Coexistence of pure and mixed states in nonlinear maps

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Abstract. Coherence and interaction are important concepts in physics. While interaction describes a relation between individual objects such as forces acting between distinguishable particles, coherent objects exist with the sole purpose of describing a single object. For example, each component of a vector provides us with only partial information. The whole picture is revealed only when the components are coherently related to their generating vector. Another example is a singlet of two spin 1/2 particles. The true nature of these two coherent particles is described by a spin-less single particle. Apparently it seems that objects can be either coherent or non-coherent but they cannot be both simultaneously. This is almost true. We show that a system can be described simultaneously as coherent and non-coherent but an observer can distinguish only one concept at a time.

1. Introduction
Although coherence and interaction seem to describe similar concepts, in fact, they are referred to as entirely different concepts. While interaction describes relations between individual objects, coherent objects exist with the sole purpose of describing a single object. This substantial difference is stressed by the inconceivable quantum measurement problem (the hard problem) [1]-[6]: Quantum entities can compose a coherent state, such as the $|+\rangle = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle)$ state in two-level systems. Although the $|1\rangle$, $|2\rangle$-states are only the components of the $|+\rangle$-vector, the essence of the states $|1\rangle$, $|2\rangle$ measurement is the ability to detect them separately as distinguishable quantities. This apparent paradox in the measurement procedure requires a drastic solution reflected through the collapse procedure [1]-[6].

Thus, at first glance it seems that defining a system as both coherent and incoherent is inconceivable unless the brutal collapse procedure is implemented. Nevertheless, in this paper we present a system for which both descriptions are simultaneously valid. For that purpose we define an observer who can interpret reality in various ways.

2. Primary Definitions
Following previous works [7, 8] we define nonlinear maps to describe the system evolution. Such a system can be related to a negative feedback system where the term negative feedback describes an evolving system that is fed back in a manner tending to balance the changes [9]. In our model the system is represented through a nonlinear regular map while this system is in contact with an environment that balanced its evolution. The environment is represented by a chaotic map. We represent the observed system map (s-map) with the iterating equation

$$x_{n+1} = f_R (x_n), \quad \forall n \quad x_n \in [0,1]$$

(1)
where all values of $x$ are within the interval $[0, 1]$ and $R$ is a control parameter that determines the map nature.

We also introduce an environment through a chaotic map [7, 8]

$$\chi_{\nu+1} = \mathcal{F}_R(\chi_\nu), \quad \forall \nu \quad \chi_\nu \in [0, 1],$$

(2)

where now $R$ is the environment map control parameter.

Many nonlinear maps behaves as follows:

For relatively low value of the control parameter, a typical map converges into a single stable point, $x[1]$, calculated as

$$x_{n+1} = x_n \equiv x[1] \Rightarrow x[1] = f^{[1]}_R(x[1])$$

(3)

where the superscript in the square brackets [1] indicates that the map converges into a single value.

In general a higher rank map experiences the Andronov Hopf bifurcation[11]. However, for $m$-splits, it is possible to redefine a map that retrieves a single stable value simply by using an $m$-step iteration using the map

$$x_{n+m} = x_n \equiv x[m] \Rightarrow x[m] = f^{[m]}_R(x[m]) \Rightarrow f^{[m]}_R = f^{[1]}_R \circ f^{[1]}_R \circ \cdots f^{[1]}_R \hspace{0.2cm} m \text{ - times} \hspace{0.2cm} \rightarrow f^{[m/2]}_R \circ f^{[m/2]}_R.$$

(4)

Let us recall that this map of rank $[m]$ possesses an $m$- dimension [7, 8, 10].

3. Batch of Maps Definition as the Representative of a Physical Phenomenon

We define the concept of a batch of maps where a batch defines the system in hand while the participating maps stand for a possible variety of descriptions.

We consider a basic map $f^{[1]}_R$ that establishes a batch of maps containing the basic map and maps of higher ranks. For example a map $f^{[1]}_R$ that converges into two final values establishes the $\{f^{[1]}_R, f^{[2]}_R\}$-batch of maps. Each batch is associated with a rank-number $\{b\}$ (b for batch) defined as the number of the final values reached by the basic map $f^{[1]}_R$.

