Analyzing a New Third Order Iterative Method for Solving Nonlinear Problems

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Authors' contributions

This work was carried out in collaboration between both authors. Author MSN designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author BS managed the analyses of the study managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

In this paper, we propose and analyse a new iterative method for solving nonlinear equations. The method is constructed by applying Adomian method to Taylor’s series expansion. Using one-way analysis of variance (ANOVA), the method is being compared with other existing methods in terms of the number of iterations and solution to convergence between the individual methods used. Numerical examples are used in the comparison to justify the efficiency of the new iterative method.

Keywords: Iterative methods; ANOVA; nonlinear equation; order of convergence.

1 Introduction

Finding zeros or roots of one variable nonlinear equations, \( f(x) = 0 \) efficiently is a significant discussion in numerical analysis and has broad span of applications in all areas of science and engineering. Choosing a single method from the numerous different methods developed by researchers to get analytic solutions of such equations is strenuous, as a result of which various researchers are continuously coming up with new
numerical methods for solving nonlinear equations. Some of these methods were developed using Newton-Raphson method (NR), Taylor's method and methods [1-3] to mention few. Presently, iterative methods are being developed by combining two or more existing methods which results in improving the convergence rate, better accuracy as well as iteration perspective for solving nonlinear equations. Nasr [4] proposed two iterative methods for solving nonlinear algebraic equations by Using Least Square Method” which was developed by combining Rafiullah and Jabeens [5] method and the least square method. The two methods have eighth and sixteenth-order rate of convergence. Napassanan and Montri [6], presented a new higher order iterative method for solving nonlinear equations. The method is based on both Halley’s method see Albeanu [7] and the predictor-corrector technique. The convergence analysis shows that the method is of seventh order. Saqibi and Iqbal [8] presented and analysed two new methods which have fourth and fifth order convergence. The methods were developed by rewriting the nonlinear equations as a coupled system and applying modified decomposition technique. The efficiency and performance of the methods were compared with those of some existing methods. Alyauma [9] developed a two- step iterative method for solving a nonlinear equation, which is derivative free. The method has a convergence of order four and the method requires three evaluations of functions per iteration.

In this paper, we introduce a new iterative method for solving nonlinear equations without second derivative. The method has a third order rate of convergence and it is developed by applying Adomian decomposition method to Taylor’s series expansion around x of higher order. In the method, the assumption that 
\[
\frac{f''(\gamma)}{f'((\gamma)} \approx m, \text{ where } m \text{ is a real number is used. This is done in order to eliminate the evaluation of second derivative in the new scheme. The presence of second derivative in an iterative scheme is usually a drawback. The new proposed scheme is then compared with other existing iterative schemes. The convergence of the proposed method is faster in many cases from the tested numerical examples.}

2 Description of Adomian Decomposition method

Adomian decomposition method is applied to solve problems in mathematics, engineering and other related fields. This method does not require any assumption or linearization to solve any given problem. The idea is as follows:

Consider the equation:
\[
Fu = g(t, x),
\]

where F is a differential operator involving linear and nonlinear terms. Rewriting the equation in operator form as
\[
Lu + Ru + Nu = g
\]

where L is the highest order derivative which is easily invertible, R is the remainder of the linear differential portion, and N is a nonlinear operator. Solving (1) and since L is invertible we get
\[
\begin{align*}
u = L^{-1}Lu & = L^{-1}g - L^{-1}Ru - L^{-1}Nu
\end{align*}
\]

Since F is taken to be a differential operator and L is linear, \(L^{-1}\) would represent integration and with the given initial or boundary conditions,

The solutions of (1) consist of approximate solutions as an infinite series
\[ u(t) = \sum_{n=0}^{\infty} u_n(t) \]

Decomposing the nonlinear term into a series of Adomian polynomials, as

\[ Nu = \sum_{n=0}^{\infty} A_n \]

Where \( A_n \) are called the Adomian polynomials depending on \( u_0, u_1, \cdots, u_n \).

To determine the Adomian polynomials, a grouping parameter, \( \lambda \) is introduced. It should be noted that \( \lambda \) is not a “smallness parameter”.

\[ u(t) = \sum_{n=0}^{\infty} \lambda^n u_n \]  \hfill (2)

And

\[ Nu(\lambda) = \sum_{n=0}^{\infty} \lambda^n A_n \]  \hfill (3)

Then

\[ A_n = \frac{1}{n!} \frac{d^n}{d \lambda^n} \left[ N \left( \sum_{i=0}^{\infty} \lambda^i u_i \right) \right] \bigg|_{\lambda=0}, \quad n = 0, 1, 2, \cdots \]

