Intermittent emission of particles from three coupled condensates in a one-dimensional lattice

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We investigate particle emission, driven by periodically modulating the interaction strength, from three coupled Bose-Einstein condensates in a one-dimensional lattice. Within perturbative analyses, which lead to the regimes of instabilities for different modes, we not only obtain two main frequencies, under which the system can emit a large particle jet, but also find that the emission is distinctly intermittent rather than continuous. The time evolution of the trapped particles exhibits a stair-like decay, and a larger drive induces a more significant intermittency. We further shed light on the dynamics of the stimulating process, and demonstrate that instead of a real suspension, the intermittency represents a build-up stage of the particles. The theoretical framework might be generalized, to the explorations on other multiple-condensate systems with analogous configurations and couplings.

I. INTRODUCTION

Cold atom experiments have enabled precise quantum coherent manipulations of interparticle interaction in many-body systems [1, 11], and ingenious control of versatile unconventional configurations [2, 3], which revealed a number of novel nonequilibrium quantum effects [4, 5]. In recent years, matter-wave jet emission resembling fireworks [6], and the follow-ups [7, 12] have attracted intensive attention. Pairwise interactions were modulated by time-periodic drive in these seminal experiments, which induced exponentially amplified excitations in a Bose-Einstein condensate, and a large number of stimulated particles rapidly escaped from the trap, leading to a burst of jets along the radial directions.

There have been several theoretical works exploring various aspects of the highly nonequilibrium phenomena [11–15], from the dynamics of the stimulating process of the observed pair emission [14, 16] and a unique single-particle emission [17], to the characterizations of high-harmonic generations [12], rapid density oscillations and typical threshold behavior [18]. Configurations with more sites inside the trap and with more than two leads would be particularly interesting, through which one can seed various initial fluctuations, and investigate the correlations between the particle emission and the leads. Accordingly, if one wanted to explore the competitions among diverse modes in the collective particle emission, extended models should be employed. From the theoretical point of view, one can configure synthetic traps, condensates and couplings based upon demands, to study the properties of the resulting particle jets.

Related interesting topics and properties, in other contexts, on the dynamics of more than one condensate have been extensively reported, such as spatio-temporal and low-energy dynamical behaviors in arrays of Bose-Einstein condensates [19–23], resonances, phase fluctuation, critical behavior and time-delayed polaritons in two coupled condensates [24, 33], and interference effects, self-trapping, chaotic behavior, quantum transition and parity effects in three coupled condensates [34–41]. Geometries with more sites and leads would somewhat make the analysis more complicated. In the present work, for simplicity, we focus on three coupled condensates in a one-dimensional (1D) lattice, and extensively explore the particle emission under periodic drive in a transparent way, which can be generalized to multiple-condensate systems sharing similar configurations and couplings. Unlike the commonly observed and widely studied continuous emission, we find the distinctly intermittent emission process, and the trapped particles exhibit a stair-like decay. In particular, we revisit the previous two-site model [18], to further clarify the intermittency.

The rest of the paper is structured as follows. In Sec. II we introduce our model and the relevant equations of motion. In Sec. III we outline the perturbative analyses with respect to different modes, and discuss the regimes of instabilities. In Sec. IV we parametrically drive the system and present the numerical results. A summary is given in Sec. V.

II. THEORETICAL MODEL

As depicted in Fig. 1 we introduce a 1D infinite lattice, where three coupled condensates labeled $a$, $b$ and $c$ are confined in a local deep trap of depth $V$. The coupling strength $J_c$ enables particles to hop back and forth among condensates, while $J_h$ and $J_l$ quantify the hopping from the trap to the leads, and the coupling between nearest-neighbour sites in each lead, respectively. We assume that only interactions between atoms which sit on the central sites are included, and excited atoms with sufficient energy can move off to infinity along the
leads, with lattice sites labeled by non-zero numbers 1, 2, ..., \infty. Such a configuration of symmetric geometry, with inhomogeneous lattice, trap and localized interactions, could be literally implemented in an experiment involving optical lattices and microtraps [2] [12] [33].

