On upper bounds for parameters related to construction of special maximum matchings

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For a graph $G$ let $L(G)$ and $l(G)$ denote the size of the largest and smallest maximum matching of a graph obtained from $G$ by removing a maximum matching of $G$. We show that $L(G) \leq 2l(G)$, and $L(G) \leq \frac{3}{2}l(G)$ provided that $G$ contains a perfect matching. We also characterize the class of graphs for which $L(G) = 2l(G)$. Our characterization implies the existence of a polynomial algorithm for testing the property $L(G) = 2l(G)$. Finally we show that it is $NP$-complete to test whether a graph $G$ containing a perfect matching satisfies $L(G) = \frac{3}{2}l(G)$.

1. Introduction

In the paper graphs are assumed to be finite, undirected, without loops or multiple edges. Let $V(G)$ and $E(G)$ denote the sets of vertices and edges of a graph $G$, respectively. If $v \in V(G)$ and $e \in E(G)$, then $e$ is said to cover $v$ if $e$ is incident to $v$. For $V' \subseteq V(G)$ and $E' \subseteq E(G)$ let $G[V']$ and $G[E']$ denote the graphs obtained from $G$ by removing $V'$ and $E'$, respectively. Moreover, let $V(E')$ denote the set of vertices of $G$ that are covered by an edge from $E'$. A subgraph $H$ of $G$ is said to be spanning for $G$, if $V(E(H)) = V(G)$.

The length of a path (cycle) is the number of its edges. A $k$-path ($k$-cycle) is a path (cycle) of length $k$. A 3-cycle is called a triangle.

A set $V' \subseteq V(G)$ ($E' \subseteq E(G)$) is said to be independent, if $V'$ ($E'$) contains no adjacent vertices (edges). An independent set of edges is called matching. A matching of $G$ is called perfect, if it covers all vertices of $G$. Let $\nu(G)$ denote the cardinality of a largest matching of $G$. A matching of $G$ is maximum, if it contains $\nu(G)$ edges.

For a positive integer $k$ and a matching $M$ of $G$, a $(2k - 1)$-path $P$ is called $M$-augmenting, if the $2^{nd}$, $4^{th}$, $6^{th}$, ..., $(2k - 2)^{th}$ edges of $P$ belong to $M$, while the endvertices of $P$ are not covered by an edge of $M$. The following theorem of Berge gives a sufficient and necessary condition for a matching to be maximum:

**Theorem 1** (Berge [2]) A matching $M$ of $G$ is maximum, if $G$ contains no $M$-augmenting path.

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For two matchings $M$ and $M'$ of $G$ consider the subgraph $H$ of $G$, where $V(H) = V(M \triangle M')$ and $E(H) = M \triangle M'$. The connected components of $H$ are called $M \triangle M'$-alternating components. Note that $M \triangle M'$ alternating components are always paths or cycles of even length. For a graph $G$ define:

$$L(G) \equiv \max \{ \nu(G \backslash F) : F \text{ is a maximum matching of } G \},$$

$$l(G) \equiv \min \{ \nu(G \backslash F) : F \text{ is a maximum matching of } G \}.$$

It is known that $L(G)$ and $l(G)$ are $NP$-hard calculable even for connected bipartite graphs $G$ with maximum degree three [4], though there are polynomial algorithms which construct a maximum matching $F$ of a tree $G$ such that $\nu(G \backslash F) = L(G)$ and $\nu(G \backslash F) = l(G)$ (to be presented in [5]).

In the same paper [5] it is shown that $L(G) \leq 2l(G)$. In the present paper we reprove this equality, and also show that $L(G) \leq \frac{3}{2}l(G)$ provided that $G$ contains a perfect matching.

A naturally arising question is the characterization of graphs $G$ with $L(G) = 2l(G)$ and the graphs $G$ with a perfect matching that satisfy $L(G) = \frac{3}{2}l(G)$. In this paper we solve these problems by giving a characterization of graphs $G$ with $L(G) = 2l(G)$ that implies the existence of a polynomial algorithm for testing this property, and by showing that it is $NP$-complete to test whether a bridgeless cubic graph $G$ satisfies $L(G) = \frac{3}{2}l(G)$. Recall that by Petersen theorem any bridgeless cubic graph contains a perfect matching.

Terms and concepts that we do not define can be found in [1, 2, 6, 8].

