THE PBW FILTRATION AND CONVEX POLYTOPES IN TYPE $B$

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ABSTRACT. We study the PBW filtration on irreducible finite-dimensional representations for the Lie algebra of type $B_n$. We prove in several cases, including all multiples of the adjoint representation and all irreducible finite-dimensional representations for $B_3$, that there exists a normal polytope such that the lattice points of this polytope parametrize a basis of the corresponding associated graded space. As a consequence we obtain several classes of examples for favourable modules and graded combinatorial character formulas.

1. Introduction

Let $\mathfrak{g}$ be a complex finite-dimensional simple Lie algebra. The PBW filtration on finite-dimensional irreducible representations of $\mathfrak{g}$ was introduced in [13] and a description of the associated graded space in terms of generators and relations has been given in type $A_n$ and $C_n$ (see [13, 14]). As a beautiful consequence the authors obtained a new class of bases parametrized by the lattice points of normal polytopes, which we call the FFL polytopes. A new class of bases for type $G_2$ is established in [16] by using different arguments.

It turned out that the PBW theory has a lot of connections to many areas of representation theory. For example, in the branch of combinatorial representation theory the FFL polytopes can be used to provide models for Kirillov–Reshetikhin crystals (see [20, 19]). Further, a purely combinatorial research shows that there exists an explicit bijection between FFL polytopes and the well-known (generalized) Gelfand–Tsetlin polytopes (see [1, Theorem 1.3]). Although Berenstein and Zelevinsky defined the $B_n$–analogue of Gelfand–Tsetlin polytopes in [4] it is much more complicated to define the $B_n$–analogue of FFL polytopes (see [1, Section 4]). One of the motivations of the present paper is to better understand (the difficulties of) the PBW filtration in this type.

In the branch of geometric representation theory the PBW filtration can be used to study flat degenerations of generalized flag varieties. The degenerate flag variety of type $A_n$ and $C_n$ respectively can be realized inside a product of Grassmanians (see [8, Theorem 2.5] and [11, Theorem 1.1]) and furthermore the degenerate flag variety is isomorphic to an appropriate Schubert variety (see [17, Theorem 1.1]). Another powerful tool of studying these varieties are favourable modules, where the properties of a favourable module are governed by the combinatorics of an associated normal polytope (see for details [12] or Section 6). It has been proved in [12] that the degenerate flag varieties associated to favourable modules have nice properties. For example, they are normal and Cohen–Macaulay and, moreover, the underlying polytope can be interpreted as the Newton–Okounkov body for the flag variety. In the same paper several classes of examples for favourable modules of type $A_n$, $C_n$ and $G_2$ respectively are provided; more classes of examples were constructed in [2, 5, 15].

Beyond these cases very little is known about the PBW filtration and whether there exists a normal polytope parametrizing a PBW basis of the associated graded space. This paper is motivated by proving the existence of such polytopes for several classes of representations of type $B_n$ and to

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construct favourable modules (see Section 6). Moreover, we use the results of [16] to describe the associated graded space for type $B_2$ in terms of generators and relations (see Section 7). Our main results are the following; we refer to Section 4 and Section 6 for the precise definition of the ingredients.

**Theorem.** Let $\mathfrak{g}$ be the Lie algebra of type $B_n$ and $\lambda = m\omega_i$ be a rectangular highest weight. There is a convex polytope $P(\lambda)$ such that the lattice points $S(\lambda)$ parametrize a generating set of $V(\lambda)$ and $\text{gr} \ V(\lambda)$ respectively. Further, if $1 \leq i \leq 3$ ($n$ arbitrary) or $1 \leq n \leq 4$ ($i$ arbitrary) we have:

1. The lattice points $S(\lambda)$ parametrize a basis of $V(\lambda)$ and $\text{gr} \ V(\lambda)$ respectively. In particular, 
   \[ \{ X^s v_\lambda \mid s \in S(\lambda) \} \]
   forms a basis of $\text{gr} \ V(\lambda)$.
2. We have $\text{gr} \ V(\lambda) \cong S(\mathfrak{n}^-)/I_{\lambda}$, where 
   \[ I_{\lambda} = S(\mathfrak{n}^-)(\mathbf{U}(\mathfrak{n}^+) \circ \text{span} \{ x^{\lambda(\beta') + 1} \mid \beta \in R^+ \}) \).
3. The character and graded $q$-character respectively is given by 
   \[ \text{ch} \ V(\lambda) = \sum_{\mu \in \mathfrak{h}^*} |S(\lambda)\mu| e^\mu, \quad \text{ch}_q \text{gr} \ V(\lambda) = \sum_{s \in S(\lambda)} e^{\lambda - \text{wt}(s)} q^{\sum s_\beta}. \]
4. We have an isomorphism of $S(\mathfrak{n}^-)$–modules for all $\ell \in \mathbb{Z}_+$:
   \[ \text{gr} \ V(\lambda + \epsilon_i \ell \omega_i) \cong S(\mathfrak{n}^-)(\mathfrak{v}_\lambda \otimes \mathfrak{v}_{\epsilon_i \ell \omega_i}) \subseteq \text{gr} \ V(\lambda) \otimes \text{gr} \ V(\epsilon_i \ell \omega_i), \]
   where $\epsilon_i = 1$ if $i \leq 2$ and $\epsilon_i = 2$ else.
5. The module $V(\epsilon_i \lambda)$ is favourable.

We conjecture that the above theorem remains true in general (see Conjecture 4.3) and we verified the cases $n \leq 8$ and $m \leq 9$ with a computer program. If the Lie algebra is of type $B_3$ we associate to any dominant integral weight $\lambda$ a normal polytope and prove that a basis of $\text{gr} \ V(\lambda)$ can be parametrized by the lattice points of this polytope. In particular, our results are the following (see Theorem 5.2).

**Theorem.** Let $\mathfrak{g}$ be the Lie algebra of type $B_3$. There is a normal polytope $P(\lambda)$ with the following properties:

1. The lattice points $S(\lambda)$ parametrize a basis of $V(\lambda)$ and $\text{gr} \ V(\lambda)$ respectively. In particular, 
   \[ \{ X^s v_\lambda \mid s \in S(\lambda) \} \]
   forms a basis of $\text{gr} \ V(\lambda)$.
2. The character and graded $q$-character respectively is given by 
   \[ \text{ch} \ V(\lambda) = \sum_{\mu \in \mathfrak{h}^*} |S(\lambda)\mu| e^\mu, \quad \text{ch}_q \text{gr} \ V(\lambda) = \sum_{s \in S(\lambda)} e^{\lambda - \text{wt}(s)} q^{\sum s_\beta}. \]
3. We have an isomorphism of $S(\mathfrak{n}^-)$–modules 
   \[ \text{gr} \ V(\lambda + \mu) \cong S(\mathfrak{n}^-)(\mathfrak{v}_\lambda \otimes \mathfrak{v}_\mu) \subseteq \text{gr} \ V(\lambda) \otimes \text{gr} \ V(\mu). \]
4. The module $V(\lambda)$ is favourable.

Our paper is organized as follows: In Section 2 we give the main notations. In Section 3 we present the PBW filtration and establish the elementary results needed in the rest of the paper. In Section 4 we introduce the notion of Dyck paths for the special odd orthogonal Lie algebra and prove in various cases a presentation for the associated graded space. In Section 5 we associate
to any dominant integral weight for $B_3$ a normal polytope parametrizing a basis of the associated graded space. In Section 6 we give classes of examples for favourable modules.

2. Preliminaries

We denote the set of complex numbers by $\mathbb{C}$ and, respectively, the set of integers, non–negative integers, and positive integers by $\mathbb{Z}$, $\mathbb{Z}_+$, and $\mathbb{N}$. Unless otherwise stated, all the vector spaces considered in this paper are $\mathbb{C}$-vector spaces and $\otimes$ stands for $\otimes_{\mathbb{C}}$.

2.1. We refer to [7, 18] for the general theory of Lie algebras. We denote by $g$ a complex finite-dimensional simple Lie algebra. We fix a Cartan subalgebra $h$ of $g$ and denote by $R$ the set of roots of $g$ with respect to $h$. For $\alpha \in R$ we denote by $\alpha^\vee$ its coroot. We fix $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a basis for $R$; the corresponding sets of positive and negative roots are denoted as usual by $R^\pm$. For $1 \leq i \leq n$, define $\omega_i \in h^*$ by $\omega_i(\alpha_j^\vee) = \delta_{i,j}$, for $1 \leq j \leq n$, where $\delta_{i,j}$ is the Kronecker’s delta symbol. The element $\omega_i$ is the fundamental weight of $g$ corresponding to the coroot $\alpha_i^\vee$. Let $Q = \oplus_{i=1}^n \mathbb{Z} \alpha_i$ be the root lattice of $R$ and $Q^+ = \oplus_{i=1}^n \mathbb{Z}_+ \alpha_i$ be the respective $\mathbb{Z}_+$-cone. The weight lattice of $R$ is denoted by $P$ and the cone of dominant weights is denoted by $P^+$. Let $\mathbb{Z}[P]$ be the integral group ring of $P$ with basis $e^\mu$, $\mu \in P$. Let $W$ be the Weyl group of $g$.

2.2. Given $\alpha \in R^+$ let $g_{\pm \alpha}$ be the corresponding root space and fix a generator $x_{\pm \alpha} \in g_{\pm \alpha}$. We define several subalgebras of $g$ that will be needed later. Let $b$ be the Borel subalgebra corresponding to $R^+$, and let $n^+$ be its nilpotent radical,

$$b = h \oplus n^+, \quad n^+ = \bigoplus_{\alpha \in R^+} g_{\pm \alpha}.$$

The Lie algebra $g$ has a triangular decomposition

$$g = n^- \oplus h \oplus n^+.$$

For a subset $\Delta - \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$ of $\Delta$ we denote by $p_{i_1, \ldots, i_s}$ the corresponding parabolic subalgebra of $g$, i.e. the Lie algebra generated by $b$ and all root spaces $g_{-\alpha}$, $\alpha \in \Delta - \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$. The maximal parabolic subalgebras correspond to subsets of the form $\Delta - \{\alpha_i\}$, $1 \leq i \leq n$. The Lie algebra $g$ contains the parabolic subalgebra as a direct factor and therefore

$$g = p_{i_1, \ldots, i_s} \oplus n^-_{i_1, \ldots, i_s}.$$

We can split off $p_{i_1, \ldots, i_s}$ and consider the nilpotent vector space complement with root space decomposition

$$n^-_{i_1, \ldots, i_s} = \bigoplus_{\alpha \in R^+_{i_1, \ldots, i_s}} g_{-\alpha}.$$

For instance, if $g$ is of type $A_n$ we have $R^+ = \{\alpha_{r,s} \mid 1 \leq r \leq s \leq n\}$ and $R^+_{i} = \{\alpha_{r,s} \in R^+ \mid r \leq i \leq s\}$ where $\alpha_{r,s} = \sum_{j=r}^{s} \alpha_j$. In the following we shall be interested in maximal parabolic subalgebras.

3. PBW filtration and graded spaces

We start by recalling some standard notation and results on the representation theory of $g$.
3.1. A $\mathfrak{g}$–module $V$ is said to be a weight module if it is $\mathfrak{h}$–semisimple, 
\[ V = \bigoplus_{\mu \in \mathfrak{h}^*} V^\mu, \quad V^\mu = \{ v \in V \mid hv = \mu(h)v, \ h \in \mathfrak{h} \}. \]

Set $\text{wt} V = \{ \mu \in \mathfrak{h}^* : V^\mu \neq 0 \}$. Given $\lambda \in P^+$, let $V(\lambda)$ be the irreducible finite–dimensional $\mathfrak{g}$–module generated by an element $v_\lambda$ with defining relations:
\[ n^+ v_\lambda = 0, \quad hv_\lambda = \lambda(h) v_\lambda, \quad x_{-\alpha}^{(\lambda(\alpha)+1)} v_\lambda = 0, \quad (3.1) \]
for all $h \in \mathfrak{h}$ and $\alpha \in R^+$. We have $\text{wt} V(\lambda) \subset \lambda - Q^+$ and $\text{wt} V(\lambda)$ is a $W$–invariant subset of $\mathfrak{h}^*$. If $\dim V^\mu < \infty$ for all $\mu \in \text{wt} V$, then we define $\text{ch} V : \mathfrak{h}^* \to \mathbb{Z}_+$, by sending $\mu \mapsto \dim V^\mu$. If $V$ is a finite set, then
\[ \text{ch} V = \sum_{\mu \in \mathfrak{h}^*} \dim V^\mu e^\mu \in \mathbb{Z}[P]. \]

3.2. A $\mathbb{Z}_+$–filtration of a vector space $V$ is a collection of subspaces $\mathbf{F} = \{ V_s \}_{s \in \mathbb{Z}_+}$, such that $V_{s-1} \subseteq V_s$ for all $s \geq 1$. We build the associated graded space with respect to the filtration $\mathbf{F}$
\[ \text{gr}^\mathbf{F} V = \bigoplus_{s \in \mathbb{Z}_+} V_s/V_{s-1}, \text{where } V_{-1} = 0. \]

In this paper we shall be interested in the PBW filtration of the irreducible module $V(\lambda)$ which we will explain now. Consider the increasing degree filtration on the universal enveloping algebra $\mathcal{U}(n^-)$:
\[ \mathcal{U}(n^-)_s = \text{span}\{ x_1 \cdots x_l \mid x_j \in n^-, l \leq s \}, \]
for example, $\mathcal{U}(n^-)_0 = \mathbb{C}$. The induced increasing filtration $V = \{ V(\lambda)_s \}_{s \in \mathbb{Z}_+}$ on $V(\lambda)$ where $V(\lambda)_s := \mathcal{U}(n^-)_s v_\lambda$ is called the PBW filtration. With respect to the PBW filtration we build the associated graded space $\text{gr} V(\lambda)$ as above. To keep the notation as simple as possible, we will write $\text{gr} V(\lambda)$ to refer to $\text{gr} V(\lambda)$. The graded $q$–character is defined as
\[ \text{ch}_q \text{gr} V(\lambda) = \sum_{\mu \in \mathfrak{h}^*} \left( \sum_{s \geq 0} (\dim V(\lambda)_s^\mu / V(\lambda)^\mu_{s-1}) q^s \right) e^\mu, \text{ where } \text{gr} V(\lambda)^\mu = \bigoplus_{s \in \mathbb{Z}_+} V(\lambda)_s^\mu / V(\lambda)^\mu_{s-1}. \]

The following is immediate:

**Lemma.** The action of $\mathcal{U}(n^-)$ on $V(\lambda)$ induces a structure of a $S(n^-)$ module on $\text{gr} V(\lambda)$. Moreover,
\[ \text{gr} V(\lambda) = S(n^-) v_\lambda \cong S(n^-)/I_\lambda, \]
for some graded Ideal $I_\lambda$. The action of $\mathcal{U}(n^+)$ on $V(\lambda)$ induces a structure of a $\mathcal{U}(n^+)$ module on $\text{gr} V(\lambda)$.

By the previous lemma, the representation $\text{gr} V(\lambda)$ is cyclic as a $S(n^-)$–module. By the PBW theorem and the defining relations (3.1) of $V(\lambda)$ we obtain the following proposition.

**Proposition.** The set
\[ \left\{ \prod_{\beta \in R^+} x_{-\beta}^{m_\beta} v_\lambda \mid m_\beta \in \mathbb{Z}_+, m_\beta \leq \lambda(\beta^\vee) \right\} \]
is a (finite) spanning set of $\text{gr} V(\lambda)$.

