On Carleman estimates with two large parameters

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Abstract. We provide a general framework for the analysis and the derivation of Carleman estimates with two large parameters. For an appropriate form of weight functions strong pseudo-convexity conditions are shown to be necessary and sufficient.

1. Introduction

Carleman estimates are an important tool in subjects in analysis of partial differential equations (PDEs) such as unique continuation, control theory, and inverse problems. They are weighted $L^2$-norm estimates of the solution of a PDE. Weights are of exponential form and involve a parameter that is taken large. The work of L. Hörmander in the late 50’s-early 60’s provided large classes of operators for which such estimates can be derived. The notions of principal normality for the operator and strong pseudo-convexity for the weight function were central in this seminal work.

If choosing a double exponential for weight function, sufficient conditions for the Carleman estimate to hold can be obtained from strong pseudo-convexity conditions. With this choice of weight function one introduces a second large parameter. Several authors have derived Carleman estimates for some operators in which the dependency upon the second large parameters is kept explicit. Such results can be very useful to address applications such as inverse problems. See for instance [1, 2, 3, 4].

We provide a general framework for the analysis and the derivation of Carleman estimates with two large parameters. Strong pseudo-convexity conditions are shown to be necessary and sufficient for the estimate to hold. Under further conditions on the operator and the weight function stronger Carleman estimates can also be achieved.

For an introduction to Carleman estimates for second-order elliptic and parabolic operator see [5].

Notation

Here $\Omega$ will always denote a bounded open subset of $\mathbb{R}^n$. When the constant $C$ is used, it refers to a constant that is independent of the different parameter, e.g. $\tau$ and $\alpha$. Its value may however change from one line to another. If we want to keep track of the value of a constant we shall use another letter.
2. A pseudo-differential calculus with two large parameters

We set $W = \mathbb{R}^n \times \mathbb{R}^n$, often referred to as phase-space. A typical element of $W$ will be $X = (x, \xi)$, with $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$.

Let $\psi \in C^\infty(\mathbb{R}^n; \mathbb{R})$ be such that

$$\psi \geq C > 0, \quad |\psi'| > 0, \quad \text{and} \quad \|\psi'\|_\infty < \infty.$$  

We then set

$$\varphi(x) = e^{\alpha \psi(x)}, \quad \text{with} \quad \alpha \geq 1.$$  

We observe that $|\varphi'| > 0$. We make the following further assumption on the function $\psi$.

**Assumption 2.1.** There exists $k \in \mathbb{N}$ such that $\sup_{\mathbb{R}^n} \psi \leq (k + 1) \inf_{\mathbb{R}^n} \psi$.

### 2.1. Metric and order function on phase-space

We consider the metric on phase-space:

$$g = \alpha^2 |dx|^2 + \frac{|d\xi|^2}{\mu^2}, \quad \text{with} \quad \mu^2 = \mu^2(x, \xi, \tau) = (\tau \alpha \varphi(x))^2 + |\xi|^2, \quad \text{and} \quad \tau \geq 1, \quad \alpha \geq 1.$$  

The first result of this section shows that this metric on $W$ defines a Weyl-Hörmander pseudo-differential calculus.

**Proposition 2.2.** The metric $g$ and the order function $\mu$ are admissible, in the sense that,

(i) $g$ satisfies the uncertainty principle, with $\lambda_h = h^{-1}_g = \alpha^{-1}\mu$.

(ii) $\mu$ and $g$ are slowly varying;

(iii) $\mu$ and $g$ are temperate.

For a presentation of the Weyl-Hörmander calculus we refer to [6], [7, Sections 18.4–6] and [8].

We may define Sobolev spaces associated with the calculus defined by the metric $g$ as in [9]. For $k, s \in \mathbb{R}$, we set

$$\mathcal{H}_{k,s}(\mathbb{R}^n) = \left\{ \text{Op}^w(\tilde{\tau}^{-s} \mu^{-k})v; \ v \in L^2 \right\}, \quad \text{and} \quad \|u\|_{k,s} = \|\text{Op}^w(\tilde{\tau}^{s} \mu^{k})u\|_{L^2(\mathbb{R}^n)}, \quad \text{if} \ u \in \mathcal{H}_{k,s}(\mathbb{R}^n),$$

with $\tilde{\tau}(x) = \tau \alpha \varphi(x)$. Note that $\|\cdot\|_{k,s}$ is a norm on $\mathcal{H}_{k,s}$.

