INDUCED AND EFFECTIVE GRAVITY THEORIES IN D=2

Alexander Sevrin¹, Kris Thielemans² and Walter Troost†

¹. Department of Physics
University of California at Berkeley
and
Theoretical Physics Group
Lawrence Berkeley Laboratory
Berkeley, CA 94720, U.S.A.

². Instituut voor Theoretische Fysica
Universiteit Leuven
Celestijnenlaan 200D, B-3001 Leuven, Belgium

Abstract

As a preparation for the study of arbitrary extensions of d = 2 gravity we present a detailed investigation of SO(N) supergravity. Induced d = 2, SO(N) supergravity is constructed by gauging a chiral, nilpotent subgroup of the OSp(N|2) Wess-Zumino-Witten model. In order to get a gauge invariant theory with the correct number of degrees of freedom, we need to introduce N free fermions. From this we derive an all order expression for the effective action. Reality of the coupling constant imposes the usual restrictions on c for N = 0 and 1. No such restrictions appear for N ≥ 2. For N = 2, 3 and 4, no renormalizations of the coupling constant beyond one loop occur. Also, the effective N = 4 gravity based upon a linear N = 4 superconformal algebra, there is no renormalization at all, i.e. the quantum theory is equal to the classical. These results are related to non-renormalization theorems for theories with extended supersymmetries. Arbitrary (super)extensions of d = 2 gravity are then analyzed. The induced theory is represented by a WZW model for which a chiral, solvable group is gauged. From this, we obtain the effective action. All order expressions for both the coupling constant renormalization and the wavefunction renormalization are given. From this we classify all extensions of d = 2 gravity for which the coupling constant gets at most a one loop renormalization. As an application of the general strategy, N = 4 theories based on D(2, 1, α) and SU(1, 1|2), all WA gravities and the N = 2 W₉ models are treated in some detail.

*This work was supported in part by the Director, Office of Energy Research, Office of High Energy and Nuclear Physics, Division of High Energy Physics of the U.S. Department of Energy under Contract DE-AC03-76SF00098 and in part by the National Science Foundation under grant PHY90-21139.

†Bevoegdverklaard Navorser NFWO,Belgium
1 Introduction

Soon after the observation of Polyakov [1, 2] of a hidden, affine, \( Sl(2, \mathbb{R}) \) symmetry in the effective \( d = 2 \) gravity theory, \( d = 2 \) gravity was obtained from a gauged (or constrained) \( Sl(2, \mathbb{R}) \) Wess-Zumino-Witten (WZW) model [3, 4]. Following the understanding of \( d = 2 \) gravity, a major effort was put into the study of other types of \( d = 2 \) gravity. Those theories are based on extensions of the Virasoro algebra. Most of these extensions are non-linearly generated algebras. These algebras have as characteristic feature that the commutator of two generators contains not only a linear combination of the generators but composites of the generators as well:

\[
[T_a, T_b] = f_{abc}T_c + V_{abc}T_cT_d + W_{abcde}T_cT_dT_e + \cdots .
\] (1.1)

Though such algebras could be considered as ordinary Lie algebras by taking the composites as new generators, such an approach is not very practical as it often leads to an uncontrollable number of new generators. Many properties of Lie algebras find their analogue in non-linearly generated algebras [5]. An important feature of these algebras is that a distinction between classical and quantum algebras has to be made. Indeed, in the classical case, the algebra is a Poisson bracket algebra and composite terms in eq. (1.1) are unambiguously defined. In the quantum case, the generators are operators on a Hilbert space and ordering ambiguities arise. This fact is reflected in e.g. the Jacobi identities which assume different forms in the classical and the quantum case [6].

A typical example of such algebras are \( W \)-algebras, which are higher spin extensions of the Virasoro algebra. Since their discovery [7], \( W \)-algebras have attracted a lot of attention and numerous applications, both in physics and mathematics, of this type of symmetry algebras have been found (for a review see [8]).

The prototype of these algebras is the \( W_3 \) algebra. It is generated by the energy-momentum tensor \( T \) and a dimension 3 current \( W \) with operator product expansions (OPE) given by:

\[
T(x)T(y) = \frac{c}{2}(x-y)^{-4} + 2(x-y)^{-2}T(y) + (x-y)^{-1}\partial T(y) + \cdots
\]

\[
T(x)W(y) = 3(x-y)^{-2}W(y) + (x-y)^{-1}\partial W(y) + \cdots
\]

\[
W(x)W(y) = \frac{c}{3}(x-y)^{-6} + 2(x-y)^{-4}T(y) + (x-y)^{-3}\partial T(y) + \cdots
\]

\[
+(x-y)^{-2}\left[2\beta \Lambda(y) + \frac{3}{10}\partial^2 T(y)\right]
\]

\[
+(x-y)^{-1}\left[\beta \partial \Lambda(y) + \frac{1}{15}\partial^3 T(y)\right] + \cdots ,
\] (1.2)
where
\[ \Lambda(x) = (TT)(x) - \frac{3}{10} \partial^2 T(x) \] (1.3)
and
\[ \beta = \frac{16}{22 + 5c}. \] (1.4)

Just as the Virasoro algebra appears as the residual symmetry of gauge fixed gravity in \( d = 2 \), the \( W_3 \)-algebra appears as the residual symmetry of gauge fixed \( W_3 \)-gravity. The induced action for \( W_3 \)-gravity in the chiral gauge is
\[ e^{-\Gamma_{\text{ind}}[h, b]} = \langle \exp \frac{1}{\pi} \int d^2 x [h(x)T(x) + b(x)W(x)] \rangle. \] (1.5)

In [9], it was shown that \( \Gamma_{\text{ind}}[h, b] \) is expanded in \( 1/c \):
\[ \Gamma_{\text{ind}}[h, b] = \sum_{i \geq 0} c^{1-i} \Gamma^{(i)}[h, b]. \] (1.6)

This is in stark contrast with induced actions for linear conformal algebras, which are proportional to \( c \). The subleading terms in \( 1/c \) in eq. (1.6) arise from a proper treatment of the composite terms in eq. (1.2). In [10], an explicit form for the classical term \( \Gamma^{(0)}_{\text{ind}}[h, b] \) was obtained through the classical reduction of an \( \text{Sl}(3, \mathbb{R}) \) Wess-Zumino-Witten model (WZW model).

The Legendre transform \( W^{(0)}[t, w] \) of \( \Gamma^{(0)}_{\text{ind}}[h, b] \) is defined by
\[ W^{(0)}[t, w] = \min_{[h, b]} \left( \Gamma^{(0)}_{\text{ind}}[h, b] - \frac{1}{12\pi} \int d^2 x \left( h t + \frac{1}{30} b w \right) \right). \] (1.7)

In [11], it was conjectured that the generating functional \( W[t, w] \) of connected Green’s functions, defined by
\[ e^{-W[t, w]} = \int [dh][db] \ e^{-\Gamma_{\text{ind}}[h, b]} + \frac{1}{12\pi} \int d^2 x \left( h t + \frac{1}{30} b w \right). \] (1.8)

is given by
\[ W[t, w] = \frac{k_c}{6} W^{(0)} \left[ Z^{(t)} t, Z^{(w)} w \right], \] (1.9)
where
\[ k_c = -\frac{1}{48} \left( 50 - c - \sqrt{(c-2)(c-98)} \right) - 3, \] (1.10)
and
\[ Z^{(t)} = \frac{1}{2(k_c + 3)}, \quad Z^{(w)} = \frac{1}{\sqrt{30\beta(k_c + 3)^{3/2}}}. \] (1.11)

This conjecture was based on a computation of the first order quantum corrections to \( W^{(0)} \) which showed that the quantum corrections split into two parts: one part contributes to the multiplicative renormalizations of \( W^{(0)}[t, w] \) while the other cancels \( \Gamma^{(1)}_{\text{ind}}[h, b] \). If true, this conjecture implies that the 1PI or effective action is simply given by
\[ \Gamma_{\text{eff}}[h, b] = \frac{k_c}{6} \Gamma^{(0)} \left[ \frac{1}{2k_c Z^{(t)} h}, \frac{30}{2k_c Z^{(w)} b} \right]. \] (1.12)

This result guarantees the integrability of \( W_3 \)-gravity as the \( \text{Sl}(3, \mathbb{R}) \) current algebra, essential to solve the theory, persists at quantum level.

Recently, this conjecture, for the case of \( W_3 \) at least, was elegantly proven through the use of a quantum Hamiltonian reduction [12]. The principle behind this is quite simple and based on observations in [3, 4]. Consider a matter system, where we denote the matter fields collectively by \( \varphi \), with as action \( S[\varphi] \) and with a set of symmetry currents, denoted by \( J[\varphi] \). The induced action is defined by
\[ e^{-\Gamma[A]} = \int [d\varphi] e^{-S[\varphi]} - \frac{1}{\pi} \int AJ[\varphi], \] (1.13)

where \( A \) is a source. Alternatively \( A \) can be viewed as a chiral gauge field. The generating functional of its connected Greens functions, which upon a Legendre transform becomes the effective action, is defined by
\[ e^{-W[\tilde{J}]} = \int [dA] e^{-\Gamma[A]} + \frac{1}{\pi} \int A\tilde{J} = \int [d\varphi] \delta(J[\varphi] - \tilde{J}) e^{-S[\varphi]}. \] (1.14)

Evaluating this functional integral is impossible in general as it involves the computation of a usually very complicated Jacobian. In the gravity case however, one can realize the matter system by WZW model for which a chiral, solvable group is gauged. In that case, the fields \( \varphi \) are fixed by the currents, and in addition the Jacobians are manageable.

In this paper we will obtain an all order expression for the effective action of an arbitrary extension of \( d = 2 \) gravity. Before studying the general case, we present a detailed study of \( SO(N) \) supergravity. This case is most instructive as it covers all subtleties encountered in the general case.
Aspects of $N = 1$ and $N = 2$ supergravity were studied in [13, 14, 15]. The supergravity theories are based on the $N$-extended $SO(N)$ superconformal algebra, which, for $N \geq 3$, is an example of a non-linearly generated algebra [16, 17]. The subalgebra of transformations, globally defined on the sphere, form an $OSp(N|2)$ algebra. A realization of the matter sector referred to above is constructed from a gauged $OSp(N|2)$ WZW model. Features of the $N = 3$ theory were examined in [18], where one-loop results for the effective action were given. We will give all loop results for the effective theory for arbitrary $N$.

The paper is organized as follows. In the next section, the main properties of induced $SO(N)$ supergravity are obtained through the study of the anomalous Ward identities. In section 3, we get the large $c$ limit of the induced and effective actions through the reduction of the $OSp(N|2)$ WZW model. In section 4, inspired by the methods of [12], an all order representation of the induced action is constructed through a quantum Hamiltonian reduction (or gauging) of the $OSp(N|2)$ WZW model. In the following section we use this representation to obtain an all order expression for the effective action. The results are checked against one loop computations. In section 6, we extend the framework to an arbitrary extension of $d = 2$ gravity. Both the coupling constant renormalization and the wavefunction renormalization are explicitly computed. As an application we briefly analyze some other $N = 4$ supergravity theories. These are based on a one parameter family of linear $N = 4$ superconformal algebras [19]. By decoupling a $U(1)$ current and 4 fermions [20], one obtains a one parameter family of non-linearly generated $N = 4$ superconformal algebras. The corresponding supergravity theories are obtained by reducing the $D(2,1,\alpha)$ WZW model. We touch upon $SU(1,1|2)$ supergravity, all classes of $WA$ gravity and the $N = 2$ extensions of $W_n$ gravity. We end by presenting some conclusions. In appendix A, we summarize several useful properties of WZW models on a supergroup. Induced gauge theories are reviewed in appendix B. In appendix C we give a few useful facts about $sl(2,R)$ embeddings.

2 $SO(N)$ Supergravity

The $N$-extended $SO(N)$ superconformal algebras, [16, 17], are generated by the energy-momentum tensor $T$, $N$ dimension 3/2 supersymmetry currents $G^a$ and an affine $SO(N)$ Lie algebra generated by currents $U^i$ where the index $i$ stands for a pair of indices $(pq)$ with $1 \leq p < q \leq N$. The operator product expansions (OPEs) are given by:

$$T(x)T(y) = \frac{c}{2}(x-y)^{-4} + 2(x-y)^{-2}T(y) + (x-y)^{-1}\partial T(y)$$

$$T(x)\Phi(y) = h_\Phi(x-y)^{-2}\Phi(y) + (x-y)^{-1}\partial \Phi(y),$$

4
\[ G^a(x)G^b(y) = \delta^{ab}(x-y)^{-3} + 2\delta^{ab}(x-y)^{-1}T(y) \]
\[ + \frac{b}{k}\lambda_{ab}^i \left( 2(x-y)^{-2}U^i(y) + (x-y)^{-1}\partial U^i(y) \right) \]
\[ + (x-y)^{-1}\gamma \Pi^i_{ab}(U^iU^j)(y), \]
\[ U^i(x)U^j(y) = -\frac{k}{2}\delta^{ij}(x-y)^{-2} + (x-y)^{-1}f_{ij}kU^k(y) \]
\[ U^i(x)G^a(y) = (x-y)^{-1}\lambda_{ab}^{i}G^b(y), \tag{2.1} \]

where
\[ c = \frac{k(6k + N^2 - 10)}{2k + N - 3} \]
\[ b = \frac{2k + N - 4}{k + N - 3} \]
\[ \gamma = \frac{2}{k + N - 3} \tag{2.2} \]

and
\[ h_\Phi = \frac{3}{2} \text{ and } 1 \text{ for } \Phi = G^a \text{ and } U^{ab}. \tag{2.3} \]

The normalizations are such that \( \lambda_{ab}^{(pq)} \equiv 1/\sqrt{2(\delta^{ab}_a \delta^{pq}_b - \delta^{ab}_b \delta^{pq}_a)} \), \( [\lambda^i, \lambda^j] = f_{ij}^{k}\lambda^k \), \( tr(\lambda^i \lambda^j) = -\delta^{ij} \), \( f_{ik} f_{jl}^{k} = -(N-2)\delta_{ij} \), and \( \Pi^i_{ab} = \Pi^j_{ba} = \lambda_{ac}^{i}\lambda_{cb}^{j} + \lambda_{ac}^{j}\lambda_{cb}^{i} + \delta_{ab}\delta^{ij} \). For \( N = 1 \) and \( N = 2 \) these are just the standard \( N = 1 \) and \( N = 2 \) superconformal algebras. For \( N \geq 3 \) the algebras contain composite terms in the \( GG \) OPE.

The induced action \( \Gamma[h, \psi, A] \) is defined as
\[ \exp(-\Gamma[h, \psi, A]) = \exp\left( -\frac{1}{\pi} \int d^2 x \left( h(x)T(x) + \psi^a(x)G_a(x) \right. \right. \]
\[ \left. \left. + A^i(x)U_i(x) \right) \right). \tag{2.4} \]

The chiral, linearized supergravity transformations:
\[ \delta h = \bar{\partial} \epsilon + \epsilon \partial h - \partial \epsilon h + 2\theta^a \psi_a, \]
\[ \delta \psi^a = \bar{\partial} \theta^a + \epsilon \partial \psi^a - \frac{1}{2}\partial \epsilon \psi^a + \frac{1}{2}\theta^a \partial h - \partial \theta^a h \]
\[ + \lambda_{ab}^{i}(\theta^b A^i - \omega^i \psi^b), \]
\[ \delta A^i = \bar{\partial} \omega^i + \epsilon \partial A^i + \frac{b}{k}\lambda_{ab}^i(\theta^a \psi^b - \theta^a \partial \psi^b) \]
\[ - f_{jk}^{i} \omega^j A^k - \partial \omega^i h, \tag{2.5} \]
are anomalous in the induced theory:

$$\delta \Gamma[h, \psi, A] = -\frac{c}{12\pi} \int \epsilon \partial^3 h - \frac{b}{2\pi} \int \theta^a \partial^2 \psi_a + \frac{k}{2\pi} \int \omega^i \partial A_i - \frac{\gamma}{\pi} \int \theta^a \psi^b \Pi^{ij}_{ab} (U^i U^j)_{\text{eff}}. \quad (2.6)$$

The last term is due to the non-linear nature of the superconformal algebras. Defining

$$u^i = -\frac{2\pi}{k} \frac{\delta \Gamma[h, \psi, A]}{\delta A^i} \quad (2.7)$$

one finds

$$\left(U^{(i} U^{j)}\right)_{\text{eff}}(x) = \left\langle \int d^2 x U^{(i} U^{j)}(x) \exp\left( -\frac{1}{\pi} \int (h T + \psi^a G_a + A^i U_a) \right) \right\rangle / \exp(-\Gamma),$$

$$= \left( \frac{k}{2} \right)^2 u^i(x) u^j(x) + \frac{k\pi}{4} \lim_{y \to x} \left( \frac{\partial u^i(x)}{\partial A^j(y)} - \frac{\partial}{\partial \delta^{(2)}(x-y)} \delta^{ij} + i \leftarrow j \right). \quad (2.8)$$

The limit in the last term of eq. (2.8) reflects the point-splitting regularization of the composite terms in eq. (2.1). One notices that in the limit $c \to \infty$, $u$ becomes $c$ independent and one has simply

$$\lim_{c \to \infty} \left( \left( \frac{2}{k} \right)^2 \left(U^{(i} U^{j)}\right)_{\text{eff}}(x) \right) = u^i(x) u^j(x). \quad (2.9)$$

Using eq. (2.8), we find that eq. (2.6) can be rewritten as:

$$\delta \Gamma[h, \psi, A] = -\frac{c}{12\pi} \int \epsilon \partial^3 h - \frac{b}{2\pi} \int \theta^a \partial^2 \psi_a + \frac{k}{2\pi} \int \omega^i \partial A_i - \frac{k^2 \gamma}{4\pi} \int \theta^a \psi^b \Pi^{ij}_{ab} u^i u^j$$

$$- \lim_{y \to x} \frac{k\gamma}{2} \int \theta^a \psi^b \Pi^{ij}_{ab} \left( \frac{\partial u^i(x)}{\partial A^j(y)} - \frac{\partial}{\partial \delta^{(2)}(x-y)} \delta^{ij} \right), \quad (2.10)$$

where the last term disappears in the large $k$ limit. The term proportional to $\int \theta^a \psi^b \Pi^{ij}_{ab} u^i u^j$ in eq. (2.10) can be absorbed by adding a field dependent term in the transformation rule for $A$:

$$\delta_{\text{extra}} A^i = -\frac{k\gamma}{2} \theta^a \psi^b \Pi^{ij}_{ab} u_j. \quad (2.11)$$

Doing this, we find that in the large $k$ limit, the anomaly reduces to the minimal one.

Introducing

$$t = \frac{12\pi}{c} \frac{\delta \Gamma[h, \psi, A, \eta]}{\delta h}, \quad g^a = \frac{2\pi}{b} \frac{\delta \Gamma[h, \psi, A, \eta]}{\delta \psi^a} \quad (2.12)$$
we obtain the Ward identities by combining eqs. (2.5) and (2.6):
\[
\partial^3 h = \nabla t - \frac{3b}{c} (\psi^a \partial + 3 \partial \psi^a) g_a + \frac{6k}{c} \partial A^i u_i ,
\]
\[
\partial^2 \psi^a = \nabla g^a - \frac{c}{3b} \psi^a t + \lambda_{ab}^i A^i g^b - \lambda_{ab}^i \left( 2 \partial \psi^b + \psi^b \partial \right) u^i
- \frac{k^2 \gamma}{2b} \Pi_{ab}^{ij} u^i u^j - \frac{k \gamma}{b} \Pi_{ab}^{ij} \lim_{y \to x} \left( \frac{\partial u^i(x)}{\partial A^j(y)} - \frac{\partial}{\partial \delta(2)(x - y) \delta^{ij}} \right)
\]
\[
\partial A^i = \nabla u^i + \frac{b}{k} \lambda_{ab}^i \psi^a g^b + f_{ij} A^j u^k ,
\]
where
\[
\nabla \Phi = \left( \bar{\partial} - \partial \Phi - h_{\Phi} (\partial h) \right) \Phi ,
\]
with
\[
h_{\Phi} = 2, \frac{3}{2}, 1 \quad \text{for} \quad \Phi = t, g^a, u^a .
\]
The Ward identities provide us with a set of functional differential equations for the induced action. Because of the explicit dependence on \( k \) of the Ward identities, the induced action is given as a \( 1/k \) expansion:
\[
\Gamma[h, \psi, A] = \sum_{i \geq 0} k^{1-i} \Gamma^{(i)}[h, \psi, A] .
\]
In the large \( k \) limit, the Ward identities become local and they are solved by \( \Gamma[h, \psi, A] = k \Gamma^{(0)}[h, \psi, A] \). As we will show in the next section, the large \( k \) limit of the induced action can be obtained from a classical reduction of an \( OSp(N|2) \) WZW model.

