Reducing the conjugacy problem for relatively hyperbolic automorphisms to peripheral components

François Dahmani, Nicholas Touikan

October 17, 2023

Abstract

We use relative hyperbolicity of mapping tori and Dehn fillings of relatively hyperbolic groups to solve the conjugacy problem between certain outer automorphisms. We reduce this problem to algorithmic problems entirely expressed in terms of the parabolic subgroups of the mapping tori. As an immediate application, we solve the conjugacy problem for the outer automorphisms of free groups whose polynomial part is piecewise inner. This proposes a path toward a full solution to the conjugacy problem for $\text{Out}(F_n)$.

Contents

1 Preliminaries
   1.1 Graphs, graphs of groups, and their fundamental groups . . . 7
   1.2 Automorphisms of graphs of groups . . . . . . . . . . . . . . . 7

2 Main reduction result
   2.1 Assumptions and statement . . . . . . . . . . . . . . . . . . . . . 9
   2.2 JSJ decompositions . . . . . . . . . . . . . . . . . . . . . . . . . . 11
   2.3 On white vertices: Congruences and Dehn fillings for lists . . . 13
       2.3.1 Peripheral structures, orbits of markings . . . . . . . . . . 13
       2.3.2 White vertices of the JSJ decompositions and peripheral structures 13
       2.3.3 Dehn fillings . . . . . . . . . . . . . . . . . . . . . . . . . . . 14
Introduction

Given an automorphism $\alpha$ of a group $G$, we wish to understand the conjugacy class of $\alpha$ in the outer automorphism group of $G$. We form the semidirect product $T_\alpha = G \rtimes \alpha \mathbb{Z}$ that we call the mapping torus of $G$ by $\alpha$. The conjugacy class of the outer automorphism $[\alpha] \in \text{Out}(G)$ is characterised by, not only $T_\alpha$, but by the specific homomorphism $T_\alpha \to T_\alpha/G \cong \mathbb{Z}$.

Dynamical properties of $\alpha$ often endow the mapping torus $T_\alpha$ with an interesting geometry. Thurston [Th] showed that if $G$ is the fundamental group of a closed orientable surface, and $\alpha$ an automorphism induced by a pseudo-Anosov mapping class, then the mapping torus is word-hyperbolic. Later Brinkmann [Br] proved an analogous result when $G$ is a free group, characterising the hyperbolic mapping tori of $G$ as being induced by atoroidal automorphisms.

A fundamental dynamical property of automorphisms is their growth: if there exist an element $g$ of $G$ for which the sequence of lengths of conjugacy classes $[\alpha^n(g)]$ grow exponentially fast in $n$, we say that $\alpha$ has exponential growth, whereas if for all $g$, the sequence is bounded by a polynomial in $n$, we say that $\alpha$ has polynomial growth.

Gautero and Lustig proposed the important insight that for exponential growth automorphisms of free groups, $G \rtimes \alpha \mathbb{Z}$ should admit a natural relatively hyperbolic structure with respect to semidirect products of subgroups of $G$ that are of polynomial growth, see [GL]. It is true even in greater generality, e.g. when $G$ is an arbitrary torsion free hyperbolic group, see [Gh, DL, DK].

Consider a free or torsion-free hyperbolic group $G$. Its polynomially growing automorphisms produce mapping tori that are not properly relatively hyperbolic [M], [D2, Prop. 1.3]. On the other end of the spectrum, if $\alpha$ is such that all non-trivial $g$ elements of $G$ give sequences $([\alpha^n(g)])_n$ whose length

2
have exponential growth rate, the mapping torus is word-hyperbolic [Br] (this is the atoroidal case). In the remaining case, i.e. if an automorphism is not polynomially growing but not all conjugacy classes are exponentially growing, then it is nevertheless exponentially growing [Le], and the mapping torus is thus properly relatively hyperbolic with respect to sub-mapping-tori of polynomially growing automorphisms on subgroups: it is a case in which one expects the difficulties and features of both the polynomial growth case, and the atoroidal case.

The conjugacy problem in Out($F$) for automorphisms of a f.g. free group $F$ that are of polynomial growth, and so-called unipotent, is solved by Feighn and Handel [FH]. For atoroidal automorphisms of free groups, it is solved in [D]. That said, solutions to the conjugacy problems for atoroidal and polynomial growth automorphisms do not combine to give a solution in the general exponential growth case: there are substantial interactions between polynomial and exponential growth within automorphisms.

We show in this paper that relative hyperbolicity provides a way to completely reduce the full conjugacy problem in Out($F$) to the study of the pieces of polynomial growth.

We start with the observation that two automorphisms $\alpha, \beta$ of $G$ are conjugate in Out($G$) if and only if there exists an isomorphism of between the mapping tori $G \times_\alpha \langle t_\alpha \rangle \to G \times_\beta \langle t_\beta \rangle$ carrying $G$ to $G$ and $t_\alpha$ into the coset $Gt_\beta$. We say that such an isomorphism preserves the fibre and the orientation of the mapping torus, because these groups are fundamental groups of (topological) mapping tori, that would fibre over the circle (if it wasn’t for the singular locus of the gluing).

If the groups $G \times_\alpha \langle t_\alpha \rangle$ and $G \times_\beta \langle t_\beta \rangle$ are relatively hyperbolic then solutions to the isomorphism problem [DT19] can certify, if $T_\alpha$ and $T_\beta$ are not abstractly isomorphic, that $\alpha$ and $\beta$ are not conjugate. It is not enough to conclude in general though, since $T_\alpha$ and $T_\beta$ could be isomorphic but not through a fibre and orientation preserving isomorphism. In the case of hyperbolic mapping tori, this caveat was treated [S, D] (see also [D2]). Our work in this paper is pushing this to a general relatively hyperbolic case.

Our main result (Theorem 2.1) is a reduction of the fibre and orientation preserving isomorphy problem for a class of relatively hyperbolic mapping tori to a small number of intrinsic problems in the class of the peripheral sub mapping tori. This way, no matter how curious the exponentially growing parts will be, the decision will be either achieved, or reduced to considerations about polynomially growing automorphisms (the polynomial parts).
While rather technical to state, (see Subsection 2.1 for definitions and the precise statement) our main result highlights several specific algorithmic problems. Some of these are standard algorithmic problems in groups and subgroups. For example, unsurprisingly, the fibre and orientation preserving isomorphpy problem for the class of peripheral sub mapping tori is one such algorithmic problem. The two last highlighted problems are a congruence problem and an orbit problem that we will come back to later.

As we mentioned, Theorem 2.1 can be worded in terms of conjugacy problems in outer automorphism groups, and, together with the main results of [Gh, DL], and [DK], it gives the following.

**Theorem.** If $F_n$ is a finitely generated free group of rank $n$, the conjugacy problem in $\text{Out}(F_n)$ can be algorithmically reduced to the set of problems for polynomially growing automorphisms of a free group of rank at most $n$, that is listed in Theorem 2.1.

If $G$ is a torsion free hyperbolic group, the conjugacy problem in $\text{Out}(G)$ can be algorithmically reduced to the set of problems for polynomially growing automorphisms of torsion free hyperbolic groups, that is listed in Theorem 2.1.

When considering whether two automorphisms of a free group $F_n$ are conjugate in $\text{Out}(F_n)$, Theorem 2.1 allows to completely evacuate the exponential growth part of these automorphisms, which is the most pervasive part, and reduce to the usually much smaller polynomial growth part.

While even the possibility of such a reduction is by no means obvious, our result identifies algorithmic problems in the polynomial part that are approachable. This will be illustrated in Section 3, where we apply the reduction to solve the conjugacy problem for all automorphisms for which the polynomial growth part is inner on some subgroups (and in further work [DT23] we will treat the unipotent linear growth case). This suggests a program to fully solve the conjugacy problem in $\text{Out}(F_n)$, in particular in view of the results of Feighn and Handel [FH] solving the conjugacy problem for unipotent polynomially growing outer automorphisms.

**Corollary.** The conjugacy problem among outer automorphisms $[\phi]$ of a f.g. free group $F$ whose maximal polynomially growing subgroups are subgroups $P$ such that there exists $\gamma_P \in F$ for which $\text{ad}_{\gamma_P} \circ \phi$ induces the identity on $P$, is solvable.

In order to explain how we proceed, let us recall first that for isomorphism problems in hyperbolic or relatively hyperbolic groups, a specific canonical
decomposition, the JSJ decomposition [RS, GL], plays a key role [S, DGr, DG10, DT19]. It splits the group as a bipartite graph of groups with vertex groups that are either maximal elementary (i.e. either maximal infinite cyclic, or peripheral in the relatively hyperbolic structure), or non-elementary relatively hyperbolic and rigid (or, possibly, surface groups, but this later case cannot happen in our setting). By [T] we may algorithmically compute such canonical decompositions, and work with them to determine whether the graphs of groups are isomorphic.

Among the four algorithmic hypotheses of Theorem 2.1, the first one gathers standard algorithmic problems in groups and subgroups and enables the computation of the JSJ decomposition using [T]. The second is, as would be expected, equivalent to the conjugacy problem for automorphisms induced in the sub-mapping tori. Hypotheses 3 and 4 are the true critical ingredients. We discuss them further here, and sketch how the whole reduction works.