**Proposition 2.3.** Let $k, k', s, s' \in \mathbb{R}$. For $a \in S(\tilde{\tau}^s \mu^k, g)$ there exist $C > 0$ and $\tau_1 > 0$ such that for all $\tau \geq \tau_1$, we have

$$\|\text{Op}^w(a)u\|_{k', s'} \leq C\|u\|_{k+k', s+s'}, \quad u \in \mathcal{H}_{k+k', s+s'}(\mathbb{R}^n).$$

### 3. Carleman estimates under strong pseudo-convexity assumptions

Let $P(x, D_x)$ be a differential operator of order $m$, with homogeneous principal symbol $p(x, \xi)$.

**Definition 3.1** (Principal normality [10, Definition 28.2.4]). The operator $P(x, D_x)$ is said to be principally normal on $\Omega$ if for some $C > 0$

$$\left| \{\mathcal{P}, p\}(x, \xi) \right| \leq C|p(x, \xi)| \|\xi\|^{m-1}, \quad x \in \overline{\Omega}, \quad \xi \in \mathbb{R}^n.$$

Elliptic operators and operators with real coefficients in the principal part are typical examples of principally normal operators.

We now revisit some consequences of pseudo-convexity and strong pseudo-convexity.
3.1. Pseudo-convexity properties and symbol estimates
Let $\psi \in C^\infty(\Omega, \mathbb{R})$. We recall the following definitions [11].

**Definition 3.2** (pseudo-convexity). We say that $\psi$ is pseudo-convex at $x \in \overline{\Omega}$ w.r.t. $P(x, D_x)$ if $\psi'(x) \neq 0$ and if

$$\forall \xi \in \mathbb{R}^n \setminus \{0\}, \ p(x, \xi) = 0 \text{ and } \{p, \psi\}(x, \xi) = 0 \Rightarrow \text{Re} \left\{ \overline{p}, \{p, \psi\} \right\}(x, \xi) > 0. \quad (\Psi_c)$$

The function $\psi$ is said to be pseudo-convex w.r.t. $\Omega$ and $p$ if $\psi' \neq 0$ in $\overline{\Omega}$ and ($\Psi_c$) is valid for all $x \in \overline{\Omega}$.

We note that

$$\{p, \psi\}(x, \xi) = \langle p'(x, \xi), \psi'(x) \rangle, \quad \text{and} \quad \text{Re} \left\{ \overline{p}, \{p, \psi\} \right\}(x, \xi) = \theta_{p, \psi}(x, \xi)$$

with $\theta_{p, \psi}(x, \xi) = \sum_{j,k} \partial^2_{x_j x_k} \psi(x) \partial_{\xi_j} \overline{p}(x, \xi) \partial_{\xi_k} p(x, \xi) + \text{Re} \sum_j \partial_{x_j} \psi(x) \{p, \partial_{\xi_j} \overline{p} \}(x, \xi)$.

**Definition 3.3** (strong pseudo-convexity). We say that $\psi$ is strongly pseudo-convex at $x \in \overline{\Omega}$ w.r.t. $p$ if

(i) $\psi$ is pseudo-convex at $x$ w.r.t. $p$;

(ii) if for all $\xi \in \mathbb{R}^n$ and $\tau > 0$,

$$p(x, \xi + i\tau \psi'(x)) = 0 \text{ and } \{p, \psi\}(x, \xi + i\tau \psi'(x)) = 0$$

$$\Rightarrow \quad \frac{1}{2i} \left\{ \overline{p}(x, \xi - i\tau \psi'(x)), p(x, \xi + i\tau \psi'(x)) \right\} > 0. \quad (s-\Psi_c)$$

The function $\psi$ is said to be strongly pseudo-convex w.r.t. $\Omega$ and $p$ if items (i) and (ii) are valid for all $x \in \overline{\Omega}$.

We note that

$$\frac{1}{2i} \left\{ \overline{p}(x, \xi - i\tau \psi'(x)), p(x, \xi + i\tau \psi'(x)) \right\} = \left\{ \text{Re} p(x, \xi + i\tau \psi'(x)), \text{Im} p(x, \xi + i\tau \psi'(x)) \right\}$$

$$= \Theta_{p, \psi}(x, \xi, \tau),$$

with

$$\Theta_{p, \psi}(x, \xi, \tau) = \tau \sum_{j,k} \partial^2_{x_j x_k} \psi(x) \partial_{\xi_j} \overline{p}(x, \xi + i\tau \psi'(x)) \partial_{\xi_k} p(x, \xi - i\tau \psi'(x))$$

$$+ \text{Im} \sum_j \partial_{x_j} p(x, \xi + i\tau \psi'(x)) \partial_{\xi_j} \overline{p}(x, \xi - i\tau \psi'(x)).$$

The following result sharpens the estimate of Proposition 28.3.3 in [10].

