A NOTE ON THE SCALAR CURVATURE ON NONCOMPACT SURFACES.

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ABSTRACT. We give a short proof of the following fact: if Σ is a connected, noncompact manifold without boundary admitting a complete metric of positive scalar curvature, then Σ is homeomorphic to \( \mathbb{R}^2 \) and the scalar curvature is integrable. In particular, there exists no complete metric of uniformly positive scalar curvature on \( \mathbb{R}^2 \).

1. Introduction

In [5], Rosenberg and Stolz proposed the following conjecture.

Conjecture 1.1. ([5, Conjecture 7.1]) A closed manifold \( X \) admits a metric of positive scalar curvature in the following cases:

- (A) \( X \times \mathbb{R} \) admits a complete metric of positive scalar curvature;
- (B) \( X \times \mathbb{R}^2 \) admits a complete metric of uniformly positive scalar curvature.

This conjecture cannot be strengthened further: by [5, Proposition 7.2], there exist complete metrics of (non-uniformly) positive scalar curvature on \( X \times \mathbb{R}^2 \) and there exist complete metrics of uniformly positive scalar curvature on \( \mathbb{R}^3 \) with arbitrarily large lower bound.

When \( X \) is one-dimensional, Part (A) of Conjecture 1.1 corresponds to the statement that the cylinder \( S^1 \times \mathbb{R} \) cannot carry a complete Riemannian metric of positive scalar curvature. On the one hand, in [3, Corollary 6.13], Gromov and Lawson proved that if \( X \) is a compact enlargeable manifold, then the cylinder \( X \times \mathbb{R} \) cannot carry a complete Riemannian metric of positive scalar curvature. Since \( S^1 \) is enlargeable, this corollary implies Part (A) of Conjecture 1.1 in the one-dimensional case. On the other hand, it is well-known that, on closed surfaces, the nonpositivity of the Euler characteristic is an obstruction to the existence of Riemannian metrics whose scalar curvature is nonnegative and positive at one point. Since \( \chi(S^1 \times \mathbb{R}) = \chi(S^1) = 0 \), it is natural to ask whether the vanishing of the Euler characteristic of \( S^1 \times \mathbb{R} \) implies that there are no metrics of positive scalar curvature on this noncompact surface.

In this note, we adopt a more general point of view. We consider a class of noncompact surfaces, which are called finitely connected. Roughly speaking, they are obtained by removing a finite number of points from a compact surface. For such surfaces, the Euler characteristic is obviously well-defined. We fix a connected, noncompact surface without boundary \( \Sigma \), and ask the following two questions.

(Q1) Suppose that \( \Sigma \) is finitely connected and \( \chi(\Sigma) \leq 0 \). Does there exist a complete Riemannian metric on \( \Sigma \) whose scalar curvature is nonnegative and positive at one point?
(Q2) Suppose that Σ is not finitely connected. Does there exist a complete Riemannian metric of nonnegative scalar curvature on Σ?

We give a negative answer to both questions. The case when Σ is finitely connected (Q1) is treated by using the classical Gauss-Bonnet theorem on a suitable family of compact, two-dimensional submanifolds with boundary of Σ. The case when Σ is not finitely connected (Q2) is a direct consequence of the classical theorem of Huber [4].

Finally, we consider the case of $\mathbb{R}^2$. In this case, the Euler characteristic is positive and the existence of metrics of positive scalar curvature is a well-known fact (cf. [5, Section 7]). Notice that $\mathbb{R}^2$ is a finitely connected surface, since it is homeomorphic to the two-sphere with one point removed. The approximation procedure used to answer Question (Q1) allows us to deduce that there is no complete Riemannian metric of uniformly positive scalar curvature on $\mathbb{R}^2$. Notice that this fact provides evidence to Part (B) of Conjecture 1.1.

2. The theorem

We start with selecting a special class of noncompact manifolds for which the Euler characteristic is well-defined.

**Definition 2.1.** We say that a smooth surface without boundary Σ is **finitely connected** if there exists a compact surface with boundary $\Omega \subset \Sigma$ such that

(i) the boundary of $\Omega$ is a disjoint union of closed simple curves $l_1, \ldots, l_p$;

(ii) the open set $\Sigma \setminus \Omega$ is a disjoint union of the cylinders $C_j := l_j \times (0, \infty)$, for $j = 1, \ldots p$.

