Kasner Asymptotics of Mixmaster Hořava-Witten Cosmology.

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(March 28, 2022)

Abstract

Bianchi type I and type IX (‘Mixmaster’) geometries are investigated within the framework of Hořava-Witten cosmology. We consider the models for which the fifth coordinate is a $S^1/Z_2$ orbifold while the four coordinates are such that the 3-space is homogeneous and has geometry of Bianchi type I or IX while the rest six dimensions have already been compactified on a Calabi-Yau space. In particular, we study Kasner-type solutions of the Bianchi I field equations and discuss Kasner asymptotics of Bianchi IX field equations. We are able to recover the isotropic 3-space solutions found by Lukas et al. Finally, we discuss if such Bianchi IX configuration can result in chaotic behaviour of these Hořava-Witten cosmologies.

PACS number(s): 98.80.Hw, 04.50.+h, 11.25.Mj, 98.80.Cq

Key words: cosmology: superstring, Hořava-Witten; chaos

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String cosmology has attracted a lot of interest recently (for a review see [1]), especially in the context of duality symmetry, which is a striking feature of the underlying string theory and justifies a kinetic-energy-driven inflation, known as pre-big-bang inflation [2]. Pre-big-bang inflation is the result of the admission of the cosmological solutions for bosonic low-energy-effective-action for strings [3]. Bosonic action is the simplest stringy action and in view of duality symmetry this action together with other superstring actions are not necessarily the right description of physics at strong coupling - M-theory. Hořava and Witten [4] proposed that the right candidate for M-theory is strongly coupled limit of $E_8 \times E_8$ heterotic superstring theory compactified on a $S^1/Z_2$ orbifold with $E_8$ gauge fields on each orbifold fixed plane. This means the gauge fields live on 10-dimensional planes while gravity can propagate in the whole 11-dimensional bulk. The idea of having extra dimensions in which only gravity can propagate has been under intensive studies recently and different scenarios (including extra time dimensions [9]) have been considered [5–8].

In this paper we will consider the models in which M-theory is compactified on an orbifold and then reduced to four dimensions using Calabi-Yau manifold [10,11]. Since the size of the orbifold is bigger than the size of the Calabi-Yau space then there was a period in the history of the universe during which the universe was five-dimensional. That means we can consider the cosmological models for which the fifth coordinate is an orbifold while the remaining four coordinates are such that the three-space is homogeneous with Bianchi type I or IX geometry. The main objective is to study Kasner-type solutions of Bianchi type I field equations and Kasner asymptotic states (as a result of Kasner-to-Kasner transitions) of Bianchi type IX field equations. The form of these solutions allows us to find out whether there is a possibility for chaotic behaviour in these Hořava-Witten cosmologies. Similar question was addressed in [12] for pre-big-bang cosmology with the answer that only finite number of chaotic oscillations are possible.

It is well known that the vacuum BIX homogeneous cosmology in general relativity is chaotic [13]. An infinite number of oscillations of the orthogonal scale factors occurs in general on any finite interval of proper time including the singularity at $t = 0$. 
If a minimally coupled, massless scalar field (e.g. the inflaton) is admitted, the situation changes. Only a finite number of spacetime oscillations can occur before the evolution is changed into a state in which all directions shrink monotonically to zero as the curvature singularity is reached and the oscillatory behaviour ceases [14]. This is also the case in 4-dimensional pre-big-bang cosmology where the role of a scalar field is played by the dilaton [12]. On the other hand, 5-dimensional vacuum Einstein solutions of Bianchi type IX do not allow chaos to occur either [15–17]. The point is that the fifth dimension plays effectively the role of a scalar field in scalar field cosmologies and stops chaotic oscillations. In 5-dimensional Hořava-Witten cosmology the situation is in some ways analogous to both of the above cases which gives some new interesting points to be made and this is the task of our paper.

