SAALSCHÜTZ’S THEOREM AND SUMMATION FORMULAE INVOLVING GENERALIZED HARMONIC NUMBERS

CHUANAN WEI

Department of Mathematics
Shanghai Normal University, Shanghai 200234, China

Abstract. In terms of the derivative operator, integral operator and Saalschütz’s theorem, two families of summation formulae involving generalized harmonic numbers are established.

1. Introduction

For a complex variable \( x \), define the shifted factorial to be

\[
(x)_0 = 0 \quad \text{and} \quad (x)_n = x(x + 1) \cdots (x + n - 1) \quad \text{with} \quad n \in \mathbb{N}.
\]

Following Andrews, Askey and Roy [2, Chapter 2], define the hypergeometric series by

\[
\sum_{k=0}^{\infty} \frac{(a_0)_k(a_1)_k \cdots (a_r)_k}{(1)_k(b_1)_k \cdots (b_s)_k} z^k,
\]

where \( \{a_i\}_{i \geq 0} \) and \( \{b_j\}_{j \geq 1} \) are complex parameters such that no zero factors appear in the denominators of the summand on the right hand side. Then Saalschütz’s theorem (cf. [2, p. 69]) can be stated as

\[
\sum_{k=0}^{n} \frac{(c-a)_k(c-b)_k}{(1)_k(c-a-b)_k} z^k.
\]

For a complex number \( x \) and a positive integer \( \ell \), define generalized harmonic numbers of \( \ell \)-order to be

\[
H^{(\ell)}_n(x) = 0 \quad \text{and} \quad H^{(\ell)}_n(x) = \sum_{k=1}^{n} \frac{1}{(x+k)^{\ell}} \quad \text{with} \quad n \in \mathbb{N}.
\]

When \( x = 0 \), they become harmonic numbers of \( \ell \)-order

\[
H^{(\ell)}_0 = 0 \quad \text{and} \quad H^{(\ell)}_n = \sum_{k=1}^{n} \frac{1}{k^{\ell}} \quad \text{with} \quad n \in \mathbb{N}.
\]

Fixing \( \ell = 1 \) in \( H^{(\ell)}_0(x) \) and \( H^{(\ell)}_n(x) \), we obtain generalized harmonic numbers

\[
H_0(x) = 0 \quad \text{and} \quad H_n(x) = \sum_{k=1}^{n} \frac{1}{x+k} \quad \text{with} \quad n \in \mathbb{N}.
\]
When $x = 0$, they reduce to classical harmonic numbers

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{with} \quad n \in \mathbb{N}.$$ 

For a differentiable function $f(x)$, define the derivative operator $D_x$ by

$$D_x f(x) = \frac{d}{dx} f(x).$$

For an integrable function $g(x)$, define the integral operator $I_x$ by

$$I_x g(x) = \int_{0}^{x} g(x) dx.$$ 

In order to explain the relation of the derivative operator and generalized harmonic numbers, we introduce the following lemma.

**Lemma 1.** Let $x$ and \{a_j, b_j, c_j, d_j\}$_{j=1}^{s}$ be all complex numbers. Then

$$D_x \prod_{j=1}^{s} a_j x + b_j \prod_{j=1}^{s} c_j x + d_j = \prod_{j=1}^{s} a_j x + b_j \prod_{j=1}^{s} c_j x + d_j.$$

**Proof.** It is not difficult to verify the case $s = 1$ of Lemma 1. Suppose that

$$D_x \prod_{j=1}^{m} a_j x + b_j c_j x + d_j = \prod_{j=1}^{m} a_j x + b_j c_j x + d_j$$

is true. We can proceed as follows:

$$D_x \prod_{j=1}^{m+1} a_j x + b_j c_j x + d_j = D_x \prod_{j=1}^{m} a_j x + b_j c_j x + d_j + \sum_{j=1}^{m} a_j x + b_j c_j x + d_j$$

$$= \prod_{j=1}^{m} a_j x + b_j c_j x + d_j + \sum_{j=1}^{m} a_j x + b_j c_j x + d_j$$

$$= \prod_{j=1}^{m} a_j x + b_j c_j x + d_j + \sum_{j=1}^{m} a_j x + b_j c_j x + d_j$$

$$= \prod_{j=1}^{m+1} a_j x + b_j c_j x + d_j + \sum_{j=1}^{m} a_j x + b_j c_j x + d_j$$