The effective action \( W[t, g, u] \) is defined by
\[
\exp \left( - W[t, g, u] \right) = \int [dh][d\psi][dA] \exp \left( - \Gamma[h, \psi, A] \right.
+ \frac{1}{4\pi} \left( \int h t + 4 \psi^a g_a - 2 A^i u_i \right) .
\]
If one defines the Legendre transform of \( \Gamma^{(0)}[h, \psi, A] \) as
\[
W^{(0)}[t, g, u] = \min_{\{h, \psi, A\}} \left( \Gamma^{(0)}[h, \psi, A] - \frac{1}{4\pi} \left( \int h t + 4 \psi^a g_a - 2 A^i u_i \right) \right)
\]
*For shortness, we will use the term 'effective action' also for the generating functional of the connected Green functions. We trust the symbol \( W \) in stead of \( \Gamma \) is enough to avoid confusion.*
we will show that \( W[t, g, u] \) can be expressed as

\[
W[t, g, u] = k_c W^{(0)}[Z^{(t)} t, Z^{(g)} g, Z^{(u)} u].
\]

(2.19)

So to leading (i.e. classical) order we have

\[
k_c \equiv k \quad Z^{(t)} \equiv Z^{(g)} \equiv Z^{(u)} \equiv \frac{1}{k}
\]

(2.20)

3 Classical Reduction of \( OSp(N|2) \)

The Lie algebra of \( OSp(N|2) \) is generated by a set of bosonic generators \( \{t_{\pm}, t_0, t_a, t_{ab} = -t_{ba} \text{ and } a, b \in \{1, \cdots, N\} \} \) which form an \( Sl(2) \times SO(N) \) Lie algebra and a set of fermionic generators \( \{t_+, t_-; a \in \{1, \cdots, N\} \} \). A Lie algebra valued field \( A = A^+_a t_a + A^0 t_0 + A^- t_m + \frac{1}{2} A^{ab} t_{ab} + A^{+a} t_+ + A^- a t_- \) where the representation matrices are in the fundamental representation, is explicitly given by:

\[
A = \begin{pmatrix}
A^0 & A^+_1 & A^+_2 & A^+_3 & \cdots & A^+_N \\
A^-_1 & A^0 & A^-_2 & A^-_3 & \cdots & A^-_N \\
A^-_2 & A^-_1 & A^1_2 & A^1_3 & \cdots & A^1_N \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
A^-_N & A^+_N & A^1_N & A^2_N & \cdots & 0
\end{pmatrix},
\]

(3.1)

and \( A^{ab} = -A^{ba} \). From this one reads off the generators of \( OSp(N|2) \) in the fundamental representations and one easily computes the (anti)commutation relations.

Given the flat connections \( A_z \) and \( u_z \), i.e.

\[
R_{z\bar{z}} = \partial A_{\bar{z}} - \bar{\partial} u_{\bar{z}} - [u_z, A_z] = 0,
\]

(3.2)

we impose the following constraint on \( u_z \)

\[
u_z \equiv \begin{pmatrix}
0 & u^+_z & u^+_1 & u^+_2 & \cdots & u^+_N \\
1 & 0 & 0 & 0 & \cdots & 0 \\
0 & -u^+_1 & 0 & u^+_2 & \cdots & u^+_N \\
0 & -u^+_2 & -u^+_1 & 0 & \cdots & u^+_N \\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
0 & -u^+_N & -u^+_1 & -u^+_2 & \cdots & 0
\end{pmatrix}.
\]

(3.3)
After we impose the constraints eq. (3.3), we find that some of the components of $R_{zz} = 0$ become algebraic equations. Indeed, $R_{zz} = R^0_{zz} = R^{-a}_{zz} = 0$ for $0 \leq a \leq N$ can be solved for $A^0_z$, $A^+_z$ and $A^{+a}_z$ giving

$$A^0_z = \frac{1}{2} \partial A^+_z$$
$$A^+_z = -\frac{1}{2} \partial^2 A^+_z + A^+_z u^+_z + A^{-a}_z u^{+a}_z$$
$$A^{+a}_z = \partial A^{-a}_z + A^{-a}_z u^{+a}_z - \sqrt{2} \lambda_{ab}^i A^{+b}_z u^i_z,$$  \hspace{1cm} (3.4)

where

$$u^i_z \equiv \frac{1}{\sqrt{2}} \lambda_{ab}^i A^{ab}_z.$$  \hspace{1cm} (3.5)

The remaining curvature conditions $R^+_{zz} = R^{+a}_{zz} = R^{-b}_{zz} = 0$ reduce now to the Ward identities eq. (2.13) in the limit $k \to \infty$ upon identifying

$$h \equiv A^+_z$$
$$\psi^a \equiv i A^{-a}_z$$
$$A^i \equiv -\sqrt{2} (A^i_z - A^+_z u^i_z)$$
$$t \equiv -2 (u^+_z + u^i_z u^i_z)$$
$$g^a \equiv i u^{+a}_z$$
$$u^i \equiv -\sqrt{2} u^i_z,$$  \hspace{1cm} (3.6)

where

$$A^i_z \equiv \frac{1}{\sqrt{2}} \lambda_{ab}^i A^{ab}_z.$$  \hspace{1cm} (3.7)

Consider now a WZW model $\kappa S^-[g]$ (for conventions, see the appendix A) on $OSp(N|2)$. We can identify

$$W^{(0)}[t = -2(\partial gg^{-1})^+ - 2(\partial gg^{-1})^i(\partial gg^{-1})^i, g^a = i(\partial gg^{-1})^{+a}, u^i = -\sqrt{2}(\partial gg^{-1})^i]$$

$$= -\frac{1}{2} S^-[g],$$  \hspace{1cm} (3.8)

where $S^-[g]$ is the WZW-model with the constraints eq. (3.3) imposed. As such, one finds to leading order

$$W[t, g, u] = kW^{(0)}[t/k, g/k, u/k] = \kappa S^-[g]$$  \hspace{1cm} (3.9)
and the level \( \kappa \) of the \( osp(N|2) \) affine Lie algebra is related to the \( SO(N) \) level \( k \) as \( k = -2\kappa \) or \( c = -6\kappa \).

Taking the Legendre transform
\[
\Gamma[h, \psi, A] = \min_{\{t, g, u\}} \left[ W[t, g, A] + \frac{1}{4\pi} \left( \int h t + 4\psi^a g_a - 2A^i u_i \right) \right],
\]
(3.10)
we find that induced action in the limit \( k \to \infty \) is given by:
\[
\Gamma[h, \psi, A] = k\Gamma^{(0)}[h, \psi, A],
\]
(3.11)
where
\[
\Gamma^{(0)}[h] = (\partial gg)^{-1}, \psi^a = i(\partial gg)^{-a}, A^i = -\sqrt{2}(\partial gg)^i(\partial gg)^{-1} + (\partial gg)^{-a}(\partial gg)^{-1})^a
\]
\[
= \frac{1}{2} S^+[g] + \frac{1}{2\pi} \int \left\{ ((\partial gg)^{-1}) + ((\partial gg)^{-1})^i(\partial gg)^{-1})^i + (\partial gg)^{-a}(\partial gg)^{-1})^a \right\},
\]
(3.12)
or
\[
\Gamma[h, \psi, A] = -\kappa S^+[g] + \frac{\kappa}{2\pi} \int (ht + 2\psi^ag^a).
\]
(3.13)
In order to obtain an explicit expression for the induced action in the \( k \to \infty \) limit, we parametrize the group element \( g \) as
\[
g \equiv e^{x^+t_+}e^{x^+t_{+a}}e^{x^0t_0}e^{\varphi^i t_i}e^{\phi^b t_b} = e^{\phi^b t_b},
\]
(3.14)
and we solve the constraints eq. (3.3) with \( u_2 = \partial gg^{-1} \). This gives
\[
x^0 = \frac{1}{2} \ln (\partial f + \partial \phi^a \phi^a)
\]
\[
x^a = (\partial f + \partial \phi^c \phi^c)^{-1/2} \left[ \exp \left( \sqrt{2} \varphi^i \lambda^i \right) \right]_{ab} \partial \phi^b
\]
\[
x^+ = -\frac{1}{2} \partial \ln (\partial f + \partial \phi^a \phi^a).
\]
(3.15)
Using the parametrization eq. (3.14) with the solutions, eq. (3.15), in eqs. (3.11) (3.12) one gets, through repeated application of the Polyakov-Wiegman formula, eq. (A.7), the explicit form of the induced action.
Finally, we can rewrite the right hand side of eq. (3.12) in the following suggestive form

$$\Gamma^{(0)}[h,\psi,A] = \frac{1}{2} S^+[e^{-zt=g}] + \frac{1}{2\pi} \int \{(\bar{\partial}g\bar{g}^{-1})^i(\partial g\bar{g}^{-1})^i\}$$ \hspace{1cm} (3.16)$$

Defining

$$\bar{g} \equiv e^{-zt=g}, \hspace{1cm} (3.17)$$

and using the Polyakov-Wiegman formula we get

$$\Gamma^{(0)}[h,\psi,A] = \frac{1}{2} S^+[\bar{g}] + \frac{1}{2\pi} \int \{h(\bar{\partial}g\bar{g}^{-1})^i(\partial g\bar{g}^{-1})^i\},$$ \hspace{1cm} (3.18)$$

where we have now that:

$$h = \left(\bar{\partial}g\bar{g}^{-1}\right)^0 + 2z \left(\bar{\partial}g\bar{g}^{-1}\right)^0 - z^2 \left(\bar{\partial}g\bar{g}^{-1}\right)^{\pm}$$

$$\psi^a = i \left(\bar{\partial}g\bar{g}^{-1}\right)^- a + iz \left(\bar{\partial}g\bar{g}^{-1}\right)^{+a}$$

$$A^i = -\sqrt{2} \left(\bar{\partial}g\bar{g}^{-1}\right)^i.$$ \hspace{1cm} (3.19)$$

This clearly explains the origins of the hidden $OSp(N|2)$ current algebra in $SO(N)$ supergravity in the light-cone gauge.

This concludes our treatment of $SO(N)$ supergravity in the $k \to \infty$ limit. In the next section, we will investigate this theory for arbitrary $k$.

4 Quantum Reduction of $OSp(N|2)$

In order to obtain the full induced action, we consider a quantum Hamiltonian reduction, \textit{i.e.} instead of imposing the constraints eq. (3.3) at the classical level we impose them as quantum conditions. Consider the subalgebras $\Pi_{\pm}osp(N|2)$ of $osp(N|2)$ generated by $\Pi_{+}osp(N|2) = \{t_+, t_a; 0 \leq a \leq N\}$ and $\Pi_{-}osp(N|2) = \{t_-, t_{-a}; 0 \leq a \leq N\}$. We call the corresponding subgroups of $OSp(N|2)$, $\Pi_{+}OSp(N|2)$ and $\Pi_{-}OSp(N|2)$. Introduce gauge fields $A_\pm \equiv \partial h_- h_{-1}^-$ and $A_\pm \equiv \bar{\partial} h_+ h_{+1}^+$ where $h_\pm \in \Pi_{\pm}OSp(N|2)$. Obviously we have that $A_\pm \in \Pi_{\pm}osp(N|2)$ and $A_\pm \in \Pi_{\pm}osp(N|2)$. The action

$$S_0 = \kappa S^-[g] - \frac{\kappa}{2\pi x} \int str A_\pm g^{-1}\bar{\partial}g + \frac{\kappa}{2\pi x} \int str A_\pm g\bar{g}^{-1} + \frac{\kappa}{2\pi x} \int str A_\pm g A_\pm g^{-1}$$ \hspace{1cm} (4.1)$$
(with $x = \frac{1}{2}$ in the fundamental representation) is invariant under

$$
\begin{align*}
  h_\pm & \rightarrow \gamma_\pm h_\pm \\
  g & \rightarrow \gamma_+ g \gamma_-^{-1},
\end{align*}
$$

(4.2)

where

$$
\gamma_\pm \in \Pi_\pm OSp(N|2).
$$

(4.3)

This can easily be shown by using the Polyakov-Wiegman formula, eq. (A.7), to bring eq. (4.1) in the form

$$
S_0 = \kappa S_0 - \kappa \int A_z - \kappa \int A_\pm.
$$

(4.4)

However, as can easily be verified, this term is not invariant under the gauge transformations eq. (4.2), which we take as our guiding principle. From the form of the non-invariance terms, one finds that invariance can be restored through the introduction of $2 \times N$ fermions, $\tau^{+a}$ and $\tau^{-a}$. The action

$$
S_1 = S_0 - \kappa \int \tau^{+a} \partial \tau^{+a} - \kappa \int \tau^{-a} \partial \tau^{-a}
$$

$$
- \kappa \int \left( A_\pm^+ + 2 \tau^{+a} A_\pm^+ \right) + \kappa \int \left( A_\pm^- - 2 \tau^{-a} A_\pm^- \right)
$$

(4.5)

is fully invariant under eq. (4.2) provided the fermions transform as

$$
\tau^{\pm a} \rightarrow \tau^{\pm a} - \eta^{\pm a},
$$

(4.6)

and we parametrized

$$
\gamma_+ \equiv \exp \left( \eta^{+a} t_+^{+a} + \eta^{-a} t_-^{-a} \right) \quad \gamma_- \equiv \exp \left( \eta^{-a} t_+^{+a} + \eta^{+a} t_-^{-a} \right)
$$

(4.7)

The equations of motion for the fields $A^+_\pm, A^{+a}_\pm, A^-_\pm, A^{-a}_\pm$ and $\tau^{\pm a}$ read:

$$
\begin{align*}
  A^+_\pm : & \quad \left( \partial g g^{-1} \right)^\pm = 1 - (g A_z g^{-1})^\pm \\
  A^{+a}_\pm : & \quad \left( \partial g g^{-1} \right)^{-a} = \tau^{+a} + (g A_z g^{-1})^{-a} \\
  A^-_\pm : & \quad \left( g^{-1} g^{-1} \right)^\pm = 1 + (g^{-1} A_z g)^\pm \\
  A^{-a}_\pm : & \quad \left( g^{-1} g^{-1} \right)^{+a} = \tau^{-a} + (g^{-1} A_z g)^{+a}.
\end{align*}
$$

(4.8)
The two main options for fixing the gauge which are usually studied are the conformal gauge and the light-cone gauge.

- **Toda (or conformal) gauge**

  We use the gauge symmetry to restrict \( g \) to \( \mathbb{R} \times SO(N) \):

  \[
g = e^{\varphi t_0} + \phi^i t_i ,
  \]

  (4.9)

  The equations of motion for \( A^\pm_z, A^+_z, A^-_z \), and \( A^{-a}_z \), eq. (4.8), become algebraic and we can perform the integration over the \( A^\pm_z \) and \( A^-_z \) fields. The action becomes:

  \[
  S_1 = -\frac{\kappa}{\pi} \int \left( \partial \varphi \partial \varphi + \tau^{+a} \partial \tau^{+a} + \tau^{-a} \partial \tau^{-a} \right) - \kappa \hat{S}_{-}[\hat{g}]
  
  + \frac{\kappa}{\pi} \int \left( e^{2\varphi} + 2e^{\varphi} \tau^{+a} \hat{g}_{ab} \tau^{-b} \right),
  \]

  (4.10)

  with

  \[
  \hat{g}_{ab} = \hat{g}_{ba}^{-1} \equiv \left[ \exp \left( \sqrt{2} \phi^j \lambda^i \right) \right]_{ab}
  \]

  (4.11)

  and where \( \hat{S}_{-}[\hat{g}] \) is an \( SO(N) \) WZW model, i.e. \( \hat{g} = \exp \phi^i t_i = \exp \sqrt{2} \phi^i \lambda_i \). The “wrong” sign in front of the \( \hat{S}_{-}[\hat{g}] \) action is due to the fact that supertraces in the \( SO(N) \) WZW action have been replaced by ordinary traces.

  This action describes the \( N \)-extended super Liouville theory or, alternatively, the \( OSp(N|2) \) Toda action. A detailed study of super Liouville theories will be presented elsewhere [21], see also [22].

- **Drinfeld-Sokolov (or light-cone) gauge**

  We put

  \[
  A^+_z = A^{-a}_z = 0 ,
  \]

  (4.12)

  thus fixing the \( \Pi_- OSp(N|2) \) gauge symmetry and fix the \( \Pi_+ OSp(N|2) \) gauge symmetry by

  \[
  \left( \partial gg^{-1} \right)^0 = \tau^{+a} = 0 .
  \]

  (4.13)

  From eq. (4.8), one sees that the Lagrange multipliers \( A^+_z \) and \( A^{+a}_z \) impose the constraints

  \[
  \left( \partial gg^{-1} \right)^0 = 1 , \quad \left( \partial gg^{-1} \right)^{-a} = \tau^{+a} .
  \]

  (4.14)
and we find that the constrained quantum current $\partial gg^{-1}$ is of the classical form given in eq. (3.13).

We now claim that the system discussed above has an $N$-extended $SO(N)$ superconformal symmetry. In other words it possesses conserved currents which satisfy the OPEs given in eq. (2.1). From now on we will essentially work in the Drinfeld-Sokolov gauge or a slight modification thereof. We first fix the $\Pi_+ OSP(N|2)$ gauge symmetry as in eq. (4.12) and modify the action to:

$$S_1 = \kappa S^{-}[g] - \frac{\kappa}{\pi} \int \tau^{+a} \partial \tau^{+a} + \frac{\kappa}{\pi} \int A_{z}^{\pm} \left( (\partial gg^{-1})^{+} - 1 \right) - \frac{2\kappa}{\pi} \int A_{z}^{+a} \left( (\partial gg^{-1})^{-a} - \tau^{+a} \right), \quad (4.15)$$

where we dropped the $-\frac{\kappa}{\pi} \int \tau^{+a} \partial \tau^{-a}$ term in the action as it plays no significant role in what follows.