The Hořava-Witten field equations are given by [4,10,11]

\[ R_{\mu}^{\nu} = \nabla_\mu \nabla_\nu \phi + \frac{\alpha_0^2}{6} g_\mu^{\nu} e^{-2 \sqrt{2} \phi} + \sqrt{2} \alpha_0 e^{-\sqrt{2} \phi} \sqrt{\frac{g}{g'}} g_{ij} \left[ g_{i\mu} g_{j\nu} - \frac{1}{2} g_{\sigma\sigma} g_{ij} \right] \left[ \delta(y) - \delta(y - \pi \lambda) \right] \]  
\[ \frac{1}{\sqrt{-g}} \partial_\mu \left( \sqrt{-g} \partial^\mu \phi \right) = -\frac{\sqrt{2}}{3} \alpha_0^2 e^{-2 \sqrt{2} \phi} + 2 \alpha \sqrt{\frac{g}{g'}} e^{-\sqrt{2} \phi} \left[ \delta(y) - \delta(y - \pi \lambda) \right], \]  

where \( \phi = 1/\sqrt{2} \ln V \) and \( V \) is a scalar field measuring the deformation of the Calabi-Yau space, \( g_{\mu\nu} \) is the five dimensional metric tensor while \( g_{ij} \) is the four dimensional metric which denotes the pull-back of the metric on five-dimensional manifold \( M_5 \) onto the orbifold fixed four-dimensional manifolds \( M_4^{(1)} \) and \( M_4^{(2)} \). In (1)-(2) we have neglected the terms which come from the three-form on the Calabi-Yau space. Actually, they will not make any qualitative change in our discussion - this is on the same footing as it was the case in pre-big-bang models [12]. In (1)-(2) \( y \in [-\pi \lambda, \pi \lambda] \) is a coordinate in the orbifold direction and the orbifold fixed planes are at \( y = 0, \pi \lambda \). \( Z_2 \) acts on \( S^1 \) by \( y \rightarrow -y \). The terms involving delta functions arise from the stress energy on the boundary planes.

Following [10] we consider cosmological models of the form

\[ ds_5^2 = -N^2(\tau, y)d\tau^2 + ds_3^2 + d^2(\tau, y)dy^2, \]  

where
\[ ds^2_3 = a^2(\tau, y)(\sigma^1)^2 + b^2(\tau, y)(\sigma^2)^2 + c^2(\tau, y)(\sigma^3)^2, \]

is a homogeneous Bianchi type IX 3-metric and the orthonormal forms \( \sigma^1, \sigma^2, \sigma^3 \) are given by

\[
\sigma^1 = \cos \psi d\theta + \sin \psi \sin \theta d\varphi, \\
\sigma^2 = \sin \psi d\theta - \cos \psi \sin \theta d\varphi, \\
\sigma^3 = d\psi + \cos \theta d\varphi,
\]

and the angular coordinates \( \psi, \theta, \varphi \) span the following ranges,

\[
0 \leq \psi \leq 4\pi, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi \leq 2\pi.
\]

Similarly as in \([10,11]\) we will look for separable solutions of the form

\[
N(\tau, y) = n(\tau)\tilde{a}(y), \\
a(\tau, y) = \alpha(\tau)\tilde{a}(y), \\
b(\tau, y) = \beta(\tau)\tilde{a}(y), \\
c(\tau, y) = \gamma(\tau)\tilde{a}(y), \\
d(\tau, y) = \delta(\tau)\tilde{a}(y), \\
V(\tau, y) = \varepsilon(\tau)\tilde{a}(y),
\]

The nonzero components of the field equations read as (an overdot means a derivative with respect to time \( \tau \) and a prime means a derivative with respect to an orbifold coordinate \( y \))

\[
\frac{\ddot{a}}{a} \left[ -\frac{\dot{a}''}{a} - \frac{\dot{a}'}{a} \left( 3\frac{\dot{a}'}{a} - \frac{\ddot{a}}{a} \right) - \frac{1}{6} \frac{\alpha_0^2 \delta^2 d^2}{\varepsilon V^2} \right] = \frac{\delta^2}{n^2} \left[ \frac{n}{n} \left( \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} \right) - \frac{\ddot{\alpha}}{\alpha} - \frac{\ddot{\beta}}{\beta} - \frac{\ddot{\gamma}}{\gamma} - \frac{\ddot{\delta}}{\delta} - \frac{1}{2} \right],
\]

\[
\frac{\ddot{a}}{a} \left[ -\frac{\dot{d}''}{a} - \frac{\dot{d}'}{a} \left( 3\frac{\dot{d}'}{a} - \frac{\ddot{d}}{a} \right) - \frac{1}{6} \frac{\alpha_0^2 \delta^2 d^2}{\varepsilon V^2} \right] = \frac{\delta^2}{n^2} \left[ \frac{n}{n} \left( \frac{\dot{\beta}}{\beta} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\delta}}{\delta} \right) - \frac{\ddot{\beta}}{\beta} - \frac{\ddot{\gamma}}{\gamma} - \frac{\ddot{\delta}}{\delta} - \frac{1}{2} \right] \left( (\beta^2 - \gamma^2)^2 - \alpha^4 \right),
\]

