Regularization of an Ill-posed Cauchy Problem for the Wave Equation (Fourier Method)

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Abstract
An ill-posed Cauchy problem for the wave equation is considered: the solution is to be determined by the Cauchy data on some part of the time-space boundary. By means of Fourier method we obtain a regularization algorithm for this problem, which is given by rather explicit formula.

**Keywords:** wave equation, ill-posed Cauchy problem, regularization algorithm.

1 Problem statement

Suppose $u(x, y, t)$ is a smooth function in $x, t \in \mathbb{R}, y \geq 0$, satisfying the wave equation and the Dirichlet boundary condition:

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0,$$

$$u|_{y=0} = 0. \quad (1)$$

We consider an ill-posed Cauchy problem for the equation (1): the function $u(x, y, t)$ is to be determined by the normal derivative $\frac{\partial u}{\partial y}$ given on the part of the boundary $\{y = 0\}$. We shall obtain a regularization procedure for determining $u(x_0, y_0, t_0)$, $y_0 > 0$, which requires the data $\frac{\partial u}{\partial y}$ on the set

$$U := \left\{ (x, 0, t) \mid |x - x_0| \leq D \left( \sqrt{y_0^2 - (t - t_0)^2} \right) + \varepsilon, |t - t_0| \leq y_0 \right\}. \quad (3)$$

Here

$$D(z) := z \cdot \sqrt{\frac{c}{c + 2z}}, \quad z \geq 0,$$

and $c, \varepsilon$ are arbitrary positive numbers. Note that the function $D(z)$ increases, so $U$ is a subset of the following rectangle

$$\{(x, 0, t) \mid |x - x_0| \leq D(y_0) + \varepsilon, |t - t_0| \leq y_0\}. \quad (4)$$

The main result of the paper is the following

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Theorem. Suppose $y_0 > 0$, $x_0, t_0 \in \mathbb{R}$. For a $C^\infty$-smooth solution of the equation (7) in the domain $\{y \geq 0\}$ with the boundary condition (2) the following relation holds true

$$u(x_0, y_0, t_0) = \lim_{h \to 0^+} \int_U dx dt \, K_h(x - x_0, y_0, t - t_0) \frac{\partial u}{\partial y}(x, 0, t),$$

(5)

where the kernel $K_h$ is defined as follows

$$K_h(x, y_0, t) := \frac{1}{2\pi^{3/2}h} \Re \left[ \frac{1}{\sqrt{c + ix}} \int_0^{\pi/2} ds \exp \left( -\frac{1}{4h(c + ix)} \left( x + i\sqrt{y_0^2 - t^2 \sin s} \right)^2 \right) \right].$$

(6)

(we chose a leaf of the square root $\sqrt{c + ix}$ in such a way that $\Re\sqrt{c + ix} > 0$).

Note that the regularization kernel $K_h$ is real and even in both variables $x$ and $t$.

In formula (5) the set of integration $U$ depends on $\varepsilon$, while the integrand does not. Besides, $K_h(x - x_0, y_0, t - t_0) \to 0$ as $h \to 0$, if $|x - x_0| > D\left(\sqrt{y_0^2 - (t - t_0)^2}\right)$, consequently the limit in the r.h.s. of (5) does not depend on $\varepsilon$ (this is discussed in sec. 5 in more details). However, the rate of convergence of the limit does depend on the choice of $\varepsilon$.

The Cauchy problem considered here was solved by R. Courant [1]: the reduction to the problem of recovering function by its spherical means was used. According to [1] the Cauchy data on the rectangle

$$\{(x, 0, t) \mid |x - x_0| \leq \delta, |t - t_0| \leq y_0\}$$

with arbitrarily small $\delta$ are sufficient to determine $u(x_0, y_0, t_0)$. These data would be sufficient if one uses the formula (5) as well: parameters $c, \varepsilon$ can be chosen arbitrarily small and $D(y_0) \to 0$ as $c \to 0$, so the rectangle (4) together with the set $U$ can have arbitrarily small size along the $x$-axis.

In [1] there is no parameter that is analogous to $c$ in (5); in fact the Cauchy data in the “infinitesimal” neighborhood of the interval $\{x = x_0, |t - t_0| \leq y_0\}$ is used (the derivatives of data and intermediate functions on the interval are calculated). At the same time for a fixed $c$ the kernel $K_h(x - x_0, y_0, t - t_0)$ does not tend to zero as $h \to 0$ if $|x - x_0| \leq D\left(\sqrt{y_0^2 - (t - t_0)^2}\right)$ (from sec. 5 one can conclude that $K_h$ grows exponentially). Hence in the r.h.s. of (5) the Cauchy data on a set of positive (2-dimensional) measure are taken into account. The dependence of stability of regularization of ill-posed problems for hyperbolic equations on the amount of data was studied in [3] (the singular value decomposition was applied): the larger amount of data provides the more stable regularization.