The superconformal currents are those functionals which are invariant, modulo the constraints eq. (4.14), under the $\Pi_+ OSP(N|2)$ gauge transformations. They are easily found by imposing the unique $\Pi_+ OSP(N|2)$ gauge transformations which yields $(\partial gg^{-1})^{+} = \tau^{+a} = 0$. This transformation brings $(\partial gg^{-1})^{-}, (\partial gg^{-1})^{+a}, (\partial gg^{-1})^{i}$ in a form that is proportional to the $SO(N)$ superconformal currents $T$ (modulo a term proportional to the $SO(N)$ Sugawara tensor), $G^{a}$ and $U^{i}$. The explicit form for the gauge invariant polynomials is

$$\begin{align*}
\tilde{J}_{z}^{+} &= J^{+} + \frac{1}{\alpha} J^{0} J_{0} - 2 J^{+a} \tau^{+a} - \sqrt{2} \lambda_{ab} i J_{i} \tau^{+a} \tau^{+b} - \partial J^{0} - \alpha \partial \tau^{+a} \tau^{+a}, \\
\tilde{J}_{z}^{+a} &= J^{+a} - \sqrt{2} \lambda_{ab} i J_{i} \tau^{+a} \tau^{+b} - J^{0} \tau^{+a} + \alpha \partial \tau^{+a} \\
\tilde{J}_{z}^{i} &= J^{i} + \frac{\alpha}{\sqrt{2}} \lambda_{ab} i \tau^{+a} \tau^{+b}, \quad (4.16)
\end{align*}$$

and we identified $J \equiv \alpha \partial gg^{-1}$ where classically $\alpha = \kappa/2$. The polynomials given above are indeed invariant modulo the constraints:

$$\begin{align*}
\delta \tilde{J}_{z}^{+} &= -\frac{\pi}{2} \left( \partial \eta_{+}^{+} + \eta_{+}^{+} \partial - \frac{4}{\kappa} \eta_{+}^{+} J_{0} \right) \frac{\delta S_{1}}{\delta A_{\pm}^{+}} + \frac{\pi}{4} \left( \eta^{+a} \partial + \partial \eta^{+a} + 2 \eta^{+a} \tau^{+a} - 2 \eta^{+b} \tau^{+b} \tau^{+a} - \frac{4}{\kappa} \eta^{+a} J_{0} \right) \frac{\delta S_{1}}{\delta A_{\pm}^{+a}} \\
\delta \tilde{J}_{z}^{+a} &= -\frac{\pi}{2} \eta^{+a} \tau^{+a} \frac{\delta S_{1}}{\delta A_{\pm}^{+}} - \frac{\pi}{4} \left( \eta_{ab}^{+} + \eta^{+a} \tau^{+b} - \eta^{+b} \tau^{+a} - \delta_{ab}^{+} \eta^{c} \tau^{+c} \right) \frac{\delta S_{1}}{\delta A_{\pm}^{+b}} \\
\delta \tilde{J}_{z}^{i} &= -\frac{\pi}{2\sqrt{2}} \lambda_{ab} i \eta^{+a} \frac{\delta S_{1}}{\delta A_{\pm}^{+b}}, \quad (4.17)
\end{align*}$$
Having found the currents realising $SO(N)$ supergravity, our aim is now to obtain the induced and effective action through this realisation, following the method used for $W_3$ in \cite{12}. To this end we couple the currents to sources $h, \psi^a$ and $A^i$, and consider the classical action $S_2$:

$$S_2 = S_1 + \frac{1}{\pi} \int \left( h T + \psi^a G_a + A^i U_i \right)$$

with $S_1$ was defined in eq. (4.15) and $T, G^a$ and $U^i$,

$$T = C_T \left( J^+_z + \frac{2}{\kappa} \bar{J}_z^i J^i_z \right)$$

$$G^a = C_G \bar{J}_{z}^{+a}$$

$$U^i = C_U \bar{J}_i$$

where $C_T = 2, C_G = 4i$ and $C_U = -2\sqrt{2}$, satisfy the classical limit of the superconformal algebra eq. (2.1).

From eq. (4.17) we get that the couplings to the sources $h, \psi^a$ and $A^i$ in the action are gauge invariant up to terms proportional to the equations of motion of $A^+_z$ and $A^+_z$. We can cancel these noninvariance terms by adding extra terms to the transformation rules of $A^+_z$ and $A^+_z$. As $A^+_z$ and $A^+_z$ appear linearly in the action, no further modifications are needed and the action is invariant under $\Pi_+ OSp(N|2)$ gauge transformations. In this way we obtain a realization of the induced action:

$$\exp -\Gamma[h, \psi, A] = \int [\delta gg^{-1}] [d\tau][dA] (\text{Vol}(\Pi_+ OSp(N|2)))^{-1} \exp -S_2[g, \tau, A]$$

provided that at quantum level the currents eq. (1.19), up to multiplicative renormalizations and terms containing ghostfields, satisfy the quantum $SO(N)$ superconformal algebra.

The gauge fixing procedure is most easily performed using the Batalin-Vilkovisky (BV) method \cite{23}. We skip the details here, a readable account of the BV method can be found in e.g. \cite{24}. For each of the fields appearing in the theory we introduce anti-fields of opposite statistics which we denote by $J^*_z, A^*_z, c^+$, etc. The solution to the master equation is given by:

$$S_{BV} = S_2 - \int J^*_z a^* \gamma^a J^*_z + \int J^0_z \left( c^+_z J_z^0 + \gamma^a J_z^{-a} \right) - \sqrt{2} \lambda a^i \int J^*_z \gamma^a J^a_z - \int J^0_z \left( \gamma^+ J_z^0 + c^+_z J^a_z + \sqrt{2} \lambda a^i \gamma^b J^i_z - \frac{\kappa}{2} \partial \gamma^a \right)$$

$$+ \int J^+_z \left( \frac{\kappa}{2} \partial c^+ - 2 c^+_z J^0_z - 2 \gamma^a J^+_a \right) - \int \gamma^a \gamma^a + \int c^+ a^+ \gamma^a$$

15
In order to compute the normalization of the currents in eq. (4.19) and \( k \) as a function of \( \kappa \) in the quantum theory, we choose a hybrid gauge to fix the \( \Pi \text{osp}(N|2) \) gauge invariance by putting \( A_\pm^a = 0 \), which is different from the Drinfeld-Sokolov gauge! This gauge choice will allow us to compute the normalization constants and \( k(\kappa) \) using operator methods. The all-order computation of these constants, using solely functional methods, seems technically not feasible.

To fix the gauge, we make a canonical transformation (of fields and antifields) such that the functional integration over the new fields has no gauge-directions. Here this transformation is very simple, it consists in interchanging the denominations ‘field’ and ‘antifield’ for the canonical pairs \( \{ A_\pm^a, A_\mp^a \} \) and \( \{ A_\pm^{*a}, A_\mp^{*a} \} \). Introducing more conventional names for the Faddeev-Popov antighosts (which is what they turn out to be, in this gauge), we put

\[
A_\pm^{*a} = \frac{2}{\pi} \beta^{-a}, \quad A_\pm^a = \frac{\pi}{2} \beta^{a*}, \quad A_\pm^{*a} = \frac{b^{-a*}}{\pi}, \quad A_\pm^a = -\pi b^{a*} \tag{4.22}
\]

The gauge fixed action is then simply obtained from \( S_{\text{BV}} \) by putting the (new) antifields to zero. One finds:

\[
S_{\text{gf}} = \kappa S^{-}[g] - \frac{\kappa}{\pi} \int \tau^a \tilde{\partial} \tau^{*a} + \frac{1}{\pi} \int b^{-} \tilde{\partial} c^{+} - \frac{2}{\pi} \int \beta^{-a} \tilde{\partial} \gamma^{+a} + \frac{1}{\pi} \int \left( h \tilde{T} + \psi^a \tilde{G}^a + A^i \tilde{U}_i \right), \tag{4.23}
\]

where \( \tilde{T}, \tilde{G}^a \) and \( \tilde{U}_i \) have precisely the form given in eq. (4.19) but with the currents \( J^\pm, J^{+a}_z, J^0_z \) and \( J^i_z \) replaced by \( \hat{J}^\pm, \hat{J}^{+a}_z, \hat{J}^0_z \) and \( \hat{J}^i_z \), given by

\[
\hat{J}^\pm_z = J^\pm_z
\]
Note that the new extended action is linear in antifields, which makes the gauge algebra in fact simpler than in the original variables, closing now even off shell. Only a few terms that will be needed later are repeated here explicitly:

$$S_{BV} = S_{gf} - \kappa b^{-a} \left( (\partial gg^{-1}) = -1 \right) - \beta^{-a} \left( \kappa (\partial gg^{-1})^{-a} - \kappa \tau^{+a} + b^= \gamma^{+a} \right) + \cdots \quad (4.25)$$

This ends the construction of the classical extended and gauge fixed action. To preserve the gauge invariance at the quantum level, it may be necessary to add quantum corrections to this extended action. We will not make a fully regularised quantum field theory computation. To make the transition to the quantum theory, we use BRST invariance as a guide. We will use OPE-techniques without specifying a regularisation underlying this method in renormalised perturbation theory.

By construction, the action, eq. (4.23) is classically BRST invariant with the BRST charge given by

$$Q = \frac{1}{2\pi i} \oint \left\{ c^+ \left( J^z - \frac{\kappa}{2} \right) + 2\gamma^{+a} \left( J^{-a} - \frac{\kappa}{2} \tau^{+a} \right) - \frac{1}{2} b^= \gamma^{+a} \gamma^{+a} \right\} \quad (4.26)$$

Using the $OSp(N|2)$ OPEs given in eq. (A.17) and

$$\tau^{+a}(x)\tau^{+b}(y) = -\frac{1}{2\kappa} \delta^{ab}(x - y)^{-1}$$
$$b^=(x)c^+(y) = (x - y)^{-1}$$
$$\beta^{-a}(x)\gamma^{+b}(y) = \frac{1}{2} \delta^{ab}(x - y)^{-1}, \quad (4.27)$$

one finds that the BRST charge is also in the quantum theory nilpotent, provided that the currents still satisfy the OPE’s of eq. (A.17). It is known that the classical relation $J = \frac{\kappa}{2} \partial gg^{-1}$ is then renormalised to $J = \frac{\alpha_{\kappa}}{2} \partial gg^{-1}$, where the analysis of WZW models in the operator formalism [25], strongly suggests that $\alpha_{\kappa} = \kappa + \tilde{h}$. From now on we do choose the value $\alpha_{\kappa} = \kappa + \tilde{h}$. It may also be noted that the hatted currents satisfy the same algebra as the unhatted ones in eq. (A.17), with only the following modifications in the central terms:

$$\hat{j}^0(x, \hat{j}^0(y) = \frac{2\kappa + 4 - N}{16} (x - y)^{-2}$$
\[ \hat{J}^i(x) \hat{J}^j(y) = \frac{\kappa + 1}{8} \delta^{ij} (x - y)^{-2} - \frac{\sqrt{2}}{4} (x - y)^{-1} f_{ij} \hat{J}^k(y). \]  

(4.28)

Now we construct the quantum corrections to the action by using the BRST invariance. The gauge fixed action for \( h = \psi^a = A^i = 0 \) is invariant as it stands. Since \( h, \psi^a \) and \( A^i \) do not transform, we determine the quantum form of the \( T, G^a \) and \( U^i \) currents by requiring them to be BRST invariant also. This results in the following currents:

\[
\begin{align*}
\tilde{T} &= C_T \left( \hat{J}^+ + 2 \tau^a \hat{J}^+_a + \frac{2}{\kappa} \hat{j}^0 \hat{j}^0 - \frac{\kappa + 1}{\kappa} \partial \hat{j}^0 + \frac{2}{\kappa} \hat{j}^i \hat{j}^i - \kappa \partial \tau^+ \right) \\
\tilde{G}^a &= C_G \left( \hat{j}^+ - \sqrt{2} \lambda_{ab} \tau^a \hat{j}^b - \tau^a \hat{j}^0 + \frac{\kappa + 1}{2} \partial \tau^+ \right) \\
\tilde{U}^i &= C_U \left( \hat{j}^i - \frac{\kappa}{2\sqrt{2}} \lambda_{ab} \tau^a \tau^b \right),
\end{align*}
\]

(4.29)

where

\[
\begin{align*}
C_T &= \frac{4\kappa}{2\kappa + 4 - N} \\
C_G &= \sqrt{\frac{32\kappa}{N - 2\kappa - 4}} \\
C_U &= -\frac{4}{\sqrt{2}},
\end{align*}
\]

(4.30)

and the bilinears are understood as regular parts in the OPE-expansions.

These currents satisfy the superconformal algebra eq. (2.1) with

\[ k = -2\kappa - 1. \]  

(4.31)

Therefore, we showed that the reduced \( OSp(N|2) \) WZW model, eq. (4.20), yields a representation of induced \( SO(N) \) supergravity at the quantum level.

To close, we also give the quantum corrections in the extended action to some other terms that are needed in the next section. In the present gauge, these correspond to terms proportional to antifields, i.e. to transformation laws. Since we know the BRST charge explicitly, the quantum transformation laws are easy to derive in this gauge. This entails the following modifications of eq.(4.25):

\[
S_{BV}^q = S_{st}^q - \bar{b}^{+ \ast} \left( \alpha_\kappa (\partial gg^{-1})^a - \kappa \right) - \beta^{- \ast} \left( \alpha_\kappa (\partial gg^{-1})^{-a} - \kappa \tau^+ a + \bar{b}^{+ \gamma} a \right) + \cdots
\]

18
\[
S^q_{\text{gf}} + \frac{A^+_z}{\pi} \left( \alpha_\kappa (\partial gg^{-1})^a = -\kappa \right) - \frac{2A^+_z}{\pi} \left( \alpha_\kappa (\partial gg^{-1})^{-a} - \kappa \tau^+ + b=\gamma^+ \right) + \cdots
\]

(4.32)

where \( S^q_{\text{gf}} \) refers to the gauge fixed action, with the modified currents of eq.(4.29), and we reverted to the original names of the ghost-variables.

From the BV viewpoint, the modification of the currents, eq.(4.29) and the transformation laws, eq.(4.32) amount to (part of) the computation of the quantum corrections \((M_i)\) to the classical extended action. Without computing \(\Delta S\) and checking the quantum BV master equation, which would require some regularisation procedure, we feel confident that \(Q^2 = 0\) guarantees the gauge invariance of the quantum theory. Therefore we will use eq.(4.32) as it stands also for a different gauge.

5 The Effective Action

5.1 All Order Results

The expression for the induced action which we obtained in previous section, is perfectly suited for an all order computation of the effective action. The effective action was defined in eq. (2.17) where \(\Gamma[h, \psi, A]\) is given in eq. (4.20). In order to compute it, we fix the \(\Pi_+ OSp(N|2)\) gauge invariance by choosing the Drinfeld-Sokolov gauge, \(\psi^0 = \psi^+ = 0\). In the BV formalism, this is achieved by turning them into antifields. So we again make a simple canonical transformation interchanging fields and antifields, this time on the canonical pairs \{(\tau^+ a, \tau^+ a^*)\}, i.e. we now put

\[
\begin{align*}
J^{0*} &= \frac{b=}{\pi} \\
J^0 &= -\pi b^{*} \\
\tau^+ a^* &= -\frac{2}{\pi} \beta^{-a} \\
\tau^+ a &= \frac{\pi}{2} \beta^{-a^*}
\end{align*}
\]

(5.1)

Note that some of the quantum corrections to the transformation laws, eq.(4.32), e.g. the fact that \(J_z = (\kappa + \bar{h})/2\partial gg^{-1}\) and the normalization of the leading terms of the superconformal currents eq. (4.30), now show up in the gauge fixed action itself.
Combining eqs. (2.17), (4.20) and (4.21), the action becomes:

\[
\exp -W[t, g, u] = \int [d(\delta gg^{-1})][d\tau][dA_z][db][dc][d\beta][d\gamma][dh][d\psi][dA] \delta (\tau^+)^a \delta (J_z^0)
\]

\[
\exp \left( -\kappa S^-[g] + \frac{\kappa}{\pi} \tau^+ a \tilde{\tau}^+ a - \frac{2}{\pi} \int A_z^\pm \left( J_z^\pm - \frac{\kappa}{2} \right) + \frac{4}{\pi} \int A_z^+ \left( J_z^- - \frac{\kappa}{2} \tau^+ a \right) \right)
\]

\[-\frac{1}{\pi} \int b^\pm \left( c^+ J_z^\pm + \gamma^+ a J_z^- a \right) - \frac{2}{\pi} \int \beta^- a \gamma^+ a
\]

\[-\frac{1}{\pi} \left( \int h \left( T - \frac{t}{4} \right) + \psi^a \left( G_a - g_a \right) + A^i \left( U_i + \frac{1}{2} u_i \right) \right)\]  

(5.2)

Passing from the Haar measure \([\delta gg^{-1}]\) to the measure \([dJ_z]\), see eq. (B.14), we pick up a Jacobian:

\[
[\delta gg^{-1}] = [dJ_z] \exp \left( (N - 4) S^-[g] \right).
\]

(5.3)

Performing the integration over the \(A_z, \tau, h, \psi\) and \(A\), we get:

\[
\exp -W[t, g, u] = \int [dJ_z][db][dc][d\beta][d\gamma] \delta \left( J_z^\pm - \frac{\kappa}{2} \right) \delta \left( J_z^- a \right) \delta \left( J_z^0 \right)
\]

\[
\delta \left( C_T J_z^\pm + \frac{2C_T}{\kappa} J_z^i J_z^i - \frac{t}{4} \right) \delta \left( C_G J_z^+ a - g^a \right) \delta \left( C_U J_z^i + \frac{u^i}{2} \right) \exp \left( -\kappa_c S^-[g] \right)
\]

\[-\frac{1}{\pi} \int b^\pm \left( c^+ J_z^\pm + \gamma^+ a J_z^- a \right) - \frac{2}{\pi} \int \beta^- a \gamma^+ a\]  

(5.4)

where

\[
\kappa_c = \kappa + 4 - N.
\]

(5.5)

Performing the integrals over \(J_z\) and then over the ghosts in eq. (5.4), we observe that the ghost contributions amount to an overall factor, which we drop. Then we get that \(W[t, g, u]\) is given by

\[
W[t, g, u] = \kappa_c S^-[g],
\]

(5.6)

where the WZW functional is constrained by:

\[
(\partial gg^{-1})^a = \frac{\kappa}{\alpha_\kappa},
\]

\[
(\partial gg^{-1})^{-a} = (\partial gg^{-1})^0 = 0
\]

20
\[(\partial gg^{-1})^\pm + \frac{\alpha_k}{\kappa} (\partial gg^{-1})^i (\partial gg^{-1})^i = \frac{t}{2\alpha_k C_T}\]

\[(\partial gg^{-1})^{-a} = \frac{2}{\alpha_k C_G} g^a\]

\[(\partial gg^{-1})^i = -\frac{1}{\alpha_k C_U} u^i\]  \hspace{1cm} (5.7)

Comparing with the results of section 3, we find that we rather need the constraint \((\partial gg^{-1})^\pm = 1\) instead of \((\partial gg^{-1})^\pm = \kappa/\alpha_k\). The conversion is easily performed through a global group transformation:

\[g \rightarrow e^{\ln \left( \sqrt{\frac{\alpha_k}{\kappa}} \right) t_0} g.\]  \hspace{1cm} (5.8)

Combining this with eqs. (2.19) and (3.8), we get

\[W[t, g, u] = -2\kappa_c W^{(0)} \left( Z^{(t)} t, Z^{(g)} g, Z^{(u)} u \right),\]  \hspace{1cm} (5.9)

where

\[\kappa_c = -\frac{1}{2} (k - 7 + 2N),\]  \hspace{1cm} (5.10)

and

\[Z^{(t)} = -\frac{\kappa}{C_T \alpha_k^2}\]

\[Z^{(g)} = \frac{2i}{C_G \alpha_k} \sqrt{\frac{\kappa}{\alpha_k}}\]

\[Z^{(u)} = \frac{\sqrt{2}}{C_U \alpha_k}\]  \hspace{1cm} (5.11)

\(W^{(0)}\) was defined in eqs. (2.18) and (3.8) and the \(C\) coefficients were given in eq. (4.30). These results are fully consistent with the large \(k\) results found in section three.

As discussed, the normalization of the currents, eq. (4.30), was computed in the previous section using operator methods. Choosing for \(\alpha_k\) the value which is found in the same formalism: \(\alpha_k = \kappa + \frac{1}{2} (4 - N)\), eq.(5.11) further simplifies to

\[Z^{(t)} = Z^{(g)} = Z^{(u)} = \frac{1}{k + N - 3}.\]  \hspace{1cm} (5.12)
Using eqs. (2.2) and (5.10), we find that the level $\kappa_c$ of the $OSp(N|2)$ affine Lie algebra as a function of the central extension $c$ is given by:

\[
12\kappa_c = -c + 37 + \frac{1}{2}N(N - 24)
- \left( (c - 13 - \frac{1}{2}N(N - 12))^2 + 6(N - 2)(N - 3)(N - 4) \right)^{1/2}.
\]

(5.13)

For $N = 0$, one gets real values for $\kappa_c$ if $0 \leq c \leq 1$ or $c \geq 25$; for $N = 1$ the allowed range is $0 \leq c \leq 3/2$ or $c \geq 27/2$ and for $N \geq 2$, there is no restriction on the range of $c$.