(11)
\[
\frac{\ddot{a}}{d} \left[ -\frac{\dddot{a}}{a} - \frac{\ddot{a}'}{a} \left( \frac{3\ddot{a}'}{a} - \frac{\dddot{a}'}{a} \right) - \frac{1}{6}\alpha_0^2 \delta^2 \frac{d^2}{V^2} - \sqrt{2\alpha_0^2} \frac{\dot{d}}{V} (\delta(y) - \delta(y - \pi \lambda)) \right] = \frac{\delta^2}{n^2} \left[ \frac{\dot{\beta}}{\beta} \left( \frac{\dot{n}}{n} - \frac{\dot{\alpha}}{\alpha} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\delta}}{\delta} \right) - \frac{\dot{\beta}}{\beta} - \frac{n^2}{2\alpha_0^2 \beta^2 \gamma^2} \left( (\alpha^2 - \gamma^2)^2 - \beta^4 \right) \right], \quad (12)
\]

\[
\frac{\ddot{a}}{d} \left[ -\frac{\dddot{a}}{a} - \frac{\ddot{a}'}{a} \left( \frac{3\ddot{a}'}{a} - \frac{\dddot{a}'}{a} \right) - \frac{1}{6}\alpha_0^2 \delta^2 \frac{d^2}{V^2} - \sqrt{2\alpha_0^2} \frac{\dot{d}}{V} (\delta(y) - \delta(y - \pi \lambda)) \right] = \frac{\delta^2}{n^2} \left[ \frac{\dot{\gamma}}{\gamma} \left( \frac{\dot{n}}{n} - \frac{\dot{\alpha}}{\alpha} - \frac{\dot{\beta}}{\beta} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\delta}}{\delta} \right) - \frac{\dot{\gamma}}{\gamma} - \frac{n^2}{2\alpha_0^2 \beta^2 \gamma^2} \left( (\alpha^2 - \beta^2)^2 - \gamma^4 \right) \right], \quad (13)
\]

\[
\frac{\ddot{a}}{d} \left[ 4 \left( \frac{\ddot{a}'}{\ddot{a}'} - \frac{\dddot{a}'}{\dddot{a}'} \right) - \frac{1}{6}\alpha_0^2 \delta^2 \frac{d^2}{V^2} - \frac{1}{2\gamma} \frac{\dot{V}^2}{V} \right] = \frac{\delta^2}{n^2} \left[ \frac{\dot{\delta}}{\delta} \left( \frac{\dot{n}}{n} - \frac{\dot{\alpha}}{\alpha} - \frac{\dot{\beta}}{\beta} - \frac{\dot{\gamma}}{\gamma} - \frac{\dot{\delta}}{\delta} \right) \right]. \quad (14)
\]

The equation of motion (2) for the scalar field \( V \) is

\[
\frac{\ddot{a}}{d} \left[ 4 \left( \frac{\ddot{V}}{\ddot{a}} \frac{\dot{V}}{\ddot{V}} - \frac{\dddot{V}}{\dddot{a}} \frac{\dot{V}}{\dddot{V}} - \frac{\dddot{V}}{\dddot{a}} \frac{\dot{V}}{\dddot{V}} \right) = \frac{\delta^2}{n^2} \left[ \frac{\dot{\epsilon}}{\epsilon} \left( \frac{\dot{\alpha}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} \right) + \frac{\ddot{\epsilon}}{\epsilon} - \frac{\epsilon^2}{\epsilon^2} \right]. \quad (15)
\]

