A Novel Method for the Analytical Solution of Partial Differential Equations Arising in Mathematical Physics

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Abstract: In this article, an efficient analytical technique, called Sumudu variational iteration method (SVIM), is used to obtain the solution of fractional partial differential equations arising in mathematical physics. The fractional derivatives are described in terms of Caputo sense. This method is the combination of the Sumudu transform (ST) and variational iteration method (VIM). The solution of the suggested technique is represented in a series form, which is convergent to the exact solution of the given problems. Furthermore, the results of the present method have shown close relations with the exact approaches of the investigated problems. Illustrative examples are discussed, showing the validity of the current method. The attractive and straightforward procedure of the present method suggests that this method can easily be extended for the solutions of other nonlinear fractional-order partial differential equations.

Keyword: Klein-Gordon equation; Sumudu transform; variational iteration method; Caputo fractional derivative.

1. Introduction

Fractional differential equations (FDEs) have gained a lot of attention of researchers due to their ability to enhance real-world issues, used in various fields of engineering and physics. Numerous physical marvels in signal processing, chemical physics, electrochemistry of corrosion, probability and statistics, acoustics and electromagnetic are precisely modeled by DEs of fractional order. Nonlinear partial differential equations (PDEs) can be considered the generalization of the differential equations of integer order. In the modern age it is impossible
to imagine modeling of many real world problems without using fractional partial differential equations (FPDEs). Indeed, fractional calculus can be called this century’s calculus because of the diversity of applications in different areas of science and technology [1-5].

Many numerical and analytical techniques have been suggested for the solutions of linear and nonlinear partial differential equations of fractional order such as homotopy analysis technique [6], variational iteration method [7,8], homotopy perturbation method [9-11], Laplace homotopy perturbation method [12], Laplace decomposition method [13], Sumudu variational iteration method [14], variation iteration transform method [15], reduce differential transform method [16], series expansion method [17], and another methods [18,19]. This paper considers the efficiency of fractional Sumudu homotopy analysis method (FSHAM) to solve time-fractional partial differential equations. The FSHAM is a graceful coupling of two powerful techniques namely homotopy analysis method and Sumudu transform methods and gives more refined convergent series solution.

2. Preliminaries

Some fractional calculus definitions and notation needed [2,4,14] in the course of this work are discussed in this section.

Definition 2.1. A real function \( u(t), t > 0, \) is said to be in the space \( C_\vartheta, \vartheta \in \mathbb{R} \) if there exists a real number \( q, (q > \vartheta), \) such that \( u(t) = t^q u_1(t), \) where \( q, \mu \in C [0, \infty), \) and it is said to be in the space \( C^m_\vartheta \) if \( u^{(m)} \in C_\vartheta, m \in \mathbb{N}. \)

Definition 2.2. The Riemann Liouville fractional integral operator of order \( \alpha \geq 0, \) of a function \( u(t) \in C_\vartheta, \vartheta \geq -1, \) is defined as

\[
I^\alpha u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} u(\tau) d\tau, & \alpha > 0, t > 0, \\ I^0 u(t) = u(t), & \alpha = 0, \end{cases}
\]

where \( \Gamma(\cdot) \) is the well-known Gamma function.

Properties of the operator \( I^\alpha, \) which we will use here, are as follows

For \( u \in C_\vartheta, \vartheta \geq -1, \alpha, \sigma \geq 0, \)

1. \( I^\alpha I^\sigma u(t) = I^{\alpha+\sigma} u(t). \)
2. \( I^\alpha I^\sigma u(t) = I^{\sigma} I^\alpha u(t) \)
3. \[ I^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} t^{\alpha+m}. \]

**Definition 2.3.** The fractional derivative of \( u(x, t) \) in the Caputo sense is defined as
\[
D^\alpha u(t) = I^{m-\alpha} D^m u(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} u^{(m)}(\tau) d\tau,
\]
For \( m - 1 < \alpha \leq m, m \in \mathbb{N}, t > 0, \phi \in C^m_1. \)
The following are the basic properties of the operator \( D^\alpha: \)
1. \( D^\alpha I^\alpha u(x, t) = u(x, t). \)
2. \( I^\alpha D^\alpha u(x, t) = u(x, t) - \sum_{k=0}^{m-1} \frac{u^{(k)}(x, 0)}{k!} t^k. \)

