Network Parameter Learning Using Nonlinear Transforms, Local Representation Goals and Local Propagation Constraints

Dimche Kostadinov 1 Behroz Razeghi 1 Slava Voloshynovskiy 1

Abstract

In this paper, we introduce a novel concept for learning of the parameters in a neural network. Our idea is grounded on modeling a learning problem that addresses a trade-off between (i) satisfying local objectives at each node and (ii) achieving desired data propagation through the network under (iii) local propagation constraints. We consider two types of nonlinear transforms which describe the network representations. One of the nonlinear transforms serves as activation function. The other one enables a locally adjusted, deviation corrective components to be included in the update of the network weights in order to enable attaining target specific representations at the last network node. Our learning principle not only provides insight into the understanding and the interpretation of the learning dynamics, but it offers theoretical guarantees over decoupled and parallel parameter estimation strategy that enables learning in synchronous and asynchronous mode. Numerical experiments validate the potential of our approach on image recognition task. The preliminary results show advantages in comparison to the state-of-the-art methods, w.r.t. the learning time and the network size while having competitive recognition accuracy.

1. Introduction

In the recent years, the multi-layer neural networks (NN) have had significant progress and advances, where impressive results were demonstrated on variety of tasks across many fields Schmidhuber (2014). Addressing estimation/learning of task-relevant, useful and information preserving representation, the main idea behind the NN learning methods lies in the concept of representing the input through increasingly more abstract layers of feature representations. Usually, to learn the output of the multi-layer NN representation, a target is defined by only one loss (cost) function, which most often is specified in a supervised manner, and is set for the representation at the last node in the network. In the most basic case, the problem related to estimation of the parameters in a feed-forward NN can be expressed in the following form:

$$\{W, \theta\} = \min_{W, \theta} l_f(Y_0, W, \theta) = \min_{W, \theta} \sum_{l=1}^{L} l_p(W_l) + \sum_{c,k}^{C,K} \sum_{l=1}^{L} f(W_{L-1} f(W_{L-2} f(... f(W_1 y_{0,{c,k}}) ...)), \theta), \tag{1}$$

where $l_f(., .)$ is a parametric cost function w.r.t. to a certain task objective, i.e., goal, with parameter $\theta \in \mathbb{R}^{M_l \times S}$, $W = \{W_1, ..., W_{L-1}\}$. $W_l \in \mathbb{R}^{M_l \times M_{l+1}}$ is the matrix of weights that connects the nodes (layers) at levels $l$ and $l+1$, $f(., .)$ is element-wise nonlinear activation function (examples include sigmoid, tanh, hyperbolic, exponential, ReLU, etc.), and $h_{l,\{c,k\}} \in \mathbb{R}^{M_l}$ is the network representation at node (layer) level $l$ for the $k$-th input data $y_{0,\{c,k\}} \in \mathbb{R}^{M_l}$ coming from class $c$, and $Y_0 = \{y_{0,\{1,1\}}, y_{0,\{1,2\}}, ..., y_{0,\{c,k\}}, \} \in \{1, ..., C\}, \{1, ..., L\}$.

The cost function in (1) defines a minimization objective that usually is not convex. In order to estimate the parameters $W$ and $\theta$, the most commonly used learning strategy boils down to iterative execution of two steps. In the first step, the data $y_{0,\{c,k\}}$ is forward propagated through the network and all $h_{l,\{c,k\}}$ are estimated recursively as:

$$h_{l,\{c,k\}} = f(W_{l-1} h_{l-1,\{c,k\}}), \forall l, c, k. \tag{2}$$

The second step relies on back-propagation Plaut et al. (1986), Lecun (1988) and Schmidhuber (2014) with a gradient-based algorithm LeCun et al. (1998), Bengio (2012) in order to update $W_l$ and $\theta$ and minimize the non-convex objective in (1). At iteration $t$, starting from the last node at level $l = L$ and using the gradient and/or second order derivative information of the objective w.r.t. the weight $W^t_l$,
the parameters $W_l^t$ are updated sequentially by back propagating the loss through the NN. The common update rule has the following form:

$$W_{l+1}^t = W_l^t - \alpha \frac{\partial l(f(Y_0, W, \theta))}{\partial W_l^t} + \beta \mathcal{V}\left(\frac{\partial l(f(Y_0, W, \theta))}{\partial W_l^t}, W_{l-1}^t, W_l^t\right),$$

where $\mathcal{V}(\partial l(f(Y_0, W, \theta))/\partial W_l^t, W_l^t, W_{l-1}^t)$ represent the second order derivative of the objective w.r.t. $W_l$ or its approximation as proposed in Bottou (2012), Shamir and Zhang (2013), Srivastava et al. (2014) and Kingma and Ba (2014).

One of the most crucial issues in the above approach is the second step. Since in the back-propagation a gradient based algorithm is used, the problem of vanishing gradient Hochreiter (1998) or the exploding gradient Pascanu et al. (2012) might lead to a non-desirable local minimum (or saddle point). On the other hand, the dependencies from the subsequent propagation do not allow parallel parameter learning per node, while an additional challenge is the interpretation of the learning dynamics during training.

In the last decade, many works Bottou (2012), Shamir and Zhang (2013), Srivastava et al. (2014), Kingma and Ba (2014), Loshchilov and Hutter (2016), Ruder (2016) Gabriel (2017), Zhu et al. (2017) have addressed issues related with the gradient based weight update. Parallel parameters updates were addressed by Jaderberg et al. (2016) and Czarnecki et al. (2017) and the methods proposed by Lee et al. (2014), Balduzzi et al. (2015), Taylor et al. (2016) and No kland (2016).