For a multi–exponent $s = (s_\beta)_{\beta \in R^+} \in \mathbb{Z}_{+}^{\lvert R^+ \rvert}$ (resp. $s = (s_\beta)_{\beta \in R^+} \in \mathbb{Z}_{+}^{\lvert R^+ \rvert}$) we denote the corresponding monomial $\prod_{\beta \in R^+} x_{-\beta}^{s_\beta}$ (resp. $\prod_{\beta \in R^+} x_{-\beta}^{s_\beta}$) for simplicity by $X^s \in S(n^-)$.

In recent years it became a popular goal to determine the $S(n^-)$–structure of the representations $\text{gr} V(\lambda)$, i.e. to describe the (graded) ideals $I_\lambda$ and furthermore to find a PBW basis for these graded
representations, favourably parametrized by the integral points of a suitable convex polytope. For the finite–dimensional Lie algebras of type \(A_n, C_n\) and \(G_2\) various results are known which we will discuss later (see [13, 14, 16]). The focus of this paper is on the Lie algebra of type \(B_n\) where many technical difficulties show up.

3.3. Let \(D \subseteq P(R^+)\) be a subset of the power set of \(R^+\). We attach to each element \(p \in D\) a non–negative integer \(M_p(\lambda)\). We consider the following polytope

\[
P(D, \lambda) = \left\{ s = (s_\beta)_{\beta \in R^+} \in \mathbb{R}^{|R^+|} \mid \forall p \in D : \sum_{\beta \in p} s_\beta \leq M_p(\lambda) \right\}. 
\]

(3.2)

The integral points of the above polytope are denoted by \(S(D, \lambda)\). The proof of part (i) of the following theorem for type \(A_n\) can be found in [13], for type \(C_n\) in [14] and for type \(G_2\) in [16]. Part (ii) is only proved for type \(A_n\) and \(C_n\), but a simple calculation shows that part (ii) for type \(G_2\) remains true (for a proof see Proposition 7.1 in the Appendix).

**Theorem.** There exists a set \(D \subseteq P(R^+)\) and suitable non–negative integers \(M_p(\lambda)\) attached to each element \(p \in D\), such that the following holds:

(i) The lattice points \(S(D, \lambda)\) parametrize a basis of \(V(\lambda)\) and \(\text{gr} \ V(\lambda)\) respectively. In particular,

\[
\{ X^s v_\lambda \mid s \in S(D, \lambda) \}
\]

forms a basis of \(\text{gr} \ V(\lambda)\).

(ii) We have

\[
I_\lambda = S(n^-)(U(n^+) \circ \text{span}\{ x_{-\beta}^{\lambda(\beta^\vee)+1} \mid \beta \in R^+ \}).
\]

We note that the order in the theorem above is important when treating the representation \(V(\lambda)\), but we can choose for any \(s \in S(D, \lambda)\) an arbitrary order of factors \(x_{-\beta}\) in the product \(X^s\), such that the set

\[
\{ X^s v_\lambda \mid s \in S(D, \lambda) \}
\]

forms a basis of \(V(\lambda)\).

**Remark.** The set \(D\) and non–negative integers \(M_p(\lambda)\) are explicitly described in these papers.

Another interesting point is to understand the geometric aspects of the PBW filtration. In [9] degenerated flag varieties have been introduced which are certain varieties in the projectivization \(\mathbb{P}(\text{gr} \ V(\lambda))\) of \(\text{gr} \ V(\lambda)\). In type \(A_n\) (see [9, 10]) and type \(C_n\) (see [11]) it has been shown that the degenerated flag varieties can be embedded into a product of Grassmanians and desingularizations are constructed. Recently in [12] the notion of favourable modules has been introduced whose properties are governed by the combinatorics of an associated polytope and it has been shown that the corresponding degenerated flag varieties have nice properties, e.g. they are projectively normal and arithmetically Cohen-Macaulay varieties (see also Section 7). Especially it has been proved that \(V(\lambda)\) for types \(A_n, C_n\) and \(G_2\) are favourable (with respect to the polytope from Theorem 3.3), where the proof of this fact uses the Minkowski sum property of these polytopes. Our aim is to obtain similar results to Theorem 3.3 for type \(B_n\) for certain dominant integral weights and, motivated by the corresponding nice geometry of favourable modules, to construct various favourable modules.
4. Dyck path, polytopes and PBW bases

The notion of Dyck paths is used in the papers [13, 14] in order to describe the set $D$ from Theorem 3.3 (and thus $S(D,\lambda)$), but appears earlier in the literature in a different context. In this section we define two types of paths (type 1 and type 2), which we also call Dyck paths to avoid deviating from the established terminology. The set of Dyck paths of type 1 is similar to the definition given in [13, 14], while the type 2 Dyck paths are unions of type 1 Dyck paths with some extra conditions and are called double Dyck paths. We shall use the Hasse diagram to provide a better imagination of Dyck paths.

4.1. To each finite partially ordered set $(S,\leq)$ we can associate a diagram, called the Hasse diagram. The vertices are given by the elements in $S$ and we draw a line segment from $x$ to $y$ whenever $y$ covers $x$, that is, whenever $x < y$ and there is no $z$ such that $x < z < y$. We consider the partial order $\leq$ on $R^+$ given by $\alpha \leq \beta$ if and only if $\alpha - \beta \in Q^+$. We shall be interested in the Hasse diagram of $(R^+,\leq)$ and $(R^+_1,\leq)$. Note that the Hasse diagram of $R^+_1$ is obtained from the Hasse diagram of $R^+$ by erasing all vertices $\alpha \in R^+\setminus R^+_1$.

**Example.** We find below the Hasse diagram of $(R^+,\leq)$ for type $A_n$ and $B_n$ respectively. The Hasse diagram for $(R^+_2,\leq)$ of type $B_n$ is highlighted in red.

![Hasse diagram](image)

4.2. For the rest of this section we fix $i \in \{1,\ldots,n\}$ and let $\lambda = m\omega_i$ for some $m \in \mathbb{Z}_+$. All roots of type $B_n$ are of the form $\alpha_p + \cdots + \alpha_q$ for some $1 \leq p \leq q \leq n$ or of the form $\alpha_p + \cdots + \alpha_{2n-q} + 2\alpha_{2n-q+1} + \cdots + 2\alpha_n$ for some $1 \leq p \leq 2n - q < n$. To keep the notation as simple as possible we define

$$\alpha_{p,q} := \begin{cases} 
\alpha_p + \cdots + \alpha_q, & \text{if } 1 \leq p \leq q \leq n \\
\alpha_p + \cdots + \alpha_{2n-q} + 2\alpha_{2n-q+1} + \cdots + 2\alpha_n, & \text{if } 1 \leq p \leq 2n - q < n
\end{cases}$$

Furthermore, we write $R^+_i(\ell)$ for $R^+_i\setminus\{\alpha_{p,q} \mid q \geq \ell\}$. We call a subset of positive roots $p = \{\beta(1),\ldots,\beta(k)\}, k \geq 1$ a Dyck path of type 1 if and only if the following two conditions are satisfied

- $\beta(1) = \alpha_{1,i}, \beta(k) = \alpha_{1,2n-i-1}$ or $\beta(1) = \alpha_{1,i+1}, \beta(k) = \alpha_{i,2n-i}$
- if $\beta(s) = \alpha_{p,q}$, then $\beta(s+1) = \alpha_{p,q+1}$ or $\beta(s+1) = \alpha_{p+1,q}$

(4.1)

The set of all type 1 Dyck paths is denoted by $D^{\text{type }1}_1$ and $D^{\text{type }1}_2$ (resp. $D^{\text{type }2}_2$) denotes the subset consisting of all type 1 Dyck paths starting at $\alpha_{1,i}$ (resp. $\alpha_{1,i+1}$). Furthermore, we call a subset of positive roots $p = \{\beta(1),\ldots,\beta(k)\}, k \geq 1$ a Dyck path of type 2 if and only if we can write $p = p_1 \cup p_2$ ($p_1 = \{\beta_1(1),\ldots,\beta_1(k_1)\}, k_1 \geq 1, p_2 = \{\beta_2(1),\ldots,\beta_2(k_2)\}, k_2 \geq 1$) with the following properties:

- $\beta_1(1) = \alpha_{1,i}, \beta_1(1) = \alpha_{2,i}$ and $\beta_1(k_1) = \alpha_{j,2n-j}, \beta_2(k_2) = \alpha_{j+1,2n-j-1}$ for some $1 \leq j < i$
- $p_1$ and $p_2$ satisfy the second property of (4.1)
- $p_1 \cap p_2 = \emptyset$
We make the following conjecture and prove various cases in this paper. We set \( D \) in the Hasse diagram of \((R^+_i, \leq)\). The set of all type 2 Dyck paths is denoted by \( D^{\text{type2}} \).

**Example.** We list all Dyck paths for \( B_4 \), \( i = 3 \). We have

\[
D^{\text{type1}} = \left\{ \{\alpha_{1,3}, \alpha_{2,3}, \alpha_{3,3}, \alpha_{3,4}\}, \{\alpha_{1,3}, \alpha_{2,3}, \alpha_{2,4}, \alpha_{3,4}\}, \{\alpha_{1,3}, \alpha_{1,4}, \alpha_{2,4}, \alpha_{3,4}\}, \{\alpha_{1,4}, \alpha_{2,4}, \alpha_{3,4}, \alpha_{3,5}\}, \{\alpha_{1,4}, \alpha_{2,5}, \alpha_{3,5}\}, \{\alpha_{1,4}, \alpha_{1,5}, \alpha_{2,5}, \alpha_{3,5}\} \right\}.
\]

\[
D^{\text{type2}} = \left\{ \{\alpha_{1,3}, \alpha_{2,3}, \alpha_{1,4}, \alpha_{2,4}, \alpha_{2,5}, \alpha_{1,6}, \alpha_{2,6}, \alpha_{3,7}\}, \{\alpha_{1,3}, \alpha_{2,3}, \alpha_{1,4}, \alpha_{2,4}, \alpha_{2,5}, \alpha_{1,5}, \alpha_{2,5}, \alpha_{1,6}, \alpha_{2,6}, \alpha_{3,5}\}, \{\alpha_{1,3}, \alpha_{2,3}, \alpha_{1,4}, \alpha_{3,3}, \alpha_{1,5}, \alpha_{3,4}, \alpha_{1,6}, \alpha_{2,6}, \alpha_{3,5}\}, \{\alpha_{1,3}, \alpha_{2,3}, \alpha_{1,4}, \alpha_{3,3}, \alpha_{2,5}, \alpha_{2,6}, \alpha_{3,5}\}, \{\alpha_{1,3}, \alpha_{2,3}, \alpha_{1,4}, \alpha_{3,3}, \alpha_{2,5}, \alpha_{2,6}, \alpha_{3,5}\} \right\}.
\]

The corresponding polytope is defined by

\[
P(D, m\omega_i) = \{ s = (s_\beta) \in \mathbb{R}^{R_+^i} \mid \forall p \in D : \sum_{\beta \in p} s_\beta \leq M_p(m\omega_i) \}, \tag{4.2}
\]

where we set

\[
M_p(m\omega_i) = \begin{cases} m & \text{if } p \in D^{\text{type1}} \\ m\omega_i(\theta^i) & \text{if } p \in D^{\text{type2}} \end{cases}
\]

We consider the polytope \( P(D, m\omega_i) \) as a subset of \( \mathbb{R}^{R_+^i} \) by requiring \( s_\beta = 0 \) for \( \beta \in R^+_i \backslash R^+_i \).

**Remark.** Note that the set \( D \) is a subset of \( \mathcal{P}(R^+_i) \) and depends therefore on \( i \) (unlike as in the \( A_n \), \( C_n \) and \( G_2 \) case). We do not expect that there exists a set \( D' \subset \mathcal{P}(R^+) \) such that the following holds: For any dominant integral weight \( \mu \) there exists non-negative integers \( M_p(\mu) \) (\( p \in D' \)) such that the integral points of the corresponding polytope (3.2) parametrize a basis of \( \text{gr} \, V(\mu) \). We rather expect that there exists a polytope parametrizing a basis of the associated graded space where the coefficients of the describing inequalities might be greater than 1. We will demonstrate this in the \( B_3 \) case (see Section 5).

4.3. For \( s \in S(D, m\omega_i) \) let \( \text{wt}(s) := \sum_{\beta \in R^+_i} s_\beta \beta \) and

\[
S(D, m\omega_i)^\mu = \{ s \in S(D, m\omega_i) \mid m\omega_i - \text{wt}(s) = \mu \}.
\]

We make the following conjecture and prove various cases in this paper. We set \( \epsilon_i = 1 \) if \( i \leq 2 \) and \( \epsilon_i = 2 \) else.

**Conjecture.** Let \( \mathfrak{g} \) be the Lie algebra of type \( B_n \) and \( 1 \leq i \leq n \).
(1) The lattice points $S(D, m\omega_i)$ parametrize a basis of $V(m\omega_i)$ and $\text{gr} V(m\omega_i)$ respectively. In particular,

\[ \{ X^s v_{m\omega_i} \mid s \in S(D, m\omega_i) \} \]

forms a basis of $\text{gr} V(m\omega_i)$.

(2) We have

\[ I_{m\omega_i} = S(n^-)((U(n^+) \circ \text{span} \{ x_{-\omega_i(\beta')}^{m+1}, x_{-\alpha_1,2n-1}^-, x_{-\beta} \mid \beta \in R^+ \setminus R_i^+ \}) \].

(3) The character and graded $q$-character respectively is given by

\[ \text{ch} V(m\omega_i) = \sum_{\mu \in h^*} |S(D, m\omega_i)^\mu| e^\mu \]

\[ \text{ch}_q \text{gr} V(m\omega_i) = \sum_{s \in S(D, m\omega_i)} e^{m\omega_i - \text{wt}(s)} q^{\sum s_\beta}. \]

(4) We have an isomorphism of $S(n^-)$–modules for all $\ell \in \mathbb{Z}_+$:

\[ \text{gr} V((m + \epsilon_i \ell)\omega_i) \cong S(n^-)(v_{m\omega_i} \otimes v_{\epsilon_i \omega_i}) \subseteq \text{gr} V(m\omega_i) \otimes \text{gr} V(\epsilon_i \omega_i). \]

**Lemma.** The proof of Conjecture 4.3 can be reduced to the following three statements:

(i) The set

\[ \{ X^s \mid s \in S(D, m\omega_i) \} \]

generates the module $S(n^-)/I_{m\omega_i}$.