The first few polynomials are given as

\[ A_0 = N(x_0), \]
\[ A_1 = x_1 N'(x_0), \]
\[ A_2 = x_2 N'(x_0) + \frac{1}{2} x_1^2 N''(x_0), \]
\[ A_3 = x_3 N'(x_0) + x_1 x_2 + \frac{1}{3!} x_1^2 N'''(x_0), \]

\vdots

2.1 Formation of the new method

Consider the nonlinear equation,

\[ f(x) = 0 \]  \hfill (4)

If \( \alpha \) is a root of (4), and \( y \) is the initial guess sufficiently close to \( \alpha \), then (4) can be rewritten using Taylor’s series, see Burden and Faires [10], so that
\begin{equation}
\begin{split}
 f ( \gamma ) + f' ( \gamma ) ( x - \gamma ) + f'' ( \gamma ) \frac{(x-\gamma)^2}{2} + g ( x ) = 0
\end{split}
\end{equation}

where \( g ( x ) \) represents the truncated part of higher order from the third term.

Rearranging (5) we get the following equation,

\begin{equation}
\begin{split}
 g ( x ) = f ( x ) - f ( \gamma ) - f' ( \gamma ) ( x - \gamma ) - f'' ( \gamma ) \frac{(x-\gamma)^2}{2}, \text{ for } f ( x ) = 0
\end{split}
\end{equation}

And from (5) we obtain

\begin{equation}
\begin{split}
 f' ( \gamma ) ( x - \gamma ) = - f ( \gamma ) - f'' ( \gamma ) \frac{(x-\gamma)^2}{2} - g ( x )
\end{split}
\end{equation}

\begin{equation}
\begin{split}
 x - \gamma = - \frac{f ( \gamma )}{f' ( \gamma )} - \frac{f'' ( \gamma ) (x-\gamma)^2}{2} - \frac{g ( x )}{f' ( \gamma )}
\end{split}
\end{equation}

\begin{equation}
\begin{split}
 x = \gamma - \frac{f ( \gamma )}{f' ( \gamma )} - m \frac{(x-\gamma)^2}{2} - \frac{g ( x )}{f' ( \gamma )},
\end{split}
\end{equation}

\{ on the assumption that \( \frac{f'' ( \gamma )}{f' ( \gamma )} = m \), where m is a real number \}

\begin{equation}
\begin{split}
 \therefore x = c + N ( x )
\end{split}
\end{equation}

where \( c = \gamma - \frac{f ( \gamma )}{f' ( \gamma )} \)

and

\begin{equation}
\begin{split}
 N ( x ) = - m \frac{(x-\gamma)^2}{2} - \frac{g ( x )}{f' ( \gamma )} \text{ is a nonlinear function.}
\end{split}
\end{equation}

Comparing this analogously with the Adomian decomposition solution series we obtain \( x = \sum_{n=0}^{\infty} x_n \), where the nonlinear function is given as \( N ( x ) = \sum_{n=0}^{\infty} A_n \)
\[ x = \sum_{n=0}^{\infty} x_n = c + \sum_{n=0}^{\infty} A_n \] from which \( x_0 = c - \frac{f(\gamma)}{f'(\gamma)}. \)

If \( x = x_0 = \gamma \) for initial guess, then from (6),

\[ g(x) = f(x) - f(\gamma) - f'(\gamma)(x - \gamma) - \frac{f''(\gamma)(x - \gamma)^2}{2} \]

\[ \Rightarrow g(x) = f(x) \]

\[ \therefore g(x_0) = f(x_0) \]

\[ N(x_0) = -m \frac{(x_0 - \gamma)^2}{2} - \frac{g(x_0)}{f'(\gamma)} = -m \frac{(x_0 - \gamma)^2}{2} - \frac{f(x_0)}{f'(\gamma)} = A_0, \]

since \( A_0 = N(x_0). \)

from (6) and (9) we have

\[ N(x) = -m \frac{(x - \gamma)^2}{2} - \frac{f(x)}{f'(\gamma)} + \frac{f(\gamma)}{f'(\gamma)} + \frac{f''(\gamma)}{f'(\gamma)}(x - \gamma) + m \frac{(x - \gamma)^2}{2}. \]  \((10)\)

Ignoring the fifth term of the R.H.S of equation (10) and differentiating, we get

\[ N'(x) = -m(x - \gamma) - \frac{f'(x)}{f'(\gamma)} + 1 = -m \left( \frac{f(\gamma)}{f'(\gamma)} - \gamma \right) - \frac{f'(x)}{f'(\gamma)} + 1, \] since \( x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}. \)

\[ \therefore N'(x_0) = 1 + m \frac{f(\gamma)}{f'(\gamma)} - \frac{f'(x_0)}{f'(\gamma)} \]

\[ A_1 = x_1 N'(x_0) = -\frac{f(x_0)}{f'(\gamma)} \left[ 1 + m \frac{f(\gamma)}{f'(\gamma)} - \frac{f'(x_0)}{f'(\gamma)} \right] = -\frac{f(x_0)}{f'(\gamma)} - m \frac{f'(x_0)}{f'^2(\gamma)} + \frac{f(x_0) f'(x_0)}{f'^2(\gamma)} = x_2. \]