**Definition 4.** The Mittag–Leffler function \( E_\alpha \) with \( \alpha > 0 \) is defined as
\[
E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(m\alpha + 1)}. \tag{3}
\]

**Definition 2.5.** The Sumudu transform is defined over the set of function
\[
A = \{ u(t) / \exists M, \omega_1, \omega_2 > 0, |u(t)| < Me^{\omega_1|t|/\omega_2}, if \ t \in (-1)^j \times [0, \infty) \}
\]
by the following formula
\[
S[u(t)] = \int_0^\infty e^{-\omega t} u(\omega t) dt, \ \omega \in (-\omega_1, \omega_2). \tag{4}
\]

**Definition 2.6.** The Sumudu transform of the Caputo fractional derivative is defined as
\[
S[D_t^\alpha u(x, t)] = \omega^{-m\alpha} S[u(x, t)] - \sum_{k=0}^{m-1} \omega^{(-m\alpha+k)} u^{(k)}(x, 0), \ m - 1 < m\alpha < m. \tag{5}
\]

3. Fractional Sumudu Variational Iteration Method (FSVIM)

Let us consider a general fractional nonlinear partial differential equation of the form:

\[
3. \]

\[ I^\alpha t^m = \frac{\Gamma(m+1)}{\Gamma(\alpha+m+1)} t^{\alpha+m}. \]
\[ D_t^\alpha u(x,t) + R[u(x,t)] + N[u(x,t)] = g(x,t), \]

(6)

with the initial condition

\[ u(x,0) = h(x) \]

(7)

We will see the whole process of the Lagrange multipliers in the case of an algebraic equation. The solution of the algebraic equation \( f(x) = 0 \) can be obtained by an iteration formula:

\[ X_{n+1} = X_n + \lambda f(X_n) \]

(8)

The optimality condition for the extreme \( \frac{\delta X_{n+1}}{\delta X_n} = 0 \)

Leads to \( \lambda = - \frac{1}{f'(X_n)} \)

(9)

Where \( \delta \) is the classical variational operator. By using initial value \( X_0 \).

We can find the approximate solution \( X_{n+1} \) by the following iterative scheme, using (8) and (9):

\[ X_{n+1} = X_n - \frac{f(X_n)}{f'(X_n)} \]

(10)

This algorithm (10) is well known as the Newton–Raphson method and has quadratic convergence.

In this paper we extend this idea to find the unknown Lagrange multiplier. In this process, first we apply the Sumudu transform to (6), and get

\[ G_n(w) = \frac{u(x,0)}{w^\alpha} + S[R[u(x,t)] + N[u(x,t)] - g(x,t)] = 0 \]

(11)

Where \( G(w) = S[u(x,t)] = \int_0^\infty e^{-t}u(wt)dt \)

Using (8), the iteration formula (11) can be written as
\[ G_{n+1}(w) = G_n(w) + \lambda(w) \left( \frac{G_n(w)}{w^\alpha} - \frac{u(x,0)}{w^\alpha} + S(R[u(x,t)] + N[u(x,t)]) - g(x,t) \right) \]  

(12)

Taking the variation of (12), which is given by

\[ \delta[G_{n+1}(w)] = \delta[G_n(w)] + \lambda(w) \delta \left( \frac{G_n(w)}{w^\alpha} - \frac{u(x,0)}{w^\alpha} + S(R[u(x,t)] + N[u(x,t)] - g(x,t)) \right) \]  

(13)

By using computation of (13), we get

\[ \delta[G_{n+1}(w)] = \delta[G_n(w)] + \lambda(w) \frac{G_n(w)}{w^\alpha} \delta[G_n(w)] + \lambda(w) \delta \left( \frac{G_n(w)}{w^\alpha} \right) \]

\[ = \delta[G_n(w)] + \lambda(w) \frac{G_n(w)}{w^\alpha} \delta[G_n(w)] \]

\[ = 0 \]  

