Characterization of commutative algebras embedded into the algebra of smooth operators

Tomasz Ciaś

Abstract

This paper deals with the noncommutative Fréchet *-algebra \( L(s', s) \) of the so-called smooth operators — linear and continuous operators acting from the space \( s' \) of slowly increasing sequences to the Fréchet space \( s \) of rapidly decreasing sequences. By a canonical identification, this algebra of smooth operators can be also seen as the algebra of the rapidly decreasing matrices. We give a full description of closed commutative *-subalgebras of this algebra and we show that every closed subspace of \( s \) with basis is isomorphic (as a Fréchet space) to some closed commutative *-subalgebra of \( L(s', s) \). As a consequence, we give some equivalent formulation of the long-standing quasi-equivalence conjecture for closed subspaces of \( s \).

1. Introduction

In this paper we consider some specific noncommutative Fréchet *-algebra with involution — known, for instance, as the algebra \( L(s', s) \) of smooth operators or as the algebra

\[
K_\infty := \left\{ (x_{j,k})_{j,k \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}^2} : \sup_{j,k \in \mathbb{N}} |x_{j,k}| j^q k^q < \infty \text{ for all } q \in \mathbb{N}_0 \right\}
\]

of rapidly decreasing matrices. Our main goal is to solve the following problems:

(A) to characterize Fréchet *-algebras isomorphic to closed commutative *-subalgebras of \( L(s', s) \) (Theorem 4);
(B) to characterize the underlying Fréchet spaces for the class of closed commutative *-subalgebras of \( L(s', s) \) (Theorem 7).

By definition, \( L(s', s) \) is the Fréchet *-algebra consisting of all linear and continuous operators from the space

\[
s' := \left\{ (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{j \in \mathbb{N}} |\xi_j| j^{-q} < \infty \text{ for some } q \in \mathbb{N}_0 \right\}
\]  

(1)

of slowly increasing sequences (this is an inductive limit of Banach spaces) to the Fréchet space

\[
s := \left\{ (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^{\mathbb{N}} : \sup_{j \in \mathbb{N}} |\xi_j| j^q < \infty \text{ for all } q \in \mathbb{N}_0 \right\}
\]  

(2)

of rapidly decreasing sequences. Since there is a natural inclusion \( L(s', s) \) into \( B(\ell_2) \), the space \( L(s', s) \) is indeed a *-algebra with composition of operators and hilbertian involution taken from \( B(\ell_2) \). Furthermore, \( L(s', s) \) with its natural topology of the uniform convergence on bounded subsets of \( s' \) is a Fréchet space.

We prove, in particular, that the class of closed commutative *-subalgebras of \( L(s', s) \) is — in the sense of an isomorphism of Fréchet *-algebras — precisely the class of Köthe sequence...
algebras closed under taking square roots (Definition 3) and at the same time isomorphic (as Fréchet spaces) to closed subspaces of $s$ (Theorem 4). This can be easily expressed by conditions on the corresponding Köthe matrices. Moreover, from the solution of problem (A) we get the solution of problem (B): the underlying Fréchet spaces for closed commutative $^*$-subalgebras of $L(s', s)$ are just closed subspaces of $s$ with basis (Theorem 7).

It has already been shown in [2, Theorem 4.8] that every infinite-dimensional closed commutative $^*$-subalgebra $X$ of $L(s', s)$ is isomorphic as a Fréchet $^*$-algebra to a Köthe sequence algebra

$$
\lambda^\infty(||P_j||_q) := \left\{ (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : \sup_{j \in \mathbb{N}} |\xi_j| \cdot ||P_j||_q < \infty \text{ for all } q \in \mathbb{N}_0 \right\},
$$

where $(P_j)_{j \in \mathbb{N}}$ is a suitable sequence of pairwise orthogonal, nonzero, self-adjoint projections belonging to $X$, and $(|| \cdot ||_q)_{q \in \mathbb{N}_0}$ is a sequence of continuous norms on $L(s', s)$ which determines the Fréchet space topology of $L(s', s)$. Hence, in order to solve problems (A) and (B), we needed to ‘understand’ matrices $(||P_j||_q)_{j \in \mathbb{N}, q \in \mathbb{N}_0}$, where $(P_j)_{j \in \mathbb{N}}$ are pairwise orthogonal, nonzero, self-adjoint projections belonging to $L(s', s)$. The key step in solving our problems is to show that for every nuclear Fréchet space with the so-called dominating Hilbert norm $|| \cdot ||$ there is a topological embedding $V : X \hookrightarrow s$ such that $||Vx||_{\ell_2} = ||x||$ for every $x \in X$ (Theorem 14).

In this context, it is worth mentioning that, by [4, Theorem 6.2], a Köthe sequence algebra of the form (3) is isomorphic to some closed $^*$-subalgebra of the algebra $s$ if and only if it is isomorphic as a Fréchet space to a complemented subspace of $s$. One would think that every closed commutative $^*$-subalgebra of $L(s', s)$ is of this type. The theorem in [4, Theorem 6.9] shows that this is not true (in the proof we give a concrete counterexample), and the present paper shows that indeed — taking into account the representation (3) — the class of the underlying Fréchet spaces for closed commutative $^*$-subalgebras of $L(s', s)$ is as big as possible, that is, as we mentioned above, it consists of all closed subspaces of $s$ with basis.

The algebra of smooth operators is a quite natural object. It can be seen as a kind of a noncommutative analogue of the commutative Fréchet algebra $s$ (look at the definition of $K_{\infty}$ above) and, in particular, it is isomorphic as a Fréchet space to $s$ (see [12, Lemma 31.1]) — one of the most significant Fréchet spaces. At the level of Fréchet spaces, the space $s$ is isomorphic to many classical spaces of smooth functions such as the Schwartz space $S(\mathbb{R})$ of smooth rapidly decreasing functions on the real line (which is a natural domain for the Fourier transform and differential operators with smooth coefficients), the space $C^\infty[-1, 1]$ of smooth functions on the interval $[-1, 1]$, the space $D(K)$ of smooth functions with support contained in a compact set $K \subset \mathbb{R}^n$, int$(K) \neq \emptyset$, or the space $C^\infty(M)$ of smooth functions on an arbitrary compact smooth manifold $M$. Finally, the space $s$ carries all the information about nuclear locally convex spaces. Indeed, the Köthe–Kômura theorem states that a locally convex space is nuclear if and only if it is isomorphic to some closed subspace of a suitable big cartesian product of the space $s$ (see [12, Theorem 29.8]). Consequently, a Fréchet space is nuclear (that is, a Fréchet space in which every unconditionally convergent series is absolutely convergent) if and only if it is isomorphic to some closed subspace of $s^\infty$ (see [10, Corollary 29.9]).

Replacing $s'$ with the space $S'(\mathbb{R})$ of tempered distributions and $s$ with $S(\mathbb{R})$, we still end up with the same Fréchet $^*$-algebra — this time consisting of all linear and continuous operators acting from $S'(\mathbb{R})$ to $S(\mathbb{R})$, where $^*$-algebra operations are inherited from $L^2(\mathbb{R})$. In fact, for every Fréchet space $E$ isomorphic to $s$ and its topological dual $E'$, the space $L(E', E)$ (with appropriate $^*$-algebra operations) of linear and continuous operators from $E'$ to $E$ is a Fréchet $^*$-algebra isomorphic to $L(s', s)$ (for details, see [3, Theorem 1.10 and Example 1.13] and [8, Theorem 2.1]).

The algebra $L(s', s)$ appears and plays a significant role in the $K$-theory of Fréchet algebras (see Blatt, Inoue and Ogi [1, Example 2.12]; Cuntz [6, p. 144; 7, pp. 64–65]; Glöckner and Langkamp [10]; Phillips [14, Definition 2.1]) and in $C^*$-dynamical systems (Elliot, Natsume
and Nest [9, Example 2.6]). Recently, Piszczek obtained several results concerning closed ideals, automatic continuity, amenability, Jordan decomposition and Grothendieck-type inequality in $K_\infty$ (see Piszczek [16–20]).

