THE FIBONACCI DECOMPOSITION OF SYMMETRIC TETRANACCI POLYNOMIALS

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Abstract. In this manuscript, we introduce (symmetric) Tetranacci polynomials $\xi_j$ as a twofold generalization of ordinary Tetranacci numbers, by considering both non unity coefficients and generic initial values in their recursive definition. The issue of these polynomials arose in condensed matter physics and the diagonalization of symmetric Toeplitz matrices having in total four non-zero off diagonals. For the latter, the symmetric Tetranacci polynomials are the basic entities of the associated eigenvectors; thus, treating the recursive structure determines the eigenvalues as well. Subsequently, we present a complete closed form expression for any symmetric Tetranacci polynomial. The key feature is a decomposition in terms of generalized Fibonacci polynomials.

1. Introduction

Undoubtedly one of the most famous sequences are the Fibonacci numbers $f_n$, defined recursively by $f_{n+1} = f_n + f_{n-1}$ for $n \geq 1$ and $f_0 = 0, f_1 = 1$ [1, 2, 3]. As noticed by Horadam in the midst of the last century, generalizations require either altered initial values or alternatively a modified recursion formula [3]. For instance, Webb and Parberry did the former and studied Fibonacci polynomials $F_n$ obeying ($n \geq 1$) $F_{n+1} = x F_n + F_{n-1}, F_0 = 0, F_1 = 1$ [4]. Half a decade later, Hoggatt Jr. and Long defined in Ref. [5] generalized Fibonacci polynomials $F_n$ as ($n \geq 1$)

$$F_{n+1} = x F_n + y F_{n-1}, \quad F_0 = 0, F_1 = 1,$$

(1.1)

while Özvatan and Pashaev substituted $F_{0,1}$ by generic initial values $G_{0,1}$ [6].

Extending the recursion range in Eq. (1.1) from two to three yields Tribonacci numbers or Tribonacci polynomials depending on the coefficients and supposing properly chosen initial values [7, 8, 9, 10, 13]. Subsequently, the first notion of Tetranacci numbers, where the next element of the sequence is formed by the previous four, appeared (to our best knowledge) in Ref. [7]. Since then, Tetranacci or Tetranacci-like sequences were studied in many variations up to modern days, cf. Refs. [11, 12, 14, 15, 16] in order to mention only a few. The most generic form of what we call hereinafter Tetranacci polynomials $t_n$ ($n \geq 0$)

$$t_{n+2} = x_1 t_{n+1} + x_0 t_n + x_{-1} t_{n-1} + x_{-2} t_{n-2}$$

(1.2)

with some initial values $t_{-2}, \ldots, t_1$ and given coefficients $x_1, \ldots, x_{-2}$ was previously presented in Ref. [12]. In contrast to Eq. (1.2), we focus on the special situation of $x_{-2} = -1$ and $x_1 = x_{-1}$ (cf. Eq. (2.1) below) but still generic $x_{0,1}$.

This particular choice of coefficients seems arbitrary; it is not. Rather, this type of polynomials appear in condensed matter theory [19] as elements of eigenvectors [17] or also as basic entities of Green’s functions in quantum transport [18]. Their appearance in solid state physics originates from the fact that we physicists consider most often particular systems, to which we refer as being ”translation invariant”. In more mathematical terms, the model’s physics is captured by (banded) Toeplitz matrices.
Although the issue of eigenvalues of banded Toeplitz was investigated formerly in more
generality [20], symmetric tridiagonal Toeplitz in particular adopt here a key role as their
eigenvector elements are Chebyshev [21] or Fibonacci polynomials [22]. As a side note, this
feature of Fibonacci/ Chebyshev recursions seems to generalize also to tridiagonal, non Toeplitz
matrices as Refs. [19, 23] suggest, supposing here that specific requirements are satisfied (cf.
Refs. [24, 25, 26]) even though the authors perhaps had not noticed that.

In case of the previously mentioned constraints on \( x_1, \ldots, x_{-2} \), we refer to symmet-
tric Tetranacci polynomials (cf. Definition 2.3 below) originating from symmetric Toeplitz/
Toeplitz-like matrices owning a bandwidth of two [17, 19]. There, they are associated to the
characteristic polynomial and the entries of eigenvectors in analogy to the tridiagonal case and
Fibonacci/ Chebyshev polynomials. Our main contribution is to present a simple and closed
form expression for generic symmetric Tetranacci polynomials. The simplicity originates from
a decomposition into generalized Fibonacci polynomials (cf. section 3). As motivation, the
fundamental conviction of physicists is that eigenstates (eigenvectors) of finite systems are
given in terms of standing waves. Their form is sinusoidal and the perhaps most evident ex-
ample are oscillations of a guitar string, which is fixed at both ends. By the knowledge that
Binet/ Binet-like forms can be reshaped into a sine function [4, 5], the stage was set.

The manuscript is organized as follows. In section 2, we formally define symmetric
Tetranacci polynomials and present the basic strategy to find their closed form expression.
Subsequently, we introduce so called basic Tetranacci polynomials and discuss a few of their
properties. In section 3, we demonstrate that specific generalized Fibonacci polynomials obey
also the Tetranacci recursion formula. Finally, we verify that any generic Tetranacci polyno-
mial can be expressed in terms of those specific solutions.

2. Generic properties of symmetric Tetranacci polynomials

**Definition 2.1.** The symmetric Tetranacci polynomial \( \xi_j \) is recursively defined by

\[
\xi_{j+2} = \zeta \xi_j - \xi_{j-2} + \eta (\xi_{j+1} + \xi_{j-1}), \quad j \in \mathbb{Z}
\]

in terms of its initial values \( \xi_i = g_i(\zeta, \eta) \in \mathbb{C} \) \( i = -2, \ldots, 1 \) and complex coefficients \( \zeta, \eta \).

Although the initial values may or may not depend themselves on \( \zeta \) and/or \( \eta \), we utilize
always the shorthand notation of \( g_{-2}, \ldots, g_1 \) and \( \xi_j \) respectively rather then to mention this
dependency explicitly. For the purpose of illustration, the first few \( \xi_j \) terms are

\[
\begin{align*}
\xi_2 &= -g_{-2} + \eta g_{-1} + \zeta g_0 + \eta g_1, \\
\xi_3 &= -\eta g_{-2} + (\eta^2 - 1) g_{-1} + \eta (\zeta + 1) g_0 + (\eta^2 + \zeta) g_1, \\
\xi_4 &= - (\eta^2 + \zeta) g_{-2} + \eta (\eta^2 + \zeta - 1) g_{-1} + (\zeta + 1) (\zeta^2 - 1 + \eta^2) g_0 \\
&\quad + \eta (\eta + 2\zeta + 1) g_1.
\end{align*}
\]

and further terms follow from Eq. (2.1). Alternatively, we may also rely on the generating
function, to which we turn next.

