Congestion-Free Rerouting of Flows on DAGs

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Abstract. We initiate the theoretical study of a fundamental practical problem: how to schedule the congestion-free rerouting of $k$ flows? The input to our problem are $k$ path pairs: for each of the $k$ unsplittable flows (of a certain demand), there is an old and a new path along which the flow should be routed. As different flows can interfere on the physical links, the updates of the different flows at the different nodes must be scheduled such that transient congestion is avoided. This optimization problem finds immediate applications, e.g., in traffic engineering in computer networks. We show that the problem is generally NP-hard already for $k = 2$ flows. Interestingly, we find that for general $k$, deciding whether an unsplittable multi-commodity flow rerouting schedule exists, is even NP-hard on DAGs. Both NP-hardness proofs are non-trivial. We then present two polynomial-time algorithms to solve the route update problem for a constant number of flows on DAGs. Both algorithms employ a decomposition of the flow graph into smaller parts which we call blocks. Based on the given block decomposition, we define a dependency graph whose properties can be leveraged to compute an optimal solution for $k = 2$ flows. For arbitrary but fixed $k$, we introduce a weaker dependency graph and present our main contribution: an elegant linear-time algorithm which solves the problem in time $2^{O(k \log k)} O(|G|)$.

1 Introduction

While congestion-aware routing problems have been studied intensively [1,6,7,8,13,14,15,17], surprisingly little is known today about how to update the routes of flows in a capacitated network in a congestion-free manner. Flow rerouting problems are not only natural and of fundamental theoretical interest, but also practically very relevant, e.g., in the context of traffic engineering problems. For two reasons: First, we currently witness a renaissance of routing based on unsplittable flows: traffic engineering mechanisms based on RSVP-TE [3] or OpenFlow resp. Software-Defined Networks (SDNs) [5,18] are

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based on unsplittable flows, to provide a better quality-of-service (compared to traditional packet-switched networks). Second, real networks are dynamic, and the ability to update routes is essential in a number of scenarios: to account for changes in the security policies, in response to new route advertisements, during maintenance (e.g., replacing a router), to support the migration of virtual machines, etc. With the advent of more programmable computer networks [9], traffic engineering is likely to become even more dynamic and adaptive, entailing more frequent route updates [10].

However, the migration of flows from their old (simple) paths to the given new (simple) paths can be challenging, due to the asynchrony introduced by imprecisions of clock synchronization protocols as well as by variations in the update times of the internal data structures of the router’s forwarding tables [12]. In particular, updates simultaneously sent out by a logically centralized SDN controller or updates scheduled for a certain time at a certain router, can take effect asynchronously. Accordingly, to avoid transient inconsistencies, the updates of the different flows at the different routers must be scheduled carefully: potentially conflicting updates should be scheduled at different times or rounds.

Figure 1 presents an example of the congestion-free rerouting problem of unsplittable flows considered in this paper.

In this paper we present a succinct definition of consistency which generalizes well-studied properties such as congestion-freedom [5] and loop-freedom [2,11,16]. We make the following technical contributions:

1. We present a non-trivial proof that deciding whether a consistent network update schedule exists is NP-hard, already for 2 flows.
2. We prove that the problem is even NP-hard on loop-free networks, i.e., on DAGs, for general $k$.
3. We present a deterministic and polynomial-time algorithm for updating $k = 2$ flows on DAGs, whenever a feasible schedule exists. The algorithm is based on a decomposition of the flow graph into so-called blocks. Using these blocks, we introduce a dependency graph on which an optimal flow migration schedule can be found efficiently.
4. For constant $k > 2$, the block decomposition of the flow graph is still useful, however, a weaker dependency graph is described. Based on this dependency graph, we present an elegant linear-time algorithm which solves the problem in time and space $2^{O(k \log k)}O(|G|)$.

2 Model and Preliminaries

We assume basic familiarity with directed graphs and we refer the reader to [4] for further reading. For a set of integers $1, \ldots, k$, we will write $[k]$. We denote a directed edge $e$ with head $v$ and tail $u$ as $e = (u, v)$. For an undirected edge $e$ between vertices $u, v$, we write $e = \{u, v\}$; $u, v$ are called endpoints of $e$.

We are given a flow network: a directed capacitated graph $G = (s, t, V, E, c)$, where $s$ is the source, $t$ the terminal, $V$ is the set of vertices with $s, t \in V$, 
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Fig. 1. Example: The congestion-free flow update problem for \( k = 2 \) \((s,t)\)-flows, red and blue, each of demand 1. Each of the two flows is initially routed along a simple path (indicated as a solid line of the respective color), and needs to be migrated to a (simple) update path (indicated as a dashed line of the respective color). Each edge in the flow graph is annotated with its current load (top) and its capacity (bottom). Note that the flow graph is a DAG. To reroute the flows from their old to their new paths in a congestion-free manner, the forwarding rules (i.e., the out-edges) of the two flows at the different nodes need to be scheduled carefully. In particular, as links \((s, w)\) and \((v, t)\) are of capacity 1 and initially saturated, an update dependency is introduced. This dependency can be expressed in terms of flow blocks: we can decompose the red (resp. the blue) flow into red (resp. blue) blocks where the old and the new flow meet. In our case, there is one red block \( r_1 \) (the old and the new path only meet at \( t \) again) and two blue blocks \( b_1 \) and \( b_2 \) (the old and the new flow first meet at \( w \) and then again at \( t \)). We observe that \( b_2 \) can only be updated after \( r_1 \) has been updated; similarly, \( r_1 \) can only be updated after \( b_1 \) has been updated. An update scheduling respecting these dependencies can be constructed as follows. We can first prepare the blocks by updating the following two out-edges which currently do not carry any flow (we use the notation \((v_i, \text{col})\) to denote an update of the rule of color “\text{col}” at node \( v_i \): \((w, \text{red})\), \((u, \text{blue})\), and \((v, \text{blue})\). Subsequently, the three blocks can be updated in a congestion-free manner in the following order: Prepare the update for all blocks in the first round. Then, update \( b_1 \) in the second round, \( r_1 \) in the third round, \( b_2 \) in the fourth round.

\[ E \subseteq V \times V \] is a set of ordered pairs known as edges, and \( c: E \rightarrow \mathbb{N} \) a capacity function assigning a capacity \( c(e) \) to every edge \( e \in E \).
An \((s,t)\)-flow \(F\) of capacity \(d \in \mathbb{N}\) is a directed path from \(s\) to \(t\) in a flow network such that \(d \leq c(e)\) for all \(e \in E(F)\).\(^1\) Given a family \(\mathcal{F}\) of \((s,t)\)-flows \(F_1, \ldots, F_k\) with demands \(d_1, \ldots, d_k\) respectively, we call \(\mathcal{F}\) a valid network flow, or simply valid, if \(c(e) \geq \sum_{i \in \mathcal{F}} d_i\).

An update flow pair \(F = (F^0, F^1, \alpha_F, d)\) consists of two \((s,t)\)-flows \(F^0\), the old flow, and \(F^1\), the update \((or \new)\) flow, each of demand \(d\) and coming with an activation label \(\alpha_F\): the label indicates whether a given edge of \(E(F^0 \cup F^1)\) is active or inactive.

Given any update flow pair \(F = (F^0, F^1, \alpha_F)\) with activation label \(\alpha_F\) and demand \(d\), we define the graph \(\alpha_F(F) = (V(F^0 \cup F^1), \{e \in E(F^0 \cup F^1) \mid e \text{ is active}\})\). The flow pair \(F\) is called transient, if \(\alpha_F(F)\) contains a unique \((s,t)\)-flow \(U_F\). If there is a family \(\mathcal{F} = \{F_1, \ldots, F_k\}\) of update flow pairs with demands \(d_1, \ldots, d_k\) respectively, we call \(\mathcal{F}\) a transient network flow, iff. the family \(\{U_{F_1}, \ldots, U_{F_k}\}\) is valid.

An update flow network now is a graph \(G = (s,t, V, E, \mathcal{F}, c)\), where \((s,t, V, E, c)\) is a flow network, and \(\mathcal{F} = \{F_1, \ldots, F_k\}\) with \(F_i = (F_i^0, F_i^1, \alpha_{F_i}, d_i)\) a family of update flow pairs, \(V = \bigcup_{i \in [k]} V(F_i^0 \cup F_i^1)\) and \(E = \bigcup_{i \in [k]} E(F_i^0 \cup F_i^1)\). We call \(G\) initial, if all edges of \(F_i^0\) are active and all edges of \(F_i^1\) that do not belong to \(F_i^0\), are inactive for all \(i \in [k]\). Similarly an update flow network \(G\) is called complete, if all edges of \(F_i^0\) are active, and all edges of \(F_i^1\) that do not belong to \(F_i^0\), are inactive for all \(i \in [k]\).

Updating Flows: Given an update flow network \(G = (s,t, V, E, \mathcal{F}, c)\), an update is a tuple \(\mu = (v, F_i) \in V \times \mathcal{F}\): the update is resolved by activating all edges \(e \in F_i^0 - F_i^1\) that have \(v\) as a tail, hence setting \(\alpha_{F_i}(e)\) to active, and deactivating all \(e' \in F_i^0 - F_i^1\) that have \(v\) as a tail. An update sequence \(\rho = (\rho_1, \ldots, \rho_\ell)\) is a partition of \(V \times \mathcal{F}\) into sets \(\rho_i, i \in [\ell]\), such that the graph \(G_{\rho_i}\) is an initial update flow network, the graph \(G_{\rho_{\ell}}\) is an update flow network obtained from \(G_0\) by resolving all updates in \(\bigcup_{i \leq \ell} \rho_i\), and \(G_\ell\) is complete.

We call \(\rho_i, i \in [\ell]\), a round. Given an update sequence \(\rho\), we denote the round in which some update \((v, F) \in V \times \mathcal{F}\) is resolved by \(\rho(v, F)\). Since the rounds are totally ordered, we will slightly abuse the notation and refer to rounds by their indices. An update sequence \(\rho\) is valid, or feasible, if every round \(\rho_i\) obeys the following consistency rule:

**Consistency Rule:** Given \(S \subseteq \rho_i\), let \(F_i^S\) be the family of update flow pairs of the update flow network \(G_i^S\), which is obtained from \(G_{i-1}\) by resolving all updates in \(S\). We require that for any \(S \subseteq \rho_i\), the family \(\{U_F \mid F \in F_i^S\}\) is a transient network flow.

