Triangular Matrix Categories I: Dualizing Varieties and Generalized One-point Extensions

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Abstract
Following Mitchell’s philosophy, in this paper we define the analogous of the triangular matrix algebra to the context of rings with several objects. Given two preadditive categories \( \mathcal{U} \) and \( \mathcal{T} \) and \( M \in \text{Mod}(\mathcal{U} \otimes \mathcal{T}^{op}) \) we construct the triangular matrix category \( \Lambda := \begin{bmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{bmatrix} \). First, we prove that \( \text{Mod}(\Lambda) \) is equivalent to a comma category \( (\text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U})) \) which is induced by a functor \( \mathcal{G} : \text{Mod}(\mathcal{U}) \to \text{Mod}(\mathcal{T}) \). One of our main results is that if \( \mathcal{U} \) and \( \mathcal{T} \) are dualizing \( K \)-varieties and \( M \in \text{Mod}(\mathcal{U} \otimes \mathcal{T}^{op}) \) satisfies certain conditions then \( \Lambda := \begin{bmatrix} \mathcal{T} & 0 \\ M & \mathcal{U} \end{bmatrix} \) is a dualizing variety (see Theorem 6.10). In particular, \( \text{mod}(\Lambda) \) has Auslander–Reiten sequences. Finally, we apply the theory developed in this paper to quivers and give a generalization of the so called one-point extension algebra.

Keywords Comma category · Dualizing varieties · Functor categories · Triangular matrix categories.

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1 Introduction

The idea that preadditive categories are rings with several objects was developed convincingly by Barry Mitchell (see [43]) who showed that a substantial amount of noncommutative ring theory is still true in this generality. Here we would like to emphasise that sometimes clarity in concepts, statements, and proofs are gained by dealing with preadditive categories, and that familiar theorems for rings come out of the natural development of category theory. For instance, the notions of radical of an preadditive category, perfect and semisimple rings, global dimensions etc, have been amply studied in the context of rings with several objects (see [18, 30–37, 44, 51, 52, 55–57]).

As an example of the power of this point of view is the approach that M. Auslander and I. Reiten gave to the study of representation theory (see for example [4–14, 16]), which gave birth to the concept of almost split sequence. There were two different approaches to the existence of almost split sequences. One was inspired by [2] and focused on showing that simple functors are finitely presented. An essential ingredient in this proof was to establish a duality between finitely presented contravariant and finitely presented covariant functors. This led to the notion of dualizing $R$-varieties, introduced and investigated in [6]. Therefore the existence of almost split sequences is proved in the context of dualizing $R$-varieties. Dualizing $R$-varieties have appeared in the context of the locally bounded $k$-categories over a field $k$, categories of graded modules over artin algebras and also in connection with covering theory. M. Auslander and I. Reiten continued a systematic study of $R$-dualizing varieties in [7–9]. One of the advantages of notion of a dualizing $R$-variety defined in [6] is that it provides a common setting for the category proj$(A)$ of finitely generated projective $A$-modules, mod$(A)$ and mod(mod$(A)$), which all play an important role in the study of an artin algebra $A$.

On the other hand, rings of the form $R = T \otimes_k U$ where $T$ and $U$ are rings and $M$ is a $T$-$U$-bimodule have appeared often in the study of the representation theory of artin rings and algebras (see for example [15, 21, 25–27]). Such rings, called triangular matrix rings, appear naturally in the study of homomorphic images of hereditary artin algebras and in the study of the decomposition of algebras and direct sum of two rings. Triangular matrix rings and their homological properties have been widely studied, (see for example [19, 22, 28, 41, 45, 58, 59]). The so-called one-point extension is a special case of the triangular matrix algebra and this types of algebras have been studied in several contexts. For example, in [29], D. Happel showed how to compute the Coxeter polynomial for $\Gamma$ from the Coxeter polynomial of $\Lambda$ and homological invariants of $M$ for a given one-point extension algebra $\Gamma = \Lambda[M]$. In [60], Zhu consider the triangular matrix algebra $\Lambda := \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ where $T$ and $U$ are quasi-hereditary algebras and he proved that under suitable conditions on $M$, $\Lambda$ is quasi-hereditary algebra.

It is well known that the category Mod$\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right)$ is equivalent to the category whose objects are triples $(A, B, f)$ where $f : M \otimes_T A \rightarrow B$ is a morphism of $U$-modules and the morphisms are pairs $(\alpha, \beta)$ with $\alpha : A \rightarrow A'$ and $\beta : B \rightarrow B'$ such that the following diagram commutes

$$
\begin{array}{ccc}
M \otimes_T A & \xrightarrow{1_M \otimes f} & M \otimes_T A' \\
\downarrow f & & \downarrow f' \\
B & \xrightarrow{\beta} & B',
\end{array}
$$
In particular, given a ring $T$ we can consider $\Lambda := \begin{bmatrix} T & 0 \\ T & T \end{bmatrix}$. Then the category of $\Lambda$-modules is equivalent to the category whose objects are triples $(A, B, f)$ where $f : A \rightarrow B$ is a morphism of $T$-modules.

In this context, in [50] Ringel and Schmidmeier studied the category of monomorphisms and epimorphisms and they proved that if $\Gamma$ is an artin algebra, the category of all the embeddings $(A \subseteq B)$ where $B$ is a finitely generated $\Gamma$-module and $A$ is a submodule of $B$, is a Krull-Schmidt category which has Auslander-Reiten sequences. Also in this direction, R.M. Villa and M. Ortiz studied the Auslander-Reiten sequences in the category of maps and also studied some contravariantly finite subcategories (see [39]).

Following Mitchell’s philosophy, in this paper we define the analogous of the triangular matrix algebra to the context of rings with several objects. Given two preadditive categories $\mathcal{U}$ and $\mathcal{T}$ and an additive functor $M$ from $\mathcal{U} \otimes \mathcal{T}^{op}$ to the category of abelian groups $\text{Ab}$, $M \in \text{Mod}(\mathcal{U} \otimes \mathcal{T}^{op})$ for short, we construct the triangular matrix category $\Lambda := \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ and several properties of $\text{Mod}(\Lambda)$, the category of additive functors from $\Lambda$ to $\text{Ab}$, are studied. For example, under certain conditions, we show that $\Lambda := \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ is a dualizing variety, this result is the analogous of the following one: $\Lambda := \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ is an artin algebra if and only if there is a commutative ring $R$ such that $T$ and $U$ are artin $R$-algebras and $M$ is finitely generated over $R$ which acts centrally on $M$ (see [17, Theorem 2.1] in page 72). We give some applications to path categories given by infinite quivers. In the part II of this work, we will give other applications as the construction of recollements and the study of functorially finite subcategories of $\text{mod}(\Lambda)$, obtaining a generalization of a result given by Chen and Zheng in [20, Theorem 4.4] and a generalization of a result due to Smałło ([54, Theorem 2.1]).

We now give a brief description of the contents on this paper.

In Section 2, we recall basic results of $\text{Mod}(\mathcal{C})$ that will be used throughout this paper. In Section 3, we construct the category of matrices $\Lambda := \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ and we prove that there is an equivalence between the comma category $(\text{Mod}(\mathcal{T}), \mathcal{GMod}(\mathcal{U}))$ and $\text{Mod}(\Lambda)$ and we compute $\text{rad}_{\Lambda}(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, \begin{bmatrix} T & 0 \\ M & U \end{bmatrix})$ in terms of the radical of $\mathcal{U}$ and $\mathcal{T}$ (see Proposition 3.8).

In Section 4, we consider $K$-varieties and define a functor $D_{\Lambda} : \text{Mod}(\Lambda) \rightarrow \text{Mod}(\Lambda^{op})$ and we describe it how it acts, when we identify $(\text{Mod}(\mathcal{T}), \mathcal{GMod}(\mathcal{U}))$ with $\text{Mod}(\Lambda)$ (see Theorem 4.11).

In Section 5, we show that there exists an adjoint pair $(\mathcal{F}, \mathcal{G})$ and we describe the finitely generated projectives in $(\text{Mod}(\mathcal{T}), \mathcal{GMod}(\mathcal{U}))$ (see Proposition 5.5). We also prove that there exists an isomorphism between the comma categories $(\mathcal{F}(\text{Mod}(\mathcal{T})), \mathcal{G}(\text{Mod}(\mathcal{U}))) \simeq (\text{Mod}(\mathcal{T}), \mathcal{G}(\text{Mod}(\mathcal{U})))$, (see Proposition 5.4) and in Section 6 we restrict that isomorphism to the category of finitely presented modules.

In Section 6, we prove that if $\mathcal{T}$ and $\mathcal{U}$ are categories with splitting idempotents, then $\Lambda$ is with splitting idempotents (see Proposition 6.8). We consider the case in which $\mathcal{U}$ and $\mathcal{T}$ are Hom-finite Krull-Schmidt $K$-varieties and we show that under this conditions $\Lambda$ is Hom-finite and Krull-Schmidt (see Proposition 6.8). Finally, we prove that if $\mathcal{U}$ and $\mathcal{T}$ are dualizing $K$-varieties and $M$ is a functor from $\mathcal{U} \otimes \mathcal{T}^{op}$ to $\text{Ab}$ satisfying that $M_{\mathcal{T}} \in \text{mod}(\mathcal{U})$ and $M_{\mathcal{U}} \in \text{mod}(\mathcal{T}^{op})$ for all $T \in \mathcal{T}^{op}$ and $U \in \mathcal{U}$, where $M_{\mathcal{T}} := M(-, T) : \mathcal{U} \rightarrow \text{Ab}$ and $M_{\mathcal{U}} := M(U, -) : \mathcal{T}^{op} \rightarrow \text{Ab}$, then $\Lambda := \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ is a dualizing variety (see Theorem 6.10). In particular, $\text{mod}(\Lambda)$ has Auslander-Reiten sequences (see Proposition 6.13).

In Section 7, we provide some applications to splitting torsion pairs which are in relation with tilting theory and path categories which are studied in [49]. In particular, we prove that given a splitting torsion pair $(\mathcal{U}, \mathcal{T})$ in a Krull-Schmidt category $\mathcal{C}$, we have that $\mathcal{C}$ is...
equivalent to a triangular matrix category and this result is kind of generalization of the one-point extension algebra.

2 Preliminaries

An arbitrary category \( \mathcal{C} \) is **skeletally small** if there is a full subcategory \( \mathcal{C}' \) of \( \mathcal{C} \) such that the class of objects of \( \mathcal{C}' \) is a set and every object of \( \mathcal{C} \) is isomorphic to an object in \( \mathcal{C}' \). We recall that a category \( \mathcal{C} \) together with an abelian group structure on each of the sets of morphisms \( \mathcal{C}(C_1, C_2) \) is called **preadditive category** provided all the composition maps \( \mathcal{C}(C, C') \times \mathcal{C}(C', C'') \rightarrow \mathcal{C}(C, C'') \) in \( \mathcal{C} \) are bilinear maps of abelian groups. A covariant functor \( F : \mathcal{C}_1 \rightarrow \mathcal{C}_2 \) between preadditive categories \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) is said to be **additive** if for each pair of objects \( C \) and \( C' \) in \( \mathcal{C}_1 \), the map \( F : \mathcal{C}_1(C, C') \rightarrow \mathcal{C}_2(F(C), F(C')) \) is a morphism of abelian groups. Let \( \mathcal{C} \) and \( \mathcal{D} \) be preadditive categories and \( \text{Ab} \) the category of abelian groups. A functor \( F : \mathcal{C} \times \mathcal{D} \rightarrow \text{Ab} \) is called **biadditive** if

\[
F(f + f', g) = F(f, g) + F(f', g) \quad \forall f, f' \in \mathcal{C}(C, C'), \forall g \in \mathcal{D}(D, D')
\]

\[
F(f, g + g') = F(f, g) + F(f, g') \quad \forall f \in \mathcal{C}(C, C'), \forall g, g' \in \mathcal{D}(D, D').
\]

If \( \mathcal{C} \) is a preadditive category we always consider its opposite category \( \mathcal{C}^{\text{op}} \) as a preadditive category by letting \( \mathcal{C}^{\text{op}}(C', C) = \mathcal{C}(C, C') \). We follow the usual convention of identifying each contravariant functor \( F \) from a category \( \mathcal{C} \) to \( \mathcal{D} \) with the covariant functor \( F^{-1} \) from \( \mathcal{C}^{\text{op}} \) to \( \mathcal{D} \). An **additive category** is a preadditive category \( \mathcal{C} \) such that every finite family of objects in \( \mathcal{C} \) has a coproduct.

2.1 The Category \( \text{Mod}(\mathcal{C}) \)

Let \( \mathcal{C} \) be a preadditive category and \( M_1, M_2 : \mathcal{C} \rightarrow \text{Ab} \) be two covariant functors. Recall that if \( \mathcal{C} \) is skeletally small, the collection of natural transformations from \( M_1 \) to \( M_2 \), denoted by \( \text{Nat}(M_1, M_2) \), is a set (see [4]). Throughout this section \( \mathcal{C} \) will be an arbitrary skeletally small preadditive category, and \( \text{Mod}(\mathcal{C}) \) will denote the **category of covariant functors** from \( \mathcal{C} \) to \( \text{Ab} \), called the category of \( \mathcal{C} \)-modules. This category has as objects the functors from \( \mathcal{C} \) to \( \text{Ab} \), and and a morphism \( f : M_1 \rightarrow M_2 \) of \( \mathcal{C} \)-modules is a natural transformation, that is, the set of morphisms \( \text{Hom}_\mathcal{C}(M_1, M_2) \) from \( M_1 \) to \( M_2 \) is given by \( \text{Nat}(M_1, M_2) \). We sometimes we will write for short, \( \mathcal{C}(\cdot, \cdot) \) instead of \( \text{Hom}_\mathcal{C}(\cdot, \cdot) \) and when it is clear from the context we will use just \( (\cdot, \cdot) \). We now recall some of properties of the category \( \text{Mod}(\mathcal{C}) \), for more details consult [4]. The category \( \text{Mod}(\mathcal{C}) \) is abelian with the following properties:

1. A sequence

\[
M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3
\]

is exact in \( \text{Mod}(\mathcal{C}) \) if and only if

\[
M_1(C) \xrightarrow{f_C} M_2(C) \xrightarrow{g_C} M_3(C)
\]

is an exact sequence of abelian groups for each \( C \) in \( \mathcal{C} \).

2. Let \( \{M_i\}_{i \in I} \) be a family of \( \mathcal{C} \)-modules indexed by the set \( I \). The \( \mathcal{C} \)-module \( \bigoplus_{i \in I} M_i \) defined by \( (\bigoplus_{i \in I} M_i)(C) = \bigoplus_{i \in I} M_i(C) \) for all \( C \) in \( \mathcal{C} \), is a direct sum for the family \( \{M_i\}_{i \in I} \) in \( \text{Mod}(\mathcal{C}) \), where \( \bigoplus_{i \in I} M_i(C) \) is the direct sum in \( \text{Ab} \) of the family of abelian groups.
The $\mathcal{C}$-module $\prod_{i \in I} M_i$ defined by $(\prod_{i \in I} M_i)(C) = \prod_{i \in I} M_i(C)$ for all $C$ in $\mathcal{C}$, is a product for the family $\{M_i\}_{i \in I}$ in $\text{Mod}(\mathcal{C})$, where $\prod_{i \in I} M_i(C)$ is the product in $\text{Ab}$.

3. For each $C$ in $\mathcal{C}$, the $\mathcal{C}$-module $(C, -)$ given by $(C, -)(X) = \mathcal{C}(C, X)$ for each $X$ in $\mathcal{C}$, has the property that for each $\mathcal{C}$-module $M$, the map $((C, -), M) \mapsto M(C)$ given by $f \mapsto f_C(1_C)$ for each $\mathcal{C}$-morphism $f : (C, -) \mapsto M$ is an isomorphism of abelian groups. We will often consider this isomorphism an identification. Hence

(a) The functor $P : \mathcal{C} \mapsto \text{Mod}(\mathcal{C})$ given by $P(C) = (C, -)$ is fully faithful.
(b) For each family $\{C_i\}_{i \in I}$ of objects in $\mathcal{C}$, the $\mathcal{C}$-module $\coprod_{i \in I} P(C_i)$ is a projective $\mathcal{C}$-module.
(c) Given a $\mathcal{C}$-module $M$, there is a family $\{C_i\}_{i \in I}$ of objects in $\mathcal{C}$ such that there is an epimorphism $\coprod_{i \in I} P(C_i) \mapsto M \mapsto 0$. We say that $M$ is finitely generated if such family is finite.
(d) A finitely generated projective $\mathcal{C}$-module is a direct summand of $\coprod_{i \in I} P(C_i)$ for some finite family of objects $\{C_i\}_{i \in I}$ in $\mathcal{C}$.

2.2 Dualizing Varieties and Krull-Schmidt Categories

Let $\mathcal{C}$ be a preadditive category, we denote by $\text{proj}(\text{Mod}(\mathcal{C}))$ the full subcategory of $\text{Mod}(\mathcal{C})$ consisting of all finitely generated projective $\mathcal{C}$-modules. Let $\mathcal{C}$ be an additive category, it is said that $\mathcal{C}$ is a category in which idempotents split if given $e : C \rightarrow C$ an idempotent endomorphism of an object $C \in \mathcal{C}$, then $e$ has a kernel in $\mathcal{C}$. It is well known that for a preadditive category $\mathcal{C}$ the category $\text{proj}(\text{Mod}(\mathcal{C}))$ is a skeletally small additive category in which idempotents split, the functor $\text{proj}(\text{Mod}(\mathcal{C}))$ given by $P(C) = (C, -)$, is fully faithful and induces by restriction an equivalence $\text{Mod}(\text{proj}(\text{Mod}(\mathcal{C}))) \simeq \text{Mod}(\mathcal{C})$.

We recall the following notion given by Auslander in [4]. A variety is a skeletally small, additive category in which idempotents split.

Given a ring $R$, we denote by $\text{Mod}(R)$ the category of left $R$-modules and by $\text{mod}(R)$ the full subcategory of $\text{Mod}(R)$ consisting of the finitely generated left $R$-modules.

To fix the notation, we recall known results on functors and categories that we use through the paper, referring for the proofs to the papers by Auslander and Reiten [3, 4, 6].

**Definition 2.1.** Let $\mathcal{C}$ be a variety. We say $\mathcal{C}$ has pseudokernels; if given a map $f : C_1 \rightarrow C_0$, there exists a map $g : C_2 \rightarrow C_1$ such that the sequence of morphisms $\mathcal{C}(-, C_2) \xrightarrow{(-, g)} \mathcal{C}(-, C_1) \xrightarrow{(-, f)} \mathcal{C}(-, C_0)$ is exact in $\text{Mod}(\mathcal{C}^{op})$.

Now, we recall some results from [6].

**Definition 2.2.** Let $R$ be a commutative artin ring. An $R$-variety $\mathcal{C}$, is a variety such that $\mathcal{C}(C_1, C_2)$ is an $R$-module, and the composition is $R$-bilinear. An $R$-variety $\mathcal{C}$ is Hom-finite, if for each pair of objects $C_1, C_2$ in $\mathcal{C}$, the $R$-module $\mathcal{C}(C_1, C_2)$ is finitely generated. We denote by $(\mathcal{C}, \text{mod}(R))$, the full subcategory of $(\mathcal{C}, \text{Mod}(R))$ consisting of the $\mathcal{C}$-modules such that for every $C$ in $\mathcal{C}$ the $R$-module $M(C)$ is finitely generated.

Suppose $\mathcal{C}$ is a Hom-finite $R$-variety. If $M : \mathcal{C} \rightarrow \text{Ab}$ is a $\mathcal{C}$-module, then for each $C \in \mathcal{C}$ the abelian group $M(C)$ has a structure of $\text{End}_\mathcal{C}(C)^{op}$-module and hence as an
\( R \)-module since \( \text{End}_G(C) \) is an \( R \)-algebra. Further if \( f : M \rightarrow M' \) is a morphism of \( C \)-modules it is easy to show that \( f_C : M(C) \rightarrow M'(C) \) is a morphism of \( R \)-modules for each \( C \in C \). Then, \( \text{Mod}(C) \) is an \( R \)-variety, which we identify with the category of covariant functors \( (C, \text{Mod}(R)) \). Moreover, the category \( (C, \text{mod}(R)) \) is abelian and the inclusion \( (C, \text{mod}(R)) \rightarrow (C, \text{Mod}(R)) \) is exact.

**Definition 2.3.** Let \( C \) be a Hom-finite \( R \)-variety. We denote by \( \text{mod}(C) \) the full subcategory of \( \text{Mod}(C) \) whose objects are the **finitely presented functors**. That is, \( M \in \text{mod}(C) \) if and only if, there exists an exact sequence in \( \text{Mod}(C) \)

\[
P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0,
\]

where \( P_1 \) and \( P_0 \) are finitely generated projective \( C \)-modules.

It can be seen that the finitely generated projectives \( P_1 \) and \( P_0 \) can be chosen as the representables. Then, a functor \( M \) is finitely presented if there exists an exact sequence

\[
C(-, C_1) \rightarrow C(-, C_0) \rightarrow M \rightarrow 0.
\]

It was proved in [23] that \( \text{mod}(C) \) is abelian if and only if \( C \) has pseudokernels. This fact was later rediscovered by Auslander in [6].

Consider the functors \( \mathbb{D}_C : (C^{op}, \text{mod}(R)) \rightarrow (C, \text{mod}(R)) \), and \( \mathbb{D}_C : (C, \text{mod}(R)) \rightarrow (C^{op}, \text{mod}(R)) \), which are defined as follows: for any object \( C \) in \( C \), \( \mathbb{D}_C(M)(C) = \text{Hom}_R(M(C), I(R/r)) \), with \( r \) the Jacobson radical of \( R \), and \( I(R/r) \) is the injective envelope of \( R/r \). The functor \( \mathbb{D}_C \) defines a duality between \( (C, \text{mod}(R)) \) and \( (C^{op}, \text{mod}(R)) \). We know that since \( C \) is Hom-finite, \( \text{mod}(C) \) is a subcategory of \( (C, \text{mod}(R)) \). Then we have the following definition due to Auslander and Reiten (see [6].).

**Definition 2.4.** A Hom-finite \( R \)-variety \( C \) is **dualizing**, if the functor \( \mathbb{D}_C : (C, \text{mod}(R)) \rightarrow (C^{op}, \text{mod}(R)) \) induces a duality between the categories \( \text{mod}(C) \) and \( \text{mod}(C^{op}) \).

It is clear from the definition that for dualizing categories \( C \) the category \( \text{mod}(C) \) has enough injectives. To finish, we recall the following definition:

**Definition 2.5.** An additive category \( C \) is **Krull-Schmidt**, if every object in \( C \) decomposes in a finite sum of objects whose endomorphism ring is local.

Assume that \( R \) is a commutative ring and \( R \) is a dualizing \( R \)-variety. Since the endomorphism ring of each object in \( C \) is an artin algebra, it follows that \( C \) is a Krull-Schmidt category [6] moreover, we have that for a dualizing variety the finitely presented functors have projective covers [4, Cor. 4.13], [38, Cor.4.4].

### 2.3 Tensor Product of Categories

If \( C \) and \( D \) are preadditive categories, B. Mitchell defined in [43] the **tensor product** \( C \otimes D \) of two preadditive categories, whose objects are those of \( C \times D \) and the abelian group of morphism from \( (C, D) \) to \( (C', D') \) is the ordinary tensor product of abelian groups \( C(C, C') \otimes_D D(D, D') \). Since that the tensor product of abelian groups is associative and
commutative and the composition in $\mathcal{C}$ and $\mathcal{D}$ is $\mathbb{Z}$-bilinear then the bilinear composition in $\mathcal{C} \otimes \mathcal{D}$ is given as follows:

$$(f_2 \otimes g_2) \circ (f_1 \otimes g_1) := (f_2 \circ f_1) \otimes (g_2 \circ g_1)$$

for all $f_1 \otimes g_1 \in \mathcal{C}(C, C') \otimes \mathcal{D}(D, D')$ and $f_2 \otimes g_2 \in \mathcal{C}(C', C'') \otimes \mathcal{D}(D', D'').$

**Remark 2.6.** If $\mathcal{C}$ and $\mathcal{D}$ are $R$-categories. The tensor product $\mathcal{C} \otimes_R \mathcal{D}$ of two $R$-categories, is the $R$-category whose objects are those of $\mathcal{C} \times \mathcal{D}$ and the abelian group of morphism from $(C, D)$ to $(C', D')$ is the ordinary tensor product of $R$-modules $\mathcal{C}(C, C') \otimes_R \mathcal{D}(D, D')$ and the composition is defined as above.

Assume that $\mathcal{C}$ and $\mathcal{D}$ are preadditive categories. Then, there exists a canonical functor $T : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D},$ given by $T((C, D)) = (C, D),$ and $T : \mathcal{C}(C, C') \times \mathcal{D}(D, D') \to \mathcal{C}(C, C') \otimes \mathcal{D}(D, D'),$ $(f, g) \mapsto f \otimes g.$

**Proposition 2.7.** Let $F : \mathcal{C} \times \mathcal{D} \to \text{Ab}$ a biadditive bifunctor. Then there exists a functor $\widehat{F} : \mathcal{C} \otimes \mathcal{D} \to \text{Ab},$ such that $F = \widehat{F}T.$

**Proof** The proof is an easy consequence of the universal property which characterizes the tensor product of abelian groups.

Whenever there is no risk of confusion, we will also denote by $F$ the functor $\widehat{F}$ mentioned in Proposition 2.7. Let $F : \mathcal{C} \otimes \mathcal{D} \to \text{Ab}.$ For all $X \in \mathcal{C}$ we have a functor $F_X : \mathcal{D} \to \text{Ab}$ associated to $F,$ given by $F_X(Y) = F(X, Y)$ and $F_X(g) = F(1_X \otimes g)$ for all $Y \in \mathcal{D}$ and $g \in \mathcal{D}(Y, Y').$ Let $\mathcal{C}$ be a preadditive category. Then, for each functor $F(-, -) : \mathcal{C}^{\text{op}} \otimes \mathcal{C} \to \text{Ab},$ we have a functor $\widehat{F}(-, -) : \mathcal{C} \otimes \mathcal{C}^{\text{op}} \to \text{Ab}$ defined by $\widehat{F}(X, Y) := F(Y, X)$ and $\widehat{F}(f, g) := F(g, f)$ for all $(X, Y) \in \mathcal{C} \times \mathcal{C}^{\text{op}}$ and $(f, g) \in \mathcal{C}(X, X') \otimes \mathcal{C}^{\text{op}}(Y, Y').$

Let $Y \in \mathcal{C}^{\text{op}},$ then it is clear that $\widehat{F}(-, Y) = F(-, Y)$ as $\mathcal{C}$-modules. By using Proposition 2.7, we summarize the above observations in the next proposition.

**Proposition 2.8.** (i) Let $\mathcal{C}$ be a preadditive category and $F : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}$ a biadditive functor. Then we have a functor $\overline{F} : \mathcal{C} \otimes \mathcal{C}^{\text{op}} \to \text{Ab}$

for which $\overline{F}(X, Y) = F(Y, X)$ for all $(X, Y) \in \mathcal{C} \otimes \mathcal{C}^{\text{op}},$ and the $\mathcal{C}$-modules $F(-, Y), F(Y, -)$ and $\overline{F}$ are isomorphic, for all $Y \in \mathcal{C}^{\text{op}}.$

(ii) Let $\mathcal{C}$ be a preadditive category and consider the bifunctors

$$\text{Hom}_C(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}, \quad \text{Ext}_C^n(-, -) : \mathcal{C}^{\text{op}} \times \mathcal{C} \to \text{Ab}, \quad n > 1.$$

Then, there exist functors

$$\overline{\text{Hom}} : \mathcal{C} \otimes \mathcal{C}^{\text{op}} \to \text{Ab}, \quad \overline{\text{Ext}}^n : \mathcal{C} \otimes \mathcal{C}^{\text{op}} \to \text{Ab}, \quad n > 1.$$

for which $\overline{\text{Hom}}(X, Y) = \text{Hom}_C(Y, X), \overline{\text{Ext}}^n(X, Y) = \text{Ext}_C^n(Y, X)$ for all $(X, Y) \in \mathcal{C} \otimes \mathcal{C}^{\text{op}},$ and we have isomorphisms of $\mathcal{C}$-modules

$$\overline{\text{Hom}}_Y \cong \text{Hom}_C(Y, -) : \mathcal{C} \to \text{Ab} \quad \text{and} \quad \overline{\text{Ext}}^n_Y \cong \text{Ext}_C^n(Y, -) : \mathcal{C} \to \text{Ab},$$

for all $Y \in \mathcal{C}^{\text{op}}.$
2.4 Quotient and Comma Category and Radical of a Category

Let $\mathcal{C}$ be a preadditive category. A two sided ideal $I(\cdot, \cdot)$ is an additive subfunctor of the two variable functor $\mathcal{C}(\cdot, \cdot) : \mathcal{C}^{op} \otimes \mathcal{C} \to \text{Ab}$ such that: (a) if $f \in I(X, Y)$ and $g \in \mathcal{C}(Y, Z)$, then $gf \in I(X, Z)$; and (b) if $f \in I(X, Y)$ and $h \in \mathcal{C}(U, X)$, then $fh \in I(U, Z)$. If $I$ is a two-sided ideal, then we can form the quotient category $\mathcal{C}/I$ whose objects are those of $\mathcal{C}$, and where $(\mathcal{C}/I)(X, Y) := \mathcal{C}(X, Y)/I(X, Y)$. Finally the composition is induced by that of $\mathcal{C}$ (see [43]). There is a canonical projection functor $\pi : \mathcal{C} \to \mathcal{C}/I$ such that:

1. $\pi(X) = X$, for all $X \in \mathcal{C}$.
2. For all $f \in \mathcal{C}(X, Y)$, $\pi(f) = f + I(X, Y) := \tilde{f}$.