![FIG. 1. Schematic of the infinite lattice. The dashed box contains three locally trapped condensates \(a, b\) and \(c\), and the sites on the leads are labeled by non-zero integers.]

This model can be described by the Hamiltonian

\[
\hat{H} = V \left( \hat{a}_0^\dagger \hat{a}_0 + \hat{b}_0^\dagger \hat{b}_0 + \hat{c}_0^\dagger \hat{c}_0 \right) + \frac{1}{2} \left[ U + g(t) \right] \left( \hat{a}_0^\dagger \hat{a}_0 \hat{b}_0^\dagger \hat{b}_0 + \hat{c}_0^\dagger \hat{c}_0 \right) - J_c \left( \hat{a}_0^\dagger \hat{c}_0 + \hat{c}_0^\dagger \hat{a}_0 + \hat{b}_0^\dagger \hat{b}_0 \right) - J_h \left( \hat{a}_0^\dagger \hat{a}_0 + \hat{b}_0^\dagger \hat{b}_0 + \hat{c}_0^\dagger \hat{c}_0 \right) - J_l \sum_{j=1}^\infty \left( \hat{a}_j^\dagger \hat{a}_{j+1} + \hat{b}_j^\dagger \hat{b}_{j+1} + \hat{b}_j^\dagger \hat{a}_{j+1} \right),
\]

where \(\hat{a}_j^\dagger\) and \(\hat{b}_j^\dagger\) are creation (annihilation) operators on the \(j\)th site to the left and right, and \(\hat{a}_0, \hat{b}_0\) and \(\hat{c}_0\) represent the central sites. The term \(U + g(t)\) characterizes the time-dependent pairwise interactions, where the on-site interaction \(U\) is constant, and \(g(t) = g \sin(\omega t)\theta(t)\) is the periodic drive, with \(g\) the drive strength, \(\omega\) the drive frequency, and \(\theta(t)\) the step function.

We can thus write down the expectation value of the Heisenberg equations of motion for the central sites at \(j = 0\) (\(h = 1\) throughout this paper)

\[
i \partial_t \hat{a}_0 &= \langle [\hat{a}_0, H] \rangle = V \hat{a}_0 + [U + g \sin(\omega t)] \hat{a}_0^2 - J_c \hat{c}_0 - J_h \hat{a}_1 \quad (2)
i \partial_t \hat{b}_0 &= \langle [\hat{b}_0, H] \rangle = V \hat{b}_0 + [U + g \sin(\omega t)] \hat{b}_0^2 - J_c \hat{c}_0 - J_h \hat{b}_1 \quad (3)
i \partial_t \hat{c}_0 &= \langle [\hat{c}_0, H] \rangle = V \hat{c}_0 + [U + g \sin(\omega t)] \hat{c}_0^2 - J_c \hat{a}_0 - J_h \hat{b}_0 \quad (4)
\]

and for the sites where \(j \geq 1\),

\[
i \partial_t \hat{a}_j &= -J_h \hat{a}_j - J_c \hat{a}_{j+1}, \quad (5)
i \partial_t \hat{b}_j &= -J_l \left( \hat{a}_{j+1} + \hat{b}_{j-1} \right), \quad (6)
i \partial_t \hat{b}_j &= -J_l \left( \hat{b}_{j+1} + \hat{b}_{j-1} \right). \quad (7)
\]

In equilibrium we begin from the ansatz \(\hat{a}_0 = \alpha e^{-\nu t}\), \(\hat{b}_0 = \beta e^{-\nu t}\), \(\hat{c}_0 = \gamma e^{-\nu t}\), where \(\alpha\), \(\beta\) and \(\gamma\) are constant. Assuming that \(\alpha_1 = \alpha e^{-\nu t} e^{-\kappa_1}\) and \(\alpha_2 = \alpha e^{-\nu t} e^{-\kappa_1 - \kappa}\), Eq. (6) is a set of linear equations, i.e., \(\alpha_j/\alpha_{j-1} = e^{-\kappa}\), thus \(\cosh \kappa = \nu/(-2J_l)\), and for \(j \geq 1\) we have explicitly \(\alpha_j = \alpha e^{-\nu t} e^{-\kappa(j-1)}\) with \(\kappa_1 = -\ln((-J_l/\nu))\). Analogous equations for \(b_j\geq 1\) are straightforward, while we only have \(c_j=0\).