Since \( x_1 = \frac{-f(x_0)}{f'(\gamma)} \) from Appendix III

The value of \( X \) is approximated by

\[ X_m = x_0 + x_1 + \cdots + x_m = x_0 + A_0 + A_1 + \cdots + A_{m-1} \]

where \( \lim_{m \to \infty} X_m = x. \)

For \( m=0 \)
\[ x = X_0 = x_0 = c = \frac{f(y)}{f'(y)} \]

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

For \( m = 1 \)

\[ x = X_1 = x_0 + x_1 = c + A_0 = \gamma - \frac{f(y)}{f'(y)} - \frac{f(x_0)}{f'(x_0)} \]

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f(x_{n+1})}{f'(x_{n+1})} \text{ where } x_{n+1}^* = x_n - \frac{f(x_n)}{f'(x_n)} \]

For \( m = 2 \)

\[ x = X_2 = x_0 + x_1 + x_2 = c + A_0 + A_1 = \gamma - \frac{f(y)}{f'(y)} - \frac{f(x_0)}{f'(x_0)} - \frac{m f(x_0) f(y)}{f'(y)} + \frac{f(x_0) f^2(x_0)}{f'^2(y)} \]

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2 \frac{f(x_{n+1})}{f'(x_{n+1})} - m \frac{f(x_0) f(y)}{f'(y)} + \frac{f(x_0) f^2(x_0)}{f'^2(y)} \]

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2 \frac{f(x_{n+1})}{f'(x_{n+1})} - m \frac{f(x_0) f(x_n)}{f'^2(x_n)} + \frac{f(x_0) f^2(x_{n+1})}{f'^2(x_{n+1})} \] (11)

3 Convergence Analysis

Theorem 3.1; Let \( \alpha \in I \) be a simple root of sufficiently differentiable function \( f : I \to R \) for an open interval \( I \). Then the New method (11), has a third order rate of convergence and satisfy the following error equation.

\[ e_{n+1} = (c_2 - m) e_n^3 + 0(e_n^4) \]

Proof

Let \( \alpha \) be a simple root of \( f(x) \) i.e. \( f(\alpha) = 0 \), \( f'(\alpha) \neq 0 \). Assume \( e_n \) to be the error at the \( nth \) iteration so that \( e_n = x_n - \alpha \), By Taylors series expansion

\[ f(x_n) = f(\alpha + e_n) \]
\[ f(x_n) = f(\alpha) + f'(\alpha)x_n + \frac{f''(\alpha)}{2!}x_n^2 + \frac{f'''(\alpha)}{3!}x_n^3 + \frac{f^{(4)}(\alpha)}{4!}x_n^4 + 0(e_n^4) \]

\[ = f'(\alpha) \left[ e_n + \frac{f''(\alpha)}{2f'(\alpha)}e_n^2 + \frac{f'''(\alpha)}{3f'(\alpha)}e_n^3 + \frac{f^{(4)}(\alpha)}{4f'(\alpha)}e_n^4 + 0(e_n^4) \right], \text{ since } f(\alpha) = 0, \]

\[ \therefore f(x_n) = f'(\alpha) \left[ e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + 0(e_n^4) \right], \tag{12} \]

\[ f'(x_n) = f'(\alpha) \left[ 1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_n^3 + 0(e_n^4) \right], \]

where \( c_j = \frac{f^{(j)}(\alpha)}{j!f'(\alpha)}, j = 2, 3, \ldots \)

i.e. \( c_2 = \frac{f^2(\alpha)}{2f'(\alpha)} \Rightarrow 2c_2 = m, \) since \( \frac{f''(\alpha)}{f'(\alpha)} \approx m \) and \( e_n = x_n - \alpha \)

\[ \therefore f'(x_n) = f'(\alpha) \left[ 1 + me_n + 3c_3e_n^2 + 4c_4e_n^3 + 0(e_n^4) \right] \tag{14} \]

and

\[ f^{(2)}(x_n) = f^{(2)}(\alpha) \left[ 1 + 2me_n + (6c_3 + m^2)e_n^2 + (6mc_4 + 8c_4)e_n^3 + (0)e_n^4 \right] \tag{15} \]

