A Gravity Dual of the Chiral Anomaly

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Abstract

We study effects associated with the chiral anomaly for a cascading $SU(N+M) \times SU(N)$ gauge theory using gauge/gravity duality. In the gravity dual the anomaly is a classical feature of the supergravity solution, and the breaking of the $U(1)$ R-symmetry down to $\mathbb{Z}_{2M}$ proceeds via the Higgs mechanism.
1 Introduction

Many supersymmetric gauge theories exhibit a classical $U(1)$ R-symmetry which is broken quantum mechanically to some discrete subgroup. In traditional quantum field theory, this symmetry breaking can be understood as an instanton effect. The purpose of this note is to explore the analogous effects in the gravity duals to a few field theories exhibiting this phenomenon.

Our analysis relies on the results of recent encouraging progress in extending the gauge theory/supergravity correspondence \cite{1,2,3} to theories with less than maximal supersymmetry, realized by configurations of D-branes at singularities. For example, we might consider a stack of $N$ coincident D3-branes located at the tip of the singular Calabi-Yau space known as the conifold \cite{4}. This system is dual to an $\mathcal{N} = 1$ supersymmetric gauge theory with gauge group $SU(N) \times SU(N)$ coupled to chiral superfields (two bifundamentals transforming in the $(N, \bar{N})$ representation, and their conjugates, which transform in the $(\bar{N}, N)$ representation.) The supergravity solution has the geometry $AdS_5 \times T^{1,1}$, which manifests geometrically the conformal invariance of the gauge theory. The Einstein manifold $T^{1,1}$ possesses a two-cycle and a three-cycle, and we can obtain interesting variations of this theory by wrapping various branes on these cycles \cite{5,6,7,8}. In particular, wrapping D5-branes on this two-cycle changes the gauge group to $SU(N) \times SU(N + M)$ and breaks the conformal symmetry.

This supergravity solution with wrapped D5-branes exhibits a host of interesting gauge theory phenomena; for example, the reduction of the five-form flux as the radial coordinate decreases corresponds to a reduction in the size of the gauge groups by a duality cascade \cite{8}. An important feature for our purposes is that the UV metric exhibits a $U(1)$ symmetry under rotations of a particular angle $\beta$ in the transverse space $T^{1,1}$, which is a geometric realization of the field theoretic R-symmetry. However, the RR potentials break the $U(1)$ symmetry. It can be shown that there is an unbroken $\mathbb{Z}_{2M}$ subgroup of this $U(1)$ by studying fractional instanton and domain wall probes in the gravity background \cite{9,10,11}. However, one should not think about this symmetry breaking as an effect of these instantons, which do not appear explicitly anywhere in the gravity dual. Rather, we argue that the field theory anomalies are present because the classical supergravity RR potentials are not invariant under the $U(1)$ symmetry. By computing the variation of the RR potentials we obtain the relevant anomaly coefficients, which agree with field theory exactly. Moreover, the anomalous breaking of the global $U(1)$ symmetry appears as spontaneous symmetry breaking in supergravity: the bulk vector field dual to the R-symmetry current of
the gauge theory acquires a mass. We will also check the anomaly coefficients for a related $\mathcal{N} = 2$ orbifold theory, again showing exact agreement.

2 The Anomaly as Non-Invariance of the UV Supergravity Solution

Let us recall a few results regarding the supergravity dual of the cascading $SU(N + M) \times SU(N)$ gauge theory. The metric is of the form [8]

$$ds_{10}^2 = h^{-1/2}(\tau)dx_1^2 + h^{1/2}(\tau)ds_6^2,$$

where $ds_6^2$ is the metric of the deformed conifold. The UV (large $\tau$) limit of this metric was found in [7]:

$$ds_{10}^2 = h^{-1/2}(r)dx_1^2 + h^{1/2}(r)(dr^2 + r^2ds_{T^1,1}^2).$$

The metric on $T^1,1$, the base of the conifold, is

$$ds_{T^1,1}^2 = \frac{1}{9}\left(2d\beta + \sum_{i=1}^2 \cos \theta_i d\phi_i\right)^2 + \frac{1}{6}\sum_{i=1}^2 \left(d\theta_i^2 + \sin^2 \theta_i d\phi_i^2\right).$$

