ADJOINT METHODS FOR OBSTACLE PROBLEMS AND WEAKLY COUPLED SYSTEMS OF PDE

F. CAGNETTI, D. GOMES, AND H.V. TRAN

Abstract. The adjoint method, recently introduced by Evans, is used to study obstacle problems, weakly coupled systems, cell problems for weakly coupled systems of Hamilton–Jacobi equations, and weakly coupled systems of obstacle type. In particular, new results about the speed of convergence of common approximation procedures are derived.

1. Introduction

In this paper we study the speed of convergence of certain approximations for obstacle problems and weakly coupled systems of Hamilton–Jacobi equations, using the Adjoint Method. This technique, recently introduced by Evans (see [Eva10], and also [Tra] and [CGT]), is a very successful tool to understand several types of degenerate PDEs. It can be applied, for instance, to Hamilton–Jacobi equations with non convex Hamiltonians, e.g. time dependent (see [Eva10]) and time independent (see [Tra]) Hamilton–Jacobi equations, to weak KAM theory (see [Fat97a, Fat97b, Fat98a, Fat98b, EG01, EG02]), and to the infinity Laplacian equation (see [ES]). We address here several applications and propose some new open questions. Further results, which will not be discussed here, can be found in [Eva10] and [CGT].

1.1. Overview of the Adjoint Method. To apply the Adjoint Method to a (non linear) PDE, one has to consider the adjoint equation associated to the linearization of the original problem. In this way it is possible to prove new estimates, which can then be used to obtain additional information on the solution of the initial PDE. In order to give an idea of the technique, we show below how this was used in the context of Aubry-Mather theory, in the periodic setting (see [CGT]).

To start with, we quote a fundamental result (see [LPV88]), stating existence and uniqueness of the effective Hamiltonian. Here with \( \mathbb{T}^n \) we denote the \( n \)-dimensional unit torus in \( \mathbb{R}^n \), \( n \in \mathbb{N} \).

Theorem 1.1 (Lions, Papanicolaou and Varadhan). Let \( H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R} \) be smooth and coercive, i.e.

\[
\lim_{|p| \to +\infty} H(x, p) = +\infty.
\]
Then, for every $P \in \mathbb{R}^n$ there exists a unique $\overline{H}(P) \in \mathbb{R}$ such that the equation

$$H(x, P + D_x u(x, P)) = \overline{H}(P)$$

(1.1)

admits a $\mathbb{Z}^n$-periodic viscosity solution $u(\cdot, P) : \mathbb{T}^n \to \mathbb{R}$.

In the spirit of Theorem 1.1 we prove in [CGT] the analogue result for an elliptic regularization of equation (1.1). See also [Gom02] for similar results.

**Theorem 1.2** (See [CGT]). Let $H : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}$ be smooth and assume that

$$\lim_{|p| \to +\infty} \left( \frac{1}{2} |H(x, p)|^2 + D_x H(x, p) \cdot p \right) = \lim_{|p| \to +\infty} \frac{H(x, p)}{|p|} = +\infty.$$ 

Then, for every $\eta > 0$ and every $P \in \mathbb{R}^n$, there exists a unique number $\overline{H}_\eta(P) \in \mathbb{R}$ such that the equation

$$-\frac{\eta^2}{2} \Delta u_\eta(x) + H(x, P + Du_\eta(x)) = \overline{H}_\eta(P)$$

(1.2)

admits a unique (up to constants) $\mathbb{Z}^n$-periodic viscosity solution. Moreover, for every $P \in \mathbb{R}^n$

$$\lim_{\eta \to 0^+} \overline{H}_\eta(P) = \overline{H}(P),$$

(up to subsequences)

where $\overline{H}(P)$ is given by Theorem 1.1. In addition, we have

$$u_\eta \to u \quad \text{uniformly},$$

(up to subsequences)

where $u$ is a $\mathbb{Z}^n$-periodic viscosity solution of (1.1).

For every $\eta > 0$ and $P \in \mathbb{R}^n$, the formal linearized operator $L^{\eta, P} : C^2(\mathbb{T}^n) \to C(\mathbb{T}^n)$ associated to equation (1.2) is defined as

$$L^{\eta, P} v(x) := -\frac{\eta^2}{2} \Delta v(x) + D_p H(x, P + Du_\eta(x)) \cdot Dv(x), \quad v \in C^2(\mathbb{T}^n).$$

As already mentioned, the main idea of the method consists in the introduction of the adjoint equation associated to $L^{\eta, P}$:

$$\begin{cases}
-\frac{\eta^2}{2} \Delta \sigma_\eta(x) - \operatorname{div}(D_p H(x, P + Du_\eta(x)) \sigma_\eta(x)) = 0, & \text{in } \mathbb{T}^n, \\
\sigma_\eta \geq 0, \text{ } \mathbb{T}^n\text{-periodic and } \int_{\mathbb{T}^n} \sigma_\eta dx = 1.
\end{cases}$$

(1.3)

Then, exploiting the properties of the solution $\sigma_\eta$ of (1.3), we can retrieve additional information about $u_\eta$. 
First of all, one can show new estimates that do not seem to be easily obtained in a classical way. As an example, define the function
\[ w \eta := |Du \eta|^2. \]
Then, \( w \eta \) satisfies
\[ DxH \cdot Du + D_p H \cdot Dw = \frac{\eta^2}{2} \Delta w - \frac{\eta^2}{2} |D^2 w|^2. \]  
(1.4)
Multiplying the above relation by \( \sigma \eta \) and integrating by parts, we eventually get
\[ \eta^2 \int_{\mathbb{T}^n} |D^2 w|^2 \sigma \eta \, dx \leq C, \]  
(1.5)
for some \( C > 0 \) independent of \( \eta \). Relation (1.5) gives information about the behavior of all the Hessian \( D^2 w \) of \( w \) in the support of \( \sigma \). We observe that, without passing to the adjoint equation, one can only conclude that
\[ \eta^2 |\Delta w|^2 \leq C, \]
thus obtaining a relation which involves just the Laplacian \( \Delta w \) of \( w \). More generally, by considering functions of the form \( w(x) = \phi(x, P + Du(x)) \) and studying the analogous of equation (1.4), one can obtain further properties, using compensated compactness based estimates (see [Eva10], [CGT]).

In addition, the Hamiltonian \( H \) is not required to be convex. When we have that \( H \) is uniformly convex in \( p \) (i.e. \( D^2_{pp} H \geq \alpha \) for some \( \alpha > 0 \)), (1.5) can be significantly improved (see [CGT]). Indeed, differentiating (1.2) twice along a generic vector \( \xi \in \mathbb{R}^n \) with \( |\xi| = 1 \) we have (here and always in the sequel, we use Einstein’s convention for repeated indices in a sum)
\[ H_{\xi \xi} + 2H_{p, p, u_{\xi x}} + H_{p, p, u_{\xi x}, u_{\xi x}, u_{\xi x}} + D_p H \cdot Du_{\xi \xi} = \frac{\eta^2}{2} \Delta u_{\xi \xi}. \]
Then, multiplying by \( \sigma \eta \) and integrating by parts we get
\[ \int_{\mathbb{T}^n} |Du_{\xi}|^2 \sigma \eta \, dx \leq C, \]
or more generally
\[ \int_{\mathbb{T}^n} |D^2 u|^2 \sigma \eta \, dx \leq C, \]
which is clearly stronger than (1.5). The differences between convex and nonconvex setting can be also observed by investigating the existence of invariant Mather measures (see [CGT]), and by studying the nature of the shocks in Hamilton–Jacobi equations (see [Eva10]).

Finally, the treatment allows to analyze the speed of convergence of \( \overline{\mathcal{H}}^\eta(P) \) to \( \overline{\mathcal{H}}(P) \). Indeed, classical arguments in elliptic regularization imply that \( u \) and \( \overline{\mathcal{H}}^\eta \) are smooth in \( \eta \) away from \( \eta = 0 \). Then, differentiating (1.2) w.r.t. \( \eta \)
\[ D_p H \cdot Du_{\eta} = \overline{\mathcal{H}}^\eta(P) + \frac{\eta^2}{2} \Delta u_{\eta} + \eta \Delta u, \]
where we denoted the differentiation w.r.t. \( \eta \) with a subscript. Again, multiplying by \( \sigma^n \) and integrating by parts we infer that

\[
|H_\eta(P)| \leq \eta \int_{\mathbb{T}^n} |\Delta u^n| \sigma^n \, dx \leq C \eta \left( \int_{\mathbb{T}^n} |D^2 u^n|^2 \sigma^n \, dx \right)^{1/2} \left( \int_{\mathbb{T}^n} \sigma^n \, dx \right)^{1/2} \leq C,
\]

where the latter inequality follows by using (1.5). Thus, we conclude that

\[
|H_\eta(P) - H(P)| \leq C \eta,
\]

which shows that the speed of convergence is \( O(\eta) \). In the uniformly convex case, this result can be improved to \( O(\eta^2) \). Let us observe that, as pointed out in [Eva10] and by Fraydoun Rezakhanlou to us, these estimates on the speed of convergence can also be obtained via Maximum Principle.

1.2. Outline of the paper. This paper contains four further sections concerning obstacle problems, weakly coupled systems, effective Hamiltonian for weakly coupled systems of Hamilton–Jacobi equations, and weakly coupled systems of obstacle type, respectively. We use a common strategy to study all these problems. For this reason, we describe in more detail our approach just in the case of the obstacle problem. In all the paper, \( U \) is an open bounded domain in \( \mathbb{R}^n \) with smooth boundary, \( n \geq 2 \). Moreover, we will denote with \( \nu \) the outer unit normal to \( \partial U \).

In Section 2 we consider an obstacle problem of the form

\[
\begin{aligned}
\max\{u - \psi, u + H(x, Du)\} &= 0 \text{ in } U, \\
u &= 0 \text{ on } \partial U,
\end{aligned}
\]

(1.6)

where \( \psi : \overline{U} \to \mathbb{R} \) and \( H : \mathbb{R}^n \times \overline{U} \to \mathbb{R} \) are smooth, with \( \psi \geq 0 \) on \( \partial U \). This equation arises naturally in Optimal Control theory, in the study of optimal stopping (see [Lio82]). In [IY90], Ishii and Yamada consider similar problems in the special case in which \( H \) is uniformly convex in the second variable.

Classically, in order to study (1.6) one first modifies the equation, by adding a perturbation term that penalizes the region where \( u > \psi \). Then, a solution is obtained as a limit of the solutions of the penalized problems. More precisely, let \( \gamma : \mathbb{R} \to [0, +\infty) \) be smooth, such that

\[
\begin{aligned}
\gamma(s) &= 0 \text{ for } s \leq 0, \quad \gamma(s) > 0 \text{ for } s > 0, \\
0 < \gamma'(s) &\leq 1 \text{ for } s > 0, \quad \lim_{s \to +\infty} \gamma(s) = +\infty,
\end{aligned}
\]

and define \( \gamma^\varepsilon : \mathbb{R} \to [0, +\infty) \) as

\[
\gamma^\varepsilon(s) := \gamma \left( \frac{s}{\varepsilon} \right), \text{ for all } s \in \mathbb{R}, \text{ for all } \varepsilon > 0.
\]

(1.7)
In some of the problems we discuss we also require \( \gamma \) to be convex in order to obtain improved results, but that will be pointed out where necessary. For every \( \varepsilon > 0 \), one can introduce the penalized PDE

\[
\begin{aligned}
    u^\varepsilon + H(x,Du^\varepsilon) + \gamma^\varepsilon(u^\varepsilon - \psi) &= \varepsilon \Delta u^\varepsilon \quad \text{in } U, \\
    u^\varepsilon &= 0 \quad \text{on } \partial U.
\end{aligned}
\]  

To avoid confusion, we stress the fact that here \( \gamma^\varepsilon(u^\varepsilon - \psi) \) stands for the composition of the function \( \gamma^\varepsilon \) with \( u^\varepsilon - \psi \). Unless otherwise stated, we will often simply write \( \gamma^\varepsilon \) and \( \gamma^\varepsilon' \) to denote \( \gamma^\varepsilon(u^\varepsilon - \psi) \) and \( \gamma^\varepsilon'(u^\varepsilon - \psi) \), respectively. Also, notice that in (1.8) the parameter \( \varepsilon \) corresponds to \( \eta^2 \) in (1.2). We made this choice in order to compare our results with existing estimates for the speed of convergence in literature.

Thanks to [Lio82], for every \( \varepsilon > 0 \) there exists a smooth solution \( u^\varepsilon \) to (1.8). It is also well known that, up to subsequences, \( u^\varepsilon \) converges uniformly to a viscosity solution \( u \) of (1.6) (see also Section 2 for further details).

In [IY90] Ishii and Yamada considered related problems when \( H(x,\cdot) \) is uniformly convex, and studied the speed of convergence of the functions \( u^\varepsilon \) to \( u \). However, both the original problem, the regularized PDE, and their methods are different from ours. To the best of our knowledge, no results are available in literature concerning non convex Hamiltonians.

We face here the problem requiring a coercivity assumption on \( H \) and a compatibility condition for equation (1.6) (see hypotheses (H2.1) and (H2.2), respectively), showing that the speed of convergence in the general case is \( O(\varepsilon^{1/2}) \).

**Theorem 1.3.** Suppose conditions (H2.1) and (H2.2) in Section 2 hold. Then, there exists a positive constant \( C \), independent of \( \varepsilon \), such that

\[
||u^\varepsilon - u||_{L^\infty} \leq C \varepsilon^{1/2}.
\]

At the end of the section, we give a dynamic and a stochastic interpretation of the problem (see Subsection 2.3 and Subsection 2.4, respectively). The proof of Theorem 1.3 consists of three steps.

**Step I: Preliminary Estimates.** We first show that

\[
\max_{x \in U} \frac{u^\varepsilon(x) - \psi(x)}{\varepsilon} \leq C,
\]  

for some constant \( C > 0 \) independent of \( \varepsilon \) (see Lemma 2.2). This allows us to prove that

\[
||u^\varepsilon||_{L^\infty}, ||Du^\varepsilon||_{L^\infty} \leq C,
\]

see Proposition 2.1.
**Step II: Adjoint Method.** We consider the formal linearization of (1.8), and then introduce the correspondent adjoint equation (see equation (2.6)). The study of this last equation for different values of the data allows us to obtain several useful estimates (see Lemma 2.3 and Lemma 2.4).

**Step III: Conclusion.** We conclude the proof of Theorem 1.3 by showing that

$$
\max_{x \in U} |u_\varepsilon(x)| \leq \frac{C}{\varepsilon^{1/2}}, \quad u_\varepsilon(x) := \frac{\partial u_\varepsilon}{\partial \varepsilon}(x),
$$

for some constant $C > 0$ independent of $\varepsilon$ (see Lemma 2.5). The most delicate part of the proof of (1.10) consists in controlling the term (see relation (2.11))

$$
\gamma_\varepsilon(s) = -\frac{s}{\varepsilon^2} \gamma'(\frac{s}{\varepsilon}), \quad s \in \mathbb{R}, \quad \gamma_\varepsilon(s) := \frac{\partial \gamma_\varepsilon}{\partial \varepsilon}(s).
$$

We underline that getting a bound for (1.11) can be extremely hard in general. In this context, this is achieved by differentiating equation (1.8) w.r.t. $\varepsilon$ (see equation (2.10)), and then by using relation (1.11), Lemma 2.3 and Lemma 2.4. This means that we overcome the problem by essentially using the Maximum Principle and the monotonicity of $\gamma_\varepsilon$ (see estimates (2.12) and (2.13)). We were not able to obtain such a bound when dealing with homogenization or singular perturbation, where also similar terms appear. We believe it would be very interesting to find the correct way to apply the Adjoint Method in these situations.

In Section 3 we study the weakly coupled system of Hamilton–Jacobi equations

$$
\begin{cases}
c_{11} u_1 + c_{12} u_2 + H_1(x, Du_1) = 0, \\
c_{21} u_1 + c_{22} u_2 + H_2(x, Du_2) = 0,
\end{cases}
$$

in $U$. (1.12)

Under some coupling assumptions on the constants (see conditions (H3.2) and (H3.3)), Engler and Lenhart [EL91], Ishii and Koike [IK91] prove existence, uniqueness and stability for the viscosity solutions $(u_1, u_2)$ of (1.12), but they do not consider any approximation of the system.

As before, we introduce perturbed problems (see (3.2)) and show that, under the same assumptions of [EL91], the speed of convergence of the corresponding solutions $(u^*_1, u^*_2)$ to $(u_1, u_2)$ is $O(\varepsilon^{1/2})$ (see Theorem 3.6). We observe that the coupling assumptions here play a crucial role and cannot be replaced. For the sake of simplicity, we just focus on a system of two equations, but the general case can be treated in a similar way.