While the aforementioned manuscripts provide means to surpass sequential updates they still fall within the realm of the concept that uses "propagated information" about the deviations from only one goal (target) at the last network node. Other alternatives that allow posing local goals on the network representation while enabling parallel update on the network weighs by including a local propagation constraints were not addressed. In this line, a network learning principle that takes into account local correction component that is based on the deviations from a local goals on the representations at a given network node and its closely connected nodes was not explored.

1.1. Learning Model Outline

In the usual problem modeling by (1), only one objective function is defined for the representations on the last node in the network and one predefined activation function is used.

In this paper, in order to allow a local decoupling per the network nodes that enables parallel update of the parameters, as well as provides a possibility to interpret and explain the learning dynamics, we describe a novel learning concept. In our problem modeling, we introduce (i) two types of nonlinear transforms per network node (ii) a local objective at each node related to the corresponding local representation goal and (iii) a local propagation constraints.

1.1.1. Sparsifying Nonlinear Transform as Activation Function

In the most simple case, analogous to the commonly used description by an activation function (2), we use a sparsifying nonlinear transforms (sNTs) Rubinstein and Elad (2014) and Ravishankar and Bresler (2014). We denote the sNT representation at node level $l$ defined w.r.t. $W_l$ or it approximation as proposed in Bottou (2012), Shamir and Zhang (2013), Srivastava et al. (2014) and Kingma and Ba (2014).

By our learning principle, we introduce local objectives (local goals) per all representations that describe the desired objective, but, the difference is that we define the functional mapping as a solution to an optimization problem, where its role is to transform a given representation into a representation with specific properties, e.g., discrimination, information preserving, local propagation constraints preserving, sparsity, compactness, robustness etc.

1.1.2. Local Representation Goals

When we propagate data forward through the network, the sNT representations might deviate from their local objectives, even if we have only one objective (e.g. at the last node in the network). Therefore, the main idea behind our learning approach is to introduce a local propagation constraints in order to compensate and correct this deviation, but, in a localized manner using the sNT representations, a corrective sparsifying nonlinear transform (c-sNT) representations and representations that exactly satisfy the local goal from the current node and the closely connected nodes.

1.1.3. Local Model for Deviation Correction

Assume that a correction vector $\nu_{l\{c,k\}} \in \mathbb{R}^{M_l}$ and a threshold parameter $\lambda_{l,1} \in \mathbb{R}_+$ are given. Denote $b_{l\{c,k\}} = \nu_{l\{c,k\}} - \lambda_{l,1}$, then similarly to the sNT, we define the c-sNT representation $y_{l\{c,k\}}$ at level $l$ as:

$$y_{l\{c,k\}} = \text{sign}(b_{l\{c,k\}}) \odot \max(|b_{l\{c,k\}}| - \lambda_{l,1}, 0).$$
We also refer to $\nu(iii)$ We provide theoretical analysis and empirically validate

1.2. Contributions

In the following, we summarize our contributions.

(i) We introduce a learning problem formulation, which to the best of our knowledge is first of this kind that:

(a) explicitly addresses a trade-off between (i) satisfying local objectives at each node related to the corresponding local representation goal and (ii) achieving desired data propagation through the network that enables attaining a targeted representations at the last node in the network under (iii) local propagation constraints

(b) offers the possibility of posing a wide class of arbitrary local goals and propagation constraints while enabling efficient estimation of the sNT, c-sNT and the network weights

(c) provides interpretation of the local learning dynamics by connecting it to a local diffusion model Kittel and Kroemer (1980) or change of the local flow.

(ii) We propose a novel learning strategies that can operate in synchronous or asynchronous mode, which we implement by an efficient algorithm with parallel execution that iterates between two stages:

1) estimation of the sNT representations and the exact goal satisfying representations and

2) estimation of the c-sNT representations and the actual network weights.

(iii) We provide theoretical analysis and empirically validate the potential of our approach. Our results demonstrate that the proposed learning principle allows targeted representations to be attained w.r.t. a goal set only at one node located anywhere in the NN.

1.3. Notations

A variable at node level $l$ has a subscript $e_l$. Scalars, vectors and matrices are denoted by usual, bold lower and bold upper case symbols as $x_l$, $\mathbf{x}_l$ and $\mathbf{X}_l$. A set of data samples from $C$ classes is denoted as $\mathbf{Y}_l = [\mathbf{Y}_{l,1},...,\mathbf{Y}_{l,C}] \in \mathbb{R}^{M_l \times CK}$. Every class $c \in \{1,...,C\}$ has $K$ samples, $\mathbf{Y}_{l,c} = [\mathbf{y}_{l,c,1},...,\mathbf{y}_{l,c,K}] \in \mathbb{R}^{M_l \times K}$. We denote the $k$-th representation from class $c$ at level $l$ as $\mathbf{y}_{l,c,k} \in \mathbb{R}^{M_l}$, $\forall c \in \{1,...,C\}$, $\forall k \in \{1,...,K\}, \forall l \in \{1,...,L\}$. The $\ell_p$-norm, nuclear norm, matrix trace and Hadamard product are denoted as $||.||_p$, $||.||_*$, $Tr()$ and $\odot$, respectively. The first order derivative of a function $C(\mathbf{Y}_l)$ w.r.t. $\mathbf{Y}_l$ is denoted as $\frac{\partial C(\mathbf{Y}_l)}{\partial \mathbf{Y}_l}$. We denote $|\mathbf{y}_{l,c,k}|$ as the vector having as elements the absolute values of the corresponding elements in $\mathbf{y}_{l,c,k}$.