We associate a batch of rank $\{b\}$ with some physical phenomenon while $\{b\}$ - the batch rank - defines the alternative ways an observer can describe this physical phenomenon. Let us elaborate using the following example. Suppose we have an observer who explores a batch of rank $\{2\}$. He finds two alternative methods to describe the system:

The first order map, $f^{[1]}_R$, describes a $1-D$ system that alternates between two values. The second rank map, $f^{[2]}_R$, leads to a categorization method by cataloging the physical phenomenon into two sections with sets terminating at $x_\alpha$ or $x_\beta$. Increasing a map rank beyond the double rank batch provides us with no further descriptions. Furthermore, the time step enlargement of the second-rank map eliminates some of the early-stage iterations. In this present example, maps with rank greater than 2 (the maps $f^{[4]}_R, f^{[8]}_R, f^{[16]}_R$, ...) behave basically as a map of rank two.

Following this logic, in the general case, we limit the number of maps in a $\{b\}$-rank batch to maps with rank $m$ smaller or equal to the batch rank $b$, that is, the batch $\left\{f^{[1]}_R, f^{[2]}_R, \ldots, f^{[b]}_R\right\}$.

**States-Model Construction.** We define the following state:

$$|n\rangle = (x_{n+1} - x_n) |0\rangle + e^{i\theta_n} \sqrt{1 - (x_{n+1} - x_n)^2} |1\rangle, \quad \theta_n = \chi_{n+1} - \chi_n$$

(5)
An observer who tracks the vector rotation can use the projective operators to describe the same $f_R$-map-iteration.

**Notation Conventions** A state composed of a $(k)$-set (selected out of the $m$-dimensions) with rank $[m]$, embedded in a batch of rank $\{b\}$ appears as

$$ |\{b\}, [m], (k), n\rangle. \quad (6) $$

Adding the environment $\{\beta \to \infty\}$-batch, the notation becomes

$$ |\{b\}, [m], (k), n; \{\beta \to \infty\}, [\mu], (\kappa), \nu\rangle. \quad (7) $$

4. **The Map Density Matrix**

An observer who observes the world is subject to the influence of two types of batches: The system $\{b\}$-batch, as was defined in eq. 1, and the environment $\{\beta\}$-batch $(\beta \to \infty)$ composed by chaotic maps according to eq. 2. All types of phases in these states representation are determined by the same map-phase relations where the chaotic phases $\theta^{[\mu]}_{\nu, (\kappa)}$ are defined as

$$ \sin (\theta^{[\mu]}_{\nu, (\kappa)}) = \left( x^{[\mu]}_{\nu, (\kappa)} - x^{[\mu]}_{\nu, (\kappa)} \right), \; \kappa = 0, 1, 2, \ldots \mu - 1, \; \mu = 1, 2, \ldots, \infty, \quad (8) $$

while in the same manner the regular phases $\phi^{[m]}_{n, (k)}$ are associated with the $x$-map by

$$ \sin (\phi^{[m]}_{n, (k)}) = \left( x^{[m]}_{n, (k)} - x^{[m]}_{n, (k)} \right), \; k = 0, 1, 2, \ldots, m - 1, \; m = 1, 2, \ldots, b \quad (9) $$

where the subscripts $k$ and $\kappa$ count the system and environment-independent sets, respectively. We now consider a specific set $(k)$ picked arbitrarily. We assume that the iterations act on the $|\theta^{(k)}\rangle$ and $|1^{(k)}\rangle$ states as a unitary operator to form the rotated state

$$ |\psi\rangle = |[m, \mu], (k, \kappa), n, \nu\rangle = \sin (\phi^{[m]}_{n, (k)}) |\theta^{(k)}\rangle + \exp \left\{ i \theta^{[\mu]}_{\nu, (\kappa)} \right\} \cos (\phi^{[m]}_{n, (k)}) |1^{(k)}\rangle. \quad (10) $$