Note that while succinct, this consistency rule models and consolidates the fundamental properties usually studied in the literature, such as congestion-freedom [5] and loop-freedom [16].

Note that we do not forbid edges \(e \in E(F_i^0 \cap F_i^1)\) and we never activate or deactivate such an edge. Starting with an initial update flow network these

\(^1\) The assumption that flows have the same \(s\) and \(t\) is for ease of presentation and comes with no loss of generality.
edges will be active and remain so until all updates are resolved. Hence there are vertices \( v \in V \) with either no outgoing edge for a given flow pair \( F \) at all, or it has an outgoing edge, but this edge is used by both the old and the update flow of \( F \). We will call such updates \((v, F)\) empty.

Empty updates do not have any impact on the actual problem since they never change any transient flow. Hence they can always be scheduled in the first round and thus w.l.o.g. we can ignore them in the following.

We can now define the \( k \)-network flow update problem: Given an initialized update flow network \( G \) with \( k \) update flow pairs, is there a feasible update sequence \( \rho \)? The corresponding optimization problem is: What is the minimum \( \ell \) such that there exists a valid update sequence \( \rho \) using exactly \( \ell \) rounds?

We conclude this section by introducing some more preliminaries. Let \( G = (s, t, V, E, F, c) \) be an acyclic update flow network, i.e., we assume that the graph \((V, E)\) is acyclic. Let \( \prec \) be a topological order on the vertices \( V = \{v_1, \ldots, v_n\} \). Let \( F_i = (F_i^o, F_i^u, \alpha_i, d) \) be an update flow pair and let \( v_1^i, \ldots, v_{\ell_i}^i \) be the induced topological order on the vertices of \( F_i^o \); analogously, let \( a_1^i, \ldots, a_{\ell_i}^i \) be the order on \( F_i^u \). Furthermore, let \( V(F_i^o) \cap V(F_i^u) = \{z_1^i, \ldots, z_{k_i}^i\} \) be ordered by \( \prec \) as well. The subgraph of \( F_i^o \cup F_i^u \) induced by the set \( \{v \in V(F_i^o \cup F_i^u) \mid z_1^i \prec v \prec z_{j+1}^i, j \in [k_i - 1]\} \) is called the \( j \)th block of the update flow pair \( F_i \), or simply the \( j \)th \( i \)-block. We will denote this block by \( b_j^i \).

For a block \( b \), we define \( \mathcal{S}(b) \) to be the start of the block, i.e., the smallest vertex w.r.t. \( \prec \); similarly, \( \mathcal{E}(b) \) is the end of the block: the largest vertex w.r.t. \( \prec \).

Let \( G = (s, t, V, E, F, c) \) be an update flow network with \( F = \{F_1, \ldots, F_k\} \) and let \( B \) be the set of its blocks. We define a binary relation \( \prec \) between two blocks as follows. For two blocks \( b_1, b_2 \in B \), where \( b_1 \) is an \( i \)-block and \( b_2 \) a \( j \)-block, \( i, j \in [k] \), we say \( b_1 \prec b_2 \) (\( b_1 \) is smaller than \( b_2 \)) if one of the following holds.

1. \( \mathcal{S}(b_1) \prec \mathcal{S}(b_2) \),
2. if \( \mathcal{S}(b_1) = \mathcal{S}(b_2) \) then \( b_1 \prec b_2 \), if \( \mathcal{E}(b_1) \prec \mathcal{E}(b_2) \),
3. if \( \mathcal{S}(b_1) = \mathcal{S}(b_2) \) and \( \mathcal{E}(b_1) = \mathcal{E}(b_2) \) then \( b_1 \prec b_2 \), if \( i < j \).

Let \( b \) be an \( i \)-block and \( F_i \) the corresponding update flow pair. For a feasible update sequence \( \rho \), we will denote the round \( \rho(\mathcal{S}(b), F_i) \) by \( \rho(b) \). We say that \( i \)-block \( b \) is updated, if all edges in \( b \cap F_i^o \) are active and all edges in \( b \cap F_i^u \setminus F_i^u \) are inactive.

### 3 NP-Hardness of 2-Flow Update in General Graphs

It is easy to see that for an update flow network with a single flow pair, feasibility is guaranteed.

**Theorem 1.** The 2-flow network update problem is NP-hard.

The proof is by reduction from 3-SAT. In what follows let \( C \) be any 3-SAT formula with \( n \) variables and \( m \) clauses. We will denote the variables as \( X_1, \ldots, X_n \).
and the clauses as $C_1, \ldots, C_n$. The resulting update flow network will be denoted as $G(C)$. Furthermore, we will assume that the variables are ordered by their indices and their appearance in each clause respects this order.

We will create 2 update flow pairs, a blue one $B = (B^O, B^U, \alpha_B, 1)$ and a red one $R = (R^O, R^U, \alpha_R, 1)$. The pair $B$ will contain gadgets corresponding to the variables. The order in which the edges of each of those gadgets are updated will correspond to assigning a value to the variable. The pair $R$ on the other hand will contain gadgets representing the clauses: they will have edges that are “blocked” by the variable edges of $B$. Therefore, we will need to update $B$ to enable the updates of $R$.

We proceed by giving a precise construction of the update flow network $G(C)$. In the following, the capacities of all edges will be 1. Since we are working with just two flows and each of those flows contains many gadgets, we give the construction of the two update flow pairs in terms of their gadgets.

1. **Clause Gadgets:** For every $i \in [n]$, we introduce eight vertices $u^i_1, u^i_2, \ldots, u^i_8$ corresponding to the clause $C_i$. The edges $(u^i_j, u^i_{j+1})$ with $j \in [7]$ are added to $R^O$ while the edges $(u^i_{j'}, u^i_{j'+3})$ for $j' \in \{1, 2, 3\}$ and $(u^i_{j'}, u^i_{j'-4})$ for $j' \in \{6, 7\}$ are added to $R^U$.

2. **Variable Gadgets:** For every $j \in [n]$, we introduce four vertices: $v^j_1, \ldots, v^j_4$.

   Let $P_j = \{p^j_1, \ldots, p^j_{k_j}\}$ denote the set of indices of the clauses containing the literal $x_j$ and $\overline{P}_j = \{\overline{p}^j_1, \ldots, \overline{p}^j_{k_j}\}$ the set of indices of the clauses containing the literal $\overline{x}_j$. Furthermore, let $\pi(i, j)$ denote the position of $x_j$ in the clause $C_i$, $i \in P_j$. Similarly, $\overline{\pi}(i', j)$ denotes the position of $\overline{x}_j$ in $C_{i'}$ where $i' \in \overline{P}_j$.

   To $B^O$ we now add the following edges for every $j \in [n]$:  
   
   i) $(u^\pi(i, j), u^\pi(i, j)+5)$, for $i \in P_j$ (these edges are shared with $R^U$),  
   
   ii) $(u^\pi(i, j)+5, u^\pi(i+1, j))$, for $i \in P_j$, $i \neq k_j$,  
   
   iii) $(v^j_1, u^\pi(i,j))$ and $(u^\pi(p^j_1,j)+5, v^j_2)$,  
   
   iv) $(u^\pi(i,j), u^\pi(i+1, j))$, for $i \in \overline{P}_j$,  
   
   v) $(u^\pi(p^j_1,j)+5, u^\pi(p^j_{i+1},j))$, for $i \in [\overline{P}_j] - 1$,  
   
   vi) $(v^j_3, u^\pi(i,j))$ and $(u^\pi(p^j_1,j)+5, v^j_4)$, and  
   
   vii) $(v^j_2, v^j_3)$.  

   On the other hand, $B^O$ will contain the edges $(v^j_1, v^j_2)$, $(v^j_3, v^j_4)$ and $(v^j_2, v^j_4)$.

3. **Blocking Edges:** The goal is to block the updates $(v^j_2, B)$ for every $j \in [n]$ until all clauses are satisfied. To do this, we introduce 4 additional vertices $w_1, w_2, z_1$ and $z_2$. Then for $R^O$, we introduce the following edges:  
   
   i) $(v^j_2, v^j_2)$ for $j \in [n]$,  
   
   ii) $(v^j_2, v^j_{2+1})$ for $j \in [n-1]$, and  
   
   iii) $(z_1, v^j_3)$ and $(v^j_2, z_2)$,
Fig. 2. Examples: A clause gadget is shown in red, the \( R^o \) edges are depicted as a solid line, and the dashed lines belong to \( R^u \). The variable gadget is shown in blue. Again, solid lines indicate the old flow and dashed lines the update flow.

while \( R^u \) contains the edges \((z_1), (w_1, w_2)\) and \((w_2, z_2)\).

In a similar fashion, \( B^o \) contains the edge \((w_1, w_2)\). For \( B^u \), we introduce the following edges:

i) \((u^i_1, u^i_2)\) for \(i \in [m]\),

ii) \((u^i_1, u^{i+1}_2)\) for \(i \in [m-1]\), and

iii) \((w_1, u^1_2)\) and \((u^n_2, w_2)\).

Fig. 3. The gadget for blocking the update \((v^j_1, B)\) for all \(j \in [n]\). Again dashed edges correspond to the update flows and solid ones to the old flows.

4. **Source and Terminal.** Finally, to complete the graph, we introduce a source \(s\) and a terminal \(t\).

For both, \( R^o \) and \( R^u \) we introduce the following edges:

i) \((s, z_1)\) and \((z_2, u^1_1)\),

ii) \((u^i_3, u^{i+1}_2)\) for \(i \in [m-1]\), and

iii) \((u^n_2, t)\).

And for \( B^o \) and \( B^u \) we complete the flows with the following edges:

i) \((s, w_1)\) and \((w_2, v^1_1)\),

ii) \((v^j_1, v^{j+1}_1)\) for \(j \in [n-1]\), and

iii) \((v^n_1, t)\).

