I. INTRODUCTION

One of the exciting subjects in physics & astronomy are relativistic stars (RS). These objects can play a major role in studying the early universe and, by understanding these compact objects, we can investigate the behavior of super-dense matter and determine the equation of state (EoS) associated with it. The number of these compact objects is large enough to affect the evolution of the universe. However, because of the severe conditions in which they function, RSs are natural laboratories for testing powerful gravitational field regimes that can barely be attained in any other area of the Universe. As a result, their interior structure cannot be easily replicated. As an example, our knowledge of today says that we consider neutron stars (NS) as one the smallest densest stars [1]. If we are going to understand these relativistic objects, we need a model that explains the physical behavior in the manner of general relativity (GR).

There are some points that we should take them as a postulate here. One is, we humans as an observer here on earth, don’t feel the gravitational effect of stars. This means the metric of stars at infinity behaves as Minkowski spacetime, better to say, their spacetime should be asymptotically flat. And by considering a suitable action for this problem, we can drive the Einstein field equations and continue by solving those field equations, we can have the energy-momentum tensor on the right-hand side, and at the left-hand side, the geometrical behavior of the star [2]. Another thing we should never forget is the conservation laws $T^\mu_{\;\nu,\nu} = 0$ and, this leads us to have a class of equation in which for the first time has introduced for neutron star by Tolman-Oppenheimer-Volkof (TOV) equations [3][4]. TOVs are showing us the pressure and gravitational pressure, and also mass-radius behavior in Schwarzschild spacetime.

Considering Birkhoff’s theorem [5] even though, we have a tendency to drop our initial assumptions that the metric is static. This implies that even a radially pulsing or collapsing star can have a static exterior metric of constant mass $M$ [6]. One conclusion we can draw from this is often that there are not any gravitational waves from pulsing spherical systems. The initial motivation for finding out modified gravity came from the discovered accelerated growth of the universe confirmed by varied observations [7]. This acceleration takes place at comparatively little distances (“Hubble flow”) and needs (in GR) non-standard cosmic fluid (dark energy) filling the universe with negative pressure however not clustered in a huge scale structure. Still, the character of dark energy is unclear. An alternate approach to dark energy downside comprises extending GR.

Theories of modified gravity are often introduced as real different from GR [8][11]. The study of relativistic stars in modified gravity is fascinating for many reasons and will represent formidable researches for such theories. First, one will reject some models that don’t permit the existence of stable star configurations. Second, there is a chance for the existence of the latest stellar structures within the framework of modified gravity, escaping the quality stellar models. The observation of such self-gravitating abnormal structures may offer robust proof for the Extended Gravity.

For example in some cases of $f(R)$-gravity like squared-gravity, we can express heavier NSs than GR [12]. These objects are often helpful in constraint-free parameters of modified-gravity theories. It increases the variations between different gravity theories within the robust attraction regime. During this regime, owing to the complexity of the fields equation, perturbative ways become an honest option to treat the matter. Numerical integration of the structure equations that describe NSs in $f(R)$-gravity, causes their mass-radius relations. However, it specializes in specific options that arise from this approach within the NS interior. During this analysis, we have to square measure about to see the behavior of stars in $f(R)$ gravity minimally, not to mention nonlinear electrodynamics that these solutions aren’t about to be asymptotically flat. Now, we have to take a step back and take a glance at nonlinear electrodynamics to check how we can describe such associate degrees during this framework.
From the start of the last century, it places many proposals of nonlinear electrodynamics (NED) forth for doing the task to alleviate the singularity of Maxwell’s resolution to the field of some extent charge. Among these ideas, the formulation of Born and Infeld was productive [13]. Non-linear electrodynamics arises once classical electrodynamics has modified owing to quantum corrections. Thus, by assumptive loop corrections in quantum electrodynamics, we can get Heisenberg-Euler electrodynamics [14], which provides thanks to the vacuum refraction development. In the 1930s, Born-Infeld gave an associate degree example of the NEDs model that with success removed the singularities thanks to purpose charges that arise in linear Maxwell theory. Soon this trend of electromagnetism applied to the theory of relativity with the hope that divergences of gravity are often eliminated similarly. Since there are square measure robust arguments in favor of magnetic sorts instead of the electrical, one’s main search has focused around the magnetic black holes. One reason against the electrical black holes is that if the model admits linear Maxwell limit, then no regular electric region exists [15]. It’s well-known that, in each classical and quantum field, Einstein’s theory of gravity is ultraviolet-incomplete (UV-incomplete).