**Proposition 3.4.** Let $P$ be principally normal on $\Omega$ and let $\psi$ be a strongly pseudo-convex function w.r.t. $\Omega$ and $P$. We set $\varphi = e^{i\psi}$ and

$$\zeta = \zeta(x, \xi, \tau) = \xi + i\tau \varphi'(x) = \xi + i\tilde{\tau}(x)\psi'(x), \quad \tilde{\tau}(x) = \tau \alpha \varphi(x).$$

There exist $C > 0$, $\tau_0 \geq 1$, $\alpha_0 > 0$ such that we have

$$\tilde{\tau}(x) \mu^{2(m-1)} \leq C \left( |p(x, \zeta)|^2 + \frac{1}{2i} \left\{ \overline{p(x, \zeta)}, p(x, \zeta) \right\} \right), \quad \tau \geq \tau_0, \quad \alpha \geq \alpha_0, \quad (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \quad (1)$$

The proof is based on a different treatment of the regions $\tau \alpha \varphi(x) \ll |\xi|$ and $\tau \alpha \varphi(x) \gg |\xi|$.
3.2. Carleman estimate  
We now state a Carleman estimate with two large parameters.

**Theorem 3.5.** Let \( P = P(x, D_x) \) be a principally normal operator in \( \Omega \) of order \( m \) and \( \psi \in C^\infty(\overline{\Omega}) \), \( \psi \geq C > 0 \) on \( \Omega \), be a strongly pseudo-convex function w.r.t. \( \Omega \) and \( P \). We set \( \varphi = e^{\alpha \psi} \). If \( X \) is an open subset, \( X \subseteq \Omega \), there exist \( C > 0, \alpha_1 > 0, \) and \( \tau_1 \geq 1 \) such that
\[
\sum_{|\beta| < m} (\tau \alpha)^{2(m-|\beta|)-1} \| e^{\alpha \psi} \partial_x^{m-|\beta|} D_x^\beta u \|_{L^2}^2 \leq C \| e^{\alpha \psi} \partial_x^\beta u \|_{L^2}^2,
\]
for all \( u \in C^\infty_c(X) \), \( \alpha \geq \alpha_1 \), and \( \tau \geq \tau_1 \).

The proof is based on the pseudo-differential calculus introduced above, by mean of the Fefferman-Phong inequality (see [12] and [7, Theorem 18.6.8]).

4. Cases of stronger estimates  
In this section we present classes of operators for which stronger Carleman estimates with two large parameters can be derived as compared to the result of Theorem 3.5.

4.1. Simple characteristics  
We introduce the map
\[
\rho : \mathbb{R}^+ \to \mathbb{C}, \quad \hat{\tau} \mapsto p(x, \xi + i\hat{\tau} \psi'(x)),
\]
We assume in this section that the operator satisfies the simple complex characteristic property:
\[
\begin{cases}
\hat{\tau} \mapsto \rho(\hat{\tau}) \text{ has simple roots} & \text{if } (x, \xi) \in \Omega \times (\mathbb{R}^n \setminus 0), \\
\hat{\tau} \mapsto \rho(\hat{\tau}) \text{ has simple nonzero roots} & \text{if } x \in \Omega \text{ and } \xi = 0.
\end{cases}
\]
Note that the case \( \xi = 0 \) is particular, as the root \( \hat{\tau} = 0 \) has of course multiplicity \( m \). Note also that we have
\[
\rho'(\hat{\tau}) = i\{p'_x(x, \xi + i\hat{\tau} \psi'(x)), \psi'(x)\} = i\{p, \psi\}(x, \xi + i\hat{\tau} \psi'(x)).
\]
With the simple-characteristic property we shall obtain a Carleman estimate with an additional power in the second large parameter \( \alpha \).

**Examples 4.1.** Evidently, first-order operators satisfy this property. Second-order elliptic operators with real coefficients also satisfy this property for any smooth function \( \psi \) such that \( \psi' \neq 0 \). Indeed, complex roots of \( \sigma \mapsto p(x, \xi + \sigma \psi'(x)) \) come in conjugated pairs. They can only be double if they are real. For the roots of \( \hat{\tau} \mapsto \rho(\hat{\tau}) \) this means \( \hat{\tau} = 0 \), that is \( p(x, \xi) = 0 \).