**Lemma 1.** Given any valid update sequence \(\rho\) for the above constructed update flow network \(G(C)\), the following conditions hold for every round \(r < \rho(w_1, B)\).
1. $r < \rho(z_1, R)$
2. For any $j \in [n]$, $v^1_j$ is a vertex of the transient network flow $U_B$ and $\rho(v^1_j, B) > r$.
3. Let $j \in [n]$, then $P_j$ and $\overline{P_j}$ are the index sets of the clauses containing the corresponding literals $x_j$ and $\overline{x_j}$. Then $U_B$ contains all edges of the form $(u^1_{\pi(i,j)}, u^1_{\pi(i,j)+5})$ for $i \in P_j$, or all the edges $(u^1_{\pi(i,j)}, u^1_{\pi(i,j)+5})$ for $i \in \overline{P_j}$ (or both).
4. The vertex $z_1$ and the $w^1_i$, for all $i \in [m]$, are contained in $U_R$.

Proof. 1. Suppose $\rho(z_1, R) \leq r$, then there is a set of updates $S \subseteq \rho(u_1, B)$ such that $(u_1, B)$ is not resolved in $G^S_{\rho(u_1, B)}$, but $(z_1, R)$ is. If $(w_1, R) \in S$, $U_R$ and $U_B$ pass through $(w_1, w_2)$ violating the capacity of 1, otherwise there is no path $U_R$ in $G^S_{\rho(u_1, B)}$.
2. The first assertion is trivially true, since the edges $(w_2, v^1_1)$ and $(v^1_i, v^1_{i+1})$ for $j \in [n-1]$ belong to both $B^o$ and $B^u$, hence $U_B$ has to always contain these edges. From Property 1 we know, that $U_R$ has to contain the $z_1-z_2$-subpath of $R^o$ and thus $U_R$ fills the capacity of the edges $(v^1_i, v^1_{j})$ for all $j \in [n]$; hence resolving $(v^1_i, B)$ for all $i \in [n]$ is impossible.
3. Let $j \in [n]$. By Property 2, $v^1_j$ is contained in $U_B$, but $\rho(v^1_j, B) > r$. Hence, if $\rho(v^1_j, B) \leq r$, then $U_B$ traverses directly from $v^1_j$ to $v^1_k$ and then follows along $B^u$ to $v_4$. Otherwise it follows along $B^o$ from $v^1_j$ to $v^1_k$. In both cases we are done.
4. This is again trivially true, since the edges $(s, z_1)$ and $(u^o_k, u^o_{i+1})$ for $i \in [m-1]$ are contained in both $R^o$ and $R^u$; thus they always have to be part of $U_R$. \hfill \Box

Proof (Proof of Theorem 1). Now we are ready to finish the proof of Theorem 1. First we will show that if $C$ is satisfiable, then there is a feasible order of updates for $G(C)$.

Let $\sigma$ be an assignment satisfying $C$. Then the update order for $G(C)$ is as follows (each item marks a round):
1. For each $j \in [n]$, if $\sigma(X_j) = 1$ then update $v^1_j$. Otherwise update $v^2_j$.
2. For each $i \in [m]$, at least one of edges $(u^1_i, u^o_k), (u^2_i, u^o_k), (u^3_i, u^o_k)$ is no longer used by $U_B$. Therefore the corresponding update of $R$ can be resolved (this follows from $\sigma$ being a satisfying assignment).
3. For each $i \in [m]$, $(u^2_i, u^o_k)$ is no longer used by $U_R$. Therefore we can resolve to blue updates along the $w_1-w_2$-subpath of $B^o$ excluding $(w_1, B)$.
4. Resolve $(w_1, B)$.
5. Resolve $(w_1, R)$ and $(w_2, R)$. (Note that now all conflicts between $B$ and $R$ have been resolved and we can finish the updates. We will now leave the state described in Lemma 1.)
6. Resolve $(z_1, R)$.
7. For each $j \in [n]$, $v^2_j$ has already been updated for exactly one $k \in \{1, 2\}$. If $k = 1$, resolve all updates of $B$ along the $u^1_i$-$u^o_k$-subpath of $B^o$ together
with \((v^3_i, B)\). Otherwise resolve \((v^3_i, B)\) together with all updates of \(B\) along the \(u^*_{1i} - u^*_{6i}\)-subpath of \(B^o\).

8. Resolve the remaining updates of \(B\).
9. Resolve all updates of \(R\) along the \(v^1_i-v^2_i\)-subpath of \(R^o\) and for each \(i \in [m]\) resolve \((u^1_i, R)\), \((u^2_i, R)\) and \((u^3_i, R)\).
10. Resolve the remaining updates of \(R\).

Now let us assume that there is a feasible update sequence \(\rho\) for \(G(C)\). We will show that \(C\) is satisfiable by constructing assignment \(\sigma\).

Let us consider the rounds \(r < \min\{\rho(w_1, R), \rho(w_1, B)\}\). Then we will use Condition 3 of Lemma 1 to assign values to variables in the following way. Let \(j \in [n]\), if \(U_B\) does not use the edges \((u^b_{i(h,j)}, u^b_{(h,j)+5})\) for all \(h \in P_j\) (or equivalently if \(v^j_i\) is updated) we set \(\sigma(X_j) := 1\). Otherwise we set \(\sigma(x) := 0\).

Now we will show that assignment \(\sigma\) satisfies \(C\). First let us notice that because we can resolve \((w_1, B)\), none of edges \((u^1_i, u^2_i)\), for any \(i \in [m]\), can be used by \(U_B\) in \(\rho(w_1, B)\). Hence, from Condition 4 of Lemma 1, we know that all vertices \(u^1_i\), for any \(i \in [n]\), and the vertex \(z_1\), are contained in \(U_R\).

Let us consider any clause \(C_i, i \in [m]\). The transient network flow \(U_R\) cannot go from \(u^1_i\) to \(u^j_{i+1}\) along \(R^o\): this would mean that edge \((u^1_i, u^2_i)\) cannot be used by \(U_B\). Therefore, for at least one of the edges \((u^1_i, u^2_i)\), \((u^2_i, u^3_i)\) and \((u^3_i, u^4_i)\), the corresponding blue update has already been resolved. This implies that there is some variable \(X_j, j \in [n]\), that appears in \(C_i\), such that, in the gadget for \(X_j\), \(U_B\) skips \(u^1_i\), for some \(h \in \{1, 2, 3\}\). This vertex is between \(v^1_i\) and \(v^2_i\), if \(C_i\) contains literal \(x_j\). In that case, we set \(\sigma(X_j) := 1\), so \(C_i\) is satisfied. Otherwise \(C_i\) contains literal \(\bar{x}_j\) and we assign \(\sigma(X_j) := 0\), so \(C_i\) is also satisfied.

\[\Box\]

4 Optimal Solution for \(k = 2\) Flows

As we have seen in the previous section, even the 2-flow update problem is computationally hard on general graphs. However, we will now show that an elegant polynomial-time solution exists for the more restricted class of Directed Acyclic Graphs (DAGs). Our algorithm is based on a dependency-graph approach, and not only finds a feasible, but also a shortest schedule (minimum number of rounds).

In the following, let \(G = (s, t, V, E, F, c)\) be an update flow network where \((V, E)\) forms a DAG and \(F = \{B, R\}\) are the two update flow pairs with \(B = (B^o, B^u, \alpha_B, d_B)\) and \(R = (R^o, R^u, \alpha_R, d_R)\). As in the previous section, we identify \(B\) with blue and \(R\) with red.

We say that an I-block \(b_1\) is dependent on an I-block \(b_2\), \(I, J \in \{B, R\}, I \neq J\), if there is an edge \(e \in (E(b_1) \cap E(I^u)) \cap (E(b_2) \cap E(J^u))\), but \(c(e) < d_I + d_J\). In fact, to update \(b_1\), we either violate the capacity condition, or we update \(b_2\) first in order to prevent congestion. In this case we write \(b_1 \rightarrow b_2\) and say that \(b_1\) requires \(b_2\).

We say a block \(b\) is a free block, if it is not dependent on any other block. A dependency graph of \(G\) is a graph \(D = (V_D, E_D)\) for which there exists a bijective
mapping \( \mu: V(D) \leftrightarrow B(G) \), and there is an edge \((v_b, v_{b'})\) in \(D\) if \(b \rightarrow b'\). Clearly, a block \(b\) is free if and only if it corresponds to a sink in \(D\).

We propose the following algorithm to check the feasibility of the flow rerouting problem.

**Algorithm 1. Feasible 2-Flow DAG Update**

**Input:** Update Flow Network \(G\)

1. Compute the dependency graph \(D\) of \(G\).
2. If there is a cycle in \(D\), return **impossible to update**.
3. While \(D \neq \emptyset\) repeat:
   i. Update all blocks which correspond to the sink vertices of \(D\) as in Algorithm 2.
   ii. Delete all of the current sink vertices from \(D\).

Recall that empty updates can always be scheduled in the first round, even if it is not possible to update an update flow network at all. So for Algorithm 1 and all following algorithms we simply assume these updates to be scheduled together with the non-empty updates of round 1.

Suppose \(\rho\) is a feasible update sequence for \(G\).

We say a \(c\)-block \(b\) w.r.t. \(\rho = (\rho_1, \ldots, \rho_\ell)\) is **updated in consecutive rounds**, if the following holds: if some of the edges of \(b\) are activated/deactivated in round \(i\) and some others in round \(j\), then for every \(i < k < j\), there is an edge of \(b\) which is activated/deactivated.

We update free blocks as follows:

**Algorithm 2. Update a Free \(i\)-Block \(b\)**

1. Resolve \((v, F_i)\) for all \(v \in F_i^u \cap b - \mathcal{J}(b)\).
2. Resolve \((\mathcal{J}(b), F_i)\).
3. Resolve \((v, F_i)\) for all \(v \in (b - F_i^u)\).

**Lemma 2.** Let \(b\) be a \(c\)-block. Then in a feasible update sequence \(\rho\), all vertices (resp. their outgoing \(c\) flow edges) in \(F_c^u \cap b - \mathcal{J}(b)\) are updated strictly before \(\mathcal{J}(b)\). Moreover, all vertices in \(b - F_c^u\) are updated strictly after \(\mathcal{J}(b)\) is updated.

