obey the transformation formula (55), the later are transformed according to Eq. (82). This separation of the kinematical and mechanical sector is characteristic property for space-time with non-zero anisotropy.

B. Fractal interaction without torsion

The transformations without structural torsion model situation when the interacting fractal structures do not spin mutually around the collision axis. In such case, the requirements (76) and (77) have to be fulfilled in consistence with the structural transformations (65). We connect therefore the mechanical variables \( p_2 \) and \( E_2 \) defined in the isolated fractal reference frame \( S_2 \) with their counterparts \( P \) and \( E \) in the reference system \( S \) by the relations
\[
p_2 = P - u \frac{u \cdot P}{u^2} (\gamma^{-1} - 1) - u E, \quad E_2 = E.
\]
(95)
Also here we require equality of the energy of a free particle in the system \( S_2 \) with its value in the reference frame \( S \) being the same fractal system disturbed by the interacting fractal object \( A \). The relation between \( p_2 \) and \( P \) differs however from Eq. (78). The inverse relation reads
\[
P = p_2 + U \frac{U \cdot p_2}{U^2} (\gamma - 1) + U E_2.
\]
(96)
The standard relativistic invariant in the system \( S_2 \) is replaced by the invariant expression
\[
\gamma^{-2}(E + U \cdot P)^2 - P^2 = m_0^2
\]
(97)
corresponding to the metrics
\[
\tilde{\eta}(u) = \left( \begin{array}{ccc}
-\delta_{ij} + u_i u_j & u_i \gamma^{-1} \\
u_j \gamma^{-1} & \gamma^{-2}
\end{array} \right).
\]
(98)
The relativistic invariant and the metrics in \( (P, E) \) space are thus identical for both cases, the interaction with and without torsion. The same is true for the corresponding relativistic transformations with respect to motion. Really, from the relations (65) we see that the transformations can be written as follows
\[
P'' = \Delta_{H}(W, U)P',
\]
(99)
\[ \Delta_{\tilde{H}}(W, U) = \tilde{H}^{-1}(u)\Lambda(v_2)\tilde{H}(u). \]  

(100)

Here \( \Lambda(v_2) \) is the same as in Eq. (83) while

\[ \tilde{H}(u) = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{\sqrt{1 - 2U \cdot W - W^2}} & \Gamma & \Gamma \pm \end{pmatrix}. \]  

(101)

When evaluating the right-hand side of the expression (100), it can be shown that the transformation matrix \( \Delta_{\tilde{H}} \) is explicit function of the anisotropy \( U \) and the vector \( W = v + u \cdot v \frac{u}{u^2} \frac{1}{\gamma - 1} \).

(102)

It can be rewritten to the form

\[ \Delta_{\tilde{H}}(W, U) = \begin{pmatrix} \delta_{ij} + G W W_j - G_+ U W_j - G_- W_i + GW^2 U_i \\ \Gamma W_j \\ 1 + \Gamma \pm \end{pmatrix}. \]  

(103)

which is identical with the matrix (86), \( \Delta_{\tilde{H}} \equiv \Delta_H \). The \( \Gamma, G, \Gamma_\pm, \text{and} G_\pm \) are the same functions of the quantity \( W \) (given here by Eq. (102)) as the expressions (60), (61), and (62), respectively. The identical relations as in the previous subsection are valid for the inverse transformations and for the composition rules of \( W \).

VI. RELATIONS OF THE KINEMATICAL AND MECHANICAL VARIABLES

Fundamental concepts of the special theory of relativity lead us to the relation between the energy/momentum of a particle and its velocity. The velocity is limited within the sphere of the radius \( c = 1 \) in every inertial system of reference and is oriented in the direction of the particle momentum. Coefficient of the proportionality between the momentum and the velocity is relativistic mass of the particle. The relativistic mass is equivalent to the particle’s energy. Change of the particle’s momentum per unit time defines force the particle is acted upon. This concerns the homogeneous and isotropic space-time. We show how the relations modify provided structural violation of the space-time isotropy characterized by the structural velocity \( u \).