(ii) We have

\[ S(D, (m + \epsilon_i \ell)\omega_i) = S(D, m\omega_i) + S(D, \epsilon_i \omega_i). \]

(iii) We have

\[ \dim V(\ell \omega_i) = |S(D, \ell \omega_i)| \text{ for } \ell \leq \epsilon_i. \]

**Proof.** Assume that part (1) of the conjecture holds. Part (3) of the conjecture follows immediately from part (1). Since $I_{m\omega_i} v_{m\omega_i} = 0$, we have a surjective map

\[ S(n^-)/I_{m\omega_i} \twoheadrightarrow \text{gr} V(m\omega_i) \]

and hence part (2) of the conjecture follows with part (1) and (i). It has been shown in [14, Proposition 3.7] (cf. also [12, Proposition 1.11]) that if $\{ X^s v_\lambda \mid s \in S(D, \lambda) \}$ is a basis of $\text{gr} V(\lambda)$ and $\{ X^s v_\mu \mid s \in S(D, \mu) \}$ is a basis of $\text{gr} V(\mu)$, then $\{ X^s (v_\lambda \otimes v_\mu), s \in S(D, \lambda) + S(D, \mu) \}$ is a linearly independent subset of $\text{gr} V(\lambda) \otimes \text{gr} V(\mu)$ and therefore also a linearly independent subset of $V(\lambda) \otimes V(\mu)$. Since we have a surjective map

\[ S(n^-)/I_{(m + \epsilon_i \ell)\omega_i} \twoheadrightarrow \text{gr} V((m + \epsilon_i \ell)\omega_i) \rightarrow S(n^-)(v_{m\omega_i} \otimes v_{\epsilon_i \omega_i}) \subseteq \text{gr} V(m\omega_i) \otimes \text{gr} V(\epsilon_i \omega_i), \]

part (4) follows from part (1) and (ii). So it remains to prove that part (1) follows from (i)–(iii). If $m \leq \epsilon_i$ we are done with (iii), so let $m > \epsilon_i$. By induction we can suppose that $S(D, (m - \epsilon_i)\omega_i)$ parametrizes a basis of $\text{gr} V((m - \epsilon_i)\omega_i)$ and by (i) and (iii) that $S(D, \epsilon_i \omega_i)$ parametrizes a basis of $\text{gr} V(\epsilon_i \omega_i)$. Thus, together with (ii), we obtain similar as above that $\{ X^s (v_{(m-\epsilon_i)\omega_i} \otimes v_{\epsilon_i \omega_i}), s \in S(D, m\omega_i) \}$ is a linearly independent subset of $V((m - \epsilon_i)\omega_i) \otimes V(\epsilon_i \omega_i)$. Since $V(m\omega_i) \cong U(n^-)(v_{(m-\epsilon_i)\omega_i} \otimes v_{\epsilon_i \omega_i})$ and $\dim V(m\omega_i) = \dim \text{gr} V(m\omega_i)$ part (1) follows. \(\square\)

Therefore it will be enough to prove the above lemma. The first part of the lemma is proved in full generality in Section 4.4 whereas the second part is proved only for several special cases ($1 \leq i \leq 3$ and $n$ arbitrary or $i$ arbitrary and $1 \leq n \leq 4$) in Section 4.5. The proof of the third part for these special cases is an easy calculation and will be omitted.
4.4. Proof of Lemma 4.3 (i). We choose a total order \( < \) on \( R^+ \):
\[
\alpha_{p,q} < \alpha_{s,t} \iff q < t \text{ or } q = t \text{ and } p > s.
\]
Interpreted in the Hasse diagram this means that we order the roots from the bottom to the top and from left to right. We extend this order to the induced homogeneous reverse lexicographic order on the monomials in \( S(n^-) \). We order the set of positive roots \( R^+ = \{\beta_1, \ldots, \beta_N\} \) with respect to \( < \)
\[
\beta_N < \beta_{N-1} < \cdots < \beta_1.
\]
The definition of the order \( < \) implies the following. Let \( \beta_\ell < \beta_p \) and \( \nu \in R^+ \), such that \( \beta_\ell - \nu \in R^+ \) and \( \beta_p - \nu \in R^+ \), then
\[
\beta_\ell - \nu < \beta_p - \nu.
\]
We define differential operators for \( \alpha \in R^+ \) on \( S(n^-) \) by:
\[
\partial_\alpha x^{-\beta} := \begin{cases} 
  x^{-\beta + \alpha}, & \text{if } \beta - \alpha \in R^+ \\
  0, & \text{else}.
\end{cases}
\]
The operators satisfy
\[
\partial_\alpha x^{-\beta} = c_{\alpha,\beta} [x_\alpha, x^{-\beta}],
\]
where \( c_{\alpha,\beta} \in \mathbb{C} \) are some non–zero constants.

**Lemma.** Let \( \sum_{r \in \mathbb{Z}_+^N} c_r X_r \in S(n^-) \) and \( \nu \in R^+ \). We set
\[
t = \max \{ r \mid \partial_\nu X_r \neq 0, c_r \neq 0 \}.
\]
Then the maximal monomial in \( \sum_{r \in \mathbb{Z}_+^N} c_r \partial_\nu X^r \) is a summand of \( \partial_\nu X^t \).

**Proof.** We express \( \partial_\nu X^t \) as a sum of monomials and let \( X^\bar{r} \) be the maximal element appearing in this expression. From the definition of the differential operators it is clear that
\[
\bar{t}_{\beta_\ell} = \begin{cases} 
  t_{\beta_\ell}, & \text{if } \ell \neq j_\ell, \beta_\ell \neq \beta_{j_\ell} - \nu \\
  t_{\beta_\ell} - 1, & \text{if } \ell = j_\ell \\
  t_{\beta_\ell} + 1, & \text{if } \beta_\ell = \beta_{j_\ell} - \nu
\end{cases}, \quad \text{where } \beta_{j_\ell} = \max_{1 \leq k \leq N} \{ \beta_k \mid \partial_\nu x^{-\beta_k} \neq 0, t_{\beta_k} \neq 0 \}.
\]
With other words, \( X^\bar{r} \) is a scalar multiple of
\[
\prod_{\ell \neq j_\ell} x_{-\beta_\ell}^{t_{\beta_\ell}} x_{-\beta_{j_\ell}}^{t_{\beta_{j_\ell}} - 1} x_{-\beta_{j_\ell} + \nu}.
\]
Moreover, let \( X^\bar{r} \) be any monomial with \( c_r \neq 0 \) and \( \partial_\nu X^\bar{r} \neq 0 \). Similar as above we denote by \( X^\bar{r} \) the maximal element which appears as a summand of \( \partial_\nu X^r \). In the rest of the proof we shall verify that \( \bar{r} > r \). Since \( t > r \) this follows immediately if \( j_t \leq j_r \). So suppose that \( j_t > j_r \) and \( \bar{r} > r \). This is only possible if \( r_{\beta_{j_t}} - 1 < t_{\beta_{j_t}} \) and \( t_{\beta_p} = r_{\beta_p} \) for \( 1 \leq p < j_t \). Therefore we can deduce from \( t > r \) that \( r_{\beta_{j_t}} = t_{\beta_{j_t}} \). It follows \( t_{\beta_{j_t}} \neq 0 \), \( \partial_\nu x^{-\beta_{j_t}} \neq 0 \) and \( \beta_{j_t} < j_{j_t} \), which is a contradiction to the choice of \( \beta_{j_t} \).

The proof of Lemma 4.3 (i) proceeds as follows. We use the above monomial order on \( S(n^-) \) and prove that any monomial \( X^s \), \( s \notin S(D, m\omega_i) \) in \( S(n^-)/I_{m\omega_i} \) can be written as a sum of monomials, where each monomial appearing in this expression is less than \( X^s \). We repeat this argument for any summand \( X^t \), \( t \notin S(D, m\omega_i) \) in this expression. After finitely many steps \( X^s \) can be written as a sum of monomials \( X^t \), \( t \in S(D, m\omega_i) \) which is exactly the statement of the lemma. So let \( X^s \), \( s \notin S(D, m\omega_i) \) be a monomial in \( S(n^-)/I_{m\omega_i} \). Then there exists a Dyck path \( p \) such that
\[
\sum_{\beta}s_{\beta} > M_p(m\omega_i).
\]
We define another multi–exponent \( r = (r_{\beta}) \) by \( r_{\beta} = s_{\beta} \) if \( \beta \in p \) and \( r_{\beta} = 0 \) if \( \beta \notin p \).
otherwise. Since we have a monomial order it will be enough to prove that $X^r$ can be written as a sum of smaller monomials. Hence the following proposition proves Lemma 4.3 (i).

**Proposition.** Let $p \in D$ and $s \in \mathbb{Z}^{\left|R^+_p\right|}$ be a multi–exponent supported on $p$, i.e. $s_{\beta} = 0$ for $\beta \notin p$. Suppose $\sum_{\beta \in p} s_{\beta} > M_p(m \omega_i)$. Then there exists constants $c_t \in \mathbb{C}, t \in \mathbb{Z}^{\left|R^+_p\right|}$ such that

$$X^s + \sum_{t < s} c_t X^t \in I_\lambda.$$

**Proof.** First we assume that $p = \{\beta(1), \ldots, \beta(k)\} \in D^\text{type 1}_2$. Note that the ideal $I_\lambda$ is stable under the action of the differential operators and $x_{-\alpha_1,2n-i} \in I_\lambda$. In the following we write simply $x_{p,q} := x_{-\alpha_p,q}$ and $s_{p,q} := s_{\alpha_p,q}$ and rewrite the monomial $x_{-\beta(1)} \cdots x_{-\beta(k)}$ as follows. We can choose a sequence of integers

$$1 = p_0 < p_1 < p_2 < \cdots < p_{r-1} < p_r = i < i + 1 = q_0 < q_1 < q_2 < \cdots < q_{r-1} \leq q_r = 2n - i$$

with $1 \leq p_\ell \leq q_\ell \leq n$ or $1 \leq p_\ell < 2n - q_\ell < n$ for all $0 \leq \ell \leq r$ such that

$$x_{-\beta(1)} \cdots x_{-\beta(k)} = x_{1,i+1} \cdots x_{p_r,q_r} x_{1,i+2} \cdots x_{p_1,q_1} x_{1,i+1+q_1} \cdots x_{p_2,q_2} x_{2,2n-i} .$$

See the picture below for a better imagination:

![Diagram](https://example.com/diagram.png)

For $0 \leq \ell \leq r$ we define $s_{p_\ell} := s_{p_\ell,q_\ell-1+1} + \cdots + s_{p_\ell,q_\ell} + s_{p_\ell+1,q_\ell} + \cdots + s_{p_r,q_r}$ and $|s| := s_{\beta(1)} + \cdots + s_{\beta(k)}$. Then

$$\partial_{\alpha_1,p_1-1}^{s_{p_1}} |s| x_{1,2n-i} = x_{1,2n-i} x_{p_1,2n-i} \in I_\lambda.$$

Since $\partial_{\alpha_1,x_{1,2n-i}} = 0$ for $1 < t \leq \ell < i$ we conclude with $p_1 < p_2 < \cdots < p_r$:

$$\partial_{\alpha_1,p_r-1}^{s_{p_r}} \cdots \partial_{\alpha_1,p_2-1}^{s_{p_2}} x_{1,2n-i} = x_{1,2n-i} x_{1,2n-i} x_{p_1,2n-i} x_{p_2,2n-i} \cdots x_{p_r,2n-i} \in I_\lambda.$$

Note that the operator $\partial_{\alpha_1,x_{1,2n-i}}$ acts non–trivially only on the $x_{p_r,2n-i}$. The choice of the order implies that the largest monomial in

$$\partial_{\alpha_1,x_{1,2n-i}}^{s_{p_1} + \cdots + s_{p_1 + 1}} |s| x_{1,2n-i} x_{p_1,2n-i} x_{p_2,2n-i} \cdots x_{p_r,2n-i}$$

(4.3)

is obtained by acting with $\partial_{\alpha_1,x_{1,2n-i}}$ only on the largest element $x_{1,2n-i}$. So the largest monomial in (4.3) with respect to $|s|$ is

$$x_{1,i+1} \cdots x_{p_1,2n-i} x_{p_2,2n-i} \cdots x_{p_r,2n-i}.$$  

(4.4)

Each of the operators $\partial_{\alpha_1,x_{1,2n-i}} \cdots \partial_{\alpha_1,x_{1,2n-i}}$ acts trivially on each $x_{p_r,2n-i}$. Since

$$\partial_{\alpha_1,x_{1,2n-i}}^{s_{p_1} + \cdots + s_{p_1 + 1}} x_{1,i+1} = x_{1,i+1} \cdots x_{p_1,i+1}$$

(4.5)
we obtain by acting with these operators on (4.4) that
\[
x_{1,i+1}^{s_{1,i+1}} \cdots x_{p_1,i+1}^{s_{p_1,i+1}} x_{p_1,2n-i}^{s_{p_1,2n-i}} \cdots x_{p_r,2n-i}^{s_{p_r,2n-i}} + \sum \text{smaller monomials} \in I_\lambda. \tag{4.5}
\]
In the next step we act with the operators \(\partial_{\alpha_{i+1,2n-\ell}}\), \(\partial_{\alpha_{i+1,2n-\ell}+1}\), \ldots, \(\partial_{\alpha_{i+1,2n-(q_0+1)}}\) on \(x_{p_1,2n-i}\) and obtain with Lemma 4.4:
\[
\partial_{\alpha_{i+1,2n-\ell}} \partial_{\alpha_{i+1,2n-\ell}+1} \cdots \partial_{\alpha_{i+1,2n-(q_0+1)}} x_{p_1,2n-i} = x_{p_1,2n-i}^{s_{p_1,2n-i}} \cdots x_{p_1,q_1}^{s_{p_1,q_1}-1} \cdots x_{p_1,q_0+1}^{s_{p_1,q_0+1}} + \sum \text{smaller monomials} \tag{4.6}
\]
Since \(x_{p_1,2n-i}\) is the maximal element with respect to \(\prec\) among the factors in the leading term of (4.5) we get by combining Lemma 4.4 and (4.6)
\[
x_{1,i+1}^{s_{1,i+1}} \cdots x_{p_1,i+1}^{s_{p_1,i+1}} x_{p_1,q_1}^{s_{p_1,q_1}-1} \cdots x_{p_1,q_0+1}^{s_{p_1,q_0+1}} x_{p_1,2n-i}^{s_{p_1,2n-i}} \cdots x_{p_r,2n-i}^{s_{p_r,2n-i}} + \sum \text{smaller monomials} \in I_\lambda. \tag{4.7}
\]
Now we act with the operators \(\partial_{\alpha_{p_2-1,p_2-1}}\), \(\partial_{\alpha_{p_1+1,p_1+1}}\), \(\partial_{\alpha_{p_1,p_1}}\):
\[
\partial_{\alpha_{p_2-1,p_2-1}} \cdots \partial_{\alpha_{p_1+1,p_1+1}} \partial_{\alpha_{p_1,p_1}} x_{p_1,q_1} = x_{p_1,q_1}^{s_{p_1,q_1}} x_{p_1,q_1-1}^{s_{p_1,q_1-1}} \cdots x_{p_1,q_0+1}^{s_{p_1,q_0+1}} x_{p_1,q_1}^{s_{p_1,q_1}+s_{p_1+1,q_1}+s_{p_2,q_1}} \cdots x_{p_1,q_1}^{s_{p_2,q_1}}. \tag{4.8}
\]
Since \(\partial_{\alpha_{p_2-1,p_2-1}}\), \(\partial_{\alpha_{p_1+1,p_1+1}}\), \(\partial_{\alpha_{p_1,p_1}}\) act trivially on each \(x_{p_1,2n-i}\) and \(x_{p_1,q_1}\) is the largest element with respect to \(\prec\) among the remaining factors in the leading term of (4.7) we get by combining (4.7) and (4.8) that the following element is the sum of strictly smaller monomials in \(S(n^-)/I_\lambda\):
\[
x_{1,i+1}^{s_{1,i+1}} \cdots x_{p_1,i+1}^{s_{p_1,i+1}} x_{p_1,q_1}^{s_{p_1,q_1}-1} x_{p_1,q_1-1}^{s_{p_1,q_1-1}} \cdots x_{p_1,q_0+1}^{s_{p_1,q_0+1}} x_{p_1,q_1}^{s_{p_1,q_1}+s_{p_1+1,q_1}+s_{p_2,q_1}} \cdots x_{p_1,q_1}^{s_{p_2,q_1}} x_{p_2,2n-i}^{s_{p_2,2n-i}} \cdots x_{p_r,2n-i}^{s_{p_r,2n-i}} + \sum \text{smaller monomials} \in I_\lambda.
\]
If we repeat the above steps with \(x_{p_2,2n-i}\), \(\ldots, x_{p_r,2n-i}\) we can deduce the proposition for \(p \in D_2^{\text{type 1}}\).