An observer who tracks the vector rotation can use the projective operators $P = |\psi\rangle \langle \psi|$ of eq. 10. Selecting some $k$-series and implementing the Birkhoff Von-Neumann ergodic theorem[12] over the time- dependent recursive maps, we assume $\nu$ - the chaotic time interval - to be much shorter than $n$ - the system time interval. Alternatively, we can use the ergodic representation and assume that the chaotic map possesses an enormous degree of freedom compared to the small regular system size. Thus, either by using the ergodic representation or through the time dependent representation we obtain:

$$ \rho_{n, (k)}^{[m]} = \left\langle P^{[m, \mu]}_{n, (k), (\kappa=1)} \right\rangle _{\text{ErgodicAverage}} = \left\langle P^{[m, \mu]}_{n, (\nu=1), (k), (\kappa)} \right\rangle _{\text{ErgodicAverage}} = \begin{pmatrix} \sin^2 (\phi^{[m]}_{n, (k)}) & 0 \\ 0 & \cos^2 (\phi^{[m]}_{n, (k)}) \end{pmatrix}. \quad (11) $$

In the general scenario a multi-dimensional projective operator always possesses the chaotic phase terms in the non-diagonal parts. Thus, after averaging over the chaotic phases we are left only with the density matrix diagonal terms.

5. **The $\{b = 1\}$-batch Converges toward a Single Pure Density Matrix**

A $\{b = 1\}$-batch possesses only the basic map $f^{[1]}_R$ that converges into a single value. According to eq. 9 the corresponding phase $\phi$ vanishes, thereby generating a pure density matrix. We can say that regardless of the random $\theta$-phase a $\{b = 1\}$-batch always corresponds with pure states density matrices. That is $\rho_{n \to \infty}^{[1]} \rightarrow |1\rangle \langle 1|$. (see eqs 11).
6. The \( \{ b = 2 \} \)-batch converges toward Pure and Mixed Density Matrices

A batch of the second rank consists of the maps \( \{ f_R^{[1]}, f_R^{[2]} \} \).

The \( m = 1 \) map embedded in a \( b = 2 \)-Batch

The first map \( f_R^{[1]} \) induces the following density matrices

\[
\rho^{[1]}_{(2), (k), n} = \begin{pmatrix} \sin^2 \left( \phi_n^{[1]}(k) \right) & 0 \\ 0 & \cos^2 \left( \phi_n^{[1]}(k) \right) \end{pmatrix} \quad k = 0, 1 \tag{12}
\]

where \( \sin \left( \phi_n^{[m]}(k) \right) = \left( x_n^{[m]} - x_n^{[m]} \right) \) while \( k = 0 \) and \( k = 1 \) denote the two distinguishable sets. After sufficiently long iterations the trigonometric functions converge to

\[
\lim_{n \to \infty} \sin \left( \phi_n^{[1]}(\alpha) \right) = - \lim_{n \to \infty} \sin \left( \phi_n^{[1]}(\beta) \right) = (x_\alpha - x_\beta). \tag{13}
\]

We then obtain the mixed-state density matrix

\[
\rho^{[1]}_{(2), (k), n} \to \infty = \begin{pmatrix} (x_\alpha - x_\beta)^2 & 0 \\ 0 & 1 - (x_\alpha - x_\beta)^2 \end{pmatrix}. \tag{14}
\]

Thus, a map of rank one embedded in a batch of rank two generates a density matrix of mixed state.

The \( m = 2 \)-map embedded in a \( b = 2 \)-batch

In the the second-rank map representation \( f_R^{[2]} \) the space splits into two parts: The \( \alpha \) part containing all sets terminating at the \( x_\alpha \)-value and the \( \beta \) part, the other sets ending at the \( x_\beta \)-value. Now there are four states divided into two categories: correlated and uncorrelated states. The uncorrelated bases of states are

\[
\begin{align*}
|1_{(\alpha)}\rangle, & |\emptyset_{(\alpha)}\rangle, & |1_{(\beta)}\rangle, & |\emptyset_{(\beta)}\rangle \\
\alpha\text{-Basis} & & \beta\text{-Basis}
\end{align*} \tag{15}
\]

with the two independent density matrices

\[
\rho^{[2]}_{(2), (\alpha), n} = \begin{pmatrix} \sin^2 \left( \phi_n^{[2]}(\alpha) \right) & 0 \\ 0 & \cos^2 \left( \phi_n^{[2]}(\alpha) \right) \end{pmatrix}, \quad \rho^{[2]}_{(2), (\beta), n} = \begin{pmatrix} \sin^2 \left( \phi_n^{[2]}(\beta) \right) & 0 \\ 0 & \cos^2 \left( \phi_n^{[2]}(\beta) \right) \end{pmatrix} \tag{16}
\]

with

\[
\sin \left( \phi_n^{[2]}(\alpha) \right) = \left( x_n^{[2]} - x_n^{[2]} \right), \quad \sin \left( \phi_n^{[2]}(\beta) \right) = \left( x_n^{[2]} - x_n^{[2]} \right). \tag{17}
\]

