A homological interpretation of Jantzen’s sum formula

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Abstract

For a split reductive algebraic group, this paper observes a homological interpretation for Weyl module multiplicities in Jantzen’s sum formula. This interpretation involves an Euler characteristic $\chi$ built from $\text{Ext}$ groups between integral Weyl modules. The new interpretation makes transparent for $GL_n$ (and conceivable for other classical groups) a certain invariance of Jantzen’s sum formula under “Howe duality” in the sense of Adamovich and Rybnikov. For $GL_n$ a simple and explicit general formula is derived for $\chi$ between an arbitrary pair of integral Weyl modules. In light of Brenti’s work on certain $R$-polynomials, this formula raises interesting questions about the possibility of relating $\text{Ext}$ groups between Weyl modules to Kazhdan-Lusztig combinatorics.

0. Introduction

Let $G_{\mathbb{Z}}$ be a split and connected reductive algebraic group scheme over $\mathbb{Z}$. For a prime number $p$, $G_p$ will denote the corresponding group scheme over $\mathbb{F}_p$, the field of $p$ elements. We will be concerned with (rational) representations of $G_p$, in the course of which we will need to use $G_{\mathbb{Z}}$. We will need some background material on these topics. For all such standard facts see the recent edition of Jantzen’s classic text [Jantzen], where one can also find the original references.

For a finite dimensional rational representation $M$ of $G_p$, its formal character is

$$ch(M) = \sum_{\mu \in X} \dim(M_\mu)e(\mu)$$

in $\mathbb{Z}[X]^W$, the Weyl group invariants in the integral group ring of the character group $X$ of a split maximal torus. A central problem is to calculate formal characters of all simple modules, which are in bijective correspondence with their highest weights. To get to the issue of interest in this paper, first let $L_\mu(\lambda)$ and $V_\mu(\lambda)$ respectively be the simple module and the Weyl module (i.e., the universal highest weight module) corresponding to a given dominant integral weight $\lambda$. Since each of the families $\{ch(L_\mu(\lambda))\}$ and $\{ch(V_\mu(\lambda))\}$ forms a basis of $\mathbb{Z}[X]^W$, and since $ch(V_\mu(\lambda))$ is known by Weyl’s character formula, the problem
stated above reduces to finding the multiplicities $a_{p,\lambda\mu}$ defined by

$$ch(L_p(\lambda)) = \sum_{\text{dominant } \mu} a_{p,\lambda\mu} ch(V_p(\mu)).$$

There is the following well-known interpretation of $a_{p,\lambda\mu}$ as an Euler characteristic [Jantzen II.6.21]

$$(*)
 a_{p,\lambda\mu} = \sum_i (-1)^i \dim Ext^i_{G_p}(V_p(\mu), L_p(\lambda)).$$

(A lot more is known about this situation. Here is a sketch for completeness, though we will not need this information later. To begin with, $a_{p,\lambda\lambda} = 1$. Using the linkage principle, $a_{p,\lambda\mu} = 0$ unless $\mu \uparrow \lambda$. Using translation functors further, we may additionally take without loss of generality $\lambda$ and $\mu$ to be $p$-regular weights, provided the prime $p$ is large enough to allow this. Then Lusztig’s conjecture says that under a further condition on $\lambda$ (namely that it be in Jantzen’s region) this multiplicity is the value at 1 of the appropriate Kazhdan-Lusztig polynomial for the affine Weyl group $W_p$ associated to $G_p$. See [Jantzen II.6-8] for these issues and [Jantzen II.B,C,H] for current status of Lusztig’s conjecture.)

The main purpose of this paper is twofold. (1) First, to observe a homological interpretation in the spirit of (*) for Weyl module multiplicities in Jantzen’s sum formula [Jantzen II.8.19]. This formula calculates $\sum_{i>0} ch(V^i_p(\lambda))$, the sum of formal characters of all proper submodules $V^i_p(\lambda)$ appearing in Jantzen’s filtration of a Weyl module $V_p(\lambda)$. The new interpretation involves an Euler characteristic $\chi$ over $G\mathbb{Z}$ built from $Ext$ groups between integral Weyl modules. (This is to be expected since Jantzen’s filtration is defined by working over $\mathbb{Z}$.) The proof rests on the same idea as that behind (*), namely a fundamental $Ext$ calculation due to Cline-Parshall-Scott-van der Kallen. Only here one uses the version over $\mathbb{Z}$ rather than the one over $\mathbb{F}_p$. An application is that for $GL_n$ the new interpretation makes transparent a certain invariance of Jantzen’s sum formula under “Howe duality” in the sense of Adamovich and Rybnikov. It further suggests that a similar invariance may be true for some other classical groups too. The new interpretation also provides some evidence for the likely importance in modular theory of $Ext$ groups over $G\mathbb{Z}$, in particular those between integral Weyl modules. (The modular analogue of $\chi$ is not so interesting for a pair of Weyl modules since it is easily seen to satisfy the orthonormal property.) With this in mind we will conduct additional analysis of the Euler characteristic $\chi$ and briefly discuss some integral $Ext$ groups. All this is done in Section 1.

(2) Second, to calculate for $GL_n$ a simple and explicit general formula for the integral Euler characteristic $\chi$ between an arbitrary pair of integral Weyl modules in terms of the associated dominant weights. This is done in Section 2. The formula is derived via a recursive procedure that employs the following ingredients. Certain Weyl filtrations from [AB1] corresponding to characteristic-free versions of Pieri-type rules, the skew representative theorem from [Kulkarni1], conjugate symmetry of $Ext$ groups between Weyl modules from [AB2] and the calculation in [Kulkarni2] of $Ext$ groups between Weyl modules for $GL_n$ whose dominant weights differ by a single root.

2
The $GL_n$ calculation of $\chi$ gives us $\sum_{i>0} ch(V_p^i(\lambda))$ in this case, which is already known in general thanks to Jantzen’s sum formula. Still, the $\chi$ calculation is of additional interest for the following reason. The answer in Jantzen’s formula is expressed as a linear combination of various $ch(V_p(\mu))$, but due to the nature of this formula, it is not readily visible which dominant weights $\mu$ occur. Moreover, the ones that do occur may repeat (unless $\lambda$ is $p$-regular, see [Jantzen II.8.19, Remark 3]), so the multiplicities of various $ch(V_p(\mu))$ cannot in general be read off easily from the formula. Thus, especially for small primes, it is interesting to have a formula for $\chi$ between Weyl modules, which does give us these multiplicities.

1. *Jantzen’s sum formula and Ext groups*

Let us proceed with the same set-up as in the introduction but the focus will shift to representations of $G_Z$ (= $G_Z$-modules). Given a dominant integral weight $\lambda$, both the Weyl module $V_p(\lambda)$ and its contravariant dual $H_p(\lambda)$ (i.e., the dual Weyl module of largest weight $\lambda$) are obtained from characteristic-free objects. So we have $\mathbb{Z}$-free $G_Z$-modules $V_Z(\lambda)$ and $H_Z(\lambda)$ such that $V_Z(\lambda) \otimes F_p = V_p(\lambda)$ and $H_Z(\lambda) \otimes F_p = H_p(\lambda)$. One has $ch(V_Z(\lambda)) = ch(H_Z(\lambda)) = ch(V_p(\lambda))$. (The definition of $ch(M)$ in the introduction applies equally well to $G_Z$-modules that are free abelian groups of finite rank, the dimension being replaced by the rank of each weight space as an abelian group.)

$V_Z(\lambda)$ and $H_Z(\lambda)$ may be realized as follows. Extend the scalars from $\mathbb{Z}$ to rational numbers $\mathbb{Q}$ to get the group scheme $G_Q$. Consider $V_Q(\lambda)$, the simple $G_Q$-module of highest weight $\lambda$. Fix a vector $v$ in the one dimensional $\lambda$-weight space $V_Q(\lambda)_\lambda$. Among the finitely many $G_Z$-stable lattices $M$ in $V_Q(\lambda)$ such that $M \cap V_Q(\lambda)_\lambda = \mathbb{Z}v$, the unique minimal one is $V_Z(\lambda)$ and the unique maximal one $H_Z(\lambda)$. For future use note that this gives a $G_Z$-equivariant injection $\phi : V_Z(\lambda) \hookrightarrow H_Z(\lambda)$. (In other words one has a nondegenerate bilinear form on $V_Z(\lambda)$ (since as an abelian group $H_Z(\lambda)$ is just the linear dual of $V_Z(\lambda)$) and this form is “contravariant” due to the way $G_Z$-action is defined on $H_Z(\lambda)$. See [Jantzen II.8.17]. We will not use this language here.)

Prior to stating and proving the main results in 1.4, we will gather some general results regarding $Ext$ groups (1.1), define and analyze the torsion Euler characteristic $\chi$ (1.2) and review the setting of Jantzen’s sum formula (1.3). In view of the likely but as yet unclear significance of integral $Ext$ groups, some additional remarks regarding $\chi$ and $Ext$ groups are offered in 1.5. Sections 1.6 discusses a connection of Jantzen’s sum formula with Howe duality in the sense of Adamovich and Rybnikov. Finally 1.7 points out possible generalizations to other settings. All $G_Z$-modules (except when scalars are extended to $\mathbb{Q}$) will be finitely generated–equivalently, finitely generated as abelian groups. For a finitely generated abelian group $M$, $M_{tor}$ will denote its torsion subgroup and $M_{fr} = M/M_{tor}$ its largest $\mathbb{Z}$-free quotient. If $M$ is a $G_Z$-module so are $M_{tor}$ and $M_{fr}$.

1.1. *Some basic results on Ext groups over $G_Z$* (two fundamental $Ext$ calculations, finiteness of $Ext$ groups in general). Let us first record two important $Ext$ calculations
due to Cline-Parshall-Scott-van der Kallen.

Theorem. [CPSvdK] For dominant weights $\lambda$ and $\mu$,
(i) $\text{Ext}^i_{G,\mathbb{Z}}(V_{\mathbb{Z}}(\mu), H_{\mathbb{Z}}(\lambda)) = 0$ unless $(\mu = \lambda$ and $i = 0$).
(ii) $\text{Ext}^i_{G,\mathbb{Z}}(V_{\mathbb{Z}}(\mu), V_{\mathbb{Z}}(\lambda)) = 0$ unless $\mu < \lambda$ or $(\mu = \lambda$ and $i = 0$).

Note. Of course $\text{Hom}_{G,\mathbb{Z}}(V_{\mathbb{Z}}(\lambda), V_{\mathbb{Z}}(\lambda)) \simeq \text{Hom}_{G,\mathbb{Z}}(V_{\mathbb{Z}}(\lambda), H_{\mathbb{Z}}(\lambda)) \simeq \mathbb{Z}$.