2. Some auxiliarly results

We will need the following:

**Theorem 2** Let $G$ be a graph. Then:

(a) for any two maximum matchings $F, F'$ of $G$, we have $\nu(G \backslash F') \leq 2\nu(G \backslash F)$;

(b) $L(G) \leq 2l(G)$;

(c) If $L(G) = 2l(G)$, $F_L, F_i$ are two maximum matchings of the graph $G$ with $\nu(G \backslash F_L) = L(G), \nu(G \backslash F_i) = l(G)$, and $H_L$ is any maximum matching of the graph $G \backslash F_L$, then:

(c1) $F_i \backslash F_L \subseteq H_L$;

(c2) $H_L \backslash F_i$ is a maximum matching of $G \backslash F_i$;

(c3) $F_L \backslash F_i$ is a maximum matching of $G \backslash F_i$;

(d) if $G$ contains a perfect matching, then $L(G) \leq \frac{3}{2}l(G)$.

**Proof.** (a) Let $H'$ be any maximum matching in the graph $G \backslash F'$. Then:

$$\nu(G \backslash F') = |H'| = |H' \cap F| + |H' \backslash F| \leq |F \backslash F'| + \nu(G \backslash F) = |F' \backslash F| + \nu(G \backslash F) \leq 2\nu(G \backslash F).$$

(b) follows from (a).
(c) Consider the proof of (a) and take $F' = F_L$, $H' = H_L$ and $F = F_I$. Since $L(G) = 2l(G)$, we must have equalities throughout, thus properties (c1)-(c3) should be true.

(d) Let $F_L, F_I$ be two perfect matchings of the graph $G$ with $\nu(G \backslash F_L) = L(G), \nu(G \backslash F_I) = l(G)$, and assume $H_L$ to be a maximum matching of the graph $G \backslash F_L$. Define:

$$X = \{ e = (u, v) \in F_L : u \text{ and } v \text{ are incident to an edge from } H_L \cap F_I \} ,$$

$$x = |X|, k = |H_L \cap F_I| ;$$

Clearly, $(H_L \backslash F_I) \cup X$ is a matching of the graph $G \backslash F_I$, therefore, taking into account that $(H_L \backslash F_I) \cap X = \emptyset$, we deduce

$$l(G) = \nu(G \backslash F_I) \geq |H_L \backslash F_I| + |X| = |H_L| - |H_L \cap F_I| + |X| = L(G) - k + x.$$ 

Since $F_L$ is a perfect matching, it covers the set $V(\nu(H_L \cap F_I)) \backslash V(X)$, which contains

$$|V(\nu(H_L \cap F_I)) \backslash V(X)| = 2 |(H_L \cap F_I)| - 2 |X| = 2k - 2x$$

vertices. Define the set $E_{F_L}$ as follows:

$$E_{F_L} = \{ e \in F_L : e \text{ covers a vertex from } V(\nu(H_L \cap F_I)) \backslash V(X) \} .$$

Clearly, $E_{F_L}$ is a matching of $G \backslash F_I$, too, and therefore

$$l(G) = \nu(G \backslash F_I) \geq |E_{F_L}| = 2k - 2x.$$ 

Let us show that

$$\max \{ L(G) - k + x, 2k - 2x \} \geq \frac{2L(G)}{3} .$$

Note that

if $x \geq k - \frac{L(G)}{3}$ then $L(G) - k + x \geq L(G) - k + k - \frac{L(G)}{3} = \frac{2L(G)}{3}$;

if $x \leq k - \frac{L(G)}{3}$ then $2k - 2x \geq \frac{2L(G)}{3}$,

thus in both cases we have $l(G) \geq \frac{2L(G)}{3}$ or

$$\frac{L(G)}{l(G)} \leq \frac{3}{2} .$$

The proof of the theorem is completed. □

**Lemma 1** (Lemma 2.20, 2.41 of [4]) Let $G$ be a graph, and assume that $u$ and $v$ are vertices of degree one sharing a neighbour $w \in V(G)$. Then:

$$L(G) = L(G \backslash \{ u, v, w \}) + 1, l(G) = l(G \backslash \{ u, v, w \}) + 1.$$ 

**Corollary 1** Let $G$ be a graph with $L(G) = 2l(G)$. Then there are no vertices $u, v$ of degree one, that are adjacent to the same vertex $w$.

**Proof.** Suppose not. Then lemma 1 and (b) of theorem imply

$$L(G) = 1 + L(G - \{ u, v, w \}) \leq 1 + 2l(G - \{ u, v, w \}) =$$

$$= 1 + 2(l(G) - 1) = 2l(G) - 1 < 2l(G)$$

a contradiction. □
3. Characterization of graphs $G$ satisfying $L(G) = 2l(G)$

Let $T$ be the set of all triangles of $G$ that contain at least two vertices of degree two. Note that any vertex of degree two lies in at most one triangle from $T$. From each triangle $t \in T$ choose a vertex $v_t$ of degree two, and define $V_1(G)$ as follows:

$$V_1(G) = \{v : d_G(v) = 1\} \cup \{v_t : t \in T\}$$

**Theorem 3** Let $G$ be a connected graph with $|V(G)| \geq 3$. Then $L(G) = 2l(G)$ if and only if

1. $G \setminus V_1(G)$ is a bipartite graph with a bipartition $(X,Y)$;

2. $|V_1(G)| = |Y|$ and any $y \in Y$ has exactly one neighbour in $V_1(G)$;

3. the graph $G \setminus V_1(G)$ contains $|X|$ vertex disjoint 2-paths.