The problem of determining of the solution of the wave equation by the boundary data in specific domains (such as ball, ellipsoid, half-space) and related problems of integral geometry were considered (besides [1] mentioned above) in [2, 4–9]. In papers [4–7] the inversion formulas were obtained, which unlike (5), require the data on the whole
boundary and on sufficiently large time interval depending on the diameter of the domain. In [8] the problem of recovering of the function in half-plane by its mean values over circles centered at the boundary of half-plane; the microlocal estimate was obtained under assumption that the function is compactly supported.

To obtain (5) we apply Fourier method to the wave equation. Applying Fourier transform in $x$, we obtain the Cauchy problem for the wave equation in the domain $y \geq 0, t \in \mathbb{R}$. The inverse Fourier transform requires a regularization (in (5) $h$ is a small parameter of regularization). Note that our problem is a particular case of the problem of integral geometry considered in [2], where also Fourier method was applied. Our goal is to obtain a regularization that requires the Cauchy data only in $U$.

Now we make some obvious simplifications of our problem. Further we suppose that $x_0 = t_0 = 0$. Thus formula (5) takes the following form

$$u(0, y_0, 0) = \lim_{h \to 0+} \int_{U} dx dt K_h(x, y_0, t) \frac{\partial u}{\partial y}(x, 0, t).$$

(7)

It is sufficient to prove formula (7) for even function $u$ in $t$. Indeed, in general case we may consider an even function $u'_1(x, y, t) = u(x, y, t) + u(x, y, -t))/2$, which satisfies (1), (2), and apply (7) to $u'$. We have $u(0, y_0, 0) = u'(0, y_0, 0)$. Since the kernel $K_h$ is even in $t$, the derivative $\frac{\partial (u-u')}{\partial y}$ is odd in $t$, and the set $U$ is symmetric with respect to $t = 0$, we have

$$\int_{U} dx dt K_h(x, y_0, t) \left( \frac{\partial u}{\partial y}(x, 0, t) - \frac{\partial u'}{\partial y}(x, 0, t) \right) = 0.$$

This implies (7) in general case.

Further we suppose $u$ to be even in $t$. From the wave equation (1) and boundary condition (2) immediately follows the relation $\partial^n u/\partial y^n = 0$ for $y = 0$ and even $n$. Therefore, an odd continuation of $u$ in $y$ belongs to $C^\infty(\mathbb{R}^3)$. We use the same notation $u$ for such a continuation.

Put $u_0 := u|_{t=0}$, $u_0 \in C^\infty(\mathbb{R}^2)$. As $u$ is even in $t$ we have $\partial u/\partial t = 0$ for $t = 0$. Let $\chi(x, y)$ be a $C^\infty$-smooth compactly supported function in $\mathbb{R}^2$ satisfying $\chi(x, y) = 1$ for $x^2 + y^2 \leq R^2$ for some $R$. Then the solution $u_\chi$ of the problem

$$\frac{\partial^2 u_\chi}{\partial t^2} - \frac{\partial^2 u_\chi}{\partial x^2} - \frac{\partial^2 u_\chi}{\partial y^2} = 0, \quad u_\chi|_{t=0} = \chi u_0, \quad \frac{\partial u_\chi}{\partial t}|_{t=0} = 0,$$

coincides with $u$ on the set

$$\{(x, y, t) \in \mathbb{R}^3 : \sqrt{x^2 + y^2} + |t| \leq R\}.$$

\[1\) In [2, Ch. 1] the function in a layer is recovered by its mean values over some family of surfaces; the latter was supposed to be invariant with respect to translation along transversal directions. Our problem reduces to recovering a function by its mean values over circles centered at the line $\{y = 0\}.$
Choosing sufficiently large $R$ we guarantee that $u_\chi(0, y_0, 0) = u(0, y_0, 0)$ and
\[
\frac{\partial u_\chi}{\partial y} = \frac{\partial u}{\partial y}
\]
on the set $\mathbb{B}$. Thus, in proof of formula (7) we may suppose that $u_0$ is compactly supported.

2 Fourier transform of the solution $u$

Here we study some properties of Fourier transform $\hat{u}(k, l, \omega)$ of the function $u(x, y, t)$. We use the following formulas for Fourier transform (and its inverse) of function $f$:
\[
\hat{f}(k) = \int_{-\infty}^{\infty} dx \ e^{-ikx} f(x), \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{ikx} \hat{f}(k),
\]

Since $u_0(x, y)$ is smooth and compactly supported its Fourier transform $\hat{u}_0(k, l)$ belongs to $S(\mathbb{R}^2)$ (Schwartz space). Further we use the following estimates ($N$ is a positive integer):
\[
|\hat{u}_0(k, l)|, \left| \frac{\partial \hat{u}_0(k, l)}{\partial k} \right|, \left| \frac{\partial \hat{u}_0(k, l)}{\partial l} \right| \leq \frac{C_N}{(1 + |k|^N)(1 + |l|^N)}. \tag{8}
\]

The function $\hat{u}$ belongs to $S'(\mathbb{R}^3)$ (tempered distributions).