Furthermore, eq. (5.13) implies that while $N = 0$ and 1 receive contributions to the renormalization of $\kappa_c$ at all loop orders, $N = 2$, 3 and 4 receive only one-loop contributions to the renormalization. For $N = 4$ an even stronger statement is possible. The $N = 4$ superconformal algebra can be linearized by adding one $U(1)$ field and four free fermions to the system. Then $c$ changes to $c_{\text{lin}} = c + 3$ and one has $c_{\text{lin}} = -6\kappa$. We conclude that in this case no renormalization at all occurs and the quantum theory is equal to the classical one. These results are a reflection of the non-renormalization theorems for theories with extended supersymmetries.

However, we want to stress here that while the value of the coupling constant renormalization is unambiguously determined, the computation of the value of the wavefunction renormalization is very delicate. If the gauged WZW model serves as a guideline [26], we expect that the precise value of the wavefunction renormalization depends on the chosen regularization scheme. As mentioned before the computations leading to the quantum effective action were performed in the operator formalism using point-splitting regularization. Within this framework, we believe that eq. (5.12) is fully consistent. This claim is further supported, as we will show next, by perturbative computations which also rely on operator methods and which give results which are fully consistent with both eqs. (5.13) and (5.12).

We want to remark here that computing the $Z$ factors using functional methods in a certain regularization scheme, probably does not simply amount to eq. (5.11) with an appropriate choice for $\alpha_{\kappa}$, as claimed in [12]. This, because the value of $C$, eq. (4.30), might also very well depend on the choice of regularization.

### 5.2 Semiclassical Evaluation

In the previous section we computed the renormalisation factors for the nonlinear $O(N)$ superconformal algebras by realising them as WZW models. For $N = 3$ and 4, these algebras can be obtained from linear ones by eliminating the dimension $\frac{1}{2}$ fields and for $N = 4$ an additional $U(1)$ factor. Also it has been shown [27] that the effective actions $W$ of the linear theories can be
obtained from the linear effective actions simply by putting to zero the spin $\frac{1}{2}$ currents. In this section we compute these effective actions for the linear theories in the semiclassical approximation. A comparison with the results for the nonlinear algebras obtained so far, eqs. (5.12,5.13) shows complete agreement through the linear-nonlinear connection established in [27].

Let us first explain the method [28, 56]. In the semiclassical approximation, the effective action is computed by a steepest descent method:

$$e^{-W[u]} = \int [dA] e^{-\Gamma[A] - \frac{1}{\pi} \int u A} \approx e^{-\Gamma[A_{cl}] - \frac{1}{\pi} u_{A_{cl}} \int [dA] \exp -\frac{1}{2} \tilde{A} \frac{\delta^2 \Gamma[A_{cl}]}{\delta A_{cl} \delta A_{cl}} \delta A_{cl}},$$

(5.14)

where $A_{cl}[u]$ is the saddle point value that solves

$$- \frac{\delta \Gamma[A]}{\delta A} = \frac{1}{\pi} u,$$

(5.15)

and $\tilde{A}$ is the fluctuation around this point. Therefore, all that has to be done is to compute a determinant:

$$W[u] \approx W_{cl}[u] + \frac{1}{2} \log \det \frac{\delta^2 \Gamma[A_{cl}]}{\delta A_{cl} \delta A_{cl}}.$$

(5.16)

To evaluate this determinant, one may use the Ward identities: schematically, they have the form

$$\overline{D}_1[A] \frac{\delta \Gamma}{\delta A} \sim \partial_2 A$$

where on the l.h.s. there is a covariant differential operator, and on the r.h.s. the term resulting from the anomaly, the symbol $\partial_2$ standing for a differential operator of possibly higher order (see for example eq. (2.13) without the non-linear term). Taking the derivative with respect to $A$, and transferring some terms to the r.h.s. one obtains

$$\overline{D}_1[A] \frac{\delta^2 \Gamma}{\delta A \delta A} \sim D_2[u],$$

(5.17)

where now there appears a covariant operator on the r.h.s. also, with $u$ and $A$ again related by eq. (5.15). The sought-after determinant is then formally the quotient of the determinants of the two covariant operators in eq. (5.17).

For the induced and effective actions of fields coupled to affine currents, the covariant operators are both simply covariant derivatives, and their determinants are known: both induce a Wess-Zumino-Witten model action. For the 2-D gravity action $A \rightarrow h$ and $u \rightarrow t$, the operator on the r.h.s. is $\partial^3 + t \partial + \partial t$ and the determinant is in [28]. The similar computation for the semiclassical approximation to $W_3$ is in [11]. From these cases, one may infer the general
structure of these determinants. In the gauge where the fields \( A \) are fixed, the operator \( \overline{D}_1[A] \) corresponds to the ghost Lagrangian: in BV language, the relevant piece of the extended action is \( A^* \overline{D}_1[A] c + b^* \lambda \) and as in section four the \( A^* \)-field is identified with the Faddeev-Popov antighost. The determinant is then given by the induced action resulting from the ghost currents. Since these form the same algebra as the original currents, with a value of the central extension that can be computed, one has

\[
\log \det \overline{D}_1[A] = k_{\text{ghost}} \Gamma^{(0)}[A]. \tag{5.18}
\]

The second determinant can similarly be expressed as a functional integral over some auxiliary \( bc \) and/or \( \beta \gamma \) system. Let us, to be concrete, take \( D_2[h] = \frac{1}{4} (\partial^3 + t \partial + \partial t) \) as an illustration. Then we have

\[
(\det D_2[t])^{1/2} = \int [d\sigma] \exp \frac{1}{8\pi} \sigma (\partial^3 \sigma + t \partial \sigma + \partial (t \sigma)) \tag{5.19}
\]

where it is sufficient to use a single fermionic integral since \( D_2 \) is antisymmetric. We can rewrite this as

\[
(\det D_2[t])^{1/2} = \exp^{-\bar{W}[t]} = \langle \exp \frac{1}{\pi} \int t H \rangle_{\sigma}
\]

where \( H = \frac{1}{4} \sigma \partial \sigma \).

The propagator of the \( \sigma \)-field fluctuations is given by

\[
\sigma(x)\sigma(y) = 2 \frac{(x - y)^2}{x - y} + \text{regular terms} \tag{5.20}
\]

and the induced action has been called \( \bar{W} \) instead of \( \Gamma \) for reasons that will be clear soon. The Lagrangian eq. (5.19) has an invariance:

\[
\begin{align*}
\delta t &= \partial^3 \omega + 2 (\partial \omega) t + \omega \partial t \\
\delta \sigma &= \omega \partial \sigma - (\partial \omega) \sigma
\end{align*}
\]

and, correspondingly, \( \bar{W}[t] \) obeys a Ward identity. This is anomalous. We find:

\[
\partial^3 \frac{\delta \bar{W}[t]}{\delta t} + 2t \partial \frac{\delta \bar{W}(t)}{\delta t} + (\partial t) \frac{\partial \bar{W}(t)}{\partial t} = -\frac{1}{\pi} \partial t. \tag{5.21}
\]

This is nothing but the usual chiral gauge conformal Ward identity ‘read backwards’, i.e. \( t \leftrightarrow \frac{\delta \Gamma[h]}{\delta h} \) and \( h \leftrightarrow \frac{\partial \bar{W}(t)}{\partial t} \). We conclude that \( \bar{W}(t) \) is proportional to the Legendre transform of \( \Gamma^{(0)} \),

\[
\bar{W}(t) = -6k' W^{(0)}(t) \tag{5.22}
\]
with \( k' = 2 \). Another way to obtain this Ward identity eq. \((5.21)\) is by evaluating the operator product of the \( H \)-operator. One finds

\[
H(z)H(0) = -\frac{k'}{4} \left( \frac{z}{\bar{z}} \right)^2 - \frac{z}{\bar{z}} H(0) - \frac{z^2}{2\bar{z}} \partial H(0) + \cdots \tag{5.23}
\]

as in \([28]\), from which eq.\((5.22)\) also follows.

Note that in \([28]\) different bosonic realisations of the algebra \((5.23)\) were used. Starting from the action \( \frac{1}{2} \phi (\partial^2 + \frac{1}{2} t^2) \phi \) one finds that \( H(\phi) = \frac{1}{4} \phi^2 \) satisfies \((5.23)\) with \( k' = -1/2 \). This realisation will appear naturally when we discuss \( N = 1 \). Another one starting from \( \phi_1 [\partial^3 + t\partial + \partial t] \phi_2 \) has \( k' = -4 \) and is a bosonic twin of the one we used. The same algebra also realises a connection with \( sl_2 \), through \([28]\)

\[
H(z) = -\frac{z^2}{2j + (\bar{z}) + zj_0 (\bar{z}) + \frac{1}{2} j} - \frac{\bar{z}}{2j}(\bar{z})
\]

The antiholomorphic components \( j^a \) of \( H(z, \bar{z}) \) generate an affine \( sl_2 \) algebra.

The upshot is that whereas the first determinant is proportional to the classical induced action \( \Gamma \), the second one is proportional to the classical effective action \( W \). The proportionality constants are pure numbers independent of the central extension of the original action. From these numbers the renormalisation factors for the quantum effective action in the semiclassical approximation follow:

\[
W[u] \simeq W_d^{(0)} \left[ \frac{u}{k} \right] - 6k'W^{(0)} \left[ \frac{u}{k} \right] - \frac{k_{\text{ghost}}}{2} \Gamma^{(0)}[A]
\]

\[
\simeq \left( k - 6k' - \frac{k_{\text{ghost}}}{2} \right) W_d^{(0)} \left[ \frac{u}{k} \right] + \frac{k_{\text{ghost}}}{2} u \frac{\partial W^{(0)}}{\partial u}\left[ \frac{u}{k} \right]
\]

\[
\simeq \left( k - 6k' - \frac{k_{\text{ghost}}}{2} \right) W_d^{(0)} \left[ \frac{u}{k} (1 + \frac{k_{\text{ghost}}}{2k}) \right].
\]

In the example of affine KM-currents, the relevant numbers are \( 6k' = -\tilde{h} \) and \( k_{\text{ghost}} = -2\tilde{h} \), with the familiar result *. For \( W_2 \)-gravity, the results of \([28]\) follow. We now turn to \( N = 1 \cdot \cdots 4 \) linear supergravities.

\[ N = 1 \]

This case has been treated also in \([28, 29]\). The induced action is

\[
e^{-\Gamma[h, \psi]} = \langle e^{-\frac{1}{8}(hT + \psi G)} \rangle
\]

where \( T \) and \( G \) generate the \( N = 1 \) superconformal algebra with central charge \( c \). The Ward identities read\[\] with \( \Gamma = c\Gamma^{(0)}\),

\[
\left[ \bar{T} - h\partial - 2(\partial h) \right] \frac{\delta \Gamma^{(0)}}{\delta h} - \frac{1}{2} [\psi \partial - 3(\partial \psi)] \frac{\delta \Gamma^{(0)}}{\delta \psi} = \frac{1}{12\pi} \partial^3 h
\]

* Different calculations give different answers for the field renormalisation factor. See the discussion at the end of the previous subsection, appendix B, and \([26]\)

† All derivatives are left derivatives.
\[
[\bar{\partial} - h\partial - \frac{3}{2}(\partial h)]\frac{\delta \Gamma^{(0)}}{\delta \psi} - \frac{1}{2}\psi \frac{\delta \Gamma^{(0)}}{\delta h} = \frac{1}{3\pi} \partial^2 \psi.
\]

(5.24)

From this we read off $\bar{D}_1$ and $D_2$ of eq. (5.17):

\[
\bar{D}_1 = \begin{pmatrix}
\bar{\partial} - h\partial - 2(\partial h) & -\frac{1}{2}\psi \bar{\partial} + \frac{3}{2}(\partial \psi)
\end{pmatrix}
\]

\[
D_2 = \frac{1}{3\pi} \begin{pmatrix}
\frac{1}{4}(\partial^3 + (\partial t) + 2\hat{t}\partial) & \frac{3}{2}\hat{g}\partial - \frac{3}{2}\hat{g}\partial
\end{pmatrix}
\]

We abbreviated $\hat{t} = -12\pi\frac{\partial \Gamma^{(0)}[\hat{t}]}{\partial \hat{t}} = t/c$ and $\hat{g} = g/c$. For ease of notation, we will drop the hats in the computation of $\det D_2$.

$\bar{D}_1$ gives rise to the ghost-realisation for $N = 1$, so we have

\[
s \det \bar{D}_1 = 15\Gamma^{(0)}(h, \psi).
\]

$D_2$ is a (super)antisymmetric operator, as can be seen by rewriting

\[
\partial^3 + (\partial t) + 2t\partial = \partial^3 + \partial t + t\partial,
\]

\[
\frac{1}{2}(\partial g) + \frac{3}{2}g\partial = \frac{1}{2}\partial g + g\partial,
\]

\[
(\partial g) + \frac{3}{2}g\partial = \frac{1}{2}\partial g + g\partial.
\]

The relevant action is then

\[
\frac{1}{\pi} \int \left(\frac{1}{8} \sigma (\partial + \partial t + t\partial)\sigma - \frac{1}{2} \sigma (\partial g)\varphi - \frac{3}{2} \sigma g\partial \varphi + \frac{1}{2} \varphi \left(\partial^2 + \frac{t}{2}\right) \varphi \right).
\]

The determinant is

\[
(s \det D_2)^{1/2} = \langle e^{-\frac{1}{4}(tH + g\Psi)} \rangle = e^{-\bar{W}(t,g)}
\]

(5.25)

where $H = \frac{1}{2}\sigma\partial\sigma + \frac{1}{4}\varphi^2$ and $\Psi = -\frac{1}{2}(\partial\sigma)\varphi + \sigma\partial\varphi$ and the average is taken in a free field sense with propagators eq.(5.20) and

\[
\langle \varphi(x)\varphi(y) \rangle = \frac{x - y}{|x - y|^2}.
\]

(5.26)

This leads to the operator product expansions

\[
H(z)H(0) = -\frac{k'}{4}\frac{z^2}{\bar{z}^2} + \frac{z}{\bar{z}}H(0) + \frac{1}{2}\frac{z^2}{\bar{z}}\partial H(0) + \cdots
\]

\[
H(z)\Psi(0) = \frac{1}{2}\frac{z}{\bar{z}}\Psi(0) + \frac{1}{2}\frac{z^2}{\bar{z}}\partial \psi(0) + \cdots
\]

\[
\Psi(z)\Psi(0) = 2k'\frac{z}{\bar{z}^2} - \frac{4}{z}H(0) - \frac{2z}{\bar{z}}\partial H(0) + \cdots
\]

(5.27)
with a value for the central extension \( k' = 2 - \frac{1}{2} \).

The resulting Ward identities for \( \tilde{W} \), eq. (5.25), are:

\[
(\partial^3 + (\partial t) + 2i\partial) \frac{\partial \tilde{W}}{\partial t} - (2(\partial g) + 6g\partial) \frac{\partial \tilde{W}}{\partial g} = -\frac{k}{2\pi} \partial_t
\]

\[
(\partial^2 + \frac{t}{2}) \frac{\partial \tilde{W}}{\partial g} + (\partial g) + \frac{3}{2}g\partial \frac{\partial \tilde{W}}{\partial t} = -\frac{2k}{\pi} \partial g.
\]

Comparing with eq. (5.24), and reverting to the proper normalisation of \( t \) and \( g \), we have

\[
\tilde{W}(\hat{t}, \hat{g}) = 6k' W^{(0)}[\hat{t}, \hat{g}] = 6k' W^{(0)}[t/c, g/c]
\]

where we used

\[
W^{(0)}(\hat{t}, \hat{g}) = \min_{(h, \psi)} \left( \Gamma^{(0)}(h, \psi) - \frac{1}{12\pi} \int h \hat{t} - \frac{1}{3\pi} \int \hat{g} \psi \right).
\]

Putting everything together in eq. (5.16) we find, for \( N = 1 \), \( k' = 3/2 \),

\[
W[t, g] \simeq cW_d^{(0)} \left[ \frac{t}{c}, \frac{g}{c} \right] - \frac{15}{2} \Gamma^{(0)}[h, \psi] - 9W^{(0)} \left[ \frac{t}{c}, \frac{g}{c} \right].
\]

Writing these results as

\[
W^{(N)}[\Phi] \simeq Z^{(N)}_W W^{(0)}[Z^{(N)}_\Phi \Phi]
\]

we have

\[
Z^{(1)}_W \simeq c - \frac{33}{2}
\]

\[
Z^{(1)}_t = Z^{(1)}_g \simeq \frac{1}{c}(1 + \frac{15}{2c}).
\]

For reference, the corresponding equations for \( N = 0 \) are \( k' = 2 \),

\[
W[t] = cW_d^{(0)}[t/c] - \frac{26}{2} \Gamma^{(0)}[h] - 12W^{(0)}[t/c]
\]

\[
Z^{(0)}_W = c - 25
\]

\[
Z^{(0)}_t = \frac{1}{c}(1 + \frac{13}{c}).
\]

These values are in complete agreement with [28, 2, 14, 29] and with our section 5.

Before going to \( N = 2 \), we comment on the technique we used to obtain eq. (5.23) and (5.27). The easiest way is expand the fields \( \sigma, \varphi \) in solutions of the free field equations:

\[
\sigma(z, \bar{z}) = \sigma^{(0)}(\bar{z}) + z\sigma^{(1)}(\bar{z}) + \frac{z^2}{2} \sigma^{(2)}(\bar{z})
\]

\[
\Psi(z, \bar{z}) = \Psi^{(0)}(\bar{z}) + z\Psi^{(1)}(\bar{z})
\]

27
and read off the OPE’s for the antiholomorphic coefficients from eqs. (5.26). Then all singular terms are given in eq. (5.27). An alternative would be, to use Wick’s method, with the contractions given by the propagators. The resulting bilocals then give, upon Taylor-expanding the same algebra as in eq. (5.27), up to terms proportional to equations of motion. This ambiguity was already present in [28], see also [11]. We have simply used the antiholomorphic mode expansion in the following calculation. A disadvantage is, that in this way one loses control over equation of motion terms.

Let us close the \( N = 1 \) case by noting that, just as the antiholomorphic modes corresponding to eq. (5.23) generate on \( \mathfrak{sl}(2) \) affine algebra, we get an affine \( \mathfrak{osp}(1|2) \) from the modes of \( H \) and \( \Psi \) of eq. (5.27).

For \( N = 2 \) the extension of the scheme above has two \( \Psi \)-fields and a free fermion \( \tau \), with \( \langle \tau(x)\tau(0) \rangle = -\frac{1}{x} \). This last field does not contribute to \( H \):

\[
H = \frac{1}{4} \sigma \partial \sigma + \frac{1}{4} \sum_{a=1}^{2} \phi_a^2,
\]

\[
\Psi_a = -\frac{1}{2} (\partial \sigma) \phi_a + \sigma \partial \phi_a - \epsilon_{ab} \phi_b \tau,
\]

\[
A = \epsilon_{ab} \partial \phi_a \phi_b + \sigma \partial \tau.
\]

Note that \( \partial A \) is proportional to the equations of motion for \( \phi_a \) and \( \tau \). Neglecting terms proportional to equations of motion, we find that in the algebra of eq. (5.27) the first two equations are supplemented with

\[
H(z)A(0) = 0 + \cdots
\]

\[
\Psi_a(z)\Psi_b(0) = \delta_{ab} \left( \frac{2k' z}{z^2} - \frac{4H(0)}{z} - \frac{2z}{z} \partial H(0) \right) + \epsilon_{ab} \frac{z}{z} A(0) + \cdots
\]

\[
A(z)\Psi_a(0) = \frac{\epsilon_{ab}}{z} \Psi_b(0) + \cdots
\]

\[
A(z)A(0) = k' \frac{z}{z^2} + \cdots, \quad (5.28)
\]

and \( k' = 2 - 2 \cdot \frac{1}{2} = 1 \). With the central charge of the ghosts being \( k_{\text{ghost}} = +6 \), we can immediately write down the \( Z \)-factors for \( N = 2 \):

\[
Z_{\mathcal{W}}^{(2)}(2) = c - 9
\]

\[
Z_{\tau}^{(2)} = Z_{\Psi}^{(2)} = Z_{a} \simeq \frac{1}{c}(1 + 3/c).
\]

The algebra of antiholomorphic coefficients of \( H, \Psi_a \) and \( U \) is now \( \mathfrak{osp}(2|2) \).
It should be remarked that for $N = 2$ (and higher) the algebra eq. (5.28) does not quite reproduce the Ward identities for the induced action. Here also, the equations of motion are involved. The difference is in the identity.

$$\frac{1}{4}(\partial^3 + 2t\partial + (\partial t))h - \frac{1}{2}((\partial g_a) + 3g_a\partial)\psi^a - (\partial A) \cdot u = \overline{\partial t}.$$ 

The last term, as noted, is proportional to equations of motion of the free part of the action of the auxiliary system, and is not be recovered from the procedure outlined above. We surmise that, as for $N = 0$ and 1, these terms do not change the result.

