Non-Abelian braiding of Majorana-like edge states and scalable topological quantum computations in electric circuits

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Majorana fermions subject to the non-Abelian braid group are believed to be the basic ingredients of future topological quantum computations. In this work, we propose to simulate Majorana fermions of the Kitaev model in electric circuits. We generate an arbitrary number of segments in a Kitaev chain which are in the topologically nontrivial phase. A pair of topological states emerge at the edges of a topological sector, generating a two-dimensional Hilbert space. We call them Majorana-like edge states. Their ground-state degeneracy is $2^N$ when the Kitaev chain contains $N$ topological sectors. It is possible to braid any pair of neighboring edge states with the aid of T-junction geometry. By calculating the Berry phase acquired by their eigenfunctions, the braiding is shown to generate one-qubit and two-qubit unitary operations. We explicitly construct Clifford quantum gates based on them. It is intriguing that we may design a network of Kitaev chains on the square and cubic lattices, where we may create an arbitrary large number of Majorana-like edge states, paving a way to scalable topological quantum computations based on electric circuits.

I. INTRODUCTION

The braiding relation plays a key role in future topological quantum computations. Majorana-fermion edge states emerging in topological superconductors are the best candidate. They emerge in the $p_x + ip_y$ topological superconductors and also in the T-junction of the Kitaev model. Majorana fermions are to be adiabatically manipulated during the braiding. However, an experimental realization of braiding of Majorana fermions still remains challenging.

Various topological phases are known to be materialized by electric circuits based on the observation that the circuit Laplacian has the same expression as the Hamiltonian in an appropriately designed system. The admittance corresponds to the energy. Very recently, we have generated Majorana-like corner states akin to those in topological superconductors and shown that the braiding of these Majorana-like corner states are possible in electric circuits. Indeed, we have derived the relation $\sigma^2 = -1$ by calculating the Berry phase, where $\sigma$ denotes a single braid of two topological corner states. It indicates that a topological corner state is an Ising anyon. Note that the relation $\sigma^2 = 1$ holds both for bosons and fermions. However, the single braid $\sigma$ is impossible in this model. This is because the braiding is controlled by an applied field and the direction of the field becomes opposite after a single braid. Another problem is that it is not clear how to braid more than two topological corner states.

In this paper, we simulate the Kitaev model in electric circuits, we generate $N$ topological sectors together with $2N$ topological edge states. All topological sectors are independent each other, leading to the $2^N$-fold ground-state degeneracy. We then carry out the braiding of edge states with the use of the T-junction. We show edge states to satisfy the braid relations by calculating the adiabatic evolution of the Berry phase associated with an eigenstate of the Hamiltonian. Hence, we may use these topological edge states in place of Majorana-fermion edge states to construct scalable topological quantum computers. We note that the two edge states of one topological sector work as one qubit. Furthermore, we demonstrate that the braiding of two edge states of a topological (trivial) sector generates one-qubit (two-qubit) unitary operation.

The paper is composed as follows. In Section II, we introduce a chain realizing the Kitaev model. We may generate a number of topological and trivial sectors side by side in one chain by controlling the system parameters locally as in Fig. II. There emerge Majorana-like edge states between the topological and trivial sectors. The wave functions $\psi_\alpha$ $(\alpha = 1, 2)$ describing these edge states are analytically constructed by making an appropriate choice of the system parameters, where the Kitaev model is converted to the Su-Schrieffer-Heeger (SSH) model. Let us focus on one topological sector together with a pair of Majorana-like edge states $(\psi_1, \psi_2)$. We show that their certain linear combinations form one qubit customarily denoted as $|0\rangle$ and $|1\rangle$. When a chain contains $N$ topological sectors, it provides us with $N$ qubits $|n_1 n_2 \cdots n_N\rangle = |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes \cdots \otimes |n_N\rangle_N$ with $n_j = 0, 1$, where the index $j$ denotes the $j$-th topological sector. The degeneracy is $2^N$.

In Section III, we investigate the braiding of edge states. First, we analyze how the eigenfunction adiabatically evolves when some system parameters are locally modified. For instance, the Berry phase develops when the superconducting phase parameter is externally modified. Then, using these results, we investigate the braiding of two edges of a topological sector and a trivial sector. Their effect is represented as one-qubit and two-qubit unitary operators, respectively. We explicitly construct Clifford quantum gates based on them. Furthermore, we derive the braiding relation. We also discuss how to initialize the qubit system to $|00\cdots 0\rangle$.

In Section IV, we investigate a realization of a Kitaev chain by electric circuits. We propose electric circuits for topological and trivial sectors, and write down circuit Laplacians to be identified with the Kitaev Hamiltonian. We then explain how to shift a Majorana-like state along the Kitaev chain, and also how to control the "superconducting" phase by electrical method. Furthermore, we point out that the one-dimensional chain may be generalized to a two-dimensional square lattice and also to a three-dimensional cubic lattice to form a network.
of topological edge states with the use of electric circuits. Section V is devoted to discussions.

II. KITAEV MODEL

The Bogoliubov-de Gennes Hamiltonian is written as
\[ \hat{H}(k) = \sum_{\mathbf{k}} \{ c^\dagger(\mathbf{k}), c(\mathbf{k}) \} \].

It is customary to refer to \( H(k) \) also as the Hamiltonian. Topological properties of the system are determined by the property of \( H(k) \).

The Kitaev \( p \)-wave topological superconductor model is the fundamental one-dimensional model hosting Majorana edge states.\(^\text{10,11,14-15} \) It is a two-band model whose Hamiltonian is
\[ H_K(k) = \frac{1}{2} \begin{pmatrix} \varepsilon_k & i\Delta e^{-i\phi} \sin k \\ -i\Delta e^{i\phi} \sin k & -\varepsilon_k \end{pmatrix}, \] (2)

with \( \varepsilon_k = -t \cos k - \mu \), where \( t, \mu, \phi \) and \( \Delta \) represent the hopping amplitude, the chemical potential, the superconducting phase and gap parameters, respectively. It is well known that the system is topological for \( |\mu| < |2t| \) and trivial for \( |\mu| > |2t| \) irrespective of \( \Delta \) provided \( \Delta \neq 0 \). A pair of Majorana-like states emerges topologically at the edges of a topological phase according to the bulk-edge correspondence. They are protected by particle-hole symmetry (PHS). In the present work, we will simulate the Kitaev model by electric circuits, where \( t, \mu, \phi \) and \( \Delta \) are locally controllable parameters by way of tuning capacitors, inductors and resistors in the corresponding circuit Laplacian.

