Modeling Non-Force-Free and Deformed Flux Ropes in Titan’s Ionosphere

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Key Points:
• Various models are fitted to 85 flux ropes that were detected in Titan’s ionosphere
• A non-force-free model is a more suitable fit statistically compared to a force-free model, implying flux ropes are dynamic at Titan
• Various deformations that can act on a flux rope are explored, and in some cases an elliptical cross-section is more appropriate than circular

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Abstract

Previous work at Titan presented a set of 85 flux ropes detected during Cassini flybys of Titan from 2005-2017. In that study a force-free model was used to determine the radii and axial magnetic field of the flux ropes. In this work we apply non-force free models. The non-force-free model shows an improvement in the number of flux ropes that can be fitted with a model, along with improved uncertainties and $\chi^2$ values. A number of asymmetries and features in the magnetometer data cannot be reproduced by either model, therefore we deform the force-free model to show that small deformations can replicate these features. One such deformation is to use an elliptical cross-section which replicates a plateau in magnetic field strength along with asymmetries on either side of the centre of the flux ropes. Additionally, we explore the properties of bending a flux rope, where we find that minimum variance analysis becomes increasingly degenerate with bending, along with a slight bend causing the switching of the axial field direction from intermediate to maximum variance direction. We conclude that the flux ropes at Titan show aspects of developing flux ropes, compared to other planetary bodies which exhibit more agreement to the force-free assumptions of mature flux ropes.

1 Introduction

Flux ropes are a magnetic phenomena found in most plasma and magnetic field regimes across the solar system and can be found in the solar wind (Burlaga et al., 1982), on the solar surface (e.g., Mouschovias & Poland, 1978), in the magnetospheres of Mercury, Earth and Saturn (e.g., C. T. Russell & Elphic, 1979; Hughes & Sibeck, 1987; Slavin et al., 2010; Jasinski et al., 2016) and in the ionospheres of Venus and Mars (e.g., C. Russell & Elphic, 1979; Vignes et al., 2004). Flux ropes are bundles of magnetic flux, which is twisted around a central axis. They are a result of an interactive and dynamic plasma and magnetic environment. Taking a cross-section, the flux rope appears to have a purely tangential field at the edges which reduces in the centre where an axial field is dominant. Flux ropes could be a diagnostic tool to uncover the dynamical interaction with the ionosphere and surrounding magnetic field. On the solar surface flux ropes are thought to be a precursor to coronal mass ejections Chen and Shibata (2000).

Titan is Saturn’s largest moon and is home to a large, extended atmosphere that is over 1000 km thick (Yelle et al., 2006) due to the lower gravity and higher density than at Earth. As such, Titan’s ionospheric electron density peak, in comparison, sits at a much higher altitude of 1100 - 1200 km (Keller et al., 1992; Ågren et al., 2009). The ionosphere is mainly formed though magnetospheric electron impacts and solar radiation (e.g., Cravens et al., 2005; Ågren et al., 2007) along with other sources. Titan has no measured internal magnetic field and as such the interaction between the ionosphere and the exterior magnetic field forms an induced magnetosphere, similar to that at Venus, in the form of draped magnetic field lines which are ‘captured’ by the ionospheric plasma (Ness et al., 1982).

Flux ropes in Titan’s ionosphere were first reported by Wei et al. (2010), where the authors find instances of increased magnetic field magnitude in Cassini magnetometer data. These magnetic field signatures were compared to a force-free flux rope model (e.g., Burlaga, 1988) to conclude that they were indeed flux ropes. The authors also compare these flux ropes to similar instances of flux ropes in Venus’ ionosphere which are considered more mature than the flux ropes at Titan. Additionally, Wei et al. (2011) recorded an unusually large peak in magnetic field magnitude during the T42 flyby which exceeded all previous measurements at Titan. This structure was shown to be a large flux rope and the authors discuss a possible source in the interaction with solar wind plasma when Saturn’s magnetopause was pushed back inside Titan’s orbit in the recent past. Additional possible formation mechanisms are also discussed in Martin et al. (Accepted).
Further to this, Martin et al. (Accepted) found 85 instances of flux ropes in the ionosphere of Titan and showed statistically that they are larger on average than those at Venus and are found in locations of higher dynamic magnetic environments. The authors also fitted a force-free flux rope model to the examples found, however they found that just over half of the flux ropes fitted adequately to the force-free assumptions and as such further study is needed to determine a more accurate model to represent the flux ropes in Titan’s ionosphere. This study presents the force-free model results along side a non-force-free approach to compare the two models. Additionally, deformations to the flux ropes are examined, such as elliptical cross sections and bent flux ropes, to examine if the flux ropes themselves are dynamic structures that require more than a simple cylindrically symmetric, stationary model to accurately portray.

