Criterion toward understanding non-constant solutions to $p$-Laplace Neumann boundary value problem

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Abstract
We consider a $p$-Laplace equation $\Delta_p V + h(V) = 0$, with an arbitrary $C^1$-nonlinearity $h$, in a bounded domain and supplemented with the Neumann boundary condition. We prove a necessary condition for zeros of $h = h(V)$ to be touched by non-constant solutions to this problem.

1. Introduction
In this note, we present an elementary proof of a certain property of constant solutions to the following Neumann boundary value problem for the general nonlinear $p$-Laplace equation
\[ \Delta_p V + h(V) = 0 \quad \text{in } \Omega, \]
\[ \frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial \Omega, \]
where $\Omega \subset \mathbb{R}^n$ with $n \geq 1$ is a bounded domain with smooth boundary $\partial \Omega$, and $\nu$ denotes the unit outer normal vector to $\partial \Omega$. Here, we consider an arbitrary function $h \in C^1(\mathbb{R})$. $\Delta_p$ is the $p$-Laplace operator defined by
\[ \Delta_p V = \text{div} \left( |\nabla V|^{p-2} \nabla V \right) \quad \text{for } V \in W^{1,p}(\Omega) \]
with $p \in (1, \infty)$.

We note that the equation (1.1) is the Euler-Lagrange equation for the variational integral
\[ J_p(v) = \frac{1}{p} \int_{\Omega} \{ |\nabla v|^p - H(v) \} \, dx, \quad H(v) = \int_0^v h(s) \, ds. \]
Hence, $V \in W^{1,p}(\Omega)$ is a weak solution of the equation (1.1) if
\[ \int_{\Omega} (|\nabla V|^{p-2} \nabla V \cdot \nabla \eta) \, dx = \int_{\Omega} h(V) \eta \, dx \]

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is satisfied for all $\eta \in C_0^1(\Omega) = \{ \eta \in C^1(\Omega) \mid \eta = 0 \text{ on } \partial \Omega \}$. If $V \in C^1(\Omega)$ satisfies (1.1) in the distribution sense, then it is called a classical solution.

For the case $h \equiv 0$ in (1.1), weak solutions of (1.1) become members of $C^1_{\text{loc}}(\Omega)$, which is the set of all locally H"older continuous functions with exponent $\alpha = \alpha(n,p)$. Moreover, there are stronger regularity results, that is, the gradient is locally H"older continuous, see [26, 2, 3, 25, 13, 24].

Concerning the boundary value problem for a $p$-Laplace equation, there are many existence results of solutions for the problem with homogeneous Dirichlet boundary condition, see for example [12, 11, 10, 21, 23, 22]. For the problem with Neumann boundary condition, there are a few systematic studies and we can find some results in series of papers [4, 5, 6, 7, 8, 9]. In [9], the existence of a positive solution $V \in C^1(\Omega)$ of (1.1) has been obtained if the nonlinear term $h$ satisfies the following hypotheses $(A_i)$--$(A_{iii})$:

$(A_i)$ there exists $c > 0$ such that
$$h(\xi) \leq c(1 + \xi^{p-1}), \quad \text{for all } \xi \geq 0;$$

$(A_{ii})$ the function $\xi \mapsto \frac{h(\xi)}{\xi^{p-1}}$ is strictly decreasing on $(0, +\infty)$;

$(A_{iii})$ $\lim_{\xi \to +\infty} \frac{h(\xi)}{\xi^{p-1}} < 0 < \lim_{\xi \to 0} \frac{h(\xi)}{\xi^{p-1}}$.

We note that 0 in $(A_{iii})$ is the first eigenvalue of the nonlinear eigenvalue problem:

$$-\Delta_p V(x) = \lambda |V(x)|^{p-2} V(x) \quad \text{in } \Omega, \quad \frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial \Omega.$$ .

