The Problem of Modelling of Economic Dynamics in Differential Form

S. I. Chernyshov V. S. Ponomarenko A. V. Voronin
Contents

1 Analysis of Harrod’s model
   1.1 Computational relations ........................................ 3
   1.2 Refinement on the content of the model ......................... 4
   1.3 Improvement of the initial model ............................... 6
   1.4 Discrete essence of the initial model .......................... 8
   1.5 Inadequacy of the initial model ............................. 10

2 Other models
   2.1 Harrod-Domar’s model .......................................... 13
   2.2 Phillips’ model and other models .............................. 15
   2.3 Differential and difference models ............................. 18
   2.4 Abstracted model .............................................. 19
   2.5 Leontief’s model ............................................... 20

3 Constructive arguments
   3.1 Leontief’s differential model ................................. 23
   3.2 Nonlinear model and an alternative ........................... 25
   3.3 Representativeness of the integral model ................. 28
Introduction

The authors have tried to analyze procedures of the construction of differential equations that are employed for modeling of macroeconomic processes. The results prove to be rather unexpected. Thus, the derivation of the differential equation of Harrod’s model is based on a linear relation between capital and income. As a result, there arises a contradiction in terms of dimension that is rooted in incorrect treatment of the fundamental notion of the infinitesimal quantity. One can overcome this contradiction by relating capital to the integral of income over a corresponding time interval. However, in this case, the solution is by no means an exponential growth but a much more realistic relation that reflects, in particular, objective finiteness of the prognostic period.

An analysis of the models of Harrod-Domar, Phillips, as well as of other models (see the well-known treatise by R. Allen), leads us to the conclusion that analogous deficiencies are, in principle, inherent in these models too. In general, the refraction in the sphere of economic dynamics of the methodology of the construction of mathematical models borrowed from the field of natural sciences, such as dynamics, electrodynamics, etc., proves to be absolutely unjustified. As a matter of fact, differential equations adequate to these models follow naturally from the consideration of an infinitesimal element. However, as regards the problems of economics, such an approach is objectively senseless. Nevertheless, economics, in its turn, has intrinsic advantages from the point of view of possibilities of mathematical modeling, which is embodied in the notion of balance. As we will show, there exist formal means to reduce Leontief’s model of “expenses-output” in its canonical interpretation to a system of linear differential equation (of, generally speaking, arbitrary order with respect to the derivatives).

At the same time, the scantiness of the arsenal of the means of linear theory that are used in representative modeling of macroeconomic processes is almost universally recognized nowadays. In this regard, we will characterize briefly those areas of systems analysis that are devoted to the construction of non-linear models that are adequate to a given “input-output” mapping. In what follows, we nonetheless note that Leontief’s model in the differential form can be elementary reduced to a Fredholm integral equation of the second kind (with respect to a vector function), whose theory and algorithms of numerical realization are as constructive as possible. In the case, when the kernel of such an equation depends on a parameter, which is quite naturally interpreted in terms of the object sphere, the spectrum of its possible solutions becomes extremely wide. We think that the development of the theory of Fredholm integral equations of the second kind, whose kernels contain parameters, and its application to the modeling of the processes of economic dynamics is rather promising.

Note on literature references in the English version: The reader should be advised that all the references to page and section numbers appearing in the text are given according to the Russian editions of corresponding literature sources.

Contact address: voronin@hmeu.edu.ua
Chapter 1

Analysis of Harrod’s model

1.1 Computational relations

Let us turn to Harrod’s model of the development of the economy represented in the book by L. V. Kantorovich and A. B. Gorstko \[1\] (pp.160, 161). In the authors’ view, ”despite the simplicity - the model takes into account only one limited factor, i.e., the capital - it can be used for a rude approximate investigation into the laws of the growth of the economy”. In the latter context, we focus our attention on the above-mentioned exclusiveness of the capital. The employed notation are explained as follows: \( Y(t) \) is the national income; \( K(t) \) is the capital (the assets); \( C(t) \) is the volume of consumption; \( S(t) \) is the volume of accumulation, and \( I(t) \) is the investment. All the quantities are dimensional. For the analysis of the model, it is reasonable to reproduce the text \[1\].

”It is assumed that the economy is functioning in such a way that the following relations are fulfilled:

\[
Y(t) = C(t) + S(t) \tag{1.1}
\]

(the national income is distributed between accumulation and consumption);

\[
S(t) = I(t) \tag{1.2}
\]

(the accumulation is equal to the investment);

\[
S(t) = \mu Y(t), \tag{1.3}
\]

where \( \mu = const \) (the accumulation constitutes a constant fraction of the national income);

\[
\dot{K}(t) = I(t) \tag{1.4}
\]

(the growth rate of the capital is equal to the investment, where \( \dot{K} = dK/dt \));

\[
K(t) = \nu Y(t), \tag{1.5}
\]
where $\nu = \text{const}$ (the capital-to-national income ratio is a constant quantity, which is an observed empirical fact).

From the equations that describe the model, it follows immediately:

$$Y(t) = \frac{1}{\mu} I(t) = \frac{1}{\mu} \dot{K}(t) = \frac{\nu}{\mu} \dot{Y}(t)$$  \hspace{1cm} (1.6)

or

$$\dot{Y}(t)/Y(t) = \mu/\nu.$$  \hspace{1cm} (1.7)

Consequently, the growth rate of the national income is equal to $\mu/\nu$, and, therefore, if one assumes that at the initial moment of time (at $t = 0$) the national income is equal to $Y_0$, the law of its change with time has the following form:

$$Y(t) = Y_0 e^{\mu t/\nu},$$  \hspace{1cm} (1.8)

which usually stands in quite good agreement with practice”.

In obtaining relations (1.6) and differential equation (1.7), a key role is played by the dependence

$$K(t) = \int_{-T}^{t} I(\eta) d\eta$$  \hspace{1cm} (1.9)

that is obvious. Here, the capital $K(t)$ can be measured only monetary equivalent whose unit will be chosen to be the dollar ($) and $t$ is the time variable measured, for definiteness, in seconds (s). Given that $dt$ has the same dimension as $t$, the function $I(t)$ represents the intensity of the investment flow measured as $$/s$. A corollary of (1.9) is just the derivative (1.4): there is simply no other derivative in the model (1.1)-(1.5).

On the basis of (1.1)-(1.3), we can conclude that $Y(t), C(t) \text{ and } S(t)$ are also intensities of the flows of the income, consumption, and accumulation ($$/s), respectively. However, there arises a contradiction, because $Y(t)$ in (1.5) is the national income per one-year period measured in $. Analogously, $t$ in (1.8) denotes the number of years (their fractions). Therefore, the exponent $\mu t/\nu$ is dimensionless. Note that we had to carry out an investigation into the dimension of the components of the model. For the reason that will be explained in what follows, one does not specify clearly the dimension of employed quantities in macroeconomics of mathematical orientation. However, this issue underlies the most important point of the whole consideration!

1.2 Refinement on the content of the model

As $Y(t)$ is the intensity of the income, relation (1.5) had to look as follows

$$K(t) = \nu \int_{t_i}^{t_i+t_*} Y(\eta) d\eta, \quad t \in [t_i, t_i+t_*],$$  \hspace{1cm} (1.10)
where \( t^* \) is the time interval of one year; \( \nu \) is the number of years during which such an income counterbalances the capital of the \( i \)-th year.

From a mathematical point of view, such a specification is of no great importance: what is actually important is the fact that \( t^* \) is finite. Behind this fact, one can easily see a transition to discrete analysis and the impossibility of deriving any differential equation! This major objective of the present investigation will be developed below.