**Proof.** Sufficiency. Let $G$ be a connected graph with $|V(G)| \geq 3$ satisfying the conditions (1)-(3). Let us show that $L(G) = 2l(G)$.

For each vertex $v$ with $d(v) = 1$ take the edge incident to it and define $F_1$ as the union of all these edges. For each vertex $v_t \in V_1(G)$ take the edge that connects $v_t$ to a vertex of degree two, and define $F_2$ as the union of all those edges. Set:

$$F = F_1 \cup F_2.$$  

Note that $F$ is a matching with $|F| = |V_1(G)| = |Y|$. Moreover, since $G$ is bipartite and $|V_1(G)| = |Y|$, the definitions of $F_1$ and $F_2$ imply that there is no $F$-augmenting path in $G$. Thus, by Berge theorem, $F$ is a maximum matching of $G$, and

$$\nu(G) = |F| = |V_1(G)| = |Y|.$$  

Observe that the graph $G \setminus F$ is a bipartite graph with $\nu(G \setminus F) \leq |X|$, thus

$$l(G) \leq \nu(G \setminus F) \leq |X|.$$  

Now, consider the $|X|$ vertex disjoint 2-paths of the graph $G \setminus V_1(G)$ guaranteed by (3). (2) implies that these 2-paths together with the $|F| = |V_1(G)| = |Y|$ edges of $F$ form $|X|$ vertex disjoint 4-paths of the graph $G$.

Consider matchings $M_1$ and $M_2$ of $G$ obtained from these 4-paths by adding the first and the third, the second and the fourth edges of these 4-paths to $M_1$ and $M_2$, respectively. Define:

$$F' = (F \setminus M_2) \cup (M_1 \setminus F).$$  

Note that $F'$ is a matching of $G$ and $|F'| = |F|$, thus $F'$ is a maximum matching of $G$. Since $F' \cap M_2 = \emptyset$, we have

$$L(G) \geq \nu(G \setminus F') \geq |M_2| = 2|X| \geq 2l(G).$$

(b) of theorem 2 implies that $L(G) = 2l(G)$.

Necessity. Now, assume that $G$ is a connected graph with $|V(G)| \geq 3$ and $L(G) = 2l(G)$. By proving a series of claims, we show that $G \setminus V_1(G)$ satisfies the conditions (1)-(3) of the theorem.
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Claim 1 For any maximum matchings $F_L, F_i$ of the graph $G$ with $\nu(G\setminus F_L) = L(G), \nu(G\setminus F_i) = l(G)$, $F_L \cup F_i$ induces a spanning subgraph, that is $V(F_L) \cup V(F_i) = V(G)$.

Proof. Suppose that there is a vertex $v \in V(G)$ that is covered neither by $F_L$ nor by $F_i$. Since $F_L$ and $F_i$ are maximum matchings of $G$, for each edge $e = (u,v)$ the vertex $u$ is incident to an edge from $F_L$ and to an edge from $F_i$.

Case 1: there is an edge $e = (u,v)$ such that $u$ is incident to an edge from $F_L \cap F_i$.

Note that $\{e\} \cup (F_L \setminus F_i)$ is a matching of $G \setminus F_i$ which contradicts (c3) of the theorem \[2\] in the latter case, if $u / \in F_i$ and to an edge $f_i \in F_i \setminus F_L$.

Let $H_L$ be any maximum matching of $G \setminus F_L$. Due to (c1) of theorem \[2\] $f_i \in H_L$. Define:

$$H'_L = (H_L \setminus \{f_i\}) \cup \{e\}.$$ 

Note that $H'_L$ is a maximum matching of $G \setminus F_L$ such that $F_i \setminus F_L$ is not a subset of $H'_L$ contradicting (c1) of theorem \[2\]. □

Claim 2 For any maximum matchings $F_L, F_i$ of the graph $G$ with $\nu(G\setminus F_L) = L(G), \nu(G\setminus F_i) = l(G)$, the alternating components $F_L \Delta F_i$ are 2-paths.

