An Efficient Adaptive Finite Element Method for Eigenvalue Problems

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Abstract

The aim of this paper is to propose an efficient adaptive finite element method for eigenvalue problems based on the multilevel correction scheme and inverse power method. This method involves solving associated boundary value problems on each adaptive partitions and very low dimensional eigenvalue problems on some special meshes which are controlled by the proposed algorithm. Since we Hence the efficiency of solving eigenvalue problems can be improved to be similar to the adaptive finite element method for the associated boundary value problems. The convergence and optimal complexity is theoretically verified and numerically demonstrated.

Keywords. Eigenvalue problem, multilevel correction method, inverse power, adaptive finite element method, convergence, optimality

AMS Subject Classification: 65F15, 65N15, 65N25, 65N30, 65N50

1 Introduction

The finite element method is one of the widely used discretization schemes for solving eigenvalue problems. The adaptive finite element method (AFEM) is a meaningful approach which can generate a sequence of optimal triangulations by refining those elements where the errors, as the local error estimators indicate, are relatively large. The AFEM is really an effective way to make efficient use of given computational resources. Since Babuška and Rheinboldt [5], the AFEM has been an active topic, many researchers are attracted to study the AFEM (see, e.g., [2, 5, 6, 9, 27, 28, 33] and the references cited therein) in the past 40 years. So far, the convergence and optimality of the AFEM for boundary value problems have been obtained and understood well (see, e.g., [8, 10, 15, 16, 25, 26, 27, 29, 30] and the references cited therein).

It is well known that the eigenvalue problem is one of the fundamental problems in computational mathematics and large scale eigenvalue problems always occur in discipline of sciences and engineering such as materials science, quantum chemistry or physics, structure mechanics, biological system, data and information fields, etc. However, it is always a very difficult task to solve high-dimensional eigenvalue problems which come from practical physical and chemistry sciences, and there is a strong demand by engineers and scientists for efficient eigenvalue solvers. Besides for the boundary value problems, the AFEM is also a very useful and efficient way for solving eigenvalue problems (see, e.g., [7, 17, 19, 21, 23, 32]). The AFEM for eigenvalue problems has been analyzed in some papers (see, e.g., [14, 18, 19] and the references cited therein). Especially, [14] give an elaborate analysis of the convergence and optimality.

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for the adaptive finite element eigenvalue computation based on the methods and results in [10]. In [18], authors also give the analysis of the convergence for the eigenvalue problems by the AFEM. The optimality in AFEM only means the scale of the discretization is optimal. But the computing efficiency of AFEM does not arrive the optimality.

In order to improve the efficiency of the AFEM for eigenvalue problems, the purpose of this paper is to propose and analyze a type of AFEM to solve the eigenvalue problems based on the adaptive refinement technique and the recent work on the multilevel correction method [22, 34, 35, 36, 20, 12]. Compared with the standard AFEM which includes solving eigenvalue problems on each refined mesh, we only needs to solve the associated linear boundary value problem on each refined mesh and some very low dimensional eigenvalue problems at some special adaptive steps which is controlled by the AFEM proposed in this paper. Furthermore, the dimension or scale of the included in the new AFEM is fixed all through the adaptive refine process. Thus, in this new scheme, the cost of solving eigenvalue problems is almost the same as solving the associated boundary value problems and the overall efficiency of eigenvalue solving can be improved. Here, we also prove the convergence and quasi-optimal complexity of the new AFEM for the eigenvalue problems.

The rest of the paper is arranged as follows. In Section 2, we shall describe some basic notation and the adaptive multilevel correction algorithm for the second order elliptic eigenvalue problem. We then give the analysis of convergence and complexity of the proposed AFEM in Section 3 and Section 4, respectively. In Section 5, some numerical experiments are presented to test the theoretical analysis. Finally, some concluding remarks are given in the last section.

2 Multilevel correction adaptive finite element method

Let $\Omega \subset \mathbb{R}^d$ ($d \geq 1$) denotes a polytopic bounded domain with Lipschitz continuous boundary. In this paper, standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms (see e.g., [1, 13]) will be used. We denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H^1_0(\Omega) = \{ v \in H^1(\Omega) : v|_{\partial \Omega} = 0 \}$, where $v|_{\partial \Omega}$ is understood in the sense of trace, $\|v\|_{s,\Omega} = \|v\|_{s,2,\Omega}$ and $\|v\|_{0,\Omega} = \|v\|_{0,2,\Omega}$. Throughout this paper, let $V := H^1_0(\Omega)$. We consider the finite element discretization on the shape regular family of nested conforming meshes $\{T_k\}$ over $\Omega$: there exists a constant $\gamma^*$ such that

$$\frac{h_T}{\rho_T} \leq \gamma^*, \quad \forall T \in \bigcup_k T_k,$$

where $h_T$ denotes the diameter of $T$ for each $T \in T_k$, and $\rho_T$ is the diameter of the biggest ball contained in $T$, $h_k := \max\{h_T : T \in T_k\}$. In this paper, the notation $E_k$ is used to denote the set of interior faces (edges or sides) of $T_k$.

2.1 Preliminaries

In this paper, we are concerned with the following second order elliptic eigenvalue problem

$$\begin{cases}
Lu := -\nabla(A \cdot \nabla u) + \varphi u & = \lambda u \quad \text{in } \Omega, \\
u & = 0 \quad \text{on } \partial \Omega,
\end{cases}$$

$$\int_{\Omega} (A \nabla u \cdot \nabla v + \varphi uv) d\Omega = 1,$$

where $A = (a_{ij})_{d \times d}$ is a symmetric positive definite matrix with $a_{ij} \in W^{1,\infty}(\Omega)$ ($i, j = 1, \cdots, d$), and $0 \leq \varphi \in L^\infty(\Omega)$.

We first define a bounded bilinear form

$$a(u, v) = \int_{\Omega} (A \nabla u \cdot \nabla v + \varphi uv) d\Omega.$$

From the properties of $A$ and $\varphi$, the bilinear form $a(\cdot, \cdot)$ is bounded over $V$

$$|a(w, v)| \leq C^2_0 \|w\|_{1,\Omega} \|v\|_{1,\Omega}, \quad \forall w, v \in V,$$
and satisfies
\[ c_\alpha \|w\|_{1, \Omega} \leq \|w\|_{a, \Omega} \leq C_\alpha \|w\|_{1, \Omega}, \] (2.1)

where the energy norm \(\| \cdot \|_{a, \Omega}\) is defined by \(\|w\|_{a, \Omega} = \sqrt{a(w, w)}\), \(c_\alpha\) and \(C_\alpha\) are positive constants. Then the corresponding variational form can be written as: Find \((\lambda, u) \in \mathbb{R} \times V\) such that \(\|u\|_{a, \Omega} = 1\) and
\[ a(u, v) = \lambda(u, v), \quad \forall v \in V. \] (2.2)

As we know, the eigenvalue problem (2.2) has a countable sequence of real eigenvalues
\[ 0 < \hat{\lambda}^1 < \hat{\lambda}^2 \leq \hat{\lambda}^3 \leq \cdots \]
and corresponding orthogonal eigenfunctions
\[ \hat{u}^1, \hat{u}^2, \hat{u}^3, \cdots, \]
which satisfy \(a(\hat{u}^i, \hat{u}^j) = \delta_{ij}, i, j = 1, 2, \cdots\). Here we use \(\hat{\lambda}^i\) and \(\hat{u}^i\) to denote the \(i\)-th exact eigenvalue and eigenfunction, respectively.

Let \(V_k \subset V\) be the corresponding family of nested finite element spaces of continuous piecewise polynomials over \(T_k\) of fixed degree \(m \geq 1\), which vanish on the boundary of \(\Omega\), and are equipped with the same norm \(\| \cdot \|_{a, \Omega}\) of space \(V\). The standard finite element discretization for (2.2) is: Find \((\tilde{\lambda}_k, \tilde{u}_k) \in \mathbb{R} \times V_k\) such that \(\|\tilde{u}_k\|_{a, \Omega} = 1\) and
\[ a(\tilde{u}_k, v_k) = \tilde{\lambda}_k(\tilde{u}_k, v_k), \quad \forall v_k \in V_k. \] (2.3)

The eigenvalues of (2.3) can also be ordered as an increasing sequence
\[ 0 < \tilde{\lambda}_1 \leq \tilde{\lambda}_2 \leq \cdots \leq \tilde{\lambda}_{n_k}, \quad n_k = \dim V_k, \]
and the corresponding orthogonal eigenfunctions
\[ \tilde{u}_1, \tilde{u}_2, \cdots, \tilde{u}_{n_k} \]
satisfying \(a(\tilde{u}_i, \tilde{u}_j) = \delta_{ij}, i, j = 1, 2, \cdots, n_k\).

Based on the finite element space \(V_k\), we define the Galerkin projection \(R_k : V \to V_k\) by
\[ a(v - R_k v, v_k) = 0, \quad \forall v \in V, \forall v_k \in V_k. \]

Then the following boundness holds
\[ \|R_k v\|_{a, \Omega} \leq \|v\|_{a, \Omega}, \quad \forall v \in V. \]

**Lemma 2.1.** (13) The duality argument leads to the following inequality
\[ \|(I - R_k)v\|_{0, \Omega} \leq C_{\alpha n} \eta_a(V_k) \|(I - R_k)v\|_{a, \Omega}, \]
where \(C_{\alpha n}\) is a positive constant and the quantity \(\eta_a(V_k)\) is defined as follows:
\[ \eta_a(V_k) = \sup_{f \in L^2(\Omega), \|f\|_{0, \Omega} = 1} \inf_{v_k \in V_k} \|L^{-1}f - v_k\|_{a, \Omega}. \] (2.4)

Let \(K : L^2(\Omega) \to V\) be the operator defined by
\[ a(K w, v) = (w, v), \quad \forall w, v \in V. \]

Then the eigenvalue problems (2.2) and (2.3) can be written as
\[ u = \lambda K u, \quad \tilde{u}_k = \tilde{\lambda}_k R_k K \tilde{u}_k. \] (2.5)
For any \( v \in V \), since \( a(Kv, Kv) = (v, Kv) \) and from (2.1), we have
\[
\|Kv\|_{a, \Omega} \leq \frac{\|v\|_{a, \Omega}}{c_a} \leq \frac{\|v\|_{\Sigma, \Omega}}{c_a^2}.
\] (2.6)

For the aim of error estimate, we define
\[ M(\hat{\lambda}^i) = \{ v \in V : v \text{ is an eigenfunction of (2.2)} \text{ corresponding to the eigenvalue } \hat{\lambda}^i \} \]
and the quantity
\[
\delta_W(\hat{\lambda}^i) = \sup_{v \in M(\hat{\lambda}^i), \|v\|_{a, \Omega} = 1} \inf_{w \in W} \|v - w\|_{a, \Omega},
\] (2.7)
where \( W \) is a finite dimensional space.

From [3, 4, 11, 13], it is known that \( \eta_a(V_k) \to 0 \), \( \delta_{V_k}(\hat{\lambda}^i) \to 0 \) as \( h_k \to 0 \) and the following error estimates of finite element method for eigenvalue problems hold.

**Lemma 2.2.** ([3] [4] [11] [13]) Let \( (\hat{\lambda}^i, \hat{\nu}^i) \in \mathbb{R} \times V_k \) be the solution of (2.2) for \( 1 \leq i \leq n_k \). Then there exist an exact eigenpair \( (\lambda^i, \nu^i) \) of (2.2) and constants \( C_{\text{ca}}, C_{\text{co}} \) and \( C_{\text{c} \lambda} \) such that
\[
\|\hat{\nu}^i_k - \hat{\nu}^i\|_{a, \Omega} \leq C_{\text{ca}} \delta_{V_k}(\hat{\lambda}^i),
\]
\[
\|\hat{\nu}^i_k - \hat{\nu}^i\|_{0, \Omega} \leq C_{\text{co}} \eta_a(V_k) \|\hat{\nu}^i_k - \hat{\nu}^i\|_{a, \Omega},
\]
\[
|\hat{\lambda}^i_k - \hat{\lambda}^i| \leq C_{\text{c} \lambda} \|\hat{\nu}^i - \hat{\nu}^i\|_{a, \Omega}.
\]

### 2.2 Adaptive multilevel correction algorithm

Now we follow the classic routine to define the a posteriori error estimator for boundary value problem with \( f \in L^2(\Omega) \) as the right hand side term. For any \( f \in L^2(\Omega) \) and \( v_k \in V_k \), let us define the element residual \( R_T(f, v_k) \) and the jump residual \( J_E(v_k) \) by
\[ R_T(f, v_k) := f - L v_k = f + \nabla \cdot (A \nabla v_k) - \varphi v_k \text{ in } T \in T_k, \]
\[ J_E(v_k) := -A \nabla v_k^+ \cdot \nu^+ - A \nabla v_k^- \cdot \nu^- = (A \nabla v_k)_E \cdot \nu_E \text{ on } E \in \mathcal{E}_k, \]
where \( E \) is the common side of elements \( T^+ \) and \( T^- \) with outward normals \( \nu^+ \) and \( \nu^- \), \( \nu_E = \nu^- \). Then we can define the local error indicator \( \eta_k(f, v_k; T) \) for the element \( T \in T_k \) by
\[
\eta_k^2(f, v_k; T) := h_T^2 \|R_T(f, v_k)\|_{0,T}^2 + \sum_{E \in \mathcal{E}_k, E \subset \partial T} h_E \|J_E(v_k)\|_{0,E}^2,
\] (2.8)
and for a submesh \( T' \subset T_k \) by
\[
\eta_k(f, v_k; T') := \left( \sum_{T \in T'} \eta_k^2(f, v_k; T) \right)^{1/2}.
\]
Thus \( \eta_k(f, v_k; T_k) \) denotes the error estimator of finite element approximation \( v_k \) with respect to \( T_k \).