2. The Learning Problem

In this section, we present our problem formulation, explain the local goal and the local propagation constraint and unveil our learning target.

We take into account one extended version of feed-forward NN. At each network node, our learning concept considers c-sNT representations $\mathbf{Y}_{l}$, sNT representations $\mathbf{U}_{l}$, representations $\mathbf{G}_{l} \in \mathbb{R}^{M_l \times CK}$ that exactly satisfy the local goal, local objectives related to the corresponding desired representations and a local propagation constraints. In the most general form, we introduce our problem formulation in the following form:

$$\hat{\Omega} = \min_{\Omega} \frac{1}{L} \sum_{l=1}^{L} \left( R_3(l) + R_2(l) + A(l) + R_4(l) + R_1(l) \right),$$

subject to $U(\mathbf{G}_l) = 0, \forall l$.

where $\Omega = \{A_0, ..., A_{L-1}, U_1, ..., U_L, Y_1, ..., Y_L, G_1, ..., G_L, B_0, ..., B_{L-1}\}$ are the network parameters used during learning. We denote $A_l$ as a forward weight, whereas $B_l \in \mathbb{R}^{M_{l+1} \times M_l}$ as a backward weight. Note that the actual network weights used during testing are $\{A_0, ..., A_{L-1}\}$ and the resulting network representation from consecutively using our sNT$^1$ are $\{U_1, ..., U_L\}$. In the following subsections, we define and explain each of the components in our problem formulation.

$^1$The sNTs are one type of nonlinear activation function (2).
2.1. Nonlinear Transform Errors and Error Vectors

The term $R_3(l) = \mathcal{L}(A_{l-1}U_{l-1}, Y_l) + \mathcal{L}(B_lU_{l+1}, Y_l) + \mathcal{L}(A_{l-1}U_{l-1}, G_l)$ models three nonlinear transform errors at node level $l$. The first two are related to the sNT representations $U_{l-1}, U_{l+1}$ and the c-sNT $Y_l$ representations and the last one is related to the $G_l$ representations. Term $\mathcal{L}(A_{l-1}U_{l-1}, Y_l) = \frac{1}{2} \|A_{l-1}U_{l-1} - Y_l\|^2_F$ measures the deviations of the gNT representations away from the linear transform representations $Q_l = A_{l-1}U_{l-1}$, whereas:

$$\text{te: } \frac{\partial \mathcal{L}(Q_l, Y_l)}{\partial Y_l} = Y_l - Q_l,$$

represent the corresponding deviation vectors. Also, the term $\mathcal{L}(A_{l-1}U_{l-1}, G_l) = \frac{1}{2} \|A_{l-1}U_{l-1} - G_l\|^2_F$ has similar role, related to $G_l$, respectively.

In addition, $\mathcal{L}(A_{l-1}U_{l-1}, Y_l)$ is related to the forward propagation of $U_{l-1}$ through $A_{l-1}$, whereas $\mathcal{L}(B_lU_{l+1}, Y_l)$ is related to the backward propagation of $U_{l+1}$ through $B_l$. We introduce the backward weights to enable regularization of the local propagation in a localized manner. We can also model $B_l = A_l^T$, but, in order to present the full potential of our approach, we consider that $B_l$ is different from $A_l^T$.

2.2. Weights Constraint

The term $R_2(l) = \mathcal{V}(A_{l-1}) + \mathcal{L}(A_l, B_l)$ models the properties of the weights that connect nodes at levels $l-1$ and $l$, as well as nodes at levels $l$ and $l+1$, where $\mathcal{V}(A_{l-1}) = \lambda_{l-1} \|A_{l-1}\|^2_F + \lambda_{l-2} \|A_{l-1}A_{l-1}^T - I\|^2_F - \lambda_{l-1} \log |\text{det } A_{l-1}^T A_{l-1}|$ and $\mathcal{L}(A_l, B_l) = \lambda_{l} \|B_l - A_l^T\|^2_F$ are used to regularize the conditioning, the coherence of $A_l$, Kostadinov et al. (2018), and the similarity between $A_l$ and $B_l$, respectively.

2.3. Sparsity Constraints

Our sparsity constraint is defined on the sNT representations $U_l \equiv \{u_{l,(1,1)}, \ldots, u_{l,(c,K)}\}$, the c-sNT representations $Y_l \equiv \{y_{l,(1,1)}, \ldots, y_{l,(c,K)}\}$ and the representations $G_l \equiv \{g_{l,(1,1)}, \ldots, g_{l,(c,K)}\}$ that exactly satisfy the specified local goal as $A(l) = \lambda_{l-1} \sum_{c=1}^C \sum_{k=1}^K \|u_{l,(c,k)}\|_1 + \|y_{l,(c,k)}\|_1 + \|g_{l,(c,k)}\|_1$.

2.4. Local Goal Constraint

Before defining the term $R_3(l)$, we first define our local goal that is explicitly set on $G_l = \{g_{l,(1,1)}, \ldots, g_{l,(c,K)}\}$. That is, knowing the corresponding labels, we express it in a form of a discrimination constraint, which is defined as $U(G_l) = \lambda_{l,0}D(G_l) = \lambda_{l,0} \sum_{c1,c1 \neq c2} \sum_{k1} \|g_{l,(c1,k1)}^t \odot g_{l,(c2,k1)}^t\|_1 + \|g_{l,(c1,k)}\odot g_{l,(c2,k)}\|_1 + \|g_{l,(c1,k1)}\odot g_{l,(c1,k2)}\|_2$, where $g_{l,(c,k)} = g_{l,(c,k)}^t = \max(0, g_{l,(c,k1)})$, and $g_{l,(c1,k1)} = \max(-g_{l,(c1,k1)}, 0)$ Kostadinov and Voloshynovskiy (2018).