Proof. It suffices to show that there is no edge $f_L \in F_L$ that is adjacent to two edges from $F_i$. Suppose that some edge $f_L \in F_L$ is adjacent to edges $f'_i, f''_i$ from $F_i$. Let $H_L$ be any maximum matching of $G \setminus F_L$. Due to (c1) of theorem \[2\] $f'_i, f''_i \in H_L$. This implies that $\{f_L\} \cup (H_L \setminus F_i)$ is a matching of $G \setminus F_i$ which contradicts (c2) of theorem \[2\]. □

Claim 3 For any maximum matchings $F_L, F_i$ of the graph $G$ with $\nu(G\setminus F_L) = L(G), \nu(G\setminus F_i) = l(G)$

(a) if $u \in V(F_i) \setminus V(F_L)$ then $d(u) = 1$ or $d(u) = 2$. Moreover, in the latter case, if $v$ and $w$ denote the two neighbours of $u$, where $(u,w) \in F_i$, then $d(w) = 2$ and $(v,w) \in F_L$.

(b) if $u \in V(F_L) \setminus V(F_i)$ then $d(u) \geq 2$.

Proof. (a) Assume that $u$ is covered by an edge $e_i \in F_i$ and $u \notin V(F_L)$. Suppose that $d(u) \geq 2$, and there is an edge $e = (u,v)$ such that $e \notin F_i$. Taking into account the claim \[1\] we need only to consider the following four cases:

Case 1: $v \in V(F_i) \setminus V(F_L)$.

This is impossible, since $F_L$ is a maximum matching.

Case 2: $v$ is covered by an edge $f \in F_L \cap F_i$;

Let $H_L$ be any maximum matching of $G \setminus F_L$. Due to (c1) of theorem \[2\] $e_i \in H_L$, thus $e \notin H_L$.

Define:

$$F'_L = (F_L \setminus \{f\}) \cup \{e\}.$$ 

Note that $F'_L$ is a maximum matching, and $H_L$ is a matching of $G \setminus F'_L$. Moreover,

$$\nu(G\setminus F'_L) \geq |H_L| = \nu(G\setminus F_L) = L(G),$$
thus $H_L$ is a maximum matching of $G\setminus F'_L$ and $\nu(G\setminus F'_L) = L(G)$. This is a contradiction because $F'_d \triangle F_l$ contains a component which is not a 2-path contradicting claim 2.

Case 3: $v$ is incident to an edge $f_L \in F_L$, $f_l \in F_l$ and $f_L \neq f_l$.

Let $H_L$ be any maximum matching of $G\setminus F_L$. Due to (c1) of theorem 2, $e_l, f_l \in H_L$. Define:

$$F'_L = (F_L \setminus \{f_L\}) \cup \{e\}.$$ 

Note that $F'_L$ is a maximum matching, and $H_L$ is a matching of $G\setminus F'_L$. Moreover,

$$\nu(G\setminus F'_L) \geq |H_L| = \nu(G\setminus F_L) = L(G),$$

thus $H_L$ is a maximum matching of $G\setminus F'_L$ and $\nu(G\setminus F'_L) = L(G)$. This is a contradiction because $F'_d \triangle F_l$ contains a component which is not a 2-path contradicting claim 2.

Case 4: $v$ is covered by an edge $e_L \in F_L$ and $v \notin V(F_l)$.

Note that if $e_L$ is not adjacent to $e_l$ then the edges $e, e_L$ and the edge $\tilde{e} \in F_l \setminus F_L$ that is adjacent to $e_L$ would form an augmenting 3-path with respect to $F_L$, which would contradict the maximality of $F_L$.

Thus it remains to consider the case when $e_L$ is adjacent to $e_l$ and $d(u) = 2$. Let $w$ be the vertex adjacent to both $e_l$ and $e_L$. Let us show that $d(w) = 2$. Let $H_L$ be any maximum matching of $G\setminus F_L$. Due to (c1) of theorem 2 $e_l \in H_L$. Define:

$$F'_L = (F_L \setminus \{e_L\}) \cup \{e\}.$$ 

Note that $F'_L$ is a maximum matching, and $H_L$ is a matching of $G\setminus F'_L$. Moreover,

$$\nu(G\setminus F'_L) \geq |H_L| = \nu(G\setminus F_L) = L(G),$$

thus $H_L$ is a maximum matching of $G\setminus F'_L$ and $\nu(G\setminus F'_L) = L(G)$. Since $d(w) \geq 3$ there is a vertex $w' \neq u, v$ such that $(w, w') \in E(G)$ and $w'$ satisfies one of the conditions of cases 1, 2 and 3 with respect to $F'_L$ and $F_l$. A contradiction. Thus $d(w) = 2$.

Clearly, $(v, w) = e_L \in F_L$.

(b) This follows from (a) of claim 3 and corollary 1. □

**Claim 4** Let $F_L, F_l$ be any maximum matchings of the graph $G$ with $\nu(G\setminus F_L) = L(G)$, $\nu(G\setminus F_l) = l(G)$. Then for any maximum matching $H_L$ of the graph $G\setminus F_L$ there is no edge of $F_L \cap F_l$ which is adjacent to two edges from $H_L$.