For \( f \in L^2(\Omega) \), we define the data oscillation as
\[
\text{osc}(f; T_k) := \left( \sum_{T \in T_k} \|h_T(f - P_T f)\|_{0,T}^2 \right)^{1/2},
\]
where \( P_T \) is the \( L^2 \)-projection operator to polynomials of some degree on \( T \). It is obvious that the following inequality holds
\[
\text{osc}(f - L v_k; T_k) \leq \eta_k(f, v_k; T_k).
\]
For convenience, we use the notation \( \eta(f; T_k) := \eta_k(f, R_k K f; T_k) \). There exist the following reliability and efficiency for the a posterior error estimator \( \eta(f; T_k) \) (see, e.g., [25] [27] [22]):
Lemma 2.3. \((2.2)\) For \(f \in L^2(\Omega)\), there exist a constant \(C_{\text{up}}\), solely depending on regularity constant \(\gamma^*\) and coercivity constant \(c_a\) in \((2.1)\), such that
\[
\|(I - R_k)Kf\|_{a, \Omega} \leq C_{\text{up}}\eta(f; T_k),
\]
and a constant \(C_{\text{low}}\), solely depending on regularity constant \(\gamma^*\) and continuity constant \(C_a\) in \((2.1)\), such that
\[
C_{\text{low}}\eta(f; T_k) \leq \|(I - R_k)Kf\|_{a, \Omega} + \text{osc}(f - LR_kKf; T_k).
\]

Before introducing our AFEM, we first introduce some modules for preparation:

- \((\mu, w) = \text{ESOLVE}(\Omega)\): Solve the eigenvalue problem \((2.3)\) in the finite element space \(W\) and output the discrete eigenpair \((\mu, w) \in \mathbb{R} \times W\).
- \(w = \text{LSOLVE}(f, W)\): Solve the linear boundary value problem in the finite element space \(W\) with the right hand side term \(f\), namely, the output \(w \in W\) satisfies the following boundary value problem
  \[
a(w, v) = (f, v), \quad \forall v \in W.
\]
- \(\{\eta_k(f, v_k; T)\}_{T \in T_k} = \text{ESTIMATE}(f, v_k, T_k)\): Compute the error indicator on each element \(T \in T_k\) as \((2.8)\).
- \(M_k = \text{MARK}(\theta, \{\eta_k(f, v_k; T)\}_{T \in T_k}, T_k)\): Construct a minimal subset \(M_k\) from \(T_k\) by selecting some elements in \(T_k\) such that
  \[
  \eta_k(f, v_k; M_k) \geq \theta \eta_k(f, v_k; T_k)
  \]
  and mark all the elements in \(M_k\).
- \((T_{k+1}, V_{k+1}) = \text{REFINE}(M_k, T_k)\): Output a conforming refinement \(T_{k+1}\) of \(T_k\) where at least all elements of \(M_k\) are refined and construct the finite element space \(V_{k+1}\) over \(T_{k+1}\).

Then we present a type of AFEM to compute the eigenvalue problem in the multilevel correction framework which is the main contribution of this paper.

### Adaptive Algorithm C

- Given parameters \(0 < \theta_1 < 1\) and \(0 < \theta_2 < 1\), a coarse mesh \(T_H\) with mesh size \(H\) and construct the finite element space \(V_H\).
- Refine the mesh \(T_H\) to obtain an initial mesh \(T_1\) and the finite element space \(V_1\) by the regular way.
- Solve an eigenvalue problem: \((\lambda_1, u_1) = \text{ESOLVE}(V_1)\).
- Set \(\bar{f}_2^{(0)} = \bar{f}_1 = \bar{u}_1 = u_1, \bar{u}_1 = \frac{u_1}{n_1}, n_1 = 1, k = 2, \ell = 1, j = 0\).
- Compute \(\{\eta_l(\bar{f}_1, \bar{u}_1; T)\}_{T \in T_1} = \text{ESTIMATE}(\bar{f}_1, \bar{u}_1, T_1)\) and set \(\eta_1 = \eta_1(\bar{f}_1, \bar{u}_1; T_1)\).

Do the following iteration:
1. $\mathcal{M}_{k-1} = \text{MARK}(\theta_1, \{\eta_{k-1}(f_{k-1}, \bar{u}_{k-1}; T)\} \in T_{k-1}, T_{k-1});$

2. $(T_k, V_k) = \text{REFINE}(\mathcal{M}_{k-1}, T_{k-1});$

3. Linear solving: $\bar{u}_k^{(j)} = \text{LSOLVE}(\bar{f}_k^{(j)}, V_k)$ and compute
$$u_k^{(j)} = \frac{\bar{u}_k^{(j)}}{\|\bar{u}_k^{(j)}\|_{a, \Omega}}, \quad \lambda_k^{(j)} = \frac{\alpha(u_k^{(j)}, u_k^{(j)})}{(u_k^{(j)}, u_k^{(j)})};$$

4. $\{\eta_k(\bar{f}_k^{(j)}, \bar{u}_k^{(j)}, T)\} \in T_k = \text{ESTIMATE}(\bar{f}_k^{(j)}, \bar{u}_k^{(j)}, T_k);$

5. • If $\eta_k(\bar{f}_k^{(j)}, \bar{u}_k^{(j)}; T_k) \leq \theta_2^{j+1}\eta_k$, then solve $(\bar{\lambda}_k^{(j)}, \bar{u}_k^{(j)}) = \text{ESOLVE}(V_H + \text{span}(u_k^{(j)}))$, set $\bar{f}_k^{(j+1)} = \bar{u}_k^{(j)}$, $j = j + 1$ and go to step 3;
   • Else, set $\bar{f}_k = \bar{f}_k^{(j)}$, $\bar{u}_k = \bar{u}_k^{(j)}$, $u_k = u_k^{(j)}$ and $\lambda_k = \lambda_k^{(j)}$;

6. • If $j > 0$, solve an eigenvalue problem $(\bar{\lambda}_k; \bar{u}_k) = \text{ESOLVE}(V_H + \text{span}(u_k))$ and set $\bar{f}_{k+1}^{(0)} = \bar{u}_k$; then set $\eta_{k+1} = \eta_k(\bar{f}_k^{(j)}, \bar{u}_k^{(j)}; T_k), n_{k+1} = k, \ell = \ell + 1, j = 0$;
   • Else, set $\bar{f}_{k+1}^{(0)} = u_k$;

7. Let $k = k + 1$ and go to step 1.

Remark 2.1. Here we use the iterative or recursive bisection (see, e.g., [24] [27]) of elements with the minimal refinement condition in the procedure REFINE. The marking strategy adopted in Adaptive Algorithm C was introduced by Dörfler [13] and Morin et al. [27].

Different from the standard AFEM for eigenvalue problems, Adaptive Algorithm C has no requirement to solve the eigenvalue problems on the adaptively refined meshes $T_k$, which can improve the efficiency since eigenvalue solving need much more computations than solving the associated linear boundary value problem. This point is the main contribution of this paper.

In Adaptive Algorithm C, we denote by index $k$ the order of nested mesh, index $j$ the order of iteration on a fixed mesh. Besides, we denote $n_k$ ($\ell > 1$) to denote the order of the mesh on which the decision condition $\eta_{n_k}(\bar{f}_k^{(0)}, \bar{u}_k^{(0)}; T_{n_k}) \leq \theta_2\eta_{k-1}$ is satisfied, where reference estimate $\eta_{k-1} = \eta_{n_{k-1}}(\bar{f}_{n_{k-1}}, \bar{u}_{n_{k-1}}; T_{n_{k-1}})$ is defined recursively, and $\ell$ is introduced to denote the order of subset $\{T_{n_k}\} \in \in N \subset \{T_k\} \in k \in N$. For easier understanding, we visualize Adaptive Algorithm C with two flow charts.

The first flow chart translates Adaptive Algorithm C to the figurative language.

The second flow chart shows the overall behavior of Adaptive Algorithm C. The processes between two dotted lines are over meshes from $T_{n_{k+1}}$ to $T_{n_{k+1}}$.

In this paper, $(\bar{\lambda}_n, \bar{u}_n)$ is the eigenpair approximation by the eigenvalue solving module ESOLVE($V$) for $\ell = 1$ and ESOLVE($V_H + \text{span}([u_n])$) for $\ell \geq 2$, $(\lambda_n^{(j)}, u_n^{(j)})$ is the eigenpair approximation by module ESOLVE($V_H + \text{span}([u_n])$) for $\ell \geq 2$ and $j \geq 0$, $(\lambda_k, u_k) = \text{ESOLVE}$ by the linear solving.

From Adaptive Algorithm C, over each mesh $T_k$ ($k \in N$) we solve a linear boundary value problem with $f_k$ as the right hand side term, and then refine the mesh based on the corresponding error indicator. The following theorem can be derived directly from the convergence of standard AFEM for boundary value problems, which has been proved by Cascon et al. [10].
Then the symbols $(2.4)$ and $(2.7)$, it is easy to show that
\[
\|(I - R_{k+1})Kf_k\|_{α,Ω}^2 + γ^2(f_k;T_{k+1}) \leq \tilde{α}^2\left(\|(I - R_k)Kf_k\|_{α,Ω}^2 + γ^2(f_k;T_k)\right).
\]

In our analysis, we also need following lemmas.

**Lemma 2.4.** (10) There exists a constant $\hat{C}_D$ depending only on data $D$ and regularity constant $γ^*$ such that

\[
\|(I - R_k)Kf\|_{α,Ω}^2 + \text{osc}^2(f - LR_kKf;T_k) \leq \hat{C}_D \inf_{υ_k ∈ V_k} (\|Kf - υ_k\|_{α,Ω}^2 + \text{osc}^2(f - LV_k;T_k)), \quad ∀f ∈ L^2(Ω).
\]

In this paper, we assume that the marking parameter $θ_1$ satisfies $θ_1 ∈ (0, θ_*)$ with $θ_*$ defined in Assumption 5.8 of [10].

**Lemma 2.5.** (10) For any function $f ∈ L^2(Ω)$, let $T_k^*$ and $V_k^*$ be a refinement of $T_k$ and the finite element space over the mesh $T_k^*$, such that the Ritz-Galerkin approximation of $Kf$ satisfies the energy decreasing property

\[
\|(I - R_k)Kf\|_{α,Ω}^2 + \text{osc}^2(f - LR_kKf;T_k) ≤ \hat{C}_D \inf_{υ_k ∈ V_k} (\|Kf - υ_k\|_{α,Ω}^2 + \text{osc}^2(f - LV_k;T_k))
\]

with $\hat{C}_D ∈ (0, \frac{1}{2})$. Then the set $T_k^*(T_k^* ∩ T_k)$ of refined elements satisfies the Dörfler property

\[
\eta_k^2(f, R_kKf;T_k ∩ (T_k^* ∩ T_k)) ≥ \tilde{θ}^2 \eta_k^2(f, R_kKf;T_k),
\]

where $\tilde{θ} = θ_*\sqrt{1 - 2\hat{C}_D}^2$.

**Remark 2.2.** In Adaptive Algorithm C, since the right hand side term $\tilde{f}_k$ of boundary value problems are always piecewise polynomials, by the definition of data oscillation, we have

\[
\text{osc}(\tilde{f}_k - L\tilde{u}_k;T_k) = \text{osc}(L\tilde{u}_k;T_k).
\]

And similar properties of $[2.16]$ will be used in the following parts of this paper.

For simplicity, we only give the analysis for the first eigenpair approximation by the adaptive multilevel correction algorithm (Adaptive Algorithm C) in this paper. Then the symbols $υ_k$, $\tilde{u}_k$, $\bar{u}_k^{(i)}$ and $u_k^{(j)}$ denote approximations for the first exact eigenfunction $\tilde{α}^1$, $λ_k$ and $λ_k^{(j)}$ are approximations to the first exact eigenvalue $\tilde{λ}^1$. Let us define the spectral projection $E : V → M(\tilde{λ}^1)$ as follows

\[
a(v - Ev, w) = 0, \quad ∀w ∈ M(\tilde{λ}^1), \forall v ∈ V.
\]

From the definitions [2.4] and [2.7], it is easy to show that $δ_{V_{M}}(\tilde{λ}^1) ≤ δ_{V_H}(\tilde{λ}^1)$ and $η_{a}(\tilde{V}_H) ≤ η_{a}(V_H)$, where $V_H ⊂ \tilde{V}_H$. Hence, the following properties are direct results of Lemma 2.4

**Corollary 2.1.** For each obtained eigenpair approximation $(\tilde{λ}_n^{(j)}, \tilde{u}_n^{(j)}) (ℓ > 1, j ≥ 0)$ and $(\tilde{λ}_{nε}, \tilde{u}_{nε})(ℓ > 1)$ in Adaptive Algorithm C, the following estimates hold

\[
\|\tilde{u}_n^{(j)} - E\tilde{u}_n^{(j)}\|_{α,Ω} ≤ C_{ea} \min \{δ_{V_H}(\tilde{λ}^1), \|\tilde{α}^1 - u_n^{(j)}\|_{α,Ω}\},
\]

\[
\|\tilde{u}_n^{(j)} - E\tilde{u}_n^{(j)}\|_{0,Ω} ≤ C_{eo}\eta_{a}(V_H)\|\tilde{u}_n^{(j)} - E\tilde{u}_n^{(j)}\|_{α,Ω},
\]

\[
\|\tilde{u}_n^{(j)} - E\tilde{u}_n^{(j)}\|_{0,Ω} ≤ C_{eo}\eta_{a}(V_H)\|\tilde{u}_n^{(j)} - E\tilde{u}_n^{(j)}\|_{α,Ω},
\]

\[
\|\tilde{u}_n^{(j)} - E\tilde{u}_n^{(j)}\|_{α,Ω} ≤ C_{eo}\eta_{a}(V_H)\|\tilde{u}_n^{(j)} - E\tilde{u}_n^{(j)}\|_{α,Ω},
\]

where $C_{ea} = \hat{C}_{ea}$ and $C_{eo} = 2\hat{C}_{eo}$. 


2.3 Error estimate for eigenfunction approximation

For \( k \in \mathbb{N}, j \geq 0, \) we define \( f_k^{(j)} \) and \( w_k^{(j)} \) by
\[
 f_k^{(j)} = \frac{\tilde{f}_k^{(j)}}{\|K \tilde{f}_k^{(j)}\|_{a,\Omega}} \quad \text{and} \quad w_k^{(j)} = K f_k^{(j)}, \tag{2.12}
\]
and \( f_k \) and \( w_k \) by
\[
 f_k = \frac{\tilde{f}_k}{\|K f_k\|_{a,\Omega}} \quad \text{and} \quad w_k = K f_k. \tag{2.13}
\]

It is obvious that \( \|w_k^{(j)}\|_{a,\Omega} = \|w_k\|_{a,\Omega} = 1. \) From step 3 of Adaptive Algorithm C, the following properties hold
\[
 u_k^{(j)} = \frac{R_k K \tilde{f}_k^{(j)}}{\|R_k K \tilde{f}_k^{(j)}\|_{a,\Omega}} = \frac{R_k w_k^{(j)}}{\|R_k w_k^{(j)}\|_{a,\Omega}} \quad \text{and} \quad u_k = \frac{R_k K \tilde{f}_k}{\|R_k K f_k\|_{a,\Omega}} = \frac{R_k w_k}{\|R_k w_k\|_{a,\Omega}}. \tag{2.14}
\]

**Lemma 2.6.** For any functions \( v \in V \) and \( w \in V \) satisfying \( \|v\|_{a,\Omega} = \|w\|_{a,\Omega} = 1, \) we have the following inequality
\[
 \|v - E v\|_{a,\Omega} \leq \inf_{\tau \in \mathbb{R}} \|w - \tau v\|_{a,\Omega} + \|w - E w\|_{a,\Omega}. \tag{2.15}
\]

**Proof.** By the definition of \( E \) in (2.11), we have \( \|v - E v\|_{a,\Omega} \leq \|v - \tau_0 E w\|_{a,\Omega} \) for any \( \tau_0 \in \mathbb{R}. \) The triangle inequality leads to the following estimate
\[
 \|v - \tau_0 E w\|_{a,\Omega} \leq \|v - \tau_0 w\|_{a,\Omega} + |\tau_0| \|w - E w\|_{a,\Omega}. \tag{2.16}
\]

We choose \( \tau_0 \) such that \( \|v - \tau_0 w\|_{a,\Omega} = \inf_{\tau \in \mathbb{R}} \|v - \tau w\|_{a,\Omega}. \) It is obvious that \( |\tau_0| \leq 1. \) Then from (2.16), we obtain
\[
 \|v - E v\|_{a,\Omega} \leq \inf_{\tau \in \mathbb{R}} \|v - \tau w\|_{a,\Omega} + \|w - E w\|_{a,\Omega} = \inf_{\tau \in \mathbb{R}} \|w - \tau v\|_{a,\Omega} + \|w - E w\|_{a,\Omega}.
\]

This is the desired result (2.15) and the proof is complete. \( \square \)

From (2.14), we know that \( \inf_{\tau \in \mathbb{R}} \|u_k^{(j)} - \tau u_k^{(j)}\|_{a,\Omega} = \|u_k^{(j)} - R_k w_k^{(j)}\|_{a,\Omega}. \) Then the following theorem is a direct result of Lemma 2.6.