The next result proves finite generation of $\text{Ext}^i_{G,\mathbb{Z}}$ (and therefore finiteness for $i > 1$). It is surely known (e.g., see [Jantzen II.4.10] for finite dimensionality of $\text{Ext}^i_{G,\mathbb{Q}}$, but does not seem to be published, so a proof is supplied. (Note. A more basic and immediate argument for finite generation than the one given is as follows. Use an appropriate Schur algebra $S$ over $\mathbb{Z}$, which is finitely generated, and appeal to Donkin’s theorem (actually to its easy extension to the integral situation) asserting isomorphism of $\text{Ext}_S$ with $\text{Ext}_{G,\mathbb{Z}}$. Instead a different argument is given below to illustrate two techniques: use of Theorem 1.1(i) and a certain strategy to build up general $G_{\mathbb{Z}}$-modules.)

Proposition. Fix finitely generated $G_{\mathbb{Z}}$-modules $M$ and $N$ and consider the abelian groups $\text{Ext}^i_{G,\mathbb{Z}}(M, N)$.
(i) These groups are all finitely generated and are 0 for large $i$.
(ii) For $i > 0$ these groups are all finite.
(iii) $\text{Hom}_{G,\mathbb{Z}}(M, N)$ is infinite iff the $G_{\mathbb{Q}}$-modules $M \otimes \mathbb{Q}$ and $N \otimes \mathbb{Q}$ have at least one isomorphic simple summand.

Proof. (i) If the statement is true for $M = \text{two of the three modules in a short exact sequence then it is clearly true for the third. (We will say that each module is "obtainable" from the other two.) The same is true for $N$. We will show that all finitely generated modules are obtainable from the family $\{V_{\mathbb{Z}}(\lambda)\}$ as well as from $\{H_{\mathbb{Z}}(\lambda)\}$. Now taking $M = V_{\mathbb{Z}}(\mu)$ and $N = H_{\mathbb{Z}}(\lambda)$ will give (i) by Theorem 1.1(i) quoted above.

$M$ is obtainable from $M_{\text{tor}}$ and $M_{\text{fr}}$. Now $M_{\text{tor}}$ has a composition series with factors $L_p(\lambda)$ for various primes $p$ and dominant weights $\lambda$. For fixed $p$ the family $\{L_p(\lambda)\}$ is well-known to be obtainable from either of the families $\{V_{\mathbb{p}}(\lambda)\}$ and $\{H_{\mathbb{p}}(\lambda)\}$, which in turn are respectively obtainable from the families $\{V_{\mathbb{Z}}(\lambda)\}$ and $\{H_{\mathbb{Z}}(\lambda)\}$ using multiplication by $p$. As for $M_{\text{fr}}$, it has a filtration whose factors are $G_{\mathbb{Z}}$-stable lattices $P$ in various $V_{\mathbb{Q}}(\lambda)$. We have lattices of the same rank $V_{\mathbb{Z}}(\lambda) \subset P \subset H_{\mathbb{Z}}(\lambda)$, resulting in torsion modules $P/V_{\mathbb{Z}}(\lambda)$ and $H_{\mathbb{Z}}(\lambda)/P$. By the argument for torsion modules $M_{\text{fr}}$ is also obtainable from either of the families $\{V_{\mathbb{Z}}(\lambda)\}$ and $\{H_{\mathbb{Z}}(\lambda)\}$. This proves (i).

(ii) In view of (i) clearly all $\text{Ext}^i$ are finite if $M$ or $N$ is torsion. So by looking at the appropriate long exact sequences derived from $0 \to M_{\text{tor}} \to M \to M_{\text{fr}}$ and $0 \to N_{\text{tor}} \to N \to N_{\text{fr}}$ it suffices to take both $M$ and $N$ to be $\mathbb{Z}$-free. Now the result follows after extending scalars to $\mathbb{Q}$ (e.g., use the universal coefficient theorem [Jantzen I.4.18] to relate $\text{Ext}_{G,\mathbb{Z}}$ and $\text{Ext}_{G,\mathbb{Q}}$), where all representations become semisimple.

Finally (iii) is clear.
1.2. The torsion Euler characteristic $\chi$. In view of Proposition 1.1 the following definition is valid.

**Definition.** For finitely generated $G_{\mathbb{Z}}$-modules $M$ and $N$ define

$$\chi(M, N) = |\text{Hom}_{G_{\mathbb{Z}}}(M, N)_{\text{tor}}| \prod_{i > 0} |\text{Ext}^{i}_{G_{\mathbb{Z}}}(M, N)|^{(-1)^i}.$$ 

**Note.** Another alternative is to simply drop the first term if the $\text{Hom}$ is infinite. In fact for the purposes of this paper that would work just as well, since in all our uses of $\chi$, $\text{Hom}_{G_{\mathbb{Z}}}(M, N)$ will be either $\mathbb{Z}$-free or torsion. But in the general definition it seems preferable to include the “finite part” $\text{Hom}_{G_{\mathbb{Z}}}(M, N)_{\text{tor}}$.

Clearly $\chi$ is multiplicative on short exact sequences of finite modules in either argument. Moreover if one of the arguments of $\chi$ is finite, $\chi$ is multiplicative on short exact sequences in the other argument. This implies that for torsion modules $M$ and $N$, $\chi(M, N)$ is always 1. *(Proof. Apply $\text{Hom}_{G_{\mathbb{Z}}}(M, -)$ to the exact sequence $0 \to V_\mathbb{Z}(\nu) \xrightarrow{p} V_\mathbb{Z}(\nu) \to V_p(\nu) \to 0$ to get $\chi(M, V_\mathbb{Z}(\nu)) = 1$. Now any finite $N$ may be obtained from the various $V_p(\nu)$.)

$\chi$ fails to be multiplicative in general because infinite $\text{Hom}$ groups exist. Let us sketch how this failure can be quantified. Apply $\text{Hom}_{G_{\mathbb{Z}}}(M, -)$ or $\text{Hom}_{G_{\mathbb{Z}}}(-, M)$ to a short exact sequence $0 \to P \to Q \to R \to 0$ of $G_{\mathbb{Z}}$-modules and take the corresponding long exact sequence. Replacing the three $\text{Hom}$ terms by their torsion parts renders the long sequence possibly inexact in two places: the last $\text{Hom}$ term and the first $\text{Ext}^1$ term. Let us make explicit the necessary adjustment to multiplicativity in two situations of interest.

**Example 1.** *(Breaking up $M, N$ into free and torsion parts.)* Apply $\text{Hom}_{G_{\mathbb{Z}}}(-, N_{\text{tor}})$ to $0 \to M_{\text{tor}} \to M \to M_{\text{fr}}$ to get $\chi(M, N_{\text{tor}}) = \chi(M_{\text{fr}}, N_{\text{tor}})$ since $\chi(M_{\text{tor}}, N_{\text{tor}}) = 1$. Apply $\text{Hom}_{G_{\mathbb{Z}}}(-, N_{\text{fr}})$ to the same sequence and use $\text{Hom}_{G_{\mathbb{Z}}}(M_{\text{tor}}, N_{\text{fr}}) = 0$ to get $\chi(M, N_{\text{fr}}) = \chi(M_{\text{tor}}, N_{\text{fr}})\chi(M_{\text{fr}}, N_{\text{fr}})$. Next apply $\text{Hom}_{G_{\mathbb{Z}}}(M, -)$ to $0 \to N_{\text{tor}} \to N \to N_{\text{fr}} \to 0$ to get a long exact sequence beginning as follows.

$$0 \to \text{Hom}_{G_{\mathbb{Z}}}(M, N_{\text{tor}}) \to \text{Hom}_{G_{\mathbb{Z}}}(M, N) \to \text{Hom}_{G_{\mathbb{Z}}}(M, N_{\text{fr}}) \to \text{Ext}^1_{G_{\mathbb{Z}}}(M, N_{\text{tor}}) \to \cdots$$

Now $\text{Hom}_{G_{\mathbb{Z}}}(M, N_{\text{tor}}) \simeq \text{Hom}_{G_{\mathbb{Z}}}(M, N)_{\text{tor}}$, giving an injection of free abelian groups of equal rank $\text{Hom}_{G_{\mathbb{Z}}}(M, N)_{\text{fr}} \hookrightarrow \text{Hom}_{G_{\mathbb{Z}}}(M, N_{\text{fr}})$. Calling the necessarily finite cardinality of the cokernel of this injection $s$, we clearly have $\chi(M, N) = \chi(M, N_{\text{tor}})\chi(M, N_{\text{fr}})s$. All in all we have

$$\chi(M, N) = \chi(M_{\text{fr}}, N_{\text{tor}})\chi(M_{\text{tor}}, N_{\text{fr}})\chi(M_{\text{fr}}, N_{\text{fr}})s.$$ 

It is easy to see that the correction factor $s$ is really necessary. Otherwise the map $\text{Hom}_{G_{\mathbb{Z}}}(M, N) \to \text{Hom}_{G_{\mathbb{Z}}}(M, N_{\text{fr}})$ would always be surjective, i.e., the exact sequence $0 \to N_{\text{tor}} \to N \to N_{\text{fr}} \to 0$ would always split (e.g., letting $M = N_{\text{fr}}$, the inverse image of $\text{id}_{N_{\text{fr}}}$ would give a splitting), which surely does not happen. To give a concrete example, it suffices to produce a non-split extension of a $\mathbb{Z}$-free $G_{\mathbb{Z}}$-module by a torsion $G_{\mathbb{Z}}$-module.
For this take $G_{\mathbb{Z}} = GL(\mathbb{Z}^n)$. Let $\Lambda^2$ and $D^2$ respectively be the second exterior and divided powers of the defining representation $\mathbb{Z}^n$. One knows from [AB2, Section 9] that $Ext^i_G(\Lambda^2, D^2)$ is $\mathbb{Z}/2\mathbb{Z}$ if $i = 1$ and vanishes for other $i$. Using this in the long exact sequence obtained by applying $Hom_{G\mathbb{Z}}(\Lambda^2, -)$ to $0 \to D^2 \to D^2 \to D^2 \otimes \mathbb{Z}/2\mathbb{Z} \to 0$ yields that $Ext^i_{G\mathbb{Z}}(\Lambda^2, D^2 \otimes \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$ if $i = 0, 1$ and vanishes otherwise. In particular we get the desired non-split extension. See Remark 1.5.(5) below for similar elementary analysis of the relationships among several kinds of $Ext$ groups in a general situation.