Recall that two bases $(x_j)_{j \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ of a Fréchet space $X$ are called quasi-equivalent if there exist a permutation $\sigma : \mathbb{N} \to \mathbb{N}$, a sequence $(\lambda_j)_{j \in \mathbb{N}}$ of nonzero scalars and a Fréchet space isomorphism $T : X \to X$ such that $Tx_{\sigma(j)} = \lambda_j y_{\sigma(j)}$ for $j \in \mathbb{N}$. It is still an open question – the so-called quasi-equivalence conjecture – whether all bases in a nuclear Fréchet space are quasi-equivalent. This conjecture was posed in 1961 by Mityagin [13, §8.3]. There were many attempts to solve this problem (see [26] and references therein); for example, Crone and Robinson [5] showed that in every nuclear Fréchet space with the so-called regular basis all bases are quasi-equivalent. In particular, all bases in $s$ are quasi-equivalent but the conjecture remains open for closed subspaces of $s$. As a consequence of Theorem 4, we obtain a characterization of closed subspaces of $s$ which have all bases quasi-equivalent (Theorem 15). We hope that our characterization will shed some light on this long-standing conjecture.

2. Preliminaries

In what follows, $\mathbb{N}$ will denote the set of natural numbers $\{1, 2, \ldots \}$ and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

By $e_j$ we denote the vector in $\mathbb{C}^\mathbb{N}$ whose $j$th coordinate equals 1, while all others equal 0.

By a Fréchet space we mean a complete metrizable locally convex space over $\mathbb{C}$ (we will not use locally convex spaces over $\mathbb{R}$). A Fréchet algebra is a Fréchet space which is an algebra with continuous multiplication. A Fréchet *-algebra is a Fréchet algebra with continuous involution.

We use the standard notation and terminology. All the notions from functional analysis are explained in [12].

We define the space of rapidly decreasing sequences as the Fréchet space

$$s := \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : |\xi|_q := \left( \sum_{j=1}^{\infty} |\xi_j|^2 j^{-2q} \right)^{1/2} < \infty \text{ for all } q \in \mathbb{N}_0 \right\}$$

with the topology corresponding to the system $(|\cdot|_q)_{q \in \mathbb{N}_0}$ of norms. The strong dual of $s$ (that is, the space of all continuous linear functionals on $s$ with the topology of uniform convergence on bounded subsets of $s$; see the definition in [12, p. 267]) can be identified with the space of slowly increasing sequences

$$s' := \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : |\xi|'_q := \left( \sum_{j=1}^{\infty} |\xi_j|^2 j^{2q} \right)^{1/2} < \infty \text{ for some } q \in \mathbb{N}_0 \right\}.$$

It is an easy exercise to show that by replacing the $\ell_2$-norms in the above definition of $s$ and $s'$ by the corresponding sup-norms — as in (1) and (2) — or $\ell_p$-norms ($1 \leq p < \infty$), we end up with the same sequence spaces.

Closed subspaces of the space $s$ can be characterized by the so-called property (DN) (see [12, Proposition 31.5]).

**Definition 1.** A Fréchet space $X$ with a fundamental system $(|| \cdot ||_q)_{q \in \mathbb{N}_0}$ of seminorms has property (DN) (see the definition in [12, p. 359]) if there is a continuous norm $|| \cdot ||$ on $X$ such that for all $q \in \mathbb{N}_0$ there exist $r \in \mathbb{N}_0$ and $C > 0$ such that

$$||x||_q^2 \leq C ||x|| ||x||_r$$

for all $x \in X$. The norm $|| \cdot ||$ is called a dominating norm.
We will also use property (DN) in the following equivalent form (see [12, Lemma 29.10]): there is a continuous norm $|| \cdot ||$ on $X$ such that for any $q \in \mathbb{N}_0$ and $\theta \in (0, 1)$ there exist $r \in \mathbb{N}_0$ and $C > 0$ such that

$$||x||_q \leq C||x||^{1-\theta}||x||^\theta_r$$

for all $x \in X$.

The algebra of smooth operators is defined as the Fréchet *-algebra $L(s', s)$ of continuous linear operators from $s'$ to $s$ with the topology of uniform convergence on bounded sets in $s'$. The *-algebra operations on $L(s', s)$ are inherited from $B(\ell_2)$ (note that every smooth operator is bounded on $\ell_2$). The algebra $L(s', s)$ can be also seen as an algebra of matrices which is sometimes more useful; it is isomorphic, via the map $x \mapsto (\langle xe_k, e_j \rangle)_{j,k \in \mathbb{N}}$, to the Fréchet *-algebra

$$K_\infty := \left\{ (x_{j,k})_{j,k \in \mathbb{N}} \in \mathbb{C}^{N^2} : \sup_{j,k \in \mathbb{N}} |x_{j,k}| j^q k^q < \infty \text{ for all } q \in \mathbb{N}_0 \right\}$$

of rapidly decreasing matrices (with matrix multiplication and matrix complex conjugation). Therefore, in this paper, one can replace each $L(s', s)$ with $'K_\infty'$. More details concerning $L(s', s)$ can be found in the introductions of [2-4].

**Definition 2.** A matrix $A = (a_{j,q})_{j \in \mathbb{N}, q \in \mathbb{N}_0}$ of nonnegative numbers such that

(i) for each $j \in \mathbb{N}$ there is $q \in \mathbb{N}_0$ such that $a_{j,q} > 0$, and

(ii) $a_{j,q} \leq a_{j,q+1}$ for $j \in \mathbb{N}$ and $q \in \mathbb{N}_0$,

is called a Köthe matrix. For $1 \leq p < \infty$ and a Köthe matrix $A$ we define the Köthe space

$$\lambda^p(A) := \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : |\xi|_{\lambda^p(A), q} := \sum_{j=1}^\infty |\xi_j| a_{j,q}^p \ < \infty \text{ for all } q \in \mathbb{N}_0 \right\},$$

and for $p = \infty$,

$$\lambda^\infty(A) := \left\{ \xi = (\xi_j)_{j \in \mathbb{N}} \in \mathbb{C}^\mathbb{N} : |\xi|_{\lambda^\infty(A), q} := \sup_{j \in \mathbb{N}} |\xi_j| a_{j,q} \ < \infty \text{ for all } q \in \mathbb{N}_0 \right\},$$

with the locally convex topology given by the seminorms $(1 \cdot |\lambda^\sigma(A), q|_{\mathbb{N}_0})$ (see the definition in [12, p. 326]). For simplicity, we will also write $\lambda^p(a_{j,q})$ (that is, only the entries of the matrix) instead of $\lambda^p(A)$.

All spaces $\lambda^p(A)$ are Fréchet spaces [12, Lemma 27.1]. By the Dynin–Mityagin theorem (see [12, Theorem 28.12]), if $\lambda^p(A)$ is nuclear (that is, for all $q$ there is $r$ such that $\sum_{j=1}^\infty a_{j,q} \frac{a_{j,r}}{a_{j,r}} < \infty$), then the sequence of vectors $(e_j)_{j \in \mathbb{N}}$ is an absolute Schauder basis of $\lambda^p(A)$. We will also use the following result: $\lambda^p(A)$ is nuclear for some $1 \leq p \leq \infty$ if and only if $\lambda^p(A) = \lambda^q(A)$ as Fréchet spaces for all $1 \leq p, q \leq \infty$ [12, Proposition 28.16].

For Köthe matrices $A = (a_{j,q})_{j \in \mathbb{N}, q \in \mathbb{N}_0}$ and $B = (b_{j,q})_{j \in \mathbb{N}, q \in \mathbb{N}_0}$ such that

$$\forall q \in \mathbb{N}_0 \ \exists r \in \mathbb{N}_0 \ \exists C > 0 \ \forall j \in \mathbb{N} \ a_{j,q} \leq C b_{j,r},$$

we write $A < B$. If $A < B$ and $B < A$, then we write $A \sim B$. For a bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$, $A_\sigma$ denotes the Köthe matrix $(a_{\sigma(j), q})_{j \in \mathbb{N}, q \in \mathbb{N}_0}$ and, moreover, $A^2$ is by definition the Köthe matrix $(a_{j,q}^2)_{j \in \mathbb{N}, q \in \mathbb{N}_0}$. 



By [15, Proposition 3.1], $\lambda^1(A)$ with pointwise multiplication is an algebra (and, clearly, if $\lambda^1(A)$ is nuclear so is $\lambda^p(A)$ for $1 < p \leq \infty$) if and only if $A \prec A^2$; in this case $\lambda^p(A)$ is called a Köthe algebra.

**Definition 3.** We say that a Köthe space $\lambda^p(A)$ is closed under taking square root if it is a Köthe algebra and for each sequence $(\xi_j)_{j \in \mathbb{N}}$ in $\lambda^p(A)$ of nonnegative scalars, the sequence $(\sqrt{\lambda_j})_{j \in \mathbb{N}}$ belongs to $\lambda^p(A)$ as well.

