Star products, duality and double Lie algebras

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Abstract

Quantization of classical systems using the star-product of symbols of observables is discussed. In the star-product scheme an analysis of dual structures is performed and a physical interpretation is proposed. At the Lie algebra level duality is shown to be connected to double Lie algebras. The analysis is specified to quantum tomography. The classical tomographic Poisson bracket is found.

Keywords: star-product, integral kernel, Planck’s constant, Poisson bracket, quantum tomography.

1 Introduction

The transition from quantum to classical mechanics has been an important research subject since the beginning of quantum mechanics (see [1, 2] for a review). A suitable setting for this problem is represented by the Wigner-Weyl-Moyal formalism where the operators corresponding to observables and the states, considered as linear functionals on the space of observables, are mapped onto functions on a suitable manifold. Such a representation for quantum mechanics has been later generalized yielding to the deformation quantization program [3]. There the operator noncommutativity is implemented by a noncommutative (star) product which is a generalization of the Moyal product [4, 5, 6]. Since then, most attention to the star-product quantization scheme has been devoted to the case where the functions (symbols of the operators) are defined on the “classical” phase space of the system [7, 8, 9, 10]. But this is not the only possibility. An interesting example which cannot be described in terms of a deformation of the commutative product of classical phase space is the so called quantum tomography. In this setting quantum states are mapped onto probability distributions which depend on a random variable $X$ representing the position in classical phase–space [11] and two additional real parameters labelling different reference frames in the phase space. This approach provides a formulation of quantum mechanics where the states are described by positive probabilities as an alternative to wave functions and density states [12, 13]. In [14, 15, 16] it has been
shown that the symplectic tomography as well as other known types of tomographic methods for measuring quantum states like optical tomography [17, 18] and spin tomography [19, 20, 21] can be formulated in a star-product approach once the star-product quantization scheme is suitably reinterpreted. This new presentation led recently [22] to find and elucidate a specific duality symmetry of the star-product quantization which has proved to be useful to clarify well known relations among important distribution functions of quantum mechanics such as the Wigner function [23], the Husimi Q-function [24], the Sudarshan–Glauber P-function [25, 26] and s-ordered quasidistributions [27], as well as relations among time–frequency quasidistributions known in signal analysis [28, 29]. Also the duality symmetry provides a tool to find new solutions to the nonlinear associativity equation for the star-product kernel if one solution of the equation is available.

The formulation of the star-product quantization scheme proposed in [14, 15, 22] is based on the properties of two sets of operators which we refer as quantizers and dequantizers, respectively. They are in fact operator-valued functions on a manifold and behave like elements of dual vector spaces. In the Weyl–Moyal star-product quantization the quantizer and the dequantizer coincide up to a constant (implying that the Moyal product is self-dual) and the manifold is the phase-space of the system. Moreover, in the limit \( \hbar \to 0 \) the commutator correctly reproduces the canonical Poisson bracket for the classical phase space. This should be possible for other quantization schemes as well; namely, knowing the dependence of the star product kernel on the deformation parameters one should recover the classical Poisson bracket in the appropriate limit. Till now for the symplectic tomographic approach [30, 31] the classical Poisson structure was not obtained in explicit form.

In this paper we focus our analysis on the duality relation between quantizers and dequantizers. As we shall see, this allows to introduce dual Lie algebra structures on the symbols, which we will investigate in some detail. Moreover, we take the occasion to clarify the physical meaning of dual quantization schemes and specifically of the dual tomographic scheme, as the one encoding the information about quantum observables, as opposed to tomograms which describe quantum states. We then analyze the classical limit in the tomographic representation. We obtain the Poisson structure for classical symplectic tomograms associated to classical observables as a suitable limit of the quantum tomography. As a related subject we discuss the nonlinearity property of the dual tomographic star-product kernel with respect to the quantizer operator. This influences the uniqueness of the corresponding Poisson structure.

The paper is organized as follows.

In Section 2 we review the star-product quantization scheme following [14, 15], together with a class of star products (\( \mathcal{K} \)-star products) which is obtained via a specific deformation procedure [15]. In Section 3 we deepen the analysis of the duality symmetry started in [22] and work out explicit examples. In section 4, starting from the observation that the anti-symmetrized star product kernel, for a discrete space of parameters, may be interpreted as the structure constants of a Lie algebra we discuss a class of solutions of the Jacobi identity. In section 5, upon reviewing the symplectic tomography scheme, we analyze the classical limit of the tomographic quantum Poisson brackets. Then we investigate the form of the integral kernel of the Moyal product in Fourier representation. As in the tomographic case, we find that the star product kernel differs from the point-wise one by a twist factor which is the exponential of the symplectic area. Finally
we summarize our results.

2 The star product formalism

Following [14, 15, 16, 22] let us consider a given manifold $M$, with coordinates $\vec{x} = (x_1, x_2, \ldots, x_N)$ and two dual sets of operators $\hat{D}(\vec{x})$ and $\hat{U}(\vec{x})$, which generate, as a continuous basis, the vector space $V$ and its dual $V^*$. The coordinates may be discrete as well. These are vector spaces of operators acting on a given Hilbert space. We call the operators $\hat{D}(\vec{x})$ quantizers and the operators $\hat{U}(\vec{x})$ dequantizers due to the following reasons. On $V^* \oplus V$, by exploiting the duality relation, it is possible to define a scalar product $<$ $|$ $>$

\[
\begin{align*}
<(a,0)|(0,b)> & = a(b) \\
<(a,0)|(c,0)> & = <(0,b)|(0,c)> = 0
\end{align*}
\]

where $(a,b) \in V^* \oplus V$. Being vector spaces of linear maps, they both carry an associative product and therefore a Lie algebra structure. Thus, $V$ and $V^*$ are assumed to both carry a Lie algebra structure coming from the associative product. By requiring that the product be adjoint invariant we may define a Lie algebra structure on $V^* \oplus V$. Therefore our considerations are closely related to what are known as metrical Lie algebras. Having a pairing between $V$ and $V^*$ we can associate with an operator $\hat{A}$ in $V$ a function $f_A(\vec{x})$ in the following way. The operator $\hat{A} \equiv (0, \hat{A})$ may be written in terms of the basis in $V$ as

\[
\hat{A} = \int d\vec{x} \ a(\vec{x}) \hat{D}(\vec{x})
\]

with

\[
a(\vec{x}) \equiv f_A(\vec{x}) = <(\hat{U}(\vec{x}),0)|(0, \hat{A})> = \text{Tr} \hat{U}^\dagger(\vec{x}) \hat{A}
\]

which may be rewritten as

\[
f_A(\vec{x}) = \text{Tr} \hat{U}^\dagger(\vec{x}) \hat{A}
\]

and the trace stays for the appropriate scalar product. Replacing (2) into (4) we get a compatibility condition between $\hat{U}(\vec{x})$ and $\hat{D}(\vec{x})$ which is just the duality relation between the two bases of $V$ and $V^*$

\[
<(\hat{U}(\vec{x}),0)|(0, \hat{D}(\vec{x}'))> = \text{Tr} \hat{U}^\dagger(\vec{x}) \hat{D}(\vec{x'}) = \delta(\vec{x} - \vec{x'})
\]

where $\delta(\vec{x} - \vec{x'})$ is a Dirac delta-function for the continuous variables, a Kronecker delta for discrete variables. So the operators $\hat{U}(\vec{x})$ associate, through the scalar product, to the operator $\hat{A}$ a function, i.e. they dequantize the quantum observable, while the role of the operators $\hat{D}(\vec{x})$ is opposite. They furnish a basis for the expansion of $\hat{A}$ with coefficients $f_A(\vec{x})$ ($\vec{x}$) plays the role of a continuous index. This means that they associate to the function $f_A(\vec{x})$ (classical observable) the operator $\hat{A}$ (quantum observable), i.e. they “quantize” the function. This association should remind us of the momentum map associated with a Lie algebra $\mathcal{G}$ acting on the cotangent bundle $T^*\mathcal{G} = \mathcal{G}^* \oplus \mathcal{G}$. On the corresponding functions we define a $*$-product instead of a point-wise product.