Based on the Jacobson radical of a ring, we recall the radical of an preadditive category. This concept goes back to work of Kelly (see [37]).

**Definition 2.9.** The (Jacobson) radical of a preadditive category $\mathcal{C}$ is the two-sided ideal $\text{rad}_\mathcal{C}$ in $\mathcal{C}$ defined by the formula

$$\text{rad}_\mathcal{C}(X, Y) = \{h \in \mathcal{C}(X, Y) \mid 1_X - gh \text{ is invertible for any } g \in \mathcal{C}(Y, X)\}$$

for all objects $X$ and $Y$ of $\mathcal{C}$.

If $\mathcal{A}$ and $\mathcal{B}$ are abelian categories and $F : \mathcal{A} \to \mathcal{B}$ is an additive functor. The comma category $(\mathcal{B}, F, \mathcal{A})$ is the category whose objects are triples $(B, f, A)$ where $f : B \to FA$; and whose morphisms between the objects $(B, f, A)$ and $(B', f', A')$ are pair $(\beta, \alpha)$ of morphisms in $\mathcal{B} \times \mathcal{A}$ such that the diagram

$$\begin{array}{ccc}
B & \xrightarrow{\beta} & B' \\
 f \downarrow & & \downarrow f' \\
FA & \xrightarrow{F\alpha} & FA'
\end{array}$$

is commutative in $\mathcal{B}$ (see [24]).

3 Triangular Matrix Categories

In all this section $\mathcal{U}$ and $\mathcal{T}$ will be preadditive categories.

**Proposition 3.1.** Let $M \in \text{Mod}(\mathcal{U} \otimes \mathcal{T}^{op})$ be. Then, there exists two covariant functors

$$E : \mathcal{T} \longrightarrow \text{Mod}(\mathcal{U})^{op}$$

$$E' : \mathcal{U} \longrightarrow \text{Mod}(\mathcal{T}^{op}).$$

**Proof**

(a) For $T \in \mathcal{T}$, we define a covariant functor $E(T) := M_T : \mathcal{U} \longrightarrow \text{Ab}$ as follows

(i) $M_T(U) := M(U, T)$, for all $U \in \mathcal{U}$.
(ii) $M_T(u) := M(u \otimes 1_T)$, for all $u \in \text{Hom}_\mathcal{U}(U, U')$.

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Now, given a morphism $t : T \to T'$ in $\mathcal{T}$ we set $E(t) := \tilde{t} : M_{T'} \to M_T$ where 
$\tilde{t} = \{ \tilde{t}_U : M_{T'}(U) \to M_T(U) \}_{U \in \mathcal{U}}$ with $[\tilde{t}_U] = M(1_U \otimes t^{op}) : M(U, T') \to M(U, T)$. It is easy to show that $E$ is a contravariant functor $E : \mathcal{T} \to \text{Mod}(\mathcal{U})$.

(b) Similarly for $U \in \mathcal{U}$ we define a contravariant the functor $E'(U) := M_U : T \to \text{Ab}$ (or a covariant functor $M_U : T^{op} \to \text{Ab}$) as follows:

(i) $M_U(T) := M(U, T)$, for all $T \in \mathcal{T}$.

(ii) $M_U(t) := M(1_U \otimes t^{op})$, for all $t \in \text{Hom}_\mathcal{T}(T, T')$.

Now, given $u \in \text{Hom}_\mathcal{U}(U, U')$ we set $E'(u) := \tilde{u} : M_U \to M_{U'}$ where $\tilde{u} = \{ [\tilde{u}]_T : M_U(T) \to M_{U'}(T) \}_{T \in T^{op}}$ (we are seeing $M_U : T^{op} \to \text{Ab}$ as a covariant functor) with $[\tilde{u}]_T = M(u \otimes 1_T) : M(U, T) \to M(U', T)$.

\[ \square \]

**Definition 3.2.** We define a covariant functor $\mathbb{G} : \text{Mod}(\mathcal{U}) \to \text{Mod}(\mathcal{T})$ as follows. Let $Y : \text{Mod}(\mathcal{U}) \to \text{Mod}(\text{Mod}(\mathcal{U})^{op})$ be the Yoneda functor $Y(B) := \text{Hom}_{\text{Mod}(\mathcal{U})}(\cdot, B)$, and consider the functor $I : \text{Mod}(\text{Mod}(\mathcal{U})^{op}) \to \text{Mod}(\mathcal{T})$, induced by $E : \mathcal{T} \to \text{Mod}(\mathcal{U})^{op}$, that is, if $H \in \text{Mod}(\text{Mod}(\mathcal{U})^{op})$, the functor $I(H) : \mathcal{T} \to \text{Ab}$ is defined by $I(H)(T) = H(M_T)$, for all $T \in \mathcal{T}$. We set

\[ \mathbb{G} := I \circ Y : \text{Mod}(\mathcal{U}) \to \text{Mod}(\mathcal{T}). \]

**Remark 3.3.** The following holds:

1. For $B \in \text{Mod}(\mathcal{U})$, $\mathbb{G}(B)(T) := \text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, B)$ for all $T \in \mathcal{T}$. Moreover, for all $t \in \text{Hom}_\mathcal{T}(T, T')$ we have that $\mathbb{G}(B)(t) := \text{Hom}_{\text{Mod}(\mathcal{U})}(\tilde{t}, B)$.

2. If $\eta : B \to B'$ is a morphism of $\mathcal{U}$-modules we have that $\mathbb{G}(\eta) : \mathbb{G}(B) \to \mathbb{G}(B')$ is such that

\[
\mathbb{G}(\eta) = \left\{ [\mathbb{G}(\eta)]_T := \text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, \eta) : \mathbb{G}(B)(T) \to \mathbb{G}(B')(T) \right\}_{T \in \mathcal{T}}.
\]

Hence we have the comma category $(\text{Mod}(\mathcal{T}), \mathbb{G}_{\text{Mod}(\mathcal{U})})$ whose objects are the triples $(A, f, B)$ with $A \in \text{Mod}(\mathcal{T}), B \in \text{Mod}(\mathcal{U})$, and $f : A \to \mathbb{G}(B)$ a morphism of $\mathcal{T}$-modules. A morphism between two objects $(A, f, B)$ and $(A', f', B')$ is a pairs of morphism $(\alpha, \beta)$ where $\alpha : A \to A'$ is a morphism of $\mathcal{T}$-modules and $\beta : B \to B'$ is a morphism of $\mathcal{U}$-modules such that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\downarrow f & & \downarrow \beta \\
\mathbb{G}(B) & \xrightarrow{\mathbb{G}(\beta)} & \mathbb{G}(B')
\end{array}
\]

commutes.
Note that, since $f : A \rightarrow \mathcal{G}(B)$ is a morphism of $\mathcal{T}$-modules, for each $t \in \text{Hom}_{\mathcal{T}}(T, T')$ the following diagram
\[
\begin{array}{ccc}
A(T) & \xrightarrow{f_T} & \mathcal{G}(B)(T) \\
\downarrow & & \downarrow \mathcal{G}(B)(t) \\
A(T') & \xrightarrow{f_{T'}} & \mathcal{G}(B)(T')
\end{array}
\]
commutes in $\text{Ab}$. Then, for all $x \in A(T)$, $f_T(x) \in \text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, B)$, that is $f_T(x)$ is a morphism of $\mathcal{U}$-modules. We denote it by
\[
f_T(x) = \left\{ [f_T(x)]_U : M_T(U) \longrightarrow B(U) \right\}_{U \in \mathcal{U}}.
\]
Let $0 \rightarrow B_1 \rightarrow B_2 \rightarrow B_3$ be an exact sequence of $\mathcal{U}$-modules. Since $Y : \text{Mod}(\mathcal{U}) \rightarrow \text{Mod}(\text{Mod}(\mathcal{U})^{op})$ is left exact, we get an exact sequence of $\text{Mod}(\mathcal{U})^{op}$-modules $0 \rightarrow Y(B_1) \rightarrow Y(B_2) \rightarrow Y(B_3)$, and as a consequence $0 \rightarrow Y(B_1)(M_T) \rightarrow Y(B_2)(M_T) \rightarrow Y(B_3)(M_T)$ is an exact sequence of abelian groups, for all $T \in \mathcal{T}$. That is, $0 \rightarrow \mathcal{G}(B_1)(T) \rightarrow \mathcal{G}(B_2)(T) \rightarrow \mathcal{G}(B_3)(T)$ is an exact sequence of abelian groups, for all $T \in \mathcal{T}$. Therefore, $0 \rightarrow \mathcal{G}(B_1) \rightarrow \mathcal{G}(B_2) \rightarrow \mathcal{G}(B_3)$ is an exact sequence of $\mathcal{T}$-modules.

It follows that $\mathcal{G}$ is left exact. Hence, the comma category $\left( \text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U}) \right)$ is abelian (see [24, Corollary 1.2] and discussion in p. 6 in [24]). We note that this will follow also from Theorem 3.17.

**Proposition 3.4.** Let $M \in \text{Mod}(\mathcal{U} \otimes \mathcal{T}^{op})$ be and $f : A \rightarrow \mathcal{G}(B)$ a morphism in $\text{Mod}(\mathcal{T})$. Let $U \in \mathcal{U}$ and $T \in \mathcal{T}$, for $m \in M(U, T)$ and $x \in A(T)$ we set
\[
m \cdot x := [f_T(x)]_U(m) \in B(U)
\]
(this product can be defined for each $f : A \rightarrow \mathcal{G}(B)$).

(a) For $m \in M(U, T')$, $t \in \text{Hom}_{\mathcal{T}}(T, T')$ and $x \in A(T)$ we set
\[
m \cdot t := M(1_U \otimes t^{op})(m), \quad t \cdot x := A(t)(x).
\]
Then we have that $(m \cdot t) \cdot x = m \cdot (t \cdot x)$.

(b) For $m \in M(U, T)$, $u \in \text{Hom}_U(U, U')$ and $z \in B(U)$ we set
\[
u \cdot m := M(u \otimes 1_T)(m), \quad u \circ z := B(u)(z).
\]
Then for $x \in A(t)$ we have that $(u \cdot m) \cdot x = u \circ (m \cdot x)$.

(c) Let $m_1 \in M(U', T')$, $m_2 \in M(U, T)$, $t \in \text{Hom}_{\mathcal{T}}(T, T')$ and $u \in \text{Hom}_U(U, U')$. For $x \in A(T)$ we have that
\[
(m_1 \cdot t + u \cdot m_2) \cdot x = (m_1 \cdot t) \cdot x + (u \cdot m_2) \cdot x = m_1 \cdot (t \cdot x) + u \circ (m_2 \cdot x).
\]
Proof

(a) Since \( f : A \rightarrow G(B) \) is a morphism of \( T \)-modules, for each \( t \in \text{Hom}_T(T', T') \), we have a commutative diagram as the given in Eq. 2. Thus, for \( x \in \text{A}(T) \) we have that

\[
\left( (G(B))(t) \circ f_T \right)(x) = \left( f_{T'} \circ A(t) \right)(x) \in \text{Hom}_{\text{Mod}(\mathcal{U})}(M_{T'}, B).
\]

Then, we have that

\[
f_{T'} \left( A(t)(x) \right) = \left( (G(B))(t) \circ f_T \right)(x) = \left( \text{Hom}_{\text{Mod}(\mathcal{U})}(\tilde{t}, B) \circ f_T \right)(x) = f_T(x) \circ \tilde{t}.
\]

Hence, for \( U \in \mathcal{U} \) we have that

\[
\left[ f_T \left( A(t)(x) \right) \right]_U = \left[ f_T(x) \right]_U \circ \tilde{t}_U = \left[ f_T(x) \right]_U \circ M(1_U \otimes t^{op}).
\]

It follows that

\[
\left[ f_T(A(t)(x)) \right]_U(m) = \left[ f_T(x) \right]_U(M(1_U \otimes t^{op})(m)), \text{ for all } m \in M(U, T'). \]  \hspace{1cm} (3)

This means that \( (m \cdot t) \cdot x = m \cdot (t \cdot x) \).

(b) Since \( f_T(x) \in \text{Hom}_{\text{Mod}(\mathcal{U})}(M_{T'}, B) \) with \( f_T(x) = \left\{ \left[ f_T(x) \right]_U : M(U, T) \rightarrow B(U) \right\} \) is a morphism of \( \mathcal{U} \)-modules, for \( u : U \rightarrow U' \) we have the following commutative diagram

\[
\begin{array}{ccc}
M_T(U) & \xrightarrow{[f_T(x)]_U} & B(U) \\
M_T(u) \downarrow & & \downarrow B(u) \\
M_T(U') & \xrightarrow{[f_T(x)]_{U'}} & B(U').
\end{array}
\]

Then, for \( m \in M_T(U) = M(U, T) \) we have that

\[
(B(u) \circ [f_T(x)]_U)(m) = \left( \left[ f_T(x) \right]_{U'} \circ M_T(u) \right)(m) = \left( \left[ f_T(x) \right]_{U'} \circ M(u \otimes 1_T) \right)(m) = \left[ f_T(x) \right]_{U'}(M(u \otimes 1_T)(m)).
\]

This means that \( u \circ (m \cdot x) = (u \cdot m) \cdot x \).

(c) Since \( m_1 \in M(U', T') \) and \( m_2 \in M(U, T) \) we have that \( m_1 \otimes t = M(1_U \otimes t_{op})(m_1) \in M(U', T) \) and \( u \otimes 1_T = M(u \otimes 1_T)(m_2) \in M(U', T) \). Now, since \( M(U', T) \) is an abelian group we can consider the element \( m' := m_1 \otimes t + u \otimes m_2 \in M(U', T) \). Then by definition, for \( x \in A(T) \) we have that

\[
m' \cdot x = \left[ f_T(x) \right]_{U'}(m').
\]

Since \( \left[ f_T(x) \right]_{U'} : M(U', T) \rightarrow B(U') \) is a morphism of abelian groups we have that

\[
\left[ f_T(x) \right]_{U'}(m') = \left[ f_T(x) \right]_{U'}(m_1 \cdot t) + \left[ f_T(x) \right]_{U'}(u \cdot m_2) = (m_1 \cdot t) \cdot x + (u \cdot m_2) \cdot x.
\]

Then, the result follows from (a) and (b).

Let \( R \) be a ring. Then there exists a preadditive category associated to \( R \), the category \( \mathcal{R} \) that has only one object, say \( a \), and \( \text{Hom}_\mathcal{R}(a, a) = R \). Then, the functor evaluation \( \text{ev}_a : \text{Mod}(\mathcal{R}) \rightarrow \text{Mod}(R), \text{ev}_a(F) = F(a) \) induces an equivalence of categories, where the scalar multiplication for the \( R \)-module \( F(a) \) is given by the map \( \cdot : R \times F(a) \rightarrow F(a), r \cdot x := F(r)(x) \forall r \in R \) and \( \forall x \in F(a) \) (see [4]).
Let $U$ and $T$ be rings and consider their respective associated categories $\mathcal{U}$ and $\mathcal{T}$. Denote by $a$ and $\star$ the unique objects in $\mathcal{U}$ and $\mathcal{T}$ respectively, that is, $\text{Hom}_\mathcal{U}(a, a) = U$ and $\text{Hom}_\mathcal{T}(\star, \star) = T$. Thus, by the above mentioned we have equivalences of categories $\text{Mod}(\mathcal{U}) \cong \text{Mod}(U)$ and $\text{Mod}(\mathcal{T}) \cong \text{Mod}(T)$. In addition, the tensor product category $\mathcal{U} \otimes T^{\text{op}}$ has only one object, the pair $(a, \star)$, and $\text{Hom}_\mathcal{U}(\mathcal{U} \otimes T^{\text{op}})((a, \star), (a, \star)) = U \otimes T^{\text{op}}$.

In this way, we have the equivalence of categories $\text{Mod}(\mathcal{U} \otimes T^{\text{op}}) \cong \text{Mod}(U \otimes T^{\text{op}})$, and given an $U-T$-bimodule $N$, it can be seen as a left $U$ $T^{\text{op}}$-module with the scalar multiplication $\cdot : U \otimes T^{\text{op}} \times N \to N$, $(u \otimes t) \cdot n = unt$. Moreover, there exists a functor $M : \mathcal{U} \otimes T^{\text{op}} \to \text{Ab}$ and an isomorphism of $U$-$T$-modules $N \cong M((a, \star))$ such that, after identifying $N$ with $M((a, \star))$,

$$um = M(u \otimes 1_\star)(m) \text{ and } mt = M(1_a \otimes t^{\text{op}})(m),$$

for all $u \in U$, $t \in T$ and $m \in M((a, \star))$.

On the other hand the **triangular matrix ring** $\Lambda = \begin{bmatrix} T & 0 \\ N & U \end{bmatrix}$ is the set consisting of matrices $\begin{bmatrix} I & 0 \\ n & u \end{bmatrix}$, $t \in T$ and $u \in U$ and $n \in N$, with the sum and multiplication

$$\begin{bmatrix} I & 0 \\ n & u \end{bmatrix} + \begin{bmatrix} I & 0 \\ m & v \end{bmatrix} := \begin{bmatrix} I & 0 \\ n + m & u + v \end{bmatrix}, \quad \begin{bmatrix} I & 0 \\ n & u \end{bmatrix} \begin{bmatrix} I & 0 \\ m & v \end{bmatrix} := \begin{bmatrix} I & 0 \\ n + m & u + v \end{bmatrix}. \quad (4)$$

Following Mitchell’s philosophy in [43], of thinking preadditive categories as rings with several objects, we want to extend the notion of triangular matrix ring to categories, so now we briefly explain how we do it.

Since the ring $\Lambda$ is built from $U$, $T$ and $N$, the idea is to build a category $\Lambda$ with only one object, say $X$, such that $\text{Hom}_\Lambda(X, X) = \Lambda$, and for which the composition of morphisms in $\Lambda$ coincides with the multiplication (4) in $\Lambda$, then get a generalization of this construction replacing the categories $\mathcal{U}$ and $\mathcal{T}$ by categories with more objects. By the above mentioned, it results natural to define $\text{Hom}_\Lambda(X, X) = \begin{bmatrix} T^{(\star, \star)} & 0 \\ M((a, \star)) \text{ U}(a, a) \end{bmatrix}$, where

$$\begin{bmatrix} T^{(\star, \star)} & 0 \\ M((a, \star)) \text{ U}(a, a) \end{bmatrix} := \left\{ \begin{bmatrix} I & 0 \\ m & u \end{bmatrix} : t \in T^{(\star, \star)}, u \in \text{U}(a, a), \text{ and } m \in M((a, \star)) \right\},$$

and the composition by

$$\circ : \begin{bmatrix} T^{(\star, \star)} & 0 \\ M((a, \star)) \text{ U}(a, a) \end{bmatrix} \times \begin{bmatrix} T^{(\star, \star)} & 0 \\ M((a, \star)) \text{ U}(a, a) \end{bmatrix} \to \begin{bmatrix} T^{(\star, \star)} & 0 \\ M((a, \star)) \text{ U}(a, a) \end{bmatrix}$$

$$\left( \begin{bmatrix} I & 0 \\ m & u \end{bmatrix}, \begin{bmatrix} I & 0 \\ m' & u' \end{bmatrix} \right) \mapsto \begin{bmatrix} I & 0 \\ m + m' & u + u' \end{bmatrix},$$

with $m't := M(1_a \otimes t^{\text{op}})(m')$ and $u'm = M(u' \otimes 1_\star)(m)$, $\forall u, u' \in U, \forall t, t' \in T$ and $\forall m, m' \in M((a, \star))$.

With the aforementioned as motivation, we give one of the most important definitions of this work.

**Definition 3.5.** Let $\mathcal{U}$ and $\mathcal{T}$ be preadditive categories and consider an additive functor $M$ from the tensor product category $\mathcal{U} \otimes T^{\text{op}}$ to the category $\text{Ab}$. We define the **triangular matrix category** $\Lambda = \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ as follows.

(a) The class of objects of this category are matrices $\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$ with $T \in \text{obj} \mathcal{T}$ and $U \in \text{obj} \mathcal{U}$.

(b) Given a pair of objects in $\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}$, $\begin{bmatrix} T' & 0 \\ M' & U' \end{bmatrix}$ in $\Lambda$ we define

$$\text{Hom}_\Lambda \left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, \begin{bmatrix} T' & 0 \\ M' & U' \end{bmatrix} \right) := \begin{bmatrix} \text{Hom}_\mathcal{T}(T, T') & 0 \\ M(U', T) \text{ Hom}_\mathcal{U}(U, U') \end{bmatrix}.$$
The composition is given by
\[
\circ : \left[ \begin{array}{c} T(T',T'') \\
M(U'',T') \end{array} \right] \times \left[ \begin{array}{c} T(T',T') \\
M(U',T') \end{array} \right] \rightarrow \left[ \begin{array}{c} T(T',T'') \\
M(U'',T) \end{array} \right]
\]
\[
= \left[ \begin{array}{ccc}
0 & 0 & 0 \\
2 & 0 & 2 \\
1 & 0 & 1 \\
\end{array} \right],
\]
where \( m_2 \cdot t_1 := M(1_{U''} \otimes t_1^{op})(m_2) \) and \( u_2 \cdot m_1 = M(u_2 \otimes 1_T)(m_1) \).

We recall that \( m_2 \cdot t_1 := M(1_{U''} \otimes t_1^{op})(m_2) \) and \( u_2 \cdot m_1 = M(u_2 \otimes 1_T)(m_1) \).

The following result is a direct consequence of Proposition 3.4 and we leave the details to the reader.

**Lemma 3.6.** The composition defined above is associative and given an object \( \left[ \begin{array}{c} T \\
M(U) \end{array} \right] \in \Lambda \), the identity morphism is given by \( \left[ \begin{array}{c} 1_T \\
0 \\
1_U \end{array} \right] \).

Now, for \( \left[ \begin{array}{c} t_1 \\
m_1 \\
u_1 \end{array} \right], \left[ \begin{array}{c} r_1 \\
m_1 \\
u_1 \end{array} \right] \in \text{Hom}_\Lambda \left[ \begin{array}{c} T \\
M(U) \end{array} \right], \left[ \begin{array}{c} T' \\
M(U') \end{array} \right] \) we define
\[
\left[ \begin{array}{c} t_1 \\
m_1 \\
u_1 \end{array} \right] + \left[ \begin{array}{c} r_1 \\
m_1 \\
u_1 \end{array} \right] := \left[ \begin{array}{c} t_1 + r_1 \\
m_1 + r_1 \\
u_1 + v_1 \end{array} \right]
\]

Then, it is clear that \( \Lambda \) is a preadditive category since \( \mathcal{T} \) and \( \mathcal{U} \) are preadditive categories and \( M(U', T) \) is an abelian group.

**Proposition 3.7.** If \( \mathcal{U} \) and \( \mathcal{T} \) have finite coproducts, then \( \Lambda \) has finite coproducts.

**Proof** It is straightforward. \( \square \)

Now, we compute the radical in \( \Lambda \).

**Proposition 3.8.** \( \text{rad}_\Lambda \left[ \begin{array}{c} T \\
M(U) \end{array} \right], \left[ \begin{array}{c} T' \\
M(U') \end{array} \right] \) \( \text{rad}_{\mathcal{U}}(U', U) \)

**Proof** Let \( \left[ \begin{array}{c} t \\
m \\
u \end{array} \right] \in \text{rad}_\Lambda \left[ \begin{array}{c} T \\
M(U) \end{array} \right], \left[ \begin{array}{c} T' \\
M(U') \end{array} \right] \) and \( t' : T' \rightarrow T \) and \( u' : U' \rightarrow U \) morphisms in \( \mathcal{T} \) and \( \mathcal{U} \) respectively. Consider \( \left[ \begin{array}{c} t' \\
u' \end{array} \right] \in \text{Hom}_\Lambda \left[ \begin{array}{c} T' \\
M(U') \end{array} \right], \left[ \begin{array}{c} T \\
M(U) \end{array} \right] \), then
\[
\left[ \begin{array}{c} 1_T \\
u' \\
u \\
u \\
u \end{array} \right] - \left[ \begin{array}{c} 0 \\
u' \\
u \\
u \\
u \end{array} \right] = \left[ \begin{array}{c} 1_T - t' \\
u \\
u \\
u \\
u \end{array} \right] \text{ is invertible in } \Lambda. \]

It follows from this that \( 1_T - t' \) and \( 1_U - u' u \) are invertibles in \( \mathcal{T} \) and \( \mathcal{U} \) respectively. Then \( t \in \text{rad}_\mathcal{T}(T, T') \) and \( u \in \text{rad}_{\mathcal{U}}(U, U') \).

Conversely, let \( t \in \text{rad}_\mathcal{T}(T, T'), u \in \text{rad}_{\mathcal{U}}(U, U') \) and \( m \in M(U', T) \). We assert that \( \left[ \begin{array}{c} t \\
m \\
u \end{array} \right] \in \text{rad}_\Lambda \left[ \begin{array}{c} T \\
M(U) \end{array} \right], \left[ \begin{array}{c} T' \\
M(U') \end{array} \right] \). Indeed, let \( \left[ \begin{array}{c} t' \\
m' \\
u' \end{array} \right] \in \text{Hom}_\Lambda \left[ \begin{array}{c} T' \\
M(U') \end{array} \right], \left[ \begin{array}{c} T \\
M(U) \end{array} \right] \). Since \( 1_T - t' \) and \( 1_U - u'u \) are invertibles, there exists \( t'' \in \text{Hom}_\mathcal{T}(T, T) \) and \( u'' \in \text{Hom}_{\mathcal{U}}(U, U) \) such that
\[
(1_T - t')t'' = t''(1_T - t') = 1_T, \quad (1_U - u'u')u'' = u''(1_U - u'u) = 1_U.
\]
Let $m'' := u'' \bullet (m' \bullet t + u' \bullet m) \bullet t'' \in M(U, T)$. Using that $(1_U - u'u)u'' = 1_U$ and that $t''(1_T - t') = 1_T$ we see that

\[
\begin{bmatrix}
1_T & 0 \\
 t' & 1_U
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m' & u
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m' & u
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
-(m' \bullet t + u' \bullet m) \bullet t'' + (1_U - u'u) \bullet m'' (1_U - u'u)u'' \\
1_T
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
= \begin{bmatrix}
1_T & 0 \\
0 & 1_U
\end{bmatrix},
\]

where the last equality is because $(1_U - u'u) \bullet m'' = (1_U - u'u) \bullet (u'' \bullet (m' \bullet t + u' \bullet m) \bullet t'')$.

Similarly, \[
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
= \begin{bmatrix}
1_T & 0 \\
0 & 1_U
\end{bmatrix}.\]

Proving that \[
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix} \in \text{rad}_A \left( \begin{bmatrix} T & 0 \\ M & U' \end{bmatrix} \right).
\]

Our main purpose in this part is to show that we have an equivalence of categories 
\[
\left( \text{Mod}(T), \text{GMod}(\mathcal{U}) \right) \simeq \text{Mod}(\Lambda).
\]

Let $(A, f, B) \in \left( \text{Mod}(T), \text{GMod}(\mathcal{U}) \right)$, that is, we have a morphism of $\mathcal{T}$-modules $f : A \rightarrow \text{G}(B)$. We can construct a functor

\[A \sqcup f \ B : A \rightarrow \text{Ab}\]

as follows.