This proves Lemma 1 inductively. \(\square\)

Setting $a_j = 1, b_j = r - j + 1, c_j = 0, d_j = j$ in Lemma 1, it is easy to find that

$$D_x \left( \binom{x + r}{s} \right) = \binom{x + r}{s} \left\{ H_r(x) - H_{r-s}(x) \right\},$$

where $r, s \in \mathbb{N}_0$ with $s \leq r$. Besides, we have the following relation:

$$D_x H_n^{(r)}(x) = -\ell H_n^{(r+1)}(x).$$
As pointed out by Richard Askey (cf. [1]), expressing harmonic numbers in accordance with differentiation of binomial coefficients can be traced back to Isacc Newton. In 2003, Paule and Schneider [7] computed the family of series:

$$W_n(\alpha) = \sum_{k=0}^{n} \binom{n}{k}^{\alpha} \{1 + \alpha(n - 2k)H_k\}$$

with $\alpha = 1, 2, 3, 4, 5$ by combining this way with Zeilberger's algorithm for definite hypergeometric sums. According to the derivative operator and the hypergeometric form of Andrews' $q$-series transformation, Krattenthaler and Rivoal [1] deduced general Paule-Schneider type identities with $\alpha$ being a positive integer. More results from differentiation of binomial coefficients can be seen in the papers [9, 13, 14, 15]. For different ways and related harmonic number identities, the reader may refer to [3, 4, 5, 8, 10, 12]. It should be mentioned that Sun [11] showed recently some congruence relations concerning harmonic numbers to us.

Inspired by the work just mentioned, we shall explore, by means of the derivative operator, integral operator and (1), closed expressions for the following two families of series:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2x-y+n+k}{k}\right) \left(\frac{y+k}{y+k}\right) \frac{H_k^{(2)}(x)}{y+k},$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{y+k}{y-k}\right) \left(\frac{y}{y+k}\right) \frac{H_k^{(\ell)}(x)}{y+k},$$

where $t \in \mathbb{N}$. In order to avoid appearance of complicated expressions, our explicit formulae are offered only for $t = 1, 2$ and $\ell = 1, 2, 3, 4$.

2. THE FIRST FAMILY OF SUMMATION FORMULAE INVOLVING GENERALIZED HARMONIC NUMBERS

**Theorem 2.** Let $x$ and $y$ be both complex numbers. Then

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{2x-y+n+k}{k}\right) \left(\frac{y+k}{y+k}\right) \frac{H_k^{(2)}(x)}{y+k} = \frac{(x-y+n)}{(x+n)} H_n^{(2)}(x) - H_n^{(2)}(x-y).$$

**Proof.** Perform the replacements $a \rightarrow 1 + z$, $b \rightarrow y$, $c \rightarrow 1 + x$ in (1) to get

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{z+k}{k}\right) \left(\frac{y+k}{y+k}\right) \frac{y}{y+k} = \frac{(x-y+n)}{(x+n)} \left[\frac{(x-z+n)}{(x-z-1+n)} H_n(x-z+n) - H_n(x)\right].$$

(2)

Applying the derivative operator $D_x$ to both sides of (2), we gain

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left(\frac{z+k}{k}\right) \left(\frac{y+k}{y+k}\right) \frac{y}{y+k} \left[H_k(y+z-x+n) - H_k(x)\right] = \frac{(x-y+n)}{(x+n)} H_n(x-z+n) - H_n(x) - H_n(x-y-z-1).$$

$$= \frac{(x-y+n)}{(x+n)} \left\{H_n(x-y) + H_n(x-z-1) - H_n(x) - H_n(x-y-z-1)\right\}.$$
The equivalent form of it reads as

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{z+k}{k} y^k}{(x+y-z-1)^k (x+y-z-2)^{k+1}} \cdot \frac{1}{\underbrace{(x+y-z-1)(x+y-z-2)\cdots(x+y-z-n+1)}_{n}}
\]

\[
= \frac{(x-y+n)(x-z+n)}{(x+n)^2} \left\{ \sum_{i=1}^{k} (x+y-z-n+1) \left[ \frac{H_n(x-y) + H_n(x-z-1)}{2x-y-z+n} - \frac{H_n(x) + H_n(x-y-z-1)}{2x-y-z+n} \right] \right\}. \quad (3)
\]