Now suppose that \(p \in D_1^{\text{type 1}}\) is of the form
\[
p = \{\alpha_{1,i}, \alpha_{2,i}, \ldots, \alpha_{\ell,i}, \alpha_{\ell,i+1}, \ldots, \alpha_{r,i+1}, \alpha_{r,i+2}, \ldots, \alpha_{i,2n-i-1}\}. \tag{4.9}
\]
We shall construct another Dyck path as follows. We set \(q = \{\alpha_{\ell,i+1}, \ldots, \alpha_{r,i+1}, \alpha_{r,i+2}, \ldots, \alpha_{i,2n-i-1}\}\). Then it is easy to see that we can find an element \(\bar{q} \in \mathcal{P}(P_i^+)\) such that the path \(\bar{q} := q \cup \bar{q} \in D_2^{\text{type 1}}\).

We define a multi–exponent \(s(\bar{q})\) by
\[
s(\bar{q})_\beta = s_\beta, \text{ if } \beta \in q, \quad s(\bar{q})_{\alpha_{1,i-1}} = s_{\alpha_{1,i}} + \cdots + s_{\alpha_{\ell,i}}, \text{ and else } s(\bar{q})_\beta = 0.
\]
By our previous calculations we get
\[
X^s(\bar{q}) + \sum \text{ct} X^t \in I_\lambda. \tag{4.10}
\]
Note that each operator \(\partial_{\alpha_{1,i-1}}\), \(\partial_{\alpha_{\ell,i-1}}\), \ldots, \(\partial_{\alpha_{r,i-1}}\) acts trivially on \(x_\beta\) for all \(\beta \in q\) and \(\partial_{\alpha_{i+1,i+1}}\) acts trivially on \(x_\beta\) for all \(\beta \in q\backslash\{\alpha_{i+1,i+1}, \ldots, \alpha_{r,i+1}\}\). Since \(x_{1,i+1} \succ x_{j,i+1}\) for all \(\ell + 1 \leq j \leq r\) the maximal element when acting with \(\partial_{\alpha_{i+1,i+1}}\) on (4.9) is obtained by acting with \(\partial_{\alpha_{i+1,i+1}}\) on \(x_{1,i+1}\). We have
\[
\partial_{\alpha_{i+1,i+1}} X^s(\bar{q}) + = x_{1,i+1}^{s_{1,i+1} + \cdots + s_{\ell,i}} X^s(q) + \sum \text{smaller monomials} \in I_\lambda, \tag{4.10}
\]
where \(s(q)\) is the multi–exponent defined by \(s(q)_\beta = s_\beta\) if \(\beta \in q\) and \(s(q)_\beta = 0\) otherwise. In the last step we act with \(\partial_{\alpha_{i-1,i-1}}\partial_{\alpha_{i-2,i-2}} \cdots \partial_{\alpha_{1,1}}\) on (4.10) and get
\[
X^s + \sum \text{ct} X^t \in I_\lambda.
\]
Now we assume that $p \in D_{\text{type} 2}$, which means that $p$ can be written as a union $p = p_1 \cup p_2$ with $p_1 = \{\beta_1(1), \ldots, \beta_1(k_1)\}$ and $p_2 = \{\beta_2(1), \ldots, \beta_2(k_2)\}$ such that $\beta_1(k_1) = \alpha_{j-1,2n-j+1}$ and $\beta_2(k_2) = \alpha_{j,2n-j}$. We have
\[ x_{1,2n-1}^{s_1(1) + \cdots + s_1(k_1)} \cdot \cdots \cdot x_{1,2n-1}^{s_2(1) + \cdots + s_2(k_2)} \in I_\lambda. \] (4.11)
We will prove the statement of the proposition by upward induction on $j \in \{2, \ldots, i\}$. If $j = 2$, we have
\[ p_1 = \{\alpha_{1,i}, \alpha_{1,i+1}, \ldots, \alpha_{1,2n-1}\} \text{ and } p_2 = \{\alpha_{2,i}, \alpha_{2,i+1}, \ldots, \alpha_{2,2n-2}\} \]
and therefore by acting on (4.11) we get
\[
\partial_{\alpha_{1,2n-i}}^{s_{1,i}} \partial_{\alpha_{1,2}}^{s_{2,2n-2}} \partial_{\alpha_{2,2n-i}}^{s_{1,1}} \partial_{\alpha_{2,2}}^{s_{2,1} + \cdots + s_{2,k_2}} x_{1,2n-1}^{s_1(1) + \cdots + s_1(k_1) + s_2(1) + \cdots + s_2(k_2)} = 
\]
\[
x_{1,2n-1}^{s_{1,1} + \cdots + s_{1,i+1}} x_{1,2n-2}^{s_{2,2n-i-1} + \cdots + s_{2,2}} \cdot x_{2,i+1}^{s_{2,i}} + \sum \text{smaller monomials} \in I_\lambda \]
and the induction begins. As before we rewrite the Dyck path as follows:
\[
x - \beta_1(1) x - \beta_1(2) \cdots x - \beta_1(k_1) = x_1 x_1, x_1, x_1, x_2, x_2, c_1 \cdots x_2, c_2 \cdots x_r c_r \]
\[
x - \beta_2(1) x - \beta_2(2) \cdots x - \beta_2(k_2) = x_2 x_3, x_3, x_3, x_4, x_4, x_4, x_4, x_5, x_5, x_5, x_5, x_5, x_5, x_5, x_5, x_5, x_5, \cdots x_{pt} q_t \]
where
\[
1 = b_0 = b_1 < b_2 < \cdots < b_r - 1 \leq b_r = j - 1, \ i = c_0 < c_1 < c_2 < \cdots < c_r - 1 \leq c_r = 2n - j + 1, \]
\[
2 = p_0 \leq p_1 < p_2 < \cdots < p_r - 1 \leq p_r = j \text{ and } i = q_0 < q_1 < q_2 < \cdots < q_t - 1 \leq q_t = 2n - j. \]
For a pictorial illustration see the picture below:

We will construct another path $\overline{p} \in D_{\text{type} 2}$. We set
\[
\overline{p}_1 = p \setminus \{\alpha_{pt,qt-1}, \alpha_{pt,qt-1} + 1, \ldots, \alpha_{pt,qt}\}. \]
Then it is easy to see that there exists a unique element $\overline{p}_2 \in \mathcal{P}(R_i^+) \text{ such that } \overline{p} = \overline{p}_1 \cup \overline{p}_2 \in D_{\text{type} 2}$ and the roots $\alpha_{j-2n-j+2}, \alpha_{j-1,2n-j+1}$ appear in $\overline{p}$. We define a multi–exponent $s(\overline{p})$ by
\[
s(\overline{p})_{\beta} = s_\beta, \text{ if } \beta \in \overline{p}_1 \setminus \{\alpha_{br-1,cr}\}, \ s(\overline{p})_{\alpha_{br-1,cr}} = s_{br-1,cr} + s_{pt,qt-1} + s_{pt,qt-1} + \cdots + s_{pt,qt}. \]
and \( s(\overline{p})_\beta = 0 \) otherwise. The induction hypothesis yields
\[
X^{s(\overline{p})} + \sum_{t < s(\overline{p})} c_t X^t \in I_\lambda. \tag{4.12}
\]

Now we want to act with suitable operators on (4.12) such that the leading term is the required monomial \( X^s \). Since \( x_{br_{r-1}c_r} \) is the maximal element in \( X^{s(\overline{p})} \) and \( \partial_{\alpha br_{r-1},j} \), \( \ldots, \partial_{\alpha br_{r-1},2n-qt-1} \) act non-trivially on \( x_{br_{r-1}c_r} \) we obtain the desired property
\[
\partial_{\alpha br_{r-1},2n-qt-1} \cdots \partial_{\alpha br_{r-1},1} X^{s(\overline{p})} + \sum_{t < s(\overline{p})} c_t \partial_{\alpha br_{r-1},2n-qt-1} \cdots \partial_{\alpha br_{r-1},1} X^t = X^s + \text{smaller monomials} \in I_\lambda.
\]

\( \square \)

4.5. Proof of Lemma 4.3 (ii) in several cases. In this section we shall prove several cases of Lemma 4.3 (ii). Consider the partial order
\[
\alpha_{j,k} \leq \alpha_{p,r} \iff (j \geq p \wedge k \geq r)
\]
and suppose we are given a multi–exponent \( s \in S(\mathbf{D}, m\omega_i) \). Let \( R^s = \{ \beta \in R^+_i(2n-i) \mid s_\beta \neq 0 \} \) and \( T^s \) the set of minimal elements in \( R^s \) with respect to \( \leq \). We define a multi–exponent \( t^s \) by \( t_\beta = 1 \), if \( \beta \in T^s \) and \( t_\beta = 0 \) otherwise and call it the multi–exponent associated to \( s \). The following lemma can be deduced from [13, Proposition 3.7].

**Lemma.** Let \( s \in S(\mathbf{D}, m\omega_i) \) such that \( s_\beta \neq 0 \) forces \( \beta \in R^+_i(2n-i-1) \) (resp. \( \beta \in (R^+_i \cap R^+_{i+1})(2n-i) \)). Then we have
\[
s - t^s \in S(\mathbf{D}, (m - 1)\omega_i).
\]

For a multi–exponent \( t \in \mathbb{Z}^{[R^+_i]} \) define
\[
\text{supp}(t) = \{ \beta \in R^+_i \mid t_\beta \neq 0 \}, \quad T(1) = \{ t \in \mathbb{Z}^{[R^+_i]} \mid t_\beta \leq 1, \forall \beta \in R^+_i \}.
\]

The following proposition proves Lemma 4.3 (ii) for \( 1 \leq i \leq 3 \).

**Proposition.** Let \( 1 \leq i \leq 3 \) and \( m \geq \epsilon_i \). Then we have
\[
S(\mathbf{D}, m\omega_i) = S(\mathbf{D}, (m - \epsilon_i)\omega_i) + S(\mathbf{D}, \epsilon_i\omega_i).
\]

**Proof.** The proof for \( i = 1 \) is straightforward since \( S(\mathbf{D}, m\omega_1) \) is determined by two inequalities. So suppose \( s \in S(\mathbf{D}, m\omega_2) \) and recall that \( \mathbf{D}^{\text{type } 2} = \{ R^+_2 \} \). We will construct a multi–exponent \( t \in S(\mathbf{D}, \omega_2) \) such that \( s - t \in S(\mathbf{D}, (m - 1)\omega_2) \). We prove the statement by induction on \( s_\theta \) and start with \( s_\theta = 0 \). Note that \( \sum_{\beta \in p} (s_\beta - t_\beta) \leq m - 1 \) for all \( p \in \mathbf{D}^{\text{type } 1} \) implies already \( s - t \in S(\mathbf{D}, (m - 1)\omega_2) \).

**Case 1:** In this case we suppose \( s_{2,2n-2} \neq 0 \). If \( s_{1,2} = s_{2,2} = 0 \) the statement follows from Lemma 4.5. So let \( t \in T(1) \) be the multi–exponent with \( \text{supp}(t) = \{ \alpha_{2,2n-2}, \alpha_{k,2} \} \), where \( k = \min\{1 \leq j \leq 2 \mid s_{j,2} \neq 0 \} \). It is easy to see that \( t \in S(\mathbf{D}, \omega_2) \) and \( s - t \in S(\mathbf{D}, (m - 1)\omega_2) \).

**Case 2:** In this case we suppose that \( s_{2,2n-2} = 0 \) and \( s_{1,2} \neq 0 \). If \( s_{1,2n-2} = 0 \) the statement follows as above from Lemma 4.5. So let \( t \in T(1) \) be the multi–exponent with \( \text{supp}(t) = \{ \alpha_{1,2}, \alpha_{1,2n-2} \} \). It is straightforward to prove that \( t \in S(\mathbf{D}, \omega_2) \) and \( s - t \in S(\mathbf{D}, (m - 1)\omega_2) \).

**Case 3:** In this case we suppose \( s_{1,2} = s_{2,2n-2} = 0 \). Again with Lemma 4.5 we can assume that \( s_{2,2} \neq 0 \) and \( s_{1,2n-2} \neq 0 \). Let \( t \in T(1) \) be the multi–exponent with \( \text{supp}(t) = \{ \alpha_{2,2}, \alpha_{1,k} \} \), where \( k = \min\{3 \leq j \leq 2n-2 \mid s_{1,j} \neq 0 \} \). It follows \( t \in S(\mathbf{D}, \omega_2) \). Suppose we are given a Dyck path
\( p \in D_{\text{type}1} \) with \( \sum_{\beta \in p} (s_\beta - t_\beta) = m \), which is only possible if \( t_\beta = 0 \) for all \( \beta \in p \). It follows that \( p \) is of the form

\[
p = \{\alpha_1, \ldots, \alpha_{1,p}, \alpha_{2,p}, \ldots, \alpha_{2,2n-3}\}, \quad \text{for some } 3 \leq p < k.
\]

Since \( s_{1,r} = 0 \) for all \( 2 \leq r < k \) we get

\[
\sum_{\beta \in p} s_\beta \leq s_{2,3} + \cdots + s_{2,2n-3} \leq s_{2,2} + s_{2,3} + \cdots + s_{2,2n-3} - 1 \leq m - 1,
\]

which is a contradiction. Hence \( s - t \in S(D, (m-1)\omega_2) \) and the induction begins.

Assume that \( s_\theta \neq 0 \) and let \( s^1 \) be the multi–exponent obtained from \( s \) by replacing \( s_\theta \) by \( s_\theta - 1 \). By induction there exists a multi–exponent \( t^1 \in S(D, \omega_2) \) such that \( r^1 = s^1 - t^1 \in S(D, (m-1)\omega_2) \). If \( \sum_{\beta \in R^+_2} t_\beta \leq 1 \) we set \( t \) to be the multi–exponent obtained from \( t^1 \) by replacing \( t_\theta \) by \( t_\theta + 1 \). Then we get \( t \in S(D, \omega_2) \) and \( s - t = r^1 \). Otherwise we set \( r \) to be the multi–exponent obtained from \( r^1 \) by replacing \( r_\theta \) by \( r_\theta + 1 \). Since \( \sum_{\beta \in R^+_2} r_\beta \leq 2m - 2 \) and therefore

\[
s = r + t^1, \quad \text{and } s - t \in S(D, (m-1)\omega_2).
\]

Now suppose that \( i = 3 \) and let \( s \in S(D, m\omega_3) \). In contrast to the \( i = 2 \) case we will construct a multi–exponent \( t \in S(D, \rho\omega_3) \) such that \( s - t \in S(D, (m-p)\omega_3) \) where \( p = 1 \) or \( p = 2 \). A similar induction argument as above shows that it is enough to prove the statement for all multi–exponents \( s \) with \( s_\theta = 0 \). Since \( s_\theta = 0 \) it is sufficient to check the defining inequalities of the polytope for all \( p \in D \setminus q \), where \( q \) is the unique type 2 Dyck path with \( \theta \in q \). In other words

\[
\sum_{\beta \in p} (s_\beta - t_\beta) < M_p((m-p)\omega_3), \quad \forall p \in D \setminus q \Rightarrow s - t \in S(D, (m-p)\omega_3).
\]

Again we consider several cases.