**Proof.** In the following, we will implicitly assume flow \(c\), and will not mention it explicitly everywhere. We will write \(F_\uparrow\) for \(F_c^u \cap b\) and \(F_\downarrow\) for \(F_c^o \cap b\). For the sake of contradiction, let \(U = \{v \in V(G) \mid \text{tail}(v) \notin \mathcal{J}(b), v \in F_\uparrow \cup b - F_\downarrow - F_c^o, \rho(v, c) > \rho(\mathcal{J}(b), c)\}\). Moreover, let \(v\) be the vertex of \(U\) which is updated the latest and \(\rho(v, c) = \max_{u \in U} \rho(u, c)\). By our condition, the update of \(v\) enables a transient flow along edges in \(F_c^u \cap b\). Hence, there now exists an \((s, t)\)-flow through \(b\) using only update edges.
Lemma 3. Consider $G$ and assume a feasible update sequence $\rho$. Then there exists a feasible update sequence $\rho'$ which updates every block in at most 3 consecutive rounds.

Proof. Let $\rho$ be a feasible update sequence with a minimum number of blocks that are not updated in 3 consecutive rounds. Furthermore let $b$ be such a $c$-block. Let $i$ be the round in which $\mathcal{S}(b)$ is updated. Then by Lemma 2, all other vertices of $F^c \cap b$ have been updated in the previous rounds. Moreover, since they do not carry any flow during these rounds, the edges can all be updated in round $i - 1$. By our assumption, we can update $\mathcal{S}(b)$ in round $i$, and hence now this is still possible.

As $\mathcal{S}(b)$ is updated in round $i$, the edges of $F^c \cap b$ do not carry any active $c$-flow in round $i + 1$ and thus we can deactivate all remaining such edges in this round. This is a contradiction to the choice of $\rho$, and hence there is always a feasible sequence $\rho$ satisfying the requirements of the lemma. In particular, Algorithm 2 is correct.

From the above two lemmas, we immediately derive a corollary regarding the optimality in terms of the number of rounds: the 3 rounds feasible update sequence.

Corollary 1. Let $b$ be any $c$-block with $|E(b \cap F^c)| \geq 2$ and $|E(b \cap F^c_v)| \geq 2$. Then it is not possible to update $b$ in less than 3 rounds: otherwise it is not possible to update $b$ in less than 2 rounds.

Next we show that if there is a cycle in the dependency graph, then it is impossible to update any flow.

Lemma 4. If there is a cycle in the dependency graph, then there is no feasible update sequence.

Proof. A cycle of length 2 in the dependency graph means that there is a $c_1$-block $b_1$ and a $c_2$-block $b_2$ whose updates mutually depend on each other: If there was a feasible update sequence, then according to Lemma 3 there would exist a feasible update sequence which either updates $b_1$ and then $b_2$ (or vice versa), or which updates them simultaneously. However, this is not possible due to the mutual dependency.

On the other hand, the dependency graph is bipartite. Therefore, every cycle of length more than two has length at least 4. So suppose that there exists a
cycle $C = b_1, b_2, \ldots, b_{2\ell - 1}, b_{2\ell}$ whose vertices correspond to blocks of the corresponding name: blocks of odd indices belong to the $c$-flow and blocks of even indices belong to the $c'$-flow. Suppose that $b_1^c$ is the smallest block of the $c$-flow in $C$. There are two cases:

1. $b_2 \leq b_{2\ell}$
2. $b_{2\ell} \leq b_2$

First suppose $b_2 < b_{2\ell}$, and therefore $\delta'(b_2) \leq \mathcal{S}(b_{2\ell})$. As $b_2$ requires $b_3$, we have $\mathcal{S}(b_3) < \delta'(b_2)$. However, as $b_1$ was the smallest $c$-block in $C$, we have $\delta'(b_1) \leq \mathcal{S}(b_3) < \delta'(b_2)$ or $\delta'(b_1) < \mathcal{S}(b_2)$. On the other hand, $b_{2\ell}$ requires $b_1$, and therefore $\mathcal{S}(b_{2\ell}) < \delta'(b_1)$. But then $\mathcal{S}(b_{2\ell}) < \delta'(b_2)$: a contradiction to our assumption.

Therefore, if there is a feasible update sequence, then $b_{2\ell} < b_2$. Moreover, $b_{2\ell} \rightarrow b_2$ so $\delta'(b_1) \leq \mathcal{S}(b_{2\ell}) < \mathcal{S}(b_2)$ and hence $\delta'(b_1) < \mathcal{S}(b_2)$. But $b_1 \rightarrow b_2$ and therefore $\mathcal{S}(b_2) < \delta'(b_1)$: a contradiction. So there cannot be a cycle of length at least 4 in the dependency graph. \hfill \square

We will now slightly alternate Algorithm 1 in order to create a new algorithm which will compute a feasible sequence $\rho$ for a given update flow network in polynomial time and which ensures that $\rho$ uses the minimum number of rounds. For any block $b$ let $c(b)$ denote its corresponding flow pair.

**Algorithm 3. Optimal 2-Flow DAG Update**

**Input:** Update Flow Network $G$

1. Compute the dependency graph $D$ of $G$.
2. If there is a cycle in $D$, return impossible to update.
3. If there is any block $b$ corresponding to a sink vertex of $D$ with $(b \cap F_{c(b)}^u) - \mathcal{S}(b) \neq \emptyset$ set $i := 2$, otherwise set $i := 1$.
4. While $D \neq \emptyset$ repeat:
   i. Schedule the update of all blocks $b$ which correspond to the sink vertices of $D$ as in Algorithm 2 for the rounds $i - 1, i, i + 1$, such that $\mathcal{S}(b)$ is updated in round $i$.
   ii. Delete all of the current sink vertices from $D$.
   iii. Set $i := i + 1$.

**Theorem 2.** An optimal (feasible) update sequence on acyclic update flow networks with exactly 2 update flow pairs can be found in linear time.

**Proof.** Let $G$ denote the given update flow network. In the following, for ease of presentation, we will slightly abuse terminology and say that “a block is updated in some round”, meaning that the block is updated in the corresponding consecutive rounds by Algorithm 2.
We proceed as follows. First, we find a block decomposition and create the dependency graph of the input instance. This takes linear time only. If there is a cycle in that graph, we output impossible (cf. Lemma 4). Otherwise, we apply Algorithm 3. As there is no cycle in the dependency graph (a property that stays invariant), in each round, either there exists a free block which is not processed yet, or everything is already updated or is in process of update. Hence, if there is a feasible solution (it may not be unique), we can find one in time $O(|G|)$.

For the optimality in terms of the number of rounds, consider two feasible update sequences. Let $\rho_{\text{Opt}}$ be the update sequence produced by Algorithm 3 and let $\rho_{\text{Opt}}'$ be a feasible update sequence that realizes the minimum number of rounds. According to Lemma 2, any block $b$ is processed only in round $\mathcal{S}(b)$.

Suppose there is a block $b'$ such that $\rho_{\text{Opt}}(b') < \rho_{\text{Opt}}'(b')$. Then let $b$ be the block with the smallest such $\rho_{\text{Opt}}(b)$. Hence, for every block $b''$ with $\rho_{\text{Opt}}(b'') < \rho_{\text{Opt}}(b)$, $\rho_{\text{Opt}}(b'') \geq \rho_{\text{Opt}}'(b'')$ holds. Since $\mathcal{S}(b)$ is updated in round $\rho_{\text{Opt}}(b)$, there are no dependencies for $b$ that are still in place in this round. Thus, according to the sequence $\rho_{\text{Opt}}$, $b$ is a sink vertex of the dependency graph after round $\rho_{\text{Opt}}(b) - 1$. Furthermore, by our previous observation, every start of some block that has been updated up to this round in the optimal sequence, and hence it is also already updated in the same round in $\rho_{\text{Opt}}$. This means that after round $\rho_{\text{Opt}}(b) - 1 < \rho_{\text{Opt}}(b) - 1$, $b$ is a sink vertex of the dependency graph of $\rho_{\text{Opt}}$ as well. Thus, Algorithm 3 would have scheduled the update of block $b$ in the rounds $\rho_{\text{Opt}}(b) - 1$, $\rho_{\text{Opt}}(b)$ and $\rho_{\text{Opt}}(b) + 1$. Contradiction.

Thus $\rho_{\text{Opt}}(b) \leq \rho_{\text{Opt}}'(b)$ for all blocks $b$. Now let $b_1, \ldots, b_\ell$ be the last blocks whose starts are updated the latest under $\rho_{\text{Opt}}$. If there is some $i \in [\ell]$ such that $|E_{b_i}^u| \geq 2$ and $|E_{b_i}^u| \geq 2$, $\rho_{\text{Opt}}$ uses exactly $\rho_{\text{Opt}}'(b_i) + 1$ rounds; otherwise it is one round less, by Corollary 1. By our previous observation, none of these blocks can start later than $\rho_{\text{Opt}}'(b_i)$ and thus $\rho_{\text{Opt}}$ uses at least as many rounds as Algorithm 3. Hence the algorithm is optimal in the number of rounds. \qed

5 Updating $k$-Flows in DAGs is NP-complete

In this section we show that if the number of flows, $k$, is part of the input, the problem remains hard even on DAGs. In fact, we prove the following theorem.

**Theorem 3.** Finding a feasible update sequence for $k$-flows is NP-complete, even if the update graph $G$ is acyclic.

To prove the theorem, we provide a polynomial time reduction from the 3-SAT problem. Let $C = C_1 \land \ldots \land C_m$ be an instance of 3-SAT with $n$ variables $X_1, \ldots, X_n$, where each variable $X_i$ appears positive ($x_i$) or negative ($\bar{x}_i$) in some clause $C_j$.

We construct an acyclic network update graph $G$ such that there is a feasible sequence of updates $\rho$ for $G$, if and only if $C$ is satisfiable by some variable assignment $\sigma$. By Lemma 3, we know that if $G$ has a feasible update sequence, then there is a feasible update sequence which updates each block in consecutive rounds.
In the following, we denote the first vertex of a directed path $p$ with $\text{head}(p)$ and the end vertex with $\text{tail}(p)$. Furthermore, we number the vertices of a path $p$ with numbers $1, \ldots, |V(p)|$, according to their order of appearance in $p$ ($\text{head}(p)$ is number 1). We will write $p(i)$ to denote the $i$th vertex in $p$.

We now describe how to construct the initial update flow network $G$.