By means of L'Hôpital rule, we achieve

\[
\lim_{z \to 2x-y+n} \frac{H_n(x-y) + H_n(x-z-1)}{2x-y-z+n} = -H_n^2(y-x-n-1) = -H_n^2(x-y),
\]

\[
\lim_{z \to 2x-y+n} \frac{H_n(x) + H_n(x-y-z-1)}{2x-y-z+n} = -H_n^2(-x-n-1) = -H_n^2(x).
\]

Taking the limit \( z \to 2x-y+n \) on both sides of (3) and using (4)-(5), we attain Theorem 2 to complete the proof. \( \square \)

Choosing \( x = p, y = q \) in Theorem 2 with \( p, q \in \mathbb{N}_0 \) and utilizing (2), we obtain the summation formula involving harmonic numbers of 2-order.

**Corollary 3.** Let \( p \) and \( q \) be both nonnegative integers satisfying \( p \geq q \). Then

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{2p-q+n+k}{k} p^k}{(p+k)^k} q + k H_{p+k}^{(2)} = \frac{(p-q+n)^2}{(p+n)^2} \left\{ H_{p-q}^{(2)} + H_{p+n}^{(2)} - H_{p-q+n}^{(2)} \right\}.
\]

**Theorem 4.** Let \( x \) and \( y \) be both complex numbers. Then

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(x+y+n+k) y^k}{(x+k)^k} \frac{(y-1)y}{(y+k-1)(y+k)} H_k^{(2)}(x)
\]

\[
= \frac{n^2 + n(1 + x-y) + (1 + x-y)^2}{(1 + x-y)^2} \left\{ \left\{ \frac{x-y+n}{n} \right\}^2 \left\{ H_n^{(2)}(x) - H_n^{(2)}(x-y) \right\} \right. + \left. \frac{n^2 + n(1 + x-y) (x+y+n)}{(1 + x-y)^2} \right\}
\]

\[
+ \frac{n^2 + n(1 + x-y) (x+y+n)}{(1 + x-y)^2} \right\}^2.
\]

**Proof.** Replace \( c \) by \( 1+c \) in (11) to get

\[
\binom{a,b,-n}{1+c,a+b-c-n} = \frac{(1+c-a)n(1+c-b)n}{(1+c)n(1+c-a-b)n}.
\]
The combination of (1) and the last equation gives
\[ 3F_2 \left[ \begin{array}{c} a, b, -n \\ 1 + c, 1 + a + b - c - n \end{array} \right] = \left\{ 1 + \frac{n(c - a - b)}{(c - a)(c - b)} \right\} \frac{(c - a)_n(c - b)_n}{(1 + c)_n(c - a - b)_n}. \] (6)

Employ the substitutions \( a \to 1 + z, \ b \to y - 1, \ c \to x \) in (6) to gain
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(z+k)}{k} \frac{(y+k)}{y+k} \frac{(y-1)y}{(y+k-1)(y+k)} \left\{ H_k(y+z-x-n) - H_k(x) \right\} \]
\[ = \frac{(x-y+1)(x-z-1) + n(x-y-z) \frac{(x-y+n)x-z-1+n}{n}(x-y-z-1+n)}{(x-y+1)(x-z-1+n) \frac{(x-n)(x-y-z-1+n)}{n}} \times \left\{ H_n(x+y) + H_n(x-z-1) - H_n(x) - H_n(x-y-z-1) \right\} \]
\[ + \frac{n(z+1)(2x-y-z+n) \frac{(x-y+n)x-z-1+n}{n}(x-y-z-1+n)}{(x-y+1)^2(x-z-1+n)^2 \frac{(x-n)(x-y-z-1+n)}{n}}. \]