For a group $H$, having congruences separating the torsion means that there exists a finite index characteristic subgroup $H_0$, such that the quotient $H \rightarrow H/H_0$ induces an homomorphism $\text{Out}(H) \rightarrow \text{Out}(H/H_0)$ whose kernel is torsion free. This happens if $H = \mathbb{Z}^n$, as it has been known since Minkowski that the principal congruence subgroup of level 3 in $GL_n(\mathbb{Z})$ is torsion free. For this reason we will also say that that $H$ is Minkowskian. This property holds if $H$ is polycyclic-by-finite, by an argument of Segal, see [DT19, Prop. 7.15], and there is evidence that this can happen in many other interesting classes of groups such as direct products of residually finite hyperbolic groups. We provide here a general result that will be useful, as Proposition 3.1. In particular we obtain:

**Corollary** (See Corollary 3.2, and the proof of Corollary 3.3). *A free group of rank $n$ is effectively Minkowskian, and its direct product with $\mathbb{Z}$ as well.*

The Minkovskian property for peripheral subgroups of $G \rtimes_a \langle t \rangle$ and $G \rtimes_b \langle t \rangle$ plays a crucial role in the approach to the isomorphism problem using Dehn fillings. A Dehn filling of a relatively hyperbolic group is a quotient by a normal subgroup generated by a subgroup of a parabolic group. This is an important construction for groups with hyperbolic features [La, O07, GM]. If the subgroup is sufficiently deep, the quotient keeps certain key properties of the initial group. By taking Dehn fillings of vertex groups of a canonical JSJ-decomposition, we can keep a record of all the finite order automorphisms of peripheral edge groups that can be induced by automorphisms of the ambient non-elementary vertex group.
The last important property is the mixed Whitehead problem in the peripheral subgroups of $G \rtimes_\alpha \langle t \rangle$ and $G \rtimes_\beta \langle t \rangle$. This is the automorphism orbit problem that decides whether two tuples of conjugacy classes of tuples of elements are in the same orbit. This is needed to decide whether the automorphisms of edge groups, found by congruences in their non-elementary adjacent vertex groups, can be matched up by an automorphism of the adjacent elementary vertex group.

Together, these two properties are used in [DT19] to decide if isomorphisms between vertex groups of canonical JSJ decompositions can be assembled into a global isomorphism of graphs of groups. Unfortunately, the methods in [DT19] cannot decide fibre and orientation preserving isomorphy.

The solution we present in this paper is to use a second set of Dehn fillings, this time with kernels in the fibre, in order to eliminate markings corresponding to a non-preserved fibre. In [DT19], all Dehn fillings were cofinite, i.e. peripheral subgroups had finite images. With this second set of Dehn fillings, peripheral subgroups will have virtually cyclic images.

There are two difficulties. Even if we find isomorphisms that preserve the fibre in each vertex group, we do not know how to find all of them in general; and there is still a part of the fibre that does not live in the vertex groups that needs to be controlled.

It turns out that a Dehn filling that is deep enough allows us to find, perhaps not all fibre-preserving isomorphisms between vertex groups, but still sufficiently many so that our combinations meet each orbit of the images of the entire fibre, modulo action by the small modular group. The small modular group is an abelian subgroup of the automorphism of the target group, whose generators can immediately be found from a relative JSJ presentation.

This, in the end, allows us to decide whether there is a fibre and orientation preserving isomorphism $G \rtimes_\alpha \langle t \rangle \to G \rtimes_\beta \langle t \rangle$, and therefore to decide whether the automorphisms $\alpha$ and $\beta$ are conjugate in $\text{Out}(G)$.

Acknowledgments. We would like to thank the referees for useful comments. We also thank Armando Martino and Stefano Francaviglia for encouraging discussions around the methods and the applications. The second named author is supported by an NSERC Discovery grant.
1 Preliminaries

1.1 Graphs, graphs of groups, and their fundamental groups

A graph $X = (X^{(0)}, X^{(1)}, \iota, \tau, -)$ is a set of vertices $X^{(0)}$, a set of oriented edges $X^{(1)}$, endowed with two maps $\iota : X^{(1)} \to X^{(0)}$, $\tau : X^{(1)} \to X^{(0)}$ and a fixed-point free involution $-$ : $X^{(1)} \to X^{(1)}$ satisfying $\iota(\bar{e}) = \tau(e)$.

Of course it is sometimes convenient to consider the geometric realisation of a graph, and the vocabulary from it.

A graph of groups $X$ is the data of a graph $X$, for each vertex $v$, a group $G_v$, for each unoriented edge $\{e, \bar{e}\}$, a group $G_e = G_{\bar{e}}$, and for each oriented edge $e$, of terminal vertex $\tau(e)$, an injective homomorphism $t_e : G_e \to G_{\tau(e)}$.

The Bass group of $X$ is the group generated by the collection of all vertex groups $G_v, v \in X^{(0)}$, and the collection of all edges $e \in X^{(1)}$, subject to the relations that, for all $e$, $ee = 1$ and that, for all $e$ and all $g \in G_e$, $\bar{e}t_e(g)e = t_e(g)$. The generators corresponding to edges are called Bass generators.

The Bass group has a natural epimorphism onto the free group on the set $X^{(1)}$ (with the identification $\bar{e} = e^{-1}$). Given $v_0$, a vertex of $X$, the fundamental group $\pi_1(X, v_0)$ is the subgroup of the Bass group whose elements map in this epimorphism to well defined loops (given by sequences of directed edges of $X$) from $v_0$ to $v_0$. Given $\tau_0$ a spanning subtree of $X$, the fundamental group $\pi_1(X, \tau_0)$ is the quotient of the Bass group obtained by identify all edges in $\tau_0$ to the identity.

An important theorem of the theory of Bass-Serre decompositions is that this quotient map induces an isomorphism from $\pi_1(X, v_0)$ to $\pi_1(X, \tau_0)$.

1.2 Automorphisms of graphs of groups

In a group, we denote conjugation as follows $a^g = g^{-1}ag = ad_g(a)$. An automorphism $\Phi$ of a graph of groups $X$ is a tuple consisting of an automorphism $\phi_X$ of the underlying graph $X$, an isomorphism $\phi_v : G_v \to G_{\phi_X(v)}$ for every vertex $v$, an isomorphism $\phi_e : G_e \to G_{\phi_X(e)}$ for every edge $e$, with $\phi_{\bar{e}} = \phi_e$, and elements $\gamma_e \in G_{\phi_X(\tau(e))}$ for every edge $e$, that satisfy the Bass diagram:
which can also be written as an equation:

$$\phi_{\tau(e)} \circ t_e = \text{ad}_{\gamma_e} \circ t_{\phi_X(e)} \circ \phi_e.$$  

One can check that an automorphism of the graph of groups $X$ extends naturally to an automorphism of the Bass group, by sending the generator $e$ to $\gamma_e^{-1} \phi_X(e) \gamma_e$. While this does not necessarily preserves the vertex $v_0$ or the spanning subtree $\tau_0$, it induces an isomorphism between $\pi_1(X, v_0)$ and $\pi_1(X, \phi_X(v_0))$. See [Ba, §2.3], [DG10, Lemmas 2.20-2.22].

The composition rule is as follows (where we replaced the notations $\phi_X, \phi'_X$ by $f, f'$ for readability)

$$(f, (\phi_v)_v, (\phi_e)_e, (\gamma_e)_e) \circ (f', (\phi'_v)_v, (\phi'_e)_e, (\gamma'_e)_e) = (f \circ f', (\phi_{f'_v}) \circ \phi_v)_v, (\phi_{f'_e}) \circ \phi_e)_e, (\phi_{f'_v(e)} \circ \phi_{f'_v(e)}) \circ \phi_{f'_e(e)}).$$

The group of all automorphisms of the form $(\phi_X, (\phi_v)_v, (\phi_e)_e, (\gamma_e)_e)$ satisfying the Bass diagram is denoted $\delta \text{ Aut}(X)$, and maps naturally to a well defined subgroup of $\text{Out}(\pi_1(X, v))$.

The subgroup $\delta_0 \text{ Aut}(X)$ consisting of automorphisms of the form

$$(\text{Id}_X, (\phi_v)_v, (\phi_e)_e, (\gamma_e)_e)$$

is of finite index in $\delta \text{ Aut}(X)$. Indeed, if $\Phi_1, \Phi_2$ define the same underlying graph isomorphism, then $\Phi_1^{-1} \Phi_2 \in \delta_0 \text{ Aut}(X)$.

**Proposition 1.1.** Consider a finite graph of groups $X$, whose vertex groups lie in a class of groups for which the isomorphism problem and the mixed Whitehead problem are solvable. Then there is an algorithm that computes a complete set of coset representatives of $\delta_0 \text{ Aut}(X)$ in $\delta \text{ Aut}(X)$.

**Proof.** Let $\phi_X : X \to X$ be a graph isomorphism. We wish to know whether it can be extended into a graph of groups automorphism in $\delta \text{ Aut}(X)$. By
hypothesis, we may determine whether for each vertex $G_v$ is isomorphic to $G_{\phi_X(v)}$, and if so, find such an isomorphism. If for one vertex $v$, the vertex groups $G_v$ and $G_{\phi_X(v)}$ are not isomorphic, then $\phi_X$ cannot be extended into an automorphism of graph of groups. We can therefore produce the finite list of these graph automorphisms.

[DT19, Proposition 4.4] then ensures that the graphs of groups are isomorphic if and only if there exists an extension adjustment (see [DT19, Definition 4.3]). The solution to the mixed Whitehead problem for vertex groups ensures that one can decide whether such an adjustment exists.

Recall that, following [D, §1.2], the small modular group of a graph of groups $X$ is the subgroup of $\delta \text{Aut}(X)$ whose elements are of the form $(\text{Id}_X, (\text{ad}_{\gamma_v}), (\text{Id}_e), \gamma_e)$, for $\gamma_v \in G_v$. We note that the Bass diagram imposes that $\gamma_v \gamma_e^{-1} \in Z_{G_v(e)}(t_{a}(G_e))$ for all edge $e$. Recall also that it is generated by Dehn twists over edges of the graph of groups $X$.

2 Main reduction result

2.1 Assumptions and statement

We now need to detail our vocabulary on mapping tori. Let $F$ be a group, $\alpha$ an automorphism of $F$, and $T_\alpha = F \rtimes \langle t_\alpha \rangle$ its mapping torus. Thus, in $T_\alpha$, if $a \in F$, one has $\text{ad}_{t_\alpha}(a) = \alpha(a)$. We will often write $t$ for $t_\alpha$ if the context allows the abuse.

The group $T_\alpha$ is equipped with a fibre and an orientation: we call $F$ the fibre of the semi-direct product $T_\alpha$, and we say that the coset $Ft$ determines the positive orientation of the semi-direct product.