**Proposition 1.** The distribution $\hat{u}$ acts on the test function $\varphi \in S(\mathbb{R}^3)$ in the following way
\[
\langle \hat{u}, \varphi \rangle = \int_{\mathbb{R}^2} dk dl \ \hat{u}_0(k, l) \frac{1}{2} \left( \varphi \left( k, l, \sqrt{k^2 + l^2} \right) + \varphi \left( k, l, -\sqrt{k^2 + l^2} \right) \right). \tag{9}
\]

**Proof.** The r.h.s. of (9) defines some distribution $\psi$ from $S'(\mathbb{R}^3)$. Now we show that the inverse Fourier transform $\check{\psi}$ is a regular function and the following relation holds true
\[
\check{\psi}(x, y, t) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} dk dl \ \hat{u}_0(k, l) \frac{1}{2} \left( e^{i(kx + ly + \sqrt{k^2 + l^2}t)} + e^{i(kx + ly - \sqrt{k^2 + l^2}t)} \right). \tag{10}
\]
(The integral in the r.h.s. absolutely converges due to (8)). For a test function $\zeta \in S(\mathbb{R}^3)$ we have
\[
\langle \check{\psi}, \zeta \rangle = \langle \psi, \zeta \rangle = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dx dy dt \ \zeta(x, y, t) \frac{1}{2} \sum_{\pm} e^{i(kx + ly \pm \sqrt{k^2 + l^2}t)} =
\]
\[
= \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dx dy dt \ \zeta(x, y, t) \int_{\mathbb{R}^2} dk dl \ \hat{u}_0(k, l) \frac{1}{2} \sum_{\pm} e^{i(kx + ly \pm \sqrt{k^2 + l^2}t)}
\]
(\(\zeta\) is the inverse Fourier transform of \(\zeta\)). This implies (10). It can be easily derived from the formula (10) that $\check{\psi}$ is the solution of the following Cauchy problem
\[
\frac{\partial^2 \check{\psi}}{\partial t^2} - \frac{\partial^2 \check{\psi}}{\partial x^2} - \frac{\partial^2 \check{\psi}}{\partial y^2} = 0, \quad \check{\psi}|_{t=0} = u_0, \quad \frac{\partial \check{\psi}}{\partial t}|_{t=0} = 0,
\]
therefore, $\check{\psi} = u$, and so $\psi = \hat{u}$. \qed
Let \( \tilde{u}(k, y, \omega) \) be Fourier transform of \( u(x, y, t) \) in \( x, t \). The function \( \tilde{u}(\cdot, y, \cdot) \) belongs to \( S'(\mathbb{R}^2) \).

**Proposition 2.** For any \( y_0 \) the function \( \tilde{u}(\cdot, y_0, \cdot) \) is regular and the following relation holds true

\[
\tilde{u}(k, y_0, \omega) = \theta(|\omega| - |k|) \frac{|\omega|}{4\pi \sqrt{\omega^2 - k^2}} \sum_{\pm} \hat{u}_0 \left( k, \pm \sqrt{\omega^2 - k^2} \right) e^{\pm iy_0 \sqrt{\omega^2 - k^2}} \tag{11}
\]

(\( \theta \) is the Heaviside function).

**Proof.** For a test function \( \varphi(k, \omega) \) belonging to \( S(\mathbb{R}^2) \) we have

\[
\langle \tilde{u}(\cdot, y_0, \cdot), \varphi \rangle = \langle u(\cdot, y_0, \cdot), \varphi \rangle = \lim_{\varepsilon \to 0} \langle u, f_\varepsilon \rangle, \tag{12}
\]

where

\[
f_\varepsilon(x, y, t) := \varphi(x, t) g_\varepsilon(y), \quad g_\varepsilon(y) := \frac{e^{-(y-y_0)^2/\varepsilon}}{\sqrt{\pi \varepsilon}},
\]

(\( g_\varepsilon \) tends to \( \delta(y - y_0) \) as \( \varepsilon \to 0 \)). Due to (9) we have

\[
\langle u, f_\varepsilon \rangle = \langle \tilde{u}, \tilde{f}_\varepsilon \rangle = \int_{\mathbb{R}^2} dkdl \, \hat{u}_0(k, l) \frac{1}{2} \sum_{\pm} \tilde{f}_\varepsilon \left( k, l, \pm \sqrt{k^2 + l^2} \right).
\]

It is easy to see that

\[
\tilde{f}_\varepsilon(k, l, \omega) \to \frac{1}{2\pi} e^{iy_0l} \varphi(k, \omega), \quad \varepsilon \to 0.
\]

Together with (12) this yields

\[
\langle \tilde{u}(\cdot, y_0, \cdot), \varphi \rangle = \int_{\mathbb{R}^2} dkd\omega \, \hat{u}_0(k, l) \frac{1}{4\pi} \sum_{\pm} e^{iy_0l} \varphi(k, \pm \sqrt{k^2 + l^2}). \tag{13}
\]