By considering the representations $G_l$ and $U_l$, beside the te vectors, we also define the local goal error (ge) vectors as:

$$\text{ge: } \frac{\partial \mathcal{G}(G_l, U_l)}{\partial U_l} = U_l - G_l,$$

which represent the deviation of the representations $U_l$ away from the representations $G_l$.

We would like $U_l$ to match $G_l$, but in order to have more freedom in modeling a wide range of goals as well as allow decoupled update per the network weights, we do not set it as explicit constraint on $U_l$. Rather, we define it as follows:

$$R_3(l) = \sum_{c=1}^C \sum_{k=1}^K \psi(r_g(c,k)), \quad r_g(c,k) = \left( \frac{\partial \mathcal{L}(q_{l,(c,k)}, y_{l,(c,k)})}{\partial y_{l,(c,k)}} \right)^T \frac{\partial \mathcal{L}(g_{l,(c,k)}, u_{l,(c,k)})}{\partial u_{l,(c,k)}},$$

where in the simplest case we let $\psi(r_g(c,k)) = r_g(c,k)$.

2.5. Local Propagation Constraint

Our local propagation constraint takes into account the deviation vectors (7) and (8) that where explained in the previous two subsection and has a diffusion Spivak (1980) and Kittel and Kroemer (1980) related form that we define as:

$$R_4(l) = \sum_{c=1}^C \sum_{k=1}^K \psi(r_p(c,k)), \quad r_p(c,k) = \left( \frac{\partial \mathcal{L}(q_{l,(c,k)}, y_{l,(c,k)})}{\partial y_{l,(c,k)}} \right)^T \nabla^2 \mathcal{G}(u_{l-1,(c,k)}, u_{l+1,(c,k)}),$$

where $\nabla^2 \mathcal{G}(U_{l-1}, U_{l+1}) = \left[ \lambda_{l,f}B_l \frac{\partial \mathcal{G}(u_{l+1}, U_{l+1})}{\partial U_{l+1}} + \lambda_{l,b}A_{l-1} \frac{\partial \mathcal{G}(u_{l-1}, U_{l-1})}{\partial U_{l-1}} \right] = F_{l,f} + F_{l,b}$ is the local diffusion term, representing the vectors for the change of the local propagation flow, and $\lambda_{l,b} \text{ and } \lambda_{l,f}$ are regularization parameters. Term $\nabla^2 \mathcal{G}(U_{l-1}, U_{l+1})$ compactly describes the deviations of the representation $Y_l$ at node level $l$ w.r.t. the propagated ge vectors $\frac{\partial \mathcal{G}(u_{l+1}, U_{l+1})}{\partial U_{l+1}}$ and $\frac{\partial \mathcal{G}(u_{l-1}, U_{l-1})}{\partial U_{l-1}}$ from node levels $l-1$ and $l+1$, through $A_{l-1}$ and $B_l$, respectively. 

2.5.1. Local Propagation Dynamics

To explain the local propagation dynamics, we analyze its influence in the learning problem. The terms

Note that when there is no local goals defined at node levels $l-1$ and $l+1$, the representations $G_{l-1}$ and $G_{l+1}$ are zero vectors, (10) regularizes the local propagation flow and takes the form as $R_3(l) = \sum \psi(l(\frac{\partial \mathcal{L}(q_{l,(c,k)}, y_{l,(c,k)})}{\partial y_{l,(c,k)}})^T \nabla \mathcal{G}(u_{l-1,(c,k)}, u_{l+1,(c,k)}))$, where $\nabla \mathcal{G}(U_{l-1}, U_{l+1}) = [\lambda_{l,f}B_l U_{l+1} + \lambda_{l,b} A_{l-1} U_{l-1}].$
Assume that we have a feed-forward network with local propagation constraints. We analyze one commonly used network as an example. To understand what exactly this alignment means w.r.t. the c-sNT representations, we described in 2.5.1 and 2.5.2. Therefore, the c-sNT representations do not contain any additional components different than the sNT representations. In that case, the achieved alignment w.r.t. the preserved local propagation flow terms is achieved between the sNT representations and the \( \{A_l\} \). This means that the used weights \( A_{l-1} \) and \( B_l \) are not adding additional deviation in the change of the local propagation flow \( \nabla^2 g(U_l, U_{l+1}) \) when we propagate the representations \( U_{l-1} \) forward through \( A_{l-1} \). In that case, the achieved alignment w.r.t. the preserved local propagation flow indicates that the c-sNT representations do not contain any additional components different than the sNT representations. Therefore, the c-sNT representations do not add additional "information", which can be used in the update of \( A_{l-1} \) or \( B_l \) to further reduce the term (10). Since, when \( \frac{\partial g(Y_l)}{\partial Y_l} \) are orthogonal to \( \nabla^2 g(U_{l-1}, U_{l+1}) \), (10) is already zero.

Let's say that the local propagation flow is preserved at every network node. In addition, let's say that at the last network node the local goal is also satisfied. Then, by propagating the data forward through the network using the sNT, the network weights \( A_l \) are estimated such that they will not add additional deviation in the consecutive estimation of the sNT representations. Thereby, this will allow to attain the desired representations at the last network node. Otherwise, if there are deviations, the corresponding components from term (10) should be used to add a locally adjusted correction element in the update of the weights and thus to enforce preservation in the change of the propagation flow.