The associative star-product of symbols $f_A(\vec{x}), f_B(\vec{x})$ associated with the operators $\hat{A}$ and $\hat{B}$ is defined as

\[
\hat{A} \hat{B} \leftrightarrow f_A(\vec{x}) \ast f_B(\vec{x}) := f_{AB}(\vec{x}).
\]
This product is associative since the product of the operators is associative, i.e.,

\[(\hat{A}\hat{B})\hat{C} = \hat{A}(\hat{B}\hat{C}) \rightarrow (f_A(\vec{x}) \ast f_B(\vec{x})) \ast f_C(\vec{x}) = f_A(\vec{x}) \ast (f_B(\vec{x}) \ast f_C(\vec{x})).\] (7)

Moreover it is nonlocal and it can be described through an integral kernel

\[f_A(\vec{x}) \ast f_B(\vec{x}) = \int_{M \times M} K(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) f_A(\vec{x}_1) f_B(\vec{x}_2) d\vec{x}_1 d\vec{x}_2\] (8)

which plays the role of the structure function for the product. Using Eqs. (4),(2) we can get the expression for the kernel in the form

\[K(\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4) = \langle \hat{U}(\vec{x}), 0 \rangle |(0, \hat{D}(\vec{x}_1)\hat{D}(\vec{x}_2)) >= \text{Tr}(\hat{U}(\vec{x})\hat{D}(\vec{x}_1)\hat{D}(\vec{x}_2)).\] (9)

We can see that this expression is linear with respect to \(\hat{U}(\vec{x})\) and quadratic with respect to \(\hat{D}(\vec{x})\). From this point of view there is an asymmetry in the kernel with respect to quantizers and dequantizers.\(^1\) The associativity condition for operator symbols implies that the kernel \(K(\vec{x}_1, \vec{x}_2, \vec{x})\) satisfies the nonlinear equation

\[\int K(\vec{x}_1, \vec{x}_2, \vec{y}) K(\vec{y}, \vec{x}_3, \vec{x}_4) d\vec{y} = \int K(\vec{x}_1, \vec{y}, \vec{x}_3, \vec{x}_4) K(\vec{x}_2, \vec{x}_3, \vec{y}) d\vec{y}.\] (10)

In connection with our previous comment on metrical Lie algebras we might say that we are dealing here with metrical Lie algebras realized via associative algebras of operators or associative algebras of functions via the star product formalism.

Let us consider as an example the two real matrix algebras

\[a = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}\] (11)

and

\[b = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}\] (12)

They commute with each other and are dual with respect to the adjoint invariant product \(\gamma\), defined on the direct sum \(a \oplus b\)

\[\gamma(A_i, B_j) = \text{Tr}(A_i J B_j J)\] (13)

with \(i = 1, 2\) labelling the bases (11), (12) and

\[J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\] (14)

Their direct sum is the Lie algebra \(\mathcal{GL}(2, \mathbb{R})\). The pairing (13) allows to associate to a given operator \(B\) in the vector space \(b\) a function, \(f_B\), which represents the symbol of \(B\) in the given scheme. Consider indeed

\[\hat{B} = x_1 B_1 + x_2 B_2\] (15)

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\(^1\)We warn the reader that in our previous articles on this subject, with a less precise notation, we used the symbol \(\hat{U}(\vec{x})\) instead than its adjoint.
then
\[
f_B(1) = \text{Tr}(A_1 JB J) = x_1
\]
\[
f_B(2) = \text{Tr}(A_2 JB J) = x_2
\]

thus implying that
\[
\hat{B} = f_B(1)B_1 + f_B(2)B_2
\]

2.1 Deformed star-products from deformed operator products

In [15] a deformed operator product was introduced (K-product) in the form
\[
\hat{A} \cdot_K \hat{B} = \hat{A} \hat{K} \hat{B}
\]
where \( \hat{K} \) is a generic operator. It satisfies the associativity condition
\[
\left( \hat{A} \cdot_K \hat{B} \right) \cdot_K \hat{C} = \hat{A} \cdot_K \left( \hat{B} \cdot_K \hat{C} \right).
\]

In matrix representation the deformed operator product provides a deformed matrix product of the form
\[
A \cdot_K B := AKB
\]

As in the undeformed case we can derive an expression for the kernel
\[
\left( f_A \ast_K f_B \right)(\vec{x}) = \int K^{(K)}(\vec{x}_1, \vec{x}_2, \vec{x}) f_A(\vec{x}_1) f_B(\vec{x}_2) d\vec{x}_1 d\vec{x}_2
\]

K-deformed products are a simple way to generate new associative products which deserves in our opinion further attention. They are a simple instance of deformed products associated with a Nijenhuis tensor [32]. In section 5 we give an example of such potentiality in the framework of metrical Lie algebras.
3 Duality symmetry of the star-product

In [22] a duality symmetry of the star-product scheme was found. Shortly, it originates from the observation that the role of the quantizer and the dequantizer can be exchanged because the compatibility condition (5) remains fulfilled. Then, one can introduce a new pair of operators \( \hat{U}'(\vec{x}) \) and \( \hat{D}'(\vec{x}) \)

\[
\hat{U}'(\vec{x}) = \hat{D}(\vec{x}), \quad \hat{D}'(\vec{x}) = \hat{U}(\vec{x}).
\]  

We say that new pair quantizer–dequantizer is dual to the initial one. The terminology is justified by the observation that

\[
(V \times V^*)^* \simeq V^* \times V^* \simeq V^* \times V
\]

when \( V^{**} = V \). This interchange corresponds to a specific symmetry of the equation for the associative star-product kernel. We have seen that the kernel \( K(\vec{x}_1, \vec{x}_2, \vec{x}) \) satisfies the associativity equation (10), a solution to which is represented by (9). But this equation admits also the dual solution

\[
K^{(d)}(\vec{x}_1, \vec{x}_2, \vec{x}) = \text{Tr}\left( \hat{U}(\vec{x}_1)\hat{U}(\vec{x}_2)\hat{D}^\dagger(\vec{x}) \right).
\]  

Solutions may be singled out by requiring particular symmetry properties; for example imposing translation invariance one finds that the only solution is the Moyal one [33].