(a) For \[
\begin{bmatrix}
T & 0 \\
M & U' \end{bmatrix} \in \Lambda \]
we set \[
\left( A \sqcup f \ B \right) \left( \begin{bmatrix} T & 0 \\ M & U' \end{bmatrix} \right) := \begin{bmatrix} A(T) \sqcup B(U) \end{bmatrix} \in \text{Ab}.
\]

(b) If \[
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix} \in \text{Hom}_A \left( \begin{bmatrix} T & 0 \\ M & U' \end{bmatrix} \right), \begin{bmatrix} T' & 0 \\ M & U' \end{bmatrix} \right) = \begin{bmatrix} \text{Hom}_{\mathcal{T}}(T, T') & 0 \\ \text{Hom}_{\mathcal{U}}(U', U') \end{bmatrix}
\]
we define the map

\[
\begin{bmatrix}
A(f) & 0 \\
B(u) & 1_U
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
= \begin{bmatrix}
A(f) & 0 \\
B(u) & 1_U
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
\begin{bmatrix}
0 & 1_U \\
m & u
\end{bmatrix}
\]

given by \[
\begin{bmatrix}
A(f) & 0 \\
B(u) & 1_U
\end{bmatrix}
\begin{bmatrix}
X \\
y
\end{bmatrix}
= \begin{bmatrix}
A(f)(x) \\
B(u)(y)
\end{bmatrix}
\text{for } (x, y) \in A(T) \sqcup B(U), \text{ where } m \cdot x := \left[ f_T(x) \right]_{U'}(m) \in B(U') \text{ (see Proposition 3.4).}
\]

**Remark 3.9.** In terms of Proposition 3.4, we have that

\[
\begin{bmatrix}
A(f) & 0 \\
B(u) & 1_U
\end{bmatrix}
\begin{bmatrix}
X \\
y
\end{bmatrix}
= \begin{bmatrix}
t^{ex} \\
x + uoy
\end{bmatrix}
\forall (x, y) \in A(T) \sqcup B(U).
\]

The following lemma tell us that $A \sqcup f B$ is a functor.

**Lemma 3.10.** Let $(A, f, B) \in \left( \text{Mod}(T), \text{GMod}(\mathcal{U}) \right)$, then $A \sqcup f B : A \rightarrow \text{Ab}$ is a functor.

**Proof** Let \[
\begin{bmatrix}
T & 0 \\
M & U' \end{bmatrix} \longrightarrow \begin{bmatrix} T' & 0 \\ M & U' \end{bmatrix}, \begin{bmatrix} T & 0 \\ M & U' \end{bmatrix} \longrightarrow \begin{bmatrix} T'' & 0 \\ M & U' \end{bmatrix}\]
and $(x, y) \in A(T) \sqcup B(U)$. 

\[ Springer \]
We have that
\[
\left((A \amalg f) B\right)\left(\begin{bmatrix} t_2 & 0 \\ m_2 & u_2 \end{bmatrix} \circ \begin{bmatrix} t_1 & 0 \\ m_1 & u_1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = (A \amalg f) B\left(\begin{bmatrix} t_2 t_1 \\ m_2 \cdot t_1 + u_2 \cdot m_1 & u_2 \cdot u_1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A(t_2 t_1) \\ m_2 \cdot t_1 + u_2 \cdot m_1 & B(u_2 \cdot u_1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A(t_2 t_1)(x) \\ (m_2 \cdot t_1 + u_2 \cdot m_1) \cdot x + B(u_2 \cdot u_1)(y) \end{bmatrix}.
\]

On the other hand,
\[
\left((A \amalg f) B\right)\left(\begin{bmatrix} t_2 & 0 \\ m_2 & u_2 \end{bmatrix}\right) \circ \left((A \amalg f) B\right)\left(\begin{bmatrix} t_1 & 0 \\ m_1 & u_1 \end{bmatrix}\right)\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} A(t_2) \\ m_2 \end{bmatrix}\begin{bmatrix} 0 \\ B(u_2) \end{bmatrix} = \begin{bmatrix} A(t_2)(A(t_1)(x)) \\ m_2 \cdot (A(t_1)(x)) + B(u_2)(m_1 \cdot x + B(u_1)(y)) \end{bmatrix} = \begin{bmatrix} A(t_2)(t_1)(x) \\ (m_2 \cdot (t_1 \cdot x) + u_2 \cdot (m_1 \cdot x) + B(u_2 \cdot u_1)(y)) \end{bmatrix}.
\]

By Proposition 3.4 (c), we conclude that \((m_2 \cdot t_1 + u_2 \cdot m_1) \cdot x = m_2 \cdot (t_1 \cdot x) + u_2 \cdot (m_1 \cdot x)\).

Proving that \((A \amalg f) B\) preserves compositions. Now, consider \([\begin{bmatrix} 1_T & 0 \\ 0 & 1_U \end{bmatrix} : [\begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \rightarrow [\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}]\).

We have that \((A \amalg f) B\)\(\left(\begin{bmatrix} 1_T & 0 \\ 0 & 1_U \end{bmatrix}\right) := [\begin{bmatrix} A(1_T) & 0 \\ 0 & B(1_U) \end{bmatrix}]\) is such that \([\begin{bmatrix} A(1_T) & 0 \\ 0 & B(1_U) \end{bmatrix}][\begin{bmatrix} x \\ y \end{bmatrix}] = [\begin{bmatrix} A(1_T)(x) \\ 0 \cdot x + B(1_U)(y) \end{bmatrix}]\) for \((x, y) \in (A) \amalg B(U),\) since \(0 \cdot x = 0.\) Then \((A \amalg f) B\)\(\left(\begin{bmatrix} 1_T & 0 \\ 0 & 1_U \end{bmatrix}\right) = 1_{A(T) \amalg B(U)}\). Proving that \((A \amalg f) B\) is a functor.

In this way we can construct a functor
\[
\mathcal{S} : \left(\text{Mod}(T), \mathcal{G}\text{Mod}(\mathcal{U})\right) \rightarrow \text{Mod}(A)
\]
which is defined as follows.

(a) For \((A, f, B) \in \left(\text{Mod}(T), \mathcal{G}\text{Mod}(\mathcal{U})\right)\) we define \(\mathcal{S}((A, f, B)) := A \amalg f B.\)

(b) If we have \((\alpha, \beta) : (A, f, B) \rightarrow (A', f', B')\) in \(\left(\text{Mod}(T), \mathcal{G}\text{Mod}(\mathcal{U})\right)\) then \(\mathcal{S}(\alpha, \beta) = \alpha \amalg \beta\) is the natural transformation
\[
\alpha \amalg \beta = \left\{\left(\alpha \amalg \beta\right) \left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right) : \alpha_T \amalg \beta_U : (A \amalg f) B \left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right) \rightarrow (A' \amalg f') B' \left(\begin{bmatrix} T' & 0 \\ M' & U' \end{bmatrix}\right)\right\}
\]

Lemma 3.11. Let \((\alpha, \beta) : (A, f, B) \rightarrow (A', f', B')\) be in \(\left(\text{Mod}(T), \mathcal{G}\text{Mod}(\mathcal{U})\right),\) then \(\alpha \amalg \beta\) is a natural transformation.

Proof Let \([\begin{bmatrix} t_1 & 0 \\ m_1 & u_1 \end{bmatrix} : [\begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \rightarrow [\begin{bmatrix} T' & 0 \\ M' & U' \end{bmatrix}] be a morphism in \(A = [\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}].\) We have to check that the following diagram commutes in \(\text{Ab}\)
\[
\begin{array}{ccc}
A(T) \amalg B(U) & \xrightarrow{\alpha_T \amalg \beta_U} & A'(T) \amalg B'(U') \\
\left[\begin{bmatrix} A(t_1) & 0 \\ m_1 & B(u_1) \end{bmatrix}\right] & \downarrow & \left[\begin{bmatrix} A'(t_1) & 0 \\ m_1 & B'(u_1) \end{bmatrix}\right] \\
A(T') \amalg B(U') & \xrightarrow{\alpha_{T'} \amalg \beta_{U'}} & A'(T') \amalg B'(U')
\end{array}
\]

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Indeed, for \((x, y) \in A(T) \amalg B(U)\) we have that
\[
\left( (\alpha_{T'}, \amalg \beta_{U'}) \circ \left[ \begin{array}{c} A(t_1) \\ m_1 \\ B(u_1) \end{array} \right] \right) \left[ \begin{array}{c} x \\ y \end{array} \right] = (\alpha_{T'}, \amalg \beta_{U'})(A(t_1)(x), m_1 \cdot x + B(u_1)(y)) = \left( \alpha_{T'}(A(t_1)(x)), \beta_{U'}(m_1 \cdot x + B(u_1)(y)) \right)
\]
We also have that
\[
\left( \left[ \begin{array}{c} A'(t_1) \\ m_1 \\ B'(u_1) \end{array} \right] \circ (\alpha_{T'} \amalg \beta_{U'}) \right) \left[ \begin{array}{c} x \\ y \end{array} \right] = \left[ \begin{array}{c} A'(t_1) \\ m_1 \\ B'(u_1) \end{array} \right] \left[ \begin{array}{c} \alpha_T(x) \\ \beta_{U'}(y) \end{array} \right] = \left( A'(t_1)(\alpha_T(x)), m_1 \cdot (\alpha_T(x)) + B'(u_1)(\beta_{U'}(y)) \right)
\]
Let us see that \(\beta_{U'}(m_1 \cdot x) = m_1 \cdot (\alpha_T(x))\). Since \(m_1 \in M(U', T), x \in A(T)\) we have that \(m_1 \cdot x = [f_T(x)]_{U'}(m_1) \in B(U')\) (see Proposition 3.4). Then \(\beta_{U'}(m_1 \cdot x) = [f_T(x)]_{U'}(m_1) \in B(U')\). On the other hand, since \(m_1 \in M(U', T), \alpha_T(x) \in A'(T)\) we have that \(m_1 \cdot (\alpha_T(x)) = [f_T'(\alpha_T(x))]_{U'}(m_1)\) (see Proposition 3.4). Consider the following commutative diagram in \(\text{Mod}(T)\) (this is because \(\langle \alpha, \beta \rangle : (A, f, B) \rightarrow (A', f', B')\) is a morphism)

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
\downarrow{f} & & \downarrow{f'} \\
\mathcal{G}(B) & \xrightarrow{\mathcal{G}(\beta)} & \mathcal{G}(B').
\end{array}
\]

Then, for each \(T \in \mathcal{T}\), the following diagram commutes in \(\text{Ab}\)

\[
\begin{array}{ccc}
A(T) & \xrightarrow{\alpha_T} & A'(T) \\
\downarrow{f_T} & & \downarrow{f'_T} \\
\mathcal{G}(B(T)) & \xrightarrow{\mathcal{G}(\beta)_T} & \mathcal{G}(B'(T)).
\end{array}
\]

Then, we have that \(f'_T(\alpha_T(x)) : M_T \rightarrow B'\) coincides with \((\mathcal{G}(\beta)_T)(f_T(x)) = \text{Hom}_{\text{Mod}(U)}(M_T, \beta)(f_T(x)) = \beta \circ f_T(x)\). In particular for \(U'\) we have that
\[
[f_T'(\alpha_T(x))]_{U'} = [f_T'(\alpha_T(x))]_{U'} : M(U', T) \rightarrow B'(U').
\]
Hence, for \(m_1 \in M(U', T)\) we have that
\[
([\beta]_{U'} \circ [f_T(x)]_{U'}) \left( m_1 \right) = [f'_T(\alpha_T(x))]_{U'} \left( m_1 \right).
\]
This means precisely that \(\beta_{U'}(m_1 \cdot x) = m_1 \cdot (\alpha_T(x))\).

On the other hand, since \(\alpha : A \rightarrow A'\) and \(\beta : B \rightarrow B'\) are natural transformations, for \(T \in \mathcal{T}\) and \(U \in \mathcal{U}\), we have that \(\alpha_T(A(t_1)(x)) = A'(t_1)(\alpha_T(x))\) and \((B'\circ u_1)(\beta_U(y)) = \beta_{U'}(B(u_1)(y)).\) Proving that
\[
\left( \alpha_T(A(t_1)(x)), \beta_{U'}(m_1 \cdot x + B'(u_1)(y)) \right) = \left( A'(t_1)(\alpha_T(x)), m_1 \cdot (\alpha_T(x)) + B'(u_1)(\beta_{U'}(y)) \right).
\]
This proves that \(\alpha \amalg \beta : A \amalg f B \rightarrow A' \amalg f' B'\) is a natural transformation.

**Proposition 3.12.** The assignment \(\mathcal{F} : \left( \text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}) \right) \rightarrow \text{Mod}(\mathcal{A})\) is a functor.

**Proof** It is straightforward.
Lemma 3.13. Let \((A, f, B), (A', f', B') \in \left(\text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U})\right)\) be. The map \(\mathcal{S} : \text{Hom}\left((A, f, B), (A', f', B')\right) \rightarrow \text{Hom}_{\text{Mod}(\Lambda)}(A \amalg_f B, A' \amalg_{f'} B')\)

is bijective. That is, \(\mathcal{S}\) is full and faithful.

Proof. Firstly, we will see that \(\mathcal{S}\) is surjective. Let \((A, f, B), (A', f', B') \in \left(\text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U})\right)\) and let \(S : A \amalg_f B \rightarrow A' \amalg_{f'} B'\) be a morphism in \(\text{Mod}(\Lambda)\) whose components are:

\[
S = \left\{ S_{\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}} : A(T) \amalg B(U) \rightarrow A'(T) \amalg B'(U) \right\}_{\begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \in \Lambda}.
\]

For \(T \in \mathcal{T}\), consider the object \(\begin{bmatrix} T & 0 \\ M & 0 \end{bmatrix} \in \Lambda\). Then we have the morphism

\[
S_{\begin{bmatrix} T & 0 \\ M & 0 \end{bmatrix}} : A(T) \amalg 0 \rightarrow A'(T) \amalg 0.
\]

Therefore, for \((x, 0) \in A(T) \amalg 0\) we have that \(S_{\begin{bmatrix} T & 0 \\ M & 0 \end{bmatrix}}(x, 0) = (x', 0)\), for some \(x' \in A'(T)\).

Hence, we define \(\alpha_T : A(T) \rightarrow A'(T)\) as \(\alpha_T(x) := x'\) for \(x \in A(T)\). It is straightforward to show that \(\alpha = \{\alpha_T : A(T) \rightarrow A'(T)\}_{T \in \mathcal{T}}\) is a morphism of \(\mathcal{T}\)-modules.

Now, for \(U \in \mathcal{U}\) we consider the morphism

\[
S_{\begin{bmatrix} 0 & 0 \\ M & U \end{bmatrix}} : 0 \amalg B(U) \rightarrow 0 \amalg B'(U).
\]

Then, we define \(\beta_U : B(U) \rightarrow B'(U)\) as follows: for \(y \in B(U)\) we have that \(\beta_U(y) := y' \in B'(U)\) is such that \(S_{\begin{bmatrix} 0 & 0 \\ M & U \end{bmatrix}}(0, y) = (0, y')\). Similarly, it can be proved that \(\beta = \{\beta_U : B(U) \rightarrow B'(U)\}\) is a morphism of \(\mathcal{U}\)-modules.

Therefore, for \(T \in \mathcal{T}\) and \(U \in \mathcal{U}\) we define \(\alpha_T \amalg \beta_U : A(T) \amalg B(U) \rightarrow A'(T) \amalg B'(U)\) as \((\alpha_T \amalg \beta_U)(x, y) := (\alpha_T(x), \beta_U(y))\). In this way, we obtain a family of morphisms in \(\text{Ab}\) as follows

\[
\alpha \amalg \beta := \left\{ (\alpha_T \amalg \beta_U) : A(T) \amalg B(U) \rightarrow A'(T) \amalg B'(U) \right\}_{\begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \in \Lambda}.
\]

We assert that \(S_{\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}} = \alpha_T \amalg \beta_U\). Indeed, for \(1_T : T \rightarrow T\) in \(\mathcal{T}\) we have the morphism

\[
\left[ \begin{bmatrix} 1_T & 0 \\ T & 0 \end{bmatrix} \right] : \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \rightarrow \begin{bmatrix} T & 0 \\ M & 0 \end{bmatrix} \in \Lambda.
\]

Then we have the following commutative diagram in \(\text{Ab}\)

\[
\begin{array}{ccc}
A(T) \amalg B(U) & \xrightarrow{S_{\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}}} & A'(T) \amalg B'(U) \\
\downarrow{\begin{bmatrix} A(1_T) & 0 \\ 0 & 0 \end{bmatrix}} & & \downarrow{\begin{bmatrix} A'(1_T) & 0 \\ 0 & 0 \end{bmatrix}} \\
A(T) \amalg 0 & \xrightarrow{S_{\begin{bmatrix} T & 0 \\ M & 0 \end{bmatrix}}} & A'(T) \amalg 0.
\end{array}
\]

For \((x, y) \in A(T) \amalg B(U)\) we consider \((x', y') = S_{\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}}(x, y)\) then we have that

\[
\left[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right] S_{\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}}(x, y) = S_{\begin{bmatrix} T & 0 \\ M & 0 \end{bmatrix}} \left[ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right](x, y).
\]
It follows that \((x', 0) = (\alpha_T(x), 0)\). Similarly, we have that \((0, y') = (0, \beta_U(y))\) and then, we conclude that

\[
S_T^{\frac{T}{M_U}}(x, y) = (x', y') = (\alpha_T(x), \beta_U(y)) = (\alpha_T \amalg \beta_U)(x, y).
\]

Proving that \(S_T^{\frac{T}{M_U}} = (\alpha_T \amalg \beta_U)\). Therefore, we have that \(S = \alpha \amalg \beta\).

Now, let us check that \((\alpha, \beta)\) is a morphism from \((A, f, B)\) to \((A', f', B')\). We have to show that for \(T \in T\) the following diagram commutes

\[
\begin{array}{ccc}
A(T) & \xrightarrow{\alpha_T} & A'(T) \\
\downarrow{f_T} & & \downarrow{f'_T} \\
\mathbb{G}(B)(T) & \xrightarrow{\mathbb{G}(\beta)_T} & \mathbb{G}(B')(T).
\end{array}
\]

Then, for \(x \in A(T)\) we have to show that \(f'_T(\alpha_T(x)) : M_T \rightarrow B'\) coincides with \((\mathbb{G}(\beta)_T)(f_T(x)) = \text{Hom}_{\text{Mod}(\mathcal{A})}(M_T, \beta)(f_T(x)) = \beta \circ f_T(x) : M_T \rightarrow B'\). In particular, we have to show that for \(U \in \mathcal{U}\), the following equality holds

\[
[\beta]_U \circ [f_T(x)]_U = [f'_T(\alpha_T(x))]_U.
\]

Since \(S : A \amalg f \rightarrow A' \amalg f' \rightarrow B'\) is a morphism in \(\text{Mod}(\mathcal{A})\), for every morphism \(\begin{bmatrix} 1_T & 0 \\ m & 1_U \end{bmatrix} : \begin{bmatrix} T \end{bmatrix}_{M_U} \rightarrow \begin{bmatrix} T \end{bmatrix}_{M_U}\) in \(\mathcal{A}\) with \(m \in M(U, T)\) we have the following commutative diagram in \(\text{Ab}\)

\[
\begin{array}{ccc}
A(T) \amalg B(U) & \xrightarrow{\alpha_T \amalg \beta_U} & A'(T) \amalg B'(U) \\
\downarrow{[1_{A(T)} \ 0 \ m \ 1_{B(U)}]} & & \downarrow{[1_{A'(T)} \ 0 \ m \ 1_{B'(U)}]} \\
A(T) \amalg B(U) & \xrightarrow{\alpha_T \amalg \beta_U} & A'(T) \amalg B'(U).
\end{array}
\]

Hence, for each \((x, y) \in A(T) \amalg B(U)\) we obtain that

\[
\left(\alpha_T \amalg \beta_U\right) \circ \begin{bmatrix} 1_{A(T)} & 0 \\ m & 1_{B(U)} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (\alpha_T \amalg \beta_U) \left(1_{A(T)}(x), m \cdot x + (1_{B(U)}(y))\right)
\]

\[
= (\alpha_T(x), \beta_U(m \cdot x + y)).
\]

On the other hand, we have that

\[
\begin{bmatrix} 1_{A'(T)} & 0 \\ m & 1_{B'(U)} \end{bmatrix} \circ (\alpha_T \amalg \beta_U) \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1_{A'(T)} & 0 \\ m & 1_{B'(U)} \end{bmatrix} \begin{bmatrix} \alpha_T(x) \\ \beta_U(y) \end{bmatrix}
\]

\[
= (\alpha_T(x), m \cdot (\alpha_T(x)) + \beta_U(y)).
\]

Then, we have that \((\alpha_T(x), \beta_U(m \cdot x + y)) = (\alpha_T(x), m \cdot (\alpha_T(x)) + \beta_U(y))\). Therefore, since \(\beta_U\) is a morphism of abelian groups, we have that

\[
\beta_U(m \cdot x) = m \cdot (\alpha_T(x))
\]

for \(m \in M(U, T)\). This means that

\[
\beta_U \circ [f_T(x)]_U(m) = [f'_T(\alpha_T(x))]_U(m) \quad \forall m \in M(U, T).
\]

That is, we have that

\[
\beta_U \circ [f_T(x)]_U = [f'_T(\alpha_T(x))]_U : M_T(U) \rightarrow B'(U)
\]
for each $U \in \mathcal{U}$. Therefore, we have that

$$f_T'(\alpha T(x)) = \beta \circ f_T(x) = \text{Hom}_{\text{Mod}(U)}(M_T, \beta)(f_T(x)) = (\mathbb{G}(\beta)_T)(f_T(x)) \quad \forall x \in A(T).$$

Therefore $(\alpha, \beta): (A, f, B) \longrightarrow (A', f', B')$ is a morphism in $\left(\text{Mod}(\mathcal{T}), \mathbb{G}\text{Mod}(\mathcal{U})\right)$ and $\mathcal{S}(\alpha, \beta) = \mathcal{S}$, proving that $\mathcal{S}$ is surjective.

Now, let suppose that $(\alpha', \beta'), (\alpha, \beta): (A, f, B) \longrightarrow (A', f', B')$ are morphisms such that and $\mathcal{S}(\alpha, \beta) = \mathcal{S}(\alpha', \beta')$. Then for each $T \in \mathcal{T}$, $U \in \mathcal{U}$ we have that $\alpha_T \sqcup \beta_U = \alpha'_T \sqcup \beta'_U$. This implies, that $\alpha_T = \alpha'_T$ and $\beta_U = \beta'_U$ and therefore $\alpha = \alpha'$ and $\beta = \beta'$. Proving that $\mathcal{S}$ is injective and therefore $\mathcal{S}$ is full and faithful. □

We can define a functor $I_1: \mathcal{T} \longrightarrow \Lambda$ defined as follows $I_1(T) := \left[ \begin{array}{c} T' \\ M \end{array} \right]$ and for a morphism $t: T \longrightarrow T'$ in $\mathcal{T}$ de define

$$I_1(t) = \left[ \begin{array}{c} t' \\ 0 \end{array} \right]: \left[ \begin{array}{c} T \\ M \end{array} \right] \longrightarrow \left[ \begin{array}{c} T' \\ M \end{array} \right].$$

In the same way, we define a functor $I_2: \mathcal{U} \longrightarrow \Lambda$. Then we have the induced morphisms

$$I_1: \text{Mod}(\Lambda) \longrightarrow \text{Mod}(\mathcal{T})$$

$$I_2: \text{Mod}(\Lambda) \longrightarrow \text{Mod}(\mathcal{U})$$

Let $C$ be a $\Lambda$-module, we denote by $C_1 = I_1(C) = C \circ I_1: \mathcal{T} \longrightarrow \text{Ab}$ and $C_2 = I_2(C) = C \circ I_2: \mathcal{U} \longrightarrow \text{Ab}.$

**Lemma 3.14.** Let $C$ be a $\Lambda$-module. Then, there exists a morphism of $\mathcal{T}$-modules

$$f : C_1 \longrightarrow \mathbb{G}(C_2).$$

**Proof** Let $C$ be a $\Lambda$-module and consider $T \in \mathcal{T}$ and $U \in \mathcal{U}$. For all $m \in M(U, T)$ we have a morphism $\overline{m} := \left[ \begin{array}{c} 0 \\ m \end{array} \right]: \left[ \begin{array}{c} T \\ M \end{array} \right] \longrightarrow \left[ \begin{array}{c} 0 \\ M \end{array} \right]$. We note that $\left[ \begin{array}{c} T \\ M \end{array} \right] = I_1(T)$ and $\left[ \begin{array}{c} 0 \\ M \end{array} \right] = I_2(U)$. Applying the $\Lambda$-module $C$ to $m$ yields a morphism of abelian groups

$$C(\overline{m}): C_1(T) = C(I_1(T)) \longrightarrow C_2(U) = C(I_2(U)).$$

We assert that $C(\overline{m})$ induces a morphism of abelian groups

$$f_T : C_1(T) \longrightarrow \mathbb{G}(C_2)(T) = \text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, C_2).$$

Indeed, for $x \in C_1(T)$ we define $f_T(x): M_T \longrightarrow C_2$ such that

$$f_T(x) = \{f_T(x)[U]: M_T(U) \longrightarrow C_2(U)\}_{U \in \mathcal{U}}$$

where $\{f_T(x)[U](m) := C(\overline{m})(x)\}$ for all $m \in M_T(U)$. It is straightforward to check that $f_T(x): M_T \longrightarrow C_2$ is a natural transformation. Now, let $t \in \text{Hom}_{\mathcal{T}}(T, T')$, we assert that the following diagram commutes

$$\begin{array}{ccc}
C_1(T) & \xrightarrow{f_T} & \mathbb{G}(C_2)(T) \\
\downarrow \text{C}_1(t) & & \downarrow \text{G}(C_2)(t) \\
C_1(T') & \xrightarrow{f_{T'}} & \mathbb{G}(C_2)(T')
\end{array}$$

Indeed, let $x \in C_1(T)$ then

$$\left(\mathbb{G}(C_2)(t) \circ f_T\right)(x) = \left(\text{Hom}_{\text{Mod}(\mathcal{U})}(\overline{t}, C_2) \circ f_T\right)(x) = f_T(x) \circ \overline{t}.$$
Then, for $U \in \mathcal{U}$ and $m' \in M(U, T')$ we have that
\[
\left( [f_T(x)]_U \circ [\tilde{f}]_U \right)(m') = [f_T(x)]_U (M(1_U \otimes r^{op})(m')) = [f_T(x)]_U (m' \bullet t)
\]
\[
= C(m' \bullet t)(x)
\]
\[
= C\left( \left[ \begin{array}{cc} 0 & 0 \\ m' & 0 \end{array} \right] \right)(x).
\]

On the other hand, $(f_{T'} \circ C_1(t))(x) = f_{T'}(C_1(t)(x)) = f_{T'}\left( C\left( \left[ \begin{array}{cc} t & 0 \\ 0 & 0 \end{array} \right] \right)(x) \right)$. Hence, for $U \in \mathcal{U}$ and $m' \in M(U, T')$ we have that
\[
\left[ f_{T'}\left( C\left( \left[ \begin{array}{cc} t & 0 \\ 0 & 0 \end{array} \right] \right)(x) \right) \right]_U (m') = C(m')\left( C\left( \left[ \begin{array}{cc} t & 0 \\ 0 & 0 \end{array} \right] \right)(x) \right)
\]
\[
= \left( C\left( \left[ \begin{array}{cc} 0 & 0 \\ m' & 0 \end{array} \right] \right) \right)\left( C\left( \left[ \begin{array}{cc} t & 0 \\ 0 & 0 \end{array} \right] \right)(x) \right)
\]
\[
= C\left( \left[ \begin{array}{cc} 0 & 0 \\ m' & 0 \end{array} \right] \right)(x).
\]

Therefore $f : C_1 \longrightarrow \mathcal{G}(C_2)$ is a morphism of $\mathcal{T}$-modules. Hence $(C_1, f, C_2) \in \left( \text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}) \right)$.