**Theorem 2.2.** Let \( u_k^{(j)} \in V \) and \( u_k \in V \) be produced by Adaptive Algorithm C, \( w_k^{(j)} \) and \( w_k \) be defined by (2.12) and (2.13), respectively. Then we have
\[
 \|u_k^{(j)} - E u_k^{(j)}\|_{a,\Omega} \leq \|(I - R_k) w_k^{(j)}\|_{a,\Omega} + \|u_k^{(j)} - E w_k^{(j)}\|_{a,\Omega}, \tag{2.17}
\]
\[
 \|u_k - E u_k\|_{a,\Omega} \leq \|(I - R_k) w_k\|_{a,\Omega} + \|u_k - E w_k\|_{a,\Omega}. \tag{2.18}
\]

Theorem 2.2 establishes a basic relation between the error estimates \( \|u_k - E u_k\|_{a,\Omega} \) of the finite element approximation produced by Adaptive Algorithm C and \( \|(I - R_k) w_k\|_{a,\Omega} \) of the associated finite element projection.

**Lemma 2.7.** For any given function \( v \in V \) satisfying \( \|v\|_{a,\Omega} = 1 \) and \( a(v, \tilde{u}^1) \geq 0, \) the following inequality holds
\[
 \|v - \tilde{u}^1\|_{a,\Omega} \leq 2 \|v - E v\|_{a,\Omega}. \tag{2.19}
\]

**Proof.** The triangle inequality implies the following estimate
\[
 \|v - \tilde{u}^1\|_{a,\Omega} \leq \|v - E v\|_{a,\Omega} + \|E v - \tilde{u}^1\|_{a,\Omega}. \tag{2.20}
\]
Then we only need to prove that
\[ \|Ev - \hat{u}^1\|_{a, \Omega} \leq \|v - Ev\|_{a, \Omega}. \]
Since \(\|Ev\|_{a, \Omega} \leq \|v\|_{a, \Omega} = 1\) and \(a(v, \hat{u}^1) \geq 0\), the following inequality holds
\[ \|Ev - \hat{u}^1\|_{a, \Omega}^2 = (1 - \|Ev\|_{a, \Omega})^2 \leq 1 - \|Ev\|_{a, \Omega}^2 = \|v - Ev\|_{a, \Omega}^2. \tag{2.21} \]

The desired result (2.19) can be deduced by combining (2.20) and (2.21). \(\Box\)

The following lemma gives the estimate for \(\|w_k^{(j)} - Ew_k^{(j)}\|_{a, \Omega}\), which is the second term in the right hand side of (2.17).

**Lemma 2.8.** The following three propositions hold

1. If \(k = n\ell \) for any \(1 \neq \ell \in \mathbb{N}\), which implies \(\tilde{f}_k^{(j)} = \tilde{u}_k^{(j)} \) \((j \geq 0)\) is produced by the eigenvalue solving module ESOLVE in step 5 of Adaptive Algorithm C, we have
   \[ \|w_{n\ell}^{(j+1)} - Ew_{n\ell}^{(j)}\|_{a, \Omega} \leq \frac{C_{wu}}{2\lambda^1}\|K\tilde{u}_{n\ell}^{(j)}\|_{a, \Omega} \eta_a(V_H)\|u_{n\ell}^{(j)} - E\tilde{u}_{n\ell}^{(j)}\|_{a, \Omega}, \tag{2.22} \]
   where
   \[ C_{wu} := \frac{4C_{ed}C_{ca}\lambda^1}{c_a}. \]

2. If \(k = n\ell + 1 \) for any \(1 \neq \ell \in \mathbb{N}\), which implies \(\tilde{f}_k^{(0)} = \tilde{u}_{k-1}\) is produced by the eigenvalue solving module ESOLVE in step 6 of Adaptive Algorithm C, we have
   \[ \|w_{n\ell+1}^{(0)} - Ew_{n\ell+1}^{(0)}\|_{a, \Omega} \leq \frac{C_{wu}}{2\lambda^1}\|K\tilde{u}_{n\ell}^{(0)}\|_{a, \Omega} \eta_a(V_H)\|u_{n\ell}^{(0)} - E\tilde{u}_{n\ell}^{(0)}\|_{a, \Omega}. \tag{2.23} \]

3. If \(n\ell + 1 < k \leq n\ell + 1\) for any \(\ell \in \mathbb{N}\), which implies \(\tilde{f}_k^{(0)} = u_{k-1}\) is produced by the boundary value problem solving module LSOLVE in step 3 of Adaptive Algorithm C, we have
   \[ \|w_k^{(0)} - Ew_k^{(0)}\|_{a, \Omega} \leq \frac{C_{wu}\eta_a(V_{k-1})\|(I - R_{k-1})w_{k-1}\|_{a, \Omega}}{2\lambda^1\|Kw_{k-1}\|_{a, \Omega}} + \|w_{k-1} - Ew_{k-1}\|_{a, \Omega}, \tag{2.24} \]
   where
   \[ C_{wu} := \frac{2C_{an}\lambda^1}{c_a}. \]

**Proof.** If \(k = n\ell \) for any \(1 \neq \ell \in \mathbb{N}\), from the algorithm definition, we have
\[ w_k^{(j+1)} = K\tilde{f}_k^{(j+1)} = \frac{K\tilde{u}_k^{(j)}}{\|K\tilde{u}_k^{(j)}\|_{a, \Omega}} \text{ for } j \geq 0. \]

By the optimality of spectral projection \(E\) in (2.11), the following inequality holds
\[ \|w_k^{(j+1)} - Ew_k^{(j+1)}\|_{a, \Omega} \leq \frac{\|w_k^{(j+1)} - E\tilde{u}_k^{(j)}\|_{a, \Omega}}{\lambda^1\|K\tilde{u}_k^{(j)}\|_{a, \Omega}}. \tag{2.25} \]

Since \(E\tilde{u}_k^{(j)} \in M(\lambda^1)\), we have \(\tilde{u}_k^{(j)} = \lambda^1 KE\tilde{u}_k^{(j)}\). Combining (2.6), (2.13), (2.25), Corollary 2.1 and Lemma 2.7 leads to the following estimates
\[ \|w_k^{(j+1)} - Ew_k^{(j+1)}\|_{a, \Omega} \leq \frac{\|K(\tilde{u}_k^{(j)} - E\tilde{u}_k^{(j)})\|_{a, \Omega}}{\|K\tilde{u}_k^{(j)}\|_{a, \Omega}} \leq \frac{\|\tilde{u}_k^{(j)} - E\tilde{u}_k^{(j)}\|_{0, \Omega}}{c_a\|K\tilde{u}_k^{(j)}\|_{a, \Omega}}. \]
Let us define \( \tau \) the following estimate holds from Lemma 2.6, the following inequality holds.

From (2.6) and Lemma 2.1, we can derive.

\[
\sum_{i=1}^{\infty} \frac{\beta_i}{\lambda_i} \|	ilde{u}_k - \tilde{u}_k^{(j)} - E\tilde{u}_k^{(j)}\|_a, \Omega 
\leq \left( \sum_{i=1}^{\infty} \frac{\beta_i}{\lambda_i} \right)^2 = 1 - \beta_1^2 = \|w_{k-1} - Ew_{k-1}\|_{a, \Omega}^2.
\]

which is the desired result (2.24).

If \( k = n_\ell + 1 \) for any \( 1 \neq \ell \in \mathbb{N} \), we can prove (2.23) with a similar procedure as above.

If \( n_\ell + 1 < k \leq n_{\ell+1} \) for any \( \ell \in \mathbb{N} \), from the algorithm definition, we have

\[
w_k^{(0)} = \frac{Ku_{k-1}}{\|Ku_{k-1}\|_{a, \Omega}}.
\]

Let us introduce an auxiliary function

\[
\tilde{w}_k = \frac{Kw_{k-1}}{\|Kw_{k-1}\|_{a, \Omega}}.
\]

From Lemma 2.6, the following inequality holds

\[
\|w_k^{(0)} - Ew_k^{(0)}\|_{a, \Omega} \leq \inf_{\tau \in \mathbb{R}} \|\tilde{w}_k - \tau w_k^{(0)}\|_{a, \Omega} + \|\tilde{w}_k - E\tilde{w}_k\|_{a, \Omega}.
\]

Let us define \( \tau_2 \) as follows

\[
\tau_2 = \frac{\|Ku_{k-1}\|_{a, \Omega} \|R_{k-1}w_{k-1}\|_{a, \Omega}}{\|Kw_{k-1}\|_{a, \Omega}}.
\]

Since

\[
u_{k-1} = \frac{R_{k-1}w_{k-1}}{\|R_{k-1}w_{k-1}\|_{a, \Omega}},
\]

the following estimate holds

\[
\inf_{\tau \in \mathbb{R}} \|\tilde{w}_k - \tau w_k^{(0)}\|_{a, \Omega} \leq \|\tilde{w}_k - \tau_2 w_k^{(0)}\|_{a, \Omega} = \frac{\|K(I - R_{k-1})w_{k-1}\|_{a, \Omega}}{\|Kw_{k-1}\|_{a, \Omega}}.
\]

From (2.29) and Lemma 2.1 we can derive

\[
\|K(I - R_{k-1})w_{k-1}\|_{a, \Omega} \leq \frac{\|K(I - R_{k-1})w_{k-1}\|_{0, \Omega}}{\|Kw_{k-1}\|_{a, \Omega}} \leq \frac{C_{an}}{c_a} \eta_a(V_{k-1}) \|I - R_{k-1}\|_{a, \Omega}.
\]

Then combining (2.29) and (2.30) leads to

\[
\inf_{\tau \in \mathbb{R}} \|\tilde{w}_k - \tau w_k^{(0)}\|_{a, \Omega} \leq \frac{C_{an}}{c_a} \eta_a(V_{k-1}) \|I - R_{k-1}\|_{a, \Omega}.
\]

We assume \( w_{k-1} \in V \) has the expansion \( w_{k-1} = \sum_{i=1}^{\infty} \beta_i \hat{u}_i \). Then

\[
Kw_{k-1} = \sum_{i=1}^{\infty} \frac{\beta_i}{\lambda_i} \hat{u}_i.
\]

By the definition of spectral projection \( E \) in (2.11), (2.27) and the fact \( \|w_{k-1}\|_{a, \Omega} = \sum_{i=1}^{\infty} \beta_i^2 = 1 \), we have

\[
\|\tilde{w}_k - E\tilde{w}_k\|_{a, \Omega}^2 = \|\tilde{u}_k\|_{a, \Omega}^2 - \|E\tilde{w}_k\|_{a, \Omega}^2 = 1 - \frac{\left( \frac{\beta_i}{\lambda_i} \right)^2}{\sum_{i=1}^{\infty} \left( \frac{1}{\lambda_i} \right)^2}.
\]

From (2.27), (2.31) and (2.33), we can obtain the desired result (2.24) and the proof is complete. \( \square \).
From Lemma 2.8 it is required to obtain the lower bounds of $K\bar{u}_{n_{\ell}}^{(j)}$, $K\bar{u}_{n_{\ell}}$, and $Kw_k$. For this aim, we first state the following lemma.

**Lemma 2.9.** For any function $v \in V$ with $\|v\|_{a,\Omega} = 1$, we have the following inequality

$$
\frac{1}{\lambda^1} - \frac{2\|v - Ev\|_{a,\Omega}}{c_a^2} \leq \|Kv\|_{a,\Omega}.
$$

(2.34)

**Proof.** First, from (2.6) and Lemma 2.7 we have

$$
\|K\bar{u}^1\|_{a,\Omega} \leq \|Kv\|_{a,\Omega} \leq \|K\bar{u}^1\|_{a,\Omega} \leq \frac{2\|v - \bar{u}^1\|_{a,\Omega}}{c_a^2} \leq \frac{2\|v - Ev\|_{a,\Omega}}{c_a^2}.
$$

(2.35)

The property (2.6) for $\bar{u}^1$ leads to the following equalities

$$
\|K\bar{u}^1\|_{a,\Omega} = \frac{\|\bar{u}^1\|_{a,\Omega}}{\lambda^1} = \frac{1}{\lambda^1}.
$$

(2.36)

Then the desired result (2.34) can be deduced from (2.35) and (2.36).

**Corollary 2.2.** When $H$ is small enough, for $\ell \in N$ and $j \geq 0$, we have

$$
\frac{1}{2\lambda^1} \leq \|K\bar{u}_{n_{\ell}}\|_{a,\Omega} \quad \text{and} \quad \frac{1}{2\lambda^1} \leq \|K\bar{u}_{n_{\ell}}^{(j)}\|_{a,\Omega}.
$$

(2.37)

**Proof.** We only prove the first inequality and the second one can be proved by a similar procedure. For any $\ell \in N$, Lemma 2.9 implies

$$
\frac{1}{\lambda^1} - \frac{2\|\bar{u}_{n_{\ell}} - E\bar{u}_{n_{\ell}}\|_{a,\Omega}}{c_a^2} \leq \|K\bar{u}_{n_{\ell}}\|_{a,\Omega}.
$$

(2.38)

From Corollary 2.1 when $H$ is small enough, the following inequalities hold

$$
\|\bar{u}_{n_{\ell}} - E\bar{u}_{n_{\ell}}\|_{a,\Omega} \leq C_a\delta V_{\theta_{\ell}}(\lambda^1) \leq \frac{c_a^2}{4\lambda^1}.
$$

(2.39)

Then from (2.38) and (2.39), we can obtain the desired result (2.37) and the proof is complete.