1. $G$ has a start vertex $s$ and a terminal vertex $t$.

2. We define $n$ variable selector flow pairs $S_1, \ldots, S_n$, where each $S_i = (S_i^o, S_i^r, \alpha_{S_i}, 1)$ defined as follows:
   (a) Variable Selector Old Flows are $n$ s, t-flows $S_1^o, \ldots, S_n^o$ defined as follows: Each consists of a directed path of length 3, where every edge in path $S_i^o$ (for $i \in [n]$) has capacity 1, except for the edge $(S_i^o(2), S_i^o(3))$, which has capacity 2.
   (b) Variable Selector Update Flows are $n$ s, t-flows $S_1^u, \ldots, S_n^u$ defined as follows: Each consists of a directed path of length 5, where the edge’s capacity of path $S_i^u$ is set as follows. $(S_i^u(2), S_i^u(3))$ has capacity 2, $(S_i^u(4), S_i^u(5))$ has capacity $m$, and the rest of its edges has capacity 1.

3. We define $m$ clause flow pairs $C_1, \ldots, C_n$, where each $C_i = (C_i^o, C_i^u, \alpha_{C_i}, 1)$ defined as follows.
   (a) Clauses Old Flows are $m$ s, t-flows $C_1^o, \ldots, C_m^o$, each of length 5, where for $i, j \in [m]$, $C_i^o(3) = C_j^o(3)$ and $C_i^o(4) = C_j^o(4)$. Otherwise they are disjoint from the above defined. The edge $(C_i^o(3), C_j^o(4))$ (for $i \in [m]$) has capacity $m$, all other edges in $C_i^o$ have capacity 1.
   (b) Clauses Update Flows are $m$ s, t-flows $C_1^u, \ldots, C_m^u$, each of length 3. Every edge in those paths has capacity 3.

4. We define a Clause Validator flow pair $V = (V^o, V^u, \alpha_V, m)$, defined as follows.
   (a) Clause Validator Old Flow is an s, t-flow $V^o$ whose path consists of edges $(s, S^o(4)), (S^o(4), S^o(5)), (S^o(5), S^o_{n+1}(4)), (S^o_n(4), S^o_n(5)), (S^o_n(5), t)$ for $i \in [n-1]$. Note that, the edge $(S^o_n(4), S^o_n(5))$ (for $i \in [n]$) also belongs to $S^o_i$. All edges of $V$ have capacity $m$.
   (b) Clause Validator Update Flow is an s, t-flow $V^u$ whose path has length 3, such that $V^u(2) = C_{1^o}(3), V^u(3) = C_{1^u}(4)$. All new edges of $V^u$ have capacity $m$.

5. We define $2n$ literal flow pairs $L_1, \ldots, L_{2n}$. Each $L_i = (L_i^o, L_i^r, \alpha_{L_i}, 1)$ is defined as follows:
   (a) Literal’s Old Flows are $2n$ s, t-flows $L_1^o, \ldots, L_n^o$ and $\bar{L}_1^o, \ldots, \bar{L}_n^o$. Suppose $x_i$ appears in clauses $C_{i_1}, \ldots, C_{i_r}$, then the path $L_i^o$ is a path of length $2\ell + 5$, where $L_i^o(2j + 1) = C_i^o(2), L_i^o(2j + 2) = C_i^o(3)$ for $j \in [\ell]$ and furthermore $L_i^o(2\ell + 3) = S_i^o(2), L_i^o(2\ell + 4) = S_i^o(3)$. On the other hand, if $\bar{x}_i$ appears in clauses $C_{i_1}, \ldots, C_{i_r}$, then $\bar{L}_i^o$ is a path of length $2\ell' + 5$ where $\bar{L}_i^o(2j + 1) = C_i^o(2), \bar{L}_i^o(2j + 2) = C_i^o(3)$ for $j \in [\ell']$, and furthermore $\bar{L}_i^o(2\ell' + 3) = S_i^u(2), \bar{L}_i^o(2\ell' + 4) = S_i^u(3)$. All new edges in $L_i^o$ (resp. $\bar{L}_i^o$) have capacity 3. Note that some of $L_i^o$’s may share common edges.
(b) **Literal’s Update Flows** are 2 \( s, t \)-flows \( L_1^s, \ldots, L_n^s \) and \( L_1^o, \ldots, L_n^o \). For \( i \in [n] \), \( L_i^s \) and \( L_i^o \) are paths of length 5 such that \( L_i^s(2) = \bar{L}_i^s(2) = S_i^o(2) \) and \( L_i^o(3) = \bar{L}_i^o(3) = S_i^o(3) \). All new edges in those paths have capacity 3.

Note that \( G \) is acyclic and every flow pair in \( G \) forms a single block. Let \( \rho \) be a feasible update sequence of \( G \). We suppose in \( \rho \), every block is updated in consecutive rounds (Lemma 2). For a single flow \( F \), we write \( \rho(F) \) for the round where the last edge of \( F \) updated.

**Lemma 5.** For \( \rho \) and \( G \), we have the following observations.

i) We either have \( \rho(L_i^s) < \rho(S_i^o) < \rho(\bar{L}_i^s) \), or \( \rho(\bar{L}_i^s) < \rho(\bar{L}_i^o) \), for all \( i \in [n] \).

ii) \( \rho(C_i^o) < \rho(V^o) \) for all \( i \in [m] \).

iii) \( \rho(S_i^o) < \rho(V^o) \) for all \( i \in [n] \).

iv) For every \( i \in [m] \) there is some \( j \in [n] \) such that \( \rho(C_i^o) < \rho(L_j^o) \) or \( \rho(C_i^o) < \rho(L_j^s) \).

v) We either have \( \rho(L_i^o) < \rho(C_i^o) < \rho(\bar{L}_i^o) \), or \( \rho(\bar{L}_i^o) < \rho(C_i^o) < \rho(L_i^o) \), for all \( i \in [m] \) and all \( j \in [n] \).

**Proof.**

i) As the capacity of the edge \( e = (S_i^o(2), S_i^o(3)) \) is 2, and both \( L_i^s, \bar{L}_i^s \) use that edge, before updating both of them, \( S_i^o \) (resp. \( S_i^o \)) should be updated. On the other hand, the edge \( e' = (S_i^o(2), S_i^o(3)) \) has capacity 2 and it is in both \( L_i^o \) and \( \bar{L}_i^o \). So to update \( S_i^o \), \( e' \) for one of the \( L_i^o, \bar{L}_i^o \) should be updated.

ii) The edge \( (V_i^o(2), V_i^o(3)) \) of \( V^o \) also belongs to all \( C_i^o \) (for \( i \in [m] \)) and its capacity is \( m \). Moreover, the demand of \( (V^o, V^o) \) is \( m \), so \( V^o \) cannot be updated unless \( C_i^o \) has been updated for all \( i \in [m] \).

iii) Every \( S_i^o \) (\( i \in [n] \)) requires the edge \( (S_i^o(4), S_i^o(5)) \), which is also used by \( V^o \), until after round \( \rho(V^o) \).

iv) This is a consequence of Observation iii and Observation ii.

v) This is a consequence of Observation iv and Observation i.

\[ \square \]

**Proof** *(Proof of Theorem 3).* Given a sequence of updates, we can check if it is feasible or not. The length of the update sequence is at most \( k \) times the size of the graph, hence, the problem clearly is in NP.

To show that the problem is complete for NP, we use a reduction from 3-SAT. Let \( C \) be as defined earlier in this section, and in polynomial time we can construct \( G \).

By the construction of \( G \), if there is a satisfying assignment \( \sigma \) for \( C \), we obtain a sequence \( \rho \) to update the flows in \( G \) as follows. In the first round, if in \( \sigma \) we have \( X_i = 1 \) for some \( i \in [n] \), update the literal flow \( L_i^o \); otherwise update the literal flow \( \bar{L}_i^o \). \( \sigma \) satisfies \( C \), therefore, for every clause \( C_i \) there is some literal flow \( L_j \) or \( \bar{L}_j \), which is already updated. Hence, for all \( i \in [m] \) the
Fig. 4. Gadget Construction for Hardness in DAGs: There are 4 types of flows: Clause flows, Literal flows, Clause Validator flow and Literal Selector flows. The edge \((S_o(2), S_o(3))\) cannot route 3 different flows \(S_o, L_i, \bar{L}_i\) at the same time. On the other hand the edge \((S_i(2), S_i(3))\) cannot route the flow \(S_i\) before updating either \(L_i\) or \(\bar{L}_i\), hence by the above observation, exactly one of the \(L_i\) or \(\bar{L}_i\)’s will be updated strictly before \(S_i\) and the other will be updated strictly after \(S_i\) was updated. Only after all Clause flows are updated, the edge \((C_k(3), C_k(4))\) can route the flow \(V\) (Clause Validator flow). A Clause flow \(C_k\) can be updated only if at least one of the Literal flows which goes along \((C_u(2), C_u(3))\) is updated. So in each clause, there should be a valid literal. On the other hand the Clause validator flow can be updated only if all Clause Selector flows are updated, this is guaranteed by the edge \((S_i(4), S_i(5))\). Hence, before updating all clauses, we are allowed to update at most one of the \(L_i\) or \(\bar{L}_i\)’s, and this corresponds to a valid satisfying assignment.

Edge \((C^n_u(3), C^n_u(4))\) incurs a load of 2 while its capacity is 3. Therefore, we can update all of the clause flows and afterward the clause validator flow \(V^o\). Next,
we can update the clause selector flows and at the end, we update the remaining half of the literal flows.