The equivalent form of it can be expressed as
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(z+k)}{k} \frac{(y+k)}{y+k} \frac{(y-1)y}{(y+k-1)(y+k)} \sum_{i=1}^{k} \frac{1}{(x+i)(y+z-x-n+i)} \]
\[ = \frac{(x-y+1)(x-z-1) + n(x-y-z) \frac{(x-y+n)x-z-1+n}{n}(x-y-z-1+n)}{(x-y+1)(x-z-1+n) \frac{(x-n)(x-y-z-1+n)}{n}} \times \left\{ H_n(x+y) + H_n(x-z-1) - H_n(x) - H_n(x-y-z-1) \right\} \]
\[ + \frac{n(z+1)(2x-y-z+n) \frac{(x-y+n)x-z-1+n}{n}(x-y-z-1+n)}{(x-y+1)^2(x-z-1+n)^2 \frac{(x-n)(x-y-z-1+n)}{n}}. \]

Taking the limit \( z \to 2x-y+n \) on both sides of the last equation and exploiting (2)-(5), we obtain Theorem (4) to finish the proof.

Selecting \( x = p, \ y = q \) in Theorem (4) with \( p, q \in \mathbb{N}_0 \) and availing (7), we obtain the summation formula involving harmonic numbers of 2-order.

**Corollary 5.** Let \( p \) and \( q \) be both nonnegative integers provided that \( p \geq q \). Then
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(2p-q+n+k)}{p+k} \frac{(q+k)}{q+k} \frac{(q-1)q}{(q+k-1)(q+k)} H_{p+k} \]
\[ = \frac{n^2 + n(1 + 2p - q) + (1 + p - q)^2 \frac{(p-q+n)^2}{p+n} \left\{ H_{p+q} + H_{p+q} - H_{p+q+n} \right\}}{(1 + p - q)^2 \frac{(p+n)^2}{p+n}} \]
\[ + \frac{n^2 + n(1 + 2p - q) \frac{(p-q+n)^2}{p+n}}{(1 + p - q)^4 \frac{(p+n)^2}{p+n}}. \]
Similarly, closed expressions for the following series
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{y+y}{y+k} H_k^{(2)}(x) \]
with \( t \geq 2 \) can also be derived. The corresponding results will not be displayed here.

3. The Second Family of Summation Formulae Involving Generalized Harmonic Numbers

Theorem 6. Let \( x \) and \( y \) be both complex numbers. Then
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{y+y}{y+k} H_k^{(2)}(x) = \frac{1}{n} \binom{n}{k} \frac{(y+y)}{(y+n)} \{ H_n(x-y) - H_n(x) \} \]

Proof. Perform the replacements \( a \to 1 + x \), \( b \to y \), \( c \to 1 + z \) in (1) to get
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(z+k)}{(z-n+k)} \frac{y+y}{z+k} y + k = \frac{1}{n} \binom{n}{k} \frac{(z-z-1+n)}{(z-z-y-1+n)} \{ H_n(z-x-y-1) - H_n(z-x-1) \} \]

Applying the derivative operator \( D_z \) to both sides of (8), we have
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(z+k)}{(z-n+k)} \frac{y+y}{z+k} y + k \{ H_k(x) - H_k(x+y-z-n) \} = \frac{1}{n} \binom{n}{k} \frac{(z-z-1+n)}{(z-z-y-1+n)} \{ H_n(z-x-y-1) - H_n(z-x-1) \} \]

The equivalent form of it reads as
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(z+k)}{(z-n+k)} \frac{y+y}{z+k} y + k \sum_{i=1}^{k} \frac{1}{(x+i)(x+y-z-n+i)} \]
\[ = \frac{1}{n} \binom{n}{k} \frac{(z-z-1+n)}{(z-z-y-1+n)} \frac{H_n(z-x-y-1) - H_n(z-x-1)}{y-z} \]

Taking the limit \( z \to y - n \) on both sides of the last equation, we gain Theorem 6 to complete the proof.

Fixing \( x = p \), \( y = q \) in Theorem 6 with \( p, q \in N_0 \) and using (8), we achieve the summation formula involving harmonic numbers of 2-order.