3. Commutative subalgebras of $L(s', s)$

Our main result — solving problem (A) — reads as follows.

**Theorem 4.** For a Köthe matrix $A = (a_{j,q})_{j \in \mathbb{N}, q \in \mathbb{N}_0}$ the following assertions are equivalent.

1. $\lambda^2(A)$ is isomorphic as a Fréchet *-algebra to some closed commutative *-subalgebra of $L(s', s)$.
2. There is a sequence $(P_j)_{j \in \mathbb{N}}$ of nonzero pairwise orthogonal self-adjoint projections belonging to $L(s', s)$ such that $A \sim (||P_j'||_{j \in \mathbb{N}, q \in \mathbb{N}_0}$.
3. There is a basic sequence $(f_j)_{j \in \mathbb{N}}$ of $s$ which is at the same time an orthonormal sequence in $\ell_2$ such that $A \sim (||f_j||_{j \in \mathbb{N}, q \in \mathbb{N}_0}$.
4. The matrix $A$ satisfies the following conditions:
   (i) $\forall q \in \mathbb{N}_0 \exists r \in \mathbb{N}_0 \sum_{j=1}^{\infty} a_{j,q} < \infty$;
   (ii) $\exists p \in \mathbb{N}_0 \exists C > 0 \forall j \in \mathbb{N} a_{j,p} \geq \frac{1}{C}$;
   (iii) $A^2 \prec A$.
5. The matrix $A$ satisfies the following conditions:
   (i) $\forall q \in \mathbb{N}_0 \exists r \in \mathbb{N}_0 \sum_{j=1}^{\infty} a_{j,q} a_{j,r} < \infty$;
   (ii) $\exists p \in \mathbb{N}_0 \forall q \in \mathbb{N}_0 \exists r \in \mathbb{N}_0, C > 0 \forall j \in \mathbb{N} a_{j,q}^2 \leq Ca_{j,p} a_{j,r}$;
   (iii) $A \sim A^2$.
6. $\lambda^2(A)$ is a nuclear biprojective Köthe algebra with property (DN).
7. $\lambda^2(A)$ is isomorphic as a Fréchet space to a closed subspace of $s$ and it is closed under taking square roots.
8. $\lambda^2(A)$ is a nuclear Fréchet space with $|| \cdot ||_{\ell_2}$ as a dominating norm.

**Remark 5.** By [2, Theorem 4.8], every infinite-dimensional closed commutative *-subalgebra of $L(s', s)$ is isomorphic as a Fréchet *-algebra to some Köthe sequence algebra, and thus the above theorem characterizes all infinite-dimensional closed commutative *-subalgebras of $L(s', s)$.

**Remark 6.** Homology theory is outside the main scope of this paper, and therefore we only recall that the a Fréchet algebra $A$ is called biprojective if the product map $\pi: A \hat{\otimes} A \to A$, $a \hat{\otimes} b \mapsto ab$, has a right inverse in the category of $A$-bimodules (see also [11, Definition IV.5.1]). The characterization of biprojectivity for Köthe algebras in terms of the corresponding Köthe matrices is due to Pirkovskii [15, Theorem 5.9].

The following theorem leads to the solution of problem (B).

**Theorem 7.** For a Köthe matrix $A = (a_{j,q})_{j \in \mathbb{N}, q \in \mathbb{N}_0}$ the following assertions are equivalent.

1. $\lambda^2(A)$ is isomorphic as a Fréchet space to some closed commutative *-subalgebra of $L(s', s)$.

(2) The matrix $A$ satisfies the following conditions:

(i) $\forall q \in \mathbb{N}_0 \exists r \in \mathbb{N}_0 \sum_{j=1}^{\infty} \frac{a_{j,q}}{r_j} < \infty$;

(ii) $\exists p \in \mathbb{N}_0 \forall q \in \mathbb{N}_0 \exists r \in \mathbb{N}_0, C > 0 \forall j \in \mathbb{N} a_{j,q}^2 \leq Ca_{j,p}a_{j,r}$.

(3) $\lambda^2(A)$ is a nuclear Fréchet space with the property (DN).

(4) $\lambda^2(A)$ is a Fréchet space isomorphic to a closed subspace of $s$.

**Proof.** It is well known that a Kőthe space $\lambda^2(A)$ is nuclear and has property (DN) if and only if $A$ satisfies conditions (i) and (ii), respectively. This shows (1)$\Rightarrow$(2)$\Rightarrow$(3), because $L(s',s)$ is $s$ if Fréchet spaces and nuclearity together with property (DN) are inherited by closed subspaces.

In order to prove (3)$\Rightarrow$(1), choose $p_0 \in \mathbb{N}_0$ for which $|\cdot|_{\lambda^2(A),p_0}$ is a dominating norm on $\lambda^2(A)$ and define the Kőthe matrix $B := (a_{j,q}/a_{j,p_0})_{j \in \mathbb{N},q \in \mathbb{N}_0}$. Let $T : \lambda^2(B) \to \lambda^2(A)$, $T\xi := (\xi_j/a_{j,p_0})_{j \in \mathbb{N}}$. Then, for $\xi \in \lambda^2(B)$ and $q \in \mathbb{N}_0$, we have $|T\xi|_{\lambda^2(A),q} = |\xi|_{\lambda^2(B),q}$, and thus $T$ is an isomorphism of Fréchet spaces. Moreover, for $q \in \mathbb{N}_0$ there exist $r \in \mathbb{N}_0$ and $C > 0$ such that

$$|\xi|_{\lambda^2(B),q}^2 = |T\xi|_{\lambda^2(A),p_0}^2 T\xi|_{\lambda^2(A),r} = C|\xi|_{\ell_2}|\xi|_{\lambda^2(B),r}, \quad \xi \in \lambda^2(B).$$

Therefore, $|\cdot|_{\ell_2}$ is a dominating norm on $\lambda^2(B)$, and the conclusion follows from Theorem 4 (implication (8)$\Rightarrow$(1)).

Finally, equivalence (3)$\Leftrightarrow$(4) follows directly from [12, Proposition 31.5].

As a direct consequence of Theorems 4, 7 and [12, Corollary 28.13 and Propositions 28.16 and 31.5] we also get the following characterization.

COROLLARY 8. For a Fréchet space $E$ the following assertions are equivalent:

(1) $E$ is isomorphic as a Fréchet space to some closed commutative $*$-subalgebra of $L(s',s)$;

(2) $E$ is isomorphic as a Fréchet space to a Kőthe space $\lambda^2(f_j_{j \in \mathbb{N}})$ for some basic sequence $(f_j)_{j \in \mathbb{N}}$ of $s$ which is at the same time an orthonormal sequence in $\ell_2$;

(3) $E$ is nuclear, has a basis and property (DN);

(4) $E$ is isomorphic as a Fréchet space to some closed subspace of $s$ with basis.

We now provide the proof of Theorem 4. We start with some lemmas involving short exact sequences of Fréchet–Hilbert spaces.

**Lemma 9.** Let

$$0 \to E \xrightarrow{j} F \xrightarrow{q} G \to 0$$

be a short exact sequence of Fréchet–Hilbert spaces. Let $\|\cdot\|_E$ be a continuous Hilbert norm on $E$ and let $\|\cdot\|_G$ be a continuous norm on $G$. Then there is a continuous Hilbert norm $\|\cdot\|_F$ on $F$ with

$$(\alpha) \quad \|x\|_E \sim \|j(x)\|_F \text{ for } x \in E;

(\beta) \quad \|q(x)\|_G \leq \inf_{y \in \ker q} \|x-y\|_F \text{ for } x \in F.$$

**Proof.** Without loss of generality, we may assume that $E$ is a closed subspace of $F$ and $j(x) = x$ for $x \in E$. By [12, Remark 22.8], there is a continuous Hilbert norm $\|\cdot\|_F$ on $F$ with $\|x\|_E \leq \|x\|_1$ for $x \in E$. Moreover, by the continuity of $q$, there is a continuous Hilbert norm $\|\cdot\|_2$ on $F$ such that

$$\inf_{z \in E} \|x-z\|_2 \leq \inf_{z \in E} \|x-z\|_2.$$

Now, let $x \in F$. Then $x = j(x) + q(x)$ and

$$\|x\|_F^2 = \|j(x)\|_F^2 + \|q(x)\|_F^2 \leq \|j(x)\|_F^2 + \|q(x)\|_F^2.$$
for $x \in F$. Then $|| \cdot || := (|| \cdot ||_F^2 + || \cdot ||_E^2)^{1/2}$ is a continuous Hilbert norm on $F$ such that

(i) $||x||_E \leq ||x||$ for $x \in E$;

(ii) $||q(x)||_G \leq \inf_{x \in E} ||x - z||$ for $x \in F$.