We define the angle $\beta$ to range from 0 to $2\pi$; it is related to the angle $\psi$ used in the previous literature by $\psi = 2\beta$. The asymptotic form of the warp factor is

$$h(r) = \frac{27\pi (\alpha')^2}{4r^4} \left[g_s N + \frac{3}{2\pi} (g_s M)^2 \ln(r/r_0) + \frac{3}{8\pi} (g_s M)^2\right].$$

For sufficiently small $r$, this metric is singular; to study IR physics in the gauge theory, one must use the full solution of [8].

The following basis of 1-forms on the compact space is convenient for calculations on the deformed conifold [12]:

$$g^1 = \frac{e^1 - e^3}{\sqrt{2}}, \quad g^2 = \frac{e^2 - e^4}{\sqrt{2}},$$

$$g^3 = \frac{e^1 + e^3}{\sqrt{2}}, \quad g^4 = \frac{e^2 + e^4}{\sqrt{2}},$$

$$g^5 = e^5,$$

where

$$e^1 \equiv -\sin \theta_1 d\phi_1, \quad e^2 \equiv d\theta_1,$$

$$e^3 \equiv \cos 2\beta \sin \theta_2 d\phi_2 - \sin 2\beta d\theta_2,$$

$$e^4 \equiv \sin 2\beta \sin \theta_2 d\phi_2 + \cos 2\beta d\theta_2,$$

$$e^5 \equiv 2d\beta + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2.$$
In terms of this basis, the Einstein metric on $T^{1,1}$ assumes the form
\[ ds^2_{T^{1,1}} = \frac{1}{9}(g^5)^2 + \frac{1}{6} \sum_{i=1}^4 (g^i)^2 . \]

The three-form fields are turned on in this background:
\[ F_3 = \frac{M\alpha'}{2} \omega_3 , \quad B_2 = \frac{3g_s M\alpha'}{2} \omega_2 \ln(r/r_0) , \]
\[ H_3 = dB_2 = \frac{3g_s M\alpha'}{2r} dr \wedge \omega_2 , \]
where
\[ \omega_3 = g^5 \wedge \omega_2 \]
\[ \omega_2 = \frac{1}{2}(\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sin \theta_2 d\theta_2 \wedge d\phi_2) . \]

A key feature of this solution is that the five-form flux is radially dependent [7]:
\[ F_5 = F_5 + *F_5 , \quad F_5 = \frac{1}{2}\pi\alpha'^2 N_{\text{eff}}(r) \omega_2 \wedge \omega_3 , \]
with
\[ N_{\text{eff}}(r) = N + \frac{3}{2\pi}g_s M^2 \ln(r/r_0) , \]
and the ten-dimensional Hodge dual is defined by $\varepsilon_{t \bar{x} y z \bar{r} 5 \theta_1 \phi_1 \theta_2 \phi_2} = \sqrt{-G_{10}}$. In [13] it was shown that
\[ \int_{S^2} \omega_2 = 4\pi , \quad \int_{S^3} \omega_3 = 8\pi^2 \]

The normalization of the R-R 3-form flux is determined by the quantization condition
\[ \frac{1}{4\pi^2\alpha'} \int_{S^3} F_3 = M . \]

With these results in hand, let us see how the chiral anomaly emerges in supergravity. The asymptotic UV metric [4,3] has a $U(1)$ symmetry associated with the rotations of the angular coordinate $\beta$, which appears as the R-symmetry of the dual gauge theory. It is crucial, however, that the background value of the R-R 2-form $C_2$ does not have this continuous symmetry. Indeed, although $F_3$ is $U(1)$ symmetric, there is no smooth global expression for $C_2$. Locally, we may write for large $r$,
\[ C_2 \rightarrow M\alpha' \beta \omega_2 . \]