Case 1: In this case we suppose \( s_{3,2n-3} \neq 0 \). Let \( t \in T(1) \) be the multi–exponent with \( \text{supp}(t) = \{\alpha_{3,2n-3}, \alpha_{k,3}\} \), where \( k = \min\{1 \leq j \leq 2 \mid s_j,3 \neq 0\} \). If \( k \) exists, it is easy to see that \( t \in S(D, \omega_3) \) and \( s - t \in S(D, (m-1)\omega_3) \). So suppose that \( s_{1,3} = s_{2,3} = 0 \). Now we consider two additional cases. First we assume that \( \sum_{k=3}^{2n-4} s_{3,k} = m \), which forces \( s_{3,3} \neq 0 \). Then we define \( t \in T(1) \) to be the multi–exponent with \( \text{supp}(t) = \{\alpha_{3,2n-3}, \alpha_{3,3}\} \). We shall prove that \( s - t \in S(D, (m-1)\omega_3) \).

For any \( p \in D_{\text{type}1} \) we obviously have \( \sum_{\beta \in p} (s_\beta - t_\beta) \leq m - 1 \). So let \( p = p_1 \cup p_1 \in D_{\text{type}2} \setminus q \). If \( \alpha_{3,3} \in p_2 \), there is nothing to show. Otherwise we get

\[
\sum_{\beta \in p_1} (s_\beta - t_\beta) + \sum_{\beta \in p_2} (s_\beta - t_\beta) \leq \sum_{\beta \in p_1} s_\beta + s_{2,3} + \sum_{k=3}^{2n-3} (s_{3,k} - t_{3,k}) \leq 2m - 1.
\]

It remains to consider the case \( \sum_{k=3}^{2n-4} s_{3,k} \leq m - 1 \). In this case we define \( t \in T(1) \) to be the multi–exponent with \( \text{supp}(t) = \{\alpha_{3,2n-3}\} \) if \( s_{1,2n-2} = s_{2,2n-2} = 0 \) and otherwise \( \text{supp}(t) = \{\alpha_{3,2n-3}, \alpha_{2,2n-2}\} \), where \( k = \max\{1 \leq j \leq 2 \mid s_{j,2n-2} \neq 0\} \). In either case \( t \in S(D, \omega_3) \) and if \( s_{1,2n-2} = s_{2,2n-2} = 0 \) or \( s_{2,2n-2} = 0 \) it is easy to verify that \( s - t \in S(D, (m-1)\omega_3) \). So suppose that \( s_{2,2n-2} = 0 \) and let \( p \in D \). If \( p \in D_{\text{type}1} \) the statement follows from the construction of \( t \) and the assumption \( \sum_{k=3}^{2n-4} s_{3,k} \leq m - 1 \). So let again \( p = p_1 \cup p_1 \in D_{\text{type}2} \setminus q \). If \( \alpha_{1,2n-2} \in p_1 \), we are done. Otherwise set

\[
\overline{p}_1 = p_1 \setminus \{\alpha_{1,3}, \alpha_{2,2n-2}\} \cup \{\alpha_{3,2n-3}\}, \quad \overline{p}_2 = p_2 \setminus \{\alpha_{2,3}\}.
\]

This yields \( \overline{p}_1, \overline{p}_2 \in D_{\text{type}1} \) and therefore

\[
\sum_{\beta \in p_1} (s_\beta - t_\beta) + \sum_{\beta \in p_2} (s_\beta - t_\beta) \leq \sum_{\beta \in \overline{p}_1} (s_\beta - t_\beta) + \sum_{\beta \in \overline{p}_2} (s_\beta - t_\beta) \leq (m - 1) + (m - 1).
\]
So from now on we can assume that \( s_{3,2n-3} = 0 \). Hence we have simplified the situation to the following
\[
\sum_{\beta \in \mathbf{p}} (s_{\beta} - t_{\beta}) \leq M_{\mathbf{p}}((m-p)\omega_3), \quad \forall \mathbf{p} \in \mathbf{D}_1^{\text{type 1}} \cup \widetilde{\mathbf{D}}_2^{\text{type 1}} \cup \mathbf{D}^{\text{type 2}} \setminus \mathbf{q} \Rightarrow \mathbf{s} - \mathbf{t} \in \mathbf{S}(\mathbf{D}, (m-p)\omega_3), \quad (4.13)
\]
where \( \widetilde{\mathbf{D}}_2^{\text{type 1}} = \{ \mathbf{p} \in \mathbf{D}_2^{\text{type 1}} | \alpha_{2,2n-3} \in \mathbf{p} \} \). Let \( \mathbf{s}' \) be the multi–exponent obtained from \( \mathbf{s} \) by setting all entries \( s_{\beta} \) with \( \beta \in R_i^+(2n-i-1) \) to zero and \( \mathbf{t}^{\mathbf{s}'} = (t_{\beta}') \) be the multi–exponent associated to \( \mathbf{s}' \). By Lemma 4.5 we obtain for all \( \mathbf{p} \in \mathbf{D}_1^{\text{type 1}} \)
\[
\sum_{\beta \in \mathbf{p}} (s_{\beta} - t_{\beta}') \leq m - 1.
\]

Note that the statement of the proposition can be easily deduced from the \( i = 2 \) case if \( t_{3,j} = 0 \) for all \( 3 \leq j \leq 2n - 4 \). So we consider the following cases which can appear.

**Case 2:** Suppose that \( \sum_{\beta} t_{\beta}' = 3 \). In this case there exists \( 3 \leq j_3 < j_2 < j_1 \leq 2n - 4 \) such that \( t_{1,j_1} = t_{2,j_2} = t_{3,j_3} = 0 \). Let \( \mathbf{p} \in \widetilde{\mathbf{D}}_2^{\text{type 1}} \) of the following form
\[
\mathbf{p} = \{ \alpha_{1,4}, \ldots, \alpha_{1,p}, \alpha_{2,p}, \ldots, \alpha_{2,2n-3}, \alpha_{3,2n-3} \}.
\]

We suppose that \( j_1 > p > j_2 \), because otherwise there is nothing to show. This yields \( s_{2,p} = \cdots = s_{2,2n-4} = 0 \) and hence
\[
\sum_{\beta \in \mathbf{p}} (s_{\beta} - t_{\beta}') \leq (s_{1,4} - t_{1,4}) + \cdots + (s_{1,2n-3} - t_{1,2n-3}) + (s_{2,2n-3} - t_{2,2n-3}) \leq m - 1.
\]

Similar arguments show
\[
\sum_{\beta \in \mathbf{p}} (s_{\beta} - t_{\beta}') \leq 2(m - 1), \quad \text{for all } \mathbf{p} \in \mathbf{D}^{\text{type 2}} \setminus \mathbf{q}.
\]

Hence Lemma 4.5 and (4.13) together imply
\[
\mathbf{s} - \mathbf{t}^{\mathbf{s}'} \in \mathbf{S}(\mathbf{D}, (m-1)\omega_3).
\]

**Case 3:** In this case we suppose \( \sum_{\beta} t_{\beta}' = 1 \). So let \( t_{3,j_3} = 1 \) for some \( 3 \leq j_3 \leq 2n - 4 \). Suppose first that \( s_{2,2n-3} \neq 0 \). Then we let \( \mathbf{t} \in \mathbf{T}(1) \) to be the multi–exponent with \( \text{supp(} \mathbf{t} \text{)} = \{ \alpha_{3,j_3}, \alpha_{2,2n-3} \} \) if \( s_{1,2n-2} = 0 \) and \( \text{supp(} \mathbf{t} \text{)} = \{ \alpha_{3,j_3}, \alpha_{2,2n-3}, \alpha_{1,2n-2} \} \) otherwise. It is straightforward to verify that \( \mathbf{t} \in \mathbf{S}(\mathbf{D}, \omega_3) \) and
\[
\mathbf{s} - \mathbf{t} \in \mathbf{S}(\mathbf{D}, (m-1)\omega_3).
\]

So suppose that \( s_{2,2n-3} = 0 \). If \( \sum_{k=4}^{2n-3} s_{1,k} \leq m - 1 \) we set \( \mathbf{t} \in \mathbf{T}(1) \) to be the multi–exponent with
\[
\text{supp(} \mathbf{t} \text{)} = \begin{cases} \{ \alpha_{3,j} \}, & \text{if } s_{1,2n-2} = 0 \\ \{ \alpha_{3,j}, \alpha_{k,2n-2} \}, & \text{otherwise} \end{cases}
\]
where \( k = \max\{1 \leq j \leq 2 \mid s_{j,2n-2} \neq 0\} \). By the assumptions and Lemma 4.5 it follows immediately \( \mathbf{t} \in \mathbf{S}(\mathbf{D}, \omega_3) \) and \( \mathbf{s} - \mathbf{t} \in \mathbf{S}(\mathbf{D}, (m-1)\omega_3) \). It remains to consider the case \( \sum_{k=4}^{2n-3} s_{1,k} = m \), which forces \( s_{1,2n-3} \neq 0 \). We set \( \mathbf{t} \in \mathbf{T}(1) \) to be the multi–exponent with \( \text{supp(} \mathbf{t} \text{)} = \{ \alpha_{3,j_3}, \alpha_{1,2n-3} \} \). We get \( \mathbf{t} \in \mathbf{S}(\mathbf{D}, \omega_3) \) and Lemma 4.5 and the construction of \( \mathbf{t} \) yields
\[
\sum_{\beta \in \mathbf{p}} (s_{\beta} - t_{\beta}) \leq m - 1, \quad \text{for all } \mathbf{p} \in \mathbf{D}^{\text{type 1}}.
\]
So let \( p = p_1 \cup p_2 \in D^{\text{type 2}} \) be a type 2 Dyck path where \( p_1 \) is of the form
\[
p_1 = \{ \alpha_{1,3}, \ldots, \alpha_{1,p}, \alpha_{2,p}, \ldots, \alpha_{2,2n-2} \}
\]
for some \( p \leq j_3 \), because otherwise there is nothing to show. Since
\[
\sum_{k=4}^{2n-3} (s_{1,k} - t_{1,k}) = m - 1, \quad \text{and} \quad \sum_{k=4}^{2n-3} (s_{1,k} - t_{1,k}) + \sum_{k=p}^{2n-3} (s_{2,k} - t_{2,k}) \leq m - 1
\]
we get
\[
\sum_{\beta \in p_1} (s_{\beta} - t_{\beta}) + \sum_{\beta \in p_2} (s_{\beta} - t_{\beta}) \leq \sum_{k=3}^{2n-2} (s_{1,k} - t_{1,k}) + (s_{2,2n-2} - t_{2,2n-2}) + \sum_{\beta \in p_2} (s_{\beta} - t_{\beta}) \leq 2(m - 1)
\]
and therefore
\[
s - t \in S(D_1(m - 1)\omega_3).
\]

Case 4: In this case we suppose that \( \sum_{\beta} t'_{\beta} = 2 \).

Case 4.1: First suppose that there exists \( 3 \leq j_3 < j_2 \leq 2n - 4 \) such that \( t'_{2,j_2} = t'_{3,j_3} = 1 \). We set \( t \in T(1) \) to be the multi–exponent with
\[
supp(t) = \begin{cases} 
\{ \alpha_{2,j_2}, \alpha_{3,j_3} \}, & \text{if } s_{1,2n-2} = s_{1,2n-3} = s_{2,2n-2} = 0 \\
\{ \alpha_{2,j_2}, \alpha_{3,j_3}, \alpha_{1,k} \}, & \text{if } s_{1,2n-2} \neq 0 \text{ or } s_{1,2n-3} \neq 0
\end{cases}
\]
where \( k = \min\{2n - 3 \leq j \leq 2n - 2 \mid s_{1,j} \neq 0 \} \). In either case it is an easy calculation to show \( t \in S(D, \omega_3) \) and \( t - t \in S(D, \omega_3) \). It remains to consider the case \( s_{1,2n-2} = s_{1,2n-3} = 0 \) and \( s_{2,2n-2} \neq 0 \). If \( \sum_{s_{\beta} \leq 2m - 1 \text{ we let } t \in T(1) \text{ to be the multi–exponent with } supp(t) = \{ \alpha_{2,j_2}, \alpha_{3,j_3} \} \text{ and again it is straightforward to show } t \in S(D, \omega_3) \text{ and } s - t \in S(D,(m - 1)\omega_3). \}
\)
Otherwise we let \( t \in T(1) \) to be the multi–exponent with \( supp(t) = \{ \alpha_{2,j_2}, \alpha_{2,2n-2} \} \). From the construction of the multi–exponent \( t \) we get
\[
\sum_{\beta \in p} (s_{\beta} - t_{\beta}) \leq M_p((m - 1)\omega_3), \quad \forall p \in D_1^{\text{type 1}} \cup D^{\text{type 2}}.
\]
So let \( p \in D_1^{\text{type 1}} \) be a path of the form
\[
p = \{ \alpha_{1,3}, \ldots, \alpha_{1,p_1}, \alpha_{2,p_1}, \ldots, \alpha_{2,2n-4} \}, \text{ for some } p_1 \leq p_2.
\]
Note that we can further assume that \( p_2 \leq j_3 \), since otherwise the statement is obvious. We extend \( p \) to the following type 2 Dyck path
\[
\overline{p} = p \cup \{ \alpha_{2,3}, \ldots, \alpha_{2,p_1-1}, \alpha_{3,2n-3} \} \cup \{ \alpha_{1,p_1+1}, \ldots, \alpha_{1,j_2}, \alpha_{2,j_2}, \ldots, \alpha_{2,2n-2} \}.
\]
By our assumptions we obtain
\[
\sum_{\beta \in \overline{p}} s_{\beta} = \sum_{\beta \in q} s_{\beta} = 2m,
\]
which implies \( s_{3,p_2} + \cdots + s_{3,j_3} \leq s_{2,p_2 + 1} + \cdots + s_{2,j_2 - 1} \) and hence
\[
\sum_{\beta \in p} (s_{\beta} - t_{\beta}) \leq (s_{1,3} - t_{1,3}) + \cdots + (s_{1,p_1} - t_{1,p_1}) + (s_{2,p_2} - t_{2,p_2}) + \cdots + (s_{2,j_2} - t_{2,j_2}) \leq m - 1
\]
Case 4.2: It remains to consider the case where there exists \( 3 \leq j_3 < j_1 \leq 2n - 4 \) such that \( t'_{1,j_1} = t'_{3,j_3} = 1 \).
Case 4.2.1: Suppose first that $\sum_{k=3}^{2n-4} s_{1,k} \leq m - 1$. If in addition $s_{2,2n-3} \neq 0$, we set $t \in T(1)$ to be the multi–exponent with
\[
\text{supp}(t) = \begin{cases} 
\{\alpha_{3,j_3}, \alpha_{2,2n-3}\}, & \text{if } s_{1,2n-2} = 0. \\
\{\alpha_{3,j_3}, \alpha_{2,2n-3}, \alpha_{1,2n-2}\}, & \text{otherwise}
\end{cases}
\]

So suppose that $s_{2,2n-3} = 0$. If $\sum_{k=4}^{2n-3} s_{1,k} = m$ (this forces $s_{1,2n-3} \neq 0$) we set $t \in T(1)$ to be the multi–exponent with $\text{supp}(t) = \{\alpha_{3,j_3}, \alpha_{1,2n-3}\}$ and otherwise we set $t \in T(1)$ to be the multi–exponent with
\[
\text{supp}(t) = \begin{cases} 
\{\alpha_{3,j_3}\}, & \text{if } s_{1,2n-2} = s_{2,2n-2} = 0. \\
\{\alpha_{3,j_3}, \alpha_{k,2n-2}\}, & \text{otherwise}
\end{cases}
\]
where $k = \max\{1 \leq j \leq 2 \mid s_{j,2n-2} \neq 0\}$. In all cases it is an easy calculation to show that $t \in S(D, \omega_3)$ and $s - t \in S(D, (m - 1)\omega_3)$.

