Asymptotic critical behavior of holographic phase transition at finite topological charge - a new quantum phase transition at finite chemical potential

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The asymptotic critical behavior at large chemical potential of the conformal field living at the AdS boundary of a four-dimensional spacetime Einstein gravity is investigated analytically. The threshold values of the chemical potential for the appearance of condensate states are discrete, equal spacing, with the gap approaches zero logarithmically in the limit $T \to 0$. Remarkably, numerical results show that, this behavior applies even for states with low quantum number, as low as for the first or second excited states of the condensate, especially on the liquid side of the black hole van der Waals-like phase transition (small or zero topological charge). We postulate that, at the exact limit $T = 0$ where the gap is zero, the spectrum of threshold chemical potentials becomes continuous, all excited states of the condensate are activated above a finite chemical potential, suggesting a new quantum phase transition as a function of the chemical potential. Previous studies have largely missed this continuous spectrum of excited states in the $T \to 0$ limit.

We start from the model of an Abelian Higgs field and a Maxwell field in the four-dimensional spacetime Einstein gravity. The bulk action is given by,

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left( R - \frac{16}{L^2} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - |\nabla \Psi - i q A| \phi^2 - m^2 |\Psi|^2 \right),$$

where $G_N$ is the Newton constant. In the uncondensed phase, the solutions to Eq. (1) are the Reissner–Nordstrom black hole (BH).

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{2,k},$$

where

$$f(r) = k - \frac{2M}{r} + \frac{Q^2}{r^2} + \frac{r^2}{L^2}.$$  

The event horizon $r_0$ is the largest solution of the equation $f(r_0) = 0$. The BH Hawking temperature $T$ is:

$$T_{\text{Hawking}} = \frac{f'(r_0)}{4\pi},$$

which is also regarded as the temperature of the conformal field at the AdS boundary.

For $k > 0$, using the entropy $S = \pi r_0^2$, pressure $P = 3/8\pi L^2$, one discovered a total analogy between the small–large BH phase transition and the liquid–gas phase transition of the van der Waals (vdW) theory [1]. In this set up, $M$ becomes the enthalpy and $k$ is interpreted as the measure of a new charge, the topological charge [2]. For $k = 1$, $L < 6$ corresponds to the liquid side, while $L > 6$ corresponds to the gas side of the vdW transition [3].

Using the ansatz, $A_\mu = (\Phi(r), 0, 0, 0)$, $\Psi = \Psi(r)$, after minimizing the action with respect to $\Phi(r)$ and $\Psi(r)$, we arrive at the equations of motion for the fields:

$$\Psi'' + \left( \frac{2}{r} + \frac{f'}{f} \right) \Psi' + \left( \frac{q^2 \phi^2}{f^2} + \frac{2}{L^2 f} \right) \Psi = 0,$$

$$\Phi'' + \frac{2}{r} \Phi' - \frac{2q^2 |\Psi|^2}{f} \Phi = 0.$$  

At the AdS boundary $r \to \infty$,

$$\Phi(r \to \infty) = \mu - \frac{\rho}{r} + ...$$

$$\Psi(r \to \infty) = \frac{\Psi^{(1)}}{r} + \frac{\Psi^{(2)}}{r^2} + ...$$

We will work in the grand canonical ensemble and fix the chemical potential $\mu$ of the conformal field living at the boundary, $\rho$ is the corresponding density [4].

At large temperature, the system has only a “trivial” normal solution:

$$\Phi(r) = \mu \left( 1 - \frac{r_0}{r} \right),$$

$$\Psi(r) = 0.$$  

As the temperature decreases below a critical value $T_c$, the system permits a non-vanishing $\Psi(r)$ condensate solution. Depending on the value of the chemical potential, as the temperature decreases even further, other solutions become permissible, which are identified as the 1st excited state, 2nd excited state, ... of the condensate [5]. This work attempts to understand critical properties of these states, especially the asymptotic behavior at large chemical potential $\mu$.