Define $|| \cdot ||_F$ by

$$||x||_F^2 = \inf_{z \in E} (||z||_E^2 + ||x - z||^2)$$

for $x \in F$.

To show condition (a), let us fix $x \in E$. Taking $z := x$ in the infimum, we get

$$||x||_F \leq ||x||_E.$$

Moreover, for $z \in E$, we obtain

$$||x||_E^2 \leq (||z||_E + ||x - z||_E)^2 = ||z||_E^2 + ||x - z||_E^2 + 2||z||_E ||x - z||_E.$$

If $||z||_E \leq ||x - z||_E$, then $||z||_E ||x - z||_E \leq ||x - z||_E^2$. Otherwise, $||z||_E ||x - z||_E \leq ||z||_E^2$, so, by (i),

$$||x||_E^2 \leq 3 (||z||_E^2 + ||x - z||_E^2) \leq 3 (||z||_E^2 + ||x - z||^2).$$

In consequence,

$$||x||_E \leq \sqrt{3} \inf_{z \in E} (||z||_E^2 + ||x - z||^2)^{1/2} = \sqrt{3}||x||_F,$$  \hspace{1cm} (4)

and thus condition (a) if fulfilled. By (ii), for $x \in F$, we get

$$||q(x)||_G^2 = \inf_{z \in E} ||x - z||^2 = \inf_{z_1 \in E} \inf_{z_2 \in E} (||z_2||_E^2 + ||x - z_1||^2) = \inf_{z_1, z_2 \in E} (||z_2||_E^2 + ||x - (z_1 + z_2)||^2)$$

$$= \inf_{z_1 \in E} \inf_{z_2 \in E} (||z_2||_E^2 +||(x - z_1) - z_2||^2) = \inf_{z \in E} ||x - z||_E^2,$$

which yields condition (b).

Now define a Hilbert norm $||(\cdot , \cdot )||_H$ on $H := E \times F$ by

$$||(x,y)||_H^2 := ||x||_E^2 + ||y||^2.$$  

Let

$$R : (H, ||(\cdot , \cdot )||_H) \to F, \quad (x, y) \mapsto x + y$$

and let

$$Q : (H, || \cdot ||_H) \to (H/\ker R, || \cdot ||_{H/\ker R})$$

be the quotient map. Then

$$||Q(x,y)||_{H/\ker R} = \inf_{(x',y') \in \ker R} ||(x,y) - (x',y')||_H = \inf_{x' \in E} ||(x,y) - (x',-x')||_H$$

$$= \inf_{x' \in E} (||x - x'||_E^2 + ||y + x'||^2)^{1/2} = \inf_{z \in E} (||z||_E^2 + ||(x + y) - z||^2)^{1/2} \hspace{1cm} (5)$$

$$= ||x + y||_F.$$

In particular,

$$|| \cdot ||_F = ||Q(0, \cdot )||_{H/\ker R},$$

hence $|| \cdot ||_F$ is a seminorm. Moreover, since $|| \cdot ||_F \leq || \cdot ||$, it follows that $|| \cdot ||_F$ is continuous.
Now assume that $\|x\|_F = 0$ for some $x \in F$. Then, by condition (\(\beta\)), $|q(x)|_G \leq \|x\|_F = 0$, and since $\|\cdot\|_G$ is a norm, we have $x \in \ker q = E$. Moreover, by (4), $\|x\|_E \leq \sqrt{3}\|x\|_F = 0$, but $\|\cdot\|_E$ is a norm, so $x = 0$. This shows that $\|\cdot\|_F$ is a norm, and therefore, by (5), $\|\cdot\|_{H/\ker R}$ is a norm as well (that is, $\ker R$ is a closed subspace of $H$; see [12, Lemma 5.10]). Since the quotient norm induced by a Hilbert norm is a Hilbert norm, the norm $\|\cdot\|_{H/\ker R}$ is hilbertian (see [21, p. 44]). Therefore, by (6), $\|\cdot\|_F$ satisfies the parallelogram law, hence $\|\cdot\|_F$ is a Hilbert norm.

The easy proof of our next lemma is left to the reader.

**Lemma 10.** Consider the diagram

$$
\begin{array}{ccc}
(Y, \|\cdot\|_Y) & \xrightarrow{j_Y} & (X, \|\cdot\|_X) \\
\downarrow & & \downarrow \\
(\tilde{Y}, \|\cdot\|_{\tilde{Y}}) & \xrightarrow{\tilde{i}} & (\tilde{X}, \|\cdot\|_{\tilde{X}})
\end{array}
$$

where

- $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are normed spaces such that $Y$ is a closed subspace of $(X, \|\cdot\|_X)$ and $\|\cdot\|_Y \sim \|\cdot\|_X|_Y$;
- $(\tilde{Y}, \|\cdot\|_{\tilde{Y}})$ and $(\tilde{X}, \|\cdot\|_{\tilde{X}})$ are the completions of $(Y, \|\cdot\|_Y)$ and $(X, \|\cdot\|_X)$, respectively;
- $j_Y$ and $j_X$ are the canonical inclusions;
- $\tilde{i}$ is the continuous linear extension of the map $j_X \circ i : Y \to \tilde{X}$.

Then $\tilde{i}(\tilde{Y})$ is a closed subspace of $(\tilde{X}, \|\cdot\|_{\tilde{X}})$.

**Lemma 11.** Let

$$
0 \longrightarrow E \xrightarrow{j} F \xrightarrow{q} G \longrightarrow 0
$$

be a short exact sequence of Fréchet–Hilbert spaces. Let $\|\cdot\|_E$ be a continuous Hilbert norm on $E$ and let us assume that $G$ has a continuous norm. Then there is a continuous Hilbert norm $\|\cdot\|$ on $F$ such that $\|j(x)\| = \|x\|_E$ for $x \in E$.

**Proof.** Without loss of generality, we may assume that $E$ is a closed subspace of $F$ and $j(x) = x$ for $x \in E$. Let $\|\cdot\|_G$ be a continuous norm on $G$. By Lemma 9, there is a continuous Hilbert norm $\|\cdot\|_F$ on $F$ such that

- (\(\alpha\)) $\|\cdot\|_E \sim \|\cdot\|_{F|_E}$ for $x \in E$;
- (\(\beta\)) $|q(x)|_G \leq \inf_{y \in \ker q} \|x - y\|_F =: \|x\|_G$ for $x \in F$.

Consider the diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & (E, \|\cdot\|_E) \xrightarrow{j} (F, \|\cdot\|_F) \xrightarrow{q} (G, \|\cdot\|_G) \longrightarrow 0 \\
\downarrow & & \downarrow \\
(E, \|\cdot\|_E) & \xrightarrow{\pi_0} & (\tilde{F}, \|\cdot\|_{\tilde{F}})
\end{array}
$$

where

- $(\tilde{E}, \|\cdot\|_{\tilde{E}})$ and $(\tilde{F}, \|\cdot\|_{\tilde{F}})$ are the completions of $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$, respectively;
• $j_E$ and $j_F$ are the canonical inclusions;
• $j$ is the continuous extension of the map $j_F \circ j : E \to \tilde{E}$.

Since, by (β), $\| \cdot \|_G$ is a norm, $E$ is a closed subspace of $(F, \| \cdot \|_F)$, and thus, by Lemma 10, $\tilde{E} := \tilde{j}(\tilde{E})$ is a closed subspace of the Hilbert space $(\tilde{F}, \| \cdot \|_{\tilde{F}})$. Then $\pi_0$ is defined to be the orthogonal projections onto $\tilde{E}$, $\kappa$ is the inverse of $\tilde{i}$ on $\tilde{E}$ and $\pi := \kappa \circ \pi_0$.