**Example 2.** ($\chi$ involving an extension of $\mathbb{Z}$-free $G_{\mathbb{Z}}$-modules.) Take an exact sequence of $\mathbb{Z}$-free $G_{\mathbb{Z}}$-modules $0 \to N \to P \to R \to 0$. Then for a $\mathbb{Z}$-free $G_{\mathbb{Z}}$-module $M$ one has exactly as in Example 1

$$\chi(M, P) = \chi(M, N) \chi(M, R) s,$$

where $s$ is the cardinality of the necessarily finite cokernel of the map of free abelian groups $Hom(M, P) \to Hom(M, R)$. Clearly $s = 1$ unless $M \otimes \mathbb{Q}, P \otimes \mathbb{Q}, R \otimes \mathbb{Q}$ all have an isomorphic simple summand. Similar considerations apply to such a short exact sequence in the first argument of $\chi$. In the recursive algorithm to compute $\chi$ between integral Weyl modules for $GL_n$ to be presented in the next section, the terminating stage of the algorithm involves exactly such pairs $(M, R)$ which have infinite $Hom$ groups. In this sense these numbers $s$ are the source of the final numerical answers for $\chi$.

1.3. **Background on formal character for torsion modules and Jantzen’s sum formula.** We need some more machinery to be able to state and prove the main result. A convenient reference for all of this background material is [Jantzen II.8].

**Definition and basic properties of $ch_{tor}$.** One can define a torsion formal character for a finite $G_{\mathbb{Z}}$-module $M$ by

$$ch_{tor}(M) = \sum_{\mu \in X} \text{div}|M_\mu| e(\mu),$$

where $\text{div}$ stands for taking the divisor of a rational number (here an integer). (Compare the definition of $v^c$ in [Jantzen II.8.12].) Clearly $ch_{tor}(L_p(\lambda)) = ch(L_p(\lambda))[p]$ and $ch_{tor}$ is additive on short exact sequences of finite $G_{\mathbb{Z}}$-modules. As $\lambda$ varies over all dominant weights and $p$ over all primes, each of the two families $\{ch_{tor}(L_p(\lambda))\}$ and $\{ch_{tor}(V_p(\lambda)) = ch_{tor}(H_p(\lambda))\}$ forms a basis of the abelian group of all torsion formal characters. Note that since $ch(V_p(\lambda)) = ch(V_{\mathbb{Z}}(\lambda))$ is independent of $p$, it makes sense to speak about the coefficient of $ch(V_{\mathbb{Z}}(\lambda))$ in $ch_{tor}(M)$, this coefficient being the divisor of a unique positive rational number (that will be 1 for all but finitely many $\lambda$). We will need these considerations while explaining the setting of Jantzen’s sum formula, which will be our next task.

**Jantzen’s filtration and Jantzen’s sum formula.** Recall the injection $\phi : V_{\mathbb{Z}}(\lambda) \hookrightarrow H_{\mathbb{Z}}(\lambda)$. Jantzen’s filtration is a descending filtration $V_p^i(\lambda)$ of $V_p(\lambda)$ defined as follows. Fixing $p$ for the moment, first let $V_{\mathbb{Z}}^i(\lambda)$ be the submodule $\phi^{-1}(p^i H_{\mathbb{Z}}(\lambda))$ of $V_{\mathbb{Z}}(\lambda)$ and then $V_p^i(\lambda)$ is the image of $V_{\mathbb{Z}}^i(\lambda)$ under the canonical map $V_{\mathbb{Z}}(\lambda) \to V_p(\lambda)$. (So in particular $V_p(\lambda)/V_p^1(\lambda) = \text{image of } \phi \otimes id_{\mathbb{F}_p}$, which is well-known to be $L_p(\lambda)$.)

6
Set $Q(\lambda) = \text{coker}(\phi)$. By a well-known argument due to Jantzen,

$$ch_{tor}Q(\lambda) = \sum_p \left( \sum_{i > 0} ch(V^i_p(\lambda)) \right) [p].$$

At the same time it is immediate (e.g., after diagonalizing $\phi$ by using suitable bases for $V^i_Z(\lambda)$ and $H^i_Z(\lambda)$) that calculating $ch_{tor}Q(\lambda)$ is equivalent to calculating the determinant of $\phi$ on each weight space. In most cases Jantzen succeeded in calculating the determinant (more precisely its $p$-adic valuation when $p$ is not small and without restriction for type A) and so proved a formula for $\sum_{i > 0} ch(V^i_p(\lambda))$, which was soon afterwards obtained in general by Andersen via a different method. See [Jantzen II.8] for a detailed explanation of all this and the formula itself. We do not need the formula here. Rather, our goal is to observe an Euler characteristic interpretation for the Weyl module multiplicities occuring in this formula. These multiplicities, denoted $b_{p,\lambda\mu}$, are defined by

$$\sum_{i > 0} ch(V^i_p(\lambda)) = \sum_{\text{dominant } \mu} b_{p,\lambda\mu} ch(V^i_Z(\mu)).$$

Note that only $\mu < \lambda$ can have nonzero coefficients on the right hand side.

1.4. A formula for $ch_{tor}$ using $\chi$ and application to the sum formula. We will interpret the coefficients $b_{p,\lambda\mu}$ in terms of $\chi$ between integral Weyl modules. This will be a consequence of the following integral analogue of (*)& in the introduction (more precisely, analogue of [Jantzen II.6.21]).

**Proposition.** For a finite $G_Z$-module $M$,

$$ch_{tor}(M) = \sum_{\text{dominant } \mu} \div(\chi(V^i_Z(\mu), M)) ch(V^i_Z(\mu)) = \sum_{\text{dominant } \mu} -\div(\chi(M, H_Z(\mu))) ch(V^i_Z(\mu)).$$

Further, the statements stay true if one replaces $V^i_Z(\mu)$ or $H_Z(\mu)$ by any $G_Z$-stable lattice inside $V_Q^i(\mu)$.

**Proof.** By additivity of $ch_{tor}$ and of the divisor of $\chi(V^i_Z(\mu), -)$ on finite $G_Z$-modules it is enough to check the first equality for $M = H^i_p(\nu)$ for each prime $p$ and each dominant integral weight $\nu$. One has $ch_{tor}(H^i_p(\nu)) = ch(V^i_Z(\nu))[p]$. For the other calculation apply $\text{Hom}_{G_Z}(V^i_Z(\mu), -)$ to $0 \rightarrow H^i_Z(\nu) \overset{p}{\rightarrow} H^i_Z(\nu) \rightarrow H^i_p(\nu) \rightarrow 0$ and use Theorem 1.1(i) above to get $\chi(V^i_Z(\mu), H^i_p(\nu)) = p^{i\mu\nu}$. The second equality is checked similarly using $M = V^i_p(\nu)$. Since $\chi$ is 1 for a pair of finite $G_Z$-modules, using multiplicativity of $\chi(\cdot, -)$ (respectively, of $\chi(M, -)$) one may replace $V^i_Z(\mu)$ (respectively $H^i_Z(\mu)$) inside $\chi$ by any $G_Z$-stable lattice inside $V^i_Q(\mu)$, thus proving the last statement.
Corollary. (Homological interpretation of Jantzen’s sum formula.) The multiplicities of $\operatorname{ch}(V_{\mathbf{Z}}(\mu))$ in Jantzen’s sum formulas for $V_p(\lambda)$ for various primes $p$ are given by

$$\sum_p b_{p,\lambda\mu}[p] = -\operatorname{div}(\chi(V_{\mathbf{Z}}(\mu), V_{\mathbf{Z}}(\lambda)).$$

Proof. Note that

$$\operatorname{ch}_{\operatorname{tor}}Q(\lambda) = \sum_p \left( \sum_{\text{dominant } \mu} b_{p,\lambda\mu}\chi(V_{\mathbf{Z}}(\mu)) \right) [p] = \sum_{\text{dominant } \mu} \left( \sum_p b_{p,\lambda\mu}[p] \right) \chi(V_{\mathbf{Z}}(\mu)),$$

Here the first line comes simply from the set-up in 1.3 and the second from the first equality in Proposition 1.4. So all we need to show is that

$$\operatorname{div}(\chi(V_{\mathbf{Z}}(\mu), Q(\lambda)) = -\operatorname{div}(\chi(V_{\mathbf{Z}}(\mu), V_{\mathbf{Z}}(\lambda)).$$

For this take the long exact sequence obtained by applying $\operatorname{Hom}_{G_{\mathbf{Z}}}(V_{\mathbf{Z}}(\mu), -)$ to $0 \to V_{\mathbf{Z}}(\lambda) \to H_{\mathbf{Z}}(\lambda) \to Q(\lambda) \to 0$. Now $\chi(V_{\mathbf{Z}}(\mu), H_{\mathbf{Z}}(\lambda)) = 1$ by using Theorem 1.1(i) once again and hence we have the desired result. (Note that the first two $\operatorname{Hom}$ terms in the long exact sequence–potentially the only infinite ones–vanish unless $\mu = \lambda$. If $\mu = \lambda$, these terms are $\mathbf{Z}$ and the map between them is an isomorphism. So multiplicativity of $\chi(V_{\mathbf{Z}}(\mu), -)$ is valid in either case. The latter case is trivial anyway, since then all terms other than the first two vanish and all three $\chi$’s are 1.)

1.5. Complementary remarks on $\chi$ and on $\operatorname{Ext}$ groups. (1) Note that the analogue over $F_p$ of $\chi$ between Weyl modules is uninteresting as it satisfies the orthonormal property. One has $\sum_{i = 0}^{\dim \operatorname{Ext}_{G_p}(V_p(\lambda), V_p(\mu))} = 1$ by [Jantzen II.6.21].