Consider for example the term \( \text{"}+\text{"} \) of sum in the integral in (13). Represent its integral as the following sum

\[
\int_{\mathbb{R}^2 \cap \{l > 0\}} + \int_{\mathbb{R}^2 \cap \{l < 0\}} dkd\omega \, \hat{u}_0(k, l) e^{iy_0l} \varphi(k, \sqrt{k^2 + l^2}).
\]

Now we make change of variables in both integrals

\[
(k, l) \mapsto (k, \omega), \quad \omega = \sqrt{k^2 + l^2}.
\]

We obtain

\[
\int_{0}^{\infty} d\omega \int_{-\omega}^{\omega} dk \, \hat{u}_0(k, \sqrt{\omega^2 - k^2}) e^{iy_0 \sqrt{\omega^2 - k^2}} \varphi(k, \omega) \frac{\omega}{\sqrt{\omega^2 - k^2}} + \int_{0}^{\infty} d\omega \int_{-\omega}^{\omega} dk \, \hat{u}_0(k, -\sqrt{\omega^2 - k^2}) e^{-iy_0 \sqrt{\omega^2 - k^2}} \varphi(k, \omega) \frac{\omega}{\sqrt{\omega^2 - k^2}}.
\]

Carrying out analogous calculations for the term \( \text{"} - \text{"} \) in the integral (13) we arrive at (11).
Since the function $u_0(x,y)$ is odd in $y$, its Fourier transform $\hat{u}_0(k,l)$ is odd in $l$. Hence the formula (11) can be written as follows

$$\hat{u}(k, y_0, \omega) = \theta(|\omega| - |k|) \frac{i|\omega|}{2\pi \sqrt{\omega^2 - k^2}} \hat{u}_0(k, \sqrt{\omega^2 - k^2}) \sin\left(y_0 \sqrt{\omega^2 - k^2}\right).$$  \hspace{1cm} (14)

Put

$$v(x,t) := \frac{\partial u}{\partial y}(x,0,t)$$ \hspace{1cm} (15)

and denote by $\tilde{v}(k,\omega)$ Fourier transform of $v(x,t)$. It follows from (11) that

$$\tilde{v}(k,\omega) = \frac{i|\omega|}{2\pi} \theta(|\omega| - |k|) \hat{u}_0(k, \sqrt{\omega^2 - k^2}).$$ \hspace{1cm} (16)

Now we recast formula (14) as follows

$$\tilde{u}(k, y_0, \omega) = \tilde{v}(k,\omega) \frac{\sin\left(y_0 \sqrt{\omega^2 - k^2}\right)}{\sqrt{\omega^2 - k^2}}.$$ \hspace{1cm} (17)

Relation (16) implies that $\tilde{v}(k,\cdot) \in L_1(\mathbb{R})$ for any $k$ and the estimate

$$\int_{-\infty}^{\infty} d\omega |	ilde{v}(k,\omega)| \leq C \frac{1}{1 + k^2}$$ \hspace{1cm} (18)

holds true, where $C$ does not depend on $k$. Indeed,

$$\int_{-\infty}^{\infty} d\omega |	ilde{v}(k,\omega)| = \frac{1}{\pi} \int_{|k|}^{\infty} d\omega \left| \hat{u}_0\left(k, \sqrt{\omega^2 - k^2}\right) \right| = \frac{1}{\pi} \int_{0}^{\infty} dl \left| \hat{u}_0(k,l) \right|.$$ \hspace{1cm} (19)

Here the first equality holds true since $u$ is even in $t$ and so $\tilde{v}$ is even in $\omega$. In the second equality we made change of variable $l = \sqrt{\omega^2 - k^2}$. Now applying the estimate (8) we arrive at (18).

The estimate (18) means that $\tilde{v} \in L_1(\mathbb{R}^2)$. Taking into account (17) and that $\tilde{u}$ and $\tilde{v}$ are supported in the set $\{|\omega| \geq |k|\}$, we have $\tilde{u}(\cdot,y_0,\cdot) \in L_1(\mathbb{R}^2)$. Applying inverse Fourier transform to $\tilde{u}$ we obtain

$$u(0,y_0,0) = \frac{1}{4\pi^2} \int_{\mathbb{R}^2} dk \, d\omega \, \frac{\sin\left(y_0 \sqrt{\omega^2 - k^2}\right)}{\sqrt{\omega^2 - k^2}} \tilde{v}(k,\omega).$$ \hspace{1cm} (19)

Further we also use the following estimate

$$\int_{-\infty}^{\infty} d\omega \left| \tilde{v}(k,\omega) - \tilde{v}(k',\omega) \right| \leq C \frac{|k - k'|}{1 + k^2}, \quad k \cdot k' \geq 0, \quad |k| \leq |k'| \leq |k| + 1.$$ \hspace{1cm} (20)