**Proposition 2.2.** The generating function \( E(t) = \sum_{n=0}^{\infty} \xi_n t^n \) of symmetric Tetranacci polyno-
mials reads

\[
E(t) = \frac{g_1 t + g_0 (1 - \eta t) + g_{-1} (\eta t^2 - t^3) - g_{-2} t^2}{1 - \eta t - \zeta t^2 - \eta t^3 + t^4}.
\]
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| $j$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|-----|-----|-----|-----|-----|-----|-----|
| $\mathcal{T}_{-2}(j)$ | $\eta$ | $1$ | $0$ | $0$ | $0$ | $-1$ |
| $\mathcal{T}_{-1}(j)$ | $\zeta$ | $0$ | $1$ | $0$ | $0$ | $\eta$ |
| $\mathcal{T}_{0}(j)$ | $\eta$ | $0$ | $0$ | $1$ | $0$ | $\zeta$ |
| $\mathcal{T}_{1}(j)$ | $-1$ | $0$ | $0$ | $0$ | $1$ | $\eta$ |

Table 1. Basic Tetranacci polynomials $\mathcal{T}_i(j)$ for $j = -3, \ldots, 2$. The central columns ($j = -2, \ldots, 1$) provide the initial values according to Eq. (2.7). The inversion point ($\bullet$) proposes the relations $\mathcal{T}_0(j) = \mathcal{T}_{-1}(-1-j)$ and $\mathcal{T}_1(j) = \mathcal{T}_{-2}(-1-j)$ for arbitrary $j$, $\eta$, $\zeta$ which is proven in Lemma 2.4. (Table from Ref. [19] with permission.)

Proof. Using the definition of the generating function $E(t) = \sum_{n=0}^{\infty} \xi_n t^n$ yields that

$$E(t) = g_0 + g_1 t + t^2 \sum_{n=0}^{\infty} \xi_{n+2} t^n$$

$$= g_0 + g_1 t + t^2 \sum_{n=0}^{\infty} [\zeta \xi_n - \xi_{n-2} + \eta (\xi_{n+1} + \xi_{n-1})] t^n$$

$$= g_0 + g_1 t + g_{-1} t^2 (\eta - t) - g_{-2} t^2 - g_0 \eta t + E(t) (\zeta t^2 - t^4 + \eta t^3 + \eta t)$$

(2.6)

is true. Here, we substituted Eq. (2.1) for the last term, and we completed properly the sums in order to provide $E(t)$; thus, solving for $E(t)$ grants Eq. (2.5).

In this article, we provide the reader with a full closed form expression for any symmetric Tetranacci polynomial. Perhaps contrary to their appearance in Eqs. (2.2)-(2.4), the closed form expression is rather simple. Since the intention of our mindset is the applicability, we aim at a specific form for $\xi_j$, namely Eq. (2.8) below, demanding the introduction of specific solutions to Eq. (2.1), hereinafter referred to as the basic Tetranacci polynomials.

Definition 2.3. The basic Tetranacci polynomials $\mathcal{T}_i(j)$ ($i = -2, \ldots, 1$) satisfy Eq. (2.1) for generic $j \in \mathbb{Z}$ and their initial values are summarized by

$$\mathcal{T}_i(j) = \delta_{ij}, \quad i, j = -2, \ldots, 1.$$  (2.7)

Here, $\delta_{ij}$ denotes the Kronecker-Delta\footnote{The Kronecker-Delta is defined as $\delta_{nl} = 1$ for $n = l$ and zero otherwise.} and we call Eq. (2.7) the selective property of $\mathcal{T}_i(j)$.

For the purpose of illustration, Table 1 presents the first few terms of $\mathcal{T}_i(j)$ while Eqs. (2.2)-(2.4) provide additional ones. The primary advantage of the basic Tetranacci polynomial resides in the fact that the arbitrary initial values of $\xi_j$ and the recursion formula Eq. (2.1) separate by means of the selective property. A similar strategy was pursued in Ref. [15] for Tetranacci numbers. Nevertheless, we then have initially to deal with four symmetric Tetranacci polynomials rather than only one.
Corollary 1. Any symmetric Tetranacci polynomial \( \xi_j \) can be written as
\[
\xi_j = \sum_{i=-2}^{1} g_i \mathcal{T}_i(j), \quad j \in \mathbb{Z}
\] (2.8)
for generic \( \eta, \zeta \in \mathbb{C} \) and complex initial values \( \xi_i = g_i, i = -2, \ldots, 1 \).

Proof. Due to the linearity of the Tetranacci recursion formula, any linear combination of solutions also satisfies Eq. (2.1); thus, the l.h.s. of Eq. (2.8) is a symmetric Tetranacci polynomial. Hence, in case that Eq. (2.8) is satisfied on the level of the initial values, this relation is true for generic integer \( j \). Indeed, we find that \( (j = -2, \ldots, 1) \)
\[
\xi_j = \sum_{i=-2}^{1} g_i \mathcal{T}_i(j) = \sum_{i=-2}^{1} g_i \delta_{ij} = g_j
\]
is correct, substituting Eq. (2.7) in the intermediate step. \( \square \)

Naturally, the description of \( \xi_j \) in terms of \( \mathcal{T}_i(j) \) is not specific for Tetranacci polynomials and small modifications in both Definition 2.3 and Eq. (2.8) extend this strategy to arbitrary (linear) recursive problems.

The basic Tetranacci polynomials inherit some specific properties originating from their particular initial values, as can be anticipated from Table 1. More importantly though, we find interconnections between \( \mathcal{T}_1 (\mathcal{T}_{-1}) \) and \( \mathcal{T}_2 (\mathcal{T}_0) \) reducing effectively the number of involved polynomials. Actually, the Lemmata 2.4, 2.5 even demonstrate that \( \mathcal{T}_{-1}, \mathcal{T}_0 \) and \( \mathcal{T}_1 \) can be constructed from \( \mathcal{T}_{-2} \).

Lemma 2.4. The basic Tetranacci polynomials \( \mathcal{T}_i(j) \) \( (i = -2, \ldots, 1) \) obey
\[
\mathcal{T}_1(j) = \mathcal{T}_{-2}(-1 - j), \quad (2.9)
\]
\[
\mathcal{T}_0(j) = \mathcal{T}_{-1}(-1 - j), \quad (2.10)
\]
\[
\mathcal{T}_1(-1 - j) = \mathcal{T}_{-2}(j), \quad (2.11)
\]
\[
\mathcal{T}_0(-1 - j) = \mathcal{T}_{-1}(j), \quad (2.12)
\]
for all \( j \in \mathbb{Z} \) and generic \( \zeta, \eta \in \mathbb{C} \).