Remark 3.15. Let $C$ be a $\Delta$-module and consider the morphism $f : C_1 \longrightarrow \mathcal{G}(C_2)$ constructed in 3.14. Since $(C_1, f, C_2) \in \left( \text{Mod}(\mathcal{T}), \mathcal{G}\text{Mod}(\mathcal{U}) \right)$, we can construct $C_1 \amalg f C_2$. We recall that

(a) For $\left[ \begin{array}{cc} t & 0 \\ M & U \end{array} \right] \in \Delta$ we have that $(C_1 \amalg f C_2)(\left[ \begin{array}{cc} t & 0 \\ M & U \end{array} \right]) := C_1(T) \amalg f C_2(U) \in \text{Ab}$.

(b) If $\left[ \begin{array}{cc} t & 0 \\ m & u \end{array} \right] \in \text{Hom}_\Delta(\left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right], \left[ \begin{array}{cc} T' & 0 \\ M' & U' \end{array} \right])$ we have that

\[
(C_1 \amalg f C_2)(\left[ \begin{array}{cc} t & 0 \\ m & u \end{array} \right]) := \left[ \begin{array}{cc} C_1(t) & 0 \\ m & C_2(u) \end{array} \right] : C_1(T) \amalg f C_2(U) \rightarrow C_1(T') \amalg f C_2(U')
\]

is given by $\left[ \begin{array}{cc} C_1(t)(x) & 0 \\ m \cdot x + C_2(u)(y) \end{array} \right] \left[ \begin{array}{cc} X \\ y \end{array} \right] = \left[ \begin{array}{cc} C_1(t)(x) \\ m \cdot x + C_2(u)(y) \end{array} \right] \left[ \begin{array}{cc} X \\ y \end{array} \right]$ for $(x, y) \in C_1(T) \amalg f C_2(U)$, where $m \cdot x := [f_T(x)]_{U'}(m) \in C_2(U')$.

By the construction of the morphism $f : C_1 \longrightarrow \mathcal{G}(C_2)$ in 3.14, we have that

\[
\overline{m} := \left[ \begin{array}{cc} 0 & 0 \\ m & 0 \end{array} \right] : \left[ \begin{array}{cc} T & 0 \\ M & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc} T & 0 \\ M & 0 \end{array} \right].
\]

Lemma 3.16. Let $C$ be a $\Delta$-module. Then $C_1 \amalg f C_2 \cong C$.

Proof (i) Let $T \in \mathcal{T}$ and $U \in \mathcal{U}$. We have sequences of morphisms in $\Delta$.

\[
\left[ \begin{array}{cc} t & 0 \\ M & 0 \end{array} \right] \quad \lambda_T := \left[ \begin{array}{cc} 1_T & 0 \\ 0 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc} t & 0 \\ M & U \end{array} \right] \quad \rho_T := \left[ \begin{array}{cc} 1_T & 0 \\ 0 & 0 \end{array} \right] \rightarrow \left[ \begin{array}{cc} t & 0 \\ M & U \end{array} \right]
\]

\[
\left[ \begin{array}{cc} 0 & 0 \\ M & U \end{array} \right] \quad \lambda_U := \left[ \begin{array}{cc} 0 & 0 \\ 0 & 1_U \end{array} \right] \rightarrow \left[ \begin{array}{cc} 0 & 0 \\ M & U \end{array} \right] \quad \rho_U := \left[ \begin{array}{cc} 0 & 0 \\ 0 & 1_U \end{array} \right] \rightarrow \left[ \begin{array}{cc} 0 & 0 \\ M & U \end{array} \right]
\]

We consider the maps

\[
C(\lambda_T) := C\left( \left[ \begin{array}{cc} 1_T & 0 \\ 0 & 0 \end{array} \right] \right) : C\left( \left[ \begin{array}{cc} T & 0 \\ M & 0 \end{array} \right] \right) = C_1(T) \rightarrow C\left( \left[ \begin{array}{cc} T & 0 \\ M & 0 \end{array} \right] \right)
\]
This produces a map
\[ \phi_{T,U} : C_1(T) \amalg C_2(U) = (C_1 \amalg f C_2)(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}) \longrightarrow C(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}) \]
defined as \( \phi_{T,U}(x, y) = C(\lambda_T)(x) + C(\lambda_U)(y) \forall (x, y) \in C_1(T) \amalg C_2(U) \).

It is routine to check that
\[ C(\rho_T) := C\left(\begin{bmatrix} 1 & T \\ 0 & 0 \end{bmatrix}\right) : C\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right) \longrightarrow C(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}) \]
is a morphism of \( \Lambda \)-modules \( C : C_1 \amalg f C_2 \longrightarrow C \). Now, we consider the maps
\[ C(\rho_U) := C\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) : C\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right) \longrightarrow C(\begin{bmatrix} 0 & 0 \\ M & U \end{bmatrix}). \]

This produces a map
\[ \psi_{T,U} : C\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right) \longrightarrow C_1(T) \amalg C_2(U) = (C_1 \amalg f C_2)(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}) \]
such that
\[ \psi_{T,U}(z) = \left( C(\rho_T)(z), C(\rho_U)(z) \right). \]

It is straightforward to check that
\[ \psi = \left\{ \begin{array}{ll}
\psi_{T,U} := C\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right) \longrightarrow C\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}\right) & \end{array}\right\} \in \Lambda \]
is a natural transformation \( \psi : C \longrightarrow C_1 \amalg f C_2 \).

Finally, it is easy to see that \( \phi \) is invertible with inverse \( \psi \). Then \( \psi : C \longrightarrow C_1 \amalg f C_2 \),
is an isomorphism. \( \square \)

**Theorem 3.17.** The functor \( \mathcal{S} : \left( \text{Mod}(T), \text{GMod}(U) \right) \longrightarrow \text{Mod}(\Lambda) \) is an equivalence of categories

**Proof** It follows from Lemmas 3.13 and 3.16. \( \square \)

Let \( U \) and \( T \) be rings and \( N \) a \( U-T \)-bimodule, and consider the triangular matrix ring \( \Lambda = \begin{bmatrix} T & 0 \\ N & U \end{bmatrix} \). Then, using the same notation that we use before Definition 3.5, we have categories \( \text{Mod}(U) \) and \( \text{Mod}(T) \) associated to \( U \) and \( T \) respectively. Then, there exists a functor \( M : U \otimes_\mathbb{Z} T^{\text{op}} \rightarrow \text{Ab} \) for which we have an isomorphism of \( U-T \)-modules \( M(a, \ast) \cong N \). Thus, the following diagram commutes up isomorphism
\[
\begin{array}{ccc}
\text{Mod}(U) & \xrightarrow{\mathcal{S}} & \text{Mod}(T) \\
\downarrow{\text{ev}_a} & & \downarrow{\text{ev}_a} \\
\text{Mod}(U) & \xrightarrow{\text{Hom}_{U}(N,-)} & \text{Mod}(T).
\end{array}
\]

Indeed, let \( B \in \text{Mod}(U) \) be. One one hand \( \text{ev}_a(\mathcal{S}(B)) = \text{Hom}_{\text{Mod}(U)}(M_*, B) \) and on the other hand we have \( \text{Hom}_{U}(N, \text{ev}_a(B)) = \text{Hom}_{U}(N, B(a)) \). Finally, the assertion follows from the isomorphism \( \text{ev}_a : \text{Hom}_{\text{Mod}(U)}(M_*, B) \rightarrow \text{Hom}_{U}(M(a, \ast), B(a)) \), since

\( \square \) Springer
$\mathcal{M}(a, \ast) \cong N$. Therefore, if we set $G = \text{Hom}_U(N, -)$, we get an equivalence of categories $(\text{Mod}(T), G\text{Mod}(U)) \cong \text{Mod}(\Lambda)$ which is well-known (see [17, Proposition 2.2]).

4 Duality Functor

In this section, we are going to construct a functor $\mathbb{D}: \text{Mod}(\Lambda) \rightarrow \text{Mod}(\Lambda^{op})$ and we will describe it when we identify $\text{Mod}(\Lambda)$ with $\left(\text{Mod}(T), G\text{Mod}(U)\right)$. This functor will be useful in the case of dualizing varieties, because we will show that under certain conditions we get a duality

$$\mathbb{D}: \text{mod}(\Lambda) \rightarrow \text{mod}(\Lambda^{op}).$$

Let $\mathcal{U}$ and $\mathcal{T}$ preadditive categories and $M \in \text{Mod}(\mathcal{U} \otimes \mathcal{T}^{op})$. We can define $\overline{M} \in \text{Mod}(\mathcal{T}^{op} \otimes \mathcal{U})$ as follows:

(a) $\overline{M}(T, U) := M(U, T)$ for all $(T, U) \in \mathcal{T}^{op} \otimes \mathcal{U}$.

(b) $\overline{M}(\alpha^{op} \otimes \beta) := M(\beta \otimes \alpha^{op})$ for all $\alpha^{op} \otimes \beta: (T', U) \rightarrow (T, U')$ where $\alpha: T \rightarrow T'$ in $\mathcal{T}$ and $\beta: U \rightarrow U'$ in $\mathcal{U}$.

Proposition 4.1. Let $\overline{M} \in \text{Mod}(\mathcal{T}^{op} \otimes \mathcal{U})$ be. Then, there exists two covariant functors $\overline{E}: \mathcal{U}^{op} \rightarrow \text{Mod}(\mathcal{T}^{op})^{op}$, $\overline{E}': \mathcal{T}^{op} \rightarrow \text{Mod}(\mathcal{U})$.

Proof For $U \in \mathcal{U}^{op}$, we define a covariant functor $\overline{E}(U) := \overline{M}_U: \mathcal{T}^{op} \rightarrow \text{Ab}$ (that is, $\overline{M}_U: \mathcal{T} \rightarrow \text{Ab}$ is a contravariant functor) as follows.

(i) $\overline{M}_U(T) := \overline{M}(T, U)$, for all $T \in \mathcal{T}^{op}$.

(ii) $\overline{M}_U(t^{op}) := \overline{M}(t^{op} \otimes 1_U)$, for all $t^{op} \in \text{Hom}_{\mathcal{T}^{op}}(T', T)$.

Now, given a morphism $u^{op}: U' \rightarrow U$ in $\mathcal{U}^{op}$ we set $\overline{E}(u^{op}) := \overline{u}^{op}: \overline{M}_U \rightarrow \overline{M}_{U'}$ where $\overline{u}^{op} = \{[\overline{u}^{op}]_T: \overline{M}_U(T) \rightarrow \overline{M}_{U'}(T)\}_{T \in \mathcal{T}^{op}}$ with $[\overline{u}^{op}]_T = \overline{M}(1_T \otimes u)$.

Similarly for $T \in \mathcal{T}^{op}$ we define a contravariant functor $\overline{E}'(T) := \overline{M}_T: \mathcal{U}^{op} \rightarrow \text{Ab}$ (that is, is a covariant functor $\overline{M}_T: \mathcal{U} \rightarrow \text{Ab}$).

Remark 4.2. We have a covariant functor $\overline{G}: \text{Mod}(\mathcal{T}^{op}) \rightarrow \text{Mod}(\mathcal{U}^{op})$. In detail, we have that the following holds.

(i) For $B \in \text{Mod}(\mathcal{T}^{op})$, $\overline{G}(B)(U) := \text{Hom}_{\text{Mod}(\mathcal{T}^{op})}(\overline{M}_U, B)$ for all $U \in \mathcal{U}^{op}$. Moreover, for all $u^{op} \in \text{Hom}_{\mathcal{U}^{op}}(U', U)$ we have that $\overline{G}(B)(u^{op}) := \text{Hom}_{\text{Mod}(\mathcal{T}^{op})}(\overline{M}_U, B)$

(ii) $\overline{G}(\eta): B \rightarrow B'$ is a morphism of $\mathcal{T}^{op}$-modules we have that $\overline{G}(\eta): \overline{G}(B) \rightarrow \overline{G}(B')$ is such that $\overline{G}(\eta) = \left\{[\overline{G}(\eta)]_U := \text{Hom}_{\text{Mod}(\mathcal{T}^{op})}(\overline{M}_U, \eta): \overline{G}(B)(U) \rightarrow \overline{G}(B')(U)\right\}_{U \in \mathcal{U}^{op}}$.

(iii) We note that $\overline{M}_U = M_U$.

Since we have that $\overline{M} \in \text{Mod}(\mathcal{T}^{op} \otimes \mathcal{U})$. Following Definition 3.5, we have the following.
Definition 4.3. We define the triangular matrix category $\Lambda = \left[ \frac{U^0}{M} T^0 \right]$ as follows.

(a) The class of objects of this category are matrices $\left[ \begin{array}{cc} U & 0 \\ M & T \end{array} \right]$ with $U \in \text{obj}(U^0)$ and $T \in \text{obj}(T^0)$.

(b) Given a pair of objects in $\left[ \begin{array}{cc} U' & 0 \\ M & T' \end{array} \right], \left[ \begin{array}{cc} U & 0 \\ M & T \end{array} \right]$ in $\Lambda$ we define

$$\text{Hom}_{\Lambda} \left( \left[ \begin{array}{cc} U' & 0 \\ M & T' \end{array} \right], \left[ \begin{array}{cc} U & 0 \\ M & T \end{array} \right] \right) := \left[ \begin{array}{cc} \text{Hom}_{U^0}(U', U) & 0 \\ \text{Hom}_{M}(T', T) & \text{Hom}_{T^0}(T', T) \end{array} \right].$$

The composition is given by

$$\circ : \left[ \begin{array}{cc} \text{Hom}_{U^0}(U', U) & 0 \\ \text{Hom}_{M}(T', T) & \text{Hom}_{T^0}(T', T) \end{array} \right] \times \left[ \begin{array}{cc} \text{Hom}_{U^0}(U'', U') & 0 \\ \text{Hom}_{M}(T'', T') & \text{Hom}_{T^0}(T'', T') \end{array} \right] \rightarrow \left[ \begin{array}{cc} \text{Hom}_{U^0}(U'', U) & 0 \\ \text{Hom}_{M}(T'', T) & \text{Hom}_{T^0}(T'', T) \end{array} \right].$$

We recall that $m_1 \circ u_2^0 := \overline{M}(1_T \otimes u_2)(m_1)$ and $t_1^0 \circ m_2 = \overline{M}(t_1^0 \otimes 1_{U^0})(m_2).

Since $m_1 \in \overline{M}(T, U') = M(U', T)$ and $m_2 \in \overline{M}(T', U'') = M(U'', T')$, and since

$$\overline{M}(1_T \otimes u_2) = M(u_2 \otimes 1_T) : M(U', T) \rightarrow M(U'', T),$$

$$\overline{M}(t_1^0 \otimes 1_{U^0}) = M(1_{U''} \otimes t_1^0) : M(U'', T') \rightarrow M(U'', T),$$

then we have that

$$m_1 \circ u_2^0 + t_1^0 \circ m_2 = M(u_2 \otimes 1_T)(m_1) + M(1_{U''} \otimes t_1^0)(m_2) = u_2 \cdot m_1 + m_2 \cdot t_1.$$

Using this equation, it is straightforward to verify the following result.

Proposition 4.4. There is an isomorphism

$$\mathbb{T} : \Lambda^0 \rightarrow \overline{\Lambda},$$

defined as $\mathbb{T}(\left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right]) = \left[ \begin{array}{cc} U & 0 \\ M & T \end{array} \right]$ and for $\left[ \begin{array}{cc} t_1 & 0 \\ m_1 & u_1 \end{array} \right] : \left[ \begin{array}{cc} T' & 0 \\ M' & U' \end{array} \right] \rightarrow \left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right]$ a morphism in $\Lambda^0$ we set

$$\mathbb{T} \left( \left[ \begin{array}{cc} t_1 & 0 \\ m_1 & u_1 \end{array} \right] \right) = \left[ \begin{array}{cc} u_1^0 & 0 \\ m_1 t_1^0 \end{array} \right] : \left[ \begin{array}{cc} U' & 0 \\ M' & T' \end{array} \right]$.

Let $K$ be a field. Now, let $C$ be a $K$-variety, we know (see Section 2.2) that $\text{Mod}(C)$ is an $K$-variety, which we identify with the category of additive covariant functors $(C, \text{Mod}(K))$. Now, we are going to consider the full subcategory of $K$-linear functors $M : C \rightarrow \text{Mod}(K)$ with images in $\text{mod}(K)$, and we identify $\text{mod}(K)$ with $K$-linear functors with images in $\text{mod}(K)$. Let $K$ be a $K$-variety, in this case we can see that the injective envelope of $K/\text{rad}(K)$ is $K$. So the duality given in the Definition 2.4, becomes

$$(C, \text{mod}(K)) := \{ M \in (C, \text{Mod}(K)) | M(C) \in \text{mod}(K) \forall C \in C \}.$$
Lemma 4.5. Let $\mathcal{U}$ and $\mathcal{T}$ be Hom-finite $K$-varieties. Then $\mathcal{U} \otimes_K \mathcal{T}^{op}$ is a Hom-finite $K$-variety.

Proof It is straightforward.

In all what follows we suppose that $\mathcal{U}$ and $\mathcal{T}$ are Hom-finite $K$-varieties and therefore $\mathcal{U} \otimes \mathcal{T}^{op}$ is also a Hom-finite $K$-variety (see Lemma 4.5). We are going to consider the dualities

$$D_\mathcal{U} : (\mathcal{U}, \text{mod}(K)) \rightarrow (\mathcal{U}^{op}, \text{mod}(K)),$$

$$D_\mathcal{T} : (\mathcal{T}, \text{mod}(K)) \rightarrow (\mathcal{T}^{op}, \text{mod}(K)).$$

We have the following technical results.

Proposition 4.6. Let $\mathcal{U}$ and $\mathcal{T}$ be Hom-finite $K$-varieties, $B \in \text{Mod}(\mathcal{U})$, $U \in \mathcal{U}^{op}$ and $s \in \text{Hom}_K(B(U), K)$. Then we have a morphism of $\mathcal{T}^{op}$-modules

$$s_{B,U} : \overline{M}_U \rightarrow D_\mathcal{T} \mathcal{G}(B).$$

That is, $s_{B,U} \in \overline{\mathcal{G}(\mathcal{T})(\mathcal{G}(B))}(U) = \text{Hom}_{\text{Mod}(\mathcal{T}^{op})}(\overline{M}_U, D_\mathcal{T} \mathcal{G}(B))$, where for each $T \in \mathcal{T}^{op}$ we have that $[s_{B,U}]_T : M(U, T) \rightarrow \text{Hom}_K(\text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, B), K)$ is such that, for $m \in M(U, T)$, $[s_{B,U}]_T(m) : \text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, B) \rightarrow K$ is defined as

$$([s_{B,U}]_T(m)) \eta := (s \circ [\eta]_U)(m) = s([\eta]_U(m)).$$

for every $\eta \in \text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, B)$.

Proof First, it is straightforward to see that we have a morphism of $\mathcal{T}$-modules $\Gamma_U : \mathcal{G}(B) \rightarrow \text{Hom}_K(-, B(U)) \circ M_U$, where for $T \in \mathcal{T}$ we have that

$$[\Gamma_U]_T : \text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, B) \rightarrow \text{Hom}_K(M(U, T), B(U))$$

is defined as

$$[\Gamma_U]_T(\beta) := \beta_U \quad \forall \beta \in \text{Hom}_{\text{Mod}(\mathcal{U})}(M_T, B).$$

On the other hand, for each $U \in \mathcal{U}^{op}$, we have a natural isomorphism of $\mathcal{T}^{op}$-modules

$$\Delta_U : M_U \rightarrow \text{Hom}_K(-, K) \circ M_U$$

where for $T \in \mathcal{T}^{op}$ and $m \in M(U, T)$ the function $[\Delta_U]_T(m) : \text{Hom}_K(M(U, T), K) \rightarrow K$ is defined as $[\Delta_U]_T(m)(f) := f(m) \forall f \in \text{Hom}_K(M(U, T), K)$.

Now, we consider the canonical function (recall that $\overline{M}_U = M_U$)

$$\text{Hom}_{\text{Mod}(\mathcal{T}^{op})}(M_U, \text{Hom}_K(-, K) \circ \text{Hom}_{\text{Mod}(\mathcal{U})}(-, B) \circ E) \xrightarrow{\Phi_U} \text{Hom}_{\text{Mod}(\mathcal{T})}(\text{Hom}_{\text{Mod}(\mathcal{U})}(-, B) \circ E, \text{Hom}_K(-, K) \circ M_U),$$

defined as follows: If $\lambda \in \text{Hom}_{\text{Mod}(\mathcal{T})}(\text{Hom}_{\text{Mod}(\mathcal{U})}(-, B) \circ E, \text{Hom}_K(-, K) \circ M_U)$, we have the natural transformation

$$\text{Hom}_K(-, K)\lambda : \text{Hom}_K(-, K) \circ \text{Hom}_K(-, K) \circ M_U \rightarrow \text{Hom}_K(-, K) \circ \text{Hom}_{\text{Mod}(\mathcal{U})}(-, B) \circ E$$
Then we define \( \Phi_U(\lambda) := (\hom_K(-, K)\lambda) \circ \Delta_U \).

Next, we consider the Yoneda functor

\[ \mathcal{Y} : \text{Mod}(K) \longrightarrow \text{Mod}(\text{Mod}(K)), \]

defined as \( \mathcal{Y}(V) := \hom_K(-, V) \) for \( V \in \text{Mod}(K) \). Since \( B(U) \in \text{Mod}(K) \), we have a function

\[ \mathcal{Y} : \hom_K(B(U), K) \longrightarrow \hom_{\text{Mod}(\text{Mod}(K))}(\hom_K(-, B(U)), \hom_K(-, K)). \]

That is, for \( s \in \hom_K(B(U), K) \) we obtain \( \mathcal{Y}(s) : \hom_K(-, B(U)) \rightarrow \hom_K(-, K) \), in \( \text{Mod}(\text{Mod}(K)) \).

Then we have the following natural transformation in \( \text{Mod}(T) \)

\[ \mathcal{Y}(s)M_U : \hom_K(-, B(U)) \circ M_U \longrightarrow \hom_K(-, K) \circ M_U. \]

By composing with the morphism \( \Gamma_U \), we get the following morphism in \( \text{Mod}(T) \)

\[ H_U := (\mathcal{Y}(s)M_U) \circ \Gamma_U : \hom_{\text{Mod}(\mathcal{U})}(-, B) \circ E \longrightarrow \hom_K(-, K) \circ E. \]

By applying \( \Phi_U \) to \( H_U \) we obtain the following

\[ \Phi_U(H_U) \in \hom_{\text{Mod}(\mathcal{T}^{\text{op}})}(M_U, \hom_K(-, K) \circ \hom_{\text{Mod}(\mathcal{U})}(-, B) \circ E). \]

We define \( S_{B, U} := \Phi_U(H_U) \). Now, it is easy to show that for each \( T \in \mathcal{T}^{\text{op}} \) the morphism \([S_{B, U}]_T \) acts as in the statement of the Proposition 4.6.

**Proposition 4.7.** Let \( \mathcal{U} \) and \( \mathcal{T} \) be Hom-finite \( K \)-varieties and let \( B \in \text{Mod}(\mathcal{U}) \) be. Then, there exists a morphism of \( \mathcal{U}^{\text{op}} \)-modules \( \Psi_B : \mathcal{D}_U(B) \longrightarrow \overline{\mathcal{G}}(\mathcal{D}_T \mathcal{G}(B)) \), where for each \( U \in \mathcal{U}^{\text{op}} \) we have that

\[ \hom_K(B(U), K) \]

\[ \downarrow \psi_B \]

\[ \hom_{\text{Mod}(\mathcal{T}^{\text{op}})}(M_U, \hom_K(-, K) \circ \hom_{\text{Mod}(\mathcal{U})}(-, B) \circ E) \]

is defined as: \([\Psi_B]_U(s) := S_{B, U} \) for every \( s \in \hom_K(B(U), K) \).

Moreover, if \( \beta : B \longrightarrow B' \) is a morphism of \( \mathcal{U} \)-modules, the following diagram commutes

\[ \mathcal{D}_U(B') \xrightarrow{\psi_B'} \overline{\mathcal{G}}(\mathcal{D}_T \mathcal{G}(B')) \]

\[ \mathcal{D}_U(B) \xrightarrow{\psi_B} \overline{\mathcal{G}}(\mathcal{D}_T \mathcal{G}(B)). \]

**Proof** Let us see that \( \Psi_B : \mathcal{D}_U(B) \longrightarrow \overline{\mathcal{G}}(\mathcal{D}_T \mathcal{G}(B)) \) is a morphism of \( \mathcal{U}^{\text{op}} \)-modules. Let \( u \in \hom_{\mathcal{U}}(U, U') \) be. Then \( \overline{u} : M_U \longrightarrow M_{U'} \) is a morphism in \( \text{Mod}(\mathcal{T}^{\text{op}}) \) and \( B(u) : B(U) \longrightarrow B(U') \) is a morphism of abelian groups. We have to show that \( L \circ [\Psi_B]_U = [\Psi_B]_U \circ \hom_K(B(u), K), \) where \( L = \hom_{\text{Mod}(\mathcal{T}^{\text{op}})}(\overline{u}, \hom_K(-, K) \circ \hom_{\text{Mod}(\mathcal{U})}(-, B) \circ E) \).
Indeed, on one side, for \( s \in \text{Hom}_K(B(U'), K) \) we have that \((L \circ [\Psi_B]_{U'})(s) = [\Psi_B]_{U'}(s) \circ \bar{u} = \mathcal{S}_{B,U'} \circ \bar{u} \). Then, for \( T \in \mathcal{T}^{op} \) and \( m \in M(U, T) \) the function

\[
[\mathcal{S}_{B,U'}]_T(M(u \otimes 1_T)(m)) : \text{Hom}_{\mathcal{U}(T)}(M_T, B) \rightarrow K
\]

is defined as

\[
(\mathcal{S}_{B,U'}]_T(M(u \otimes 1_T)(m))(\eta) = (s \circ [\eta]_{U'})(M(u \otimes 1_T)(m)) = s\left([\eta]_{U'}(M(u \otimes 1_T)(m))\right),
\]

for every \( \eta \in \text{Hom}_{\mathcal{U}(T)}(M_T, B) \) (recall that \([\bar{u}]_T = M(u \otimes 1_T)\)).

On the other side, \(([\Psi_B]_{U'} \circ J)(s) = [\Psi_B]_{U'}(J(s)) = [\Psi_B]_{U}(s \circ B(u))\). Let us denote \( s' := s \circ B(u) \), hence \([\Psi_B]_{U}(s \circ B(u)) = [\Psi_B]_{U}(s') := \mathcal{S}_{B,U'}\).

Then, for \( T \in \mathcal{T}^{op} \) and \( m \in M(U, T) \), the function

\[
[\mathcal{S}_{B,U'}]_T(m) : \text{Hom}_{\mathcal{U}(T)}(M_T, B) \rightarrow K
\]

is defined as

\[
([\mathcal{S}_{B,U'}]_T(m))(\eta) := \left(s' \circ [\eta]_U\right)(m) = s'(\eta_U(m)) = (s \circ B(u))([\eta]_U(m))
\]

for every \( \eta \in \text{Hom}_{\mathcal{U}(T)}(M_T, B) \).

Since \( \eta \in \text{Hom}_{\mathcal{U}(T)}(M_T, B) \), we have that \( B(u) \circ [\eta]_U = [\eta]_U \circ M(u \otimes 1_T) \). Therefore, for \( m \in M(U, T) \) we have that \( B(u)([\eta]_U(m)) = [\eta]_U(M(u \otimes 1_T)(m)) \). Proving that \( \Psi_B : \mathcal{D}_{\mathcal{U}(T)}(B) \rightarrow \mathcal{G}(\mathcal{D}_T \mathcal{G}(B)) \) is a morphism of \( \mathcal{U}^{op} \)-modules.

Let \( \beta : B \rightarrow B' \) be a morphism of \( \mathcal{U} \)-modules, in order to show that the diagram in the statement of this Proposition commutes, we have to show that for \( U \in \mathcal{U}^{op} \) the following equality holds: \( \mathcal{G}(\mathcal{D}_T \mathcal{G}(B')) \circ \Psi_{B'} \mid U = [\Psi_B] \mid U \circ [\mathcal{D}_T \mathcal{G}(\beta)] \).