We start with the relations (45) valid in an isolated fractal reference frame \( S_2 \). The corresponding relations in the reference system \( S \) are different. Exploiting Eqs. (51) and (78), or (51) and (95) we get

\[ P = M(W + U), \quad E = M(1 - U \cdot W). \]  

(104)

where

\[ M = \Gamma m_0, \quad \Gamma = \frac{1}{\sqrt{1 - 2U \cdot W - W^2}}. \]  

(105)

Form of these expressions is the same for fractal interaction with or without torsion, respectively. The \( W \) depends, however, in both cases on the motion velocity \( v \) in a different way. The expressions (104) are form invariant with respect to motion. This allows to consider the quantity \( M \) as meaningful generalization of the relativistic mass of a free particle in space-time with the structural anisotropy \( u \). We see from the above relations that the energy \( E \) and the mass \( M \) become independent. The energy is always larger than the particle’s rest mass \( m_0 \), \( E \geq m_0 \). Minimum of the energy is acquired for \( W = 0 \) and thus for the non-zero value of \( P = P_0 \),

\[ E_{\text{min}} = E(P_0) = m_0, \quad P_0 = m_0 \frac{u}{\sqrt{1 - u^2}}. \]  

(106)

This is in consistence with the requirement (60), because for both cases, the fractal interaction with and/or without torsion, Eqs. (55) and (102) give the zero motion velocity \( v = 0 \) when \( W = 0 \). Contrary to the energy \( E \), the mass
The minimal value of a particle’s mass is acquired for \( P = 0 \) and thus for the non-zero value of \( W = W_0 \),

\[
M_{\text{min}} = M(W_0) \leq m_0, \quad W_0 = -U.
\]  

This corresponds to the minimum of the factor \( \Gamma \) and to the minimal value of \( t \). Consequently, the requirement (77) is thus fulfilled as well. The minimal mass depends on the value of the space-time anisotropy. Formal relation between the rest mass \( m_0 \) and the minimal mass \( M_{\text{min}} \) can be written in the form

\[
m_0 = \gamma M_{\text{min}}.
\]  

From the above relations we conclude that, in space-time with structural anisotropy, the energy and mass of a free particle become independent quantities.

We shall demonstrate this in a more formal way. The energy \( E \) of a particle with the rest mass \( m_0 \) is function of two independent quantities, \( U \) and \( P \). This can be obtained when solving the invariant (80) or (97) with respect to \( E \). One gets

\[
E(U, P) = \sqrt{1+U^2} \sqrt{P^2 + m_0^2} - U \cdot P.
\]  

Note that this relation, when expressed in terms of \( u \),

\[
E = \gamma \left( \sqrt{P^2 + m_0^2} - u \cdot P \right),
\]  

represents the relativistic transformation of energy between the isolated fractal reference frames \( S_1 \) and \( S_2 \). When calculating partial derivatives of the energy \( E(U, P) \) with respect to \( P \) and \( U \), one obtains

\[
\frac{\partial E}{\partial P_i} = W_i, \quad \frac{\partial E}{\partial U_i} = -MW_i.
\]  

Because \( U_i \) are space components of the space-time structural four-velocity, we can consider the above two partial differential equations as an analogue of the Hamilton equations. First of them serves as definition of \( W \). In this way the vector \( W \) is defined as partial derivative which involves the mechanical variables \( E \) and \( P \). Therefore, we will refer to it as "mechanical velocity". By this name we want to distinguish the mechanical velocity \( W \) from the "kinematical velocity" \( v \) which is defined by means of pure kinematical variables \( r \) and \( t \). Unlike the structural velocity \( u \), both kinematical and mechanical velocity are quantities reflecting amount of motion. The explicit relations between them (Eqs. (85) and (102)) depend on the way the space-time anisotropy is induced by the interaction.

The second equation in (111) can be exploited by the independent definition of the particle mass. We can define namely

\[
M = \frac{\partial E}{\partial U_i} \left( \frac{\partial E}{\partial P_i} \right)^{-1}.
\]  

Using this definition and exploiting Eq. (109), we obtain the formula

\[
M(U, P) = \sqrt{\frac{P^2 + m_0^2}{1 + U^2}}
\]  

which depends on the magnitudes of both \( U \) and \( P \). Inserting this expression into the first equation (111), we get

\[
\frac{P}{M} - U = W.
\]  

From here the relations (104) and (105) follow immediately. When considering the vector \( U \) as a scale dependent fluctuating anisotropy parameter of space-time, the mass of a particle can be treated as a quantity proportional to the change of particle’s energy with these space-time fluctuations. At small scales the characteristic size of fluctuations increases and the mass decreases. On the contrary when the resolution decreases, the fluctuations become negligible. For small \( U \to 0 \) the change of energy with fluctuations remains finite

\[
\frac{\partial E}{\partial U_i} \to -p_i,
\]  

where
allowing for smooth limit

\[ M \to \frac{m_0}{\sqrt{1-v^2}} \]  \hspace{1cm} (116)