The analogue of Eq. (10) when continuous indices are replaced by discrete N-dimensional ones reads [15]

\[
\sum_{m=1}^{N} K_{nm}^m K_{sk}^s = \sum_{m=1}^{N} K_{ms}^m K_{nk}^n
\]

The associativity equation (10) has a hidden symmetry, the scaling transform. Namely, given a solution, \( K(\vec{x}_1, \vec{x}_2, \vec{x}_3) \), of the equation (10) the new kernel

\[
K_\lambda(\vec{x}_1, \vec{x}_2, \vec{x}_3) = \lambda K(\vec{x}_1, \vec{x}_2, \vec{x}_3)
\]

is also a solution. The scaling transform of the kernel can be induced transforming the quantizer and dequantizer as

\[
\hat{U}(\vec{x}) \rightarrow \hat{U}_\lambda(\vec{x}) = \lambda \hat{U}(\vec{x}),
\]

\[
\hat{D}(\vec{x}) \rightarrow \hat{D}_\lambda(\vec{x}) = \lambda^{-1} \hat{D}(\vec{x})
\]

in such a way to preserve the metrical condition (5). The symmetry of Eq. (10) can be described by a transformation kernel. In fact, given a pair \( \left( \hat{U}(\vec{x}), \hat{D}(\vec{x}) \right) \) and another pair \( \left( \hat{U}_1(\vec{y}), \hat{D}_1(\vec{y}) \right) \) which provides a new symbol for a given operator \( \hat{A} \)

\[
f_A(\vec{y}) = \text{Tr}\left( \hat{U}_1(\vec{y})\hat{A} \right)
\]

and

\[
\hat{A} = \int f_A(\vec{y})\hat{D}_1(\vec{y})d\vec{y},
\]
it is possible to establish the relations

\[ f_A(\vec{y}) = \int f_A(\vec{x})K_1(\vec{x}, \vec{y})d\vec{x}, \quad (34) \]

and

\[ f_A(\vec{x}) = \int f_A(\vec{y})K_2(\vec{y}, \vec{x})d\vec{y} \quad (35) \]

with

\[
K_1(\vec{x}, \vec{y}) = \text{Tr} \left( \hat{D}(\vec{y}) \hat{D}(\vec{x}) \right),
\]

\[
K_2(\vec{y}, \vec{x}) = \text{Tr} \left( \hat{U}(\vec{y}) \hat{U}(\vec{x}) \right). \quad (36)
\]

For the dual pair \( \left( \hat{U}_1(\vec{x}) = \hat{D}(\vec{x}), \quad \hat{D}_1(\vec{x}) = \hat{U}(\vec{x}) \right) \) the dual symbol of an operator \( \hat{A} \) reads

\[ f^{(d)}_A(\vec{x}) = \text{Tr} \left( \hat{D}(\vec{x}) \hat{A} \right) \quad (37) \]

and the operator reconstruction formula is

\[ \hat{A} = \int \left( \text{Tr} \left( \hat{D}(\vec{x}) \hat{A} \right) \right) \hat{U}(\vec{x})d\vec{x}. \quad (38) \]

Then equations (36) imply that

\[ f^{(d)}_A(\vec{x}) = \int f_A(\vec{x}_1)(\text{Tr}(\hat{D}(\vec{x})\hat{D}(\vec{x}_1)))d\vec{x}_1. \quad (39) \]

The dual kernel of Eq. (27), corresponding to the transformed pair of quantizer and dequantizer, reflects the symmetry of equation (10). Thus, the generic formulae (36), (37) read, for the transition from the initial star-product to its dual one,

\[
K_1(\vec{x}, \vec{y}) = \text{Tr} \left( \hat{D}(\vec{y}) \hat{D}(\vec{x}) \right),
\]

\[
K_2(\vec{y}, \vec{x}) = \text{Tr} \left( \hat{U}(\vec{y}) \hat{U}(\vec{x}) \right). \quad (40)
\]

It is worth noticing that a dual may be introduced for the deformed star product as well. The kernel which is dual to the deformed one (24) is represented by

\[
K^{(d)}_d(\vec{x}_1, \vec{x}_2, \vec{x}) = \text{Tr} \left( \hat{U}(\vec{x}_1) \hat{K} \hat{D}(\vec{y}) \hat{U}(\vec{x}_2) \hat{D}(\vec{x}) \right). \quad (41)
\]

Also in this case the associativity (10) may be easily verified.

### 3.1 The dual star product and quantum observables

To better understand the physical meaning of the dual star-product let us consider the mean value of a quantum observable \( \hat{A} \). In tomographic representation (see below Section 5) this was shown to be

\[ \langle \hat{A} \rangle = \text{Tr} \hat{A} = \int w(X, \mu, \nu)f_A(X, \mu, \nu)dX \ d\mu \ d\nu \quad (42) \]
where, comparing with standard definitions of the mean value, the function $f_A(X, \mu, \nu)$ can be seen to be given by

$$f_A(X, \mu, \nu) = \text{Tr} \frac{1}{2\pi} \exp[i(X - \mu \hat{q} - \nu \hat{p})].$$  \hspace{1cm} (43)

This is nothing but the symbol of the observable $\hat{A}$ in the \textit{dual} tomographic scheme, namely,

$$f_A(X, \mu, \nu) \equiv f_d^A(X, \mu, \nu) = \text{Tr} \left( \hat{D}(X, \mu, \nu) \hat{A} \right),$$  \hspace{1cm} (44)

while the tomogram $w(X, \mu, \nu)$ is the symbol of the density matrix, in tomographic representation. Therefore, according to (42), the mean value of an observable $\hat{A}$ is given by the product of the symbol of the density matrix and the symbol of the observable in the dual representation, appropriately integrated. In fact, it can be shown that this observation is true in general. For any star-product scheme the mean value of an observable $\hat{A}$ can be always written as

$$\langle \hat{A} \rangle = \text{Tr}(\hat{\rho} \hat{A}) = \int f_\rho(x) f_d^A(x) d\vec{x}$$  \hspace{1cm} (45)

where $f_\rho$ is the symbol of $\hat{\rho}$ in a given quantization scheme, while $f_d^A$ is the symbol of the observable $\hat{A}$ in the dual scheme. Eq. (44) is easily verified substituting back the definitions of $f_\rho$ and $f_d^A$ and using the compatibility condition (5). This consideration points out the different nature of density states and observables. Though both are hermitian operators, they possess a different interpretation within the operator algebra, the former being linear functionals over the latter. Indeed, states are a convex body in $G^*(U)$, the dual of the Lie algebra $G(U)$ of the unitary group $U$. The duality transformation is again applied to the ambient space of states and observables $G^*(U) \times G(U) \rightarrow G^{**}(U) \times G^*(U)$. In the star product approach, their symbols are related to dual quantum schemes. In self-dual schemes (Weyl-Moyal-Wigner) this difference is hidden.

### 4 Class of solutions of the Jacobi identity

It is well known that, as any other Lie bracket defined via an associative product, the commutator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$  \hspace{1cm} (46)

satisfies the Jacobi identity

$$\left[ [\hat{A}, \hat{B}], \hat{C} \right] + \left[ [\hat{B}, \hat{C}], \hat{A} \right] + \left[ [\hat{C}, \hat{A}], \hat{B} \right] = 0.$$  \hspace{1cm} (47)

This in turn gives rise to an corresponding identity for the symbols of the operators. To this, following \cite{15} let us introduce the symbol of the commutator

$$f_{[A,B]}(\vec{x}) = \text{Tr}\left([\hat{A}, \hat{B}]\hat{U}(\vec{x})\right).$$  \hspace{1cm} (48)

One has

$$f_{[A,B]}(\vec{x}) = -f_{[B,A]}(\vec{x}).$$  \hspace{1cm} (49)