Elphic and Russell (1983a, 1983b) described a flux rope as a discrete individual excursion of magnetic field, where a peak in magnetic field magnitude is larger than the surrounding magnetic field. Using this description, flux ropes are detected using the Cassini magnetometer (Dougherty et al., 2004) data set during all of Cassini’s Titan flybys. An example of two altitude plots with single flux ropes on flybys T30 and T84 are shown in figure 1 of the companion paper Martin et al. (Accepted).

Historically, most studies (e.g., C. Russell & Elphic, 1979; Vignes et al., 2004; Wei et al., 2010; Jasinski et al., 2016; Martin et al., Accepted) rely on minimum variance analysis (MVA) (Sonnerup & Cahill Jr, 1967) to rotate the magnetic field into a flux rope aligned cylindrical coordinate system. Some studies have circumvented MVA by explicitly including the orientation angles as free parameters in the mode (e.g., Hidalgo et al., 2002) or using geometrical assumptions (e.g., Li et al., 2016). However, MVA remains the most convenient method for automatic reorientation of the coordinate systems.

Martin et al. (Accepted) discussed that in a pure mathematical sense, the axial field direction is found as the intermediate variance direction, and the tangential field direction is found mainly in the maximum field direction, where the minimum variance direction is a constant zero. At Titan, however, the axial field is commonly found as the maximum variance direction and the tangential field is found as the intermediate direction. In this study we will present a force-free model using orientation angles rather than the use of MVA, along with a bent flux rope model which assesses the validity of MVA as we diverge from a pure mathematical explanation of how MVA operates in non-ideal situations. Additionally, we develop a force-free elliptical flux rope model to discuss the validity of the circular cross-section assumption of other models, along with a comparison of purely force-free and non-force-free models.

2 Model Comparison

The force-free model assumes that j×B force density is equal to zero and as such any currents present are field aligned or zero valued. This also means that then magnetic tension force in the flux rope \( \frac{B^2}{2\mu_0R_c} \) is balanced with the magnetic pressure force \( \frac{B^2}{2\mu_0} \), where B is magnetic field magnitude, \( \mu_0 \) is the permeability of free space and \( R_c \) is the radius of curvature. (Osherovich et al., 1995) emphasises that the force-free assumptions shows that the flux rope is in the lowest energy state, and hence is in equilibrium or mature. This idea is further discussed by (Wei et al., 2010), inferring that a developing flux rope would not appear as force-free.

As described earlier, MVA can be used to rotate the magnetic field data into flux rope coordinates, where the variance directions at Titan are different to those found at other planetary bodies (i.e., Mercury Slavin et al., 2010). There are numerous caveats in using MVA and as such the success of fitting a force-free model based on the use of MVA is dependent on the degeneracy of the variance directions (Sonnerup & Cahill Jr,
1967) and the path of the spacecraft through the structure (Xiao et al., 2004). As such if the variance directions are degenerate then MVA is unable to rotate the magnetic field data into the flux rope coordinate system. However, one can include orientation angles in the fitting process to possibly improve the success rate (e.g., Hidalgo et al., 2002).

In cylindrical coordinates, the force-free model reads:

\[ B_A = B_0 J_0(\alpha R) + b_0, \]

\[ B_T = H B_0 J_1(\alpha R), \]

\[ B_R = 0, \]

where, \( B_0 \) is the central magnetic field, \( J_0 \) and \( J_1 \) are the zeroth and first order Bessel functions, \( \alpha \) is the first root of the zeroth order Bessel function, 2.4048. \( R \) is the radial distance to the centre of the flux rope, \( b_0 \) is a magnetic offset and \( H \) is the handedness of the flux rope or which way the flux rope twists around the center and takes the value 1 for right-handed ropes, and -1 for left-handed ropes.