From equations (12) and (14) we have

\[ \frac{f(x_n)}{f'(x_n)} = \left\{ e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + 0(e_n^4) \right\} \left\{ 1 + me_n + 3c_3e_n^2 + 4c_4e_n^3 + 0(e_n^4) \right\}^{-1} \]

\[ = e_n + (c_2 - m)e_n^2 + (-mc_2 - 2c_3 + m^2)e_n^3 + 0(e_n^4) \]

\[ \therefore \frac{f(x_n)}{f'(x_n)} = e_n + (c_2 - m)e_n^2 + (-mc_2 - 2c_3 + m^2)e_n^3 + 0(e_n^4) \]

but

\[ x_n - \frac{f(x_n)}{f'(x_n)} = x_{n+1} = \alpha + e_n + \left[ e_n + (c_2 - m)e_n^2 + (-mc_2 - 2c_3 + m^2)e_n^3 + 0(e_n^4) \right] \tag{16} \]

and so

\[ x_{n+1} = \alpha + (mc_2 + 2c_3 - m^2)e_n^2 - (c_2 - m)e_n^3 + 0(e_n^4) \]
Thus the error for $x_{n+1}^*$ is

\[ (mc + 2c - m^2)e_n^3 - (c - m)e_n^2 + 0(e_n^4) \]  

(17)

Using equation (12), we can obtain the value of $f(x_{n+1}^*)$ by substituting $e_n$ in equation (17), and we get:

\[ f(x_{n+1}^*) = f'(\alpha) \left[ -(c - m)e_n^2 + (mc + 2c - m^2)e_n^3 + c, \left[ -(c - m)e_n^2 + (mc + 2c - m^2)e_n^3 \right]^2 + 0(e_n^4) \right] \]

\[ \Rightarrow f(x_{n+1}^*) = f'(\alpha) \left[ -(c - m)e_n^2 + (mc + 2c - m^2)e_n^3 + 0(e_n^4) \right] \]  

(18)

Similarly, using equation (14), we can obtain the value of $f'(x_{n+1}^*)$, substituting $e_n$ in equation (17).

This gives:

\[ f'(x_{n+1}^*) = f'(\alpha) \left[ 1 - m [ (c - m)e_n^2 + (mc + 2c - m^2)e_n^3 ] + 0(e_n^4) \right] \]

\[ \Rightarrow f'(x_{n+1}^*) = f'(\alpha) \left[ 1 - [ (mc - m^2)e_n^2 + (m^2c + 2mc - m^2)e_n^3 ] + 0(e_n^4) \right] \]  

(19)

Using equations (14) and (18) we get

\[ \frac{f(x_{n+1}^*)}{f'(x_n)} = \frac{-(c - m)e_n^2 + (mc + 2c - m^2)e_n^3 + 0(e_n^4)}{1 + me_n^2 + 3c_2e_n^3 + 4c_3e_n^2 + 0(e_n^4)} \]

\[ = \left( m(m) + 2c - 2m^2 \right)e_n^3 - (c - m)e_n^2 + 0(e_n^4) \]  

{since $2c_2 = m$ see (13)}

\[ = (2c - m^2)e_n^3 - (c - m)e_n^2 + 0(e_n^4) \]

\[ \Rightarrow 2 \frac{f(x_{n+1}^*)}{f'(x_n)} = 2 \left[ (2c - m^2)e_n^3 - (c - m)e_n^2 + 0(e_n^4) \right] \]

\[ = (4c - 2m^2)e_n^3 - (2c - 2m)e_n^2 + 0(e_n^4) \]

\[ = (4c - 2m^2)e_n^3 - (m - 2m)e_n^2 + 0(e_n^4) \]

\[ \therefore 2 \frac{f(x_{n+1}^*)}{f'(x_n)} = (4c - 2m^2)e_n^3 + me_n^2 + 0(e_n^4) \]  

(20)
Using equations (12), (15) and (18) we get

\[
\frac{f(x_n) f(x_{n+1}^*)}{f''(x_n)} = \frac{e_n + c_2 e_n^2 + c_3 e_n^3 + 0(e_n^4)}{1 + (6mc_1 + 8c_4) e_n^2 + (6c_3 + m^2) e_n^3 + 2mc_m + (0)e_n^4}
\]

\[
\therefore m \frac{f(x_n) f(x_{n+1}^*)}{f''(x_n)} = (m^2 - mc_2) e_n^3 + (0)e_n^4
\]

(21)

Again, using equations (18), (19) and (15) we get

\[
\frac{f(x_{n+1}^*) f'(x_{n+1}^*)}{f''(x_n)} = \frac{- (c_2 - m) e_n^2 + (mc_2 + 2c_3 - m^2) e_n + 0(e_n^4)}{1 + (6mc_1 + 8c_4) e_n^2 + (6c_3 + m^2) e_n^3 + 2mc_m + (0)e_n^4}
\]

\[
= (m - c_2) e_n^2 + (mc_2 + 2c_3 - 2m^2) e_n^3 + (0)e_n^4
\]

\[
\therefore \frac{f(x_{n+1}^*) f'(x_{n+1}^*)}{f''(x_n)} = (m - c_2) e_n^2 + (mc_2 + 2c_3 - 2m^2) e_n^3 + (0)e_n^4
\]