Section 4 is devoted to an analog of the effective Hamiltonian problem (1.1) introduced by Lions, Papanicolaou and Varadhan [LPV88], which is the following weakly coupled system of
Hamilton–Jacobi equations:
\[
\begin{align*}
    c_1 u_1 - c_1 u_2 + H_1(x, Du_1) &= \overline{H}_1 \\
    -c_2 u_1 + c_2 u_2 + H_2(x, Du_1) &= \overline{H}_2 
\end{align*}
\]  
(1.13)

Here \(c_1\) and \(c_2\) are positive constants and \(H_1, H_2 : \mathbb{T}^n \times \mathbb{R}^n \to \mathbb{R}\) are smooth, while \(u_1, u_2 : \mathbb{T}^n \to \mathbb{R}\) and \(\overline{H}_1, \overline{H}_2 \in \mathbb{R}\) are unknowns. Systems of this type have been studied by Camilli, Loreti and Yamada in [CL08] and [CLY09], for uniformly convex Hamiltonians in a bounded domain. They arise naturally in optimal control and in large deviation theory for random evolution processes.

As in [CL08], we assume the system to be quasi-monotone and not necessarily monotone (see condition (H4.2)). In this context, by monotonicity we mean exactly the coupling assumptions of Engler and Lenhart in [EL91]. Moreover, we also require \(H_1, H_2\) to be coercive (see condition (H4.1)). As it happens for the cell problem in the framework of weak KAM theory, there is no hope here of a general uniqueness result for \((u_1, u_2)\), even modulo the addition of constants.

In the spirit of Theorem 1.1, studying a perturbation of (1.13) (see (4.2)) we prove that there exist \(H_1, H_2 \in \mathbb{R}\) such that the system above admits viscosity solutions \(u_1, u_2\). Notice that \(H_1, H_2\) are not unique as well, but \(c_2 \overline{H}_1 + c_1 \overline{H}_2\) is. More precisely, we show that there exists a unique \(\mu \in \mathbb{R}\) such that
\[
c_2 \overline{H}_1 + c_1 \overline{H}_2 = \mu,
\]
for every pair \((\overline{H}_1, \overline{H}_2) \in \mathbb{R}^2\) for which viscosity solutions \(u_1, u_2\) exist (see Theorem 4.2). This result is the analog of Theorem 1.1 by Lions, Papanicolaou and Varadhan in the case of systems.

Finally, we prove that the speed of convergence of \((\varepsilon u_1^\varepsilon, \varepsilon u_2^\varepsilon)\) to \((\overline{H}_1, \overline{H}_2)\) is \(O(\varepsilon)\) (see Theorem 4.4), where \(u_1^\varepsilon\) and \(u_2^\varepsilon\) are the solutions of the approximating problem (4.2).

In Section 5 we conclude the paper with the study of weakly coupled systems of obstacle type, namely
\[
\begin{align*}
    \max\{u_1 - u_2 - \psi_1, u_1 + H_1(x, Du_1)\} &= 0 \quad \text{in } U, \\
    \max\{u_2 - u_1 - \psi_2, u_2 + H_2(x, Du_2)\} &= 0 \quad \text{in } U. 
\end{align*}
\]  
(1.14)

Problems of this type appeared in [CDE84] and [CLY09]. Here \(H_1, H_2 : \overline{U} \times \mathbb{R}^n \to \mathbb{R}\) and \(\psi_1, \psi_2 : \overline{U} \to \mathbb{R}\) are smooth, with \(\psi_1, \psi_2 \geq \alpha > 0\).

In this case, although the two equations in (1.14) are coupled just through the difference \(u_1 - u_2\) (weakly coupled system), the problem turns out to be considerably more difficult than the corresponding scalar equation (1.6). Indeed, we cannot show now the analogous of estimate (1.9) as in Section 2.

For this reason, the hypotheses we require are stronger than in the scalar case. Together with the usual hypotheses of coercivity and compatibility (see conditions (H5.2) and (H5.4)), we have
2. Obstacle problem

In this section, we study the following obstacle problem

\[
\begin{align*}
\max\{u - \psi, u + H(x, Du)\} &= 0 \quad \text{in } U, \\
u &= 0 \quad \text{on } \partial U,
\end{align*}
\]

where \(\psi: \overline{U} \to \mathbb{R}\) and \(H: \mathbb{R}^n \times \overline{U} \to \mathbb{R}\) are smooth, with \(\psi \geq 0\) on \(\partial U\). We also assume that

(H2.1) \(\lim_{|p| \to +\infty} \left( \frac{1}{2}|H(x, p)|^2 + D_x H(x, p) \cdot p \right) = \lim_{|p| \to +\infty} \frac{H(x, p)}{|p|} = +\infty\) uniformly in \(x \in \overline{U}\);

(H2.2) there exists a function \(\Phi \in C^2(U) \cap C^1(\overline{U})\) such that \(\Phi \leq \psi\) on \(U\), \(\Phi = 0\) on \(\partial U\) and \(\Phi + H(x, D\Phi) < 0\) in \(\overline{U}\).

We observe that in the classical case \(H(x, p) = \mathcal{H}(p) + V(x)\) with

\[\lim_{|p| \to +\infty} \frac{\mathcal{H}(p)}{|p|} = +\infty,\]

or when \(H\) is superlinear in \(p\) and \(|D_x H(x, p)| \leq C(1 + |p|)\), then we immediately have (H2.1).

Assumption (H2.2) (stating, in particular, that \(\Phi\) is a sub-solution of (2.1)), will be used to derive the existence of solutions of (2.1), and to give a uniform bound for the gradient of solutions of the penalized equation below.

2.1. The classical approach. For every \(\varepsilon > 0\), the penalized PDE is the equation given by

\[
\begin{align*}
u^\varepsilon + H(x, D\nu^\varepsilon) + \gamma^\varepsilon (u^\varepsilon - \psi) &= \varepsilon \Delta \nu^\varepsilon \quad \text{in } U, \\
u^\varepsilon &= 0 \quad \text{on } \partial U,
\end{align*}
\]

where \(\gamma^\varepsilon\) is defined by (1.7). From [Lio82] it follows that under conditions (H2.1) and (H2.2), for every \(\varepsilon > 0\) there exists a smooth solution \(u^\varepsilon\) to (2.2). The first result we establish is a uniform bound for the \(C^1\)-norm of the sequence \(\{u^\varepsilon\}\).

Proposition 2.1. There exists a positive constant \(C\), independent of \(\varepsilon\), such that

\[||u^\varepsilon||_{L^\infty}, ||D\nu^\varepsilon||_{L^\infty} \leq C.\]

In order to prove Proposition 2.1 we need the following fundamental lemma:
Lemma 2.2. There exists a constant $C > 0$, independent of $\varepsilon$, such that
\[
\max_{x \in \overline{U}} \gamma^\varepsilon(u^\varepsilon - \psi) \leq C, \quad \max_{x \in \overline{U}} \frac{u^\varepsilon - \psi}{\varepsilon} \leq C.
\]

Proof. We only need to show that $\max_{x \in \overline{U}} \gamma^\varepsilon(u^\varepsilon - \psi) \leq C$, since then the second estimate follows directly by the definition of $\gamma^\varepsilon$. Since $u^\varepsilon - \psi \leq 0$ on $\partial U$, we have $\max_{x \in \partial U} \gamma^\varepsilon(u^\varepsilon - \psi) = 0$.

Now, if $\max_{x \in \overline{U}} \gamma^\varepsilon(u^\varepsilon - \psi) = 0$, then we are done. Thus, let us assume that there exists $x_1 \in U$ such that $\max_{x \in \overline{U}} \gamma^\varepsilon(u^\varepsilon - \psi) = \gamma^\varepsilon(u^\varepsilon - \psi)(x_1) > 0$. Since $\gamma^\varepsilon$ is increasing, we also have $\max_{x \in U}(u^\varepsilon - \psi) = u^\varepsilon(x_1) - \psi(x_1)$. Thus, using (2.2), by the Maximum principle
\[
(u^\varepsilon(x_1) - \psi(x_1)) + \gamma^\varepsilon(u^\varepsilon(x_1) - \psi(x_1)) = \varepsilon \Delta u^\varepsilon(x_1) - H(x_1,Du^\varepsilon(x_1)) - \psi(x_1)
\]
\[
\leq \varepsilon \Delta \psi(x_1) - H(x_1,D\psi(x_1)) - \psi(x_1).
\]
Since $u^\varepsilon(x_1) - \psi(x_1) > 0$,
\[
\gamma^\varepsilon(u^\varepsilon(x_1) - \psi(x_1)) \leq \max_{x \in \overline{U}}(|\Delta \psi| + |H(x,D\psi)| + |\psi(x)|) \leq C,
\]
for any $\varepsilon < 1$, and this concludes the proof. \qed

Proof of Proposition 2.1 Suppose there exists $x_0 \in U$ such that $u^\varepsilon(x_0) = \max_{x \in \overline{U}} u^\varepsilon(x)$. Then, since $\Delta u^\varepsilon(x_0) \leq 0$ and using the fact that $\gamma^\varepsilon \geq 0$
\[
u^\varepsilon(x_0) = \varepsilon \Delta u^\varepsilon(x_0) - H(x_0,0) - \gamma^\varepsilon(u^\varepsilon(x_0) - \psi(x_0))
\]
\leq -H(x_0,0) \leq \max_{x \in \overline{U}} (-H(x,0)) \leq C.

Let now $x_1 \in U$ be such that $u^\varepsilon(x_1) = \min_{x \in \overline{U}} u^\varepsilon(x_1)$. Then, using Lemma 2.2
\[
u^\varepsilon(x_1) = \varepsilon \Delta u^\varepsilon(x_1) - H(x_1,0) - \gamma^\varepsilon(u^\varepsilon(x_1) - \psi(x_1))
\]
\geq -H(x_1,0) - \gamma^\varepsilon(u^\varepsilon(x_1) - \psi(x_1))
\geq \min_{x \in \overline{U}} (-H(x,0) - \gamma^\varepsilon(u^\varepsilon(x) - \psi(x))) \geq -C.

This shows that $\|u^\varepsilon\|_{L^\infty}$ is bounded.

To prove that $\|Du^\varepsilon\|_{L^\infty}$ is bounded independently of $\varepsilon$, we first need to prove that $\|Du^\varepsilon\|_{L^\infty(\partial U)}$ is bounded by constructing appropriate barriers.

Let $\Phi$ be as in (H2.2). For $\varepsilon$ small enough, we have that
\[
\Phi + H(x,D\Phi) + \gamma^\varepsilon(\Phi - \psi) < \varepsilon \Delta \Phi,
\]
and $\Phi = 0$ on $\partial U$. Therefore, $\Phi$ is a sub-solution of (2.2). By the comparison principle, $u^\varepsilon \geq \Phi$ in $U$. 

\[\text{ADJOINT METHODS FOR OBSTACLE PROBLEMS AND WEAKLY COUPLED SYSTEMS OF PDE 9}\]
Let now \( d(x) = \text{dist}(x, \partial U) \). It is well-known that for some \( \delta > 0 \) \( d \in C^2(U_\delta) \) and \( |Dd| = 1 \) in \( U_\delta \), where \( U_\delta := \{ x \in U : d(x) < \delta \} \). For \( \mu > 0 \) large enough, the uniform bound on \( |u^\varepsilon|_{L^\infty} \) yields \( v := \mu d \geq u^\varepsilon \) on \( \partial U_\delta \). Assumption (H2.1) then implies

\[
v + H(x, Du) + \gamma^\varepsilon(v - \psi) - \varepsilon \Delta v \geq H(x, \mu Dd) - C\mu \geq 0,
\]

for \( \mu \) is sufficiently large. So the comparison principle gives us that \( \Phi \leq u^\varepsilon \leq v \) in \( U_\delta \). Thus, since \( \nu \) is the outer unit normal to \( \partial U \), and \( \Phi = u^\varepsilon = v = 0 \) on \( \partial U \), we have

\[
\frac{\partial v}{\partial \nu}(x) \leq \frac{\partial u^\varepsilon}{\partial \nu}(x) \leq \frac{\partial \Phi}{\partial \nu}(x), \quad \text{for } x \in \partial U.
\]

Hence, we obtain \( \|Du^\varepsilon\|_{L^\infty(\partial U)} \leq C \).

Next, let us set \( w^\varepsilon = \frac{|Du^\varepsilon|^2}{2} \). By a direct computation one can see that

\[
2(1 + (\gamma^\varepsilon'))w^\varepsilon + D_p H \cdot Dw^\varepsilon + D_\nu H \cdot Du^\varepsilon - (\gamma^\varepsilon')Du^\varepsilon \cdot D\psi = \varepsilon \Delta w^\varepsilon - \varepsilon |D^2 u^\varepsilon|^2. \tag{2.3}
\]

If \( \|Du^\varepsilon\|_{L^\infty} \leq \max(\|D\psi\|_{L^\infty}, \|Du^\varepsilon\|_{L^\infty(\partial U)}) \) then we are done.

Otherwise, max(\( \|D\psi\|_{L^\infty}, \|Du^\varepsilon\|_{L^\infty(\partial U)} \) \( \|Du^\varepsilon\|_{L^\infty} \). We can choose \( x_2 \in U \) such that \( w^\varepsilon(x_2) = \max_{x \in U} w^\varepsilon(x) \). Then, using (2.3)

\[
\varepsilon|D^2 u^\varepsilon|^2(x_2) = \varepsilon \Delta w^\varepsilon(x_2) - 2w^\varepsilon(x_2) - D_\nu H(x_2, Du^\varepsilon(x_2)) \cdot Du^\varepsilon(x_2)
\]

\[
+ (\gamma^\varepsilon')Du^\varepsilon(x_2) \cdot D\psi(x_2) - |Du^\varepsilon|^2(x_2)) \leq -D_\nu H(x_2, Du^\varepsilon(x_2)) \cdot Du^\varepsilon(x_2). \tag{2.4}
\]

Moreover, for \( \varepsilon \) sufficiently small we have

\[
\varepsilon|D^2 u^\varepsilon|^2(x_2) \geq \varepsilon^2 |\Delta u^\varepsilon(x_2)|^2 = |u^\varepsilon(x_2) + H(x_2, Du^\varepsilon(x_2)) + \gamma^\varepsilon(u^\varepsilon(x_2) - \psi(x_2))|^2
\]

\[
\geq \frac{1}{2}|H(x_2, Du^\varepsilon(x_2))|^2 - C, \tag{2.5}
\]

where we use Lemma 2.2 for the last inequality. Collecting (2.4) and (2.5)

\[
\frac{1}{2}|H(x_2, Du^\varepsilon(x_2))|^2 + D_\nu H(x_2, Du^\varepsilon(x_2)) \cdot Du^\varepsilon(x_2) \leq C.
\]

Recalling hypothesis (H2.1), we must have

\[
\|Du^\varepsilon\|_{L^\infty} = |Du^\varepsilon(x_2)| \leq C.
\]

\[\square\]

Thanks to Proposition 2.1 one can show that, up to subsequences, \( u^\varepsilon \) converges uniformly to a viscosity solution \( u \) of the obstacle problem (2.1).
2.2. Proof of Theorem 1.3. We now study the speed of convergence.

To prove our theorem we need several steps.

Adjoint method: The formal linearized operator $L^\varepsilon : C^2(U) \rightarrow C(U)$ corresponding to (2.2) is given by

$$L^\varepsilon z := (1 + (\gamma^\varepsilon)'z + D_pH \cdot Dz - \varepsilon \Delta z.$$ 

We will now introduce the adjoint PDE corresponding to $L^\varepsilon$. Let $x_0 \in U$ be fixed. We denote by $\sigma^\varepsilon$ the solution of:

$$
\begin{cases}
(1 + (\gamma^\varepsilon)'\sigma^\varepsilon - \text{div}(D_pH\sigma^\varepsilon)) = \varepsilon \Delta \sigma^\varepsilon + \delta_{x_0}, & \text{in } U, \\
\sigma^\varepsilon = 0, & \text{on } \partial U,
\end{cases}
$$

(2.6)

where $\delta_{x_0}$ stands for the Dirac measure concentrated in $x_0$. In order to show existence and uniqueness of $\sigma^\varepsilon$, we have to pass to a further adjoint equation. Let $f \in C(U)$ be fixed. Then, we denote by $v$ the solution to

$$
\begin{cases}
(1 + (\gamma^\varepsilon)'v + D_pH \cdot Dv = \varepsilon \Delta v + f, & \text{in } U, \\
v = 0, & \text{on } \partial U.
\end{cases}
$$

(2.7)

When $f \equiv 0$, by using the Maximum Principle one can show that $v \equiv 0$ is the unique solution to (2.7). Thus, by the Fredholm Alternative we infer that (2.6) admits a unique solution $\sigma^\varepsilon$.