Consider now $T_\alpha$ and $T_\beta$ the mapping tori of the same group $F$, by automorphisms $\alpha$ and $\beta$. If $\psi: T_\alpha \rightarrow T_\beta$ is an isomorphism, we say that it is fibre preserving if $\psi(F) = F$, and in this case we say that it is orientation preserving if $\psi(t_\alpha) \in Ft_\beta$.

For a subgroup $H$ of $F$ whose conjugacy class is preserved by some power of $\alpha$, the sub-mapping torus of $H$ is obtained as follows. Let $k > 0$ the smallest integer such that there exists $a \in F$ for which $\alpha^k(H) = \text{ad}_a(H)$. The sub-mapping torus of $H$ is the subgroup $\langle H, t_\alpha^k a^{-1} \rangle$, of $T_\alpha$. It does not depend on the choice of $a$, and is isomorphic to $H \rtimes_{(\text{ad}_{a^{-1} \circ \alpha^k})} \langle t_\alpha^k a^{-1} \rangle$.

We now make the list of algorithmic problems in groups, in order to state our main result. Following [T], we say that a class of groups is hereditarily...
algorithmic tractable, if

- the class is closed under taking finitely generated subgroups, and is effectively coherent: finitely generated subgroups are finitely presentable, and a presentation is computable,
- the presentations of the groups in the class are recursively enumerable,
- the class has uniform solution to the conjugacy problem,
- the class has uniform solution to the generation problem.

The interest of this definition, in this paper, is that it allows the computation of the canonical JSJ splitting which we will use later. We indicate that any other way of computing the canonical JSJ can be substituted to this assumption, without changing the argument.

We need two more algorithmic problems. Let $F$ be a group, $\alpha_0$ an automorphism of $F$, and $T_{\alpha_0}$ the mapping torus. Let $H$ be a finitely generated subgroup of $T_{\alpha_0}$.

- Following [DT19], we say that congruences separate torsion effectively in $H$, if there is a computable finite index subgroup $N$ of $H$, that is characteristic, and such the map from $\text{Out}(H)$ to $\text{Out}(H/N)$ has torsion free kernel. We also say that $H$ is effectively Minkowskian.

- We say that $T_{\alpha_0}$ has solvable fibre-and-orientation preserving mixed Whitehead problem if there is an algorithm solving the orbit problem for tuples of conjugacy classes of tuples of elements in $T_{\alpha_0}$, under the action of $\text{Out}_{T_{\alpha_0}}(\mathbb{T}_{\alpha_0})$, the subgroup of $\text{Out}(T_{\alpha_0})$ that preserves the fibre and the orientation.

A group is small if it does not contain a non-abelian free subgroup. We may now state our main result.

**Theorem 2.1.** Let $F$ be a finitely generated, torsion-free group. Consider the class $\mathcal{T}$ of mapping tori of $F$, of the form $\mathbb{T}_\alpha = F \rtimes_{\alpha} \mathbb{Z}$, that are relatively hyperbolic with respect to a collection $\mathcal{P}_\alpha$ of sub-mapping tori such that

1. the groups of $\mathcal{P}_\alpha$ belong to an hereditarily algorithmically tractable class of groups, and their small subgroups are finitely generated
2. the fibre and orientation preserving isomorphism problem is solvable for a class of mapping tori that contain the sub-mapping tori in \( \mathcal{P}_\alpha \).

3. the groups in \( \mathcal{P}_\alpha \) belong to a (possibly larger) class of groups stable for taking finitely generated subgroups, for which congruences separate the torsion effectively.

4. the groups in \( \mathcal{P}_\alpha \) form a class of groups for which the fibre-and-orientation preserving mixed Whitehead problem is solvable.

Then in the class \( \mathcal{T} \), the fibre and orientation preserving isomorphism problem is solvable.

Remark 2.2. It will actually be sufficient to assume the third point for the subgroups of the groups in \( \mathcal{P}_\alpha \) that are peripheral in the rigid vertex groups of a peripheral JSJ decomposition of \( T_\alpha \), and stabilize some edge. In practice, these subgroups groups may be much simpler than those in \( \mathcal{P}_\alpha \) (e.g. free abelian), which makes the reduction easier to apply.

Remark 2.3. The mixed Whitehead problem is used for tuples that are generating sets of adjacent edge groups in peripheral vertex groups of peripheral JSJ decomposition of \( T_\alpha \). Any partial solution that covers such cases is enough to have the conclusion.

Remark 2.4. As already mentioned, the assumption of belonging to an hereditarily algorithmically tractable class of groups can be replaced by the ability to compute the JSJ splitting with edge groups in this collection.

In this statement, \( F \) is not necessary a free group (\( F \) stands for Fibre).

We call a subgroup of \( T_\alpha \) elementary if it is either cyclic or a subgroup of a group in \( \mathcal{P}_\alpha \). For the rest of the paper we consider \( F \), \( T_\alpha \) and \( T_\beta \) as in the statement of Theorem 2.1, and aim to determine whether they are isomorphic via a fibre and orientation preserving isomorphism. We will sometimes write \( t \) for either \( t_\alpha \) or \( t_\beta \), when the context is clear.

### 2.2 JSJ decompositions

First we record that no splitting of a mapping torus \( T_\alpha \) of a finitely generated, torsion-free group \( F \) has a vertex group that quadratically hanging (QH) also known as hanging fuchsian.
Proposition 2.5 ([D, Proposition 2.11]). If $\mathbb{T}_\alpha$ acts co-finitely on a tree $T$ and if $G_v$ is the stabilizer of $v$, and if $\mathcal{P}_v$ is the peripheral structure on $G_v$ given by the collection of adjacent edge stabilizers, then $(G_v, \mathcal{P}_v)$ is not isomorphic to the fundamental group of a surface, with the peripheral structure of its boundary components.

Observe that the reference given only treats trees with cyclic edge groups, but one can apply it to the tree $\check{T}$ obtained from $T$ by collapsing all edges with non-cyclic stabilizer: if there exists a vertex of $T$ whose group $(G_v, \mathcal{P}_v)$ is isomorphic to the fundamental group of a surface, with the peripheral structure of its boundary components, then its star would be unchanged in the collapse $\check{T}$, and its image in $\check{T}$ would have same stabilizer. Now one can apply [D, Proposition 2.11] to obtain that the later cannot be a surface group, hence a contradiction.

There is a specific $\mathbb{T}_\alpha$-tree that is called the JSJ-tree of the relatively hyperbolic group $(\mathbb{T}_\alpha, \mathcal{P}_\alpha)$ that is invariant by automorphisms of $\mathbb{T}_\alpha$. We refer the reader to [GL], and to [DT19, Theorem 3.22] for a convenient characterisation.

Recall that Bass-Serre theory dualises actions on trees and decompositions into graphs of groups (or splittings). We call a splitting essential if there is no valence 1 vertex, whose group is equal to its adjacent edge group. The following is an immediate consequence of [DT19, Theorem 3.22] and Proposition 2.5 above, which guarantees the absence of QH vertex groups.

Proposition 2.6. The JSJ decomposition of $\mathbb{T}_\alpha$ is the unique essential splitting that is a bipartite graph of groups, with white vertices, and black vertices, such that black vertices are maximal elementary, white vertices are rigid (in the sense that they have not further compatible elementary splitting), and such that for any two different edges adjacent to a white vertex $w$, the image of the two edge groups in the vertex group $G_w$ are not conjugate in $G_w$ into the same maximal elementary subgroup of $G_w$.

Proposition 2.7. Assume that the class of peripheral subgroups in $\mathcal{P}_\alpha$ belong to a hereditarily algorithmically tractable class of groups. Then, there exists an algorithm that, given $\mathbb{T}_\alpha$, computes the JSJ splitting of $\mathbb{T}_\alpha$.

The algorithm is actually as explicit as the given algorithms in the assumptions. The proposition follows from [DT19, Theorem 1.4] (after observing that $\mathbb{T}_\alpha$ is always one-ended and torsion-free).
2.3 On white vertices: Congruences and Dehn fillings for lists

In the following, the graphs of groups $J_\alpha$, $J_\beta$ are the JSJ decompositions of $T_\alpha$, $T_\beta$ respectively. Let $J_\alpha$ and $J_\beta$ the underlying graphs of these graph of groups. We assume that we are given an isomorphism of graphs $\phi_J : J_\alpha \to J_\beta$.

2.3.1 Peripheral structures, orbits of markings

All our tuples will be ordered (and finite), while sets are not ordered. Let $G$ be a group. A peripheral structure on $G$ is a finite set of conjugacy classes of subgroups. An ordered peripheral structure is a tuple of conjugacy classes of subgroups. A marked peripheral structure is a finite set of conjugacy classes of tuples of elements, while a marked ordered peripheral structure is a tuple of conjugacy classes of tuples of elements. For each marked structure, there is an associated unmarked structure, by taking the subgroups generated by the tuples of elements.

We sometimes abuse terminology by saying that a peripheral structure is the (usually infinite) set of all subgroups whose conjugates belong to the given finitely-many conjugacy classes.

Let $G$ be a group endowed with a peripheral structure $\mathcal{E}$, and $G'$ another group.

We say that two isomorphisms $\phi, \psi : G \to G'$ peripherally coincide on a peripheral structure $\mathcal{E}$ if for every subgroup $E \in \mathcal{E}$, there exists $h = h(E) \in G'$ such that the restriction $\psi|_E$ differs from $\phi|_E$ by the (post)-conjugation by $h$ in $G'$.

2.3.2 White vertices of the JSJ decompositions and peripheral structures

We apply this setting to the white vertices of the JSJ decompositions.

Consider a white vertex $w$ of the JSJ decomposition $\mathbb{J}_\alpha$. We endow its group $G_w$ with several peripheral structures. First there is $\mathcal{P}(w)$ the relatively hyperbolic peripheral structure coming from that of the ambient relatively hyperbolic group: it consists of the subgroups that are intersection of $G_w$ with maximal parabolic subgroups of $T_\alpha$ in $\mathcal{P}_\alpha$.

Second there is $\mathcal{E}_2(w)$ a cyclic peripheral structure coming from the cyclic edge groups that are not subgroups of groups in $\mathcal{P}(w)$. It is the collection of
conjugacy classes of subgroups of $G_w$ that are cyclic, conjugate to an adjacent edge group in $\mathbb{J}_\alpha$, and not subgroups of groups in $\mathcal{P}(w)$. We say that this peripheral structure is transverse (to $\mathcal{P}(w)$).