Without losing generality, we set $Q = q = 1$. To simplify formulas, the coordinate is changed to $z = r_0/r$, ...
and dimensionless quantities are used:

\[ f(z) = \frac{\Psi'(0)}{\rho} = k \frac{2M}{z} + z^2 + \frac{1}{L^2 z_h^4}, \]

\[ \Phi(z) = \Phi(r = 1/z)/z_h^3, \]

\[ \Psi(z) = r \Psi(r = 1/z)/z_h^2, \]

here \( z_h = 1/r_0, T = 4 \pi T_{\text{Hawking}}/z_h^3 = -f'(1); \) \( \Psi^{(1)}, \Psi^{(2)}, \mu, \rho \) are also rescaled accordingly (not listed here to avoid crowding). In the rest of the paper, the dimensionless version of these quantities is always assumed, unless explicitly stated otherwise. The AdS boundary is now at \( z = 0, \) the BH event horizon is at \( z = 1. \) For convenience, the field \( \Psi(r) \) is not only rescaled, but also changed so that the boundary conditions at the AdS boundary are:

\[ \Psi(z \to 0) = \Psi^{(1)} + \Psi^{(2)} z + \cdots, \]

\[ \Phi(z \to 0) = \mu - \rho z + \cdots. \]

Because \( z = 1 \) is the smallest positive solution of \( f(z), \) the latter can be rewritten in a more compact form:

\[ f(z) = z^2 (z - 1) \left[ (z^2 + 2bz + b) - z^2T \right], \]

where \( b = 1/L^2 z_h^4. \) The dimensionless equations of motion are:

\[ \Phi'' + \frac{F'}{F} \Phi' + \frac{zF' - 2F + 2b}{z^2F} \Phi + \frac{\Phi^2}{F^2} = 0, \]

\[ \Phi'' - 2\Phi^2 \Phi/F = 0, \]

with \( F(z) = z^2 f(z). \) For condensate solution at low temperature, the expectation value of the operator, \( \langle O \rangle = \sqrt{2} \Psi^{(\delta)} (\delta = 1 \text{ or } 2) \) is non-zero. Analytical expressions are not available and one typically resorts to numerical solution. In Fig. 1 condensate solutions for \( k = 1, L = 1, \mu = 8, \) and \( T = 0.1T_c \) are shown as a typical example. As the temperature decreases below the critical temperature \( T_c = 1.699, \) the condensate ground state is available. Below the critical temperature \( T_c = 0.295, \) the first excited state is available, the field \( \Psi(z) \) has one node (zero) in the interval \((0, 1). \) Finally, below the critical temperature \( T_c = 0.0806, \) the second excited state is available, \( \Psi(z) \) has two nodes.

Furthermore, in numerical solutions, it is easy to overlook various hidden symmetries in the system. In this paper, we attempt to investigate behavior of the system near criticality using semi-analytical methods, which reveals several interesting features of the system.

The first question one asks is, given the chemical potential \( \mu, \) how many condensate solutions are permitted, and what are the corresponding critical transition temperatures? Near criticality, \( \langle O \rangle \) are small. One does not need to solve the full set of equations of motion to obtain the critical temperature \([6]. \) This analysis also provides a check on our formulation by comparing with previous results obtained by directly solving the full system of equations of motion.

Indeed, near criticality \( T \approx T_c, \) the condensate solution is very small, \( \langle O \rangle \approx 0 \) and \( \Phi(z) \approx \mu/(1-z). \) The equation for \( \Psi \) becomes:

\[ \Psi'' + \frac{F'}{F} \Psi' + \frac{zF' - 2F + 2b}{z^2F} \Phi + \frac{\mu^2(1-z)^2}{F^2} \Psi = 0, \]

Equation \([19] \) is to be solved with boundary conditions depending on the type of condensate. At a given \( \mu, \) the procedure for numerical calculation of the critical transition temperature \( T_c \) is following. Since the equation is linear, one sets \( \Psi(1) = 1, \Psi'(1) = (2b-T)/T. \) Numerically integrating the equation from \( z = 1 \) to \( z = 0, \) one obtains unique boundary values \( \Psi(0) = \Psi^{(1)}(\mu,T) \) and \( \Psi'(0) = \Psi^{(2)}(\mu,T). \) For \( \langle O \rangle \) condensate, for example, by varying \( T, \) every time \( \Psi^{(2)}(\mu,T) \) crosses zero, the corresponding \( T \) is the critical temperature of different states of the condensate.