Proof. Notice that once the validity of Eq. (2.9) (Eq. (2.10)) is shown, Eq. (2.11) (Eq. (2.12)) follows automatically by setting \( l := -1 - j \) and renaming \( l \to j \) afterwards. Since the proofs of Eqs. (2.9), (2.10) are similar, we focus only on the former. The presented values in Table 1 imply the validity of Eq. (2.9) already for \( j = -2, -1, 0 \): \( \mathcal{T}_1(j) = \mathcal{T}_{-2}(-1 - j) = 0 \), and at \( j = -3 \): \( \mathcal{T}_1(j) = \mathcal{T}_{-2}(-1 - j) = -1 \). Assuming that Eq. (2.9) holds already for some integers \( n - 2, n - 1, n, n + 1 \), we are left to demonstrate Eq. (2.9) at \( n + 2 \) \( (n - 3) \) for increasing (decreasing) indices. Since \( \mathcal{T}_1(j), \mathcal{T}_{-2}(j) \) are symmetric Tetranacci polynomials, Eq. (2.1) gives
\[
\mathcal{T}_1(n + 2) = \zeta \mathcal{T}_1(n) - \mathcal{T}_1(n - 2) + \eta \left[ \mathcal{T}_1(n + 1) + \mathcal{T}_1(n - 1) \right]
\] (2.13)
at \( j = n \). Similarly at \( j = -1 - n \), we find
\[
\mathcal{T}_{-2}(-3 - n) = \zeta \mathcal{T}_{-2}(-1 - n) - \mathcal{T}_{-2}(1 - n) + \eta \left[ \mathcal{T}_{-2}(-2 - n) + \mathcal{T}_{-2}(-n) \right],
\] (2.14)
after reordering the terms. Due to our assumption, we find that Eqs. (2.13), (2.14) are identical which is equivalent to \( \mathcal{T}_1(j) = \mathcal{T}_{-2}(-1 - j) \) at \( j = n + 2 \). The demonstration for decreasing indices, i.e. for \( n - 3 \), is carried out analogously by exchanging the \( j + 2 \) and \( j - 2 \) terms in Eq. (2.1). \( \square \)
The correctness of Eq. (2.17) is a direct consequence of Eqs. (2.10), (2.15), (2.16): while Eq. (2.18) follows from Eqs. (2.12), (2.15): of an arbitrary Tetranacci polynomial \(T\) matrices of bandwidth two. As long as Eq. (2.22) is valid, this relation is consistent with

\[
T_{-2}(n + 2) = \zeta T_{-2}(n) - T_{-2}(n - 2) + \eta [T_{-2}(n + 1) + T_{-2}(n - 1)],
\]

\[
T_{-2}(-n - 2) = \zeta T_{-2}(-n) - T_{-2}(2 - n) + \eta [T_{-2}(-1 - n) + T_{-2}(1 - n)],
\]

and, due to our assumption, the two expressions differ only by a sign, i.e. Eq. (2.15) is valid. Next, we focus on Eq. (2.16). \(T_{-2}(j - 1)\) is apparently a solution to Eq. (2.1), since \(T_{-2}(j)\) is a symmetric Tetranacci polynomial and the linearity of Eq. (2.1) guarantees (2.16) to be correct. According to Table 1, this is indeed true:

\[
\begin{align*}
j = -2: & \quad T_{-2}(-3) - \eta T_{-2}(-2) = \eta - \eta = 0 \equiv T_{-1}(-2), \\
j = -1: & \quad T_{-2}(-2) - \eta T_{-2}(-1) = 1 - 0 = 1 \equiv T_{-1}(-1), \\
j = 0: & \quad T_{-2}(-1) - \eta T_{-2}(0) = 0 - 0 = 0 \equiv T_{-1}(0), \\
j = 1: & \quad T_{-2}(0) - \eta T_{-2}(1) = 0 - 0 = 0 \equiv T_{-1}(1).
\end{align*}
\]

The correctness of Eq. (2.17) is a direct consequence of Eqs. (2.10), (2.15), (2.16):

\[
T_0(j) = T_{-1}(-1 - j) = T_{-2}(-2 - j) - \eta T_{-2}(-1 - j) = \eta T_{-2}(j + 1) - T_{-2}(2 + j),
\]

while Eq. (2.18) follows from Eqs. (2.12), (2.15):

\[
T_1(j) = T_{-2}(-1 - j) = -T_{-2}(j + 1).
\]

In the view of Corollary 1 and the Lemmata 2.4, 2.5, the closed form expression (Eq. (2.8)) of an arbitrary Tetranacci polynomial \(\xi_j\) demands merely the one of \(T_{-2}\). Yet the final result for \(T_{-2}\) (cf. Theorem 3.5 below) requires some preparation.

Furthermore, Eq. (2.8) is also interesting when studying algebraic properties of \(\xi_j\). For instance, we have that (\(j \in \mathbb{Z}\))

\[
\xi_{1-j} = g_{-2} T_1(j) + g_{-1} T_0(j) + g_0 T_{-1}(j) + g_1 T_{-2}(j)
\]

(2.21)
as follows by imposing Eqs. (2.9), (2.12) on Eq. (2.8). Thus, \(\xi_{1-j} = \xi_j\) holds in case that either \(g_{-2} = g_1\) and \(g_{-1} = g_0 = 0\) is true or alternatively that \(g_{-1} = g_0\) and \(g_{-2} = g_1 = 0\) is satisfied. Similar properties of \(\xi_j\) may follow, once they have been proven for the basic Tetranacci polynomials.

Although the next statement is rather trivial from the mathematical point of view, i.e. that \(\xi_j\) can be written as linear combination of complex entities (Eq. (2.22)), the Lemma summarizes (to our best knowledge) all relations necessary to diagonalize symmetric Toeplitz matrices of bandwidth two. As long as Eq. (2.22) is valid, this relation is consistent with
the famous in solid state physics Bloch’s theorem without touching further details [19, 27]. Nevertheless, the consequences imposed by the quantities $S_{1,2}$ defined in Lemma 2.6 below are more important for us.

**Lemma 2.6.** Any symmetric Tetranacci polynomial can be expressed as

$$\xi_j = A e^{i\theta_1 j} + B e^{-i\theta_1 j} + C e^{i\theta_2 j} + D e^{-i\theta_2 j},$$

provided that $S_1 \neq \pm S_2$ and $S_{1,2}^2 \neq 4$ hold true, where $S_{1,2} = (\eta \pm \sqrt{\eta^2 + 4(\zeta + 2)}) / 2$. In Eq. (2.22), we introduced $\theta_{1,2} \in \mathbb{C}$ defined by $2 \cos(\theta_{1,2}) := S_{1,2}$. The coefficients $A, B, C, D$ are set by the initial values $\xi_i = g_i, i = -2, \ldots, 1$.