$\boxed{N = 3, 4}$

The determinant leads to consider the Lagrangian

$$L = \frac{1}{8}\sigma \partial^3 \sigma + \frac{t}{4}\sigma \partial \sigma - \frac{1}{2}\sigma \partial g \cdot \varphi - \frac{3}{2}\sigma g \cdot \varphi + \sigma u \cdot \tau - \frac{\sigma}{2}(g\bar{\varphi} - \partial q\bar{q})$$

$$\frac{1}{2}\varphi \cdot \partial^2 \varphi + \frac{t}{4}\varphi \cdot \varphi - u \cdot \varphi \wedge \partial \varphi + \varphi \cdot g \wedge \tau - q\varphi \cdot \partial \tau + u \cdot \varphi \bar{q}$$

$$- \frac{1}{2}\tau \cdot \partial \tau - \frac{1}{2}(1 + \tau)$$

$$= \frac{1}{8}\sigma \partial^3 \sigma + \frac{1}{2}\varphi \cdot \partial^2 \varphi - \frac{1}{2}\tau \partial \tau - \frac{1}{2}\bar{q}^2 + tH + g \cdot \Psi + u \cdot A + q \cdot \Theta$$

where $\sigma, t, q$ and $\bar{q}$ are $O(3)$ scalars and $g, \varphi, u, \tau$ are $O(3)$ vectors. It has the following invariances, with scalar parameters $\omega$ and $\beta$ and vectors $\theta$ and $\alpha$:

- $\delta t = \partial^3 \omega + \omega \partial t + 2(\partial \omega) t$

  $\delta \chi = \omega \partial \chi + j(\partial \omega) \chi$

  with $j = \frac{3}{2}, 1, \frac{1}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}$ for $\chi = g^a, u^a, q, \sigma, \varphi^a, \tau^a, \bar{q}$

- $\delta t = -2(3\partial \theta \cdot g + \theta \cdot \partial g)$

  $\delta g = \partial^2 \theta - \theta \wedge \partial u - 2\partial \theta \wedge u + \frac{t}{2}\theta$

  $\delta \varphi = -\theta \wedge \tau - \frac{1}{2}\theta \partial \sigma + \partial \theta \sigma$

  $\delta u = -\theta \wedge g + \partial q \theta + q \partial \theta$

  $\delta \tau = -\varphi \wedge \partial \theta + \partial \varphi \wedge \theta + \theta \bar{q}$

  $\delta q = \theta u$

- $\delta t = -4u \cdot \partial \alpha$

  $\delta \sigma = 0$

  $\delta \varphi = \varphi \wedge \alpha$

  $\delta \tau = \sigma \partial \alpha + \tau \wedge \alpha$

  $\delta q = 0$
\[\delta t = -2q\theta \beta + 2(\partial q)\beta\]
\[\delta g = -u\beta\]
\[\delta u = 0\]
\[\delta q = \beta\]
\[\delta \sigma = 0\]
\[\delta \varphi = 0\]
\[\delta \tau = -\varphi\beta\]
\[\delta \overline{q} = +\frac{1}{2}\partial \sigma \beta + \sigma \partial \beta\]

Note that \(\overline{q}\) is an auxiliary field. Again working up to equations of motion as before, one finds that \(H, \Psi, A\) and \(\Theta\) form a closed operator algebra that extends eq. (5.28) straightforwardly. The resulting anomalous Ward identities again afford the conclusion that the functional determinant is proportional to the appropriate effective action. For \(N = 4\), we refrain from writing out the action and transformation laws, but the same results are valid. The algebra of eq. (5.28) only changes in that more \(\Psi_a\) and \(A\) fields are present. The value of the central charge \(k^{(N)}\) in that algebra can most simply be obtained from \(H(z)H(0)\), since only \(\sigma\) and \(\varphi_a\) fields contribute to it:

\[k^{(N)} = 2 - \frac{N}{2}\]

The resulting antiholomorphic coefficients constitute an affinisation of the \(osp(N|2)\) superalgebra: the spin \(\frac{1}{2}\) field \(\Theta\) contributes no antiholomorphic modes.

The ghost system central charges vanish for \(N = 3, 4\). As a result, for \(N = 3\),

\[Z_W^{(3)} = c - 3,\]
\[Z_t^{(3)} \simeq \frac{1}{c},\]

and for \(N = 4\), all \(Z\)-factors are equal to their classical values.

Now we compare these results for \(Z_W\) with the results of eqs. (5.11, 5.13) for the nonlinear algebras, using the result of [27]. According to [27], the respective effective actions are equal upon putting the appropriate currents to zero. Recall that the linear algebras reduce to the nonlinear ones when eliminating [20] one spin \(1/2\) field for \(N = 3\), and four spin \(1/2\) fields and one spin \(1\) field for \(N = 4\). In the process, the central charge is modified:

\[\hat{c}^{(3)}_{\text{nonlinear}} = c^{(3)}_{\text{linear}} - 1/2\]
\[\hat{c}^{(4)}_{\text{nonlinear}} = c^{(4)}_{\text{linear}} - 3.\]

With these substitutions, the agreement with eq. (5.13) is complete, both for the overall renormalisation factor and for the field renormalisations.

For \(N = 3\) a similar computation was made [18] directly on the theory based on the nonlinear algebra, using Feynman diagrams to compute the determinants. In that case the classical approximation is not linear in \(c\), but can be written as a power series. The determinant replacing
Our $\det D_1$ is not directly proportional to the induced action $\Gamma^{(0)}$ as in eq. (5.18). In fact, this part vanishes since $k_{\text{ghost}} = 0$ for $N = 3$. Instead, it contains extra terms. These terms are computed in [18]. They cancel the non-leading terms of the classical induced action, at least to the extent they are relevant here (next-to-leading order). A similar cancellation was also observed in the computation of the $W_3$ effective action [14]. The non-leading contribution and its cancellation with some of the loop contributions seems to have been overlooked in [18]. We have recomputed the renormalisation factors for the nonlinear algebra with the method of [18], taking into account the non-leading terms also. We again find agreement with the results obtained above. Note in particular that all field renormalisation factors are equal. This alternative computation of the determinants, using Feynmann diagrams, implicitly confirms our treatment of equation of motion terms in the Ward identities.

6 The General Case

Given a (super) affine Lie algebra $\hat{g}$ of level $\kappa$, we call the finite dimensional subalgebra $\bar{g}$. To every inequivalent, nontrivial embedding of $sl(2, \mathbb{R})$ in $\bar{g}$ there corresponds a certain extension of $d = 2$ gravity [36]. In this section we will develop the general procedure to construct the effective theory. Details about $sl(2, \mathbb{R})$ embeddings can be found in appendix C.

A nontrivial embedding of $sl(2, \mathbb{R})$ in $\bar{g}$ is given. In [39], it was shown that by constraining $\partial gg^{-1}$ as

$$\partial gg^{-1} = e_- + u_z,$$  

(6.1)

where

$$u_z \in \text{Ker ad} e_+,$$  

(6.2)

the flat connection condition reduces to the large $\kappa$ limit of the Ward identities of an extension of the Virasoro algebra. For every spin $j$ $sl(2, \mathbb{R})$ multiplet one has a current with conformal dimension given by $j + 1$. We will now implement these constraints through a gauged WZW model.

One easily checks that

$$\dim \text{Ker ad} e_+ = \left( \dim \bar{g} + \dim \Pi_{+1/2} \bar{g} \right) - 2 \left( \dim \Pi_{+1} \bar{g} + \dim \Pi_{+1/2} \bar{g} \right),$$  

(6.3)

*This is merely rephrasing the fact that to every inequivalent, nontrivial embedding of $sl(2, \mathbb{R})$ in $\bar{g}$, one can associate an extension of the Virasoro algebra.
which, in the case that \( \Pi_{+1/2}\bar{g} \neq \emptyset \), strongly suggests the need to introduce \( \dim \Pi_{+1/2}\bar{g} \) auxiliary fields \( \tau \in \Pi_{-1/2}\bar{g} \). When \( \bar{g} \) is a purely bosonic algebra, it was shown in e.g. [31, 32] that there is no need for the auxiliary fields. This is due to the fact that in that case \( \dim \Pi_{+1/2}\bar{g} \) is always even. The branching always contains a \( U(1) \) generator under which the multiplets with \( j \) halfinteger can be split into two subspaces having opposite eigenvalues under the action of the \( U(1) \) symmetry. Instead of taking \( \Pi_{+}\bar{g} \) as the gauge group, one splits \( \Pi_{+1/2}\bar{g} \) according to the \( U(1) \) chirality as \( \Pi^B_{+1/2}\bar{g} = \Pi^B_{+1/2}\bar{g} + \Pi^B_{-1/2}\bar{g} \) and one takes \( \Pi^B_{\geq+1}\bar{g} + \Pi^B_{+1/2}\bar{g} \) as gauge group. The other constraints are then regained as gauge fixing conditions. As the example of \( N=1 \) supergravity already shows, the introduction of extra fields is in certain cases unavoidable. In order to give a unified description, we always introduce extra fields whenever representations of half-integral dimension occur.

Introduce the gauge fields \( A_z \in \Pi_{+}\bar{g} \) and the “auxiliary” fields \( \tau \in \Pi^F_{-1/2}\bar{g} \), \( r \in \Pi^B_{-1/2}\bar{g} \) and \( \bar{r} \in \Pi^B_{-1/2}\bar{g} \) where we denote

\[
\begin{align*}
    r & \equiv \sum_{j,\alpha} r^{(j,\alpha,\beta=1)} t_{(j-1/2,\alpha)} \\
    \bar{r} & \equiv \sum_{j,\alpha,\beta=-1} \bar{t}^{(j,\alpha)} \bar{t}_{(j-1/2,\alpha)} \\
    \tau & \equiv \sum_{j',\alpha'} \tilde{\tau}^{(j',\alpha')} \tilde{t}_{(j'-1/2,\alpha')}, \quad (6.4)
\end{align*}
\]

and \( t_{(j-1/2,\alpha)} \in \Pi^B_{+1/2}\bar{g} \), \( \bar{t}_{(j-1/2,\alpha)} \in \Pi^B_{-1/2}\bar{g} \) and \( \tilde{t}_{(j'-1/2,\alpha')} \in \Pi^F_{-1/2}\bar{g} \).

Using elementary properties of \( sl(2,R) \) representations and some of the results in appendix C, one shows that the action \( S_1 \):

\[
S_1 = \kappa S^{-}[g] + \frac{\kappa}{2\pi} \int \text{str} A_z \left( \partial g g^{-1} - e_- - r - \bar{r} - \tau \right)
- \sum_{j',\alpha'} \frac{\kappa y}{(j' + \frac{1}{2})\pi} \int \tau^{(j',\alpha')} \bar{\partial} \tau^{(j',\alpha')} - \sum_{j,\alpha} \frac{2\kappa y}{(j + \frac{1}{2})\pi} \int \bar{\tau}^{(j,\alpha)} \bar{\partial} \tau^{(j,\alpha)}, \quad (6.5)
\]

is indeed invariant under gauge transformations with parameters \( \Pi_{+}\bar{g} \), provided \( A_z, r, \bar{r} \) and \( \tau \) transform as

\[
\begin{align*}
    \delta A_z & = \bar{\partial} \xi + [\xi, A_z] \\
    \delta r & = \left[ \Pi_{+1/2}^B \eta, e_- \right] \\
    \delta \bar{r} & = \left[ \Pi_{-1/2}^B \eta, e_- \right]
\end{align*}
\]

\(^1\)Note that even in the supersymmetric case, auxiliary fields can sometimes be avoided. One example is \( N=2 \) supergravity where the \( U(1) \) symmetry can be used to restrict the gauge group.
\[ \delta \tau = \left[ \Pi^{F}_{+1/2} \eta, e_- \right] \] (6.6)

and \( \eta \in \Pi_{+\bar{g}} \).

Using exactly the same methods as in section 5, we can construct the polynomials \( T \equiv \sum_{j,\alpha_j} T^{(j,\alpha_j)} t^{(j,\alpha_j)} \) which are gauge invariant modulo the field equations of the gauge fields:

\[ T \equiv e^{\ln \sqrt{C} e_0 \Pi_{\text{ker} \, ad} + J_z e^{-\ln \sqrt{C} e_0} + \cdots} \] (6.7)

and the normalization constant \( C \) is determined by the requirement that \( T^{(1,0)} \) generates the Virasoro algebra\(^*\). We couple these currents to sources and modify the action to

\[ S_2 = S_1 + \frac{1}{4\pi xy} \int \text{str} h T, \] (6.8)

where the sources \( h \) are given by

\[ h \equiv \sum_{j,\alpha_j} h^{(j,\alpha_j)} t^{(j,\alpha_j)}. \] (6.9)

The action \( S_2 \) is gauge invariant provided we modify the transformation rules for the gauge fields suitably. These modifications are proportional to the \( h \)-fields. One then proceeds by solving the BV master equation. This will introduce ghost fields \( c \in \Pi_{+\bar{g}} \). Though we cannot give the solution of the BV master equation in its full generality, we have enough information to proceed and choose a gauge. The chosen gauge corresponds to putting \( A_{\bar{z}} = 0 \) and will allow us to determine the normalization of the currents and the value of the central extension \( c \). As in section 4, this gauge choice is accomplished by changing \( A_{\bar{z}}^* \) into a field, the antighost \( b \in \Pi_{-\bar{g}} \), and \( A_{\bar{z}} \) into an antifield \( b^\ast \). The action \( S_1 \), eq. (6.5), together with its gauge transformation rules, is just sufficient to determine the gauge fixed action to be

\[ S_{gf} = \kappa S^{-[g]} - \sum_{j' \alpha'} \left( \frac{ky}{2} \right) \int \tau^{(j',\alpha')} \bar{\partial} \tau^{(j',\alpha')} - \sum_{j \alpha} \left( \frac{2ky}{2} \right) \int \bar{r}^{(j,\alpha)} \partial r^{(j,\alpha)} \]

\[ + \frac{1}{2\pi x} \int \text{str} b \partial c + \frac{1}{4\pi xy} \int \text{str} h \hat{T}, \] (6.10)

where \( \hat{T} \) will be discussed shortly. The BRST charge also follows, since the transformation laws of all relevant fields are known, including the \( b \)-ghosts which are known explicitly from the term proportional to \( A_{\bar{z}} \) in eq. (6.5):

\[ Q = \frac{1}{4\pi i x} \int \text{str} \left\{ c \left( J_z - \frac{\kappa}{2} (e_- + \tau + r + \bar{r}) \right) + \frac{1}{2} b c c \right\}. \] (6.11)

\(^*\)Note that this normalization differs by constant factors from the one used in the study of SO(N) supergravity.
It is nilpotent.

For $h = 0$ the action is BRST invariant. In order to guarantee BRST invariance for $h \neq 0$, the currents $\hat{T}$ themselves have to be BRST invariant. For these currents we have not given an explicit form. In fact they absorb in the present gauge all complications arising from the non-closure terms in the extended action. However, the requirement of BRST invariance determines them up to BRST exact pieces. As we will see next, even this ambiguity can be eliminated by considering a reduced BRST complex.

We now study the BRST cohomology in some detail. We will use methods inspired by [34, 35] and without explicitly mentioning, several results from [36]. However the presence of the auxiliary fields $\{\tau, r, \bar{r}\}$ considerably complicates the analysis. Our arguments will be somewhat heuristic and we postpone a rigorous derivation to a future publication.

We split the BRST charge into three parts $Q = Q_0 + Q_1 + Q_2$, where

$$Q_0 = -\frac{\kappa}{8\pi i x} \oint strc e_-, \quad Q_1 = -\frac{\kappa}{8\pi i x} \oint strc (\tau + r + \bar{r}). \quad (6.12)$$

One has $Q_0^2 = Q_2^2 = \{Q_0, Q_1\} = \{Q_1, Q_2\} = 0$ and

$$Q_1^2 = -\{Q_0, Q_2\} = \frac{\kappa}{32\pi i x} \oint str \left\{ c \left[ \Pi_{1/2} c, e_- \right] \right\}. \quad (6.13)$$

The action of $Q = Q_0, Q_1$ and $Q_2$ on the basic fields is given by

$$Q_0 : \begin{array}{l} b \rightarrow -\frac{\kappa}{2} e_- \\
c \rightarrow 0 \\
\hat{J}_z \rightarrow -\frac{\kappa}{2} [e_-, c] \\
r \rightarrow 0 \\
\bar{r} \rightarrow 0 \\
\tau \rightarrow 0 \end{array} \quad Q_1 : \begin{array}{l} b \rightarrow -\frac{\kappa}{2} (\tau + r + \bar{r}) \\
c \rightarrow 0 \\
\hat{J}_z \rightarrow -\frac{\kappa}{2} [\tau + r + \bar{r}, c] \\
r \rightarrow 0 \\
\bar{r} \rightarrow 0 \\
\tau \rightarrow 0 \end{array} \quad Q_2 : \begin{array}{l} b \rightarrow \Pi_- \hat{J}_z \\
c \rightarrow \frac{1}{2} e_- \\
\hat{J}_z \rightarrow \frac{1}{2} [c, \Pi_- \hat{J}_z] + \frac{\kappa}{8} \partial c \\
\frac{1}{2} [\Pi_- (t^a), [\Pi_+ (t^a), \partial c] \end{array} \quad (6.14)$$

where

$$\hat{J}_z \equiv J_z + \frac{1}{2} \{b, c\} \quad (6.15)$$

and $\Pi_- \equiv 1 - \Pi_-$ and $[A, B]$ stands for

$$[A, B] = (-)^{ab} \left( A^a B^b \right) f_{ab} c \partial c, \quad (6.16)$$

34
where \((A^a B^b)\) is a regularized product.

Consider the algebra \(\mathcal{A}\) generated by the basic fields \(\{b, \hat{J}_z, \tau, r, \bar{r}, c\}\), which consists of all regularized products\(^3\) of the basic fields and their derivatives modulo the usual relations \([37, 38]\) between different orderings, derivatives, etc. We assign ghostnumber \(-1\) to \(b\), \(1\) to \(c\) and \(0\) to all other fields. Through this \(\mathcal{A}\) acquires a grading \(\mathcal{A} = \bigoplus_{n \in \mathbb{Z}} \mathcal{A}_n\) where the elements of \(\mathcal{A}_n\) have ghostnumber \(n\). The BRST charge \(Q\) is a map from \(\mathcal{A}_n\) to \(\mathcal{A}_{n+1}\) and it acts as a derivation on a regularized product of fields. In view of the application at hand, we want to study the cohomology of this complex at ghostnumber zero, \(H^0(\mathcal{A})\).

The cohomology of the subcomplex \(\mathcal{A}^{(1)}\), generated by \(\{b, \Pi_\tau \hat{J}_z - \frac{\pi}{2} (\tau + r + \bar{r})\}\) is \(H^*(\mathcal{A}^{(1)}) = \mathbb{C}\). As such, we have that \(H^*(\mathcal{A}) = H^*(\bar{\mathcal{A}})\) where \(\bar{\mathcal{A}} = \mathcal{A}/\mathcal{A}^{(1)}\) which we choose to be generated by \(\{\Pi_\tau \hat{J}_z, \tau, r, \bar{r}, c\}\).