**Proof.** Due to (c3) of theorem 2 any edge from $H_L$ that is incident to a vertex covered by an edge of $F_L \cap F_l$ is also incident to a vertex from $V(F_L) \setminus V(F_l)$. If there were an edge $e \in F_L \cap F_l$ which is adjacent to two edges $h_L, h'_l \in H_L$, then due to (c1) of theorem 2 and (a) of claim 3 we would have an augmenting 7-path with respect to $F_L$, which would contradict the maximality of $F_L$. □

**Claim 5** (1) for any maximum matchings $F_L, F_l$ of the graph $G$ with $\nu(G\setminus F_L) = L(G)$, $\nu(G\setminus F_l) = l(G)$, we have $(V(F_L) \setminus V(F_l)) \cap V_f(G) = \emptyset$;
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(2) there is a maximum matching \( F_1 \) of \( G \) with \( \nu(G\setminus F_1) = l(G) \) and a maximum matching \( F_L \) of the graph \( G \) with \( \nu(G\setminus F_L) = L(G) \), such that \( V_1(G) \subseteq V(F_L \cap F_1) \cup (V(F_1)\setminus V(F_L)) \).

Proof. (1) On the opposite assumption, consider a vertex \( x \in (V(F_L)\setminus V(F_1)) \cap V_1(G) \). Since \( x \in V_1(G) \) then \( d(x) \leq 2 \). On the other hand, (b) of claim \( \text{[3]} \) implies that \( d(x) \geq 2 \), thus \( d(x) = 2 \). Then there are vertices \( y,z \) such that \( (x,y) \in F_L \) and \( (z,y) \in F_1 \). Note that due to (a) of claim \( \text{[3]} \) we have \( d(y) \leq 2 \). Let us show that \( d(y) = 1 \). Suppose that \( d(y) = 2 \). Then due to (a) of claim \( \text{[3]} \) we have that \( d(z) = 2 \), thus \( G \) is the triangle, which is a contradiction, since \( G \) does not satisfy \( L(G) = 2l(G) \).

Thus \( d(y) = 1 \). Since \( x \in V_1(G) \), we imply that there is a vertex \( w \) with \( d(w) = 2 \) such that \( w,x,z \) form a triangle. Note that \( w \) is covered neither by \( F_L \) nor by \( F_1 \), which contradicts claim \( \text{[3]} \).

(2) Let \( e_t \) be an edge of a triangle \( t \in T \) connecting the vertex \( v_t \in V_1(G) \) to a vertex of degree two. Let us show that there is a maximum matching \( F_t \) of \( G \) with \( \nu(G\setminus F_t) = l(G) \) such that \( e_t \in F_t \) for each \( t \in T \).

Choose a maximum matching \( F_t \) of \( G \) with \( \nu(G\setminus F_t) = l(G) \) that contains as many edges \( e_t \) as possible. Let us show that \( F_t \) contains all edges \( e_t \). Suppose that there is \( t_0 \in T \) such that \( e_{t_0} \notin F_t \). Define:

\[
F'_t = (F_t \setminus \{e\}) \cup \{e_{t_0}\},
\]

where \( e \) is the edge of \( F_t \) that is adjacent to \( e_{t_0} \). Note that

\[
\nu(G\setminus F'_t) \leq \nu(G\setminus F_t) = l(G),
\]

thus \( F'_t \) is a maximum matching of \( G \) with \( \nu(G\setminus F_t) = l(G) \). Note that \( F'_t \) contains more edges \( e_t \) than does \( F_t \) which contradicts the choice of \( F_t \).

Thus, there is a maximum matching \( F_t \) of \( G \) with \( \nu(G\setminus F_t) = l(G) \) such that \( e_t \in F_t \) for all \( t \in T \). Now, for this maximum matching \( F_t \) of \( G \) choose a maximum matching \( F_L \) of the graph \( G \) with \( \nu(G\setminus F_L) = L(G) \), such that \( V(F_L \cap F_t) \cup (V(F_t)\setminus V(F_L)) \) covers maximum number of vertices from \( V_1(G) \). Let us show that \( V(F_1) \subseteq V(F_L \cap F_t) \cup (V(F_t)\setminus V(F_L)) \).

Suppose that there is a vertex \( x \in V_1(G) \) such that \( x \notin V(F_L \cap F_t) \cup (V(F_t)\setminus V(F_L)) \). Note that due to claim \( \text{[1]} \) and (b) of claim \( \text{[3]} \) any vertex of degree one is either incident to an edge from \( F_L \cap F_t \) or to an edge \( V(F_t)\setminus V(F_L) \). Thus due to definition of \( V_1(G) \), \( d(x) = 2 \) and if \( y \) and \( z \) denote the two neighbors of \( x \), then \( d(y) = 2 \) and \( (y,z) \in E(G) \).