This expression is not single-valued as a function of the angular variable $\beta$, but it is single-valued up to a gauge transformation, so that $F_3 = dC_2$ is single-valued. In
fact, $F_3$ is completely independent of $\beta$. Because of the explicit $\beta$ dependence, $C_2$ is not $U(1)$-invariant. Under the transformation $\beta \to \beta + \epsilon$,

$$C_2 \to C_2 + M \alpha' \epsilon \omega_2. \quad (17)$$

A gauge transformation can shift $C_2/(4\pi^2\alpha')$ by an arbitrary integer multiple of $\omega_2/(4\pi)$, so $\beta \to \beta + \epsilon$ is a symmetry precisely if $\epsilon$ is an integer multiple of $\pi/M$; because $\epsilon$ is anyway only defined mod $2\pi$, a $\mathbb{Z}_{2M}$ subgroup of the $U(1)$ leaves fixed the asymptotic values of the fields, and thus corresponds to a symmetry of the system. This $\mathbb{Z}_{2M}$ respects the asymptotic values of the fields, but in the solution found in [8], it is spontaneously broken in the IR to $\mathbb{Z}_2$, generated by $(-1)^F$, since the full solution does not have $\mathbb{Z}_{2M}$ symmetry. (In that solution, $\mathbb{Z}_{2M}$ is broken to $\mathbb{Z}_2$ by the deformation parameter of the conifold.) The analogous statement in field theory is that instantons break the $U(1)$ down to $\mathbb{Z}_2$, which is then spontaneously broken to $\mathbb{Z}_2$.

The way that the asymptotic behavior of $C_2$ transforms under the $U(1)$ generator is dual to the way that in field theory a $U(1)_R$ transformation shifts the $\Theta$-angles of the two gauge groups by opposite amounts. The $\Theta$-angles are given by

$$\Theta_1 - \Theta_2 = \frac{1}{\pi \alpha'} \int_{S^2} C_2, \quad \Theta_1 + \Theta_2 \sim C, \quad (18)$$

where $C$ is the RR scalar, which vanishes for the case under consideration. Using the fact that $\int_{S^2} \omega_2 = 4\pi$, we find that the small $U(1)$ rotation induces

$$\Theta_1 = -\Theta_2 = 2M\epsilon. \quad (19)$$

We can compare (19) with our expectations from the field theory. The conventionally normalized $\Theta$ terms for the gauge theory action are

$$\int d^4x \left( \frac{\Theta_1}{32\pi^2} F^a_{ij} \tilde{F}^{aij} + \frac{\Theta_2}{32\pi^2} G^b_{ij} \tilde{G}^{bij} \right), \quad (20)$$

where $F^a_{ij}$ and $G^b_{ij}$ are the field strengths of $SU(N+M)$ and $SU(N)$ respectively. If we assume that $\epsilon$ is a function of the 4 world volume coordinates $x^i$, then the terms linear in $\epsilon$ in the dual gauge theory are

$$\int d^4x \left[ -\epsilon \partial_i J^i + \frac{M\epsilon}{16\pi^2} (F^a_{ij} \tilde{F}^{aij} - G^b_{ij} \tilde{G}^{bij}) \right], \quad (21)$$
where \( J^i \) is the chiral \( R \)-current. The appearance of the second term is due to the non-invariance of \( C_2 \) under the \( U(1) \) rotation. Varying with respect to \( \epsilon \), we therefore find the equation
\[
\partial_i J^i = \frac{M}{16\pi^2} (F^a_{ij} \tilde{F}^{a\ij} - G^b_{ij} \tilde{G}^{bij}).
\] (22)
This is precisely the anomaly equation for this theory. Indeed, the effective number of flavors for the \( SU(N + M) \) factor is \( 2N \), and each one carries \( R \)-charge \( 1/2 \). The chiral fermions which are their superpartners have \( R \)-charge \( -1/2 \) while the gluinos have \( R \)-charge \( 1 \). Therefore, the anomaly coefficient is \( \frac{M}{16\pi^2} \). An equivalent calculation for the \( SU(N) \) gauge group with \( 2(N + M) \) flavors produces the opposite anomaly, in agreement with the holographic result (21).

