CORSON REFLECTIONS

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Abstract. A reflection principle for Corson compacta holds in the
forcing extension obtained by Levy-collapsing a supercompact cardinal
to $\aleph_2$. In this model, a compact Hausdorff space is Corson if and only
if all of its continuous images of weight $\aleph_1$ are Corson compact. We
use the Gelfand–Naimark duality, and our results are stated in terms of
unital abelian $C^*$-algebras.

Before starting, we should thank Alan Dow for pointing our attention
to [2]. Our use of $C^*$-algebras is closely related to Bandlow’s use of large
Hilbert cubes. Similar methods have been used in [3], [1], [10], [4], [8], [9],
[15], [11], and it is possible that the $C^*$-algebraic vantage point may yield
additional applications. A paper of Kunen ([15]) contains a closely related
analysis of Corson compact spaces. Although our main theorem is not a
logical consequence of Bandlow’s and Kunen’s results, most of the ideas
are contained in their papers. Since we were not aware of these results,
the present paper should be considered as a survey rather than a research
article.

A compact Hausdorff space $X$ is a Corson compactum (or shortly, Corson)
if it is homeomorphic to a subspace of some Tychonoff cube $[0, 1]^\kappa$ which has
the property that for every $\xi < \kappa$ the set $\{x \in X : x(\xi) \neq 0\}$ is countable.

Every metrizable compactum is homeomorphic to a subspace of $[0, 1]^\omega$ and
therefore Corson. In [18] it was proved that if there exists a non-reflecting
stationary subset of cofinality $\omega$ ordinals in $\omega_2$, then there exists a compact
Hausdorff space $X$ all of whose continuous images of weight $\aleph_1$ are Corson
(and even uniform Eberlein; see [15], but $X$ is not Corson.

Theorem 1. Suppose $\kappa$ is a supercompact cardinal. Then the following re-
flection statement holds in $\mathbb{V}^{\text{Coll}(\aleph_1, < \kappa)}$: If $X$ is a compact Hausdorff space,
then all continuous images of $X$ of weight at most $\aleph_1$ are Corson compact
if and only if $X$ is Corson. The same principle follows from Martin’s Max-
imum.

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1. **C*-algebras and Elementary Submodels**

1.1. **Background on C*-algebras.** We quickly review the required results on C*-algebras, and unital, abelian C*-algebras in particular. For additional information see e.g., [20] or [5].

A C*-algebra is a complex Banach algebra with involution which is isomorphic to a norm-closed, self-adjoint, algebra of bounded linear operators on a complex Hilbert space. If \( C \) is a C*-algebra and \( Z \subseteq C \) then \( C^*(Z) \) denotes the C*-subalgebra of \( C \) generated by \( Z \). We write \( C^*(a, Z) \) for \( C^*(\{a\} \cup Z) \).

1.1.1. **Positivity.** An element \( a \) of a C*-algebra \( A \) is self-adjoint if \( a = a^* \) and positive if it is self-adjoint and its spectrum is included in \([0, \infty)\). A standard argument (see [5, II.3.1.3(ii)]) shows that \( a \) is positive if and only if \( a = b^*b \) for some \( b \in A \). It is common to write \( a \geq 0 \) for '\( a \) is positive'.

For a C*-algebra \( A \) we write
\[
A_{\text{sa}} = \{ a \in A : a = a^* \},
\]
\[
A_+ = \{ a \in A : a \geq 0 \},
\]
\[
A_{+,1} = \{ a \in A : 0 \leq a \leq 1, \| a \| = 1 \}.
\]

On \( A_{\text{sa}} \) one defines partial ordering by letting \( a \leq b \) if and only if \( b - a \) is positive.

1.1.2. **States.** A continuous linear functional \( \varphi \) on a C*-algebra \( C \) is positive if \( \varphi(a) \geq 0 \) for every \( a \in C_+ \). A positive functional \( \varphi \) is a state if \( \| \varphi \| = 1 \). If \( C \) is a unital C*-algebra, with unit denoted \( 1_C \), then a functional \( \varphi \) is positive if and only if \( \varphi(1) = \| \varphi \| \). The space of all states of a unital C*-algebra \( C \) is convex and weak*-compact.

A proof of the following a well-known property of states is included for reader's convenience.

**Lemma 1.1.** Suppose that \( \varphi \) is a state of a C*-algebra \( A \) and \( 0 \leq a \).

If \( \varphi(a) = 0 \) then \( \varphi(ab) = 0 \) for all \( b \).

If \( \varphi(a) = \| a \| \) then \( \varphi(ab) = \varphi(ba) = \| a \| \| \varphi(b) \| \) for all \( b \).

**Proof.** Fix \( a \in A_{+,1} \) such that \( \varphi(a) = 1 \). Then \( 0 \leq 1 - a \leq 1 \) and therefore \( (1 - a)^2 \leq 1 \). Since \( \varphi \) is positive, it satisfies \( \varphi((1 - a)^2) \leq \varphi(1 - a) = 0 \). As it satisfies the Cauchy–Schwarz inequality, \( \| \varphi(c^*c) \| \leq \varphi(c^*c)\varphi(d^*d) \) for all \( c \) and \( d \), we have
\[
\varphi(b(1 - a)) \leq \varphi((1 - a)^2)\varphi(b^*b) = 0.
\]

By simplifying the left-hand side, this implies \( \varphi(b) = \varphi(ba) \). The other equalities follow by a similar argument and by rescaling. \( \Box \)
1.1.3. Characters. A character of a unital C*-algebra is a unital *-homomorphism \( \varphi : C \to \mathbb{C} \). The invertible elements of a unital C*-algebra form an open set, the kernel of any character is norm-closed. Characters of C*-algebras are therefore automatically continuous. Since every *-homomorphism is positive and of norm at most 1, every character is a state.