Note that elliptic operators of order higher than 2 may however not satisfy this property, e.g. the bilaplacian operator \( \Delta^2 \). An example of an elliptic operator of order greater than two satisfying the simple-characteristic property (3) is \( D^4_x + D^4_y \) in \( \Omega \subset \mathbb{R}^2 \) for any function \( \psi \) whose gradient does not vanish in \( \overline{\Omega} \).

An example of a second-order operator that is not elliptic satisfying the simple-characteristic property (3) is \( \frac{1}{2} D^2_x + D_x D_y \) in \( \Omega \subset \mathbb{R}^2 \), with \( \psi(x) = \frac{1}{2}(x - x_0)^2 \), where \( x_0 \) is chosen such that \( |\psi'| \neq 0 \) in \( \overline{\Omega} \), i.e., \( x_0 \notin \overline{\Omega} \).

Note that the examples we have just given are principally normal (see definition 3.1) as they have real coefficients.
Remark 4.2. The simple-characteristic property (3) implies that $\psi$ is strongly pseudo-convex with respect to $\Omega$ and $P$.

**Proposition 4.3.** Let $P$ be principally normal on $\Omega$ and let $\psi$ be such that the simple-characteristic property (3) is fulfilled. We set $\varphi = e^{\alpha \psi}$ and

$$
\zeta = \zeta(x, \xi, \tau) = \xi + i \tau \varphi'(x) = \xi + i \tilde{\tau}(x) \psi'(x), \quad \tilde{\tau}(x) = \tau \alpha \varphi(x).
$$

There exist $C > 0$, $\tau_0 \geq 1$, $\alpha_0 > 0$, and $\nu_0$ such that we have

$$
\tilde{\tau}(x)^2 \mu^{2(m-1)} \leq C \left( \nu |p(x, \zeta)|^2 + \frac{\tau \varphi(x)}{2i} \left\{ p(x, \zeta), p(x, \xi) \right\} \right),
$$

$$
\tau \geq \tau_0, \quad \alpha \geq \alpha_0, \quad \nu \geq \nu_0, \quad (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \quad (5)
$$

With Proposition 4.3 we obtain the following Carleman estimate.

**Theorem 4.4.** Let $P = P(x, D_x)$ be a principally normal operator in $\Omega$ of order $m$ and $\psi \in C^\infty(\overline{\Omega})$, $\psi \geq C > 0$ on $\Omega$, a function such that the simple-characteristic property (3) is fulfilled. We set $\varphi = e^{\alpha \psi}$. If $X$ is an open subset, $X \Subset \Omega$, there exist $C > 0$, $\alpha_1 > 0$, and $\tau_1 \geq 1$ such that

$$
\alpha \sum_{|\beta| < m} (\tau \alpha)^{2(m-|\beta|)-1} \| \varphi^{m-|\beta|} - \frac{1}{2i} e^{\tau \varphi} D_x^\beta u \|_{L^2}^2 \leq C \| e^{\tau \varphi} Pu \|_{L^2}^2,
$$

for all $u \in C^\infty_c(X)$, $\alpha \geq \alpha_1$, and $\tau \geq \tau_1$.

We observe that we have gained a factor $\alpha$ on the l.h.s. in contrast to the Carleman estimate of Theorem 3.5.

### 4.2. Elliptic operators

For elliptic operators stronger results can also be achieved. First we shall consider general elliptic operators and second we shall consider elliptic operators with simple characteristics.

#### 4.2.1. General elliptic operators under strong pseudo-convexity condition

Let $A$ be an elliptic operator of order $m$. We first note that if $\psi$ is a smooth function such that $|\psi'| > 0$ on $\overline{\Omega}$, then it is a pseudo-convex function on $\Omega$ for the operator $A$.