At the same time, one can express the capital \( K(t) \) via the intensity of the income \( Y(t) \) in a way that is different from what is done in (1.10): namely, based on (1.2) and (1.3), one can employ an analog of (1.9). Consider relation (1.5) strictly adhering to the definition [1]: "\( \nu = \text{const} \) (the capital-to-national income ratio is a constant quantity, which is an observed empirical fact)." In this regard, let us draw attention to the informativeness of the coefficient \( \nu \) (in contrast, \( \mu \) is purely primitive). The essence is that the "observed fact" characterizes a basic coefficient \( \nu^* \) for the time interval \( t^* \) during which the income is measured. For \( t^* = 1 \) year it is reasonable to set \( \nu^* = 10 \). Accordingly, for a certain time interval \( \Delta t \), the coefficient is given by

\[
\nu = \nu^* t^*/\Delta t. \tag{1.11}
\]

At that, the flow \( Y(t) \) (/s) creates, in the period of time \( \Delta t \), the volume of the income ($) equal to

\[
\int_{t}^{t+\Delta t} Y(\eta) \, d\eta \tag{1.12}
\]

that, by the definition [1], is proportional to the capital \( K(t+\Delta t) \) with the coefficient \( \nu \); see (1.11). Accordingly,

\[
K(t+\Delta t) = \nu \int_{t}^{t+\Delta t} Y(\eta) \, d\eta; \quad K(t+2\Delta t) = \nu \int_{t}^{t+2\Delta t} Y(\eta) \, d\eta.
\]

After differentiation,

\[
\dot{K}(t+\Delta t) = \nu [Y(t+\Delta t) - Y(t)]; \quad \dot{K}(t+2\Delta t) = \nu [Y(t+2\Delta t) - Y(t)],
\]

and, as a result of subtraction,

\[
\dot{K}(t+2\Delta t) - \dot{K}(t+\Delta t) = \nu [Y(t+2\Delta t) - Y(t+\Delta t)].
\]

Assuming that the interval \( \Delta t \) is sufficiently small, we use a Taylor expansion:

\[
\dot{K}(t+\Delta t) = \ddot{K}(t) + \dddot{K}(t) \, \Delta t + \ldots; \quad \dot{Y}(t+\Delta t) = \ddot{Y}(t) + \dddot{Y}(t) \, \Delta t + \ldots,
\]

where we have preserved the terms containing the first power of \( \Delta t \) and \( 2\Delta t \). We get:

\[
\dddot{K}(t) = \nu \dddot{Y}(t). \tag{1.13}
\]
Thus, the considered model is reduced to the set of relations (1.1)-(1.4) and (1.13). From these relations, it follows
\[ K'(t) = \mu \dot{Y}(t) = \nu \dot{Y}(t), \]
which just yields \( \mu = \nu \), or, in the case \( \mu \neq \nu \),
\[ Y(t) = c/(\mu - \nu), \]
where \( c \) is a constant. As regards the derivative \( \dot{Y}(t) \) that, as in (1.6), leads to a differential equation, it has treacherously escaped. In general, the solution to the considered problem by means of (1.12) can only be trivial.

In other words, in order to equate the function \( Y(t) \) (with the coefficient) to its derivative in (1.7), obtaining the exponential growth (1.8), it was absolutely necessary to resort to the surrogate (1.5) instead of (1.10). However, the "derivative" can be understood. Having used (1.12), we have essentially obtained an analog of the interrelation between the capital and the flow (1.9) without complementing the model at the level of a quantitative content. These arguments are of heuristic nature.

1.3 Improvement of the initial model

Let us act in a different way, namely, by considering the time interval from 0 to \( t \). A priori, we assume that \( \Delta t \) is small. The income during this period is equal to
\[ \int_0^t Y(\eta) \, d\eta, \]
and, by analogy with (1.11),
\[ \nu = \nu_* t_*/t; \quad (1.14) \]
however, the assumption that \( \nu = \text{const} \) in relation (1.6) should be rejected. Indeed, it contains a contradiction resulting from the fact that \( \nu \) is related to a year [1] whose duration \( t_* \) is related to (1.10). In order to remain within the framework of continuous analysis, we just use the rule of proportion. Thus,
\[ K(t) = \nu \int_0^t Y(\eta) \, d\eta = \frac{\nu_* t_*}{t} \int_0^t Y(\eta) \, d\eta. \quad (1.15) \]

By the way, for \( t \to 0 \), the coefficient \( \nu \to \infty \), which is quite reasonable. Simultaneously, equation (1.15) takes the form \( K(0) = \nu_* t_* Y(0) \), and it can be easily compared to (1.5) and (1.10) with regard to dimension. From (1.15), it follows that
\[ \dot{K}(t) = -\frac{\nu_* t_*}{t^2} \int_0^t Y(\eta) \, d\eta + \frac{\nu_* t_*}{t} Y(t), \quad (1.16) \]
and, obviously, \( \dot{K}(t) \to \infty \) for \( t \to 0 \). By (1.3), \( I(t) \to \infty \) as well. This peculiarity will be eliminated in what follows by the cancellation of \( t \).

From relations (1.2), (1.3), we get

\[
\dot{K}(t) = \mu Y(t),
\]

which, combined with (1.10), yields the equation

\[
\left(1 - \frac{\mu t}{\nu_* t_*}\right) \dot{Y}(t) = \frac{2\mu}{\nu_* t_*} t Y(t), \quad t > 0.
\]

Making a change of the variable

\[
\hat{t} = t/t_*,
\]

by

\[
\frac{dY(t)}{dt} = \frac{dY(\hat{t})}{d\hat{t}} \frac{d\hat{t}}{dt} = \dot{Y}(\hat{t})/t_*,
\]

we get

\[
\dot{Y}(\hat{t}) - \frac{2\sigma}{1 - \sigma \hat{t}} Y(\hat{t}) = 0, \quad \sigma = \frac{\mu}{\nu_*}.
\]

The solution to this equation has the form

\[
Y(\hat{t}) = \frac{Y_0}{(1 - \sigma \hat{t})^\sigma}.
\]

On this basis, the following conclusions can be drawn:

- if \( \mu = 0 \) (the absence of investment) or \( \nu_* \to \infty \) in (1.18), then \( Y(\hat{t}) = Y_0 \), which is not unreasonable;
- the time interval of a reasonable forecast is reflected, because for \( \hat{t} = \sigma^{-1} = \nu_*/\mu \) the income function (1.19) becomes senseless;
- generally speaking, such a forecast is inherent in a reliable model, which should be contrasted with an infinite growth of the function (1.8);
- under the interpretation that \( \mu = 0.5 \) (consumption and accumulation equally share the income) and \( \nu_* = 10 \) years, the period of a conditionally reliable forecast is also equal to 10 years, if it is set equal to \( 0.5\sigma^{-1} \). This seems to be realistic.

It should be emphasized that \( Y_0 \) in (1.19) is the intensity of the income for \( t = 0 \) (\$/s) rather than an income per year as in (1.8). Note that the coefficient \( \nu \) (as the number of years) is interpreted in sections 1.1 and 1.2, 1.3 as dimensional and dimensionless, respectively. This fact is a consequence of the observed in section 1.1 contradiction between the interpretation of \( Y(t) \) as the intensity of the income, according to (1.1)–(1.3), and the volume of the income, according to (1.5).

Thus, based on (1.14), we have obtained a rather satisfactory result. It is stipulated by qualitative different dependence of the capital \( K(t) \) on the flows \( I(t) \) and \( Y(t) \): see (1.9) and (1.15), respectively. Namely, there is a variable coefficient \( t^{-1} \) in (1.15).
1.4 Discrete essence of the initial model

Under the interpretation of \( Y (t), C (t), S (t), \) and \( I (t) \) as flows \(($/s)\), the model is represented by relations (1.1)–(1.4) and (1.10). Using the definite integral in its simplest interpretation, we can represent (1.10) as follows:

\[
K (t) = \nu \int_{t_i}^{t} Y (\eta) d\eta = \nu t_s Y (t_i), \quad t \in [t_i, t_i + t_s], \quad i = 0, 1, \ldots
\]  

(1.20)

A change of the variables, by (1.17), leads to the following relations:

\[
\begin{align*}
Y (\hat{t}) &= C (\hat{t}) + S (\hat{t}); \quad S (\hat{t}) = I (\hat{t}); \quad S (\hat{t}) = \mu Y (\hat{t}); \\
\dot{K} (\hat{t}) &= t_s I (\hat{t}); \\
K (\hat{t}) &= \nu t_s Y (\hat{t}_i); \quad t \in [\hat{t}_i, \hat{t}_i + 1], \quad \hat{t}_i = 0, 1, \ldots
\end{align*}
\]  

(1.21–1.23)

where \( Y (\hat{t}), C (\hat{t}), S (\hat{t}), \) and \( I (\hat{t}) \) are intensities of the flows measured in $/s.

