Heisenberg-picture approach to the evolution of the scalar fields in an expanding universe

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Abstract

We present the Heisenberg-picture approach to the quantum evolution of the scalar fields in an expanding FRW universe which incorporates relatively simply the initial quantum conditions such as the vacuum state, the thermal equilibrium state, and the coherent state. We calculate the Wightman function, two-point function, and correlation function of a massive scalar field. We find the quantum evolution of fluctuations of a self-interacting field perturbatively and discuss the renormalization of field equations.

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I. INTRODUCTION

Quantum scalar fields such as the inflaton or Higgs fields in an expanding universe are characterized by particle production and *nonequilibrium* state during a rapidly expanding period. Path integral and canonical method are two typical methods to describe the quantum evolution and oftentimes to incorporate an initial condition. Path-integral formalism requires a modification for the *nonequilibrium* process. The real-time formalism, the imaginary-time formalism, and the complex-time formalism are the field theoretic methods to describe such a *nonequilibrium* quantum evolution of the field from an initial thermal equilibrium. The real-time formalism has been frequently used to evaluate the finite temperature Green’s function in the expanding FRW universe [1–3]. The canonical method based on the Hamiltonian density operator also has been widely used in which the functional Schrödinger equation describes the quantum evolution of the Higgs or scalar field in the expanding universe [4–6]. Functional Schrödinger-picture has been developed to find the wave function, bearing a close parallel to the Schrödinger-picture of quantum mechanics.

The purpose of this paper is to present the Heisenberg-picture approach to describe the quantum evolution of the scalar fields in the expanding FRW universe. The Heisenberg-picture has the advantage in incorporating an arbitrary initial quantum condition in a particularly simple form. For this purpose we reinterpret the quantum operators, the so-called generalized invariants [7], for a time-dependent quantum system as constant operators in the Heisenberg-picture, and in terms of which we find the quantum field in the Heisenberg-picture. Extensive studies have already been done to find and to make use of these invariants for time-dependent oscillators. A remarkable point is that the exact quantum states are the eigenstates of these invariant operators up to time-dependent phase factors in the Schrödinger-picture. We applied the generalized invariants to the Schrödinger-picture description of the massive scalar field in the expanding FRW universe [8,9] and Gao *et al* also applied independently the Schrödinger-picture to both the massive scalar field and the self-interacting scalar field in Ref. [10]. Recently, the Heisenberg-picture, which has been sel-
domly used, however, was employed to find the exact quantum evolution for the same quantum systems \[\text{[1]}\]. Therefore, it would be worthy to apply the Heisenberg-picture approach to the scalar fields in the expanding FRW universe, which can be regarded as an infinite system of decoupled time-dependent harmonic oscillators and coupled time-dependent anharmonic oscillators for the massive scalar field and the interacting scalar field, respectively.

II. MASSIVE SCALAR FIELD

As the simplest model, we consider a massive scalar field in the expanding FRW universe with the metric

\[
ds^2 = -dt^2 + a(t)^2 dx^2. \tag{1}
\]

The action is given by

\[
S = \int dt d^3x \frac{a^3}{2} \left[ \dot{\phi}^2 - \frac{1}{a^2} (\nabla \phi)^2 - \left( m^2 + \frac{\xi}{6} R \right) \phi^2 \right] \tag{2}
\]

where \(\xi = 0\) for the minimal coupling and \(\xi = \frac{1}{6}\) for the conformal coupling. We Fourier-transform the scalar field

\[
\phi(x, t) = \frac{1}{(2\pi)^{3/2}} \int d^3k \phi_k(t) e^{i k \cdot x} \tag{3}
\]

and take the linear combinations

\[
\phi^{(+)}_k(t) = \frac{1}{2} (\phi_k(t) + \phi_{-k}(t)),
\]

\[
\phi^{(-)}_k(t) = i \frac{1}{2} (\phi_k(t) - \phi_{-k}(t)). \tag{4}
\]