1.1.4. The Gelfand–Naimark duality. The category of compact Hausdorff spaces with respect to continuous maps as morphisms is equivalent to the category of unital, abelian C*-algebras with respect to *-homomorphisms (i.e., homomorphisms which respect the adjoint operation) as morphisms. Given a compact Hausdorff space \( X \), the C*-algebra associated with it is the *-algebra of all complex continuous functions on \( X \). To a continuous map \( f : X \to Y \) one associates the *-homomorphism

\[
C(Y) \ni a \mapsto a \circ f \in C(X).
\]

The inverse functor is defined as follows. If \( A \) is a unital abelian C*-algebra, then the space \( X \) of characters of \( A \) is compact in the weak*-topology and it separates points of \( A \). The Gelfand transform identifies \( A \) with \( C(X) \). By the Riesz Representation Theorem, the states on \( C(X) \) are in a bijective correspondence with the regular Radon probability measures on \( X \), via the correspondence \( \varphi \mapsto \mu_\varphi \) where

\[
\varphi(a) = \int_X a(x) d\mu_\varphi(x).
\]

Every unital *-homomorphism \( \Phi : C(X) \to C(Y) \) is of the form \( a \mapsto a \circ f \) for a continuous function \( f : Y \to X \).

The space of states of \( C \) is denoted \( S(C) \). The pure states are the extreme points of \( S(C) \). A state \( \varphi \) on \( C(X) \) is pure if and only if the associated probability measure \( \mu_\varphi \) is a point-mass measure (i.e., a measure that concentrates on a single point). In this case \( \varphi \) agrees with the evaluation functional \( a \mapsto a(x) \) for \( x \in X \). The space of pure states of \( C \), also known as the spectrum of \( C \), is denoted \( P(C) \).

When \( C \) is abelian, then a state is pure if and only if it is a character.

We will also need the following well-known lemma (the weight of a topological space is the smallest cardinality of its basis, and the density character of a metric space is the minimal cardinality of a dense subspace).

**Lemma 1.2.** The weight of an infinite compact Hausdorff space \( X \) is equal to the density character of \( C(X) \).

**Proof.** The weight of an infinite compact Hausdorff space \( X \) is equal to the minimal cardinal \( \kappa \) such that \( X \) is homeomorphic to a subspace of \([0, 1]^\kappa\). The density character of \( C(X) \) is equal to the cardinality of a minimal generating set for \( C(X) \), which is by the Stone–Weierstrass theorem equal to the minimal cardinality of a subset of \( C(X) \) that separates points of \( X \).
But this is the minimal $\kappa$ such that $X$ is homeomorphic to a subspace of $[0, 1]^\kappa$. \hfill $\square$

1.1.5. *Continuous Functional Calculus.* The spectrum of an element $a$ of a unital $C^*$-algebra $C$, i.e.

$$\text{sp}(a) = \{ \lambda \in \mathbb{C} : \lambda - a \text{ is not invertible} \}.$$ 

Two nontrivial facts deserve mention. The spectrum of any operator is a nonempty compact subset of the field of complex numbers. Second, if $A \subseteq B$ are unital $C^*$-algebras with the same unit and $a \in A$, then $\text{sp}(a)$ as computed in $A$ is equal to $\text{sp}(a)$ as computed in $B$. A normal element $a$ is positive if and only if its spectrum is included in $[0, \infty)$. If $a \in C(X)$ then clearly $\text{sp}(a)$ is equal to the range of $a$. An element $a$ of a $C^*$-algebra is normal if $aa^* = a^*a$. The continuous functional calculus asserts that $C(\text{sp}(a)) \cong C^*(a, 1)$, via the isomorphism defined by $f \mapsto f(a)$.

1.2. *Corson compacta.* We will need the following standard properties of Corson compacta.

1. Every closed subspace of a Corson compactum is a Corson compactum.
2. Every Corson compactum $X$ has the following property. If $Z \subseteq X$ and $x$ is an accumulation point of $Z$, then there exists a sequence $z_n$, for $n \in \mathbb{N}$, such that $\lim_n z_n = x$. A space with this property is said to be Fréchet or Fréchet–Urysohn.
3. Every continuous image of a Corson compactum is a Corson compactum.

The third property is [19, Theorem 6.2], and the first two are straightforward.

1.3. *Corson compacta and $C^*$-algebras.* The following, almost tautological, lemma provides reformulation of our results in terms of $C^*$-algebras.

**Lemma 1.3.** If $X$ is a compact Hausdorff space then $X$ is Corson compact if and only if there are a cardinal $\kappa$ and a family $a_\alpha$, $\alpha < \kappa$, in $C = C(X)$ such that the following conditions hold (see [1.1.4] for the notation $\leq$).

1. For every $\alpha$ we have $0 \leq a_\alpha \leq 1$.
2. For every pure state $\varphi$ on $C$ the set $\{ \alpha < \kappa : \varphi(a_\alpha) \neq 0 \}$ is countable.
3. The $C^*$-algebra $C$ is generated by $\{a_\alpha : \alpha < \kappa\}$ and $1$.

**Proof.** Suppose $X$ is a Corson compactum. Therefore we may identify $X$ with a subspace of a Hilbert cube $[0, 1]^\kappa$ such that $\{ \alpha < \kappa : x(\alpha) \neq 0 \}$ is countable for all $x \in X$. Then for $\alpha < \kappa$ the projection $a_\alpha$ to the $\alpha$th coordinate is a continuous function from $X$ into $[0, 1]$. These functions separate the points of $X$, and therefore the complex Stone–Weierstrass theorem implies $C(X) \cong C^*(\{1\} \cup \{a_\lambda : \lambda < \kappa\})$. The condition (2) is clearly equivalent.

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1Since $C$ is unital, we identify the scalar multiples of its unit $1_C$ with the complex numbers.
to the assertion that for every $\lambda < \kappa$ the set $\{x \in X : x(\lambda) \neq 0\}$ is countable. Since the pure states of $C(X)$ are exactly the evaluation functions at the points of $X$ (see §1.1.4), this proves that every pure state $\varphi$ vanishes at all but countably many of $a_\lambda$.

Conversely, if some family $a_\alpha$, for $\alpha < \kappa$, in $C = C(X)$ satisfies (1)–(3), then the function from $X = \mathcal{P}(C(X))$ defined by

$$\varphi \mapsto \langle \varphi(a_\alpha) : \alpha < \kappa \rangle$$

is a homeomorphism onto a Corson compact subspace of $[0,1]^\kappa$. $\square$

An proof analogous to that of Lemma 1.3 gives the following.