(2) $\chi$ for some other pairs of modules. One may contemplate $\chi(A, B)$ where $A$ and $B$ are $G_{\mathbf{Z}}$-modules from the three families $\{V_{\mathbf{Z}}(\lambda)\}, \{H_{\mathbf{Z}}(\lambda)\}$ and $\{Q(\lambda)\}$. Using information obtained so far, it is straightforward to analyze the nine possibilities. By Theorem 1.1(i), $\chi(V_{\mathbf{Z}}(\mu), H_{\mathbf{Z}}(\lambda)) = 1$. $\chi(Q(\mu), Q(\lambda)) = 1$ since both arguments are torsion modules. $\chi(V_{\mathbf{Z}}(\mu), V_{\mathbf{Z}}(\lambda)) = \chi(H_{\mathbf{Z}}(\lambda), H_{\mathbf{Z}}(\mu))$ (by contravariant duality) = $1/\chi(V_{\mathbf{Z}}(\mu), Q(\lambda))$ (by Theorem 1.1(i)) = $\chi(Q(\lambda), H_{\mathbf{Z}}(\mu)) = 1/\chi(H_{\mathbf{Z}}(\mu), Q(\lambda)) = \chi(Q(\mu), V_{\mathbf{Z}}(\mu))$ (last three expressions by Proposition 1.4). This leaves $\chi(H_{\mathbf{Z}}(\mu), V_{\mathbf{Z}}(\lambda))$. To relate this to previous cases apply $\operatorname{Hom}_{G_{\mathbf{Z}}}(-, V_{\mathbf{Z}}(\lambda))$ to $0 \to V_{\mathbf{Z}}(\mu) \to H_{\mathbf{Z}}(\mu) \to Q(\mu) \to 0$ and use known information to get $\chi(H_{\mathbf{Z}}(\mu), V_{\mathbf{Z}}(\lambda)) = \chi(V_{\mathbf{Z}}(\mu), V_{\mathbf{Z}}(\lambda)) = \chi(V_{\mathbf{Z}}(\mu), V_{\mathbf{Z}}(\lambda))$. The symmetry in $\lambda$ and $\mu$ is to be expected in view of contravariant duality. Note that by Theorem 1.1(ii) at most one of the two factors on the right hand side may be different from 1.

(3) A certain skew-symmetry of $\chi$. It follows from Proposition 1.4 that for a torsion $G_{\mathbf{Z}}$-module $M$, $\chi(M, V_{\mathbf{Z}}(\mu))\chi(V_{\mathbf{Z}}(\mu), M) = 1$. Using multiplicativity of $\chi(M, -)$ as well as of $\chi(-, M)$, and since $\chi$ is 1 for a pair of torsion modules, one may replace $V_{\mathbf{Z}}(\mu)$ by any
$G_{\mathbb{Z}}$-module $N$ via a route similar to the one followed in the proof of Proposition 1.1(i). So for a torsion $G_{\mathbb{Z}}$-module $M$ and for any $G_{\mathbb{Z}}$-module $N$, one has $\chi(M, N)\chi(N, M) = 1$. This is not true in general if $M$ is not torsion, e.g., when $M$ and $N$ are both integral Weyl modules in view of Corollary 1.4 and Theorem 1.1(ii).

(4) Of course it is much more interesting (and harder) to calculate the $\text{Ext}$ groups themselves rather than merely calculating $\chi$. Here is a reason why the $\text{Ext}$ groups between integral Weyl modules are likely to be important. Note that we made essential use of Theorem 1.1(i) only “up to Euler characteristic,” not in its full strength. This theorem gives $\text{Ext}^{i+1}_{G_{\mathbb{Z}}}(V_{\mathbb{Z}}(\mu), V_{\mathbb{Z}}(\lambda)) \simeq \text{Ext}^i_{G_{\mathbb{Z}}}(V_{\mathbb{Z}}(\mu), Q(\lambda))$ and clearly the structure of $Q(\lambda)$ is intimately related to Jantzen’s filtration itself. It would be very interesting if one can relate individual $\text{Ext}$ groups between integral Weyl modules to Jantzen’s filtration. It seems reasonable to speculate that Kazhdan-Lusztig type combinatorics should come into play in such a relationship. (There is already a hint to this effect for type $A$. See Remark 2 after the proof of Theorem 2.3.) For instance one can ask the following question. Consider the alternating sum obtained by taking the divisor of the defining expression for $\chi(V_{\mathbb{Z}}(\mu), V_{\mathbb{Z}}(\lambda))$. What is the polynomial obtained by replacing $-1$ in this expression by an indeterminate $q$? (Unlike over $\mathbb{F}_p$, this is a weaker question than knowing the individual $\text{Ext}$ groups since in general the structure of an abelian group is not determined by its size.)

(5) Elementary observations relating $\text{Ext}_{G_{\mathbb{Z}}}$ and $\text{Ext}_{G_p}$. Consider a $G_p$-module $L$ and a $\mathbb{Z}$-free $G_{\mathbb{Z}}$-module $M$. First we have the isomorphism of $\mathbb{F}_p$-vector spaces (compare [McNinch, Lemma 3.1.1b]) $\text{Ext}^i_{G_{\mathbb{Z}}}(M, L) \simeq \text{Ext}^i_{G_p}(M \otimes \mathbb{F}_p, L)$. Proof. Using $^*$ to denote the linear dual over $\mathbb{Z}$, [Jantzen I.4.2(1) and I.4.4] give $\text{Ext}^i_{G_{\mathbb{Z}}}(M, L) \simeq H^i(G_{\mathbb{Z}}, M^* \otimes L)$ and $\text{Ext}^i_{G_p}(M \otimes \mathbb{F}_p, L) \simeq H^i(G_p, M^* \otimes \mathbb{F}_p \otimes L)$. Now the Hochschild complexes computing both group cohomologies are isomorphic. Alternatively, take a projective resolution $P. \to M$ over an appropriate Schur algebra $S_{\mathbb{Z}}$. Tensoring by $\mathbb{F}_p$ gives a projective resolution of $M \otimes \mathbb{F}_p$ over the Schur algebra $S_p = S_{\mathbb{Z}} \otimes \mathbb{F}_p$. Applying the appropriate $\text{Hom}$ to each resolution gives isomorphic complexes computing the required $\text{Ext}$ groups.

Next, it is easy to relate these isomorphic $\text{Ext}$ groups to $\text{Ext}^i_{G_{\mathbb{Z}}}(M \otimes \mathbb{F}_p, L)$. Apply $\text{Hom}_{G_{\mathbb{Z}}}(\cdot, L)$ to the short exact sequence $0 \to M \overset{p}{\to} M \to M \otimes \mathbb{F}_p \to 0$. The resulting long exact sequence easily breaks up into short exact sequences, giving the isomorphism of $\mathbb{F}_p$-vector spaces $\text{Ext}^i_{G_{\mathbb{Z}}}(M \otimes \mathbb{F}_p, L) \simeq \text{Ext}^{i-1}_{G_{\mathbb{Z}}}(M, L) \oplus \text{Ext}^i_{G_{\mathbb{Z}}}(M, L)$.

One can say more in the following special situation. Let $L = N \otimes \mathbb{F}_p$, where $N$ is a $\mathbb{Z}$-free $G_{\mathbb{Z}}$-module. Suppose the localization at prime $p$ of the abelian group $\text{Ext}^i_{G_{\mathbb{Z}}}(M, N)$ is a direct sum of $n_i$ cyclic groups. Using the short exact sequences $0 \to M \overset{p}{\to} M \to M \otimes \mathbb{F}_p \to 0$ and $0 \to N \overset{p}{\to} N \to N \otimes \mathbb{F}_p \to 0$ as in the previous paragraph, one gets the following equalities. $\dim \text{Ext}^i_{G_{\mathbb{Z}}}(M \otimes \mathbb{F}_p, N) = n_{i-1} + n_i$. $\dim \text{Ext}^i_{G_p}(M, N \otimes \mathbb{F}_p) = \dim \text{Ext}^i_{G_p}(M \otimes \mathbb{F}_p, N \otimes \mathbb{F}_p) = n_i + n_{i+1}$. (The preceding equality can also be seen from the Universal Coefficient Theorem [Jantzen I.4.18a]) $\dim \text{Ext}^i_{G_{\mathbb{Z}}}(M \otimes \mathbb{F}_p, N \otimes \mathbb{F}_p) = n_{i-1} + 2n_i + n_{i+1}$.

The foregoing considerations may be of particular interest in two cases because of the
significance of the $Ext$ groups involved. Take $M = V_{\mathbb{Z}}(\mu)$ in each case and in turn let $L = L_p(\lambda)$ (see (*) in the Introduction) or $L = V_{p}(\lambda)$ (in view of the previous remark).

1.6. **Symmetry of sum formula under Howe duality and complements.** Consider Young diagrams of two partitions $\lambda$ and $\mu$. These may be considered to be two dominant weights for $GL(F)$, where $F$ is a free abelian group of rank at least as much as the number of rows in $\lambda$ as well as $\mu$. Let $\tilde{\lambda}$ (respectively, $\tilde{\mu}$) be the Young diagram obtained by transposing the rows and columns of $\lambda$ (respectively, $\mu$). $\tilde{\lambda}$, $\tilde{\mu}$ are dominant weights for $GL(E)$, where $E$ is a free abelian group of rank at least as much as the number of rows in $\tilde{\lambda}$ as well as $\tilde{\mu}$. (One could also work with a single group of large enough rank, or in the category of polynomial functors without having to worry about the rank at all.) Now the functor $\Omega$ from [AB2, Section 7] (a characteristic-free form of Howe duality, in the sense similar to [AR]) combined with contravariant duality gives the following “conjugate symmetry” of $Ext$ groups between Weyl modules.

$$\text{Ext}^i_{GL(F)}(V_{\mathbb{Z}}(\mu), V_{\mathbb{Z}}(\lambda)) \simeq \text{Ext}^i_{GL(E)}(V_{\mathbb{Z}}(\tilde{\lambda}), V_{\mathbb{Z}}(\tilde{\mu})).$$

Still working with $GL(F)$, a more elementary symmetry of these $Ext$ groups is as follows. Enclose Young diagrams of $\lambda$ and $\mu$ in a rectangle of height $\text{rank}(F)$ and of suitable width $w$. Let $\lambda^c$ and $\mu^c$ respectively be the partitions obtained by taking the complements of $\lambda$ and $\mu$ within this rectangle. One has (e.g., by [ABW, II.4]) $V_{\mathbb{Z}}(\mu) = H_{\mathbb{Z}}(\mu^c)^* \otimes (\text{det})^w$ and likewise for $\lambda$. (Here $^*$ denotes ordinary linear dual of a representation, where one uses the group antiautomorphism taking $g$ to $g^{-1}$ to convert the natural right action into a left one.) Now

$$\text{Ext}^i_{GL(F)}(V_{\mathbb{Z}}(\mu), V_{\mathbb{Z}}(\lambda)) \simeq \text{Ext}^i_{GL(F)}(H_{\mathbb{Z}}(\mu^c)^* \otimes (\text{det})^w, H_{\mathbb{Z}}(\lambda^c)^* \otimes (\text{det})^w)$$

$$\simeq \text{Ext}^i_{GL(F)}(H_{\mathbb{Z}}(\mu^c)^*, H_{\mathbb{Z}}(\lambda^c)^*)$$

$$\simeq \text{Ext}^i_{GL(F)}(H_{\mathbb{Z}}(\lambda^c), H_{\mathbb{Z}}(\mu^c))$$

$$\simeq \text{Ext}^i_{GL(F)}(V_{\mathbb{Z}}(\mu^c), V_{\mathbb{Z}}(\lambda^c)).$$

Here the second line comes from canceling the determinant, the third by using linear duality and the last by using contravariant duality.