This is then updated to include the orientation angles, \( \gamma, \eta \) and \( \nu \), which describe the orientation of the centre of the flux rope axis in TIIS (Titan Ionospheric Interaction System) coordinates. These angles represent the three Euler angles required to rotate a coordinate system. In the model, \( R \) is an input vector describing the radial distance from the centre of the flux rope, as we do not know \( R \), we use a proxy \( u \) which ranges from -1 to 1 along Cassini’s trajectory to estimate \( \alpha R \) in the following equation.

\[ \alpha R = 2.40 \left( \sqrt{\left( \frac{Y_0}{R_0} \right)^2 + u^2 \left( 1 - \left( \frac{Y_0}{R_0} \right)^2 \right)} \right), \]

where \( Y_0 \) is the distance to closest approach, \( R_0 \) is the radius and \( \frac{Y_0}{R_0} \) is the impact factor which equals 1 at the flux rope edge and 0 at the center (Lepping et al., 2017). As \( Y_0 \) and \( R_0 \) are both dependent on \( R_0 \), this means that we must fit the impact factor rather than the two separately and the radius must be found geometrically using the following equation, the spacecraft velocity \( V \) (assuming a stationary flux rope), the angle between the flux rope axis and the trajectory subtracted from 90° (\( \phi \)), and time in seconds (\( t \)).

\[ R_0 = \frac{V t \cos(\phi)}{2 \sin \left( \arccos \left( \frac{\lambda}{Y_0} \right) \right)). \]

Alternatively to the previous force-free study (Martin et al., Accepted) which uses a non-linear least-squares fitting technique, this model is fitted using the Bayesian regression concept where each parameter is given an initial probability distribution, which is then sampled and a \( \chi^2 \) value for one sample of each distribution is found. If the \( \chi^2 \) is below the acceptable threshold then the sample is retained, if it is above it is discarded. This is repeated until 10,000 samples give a posterior distribution of acceptable \( \chi^2 \) for each parameter. This process allows easy control over an initial test space for each parameter, probes the entire \( \chi^2 \) and avoids the caveats of using a least-squares fitting algorithm. The modal value of each parameter’s posterior distribution is taken as the fitted value and a credible interval is found where 30% of all of the posterior distribution is inside the interval and is considered the uncertainty in the parameter (this value is comparable to a standard deviation estimate of uncertainty).

Force-free model results are shown statistically in figure 2, along with two examples of fitted flux ropes in figures 3 & 4 in blue.

The non-force-free model is based on the Hidalgo et al. (2002) model, used to determine size and currents inside a magnetic cloud in the solar wind. This model used geometry of the spacecraft trajectory and predictable direction of propagation of magnetic
clouds to fit angles of rotation, and as such does not require MVA. Nieves-Chinchilla et al. (2016) generalised the model into the flux rope coordinate system by fitting the currents as a polynomial expansion. Thus, the currents are fitted and as such may not be restricted to only the parallel direction or a zero value.

\[ B_R = 0, \]  

\[ B_A = B_A^0 + \mu_0 \int_0^r j_T(r)dr = B_A^0 + \mu_0 \sum_{n=1}^{\infty} \frac{1}{n+1} r^{n+1}, \]  

\[ B_T = -\frac{\mu_0}{r} \int_0^r r j_A(r)dr = -\mu_0 \sum_{m=0}^{\infty} \beta_m \frac{r^{m+1}}{m+2}, \]

where \( B_A^0 \) is a boundary condition of the flux rope where in this study an infinitely tangential field is found at the flux rope radius, \( B_A^0 = \mu_0 \sum_{n=1}^{\infty} \alpha_n \frac{1}{n+1} R^{n+1} \), where \( R \) is the flux rope radius, \( r \) is radial distance, \( j_T(r) \) and \( j_A(r) \) are the tangential and axial current densities which are modelled as polynomial expansions \( j = \sum_{m=0}^{\infty} \beta_m r^m e_A - \sum_{n=1}^{\infty} \alpha_n r^n e_T \) with polynomial coefficients \( \alpha_n \) and \( \beta_m \). \( \mu_0 \) is the permeability of free space. Handedness of the rope is included in the sign on \( j_A \) where +1 is right and -1 is left-handed.