One can rewrite the symbol in the form

$$f_{[A,B]}(\vec{x}) = \left(f_A * f_B\right)(\vec{x}) - \left(f_B * f_A\right)(\vec{x}).$$  \hspace{1cm} (50)
The integral form of this relation reads
\[
f_{[A,B]}(\vec{x}) = \int K_-(\vec{x}_1, \vec{x}_2) f_A(x_1) f_B(x_2) d\vec{x}_1 d\vec{x}_2
\] (51)
where
\[
K_-(\vec{x}_1, \vec{x}_2, \vec{x}) = K(\vec{x}_1, \vec{x}_2, \vec{x}) - K(\vec{x}_2, \vec{x}_1, \vec{x}) = \text{Tr}\left( [\hat{D}(\vec{x}_1), \hat{D}(\vec{x}_2)] \hat{U}(\vec{x}) \right). \quad (52)
\]
Due to Eqs. (47) the kernel (52) inherits a Jacobi identity for the Lie bracket on the symbols
\[
f_{[[A,B],[C]]}(\vec{x}) + f_{[[B,C],[A]]}(\vec{x}) + f_{[[C,A],[B]]}(\vec{x}) = 0. \quad (53)
\]
In terms of the kernel (52) this identity reads
\[
\int \left( K_-(\vec{x}_1, \vec{x}_2, \vec{y}) K_-(\vec{y}, \vec{x}_3, \vec{x}_4) + K_-(\vec{x}_2, \vec{x}_3, \vec{y}) K_-(\vec{y}, \vec{x}_1, \vec{x}_4) \\
+ K_-(\vec{x}_3, \vec{x}_1, \vec{y}) K_-(\vec{y}, \vec{x}_2, \vec{x}_4) \right) d\vec{y} = 0 \quad (54)
\]
which is the anti-symmetrized version of Eq. (10). When the parameters are discrete, this equation provides the Jacobi identity for the structure constants of a Lie algebra. In fact, for a \( N \) dimensional Lie algebra with generators \( \hat{L}_i \) the commutation relations read
\[
[\hat{L}_i, \hat{L}_j] = C_{ij}^l \hat{L}_l \quad (55)
\]
where the sum over repeated indices is understood. The Jacobi identity reads here
\[
[[\hat{L}_i, \hat{L}_j], \hat{L}_k] + [[\hat{L}_j, \hat{L}_k], \hat{L}_i] + [[\hat{L}_k, \hat{L}_i], \hat{L}_j] = 0 \quad (56)
\]
and using (55) we get
\[
C_{ij}^l C_{lk}^m \hat{L}_m + C_{jk}^l C_{li}^m \hat{L}_m + C_{ki}^l C_{lj}^m \hat{L}_m = 0. \quad (57)
\]
Since the generators \( \hat{L}_m \) are a basis in the Lie algebra, this implies that the structure constants satisfy the quadratic equations (Jacobi identity)
\[
\sum_{l=1}^{N} C_{ij}^l C_{lk}^m + C_{jk}^l C_{li}^m + C_{ki}^l C_{lj}^m = 0 \quad (58)
\]
where we have restored the summation symbol to compare with (54). Changing notation
\[
C \rightarrow K_-, \quad i \rightarrow \vec{x}_1, \quad j \rightarrow \vec{x}_2, \quad l \rightarrow \vec{y}, \quad k \rightarrow \vec{x}_3, \quad m \rightarrow \vec{x}_4 \quad (59)
\]
and replacing the sum over \( l \) with an integral over \( \vec{y} \) one can see that (59) is identical to (51). Thus, writing the Jacobi identity as an algebraic equation for the structure constants of a Lie algebra, one can ask what are the solutions (possibly all) for these equations from a novel point of view. Namely the problem is translated into the search of compatible pairs of quantizers \( \hat{D}(\vec{x}) \) and dequantizers \( \hat{U}(\vec{x}) \); solutions of the Jacobi identity will then be given in the form (52).

We also notice that the dual formula provides other solutions of the Jacobi identity
\[
K_{-(d)}(\vec{x}_1, \vec{x}_2, \vec{x}) = \text{Tr}\left( [\hat{U}(\vec{x}_1), \hat{U}(\vec{x}_2)] \hat{D}^d(\vec{x}) \right). \quad (60)
\]
Moreover, since any solution of the associativity equation for the kernel of the star-product determines a solution of the Jacobi identity for the structure constants, one can use the $K$-deformed kernels \( K^{(k)} \) and \( K^{(kd)} \) to obtain yet other solutions of the Jacobi identity. In fact, the kernels given by

\[
K^{(k)}(\vec{x}_1, \vec{x}_2, \vec{x}) = Tr\left( [\hat{D}(\vec{x}_1), \hat{D}(\vec{x}_2)]K \hat{U}^{\dagger}(\vec{x}) \right),
\]

and

\[
K^{(kd)}(\vec{x}_1, \vec{x}_2, \vec{x}) = Tr\left( [\hat{U}(\vec{x}_1), \hat{U}(\vec{x}_2)]K \hat{D}^{\dagger}(\vec{x}) \right),
\]

where

\[
[\hat{D}(\vec{x}_1), \hat{D}(\vec{x}_2)]_K = \hat{D}(\vec{x}_1)\hat{D}(\vec{x}_2) - \hat{D}(\vec{x}_2)\hat{D}(\vec{x}_1)
\]

and

\[
[\hat{U}(\vec{x}_1), \hat{U}(\vec{x}_2)]_K = \hat{U}(\vec{x}_1)\hat{U}(\vec{x}_2) - \hat{U}(\vec{x}_2)\hat{U}(\vec{x}_1)
\]

satisfy \( (\ref{eq:54}) \) and \( [\ , \ ]_K \) is the $K$-deformed commutator. Of course, because this deformation does not provide us with the most general associative algebras, we may use alternative deformations along the lines of \( \text{[32]} \).

Let us note that, in finite dimensions, on using Ado’s theorem on the possibility of realizing any Lie algebra as an algebra of matrices and using the cotangent bundle of the identified algebra of matrices, in principle we may always realize this program. The relevance of our proposal relies on the possibility of constructing, by the present procedure, infinite dimensional double Lie algebras.

### 4.1 An illustrative example: the U(2) Lie algebra

To see our previous constructions at work let us consider a well known example, the $U(2)$ Lie algebra. Proceeding backwards from the Lie algebra we want to show how the corresponding quantizer and dequantizer look like. Then, exchanging their role as in Eq. \( (\ref{eq:60}) \) we will find a new Lie algebra which is dual to the starting one. These being compatible as Lie algebras, their sum is still a Lie algebra, known as the double (see for example \[ \text{[34]} \]).

Let us consider the structure constants of the group $U(2)$

\[
C^k_{ij} = \begin{cases} 
\varepsilon_{ijk}, & i, j, k = 1, 2, 3 \\
0 & \text{otherwise.}
\end{cases}
\]

The generators of $U(2)$ may be given in terms of Pauli matrices plus the $2 \times 2$ identity matrix

\[
S_j = \frac{\sigma_j}{2}, \quad j \neq 0 \quad S_0 = \sigma_0
\]

with

\[
\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

We recall their commutation rules

\[
[\sigma_0, \sigma_j] = 0; \quad [\sigma_j, \sigma_k] = 2i\varepsilon_{jkm}\sigma_m;
\]
while the associative product obeys
\[ \sigma_i \sigma_0 = \sigma_0 \sigma_i = \sigma_i; \quad \sigma_j \sigma_k = \delta_{jk} \sigma_0 + i \varepsilon_{jkm} \sigma_m, \quad j, k, m = 1, 2, 3, \] (69)
(here we sum over the index \( m \)). As for the traces
\[ \text{Tr} \sigma_0 = 2, \quad \text{Tr} \sigma_j = 0. \] (70)
Thus one has
\[ [S_j, S_k] = i \varepsilon_{jkm} S_m. \] (71)
Now we select a basis in the vector space underlying \( \mathcal{U}(2) \) and a basis for its dual vector space. This selection may be operated choosing a non-degenerate \((0,2)\)-tensor on \( \mathcal{U}(2) \), which realizes the pairing between the two vector spaces. If the pairing between \( \mathcal{U}(2) \) and its dual is provided by the twice the trace,
\[ <, > = 2 \text{Tr} \] (72)
the dequantizer and the quantizer may be seen to be respectively
\[ \hat{U}(x) = \{ \hat{U}(0) = \frac{1}{2} \sigma_0, \quad \hat{U}(1) = \frac{1}{2} \sigma_1, \quad \hat{U}(2) = \frac{1}{2} \sigma_2, \quad \hat{U}(3) = \frac{1}{2} \sigma_3 \} \] (73)
and
\[ \hat{D}(x) = \hat{U}(x) \] (74)
where we have introduced the notation \( x = (0,1,2,3) \). Indeed, they satisfy the compatibility condition
\[ <(\hat{U}(j),0),(0,\hat{D}(k))> = 2 \text{Tr} \left( \hat{U}^\dagger(j) \hat{D}(k) \right) = \frac{1}{2} \text{Tr} \sigma_j \sigma_k = \delta_{jk}, \] (75)
where \( \delta_{jk} \) is the Kronecker delta-function and reproduce the correct structure constants by means of the kernel
\[ K_-(j,k,m) = \frac{1}{4} \text{Tr} ([\sigma_j, \sigma_k] \sigma_m); \quad (j, k, m = 0, 1, 2, 3). \] (76)
The symbol of any operator represented by a matrix \( A \) reads
\[ f_A(x) = 2 \text{ Tr}(\hat{U}(x)A) \] (77)
Thus one has
\[ f_A(0) = \text{Tr} A, \quad f_A(1) = \text{Tr} A \sigma_1, \quad f_A(2) = \text{Tr} A \sigma_2, \quad f_A(3) = \text{Tr} A \sigma_3. \] (78)
while the star product is given by the kernel \( \hat{K}_d \) with \( \hat{U}(x) \) and \( \hat{D}(x) \) given by (73), (74).