The upshot of the calculation presented above is that the chiral anomaly of the \( SU(N + M) \times SU(N) \) gauge theory is encoded in the ultraviolet (large \( r \)) behavior of the dual classical supergravity solution. No additional fractional D-instanton effects are needed to explain the anomaly. Thus, as often occurs in the gauge/gravity duality, a quantum effect on the gauge theory side turns into a classical effect in supergravity.

3 The Anomaly as Spontaneous Symmetry Breaking in \( AdS_5 \)

Let us look for a deeper understanding of the anomaly from the dual gravity point of view. On the gauge theory side, the \( R \)-symmetry is global, but in the gravity dual it as usual becomes a gauge symmetry, which must not be anomalous, or the theory would not make sense at all. Rather, we will find that the gauge symmetry is spontaneously broken: the 5-d vector field dual to the \( R \)-current of the gauge theory ‘eats’ the scalar dual to the difference of the theta angles and acquires a mass. ¹ A closely related mechanism was observed in studies of RG flows from the dual gravity point of view [15, 16, 17]. There \( R \)-current conservation was violated not through anomalies but by turning on relevant perturbations or expectation values for fields. In these cases it was shown [13, 16, 17] that the 5-d vector field dual to the \( R \)-current acquires a mass through the Higgs mechanism. We will show that symmetry breaking through anomalies can also have the bulk Higgs mechanism as its dual.

In the absence of fractional branes there are no background three-form fluxes, so the \( U(1) \) \( R \)-symmetry is a true symmetry of the field theory. Because the \( R \)-

¹ The connection between anomalies in a D-brane field theory and spontaneous symmetry breaking in string theory was previously noted in [14] (and probably elsewhere in the literature).
symmetry is realized geometrically by invariance under a rigid shift of the angle $\beta$, it becomes a local symmetry in the full gravity theory, and the associated gauge fields $A = A_\mu dx^\mu$ appear as fluctuations of the ten-dimensional metric and RR four-form potential \[19, 20\]. The natural metric ansatz is of the familiar Kaluza-Klein form:

$$ds^2 = h(r)^{-1/2} (dx_5 dx^n) + h(r)^{1/2} r^2 \left[ \frac{dr^2}{r^2} + \frac{1}{9} (g^5 - 2A)^2 + \frac{1}{6} \sum_{r=1}^4 (g^r)^2 \right],$$

(23)

where $h(r) = L^4/r^4$, and $L^4 = \frac{27}{16}(4\pi\alpha'^2 g_s N)$. It is convenient to define the one-form $\chi = g^5 - 2A$, which is invariant under the combined gauge transformations

$$\beta \to \beta + \lambda, \quad A \to A + d\lambda.$$  

(24)

The equations of motion for the field $A_\mu$ appear as the $\chi_\mu$ components of Einstein’s equations,

$$R_{MN} = \frac{g_s^2}{4} \tilde{F}_{MPQRS} \tilde{F}^{PQRS}_N.$$  

(25)

The five-form flux will also fluctuate when we activate the Kaluza-Klein gauge field; indeed, the unperturbed $\tilde{F}_5$ of \(12\) is not self-dual with respect to the gauged metric \(23\). An appropriate ansatz to linear order in $A$ is

$$\tilde{F}_5 = dC_4 = \frac{1}{g_s} d^4 x \wedge dh^{-1} + \frac{\pi \alpha'^2 N}{4} \left[ \chi \wedge g^1 \wedge g^2 \wedge g^3 \wedge g^4 \\
- dA \wedge g^5 \wedge dg^5 + \frac{3}{L} \star_5 dA \wedge dg^5 \right].$$