In order to separate Equations (10)-(15) one has to make a choice \( \delta = \epsilon \) and one additionally choose a gauge in the form \( n = 1 \) as in Ref. [14]. One gets the following set of time-dependent field equations (note that equations (14) and (15) become identical so we reduce the number of equations to five [1])

\[
\frac{\ddot{a}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} = -\frac{1}{2} \frac{\delta^2}{a^2}, \quad (16)
\]

\[
\frac{\ddot{a}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} = \frac{1}{2\alpha_0^2 \beta^2 \gamma^2} \left[ (\beta^2 - \gamma^2)^2 - a^4 \right], \quad (17)
\]

\[
\frac{\ddot{a}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} = \frac{1}{2\alpha_0^2 \beta^2 \gamma^2} \left[ (\alpha^2 - \gamma^2)^2 - \beta^4 \right], \quad (18)
\]

\[
\frac{\ddot{a}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} = \frac{1}{2\alpha_0^2 \beta^2 \gamma^2} \left[ (\alpha^2 - \beta^2)^2 - \gamma^4 \right], \quad (19)
\]

\[
\frac{\ddot{a}}{\alpha} + \frac{\dot{\beta}}{\beta} + \frac{\dot{\gamma}}{\gamma} + \frac{\dot{\delta}}{\delta} = 0, \quad (20)
\]

which, except for the right-hand side of Eq.(16), is the same set as the set of equations (3.7)-(3.11) of [12] for bosonic low-energy-effective-action cosmology in string frame, provided we

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\(^1\)This, of course, reduces the generality of our discussion in the context of [14] since we identify an extra dimension with a scalar field.
take dilaton field $\phi$ as defined in [12] to be equal to $-(\ln \delta)$ and also neglect axion (i.e. take $A = 0$ in Ref. [12]).

A new time coordinate is introduced to simplify the field equations by (compare Eq. (3.12) of [12])

$$d\eta = \frac{d\tau}{\alpha \beta \gamma \delta}.$$  \hspace{1cm} (21)

From now on we will use the notation $(...)_{,\eta} = d/d\eta$. To further simplify the equations we additionally define

$$\tilde{\alpha} = \ln \alpha \hspace{1cm} \tilde{\beta} = \ln \beta \hspace{1cm} \tilde{\gamma} = \ln \gamma \hspace{1cm} \tilde{\delta} = \ln \delta,$$  \hspace{1cm} (22)

so that the set of equations (16)-(20) reads as

$$\left(\tilde{\alpha} + \tilde{\beta} + \tilde{\gamma} + \tilde{\delta}\right)_{,\eta} + \frac{1}{2}\tilde{\delta}_{,\eta} = 2\left(\tilde{\alpha}_{,\eta} \tilde{\beta}_{,n} + \tilde{\alpha}_{,n} \tilde{\gamma}_{,n} + \tilde{\beta}_{,n} \tilde{\gamma}_{,n}\right) + 2\left(\tilde{\alpha}_{,n} + \tilde{\beta}_{,n} + \tilde{\gamma}_{,n}\right) \tilde{\delta}_{,\eta},$$  \hspace{1cm} (23)

$$2\tilde{\alpha}_{,\eta\eta} = \left[\left(\beta^2 - \gamma^2\right)^2 - \alpha^4\right] \delta^2,$$  \hspace{1cm} (24)

$$2\tilde{\beta}_{,\eta\eta} = \left[\left(\alpha^2 - \gamma^2\right)^2 - \beta^4\right] \delta^2,$$  \hspace{1cm} (25)

$$2\tilde{\gamma}_{,\eta\eta} = \left[\left(\beta^2 - \gamma^2\right)^2 - \alpha^4\right] \delta^2,$$  \hspace{1cm} (26)

$$\tilde{\delta}_{,\eta\eta} = 0.$$  \hspace{1cm} (27)

These equations are the same as pre-big-bang cosmology Mixmaster equations in string frame (3.19)-(3.22) of Ref. [12] if we take $\tilde{\delta} = -\phi = -M \eta + \text{const}$ or as 5-dimensional vacuum Mixmaster equations (31)-(32) of Ref. [17].

Now we consider suitable initial conditions expressed in terms of the Kasner parameters and discuss the general behaviour of Bianchi type IX Hořava-Witten cosmology on the approach to singularity.