(14)

Hence, from (14) we get

\[ \lambda(w) = -w^\alpha \]  

(15)

By applying the inverse sumudu transform, \( S^{-1} \) to (12), after putting the value of \( \lambda(w) \), we get

\[ u_{n+1}(x,t) = S^{-1}(u_n(x,0)) - S^{-1}(w^\alpha[S[R[u_n(x,t)] + N[u_n(x,t)] - g(x,t)]) \]  

(16)

Consequently the approximate solution may be procured by using

\[ u(x,t) = \lim_{n \to \infty} u_n(x,t) \]  

(17)
4. Application

Example 4.1. Solve the homogeneous KdV equation

\[ D_t^\alpha u(x,t) - 6uu_x + u_{xxx} = 0 \] \hspace{1cm} (18)

With condition \( u(x,0) = 6x \) \hspace{1cm} (19)

\[ S[D_t^\alpha u(x,t) - 6uu_x + u_{xxx}] = S[0] \]

\[ S[D_t^\alpha u(x,t)] = S[6uu_x - u_{xxx}] \]

\[ \frac{S[u_{n+1}(x,t)]}{w^\alpha} - \frac{u_n(x,0)}{w^\alpha} = S[6uu_x - u_{xxx}] \]

\[ S[u_{n+1}(x,t)] = u_n(x,0) + w^\alpha[S[6uu_x - u_{xxx}]] \]

\[ u_{n+1}(x,t) = S^{-1}(u_n(x,0)) + S^{-1}\left(w^\alpha S[6u_0 \frac{\partial}{\partial x} u_n - \frac{\partial^3}{\partial x^3} u_n]\right) \] \hspace{1cm} (20)

The initial iteration \( u_0(x,t) \) is given as follows

\[ u_0(x,t) = u(x,0) = 6x \] \hspace{1cm} (21)

Now, we get the first approximation namely

\[ u_1(x,t) = S^{-1}(u_0(x,0)) + S^{-1}\left(w^\alpha S[6u_0 \frac{\partial}{\partial x} u_0 - \frac{\partial^3}{\partial x^3} u_0]\right) \]

\[ = S^{-1}(6x) + S^{-1}(w^\alpha S[(6)(6x)(6) - (0)]) \]

\[ = 6x + S^{-1}(w^\alpha S[6^3x]) \]

\[ = 6x + S^{-1}(w^\alpha[6^3x]) \]

\[ = 6x + S^{-1}(6^3x w^\alpha) \]

\[ = 6x + 6^3x \frac{t^\alpha}{\Gamma(\alpha+1)} \] \hspace{1cm} (22)
\[ u_2(x,t) = S^{-1}(u_1(x,0)) + S^{-1}\left( w^\alpha \left[ S\left[ 6u_1 \frac{\partial}{\partial x} u_1 - \frac{\partial^3}{\partial x^3} u_1 \right] \right] \right) \]

\[ = S^{-1}(6x) + S^{-1}\left( w^\alpha \left[ S\left[ (6x + 6^3x \frac{t^\alpha}{\Gamma(\alpha+1)}) + 6^3x \frac{t^\alpha}{\Gamma(\alpha+1)} \right] \right] \right) \]

\[ = 6x + S^{-1}\left( w^\alpha \left[ 6^3x + 6^5x \frac{t^\alpha}{\Gamma(\alpha+1)} + 6^5x \frac{t^\alpha}{\Gamma(\alpha+1)} + 6^7x \frac{t^\alpha}{\Gamma(\alpha+1)} \right] \right) \]

\[ = 6x + S^{-1}\left( w^\alpha \left[ 3^3x + 2\left( 6^5x w^\alpha \right) + 6^7x \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} w^{2\alpha} \right] \right) \]

\[ = 6x + S^{-1}\left( w^\alpha \left[ 3^3x + 2\left( 5^5x w^\alpha \right) + 6^7x \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} w^{3\alpha} \right] \right) \]

\[ = 6x + 3^3x \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\left( 5^5x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right) + 6^7x \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \cdot \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \] \hspace{1cm} (23)

\[ \vdots \]

and so on

Then we have

\[ u(x,t) = 6x + 3^3x \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\left( 5^5x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \right) + 6^7x \frac{\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)} \cdot \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} \] \hspace{1cm} (24)