Case 4.2.2: Now we suppose that $\sum_{k=3}^{2n-4} s_{1,k} = m$. If $s_{2,2n-3} = 0$, we set $t \in T(1)$ to be the multi–exponent with $\text{supp}(t) = \{\alpha_{3,j_3}, \alpha_{1,2n-3}\}$. Then the statement can be easily deduced. So suppose from now on that $s_{2,2n-3} \neq 0$. This forces also that $s_{1,3} \neq 0$, because otherwise
\[
\sum_{k=4}^{2n-4} s_{1,k} + s_{2,2n-3} = m + s_{2,2n-3} > m.
\]

If in addition $s_{1,2n-2} = 0$, then we can define $t \in T(1)$ to be the multi–exponent with $\text{supp}(t) = \{\alpha_{2,2n-3}, \alpha_{1,3}\}$ and the statement follows easily. So we can assume that $s_{1,2n-2}$ is also non–zero. This is the only case where there is no multi–exponent $t \in S(D, \omega_3)$ such that $s - t \in S(D, (m - 1)\omega_3)$. We shall define a multi–exponent $t \in S(D, 2\omega_3)$ such that $s - t \in S(D, (m - 2)\omega_3)$. Let $t$ be the multi–exponent with $\text{supp}(t) = \{\alpha_{3,j_1}, \alpha_{1,j_1}, \alpha_{1,3}, \alpha_{1,2n-2}, \alpha_{2,2n-3}\}$. Obviously we have $t \in S(D, 2\omega_3)$. If $p \in D_1^{type 1}$, then we can also deduce immediately
\[
\sum_{\beta \in p} (s_{\beta} - t_{\beta}) \leq m - 2.
\]

So let $p \in D_2^{type 1}$. There is only something to prove if $p$ is of the following form
\[
p = \{\alpha_{1,4}, \ldots, \alpha_{1,p}, \alpha_{2,p}, \ldots, \alpha_{2,2n-3}, \alpha_{3,2n-3}\}, \quad \text{for some where } p \leq j_3.
\]

Since
\[
s_{1,3} + \cdots + s_{1,p} + s_{2,p} + \cdots + s_{2,j_3} \leq m - s_{3,j_3} \leq m - 1 < \sum_{k=3}^{2n-4} s_{1,k}
\]
we obtain
\[
s_{1,4} + \cdots + s_{1,p} + s_{2,p} + \cdots + s_{2,j_3} < s_{1,4} + \cdots + s_{1,2n-4}
\]
and therefore
\[
\sum_{\beta \in p} (s_{\beta} - t_{\beta}) \leq \sum_{k=4}^{2n-3} (s_{1,k} - t_{1,k}) + (s_{2,2n-3} - t_{2,2n-3}) = \sum_{k=4}^{2n-3} s_{1,k} + s_{2,2n-3} - 2 \leq m - 2.
\]

Let $p = p_1 \cup p_2 \in D^{type 2}$ be a type 2 Dyck path. There is only something to show if $p_1$ is of the form
\[
p_1 = \{\alpha_{1,3}, \ldots, \alpha_{1,p}, \alpha_{2,p}, \ldots, \alpha_{2,2n-2}\}, \quad \text{for some where } p \leq j_3.
\]
We get similar as above
\[
\sum_{\beta \in \mathcal{P}} (s_\beta - t_\beta) \leq \sum_{k=3}^{2n-3} (s_{1,k} - t_{1,k}) + (s_{2,2n-3} - t_{2,2n-3}) + (s_{2,2n-2} - t_{2,2n-2}) + \sum_{\beta \in \mathcal{P}_2} (s_\beta - t_\beta) \leq 2(m - 2).
\]

\[\square\]

In order to cover the remaining special cases, we shall prove Lemma 4.3 (ii) for \( n = i = 4 \). Let \( s \in S(D, m\omega_4) \). We will prove the Minkowski property by induction on \( s_{4,4} + s_{1,1} \). If \( s_{4,4} = s_{1,1} = 0 \), we consider two cases.

**Case 1:** In this case we suppose that \( s_{1,6}, s_{2,5} \) and \( s_{3,4} \) are non–zero. Then we define \( t \in S(D, 2\omega_4) \) to be the multi–exponent with \( t_{1,6} = t_{2,5} = s_{3,4} = 1 \) and 0 else. It is immediate that the difference \( s - t \in S(D, (m - 2)\omega_4) \).

**Case 2:** In this case we suppose that one of the entries \( s_{1,6}, s_{2,5} \) or \( s_{3,4} \) is zero. Then there is a Dyck path \( \mathbf{p} \) such that \( s \) is supported on \( \mathbf{p} \) and the statement is immediate.

So suppose that either \( s_{4,4} \neq 0 \) or \( s_{1,1} \neq 0 \). The proof in both cases is similar, so that we can assume \( s_{4,4} \neq 0 \). We set \( s^1 \) to be the multi–exponent obtained from \( s \) by replacing \( s_{4,4} \) by \( s_{4,4} - 1 \). By induction we can find \( t^1 \in S(D, 2\omega_4) \) such that \( s^1 - t^1 \in S(D, (m - 2)\omega_4) \). Now we define \( t \) to be the multi–exponent obtained from \( t^1 \) by replacing \( t_{4,4} \) by \( t_{4,4} + 1 \) if the resulting element stays in \( S(D, 2\omega_4) \) and otherwise we set \( t = t^1 \). In either case \( s - t \in S(D, (m - 2)\omega_4) \).

**Remark.**

1. The set \( S(D, m\omega_1) \) does not satisfy the usual Minkowski sum property in general, e.g. the element \( (m_\beta) \in S(D, 2\omega_4) \) \((n = 4)\) with \( m_\beta = 1 \) for \( \beta \in \{\alpha_{1,6}, \alpha_{2,5}, \alpha_{3,4}\} \) and else 0 is not contained in \( S(D, \omega_4) + S(D, \omega_4) \). Another example is the element \( (m_\beta) \in S(D, 2\omega_3) \) \((n = 4)\) with \( m_\beta = 1 \) for \( \beta \in \{\alpha_{1,3}, \alpha_{1,4}, \alpha_{1,6}, \alpha_{2,5}, \alpha_{3,3}\} \) and else 0.
2. The polytope \( P(D, \epsilon m\omega_i) \) is defined by inequalities with integer coefficients and hence the Minkowski property in Lemma 4.3 (ii) ensures that \( P(D, \epsilon m\omega_i) \) is a normal polytope for \( 1 \leq i \leq 3 \) and \( n \) arbitrary or \( i \) arbitrary and \( 1 \leq n \leq 4 \). The proof is exactly the same as in [12, Lemma 8.7].

Summarizing, we have proved Conjecture 4.3 for arbitrary \( n \) and \( 1 \leq i \leq 3 \) or arbitrary \( i \) and \( 1 \leq n \leq 4 \). Moreover the proof of the general case can be reduced to the proof of Lemma 4.3 (ii) and Lemma 4.3 (iii).

5. Dyck path, polytopes and PBW bases for \( \mathfrak{so}_7 \)

If the Lie algebra is of type \( B_3 \) we shall associate to any dominant integral weight \( \lambda \) a normal polytope and prove that a basis of \( \text{gr} V(\lambda) \) can be parametrized by the lattice points of this polytope. We emphasize at this point that the polytopes we will define for \( B_3 \) are quasi compatible with the polytopes defined in Section 4.2; see Remark 5.1 for more details.

**5.1.** We use the following abbreviations:

\[\begin{align*}
\beta_1 &:= \alpha_{1,5}, \\
\beta_2 &:= \alpha_{1,4}, \\
\beta_3 &:= \alpha_{2,4}, \\
\beta_4 &:= \alpha_{1,3}, \\
\beta_5 &:= \alpha_{2,3}, \\
\beta_6 &:= \alpha_{1,2}, \\
\beta_7 &:= \alpha_{2,2}, \\
\beta_8 &:= \alpha_{3,3}, \\
\beta_9 &:= \alpha_{1,1}.
\end{align*}\]

Let \( \lambda = m_1\omega_1 + m_2\omega_2 + m_3\omega_3 \), \( s_i := s_{\beta_i} \) for \( 1 \leq i \leq 9 \) and set \( (a, b, c) := am_1 + bm_2 + cm_3 \). We denote by \( P(\lambda) \subseteq \mathbb{R}^9_+ \) the polytope determined by the following inequalities:
Theorem. For the rest of this section we prove the following theorem.

5.2. For the rest of this section we prove the following theorem.

**Theorem.** Let $\mathfrak{g}$ be of type $B_3$.

1. The lattice points $S(\lambda)$ parametrize a basis of $V(\lambda)$ and $\text{gr } V(\lambda)$ respectively. In particular,
   \[
   \{X^s v_\lambda \mid s \in S(\lambda)\}
   \]
   forms a basis of $\text{gr } V(\lambda)$.

2. The character and graded $q$-character respectively is given by
   \[
   \text{ch } V(\lambda) = \sum_{\mu \in \mathfrak{h}^*} |S(\lambda)^\mu| e^{\mu},
   \]
   \[
   \text{ch}_q \text{gr } V(\lambda) = \sum_{s \in S(\lambda)} e^{\lambda - \text{wt}(s)} q^{\sum s_\beta}.
   \]

3. We have an isomorphism of $S(\mathfrak{n}^-)$–modules
   \[
   \text{gr } V(\lambda + \mu) \cong S(\mathfrak{n}^-)(v_\lambda \otimes v_\mu) \subseteq \text{gr } V(\lambda) \otimes \text{gr } V(\mu)
   \]
   As in Section 4 we can deduce the above theorem from the following lemma.

**Lemma.**

(i) Let $\lambda, \mu \in P^+$. We have
   \[
   S(\lambda + \mu) = S(\lambda) + S(\mu)
   \]

(ii) For all $\lambda \in P^+$:
   \[
   \dim V(\lambda) = |S(\lambda)|
   \]
The proof of Lemma 5.2 (i) is given in Section 5.3 and the proof of Lemma 5.2 (ii) can be found in Section 5.4.

5.3. Proof of Lemma 5.2 (i). For this part of the lemma it is enough to prove that $S(\lambda) = S(\lambda - \omega_j) + S(\omega_j)$ where $j$ is the minimal integer such that $\lambda(\alpha_j^\vee) \neq 0$. If $j = 3$, many of the inequalities are redundant and the polytope can be simply described by the inequalities

$$s_1 + s_2 + s_3 + s_4 + s_5 \leq (0, 0, 1), \quad s_2 + s_3 + s_4 + s_5 + s_8 \leq (0, 0, 1).$$

The proof of the lemma in that case is obvious. If $j = 2$, there are again redundant inequalities and the polytope can simply described by the inequalities (1) -- (4), (7), (9) -- (10) and (15) -- (16). A straightforward calculation proves the proposition in that case. So let $j = 1$ and $s = (s_i)_{1 \leq i \leq 9} \in S(\lambda)$. We will consider several cases.

Case 1: Assume that $s_9 \neq 0$ and let $t = (t_i)_{1 \leq i \leq 9}$ be the multi–exponent given by $t_9 = 1$ and $t_j = 0$ otherwise. It follows immediately $t \in S(\omega_1)$ and $s - t \in S(\lambda - \omega_1)$.

Case 2: In this case we suppose that $s_9 = 0$ and $s_2, s_6 \neq 0$.

Case 2.1: If in addition $s_3 + s_4 + s_5 + s_8 < (1, 1, 1)$ we let $t = (t_i)_{1 \leq i \leq 9}$ to be the multi–exponent given by $t_6 = t_2 = 1$ and $t_j = 0$ otherwise. It is easy to show that $t \in S(\omega_1)$ and $s - t \in S(\lambda - \omega_1)$, since $s - t \notin S(\lambda - \omega_1)$ is only possible if inequality (2) is violated.

Case 2.2: Now we suppose that $s_3 + s_4 + s_5 + s_8 = (1, 1, 1)$. Together with (5) we obtain $s_4 \geq m_1 > 0$. We let $t = (t_i)_{1 \leq i \leq 9}$ to be the multi–exponent with $t_4 = 1$ and $t_j = 0$ otherwise. Suppose that $s - t \notin S(\lambda - \omega_1)$, which is only possible if (4), (7), (15) or (16) is violated. Assume that (4) is violated, which means $s_5 + s_6 + s_7 + s_8 = (1, 1, 1)$. We obtain

$$s_3 + s_4 + s_5 + s_8 + s_5 + s_6 + s_7 + s_8 = s_3 + s_4 + 2s_5 + s_6 + s_7 + 2s_8 = (2, 2, 2),$$

which is a contradiction to (19). Assume that (7) is violated, which means $s_6 + s_7 = (1, 1, 0)$. We get

$$s_3 + s_4 + s_5 + s_8 + s_6 + s_7 = (2, 2, 1),$$

which is a contradiction to (11). In the remaining two cases (inequality (15) and (16) respectively is violated) we obtain similarly contradictions to (17) and (18) respectively.

Case 3: Assume that $s_2 = s_9 = 0$ and $s_6 \neq 0$. In this case many equalities are redundant. In particular, for a multi–exponent $t$ with $t_j \leq s_j$ for $1 \leq j \leq 9$ we have $s - t \notin S(\lambda - \omega_1)$ if and only if $s - t$ satisfies (2) -- (11), (13) and (19).

Case 3.1: If in addition $s_3 + s_4 + s_5 + s_8 < (1, 1, 1)$ we let $t = (t_i)_{1 \leq i \leq 9}$ to be the multi–exponent given by $t_6 = 1$ and $t_j = 0$ otherwise. It is straightforward to check that $t \in S(\omega_1)$ and $s - t \in S(\lambda - \omega_1)$.

Case 3.2: If $s_3 + s_4 + s_5 + s_8 = (1, 1, 1)$ we let $t = (t_i)_{1 \leq i \leq 9}$ to be the multi–exponent with $t_4 = 1$ and $t_j = 0$ otherwise. The desired property can be checked similarly as in Case 2.2.

Case 4: Assume that $s_6 = s_9 = 0$ and $s_2 \neq 0$. This case works similar to Case 3 and will be omitted.

Case 5: In this case we suppose $s_6 = s_9 = s_2 = 0$ and simplify further the defining inequalities of the polytope. As in Case 3, for a multi–exponent $t$ with $t_j \leq s_j$ for $1 \leq j \leq 9$ we have $s - t \notin S(\lambda - \omega_1)$ if and only if $s - t$ satisfies (2), (5), (6), (8) -- (11), (13) and (19).

Case 5.1: We suppose that $s_4 \neq 0$ and let $t = (t_i)_{1 \leq i \leq 9}$ to be the multi–exponent given by $t_4 = 1$ and $t_j = 0$ otherwise. The desired property follows immediately.
Case 5.2: Let $s_4 = 0$. Then again we can simplify the inequalities and obtain that $s - t \in S(\lambda - \omega_1)$ if and only if $s - t$ satisfies (5), (6), (8) – (10), and (13).

Case 5.2.1: If $s_1 = 0$ we already have $s \in S(\lambda - m_1 \omega_1)$. If $s_1 \neq 0$, let $t = (t_i)_{1 \leq i \leq 9}$ be the multi-exponent with $t_1 = 1$ and $t_j = 0$ otherwise. It follows immediately $t \in S(\omega_1)$ and $s - t \in S(\lambda - \omega_1)$.