The Kasner solutions are obtained as approximate solutions of the equations (16)-(20) when the right-hand sides (describing the curvature anisotropies) are neglected. In terms of $\tau$-time, they are

$$\alpha = \alpha_0 \tau^{p_1},$$
\[ \beta = \beta_0 \tau^{p_2}, \]  
\[ \gamma = \gamma_0 \tau^{p_3}, \]  
\[ \delta = \delta_0 \tau^{p_4}, \]  
while

\[ \tilde{\delta} = -\ln \delta_0 - p_4 \ln \tau. \]  

From (23)-(27) we have the following algebraic conditions for the Kasner indices, \( p_i \):

\[ p_1 + p_2 + p_3 + p_4 = 1, \]  
and

\[ p_1^2 + p_2^2 + p_3^2 + \frac{3}{2} p_4^2 = 1. \]  

This, in particular, proves that the isotropic Friedmann case as obtained by Lukas et al. [10] is given by

\[ p_1 = p_2 = p_3 = p_+ = \frac{3}{11} \pm \frac{4}{11 \sqrt{3}}, \]  
\[ p_4 = q_\pm = \frac{2}{11} \pm \frac{4 \sqrt{3}}{11}. \]  

The meaning of such isotropic solutions was discussed in Ref. [10] (note that numerically \( p_+ = 0.48, p_- = 0.06 \) and \( q_- = -0.45, q_+ = 0.81 \)). In fact, there are two branches each one for negative and positive values of time coordinate \( \tau \) (negative values of time can be achieved by taking \(-\tau\) instead of \(+\tau\) in (28), or, simply by taking the modulus). If \( \tau < 0 \) one has \((-\) branch and if \( \tau > 0 \) one has \((+) \) branch. For \((-\) branch both the worldvolume (of 3-dimensional space) and the orbifold contract for \( p_- \) and \( q_+ \) while the worldvolume contracts and the orbifold expands (superinflationary) for \( p_+ \) and \( q_- \). For \((+) \) branch the worldvolume and the orbifold expand for \( p_- \) and \( q_+ \) while the wordvolume expands and the orbifold contracts for \( p_+ \) and \( q_- \).

Notice that the conditions (30)-(31) are different from the conditions which emerge in pre-big-bang cosmology where the role of the fifth coordinate is played by the dilaton.
(see Eqs.(3.50)-(3.51) of [12]). They are also different from Mixmaster Kaluza-Klein five-dimensional models where the homogeneity group acts on four-dimensional hypersurfaces of constant time (see Eq.(33) of [17]). The reason for that is simply the fact the fifth coordinate in Hořava-Witten cosmology is an orbifold. The isotropization of the models under consideration means that the Kasner indices reach the values (32)-(33). Finally, the isotropization of 5-dimensional Kaluza-Klein models in supergravity as first considered by Chodos and Detweiler [18] would require the different values of the Kasner indices namely $p_1 = p_2 = p_3 = -p_4 = 1/2$ which fulfill the conditions (30) and (31) without a factor $3/2$ in front of $p_4$ in (31).

Having given the conditions (30)-(31), one can express the indices $p_2$ and $p_3$ by using $p_1$ and $p_4$, i.e.,

$$p_2 = \frac{1}{2} \left[ (1 - p_1 - p_4) - \sqrt{-3p_1^2 + 2p_1 (1 - p_4) + 1 + 2p_4 (1 - 2p_4)} \right],$$

$$p_3 = \frac{1}{2} \left[ (1 - p_1 - p_4) + \sqrt{-3p_1^2 + 2p_1 (1 - p_4) + 1 + 2p_4 (1 - 2p_4)} \right].$$

(34)

Since the expression under the square root in (34) should be nonnegative, one can extract the restriction on the permissible values of $p_4$ which is

$$q_- \leq p_4 \leq q_+.$$  

(35)

Some particular choices are of interest. If one takes $p_4 = 0$ one recovers vacuum general relativity limit with Kasner indices $-1/3 \leq p_1 \leq 0, 0 \leq p_2 \leq 2/3, 2/3 \leq p_3 \leq 1$. The range dividing case is for $p_4 = 2/11$ with the following ordering of the Kasner indices

$$\frac{3}{11} - \frac{4}{11\sqrt{3}} \sqrt{11} \leq p_1 \leq \frac{3}{11} - \frac{2}{11\sqrt{3}} \sqrt{11},$$

$$\frac{3}{11} - \frac{2}{11\sqrt{3}} \sqrt{11} \leq p_2 \leq \frac{3}{11} + \frac{2}{11\sqrt{3}} \sqrt{11},$$

$$\frac{3}{11} + \frac{2}{11\sqrt{3}} \sqrt{11} \leq p_3 \leq \frac{3}{11} + \frac{4}{11\sqrt{3}} \sqrt{11}.$$  