**Lemma 1.4.** If $X$ is a compact Hausdorff space then $X$ is Eberlein compact if and only if there are a cardinal $\kappa$ and a family $a_\alpha$, $\alpha < \kappa$, in $C = C(X)$ such that the following hold (see §1.1.1 for the notation $\leq$).

1. For every $\alpha$ we have $0 \leq a_\alpha \leq 1$.
2. The set $\{\alpha < \kappa : \varphi(a_\alpha) > \varepsilon\}$ is finite for every pure state $\varphi$ on $C$ and every $\varepsilon > 0$.
3. The $C^*$-algebra $C$ is generated by $\{a_\alpha : \alpha < \kappa\}$ and $1$. $\square$

This is a good moment to introduce the following lemma, needed in the proof of Theorem 1.

**Lemma 1.5.** If $X$ is a Corson compactum of weight at most $\aleph_1$ then $|X| \leq 2^{\aleph_0}$. In particular, if the Continuum Hypothesis holds, then $|X| \leq \aleph_1$.

**Proof.** Let $\kappa$ be a cardinal such that $X$ is homeomorphic to a subspace of $[0,1]^{\kappa}$ such that

$$\text{supp}(x) = \{\alpha : x(\alpha) \neq 0\}$$

is countable for all $x \in X$. Since $X$ has weight at most $\aleph_1$, there exists $S \subseteq \kappa$ such that the evaluation functions at the points of $S$ separate the points of $X$, and therefore $X$ is homeomorphic to a subset of $[0,1]^S$ as in the definition of Corson compacta. For an ordinal $\beta < \aleph_1$ let $X_\beta = \{x \in X : \text{supp}(x) \subseteq \beta\}$ (with $\text{supp}(x) \subseteq \aleph_1$). Then $|X_\beta| \leq ||[0,1]|^\beta = 2^{\aleph_0}$. Since $X$ is Corson, we have $X = \bigcup_\beta X_\beta$ and therefore $|X| \leq \aleph_12^{\aleph_0} = 2^{\aleph_0}$, as required. The second sentence follows immediately. $\square$
Suppose that $C$ is a $C^*$-algebra (unital and not necessarily abelian) and $M$ is an elementary submodel (not necessarily countable) of a large enough $H_\theta$ such that $C \in M$. We write
\[ C_M = C^*(M \cap C), \]
\[ P(C)^M = P(C) \cap M. \]

For $a \in C$ and $Y \subseteq S(C)$ write
\[ \|a\|_Y = \sup_{\varphi \in Y} |\varphi(a)| \]
and let $\|a\|_M = \|a\|_{M \cap P(C)}$.

By a minor abuse of notation, we denote the annihilator of $P(C) \cap M$ in $C$ by $M^\perp \cap C$, so that
\[ M^\perp \cap C = \{ a \in C : \|a\|_M = 0 \}. \]

The following is a consequence of the definitions.

**Lemma 2.2.** If $M$ and $C = C(X)$ are as in Definition 2.1 then we have
\[ M^\perp \cap C = \{ a \in C : a(x) = 0 \text{ for all } x \in X \cap M \}. \]

**Lemma 2.3.** If $M$, $C = C(X)$, and $Y \subseteq S(X)$, are as in Definition 2.1 then $\| \cdot \|_Y$ (and $\| \cdot \|_M$ in particular) is a seminorm majorized by $\| \cdot \|$ on $C$.

**Proof.** For every state $\varphi$ of $C$ we have $\| \varphi \| = 1$ and therefore $|\varphi(\cdot)|$ is a seminorm majorized by $\| \cdot \|$. Therefore $\| \cdot \|_Y$ is the supremum of a family of seminorms majorized by $\| \cdot \|$. \hfill $\square$

An order ideal in a $C^*$-algebra $A$ is a subset $\mathcal{A}$ of $A_+$ that is a cone (i.e., closed under the multiplication by positive scalars and addition) and hereditary (i.e., if $a \in \mathcal{A}$ and $0 \leq b \leq a$, then $b \in \mathcal{A}$).

**Lemma 2.4.** Suppose that $A$ is a (not necessarily commutative) $C^*$-algebra and $M$ is an elementary submodel of a large enough $H_\theta$ such that $A \in M$. Then the annihilator of $P(A)^M$,
\[ (P(A) \cap M)^\perp = \{ a \in A : \|a\|_M = 0 \} \]
is a norm-closed subspace of $A$ and its positive cone, $((P(A) \cap M)^\perp)_+$, is a norm-closed order ideal in $A$.

If $A$ is in addition abelian, then $(P(A) \cap M)^\perp$ is an ideal of $A$, and the quotient $C/((P(A) \cap M)^\perp)$ is isomorphic to $C(M \cap X)$, where $M \cap X$ is considered with the subspace topology.

**Proof.** For the first part, the annihilator of any subset of the dual space of a Banach space $Z$ is a norm-closed subspace of $Z$.

To prove the second part, note that the zero set $\{ a \in A_+ : \varphi(a) = 0 \}$ of any state $\varphi$ is a norm-closed cone. Hence $(P(A) \cap M)^\perp$ is an intersection of a family of cones, and therefore a norm-closed cone itself.
A $C^*$-subalgebra of a $C^*$-algebra is a left ideal if and only if its positive part is an order ideal (this is a result of Effros, see [21, Theorem 1.5.2]).

Now suppose $A$ is abelian. Then every left ideal of $A$ is an ideal of $A$. It remains to prove $C/(\mathcal{P}(A) \cap M) \cong C(M \cap X)$. Consider the *-homomorphism $\pi_M : C \to C(M \cap X)$ defined by

$$\pi_M(a) = a \upharpoonright M \cap X.$$  

This is a surjection of $C$ onto $C(M \cap X)$, and its kernel is equal to $\{a \in C : a \upharpoonright M \cap X = 0\} = (\mathcal{P}(A) \cap M) \perp$. □

By Lemma 2.4, if $C$ is a unital abelian $C^*$-algebra and $M \prec H_\theta$ has $C$ as an element, then we have an exact sequence

$$0 \to M \perp \cap C \to C \xrightarrow{\pi_M} C(M \cap X) \to 0$$

What is the relation of $C_M$ to the algebras in this exact sequence? With $\iota_M : C_M \to C$ denoting the inclusion map, we have the following commutative diagram.

\[
\begin{array}{ccc}
M \perp \cap C & \xrightarrow{\pi_M} & C(M \cap X) \\
\downarrow{\iota_M} & & \downarrow{\pi_M \circ \iota_M} \\
C_M & & \\
\end{array}
\]

A model $M$ that satisfies any of the equivalent conditions in Lemma 2.5 is said to split $X$. In §4 we will discuss this notion in some depth (not needed in the proof of Theorem 1).