Remark. The polytope $P(\lambda)$ is defined by inequalities with integer coefficients and hence the Minkowski property in Lemma 5.2 (i) ensures that $P(\lambda)$ is a normal polytope. The proof is exactly the same as in [12, Lemma 8.7].

5.4. Proof of Lemma 5.2 (ii). We consider the convex lattice polytopes $P_i := P(\omega_i) \subseteq \mathbb{R}_+^n$ for $1 \leq i \leq 3$. By [3, Problem 3, pg. 164] there exists a 3–variate polynomial $E(T_1, T_2, T_3)$ of total degree $\leq 9$ such that

$$E(m_1, m_2, m_3) = \|(m_1 P_1 + m_2 P_2 + m_3 P_3) \cap \mathbb{Z}_+^n\|,$$

for non–negative integers $m_1, m_2, m_3$.

By Lemma 5.2 (i) we get

$$E(m_1, m_2, m_3) = |S(\lambda)|,$$

and by Weyl’s dimension formula, we know that there is another 3–variate polynomial $W(T_1, T_2, T_3)$ of total degree $\leq 9$ such that

$$W(m_1, m_2, m_3) = \dim V(\lambda).$$

Hence it will be enough to prove that both polynomials coincide. In the Appendix we prove that $E(\lambda_0, \lambda_1, \lambda_2) = W(\lambda_0, \lambda_1, \lambda_2)$ for all $(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}_+^3$ with $\lambda_0 + \lambda_1 + \lambda_2 \leq 9$. We claim that this fact already implies $E(T_1, T_2, T_3) = W(T_1, T_2, T_3)$. Let $I = \{(\lambda_0, \lambda_1, \lambda_2) \in \mathbb{Z}_+^3 \mid \lambda_0 + \lambda_1 + \lambda_2 \leq 9\}$ and write

$$E(T_1, T_2, T_3) = \sum_{(n,m,k) \in I} e_{n,m,k} T_1^n T_2^m T_3^k, \quad W(T_1, T_2, T_3) = \sum_{(n,m,k) \in I} w_{n,m,k} T_1^n T_2^m T_3^k$$

We obtain with our assumption that

$$\sum_{(n,m,k) \in I} \left(e_{n,m,k} - w_{n,m,k}\right) \lambda_0^n \lambda_1^m \lambda_2^k = 0.$$

We can translate this into a system of linear equations where the underlying matrix is given by

$$(\lambda_0^{\mu_0} \lambda_1^{\mu_1} \lambda_2^{\mu_2})_{\lambda, \mu \in I}.$$

This matrix is invertible by [6, Theorem 1] and therefore the claim is proven.

6. Construction of favourable modules

In [12] the notion of favourable modules has been introduced and several classes of examples for type $A_n, C_n$ and $G_2$ have been discussed. This section is dedicated to give further examples of favourable modules in type $B_n$. Let us first recall the definition.

6.1. We fix an ordered basis $\{x_1, \ldots, x_N\}$ of $\mathfrak{n}^-$ and an induced homogeneous lexicographic order $<$ on the monomials in $\{x_1, \ldots, x_N\}$. Let $M$ be any finite–dimensional cyclic $\mathcal{U}(\mathfrak{n}^-)$–module with cyclic vector $v_M$ and let

$$X^s v_M = x_1^{s_1} \cdots x_N^{s_N} v_M \in M,$$

where $s \in \mathbb{Z}_+^N$ is a multi–exponent.

Definition. A pair $(M, s)$ is called essential if

$$X^s v_M \notin \text{span}\{X^q v_M \mid q < s\}.$$
If the pair \((M, s)\) is essential, then \(s\) is called an essential multi-exponent and \(X^s\) is called an essential monomial in \(M\). The set of all essential monomials are denoted by \(\text{es}(M) \subseteq \mathbb{Z}_+^N\). We introduce subspaces \(F_s(M)^- \subseteq F_s(M) \subseteq M\):

\[
F_s(M)^- = \text{span}\{X^q v_M \mid q < s\}, \quad F_s(M) = \text{span}\{X^q v_M \mid q \leq s\}.
\]

These subspaces define an increasing filtration on \(M\) and the associated graded space with respect to this filtration is defined by

\[
M^t = \bigoplus_{s \in \mathbb{Z}_+^N} F_s(M)^{-}/F_s(M).
\]

Similar as in Section 3 we can define the PBW filtration on \(M\) and the associated graded space \(\text{gr} M\) with respect to the PBW filtration. The following proposition follows from the construction of \(M^t\) and \(\text{gr} M\) (see also [12, Proposition 1.5]).

**Proposition.** The set \(\{X^s \mid s \in \text{es}(M)\}\) forms a basis of \(M^t\), \(\text{gr} M\) and \(M\).

6.2. We recall the definition of favourable modules.

**Definition.** We say that a finite–dimensional cyclic \(U(n^-)\)-module \(M\) is favourable if there exists an ordered basis \(x_1, \ldots, x_N\) of \(n^-\) and an induced homogeneous monomial order on the PBW basis such that

- There exists a normal polytope \(P(M) \subseteq \mathbb{R}^N\) such that \(\text{es}(M)\) is exactly the set \(S(M)\) of lattice points in \(P(M)\).
- \(\forall k \in \mathbb{N} : \text{dim } U(n^-)(v_M \otimes \cdots \otimes v_M) = |S(M) + \cdots + S(M)|\).

Let \(N\) be a complex algebraic unipotent group such that \(n^-\) is the corresponding Lie algebra. Similarly on the group level, we have a commutative unipotent group \(\text{gr} N\) with Lie algebra \(\text{gr} n^-\) acting on \(\text{gr} M\) and \(M^t\). We associate to the action of the unipotent groups projective varieties, which are called flag varieties in analogy to the classical highest weight orbits (see [12] for details)

\[
\mathfrak{F}(M) = \mathbb{N}[v_M] \subseteq \mathbb{P}(M), \quad \mathfrak{F}(\text{gr} M) = \text{gr} \mathbb{N}[v_M] \subseteq \mathbb{P}(\text{gr} M), \quad \mathfrak{F}(M^t) = \text{gr} \mathbb{N}[v_M] \subseteq \mathbb{P}(M^t).
\]

The following theorem proved in [12] gives a motivation for constructing favourable modules by showing that the flag varieties associated to favourable modules have nice properties.

**Theorem.** Let \(M\) be a favourable \(n^-\)-module.

1. \(\mathfrak{F}(M^t) \subseteq \mathbb{P}(M^t)\) is a toric variety.
2. There exists a flat degeneration of \(\mathfrak{F}(M)\) into \(\mathfrak{F}(\text{gr} M)\), and for both there exists a flat degeneration into \(\mathfrak{F}(M^t)\).
3. The projective flag varieties \(\mathfrak{F}(M) \subseteq \mathbb{P}(M)\) and its abelianized versions \(\mathfrak{F}(\text{gr} M) \subseteq \mathbb{P}(\text{gr} M)\) and \(\mathfrak{F}(M^t) \subseteq \mathbb{P}(M^t)\) are projectively normal and arithmetically Cohen–Macaulay varieties.
4. The polytope \(P(M)\) is the Newton–Okounkov body for the flag variety and its abelianized version, i.e. \(\Delta(\mathfrak{F}(M)) = P(M) = \Delta(\mathfrak{F}(\text{gr} M))\).

6.3. In [12, Section 8] the authors provided concrete classes of examples of favourable modules for the types \(A_n\), \(C_n\) and \(G_2\). The following theorem gives us classes of examples of favourable modules in type \(B_n\) (including multiples of the adjoint representation).

**Theorem.** Let \(\mathfrak{g}\) be the Lie algebra of type \(B_n\) and \(\lambda\) be a dominant integral weight satisfying one of the following

1. \(n = 3\) and \(\lambda\) is arbitrary
(2) $n$ is arbitrary and $\lambda = m\omega_1$ or $\lambda = m\omega_2$

(3) $n$ is arbitrary and $\lambda = 2n\omega_3$ or $n = 4$ and $\lambda = 2m\omega_4$

Then there exists an ordered basis on $n^-$ and an induced homogeneous monomial order on the PBW basis such that $V(\lambda)$ is a favourable $n^-$-module.

Proof. We will show that $V(\lambda)$ satisfies the properties from Definition 6.2. We consider the appropriate polytopes from (4.2) and $P(\lambda)$ from Section 5. These polytopes are normal by Remark 4.5 and Remark 5.3 and therefore the natural candidates for showing the properties from Definition 6.2. For simplicity we will denote these polytopes by $P(\lambda)$ since it will be clear from the context which polytope we mean. The second property follows immediately since on the one hand $U(n^-) (v_{\alpha} \otimes \cdots \otimes v_{\alpha}) \cong V(k\lambda)$ and on the other hand the $k$-fold Minkowski sum parametrizes a basis of $V(k\lambda)$ by Theorem 5.2 (1), Lemma 5.2 (i), Conjecture 4.3 (1) (which is proved in these cases) and Lemma 4.3 (ii). Hence it remains to prove that $\text{es}(V(\lambda))$ (with respect to a fixed order) is exactly the set $S(\lambda)$. Let $\lambda = \sum_{j=1}^{n} m_j a_{j}\omega_j$. By [12, Proposition 1.11] we know that

$$\text{es}(V(\lambda)) \supseteq \text{es}(V(a_1\omega_1)) + \cdots + \text{es}(V(a_1\omega_1)) + \cdots + \text{es}(V(a_n\omega_n)) + \cdots + \text{es}(V(a_n\omega_n)).$$

(6.1)

and hence it is enough to show that there exists an ordered basis on $n^-$ and an induced homogeneous monomial order on a PBW basis such that $\text{es}(V(a_j\omega_j)) = S(a_j\omega_j)$ for all $j$ with $m_j \neq 0$ (recall from Proposition 6.1 that $|\text{es}(V(\lambda))| = |S(\lambda)|$). Suppose first that we are in case (2) or (3) (then $a_j = 1$ and $a_k = 0$ for all $k \neq j$ in case (2) and in case (3) we have $a_3 = 2$ respectively $a_4 = 2$ and $a_k = 0$ else). Then we choose the order given in Section 4.4 (we ordered the roots in the Hasse diagram from the bottom to the top and from left to right) and the induced homogeneous reverse lexicographic order on a PBW basis. By our results we obtain for $s \notin S(a_j\omega_j)$ that

$$X^s v_{a_j\omega_j} \in \text{span}\{X^q v_{a_j\omega_j} \mid q < s\}$$

and hence $\text{es}(V(a_j\omega_j)) \subseteq S(a_j\omega_j)$. Since these sets have the same cardinality we are done. Suppose now that we are in case (1) ($a_j = 1$ for all $j$). Then we choose the following order on the positive roots

$$\beta_7 \succ \beta_6 \succ \beta_5 \succ \beta_4 \succ \beta_3 \succ \beta_2 \succ \beta_1.$$ 

Similar as in Section 4.4 we can prove for all $s \notin S(\omega_j)$ that

$$X^s v_{\omega_j} \in \text{span}\{X^q v_{\omega_j} \mid q < s\},$$

which finishes the proof of the theorem.

\[ \square \]

7. Appendix

In this section we want to complete the proof of Lemma 5.2 (ii) and give a proof of the second part of Theorem 3.3 for type $G_2$.

7.1. We consider the Lie algebra of type $G_2$ and the following order on the positive roots:

$$\beta_1 := 3\alpha_1 + 2\alpha_2 \succ \beta_2 := 3\alpha_1 + \alpha_2 \succ \beta_3 := 2\alpha_1 + \alpha_2 \succ \beta_4 := \alpha_1 + \alpha_2 \succ \beta_5 := \alpha_2 \succ \beta_6 := \alpha_1.$$ 

As before, we extend the above order to the induced homogeneous reverse lexicographic order on the monomials in $S(n^-)$. The order is chosen in a way such that Lemma 4.4 can be applied. Let $\lambda = m_1\omega_1 + m_2\omega_2$, $s_i := s_{\beta_i}$ for $1 \leq i \leq 6$ and set $(a, b) := am_1 + bm_2$. It has been proved in [16] that the lattice points $S(\lambda)$ of the following polytope $P(\lambda)$ parametrize a basis of $\text{gr} V(\lambda)$:
that there exists we will simply show that any multi–exponent the desired further we apply with $\partial$ for all $(\lambda, s)$ constants $C$ we have gr $V(\lambda) \cong S(n^-) \setminus I_\lambda$, where $I_\lambda = S(n^-)(U(n^+) \circ \text{span}\{x^{\lambda(\beta^n)^{-1}} \mid \beta \in R^+\})$.

Proof. Since we have a surjective map $S(n^-)/I_\lambda \rightarrow gr V(\lambda)$, it will be enough to show by the result of [16] that the set $\{X^s v_\lambda \mid s \in S(\lambda)\}$ generates $S(n^-)/I_\lambda$. As in Section 4 we will simply show that any multi–exponent $s$ violating on of the inequalities (1) – (7) can be written as a sum of strictly smaller monomials. It means there exists constants $c_t \in \mathbb{C}$ such that

$$X^s + \sum_{t \prec s} c_t X^t \in I_\lambda.$$  

The proof for all inequalities is similar and therefore we provide the proof only when $s$ violates (7). So let $s$ be a multi–exponent with $s_1 = 0$ and $s_2 + s_3 + s_4 + s_5 + s_6 > (1, 2)$. We apply the operators $\partial_{\beta_3}^{s_4 + s_6} \partial_{\beta_2}^{s_5}$ on $X_{\beta_1}$ and obtain

$$\partial_{\beta_3}^{s_4 + s_6} \partial_{\beta_2}^{s_5} X_{\beta_1} X_{\beta_3} X_{\beta_4} X_{\beta_5} \in I_\lambda,$$

for some non–zero constant $c \in \mathbb{C}$.

Further we apply with $\partial_{\beta_2}^{s_2} \partial_{\beta_4}^{s_3}$ on $X_{\beta_1} X_{\beta_2} X_{\beta_3} X_{\beta_4} X_{\beta_5}$ and obtain with Lemma 4.4 that there exists constants $c_t \in \mathbb{C}$ such that

$$\partial_{\beta_2}^{s_2} \partial_{\beta_4}^{s_3} X_{\beta_1} X_{\beta_2} X_{\beta_3} X_{\beta_4} X_{\beta_5} = X_{\beta_2} X_{\beta_3} X_{\beta_4} X_{\beta_5} + \sum_{t \prec s} c_t X^t \in I_\lambda.$$

Finally, we act with the operator $\partial_{\beta_3}^{s_6}$ on (7.1) and get once more with Lemma 4.4 the desired property.