Moreover, one can also prove that $\sigma^\varepsilon \in C^\infty(U \setminus \{x_0\})$. Some additional properties of $\sigma^\varepsilon$ are given by the following lemma.

**Lemma 2.3** (Properties of $\sigma^\varepsilon$). Let $\nu$ denote the outer unit normal to $\partial U$. Then,

(i) $\sigma^\varepsilon \geq 0$ on $\overline{U}$. In particular, $\frac{\partial \sigma^\varepsilon}{\partial \nu} \leq 0$ on $\partial U$.

(ii) The following equality holds:

$$
\int_U (1 + (\gamma^\varepsilon)' \sigma^\varepsilon) \, dx = 1 + \varepsilon \int_{\partial U} \frac{\partial \sigma^\varepsilon}{\partial \nu} \, dS.
$$

In particular,

$$
\varepsilon \int_{\partial U} \left| \frac{\partial \sigma^\varepsilon}{\partial \nu} \right| \, dS \leq 1.
$$

**Proof.** First of all, consider equation (2.7) and observe that

$$
f \geq 0 \implies v \geq 0.
$$

(2.8)

Indeed, assume $f \geq 0$ and let $\overline{x} \in \overline{U}$ be such that

$$
v(\overline{x}) = \min_{x \in \overline{U}} v(x).
$$
We can assume that \( x \in U \), since otherwise clearly \( v \geq 0 \). Then, for every \( x \in U \)
\[
((1 + (\gamma^\varepsilon)' )v(x) = \varepsilon \Delta v(x) + f(x) \geq 0,
\]
and (2.8) follows, since \( 1 + (\gamma^\varepsilon)' > 0 \).

Now, multiply equation (2.6) by \( v \) and integrate by parts, obtaining
\[
\int_U f \sigma^\varepsilon \, dx = v(x_0).
\]
Taking into account (2.8), from last relation we infer that
\[
\int_U f \sigma^\varepsilon \, dx \geq 0 \quad \text{for every} \quad f \geq 0,
\]
and this implies \( \sigma^\varepsilon \geq 0 \).

To prove (ii), we integrate (2.6) over \( U \), to get
\[
\int_U (1 + (\gamma^\varepsilon)' )\sigma^\varepsilon \, dx = \int_U \text{div}(D_p H \sigma^\varepsilon ) \, dx + \varepsilon \int_U \Delta \sigma^\varepsilon \, dx + 1
\]
\[
= \int_{\partial U} (D_p H \cdot \nu )\sigma^\varepsilon \, dS + \varepsilon \int_{\partial U} \partial \sigma^\varepsilon / \partial \nu \, dS + 1 = \varepsilon \int_{\partial U} \partial \sigma^\varepsilon / \partial \nu \, dS + 1,
\]
where we used the fact that \( \sigma^\varepsilon = 0 \) on \( \partial U \).
\[\Box\]

Using the adjoint equation, we have the following new estimate.

**Lemma 2.4.** There exists \( C > 0 \), independent of \( \varepsilon > 0 \), such that
\[
\frac{1}{2} \int_U (1 + (\gamma^\varepsilon)' )|Du^\varepsilon|^2 \sigma^\varepsilon \, dx + \varepsilon \int_U |D^2 u^\varepsilon|^2 \sigma^\varepsilon \, dx \leq C. \quad (2.9)
\]

**Proof.** Multiplying (2.6) by \( \sigma^\varepsilon \) and integrating by parts, using equation (2.6) we get
\[
\frac{1}{2} \int_U (1 + (\gamma^\varepsilon)' )|Du^\varepsilon|^2 \sigma^\varepsilon \, dx + \varepsilon \int_U |D^2 u^\varepsilon|^2 \sigma^\varepsilon \, dx
\]
\[
= -w^\varepsilon(x_0) - \int_U [D_x H \cdot Du^\varepsilon - (\gamma^\varepsilon)' D\psi \cdot Du^\varepsilon] \sigma^\varepsilon \, dx - \varepsilon \int_{\partial U} w^\varepsilon \partial \sigma^\varepsilon / \partial \nu \, dS.
\]

Thanks to Lemma 2.3 and Proposition 2.1, the conclusion follows. \[\Box\]

Relation (2.9) shows that we have a good control of the Hessian \( D^2 u^\varepsilon \) in the support of \( \sigma^\varepsilon \).

We finally have the following result, which immediately implies Theorem 1.3.

**Lemma 2.5.** There exists \( C > 0 \), independent of \( \varepsilon \), such that
\[
\max_{x \in \overline{U}} |u^\varepsilon(x)| \leq \frac{C}{\varepsilon^{1/2}}.
\]
Proof. Differentiating (2.2) w.r.t. \( \varepsilon \) we get

\[
(1 + (\gamma^\varepsilon)' u^\varepsilon + D_p H \cdot D u^\varepsilon + \gamma^\varepsilon = \varepsilon \Delta u^\varepsilon + \Delta u^\varepsilon, \quad \text{in } U.
\]  

(2.10)

In addition, we have \( u^\varepsilon(x) = 0 \) for all \( x \in \partial U \), since \( u^\varepsilon(x) = 0 \) on \( \partial U \) for every \( \varepsilon \). So, we may assume that there exists \( x_2 \in U \) such that \( |u^\varepsilon(x_2)| = \max_{x \in U} |u^\varepsilon(x)| \).

Consider the adjoint equation (2.6), and choose \( x_0 = x_2 \). Multiplying by \( \sigma^\varepsilon \) both sides of (2.10) and integrating by parts,

\[
|u^\varepsilon(x_2)| = - \int_U \gamma^\varepsilon \sigma^\varepsilon dx + \int_U \Delta u^\varepsilon \sigma^\varepsilon dx.
\]  

(2.11)

By Lemma 2.2,

\[
|\gamma^\varepsilon| = \left| \frac{u^\varepsilon - \psi}{\varepsilon} \gamma' \left( \frac{u^\varepsilon - \psi}{\varepsilon} \right) \right| = \left| \frac{u^\varepsilon - \psi}{\varepsilon} (\gamma^\varepsilon)' (u^\varepsilon - \psi) \right| \leq C(\gamma^\varepsilon)'.
\]  

(2.12)

Hence, thanks to Lemma 2.3

\[
\int_U |\gamma^\varepsilon| \sigma^\varepsilon dx \leq C \int_U (\gamma^\varepsilon)' \sigma^\varepsilon dx \leq C,
\]  

(2.13)

while using (2.9)

\[
\int_U |\Delta u^\varepsilon| \sigma^\varepsilon dx \leq \left( \int_U |\Delta u^\varepsilon|^2 \sigma^\varepsilon dx \right)^{1/2} \left( \int_U \sigma^\varepsilon dx \right)^{1/2}
\]

\[
\leq C \left( \int_U |D^2 u^\varepsilon|^2 \sigma^\varepsilon dx \right)^{1/2} \left( \int_U \sigma^\varepsilon dx \right)^{1/2} \leq \frac{C}{\varepsilon^{1/2}}.
\]  

(2.14)

Thus, by (2.11), (2.13) and (2.14)

\[
|u^\varepsilon(x_2)| \leq \frac{C}{\varepsilon^{1/2}}, \quad \text{for } \varepsilon < 1.
\]  

(2.15)

\[\square\]

2.3. Dynamic interpretation. We give now a dynamic interpretation of the measure \( \sigma^\varepsilon \). Thanks to the properties given by Lemma 2.3 and arguing as in [Eva10] and [CGT], we have the following theorem.

**Theorem 2.6.** There exist

(i) a measure \( \mu \) on \( U \times \mathbb{R}^n \), such that

\[
\int_U \phi(x, Du^\varepsilon) \sigma^\varepsilon dx \rightarrow \int_{U \times \mathbb{R}^n} \phi(x, p) d\mu, \quad \forall \phi \in C(\overline{U} \times \mathbb{R}^n);
\]
(ii) a measure $\gamma_1$ on $\overline{U} \times \mathbb{R}^n$, and a measure $\gamma_2$ on $\partial U \times \mathbb{R}^n$, such that

$$\int_{U} (1 + (\gamma^\varepsilon)' \phi(x, Du^\varepsilon) \sigma^\varepsilon \, dx \to \int_{U \times \mathbb{R}^n} \phi(x, p) \, d\gamma_1, \quad \forall \phi \in C(\overline{U} \times \mathbb{R}^n),$$

and

$$-\varepsilon \int_{\partial U} \frac{\partial \sigma^\varepsilon}{\partial \nu} \phi(x, Du^\varepsilon) \, dS \to \int_{\partial U \times \mathbb{R}^n} \phi(x, p) \, d\gamma_2, \quad \forall \phi \in C(\partial U \times \mathbb{R}^n),$$

with the property that $\gamma_1(U \times \mathbb{R}^n) + \gamma_2(\partial U \times \mathbb{R}^n) = 1$;

(iii) a nonnegative definite matrix of measures $(m_{jk})$, that we call matrix of dissipation measures, such that

$$\varepsilon \int_{U} \phi(x, Du^\varepsilon) u^\varepsilon_{ix, jx} u^\varepsilon_{ix, xj} \sigma^\varepsilon \, dx \to \int_{U \times \mathbb{R}^n} \phi(x, p) \, dm_{jk}, \quad \forall \phi \in C(\overline{U} \times \mathbb{R}^n), \quad \forall j, k = 1, \ldots, n,$

where we used Einstein summation convention;

(iv) a compact set $K \subset \mathbb{R}^n$ such that

$$\text{supp } \mu, \text{supp } \gamma_1, \text{supp } m_{jk} \subset \overline{U} \times K, \quad \text{supp } \gamma_2 \subset \partial U \times K.$$

Next theorem gives a relation involving the measures $\mu, \gamma_1, \gamma_2, m_{kj}$.

**Theorem 2.7.** For any $\phi \in C^1(\overline{U} \times \mathbb{R}^n)$ with $\phi(x, \cdot) \in C^2(\mathbb{R}^n)$ and for any $x \in \overline{U}$, we have

$$\lim_{\varepsilon \to 0^+} \phi(x_0, Du^\varepsilon(x_0)) = \int_{\overline{U} \times \mathbb{R}^n} (D_p \phi \cdot (D\psi - p) + \phi) \, d\gamma_1 + \int_{\partial U \times \mathbb{R}^n} \phi \, d\gamma_2$$

$$+ \int_{U \times \mathbb{R}^n} \{H, \phi\} \cdot D_p \phi \, d\mu - \int_{U \times \mathbb{R}^n} \phi_{p, p_k} \, dm_{jk},$$

where the symbol $\{\cdot, \cdot\}$ stands for the Poisson bracket, that is

$$\{F, G\} := D_p F \cdot D_x G - D_x F \cdot D_p G, \quad \forall F, G \in C^1(\overline{U} \times \mathbb{R}^n).$$

In particular, if $\phi(x, p) = \phi(x)$ then

$$\int_{U \times \mathbb{R}^n} \phi \, d\gamma_1 + \int_{\partial U \times \mathbb{R}^n} \phi \, d\gamma_2 + \int_{U \times \mathbb{R}^n} \{H, \phi\} \, d\mu = \phi(x_0).$$

**Proof.** Let us set $\varphi^\varepsilon(x) := \phi(x, Du^\varepsilon(x))$ for every $x \in \overline{U}$. Then

$$\varphi^\varepsilon_{x_i} = \phi_{x_i} + \phi_{p_j} u^\varepsilon_{x_i, xj}, \quad i = 1, \ldots, n,$$

and

$$\Delta \varphi^\varepsilon = \Delta_x \phi + 2\phi_{x_ip_j} u^\varepsilon_{x_i, xj} + D_p \phi \cdot D(\Delta u^\varepsilon) + \phi_{p_j p_k} u^\varepsilon_{x_i, xj} u^\varepsilon_{x_i, x_k},$$

where we used the notation $\Delta_x \phi = \sum_i \phi_{x_i, x_i}$. Differentiating (2.2) w.r.t. $x$ and computing the scalar product by $D_p \phi$, we get

$$(1 + (\gamma^\varepsilon)') D_p \phi \cdot Du^\varepsilon - (\gamma^\varepsilon)' D_p \phi \cdot D\psi + D_x H \cdot D_p \phi + H_{p_j} \phi_{p_k} u^\varepsilon_{x_i, xj} = \varepsilon D_p \phi \cdot D(\Delta u^\varepsilon).$$
Thanks to the above calculation on $\varphi^\varepsilon_{x_i}$ and $\Delta \varphi^\varepsilon$,

$$(1 + (\gamma^\varepsilon)^') D_p \phi \cdot Du^\varepsilon - (\gamma^\varepsilon)' D_p \phi \cdot D\psi + D_x H \cdot D_p \phi + H_{p_1}(\varphi^\varepsilon_{x_j} - \phi_{x_j})$$

$$= \varepsilon (\Delta \varphi^\varepsilon - \Delta_x \phi - 2\phi_{x_i} u^\varepsilon_{x_i,j} - \phi_{p_i p_k} u^\varepsilon_{x_i,j,x_k}).$$

Hence, adding and subtracting the term $D_p \phi \cdot D\psi$

$$(1 + (\gamma^\varepsilon)^') D_p \phi \cdot (Du^\varepsilon - D\psi) + D_p \phi \cdot D\psi - \{H, \phi\} + D_p H \cdot D\varphi^\varepsilon$$

$$= \varepsilon (\Delta \varphi^\varepsilon - \varepsilon (\Delta_x \phi + 2\phi_{x_i} u^\varepsilon_{x_i,j}) - \varepsilon \phi_{p_i p_k} u^\varepsilon_{x_i,j,x_k}).$$

(2.16)

Multiplying (2.16) by $\sigma^\varepsilon$ and integrating by parts over $U$,

$$\int_U (D_p \phi \cdot (Du^\varepsilon - D\psi) - \varphi^\varepsilon)(1 + (\gamma^\varepsilon)^') \sigma^\varepsilon \, dx + \int_U (D_p \phi \cdot D\psi - \{H, \phi\}) \sigma^\varepsilon \, dx$$

$$- \int_U \varphi^\varepsilon \text{div}(D_p H \sigma^\varepsilon) \, dx - \int_U \varphi^\varepsilon \Delta \sigma^\varepsilon \, dx + \int_U \varphi^\varepsilon (1 + (\gamma^\varepsilon)^') \sigma^\varepsilon \, dx$$

$$= -\varepsilon \int_{\partial U} \varphi^\varepsilon \frac{\partial \sigma^\varepsilon}{\partial \nu} \, dS - \int_U \varepsilon (\Delta_x \phi + 2\phi_{x_i} u^\varepsilon_{x_i,j}) \sigma^\varepsilon \, dx - \varepsilon \int_U \phi_{p_i p_k} u^\varepsilon_{x_i,j,x_k} \sigma^\varepsilon \, dx. \quad (2.17)$$

Recalling equation (2.6) and the definition of $\varphi^\varepsilon$,

$$\int_U (D_p \phi \cdot (Du^\varepsilon - D\psi) - \varphi^\varepsilon)(1 + (\gamma^\varepsilon)^') \sigma^\varepsilon \, dx$$

$$+ \int_U (D_p \phi \cdot D\psi - \{H, \phi\}) \sigma^\varepsilon \, dx + \phi(x_0, Du^\varepsilon(x_0)) + \varepsilon \int_{\partial U} \varphi^\varepsilon \frac{\partial \sigma^\varepsilon}{\partial \nu} \, dS$$

$$= - \int_U \varepsilon (\Delta_x \phi + 2\phi_{x_i} u^\varepsilon_{x_i,j}) \sigma^\varepsilon \, dx - \varepsilon \int_U \phi_{p_i p_k} u^\varepsilon_{x_i,j,x_k} \sigma^\varepsilon \, dx. \quad (2.18)$$

Thanks to Hölder inequality and using (2.10),

$$\left| \lim_{\varepsilon \to 0^+} \int_U \varepsilon (\Delta_x \phi + 2\phi_{x_i} u^\varepsilon_{x_i,j}) \sigma^\varepsilon \, dx \right| \leq \varepsilon \lim_{\varepsilon \to 0^+} \int_U C(1 + |Du^\varepsilon|) \sigma^\varepsilon \, dx \leq \lim_{\varepsilon \to 0^+} C \sqrt{\varepsilon} = 0. \quad (2.19)$$

Letting $\varepsilon \to 0^+$, using Theorem 2.6 and relation (2.18) we finally get

$$\lim_{\varepsilon \to 0^+} \phi(x_0, Du^\varepsilon(x_0)) = \int_{U \times \mathbb{R}^n} (D_p \phi \cdot (D\psi - p) + \phi) \, d\gamma_1 + \int_{\partial U \times \mathbb{R}^n} \phi \, d\gamma_2$$

$$+ \int_{U \times \mathbb{R}^n} \{H, \phi\} - D_p \phi \cdot D\psi \, d\mu - \int_{U \times \mathbb{R}^n} \phi_{p_i p_k} \, dm_{jk}. \quad (2.20)$$