**Lemma 2.10.** When $H$ is small enough, if $\bar{u}_{n_{\ell}}^{(i)}$ $(0 \leq i \leq j)$ produced by Adaptive Algorithm $C$ satisfies

$$
\eta_{n_{\ell}}(\bar{f}_{n_{\ell}}^{(i)}, \bar{u}_{n_{\ell}}^{(i)}; T_{n_{\ell}}) \leq \theta_{\ell}^{i+1} \eta_{\ell-1}, \quad 0 \leq i \leq j,
$$

(2.40)

we have the following property

$$
\|u_{n_{\ell}}^{(j)} - Ev_{n_{\ell}}^{(j)}\|_{a,\Omega} \leq 4\theta_{\ell}^{j+1} \lambda C_{\text{up}}\eta_{\ell-1} + (C_wu_{\ell}a(V_H))^{\ell} \|u_{n_{\ell}}^{(0)} - Ev_{n_{\ell}}^{(0)}\|_{a,\Omega}.
$$

(2.41)

**Proof.** We prove (2.41) by the induction method. It is obvious that (2.41) holds for $j = 0$. Now we suppose (2.41) holds for the case $j - 1$ and consider the case $j$. From Theorem 2.2, Lemmas 2.3 2.8 and Corollary 2.2 we have

$$
\|u_{n_{\ell}}^{(j)} - Ev_{n_{\ell}}^{(j)}\|_{a,\Omega} \leq \|(I - R_{n_{\ell}})w_{n_{\ell}}^{(j)}\|_{a,\Omega} + \|w_{n_{\ell}}^{(j)} - Ev_{n_{\ell}}^{(j)}\|_{a,\Omega} \leq C_{\text{up}}\eta(f_{n_{\ell}}^{(j)}, T_{n_{\ell}}) + C_wu_{\ell}a(V_H)\|u_{n_{\ell}}^{(j-1)} - Ev_{n_{\ell}}^{(j-1)}\|_{a,\Omega}.
$$

(2.42)

Note that $\bar{f}_{n_{\ell}}^{(j)} = \bar{u}_{n_{\ell}}^{(j-1)}$ and $\|\bar{u}_{n_{\ell}}^{(j-1)}\|_{a,\Omega} = 1$. Then from (2.37) and (2.40), we can derive

$$
\eta(f_{n_{\ell}}^{(j)}; T_{n_{\ell}}) = \frac{1}{\|Kf_{n_{\ell}}^{(j)}\|_{a,\Omega}} \eta(f_{n_{\ell}}^{(j)}; T_{n_{\ell}}) \leq 2\lambda\theta_{\ell}^{j+1} \eta_{\ell-1}.
$$

(2.43)
Since $H$ is small enough, the inequality $2C_{wu}n_\Omega(V_H) \leq \theta_2$ holds. Combining (2.42) and (2.43) leads to
\[
\|u_n^{(j)} - E\eta_{n_\Omega}^{(j)}\|_{a,\Omega} \leq 2C_{up}\hat{\lambda}_2^{j+1}n_{\ell-1} + C_{wu}n_\Omega(V_H)\left(4\theta_2^2\hat{\lambda}_1 C_{up}n_{\ell-1} + (C_{wu}n_\Omega(V_H))^j\|u_n^{(0)} - E\eta_{n_\Omega}^{(0)}\|_{a,\Omega}\right)
= (2\theta_2 + 4C_{wu}n_\Omega(V_H))\theta_2^2\hat{\lambda}_1 C_{up}n_{\ell-1} + (C_{wu}n_\Omega(V_H))^j\|u_n^{(0)} - E\eta_{n_\Omega}^{(0)}\|_{a,\Omega}
\leq 4\theta_2^{j+1}\hat{\lambda}_1 C_{up}n_{\ell-1} + (C_{wu}n_\Omega(V_H))^j\|u_n^{(0)} - E\eta_{n_\Omega}^{(0)}\|_{a,\Omega},
\]
which means (2.41) holds for the case $j$ and the proof is complete.

In this paper, we assume $\hat{u}^1$ is not a piecewise polynomial. Then from Lemma 2.10, we can conclude that there exists $j_{n_\ell} \in \mathbb{N}$ such that
\[
\eta(f^{(j_{n_\ell})};\hat{T}_{n_\ell}) > \theta_2^{j_{n_\ell}+1}n_{\ell-1}. \tag{2.44}
\]
Otherwise, from (2.41), we have
\[
\lim_{j \to +\infty} \|u_n^{(j)} - E\eta_{n_\Omega}^{(j)}\|_{a,\Omega} = 0,
\]
which implies $\hat{u}^1 \in V_{n_\ell}$.

As for the $k$-th level mesh satisfying $n_\ell < k < n_{\ell+1}$ for any $\ell \in \mathbb{N}$, it is easy to know that $\bar{f}_k = \bar{f}_k^{(0)}$ and
\[
\eta(\bar{f}_k^{(0)}; T_k) > \theta_2 \eta_k.
\]

Thus we define $j_k = 0$ in this case. Then from Adaptive Algorithm $C$, we know
\[
\bar{f}_k = \bar{f}_k^{(j_k)}, \quad \bar{u}_k = \bar{u}_k^{(j_k)}, \quad f_k = f_k^{(j_k)}, \quad w_k = w_k^{(j_k)} \quad \text{and} \quad u_k = u_k^{(j_k)}.
\]
Moreover, on mesh $T_{n_\ell}$ we have
\[
\eta(\bar{f}_k^{(j_{n_\ell}-1)}; T_{n_\ell}) \leq \theta_2^{j_{n_\ell}}\eta_{n_\ell-1} \leq \theta_2^{j_{n_\ell}-1}\eta(\bar{f}_k^{(j_{n_\ell}-1)}; T_{n_\ell-1}),
\]
which suggests that the number of iterations depends on the ratio of discretization errors on meshes $T_{n_\ell}$ and $T_{n_\ell-1}$. In general case, we assume these ratios are bounded by a constant and the numbers of iterations are bounded too.

## 3 Convergence analysis

In this section, we deduce the convergence of Adaptive Algorithm $C$ with the help of results introduced in Section 2.

### 3.1 Properties of the first two levels

This subsection is only concerned with the first two levels of Adaptive Algorithm $C$ and shows some properties. From the definition of Adaptive Algorithm $C$, we have
\[
a(u_1, v_1) = \lambda_1(u_1, v_1), \quad \forall v_1 \in V_1
\]
and
\[
a(\bar{u}_2^{(0)}, v_2) = (u_2, v_2), \quad \forall v_2 \in V_2.
\]
Since $\bar{f}_1 = u_1 = \bar{u}_1 = f_2^{(0)}$ and the definition (2.13), the following properties hold
\[
f_1 = f_2^{(0)} = \frac{u_1}{\|Ku_1\|_{a,\Omega}} \quad \text{and} \quad w_1 = w_2^{(0)} = \frac{Ku_1}{\|Ku_1\|_{a,\Omega}}.
\]

First, we state error estimates for $\|w_1 - Ew_1\|_{a,\Omega}$ and $\|w_2^{(0)} - Ew_2^{(0)}\|_{a,\Omega}$ in the following lemma.
Lemma 3.1. When the mesh size $H$ is small enough, we have following estimates for the first two levels

$$
\| w_1 - Ew_1 \|_{a, \Omega} \leq C_{w,1} \eta_a (V_H) \eta (f_1; T_1),
$$  
(3.1)

$$
\| w_2^{(0)} - Ew_2^{(0)} \|_{a, \Omega} \leq C_{w,1} \eta_a (V_H) \eta (f_1; T_1),
$$  
(3.2)

where

$$
C_{w,1} = \frac{4 \lambda^1 C_{e0} C_{up}}{c_a}.
$$  

Proof. From (2.37) for $\ell = 1$ and analogous to (2.26), we can derive

$$
\| w_1 - Ew_1 \|_{a, \Omega} \leq \frac{C_{e0}}{c_a \| Ku_1 \|_{a, \Omega}} \eta_a (V_1) \| u_1 - E u_1 \|_{a, \Omega} \leq \frac{C_{e0}}{c_a \| Ku_1 \|_{a, \Omega}} \eta_a (V_H) \| u_1 - E u_1 \|_{a, \Omega}.
$$  
(3.3)

From Theorem 2.2 the following inequality holds

$$
\| u_1 - E u_1 \|_{a, \Omega} \leq \| u_1 - R_1 w_1 \|_{a, \Omega} + \| w_1 - E w_1 \|_{a, \Omega}.
$$  
(3.4)

Let us choose $H$ to be small enough such that

$$
2 \lambda^1 C_{e0} \eta_a (V_H) \leq \frac{c_a}{2}.
$$  
(3.5)

Combining (3.3), (3.4) and (3.5) leads to

$$
\| w_1 - Ew_1 \|_{a, \Omega} \leq \frac{2 \lambda^1 C_{e0}}{c_a - 2 \lambda^1 C_{e0} \eta_a (V_H)} \eta_a (V_H) \| u_1 - R_1 w_1 \|_{a, \Omega} \leq \frac{4 \lambda^1 C_{e0}}{c_a} \eta_a (V_H) \| u_1 - R_1 w_1 \|_{a, \Omega}.
$$  
(3.6)

From Lemma 2.3 and (3.1), we obtain the desired result (3.1). Since $w_2^{(0)} = w_1$, $\| w_2^{(0)} - Ew_2^{(0)} \|_{a, \Omega}$ has the same estimate (3.2).

Lemma 3.2. When the mesh size $H$ is small enough, we have the following estimate for the second level

$$
\| w_2 - Ew_2 \|_{a, \Omega} \leq C_{w,2} \eta_a (V_H) \eta (f_2; T_2),
$$  
(3.7)

where

$$
C_{w,2} = \max \left\{ \frac{C_{w,1}}{\theta_2}, \frac{8 C_{wu} \lambda^2 C_{up}}{\theta_2 c_a^2} \right\}.
$$

Proof. We prove the assertion case by case.

Case 1: If $\eta (\tilde{f}_2^{(0)}; T_2) > \theta_2 \eta (\tilde{f}_1; T_1)$, then from step 5 of Adaptive Algorithm $C$ and definition (2.13), we have $\tilde{f}_2 = \tilde{f}_2^{(0)} = \tilde{f}_1$ and $w_2 = w_2^{(0)}$. Thus, the following inequality holds

$$
\eta (f_2; T_2) = \eta (\tilde{f}_2^{(0)}; T_2) > \frac{\theta_2 \eta (\tilde{f}_1; T_1)}{\| K \tilde{f}_2^{(0)} \|_{a, \Omega}} = \frac{\theta_2 \eta (f_1; T_1)}{\| K f_1 \|_{a, \Omega}} = \theta_2 \eta (f_1; T_1).
$$  
(3.8)

From Lemma 3.1 and (3.8), we have

$$
\| w_2 - Ew_2 \|_{a, \Omega} \leq C_{w,1} \eta_a (V_H) \eta (f_1; T_1) < \frac{C_{w,1}}{\theta_2} \eta_a (V_H) \eta (f_2; T_2).
$$  
(3.9)

Case 2: If $\eta (\tilde{f}_2^{(0)}; T_2) \leq \theta_2 \eta (\tilde{f}_1; T_1)$, then we have $j_2 > 0$ and $n_2 = 2$. From Lemma 2.10 inequalities (2.22), (2.37), (2.41) and (2.44), we can derive

$$
\| w_2^{(j_2)} - Ew_2^{(j_2)} \|_{a, \Omega} \leq C_{wu} \eta_a (V_H) \| u_2^{(j_2-1)} - E u_2^{(j_2-1)} \|_{a, \Omega}
$$

Since $\eta (\tilde{f}_2^{(0)}; T_2) \leq \theta_2 \eta (\tilde{f}_1; T_1)$, we have $\| u_2^{(j_2-1)} - E u_2^{(j_2-1)} \|_{a, \Omega} \leq C_{wu} \eta_a (V_H) \| u_2^{(j_2-1)} - E u_2^{(j_2-1)} \|_{a, \Omega}$.
cases of the corresponding linear boundary value problem. There exists a constant Lemma 3.3. 

As for reference indicator as \( \theta \) give an uniform lower bound \( \eta \). We use the fact that the mesh size \( a \) to the following bounds for \( K \bar{f}_1 \| a,\Omega \) and \( \bar{K} \bar{f}_2 \| a,\Omega \)

\[
\frac{1}{2\lambda^2} \leq \| K \bar{f}_1 \| a,\Omega \leq \| \bar{f}_1 \| a,\Omega = \frac{1}{c^2 a}, \quad \frac{1}{2\lambda^2} \leq \| K \bar{f}_2 \| a,\Omega \leq \frac{1}{c^2 a}. \tag{3.11}
\]

By a similar procedure as (3.8), we can derive

\[
\eta(f_2^0; \bar{T}_2) \leq \theta_2 \eta(f_1; \bar{T}_1). \tag{3.12}
\]

From (2.17), (3.2), (3.11), (3.12) and Lemma 2.3, we have

\[
\begin{align*}
\| u_2^{(0)} - E u_2^{(0)} \| a,\Omega & \leq \| (I - R_2) u_2^{(0)} \| a,\Omega + \| u_2^{(0)} - E u_2^{(0)} \| a,\Omega \\
& \leq C_{\text{up}} \eta(f_2^{(0)}; \bar{T}_2) + C_{\text{wu}} \eta_a(V_H) \eta(f_1; \bar{T}_1) \leq (C_{\text{up}} \theta_2 + C_{\text{wu}} \eta_a(V_H)) \eta(f_1; \bar{T}_1) \\
& \leq 2(C_{\text{up}} \theta_2 + C_{\text{wu}} \eta_a(V_H)) \lambda^1 \eta(\bar{f}_2^1; \bar{T}_1). \tag{3.13}
\end{align*}
\]

Thus, combining (2.44) and (3.13) leads to following inequalities

\[
\begin{align*}
& (C_{\text{wu}} \eta_a(V_H))^j_{\bar{T}_2} \| u_2^{(0)} - E u_2^{(0)} \| a,\Omega \\
& \leq 2(C_{\text{up}} \theta_2 + C_{\text{wu}} \eta_a(V_H)) \lambda^1 \frac{C_{\text{wu}} \eta_a(V_H)}{\theta_2} \left( \frac{C_{\text{wu}} \eta_a(V_H)}{\theta_2} \right)^j_{\bar{T}_2} \| \bar{f}_2^1 \| a,\Omega \\
& \leq \frac{4C_{\text{wu}} \lambda^1 C_{\text{up}} \eta_a(V_H)}{\theta_2} \eta(\bar{f}_2^1; \bar{T}_2), \tag{3.14}
\end{align*}
\]

where we use the fact that the mesh size \( H \) is large enough such that \( C_{\text{wu}} \eta_a(V_H) \leq C_{\text{up}} \theta_2 \) and \( 2C_{\text{wu}} \eta_a(V_H) \leq \theta_2 \). From (3.10), (3.11) and (3.14), the following estimates hold

\[
\| u_2^{(j2)} - E u_2^{(j2)} \| a,\Omega \leq 8C_{\text{wu}} \lambda^1 \frac{C_{\text{up}} \eta_a(V_H)}{\theta_2} \eta(\bar{f}_2^{(j2)}; \bar{T}_2) \leq \frac{8C_{\text{wu}} \lambda^1 C_{\text{up}}}{\theta_2^2 c^2 a} \eta_a(V_H) \eta(\bar{f}_2^{(j2)}; \bar{T}_2). \tag{3.15}
\]