It follows immediately from (1.6.1), (1.6.2) and Corollary 1.4 that for general linear groups Jantzen’s sum formula is stable under Howe duality and under complements. More precisely one has the following.

**Corollary.** For the general linear groups (in the terminology of 1.3),

$$b_{p,\lambda\mu} = b_{p,\tilde{\mu}\tilde{\lambda}} = b_{p,\lambda^c\mu^c}.$$

Adamovich and Rybnikov [AR] have constructed Howe duality functors in positive characteristic for several other pairs of classical groups. Their set-up gives an isomorphism of $Ext$ groups analogous to a combination of (1.6.1) and (1.6.2), but in characteristic $p$. It is not clear that their isomorphism is valid while working over $\mathbb{Z}$. The groups
$\text{Ext}^i_{GZ}(V\mathbb{Z}(\mu), V\mathbb{Z}(\lambda))$ for all $i$ determine the corresponding modular $\text{Ext}$ groups (via, e.g., [Jantzen I.4.18a]), but not vice versa. Still the work of Adamovich and Rybnikov makes it reasonable to ask whether Jantzen’s sum formulas are stable under Howe duality for any other pairs of groups besides the general linear groups. This should (at least in principle) be verifiable directly from the known sum formulas.

Notes. (1) [McNinch] proves the following interesting connection within this cluster of ideas. Howe duality in [AR] carries Jantzen’s filtrations to Andersen’s “tilting filtrations.”

(2) Here is yet another connection between Andersen’s tilting filtrations and Jantzen’s filtrations. (This is entirely independent of Howe duality, but is still mentioned here for completeness.) Using the results in this section, it is possible to derive Andersen’s “tilting sum formula” as a formal consequence of Jantzen’s sum formula. Currently this is known only if the characteristic is not small, since the original proof uses regular weights. The new proof will be explained in [Kulkarni3].

1.7. Generalization to highest weight categories. The arguments in this section used certain objects (such as simple, Weyl and dual Weyl modules indexed by a suitable partially ordered set) and facts about these objects (such as suitable vanishing properties) that are available in other situations as well. The natural home for the needed objects and vanishing results is the axiomatic notion of a highest weight category (= representations of a quasi-hereditary algebra) due to Cline-Parshall-Scott. Additionally one needs a suitable model of such a category over a principal ideal domain from which the category of interest is obtained by reduction modulo a prime. See for example the set-up in [McNinch]. It should be straightforward to carry over the results in this section (except 1.6) to such an axiomatic setting. In particular analogues of the main results should hold in the following situations: the BGG category $\mathcal{O}$ (using the version constructed by Gabber-Joseph) and representations of quantum groups at a root of unity. (One has Jantzen’s filtrations and sum formulas in these situations as well.) Because of the focused nature of interest in this paper, we will not carry out any of these extensions here.

2. A formula for $\chi(V\mathbb{Z}(\mu), V\mathbb{Z}(\lambda))$ for the general linear group

Henceforth let $G\mathbb{Z} = GL(F)$, where $F$ is a free abelian group of finite rank. The rank of $F$ will be essentially immaterial (see below for the precise statement). We will calculate $\chi(V\mathbb{Z}(\mu), V\mathbb{Z}(\lambda))$ for arbitrary dominant weights $\lambda$ and $\mu$ and so obtain for this group a version of Jantzen’s sum formula (in view of Corollary 1.4). Since the argument has some combinatorial intricacy, it will be broken up as follows. In 2.1 we will develop an algorithm to calculate the desired $\chi$. The algorithm will be illustrated in 2.2 by working out two simple classes of examples, after which we will derive the general formula in 2.3. But first let us (slightly) reformulate the problem so as to arrive at the setting in which the algorithm will take place.

Each of the dominant weights $\lambda$ and $\mu$ may be identified with a weakly decreasing set of $\text{rank}(F)$ integers. By tensoring with the determinant enough times, we may take these
integers to be nonnegative without loss of generality. Thus $\lambda$ and $\mu$ may be taken to be just partitions with at most $\text{rank}(F)$ rows. We will freely identify partitions with their Young diagrams.

Now an arbitrary partition may be considered a dominant weight for any $GL(\mathbb{Z}^n)$ such that $n \geq$ the number of rows in that partition. As soon as this condition on $n$ is met for both $\lambda$ and $\mu$, $\chi(V_{\mathbb{Z}}(\mu), V_{\mathbb{Z}}(\lambda))$ is independent of $n$ for the following reason. Let $N > n \geq$ the number of rows in $\lambda$ as well as those in $\mu$. For the moment let $\lambda_N$ (respectively $\lambda_n$) denote the dominant weight corresponding to the partition $\lambda$ for the group $GL(\mathbb{Z}^N)$ (respectively $GL(\mathbb{Z}^n)$) and likewise define $\mu_n, \mu_N$. Then one has $\text{Ext}^i_{GL(\mathbb{Z}^n)}(V_{\mathbb{Z}}(\mu_n), V_{\mathbb{Z}}(\lambda_n)) \simeq \text{Ext}^i_{GL(\mathbb{Z}^N)}(V_{\mathbb{Z}}(\mu_N), V_{\mathbb{Z}}(\lambda_N))$ by, e.g., [Kulkarni2, Proposition 1.1]. So henceforth we will deal with pairs of arbitrary partitions, always assuming that we are working over an appropriate $GL(F)$, i.e., one for which $\text{rank}(F)$ is large enough for the partitions in question to be valid weights.

2.1. An algorithm to compute $\chi(V_{\mathbb{Z}}(\mu), V_{\mathbb{Z}}(\lambda))$ for $GL(F)$. Let us begin by setting up an induction and by making some simple reductions.

(1) If $\mu \not< \lambda$ then $\chi(V_{\mathbb{Z}}(\mu), V_{\mathbb{Z}}(\lambda)) = 1$ by Theorem 1.1(ii). Therefore we may assume that $\mu < \lambda$ and we may further induct on the number of steps by which $\mu$ and $\lambda$ differ in (any linearization of) the dominance partial order. The base case of this induction is known due to [Kulkarni2, Theorem 2.1] and is subsumed in (2.1.3) below.

(2) Note that $\mu < \lambda$ means in particular that $\lambda$ and $\mu$ have the same degree, i.e., the same number of boxes. (Of course the necessity of this condition for nontriviality of $\chi$ is clear even before by considering the action of the center of $GL(F)$.) Let us induct on this degree as well. In degree 1 there is only one Weyl module, namely $F$, and $\chi(F, F)$ is 1 by either part of Theorem 1.1.

(3) If the first rows (or columns) of $\mu$ and $\lambda$ have the same length then by [Kulkarni2, Proposition 1.2] we may strip them off without affecting the $\text{Ext}$ groups and appeal to induction. So we may assume that $\mu$ and $\lambda$ differ in the lengths of their first rows and also in the lengths of their first columns.

Now let us proceed to the main inductive step. Let $\lambda'$ be the partition obtained by removing the first column of $\lambda$ and call the length of this column $t$. By reductions (1) and (3), the first column of $\mu$ is longer than that of $\lambda$. Removing the first $t$ boxes in this column gives the skew partition $\mu/1^t$. The partition $1^t$ is just a single column of length $t$ and the corresponding Weyl module is $\Lambda^t(F) =$ the $t$-th exterior power of the defining representation $F$. $\Lambda^t(F)$ is also the dual Weyl module corresponding to $1^t$. Consider the following equality given by the Skew Representative Theorem from [Kulkarni1].

$$(2.1.1) \quad \chi(V_{\mathbb{Z}}(\mu), \Lambda^t(F) \otimes V_{\mathbb{Z}}(\lambda')) = \chi(V_{\mathbb{Z}}(\mu/1^t), V_{\mathbb{Z}}(\lambda')),$$

where $V_{\mathbb{Z}}(\mu/1^t)$ is the skew Weyl module corresponding to $\mu/1^t$ defined in [ABW], where it is denoted $K_{\mu/1^t}(F)$. Let us analyze both sides of the equality (2.1.1) using the almost multiplicativity of $\chi$. 

12
On the right hand side, \( V_{Z}(\mu/1^{t}) \) has a Weyl filtration with factors \( V_{Z}(\mu^{1}), \ldots, V_{Z}(\mu^{k}) \) where \( \mu^{i} \) are all the partitions contained in \( \mu \) such that \( \mu/\mu^{i} \) consists of \( t \) boxes no two of which are in the same row. In other words, \( \mu^{i} \) are all the partitions obtainable by removing \( t \) boxes from the rightmost border strip of \( \mu \). This is the characteristic-free skew Pieri rule from [AB1, Section 3], where one can also find a more formal statement and a proof in the contravariant dual case of Schur modules, which carries over easily to ours. By induction on the degree \( \chi(V_{Z}(\mu^{i}), V_{Z}(\lambda^{j})) \) are all known.

By the characteristic-free Pieri rule in [AB1, Section 3], \( \Lambda^{t}(F) \otimes V_{Z}(\lambda') \) has a Weyl filtration with factors \( V_{Z}(\lambda^{1}), V_{Z}(\lambda^{2}), \ldots, V_{Z}(\lambda^{m}) \), where \( \lambda^{j} \) are all the partitions containing \( \lambda' \) such that \( \lambda^{j}/\lambda' \) consists of \( t \) boxes no two of which are in the same row. Clearly these include \( \lambda = \lambda^{1} \) (say), and \( \lambda/\lambda^{j} \) for \( j > 1 \). So by induction on the dominance order \( \chi(V_{Z}(\mu), V_{Z}(\lambda^{j})) \) are all known for \( j > 1 \).

**Case 1.** If \( Hom(V_{Z}(\mu), \Lambda^{t}(F) \otimes V_{Z}(\lambda')) \simeq Hom(V_{Z}(\mu/1^{t}), V_{Z}(\lambda')) \) is zero, i.e., if none of the \( \lambda^{j} \) equals \( \mu \), i.e., if none of the \( \mu^{i} \) equals \( \lambda' \), then \( \chi \) is multiplicative as we glue the \( V_{Z}(\lambda^{j}) \) together to get \( \Lambda^{t}(F) \otimes V_{Z}(\lambda') \) (respectively, the \( V_{Z}(\mu^{i}) \) to get \( V_{Z}(\mu/1^{t}) \)) because all the \( Hom \) terms in the associated long exact sequences vanish. Therefore we get the following equations.

\[
\chi(V_{Z}(\mu), \Lambda^{t}(F) \otimes V_{Z}(\lambda')) = \prod_{j} \chi(V_{Z}(\mu), V_{Z}(\lambda^{j})).
\]

(2.1.2)

\[
\chi(V_{Z}(\mu/1^{t}), V_{Z}(\lambda')) = \prod_{i} \chi(V_{Z}(\mu^{i}), V_{Z}(\lambda^{j})).
\]

Now one easily finds \( \chi(V_{Z}(\mu), V_{Z}(\lambda)) \) from (2.1.1) and (2.1.2).