**Proof.** The announced result is found straightforwardly by the power law ansatz $\xi_j \propto r^j (r \neq 0)$ on Eq. (2.1). Substituting the ansatz and after dividing by $r^j \neq 0$, we arrive at the characteristic equation:

$$r^2 + \frac{1}{r^2} - \zeta \eta \left( r + \frac{1}{r} \right) = 0.$$  

(2.23)

Its peculiar form suggests to introduce the variable $S := r + r^{-1}$, granting in turn the quadratic equation $S^2 - \eta S - \zeta - 2 = 0$, whose zeros are

$$S_{1,2} = \frac{\eta \pm \sqrt{\eta^2 + 4(\zeta + 2)}}{2}.$$  

(2.24)

Solving $S = r + r^{-1}$ at $S = S_{1,2}$ for $r$ yields

$$r_{\pm l} = \frac{S_l \pm \sqrt{S_l^2 - 4}}{2}, \quad l = 1, 2,$$

(2.25)

having the properties $r_{+l}r_{-l} = 1$ and $r_{+l} + r_{-l} = S_l$ for $l = 1, 2$. In case all four roots $r_{\pm 1}, r_{\pm 2}$ are non-degenerate, we can express $\xi_j$ as their linear combination:

$$\xi_j = A r_{+1}^j + B r_{-1}^j + C r_{+2}^j + D r_{-2}^j.$$  

(2.26)

This requires $S_1 \neq \pm S_2$ and $S_{1,2}^2 \neq 4$. The coefficients $A, B, C, D$ are to be set by the initial values $g_{-2}, \ldots, g_1$ of $\xi_j$ and introducing $\theta_{1,2}$ via $2 \cos(\theta_{1,2}) := S_{1,2}$ grants $r_{\pm l} = \exp(\pm i\theta_l)$. □

Any degeneracy of the roots $r_{\pm 1,2}$ alters Eq. (2.22) qualitatively, i.e. the closed form expression for $\xi_j$ will change, and we refer here to appendix [A] for further details. Perhaps contrary to the impression of the reader that we apply next Lemma 2.6 to determine $T_{-2}$ or $\xi_{-2}$ resulting in a Binet-like form, similar to the one for Tetranacci numbers in Ref. [12], we follow a different strategy. In fact specific generalized Fibonacci polynomials also satisfy Eq. (2.1) out of which $T_{-2}$ can be constructed. This aim demands still to distinguish the cases of: i) $S_1 \neq S_1$, ii) $S_1 = S_2$, but $S_{1,2}^2 \neq 4$ and iii) $S_1 = S_2, S_{1,2}^2 = 4$.

### 3. The Fibonacci decomposition

Generalized Fibonacci polynomials, defined here according to Eq. (1.1) (cf. Ref. [5]), are closely related to symmetric Tetranacci polynomials. The trivial limit is $\eta = 0$, where Eq. (2.1) simplifies to $\xi_{j+2} = \xi_j - \xi_{j-2}$, thus, separating even and odd indices $j$. Defining now $v_l := \xi_{2l}$ ($u_l := \xi_{2l+1}$) grants then $u_l = \zeta u_l - u_{l-1}$, $v_l = \zeta v_l - v_{l-1}$. Yet, even for $\eta \neq 0$ we find that specific symmetric Tetranacci polynomials obey simultaneously a two term recursion formula.
The generalized Fibonacci polynomial

**Theorem 3.1.** The generalized Fibonacci polynomial \( \varphi_l(j) \) \((l = 1, 2)\), set by

\[
\varphi_l(j + 1) = S_l \varphi_l(j) - \varphi_l(j - 1), \quad j \in \mathbb{Z}
\]  

(3.1)

with \( S_{1,2} = (\eta \pm \sqrt{\eta^2 + 4(\zeta + 2)})/2 \) and initial values \( \varphi_l(0) = 0, \varphi_l(1) = 1 \) is a symmetric Tetranacci polynomial.

**Proof.** For the sake of clarity, we suppress the index \( l = 1, 2 \) in the following. The proposed statement follows straightforwardly by assuming initially that \( \varphi_l(j) \) obeys \( \varphi_l(j + 1) = x \varphi_l(j) + y \varphi_l(j - 1) \) \((j \in \mathbb{Z})\) for arbitrary (complex) \( x, y \) and initial values \( f_0, f_1 \). In order to be a symmetric Tetranacci polynomial, \( \varphi_l(j) \) has to satisfy also Eq. (2.1). Replacing in Eq. (2.1) all terms carrying the indices \( j + 2, j + 1 \) grant

\[
(x^2 + y - \eta x - \zeta) \varphi(j) = (\eta y + \eta - xy) \varphi(j - 1) - \varphi(j - 2).
\]  

(3.2)

Comparing the coefficients between Eq. (3.2) and our ansatz for \( \varphi \) yields \( y = -1 \) immediately. In turn, \( \eta y + \eta - xy = x \) holds without restrictions on \( x \). Instead, the latter is set by \( 1 = x^2 + y - \eta x - \zeta \equiv x^2 + \eta x - \zeta - 1 \) after substituting \( y \). The associated quadratic equation has the two roots \( S_{1,2} = (\eta \pm \sqrt{\eta^2 + 4(\zeta + 2)})/2 \). Thus, \( \varphi(j) \) obeys Eq. (2.1) for the announced coefficients. As \( \varphi(j) \) is well defined by \( f_0, f_1 \) and Eq. (3.1), the initial values \( \varphi(-2), \varphi(-1), \varphi(0) = f_0, \varphi(1) = f_1 \) for the Tetranacci recursion in Eq. (2.1) are fixed. Hence, \( \varphi(j) \) satisfies the definition of symmetric Tetranacci polynomials. Without loss of generality, we choose \( f_0 = 0 \) and \( f_1 = 1 \).

Notice that Theorem 3.1 is an implication: An arbitrary symmetric Tetranacci polynomial \( \xi_j \) will not obey Eq. (3.1) due to its generic initial values \( g_{-2}, \ldots, g_1 \). For instance, we may choose \( g_{-2} = \varphi_1(-2) + \epsilon, g_{-1} = \varphi_1(-1) \) and \( g_0 = \varphi_1(0) = 0, g_1 = \varphi_1(1) = 1 \) for \( \epsilon > 0 \).

Further, Table 2 exposes the first few values of \( \varphi_{1,2} \), from where we deduce the next proposition before we turn to their closed form expression.

**Proposition 3.2.** The polynomials \( \varphi_l(j) \) \((l = 1, 2)\) satisfy \( \varphi_l(j) = -\varphi_l(-j) \) for all \( j \in \mathbb{Z} \).