Corollary 7. Let \( p \) and \( q \) be both nonnegative integers satisfying \( p \geq q \geq n \). Then
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{y+y}{y+k} H_k^{(2)}(x) = \frac{1}{n} \binom{n}{k} \frac{(y+y)}{(y+n)} \{ H_{p-q+n} - H_{p+n} - H_{p-q} + H_p \} \]

Theorem 8. Let \( x \) and \( y \) be both complex numbers. Then
\[ \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{y+y}{y+k} H_k(x) = \frac{1}{n} \binom{n}{k} \frac{(y+y)}{(y+n)} \left( 1 - \frac{(x+y+n)}{(x+n)} \right) \]
Proof. Applying the integral operator $I_x$ to both sides of Theorem 6, we attain

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{y}{y+k} \{H_k - H_k(x)\} = \frac{(-1)^n}{n} \frac{(x-y+n)}{n} x^n.
$$

Take the limit $x \to \infty$ on both sides of (9) to deduce

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{y}{y+k} H_k = \frac{(-1)^n}{n} \frac{1}{n} \left\{ 1 - \frac{(p-q+n)}{(p+n)} \right\}.
$$

The difference of (9) and the last equation creates Theorem 8.

Setting $x = p$, $y = q$ in Theorem 8 with $p, q \in \mathbb{N}_0$ and utilizing 9, we obtain the summation formula involving harmonic numbers.

**Corollary 9.** Let $p$ and $q$ be both nonnegative integers provided that $q \geq n$. Then

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(q+k)}{(q-n+k)} \frac{q}{q+k} H_{p+k} = \frac{(-1)^n}{n} \frac{1}{n} \left\{ 1 - \frac{(p-q+n)}{(p+n)} \right\}.
$$

Applying the derivative operator $D_x$ to both sides of Theorem 8, we get the summation formula involving generalized harmonic numbers of 3-order.

**Theorem 10.** Let $x$ and $y$ be both complex numbers. Then

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{y}{y+k} H^{(3)}_k(x) = \frac{(-1)^n}{2n} \frac{(x-y+n)}{n} \left( \frac{y}{n} \right)
$$

$$
\times \left\{ [H^{(2)}_n(x-y) - H^{(2)}_n(x)] - [H_n(x-y) - H_n(x)]^2 \right\}.
$$

Choosing $x = p$, $y = q$ in Theorem 10 with $p, q \in \mathbb{N}_0$ and exploiting 8, we gain the summation formula involving harmonic numbers of 3-order.

**Corollary 11.** Let $p$ and $q$ be both nonnegative integers satisfying $p \geq q \geq n$. Then

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(q+k)}{(q-n+k)} \frac{q}{q+k} H^{(3)}_{p+k} = \frac{(-1)^n}{2n} \frac{(p-q+n)}{(p+n)} \left( \frac{q}{n} \right)
$$

$$
\times \left\{ [H^{(2)}_{p-q+n} - H^{(2)}_{p+n} - H^{(2)}_{p+q} + H^{(2)}_p] - [H_{p-q+n} - H_{p+n} - H_{p+q} + H_p] \right\}.
$$

Applying the derivative operator $D_x$ to both sides of Theorem 10, we achieve the summation formula involving generalized harmonic numbers of 4-order.

**Theorem 12.** Let $x$ and $y$ be both complex numbers. Then

$$
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{y}{y+k} H^{(4)}_k(x) = \frac{(-1)^n}{6n} \frac{(x-y+n)}{n} \left( \frac{y}{n} \right)
$$

$$
\times \left\{ [H_n(x-y) - H_n(x)]^3 + 2[H_n^{(3)}(x-y) - H_n^{(3)}(x)]
$$

$$
-3[H_n(x-y) - H_n(x)] [H_n^{(2)}(x-y) - H_n^{(2)}(x)] \right\}.
$$
Selecting $x = p$, $y = q$ in Theorem 12 with $p, q \in \mathbb{N}_0$ and availing (8), we attain the summation formula involving harmonic numbers of 4-order.

**Corollary 13.** Let $p$ and $q$ be both nonnegative integers provided that $p \geq q \geq n$. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{y+k}{q+n+k} q + k H_{p+k}^{(4)} = \frac{(-1)^n}{6n} \binom{p+q+n}{q+n} \sum_{k=0}^{n} \frac{1}{y+k} \binom{p+q+n}{q+n}
\]
\[
\times \left\{ [H_{p+q+n} - H_{p+n} - H_{p-q} + H_p]^3 + 2 [H_{p+q+n} - H_{p+n} - H_{p-q} + H_p]^3 - 3 H_{p+q+n} - H_{p+n} - H_{p-q} + H_p \right\}
\]