2.4. Stochastic process interpretation. In this section we show that the problem could have been approached also by using stochastic processes. Let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $w_t$ be a $n$-dimensional Brownian motion on $\Omega$. Let $\varepsilon > 0$, and let $u^\varepsilon$ be a solution of (2.2). Let $T > 0$ and consider the solution $x^\varepsilon : [0, T] \to \mathbb{R}^n$ of the SDE

$$\begin{cases}
\frac{dx^\varepsilon}{dt} = -D_p H(x^\varepsilon, Du^\varepsilon(x^\varepsilon)) \, dt + \sqrt{2\varepsilon} \, dw_t, \\
x^\varepsilon(0) = \tau,
\end{cases} \quad (2.19)$$
with \( \mathcal{F} \in U \) arbitrary. Accordingly, the momentum variable is defined as

\[
p^\varepsilon(t) = Du^\varepsilon(x^\varepsilon(t)).
\]

We then define the exit time as

\[
T^\varepsilon_{\partial U} := \inf \{ t \in (0, T] : x^\varepsilon(t) \in \partial U \}.
\]

Let us now recall some basic facts about stochastic calculus. Suppose \( \mathcal{Z} : [0, T] \to \mathbb{R}^n \) is a solution to the SDE:

\[
d\mathcal{Z}_t = a_i \, dt + b_{ij} \, w^j_t, \quad i = 1, \ldots, n,
\]

with \( a_i \) and \( b_{ij} \) bounded and progressively measurable processes. Let \( \phi : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \) be a smooth function. Then, \( \phi(\mathcal{Z}, t) \) satisfies the Itô formula:

\[
d\phi = \phi_{z_i} dz^i + \left( \phi_t + \frac{1}{2} \sum_{j,k} b_{ij} b_{jk} \phi_{z_i z_k} \right) dt.
\]

An integrated version of the Itô formula is the Dynkin's formula:

\[
E[\phi(\mathcal{Z}(T)) - \phi(\mathcal{Z}(0))] = E \left[ \int_0^T \left( a_i \, D\mathcal{Z}_t, \phi(\mathcal{Z}(t)) + \frac{1}{2} \sum_{j,k} b_{ij} b_{jk} D^2 \phi(\mathcal{Z}(t)) \right) \, dt \right].
\]

In the present situation, we have

\[
a_i = -D_{p_i} H(x^\varepsilon, D_x u^\varepsilon), \quad b_{ij} = \sqrt{2 \varepsilon} \delta_{ij}.
\]

Hence, recalling (2.19) and (2.20)

\[
dp = D^2 u^\varepsilon \, dx^\varepsilon + \varepsilon \Delta(Du^\varepsilon)dt
\]

\[
= \left( -D^2 u^\varepsilon D_p H + \varepsilon \Delta(Du^\varepsilon) \right) dt + \sqrt{2 \varepsilon} D^2 u^\varepsilon \, dw_t
\]

\[
= \left( D_x H + (1 + (\gamma^\varepsilon))' p^\varepsilon - (\gamma^\varepsilon)' D\psi \right) dt + \sqrt{2 \varepsilon} D^2 u^\varepsilon \, dw_t,
\]

where in the last equality we used the identity obtained by differentiating (2.2) with respect to \( x \).

Thus, \((x^\varepsilon, p^\varepsilon)\) satisfies the following stochastic version of the Hamiltonian dynamics:

\[
\begin{align*}
\frac{dx^\varepsilon}{dt} &= -D_p H(x^\varepsilon, p^\varepsilon)\ dt + \sqrt{2 \varepsilon} \, dw_t, \\
\frac{dp^\varepsilon}{dt} &= \left( D_x H(x^\varepsilon, p^\varepsilon) + (1 + (\gamma^\varepsilon))' p^\varepsilon - (\gamma^\varepsilon)' D\psi \right) dt + \sqrt{2 \varepsilon} D^2 u^\varepsilon \, dw_t.
\end{align*}
\]

We are now going to study the behavior of the solutions \( u^\varepsilon \) of equation (2.2) along the trajectory \((x^\varepsilon(\cdot), p^\varepsilon(\cdot))\). Let \( \phi \in C^1(\overline{U} \times \mathbb{R}^n) \). Thanks to the Itô formula and (2.21), the differential of the
Thus, by Dynkin’s formula we have the following equality
\[
\phi(x) = D_x\phi \cdot dx + D_p\phi \cdot dp + \varepsilon \left( \Delta_x \phi + 2\varepsilon \phi_{x,p} u_{x,j} + \phi_{p,p} u_{x,x} u_{x,x} \right) dt
\]
which we may also write as
\[
\phi(x) = \left( \{\phi, H\} + (1 + (\gamma^\varepsilon)'D_p\phi \cdot D\psi - (\gamma^\varepsilon)'D_p\phi \cdot D\psi \right) dt
\]
and
\[
\phi(x) = \left( \{\phi, H\} + (1 + (\gamma^\varepsilon)'D_p\phi \cdot (p^\varepsilon - D\psi) + D_p\phi \cdot D\psi \right) dt
\]
Thus, by Dynkin’s formula we have the following equality
\[
E[\phi(x) - \phi(x_0, Du)] = \int_0^{T_{u'}} \left( D_p\phi \cdot D\psi - \phi \right) dt
\]
which we may also write as
\[
E\left[ \int_0^{T_{u'}} (D_p\phi \cdot (p^\varepsilon - D\psi) - \phi) \left( 1 + (\gamma^\varepsilon)' \right) dt \right]
\]
Relation (2.22) is the analogous of (2.17), and can be as well used (together with suitable estimates) to prove Theorem 3.3

3. Weakly coupled systems of Hamilton–Jacobi equations

We study now the model of monotone weakly coupled systems of Hamilton-Jacobi equations considered by Engler and Lenhart [EL91], and by Ishii and Koike [IK91]. For the sake of simplicity, we will just focus on the following system of two equations:

\[
\begin{cases}
  c_{11}u_1 + c_{12}u_2 + H_1(x, Du_1) = 0 & \text{in } U, \\
  c_{21}u_1 + c_{22}u_2 + H_2(x, Du_2) = 0
\end{cases}
\]
with boundary conditions \( u_1 = u_2 = 0 \) on \( \partial U \). The general case of more equations or arbitrary boundary data can be treated in a similar way.

We assume that the Hamiltonians \( H_1, H_2 : U \times \mathbb{R}^n \to \mathbb{R} \) are smooth satisfying
We observe that, as a consequence, we also have for every $j = 1, 2$. Following [EL91] and [IK91], we suppose further that

(H3.2) $c_{12}, c_{21} \leq 0$;

(H3.3) there exists $\alpha > 0$ such that $c_{11} + c_{12}, c_{21} + c_{22} \geq \alpha > 0$.

We observe that, as a consequence, we also have $c_{11}, c_{22} > 0$. Finally, we require that

(H3.4) There exist $\Phi_1, \Phi_2 \in C^2(U) \cap C^1(\overline{U})$ with $\Phi_j = 0$ on $\partial U$ ($j = 1, 2$), and such that

\[
\begin{cases}
  c_{11}\Phi_1 + c_{12}\Phi_2 + H_1(x, D\Phi_1) < 0 & \text{in } U, \\
  c_{22}\Phi_2 + c_{21}\Phi_1 + H_2(x, D\Phi_2) < 0 & \text{in } U.
\end{cases}
\]

Thanks to these conditions, the Maximum Principle can be applied and existence, comparison and uniqueness results hold true, as stated in [EL91].

We consider now the following regularized system (here $\varepsilon > 0$):

\[
\begin{cases}
  c_{11}\varepsilon u_1^\varepsilon + c_{12}\varepsilon u_2^\varepsilon + H_1(x, D\varepsilon u_1^\varepsilon) = \varepsilon \Delta u_1^\varepsilon & \text{in } U, \\
  c_{21}\varepsilon u_1^\varepsilon + c_{22}\varepsilon u_2^\varepsilon + H_2(x, D\varepsilon u_2^\varepsilon) = \varepsilon \Delta u_2^\varepsilon & \text{in } U,
\end{cases}
\]  

(3.2)

with boundary conditions $u_1^\varepsilon = u_2^\varepsilon = 0$ on $\partial U$.

Conditions (H3.1), (H3.2), and (H3.3) yield the existence and uniqueness of the pair of solutions $(u_1^\varepsilon, u_2^\varepsilon)$ in (3.2).

**Lemma 3.1.** There exists a positive constant $C$, independent of $\varepsilon$, such that

\[ ||u_1^\varepsilon||_{L^\infty}, ||u_2^\varepsilon||_{L^\infty} \leq C. \]

**Proof.** First of all observe that $u_1^\varepsilon = u_2^\varepsilon = 0$ on $\partial U$ for every $\varepsilon$. Thus, it will be sufficient to show that $u_1^\varepsilon$ and $u_2^\varepsilon$ are bounded in the interior of $U$.

Let us assume that there exists $\overline{x} \in U$ such that

\[ \max_{j=1,2} \max_{x \in \overline{U}} u_j^\varepsilon(x) = u_1^\varepsilon(\overline{x}). \]

We have

\[ \alpha u_1^\varepsilon(\overline{x}) \leq c_{11} u_1^\varepsilon(\overline{x}) + c_{12} u_2^\varepsilon(\overline{x}) \leq -H_1(\overline{x}, 0) \leq \max_{x \in \overline{U}} (-H_1(x, 0)), \]

where we used (H3.3). Analogously, if $\hat{x} \in U$ is such that

\[ \min_{j=1,2} \min_{x \in \overline{U}} u_j^\varepsilon(x) = u_1^\varepsilon(\hat{x}), \]

then

\[ u_1^\varepsilon(\hat{x}) \geq \frac{c_{11}}{c_{11} + c_{12}} u_1^\varepsilon(\hat{x}) + \frac{c_{12}}{c_{11} + c_{12}} u_2^\varepsilon(\hat{x}) \geq \frac{H_1(\hat{x}, 0)}{c_{11} + c_{12}} \geq \frac{1}{c_{11} + c_{12}} \min_{x \in \overline{U}} (-H_1(x, 0)). \]
Lemma 3.2. There exists a positive constant $C$, independent of $\varepsilon$, such that

$$||Du_1^\varepsilon||_{L^\infty}, ||Du_2^\varepsilon||_{L^\infty} \leq C.$$  

Proof. We will argue as in the proof of Proposition 2.1.

Step I: Bound on $\partial U$. We shall first show that

$$\max_{j=1,2} \max_{x \in \partial U} |Du_j^\varepsilon(x)| \leq C,$$

for some constant $C$ independent of $\varepsilon$. As it was done in Section 2, we are going to construct appropriate barriers. For $\varepsilon$ small enough, assumption (H3.4) implies that

$$\begin{cases}
  c_{11}\Phi_1 + c_{12}\Phi_2 + H_1(x, Du_1) < \varepsilon \Delta \Phi_1 & \text{in } U, \\
  c_{22}\Phi_2 + c_{21}\Phi_1 + H_2(x, Du_2) < \varepsilon \Delta \Phi_2 & \text{in } U,
\end{cases}$$

and $\Phi_1 = \Phi_2 = 0$ on $\partial U$. Therefore, $(\Phi_1, \Phi_2)$ is a sub-solution of (3.2). By the comparison principle, $u_j^\varepsilon \geq \Phi_j$ in $U$, $j = 1, 2$.

Let $d(x)$, $\delta$, and $U_0$ be as in the proof of Proposition 2.1. For $\mu > 0$ large enough, the uniform bounds on $||u_1^\varepsilon||_{L^\infty}$ and $||u_2^\varepsilon||_{L^\infty}$ yield $v := \mu d \geq u_j^\varepsilon$ on $\partial U_0$, $j = 1, 2$, so that

$$\begin{cases}
  (c_{11} + c_{12})v + H_1(x, Dv) - \varepsilon \Delta v \geq H_1(x, \mu Dd) - \mu C & \text{in } U, \\
  (c_{21} + c_{22})v + H_2(x, Dv) - \varepsilon \Delta v \geq H_2(x, \mu Dd) - \mu C & \text{in } U.
\end{cases}$$

Now, we have $\Phi_j = u_j^\varepsilon = v = 0$ on $\partial U$. Also, thanks to assumption (H3.1), for $\mu > 0$ large enough

$$\begin{cases}
  (c_{11} + c_{12})v + H_1(x, Dv) - \varepsilon \Delta v \geq 0 & \text{in } U, \\
  (c_{21} + c_{22})v + H_2(x, Dv) - \varepsilon \Delta v \geq 0 & \text{in } U,
\end{cases}$$

that is, the pair $(v, v)$ is a super-solution for the system (3.2). Thus, the comparison principle gives us that $\Phi_j \leq u_j^\varepsilon \leq v$ in $U_0$, $j = 1, 2$. Then, from the fact that $\Phi_j = u_j^\varepsilon = v = 0$ on $\partial U$ we get

$$\frac{\partial v}{\partial \nu}(x) \leq \frac{\partial u_j^\varepsilon}{\partial \nu}(x) \leq \frac{\partial \Phi_j}{\partial \nu}(x), \quad \text{for } x \in \partial U, \quad j = 1, 2.$$

Hence, we obtain $||Du_j^\varepsilon||_{L^\infty(\partial U)} \leq C$, $j = 1, 2$.

Step II: Bound on $U$.

Setting $u_j^\varepsilon = \frac{|Du_j^\varepsilon|^2}{2}$, $j = 1, 2$, by a direct computation we have that

$$\begin{cases}
  2c_{11}u_1^\varepsilon + D_p H_1 \cdot Du_1^\varepsilon + c_{12}Du_1^\varepsilon \cdot Du_2^\varepsilon + D_x H_1 \cdot Du_1^\varepsilon = \varepsilon \Delta u_1^\varepsilon - \varepsilon |D^2 u_1^\varepsilon|^2, \\
  2c_{22}u_2^\varepsilon + D_p H_2 \cdot Du_2^\varepsilon + c_{21}Du_1^\varepsilon \cdot Du_2^\varepsilon + D_x H_2 \cdot Du_2^\varepsilon = \varepsilon \Delta u_2^\varepsilon - \varepsilon |D^2 u_2^\varepsilon|^2. \quad (3.3)
\end{cases}$$

Assume now that there exists $\hat{x} \in U$ such that
\[
\max_{j=1,2} \max_{x \in \Gamma^j} w_j^\varepsilon(x) = w_1^\varepsilon(\hat{x}).
\]
Then, we have
\[
\varepsilon|D^2 u_1^\varepsilon|^2(\hat{x}) = \varepsilon\Delta w_1^\varepsilon(\hat{x}) - 2c_{11} w_1^\varepsilon(\hat{x}) - c_{12} Du_1^\varepsilon(\hat{x}) \cdot Du_2^\varepsilon(\hat{x}) - D_x H_1 \cdot Du_1^\varepsilon(\hat{x})
\]
\[
\leq -2(c_{11} + c_{12}) w_1^\varepsilon(\hat{x}) - D_x H_1 \cdot Du_1^\varepsilon(\hat{x}) \leq -D_x H_1 \cdot Du_1^\varepsilon(\hat{x}).
\]
Now, for $\varepsilon$ sufficiently small
\[
\varepsilon|D^2 u_1^\varepsilon(\hat{x})|^2 \geq \varepsilon^2|\Delta u_1^\varepsilon(\hat{x})|^2 = [c_{11} u_1^\varepsilon(\hat{x}) + c_{12} u_2^\varepsilon(\hat{x}) + H_1(\hat{x}, Du_1^\varepsilon(\hat{x}))]^2
\]
\[
\geq \frac{1}{2} H_1(\hat{x}, Du_1^\varepsilon(\hat{x}))^2 - C.
\]
Collecting the last two relations we have
\[
\frac{1}{2} H_1(\hat{x}, Du_1^\varepsilon(\hat{x}))^2 + D_x H_1(\hat{x}, Du_1^\varepsilon(\hat{x})) \cdot Du_1^\varepsilon(\hat{x}) \leq C.
\]
Recalling condition (H3.1) the conclusion follows.