Next we observe that \(\bar{\mathcal{A}}\) has yet another subcomplex \(\mathcal{A}^{(2)}\) generated by \(\{\tau, r, \bar{r}, \Pi_{1/2} c\}\). Its cohomology is \(H^*(\mathcal{A}^{(2)}) = \mathbb{C}\). Though we have that \(H^*(\bar{\mathcal{A}}) = H^*(\bar{\mathcal{A}})\) where \(\bar{\mathcal{A}} = \bar{\mathcal{A}}/\mathcal{A}^{(2)}\), it is not straightforward to find a set of generators for \(\bar{\mathcal{A}}\) on which the action of \(Q\) closes in the strong sense. In other words we need to find fields \(\Pi_\tau \hat{J}_z\) and \(\Pi_{1/2} \bar{c}\) which generate the algebra \(\bar{\mathcal{A}}\) and on which the action of \(Q\) closes. In order to do this we introduce \(\tau', r'\) and \(\bar{r}'\):

\[
\tau' = \sum_{j, \alpha} (-)^{j+\frac{1}{2}} \frac{2}{j + \frac{1}{2}} r(j, \alpha) t(j + \frac{1}{2}, \alpha),
\]

\[
r' = \sum_{j, \alpha, \beta=1} (-)^{j+\frac{1}{2}} \frac{2}{j + \frac{1}{2}} r(j, \alpha) t(j + \frac{1}{2}, \alpha),
\]

\[
\bar{r}' = \sum_{j, \alpha, \beta=-1} (-)^{j+\frac{1}{2}} \frac{2}{j + \frac{1}{2}} r(j, \alpha) t(j + \frac{1}{2}, \alpha).
\]

These fields transform in a very simple way under \(Q\): \(\tau' \rightarrow \Pi_{1/2}^{-c}, r' \rightarrow \Pi_{1/2}^c\) and \(\bar{r}' \rightarrow \Pi_{1/2}^{-c}\). Using this, one can now recursively construct \(\Pi_{1/2} \bar{c}\). One finds e.g. \(\Pi_1 \bar{c} = \Pi_1 c - \frac{1}{4} \text{ad}(\tau' + r' + \bar{r}')\Pi_{1/2} c\), \(\Pi_{3/2} \bar{c} = \Pi_{3/2} c - \frac{1}{2} \text{ad}(\tau' + r' + \bar{r}')\Pi_1 \bar{c} - \frac{1}{24} (\text{ad}(\tau' + r' + \bar{r}'))^2 \Pi_{1/2} c\), etc... So in general, we will have

\[
\Pi_m \bar{c} = \sum_{n=0}^{2m-1} (-)^n a_n^{(m)} (\text{ad}(\tau' + r' + \bar{r}'))^n \Pi_{m-n/2} c,
\]

where \(a_0^{(m)} = 1\), and one has e.g. \(a_1^{(2)} = 1/2\), \(a_2^{(2)} = 1/8\) and \(a_3^{(2)} = 1/64\). One proceeds in a similar manner for the construction of \(\Pi_{1/2} \hat{J}_z\). One finds e.g. \(\Pi_0 \hat{J}_z = \Pi_0 \hat{J}_z + \frac{\pi}{8} [\tau, \tau']\).

On the reduced complex \(\bar{\mathcal{A}}\) we get from eq. \((1.13)\) that \(Q_0\) and \(Q_1 + Q_2\) yield a double complex in a weak sense. We conjecture that using these observations one can show that the cohomology
of \( Q \) is isomorphic to that of \( Q_0 \) on \( \tilde{A} \). The cohomology of \( Q_0 \) is given by \( \{ \tilde{J}_z^{\text{HW}} \} = \{ \tilde{J}_z \} \cap \{ \ker e_+ \} \). This conjecture is strongly supported by the results of section four, and those of [35]. We denote the representant of the full cohomology by \( \check{J}_z = \tilde{J}_z^{\text{HW}} + \cdots \). Combining this with eq. (6.7), we get that

\[
T = e \ln \sqrt{C} e_0 \check{J}_z e - \ln \sqrt{C} e_0 .
\] (6.19)

A priori, one can only say that the operator algebra of \( \{ T^{(j_\alpha)} \} \) closes modulo BRST exact terms. However, as the cohomology was computed on a reduced complex which has no fields with negative ghost number, there are no BRST exact terms at ghostnumber 0 in the reduced complex.

Computing \( T \) explicitly has obviously to be done on a case by case basis. However the energy momentum tensor can be explicitly constructed. One easily checks that

\[
T_{\text{EM}} = \frac{1}{x(k + h)} \text{str} J_z J_z - \frac{1}{8xy} \text{str} e_0 \partial J_z + \sum_{j, \alpha_j} \frac{\kappa y}{j + \frac{1}{2}} r^{j_\alpha} \partial r^{j_\alpha} - \sum_{j, \alpha_j} \frac{\kappa y}{j + \frac{1}{2}} \tilde{r}^{j_\alpha} \partial \tilde{r}^{j_\alpha} + \frac{1}{4x} \text{str} b[e_0, \partial c] + \frac{1}{2x} \text{str} b\partial c + \frac{1}{4x} \text{str} b[e_0, c],
\] (6.20)

where \( e_0 = t_{(10,0)} \), is indeed BRST invariant. This current differs from the currents previously studied by a BRST exact term:

\[
T_{\text{exact}} = Q \left( -\frac{2}{x(k + h)} \text{str} \tilde{J}_z \right)
\] (6.21)

and we get:

\[
T^{(1,0)} = T + T_{\text{exact}} = \frac{\kappa}{x(k + h)} \text{str} (e_- + \tau + r + \tilde{r}) \tilde{J}_z + \text{more} ... \] (6.22)

From this we read off that the normalization constant \( C \) is given by

\[
C = \frac{4y\kappa}{\sqrt{2(k + h)}}.
\] (6.23)

Eq. (6.22) can easily be obtained from eq. (6.20) by supplementing eq. (6.20) with the constraints

\[
\Pi_- \left( \partial gg^{-1} - e_- - r - \tilde{r} - \tau \right) = 0.
\] (6.24)

36
We now analyze the various parts of the energy-momentum tensor, eq. (6.20). The fact that it differs by a BRST exact form from the “true” energy-momentum tensor does not change the value of the central charge. The first term is the Sugawara tensor for \( \hat{g} \) with central charge:

\[
c_{\text{Sug}} = \frac{\kappa (d_B - d_F)}{\kappa + \tilde{h}}.
\] (6.25)

The second term is the so-called improvement term. Given an affine current \( J \in \Pi_{-m}\bar{g}, m > 0 \), it takes care that the conformal dimension of \( J \) is given by \(-m + 1\). Its contribution to the central charge is

\[
c_{\text{imp}} = -6y \kappa.
\] (6.26)

The next term is the energy-momentum tensor for the auxiliary \( \tau \) fields with central charge

\[
c_{\tau} = \frac{1}{2} \dim \left( \Pi_{1/2}^{F} \bar{g} \right),
\] (6.27)

and the two next terms give the energy-momentum tensor for the auxiliary \( r \) and \( \bar{r} \) fields with central charge

\[
c_{r\bar{r}} = -\frac{1}{2} \dim \left( \Pi_{1/2}^{B} \bar{g} \right).
\] (6.28)

Finally, the last terms form the energy-momentum tensor for the ghost-antighost system. To each generator \( t \) of the gauge group, \( t \in \Pi_{m}\bar{g} \) where \( m > 0 \), we associated a ghost \( c \in \Pi_{m}\bar{g} \) with conformal dimension \( m \) and an anti-ghost \( b \in \Pi_{-m}\bar{g} \) with conformal dimension \(-m - 1\). Such a pair contributes \( \mp (12m^2 - 12m + 2) \) where we have \(- (+ \text{ resp.}) \) if \( b \) and \( c \) are fermionic (bosonic resp.). As such the total contribution to the central charge coming from the ghosts is given by

\[
c_{\text{ghost}} = -\sum_{j,\alpha} (-)^{(\alpha)} 2j(2j^2 - 1) + \frac{1}{2} \left( \dim \left( \Pi_{1/2}^{B} \bar{g} \right) - \dim \left( \Pi_{1/2}^{F} \bar{g} \right) \right)
\] (6.29)

Adding all of this together, we obtain the full expression for the total central charge \( c \) as a function of the level \( \kappa \):

\[
c = c_{\text{Sug}} + c_{\text{imp}} + c_{\tau} + c_{r\bar{r}} + c_{\text{ghost}},
\] (6.30)

where the individual contributions are given in eqs. (6.25-6.29). Using the explicit form for the index of embedding \( y \) given in eq. (C.3) and some elementary combinatorics, we can rewrite eq. (6.30) in the following, very recognizable form:

\[
c = \frac{1}{2} c_{\text{crit}} - \frac{(d_B - d_F)\tilde{h}}{\kappa + \tilde{h}} - 6y(\kappa + \tilde{h}),
\] (6.31)
where $c_{\text{crit}}$ is the critical value of the central charge for the extension of the Virasoro algebra under consideration:

$$c_{\text{crit}} = \sum_{j,\alpha} (-)^{(\alpha_j)} (12j^2 + 12j + 2). \quad (6.32)$$

We now turn to the effective action, which we determine along the same lines as those followed in section 5.1. The effective action is given by:

$$\exp -W[\tilde{T}] = \int [\delta g g^{-1}] [d\tau] [dr] [d\bar{r}] [dA_\bar{z}] (\text{Vol} (\Pi_+ \bar{g}))^{-1} \exp - \left( S_1 + \frac{1}{4\pi xy} \int \text{strh} \left( T - \tilde{T} \right) \right), \quad (6.33)$$

where the sources $\tilde{T}$ are given by

$$\tilde{T} \equiv \sum_{j,\alpha} \tilde{T}^{(j,\alpha)} t_{(jj,\alpha_j)}. \quad (6.34)$$

The road to follow is now precisely analogous to section 5.1. Choosing the Drinfeld-Sokolov gauge, we obtain an explicit expression for the effective action:

$$W[\tilde{T}] = \kappa_c S_-[g] \quad (6.35)$$

where

$$\kappa_c = \kappa + 2\tilde{h} \quad (6.36)$$

and from eq. (6.31) we get the central extension as a function of the level

$$c = \frac{1}{2} c_{\text{crit}} - \frac{(d_B - d_F)\tilde{h}}{\kappa_c - \tilde{h}} - 6y(\kappa_c - \tilde{h}), \quad (6.37)$$

or more usefully, the level as a function of the central charge:

$$12y\kappa_c = 12y\tilde{h} - \left( c - \frac{1}{2} c_{\text{crit}} \right) - \sqrt{\left( c - \frac{1}{2} c_{\text{crit}} \right)^2 - 24(d_B - d_F)\tilde{h}y} \quad (6.38)$$

Eqs. (6.37) or (6.38) provide us with all-order expressions for the coupling constant renormalization. We now turn to the wavefunction renormalization.

The WZW model in eq. (6.35) is constrained by

$$\partial gg^{-1} + \frac{\alpha_e}{\kappa} \frac{1}{4xy} \text{str} \left\{ \Pi_{NA} (\partial gg^{-1}) \Pi_{NA} (\partial gg^{-1}) \right\} e_+ = \frac{\kappa}{\alpha_e} e_+ + \frac{2}{\alpha_e} e - \ln \sqrt{C} e_0 \tilde{T} e \ln \sqrt{C} e_0, \quad (6.39)$$
where $\Pi_{NA} \bar{g}$ are those elements of $\Pi_0 \bar{g}$ which do not belong to the Cartan subalgebra of $\bar{g}$, i.e. the centralizer of $sl(2, \mathbb{R})$ in $\bar{g}$. Performing a global group transformation

$$g \rightarrow e^{\ln \left( \sqrt{\frac{\alpha_\kappa}{\kappa}} \right) e_0} g.$$ 

(6.40)

we bring the constraints in the standard form eq. (6.1):

$$\partial gg^{-1} + \frac{1}{4xy} \text{str} \left\{ \Pi_{NA} \left( \partial gg^{-1} \right) \Pi_{NA} \left( \partial gg^{-1} \right) \right\} e_+ = e_- + \sum_{j,\alpha_j} \frac{2\kappa_j}{\alpha_{\kappa} + 1} T^{(j\alpha_j)} t_{(jj,\alpha_j)}. \quad (6.41)$$

All computations were done using operator methods, so again $\alpha_\kappa$ is given by $\alpha_\kappa = \kappa + \tilde{h}$. With this choice we get the following final expression for the constraints:

$$\partial gg^{-1} + \frac{1}{4xy} \text{str} \left\{ \Pi_{NA} \left( \partial gg^{-1} \right) \Pi_{NA} \left( \partial gg^{-1} \right) \right\} e_+ = e_- + \frac{1}{\kappa + \tilde{h}} \sum_{j,\alpha_j} \frac{1}{2j+1} y^j T^{(j\alpha_j)} t_{(jj,\alpha_j)}, \quad (6.42)$$

which gives us, for the chosen normalization, eq. (6.7), of the conformal currents, the wavefunction renormalization to all orders.

From eq. (6.38), we deduce that for generic values of $\kappa$, no renormalization of the coupling constant beyond one loop occurs if and only if either $d_B = d_F$ or $\tilde{h} = 0$ (or both). Both cases are only possible for superalgebras. We get $d_B = d_F$ for $su(m \pm 1|m)$, $osp(m|m)$ and $osp(m + 1|m)$. The quadratic Casimir in the adjoint representation vanishes, i.e. $\tilde{h} = 0$, for $su(m|m)$, $osp(m + 2|m)$ and $D(2,1,\alpha)$. Note that $P(m)$ and $Q(m)$ have not been considered, since the absence of an invariant metric implies that no WZW models exist for them. The non-renormalization of the couplings is reminiscent of nonrenormalization theorems [39] for extended supersymmetry. These imply that under suitable circumstances at most one loop corrections to the coupling constants are present (the wave function renormalization may have higher order contributions). Comparing our list with the tabulation [40] of super-$W$ algebras obtained from a (classical) reduction of superalgebras, we find that many of them, though not all, have $N = 2$ supersymmetry. For instance, there is an $sl_2$ embedding in $osp(3|2)$ which gives the super-$W_2$ algebra of [11], which contains four fields (dimensions $5/2, 2, 2, 3/2$) and no $N = 2$. Also, it seems that all superalgebras based on the reduction of the unitary superalgebras $su(m|n)$ contain an $N = 2$ subalgebra, whereas our list contains only the series $|m - n| \leq 1$. Clearly, the structural reason behind the lack of renormalization beyond one loop remains to be clarified [42].

39
7 Examples

As an application of the general framework developed in previous section, we briefly study a few examples.

7.1 Other $N = 4$ Supergravities

The $N = 4$ algebra given in eq. (2.1) is only a special case of a one parameter family of $N=4$ algebras. This one-parameter family of $N = 4$ superconformal algebras occurs both in a linearized version, discovered in [19], generalizing the Ademollo et al. [43] $N = 4$ algebra and a non-linearly generated version, generalizing the Knizhnik-Bershadsky $SO(4)$ [16, 17] algebra, discovered in [20].

The latter differs essentially from the $N = 4$ algebra given in eq. (2.1) by the GG OPE for which at the right hand side, the $so(4)$ current algebra gets broken to two commuting $su(2)$ current algebras, the relative strength of which is determined by a parameter $\alpha \equiv k_+/k_-$, where $k_\pm$ are the levels of the two $su(2)$ current algebras. The central charge $c$ of the superconformal algebra is given by

$$c = \frac{6k_+k_-}{k_+ + k_-} - 3.$$  (7.1)

For $k_+ = k_-$ the superconformal algebra is isomorphic to eq. (2.1) for $N = 4$. The subalgebra of transformations globally defined on the sphere is on-shell isomorphic to $D(2, 1, \alpha)$. The resulting effective supergravity theory is given by a constrained $D(2, 1, \alpha)$ WZW model. The methods and results are very similar to those used and obtained for $osp(4|2)$ in sections 4 and 5. E.g. the level as function of the central charge is precisely given by eq. (5.13) for $N = 4$.

Again the superconformal algebra can be linearized by adding 4 free fermions and a $U(1)$ current. Using the results of [27], we find that the level of the $D(2, 1, \alpha)$ current algebra is related to the central extension $c_{\text{lin}} = c + 3$ by

$$\kappa_c = -2c_{\text{lin}}.$$  (7.2)

Finally, there is one more “standard” $N = 4$ superconformal algebra. It can be obtained by folding or twisting [19] the previously mentioned linear $N = 4$ algebras. The resulting superconformal algebra has besides the energy-momentum tensor, 4 dimension 3/2 supercurrents and an $so(3)$ affine Lie algebra. After the twist the algebra acquires a central charge $\tilde{c}$:

$$\tilde{c} = 6k_+.$$  (7.3)
and the level of the $so(3)$ current algebra is given by $2k_+$ (the factor 2 explains why we called this an $so(3)$ algebra).

The subalgebra of transformations globally defined on the sphere is isomorphic to $su(1, 1|2)$. As such we expect that the corresponding effective supergravity theory is given by a constrained $su(1, 1|2)$ WZW model. For the principal embedding of $sl(2, \mathbb{R})$ in $su(1, 1|2)$, one finds that the adjoint representation of $su(1, 1|2)$ branches to a $j = 1, 4 \ j = 1/2$ and $3 \ j = 0$ (generating an $so(3)$ subalgebra) irreducible representations of $sl(2, \mathbb{R})$. The critical dimension is $c_{\text{crit}} = -12$ and due to the nature of the embedding we find $y = 1$ (Formula (C.3) can not be used, but the value is obvious). Considering the principal embedding of $sl(2, \mathbb{R})$ in $su(1, 1|m)$ where one finds, using eq. (C.3) that $y = 1$ for every $m$. Taking all of this together, we find

$$- 6\kappa = \tilde{c} + 1.$$  \hfill (7.4)

The factor 1 at the rhs combined with the results of [27], suggest that adding a $U(1)$ current to the superconformal algebra, will again yield a theory where the coupling constant does not get renormalized. One such a theory which comes to mind is the one where we realize the $N = 4$ superconformal algebra in terms of 2 complex fermions and 4 “symplectic” bosons [44]. For this particular theory one gets $\kappa = 0$. As such, this theory will only have a finite number of degrees of freedom. A further study of this extended topological gravity would be interesting.

7.2 $W$ Gravity

The $WA$ algebras, which are usually called $W$ algebras, are the most studied non-linearly generated conformal algebras. They are characterized by the fact that the subalgebra of transformations, globally defined on the sphere, forms on-shell, i.e. putting the non-linear terms to zero, an $sl(n, \mathbb{R})$ algebra. For given $n$, the different $W$ algebras are classified by the inequivalent, non-trivial embeddings of $sl(2, \mathbb{R})$ in $sl(n, \mathbb{R})$. And the procedure of section six can be applied to construct both the induced action, i.e. essentially realize the $WA$ algebra in terms of a gauged $sl(n, \mathbb{R})$ WZW model, and the effective action.

It is known [45] that the inequivalent embeddings of $sl(2, \mathbb{R})$ in $sl(n, \mathbb{R})$ are completely characterized by the branching of the fundamental representation of $sl(n, \mathbb{R})$, $n$ in irreducible representations of $sl(2, \mathbb{R})$:

$$\mathfrak{n} = \bigoplus_{j \in \frac{1}{2}\mathbb{N}} n_j \cdot [2j + 1].$$ \hfill (7.5)

*In other words, they are in one to one correspondence with the partitions of $n$. 
As we only consider non-trivial embeddings, we have to supplement eq. \((7.5)\) with the condition:

\[
q \equiv \sum_{j \in \mathbb{N}} n_j < n. \tag{7.6}
\]

The “standard” \(W\) algebras, correspond to the partition for which the only non vanishing \(n_j\) is \(n_{(n-1)/2} = 1\).