There is an obvious contradiction related to the fact that the functions in (1.21)–(1.23) cannot, on the one hand, be discrete and, on the other hand, represent intensities of continuous flows. However, it is impossible to satisfy (1.23) continuously, within the framework of the model (1.1)–(1.4): see section 1.2. By analogy with (1.23), one is only left with the option to represent relations (1.21) and (1.22) in a discrete form:

\[
\begin{align*}
\hat{Y} (\hat{t}_i) &= \hat{C} (\hat{t}_i) + \hat{S} (\hat{t}_i); \quad \hat{S} (\hat{t}_i) = \hat{I} (\hat{t}_i); \quad \hat{S} (\hat{t}_i) = \mu \hat{Y} (\hat{t}_i); \\
\dot{\hat{K}} (\hat{t}_i) &= \hat{I} (\hat{t}_i),
\end{align*}
\]  

(1.24–1.25)

where

\[
\begin{align*}
\hat{Y} (\hat{t}_i) &= t_s Y (\hat{t}_i); \quad \hat{C} (\hat{t}_i) = t_s C (\hat{t}_i); \quad \hat{S} (\hat{t}_i) = t_s S (\hat{t}_i); \quad \hat{I} (\hat{t}_i) = t_s I (\hat{t}_i), \\
&\quad i = 0, 1, \ldots, n
\end{align*}
\]  

(1.26)

have the dimension of capital, which is quite unambiguously said in [1]: see section 1.1.

At the same time, under inaccurate treatment, such an approach stands in disagreement with the rules of the calculus of infinitesimal that form the basis of differential models. Suppose that investments in years \( \hat{t}_i \) and \( \hat{t}_i + 1 \) are given by \( I_i \) and \( I_{i+1} \), respectively. Then, the capital is

\[
K_i = K_{i-1} + I_i; \quad K_{i+1} = K_{i-1} + I_i + I_{i+1}.
\]

However, the definition \( \dot{K} (\hat{t}_i) = I_{i+1} - I_i \) in discrete analysis is illegitimate.

Should the techniques of continuous analysis be formally extended to a transformation of relations (1.23)–(1.25), then, as it is said about (1.6) (see section 1.1), it "follows immediately" that

\[
\frac{d\hat{Y}}{dt} (\hat{t}_i) / \hat{Y} (\hat{t}_i) = \mu / \nu,
\]  

(1.27)
and, accordingly,

\[ \tilde{Y}(\hat{t}_i) = \tilde{Y}_0 e^{\hat{t}_i/\nu}, \]  

(1.28)

which should be exactly the interpretation of the solution (1.8).

Thus, all the functions in (1.23)-(1.25) are discontinuous at \( \hat{t} = \hat{t}_i \). Otherwise, the system of these relations is degenerate, and the solution is trivial. As a matter of fact, discreteness stipulates a combination of factors: \( Y(t) \neq Y_0 \); the interval \( t_* \) in (1.20) is finite.

The function \( K(\hat{t}) \) is also discontinuous, because, by (1.9), (1.17) and (1.26),

\[ K(\hat{t}) = \int_{-T/t}^{\hat{t}} \hat{I}(\eta) \, d\eta = K_0 + \sum_{i=0}^{n} \hat{I}(\hat{t}_i), \quad 0 \leq \hat{t} < \hat{t}_{n+1}, \]  

(1.29)

where

\[ K_0 = \int_{-T/t}^{0} \hat{I}(\eta) \, d\eta. \]

As a consequence, this function is not differentiable in a usual sense at \( \hat{t} = \hat{t}_i \).

However, by the existence of (1.28), the discontinuities of \( K(\hat{t}) \) at \( \hat{t} = \hat{t}_i \) in (1.29) are finite. Namely, they are analogous to discontinuities of the function \( Y(\hat{t}) \) with the coefficient \( \nu \). Summarizing, we arrive at the conclusion that the function (1.29) can be understood only as a generalized function, and, accordingly \([2]\) (pp. 57-60),

\[ \dot{K}(\hat{t}) = \sum_{i=0}^{n} \hat{I}(\hat{t}_i) \delta(\hat{t} - \hat{t}_i), \]  

(1.30)

where \( \delta(\hat{t}) \) is Dirac’s delta-function, such that

\[ \delta(\hat{t}) = \begin{cases} 0, & \hat{t} \neq 0; \\ \infty, & \hat{t} = 0; \end{cases} \quad \int_{-\infty}^{\infty} \delta(\eta) \, d\eta = 1. \]

By analogy with (1.30) and (1.28), in (1.27),

\[ \tilde{Y}(\hat{t}) = \frac{1}{\nu} \left[ K_0 + \sum_{i=0}^{n} \hat{I}(\hat{t}_i) \right], \quad 0 \leq \hat{t} < \hat{t}_{n+1}, \]

and the functions \( \tilde{C}(\hat{t}), \tilde{S}(\hat{t}), \) and \( \tilde{I}(\hat{t}) \) in (1.24) and (1.25) have the same structure: in other words, all of them are generalized functions. At that, the ordinary differential equation with constant coefficients (1.27) has, indeed, a solution of the form (1.8) in the class of generalized functions \([3]\) (pp. 60, 61).
Can one, based on these arguments, draw a conclusion that
\[
\check{Y} (\hat{t}) = Y (\hat{t}) ,
\]  
(1.31)
where \( Y (\hat{t}) \) is determined by expression \( \text{(1.28)} \), and that the model \( \text{(1.1)}-\text{(1.5)} \) objectively reflects the dynamics of the macroeconomic growth?

### 1.5 Inadequacy of the initial model

In the explanation to \( \text{(1.5)} \), it is said: "the capital-to-national income ratio is a constant quantity" [1]. However, this fact predetermines the solution of the problem. Indeed, for \( \hat{t} = 0, 1, \ldots, \) we have, respectively: \( K_0, Y_0 = K_0 / \nu; \)

\[
I_0 = K_0 \mu / \nu; \quad K_1 = K_0 (1 + \alpha); \quad \check{Y}_1 = K_1 / \nu; \quad \check{I}_1 = K_1 \mu / \nu; \quad \alpha = \mu / \nu;
\]

\[
K_2 = K_0 [1 + (1 + \alpha) \alpha] = K_0 \left(1 + \alpha + \alpha^2 \right); \quad \check{Y}_2 = K_2 / \nu; \quad \check{I}_2 = K_2 \mu / \nu; \ldots;
\]

\[
K_n = K_0 \sum_{i=0}^{n} \alpha^i; \quad \check{Y}_n = K_n / \nu; \quad \check{I}_n = K_n \mu / \nu. \quad (1.32)
\]

As \( \alpha < 1 \) (this follows from the content of the model, which is, mathematically, insignificant), using the formula for a decreasing geometrical progression, we get:

\[
\check{Y}_n = K_0 \frac{1 - \alpha^{n+1}}{\nu (1 - \alpha)} = \check{Y}_0 \frac{1 - \alpha^{n+1}}{1 - \alpha} = \check{Y}_0 \frac{1 - (\mu / \nu)^{n+1}}{1 - \mu / \nu}, \quad n = 0, 1, \ldots. \quad (1.33)
\]

Such a result contradicts the solution \( \text{(1.28)} \) that, for \( \hat{t} = \hat{t}_n = n \), has the form

\[
\check{Y}_n = \check{Y}_0 e^{\alpha n} = \check{Y}_0 e^{\mu n / \nu}, \quad (1.34)
\]

because the equality of expressions \( \text{(1.33)} \) and \( \text{(1.34)} \) would mean that

\[
e^{\alpha n} = \frac{1 - \alpha^{n+1}}{1 - \alpha}, \quad \alpha n = \ln \frac{1 - \alpha^{n+1}}{1 - \alpha} \quad (1.35)
\]

for any \( n = 1, 2, \ldots \) and \( 0 < \alpha < 1 \), whereas the cases \( n = 0 \) and \( \alpha = 0 \) are trivial.