**Theorem 14.** Let $x$ and $y$ be both complex numbers. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{y+n+k} \binom{y+k}{y+n+k} \binom{y-1}{y+n}(y+k) \frac{H_k^{(2)}(x)}{n(n-1)(1+x-y)}
\]
\[
\times \left\{ H_n(x) - H_n(x-y) + \frac{ny}{(1+x-y)(1+x-y+ny)} \right\}
\]

**Proof.** Employ the substitutions $a \rightarrow 1 + x$, $b \rightarrow y - 1$, $c \rightarrow z$ in (10) to obtain
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{y+n+k} \binom{y+k}{y+n+k} \binom{y-1}{y+n}(y+k) \frac{H_k^{(2)}(x)}{n(n-1)(1+x-y)}
\]
\[
= \frac{(z-x-1)(z-y+1)+n(z-x-y)}{(z-x-1+n)(z-y+1)} \frac{(z-x-1+n)(z-y+1)}{(z-x-1+n)(z-y+1)} \frac{(z-x-1+n)(z-y+1)}{(z-x-1+n)(z-y+1)}
\]

Applying the derivative operator $D_x$ to both sides of (10), we get
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{y+n+k} \binom{y+k}{y+n+k} \binom{y-1}{y+n}(y+k) \frac{H_k(x) - H_k(x+y-z-n)}{(z-x-1+n)(z-y+1)}
\]
\[
= \frac{(z-x-1)(z-y+1)+n(z-x-y)}{(z-x-1+n)(z-y+1)} \frac{(z-x-1+n)(z-y+1)}{(z-x-1+n)(z-y+1)} \frac{(z-x-1+n)(z-y+1)}{(z-x-1+n)(z-y+1)}
\]

Its equivalent form can be written as
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{x+k}{y+n+k} \binom{y+k}{y+n+k} \binom{y-1}{y+n}(y+k) \sum_{i=1}^{k} \frac{1}{(x+i)(y+z-n+i)}
\]
\[
= \frac{(z-x-1)(z-y+1)+n(z-x-y)}{(z-x-1+n)(z-y+1)} \frac{(z-x-1+n)(z-y+1)}{(z-x-1+n)(z-y+1)} \frac{(z-x-1+n)(z-y+1)}{(z-x-1+n)(z-y+1)}
\]

Taking the limit $z \rightarrow y - n$ on both sides of the last equation, we gain Theorem 14 to finish the proof.
Fixing \( x = p, y = q \) in Theorem 14 with \( p, q \in \mathbb{N}_0 \) and using (10), we achieve the summation formula involving harmonic numbers of 2-order.

**Corollary 15.** Let \( p \) and \( q \) be both nonnegative integers satisfying \( p \geq q \geq n \). Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(q+k)}{(q-n+k)} \frac{(q-1)q}{(q+k-1)(q+k)} H^{(2)}_{p+k} = \frac{(-1)^n(1+p-q+q)}{n(n-1)(1+p-q)} \left( \frac{p-q+n}{n} \right) \frac{1}{(1+p-q)(1+p-q+q)}.
\]

**Theorem 16.** Let \( x \) and \( y \) be both complex numbers. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{(y-1)y}{(y+k-1)(y+k)} H_k(x) = \frac{(-1)^n(1+x-y+ny)}{n(n-1)(1+x-y)} \left( \frac{x-y+n}{n} \right) \frac{1}{(1+x-y)(x-y+n)}.
\]

**Proof.** Applying the integral operator \( \mathcal{I}_x \) to both sides of Theorem 14 we attain
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{(y-1)y}{(y+k-1)(y+k)} \left\{ H_k - H_k(x) \right\} = \frac{(-1)^{n+1}(1+x-y+ny)}{n(n-1)(1+x-y)} \left( \frac{x-y+n}{n} \right) \frac{1}{(1+x-y)(x-y+n)}.
\]

Take the limit \( x \to \infty \) on both sides of (11) to derive
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y-n+k)} \frac{(y-1)y}{(y+k-1)(y+k)} H_k = \frac{(-1)^n(1-y+ny)}{n(n-1)(1-y)} \left( \frac{y+n}{n} \right) \frac{1}{n(n-1)} - \frac{(-1)^n}{n(n-1)} \left( \frac{1}{n} \right).
\]

The difference of (11) and the last equation produces Theorem 16. \(\square\)

Setting \( x = p, y = q \) in Theorem 16 with \( p, q \in \mathbb{N}_0 \) and utilizing (10), we obtain the summation formula involving harmonic numbers.