(22)

Finally, using equations (16), (20), (21) and (22) in (11) we get

\[
x_{n+1} = \alpha + (mc_2 + 2c_3 - m^2) e_n^3 - (c_2 - m) e_n^2 - (4c_3 - 2m^2) e_n^3 - mc_2 e_n^4 - (m^2 - mc_2) e_n^4 + (m - c_2) e_n^4
\]

\[
= \alpha + (3mc_2 - 2m^2) e_n^3 - (2c_2 + m) e_n^2 + 0(e_n^4)
\]

\[
\Rightarrow x_{n+1} = \alpha = (mc_2 - m^2) e_n^3 + 0(e_n^4)
\]

\[
\therefore e_{n+1} = (mc_2 - m^2) e_n^3 + 0(e_n^4)
\]

(23)

equation (23) proves that the New method (11) has a third order rate of convergence.

To demonstrate the behavior and effectiveness of the new method, a comparison between the new method and other good and competent methods was conducted using fifty distinct problems.

The contrast is based on the number of iterations that every method takes before the solution is reached. The methods used in the comparison are:

i. The New method

ii. Noor’s method [11]
The complete fifty problems included in the analysis are in Appendix I, while the details of the computations are in Appendix II.

### 3.1 Computation and analysis of results

Computation of $\frac{f''(x)}{f'(x)} \approx m \in \mathbb{R}$

To use the new iterative method, we need to get the value of $m$. We approximate $m$ from $\frac{f''(x)}{f'(x)}$ where $x$ is the initial point. For instance, to obtain the value of $m$ for the first problem in Appendix I,

\[ i.e., f(x) = \sin^2 x - x^2 + 1, \text{ at } x = 1.3 \]

we obtain

\[ m \approx \frac{f''(x)}{f'(x)} = \frac{2 \cos^2 x - 2 \sin^2 x - 2}{2 \sin x \cos x - 2x} = 1.781616671 \text{ at } x = 1.3 \text{ where } x \text{ is in radians.} \]

which is substituted into equation (11) to get

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - 2 \frac{f(x_{n+1})}{f'(x_n)} - m \frac{f(x_{n+1}) f'(x_n)}{f''(x_n)} + \frac{f'(x_{n+1}) f''(x_n)}{f'''(x_n)} \]

Statistical analysis was carried out from the numerical data to ascertain the findings. A one-way ANOVA test was conducted and the following results were obtained taking into account, 95 percent confidence interval. As a result, we find a significant difference in the number of iterations between the methods analysed, since $P = 0.00 < 0.05$, see Table 1. Also from Table 2, since $P = 0.998 > 0.05$, for the solutions, we conclude that there is no significant difference in the average solutions obtained by the methods used.

| Sum of Squares | Df | Mean Square | F     | Sig. |
|----------------|----|-------------|-------|------|
| Between Groups | 159.532 | 5 | 31.906 | 17.910 | 0.000 |
| Within Groups  | 479.210 | 269 | 1.781 |
| Total          | 638.742 | 274 |       |

| Sum of Squares | Df | Mean Square | F     | Sig. |
|----------------|----|-------------|-------|------|
| Between Groups | 2.340 | 5 | 0.468 | 0.056 | 0.998 |
| Within Groups  | 2229.456 | 269 | 8.288 |
| Total          | 2231.796 | 274 |       |
Test (Duncan Multiple range) and from the results obtained, which are in Table 3, we see that Chun’s method, Basto et al’s method and the New method have the least number of iterations all in the first homogeneous subset and Newton Raphson method in the second homogeneous subset. In the third homogeneous subset we have Ndayawo and Sani’s method and Noor’s method, with higher number of iterations.

Table 3. Homogeneous subsets (Post Hoc Test) showing number of iterations to convergence in respect of the methods

| Methods                | N | Subset for alpha = 0.05 |
|------------------------|---|------------------------|
|                        |   | 1  | 2  | 3  |
| Chun                   | 49 | 2.69 |    |    |
| Basto                  | 45 | 2.84 |    |    |
| New Method             | 50 | 2.94 |    |    |
| Newton Raphson         | 46 |       | 3.72 |    |
| Ndayawo & Sani         | 45 |       |     | 4.29 |
| Noor                   | 40 |       |     | 4.75 |
| Sig.                   |   | 0.411 | 1.000 | 0.100 |

As can be seen from Table 4 below, the descriptive statistics, shows the mean, standard deviation and standard error of the number of iterations to convergence in respect of each of the methods.