We consider a chain realizing the Kitaev model. A chain need not be straight; it can bend or even branch off. We control the system parameters locally so as to generate several topological sectors with \( |\mu| < |2t| \) sandwiched by trivial sectors with \( |\mu| > |2t| \) as in Fig.\( [1] \) It is convenient to choose the parameters such that
\[ \Delta = t, \quad \mu = 0 \] (3)
to generate a trivial sector, and
\[ \Delta = t, \quad \mu = 4t \] (4)
to generate a topological sector, because analytical solutions describing the Majorana-like edge states are constructed.

There emerges a pair of Majorana-like states at the edges of a topological sector, which we use as one qubit. The key process is to braid two of these edge states. We analyze the following two basic braids illustrated in Fig.\( [1] \): (i) We braid two edge states of one topological sector, which turns out to be a unitary operation in one qubit. (ii) We braid two edge states of one trivial sector, which turns out to be a unitary operation in two qubits.

The Kitaev model \( (2) \) is reduced to
\[ H^y_K(k) = \frac{1}{2} \varepsilon_k \sigma_z - \frac{1}{2} \Delta \sigma_y \sin k \] (5)
for \( \phi = 0 \), and
\[ H^x_K(k) = \frac{1}{2} \varepsilon_k \sigma_z + \frac{1}{2} \Delta \sigma_x \sin k \] (6)
for \( \phi = \pi/2 \). We have previously shown\(^\text{22} \) that the \( H^y_K \) model \( (5) \) is well simulated by the circuit Laplacian for a certain pure \( LC \) circuit and that the \( H^x_K \) model \( (6) \) is by an additional use of operational amplifiers. In the present work, we start with the \( H^y_K \) model \( (5) \), where \( \phi = 0 \). However, in order to braid two edge states it is necessary to use the degree of freedom associated with the "superconducting" phase \( \phi \). We propose an electric circuit to control it in Section V.

A. SSH model

In order to obtain analytic solutions of the zero-energy states, we make a unitary transformation of the Kitaev model \( (2) \) as \( H'_K(k) = U_{KS} H_K(k) U_{KS}^{-1} \) with
\[ U_K = \frac{1}{\sqrt{2}} \begin{pmatrix} -ie^{i\phi/2} & ie^{-i\phi/2} \\ e^{i\phi/2} & e^{-i\phi/2} \end{pmatrix}, \] (7)
and obtain
\[ H'_K(k) = \frac{1}{2} \begin{pmatrix} 0 & -i\varepsilon_k + \Delta \sin k \\ i\varepsilon_k + \Delta \sin k & 0 \end{pmatrix}. \] (8)

With the choice \( (3) \) of the parameters, it is simplified as
\[ H'_K(k) = \frac{1}{2} \begin{pmatrix} 0 & te^{ik} \\ te^{-ik} & 0 \end{pmatrix}, \] (9)
which is identical to the SSH model,
\[ H_{SSH} = \begin{pmatrix} 0 & t_a + t_b e^{ik} \\ t_a + t_b e^{-ik} & 0 \end{pmatrix}, \] (10)
where \( t_a = 0 \) and \( t_b = it/2 \).

B. Zero-energy solution

We consider one topological sector containing \( N \) sites. The zero-energy solutions of the \( H_{SSH} \) model \( (10) \) are explicitly given in the coordinate space by
\[ |\psi'_1\rangle = (1, 0, \cdots, 0), \quad |\psi'_2\rangle = (0, \cdots, 0, 1), \] (11)
which are $2N$ component vectors.

By making the inverse unitary transformation, we obtain the zero-energy solutions in the original Kitaev Hamiltonian \( H_K \) as

\[
|\psi_1\rangle = U_{KS}^{-1}|\psi'_1\rangle = \frac{1}{\sqrt{2}} \left\{ e^{-i\phi/2}, 0, \ldots, 0; -ie^{i\phi/2}, 0, \ldots, 0 \right\},
\]

\[
|\psi_2\rangle = U_{KS}^{-1}|\psi'_2\rangle = \frac{1}{\sqrt{2}} \left\{ 0, \ldots, 0, e^{-i\phi/2}, 0, \ldots, 0, e^{i\phi/2} \right\},
\]

where \( \phi \) is the "superconducting" phase in the Hamiltonian \( H_{KS} \). We refer to the first (last) component part as the electron (hole) sector in accord with the Nambu operator \( \Gamma \). It is seen that \( |\psi_1\rangle \) and \( |\psi_2\rangle \) are perfectly localized at the left edge and the right edge, respectively.

C. One-qubit state

We analyze a Kitaev chain containing one topological sector, where there are two topological edge states \( |\psi_1\rangle \) and \( |\psi_2\rangle \). Any linear combination of these two states are degenerate at zero energy,

\[
|\psi\rangle = \alpha |\psi_1\rangle + \beta |\psi_2\rangle,
\]

where \( \alpha \) and \( \beta \) are complex values satisfying \( |\alpha|^2 + |\beta|^2 = 1 \).

We construct a set of orthogonal states \( |0\rangle \) and \( |1\rangle \) given by

\[
\begin{pmatrix}
|0\rangle \\
|1\rangle
\end{pmatrix} = U_{\text{basis}}
\begin{pmatrix}
|\psi_1\rangle \\
|\psi_2\rangle
\end{pmatrix},
\]

where

\[
U_{\text{basis}} = \frac{1}{\sqrt{2}}
\begin{pmatrix}
1 & i \\
1 & -i
\end{pmatrix}.
\]

The wave functions read

\[
|n\rangle = \frac{1}{2} \left\{ i e^{-i\phi/2}, \ldots, 0, i (-1)^n e^{-i\phi/2}; -i e^{i\phi/2}, \ldots, 0, i (-1)^n e^{i\phi/2} \right\},
\]

where \( n = 0, 1 \). The set of states, \( |0\rangle \) and \( |1\rangle \), may be used to form one qubit in quantum computation.