The combination of (3.9) and (3.15) leads to the desired result (3.7). \( \Box \)

**Remark 3.1.** The above proof indicates that the keys to deduce the result \( \| \bar{f}_1 \| a,\Omega \) are the property \( \| K \bar{f}_1 \| a,\Omega \) and the bounds of \( \| K \bar{f}_2 \| a,\Omega \) and \( \| \bar{K} \bar{f}_2 \| a,\Omega \). In what follows, we will prove a similar property as (3.8) for general cases, namely, the estimate of \( \| u_2^{(0)} - E u_2^{(0)} \| a,\Omega \) is controlled by the estimate \( \eta_a(V_H) \eta(f_{n_1}; T_{n_1}) \) for \( n_1 < k < n_{\ell+1} \). The role of parameter \( \theta_2 \) in the Adaptive Algorithm 3 is to give an uniform lower bound \( \theta_2 \eta(f_{n_1}; T_{n_1}) \) of the error estimators \( \eta(f_k; T_k) \), whenever \( n_1 < k < n_{\ell+1} \). As for \( n_{\ell+1} \)-th level, the eigenvalue problem solving is designed to reduce the ratio of nonlinear error \( \| w_n^{(j)} - E w_n^{(j)} \| a,\Omega \) to discretization error \( \| w_{n+1}^{(j)} - R_{n+1} w_{n+1}^{(j)} \| a,\Omega \) to a small value and reset the reference indicator as \( \eta_{n+1} := \eta(f_{n_{\ell+1}}; T_{n_{\ell+1}}) \). As a result, we will prove that \( \| w_k - E w_k \| a,\Omega \) is a high order term with \( \| w_k - E w_k \| a,\Omega \) to the posterior error estimate of the corresponding linear boundary value problem.

The following lemmas will be used in our analysis to estimate \( \| (I - R_2) w_2 \| a,\Omega \).

**Lemma 3.3.** (3.12) There exists a constant \( C_{oa} \) which depends on \( A \), regularity constant \( \gamma^* \) and coefficient \( \varphi \), such that

\[
\text{osc}(Lv_k; T_k) \leq C_{oa} \| v_k \| a,\Omega, \quad \forall v_k \in V_k.
\]
Lemma 3.4. The following inequality holds
\[ \eta(f; \mathcal{T}_k) - \eta(g; \mathcal{T}_k) \leq \eta(f - g; \mathcal{T}_k), \quad \forall f, g \in L^2(\Omega). \]  \hfill (3.16)

Proof. For any element \( T \in \mathcal{T}_k \), we have
\[
\begin{align*}
\eta_k(f, R_kKf; T) - \eta_k(g, R_kKg; T) &= \left( h_T^2 \| f - LR_kKf \|_{0,T}^2 + \sum_{E \in \mathcal{E}_k, E \subset \partial T} h_E \| [\nabla(R_kKf)]_E \cdot \nu_E \|_{0,E}^2 \right)^{\frac{1}{2}} \\
&\quad - \left( h_T^2 \| g - LR_kKg \|_{0,T}^2 + \sum_{E \in \mathcal{E}_k, E \subset \partial T} h_E \| [\nabla(R_kKg)]_E \cdot \nu_E \|_{0,E}^2 \right)^{\frac{1}{2}} \\
&\leq \left( \left( \sum_{T \in \mathcal{T}_k} \eta_k(f, R_kKf; T) \right)^{\frac{1}{2}} - \left( \sum_{T \in \mathcal{T}_k} \eta_k(g, R_kKg; T) \right)^{\frac{1}{2}} \right) \\
&\quad + \sum_{E \in \mathcal{E}_k, E \subset \partial T} h_E \left( \left( \| [\nabla(R_kKf)]_E \cdot \nu_E \|_{0,E} - \| [\nabla(R_kKg)]_E \cdot \nu_E \|_{0,E} \right)^2 \right)^{\frac{1}{2}} \\
&\leq \eta_k(f - g, R_kK(f - g); T).
\end{align*}
\]

The previous inequality leads to the following inequalities
\[
\begin{align*}
\eta(f; \mathcal{T}_k) - \eta(g; \mathcal{T}_k) &= \left( \sum_{T \in \mathcal{T}_k} \eta_k(f, R_kKf; T) \right)^{\frac{1}{2}} - \left( \sum_{T \in \mathcal{T}_k} \eta_k(g, R_kKg; T) \right)^{\frac{1}{2}} \\
&\leq \left( \sum_{T \in \mathcal{T}_k} \eta_k(f, R_kKf; T) - \eta_k(g, R_kKg; T) \right)^{\frac{1}{2}} \\
&\leq \left( \sum_{T \in \mathcal{T}_k} \eta_k(f - g, R_kK(f - g); T) \right)^{\frac{1}{2}} = \eta(f - g; \mathcal{T}_k).
\end{align*}
\]

This is the desired result (3.16) and the proof is complete. \( \square \)

Lemma 3.5. The following inequality holds
\[ \eta(f; \mathcal{T}_k) - \eta(g; \mathcal{T}_k) \leq \frac{1 + C_{\text{osc}}}{C_{\text{low}}} \| K(f - g) \|_{a,\Omega}, \quad \forall f, g \in V_k. \]  \hfill (3.17)

Proof. From Lemmas 2.3, 3.3 and Remark 2.2 we have
\[
\begin{align*}
\eta(f; \mathcal{T}_k) - \eta(g; \mathcal{T}_k) \leq \eta(f - g; \mathcal{T}_k) &\leq \frac{1}{C_{\text{low}}} \left( \| (R_k - I)K(f - g) \|_{a,\Omega} + \text{osc}(LR_kK(f - g); \mathcal{T}_k) \right) \\
&\leq \frac{1 + C_{\text{osc}}}{C_{\text{low}}} \| K(f - g) \|_{a,\Omega},
\end{align*}
\]

which is the desired result (3.17) and the proof is complete. \( \square \)

To estimate \( \| (I - R_2)w_2 \|_{a,\Omega} \), we establish a relation between the finite element error estimates of two boundary value problems with the right hand side terms \( f_1 \) and \( f_2 \), respectively.

Lemma 3.6. The following relation holds
\[ \| (I - R_2)w_2 \|_{a,\Omega}^2 + \gamma \eta^2(f_2; \mathcal{T}_2) \leq \alpha_1^2 \left( \| (I - R_1)w_1 \|_{a,\Omega}^2 + \gamma \eta^2(f_1; \mathcal{T}_1) \right), \]  \hfill (3.18)

where
\[
\alpha_1^2 = \frac{(1 + \delta)\alpha^2 + \frac{8C_{a,n}^2C_{\text{osc}}}{\gamma} \left( 1 + \frac{1}{\delta} \right) \eta_0^2(V_H)}{1 - \frac{8C_{a,n}^2C_{\text{osc}}}{\gamma} \left( 1 + \frac{1}{\delta} \right) \eta_0^2(V_H)} \]  \hfill (3.19)
\[ \delta > 0, \quad C_{\omega} = 1 + \gamma \left( \frac{1 + C_{\omega}}{C_{\text{low}}} \right)^2. \]  

**Proof.** First, from the definition of \( R_2 \) and triangle inequality, the following inequalities hold
\[
\|(I - R_2)w_2\|_{a, \Omega} \leq \|(I - R_2)w_1\|_{a, \Omega} + \|(I - R_2)(w_1 - w_2)\|_{a, \Omega} \\
\leq \|(I - R_2)w_1\|_{a, \Omega} + \|w_1 - w_2\|_{a, \Omega}. \tag{3.21}
\]

From Lemma 3.6, we have
\[
\eta(f_2; T_2) \leq \|w_1 - w_2\|_{a, \Omega}. \tag{3.22}
\]

Combining (3.21) and (3.22) leads to
\[
\|(I - R_2)w_2\|_{a, \Omega} + \gamma \eta^2(f_2; T_2) \leq (1 + \delta)(\|(I - R_2)w_1\|_{a, \Omega} + \gamma \eta^2(f_1; T_2)) \\
+ \left(1 + \frac{1}{\delta}\right)C_{\omega} \|w_1 - w_2\|_{a, \Omega}, \tag{3.23}
\]

where \( \delta > 0 \) and \( C_{\omega} \) is defined by (3.20). For the last term of (3.23), by triangle inequality and Lemma 2.7, we have
\[
\|w_1 - w_2\|_{a, \Omega} \leq 2\|w_1 - Ew_1\|_{a, \Omega} + 2\|w_2 - Ew_2\|_{a, \Omega}. \tag{3.24}
\]

Then combining (3.1), (3.7) and (3.24) leads to
\[
\|w_1 - w_2\|_{a, \Omega} \leq \left(2C_{w, 1} \eta_a(VH)\eta(f_1; T_1) + 2C_{w, 2} \eta_a(VH)\eta(f_2; T_2)\right)^2 \\
\leq 8C_{w, 1}^2 \eta_a^2(VH)\eta^2(f_1; T_1) + 8C_{w, 2}^2 \eta_a^2(VH)\eta^2(f_2; T_2). \tag{3.25}
\]

From (3.22) and (3.25), the following inequalities hold
\[
\left(1 - \frac{8C_{w, 2} C_{\omega}}{\gamma} \left(1 + \frac{1}{\delta}\right)\eta_a^2(VH)\right)(\|(I - R_2)w_2\|_{a, \Omega}^2 + \gamma \eta^2(f_2; T_2)) \\
\leq \|(I - R_2)w_2\|_{a, \Omega}^2 + \gamma \eta^2(f_2; T_2) - 8C_{\omega} \left(1 + \frac{1}{\delta}\right)\eta_a^2(VH)C_{w, 2} \eta^2(f_2; T_2) \\
\leq (1 + \delta)(\|(I - R_2)w_1\|_{a, \Omega}^2 + \gamma \eta^2(f_1; T_2)) + 8C_{\omega} \left(1 + \frac{1}{\delta}\right)\eta_a^2(VH)C_{w, 2} \eta^2(f_1; T_1). \tag{3.26}
\]

Using Theorem 2.1 and the definitions
\[
f_1 = \frac{\tilde{f}_1}{\|K\tilde{f}_1\|_{a, \Omega}}, \quad w_1 = \frac{K\tilde{f}_1}{\|K\tilde{f}_1\|_{a, \Omega}},
\]

we have
\[
\|(I - R_2)w_1\|_{a, \Omega}^2 + \gamma \eta^2(f_1; T_2) \leq \hat{\alpha}^2(\|(I - R_1)w_1\|_{a, \Omega}^2 + \gamma \eta^2(f_1; T_1)), \tag{3.27}
\]

where \( \hat{\alpha} \in (0, 1) \). Combining (3.19), (3.26) and (3.27) leads to the desired result (3.18) and the proof is complete. \( \square \)

By the definition of \( \alpha_1 \) in (3.18), we can choose \( \delta \) and \( H \) small enough such that
\[ \hat{\alpha} < \alpha_1 < 1. \]

Furthermore, for a fixed constant \( \alpha \) satisfying \( \hat{\alpha} < \alpha < 1 \), we can choose \( \delta \) and \( H \) small enough such that
\[ \hat{\alpha} < \alpha_1 < \alpha < 1. \]

From Lemma 3.6, when \( H \) is small enough, we have
\[
\|(I - R_2)w_2\|_{a, \Omega}^2 + \gamma \eta^2(f_2; T_2) \leq \alpha^2(\|(I - R_1)w_1\|_{a, \Omega}^2 + \gamma \eta^2(f_1; T_1)). \tag{3.28}
\]
3.2 Induction procedure

In this subsection, the convergence for the general levels will be deduced. These results are provided as the preparation for the analysis of the convergence by the induction method for Adaptive Algorithm C proposed in this paper. For this aim, we define the following notation

\[
\tilde{C}_w^{(0)} = \frac{C_{wu} \alpha(C_{up}^2 + \gamma)^2}{(1 - \alpha^2)} + 2C_{wu}C_{up}, \tag{3.29}
\]

\[
\tilde{C}_w = \frac{2\tilde{C}_w^{(0)}\lambda^1}{\theta_2 c_{a}}, \tag{3.30}
\]

\[
C_w = \max \left\{ C_{w,1}, C_{w,2}, \tilde{C}_w \right\}. \tag{3.31}
\]

For the induction procedure, from (3.28), Lemmas 3.1 and 3.2 in the previous section, it is natural to suppose that

\[
\|(I - R_{k+1})w_{k+1}\|_{a,\Omega}^2 + \gamma \eta^2(f_{k+1}; T_{k+1}) \leq \alpha^2 \left(\|(I - R_k)w_k\|_{a,\Omega}^2 + \gamma \eta^2(f_k; T_k)\right) \tag{3.32}
\]

holds for \(1 \leq k \leq n\), and

\[
\|w_k - Ew_k\|_{a,\Omega} \leq C_w \eta_a(V_H)\eta(f_k; T_k) \tag{3.33}
\]

holds for \(1 \leq k \leq n + 1\).

Since \(C_{w,1} \leq C_w\) and \(C_{w,2} \leq C_w\), the properties (3.32), (3.33) hold for \(n = 1\) from Lemmas 3.1 and 3.2 and (3.28). Now we come to prove (3.32) holds for \(k = n + 1\) and (3.33) holds for \(k = n + 2\). In this subsection, we suppose \(n\ell < n + 2 \leq n\ell + 1\) for some \(\ell \in \mathbb{N}\).