The independence of the energy \( E \) and the mass \( M \) of a particle is only one of the consequences in the anisotropic space-time. For the same reasons we have to distinguish the particle’s momentum

\[ p \equiv Mv \]  \hspace{1cm} (117)

from the ”impulse” \( P \) of the particle standing in the text in upper case notation. The momentum \( p \) (or more precisely the kinematical momentum) is product of the particle’s mass \( M \) and its kinematical velocity \( v \). It satisfies the standard dispersion relation between the energy \( E \) and the rest mass \( m_0 \),

\[ E^2 = (Mv)^2 + m_0^2. \]  \hspace{1cm} (118)

In classical mechanics, when the velocity of a particle and therefore its momentum are constant in time, this indicates that the particle is free. If, however, the momentum of the particle changes with time, the particle is said to be acted upon by a force. In anisotropic space-time force, work and the kinetic energy is directly connected with the ”impulse” \( P \) of the particle. The force acting on the particle is equal to change of the particle’s ”impulse” \( P \) per unit time,

\[ F = \frac{dP}{dt}. \]  \hspace{1cm} (119)

The connection between \( F \) defined in space-time with structural anisotropy \( u \) and the force \( F_2 \) expressed in the isolated fractal reference system \( S_2 \) reads

\[ F_2 = \frac{dp_2}{dt_2} = \frac{1}{1-u \cdot v} \left( F \gamma^{-1} - \sigma u \times F - u \frac{dE}{dt} \right) \]  \hspace{1cm} (120)

or

\[ F_2 = \frac{dp_2}{dt_2} = \frac{1}{1-u \cdot v} \left( F + u \frac{u \cdot F}{u^2} (\gamma^{-1} - 1) - u \frac{dE}{dt} \right), \]  \hspace{1cm} (121)

in dependence the fractal interaction is with or without torsion, respectively. In standard mechanics, the work \( A \) done by a force per unit time is defined as scalar product of the force and velocity. This implies the same definition in any isolated fractal reference frame, in particular in \( S_2 \),

\[ A_2 = v_2 \cdot F_2. \]  \hspace{1cm} (122)

On the other side, the work \( A_2 \) equals to change of the kinetic energy \( T_2 = E_2 - m_0 \) per unit time,

\[ A_2 = \frac{dT_2}{dt_2} = \frac{dE_2}{dt_2}. \]  \hspace{1cm} (123)

Realizing that

\[ \frac{dE_2}{dt_2} = \frac{1}{1-u \cdot v} \frac{dE}{dt}, \]  \hspace{1cm} (124)

and exploiting Eqs. (120) or (121), we get

\[ \frac{dE}{dt} = W \cdot F, \]  \hspace{1cm} (125)

where \( W \) is given by Eqs. (85) or (102), respectively. This relation states that, in space-time with the structural anisotropy \( u \), the change of energy per unit time is equal to the scalar product of the acting force \( F \) and the mechanical velocity \( W \). Inserting here the expression (119) for \( F \), we arrive again at the first equation (111).

One can proceed in the reverse order and define the kinetic energy \( T \) by the equation

\[ \frac{dT}{dt} = A = W \cdot F \]  \hspace{1cm} (126)
which means that change of the kinetic energy per unit time is equal to the work $A$. Using (119) and (104) the right-hand side of Eq. (126) may be rewritten to the form

$$A = \left[ W \frac{d}{dt} M(W + U) \right] = M \left( W \cdot \frac{dW}{dt} + U \frac{dU}{dt} + W \frac{dW}{dt} \right) \left[ \frac{(W^2 + U \cdot W)}{(1 - 2U \cdot W - W^2)} \right] = \frac{d}{dt} \left[ M(1 - U \cdot W) \right]. \tag{127}$$

Here we have used the identity

$$W \cdot \frac{dW}{dt} = W \frac{dW}{dt}. \tag{128}$$

Inserting expression (127) to the right-hand side of Eq. (126) and integrating over $t$, we obtain for the kinetic energy of a particle

$$T = \frac{m_0(1 - U \cdot W)}{\sqrt{1 - 2U \cdot W - W^2}} - m_0. \tag{129}$$

The integration constant is chosen so that $T(W = 0) = 0$. For small values of $W$ compared with unity, we can make an expansion in terms of $W$. We get to a first approximation

$$T = \frac{1}{2} m_0 \left( \frac{W^2}{v^2} \right) = \frac{1}{2} m_0 v^2. \tag{130}$$

Last equality between $W^2$ and $v^2$ follows from both Eqs. (85) and (102) simultaneously. For small velocities, we have obtained standard expression for the kinetic energy in terms of the kinematical velocity $v$.