(36)

However, we are interested in knowing whether the curvature terms on the right-hand side of the field equations (24)-(26) really increase as $\eta \to -\infty$ ($\tau \to 0$ - approach to singularity for (+) branch) since from (21) and (28) we get
\[ \eta = \eta_0 + \ln \tau, \quad (37) \]

and \( \eta_0 = \text{const.} \) This would require either \( \alpha^4 \delta^2, \beta^4 \delta^2, \) or \( \gamma^4 \delta^2 \) to increase if the transition to another Kasner epoch is to occur \[17,16\]. Since

\[ \alpha^4 \delta^2 \propto \tau^{(2p_1 + p_4)} = \tau^{(1 + p_1 - p_2 - p_3)}, \]

\[ \beta^4 \delta^2 \propto \tau^{(2p_2 + p_4)} = \tau^{(1 + p_2 - p_3 - p_1)}, \quad (38) \]

\[ \gamma^4 \delta^2 \propto \tau^{(2p_3 + p_4)} = \tau^{(1 + p_3 - p_1 - p_2)}, \]

we need one of the following three conditions to be fulfilled

\[ 2p_1 + p_4 = 1 + p_1 - p_2 - p_3 < 0, \]

\[ 2p_2 + p_4 = 1 + p_2 - p_3 - p_1 < 0, \quad (39) \]

\[ 2p_3 + p_4 = 1 + p_3 - p_1 - p_2 < 0. \]

From Kasner conditions (30)-(31) we are free to choose only two parameters so that we can write

\[ p_4 = 1 - p_1 - p_2 - p_3, \]

\[ p_3 = \frac{3}{5} (1 - p_1 - p_2) \pm \frac{2}{5} \left[ -4 \left( p_1^2 + p_2^2 \right) - 3p_1p_2 + 3p_1 + 3p_2 + 1 \right]^\frac{1}{2}, \quad (40) \]

which gives also the condition for \( p_3 \) to be real as follows

\[ 1 - 4 \left( p_1^2 + p_2^2 \right) - 3p_1p_2 + 3p_1 + 3p_2 \geq 0. \quad (41) \]

In (40) we take the + sign for \( p_4 < 2/11 \) and the − sign for \( p_4 > 2/11 \). The conditions (39) become (compare \[?\])

\[ p_1^2 + 4p_2^2 - p_1p_2 + p_2 - p_1 < 0, \]

\[ 4p_1^2 + p_2^2 - p_1p_2 + p_1 - p_2 < 0, \quad (42) \]

\[ 4 \left( p_1^2 + p_2^2 \right) + 7p_1p_2 - 7 (p_1 + p_2) + 3 > 0. \]

One should remind here that for (-) branch we have to take \((-\tau)\) in (28), (29) and (38) which leads to the same conditions (42).
The plot of the conditions (41)-(42) is given in Fig.1. The chaotic oscillations (Kasner-to-Kasner transitions) can start in any region except the narrow range surrounding the isotropic points $p_1 = p_2 = p_+(q = q_-)$ and $p_1 = p_2 = p_-(q = q_+)$. However, such chaotic oscillations would continue indefinitely provided there were no such regions at all (this is the case of vacuum general relativity, for example). Here, once the Kasner parameters fall into the region surrounding the isotropic Friedmann $p_+$ or $p_-$ solutions of Lukas et al. [10] the chaotic oscillations cease so that there is no chaos in such Hořava-Witten cosmologies.

Acknowledgments

This work was supported by the Polish Research Committee (KBN) grant No 2 PO3B 105 16.
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FIG. 1. The range of Kasner indices $p_1$ and $p_2$ which fulfill the conditions (41)-(42). The appearance of the isotropic Friedmann cases at $p_1 = p_2 = p_-$ and $p_1 = p_2 = p_+$ prevents chaotic oscillations in the shaded region that surrounds them. For the values of Kasner indices $p_1$ and $p_2$ from that region chaotic oscillations are not possible to begin. On the other hand, even if after some number of oscillations from one Kasner epoch to the other, the values of $p_1$ and $p_2$ will fall into that region, the chaotic oscillations of the scale factors stop which reflects nonchaotic behaviour of such Bianchi IX Hořava-Witten cosmologies.