The main motivation for this work comes from the following observation. The problem (1.1), particularly in the case $p = 2$, arises in an analysis of models from biology, physics, and other different fields of sciences. If $V \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of problem (1.1) with $p = 2$, then integrating the equation and using the boundary condition, we obtain $\int_{\Omega} h(V(x)) \, dx = 0$. Hence, there exists $x_0 \in \overline{\Omega}$ such that $h(V(x_0)) = 0$. In other words, the number $a_0 = V(x_0)$ is a constant solution of problem (1.1). In such a case, we shall say that the non-constant solution $V = V(x)$ touches the constant solution $\overline{V} = a_0$. Hence, the property described above says that each non-constant solution of problem (1.1) with $p = 2$ has to touch at least one constant solution. We would like to consider that the same property holds true or not for classical solutions of the problem (1.1) with $p \in (1, \infty)$.

In this work, we present a necessary condition for certain constant solutions of problem (1.1) to be touched by a non-constant classical solution $V \in C^1(\overline{\Omega})$. As a consequence, we obtain a simple method which leads to a priori estimates of solutions to problem (1.1).

2. Results and examples

We begin by formulating our standing assumptions:
1. $\Omega$ satisfies an interior sphere condition, that is, for any $y \in \partial \Omega$, there exists a ball $B \subset \Omega$ with $y \in \partial B$;

2. the function $h \in C^1(\mathbb{R})$ is arbitrary such that (1.1) has a classical solution in $C^1(\overline{\Omega})$;

3. we consider non-constant classical solutions $V \in C^1(\overline{\Omega})$.

In the following, we say that $a \in \mathbb{R}$ is a non-degenerate zero of $h$ if $h(a) = 0$ and $h'(a) \neq 0$.

First, we state our main theorem.

**Theorem 1.** Let $V \in C^1(\overline{\Omega})$ be a non-constant classical solution of problem (1.1). Denote by $a_0 \in \mathbb{R}$ a zero of $h$ which is the biggest one touched by $V$, and assume that $a_0$ is non-degenerate.

(i) If $\max_{x \in \overline{\Omega}} V(x) > a_0$, then $h'(a_0) > 0$;

(ii) If $\max_{x \in \overline{\Omega}} V(x) = a_0$, then $h'(a_0) > 0$ provided that $1 < p \leq 2$.

We postpone a proof of Theorem 1 to the next section. For the case of $p > 2$ and $\max_{x \in \overline{\Omega}} V(x) = a_0$, which is a remaining case of (ii) in Theorem 1, we show the existence of solutions satisfying that $V(x_0) = a_0$ for $x_0 \in \overline{\Omega}$ and $h'(a_0) < 0$ in Section 5.

In the following corollary, we consider solutions of (1.1) which touch more than one zero of the function $h$.

**Corollary 2.** Let $b \in \mathbb{R}$ be a zero of $h$ which is the smallest one touched by $V \in C^1(\overline{\Omega})$. Assume that $b$ is non-degenerate zero. Then we obtain $h'(b) > 0$ under each assumption of (i) and (ii) in Theorem 1.

**Proof.** Here, it suffices to apply Theorem 1 with the function $\tilde{V}(x) = -V(x)$ which is a solution of equation

$$\Delta_p \tilde{V} + \tilde{h}(\tilde{V}) = 0,$$

where $\tilde{h}(s) = -h(s)$. In this case, the number $\tilde{b} = -b$ is the biggest zero of $\tilde{h}$ which is touched by $\tilde{V}$. Moreover, $\left.\frac{d}{ds} \tilde{h}\right|_{s=b} = \left.\frac{d}{ds} h\right|_{s=b}$. \hfill $\square$

We conclude this section with examples which illustrate the theorem and the corollary stated above.