D. Multi-qubit state

The many-body Hilbert space is essential for scalable quantum computation. The \( j \)-th topological sector produces two topological edge states \( |\psi_{2j-1}\rangle, |\psi_{2j}\rangle \) as in Fig 1. They are independent and degenerate at zero energy, forming a two-dimensional Hilbert space \( \mathcal{H}_j \). When there are \( N \) topological sectors, the total Hilbert space is the direct product of \( N \) Hilbert spaces, that is \( \bigotimes_j \mathcal{H}_j \). All these states are degenerate at zero energy. Namely, the ground-state degeneracy is \( 2^N \) as in the case of topological superconductors.

First, we consider a Kitaev chain composed of two topological sectors and three trivial sectors, where there are four edges states. The many-body ground states are given by the direct product in the qubit basis,

\[
|00\rangle = |0\rangle_1 \otimes |0\rangle_2, \quad |01\rangle = |0\rangle_1 \otimes |1\rangle_2,
\]

\[
|10\rangle = |1\rangle_1 \otimes |0\rangle_2, \quad |11\rangle = |1\rangle_1 \otimes |1\rangle_2,
\]

where the subscript \( j = 1, 2 \) labels the topological sectors.

The generalization to the system containing \( N \) topological sectors is straightforward. The many-body ground states are given by the direct product,

\[
|n_1 n_2 \cdots n_N\rangle = |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes \cdots \otimes |n_N\rangle_N
\]

with \( n_j = 0, 1 \), where the index \( j \) denotes the \( j \)-th topological sector. The degeneracy is \( 2^N \) as in the case of the Kitaev topological superconductor model.

III. BRAIDING PROCESS

A. Berry phase

We investigate how a set of eigenfunctions describing a pair of edge states adiabatically evolves when a system parameter is locally modified from an initial value \( \Omega_0 \) to another values \( \Omega \). A pair of edge states may be that of a topological sector or that of a trivial sector. Examples are \( (|\psi_1\rangle, |\psi_2\rangle) \) of a topological sector, and \( (|\psi_2\rangle, |\psi_3\rangle) \) of a trivial sector in Fig 1.

Let \( |\psi_\beta\rangle_{\text{initial}} \) be the eigenfunction at the initial point. It evolves as \( |\psi_\alpha\rangle (\Omega) = U^{\alpha\beta} (\Omega, \Omega_0) |\psi_\beta\rangle_{\text{initial}} \),

where

\[
U (\Omega, \Omega_0) = e^{i\Gamma (\Omega, \Omega_0)},
\]

with \( \Gamma (\Omega, \Omega_0) \) the Berry phase defined by

\[
\Gamma_{\alpha\beta} (\Omega, \Omega_0) = -i \int_{\Omega_0}^{\Omega} \langle \psi_\alpha | d\psi_\beta \rangle.
\]

It depends on the path along which the edge state moves. Here, \( \psi_\alpha \) is the electron-sector part of the eigen function in \( \mathcal{H}_j \). The Berry phase accumulation is opposite between the electron and hole sectors, as follows from the PHS.

We consider two basic examples. First, we control the phase \( \phi \) locally. Let us choose \( \phi = 0 \) for the initial state. When it increases from \( \phi = 0 \) to \( \phi = \Phi \), the Berry phase is

\[
\Gamma_{\alpha\beta} (\Phi) = -i \int_0^{\Phi} \langle \psi_\alpha | \partial_\phi | \psi_\beta \rangle d\phi = \delta_{\alpha\beta} \frac{\Phi}{4},
\]

where we have used \( \langle \psi_\alpha | \partial_\phi | \psi_\beta \rangle = (i/4) \delta_{\alpha\beta} \).

Second, we control the length of a topological sector by tuning locally the chemical potential \( \mu \). We obtain \( \Gamma_{\alpha\beta} = 0 \)
We cannot braid two edges with the use of a single chain. This problem has been solved by considering a T-junction\cite{17} as shown in Fig.2. We consider a T-junction with three legs (named 1, 2, and 3) made of Kitaev chains. We set $\phi = 0$ in all legs. Such a structure is easily designed in electric circuits as we show later. We prepare the initial state [Fig.2(a)], which consists of the two horizontal legs 1 and 2 made topological and the vertical leg 3 made trivial. As we have already stated, it is possible to make a portion of a chain topological by controlling the chemical potential $\mu$ locally.

We braid two edges following the eight steps from (a) to (i) in the process as shown in Fig.2, where the initial and final states in Fig.2(a) and (i) are identical.

Two edge states emerge at the left and right hands of the horizontal line as indicated in Fig.2(a).

(1: a→b) We move the edge on leg 2 toward the T-junction, by making the topological sector on leg 2 shorter. This process is done as explained in eq.(6) analytically, and by making the use of topology-control switch (TCS) in Fig.7 in electric circuits.

(2: b→c) When the edge reaches at the junction, we turn the trivial sector on leg 2 topological gradually so that the edge moves upward.

(3: c→d) When leg 3 becomes topological entirely, we move the edge on leg 1 toward the T-junction.

(4: d→e) When the edge on leg 3 reaches at the junction, we rotate the phase of leg 3 from $\phi_{ini} = 0$ to $\phi_{fin} = \pi$. This process is done as explained in eq.(6) analytically, and by making the use of phase-control switch (PCS) in Fig.7 in electric circuits.

(5: e→f) When the phase of leg 3 becomes $\phi = \pi$, we move the edge move right on leg 2.

(6: f→g) When leg 2 becomes topological entirely, we move the edge on leg 3 downward.

(7: g→h) When the edge on leg 3 reaches at the junction, we move it leftward on leg 1.