**Proof.** Eq. (3.1) and the initial values \( \varphi_l(0) = 0, \varphi_l(1) = 1 \) yield \( \varphi_l(-1) = -1 \). Thus, the statement is already correct for \( j = 0, 1 \). Assuming that \( \varphi_l(j) = -\varphi_l(-j) \) holds already at \( n, n + 1 (n \in \mathbb{N}_0) \), Eq. (3.1) states that

\[
\varphi_l(n + 2) = S \varphi_l(n + 1) - \varphi_l(n) = -[S \varphi_l(-n - 1) - \varphi_l(-n)]
\]  

(3.3)

is true. Next, we exchange the \( \varphi_l(j + 1), \varphi_l(j - 1) \) terms in Eq. (3.1), since \( n \geq 0 \) implies \( -n - 1 < -n \), and we thus identify \( S \varphi_l(-n - 1) - \varphi_l(-n) = \varphi_l(-n - 2) \).

**Proposition 3.3.** The explicit closed form expression for \( \varphi_l(j) \) \((l = 1, 2, j \in \mathbb{Z})\)

\[
\varphi_l(j) = \frac{r_j^+ + r_{-j}^-}{r_j^+ - r_{-j}^-}
\]  

(3.4)

\[
\begin{array}{c|c|c|c|c|c|c}
 j & -3 & -2 & -1 & 0 & 1 & 2 & 3 \\
 \varphi_l(j) & -(S_l^2 - 1) & -S_l & -1 & 0 & 1 & S_l & S_l^2 - 1 \\
\end{array}
\]

Table 2. The first terms of \( \varphi_l(j) \) \((l = 1, 2)\) for \( j = -3, \ldots, 3 \). (Table from Ref. [19] with permission.)
is Binet-like whenever \(S^2_t \notin \{0, 4\}\), which becomes \(\varphi_l = \sin(\theta_l j)/\sin(\theta_l)\) in terms of \(\theta_{1,2}\) \[^{4, 5}\]. The special situation of \(S^2_t = 4\) yields

\[
\varphi_l(j) = j \left( \frac{S_t}{2} \right)^{j+1}
\]

(3.5)

while \(S_t = 0\) implies \((k \in \mathbb{Z})\)

\[
\varphi_l(j) = \begin{cases} 
(-1)^{k}, & j = 2k + 1 \\
0, & j = 2k 
\end{cases}.
\]

(3.6)

\[\text{Proof.}\] First, we focus on \(S^2_t \notin \{0, 4\}\). Although Eq. (3.4) and its version in terms of \(\theta_l\) is known as the closed form expression for generalized Fibonacci polynomials (cf. Refs. \[^{4, 5}\]), we re-derive it for completeness and in order to better demonstrate the changes imposed by \(S^2_t = 4\) afterwards. Using the ansatz \(\varphi_l \propto r_j^j (r \neq 0)\) on Eq. (3.1) we find \(r^2 - S_t r + 1 = 0\) after dividing by \(r^{j-1}\). The two roots are \(r_{\pm j} = (S_t \pm \sqrt{S^2_t - 4})/2\) from Eq. (2.25) in Lemma 2.6 and \(S^2_t \notin \{0, 4\}\) implies \(\varphi_l(j) = \alpha r_{+j}^j + \beta r_{-j}^j\). The coefficients \(\alpha, \beta\) are set by \(\varphi_l(0) = 0\) and \(\varphi_l(1) = 1\) as \(r_{+j} = r_{-j}\), \(\beta = -\alpha\) granting Eq. (3.4). Introducing \(\theta_l\) by \(2\cos(\theta_l) = S_t\) turns Eq. (3.4) into \(\sin(\theta_l j)/\sin(\theta_l)\).

For \(S^2_t = 4\), the two roots \(r_{\pm j}\) become degenerate: \(r_{+j} = r_{-j} = S_t/2\). Hence, the linear combination \(\varphi_l(j) = \alpha r_{+j}^j + \beta r_{-j}^j = \tilde{\alpha} r_{+j}^j\) becomes insufficient to properly account for the two initial values of \(\varphi_l(j)\). Since the recursion formula Eq. (3.1) does not change qualitatively at \(S^2_t = 4\), one misses the second solution \(j r_{+j}^j\). Substituting the ansatz \(\varphi_l(j) \propto j r_{+j}^j\) into Eq. (3.1) and reordering according to powers in \(j\) grants

\[
j (r_{+j}^2 - S_t r_{+j} + 1) + r_{+j}^2 - 1 = 0.
\]

(3.7)

where the bracket vanishes since \(r_{+j}\) is the root of \(r^2 - S_t r + 1\). Since \(S^2_t = 4\) imposes \(r_{+j} = S_t/2 = \pm 1, i.e. r_{+j}^2 - 1 = 0\) also the second term vanishes. Thus, \(\varphi_l(j) = \tilde{\alpha} r_{+j}^j = (\tilde{\alpha} + \tilde{\beta} j) r_{+j}^j\) is true. The initial values of \(\varphi_l(j)\) set \(\tilde{\alpha} = 0, \tilde{\beta} = 1/r_{+j} = r_{+j}\) and Eq. (3.5) is found.

For \(S_t = 0\), Eq. (3.1) simplifies to \(\varphi_l(j + 1) = -\varphi_l(j - 1)\) and the initial values \(\varphi_l(0) = 0, \varphi_l(1) = 1\) imply directly Eq. (3.6). \(\square\)

Similar as for "ordinary" symmetric Tetranacci polynomials \(\xi_j\), the situation of \(S_1 = S_2\) offers further special solutions to Eq. (2.1) apart from only \(\varphi_{1,2}(j)\) (cf. Lemma 2.6 or also appendix A). The following lemma is the last intermediate step, before we finally turn to one of the main results of the article: The decomposition of \(T_2(j)\) (and thus any symmetric Tetranacci polynomial) in terms of the generalized Fibonacci polynomials \(\varphi_{1,2}(j)\).