**Example 3.** We consider the boundary value problem

$$\varepsilon^2 \Delta_p V - V + |V|^{q-1} V = 0 \quad \text{in } \Omega,$$

$$\frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial \Omega,$$

with $\varepsilon > 0$ and $q > 1$. The problem (2.1) with $p = 2$ was considered e.g. in a series of papers [20, 14, 17, 18]. When $p = 2$ and $1 < q < \frac{n+2}{n-2}$ for $n \geq 3$, and $p = 2$ and
1 < q < \infty \text{ for } n = 1, 2, \text{ it was shown by using the variational method that problem (2.1) has a positive solution } u_\varepsilon, \text{ so called a least-energy solution, for sufficiently small } \varepsilon > 0. \text{ Moreover, it was proved that this least energy solution has its only maximum point located on } \partial\Omega. \text{ We refer the reader to } [16] \text{ for more comments and references on this problem with } p = 2. \text{ For } 1 < p < 2, \text{ the nonlinear term } h(V) = -V + |V|^{q-1}V \text{ satisfies the hypotheses (A_i)-(A_{iii}) with } 0 \leq q \leq p - 1. \text{ Then, the problem (2.1) has a positive classical solution } V \in C^1(\Omega).

Assume that there exists a non-constant solution } V \in C^1(\Omega), \text{ which is not necessarily positive. Here, the functions } h(V) = -V + |V|^{q-1}V \text{ has three non-degenerate zeros } V(x) \in \{-1, 0, 1\}. \text{ It is clear from Theorem 1 and Corollary 2 that the solution has to touch either 1 or } -1, \text{ because } h'(\pm1) = -1 + p > 0. \text{ Since } h'(0) = -1, \text{ if } V(x) \text{ touches 0, then it has to touch both numbers } -1 \text{ and } 1. \text{ If } V(x) \text{ is a positive non-constant solution of (2.1), then } \max_{x \in \Omega} V(x) > 1.

Example 4. \text{ Next, we consider a } p \text{-Laplace equation with a bistable nonlinearity }
\varepsilon^2 \Delta_p V + V(1 - |V|^{q-1}) = 0 \quad \text{in } \Omega,
\frac{\partial V}{\partial \nu} = 0 \quad \text{on } \partial\Omega, (2.2)
\text{ where } \varepsilon > 0 \text{ and } q > 1. \text{ If } p = 2 \text{ and } q = 3, \text{ this is the boundary value problem for the Allen-Cahn equation, for which questions on the existence of non-constant solutions has been answered in [1, 15] and in references therein.}

Let } p \in [2, \infty). \text{ Then, the nonlinear term } h(V) = V(1 - |V|^{q-1}) \text{ satisfies the hypotheses (A_i)-(A_{iii}). In this case, the problem (2.2) has a positive classical solution } V \in C^1(\Omega). \text{ On the other hand, at the roots of the function } h(V) = 0, \text{ we have }
h'(-1) < 0, \quad h'(0) > 0, \quad h'(1) < 0.
\text{ Thus, every non-constant solution } V \in C^1(\Omega) \text{ of problem (2.2) for any } p \in (1, \infty) \text{ has to satisfy }
-1 \leq V(x) \leq 1 \quad \text{for all } x \in \Omega \quad \text{and} \quad V(x_0) = 0 \quad \text{for some } x_0 \in \Omega.
\text{ Therefore, if there exists a positive solution, then it should be a constant solution } V(x) \equiv 1.

3. Preliminaries

It is sometimes useful to consider weak supersolutions and weak subsolutions of a p-Laplace equation.

Definition 5. \text{ We say } u \in W^{1,p}_{loc}(\Omega) \text{ is a weak supersolution of a } p \text{-Laplace equation if } u \text{ satisfies }
\int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla \eta dx \geq 0 \quad (3.1)
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for all $\eta \in C^1_0(\Omega)$ with $\eta \geq 0$. If $u$ satisfies the reversed inequality of (3.1), that is,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \eta \, dx \leq 0,$$

then it is called a weak subsolution.

If we write

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) \leq 0,$$

then we promise that it denotes the inequality (3.1). For the reversed inequality above, it corresponds to (3.2).