As an example, Fig. 2 shows \( \Psi^{(2)}(\mu,T) \) as function of \( T \) for the case \( k = 1, L = 1, \) for two different values of the chemical potential \( \mu = 0.3 \) and 8. At \( \mu = 0.3, \) \( \Psi^{(2)}(\mu,T) \) never crosses zero. This means that the system is always in the normal state, no phase transition at finite \( T. \) At \( \mu = 8, \) \( \Psi^{(2)}(\mu,T) \) crosses zero three times. This means that, at this chemical potential, the system goes through phase transition to the ground, 1st excited, and 2nd excited states at \( T = 1.699, 0.295, \) and 0.0806 respectively. These values exactly match critical temperatures.
obtained by directly solving the full system of equations of motion [7].

As demonstrated, for small $\mu$, the system permits no condensate solutions. As $\mu$ increases beyond a threshold $\mu_0$, a condensate solution is permitted below certain temperature $T_c$ where the expectation value of the operator is non-zero, $\langle O_{\delta} \rangle \neq 0$. As $\mu$ increases further, more solutions are permitted at lower $T_c$ which one identifies as excited states of the condensate. Therefore, the next question one would like to ask is, at which threshold value $\mu_0$, the ground state solution is available; at which $\mu_1$, the first excited state solution is available, ... .

Clearly, at each threshold $\mu$ where a new solution (new condensate state) appears, its critical temperature $T_c$ is zero. As $\mu$ increases, $T_c$ increases from zero. Therefore, we seek to investigate the criticality $\Psi(z)$ equation, Eq. [19], in the limit $T \rightarrow 0$. Before going to analytical discussions, let us give typical examples of threshold $\mu$ behaviors obtained from numerical solution. Fig. 3 shows the ground state and $n - 1$ excited states. For $\mu < \mu_0$, no non-trivial solutions are available, the system cannot undergo phase transition and the normal state prevails all the way to $T = 0$ limit. The rest of this paper concerns with explicit estimates for these eigenvalues.

We introduce new coordinate $x$ and field $\omega(x)$:

$$x = \int_0^z ds \left(1 - s/F(s)\right),$$  \hspace{1cm} (22)

$$\omega(x) = \Psi(z)\sqrt{1 - z},$$  \hspace{1cm} (23)

the $S$–$L$ equation becomes,

$$-\omega''(x) + V(x) \omega(x) = \mu^2 \omega(x),$$  \hspace{1cm} (24)

which is exactly the one-dimensional Schrödinger equation for a unit mass particle moving in the “potential”:

$$V(z(x)) = -\frac{z(2z - 1) + 2b - T}{(1 - z)^2}$$

The Schrödinger equation is to be solved with appropriate boundary conditions. Because $F(z) > 0$, the relation, Eq. (22), gives one-to-one mapping between $x$ and $z$. The interval $z \in [0, 1)$ maps into $x \in [0, c]$ where

$$c = \int_0^1 ds \left(1 - s/F(s)\right),$$  \hspace{1cm} (26)

is positive and finite at finite $T$, and grows logarithmically with $1/T$ as $T \rightarrow 0$. For analytical solution, it is more appropriate to specify boundary conditions at the regular $x = 0$ boundary, requiring only that $\Psi(1)$ is finite (remind that we are working with a linear equation). Thus, $\omega(c) = 0$, a common physical requirement for Schrödinger equations at singular boundaries.

Boundary conditions at $x = 0$ depend on the type of condensate operator one wishes to study:

$$\omega(0) = \Psi(0) = 1 \text{ or 0},$$

$$\omega'(0) = b[\Psi'(0) - \Psi(0)/2] = -b/2 \text{ or } b.$$  \hspace{1cm} (28)

for $\langle O_1 \rangle$ or $\langle O_2 \rangle$ condensate.