(8: h→i) When leg 1 becomes topological entirely, we rotate the phase of the trivial sector on leg 3 from $\phi_{ini} = \pi$ to $\phi_{fin} = 0$ on leg 3.

We present numerical results of this braiding process in Fig.3 which demonstrates that the process proceeds smoothly. It is confirmed that the energy of the edge states remains zero during the process and that the edge states are well separated from all other states as in Fig.3(b).

They describe the edge states (12) when $x = 0$. As $x$ increases from $x = 0$ to $x = \pi/2$, the edge moves just by one site. Then we find
\[
\langle \psi_\alpha | d \psi_\beta \rangle = \langle \psi_\alpha | \partial_x | \psi_\beta \rangle | dx = 0.
\]

B. Braiding of two edges of a topological sector

We cannot braid two edges with the use of a single chain. This problem has been solved by considering a T-junction\cite{17} as shown in Fig.2. We consider a T-junction with three legs (named 1, 2, and 3) made of Kitaev chains. We set $\phi = 0$ in all legs. Such a structure is easily designed in electric circuits as we show later. We prepare the initial state [Fig.2(a)], which consists of the two horizontal legs 1 and 2 made topological and the vertical leg 3 made trivial. As we have already stated, it is possible to make a portion of a chain topological by controlling the chemical potential $\mu$ locally.

We braid two edges following the eight steps from (a) to (i) in the process as shown in Fig.2, where the initial and final states in Fig.2(a) and (i) are identical.

Two edge states emerge at the left and right hands of the horizontal line as indicated in Fig.2(a).

(1: a→b) We move the edge on leg 2 toward the T-junction, by making the topological sector on leg 2 shorter. This process is done as explained in eq.(6) analytically, and by making the use of topology-control switch (TCS) in Fig.7 in electric circuits.

(2: b→c) When the edge reaches at the junction, we turn the trivial sector on leg 2 topological gradually so that the edge moves upward.

(3: c→d) When leg 3 becomes topological entirely, we move the edge on leg 1 toward the T-junction.

(4: d→e) When the edge on leg 3 reaches at the junction, we rotate the phase of leg 3 from $\phi_{ini} = 0$ to $\phi_{fin} = \pi$. This process is done as explained in eq.(6) analytically, and by making the use of phase-control switch (PCS) in Fig.7 in electric circuits.

(5: e→f) When the phase of leg 3 becomes $\phi = \pi$, we move the edge move right on leg 2.

(6: f→g) When leg 2 becomes topological entirely, we move the edge on leg 3 downward.

(7: g→h) When the edge on leg 3 reaches at the junction, we move it leftward on leg 1.

(8: h→i) When leg 1 becomes topological entirely, we rotate the phase of the trivial sector on leg 3 from $\phi_{ini} = \pi$ to $\phi_{fin} = 0$ on leg 3.

We present numerical results of this braiding process in Fig.3 which demonstrates that the process proceeds smoothly. It is confirmed that the energy of the edge states remains zero during the process and that the edge states are well separated from all other states as in Fig.3(b).

It is also confirmed that the Berry phase $\pi/4$ is acquired in the process (4: d→e) in Fig.3(b) and it follows from (20) that
\[
U_{\alpha\beta}(d \rightarrow e) = e^{i\pi/4} \delta_{\alpha\beta},
\]
The braiding process acts as a phase-shift gate in the qubit representation. It follows that
\[ U_{12}^2 = -i\sigma_z, \] (30)
and hence \( U_{12} \) is proportional to the square root of the \( Z \) gate,
\[ U_{12} = \sqrt{-i\sigma_z}, \]
(31)
which is the \( \sqrt{\sigma_Z} \) gate[16].

C. Braiding of two edges of a trivial sector

We discuss the braiding of two edges of a trivial sector. The braiding process is similar to the previous case: It occurs following nine steps from (a) to (i), as illustrated in Fig. 4. A new configuration is Fig. 4(d), where all three legs are topological. It is understood that the point at the junction is the edge of the vertical topological sector. A phase rotation by \( \pi \) occurs on a topological sector from Fig. 4(d) to (e) with a contributing to the Berry phase, and on a trivial sector from Fig. 4(h) to (i) with no contribution to the Berry phase. We show numerical results of this braiding process in Fig. 5, which demonstrates that the process proceeds smoothly.

The braiding process involves two topological sectors, and hence it is a two-qubit operation. The two-qubit basis is \( |n_1n_2\rangle \) as given by (17), the first qubit \( n_1 \) is formed by \( \psi_1 \) and \( \psi_2 \), while the second qubit \( n_2 \) is formed by \( \psi_3 \) and \( \psi_4 \).

However, since the braiding occurs for a pair \( \psi_2 \) and \( \psi_3 \), it is convenient to consider two qubits \( (\psi_2, \psi_3) \) and \( (\psi_1, \psi_4) \). Hence, we introduce a new two-qubit basis \( |n_+n_-\rangle \), where \( n_+ = 0,1 \) is a qubit formed by \( \psi_2 \) and \( \psi_3 \), while \( n_- = 0,1 \) is a qubit formed by \( \psi_1 \) and \( \psi_4 \). The transformation is given by the fusion matrix of the Ising anyon[40–42],
\[ E_{12} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \] (32)
The inverse relation reads
\[ E_{12}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}, \] (33)
Here, we recall that the formula (29) acts on \( |0\rangle \) and \( |1\rangle \) for the exchange of \( \psi_1 \) and \( \psi_2 \). Now, the corresponding exchange operator is \( U_{23} \) for the exchange of \( \psi_2 \) and \( \psi_3 \). Since it acts on \( |0n_-\rangle \) and \( |1n_-\rangle \) irrespective of the second component \( n_- \), we obtain
\[ U_{23} \begin{pmatrix} |00\rangle \\ |11\rangle \end{pmatrix} = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} |00\rangle \\ |11\rangle \end{pmatrix}. \] (34)
and
\[ U_{23} \left( \begin{array}{c} |01\rangle \\ |10\rangle \end{array} \right) = \left( \begin{array}{cc} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{array} \right) \left( \begin{array}{c} |01\rangle \\ |10\rangle \end{array} \right). \] (35)