**Lemma 2.5.** Suppose $X \in H_\theta$ is a compact Hausdorff space, $M \prec H_\theta$, and $X \in M$. Then the following are equivalent

1. $C^*(C_M, M \perp \cap C) = C$.
2. If $x$ and $y$ are in $\overline{M \cap X}$ and satisfy $a(x) = a(y)$ for all $a \in M \cap C$, then $x = y$.
3. $\pi_M \circ \iota_M$ is a surjection of $C_M$ onto $C(M \cap X)$, and the exact sequence $0 \to M \perp \cap C \to C(X) \to C(M \cap X) \to 0$ splits.

**Proof.** $\text{(3)} \to \text{(1)}$: Suppose $\pi_M \circ \iota_M$ is a surjection. Let us first show that this implies that the exact sequence in (3) splits. Since $\overline{M \cap X}$ separates points of $C_M$, $\pi_M \circ \iota_M$ is an injection and therefore an isomorphism between $C_M$ and $C(M \cap X)$. With $\theta : C(M \cap X) \to C_M$ denoting the inverse of $\pi_M \circ \iota_M$, we have a split exact sequence

$$0 \to M \perp \cap C \xrightarrow{\iota_M \circ \theta} C(M \cap X) \to 0$$

$\square$
In order to prove (1), fix $a \in C$ and let $a_1 = (\iota_M \circ \theta)(a)$. Then $a_1 \in C_M$, $a_0 = a - a_1$ belongs to $M^\perp \cap C$, and therefore $a = a_0 + a_1$ belongs to $C(C_M, M^\perp \cap C)$.

(1) $\implies$ (2): Suppose that (2) fails and fix distinct $x$ and $y$ in $\overline{M \cap X}$ such that $a(x) = a(y)$ for all $a \in M \cap C$. Since $x \neq y$, there exists $b \in C$ such that $b(x) \neq b(y)$. But every $c \in (\overline{M \cap X})^\perp$ satisfies $c(x) = 0 = c(y)$, and therefore $b \notin C^*(M \cap C, (\overline{M \cap X})^\perp)$, showing that (1) fails.

(2) $\implies$ (3) The assumption asserts that the elements of $C_M$ separate points of $\overline{M \cap X}$. Therefore $(\pi_M \circ \iota_M)(C_M)$ is a norm-closed, self-adjoint, subalgebra of $C(\overline{M \cap X})$ that separates points and contains all constant functions. By the complex Stone–Weierstrass theorem (e.g., [22, Theorem 4.3.4]), it is equal to $C(\overline{M \cap X})$.

\textbf{Lemma 2.6.} Suppose $X$ is a compact Hausdorff space and $\kappa$ is a cardinal such that every continuous image of $X$ of weight at most $\kappa$ is Corson.

(1) If $\theta$ is a regular cardinal such that $X \in H_\theta$ and $M \ll H_\theta$ is internally approachable and satisfies $X \in M$ and $|M| = \kappa$, then $X \cap M$ is a closed Corson compact subspace of $X$.

(2) If $\theta$ and $M$ are as in (1), then $M$ splits $X$.

(3) If $Y$ is a subspace of $X$ of cardinality not greater than $\kappa$, then the closure of $Y$ is a Corson compactum.

\textbf{Proof.} (1) Since every Corson compactum is Fréchet, the assumptions imply that every continuous image of $X$ of weight at most $\kappa$ is Fréchet. The pure state space of $C_M$ is a continuous image of $X$. Its weight is equal to the density character of $C_M$ (Lemma 1.2), which is not greater than $|M| = \kappa$. Therefore the pure state space of $C_M$ is Fréchet.

Fix $x \in \overline{M \cap X}$. Let $N \ll H_\theta$ be internally approachable and such that $M \subseteq N$, $x \in N$ and $|N| = \kappa$. Then the pure state space of $C_N$ is Fréchet, as a continuous image of $X$ of weight at most $\kappa$. This implies that there exists a sequence $y_n$, for $n \in \mathbb{N}$, in $M \cap X$ such that in the weak*-topology on $\mathcal{P}(C_N)$ we have $\lim_n y_n = x$.

This means that for all $a \in N \cap C$ we have $\lim_n a(y_n) = a(x)$. Since $N$ is internally approachable, the sequence $(y_n)$ is included in some $N' \ll N$. By elementarity, some sequence in $N$ has the same property. We may assume that this sequence is $(y_n)$. By elementarity again, $\lim_n a(y_n) = a(x)$ for all $a \in C$. But since $M$ is internally approachable, by the same argument as before we may assume that the sequence $(y_n)$ belongs to $M$. By elementarity there exists $z \in M$ such that for all $a \in C \cap M$ we have $\lim_n a(y_n) = a(z)$. Again by elementarity, for all $a \in C$ we have $\lim_n a(y_n) = a(z)$. Since $C$ separates points of $X$, this implies $z = x$ and therefore $x \in M$.

Since $x$ was an arbitrary element of $\overline{M \cap X}$, this implies that $M \cap X$ is a closed subspace of $X$. This space is homeomorphic to the space of all pure states of $C_M$, which is a continuous image of $X$ of weight $\kappa$. By our assumption, it is a Corson compactum.

(2) By (1), $M$ satisfies Lemma 2.5 (3).
(3) Fix $M \prec H_\theta$ such that $X \subseteq M$, $Y \subseteq M$, $M^\omega \subseteq M$, and $|M| = \kappa$. The closure of $Y$ is included in $M \cap X$. By (1), $M \cap X = M \cap X$ is Corson compact. Since a closed subspace of a Corson compactum is a Corson compactum, the conclusion follows. \qed

The following proposition ought to be well-known.

**Proposition 2.7.** Suppose that $X$ is a compact Hausdorff space such that every continuous image of $X$ of weight at most $2^{\aleph_0}$ is Fréchet. Then $X$ is Fréchet.