Using an analog of \( \text{(1.35)} \) for \( \hat{t} = \hat{t}_{n+1} \), we arrive at the equation

\[
\alpha = \ln \frac{1 - \alpha^{n+1}}{1 - \alpha^n} \quad (1.36)
\]

that is satisfied by \( \alpha = 1 \). However, in this case, \( \text{(1.32)} \) does not represent any progression, and, instead of \( \text{(1.33)} \) and \( \text{(1.35)} \), we get:

\[
\check{Y}_n = (n + 1) \check{Y}_0; \quad e^n = n + 1.
\]
This equation cannot be satisfied for any \( n = 1, 2, \ldots \), and it becomes apparent that the values of \( \tilde{Y}_n \) determined by formulas (1.33) and (1.34) are cardinally different. Here, one cannot assume any approximation of the exponential dependence (1.34) by means of the rational representation (1.33): see [4] (pp. 19-24).

As a matter of fact, the same conclusion can be easily drawn by substituting concrete values of \( n \) and \( \alpha \) into (1.36).

However, a contradiction of the considered model is obvious:
- on the one hand, the function \( \tilde{Y} (\hat{t}) \) determined by expression (1.28) satisfies equation (1.27);
- on the other hand, the same function in the form (1.34) is rather remote from (1.33) that is, in essence, a programmed solution of the same problem in the difference formulation.

Certainly, an explanation is related to the fact that both the functions \( \tilde{Y} (\hat{t}) \) from section 1.4 and, accordingly, \( Y (t) \) in (1.7) and (1.8) can only be generalized functions. In this situation, the equality (1.33) is impossible in principle, because, in reality, \( \tilde{Y} (\hat{t}) \) is a number appearing as a result of an operation on the function \( Y (\hat{t}) \). In other words, there exists only a correspondence of the form
\[
\tilde{Y} (t) \sim \int_0^{t_n} Y (\eta) \varphi (\eta) d\eta,
\]
where \( \varphi (\hat{t}) \) is an infinitely differentiable functions on the interval \([0, \hat{t}_n] \) except for its boundaries: it is called finite (alias trial or support) function.

For example,
\[
\varphi (\hat{t}) = \begin{cases} \sin (\pi \hat{t}/\hat{t}_n), & \hat{t} \in [0, \hat{t}_n] \\ 0, & 0 > \hat{t} > \hat{t}_n, \end{cases}
\]
\( \varphi (\hat{t}_n/2) = 1. \) Moreover, the derivative in (1.4) is understood as follows [3] (p. 34):
\[
\int_0^{t_n} K (\eta) \varphi (\eta) d\eta = - \int_0^{t_n} K (\eta) \varphi (\eta) d\eta.
\]

As regards expressions (1.8) and (1.28), they can be associated only with the function \( \tilde{Y} (\hat{t}) \) from (1.37), which by no means facilitates evaluation of \( Y (\hat{t}) \) at the points \( \hat{t} = \hat{t}_n \). "But how can one define the integral of the product of a generalized function and a trial function, if one cannot work with values of the function at separate points? The answer is simple: in this case, one should define the integral axiomatically rather than constructively" [5] (pp. 10, 11). According to an alternative theory [6], generalized functions are introduced as specially defined limits of series of continuous functions. From the point of view of reanimating the model (1.1)-(1.5), they are also useless.

I. M. Gelfand and G. E. Shilov point out: "In the solution of concrete problems of mathematical physics, the delta-function (as well as other singular functions) appear, as a rule, only at Intermediate stages. In the final answer,
singular functions are either altogether absent or figure under the sign of integration in a product with some sufficiently good function. Thus, there is no direct need to answer the question what a singular function is in itself. It is sufficient to answer the question what the integral of a product of a singular function and a 'good' function is. ... In other words, we relate any singular function to a functional that puts this singular function and any 'sufficiently good' function into correspondence with a certain number” [3] (pp. 14, 15).

However, why should we at all mention generalized functions, if their techniques are useless in our case? The whole point is that, only for these functions, the transformations leading to equation (1.7) and, accordingly, its solution are legitimate. The solution of (1.7), understood in the classical sense, is continuous: see, e.g., [7] (pp.28, 29). At the same time, relations (1.1)-(1.5) are fundamentally irresolvable in the class of continuous functions. One can draw a conclusion that they are inadequate to the process of the macroeconomic growth that is the subject of the model! What is more, the following arguments prove to be completely unsuitable:

- macroeconomics operates the scales of decades;
- against this background, an income per year is only a small step;
- the function (1.8) smooths out such steps, which reflects the dynamics of growth on the whole.

However, we have here not only just "steps": as a matter of fact, they induce a cardinal change of the type of the equation inherent in the problem. The inadequacy of the problem is stipulated, in the first place, by relation (1.5) that provides it with the property of discreteness. In this case, the function \( Y(t) \) in (1.1)-(1.5) embodies fundamentally different qualities: the intensity of the flow and its scale on a finite time interval. The role of relation (1.5) is rather pragmatic: namely, as the derivative (1.4) is objective, this relation relates it in a proportional way to the function, which leads to the differential equation (1.7).
Chapter 2

Other models

2.1 Harrod-Domar’s model

In the work by R. Allen [8] (pp. 75-78), this model and the resulting dynamics of the growth are represented in a dimensionless form:

\[ \ddot{Y}(\tilde{t}) = \dot{C}(\tilde{t}) + \ddot{I}(\tilde{t}); \quad \dot{C}(\tilde{t}) = (1 - \mu) \ddot{Y}(\tilde{t}); \quad \dot{I}(\tilde{t}) = \nu \frac{d\ddot{Y}(\tilde{t})}{dt}, \quad (2.1) \]

where \(0 < \mu < 1; \nu > 0\). The author emphasizes that \(\ddot{Y}(\tilde{t}), \dot{C}(\tilde{t}),\) and \(\ddot{I}(\tilde{t})\) are “functions of continuously changing time”. The solution of the problem is obtained in the form

\[ \ddot{Y}(\tilde{t}) = \dot{Y}_0 e^{\mu \tilde{t}/\nu}. \quad (2.2) \]

For greater clearness, it is reasonable to transform (2.1) and (2.2) into dimensional notation:

\[ t = t_0 \tilde{t}; \quad Y(t) = Y_0 \ddot{Y}(\tilde{t}); \quad C(t) = C_0 \dot{C}(\tilde{t}); \quad I(t) = I_0 \ddot{I}(\tilde{t}), \quad (2.3) \]

where \(Y_0, C_0,\) and \(I_0\) are the intensities of corresponding flows (\$/s).

We get:

\[ Y(t) = k_1 C(t) + k_2 I(t); \quad k_1 C(t) = (1 - \mu) Y(t); \quad (2.4) \]

\[ k_2 I(t) = \nu t_0 \dot{Y}_0, \quad (2.5) \]

where \(k_1 = Y_0/C_0; k_2 = Y_0/I_0,\) which leads to the differential equation

\[ \nu t_0 \ddot{Y}(t) = \mu Y(t), \quad (2.6) \]

whose solution is

\[ Y(t) = Y_0 e^{\mu t/\nu t_0}. \quad (2.7) \]

However, let us draw attention to the fact that the confusion of double-faced character of the function \(Y(t)\) from the previous section has disappeared, and
the situation has become completely transparent. Indeed, all the functions in (2.4) are intensities of the flows; one cannot raise any objections against the derivation of equation (2.6) and its solution (2.7).