Then, we determine how \( U_{23} \) acts on the original basis. We use (33) to find
\[ U_{23} |00\rangle = \frac{1}{\sqrt{2}} \left( U_{23}|00\rangle - U_{23}|11\rangle \right). \] (36)

Then, we use (34) to derive
\[ U_{23} |00\rangle = \frac{1}{\sqrt{2}} \left( e^{-i\pi/4}|00\rangle - e^{i\pi/4}|11\rangle \right). \] (37)

Finally, we use (32) to find
\[ U_{23} |00\rangle = \frac{1}{\sqrt{2}} (|00\rangle - i|11\rangle). \] (38)

We may carry out similar analysis for \( U_{23} |11\rangle, U_{23} |01\rangle \) and \( U_{23} |10\rangle \). The results are summarized as
\[ U_{23} \left( \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \right) = \frac{1}{\sqrt{2}} \left( \begin{array}{cccc} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ -i & 0 & 0 & 1 \end{array} \right) \left( \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \right). \] (39)

It follows that
\[ U_{23}^2 = i\sigma_x. \] (40)

The operation is a square root of the NOT gate, that is the square root of the NOT gate with an imaginary phase factor.
\[ U_{23} = \sqrt{i} \sqrt{\sigma_x}. \] (41)

All these results are exactly the same as those in the two-qubit operation based on Majorana fermions.

D. Entangled states

We proceed to show that an entangled state is generated by a two-qubit operation. For example, we have
\[ U_{23} |00\rangle = |00\rangle - i |11\rangle. \] (42)

This is an entangled states. Let us prove it. If it is not, the final state should be a pure state and written as
\[ U_{23} |00\rangle = (\alpha_1 |0\rangle + \beta_1 |1\rangle) \otimes (\alpha_2 |0\rangle + \beta_2 |1\rangle) \]
\[ = \alpha_1 \alpha_2 |00\rangle + \alpha_1 \beta_2 |01\rangle + \beta_1 \alpha_2 |10\rangle + \beta_1 \beta_2 |11\rangle. \] (43)

It follows from (42) that \( \alpha_1 \alpha_2 \neq 0 \) and \( \beta_1 \beta_2 \neq 0 \), and hence \( \alpha_1 \alpha_2 \beta_1 \beta_2 \neq 0 \), which yields \( \alpha_1 \beta_2 \neq 0 \) and \( \beta_1 \alpha_2 \neq 0 \). This contradicts (42). Namely, an entangled state is produced by a braiding of two edges of a trivial sector.

E. Braiding relations

We explore the braiding relations. We have explicitly constructed \( U_{12} \) and \( U_{23} \). It is obvious that \( U_{2j-1,2j} \) and \( U_{2j,2j+1} \) have the same expressions as these, for \( j = 1, 2, 3, \ldots \). By using the matrix representation of these braiding operations, we can check that
\[ U_{j-1,j} U_{j,j+1} U_{j-1,j} = U_{j,j+1} U_{j-1,j} U_{j,j+1}, \]
\[ U_{j,j+1} U_{j',j'+1} = U_{j',j'+1} U_{j,j+1} \quad \text{for} \quad |j - j'| \geq 2, \]
\[ U_{j-1,j} U_{j,j+1} \neq U_{j,j+1} U_{j-1,j}, \] (44)

which are the braiding relations.

F. Clifford gates

We have constructed the \( \sqrt{\sigma_X} \) and \( \sqrt{\sigma_Z} \) gates in (41) and (42), respectively. The \( \sqrt{\sigma_Y} \) gate is constructed by their successive operations as
\[ \sqrt{\sigma_Y} = \sqrt{i} \sqrt{\sigma_X} \sqrt{\sigma_Z} = \frac{1}{\sqrt{2}} U_{23} U_{12}. \] (45)

These sets construct the Pauli gates. On the other hand, the Hadamard gate is constructed by
\[ iU_{12} U_{23} U_{12} = iU_{23} U_{12} U_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \] (46)
because the set of \( |00\rangle \) and \( |11\rangle \) and the set of \( |01\rangle \) and \( |10\rangle \) are independent in the braiding operation.

G. Initialization

It is standard to start with the pure state \( |00 \cdots 0\rangle \) to carry out quantum computation. Such a pure state can be prepared as follows. We first tune the "superconducting" gap \( \Delta \) slightly larger than the hopping amplitude \( t, \Delta > t \), in a topological region. Although the two-fold degeneracy of the state \( |0\rangle \) and the state \( |1\rangle \) is intact in an infinitely long system due to the PHS, it is broken in a finite system because there is a mixing...
between the two Majorana-like states. The energy of the state \(|0\rangle\) becomes lower than that of the state \(|1\rangle\), as numerically shown in Fig.6 for a finite chain with length 4. Thus, we can choose the state \(|0\rangle\). By doing this setup for all topological phases, we can construct the pure state \(|00\cdots0\rangle\).

IV. ELECTRIC-CIRCUIT REALIZATION

We make a concise review how to simulate the Kitaev chain by an electric circuit. We start with the Kitaev Hamiltonian \(H_K\), where \(\phi = 0\): See Fig.7(a). We use two main wires (red and blue) to represent a two-band model as in Fig.7(b): One wire consists of capacitors \(C\) in series, implementing the electron band, while the other wire consists of inductors \(L\) in series, implementing the hole band. The hopping parameters are opposite between the electron and hole bands, which are represented by capacitors and inductors. Indeed, they contribute the terms proportional to \(i\omega C\) and \(1/(i\omega L)\) to the circuit Laplacian \(J_{ab}(\omega)\) in \(48\), respectively, where \(\omega\) is the frequency of the AC current. We then introduce pairing interactions between them, by crosslinking the two main wires with the use of capacitors \(C_X\) and inductors \(L_X\), as shown in green in Fig.7(b).

Each node in the main wire is connected to the ground via a capacitor \(C_0\) or an inductor \(L_0\) as shown in Fig.7(b). The setting makes the system topological or trivial, as we now explain.