Third, there is the non-transverse edge peripheral structure $\mathcal{E}_\mathcal{P}(w)$ consisting of edge groups that are subgroups of groups in $\mathcal{P}(w)$: they are parabolic in the relatively hyperbolic structure.

These three peripheral structures are unmarked but ordered (by choice of an order on the set of edges of $\mathbb{J}_\alpha$ around each vertex): they correspond to an ordered finite family of conjugacy classes of subgroups, but they do not correspond to classes of generating sets of these subgroups. However, $\mathcal{E}_\mathcal{Z}(w)$ is naturally marked: each of its subgroups has a unique generator with positive orientation.

We denote by $\mathcal{E}_w$ the ordered concatenation of the peripheral structures $\mathcal{E}_\mathcal{Z}(w)$ and $\mathcal{E}_\mathcal{P}(w)$. It is the peripheral structure of adjacent edge groups of $G_w$. The group $\text{Out}(G_w, \mathcal{E}_w)$ acts on the set of markings of $\mathcal{E}_\mathcal{Z}(w)$ and $\mathcal{E}_\mathcal{P}(w)$.

### 2.3.3 Dehn fillings

We need now to discuss Dehn fillings, Dehn kernels, and the $\alpha$-certification property, given in [D2].

Given a relatively hyperbolic group $(G, \mathcal{P})$, with a choice of conjugacy representatives $\{P_i\}$ of the peripheral structure $\mathcal{P}$, a Dehn kernel is a normal subgroup $K$ of $G$ normally generated by subgroups $N_i$ of the peripheral groups $P_i$: in other words $K = \langle \bigcup N_i \rangle_G$. The quotient of $G$ by the Dehn kernel $K$ is a Dehn filling of $G$.

The Dehn filling theorem [O07] states that there exists a finite set $S$ of $G\setminus\{1\}$ such that, whenever $N_i \cap S$ is empty for each $i$, the group $G/K$ is hyperbolic relative to the injective images of $P_i/N_i$.

If one is given a quotient $G \twoheadrightarrow \mathbb{Z}$, we call its kernel $F$ a fibre for $G$, and we say that a Dehn kernel is in the fibre (or is fibered) with respect to this quotient, if each $N_i$ is contained in $F$. We will say a Dehn filling fibered if it is obtained by quotienting by a fibered Dehn kernel.

When $G$ maps onto $\mathbb{Z}$ as before, with fibre $F$, two sequences of Dehn kernels will be of interest to us.

First, for each $m \geq 0$, the kernel $K^{(m)}$ is defined by setting $N_i^{(m)}$ to be the intersection of all index $\leq m$ subgroups of $P_i$; observe that they are not fibered Dehn kernels. We will denote $G/K^{(m)}$ by $\overline{G}^{(m)}$.
Second, for each $m \geq 0$, the kernel $K^{(m,f)}$ is defined by setting $N^{(m,f)}_i$ to be the intersection $N_i^{(m)} \cap F$, where $F$ is the given fibre. We will denote $G/K^{(m,f)}$ by $G^{(m,f)}$. These Dehn kernels are thus fibered. Observe that the Dehn filling theorem gives the following (recall that peripheral subgroups of finitely generated relatively hyperbolic groups are finitely generated).

**Proposition 2.8.** If $G$ is a finitely generated relatively hyperbolic group, that is residually finite, with a fibre $F$, then for every sufficiently large $m$, the group $G/K^{(m)}$ is hyperbolic relative to the subgroups $P_i/N_i^{(m)}$, which are finite, and the group $G/K^{(m,f)}$ is hyperbolic relative to the subgroups $P_i/N_i^{(m,f)}$, which are virtually cyclic.

In both cases, $G/K^{(m)}$ and $G/K^{(m,f)}$ are word-hyperbolic. Moreover, in the second case, the image of $P_i/N_i^{(m,f)}$ in $G/K^{(m,f)}$ is a subgroup that is its own normaliser.

Only the last conclusion requires an explanation: any infinite maximal parabolic subgroup of a relatively hyperbolic group is its own normaliser, by [O04, Thm 1.14]. The Dehn filling theorem actually says that the image of $P_i/N_i^{(m,f)}$ is a maximal parabolic subgroup of a relatively hyperbolic structure and it is infinite. The conclusion follows.

Consider a white vertex $w$ of the JSJ decomposition $\mathbb{J}_\alpha$, and its group $G_w$, and its relatively hyperbolic peripheral structure $\mathcal{P}(w)$.

The considerations above apply to $(G_w, \mathcal{P}(w))$ since it is residually finite, relatively hyperbolic, and endowed with a natural quotient onto $\mathbb{Z}$.

For fibered Dehn fillings, we must introduce some conditions and find Dehn fillings that satisfy them. In particular we introduce the $\alpha$-certification of Dehn fillings, already used in [D2], that will permit, first the use of [D2, Prop. 2.4] (that we can collect a list of automorphisms for which the images of edge groups through Dehn fillings is well-behaved), and later, to prove Proposition 2.14 (through Lemma 2.16) that collects a list of so-called fibre-controlled automorphisms (see the definition before the said Proposition).

We call a Dehn filling $G^{(m,f)}_w$ a **resolving Dehn filling** if all the following conditions are fulfilled:

- the quotient is hyperbolic and rigid (in the sense that it has no elementary splitting and it is not a virtual surface group), and for any $P_i$ in $\mathcal{P}(w)$, its image in $G^{(m,f)}_w$ is naturally isomorphic to $P_i/N_i^{(m,f)}$. 

15
and the Dehn filling is $\alpha$-certified as defined in [D2], that is: for each (cyclic) subgroup $C$ in $\mathcal{E}_Z(w)$, the centraliser of the image of $C$ in the quotient is equal to the image of the centraliser of $C$.

Observe that the kernel of the quotient map $G_w \to \overline{G_w}^{(m,f)}$ is in the fiber, hence the peripheral subgroups in $\mathcal{E}_Z(w)$ (that are transverse to $\mathcal{P}(w)$) map injectively in the quotient.

**Lemma 2.9.** If all small subgroups of the fibre of the peripheral subgroups of $G_w$ are finitely generated, then there is an algorithm that, given a resolving Dehn filling of $G_w$, terminates and produces a proof that it is resolving. Moreover, for all $m_0$, there are resolving Dehn fillings of $G_w$ obtained by a choice of $m$ larger than $m_0$.

**Proof.** By the Dehn filling theorem, for sufficiently deep Dehn fillings, the quotient is hyperbolic relative to virtually cyclic subgroups, hence word-hyperbolic, and the peripheral quotients inject. By [D2, Lemma 2.12], sufficiently deep Dehn fillings satisfy the $\alpha$-certification property.

Let us argue that they are rigid. For that we will show that they have no peripheral splitting (i.e. no splitting relative to the peripheral structure $\mathcal{P}(w) \cup \mathcal{E}_Z(w)$, over some parabolic subgroups), and no splitting relative to their parabolic peripheral structure, over a maximal cyclic subgroup. We need a short digression in order to use Groves and Manning’s result [GM, Theorem 1.8]. Any small subgroup $U$ of $G_w$ in the decomposition has to intersect the fibre as a small group, hence in the peripheral structure. By assumption, this intersection is finitely generated. Therefore, $U$ is finitely generated, since it is (f.g. small)-by-cyclic. Also, the group $G_w$, as a semidirect product of a finitely generated group with $\mathbb{Z}$, is one-ended. We can therefore use [GM, Theorem 1.8] to conclude that sufficiently deep Dehn fillings have no peripheral splittings, for the peripheral structure $\mathcal{P}(w) \cup \mathcal{E}_Z(w)$. Finally, in order to check that they don’t have splittings relative to the peripheral structure $\mathcal{P}(w) \cup \mathcal{E}_Z(w)$ over a maximal cyclic subgroup, we invoke [DG18]: if a sequence of such Dehn fillings all had such a splitting, considering a diagonal sequence of Dehn twists over these splittings, one gets a contradiction to [DG18, Corollary 5.10 (of Proposition 5.8)], applied to $G' = G = G_w$. Therefore, sufficiently deep Dehn fillings of $G_w$ are rigid.

All the properties are algorithmically verified, by [P] (for word hyperbolicity), [DG10, Corollary 3.4] (for rigidity), and [DG10, Lemma 2.8] (for the computation of centralizers, and identification of the images of the peripheral quotients).
2.3.4 Back to lists for white vertices

We come back to our context, in which $w$ is a white vertex of the canonical JSJ decomposition of $T_\alpha$, and $w'$ is its image by some graph isomorphism $\phi_j$.

Observe that the groups $G_w$ and $G_{w'}$ are finitely presented, relatively hyperbolic with respect to the structures $\mathcal{P}(w), \mathcal{P}(w')$, which consist of infinite groups, they are residually finite, and admit no peripheral splitting over an elementary group. We may therefore apply [DT19, Proposition 5.1] in order to obtain the following.

**Proposition 2.10** (See [DT19, Proposition 5.1]). Assume that congruences effectively separate the torsion in the peripheral subgroups of $G_w$.

Given a marking of $\mathcal{E}_Z(w) \cup \mathcal{E}_P(w)$, its orbit under the group $\text{Out}(G_w, \mathcal{E}_w)$ is computable.

In order to prove this Proposition, Dehn fillings were used such that the corresponding Dehn kernels were finite index subgroups of the maximal parabolic subgroups, and, in particular, it is important that these subgroups are chosen deep enough to ensure that Dehn fillings fulfil the Minkowski property: they are congruences that separate the torsion (in $\text{Out}(G_w)$). We do not give the detail of the argument, since it is covered by the case treated in [DT19, Proposition 5.1].

In the next Proposition, we will use Dehn fillings with fibered Dehn kernels (i.e. lying in the fibre). It will not be important whether or not they fulfil the Minkowski property.