Nevertheless, expression (2.7) constitutes a clear proof of the inadequacy of this model. The main point is the dependence of $Y(t)$ on the dimension of the parameter $t_0$ from (2.3) that, by definition, is chosen arbitrarily. In other words, a mathematical model, under the condition of its reliability, cannot depend on the scale of time.

Accordingly, relations (2.4) and (2.5) do not contain objective meaning. Generally speaking, the first two of them are quite elementary, whereas (2.5), the so-called accelerator, essentially introduces a derivative. On can assume that, in this case, the problem was solved the other way, namely, in relationship to obtaining differential relations of the form (1.6).

Note that the model (2.1) does not employ the function $K(t)$, the capital, whose special role is pointed out at the beginning of section 1.1 with reference to [1]. Indeed, the dependence (1.9), as well as its corollary (1.4), can be called fundamental. The substitution of $I(t)$ from (1.4) into (2.5) yields:

$$k_2 \dot{K}(t) = \nu t_0 \dot{Y}_0,$$

from which we get

$$K_1(t) = (\nu/k_2) t_0 Y_1(t),$$

(2.8)

where

$$K_1(t) = K(t) - K_0; \ Y_1(t) = Y(t) - Y_0$$

are, respectively, an increase of the capital during the period of time from 0 and $t$ and the intensity of the income at the moment $t$.

How can the capital in (1.8) be equal to the product (with a certain coefficient) of the intensity of the income at the final moment of the term of accumulation and an arbitrary period of time? However, relation (2.8) is, practically, an analogue of (1.5) if one assumes $t_0 = t^* = 1$ year. Accordingly, there appears discreteness (see section 1.4) that in the same way leads to a conclusion that the considered model is inadequate (see section 1.5)!

It should be noted that R. Allen himself does not touch on the issue of dimension. It can be firmly grasped only on the basis of numerical examples. In this sense, the situation with the model of section 1.1 was more complicated. By definition [8] (p. 193),

$$\bar{I}(\bar{t}) = \frac{d\bar{K}(\bar{t})}{d\bar{t}}.$$

(2.9)

Accordingly, for $\dot{K}(\bar{t}) = K(t)/K_0$, by analogy with (2.8), to reduce (2.9) to the form (1.4), it is necessary that $t_0 = K_0/I_0$. In this case, however, instead of the coefficient $\nu$, we get $\mu \nu^2$ in (1.5). If only the variable $t$ is dimensionless, instead of (1.4), we get a relation of the type (1.22). This list can be continued.

As is pointed out by T. Puu [9] (p. 76), exactly R. Harrod is the author of the idea of the formulation of the macroeconomic model in continuous time by means of a differential equation (1948). At the same time, an analysis of
the basic work by P. Harrod [10], also published in 1948, does not confirm this fact. Computational relations in this work are given in a discrete form. On the basis of the same approach the model of Harrod, as well as the models of Domar, of Solow, and of Samuelson and Hicks, are considered by A. Lusse [11] (pp. 163-166, 172-179, 180-182). L. Stoleru is of the same opinion about Harrod’s model [12] (pp. 272-277). On the whole, one can assume that it is not R. Harrod to whom dubious merit of inventing the above-discussed models should be attributed.

2.2 Phillips’ model and other models

$K (\bar{t})$ is also absent from the accelerator-multiplier model of Phillips considered by R. Allen. We restrict ourselves to the equations of the general solution [8] (pp. 81, 82):

$$\frac{d \bar{I} (\bar{t})}{d \bar{t}} = -\kappa \left[ \bar{I} (\bar{t}) - \nu \frac{d \bar{Y} (\bar{t})}{d \bar{t}} \right] ; \quad \bar{Z} (\bar{t}) = (1 - \mu) \bar{Y} (\bar{t}) + \bar{I} (\bar{t}) ;$$  \hspace{1cm} (2.10)

$$\frac{d \bar{Y} (\bar{t})}{d \bar{t}} = -\lambda \left[ \bar{Y} (\bar{t}) - \bar{Z} (\bar{t}) \right],$$

where, in addition to the functions employed above, $\bar{Z} (\bar{t})$ is the intensity of cumulative demand. The coefficients are specified as follows:

- $\kappa$ is the rate of reaction, i.e., the inverse of a constant lag of investment;
- $\nu$ is the factor of the accelerator power;
- $\mu$ is the multiplier;
- $\lambda$ is the rate of the influence of the production (income) on the demand.

(Note that, in contrast to the model [1] where in five relations we had only one derivative, here, in only three relations we have three derivatives.)

In the variables (2.3), the system of equations (2.10) takes the form

$$t_0 \dot{I} (t) = -\kappa \left[ I (t) - \left( \nu t_0/k_2 \right) \bar{Y} (t) \right] ; \quad k_3 \bar{Z} (t) = (1 - \mu) Y (t) + k_2 I (t) ;$$  \hspace{1cm} (2.11)

$$t_0 \bar{Y} (t) = -\lambda [\bar{Y} (t) - \bar{Z} (t)],$$

where $Z (t) = Z_0 \bar{Z} (\bar{t})$; $k_3 = Y_0/Z_0$.

Changing the variables according to (2.17), we reduce the problem to the solution of the second-order ordinary differential equation

$$\ddot{Y} (t) + a \dot{Y} + b Y (t) = 0,$$  \hspace{1cm} (2.12)

where $a = a_1/\rho$; $b = b_1/\rho^2$,

$$a_1 = \kappa + \mu \lambda - \kappa \nu \lambda; \quad b_1 = \kappa \nu \lambda; \quad \rho = t_0/t_*. $$

It has the form

$$Y (\bar{t}) = c_1 e^{p_1 \bar{t}} + c_2 e^{p_2 \bar{t}},$$  \hspace{1cm} (2.13)

15
where $c_1$ and $c_2$ are arbitrary constants;

$$p_{1,2} = \frac{1}{2\rho} \left[ a_1 \pm \sqrt{a_1^2 - 4b_1} \right]$$

are the roots of the quadratic equation $p^2 + ap + b = 0$.

As in section 2.1 the transformations of the flows are quite correct. However, the situation is completely analogous to that considered in section 2.1. Because of the presence of the parameter $\rho = t_0/t_*$ in (2.13), the solution depends on the choice of the scale of time, which makes the model inadequate. Otherwise, with the help of $\rho$, we could a priori set the periods of oscillations.

Using (1.4), we express $I(t)$ in (2.11) via $K(t)$:

$$t_0 \ddot{K}(t) = -\kappa \left[ \dot{K}(t) - (\nu t_0/k_2) \dot{Y}(t) \right]; \quad k_3 Z(t) = (1 - \mu) Y(t) + k_2 \ddot{K}(t),$$

which yields

$$t_0 \ddot{Y}(t) = -\lambda [Y(t) - k_3 Z(t)],$$

As a result of the substitution

$$Y(t) = -\frac{k_2}{k_2 \nu \mu \lambda} \left[ t_0 \ddot{K}(t) + \kappa (1 - \nu \lambda) \dot{K}(t) \right]$$

and some simple transformations, we get

$$t_0^2 \dddot{K}(t) + t_0 (\kappa + \mu \lambda - \kappa \nu \lambda) \ddot{K}(t) + \kappa \nu \lambda \dot{K}(t) = 0. \quad (2.14)$$

Here, absolutely inappropriate dependence of the solution on the scale of time is obvious. However, let us put aside this cardinal issue and assume that $t_0 = t_*$, as a law of Nature. It is not difficult to find that the coefficients of equations (2.12) and (2.14) completely coincide. At the same time, on the basis of (1.4), equation (2.14) is a second-order differential equation with respect to the investment $I(t)$.

Accordingly, the functions $Y(t)$ and $I(t)$ may differ from each other only in the constants $c_1$ and $c_2$ in the representation of the solutions of the type (2.13). To find them, we need initial conditions

$$Y(0) = Y_0; \quad \dot{Y}(0) = \dot{Y}_0; \quad I(0) = I_0; \quad \dot{I}(0) = \dot{I}_0$$

that are rather ephemeral. Combined with abstract character of the coefficients of equations (2.10), this fact makes the model practically useless.