**Proposition 4.5.** Let $A$ be elliptic on $\Omega$ and let $\psi$ satisfy (s-$\Psi c$) (point ii in Definition 3.3) for all $x \in \Omega$. We set $\varphi = e^{\alpha \psi}$ and

$$
\zeta = \zeta(x, \xi, \tau) = \xi + i \tau \varphi'(x) = \xi + i \tilde{\tau}(x) \psi'(x), \quad \tilde{\tau}(x) = \tau \alpha \varphi(x)
$$

There exist $C > 0$, $\tau_0 \geq 1$, $\alpha_0 > 0$ such that we have

$$
(\tau \alpha \varphi)^{-1} \mu^{2m} \leq C \left( |p(x, \zeta)|^2 + \frac{1}{2i} \left\{ p(x, \zeta), p(x, \xi) \right\} \right), \quad \tau \geq \tau_0, \quad \alpha \geq \alpha_0, \quad (x, \xi) \in \overline{\Omega} \times \mathbb{R}^n. \quad (6)
$$

With Proposition 4.5 we then obtain the following Carleman estimate.

**Theorem 4.6.** Let $P = P(x, D_x)$ be an elliptic operator in $\Omega$ of order $m$ and $\psi \in C^\infty(\overline{\Omega})$, $\psi \geq C > 0$ on $\Omega$, be a strongly pseudo-convex function w.r.t. $\Omega$ and $P$. We set $\varphi = e^{\alpha \psi}$. If $X$ is an open subset, $X \Subset \Omega$, there exist $C > 0$, $\alpha_1 > 0$, and $\tau_1 \geq 1$ such that

$$
\sum_{|\beta| \leq m} (\tau \alpha)^{2(m-|\beta|)-1} \| \varphi^{m-|\beta|} - \frac{1}{2i} e^{\tau \varphi} D_x^\beta u \|_{L^2}^2 \leq C \| e^{\tau \varphi} Pu \|_{L^2}^2,
$$

for all $u \in C^\infty_c(X)$, $\alpha \geq \alpha_1$, and $\tau \geq \tau_1$. 


We observe that we have gained an additional term in the sum on the l.h.s. in contrast to the Carleman estimate of Theorem 3.5.

**Remark 4.7.** From an estimate of the form of (7) by fixing the values of \( \tau \) and \( \alpha \) we obtain

\[
\|u\|_{H^m(\mathbb{R}^n)} = \sum_{|\beta| \leq m} \|D_x^\beta u\|_{L^2} \leq C\|Pu\|_{L^2},
\]

which implies that \( P \) is elliptic. The additional term we have obtained in the previous theorem is thus a privilege of elliptic operators.

### 4.2.2. Elliptic operators with simple characteristics

Combining the simple characteristic and elliptic properties we obtain the following estimate.

**Proposition 4.8.** Let \( p(x, \xi) \) and \( \psi(x) \) be such that (3) holds. Assume moreover that \( p(x, \xi) \) is elliptic. We set \( \varphi = e^{\alpha \psi} \) and

\[
\zeta = \zeta(x, \xi, \tau) = \xi + i\tau \varphi'(x) = \xi + i\tau \alpha \psi'(x) \varphi(x).
\]

There exist \( C > 0, \tau_0 \geq 1, \alpha_0 > 0, \nu_0 > 0, \) such that we have

\[
\mu^{2m} \leq C \left( \nu p(x, \xi)^2 + \frac{\tau \varphi(x)}{2i} \left\{ p(x, \xi), p(x, \xi) \right\} \right), \quad \tau \geq \tau_0, \alpha \geq \alpha_0, \nu \geq \nu_0, (x, \xi) \in \Omega \times \mathbb{R}^n.
\]

(8)

With Proposition 4.8 we then obtain the following Carleman estimate.

**Theorem 4.9.** Let \( P = P(x, D_x) \) be an elliptic differential operator in \( \Omega \) of order \( m \) and \( \psi \in C^\infty(\Omega) \), be such that the simple-characteristic property (3) holds. We set \( \varphi = e^{\alpha \psi} \). If \( X \) is an open subset, \( X \subseteq \Omega \), there exist \( C > 0, \alpha_1 > 0 \), and \( \tau_1 \geq 1 \) such that

\[
\alpha \sum_{|\beta| \leq m} (\tau \alpha)^{2(m-|\beta|)-1} \left\{ \varphi^{m-|\beta|} - \frac{1}{2} e^{\tau \varphi} D_x^\beta u \right\}_{L^2}^2 \leq C \left\{ e^{\tau \varphi} Pu \right\}_{L^2}^2,
\]

for all \( u \in C_0^\infty(X) \), \( \alpha \geq \alpha_1 \), and \( \tau \geq \tau_1 \).

We observe that we have gained a factor \( \alpha \) and an additional term in the sum on the l.h.s. in contrast to the Carleman estimate of Theorem 3.5.