Let us define

$$||x||^2 := ||(\pi \circ j_F)(x)||_E^2 + ||q(x)||_G^2$$

for $x \in F$. Clearly, $|| \cdot ||$ is a continuous seminorm on $F$. Next, for $x \in E$, we get

$$||x|| = ||(\pi \circ j_F)(x)||_E = ||j_E(x)||_{\tilde{E}} = ||x||_E,$$

hence $|| \cdot ||_E = || \cdot ||_F$. For $x \in F$ with $||x|| = 0$ we have $||q(x)||_G = 0$, and thus $x \in E$. Then we have also $||x||_E = ||x|| = 0$ so $x = 0$. This shows that $|| \cdot ||$ is a norm.

Finally, since $|| \cdot ||_E$ and $|| \cdot ||_G$ are Hilbert norms, $|| \cdot ||$ is a Hilbert norm as well. □

The proof of the following lemma is a slight modification of Vogt’s proof of [24, Lemma 3.1]. For the convenience of the reader we present here the full argument.

**Lemma 12.** Let

$$0 \longrightarrow E \overset{j}{\longrightarrow} F \overset{q}{\longrightarrow} G \longrightarrow 0$$

be a short exact sequence of Fréchet–Hilbert spaces. Let $|| \cdot ||_E$ be dominating Hilbert norm on $E$ and let us assume that $G$ has property (DN). Then there is a dominating Hilbert norm $|| \cdot ||$ on $F$ such that $||j(x)|| = ||x||_E$ for $x \in E$.

**Proof.** Without loss of generality, we may assume that $E$ is a closed subspace of $F$ and $j(x) = x$ for $x \in E$. By Lemma 11, there is continuous Hilbert norm $|| \cdot ||_F$ on $F$ such that $|| \cdot ||_{F|E} = || \cdot ||_E$ and, by assumption, there is a dominating Hilbert norm $|| \cdot ||_G$ on $G$. Define

$$||x||^2 := ||x||_E^2 + ||qx||_G^2$$

for $x \in F$ and let $U_0 := \{x \in E : ||x|| < 1\}$. Then, clearly, $|| \cdot ||$ is a continuous Hilbert norm on $F$. We shall show that $|| \cdot ||$ is a dominating norm.

For an absolutely convex zero neighborhood $U$ in $F$, let $|| \cdot ||_U$ denote the Minkowski functional for $U$ and let $|| \cdot ||_G : G \to [0, \infty)$ be defined by $||qx||_G := \inf\{||x + y||_U : y \in E\}$ for $x \in F$.

Since $|| \cdot ||_E$ is a dominating norm on $E$, for each absolutely convex zero neighborhood $U$ in $F$ there is an absolutely convex zero neighborhood $W$ in $F$ with $W \subseteq U \cap U_0$ such that

$$||z||_U \leq ||z||_E^{2/3}||z||_W^{1/3},$$

for all $z \in E$. Since $|| \cdot ||_G$ is a dominating norm on $G$, there is an absolutely convex zero neighborhood $V$ in $F$ with $V \subseteq W$ such that

$$||qx||_W \leq ||qx||_G^{3/4}||qx||_V^{1/4} \leq ||x||_E^{3/4}||x||_V^{1/4},$$

for all $x \in F$.

Let us fix $x \in F$ and $\varepsilon > 0$. By the very definition of the norm $|| \cdot ||_W$, there is $y \in F$ such that

$$qx = qy \quad \text{and} \quad ||y||_W \leq ||qx||_W + \varepsilon,$$

and thus (8) yields

$$||y||_W \leq ||x||_E^{3/4}||x||_V^{1/4} + \varepsilon.$$
Hence, for $z := x - y$, we have
\[ \|z\|_E \leq \|x\|_F + \|y\|_F \leq \|x\| + \|y\|_W \leq \|x\| + \|x|^{3/4}\|x\|_{V'}^{1/4} + \varepsilon, \]
and since
\[ \|x\| = \|x|^{3/4}\|x\|_{V'}^{1/4} \leq \|x|^{3/4}\|x\|_{V'}, \]
it follows that
\[ \|z\|_E \leq 2\|x|^{3/4}\|x\|_{V'}^{1/4} + \varepsilon. \] (10)
Moreover, by (9),
\[ \|z\|_2 \leq \|x\|_W + \|y\|_W \leq \|x\|_V + \|x|^{3/4}\|x\|_{V'}^{1/4} + \varepsilon \leq 2\|x\|_V + \varepsilon. \]
Now, since for $a, b \geq 0$ we have $(a + b)^{2/3} \leq 3(a^{2/3} + b^{2/3})$, it follows from (7) and (10) that
\[ \|z\|_U \leq \left(2\|x|^{3/4}\|x\|_{V'}^{1/4} + \varepsilon\right)^{2/3} \left(2\|x\|_V + \varepsilon\right)^{1/3} \leq \left[3\left(2\|x|^{3/4}\|x\|_{V'}^{1/4}\right)^{2/3} + \varepsilon^{2/3}\right] \left[(2\|x\|_V)^{1/3} + \varepsilon^{1/3}\right] = \left(3\sqrt[3]{4}\|x|^{1/2}\|x\|_{V'}^{1/4} + 3\varepsilon^{2/3}\right) \left(\sqrt[3]{2}\|x\|_V^{1/3} + \varepsilon^{1/3}\right) = 6\|x|^{1/2}\|x\|_{V'}^{1/4} + h(\varepsilon), \] (11)
where $h(\varepsilon) \to 0$ when $\varepsilon \to 0$.
Next, by (9),
\[ \|y\|_U \leq \|y\|_W \leq \|x|^{3/4}\|x\|_{V'}^{1/4} + \varepsilon = \|x|^{1/2}\|x\|_{V'}^{1/4} \left(\|x\|_V^{1/4} + \varepsilon\right) \leq \|x|^{1/2}\|x\|_{V'}^{1/4} + \varepsilon. \] (12)
Consequently, by (11),
\[ \|x\|_U \leq \|y\|_U + \|z\|_U \leq 7\|x|^{1/2}\|x\|_{V'}^{1/4} + h(\varepsilon) + \varepsilon, \]
and finally, since $h(\varepsilon) + \varepsilon$ can be chosen arbitrary small, we have
\[ \|x\|_{U^2}^2 \leq 49\|x\| \|x\|_V, \]
which shows that $\|\cdot\|$ is a dominating norm on $F$. \hfill $\Box$

**Corollary 13.** Let $E$ be a closed subspace of $s$ such that $s/E$ has a dominating norm. Then for every dominating Hilbert norm $\|\cdot\|_E$ on $E$ there is a dominating Hilbert norm $\|\cdot\|$ on $s$ such that the restriction of $\|\cdot\|$ to $E$ is equal to $\|\cdot\|_E$.

**Proof.** The result follows by applying Lemma 12 to the canonical short exact sequence
\[ 0 \to E \hookrightarrow s \to s/E \to 0. \] \hfill $\Box$

Corollary 13 together with some results of Vogt [25, Corollary 7.6; 23, Proposition 3.3(a)] allows us to give a stronger version of the well-known theorem of Vogt characterizing closed subspaces of $s$ in terms of nuclearity and property (DN) (see [22, Satz 1.7] or [12, Proposition 31.5]).

**Theorem 14.** For a Fréchet space $X$ and a seminorm $\|\cdot\| : X \to [0, \infty)$ the following assertions are equivalent.

\[ \text{Proof. } \]
(1) There is a topological embedding $V : X \hookrightarrow s$ such that $\| Vx \|_{\ell_2} = \| x \|$ for all $x \in X$.
(2) $X$ is nuclear and $\| \cdot \|$ is a dominating Hilbert norm on $X$.

Proof. $(\Rightarrow)$ By assumption, $X$ is isomorphic to a closed subspace of $s$, so it is nuclear. Moreover, since $\| \cdot \|_{\ell_2}$ is a dominating Hilbert norm on $s$, it is easily seen that $\| \cdot \|$ is a dominating Hilbert norm on $X$.

$(\Leftarrow)$ Since $X$ is a nuclear Fréchet space with property (DN), by [12, Proposition 31.5], it is isomorphic to some closed subspace $E$ of $s$. Without loss of generality, we may assume that $s/E$ has property (DN). Indeed, by [23, Proposition 3.3(a)], for an arbitrary closed subspace $E_0$ of $s$ isomorphic to $X$, there exist a closed subspace $F_0$ of $s$ and an exact sequence

$$0 \longrightarrow E_0 \xrightarrow{\iota_0} s \xrightarrow{q_0} F_0 \longrightarrow 0$$

with continuous linear maps. Then $q_0$ induces an isomorphism $\overline{q_0} : s/\ker q_0 \rightarrow F_0$ (see [12, Proposition 22.11]), and thus

$$s/\iota_0(E_0) = s/\ker q_0 \cong F_0.$$

Hence $E := \iota_0(E_0)$ is a closed subspace of $s$ such that $s/E$ has property (DN), and by the open mapping theorem $E \cong E_0$.