Then the \(x\)-integration yields

\[
\int d^3x \phi^2(x, t) := \sum_\alpha \phi^2_\alpha(t) = \int d^3k \phi_k(t) \phi_{-k}(t)
\]

\[
\int d^3x \pi^2(x, t) := \sum_\alpha \pi^2_\alpha(t) = \int d^3k \pi_k(t) \pi_{-k}(t) \tag{5}
\]

where \(\alpha\) denotes \((\pm, k)\). Here \(\phi^{(+)}\) and \(\phi^{(-)}\) should be treated as independent variables corresponding to the cosine and sine modes in the box normalization. Thus one obtains the Hamiltonian of mode-decomposed time-dependent harmonic oscillators:
\[ H(t) = \sum_{\alpha} H_\alpha := \sum_{\alpha} \left[ \frac{\tau_\alpha^2}{2a^3(t)} + \frac{a^3(t)\omega_\alpha^2(t)}{2} \phi_\alpha^2 \right] \quad (6) \]

where

\[ \omega_\alpha^2(t) = m^2 + \frac{k^2}{a^2} + \xi R, \quad (7) \]

is the time-dependent frequency squared. Due to the complete mode-decomposition, the quantum states is a product of quantum state for each mode:

\[ |\Psi(t)\rangle = \prod_\alpha |\psi_\alpha(t)\rangle. \quad (8) \]

Each quantum state obeys separately the Schrödinger equation

\[ i \frac{\partial}{\partial t} |\psi_\alpha(t)\rangle = \hat{H}_\alpha(t)|\psi_\alpha(t)\rangle. \quad (9) \]

As pioneered by Lewis and Riesenfeld \cite{7}, the exact quantum states of a time-dependent quantum system can be found in the Schrödinger-picture as the eigenstates of the generalized invariant operators. We make use of these operators as a method to find the quantum evolution of the scalar fields. The invariant operator for the \( \alpha \)-th mode harmonic oscillator satisfies

\[ \frac{\partial}{\partial t} \hat{I}_\alpha - i[\hat{I}_\alpha, \hat{H}_\alpha] = 0. \quad (10) \]

There are many operators that satisfy Eq. (10). An invariant operator quadratic in momentum and position was found explicitly for a time-dependent harmonic oscillator \cite{7}. In the Schrödinger-picture, a pair of the invariant operators that are linear in momentum and position, and can be interpreted as the time-dependent creation and annihilation operators were found \cite{8} and were applied to the massive scalar field in the FRW universe \cite{9}. However, it is our observation that these invariant operators have a simple interpretation in the Heisenberg-picture than in the Schrödinger-picture. We note that the time-dependent invariant operators in the Schrödinger-picture

\[ \hat{B}_\alpha^\dagger(t) = -i[\phi_\alpha(t)\hat{\pi}_\alpha - a^3\dot{\phi}_\alpha(t)\hat{\phi}_\alpha], \]

\[ \hat{B}_\alpha(t) = i[\phi_\alpha^*(t)\hat{\pi}_\alpha - a^3\dot{\phi}_\alpha^*(t)\hat{\phi}_\alpha], \quad (11) \]
where $\phi_\alpha(t)$ and $\phi^*_\alpha(t)$ are the complex solutions of the classical field equation

$$\ddot{\phi}_\alpha(t) + 3 \frac{\dot{a}(t)}{a(t)} \dot{\phi}_\alpha(t) + \omega^2_\alpha(t) \phi_\alpha(t) = 0,$$

(12)
can be interpreted as the time-independent operators in the Heisenberg-picture

$$\frac{d}{dt} \hat{B}_{\alpha,H}(t) = \frac{\partial}{\partial t} \hat{B}_{\alpha,H} - i [\hat{B}_{\alpha,H}, \hat{H}_{\alpha,H}] = 0.$$  

(13)

In order to have the usual commutation relation $[\hat{B}_{\alpha,H}, \hat{B}^\dagger_{\alpha,H}] = 1$, the classical solutions satisfy the boundary condition

$$ia^3(\dot{\phi}_\alpha(t)\phi^*_\alpha(t) - \phi_\alpha(t)\dot{\phi}^*_\alpha(t)) = 1.$$  