We say that a noncompact surface without boundary Σ is **infinitely connected** if it is not finitely connected.

Notice that equivalently a noncompact surface without boundary Σ is finitely connected if it is homeomorphic to a closed surface $\Sigma_1$ with $p$ points removed. In this case, the singular homology groups of Σ have finite rank and the Euler characteristic $\chi(\Sigma)$ of Σ is given by the formula

$$\chi(\Sigma) = \chi(\Sigma_1) - p.$$  \hspace{1cm} (2.1)

For the basic notions on finitely connected surfaces, we refer to [8, Section 2.1].

**Remark 2.2.** Let Σ be a connected, finitely connected, noncompact surface without boundary. If $\chi(\Sigma) > 0$, Formula (2.1) implies that actually $\chi(\Sigma) = 1$ and Σ is homeomorphic to the two-sphere with a single point removed, i.e. it is homeomorphic to $\mathbb{R}^2$.

**Theorem 2.3.** Let Σ be a connected, noncompact surface without boundary. Then:

(I) if Σ is infinitely connected, then Σ cannot carry a complete Riemannian metric of nonnegative scalar curvature;

(II) if Σ is finitely connected and $\chi(\Sigma) \leq 0$, then Σ cannot carry a complete Riemannian metric whose scalar curvature is nonnegative and strictly positive at one point;

(III) if Σ is homeomorphic to a two-sphere with a single point removed, then Σ cannot carry a complete Riemannian metric of uniformly positive scalar curvature.
In particular, Case (II) of this theorem directly implies the following corollary, giving a positive answer to our initial question.

**Corollary 2.4.** The cylinder $S^1 \times \mathbb{R}$ doesn’t admit a complete Riemannian metric whose scalar curvature is nonnegative and strictly positive at one point.

**Remark 2.5.** Notice that from [3, Corollary 6.13] it follows that there is no metric of strictly positive scalar curvature on $S^1 \times \mathbb{R}$. Therefore, Corollary 2.4 is slightly stronger and cannot be entirely deduced from [3, Corollary 6.13].

In order to prove Theorem 2.3, we first need to define the total curvature of a noncompact Riemannian surface $(\Sigma, g)$ (for more details on the notion of total curvature for noncompact surfaces, see [8, Section 2.1]). Let $\text{scal}(g)$ be the scalar curvature of $g$, and $dA_g$ the area element.

**Definition 2.6.** We say that a Riemannian surface $(\Sigma, g)$ admits total curvature if

$$\int_{\Sigma} \text{scal}(g)_{+} \, dA_g < \infty \quad \text{or} \quad \int_{\Sigma} \text{scal}(g)_{-} \, dA_g < \infty,$$

where $\text{scal}(g)_{+} = \max\{\text{scal}(g), 0\}$ and $\text{scal}(g)_{-} := \max(-\text{scal}(g), 0)$. In this case, the extended real number

$$c(\Sigma; g) := \frac{1}{2} \int_{\Sigma} \text{scal}(g) \, dA_g = \frac{1}{2} \int_{\Sigma} \text{scal}(g)_{+} \, dA_g - \frac{1}{2} \int_{\Sigma} \text{scal}(g)_{-} \, dA_g \in [-\infty, \infty] \quad (2.2)$$

is called the total curvature of $(\Sigma, g)$.

**Remark 2.7.** The total curvature is usually defined in terms of the integral of the Gaussian curvature $K$ (cf. [8, Definition 2.1.3]). Since $\text{scal}(g) = 2K$, the two definitions are equivalent. The choice of using the scalar curvature instead of the Gaussian curvature is due to the fact that the former is the main object studied in this note.

In the next lemma we select a class of noncompact Riemannian surfaces that admit total curvature. Let us start with a definition.

**Definition 2.8.** We say that the scalar curvature of a Riemannian metric $g$ on a surface $\Sigma$ is nonnegative at infinity if $\text{scal}(g) \geq 0$ outside of a compact set $K \subset \Sigma$.

**Lemma 2.9.** Let $(\Sigma, g)$ be a complete Riemannian surface. If the scalar curvature of $g$ is nonnegative at infinity, then $(\Sigma, g)$ admits total curvature $c(\Sigma; g)$ ranging over the interval $(-\infty, +\infty]$.