**Case 2.** If Case 1 does not happen then it is clear from the Pieri rules that \( \chi \) fails to be multiplicative at exactly one stage in gluing. By the work done in Example 2 in 1.2, the equations (2.1.2) now need to be modified by writing certain correction factors on their right hand sides. At first glance this seems to make it necessary to compute these integers every time we are in Case 2. The only way I know of directly doing this computation involves finding explicit generators of certain \( Hom \) groups. This is theoretically possible, but quite hard in practice, even in the relatively simple case treated in [Kulkarni2, Theorem 2.1], as seen in the proofs of Lemmas A–C there.

Fortunately we can wriggle out of this difficulty in most cases by analyzing the combinatorics of Pieri’s rule and by using conjugate symmetry of \( Ext \) groups (1.6.1). Vizualizing the dominant weights occuring in \( \Lambda^{t}(F) \otimes V_{Z}(\lambda') \) reveals that if one encounters Case 2 then \( \mu \) must be obtainable from \( \lambda \) by removing some of the boxes from the rightmost border strip of \( \lambda \) and placing them at the bottom of the first column.

Now (1.6.1) gives \( \chi(V_{Z}(\mu), V_{Z}(\lambda)) = \chi(V_{Z}(\bar{\mu}), V_{Z}(\bar{\lambda})) \). So if we encounter Case 2, we may discard the pair \((\mu, \lambda)\) of dominant weights and work with the pair \((\bar{\lambda}, \bar{\mu})\) instead. A little further thought shows that the only way \((\bar{\lambda}, \bar{\mu})\) will also lead to Case 2 is when \( \mu \) has a single box in its last row and \( \lambda \) is obtained by removing this box and placing...
it at the end of the first row. (Recall that by the reduction (3), \( \mu \) and \( \lambda \) must differ in their first rows as well as in their first columns.) So \( \lambda - \mu = \) a positive root \( \alpha \). But this is precisely the situation treated by [Kulkarni2, Theorem 2.1]. This theorem gives the following calculation of all Ext groups for such pairs of Weyl modules. Ext\(^1\) is cyclic of order \( \langle \mu + \rho, \alpha \rangle + 1 \) = hook length of the box in the first row and first column of \( \mu \) (or \( \lambda \)), and all other Ext groups vanish. So for \( \lambda = \mu + \alpha \) one has

\[
(2.1.3) \quad \chi(VZ(\mu), VZ(\lambda)) = \frac{1}{\langle \mu + \rho, \alpha \rangle + 1}.
\]

This ends the recursion and completes the description of the algorithm.

**Remarks.** (1) For future use note the following extra flexibility that may be built into the above algorithm. \( t \) may be chosen to be the length of any column in \( \lambda \). Then \( \lambda' \) would be the partition obtained by deleting the rightmost box in each of the top \( t \) rows of \( \lambda \). The induction goes through just as before with the following differences. For one, different sets of \( \mu_i \) and \( \lambda_j \) are involved in the computation. Moreover the characterization of when one encounters Case 2 is no longer as clean. Nor is the escape by use of conjugate symmetry guaranteed if one is required to use a given \( t \). But of course one may then use a different \( t \) (e.g., just follow the algorithm above). The point is that sometimes choosing a different \( t \) will make the calculation simpler. For example choosing \( t = 1 \) (of course the last column of \( \lambda \) must be of length 1 for that) may considerably simplify the combinatorics for large partitions. Note that [Kulkarni2, Theorem 2.1] was proved by taking \( t = 1 \). In the proof of that theorem recursion could be controlled sufficiently to give all Ext groups, not just \( \chi \).

(2) (Relating the algorithm to symmetries of Ext groups between Weyl modules.) By (1.6.1), (1.6.2) and using the terminology there we have

\[
\chi(VZ(\mu^c), VZ(\lambda^c)) = \chi(VZ(\mu), VZ(\lambda)) = \chi(VZ(\lambda), VZ(\mu^c)).
\]

Under the isomorphism (1.6.2), using the algorithm for the pair of partitions \( (\mu, \lambda) \) by splitting off the first column of \( \lambda \) is seen to be tantamount to using the algorithm for the pair \( (\mu^c, \lambda^c) \) by splitting off the last column of \( \lambda^c \). Similarly one can devise a version of the algorithm that is consistent with the second equality. In place of (2.1.1) this version will rely on a procedure that splits off the top row of \( \mu \) rather than the first column of \( \lambda \). Later both of these symmetries will be very useful for substantial reductions while proving a general formula for \( \chi(VZ(\mu), VZ(\lambda)) \).

2.2. **Examples.** Before stating and proving the formula for \( \chi \) for a general pair of partitions, it will be instructive to see the above algorithm in action in some simple examples.

**Example 1.** \((GL_2)\) Let us compute \( \chi(VZ(\mu), VZ(\lambda)) \), where \( \mu \) and \( \lambda \) are partitions with at most two parts. By the initial reductions the general case immediately reduces to the following: \( \mu = (a, b) \), \( \lambda = (a + b, 0) \), where \( a \geq b > 0 \). For convenience let us denote \( \chi(VZ(\mu), VZ(\lambda)) \) by \( \chi[a, b] \). Clearly the only choice for \( t \) here is \( t = 1 \). Unless \( b = 1 \), the
algorithm does not lead to Case 2. Using (2.1.1), (2.1.2) and reduction (3) one gets the following equations.

\[
\chi[a, b] \chi[a - 1, b - 1] = \chi[a, b - 1] \chi[a - 1, b] \quad \text{if } a > b.
\]
\[
\chi[a, a] \chi[a - 1, a - 1] = \chi[a, a - 1].
\]

If \( b = 1 \) one has \( \chi[a, 1] = \frac{1}{a+1} \). Now one easily derives that

\[
\chi[a, b] = \frac{b}{a+1}.
\]

**Example 2.** (\( \chi \) for two hook partitions.) Let \( \mu = (a, 1^b), \lambda = (a+s, 1^{b-s}) \), i.e., both diagrams have the shape of a hook. For convenience let us denote \( \chi(V_Z(\mu), V_Z(\lambda)) \) by \( \chi[a, b, s] \). As in the algorithm, let us choose \( t = b + 1 - s \), i.e. the length of the first column of \( \lambda \) (The reason for this choice will be clear soon.) If \( s \neq 1 \), the algorithm does not lead to Case 2. In this case one gets the following equation using (2.1.1) and (2.1.2).

\[
\chi[a, b, s] \chi[a, b, s - 1] = \chi[a, s - 1, s - 1] \chi[a - 1, s, s].
\]

If \( s = 1 \) one has \( \chi[a, b, 1] = \frac{1}{a+b} \). Now one easily derives that

\[
\chi[a, b, s] = \left( \frac{a+b}{s} \right)^{(-1)^s}.
\]

Notice that for this calculation, we only needed to deal with pairs of hooks i.e., no other partitions appeared in recursion. A little thought reveals that if we had chosen \( t = 1 \), this would not have been the case and the computation would have been much more unwieldy.

### 2.3. A formula for \( \chi(V_Z(\mu), V_Z(\lambda)) \).

It turns out that there is a fairly simple formula for \( \chi(V_Z(\mu), V_Z(\lambda)) \) in terms of the geometry of the diagrams of the involved partitions. Once the formula is guessed (after working out several classes of examples like the ones shown above), it is not so hard to prove by judicious use of the algorithm and careful bookkeeping. Before stating the final answer in the next theorem, let us set up some terminology. A skew partition is *connected* if a rook can go from any box in it to any other by ordinary chess moves. A *skew hook* (also called ribbon or a border strip) is a skew partition not containing a 2 by 2 square. We may call a skew partition failing this condition *overconnected*. Thus a skew partition will fail to be a connected skew hook by being disconnected or overconnected or both. We could call a connected skew hook a *snake*. The “right endpoint” (respectively, “left endpoint”) of a skew partition will mean the rightmost box in its top row (respectively, the leftmost box in its bottom row). This terminology will be used mainly when the skew partition is a skew hook.

**Theorem.** Let \( \mu \) and \( \lambda \) be arbitrary partitions. Considering \( \mu \) and \( \lambda \) as sets of appropriately situated boxes in a plane, let \( \nu \) be the partition \( \lambda \cap \mu \). We will say that \( \mu \) and \( \lambda \) differ by connected skew hooks if the skew partitions \( \mu/\nu \) and \( \lambda/\nu \) are both connected skew
hooks. Then one has the following. (1) $\chi(V_Z(\mu), V_Z(\lambda))$ is 1 unless $\mu < \lambda$ and moreover $\mu$ and $\lambda$ differ by connected skew hooks. (2) If $\mu$ and $\lambda$ do differ by connected skew hooks and $\mu < \lambda$ as well, then

$$\chi(V_Z(\mu), V_Z(\lambda)) = \left( \frac{\ell}{d} \right)^{-r},$$

where the symbols have meanings as follows. $\ell$ is the the equal number of boxes in skew hooks $\mu/\nu$ and $\lambda/\nu$. We will call this the length of the skew hook, not to be confused with the number of rows in the skew hook. $d$ is the total number of right and up moves that each box in $\mu/\nu$ has to make (i.e., the distance this snake has to cover by sliding along the border of the diagram) in order to transform $\mu$ into $\lambda$. $r$ is the sum of the number of nonempty rows in $\mu/\nu$ and that in $\lambda/\nu$. By drawing pictures, the numbers $\ell$ and $d$ can also be seen to be the following hook lengths. $\ell$ is the hook length of the box in $\lambda$ that is in the same row as the right endpoint of $\lambda/\nu$ and the same column as the left endpoint of $\lambda/\nu$. The previous sentence stays valid after replacing each $\lambda$ by $\mu$. $d$ is the hook length of the box in $\mu$ that is in the same row as the left endpoint of $\lambda/\nu$ and the same column as the left endpoint of $\mu/\nu$. $d$ is also the hook length of the box in $\lambda$ that is in the same row as the right endpoint of $\lambda/\nu$ and the same column as the right endpoint of $\mu/\nu$.