(14)

We denote these by $\hat{A}^\dagger_{\alpha}$ and $\hat{A}_\alpha$, respectively. Thus, we find the position and momentum operators in the Heisenberg-picture

$$\hat{\phi}_{\alpha,H}(t) = \phi_\alpha(t)\hat{A}_\alpha + \phi^*_\alpha(t)\hat{A}^\dagger_{\alpha},$$

$$\hat{\pi}_{\alpha,H}(t) = a^3[\dot{\phi}_\alpha(t)\hat{A}_\alpha + \dot{\phi}^*_\alpha(t)\hat{A}^\dagger_{\alpha}].$$

(15)

The Fock space of each mode is constructed from the vacuum state

$$\hat{A}_\alpha|0_\alpha, t_0\rangle = 0.$$  

(16)

The vacuum state of the scalar field is the product of the vacuum state of each mode

$$|0, t_0\rangle = \prod_\alpha |0_\alpha, t_0\rangle.$$  

(17)

To apply this Heisenberg-picture to the massive Higgs field and to incorporate the initial quantum conditions, we choose $\hat{A}^\dagger_{\alpha}$ and $\hat{A}_\alpha$ that diagonalize the Hamiltonian at an initial time $t_0$

$$\hat{H}_\alpha(t_0) = \omega_\alpha(t_0)(\hat{A}^\dagger_{\alpha}\hat{A}_\alpha + \frac{1}{2}).$$

(18)

The expectation value of any operator in the Heisenberg-picture is evaluated as

$$\langle \hat{O} \rangle = \langle \hat{O}_H \rangle_H.$$  

(19)

Below we consider separately three different types of initial conditions.
A. Vacuum State

Assuming that the initial quantum state be the \( n \)th number state of the Hamiltonian, we find the Wightman function of each mode

\[
\langle n_\alpha, t_0 | \hat{\phi}_\alpha(t) \hat{\phi}_\alpha(t') | n_\alpha, t_0 \rangle = n_\alpha \left( \phi^*_\alpha(t) \phi_\alpha(t') + \phi_\alpha(t) \phi^*_\alpha(t') \right) + \phi_\alpha(t) \phi^*_\alpha(t').
\] (20)

From Eq. (20) we find the Wightman function for the scalar field

\[
\langle \hat{\phi}(x, t) \hat{\phi}(x', t') \rangle^{(\text{vac})} = \frac{1}{(2\pi)^3} \int d^3k e^{i k \cdot (x-x')} \phi_k(t) \phi^*_k(t')
\] (21)

The two-point function at an equal time follows

\[
\langle \hat{\phi}(x, t) \hat{\phi}(x', t) \rangle^{(\text{vac})} = \frac{1}{(2\pi)^3} \int d^3k e^{i k \cdot (x-x')} |\phi_k(t)|^2,
\] (22)

and so does the correlation function

\[
\langle \hat{\phi}^2(x, t) \rangle^{(\text{vac})} = \frac{1}{(2\pi)^3} \int d^3k |\phi_k(t)|^2.
\] (23)

B. Thermal Equilibrium

Next we consider a thermal equilibrium as an initial state. The initial thermal equilibrium is described by the density matrix

\[
\hat{\rho}_{\alpha, I}(t_0) = \frac{1}{Z_{\alpha, I}} e^{-\beta \hat{H}_\alpha(t_0)},
\] (24)

where

\[
Z_{\alpha, I} = \text{Tr}(e^{-\beta \hat{H}_\alpha(t_0)}) = \left(2 \sinh \frac{\beta \omega_\alpha(t_0)}{2}\right)^{-1}.
\] (25)