**Proof.** Fix $Z \subseteq X$ and $x \in \overline{Z}$. In order to find a sequence in $Z$ that converges to $x$, fix a large enough regular cardinal $\theta$ and $M \prec H_\theta$ that contains $X, Z$, and $x$ and such that $M^\omega \subseteq M$ and $|M| = 2^{\aleph_0}$. Then the pure state space of $C_M$ is, being a continuous image of $X$ of weight at most $2^{\aleph_0}$, Fréchet. We can identify all $z \in Z \cap M$ and $x$ with pure states of $C_M$.

**Claim 2.8.** In the weak*-topology induced by $C_M$, $x$ is an accumulation point of $Z \cap M$.

**Proof.** In the weak*-topology induced by $C$, $x$ is an accumulation point of $Z$. This means that for all $n \geq 1$ and all $a_j \in C$, for $j < n$, the $n$-tuple $(a_j(x) : j < n)$ is an accumulation point of $\{(a_j(z) : j < n) : z \in Z\}$. Since $M \prec H_\theta$, the following holds for all $n$:

For all $a_j \in C(X) \cap M$, for $j < n$, the $n$-tuple $(a_j(x) : j < n)$ is an accumulation point of $\{(a_j(z) : j < n) : z \in Z \cap M\}$.

Since $C(X) \cap M$ is dense in $C_M$, $x$ is an accumulation point of $Z \cap M$ in the weak*-topology induced by $C_M$. \qed

Let $z_n$, for $n \in \mathbb{N}$, be a sequence in $Z \cap M$ that converges to $x$ in the weak*-topology of $C_M$. This sequence belongs to $M$, since $M^\omega \subseteq M$. By elementarity, $\lim_n a(z_n) = a(x)$ for all $a \in C$. This implies $\lim_n z_n = x$ in the topology of $X$.

Since $Z$ and $x$ were arbitrary, this proves that $X$ is Fréchet. \qed

The following result appears as [12, Exercise 2.4G] and we include a proof for reader’s convenience.

**Lemma 2.9.** A continuous image of a compact, Hausdorff, and Fréchet space is Fréchet.

**Proof.** Suppose $X$ is Fréchet and $f : X \to Y$ is a surjection. Fix $Z \subseteq Y$ and an accumulation point $z$ of $Z$. Let $Z' = f^{-1}\{(z)\}$ and let $T = f^{-1}\{x\}$. We claim that $\overline{Z'} \cap T \neq \emptyset$. Assume otherwise, and for every $z \in T$ fix an open neighbourhood $u_z$ disjoint from $Z'$. Then $U = \bigcup_{z \in T} u_z$ is an open cover of $T$ disjoint from $Z'$, and $f[X \setminus U]$ is a compact subset of $Y$ containing $Z$ that $x$ does not belong to; contradiction.

Therefore $\overline{Z'} \cap T \neq \emptyset$. Since $X$ is Fréchet, there exists a sequence $(z'_n)$ in $Z'$ such that $x' = \lim_n z'_n$ belongs to $T$. Then $f(z'_n) \in Z$ for all $n$ and $\lim_n f(z'_n) = x$. 


Since \( Z \) and \( x \) were arbitrary, this proves that \( Y \) is Fréchet. \( \square \)

3. Proof of Theorem \([1]\)

Recall that \( N \prec H_\theta \) is **internally approachable** if it is equal to the union of an increasing \( \omega_1 \)-chain of countable elementary submodels each of whose proper initial segments belongs to the next model in the sequence. Consider the following reflection principle.

(R) If \( \theta \) is an uncountable regular cardinal and \( S \subseteq \mathcal{P}_{\aleph_1}(H_\theta) \) is stationary, then there exists an internally approachable \( N \prec H_\theta \) of cardinality \( \aleph_1 \) such that \( S \cap \mathcal{P}_{\aleph_1}(N) \) is stationary in \( \mathcal{P}_{\aleph_1}(N) \).

This principle was introduced and proved to follow from MM in [14, Theorem 13]. Requiring \( N \) to be an elementary submodel of \( H_\theta \) is not a loss of generality, since if every stationary subset of \( \mathcal{P}_{\aleph_1}(H_\theta) \) reflects to \( \mathcal{P}_{\aleph_1}(Z) \) for some \( Z \in \mathcal{P}_{\aleph_1}(H_\theta) \), then every stationary subset of \( \mathcal{P}_{\aleph_1}(H_\theta) \) reflects to a stationary set of \( Z \in \mathcal{P}_{\aleph_1}(H_\theta) \).

Proposition 3.1 is standard, but we could not find it in the literature. A proof is included for reader’s convenience.

**Proposition 3.1.** If \( \kappa \) is a supercompact cardinal then \( \text{Coll}(\aleph_1, \kappa) \) forces (R).

**Proof.** We will prove a strengthening in which the model \( N \) is required to be closed under \( \omega \)-sequences. Let \( V[G] \) be a forcing extension by \( \text{Coll}(\aleph_1, \kappa) \). Suppose \( \theta \) is an uncountable regular cardinal and \( S \subseteq \mathcal{P}_{\aleph_1}(H_\theta) \) is stationary. Let \( j: V \rightarrow N \) be an elementary embedding with critical point \( \kappa \) such that \( j(\kappa) > \theta \) and \( N \) is closed under \( 2^{<\theta} \)-sequences. (We will be using \( |H_\theta| = 2^{<\theta} \).)