\hfill \Box

**Adjoint method.** At this point, we introduce the adjoint of the linearization of system (3.2).

The linearized operator corresponding to (3.2) is
\[
L^\varepsilon(z_1, z_2) := \begin{cases} 
D_p H_1(x, Du_1^\varepsilon) \cdot D z_1 + c_{11} z_1 + c_{12} z_2 - \varepsilon \Delta z_1, \\
D_p H_2(x, Du_2^\varepsilon) \cdot D z_2 + c_{22} z_2 + c_{21} z_1 - \varepsilon \Delta z_2.
\end{cases}
\]
Let us now identify the adjoint operator $(L^\varepsilon)^*$. For every $\nu^1, \nu^2 \in C_0^\infty(U)$ we have
\[
\langle (L^\varepsilon)^*(\nu^1, \nu^2), (z_1, z_2) \rangle := \langle (\nu^1, \nu^2), L^\varepsilon(z_1, z_2) \rangle
\]
\[
= \langle \nu^1, [L^\varepsilon(z_1, z_2)]_1 \rangle + \langle \nu^2, [L^\varepsilon(z_1, z_2)]_2 \rangle
\]
\[
= \int_U [D_p H_1(x, Du_1^\varepsilon) \cdot D z_1 + c_{11} z_1 + c_{12} z_2 - \varepsilon \Delta z_1] \nu^1 \, dx
\]
\[
+ \int_U [D_p H_2(x, Du_2^\varepsilon) \cdot D z_2 + c_{22} z_2 + c_{21} z_1 - \varepsilon \Delta z_2] \nu^2 \, dx
\]
\[
= \int_U [-\text{div}(D_p H_1 \nu^1) + c_{11} \nu^1 + c_{21} \nu^2 - \varepsilon \Delta \nu^1] \, z_1 \, dx
\]
\[
+ \int_U [-\text{div}(D_p H_2 \nu^2) + c_{22} \nu^2 + c_{12} \nu^1 - \varepsilon \Delta \nu^2] \, z_2 \, dx.
\]
Then, the adjoint equations are:
\[
\begin{cases}
-\text{div}(D_p H_1 \sigma^{1, \varepsilon}) + c_{11} \sigma^{1, \varepsilon} + c_{21} \sigma^{2, \varepsilon} = \varepsilon \Delta \sigma^{1, \varepsilon} + (2 - i) \delta_{x_0} & \text{in } U, \\
-\text{div}(D_p H_2 \sigma^{2, \varepsilon}) + c_{22} \sigma^{2, \varepsilon} + c_{12} \sigma^{1, \varepsilon} = \varepsilon \Delta \sigma^{2, \varepsilon} + (i - 1) \delta_{x_0} & \text{in } U,
\end{cases}
\]
with boundary conditions

\[
\begin{cases}
\sigma^{1,\varepsilon} = 0 & \text{on } \partial U, \\
\sigma^{2,\varepsilon} = 0 & \text{on } \partial U,
\end{cases}
\]

where \(i \in \{1, 2\}\) and \(x_0 \in U\) will be chosen later. Existence and uniqueness of \(\sigma^{1,\varepsilon}\) and \(\sigma^{2,\varepsilon}\) follow by Fredholm alternative, by arguing as in Section 2, and we have \(\sigma^{1,\varepsilon}, \sigma^{2,\varepsilon} \in C^\infty(U \setminus \{x_0\})\). We study now further properties of \(\sigma^{1,\varepsilon}\) and \(\sigma^{2,\varepsilon}\).

**Lemma 3.3** (Properties of \(\sigma^{1,\varepsilon}, \sigma^{2,\varepsilon}\)). Let \(\nu\) be the outer unit normal to \(\partial U\). Then

(i) \(\sigma^{\cdot,\varepsilon} \geq 0\) on \(\bar{U}\). In particular, \(\frac{\partial \sigma^{\cdot,\varepsilon}}{\partial \nu} \leq 0\) on \(\partial U\) \((j = 1, 2)\).

(ii) The following equality holds:

\[
\sum_{j=1}^{2} \left( \int_U (c_{j1} + c_{j2})\sigma^{j,\varepsilon} \, dx - \varepsilon \int_{\partial U} \frac{\partial \sigma^{j,\varepsilon}}{\partial \nu} \, dS \right) = 1.
\]

In particular,

\[
\sum_{j=1}^{2} \int_U (c_{j1} + c_{j2})\sigma^{j,\varepsilon} \, dx \leq 1.
\]

**Proof.** First of all, we consider the adjoint of equation (3.4):

\[
\begin{cases}
D_p H_1(x, Du^\varepsilon) \cdot Dz_1 + c_{11} z_1 + c_{12} z_2 - \varepsilon \Delta z_1 = f_1, \\
D_p H_2(x, Du^\varepsilon) \cdot Dz_2 + c_{22} z_2 + c_{21} z_1 - \varepsilon \Delta z_2 = f_2,
\end{cases}
\]

where \(f_1, f_2 \in C(U)\), with boundary conditions \(z_1 = z_2 = 0\) on \(\partial U\). Note that

\[
f_1, f_2 \geq 0 \implies \min_{x \in \overline{U}} z_j(x) \geq 0.
\]

Indeed, if the minimum is achieved for some \(\overline{x} \in \partial U\), then clearly \(z_1, z_2 \geq 0\). Otherwise, assume

\[
\min_{x \in \overline{U}} z_j(x) = z_1(\overline{x}),
\]

for some \(\overline{x} \in U\). Using condition (H3.2)

\[
(c_{11} + c_{12})z_1(\overline{x}) \geq c_{11} z_1(\overline{x}) + c_{12} z_2(\overline{x}) + \varepsilon \Delta z_1(\overline{x}) + f_1(\overline{x}) \geq 0.
\]

Thanks to (H3.3), (3.6) follows.

Let us now multiply \((3.4)_1\) and \((3.4)_2\) by the solutions \(z_1\) and \(z_2\) of \((3.5)\). Adding up the relations obtained we have

\[
\int_U f_1 \sigma^{1,\varepsilon} \, dx + \int_U f_2 \sigma^{2,\varepsilon} \, dx = (2 - i)z_1(x_0) + (1 - i)z_2(x_0).
\]

Thanks to (3.6), from last relation we conclude that

\[
\int_U f_1 \sigma^{1,\varepsilon} \, dx + \int_U f_2 \sigma^{2,\varepsilon} \, dx \geq 0, \quad \text{for every } f_1, f_2 \geq 0,
\]
and this implies that $\sigma^{1,\varepsilon}, \sigma^{2,\varepsilon} \geq 0$. To prove (ii), it is sufficient to integrate equations (3.3) and (3.4) over $U$, and to add up the two relations obtained.

**Lemma 3.3.** There exists a constant $C > 0$, independent of $\varepsilon$, such that

$$
\varepsilon \int_U |D^2u_1^n| \sigma^{1,\varepsilon} dx + \varepsilon \int_U |D^2u_2^n| \sigma^{2,\varepsilon} dx \leq C.
$$

**Proof.** To show the lemma, one has first to multiply equations (3.3) and (3.4) by $\sigma^{1,\varepsilon}$ and $\sigma^{2,\varepsilon}$ respectively. Then, adding up the relations obtained and using (3.4), thanks to Lemma 3.1 and Lemma 3.2 the conclusion follows.

We now give the last lemma needed to estimate the speed of convergence. Here we use the notation $u_{\varepsilon j}(x) := \partial u_{\varepsilon j}(x)/\partial \varepsilon$, $j = 1, 2$.

**Lemma 3.5.** There exists a constant $C > 0$, independent of $\varepsilon$, such that

$$
\max_{j=1,2} \max_{x \in \Omega} |u_{\varepsilon j}(x)| \leq \frac{C}{\varepsilon^{1/2}}.
$$

**Proof.** Differentiating (3.2) w.r.t. $\varepsilon$ we obtain the system

$$
\begin{align*}
\begin{cases}
\begin{align*}
c_{11}u_{1,\varepsilon} + c_{12}u_{2,\varepsilon} + D_pH_1 \cdot Du_{1,\varepsilon} &= \varepsilon \Delta u_{1,\varepsilon} + Du_{1,\varepsilon}, \\
c_{21}u_{1,\varepsilon} + c_{22}u_{2,\varepsilon} + D_pH_2 \cdot Du_{2,\varepsilon} &= \varepsilon \Delta u_{2,\varepsilon} + Du_{2,\varepsilon}.
\end{align*}
\end{cases}
\end{align*}
$$

(3.7)

Since $u_{1,\varepsilon} = u_{2,\varepsilon} = 0$ on $\partial U$, we have

$$
\max_{x \in \partial U} u_{1,\varepsilon}(x) = \max_{x \in \partial U} u_{2,\varepsilon}(x) = 0.
$$

Assume now that there exists $\tilde{x} \in U$ such that

$$
\max_{j=1,2} \max_{x \in U} |u_{\varepsilon j}(x)| = |u_{\varepsilon}(\tilde{x})|,
$$

and let $\sigma^{1,\varepsilon}, \sigma^{2,\varepsilon}$ be the solutions of system (3.4) with $i = 1$ and $x_0 = \tilde{x}$.

Multiplying equations (3.4) and (3.7) by $\sigma^{1,\varepsilon}$ and $\sigma^{2,\varepsilon}$ respectively and adding up, thanks to (3.4) we obtain

$$
u_{\varepsilon j}(\tilde{x}) = \int_U \Delta u_{\varepsilon j} \sigma^{1,\varepsilon} dx + \int_U \Delta u_{\varepsilon j} \sigma^{2,\varepsilon} dx.
$$

Thanks to Lemma 3.4 and repeating the chain of inequalities in (2.14) one can show that

$$
\int_U \Delta u_{\varepsilon j}^{1,\varepsilon} dx \leq \frac{C}{\varepsilon^{1/2}}, \quad j = 1, 2,
$$

and from this the conclusion follows.

We can now prove the following result on the speed of convergence.
Theorem 3.6. There exists $C > 0$, independent of $\varepsilon$, such that
\[ \|u_1^\varepsilon - u_1\|_{L^\infty}, \|u_2^\varepsilon - u_2\|_{L^\infty} \leq C\varepsilon^{1/2}. \]

Proof. The theorem is a direct consequence of Lemma 3.5. \hfill \Box

4. Cell problem for Weakly coupled system of Hamilton–Jacobi equations

In this section, we study the following weakly coupled systems of Hamilton–Jacobi equations:
\[
\begin{aligned}
&c_1u_1 - c_1u_2 + H_1(x, Du_1) = \overline{H}_1 \quad \text{in } \mathbb{T}^n, \\
&-c_2u_1 + c_2u_2 + H_2(x, Du_2) = \overline{H}_2
\end{aligned}
\]
which is the analog of the cell problem for single equation introduced by Lions, Papanicolaou, and Varadhan [LPV88]. We will assume that $H_1, H_2 \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n)$, and:
\[
\begin{aligned}
&\text{(H1)} \quad \lim_{|p| \to +\infty} \left( \frac{1}{2} |H_j(x, p)|^2 + D_x H_j(x, p) \cdot p - 8nc_p^2 |p|^2 \right) = +\infty \text{ uniformly in } x \in \mathbb{T}^n, j = 1, 2; \\
&\text{(H2)} \quad c_1, c_2 > 0.
\end{aligned}
\]

It is easy to see that the coefficients of $u_1, u_2$ in this system do not satisfy the coupling assumptions of the previous section. Indeed, as it happens for the cell problem in the context of weak KAM theory, there is no hope of a uniqueness result for (4.1).

To find the effective Hamiltonians $\overline{H}_1, \overline{H}_2$ we use the same arguments as in [17a]. First, for every $\varepsilon > 0$, let us consider the following regularized system:
\[
\begin{aligned}
&(c_1 + \varepsilon)u_1^\varepsilon - c_1u_2^\varepsilon + H_1(x, Du_1^\varepsilon) = \varepsilon^2 \Delta u_1^\varepsilon \quad \text{in } \mathbb{T}^n, \\
&(c_2 + \varepsilon)u_2^\varepsilon - c_2u_1^\varepsilon + H_2(x, Du_2^\varepsilon) = \varepsilon^2 \Delta u_2^\varepsilon
\end{aligned}
\]

For every $\varepsilon > 0$ fixed, the coefficients of this new system satisfy the coupling assumptions (H3.2) and (H3.3) of the previous section. Thus, (4.2) admits a unique pair of smooth solutions $u_1^\varepsilon, u_2^\varepsilon$.

In particular, this implies that $u_1^\varepsilon$ and $u_2^\varepsilon$ are $\mathbb{T}^n$-periodic.

The following result gives some a priori estimates.

Theorem 4.1. There exists $C > 0$, independent of $\varepsilon$, such that
\[ \|\varepsilon u_1^\varepsilon\|_{L^\infty}, \|\varepsilon u_2^\varepsilon\|_{L^\infty}, \|Du_1^\varepsilon\|_{L^\infty}, \|Du_2^\varepsilon\|_{L^\infty} \leq C. \]

Proof. Our proof is based on the Maximum Principle. Without loss of generality, we may assume that
\[ \max_{x \in \mathbb{T}^n} \{\varepsilon u_j^\varepsilon(x)\} = \varepsilon u_1^\varepsilon(x_0^\varepsilon), \]
for some $x_0^\varepsilon \in \mathbb{T}^n$. Applying the Maximum Principle to the first equation of (4.2),
\[ \varepsilon u_1^\varepsilon(x_0^\varepsilon) \leq (c_1 + \varepsilon)u_1^\varepsilon(x_0^\varepsilon) - c_1u_2^\varepsilon(x_0^\varepsilon) \leq -H_1(x_0^\varepsilon, 0) \leq C, \tag{4.3} \]
and this shows the existence of a bound from above for $\varepsilon u_1^\varepsilon$ and $\varepsilon u_2^\varepsilon$. Using a similar argument one can show that there is also a bound from below, so that

$$||\varepsilon u_1^\varepsilon||_{L^\infty}, ||\varepsilon u_2^\varepsilon||_{L^\infty} \leq C. \quad (4.4)$$

We observe that the previous inequality doesn’t provide any bound for the difference $u_1^\varepsilon(x) - u_2^\varepsilon(x)$ in a generic point $x \in \mathbb{T}^n$. Nevertheless, thanks to (4.4) we have

$$u_1^\varepsilon(x_0^\varepsilon) - u_2^\varepsilon(x_0^\varepsilon) \leq -\frac{1}{c_1}H_1(x_0^\varepsilon, 0) - \frac{\varepsilon}{c_1}u_1^\varepsilon(x_0^\varepsilon) \leq C. \quad (4.5)$$

Then, (4.5) and (4.4) imply that

$$c_1|u_1^\varepsilon(x_0^\varepsilon) - u_2^\varepsilon(x_0^\varepsilon)| \leq C. \quad (4.6)$$

In order to find a bound for the gradients, let us set $w_j^\varepsilon = \frac{|Du_j^\varepsilon|^2}{2}$, $j = 1, 2$. Then, by a direct computation one can see that

$$\begin{cases}
2(c_1 + \varepsilon)w_1^\varepsilon + D_pH_1 \cdot Dw_1^\varepsilon - c_1Du_1^\varepsilon \cdot Du_2^\varepsilon + D_xH_1 \cdot Du_1^\varepsilon = \varepsilon^2\Delta w_1^\varepsilon - \varepsilon^2|D^2u_1^\varepsilon|^2 \\
2(c_2 + \varepsilon)w_2^\varepsilon + D_pH_2 \cdot Dw_2^\varepsilon - c_2Du_1^\varepsilon \cdot Du_2^\varepsilon + D_xH_2 \cdot Du_2^\varepsilon = \varepsilon^2\Delta w_2^\varepsilon - \varepsilon^2|D^2u_2^\varepsilon|^2
\end{cases} \quad \text{in } \mathbb{T}^n.
$$

Without loss of generality, we may assume that there exists $x_1^\varepsilon \in \mathbb{T}^n$ such that

$$\max_{j=1,2} \max_{x \in \mathbb{T}^n} \{w_j^\varepsilon(x)\} = w_1^\varepsilon(x_1^\varepsilon).$$

Then, by the Maximum Principle

$$\varepsilon^2|D^2u_1^\varepsilon(x_1^\varepsilon)|^2 \leq -2(c_1 + \varepsilon)w_1^\varepsilon(x_1^\varepsilon) + c_1Du_1^\varepsilon(x_1^\varepsilon) \cdot Du_2^\varepsilon(x_1^\varepsilon) - D_xH_1 \cdot Du_1^\varepsilon(x_1^\varepsilon)$$

$$\leq -D_xH_1 \cdot Du_1^\varepsilon(x_1^\varepsilon). \quad (4.7)$$

Moreover, for $\varepsilon$ sufficiently small

$$\varepsilon^2|D^2u_1^\varepsilon(x_1^\varepsilon)|^2 \geq \varepsilon^4(\Delta u_1^\varepsilon(x_1^\varepsilon))^2 = [H_1(x_1^\varepsilon, Du_1^\varepsilon(x_1^\varepsilon)) + (c_1 + \varepsilon)u_1^\varepsilon(x_1^\varepsilon) - c_1u_2^\varepsilon(x_1^\varepsilon)]^2. \quad (4.8)$$

Also, thanks to (4.4) and (4.6)

$$|(c_1 + \varepsilon)u_1^\varepsilon(x_1^\varepsilon) - c_1u_2^\varepsilon(x_1^\varepsilon)|$$

$$\leq \varepsilon|u_2^\varepsilon(x_1^\varepsilon)| + c_1|u_1^\varepsilon(x_1^\varepsilon) - u_1^\varepsilon(x_0^\varepsilon)| + c_1|u_2^\varepsilon(x_1^\varepsilon) - u_2^\varepsilon(x_0^\varepsilon)| + c_1|u_1^\varepsilon(x_0^\varepsilon) - u_2^\varepsilon(x_0^\varepsilon)|$$

$$\leq C + c_1|u_1^\varepsilon(x_1^\varepsilon) - u_1^\varepsilon(x_0^\varepsilon)| + c_1|u_2^\varepsilon(x_1^\varepsilon) - u_2^\varepsilon(x_0^\varepsilon)|$$

$$\leq C + 2c_1|Du_1^\varepsilon(x_1^\varepsilon)||x_1^\varepsilon - x_0^\varepsilon| \leq C + 2c_1\sqrt{\varepsilon}Dw_1^\varepsilon(x_1^\varepsilon)|,$$
H where we used the fact that the diameter of $\mathbb{T}^n$ is $\sqrt{n}$. Last relation, together with (4.8), gives that
\[
\varepsilon^2 |D^2 u_1(x_1)|^2 \geq [H_1(x_1, Du_1(x_1)) + (c_1 + \varepsilon)u_1(x_1) - c_1 u_2^2(x_1)]^2 \\
\geq \frac{1}{2} [H_1(x_1, Du_1(x_1))]^2 - C - 8nc_1^2 |Du_1(x_1)|^2.
\]
Using last inequality and (4.7) we have
\[
\frac{1}{2} |H_1(x_1, Du_1(x_1))|^2 + D_2 H_1 : Du_1(x_1) - 8nc_1^2 |Du_1(x_1)|^2 \leq C.
\]
This, thanks to condition (H1), gives the conclusion. \qed

Thanks to Theorem 4.1 up to subsequences,
\[
\varepsilon u_1 \to \overline{H}_1, \quad \varepsilon u_2 \to \overline{H}_2, \quad \text{uniformly in } \mathbb{T}^n,
\]
for some constants $\overline{H}_1, \overline{H}_2 \in \mathbb{R}$. Furthermore, still up to subsequences,
\[
u_1^\varepsilon - \min_{\mathbb{T}^n} u_1^\varepsilon \to u_1, \quad \nu_2^\varepsilon - \min_{\mathbb{T}^n} u_2^\varepsilon \to u_2, \quad \text{uniformly in } \mathbb{T}^n,
\]
where $u_1$ and $u_2$ are viscosity solutions of (4.1).