The expectation value of the operator is given by
\[
\langle \hat{O}_H \rangle^{(\text{therm})} = \frac{1}{Z_{\alpha,I}} \text{Tr}(e^{-\beta \hat{H}_\alpha(t_0)} \hat{O}_H) = \frac{1}{Z_{\alpha,I}} \sum_{n_\alpha} \langle n_\alpha | e^{-\beta \hat{H}_\alpha(t_0)} \hat{O}_H | n_\alpha \rangle
\]  

(26)

It is straightforward to evaluate the Wightman function of each mode

\[
\langle \hat{\phi}_\alpha(t) \hat{\phi}_\alpha(t') \rangle^{(\text{therm})} = \phi^*_\alpha(t) \phi_\alpha(t') \frac{1}{Z_{\alpha,I}} \sum_{n_\alpha} \langle \hat{A}^\dagger_\alpha \hat{A}_\alpha \rangle e^{-\beta \omega_\alpha(t_0)}(n_\alpha + \frac{1}{2})
\]

\[+ \phi_\alpha(t) \phi^*_\alpha(t') \frac{1}{Z_{\alpha,I}} \sum_{n_\alpha} \langle \hat{A}_\alpha \hat{A}^\dagger_\alpha \rangle e^{-\beta \omega_\alpha(t_0)}(n_\alpha + \frac{1}{2})
\]

\[= \frac{1}{e^{\beta \omega_\alpha(t_0)} - 1} \phi_\alpha(t) \phi_\alpha(t') + \frac{1}{e^{\beta \omega_\alpha(t_0)} - 1} \phi_\alpha(t) \phi^*_\alpha(t').
\]  

(27)

The Wightman function of the scalar field is obtained by summing over modes

\[
\langle \hat{\phi}(\mathbf{x},t) \hat{\phi}(\mathbf{x}',t') \rangle^{(\text{therm})} = \frac{1}{(2\pi)^3} \int d^3k e^{i \mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')}
\]

\[\left[ \phi^*_k(t) \phi_k(t')
\]

\[+ \frac{1}{e^{\beta \omega_k(t_0)} - 1} \left( \phi^*_k(t) \phi_k(t') + \phi_k(t) \phi^*_k(t') \right) \right]
\]  

(28)

The first term is the vacuum Green’s function and the second term is the thermal correction [2]. The two-point function and correlation function can be found by taking the coincident limits \( t = t' \) and \( \mathbf{x} = \mathbf{x}' \), \( t = t' \), respectively.

### C. Coherent State

As the last initial condition, we consider the coherent state described by the density operator

\[
\hat{\rho}_{\alpha,II} = \frac{1}{Z_{\alpha,II}} e^{-\beta [\omega_\alpha(t_0) \hat{A}^\dagger_\alpha \hat{A}_\alpha + \gamma_\alpha \hat{A}^\dagger_\alpha + \gamma^{*}_\alpha \hat{A}_\alpha + \epsilon_\alpha]},
\]

(29)

where \( \epsilon_\alpha = |\gamma_\alpha|^2 / \omega_\alpha + \omega_\alpha / 2 \). The density operator can be unitarily transformed into the canonical form [24] by acting the displacement operator

\[\hat{D}_\alpha = e^{-\frac{\gamma_\alpha}{\omega_\alpha(t_0)} \hat{A}^\dagger_\alpha + \frac{\gamma^{*}_\alpha}{\omega_\alpha(t_0)} \hat{A}_\alpha}
\]

(30)

and so the partition function becomes
\[ Z_{\alpha,\beta} = \frac{e^{-\beta \epsilon}}{1 - e^{-\beta \omega(t)}}. \]  

(31)

We find the Wightman function of the scalar field

\[
(\hat{\phi}(x, t)\hat{\phi}(x', t'))^{(\text{coh})} = \frac{1}{(2\pi)^3} \int d^3k e^{ik(x-x')} \left[ \phi_k(t)\phi_k^*(t') + \frac{1}{e^{\beta \omega(t)}} - 1 \left( \phi_k^*(t)\phi_k(t') \right) + \frac{1}{(2\pi)^3} \int d^3k \phi_{k,c}(t)\phi_{k,c}^*(t') \right]
\]

(32)

where

\[
\phi_{k,c} = \frac{\phi_k^*(t)\gamma_k + \phi_k(t)\gamma_k^*}{\omega_k}
\]

(33)

is the classical field corresponding to the coherent state.