**Proof.** It suffices to notice that in this case $\text{scal}(g)_{-}$ is compactly supported. \hfill \Box

2.10. **Infinitely connected surfaces.** For infinitely connected surfaces we have the following remarkable result of Huber (see [4] and [8, Theorem 2.2.2]).

**Theorem 2.11** (Huber). If a connected, infinitely connected, complete Riemannian surface without boundary $(\Sigma, g)$ admits total curvature, then $c(\Sigma; g) = -\infty$.

From Lemma 2.9 and Theorem 2.11 we directly deduce the following consequence.
Corollary 2.12. A connected, infinitely connected, noncompact surface without boundary Σ cannot carry a complete Riemannian metric whose scalar curvature is nonnegative at infinity.

Remark 2.13. Corollary 2.12 implies Case (I) of Theorem 2.3.

2.14. Finitely connected surfaces. As we already observed, when Σ is finitely connected the Euler characteristic χ(Σ) is well-defined. Moreover, if g is a Riemannian metric on Σ and (Σ, g) admits total curvature c(Σ; g), then χ(Σ) and c(Σ; g) are related by the following Gauss-Bonnet inequality due to Cohn-Vossen (cf. [11] and [8, Theorem 2.2.1]).

Theorem 2.15 (Cohn-Vossen). Let (Σ, g) be a connected, finitely connected, complete Riemannian surface without boundary. If (Σ, g) admits finite total curvature, then

\[ 2\pi \chi(\Sigma) \geq c(\Sigma; g). \quad (2.3) \]

When the scalar curvature scal(g) is nonnegative at infinity, by Lemma 2.10 (Σ, g) admits total curvature. In the next proposition, we also show that it must be finite.

Proposition 2.16. Let (Σ, g) be a connected, finitely connected, complete Riemannian surface without boundary. If the scalar curvature of g is nonnegative at infinity, then the total curvature c(Σ; g) is finite.

Remark 2.17. Our proof of this proposition also provides a direct proof of the Gauss-Bonnet Inequality 2.3 when the scalar curvature is nonnegative at infinity (cf. Inequality 2.11).

From Theorem 2.15 and Proposition 2.16 we directly deduce the following two corollaries.

Corollary 2.18. Let Σ be a connected, finitely connected, noncompact surface without boundary. If χ(Σ) ≤ 0, then Σ cannot carry a complete Riemannian metric whose scalar curvature is nonnegative and strictly positive at one point.

Remark 2.19. Corollary 2.18 corresponds to Case (II) of Theorem 2.3.

Corollary 2.20. Let Σ be a surface homeomorphic to \( \mathbb{R}^2 \) and let g be a complete Riemannian metric on Σ of nonnegative scalar curvature. Then the scalar curvature of g is integrable. In particular, there exists no complete Riemannian metric of uniformly positive scalar curvature on \( \mathbb{R}^2 \).

Remark 2.21. Corollary 2.20 implies Case (III) of Theorem 2.3.

Before presenting our proof of Proposition 2.16 we reduce the nonorientable case to the orientable one.

Lemma 2.22. If Proposition 2.16 holds for all orientable Σ, then it also holds when Σ is nonorientable.

Proof. Let Σ be a connected, finitely connected, noncompact, nonorientable surface without boundary and let \( \tilde{\Sigma} \) be its orientable double cover. Notice that \( \tilde{\Sigma} \) is also a connected, finitely
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Let $g$ be a complete Riemannian metric on $\Sigma$ whose scalar curvature is nonnegative at infinity. The metric $g$ lifts to a complete $\mathbb{Z}_2$-invariant metric $\tilde{g}$ on $\tilde{\Sigma}$. Since the scalar curvature is local, $\text{scal}(\tilde{g})$ coincides with the lift of $\text{scal}(g)$ to $\tilde{\Sigma}$. Therefore, $\text{scal}(\tilde{g})$ is also nonnegative at infinity and, by Proposition 2.16, the total curvature $c(\tilde{\Sigma}; \tilde{g})$ is finite.