**Proof.** Recall the following from 2.1. By Theorem 1.1(ii) we may assume that $\mu < \lambda$. We may also assume that $\lambda$ has a strictly longer first row and strictly shorter first column than $\mu$. So the entire last column of $\lambda$ is missing in $\mu$ and the entire last row of $\mu$ is missing in $\lambda$. We will induct on the equal number of boxes in $\mu$ and $\lambda$ and on the number of steps by which $\lambda$ and $\mu$ differ under (a linearization of) the dominance partial order. Note that the base case of each induction is already known. By (2.1.3) we may additionally assume that $\lambda/\nu$ (and $\mu/\nu$) contains more than one box. We will follow the algorithm as extended in the first remark following it. More precisely we will always use $t = \ell$ the length of the last column of $\lambda$. It will be clear that the difficulty with infinite $\text{Hom}$ encountered in Case 2 in 2.1 never arises (partly due to the use of (2.1.3) at the outset).

The proof will consist of a series of reductions and one crucial calculation requiring treatment of several cases. Throughout it will be easier to follow the combinatorial arguments by visualizing the diagrams of $\lambda$ and $\mu$ with $\lambda/\nu$ and $\mu/\nu$ (the set differences between $\lambda$ and $\mu$) “colored” differently from $\nu$.

**Step 1.** (Getting rid of almost all overconnectedness.) Suppose that $\lambda/\nu$ contains a 2 by 2 square (i.e., is overconnected) and that $\lambda$ has at least one column strictly to the right of this 2 by 2 square. When one runs the main step of the algorithm with $t = \ell$ the length of such a column (say the last one), for each resulting pair $(\mu, \lambda')$ the skew partition $\lambda_i/(\lambda_i \cap \mu)$ will stay overconnected. This is simply because $\lambda_i$ is obtained by moving down some boxes to the right of the “excess 2 by 2 square,” which will clearly keep this square intact. Similarly for each pair $(\mu', \lambda')$ the skew partition $\lambda'/(\lambda' \cap \mu_i)$ will also stay overconnected. Now equations (2.1.1), (2.1.2) and induction give the desired triviality of $\chi(V_Z(\mu), V_Z(\lambda))$. By conjugate symmetry (1.6.1), one also gets the triviality of $\chi(V_Z(\mu), V_Z(\lambda))$ when $\mu/\nu$ contains a 2 by 2 square and $\mu$ has at least one row strictly below this 2 by 2 square.
Enclose $\mu$ and $\lambda$ in a rectangle and recall the symmetry of $\text{Ext}$ under complements (1.6.2). Since any 2 by 2 square in $\lambda/\nu$ is present is $\lambda$ and absent in $\mu$, the same square will be present in $\mu^c$ and absent in $\lambda^c$. If such an excess 2 by 2 square occurs in $\lambda$ and is situated below the top row of $\lambda$, the corresponding excess square in $\mu^c$ must be situated above the bottom row of $\mu^c$. Now applying the last sentence of the previous paragraph to the pair $(\mu^c, \lambda^c)$ and using (1.6.2) we get that $\chi(V_Z(\mu), V_Z(\lambda)) = 1$. Conjugate symmetry (1.6.1) gives the same conclusion if $\mu/\nu$ contains a 2 by 2 square that is situated to the right of the first column of $\mu$.

By the previous two paragraphs we may assume the following. $\lambda/\nu$ contains at most one 2 by 2 square and such a square must occur in the last two columns and the first two rows of $\lambda$. Similarly $\mu/\nu$ contains at most one 2 by 2 square and such a square must occur in the first two columns and the last two rows of $\mu$. Henceforth these assumptions will be in force throughout the proof.

**Step 2.** (The crucial calculation.) Suppose the last column of $\lambda$ (which is absent from $\mu$ by our reduction) consists of exactly one box. We will run the algorithm with $t = 1$ and show by induction that the claimed formula holds. By Step 1, $\lambda/\nu$ is a (possibly disconnected) skew hook. We will treat three cases.

**Case A.** Suppose $\lambda/\nu$ and $\mu/\nu$ are both connected skew hooks. We will analyze both equations in (2.1.2). In the first equation it is easy to see by induction that there are at most three partitions $\lambda^j$ other than $\lambda$ that lead to nontrivial $\chi(V_Z(\mu), V_Z(\lambda^j))$. These are described in cases (A.1.i) through (A.1.iii) below. For ease of description let the right endpoint of $\mu/\nu$ be in $u$-th row and $v$-th column of $\mu$. Let the left endpoint of $\lambda/\nu$ is in $x$-th row and $y$-th column of $\lambda$.

(A.1.i) When $\lambda^j$ is obtained by adding a box to $\lambda'$ immediately below the left endpoint of $\lambda/\nu$, i.e., in $(x + 1)$-th row and $y$-th column. This gives a valid partition precisely when the $x$-th and $(x + 1)$-th rows of $\nu$ are of equal length $y - 1$. In that case $\lambda^j \cap \mu = \nu$ and moreover $\lambda^j$ and $\mu$ still differ by connected skew hooks of length $\ell$. But the distance needed to slide one of these skew hooks into another is one less than before and $\lambda^j/\nu$ has one more row than $\lambda/\nu$. By induction

$$\chi(V_Z(\mu), V_Z(\lambda^j)) = \left( \frac{\ell}{d - 1} \right)^{(-1)^{r+1}}.$$

(A.1.ii) When $\lambda^j$ is obtained by adding a box to $\lambda'$ in the position of the left endpoint of $\mu/\nu$. This gives a valid partition precisely when $\mu$ has exactly one row more than $\lambda$. In a manner similar to (A.1.i) it is easy to see that in this case we have

$$\chi(V_Z(\mu), V_Z(\lambda^j)) = \left( \frac{\ell - 1}{d - 1} \right)^{(-1)^r}.$$

(A.1.iii) When $\lambda^j$ is obtained by adding a box to $\lambda'$ in the position of the right endpoint of $\mu/\nu$, i.e., in $u$-th row and $v$-th column. This procedure gives a valid partition precisely
when in $\mu/\nu$ this right endpoint is the only box in its row, i.e., when the $u$-th row of $\nu$ has length $v - 1$. In a manner similar to (A.1.i) and (A.1.ii) it is easy to see that in this case we have

$$\chi(V_Z(\mu), V_Z(\lambda^j)) = \left(\frac{\ell - 1}{d}\right)^{(1)d - 1}.$$

The analysis of the second equation is very similar. Here we need to look for the $\mu^i$ that will give nontrivial $\chi(V_Z(\mu^i), V_Z(\lambda^j))$. There are at most three possibilities, which are listed below.

(A.2.i) When $\mu^i$ is obtained from $\mu$ by removing the box immediately to the left of the left endpoint of $\lambda/\nu$, i.e., the one in $x$-th row and $(y-1)$-th column. This gives a valid partition precisely when $x$-th and $(x+1)$-th rows of $\nu$ are of unequal lengths, i.e., when (A.1.i) is not possible. In that case $\lambda'$ and $\mu^i$ still differ by connected skew hooks of length $\ell$; the distance needed to slide one of these skew hooks into another is one less than before; and the number of rows in each skew hook stays unchanged. By induction

$$\chi(V_Z(\mu^i), V_Z(\lambda')) = \left(\frac{\ell}{d - 1}\right)^{(1)d - 1}.$$

(A.2.ii) When $\mu^i$ is obtained from $\mu$ by removing the box at the left endpoint of $\mu/\nu$. It is easy to see that this gives a valid partition precisely when (A.1.ii) is not possible. Moreover, in that case $\chi(V_Z(\mu^i), V_Z(\lambda'))$ is the reciprocal of the number obtained in (A.1.ii).

(A.2.iii) When $\mu^i$ is obtained from $\mu$ by removing the box at the right endpoint of $\mu/\nu$. Again this gives a valid partition precisely when (A.1.iii) is not possible. In that case $\chi(V_Z(\mu^i), V_Z(\lambda'))$ is the reciprocal of the number obtained in (A.1.iii).

Note that a fourth $\mu^i$ seems possible at first glance. Namely one could try to take off a box from $\mu$ so as to somehow augment the skew hook $\lambda/\nu$ at its right endpoint (similar to the way it was augmented on the left in (A.2.i)). But this is clearly seen to be impossible. We will encounter such a possibility later in Case C.

So we may as well suppose that none of the cases (A.2.i) through (A.2.iii) occurs and thus all of the cases (A.1.i) though (A.1.iii) do. The desired claim is now immediate from (2.1.1) and the following trivial calculation.

$$\left(\frac{\ell}{d}\right) \left(\frac{d - 1}{\ell}\right) \left(\frac{\ell - 1}{d - 1}\right) \left(\frac{d}{\ell - 1}\right) = 1.$$

Case B. Suppose $\lambda/\nu$ is a connected skew hook but $\mu/\nu$ is not a connected skew hook. Proceeding in a manner similar to Case A, it is easy to see that there are only two ways in which one of the equations in (2.1.2) could yield a lower term with nontrivial $\chi$. (If every lower $\chi$ in both equations is trivial, the desired result, namely triviality of $\chi(V_Z(\mu), V_Z(\lambda))$, is immediate.)
(B.1) If $\mu/\nu$ contains a 2 by 2 square (by Step 1 necessarily only one, occurring in the last two rows and the first two columns of $\mu$), then in each equation there is at most one nontrivial term possible. In the first equation this occurs when $\lambda^j$ is obtained by adding a box to $\lambda'$ in the position of the top left box of the 2 by 2 square. In the second equation this occurs when $\mu^i$ is obtained from $\mu$ by removing the bottom right box of this 2 by 2 square. These two possibilities are easily seen to give the same value of $\chi$. So they cancel each other in (2.1.1) and give the desired triviality of $\chi(V_Z(\mu), V_Z(\lambda))$.

(B.2) If $\mu/\nu$ does not contain a 2 by 2 square, it must be a disconnected skew hook. It is easy to see that the only way to get a nontrivial term in the recursion is when $\mu/\nu$ has two connected components and one of the components is a single box. In this case each equation has exactly one nontrivial lower term and again these two terms cancel each other. (In the first equation get $\lambda^j$ by adding a box to $\lambda'$ in place of the isolated box in $\mu/\nu$. In the second equation get $\mu^i$ by removing the same isolated box from $\mu$.)

Case C. Suppose $\lambda/\nu$ is a disconnected skew hook. Proceeding as before, we will analyze the ways in which one of the equations in (2.1.2) could yield a nontrivial lower $\chi$.