### III. SELF-INTERACTING SCALAR FIELD

We now turn to the case of a self-interacting scalar field. The Hamiltonian takes the form

\[
H(t) = \int d^3x \left[ \frac{\pi^2}{2a^3} + a^3 \left( \frac{1}{2a^2}(\nabla \phi)^2 + V(\phi) \right) \right]
\]

(34)

We consider the potential of a general power law

\[
V_1(\phi) = \sum_n \frac{\lambda_n}{n!} \phi^n.
\]

(35)

and the potential of a particular cosmological interest

\[
V_2(\phi) = \frac{m^2}{2} \phi^2 + \frac{\xi}{2} r^2 \phi^2 + \frac{\lambda}{4} \phi^4.
\]

(36)

Unlike the free massive scalar field, the Hamiltonian cannot be simply decomposed as a sum over modes, but it is a coupled system of modes. Thus, the canonical approach to the full quantum evolution of the fields is extremely difficult to carry out explicitly. However, we may find the quantum evolution of fluctuations around the classical background field and include parts of the quantum back-reaction of these fluctuations. By dividing the scalar field into the classical background field and the fluctuations...
\[ \phi = \phi_c + \varphi \]  

such that

\[ \langle \hat{\varphi} \rangle = \varphi_c(t), \quad \langle \hat{\dot{\varphi}} \rangle = 0, \]  

and by denoting \( \pi_{\phi_c} = a^3 \dot{\phi_c}, \pi_{\varphi} = a^3 \dot{\varphi} \), we rewrite the Hamiltonian as

\[ H(t) = \int d^3x \left[ \left( \frac{\pi_{\phi_c}^2}{2a^3} + \frac{a}{2} (\nabla \varphi)^2 + a^3 V(\varphi) \right) \right. \]
\[ + \left( \frac{\pi_{\varphi}^2}{2a^3} + \frac{a}{2} (\nabla \varphi)^2 + \frac{a^3}{2!} \frac{\delta^2 V(\varphi_c)}{\delta \varphi_c^2} \varphi^2 \right) \]
\[ + \left( \frac{\pi_{\phi_c} \pi_{\varphi}}{a^3} + a \nabla \varphi \cdot \nabla \varphi + a^3 \frac{\delta^3 V(\varphi_c)}{\delta \varphi_c^3} \varphi^3 \right. \]
\[ + \left. \frac{a^3}{3!} \frac{\delta^3 V(\varphi_c)}{\delta \varphi_c^3} \varphi^3 \right] \]  

(39)

The canonical formalism is difficult to apply to the Hamiltonian (39), as a whole, consisted of the classical background field and the fluctuations coupled each other. In this paper we treat the background field \( \phi_c \) as classical but evolve the fluctuations \( \varphi \) quantum mechanically. It is, however, still difficult to proceed further because we do not yet know the exact quantum states for the sub-Hamiltonian of fluctuations, which is beyond quadratic.

As a constructive way to find the quantum evolution explicitly, we truncate the Hamiltonian for the fluctuations at the quadratic order:

\[ H(0)(\varphi, t) = \int d^3x \left[ \frac{\pi_{\varphi}^2}{2a^3} + \frac{a^3}{2a^2} (\nabla \varphi)^2 + \frac{1}{2!} \frac{\delta^2 V(\varphi_c)}{\delta \varphi_c^2} \varphi^2 \right]. \]  

(40)

Then we evolve the full fluctuations perturbatively. Then, as in the massive case, we decompose the field by modes to get the Hamiltonian as a sum of time-dependent harmonic oscillators now with the frequency squared

\[ \omega^2(t) = \frac{\delta^2 V(\varphi_c)}{\delta \varphi_c^2} + \frac{k^2}{a^2}. \]  