Put \( \alpha = 1 \)

\[ u(x,t) = 6x + 3^3x t + 2\left( 5^5x \frac{t^2}{2} \right) + 6^7x \frac{2}{1} \cdot \frac{t^3}{3!} \]

\[ = 6x + 3^3x t + 6^5x t^2 + \frac{1}{3} 6^7x t^3 \]

\[ = 6x\left[ 1 + (6^2t)^1 + (6^2t)^2 + (6^2t)^3 + (6^2t)^4 \ldots \right] \]

\[ = \frac{6x}{1-6^2t} \] \hspace{1cm} (25)
Example 4.2. Consider the following nonlinear time-fractional Klein-Gordon equation:

\[ D_t^\alpha u(x,t) = u_{xx} - u^2 \quad \text{(26)} \]

\[ u(x,0) = 1 + \sin(x) \quad 0 < \alpha \leq 1 \quad , t \geq 0 \quad \text{(27)} \]

\[ S[D_t^\alpha u_{n+1}(x,t)] = S[u_{nxx} - u_n^2] \]

\[ \frac{S[u_{n+1}(x,t)]}{w^\alpha} - \frac{u_n(x,0)}{w^\alpha} = S[u_{nxx} - u_n^2] \]

\[ S[u_{n+1}(x,t)] = u_n(x,0) + w^\alpha \left[ S[u_{nxx} - u_n^2] \right] \]

\[ u_{n+1}(x,t) = S^{-1}[u_n(x,0)] + S^{-1}(w^\alpha[S[u_{nxx} - u_n^2]]) \]

\[ u_{n+1}(x,t) = S^{-1}[u_n(x,0)] + S^{-1}\left( w^\alpha \left[ S[\frac{\partial^2}{\partial x^2} u_n(x,t) - u_n^2(x,t)] \right] \right) \quad \text{(28)} \]

The initial iteration \( u_0(x,t) \) is given as follows.

\[ u_0(x,t) = u(x,0) = 1 + \sin(x) \quad \text{(29)} \]

Now, we get the first approximation namely

\[ u_1(x,t) = S^{-1}[u_0(x,0)] + S^{-1}\left( w^\alpha \left[ S[\frac{\partial^2}{\partial x^2} u_0(x,t) - u_n^2(x,t)] \right] \right) \]

\[ = S^{-1}[1 + \sin(x)] + S^{-1}\left( w^\alpha \left[ S[\frac{\partial^2}{\partial x^2} (1 + \sin(x)) - (1 + \sin(x))^2] \right] \right) \]

\[ = 1 + \sin(x) + S^{-1}(w^\alpha[S[-\sin(x) - 1 - 2 \sin(x) - \sin^2(x)]]) \]

\[ = 1 + \sin(x) + S^{-1}(w^\alpha[-1 - 3 \sin(x) - \sin^2(x)]) \]

\[ = 1 + \sin(x) + S^{-1}(w^\alpha[-w^\alpha 3 \sin(x) - w^\alpha \sin^2(x)]) \]

\[ = 1 + \sin(x) - \frac{\tau^\alpha}{\Gamma(\alpha+1)} - \frac{\tau^\alpha}{\Gamma(\alpha+1)} 3 \sin(x) - \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sin^2(x) \]
\[ u_2(x, t) = S^{-1} [u_1(x, 0)] + S^{-1} \left( w^\alpha \left[ S \left( \frac{\partial^2}{\partial x^2} u_1(x, t) - u_1^2(x, t) \right) \right] \right) \]

\[ = S^{-1} [1 + \sin(x)] + S^{-1} \left( w^\alpha \left[ S \left( -\sin(x) - \frac{t^\alpha}{\Gamma(\alpha+1)} \left\{ -3 \sin(x) - 
\right. \right. \right. \]