A. Circuit Laplacian

Electric circuits are characterized by the Kirchhoff’s current law,
\[
\frac{d}{dt} I_a = \sum_b C_{ab} \frac{d^2}{dt^2} (V_a - V_b) + \frac{1}{L_0} V_a + \sum_b \frac{1}{L_{ab}} (V_a - V_b) + C_0 \frac{d^2}{dt^2} V_a , \tag{47}
\]
where \(I_a\) is the current between node \(a\) and the ground, \(V_a\) is the voltage at node \(a\), \(C_{ab}\) is the capacitance and \(L_{ab}\) is the inductance between nodes \(a\) and \(b\), and the sum is taken over all adjacent nodes \(b\), while \(L_0\) is the inductance and \(C_0\) is the capacitance between node \(a\) and the ground.

When we apply an AC voltage \(V(t) = V(0) e^{i\omega t}\), the Kirchhoff current law leads to the formula,
\[
I_a(\omega) = \sum_b J_{ab}(\omega) V_b(\omega) , \tag{48}
\]
where the sum is taken over all adjacent nodes \(b\). Here, \(J_{ab}(\omega)\) is called the circuit Laplacian, which we express as
\[
J = \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix} . \tag{49}
\]
We equating this with the Kitaev Hamiltonian \(2\),
\[
J_{ab}(\omega) = i\omega H_{ab}(\omega) . \tag{50}
\]

The parameters in electric circuits are determined by this formula so as to realize the Kitaev model.

B. Topological and trivial phases

It is necessary to construct the circuit Laplacians for the topological and trivial phases separately in Fig.7. Note that we have assumed \(\phi = 0\).

(i) We first study the circuit given in the right-hand side of Fig.7(b). We shall show that it describes the topological phase. Analyzing the Kirchhoff current law for the circuit, we

FIG. 7: (a) A portion of a Kitaev chain containing a trivial sector and a topological sector. A topological edge state emerges at the boundary between the two sectors. (b) This Kitaev chain is simulated by a set of two wires containing capacitors \(C\) (red) and inductors \(L\) (blue). Each node is connected to the ground via a capacitor \(C_0\) or an inductor \(L_0\) in a certain way to realize the trivial and topological sectors. In the model with \(\phi = 0\), the two main wires are crosslinked by a circuit composed of capacitors \((C_X)\) and inductors \((L_X)\). (c) The same Kitaev chain is illustrated in terms of the TCS (topology-control switch) and the PCS (phase-control switch) with \(\phi = 0\). (d) Illustration of TCS. (e) Illustration of PCS at \(\phi = 0\).
which is given by setting $C_0 = 0$ in (54). Then, (51) are modified as
\begin{align*}
f_1 &= -2C \cos k + 2C, \\
f_2 &= 2(\omega^2 L)^{-1} \cos k - 2(\omega^2 L)^{-1},
\end{align*}
by setting $C_0 = 0$ and $L_0 \to \infty$. Then, the chemical potential is given by
\[
\mu = -2C.
\] (58)

The system is precisely at the topological phase-transition point $|\mu| = |2t|$, since the condition $\mu = 2t$ is satisfied.

(iii) Finally, we study the circuit given in the left-hand side of Fig.7(b), which is obtained by interchanging $C_0$ and $L_0$ in the right-hand side of the same figure. The circuit Laplacian is given by
\[
J = [2C(1 - \cos k) + C_0] \sigma_z + 2C_X \sigma_y \sin k,
\] (59)
instead of (54), and we obtain
\begin{align*}
f_1 &= -2C \cos k + 2C + C_0, \\
f_2 &= 2(\omega^2 L)^{-1} \cos k - 2(\omega^2 L)^{-1} - (\omega^2 L)^{-1},
\end{align*}
(60)
instead of (51). All other equations are unmodified except that the chemical potential is given by
\[
\mu = -2C - C_0.
\] (61)

The system is in the trivial phase since $|\mu| > |2t|$ is satisfied.

**C. TCS (topology-control switch)**

We have shown that the topological (trivial) sector is realized in the right-hand (left-hand) side of Fig.7(b). These two sectors are switched from one to another by interchanging inductors $L_0$ and capacitors $C_0$. It is remarkable that we can make a portion of the chain topological or trivial simply by the interchange of $L_0$ and $C_0$. It is convenient to introduce the symbol of TCS to represent this operation, as explained in Fig.7(d). With the use of the TCS, Fig.7(b) is rewritten as Fig.7(e).

Now, recall that a Majorana-like state emerges at the edge site of a topological sector adjacent to a trivial sector. During the braiding process it is necessary to shift this edge state, which is made possible by the TCS. Namely, a portion of the Kitaev chain becomes topological (trivial) by turning on (off) TCS there. In other words, the edge state shifts by turning on (off) TCS appropriately.

**D. PCS (phase-control switch)**

As we have stated, Fig.7(b) is for the Kitaev $H_K^{\mu}$ model (5), by setting $\phi = 0$ in the Kitaev $H_K$ model (2). This phase choice of $\phi = 0$ is made by the setting of the pairing interactions between the two main wires shown in green in Fig.7(b).
FIG. 9: (a) Illustrations of an electric circuit for a T-junction in terms of TCS and PCS. The horizontal part is essentially the same as the one in Fig 7 except that it is entirely in the trivial phase. This circuit is for the configurations in Fig 2(d) and (e), where the horizontal legs 1 and 2 are trivial while the vertical leg 3 is topological.

We have made this point explicit in Fig 7(c) and (e), which is equivalent to Fig 7(b), by introducing the symbol of PCS at φ = 0. It is composed of the capacitor C_X and the inductor L_X.

We include the phase degree of freedom associated with φ. We have considered the case with φ = π/2 in a previous work [52], where we have used operational amplifiers R_X. An operational amplifier illustrated in Fig 8(f) acts as a negative impedance converter with current inversion [52]. In the operational amplifier, the resistance depends on the current flowing direction: R_X for the forward flow and −R_X for the backward flow with the convention that R_X > 0.