(C.1) If $\lambda'/\nu$ is a disconnected skew hook as well, then it is easy to see that the only way one could get nontrivial lower terms in either equation is when the all of the following three conditions hold. $\lambda'/\nu$ must consist of exactly two connected components. Moreover the gap between the components must be exactly one box, i.e., adding just one box to $\lambda'/\nu$, say in $a$-th row and $b$-th column of $\lambda'$, should make it a connected skew hook. And $\mu/\nu$ must be a connected skew hook. In that case there is exactly one nontrivial term in each equation and once again these cancel each other. ($\lambda^j$ is obtained by adding a box to $\lambda'$ in place of the isolated box in $\mu/\nu$. In the second equation obtain $\mu^i$ from $\mu$ by deleting the box in $(a - 1)$-th row and $(b - 1)$-th column.)

(C.2) So suppose now that $\lambda'/\nu$ is a connected skew hook (i.e., $\lambda/\nu$ has exactly two connected components and one of the components consists of the last box in the first row of $\lambda$.) Now if $\mu/\nu$ fails to be a connected skew hook, it is easy to see that the entire analysis in Case B carries over. (After changing $\lambda$ to $\lambda'$ in the opening sentence of Case B, the rest applies verbatim.)

(C.3) So we may suppose that $\mu/\nu$ and $\lambda'/\nu$ are both connected skew hooks. The analysis here is entirely parallel to that in Case A with the following crucial difference. There is a fourth nontrivial term possible in each equation in (2.1.2). In the first equation one could get a $\lambda^j$ by adding a box to $\lambda'$ immediately to the right of the right endpoint of $\lambda'/\nu$. (In Case A this would just give $\lambda^j = \lambda$, but here we get a lower term.) In the second equation one could get another $\mu^i$ from $\mu$ by deleting the box immediately above the right endpoint of $\lambda'/\nu$. (This is the putative fourth possibility discussed above immediately after (A.2.iii), which could not occur there.) Further, just as for the three pairs of possibilities in Case A, exactly one of this new pair of possibilities will actually take place. It is easy to see that the same calculation as in Case A gives the desired triviality of $\chi(V_Z(\mu), V_Z(\lambda))$.

Clearly cases A, B and C exhaust all possibilities when the last column of $\lambda$ has
length 1 (in presence of the reductions that were made previously and which will be in force throughout the proof). This finishes Step 2.

Step 3. (Further reductions.) If the last row of \( \mu \) (which is absent from \( \lambda \) by earlier reduction) contains exactly one box we will be done by Step 2 and conjugate symmetry (1.6.1). So henceforth we will assume that the last column of \( \lambda \) as well as the last row of \( \mu \) (each of which is entirely missing from the other partition) have lengths greater than 1.

Now suppose that the first column of \( \mu \) has exactly one more box than the first column of \( \lambda \). Then one reduces to the case in Step 2 as follows. In view of the discussion near the beginning of Section 2 we may take \( n \) (the rank of the defining representation of \( GL(F) \)) to be the number of rows in \( \mu \). Now by (1.6.2) one may replace the pair of partitions \((\mu, \lambda)\) by the pair \((\mu^c, \lambda^c)\), which is covered by Step 2. So henceforth we will assume that the first column of \( \mu \) has at least two boxes more than the first column of \( \lambda \), i.e., that the entire last two rows of \( \mu \) are missing in \( \lambda \). By conjugate symmetry (1.6.1) we may assume that the entire last two columns of \( \lambda \) are missing in \( \mu \).

Step 4. (The last case.) Combining Step 1 and Step 3 we may and will assume the following. Each of \( \lambda/\nu \) as well as \( \mu/\nu \) contains exactly one 2 by 2 square and these squares are situated as described at the end of Step 1. In particular note that the last column of \( \lambda \) contains exactly two boxes, as does the last row of \( \mu \). We will run the algorithm in 2.1 with \( t = 2 \) and prove that \( \chi(V_{Z}(\mu), V_{Z}(\lambda)) \) is 1.

Consider all pairs of partitions \((\mu, \lambda^j)\) (except \((\mu, \lambda)\)) that occur in the first equation in (2.1.2). For such a pair to lead to nontrivial \( \chi \), one of the two removed boxes in the last column of \( \lambda \) must be added to the bottom of the first column of \( \lambda^j \). This is the only way to remove the overconnectedness in \( \lambda/\nu \) and \( \mu/\nu \). So essentially one has to deal with placement of just one box. Similarly for a pair \((\mu^i, \lambda')\) to lead to nontrivial \( \chi \) in the second equation in (2.1.2), the second box in the last row of \( \mu \) must be missing in \( \mu^i \). Thus essentially one has to deal with removal of only one box in \( \mu \). Thus we have a situation very similar to the one handled in Step 2 and one can imitate the cases there. Instead let us take a shortcut.

For future need let us separately treat the case when each of \( \lambda/\nu \) and \( \mu/\nu \) contains exactly four boxes, i.e., is a 2 by 2 square. Using preceding discussion, one easily verifies the desired result by an explicit calculation that is similar to but simpler than Case A in Step 2. So from now on suppose that each of \( \lambda/\nu \) and \( \mu/\nu \) contains more than four boxes.

Let \( \xi \) be the partition obtained from \( \lambda \) by removing the two boxes in the last column and adding one box at the bottom of the first column. One has

\[
\chi(V_{Z}(\mu), V_{Z}(\lambda') \otimes \Lambda^2(F)) = \chi(V_{Z}(\mu), V_{Z}(\lambda)) \chi(V_{Z}(\mu), V_{Z}(\xi) \otimes F)
\]

for the following reason. Recasting earlier discussion, any lower \( \lambda^j \) that leads to nontrivial \( \chi(V_{Z}(\mu), V_{Z}(\lambda^j)) \) must contain \( \xi \) as a subset. Thus, apart from \( \lambda^1 = \lambda \), it suffices to consider only those \( \lambda^j \) that are obtained by adding one box to \( \xi \). This is clearly a subset of the set of partitions \( \xi^k \) such that \( V_{Z}(\xi^k) \) is a filtration factor in a Weyl filtration of \( V_{Z}(\xi) \otimes F \). In
fact the two sets will be the same unless the first two columns of $\lambda$ have equal lengths, in which case there will be exactly one extra $\xi^k$ obtained by adding a box in the last row of $\xi$. (This partition will not occur among the $\lambda^j$ since it amounts to adding two boxes to $\lambda'$ in the same row.) However in this case $\mu/(\mu \cap \xi^k)$ will be disconnected since one of the connected components will be the two boxes in the last row and there must be at least three boxes in $\mu/(\mu \cap \xi^k)$ thanks to the case treated separately in the previous paragraph. Thus for the extra $\xi^k$, one has $\chi(V_{Z}(\mu), V_{Z}(\xi^k)) = 1$. This proves the claimed equality.

Now using (2.1.1) one has

$$\chi(V_{Z}(\mu), V_{Z}(\lambda')) = \chi(V_{Z}(\mu'), V_{Z}(\lambda')).$$  
$$\chi(V_{Z}(\mu), V_{Z}(\xi) \otimes F) = \chi(V_{Z}(\mu/1), V_{Z}(\xi)).$$

We will be done once we show that the right hand sides of the preceding two equations are the equal. To see this let $\pi$ be the partition obtained by deleting the second box from the last row of $\mu$. Recasting earlier discussion, any $\mu^i$ that leads to nontrivial $\chi(V_{Z}(\mu^i), V_{Z}(\lambda'))$ must be obtainable by removing one box from $\pi$. Thus while calculating $\chi(V_{Z}(\mu/1^2), V_{Z}(\lambda'))$ it suffices to consider only those $\mu^i$ that are obtained by deleting one box from $\pi$. We will set up an almost pairing between the set $S$ of such partitions and the set $T$ of all partitions $\bar{\mu}$ such that $V_{Z}(\bar{\mu})$ is a filtration factor in a Weyl filtration of $V_{Z} (\mu/1)$. Suppose $\mu^i \in S$ is obtained by removing a certain box from $\pi$. Define a map $f$ from $S$ to $T$ by sending $\mu^i$ to the partition obtained by removing the same box from $\mu$. Before examining the failure of $f$ to be a pairing, let us note that

$$\chi(V_{Z}(\mu^i), V_{Z}(\lambda')) = \chi(V_{Z}(f(\mu^i)), V_{Z}(\xi)).$$

To see this one easily verifies pictorially that $\lambda'/(\lambda' \cap \mu^i) = \xi/(\xi \cap f(\mu^i))$ and that the only difference between $\mu^i/(\lambda' \cap \mu^i)$ and $f(\mu^i)/(\xi \cap f(\mu^i))$ is in the arrangement of the three boxes at the left endpoint, where the former has the same shape as the partition (2,1) and the latter has the same shape as the skew partition (2,2)/(1). Clearly this difference is irrelevant for the formula asserted in Theorem 2.3. Further, the set differences between each pair of partitions have the same relative position. The stated equality follows by induction on the degree.

Now let us deal with the failure of $f$ to be a pairing. This can happen in two ways. (1) $f$ will not yield the partition $\bar{\mu}$ in $T$ that is obtained by removing the second box in the last row of $\mu$. However for this partition $\chi(V_{Z}(\bar{\mu}), V_{Z}(\xi))$ is trivial since $\bar{\mu}/(\bar{\mu} \cap \xi)$ is disconnected in the last two rows. So we may ignore this failure. (2) $f(\mu^i)$ will be undefined if the last but one row of $\mu$ consists of exactly two boxes and $\mu^i$ is obtained by removing the second of these from $\pi$. (Removing this box from $\mu$ will not give a partition.) However in this case $\chi(V_{Z}(\mu^i), V_{Z}(\lambda'))$ is trivial since $\mu^i/(\mu^i \cap \lambda')$, which contains at least three boxes thanks to the special case treated above, is disconnected. So we may ignore this failure as well. This finishes the proof of Theorem 2.3.

Remarks. 1) Certain products of the numbers on the right hand side of the formula in Theorem 2.3 have already appeared in the literature on Jantzen’s sum formula. [JM]
shows that the determinant of the Gram matrix for a Specht module for the symmetric group is such a product. (I am grateful to Arun Ram for pointing out this fact and for supplying the reference.)

2) Brenti’s recent work on certain parabolic Kazhdan-Lusztig and $R$-polynomials for the symmetric group also involves connected skew hooks, see [Brenti]. Theorem 2.3 and Brenti’s work together suggest that at least for type A, $\text{Ext}$ groups between Weyl modules should be somehow related to Kazhdan-Lusztig combinatorics. In the BGG category a connection between ordinary $R$-polynomials for the Weyl group and $\text{Ext}$ groups between Verma modules was suggested by Gabber and Joseph, but their guess was found to be false by Boe. Nonetheless in light of new evidence it seems that there may well be a relationship between appropriate $R$-polynomials and $\text{Ext}$ groups between Weyl modules. It would be very interesting to find a precise connection.

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