Indeed, we note that for \( U \in \mathcal{U}^{op} \)

\[
\left[\mathcal{G}(\mathcal{D}_T \mathcal{G}(\beta))\right]_U : \mathcal{G}(\mathcal{D}_T \mathcal{G}(B'))(U) \rightarrow \mathcal{G}(\mathcal{D}_T \mathcal{G}(B))(U)
\]

is defined as \( \left[\mathcal{G}(\mathcal{D}_T \mathcal{G}(\beta))\right]_U(\delta) := \text{Hom}_{\mathcal{U}(\mathcal{T}^{op})}(\overline{M_U}, \mathcal{D}_T \mathcal{G}(\beta)) \circ \delta = \mathcal{D}_T \mathcal{G}(\beta) \circ \delta \) for \( \delta \in \mathcal{G}(\mathcal{D}_T \mathcal{G}(B'))(U) \). Now, for each \( T \in \mathcal{T}^{op} \) we have

\[
\left[\mathcal{D}_T \mathcal{G}(\beta)\right]_T : \text{Hom}_K(\text{Hom}_{\mathcal{U}(T)}(M_T, B'), K) \rightarrow \text{Hom}_K(\text{Hom}_{\mathcal{U}(T)}(M_T, B), K),
\]

where for \( \lambda \in \text{Hom}_K(\text{Hom}_{\mathcal{U}(T)}(M_T, B'), K) \) and \( \eta \in \text{Hom}_{\mathcal{U}(T)}(M_T, B) \) we obtain that

\[
\left(\left[\mathcal{D}_T \mathcal{G}(\beta)\right]_T(\lambda)\right)(\eta) := \lambda(\beta \circ \eta).
\]

For \( s : B'(U) \rightarrow K \) we have that \( [\Psi_{B'}]_U(s) := \mathcal{S}_{B',U} \). Then

\[
\left(\left[\mathcal{D}_T \mathcal{G}(\beta)\right]_U \circ [\Psi_{B'}]_U\right)(s) = (\mathcal{D}_T \mathcal{G}(\beta)) \circ \mathcal{S}_{B',U}
\]

Then for \( T \in \mathcal{T}^{op} \) and \( m \in M(U, T) \) the function

\[
\left[\mathcal{D}_T \mathcal{G}(\beta)\right]_T\left([\mathcal{S}_{B',U}]_T(m)\right) : \text{Hom}_{\mathcal{U}(T)}(M_T, B) \rightarrow K
\]

is defined as

\[
\left([\mathcal{D}_T \mathcal{G}(\beta)]_T([\mathcal{S}_{B',U}]_T(m))\right)(\eta) = \left([\mathcal{S}_{B',U}]_T(m)\right)(\beta \circ \eta)
\]

\[
= s\left(\beta \circ \eta \right)U(m)
\]

\[
= s\left(\beta \circ \eta \right)(m) \quad \forall \eta \in \text{Hom}_{\mathcal{U}(T)}(M_T, B)
\]
On the other hand, we consider $s' := [D_{U}(\beta)]_{U}(s) = s \circ [\beta]_{U}$. In this case, we obtain that $\Psi_{B}(s') := S'_{B, U}$. Therefore, for each $T \in T^{op}$ and $m \in M(U, T)$ the function
\[
[S'_{B, U}]_{T}(m) : \text{Hom}_{\text{Mod}(U)}(M_{T}, B) \to K
\]
is defined as
\[
(s'([\eta]_{U}(m))) := s(\beta)_{U}([\eta]_{U}(m)) = s([\eta]_{U} \circ [\beta]_{U})(m)
\]
for every $\eta \in \text{Hom}_{\text{Mod}(U)}(M_{T}, B)$. Proving that the required equality holds.

**Proposition 4.8.** Let $U$ and $T$ be Hom-finite $K$-varieties. Then, there exists a contravariant functor $\Psi : \text{Mod}(U) \to \left(\text{Mod}(U^{op}), \overline{G}(\text{Mod}(T^{op}))\right)$.

**Proof** For $B \in \text{Mod}(U)$ we define $\Psi(B) := \left(D_{U}(B), D_{T}G(B)\right)$. For $\beta : B \to B'$ we have that
\[
\Psi(\beta) := (D_{U}(\beta), D_{T}G(\beta)) : \left(D_{U}(B'), \Psi_{B'}(D_{U}(B')) \to \left(D_{U}(B), \Psi_{B}(D_{U}(B))\right)\right).
\]
By Proposition 4.7, it follows that $\Psi(\beta)$ is a morphism in $\left(\text{Mod}(U^{op}), \overline{G}(\text{Mod}(T^{op}))\right)$.

**Proposition 4.9.** Let $U$ and $T$ be Hom-finite $K$-varieties. Then, there exists a contravariant functor $\widehat{\Theta} : \left(\text{Mod}(T), G(\text{Mod}(U))\right) \to \left(\text{Mod}(U^{op}), \overline{G}(\text{Mod}(T^{op}))\right)$ which defined as follows:

(a) For $f : A \to G(B)$ a morphism of $T$-modules. We define
\[
\overline{f} = \widehat{\Theta}(A, f, B) := \Theta(A, f, B) \circ \Psi_{B} : D_{U}(B) \to \overline{G}(D_{T}A).
\]
That is, $\overline{f}$ is the following composition
\[
D_{U}(B) \xrightarrow{\Psi_{B}} \overline{G}(D_{T}G(B)) \xrightarrow{\overline{G}(D_{T}(f))} \overline{G}(D_{T}A).
\]

(b) If $(\alpha, \beta) : (A, f, B) \to (A', f', B')$ then $\widehat{\Theta}(\alpha, \beta) = \left(D_{U}(\beta), D_{T}(\alpha)\right)$.

**Proof** Consider $\overline{G} \circ D_{T} : \text{Mod}(T) \to \text{Mod}(U^{op})$. This induces a contravariant functor $\Theta : \left(\text{Mod}(T), G(\text{Mod}(U))\right) \to \left(\text{Mod}(U^{op}), \overline{G}(\text{Mod}(T^{op}))\right)$ as follows. Let $f : A \to G(B)$ a morphism of $T$-modules. Then we have a morphism of $U^{op}$-modules $\overline{G}(D_{T}(f)) : \overline{G}(D_{T}G(B)) \to \overline{G}(D_{T}A)$. Thus we define
\[
\Theta(A, f, B) := \left(\overline{G}(D_{T}G(B)), \overline{G}(D_{T}(f)), D_{T}A\right).
\]

If $(\alpha, \beta) : (A, f, B) \to (A', f', B')$ is a morphism in $\left(\text{Mod}(T), G(\text{Mod}(U))\right)$ we set
\[
\Theta(\alpha, \beta) := \left(\overline{G}D_{T}G(\beta), D_{T}(\alpha)\right).
\]
It is easy to show that $\Theta$ is a contravariant functor. Now, by using the functor $\Psi$ in Proposition 4.8, it is straightforward to see that $\widehat{\Theta}$ is a contravariant functor.
Remark 4.10. Consider the morphism \( \overline{f} : \mathcal{D}(\mathcal{U}(B)) \longrightarrow \mathcal{G}(\mathcal{D}_{\mathcal{T}} A) \) given in 4.9. For \( U \in \mathcal{U}^{op} \), we have the morphism \( \left[ \overline{f} \right]_U : \mathcal{D}(\mathcal{U}(B)(U)) = \text{Hom}_K(B(U), K) \longrightarrow \mathcal{G}(\mathcal{D}_{\mathcal{T}} A)(U) \).

Therefore, for \( s \in \text{Hom}_K(B(U), K) \) and \( T \in \mathcal{T}^{op} \), we have the function

\[
\left[ \left[ \overline{f} \right]_U (s) \right]_T : \overline{M}_U(T) \longrightarrow \mathcal{D}_{\mathcal{T}} A(T),
\]

given as follows: for \( m \in M(U, T) \) the function \( \left[ \left[ \overline{f} \right]_U (s) \right]_T (m) : A(T) \longrightarrow K \) is defined as \( \left( \left[ \overline{f} \right]_U (s) \right)_{T} (m) (x) := s \left( \left[ f_T(x) \right]_U (m) \right) \forall x \in A(T) \).

Proof Indeed, by definition we have that \( \left[ \overline{f} \right]_U = \left[ \mathcal{G}(\mathcal{D}_{\mathcal{T}} f) \right]_U \circ \left[ \Psi_B \right]_U \). Then, for \( s : B(U) \longrightarrow K \) we have that \( \left[ \mathcal{G}(\mathcal{D}_{\mathcal{T}} f) \right]_U \circ \left[ \Psi_B \right]_U (s) = \left[ \mathcal{G}(\mathcal{D}_{\mathcal{T}} f) \right]_U (\mathcal{S}_{B,U}) \).

We recall that for \( \alpha \in \text{Hom}_{\text{Mod}(\mathcal{T}^{op})}(M_U, \mathcal{D}_{\mathcal{T}} (\mathcal{G}(B))) \) if follows that\[ \mathcal{G}([\eta])_U (\alpha) = \mathcal{D}_{\mathcal{T}} (f) \circ \alpha, \]
where \( \mathcal{D}_{\mathcal{T}} (f) \) is the following morphism of \( \mathcal{T}^{op} \)-modules \( \mathcal{D}_{\mathcal{T}} (f) : \mathcal{D}_{\mathcal{T}} \mathcal{G}(B) \longrightarrow \mathcal{D}_{\mathcal{T}} (A) \).

Therefore,
\[
\left[ \mathcal{G}(\mathcal{D}_{\mathcal{T}} f) \right]_U \circ \left[ \Psi_B \right]_U (s) = \left[ \mathcal{G}(\mathcal{D}_{\mathcal{T}} f) \right]_U (\mathcal{S}_{B,U}) = \mathcal{D}_{\mathcal{T}} (f) \circ \mathcal{S}_{B,U} : \overline{M}_U \longrightarrow \mathcal{D}_{\mathcal{T}} (A)
\]

For \( T \in \mathcal{T}^{op} \), the function \( \mathcal{D}_{\mathcal{T}} (f) \circ \mathcal{S}_{B,U} : \mathcal{M}(U, T) \longrightarrow \text{Hom}_K(A(T), K) \), is given as follows: \( \mathcal{D}_{\mathcal{T}} (f) \circ \mathcal{S}_{B,U} (m) = \mathcal{D}_{\mathcal{T}} (f)_{T} (\mathcal{S}_{B,U}_{T} (m)) \forall m \in \overline{M}(U, T) := M(U, T) \), where \( \mathcal{S}_{B,U}_{T} (m) : \text{Hom}_{\text{Mod}(\mathcal{T}^{op})}(M_T, B) \longrightarrow K \).

Then \( \mathcal{D}_{\mathcal{T}} (f) \circ \mathcal{S}_{B,U} (m) = \mathcal{D}_{\mathcal{T}} (f)_{T} (\mathcal{S}_{B,U}_{T} (m)) = \left( \mathcal{S}_{B,U}_{T} (m) \right) \circ f_T \). Hence, for \( x \in A(T) \) we obtain that \( \mathcal{D}_{\mathcal{T}} (f) \circ \mathcal{S}_{B,U} (m) (x) = s \left( \left[ f_T(x) \right]_{U} (m) \right) \).

Let \( \mathcal{U} \) and \( \mathcal{T} \) be Hom-finite \( K \)-varieties. Now, consider the isomorphism \( \psi : \mathcal{A}^{op} \longrightarrow A \), given in Proposition 4.4. Then, there exists a functor \( \mathcal{T}^{*} : \text{Mod}(A) \longrightarrow \text{Mod}(\mathcal{A}^{op}) \). Also, we have an equivalence \( \mathcal{F} : \left( \text{Mod}(\mathcal{U}^{op}), \mathcal{G}(\text{Mod}(\mathcal{T}^{op})) \right) \longrightarrow \text{Mod}(\overline{A}) \) constructed in Theorem 3.17 (for the case of \( \overline{A} \)). Thus, the composition of functors \( \mathcal{T}^{*} \circ \mathcal{F} \) give us an equivalence of categories. Moreover, we have the pair of functors \( \mathcal{T}^{*} \circ \mathcal{F} \circ \mathcal{G} \) and \( \mathcal{D}_{\mathcal{A}} \circ \mathcal{F} \) from the category \( \left( \text{Mod}(\mathcal{T}), \mathcal{G}(\text{Mod}(\mathcal{U})) \right) \) to the category \( \text{Mod}(\mathcal{A}^{op}) \). So, it is interesting to see that these functors are not the same, but in essence they are, as we will see in a moment, and therefore they preserve the information of the category \( \left( \text{Mod}(\mathcal{T}), \mathcal{G}(\text{Mod}(\mathcal{U})) \right) \) in the same way. The following result will be important when showing that \( A \) is a dualizing variety (see Theorem 6.10).

Now we establish one of the main results of this section.

Theorem 4.11. Let \( \mathcal{U} \) and \( \mathcal{T} \) be Hom-finite \( K \)-varieties. Consider the contravariant functors
\[
\mathcal{T}^{*} \circ \mathcal{F} \circ \mathcal{G}, \quad \mathcal{D}_{\mathcal{A}} \circ \mathcal{F} : \left( \text{Mod}(\mathcal{T}), \mathcal{G}(\text{Mod}(\mathcal{U})) \right) \longrightarrow \text{Mod}(\mathcal{A}^{op}).
\]
Then, there exists an isomorphism $\nu : T^* \circ \mathcal{S} \circ \hat{\Theta} \longrightarrow D_A \circ \mathcal{S}$. That is, the following diagram is commutative up to the isomorphism $\nu$

$$
\begin{array}{ccc}
\text{Mod}(T), \mathcal{G} \text{Mod}(U) & \xrightarrow{\nu} & \text{Mod}(A) \\
\mathcal{S} & \downarrow & \mu \\
\text{Mod}(U^{op}) \mathcal{G} \text{Mod}(T^{op}) & \xrightarrow{T^* \circ \mathcal{S}} & \text{Mod}(A^{op}).
\end{array}
$$

Proof. Let $(A, f, B)$ be an object in $\left(\text{Mod}(T), \mathcal{G} \text{Mod}(U)\right)$. For $\left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \in \Lambda^{op}$ we have an isomorphism in $\text{Mod}(K)$

$$
\left[ v_{(A, f, B)} \right] \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] : (\mathbb{D}_U B)(U) \amalg (\mathbb{D}_T A)(T) \longrightarrow \text{Hom}_K(A(T) \amalg B(U), K)
$$

given as follows: let $(s', w') \in (\mathbb{D}_U B)(U) \amalg (\mathbb{D}_T A)(T)$ be, then $\left[ v_{(A, f, B)} \right] \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] (s', w') = (w', s') : A(T) \amalg B(U) \longrightarrow K$ is defined by $((w', s'))(x, y) := w'(x) + s'(y), \forall (x, y) \in A(T) \amalg B(U)$.

The inverse $v^{-1} : D_A \circ \mathcal{S} \longrightarrow T^* \circ \mathcal{S} \circ \hat{\Theta}$ of the morphism $\nu$ is defined by

$$
\left[ v^{-1}_{(A, f, B)} \right] \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] : \text{Hom}_K(A(T) \amalg B(U), K) \longrightarrow (\mathbb{D}_U B)(U) \amalg (\mathbb{D}_T A)(T),
$$

which is given as follows: let $(u, v) : A(T) \amalg B(U) \longrightarrow K$ defined by $((u, v))(x, y) := u(x) + v(y), \forall (x, y) \in A(T) \amalg B(U)$. Then, set $\left[ v^{-1}_{(A, f, B)} \right] \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] (u, v) := (u, v)$.

Now, let $(A, f, B)$ be an object in $\left(\text{Mod}(T), \mathcal{G} \text{Mod}(U)\right)$. Let us just see that $v_{(A, f, B)} : (T^* \circ \mathcal{S} \circ \hat{\Theta})(A, f, B) \longrightarrow (\mathbb{D}_A \circ \mathcal{S})(A, f, B)$, is a morphism in $\text{Mod}(\Lambda^{op})$. A similar argument serves to see that $v^{-1}_{(A, f, B)} : (\mathbb{D}_A \circ \mathcal{S})(A, f, B) \longrightarrow (T^* \circ \mathcal{S} \circ \hat{\Theta})(A, f, B)$, is a morphism in $\text{Mod}(\Lambda^{op})$. Indeed, for $\left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \in \Lambda^{op}$ we have that

$$
(T^* \circ \mathcal{S} \circ \hat{\Theta})(A, f, B)\left( \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \right) = T^* (\mathbb{D}_U B) \amalg T (\mathbb{D}_T A) \left( \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \right) = (\mathbb{D}_U B) \amalg T (\mathbb{D}_T A) \left( \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \right) = \text{Hom}_K(B(U), K) \amalg \text{Hom}_K(A(T), K).
$$

and

$$
\left( (\mathbb{D}_A \circ \mathcal{S})(A, f, B) \right) \left( \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \right) = \left( \mathbb{D}_A(A \amalg F B) \right) \left( \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \right) = \text{Hom}_K(A(T) \amalg B(U), K).
$$
Let \( \begin{bmatrix} t & 0 \\ m & u \end{bmatrix} : \begin{bmatrix} T' & 0 \\ M & U' \end{bmatrix} \longrightarrow \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \) be a morphism in \( \Lambda^{op} \) with \( m \in M(U', T) \).

We have to show that the following diagram commutes in \( \text{Ab} \)

\[
\begin{array}{ccc}
\text{Hom}_K(B(U'), K) \oplus \text{Hom}_K(A(T'), K) & \xrightarrow{\begin{bmatrix} v_{(A, f, B)} & T' & 0 \\ M & U' \end{bmatrix}} & \text{Hom}_K(A(T') \oplus B(U'), K) \\
\text{Hom}_K(B(U), K) \oplus \text{Hom}_K(A(T), K) & \xrightarrow{\begin{bmatrix} v_{(A, f, B)} & T & 0 \\ M & U \end{bmatrix}} & \text{Hom}_K(A(T) \oplus B(U), K)
\end{array}
\]

Indeed, let \((s, w) \in \text{Hom}_K(B(U'), K) \oplus \text{Hom}_K(A(T'), K)\). Then

\[
\left( \mathcal{D}_{\mathcal{U}}(B) \mathcal{D}_{\mathcal{T}}(A) \right) \begin{bmatrix} u^p & 0 \\ m & t^p \end{bmatrix} \begin{bmatrix} s \\ w \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{\mathcal{T}}(B)(w^p)(s) \\ m \cdot s + \mathcal{T}(A)(t^p)(w) \end{bmatrix}.
\]

Then for \((x, y) \in A(T) \oplus B(U)\) we have that

\[
\left( \begin{bmatrix} v_{(A, f, B)} & T & 0 \\ M & U \end{bmatrix} \right) \begin{bmatrix} u^p & 0 \\ m & t^p \end{bmatrix} \begin{bmatrix} s \\ w \end{bmatrix} = \begin{bmatrix} \mathcal{D}_{\mathcal{U}}(B)(u^p)(s) \\ m \cdot s + \mathcal{T}(A)(t^p)(w) \end{bmatrix} = (x, y)
\]

On the other hand, \(\mathcal{D}_{\mathcal{A}}(A \sqcup_f B) \begin{bmatrix} t & 0 \\ m & u \end{bmatrix}^{op} = \text{Hom}_K(\begin{bmatrix} A(t) & 0 \\ m & B(u) \end{bmatrix}, K)\). Hence, for \((s, w) \in \text{Hom}_K(B(U'), K) \oplus \text{Hom}_K(A(T'), K)\) we get that

\[
\left( \mathcal{D}_{\mathcal{A}}(A \sqcup_f B) \begin{bmatrix} t & 0 \\ m & u \end{bmatrix}^{op} \right) \begin{bmatrix} v_{(A, f, B)} & T' & 0 \\ M & U' \end{bmatrix} \begin{bmatrix} s \\ w \end{bmatrix} = (w, s) \\
= \text{Hom}_K(\begin{bmatrix} A(t) & 0 \\ m & B(u) \end{bmatrix}, K) \begin{bmatrix} w \\ s \end{bmatrix} \begin{bmatrix} A(t) & 0 \\ m & B(u) \end{bmatrix}
\]

Then for \((x, y) \in A(T) \oplus B(U)\), we have that

\[
\begin{aligned}
((w, s) \circ \begin{bmatrix} A(t) & 0 \\ m & B(u) \end{bmatrix})((x, y)) &= (w, s)((A(t)(x)) \\ m \cdot x + B(u)(y))) \\
&= w(A(t)(x)) + s(m \cdot x + B(u)(y)) \\
&= w(A(t)(x)) + s(m \cdot x) + s(B(u)(y)).
\end{aligned}
\]

Then

\[
\begin{bmatrix} v_{(A, f, B)} & T & 0 \\ M & U \end{bmatrix} \begin{bmatrix} \mathcal{D}_{\mathcal{U}}(B)(w^p)(s) \\ m \cdot s + \mathcal{T}(A)(t^p)(w) \end{bmatrix} = \text{Hom}_K(\begin{bmatrix} A(t) & 0 \\ m & B(u) \end{bmatrix}, K) \begin{bmatrix} w \\ s \end{bmatrix}.
\]

Proving that the required diagram commutes thus \(v_{(A, f, B)} : \left( \mathcal{T}^* \circ \mathcal{S} \circ \mathcal{O} \right)(A, f, B) \longrightarrow (\mathcal{D} \circ \mathcal{S})(A, f, B)\), is an isomorphism in \(\text{Mod}(\Lambda^{op})\).
Now, let \((\alpha, \beta) : (A, f, B) \to (A', f', B')\) be a morphism in \(\text{Mod}(\mathcal{T}, \mathcal{G}(\text{Mod}(\mathcal{U})))\).

We leave to the reader to verify that the following diagram commutes in \(\text{Mod}(A^{\text{op}})\)

\[
\begin{array}{c}
\text{T}^* \circ \overline{S} \circ \widehat{\Theta} \quad (A', f', B') \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
\text{T}^* \circ \overline{S} \circ \widehat{\Theta} \quad (A, f, B) \\
\end{array}
\begin{array}{c}
\nu(A', f', B') \\
\nu(A, f, B) \\
\nu(A, f, B)
\end{array}
\begin{array}{c}
\text{D}_A \circ \overline{S} \quad (A', f', B') \\
\text{D}_A \circ \overline{S} \quad (A, f, B) \\
\text{D}_A \circ \overline{S} \quad (A, f, B)
\end{array}
\]

Therefore

\[
\nu := \left\{ \nu(A, f, B); \left( \text{T}^* \circ \overline{S} \circ \widehat{\Theta} \right)(A, f, B) \to \left( \text{D}_A \circ \overline{S} \right)(A, f, B) \right\}_{(A, f, B) \in \left( \text{Mod}(\mathcal{T}), \mathcal{G}(\text{Mod}(\mathcal{U})) \right)}
\]

defines an isomorphism \(\nu : \text{T}^* \circ \overline{S} \circ \widehat{\Theta} \to \text{D}_A \circ \overline{S}\).

\section{5 An Adjoint Pair}

In this section we will see that the functor \(\mathcal{G}\) has a left adjoint \(\mathcal{F}\) and as a consequence we get an isomorphism of comma categories

\[
\left( \mathcal{F}(\text{Mod}(\mathcal{T})), \text{Mod}(\mathcal{U}) \right) \simeq \left( \text{Mod}(\mathcal{T}), \mathcal{G}(\text{Mod}(\mathcal{U})) \right).
\]  

The above result is well known for general comma categories (see for example [24, Proposition 1.11]), but for the benefit of the reader we will give a proof of the isomorphism (5) in Proposition 5.4, following [24, Proposition 1.11].

First, let us consider the contravariant functor \(E : \mathcal{T} \to \text{Mod}(\mathcal{U})\) given in Definition 3.1. By [47, Theorem 6.3] in page 101, there exists a unique functor (up to isomorphism of functors) \(\mathcal{F} : \text{Mod}(\mathcal{T}) \to \text{Mod}(\mathcal{U})\) which commutes with direct limits and such that

(a) \(\mathcal{F} \circ Y \simeq E\), where \(Y : \mathcal{T} \to \text{Mod}(\mathcal{T})\) is the Yoneda functor.

(b) \(\mathcal{F}\) has a right adjoint.

Using the two properties above we will see how the functor \(\mathcal{F}\) is defined. Let \(A \in \text{Mod}(\mathcal{T})\), then there exists an exact sequence of \(\mathcal{T}\)-modules \(\mathcal{L}_i Y(T_i) \to \mathcal{L}_j Y(T_j) \to A \to 0\). Since \(\mathcal{F}\) has a right adjoint, it is right exact, and after applying \(\mathcal{F}\) we obtain the exact sequence \(\mathcal{L}_i (\mathcal{F} \circ Y(T_i)) \to \mathcal{L}_j (\mathcal{F} \circ Y(T_j)) \to \mathcal{F}(A) \to 0\). Since \(\mathcal{F} \circ Y \simeq E\) we have that \((\mathcal{F} \circ Y)(T) \simeq E(T) = M_T\) for each \(T \in \mathcal{T}\), then there exists a morphism \(\psi : F(A) \to \text{Coker}(\mathcal{L}_i T_{ij})\), which is an isomorphism, such that the following diagram commutes

\[
\begin{array}{c}
\mathcal{L}_i (\mathcal{F} \circ Y)(T_i) \\
\mathcal{L}_i M_{T_i}
\end{array}
\begin{array}{c}
\mathcal{L}_j (\mathcal{F} \circ Y)(T_j) \\
\mathcal{L}_j M_{T_j}
\end{array}
\begin{array}{c}
F(A) \\
\text{Coker}(\mathcal{L}_i T_{ij})
\end{array}
\begin{array}{c}
\approx \\
\approx \\
\psi
\end{array}
\]

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By the property (b) above we have that \( F \) has a right adjoint, assume that \( G \) is a right adjoint of \( F \). Then, for every \( A \in \text{Mod}(\mathcal{T}) \) and \( B \in \text{Mod}(\mathcal{U}) \) there exists a natural isomorphism

\[
\varphi_{A,B} : \text{Hom}_{\text{Mod}(\mathcal{U})}(F(A), B) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{T})}(A, G(B)).
\]

We claim that \( G \cong G \). Indeed, let \( T \in \mathcal{T} \) and set \( A := Y(T) = \text{Hom}_{\mathcal{T}}(T, -) \), then by the above isomorphism and the Yoneda lemma we have

\[
\text{Hom}_{\text{Mod}(\mathcal{U})}(E(T), B) \cong \text{Hom}_{\text{Mod}(\mathcal{U})}(F(Y(T)), B) \\
\cong \text{Hom}_{\text{Mod}(\mathcal{T})}(A, G(B)) \\
\cong \text{Hom}_{\text{Mod}(\mathcal{T})}(\text{Hom}_{\mathcal{T}}(T, -), G(B)) \\
\cong G(B)(T),
\]

therefore \( G(B)(T) = \text{Hom}_{\text{Mod}(\mathcal{U})}(M_{T, B}) = \text{Hom}_{\text{Mod}(\mathcal{U})}(E(T), B) \cong G(B)(T) \).

In the same way, considering the contravariant functor \( \overline{E} : \mathcal{U}^{op} \rightarrow \text{Mod}(\mathcal{T}^{op}) \) defined in 4.1, there exists a unique functor (up to isomorphism of functors) \( \overline{F} : \text{Mod}(\mathcal{U}) \rightarrow \text{Mod}(\mathcal{T}) \) which commutes with direct limits and such that

(a) \( \overline{F} \circ Y \simeq \overline{E} \), where \( Y : \mathcal{U}^{op} \rightarrow \text{Mod}(\mathcal{U}) \) is the Yoneda functor.

(b) \( \overline{F} \) has a right adjoint.

Moreover, the adjoint of \( \overline{F} \) is our functor \( \overline{G} : \text{Mod}(\mathcal{T}^{op}) \rightarrow \text{Mod}(\mathcal{U}^{op}) \). That is, we have that the pair \( (\overline{F}, \overline{G}) \) is an adjoint pair. That is, for every \( A \in \text{Mod}(\mathcal{U}^{op}) \) and \( B \in \text{Mod}(\mathcal{T}^{op}) \) there exists a natural isomorphism

\[
\overline{\varphi}_{A,B} : \text{Hom}_{\text{Mod}(\mathcal{T}^{op})}(\overline{F}(A), B) \rightarrow \text{Hom}_{\text{Mod}(\mathcal{U}^{op})}(A, \overline{G}(B)).
\]

Now, consider \( \mathcal{D}_{\mathcal{U}^{op}} : \text{Mod}(\mathcal{U}^{op}) \rightarrow \text{Mod}(\mathcal{U}) \) and \( \mathcal{D}_{\mathcal{T}} : \text{Mod}(\mathcal{T}) \rightarrow \text{Mod}(\mathcal{T}^{op}) \).