We prepare some notations. The positive and negative parts of a function are defined by

$$f^+(x) = \max\{f(x), 0\}, \quad f^-(x) = \max\{-f(x), 0\}.$$  

It is clear that $f = f^+ - f^-$ and $|f| = f^+ + f^-$. For $x_0 \in \mathbb{R}^n$ and $r > 0$, $B_r(x_0)$ denotes a ball defined by

$$B_r(x_0) = \{x \in \mathbb{R}^n \mid |x - x_0| < r\}.$$  

If $x_0 = 0$, then we simply write $B_r$.

Next, we introduce the maximum principle, the Hopf boundary lemma and the Harnack inequality for solutions of a elliptic equation in divergence form.

We consider the inequality

$$\text{div} \left( |\nabla u|^{p-2} \nabla u \right) + G(x, u, \nabla u) \leq 0 \quad \text{in} \ \Omega,$$  

where $G(x, z, \xi) \in L^\infty(\Omega \times [0, d])$ satisfies, for $\kappa > 0$, that

$$G(x, z, \xi) \geq -\kappa |\xi|^{p-1} - f(z)$$  

for $x \in \Omega$, $z \geq 0$ and all $\xi \in \mathbb{R}^n$ with $|\xi| \leq 1$. The function $f$ is in $C(\mathbb{R}^+ \cup \{0\})$ and assumed to satisfy

$$f(0) = 0, \text{ and } f \text{ is non-decreasing on some interval } (0, \delta), \delta > 0.$$  

We define functions $F$ and $G$ by

$$F(s) = \int_0^s f(u) \, du,$$

and

$$\Phi(s) = \frac{p-1}{p} s^p.$$  

**Theorem 6** (Theorem 5.3.1 in [19]). Let (3.4) and (3.5) be satisfied. If $u \in C^1(\Omega)$ with $u \geq 0$ in $\Omega$ satisfies (3.3) and $u(x_0) = 0$ for some $x_0 \in \Omega$, then $u \equiv 0$ provided that either $f \equiv 0$ in $[0, d]$ with $d > 0$ or the following holds:

$$\lim_{\varepsilon \to +0} \int_{\varepsilon}^{\delta} \frac{ds}{\Phi^{-1}(F(s))} = \infty.$$  

(3.6)
Theorem 7 (Theorem 5.5.1 in [19]). Let (3.4) and (3.5) be satisfied, and assume that either $f \equiv 0$ in $[0, d]$ with $d > 0$ or (3.6) is satisfied. If $u \in C^1(\Omega)$ satisfies (3.3) with $u > 0$ in $\Omega$ and $u(y) = 0$ for some $y \in \partial \Omega$, then
\[
\frac{\partial u}{\partial \nu}(y) < 0.
\] (3.7)

4. Proof of Theorem 1

In this section, we denote by $x_0 \in \Omega$ a point such that $V(x_0) = a_0$, where $a_0$ is a zero of $h$. Moreover, we assume that $V(x)$ does not touch the bigger zero of $h$ than $a_0$ if there are some zeros of $h$. Since $\Omega$ is a bounded set, we can find $x_M \in \Omega$ such that
\[
a_M \equiv V(x_M) = \max_{x \in \Omega} V(x).
\]

In the following, we let $U(x) = a_M - V(x)$. Then, we see that $U$ is a weak solution of the problem
\[
\Delta_p U + k(U) = 0 \quad \text{in } \Omega, \quad \frac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial \Omega,
\] (4.1)
where $k(U) = -h(a_M - U)$. Moreover, we have that $U(x) \geq 0$ for $x \in \Omega$ and $U(x_M) = 0$.

The proof of Theorem 1 is based on Theorems 6 and 7. We discuss the cases $a_0 < a_M$ and $a_0 = a_M$, separately. In the following, we suppose $h'(a_0) < 0$ and show this hypothesis leads to a contradiction.