What about other choices of the dual basis? The dual vector space, when the pairing is given by the trace, is endowed with a Lie algebra structure which is again \( \mathcal{U}(2) \), namely, with our choices, \( \mathcal{U}(2) \) is selfdual. Indeed, having chosen quantizer and dequantizer to coincide, the dual antisymmetric kernel may be seen to be
\[ K_-(j,k,m) = K_-(j,k,m) = \frac{1}{4} \text{Tr} ([\sigma_j, \sigma_k] \sigma_m) \] (79)
which is (in view of (71))

\[ K_d^{-}(j, k, m) = \begin{cases} \varepsilon_{jkm} & j, k, m=1, 2, 3 \\ 0 & \text{otherwise.} \end{cases} \]  

(80)

Thus we get back the $U(2)$ structure constants.

In $V^* \times V$ we may choose a different dual basis. For instance we may choose dual bases like

\[ \hat{D}(j) = -\frac{i}{2} \sigma_j \]  

(81)

\[ \hat{U}(j) = \frac{1}{2} (\sigma_j + i\varepsilon_{jk3}\sigma_k) \]  

(82)

with the pairing given by

\[ <(\hat{U}(j), 0), (0, \hat{D}(k))> = 2\text{Im} \text{Tr} (\hat{U}^\dagger(j)\hat{D}^\dagger(k)) = \delta_{jk} \]  

(83)

Indeed, the antisymmetric kernel

\[ K_d^{-}(ijk) = 2\text{Im} \text{Tr} [\hat{D}(i), \hat{D}(j)]\hat{U}^\dagger(k) \]  

(84)

reproduces the $SU(2)$ structure constants. Its dual may be easily calculated yielding

\[ K_d^d(ijk) = 2\text{Im} \text{Tr} [\hat{U}(i), \hat{U}(j)]\hat{D}^\dagger(k) = \varepsilon_{ijl}\varepsilon^{lk3}. \]  

(85)

These can be recognized to be the structure constants of the $SB(2, \mathbb{C})$ Lie algebra, namely the algebra of the $2 \times 2$ upper triangular complex matrices. The resulting algebra $V^* \oplus V = SB(2, \mathbb{C}) \oplus SU(2)$ is isomorphic to the metrical double Lie algebra $SL(2, \mathbb{C})$ with Manin decomposition (see for example [35]).

### 4.2 K-deformed products: examples

At the beginning of the section we have argued that also the deformed kernel (61) satisfies an associativity condition which provides the Jacobi identity for the structure constants of a Lie algebra. Therefore deformations may be put to work to generate different Lie algebras starting from a given one.

In this section we give an explicit example of this statement showing that, starting from the $SO(3)$ Lie algebra we recover all unimodular 3-d Lie algebras, along the lines previously illustrated in [36] (for a review see [37], [38]).

There, the Lie algebras under consideration where realized in terms of Poisson brackets defined on their dual. More precisely it was observed that any real finite-dimensional Lie algebra $\mathcal{G}$ with Lie bracket $[\cdot, \cdot]$ defines in a natural way a Poisson structure $\{\cdot, \cdot\}$, on the dual space $\mathcal{G}^*$ of $\mathcal{G}$. One is allowed to think of $\mathcal{G}$ as a subset of linear functions within the ring of smooth functions $C^\infty(\mathcal{G}^*)$. Choosing a linear basis $\{E_i\}_1^n$ of $\mathcal{G}$, and identifying them with linear coordinate functions $x_i$ on $\mathcal{G}^*$ by means of $x_i(x) = <x, E_i>$ for all $x \in \mathcal{G}^*$, we define the fundamental brackets on $\mathcal{G}^*$ by the expression $\{x_i, x_j\}_\mathcal{G} = c_{ij}^k x_k$ where $[E_i, E_j] = c_{ij}^k E_k$ and $c_{ij}^k$ denote
the structure constants of the Lie algebra. On this basis the equivalence classes of all three
dimensional Lie algebras are seen to be characterized by the Casimir form
\[ \alpha = h(x_1 dx_2 - x_2 dx_1) + \frac{1}{2} d(ax_1^2 + bx_2^2 + cx_3^2) \]  
by means of
\[ \{x_j, x_k\} = \varepsilon_{jkl} \left( \frac{\partial}{\partial x^l} \right) \]
with the real parameters \( h, a, b, c \) appropriately selected. This yields the Poisson brackets
\[ \{x_1, x_2\} = cx_3, \quad \{x_2, x_3\} = ax_1 - hx_2, \quad \{x_3, x_1\} = bx_2 + hx_1 \]
with the Jacobi identity encoded by
\[ d\alpha \wedge \alpha = 2h c x_3 dx_1 \wedge dx_2 \wedge dx_3 = 0 \]
which in turn holds true if and only if \( hc = 0 \). Thus we have two essentially different classes of
algebras: those corresponding to a closed Casimir form \( (h = 0 \), case A), and \( (c = 0 \), case B),
those corresponding to a Casimir form which is not closed.

In case A the parameters \( a, b, c \), when different from zero, may be all normalized to modulus
one. This grouping includes six different isomorphism classes of Lie algebras:

A.1 \( su(2) \simeq so(3) \) with \( a, b, c \) all different from 0 and of the same sign. A basis can be chosen
so that \( a = b = c = 1 \).

A.2 \( e(2) \), the algebra of the Euclidean group in two dimensions, which may be obtained by
contraction from the previous class, say \( a \to 0 \).

A.3 \( sl(2, \mathbb{R}) \simeq su(1, 1) \simeq so(2, 1) \), with \( a, b, c \) all different from 0 and one of them of different
sign.

A.4 \( iso(1, 1) \), the Poincaré algebra in two dimensions, which may be obtained by contraction
from the previous algebra.

A.5 \( h(1) \), the Heisenberg-Weyl algebra, with only one parameter different from zero, for example \( c > 0 \). It may be obtained by further contraction from both \( e(2) \) and \( iso(1, 1) \).

A.6 The abelian algebra with \( a = b = c = 0 \).

All these brackets may be written in the form
\[ \{x_i, x_j\} = \varepsilon_{ijl} \frac{\partial C}{\partial x_l} = \varepsilon_{ijl} dC \left( \frac{\partial}{\partial x_l} \right) \]
where
\[ \alpha = dC; \quad C = \frac{1}{2} (ax_1^2 + bx_2^2 + cx_3^2) \]
The second case, with \( \alpha \) not exact, includes four families of Lie algebras. We recall that all
algebras of type B have \( c = 0 \) because of the Jacobi identity. One has
B.1 $h = 1, a = b = 0$, that is $sb(2, \mathbb{C})$, the Lie algebra of the group of $2 \times 2$ upper(lower) triangular complex matrices with unit determinant.