2.5.2. TRADE-OFFS AND LEARNING TARGET

If a local goal constraint (9) is included, satisfying its objective adds additional deviation. Therefore, at one network node, we have a trade-off between satisfying a local goal and a local propagation constraint (10) while over all NN nodes we have a trade-off between (i) satisfying a local goal, (ii) satisfying a local propagation constraint and (iii) achieving desired data propagation through the NN that enables attaining targeted representations at the last NN node.

In relation to the network parameters \( \mathcal{P} = \{P_1, ..., P_L\} \) and \( \mathcal{S} = \{S_1, ..., S_L\} \) that are used for training, our learning problem (6) targets to estimate the parameter set \( \mathcal{S} \) for the sNTs that approximate the parameter set \( \mathcal{P} \) of the c-sNTs. One sNT that is defined by \( S_l = \{A_l, \tau_l\} \) approximates one set of c-sNTs that is defined by \( P_l = \{P_{l,(1,1)}, ..., P_{l,(C,K)}\} \). That is, for every node at level \( l \), given \( \tau_l \), we would like to estimate \( A_l \) such that the c-sNT representations \( Y_l \) become equal to the sNT representations \( U_l \) while our local goals and local propagation constraints are satisfied. An illustration of the learning dynamics as well as the involved trade-offs is given in Figure 1.

3. The Learning Strategy

This section presents the solution to (6) in synchronous and asynchronous scheduling regime by essentially using two variants of one learning principle.

3.1. The Learning Algorithm

Our learning algorithm iteratively updates the network parameters in two stages. Stage one updates the sNT representations \( U_l \) and the exact goal satisfying representations \( G_l \) while stage two estimates the c-sNT representations \( Y_l \) and the weights \( A_{l-1} \) and \( B_l \).
Network Parameter Learning with Local Propagation Constraints

3.1. Stage One

Given the weights $A_l$, this stage computes $U_l$ and $G_l$.

**Estimating $U_l$ Approximately by Discarding Constraints and Coupling** In this stage, we let $U_l = Y_l$, fix all the variables in problem (6) except $U_l$, disregard the local goal, the local propagation constraint and the coupling over two representations at levels $l-1$ and $l$, then per node level $l$, problem (6) reduces to:

$$
\hat{U}_l = \min_{U_l} \mathcal{L}(Q_l, U_l) + \lambda_{l,1} \sum_{c=1}^{K} \sum_{k=1}^{C} \| u_{l,(c,k)} \|_1, \tag{11}
$$

where the solution per single $\hat{u}_{l,(c,k)}$ is exactly the sNT (4). Therefore, computing $U_l$ by propagating forward through the network with consecutive execution of the sNT, is in fact an approximative solution w.r.t. (6).

**Estimating $G_l$ Approximately by Discarding Local Goal and Local Propagation Constants** Given $Q_l = A_{l-1}U_{l-1}$, if we disregard the local propagation constraint and the local goal constraint, per node level $l$, $G_l$ are defined as the solution of an optimization problem where $G_l$ has to be close to the linear transform representations $Q_l = A_{l-1}U_{l-1}$ under the sparsity constraint and the discrimination\(^4\) constraint, i.e.:

$$
\hat{G}_l = \min_{G_l} \mathcal{L}(Q_l, G_l) + \lambda_{l,1} \sum_{c=1}^{K} \sum_{k=1}^{C} \| g_{l,(c,k)} \|_1, \tag{12}
$$

subject to $\mathcal{U}(G_l) = 0$.

In Appendix A, we give an iterative solution to (12) with closed form updates at the iterative steps.

3.1.2. Stage Two

Given all of the currently estimated $U_l$ and $G_l$, note that (6) decomposes over subproblems that are separable per every parameter subset $\varsigma_l = \{ Y_l, A_{l-1}, B_l \}$, (Figure 2). This allows parallel update on all subsets $\varsigma_l$ of network parameters,

since the parameter set $\varsigma_l$ does not share parameters with any other $\varsigma_2$, i.e., $\varsigma_l \cap \varsigma_2 = \emptyset$, $\forall l \neq l_2$. The learning subproblems per the decoupled sets $\varsigma_l$ have one common form. In the following, we present it and give the solution.

Let all the variables in (6) be fixed except $\varsigma_l = \{ Y_l, A_{l-1}, B_l \}$, then (6) reduces to the following problem:

$$
\{ \hat{Y}_l, \hat{A}_{l-1}, \hat{B}_l \} = \min_{\{ Y_l, A_{l-1}, B_l \}} \sum_{j=1}^{4} R_j(l) + A(l). \tag{13}
$$

**Estimating $\varsigma_l = \{ Y_l, A_{l-1}, B_l \}$ Exactly** Problem (13) is still non-convex. Nevertheless, to solve (13), we propose an alternating block coordinate descend algorithm, where we iteratively update one variable from the set of variables $\varsigma_l = \{ Y_l, A_{l-1}, B_l \}$ while keeping the rest fixed. It has three steps: (i) estimation of the c-sNT representation $Y_l$, (ii) estimation of the forward weights $A_{l-1}$ and (iii) estimation of the backward weights $B_l$. In the following, we explain the steps of the proposed solution.

- **c-sNT Representation Estimation** Let all the variables in problem (13) be given except $y_{l,(c,k)}$ then (13) reduces to the following constrained projection problem:

$$
\hat{y} = \arg \min_{y} \frac{1}{2} \| q - y \|_2^2 + \nu^T y + \lambda_{l,1} 1^T |y|, \tag{14}
$$

where:

$$
q = A_{l-1} u_{l-1,(c,k)}, \quad \nu = p_{l,(c,k)} + t_{l,(c,k)}, \quad t_{l,(c,k)} = \frac{\partial \mathcal{L}(g_{l,(c,k)}, u_{l,(c,k)})}{\partial u_{l,(c,k)}}, \quad p_{l,(c,k)} = \nabla^2 \mathcal{G}(u_{l-1,(c,k)}; u_{l+1,(c,k)}) ,
$$

and it has a closed form solution which exactly matches the expression for the c-sNT (5). The proof is given in Appendix B. In addition, note that by (5) all $y_{l,(c,k)}$ at node level $l$ can be computed in parallel.