Case I: $a_0 < a_M$

First, we assume that $x_M \in \Omega$. If $h(V(x_M)) \geq 0$, then there exists a $x_1 \in \Omega$ such that $a_1 = V(x_1) > a_0$ and $h(a_1) = 0$. This is a contradiction because $V(x)$ does not touch any zeros bigger than $a_0$. Thus, we have $h(V(x_M)) < 0$. Since $V$ and $h$ are continuous, there exists $r > 0$ such that $h(V(x)) < 0$ for all $x \in B_r(x_M)$ where $B_r(x_M) = \{x \in \Omega \mid |x - x_M| < r\} \subset \Omega$.

Since $k(U(x)) > 0$ for all $x \in B_r(x_M)$ in (4.1), it is easily to see that $U$ becomes a weak supersolution of a $p$-Laplace equation in $B_r(x_M)$, that is, $U$ satisfies
\[
\int_{B_r(x_M)} |\nabla U|^{p-2} \nabla U \cdot \nabla \eta \, dx = \int_{B_r(x_M)} k(U)\eta \, dx \geq 0
\]
for all $\eta \in C_0^1(B_r(x_M))$ with $\eta \geq 0$. Since $U(x_M) = 0$, we use Theorem 6 with $G \equiv 0$ to obtain that $U \equiv 0$ in $B_r(x_M)$. By the standard argument, we see that $U \equiv 0$ in $\Omega$. This is a contradiction.

Next, we assume that $x_M \in \partial \Omega$. From the assumption, there exists a ball $B_r \subset \Omega$ with $x_M \in \partial B_r$ such that
\[
U(x) > U(x_M) = 0 \quad \text{for } x \in B_r.
\]
Since $U$ is a weak supersolution of a $p$-Laplace equation in $B_r$, it follows from Theorem 7 that
\[
\frac{\partial U}{\partial \nu}(x_M) < 0.
\]
This is a contradiction because \( U \) satisfies a homogeneous Neumann boundary condition.

**Case II:** \( a_0 = a_M \)

Let \( 1 < p \leq 2 \). For the function \( k \) in (4.1), we have that

\[
k(0) = 0 \quad \text{and} \quad k'(0) < 0.
\]

We assume that \( x_0 \in \Omega \). By the continuity of \( k(U(x)) \) and by (4.2), there exists an open neighborhood \( B \subset \Omega \) of \( x_0 \) such that

\[
k'(U(x)) \leq 0 \quad \text{for all} \ x \in B.
\]

Now, we use the well-known formula to obtain that

\[
k(U(x)) = k(U(x)) - k(0) = \int_0^1 \frac{d}{ds} k(sU(x)) \, ds = U(x) \cdot \int_0^1 k'(sU(x)) \, ds
\]

and this implies that

\[
c(x) \equiv \int_0^1 k'(sU(x)) \, ds \leq 0 \quad \text{for all} \ x \in B.
\]

Thus, \( U \) satisfies that

\[
\text{div} \left( |\nabla U|^{p-2} \nabla U \right) + c(x) U = 0.
\]

Since there exists \( \delta > 0 \) such that \(-\delta < c(x) \leq 0\) because \( U \in C^1(\overline{\Omega}) \) and \( k \in C^1(\mathbb{R}) \) from the assumption, the condition (3.4) holds. Hence, we can apply Theorem 6 to (4.5) and obtain that \( U \equiv 0 \) provided that (3.6) is satisfied. Here, we see, for \( 1 < p \leq 2 \), that (3.6) is

\[
\lim_{\varepsilon \to 0} \int_{\varepsilon}^{\delta} s^{-2/p} \, ds = \infty.
\]

This derives a contradiction.

Next, we assume that \( x_0 \in \partial \Omega \). As before, we find a ball \( B \subseteq \Omega \) such that \( x_0 \in \partial B \) and

\[
k'(U(x)) < 0 \quad \text{for all} \ x \in B.
\]

Hence, again using (4.4), we have that \( U \) is a solution of (4.5) in \( B \) with \(-\delta < c(x) \leq 0\).