In general, $\overline{H}_1$ and $\overline{H}_2$ are not unique. Indeed, let $u_1, u_2$ be viscosity solutions of (4.1). Then, for every pair of constants $C_1, C_2$, the functions $\tilde{u}_1 := u_1 + C_1$ and $\tilde{u}_2 := u_2 + C_2$ are still viscosity solutions of (4.1), with new effective Hamiltonians
\[
\tilde{H}_1 = \overline{H}_1 + c_1 (C_1 - C_2), \quad \tilde{H}_2 = \overline{H}_2 + c_2 (C_2 - C_1).
\]
Anyway, we have $c_2 \overline{H}_1 + c_1 \overline{H}_2 = c_2 \tilde{H}_1 + c_1 \tilde{H}_2$. This suggests that, although $\overline{H}_1$ and $\overline{H}_2$ may vary, the expression $c_2 \overline{H}_1 + c_1 \overline{H}_2$ is unique. Next theorem shows that this is the case.

**Theorem 4.2.** There exists a constant $\mu \in \mathbb{R}$ such that
\[
c_2 \overline{H}_1 + c_1 \overline{H}_2 = \mu,
\]
for every pair $(\overline{H}_1, \overline{H}_2) \in \mathbb{R}^2$ such that the system (4.1) admits viscosity solutions $u_1, u_2$.

**Proof.** Without loss of generality, we may assume $c_1 = c_2 = 1$. Suppose, by contradiction, that there exist two pairs $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ and $(\mu_1, \mu_2) \in \mathbb{R}^2$, and four functions $u_1, u_2, \tilde{u}_1, \tilde{u}_2 \in C(\mathbb{T}^n)$ such that $\lambda_1 + \lambda_2 < \mu_1 + \mu_2$ and
\[
\begin{cases}
u_1 - u_2 + H_1(x, Du_1) = \lambda_1 \\
u_1 + u_2 + H_2(x, Du_2) = \lambda_2
\end{cases}
\]
and
\[
\begin{align*}
\tilde{u}_1 - \tilde{u}_2 + H_1(x, Du_1) &= \mu_1 \\
-\tilde{u}_1 + \tilde{u}_2 + H_2(x, Du_2) &= \mu_2
\end{align*}
\] in \( T^n \).
\]

By possibly substituting \( u_1 \) and \( u_2 \) with functions \( \tilde{u}_1 := u_1 + C_1 \) and \( \tilde{u}_2 := u_2 + C_2 \), for suitable constants \( C_1 \) and \( C_2 \), we may always assume that \( \lambda_1 < \mu_1, \lambda_2 < \mu_2 \).

In the same way, by a further substitution \( \bar{u}_1 := u_1 + C_3 \), \( \bar{u}_2 := u_2 + C_3 \), with \( C_3 > 0 \) large enough, we may assume that \( u_1 > \bar{u}_1, u_2 > \bar{u}_2 \). Then, there exists \( \varepsilon > 0 \) small enough such that
\[
\begin{align*}
(\varepsilon + 1)u_1 - u_2 + H_1(x, Du_1) &< (\varepsilon + 1)\tilde{u}_1 - \tilde{u}_2 + H_1(x, D\tilde{u}_1) \\
(\varepsilon + 1)u_2 - u_1 + H_2(x, Du_2) &< (\varepsilon + 1)\tilde{u}_2 - \tilde{u}_1 + H_2(x, D\tilde{u}_2)
\end{align*}
\] in \( T^n \).

Observe that the coefficients of the last system satisfy the coupling assumptions (H3.2) and (H3.3). Hence, applying the comparison theorem in [EL91] and [IK91], we conclude that \( u_1 < \tilde{u}_1 \) and \( u_2 < \tilde{u}_2 \), which gives a contradiction.

\[
\square
\]

In the sequel, all the functions will be regarded as functions defined in the whole \( \mathbb{R}^n \) and \( \mathbb{Z}^n \)-periodic. Next lemma provides some a priori bounds on \( u_1^\varepsilon \) and \( u_2^\varepsilon \).

**Lemma 4.3.** There exists a constant \( C > 0 \), independent of \( \varepsilon \), such that
\[
|u_1^\varepsilon(x) - u_2^\varepsilon(y)|, |u_2^\varepsilon(x) - u_2^\varepsilon(y)|, |u_1^\varepsilon(x) - u_1^\varepsilon(y)| \leq C, \quad x, y \in \mathbb{R}^n.
\]

**Proof.** The first two inequalities follow from the periodicity of \( u_1^\varepsilon \) and \( u_2^\varepsilon \), and from the fact that \( Du_1^\varepsilon \) and \( Du_2^\varepsilon \) are bounded.

Let us now show the last inequality. Without loss of generality, we may assume that there exists \( x_0 \in T^n \) such that
\[
\max_{j=1,2}{\max_{x \in T^n}} \{ u_j^\varepsilon(x) \} = u_1^\varepsilon(x_0^\varepsilon).
\]

Combining the second inequality of the lemma with (1.3),
\[
u_1^\varepsilon(x) - u_2^\varepsilon(y) \leq u_1^\varepsilon(x_0^\varepsilon) - u_2^\varepsilon(x_0^\varepsilon) + u_2^\varepsilon(x_0^\varepsilon) - u_2^\varepsilon(y) \leq C, \quad x, y \in \mathbb{R}^n.
\]

The proof can be concluded by repeating the same argument for \( \min_{j=1,2}{\max_{x \in T^n}} \{ u_j^\varepsilon(x) \} \).

\[
\square
\]

The following is the main theorem of the section. See also [Tran] for similar results.

**Theorem 4.4.** There exists a constant \( C > 0 \), independent of \( \varepsilon \), such that
\[
\|\varepsilon u_1^\varepsilon - \overline{H}_1\|_{L^\infty}, \|\varepsilon u_2^\varepsilon - \overline{H}_2\|_{L^\infty} \leq C\varepsilon.
\]
Lemma 4.6. There exists a constant $x$ proof.

Proof. Differentiating (4.2) w.r.t.

Lemma 4.7. Where we set $u_{x}$ (Properties of Lemma 4.5)

The next two lemmas can be proven by using the same ways as in Lemma 3.3 and Lemma 3.4.

Lemma 4.6. There exists a constant $C > 0$, independent of $\varepsilon$, such that

$$
\varepsilon^2 \int_{\mathbb{R}^n} |D^2 u_1^\varepsilon| \, dx \leq C,
$$

$$
\varepsilon^2 \int_{\mathbb{R}^n} |D^2 u_2^\varepsilon| \, dx \leq C.
$$

Finally, next lemma allows us to prove Theorem 1.4.

Lemma 4.7. There exists a constant $C > 0$, independent of $\varepsilon$, such that

$$
\max_{x \in \mathbb{T}^n} |(\varepsilon u_1^\varepsilon)|, \quad \max_{x \in \mathbb{T}^n} |(\varepsilon u_2^\varepsilon)| \leq C.
$$

Proof. Differentiating (4.2) w.r.t. $\varepsilon$,

$$
\begin{align*}
D_p H_1 \cdot Du_1^\varepsilon + (c_1 + \varepsilon) u_1^\varepsilon + u_1^\varepsilon - c_1 u_2^\varepsilon, & = \varepsilon^2 \Delta u_1^\varepsilon + 2 \varepsilon \Delta u_1^\varepsilon, \\
D_p H_2 \cdot Du_2^\varepsilon + (c_2 + \varepsilon) u_2^\varepsilon + u_2^\varepsilon - c_2 u_1^\varepsilon, & = \varepsilon^2 \Delta u_2^\varepsilon + 2 \varepsilon \Delta u_2^\varepsilon,
\end{align*}
$$

where we set $u_{j,\varepsilon} := \partial u_j^\varepsilon / \partial \varepsilon$, $j = 1, 2$. Without loss of generality, we may assume that there exists $x_2 \in \mathbb{T}^n$ such that

$$
\max_{j=1,2} (\varepsilon u_j^\varepsilon(x))_{x} = \max_{x \in \mathbb{T}^n} \{ (\varepsilon u_{j,\varepsilon}(x) + u_j^\varepsilon(x) \} = \varepsilon u_{j,\varepsilon}(x_2) + u_j^\varepsilon(x_2).
$$

Choosing $x_0 = x_2$ in the adjoint equation (4.9), and repeating the same steps as in Theorem 2.5, we get

$$
\varepsilon u_{1,\varepsilon}(x_2) + \int_{\mathbb{T}^n} u_1^\varepsilon \sigma^1, \varepsilon \, dx + \int_{\mathbb{T}^n} u_2^\varepsilon \sigma^2, \varepsilon \, dx
\leq 2 \varepsilon \int_{\mathbb{T}^n} |\Delta u_1^\varepsilon| 1^\varepsilon \, dx + 2 \varepsilon \int_{\mathbb{T}^n} |\Delta u_2^\varepsilon| 2^\varepsilon \, dx \leq C,
$$

(4.10)
where the latter inequality follows by repeating the chain of inequalities in (2.14) and thanks to Lemma 4.6. Using Lemma 4.3 and property (ii) of Lemma 4.5 we have

\[
\left| \int_{\mathbb{T}^n} u_1(\sigma)^{1,\varepsilon} dx + \int_{\mathbb{T}^n} u_2(\sigma)^{2,\varepsilon} dx - u_1(x_2) \right| = \left| \int_{\mathbb{T}^n} (u_1(\sigma) - u_1(x_2))^{1,\varepsilon} dx + \int_{\mathbb{T}^n} (u_2(\sigma) - u_1(x_2))^{2,\varepsilon} dx \right| \leq C.
\]

In view of the previous inequality, (4.10) becomes

\[
\varepsilon u_1(x_2) + u_1(x_2) \leq C,
\]

thus giving the bound from above. The same argument, applied to \( \min_{j=1,2} (\varepsilon u_j(x))_x \), allows to prove the bound from below. \( \Box \)

**Proof of Theorem 4.4** The theorem immediately follows by using Lemma 4.7. \( \Box \)

### 5. Weakly coupled systems of obstacle type

In this last section we apply the Adjoint Method to weakly coupled systems of obstacle type. Let \( H_1, H_2 : \overline{U} \times \mathbb{R}^n \to \mathbb{R} \) be smooth Hamiltonians, and let \( \psi_1, \psi_2 : \overline{U} \to \mathbb{R} \) be smooth functions describing the obstacles. We assume that there exists \( \alpha > 0 \) such that

\[
\psi_1, \psi_2 \geq \alpha \quad \text{in} \quad \overline{U}, \tag{5.1}
\]

and consider the system

\[
\begin{cases}
\max \{ u_1 - u_2 - \psi_1, u_1 + H_1(x, Du_1) \} = 0 \quad \text{in} \ U, \\
\max \{ u_2 - u_1 - \psi_2, u_2 + H_2(x, Du_2) \} = 0 \quad \text{in} \ U, \tag{5.2}
\end{cases}
\]

with boundary conditions \( u_1 |_{\partial U} = u_2 |_{\partial U} = 0 \). We observe that (5.1) guarantees the compatibility of the boundary conditions, since \( \psi_1, \psi_2 > 0 \) on \( \partial U \).

Although the two equations in (5.3) are coupled just through the difference \( u_1 - u_2 \), this problem turns out to be more difficult than the correspondent scalar equation (2.1) studied in Section 2. For this reason, the hypotheses we require now are stronger. We assume that

(H5.1) \( H_j(x, \cdot) \) is convex for every \( x \in \overline{U}, j = 1, 2. \)

(H5.2) Superlinearity in \( p \):

\[
\lim_{|p| \to \infty} \frac{H_j(x, p)}{|p|} = +\infty \quad \text{uniformly in} \ x, \ j = 1, 2.
\]

(H5.3) \( |D_x H_j(x, p)| \leq C \) for each \( (x, p) \in \overline{U} \times \mathbb{R}^n, j = 1, 2. \)
Let $\gamma$ be convex. We approximate (5.3) by the following system

\[
\begin{aligned}
  u_1^\varepsilon + H_1(x, Du_1^\varepsilon) + \gamma'(u_1^\varepsilon - u_2^\varepsilon - \psi_1) &= \varepsilon \Delta u_1^\varepsilon \quad \text{in } U, \\
  u_2^\varepsilon + H_2(x, Du_2^\varepsilon) + \gamma'(u_2^\varepsilon - u_1^\varepsilon - \psi_2) &= \varepsilon \Delta u_2^\varepsilon \quad \text{in } U.
\end{aligned}
\]

(5.3)

We are now ready to state the main result of the section.

**Theorem 5.1.** There exists a positive constant $C$, independent of $\varepsilon$, such that

\[
\max\{|u_1^\varepsilon - u_1|_{L^\infty}, |u_2^\varepsilon - u_2|_{L^\infty}\} \leq C \varepsilon^{1/2}.
\]

In order to prove the theorem we need several lemmas. In the sequel, we shall use the notation

\[
\theta_1^\varepsilon := u_1^\varepsilon - u_2^\varepsilon - \psi_1, \quad \theta_2^\varepsilon := u_2^\varepsilon - u_1^\varepsilon - \psi_2.
\]

The linearized operator corresponding to (5.3) is

\[
L^\varepsilon(z_1, z_2) := \begin{cases}
  z_1 + D_p H_1(x, Du_1^\varepsilon) \cdot Dz_1 + (\gamma')' \varepsilon (z_1 - z_2) - \varepsilon \Delta z_1, \\
  z_2 + D_p H_2(x, Du_2^\varepsilon) \cdot Dz_2 + (\gamma')' \varepsilon (z_2 - z_1) - \varepsilon \Delta z_2.
\end{cases}
\]

Then, the adjoint equations are:

\[
\begin{aligned}
  (1 + (\gamma')' \varepsilon) \sigma^{1,\varepsilon} - \text{div}(D_p H_1 \sigma^{1,\varepsilon}) - (\gamma')' \varepsilon \sigma^{2,\varepsilon} &= \varepsilon \Delta \sigma^{1,\varepsilon} + (2 - i) \delta x_0 \quad \text{in } U, \\
  (1 + (\gamma')' \varepsilon) \sigma^{2,\varepsilon} - \text{div}(D_p H_2 \sigma^{2,\varepsilon}) - (\gamma')' \varepsilon \sigma^{1,\varepsilon} &= \varepsilon \Delta \sigma^{2,\varepsilon} + (i - 1) \delta x_0 \quad \text{in } U,
\end{aligned}
\]

(5.4)

with boundary conditions

\[
\begin{aligned}
  \sigma^{1,\varepsilon} &= 0 \quad \text{on } \partial U, \\
  \sigma^{2,\varepsilon} &= 0 \quad \text{on } \partial U,
\end{aligned}
\]

where $i \in \{1, 2\}$ and $x_0 \in U$ will be chosen later. By repeating what was done in Section 2 we get the existence and uniqueness of $\sigma^{1,\varepsilon}$ and $\sigma^{2,\varepsilon}$ by Fredholm alternative. Furthermore, $\sigma^{1,\varepsilon}$ and $\sigma^{2,\varepsilon}$ are well defined and $\sigma^{1,\varepsilon}, \sigma^{2,\varepsilon} \in C^\infty(U \setminus \{x_0\})$. In order to derive further properties of $\sigma^{1,\varepsilon}$ and $\sigma^{2,\varepsilon}$, we need the following useful formulas.