From now on, we work in a given embedding. The branching of the adjoint representation of \(sl(n, \mathbb{R})\), \(n^2 - 1\), into irreducible representations of \(sl(2, \mathbb{R})\) follows immediately from \(n \otimes \overline{n} = 1 \oplus n^2 - 1\). We get

\[
1 \oplus n^2 - 1 = \bigoplus_j n_j^2 \cdot \bigoplus_{J=0}^{2j} [2J + 1] \oplus \bigoplus_{j>j'} 2n_jn_{j'} \cdot \bigoplus_{J=j-j'} [2J + 1]. \tag{7.7}
\]

From eq. \((7.7)\), we can read the field content of the corresponding \(WA_n\) algebra. The algebra has \(qn - 1 - 2\sum_{j>j'} n_jn_{j'}(j - j')\) currents, with conformal dimensions given by \(J + 1\), where \(J\) labels the \(sl(2, \mathbb{R})\) representations in the branching of \(n^2 - 1\). The affine subalgebra of the conformal algebra, is determined by the centralizer \(\Pi_{NA}sl(n, \mathbb{R})\), of \(sl(2, \mathbb{R})\) in \(sl(n, \mathbb{R})\):

\[
\Pi_{NA}sl(n, \mathbb{R}) = \bigoplus_{j \in \mathbb{N}} sl(n_j, \mathbb{R}) \oplus (r - 1) \cdot u(1), \tag{7.8}
\]

where \(r\) is the number of different values of \(j\) for which \(n_j \neq 0\). Combining eq. \((7.5)\) with the well-known action of \(sl(2, \mathbb{R})\) on irreducible representations:

\[
e_0|jm> = 2m|jm>,
\]

\[
e_\pm|jm> = \sqrt{(j \mp m)(j \pm m + 1)}|jm \pm 1>, \tag{7.9}
\]

we immediately obtain the explicit expression of the \(sl(2, \mathbb{R})\) generators in terms of \(n \times n\) matrices.

The only two quantities which remain to be computed is the index of embedding \(y\) and the critical dimension \(c_{\text{crit}}\). The index of embedding, eq. \((C.3)\), is very easily computed using the \(sl(2)\) characters \(\chi_J(\theta) = \sin((2J + 1)\theta)/\sin(\theta)\) and using the fact that

\[
\lim_{\theta \to 0} \frac{\partial^2 \chi_J(\theta)}{\partial^2 \theta} = -\frac{4}{3} J(J + 1)(2J + 1). \tag{7.10}
\]
One finds the following, very simple expression:

\[ y = \frac{2}{3} \sum_{j \in \mathbb{N}} n_j j(j + 1)(2j + 1). \] (7.11)

The critical dimension is obtained through direct computation, i.e. combining eqs. \((7.7)\) and \((6.32)\):

\[ c_{\text{crit}} = 6qy + 2qn - 2 + 12n \sum_j n_j j(j + 1) + 4 \sum_{j > j'} n_j n_{j'} (1 - 2(j - j')^2). \] (7.12)

Using this, \( \tilde{h} = n, d_B = n^2 - 1 \) and \( d_F = 0 \), one finds from eq. \((6.38)\) the level as a function of the central charge.

All these ingredients combined fully determine the effective action in terms of a constrained \( sl(n, \mathbb{R}) \) WZW model.

Comparing these results to some of the known ones, like \( W_3^{(1)} \) gravity in \([5, 11, 12]\) and \( W_3^{(2)} 2 \) gravity in \([46, 35]\), we find complete agreement.

### 7.3 \( N = 2 \) \( W_n \) Gravity

The principal embedding of \( sl(2, \mathbb{R}) \) in \( su(n|n - 1) \) yields \( N = 2 \) \( W_n \) algebras. Twisted versions of these systems are very relevant in the study of non-supersymmetric \( W_n \) strings \([17]\). \( sl(2, \mathbb{R}) \) gets embedded in the \( su(n) \) subalgebra such that the \( [n] \) of \( su(n) \) branches to the \( [n] \) of \( sl(2, \mathbb{R}) \). The adjoint of \( su(n|n - 1) \) branches to \( n \) “\( N = 2 \) multiplets”: \((j, j + 1/2, j + 1/2, j + 1)\) where \( j \in \{0, 2, \cdots, n - 2\}\):

\[ \text{adjoint}(su(n|n - 1)) = \bigoplus_{j=0}^{n-2} \{[2j + 1] + 2 \cdot [2j + 2] + [2j + 4]\}. \] (7.13)

In a \((j, j + 1/2, j + 1/2, j + 1)\) multiplet, the first irreducible representation \( j \) arises from the bosonic \( su(n - 1) + u(1) \) factor of \( su(n|n - 1) \), while the two \( j + 1/2 \) irreducible representations are fermionic. The \( j + 1 \) irreducible representation comes from the \( su(n) \) subalgebra. These superconformal algebras contain as well the \( W_n \) algebra as a subalgebra as the \( N=2 \) Virasoro algebra and as such deserve the name of \( N = 2 \ W_n \) algebras.

The critical dimension is easily computed and gives \( c_{\text{crit}} = 6n - 6 \). The index of embedding was computed in previous section and gives

\[ y = \frac{1}{6} n(n - 1)(n + 1). \] (7.14)
Combining this we find

\[ c = 3(n - 1) - n(n - 1)(n + 1)(\kappa_c - 1). \]  

(7.15)

In section 6 we classified all superalgebras which give rise to extensions of \( d = 2 \) gravity, where no coupling constant renormalization beyond one loop occurs. The example we just analyzed is a member of this class. These theories do indeed have an \( N = 2 \) supersymmetry.

8 Conclusions

Inequivalent, nontrivial embeddings of \( sl(2, \mathbb{R}) \) in a (super) Lie algebra are associated to extensions of the Virasoro algebra. These extended conformal algebras were realized in section 6 by considering a WZW model in which a chiral, solvable group is gauged. This description forms a perfect starting point to study the associated extension of \( d = 2 \) gravity in the chiral gauge. These extensions assume various forms: higher spin gauge fields, more gravitons, fermionic fields, Yang-Mills type (super)symmetries, ... The description in terms of a gauged WZW model allowed us to obtain an all order expression for the effective action. The effective action turned out to be a constrained WZW model and we gave all-order expressions for the coupling constant renormalization and the wavefunction renormalization.

In sections 2-5 we presented a detailed study of \( SO(N) \) supergravity. The cases \( N = 2 \), \( N = 3 \) and \( N = 4 \) have the particular feature that the coupling constant does not get renormalized beyond one loop. If one considers \( N = 4 \) supergravity based on the linear \( N = 4 \) superconformal algebra, one finds that no renormalization at all occurs!

This is consistent with the non-renormalization theorems [39] for extended supersymmetry. While these theorems do not say anything on the wavefunction renormalization, they predict at most a one loop renormalization for the coupling constant. The one loop contribution is basically due to the infinite tower of ghosts which arises when one expresses the constrained superfields in terms of unconstrained ones (i.e. solving the constraints). In the case of \( N = 4 \), it can occur that even the one loop contribution vanishes. This clearly occurs here. A detailed study of the non-renormalization effects requires a superspace formulation of these theories.

The \( SO(2) \), \( SO(3) \) and \( SO(4) \) supergravity models are not isolated cases. In section 6 we derived an all-order expression for the coupling constant renormalization, eqs. (6.37,6.38). From this we obtained an exhaustive list of all the models where this phenomenon occurs. Not all of these models possess an \( N = 2 \) supersymmetry as part of the total symmetry. Some of them are characterized by a super-\( W_2 \) structure. This superconformal algebra has currents of dimension \( 3/2, 2, 2 \) and \( 5/2 \). Though we do recognize some kind of \( N = 2 \) supermultiplet here, the algebra
is definitely not the $N = 2$ algebra. A further study of these structures is underway.

Let us now address the question of non-critical strings. Given an embedding of $sl(2, \mathbb{R})$ in $\bar{g}$, we consider the corresponding $(p, q)$ minimal model as the matter sector of the string theory. Its central charge $c_M$ is given by eq. (6.31), where $\kappa_M + \bar{\kappa} = p/q$. In order to cancel the conformal anomaly, we need to supplement the matter sector by a gauge sector whose central charge $c_L$ is again given by eq. (6.31) but now $\kappa_L + \bar{\kappa} = -p/q$. The corresponding $W$ string is now determined by currents

$$T_{\text{tot}} = T_M + T_L$$

where $T_M$ and $T_L$ are of the form given in eq. (6.19) and a BRST charge of the form

$$Q = \frac{1}{2\pi i} \oint strc(T_{\text{tot}} + \frac{1}{2} T_{\text{ghost}}),$$

where the ghost system contributes $-c_{\text{crit}}$ to the central charge. Of course the whole problem is to construct the ghost system.

Ultimately, the most straightforward way to construct $W$ strings is departing from a matter sector covariantly coupled to the gravitation theory. The matter-sector is most elegantly described by a gauged WZW model in the “conformal” gauge, as in this gauge it has the extended conformal symmetry for both the left and the right movers. In order to couple the matter action covariantly to the extended gravity, one can again use WZW like techniques (in some sense a generalization of the results of [48, 49]). In a future publication we will present a detailed analysis of such coupled, gauged WZW models.

One issue to be resolved is the rigorous proof of the conjecture on the computation of the reduced cohomology in section 6. Once this is done, one can study the question whether in the presence of auxiliary fields the Feigin-Fuchs type free field realization of $T$ is, as in the purely bosonic case, obtained by putting all fields but $\Pi, \hat{J}_z, \tau, \tau$ and $\tau'$ to zero in eq. (6.19).

A most challenging problem is the understanding of the geometry behind the extensions of $d = 2$ gravity. By geometry, we mean something very simple. The Virasoro algebra appears as the algebra of residual symmetry after gauge fixing a theory invariant under general coordinate transformations in $d = 2$. The question is whether a similar statement can be made for extensions of $d = 2$ gravity. In particular, the geometric significance of the non-linearities in the extensions of the Virasoro algebra remain to be understood. However, the $N = 3$ and $N = 4$ supersymmetric theories might provide clues for the solution of this problem. As we mentioned earlier, the non-linearly generated $N = 3$ and $N = 4$ algebras can be linearized by adding free fields to the system. In the linear case, as both $N = 3$ and $N = 4$ superspace
have been constructed, the geometry is well understood. The relation between the linear and
non-linear algebras might enable one to learn something about the geometry of the non-linear
algebras. Finally, a most exciting application of the methods developed in this paper would be
the study of reductions of continuum algebras \[50\] which would presumably lead to integrable
theories in \(d > 2\)! Work in these directions is in progress.

Acknowledgements: We would like to thank Orlando Alvarez, Jan De Boer, Alex Deckmyn,
Wolfgang Lerche, Micha Savelev, Kareljan Schoutens, Wati Taylor, Jean Thierry-Mieg, Peter
van Nieuwenhuizen, and especially Martin Roček for many illuminating discussions.

A Wess-Zumino-Witten Models

In this appendix we give some essential properties of WZW models. We start by summarizing
some properties of super Lie algebras. Given a super Lie algebra with generators \(\{t_a; a \in \{1, \cdots, d_B + d_F\}\}\), where \(d_B\) (\(d_F\)) is the number of bosonic (fermionic) generators, we denote the
(anti)commutation relations by

\[
[t_a, t_b] = t_a t_b - (-)^{(a)(b)} t_b t_a = f_{ab}^{\hspace{1cm}c} t_c,
\]

where for \(t_a\), \((a) = 0\) (1) when \(t_a\) is bosonic (fermionic). We adopt the convention that \([A, B]\)
stands for the anticommutator if both \(A\) and \(B\) are fermionic, else it is a commutator. We also
take \(X t_a = (-)^{\langle X \rangle (a)} t_a X\) where \(X\) is not Lie algebra valued. From the Jacobi identities one
shows that the adjoint representation is given by

\[
[t_a]_b^c \equiv f_{ba}^{\hspace{1cm}c}.
\]

The Killing metric \(g_{ab}\) is defined by

\[
f_{ca}^d f_{db}^{\hspace{1cm}c} (-)^{(c)} = -\tilde{h} g_{ab},
\]

where \(\tilde{h}\) is the dual Coxeter number. Though this is perfect for ordinary Lie algebras, this is
not sufficient for super algebras as the dual Coxeter number might vanish in this case\[\] More

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]

\[\]\[\]
generally we have
\[ str(t_at_b) \equiv [t_a]_\alpha \beta [t_b]_\beta \alpha (-)\alpha \equiv -x g_{ab} \] (A.4)

where \( x \) is the index of the representation. Obviously we have \( x = \tilde{h} \) in the adjoint representation. A contraction runs from upper left to lower right, \( e.g. \ A^a B_a \). The Killing metric is used to raise and lower indices according to this convention (implying \( g^{ac} g_{bc} = \delta^a_b \)):

\[ A^a = g^{ab} A_b \quad A_a = A^b g_{ba}. \] (A.5)

We tabulate some properties of the (super) Lie algebras which appear in this paper:

| algebra          | bosonic subalgebra | \( d_B \) | \( d_F \) | \( \tilde{h} \) | \( x_{fun} \) |
|------------------|---------------------|-----------|-----------|-------------|-------------|
| \( sl(n) \)     | \( sl(n) \)        | \( n^2-1 \) | 0         | \( n \)     | \( \frac{1}{2} \) |
| \( so(n) \)     | \( so(n) \)        | \( \frac{1}{2}n(n-1) \) | 0         | \( n-2 \)   | \( 1 \)     |
| \( osp(n|2) \)   | \( sl(2) + so(n) \) | \( \frac{1}{2}(n^2-n+6) \) | 2n        | \( \frac{1}{2}(4-n) \) | \( \frac{1}{2} \) |
| \( D(2,1,\alpha) \) | \( sl(2) + su(2) + su(2) \) | 9         | 8         | 0           | 0           |
| \( su(1,1|2) \)  | \( sl(2) + su(2) \) | 6         | 8         | 0           | \( \frac{1}{2} \) |
| \( su(m|n) \)    | \( su(m) + su(n) + u(1) \) | \( m^2+n^2-1 \) | 2mn       | \( m-n \)   | \( \frac{1}{2} \) |
| \( m \neq n \)  |                     |           |           |             |             |

We denoted by \( x_{fun} \), the index of the fundamental (defining) representation. For \( D(2,1,\alpha) \), the size of the fundamental representation depends on \( \alpha \). The smallest representation which exists for all values of \( \alpha \) is the adjoint representation.

The WZW action \( \kappa S^+[g] \) is given by

\[ \kappa S^+[g] = \frac{\kappa}{4\pi x} \int d^2 x \str \{ \partial g^{-1} \partial g \} + \frac{\kappa}{12\pi x} \int d^3 x \epsilon^{\alpha\beta\gamma} \str \{ g,\alpha g^{-1} g,\beta g^{-1} g,\gamma g^{-1} \}. \] (A.6)

It satisfies the Polyakov-Wiegman identity \([51]\),

\[ S^+[hg] = S^+[h] + S^+[g] - \frac{1}{2\pi x} \int \str (h^{-1} \partial h \bar{g}g^{-1}), \] (A.7)

which is obtained through direct computation. We also introduce a functional \( S^-[g] \) which is defined by

\[ S^-[g] = S^+[g^{-1}]. \] (A.8)

47
The equations of motion follow from
\[ \delta S^+[g] = \frac{1}{2\pi x} \int \text{str} \left\{ \partial (g^{-1} \partial g) g^{-1} \delta g \right\} \]
\[ = \frac{1}{2\pi x} \int \text{str} \left\{ \partial (\partial gg^{-1}) \delta gg^{-1} \right\}, \quad (A.9) \]
which is solved by putting \( g \equiv g(\bar{z}) g(z) \) where \( \partial g(\bar{z}) = \partial g(z) = 0 \). As such we get that the currents
\[ J_z = -\frac{\kappa}{2} g^{-1} \partial g \]
\[ J_{\bar{z}} = \frac{\kappa}{2} \bar{\partial} gg^{-1} \]
(A.10)
are conserved. This implies the affine symmetries
\[ \delta J^a_z = -\frac{\kappa}{2} \partial \eta^a - (-)^{(b)(c)} f^a_{bc} \eta^b J^c_{\bar{z}} \]
\[ \delta J^a_{\bar{z}} = \frac{\kappa}{2} \bar{\partial} \eta^a + (-)^{(b)(c)} f^a_{bc} \bar{\eta}^b J^c_z \]
(A.11)
where
\[ \bar{\partial} \eta^a = \partial \bar{\eta}^a = 0. \]
(A.12)
From
\[ \delta J^a_z(x) = \frac{1}{2\pi i} \oint_x dy J^b_z(y) \eta_b(y) J^a_z(x), \]
(A.13)
we get the OPE of an affine Lie algebra of level \( \kappa \):
\[ J^a_z(x) J^b_z(y) = -\frac{\kappa}{2} g^{ab} (x - y)^{-2} + (x - y)^{-1} (-)^{(c)} f^{ab} c J^c_z(y) + \cdots, \]
(A.14)
and similarly for \( J_{\bar{z}} \). The Sugawara tensor is given by
\[ T = \frac{1}{x (\kappa + \hbar)} \text{str} J_z J_{\bar{z}}, \]
(A.15)
and it satisfies the Virasoro algebra with the central extension given by:
\[ c = \frac{k(d_B - d_F)}{k + \hbar}. \]
(A.16)
As an example we list the OPEs for $OSp(N|2)$, which will be used extensively throughout section four.

\[
\begin{align*}
J^0(x)J^0(y) &= \frac{K}{8} (x - y)^{-2} \\
J^\pm(x)J^\mp(y) &= \frac{K}{4} (x - y)^{-2} + (x - y)^{-1} J^0(y) \\
J^0(x)J^\pm(y) &= \pm \frac{1}{2} (x - y)^{-1} J^\mp(y) \\
J^0(x)J^{\pm a}(y) &= \pm \frac{1}{4} (x - y)^{-1} J^{\pm a}(y) \\
J^\pm(x)J^{\mp a}(y) &= \frac{1}{2} (x - y)^{-1} J^{\pm a}(y) \\
J^i(x)J^j(y) &= \frac{K}{8} \delta^{ij} (x - y)^{-2} - \frac{\sqrt{2}}{4} (x - y)^{-1} f_{ijk} J^k(y) \\
J^i(x)J^{\pm a}(y) &= \frac{\sqrt{2}}{4} (x - y)^{-1} \delta_{ab} J^{\pm b} \\
J^{\pm a}(x)J^{- b}(y) &= \frac{K}{8} \delta^{ab} (x - y)^{-2} + \frac{1}{4} \delta^{ab} (x - y)^{-1} J^0(y) + \frac{\sqrt{2}}{4} (x - y)^{-1} \delta_{ab} J^i(y) \\
J^{\pm a}(x)J^{\pm b}(y) &= \pm \frac{1}{4} \delta^{ab} (x - y)^{-1} J^{\pm}(y),
\end{align*}
\]

where the metric we used is given by

\[
g_{++} = -2, \quad g_{00} = -4, \quad g_{+a-b} = -4 \delta_{ab}, \quad g_{ij} = -4 \delta_{ij}.
\]

Note that the $J_i J_j$ OPE can be rewritten as $J_i(x)J_j(y) = 2K\delta_{ij}(x - y)^{-2} + \sqrt{2}(x - y)^{-1} f_{ijk} J_k$, from which we observe, as was to be expected, that the $SO(N)$ level is even and negative.

## B Induced Gauge Theories

In this appendix we review some basic and well-known results on gauged WZW models [52, 51, 53, 54].