Now, we apply Theorem 7 to the equation (4.5). Since (4.6) is satisfied, it follows from Theorem 7 that, for a non-constant solution \( U \) of (4.5) satisfying \( U(x) > U(x_0) = 0 \) in \( B \), we have

\[
\frac{\partial U(x_0)}{\partial \nu} < 0.
\]

Therefore, we get a contradiction.
5. Monotone solutions for bistable nonlinear case

We consider the following problem which is the problem (2.2) in one-dimensional case:

\[
\begin{align*}
\varepsilon^2 (|V'|^{p-2}V')' + V (1 - |V|^q) &= 0, \quad x \in (-1, 1) \\
V'(-1) &= V'(1) = 0,
\end{align*}
\]

(5.1)

where \( p > 2 \) and \( q \geq 2 \). Let \( h(V) = V (1 - |V|^q) \). Then, we have already seen in Example 4 that the roots of \( h(V) = 0 \) are \(-1, 0, 1\) and it is satisfied

\[
h'(-1) < 0, \quad h'(0) > 0, \quad h'(1) < 0.
\]

From Theorem 1, a non-constant classical solution of (5.1) with \( p \geq 2 \) cannot touch \(-1\) and \(1\). However, for the case \( p > 2 \), the problem (5.1) can have a solution \( V(x) \) satisfying \( V(x_0) = 1 \) for some \( x_0 \in [-1, 1] \). In order to prove the existence of such solutions, we will construct a solution of the problem (5.1) satisfying

\[-1 < V(x) < 1 \quad \text{for } x \in (-1, 1), \quad \text{and} \quad V(-1) = -1, \quad V(1) = 1,\]

which attains \(1\) and \(-1\) at the boundary points of the domain.

In the following, we use some ideas from [23] which treats Dirichlet boundary problems.

Letting \( \psi = |w'|^{p-2}w' \), we consider the following problem:

\[
\begin{align*}
\varepsilon \psi' + h(w) &= 0, \quad x \in (0, \infty), \\
w(0) &= 0, \quad \psi(0) = \alpha.
\end{align*}
\]

(5.2)

(5.3)

Here \( \alpha \) is a parameter. We will find a solution satisfying \( w(1) = 1 \) and \( \psi(1) = 0 \) for some \( \alpha \).

Integrating both sides of (5.2) with respect to \( x \) after multiplying them by \( w' \), we obtain that

\[
\varepsilon \frac{p-1}{p} \int_0^x (|\psi|^{p/(p-1)})' \, dx + \int_0^w h(s) \, ds = 0,
\]

where we have used \( w' = \psi |\psi|^{-(p-2)/(p-1)} \). Therefore, noting (5.3), we see that

\[
|\psi|^{p/(p-1)} = |\alpha|^{p/(p-1)} - \frac{p}{\varepsilon (p-1)} H(w), \quad H(w) = \int_0^w h(s) \, ds.
\]

(5.4)

Since

\[
H(w) = \int_0^w s(1 - |s|^q) \, ds = \frac{1}{q+2} w^2 \left( \frac{q+2}{2} - |w|^q \right),
\]

it follows from Figure 1 that there exists a \( x^* \in (0, \infty) \) such that we can have a solution of (5.2)–(5.3) with \( w(x^*) = 1 \) and \( \psi(x^*) = 0 \) if and only if \( \alpha = \alpha_\pm \), where

\[
\alpha_\pm = \pm \left( \frac{p}{\varepsilon (p-1)} H(1) \right)^{(p-1)/p} = \pm \left( \frac{pq}{2\varepsilon (p-1)(q+2)} \right)^{(p-1)/p}.
\]