For simplicity, we take the limit of \(U = 0\) [13], which leads to the nonlinear integrodifferential equations,

\[
i \partial_t \hat{a}_0 &= V \hat{a}_0 + g \sin(\omega t) \hat{a}_0^2 - J_c \hat{c}_0 + \int_0^\infty G_{11}(t-\tau) a_0(\tau) d\tau, \quad (9)
i \partial_t \hat{b}_0 &= V \hat{b}_0 + g \sin(\omega t) \hat{b}_0^2 - J_c \hat{c}_0 + \int_0^\infty G_{11}(t-\tau) b_0(\tau) d\tau, \quad (10)
i \partial_t \hat{c}_0 &= V \hat{c}_0 + g \sin(\omega t) \hat{c}_0^2 - J_c \hat{a}_0 - \hat{b}_0, \quad (11)
\]

where \(G_{11}(t)\) is the Green’s function [17]

\[
G_{11}(t) = i \nu^2 J_j(2J_j t) / (2J_j t) \theta(t), \quad (12)
\]

with \(J_n(z)\) the Bessel function of the first kind. We will work at the perturbative level, to largely simplify the analysis. Note, that such a treatment cannot well capture the nonlinear dynamical behaviors with large drive strength \(g\) and large coupling strength \(J_h\).

### III. Perturbative Solutions

We are now in position to perturbatively solve Eqs. (9) through (11). In the limit where both \(g\) and \(J_h\) are small, it is pretty straightforward to analytically analyze this model. We begin by finding the solutions with \(g = 0\). With the ansatz in equilibrium, the equations become

\[
\begin{pmatrix}
\nu - V - J_c^2 G_{11}(\nu) & 0 & \nu - V - J_c^2 G_{11}(\nu) \\
0 & -J_c & 0 \\
\nu - V & J_c & \nu - V
\end{pmatrix}
\begin{pmatrix}
\alpha \\
\beta \\
\gamma
\end{pmatrix} = 0.
\]

we directly get the “M” mode with \(\gamma_M = -\sqrt{2\alpha}\) and \(\nu_M^{(0)} = V + \sqrt{2J_c}\), as well as the “P” mode with \(\gamma_P = \sqrt{2\alpha}\)

It is obvious that the matrix equation has three eigen-solutions, i.e., there are three modes in the system. From the symmetric case of \(\alpha = \beta\), to the zeroth order of \(J_h\)
and \( \nu_p^{(0)} = V - \sqrt{2}J_c \). To the second order of \( J_h \), simply substituting related \( \gamma \) and \( \nu \) into the equation leads to

\[
\nu_M^{(2)} = V + \sqrt{2}J_c + \frac{J_h^2}{2J_c^2} \left( V + \sqrt{2}J_c - i\sqrt{4J_c^2 - (V + \sqrt{2}J_c)^2} \right) \tag{14}
\]

\[
\nu_P^{(2)} = V - \sqrt{2}J_c + \frac{J_h^2}{2J_c^2} \left( V - \sqrt{2}J_c - i\sqrt{4J_c^2 - (V - \sqrt{2}J_c)^2} \right) \tag{15}
\]

As for the antisymmetric case of \( \alpha = -\beta \), similarly we can obtain the “Z” mode with \( \nu_Z^{(0)} = V \), and

\[
\nu_Z^{(2)} = V + \frac{J_h^2}{2J_c^2} (V - i\sqrt{4J_c^2 - V^2}). \tag{16}
\]

To observe significant particle emission, we need to be in the regime where the “P” mode and the “Z” mode are stable but the “M” mode is damped, i.e., \(|V - \sqrt{2}J_c| > 2J_c\), \( |V| > 2J_c \) and \(|V + \sqrt{2}J_c| < 2J_c\). Specializing to the case that \( V = -|V| < 0 \), we reach the constrains of the allowed values for the trapping potential as \(-\sqrt{2}J_c < V < -2J_c\), under which the particles in the “M” mode can be excited and decay into jets.