With the familiar Schrödinger Eq. [24], we understand much better physical behaviors of the system using known results from non-relativistic quantum mechanics. Fig. 4 shows the typical structure of the potential $V(x)$. From this, we can make many qualitative discussions. Firstly, near the singular boundary $x = c$,

$$V(x \rightarrow c) \simeq -\frac{(5b - 2T - 1)/4}{c-x} - \frac{1/4}{(c-x)^2}.$$  \hspace{1cm} (29)

The dominant term near $x = c$ is the famous $-\alpha/x^2$ potential [8]. It is well-known that if the range of $x$ is
infinite, the solution is unstable (not bounded from below, no ground state) unless \( \alpha < 1/4 \) and, when this is the case, all eigenvalues are positive and the spectrum is continuous. In our case, \( \alpha \) is exactly the limiting value 1/4. This exact coincidence is not well understood yet. It acts as a topological constant, inherent in the geometric structure of the BH spacetime.

Because \( \alpha = 1/4 \), rigorous mathematical study [9] shows that our equation can be regularized to an equivalent regular S–L one. All general properties of the regular S–L equation are applicable in our case. Most importantly, the spectrum and eigenfunctions are of the type given in Eq. (21). The regularization requires introduction of regularization parameters. However, they are regularization-scheme dependent, irrelevant to our problem, and do not influence physical quantities.

Secondly, one sees from Fig. 4 that \( V(x) \) is bounded from above, \( V(x) \leq V_{\text{max}} \). Therefore, one can obtain analytical expressions in the limit of asymptotically large eigenvalues \( \mu_n \). At \( \mu^2 \gg V_{\text{max}} \), our equation becomes:

\[
\omega_{xx} + \mu^2 \omega = 0. \tag{30}
\]

This is the Schrödinger equation for a free particle in an infinite square-well potential. The general solution is \( \omega(x) = Ae^{ix} + Be^{-ix} \). Since the range of \( x \) is finite, the wave number is quantized:

\[
\mu_n = \frac{n \pi}{c} + \text{const.} \tag{31}
\]

This is our most important result. To the leading order, this spectrum does not depend on specific potential \( V(x) \) or specific boundary conditions. These details only contribute a small (position-dependent) phase shift in the eigenfunctions, and is different for different types of condensate.

Remind that \( c \) diverges logarithmically with \( 1/T \). As \( T \to 0 \), Eq. (31) shows that the spectrum of eigenvalues with high quantum number \( n \) remains discrete, but the gap between \( \mu_n \) shrinks to an infinitesimal value. This explains non-saturation behaviors observed in Fig. 3. Surprisingly, this asymptotic behavior seems to apply even for relatively low \( n \) in those figures, as low as \( n \geq 2 \). Note

that, for \( L = 9 \) case (the gas side of the vdW BH transition), threshold values \( \mu \) for three lowest energy states immediately saturate, the gap between them does not shrink with decreasing \( T \). On the other hand, for \( L = 1 \) case (the liquid side of the vdW BH transition), the gap between the 1st excited state and the ground state keeps shrinking. This might correlate with the fact that we do not see gap in the conductivity at low temperature for excited states in the gas side of the vdW transition [7].

The consequence of this observation is more profound. We postulate that at the exact limit \( T = 0 \) where the gap at large \( n \) is zero, all higher excited states of the condensate are activated beyond a finite chemical potential (approximately \( \mu_1 \) or \( \mu_2 \) depending on liquid or gas side of the BH transition), suggesting a potentially new quantum phase transition as function of the chemical potential. This continuous spectrum at the \( T \to 0 \) limit has largely been missed in previous studies of the excited states of the condensate.

Finally, Eq. (29) shows that the range of the singular \( -1/x^2 \) term is within \( 1/(5b - 1) \) of the “origin” \( x = c \) [10]. As \( T \to 0 \), \( c \to \infty \), this localized region becomes parametrically small. Therefore, we regard this term as a singular potential that “pins” the wave function to zero at the origin, \( \omega(c) = 0 \). Away from this parametrically small region, the Coulomb-like \(-1/x \) term in Eq. (29) dominates. Its spectrum is [11]:

\[
\mu_n^2 \approx -(5b - 1 - 2T)^2/64n^2. \tag{32}
\]

To avoid negative values for \( \mu^2 \) in the ground state in the original problem, one needs the constant shift \( V_{\text{max}} \geq (5b - 1)^2/64 \), which is satisfied for all \( L, k \) parameters [10], thus there are no negative eigenvalues in our problem.