On the other hand, if there is a valid update sequence \( \rho \) for flows in \( G \), by Lemma 5 Observation \( v \), there are exactly \( n \) literal flows that have to be updated, before we can update \( C_i \). To be more precise, for every \( j \in [n] \), either \( L_j \) or \( \overline{L}_j \) has to be updated, but never both. If \( L_j \) is one of those first \( n \) literal flows to be updated for some \( j \in [n] \), we set \( X_j := 1 \); otherwise \( \overline{L}_j \) is to be updated and we set \( X_j := 0 \). Since these choices are guaranteed to be unique for every \( j \in [n] \), this gives us an assignment \( \sigma \). After these \( n \) literal flows are updated, we are able to update the clause flows, since \( \rho \) is a valid update sequence. This means in particular, that for every clause \( C_i \), \( i \in [m] \), there is at least one literal which is set to true. Hence \( \sigma \) satisfies \( C \) and therefore solving the network update problem on DAGs, is as hard as solving the 3-SAT problem. \( \square \)

6 Linear Time Algorithm for Constant Number of Flows on DAGs

We have seen that for an arbitrary number of flows, the problem is hard even on DAGs. In this section, we show that if the number of flows is a constant \( k \), then a solution can be computed in linear time. More precisely, we describe an algorithm to solve the network update problem on DAGs in time \( 2^{O(k \log k)} O(|G|) \), for arbitrary \( k \). In the remainder of this section, we assume that every block has at least 3 vertices (otherwise, postponing such block updates will not affect the solution).

We say a block \( b_1 \) touches a block \( b_2 \) (denoted by \( b_1 \succ b_2 \)) if there is a vertex \( v \in b_1 \) such that \( \mathcal{S}(b_2) \prec v \prec \mathcal{E}(b_2) \), or there is a vertex \( u \in b_2 \) such that \( \mathcal{S}(b_1) \prec v \prec \mathcal{E}(b_1) \). If \( b_1 \) does not touch \( b_2 \), we write \( b_1 \not\succ b_2 \). Clearly, the relation is symmetric, i.e., if \( b_1 \succ b_2 \) then \( b_2 \succ b_1 \).

For some intuition, consider a drawing of \( G \) which orders vertices w.r.t. \( \prec \) in a line. Project every edge on that line as well. Then two blocks touch each other if they have a common segment on that projection.

Algorithm and Proof Sketch: Before delving into details, we provide the main ideas behind our algorithm. We can think about the update problem on DAGs as follows. Our goal is to compute a feasible update order for the (out-)edges of the graph. There are at most \( k \) flows to be updated for each edge, resulting in \( k! \) possible orders and hence a brute force complexity of \( O((k!)^{|G|}) \) for the entire problem. We can reduce this complexity by considering blocks instead of edges. Let \( \text{TouchSeq}(b) \) contain all feasible update sequences for the blocks that touch \( b \); still a (too) large number, but let us consider them for now. For two distinct blocks \( b, b' \), we say that two sequences \( s \in \text{TouchSeq}(b), s' \in \text{TouchSeq}(b') \) are consistent, if the order of any common pair of blocks is the same in both \( s, s' \).

It is clear that if for some block \( b \), \( \text{TouchSeq}(b) = \emptyset \), there is no feasible update sequence for \( G \): \( b \) cannot be updated.

We now create a graph \( H \) whose vertices correspond to elements of \( \text{TouchSeq}(b) \), for all \( b \in B \). Connect all pairs of vertices originating from the
same TouchSeq(b). Connect all pairs of vertices if they correspond to inconsistent elements of different TouchSeq(b). If (and only if) we find an independent set of size $|B|$ in the resulting graph, the update orders corresponding to those vertices are mutually consistent: we can update the entire network according to those orders. In other words, the update problem can be reduced to finding an independent set in the graph $H$.

However, there are two main issues with this approach. First, $H$ can be very large. A single TouchSeq(b) can have exponentially many elements. Accordingly, we observe that we can assume a slightly different perspective on our problem: we linearize the lists TouchSeq(b) and define them sequentially, bounding their size by a function of $k$ (the number of flows). The second issue is that finding a maximum independent set in $H$ is hard. The problem is equivalent to finding a clique in the complement of $H$, a $|B|$-partite graph where every partition has bounded cardinality. We prove that for an $n$-partite graph where every partition has bounded cardinality, finding an $n$-clique is NP-complete. So, in order to solve the problem, we either should reduce the number of partitions in $H$ (but we cannot) or modify $H$ to some other graph, further reducing the complexity of the problem. We do the latter by trimming $H$ and removing some extra edges, turning the graph into a very simple one: a graph of bounded path width. Then, by standard dynamic programming, we find the independent set of size $|B|$ in the trimmed version of $H$: this independent set matches the independent set $I$ of size $|B|$ in $H$ (if it exists). At the end, reconstructing a correct update order sequence from $I$ needs some effort. As we have reduced the size of TouchSeq(b) and while not all possible update orders of all blocks occur, we show that they suffice to cover all possible feasible solutions. We provide a way to construct a valid update order accordingly.

With these intuitions in mind, we now present a rigorous analysis. Let $\pi_{S_1} = (a_1, \ldots, a_{\ell_1})$ and $\pi_{S_2} = (a'_1, \ldots, a'_{\ell_2})$ be permutations of sets $S_1$ and $S_2$. We define the core of $\pi_{S_1}$ and $\pi_{S_2}$ as $\text{core}(\pi_{S_1}, \pi_{S_2}) := S_1 \cap S_2$. We say that two permutations $\pi_1$ and $\pi_2$ are consistent, $\pi_1 \approx \pi_2$, if there is a permutation $\pi$ of symbols of $\text{core}(\pi_1, \pi_2)$ such that $\pi$ is a subsequence of both $\pi_1$ and $\pi_2$.

The Weak Dependency Graph, simply called dependency graph in the following, of a set of permutations is a labeled graph defined recursively as follows. The dependency graph of a single permutation $\pi = (a_1, \ldots, a_{\ell})$, denoted by $G_{\pi}$, is a directed path $v_1, \ldots, v_{\ell}$, and the label of the vertex $v_i \in V(G_{\pi})$ is the element $a$ with $\pi(a) = i$. We denote by $\text{Labels}(G_{\pi})$ the set of all labels of $G_{\pi}$.

Let $G_B$ be a dependency graph of the set of permutations $\Pi$ and $G_{B'}$ the dependency graph of the set $\Pi'$. Then, their union (by identifying the same vertices) forms the dependency graph $G_{\Pi \cup \Pi'}$ of the set $\Pi \cup \Pi'$. Note that such a dependency graph is not necessarily acyclic.

We call a permutation $\pi$ of blocks of a subset $B' \subseteq B$ congestion free, if the following holds: it is possible to update the blocks in $\pi$ in the graph $G_B$ (the graph on the union of blocks in $B$), in order of their appearance in $\pi$, without violating any edge capacities in $G_B$. Note that we do not respect all conditions of our Consistency Rule here.
Fig. 5. Example: The weak dependency graph of three pairwise consistent permutations \(\pi_{\text{blue}}, \pi_{\text{green}}\) and \(\pi_{\text{red}}\). Each pair of those permutation has exactly one vertex in common and with this the cycle \((a, b, c)\) is created. With such cycles being possible a weak dependency graph does not necessarily contain sink vertices. To get rid of them, we certainly need some more refinement.

Lemma 6. Let \(\pi\) be a permutation of the set \(B_1 \subseteq B\). Whether \(\pi\) is congestion free can be determined in time \(O(|B_1| \cdot |G|)\).

Proof. In the order of \(\pi\), perform Algorithm 2. If it fails, i.e., if it violates congestion freedom for some edges, \(\pi\) is not a congestion free permutation. \(\Box\)

The smaller relation defines a total order on all blocks in \(G\). Let \(B = \{b_1, \ldots, b_{|B|}\}\) and suppose the order is \(b_1 < \ldots < b_{|B|}\).

Construction of \(H\): We recursively construct a labeled graph \(H\) from the blocks of \(G\) as follows.

1. Set \(H := \emptyset\), \(B' := B\), TouchList := \(\emptyset\).
2. For \(i := 1, \ldots, |B|\) do
   - Let \(B'_{|B|−i+1} := \{b'_1, \ldots, b'_t\}\) be the set of blocks in \(B'\) which touch \(b_{|B|−i+1}\).
   - Let \(\pi := \{\pi_1, \ldots, \pi_t\}\) be the set of congestion free permutations of \(B'_{|B|−i+1}\).
   - Set TouchList\((b) := \emptyset\).
   - For \(i \in [t]\) create a vertex \(v_{\pi_i}\), with \(\text{Label}(v_{\pi_i}) = \pi_i\) and set \(\text{TouchList}(b) := \text{TouchList}(b) \cup v_{\pi_i}\).
   - Set \(H := H \cup \text{TouchList}(b)\).
   - Add edges between all pairs of vertices in \(H[\text{TouchList}(b)]\).
   - Add an edge between every pair of vertices \(v \in H[\text{TouchList}(b)]\) and \(u \in V(H) - \text{TouchList}(b)\) if the labels of \(v\) and \(u\) are inconsistent.
   - Set \(B' := B' - b_{|B|−i+1}\).

Lemma 7. For Item (ii1) of the construction of \(H\), it holds that \(t \leq k\).

Proof. Suppose for the sake of contradiction that \(t\) is bigger than \(k\). So there are \(c\)-blocks \(b, b'\) (where \(b_{|B|−i+1}\) corresponds to a flow pair different from \(c\)) that touch \(b_{|B|−i+1}\). But then one of \(\mathcal{S}(b)\) or \(\mathcal{S}(b')\) is strictly larger than \(\mathcal{S}(b_{|B|−i+1})\). This contradicts our choice of \(b_{|B|−i+1}\) in that we deleted larger blocks from \(B'\) in Item (ii8). \(\Box\)
Fig. 6. Example: The graph $H$ consists of vertex sets $V_b$, $i \in [|B|]$, where each such partition contains all congestion free sequences of the at most $k$ iteratively chosen touching blocks. In the whole graph, we then create edges between the vertices of two such partitions if and only if the corresponding sequences are inconsistent with each other, as seen in the three highlighted sequences. Later we will distinguish between such edges connecting vertices of neighboring partitions (w.r.t. the topological order of their corresponding blocks), $V_b$ and $V_{b+1}$, and partitions that are further away, $V_b$ and $V_{b'}$. Edges of the latter type, depicted as red in the figure, are called long edges and will be deleted in the trimming process of $H$.

Lemma 8. Let $b_{j_1}, b_{j_2}, b_{j_3}$ be three blocks (w.r.t. $<$) where $j_1 < j_2 < j_3$. Let $b_z$ be another block such that $z \notin \{j_1, j_2, j_3\}$. If in the process of constructing $H$, $b_z$ is in the touch list of both $b_{j_1}$ and $b_{j_3}$, then it is also in the touch list of $b_{j_2}$.