Since \( x \notin V(F_L \cap F_t) \), we have that \( (x,y) \notin F_L \), and since \( x \notin (V(F_t)\setminus V(F_L)) \), we have that \( (y,z) \notin F_L \), thus \( (x,z) \in F_L \), as \( F_t \) is a maximum matching. Let \( H_L \) be any maximum matching of \( G\setminus F_L \). As \( L(G) = 2l(G) \), we have \( (x,y) \in H_L \) ((c1) of theorem \( \text{[2]} \)). Define:

\[
F'_L = (F_L \setminus \{(x,z)\}) \cup \{(y,z)\}.
\]

Note that \( F'_L \) is a maximum matching of \( G \), \( H_L \) is a matching of \( G\setminus F_L \), thus

\[
\nu(G\setminus F'_L) \geq |H_L| = \nu(G\setminus F_L) = L(G).
\]

Therefore \( F'_L \) is a maximum matching of \( G \) with \( \nu(G\setminus F'_L) = L(G) \). Now, observe that \( V(F'_L \cap F_t) \cup (V(F_t)\setminus V(F'_L)) \) covers more vertices than does \( V(F_L \cap F_t) \cup (V(F_t)\setminus V(F_L)) \) which contradicts the choice of \( F_L \). The proof of the claim \( \text{[3]} \) is completed. \(\square\)
Claim 6. For any maximum matchings \( F_L, F_i \) of the graph \( G \) with \( \nu(G \setminus F_L) = L(G) \), \( \nu(G \setminus F_i) = l(G) \), we have

1. \( V(F_L) \setminus V(F_i) \) is an independent set;
2. no edge of \( G \) connects two vertices that are covered by both \( F_L \setminus F_i \) and \( F_i \setminus F_L \);
3. no edge of \( G \) is adjacent to two different edges from \( F_L \cap F_i \);
4. no edge of \( G \) connects a vertex covered by \( F_L \cap F_i \) to a vertex covered by both \( F_L \setminus F_i \) and \( F_i \setminus F_L \);
5. if \( (u, v) \in F_L \cap F_i \) then either \( u \in V_i(G) \) or \( v \in V_i(G) \).

Proof. (1) There is no edge of \( G \) connecting two vertices from \( V(F_L) \setminus V(F_i) \) since \( F_i \) is a maximum matching.

(2) follows from (c1) and (c2) of theorem 2.

(3) follows from (c3) of theorem 2.

(4) Suppose that there is an edge \( e = (y_1, y_2) \), such that \( y_1 \) is covered by \( F_L \cap F_i \) and \( y_2 \) is covered by both \( F_L \setminus F_i \) and \( F_i \setminus F_L \). Consider a maximum matching \( H_L \) of the graph \( G \setminus F_L \). Note that \( y_1 \) must be incident to an edge from \( H_L \), as otherwise we could replace the edge of \( H_L \) that is adjacent to \( e \) and belongs also to \( F_i \setminus F_L \) ((c1) of theorem 2) by the edge \( e \) to obtain a new maximum matching \( H'_L \) of the graph \( G \setminus F_L \) which would not satisfy (c1) of theorem 2.

So let \( y_1 \) be incident to an edge \( h_L \in H_L \), which connects \( y_1 \) with a vertex \( x \in V(F_L) \setminus V(F_i) \). Note that due to claim 4 (c1) of theorem 2 and (a) of claim 3 the edge \( h_L \) lies on an \( H_L - F_L \) alternating 4-path \( P \). Define:

\[
F'_L = (F_L \setminus E(P)) \cup (H_L \cap E(P)),
\]
\[
H'_L = (H_L \setminus E(P)) \cup (F_L \cap E(P)).
\]

Note that \( F'_L \) is a maximum matching of \( G \), \( H'_L \) is a matching of \( G \setminus F'_L \) of cardinality \( |H_L| \), and

\[\nu(G \setminus F'_L) \geq |H'_L| = |H_L| = \nu(G \setminus F_L) = L(G),\]

thus \( H'_L \) is a maximum matching of \( G \setminus F'_L \) and \( \nu(G \setminus F'_L) = L(G) \). This is a contradiction since the edge \( e \) connects two vertices which are covered by \( F'_L \setminus F_i \) and \( F_i \setminus F'_L \) ((2) of claim 4).