By standard methods ([6, Proposition 9.1]), \( j \) can be extended to an elementary embedding (also denoted \( j \))

\[
j: V[G] \rightarrow N[G_1]
\]

for an \( N \)-generic filter \( G_1 \subseteq \text{Coll}(\aleph_1, j(\kappa)) \) such that

\[
G_1 \cap \text{Coll}(\aleph_1, j(\kappa)) = G.
\]

Let \( Z = (H_\theta)^{V[G]} \). Since \( \theta \geq \kappa \) is regular and \( \text{Coll}(\aleph_1, \kappa) \) has \( \kappa \)-cc, \( H_\theta^{V[G]} \) is a \( \text{Coll}(\aleph_1, \kappa) \)-name for \( Z \). As \( N \) is closed under \( 2^{<\theta} \)-sequences and \( \text{Coll}(\aleph_1, \kappa) \) has \( \kappa \)-cc, \( Y = j^*Z \) belongs to \( N[G_1] \). It is an elementary submodel of \( j(Z) = H_{j(\theta)} \). Since \( \text{Coll}(\aleph_1, j(\kappa)) \) is \( \aleph_1 \)-closed, \( \mathcal{P}_{\aleph_1}(Y) \subseteq Y \). In \( V[G] \) it holds that \( S \) is a stationary subset of \( \mathcal{P}_{\aleph_1}(Z) \), and the quotient forcing \( \text{Coll}(\aleph_1, j(\kappa)) \) is \( \aleph_1 \)-closed. Therefore \( S \) remains a stationary subset of \( \mathcal{P}_{\aleph_1}(Z) \) in \( N[G_1] \). But this is equivalent to \( j^*S \) being stationary in \( \mathcal{P}_{\aleph_1}(Y) \).

In \( N[G_1] \) we therefore have \( |Y| = \aleph_1, Y^\omega \subseteq Y \), and \( j(S) \) reflects to \( Y \). By elementarity, in \( V \) there exists \( X \prec H_\theta \) of cardinality \( \aleph_1 \) such that \( S \cap \mathcal{P}_{\aleph_1}(Y) \) is stationary in \( \mathcal{P}_{\aleph_1}(X) \) and \( X^\omega \subseteq X \).

Since \( \theta \) and \( S \) were arbitrary, we have proved that (R) holds in \( V[G] \). \( \square \)
Lemma 3.2. Suppose that \((R)\) holds and \(X\) is a compact Hausdorff space such that every continuous image of \(X\) of weight not greater than \(\aleph_1\) is Corson. If \(\theta\) is a regular cardinal such that \(X \in H_\theta\) then the set
\[
D = \{M \in \mathcal{P}_{\aleph_1}(H_\theta) : X \in M, M \text{ does not split } X\}
\]
is nonstationary.

Proof. Assume otherwise. By \((R)\) fix an internally approachable \(N \in \mathcal{P}_{\aleph_1}(H_\theta)\) such that \(D \cap \mathcal{P}_{\aleph_1}(N)\) is stationary in \(\mathcal{P}_{\aleph_1}(N)\). By Lemma 3.2 \(N \cap X\) is a Corson compactum and \(N\) splits \(X\).

By Lemma 1.3 we can fix \(a_\alpha\), for \(\alpha < \aleph_1\), that together with 1 generate \(C_N\) so that the set
\[
Z(x) = \{\alpha < \aleph_1 : a_\alpha(x) \neq 0\}
\]
is countable for all \(x \in N \cap X\).

At this point we cannot assert that this sequence of generators belongs to \(N\). Choose a continuous \(\aleph_1\)-sequence \(M_\alpha\), for \(\alpha < \aleph_1\), of countable elementary submodels of the expanded structure \((\aleph_1, (a_\alpha : \alpha < \aleph_1))\) such that \(N = \bigcup_\alpha M_\alpha\). Then for every \(M_\alpha\) and every \(\beta \in M_\alpha \cap \alpha\) we have \(a_\beta \in M_\alpha\). Also, for every \(x \in M_\alpha \cap X\) the countable set \(Z(x)\) belongs to \(M_\alpha\).

Since \(S \cap N\) is stationary, there exists \(\alpha\) such that \(M_\alpha \in S\). As \(M_\alpha\) does not split \(X\), some distinct \(x\) and \(y\) in \(M_\alpha \cap X\) satisfy \(a_\alpha(x) = a_\alpha(y)\) for all \(a \in M_\alpha \cap C\). As \(x \neq y\) and \(N\) splits \(X\), there exists \(\gamma < \aleph_1\) such that \(a_\gamma(x) \neq a_\gamma(y)\); by the previous line, \(a_\gamma \notin M_\alpha\) and we have \(\gamma \in \aleph_1 \setminus M_\alpha\).

Since \(N \cap X\) is Fréchet, there are \(x_n\) and \(y_n\), for \(n < \omega\), in \(M_\alpha \cap X\) such that, in the weak*-topology of \(C(X)\), we have \(\lim_n x_n = x\) and \(\lim_n y_n = y\). Since the sequences \((x_n)\) and \((y_n)\) belong to \(N\), by elementarity we have \(\lim_n x_n = x\) and \(\lim_n y_n = y\) in the weak*-topology on \(C\). Since \(Z(x_n) \subseteq M_\alpha\) and \(Z(y_n) \subseteq M_\alpha\), every \(\gamma \in \aleph_1 \setminus M_\alpha\) and every \(n < \omega\) satisfy \(a_\gamma(x_n) = 0\) and therefore \(a_\gamma(x) = 0\). Similarly, \(a_\gamma(y_n) = 0\) for all \(n\) and therefore \(\gamma(x) = 0\); contradiction. \(\square\)

Lemma 3.3. Suppose that \(\theta\) is a regular cardinal, \(X \in H_\theta\) is a compact, Fréchet, Hausdorff space, and the set
\[
K = \{M \in \mathcal{P}_{\aleph_1}(H_\theta) : X \in M \text{ and } M \text{ splits } X\}
\]
includes a club. If \(f : H_\theta^{\leq \omega} \to H_\theta\) is such that every \(M \in \mathcal{P}_{\aleph_1}(H_\theta)\) closed under \(f\) is in \(K\), then every \(N < H_\theta\) closed under \(f\) and such that \(X \in N\) splits \(X\).

Proof. Fix \(N < H_\theta\) such that \(X \in N\) and \(N\) is closed under \(f\). Suppose that \(x\) and \(y\) are distinct points of \(N \cap X\). Since \(X\) is Fréchet, there are sequences \((x_n)\) and \((y_n)\) in \(N \cap X\) converging to \(x\) and \(y\), respectively. Fix a countable \(M < N\) closed under \(f\) and such that \(X\) and all \(x_n\) and all \(y_n\) belong to \(M\). Then \(x\) and \(y\) belong to \(M \cap X\). Since \(M\) splits \(X\), there is \(a \in C_M\) such that \(a(x) \neq a(y)\). Since \(M \subseteq N\), we have \(a \in N\).
Because $x$ and $y$ were arbitrary distinct points of $N \cap X$, we conclude that $N$ splits $X$. □

Lemma 3.4. Suppose $X \in H_\theta$ is a compact, Fréchet, Hausdorff space and $M_\alpha$, for $\alpha < \lambda$, is a continuous chain of elementary submodels with the following properties for all $\alpha < \lambda$:

1. $X \in M_0$.
2. $M_\alpha$ splits $X$.
3. $M_\alpha \cap X$ is a Corson compact subspace of $X$.
4. $\bigcup_{\alpha < \lambda} M_\alpha \supseteq X$.