### 5. Necessary conditions on the weight function

Starting from Carleman estimates L. Hörmander derived necessary conditions on the weight function \([11, 10]\). We apply the same program in the case of Carleman estimates with two large parameters and weight functions of the form \( \varphi = e^{\alpha \psi} \).

**Lemma 5.1.** Let \( P \) be a differential operator of order \( m \) with smooth principal symbol \( p(x, \xi) \) and let \( \psi \in C^\infty(\Omega) \) be such that the following estimate holds,

\[
\sum_{|\beta| < m} (\tau \alpha)^{2(m-|\beta|)-1} \left\{ \varphi^{m-|\beta|} - \frac{1}{2} e^{\tau \varphi} D_x^\beta u \right\}_{L^2}^2 \leq K \left\{ e^{\tau \varphi} Pu \right\}_{L^2}^2, \quad \varphi = e^{\alpha \psi},
\]

(9)

for \( \tau \geq \tau_0 > 0 \) and \( \alpha \geq \alpha_0 > 0 \) and \( u \in C_0^\infty(\Omega) \). We then have

\[
\sum_{|\beta| < m} (\alpha \varphi(x))^{2(m-|\beta|)-1} |\zeta|^{2} \leq \frac{K}{\iota} \left\{ p(x, \zeta), p(x, \zeta) \right\}, \quad \zeta = \xi + i\tau \varphi'(x) = \xi + i\tau \alpha \psi'(x) \varphi(x),
\]

(10)

for \( \tau > 0 \) and \( \alpha \geq \alpha_0 \), if \( p(x, \zeta) = 0 \). If \( m \geq 2 \) we have \( \psi' \neq 0 \) in \( \Omega \). Moreover \( p(x, \xi) \) does not vanish at second order at any point of \( T^* \Omega \) \( \setminus \) \( 0 \).
This lemma is the counterpart with two large parameters of Theorem 28.2.1 in [10]. The proof is along the same lines.

**Lemma 5.2.** If the sum in (9) is replaced by \( \sum_{|\beta| \leq m} (\text{resp. } \alpha \sum_{|\beta| < m} \text{ or } \alpha \sum_{|\beta| \leq m}) \) then the same is true for (10).

We next see that the strong pseudo-convexity condition on the function \( \psi \) is in fact necessary for the Carleman estimate (9) to hold.

**Theorem 5.3.** Consider the same assumption as in Lemma 5.1 and assume further that \( P \) is principally normal. In the case of a first-order operator, further assume that \( \psi' \neq 0 \) in \( \Omega \). Then the function \( \psi \) is strongly pseudo-convex w.r.t. \( P \) and \( \Omega \).

Along with Theorem 3.5 we then obtain that the strong pseudo-convexity of \( \psi \) is necessary and sufficient for the Carleman estimate (9) to hold. This result is in contrast with L. Hörmander’s work where there is a gap between the necessary and the sufficient conditions on the weight function \( \varphi \) to have a Carleman estimate (compare Theorem 28.2.1 and Theorem 28.2.3 in [10]). Here, of course we impose a particular structure on \( \varphi \), viz. \( \varphi = e^{\alpha \psi} \).

We also obtain necessary conditions for a Carleman estimate as in Theorem 4.4 to hold.

**Theorem 5.4.** Let \( P \) be a differential operator of order \( m \) with smooth principal symbol \( p(x, \xi) \) and let \( \psi \in C^\infty_c(\Omega) \) be such that the following estimate holds,

\[
\alpha \sum_{|\beta| < m} (\tau \alpha)^{2(m-|\beta|)-1} \| \varphi^{m-|\beta|} \|_{L^2} \leq K \| e^{\tau \varphi} D_u^2 u \|_{L^2} \leq K \| e^{\tau \varphi} Pu \|_{L^2}, \quad \varphi = e^{\alpha \psi},
\]

(11)

for \( \tau \geq \tau_0 > 0 \) and \( \alpha \geq \alpha_0 > 0 \) and \( u \in C^\infty_c(\Omega) \). Assume further that \( P \) is principally normal. Then the function \( \psi \) is such that \( \psi'(x) \neq 0 \) in \( \Omega \) and the simple-characteristic property (3) holds.

Note that the case of a first-order operator, the stronger estimate (11) implies \( \psi' \neq 0 \) as opposed to the “regular” Carleman estimate with two large parameters (9).

With Theorem 4.4 we see that, in the case of principally normal operators, the simple-characteristic property (3) is a necessary and sufficient condition for the Carleman estimate (11) to hold.

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