**Lemma 3.4.** Only in case that both the constraints \(S_1 = S_2\) and \(S^2_1 \neq 0 (S^2_1 = 4)\) are met, \(j \varphi_{1,2}(j)\) \((j^2 \varphi_{1,2}(j))\) are symmetric Tetranacci polynomials.

\[\text{Proof.}\] The situation of \(S_1 = S_2\) implies \(\varphi_1(j) = \varphi_2(j)\) for all \(j \in \mathbb{Z}\) (cf. Theorem 3.1) and thus; we demonstrate the statement for only \(j \varphi_1(j)\) and \(j^2 \varphi_1(j)\). After substituting \(j \varphi_1(j)\) into Eq. (2.1), and reordering, we arrive at

\[
0 = j \left\{ \varphi_1(j + 2) - \varphi_1(j) \right\} + 2 \left[ \varphi_1(j + 2) - \varphi_1(j - 2) \right] - \eta \left[ \varphi_1(j + 1) - \varphi_1(j - 1) \right] = 2 \left[ \varphi_1(j + 2) - \varphi_1(j - 2) \right] - \eta \left[ \varphi_1(j + 1) - \varphi_1(j - 1) \right],
\]

(3.8)
where the curly bracket is identically zero, since \( \varphi_1(j) \) is a symmetric Tetranacci polynomial (cf. Theorem 3.1). So far, we have not imposed \( S_1 = S_2 \). According to Eq. (2.24), we find \( S_1 = \eta/2 \) and using Eq. (3.1) twice shows that Eq. (3.8) is indeed satisfied

\[
2 \left[ \varphi_1(j + 2) - \varphi_1(j - 2) \right] - \eta \left[ \varphi_1(j + 1) - \varphi_1(j - 1) \right] \\
= 2 \left[ \varphi_1(j + 2) - S_1 \varphi_1(j + 1) \right] + 2 \left[ S_1 \varphi_1(j - 1) - \varphi_1(j - 2) \right] = 0,
\]

i.e. \( j \varphi_1(j) \) is a symmetric Tetranacci polynomial. Notice that the constraint \( S_1 = S_2 \) is essential, otherwise only \( \eta = S_1 + S_2 \) is correct. Then we may write \( S_1 = S_2 + \epsilon (\epsilon \neq 0) \) and Eq. (3.8) becomes invalid.

Next, in order for \( j^2 \varphi_1(j) \) to obey Eq. (3.8) the additional constraint \( S_1^2 = 4 \) is mandatory. Substituting \( j^2 \varphi_1(j) \) into Eq. (2.1) and reordering the terms afterwards grants

\[
0 = j^2 \left[ \varphi_1(j + 2) - \zeta \varphi_1(j) + \varphi(j - 2) - \eta \left[ \varphi_1(j - 1) + \varphi_1(j + 1) \right] \right] \\
+ 2j \left[ \varphi_1(j + 2) - \varphi_1(j - 2) \right] - \eta \left[ \varphi_1(j + 1) - \varphi_1(j - 1) \right] \\
+ 4 \varphi_1(j + 2) + 4 \varphi_1(j - 2) - \eta \left[ \varphi_1(j + 1) + \varphi_1(j - 1) \right] \\
= 4 \varphi_1(j + 2) + 4 \varphi_1(j - 2) - \eta \left[ \varphi_1(j + 1) + \varphi_1(j - 1) \right],
\]

where the first two lines drop since \( \varphi_1(j) \), \( j \varphi_1(j) \) are symmetric Tetranacci polynomials. Due to Eq. (3.1) and \( S_1 = \eta/2 \), we find

\[
4 \varphi_1(j + 2) + 4 \varphi_1(j - 2) - \eta \left[ \varphi_1(j + 1) + \varphi_1(j - 1) \right] \\
= 4 \left[ S_1 \varphi_1(j + 1) - \varphi_1(j) \right] + 4 \left[ S_1 \varphi_1(j - 1) - \varphi_1(j) \right] - \eta S_1 \varphi_1(j) \\
= 2 \left[ 2S_1^2 - S_1 \frac{\eta}{2} - 4 \right] \varphi_1(j) \\
= 2 \left[ S_1^2 - 4 \right] \varphi_1(j),
\]

being zero only for \( S_1^2 = 4 \) at generic \( j \in \mathbb{Z} \).

Next, we construct \( T_{-2} \) in terms of \( \varphi_{1,2}(j) \) and \( j_1 \varphi_{1,2}(j) \), \( j^2 \varphi_{1,2}(j) \) when the announced conditions are satisfied.

**Theorem 3.5.** The closed form expression for the basic Tetranacci polynomial \( T_{-2}(j) \) is

\[
T_{-2}(j) = \begin{cases} 
\frac{\varphi_2(j) - \varphi_1(j)}{S_1 - S_2}, & S_1 \neq S_2 \\
\frac{(1-j)\varphi_1(j+1) + (1+j)\varphi_1(j-1)}{S_1^2 - 4}, & S_1 = S_2, S_1^2 \neq 4 \\
\frac{S_1(1-j^2)}{12} \varphi_1(j), & S_1 = S_2, S_1^2 = 4.
\end{cases}
\]

**Proof.** For \( S_1 \neq S_2 \), we have that \( \varphi_{1,2}(j) \) satisfy Eq. (2.1), while \( j_1 \varphi_{1,2}(j) \), \( j^2 \varphi_{1,2}(j) \) do not (cf. Theorem 3.1 and Lemma 3.4). Since \( S_1 \neq S_2 \) also implies \( \varphi_1(j) \neq \varphi_2(j) \) (cf. Eq. (3.1)) for all \( j \in \mathbb{Z} \setminus \{0, 1\} \) and due to the linearity of Eq. (2.1), \( [\varphi_2(j) - \varphi_1(j)]/[S_1 - S_2] \) is a non-trivial solution to Eq. (2.1). Hence, the statement is correct provided that \( [\varphi_2(j) - \varphi_1(j)]/[S_1 - S_2] \)
Proposition 3.6. The closed form expression of $T_{-2}(j)$ by applying the Lemmata 2.4, 2.5.

$$
j = -2 : \quad \frac{\varphi_2(-2) - \varphi_1(-2)}{S_1 - S_2} = \frac{-S_2 - (-S_1)}{S_1 - S_2} = 1 \equiv T_{-2}(-2),$$
$$
j = -1 : \quad \frac{\varphi_2(-1) - \varphi_1(-1)}{S_1 - S_2} = \frac{-1 - (-1)}{S_1 - S_2} = 0 \equiv T_{-2}(-1),$$
$$
j = 0 : \quad \frac{\varphi_2(0) - \varphi_1(0)}{S_1 - S_2} = \frac{0 - 0}{S_1 - S_2} = 0 \equiv T_{-2}(0),$$
$$
j = 1 : \quad \frac{\varphi_2(1) - \varphi_1(1)}{S_1 - S_2} = \frac{1 - 1}{S_1 - S_2} = 0 \equiv T_{-2}(1),$$

holds true. Turning to the case of $S_1 = S_2$ but $S_1^2 \neq 4$, we first find $\varphi_1(j) = \varphi_2(j)$ (cf. Eq. (3.1)) for all $j \in \mathbb{Z}$, but $j\varphi_1(j)$ now satisfies Eq. (2.1) due to Lemma (3.4). Further, also $\varphi_1(j \pm 1)$, $(1 \pm j)\varphi_1(j \pm 1)$ are apparently solutions to Eq. (2.1), such that also