(26)

The five-dimensional Hodge dual $\star_5$ is defined with respect to the AdS$_5$ metric $ds_5^2 = h^{-1/2} dx_5 dx^n + h^{1/2} dr^2$. It is straightforward to show that the supergravity field equation $d\tilde{F}_5 = 0$ implies that the field $A$ satisfies the equation of motion for a massless vector field in AdS$_5$ space:

$$d \star_5 dA = 0.$$  

(27)

Using the identity $dg^5 \wedge dg^5 = -2 g^1 \wedge g^2 \wedge g^3 \wedge g^4$, we can check that the expression for $C_4$ is

$$C_4 = \frac{1}{g_s} h^{-1} d^4 x + \frac{\pi \alpha'^2 N}{2} \left[ \beta g^1 \wedge g^2 \wedge g^3 \wedge g^4 - \frac{1}{2} A \wedge dg^5 \wedge g^5 \\
- \frac{3}{2r} h^{-1/4} \star_5 dA \wedge g^5 \right].$$

(28)

This expression was independently derived by D. Berenstein.

\[\text{This expression was independently derived by D. Berenstein.}\]
Another way to see that $A$ is a massless vector in $AdS_5$ is to consider the Ricci scalar for the metric (23)

$$R = R(A = 0) - \frac{h^{1/2}r^2}{9} F_{\mu\nu} F^{\mu\nu}$$

so that on reduction from ten dimensions the five-dimensional supergravity action will contain the action for a massless vector field.

The story changes when we add wrapped D5-branes. As described in Section 2, the 5-branes introduce $M$ units of RR flux through the three-cycle of $T^{1,1}$. Now, the new wrinkle is that the RR three-form flux of (8) is not gauge-invariant with respect to shifts of $\beta$ (24). To restore the gauge invariance, we introduce a new field $\theta \sim \int_{S^2} C_2$:

$$F_3 = dC_2 = \frac{M\alpha'}{2} \left( g^5 + 2\partial_\mu \theta dx^\mu \right) \wedge \omega_2$$

so that $F_3$ is invariant under the gauge transformation $\beta \to \beta + \lambda, \theta \to \theta - \lambda$. Let us also define $W_\mu = A_\mu + \partial_\mu \theta$. In terms of the gauge invariant forms $\chi$ and $W = W_\mu dx^\mu$,

$$F_3 = \frac{M\alpha'}{2} (\chi + 2W) \wedge \omega_2.$$

From (31) we can immediately see how the anomaly will appear in the gravity dual. Assuming that the NS-NS three form is still given by (9), we find that up to terms of order $g_s M^2/N$ the three-form equation implies

$$d \star 5 W = 0 \Rightarrow \frac{L^2}{r^2} \partial_i W^i + \frac{1}{r^5} \partial_r r^5 W_r = 0$$

which is just what one would expect for a massive vector field in five dimensions. To a four dimensional observer, however, a massive vector field would satisfy $\partial_i W^i = 0$. Thus in the field theory one cannot interpret the $U(1)$ symmetry breaking as being spontaneous, and the additional $W_r$ term in (32) appears in four dimensions to be an anomaly.