For \( n \to \infty \), both matrices converge into the double pure-state matrices \( |1_{(\alpha)}\rangle \langle 1_{(\alpha)}| \) and \( |1_{(\beta)}\rangle \langle 1_{(\beta)}| \).

The correlated-states density matrix

Correlated states are the basis of states that span a space obtained from a direct product between the \( \alpha \) and \( \beta \)-spaces

\[
|\emptyset_{(\beta)}\rangle |1_{(\alpha)}\rangle \quad |\emptyset_{(\alpha)}\rangle |1_{(\beta)}\rangle, \quad |1_{(\beta)}\rangle |\emptyset_{(\alpha)}\rangle \quad |1_{(\beta)}\rangle |1_{(\beta)}\rangle, \quad |1_{(\beta)}\rangle |\emptyset_{(\alpha)}\rangle \quad |1_{(\beta)}\rangle |1_{(\alpha)}\rangle, \quad |1_{(\beta)}\rangle |1_{(\beta)}\rangle \tag{18}
\]

The density of states is composed as follows. We associate each \( |0\rangle\)-state with its compatible \( \sin \)term as a coefficient (with the appropriate \( \alpha \) or \( \beta \) phases) while the coefficient of a \( |1\rangle\)-type-state
is of the appropriate \( \cos \) form. As an example observe the term \( \sin^2 \left( \phi_{n,\beta}^{[2]} \right) \left| \emptyset^{(\beta)} \right> \). Thus, assuming only diagonal terms we obtain the correlated-states density operator

\[
\rho_{[2]}^{[2]} = \sin^2 \left( \phi_{n,\beta}^{[2]} \right) \sin^2 \left( \phi_{n,\alpha}^{[2]} \right) \left| \emptyset^{(\beta)} , \emptyset^{(\alpha)} \right> \left< \emptyset^{(\alpha)} , \emptyset^{(\beta)} \right| + \cos^2 \left( \phi_{n,\beta}^{[2]} \right) \sin^2 \left( \phi_{n,\alpha}^{[2]} \right) \left| 1^{(\beta)} , \emptyset^{(\alpha)} \right> \left< \emptyset^{(\alpha)} , 1^{(\beta)} \right| + \\
\sin^2 \left( \phi_{n,\beta}^{[2]} \right) \cos^2 \left( \phi_{n,\alpha}^{[2]} \right) \left| \emptyset^{(\beta)} , 1^{(\alpha)} \right> \left< 1^{(\alpha)} , \emptyset^{(\beta)} \right| + \cos^2 \left( \phi_{n,\beta}^{[2]} \right) \cos^2 \left( \phi_{n,\alpha}^{[2]} \right) \left| 1^{(\beta)} , 1^{(\alpha)} \right> \left< 1^{(\alpha)} , 1^{(\beta)} \right|
\]

Unlike the uncorrelated density matrices defined in eqs. 16 that converged into the two pure-states density matrices \( \left| 1^{(\alpha)} \right> \left< 1^{(\alpha)} \right| \) and \( \left| 1^{(\beta)} \right> \left< 1^{(\beta)} \right| \), now the correlated-states density matrix converges only to the single pure-states density matrix \( \left| 1^{(\alpha)} , 1^{(\beta)} \right> \left< 1^{(\alpha)} , 1^{(\beta)} \right| \).

7. Summary-The General Scenario

Suppose we have a batch of rank \( \{ b \} \). All maps below the batch rank generate only mixed-state density matrices. However, when the batch and a map share the same rank number, we obtain pure states density matrix. Thus, we obtained a system which can be interpreted as a mixed or pure system.

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