**Proposition 2.11.** Consider $w$ a white vertex in the graph of groups $\mathcal{J}_\alpha$, and $w' = \phi_j(w)$ a white vertex in the in graph of groups $\mathcal{J}_\beta$. Consider the groups $G_w, G_{w'}$ with their (unmarked ordered) peripheral structures $\mathcal{P}(w), \mathcal{E}_Z(w)$ and $\mathcal{E}_P(w), \mathcal{E}_Z(w')$ and $\mathcal{E}_P(w')$. Choose a marking $(\mathcal{E}_Z(w))_m, (\mathcal{E}_P(w))_m$.

- It is decidable whether there exists a fibre-and-orientation preserving isomorphism from $(G_w, \mathcal{E}_Z(w), \mathcal{E}_P(w))$ to $(G_{w'}, \mathcal{E}_Z(w'), \mathcal{E}_P(w'))$.

- If there exists such an isomorphism, let $(\mathcal{E}_Z(w'))_m, (\mathcal{E}_P(w'))_m$ be the image in $G_{w'}$ of the chosen marking in $G_w$. One can compute the finite orbit of $(\mathcal{E}_Z(w'))_m, (\mathcal{E}_P(w'))_m$ by $\text{Out}(G_{w'}, \mathcal{E}_{w'})$. 
• For any marking $\mathcal{E}_Z(w')_{m'}, \mathcal{E}_P(w')_{m'}$ in this orbit, it is decidable whether there exists a fibre-and-orientation preserving isomorphism

$$(G_w, (\mathcal{E}_Z(w))_m, (\mathcal{E}_P(w))_m) \to (G_{w'}, \mathcal{E}_Z(w')_{m'}, \mathcal{E}_P(w')_{m'}).$$

(We then say that $\mathcal{E}_Z(w')_{m'}, \mathcal{E}_P(w')_{m'}$ is an admissible marking).

• For all markings $\mathcal{E}_Z(w')_{m'}, \mathcal{E}_P(w')_{m'}$ an algorithm computes a list $\mathcal{L}_{m,m'}$ of fibre-and-orientation preserving isomorphisms of the form

$$(G_w, \mathcal{E}_Z(w)_m, \mathcal{E}_P(w)_m) \to (G_{w'}, \mathcal{E}_Z(w')_{m'}, \mathcal{E}_P(w')_{m'}),$$

such that the following holds:

for any fibre-and-orientation preserving isomorphism

$$\psi : (G_w, \mathcal{E}_Z(w)_m, \mathcal{E}_P(w)_m) \to (G'_{w'}, \mathcal{E}_Z(w')_{m'}, \mathcal{E}_P(w')_{m'}),$$

there exists a resolving Dehn kernel $K'$ of $G_{w'}$, and $g \in G_{w'}$, and an element $\phi \in \mathcal{L}_{m,m'}$ such that $ad_g \circ \psi$ and $\phi$ coincide in the quotient $G_{w'}/K'$.

Observe that in the third statement, the peripheral structures are marked. From the second point we already know that there exists an isomorphism from $G_w$ to $G_{w'}$ intertwining the markings, but we have no guarantee that this isomorphism can be taken fibre-and-orientation preserving. Thanks to the third point we will know whether it can or not.

The first and last points of the statement of Proposition 2.11 make a reformulation of [D2, Proposition 2.4], to which we refer, but that we don't syntactically reproduce. This [D2, Proposition 2.4] gives an algorithm that, given relatively hyperbolic groups without any non-trivial peripheral splitting (relative to their transversal peripheral structures), terminates and provides a certificate of non-isomorphy, or a collection of isomorphisms as in the last point, or a certain cyclic splitting of one of the groups. Observe that in our case, there is no non-trivial peripheral splitting or cyclic splitting (relative to their transversal peripheral structures), because we consider white vertex groups in a JSJ decomposition.

\[\text{that is: for all } h \in G_w, \text{ there exists } z_h \in K' \text{ for which } \psi(h)g = \phi(h)z_h.\]
Proof. The first assertion is thus ensured by [D2, Proposition 2.4]. The second is the previous Proposition 2.10. The third assertion is an application of [DG18, Proposition 5.5] used with the constraint that the markings must be intertwined, as we detail now. If there is no such isomorphism, [DG18, Propoposition 5.5] ensures that, in some characteristic fibered Dehn fillings $\overline{G_w^{(m,f)}}, \overline{G_{w'}^{(m,f)}}$ of $G_w$ and $G_{w'}$ (with same $m$), this will be apparent.

However, in such a Dehn filling, all parabolic subgroups have become virtually cyclic, and therefore, by the Dehn filling theorem, the Dehn filling is word-hyperbolic.

Therefore, for a fixed pair of corresponding fibered Dehn fillings as above, by the solvability of the isomorphism problem for hyperbolic groups with marked peripheral structure [DG10, Thm 8.1], it is decidable whether or not there is such isomorphism between these Dehn fillings.

Enumerating the characteristic Dehn fillings in the fibre, and checking for each corresponding pairs of them this absence allows to detect if there exists indeed a pair for which there is no isomorphism. On the other hand, if there is an isomorphism of the correct form between $G_w$ and $G_{w'}$, it will be found by enumeration.

The fourth point is again treated in [D2, Proposition 2.4].

\[ \square \]

Proposition 2.12. Consider $w$ a white vertex in the graph of groups $\mathcal{J}_\alpha$, and $w' = \phi_f(w)$, and $G_w$ and $G_{w'}$ their groups with (unmarked ordered) peripheral structures $\mathcal{P}(w), \mathcal{E}_Z(w), \mathcal{E}_P(w)$, and $\mathcal{P}(w'), \mathcal{E}_Z(w'), \mathcal{E}_P(w')$. One can compute a finite list $\mathcal{L}_w$ that

- is empty if and only if
  $$(G_w, \mathcal{P}(w), \mathcal{E}_Z(w), \mathcal{E}_P(w)), \text{ and } (G_w, \mathcal{P}(w'), \mathcal{E}_Z(w'), \mathcal{E}_P(w'))$$
  are not isomorphic by a fibre and orientation preserving isomorphism,

- contains fibre and orientation preserving isomorphisms
  $$(G_w, \mathcal{P}(w), \mathcal{E}_Z(w), \mathcal{E}_P(w)) \rightarrow (G_w, \mathcal{P}(w'), \mathcal{E}_Z(w'), \mathcal{E}_P(w'))$$

- is such that, for all isomorphisms
  $$\psi : (G_w, \mathcal{P}(w), \mathcal{E}_Z(w), \mathcal{E}_P(w)) \rightarrow (G_w, \mathcal{P}(w'), \mathcal{E}_Z(w'), \mathcal{E}_P(w'))$$
  there is $\phi$ in $\mathcal{L}_w$, $g \in G_{w'}$ and a resolving Dehn kernel $K'$ of $G_{w'}$ such that for all $h \in G_w$, $\psi^g(h) \in \phi(h)K'$.
Proof. By Proposition 2.11 one can decide whether the two groups with (unmarked ordered) peripheral structures

\[(G_w, \mathcal{P}(w), \mathcal{E}_Z(w), \mathcal{E}_\mathcal{P}(w)) \text{ and } (G_{w'}, \mathcal{P}(w'), \mathcal{E}_Z(w'), \mathcal{E}_\mathcal{P}(w'))\]

are fibre and orientation preserving isomorphic or not, and if so, once a marking \(m\) on \((\mathcal{E}_Z(w), \mathcal{E}_\mathcal{P}(w))\) is chosen, one can compute all admissible markings (in the sense of Proposition 2.11) \(m'\) on \((\mathcal{E}_Z(w'), \mathcal{E}_\mathcal{P}(w'))\). Now \(L_w = \bigcup_{m'} L_{m, m'}\), whose computability is given by Proposition 2.11. It clearly satisfies the two first points of the conclusion of Proposition 2.12. The last point of the Proposition 2.11 ensures the third point, and thus proves Proposition 2.12.

\[\square\]

2.4 On black vertices: Navigating the Mixed Whitehead Problem

Recall that we are considering a fixed isomorphism \(\phi_J : J_\alpha \to J_\beta\) of the graphs underlying the JSJ decompositions of \(G\) and \(G'\). We first mark the images of edge groups in white vertices of \(G\). We make a list exhausting all possible matching markings of the corresponding white vertices in \(G'\) (by isomorphisms preserving fibre and orientation). We pull back these markings on the edge groups (through the attachment maps toward the white groups), and then we push the markings on the black groups, through the attachment maps of the reversed orientation edges, in \(G'\) and in \(G\).

We thus have, for each choice of white matching marking, several markings of each black vertex, coming from the adjacent white vertices. Choosing an order between neighbors, that is matched by the graph isomorphism, we create the ordered tuple of these markings. Call these tuples compounded markings. We have them for each black vertex of \(G\), of \(G'\), for each choice of white matching marking. We are moreover given an unmarked isomorphism between the matched black vertices, that preserves fibre and orientation, by assumption (2) in the statement of Theorem 2.1. We want to know whether one can post-compose these isomorphisms with automorphisms of the black vertices that preserve fibre and orientation, and that send the image of the compounded marking of each \(G_b\) to the compounded marking of \(G'_{b'}\). This is exactly the fibre-and-orientation preserving mixed Whitehead problem, as explained in the next proposition.
If \( b \) is a black vertex, we denote its image \( \phi_J(b) \) by \( b' \). By assumption, we can decide whether there exists a fibre-and-orientation preserving isomorphism between \( G_b \) and \( G_{b'} \). If there is none, we cannot promote \( \phi_J \) into an isomorphism of graphs of groups.

In the following, we will assume there is at least one such isomorphism and we shall denote it \( \phi^{(0)}_b \).

**Proposition 2.13.** Assume the fibre preserving mixed Whitehead problem has a solution for groups in \( \mathcal{P}_\beta \).

Then there exists an algorithm such that, given \( \phi^{(0)}_b \), and given, for each edge \( e_j = (b, w_j) \) adjacent to \( b \), a fibre preserving, and peripheral structures preserving isomorphism \( \phi_{w_j} : G'_{w_j} \to G'_{w'_j} \), will indicate whether there is an isomorphism \( G_b \to G_{b'} \) that preserves the fibre, the orientation, the unmarked ordered peripheral structures, and that peripherally coincides with \( \phi_{w_j} \circ t_{e_j}^{-1} \circ t_{e_j}^{-1} \) (i.e. that, for all \( j \), in restriction to \( t_{e_j}(G_{e_j}) \), coincide to this map postcomposed by a conjugation by an element \( p_{e_j} \) of \( G'_{w_j} \)).