\[ \left. \left. \left. 2\sin^2(x) + 2\cos^2(x) \right\} - 1 - 2 \sin(x) - \sin^2(x) + 2 \frac{\Gamma(\alpha+1)w^\alpha}{\Gamma(\alpha+1)} [1 + \right. \right. \right. \]

\[ \left. \left. \left. 4 \sin(x) + 4\sin^2(x) + \sin^3(x) \right] - \frac{\Gamma(2\alpha+1)w^{2\alpha}}{\Gamma^2(\alpha+1)\Gamma(2\alpha+1)} [1 + 6 \sin(x) + \right. \right. \right. \]

\[ \left. \left. \left. 9\sin^2(x) + 2\sin^2(x) + 6\sin^3(x) + \sin^4(x) \right]\right]\right] \right) \]

\[ = 1 + \sin(x) + S^{-1} \left[ -w^\alpha \sin(x) - w^{2\alpha} \left\{ -3 \sin(x) - 2\sin^2(x) + \right. \right. \right. \]

\[ \left. \left. \left. 2 - 2\sin^2(x) \right\} - w^\alpha - w^\alpha 2 \sin(x) - w^\alpha \sin^2(x) + 2w^{2\alpha} [1 + \right. \right. \right. \]

\[ \left. \left. \left. 4 \sin(x) + 4\sin^2(x) + \sin^3(x) \right] - \frac{w^{3\alpha}}{\Gamma^2(\alpha+1)} [1 + 6 \sin(x) + \right. \right. \right. \]

\[ \left. \left. \left. 9\sin^2(x) + 2\sin^2(x) + 6\sin^3(x) + \sin^4(x) \right]\right]\right] \right) \]

\[ = 1 + \sin(x) - \frac{t^\alpha}{\Gamma(\alpha+1)} \sin(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} 3\sin(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} 2\sin^2(x) - \]

\[ 2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} 2\sin^2(x) - \frac{t^\alpha}{\Gamma(\alpha+1)} - \frac{t^\alpha}{\Gamma(\alpha+1)} 2 \sin(x) - \]

\[ \frac{t^\alpha}{\Gamma(\alpha+1)} \sin^2(x) + 2 \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} 8 \sin(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} 8\sin^2(x) + \]
\[
\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} 2\sin^2(x) - \frac{1}{\Gamma^2(\alpha+1)} \cdot \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} [1 + 6 \sin(x) + 9\sin^2(x) + 2\sin^2(x) + 6\sin^3(x) + \sin^4(x)]
\]

\[
= 1 + \sin(x) - \frac{t^\alpha}{\Gamma(\alpha+1)} [1 + 3 \sin(x) + \sin^2(x)] + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} [11 \sin(x) + 12\sin^2(x) + 2\sin^3(x)] - \frac{1}{\Gamma^2(\alpha+1)} \cdot \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} [1 + 6 \sin(x) + 11\sin^2(x) + 6\sin^3(x) + \sin^4(x)]
\]

and so on

\[
u(x, t) = 1 + \sin(x) - \frac{t^\alpha}{\Gamma(\alpha+1)} [1 + 3 \sin(x) + \sin^2(x)] + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} [11 \sin(x) + 12\sin^2(x) + 2\sin^3(x)] - \frac{1}{\Gamma^2(\alpha+1)} \cdot \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} [1 + 6 \sin(x) + 11\sin^2(x) + 6\sin^3(x) + \sin^4(x)]
\]

Put \(\alpha = 1\)

\[
u(x, t) = 1 + \sin(x) - \frac{t}{1} [1 + 3 \sin(x) + \sin^2(x)] + \frac{t^2}{2!} [11 \sin(x) + 12\sin^2(x) + 2\sin^3(x)] - \frac{t^3}{3!} [1 + 6 \sin(x) + 11\sin^2(x) + 6\sin^3(x) + \sin^4(x)]
\]

**Example 4.3.** Consider the following system of nonlinear time-fractional PDEs:

\[
D^\alpha_t u - v_x + v + u = 0
\]

\[
D^\beta_t v - u_x + v + u = 0
\]