Besides, the considered model ignores, in fact, the function $K(t)$, i.e., the capital, whose exclusiveness as a unique "restricted factor" is emphasized by the authors of [1]: see section 1.1. Indeed, dynamic process, by virtue of objective circumstances, should be related to a comparatively stable factor, because otherwise, figuratively speaking, a reference point is lost.
Obviously, even under the condition of a "law of Nature", the model \(2.10\) has no rehabilitation potential. In other words, it cannot be converted even into a conditionally adequate one that would be analogous to the transformations of Harrod-Domar’s model considered in section 1.3.

In light of the above, let us turn to the same model of Phillips in the interpretation of L. Bergstom \[13\] (pp. 40, 41):

\[
\bar{C}(\bar{t}) = (1 - \mu) \bar{Y}(\bar{t}) ; \quad \frac{d\bar{Y}(\bar{t})}{dt} = \lambda \left[ \bar{C}(\bar{t}) + \frac{d\bar{K}(\bar{t})}{dt} - \bar{Y}(\bar{t}) \right] \\
\frac{d\bar{K}(\bar{t})}{dt} = \gamma \left[ \nu \bar{Y}(\bar{t}) - \bar{K}(\bar{t}) \right], \quad (2.15)
\]

where \(Y(\bar{t})\) is a factual net income or output; \(C(\bar{t})\) is factual consumption; \(K(\bar{t})\) is the volume of the capital; \(\mu, \nu, \gamma, \lambda\) are positive constants \((\mu < 1)\).

It is explained: "As the capital represents the 'stock', whereas output is an 'outflow', the quantity \(\nu\) is inversely proportional to a chosen unit of time. Thus, for example, when time in measured in months, \(\nu\) is 12 times greater than in the case when time is measured in years."

The problem is reduced to the solution of the equation

\[
\frac{d^2\bar{K}(\bar{t})}{dt^2} + (\gamma + \mu\lambda - \nu\gamma\lambda) \frac{d\bar{K}(\bar{t})}{dt} + \mu\gamma\lambda\bar{K}(\bar{t}) = 0
\]

[a comparison with \(2.12\) yields \(\gamma = \kappa\)]. However, the model contains the fundamental drawback that has been considered in detail above. In other words, equation \(2.15\) is nothing but an analogue of \(1.5\). Taking into account \(2.3\), for \(K(t) = K_0 \bar{K}(\bar{t})\), we again arrive at the association of the capital (\($) with the intensity of the flow \(Y(t)\) (\$/s). Nevertheless, as follows from the above-mentioned quotation, the author of \[13\] fully admits a possibility of differentiating discrete flows.

Having worked out certain techniques, we can a priori identify the components of macroeconomic models that contain contradictions. Thus, a "simple" version of Goodwin’s model \[8\] (pp. 193, 194) involves the equation

\[
\bar{K}(\bar{t}) = \nu \bar{Y}(\bar{t}) + a\bar{t}:
\]

this situation is quite analogous to \(2.14\).

From the point of view of the diagnosis, an "early" version of Kaletsky’s model \[8\] (pp. 199-201), practically, does not differ much from the above. It is based on the relation

\[
\bar{I}(\bar{t}) = a (1 - c) \bar{Y}(\bar{t}) - k \bar{K}(\bar{t}) + \varepsilon,
\]

where \(a\) and \(k\) are dimensionless constants, or, taking into account a lag,

\[
\bar{Y}(\bar{t}) = \frac{1}{\theta (1 - c)} \left[ \bar{K}(\bar{t} + \theta) - \bar{K}(\bar{t}) \right] + \frac{A}{1 - c}.
\]
Its "later" version [8] (pp. 205, 206), based on the equation

\[ \bar{B} (\bar{t}) = a (1 - c) \bar{Y} (\bar{t}) + \nu_2 \frac{d\bar{Y} (\bar{t})}{dt} - k \frac{dK (\bar{t})}{dt} + \varepsilon, \]

where \( \nu_2 \) and \( k \) are constants, mathematically, has practically the same defects as (2.10).

Phillips’ multiplier model [8] (p. 79)

\[ \bar{Z} (\bar{t}) = (1 - \mu) \bar{Y} (\bar{t}) ; \frac{d\bar{Y} (\bar{t})}{dt} = -\lambda \left[ \bar{Y} (\bar{t}) - \bar{Z} (\bar{t}) \right] \]

is directly subject to the criticism of section 2.1. Note that the model is designed in such a way that it is impossible to access the function \( \bar{K} (\bar{t}) \), i.e., the capital.

### 2.3 Differential and difference models

Of interest are the arguments of T. Puu: ”Samuelson’s invention (1939) of the business cycle that combines the principles of the interaction between a multiplier and of an accelerator, is, without any doubt, a tremendous event. The fact that a combination of such simple reasons, i.e., the buyers’ expenditure of a certain part of their income on consumption and the producers’ preservation of a fixed relation between the capital and the production volume induced cyclic changes, was simple, surprising and, at the same time, convincing. This model has, so to say, scientific elegance. By the way, it should be noted that its economic prerequisite was the macroeconomic approach of Keynes.

The initial model had been proposed as a process with discrete time, that is as a difference equation; later, it was rather skillfully developed in detail by Hicks (1950). Harrod (1948) arrived at an idea to formulate this process in continuous time, as a differential equation. The obtained system, instead of generating cycles, caused a growth; nevertheless, he clearly realized that the process of development was like balance on the edge of the blade submerged in surrounding instability.

This established a tradition for several decades. Business cycles were formed on the basis of difference equations, whereas development was formed with the help of differential equations. Today, we could say that Samuelson and Hicks chose by chance a second-order process, whereas Harrod decided in favour of a first-order process. We could also admit that a choice between discrete and continuous modelling does not change anything. Dynamic processes of any order can be formulated in the form of difference equations and in the form of differential equations.

A choice of the type of the model (discrete as compared with continuous) can be regarded as a matter of pure convenience. For analytical purposes, when we desire to apply theorems from vast literature on differential equations, a continuous approach is preferable. However, when we want to apply this model to an experimental time series that is inevitably discrete, we have to use discretization” [9] (pp. 76, 77).
A. Bergstrom’s opinion about the freedom of choice of the model is analogous: "One of the most important methodological problems of constructing economic models is the question what equations should be used to describe such models: differential or finite-difference equations. Although many individual decisions are made in regular intervals of time (say, once a week or once a month), variables observed by the econometricist represent a result of many particular decisions made by different individuals at different points of time. Moreover, the intervals of the observation of most economic variables are considerably larger than the intervals between decision-making represented by these variables. These facts lead to the idea that variables of a typical economic model should be regarded as continuous functions of time and that such a model should be described by differential equations. ...

One more argument in favor of representing models in the form of differential equations is that, even in the absence of continuous observations of economic variables, the predictable continuous trajectories of changes of these variables may prove to be of considerable value. Let us assume, for instance, that, in the company’s view, the volume of sales of its products is closely related to the national income of the country. Then, to forecast the sales, it is very useful to have a predicted continuous trajectory of a change of the national income, although measurements of this variable are carried out only once a year. A continuous model allows one to obtain such a prediction using discrete measurements of economic variables over the past period of time" [13] (pp. 8-11).

The position of J. Casti is alternative: "In discrete time, the dynamics of the system can be described by means of difference relations. The most important property of such a description is that it gives us an idea of the behavior of the system in a certain local neighborhood of the current state. At that, it is implicitly assumed that local information can be somehow ‘unified’, which allows us to understand global (in time and space) behavior of the system. Such an approach proved to be sufficiently justified for an analysis of many physical and technical problems. However, the possibility of its application in the case of less studied problems, especially of systems of social-economic nature, is by no means obvious" [14] (pp. 17, 18).