The boundary condition can be changed to give a smoother transition into a surrounding medium, or the give the magnetic field at the flux rope radius a larger axial direction, however we retain the assumption of tangential field at the radius and a general discussion of changes in this parameter are discussed further in Nieves-Chinchilla et al. (2016). The maximum central field strength can be calculated using parameters fitted for each flux rope at radius \( r = 0 \). The fitting method used is the Bayesian regression method described above, including the three Euler angles to rotate into a flux rope coordinate system. Uncertainties in each parameter are estimated through the use of a credible interval of 30%.

The order of the polynomials were determined by trial-and-error. Expansions above 3rd for the axial current, and 4th order for the tangential current were found to produce much smaller incremental steps in lowering \( \chi^2 \), which were outweighed by the added computation time. As such, 3rd and 4th order polynomials were found to be the most economical in time when automating the process.

Non-force-free model results are shown statistically in figure 2, along with two examples of fitted flux ropes in figures 3 & 4 in red. Position, central magnetic field and size derived from the non-force-free model is displayed in figure 1, where little spatial relationship is displayed over the two fitted variables. Figures 3 & 4 show the fitting of the force-free model (blue) and the non-force-free model (red) to the Cassini magnetometer data (black). The figures show the axial and tangential magnetic field and current density, along with a computed \( j \times B \) force density. The force-free model shows near zero values current densities and \( j \times B \) force density, where the uncertainty areas have been removed as they cover the whole window, showing that these parameters are valued at zero within uncertainties.

Both models are symmetrical around the peak in the axial field direction, however the non-force-free model appears to fit within uncertainties to the magnetometer data to a better degree. This is reflected in the \( \chi^2 \) value where the force-free gives values of 6.9364 nT for Figure 3 and 3.5182 nT for Figure 4. Both are acceptable values of fit (\( \chi^2 < 5 \), however, the non-force-free model gives values of 0.39752 and 0.47076 nT, which shows that the model is over-fitting (uncertainty in the model is larger than the average difference between the model and the data).

The non-force-free model shows much larger (and non-zero) values within uncertainties in current density values along with \( j \times B \). The current densities can be con-
sidered near field aligned in these examples as axial current density shows a peak in magnitude near the magnetic field peak in figure 3. However, the magnitude is much smaller than the tangential current density. In figure 4 the axial current density declines across the whole flux rope, however the tangential current density shows a similar relation with radius.

Both figures show a quasi-sinusoidal relationship with radius for the \( \mathbf{j} \times \mathbf{B} \) force density, where the edges of the flux rope show the highest magnitude in figure 4 but at half the radius in figure 3, where the force density is expected to be in the radially outward direction.

Figure 2 shows the statistical comparison between the force-free (blue) and non-force-free (red) models for all flux ropes that are fitted (49 for force-free, 84 for non-force-free). The figure shows that on average the models give similar results of 1-15 nT for axial magnetic field, 50-500 km for radius both with larger ranges. However, Figures (3 & 4) show two examples where the difference in values is large. These values are summarised in table 1 and suggest that the closest approach of Cassini to the centre of the flux rope appears to strongly affect the agreement of the models.

| Method | Figure 3: 12/05/2007 | Figure 4: 16/04/2005 |
|--------|---------------------|---------------------|
|        | Radius [km]          | Magnetic Field [nT] | CA    | Radius [km]          | Magnetic Field [nT] | CA    |
| FF     | 53 ± 7               | 6.6 ± 2.3           | 0.7   | 115 ± 11             | 13.5 ± 2.2          | 0.3   |
| NFF    | 132 ± 11             | 9.6 ± 3.5           | 0.7   | 119 ± 11             | 14.1 ± 3.4          | 0.3   |

Table 1. Comparison of parameters corresponding to figures 3 and 4.