### VII. SPACE-TIME ANISOTROPY AND KLEIN-GORDON EQUATION

The relations (104) and (118) between the kinematical and mechanical variables are form invariant with respect to the motion transformations (55) and (82), or (55) and (99), respectively. The transformations preserve the quadratic forms (53) and (80). Another relation which is invariant under these transformations is action of a free particle. The action in space-time with anisotropy characterized by the structural velocity $u$ can be written in the form

$$S_u = -E t + \mathbf{r} \cdot \mathbf{P} \gamma^{-1} + \sigma \mathbf{u} \cdot (\mathbf{r} \times \mathbf{P}), \tag{131}$$

or

$$S_u = -E t + \mathbf{r} \cdot \mathbf{P} + \frac{(\mathbf{r} \cdot \mathbf{u})(\mathbf{u} \cdot \mathbf{P})}{u^2} (\gamma^{-1} - 1) \tag{132}$$

in dependence on type of the fractal interaction. Using the mechanical velocities (85) and (102), we can define the vectors $\mathbf{X}(\mathbf{r}, \mathbf{u}) = \mathbf{W} t$. In terms of $\mathbf{X}$, both expressions (131) and (132) read

$$S_u = -\tau m_0 = -E t + \mathbf{P} \cdot \mathbf{X}(\mathbf{r}, \mathbf{u}). \tag{133}$$

Having determined the action, we re-examine the Klein-Gordon equation for a free particle. The corresponding d’Alambertian operator is modified in the metrics (52) as follows

$$\Box_u = \partial^i \left[ \eta(u) \right]^{-1} \partial^i \tag{134}$$

where

$$\left[ \eta(u) \right]^{-1} = \begin{pmatrix} -\delta_{ij} & -u_i \\ -u_j & 1 - u^2 \end{pmatrix}. \tag{135}$$

Here $\partial = (\partial_r, \partial_0)$ is four-derivative with respect to the four-coordinates $r = (\mathbf{r}, t)$. If we introduce the covariant derivatives

$$D_0 = \partial_0, \quad D = \partial + u \partial_0, \tag{136}$$

the explicit form of the operator (134) can be written in the way

$$\Box_u = \partial^2_0 - (\partial + u \partial_0)^2 \equiv D_0^2 - D^2. \tag{137}$$
One can check validity of the canonical operator equations
\[ iD_0 \psi_u = E \psi_u, \quad -iD \psi_u = M v \psi_u \] (138)
for the solution \( \psi_u = \exp(iS_u) \), where \( S_u \) is given by Eq. (133). The corresponding modified Klein-Gordon equation
\[ -\Box_u \psi_u = m_0^2 \psi_u \] (139)
leads to the invariant relation \( \Box \) between the energy \( E \) and the (kinematical) momentum \( p = M v \) of a free particle with the rest mass \( m_0 \). This relation remains valid for arbitrary non-zero space-time structural velocity \( u \). Invariance of the d’Alambertian operator and thus form-invariance of the Klein-Gordon equation with respect to the motion relativistic transformations is seen from the expression
\[ \Box_u = \partial^\dagger [\eta(u)]^{-1} \partial = \partial^\dagger [\eta(u)]^{-1} \partial' = \Box_u. \] (140)
Here we have exploited the decomposition
\[ [\eta(u)]^{-1} = \Delta_D [\eta(u)]^{-1} \Delta_D^\dagger \] (141)
and the corresponding transformation property
\[ \partial' = (\Delta_D^\dagger)^{-1} \partial. \] (142)
The above relations reflect motion invariance of the Klein-Gordon equation expressed in terms of the covariant derivatives. Form of the covariant derivatives suggests to consider the space-time anisotropy inherent to the very structure of space-time. This should include elementary quantum fields as possible source of the space-time fluctuations and requires further study which lies beyond the scope of this work.