The empirical expectation of $E[\nu^T y]$ induced by the local goal constraint and the local propagation constraint can also be considered as the empirical risk for the sNT (4), since when $\sum_{c=1}^{C} \sum_{k=1}^{K} \nu_{l,(c,k)} y_{l,(c,k)} = 0$, at layer $l$, the c-sNTs (5) do not carry additional “information” different then the one in sNT (4), while when $\nu_{l,(c,k)} = 0$, the corresponding c-sNT reduces to the sNT.

- **Forward Weights Update** Let all the variables in problem (13) be given except $A_{l-1}$ then (13) reduces to the following problem:

$$
A_{l-1} = \arg \min_{A_{l-1}} \| A_{l-1} \mathbf{s}_{l-1} - W_l \|_F^2 + R_2(l), \tag{16}
$$

---

\(^4\)In general, one might model different goals for the representations $G_l$ by defining a corresponding function $\mathcal{U}(G_l)$. 
where we assume that:

$$S_{l-1} S_{l-1}^T = U_{l-1} \left( U_{l-1} + \lambda_{l,b} \frac{\partial G(G_{l-1}, U_{l-1})}{\partial U_{l-1}} \right)^T,$$

$$W_l = Y_l - \frac{\partial G(G_l, U_l)}{\partial U_l} - \lambda_{l,f} \frac{\partial G(G_{l+1}, U_{l+1})}{\partial U_{l+1}}.$$

We give the derivation of (17) in Appendix C. To solve (16) for $A_{l-1} \in \mathbb{R}^{M_l \times M_{l-1}}$, $M_l \geq M_{l-1}$, we use the approximate closed form solution of (Kostadinov et al., 2018).

### 3.3. Local Minimum Solution Guarantee

The next result shows that with arbitrarily small error we can find a local minimum solution to (6) for $B_l = A_l^T$.

**Theorem 1** Given any data set $Y_0$, there exists $\omega = \{\lambda_{l,b}, \lambda_{l,f}\}$, $\lambda_{l,b} > 0, \lambda_{l,f} > 0$ and a learning algorithm for a $L$-node transform-based network with a goal set on one node at level $l_G$ such that the algorithm after $t > S$ iteration learns all $A_l, l \in \{0, ..., L-1\}$ with $G(D_L, U_L) = \epsilon$, where $D_L \in \mathbb{R}^{M_L \times CK}$ are the resulting representations of the propagated goal representations $G_l$ through the network from node level $l_G + 1$, and $\epsilon > 0$ is arbitrarily small constant.

The proof is given in Appendix D.

**Remark** The result by Theorem 1 reveals the possibility to attain desirable representations $U_L$ at level $L$ while only setting one local representation goal on one node at level $l_G \in \{1, ..., L\}$.

### 4. Numerical Evaluation

We present preliminary numerical evaluation of our learning strategy that is applied on a fully connected feed forward network, i.e., (6), with square weights $A_l, B_l \in \mathbb{R}^{N \times N}$, where $B_l = A_l^T$.

### 4.1. Data, Evaluated NNs, and Learning/Testing Setup

**Used Data and Evaluated Networks** The used data sets are MNIST and Fashion-MNIST. All the images from the data sets are downsampled to resolution $28 \times 28$, and are normalized to unit variance. We analyze 12 different networks, 6 per database. Per one database 4 networks have 6 nodes and additional 2 have 4 nodes. The networks are trained in synchronous syn and asynchronous mode asyn. For the 6-node networks trained in syn, 2 of them have a goal defined at the last node $L$ (asyn$_{n[6]g[6]}$) and for the remaining 2 the goal is set on node at the middle in the network at level 3 (asyn$_{n[6]g[3]}$). For the 4-node network the goal is set at node level 4 (asyn$_{n[4]g[4]}$). Similarly for the asyn mode, we denote the networks as (asyn$_{n[6]g[6]}$), (asyn$_{n[6]g[3]}$) and (asyn$_{n[4]g[4]}$).

**Scheduling Regime Setup for Network Learning** The asynchronous mode is implemented by using $L$ random draws $\phi \in \{-1, 1\}^L$, as the number of nodes, from a Bernoulli distribution. If the realization is $1$, $\phi(l) = 1$, we use $A_l^f$ in the forward pass (stage one) and we update the corresponding set of variables $s_l$ (stage two). If the realization is $-1$, $\phi(l) = -1$, then we do not use $A_l^f$, but instead we use $A_l^{f-1}$ for stage one and in stage two we do not update the corresponding set $s_l$. The synchronous mode is implemented by taking into account all $A_l^f$.

An on-line variant is used for the update of $A_l$ w.r.t. a
As well as their learning time, we refer to the original manuscripts (Schmidhuber, 2012) and (Phaye et al., 2018). The results are shown in Tables 1, 2 and 3. The networks trained using the proposed approach on both of the used databases achieve competitive to state-of-the-art recognition performance w.r.t. results reported by (Schmidhuber, 2012) and (Phaye et al., 2018)\(^a\) and \(^b\). More importantly, we point out that our networks have small number of parameters, i.e., 6 networks with 6 nodes having 6 weights with dimensionality \(784 \times 784\) and 4 networks with 4 nodes having 4 weights with dimensionality \(784 \times 784\). Whereas the learning time for \(L = 6\) node network is \(\sim 3.5\) hours, on a PC that has Intel Xeon(R) 3.60GHz CPU and 32G RAM memory when using not optimized Matlab code that implements the sequential variant of the proposed algorithm. We expect a parallel implementation of the proposed algorithm to provide \(\sim L \times \) speedup, which would reduce the learning time to less then half an hour in our not optimized Matlab code.