To prove (20) we suppose $0 \leq k \leq k' \leq k + 1$. We have

$$\int_{-\infty}^{\infty} d\omega \left| \tilde{v}(k,\omega) - \tilde{v}(k',\omega) \right| = \frac{1}{\pi} \int_{k}^{k'} d\omega \left| \hat{u}_0\left(k, \sqrt{\omega^2 - k^2}\right) \right| +$$

$$+ \frac{1}{\pi} \int_{k'}^{\infty} d\omega \left| \hat{u}_0\left(k, \sqrt{\omega^2 - k^2}\right) - \hat{u}_0\left(k', \sqrt{\omega^2 - k'^2}\right) \right|.$$ \hspace{1cm} (21)
The first integral is estimated by the r.h.s. of (20) in view of (8). In the second integral we make change of variable \( l = \sqrt{\omega^2 - k^2} \):

\[
\int_0^\infty dl \ | \hat{u}_0\left(k, \sqrt{l^2 + k'^2 - k^2}\right) - \hat{u}_0(k', l)| \leq \int_0^\infty dl \ | \hat{u}_0\left(k, \sqrt{l^2 + k'^2 - k^2}\right) - \hat{u}_0(k, l)| + \int_0^\infty dl \ | \hat{u}_0(k, l) - \hat{u}_0(k', l)|. \quad (21)
\]

Now we estimate the second integral in the obtained expression using the estimate of \( \hat{u}_0/k \) in (8):

\[
\int_0^\infty dl \ | \hat{u}_0(k, l) - \hat{u}_0(k', l)| \leq \int_0^\infty dl \ (k' - k) \frac{C}{(1 + k^3)(1 + l^3)} \leq \frac{C(k' - k)}{1 + k^3}.
\]

The first integral in the r.h.s. of (21) can be estimated with the help of the estimate of \( \hat{u}_0/l \) in (8):

\[
\int_0^\infty dl \ | \hat{u}_0\left(k, \sqrt{l^2 + k'^2 - k^2}\right) - \hat{u}_0(k, l)| \leq \int_0^\infty dl \ (k' - k) \frac{C}{(1 + k^3)(1 + l^3)} \leq \frac{C(k' - k)}{1 + k^3} \int_0^\infty dl \ \frac{l}{(1 + l^3)(\sqrt{l^2 + k'^2 - k^2} + l)}.
\]

Here the integrand can be estimated by \( 1/(1 + l^3) \). The factor before the integral is estimated by the r.h.s. of (20).

3 Another representation of the kernel \( K_h \)

Rewrite (6) in the following form

\[
K_h(x, y_0, t) = \frac{1}{2\pi} \sum_{\pm} \int_0^{\pi/2} ds \frac{1}{2\sqrt{\pi h(c \pm ix)}} \exp \left( -\frac{1}{4h(c \pm ix)} \left( x \pm i \sqrt{y_0^2 - t^2 \sin s} \right)^2 \right).
\]

The integrand is equal to

\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{-ikx-\frac{h}{2}(c\pm ix)\pm k\sqrt{y_0^2-t^2}\sin s}
\]

(the integral is absolutely convergent, since \( c, h > 0 \)). This yields

\[
K_h(x, y_0, t) = \frac{1}{4\pi} \sum_{\pm} \int_{-\infty}^{\infty} dk \ e^{-ikx-\frac{hk^2}{2}(c\pm ix)} H \left( \pm k \sqrt{y_0^2 - t^2} \right), \quad (22)
\]
where

\[ H(z) := \frac{1}{\pi} \int_{0}^{\pi/2} ds \, e^{z \sin s}. \]

Define \( G_{\pm}(k, \omega) \) as the inverse Fourier transform in \( t \) of the function

\[ \theta(y_0 - |t|) H \left( \pm k \sqrt{y_0^2 - t^2} \right). \]

We do not indicate the dependence of \( G_{\pm} \) on \( y_0 \) explicitly. To study the functions \( G_{\pm} \) we need the following relations:

\[ H(z) + H(-z) = J_0(iz) \quad \text{(23)} \]

\((J_0 \text{ is the Bessel function)},\)

\[ \frac{1}{2\pi} \int_{-y_0}^{y_0} dt \, e^{i\omega t} \frac{1}{2} J_0 \left( ik \sqrt{y_0^2 - t^2} \right) = \frac{\sin (y_0 \sqrt{\omega^2 - k^2})}{2\pi \sqrt{\omega^2 - k^2}}. \quad \text{(24)} \]

The equality (23) follows from the definition of \( H \) and the following representation for the Bessel function [10]

\[ J_0(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} ds \, e^{iz \sin s}. \]

The equality (24) is just the formula for the inverse Fourier transform in \( t \) of the function

\[ \Phi(y_0, t) := \frac{1}{2} \theta(y_0 - |t|) J_0 \left( ik \sqrt{y_0^2 - t^2} \right). \]

The function \( \Phi \) is the solution of the following Cauchy problem

\[ \frac{\partial^2 \Phi}{\partial y_0^2} - \frac{\partial^2 \Phi}{\partial t^2} - k^2 \Phi = 0, \quad \Phi|_{y_0=0} = 0, \quad \frac{\partial \Phi}{\partial y_0}|_{y_0=0} = \delta(t), \]

where the “time” variable is \( y_0 \). This can be checked directly or deduced from the results of [1] Ch. V]. Hence the inverse Fourier transform \( \tilde{\Phi}(y_0, \omega) \) in \( t \) of \( \Phi(y_0, t) \) satisfies

\[ \frac{\partial^2 \tilde{\Phi}}{\partial y_0^2} + (\omega^2 - k^2) \tilde{\Phi} = 0, \quad \tilde{\Phi}|_{y_0=0} = 0, \quad \frac{\partial \tilde{\Phi}}{\partial y_0}|_{y_0=0} = \frac{1}{2\pi}, \]

which implies (24).