**VIII. CONCLUSIONS**

The questions addressed in the paper were stimulated by fractal properties of the \( z \)-scaling observed in the inclusive reactions at high energies. In the present status of our investigations the fractality was considered with respect to the constituent (parton) content of the colliding hadrons and nuclei. We suppose that constituents of these extended objects are composed of smaller constituents which in turn are built of even smaller constituents forming thus a structure typical for fractals. The scaling variable \( z \) was constructed as a fractal measure connecting kinematics of the constituent interactions with the anomalous fractal dimensions \( \delta_1 \) and \( \delta_2 \) of the objects colliding at high energies. Value of \( z \) depends on resolution at which the underlying interaction of constituents can be singled out of the reaction. Insisting on the minimal possible resolution, we have obtained relativistic transformations as functions of the resolution dependent structural velocity \( u \). The generalization of these transformations to 3+1 dimensions includes two separate situations; the interaction of fractals with and without mutual torsion. The obtained transformations were interpreted as special realization of structural relativity. Considered realization of the relativistic principle was formulated with respect to the isolated structural reference frames associated with the isolated fractal objects of various anomalous dimensions. More generally, in view of intrinsic relation between the fractality of the interacting objects and the fractal structure of space-time, the isolated reference systems of structural relativity have been considered as attributed to the very structure of the (QCD) vacuum as well.

Motivated by many investigations concerning fractal properties of space-time, we have proceeded beyond the isolated fractal systems. This concerns our working hypothesis that interaction of fractal objects with different anomalous fractal dimensions can induce structural anisotropy of space-time. We have demonstrated that this hypothesis leads to asymmetry between the relativistic kinematics and relativistic mechanics. The kinematical sector was parametrized by the four-coordinates \( r = (r, t) \), the kinematical velocities \( v = dr/dt \), the structural velocities \( u \) and by their composition laws. In the mechanical sector enter the energy \( E \) of a particle, its impulse \( P \), mechanical velocity \( W \) and their transformation properties which are explicit functions of the structural anisotropy \( U \). Due to such splitting, the space-time anisotropy would lead to separation of the particle’s mass \( M \) from its energy \( E \) and also to separation of the particle’s mechanical momentum \( M W \) from its impulse \( P \). Both independent quantities, \( M \) and \( M W \), where shown to enter into two equations which are analogue of Hamiltonian equations in Newtonian mechanics. Connection between the kinematical and mechanical sector is characterized by the quantities such as kinematical momentum \( p = Mv \), force \( F = dP/dt \), work \( A = WF \) done by the force by unit time, and the relation between the kinematical and mechanical velocities \( v \) and \( W \).
In the considered special realization of space-time structural relativity, the space-time structural anisotropy $U$ (or the structural velocity $u$) was treated as a relative quantity governed by relativistic principles. The formulation suggests that space-time anisotropy can be induced in the ultra-relativistic collisions of structural objects such as hadrons and nuclei. The anisotropy is function of the anomalous fractal dimensions of the colliding objects. The fractal dimensions characterize constituent fractal-like hadronic sub-structure which seems to be universal property of hadronic matter revealed at high energies. Presented approach to the $z$ scaling shows that the observed regularity has relevance to fundamental principles of physics at small scales. More detailed study of the fractal aspects of $z$-scaling, both theoretical and experimental, can give better understanding of the structure of hadrons and nuclei, interaction of their constituents and particle formation in the domain tested by large accelerators of hadrons and nuclei.

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X. APPENDIX

In the theory of special relativity, the spatial symmetry lies at the root of the standard pseudo-Euclidean metric. Uni-directional breakdown of this symmetry compatible with the relativistic methods was considered within the framework of Finsler geometry [22, 23]. Spatial anisotropy is expressed in terms of the Finslerian parameter $g$ which is treated as an universal constant of pure geometrical origin. It characterizes degree of Finslerian non-Riemannianity of space-time. For small values of $g$, the Finsler-relativistic metric function $F(g, R)$ and the associated Hamiltonian function $H(g, P)$ [24] can be approximated as follows

\[
F(g, R) = |T + g_+ |R||^{(1+u)/2}|T + g_- |R||^{(1-u)/2} \sim (T^2 - R^2 - gT|R|)^{1/2}, \tag{143}
\]

\[
H(g, P) = |P_0 - g_+ |P||^{(1+u)/2}|P_0 - g_- |P||^{(1-u)/2} \sim (P_0^2 - P^2 + gP_0|P|)^{1/2} \tag{144}
\]

where we have used the notation

\[
u = \frac{g/2}{\sqrt{1+(g/2)^2}} \tag{145}\]