3 Deviations from Cylindrical Symmetry

In the confines of this study, we define a bent flux rope as a flux rope that does not have a straight axis. In both the previously discussed models and in Martin et al. (Accepted) the assumption is that the axis of the flux rope does not move or diverge from a straight configuration during Cassini’s fly-through. However, we often find that an asymmetry in the magnetic field data is present where both axial and tangential field is skewed where one side has a steeper gradient than the other towards the peak - where the peak is assumed to be the centre of the flux rope. Figure 5 shows a diagrammatic sketch of a bent flux rope, with a cut out to show the increasingly axial field, along with the expected magnetic field signatures for the shown fly-through.

To model a bent flux rope, we utilise the force-free model described in a previous section which is then deformed using the Tsyganenko (1998) general deformation method. The force-free model to obtain an undeformed magnetic field which is then deformed spatially by a parabola in the x-direction where the original z-axis is the undeformed flux rope axis,

\[
z = ax^2, \tag{9}\]

and a is the leading co-efficient of the polynomial, describing the extent of the bending. The normal to the z-axis is then calculated by finding the derivatives of the above equation:

\[
\frac{dz}{dx} = 2a(x - c), \tag{10}\]
\[
\frac{dz}{dy} = 2a(y - c), \tag{11}\]
where \( c \) is the offset from the y-axis that the flux rope has moved at the height of the parabola. To deform using the general deformation method, a normalised normal vector \((n_x, n_y, n_z)\), Y-vector \((Y_x, Y_y, Y_z)\) and X-vector \((X_x, X_y, X_z)\) are found from the above equations, the undeformed y-axis and the cross product of the normal and Y-vector, respectively. Hence, we can now build the new deformed coordinate system as:

\[
\begin{align*}
x^* &= xX_x + yX_y + zX_z \\
y^* &= xY_x + yY_y + zY_z \\
z^* &= xn_x + yn_y + zn_z
\end{align*}
\] (12)

We can now form the new magnetic field in the undeformed coordinate system denoted with an asterisk.

\[
\begin{align*}
B_z^* &= B_0 J_0(\alpha r^*) + b_0, \\
B_x^* &= -H B_0 J_1(\alpha r^*) \sin(\phi), \\
B_y^* &= H B_0 J_1(\alpha r^*) \cos(\phi),
\end{align*}
\] (15, 16, 17)

where \( r^* = \sqrt{x^*^2 + y^*^2} \), and \( \phi \) is calculated as the angle of the simulated spacecraft from the x-axis for each position to convert the force-free cylindrical model to a Cartesian coordinate system. The transformation matrix is then formed to give the new magnetic field in the deformed system \( B' \) where the full expansion of \( T \) is found in general terms in Tsyganenko (1998).

\[
B' = TB^*
\] (18)

An example of a modelled deformed (red) and undeformed (grey) force-free flux rope is shown in figure 6, where it is shown that the asymmetry in the z-component of magnetic field is reproduced in Cartesian coordinates. However, the x- and y-components do not show any asymmetry and both cross the expected centre of the flux rope at 0 seconds.

Discussed earlier is the dependence on either MVA or angles in fitting any model to flux ropes. However, analysis of flux ropes at different planetary bodies find the variance directions as different components of the magnetic field. When using MVA on a modelled force-free flux rope, we find that the component tangential to the flux rope is in the direction of maximum variance (blue) and the component along the axis of the flux rope is in the intermediate direction (yellow). Given the minimum and radial (green) are valued at zero.

This is not the case at Titan. Adding a slight bend in the flux rope, (shown in figure 7) we find that the maximum variance direction (blue) is now nearest aligned to the axial direction in a Cartesian sense. Hence, MVA is highly sensitive to small changes in the flux rope geometry and as such may lead to problems when fitting models in this manner. The fitting of orientation angles with associated uncertainty analysis is a physically superior method of attempting to fit the force-free model, which results in statistically better fittings.

An ideal flux rope is usually assumed to have a circular cross-section, and previously only a few authors have considered the possibilities of elliptical cross-sections (e.g., Vandas & Romashets, 2017; Nieves-Chinchilla et al., 2018), however none of these are for flux ropes in an ionosphere. There are still a number of features in magnetometer data that are unable to be replicated with the models shown above, and as such we also investigate the ellipticity of the flux rope as a factor that may be causing these deviations from perfect circular force-free flux ropes.