(5) Suppose that \( e = (u, v) \in F_L \cap F_i \). Since \( G \) is connected and \( |V| \geq 3 \), we, without loss of generality, may assume that \( d(v) \geq 2 \), and there is \( w \in V(G) \), \( w \neq u \) such that \( (w, v) \in E(G) \). Consider a maximum matching \( H_L \) of the graph \( G \setminus F_L \). Note that \( v \) must be incident to an edge from \( H_L \), as otherwise we could replace the edge of \( H_L \) that is incident to \( w \) \((H_L \) is a maximum matching of \( G \setminus F_L \)) by the edge \( (w, v) \) to obtain a new maximum matching \( H'_L \) of the graph \( G \setminus F_L \) such that \( v \) is incident to an edge from \( H'_L \).

So we may assume that there is an edge \( (v, q) \in H_L \), \( q \neq u \). Note that due to claim 4 (c1) of theorem 2 and (a) of claim 3 the edge \( (q, w) \) lies on an \( H_L - F_L \) alternating 4-path.
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P. Define:

\[ F'_L = (F_L \setminus E(P)) \cup (H_L \cap E(P)), \]
\[ H'_L = (H_L \setminus E(P)) \cup (F_L \cap E(P)). \]

Note that \( F'_L \) is a maximum matching of \( G \), \( H'_L \) is a matching of \( G \setminus F'_L \) of cardinality \( |H_L| \), and

\[ \nu(G \setminus F'_L) \geq |H'_L| = |H_L| = \nu(G \setminus F_L) = L(G), \]

thus \( H'_L \) is a maximum matching of \( G \setminus F'_L \) and \( \nu(G \setminus F'_L) = L(G) \). Since \( u \in V(F_i) \setminus V(F'_L) \) (a) of claim 3 implies that either \( d(u) = 1 \) and therefore \( u \in V_i(G) \), or \( d(u) = d(v) = 2 \) and therefore either \( u \in V_i(G) \) or \( v \in V_i(G) \). Proof of the claim is completed. □

We are ready to complete the proof of the theorem. Take any maximum matchings \( F_L, F_i \) of the graph \( G \) guaranteed by the (2) of claim and consider the following partition of \( V(G \setminus V_i(G)) = V(G) \setminus V_i(G) \):

\[ X = X(F_L, F_i) = V(F_L) \setminus V(F_i), Y = Y(F_L, F_i) = V(G) \setminus (V_i(G) \cup X). \]

Claim 6 implies that \( X \) and \( Y \) are independent sets of vertices of \( G \setminus V_i(G) \), thus \( G \setminus V_i(G) \) is a bipartite graph with a bipartition \((X, Y)\).

The choice of maximum matchings \( F_L, F_i \), (a) of claim 3, (5) of claim 6 and the definition of the set \( Y \) imply (2) of the theorem.

Let us show that it satisfies (3), too.

Consider the alternating 2-paths of

\[ (H_L \setminus F_i) \Delta (F_L \setminus F_i). \]

(c2), (c3) of theorem and the definition of the set \( X \) imply that there are \(|X|\) such 2-paths. Moreover, these 2-paths are in fact 2-paths of the graph \( G \setminus V_i(G) \). Thus \( G \) satisfies (3) of the theorem. The proof of the theorem 3 is completed. □

Corollary 2 The property of a graph \( L(G) = 2l(G) \) can be tested in polynomial time.

Proof. First of all note that the property \( L(G) = 2l(G) \) is additive, that is, a graph satisfies this property if and only if all its connected components do. Thus we can concentrate only on connected graphs.

All connected graphs with \(|V(G)| \leq 2\) satisfy the equality \( L(G) = 2l(G) \), thus we can assume that \(|V(G)| \geq 3\).

Next, we construct a set \( V_i(G) \), which can be done in linear time. Now, we need to check whether the graph \( G \setminus V_i(G) \) satisfies the conditions (1)-(3) of the theorem.

It is well-known that the properties (1) and (2) can be checked in polynomial time, so we will consider only the testing of (3).

From a graph \( G \setminus V_i(G) \) with a bipartition \((X, Y)\) we construct a network \( \tilde{G} \) with new vertices \( s \) and \( t \). The arcs of \( \tilde{G} \) are defined as follows:

- connect \( s \) to every vertex of \( X \) with an arc of capacity 2;
• connect every vertex of $Y$ to $t$ by an arc of capacity 1;
• for every edge $(x, y) \in E(G)$, $x \in X$, $y \in Y$ add an arc connecting the vertex $x$ to the vertex $y$ which has capacity 1.

Note that
• the value of the maximum $s-t$ flow in $\vec{G}$ is no more than $2 |X|$ (the capacity of the cut $(S, \bar{S})$, where $S = \{s\}$, $\bar{S} = V(\vec{G}) \setminus S$, is $2 |X|$);
• the value of the maximum $s-t$ flow in $\vec{G}$ is $2 |X|$ if and only if the graph $G \setminus V_1(G)$ contains $|X|$ vertex disjoint 2-paths, thus (3) also can be tested in polynomial time. □

**Remark 1** Recently Monnot and Toulouse in [7] proved that 2-path partition problem remains $NP$-complete even for bipartite graphs of maximum degree three. Fortunately, in theorem 3 we are dealing with a special case of this problem which enables us to present a polynomial algorithm in corollary 2.