### IV. NUMERICAL RESULTS

Pair atoms excited out of the trap share half of the drive energy quanta, and move along the leads in different directions, leading to particle jets. In the following we will parametrically excite the system by modulating the interaction strength, and numerically solving Eqs. (9) through (11). We assume that for \( t < 0 \) the system is in equilibrium, while the perturbation is turned on at time \( t = 0 \), taking \( g(t) = g \sin(\omega t) \). Without any loss of generality we seed the modes by making \( \alpha \approx \beta = 1 \), with a slight difference in the calculations, and from the related “M” mode we take \( \gamma = \sqrt{2} \), i.e., the particles are initially trapped in the most stable “P” mode, before the system meets the conditions of collective emission under weak drives and proper frequencies. Though we preserve the scale of energy unit \( J_c \) in some discussions as appropriate, in the numerics it is fixed as \( J_c = 1 \).

We first compare the analytical and numerical regimes for exciting the system, as shown in Fig. 2. For relatively small \( J_h \), the results are in well agreement for all the three modes. When \( J_h \) is larger \( (J_h > 0.5) \), the deviations grow with the increasing \( J_h \), under which circumstance the perturbative analysis may be inapplicable. For accuracy, we will mainly take \( J_h = 0.1 \) hereafter.

Fig. 3 shows the short-time evolution of the trapped particles \( N_{\text{tot}} = |c_0|^2 + |h_0|^2 + |c_0|^2 \) under different drive frequencies \( \omega \), when the drive strength \( g \) and the coupling strength \( J_h \) are both small. As can be plainly seen, the system keeps fairly stable under generic frequencies in the very beginning for a short period of time, until the case with frequency \( \omega = 2\sqrt{2}J_c \) firstly decays significantly at about \( t = 25 \). For the case with \( \omega = 4\sqrt{2}J_c \), at about \( t = 90 \) the system also starts to decay rapidly, but emit a somewhat less portion of particles than that of \( \omega = 2\sqrt{2}J_c \). They can thus be termed as the “main frequencies”.

Color bars denote the average rate at which the particles are emitted from the trap. Energies are in units of \( J_c \), and times are in units of \( \hbar/J_c \).

FIG. 2. (color online) Comparisons of the analytical (Dashed lines) and numerical regimes for the three modes. The analytical results are coming from Eqs. (14), (15) and (16), and the numerical results are obtained by solving Eqs. (9) through (11). Here, the drive strength is \( g = 0.1 \), the simulation time is \( t = 25 \), and the coupling strengths are \( J_h = 1 \) and \( J_c = 1 \). Color bars denote the average rate at which the particles are emitted from the trap. Energies are in units of \( J_c \), and times are in units of \( \hbar/J_c \).

FIG. 3. (color online) Total particle number as a function of time under different drive frequencies. Here, the trapping potential is \( V = -3 \), the drive strength is \( g = 0.1 \), and the coupling strengths are \( J_h = 0.1 \) and \( J_c = 1 \). Energies are in units of \( J_c \), and times are in units of \( \hbar/J_c \).
account, which means that by choosing proper trapping potential the system is seeded at the lowest “P” mode of \( \nu_P = V - \sqrt{2} J_c \), while under typical drive frequency \( \omega = 2[(V + \sqrt{2} J_c) - (V - \sqrt{2} J_c)] = 4\sqrt{2} J_c \), it grows exponentially to “M” mode of \( \nu_M = V + \sqrt{2} J_c \), before decaying into significant jets.

**FIG. 4.** (color online) Density distribution of particles on the \( j \)th site of each site as a function of time. Here, the trapping potential is \( V = -3 \), the coupling strengths are \( J_h = 0.1 \) and \( J_l = 1 \), and the drive frequency is \( \omega = 4\sqrt{2} J_c \). Energies are in units of \( J_c \), and times are in units of \( h/J_c \).