**Proof.** For each edge group \( G_e = G_{e'} \), take a generating set: it defines a marking of the attachment subgroups of adjacent vertex groups. Given \( \phi^{(0)}_b \), one looks for a fibre and orientation preserving automorphism of \( G_{b'} \) that sends, for each adjacent edge \( e'_j \), the conjugacy classes of the marking of \( t_{e_j}(G_{e_j}) \), on the marking of \( t_{e_j}(i_{e_j}^{-1}(\phi_{w_j}(t_{e_j}(G_{e_j})))) \).

Therefore, the problem to solve is a reformulation of the fibre-and-orientation preserving mixed Whitehead problem in \( G_{b'} \), which is given by assumption. \( \square \)

### 2.5 Parts of the fibre in vertex groups

Recall that we want to determine whether there exists an isomorphism \( \mathbb{T}_\alpha \to \mathbb{T}_\beta \) that is fibre and orientation preserving.

Recall that \( \mathbb{J}_\alpha \) and \( \mathbb{J}_\beta \) are the canonical JSJ decompositions of \( \mathbb{T}_\alpha \) and \( \mathbb{T}_\beta \). We say that an isomorphism of graphs of groups \( \Psi : \mathbb{J}_\alpha \to \mathbb{J}_\beta \), noted \( \Psi = (\psi_J, (\psi_v)_v, (\psi_e)_e) \) for \( \gamma_e \in G_\tau(\psi_J(e)) \) is fibre and orientation preserving for the vertices, if each \( \psi_v \) is fibre preserving, and at least one of \( \psi_v \) or \( \psi_e \) is orientation preserving (it is easy to see that it forces all \( \psi_v \) to be orientation preserving).

In spite of these conditions, it is not automatic that the whole fibre is preserved by \( \Psi \). This motivates the following definition.

---

21
Recall that we introduced, in Section 1.2, the definition of small modular group, as a subgroup of the automorphism group of a graph of groups. We say that an isomorphism Ψ as described above is fibre-controlled if there exists an isomorphism \( \Psi^{(0)} = (\psi, (\psi_v^{(0)}), (\psi_e^{(0)})) \) from \( J_\alpha \) to \( J_\beta \), that is the composition of a fibre and orientation preserving isomorphism, with an element of the small modular group of \( J_\beta \), and such that, there is a resolving Dehn filling of \( T_\beta \) in which the image of \( \psi_v \) and \( \psi_v^{(0)} \) coincide, and the images of \( \psi_e \) and \( \psi_e^{(0)} \) coincide too. Thus, in general the collections \( (\psi_v^{(0)}), (\psi_e^{(0)}) \) will be different from \( (\psi_v), (\psi_e) \), but will agree in a resolving Dehn filling.

**Proposition 2.14.** There exists an algorithm that, given \( T_\alpha, T_\beta, \) and \( J_\alpha, J_\beta \) as above, terminates and with an output \( O \) such that

- if there exists an isomorphism \( T_\alpha \to T_\beta \) that is fibre and orientation preserving, the algorithm outputs \( O \), a non-empty finite collection of isomorphisms of graphs of groups \( J_\alpha \to J_\beta \), that are fibre and orientation preserving for the vertices. Furthermore, at least one isomorphism in \( O \) is fibre-controlled.

- if there no such fibre and orientation preserving isomorphisms \( T_\alpha \to T_\beta \), but if there exists \( \Psi : J_\alpha \to J_\beta \) that is fibre and orientation preserving for the vertices, the algorithm outputs \( O \), a non-empty finite collection of such isomorphisms of graphs of groups.

- If there is no \( \Psi : J_\alpha \to J_\beta \) that is fibre and orientation preserving for the vertices, the algorithm outputs \( O \), the empty set.

**Proof.** Again we work with a fixed graph isomorphism \( \psi_J : J_\alpha \to J_\beta \) between the underlying graphs \( J_\alpha, J_\beta \).

Let \( W \) be the set of white vertices of \( J_\alpha \), and \( B \) the set of black vertices of \( J_\alpha \).

For all white vertices \( w \in W \), we use Proposition 2.12 to get lists \( \mathcal{L}_w \) of isomorphisms from \( G_w \) to \( G_{\psi_J(w)} \) that are fibre, orientation, and peripheral structure preserving, and that satisfy the conditions of the third point of Proposition 2.12. If one of them is empty, we may discard \( \psi_J \), hence we assume that all are non-empty.

For every \( b \in B \), one may decide whether there exists a fibre-and-orientation preserving isomorphism \( \phi_b^{(0)} : G_b \to G_{\psi_J(b)} \). If for some \( b \) there is none, we may discard \( \psi_J \), otherwise, we compute such an isomorphism for each \( b \in B \).
For every tuple $(\phi_w)_{w \in \mathcal{W}} \in \prod_{w \in \mathcal{W}} \mathcal{L}_w$, we use Proposition 2.13 for each black vertex in $\mathcal{B}$ to establish whether there exists, for each black vertex $b$, a fibre and orientation preserving isomorphism $G_b \to G'_w$ that agrees with the marking of the neighboring white vertices (in the sense of Proposition 2.13).

If, given $(\phi_w)_{w \in \mathcal{W}} \in \prod_{w \in \mathcal{W}} \mathcal{L}_w$, for some $b \in \mathcal{B}$, it is revealed that it is impossible to find such an isomorphism, then again we may discard this tuple. Otherwise, we compute such isomorphisms $\phi_b : G_b \to G_{\phi_j(b)}$ for each $b \in \mathcal{B}$.

Assume that one has found, for the given $\psi_J$, and a given $(\phi_w)_{w \in \mathcal{W}} \in \prod_{w \in \mathcal{W}} \mathcal{L}_w$, isomorphisms $(\phi_b)_{b \in \mathcal{B}}$ as above. Then one has a complete collection, $(\phi_v)_v$ of isomorphisms of vertex groups. For each edge $\{e, \bar{e}\}$, selecting an orientation unambiguously fixes $\tau(e)$, and by restriction of $\phi_{\tau(e)}$ one may define $\phi_e$. One also defines $\gamma_e$ for each $e$ such that $\tau(e) = b \in \mathcal{B}$ as the element $p_e$ given by Proposition 2.13, and for $\tau(e) = w \in \mathcal{W}$, as an element that conjugates $\phi_w(t_e(G_e))$ to $t_{\psi_J(e)}(G_{\psi_J(e)})$ (which exists since $\phi_w$ preserves the ordered peripheral structure).

One thus obtains that $(\psi_J, (\phi_w), (\phi_e), (\gamma_e))$ is a graph of groups isomorphism, as it satisfies the commutativity of Bass diagram (see Section 1.2).

The algorithm then either has discarded $\psi_J$, or, for each tuple $(\phi_w)_{w \in \mathcal{W}} \in \prod_{w \in \mathcal{W}} \mathcal{L}_w$, outputs a graph of groups isomorphism $(\psi_J, (\phi_w), (\phi_e), (\gamma_e))$ for this $\psi_J$, if one exists. Thus the eventual output $\mathcal{O}$ of the algorithm is a finite collection of such isomorphisms, for $\psi_J$ and for $(\phi_w)_{w \in \mathcal{W}} \in \prod_{w \in \mathcal{W}} \mathcal{L}_w$ ranging over the set of graph isomorphisms for which the process has been completed. We call this collection the selected isomorphisms.

We need three lemmas. The first is an observation.

**Lemma 2.15.** Assume that there is $\Psi^{(0)} = (\psi_J, (\psi_w), (\psi_e), (\gamma_e))$ an isomorphism of graph of groups that preserves the fibre and the orientation in the vertices.

Then there exists, for each white vertex $w$, an isomorphism $\phi_w \in \mathcal{L}_w$, that satisfies the conclusion of Proposition 2.12 for $\psi = \psi_w$, and for each black vertex $b$, an isomorphism $\phi_b$ as above, agreeing on its neighboring edge groups with the isomorphisms of its adjacent white vertices. In particular, the output $\mathcal{O}$ is non-empty.

**Proof.** The existence of $\phi_w$ is precisely Proposition 2.12, and determines the edge isomorphisms $\phi_e$. The isomorphisms $\phi_b$ can be taken to be the $\psi_b$ since $\phi_w$ agrees with $\psi_w$ on its adjacent edge groups. Finally, the elements $(\gamma_e)_e$...
can be taken to be \((\gamma_e^{(0)})_e\) since the Bass diagram (of Section 1.2) depends only on the restriction of the isomorphisms to edge groups. 

The second is the following. It is similar to [D2, Lemma 2.7, 2.8].

**Lemma 2.16.** If there is \(\Psi\), an isomorphism of graphs of groups, that preserves the fibre and the orientation in the whole groups \(T_\alpha, T_\beta\), then there is a selected isomorphism of graph of groups, that, not only preserves the fibre and orientation in the vertices, but also is fibre-controlled.

**Proof.** Let \(\Psi = (\psi_J, (\psi_v)_v, (\psi_e)_e, \gamma_e)\) as in the statement. By the composition rule of isomorphisms of graphs of groups, we have

\[
\Psi^{-1} = (\psi_J^{-1}, (\psi_J^{-1(v)}(\psi_{J^{-1(e)}(\gamma_{J^{-1(e)})^{-1}}))e)\).
\]

By construction of \(\mathcal{O}\), there is a selected isomorphism \(\Phi = (\psi_J, (\phi_v)_v, (\phi_e)_e, \gamma_e)\), and a resolving Dehn filling of \(T_\beta\), such that in this Dehn filling, all \(\psi_v\) and \(\phi_v\) coincide, up to conjugation in \(G_{\psi_J(v)}\), and moreover, \(\Phi\) (as any element of \(\mathcal{O}\)) preserves the fibre and the orientation in the vertices.