Comparing the approximate expressions for the Finsler metric function [143] and the associate Hamilton function [141] with the invariants [52] and [51], one can judge to the correspondence $g = 2U$ between the Finslerian parameter $g$ and the quantity $U$ given by Eq. [14]. This is seen from the terms proportional to $gT$ and $gP_0$, respectively. While both parameters $g$ and $U$ characterize space-time isotropy violation, there are substantial differences in the form of the metric invariants even for their small values. In the Finslerian case, the light front is approximated by a sphere with the radius $\sqrt{1+(g/2)^2}$. The sphere is shifted in the direction of the isotropy breakdown by a value $g/2$. Spherical form of the light front involves deformation of the scales perpendicular to motion whenever $g \neq 0$ and thus results in violation of the spatial isotropy. Similar concerns the Hamiltonian associate function [141] and the accessible range of the corresponding ratio $P/E$. Unlike the Finslerian metric forms, the invariant [51] is different. The light front forms an ellipse with one focus situated in the point $v = 0$. The elliptical form preserves the scales perpendicular to motion even for $U \neq 0$. In the situation we consider, the spatial isotropy is thus not violated. Similar holds for the invariant [50] and the accessible range of the corresponding parameter $P/E$.

The important point in both approaches is that, in the regions of $g \neq 0$ ($U \neq 0$), the light velocity value should be anisotropic in whatever inertial reference system. Standard interpretation of the Michelson-Morley-type experiments (including optical interferometer experiments [24, 25, 26] and modern high-precision laser experiments [27, 28]) seems to be however negative with this respect. The experiments steadily reproduce "no fringe shift" and, therefore, do not support any deviation which would point to even tiny portion of the anisotropic spread of light. In the Finslerian treatment, "null result" of these experiments was interpreted [24] as a "possible conspiracy of Nature" in the compensation of two effects: the light-velocity anisotropy and the standard spatial length anisotropy. Possibility of such compensation was shown to the first order of accuracy with neglecting the second-order relativistic effects. We show bellow that the Michelson-Morley-type experiments alone do not imply absolute absence of anisotropy in light propagation in our approach. The argumentation includes arbitrary accuracy of relativistic effects.

Let us consider experiment with the interferometer having two perpendicular arms of the length $d_I$ and $d_{II}$, respectively. Suppose the light beam from a light source is divided into two rays, $I$ and $II$, traveling perpendicular to each other along the arms. The mirrors placed on the ends of the spectrometer arms reflect the light back to the
telescope where the rays interfere with each other. Assuming the apparatus is placed in a region where the propagation of light is not isotropic one could expect existence of a phase difference $\Delta t$ between the rays I and II which is due to the anisotropy. When the apparatus is rotated through an angle of $90^\circ$, the orientation of the spectrometer arms is interchanged and the phase difference becomes $-\Delta t$. According to our standard intuition, such rotation of the apparatus should cause a shift of the interference fringes between the two rays. We show however, that this must not to be the case for any non-zero value of the space-time anisotropy even up to the arbitrary order of experimental accuracy. Suppose there exists a space-time anisotropy $u$ induced by some reasons. Let us assume that the anisotropy results in the metric changes (22) associated with deformation of the spherical light front. In this case, the light front becomes an ellipsoid (71) with one focus in the point where the light was emitted (Fig.1). The time $t_I$ and $t_{II}$ which the light rays take to travel in spectrometer arms I and II can be expressed as follows

$$t_I = d_I \left( \frac{1}{v_1(\phi)} + \frac{1}{v_2(\phi)} \right), \quad t_{II} = d_{II} \left( \frac{1}{v_3(\phi)} + \frac{1}{v_4(\phi)} \right)$$

(146)

where the angle $\phi$ describes orientation of the spectrometer with respect to the space-time anisotropy $u$. Because of the anisotropy, the velocities of light propagation in different directions $v_i(\phi)$ are not equal and depend on the orientation of the spectrometer (Fig.1). On the other hand, the spatial distances (lengths of spectrometer arms) do not depend on the orientation of the spectrometer (metric (22)). This follows from the known fact that the spatial geometry is not simply given by the spatial part $\eta^{ij}$ of the four dimensional metric $\eta_{\mu\nu}(u)$. The metric tensor $\eta^*_{ij}$ which determines the spatial geometry is given by [19]

$$\eta^*_{ij} = -\eta_{ij} + \eta^*_{i} \eta^*_{j}, \quad \eta^*_{i} = \frac{\eta_{00}}{\sqrt{\eta_{00}}}$$

(147)

In the case of the four-dimensional metric (22), the spatial metric reads

$$\eta^*_{ij} = \delta_{ij}.$$ 

(148)

Therefore, lengths of the spectrometer arms (the distances $d_I$ and $d_{II}$) are invariant under space rotations and thus do not depend on the angle $\phi$. Note that the same holds for the metrics (21), $\tilde{\eta}^*_{ij} = \delta_{ij}$.