Subject to initial conditions

\[
u(x, 0) = \sinh(x)
\]

\[
v(x, 0) = \cosh(x)
\]
\[ S[D_t^\alpha u_{n+1} - v_{nx} + v_n + u_n] = S[0] \]
\[ S[D_t^B v_{n+1} - u_{nx} + v_n + u_n] = S[0] \]
\[ S[\frac{u_{n+1}(x, t)}{w^\alpha}] - \frac{u_n(x, 0)}{w^\alpha} = S[v_{nx} - v_n - u_n] \]
\[ S[\frac{v_{n+1}(x, t)}{w^B}] - \frac{v_n(x, 0)}{w^B} = S[u_{nx} - v_n - u_n] \]
\[ S[u_{n+1}(x, t)] = u_n(x, 0) + w^\alpha [S[v_{nx} - v_n - u_n]] \]
\[ S[v_{n+1}(x, t)] = v_n(x, 0) + w^B [S[u_{nx} - v_n - u_n]] \]
\[ u_{n+1}(x, t) = S^{-1} [u_n(x, 0)] + S^{-1} (w^\alpha [S[v_{nx} - v_n - u_n]]) \]
\[ v_{n+1}(x, t) = S^{-1} [v_n(x, 0)] + S^{-1} (w^B [S[u_{nx} - v_n - u_n]]) \]  
\[ (36) \]

The initial iterations \( u_0(x, t) \) and \( v_0(x, t) \) are given as
\[ u_0(x, t) = u(x, 0) = \sinh(x) \]
\[ v_0(x, t) = v(x, 0) = \cosh(x) \]  
\[ (37) \]

Hence, we obtain the first approximation, namely
\[ u_1(x, t) = S^{-1} [u_0(x, 0)] + S^{-1} (w^\alpha [S[v_{0x} - v_0 - u_0]]) \]
\[ v_1(x, t) = S^{-1} [v_0(x, 0)] + S^{-1} (w^B [S[u_{0x} - v_0 - u_0]]) \]
\[ u_1(x, t) = S^{-1} [\sinh(x)] + S^{-1} (w^\alpha [S[\sinh(x) - \cosh(x) - \sinh(x)]) \]
\[ v_1(x, t) = S^{-1} [\cosh(x)] + S^{-1} (w^B [S[\cosh(x) - \cosh(x) - \sinh(x)]]) \]
\[ u_1(x, t) = \sinh(x) + S^{-1} (w^\alpha [S[\cosh(x)]) \]
\[ v_1(x, t) = \cosh(x) + S^{-1} (w^B [S[\sinh(x)]) \]
\[ u_2(x, t) = S^{-1}[u_1(x, 0)] + S^{-1}(w^\alpha[S[v_1x - v_1 - u_1]]) \]

\[ v_2(x, t) = S^{-1}[v_1(x, 0)] + S^{-1}(w^B[S[u_1x - v_1 - u_1]]) \]