**Lemma 3.7.** When the mesh size \(H\) and \(h_1\) are small enough, the following bounds hold

\[
\|Ku_k\|_{a,\Omega} \geq \frac{1}{2\lambda^1} \quad \text{and} \quad \|Kw_k\|_{a,\Omega} \geq \frac{1}{2\lambda^1}, \quad \text{for} \quad k \leq n + 1. \tag{3.34}
\]

**Proof.** By Theorem 2.2 the following inequality holds

\[
\|u_k - Eu_k\|_{a,\Omega} \leq \|(I - R_k)w_k\|_{a,\Omega} + \|w_k - Ew_k\|_{a,\Omega}. \tag{3.35}
\]

Since the mesh size \(H\) is small enough, (3.32) holds for \(k \leq n\) and (3.33) holds for \(k \leq n + 1\), we can derive

\[
\|u_k - Eu_k\|_{a,\Omega} \leq \|(I - R_k)w_k\|_{a,\Omega} + C_w \eta_a(V_H)\eta(f_k; T_k)
\]

\[
\leq \sqrt{2}\left(\|(I - R_k)w_k\|_{a,\Omega}^2 + \gamma \eta^2(f_k; T_k)\right)^{\frac{1}{2}} \leq \sqrt{2}\left(\|(I - R_1)w_1\|_{a,\Omega}^2 + \gamma \eta^2(f_1; T_1)\right)^{\frac{1}{2}}. \tag{3.36}
\]

Since the initial mesh size \(h_1\) is small enough, the following estimate holds

\[
\sqrt{2}\left(\|(I - R_1)w_1\|_{a,\Omega}^2 + \gamma \eta^2(f_1; T_1)\right)^{\frac{1}{2}} \leq \frac{c_{a}^2}{4\lambda^1}, \tag{3.37}
\]

which combined with (3.36) leads to

\[
\|u_k - Eu_k\|_{a,\Omega} \leq \frac{c_{a}^2}{4\lambda^1}. \tag{3.38}
\]

Similarly, the combination of (3.33), (3.36) and (3.37) implies

\[
\|w_k - Ew_k\|_{a,\Omega} \leq \frac{c_{a}^2}{4\lambda^1}. \tag{3.39}
\]

Then from Lemma 2.2, (3.38) and (3.39), we can obtain that the estimate (3.34) holds for \(k \leq n + 1\).  \(\Box\)
From Remark 3.3 in order to show (3.33) holds for \( k = n + 2 \), we need the bounds for \( \| K \tilde{w}_{n+2} \|_{a, \Omega} \), \( \| K \tilde{f}_{n+2} \|_{a, \Omega} \), and \( \| K \tilde{f}_{n+2} \|_{a, \Omega} \).

Note that \( \tilde{f}_{n_k} = \tilde{u}_{n_k}^{(j-1)} \) and \( \| \tilde{u}_{n_k}^{(j-1)} \|_{a, \Omega} = 1 \). From (2.6) and (2.37), we have the following bound for \( \| K \tilde{f}_{n_k} \|_{a, \Omega} \)

\[
\frac{1}{2\lambda^1} \leq \| K \tilde{f}_{n_k} \|_{a, \Omega} \leq \frac{\| f_{n_k} \|_{a, \Omega}}{\epsilon^2 a} = \frac{1}{\epsilon^2 a}. \tag{3.40}
\]

If \( n + 2 = n_k + 1 \), then \( \tilde{f}_{n+2} = \tilde{u}_{n_k} \) and \( \| \tilde{u}_{n_k} \|_{a, \Omega} = 1 \). Using (2.6) and (2.37) again, we have the following bound for \( \| K \tilde{f}_{n+2} \|_{a, \Omega} \)

\[
\frac{1}{2\lambda^1} \leq \| K \tilde{f}_{n+2} \|_{a, \Omega} \leq \frac{1}{\epsilon^2 a}. \tag{3.41}
\]

Else, if \( n_k + 1 < n + 2 \leq n_k + 1 \), we know that \( \tilde{f}_{n+2} = \tilde{u}_{n_k+1} \) and \( \| \tilde{u}_{n_k+1} \|_{a, \Omega} = 1 \). From (2.6) and (2.37), we still have the result (3.41).

Now, it remains to show the bound for \( \| K \tilde{w}_{n+2} \|_{a, \Omega} \). If \( n_k < n + 2 < n_k + 1 \), we already obtain the same estimate as (3.31) since \( \tilde{f}_{n+2} = \tilde{f}_{n+2}^{(0)} \) in this case. Else, in the case that \( n + 2 = n_k + 1 \), we know \( \tilde{f}_{n+2} = \tilde{u}_{n_k+1}^{(j-1)} \) and \( \| \tilde{u}_{n_k+1}^{(j-1)} \|_{a, \Omega} = 1 \). The following inequality can be deduced by using (2.6) and (2.37) again

\[
\frac{1}{2\lambda^1} \leq \| K \tilde{f}_{n+2} \|_{a, \Omega} \leq \frac{1}{\epsilon^2 a}. \tag{3.42}
\]

**Lemma 3.8.** When \( H \) is small enough, the following estimate holds

\[
\| w_{n+2}^{(0)} - E w_{n+2}^{(0)} \|_{a, \Omega} \leq C_w^{(0)} \eta_{a}(V_H) \eta(f_{n_k}; T_{n_k}), \tag{3.43}
\]

where constant \( C_w^{(0)} \) is defined by (3.29).

**Proof.** Since (3.33) holds for \( k = n_k \), using Lemma 2.26 and Theorem 2.2, we have

\[
\| u_{n_k} - E u_{n_k} \|_{a, \Omega} \leq \| (I - R_{n_k}) w_{n_k} \|_{a, \Omega} + \| w_{n_k} - E w_{n_k} \|_{a, \Omega} \leq \left( C_{up} + C_w \eta_{a}(V_H) \right) \eta(f_{n_k}; T_{n_k}). \tag{3.44}
\]

According to Lemma 2.26, 2.37, and 3.41, the following inequalities hold

\[
\| w_{n+1+1}^{(0)} - E w_{n+1+1}^{(0)} \|_{a, \Omega} \leq C_{wu} \eta_{a}(V_H) \| u_{n_k} - E u_{n_k} \|_{a, \Omega} \leq C_{wu} \left( C_{up} + C_w \eta_{a}(V_H) \right) \eta(f_{n_k}; T_{n_k}) \leq 2 C_{wu} C_{up} \eta_{a}(V_H) \eta(f_{n_k}; T_{n_k}), \tag{3.45}
\]

when \( H \) is small enough. From (3.45), we already prove (3.43) for case \( n + 2 = n_k + 1 \).

Now we consider the case \( n + 2 > n_k + 1 \). For \( n_k + 1 \leq i < n + 2 \leq n_k + 1 \), we observe that \( j_i = 0 \) and \( w_i = w_i^{(0)} \). Thus, from Lemma 2.26, (3.33) and recursive argument, \( \| w_{n+2}^{(0)} - E w_{n+2}^{(0)} \|_{a, \Omega} \) has following estimates

\[
\| w_{n+2}^{(0)} - E w_{n+2}^{(0)} \|_{a, \Omega} \leq C_{wu} \| \tilde{u}_{n_k+1} \|_{a, \Omega} \| (I - R_{n_k+1}) w_{n+1} \|_{a, \Omega} + \| w_{n_k+1}^{(0)} - E w_{n_k+1}^{(0)} \|_{a, \Omega} \leq C_{wu} \sum_{i=n_k+1}^{n+1} \eta_{a}(V_i) \| (I - R_{n_k+1}) w_i \|_{a, \Omega} + \| w_{n_k+1}^{(0)} - E w_{n_k+1}^{(0)} \|_{a, \Omega}. \tag{3.46}
\]

By using (3.45) and the following fact

\[
\eta_{a}(V_i) \leq \eta_{a}(V_H), \quad \forall i \in \mathbb{N},
\]
the following inequalities hold
\[
\sum_{i=n_{e}+1}^{n+1} \eta_{a}(V_{i}) \| (I - R_{i})w_{i} \|_{a, \Omega} \leq \eta_{a}(V_{H}) \sum_{i=n_{e}+1}^{n+1} \| (I - R_{i})w_{i} \|_{a, \Omega} \\
\leq \eta_{a}(V_{H}) \sum_{i=n_{e}+1}^{n+1} \left( \| (I - R_{i})w_{i} \|_{a, \Omega}^{2} + \gamma \eta^{2}(f_{i}; T_{i}) \right)^{\frac{1}{2}}. 
\]
(3.47)

Since (3.32) holds for \( k \leq n \), from Lemma 2.3, (2.12) and (3.47), we have
\[
\sum_{i=n_{e}+1}^{n+1} \eta_{a}(V_{i}) \| (I - R_{i})w_{i} \|_{a, \Omega} \leq \eta_{a}(V_{H}) \left( \| (I - R_{n_{e}})w_{n_{e}} \|_{a, \Omega}^{2} + \gamma \eta^{2}(f_{n_{e}}; T_{n_{e}}) \right)^{\frac{1}{2}} \sum_{i=1}^{n+1-n_{e}} \alpha^{i} \\
\leq \alpha(C_{w}^{2} + \gamma) \frac{1 - \alpha}{1 - \alpha} \eta_{a}(V_{H}) \eta(f_{n_{e}}; T_{n_{e}}). 
\]
(3.48)

Combining (3.45), (3.46), (3.48) and the definition (3.29) leads to the desired result (3.43).

So far, we already have the bounds for \( \| Kf_{n_{e}} \|_{a, \Omega} \), \( \| Kf_{n_{e}+2}^{(0)} \|_{a, \Omega} \), \( \| Kf_{n_{e}+2} \|_{a, \Omega} \) in (3.40), (3.41) and (3.42), respectively, as well as the estimate (3.43). Recall Remark 3.1 and the proofs of Lemmas 3.2 and 3.6 the following two lemmas can be deduced by a similar procedure.

**Lemma 3.9.** When \( H \) is small enough, the following estimate holds
\[
\| w_{n+2} - Ew_{n+2} \|_{a, \Omega} \leq C_{w} \eta_{a}(V_{H}) \eta(f_{n+2}; T_{n+2}), 
\]
(3.49)
where the constant \( C_{w} \) is defined by (3.31).

**Lemma 3.10.** When \( H \) is small enough, the following relation holds
\[
\| (I - R_{n_{e}})w_{n+2} \|_{a, \Omega}^{2} + \gamma \eta^{2}(f_{n+2}; T_{n+2}) \leq \alpha^{2} \| (I - R_{n_{e}+1})w_{n+1} \|_{a, \Omega}^{2} + \gamma \eta^{2}(f_{n+1}; T_{n+1}) 
\]
(3.50)

### 3.3 Main results of convergence

From the discussion in the last two subsections, we can conclude the convergence result of Adaptive Algorithm \( C \).

**Theorem 3.1.** Let \( \{ u_{k} \}_{k \in \mathbb{N}} \) and \( \{ T_{k} \}_{k \in \mathbb{N}} \) be produced by Adaptive Algorithm \( C \), \( w_{k} \) and \( f_{k} \) be defined by (2.13). When \( H \) is sufficiently small, for \( k \in \mathbb{N} \), the following inequalities hold
\[
\| w_{k} - Ew_{k} \|_{a, \Omega} \leq \alpha \eta_{a}(V_{H}) \eta(f_{k}; T_{k}), 
\]
(3.51)
\[
\| (I - R_{k+1})w_{k+1} \|_{a, \Omega}^{2} + \gamma \eta^{2}(f_{k+1}; T_{k+1}) \leq \alpha^{2} \| (I - R_{k})w_{k} \|_{a, \Omega}^{2} + \gamma \eta^{2}(f_{k}; T_{k}) 
\]
(3.52)
where \( \alpha \) satisfies \( \hat{\alpha} < \alpha < 1 \).

**Proof.** Based on the results in Sections 3.1 and 3.2, we give the proof by the induction method.

First, combining (3.1), (3.7) and (3.28) in Section 3.1 leads to that (3.51) holds for \( k \leq 2 \), (3.52) holds for \( k = 1 \).

Second, for \( n \in \mathbb{N} \), we suppose (3.51) holds for \( k \leq n+1 \) and (3.52) holds for \( k \leq n \). Then from (3.49), the desired result (3.51) holds for \( k = n + 2 \). From (3.50), we obtain (3.52) holds for \( k = n + 1 \).

Finally, we can deduce assertions (3.51) and (3.52) hold for all \( k \in \mathbb{N} \) by the induction method and the proof is complete. □

Now we state a useful Rayleigh quotient expansion for the eigenvalue which is expressed by the error of the eigenfunction approximation (see [22]).
Lemma 3.11. For any \( w \in V \), we have
\[
\frac{a(w, w)}{(w, w)} - \hat{\lambda} = \frac{a(w - \hat{u}^1, w - \hat{u}^1)}{(w, w)} - \hat{\lambda} (w - \hat{u}^1, w - \hat{u}^1).
\]

Using Theorem 3.3.1 and Lemma 3.11, we state the convergence property of eigenpair approximation \( (\lambda_k, u_k) \) produced by Adaptive Algorithm C as follows.

Theorem 3.2. When \( H \) is small enough, there exist constants \( C^u_{\text{con}} \) and \( C^\lambda_{\text{con}} \) such that the following inequalities hold
\[
\|u_k - Eu_k\|_{a, \Omega}^2 \leq C^u_{\text{con}} \alpha^{2(k-1)},
\]
\[
|\lambda_k - \hat{\lambda}| \leq C^\lambda_{\text{con}} \alpha^{2(k-1)},
\]
where
\[
C^u_{\text{con}} = 3 \left( \|w_1 - R_1 w_1\|_{a, \Omega}^2 + \gamma \eta^2(f_1; T_1) \right) \quad \text{and} \quad C^\lambda_{\text{con}} = 8 \hat{\lambda}^4 C^u_{\text{con}}.
\]

Proof. From Theorems 2.7 and 3.1 we have
\[
\|u_k - Eu_k\|_{a, \Omega}^2 \leq 2\|w_k - R_k w_k\|_{a, \Omega}^2 + 2\|w_k - Eu_k\|_{a, \Omega}^2
\]
\[
\leq 2\|w_k - R_k w_k\|_{a, \Omega}^2 + 2C^2 w_\alpha^2 (V_H) \eta^2(f_k; T_k)
\]
\[
\leq \left( 2 + \frac{2C^2 w_\alpha^2 (V_H)}{\gamma} \right) (\|w_k - R_k w_k\|_{a, \Omega}^2 + \gamma \eta^2(f_k; T_k))
\]
\[
\leq 3\alpha^{2(k-1)} (\|w_1 - R_1 w_1\|_{a, \Omega}^2 + \gamma \eta^2(f_1; T_1)),
\]
when \( H \) is small enough. Then the desired assertion (3.53) can be deduced by the definition of \( C^u_{\text{con}} \) in (3.55).

From min-max principle, Lemmas 2.7 and 3.11 we have
\[
0 \leq \lambda_k - \hat{\lambda} = \frac{\|u_k - \hat{u}^1\|_{a, \Omega}^2}{\|u_k\|_{a, \Omega}^2} - \frac{\hat{\lambda} \|u_k - \hat{u}^1\|_{a, \Omega}^2}{\|u_k\|_{a, \Omega}^2} \leq 4 \|u_k - Eu_k\|_{a, \Omega}^2.
\]
(3.56)

The mesh size \( H \) is chosen to be small enough such that \( \|u_k - Eu_k\|_{a, \Omega}^2 \leq C^u_{\text{con}} \leq 1/8 \) holds for all \( k \in \mathbb{N} \). From (3.56) and the fact \( \lambda_k \|u_k\|_{a, \Omega}^2 = \|u_k\|_{a, \Omega}^2 = 1 \), the following estimates hold
\[
\lambda_k - \hat{\lambda} \leq \frac{\lambda_k}{2} \quad \text{or} \quad \lambda_k \leq 2\hat{\lambda}.
\]
(3.57)

By using (3.56) and (3.57), we can obtain
\[
\lambda_k - \hat{\lambda} \leq 4\lambda_k \|u_k - Eu_k\|_{a, \Omega}^2 \leq 8\hat{\lambda}^4 C^u_{\text{con}} \alpha^{2(k-1)}.
\]
Then the desired assertion (3.54) can be deduced by defining \( C^\lambda_{\text{con}} := 8\hat{\lambda}^4 C^u_{\text{con}} \) and the proof is complete. \( \square \)

4 Complexity analysis

In this section, the complexity analysis of Adaptive Algorithm C will be provided. In order to state the results of complexity estimate, we need the following assumption.