Then $X$ is Corson compact.

Proof. We write $C = C(X)$, and we also write $C_\alpha$ in place of $C(M_\alpha)$. Let us say that a sequence $(a_\alpha)_{\alpha < \kappa}$ in a unital, abelian, $C^*$-algebra $D$ that satisfies (1–3) of Lemma 1.3 is a sequence of Corson generators for $D$. We will choose a sequence of Corson generators for $C$ in blocks, by recursion on $\lambda$.

First choose a sequence $A_0$ of Corson generators for $C_0$. For $\alpha < \lambda$, let $D_{\alpha+1} = C^*(1, M_\alpha \cap C_{\alpha+1})$.

This is a unital $C^*$-subalgebra of $C_{\alpha+1}$. Since the pure state space of $C_{\alpha+1}$ is Corson and every continuous image of a Corson compact space is Corson, the pure state space of $D_{\alpha+1}$ is Corson. We can therefore choose a sequence $A_{\alpha+1}$ of Corson generators for $D_{\alpha+1}$. For a limit ordinal $\alpha$, let $A_\alpha = \emptyset$.

We claim that $A = \bigcup_\alpha A_\alpha$ is a sequence of Corson generators for $C(X)$.

First we prove that $C^*(A \cup \{1\}) = C$. Towards this end, we use induction on $\alpha \leq \kappa$ to prove that $C^*(\bigcup_{\beta \leq \alpha} A_\beta \cup \{1\}) = C_\alpha$.

This is true for $\alpha = 0$. Suppose that the assertion is true for all $\beta < \alpha$. Consider the case when $\alpha$ is a successor ordinal, say $\alpha = \beta + 1$. Since $M_\beta$ splits $X$, we have $C_{\beta+1} = C^*(C_\beta, M_\beta^\perp) = C^*(A_{\beta+1} \cup \bigcup_{\gamma \leq \beta} A_\gamma)$ as required. If $\alpha$ is a limit ordinal, then $\bigcup_{\beta < \alpha} C_\beta$ is dense in $C_\alpha$, and $A_\alpha = \bigcup_{\beta < \alpha} C_\beta$ generates $C_\alpha$.

Therefore $C^*(A) = C$, and it remains to prove that the set $Z(x) = \{a \in A : a(x) \neq 0\}$ is countable for every $x \in X$. Assume otherwise and fix $x$ such that $Z(x)$ is uncountable. Since $A_\alpha$ is a sequence of Corson generators for every $\alpha$, $Z(x) \cap A_\alpha$ is countable for all $\alpha$. Therefore the set $\{\alpha : Z(x) \cap A_\alpha \neq \emptyset\}$ is uncountable. Let $\beta$ be the least limit point of this set of cofinality $\aleph_1$. Let $\varphi$ be the pure state of $C_\beta$ corresponding to $x$, i.e., $\varphi(a) = a(x)$ for $a \in C_\beta$. Lemma 2 implies that $\mathcal{P}(C_\beta)$ is Fréchet. Since $\operatorname{cof}(\beta)$ is uncountable, there exists $\gamma < \beta$ such that $\varphi$ belongs to $M_\gamma \cap X$. By construction, this implies that $A_\delta$ is annihilated by $z$ for all $\alpha \leq \delta < \beta$; contradiction. □
A discussion on why Lemma 3.4 does not apply to the space constructed in [18] is given in Example 4.3.

Proof of Theorem 1. By Proposition 3.1, (R) holds in $V^\text{Coll}(\aleph_1, < \kappa)$. This model also satisfies the Continuum Hypothesis. By induction on $\lambda$ we will prove that if $X$ is a compact Hausdorff space of weight $\lambda$ such that all continuous images of $X$ of weight at most $\aleph_1$ are Corson, then $X$ is Corson.

Suppose that the inductive hypothesis is true for all compact Hausdorff spaces of weight less than $\lambda$. Then $\lambda \geq \aleph_2$. Fix a compact Hausdorff space $X$ such that all continuous images of $X$ of weight less than $\lambda$ are Corson.

Since every Corson space is Fréchet and since $2^{\aleph_0} = \aleph_1 < \lambda$, Proposition 2.7 implies that $X$ is Fréchet.

Fix a regular $\theta$ such that $X$ and $C(X)$ belong to $H_\theta$. By Lemma 3.2, the set

$$K = \{ M \in \mathcal{P}_{\aleph_1}(H_\theta) : X \in M, M \text{ splits } X \}$$

includes a club. By Kueker’s theorem (see e.g., [13, Theorem 3.4]), there exists $f : H_\theta^\omega \to H_\theta$ such that every $M \in \mathcal{P}_{\aleph_1}(H_\theta)$ closed under $f$ belongs to $K$. By Lemma 3.3 if $M \prec H_\theta$, $X \in M$, and $M$ is closed under $f$, then $M$ splits $X$. Hence if in addition $|M| < \lambda$, then $M \cap X$ is a Corson compact subspace of $X$.

Let $M_\alpha$, for $\alpha < \lambda$, be an increasing chain of elementary submodels of $H_\theta$ closed under $f$ such that $X \in M_0$ and $|M_\alpha| < \lambda$ for all $\alpha$. By the previous paragraph, these models satisfy the assumptions of Lemma 3.4. Since in addition $X$ is Fréchet, Lemma 3.4 implies that $X$ is Corson. \hfill \Box

4. Some remarks on splitting models

In the concluding section we collect a few observations on the notion of splitting models not required in the proofs of our main results. In all of the following examples, $\theta$ is a large enough regular cardinal so that $X \in H_\theta$.