Basto et al’s method has the least standard deviation of 0.796 and standard error of 0.119, but five of the tested problems diverge, which is followed by Newton-Raphson’s method with a standard deviation of 1.167 and standard error of 0.172, and four of tested problems did not converge. Next with a standard deviation of 1.185 and a standard error of 0.168 is the New method and none of the tested problems diverge. Then Chun’s method with a standard deviation of 1.294 and standard error of 0.185 and one of the tested problems diverges.

Table 4. Descriptive statistics for each method in respect of number of iterations to convergence

| No    | Mean | Standard Deviation | Standard Error | 95% Confidence Interval for Mean | Min | Max |
|-------|------|--------------------|----------------|---------------------------------|-----|-----|
|       |      |                    |                | Lower Bound                      |     |     |
| Noor  | 40   | 4.75               | 1.645          | 0.260                           | 2   | 9   |
| Newton| 46   | 3.72               | 1.167          | 0.172                           | 1   | 7   |
| Raphson|      |                    |                |                                 |     |     |
| Chun  | 49   | 2.69               | 1.294          | 0.185                           | 1   | 8   |
| Ndayawo & Sani | 45 | 4.29               | 1.753          | 0.261                           | 2   | 11  |
| Basto | 45   | 2.84               | 0.796          | 0.119                           | 2   | 5   |
| New Method | 50 | 2.94               | 1.185          | 0.168                           | 1   | 6   |
| Total | 275  | 3.49               | 1.527          | 0.092                           | 1   | 11  |

4 Conclusion

In this paper, we present a new iterative method for solving nonlinear equations. The new method is compared with four other existing good methods for the number of iterations to convergence. There are fifty problems used in the comparison which are given in Appendix I, while the complete results of the number of iterations in respect of each method are in Appendix II. From the analysis of variance results obtained, it shows that there is a significant difference between the number of iterations to convergence obtained from the methods used. The New method converges to all the fifty solutions, Chun’s method converges to forty-
nine out of the fifty solutions while there is divergence in four of the solutions in the case Newton Raphson. For the Basto et al’s method, five of the solutions of the test problems diverge. Lastly, for Noor’s method, solutions of ten out of the fifty tested problems diverge.

As for the number of iterations, Chun’s method has the smallest mean of 2.69 followed by Basto’s method and New method with 2.84 and 2.94 respectively. In the case of the standard deviation, Basto’s method has the best standard deviation of 0.796 followed by Newton Raphson’s method and New method with standard deviations of 1.167 and 1.307 respectively. Thus the New method is comparatively good in all the phases being considered.

**Competing Interests**

Authors have declared that no competing interests exist.

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Appendix I. Fifty problems used in the comparison

Fifty different Tested Problems

1. \( \sin^2 x - x^2 + 1 = 0 \)
2. \( x^2 - e^x - 3x + 2 = 0 \)
3. \( \cos x - x = 0 \)
4. \( (x-1)^3 - 1 = 0 \)
5. \( x^3 - 10 = 0 \)
6. \( xe^x - \sin^2 x + 3cos x - x = 0 \)
7. \( xe^x - \sin^2 x + 3cos x + 5 = 0 \)
8. \( e^{x^3 + 7x - 30} - 1 = 0 \)
9. \( e^{2x} + 2x + 0.1 = 0 \)
10. \( 2x^3 - x^2 - 7x + 6 = 0 \)
11. \( x^3 - 9x + 1 = 0 \)
12. \( x^3 - 3x - 1.06 = 0 \)
13. \( x^3 - 6x + 4 = 0 \)
14. \( 2x - 3\sin x - 5 = 0 \)
15. \( x^3 - 3x + 1 = 0 \)
16. \( 3x - \ln x - 16 = 0 \)
17. \( \cos x - 2x + 3 = 0 \)
18. \( x + \ln x - 2 = 0 \)
19. \( x^3 - x - 0.1 = 0 \)
20. \( x^4 - 12x + 7 = 0 \)
21. \( x + \sin x + 0.5 = 0 \)
22. \( x - 2 - e^{-x} = 0 \)
23. \( x^3 + 4x^2 + 8x + 8 = 0 \)
24. \( 2x - \ln x - 7 = 0 \)
25. \( x^2 - 1.25x + 0.25 = 0 \)
26. \( x^2 + 5.15x + 2.37 = 0 \)
27. \( x^4 - 11x + 8 = 0 \)
28. \( \ln x - x + 3 = 0 \)
29. \( e^x - x - 3 = 0 \)
30. \( 2x^2 - 12x + 11 = 0 \)
31. \( x^4 - 5x^2 + x + 3 = 0 \)
32. \( x^3 + 4x - 9 = 0 \)
33. \( x^7 - 3x^6 + 3x^5 - 13x^4 + 43x^3 - 57x^2 + 33x - 7 = 0 \)
34. \( \sin x - \frac{x}{2} = 0 \)
35. \( x^5 + x - 1000 = 0 \)
36. \( \sqrt{x} - \frac{1}{x} - 3 = 0 \)
37. \( \ln x + \sqrt{x} - 5 = 0 \)
38. \( e^x \sin x - 2x - 5 = 0 \)
39. \( e^{-x} - \cos x = 0 \)
40. \( \cos^2 x - \frac{x}{5} = 0 \)
41. \( (1 + \cos x)(e^x - 2) = 0 \)
42. \( e^{-x} + \sin x - 1 = 0 \)
43. \( xe^{-x} - 0.1 = 0 \)
44. \( x^2 + \sin x + x = 0 \)
45. \( \sin(2 \cos x) - 1 - x^2 + e^{\sin x} = 0 \)
46. \( x^6 - 10x^3 + x^2 - x + 3 = 0 \)
47. \( x^4 - x^3 + 11x - 7 = 0 \)