\[
\begin{align*}
&= \sinh(x) + S^{-1}(-w^\alpha \cosh(x)) \\
&= \cosh(x) + S^{-1}(-w^B \sinh(x)) \\
&= \sinh(x) - \frac{t^\alpha}{\Gamma(\alpha+1)} \cosh(x) \\
&= \cosh(x) - \frac{t^B}{\Gamma(B+1)} \sinh(x) \\
&= \sinh(x) + S^{-1}(w^\alpha[x - \Gamma(\alpha+1)w^\alpha \cosh(x)]) \\
&= \cosh(x) + S^{-1}(w^B[x - \Gamma(\alpha+1)w^\alpha \cosh(x)]) \\
&= \sinh(x) + S^{-1}[w^\alpha \cosh(x) - \frac{\Gamma(B+1)w^B}{\Gamma(B+1)} \sinh(x)] \\
&= \cosh(x) + S^{-1}[w^B \cosh(x) - \frac{\Gamma(\alpha+1)w^\alpha}{\Gamma(\alpha+1)} \sinh(x)] \\
&= \sinh(x) + S^{-1}[w^\alpha + B \cosh(x) - w^\alpha \cosh(x) + w^\alpha + B \sinh(x) + w^{2\alpha} \cosh(x)] \\
&= \cosh(x) + S^{-1}[w^\alpha + B \sinh(x) + w^{2B} \sinh(x) - w^B \sinh(x) + w^{\alpha+B} \cosh(x)] 
\]
\begin{align}
&= \sinh(x) - \frac{t^{\alpha+B}}{\Gamma(\alpha+B+1)} \cosh(x) - \frac{t^{\alpha}}{\Gamma(\alpha+1)} \cosh(x) + \frac{t^{\alpha+B}}{\Gamma(\alpha+B+1)} \sinh(x) + \\
&\quad \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \cosh(x) \\
&= \cosh(x) - \frac{t^{\alpha+B}}{\Gamma(\alpha+B+1)} \sinh(x) + \frac{t^{2B}}{\Gamma(2B+1)} \sinh(x) - \frac{t^{B}}{\Gamma(B+1)} \sinh(x) + \\
&\quad \frac{t^{\alpha+B}}{\Gamma(\alpha+B+1)} \cosh(x) \\
\end{align}
(39)

\begin{align}
u(x,t) &= \sinh(x) - \frac{t^{\alpha+B}}{\Gamma(\alpha+B+1)} \cosh(x) - \frac{t^{\alpha}}{\Gamma(\alpha+1)} \cosh(x) + \\
&\quad \frac{t^{\alpha+B}}{\Gamma(\alpha+B+1)} \sinh(x) + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \cosh(x) \\
\end{align}
(40)

\begin{align}
v(x,t) &= \cosh(x) - \frac{t^{\alpha+B}}{\Gamma(\alpha+B+1)} \sinh(x) + \frac{t^{2B}}{\Gamma(2B+1)} \sinh(x) - \frac{t^{B}}{\Gamma(B+1)} \sinh(x) + \\
&\quad \frac{t^{\alpha+B}}{\Gamma(\alpha+B+1)} \cosh(x) \\
\end{align}

Put \( \alpha = B = 1 \)

\begin{align}
u(x,t) &= \sinh(x) - \frac{t^{2}}{2!} \cosh(x) - \frac{t}{1!} \cosh(x) + \frac{t^{2}}{2!} \sinh(x) + \frac{t^{2}}{2!} \cosh(x) \\
v(x,t) &= \cosh(x) - \frac{t^{2}}{2!} \sinh(x) + \frac{t^{2}}{2!} \sinh(x) - \frac{t}{1!} \sinh(x) + \frac{t^{2}}{2!} \cosh(x) \\
u(x,t) &= \sinh(x) - \frac{t}{1!} \cosh(x) + \frac{t^{2}}{2!} \sinh(x) \\
v(x,t) &= \cosh(x) - \frac{t}{1!} \sinh(x) + \frac{t^{2}}{2!} \cosh(x) \\
u(x,t) &= \sinh(x) \left[ 1 + \frac{t^{2}}{2!} + \cdots \right] - \cosh(x) \left[ t + \frac{t^{3}}{3!} + \cdots \right] \\
v(x,t) &= \cosh(x) \left[ 1 + \frac{t^{2}}{2!} + \cdots \right] - \sinh(x) \left[ t + \frac{t^{3}}{3!} + \cdots \right] \\
\end{align}
(41)
This solution is equivalent to the exact solution is closed form:

\[ u(x, t) = \sinh(x - t) \]
\[ v(x, t) = \cosh(x - t) \]  
\[(42)\]

5. Conclusion

In this study, the Sumudu variational iteration method has been implemented successfully to find the approximate solutions of the time-fractional partial differential equations. The analytical method gives a series solution which converges rapidly to the exact solution. From the obtained results, it is clear that the SVIM yields very accurate solutions using only a few iterates. As a result, the conclusion that comes through this work is that fractional SVIM can be applied to other fractional partial differential equations of higher order, due to the efficiency and flexibility in the application as can be seen in the proposed examples.

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