Evaluating the Bessel functions described in the force-free model in elliptical coordinates is not trivial and are of the form of Mathieu functions which currently only
have a complex valued solution. As such, we assume some simple geometries and avoid extended use of elliptical co-ordinates in the following solution.

At the centre of an ellipse, a central line ($2c$ in figure 8) is constructed. The ellipse has a semi-major axis $a$ and semi-minor axis $b$ where $a = b + c$. A fly-through Cassini trajectory is then simulated, where each position $P$ has a unique radial distance $r$ from the central line. This radial distance is not valued as the distance from the ellipse centre, or from the nearest point on the central line. It is found as the distance along a line extrapolated from the ellipse edge at right-angles to a tangential line, through $P$ to the central line (see $r$ in figure 8).

Hence, this value of $r$ is then used in the elliptical evaluation of the Bessel functions:

$$B_z = B_0 J_0(\alpha r), \quad (19)$$

$$B_v = H B_0 J_1(\alpha r), \quad (20)$$

$$B_u = 0, \quad (21)$$

where $\alpha r$ is evaluated in the same way as the circular force-free model, this model is shown in elliptical coordinates $(u, v, z)$, which are radial, tangential and axial equivalent respectively. The magnetic offset $b_0$ is not presented in this analysis for simplicity, however can be easily implemented if need be.

Figure 9 shows a flux rope example from T29 at 21:34 on 26/04/2007 at 13.7 SLT in black along with the fitted model of the elliptical flux rope. This elliptical flux rope has semi-major axis of 250 km and semi-minor axis of 150 km, and both the undeformed force-free and non-force-free model were unable to fit to this flux rope with adequate $\chi^2$ (6.7 and 6.5 nT respectively). Uncertainties are comparable for each fitting method, hence the lower value of 3.8 nT for the elliptical force-free model fitting can be considered the best fit from the selected models, and it is likely that this flux rope does not have a circular cross-section.

One example is shown here to emphasise the asymmetrical and plateau properties of the different components of magnetic field and how these could be caused by an elliptical flux rope cross-section. A schematic of the trajectory Cassini could take through an elliptical flux rope to produce a similar magnetic signature is shown in figure 10. However, a number of flux ropes show signs of a flat-top in axial field or an asymmetry that can not be reproduced by bending alone, and as such we show that some ellipticity is common in flux ropes at Titan.

4 Discussion

In this study, we have examined the differences of fitting a force-free and a non-force-free model along with exploring different deformations that can be made to the force-free model which will allow improved fitting and simulation of some magnetic field signatures. A full discussion of the force-free model alone is given in Martin et al. (Accepted), and as such we will restrict this discussion to the non-force-free model and their comparison.

The non-force-free model utilises a polynomial expansion of the current density in the flux rope, and as such allows for much lower $\chi^2$ results and smaller uncertainties. The model uses the 3rd order for the axial field and the 4th order for the tangential field, determined to be the optimum balance between improving the fitting and time spent fitting. Increases above the 3rd and 4th orders does not radically improve or change the fitted parameters.

From a total of 85 flux ropes, 84 are fitted with a $\chi^2$ probability of 5% or less relating to an MSE of 0.5 nT or less, which is the same criteria for a good fit for the force-
free model. One flux rope is unable to be fitted within the restrictions, which is also a flux rope which is unable to be fitted with the force-free assumptions. The remaining 84 flux ropes give a range of axial magnetic field values of 1-5 nT with a large tail up to 40 nT, and flux rope radii of 50-350 km, again with a large tail up to 1500 km.

In comparison, these ranges are smaller than the ranges given from the use of the force-free model which has a number of larger values, this however, may be a consequence of the far lower number of flux ropes fitted with the force-free model. We show that the two models give similar statistical views of the flux rope parameters, however figures 3 & 4 show examples where both models fit the magnetometer data. With the non-force-free model giving much improved $\chi^2$ values with comparable uncertainties in both examples.

Table 1 compares the individual parameters retrieved from both models for a pair of example flux ropes, where the second example has a much smaller closest approach, and values from the two models overlap in uncertainties. The first example shows very different parameters for a much larger closest approach. It appears that the difference in individual parameters from the two models may be dependent on the closest approach value.