Therefore we get

\[
\mathcal{D}_{\mathcal{T}} \circ \mathcal{G} \circ \mathcal{D}_{\mathcal{U}^{op}} : \text{Mod}(\mathcal{U}^{op}) \rightarrow \text{Mod}(\mathcal{T}^{op}).
\]

**Proposition 5.2.** Let \( \mathcal{U} \) and \( \mathcal{T} \) be Hom-finite \( K \)-varieties. There exist natural isomorphisms

\[
\Psi : \overline{F} \rightarrow \mathcal{D}_{\mathcal{T}} \circ \mathcal{G} \circ \mathcal{D}_{\mathcal{U}^{op}}, \quad \Xi : \overline{F} \rightarrow \mathcal{D}_{\mathcal{U}^{op}} \circ \overline{G} \circ \mathcal{D}_{\mathcal{T}}.
\]

**Proof** We will prove just the first isomorphism, since the second is analogous.

Let us see that there exists an isomorphism of \( \mathcal{T}^{op} \)-modules

\[
\Psi_{\text{Hom}_{\mathcal{U}}(-, U)} : M_U = \overline{F}(\text{Hom}_{\mathcal{U}}(-, U)) \rightarrow \left( \mathcal{D}_{\mathcal{T}} \circ \mathcal{G} \circ \mathcal{D}_{\mathcal{U}^{op}} \right)(\text{Hom}_{\mathcal{U}}(-, U)).
\]

Let \( T \in \mathcal{T}^{op} \) be, we want

\[
\left[ \Psi_{\text{Hom}_{\mathcal{U}}(-, U)} \right]_{T} : M(U, T) \rightarrow \text{Hom}_{K} \left( \text{Hom}_{\text{Mod}(\mathcal{U})}(M_{T, \mathcal{U}^{op}}(\text{Hom}_{\mathcal{U}}(-, U)), K) \right).
\]

Therefore, for \( m \in M(U, T) \) we define

\[
\left[ \Psi_{\text{Hom}_{\mathcal{U}}(-, U)} \right]_{T}(m) : \text{Hom}_{\text{Mod}(\mathcal{U})}(M_{T, \mathcal{U}^{op}}(\text{Hom}_{\mathcal{U}}(-, U))) \rightarrow K,
\]

as follows:

\[
\left( \left[ \Psi_{\text{Hom}_{\mathcal{U}}(-, U)} \right]_{T}(m) \right)(\eta) := \left( \mathcal{D}_{\mathcal{U}}(\eta) \circ \Gamma_{\text{Hom}_{\mathcal{U}}(-, U)} \right)_{U}(1_{U})(m) = \left( \eta_{U}(m) \right)(1_{U})
\]

\( \Box \) Springer
for every \( \eta : M_T \to \mathcal{D}_{\mathcal{U}^{op}}(\text{Hom}_{\mathcal{U}}(\cdot, U)) \).

Let \( r^{op} : T' \to T \) be a morphism in \( T^{op} \). It is straightforward to verify that the following diagram commutes

\[
\begin{array}{ccc}
M(U, T') & \xrightarrow{[\Psi_{\text{Hom}_{\mathcal{U}}(\cdot, U)}]_{T'}} & \left( (\mathcal{D}_T \circ G \circ \mathcal{D}_{\mathcal{U}^{op}})(\text{Hom}_{\mathcal{U}}(\cdot, U)) \right)(T') \\
M(U, T) & \xrightarrow{[\Psi_{\text{Hom}_{\mathcal{U}}(\cdot, U)}]_{T}} & \left( (\mathcal{D}_T \circ G \circ \mathcal{D}_{\mathcal{U}^{op}})(\text{Hom}_{\mathcal{U}}(\cdot, U)) \right)(T)
\end{array}
\]

Then

\[
\Psi_{\text{Hom}_{\mathcal{U}}(\cdot, U)} : M_U = \bar{F}(\text{Hom}_{\mathcal{U}}(\cdot, U)) \to \left( (\mathcal{D}_T \circ G \circ \mathcal{D}_{\mathcal{U}^{op}})(\text{Hom}_{\mathcal{U}}(\cdot, U)) \right)(T)
\]

is an isomorphism of \( T^{op} \)-modules.

Now, we have to show that if \( \text{Hom}_{\mathcal{U}}(\cdot, U) : \text{Hom}_{\mathcal{U}}(\cdot, U) \to \text{Hom}_{\mathcal{U}}(\cdot, U') \) is a morphism in \( \text{Mod}(\mathcal{U}^{op}) \), then it is not hard to see that for each \( T \in T^{op} \) the following diagram commutes

\[
\begin{array}{ccc}
M(U, T) = \bar{F}(\text{Hom}_{\mathcal{U}}(\cdot, U))(T) & \xrightarrow{[\Psi_{\text{Hom}_{\mathcal{U}}(\cdot, U)}]_{T}} & \left( (\mathcal{D}_T \circ G \circ \mathcal{D}_{\mathcal{U}^{op}})(\text{Hom}_{\mathcal{U}}(\cdot, U)) \right)(T) \\
M(U', T) = \bar{F}(\text{Hom}_{\mathcal{U}}(\cdot, U'))(T) & \xrightarrow{[\Psi_{\text{Hom}_{\mathcal{U}}(\cdot, U')}]} & \left( (\mathcal{D}_T \circ G \circ \mathcal{D}_{\mathcal{U}^{op}})(\text{Hom}_{\mathcal{U}}(\cdot, U')) \right)(T)
\end{array}
\]

where \( L = \left[ (\mathcal{D}_T \circ G \circ \mathcal{D}_{\mathcal{U}^{op}})(\text{Hom}_{\mathcal{U}}(\cdot, U)) \right]_T \).

Therefore, we have an isomorphism

\[
\Psi : \bar{F} \mid_{\text{proj} \text{Mod}(\mathcal{U}^{op})} \to (\mathcal{D}_T \circ G \circ \mathcal{D}_{\mathcal{U}^{op}}) \mid_{\text{proj} \text{Mod}(\mathcal{U}^{op})}.
\]

Now, the collection of objects in \( \text{proj} \text{Mod}(\mathcal{U}^{op}) \) generates \( \text{Mod}(\mathcal{U}^{op}) \) and the functor \( \bar{F} \) commutes with direct limits. Then, by [42, Chap. IV, Theorem 5.4] and [42, Chap. IV, Corollary 5.5], we can extend \( \Psi \) to an isomorphism

\[
\Psi : \bar{F} \to \mathcal{D}_T \circ G \circ \mathcal{D}_{\mathcal{U}^{op}}.
\]

**Corollary 5.3.** Let \( \mathcal{U} \) and \( T \) be Hom-finite \( K \)-varieties

(i) For every \( B \in \text{Mod}(\mathcal{U}^{op}) \) there exists an isomorphism \( \mathcal{D}_{T^{op}}(\bar{F}(B)) \simeq \mathcal{G}_{\mathcal{U}^{op}}(B) \).

(ii) For every \( A \in \text{Mod}(T) \) there exists an isomorphism \( \mathcal{D}_{\mathcal{U}}(\bar{F}(A)) \simeq \mathcal{G}_{\mathcal{U}}(A) \).

**Proof** The isomorphism (i) follows after applying \( \mathcal{D}_{T^{op}} \) to the isomorphism \( \Psi \), given in 5.2. The proof of (ii) is similar. \( \square \)

Since for every \( A \in \text{Mod}(T) \) and \( B \in \text{Mod}(\mathcal{U}) \) there exists a natural isomorphism

\[
\varphi_{A,B} : \text{Hom}_{\text{Mod}(T)}(\bar{F}(A), B) \to \text{Hom}_{\text{Mod}(\mathcal{U})}(A, \mathcal{G}(B)),
\]

we have the comma category \( \left( \bar{F}(\text{Mod}(T)), \text{Mod}(\mathcal{U}) \right) \) whose objects are the triples \((A, g, B)\) with \( A \in \text{Mod}(T), B \in \text{Mod}(\mathcal{U}) \) and \( g : \bar{F}(A) \to B \) a morphism of \( \mathcal{U} \)-modules. A morphism between two objects \((A, g, B)\) and \((A', g', B')\) is a pairs of morphism \((\alpha, \beta)\)
where \( \alpha : A \to A' \) is a morphism of \( \mathcal{T} \)-modules and \( \beta : B \to B' \) is a morphism of \( \mathcal{U} \)-modules and such that the diagram

\[
\begin{array}{ccc}
\mathbb{F}(A) & \xrightarrow{F(\alpha)} & \mathbb{F}(A') \\
\downarrow g & & \downarrow g' \\
B & \xrightarrow{\beta} & B'
\end{array}
\]

commutes.

**Proposition 5.4.** Let \( \mathcal{U} \) and \( \mathcal{T} \) be Hom-finite \( K \)-varieties.

(a) There exists an isomorphism

\[
H : \left( \mathbb{F}(\text{Mod}(\mathcal{T})), \text{Mod}(\mathcal{U}) \right) \to \left( \text{Mod}(\mathcal{T}), \mathbb{G}(\text{Mod}(\mathcal{U})) \right).
\]

(b) There exists an isomorphism

\[
\overline{H} : \left( \mathbb{F}(\text{Mod}(\mathcal{U}^{op})), \text{Mod}(\mathcal{T}^{op}) \right) \to \left( \text{Mod}(\mathcal{U}^{op}), \overline{\mathbb{G}}(\text{Mod}(\mathcal{T}^{op})) \right).
\]

**Proof** We define a functor \( H : \left( \mathbb{F}(\text{Mod}(\mathcal{T})), \text{Mod}(\mathcal{U}) \right) \to \left( \text{Mod}(\mathcal{T}), \mathbb{G}(\text{Mod}(\mathcal{U})) \right) \) by \( H(A, g, B) = (A, \varphi_{A, B}(g), B) \) on objects and \( H((\alpha, \beta)) = (\alpha, \beta) \) on morphisms. Clearly the functor \( H' : \left( \text{Mod}(\mathcal{T}), \mathbb{G}(\text{Mod}(\mathcal{U})) \right) \to \left( \mathbb{F}(\text{Mod}(\mathcal{T})), \text{Mod}(\mathcal{U}) \right) \), given by \( H'(A, g, B) = (A, \varphi_{A, B}^{-1}(g), B) \) on objects and \( H'((\alpha, \beta)) = (\alpha, \beta) \) on morphisms is an inverse of \( H \).

We will see in Proposition 6.8 that \( \Delta \) is an additive category with splitting idempotents. Thus, the finitely generated projective functors in \( \text{Mod}(\Delta) \) are the representable ones (see [4, Proposition 2.2]). The following result characterizes the projective objects in the category \( \left( \text{Mod}(\mathcal{T}), \mathbb{G}\text{Mod}(\mathcal{U}) \right) \).

**Proposition 5.5.** The subcategory \( \text{proj}(\text{Mod}(\Delta)) \) of \( \text{Mod}(\Delta) \) consisting of finitely generated projective \( \Delta \)-modules is equivalent to the subcategory of \( \left( \text{Mod}(\mathcal{T}), \mathbb{G}\text{Mod}(\mathcal{U}) \right) \) consisting of morphism of \( \mathcal{T} \)-modules

\[
g : \text{Hom}_{\mathcal{T}}(T, -) \to \mathbb{G}(M_{\mathcal{T}} \sqcup \text{Hom}_{\mathcal{U}}(U, -))
\]

given by \( g := \{ [g]_{T'} : \text{Hom}_{\mathcal{T}}(T, T') \to \text{Hom}_{\mathcal{U}}(M_{T'}, M_{\mathcal{T}} \sqcup \text{Hom}_{\mathcal{U}}(U, -)) \}_{T' \in \mathcal{T}} \), with \( [g]_{T'}(t) : M_{T'} \to M_{\mathcal{T}} \sqcup \text{Hom}_{\mathcal{U}}(U, -) \) for all \( t \in \text{Hom}_{\mathcal{T}}(T, T') \), where for \( U' \in \mathcal{U} \), the map \( \left[ \begin{bmatrix} 0 \\ -I \end{bmatrix} \right]_{U'} : M(U', T') \to M(U', T) \sqcup \text{Hom}_{\mathcal{U}}(U, U') \) is defined by

\[
\left[ \begin{bmatrix} 0 \\ -I \end{bmatrix} \right]_{U'}(m) := (M(1_U \otimes t^{op})(m), 0) = (m \circ t, 0), \forall m \in M(U', T).
\]

**Proof** Consider the projective \( \Delta \)-module, \( P := \text{Hom}_{\Delta} \left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, - \right) : \Delta \to \text{Ab} \). We will show that

\[
P \cong \text{Hom}_{\mathcal{T}}(T, -) \sqcup (M_{\mathcal{T}} \sqcup \text{Hom}_{\mathcal{U}}(U, -)).
\]

\( \square \) Springer
For this, we are going to use the notation of Lemma 3.14 and Remark 3.15. Firstly, we define a morphism of $\mathcal{T}$-modules $\alpha : \text{Hom}_T(T, -) \rightarrow P_1$. Indeed, for $T' \in \mathcal{T}$ we define

$$
\alpha_{T'} : \text{Hom}_T(T, T') \rightarrow P_1(T') = \text{Hom}_A\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, \begin{bmatrix} T' & 0 \\ M & 0 \end{bmatrix}\right)
$$

as $\alpha_{T'}(t) = \begin{bmatrix} \delta & 0 \\ 0 & 0 \end{bmatrix}$. It is straightforward that $\alpha$ is a morphism of $\mathcal{T}$-modules.

Secondly, we define a morphism of $\mathcal{U}$-modules $\beta : \text{Hom}_U(U', T) \rightarrow P_1(U') = \text{Hom}_A\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ M & U' \end{bmatrix}\right)$ as follows. For $U' \in \mathcal{U}$

$$
\beta_{U'} : \text{Hom}_U(U', T) \rightarrow P_2(U') = \text{Hom}_A\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ M & U' \end{bmatrix}\right)
$$

is defined as $\beta_{U'}(m, u) = \begin{bmatrix} 0 & 0 \\ m & u \end{bmatrix}$. It is straightforward that $\beta$ is a morphism of $\mathcal{U}$-modules.

Finally, we have to show that for each $T' \in \mathcal{T}$ the following diagram commutes

$$
\begin{array}{ccc}
\text{Hom}(T, T') & \xrightarrow{[\alpha]_{T'}} & \text{Hom}_A\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, \begin{bmatrix} T' & 0 \\ M & 0 \end{bmatrix}\right) \\
\text{Hom}_U(M_{T'}, M_T \amalg \text{Hom}_U(U, -)) & \xrightarrow{[f]_{T'}} & \text{Hom}_U(M_{T'}, P_2) \\
\end{array}
$$

Indeed, let $t : T \rightarrow T'$, then $[\alpha]_{T'}(t) : \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \rightarrow \begin{bmatrix} T' & 0 \\ M & 0 \end{bmatrix}$. Then, for $U' \in \mathcal{U}$ we have that

$$
\left([f]_{T'}([\alpha]_{T'}(t))\right)_{U'} : \text{Hom}_U(U', T') \rightarrow P_2(U') = \text{Hom}_A\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ M & U' \end{bmatrix}\right)
$$

is defined as

$$
\left([f]_{T'}([\alpha]_{T'}(t))\right)_{U'}(m) = P(\overline{m})([\alpha]_{T'}(t)) = \overline{m} \circ ([\alpha]_{T'}(t)) = \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}
$$

for $m \in M(U', T')$, where $\overline{m} = \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix} : \begin{bmatrix} T' & 0 \\ M & U' \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 \\ M & U' \end{bmatrix}$. On the other hand, we get $[g]_{T'}(t) : \begin{bmatrix} 0 & 0 \\ T' & 0 \end{bmatrix} : \text{Hom}_U(U', -) \rightarrow \text{Hom}_U(M_{T'}, P_2)$ and for $U' \in \mathcal{U}$ we have that

$$
\left([\beta \circ ([g]_{T'}(t))\right)_{U'} : \text{Hom}_U(U', T') \rightarrow P_2(U') = \text{Hom}_A\left(\begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ M & U' \end{bmatrix}\right)
$$

is defined as $\beta \circ ([g]_{T'}(t))_{U'}(m) = \beta_{U'}(m \cdot t, 0) = \begin{bmatrix} 0 & 0 \\ m & 0 \end{bmatrix}$ for $m \in M(U', T')$. Therefore the required diagram commutes. Since $\alpha$ and $\beta$ are isomorphism we get that

$$(\alpha, \beta) : \left(\text{Hom}_T(T, -), g, M_T \amalg \text{Hom}_U(U, -)\right) \rightarrow (P_1, f, P_2)$$

is an isomorphism. Thus, $P \cong P_1 \amalg f \cong (\text{Hom}_T(T, -)) \amalg g \left(M_T \amalg \text{Hom}_U(U, -)\right)$. □

Note that the above result can be obtained from a general fact for comma categories, the interested reader can see [24, Corollary 1.6] for more details.

**Proposition 5.6.** A sequence of maps

$$
0 \rightarrow (A, f, B) \xrightarrow{\alpha \beta} (A', f', B') \xrightarrow{\alpha' \beta'} (A'', f'', B'') \rightarrow 0
$$

is exact in $(\text{Mod}(T), \text{GMod}(U))$ if and only if the following are exact sequences

$$
0 \rightarrow A \xrightarrow{\alpha} A' \xrightarrow{\alpha} A \rightarrow 0, \quad 0 \rightarrow B \xrightarrow{\beta} B' \xrightarrow{\beta} B'' \rightarrow 0
$$
in $\text{Mod}(T)$ and $\text{Mod}(U)$ respectively.
Proof The proof is based on the abelian structure of \( \left( \text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U}) \right) \) and it can be proven by following the ideas in the discussion in page 6 in [24] and [24, Corollary 1.2]. □

**Proposition 5.7.** Let \((A, f, B)\) be an object in \( \left( \text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U}) \right) \) and let \( \alpha : \text{Hom}_\mathcal{T}(T, -) \rightarrow A \) an epimorphism in \( \text{Mod}(\mathcal{T}) \) and \( \beta : \text{Hom}_\mathcal{U}(U, -) \rightarrow B \) an epimorphism in \( \text{Mod}(\mathcal{U}) \). Then there exists an epimorphism in \( \text{Mod}(T_{\mathcal{U}}) \)

\[
\Gamma : P = \text{Hom}_A \left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix}, - \right) \rightarrow S(A, f, B).
\]

Proof Let us construct an epimorphism \((\theta, \psi) : \left( \text{Hom}_\mathcal{T}(T, -), g, M_{\mathcal{T}} \amalg \text{Hom}_\mathcal{U}(U, -) \right) \rightarrow (A, f, B)\). That is, we want morphisms \( \theta : \text{Hom}_\mathcal{T}(T, -) \rightarrow A \) and \( \psi : M_{\mathcal{T}} \amalg \text{Hom}_\mathcal{U}(U, -) \rightarrow B \) such that the following diagram commutes

\[
\begin{array}{ccc}
\text{Hom}_\mathcal{T}(T, -) & \xrightarrow{\theta} & A \\
g \downarrow & & \downarrow f \\
\mathbb{G} \left( M_{\mathcal{T}} \amalg \text{Hom}_\mathcal{U}(U, -) \right) & \xrightarrow{\mathbb{G}(\psi)} & \mathbb{G}(B).
\end{array}
\]

First, we will construct \( \psi : M_{\mathcal{T}} \amalg \text{Hom}_\mathcal{U}(U, -) \rightarrow B \). We have \( \rho := f_T \left( \alpha_T(1_T) \right) : M_{\mathcal{T}} \rightarrow B \). Therefore, we define

\[
\psi := (\rho, \beta) : M_{\mathcal{T}} \amalg \text{Hom}_\mathcal{U}(U, -) \rightarrow B
\]

We note that for each \( U' \in \mathcal{U} \) we have that

\[
[\psi]_{U'}(m, u) = \rho_{U'}(m) + \beta_{U'}(u) = \left[ f_T \left( \alpha_T(1_T) \right) \right]_{U'}(m) + \beta_{U'}(u)
\]

for every \((m, u) \in M(U', T) \amalg \text{Hom}_\mathcal{U}(U, U')\). Now, since \( \beta \) is an epimorphism we get that \( \psi \) is an epimorphism.

We define \( \theta := \alpha \). Let us check that the following diagrammm commutes for each \( T' \in \mathcal{T} \)

\[
\begin{array}{ccc}
\text{Hom}(T, T') & \xrightarrow{[\alpha]_{T'}} & A(T') \\
[g]_{T'} \downarrow & & \downarrow f_{T'} \\
\text{Hom}_{\text{Mod}(\mathcal{U})} \left( M_{T'}, M_{\mathcal{T}} \amalg \text{Hom}_\mathcal{U}(U, -) \right) & \xrightarrow{[\mathbb{G}(\psi)]_{T'}} & \text{Hom}_{\text{Mod}(\mathcal{U})} \left( M_{T'}, B \right)
\end{array}
\]

Indeed, for \( t : T \rightarrow T' \) we have \([g]_{T'}(t) = \begin{bmatrix} \tilde{t} & 0 \end{bmatrix} : M_{T'} \rightarrow M_{T} \amalg \text{Hom}_\mathcal{U}(U, -) \) and for \( U' \in \mathcal{U} \) we get \([\mathbb{G}(\psi)]_{T'}([g]_{T'}(t)) \left( [\psi]_U \circ \begin{bmatrix} \tilde{t} & 0 \end{bmatrix} \right)_{U'} : M(U', T') \rightarrow B(U')\). Then, for \( m \in M(U', T') \) we obtain that

\[
\left( [\psi]_{U'} \circ \begin{bmatrix} \tilde{t} \\ 0 \end{bmatrix} \right)_{U'}(m) = [\psi]_{U'} \left( \begin{bmatrix} \tilde{t} \\ 0 \end{bmatrix} \right)_{U'}(m) = [f_T \left( \alpha_T(1_T) \right)]_{U'}(m \bullet t).
\]
On the other hand, since $\alpha : \text{Hom}_\mathcal{T}(T, -) \to A$ and $f : A \to \mathcal{G}(B)$ are morphisms of $\mathcal{T}$-modules we have the following commutative diagram in $\mathbf{Ab}$

$$
\begin{array}{cccccc}
\text{Hom}_\mathcal{T}(T, T') & \xrightarrow{\alpha} & A(T) & \xrightarrow{f} & \text{Hom}_{\mathcal{Mod}(\mathcal{U})}(M_T, B) \\
\downarrow \text{Hom}_\mathcal{T}(T, t) & & \downarrow A(t) & & \downarrow \text{Hom}_{\mathcal{Mod}(\mathcal{U})}(\mathcal{T}, B) \\
\text{Hom}_\mathcal{T}(T, T') & \xrightarrow{\alpha_t} & A(T') & \xrightarrow{f_t'} & \text{Hom}_{\mathcal{Mod}(\mathcal{U})}(M_{T'}, B).
\end{array}
$$

Therefore, we get that $\text{Hom}_\mathcal{T}(T, t)(1_T) = t$. In this way we obtain that

$$
f_t'(\alpha_T(t)) = f_t'\left(\alpha_T\left(\text{Hom}_\mathcal{T}(T, t)(1_T)\right)\right) = \text{Hom}_{\mathcal{Mod}(\mathcal{U})}(\mathcal{T}, B)\left(f_T\left(\alpha_T(1_T)\right)\right)
$$

$$
= f_T\left(\alpha_T(1_T)\right) \circ \mathcal{T}.
$$

Then, for $U' \in \mathcal{U}$ and $m \in M(U', T')$ we have that

$$
\left[f_T'(\alpha_T(t))\right]_{U'}(m) = \left[\left[f_T\left(\alpha_T(1_T)\right)\right]_{U'} \circ \mathcal{T}\right]_{U'}(m)
$$

$$
= \left[f_T\left(\alpha_T(1_T)\right)\right]_{U'}\left[\mathcal{T}\right]_{U'}(m)
$$

$$
= \left[f_T\left(\alpha_T(1_T)\right)\right]_{U'}(m \cdot t).
$$

Proving that the required diagram commutes.

Now, since $\alpha$ and $\beta$ are epimorphisms, it follows by Proposition 5.6 that

$$\mathcal{S}(\alpha, \psi) : \mathcal{S}\left(\text{Hom}_\mathcal{T}(T, -), g, M_T \mathcal{H} \text{Hom}_\mathcal{U}(U, -)\right) \to \mathcal{S}(A, f, B)$$

is an epimorphism. \hfill \Box

Let $C$ be a $K$-variety. Recall from Section 2.2 that we denote by $\text{mod}(C)$ the full subcategory of $\text{Mod}(C)$ whose objects are the finitely presented functors. That is, $M \in \text{mod}(C)$ if and only if, there exists an exact sequence in $\text{Mod}(C)$

$$
\text{Hom}_C(C_1, -) \longrightarrow \text{Hom}_C(C_0, -) \longrightarrow M \longrightarrow 0,
$$

with $C_1, C_0 \in C$.

**Lemma 5.8.** Let $\mathcal{U}$ and $\mathcal{T}$ be Hom-finite $K$-varieties $M \in \text{Mod}(\mathcal{U} \otimes \mathcal{T}^{op})$ and $\mathcal{F}$ the left adjoint to $\mathcal{G}$. Then,

(i) $\mathcal{F}(\text{Hom}_\mathcal{T}(T, -)) \cong M_T$.

(ii) Assume $M_T \in \text{mod}(\mathcal{U})$ for all $T \in \mathcal{T}$. Then $A \in \text{mod}(\mathcal{T})$ implies $\mathcal{F}(A) \in \text{mod}(\mathcal{U})$.

**Proof** (i) Since $\mathcal{F} \circ Y \simeq E$ where $Y : \mathcal{T} \to \text{Mod}(\mathcal{T})$ is the Yoneda functor, we get that $M_T = E(T) \simeq (\mathcal{F} \circ Y)(E) = \mathcal{F}(\text{Hom}_\mathcal{T}(T, -))$.

(ii) Suppose that $A \in \text{mod}(\mathcal{T})$. Then, there exists an exact sequence

$$
\text{Hom}_\mathcal{T}(T_1, -) \longrightarrow \text{Hom}_\mathcal{T}(T_0, -) \longrightarrow A \longrightarrow 0
$$

Since $(\mathcal{F}, \mathcal{G})$ is an adjoint pair we have that $\mathcal{F}$ is right exact. Then we get

$$
M_{T_1} \longrightarrow M_{T_0} \longrightarrow \mathcal{F}(A) \longrightarrow 0
$$

Since $M_{T_0}, M_{T_1} \in \text{mod}(\mathcal{U})$, by [4, Proposition 4.2 (b)], we get that $\mathcal{F}(A) \in \text{mod}(\mathcal{U})$. \hfill \Box
Lemma 5.9. Let \( \mathcal{U} \) and \( \mathcal{T} \) be Hom-finite \( K \)-varieties \( \overline{M} \in \text{Mod} (\mathcal{T}^{\text{op}} \otimes \mathcal{U}) \) and \( \overline{F} \) the left adjoint to \( \overline{G} \). Then

(i) \( \overline{F} (\text{Hom}_{\mathcal{U}} (-, U)) \cong M_U \)
(ii) Assume \( M_U \in \text{mod} (\mathcal{T}^{\text{op}}) \) for all \( U \in \mathcal{U} \). Then \( B \in \text{mod} (\mathcal{U}^{\text{op}}) \) implies \( \overline{F} (B) \in \text{mod} (\mathcal{T}^{\text{op}}) \).

Proof The same proof as in Lemma 5.8.