Let $T : X \rightarrow E$ be an isomorphism and set $\| x \|_E := \| T^{-1}x \|$ for $x \in E$. An easy calculation shows that $\| \cdot \|_E$ is a dominating Hilbert norm on $E$. Hence, by Corollary 13, there is a dominating Hilbert norm $\| \cdot \|_s$ on $s$ such that $\| \cdot \|_s|_E = \| \cdot \|_E$. Moreover, by [25, Corollary 7.6], there is an automorphism $U$ of $s$ such that $\| Ux \|_{\ell_2} = \| x \|_s$ for $x \in s$.

Now, for $V := U \circ T$ and $x \in X$, we get

$$\| Vx \|_{\ell_2} = \| U(Tx) \|_{\ell_2} = \| Tx \|_s = \| Tx \|_E = \| x \|,$$

which proves the theorem. \qed

Finally, we are ready to proof our main result.

Proof of Theorem 4. $(3) \Rightarrow (1)$. We need only note that, by [4, Proposition 4.2], $\lambda^2(A) \cong \lambda^\infty(\| f_j \|_q)$ as Fréchet $^*$-algebras and $\lambda^\infty(\| f_j \|_q)$ is isomorphic as a Fréchet $^*$-algebra to the closed $^*$-subalgebra of $L(s', s)$ generated by the sequence $\{ f_j \}_{j \in \mathbb{N}}$.

$(1) \Rightarrow (2)$. Let $E$ be a closed commutative $^*$-subalgebra of $L(s', s)$ isomorphic to the Fréchet $^*$-algebra $\lambda^2(A)$. By [2, Theorem 4.8] and the nuclearity of $E$, there is a sequence $(Q_j)_{j \in \mathbb{N}}$ of nonzero pairwise orthogonal self-adjoint projections belonging to $L(s', s)$ such that $\lambda^2(A) \cong E \cong \lambda^2(\| Q_j \|_q)$ as Fréchet $^*$-algebras. Therefore, according to [4, Proposition 4.2], there is a bijection $\sigma : \mathbb{N} \rightarrow \mathbb{N}$ such that for $P_j := Q_{\sigma(j)}$ ($j \in \mathbb{N}$), we have $A \sim \{ \| P_j \|_q \}_{j \in \mathbb{N}, q \in \mathbb{N}_0}$.

$(2) \Rightarrow (4)$. By [2, Proposition 4.7] and [12, Lemma 27.25], $\lambda^2(\| P_j \|_q)$ is isomorphic to the closed $^*$-subalgebra of $L(s', s)$ generated by the sequence $(P_j)_{j \in \mathbb{N}}$. Therefore, the space $\lambda^2(A)$ (which is isomorphic as a Fréchet $^*$-algebra to $\lambda^2(\| P_j \|_q)$) is nuclear, and consequently, the matrix $A$ satisfies condition (i). Moreover, it is easy to check that the matrix $\{ \| P_j \|_q \}_{j \in \mathbb{N}, q \in \mathbb{N}_0}$ fulfills conditions (ii) and (iii), and thus (ii) and (iii) are also satisfied by the matrix $A$ (apply [4, Proposition 4.2]).

$(4) \Rightarrow (5)$. By conditions (ii) and (iii), we get

$$a_{j,r} \leq C a_{j,p} a_{j,r} \leq C a_{j,max\{p,r\}}^2$$

and

$$a_{j,q}^2 \leq C a_{j,r} a_{j,r} \leq C C a_{j,p} a_{j,r}$$

with appropriate quantifiers and constants.
(5) $\iff$ (6). It is well known that conditions (i) and (ii) are equivalent to nuclearity and property (DN), respectively. By [15, Proposition 3.1 and Theorem 5.9], $\lambda^\infty(A)$ is a biprojective Köthe algebra if and only if $A$ satisfies condition (iii).

(5) $\iff$ (7). By [12, Proposition 31.5], conditions (i) and (ii) are equivalent to the fact that $\lambda^2(A)$ is isomorphic to a closed subspace of $s$ and, by [15, Proposition 3.1], $\lambda^\infty(A)$ is an algebra if and only if $A \sim A^2$. Moreover, by [15, Lemmas 7.7 and 7.9], conditions (i) and (iii) imply that the square root of each nonnegative element of $\lambda^\infty(A)$ belongs to $\lambda^2(A)$. On the other hand, if $\lambda^2(A)$ is closed under taking square root and nuclear (note that closed subspaces of $s$ are nuclear) then, by [12, Proposition 28.16], $\lambda^2(A) \subset \lambda^2(A^2)$. Consequently, by the closed graph theorem, the identity map from $\lambda^2(A)$ to $\lambda^2(A^2)$ is continuous, and hence $A^2 \sim A$.

(5) $\Rightarrow$ (8). Nuclearity is guaranteed by condition (i). For an index $p$ from condition (ii) take, according to (i), an index $r$ and a constant $C_0 > 0$ such that $\sum_{j=1}^{\infty} \frac{a_{j,p}}{a_{j,r}} < C_0$. By (ii) and (iii) (more precisely, by $A \sim A^2$), there are $q_1, q_2, q_3 \in N_0$ and constants $C_1, C_2, C_3 > 0$ such that

$$a_{j,r} \leq C_1a_{j,q_1}^2 \leq C_2a_{j,p}a_{j,q_2} \leq C_3a_{j,p}a_{j,q_3}.$$ 

Then for $\xi \in \lambda^2(A)$ we obtain

$$||\xi||_{\ell_2} \leq C_3 \sum_{j=1}^{\infty} \frac{|\xi_j|^2a_{j,p}a_{j,q_3}}{a_{j,r}} \leq C_3 \sum_{j=1}^{\infty} \frac{a_{j,p}}{a_{j,r}} \sup_{j \in N} |\xi_j|^2a_{j,q_3}^2 \leq C_0C_3||\xi||^2_{\lambda^2(A),q_3} < \infty,$$

and thus $|| \cdot ||_{\ell_2}$ is a norm on $\lambda^2(A)$.

Moreover, the Cauchy–Schwarz inequality and condition (iii) (more precisely, relation $A^2 \sim A$) imply that for all $q \in N_0$ there are $r \in N_0$ and $C > 0$ such that

$$||\xi||_{\lambda^2(A),q} := \sum_{j \in N} |\xi_j|^2a_{j,q} \leq C \left( \sum_{j \in N} |\xi_j|^2 \right)^{1/2} \cdot \left( \sum_{j \in N} |\xi_j|^4a_{j,q}^4 \right)^{1/2} \leq C||\xi||_{\ell_2} \left( \sum_{j \in N} |\xi_j|^2a_{j,r}^2 \right)^{1/2}$$

$$= C||\xi||_{\ell_2}||\xi||_{\lambda^2(A),r},$$

which shows that $|| \cdot ||_{\ell_2}$ is a dominating norm on $\lambda^2(A)$.

(8) $\Rightarrow$ (3). By Theorem 14, there is a topological embedding $V : \lambda^2(A) \hookrightarrow s$ such that $||V\xi||_{\ell_2} = ||\xi||_{\ell_2}$ for all $\xi \in \lambda^2(A)$. Set $f_j := V\xi_j$ for $j \in N$. Then $(f_j)_{j \in N} \subset s$ is an orthonormal sequence in $\ell_2$ and a Schauder basis of $\text{im}V$. Therefore, $\Phi : \text{im}V \to \lambda^2(|f_j|_q)$ defined by $\Phi f_j := \xi_j$ for $j \in N$ is a Fréchet space isomorphism (see [12, Lemma 27.25]), and so is $\Phi \circ V : \lambda^2(A) \to \lambda^2(|f_j|_q)$. But $(\Phi \circ V)e_j := \xi_j$ for $j \in N$, hence $\lambda^2(A) = \lambda^2(|f_j|_q)$ as Fréchet $^*$-algebras, whence $A \sim (|f_j|_q)_{j \in N,q \in N_0}$.