**Proposition 3.** The following inequalities hold true

1. \(|G_{\pm}(k, \omega)| \leq C \), if \( k \geq 0; \)
2. \(|G_{\pm}(k, \omega)| \leq C \), if \(|k| \leq |\omega|; \)

(constant \( C \) is independent of \( k, \omega \).)
Proof. By the definition of \( H \) we have

\[
G_{\pm}(k, \omega) = \frac{1}{2\pi^2} \int_{-y_0}^{y_0} dt \int_0^{\pi/2} ds \, e^{\pm k \sqrt{y_0^2 - t^2} \sin s}.
\]

This leads to the first inequality of the Proposition. Due to (24) and (23) we have

\[
G_{+}(k, \omega) + G_{-}(k, \omega) = \frac{\sin(y_0 \sqrt{\omega^2 - k^2})}{\pi \sqrt{\omega^2 - k^2}}. \tag{25}
\]

The r.h.s. is bounded if \(|k| \leq |\omega|\); together with the first inequality this implies the second inequality of the Proposition.

4 “Nonlocal” version of the formula (7)

In this section we prove the relation

\[
\lim_{h \to 0^+} \frac{1}{4\pi} \int_{-\infty}^{\infty} dx \, v(x, t) \int_{-\infty}^{\infty} \frac{dke}{\pi} \hat{V}(k) \hat{G}(k, \omega) \tilde{\varphi}(k \pm h k^2, t) = \int_{-\infty}^{\infty} dx \, K_k(x, y_0, t) v(x, t), \tag{26}
\]

which differs from (7) in the set of integration (recall that \( v \) was defined by (15)).

To prove the equality (26) we show that the r.h.s. coincides with the r.h.s. of (19). In view of (22) for the integral in \( x \) in the r.h.s. of (26) we have

\[
\sum_{\pm} \frac{1}{4\pi} \int_{-\infty}^{\infty} dx \, v(x, t) \int_{-\infty}^{\infty} dk \, e^{-ikx - h \cdot k^2(\pm \cdot x)} H \left( \pm k \sqrt{y_0^2 - t^2} \right) =
\]

\[
= \sum_{\pm} \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \, e^{-h k^2} H \left( \pm k \sqrt{y_0^2 - t^2} \right) \int_{-\infty}^{\infty} dx \, e^{-ikx \mp h \cdot k^2 \cdot x} v(x, t) =
\]

\[
= \sum_{\pm} \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \, e^{-h k^2} H \left( \pm k \sqrt{y_0^2 - t^2} \right) \hat{\varphi}(k \pm h k^2, t).
\]

Here \( \hat{\varphi}(k, t) \) is Fourier transform of \( v(x, t) \) in \( x \). We can change the order of integration in the second equality, since \( v(\cdot, t) \) is compactly supported. Now for the integral in \( t \) in the r.h.s. of (26) we have

\[
\sum_{\pm} \frac{1}{4\pi} \int_{-y_0}^{y_0} dt \int_{-\infty}^{\infty} dk \, e^{-h k^2} H \left( \pm k \sqrt{y_0^2 - t^2} \right) \hat{\varphi}(k \pm h k^2, t) =
\]

\[
= \sum_{\pm} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-h k^2} \int_{-y_0}^{y_0} dt \, H \left( \pm k \sqrt{y_0^2 - t^2} \right) \hat{\varphi}(k \pm h k^2, t). \tag{27}
\]

The function \( \hat{\varphi}(k, \cdot) \) equals the inverse Fourier transform of \( \hat{\tilde{\varphi}}(k, \cdot) \), which belongs to \( L_2(\mathbb{R}) \) (the latter can be proved analogously to (18)). Hence \( \hat{\varphi}(k, \cdot) \in L_2(\mathbb{R}) \) and for the integral in \( t \) in the r.h.s. of (27) we have

\[
\int_{-y_0}^{y_0} dt \, H \left( \pm k \sqrt{y_0^2 - t^2} \right) \hat{\varphi}(k \pm h k^2, t) = \int_{-\infty}^{\infty} d\omega \, G_{\pm}(k, \omega) \tilde{\varphi}(k \pm h k^2, \omega).
\]
The expression obtained in (27) can be written as follows

\[
\sum_{\pm} \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \ e^{-hck^2} \int_{-\infty}^{\infty} d\omega \ G_{\pm}(k, \omega) \ \tilde{v}(k \pm h k^2, \omega).
\]  

We show that this tends to

\[
\sum_{\pm} \frac{1}{4\pi} \int_{-\infty}^{\infty} dk \ \int_{-\infty}^{\infty} d\omega \ G_{\pm}(k, \omega) \ \tilde{v}(k, \omega)
\]

as \( h \to 0 \), which is equal to (19) in view of (25).