Another way to see that the vector field becomes massive is to compute its equation of motion. To do this calculation precisely, we should derive the $\chi\mu$ components of Einstein’s equations, and also find the appropriate expressions for the five-form and metric up to quadratic order in $g_s M$ and linear order in fluctuations. This approach is somewhat nontrivial. A more heuristic approach is to consider the type IIB supergravity action to quadratic order in $W$, ignoring the 5-form field strength.
contributions:

\[ S = -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G_{10}} \left[ R_{10} - \frac{g_s^2}{12} |F_3|^2 \right] + \ldots \]  
\[ \sim -\frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{-G_{10}} \left[ -\frac{h^{1/2}r^2}{9} F_{\mu\nu} F^{\mu\nu} - \left( \frac{g_s M\alpha'}{2} \right)^2 \frac{36}{hr^4} W_\mu W^\mu \right] + \ldots \]  

This is clearly the action for a massive vector field, which has as its equation of motion

\[ \partial_\mu (hr^7 F^{\mu\nu}) = \tilde{m}^2 hr^7 W^\nu \]  

which in differential form notation is

\[ d\left(\frac{h^{7/4}r^7 *_5 dW}{hr^7} \right) = -\tilde{m}^2 h^{7/4}r^7 *_5 W. \]  

Here the mass-squared is given by

\[ \tilde{m}^2 = \left( \frac{g_s M\alpha'}{2} \right)^2 \frac{81}{2h^{3/2}r^6}. \]  

This result, however, ignores the subtlety of the type IIB action in presence of the self-dual 5-form field. A more precise calculation \cite{18}, which takes the mixing into account, gives instead the following equation for the transverse vector modes:

\[ \left( \frac{1}{hr^7} \partial_r hr^7 \partial_r + h \partial_i \partial_i - \frac{(9 M\alpha')^4}{64 h^2 r^{10}} \right) W_i = 0, \]  

This shows that the 10-d mass actually appears at a higher order in perturbation theory compared to the result (36) that ignores the mixing with the 5-form. \footnote{We are grateful to M. Krasnitz for correcting an error in a previous version of this paper.}

Let us compare this result to earlier work. In \cite{15, 16, 17} it was shown that the 5-d vector field associated with a $U(1)_R$ symmetry acquires a mass in the presence of a symmetry-breaking relevant perturbation, and that this mass is related in a simple way to the warp factor of the geometry. \footnote{We are grateful to O. DeWolfe and K. Skenderis for pointing out the relevance of this work to the present calculation.} It is conventional to write the 5-d gauged supergravity metric in the form

\[ \tilde{G}_{\mu \nu} dx^\mu dx^\nu = e^{2T(q)} \eta_{ij} dx^i dx^j + dq^2. \]  

The result of \cite{15} is that $m^2 = -2T''$. To relate the 5-d metric (38) to the 10-d metric \cite{1} we must normalize the 5-d metric so that the graviton has a canonical kinetic term. Doing this carefully we find

\[ \tilde{G}_{\mu \nu} dx^\mu dx^\nu = \left( hr^4/L^4 \right)^{5/6} \left( h^{-1/2} \eta_{ij} dx^i dx^j + h^{1/2} dr^2 \right). \]
The factor \((hr^4/L^4)^{5/6}\) arises due to the radial dependence of the size of \(T^{1,1}\) through the usual Kaluza–Klein reduction. The radial variables \(q\) and \(r\) are related at leading order in \(g_s M^2/N\), by

\[
\log(r) \sim \frac{q}{L} - \frac{g_s M^2}{2\pi N} \left(\frac{q}{L}\right)^2. \tag{40}
\]

We can also show that \(-2T = -2\log(r) + \text{(terms which do not affect the mass to leading order in } g_s M^2/N)\), so now computing the mass-squared by the prescription of [13] we obtain

\[
m^2 = \frac{4}{\alpha'\left(3\pi\right)^{3/2}} \frac{(g_s M)^2}{(g_s N)^{3/2}}. \tag{41}
\]

where this mass applies to a vector field \(V\) with a canonical kinetic term for the metric (39). For these calculations it is convenient to work with the transverse 4-d vector modes \(V_i\) and to decouple the longitudinal modes such as \(V_r\). The equation of motion of \(V\) is

\[
(e^{-2T} \frac{\partial}{\partial q} e^{2T} \frac{\partial}{\partial q} + e^{-2T} \partial_i \partial_i - m^2) V_i = 0. \tag{42}
\]

In fact, this equation follows from (37) after a rescaling \[18\]

\[
V_i = (hr^4/L^4)^{2/3} W_i. \tag{43}
\]

The nonvanishing vector mass is consistent with gauge invariance because the massless vector field \(A\) has eaten the scalar field \(\theta\), spontaneously breaking the gauge symmetry, as advertised. It is interesting that the anomaly appears as a bulk effect in AdS space, in contrast to some earlier examples [3, 21] where anomalies arose from boundary terms.