We exploit now the following geometrical property of the velocity ellipsoid (21). While the sections $v_i(\phi)$ connecting any point of the ellipsoid with its focus depend on their orientation $\phi$, the combinations

$$\frac{1}{v_1(\phi)} + \frac{1}{v_2(\phi)} = \frac{2a}{b^2}, \quad \frac{1}{v_3(\phi)} + \frac{1}{v_4(\phi)} = \frac{2a}{b^2}$$

(149)

are rotationally invariant i.e. do not depend on the angle $\phi$. Here

$$a = \gamma^2, \quad b = \gamma$$

(150)

are major and minor semi-axis of the ellipsoid (21), respectively. After inserting expressions (149) into Eq. (146), we get

$$t_I = 2d_I, \quad t_{II} = 2d_{II}.$$ 

(151)

These relations connect time the light rays take to travel in the spectrometer arms with the lengths $d_I$ and $d_{II}$ to the point they interfere. Both expressions are rotationally invariant. Therefore, rotation of the spectrometer apparatus can not cause any shift of the interference fringes even for $u \neq 0$.

In order to show that the invariance (149) is not accidental we will discuss more complicated case. Let us consider a tree mirrors set-up which reflect the rays of light along the sides of a triangle $ABC$. For definiteness consider the triangle depicted by the full lines in Fig.2a. Suppose a light signal is emitted in the point $A$ and then travels along the path $d_1$, $d_2$, and $d_3$. The corresponding time interval

$$t_{ABC} = \frac{d_1}{v_1(\phi)} + \frac{d_2}{v_2(\phi)} + \frac{d_3}{v_3(\phi)}$$

(152)

is function of the velocities $v_1(\phi)$, $v_2(\phi)$, and $v_3(\phi)$. The velocities are shown on the velocity diagram in Fig.2b. They depend on the orientation $\phi$ of the triangle $ABC$ relative to the space-time anisotropy $u$. We show that if this experimental set-up rotates, the time $t_{ABC}$ remains invariant, though values of the velocities $v_i(\phi)$ change during such rotations. The $\phi$ invariance of the expression (152) follows from the specific geometrical property of any rotational
ellipsoid which we outline below. Let us denote the internal angles of the triangle $ABC$ as $\alpha_1$, $\alpha_2$, and $\alpha_3$ (Fig. 2a). They comply the elementary geometrical property of the constant ratio

$$\frac{d_1}{\sin \alpha_1} = \frac{d_2}{\sin \alpha_2} = \frac{d_3}{\sin \alpha_3} \equiv d_{ABC}$$

which we denote as $d_{ABC}$. The angles among the corresponding velocities $v_1(\phi)$, $v_2(\phi)$, and $v_3(\phi)$ are shown in Fig. 2b and are indicated by $\beta_1$, $\beta_2$, and $\beta_3$, respectively. In the considered mirror setup, the angles are fixed by the relations

$$\beta_i = \pi - \alpha_i, \quad i = 1, 2, 3.$$  

Because of spatial rotational invariance (148), the angles $\alpha_1$, $\alpha_2$, as well as the distances $d_i$ do not depend on the rotation of the apparatus as the whole and thus do not depend on the angle $\phi$. Therefore Eq. (152) takes the form

$$t_{ABC} = d_{ABC} \left( \frac{\sin \beta_1}{v_1(\phi)} + \frac{\sin \beta_2}{v_2(\phi)} + \frac{\sin \beta_3}{v_3(\phi)} \right).$$