To calculate the limit of (28) we estimate the integral

\[
\int_{-\infty}^{\infty} dk \ \int_{-\infty}^{\infty} d\omega \ e^{-hck^2} |\tilde{v}(k \pm h k^2, \omega) - \tilde{v}(k, \omega)| \cdot |G_{\pm}(k, \omega)|.
\]

Consider the case “–” (the other case can be considered analogously). First we inspect the integral over the set \( 0 \leq k < \infty, \ \omega \in \mathbb{R} \). The function \( G_{-}(k, \omega) \) can be estimated by the constant \( C \) due to the inequality (1) of Proposition 3. So we need to estimate the integral

\[
\int_{0}^{h^{-\gamma}} dk \ \int_{-\infty}^{\infty} d\omega \ e^{-hck^2} |\tilde{v}(k \pm h k^2, \omega) - \tilde{v}(k, \omega)| =
\]

\[
= \int_{0}^{h^{-\gamma}} dk \ \int_{-\infty}^{\infty} d\omega \ e^{-hck^2} |\tilde{v}(k \pm h k^2, \omega) - \tilde{v}(k, \omega)|,
\]

where \( 0 < \gamma < 1/2 \). For the first integral in the r.h.s. we have

\[
\int_{0}^{h^{-\gamma}} dk \ \int_{-\infty}^{\infty} d\omega \ e^{-hck^2} |\tilde{v}(k \pm h k^2, \omega) - \tilde{v}(k, \omega)| \leq
\]

\[
\leq \int_{0}^{h^{-\gamma}} dk \ \int_{-\infty}^{\infty} d\omega \ e^{-hck^2} |\tilde{v}(k \pm h k^2, \omega) - \tilde{v}(k, \omega)| + (1 - e^{-hck^2}) |\tilde{v}(k, \omega)|.
\]

If \( 0 \leq k < h^{-\gamma} \) then \( hck^2 \leq ch^{1-2\gamma} \), so \( 1 - e^{-hck^2} \leq Ch^{1-2\gamma} \). Combining this with (18), we obtain

\[
\int_{0}^{h^{-\gamma}} dk \ \int_{-\infty}^{\infty} d\omega \ (1 - e^{-hck^2}) |\tilde{v}(k, \omega)| \leq Ch^{1-2\gamma} \to 0, \ \ h \to 0.
\]

Next in view of (20) we have

\[
\int_{0}^{h^{-\gamma}} dk \ \int_{-\infty}^{\infty} d\omega \ |\tilde{v}(k \pm h k^2, \omega) - \tilde{v}(k, \omega)| \leq \int_{0}^{h^{-\gamma}} dk \ C \cdot h k^2 \frac{1}{1 + (k - h k^2)^2}.
\]

The inequality (20) is applicable if \( h \leq 1 \), since on the set of integration we have \( h k^2 \leq h^{1-2\gamma} \leq 1 \). Besides, for sufficiently small \( h \) the inequality \( k - h k^2 \geq k/2 \) holds true, hence the integral obtained above can be estimated by

\[
C h^{1-2\gamma} \int_{0}^{h^{-\gamma}} \frac{dk}{1 + k^2} \to 0, \ \ h \to 0.
\]
Next the second integral in the r.h.s. of (30) is majorized by
\[
\int_{h^{-\gamma}}^{h^{-\beta}} dk \int_{-\infty}^{\infty} d\omega \left[ e^{-hck^2} |\tilde{v}(k - hk^2, \omega)| + |\tilde{v}(k, \omega)| \right].
\]
Here the integral of $|\tilde{v}(k, \omega)|$ tends to zero as $h \to 0$ due to (18). The integral of $e^{-hck^2} |\tilde{v}(k - hk^2, \omega)|$ equals the sum of two integrals over the intervals $h^{-\gamma} < k < h^{-\beta}$ and $h^{-\beta} < k < \infty$, where $1/2 < \beta < 1$. In case $h^{-\gamma} < k < h^{-\beta}$ for sufficiently small $h$ we use the inequality $k - hk^2 \geq k/2$ (which follows from $\beta < 1$) and the inequality (18):
\[
\int_{h^{-\gamma}}^{h^{-\beta}} dk \int_{-\infty}^{\infty} d\omega \left| \tilde{v}(k - hk^2, \omega) \right| \leq \int_{h^{-\gamma}}^{h^{-\beta}} dk \frac{C}{1 + k^2} \to 0, \quad h \to 0.
\]
For $h^{-\beta} < k < \infty$ we simply estimate the integral of $\tilde{v}$ in $\omega$ by constant in accordance with (18):
\[
\int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\omega \left| e^{-hck^2} \tilde{v}(k - hk^2, \omega) \right| \leq C \int_{h^{-\beta}}^{\infty} dk e^{-hck^2} = Ch^{-\beta} \int_1^{\infty} ds e^{-h^{1-2\beta}cs^2} \leq Ch^{-\beta} \int_1^{\infty} ds e^{-h^{1-2\beta}cs} = Ch^{-\beta - 1} e^{-h^{1-2\beta}}.
\]
The obtained majorant tends to zero as $h \to 0$, since $\beta > 1/2$.