In addition, in light of the material of sections 1.1.-2.2, we have to refer to the remark of R. Allen of somewhat ambiguous meaning: "It should be noted that mathematical economics belongs to applied mathematics: it embodies a union of mathematics and economics. Any in the least bit interesting results in the field of mathematical economics can be provided only by an economist that uses mathematical techniques” [8] (p. 19).

2.4 Abstracted model

However, the above-quoted arguments [13], as well as partly [9], implicitly imply an approximation of the behavior of the considered system that, in its turn, is a solution to an appropriate differential equation. In other words, the mathematical model is not constructed in an immediate relationship to an objective
content of the process. This fact motivated the choice of the title of the section.

As an example, we turn our attention to a simplified version of the model of long waves [15] (pp. 84, 85):

\[
\dot{x}(t) = -p [x(t) - qy(t)]; \quad \dot{y}(t) = -r [y(t) - sz(t)]; \quad z(t) = x(t) - y(t),
\]

where \( x(t) \) is the rate of an increase in labor productivity; \( y(t) \) is the rate of an increase in capital endowment; \( z(t) \) is the rate of an increase in the profit rate; \( p, q, r, \) and \( s \) are structural coefficients that can perform the following functions:

- adaptation of the model to the behavior of the observed system;
- estimation of related factors of the reliability of the model;
- forecasting of the behavior of the system (under the condition of adequacy of the model).

In [15], it is pointed out: "The system of differential equations with positive and negative feedbacks employed by the authors describes a process analogous to a simple mechanical system such as, e.g., a pendulum, a spring, elastic constructions with a damper, etc. However, there exists a considerable difference between economic and mechanical systems. In the simplest mechanics, elasticity and damping forces are usually independent of the system, whereas in economics all the factors are mutually interrelated."

In particular, the following is established. Regular cycles appear for \( s = -2 \) and \( p = r \). In the interval \( p \in [0.10, 0.12] \), periodic undamped oscillations appear whose duration is 50-60 years. Cycles of twenty years appear when \( p \) and \( r \) increase up to 0.34. A more complete version of the model was tested using the statistics of time series of the USA during the period from 1989 to 1982. The following values of the "adaptation coefficients" have been obtained: \( p = 0.048 \) and \( r = 0.25 \), which corresponds to damped oscillations with the period of 53.7 years.

The considerable difference between the coefficients \( p \) and \( r \) is objectively interpreted. However, the sensitivity of this model to a small change in the coefficients is apparent. From this it follows that the model, represented by a system of two first-order differential equations with constant coefficients, has rather limited the potential of the description of macroeconomic processes.

### 2.5 Leontief’s model

In R. Allen’s view, "the economist can learn much from the engineer: both the ways of using mathematical methods and the ability to pose technical problems" [8] (p.19). He can be well understood: indeed, equation (2.12) of Phillips’ model contains four absolutely abstract coefficients \( \kappa, \nu, \mu, \) and \( \lambda \). In addition, we have "independent investment and consumption expenditures". They represent the free term of the equation and, accordingly, should be given. It goes without saying, that the fundamental defect caused by the dependence of the solution on a choice of the scale of time is also present: see sections 2.1 and 2.2.
However, equation (2.12) is widely used in technics. Thus, it describes free oscillations of the mass suspended to a spring under the condition of viscous drag. All the parameters of such a system are concrete, they can be measured, and the same concerns the external force in the case of forced oscillation. The differential equation is rigorously derived on the basis of the laws of mechanics: see, e.g., [16] (pp. 43-49).

So, what can be learnt from the engineer, if we aiming at developing an analogous approach to mathematical modelling in macroeconomics? It seems that anything like that is impossible by definition. In economics, there are no laws for idealized objects that can be put into correspondence with a material point. However, economics, in its turn, has great advantage over mechanics that is embodied in the equation of the balance of financial flows!

Namely, Leontief’s model is brilliant both with respect to its simplicity and efficiency related to the development of computer technologies. For functions that depend discretely on time, this model was studied by M. Morishima. It was pointed out: "So-called Leontief’s model, initially static by its nature, is usually transformed to a dynamic one by the insertion of consumption and output lags, of the lifetime of the means of production, of accelerators, of the growth of final demand, of structural changes caused by technological innovations, etc.” [17] (p. 77).

One can draw a conclusion that the possibilities of mathematical techniques of mechanics and economics, figuratively speaking, have different vectors. The main thing is their reasonable application rather than following the principles of blind imitation of approaches from those fields of knowledge that seem to be the most mathematized ones from a superficial point of view. By the way, although the engineer is aware of exact solutions for a bar, a plate, and other elements of this kind, he experiences considerable difficulties when designing unique constructions. He has no restrictive criterion for the whole set of elements. The balance is absent!

In light of the above, proceeding to the constructive part of our consideration, we consider, following [18] (pp. 40-45), the dynamic of a system defined as

\[ X(t) = F(X(t - \tau)) , \]  

(2.16)

where \( X(t) \) is a vector function of a set of flows \( x_i(t) \); \( \tau_m = \max \{\tau_i\} \) is a maximum delay (lag) during which a change of the state can be taken to be linear.

The dynamics of the system is characterized by the derivative of the function (2.16):

\[ \dot{X}(t) = \frac{dF}{dX} \dot{X}(t - \tau) + \frac{dF}{dt} X(t - \tau) . \]  

(2.17)

From here, taking into account that \( \tau_m \) is small, we can get:

\[ \dot{X}(t) = \frac{1}{\tau_m} \frac{dF}{dX} [X(t) - X(t - \tau)] + \frac{dF}{dt} X(t - \tau) , \]  

(2.18)

or

\[ \dot{X}(t) = G(t, X(t), X(t - \tau)) . \]  

(2.19)
At the same time, because of the presence in (2.18) of the small factor $\tau_m$ by the derivative, equation (2.19) belongs to the class of singularly perturbed equations. Both the study of such equations and their numerical realization require the use of techniques from a rather special arsenal: see [19] (sections 2, 7) and also [20], [21] (section 1).

In the next section, we present a method of the reduction of Leontief’s balance model to a differential form which is alternative to the use of (2.16) and (2.17) that lead to (2.19). However, if the function $F(X)$ in (2.16) is linear, equation (2.18) becomes meaningless by virtue of its triviality, and, accordingly, the derivation of a differential equation proves to be fundamentally impossible. In this regard, we note that the issue of limitations of a linear theory has been indirectly touched on also in section 2.4. This fact is related to the content of section 3.2.
Chapter 3

Constructive arguments

3.1 Leontief’s differential model

Consider a system of balance equations:

\[ x_i(t) = \sum_{j=1}^{n} a_{ij} x_j(t) + c_i(t), \quad i = 1, 2, \ldots, n; \quad 0 \leq t \leq t_0 \]  

(3.1)

that can be interpreted, e.g., as follows: \( x_i(t) \) are the flows of the volumes of output \$(/s\)$; \( a_{ij} \) is a part of the commodity \( i \) used in the production of the commodity \( j \); \( c_i(t) > 0 \) is the flow of an external demand for the commodity \( i \); \( t \) is the variable of time \( (s) \). In what follows, the statement of the problem will be complemented. In matrix notation, the system of equations (3.1) takes the form

\[ X(t) = AX(t) + C(t), \quad 0 \leq t \leq t_0. \]  

(3.2)

In this case, it is reasonable to assume that the matrix \( A \) satisfies the conditions of Metzler’s theorem [17] (p. 34), and, accordingly,

\[ \sum_{j=1}^{n} a_{ij} \leq 1, \quad i = 1, 2, \ldots, n, \]

where we have a strict inequality for at least one of the sums. Then, equation (3.2) can be solved by the method of simple iterations [22] (pp. 120-122):

\[ X_{s+1}(t) = AX_s(t) + C(t), \quad s = 0, 1, \ldots, \]  

(3.3)

where, in theory, the initial element can be chosen arbitrarily.