(41)

The quantum back-reaction of the fluctuations to the classical background field can be evaluated by taking the quantum expectation values of fluctuation operators in the Hamiltonian:

\[ H(\phi_c, t) = \int d^3x \left[ \frac{\pi_{\phi_c}^2}{2a^3} + a^3 \left( \frac{1}{2a^2} (\nabla \varphi)^2 + \frac{1}{2!} \frac{\delta^2 V(\varphi_c)}{\delta \varphi_c^2} \varphi^2 \right) \right. \]
\[ + \left. \frac{1}{4!} \frac{\delta^4 V(\varphi_c)}{\delta \varphi_c^4} \langle \varphi^4 \rangle_0 + \cdots \right]. \]  

(42)
In the above equation the expectation value of any odd power of the momentum and field of quantum fluctuations such as \( \langle \hat{\pi}_\phi \hat{\pi}_\phi \rangle, \langle \nabla \hat{\phi}_c \cdot \nabla \hat{\phi} \rangle, \langle \hat{\phi} \rangle, \langle \hat{\phi}^3 \rangle, \cdots \), vanishes for the initial vacuum state and thermal equilibrium state. In these cases we see that

\[
V_{eff}(\phi_c) = V(\phi_c) + \frac{1}{2!} \frac{\delta^2 V(\phi_c)}{\delta \phi_c^2} \langle \hat{\phi}^2 \rangle(0) + \frac{1}{4!} \frac{\delta^4 V(\phi_c)}{\delta \phi_c^4} \langle \hat{\phi}^4 \rangle(0) + \cdots
\]  

is an effective potential. The Hamilton equations for the classical field equal to

\[
\ddot{\phi}_c + 3\frac{\dot{a}}{a} \dot{\phi}_c + \frac{\partial}{\partial \phi_c} V_{eff}(\phi_c) = 0.
\]  

**IV. RENORMALIZATION**

It is to be noted that \( \langle \hat{\phi}^2 \rangle(0), \langle \hat{\phi}^4 \rangle(0), \) etc., contain infinite contributions and require the renormalization of the coupling constants \( \lambda_n \). Following Ref. [3], we do this by introducing the counter-terms \( \delta \lambda_n \) in order to cancel the infinite contributions \( \langle \hat{\phi}^2 \rangle(0), \langle \hat{\phi}^4 \rangle(0), \) and etc.

The renormalized field equation for the classical background field is

\[
\ddot{\phi}_c + 3\frac{\dot{a}}{a} \dot{\phi}_c + \left( \sum_{n=1} \frac{\lambda_n^{(ren)}}{(n-1)!} \phi_c^{n-2} + \sum_{n=2} \frac{\lambda_n^{(ren)}}{(n-2)!} \phi_c^{n-3} \langle \hat{\phi}^2 \rangle(0) \right) 
+ \sum_{n=4} \frac{\lambda_n^{(ren)}}{(n-4)!} \phi_c^{n-5} \langle \hat{\phi}^4 \rangle(0) + \cdots \right) \phi_c = 0,
\]  

where \( \lambda_n^{(ren)} \) are the renormalized coupling constants. From the Appendix we find the expectation value of quantum fluctuations with the initial vacuum state

\[
\langle \hat{\phi}^{2n} \rangle_{(0)}^{(vac)} = \frac{(2n)!}{2^n n!} \frac{1}{(2\pi)^3} \int d^3k (\hat{\phi}_k^* \hat{\phi}_k)^n
\]  

and with the initial thermal equilibrium state

\[
\langle \hat{\phi}^{2n} \rangle_{(0)}^{(therm)} = \frac{(2n)!}{2^n n!} \frac{1}{(2\pi)^3} \int d^3k (\hat{\phi}_k^* \hat{\phi}_k)^n \left( 1 + \frac{2}{e^{\beta \omega_n(t_0)} - 1} \right),
\]  