Example 4.1. There exists a compact Hausdorff space $X$ such that no countable $M \prec H_\theta$ splits $X$. Take for example $X = \beta\mathbb{N} \setminus \mathbb{N}$, the Čech–Stone remainder of $\mathbb{N}$. We only need to know that if $Z$ is a countable discrete subset of $X$, then the closure of $Z$ is homeomorphic to $\beta\mathbb{N}$ and therefore of cardinality $2^{2^\omega}$.

Suppose $M \prec H_\theta$ is countable (and certainly $X = \beta\mathbb{N} \setminus \mathbb{N} \in M$ if $\theta$ is large enough). A counting argument shows that (3) of Lemma 2.5 fails. First, $|M \cap X| = 2^{2^\omega}$. Second, $C_M$ is separable and therefore $|\mathcal{P}(C_M)| \leq 2^{2^\omega}$.

Example 4.2. There is a compact, Hausdorff, Fréchet (even first countable), and separable space $X$ such that no countable $M \prec H_\theta$ splits $X$. Let $X$ be $[0, 1] \times \{0, 1\}$ with the lexicographical ordering and the order topology. It is easy to check that this space is compact, Hausdorff, first countable, and separable. Suppose $M \prec H_\theta$ is countable. Then $M \cap X = X$ since a countable dense set of $X$ belongs to (and is therefore a subset of) $M$. Choose
$x \in [0,1] \setminus M$. Then $(x,0)$ and $(x,1)$ are not separated by the elements of $C_M$, and therefore $M$ fails (2) of Lemma 2.5.

Our last example is more specific and most relevant to Theorem I.

**Example 4.3.** There exists a compact Hausdorff space $X$ and an increasing sequence of elementary submodels $M_n \prec H_\theta$ such that each $M_n$ splits $X$ but $\bigcup_n M_n$ does not split $X$. This space is based on [18], and we include the relevant details for reader’s convenience.

Suppose $\lambda$ is an ordinal, $S \subseteq \lambda$ and every $\alpha \in S$ is a limit ordinal of cofinality $\omega$. For each $\alpha \in S$ fix a strictly increasing sequence $p_n(\alpha)$, for $n \in \mathbb{N}$, of ordinals such that $\sup_n p_n(\alpha) = \alpha$. Let $A_\alpha = \{p_n(\alpha) : n \in \mathbb{N}\}$ and let $\mathcal{A}(S)$ be the Boolean algebra of subsets of $Z = \bigcup_{\alpha \in S} A_\alpha$ generated by all finite subsets of $Z$ and $\{A_\alpha : \alpha \in S\}$.

Let $X(S)$ be the Stone space of $\mathcal{A}(S)$, i.e., the space of ultrafilters of $\mathcal{A}$. It is a compact Hausdorff space. Each ordinal $\xi < \lambda$ corresponds to a principal ultrafilter which is an isolated point $x(\xi)$ of $X$. Each $\alpha \in S$ corresponds to a unique nonprincipal ultrafilter $y(\alpha)$ that concentrates on $A_\alpha$. It satisfies $y(\alpha) = \lim_n x(p_n(\alpha))$. Finally, a unique ultrafilter $z$ in $X(\alpha)$ does not concentrate on any countable set.

If $S$ is stationary then $X$ is not Corson (this was proved for $\lambda = \omega_2$ in [18, Lemma 3.3], but the proof of the general case is identical). On the other hand, if $|\lambda| \leq \aleph_1$ and $S \cap \gamma$ is nonstationary for all $\gamma < \lambda$, then $\mathcal{A}(S)$ is uniform Eberlein compact ([18, Lemma 3.5]).

Therefore the existence of a nonreflecting stationary set $S \subseteq S_\omega^1$ implies that there exists a space $X = X(S)$ all of whose continuous images of weight not greater than $\aleph_1$ are uniform Eberlein compacta, but $X$ is not Corson.

Since $X(S)$ is the Stone space of the Boolean algebra $\mathcal{A}(S)$, the relevant $C^*$-algebra $C(X)$ has a dense subset consisting of continuous functions with finite range (i.e., ‘step functions’) and $C(X)$ therefore does not provide any more information than $\mathcal{A}(S)$. We will therefore work with $\mathcal{A}(S)$ in the following.

Now suppose $M \prec H_\theta$ is such that $\lambda$ and $S$ belong to $M$. Then

$$M \cap X(S) = \{x(\xi) : \xi \in M\} \cup \{y(\alpha) : \alpha \in M\} \cup \{z\}$$

and (note that $\alpha \in M$ implies $A_\alpha \subseteq M$)

$$M \cap X(S) = \{x(\xi) : \xi \in \overline{M}\} \cup \{y(\alpha) : A_\alpha \cap M \text{ is infinite}\} \cup \{z\}.$$  

Therefore if $M \cap \lambda$ is an ordinal and it belongs to $S$, then $y(M \cap \lambda)$ belongs to $\overline{M \cap X(S)}$ but not to $M \cap X(S)$. In particular, every $M \prec H_\theta$ such that $M \prec \lambda$ is an ordinal of uncountable cofinality splits $X(S)$.

Now suppose that $S \subseteq S_\omega^1$ is stationary. We can then choose an increasing sequence $M_n \prec H_\theta$ for $n \in \mathbb{N}$ such that $M_n \cap \omega_2$ is an ordinal of cofinality $\omega_1$, but $M = \bigcup_n M_n$ satisfies $M \cap \omega_2 \in S$. Then each $M_n$ splits $X(S)$, but $M$ does not.

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2Here $S_\omega^1$ denotes the set of all ordinals below $\omega_2$ of cofinality $\omega$. 

5. Concluding Remarks

A compact Hausdorff space $X$ is an Eberlein compactum (or shortly, Eberlein) if it is homeomorphic to a subspace of some Tychonoff cube $[0,1]^\kappa$ which has the property that for every $\xi < \kappa$ and every $\varepsilon > 0$ the set $\{x \in X : x(\xi) > \varepsilon\}$ is finite. Every Eberlein compactum is clearly Corson. We do not know whether analog of Theorem 1 holds for Eberlein compacta. It is not difficult to prove that it holds for strong Eberlein compacta, and this can be easily extracted from [15, Corollary 3.3].

Our original proof of (a special case of) Theorem 1 used a very strong reflection principle obtained by adapting the proof of [17, Theorem 1, (a) $\Rightarrow$ (c)].

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