It remains to relate $c(\Sigma; g)$ and $c(\tilde{\Sigma}; \tilde{g})$. Let $U$ be an open subset of $\Sigma$ such that $\Sigma \setminus U$ has measure zero and the restriction of the covering $\tilde{\Sigma} \to \Sigma$ is trivial. Choose a suitable lift $\tilde{U}$ of $U$ to $\tilde{\Sigma}$ in a way that $\tilde{U}$ is a fundamental domain for the action of $\mathbb{Z}_2 = \{0, 1\}$. This means that $\tilde{U} \cap (1 \cdot \tilde{U}) = \emptyset$ and that the set $\tilde{\Sigma} \setminus (\tilde{U} \times \mathbb{Z}_2)$ has measure zero. Since $\text{scal}(\tilde{g})$ is the lift of $\text{scal}(g)$ and the area elements $dA_g$ and $dA_{\tilde{g}}$ coincide locally, we have

$$\begin{aligned}
c(\tilde{\Sigma}; \tilde{g}) &= \int_{\tilde{\Sigma}} \text{scal}(\tilde{g}) \, dA_{\tilde{g}} = \int_{\tilde{U}} \text{scal}(\tilde{g}) \, dA_{\tilde{g}} + \int_{1 \cdot \tilde{U}} \text{scal}(\tilde{g}) \, dA_{\tilde{g}} \\
&= 2 \cdot \int_{\Sigma} \text{scal}(g) \, dA_g = 2 \cdot c(\Sigma; g).
\end{aligned}$$

Therefore, $c(\Sigma; g)$ is also finite. \hfill \square

2.23. The approximation procedure. In this last subsection we present the proof of Proposition 2.16.

Let $\Sigma$ be a noncompact, connected, finitely connected surface without boundary. Let $\Omega \subset \Sigma$ be a compact submanifold with boundary such that the boundary $\partial \Omega$ of $\Omega$ consists of $p$-copies of $S^1$ and $\Sigma \setminus \Omega = \bigsqcup_{j=1}^p C_j$, where each $C_j$ is a copy of the cylinder $S^1 \times (0, \infty)$. We also assume $\Sigma$ is orientable and pick an orientation.

Let $\Sigma_h$ be the compact surface with boundary obtained by truncating the cylindrical ends of $\Sigma$ at the height $h$. This means that the boundary $\partial \Sigma_h$ of $\Sigma_h$ is the disjoint union of $p$ copies of $S^1$ and $\Sigma \setminus \Sigma_h = \bigsqcup_{j=1}^p \{S^1 \times (h, \infty)\}$. The total geodesic curvature of $\Sigma_h$ is defined by

$$\lambda(h) := \int_{\partial \Sigma_h} K_g,$$

where the boundary $\partial \Sigma_h$ is positively oriented with respect to the given orientation of $\Sigma$ and $K_g$ is the geodesic curvature of $\partial \Sigma_h$ (see [2], Section 4-4, Definition 10).

Lemma 2.24. Let $(\Sigma, g)$ be a connected, finitely connected, orientable, complete Riemannian surface without boundary. If the scalar curvature of $g$ is nonnegative at infinity, then $\lambda(h)$ converges to a nonnegative number as $h$ goes to infinity.

Remark 2.25. Our proof of this lemma is based on the fact that we can choose on each cylindrical end $C_j$ suitable coordinates which simplify the components of the metric. Such coordinates were first used by S. Rosenberg in [6] to provide a short proof of Theorem 2.15.
2.26. **Proof of Lemma 2.24.** Since \( \Sigma_h \) is a retract of \( \Sigma \), using the Gauss-Bonnet theorem (see \[2\] Section 4-5) on \( \Sigma_h \) we obtain

\[
2\pi \chi(\Sigma) = 2\pi \chi(\Sigma_h) = c(\Sigma_h; g) + \lambda(h). \tag{2.5}
\]

Since \( \text{scal}(g) \) is nonnegative at infinity, there exists \( h_1 > 0 \) such that \( \text{scal}(g) \geq 0 \) on \( \Sigma \setminus \Sigma_{h_1} \). From (2.5) we deduce that the function

\[
\lambda(h) = 2\pi \chi(\Sigma) - c(\Sigma_h; g) \tag{2.6}
\]

is nonincreasing on the interval \((h_1, \infty)\), so that the extended real number

\[
L := \lim_{h \to \infty} \lambda(h) \in [-\infty, +\infty) \tag{2.7}
\]

is well-defined. To conclude the proof, it remains to show that we must have \( L \geq 0 \).