B.2 $h = 1, a = 0, b = 1$.

B.3 $h \neq 0, a = 1, b = -1$.

B.4 $h \neq 0, a = b = 1$.

Cases B.3 and B.4 are one-parameter families, as it is impossible to put all parameters equal to one with a similarity transformation. We will refer collectively to the previous four classes of algebras as $\mathcal{G}_h$.

The K-deformation procedure provides a tool to derive all the three dimensional Lie algebras above. In fact, as for the $SU(2)$ Lie algebra the pair $\hat{U}(x), \hat{D}(x)$ which generate the $SO(3)$ Lie algebra can be chosen to be

$$\hat{U}(x) \equiv U(j) = L_j, \quad \hat{D}(x) \equiv D(j) = \frac{1}{2} U(j)$$

Thus the general scheme of K deformed structure constants can be applied. Let us consider the generators of $SO(3)$

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and a $3 \times 3$ symmetric matrix $K$

$$K = \begin{pmatrix} \lambda_1 & \mu_1 & \mu_3 \\ \mu_1 & \lambda_2 & \mu_2 \\ \mu_3 & \mu_2 & \lambda_3 \end{pmatrix}$$

Using the K deformed Lie bracket $[L_i, L_j]_K \equiv L_i KL_j - L_j KL_i$ we obtain the algebra

$$[L_1, L_2]_K = \mu_3 L_1 + \mu_2 L_2 + \lambda_3 L_3$$
$$[L_2, L_3]_K = \lambda_1 L_1 + \mu_1 L_2 + \mu_3 L_3$$
$$[L_3, L_1]_K = \mu_1 L_1 + \lambda_2 L_2 + \mu_2 L_3$$

It can be verified that all type A algebras are obtained by suitable choices of the parameters. Indeed, the Lie algebra represented by (95) may be written in terms of Poisson brackets by setting

$$dC_K = (dx_1 \ dx_2 \ dx_3) \begin{pmatrix} \lambda_1 & \mu_1 & \mu_3 \\ \mu_1 & \lambda_2 & \mu_2 \\ \mu_3 & \mu_2 & \lambda_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Then $\{x_i, x_j\} = \varepsilon_{ijl} dC_K (\frac{\partial}{\partial x_l})$. On using a diagonalizing transformation for $K$ we may reduce $dC_K$ to the form (91). This analysis shows that the quadratic form associated with $K$ is a Casimir for the Lie algebra we are going to define.

The algebras of type B do not possess a Casimir function, but the procedure to obtain all of them via a $K$ deformation may still be applied. Of course, we have to start from a type B
algebra to generate all the others. We consider the Lie algebra $B.4$ of our classification. The


generators may be chosen in the form

\[
X_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} h & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}
\]  

(97)

The other Lie algebras may be obtained, as previously, $K$-deforming the commutator with a

suitable $K$ matrix. Imposing Jacobi identity this may be checked to be of the form

\[
K = \begin{pmatrix} \alpha & \beta & \gamma \\ 0 & \epsilon & \phi \\ 0 & \zeta & \iota \end{pmatrix}
\]  

(98)

All type B algebras are thus reproduced with an appropriate choice of the parameters. For

example, the type B.1 algebra $SB(2, C)$ is obtained with $\epsilon = \iota = 0$ and $h\alpha = 1 - \phi = 1 + \zeta$.

However, our duality approach, via the formula (62), opens up other possibilities.

To provide an example in infinite dimensions we consider a deformation of the Weyl-product.

Using the pair $\hat{U}(q, p)$ and $\hat{D}(q, p)$ for the Weyl product as in [22] one can define the structure

constants of an infinite-dimensional Lie algebra as

\[
K - (q_1, p_1, q_2, p_2, q_3, p_3) = \frac{2}{\pi} \text{Tr} \left[ \left( \hat{D}(2\alpha_1) f(a^+ a) \hat{D}(-2\alpha_2) \
- \hat{D}(2\alpha_2) f(a^+ a) \hat{D}(-2\alpha_1) \right) \hat{D}(2\alpha_3)(-1)^{a^+ a} \right],
\]  

(99)

where

\[
\alpha_m = \frac{q_m + ip_m}{\sqrt{2}}, \quad m = 1, 2, 3.
\]

Here $f(a^+ a)$ plays the role of the deformation operator $\hat{K}$, $\hat{D}(\alpha)$ is the Weyl system operator.

Thus we find a deformation using the formalism of nonlinear $f$-deformed oscillators [39]. We

shall elaborate on the duality emerging from this example elsewhere.

5 The tomographic setting and its classical limit

With respect to quasi-distributions on phase-space, tomograms have the property of being posi-

tive both for classical and quantum systems, therefore they furnish a convenient setting to

analyze the quantum-classical transition, because we deal with the same kind of objects. Let us

review the construction of the symplectic tomographic map using explicitly the Planck’s con-

stant $\hbar$ as deformation parameter. In this case we choose $\vec{x} = (x_1, x_2, x_3)$ with $x_1 = X, x_2 =

\mu, x_3 = \nu \in \mathbb{R}$. The dequantizer is taken in the form

\[
\hat{U}(\vec{x}) := \hat{U}(X, \mu, \nu) = \delta(X - \mu \hat{q} - \nu \hat{p}).
\]  

(100)

Here $\hat{q}$ and $\hat{p}$ are position and momentum operators. The random variable $X$ describes the

position of a particle in a reference frame in its phase-space. But this reference frame is squeezed

(parameter $s$) and rotated (angle $\theta$), so that

\[
\mu = s \cos \theta, \quad \nu = s^{-1} \sin \theta.
\]  

(101)
The quantizer operator is given by
\[
\hat{D}(\vec{x}) = \hat{D}(X, \mu, \nu) = \frac{\hbar}{2\pi} \exp[i(X - \mu \hat{q} - \nu \hat{p})].
\] (102)

One can check that this choice of \( \hat{U}(\vec{x}) \), \( \hat{D}(\vec{x}) \) fulfills the compatibility condition \( \hat{w}_{\mu} \).

\[
\text{Tr} \left( \hat{U}(X, \mu, \nu) \hat{D}(X', \mu', \nu') \right) = \delta(X - X') \delta(\mu - \mu') \delta(\nu - \nu').
\] (103)

Thus, to operators \( \hat{A} \) in the Hilbert space \( \mathcal{H} \) one can associate the function (tomographic symbol or tomogram)
\[
w_{A}(X, \mu, \nu) = \text{Tr} \hat{A} \delta(X - \mu \hat{q} - \nu \hat{p}).
\] (104)

Conversely, the tomogram allows to reconstruct the operator via the relation
\[
\hat{A} = \frac{\hbar}{2\pi} \int w_{A}(X, \mu, \nu) \exp[i(X - \mu \hat{q} - \nu \hat{p})] dX d\mu d\nu.
\] (105)

The tomogram of a pure state with wave function \( \psi(x) \) can be calculated as in \( (105) \), with the operator \( \hat{A} \) replaced by the density operator \( \hat{\rho} = (|\psi\rangle \langle \psi|)/\langle \psi | \psi \rangle > 

\[
w_{\psi}(X, \mu, \nu) = \frac{1}{2\pi h |\nu|} \left| \int \psi(y) \exp\left(\frac{i\mu y^2}{2\nu \hbar} - \frac{iX y}{\hbar \nu}\right) dy \right|^2.
\] (106)