4. Quasi-equivalence conjecture

In this final section we show how commutative subalgebras of $\mathcal{L}(s',s)$ are connected with the so-called quasi-equivalence conjecture stating that all bases in a nuclear Fréchet space are quasi-equivalent (see [13, §8.3; 26] and references therein). Here we only consider the case (still unsolved) of closed subspaces of $s$, that is, nuclear Fréchet spaces with property (DN). Let us recall that two bases $(f_j)_{j \in N}$ and $(g_j)_{j \in N}$ of a Fréchet space $X$ are called quasi-equivalent if there exist a bijection $\sigma : N \to N$ and a sequence $(\lambda_j)_{j \in N}$ of nonzero scalars such that the operator $T : X \to X$ defined by $Tf_j = \lambda_jg_{\sigma(j)}$ is a Fréchet space isomorphism.

Theorem 15. Let $X$ be an infinite-dimensional closed subspace of $s$. Then the following assertions are equivalent.
(i) For every two closed commutative *-subalgebras $E, F$ of $\mathcal{L}(s', s)$, if $E \cong F \cong X$ as Fréchet spaces then $E \cong F$ as Fréchet *-algebras.

(ii) For every two basic sequences $(f_j)_{j \in \mathbb{N}}$ and $(g_j)_{j \in \mathbb{N}}$ of $s$ which are at the same time orthonormal sequences in $\ell_2$, if $\lambda^2(|f_j|_q) \cong \lambda^2(|g_j|_q) \cong X$ as Fréchet spaces then there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\lambda^2(|f_j|_q)_{j \in \mathbb{N}, q \in \mathbb{N}_0} \cong \lambda^2(|g_{\sigma(j)}|_q)_{j \in \mathbb{N}, q \in \mathbb{N}_0}$.

(iii) For every two Köthe matrices $A$ and $B$ such that $A \cong A^2$ and $B \cong B^2$, if $\lambda^2(A) \cong \lambda^2(B) \cong X$ as Fréchet spaces then there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $A \cong B_{\sigma}$.

(iv) In $X$ all bases are quasi-equivalent.

Proof. (i)$\Leftrightarrow$(ii). If $E$ and $F$ are closed commutative *-subalgebras of $\mathcal{L}(s', s)$ then, by Theorem 4, there are basic sequences $(f_j)_{j \in \mathbb{N}}$ and $(g_j)_{j \in \mathbb{N}}$ of $s$ which are at the same time orthonormal in $\ell_2$ such that $E \cong \lambda^2(|f_j|_q)$ and $F \cong \lambda^2(|g_j|_q)$ as Fréchet *-algebras. Moreover, for each such sequence $(f_j)_{j \in \mathbb{N}}$, $\lambda^2(|f_j|_q)$ is isomorphic as a Fréchet *-algebra to a closed commutative *-subalgebra of $\mathcal{L}(s', s)$. Therefore, assuming $\lambda^2(|f_j|_q) \cong \lambda^2(|g_j|_q) \cong X$ as Fréchet spaces, it follows from (i) and [4, Proposition 4.2] that $\lambda^2(|f_j|_q)_{j \in \mathbb{N}, q \in \mathbb{N}_0} \cong \lambda^2(|g_{\sigma(j)}|_q)_{j \in \mathbb{N}, q \in \mathbb{N}_0}$.

Conversely, if we assume that $E \cong F \cong X$ as Fréchet spaces then, by (ii) and again by [4, Proposition 4.2], we get that $E \cong F$ as Fréchet *-algebras.

(i)$\Leftrightarrow$(iii). This is an immediate consequence of Theorem 4 and [4, Proposition 4.2].

(iii)$\Rightarrow$(iv). Let $(x_j)_{j \in \mathbb{N}}, (y_j)_{j \in \mathbb{N}}$ be two bases in $X$. Then

$$\lambda^2(A) \cong \lambda^2(B) \cong X$$

as Fréchet spaces, where

$$A := \left( \frac{|x_j|_q}{\|x_j\|_{\ell_2}} \right)_{j \in \mathbb{N}, q \in \mathbb{N}_0} \quad \text{and} \quad B := \left( \frac{|y_j|_q}{\|y_j\|_{\ell_2}} \right)_{j \in \mathbb{N}, q \in \mathbb{N}_0}.$$

Then for all $q \in \mathbb{N}_0$,

$$\left( \frac{|x_j|_q}{\|x_j\|_{\ell_2}} \right)^2 \leq \frac{\|x_j\|_{\ell_2} |x_j|_{2q}}{\|x_j\|_{\ell_2}^2} = \frac{|x_j|_{2q}}{\|x_j\|_{\ell_2}^2}$$

so $A^2 \prec A$. Moreover, since

$$\frac{|x_j|_q}{\|x_j\|_{\ell_2}} \geq 1$$

for all $q \in \mathbb{N}_0$, we have also $A \prec A^2$, and thus $A \cong A^2$. We show similarly that $B \cong B^2$. Therefore, by (iii), there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $A \cong B_{\sigma}$. This means that

$$\forall q \in \mathbb{N}_0 \ \exists r \in \mathbb{N}_0 \ \exists C > 0 \ \forall j \in \mathbb{N} \ \ |x_j|_q \leq C \lambda_j |y_{\sigma(j)}|_r$$

and

$$\forall q' \in \mathbb{N}_0 \ \exists r' \in \mathbb{N}_0 \ \exists C' > 0 \ \forall j \in \mathbb{N} \ \ |x_j|_{q'} \leq C' |x_j|_q,$$

where $\lambda_j := \|x_j\|_{\ell_2}/\|y_{\sigma(j)}\|_{\ell_2}$. Hence $x_j \mapsto \lambda_j y_{\sigma(j)}$, $j \in \mathbb{N}$, defines an isomorphism of $X$, that is, the bases $(x_j)_{j \in \mathbb{N}}$ and $(y_j)_{j \in \mathbb{N}}$ are quasi-equivalent.

(iv)$\Rightarrow$(ii). Assume that in $X$ all bases are quasi-equivalent. Let $(f_j)_{j \in \mathbb{N}}$ and $(g_j)_{j \in \mathbb{N}}$ be basic sequences of $s$ which are at the same time orthonormal sequences of $\ell_2$ and such that $\lambda^2(|f_j|_q) \cong \lambda^2(|g_j|_q) \cong X$ as Fréchet spaces. Let $F$ and $G$ be closed linear span in $s$ of $(f_j)_{j \in \mathbb{N}}$ and $(g_j)_{j \in \mathbb{N}}$, respectively. Then $F \cong G \cong X$ as Fréchet spaces; let $T : G \to F$ be an isomorphism. Clearly, $(F_j)_{j \in \mathbb{N}}$, $F_j := Tg_j$, is a basis in $F$ which is, by assumption quasi-equivalent to $(f_j)_{j \in \mathbb{N}}$. Consequently, there exist a sequence $(\lambda_j)_{j \in \mathbb{N}}$ of nonzero scalars and a
bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $S : F \to F$, defined by sending $F_{\sigma(j)}$ to $\lambda_j^{-1}f_j$, is an isomorphism of Fréchet spaces.

Now, let $V := ST : G \to F$ and define $|| \cdot ||_V : F \to [0, \infty)$ by $||V\xi||_V := ||\xi||_{\ell_2}$. Since $|| \cdot ||_{\ell_2}$ is a dominating Hilbert norm on $G$, $|| \cdot ||_V$ is a dominating Hilbert norm on $F$ and, consequently, $(\lambda_j^{-1}f_j)_{j \in \mathbb{N}} = (Vg_{\sigma(j)})_{j \in \mathbb{N}}$ is an orthonormal sequence in $F$ with respect to the Hilbert norm $|| \cdot ||_V$. In particular, $||f_j||_V = |\lambda_j|$ for $j \in \mathbb{N}$ and thus, $|| \cdot ||_V$ being a dominating norm on $F$,

$$\forall q \in \mathbb{N}_0 \exists r \in \mathbb{N}_0, C > 0 \forall j \quad |f_j|_q \leq C||f_j||_V |f_j|_r = C|\lambda_j| |f_j|_r. \quad (13)$$

Moreover, since $|| \cdot ||_V$ is a continuous norm on $F$,

$$\exists r_0 \in \mathbb{N}_0, C_0 > 0 \forall j \quad |\lambda_j| = ||f_j||_V \leq C_0 |f_j|_r_0.$$  