$$(1 - j)\varphi_1(j + 1) + (1 + j)\varphi_1(j - 1) = 2\varphi_1(j + 1) - (j + 1)\varphi_1(j + 1) + 2\varphi_1(j - 1) + (j - 1)\varphi_1(j - 1)$$

satisfies Eq. (2.1). In addition, we find (cf. Table 2)

$$
j = -2 : \quad \frac{3\varphi_1(-1) - \varphi_1(-3)}{S_1^2 - 4} = \frac{-3 + (S_1^2 - 1)}{S_1^2 - 4} = 1 \equiv T_{-2}(-2),$$
$$
j = -1 : \quad \frac{2\varphi_1(0) + 0\varphi_1(-2)}{S_1^2 - 4} = \frac{0 + 0}{S_1^2 - 4} = 0 \equiv T_{-2}(-1),$$
$$
j = 0 : \quad \frac{\varphi_1(1) + \varphi_1(-1)}{S_1^2 - 4} = \frac{1 - 1}{S_1^2 - 4} = 0 \equiv T_{-2}(0),$$
$$
j = 1 : \quad \frac{0\varphi_1(2) + 2\varphi_1(0)}{S_1^2 - 4} = \frac{0 + 0}{S_1^2 - 4} = 0 \equiv T_{-2}(1),$$

i.e. $T_{-2}(j)$ is properly constructed. Finally, in case of $S_1 = S_2$ and $S_1^2 = 4$ also $j^2\varphi_1(j)$ satisfies Eq. (2.1) (Lemma 3.4). Apparently, $(1 - j^2)\varphi_1(j)$ vanishes at $j = \pm 1$ and also for $j = 0$ due to $\varphi_1(0) = 0$. For $j = -2$, we find

$$
\frac{S_1(1 - 4)}{12} \varphi_1(-2) = \frac{3S_1^2}{12} = 1 \equiv T_{-2}(-2),
$$

and the statement is correct. \square

Since the expression for $T_{-2}(j)$ is known to us, we can next construct the remaining basic Tetranacci polynomials by applying the Lemmata 2.4, 2.5.

Proposition 3.6. The closed form expression of $T_{-1}(j)$, $T_0(j)$, $T_1(j)$ read

$$T_{-1}(j) = \frac{\varphi_1(j + 1) - \varphi_2(j + 1) + S_2\varphi_1(j) - S_1\varphi_2(j)}{S_1 - S_2}, \quad \text{(3.13)}$$
$$T_0(j) = \frac{S_1\varphi_2(j + 1) - S_2\varphi_1(j + 1) + \varphi_2(j) - \varphi_1(j)}{S_1 - S_2}, \quad \text{(3.14)}$$
$$T_1(j) = \frac{\varphi_1(j + 1) - \varphi_2(j + 1)}{S_1 - S_2}, \quad \text{(3.15)}$$
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for all integer $j$ and supposing here $S_1 \neq S_2$ to be true. The results in case of $S_1 = S_2$ are presented in appendix B.

Proof. The displayed formulae follow directly by substituting Eq. (3.11) into the relations from Lemmata 2.4, 2.5 and exploiting the properties of $\varphi_{1,2}(j)$ drawn in Proposition 3.2 and Theorem 3.1. Alternatively, the Eqs. (3.13) - (3.15) are apparently linear combinations of solutions to the recursion formula in Eq. (2.1) and one is left to demonstrate the respective selective property, which we delegate as exercise to the reader.

4. Discussion and outlook

In the beginning of this manuscript, we promised to provide a rather simple closed form expression for $\xi_j$. On first glance of $T_i(j)$ ($i = -2, \ldots , 1$) in Theorem 3.5 and Proposition 3.6 this seems wrong. However, substituting the basic Tetranacci polynomials into Eq. (1) yields ($j \in \mathbb{Z}$)

$$\xi_j = \varphi_2(j) \frac{g_{-2} - S_1 g_{-1} + g_0}{S_1 - S_2} - \varphi_1(j) \frac{g_{-2} - S_2 g_{-1} + g_0}{S_1 - S_2} + \varphi_1(j + 1) \frac{g_{-1} - S_2 g_0 + g_1}{S_1 - S_2} - \varphi_2(j + 1) \frac{g_{-1} - S_1 g_0 + g_1}{S_1 - S_2} \quad (4.1)$$

in case of $S_1 \neq S_2$. Although to our best knowledge only the situation of $S_1 \neq S_2$ is relevant for applications in physics or for the diagonalization of Toeplitz matrices owning a bandwith of two \cite{17, 18, 19}, similar expressions as Eq. (4.1) can be anticipated also for $S_1 = S_2$, $S_1^2 \neq 4$ and $S_1 = S_2$, $S_2^2 = 4$ from appendix B. Therefore, and in view of Theorem 3.5 and Lemma 2.5 we demonstrated explicitly the decomposition of a generic symmetric Tetranacci polynomial $\xi_j$ in terms of the generalized Fibonacci polynomials $\varphi_{1,2}$.

Contrary those previous works (cf. Ref. \cite{17, 18, 19}), we both generalized and simplified the presented results beyond the limitation of $S_1 \neq S_2$. Based on the rigorous mathematical approach, the shown results are generic and thus generalize beyond a concrete physical systems, extending in turn earlier limitations.

Of further interest for physics, is the substitution $\varphi_{1,2}(j) = \sin(\theta_{1,2}j)/\sin(\theta_{1,2})$ ($S_{1,2} \notin \{0, 4\}$, cf. Proposition 3.3), revealing that $\xi_j$ is indeed a combination of standing waves. Notice that in case of equidistant spaced systems owning a lattice constant $d$, we rather write $\theta_{1,2} = k_{1,2}d$ in terms of wavevectors $k_{1,2}$. In case a boundary condition is applied the values of $k_{1,2}$ become quantized accordingly, without touching further details \cite{19}. We provide an overview on the application of Tetranacci polynomials in condensed matter physics elsewhere, where the present manuscript is used as toolbox.