Figure 11 shows the difference in fitted radii of the flux ropes against the value of closest approach. We can see that with small closest approach values the difference between the radii given by the models is of a much smaller order. It is apparent that a greater closest approach value allows a higher probability of a large disparity between the two models and as such we can assume that either model has a strong uncertainty dependence on closest approach value. Additionally, MVA, though not used in this, has been shown to have a similar dependence upon the closest approach values (Xiao et al., 2004).

To enable an accurate comparison, both models were fitted using the Bayesian regression method described earlier. We describe the quality of each fit using the $\chi^2$ parameter, however it is important to calculate and consider the uncertainties on the fitted values. These uncertainties are derived from the square root of the covariance matrix diagonal. These values represent the uncertainty of each corresponding parameter that is fitted, and most fittings are found to have uncertainty values of 5-10%. Larger uncertainties are found, however they are usually comparable to the uncertainties in the alternative model and as such we can assume that the $\chi^2$ parameter is representative of the goodness of fit.

Current densities and $j \times B$ force density are derived from the magnetic field modelled or from the model itself for non-force-free. $j \times B$ force density along with field aligned currents and current radial to the flux rope are shown for both examples when using the non-force-free model. Both examples show zero current density and $j \times B$ force density when fitted with a force-free model, which hold for the assumptions of that model. However, as the non-force-free model is fitted better, we may then assume that there are quasi-field aligned currents along with a $j \times B$ force density in the flux ropes at Titan.

This leads to the conclusion that the force-free model is a satisfactory fit for a portion of flux ropes at Titan, but a much larger number are fitted, and fitted better, by the non-force-free model. Therefore, the flux ropes at Titan are more likely to adhere to non-force-free assumptions. The implications of which for Titan's magnetosphere as a whole can be considerable.

Non-force-free structures imply not only the large scale dynamics, such as those caused by the surrounding environments and fossil fields, but small scale dynamics as well. The results suggest that these are evolving flux ropes which may be a sign that Titan’s highly variable and dynamic environment is an ideal initiator of flux ropes but does not allow them to mature fully before they are disrupted by the upstream conditions at Titan. The effects of non-steady-state conditions in Titan’s environment are similarly
concluded by Cowee et al. (2010), where certain plasma instabilities are unable to evolve due to the changes in the upstream conditions at much shorter timescales. Reconnection could also be initiated by these small scale re-configurations of the magnetic field if the plasma conditions inside the flux rope are not force-free.

Additionally, we comment that spatial and temporal changes, such as acceleration or deceleration of the flux rope during detection, can also cause asymmetries and changes in the magnetic signature of the flux ropes. On the conclusion that a flux rope may not be force-free, one may expect some expansion or contraction and this too may change the magnetic signatures. These features are outside of the remit of this study, but an area for future research.

There is no a priori reason why a flux rope has to be perfectly straight and so we have developed a model for a bent flux rope. These bent flux ropes allow certain observed asymmetries to be modelled reasonably well. We also found that this affected the orientations of the flux rope as obtained from MVA. A corollary of this is that the application of MVA to observed flux ropes might be systematically affected if the rope is bent. A small bend can change the orientation of the tangential and axial field directions which may have implications for models of flux ropes in other environments where a flux rope may be bent.

The assumption that flux ropes are cylindrically symmetric is common, however, physically it is unlikely that all flux ropes will be perfectly cylindrical. To that effect, the method shown here gives a simple and easy to implement addition to the common force-free Bessel function method to fit to and test the elliptical nature of a flux rope.

5 Summary

In this paper we have fitted asymmetrical and non-force free flux rope models to Cassini observations of flux ropes in Titan’s ionosphere. These models were fitted using Bayesian regression. We have also specifically investigated the role of MVA in modelling flux ropes and found found higher quality fits when incorporating the orientation of the flux rope as explicit fit parameters. However, in the Bayesian framework this does not preclude using MVA to inform the priors on the orientation angles.

The non-force free model was adapted from Hidalgo et al. (2002) and uses polynomial expansions for the axial and radial current density in the flux rope. These models were found to provide superior fits compared to force-free flux ropes (when accounting for the additional free parameters). This leads to our conclusion that the flux ropes at Titan are generally not force-free and in an evolving state.