If we comprehensively consider the long-time dependence of the particles on every site of each lead, the density distributions will be straightforward, as demonstrated in Fig. 4. For a relatively small drive \( (g = 0.05) \), only a very small amount of particles are excited in the beginning, and most of the particles are still trapped. A continuous emission occurs at times \( 200 < t < 350 \), with a large number of particles ejecting into the leads of the left and right. However, the emission seems to suddenly suspend with a “pause” at about \( t = 400 \), and lasts until \( t = 700 \) when the emission restarts, with a comparatively smaller decay rate. As for \( g = 0.1 \), the response of the system is analogous to the case of \( g = 0.05 \), while the emission emerges earlier, and it exhibits a second pause between \( t = 200 \) and \( t = 300 \), a third pause between \( t = 400 \) and \( t = 600 \), and a fourth pause at time \( t > 750 \). There are more pauses and more complicated situations for drives \( g = 0.2 \) and \( g = 0.3 \).

For clarity, we present in Fig. 5 more quantitatively the intermittency of the emission. One can see that the decays of the trapped particles are stair-like. For drive \( g = 0.05 \) the total particle number \( N_{tot} \) changes only a bit at \( t < 150 \), while it decays significantly afterwards, and a “platform” appears between \( t = 350 \) and \( t = 550 \), during which \( N_{tot} \) remains almost unchanged, indicating that particle emission is retardary. Subsequently the system reemits particles with a much smaller rate. With respect to \( g = 0.1 \), the \( N_{tot} \) also remain nearly unchanged in the beginning \( (t < 100) \), until it decays rapidly to the second, the third and the fourth platform, respectively. Its third platform has a right shift, with less remaining particles, compared with that of \( g = 0.05 \) (the comparison of the insets). When larger drives are exerted, i.e., \( g = 0.2 \) and \( g = 0.3 \), there would be more and more platforms, where the times of appearance are staggered. Note that the duration of first platform of \( g = 0.2 \) is much shorter than that of \( g = 0.05 \) and \( g = 0.1 \), while the first one of \( g = 0.3 \) is the shortest among these drives, indicating an earlier emission under a larger drive. In the case of a single drive, e.g. \( g = 0.3 \), the latter platforms will last for longer times than the former ones.

**FIG. 6.** (color online) The sketch of the emission process.

The intermittent phenomena manifested might be tentatively speculated as follows: Since there are three different modes, i.e., the system has three energy levels,
when the proper trapping potential is chosen, trapped particles are restricted in the ground state corresponding to the lowest “P” mode at \( t < 0 \). Once weak drive with related frequency is exerted, particles are not directly excited out, but that promoted to the dominant “M” mode, while building up in the corresponding energy level. Therefore, we see a short platform in the first certain period of time, and very few particles escape from the trap. To some extent, particles in this higher energy level start to emit instantly, until that the excited particles cannot adequately maintain the dramatic emission with an emergent “pause”, corresponding to the platforms in Fig. 5. It is clear that the build-up and the emission are simultaneous, and instead of a real suspension of the excitation, the platform mainly corresponds to another build-up stage. However, the emission is explicitly transient, which is much faster than the progressive build-up process. When insufficient particles are left in the higher energy level, the emission retards, i.e., the platform appears. With the decrease of the particles in the ground state, subsequent build-up processes will probably take longer times, leading to a longer duration of the latter platforms than the former ones. In addition, the accumulation and ejection of particles in the higher energy level would be much quicker under larger drives, and hence multiple build-up processes and reemissions appear. The sketch is shown in Fig. 6.

![FIG. 7. (color online) Time evolution of the particle imbalance \(|a_0|^2 - |b_0|^2| \) with respect to typical platforms. Upper panel: The first platforms. Lower panel: The typical platforms correspond to the insets in Fig. 5. Here, the trapping potential is \( V = -3 \), the coupling strengths are \( J_h = 0.1 \) and \( J_l = 1 \), and the drive frequency is \( \omega = 4\sqrt{2}J_c \). Energies are in units of \( J_c \), and times are in units of \( h/J_c \).](image)