Returning to the abstract mathematical view, a similar strategy of decomposition is perhaps possible also for more advanced recursive defined sequences since one can increase the recursion range by substituting a recursion formula into itself. Further, we may ask whether or not the basic Tetranacci polynomials are linear independent. This issue is motivated by Eq. (2.8) mimicking the expression of a vector in a 4d space in terms of the associated basis vectors. Finally, we speculate about orthogonality. The reasons for this are, first that $\varphi_{1,2}(j)$ can be seen also as Chebyshev polynomials of the second kind \cite{28}, which obey an orthogonality criterion. Second, the eigenvectors of distinct eigenvalues of hermitian matrices are orthogonal and since $\xi_j$ is an eigenvector element associated to (real valued) symmetric Toeplitz matrices of bandwith two, there is certainly an orthogonality criterion.
5. ACKNOWLEDGEMENT

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APPENDIX A. DEGENERATED ROOTS $r_{\pm 1,2}$

Proposition A.1. Any degeneracy of the roots $r_{\pm 1}$ is reflected by additional solutions to Eq. (2.1). A generic symmetric Tetranacci polynomial $\xi_j$ is set up by linear combinations of

$$r^j_{\pm 1}, j r^j_{\pm 1}, \ S_1 = S_2, S^2_1 \neq 4,$$

$$r^j_{\pm 1}, j r^j_{\pm 1}, j^2 r^j_{\pm 1}, j^3 r^j_{\pm 1}, \ S_1 = S_2, S^2_1 = 4.$$  (A.2)

Proof. We focus first on $S_1 = S_2, S^2_1 \neq 4$, where Eq. (2.25) implies $r_{\pm 1} = r_{\pm 2}$. Thus, the two additional solutions $j r^j_{\pm 1}$ to Eq. (2.1) are mandatory in order to determine a generic symmetric Tetranacci polynomial. Substituting $\xi_j \propto j r^j_{\pm 1}$ into Eq. (2.1), dividing by $j^3_{\pm 1}$ or 0 and reordering according to powers in $j$ grants

$$0 = j \left[r^4_{\pm 1} - \xi r^2_{\pm 1} + 1 - \eta (r^3_{\pm 1} + r_{\pm 1}) \right] + 2 r^4_{\pm 1} - 2 - \eta (r^3_{\pm 1} - r_{\pm 1}).$$  (A.3)

Here, the first bracket vanishes since $r_{\pm 1}$ satisfies Eq. (2.1). As we shall see, Eq. (A.3) holds only for $S_1 = S_2$. Due to Eq. (2.24), $S_1 = S_2$ implies $\eta = 2S_1$ and using $S_1 = r_{+1} + r_{-1}$ (cf. Eq. (2.25)), we arrive at

$$0 = 2 r^4_{\pm 1} - 2 - \eta (r^3_{\pm 1} - r_{\pm 1}) = 2 r^4_{\pm 1} - 2 - 2(r_{+1} + r_{-1}) (r^3_{\pm 1} - r_{\pm 1}),$$

(A.4)

Since $r_{+1}r_{-1} = 1$ is true always, Eq. (A.3) is indeed satisfied and $\xi_j \propto j r^j_{\pm 1}$ satisfies Eq. (2.1).

Next, we turn to $S_1 = S_2, S^2_1 = 4$. Following Eq. (2.25), we have $r_{+1} = r_{-1} = r_{+2} = r_{-2}$. Yet, we know already that $j r^j_{\pm 1}$ is a solution of Eq. (2.1) due to $S_1 = S_2$. First we demonstrate that $\xi_j \propto j^2 r^j_{+1}$ is another solution. Upon inserting, dividing by $r^j_{+1}$ and ordering all terms into powers of $j$, we arrive at

$$0 = j^2 \left[r^4_{\pm 1} - \xi r^2_{\pm 1} + 1 - \eta (r^3_{\pm 1} + r_{\pm 1}) \right] + 2 j \left[2 r^4_{\pm 1} - 2 - \eta (r^3_{\pm 1} - r_{\pm 1}) \right],$$
$$+ 4 r^4_{\pm 1} + 4 - \eta (r^3_{\pm 1} + r_{\pm 1}).$$

(A.5)

Here, the brackets in the first line vanish as demonstrated before. Since Eq. (2.25) implies $r_{+1} = S_1/2$ at $S_1 = S_2, S^2_1 = 4$, we indeed find that

$$4 r^4_{\pm 1} + 4 - \eta (r^3_{\pm 1} + r_{\pm 1}) = 4 \left( \frac{S_1}{2} \right)^4 - 4 S_1 \left( \frac{S_1}{2} \right)^3 + 4 - S^2_1 = 0$$

(A.6)

is satisfied. Thus, $j^2 r^j_{+1}$ is a solution to Eq. (2.1). Next, $j^3 r^j_{+1}$ is also a solution to Eq. (2.1). After inserting, dividing by $r^j_{+1}$, the terms associated to $j^3, j^2, j$ drop similarly as before.
Thus, one arrives at the condition
\[
0 = 8r_+^4 - 8 - \eta r_+^3 + \eta r_+ = 8 \left( \frac{S_1}{2} \right)^4 - 8 - \eta r_+ \left( r_+^2 - 1 \right)
\]
\[
= 8 \left( \frac{S_1}{2} \right)^4 - 8 - \eta r_+ \left( \frac{S_1^2}{4} - 1 \right) = 0,
\]
which is indeed satisfied due to \( r_+ = S_1/2 \) and \( S_1^2 = 4 \). Hence, \( j^3 r_+^j \) satisfies Eq. (2.1). □

Appendix B. Formulae of \( T_{-1}(j), T_0(j), T_1(j) \) for degenerate roots

For \( S_1 = S_2 \) but \( S_1 \neq 4 \), we have \( (j \in \mathbb{Z}) \)
\[
T_{-1}(j) = \frac{2(S_1^2 - 1)(j - 1) \varphi_1(j) - 3(j + 1)S_1 \varphi_1(j + 2)}{S_1^2 - 4},
\]
\[
T_0(j) = \frac{2(S_1^2 - 1)(j + 1) \varphi_1(j + 1) - 3(j + 1)S_1 \varphi_1(j + 2)}{S_1^2 - 4},
\]
\[
T_1(j) = \frac{j \varphi_1(j + 2) - (j + 2) \varphi_1(j)}{S_1^2 - 4}
\]
while we find \( (j \in \mathbb{Z}) \)
\[
T_{-1}(j) = S_1 \frac{(2 - j) j \varphi_1(j - 1) + 2S_1(j^2 - 1) \varphi_1(j)}{12},
\]
\[
T_0(j) = S_1 \frac{(3 + j)(1 + j) \varphi_1(j + 2) - S_1(j + 2)j \varphi_1(j + 1)}{12},
\]
\[
T_1(j) = S_1 \frac{(2 + j) j \varphi_1(j + 1)}{12}
\]
in case of \( S_1 = S_2 \) and \( S_1 = 4 \)

Proof. The displayed formulae follow directly by substituting Eq. (3.11) into the relations from Lemmata 2.4, 2.5 and exploiting the properties of \( \varphi_1, \varphi_2 \) drawn in Proposition 3.2 and Theorem 3.1. Alternatively, the Eqs. (3.13) - (3.15) are linear combinations of solutions (cf. Lemma 3.4) to the recursion formula in Eq. (2.1) and one is left to demonstrate the respective selective property, which we delegate as exercise to the reader. □

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