We illustrate PCS at φ = 0, 0 < φ < π/2, π/2, π/2 < φ < π and φ = π in Fig 8(a)~(e). We explain how the circuit Laplacian and the electric circuit are modified for each case. The structure of PCS is determined only by modifying the the pairing interactions between the two main wires. Hence, the diagonal components f_1 and f_2 are not affected in the circuit Laplacian [49].

(i) At φ = π, PCS is shown in Fig 8(a). The capacitors C_X and the inductors L_X are interchanged as compared with that at φ = 0. The circuit Laplacian is given by replacing (52) with

\[
\begin{align*}
g_1 &= (\omega^2 L_X)^{-1} e^{i k} - C_X e^{-i k}, \\
g_2 &= -C_X e^{i k} + (\omega^2 L_X)^{-1} e^{-i k}.
\end{align*}
\] (62)

(ii) At φ = π/2, PCS is shown in Fig 8(c). The circuit is constructed with the use of operational amplifiers only, and the circuit Laplacian is given by

\[
g_1 = g_2 = 2(\omega R_X)^{-1} \sin k.
\] (63)

(iii) For 0 ≤ φ ≤ π/2, PCS is shown in Fig 8(b). It is necessary to use C_X, L_X and R_X as a function of φ to generate the Kitaev model with φ. Although we consider the topological phase, similar analysis is made with respect to the trivial phase. The circuit Laplacian is given by replacing (52) with

\[
\begin{align*}
g_1 &= -C_X e^{i k} + (\omega^2 L_X)^{-1} e^{-i k} + 2(\omega R_X)^{-1} \sin k, \\
g_2 &= (\omega^2 L_X)^{-1} e^{i k} - C_X e^{-i k} + 2(\omega R_X)^{-1} \sin k.
\end{align*}
\] (64)

By requiring (65) and

\[
C_X^\phi = C_X |\cos \phi|, \quad L_X^\phi = \frac{L_X}{|\cos \phi|}, \quad R_X^\phi = \frac{R_X}{|\sin \phi|},
\] (65)

the circuit Laplacian is reduced to

\[
J = [2C(1 - \cos k) - C_0] \sigma_z + 2 \left( \frac{\sqrt{LC}}{R_X^\phi} \sigma_x + C_X^\phi \sigma_y \right) \sin k.
\] (66)

Formulas (65) are valid for arbitrary φ. In particular, when we set φ → 0, all these equations are reduced to those in Subsection IV C. On the other hand, when we set φ → π/2, (65) are reduced to (63).

(iv) For π/2 ≤ φ ≤ π, PCS is shown in Fig 8(d). We also consider the topological phase explicitly. The circuit Laplacian is given by replacing (52) with

\[
\begin{align*}
g_1 &= (\omega^2 L_X^\phi)^{-1} e^{i k} - C_X^\phi e^{-i k} + 2(\omega R_X^\phi)^{-1} \sin k, \\
g_2 &= -C_X^\phi e^{i k} + (\omega^2 L_X^\phi)^{-1} e^{-i k} + 2(\omega R_X^\phi)^{-1} \sin k.
\end{align*}
\] (67)

The circuit Laplacian is reduced to

\[
J = [2C(1 - \cos k) - C_0] \sigma_z + 2 \left( \frac{\sqrt{LC}}{R_X^\phi} \sigma_x + C_X^\phi \sigma_y \right) \sin k,
\] (68)

by requiring (65). We note that, when we set φ → π/2, (67) are reduced to (63).

Consequently, when we use C_X^\phi, L_X^\phi and R_X^\phi defined by (65) within PCS, the Kitaev model with arbitral φ is realized in electric circuits.

E. T-junction

A generalization to the T-junction is straightforward as in Fig 9. A vertical chain has been added to the horizontal chain explained in Fig 7(c). This circuit is for the configurations in...
FIG. 10: Non-Abelian braiding of Majorana-like states deposited on a square network. Topological sectors (cyan) are created on horizontal parallel Kitaev chains, which are connected by vertical parallel Kitaev chains. A crossroads may be used as a T-junction for two edges to braid. In this example two edges A (in red) and B (in blue) on different parallel chains are braided by following the steps (a), (b), (c) and (d).

During braiding process, it is necessary to control $\phi$ continuously from $\phi = 0$ to $\phi = \pi$ to proceed from (d) to (e) in Fig.2 and Fig.4. A similar control is necessary from (h) to (i) in these figures. Such an operation is made possible by using a rotary switch tuning variable parameters $R$, $C$, and $L$ within the PCS according to the formula (65).

F. Braiding on square and cubic lattices

By generalizing a one-dimensional array of the T junctions to the two dimensions, we propose to braid Majorana-like edge states on a square lattice, which is illustrated in Fig.10. In the same way, we can generalize them to the three dimensions, where Majorana-like edge states are deposited on a cube.

V. DISCUSSIONS

We have proposed an electric-circuit realization of a Kitaev chain, where the braiding of topological edge states is possible. In particular, we have derived the one-qubit (two-qubit) gate resulting from the braiding of a pair of edge states of a topological (trivial) sector. The results agree precisely with those obtained based on Majorana fermions. Consequently, quantum computation based on our electric circuits will be entirely equivalent to the standard quantum computation based on Majorana fermions. A merit of electric circuit realization is that we can exactly reach the critical point $t = \Delta$, which is practically impossible in topological superconductors.

We have shown that scalable topological quantum computations are possible in electric circuits provided PHS is intact. In actual electric circuits, PHS will be broken weakly due to the randomness, which acts as the on-site potential randomness. We show the energy spectrum evolution in the presence of the 10% on-site potential randomness in Fig.11(a). Although the edge states acquire slight non-zero values, they are well separated from the bulk spectrum. They evolve smoothly as a function of time since the randomness is solely determined by sample elements which are fixed during time evolution. We show the Berry phase accumulation in Fig.11(b). It is well quantized even when the PHS is broken. Furthermore, it is possible to make fine tuning of sample elements in the electric circuit as precisely as required for a practical degeneracy of edge states. We should mention that, once fine tuning is made, we can use it for good.