48. \( x^3 - \cos x + 2 = 0 \)

49. \( \sqrt{x - \cos x} = 0 \)

50. \( \ln x - x^3 + 2 \sin x = 0 \)

Problems 1-8 are from Noor [11], problems 9-34 are from Ndayawo and Sani [14], problems 35-38 are from Chun and Neta [15], problems 39-42 are from Kumar et al [16] and problems 42-50 are from Soleymani [17].
## APPENDIX II. Comparison between number of iterations

| $f''(x)$ | Initial point $x_0$ | $x_n$ (SOLUTION) | Noor | Nr | Chun | Basto | Ndayawo & Sani | New Method |
|----------|---------------------|------------------|-------|----|------|-------|----------------|------------|
| $f''(x) \approx m$ | $x_0$ |                     |       |    |      |       |                |            |
| 1        | 1.781616671         | 1.3              | 1.404491649 | 4  | 4  | 3    | 2     | 3            | 3          |
| 2        | 0.8434823573        | 2                | 0.257330285 | 9  | 5  | 4    | 5     | 6            | 4          |
| 3        | -0.06469185661      | 1.7              | 0.739085133 | 5  | 4  | 4    | 3     | 6            | 3          |
| 4        | 0.8                 | 3.5              | 2                | 6  | 6  | 4    | 3     | 5            | 3          |
| 5        | 1.3333333333       | 1.5              | 2.15443469      | 5  | 5  | 5    | 3     | 4            | 5          |
| 6        | -4.874055511        | -2               | -0.915306601    | 5  | 4  | 6    | DIV   | DIV          | 5          |
| 7        | -4.864176091        | -2               | -1.207647827    | 5  | DIV| 3    | DIV   | 4            | 4          |
| 8        | 14.14285714         | 3.5              | 3                | DIV | 8  | DIV  | 11    | 6            |            |
| 9        | 0.9500416252        | -0.05            | -0.31584581     | 4  | 4  | 2    | 3     | 3            | 3          |
| 10       | -1.94283593         | 0.86             | 1                | 5  | 4  | 2    | 3     | 4            | 6          |
| 11       | -0.0736303089       | 0.11             | 0.111264158     | 2  | 2  | 1    | 2     | 2            | 1          |
| 12       | 0.7977207977        | -0.35            | -0.370252219    | 3  | 3  | 2    | 4     | 2            | 1          |
| 13       | -0.8639030366       | 0.67             | 0.732050808     | 3  | 4  | DIV  | 2     | 3            | 2          |
| 14       | 0.4077312625        | 2.5              | 2.88323087      | DIV | 3  | 3    | 3     | 4            | 3          |
| 15       | -0.7406576142       | 0.33             | 0.34729635      | 3  | 3  | 2    | 4     | 3            | 2          |
| 16       | 0.01266303659       | 5.3              | 5.926476625     | DIV | 2  | 2    | 2     | 4            | 2          |
| 17       | 0.02359377230       | 1.5              | 1.523592933     | 5  | 3  | 1    | 2     | 3            | 2          |
| 18       | -0.1666666667       | 2                | 1.557145599     | 7  | 3  | 2    | 3     | 5            | 2          |
| 19       | 0.6185567010        | -0.1             | -0.101031258    | 2  | 2  | 2    | 2     | 2            | 2          |
| 20       | -0.3598004626       | 0.58             | 0.59365854      | 3  | 4  | 2    | 3     | 3            | 2          |
| 21       | 0.2553419212        | -0.5             | -0.251318625    | 7  | 3  | 2    | 3     | 4            | 2          |
| 22       | -0.1192092220       | 2                | 2.120028239     | 5  | 3  | 3    | 2     | 3            | 2          |
| 23       | 0.6666666667        | -1               | -2               | 5  | 1  | 1    | 4     | 7            | 1          |
| 24       | 0.04761904762       | 3.5              | 4.219906484     | DIV | 3  | 3    | 2     | 4            | 3          |
| 25       | -2.352941176        | 0.2              | 0.25             | DIV | 3  | 2    | 3     | 4            | 2          |
| 26       | 0.4210526316        | -0.2             | -0.510871855    | 4  | 4  | 2    | 4     | 4            | 4          |