Let us now exploit the following geometrical property valid for any rotational ellipsoid. Consider the ellipse which forms intersection of the ellipsoid with a plane passing through its focus. This focus is common focus for the ellipse

$$ABC$$

and the plane determined by the orientations of the ray velocities passing through the arms of the triangle $ABC$. The ray velocities mark out three different points on such ellipse. Let us denote the sections connecting the focus of the ellipse with these points by $A$, $B$, and $C$, respectively (Fig. 2b). One can convince itself that, while the magnitudes of the ray velocities $v_i(\phi)$ are functions of the angle $\phi$, the combination

$$\frac{\sin \beta_1}{v_1(\phi)} + \frac{\sin \beta_2}{v_2(\phi)} + \frac{\sin \beta_3}{v_3(\phi)} = \frac{a}{b^2} (\sin \beta_1 + \sin \beta_2 + \sin \beta_3)$$

does not depend on $\phi$. The symbols $a$ and $b$ are given by (156) and denote the major and minor semi-axes of the ellipse or the ellipsoid, respectively. Here we have in mind the velocity ellipsoid (71) in space-time with the anisotropy $\mathbf{u}$ and the plane determined by the orientations of the ray velocities passing through the arms of the triangle $ABC$. The ray velocities mark out three different points on such ellipse. Let us denote the sections connecting the focus of the ellipse with these points by $v_1(\phi)$, $v_2(\phi)$, and $v_3(\phi)$, respectively. Searching for the above remarkable geometrical property of the rotational ellipsoids was inspired by pure physical reasons and shows how physics and geometry are tightly interconnected. The relation (157) represents continuous generalization of the expressions (149). Really, if we identify the point $A$ with the point $B$ of the triangle $ABC$, it degenerates into the abscissa $AB$. In this case $d_3 = 0$, $\alpha_3 = 0$, $\alpha_1 = \alpha_2 = \pi/2$ and the relation (157) becomes identical with Eq. (149). Now it remains to exploit Eqs. (150), (153), (155), (157), and one gets the expression

$$t_{ABC} = d_1 + d_2 + d_3$$

which does not depend on the orientation $\phi$. Therefore, time $t_{ABC}$ the light rays take to travel along the triangle $ABC$ does not depend on its orientation with respect to the space-time anisotropy $\mathbf{u}$. As a consequence, arbitrary rotation of a three mirror set-up will not cause any shift of the interference fringes of light even for $\mathbf{u} \neq 0$.

Let us now imagine the light signal traveling along the triangles $ABC$ and $ACD$ depicted in Fig. 2a in the following order. The signal is emitted in the point $A$ and travels along the lines $d_1$, $d_2$, and $d_3$. The signal is partially reflected back in the point $A$ and then travels the distances $d_4$, $d_5$, and $d_6$. As follows from the above considerations applied to both triangles $ABC$ and $ACD$ separately, the light takes to travel the whole path during time

$$t_{ABC} + t_{ACD} = d_1 + d_2 + d_3 + d_4 + d_5 + d_6 = t_{out} + t_{int}$$

which does not depend on the particular choice of the angle $\phi$. Here we have denoted by $t_{int}$ time the light ray travels along the internal line $CA$ to and fro. According to Eq. (154), $t_{int}$ depends on the distance $d_{CA}$ through the rotational invariant relation $t_{int} = d_3 + d_4 = 2d_{CA}$. Consequently, the expression

$$t_{out} = d_1 + d_2 + d_5 + d_6$$

possess the rotational symmetry and do not depend on the space-time anisotropy $\mathbf{u}$, as well. It is possible to think of various trajectories from the point $A$ to the point $B$ corresponding to various experimental arrangements
of interference experiments. The "null result" of the interference fringe shift with rotations can be shown for such trajectories similarly.

This appendix should be understood so that we do not advocate the anisotropic spread of light in general. We point here only to the fact that Michelson-Morley-type interferometer experiments do not contradict to particular situations in which the anisotropy of light propagation in space-time could not be a priori excluded. This concerns not only small scale structures in particle physics but also space-time features at cosmological distances. There exists new theoretical studies [30], suggesting eccentric expansion of the universe resulting from its arrangements at large scales. These investigations point to possible non-sphericity of the universe which at some $t$ could evolve into an ellipsoid.

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FIG. 1: The velocity diagram in space-time with the structural anisotropy $\mathbf{u}$. The lines I. and II. correspond to the orientation of the spectrometer arms in the Michelson's experiment.
FIG. 2: (a) The space diagram of a multi-mirror setup. The mirrors are considered in the points \( A, B, C, \) and \( D \) reflecting the light signal along the sketched lines. (b) The velocity diagram corresponding to the mirrors setup shown in Fig.2a.