We may discuss about adiabatic manipulation in electric circuits. The circuit Laplacian is identical to the Hamiltonian in the presence of the background AC current. Its frequency $\omega$ gives the time scale of the system. The manipulation is said adiabatic provided the characteristic time scale of the braiding process is much larger than the time scale of the system.

Some ingenuity might be necessary in actual implementation of integrated circuits. Varactor (variable capacitance diode) will be useful to control the capacitance. Inductors may be displaced by simulated inductors with the use of operational amplifiers. Our results could be a basis of future topological quantum computations based on electric circuits.

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Appendix A: Correspondence between the eigen function and fermion operators in the Kitaev model

We review the Kitaev topological superconductor model together with topological Majorana states since our aim is to simulate them in electric circuits. We show how the fermion formalism and the eigen-function formalism are identified.
The Kitaev model is defined by the Hamiltonian \[ H = -\mu \sum_x c_x^\dagger c_x - \frac{\mu}{2} \sum_x \left( c_{x+1}^\dagger c_x + c_x^\dagger c_{x+1} \right) - \frac{1}{2} \sum_x \left( \Delta e^{i\phi} c_x c_{x+1} + \Delta e^{-i\phi} c_{x+1}^\dagger c_x \right), \] (A1)

where \( c_x \) is the annihilation operator at the site \( x \). The parameters \( \mu, \phi \) and \( \Delta \) correspond precisely to those in the Hamiltonian (2). In the momentum space, we obtain \( \hat{H}(k) = \Psi^\dagger(k) \hat{H}(k) \Psi(k) \), where the Nambu operator is given by (1) and \( \hat{H}(k) \) is given by (2).

We define the Majorana operators \( \gamma_{A,x} \) and \( \gamma_{B,x} \) by

\[
\gamma_{A,x} = -i \frac{e^{i\phi/2}}{2} c_x + i \frac{e^{-i\phi/2}}{2} c_x^\dagger, \\
\gamma_{B,x} = \frac{e^{-i\phi/2}}{2} \left( \gamma_{B,x} + i \gamma_{A,x} \right), \\
\gamma_{B,x} = -i \frac{e^{i\phi/2}}{2} \left( \gamma_{B,x} - i \gamma_{A,x} \right). \quad \text{(A2)}
\]

The Hamiltonian is rewritten in terms of the Majorana operators as

\[
H = -\mu \sum_{x=1}^{L-1} (1 + \gamma_{B,x} \gamma_{A,x}) - \frac{i}{4} \sum_{x=1}^{L-1} \left[ (\Delta + t) \gamma_{B,x} \gamma_{A,x+1} + (\Delta - t) \gamma_{A,x} \gamma_{B,x+1} \right]. \quad \text{(A4)}
\]

which corresponds to (3). We study the case \( \mu = 0 \) and \( t = \Delta \neq 0 \), where the system is topological. The Hamiltonian is simplified as

\[
H = -\frac{t}{2} \sum_{x=1}^{L-1} \gamma_{B,x} \gamma_{A,x+1}, \quad \text{(A5)}
\]

which corresponds to (9). For a finite chain of length \( L \), there emerge two zero-energy topological edge states \( |\varphi_A\rangle \) and \( |\varphi_B\rangle \) at the site \( x = 1 \) and \( x = L \), where they satisfy \( \gamma_{A,1} |\varphi_A\rangle = 0 \) and \( \gamma_{B,L} |\varphi_B\rangle = 0 \).

We label \( \gamma_1 \equiv \gamma_{A,1}, \quad \gamma_2 \equiv \gamma_{B,L} \) (A6) and introduce a non-local fermion operator as \( f = \frac{1}{2} \left( \gamma_1 + i \gamma_2 \right) \). We have the relation

\[
\left( f \right) \left( f^\dagger \right) = \frac{1}{2} \left( 1 + i \right) \left( \gamma_1 \right) \left( \gamma_2 \right), \quad \text{(A7)}
\]

which corresponds to (14) and (15). This definition is related to eq.(13). The qubit states are introduced by

\[
f |0\rangle = 0, \quad f^\dagger |1\rangle = 0, \quad |1\rangle = f^\dagger |0\rangle, \quad |0\rangle = f |1\rangle. \quad \text{(A8)}
\]

This set of states \( |0\rangle \) and \( |1\rangle \) constitutes one qubit for application to topological quantum computers.

The fermion number is defined by

\[
N \equiv f^\dagger f = \frac{1 + i \gamma_1 \gamma_2}{2}, \quad \text{(A9)}
\]

whose eigenvalue must be \( N = 0, 1 \).

We define the parity operation

\[
P_{ij} = i \gamma_i \gamma_j, \quad \text{(A10)}
\]

whose eigenvalues are \( \pm 1 \) since the corresponding number operator has eigenvalue

\[
N_{ij} \equiv \frac{1 + P_{ij}}{2} = 0, 1. \quad \text{(A11)}
\]

The eigenvalue 1 corresponds to the parity even and \(-1 \) corresponds to the parity odd. The two qubit states are similarly defined by

\[
f_1 |00\rangle = 0, \quad f_2 |00\rangle = 0, \quad \text{(A12)}
\]

and

\[
|10\rangle = f_1^\dagger |00\rangle, \quad |01\rangle = f_2^\dagger |00\rangle, \quad |11\rangle = f_1^\dagger f_2^\dagger |00\rangle. \quad \text{(A13)}
\]

In the same way, the multi-qubit states are defined.

The braiding operator is defined by

\[
U_{ij} = \exp \left[ \frac{\pi}{4} \gamma_j \gamma_i \right] = \frac{1}{\sqrt{2}} \left( 1 + \gamma_j \gamma_i \right). \quad \text{(A14)}
\]

It acts on \( \gamma_i \) as

\[
U_{ij} \gamma_i U_{ij}^{-1} = \gamma_j, \quad U_{ij} \gamma_j U_{ij}^{-1} = -\gamma_i, \quad \text{(A15)}
\]

which corresponds to (27). The parity is conserved during the braiding process since \([U_{ij}, P_{ij}] = 0\).