6 Dualizing Varieties

The purpose of this section is to show that under certain conditions on \( \mathcal{U} \) and \( \mathcal{T} \), we can restrict the diagram in the Theorem 4.11 to the category of finitely presented modules and as a consequence we will have that \( \Lambda := \left[ \begin{array}{cc} \mathcal{T} & 0 \\ M & \mathcal{U} \end{array} \right] \) is dualizing. In order to give a description for the category \( \text{mod} (\Lambda) \) inside \( \left( \text{Mod} (\mathcal{T}), \text{GMod} (\mathcal{U}) \right) \), we will give conditions on the functor \( M \) and on the categories \( \mathcal{U} \) and \( \mathcal{T} \) to obtain a subcategory \( \mathcal{X} \subset \left( \text{Mod} (\mathcal{T}), \text{GMod} (\mathcal{U}) \right) \) for which the restriction functor \( S |_{\mathcal{X}} : \mathcal{X} \to \text{Mod} (\Lambda) \) induces an equivalence of subcategories between \( \mathcal{X} \) and \( \text{mod} (\Lambda) \). From now on, we will assume the \( \mathcal{U} \) and \( \mathcal{T} \) are dualizing \( K \)-varieties and in this case we are going to take \( M \in \text{Mod} (\mathcal{U} \otimes_K \mathcal{T}^{\text{op}}) \) (see Remark 2.6). We recall that a Hom-finite \( K \)-variety \( \mathcal{U} \) is a dualizing \( K \)-variety if the duality \( \mathcal{D}_\mathcal{U} : (\mathcal{C}, \text{mod}(\mathcal{K})) \to (\mathcal{C}^{\text{op}}, \text{mod}(\mathcal{K})) \) restricts to a duality

\[
\mathcal{D}_\mathcal{U} : \text{mod} (\mathcal{U}) \to \text{mod} (\mathcal{U}^{\text{op}}).
\]

Assume that \( \mathcal{U} \) and \( \mathcal{T} \) are dualizing varieties and that \( M_T \in \text{mod} (\mathcal{U}) \), for all \( T \in \mathcal{T} \) and that \( M_U \in \text{mod} (\mathcal{T}^{\text{op}}) \). Then \( \text{mod} (\mathcal{U}) \) and \( \text{mod} (\mathcal{T}) \) are abelian categories, and by Lemma 5.8 the restriction \( \overline{F}^* := \overline{F} |_{\text{mod} (\mathcal{T})} : \text{mod} (\mathcal{T}) \to \text{mod} (\mathcal{U}) \) has image in \( \text{mod} (\mathcal{U}) \). Similarly, by Lemma 5.9 we have a functor \( \overline{F}^* := \overline{F} |_{\text{mod} (\mathcal{U}^{\text{op}})} : \text{mod} (\mathcal{U}^{\text{op}}) \to \text{mod} (\mathcal{T}^{\text{op}}) \).

Lemma 6.1. Let \( \mathcal{U} \) and \( \mathcal{T} \) be dualizing \( K \)-varieties and \( M \in \text{Mod} (\mathcal{U} \otimes_K \mathcal{T}^{\text{op}}) \). Assume that \( M_T \in \text{mod} (\mathcal{U}) \) for all \( T \in \mathcal{T} \) and \( M_U \in \text{mod} (\mathcal{T}^{\text{op}}) \) for all \( U \in \mathcal{U} \). Then for all \( B \in \text{mod} (\mathcal{U}) \) we get that \( \overline{G} (B) \in \text{mod} (\mathcal{T}) \). That is if \( \overline{G}^* := \overline{G} |_{\text{mod} (\mathcal{U})} \) we have that

\[
\overline{G}^* : \text{mod} (\mathcal{U}) \to \text{mod} (\mathcal{T}).
\]

In the same way we have if \( \overline{G}^* := \overline{G} |_{\text{mod} (\mathcal{T}^{\text{op}})} \) we have that

\[
\overline{G}^* : \text{mod} (\mathcal{T}^{\text{op}}) \to \text{mod} (\mathcal{U}^{\text{op}}).
\]

Proof Since \( \mathcal{U} \) is dualizing we have that \( B \simeq \mathcal{D}_{\mathcal{U}} (\mathcal{D}_{\mathcal{U}} (B)) \) with \( \mathcal{D}_{\mathcal{U}} (B) \in \text{mod} (\mathcal{U}^{\text{op}}) \). Then by Corollary 5.3 (i), we have that \( \overline{G} (B) \simeq (\mathcal{D}_{\mathcal{U}}^{\text{op}} \mathcal{D}_{\mathcal{U}} ) (B) \simeq (\mathcal{D}_{\mathcal{T}^{\text{op}}} \overline{F}) \mathcal{D}_{\mathcal{U}} (B) \).

Since \( \mathcal{D}_{\mathcal{U}} (B) \in \text{mod} (\mathcal{U}^{\text{op}}) \) we have by Lemma 5.9, that \( \overline{F} (\mathcal{D}_{\mathcal{U}} (B)) \in \text{mod} (\mathcal{T}^{\text{op}}) \). Since \( \mathcal{T} \) is dualizing we get that \( \overline{G} (B) \simeq \mathcal{D}_{\mathcal{T}^{\text{op}}} (\overline{F} (\mathcal{D}_{\mathcal{U}} (B))) \in \text{mod} (\mathcal{T}) \).

We denote by \( \left( \text{mod} (\mathcal{T}), \text{Gmod} (\mathcal{U}) \right) \) the full subcategory of \( \left( \text{Mod} (\mathcal{T}), \text{GMod} (\mathcal{U}) \right) \) whose objects are the morphisms of \( \mathcal{T} \)-modules \( A \xrightarrow{f} \overline{G} (B) \) for which \( A \in \text{mod} (\mathcal{T}) \) and \( B \in \text{mod} (\mathcal{U}) \). Similarly, we denote by \( \left( \overline{F}^{\text{op}} (\text{mod} (\mathcal{T})), \text{mod} (\mathcal{U}) \right) \) the full subcategory of \( \left( \overline{F} (\text{Mod} (\mathcal{T})), \text{Mod} (\mathcal{U}) \right) \) whose objects are the morphisms of \( \mathcal{U} \)-modules \( \overline{F} (A) \xrightarrow{g} B \).
for which \( A \in \mod(\mathcal{T}) \) and \( B \in \mod(\mathcal{U}) \). Now, we can restrict the morphism given in Proposition 5.4 to subcategories.

**Proposition 6.2.** Let \( \mathcal{U} \) and \( \mathcal{T} \) be dualizing \( K \)-varieties and assume that \( M \in \mod(\mathcal{U} \otimes_K \mathcal{T}^{op}) \) satisfies that \( M_T \in \mod(\mathcal{U}) \) and \( M_U \in \mod(\mathcal{T}^{op}) \) for all \( T \in \mathcal{T} \) and \( U \in \mathcal{U}^{op} \). Then, there exists equivalences

\[
\left( \mathbb{F}^* (\mod(\mathcal{T})), \mod(\mathcal{U}) \right) \longrightarrow \left( \mod(\mathcal{T}), \mathbb{G}^* (\mod(\mathcal{U})) \right)
\]

\[
\left( \mathbb{F}^* (\mod(\mathcal{U}^{op})), \mod(\mathcal{T}^{op}) \right) \longrightarrow \left( \mod(\mathcal{U}^{op}), \mathbb{G}^* (\mod(\mathcal{U}^{op})) \right).
\]

**Proof** It follows from Proposition 5.4. \( \square \)

**Proposition 6.3.** Let \( \mathcal{U} \) and \( \mathcal{T} \) dualizing varieties and \( M \in \mod(\mathcal{U} \otimes_K \mathcal{T}^{op}) \). Assume that \( M_T \in \mod(\mathcal{U}) \) and \( M_U \in \mod(\mathcal{T}^{op}) \) for all \( T \in \mathcal{T} \) and \( U \in \mathcal{U}^{op} \). Then we get the functor \( \mathcal{S} |_{\mod(\mathcal{T}), \mathbb{G}mod(\mathcal{U})} : \left( \mod(\mathcal{T}), \mathbb{G}mod(\mathcal{U}) \right) \longrightarrow \mod(\Lambda) \) which is an equivalence.

**Proof** By Lemma 6.1 we have that \( \mathbb{G}mod(\mathcal{U}) \subseteq \mod(\mathcal{T}) \). Let \( (A, f, B) \in \left( \mod(\mathcal{T}), \mathbb{G}mod(\mathcal{U}) \right) \). Then \( A \in \mod(\mathcal{T}) \) and \( B \in \mod(\mathcal{U}) \). Hence, there exist exact sequences

\[
\begin{align*}
\text{Hom}_\mathcal{T}(T_1, -) & \xrightarrow{\alpha'} \text{Hom}_\mathcal{T}(T_0, -) \xrightarrow{\alpha} A \longrightarrow 0 \\
\text{Hom}_\mathcal{U}(U_1, -) & \xrightarrow{\beta'} \text{Hom}_\mathcal{U}(U_0, -) \xrightarrow{\beta} B \longrightarrow 0
\end{align*}
\]

By Propositon 5.7, we can construct an epimorphism

\[
(\alpha, \psi) : \left( \text{Hom}_\mathcal{T}(T_0, -), g, M_{T_0} \amalg \text{Hom}_\mathcal{U}(U_0, -) \right) \longrightarrow (A, f, B)
\]

in \( \left( \mod(\mathcal{T}), \mathbb{G}mod(\mathcal{U}) \right) \). Therefore we can construct an exact sequence

\[
\text{Hom}_A \left( \left[ \begin{array}{c} T_0 \\ M U \end{array} \right], - \right) \longrightarrow \text{Hom}_A \left( \left[ \begin{array}{c} T_0 \\ M U_0 \end{array} \right], - \right) \longrightarrow \mathcal{S}(A, f, B) \longrightarrow 0
\]

Proving that \( \mathcal{S}(A, f, B) \in \mod(\Lambda) \).

Now, since \( \mathcal{S} \) is full and faithfull, we have that its restriction is also full and faithfull. Let us see that \( \mathcal{S} |_{\mod(\mathcal{T}), \mathbb{G}mod(\mathcal{U})} : \left( \mod(\mathcal{T}), \mathbb{G}mod(\mathcal{U}) \right) \longrightarrow \mod(\Lambda) \) is essentially surjective.

Let us consider an exact sequence \( Q \xrightarrow{F} P \xrightarrow{G} M \longrightarrow 0 \) where \( M \cong \mathcal{S}(M_1, f, M_2) \) for some \( A \in \mod(\mathcal{T}) \), \( B \in \mod(\mathcal{U}) \) and \( Q = \text{Hom}_A \left( \left[ \begin{array}{c} T_0 \\ M U_1 \end{array} \right], - \right) \) and \( P = \text{Hom}_A \left( \left[ \begin{array}{c} T_0 \\ M U_0 \end{array} \right], - \right) \). Then we have exact sequence

\[
\begin{align*}
\left( \text{Hom}_\mathcal{T}(T_1, -) \amalg (MT_1 \amalg \text{Hom}_\mathcal{U}(U_1, -)) \right) \xrightarrow{g_1} \\
\left( \text{Hom}_\mathcal{T}(T_0, -) \amalg (MT_0 \amalg \text{Hom}_\mathcal{U}(U_0, -)) \right) \longrightarrow (M_1, f, M_2) \longrightarrow 0.
\end{align*}
\]
By Proposition 5.6, we have the following exact sequences

\[\text{Hom}^1(T_1, -) \longrightarrow \text{Hom}^0(T_0, -) \longrightarrow M_1 \longrightarrow 0\]

\[M_{T_1} \sqcup \text{Hom}_U(U_1, -) \longrightarrow M_{T_0} \sqcup \text{Hom}_U(U_0, -) \longrightarrow M_2 \longrightarrow 0\]

Now, since \(M_{T_1}, M_{T_2} \in \text{mod}(U)\), we conclude by [4, Proposition 4.2 (a) and (d)] that \(M_{T_1} \sqcup \text{Hom}_U(U_1, -), M_{T_0} \sqcup \text{Hom}_U(U_0, -) \in \text{mod}(U)\). By [4, Proposition 4.2 (b)], we have that \(M_2 \in \text{mod}(U)\). By the above exact sequence we have that \(M_1 \in \text{mod}(T)\).

Proving that \((M_1, f, M_2) \in \left(\text{mod}(T), \text{Gmod}(U)\right)\) and \(M \simeq S(M_1, f, M_2)\). Proving that \(S\vert_{(\text{mod}(T), \text{Gmod}(U))} : \left(\text{mod}(T), \text{Gmod}(U)\right) \longrightarrow \text{mod}(A)\) is essentially surjective. Therefore it is an equivalence.

\begin{flushright}
\(\square\)
\end{flushright}

\textbf{Proposition 6.4.} Consider the functor given in Proposition 4.9:

\[\hat{\Theta} : \left(\text{Mod}(T), \text{G}(\text{Mod}(U))\right) \longrightarrow \left(\text{Mod}(U^{op}), \overline{\text{G}}(\text{Mod}(T^{op}))\right).\]

Suppose that \(U\) and \(T\) are dualizing \(K\)-varieties and that \(M \in \text{Mod}(U \otimes_K T^{op})\) satisfies that \(M_T \in \text{mod}(U)\) and \(M_U \in \text{mod}(T^{op})\) for all \(T \in T\) and \(U \in U^{op}\). Then, we get a functor

\[\hat{\Theta}^* := \hat{\Theta} \vert_{(\text{mod}(T), \text{G}(\text{mod}(U)))} : \left(\text{mod}(T), \text{G}(\text{mod}(U))\right) \longrightarrow \left(\text{mod}(U^{op}), \overline{\text{G}}(\text{mod}(T^{op}))\right).\]

\textbf{Proof} We recall that for \(f : A \longrightarrow \text{G}(B)\) a morphism of \(T\)-modules, we have that

\[\overline{f} = \hat{\Theta}(A, f, B) := \Theta(A, f, B) \circ \Psi_B : \text{D}_U(B) \longrightarrow \overline{\text{D}}_T(A).\]

Now, if \(A \in \text{mod}(T)\) we have that \(\overline{\text{D}}_T(A) \in \text{mod}(T^{op})\) and by Lemma 6.1 we get that \(\overline{\text{D}}_U(B) \in \text{mod}(U^{op})\) if \(B \in \text{mod}(U)\). Therefore \(\hat{\Theta}(A, f, B) \in \left(\text{mod}(U^{op}), \overline{\text{G}}(\text{mod}(T^{op}))\right)\) if \((A, f, B) \in \left(\text{mod}(T), \text{G}(\text{mod}(U))\right)\).

\begin{flushright}
\(\square\)
\end{flushright}

\textbf{Remark 6.5.} Let \(U\) and \(T\) be \(K\)-varieties and \(M \in \text{Mod}(U \otimes_K T^{op})\).

(a) For each \(U \in U\) and \(T \in T\) there exists ring morphisms

\[\varphi : K \longrightarrow \text{End}_U(U) \quad k \mapsto k1_U, \quad \psi : K \longrightarrow \text{End}_T(T) \quad k \mapsto k1_T,\]

such that the structure of \(K\)-vector spaces on \(\text{End}_U(U)\) and \(\text{End}_T(T)\) induced by \(\varphi\) and \(\psi\) respectively, is the same as the one given by the fact that \(U\) and \(T\) are \(K\)-varieties. Since \(M(U, T)\) is an \(\text{End}_U(U)\)-\(\text{End}_T(T)\) bimodule, it is easy to show that the structure of \(K\)-vector space on \(M(U, T)\) induced by \(\text{End}_U(U)\) via \(\varphi\) is the same as the induced by \(\text{End}_T(T)\) via \(\psi\) (this is because \(k1_U \otimes 1_T = 1_U \otimes k1_T\)).

(b) Suppose that \(M_T \in \text{mod}(U)\) and \(U\) is Hom-finite, then \(M(U, T)\) is a finite dimensional \(K\)-vector space.

\textbf{Lemma 6.6.} Let \(U\) and \(T\) be Hom-finite \(K\)-varieties and suppose that \(M \in \text{Mod}(U \otimes_K T^{op})\) satisfies that \(M_T \in \text{mod}(U) \forall T \in T\). Then \(\Gamma' = \text{End}_A\left(\begin{bmatrix} T & 0 \\ 0 & M(U, T) \end{bmatrix}\right) := \begin{bmatrix} \text{Hom}_T(T, T) & 0 \\ M(U, T) & \text{Hom}_U(U, U) \end{bmatrix}\) is an artin \(K\)-algebra.
Proof. We have the ring morphism \( \varphi : K \to \text{Hom}_T(T, T) \) given by \( \varphi(\lambda) = \lambda 1_T \) which made \( \text{End}_T(T) \) into an artin \( K \)-algebra. Similarly, \( \psi : K \to \text{Hom}_U(U, U) \) given by \( \psi(\lambda) = \lambda 1_U \) is a ring morphism which made \( \text{End}_U(U) \) into an artin \( K \)-algebra. We note that the structure of vector spaces induced to the rings of endomorphisms is the same as the given by the definition that \( U \) and \( T \) are Hom-finite. By Remark 6.5 (b), we have that \( \text{M}(U, T) \) is finitely generated \( K \)-module. Then by [17, Proposition 2.1] on page 72, we have that \( \Gamma \) is an artin \( K \)-algebra via the morphism

\[
\Phi : K \to \begin{bmatrix} \text{Hom}_T(T, T) & \text{Hom}_U(U, U) \\ M(U, T) & \text{Hom}_U(U, U) \end{bmatrix}, \quad \lambda \mapsto \begin{bmatrix} \varphi(\lambda) & 0 \\ 0 & \psi(\lambda) \end{bmatrix} = \begin{bmatrix} \lambda 1_T & 0 \\ 0 & \lambda 1_U \end{bmatrix}.
\]

We note that

\[
\lambda \cdot \begin{bmatrix} t \\ m \\ u \end{bmatrix} := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} t \\ m \\ u \end{bmatrix} = \begin{bmatrix} (\lambda 1_T) (t) & 0 \\ 0 & (\lambda 1_U) (m) (\lambda 1_U) (u) \end{bmatrix} = \begin{bmatrix} 1_T \circ (\lambda 1_T) & 0 \\ 0 & (\lambda 1_U) \circ m 1_U \circ (\lambda 1_U) \end{bmatrix} = \begin{bmatrix} \lambda t & 0 \\ 0 & \lambda m \lambda u \end{bmatrix},
\]

where \( t \in \text{End}_T(T) \) and \( u \in \text{End}_U(U) \).

Lemma 6.7. Let \( U \) and \( T \) be Hom-finite \( K \)-varieties and suppose that \( M \in \text{Mod}(U \otimes_K T^{op}) \) satisfies that \( \text{M}(T) \in \text{mod}(U) \) for all \( T \in T \). Then \( \Lambda = \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \) is a Hom-finite \( K \)-variety.

Proof. Let us consider \( \text{Hom}_A \left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \), and \( A := \text{End}_A \left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \). We have that \( \text{Hom}_A \left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \) is a \( K \)-vector space as follows:

Let \( \begin{bmatrix} t \\ m \\ u \end{bmatrix} \in \text{Hom}_A \left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) \) and \( \lambda \in K \), then

\[
\lambda \cdot \begin{bmatrix} t \\ m \\ u \end{bmatrix} := \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \\ 0 & \lambda \end{bmatrix} \circ \begin{bmatrix} t \\ m \\ u \end{bmatrix} = \begin{bmatrix} (\lambda 1_T) (t) & 0 \\ 0 & (\lambda 1_U) (m) (\lambda 1_U) (u) \end{bmatrix} = \begin{bmatrix} 1_T \circ (\lambda 1_T) & 0 \\ 0 & (\lambda 1_U) \circ m 1_U \circ (\lambda 1_U) \end{bmatrix} = \begin{bmatrix} \lambda t & 0 \\ 0 & \lambda m \lambda u \end{bmatrix}.
\]

Since \( \text{Hom}_U(U, U), \text{Hom}_T(T, T), \) and \( \text{M}(U, T) \) are finite dimensional \( K \)-vector spaces we conclude that \( \text{Hom}_A \left( \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \right) := \text{Hom}_T(T, T) \otimes \text{mod}(U) \) is a finite dimensional \( K \)-vector space.

Proposition 6.8. Assume that \( T \) and \( U \) are additive categories with splitting idempotents. Then \( \Lambda \) is an additive category with splitting idempotents.

Proof. Let \( \begin{bmatrix} t \circ u \\ u \circ u \end{bmatrix} \to \begin{bmatrix} T & 0 \\ M & U \end{bmatrix} \) be an idempotent morphism with \( t \in \text{Hom}_T(T, T) \) and \( u \in \text{Hom}_U(U, U) \) and \( m \in \text{M}(U, T) \). Then

\[
\begin{bmatrix} t \circ u \\ u \circ u \end{bmatrix} = \begin{bmatrix} t \\ m \\ u \end{bmatrix} = \begin{bmatrix} t^2 \\ m \circ u \circ u \end{bmatrix}.
\]

Then \( m = m \circ (t \circ u) \circ u = M(1_U \otimes t^{op})(m) + M(u \otimes 1_T)(m), t^2 = t \) and \( u^2 = u \). Let \( \mu : L \to T \) be the kernel of \( t : T \to T \) and \( \nu : K \to U \) the kernel of \( u : U \to U \) (they exists because \( U \) and \( T \) are with split idempotents). Then \( 0 = t \mu \) and therefore \( 0 = M(1_U \otimes (t \mu)^{op}) := M(1_U \otimes \mu^{op})(m). \) Hence, \( 0 = (M(1_U \otimes \mu^{op}) \circ M(1_U \otimes t^{op}))(m) = (m \circ t) \circ \mu. \) Therefore

\[
m \circ \mu = (m \circ t + u \circ m) \circ \mu = (m \circ t) \circ \mu + (u \circ m) \circ \mu = (u \circ m) \circ \mu \in M(U, L).
\]
We claim that \[ \left[ \begin{array}{c} \mu \\ -m \cdot \mu \\ v \end{array} \right] : \left[ \begin{array}{c} L \\ M \\ K \end{array} \right] \rightarrow \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \] is the kernel of \[ \left[ \begin{array}{c} t \\ m \cdot u \\ \beta \end{array} \right] : \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \rightarrow \left[ \begin{array}{c} T \\ M \\ U \end{array} \right]. \] Indeed,

(a) First we note that

\[ \left[ \begin{array}{c} t \\ 0 \\ u \\ v \\ \end{array} \right] \left[ \begin{array}{c} \mu \\ -m \cdot \mu \\ v \end{array} \right] = \left[ \begin{array}{c} t \mu \\ m \cdot \mu + u (m \cdot \mu) \\ v \\ \end{array} \right] = \left[ \begin{array}{c} t \mu \\ (m \cdot \mu + u (m \cdot \mu)) \\ v \\ \end{array} \right] = \left[ \begin{array}{c} t \mu \\ m \cdot \mu - u (m \cdot \mu) \end{array} \right] \left[ \begin{array}{c} 0 \\ 0 \\ v \\ 0 \\ \end{array} \right]. \]

(b) Consider \[ \left[ \begin{array}{c} \alpha \\ 0 \\ \beta \end{array} \right] : \left[ \begin{array}{c} T' \\ M \\ U' \end{array} \right] \rightarrow \left[ \begin{array}{c} T \\ M \\ U \end{array} \right] \] with \( n \in M(U, T') \) and \( \alpha : T' \rightarrow T \) and \( \beta : U' \rightarrow U \), such that

\[ \left[ \begin{array}{c} 0 \\ m \\ \beta \end{array} \right] = \left[ \begin{array}{c} t \\ 0 \\ \alpha \\ \beta \end{array} \right] = \left[ \begin{array}{c} t \alpha \\ m \cdot \alpha + u \cdot n \cdot \beta \end{array} \right]. \]

Then \( m \cdot \alpha = -u \cdot n \in M(U, T'). \)

Consider \( v : K \rightarrow U \) then \( v \cdot (m \cdot \alpha) = v \cdot (-u \cdot n) \in M(K, T'). \) We want a morphism \[ \alpha' \beta' : T' \rightarrow T \] such that

\[ \left[ \begin{array}{c} \alpha' \\ 0 \\ \beta' \end{array} \right] = \left[ \begin{array}{c} \mu \alpha' \\ -m \cdot \mu \\ v \end{array} \right] \left[ \begin{array}{c} \alpha' \\ 0 \\ \beta' \end{array} \right] = \left[ \begin{array}{c} \mu \alpha' \\ -m \cdot \mu + \alpha' \cdot \beta' \end{array} \right] = \left[ \begin{array}{c} \mu \alpha' \\ -m \cdot \mu + \alpha' \cdot \beta' \end{array} \right]. \]

For this, consider \( v' : K' \rightarrow U \) the kernel of \( 1_U - u \). Since idempotents split, we have that \( U \cong K' \oplus K \) with \( v : K \rightarrow U \) and \( v' : K' \rightarrow U \) the natural inclusions. In particular, there exists \( p : U \rightarrow K \) and \( p' : U \rightarrow K' \) such that

\[ 1_U = vp + v'p', \quad pv = 1_K, \quad p'v' = 1_{K'}. \]

It can be seen that \( u = v'p' \).

From this we have that \( n = v \cdot (p \cdot n) + v' \cdot (p' \cdot n) \). We set \( m' := p \cdot n \in M(K, T'). \)

Then we have that

\[ -(m \cdot \mu) \cdot \alpha' = -(m \cdot (\mu \circ \alpha')). \]

On the other hand

\[ v \cdot m' = v \cdot (p \cdot n) = n - v' \cdot (p' \cdot n) = n - v' \cdot p' \cdot n \]

Therefore

\[ (-(m \cdot \mu)) \cdot \alpha' + v \cdot m' = n \]

and \( \alpha = \mu \alpha' \) and \( \beta = v \beta' \). Proving that \( \left[ \begin{array}{c} \alpha' \\ 0 \\ \beta' \end{array} \right] \) is the required morphism.

Uniqueness. Suppose that \( \left[ \begin{array}{c} \alpha'' \\ 0 \\ \beta'' \end{array} \right] \) is such that

\[ \left[ \begin{array}{c} \mu \alpha'' \\ -m \cdot \mu \end{array} \right] \left[ \begin{array}{c} \alpha'' \\ 0 \\ \beta'' \end{array} \right] = \left[ \begin{array}{c} \mu \alpha'' \\ -m \cdot \mu + \alpha'' \cdot \beta'' \end{array} \right] = \left[ \begin{array}{c} \alpha \\ 0 \\ \beta \end{array} \right]. \]

From this we have that \( \alpha' = \alpha'' \) and \( \beta' = \beta'' \) and then \( v \cdot m' = v \cdot m'' \). Composing with \( p : U \rightarrow K \) we have that

\[ m' = 1_K \cdot m' = (p \circ v) \cdot m' = p \cdot (v \cdot m') = p \cdot (v \cdot m') = (p \circ v) \cdot m'' = 1_K \cdot m'' = m''. \]

Proving the uniqueness. Therefore \( \Lambda \) is a category with splitting idempotents.

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\textbf{Proposition 6.9.} Let \( \mathcal{U} \) and \( \mathcal{T} \) be Hom-finite \( K \)-varieties which are Krull-Schmidt and \( M \in \text{Mod}(\mathcal{U} \otimes_K \mathcal{T}^{op}) \) satisfies that \( M_T \in \text{mod}(\mathcal{U}) \) for all \( T \in \mathcal{T} \).
(a) Then $\Lambda = \left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right]$ is a Hom-finite $K$-variety and Krull-Schmidt.

(b) Then $\text{mod}(\Lambda)$ is Hom-finite $K$-variety and Krull-Schmidt.

Proof (a) By Lemma 6.6, $\Gamma = \text{End}_A \left( \left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right] \right) := \left[ \begin{array}{cc} \text{Hom}_T(T, T) & 0 \\ \text{Hom}_M(U, U) & \text{Hom}_U(U, U) \end{array} \right]$ is an artin $K$-algebra for every $\left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right] \in \Lambda$ and therefore semiperfect. Since $\mathcal{U}$ and $\mathcal{T}$ are Krull-Schmidt, we have that $\mathcal{U}$ and $\mathcal{T}$ are with splitting idempotents (see [38, Corollary 4.4]). By Propositions 6.8 and 3.7, we have that $\Lambda$ is an additive category with splitting idempotents. Therefore $\Lambda$ is Krull-Schmidt (see [38, Corollary 4.4]).

(b) It follows from item (a) and [40, Proposition 2.4].

Finally we have the following result that give us that $\Lambda = \left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right]$ is a dualizing $K$-variety. We can not see directly that the functor $\mathcal{D}_\Lambda : \text{Mod}(\Lambda) \to \text{Mod}(\Lambda)$ restricts well to the category of finitely presented modules. So we use Theorem 4.11 to identify $\mathcal{D}_\Lambda$ with $\hat{\Theta}$ and we use this functor to prove the following result.