| 27       | -0.6771332110       | 0.73             | 0.757149516     | 3  | 3  | 4    | 2     | 4            | 5          |
| 28       | 0.1666666667        | 3                | 4.505241496     | DIV | 5  | 3    | 3     | 7            | 3          |
| $f''(x)$ | Initial point ($x_0$) | $x_n$ (SOLUTION) | Noor | Nr | Chun | Basto | Ndayawo & Sani | New Method |
|-----------|----------------------|------------------|------|----|------|-------|---------------|------------|
| $f''(x)$ | $x_0$ |  |  |   |      |       |               |            |
| 29        | -0.05239569650       | -3               | -2.947530903 | 6   | 2   | 1     | 2     | 3             | 2          |
| 30        | -0.4807692308        | 0.92             | 1.129171307  | 5   | 4   | 2     | 3     | 5             | 2          |
| 31        | -1.272727273         | -3               | -2.198691243 | 7   | 6   | 3     | 4     | 6             | 4          |
| 32        | 0.7035830619         | 2.25             | 1.464595701  | 4   | 5   | 3     | 3     | 4             | 3          |
| 33        | -4.639816912         | 0.21             | 0.593685833  | DIV | 7   | 5     | 3     | 8             | 5          |
| 34        | 0.9992193599         | 2                | 1.895494267  | 3   | 3   | 2     | 2     | 3             | 2          |
| 35        | 0.9992193599         | 4                | 3.977899394  | 3   | 3   | 2     | 2     | 2             | 2          |
| 36        | -1.500000000         | 1                | 9.633595562  | DIV | 5   | 3     | DIV   | DIV           | 4          |
| 37        | -0.2559830602        | 3                | 8.309432693  | DIV | 5   | 3     | 5     | DIV           | 3          |
| 38        | -0.1883330467        | -1               | -2.52324523  | 9   | 3   | 2     | 3     | 6             | 3          |
| 39        | 0.3794947184         | 1.5              | 1.29269572   | 4   | 4   | 2     | 3     | 5             | 3          |
| 40        | 1.037575341          | 0.5              | 1.085982678  | 4   | 4   | 2     | 3     | 4             | 3          |
| 41        | -0.04148371676       | 0.9              | 0.693147181  | 7   | 3   | 2     | 3     | 4             | 2          |
| 42        | 0.8420313256         | 2.3              | 2.076831274  | 4   | 4   | 2     | 2     | 3             | 2          |
| 43        | -1.909090909         | -0.1             | 0.111832559  | 5   | 4   | 3     | 3     | 5             | 2          |
| 44        | 0.6670275325         | 0.3              | 0.00E+00     | DIV | 5   | 3     | DIV   | DIV           | 4          |
| 45        | -1.548924794         | -0.82            | 0.8135737292 | 3   | 3   | 2     | 2     | 4             | 2          |
| 46        | 1.984922746          | 0.81             | 0.658604871  | 6   | DIV | 3     | 3     | 4             | 3          |
| 47        | 0.3761100470         | 0.9              | 0.6450239555 | 5   | 3   | 2     | 4     | 5             | 3          |
| 48        | -2.244201985         | -1.1             | -1.172577964 | 4   | 4   | 2     | 3     | 4             | 3          |
| 49        | 0.07049049813        | 1.3              | 0.6417143709 | DIV | 4   | 2     | 3     | 6             | 2          |
| 50        | 2.254786488          | 1.4              | 1.297997743  | 4   | DIV | 2     | 4     | DIV           | 3          |

**DIV**: Diverging  
*Newton Raphson’s method: NR*
APPENDIX III

If $x = x_0 = \gamma$ for initial guess

Then from equation (9), we have

$$N(x_0) = -m \left( \frac{(x_0 - \gamma)^2}{2} \right) - \frac{g(x_0)}{f'(\gamma)} = -m \left( \frac{(x_0 - \gamma)^2}{2} \right) - \frac{f(x_0)}{f'(\gamma)}$$

at initial guess

$$\Rightarrow N(x_0) = -m \left( \frac{(x_0 - x_0)^2}{2} - \frac{f(x_0)}{f'(\gamma)} \right)$$

$$\therefore N(x_0) = x_i = \frac{f(x_0)}{f'(\gamma)}$$

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