**Corollary 17.** Let \( p \) and \( q \) be both nonnegative integers provided that \( q \geq n \). Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(q+k)}{(q-n+k)} \frac{(q-1)q}{(q+k-1)(q+k)} H^{(2)}_{p+k} = \frac{(-1)^n(1+p-q+q)}{n(n-1)(1+p-q)} \left( \frac{p-q+n}{n} \right) \frac{1}{(1+p-q)(1+p-q+q)}.
\]

Applying the derivative operator \( \mathcal{D}_x \) to both sides of Theorem 14 we get the summation formula involving generalized harmonic numbers of 3-order.
Theorem 18. Let \( x \) and \( y \) be both complex numbers. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{y} \frac{(y-1)y}{y+k-1}(y+k) H^{(3)}_k(x) = \frac{(-1)^n(1 + x - y + ny)}{2n(n-1)(1 + x - y)} \left\{ A_n(x, y) + B_n(x, y) \right\},
\]
where the two symbols on the right hand side stand for
\[
A_n(x, y) = \left[ H^{(2)}_n(x) - H^{(2)}_n(x-y) \right] + \frac{2ny}{(1 + x - y)^2(1 + x - y + ny)},
\]
\[
B_n(x, y) = \left[ H_n(x) - H_n(x-y) \right] \left[ H_n(x) - H_n(x-y) + \frac{2ny}{(1 + x - y)^2(1 + x - y + ny)} \right].
\]
Choosing \( x = p \), \( y = q \) in Theorem 18 with \( p, q \in \mathbb{N}_0 \) and exploiting (10), we gain the summation formula involving harmonic numbers of 3-order.

Corollary 19. Let \( p \) and \( q \) be both nonnegative integers satisfying \( p \geq q \geq n \). Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(q+k)}{(q+n)} (q-1)q \frac{H^{(3)}_{p+k}}{(q+k-1)(q+k)} = \frac{(-1)^n(1 + p - q + nq)}{2n(n-1)(1 + p - q) (p+q+n)} \left\{ C_n(x, y) + D_n(x, y) \right\},
\]
where the corresponding expressions are
\[
C_n(p, q) = \left[ H^{(2)}_{p+n} - H^{(2)}_{p+n-q} - H^{(2)}_p + H^{(2)}_{p-q} \right] + \frac{2nq}{(1 + p - q)^2(1 + p - q + nq)},
\]
\[
D_n(p, q) = \left[ H_{p+n} - H_{p+n-q} - H_p + H_{p-q} \right] \times \left[ H_{p+n} - H_{p+n-q} - H_p + H_{p-q} + \frac{2nq}{(1 + p - q)(1 + p - q + nq)} \right].
\]

Applying the derivative operator \( D_n \) to both sides of Theorem 18, we achieve the summation formula involving generalized harmonic numbers of 4-order.