**Theorem 6.10.** Suppose that $\mathcal{U}$ and $\mathcal{T}$ are dualizing $K$-varieties, $M \in \text{Mod}(\mathcal{U} \otimes_K \mathcal{T}^{op})$ satisfies that $M_T \in \text{mod}(\mathcal{U})$ and $M_U \in \text{mod}(\mathcal{T}^{op})$ for all $T \in \mathcal{T}$ and $U \in \mathcal{U}^{op}$. Consider the contravariant functors

$$\mathcal{T}^* \circ \mathcal{F} \circ \hat{\Theta}^*, \quad \mathcal{D}_\Lambda \circ \mathcal{S} : \left( \text{mod}(\mathcal{T}), \mathcal{G}(\text{mod}(\mathcal{U})) \right) \to \text{mod}(\Lambda^{op}).$$

Then, there exists an isomorphism $\nu : \mathcal{T}^* \circ \mathcal{F} \circ \hat{\Theta}^* \to \mathcal{D}_\Lambda \circ \mathcal{S}$, such that the following diagram is commutative up to the isomorphism $\nu$

$$
\begin{array}{ccc}
\left( \text{mod}(\mathcal{T}), \mathcal{G}(\text{mod}(\mathcal{U})) \right) & \xrightarrow{\mathcal{S}|_{\text{mod}(\mathcal{T}), \mathcal{G}(\text{mod}(\mathcal{U}))}} & \text{mod}(\Lambda) \\
\downarrow & & \downarrow \\
\left( \text{mod}(\mathcal{U}^{op}), \mathcal{G}(\text{mod}(\mathcal{T}^{op})) \right) & \xrightarrow{\nu} & \text{mod}(\Lambda^{op}).
\end{array}
$$

In particular, $\Lambda$ is dualizing $K$-variety.

Proof First, by Proposition 6.8(b) we get that $\Lambda$ is a Hom-finite $K$-variety. By Propositions 6.3 and 6.4 we can restrict the diagram in Theorem 4.11. Therefore, the image of $(\mathcal{D}_\Lambda)|_{\text{mod}(\Lambda)}$ coincides, up isomorphism, with the image of the composition of functors $(\mathcal{T}^* \circ \mathcal{F})|_{\text{mod}(\mathcal{U}^{op}), \mathcal{G}(\text{mod}(\mathcal{T}^{op}))} \circ \hat{\Theta} : (\text{mod}(\mathcal{T}), \mathcal{G}(\text{mod}(\mathcal{U})))$, which is contained in $\text{mod}(\Lambda^{op})$, then we have the functor

$$(\mathcal{D}_\Lambda)|_{\text{mod}(\Lambda)} : \text{mod}(\Lambda) \to \text{mod}(\Lambda^{op}).$$

In the same way we have that

$$(\mathcal{D}_{\Lambda^{op}})|_{\text{mod}(\Lambda^{op})} : \text{mod}(\Lambda^{op}) \to \text{mod}(\Lambda),$$

and since $\mathcal{D}_{\Lambda^{op}}$ and $\mathcal{D}_\Lambda$ are quasi inverse from each other, we have that the previous are quasi inverse. Therefore $\Lambda$ is dualizing.

We have the following result that give us a way to construct dualizing varieties from others.
Corollary 6.11. Let $\mathcal{C}$ be a dualizing $K$-variety with duality $\mathbb{D}_\mathcal{C} : \text{mod}(\mathcal{C}) \longrightarrow \text{mod}(\mathcal{C}^{op})$. Then the following statements hold.

(a) The triangular matrix category $\begin{bmatrix} \mathcal{C} & 0 \\ \text{Hom} \ C & \end{bmatrix}$ is dualizing.

(b) Let $\mathcal{C}$ be an abelian category with enough projectives and let $n \geq 1$. Suppose that $\text{Ext}_\mathcal{C}^n(-, C) \in \text{mod}(\mathcal{C}^{op})$ and $\text{Ext}_\mathcal{C}^n(C, -) \in \text{mod}(\mathcal{C})$ for every $C \in \mathcal{C}$. Then the triangular matrix category $\begin{bmatrix} \mathcal{C} & 0 \\ \text{Ext}_\mathcal{C}^n & \end{bmatrix}$ is dualizing.

Moreover, $\mathbb{D}_{\mathcal{C}^{op}} \overline{\mathbb{F}}(B) \simeq L^n(\mathbb{D}_{\mathcal{C}^{op}}(B))$ if $\mathbb{D}_{\mathcal{C}^{op}}(B) \in \text{mod}(\mathcal{C}^{op})$ is right exact, where $L^n \mathbb{D}_{\mathcal{C}^{op}}(B)$ denotes the $n$-th left derived functor of $\mathbb{D}_{\mathcal{C}^{op}}(B)$. Similarly, $\mathbb{D}_\mathcal{C} \mathbb{F}(A) \simeq L^n(\mathbb{D}_\mathcal{C}(A))$ if $\mathbb{D}_\mathcal{C}(A) \in \text{mod}(\mathcal{C})$ is right exact.

(c) Suppose that $\mathcal{C}$ is an abelian category with enough projectives. Then the triangular matrix category $\begin{bmatrix} \mathcal{C} & 0 \\ \text{Ext}^1 & \end{bmatrix}$ is dualizing. Moreover $\mathbb{D}_{\mathcal{C}^{op}} \overline{\mathbb{F}}(B) \simeq L^1(\mathbb{D}_{\mathcal{C}^{op}}(B))$ if $\mathbb{D}_{\mathcal{C}^{op}}(B) \in \text{mod}(\mathcal{C}^{op})$ is right exact and $\mathbb{D}_\mathcal{C} \mathbb{F}(A) \simeq L^1(\mathbb{D}_\mathcal{C}(A))$ if $\mathbb{D}_\mathcal{C}(A) \in \text{mod}(\mathcal{C})$ is right exact.

Proof

(a) By Proposition 2.8 we have that $\overline{\text{Hom}}{C} \cong \text{Hom}_\mathcal{C}(C, -)$ and $\overline{\text{Hom}}{C'} \cong \text{Hom}_\mathcal{C}(-, C')$, for all $(C, C') \in \mathcal{C} \otimes \mathcal{C}^{op}$. The result follows from Theorem 6.10.

(b) By Proposition 2.8 we have $\overline{\text{Ext}}^n{C} \cong \text{Ext}_\mathcal{C}^n(C, -)$ and $\overline{\text{Ext}}^n{C'} \cong \text{Ext}_\mathcal{C}^n(-, C')$, for all $(C, C') \in \mathcal{C} \otimes \mathcal{C}^{op}$. The result follows from Theorem 6.10.

On the other hand, by Corollary 5.3 (i), we have an isomorphism $\mathbb{D}_{\mathcal{C}^{op}} \overline{\mathbb{F}}(B)(C) \cong \text{Hom}_{\mathcal{Mod}(\mathcal{C})}(\overline{\text{Ext}}^n{C}(C, -), \mathbb{D}_{\mathcal{C}^{op}}(B)) \cong L^n(\mathbb{D}_{\mathcal{C}^{op}}(B))$ (the last isomorphism is by [46, Theorem 1.4]). The other isomorphism is analogous.

(c) Suppose that $\mathcal{C}$ is an abelian category with enough projectives. Since $\mathcal{C}$ is dualizing we have that $\mathcal{C}$ has enough injectives. For every $C \in \mathcal{C}$ there exists an exact sequence $0 \longrightarrow K \longrightarrow P \longrightarrow C \longrightarrow 0$ with $P$ projective. Then we get the exact sequence in $\text{mod}(\mathcal{C})$

$0 \rightarrow \text{Hom}_\mathcal{C}(C, -) \rightarrow \text{Hom}_\mathcal{C}(P, -) \rightarrow \text{Hom}_\mathcal{C}(K, -) \rightarrow \text{Ext}_\mathcal{C}^1(C, -) \rightarrow 0$.

It follows that $\text{Ext}_\mathcal{C}^1(C, -) \in \text{mod}(\mathcal{C})$. Similarly, for $C \in \mathcal{C}$ get the an exact sequence in $\text{mod}(\mathcal{C}^{op})$

$0 \rightarrow \text{Hom}_\mathcal{C}(-, C) \rightarrow \text{Hom}_\mathcal{C}(-, I) \rightarrow \text{Hom}_\mathcal{C}(-, L) \rightarrow \text{Ext}_\mathcal{C}^1(-, C) \rightarrow 0$,

where $I$ is injective. Hence, the result follows from item (b).

Corollary 6.12. Let $A$ be an artin algebra and consider $\mathcal{C} = \text{mod}(A)$. Then the triangular matrix categories $\begin{bmatrix} \mathcal{C} & 0 \\ \text{Hom} \ C & \end{bmatrix}$ and $\begin{bmatrix} \mathcal{C} & 0 \\ \text{Ext}^1 & \end{bmatrix}$ are dualizing.

Proposition 6.13. Suppose that $\mathcal{U}$ and $\mathcal{T}$ are dualizing $K$-varieties, $M \in \text{Mod}(\mathcal{U} \otimes_K \mathcal{T}^{op})$ satisfies that $M_T \in \text{mod}(\mathcal{U})$ and $M_U \in \text{mod}(\mathcal{T}^{op})$ for all $T \in \mathcal{T}$ and $U \in \mathcal{U}^{op}$. Then there are almost split sequences in $\text{mod}(A)$.
Proof. If follows from Theorem 6.10 and [48, Theorem 7.1.3].

7 Examples and Applications

In this section we give some applications to splitting torsion pairs which are in relation with tilting theory (see [1]) and path categories which are studied in [49] and we give a kind of generalization of the one-point extension algebra.

Let $U$ and $T$ be rings and $M$ a $U$-$T$-module. If $T$ is a division ring the triangular matrix ring $\Lambda := \left[ \begin{array}{cc} T & 0 \\ M & U \end{array} \right]$ is called a one-point extension of $U$ by the bimodule $M$ (see discussion before Definition 3.5 for the general construction of triangular matrix rings). Now, we recall the reason for this terminology. Given a finite dimensional $K$-algebra $\Lambda := KQ/I$. Let $i$ a source in $Q$ and $\bar{e}_i$ the corresponding idempotent in $\Lambda$. Since there are no nontrivial paths ending in $i$, we have $\bar{e}_i \Lambda \bar{e}_i \simeq K$ and $\bar{e}_i \Lambda (1 - \bar{e}_i) = 0$. If $Q'$ denote the quiver that we obtain by removing the vertex $i$ and $I'$ denote the relations in $I$ removing the ones which start in $i$, then $(1 - \bar{e}_i) \Lambda (1 - \bar{e}_i) \simeq KQ'/I'$. So $\Lambda = KQ/I$ is obtained from $\Lambda' := KQ'/I'$ by adding one vertex $i$, together with arrows and relations starting in $i$. Then $\Lambda \cong \left[ \begin{array}{cc} K & 0 \\ (1-\bar{e}_i) \Lambda \bar{e}_i & \Lambda' \end{array} \right]$. So $\Lambda$ is the one-point extension of $\Lambda'$.

In order to extend the notion of one-point extension for finite dimensional algebras to categories, we consider path categories. Let $Q = (Q_1, Q_0)$ be a (possibly infinite) quiver. Recall that the path category $KQ$ is an additive category, with indecomposable objects the vertices, and given $a, b \in Q_0$, the set of the maps $\text{Hom}_{KQ}(a, b)$ is given by the $K$-vector space with basis the set of all paths from $a$ to $b$. The composition of maps is induced from the usual composition of paths.

Let $T = \{x \in Q_0 \mid x$ is a source $\}$ and let $U = Q_0 - T$, and consider $\mathcal{U} = \text{add}(U)$ and $\mathcal{T} = \text{add}(T)$. We show that there exists a functor $M : \mathcal{U} \otimes T^{op} \to \text{Ab}$ such that the categories $KQ$ and $\left[ \begin{array}{cc} T & 0 \\ \text{Hom} & U \end{array} \right]$ are equivalent.

To achieve this, we consider the following setting. Let $\mathcal{C}$ be a Krull-Schmidt category and $(\mathcal{U}, \mathcal{T})$ a pair of additive full subcategories of $\mathcal{C}$. It is said that $(\mathcal{U}, \mathcal{T})$ is a splitting torsion pair if

(i) For all $X \in \text{ind}(\mathcal{C})$, then either $X \in \mathcal{U}$ or $X \in \mathcal{T}$.

(ii) $\text{Hom}_\mathcal{C}(X, -)|_\mathcal{T} = 0$ for all $X \in \mathcal{U}$.

Firstly, we get the following result that tell us that if $\mathcal{C}$ has a splitting torsion pair, then $\mathcal{C}$ can be seen as a triangular matrix category.

Proposition 7.1. Let $(\mathcal{U}, \mathcal{T})$ be a splitting torsion pair in $\mathcal{C}$. Then we have an equivalence of categories

$\mathcal{C} \cong \left[ \begin{array}{cc} \mathcal{T} & 0 \\ \text{Hom}_\mathcal{C} & \mathcal{U} \end{array} \right]$.

Here without danger to cause confusion $\text{Hom}$ denotes the restriction of $\text{Hom} : \mathcal{C} \otimes C^{op} \to \text{Ab}$ to the subcategory $\mathcal{U} \otimes T^{op}$ of $\mathcal{C} \otimes C^{op}$ (see Proposition 2.8), and as a consequence we have an equivalence of categories

$\text{Mod}(\mathcal{T}), \text{GMod}(\mathcal{U})) \cong \text{Mod}(\mathcal{C}),$

where $\text{G} : \text{Mod}(\mathcal{U}) \to \text{Mod}(\mathcal{T})$ is the functor defined by $\text{G}(B)(T) = \text{Hom}_\text{Mod}(\mathcal{C})(\text{Hom}_\mathcal{C}(T, -)|_\mathcal{U}, B), \forall B \in \text{Mod}(\mathcal{U}), \forall T \in \mathcal{T}$. 

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Proof Let \( X \in \mathcal{C} \). Then \( X \) decomposes as \( X = X_1 \oplus X_2 \) with \( X_1 \in \mathcal{T} \) and \( X_2 \in \mathcal{U} \). We define a functor \( H : \mathcal{C} \to \bigg[ \frac{T}{\text{Hom}_{\mathcal{U}}} \bigg] \) in objects by \( H(X) = \bigg[ \frac{X_1}{\text{Hom}_{X_2}} \bigg] \). Let \( X, Y \in \mathcal{C} \). Then \( X = X_1 \oplus X_2 \) and \( Y = Y_1 \oplus Y_2 \) with \( X_1, Y_1 \in \mathcal{T} \) and \( X_2, Y_2 \in \mathcal{U} \). Let \( f \in \text{Hom}_\mathcal{C}(X, Y) \). Since \( \text{Hom}_\mathcal{C}(X_2, Y_1) = 0 \) and \( \text{Hom}_\mathcal{C}(Y_2, X_1) \cong \text{Hom}_\mathcal{C}(X_1, Y_2) \), we have an isomorphism of abelian groups

\[
\text{Hom}_\mathcal{C}(X, Y) \cong \text{Hom}_\mathcal{C}(X_1, Y_1) \oplus \text{Hom}_\mathcal{C}(Y_2, X_1) \oplus \text{Hom}_\mathcal{C}(X_2, Y_2) \oplus 0
\]

\( f \mapsto (f_{11}, f_{21}, f_{22}, 0) \).

Thus, we get an isomorphism

\[
H : \text{Hom}_\mathcal{C}(X, Y) \to \bigg[ \frac{\text{Hom}_{\mathcal{T}}(X_1, Y_1)}{\text{Hom}_{\mathcal{U}}(Y_2, X_1)} \bigg]. \quad f \mapsto \begin{bmatrix} f_{11} \\ f_{21} \\ f_{22} \end{bmatrix}.
\]

That is, we have an isomorphism \( H : \text{Hom}_\mathcal{C}(X, Y) \to \bigg[ \frac{T}{\text{Hom}_{\mathcal{U}}} \bigg](H(X), H(Y)) \).

We have the following immediate consequence of Proposition 7.1.

**Corollary 7.2.** Let \( Q = (Q_1, Q_0) \) be a quiver and \( KQ \) its path category. Given the sets \( T = \{ x \in Q_0 \mid x \text{ is a source} \} \) and \( U = Q_0 - T \), consider the subcategories of \( KQ \), \( \mathcal{U} = \text{add}(U) \) and \( \mathcal{T} = \text{add}(T) \). Then \((\mathcal{U}, \mathcal{T})\) is a splitting torsion theory in \( KQ \) and we have an equivalence of categories

\[
KQ \cong \bigg[ \frac{T}{\text{Hom}_{KQ} \mathcal{U}} \bigg]. \tag{6}
\]

As a concrete example, consider the following quiver \( Q = (Q_0, Q_1) \) with set of vertices \( Q_0 = \{ u_i, t_i : i \in \mathbb{Z} \} \). As above, if \( U = \{ u_i : i \in \mathbb{Z} \} \) and \( T = \{ t_i : i \in \mathbb{Z} \} \), and we consider \( \mathcal{U} = \text{add}(U) \) and \( \mathcal{T} = \text{add}(T) \), then we have an equivalence of categories given in (6).

\[
\cdots \quad u_{i-1} \quad \cdots \quad t_{i-1} \quad \cdots \quad u_i \quad \cdots \quad t_i \quad \cdots \quad u_{i+1} \quad \cdots \quad t_{i+1} \quad \cdots
\]

**Remark 7.3.** It is worth to mention that the previous construction works when we consider quivers \( Q \) with relations \( I = (\rho_i \mid i) \), since in this case the path category \( KQ/I \) is an additive and Krull-Schmidt \( K \)-variety.

Inspired in [53] we will describe a triangular matrix category \( \Lambda \) that comes from a path category and it is related to \( \text{Ch}(\text{Mod}(K)) \), the category of chain complexes in \( \text{Mod}(K) \), which is really important when doing homological algebra in \( \text{Mod}(K) \). In Section 5.1 in part II of this work we will show that for this triangular matrix category \( \Lambda \) the maps category of the category \( \text{Ch}(\text{Mod}(K)) \) coincides with \( \text{Mod}(\Lambda) \).

Let us construct the example. Consider \( \Lambda = (\Delta_0, \Delta_1) \) a quiver with \( \Delta_0 = \mathbb{Z} \) and \( \Delta_1 = \{ \alpha_i : i \to i + 1 \mid i \in \mathbb{Z} \} \), with the set of relations \( \rho = \{ \alpha_i \alpha_{i-1} \mid i \in \mathbb{Z} \} \)

\[
\cdots \quad i - 1 \xrightarrow{\alpha_{i-1}} i \xrightarrow{\alpha_i} i + 1 \xrightarrow{\alpha_i} \cdots
\]

and consider the path category \( \mathcal{C} = K\Delta / \langle \rho \rangle \). Then the category \( \text{Mod}(\mathcal{C}) \) is equivalent to \( \text{Ch}(\text{Mod}(K)) \).
On the other hand, let $\tilde{\Delta} = (\tilde{\Delta}_0, \tilde{\Delta}_1)$ be the quiver with $\tilde{\Delta}_0 = (\Delta_0 \times \{1\}) \cup (\Delta_0 \times \{2\})$ and $\tilde{\Delta}_1 = (\Delta_1 \times \{1\}) \cup (\Delta_1 \times \{2\}) \cup \{\beta_i : (i, 1) \rightarrow (i, 2)\}_{i \in \mathbb{Z}} \cup \{\gamma_i : (i, 1) \rightarrow (i + 1, 2)\}_{i \in \mathbb{Z}}$ with relations given by the set

$$\tilde{\rho} = \{(\alpha_{i+1}, 1)(\alpha_i, 1), (\alpha_{i+1}, 2)(\alpha_i, 2), (\alpha_i, 2)\beta_i - \gamma_i, \beta_{i+1}(\alpha_i, 1) - \gamma_i\}_{i \in \mathbb{Z}}.$$

Then we can construct the path category $K\tilde{\Delta}/\langle \tilde{\rho} \rangle$. We will show that considering $K\Delta/\langle \rho \rangle \otimes (K\Delta/\langle \rho \rangle)^{op} \rightarrow \text{Mod}(K)$ we get that the category $\Lambda = \left[ \begin{array}{ccc} K\Delta/\langle \rho \rangle & 0 \\ \text{Hom} & K\Delta/\langle \rho \rangle \end{array} \right]$ is equivalent to the category $K\tilde{\Delta}/\langle \tilde{\rho} \rangle$. First, we note that we have two inclusion functors $\Phi_1, \Phi_2 : K\Delta/\langle \rho \rangle \rightarrow K\tilde{\Delta}/\langle \tilde{\rho} \rangle$ defined as follows: for $i \in \Delta$ and $\alpha_i : i \rightarrow i + 1$ we set $\Phi_1(i) = (i, 1)$ and $\Phi_1(\alpha_i) = (\alpha_i, 1)$; and $\Phi_2(i) = (i, 2)$ and $\Phi_2(\alpha_i) = (\alpha_i, 2)$.

Now, we establish a functor $\Phi : \Lambda \rightarrow K\tilde{\Delta}/\langle \tilde{\rho} \rangle$ on objects by $\Phi\left(\left[ \begin{array}{ccc} i \\ \text{Hom} \\ j \end{array} \right] \right) = (i, 1) \oplus (j, 2)$, for all $i, j \in \mathbb{Z}$. In order to define $\Phi$ on morphisms, we note that

$$\text{Hom}_\Lambda\left(\left[ \begin{array}{ccc} i \\ \text{Hom} \\ j \end{array} \right], \left[ \begin{array}{ccc} i' \\ \text{Hom} \\ j' \end{array} \right] \right) = \left[ \begin{array}{ccc} \text{Hom}_{K\Delta/\langle \rho \rangle}(i, i') & 0 \\ \text{Hom}_{K\Delta/\langle \rho \rangle}(i, j') & \text{Hom}_{K\Delta/\langle \rho \rangle}(j, j') \end{array} \right]$$

and

$$\text{Hom}_{K\tilde{\Delta}/\langle \tilde{\rho} \rangle}\left(\left(\begin{array}{ccc} i, 1 \oplus (j, 2), (i', 1) \oplus (j', 2) \end{array}\right), \left(\begin{array}{ccc} i, 1 \oplus (j, 2), (i', 1) \oplus (j', 2) \end{array}\right)\right) = \left(\begin{array}{ccc} \text{Hom}_{K\tilde{\Delta}/\langle \tilde{\rho} \rangle}(i, 1), (i', 1) & 0 \\ \text{Hom}_{K\tilde{\Delta}/\langle \tilde{\rho} \rangle}(i, 1), (j', 2) & \text{Hom}_{K\tilde{\Delta}/\langle \tilde{\rho} \rangle}(j, 2), (j', 2) \end{array}\right) \right.$$}

for all $i, i', j, j' \in \mathbb{Z}$.

Note that $\text{Hom}_{K\Delta/\langle \rho \rangle}(i, i') = \text{Hom}_{K\Delta/\langle \rho \rangle}(j, j') = \text{Hom}_{K\Delta/\langle \rho \rangle}(i, j') = 0$ unless $i' = i, j' = j$, and $\text{Hom}_{K\Delta/\langle \rho \rangle}(i, 1), (i', 1)) = \text{Hom}_{K\Delta/\langle \rho \rangle}(i, 1), (j', 2)) = \text{Hom}_{K\Delta/\langle \rho \rangle}(j, 2), (j', 2)) = 0$ unless $i' = i, j' = j, i = 0, 1$.

We will define a morphism of abelian groups

$$\Phi : \text{Hom}_\Lambda\left(\left[ \begin{array}{ccc} i \\ \text{Hom} \\ j \end{array} \right], \left[ \begin{array}{ccc} i' \\ \text{Hom} \\ j' \end{array} \right] \right) \rightarrow \text{Hom}_{K\tilde{\Delta}/\langle \tilde{\rho} \rangle}\left(\left(\begin{array}{ccc} i, 1 \oplus (j, 2), (i', 1) \oplus (j', 2) \end{array}\right), \left(\begin{array}{ccc} i, 1 \oplus (j, 2), (i', 1) \oplus (j', 2) \end{array}\right)\right).$$

as follows:

- If $j' - i = 0$, and $i' - i, j' - j \in \{0, 1\}$. Consider

$$\left[ \begin{array}{ccc} \xi \\ \eta_j \end{array} \right] \in \text{Hom}_\Lambda\left(\left[ \begin{array}{ccc} i \\ \text{Hom} \\ j \end{array} \right], \left[ \begin{array}{ccc} i' \\ \text{Hom} \\ j' \end{array} \right] \right) = \left[ \begin{array}{ccc} \text{Hom}_{K\Delta/\langle \rho \rangle}(i, i') & 0 \\ \text{Hom}_{K\Delta/\langle \rho \rangle}(i, j') & \text{Hom}_{K\Delta/\langle \rho \rangle}(j, j') \end{array} \right],$$

since $\theta_i \in \text{Hom}_{K\Delta/\langle \rho \rangle}(i, i') = K\lambda_i$ we have that $\theta_i = \lambda \lambda_i$ for some $\lambda \in K$, thus we set

$$\Phi\left(\left[ \begin{array}{ccc} \xi \\ \eta_j \end{array} \right] \right) = \left[ \begin{array}{ccc} \Phi_1(\xi) & 0 \\ \lambda \beta_i & \Phi_2(\eta_j) \end{array} \right].
If \( j' - i = 1 \), and \( i' - i, j' - j \in \{0, 1\} \). Consider
\[
\begin{bmatrix}
\xi_i & 0 \\
\delta_i & \eta_j
\end{bmatrix} \in \text{Hom}_A \left( \left[ \begin{array}{c}
\tiny i \\
\text{Hom} \ j
\end{array} \right], \left[ \begin{array}{c}
\tiny i' \\
\text{Hom} \ j'
\end{array} \right] \right) = \text{Hom}_K A/\rho(i,i') \text{Hom}_K A/\rho(j,j+1),
\]
since \( \delta_i \in \text{Hom}_K A/\rho(i,i+1) = K\alpha_i \) we have that \( \delta_i = \lambda \alpha_i \) for some \( \lambda \in K \), thus we set
\[
\Phi \left( \begin{bmatrix}
\xi_i & 0 \\
\delta_i & \eta_j
\end{bmatrix} \right) = \begin{bmatrix}
\Phi_1(\xi_i) & 0 \\
\lambda \gamma_i & \Phi_2(\eta_j)
\end{bmatrix}.
\]
If \( j' - 1 \notin \{0, 1\} \). In this case, we set \( \Phi = 0 \).

In order to check that \( \Phi \) is a functor, consider the morphisms
\[
\begin{bmatrix}
\xi_i & 0 \\
\delta_i & \eta_j
\end{bmatrix} \in \text{Hom}_A \left( \left[ \begin{array}{c}
\tiny i \\
\text{Hom} \ j
\end{array} \right], \left[ \begin{array}{c}
\tiny i' \\
\text{Hom} \ j'
\end{array} \right] \right) = \text{Hom}_A \left( \left[ \begin{array}{c}
\tiny a_{i'} \\
\text{Hom} \ j'
\end{array} \right], \left[ \begin{array}{c}
\tiny 0 \\
\text{Hom} \ j
\end{array} \right] \right)
\]
Then, we have that
\[
\begin{bmatrix}
\xi_i & 0 \\
\delta_i & \eta_j
\end{bmatrix} \begin{bmatrix}
\xi_i & 0 \\
\delta_i & \eta_j
\end{bmatrix} = \begin{bmatrix}
\delta_i \cdot \xi_i & 0 \\
\delta_i \cdot \eta_j
\end{bmatrix} \begin{bmatrix}
\xi_i & 0 \\
\delta_i & \eta_j
\end{bmatrix} = \begin{bmatrix}
\delta_i \cdot \xi_i & 0 \\
\delta_i \cdot \eta_j
\end{bmatrix} \in \text{Hom}_A \left( \left[ \begin{array}{c}
\tiny a_{i'} \\
\text{Hom} \ j'
\end{array} \right], \left[ \begin{array}{c}
\tiny 0 \\
\text{Hom} \ j
\end{array} \right] \right).
\]
In order to prove that \( \Phi \) is a functor we have several cases. These cases are straightforward but bite tedious and we left the details to the reader. Then, \( \Phi : \Lambda \rightarrow K\widetilde{\Delta}/(\rho) \) is a functor. Now, it is easy to show that
\[
\Phi : \text{Hom}_A \left( \left[ \begin{array}{c}
\tiny i \\
\text{Hom} \ j
\end{array} \right], \left[ \begin{array}{c}
\tiny i' \\
\text{Hom} \ j'
\end{array} \right] \right) \rightarrow \text{Hom}_K A/\rho(i, 1) \oplus (j, 2), (i', 1) \oplus (j', 2)
\]
is an isomorphism of abelian groups. Since \( \Phi \) is clearly essentially surjective we conclude that \( \Phi \) is an equivalence. Hence, we have proved that the triangular matrix category \( \Lambda \) is equivalent to the path category \( K\widetilde{\Delta}/(\rho) \). In part II of this work, we will see that the maps category of the category \( \text{Ch} \text{(Mod}(K)) \) coincides with \( \text{Mod}(\Lambda) \) where \( \Lambda \) is the matrix triangular category constructed in this example.

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