**Lemma 5.2.** For every $\varphi_1, \varphi_2 \in C^2(U)$ we have

\[
(2 - i) \varphi_1(x_0) = -\varepsilon \int_{\partial U} \frac{\partial \sigma^{1,\varepsilon}}{\partial \nu} \varphi_1 dS - \int_U (\gamma')' \varepsilon \varphi_1 \sigma^{2,\varepsilon} dx \\
+ \int_U [(1 + (\gamma')' \varepsilon) \varphi_1 + D_p H_1 \cdot D \varphi_1 - \varepsilon \Delta \varphi_1] \sigma^{1,\varepsilon} dx,
\]

(5.5)
and
\[(i - 1)\varphi_2(x_0) = -\varepsilon \int_{\partial U} \frac{\partial \sigma^{1,\varepsilon}}{\partial \nu} \varphi_2 \, dS - \int_{U} (\gamma^\varepsilon)' \varphi_1 \varphi_2 \sigma^{1,\varepsilon} \, dx + \int_{U} \left[ (1 + (\gamma^\varepsilon)' \varphi_2) + D_p H_2 \cdot D \varphi_2 - \varepsilon \Delta \varphi_2 \right] \sigma^{2,\varepsilon} \, dx, \tag{5.6}\]
where \(\nu\) is the outer unit normal to \(\partial U\).

**Proof.** The conclusion follows by simply multiplying by \(\varphi_j\) \((j = 1, 2)\) the two equations in (5.4) and integrating by parts. \(\square\)

We can now prove the analogous of Lemma 2.3.

**Lemma 5.3 (Properties of \(\sigma^{1,\varepsilon}, \sigma^{2,\varepsilon}\)).** Let \(\nu\) be the outer unit normal to \(\partial U\). Then
\[\begin{align*}
(i) & \quad \sigma^{j,\varepsilon} \geq 0 \text{ on } \overline{U}. \quad \text{In particular, } \frac{\partial \sigma^{j,\varepsilon}}{\partial \nu} \leq 0 \text{ on } \partial U \quad (j = 1, 2). \\
(ii) & \quad \text{The following equality holds:} \\
& \quad \sum_{j=1}^{2} \left( \int_{U} \sigma^{j,\varepsilon} \, dx - \varepsilon \int_{\partial U} \frac{\partial \sigma^{j,\varepsilon}}{\partial \nu} \, dS \right) = 1.
\end{align*}\]

In particular,
\[\sum_{j=1}^{2} \int_{U} \sigma^{j,\varepsilon} \, dx \leq 1.\]

**Proof.** First of all, we consider the adjoint of equation (5.4):
\[\begin{cases}
z_1 + D_p H_1(x, Du^1) \cdot Dz_1 + (\gamma^\varepsilon)' \varphi_1 (z_1 - z_2) - \varepsilon \Delta z_1 = f_1, \\
z_2 + D_p H_2(x, Du^2) \cdot Dz_2 + (\gamma^\varepsilon)' \varphi_2 (z_2 - z_1) - \varepsilon \Delta z_2 = f_2,
\end{cases}\]
where \(f_1, f_2 \in C(U)\), with boundary conditions \(z_1 = z_2 = 0\) on \(\partial U\). Note that
\[f_1, f_2 \geq 0 \implies \min_{j=1,2} z_j(x) \geq 0. \tag{5.8}\]

Indeed, assume
\[\min_{j=1,2} z_j(x) = z_1(\overline{x}), \quad \text{for some } \overline{x} \in \overline{U}.\]

Then,
\[z_1(\overline{x}) \geq z_1(\overline{x}) + (\gamma^\varepsilon)' \varphi_1 (z_1(\overline{x}) - z_2(\overline{x})) + \varepsilon \Delta z_1(\overline{x}) + f_1(\overline{x}) \geq 0.\]

Adding up relations (5.5) and (5.6) with \(\varphi_1 = z_1\) and \(\varphi_2 = z_2\) we get
\[\int_{U} f_1 \sigma^{1,\varepsilon} \, dx + \int_{U} f_2 \sigma^{2,\varepsilon} \, dx = (2 - i)z_1(x_0) + (1 - i)z_2(x_0).\]

Thanks to (5.8), from last relation we conclude that
\[f_1, f_2 \geq 0 \implies \int_{U} f_1 \sigma^{1,\varepsilon} \, dx + \int_{U} f_2 \sigma^{2,\varepsilon} \, dx \geq 0,\]
and this implies that $\sigma^{1,\varepsilon}, \sigma^{2,\varepsilon} \geq 0$.

The proof of property (ii) follows by choosing $\varphi_1 = \varphi_2 \equiv 1$ in (5.5) and (5.6), and summing up the relations obtained. \end{proof}

We are now able to prove a uniform bound on $u^\varepsilon_1$ and $u^\varepsilon_2$.

**Lemma 5.4.** There exists a positive constant $C$, independent of $\varepsilon$, such that

$$\|u^\varepsilon_1\|_{L^\infty}, \|u^\varepsilon_2\|_{L^\infty} \leq C.$$

**Proof.** If the maximum $\max_{j=1,2} \{u^\varepsilon_j(x)\}$ is attained on the boundary, then

$$\max_{j=1,2} \{u^\varepsilon_j(x)\} = \max_{x \in \partial U} \{u^\varepsilon_j(x)\} = 0.$$

Otherwise, assume that there exists $x^* \in U$ be such that

$$\max_{j=1,2} \{u^\varepsilon_j(x)\} = u^\varepsilon_1(x^*).$$

Then,

$$u^\varepsilon_1(x^*) = \varepsilon \Delta u^\varepsilon_1(x^*) - H_1(x^*, 0) - \gamma^\varepsilon (u^\varepsilon_1(x^*) - u^\varepsilon_2(x^*) - \psi_1(x^*))$$

$$\leq - H_1(x^*, 0) \leq \sup_{x \in U} (-H_1(x, 0)) \leq C.$$

Analogously, to prove the lower bound suppose that

$$\min_{j=1,2} \{u^\varepsilon_j(x)\} = u^\varepsilon_1(\hat{x}),$$

for some $\hat{x} \in U$. Then, $u^\varepsilon_1(\hat{x}) - u^\varepsilon_2(\hat{x}) - \psi_1(\hat{x}) < 0$ and so

$$u^\varepsilon_1(\hat{x}) = \varepsilon \Delta u^\varepsilon_1(\hat{x}) - H_1(\hat{x}, 0) - \gamma^\varepsilon (u^\varepsilon_1(\hat{x}) - u^\varepsilon_2(\hat{x}) - \psi_1(\hat{x}))$$

$$= \varepsilon \Delta u^\varepsilon_1(\hat{x}) - H_1(\hat{x}, 0) \geq -H_1(\hat{x}, 0) \geq \inf_{x \in U} (-H_1(x, 0)) \geq C.$$

\end{proof}

Next lemma will be used to give a uniform bound for $Du^\varepsilon_1$ and $Du^\varepsilon_2$.

**Lemma 5.5.** We have

$$\int_U (\gamma^\varepsilon)^\prime \left| \sigma_{\varepsilon} \right|_{\sigma_{\varepsilon}} \sigma^{1,\varepsilon} \, dx + \int_U (\gamma^\varepsilon)^\prime \left| \sigma_{\varepsilon} \right|_{\sigma_{\varepsilon}} \sigma^{2,\varepsilon} \, dx \leq C,$$

where $C$ is a positive constant independent of $\varepsilon$.\end{lemma}
Proof. First of all, observe that condition (5.1) implies that

\[
H_j(x, p) - D_p H_j(x, p) \cdot p \leq H_j(x, 0), \quad \text{for every } (x, p) \in \mathcal{U} \times \mathbb{R}^n, \quad j = 1, 2. \tag{5.9}
\]

In the same way, the convexity of \( \gamma \) implies

\[
\gamma^\varepsilon(s) - [(\gamma^\varepsilon)'(s)] s = \gamma \left( \frac{s}{\varepsilon} \right) - \left( \gamma' \left( \frac{s}{\varepsilon} \right) \right) \frac{s}{\varepsilon} \leq \gamma(0) = 0. \tag{5.10}
\]

Equation (5.1) gives

\[
0 = u_1^\varepsilon + H_1(x, Du_1^\varepsilon) + \gamma^\varepsilon \mid \partial_1 - \varepsilon \Delta u_1^\varepsilon
\]

\[
= u_1^\varepsilon + D_p H_1(x, Du_1^\varepsilon) \cdot Du_1^\varepsilon - \varepsilon \Delta u_1^\varepsilon + H_1(x, Du_1^\varepsilon) - D_p H_1(x, Du_1^\varepsilon) \cdot Du_1^\varepsilon
\]

\[
+ \gamma^\varepsilon \mid \partial_1 - (\gamma^\varepsilon)' \mid \partial_1 \theta_1^1 + (\gamma^\varepsilon)' \mid \partial_1 (u_1^\varepsilon - u_2^\varepsilon) - (\gamma^\varepsilon)' \mid \partial_1 \psi_1.
\]

Multiplying last relation by \( \sigma^{1, \varepsilon} \), integrating and using (5.9) and (5.10)

\[
\int_U (\gamma^\varepsilon)' \mid \partial_2 \psi_1 \sigma^{1, \varepsilon} dx = \int_U [H_1(x, Du_1^\varepsilon) - D_p H_1(x, Du_1^\varepsilon) \cdot Du_1^\varepsilon] \sigma^{1, \varepsilon} dx
\]

\[
+ \int_U [\gamma^\varepsilon \mid \partial_1 - (\gamma^\varepsilon)' \mid \partial_1 \theta_1^1] \sigma^{1, \varepsilon} dx
\]

\[
+ \int_U \left[ (1 + (\gamma^\varepsilon)' \mid \partial_1 \right) u_1^\varepsilon + D_p H_1(x, Du_1^\varepsilon) \cdot Du_1^\varepsilon - \varepsilon \Delta u_1^\varepsilon - (\gamma^\varepsilon)' \mid \partial_1 \mid u_2^\varepsilon \left] \sigma^{1, \varepsilon} dx
\]

\[
\leq \int_U H_1(x, 0) \sigma^{1, \varepsilon} dx
\]

\[
+ \int_U \left[ (1 + (\gamma^\varepsilon)' \mid \partial_1 \right) u_1^\varepsilon + D_p H_1(x, Du_1^\varepsilon) \cdot Du_1^\varepsilon - \varepsilon \Delta u_1^\varepsilon - (\gamma^\varepsilon)' \mid \partial_1 \mid u_2^\varepsilon \left] \sigma^{1, \varepsilon} dx.
\]

Analogously,

\[
\int_U (\gamma^\varepsilon)' \mid \partial_2 \psi_2 \sigma^{2, \varepsilon} dx \leq \int_U H_2(x, 0) \sigma^{2, \varepsilon} dx
\]

\[
+ \int_U \left[ (1 + (\gamma^\varepsilon)' \mid \partial_2 \right) u_2^\varepsilon + D_p H_2(x, Du_2^\varepsilon) \cdot Du_2^\varepsilon - \varepsilon \Delta u_2^\varepsilon - (\gamma^\varepsilon)' \mid \partial_2 \mid u_2^\varepsilon \left] \sigma^{2, \varepsilon} dx.
\]

Summing up the last two relations and using (5.5) and (5.6)

\[
\int_U (\gamma^\varepsilon)' \mid \partial_1 \psi_1 \sigma^{1, \varepsilon} dx + \int_U (\gamma^\varepsilon)' \mid \partial_2 \psi_2 \sigma^{2, \varepsilon} dx \leq (2 - \alpha) u_1^\varepsilon(x_0) + (i - 1) u_2^\varepsilon(x_0)
\]

\[
+ \|H_1(\cdot, 0)\|_{L^\infty} \int_U \sigma^{1, \varepsilon} dx + \|H_2(\cdot, 0)\|_{L^\infty} \int_U \sigma^{2, \varepsilon} dx.
\]

Thus,

\[
\int_U (\gamma^\varepsilon)' \mid \partial_1 \sigma^{1, \varepsilon} dx + \int_U (\gamma^\varepsilon)' \mid \partial_2 \sigma^{2, \varepsilon} dx \leq \frac{2 - \alpha}{2} u_1^\varepsilon(x_0) + \frac{i - 1}{\alpha} u_2^\varepsilon(x_0)
\]

\[
+ \|H_1(\cdot, 0)\|_{L^\infty} \int_U \sigma^{1, \varepsilon} dx + \frac{\|H_2(\cdot, 0)\|_{L^\infty}}{\alpha} \int_U \sigma^{2, \varepsilon} dx \leq C,
\]

where we used (5.1), Lemma 5.3 and Lemma 5.4.

We can finally show the existence of a uniform bound for the gradients of \( u_1^\varepsilon \) and \( u_2^\varepsilon \).
Lemma 5.6. There exists a positive constant $C$, independent of $\varepsilon$, such that

$$\|Du_1^j\|_{L^\infty}, \|Du_2^j\|_{L^\infty} \leq C.$$ 

Proof. Step I: Bound on $\partial U$.

As it was done in Section 2 we are going to construct appropriate barriers. For $\varepsilon$ small enough, assumption (H5.4) implies that

$$\|\partial u_\ast\|_{L^\infty} \leq C.$$ 

Hence, we obtain

$$\Phi_1 = \Phi_2 = 0 \text{ on } \partial U.$$ 

Therefore, $(\Phi_1, \Phi_2)$ is a sub-solution of (5.3). By the comparison principle, $u_j^\varepsilon \geq \Phi_j$ in $U$, $j = 1, 2$.

Let $d(\chi)$, $\delta$, and $U_3$ be as in the proof of Proposition 2.1. For $\mu > 0$ large enough, the uniform bounds of $\|u_1^\varepsilon\|_{L^\infty}$ and $\|u_2^\varepsilon\|_{L^\infty}$ yield $v := \mu d \geq u_j^\varepsilon$ on $\partial U_3$, $j = 1, 2$, so that

$$\begin{aligned}
&v + H_1(x, Du) + \gamma^\varepsilon (v - v - \psi_1) - \varepsilon \Delta v = v + H_1(x, Du) - \varepsilon \Delta v \geq H_1(x, \mu Dd) - \mu C \quad \text{in } U, \\
v + H_2(x, Du) + \gamma^\varepsilon (v - v - \psi_2) - \varepsilon \Delta v = v + H_2(x, Du) - \varepsilon \Delta v \geq H_2(x, \mu Dd) - \mu C \quad \text{in } U.
\end{aligned}$$

Now, we have $\Phi_j = u_j^\varepsilon = v = 0$ on $\partial U$. Also, thanks to assumption (H5.2), for $\mu > 0$ large enough

$$\begin{aligned}
&v + H_1(x, Du) + \gamma^\varepsilon (v - v - \psi_1) - \varepsilon \Delta v \geq 0 \quad \text{in } U, \\
v + H_2(x, Du) + \gamma^\varepsilon (v - v - \psi_2) - \varepsilon \Delta v \geq 0 \quad \text{in } U,
\end{aligned}$$

that is, the pair $(v, v)$ is a super-solution for the system (5.3). Thus, the comparison principle gives us that $\Phi_j \leq u_j^\varepsilon \leq v_j$ in $U_3$. Then, from the fact that $\Phi_j = u_j^\varepsilon = v = 0$ on $\partial U$ we get

$$\frac{\partial v}{\partial \nu}(x) \leq \frac{\partial u_j^\varepsilon}{\partial \nu}(x) \leq \frac{\partial \Phi_j}{\partial \nu}(x), \quad \text{for } x \in \partial U.$$ 

Hence, we obtain $\|Du_j^\varepsilon\|_{L^\infty(\partial U)} \leq C$, $j = 1, 2$.

Step II: Bound on $U$.

Assume now that there exists $\bar{x} \in U$ such that

$$\max_{j=1,2} w_j^\varepsilon(x) = w_1^\varepsilon(\bar{x}),$$

where $w_j^\varepsilon(x) := \frac{1}{2} |Du_j^\varepsilon|^2$, $j = 1, 2$.