To verify the above hypotheses, we take drive strengths \( g = 0.05 \) and \( g = 0.1 \) as examples in Figs. 7 and 8, which illustrate the particle imbalance \(|a_0|^2 - |b_0|^2| \) with respect to certain platforms, and the comparative decay rates, respectively. For their first platforms, the imbalance of drive \( g = 0.05 \) grows \((0 < t < 175)\) before decreases with time, while the case of \( g = 0.1 \) undergoes a shorter period \((0 < t < 90)\), which means that the response of the system is faster under a larger drive, as expected, such that more particles are excited to the higher energy level and meet the condition of emission earlier. As for the typical platforms corresponding to the insets in Fig. 5, the imbalances also increase at first and then decrease with time, thus within the periods particles are continuously promoted from the ground state to the higher energy level rather than a real pause, until the system restarts the emission afterwards. For a single drive, since a number of particles have already emitted out of the trap, the latter platforms (lower panel) bear longer than the former ones (upper panel), i.e., it takes a longer time for the system to be in the regime of reemission. The system, at this stage, mainly exhibits property of build-up, corresponding to multiple platforms. Concerning the decay rate in Fig. 8 both cases shortly reach their maximum value before decreasing to almost zero, and then increase again to a smaller value. Within a certain period of time, the case of \( g = 0.1 \) has a larger decay rate and ejects all particles earlier than that of \( g = 0.05 \), but the second maxima of \( g = 0.1 \) is somewhat less than the maximum of \( g = 0.05 \).

![FIG. 8. Decay rate vs. time for the drive strengths of \( g = 0.05 \) and \( g = 0.1 \), respectively. The trapping potential is \( V = -3 \), the coupling strengths are \( J_h = 0.1 \) and \( J_l = 1 \), and the drive frequency is \( \omega = 4\sqrt{2}J_c \). Energies are in units of \( J_c \), and times are in units of \( h/J_c \).](image)

We further calculate the relation between the drive strengths and the number of platforms. Since large \( g \) may induce a series of higher-order behaviors, and make the perturbative analysis to be invalid, we plainly restrict ourselves to \( g \leq 0.5 \), and obtain an stair-like increase in Fig. 9. As the number of platforms cannot be non-integers, i.e., the vertical coordinates changes discontinuously while varying the drive strengths continuously, red dashed lines are used to connect the numerical scatterers. The two insets illustrate the details of the inflection points within \( 0 < g < 0.1 \). The first jump emerges between 0.02 and 0.021, and their corresponding number of platforms are 1 and 2, respectively, while the second one appears between 0.054 and 0.055 with corresponding
numbers of 2 and 3. Moreover, numbers for the intervals of two inflection points should also be integers. The drive-step here is chosen to be 0.001, we are incapable of capturing particular relations with smaller steps. The results, however, sufficiently demonstrate the dependence of the number of platforms on the drive strengths, and hence illustrate the intermittency. If suitable driving conditions and a long enough simulation time were exerted, the case under the other main frequency would exhibit similar intermittent decays.

V. SUMMARY AND OUTLOOK

We have considered a 1D infinite lattice model, which contains three coupled condensates, to investigate the collective particle emission and the competition between different modes. We modulate the interaction strength periodically, and use perturbative and numerical calculations to analyze the system.

In our model, by perturbatively analyzing different modes of the system, which leads to the conditions of being stable and stimulated, and choosing proper trapping potential we are able to obtain two main frequencies, under which the system emits plenty of particles, and find that the emission is intermittent. The trapped particles exhibits a stair-like decay, where a larger drive induces a more significant intermittency. The intermittency can be thought of as the build-up stage of the particles. When weak drive with related frequency is exerted, particles in the ground state are excited to the corresponding higher energy level, and subsequently accumulate. To some extent the particles emit out rapidly, leading to pair jets. The emission and the build-up are simultaneous, while it takes longer for the progressive build-up than that of the transient emission. When the excited particles cannot maintain the emission, and the rate slows down, resembling a pause in the decay of total trapped particles. At this moment, particles in the ground state are still continuously promoted to the higher energy levels. Under a larger drive, the particles in the higher energy level would meet the conditions of emission earlier, inducing multiple processes of build-up and reemission.

It is interesting to anticipate, that for other infinite lattice models with similar configurations and couplings,
there will also be intermittent emissions, which would be useful for further researches on multiple-condensate systems. Since the couplings of the particles and the condensates are faithfully complicated, simplified models can only capture the main characteristics, and more specific investigations are in need.

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