The existence of singularities is that the major downside during this theory, e.g. solutions of Einstein’s equations like Schwarzschild, Reissner-Nordström, and Kerr metric, have curvature singularities at the origin. So one believes that the modification of this theory is workable in those regions where the curvature is extremely high [16–19]. Remember what happens to matter that falls down a black hole: It vanishes into a spacetime singularity; it’s a hint that the classical theory, which doesn’t take quantum theory into account, is incomplete. We can assume a spherically symmetric metric describing a region that is made once a null shell of mass $M$ collapses. It exists for a unit lifetime, so ends thanks to the collapse of another shell having mass - $M$.

Such a region has been named a sandwich region and has been represented by the Hayward metric [20]. The answer exists entirely within the presence of a non-linear magnetic field. In alternative words it’s the limit of standard Hayward region happiness to a modified theory [21] [22]. For a potential physical interpretation of the Hayward spacetime among NED, one could consult as there’s a constraint on the magnetic field $B$ and also the parameter $\beta$. Another advantage of this model over others is that it admits the Schwarzschild limit as a basis, whereas the exponential resolution so as that the unitarity principle holds( $B < \frac{0.66}{\sqrt{3}}$ or $\beta < 10^{-23}$). As an interesting fact, this allows us to have a star with a very strong magnetic field ( $\sim 10^{15}G$) that takes our attention to the class of NSs, the Magnetars [16–19].

In this paper, in Section II, the answer of Einstein equation fields in Hayward metric with NED are studied, and from there, the generalized sort of equations for this spacetime are found. After that, in Section III, the mass–radius relations for magnetically charged stars are analyzed and their result and behavior are compared with the conventional TOV. The conclusions are presented in Section IV.

II. MODIFIED TOV IN HAYWARD METRIC IN THE FRAMEWORK OF NED

The action with Maxwell Lagrangian in 4-dimensions given by

$$S = \int d^4x\sqrt{-g}\left[\frac{R}{2\kappa^2} + \mathcal{L}(F)\right]$$

(1)

where $\mathcal{L}(F)$ is the non-linear electrodynamics Lagrangian (NED) in terms of $F = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} = 0.5(B^2 - E^2)$. As we mention before, $B = \frac{\beta}{\sqrt{3}}$, and $\beta$ is a free parameter. There are some models for describing NED such as exponential form, $\text{arcsin}$ Lagrangian, Logarithmic and etc. [16–19]. Here, $R$ is Ricci scalar and invariant (also $\kappa^2 = 8\pi G$ and we consider $c = 1$). The Maxwell 2-form introduced for a pure magnetic field in form of below,

$$\mathbf{F} = p\sin\theta d\theta \wedge d\phi.$$ (2)

The nonlinear Maxwell equation reads

$$d\left(\ast \mathbf{F} \frac{\partial \mathcal{L}}{\partial \mathbf{F}}\right) = 0.$$ (3)

We define our NED model as form of below

$$\mathcal{L} = -\frac{\mathbf{F}}{1 + \sqrt{\mathbf{F}}}.$$ (4)

For the pure magnetic field Maxwell invariant is $\mathcal{F} = \frac{\mathbf{F}^2}{\mathbf{F}^2}$, where $p$ is a magnetic charge quantity or on the other hand is a sign of magnetic monopole.

For the static spherically symmetric metric of spacetime we are supposing the Hayward one

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{A(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

(5)

where $A(r)$ is the metric function. We can drive the Einstein equation field if we make a variation on our action Eq. (1) with respect to our metric $g_{\mu\nu}$ that can be written as

$$f'(R)R^\mu_\mu + \left(\Box f'(R) - \frac{1}{2}f(R)\right)\delta^\nu_\mu - \nabla^\nu \nabla_\mu f'(R) = \kappa T^\nu_\mu$$

(6)

The non-zero components of energy–momentum tensor can be given as

$$T^\nu_\mu = \mathcal{L}_{\mu}^\nu - F_{\mu\lambda}F^{\rho\lambda}\frac{\partial \mathcal{L}}{\partial F^{\rho\nu}}$$

$$T^r_r = T^t_t = -\frac{1}{4\pi}\mathcal{L}$$

$$T^\theta_\theta = T^\phi_\phi = \frac{1}{4\pi}(\mathcal{L} - 2\mathcal{F}\mathcal{L}_F)$$