Assumption 4.1. There exist an order \( s > 0 \) and a constant \( M > 0 \) such that
\[
M = \sup_{\varepsilon > 0} \left\{ \inf_{T_1 \leq T \leq T_1} \inf_{u \in V, \|u\|_{a, \Omega} = 1} \left( \inf_{u \in E_{u, \varepsilon} \subset \inf \text{osc}^2 (L u, T_T) \leq \varepsilon} \right) \right\} (#T - #T)_{s} < \infty,
\]
where \( T_1 \leq T \) means \( T_1 \) is a conforming refinement of \( T \) and \( #T \) denotes the number of elements in the mesh \( T \).
Now, it is time to consider the complexity of Adaptive Algorithm C. First, the number of refined elements has the following upper bound.

**Lemma 4.1.** When $H$ is small enough, the number of elements in $\mathcal{M}_k$ has the following estimate

$$\# \mathcal{M}_k \leq C_M \left( \left\| (I - R_k)w_k \right\|^2 \Omega + \text{osc}^2 (LR_k w_k; T_k) \right)^{\frac{1}{2}}, \quad (4.1)$$

where

$$C_M = \left( \frac{\bar{\varepsilon}_0^2}{16 \hat{C}_D} \right)^{\frac{1}{2}} M^{\frac{1}{2}}.$$

**Proof.** We choose $\bar{\varepsilon}_0$ in Lemma 2.6 small enough such that $\hat{\theta} = \theta_1 \sqrt{1 - 2 \bar{\varepsilon}_0^2} \geq \theta_1$ and $\varepsilon$ to be

$$\varepsilon := \left( \frac{\bar{\varepsilon}_0^2}{16 \hat{C}_D} \right)^{\frac{1}{2}} \left( \left\| (I - R_k)w_k \right\|^2 \Omega + \text{osc}^2 (LR_k w_k; T_k) \right)^{\frac{1}{2}}. \quad (4.2)$$

Let $\mathcal{T}_\varepsilon$ be a conforming refinement of $\mathcal{T}_1$ with minimum degrees of freedom satisfying

$$\left\| u_\varepsilon - Eu_\varepsilon \right\|^2 \Omega + \text{osc}^2 (Lu_\varepsilon; T_\varepsilon) \leq \varepsilon^2, \quad (4.3)$$

where $u_\varepsilon \in V_\varepsilon$ satisfies $\left\| u_\varepsilon \right\|^2 \Omega = 1$. From Assumption 4.1 we can get that

$$\# \mathcal{T}_\varepsilon - \# \mathcal{T}_1 \leq \varepsilon^{-\frac{1}{2}} M^{\frac{1}{2}}. \quad (4.4)$$

Let $\mathcal{T}_{k,+}$ be the smallest common refinement of $\mathcal{T}_k$ and $\mathcal{T}_\varepsilon$. Since both $\mathcal{T}_k$ and $\mathcal{T}_\varepsilon$ are conforming, $\mathcal{T}_{k,+}$ is conforming, and the number of elements in $\mathcal{T}_{k,+}$ that are not in $\mathcal{T}_k$ is less than the number of elements that must be added from $\mathcal{T}_1$ to $\mathcal{T}_\varepsilon$, i.e.,

$$\# \mathcal{T}_{k,+} - \# \mathcal{T}_k \leq \# \mathcal{T}_\varepsilon - \# \mathcal{T}_1.$$ 

We conclude from Lemma 2.4 there exists a constant $\hat{C}_D$ such that

$$\left\| (I - R_{k,+})w_k \right\|^2 \Omega + \text{osc}^2 (LR_{k,+} w_k; T_{k,+}) \leq \hat{C}_D \left( \left\| w_k - u_\varepsilon \right\|^2 \Omega + \text{osc}^2 (Lu_\varepsilon; T_{k,+}) \right), \quad (4.5)$$

where $R_{k,+}$ denotes the Galerkin projection onto the finite element space $V_{k,+}$ over $\mathcal{T}_{k,+}$. By applying Lemma 2.7 Theorem 3.1 and triangle inequality, we have the following estimates for the first term of (4.5)

$$\left\| w_k - u_\varepsilon \right\|^2 \Omega \leq 2 \left\| w_k - \hat{u} \right\|^2 \Omega + 2 \left\| \hat{u} - u_\varepsilon \right\|^2 \Omega \leq 8 \left\| w_k - Eu_\varepsilon \right\|^2 \Omega + 8 \left\| u_\varepsilon \right\|^2 \Omega \leq 8C_w^2 \eta_0^2 (V_H) \eta^2 (f_k; T_k) + 8 \left\| u_\varepsilon - Eu_\varepsilon \right\|^2 \Omega. \quad (4.6)$$

The lower bound (2.9) in Lemma 2.3 implies

$$\eta^2 (f_k; T_k) \leq \frac{1}{C_{low}^2} \left( \left\| (I - R_k)w_k \right\|^2 \Omega + \text{osc}^2 (LR_k w_k; T_k) \right)^2 \leq \frac{2}{C_{low}^2} \left( \left\| (I - R_k)w_k \right\|^2 \Omega + \text{osc}^2 (LR_k w_k; T_k) \right). \quad (4.7)$$

From (4.2), (4.3), (4.5), (4.6) and (4.7), when $H$ is small enough, we obtain

$$\left\| (I - R_{k,+})w_k \right\|^2 \Omega + \text{osc}^2 (LR_{k,+} w_k; T_{k,+}) \leq \hat{C}_D \left( 8 \left\| u_\varepsilon - Eu_\varepsilon \right\|^2 \Omega + \text{osc}^2 (Lu_\varepsilon; T_\varepsilon) + 8C_w^2 \eta_0^2 (V_H) \eta^2 (f_k; T_k) \right)$$
Since (4.2), (4.4) and the marking step selects a minimum set \( M \) we know that the value \( \#M \) Thus combining Lemma 2.5 and (4.8) leads to the following estimate for refinement. Then there exists a constant \( C \) where starting from \( T_1 \) and \( \mathcal{T}_k \) is small enough, for \( \eta \) \( \|u_k\|_{a, \Omega} \leq L \) and \( \mathcal{M}_k \) which is marked

\[
\leq \hat{C}_{D} \left( 8 \alpha^2 + \frac{16 C_{D}^{2} \eta_{H}^{2}(V_{H})}{C_{\text{low}}^{2}} \left( \| (I - R_{k}w_{k}) \|_{a, \Omega}^2 + \text{osc}^2 (LR_{k}w_{k}; \mathcal{T}_k) \right) \right)
\]

\[
= \left( \frac{\tilde{C}_{0}^{2}}{2} + \frac{16 \hat{C}_{D}^{2} \eta_{H}^{2}(V_{H})}{C_{\text{low}}^{2}} \right) \left( \| (I - R_{k}w_{k}) \|_{a, \Omega}^2 + \text{osc}^2 (LR_{k}w_{k}; \mathcal{T}_k) \right)
\]

\[
\leq \tilde{C}_{0}^{2} \left( \| (I - R_{k}w_{k}) \|_{a, \Omega}^2 + \text{osc}^2 (LR_{k}w_{k}; \mathcal{T}_k) \right).
\]

(4.8)

Thus the following inequality holds

\[
\eta_{k}^{2} (f_{k}, R_{k}K f_{k}; \mathcal{T}_k \setminus (\mathcal{T}_{k+} \cap \mathcal{T}_k)) \geq \theta_{1} \eta_{k}^{2} (\tilde{f}_{k}, \tilde{u}_{k}; \mathcal{T}_k).
\]

Since \( \bar{u}_{k} = R_{k}K \tilde{f}_{k} \) and

\[
f_{k} = \frac{\tilde{f}_{k}}{\| K \tilde{f}_{k} \|_{a, \Omega}},
\]

the following inequality holds

\[
\eta_{k}^{2} (\tilde{f}_{k}, \tilde{u}_{k}; \mathcal{T}_k \setminus (\mathcal{T}_{k+} \cap \mathcal{T}_k)) \geq \theta_{1} \eta_{k}^{2} (\tilde{f}_{k}, \tilde{u}_{k}; \mathcal{T}_k).
\]

Since (4.2), (4.4) and the marking step selects a minimum set \( \mathcal{M}_k \) satisfying

\[
\eta_{k}^{2} (\tilde{f}_{k}, \tilde{u}_{k}; \mathcal{M}_k) \geq \theta_{1} \eta_{k}^{2} (\tilde{f}_{k}, \tilde{u}_{k}; \mathcal{T}_k),
\]

we know that the value \( \#\mathcal{M}_k \) satisfies

\[
\#\mathcal{M}_k \leq \#(\mathcal{T}_k \setminus (\mathcal{T}_{k+} \cap \mathcal{T}_k)) \leq \#\mathcal{T}_{k+} - \#\mathcal{T}_k \leq \#\mathcal{T}_k - \#\mathcal{T}_1
\]

\[
\leq \left( \frac{\tilde{C}_{0}^{2}}{16 C_{D}} \right)^{- \theta_{1}^2} M^{2} \left( \| (I - R_{k}w_{k}) \|_{a, \Omega}^2 + \text{osc}^2 (LR_{k}w_{k}; \mathcal{T}_k) \right)^{- \frac{1}{2}}.
\]

This is the desired result (4.1) and the proof is complete. \( \square \)

In our analysis, we also need the following result (see, e.g., [10, 11, 28, 29, 30]).

**Proposition 4.1.** ([10, Lemma 2.3]) (Complexity of REFINE) Let \( \{ \mathcal{T}_k \}_{k \in \mathbb{N}} \) be a sequence of conforming nested partitions generated by REFINE starting from \( \mathcal{T}_1 \), \( \mathcal{M}_k \) be the set of elements of \( \mathcal{T}_k \) which is marked for refinement. Then there exists a constant \( C_{\#} \) depending solely on \( \mathcal{T}_1 \) such that

\[
\#\mathcal{T}_{k+1} - \#\mathcal{T}_1 \leq C_{\#} \sum_{i=1}^{k} \#\mathcal{M}_i.
\]

Now, it is time to show the quasi-optimality by estimating \( \#\mathcal{T}_k - \#\mathcal{T}_1 \) for \( k \in \mathbb{N} \).

**Theorem 4.1.** When \( H \) is small enough, for \( k \in \mathbb{N} \), the following inequality holds

\[
\| u_k - Eu_k \|_{a, \Omega}^2 + \text{osc}^2 (Lu_k; \mathcal{T}_k) \leq C_{c} (\# \mathcal{T}_k - \# \mathcal{T}_1)^{-2s},
\]

where

\[
C_{c} = C_{\#}^{2s} C_{\mathcal{M}}^{2s} \left( 1 + \frac{2 \gamma^{2}}{C_{\text{low}}^{2}} \right) \left( 1 - \alpha^{2} \right)^{-2s} \left( 2 + \frac{5}{\gamma} \right).
\]

**Proof.** From (4.1), we have the following inequality for \( f_{k} \in \mathbb{N} \)

\[
\| (I - R_{k}) w_{k} \|_{a, \Omega}^2 + \gamma \eta^{2} (f_{k}; \mathcal{T}_k) \leq \left( 1 + \frac{2 \gamma}{C_{\text{low}}^{2}} \right) \left( \| (I - R_{k}) w_{k} \|_{a, \Omega}^2 + \text{osc}^2 (LR_{k}w_{k}; \mathcal{T}_k) \right).
\]
Combining Lemma 3.1 and Proposition 3.1 leads to following estimates:

\[
\begin{align*}
\sum_{i=1}^{k-1} \| (I - R_i) w_i \|^2_{a,\Omega} + \gamma \eta^2 (f_i; T_i) \leq \sum_{i=1}^{k-1} \| (I - R_k) w_k \|^2_{a,\Omega} + \gamma \eta^2 (f_k; T_k) \\
\leq C \| M \| \| R_k w_k \|^2_{a,\Omega} + 2 C \| M \| \| a,\Omega \| \| w_k \|^2_{a,\Omega} + \gamma \eta^2 (f_k; T_k).
\end{align*}
\]

From Theorem 3.1, the following inequalities hold:

\[
\sum_{i=1}^{k-1} \left( \frac{1}{(I - R_i) w_i} + \frac{1}{(I - R_k) w_k} \right) \leq \sum_{i=1}^{k-1} \left( \frac{1}{(I - R_k) w_k} + \gamma \eta^2 (f_k; T_k) \right) \leq \sum_{i=1}^{k-1} \frac{1}{(I - R_k) w_k}.
\]

The following estimates can be deduced from Theorems 2.2 and 3.1:

\[
\begin{align*}
\| u_k - E u_k \|_{a,\Omega} & \leq 2 \| w_k - R_k w_k \|_{a,\Omega} + 2 \| w_k - E w_k \|_{a,\Omega} \\
& \leq 2 \| w_k - R_k w_k \|_{a,\Omega} + 2 C \| M \| \| w_k \|^2_{a,\Omega} + \gamma \eta^2 (f_k; T_k).
\end{align*}
\]

By applying Theorem 3.1 again, we have the lower bound of \( \| R_k w_k \|_{a,\Omega} \):

\[
\| R_k w_k \|_{a,\Omega} \geq \| w_k \|_{a,\Omega} - 1 - \left( \gamma \eta^2 (f_k; T_k) \right)^{1/2} \geq \frac{1}{2},
\]

which implies the following estimate:

\[
\text{osc}(Lu_k; T_k) = \text{osc}(LR_k w_k; T_k) \leq 2 \text{osc}(LR_k w_k; T_k) \leq 2 \gamma \eta (f_k; T_k).
\]

When \( H \) is small enough, combining (5.12) and (5.13) leads to the following inequalities:

\[
\| u_k - E u_k \|_{a,\Omega} + \text{osc}(Lu_k; T_k) \leq 2 \left( \| w_k - R_k w_k \|_{a,\Omega} + \left( 2 C \| w_k \|^2_{a,\Omega} + \gamma \eta^2 (f_k; T_k) \right) \right)
\]

\[
\leq \left( 2 + \frac{2 C \| w_k \|^2_{a,\Omega}}{\gamma} + \frac{4}{\gamma} \right) \left( 2 + \frac{5}{\gamma} \right) \left( \| w_k - R_k w_k \|_{a,\Omega} + \gamma \eta^2 (f_k; T_k) \right).
\]

Then from (4.10), (4.11) and (5.13), we have:

\[
\left( \# T_k - \# T_i \right) \leq C \left( \| u_k - E u_k \|_{a,\Omega} + \text{osc}(Lu_k; T_k) \right)^{-\frac{1}{2}}.
\]

This is equivalent to the desired result (1.9) and the proof is complete.