(5.5)
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It is remained to show that \( x^* = 1 \). Now, we consider the case \( \alpha_+ > 0 \) and \( x > 0 \) is small. Then, \( \psi(x) > 0 \). Moreover, we assume \( w(x) > 0 \). Then, for \( 0 < \alpha \leq \alpha_+ \), there exists \( b_\alpha \) such that each solution of (5.2)–(5.3) satisfies \( w(x_\alpha) = b_\alpha > 0 \) and \( \psi(x_\alpha) = 0 \) for some \( x_\alpha \in (0, \infty) \). Note that \( b_\alpha \to 1 \) as \( \alpha \to \alpha_+ \). Differentiating both sides of (5.4) with respect to \( x \), we obtain that

\[
\frac{p}{p-1} \frac{\psi^{1/(p-1)} d\psi}{dx} = -\frac{p}{\varepsilon(p-1)} h(w) \frac{dw}{dx}.
\]

Since, using (5.4) again, we have

\[
\psi^{1/(p-1)} = \left( \frac{\alpha^{p/(p-1)} - \frac{p}{\varepsilon(p-1)} H(w)}{\alpha^{p/(p-1)} - \frac{p}{\varepsilon(p-1)} H(w)} \right)^{1/p},
\]

the equation (5.2) becomes

\[
\frac{dw}{dx} = \left( \frac{\alpha^{p/(p-1)} - \frac{p}{\varepsilon(p-1)} H(w)}{\alpha^{p/(p-1)} - \frac{p}{\varepsilon(p-1)} H(w)} \right)^{1/p}.
\]

Here we see that \( \alpha^{p/(p-1)} = \frac{p}{\varepsilon(p-1)} H(b_\alpha) \) from (5.4). Integrating the both sides of (5.6) with respect to \( x \) from 0 to \( x_\alpha \), we obtain that

\[
x_\alpha = \left( \frac{\varepsilon(p-1)}{p} \right)^{1/p} \int_0^{b_\alpha} (H(b_\alpha) - H(w))^{-1/p} dw.
\]

Since the function \( \alpha \mapsto b_\alpha \) is one to one, we define

\[
I(a) = \int_0^a (H(a) - H(w))^{-1/p} dw, \quad a \in (0, 1]
\]
and show that there exists $\varepsilon > 0$ such that

$$I(1) = \left(\frac{\varepsilon (p - 1)}{p}\right)^{-1/p}. \quad (5.9)$$

Letting $w = au$, we calculate $I(a)$ and obtain

$$I(a) = \int_0^1 (H(a) - H(au))^{-1/p} a \, du$$

$$= a \int_0^1 \left\{ \int_{au}^a w(1 - w^q) \, dw \right\}^{-1/p} \, du$$

$$= a^{1 - \frac{2}{p}} \int_0^1 \left\{ \int_{u}^1 s(1 - (as)^q) \, ds \right\}^{-1/p} \, du$$

$$= a^{1 - \frac{2}{p}} \int_0^1 \left\{ \frac{q + 2(1 - a^q)}{2(q + 2)} - \frac{u^2}{2} + \frac{a^q}{q + 2} u^{q + 2} \right\}^{-1/p} \, du.$$

Define the function $g$ as

$$g(u, a) = \frac{q + 2(1 - a^q)}{2(q + 2)} - \frac{u^2}{2} + \frac{a^q}{q + 2} u^{q + 2}.$$

Then, it satisfies

$$g(0, a) \geq \frac{q}{2(q + 2)} > 0 \quad \text{and} \quad g(1, a) = 0 \quad \text{for all} \quad a \in (0, 1].$$