The appearance of a mass implies that the \(R\)-current operator should acquire an anomalous dimension. From (41) it follows that

\[
(mL)^2 = \frac{2(g_s M)^2}{\pi(g_s N)}. \tag{44}
\]

Using the AdS/CFT correspondence (perhaps naively, as the KT solution is not asymptotically AdS) we find that the dimension of the current \(J^\mu\) dual to the vector field \(W^\mu\) is

\[
\Delta = 2 + \sqrt{1 + (mL)^2}. \tag{45}
\]

Therefore, the anomalous dimension of the current is

\[
\Delta - 3 \approx (mL)^2/2 = \frac{(g_s M)^2}{\pi(g_s N)}. \tag{46}
\]
We can obtain a rough understanding of this result by considering the relevant weak coupling calculation in the gauge theory. The leading correction to the current-current two-point function comes from the three-loop Feynman diagram composed of two triangle diagrams glued together, and the resulting anomalous dimension $\gamma_J$ is quadratic in $M$ and $N$. $\gamma_J$ must vanish when $M = 0$, and it must be invariant under the map $M \to -M$, $N \to N + M$, which simply interchanges the two gauge groups. Thus, the lowest order piece of the anomalous dimension will be of order $(g_s M)^2$. Our supergravity calculation predicts that this anomalous dimension is corrected at large $g_s N$ by an extra factor of $1/(g_s N)$. Of course, it would be interesting to understand this result better from the gauge theory point of view.

4 The $\mathcal{N}=2$ Supersymmetric $Z_2$ Orbifold

Encouraged by the agreement of field theory and supergravity on the conifold, let us examine another example to see how the same physical ideas apply in a different system. In this section we will study the $\mathcal{N}=2$ version of the conifold theory; it has gauge group $SU(N+M) \times SU(N)$ and is dual to a supergravity solution on an orbifold $S^5/Z_2$ [5, 6, 22, 23]. (After the completion of this work, we learned of a very similar analysis of this orbifold system which appeared earlier in [24].) The supergravity solution may be constructed as follows. We start with the space $\mathbb{R}^{1,5} \times \mathbb{R}^4/Z_2$ where the orbifold is given by the identification $x_{6,7,8,9} \sim -x_{6,7,8,9}$. Then we add $N$ coincident D3-branes, which we choose to be tangent to the 0123 directions; the resulting space has the geometry $AdS_5 \times S^5/Z_2$. To add fractional branes, we may take $M$ D5-branes and wrap them on the vanishing two-cycle of the orbifold $\mathbb{R}^4/Z_2$. These fractional branes are “pinned” to the orbifold fixed plane.

It is possible to identify the corresponding gauge theory by standard orbifold techniques [23]. The field content is in fact almost identical to that of the conifold theory, but there is an additional pair of adjoint chiral multiplets corresponding to the motion of D-branes along the orbifold fixed plane. These extra multiplets combine with the vector multiplet in the $\mathcal{N}=1$ theory to form an $\mathcal{N}=2$ vector multiplet. It is convenient to define $(x_4 + i x_5)/(2 \pi \alpha') \equiv \Phi = |\Phi| e^{i \beta}$. Rotations of the phase of $\Phi$ are dual to the $U(1)$ R-symmetry in the gauge theory.