### 5 Numerical experiments

In this section, we investigate the numerical performance of Adaptive Algorithm C for the second order elliptic eigenvalue problems by four numerical examples. Here, the well known implicitly restarted Lanczos method, which is included in the package ARPACK, is adopted for solving the concerned generalized eigenvalue problems and the geometric multigrid (GMG) method is selected as the linear solver for the boundary value problems. All through these papers, we choose \( T_H = T_1 \).

**Example 1.** In this example, we consider the following eigenvalue problem

\[
\begin{align*}
\left\{ \begin{array}{l}
-\frac{1}{2} \Delta u + \frac{1}{2} |x|^2 u &= \lambda u & \text{in } \Omega, \\
u &= 0 & \text{on } \partial \Omega,
\end{array} \right.
\]

\[
\| u \|_{a,\Omega} = 1,
\]

\[
(5.1)
\]
where $\Omega \subset \mathbb{R}^2$ and $|x| = \sqrt{x_1^2 + x_2^2}$. The first eigenvalue of (5.1) is $\lambda = 1$ and the associated eigenfunction is $u = \kappa e^{-|x|^2/2}$ with any nonzero constant $\kappa$. In our computation, we set $\Omega = (-5, 5) \times (-5, 5)$.

The eigenvalue problem is solved by Adaptive Algorithm $C$ with the parameters $\theta_1 = 0.4$ and $\theta_2 = 0.6$ for linear element and $\theta_1 = 0.4$ and $\theta_2 = 0.4$ for quadratic element. Here, we check the numerical results for the first eigenvalue approximations. Figures 1 shows the triangulations by Adaptive Algorithm $C$ with the linear and quadratic finite element methods, respectively. Figure 2 gives the corresponding numerical results for the first 20 adaptive iterations with linear finite element method. In order to show the efficiency of Adaptive Algorithm $C$ more clearly, we compare the results with those obtained with direct AFEM. Similarly, Figure 3 shows the numerical results for the first 22 adaptive iterations with quadratic finite element method.

![Mesh after 15 iterations](image1)
![Mesh after 17 iterations](image2)

Figure 1: The triangulations after adaptive iterations for Example 1 by the linear element (left) and the quadratic element (right)

![Eigenvalue errors](image3)
![Eigenfunction errors](image4)

Figure 2: The errors of the eigenvalue and the associated eigenfunction approximations by Adaptive Algorithm $C$ and direct AFEM for Example 1 with the linear element

It is observed from Figures 2 and 3 the approximations of eigenvalue as well as eigenfunction have the optimal convergence rate which coincides with our theory. With Adaptive Algorithm $C$, we only need to solve boundary value problems on each adaptively refined meshes and small scale eigenvalue problems on the low dimensional space $V_H + \text{span}\{u_k\}$ when $k = 1, 5, 8, 11, 14, 17, 20$ ($\# T_k = [64, 122, 296, 884, 2846, 10044, 36070] \text{ and } j_k = 1, 1, 1, 1, 1, 1, 1$) and $k = 1, 4, 8, 11, 14, 17, 20$ ($\# T_k = [64, 86, 236, 654, 1894, 5810, 18876] \text{ and } j_k = 1, 1, 1, 1, 1, 1, 1$) for linear element and quadratic element, respectively. But the accuracy obtained by Adaptive Algorithm $C$ is almost the same as the standard AFEM which validate the efficiency of Adaptive Algorithm $C$. 

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Adaptive Algorithm produced by eigenfunction approximations. Figures 7 and 8 show the corresponding a posteriori error estimator eigenvalue problems on the low dimensional space $V$. In these cases, the meshes are $(−1,1) \times (−1,1) \setminus \{0,1\} \times (−1,0)$. Since $\Omega$ has a reentrant corner, eigenfunctions with singularities are expected. The convergence order for eigenvalue approximations is less than 2 by the linear finite element method which is the order predicted by the theory for regular eigenfunctions.

Here, we give the numerical results of Adaptive Algorithm $C$ with parameters $\theta_1 = 0.4$ and $\theta_2 = 0.6$ for linear element and $\theta_1 = 0.4$ and $\theta_2 = 0.4$ for quadratic element, respectively. First, we investigate the numerical results for the first eigenvalue approximations. The exact eigenvalue is not known, an adequately accurate approximation $\lambda = 9.6397238440219$ is chosen as the exact first eigenvalue for our numerical tests. Figure 4 shows the triangulations after adaptive iterations with the linear and quadratic finite element methods, respectively. Figures 5 and 6 give the corresponding numerical results. In order to show the efficiency of Adaptive Algorithm $C$ more clearly, we also compare the results with those obtained by direct AFEM. With Adaptive Algorithm $C$, it is only required to solve boundary value problems on the adaptively refined triangulations and small scale eigenvalue problems on the low dimensional space $V_H + \text{span}\{u_k\}$ when the numbers of elements of the meshes are $[96, 266, 808, 2508, 6292, 28276, 94922]$ ($k = 1, 4, 7, 10, 13, 16, 19$ and $j_k = 1, 1, 1, 1, 1, 1, 1$) for linear element and $[96, 138, 380, 1090, 2852, 7164, 18484]$ ($k = 1, 6, 11, 16, 20, 24, 28$ and $j_k = 1, 1, 1, 1, 1, 1, 1$) for quadratic element, respectively.

We also test Adaptive Algorithm $C$ for the first 5 eigenvalue approximations and their associated eigenfunction approximations. Figures 7 and 8 show the corresponding a posteriori error estimator produced by Adaptive Algorithm $C$ and direct AFEM with the linear and quadratic finite element methods, respectively. In these cases, Adaptive Algorithm $C$ only need to solve the small scale eigenvalue problems on the low dimensional space $V_H + \text{span}\{u_k\}$ when the numbers of elements of the meshes are $[96, 328, 1130, 4124, 15832, 62802]$ ($k = 1, 4, 7, 10, 13, 16$ and $j_k = 1, 1, 1, 1, 1, 1$) for linear element and $[96, 192, 690, 2086, 6686, 16832]$ ($k = 1, 5, 10, 15, 20, 24$ and $j_k = 1, 1, 1, 1, 1, 1$) for quadratic element, respectively.

From Figures 5, 6, 7 and 8 we can find the approximations of eigenvalues as well as eigenfunctions have the optimal convergence rate as the direct AFEM which coincides with our theory.

Example 2. In the second example, we consider the Laplace eigenvalue problem on the $L$-shape domain

$$
\begin{align*}
-\Delta u &= \lambda u \quad \text{in } \Omega, \\
\|u\|_{a, \Omega} &= 1, \\
\end{align*}
$$

(5.2)

where $\Omega = (−1,1) \times (−1,1) \setminus \{0,1\} \times (−1,0)$. Since $\Omega$ has a reentrant corner, eigenfunctions with singularities are expected. The convergence order for eigenvalue approximations is less than 2 by the linear finite element method which is the order predicted by the theory for regular eigenfunctions.

In this example, we consider the following second order elliptic eigenvalue problem

$$
\begin{align*}
\nabla (A \nabla u) + \varphi u &= \lambda u \quad \text{in } \Omega, \\
\|\nabla u\|_{a, \Omega} &= 1, \\
\end{align*}
$$

(5.3)
We give the numerical results by Adaptive Algorithm $C$ with parameters $\theta_1 = 0.4$ and $\theta_2 = 0.6$ for linear element and $\theta_1 = 0.4$ and $\theta_2 = 0.4$ for quadratic element, respectively. We first investigate the numerical results for the first eigenvalue approximations. Since the exact eigenvalue is not known neither, we choose an adequately accurate approximation $\lambda = 15.134144021256400$ as the exact eigenvalue for our numerical tests. Figure 4 shows the triangulations after adaptive iterations by the linear and quadratic finite element methods, respectively. Figures 10 and 11 give the corresponding numerical results by the linear and quadratic finite element methods, respectively. Similarly, we also compare the results with those obtained with direct AFEM. It is only required to solve the small scale eigenvalue problem on the low dimensional space $V_H + \text{span}\{u_k\}$ when the numbers of elements of the meshes are $[96, 243, 1034, 3282, 10870, 37030, 128259] \ (k = 1, 4, 8, 11, 14, 17, 20)$ and $[96, 180, 530, 1571, 5026, 15534] \ (k = 1, 6, 11, 16, 21, 26)$ for linear element and $[96, 180, 530, 1571, 5026, 15534] \ (k = 1, 6, 11, 16, 21, 26)$ for quadratic element, respectively.

We also test Adaptive Algorithm $C$ for 5 smallest eigenvalue approximations and their associ-
Figure 6: The errors of the smallest eigenvalue approximations and the a posteriori errors of the associated eigenfunction approximations by Adaptive Algorithm C and direct AFEM for Example 2 with the quadratic element.

Figure 7: The a posteriori error estimates of the eigenfunction approximations by Adaptive Algorithm C and direct AFEM for Example 2 with the linear element.

Example 4. In the last example, we consider the Laplace eigenvalue problem on three dimensional nonconvex domain

\[
-\Delta u = \lambda u \quad \text{in } \Omega, \\
\|u\|_{a,\Omega} = 1, \\
\tag{5.4}
\]

where \(\Omega = (-1,1)^3 \setminus [0,1)^3\). Similarly, eigenfunctions with singularities are expected due to the nonconvex property.

In this example, we give the numerical results of Adaptive Algorithm C with parameters \(\theta_1 = 0.4\) and \(\theta_2 = 0.7\) for linear element and \(\theta_1 = 0.4\) and \(\theta_2 = 0.4\) for quadratic element, respectively. First we
investigate the numerical results for the first eigenvalue approximation. Since the exact eigenvalue is not known, an adequately accurate approximation on finer finite element space is chosen as the exact first eigenvalue for numerical tests. Figure 13 and 16 show the triangulations after adaptive iterations with the linear and quadratic finite element methods, respectively. Figures 15 and 17 give the corresponding numerical results. In order to show the efficiency of Adaptive Algorithm C more clearly, we also compare the results with those obtained by direct AFEM. With Adaptive Algorithm C, it is only required to solve small scale eigenvalue problems in the low dimensional space $V_H + \text{span}\{u_k\}$ when the the numbers of elements of the mesh are $[2688, 7634, 20586, 78102, 287442, 1001202]$ $(k = 1, 5, 8, 12, 16, 20$ and $j_k = 1, 1, 1, 1, 1, 1)$ for linear element and $[2688, 4020, 12144, 57188, 264576]$ $(k = 1, 5, 9, 14, 19$ and $j_k = 1, 1, 1, 1, 1)$ for quadratic element, respectively.

In order to show the efficiency of Adaptive Algorithm C more clearly, we compare the computational time (in second) of Adaptive Algorithm C with that of direct AFEM by linear element. Figure 18 shows the corresponding CPU time results, which shows Adaptive Algorithm C has higher efficiency than the direct AFEM.

Furthermore, we also test Adaptive Algorithm C for the smallest 5 eigenvalue and their associated eigenfunctions. Figure 19 shows the a posteriori error estimators produced by Adaptive Algorithm C and direct AFEM with the linear finite element method. In these cases, Adaptive Algorithm C only solve the small scale eigenvalue problems on the low dimensional space $V_H + \text{span}\{u_k\}$ when the the
Figure 10: The errors of the smallest eigenvalue and the associated eigenfunction approximations by Adaptive Algorithm C and direct AFEM for Example 3 with the linear element

Figure 11: The errors of the smallest eigenvalue and the associated eigenfunction approximations by Adaptive Algorithm C and direct AFEM for Example 3 with the quadratic element

numbers of elements are \([21504, 67846, 173182, 584308, 2218702]\) \((k = 1, 5, 8, 12, 17\) and \(j_k = 1, 1, 1, 1, 1)\).

6 Concluding remarks

In this paper, we present an efficient AFEM for eigenvalue problems based on multilevel correction scheme and the adaptive refinement technique. The most important contribution of this new AFEM is that there is no eigenvalue solving in the adaptively refined meshes which need much more computation than solving the corresponding linear boundary value problems. Furthermore, the convergence and quasi-optimal complexity have also been proved based on a relation between the eigenvalue problem and the associated boundary value problem (see Theorems 3.2 and 4.1). Some numerical experiments for both simple and multiple eigenvalue cases are provided to demonstrate the efficiency of the proposed AFEM for eigenvalue problems. It is obvious that this efficient AFEM can be extended to the nonlinear eigenvalue problems and also other type of nonlinear problems which will be our future work.

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### Table 1: Eigenfunction Errors

| Errors | Number of Elements |
|--------|-------------------|
| Adaptive Algorithm C |  |
| the 1-st eigenfunction |  |
| the 2-nd eigenfunction |  |
| the 3-rd eigenfunction |  |
| the 4-th eigenfunction |  |
| the 5-th eigenfunction |  |
| direct AFEM |  |
| the 1-st eigenfunction |  |
| the 2-nd eigenfunction |  |
| the 3-rd eigenfunction |  |
| the 4-th eigenfunction |  |
| the 5-th eigenfunction |  |

### Figure 12: The a posteriori error estimates of the eigenfunction approximations by Adaptive Algorithm C and direct AFEM for Example 3 with the linear element

### Figure 13: The a posteriori error estimates of the eigenfunction approximations by Adaptive Algorithm C and direct AFEM for Example 3 with the quadratic element

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Figure 14: The triangulations after adaptive iterations and section for Example 4 by the linear element

| Number of elements | Errors | Errors of Adaptive Algorithm C | Errors of direct AFEM | slope=-2/3 |
|-------------------|--------|-------------------------------|-----------------------|------------|
| 10^0              | 10^1   |                               |                       |            |
| 10^1              | 10^2   |                               |                       |            |
| 10^2              | 10^3   |                               |                       |            |

Figure 15: The errors of the smallest eigenvalue and the associated eigenfunction approximations by Adaptive Algorithm C and direct AFEM for Example 4 with the linear element

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Figure 16: The triangulations after adaptive iterations and section for Example 4 by the quadratic element

Figure 17: The errors of the smallest eigenvalue and the associated eigenfunction approximations by Adaptive Algorithm C and direct AFEM for Example 4 with the quadratic element

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Figure 18: The computational time by Adaptive Algorithm C and direct AFEM for Example 4 with the linear element.

Figure 19: The a posteriori error estimates of the eigenpair approximations by Adaptive Algorithm C and direct AFEM for Example 4 with the linear element.

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