According to \( (\ref{103}) \) the star-product of two tomograms is
\[
w_{A} * w_{B}(x) = \int w_{A}(x_1) w_{B}(x_2) K(x_1, x_2, x) \hbar^2 dx_1 dx_2
\] (107)

where we have introduce the collective variables \( x = (X, \mu, \nu) \), \( x_i = (X_i, \mu_i, \nu_i) \) and, following Eq. \( (\ref{110}) \), the kernel is given by
\[
K(x_1, x_2, x) = \text{Tr} \left( \hat{D}(x_1) \hat{D}(x_2) \hat{U}(x) \right),
\] (108)

which explicitly reads
\[
K(x_1, x_2, x) = \text{Tr} \left( \frac{\hbar^2}{4\pi^2} \exp(iX_1 + iX_2 - i\mu_1 \hat{q}_1 + i\nu_1 \hat{p}_1) \right)
\times\exp(-i\mu_2 \hat{q}_2 - i\nu_2 \hat{p}_2) \delta(X - \mu \hat{q} - \nu \hat{p})
\]
\[
= \frac{\hbar^2}{4\pi^2} \delta(\nu(\mu_1 + \mu_2) - \mu(\nu_1 + \nu_2)) \times \exp \left( iX_1 + iX_2 + \frac{i\hbar}{2} (\nu_1 \mu_2 - \nu_2 \mu_1) - i \frac{(\nu_1 + \nu_2)X_3}{\nu} \right).
\] (109)

Notice that we have included the factor \( \hbar^2 \) in the measure, not in the kernel, in such a way to obtain a kernel with the same dimensions of its classical analogue, as we will see below. The kernel is the product of two factors. The first one
\[
K_{cl}(x_1, x_2, x_3) = \frac{1}{4\pi^2} \delta(\nu_3(\mu_1 + \mu_2) - \mu_3(\nu_1 + \nu_2)) \times \exp \left( iX_1 + iX_2 - i \frac{(\nu_1 + \nu_2)X_3}{\nu_3} \right)
\] (110)

furnishes the point-wise product of functions in phase space as we will see explicitly in a moment. The second factor
\[
f(h) = \exp \left[ \frac{ih}{2} (\nu_1 \mu_2 - \nu_2 \mu_1) \right].
\] (111)

is antisymmetric and depends explicitly on the Planck constant. This factorization shows very clearly that quantum mechanics modifies the structure functions of the classical associative product by means of a factor proportional to the exponential of a symplectic area.
5.1 The classical product in tomographic representation

Given two functions \( A(q,p) \) and \( B(q,p) \) on the classical phase space their pointwise product has the integral representation

\[
A(p,q) \cdot B(p,q) = \int A(p_1, q_1) B(p_2, q_2) \delta(q - q_1) \delta(p - p_2) \delta(q - q_2) \delta(p - p_1) dq_1 dq_2 dp_1 dp_2 \tag{112}
\]

where the kernel reads

\[
K(q_1, p_1, q_2, p_2) = \delta(q - q_1) \delta(q - q_2) \delta(p - p_1) \delta(p - p_2). \tag{113}
\]

This is symmetric with respect to permutations \( 1 \leftrightarrow 2 \). What is the kernel of this product in the tomographic representation? To this, let us introduce the tomographic symbols of the functions \( A(q,p) \) and \( B(q,p) \). They are respectively given by the Radon transform

\[
w_A(X, \mu, \nu) = \int A(q,p) \delta(X - \mu q - \nu p) dq dp \tag{114}
\]

This is invertible with inverse

\[
A(q,p) = \frac{1}{2\pi} \int w_A(X, \mu, \nu) \exp[i(X - \mu q - \nu p)] dX d\mu d\nu \tag{115}
\]

\[
B(q,p) = \frac{1}{2\pi} \int w_B(X, \mu, \nu) \exp[i(X - \mu q - \nu p)] dX d\mu d\nu.
\]

Using (114) the classical tomogram associated to the product \( A \cdot B \) reads

\[
w_{A \cdot B} = \int A(q,p) \cdot B(q,p) \delta(X - \mu q - \nu p) dq dp \tag{116}
\]

Then, on using (112), (115), it can be easily checked that the commutative pointwise product Eq. (112) induces for the tomograms a nonlocal commutative product whose kernel is exactly Eq. (110), as announced.

5.2 Poisson brackets in the tomographic representation

As we have seen, quantum mechanical corrections to the pointwise product are contained in the twist factor of the star product. From Eq. (109), (111), the kernel of the pointwise product in tomographic representation acquires, up to the first order in \( \hbar \), a correcting factor. It reads

\[
P(x_1, x_2, x_3) = \frac{i \hbar}{2} (\nu_1 \mu_2 - \nu_2 \mu_1) K_{cl}(X_1, \mu_1, \nu_1, X_2, \mu_2, \nu_2, X_3, \mu_3, \nu_3) \tag{117}
\]

with \( K_{cl} \) given by (110). Thus, the kernel corresponding to the classical Poisson bracket in tomographic representation is given by

\[
P(x_1, x_2, x_3) = \lim_{\hbar \to 0} \frac{1}{\hbar} \left[ K(x_1, x_2, x_3) - K(x_2, x_1, x_3) \right]
\]

\[
= (\nu_1 \mu_2 - \nu_2 \mu_1) \frac{1}{4\pi^2} \delta \left( \nu_3 (\mu_1 + \mu_2) - \mu_3 (\nu_1 + \nu_2) \right) \exp \left( iX_1 + iX_2 - i \frac{(\nu_1 + \nu_2) X_3}{\nu_3} \right). \tag{118}
\]
In order to convince ourselves that (118) really represents the canonical Poisson bracket on 
the phase space \( \mathbb{R}^{2n} \) we may derive the Poisson bracket in tomographic representa 
tion directly from the definition and then compare with (118).

¿From the inverse Radon transform (115) we have
\[
\begin{align*}
\frac{\partial A(q,p)}{\partial q} &= \frac{1}{2\pi} \int w_A(X,\mu,\nu)(-i\mu) \exp[i(X - \mu q - \nu p)]dXd\mu d\nu \\
\frac{\partial A(q,p)}{\partial p} &= \frac{1}{2\pi} \int w_A(X,\mu,\nu)(-i\nu) \exp[i(X - \mu q - \nu p)]dXd\mu d\nu
\end{align*}
\]
(119)
The tomogram associated to the Poisson bracket
\[
\{A(q,p), B(q,p)\} = \frac{\partial A(q,p)}{\partial q} \frac{\partial B(q,p)}{\partial p} - \frac{\partial A(q,p)}{\partial p} \frac{\partial B(q,p)}{\partial q}
\]
(120)
reads
\[
w_{\{A,B\}} = \int \{A(q,p), B(q,p)\} \delta(X - \mu q - \nu p) \frac{dqdp}{2\pi}
\]
(121)
Therefore, upon substituting the inverse Radon transform of derivatives (119), it may be easily 
checked that
\[
w_{\{A,B\}}(x) = \int w_A(x_1)w_A(x_2)P(x_1,x_2,x)d x_1 d x_2
\]
(122)
where the kernel \( P \) coincides with our previous result, (118) and we recall that \( dx_i \equiv dX_i d\mu_i d\nu_i \).

## 5.3 The Fourier representation

We have just seen that the tomographic star product, when regarded from the point of view of its 
integral kernel, only differs from the classical one by a twist factor proportional to a symplectic 
area. Here we want to show that this is not a peculiarity of the tomographic product, but the 
same happens for the Moyal product.