By a direct computation one can see that

$$2(1 + (\gamma^\varepsilon)' |\phi_1'| w_1^\varepsilon + D_p H_1 \cdot Du_1^\varepsilon + D_s H_1 \cdot Du_1^\varepsilon - (\gamma^\varepsilon)' |\phi_1'| Du_1^\varepsilon \cdot (D \psi_1 + Du_2^\varepsilon) = \varepsilon \Delta w_1^\varepsilon - \varepsilon |D^2 u_1^\varepsilon|^2.$$
Multiplying last relation by $\sigma^{1,\varepsilon}$ and integrating over $U$

$$
2 \int_U w_1^\varepsilon \sigma^{1,\varepsilon} \, dx + \int_U D_x H_1 \cdot D w_1^\varepsilon \sigma^{1,\varepsilon} \, dx - \varepsilon \int_U \Delta w_1^\varepsilon \sigma^{1,\varepsilon} \, dx + \int_U \varepsilon |D^2 u_1^\varepsilon|^2 \sigma^{1,\varepsilon} \, dx
$$

$$
+ \int_U D_x H_1 \cdot D w_1^\varepsilon \sigma^{1,\varepsilon} \, dx + \frac{1}{2} \int_U (\gamma^\varepsilon)' \, |\partial_x^2 \psi_1^\varepsilon| \left| |D u_1^\varepsilon|^2 + |D u_1^\varepsilon - D u_2^\varepsilon|^2 - |D u_2^\varepsilon|^2 \right| \sigma^{1,\varepsilon} \, dx
$$

$$
- \int_U (\gamma^\varepsilon)' \, |\partial_x^2 \psi_1^\varepsilon| \sigma^{1,\varepsilon} \, dx = 0. \tag{5.11}
$$

Then, using equation (5.5) with $i = 1$ and $x_0 = \tilde{x}$

$$
\int_U w_1^\varepsilon \sigma^{1,\varepsilon} \, dx + \int_U \varepsilon |D^2 u_1^\varepsilon|^2 \sigma^{1,\varepsilon} \, dx - \int_U (\gamma^\varepsilon)' \, |\partial_x^2 \psi_1^\varepsilon| \sigma^{1,\varepsilon} \, dx + \int_U D_x H_1 \cdot D w_1^\varepsilon \sigma^{1,\varepsilon} \, dx
$$

$$
+ \frac{1}{2} \int_U (\gamma^\varepsilon)' \, |\partial_x^2 \psi_1^\varepsilon| \left| |D u_1^\varepsilon|^2 + |D u_1^\varepsilon - D u_2^\varepsilon|^2 - |D u_2^\varepsilon|^2 \right| \sigma^{1,\varepsilon} \, dx \tag{5.12}
$$

$$
w_1^\varepsilon(\tilde{x}) + \varepsilon \int_{\partial U} \frac{\partial \sigma^{1,\varepsilon}}{\partial \nu} w_1^\varepsilon \, dS + \int_U (\gamma^\varepsilon)' \, |\partial_x^2 w_1^\varepsilon| \sigma^{2,\varepsilon} \, dx = 0,
$$

which implies

$$
w_1^\varepsilon(\tilde{x}) - \int_U (\gamma^\varepsilon)' \, |\partial_x^1 w_2^\varepsilon| \sigma^{1,\varepsilon} \, dx + \int_U (\gamma^\varepsilon)' \, |\partial_x^2 w_1^\varepsilon| \sigma^{2,\varepsilon} \, dx
$$

$$
\leq \int_U (\gamma^\varepsilon)' \, |\partial_x^1 D u_1^\varepsilon \cdot D \psi_1^\varepsilon \sigma^{1,\varepsilon} \, dx - \int_U D_x H_1 \cdot D u_1^\varepsilon \sigma^{1,\varepsilon} \, dx - \varepsilon \int_{\partial U} \frac{\partial \sigma^{1,\varepsilon}}{\partial \nu} w_1^\varepsilon \, dS.
$$

Let now $\eta > 0$ be a constant to be chosen later. Using Step I and Lemmas 5.3 and 5.5 thanks to Young’s inequality

$$
w_1^\varepsilon(\tilde{x}) - \int_U (\gamma^\varepsilon)' \, |\partial_x^1 w_2^\varepsilon| \sigma^{1,\varepsilon} \, dx + \int_U (\gamma^\varepsilon)' \, |\partial_x^2 w_1^\varepsilon| \sigma^{2,\varepsilon} \, dx
$$

$$
\leq \int_U (\gamma^\varepsilon)' \, |\partial_x^1 \left[ \eta^2 w_1^\varepsilon(\tilde{x}) + \frac{\|D \psi_1\|_{L^\infty}}{2 \eta^2} \right] \sigma^{1,\varepsilon} \, dx
$$

$$
+ \int_U \left[ \eta^2 w_1^\varepsilon(\tilde{x}) + \frac{\|D_x H_1\|_{L^\infty}}{2 \eta^2} \right] \sigma^{1,\varepsilon} \, dx + C \tag{5.13}
$$

$$
\leq \eta^2 (C + 1) w_1^\varepsilon(\tilde{x}) + C \left( 1 + \frac{1}{\eta^2} \right).
$$

In the same way, considering the analogous of equation (5.11) for the function $w_2^\varepsilon$ (recalling that in 5.6 we chose $i = 1$) we can obtain the following inequality :

$$
- \int_U (\gamma^\varepsilon)' \, |\partial_x^2 w_1^\varepsilon| \sigma^{2,\varepsilon} \, dx + \int_U (\gamma^\varepsilon)' \, |\partial_x^1 w_2^\varepsilon| \sigma^{2,\varepsilon} \, dx \leq \eta^2 (C + 1) w_1^\varepsilon(\tilde{x}) + C \left( 1 + \frac{1}{\eta^2} \right), \tag{5.14}
$$

where we also used the fact that $\|w_2^\varepsilon\|_{L^\infty} \leq w_1^\varepsilon(\tilde{x})$. Summing inequalities (5.13) and (5.14) and choosing $\eta > 0$ small enough the conclusion follows. 

Next lemma gives a control of the Hessians $D^2 u_1^\varepsilon$ and $D^2 u_2^\varepsilon$ in the support of $\sigma^{1,\varepsilon}$ and $\sigma^{2,\varepsilon}$ respectively.
Lemma 5.7. There exists a positive constant \( C \), independent of \( \varepsilon \), such that
\[
\sup_{j=1,2} \int_U \varepsilon |D^2 u_j^\varepsilon|^2 \sigma_j \varepsilon \, dx \leq C.
\]

Proof. The bound of the Hessian of \( D^2 u_1^\varepsilon \) comes from identity (5.12), together with Lemma 5.6. The other bound can be obtained in a similar way. \[\square\]

We can finally prove the analogous of Lemma 2.2.

Lemma 5.8. There exists a positive constant, independent of \( \varepsilon \), such that
\[
\max_{j=1,2} \max_{x \in U^\varepsilon_j} |\varepsilon u_j^\varepsilon(x)| \leq C,
\]
\[
\max_{j=1,2} \max_{x \in \partial U^\varepsilon_j} |\varepsilon u_j^\varepsilon(x)| = 0.
\]

Proof. It will be enough to prove the second inequality, since the first one will follow by the definition of \( \gamma^\varepsilon \). If the maximum is attained at the boundary, then
\[
\max_{j=1,2} \max_{x \in U^\varepsilon_j} |\varepsilon u_j^\varepsilon(x)| = \max_{j=1,2} \max_{x \in \partial U^\varepsilon_j} |\varepsilon u_j^\varepsilon(x)| = 0.
\]
Otherwise, let us assume that there exists \( x_1 \in U \) such that
\[
\max_{j=1,2} \max_{x \in U^\varepsilon_j} |\varepsilon u_j^\varepsilon(x)| = \gamma^\varepsilon(\theta_j^\varepsilon(x_1)) > 0,
\]
\[
\gamma^\varepsilon(\theta_2^\varepsilon(x_1)) = 0.
\]
Since \( \gamma^\varepsilon \) is increasing and \( \gamma^\varepsilon(z) > 0 \) if and only if \( z > 0 \), we also have \( \max_{x \in U^\varepsilon_j} \theta_j^\varepsilon(x) = \theta_j^\varepsilon(x_1) > 0 \).

Evaluating the two equations in (5.3) at \( x_1 \) and subtracting the second one from the first one,
\[
\theta_j^\varepsilon(x_1) + \gamma^\varepsilon(\theta_j^\varepsilon(x_1)) = \varepsilon \Delta u_j^\varepsilon(x_1) - \varepsilon \Delta u_2^\varepsilon(x_1) - H_1(x_1, Du_1^\varepsilon(x_1)) + H_2(x_1, Du_2^\varepsilon(x_1)) - \psi_1(x_1)
\]
\[
\leq \varepsilon \Delta \psi_1(x_1) - H_1(x_1, Du_1^\varepsilon(x_1)) + H_2(x_1, Du_2^\varepsilon(x_1)) - \psi_1(x_1)
\]
\[
\leq \| \Delta \psi_1(\cdot) \|_{L^\infty} + \| H_1(\cdot, Du_1^\varepsilon(\cdot)) \|_{L^\infty} + \| H_2(\cdot, Du_2^\varepsilon(\cdot)) \|_{L^\infty} + \| \psi_1(\cdot) \|_{L^\infty} \leq C,
\]
where we the last inequality follows from Lemma 5.6. \[\square\]

We now set for every \( \varepsilon \in (0,1) \)
\[
u_j^\varepsilon(x) := \frac{\partial u_j^\varepsilon}{\partial \varepsilon}(x), \quad x \in U, j = 1,2.
\]

The next lemma gives a uniform bound for \( u_1^\varepsilon \) and \( u_2^\varepsilon \), thus concluding the proof of Theorem 5.1.

Lemma 5.9. There exists a positive constant \( C > 0 \) such that
\[
\max_{j=1,2} \max_{x \in U} |u_j^\varepsilon(x)| \leq \frac{C}{\varepsilon^{1/2}}.
\]
Proof. If the above maximum is attained at the boundary, then
\[
\max_{j=1,2} \max_{x \in \partial U} |u_{j,\varepsilon}(x)| = \max_{j=1,2} |u_{j,\varepsilon}(x)| = 0,
\]
since \(u_{1,\varepsilon} = u_{2,\varepsilon} = 0\) on \(\partial U\). Otherwise, assume that there exists \(x \in U\) such that
\[
\max_{j=1,2} |u_{j,\varepsilon}(x)| = |u_{1,\varepsilon}(x)|.
\]
Differentiating (5.3) w.r.t. \(\varepsilon\) we have
\[
\begin{aligned}
&\left(1 + (\gamma^{\varepsilon})'|\sigma|\right)u_{1,\varepsilon} + D_{p} H_{1} \cdot Du_{1,\varepsilon} - (\gamma^{\varepsilon})'|\sigma| u_{2,\varepsilon} + \gamma^{\varepsilon}|\sigma| = \varepsilon \Delta u_{1,\varepsilon} + \Delta u_{1,\varepsilon}^{\varepsilon} \quad \text{in } U, \\
&\left(1 + (\gamma^{\varepsilon})'|\sigma|\right)u_{2,\varepsilon} + D_{p} H_{2} \cdot Du_{2,\varepsilon} - (\gamma^{\varepsilon})'|\sigma| u_{1,\varepsilon} + \gamma^{\varepsilon}|\sigma| = \varepsilon \Delta u_{2,\varepsilon} + \Delta u_{2,\varepsilon}^{\varepsilon} \quad \text{in } U.
\end{aligned}
\]
(5.15)

Let \(\sigma^{1,\varepsilon}\) and \(\sigma^{2,\varepsilon}\) be the solutions to system (5.4) with \(i = 1\) and \(x_{0} = \varpi\). Multiplying (5.15)\(_{1}\) and (5.15)\(_{2}\) by \(\sigma^{1,\varepsilon}\) and \(\sigma^{2,\varepsilon}\) respectively, integrating by parts and adding up the two relations obtained we have
\[
u_{1,\varepsilon}(\varpi) = \sum_{j=1}^{2} \left( \int_{U} \Delta u_{j,\varepsilon} \sigma^{j,\varepsilon} \, dx - \int_{U} \gamma^{\varepsilon}|\sigma| \sigma^{j,\varepsilon} \, dx \right).
\]
Thus,
\[
|u_{1,\varepsilon}(\varpi)| \leq \sum_{j=1}^{2} \left( \int_{U} |\Delta u_{j,\varepsilon}| \sigma^{j,\varepsilon} \, dx + \int_{U} |\gamma^{\varepsilon}| \sigma \sigma^{j,\varepsilon} \, dx \right).
\]
At this point, the proof can be easily concluded by repeating what was done in Section 2 showing relations (2.12)–(2.15).

6. Acknowledgments

The authors are grateful to Craig Evans and Fraydoun Rezakhanlou for very useful discussions on the subject of the paper. Diogo Gomes was partially supported by CAMGSD/IST through FCT Program POCTI - FEDER and by grants PTDC/EEA-ACR/67020/2006, PTDC/MAT/69635/2006, and UTAustin/MAT/0057/2008. Filippo Cagnetti was partially supported by FCT through the CMU/Portugal program. Hung Tran was supported in part by VEF fellowship.

References

[CDE84] I. Capuzzo-Dolcetta and L. C. Evans. Optimal switching for ordinary differential equations. SIAM J. Control Optim., 22(1):143–161, 1984.

[CGT] F. Cagnetti, D. Gomes, and H. V. Tran. Aubry-Mather measures in the non convex setting. submitted.

[CL08] F. Camilli and P. Loreti. Comparison results for a class of weakly coupled systems of eikonal equations. Hokkaido Math. J., 37(2):349–362, 2008.

[CLY09] F. Camilli, P. Loreti, and N. Yamada. Systems of convex Hamilton-Jacobi equations with implicit obstacles and the obstacle problem. Commun. Pure Appl. Anal., 8(4):1291–1302, 2009.
ADJOINT METHODS FOR OBSTACLE PROBLEMS AND WEAKLY COUPLED SYSTEMS OF PDE

L. C. Evans and D. Gomes. Effective Hamiltonians and averaging for Hamiltonian dynamics. I. Arch. Ration. Mech. Anal., 157(1):1–33, 2001.

L. C. Evans and D. Gomes. Effective Hamiltonians and averaging for Hamiltonian dynamics. II. Arch. Ration. Mech. Anal., 161(4):271–305, 2002.

H. Engler and S. M. Lenhart. Viscosity solutions for weakly coupled systems of Hamilton-Jacobi equations. Proc. London Math. Soc. (3), 63(1):212–240, 1991.

L. C. Evans and C. K. Smart. Adjoint methods for the infinity laplacian PDE. Preprint.

L. C. Evans. Adjoint and compensated compactness methods for Hamilton-Jacobi PDE. Arch. Ration. Mech. Anal., 197(3):1053–1088, 2010.

A. Fathi. Solutions KAM faibles conjuguées et barrières de Peierls. C. R. Acad. Sci. Paris Sér. I Math., 325(6):649–652, 1997.

A. Fathi. Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens. C. R. Acad. Sci. Paris Sér. I Math., 324(9):1043–1046, 1997.

A. Fathi. Orbites hétéroclines et ensemble de Peierls. C. R. Acad. Sci. Paris Sér. I Math., 326(10):1213–1216, 1998.

A. Fathi. Sur la convergence du semi-groupe de Lax-Oleinik. C. R. Acad. Sci. Paris Sér. I Math., 327(3):267–270, 1998.

D. Gomes. A stochastic analogue of Aubry-Mather theory. Nonlinearity, 15(3):581–603, 2002.

H. Ishii and S. Koike. Viscosity solutions for monotone systems of second-order elliptic PDEs. Comm. Partial Differential Equations, 16(6-7):1095–1128, 1991.

K. Ishii and N. Yamada. On the rate of convergence of solutions for the singular perturbations of gradient obstacle problems. Funkcial. Ekvac., 33(3):551–562, 1990.

P. L. Lions. Generalized solutions of Hamilton-Jacobi equations, volume 69 of Research Notes in Mathematics. Pitman (Advanced Publishing Program), Boston, Mass., 1982.

P. L. Lions, G. Papanicolaou, and S. R. S. Varadhan. Homogenization of Hamilton-Jacobi equations. Preliminary Version, 1988.

H. V. Tran. Adjoint methods for static Hamilton–Jacobi equations. Calc. Var. Partial Differential Equations, to appear.

E-mail address, Filippo Cagnetti: cagnetti@math.ist.utl.pt
E-mail address, Diogo Gomes: dgomes@math.ist.utl.pt
E-mail address, Hung Vinh Tran: tvhung@math.berkeley.edu