(7)
where we can write
\[ \nabla^I \nabla_I f_R = \nabla^r \nabla_r f_R \]
and we have \( R^t_t = R^r_r \) and \( T^t_t = T^r_r \) this implies
\[ A f_{R,r,r} = 0. \]
For \( \theta \) and \( \phi \) components of the Eq. \([6]\) we have
\[ f_R R^\theta_\theta + \left( \Box f_R - \frac{1}{2} f \right) - \nabla^\theta \nabla_\theta f_R = \kappa T^\theta_\theta \]
(12)
in which we can show that
\[ \nabla^\theta \nabla_\theta f_R = \frac{A f_{R,r}}{r} \]
where
\[ R^t_t = R^r_r = - \frac{r A'' + 2 A'}{2r} \]
(15)
and
\[ R^\theta_\theta = - \frac{r A' - 1 - A}{r^2} \]
(16)
This we take us to define our \( f'(R) \) as
\[ f'(R) = 1 + c_t r \]
(17)
with respect to these facts, we can drive Ricci scalar in terms of metric function and by solving this relation the metric function \( A(r) \) in Hayward metric is
\[ A(r) = 1 - \frac{2 M_0 r^2}{r^3 + 2 M_0 l^2} \]
(18)
where \( l \) is representing the Planck length, and Eq. \([18]\) is the solution of the Einstein’s field equations in a specific modified theory of gravity without source i.e., \( \mathcal{L} = 0 \). Herein, \( M_0 \) is the rest mass \([21]\). As we can see in Eq. \([18]\), our metric is asymptotically flat which allows us to use it for describing a model for stars. Having \( \mathcal{L} \neq 0 \) changes the solutions up to a new mass function for \( M \) which can be written as
\[ M = \int r T^t_t dr + M_0 \]
(19)
and \( M_0 = \text{constant} \), is the non-magnetic mass of the black hole. Therefore
\[ M(r) = \frac{p^{3/2}}{2^{3/4} \sqrt{\beta}} \tan^{-1}\left( \frac{2^{1/4} r}{\sqrt{\beta}} \right) + M_0 \]
(20)
now the metric function which takes the new form of
\[ A(r) = 1 - \frac{2 G M(r) r^2}{r^3 + 2 G M(r) l^2} \]
(21)
We can represent our metric function at two limits given by
\[ \lim_{r \to 0} A(r) = 1 - \frac{r^2}{l^2} + \frac{r^5}{2 l^4 G M_0} - \frac{p^2 \sqrt{\beta}}{4 l^4 G M_0 \sqrt{\beta}} r^6 + \mathcal{O}(r^7) \]
and
\[ \lim_{r \to \infty} A(r) = 1 - \frac{2 G \mu}{r} + \frac{G p^2}{r^2} + \frac{G \sqrt{2} (12 \sqrt{2} G l^2 \mu - p^3 \sqrt{\beta})}{6 r^4} \]
(23)
in which \( \mu = M_0 + \frac{r \sqrt{\beta}}{2^{3/4}} \) is the new Schwarzschild mass of the solution. In order to derive the TOV equation we need to consider the conservation law \( T^\mu_{\nu,r} = 0 \) which is concomitant with Bianchi’s identities \([2]\). Now solving the new type of TOV will be taking for but we should say that if \( l \) goes to zero we will have normal TOV. Considering all of the conditions reach us to a new form of the TOV. As it is clear if \( l \to 0 \) we will have again the Schwarzschild case of TOV \([3, 4]\).

\[ \frac{dP}{dr} = -(\epsilon + P) \frac{r}{r^3 + 2 M(r) l^2} \frac{d}{dr} \left( \frac{4 \pi r^6 P - 4 (M(r))^2 l^2 + M(r) r^3}{(r^3 + 2 M(r) l^2) (2 M(r) l^2 - 2 M(r) r^2 + r^3)} \right) \]
(24)

\[ \frac{d m}{d r} = - \frac{r}{4 \pi l^2} \]
(25)
where \( P \) is the pressure, \( \rho \) is density and \( \epsilon (r) \) represents the energy density.

III. ANALYSIS OF THE MAGNETIC STARS IN NED

A. Pure Magnetic field \( (l = 0) \)

In order to study the effects of the NED term, we decide to use two kinds of approaches to find the mass-radius relation for the magnetars stars. The first case is related to the use of a very simple choice for the EoS, namely the polytropic equation of state. The relation of pressure and energy density for this equation is given by
\[ P = K \epsilon^\gamma \]
(CASE 1)
(26)
where \( K = 1.475 \times 10^{-3} \text{[fm}^3/\text{MeV}]^{2/3} \) is polytropic constant, and \( \gamma \) is the polytropic index. Here \( \gamma \) is equal to
In this approach, the energy density and pressure relation, is used as an input to solve the coupled equations Eq. (24), and Eq. (25). These equations have to do with the modified TOV.

Concern to the second case, the dependence on the magnetic field is taking account directly in the equation of state, following the procedure given by Ref. [25]. This means that we can use the Eq. (7) to drive the density of magnetized matter and adding the amount of this density to rest mass density in EoS, and finally using the conventional TOV (with the polytropic EoS in the Schwarzschild metric) to find the mass-radius relation.