### Table 3. The size of the largest networks \(\text{syn}[n][g][6]\) and \(\text{asyn}[n][g][6]\) that were evaluated using our learning algorithm.

|                  | MNIST | F-MNIST |
|------------------|-------|---------|
| Num. of connections | \(6 \times N^2\) | \(6 \times N^2\) |
|                  |       |         |

### Table 1. Comparative result for the recognition accuracy between the the feed-forward network learned using the proposed algorithm under synchronous and asynchronous update scheme and \(^a\) (Schmidhuber, 2012) and \(^b\) (Phaye et al., 2018).

|                  | MNIST | F-MNIST |
|------------------|-------|---------|
| Acc. [%]         |       |         |
| \(\text{syn}[n][g][4]\) | 99.77\(^a\) | 94.65\(^b\) |
| \(\text{syn}[n][g][6]\) | 99.28 | 93.23 |
| \(\text{syn}[n][g][3]\) | 98.98 | 91.98 |
| \(\text{asyn}[n][g][4]\) | 98.57 | 91.07 |
| \(\text{asyn}[n][g][6]\) | 99.01 | 93.15 |
| \(\text{asyn}[n][g][3]\) | 98.91 | 92.03 |

### Table 2. Comparative result for the learning time in hours between the proposed algorithm under synchronous and asynchronous update scheme and \(^a\) (Schmidhuber, 2012) and \(^b\) (Phaye et al., 2018).

|                  | MNIST | F-MNIST |
|------------------|-------|---------|
| t[h]             |       |         |
| \(\text{syn}[n][g][6]\) | 6 \(\times\) .5 | 6 \(\times\) .5 |
| \(\text{asyn}[n][g][6]\) | 6 \(\times\) .6 | 6 \(\times\) .7 |

### 4.2. Evaluation Summary

The results are shown in Tables 1, 2 and 3. The networks trained using the proposed approach on both of the used databases achieve competitive to state-of-the-art recognition performance w.r.t. results reported by (Schmidhuber, 2012) and (Phaye et al., 2018)\(^a\). More importantly, we point out that our networks have small number of parameters, i.e., 6 networks with 6 nodes having 6 weights with dimensionality \(784 \times 784\) and 4 networks with 4 nodes having 4 weights with dimensionality \(784 \times 784\). Whereas the learning time for \(L = 6\) node network is \(\sim 3.5\) hours, on a PC that has Intel Xeon(R) 3.60GHz CPU and 32G RAM memory when using not optimized Matlab code that implements the sequential variant of the proposed algorithm. We expect a parallel implementation of the proposed algorithm to provide \(\sim L \times \) speedup, which would reduce the learning time to less then half an hour in our not optimized Matlab code.

### 5. Conclusion

In this paper, we introduced a novel learning problem formulation for estimating the network parameters. We presented insights, as well as unfolded new interpretations of the learning dynamics w.r.t. the proposed local propagation. We proposed a two stage learning strategy, which allows the network parameters to be updated in synchronous or asynchronous scheduling mode. We implemented it by an efficient algorithm that enables parallel execution of the learning stages. While in the first stage, our estimates are computed approximately, in the second stage, our estimates are computed exactly. Moreover, in the second stage, the solutions to the decoupled problems, have a local convergence guarantee.

We showed theoretically that by learning with a local propagation constraint, we can achieve desired data propagation through the network that enables attaining a targeted representations at the last node in the network. We empirically validated our approach. The preliminary numerical evaluation of the proposed learning principle was promising. On the used publicly available databases the feed-forward network trained using our learning principle provided comparable results w.r.t. the state-of-the-art methods, while having a small number of parameters and low computational cost.

The information-theoretic analysis on the fundamental limit in the trade-off between the local propagation, the local goal and the global data propagation flow as well as the study on "technical" goals, e.g., goals that add to the acceleration in convergence of the learning is one future direction. Performance evaluation on other and large data sets, together with comparative evaluation for other activation functions, goals (e.g., reconstruction, discrimination, robustness, compression, privacy and security related goals) or a combination of them under different penalties \(\psi\), is another future direction.
We point out that for other network architectures as well as for multi-path network a similar problem formulation could be considered. Moreover, by adopting the presented approach, similar solutions could also be derived. In fact, our algorithm, is applicable for network defined as a directed graph, i.e., a network where the propagation flow is specified and known.

The proposed learning principle allows us by only changing the constraints on the propagation flow to influence on the properties of all hidden and output representations. In this line, the next frontier towards the ultimate machine intelligence could be seen in unsupervised self-driven goals, propagation flows and self-configuration. Where the network will learn what will be the goals, what kind of constraints on the propagation flow is required to reach that goal and how many nodes are required.

References

David Balduzzi, Hastagiri Vanchinathan, and Joachim Buhmann. Kickback cuts backprop’s red-tape: Biologically plausible credit assignment in neural networks. In Proceedings of the Twenty-Ninth AAAI Conference on Artificial Intelligence, AAAI’15, pages 485–491. AAAI Press, 2015.

Yoshua Bengio. Practical recommendations for gradient-based training of deep architectures. CoRR, abs/1206.5533, 2012.