Consider the induced action, $\Gamma[A_\bar{z}]$, for the gauge fields $A_\bar{z}$,

\[
e^{-\Gamma[A_\bar{z}]} = \langle \exp \left( -\frac{1}{\pi x} \int d^2x \text{ str} \{ J_\bar{z}(x) A_\bar{z}(x) \} \right) \rangle,
\]

\[A.17\]
where \( J_z \) satisfies eq. (A.14). The gauge transformations

\[
\delta A_z = \partial \eta + [\eta, A_z], \tag{B.2}
\]

are anomalous

\[
\delta \Gamma[A_z] = -\frac{k^2}{2\pi x} \int d^2 x \text{str} \{ \eta \partial A_z \}. \tag{B.3}
\]

Defining

\[
u^a_z(x) = -\frac{2\pi}{k} g^{ab} \frac{\delta \Gamma[A_z]}{\delta A^b_z(x)}, \tag{B.4}\]

we deduce from eqs. (B.3) and (B.2) the following Ward identity

\[
\bar{\partial} u_z - [A_z, u_z] = \partial A_z. \tag{B.5}
\]

The Ward identity is independent of \( \kappa \), therefore

\[
\Gamma[A_z] = \kappa \Gamma^{(0)}[A_z], \tag{B.6}
\]

where \( \Gamma^{(0)}[A_z] \) is independent of \( \kappa \). In \cite{51,52}, it was observed that eq. (B.3) states that the curvature for the Yang-Mills fields \( \{A, u\} \) vanishes. This condition is solved by parametrizing \( A_z \) as \( A_z \equiv \bar{\partial} gg^{-1} \) and \( u_z \) as \( u_z \equiv \partial gg^{-1} \). Introducing the WZW functional \( S^+[g] \), we easily find that \( \Gamma^{(0)}[A_z] \) is given by

\[
\Gamma^{(0)}[A_z] = \bar{\partial} gg^{-1} = -S^+[g]. \tag{B.7}
\]

We could as well have performed the previous analysis starting from an anti-holomorphic affine Lie algebra. The induced action is then given by

\[
e^{-\mathcal{T}[A_z]} = \langle \exp \frac{1}{\pi x} \int d^2 x \text{str} \{ J_z(x)A_z(x) \} \rangle, \tag{B.8}
\]

where

\[
\mathcal{T}[A_z] = \bar{\partial} gg^{-1} = -\kappa S^+[g^{-1}] \equiv -\kappa S^-[g]. \tag{B.9}
\]

We now consider the generating functional of connected Greens functions with propagating \( A_z \) fields, \( W[u_z] \):

\[
e^{-W[u_z]} = \int [dA_z] e^{-\Gamma[A_z]} + \frac{1}{2\pi x} \int \text{str}(u_z A_z) \tag{B.10}
\]

\footnote{The source \( u_z \) is obviously different from \( u_z \) defined in eq. (B.4). We hope that this does not cause any confusion.}

50
The Legendre transform of $\Gamma^{(0)}[A_{\bar{z}}]$, 

$$W^{(0)}[u_{\bar{z}}] = \min_{\{A_{\bar{z}}\}} \left( \Gamma^{(0)}[A_{\bar{z}}] - \frac{1}{2\pi x} \int str (u_{\bar{z}} A_{\bar{z}}) \right)$$  \hspace{1cm} (B.11)$$

is explicitly given by 

$$W^{(0)}[u_{\bar{z}} \equiv \partial gg^{-1}] = S^{-}[g].$$  \hspace{1cm} (B.12)$$

It can be argued in several different ways that $W(u_{\bar{z}})$ is given by 

$$W(u_{\bar{z}}) = \hat{\kappa} W^{(0)}[Z_{\kappa} u_{\bar{z}}].$$  \hspace{1cm} (B.13)$$

The different computations of the renormalised coupling $\hat{\kappa}$ agree with each other, but the values of the current renormalisation factor $Z_{\kappa}$ differ. First we present two different arguments leading to the value used in the text. Then, for completeness, we also sketch two other lines of reasoning.

In eq.(B.10), we parametrise the integration variable by $A_{\bar{z}} = \bar{\partial} gg^{-1}$. For the Jacobian, we use 

$$[dA_{\bar{z}}] = [dgg^{-1}] det \bar{\mathbf{D}}[A_{\bar{z}} \equiv \partial gg^{-1}] = [dgg^{-1}] \exp 2\tilde{\hbar} S^+[g].$$  \hspace{1cm} (B.14)$$

and obtain 

$$e^{-W[u_{\bar{z}}]} = \int [dgg^{-1}] e^{(\kappa + 2\tilde{\hbar}) S^+[g] + \frac{1}{2\pi x} \int str(u_{\bar{z}} \bar{\partial} gg^{-1})}.$$  \hspace{1cm} (B.15)$$

Since the currents $\bar{\partial} gg^{-1}$ form an antiholomorphic affine Lie algebra with level $K = -(\kappa + 2\tilde{\hbar})$, $W[u_{\bar{z}}]$ is the corresponding induced action, cfr. eq. (B.8). The only remaining problem is to identify the proportionality constant in $\bar{\partial} gg^{-1} \propto J_{\bar{z}}$. Given a current algebra of level $K$, we put 

$$J_{\bar{z}} = \frac{\alpha K}{2} \bar{\partial} gg^{-1}.$$  \hspace{1cm} (B.16)$$

So we conclude that 

$$W[u_{\bar{z}}] = (\kappa + 2\tilde{\hbar}) W^{(0)} \left[ -\frac{u_{\bar{z}}}{\alpha - \kappa + 2\tilde{\hbar}} \right]$$  \hspace{1cm} (B.17)$$

and we identify 

$$\hat{\kappa} = \kappa + 2\tilde{\hbar}$$

$$Z_{\kappa} = -\frac{1}{\alpha - \kappa - 2\tilde{\hbar}}.$$  \hspace{1cm} (B.18)$$

51
Using OPE techniques, it is argued in [25] that for a current algebra of level $K$, the conventionally normalised currents are $J_z = \frac{K + \tilde{h}}{2} \partial gg^{-1}$. This follows from consistency requirements in the operator product algebra of the currents with $g$. In our case we have that $K = -(\kappa + 2\tilde{h})$, so accepting this argument we find

$$\alpha_\kappa = -\kappa - \tilde{h}$$  \hspace{1cm} (B.19)

and

$$Z_\kappa = \frac{1}{\kappa + \tilde{h}}.$$  \hspace{1cm} (B.20)

This is the value used in the main text. It is compatible with the value found in perturbation theory using the method described in section 5.2. This conclusion follows immediately from eq.(5.24), so the reasoning will not be repeated here.

A different argument rests on the invariance of the Haar measure and the Polyakov-Wiegmann formula. Parametrizing $u_z$ as

$$u_z \equiv (\kappa + 2\tilde{h})\partial hh^{-1}$$  \hspace{1cm} (B.21)

we find, using eq.(A.7), that eq.(B.15) becomes:

$$e^{-W[u_z]} = e^{-(\kappa + 2\tilde{h})S^- [h]} \int [dg g^{-1}] e^{(\kappa + 2\tilde{h})S^+[h^{-1}g]}.$$  \hspace{1cm} (B.22)

Assuming that we use a regulator which leaves the Haar measure invariant, we can drop the functional integral and we find that

$$Z_\kappa = \frac{1}{\kappa + 2\tilde{h}}.$$  \hspace{1cm} (B.23)

This corresponds to the classical value for $\alpha_K$: $\alpha_K = K$.

It is not difficult to reproduce this value by setting up the semiclassical computation differently. In fact, if in the perturbative calculation we factorize the determinants as follows:

$$\text{det} \left( \frac{D[u_z/\kappa]}{D[A_z^\dagger (u_z)]} \right) = \frac{\text{det} D[u_z/\kappa] \overline{D}[A_z^\dagger (u_z)]}{(\text{det} \overline{D}[A_z^\dagger (u_z)])^2},$$  \hspace{1cm} (B.24)

and we compute the determinants in the numerator with a vector gauge invariant regulator, we find eq. (B.23) back.

This leads us to yet another method to compute $W[u_z]$: the KPZ approach, [2]. The covariant induced action (which can be obtained from the computation of e.g. $\text{det} \left( \overline{D}[A_z]D[A_z] \right)$ with a
gauge invariant Pauli-Villars regulator) is given by

\[ \Gamma[A_\bar{z}] + \bar{\Gamma}[A_z] - \frac{\kappa}{2\pi x} \int \text{str} \{ A_\bar{z} A_z \}, \]

(B.25)

where \( A_\bar{z} \) transforms as in eq. (B.2) and \( \delta A_z = \partial \eta + [\eta, A_z] \). Now we want to use (B.25) as a quantum action. Using the gauge freedom to fix \( A_z \equiv \hat{A}_z \) yields the gauge fixed, BRST-invariant action:

\[ \Gamma[A_\bar{z}, \hat{A}_z, b, c] = \Gamma[A_\bar{z}] + \bar{\Gamma}[\hat{A}_z] - \frac{\kappa}{2\pi x} \int \text{str} \{ \hat{A}_z A_\bar{z} \} - \frac{\kappa}{2\pi x} \int bD[\hat{A}_z]c. \]

(B.26)

Assuming that the vectorial gauge symmetry is not anomalous (which may be guaranteed by the existence of a nilpotent BRST charge) implies that \( W[\hat{A}_z], e^{-W[\hat{A}_z]} = \int [dA_\bar{z}][db][dc]e^{-\Gamma[A_\bar{z}, \hat{A}_z, b, c]} \)

(B.27)

does not depend on the gauge, i.e. is independent of the value of \( \hat{A}_z \). Performing the integral over the ghost fields using again the same determinant as in eq. (B.14) yields

\[ \int [dA_\bar{z}]e^{-\Gamma[A_\bar{z}] + \frac{\kappa}{2\pi x} \int \{ A_\bar{z} \hat{A}_z \}} = e^{-(\kappa + 2\tilde{h})S^{-}[g]}, \]

(B.28)

where \( \hat{A}_z = \partial gg'^{-1} \). This corresponds to the value \( Z_K = 1/K \).

C  \( sl(2,\mathbb{R}) \) Embeddings

Consider an embedding of \( sl(2,\mathbb{R}) \) in a (super) Lie algebra \( \bar{g} \). The adjoint representation of \( \bar{g} \) branches into irreducible representations of \( sl(2,\mathbb{R}) \). For a given embedding, we denote the generators of \( \bar{g} \) by \( t_{(jm,\alpha_j)} \) where \( j, 2j \in \mathbb{N} \) labels the irreducible representation of \( sl(2,\mathbb{R}) \), \( m \) runs from \(-j\) to \( j \) and \( \alpha_j \) counts the multiplicity of the irreducible representation \( j \) in the branching. The \( sl(2,\mathbb{R}) \) generators \( e_\pm \) and \( e_0 \) are denoted by \( e_\pm \equiv t_{(1\pm1,0)}/\sqrt{2} \) and \( e_0 \equiv t_{(10,0)} \). The \( sl(2,\mathbb{R}) \) algebra is given by \( [e_0, e_\pm] = \pm 2e_\pm \) and \( [e_+, e_-] = e_0 \). The action of the \( sl(2,\mathbb{R}) \) algebra on the other generators is given by

\[
\begin{align*}
[e_0, t_{(jm,\alpha_j)}] &= 2m t_{(jm,\alpha_j)} \\
[e_+, t_{(jm,\alpha_j)}] &= (-)^{j+m} \sqrt{(j-m)(j+m+1)} t_{(jm+1,\alpha_j)} \\
[e_-, t_{(jm,\alpha_j)}] &= (-)^{j+m-1} \sqrt{(j-m+1)(j+m)} t_{(jm-1,\alpha_j)}
\end{align*}
\]

(C.1)
Computing $\text{str}([e_+, t_{(j-1/2)}][e_-, t_{(j+1/2)}])$ in two different ways, we conclude that, as the metric should be non-degenerate, bosonic generators of half-integer spin always occur in pairs. This reflects the fact that whenever bosonic multiplets of half-integer spin occur, the branching contains a $U(1)$ generator under which the bosonic multiplets with $j$ half-integer can be split into two subspaces having opposite eigenvalues under the action of the $U(1)$ symmetry. We use the notation $t_{(jm,\alpha_j,\beta_j)}$, where $\beta_j$ is the $U(1)$ eigenvalue. The generators are normalized such that

$$\text{str}\left(t_{(jm,\alpha_j,\beta_j)}t_{(j',m',\alpha'_j,\beta'_j)}\right) = (-)^{2j(j-m)}(-)^{(\beta_j)}4\delta_{j,j'}\delta_{m,m'}\delta_{\alpha_j\alpha'_j}\delta_{\beta_j+\beta'_j}, \quad (C.2)$$

where $(-)^{\beta_j} = +1 (-1)$ if $t_{(jm,\alpha_j)}$ for $t$ bosonic and $j$ half-integer has positive (negative) chirality under the $U(1)$ and $y$ is the index of embedding which is given by

$$y = \frac{1}{3\hbar}\sum_{j,\alpha_j}(-)^{(\alpha_j)}j(j+1)(2j+1) \quad (C.3)$$

and $(-)^{(\alpha_j)} = +1 (-1)$ if $t_{(jm,\alpha_j)}$ is bosonic (fermionic).

In section six, we need the subalgebras $\Pi_+\bar{g}$ and $\Pi_{\geq+1}\bar{g}$:

$$\Pi_+\bar{g} = \{t_{(jm,\alpha)}|m > 0; \forall j, \alpha\}$$

$$\Pi_{\geq+1}\bar{g} = \{t_{(jm,\alpha)}|m \geq 1; \forall j, \alpha\}, \quad (C.4)$$

and we define $\Pi_{+1/2}\bar{g}$ as

$$\Pi_{+1/2}\bar{g} = \{t_{(j^{1/2},\alpha)}|\forall j, \alpha\}. \quad (C.5)$$

Analogous definitions hold for $\Pi_-\bar{g}$, $\Pi_{\leq-1}\bar{g}$ and $\Pi_{-1/2}\bar{g}$. Furthermore, we introduce upper indices $B$ and $F$ to distinguish bosonic from fermionic generators, e.g. $\Pi^B_{+1/2}\bar{g} = \{t \in \Pi_{+1/2}\bar{g}|t$ bosonic$\}$ and $\Pi^F_{+1/2}\bar{g} = \{t \in \Pi_{+1/2}\bar{g}|t$ fermionic$\}$. Finally $\Pi_{+1/2}\bar{g}$ is decomposed according to the $U(1)$ chirality as $\Pi^B_{+1/2}\bar{g} = \Pi^B_{+1/2}\bar{g} + \Pi^B_{+1/2}\bar{g}$.

References

[1] A. M. Polyakov, Mod. Phys. Lett. A2 (1987) 893

[2] V.G. Knizhnik, A.M. Polyakov and A.B. Zamolodchikov, Mod.Phys.Lett. A3 (1988) 819

$\dagger$Throughout section 6, we will use the symbol $\Pi$ for a projection operator acting on the Lie algebra $\bar{g}$, the subindex indicates which part of the algebra survives.
[3] A. Alekseev and S. Shatashvili, Nucl. Phys. B323 (1989) 719

[4] M. Bershadsky and H. Ooguri, Comm. Math. Phys. 126 (1989) 49

[5] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, in Proceedings of the Jan. 1991 Miami Workshop on Quantum Field Theory, Statistical Mechanics, Quantum Groups and Topology, (Plenum, 1991)

[6] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Comm. Math. Phys. 124 (1989) 87

[7] A.B. Zamolodchikov, Theor.Math.Phys. 63 (1985) 1205

[8] P. Bouwknegt and K. Schoutens, preprint CERN-TH.6583/92 and ITP-SB-92-23, to be published in Phys. Rep.

[9] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B364 (1991) 584

[10] H. Ooguri, K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Comm. Math. Phys. 145 (1992) 515

[11] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, Nucl. Phys. B371 (1992) 315

[12] J. de Boer and J. Goeree, Utrecht preprint THU-92/33

[13] M. T. Grisaru and R. M. Xu, Phys. Lett. 205B (1988) 486.

[14] A. Polyakov and A. B. Zamolodchikov, Mod. Phys. Lett. A3 (1988) 1213.

[15] M. Bershadsky and H. Ooguri, Phys. Lett. 229B (1989) 374.

[16] V. G. Knizhnik, Theor. Math. Phys. 66 (1986) 68.

[17] M. Bershadsky, Phys. Lett. 174B (1986) 285.

[18] G.W. Delius, M.T. Grisaru and P. van Nieuwenhuizen, preprint CERN-TH.6458/92

[19] K. Schoutens, Nucl. Phys. B295[FS21] (1988) 634
A. Sevrin, W. Troost and A. Van Proeyen, Phys. Lett. B208 (1988) 447

[20] P. Goddard and A. Schwimmer, Phys. Lett. 214B (1988) 209

[21] M. Roček and A. Sevrin, in preparation.

[22] F. Delduc, E. Ragoucy and P. Sorba, Comm. Math. Phys. 146 (1992) 403.
[23] J.A. Batalin and G.A. Vilkovisky, Phys. Rev. **D28** (1983) 2567; **D30** (1984) 508; Nucl. Phys. **B234** (1984) 106.

[24] A. van Proeyen, preprint-KUL-TF-91/35, in the proceedings of “Strings and Symmetries 1991,” World Scientific; W. Troost, P. van Nieuwenhuizen and A. van Proeyen, Nucl. Phys. **B333** (1990) 727; W. Troost and A. Van Proeyen, *An introduction to Batalin-Vilkovisky Lagrangian Quantisation*, Leuven University Press, in preparation.

[25] V.G. Knizhnik and A.B. Zamolodchikov, Nucl. Phys. **B247** (1984) 83

[26] A. Sevrin, R. Siebelink and W. Troost, in preparation.

[27] A. Sevrin, K. Thielemans and W. Troost, preprint LBL-33778, UCB-PTH-93/07, KUL-TF-93/10

[28] A.B. Zamolodchikov, preprint ITEP 84-89 (1989)

[29] K.A. Meissner and J. Pawełczyk, Mod. Phys. Lett. **A5** (1990) 763

[30] F. A. Bais, T. Tjin and P. van Driel, Nucl. Phys. **B357** (1991) 632.

[31] L. Feher, L. O’Raifeartaigh, O. Ruelle, I. Tsutsui and A. Wipf, DIAS -STP -91, UdeM -LPN -TH -71/91

[32] A. Deckmyn, Phys. Lett. **298B** (1993) 318.

[33] B. L. Feigin and E. Frenkel, Phys. Lett. **246B** (1990) 75.

[34] J. de Boer and T. Tjin, preprint THU-92-32.

[35] J. de Boer and T. Tjin, preprint THU-93-05.

[36] R. Bott and L. W. Tu, *Differential Forms in Algebraic Topology*, Springer Verlag, 1986

[37] F.A. Bais, P. Bouwknegt, M. Surridge and K. Schoutens, Nucl. Phys. **B304** (1988) 348

[38] A. Sevrin, W. Troost, A. Van Proeyen and P. Spindel, Nucl. Phys. **B311** (1988) 465

[39] M. T. Grisatu, W. Siegel and M. Roček, Nucl. Phys. **B159** (1979) 429; *Superspace*, S. J. Gates, M. T. Grisatu, W. Siegel and M. Roček, Benjamin/Cummings pub. comp. 1983, p358.

[40] L. Frappat, E. Ragoucy and P. Sorba, preprint ENSLAPP-AL-391/92, july 1992.
[41] J. M. Figueroa-O’Farrill and S. Schrans, Phys. Lett. B257 (1991) 69

[42] A. Deckmyn, A. Sevrin and W. Troost, work in progress

[43] M. Ademollo et al., Phys. Lett. 62B (1976) 105; Nucl. Phys. B111 (1976) 77; Nucl. Phys. B114 (1976) 297

[44] P. Goddard, D. Olive and G. Waterson, Comm. Math. Phys. 112 (1987) 591.

[45] E. B. Dynkin, Amer. Math. Soc. Transl. 6 (1967) 111

[46] M. Bershadsky, Comm. Math. Phys. 139 (1991) 71

[47] M. Bershadsky, W. Lerche, D. Nemeschansky and N.P. Warner, preprint CERN-TH.6694/92 and W. Lerche, preprint CERN-TH.6812/93

[48] H. Verlinde, Nucl. Phys. B337 (1990) 652.

[49] J. de Boer and J. Goeree, preprint THU-92-14.

[50] M.V. Savelev and A.M. Vershik Comm. Math. Phys. 126 (1989) 367

[51] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. 131B (1983) 121; Phys. Lett. 141B (1984) 223.

[52] E. Witten, Comm. Math. Phys. 92 (1984) 455

[53] O. Alvarez, Nucl. Phys. B238 (1984) 61

[54] P. Di Vecchia and P. Rossi, Phys. Lett. 140B (1984) 344; P. Di Vecchia, B. Durhuus and J. L. Petersen, Phys. Lett. 144B (1984) 245.

[55] A. M. Polyakov, Int. Jour. Mod. Phys. A5 (1990) 833

[56] K. Schoutens, A. Sevrin and P. van Nieuwenhuizen, in the proceedings of the Stony Brook conference Strings and Symmetries 1991, (World Scientific, 1992).