We use the below condition to see the NED effect on TOV

\[ \epsilon' = \alpha \epsilon \quad \text{(CASE 2)} \]  

(27)

where \( \epsilon' \) is the charge density, and \( \alpha \) is the charge fraction, see Ref. [25].

In Fig. 1, the behavior of the charge density in function of the magnetic charge \( \beta \) is shown for different values of the magnetic field. Note that for each value of \( B \) we have an \( \alpha \) value associated. The value of the constant \( \alpha \) increases with decreasing \( \beta \). These values are shown in Table I.

If we increase the magnetic field we have an increase in the charge density for the same magnetic charge.

![Energy density of magnetic mass in function of magnetic charge for different values of magnetic field](image)

**FIG. 1.** energy density of magnetic mass in function of magnetic charge for different values of magnetic field. Here, \( \beta \) is in unity of \( \times 10^{-23} \text{T}^{-2} \).

| \( \beta (\times 10^{-23} \text{T}^{-2}) \) | \( \alpha \) | \( M_T (\text{M}_\odot) \) | \( M_0 (\text{M}_\odot) \) | \( R \) (km) |
|---|---|---|---|---|
| 0 | 1.420 | 11.58 |
| 1 | 1.976 | 9.170 |
| 0.1 | 1.975 | 9.154 |

**TABLE I.** Values for the constant \( \beta \), as well as the mass \( (M) \), and radius \( (R) \) for the maximum star for the CASE 1.

**B. relativistic star in Hayward Metric \( (l \neq 0) \)**

At this case, we consider a case that TOV equation modifies with \( l \) parameter which exist in the metric function of the Hayward metric eq. 21. Once more by using the eq. 24 while at this case \( \beta \rightarrow 0 \), we calculate the relation between the mass versus radius of the star with Polytropic EoS eq. 26. Here, the change of M vs R appear at small values of R, so in fig. 3 we zoomed at small distances to see the effects of \( l \) on the star. As we can see here, by increasing the \( l \) we have more values for maximum mass and we may see some instability at high \( l \) values.

At the final situation, we consider a condition which we have NED and \( l \) parameter at the same time. For this case, we try for \( \beta = 10^{-23} \text{T}^{-2} \) and \( l' = 7l \). We can see the comparing two cases in fig4.

**IV. CONCLUSION**

In this research, once we tried to generalize the TOV equations regarding to have a quantum gravity effect and at the next step, we looked after how this new form of TOV can change if we couple our metric with nonlinear electrodynamics(NED) field. We considered the Hayward metric because this framework is a kind of modified theory of gravity that is asymptotically flat and allows us to construct our metrics with NED. We can couple NED with other modified metrics such \( f(R) \) or other common metric but in those case, we don’t have asymptotically flat behavior.

The next approach to this work was to see the NED effect. As we expected, the maximum limit of mass got increased when we have such an effect coupled to our
FIG. 2. Mass versus radius relation for both cases studied in this work. CASE 1 (left panel), and CASE 2 (right panel). In both panels, the blue curve means the conventional TOV ($l \rightarrow 0$), and $\beta$ is in unity of $\times 10^{-23} T^{-2}$.

FIG. 3. Mass versus radius for different values of modified Planck length ($l' = 0, 1, 2, ..., 7$). By increasing the $l'$, the corresponding mass regarded the radius will increasing by decreasing radius. In addition, there is an instability in high values of $l'$.

FIG. 4. Comparing the mass versus the radius for two different cases that we have studied. once, when we modify metric only when $l' \neq 0$ (solid line) and, when both value of $\beta$ and $l'$ are not zero (dotted line).

We have shown that the magnetic field affects both matter and geometry because we add this field into our action. Using the magnetic mass to variate the EoS seems is not a true way to study this case and modifying the TOV gives us more realistic results for NS. The reason is TOV is changing due to geometry and matter spacetime. But the interesting point in this study is we have a pure magnetic charge in our metric function, which means our object is consists of a magnetic monopole. NED effect directly let us reach the higher limits of magnetic field for an NS in orders of ($\sim 10^{15}$ G). This means which this modified metric we can explain a case of highly magnetized NSs which they known as magnetars with a higher limit for the mass [23, 26, 27].

When energy density exceeds the Planck scale, or when the prevailing quantum effect at high density is a heavy strain, necessary to counterbalance weight and reverse collapse, we are searching for quantum gravitational effects as we can see in most of the figures of mass-radius relation when the central energy density increased. Another thing about this research is having a Hayward metric allowing us to define another class of stars, which we call Planck Stars [28].

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