Now, since

$$\forall q \in \mathbb{N}_0 \exists r \in \mathbb{N}_0, C > 0 \forall j \in \mathbb{N} \quad |f_j|_q^2 \leq C|f_j|_r,$$

it follows that

$$\forall q \in \mathbb{N}_0 \exists r \in \mathbb{N}_0, C > 0 \forall j \quad |\lambda_j| |f_j|_q \leq C_0 |f_j|_{r_0} |f_j|_q \leq C_0 |f_j|_{\max(r_0,q)} \leq C_0 |f_j|_r. \quad (14)$$

Finally, let $W : F \to F$ be defined by $f_j \mapsto \lambda_j f_j$ for $j \in \mathbb{N}$. Then, by (13) and (14), $W$ is an automorphism of the Fréchet space $F$, and thus $WV$ is an isomorphism of Fréchet spaces which sends $g_{\sigma(j)}$ to $f_j$. This, clearly, implies that $(|f_j|_q)_{q \in \mathbb{N}, q \in \mathbb{N}_0} \sim (|g_{\sigma(j)}|_q)_{q \in \mathbb{N}, q \in \mathbb{N}_0}$. \hfill $\Box$

For a monotonically increasing sequence $\alpha = (\alpha_j)_{j \in \mathbb{N}}$ in $[0, \infty)$ such that $\lim_{j \to \infty} \alpha_j = \infty$ we define the power series space of infinite type,

$$\Lambda_\infty(\alpha) := \left\{ (\xi_j)_{j \in \mathbb{N}} \subseteq \mathbb{C}^\mathbb{N} : \sum_{j=1}^\infty |\xi_j|^2 e^{2\alpha_j q} < \infty \text{ for all } q \in \mathbb{N}_0 \right\}.$$  

It appears that the space $\Lambda_\infty(\alpha)$ is nuclear if and only if $\sup_{j \in \mathbb{N}} \frac{\log j}{\alpha_j} < \infty$ (see [12, Proposition 29.6]). As a consequence of Theorem 15, we get another proof of the following well-known result of Mityagin [13, Theorem 12].

**Corollary 16** (Mityagin). In nuclear power series space of infinite type $\Lambda_\infty(\alpha)$ all bases are quasi-equivalent.

**Proof.** Let $E, F$ be closed commutative *-subalgebras of $\mathcal{L}(s', s)$ isomorphic as Fréchet spaces to $\Lambda_\infty(\alpha)$. Then, by [4, Corollary 6.10], $E$ and $F$ are isomorphic to $\Lambda_\infty(\alpha)$ as a Fréchet *-algebra, and thus, by Theorem 15, all bases in $\Lambda_\infty(\alpha)$ are quasi-equivalent. \hfill $\Box$

Since, by Corollary 8, every closed subspace of $s$ with basis is isomorphic as a Fréchet space to some closed commutative *-subalgebra of $\mathcal{L}(s', s)$, we can rewrite Theorem 15 in the following way.

**Corollary 17.** The following assertions are equivalent.

(i) For every two closed commutative *-subalgebras $E, F$ of $\mathcal{L}(s', s)$, if $E \cong F$ as Fréchet spaces then $E \cong F$ as Fréchet *-algebras.

(ii) For every two basic sequences $(f_j)_{k \in \mathbb{N}}$ and $(g_j)_{k \in \mathbb{N}}$ of $s$ which are at the same time orthonormal sequences in $\ell_2$, if $\lambda^2(|f_j|_q) \cong \lambda^2(|g_j|_q)$ as Fréchet spaces then there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $(|f_j|_q)_{q \in \mathbb{N}, q \in \mathbb{N}_0} \sim (|g_{\sigma(j)}|_q)_{q \in \mathbb{N}, q \in \mathbb{N}_0}$.

(iii) For every two Köthe matrices $A$ and $B$ such that $A \sim A^2$ and $B \sim B^2$, if $\lambda^2(A) \cong \lambda^2(B)$ as Fréchet spaces then there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $A \sim B_\sigma$.

(iv) In every closed subspace of $s$ all bases are quasi-equivalent.
Acknowledgements. I cordially thank Leonhard Frerick for many valuable suggestions and encouraging me to solve the problems raised in the present paper.

References

1. S. J. Bhatt, A. Inoue and H. Ogi, ‘Spectral invariance, $K$-theory isomorphism and an application to the differential structure of $C^*$-algebras’, J. Operator Theory 49 (2003) 389–405.
2. T. Ciaś, ‘On the algebra of smooth operators’, Studia Math. 218 (2013) 145–166.
3. T. Ciaś, ‘Algebra of smooth operators’, PhD thesis, Adam Mickiewicz University, Poznań, 2014. Available at https://repozytorium.amu.edu.pl/jspui/bitstream/10593/10958/1/phd_thesis_TCias.pdf
4. T. Ciaś, ‘Commutative subalgebras of the algebra of smooth operators’, Monatsh. Math. 179 (2016) 203–207.
5. J. Cuntz, ‘Bivariante $K$-Theorie für lokalkonvexe Algebren und der Chern-Connes-Charakter’, Doc. Math. 2 (1997) 139–182.
6. J. Cuntz, ‘Cyclic theory and the bivariant Chern-Connes character’, Noncommutative geometry, Lecture Notes in Mathematics 1831 (eds S. Doplchiter and R. Longo; Springer, Berlin, 2004) 73–135.
7. P. Domanski, Algebra of smooth operators. Unpublished note available at www.staff.amu.edu.pl/~domanski/salgebra1.pdf.
8. G. A. Elliot, T. Natsume and R. Nest, ‘Cyclic cohomology for one-parameter smooth crossed products’, Acta Math. 160 (1998) 285–305.
9. H. Glöckner and B. Langkamp, ‘Topological algebras of rapidly decreasing matrices and generalizations’, Topology Appl. 159 (2012) 2420–2422.
10. A. Yu. Helemskii, The homology of Banach and topological algebras, Mathematics and Its Applications (Soviet Series) 41 (Kluver Academic Publishers, Dordrecht, 1989).
11. K. Piszczek, ‘One-sided ideals of the non-commutative Schwartz space’, Monatsh. Math. 178 (2015) 599–610.
12. K. Piszczek, ‘A Jordan-like decomposition in the noncommutative Schwartz space’, Bull. Aust. Math. Soc. 91 (2015) 322–330.
13. K. Piszczek, ‘Automatic continuity and amenability in the non-commutative Schwartz space’, J. Math. Anal. Appl. 432 (2015) 954–964.
14. K. Piszczek, ‘Corrigendum to “Automatic continuity and amenability in the non-commutative Schwartz space”’ [J. Math. Anal. Appl. 432 (2015) 954–964], J. Math. Anal. Appl. 435 (2016) 1015–1016.
15. K. Piszczek, ‘The noncommutative Schwartz space is weakly amenable’, Glasgow Math. J., doi: 10.1017/S0017089516000264.
16. H. H. Schaefer, Topological vector spaces, third printing corrected, Graduate Texts in Mathematics 3 (Springer, New York, 1971).
17. D. Vogt, ‘Charakterisierung der Unterräume von $s$’, Math. Z. 155 (1977) 109–117.
18. D. Vogt, ‘Subspaces and quotient spaces of (s)’, Functional analysis: surveys and recent results, North-Holland Math. Studies 27 (North-Holland, Amsterdam, 1977) 167–187.
19. D. Vogt, ‘Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen’, Manuscripta Math. 37 (1982) 269–301.
20. D. Vogt, ‘Unitary endomorphisms of power series spaces’, Preprint, 2010, http://www2.math.uni-wuppertal.de/~vogt/preprints/Drag.pdf.
21. H. Schoeffler, Topological vector spaces, third printing corrected, Graduate Texts in Mathematics 3 (Springer, New York, 1971).
22. D. Vogt, ‘Charakterisierung der Unterräume von $s$’, Math. Z. 155 (1977) 109–117.
23. D. Vogt, ‘Subspaces and quotient spaces of (s)’, Functional analysis: surveys and recent results, North-Holland Math. Studies 27 (North-Holland, Amsterdam, 1977) 167–187.
24. D. Vogt, ‘Eine Charakterisierung der Potenzreihenräume von endlichem Typ und ihre Folgerungen’, Manuscripta Math. 37 (1982) 269–301.
25. D. Vogt, ‘Unitary endomorphisms of power series spaces’, Preprint, 2010, http://www2.math.uni-wuppertal.de/~vogt/preprints/Drag.pdf.
26. N. Zobin, ‘Some remarks on quasi-equivalence of bases in Fréchet spaces’, Linear Algebra Appl. 307 (2000) 47–67.