To compute the anomaly for the $SU(N+M)$ gauge factor, notice that there are now $2N + (N + M) = 3N + M$ effective flavors, whose fermionic components have R-charge $-1/3$. Combining this with the contribution from the gluinos, we find that the anomaly coefficient is $\frac{2M/3}{16\pi^2}$. We would like to compare this with a computation...
from supergravity. Equations (16), (17) are also satisfied for the orbifold. To identify properly the relation between $\beta$ and the R-symmetry, note that the field $\Phi$ has R-charge $2/3$; thus a shift of $\beta \rightarrow \beta + \epsilon$ actually shifts the $U(1)_R$ by $\frac{3}{2}\epsilon$. This will change the first term in (21) by a factor of $\frac{3}{2}$ and give an anomaly coefficient $\frac{2M/3}{16\pi^2}$ in agreement with the gauge theory expectation.

A very interesting generalization of this theory was studied by Graña and Polchinski [26], and also by Bertolini et al [24], who added D7-branes wrapped on the 01236789 directions; an analogous solution with D7-branes on the conifold is not currently known. The extra D7-branes allow excitations of 3-7 strings and, depending on how the 7-branes are wrapped, will add $N_{7+}$ flavors coupled to the $SU(N+M)$ gauge group and $N_{7-}$ flavors coupled to the $SU(N)$ gauge group. The total number of 7-branes is $N_{7+} + N_{7-}$. The fermionic components of these flavors also have R-charge $-1/3$. For a small rotation $\beta \rightarrow \beta + \epsilon$, the corresponding $\Theta$ terms are

$$
\Theta_1 = 2\epsilon(M - \frac{1}{2}N_{7+})
$$

(47)

$$
\Theta_2 = 2\epsilon(-M - \frac{1}{2}N_{7-}).
$$

(48)

and the associated anomaly coefficient is $\frac{1}{16\pi^2}(2M/3 - N_{7+}/6 + N_{7-}/6)$.

We can reproduce the same result for the anomaly by a supergravity calculation, using the results of [22] and [26, 24]. It is helpful to think about the D3-branes on the orbifold fixed plane as a combination of a wrapped D5-brane and anti-D5-brane, each of which carries half a unit of D3-brane charge. By considering the Chern-Simons term in the action for a probe 5-brane

$$
\pm \mu_5 \int (2\pi \alpha') F_2^\pm \wedge C_4 = \pm \frac{\mu_5}{2\pi} \int F_2^\pm \wedge C_4 = \frac{1}{2}\mu_3 \int C_4,
$$

(49)

we see that the field strength on the 5-brane worldvolume will satisfy

$$
F_2^\pm = \pm \frac{1}{4} \omega_2
$$

(50)

where the upper sign refers to a D5 and the lower sign to an anti-D5. Now let us add the D7-branes. The supergravity solution has RR scalar and two-form potentials given by [26, 24]

$$
C_0 = -\frac{\beta}{2\pi}(N_{7+} + N_{7-})
$$

(51)

$$
C_2 = \alpha' \beta \omega_2(M - \frac{N_{7+} - N_{7-}}{4}).
$$

(52)
We have chosen the signs in (52) to differ from those of [24, 26, 27] by a conventional overall minus sign. To find the \( \Theta \) terms for the dual gauge theory, we just need to look at the Chern-Simons terms in the actions for a probe D5-brane and anti-D5-brane. For a D5-brane (whose excitations are in the \( SU(N+M) \) gauge group) we find that
\[
\frac{1}{2\pi \alpha'} \int (C_2 + 2\pi \alpha' C_0 F_2^{\perp}) = 2\beta(M - \frac{1}{2} N \gamma_7).
\]
(53)

Comparison with (18) and (19) shows that gravity reproduces the field theory expectation for \( \Theta_1 \) given in equation (47). The analogous computation for an anti-D5-brane will reproduce (18). Thus the anomaly as computed from supergravity agrees exactly with the field theory calculation.

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