To this, let us derive the star product kernel for the Moyal product in Fourier representation. 
The Moyal product of two Weyl symbols is given by the well known asymptotic formula
\[
A(q,p) \ast B(q,p) = A(q,p) \exp \left[ i \left( \frac{\partial}{\partial q} \frac{\partial}{\partial p} - \frac{\partial}{\partial p} \frac{\partial}{\partial q} \right) \right] B(q,p)
\]
(123)
where \( h = 1 \). Derivatives with a left arrow are understood to act on the left while those with a 
right arrow act on the right, therefore we have
\[
A \ast B = \exp \left[ i \left( \frac{\partial}{\partial q_1} \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \frac{\partial}{\partial q_2} \right) \right] A(q_1,p_1)B(q_2,p_2) \bigg| \begin{array}{c}
q \equiv q_1 = q_2 \\
p \equiv p_1 = p_2
\end{array}
\]
\[
= \left[ A(q_1,p_1)B(q_2,p_2) + i \left( \frac{\partial A(q_1,p_1)}{\partial q_1} \frac{\partial B(q_2,p_2)}{\partial p_2} - \frac{\partial A(q_1,p_1)}{\partial p_1} \frac{\partial B(q_2,p_2)}{\partial q_2} \right) \right]
\]
\[
- \frac{1}{2!} \left( \frac{\partial}{\partial q_1} \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \frac{\partial}{\partial q_2} \right) A(q_1,p_1)B(q_2,p_2) + \cdots \bigg| \begin{array}{c}
q \equiv q_1 = q_2 \\
p \equiv p_1 = p_2
\end{array}
\]
(124)
This may be rewritten in the form

\[ A(q, p) \ast B(q, p) = \int A(q_1, p_1)B(q_2, p_2)K(q_1, p_1, q_2, p_2, q, p) \, dq_1 \, dq_2 \, dp_1 \, dp_2. \tag{125} \]

where \( K(q_1, p_1, q_2, p_2, q, p) \) is the Grönewold kernel \[4\]

\[ K(q_1, p_1, q_2, p_2, q, p) = \frac{1}{\pi^2} \exp[2i(p_2q_1 - p_1q_2 + pq_2 - p_2q + p_1q - pq_1)]. \tag{126} \]

The argument of the exponential can be seen to coincide with the symplectic area of the triangle with vertices \((p, q), (p_1, q_1), (p_2, q_2)\), if rewritten as \((p_1 - p)(q_2 - q) - (q_1 - q)(p_2 - p)\). Now, let us consider the star-product \((123)\) in Fourier representation. This form corresponds to definition of Weyl symbol in terms of the operator matrix \( \tilde{A}(x, x') \) in position representation

\[ A(q, p) = \int \tilde{A}(x = q + \frac{u}{2}, x' = q - \frac{u}{2})e^{-i pu} \, du \tag{127} \]

and \( B(q, p) \) in terms of the matrix \( \tilde{B}(x, x') \) in position representation

\[ B(q, p) = \int \tilde{B}(x = q + \frac{u}{2}, x' = q - \frac{u}{2})e^{-i pu} \, du. \tag{128} \]

The product of matrices \( \tilde{A}, \tilde{B} \) is given by

\[ (\tilde{A} \, \tilde{B})(x, y) = \int \tilde{A}(x, x')\tilde{B}(x', y) \, dx'. \tag{129} \]

The star-product \((123)\) of two Weyl symbols in Fourier representation can be written in the form

\[ A(\mu, \nu) \ast B(\mu, \nu) = \int A(\mu_1, \nu_1)B(\mu_2, \nu_2)K(\mu_1, \nu_1, \mu_2, \nu_2, \mu, \nu) \, d\mu_1 \, d\mu_2 \, d\nu_1 \, d\nu_2, \tag{130} \]

with the kernel equal to

\[ K(\mu_1, \nu_1, \mu_2, \nu_2, \mu, \nu) = \frac{1}{2\pi} \exp \left[ i \left( \frac{1}{2} (\nu_1\mu_2 - \nu_2\mu_1) \right) \right] \times \delta(\mu - \mu_1 - \mu_2)\delta(\nu - \nu_1 - \nu_2). \tag{131} \]

This is the product of a twist factor represented by the exponential of the symplectic area and delta functions which, as we will point out in a moment, correspond to the pointwise product contribution. The kernel contains an antisymmetric and a symmetric part with respect to permutation 1 \(\leftrightarrow\) 2. The antisymmetric term is determined by the exponent of the symplectic area in the \(\mu - \nu\) plane. So the whole kernel has the form of twisted star product.

As an example we consider the Weyl symbol of the unity operator

\[ A_1(q, p) = 1, \tag{132} \]

and calculate the star product \( A_1(\mu, \nu) \ast A_1(\mu, \nu) \) with

\[ A_1(\mu, \nu) = \frac{1}{2\pi} \int \exp \left[ -i(\mu q + \nu p) \right] dq \, dp = 2\pi \delta(\mu)\delta(\nu) \tag{133} \]
the Fourier component of 1. From Eqs. (130), (131) we obtain

$$A_1(\mu, \nu) \cdot A_1(\mu, \nu) = (2\pi)^2 \int \delta(\mu_1)\delta(\nu_1)\delta(\mu_2)\delta(\nu_2) \frac{1}{2\pi} \exp \left[ \frac{i}{2}(\nu_1\mu_2 - \nu_2\mu_1) \right]$$

$$\times \delta(\mu - \mu_1 - \mu_2)\delta(\nu - \nu_1 - \nu_2) d\mu_1 d\mu_2 d\nu_1 d\nu_2 = 2\pi\delta(\mu)\delta(\nu) = A_1(\mu, \nu).$$

(134)

As for the pointwise commutative product (112) its Fourier representation is easily seen to be given by a nonlocal commutative product with kernel

$$K_F(\mu_1, \nu_1, \mu_2, \nu_2, \mu, \nu) = \frac{1}{2\pi} \delta(\mu - \mu_1 - \mu_2)\delta(\nu - \nu_1 - \nu_2)$$

(135)

once we consider the Fourier transforms of the functions $A(q, p)$ and $B(q, p)$

$$A(\mu, \nu) = \frac{1}{2\pi} \int A(q, p) \exp[-i(\mu q + \nu p)] dq dp,$$

(136)

$$B(\mu, \nu) = \frac{1}{2\pi} \int B(q, p) \exp[-i(\mu q + \nu p)] dq dp.$$  

(137)

Again we find that, by working on phase-space, the classical point-wise product and the quantum noncommutative product, when seen through a kernel function, show their difference in the exponential of the symplectic area. We trust that these aspects will turn out to be helpful when we try to generalize the construction to general Lie groups replacing the Abelian vector groups we are using here.

Previous considerations should make clear that we have paved the way to deal with double Lie algebras at the quantum level. These aspects shall be taken up elsewhere.

6 Conclusions

We summarize the main results of our work. Deforming the associative product on the space of matrices and using the duality symmetry we found a class of solutions for the associativity equation of the product kernel in factorized form (e.g. formulae (60), (61)). We then used this solution to find a class of solutions for the Lie algebra Jacobi identity.

We exploited the duality symmetry for the case of the tomographic star product and we suggested a physical interpretation for the dual.

A relevant aspect of this paper is the definition of a quantum Poisson bracket on tomograms along with its classical limit. This is achieved observing that with any Wigner function we can associate a tomogram in an invertible way. The product on Wigner functions is associated with a product on tomograms, therefore it induces a skew-symmetric bracket. In the classical limit the Moyal bracket gives rise to a Poisson bracket on phase-space. In the analogue classical limit on tomograms this kernel does not have the form of a bidifferential operator on $\mathbb{R}^3$.

It would be interesting to clarify all structure constants that are solutions of Eq. (10) and can be obtained by the factorization formula (9). We are presently working on this subject.
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