Proof. Let us suppose that $\mathcal{S}(b_{j_1}) \neq \mathcal{S}(b_{j_2}) \neq \mathcal{S}(b_{j_3})$. We know that $\mathcal{S}(b_z) \prec \mathcal{S}(b_{j_1})$ as otherwise, in the process of creating $H$, we eliminate $b_z$ before we process $b_{j_1}$; it would hence not appear in the touch list of $b_{j_1}$. As $b_z \succ b_{j_3}$, there is a vertex $v \in b_z$ where $\mathcal{S}(b_{j_3}) \prec v$. But by our choice of elimination order: $\mathcal{S}(b_{j_1}) \prec \mathcal{S}(b_{j_2}) \prec v \prec \mathcal{S}(b_z)$, and on the other hand: $\mathcal{S}(b_z) \prec \mathcal{S}(b_{j_3} \prec \mathcal{S}(b_{j_2})$. Thus, $\mathcal{S}(b_z) \prec \mathcal{S}(b_{j_2}) \prec v$. Therefore, $b_z$ touches $b_{j_2}$. If some of the start vertices are the same, a similar case distinction applies. \qed

For an illustration of the property described in Lemma 8, see Figure 7: it refers to the weak dependency graph of Figure 5. This example also points out the problem with directed cycles in the weak dependency graph and the property of Lemma 8. We will utilize this in the next lemma and it will be key to our algorithm.

Lemma 9. Let $I = \{v_{\pi_1}, \ldots, v_{\pi_\ell}\}$ be an independent set in $H$. Then the dependency graph $G^+_{\Pi}$, where $\Pi = \{\pi_1, \ldots, \pi_\ell\}$, is acyclic.

Proof. Instead of working on $G_{\Pi}$, we can now work on a supergraph $G^+_{\Pi}$ of $G_{\Pi}$, which is defined as follows. For each sequence $\pi_i$, $v_{\pi_i} \in I$, create a graph $G^+_{\pi_i}$.
We claim \(\text{TouchList}(\pi_b)\) These permutations obey Lemma 8. Taking the three permutations from the example in Figure 5, we can see that the lemma forces \(a\) to be in the green permutation as well. Assuming consistency, this would mean \(a\) to come before \(b\) and after \(c\). Hence \(a <_{\pi_{\text{green}}} b\) and \(b <_{\pi_{\text{green}}} a\), a contradiction. So if our permutations are derived from \(H\) and are consistent, we will show that cycles cannot occur in their weak dependency graph.

which is the transitive closure of \(G_{\pi_i}\). Then we take the union of all \(G_{\pi_i}^+\)'s (for \(i \in [\ell]\)) by identifying the vertices of the same label, to obtain \(G_H^+\). It is clear that \(G_H \subseteq G_H^+\). We prove that there is no cycle in \(G_H^+\). By case analysis, it is easy to verify that there is no cycle of length at most 3 in \(G_H^+\). For the sake of contradiction, suppose \(G_H^+\) contains a cycle and let \(C = (a_1, \ldots, a_n) \subseteq G_H^+\) be a shortest cycle. We know \(n \geq 4\).

The vertices of each edge \((a_i, a_{i+1})\) \((a_i, a_{i+1})\in C\) appear as a subsequence in a label of a vertex \(v\in \text{TouchList}(b_i)\cap I\), for some \(b_i \in B\), in the same order as induced by the directed edge. We call a subsequence \((a_i, a_{i+1})\) a representation of the corresponding \(b_i\). There are \(n\) edges in \(C\), and therefore there are at most \(n\) such \(\text{TouchList}(b_i)\)'s. We denote the union of their corresponding blocks with a set \(B^I = \{b_1', \ldots, b_n'\}\). As \(C\) is the shortest cycle and \(G_H^+\) is created on the transitive closure of each sequence, \(|B^I| = n\) holds. In the following as we consider the cycle \(C\), whenever we write any index \(i\) we consider it w.r.t. cyclic order (in fact \(i \mod |C| + 1\)).

Suppose \(b_1' < b_2' < \ldots < b_n'\), where \((a_i, a_{i+1})\) is a representative of \(\text{TouchList}(b_i)\). Clearly \((a_{i-1}, a_i)\) is a representative of \(b_j^I\) for some \(j \in [n], j \neq 1\). We claim \(j = 2\). Suppose not, then there is a block \(b_h^I\) with \(b_1' < b_h' < b_2'\) and, by Lemma 8, \(a\) appears in \(b_h^I\) as well. But now either \((a_{i+1}, a_{i-1})\) is a subsequence of \(b\), which would give us a cycle of length 3 in \(G_H^+\), a contradiction, or there is a vertex \(a \in C\) with \(a \in \text{LabelTouchList}(b_h)\cap I\) and \(a \notin \{a_{i-1}, a_{i+1}, a_i\}\). But then either there is an edge \((a_i, a)\) or \((a, a_i)\) \((a, a_i)\) in \(G_H^+\) (as both of them will appear in touching blocks of \(b_h\)): either would give us a shorter cycle. So \(j = 2\). As we continue like this, \((a_{i+1}, a_{i+2})\) must be a representative of some \(b_t^I\) where \(t > 2\). But then considering \(b_1', b_2', b_t^I\), with a similar argument, we either have a cycle of length 3, or we can find a shorter cycle in \(G_H^+\) as seen before. Thus there is no cycle in \(G_H^+\) and hence there is no cycle in \(G_H\), as claimed. \(\square\)
The following lemma establishes the link between independent sets in $H$ and feasible update sequences of the corresponding update flow network $G$.

**Lemma 10.** There is a feasible sequence of updates for an update network $G$ on $k$ flow pairs, if and only if there is an independent set of size $|B|$ in $H$.

**Proof.** First we prove that if there is a sequence of feasible updates $\rho$, then there is an independent set of size $|B|$ in $H$. Suppose $\rho$ is a feasible sequence of updates of blocks. For a block $b$, let $B[b] = \{b'_1, \ldots, b'_r\}$ be the set of blocks that touch $b$. Let $\pi_b$ be the reverse order of updates of blocks in $B[b]$ w.r.t. $\rho$. In fact, $\rho$ updates $b'_1$ first, then $b'_2$, then $\ldots$, $b'_r$, then $\pi_b = b'_r \ldots b'_1$.

For every two blocks $b, b' \in B$, we have $\pi_b \approx \pi_{b'}$. From every set of vertices $\text{TouchList}(b)$, for $b \in B$, let $v_b^\ell$ be a vertex such that $\text{Label}(v_b^\ell)$ is a subsequence of $\pi_b$. Recall that, for labels of vertices in $V_b$, we take all possible congestion free permutations of blocks that touch $b$ in $B'$. So the vertex $v_b^\ell$ exists. Put $v_b^\ell$ in $I$. The label of every pair of vertices in $I$ are consistent, as their supersequences were consistent, so $I$ is an independent set and furthermore $|I| = |B|$.

For the other direction, suppose there is an independent set of vertices $I$ of size $|B|$ in $H$. It is clear that there is only one vertex from each of the $\text{TouchList}(b)$, for $b \in B$, in $I$.

Let us define the dependency graph of $I = \{v_b \mid b \in B\}$ as the dependency graph $D := \bigcup_{b \in B} G_{\text{Label}(v_b)}$. By Lemma 9, we know that $D$ is a DAG, and thus it contains at least one sink vertex. We update blocks which correspond to sink vertices of $D$ in parallel and we remove those vertices from $D$ after they are updated. Then we proceed recursively, until there is no vertex in $D$, similarly to Algorithm 1 for $2$-flows. We claim that this gives a valid sequence of updates for all blocks.

Suppose there is a sink vertex whose corresponding block $b$ cannot be updated. Then there is an edge $e \in E(b)$ (edges of $b$) which cannot be activated (updated). Edge $e$ cannot be updated because some other blocks are incident to $e$ and route flows: updating $b$ would violate a capacity constraint. These other blocks must have been updated already by our algorithm: otherwise the label corresponding to $b$ is an invalid congestion free label. Suppose those other blocks are updated in the order $b'_1, b'_2, \ldots, b'_r$ by the above algorithm. Among $b, b'_1, \ldots, b'_r$, there is a block $b'$ which is the largest one (w.r.t. $\prec$). In the construction of $H$, we know that $\text{TouchList}(b') \neq \emptyset$, as otherwise $I$ was not of size $|B|$. Also suppose $v \in \text{TouchList}(b') \cap I$, then in the label of $v$, we have a subsequence $b''_1, \ldots, b''_{\ell + 1}$, where $b''_{\ell} \in \{b, b'_1, \ldots, b'_r\}$: in the iteration where we create $\text{TouchList}(b')$, $b'$ touches all those blocks. We claim that the permutations $\pi_1 = b''_1, \ldots, b''_{\ell + 1}$ and $\pi_2 = b'_1, \ldots, b'_r$ are exactly the same, which would contradict our assumption that $e$ cannot be updated: $\pi_1$ is a subsequence of the congestion free permutation $\text{Label}(v)$.

Suppose $\pi_1 \neq \pi_2$, then there are two blocks $b''_1, b''_2$ with $\pi_1(b''_1) < \pi_1(b''_2)$ and $\pi_2(b''_1) < \pi_2(b''_2)$. Since both, $b''_1$ and $b''_2$, will appear in $\text{Label}(v)$, there is a directed path from $b''_2$ to $b''_1$ in $D$. Then our algorithm cannot choose $b''_2$ as a sink vertex before updating $b''_1$: a contradiction. Hence, the sequence of updates
we provided by deleting the sink vertices, is a valid sequence of updates if $I$ is an independent set of size $|B|$. □

With Lemma 10, the update problem boils down to finding an independent set of size $|B|$ in $H$. However, this reduction does not suffice yet to solve our problem in polynomial time, as we will show next.

Finding an independent set of size $|B|$ in $H$ is equivalent to finding a clique of size $|B|$ in its complement. The complement of $H$ is a $|B|$-partite graph where every partition has cardinality $\leq k!$. In general, it is computationally hard to find such a clique.