Léon Bottou. Stochastic gradient descent tricks. In Neural Networks: Tricks of the Trade - Second Edition, pages 421–436. 2012.

Corinna Cortes and Vladimir Vapnik. Support-vector networks. Mach. Learn., 20(3):273–297, September 1995.

Wojciech Marian Czarnecki, Grzegorz Swirszcz, Max Jaderberg, Simon Osindero, Oriol Vinyals, and Koray Kavukcuoglu. Understanding synthetic gradients and decoupled neural interfaces. CoRR, abs/1703.00522, 2017.

Goh Gabriel. Why momentum really works. Distill, abs/7828, 2017.

Sepp Hochreiter. The vanishing gradient problem during learning recurrent neural nets and problem solutions. Int. J. Uncertain. Fuzziness Knowl.-Based Syst., 6(2):107–116, April 1998.

Max Jaderberg, Wojciech Marian Czarnecki, Simon Osindero, Oriol Vinyals, Alex Graves, and Koray Kavukcuoglu. Decoupled neural interfaces using synthetic gradients. CoRR, abs/1608.05343, 2016.

Diederik P. Kingma and Jimmy Ba. Adam: A method for stochastic optimization. CoRR, abs/1412.6980, 2014.

Charles Kittel and Herbert Kroemer. Thermal physics. W.H. Freeman, 2nd ed edition, 1980.

Dimche Kostadinov and Slava Voloshynovskiy. Learning non-linear transform with discriminative and minimum information loss priors, 2018. URL https://openreview.net/pdf?id=SJzmJEq6W.

Dimche Kostadinov, Slava Voloshynovskiy, and Sohrab Ferdowsi. Learning overcomplete and sparsifying transform with approximate and exact closed form solutions. In 7-th European Workshop on Visual Information Processing (EUVIP), Tampere, Finland, November 2018.

Y. Lecun. A theoretical framework for back-propagation. 1988.

Yann LeCun, Léon Bottou, Genevieve B. Orr, and Klaus-Robert Müller. Efficient backprop. In Neural Networks: Tricks of the Trade, This Book is an Outgrowth of a 1996 NIPS Workshop, pages 9–50, London, UK, UK, 1998. Springer-Verlag.

Dong-Hyun Lee, Saizheng Zhang, Antoine Biard, and Yoshua Bengio. Target propagation. CoRR, abs/1412.7525, 2014.

Ilya Loshchilov and Frank Hutter. SGDR: stochastic gradient descent with restarts. CoRR, abs/1608.03983, 2016.

Arild Nø kland. Direct feedback alignment provides learning in deep neural networks. In D. D. Lee, M. Sugiyama, U. V. Luxburg, I. Guyon, and R. Garnett, editors, Advances in Neural Information Processing Systems 29, pages 1037–1045. Curran Associates, Inc., 2016.

Razvan Pascanu, Tomas Mikolov, and Yoshua Bengio. Understanding the exploding gradient problem. CoRR, 2012.

Sai Samarth R. Phaye, Apoorva Sikka, Abhinav Dhall, and Deepi R. Bathula. Dense and diverse capsule networks: Making the capsules learn better. CoRR, abs/1805.04001, 2018. URL http://arxiv.org/abs/1805.04001.

D. C. Plaut, S. J. Nowlan, and G. E. Hinton. Experiments on learning back propagation. Technical Report CMU–CS–86–126, Carnegie–Mellon University, Pittsburgh, PA, 1986.

Saiprasad Ravishankar and Yoram Bresler. Doubly sparse transform learning with convergence guarantees. In IEEE ICASSP 2014, Florence, Italy, May 4-9, 2014, pages 5262–5266, 2014.

Ron Rubinstein and Michael Elad. Dictionary learning for analysis-synthesis thresholding. IEEE Trans. Signal Processing, 62(22):5962–5972, 2014.
Sebastian Ruder. An overview of gradient descent optimization algorithms. *CoRR*, abs/1609.04747, 2016.

Jurgen Schmidhuber. Multi-column deep neural networks for image classification. In *Proceedings of the 2012 IEEE Conference on Computer Vision and Pattern Recognition (CVPR)*, CVPR ’12, pages 3642–3649, Washington, DC, USA, 2012. IEEE Computer Society. ISBN 978-1-4673-1226-4.

Jürgen Schmidhuber. Deep learning in neural networks: An overview. *CoRR*, abs/1404.7828, 2014.

Ohad Shamir and Tong Zhang. Stochastic gradient descent for non-smooth optimization: Convergence results and optimal averaging schemes. In *Proceedings of the 30th International Conference on Machine Learning, ICML 2013, Atlanta, GA, USA, 16-21 June 2013*, pages 71–79, 2013.

M. Spivak. *Calculus*. Addison-Wesley world student series. Publish or Perish, 1980. URL https://books.google.ch/books?id=-mwPAQAAMAAJ.

Nitish Srivastava, Geoffrey Hinton, Alex Krizhevsky, Ilya Sutskever, and Ruslan Salakhutdinov. Dropout: A simple way to prevent neural networks from overfitting. *J. Mach. Learn. Res.*, 15(1):1929–1958, January 2014.

Gavin Taylor, Ryan Burmeister, Zheng Xu, Bharat Singh, Ankit Patel, and Tom Goldstein. Training neural networks without gradients: A scalable ADMM approach. *CoRR*, abs/1605.02026, 2016.

An Zhu, Yu Meng, and Changjiang Zhang. An improved adam algorithm using look-ahead. In *Proceedings of the 2017 International Conference on Deep Learning Technologies, ICDLT ’17*, pages 19–22, New York, NY, USA, 2017. ACM.