7.2. In order to complete the proof of Lemma 5.2 (ii) we show that $E(\lambda_0, \lambda_1, \lambda_2) = W(\lambda_0, \lambda_1, \lambda_2)$ for all $(\lambda_0, \lambda_1, \lambda_2) \in I$, where $E(T_1, T_2, T_3)$ and $W(T_1, T_2, T_3)$ are the 3–variate polynomials defined in Section 5.4. From Weyl’s dimension formula we know that

$$W(\lambda_0, \lambda_1, \lambda_2) = \frac{1}{720} (\lambda_0 + 1)(\lambda_1 + 1)(\lambda_2 + 1)(\lambda_0 + \lambda_1 + \lambda_2 + 4)(2\lambda_0 + 2\lambda_1 + \lambda_2 + 5),$$

$$\lambda_0 + \lambda_1 + \lambda_2 + 3)(\lambda_0 + \lambda_1 + 2)(\lambda_1 + \lambda_2 + 2)(2\lambda_1 + \lambda_2 + 3).$$

In order to calculate $E(\lambda_0, \lambda_1, \lambda_2)$ we used the software Java and the following code:

```java
public class B3{
    static int dim = 0;
    public static void main(String[] args){
        int m1, m2, m3 = 0;
        for(m1 = 0; m1 <= 9; m1++){
            for(m2 = 0; m2 <= 9; m2++){
                for(m3 = 0; m3 <= 9; m3++){
                    if(m1 + m2 + m3 <= 9){
```
System.out.println(dim
Now an easy comparison between these numbers led to the following table:

| s1 | s2 | s3 | s4 | s5 | s6 | s7 | s8 | s9 |
|----|----|----|----|----|----|----|----|----|
| 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  | 0  |

The PBW filtration and convex polytopes in type B
| $(\lambda_0, \lambda_1, \lambda_2)$  | $W(\lambda_0, \lambda_1, \lambda_2)$ | $E(\lambda_0, \lambda_1, \lambda_2)$ |
|-----------------|-----------------|-----------------|
| $(0, 0, 0)$ | 1 | 1 |
| $(0, 0, 1)$ | 8 | 8 |
| $(0, 0, 2)$ | 35 | 35 |
| $(0, 0, 3)$ | 112 | 112 |
| $(0, 0, 4)$ | 294 | 294 |
| $(0, 0, 5)$ | 672 | 672 |
| $(0, 0, 6)$ | 1386 | 1386 |
| $(0, 0, 7)$ | 2640 | 2640 |
| $(0, 0, 8)$ | 4719 | 4719 |
| $(0, 0, 9)$ | 8008 | 8008 |
| $(0, 1, 0)$ | 21 | 21 |
| $(0, 1, 1)$ | 112 | 112 |
| $(0, 1, 2)$ | 378 | 378 |
| $(0, 1, 3)$ | 1008 | 1008 |
| $(0, 1, 4)$ | 2310 | 2310 |
| $(0, 1, 5)$ | 4752 | 4752 |
| $(0, 1, 6)$ | 9009 | 9009 |
| $(0, 1, 7)$ | 16016 | 16016 |
| $(0, 1, 8)$ | 27027 | 27027 |
| $(0, 2, 0)$ | 168 | 168 |
| $(0, 2, 1)$ | 720 | 720 |
| $(0, 2, 2)$ | 2079 | 2079 |
| $(0, 2, 3)$ | 4928 | 4928 |
| $(0, 2, 4)$ | 10296 | 10296 |
| $(0, 2, 5)$ | 19656 | 19656 |
| $(0, 2, 6)$ | 35035 | 35035 |
| $(0, 2, 7)$ | 59136 | 59136 |
| $(0, 3, 0)$ | 825 | 825 |
| $(0, 3, 1)$ | 3080 | 3080 |
| $(0, 3, 2)$ | 8008 | 8008 |
| $(0, 3, 3)$ | 17472 | 17472 |
| $(0, 3, 4)$ | 34125 | 34125 |
| $(0, 3, 5)$ | 61600 | 61600 |
| $(0, 3, 6)$ | 104720 | 104720 |
| $(0, 4, 0)$ | 3003 | 3003 |
| $(0, 4, 1)$ | 10192 | 10192 |
| $(0, 4, 2)$ | 24570 | 24570 |
| $(0, 4, 3)$ | 50400 | 50400 |
| $(0, 4, 4)$ | 93500 | 93500 |
| $(0, 4, 5)$ | 161568 | 161568 |
| $(0, 5, 0)$ | 8918 | 8918 |
| $(0, 5, 1)$ | 28224 | 28224 |
| $(0, 5, 2)$ | 64260 | 64260 |
| $(0, 5, 3)$ | 125664 | 125664 |
| $(0, 5, 4)$ | 223839 | 223839 |
| $(0, 6, 0)$ | 22848 | 22848 |
| $(\lambda_0, \lambda_1, \lambda_2)$ | $W(\lambda_0, \lambda_1, \lambda_2)$ | $E(\lambda_0, \lambda_1, \lambda_2)$ |
|--------------------------|--------------------------|--------------------------|
| (0, 6, 1)                | 68544                    | 68544                    |
| (0, 6, 2)                | 149226                   | 149226                   |
| (0, 6, 3)                | 280896                   | 280896                   |
| (0, 7, 0)                | 52326                    | 52326                    |
| (0, 7, 1)                | 150480                   | 150480                   |
| (0, 7, 2)                | 316008                   | 316008                   |
| (0, 8, 0)                | 109725                   | 109725                   |
| (0, 8, 1)                | 304920                   | 304920                   |
| (0, 9, 0)                | 214291                   | 214291                   |
| (1, 0, 0)                | 7                        | 7                        |
| (1, 0, 1)                | 48                       | 48                       |
| (1, 0, 2)                | 189                      | 189                      |
| (1, 0, 3)                | 560                      | 560                      |
| (1, 0, 4)                | 1386                     | 1386                     |
| (1, 0, 5)                | 3024                     | 3024                     |
| (1, 0, 6)                | 6006                     | 6006                     |
| (1, 0, 7)                | 11088                    | 11088                    |
| (1, 0, 8)                | 19305                    | 19305                    |
| (1, 1, 0)                | 105                      | 105                      |
| (1, 1, 1)                | 512                      | 512                      |
| (1, 1, 2)                | 1617                     | 1617                     |
| (1, 1, 3)                | 4096                     | 4096                     |
| (1, 1, 4)                | 9009                     | 9009                     |
| (1, 1, 5)                | 17920                    | 17920                    |
| (1, 1, 6)                | 33033                    | 33033                    |
| (1, 1, 7)                | 57344                    | 57344                    |
| (1, 2, 0)                | 693                      | 693                      |
| (1, 2, 1)                | 2800                     | 2800                     |
| (1, 2, 2)                | 7722                     | 7722                     |
| (1, 2, 3)                | 17640                    | 17640                    |
| (1, 2, 4)                | 35750                    | 35750                    |
| (1, 2, 5)                | 66528                    | 66528                    |
| (1, 2, 6)                | 116025                   | 116025                   |
| (1, 3, 0)                | 3003                     | 3003                     |
| (1, 3, 1)                | 10752                    | 10752                    |
| (1, 3, 2)                | 27027                    | 27027                    |
| (1, 3, 3)                | 57344                    | 57344                    |
| (1, 3, 4)                | 109395                   | 109395                   |
| (1, 3, 5)                | 193536                   | 193536                   |
| (1, 4, 0)                | 10010                    | 10010                    |
| (1, 4, 1)                | 32928                    | 32928                    |
| (1, 4, 2)                | 77350                    | 77350                    |
| (1, 4, 3)                | 155232                   | 155232                   |
| (1, 4, 4)                | 282625                   | 282625                   |
| (1, 5, 0)                | 27846                    | 27846                    |
| (1, 5, 1)                | 86016                    | 86016                    |
| $(\lambda_0, \lambda_1, \lambda_2)$ | $W(\lambda_0, \lambda_1, \lambda_2)$ | $E(\lambda_0, \lambda_1, \lambda_2)$ |
|-----------------|-----------------|-----------------|
| $(1, 5, 2)$     | 191862          | 191862          |
| $(1, 5, 3)$     | 368640          | 368640          |
| $(1, 6, 0)$     | 67830           | 67830           |
| $(1, 6, 1)$     | 199584          | 199584          |
| $(1, 6, 2)$     | 427329          | 427329          |
| $(1, 7, 0)$     | 149226          | 149226          |
| $(1, 7, 1)$     | 422400          | 422400          |
| $(1, 8, 0)$     | 302841          | 302841          |
| $(2, 0, 0)$     | 27              | 27              |
| $(2, 0, 1)$     | 168             | 168             |
| $(2, 0, 2)$     | 616             | 616             |
| $(2, 0, 3)$     | 1728            | 1728            |
| $(2, 0, 4)$     | 4095            | 4095            |
| $(2, 0, 5)$     | 8624            | 8624            |
| $(2, 0, 6)$     | 16632           | 16632           |
| $(2, 0, 7)$     | 29952           | 29952           |
| $(2, 1, 0)$     | 330             | 330             |
| $(2, 1, 1)$     | 1512            | 1512            |
| $(2, 1, 2)$     | 4550            | 4550            |
| $(2, 1, 3)$     | 11088           | 11088           |
| $(2, 1, 4)$     | 23025           | 23025           |
| $(2, 1, 5)$     | 45760           | 45760           |
| $(2, 1, 6)$     | 82467           | 82467           |
| $(2, 2, 0)$     | 1911            | 1911            |
| $(2, 2, 1)$     | 7392            | 7392            |
| $(2, 2, 2)$     | 19683           | 19683           |
| $(2, 2, 3)$     | 43680           | 43680           |
| $(2, 2, 4)$     | 86394           | 86394           |
| $(2, 2, 5)$     | 157464          | 157464          |
| $(2, 3, 0)$     | 7560            | 7560            |
| $(2, 3, 1)$     | 26208           | 26208           |
| $(2, 3, 2)$     | 64141           | 64141           |
| $(2, 3, 3)$     | 133056          | 133056          |
| $(2, 3, 4)$     | 248976          | 248976          |
| $(2, 4, 0)$     | 23562           | 23562           |
| $(2, 4, 1)$     | 75600           | 75600           |
| $(2, 4, 2)$     | 173888          | 173888          |
| $(2, 4, 3)$     | 342720          | 342720          |
| $(2, 5, 0)$     | 62244           | 62244           |
| $(2, 5, 1)$     | 188496          | 188496          |
| $(2, 5, 2)$     | 413343          | 413343          |
| $(2, 6, 0)$     | 145530          | 145530          |
| $(2, 6, 1)$     | 421344          | 421344          |
| $(2, 7, 0)$     | 309672          | 309672          |
| $(3, 0, 0)$     | 77              | 77              |
| $(3, 0, 1)$     | 448             | 448             |
| $(3, 0, 2)$     | 1560            | 1560            |
| $(\lambda_0, \lambda_1, \lambda_2)$ | $W(\lambda_0, \lambda_1, \lambda_2)$ | $E(\lambda_0, \lambda_1, \lambda_2)$ |
|---------------------|---------------------|---------------------|
| (3, 0, 3)            | 4200                | 4200                |
| (3, 0, 4)            | 9625                | 9625                |
| (3, 0, 5)            | 19712               | 19712               |
| (3, 0, 6)            | 37128               | 37128               |
| (3, 1, 0)            | 819                 | 819                 |
| (3, 1, 1)            | 3584                | 3584                |
| (3, 1, 2)            | 10395               | 10395               |
| (3, 1, 3)            | 24576               | 24576               |
| (3, 1, 4)            | 51051               | 51051               |
| (3, 1, 5)            | 96768               | 96768               |
| (3, 2, 0)            | 4312                | 4312                |
| (3, 2, 1)            | 16128               | 16128               |
| (3, 2, 2)            | 41769               | 41769               |
| (3, 2, 3)            | 90552               | 90552               |
| (3, 2, 4)            | 175560              | 175560              |
| (3, 3, 0)            | 15912               | 15912               |
| (3, 3, 1)            | 53760               | 53760               |
| (3, 3, 2)            | 128744              | 128744              |
| (3, 3, 3)            | 262144              | 262144              |
| (3, 4, 0)            | 47025               | 47025               |
| (3, 4, 1)            | 147840              | 147840              |
| (3, 4, 2)            | 334152              | 334152              |
| (3, 5, 0)            | 119119              | 119119              |
| (3, 5, 1)            | 354816              | 354816              |
| (3, 6, 0)            | 269192              | 269192              |
| (4, 0, 0)            | 182                 | 182                 |
| (4, 0, 1)            | 1008                | 1008                |
| (4, 0, 2)            | 3375                | 3375                |
| (4, 0, 3)            | 8800                | 8800                |
| (4, 0, 4)            | 19635               | 19635               |
| (4, 0, 5)            | 39312               | 39312               |
| (4, 1, 0)            | 1750                | 1750                |
| (4, 1, 1)            | 7392                | 7392                |
| (4, 1, 2)            | 20825               | 20825               |
| (4, 1, 3)            | 48048               | 48048               |
| (4, 1, 4)            | 97755               | 97755               |
| (4, 2, 0)            | 8568                | 8568                |
| (4, 2, 1)            | 31200               | 31200               |
| (4, 2, 2)            | 79002               | 79002               |
| (4, 2, 3)            | 168000              | 168000              |
| (4, 3, 0)            | 29925               | 29925               |
| (4, 3, 1)            | 99000               | 99000               |
| (4, 3, 2)            | 232848              | 232848              |
| (4, 4, 0)            | 84700               | 84700               |
| (4, 4, 1)            | 261800              | 261800              |
| $(\lambda_0, \lambda_1, \lambda_2)$ | $W(\lambda_0, \lambda_1, \lambda_2)$ | $E(\lambda_0, \lambda_1, \lambda_2)$ |
|-----------------------------|-----------------------------|-----------------------------|
| $(4, 5, 0)$                 | 207207                      | 207207                      |
| $(5, 0, 0)$                 | 378                         | 378                         |
| $(5, 0, 1)$                 | 2016                        | 2016                        |
| $(5, 0, 2)$                 | 6545                        | 6545                        |
| $(5, 0, 3)$                 | 16632                       | 16632                       |
| $(5, 0, 4)$                 | 36309                       | 36309                       |
| $(5, 1, 0)$                 | 3366                        | 3366                        |
| $(5, 1, 1)$                 | 13824                       | 13824                       |
| $(5, 1, 2)$                 | 38038                       | 38038                       |
| $(5, 1, 3)$                 | 86016                       | 86016                       |
| $(5, 2, 0)$                 | 15561                       | 15561                       |
| $(5, 2, 1)$                 | 55440                       | 55440                       |
| $(5, 2, 2)$                 | 137781                      | 137781                      |
| $(5, 3, 0)$                 | 51975                       | 51975                       |
| $(5, 3, 1)$                 | 168960                      | 168960                      |
| $(5, 4, 0)$                 | 141933                      | 141933                      |
| $(6, 0, 0)$                 | 714                         | 714                         |
| $(6, 0, 1)$                 | 3696                        | 3696                        |
| $(6, 0, 2)$                 | 11704                       | 11704                       |
| $(6, 0, 3)$                 | 29120                       | 29120                       |
| $(6, 1, 0)$                 | 5985                        | 5985                        |
| $(6, 1, 1)$                 | 24024                       | 24024                       |
| $(6, 1, 2)$                 | 64827                       | 64827                       |
| $(6, 2, 0)$                 | 26411                       | 26411                       |
| $(6, 2, 1)$                 | 92400                       | 92400                       |
| $(6, 3, 0)$                 | 85008                       | 85008                       |
| $(7, 0, 0)$                 | 1254                        | 1254                        |
| $(7, 0, 1)$                 | 6336                        | 6336                        |
| $(7, 0, 2)$                 | 19656                       | 19656                       |
| $(7, 1, 0)$                 | 10010                       | 10010                       |
| $(7, 1, 1)$                 | 39424                       | 39424                       |
| $(7, 2, 0)$                 | 42504                       | 42504                       |
| $(8, 0, 0)$                 | 2079                        | 2079                        |
| $(8, 0, 1)$                 | 10296                       | 10296                       |
| $(8, 1, 0)$                 | 15939                       | 15939                       |
| $(9, 0, 0)$                 | 3289                        | 3289                        |

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