Theorem 20. Let \( x \) and \( y \) be both complex numbers. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(y+k)}{(y+n+k)} (y-1)y \frac{H^{(4)}_k(x)}{y+k-1}(y+k) = \frac{(-1)^n(x-y)}{6n(n-1)(1 + x - y)} \left\{ E_n(x, y) + F_n(x, y) + G_n(x, y) \right\},
\]
where the three symbols on the right hand side stand for
\[
E_n(x, y) = [H_n(x) - H_n(x-y)]^3 + 2[H^{(3)}_n(x) - H^{(3)}_n(x-y)],
\]
\[
F_n(x, y) = [H_n(x) - H_n(x-y)]^2 + [H^{(2)}_n(x) - H^{(2)}_n(x-y)],
\]
\[
G_n(x, y) = \frac{6ny}{(1 + x - y)^2} [H_n(x) - H_n(x-y)] + \frac{6ny}{(1 + x - y)^2}.
\]
Selecting \( x = p \), \( y = q \) in Theorem 20 with \( p, q \in \mathbb{N}_0 \) and availing (10), we attain the summation formula involving harmonic numbers of 4-order.
Corollary 21. Let $p$ and $q$ be both nonnegative integers provided that $p \geq q \geq n$. Then
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(q+k)}{(q-k)} \binom{q}{k} \frac{(q-1)q}{(q+k-1)(q+k)} H_{p+k}^{(4)} \\
= \frac{(-1)^n}{6n(n-1)(1+p-q)} \binom{p-q+n}{n} \\
\times \left\{ (1+p-q+nq)U_n(p,q) + \frac{3nq}{1+p-q} V_n(p,q) + W_n(p,q) \right\},
\]
where the corresponding expressions are
\[
U_n(p,q) = \left[ H_{p+n} - H_{p+q+n} - H_p + H_{p-q} \right] + 2 \left[ H_{p+n}^{(3)} - H_{p+q+n}^{(3)} - H_p^{(3)} + H_{p-q}^{(3)} \right] \\
+ 3 \left[ H_{p+n} - H_{p+q+n} - H_p + H_{p-q} \right] \left[ H_{p+n}^{(2)} - H_{p+q+n}^{(2)} - H_p^{(2)} + H_{p-q}^{(2)} \right],
\]
\[
V_n(p,q) = \left[ H_{p+n} - H_{p+q+n} - H_p + H_{p-q} \right]^2 + \left[ H_{p+n}^{(2)} - H_{p+q+n}^{(2)} - H_p^{(2)} + H_{p-q}^{(2)} \right],
\]
\[
W_n(p,q) = \frac{6nq}{(1+p-q)^3} \left[ H_{p+n} - H_{p+q+n} - H_p + H_{p-q} \right] + \frac{6nq}{(1+p-q)^3}.
\]

Closed expressions for the following series
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{(q+k)}{(q-k)} \binom{q}{k} \frac{(q)}{\binom{k}{\ell}} H_k^{(\ell)}(x)
\]
with $t \geq 2$ and $\ell \geq 5$ can also be given in the same way. The corresponding conclusions
will not be laid out in the paper.

Acknowledgments

The work is supported by the National Natural Science Foundation of China (No. 11301120).

References

[1] G. E. Andrews, K. Uchimura, Identities in combinatorics IV: differentiation and harmonic numbers, Utilitas Math. 28 (1985), 265–269.
[2] G.E. Andrews, R. Askey, R. Roy, Special Functions, Cambridge University Press, Cambridge, 2000.
[3] Y. Chen, Q. Hou, H. Jin, The Abel-Zeilberger algorithm, Electron. J. Comb. 18 (2011), #P17.
[4] C. Krattenthaler, T. Rivoal, Hypergéométrie et fonction zéta de Riemann, Mem. Amer. Math. Soc. 186, no. 875, Providence, R. I., 2007.
[5] M.J. Kronenburg, Some generalized harmonic number identities, arXiv:1103.5430v2 [math.NT], 2012.
[6] M.J. Kronenburg, On two types of harmonic number identities, arXiv:1202.3981v2 [math.NT], 2012.
[7] P. Paule, C. Schneider, Computer proofs of a new family of harmonic number identities, Adv. Appl. Math. 31 (2003), 359–378.
[8] C. Schneider, Symbolic summation assists Combinatorics, Sém. Lothar. Combin. 56 (2006), Article B56b.
[9] A. Sofo, Sums of derivatives of binomial coefficients, Adv. Appl. Math. 42 (2009), 123–134.
[10] A. Sofo, Quadratic alternating harmonic number sums, J. Number Theory 154 (2015), 144–159.
[11] Z. Sun, Arithmetic theory of harmonic numbers, Proc. Amer. Math. Soc. 140 (2012), 415–428.
[12] W. Wang, Riordan arrays and harmonic number identities, Comput. Math. Appl. 60 (2010), 1494–1509.
[13] W. Wang, C. Jia, Harmonic number identities via the Newton-Andrews method, Ramanujan J. 35 (2014), 263–285.
[14] C. Wei, D. Gong, Q. Yan, Telescoping method, derivative operators and harmonic number identities, Integral Transforms Spec. Funct. 25 (2014), 203–214.
[15] J. Wang, C. Wei, Derivative operator and summation formulae involving generalized harmonic numbers, J. Math. Anal. Appl. 434 (2016) 315–341.