where each mode of the fluctuations obeys the classical equation

\[
\ddot{\phi}_\alpha + 3\frac{\dot{a}}{a} \dot{\phi}_\alpha + \frac{k^2}{a^2} + \left( \sum_{n=2} \frac{\lambda_n^{(ren)}}{(n-2)!} \phi_c^{n-2} \right) \phi_c^\alpha = 0.
\]
We do explicitly for the potential $V_2$ by introducing the counter-terms $\delta m^2, \delta \xi$ and $\delta \lambda$. The renormalized field equation for the classical background field reads

\[
\ddot{\phi}_c + 3 \frac{\dot{a}}{a} \dot{\phi}_c + \left( m^{2(\text{ren})} + \delta m^2 + (\xi^{(\text{ren})} + \delta \xi) \mathcal{R} + (\lambda^{(\text{ren})} + \delta \lambda) \phi_c^2 \right) \\
+ 3 (\lambda^{(\text{ren})} + \delta \lambda) \langle \dot{\hat{\varphi}}^2 \rangle \langle 0 \rangle \phi_c = 0. \tag{49}
\]

The classical field equations for the fluctuations involve only the renormalized coupling constants

\[
\ddot{\varphi}_\alpha + 3 \frac{\dot{a}}{a} \dot{\varphi}_\alpha + \left( \frac{k^2}{a^2} + m^{2(\text{ren})} + \xi^{(\text{ren})} \mathcal{R} + 3 \lambda^{(\text{ren})} \phi_c^2 \right) \varphi_\alpha = 0. \tag{50}
\]

The Wightman function of quantum fluctuations at tree level can be found from the quantum evolution of $\varphi$ using the truncated Hamiltonian (40) as in Sec. II. Higher order contributions to the Wightman function can be found perturbatively by solving the classical background field equation (45) and by substituting the classical background field into the Hamiltonian (48), and by repeating the procedure.

\section{V. CONCLUSION}

In this paper we presented the Heisenberg-picture approach to the quantum evolution of scalar fields in an expanding FRW universe. The quantum evolution of scalar fields is described by the time-dependent functional Schrödinger equation. As a methodology to find the wave functions (quantum state) of the functional Schrödinger equation we made use of the quantum operators called generalized invariants, which have been used frequently in time-dependent quantum systems. These generalized invariants have been applied mostly in the Schrödinger-picture. Recently the Heisenberg-picture has been used to find the quantum evolution of a time-dependent quantum system [11].

The mode-decomposed Hamiltonian of a massive scalar field in the expanding FRW universe is a collection of time-dependent harmonic oscillators. The generalized invariants that may be used as the creation and annihilation operators have been introduced [8] and
used to find the quantum evolution of the scalar field in the Schrödinger-picture \[9\]. In
the Schrödinger-picture, however, it is rather difficult to incorporate the initial quantum
conditions. It is our observation that it is relatively easy to incorporate an arbitrary ini-
tial quantum condition and that the generalized invariants have a simple interpretation as
constant operators in the Heisenberg-picture \[11\]. We found exactly the quantum evolution
of the massive scalar field incorporating the initial quantum conditions such as the vacuum
state, the thermal equilibrium state and the coherent state in terms of the classical solutions
and calculated the Wightman function, the two-point function, and the correlation function
incorporating the same initial conditions. We also found perturbatively the quantum evolu-
tion of a self-interacting scalar field by dividing the scalar field into a classical background
field and fluctuations and by handling the fluctuations quantum mechanically. We found
the renormalized field equation for the classical background field equation.