4. $NP$-completeness of testing $L(G) = \frac{3}{2}l(G)$ in the class of bridgeless cubic graphs

The reader may think that a result analogous to corollary 2 can be proved for the property $L(G) = \frac{3}{2}l(G)$ in the class of graphs containing a perfect matching. Unfortunately this fails already in the class of bridgeless cubic graphs, which by the well-known theorem of Petersen are known to possess a perfect matching.

**Theorem 4** It is $NP$-complete to test the property $L(G) = \frac{3}{2}l(G)$ in the class of bridgeless cubic graphs.

**Proof.** Clearly, the problem of testing the property $L(G) = \frac{3}{2}l(G)$ for graphs containing a perfect matching is in $NP$, since if we are given perfect matchings $F_L, F_i$ of the graph $G$ with $\nu(G \setminus F_L) = L(G), \nu(G \setminus F_i) = l(G)$ then we can calculate $L(G)$ and $l(G)$ in polynomial time.

We will use the well-known 3-edge-coloring problem ([3]) to establish the $NP$-completeness of our problem.

Let $G$ be a bridgeless cubic graph. Consider a bridgeless cubic graph $G_{\Delta}$ obtained from $G$ by replacing every vertex of $G$ by a triangle. We claim that $G$ is 3-edge-colorable if and only if $L(G_{\Delta}) = \frac{3}{2}l(G_{\Delta})$.

Suppose that $G$ is 3-edge-colorable. Then $G_{\Delta}$ is also 3-edge-colorable, which means that $G_{\Delta}$ contains two edge disjoint perfect matchings $F$ and $F'$. This implies that

$$L(G_{\Delta}) \geq \nu(G_{\Delta} \setminus F) \geq |F'| = \frac{|V(G_{\Delta})|}{2},$$

On the other hand, the set $E(G)$ forms a perfect matching of $G_{\Delta}$, and

$$l(G_{\Delta}) \leq \nu(G_{\Delta} \setminus E(G)) = \frac{|V(G_{\Delta})|}{3},$$

Thus $L(G_{\Delta}) = \frac{3}{2}l(G_{\Delta})$.

Let's prove the reverse direction. Suppose $L(G_{\Delta}) = \frac{3}{2}l(G_{\Delta})$. Then $G_{\Delta}$ is 3-edge-colorable, which means that $G_{\Delta}$ contains two edge disjoint perfect matchings $F$ and $F'$. This implies that

$$L(G_{\Delta}) \geq \nu(G_{\Delta} \setminus F) \geq |F'| = \frac{|V(G_{\Delta})|}{2},$$

On the other hand, the set $E(G)$ forms a perfect matching of $G_{\Delta}$, and

$$l(G_{\Delta}) \leq \nu(G_{\Delta} \setminus E(G)) = \frac{|V(G_{\Delta})|}{3},$$

Thus $L(G_{\Delta}) = \frac{3}{2}l(G_{\Delta})$.

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since every component of $G \setminus E(G)$ is a triangle. Thus:

\[
\frac{L(G_\Delta)}{l(G_\Delta)} \geq \frac{3}{2},
\]

(d) of theorem 2 implies that $\frac{L(G_\Delta)}{l(G_\Delta)} = \frac{3}{2}$.

Now assume that $\frac{L(G_\Delta)}{l(G_\Delta)} = \frac{3}{2}$. Note that for every perfect matching $F$ of the graph $G_\Delta$ the graph $G_\Delta \setminus F$ is a 2-factor, therefore

\[
L(G_\Delta) = \frac{|V(G_\Delta)| - w(G_\Delta)}{2},
\]
\[
l(G_\Delta) = \frac{|V(G_\Delta)| - W(G_\Delta)}{2}
\]

where $w(G_\Delta)$ and $W(G_\Delta)$ denote the minimum and maximum number of odd cycles in a 2-factor of $G_\Delta$, respectively. Since $\frac{L(G_\Delta)}{l(G_\Delta)} = \frac{3}{2}$ we have

\[
W(G_\Delta) = \frac{|V(G_\Delta)| + 2w(G_\Delta)}{3}.
\]

Taking into account that $W(G_\Delta) \leq \frac{|V(G_\Delta)|}{3}$, we have:

\[
W(G_\Delta) = \frac{|V(G_\Delta)|}{3},
\]
\[
w(G_\Delta) = 0.
\]

Note that $w(G_\Delta) = 0$ means that $G_\Delta$ is 3-edge-colorable, which in its turn implies that $G$ is 3-edge-colorable. The proof of the theorem is completed. □

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