Consider the composition \(\Phi \circ \Psi^{-1} : T_\beta \to T_\beta\). It is an automorphism of \(\mathbb{J}_\beta\), and it can be written as

\[
\Phi \circ \Psi^{-1} = (\text{Id}_{\mathbb{J}_\beta}, (\epsilon_v \circ \text{ad}_{g_v})_v, (\epsilon_e \circ \text{ad}_{g_e})_e, (\eta_e)_e)
\]

in which \(\epsilon_v\) is a fibre-and-orientation preserving automorphism of \(G_v\), that induces the identity in \(G(v)\), and \(g_v\) is an element of \(G(v)\), and similarly for \(\epsilon_e\) and \(g_e\). For the record, \(\eta_e = \phi_{\psi_J^{-1}(\gamma_{J^{-1(e)})^{-1}})\gamma_{J^{-1(e)}}\), but this plays no role in the argument.

We need to show that some post-composition of this automorphism with an element of the small modular group is fibre and orientation preserving. In other words, we need to show that there is a fibre and orientation preserving automorphism \(\Upsilon\) such that \(\Upsilon \circ \Phi \circ \Psi^{-1}\) is in the small modular group.

We will consider \(\Upsilon\) of the form \(\Upsilon = (\text{Id}, (\epsilon_{v^{-1}}v), (\epsilon_{e^{-1}}e), (y_e)_e)\), where the \(y_e\) are yet unknown. It suffices then to show that there is such an automorphism with the \(y_e\) such that \(\Upsilon\) preserves the fibre and orientation.

We begin to prove that there is such an automorphism \(\Upsilon\) with all \(y_e\) in the fibre (compare to [D2, Lemma 2.7]).

\(\Upsilon\) being an automorphism of graphs of groups, it is obvious that each \(\epsilon_{v^{-1}}\) sends each adjacent edge group to a conjugate of itself, by a conjugator in \(G_v\).

Since \(\epsilon_{v^{-1}}\) induces the identity on a resolving Dehn filling, such a conjugator
must be in the pre-image of the centralizer of the image of the edge group. However, because the Dehn filling is resolving, the image of the centralizer of the edge group in $G_v$ is the centralizer of the image of the edge group in $\overline{G_v}$. The centralizer of the edge group in $G_v$ is not other than the edge group itself in $G_v$, therefore, the conjugator that we considered is a product of an element of the edge group in $G_v$ and an element of the Dehn kernel. Therefore, it can be chosen in the Dehn kernel, hence in the fibre.

Now, it suffices to choose the $y_e$ to be these conjugating elements that send the edge groups to their images by $\epsilon_v^{-1}$, and are in the fibre, and then to choose $\epsilon_e$ the induced automorphism of the edge groups, in order to make the Bass diagrams commute. Thus, the elements $y_e$ are in the fibre.

We check that this choice is fibre and orientation preserving. (Compare to [D2, Lemma 2.8])

To see this, let the fibre $F$ act on the tree $J_\beta$. This action defines a free decomposition of $F$ as a graph of groups, hence $F$ is generated by its intersection with the vertex groups, and the Bass generators (corresponding to edges of the graph of groups. The automorphism $\Upsilon$ preserve the cosets of the fibre in the vertex groups, and sends the Bass generator $e$ to $y_e^{-1}ey_e$.

Since $y_e$ and $y_\bar{e}$ are in the fibre, the image of $e$ is in the same coset of the fibre as $e$. It follows that the generators of $F$ given by the graph of groups decomposition are sent in $F$, thus $\Upsilon$ is fibre preserving.

Finally, it is orientation preserving because it is so on any edge groups, so it must be the case for the whole group. The Lemma is proved.

Lemma 2.17. If there is no isomorphism of graphs of groups from $J_\alpha$ to $J_\beta$ that is fibre and orientation preserving for the vertices, then the output $\mathcal{O}$ is empty.

Proof. By construction of the output $\mathcal{O}$ (see before Lemma 2.15), its elements are isomorphisms of graphs of groups that when restricted to white vertices are in lists $\mathcal{L}_w$, hence fiber and orientation preserving for the vertices (by Proposition 2.12). The lemma follows.

The three lemmas above together ensure the proposition holds.
2.6 The rest of the fibre and completing the proof of Theorem 2.1.

We now finish the proof of Theorem 2.1. We assume that the output of the algorithm of Proposition 2.14 is a non-empty set $\mathcal{O}$ of isomorphisms of graphs of groups, that are fibre and orientation preserving for the vertices. However, we do not yet know whether we are in the first or the second case of the conclusion of the Proposition 2.14.

Consider $\{h_1, \ldots, h_s\}$, a generating set of the fibre $F$ of $T_\alpha$.

**Proposition 2.18.** There exists a fibre and orientation preserving automorphism from $T_\alpha$ to $T_\beta$ if and only if, there is an element $\Phi$ of $\mathcal{O}$, and an element $\eta$ of the small modular group of $J_\beta$ that sends $\Phi(h_1), \ldots, \Phi(h_s)$ inside the fibre of $T_\beta$, and such that $\eta \circ \Phi$ is fibre and orientation preserving on the vertices.

**Proof.** If there is such an element $\Phi$, and such an $\eta$, then $\eta \circ \Phi$ is fibre and orientation preserving, because the fibre in $T_\alpha$ is generated by the fibre in the vertices, and the elements $h_1, \ldots, h_s$, and all this generating set is indeed sent to the fibre of $T_\beta$.

Assume now conversely that there exists a fibre and orientation preserving automorphism from $T_\alpha$ to $T_\beta$. Then we are in the first case of Proposition 2.14. There exists a fibre-controlled isomorphism

$$\Phi = (\phi_J, (\phi_v)_v, (\phi_e)_e, (\gamma_e)_e) \in \mathcal{O}.$$ 

By definition, there is also an isomorphism

$$\Phi^{(0)} = (\phi_J, (\phi_v^{(0)})_v, (\phi_e^{(0)})_e, (\gamma_e^{(0)})_e),$$

for which $\phi_v$ and $\phi_v^{(0)}$ coincide on a resolving Dehn filling, and such that some post-composition with an element of the small modular group is fibre and orientation preserving.

Therefore, there exists $\eta$ in the the small modular group of $J_\beta$ that sends $\Phi(h_1), \ldots, \Phi(h_s)$ inside the fibre of $T_\beta$, and such that $\eta \circ \Phi$ is fibre and orientation preserving on the vertices. 

\qed
It follows from Proposition 2.18 that we reduced the problem to deciding, for each \( \Phi \) in \( \mathcal{O} \), whether there exists an element \( \eta \) in the small modular group of \( \mathbb{J}_\beta \), sending \( \Phi(h_1), \ldots, \Phi(h_s) \) into the fibre of \( T_\beta \).

Given \( \Phi \) in \( \mathcal{O} \), the problem of deciding whether such an element \( \eta \) of the small modular group as described above exists is treated by interpreting it in the cohomology \( \mathbb{Z} \)-module \( H^1(\mathbb{T}_\beta, \mathbb{Z}) = \text{Hom}(\mathbb{T}_\beta, \mathbb{Z}) \): each Dehn-twist over an edge of \( \mathbb{J}_\beta \) (i.e. each generator of the small modular group for the generating family proposed in Section 1.2) acts as a transvection on the module. Let \( \bar{F} \) be the image of \( F \) in \( H^1(\mathbb{T}_\beta, \mathbb{Z}) \). Because the fibre \( F \) is the kernel of a homomorphism \( \mathbb{T}_\beta \rightarrow \mathbb{Z} \), for all \( g \in \mathbb{T}_\beta, g \in F \) if and only if, \( \bar{g} \in \bar{F} \).

Each image \( \Phi(h_i) \) is the sum of an element of \( \bar{F} \) and of a multiple of \( \bar{t} \), the image of \( t \). The transvections apply some translation in projection on the line generated by \( t \). The existence of some element in the small modular group sending all \( \Phi(h_i) \) in \( \bar{F} \) is therefore encoded in an explicit system of linear diophantine equations, whose existence of solution is thus decidable. We thus have proved Theorem 2.1.

### 3 An application

We start with a sufficient condition for congruences to effectively separate torsion.

**Proposition 3.1.** Let \( G \) be a finitely presented group such that the following hold:

- \( G \) is conjugacy separable,
- for any \( \alpha \in \text{Aut}(G) \) such that the image \([\alpha] \in \text{Out}(G)\) is non-trivial and of finite order, there exists \( g_\alpha \in G \) such that \( g_\alpha \) and \( \alpha(g_\alpha) \) are non-conjugate in \( G \), (i.e. \( G \) has no pointwise inner finite-order outer automorphisms), and
- we are given a finite list \( \{\alpha_1, \ldots, \alpha_k\} \subset \text{Aut}(G) \) containing a representative of the conjugacy class of every finite order element of \( \text{Out}(G) \).

Then \( G \) is effectively Minkowskian.

**Proof.** Let \( \{\alpha_1, \ldots, \alpha_k\} \subset \text{Aut}(G) \) be a list of representatives of the conjugacy classes of finite order elements in \( \text{Out}(G) \). Since \( G \) is finitely presented,
we can enumerate $G$ as well as all its finite quotients. For each $\alpha_i$ in our list, by this enumeration we can find some $g_{\alpha_i} \in G$ and a finite quotient of $G$ in which the images $\alpha_i(g_{\alpha_i})$ and $g_{\alpha_i}$ are not conjugate. It follows that we can find finite index characteristic subgroup $K \leq G$ in which the image $\bar{\alpha}_i$ in $\Aut(G/K)$ of each automorphism in our list is not inner, and the result follows.

Finitely generated free groups are conjugacy separable, by a result of Baumslag [Bau], and their non-trivial outer-automorphisms are never point-wise inner, by a result of Grossman [Gr]. By Culler’s Realization Theorem [Cu], to obtain a list of all representatives for each finite order outer automorphism, it is sufficient to enumerate the finite number of homeomorphism types of graphs with vertices of degree at least 3 whose fundamental group is $F_n$, and, for each such graph graph, to enumerate the graph’s symmetry group (which is finite). This immediately gives.