Moreover, we see

$$\frac{\partial g}{\partial u}(u, a) = -u(1 - a^q u^q) < 0 \quad \text{for} \quad u \in (0, 1), \quad a \in (0, 1],$$

and

$$\frac{\partial^2 g}{\partial u^2}(u, a) = -1 + (q + 1)a^q u^q > 0 \quad \text{for} \quad u \in (0, 1),$$

provided that $a > 1/(q + 1)^{1/q}$. Since

$$\lim_{u \to 1^{-}} \frac{g(u, 1)}{(1 - u)^2} = \frac{q}{2} > 0,$$

it follows that $g(u, 1) = O\left((1 - u)^2\right)$ as $u \to 1$. Therefore, we obtain that there exists $C > 0$ such that

$$g(u, a)^{-1/p} \leq C(1 - u)^{-2/p} \quad \text{for} \quad \left(\frac{1}{q + 1}\right)^{1/q} < a \leq 1$$

in consideration of concavity and convexity of functions. Hence, it is satisfied that

$$\lim_{a \to 1^{-}} I(a) = I(1) = \int_0^1 g(u, 1)^{-1/p} \, du < \infty.$$
Now, we will show (5.9). The function \( m(\varepsilon) = \varepsilon^{-1/p} \) is a continuous and monotone decreasing function of \( \varepsilon \in (0, \infty) \) so that \( \lim_{\varepsilon \to +0} m(\varepsilon) = \infty \) and \( \lim_{\varepsilon \to \infty} m(\varepsilon) = 0 \). Therefore, there exists \( \varepsilon_0 > 0 \) such that

\[
I(1) = \left( \frac{\varepsilon_0(p - 1)}{p} \right)^{-1/p}.
\]

Consequently, we have obtained a solution of (5.2)–(5.3) satisfying

\[
w(0) = 0, \quad \psi(0) = \alpha_+, \quad w(1) = 1, \quad \psi(1) = 0. \quad (5.10)
\]

If \( w \) is the solution of (5.2)–(5.3), then \( z(x) = -w(-x) \) is a solution of the problem

\[
\begin{align*}
\varepsilon_0 \frac{d\tilde{\psi}}{dx} + h(z) &= 0, \quad x \in (-\infty, 0), \\
z(0) &= 0, \quad \tilde{\psi}(0) = \alpha_+, \\
z(-1) &= -1, \quad \tilde{\psi}(-1) = 0.
\end{align*}
\]

Letting

\[
V(x) = \begin{cases} 
 w(x) & x \in [0, 1], \\
z(x) & x \in [-1, 0],
\end{cases}
\]

we see that this is a solution of the problem (5.1) with \( \varepsilon = \varepsilon_0 \) satisfying \( V(1) = 1 \) and \( V(-1) = -1 \), which is the desired solution.

### 5.1. Flatness of solutions at the boundary

We see that there exists a solution of (5.1) which has flat parts around the boundary.

Fix \( 0 < \xi < 1 \) arbitrarily. We consider the existence of a solution of (5.2)–(5.3) satisfying \( w(\xi) = 1 \) and \( \psi(\xi) = 0 \). According to the same procedure as that in the previous case, we obtain \( \alpha = \alpha_0 \) which is given by (5.5). Moreover, the existence of such solution can be shown if there exists \( \varepsilon > 0 \) such that

\[
I(1) = \xi \left( \frac{\varepsilon(p - 1)}{p} \right)^{-1/p},
\]

where the function \( I(a) \) is defined by (5.8) for \( a \in (0, 1] \). It is easily seen that this is satisfied if \( \varepsilon = \varepsilon_0 \xi^p \). Hence, we have a solution of (5.2)–(5.3) satisfying

\[
w(0) = 0, \quad \psi(0) = \alpha_+, \quad w(\xi) = 1, \quad \psi(\xi) = 0.
\]

Letting \( z(x) = -w(-x) \), which is defined for \( -\xi \leq x \leq 0 \), we obtain a solution of (5.1) with \( \varepsilon = \varepsilon_0 \xi^p \):

\[
V(x) = \begin{cases} 
 1 & (\xi < x \leq 1), \\
w(x) & (0 < x \leq \xi), \\
z(x) & (-\xi < x \leq 0), \\
-1 & (-1 \leq x \leq -\xi).
\end{cases}
\]
This is monotone increasing function on \([-1, 1]\) and the maximum of \(V\) is attained not only at the boundary but also at inner parts of the domain \((-1, 1)\).

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