In order to get more information on the function \( \lambda(h) \) and the number \( L \), we compute a local expression for \( K_g \). As observed in \[6\], we can choose coordinates \((t_j, \theta_j)\) on the cylindrical end \( C_j \) in a way that

(i) for all \( P \in C_j \), the basis

\[
\left\{ \frac{\partial}{\partial t_j} \bigg|_P, \frac{\partial}{\partial \theta_j} \bigg|_P \right\}
\]

of the tangent space \( T_P C_j \cong \mathbb{R} \oplus \mathbb{R} \) is positively oriented;

(ii) the metric \( g \), restricted to the cylindrical end \( C_j \), is of the form

\[
g(t_j, \theta_j) = dt_j^2 + G_j(t_j, \theta_j) d\theta_j^2, \quad (t_j, \theta_j) \in C_j,
\]

where \( G_j : C_j \to (0, \infty) \) is a smooth function.

For the rigorous construction of the coordinates \((t_j, \theta_j)\), we refer to \[7\] page 747. With this choice, the curve \( \gamma_j^h(s) = (t_j(s), \theta_j(s)) = (h, s) \) parametrizes \( \partial \Sigma_h \cap C_j \) with positive orientation (see \[2\] pp. 267-268). Moreover, by \[2\] Section 4-4, Proposition 3, the geodesic curvature \( K_g \) of \( \gamma_j^h \) takes the form

\[
K_g = \frac{1}{2 \sqrt{G_j}} \frac{\partial G_j}{\partial t_j} \frac{d \theta_j}{ds} = \frac{\partial}{\partial t_j} \sqrt{G_j}, \tag{2.8}
\]

from which

\[
\int_{\partial \Sigma_h \cap C_j} K_g = \int_0^{2\pi} \left( \frac{\partial}{\partial t_j} \sqrt{G_j} \right)(h, s) \, ds = \frac{d}{dh} \int_0^{2\pi} \sqrt{G_j(h, s)} \, ds.
\]

Hence,

\[
\lambda(h) = \mu'(h), \tag{2.9}
\]

where

\[
\mu(h) := \sum_{j=1}^p \int_0^{2\pi} \sqrt{G_j(s, h)} \, ds. \tag{2.10}
\]

Finally, we use Equation (2.9) to deduce that the number \( L \) defined by (2.7) must be nonnegative. Suppose indeed that \( L < 0 \). Then by Equation (2.9) it follows that \( \lim_{h \to \infty} \mu(h) = -\infty \),
which is impossible, since by the expression (2.10), $\mu(h)$ is a strictly positive function. Therefore, we must have $L \geq 0$, which concludes the proof. □

2.27. Proof of Proposition 2.16. Let $\Sigma$, $g$ be as in the hypothesis of Proposition 2.16. By Lemma 2.22, we assume $\Sigma$ is orientable. From (2.5), we have

$$c(\Sigma_h; g) = 2\pi \chi(\Sigma) - \lambda(h).$$

Taking the limit for $h \to \infty$ and using Lemma 2.24, we deduce that

$$c(\Sigma; g) = 2\pi \chi(\Sigma) - L \leq 2\pi \chi(\Sigma).$$

Therefore, $c(\Sigma; g)$ is finite and satisfies the Gauss-Bonnet Inequality (2.3). □

References

[1] S. Cohn-Vossen. Kürzeste Wege und totalkrümmung auf Flächen. Compositio Math., (2):69–133, 1935.
[2] Manfredo P. do Carmo. Differential Geometry of Curves and Surfaces. Prentice-Hall, 1976.
[3] M. Gromov and H.B. Lawson. Positive scalar curvature and the Dirac operator on complete riemannian manifolds. Publ. Math. Inst. Hautes Études Sci., 58(1):83–196, 1983.
[4] A. Huber. On subharmonic functions and differential geometry in the large. Comment. Math. Helv., (32):13–72, 1952.
[5] J. Rosenberg and S. Stolz. Manifolds of positive scalar curvature, volume 27 of Math. Sci. Res. Inst. Publ. Springer, New York, 1994.
[6] S. Rosenberg. Gauss-Bonnet theorems for noncompact surfaces. Proc. Amer. Math. Soc., 86(1):184–185, 1982.
[7] S. Rosenberg. On the gauss-bonnet theorem for complete manifolds. Trans. Amer. Math. Soc., 287(2):745–753, 1985.
[8] K. Shiohama, T. Shiroya, and M. Tanaka. The Geometry of Total Curvature on Complete Open Surfaces. Cambridge Tracts in Mathematics. Cambridge University Press, 2003.

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