We explored two sources of flux rope asymmetry, bending and ellipticity, and developed quantitative modes that were fitted to the data. These bends introduced asymmetries in the axial and the radial magnetic field. We specifically found that the presence of a small bend in the flux rope would change the result of an MVA analysis of the flux rope, and so one would conclude that the orientation was quite different to reality. This justifies our approach (following (Nieves-Chinchilla et al., 2016)) of including the orientation angles as free parameters in the model. Ellipticity was introduced analytically into a force-free flux rope model and was found to produce asymmetries in the radial, axial and tangential magnetic fields. A case study was presented where the elliptical model fitted better than the circular force-free and non-force-free models. These flux ropes have a "plateau" feature in the axial magnetic field that is modelled well by the elliptical model, although we note that this may also be caused by a temporal change.

These results show that a significant proportion of flux ropes at Titan are non-force-free or deformed, suggesting these are evolving flux ropes which may be a sign that Titan’s highly variable and dynamic environment is an ideal initiator of flux ropes but does
not allow them to mature fully before they are disrupted by the surrounding magnetosphere.

These conclusions have implications for other planetary bodies, such as Mercury, where flux ropes are formed and then travel from their source to significant down-tail distances in seconds (DiBraccio et al., 2015) The models described here can be applied in many different solar system and plasma contexts.

Acknowledgments
CJM was funded by a Faculty of Science and Technology studentship from Lancaster University. CSA was funded by a Royal Society Research Fellowship. CJM, CSA and SVB were funded by STFC grant number ST/R000816/1. Cassini MAG data used in this study may be obtained from the Planetary Data System (http://pds.nasa.gov/).

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Figure 1. Figure showing the position of flux ropes at Titan in the TIIS coordinate system. The size of each point is determined by the radius found using the non-force-free model (a,b,c) and the force-free model (d,e,f), where a 200 km example is shown by the key. Titan’s outline is shown in black and each flux rope is a circle coloured by magnetic field strength in the centre of the flux rope. Flux ropes which were not fitted by either model are shown in grey with a 100 km equivalent radius.
Figure 2. Figure comparing a fit of the FF model (blue) and the NFF model (red). a) maximum magnetic field b) flux rope radius and c) flux content.
Figure 3. Figure comparing a fit of the FF model (blue) and the NFF model (red) with corresponding uncertainty bounds (shaded regions in corresponding colours). The figure shows axial magnetic field, tangential magnetic field, axial current density, tangential current density and force density where data is in black. The corresponding $\chi^2$ values are shown for each fit. This flux rope is found at 13.6 SLT on T30 at 20:07 on 12/05/2007.
Figure 4. Figure comparing a fit of the FF model (blue) and the NFF model (red) with corresponding uncertainty bounds (shaded regions in corresponding colours). The figure shows axial magnetic field, tangential magnetic field, axial current density, tangential current density and force density where data is in black. The corresponding $\chi^2$ values are shown for each fit. This flux rope is found at 5.3 SLT on T5 at 19:01 on 16/04/2005.
Figure 5. Figure showing diagram of a bent flux rope, with simulated fly-through and expected cylindrical magnetic field components.
Figure 6. Figure showing a comparison of an undeformed flux rope (grey) and a deformed flux rope (red) where the components are total field (thick solid), axial (dotted), y (dash-dot) and x (thin solid).

Figure 7. Figure showing a comparison of an undeformed flux rope (left) and a deformed flux rope (right) where MVA is used on both and give maximum (blue), intermediate (yellow) and minimum (green) variance directions averaged for the whole fly-through.
Figure 8. Figure showing the cross-section of an elliptical flux rope with model parameters labeled.
Figure 9. Figure showing magnetometer data (black) fitted with the elliptical flux rope model (red) in Cartesian coordinates.
Figure 10. Figure showing schematic of set up of trajectory and elliptical flux rope corresponding to fitted magnetometer data in figure 9, where the red dashed line is the expected trajectory, the blue quiver is the model field direction in the x-y plane. The grey shaded area is inside the flux rope and the black solid line is the edge of the flux rope. A dashed grey line shows the central line as described in figure 8 with two grey dots showing the foci of the ellipse.
Figure 11. Figure showing the increased probability of a disparity between models and CA value.