**Lemma 11.** Finding an $m$-clique in an $m$-partite graph, where every partition has cardinality at most 3, is hard.

**Proof.** We provide a polynomial time reduction from 3-SAT. Let $C = C_1 \land C_2 \land \ldots \land C_m$ be an instance of 3-SAT with $n$ variables $X_1, \ldots, X_n$. We denote positive appearances of $X_i$ as a literal $x_i$ and negative appearance as a literal $\overline{x}_i$ for $i \in [m]$. So we have at most 2$m$ different literals $x_1, \ldots, x_n, \overline{x}_1, \ldots, \overline{x}_n$. Create an $m$-partite graph $G$ as follows. Set $G$ to be an empty graph. Let $C_i = l_{i1}, l_{i2}, l_{i3}$ be a clause for $i \in [m]$, then add vertices $v^{i1}_{l_{i1}}, v^{i2}_{l_{i2}}, v^{i3}_{l_{i3}}$ to $G$ as partition $p_i$. Note that $l_{i1} = x_i$ or $l_{i2} = \overline{x}_i$ for some $t \in [n]$. Add an edge between each pair of vertices $v^i_{l_{ij}}, v^j_{l_{jk}}$ for $i, j \in [m], i \neq j$ if $x = x_i$ for some $t \in [n]$ and $y \neq \overline{x}_i$ or if $x = \overline{x}_i$ and $y \neq x_i$. It is clear that $G$ now is an $m$-partite graph with exactly 3 vertices in each partition.

**Claim:** There is a satisfying assignment $\sigma$ for $C$ if, and only if, there is an $m$-clique in $G$.

Define a vertex set $K = \emptyset$. Let $\sigma$ be a satisfying assignment. Then from each clause $C_i$ for $i \in [m]$, there is a literal $l_{ij}$ which is set to true in $\sigma$. We take all vertices of $G$ of the form $v^i_{l_{ij}}$ and add it to $K$. The subgraph $G[K]$ forms a clique of size $m$. On the other hand suppose we have an $m$-clique $K_m$ as a subgraph of $G$. Then, clearly from each partition $p_i$, there exists exactly one vertex $v^i_{l_{ij}}$ which is in $K_m$. We set the literal $l_{ij}$ to true. This gives a valid satisfying assignment for $C$. □

Now we trim $H$ to avoid the above problem. Again we will use the special properties of the touching relation of blocks. We say that some edge $e \in E(H)$ is long, if one end of $e$ is in block type $V_{b_i}$, and the other in block type $V_{b_j}$ where $j > i + 1$. The length of $e$ is $j - i$. Delete all long edges from $H$ to obtain the graph $R_H$.

**Lemma 12.** $H$ has an independent set $I$ of size $|B|$ if, and only if, $I$ is also an independent set in $R_H$.

**Proof.** One direction is clear: if $I$ is an independent set of size $|B|$ in $H$, then it is an independent set of size $|B|$ in $R_H$. On the other hand, suppose $I$ is an independent set of size $|B|$ in $R_H$. Then for the sake of contradiction, suppose there are vertices $u, v \in I$ and an edge $e = \{u, v\} \in E(H)$, where $e$ has the
shortest length among all possible long edges in $H[I]$. Let us assume that $u \in V_{b_j}, v \in V_{b_i}$ where $j > i + 1$. Suppose from each $V_{b_i}$ for $i \leq \ell \leq j$, we have $v_{b_i} \in I$, where $v_{b_i} = u, v_{b_j} = v$. Clearly as $I$ is of size $|B|$ there should be exactly one vertex from each $V_{b_i}$. We know $\text{core} \,(\text{Label}(u), \text{Label}(v)) \neq \emptyset$ as otherwise the edge $\{u, v\} \notin E(H)$. On the other hand, as $e$ is the smallest long edge which connects vertices of $I$, then there is no long edge between $v_{b_i}$ and $v_{b_i-1}$ in $H$. That means $\text{Label}(v_{b_i}) \approx \text{Label}(v_{b_{i-1}})$ but then as $\text{Label}(v_{b_i}) \neq \text{Label}(v_{b_{i-1}})$ and by Lemma 8 we know that $\text{core}(\text{Label}(u), \text{Label}(v)) \subseteq \text{Label}(v_{b_{i-1}})$, so $\text{Label}(v_{b_i}) \neq \text{Label}(v_{b_{i-1}})$. Therefore, there is an edge between $v_{b_i}$ and $v_{b_{i-1}}$: a contradiction, by our choice of $I$ in $R_h$. □

Now we have the following main theorem as a corollary of the previous lemmas.

**Theorem 4.** There is a linear time FPT algorithm which finds a solution for the network update problem, whenever it exists, and otherwise outputs there is no feasible solution for a given instance.

**Proof.** Recall that we can resolve all empty updates in the first round, so we ignore them in the following.

First directly construct the graph $R_H$ in time $O(k! \cdot |G|)$. Then we find an independent set of size $|B|$ (or we output there is no such set) by dynamic programing as follows. Define a function $f : [[B]] \times V(R_H) \rightarrow 2^{|R_H|}$ as follows.

1. Set $f(i, v) := \emptyset$ for all $i \in [[B]], v \in V(R_H)$.
2. Set $f(1, v) := v$ for all $v \in \text{TouchList}(b_1)$.
3. For $2 \leq i \leq |B|$ do
   a. For all $v \in \text{TouchList}(b_i)$
      i. If there is a vertex $u \in \text{TouchList}(b_{i-1})$ and $|f(i - 1, u)| = i - 1$ and $\{u, v\} \notin E(R_H)$ then $f(i, v) := f(i - 1, u) \cup \{v\}$, 
      ii. otherwise set $f(i, v) := \emptyset$
4. If $\exists v \in \text{TouchList}(b_B)$ where $|f(|B|, v)| = |B|$ then output $f(|B|, v)$,
5. otherwise output there is no such independent set and output the instance is infeasible.

If the above algorithm finds an independent set of size $|B|$, then, using the algorithm provided in the proof of Lemma 10, find the corresponding update sequence. The whole algorithm runs in $O(k^2 \cdot |G|)$ which is $2^{O(k \log k)}O(|G|)$. □

### 7 Conclusion

This paper initiated the study of a natural and fundamental problem: the congestion-free rerouting of unsplittable flows. The main open question of our work concerns the optimality of our algorithm for fixed $k$. In particular, it would be interesting to see whether some modifications of our algorithm provides an approximation guarantee in terms of the required number of update rounds. More generally, it will be interesting to chart a more comprehensive landscape of the
computational complexity for the network update problem, also for other graph families and from a randomized algorithm’s perspective.

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References

1. Saeed Akhoondian Amiri, Stephan Kreutzer, Dániel Marx, and Roman Rabinovich. Routing with congestion in acyclic digraphs. In Piotr Faliszewski, Anca Muscholl, and Rolf Niedermeier, editors, 41st International Symposium on Mathematical Foundations of Computer Science, MFCS, volume 58 of LIPIcs, pages 7:1–7:11. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, 2016.

2. Saeed Akhoondian Amiri, Arne Ludwig, Jan Marcinkowski, and Stefan Schmid. Transiently consistent sdn updates: Being greedy is hard. In 23rd International Colloquium on Structural Information and Communication Complexity, SIROCCO, 2016.

3. D. Awduche, L. Berger, D. Gan, T. Li, V. Srinivasan, and G. Swallow. Rsvp-te: Extensions to rsvp for lsp tunnels. In RFC 3209, 2001.

4. Jørgen Bang-Jensen and Gregory Gutin. Digraphs - theory, algorithms and applications. Springer, 2002.

5. Sebastian Brandt, Klaus-Tycho Förster, and Roger Wattenhofer. On consistent migration of flows in sdns. In 35th Annual IEEE International Conference on Computer Communications, INFOCOM, pages 1–9, 2016.

6. Chandra Chekuri, Alina Ene, and Marcin Pilipczuk. Constant congestion routing of symmetric demands in planar directed graphs. In 43rd International Colloquium on Automata, Languages, and Programming, ICALP, 2016.

7. Chandra Chekuri, Sreeram Kannan, Adnan Raja, and Pramod Viswanath. Multicommodity flows and cuts in polymatroidal networks. SIAM J. Comput., 44(4):912–943, 2015.

8. Shimon Even, Alon Itai, and Adi Shamir. On the complexity of timetable and multicommodity flow problems. SIAM J. Comput., 5(4):691–703, 1976.

9. Nick Feamster, Jennifer Rexford, and Ellen Zegura. The road to sdn: an intellectual history of programmable networks. ACM SIGCOMM Computer Communication Review, CCR, 44(2):87–98, 2014.

10. Klaus-Tycho Foerster, Stefan Schmid, and Stefano Vissicchio. Survey of consistent network updates. In arXiv Technical Report, 2016, 2016.

11. Klaus-Tycho Förster, Ratul Mahajan, and Roger Wattenhofer. Consistent Updates in Software Defined Networks: On Dependencies, Loop Freedom, and Blackholes. In 15th IFIP Networking, 2016.

12. Xin Jin, Hongqiang Liu, Rohan Gandhi, Srikanth Kandula, Ratul Mahajan, Jennifer Rexford, Roger Wattenhofer, and Ming Zhang. Dionysus: Dynamic scheduling of network updates. In ACM SIGCOMM 2014, 2014.

13. Ken-ichi Kawarabayashi, Yusuke Kobayashi, and Stephan Kreutzer. An excluded half-integral grid theorem for digraphs and the directed disjoint paths problem. In Symposium on Theory of Computing, STOC, pages 70–78, 2014.

14. Jon M. Kleinberg. Decision algorithms for unsplittable flow and the half-disjoint paths problem. In 30th Annual ACM Symposium on Theory of Computing, STOC, 1998.
15. Frank Thomson Leighton and Satish Rao. Multicommodity max-flow min-cut theorems and their use in designing approximation algorithms. *J. ACM*, 46(6):787–832, 1999.

16. Arne Ludwig, Jan Marcinkowski, and Stefan Schmid. Scheduling loop-free network updates: It’s good to relax! In *ACM Symposium on Principles of Distributed Computing, PODC*, 2015.

17. Martin Skutella. Approximating the single source unsplittable min-cost flow problem. *Math. Program.*, 91(3):493–514, 2002.

18. Amin Vahdat, David Clark, and Jennifer Rexford. A purpose-built global network: Google’s move to SDN. *ACM Queue*, 13(8):100, 2015.