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APPENDIX A: EXPECTATION VALUES

In this appendix we evaluate the expectation values of the quantum fluctuations with
respect to the initial vacuum state and the initial thermal equilibrium state. Recollect the
position operator of each mode of the scalar field

\[ \hat{\varphi}_\alpha(t) = \varphi_\alpha(t) \hat{A}_\alpha + \varphi^*_\alpha(t) \hat{A}^\dagger_\alpha, \] (A1)

We put the operator in the normal-ordered form
\[
\varphi_{\alpha}^{2n} = \sum_{k} \frac{(2n)!}{(2k)!} \frac{(2n-2k)!}{2^k} \frac{1}{k!} (\varphi^*_{\alpha}(t) \varphi_{\alpha}(t))^k \\
\times (\varphi_{\alpha}(t) \hat{A}_\alpha + \varphi^*_{\alpha}(t) \hat{A}^\dagger_{\alpha})^{2n-2k}.
\]

(A2)

to get the vacuum expectation value
\[
\langle \varphi_{\alpha}^{2n} \rangle_{(0)}^{\text{(ren)}} = \frac{(2n)!}{2^n n!} (\varphi_{\alpha}(t) \varphi_{\alpha}(t))^n.
\]

(A3)

In order to evaluate the expectation value with respect to the initial thermal equilibrium state, we make use of the following theorem [12]
\[
\langle f(\hat{a}, \hat{a}^\dagger) \rangle^{\text{(therm)}} = (1 - e^{-\omega}) \text{Tr} f(\hat{a}, \hat{a}^\dagger) e^{-\omega \hat{a}^\dagger \hat{a}}
\]
\[
= \langle 0, 0 \rvert f[\sqrt{1+n\hat{a}^\dagger \hat{a}}, \sqrt{1+n\hat{a}^\dagger + \sqrt{n\hat{c}^\dagger \hat{c}}}] \rvert 0, 0 \rangle
\]

(A4)

where
\[
\bar{n} = \frac{1}{e^\omega - 1}
\]

(A5)

and \(\hat{c}\) and \(\hat{c}^\dagger\) are the bosonic operators which commute with \(\hat{a}\) and \(\hat{a}^\dagger\), and \(\langle 0, 0 \rangle\) is the vacuum state for \(\hat{a}\) and \(\hat{c}\). Use the theorem to derive the expectation value
\[
\langle \varphi_{\alpha}^{2n} \rangle^{\text{(therm)}} = \langle 0, 0 \rvert \left[ \sqrt{1+n\varphi_{\alpha}(t) \hat{C}_\alpha + \varphi_{\alpha}(t) \hat{C}_{\alpha}^\dagger} + \sqrt{n}(\varphi_{\alpha}(t) \hat{C}_\alpha + \varphi_{\alpha}(t) \hat{C}_{\alpha}^\dagger) \right]^{2n} \rvert 0, 0 \rangle
\]
\[
= \sum_{k=0}^{n} \frac{(2n)!}{2^n k!} \left( \sqrt{1+n\varphi_{\alpha}(t) \hat{C}_\alpha + \varphi_{\alpha}(t) \hat{C}_{\alpha}^\dagger} \right)^{2k} \left[ \sqrt{n}(\varphi_{\alpha}(t) \hat{C}_\alpha + \varphi_{\alpha}(t) \hat{C}_{\alpha}^\dagger) \right]^{2n-2k} \rvert 0, 0 \rangle
\]
\[
= \sum_{k=0}^{n} \frac{(2n)!}{2^n k!} \frac{(2k)!}{2^{k} k!} (\sqrt{1+n})^{2k} (\varphi_{\alpha}(t) \varphi_{\alpha}(t))^k \frac{(2n-2k)!}{2^{n-k}(n-k)!} (\sqrt{n})^{2n-2k} (\varphi_{\alpha}(t) \varphi_{\alpha}(t))^{n-k}
\]
\[
= \frac{(2n)!}{2^n n!} (\varphi_{\alpha}(t) \varphi_{\alpha}(t))^n \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} (1 + \bar{n})^k \bar{n}^{n-k}
\]
\[
= \frac{(2n)!}{2^n n!} (\varphi_{\alpha}(t) \varphi_{\alpha}(t))^n (1 + 2\bar{n})^n,
\]

(A6)

where \(\hat{C}_\alpha\) and \(\hat{C}_{\alpha}^\dagger\) are the bosonic operators which commute with \(\hat{A}_\alpha\) and \(\hat{A}_{\alpha}^\dagger\).
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