To estimate the expression (29) it remains to consider the corresponding integral over the set $-\infty < k < 0, \omega \in \mathbb{R}$. In this case the function $G_-(k, \omega)$ is majorized by constant $C$ as well. Indeed, the integrand vanishes if $|\omega| < |k|$, which follows from (16). From the other hand, if $|\omega| > |k|$ the inequality (2) of Proposition 3 holds true. Further
\[
\int_{-\infty}^{0} dk \int_{-\infty}^{\infty} d\omega \left| e^{-hck^2} \tilde{v}(k - hk^2, \omega) - \tilde{v}(k, \omega) \right| \leq C \int_{-\infty}^{\infty} d\omega \left| e^{-hck^2} \tilde{v}(k - hk^2, \omega) - \tilde{v}(k, \omega) \right| + \int_{-\infty}^{h^{-\gamma}} dk \int_{-\infty}^{\infty} d\omega \left( |\tilde{v}(k - hk^2, \omega)| + |\tilde{v}(k, \omega)| \right).
\]
Here $0 < \gamma < 1/2$. The second integral in the r.h.s. tends to zero as $h \to 0$ in view of the inequality $|k - hk^2| > |k|$ and estimate (18). The first integral can be estimated similarly to the first integral in the r.h.s. of (30).

The relation (26) is now proved.

5 Derivation of formula (7)

In view of (26) to prove formula (7) it remains to show that
\[
\lim_{h \to 0+} \int_{-\infty}^{y_0} dt \int_{|x| > D(\sqrt{y_0 - t^2}) + \varepsilon} dx K_h(x, y_0, t) v(x, t) = 0.
\]

(31)
Recall that we suppose \( u_0 \) to be compactly supported. This means that for some \( d \) the inequalities \(-y_0 \leq t \leq y_0, x^2 + y^2 > d^2\) imply that \( u(x, y, t) = 0\). Therefore, if \(-y_0 \leq t \leq y_0, |x| > d\), we have \( v(x, t) = 0\), and thus to estimate the integral in (31) we need to estimate \( K_h(x, y_0, t) \) on the set

\[
D \left( \sqrt{y_0^2 - t^2} \right) + \varepsilon \leq |x| \leq d. \tag{32}
\]

We prove the following inequality

\[
|K_h(x, y_0, t)| \leq C h^{-1/2} e^{-a \varepsilon^2 / h}, \tag{33}
\]

where \( x, t \) satisfy (32), \( 0 < \varepsilon \leq d \) (if \( \varepsilon > d \), then the set (32) is empty), \( h > 0 \), and \( C, a \) are positive constants independent of \( x, t, h, \varepsilon \).

Denote by \( F \) the exponent in the integral in (31). Also put \( z := \sqrt{y_0^2 - t^2}, \sigma := \sin s\). We have

\[
\text{Re} F = \frac{c(-x^2 + \varepsilon^2 \sigma^2) - 2x^2z \sigma}{4h(c^2 + x^2)}.
\]

The function \( \text{Re} F \) is convex in \( \sigma \), so it satisfies the inequality

\[
\text{Re} F \leq \max(\text{Re} F|_{\sigma = 0}, \text{Re} F|_{\sigma = 1})
\]

on the interval \( 0 \leq \sigma \leq 1 \). Since \( |x| \geq \varepsilon \), we have

\[
\text{Re} F|_{\sigma = 0} = \frac{-cx^2}{4h(c^2 + x^2)} \leq \frac{-c\varepsilon^2}{4h(c^2 + \varepsilon^2)} \leq \frac{-c\varepsilon^2}{4h(c^2 + d^2)}.
\]

Next

\[
c\varepsilon^2 - x^2(2z + c) = (2z + c)(D(z)^2 - x^2) \leq -c(x^2 - D(z)^2) < -c\varepsilon(|x| + D(z)) \leq -c\varepsilon^2.
\]

Therefore,

\[
\text{Re} F|_{\sigma = 1} \leq \frac{-c\varepsilon^2}{4h(c^2 + d^2)}.
\]

We proved that if \( 0 \leq \sigma \leq 1 \), then

\[
\text{Re} F \leq -\frac{a\varepsilon^2}{h}, \quad a = \frac{c}{4(c^2 + d^2)}.
\]

This implies (33), and the relation (31) now follows.

The relations (26) and (33) lead to (7).
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