**Corollary 3.2.** $F_n$, the free group of rank $n$, is effectively Minkowskian.

### 3.1 Unipotent non-growing parabolics

We finish by mentioning a direct application.

**Corollary 3.3.** Let $F$ be a free group, and $\mathcal{O}_\mathcal{A}_0$ the subset of its outer-automorphism group consisting of classes of automorphisms $\phi$ whose maximal polynomially growing subgroups in $F$ are subgroups $P$ such that there exists $\gamma_P \in F$ for which $\ad_{\gamma_P} \circ \phi$ induces the identity on $P$. Then, the conjugacy problem in $\Out(F)$ for elements of $\mathcal{O}_\mathcal{A}_0$ is decidable: given $\phi_1, \phi_2$ defining elements in $\mathcal{O}_\mathcal{A}_0$, one can decide whether there is an automorphism $\psi \in \Aut(F)$ such that $[\psi^{-1} \circ \phi_1 \circ \psi] = [\phi_2]$.

**Proof.** First, recall two automorphisms $\phi_1, \phi_2$ of $F$, are conjugate in $\Out(F)$ if and only if there exists a fibre-and-orientation preserving isomorphism from $T_{\phi_1} = F \rtimes \langle t_{\phi_1} \rangle$ and $T_{\phi_2} = F \rtimes \langle t_{\phi_2} \rangle$. Second, according to [DL], these groups are relatively hyperbolic with respect to the sub-mapping tori of the maximal polynomially growing subgroups. Moreover, from [Le], we know that maximal polynomially growing subgroups are finitely generated. The peripheral sub-mapping tori are therefore direct products of finitely generated free groups, with $\mathbb{Z}$. Let us call $\mathcal{C}$ this class group. Observe that finitely generated subgroups of groups in $\mathcal{C}$ are either free or in $\mathcal{C}$ (according to
whether the restriction of the quotient map by the cyclic factor is injective or not on them).

Using Theorem 2.1 it remains to see whether one can solve the different algorithmic problems in \( C \) or in free groups.

First, consider the algorithmic properties that define hereditarily algorithmic tractable classes of groups. The recursive enumerability of presentations, the uniform conjugacy problem, the uniform generation problem are classical in free groups, and in the class \( C \).

For groups in \( C \), the fibre-and-orientation preserving isomorphism problem is solvable, since it is the question of the rank of the direct factor that is free.

Take \( G \) in the class \( C \). Either \( G \cong \mathbb{Z}^2 \) or there is a unique cyclic subgroup \( C \) for which \( G = H \times C \). Observe that \( H \) is not unique, but all such factors are free subgroups of \( G \) of same rank. Given \( G \) in \( C \) with a fibre and an orientation, the fibre-and-orientation preserving automorphism group is thus isomorphic to \( \text{Aut}(H) \). The mixed Whitehead problem in \( G \) by the fibre and orientation preserving automorphism group then easily reduces to the mixed Whitehead problem in \( H \), which is solvable by Whitehead’s algorithm.

It remains to see that in \( C \), congruences effectively separate the torsion. It is Minkowski’s theorem if \( G \cong \mathbb{Z}^2 \). Take \( G = F \times \mathbb{Z} \), where \( F \) is a non-abelian free group.

A non-trivial torsion element in \( \text{Out}(G) \) either has even order, inducing a flip of orientation of the center, or maps to a non-trivial torsion element in \( \text{Out}(F) \) by quotient by the characteristic subgroup \( \mathbb{Z} \) (the center). Indeed, assume the contrary. Write \( G = F \times C \) with \( C = \langle c \rangle \) the infinite cyclic center, and \( \alpha \in \text{Aut}(G) \) such that \( \bar{\alpha} : H \to H \) is inner: \( \bar{\alpha} = \text{ad}_{h_0} \). Since \( c \) is central, for all \( h \in H \), there is \( n_h \) such that \( \alpha(h) = \text{ad}_{h_0}(h)c^{n_h} \), and if \( \alpha \) is not inner, there is \( h \) such that \( n_h \neq 0 \), and assume it positive. Also observe that \( \alpha(c) = c^{\pm 1} \). If \( \alpha(c) = c^{-1} \), we are in the first case of the claim. Thus assume that \( \alpha(c) = c \). It follows (still using that \( c \) is central) that \( \alpha^k(h) = \text{ad}_{h_0^k}(h)c^{kn_h} \), and it is never inner, and thus \( \alpha \) does not have finite order.

Now, the free group \( F \) itself has congruences effectively separating the torsion by Corollary 3.2. It follows that one can effectively find congruence separating all non-trivial torsion elements in \( \text{Out}(G) \) except possibly those of even order inducing a flip of orientation of the center. Those later ones are easily separated, by choosing a finite quotient of \( G \) on which the center \( C \) maps on a subgroup of order \( \geq 3 \).
References

[Ba] Hyman Bass, Covering theory for graphs of groups. J. Pure Appl. Algebra, 89(1-2):3-47, 1993.

[BJ] Hyman Bass and Renfang Jiang Automorphism groups of tree actions and of graphs of groups. J. Pure Appl. Algebra, 112(2):109-155, 1996.

[Bau] , Gilbert Baumslag, Residual nilpotence and relations in free groups. J. Algebra 2, 271-282, 1965.

[Br] Peter Brinkmann, Hyperbolic Automorphisms of Free Groups. Geom. Funct. Anal. 10 (2000), no. 5, 1071-1089.

[Cu] Culler, Marc, Finite groups of outer automorphisms of a free group. Contributions to group theory 33 (1984): 197-207.

[DGr] François Dahmani and Daniel Groves, The isomorphism problem for toral relatively hyperbolic groups, Publ. Math. I HÉ S. 107 no.1 (2008) 211-290.

[DG10] François Dahmani and Vincent Guirardel, The isomorphism problem for all hyperbolic groups, Geom. Funct. Anal. 21 no.2 (2011) 223-300.

[D] François Dahmani, On suspensions and conjugacy of hyperbolic automorphisms, Trans. Amer. Math. Soc. 368 (2016) 5565-5577.

[D2] François Dahmani, On suspensions and conjugacy of a few more automorphisms of free groups, Adv. Stud. Pure Math. 73 (2017) Hyperbolic Geometry and Geometric Group Theory, pp. 135-158.

[DG18] François Dahmani and Vincent Guirardel, Recognizing a relatively hyperbolic group by its Dehn Fillings, Duke Math J. 167 no. 12. (2018) 2189-2241

[DT19] François Dahmani and Nicholas Touikan, Deciding isomorphy using Dehn Fillings: the splitting case, Invent. Math. 215, no1, (2019) 81-169.

[DT23] François Dahmani and Nicholas Touikan, Unipotent linear suspensions of free groups, arXiv:2305.11274.

[DL] François Dahmani and Ruoyu Li, Relative hyperbolicity for automorphisms of free products, J. Topol. Anal. 14, No. 1, 55-92 (2022).

[DK] François Dahmani and Suraj Krishna, Relative hyperbolicity of hyperbolic-by-cyclic groups, Groups Geom. Dyn. 17, No. 2, 403-426 (2023).
[FH] Mark Feighn and Michael Handel, The Conjugacy Problem for UPG elements of Out \((F_n)\), arXiv:1906.04147

[GL] François Gautero, Martin Lustig, The mapping-torus of a free group automorphism is hyperbolic relative to the canonical subgroups of polynomial growth, preprint arXiv:0707.0822.

[Gh] Pritam Ghosh, Relative hyperbolicity of free-by-cyclic extensions, Compos. Math. 159, No. 1, 153-183 (2023).

[Gr] Edna Grossman, On the residual finiteness of certain mapping class groups. J. Lond. Math. Soc., II. Ser. 9, 160-164 (1974).

[GM] Daniel Groves and Jason Manning, Dehn fillings and elementary splittings, Trans. Amer. Math. Soc. 370 (2018), 3017-3051

[GL] Vincent Guirardel and Gilbert Levitt, JSJ decompositions of groups. Astérisque No. 395 (2017), vii+165 pp.

[La] Marc Lackenby, Word hyperbolic Dehn surgery, Invent. Math. 140, (2000), 243–282.

[Le] Gilbert Levitt, Counting growth types of automorphisms of free groups, Geom. Funct. Anal. 19, 1119 (2009).

[M] Nataša Macura, Detour functions and quasi-isometries. Q. J. Math. 53, No. 2, 207-239 (2002).

[O04] Denis Osin, Relatively hyperbolic groups: intrinsic geometry, algebraic properties, and algorithmic problems. Mem. Amer. Math. Soc. 179, vii+100 (2006)

[O07] Denis Osin, Peripheral fillings of relatively hyperbolic groups, Invent. Math. 167 (2007), no. 2, 295–326.

[P] Panos Papasoglu, An algorithm detecting hyperbolicity. In Geometric and computational perspectives on infinite groups (Minneapolis, MN and New Brunswick, NJ, 1994), volume 25 of DIMACS Ser. Discrete Math. Theoret. Comput. Sci., pages 193–200. Amer. Math. Soc., Providence, RI, 1996.

[RS] Eliyahu Rips and Zlil Sela, Cyclic splittings of finitely presented groups and the canonical JSJ decomposition, Ann. Math. (2), 146 (1997), 53–104.

[S] Zlil Sela, The isomorphism problem for hyperbolic groups I, Annals of Math (2), 141 (1995), 217–283.
[Th] William Thurston, Hyperbolic structures on 3-manifolds, ii: Surface groups and 3-manifolds which fiber over the circle, arXiv:math/9801045.

[T] Nicholas Touikan, Detecting geometric splittings in finitely presented groups, Trans. Amer. Math. Soc. 370 (2018), 5635-5704

François Dahmani, Univ. Grenoble Alpes, Institut Fourier, 38000 Grenoble, France.
e-mail. francois.dahmani@univ-grenoble-alpes.fr
https://www-fourier.univ-grenoble-alpes.fr/~dahmani

Nicholas Touikan, Department of Mathematics and Statistics, University of New Brunswick, Fredericton, New Brunswick, Canada.
e-mail. ntouikan@unb.ca
https://ntouikan.ext.unb.ca