As one last comment, let us study the system below but close to the critical temperature to obtain the critical exponent. Setting \( \Psi_n(z) = \langle O_\delta \rangle z^{(\delta - 1)\eta_n(z)} \), from the equation for \( \Phi(z) \) we have

\[
\Phi'' = 2\Psi_n^2 \Phi/F = 2 \langle O_\delta \rangle^2 z^{2(\delta - 1)\eta_n^2} \Phi/F. \tag{33}
\]

Near \( T_c \), \( \langle O_\delta \rangle \) is very small, we expand in this parameter:

\[
\Phi(z) \simeq \mu(1 - z) + \langle O_\delta \rangle^2 \chi(z), \tag{34}
\]

with \( \chi(0) = \chi(1) = 0 \), \( \chi(z) < 0 \). Substituting into Eq. (33), keeping only the leading terms, one gets

\[
\chi(z) = \chi'(0)z + \mu \int_0^z dz_1 \int_0^{z_1} ds \frac{2s^{2(\delta - 1)}\eta_n^2(s)}{F(s)/(1-s)} \tag{35}
\]

The condition \( \chi(1) = 0 \) gives

\[
\chi'(0) = -C_n = \mu \int_0^1 dz_1 \int_0^{z_1} ds \frac{2s^{2(\delta - 1)}\eta_n^2(s)}{F(s)/(s - 1)}. \tag{36}
\]
By this definition, the constant $C_n$ is finite and positive. From the limit $\Phi(z \to 0) \simeq \mu - \rho z$, we have

$$\mu - \rho z = \mu(1 - z) - \langle O_\delta \rangle C_n z,$$

(37)

$$\langle O_\delta \rangle \simeq \sqrt{\mu/C_n} (1 - T/T_c)^{1/2}.$$

(38)

Here we use the fact that near $T_c$, $\rho \simeq \mu T_c/T$. Eq. (38) shows that the condensates obey universal mean-field critical exponent $1/2$ for all ground and excited states.

In conclusion, by considering holographic phase transition near criticality, excited states of the condensate of the conformal field living at the AdS boundary at large quantum number are investigated. The spectrum of threshold chemical potentials of the states are discrete, asymptotically equal spacing, with the gap between consecutive values shrinks to zero logarithmically in the limit $T \to 0$. Surprisingly, numerical solution shows that this asymptotic behavior is valid for the quantum number $n$ as low as 2 or 3 depending on the nature of the small—large BH vdW-like transition. We postulate that, at the exact limit $T = 0$ where the gap vanishes, all excited states are activated beyond a finite chemical potential, suggesting a potentially new quantum phase transition, a fact which has been largely missed in previous studies of excited states of the condensate.

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[1] M. Ammon and J. Erdmenger, *Gauge/gravity duality: Foundations and applications* (Cambridge University Press, 2015).
[2] D. Kubizňák, R. B. Mann, and M. Teo, Class. and Quantum Gravity 34, 063001 (2017).
[3] T. H. Phat and T. T. Nguyen, Eur. Phys. J. C 81, 428 (2021).
[4] C. Herzog, P. Kovtun, and D. Son, Phys. Rev. D 79, 066002 (2009).
[5] Y.-Q. Wang, T.-T. Hu, Y.-X. Liu, J. Yang, and L. Zhao, J. High Energy Phys. 2020, 13.
[6] G. Siopsis and J. Therrien, J. High Energy Phys. 2010 (5), 1.
[7] T. H. Phat and T. T. Nguyen, On the triplet of holographic phase transition (2021), arXiv:2110.08757 [hep-th].
[8] A. M. Essin and D. J. Griffiths, Am. J. Phys. 74, 109 (2006).
[9] A. Zettl, *Sturm-Liouville theory*, 121 (American Mathematical Soc., 2010).
[10] There is an interesting condition $5b > 1$ or $Lk < 2/\sqrt{5}$. For $Lk > 2/\sqrt{5}$, the Coulomb term is repulsive.
[11] H. N. Yepez, C. Vargas, and A. S. Brito, Eur. J. Phys. 8, 189 (1987).