However, we have the initial condition \( X(0) = X_0 \), and, accordingly, the iterations (3.3) have, in reality, the form

\[ X(t_{s+1}) = AX(t_s) + C(t_s), \]  

(3.4)
where \( t_{s+1} = t_s + \Delta t \), and there emerges the issue of choosing the step \( \Delta t \) that corresponds to the factors of changes in the vector function \( X (t) \).

To approximate this function, one employs a Taylor series expansion and usually retains the first-order derivative [23] (pp. 32, 33). However, if we only want to capitalize on the idea of the outlined approach and, instead of (3.4), consider the relation

\[
X (t + t_0) = AX (t) + C (t), \quad 0 \leq t \leq t_0,
\]

where the time interval \( t_0 \) is sufficiently large, we will have to retain in the Taylor series

\[
X (t + t_0) = X (t) + t_0 \dot{X} (t) + \frac{1}{2} t_0^2 \ddot{X} (t) + \frac{1}{6} t_0^3 \dddot{X} (t) + \ldots
\]

(3.6)

more terms.

If, as an illustration, we restrict ourselves to three terms and use the notation

\[
A = E - B,
\]

where

\[
B = \begin{pmatrix}
  1 - a_{11} & a_{12} & \ldots & a_{1n} \\
  a_{21} & 1 - a_{22} & \ldots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \ldots & 1 - a_{nn}
\end{pmatrix},
\]

(3.7)

and

\[
E = \begin{pmatrix}
  1 & 0 & \ldots & 0 \\
  0 & 1 & \ldots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \ldots & 1
\end{pmatrix}
\]

is a unit matrix, after substitution into (3.5), we obtain

\[
t_0 \dot{X} (t) + 0.5 t_0^2 \ddot{X} (t) = -BX (t) + C (t)
\]

(3.8)

that under a change of the variable \( \tilde{t} = t/t_0 \) becomes

\[
0.5 \ddot{X} (\tilde{t}) + \dot{X} (\tilde{t}) + BX (\tilde{t}) = C (\tilde{t}), \quad 0 \leq \tilde{t} \leq 1.
\]

(3.9)

As a matter of fact, a derivation of this equation has been the aim of the previous transformations. In the solution of this equation, one uses initial condition of the form

\[
X (0) = X_0, \quad \dot{X} (0) = \dot{X}_0,
\]

(3.10)

and, despite its matrix form, the procedure of numerical realization is quite standard [24] (section 10). This reference also gives the solution by quadrature of the first-order differential equation with the elements of the matrix (3.7) depending on time, i.e., for \( a_{ij} = a_{ij} (t) \). In principle, this result can be easily extended to the case of the second order as well as of higher order: see, e.g., [7] (section 5).

24
However, one may ask whether we are resorting to a double standard here because the procedure of the reduction of (3.8) and (2.11) to the forms (3.9) and (2.12), respectively, is the same. Moreover, why the dependence of the solution on the time scale has not been pointed out in this case? This is a very important issue resulting from the use of the Taylor series (3.6). The number of retained terms in this series directly depends on the value of $t_0$.

The difference lies in the fact how this dependence is established: by a trial-and-error method or by a profound analysis. As regards the consideration of sections 2.1 and 2.2, there was no criterion of this kind there.

In the formulation (3.1), as the function $C(\bar{t})$ is not given, the demand is prevailing and, in practice, the output may fail to meet it. To avoid this situation, we can multiply $C(\bar{t})$ by a factor $0 < \alpha < 1$ defined, e.g., by the condition

$$\int_0^1 X(\eta) \, d\eta = X_*,$$

where $X_*$ characterizes the demand for internal output. To carry out analytical studies for forecast purposes, one can complement the vector $C(\bar{t})$ by a component that nonlinearly depends on $X(\bar{t})$.

3.2 Nonlinear model and an alternative

"The author has arrived at a final conclusion that linear dynamic modelling yields very little. This is caused by the scarcity of the set of alternatives, i.e., either damped or explosive motion, that are related to linear models. Therefore, in the present study, we focus mainly on nonlinearity" [9] (p. 7). In what follows, we show that nonlinearity has its own alternatives.

In our view, the work of A. van der Schaft [25] is of great interest to mathematical modelling. It contains information about the orientation of system theory concerned with the realization of nonlinear dynamic models. Thus realization implies the construction of a system of equations

$$\dot{x} = f(x, u), \quad x(0) = x_0, \quad y = h(x, u), \quad (3.11)$$

where $x(t)$, $y(t)$, and $u(t)$ are vector functions, that is adequate to a given input-output mapping

$$y(t) = F(u(\tau); 0 \leq \tau \leq t). \quad (3.12)$$

In other words, corresponding measurements can be carried out with a shift in time, which is rather important. All the quantities are dimensionless.

As is pointed out by J. Casti, the problem of realization consists in the construction, as far as possible, of a compact model that stands in agreement with observed data [14].

Note that, for analytical studies of the nonlinear model (3.11), there exist efficient mathematical techniques dating back to the works of A. M. Lyapunov.
and A. Poincaré. The monograph [26] is devoted to the adaptation of these techniques to the problems of macroeconomics.

The author of [25] has formulated an original part of his work in such a way: "We consider systems of smooth nonlinear differential and algebraic equations where certain variables are singled out as 'external' ones. The problem of realization amounts to the substitution of an implicit high-order differential equation by explicit first-order differential equations and algebraic equations by means of a mapping in terms of external variables".

Thus, instead of (3.12), a relationship between the vectors at the input and output of the system, i.e., $u(t)$ and $y(t)$, respectively, is implied in the form of a nonlinear high-order differential equation. Certainly, with regard to the problem we are considering, such formulation is absolutely irrelevant. Indeed, in the course of consideration, we have been trying to derive a differential equation of at least the first order, and we have been discussing the correctness of the application of this procedure in the works on macroeconomics that are considered to be classic. At the same time, the mapping (3.12) is rather organically related to the factors of the balance of financial flows, to the realia of corresponding measurements as well as to other factors of this kind. It may seem that one can be satisfied with this fact. Why have we then referred to the above quotation?

The reason is that the text [25] implies an alternative way of the construction of macroeconomic models. Indeed, as a prerequisite of further transformations, here appears an input-output mapping that is represented by a system of linear differential equations

$$D \left( \frac{d}{dt} \right) y(t) = N \left( \frac{d}{dt} \right) u(t),$$

where $D(s)$ and $N(s)$ are polynomial matrices of "appropriate dimensions", subject to a number of loose requirements. This case is matched by the model (3.11) under a linear approximation:

$$\dot{x} = Ax + Bu; \quad Y = Cx + Du,$$

where $A$, $B$, and $C$ are corresponding matrices.

Consider, however, an ordinary differential equation, with constant coefficients as in (3.13), of order $n$:

$$z^{(n)} + a_{n-1}z^{(n-1)} + \ldots + a_1 z = 0,$$

where $z^{(n)} = (d/dt)^n z(t)$.

Using a formal notation

$$z^{(n)}(t) = \varphi(t),$$

we get

$$z(t) = \frac{1}{(n-1)!} \int_0^t (t - \eta)^{n-1} \varphi(\eta) \, d\eta + \sum_{i=1}^{n} c_i \frac{t^{i-1}}{(i-1)!}.$$  

(3.15)
Here, $c_i$ are the constants of integration defined by conditions at the input-output of the system, that is, at $t = 0$ and, for definiteness, at $t = 1$.

The substitution of the function (3.15) and of its derivatives into (3.14) leads to a Fredholm integral equation of the second kind in the canonical form:

$$\varphi(t) = \lambda \int_0^1 k(t, \eta) \varphi(\eta) \, d\eta + q(t), \quad t \in [0, 1],$$  

(3.16)

where $\lambda$ is a parameter discussed below. As a matter of fact, we have described a classic method of the reduction of an ordinary differential equation to an integral equation that is given in numerous publications: see, e.g., [27] (pp. 31-33).

Note that the use in (3.16) of the notation

$$U(\bar{t}) = \ddot{X}(\bar{t})$$

yields

$$X(\bar{t}) = \int_0^{\bar{t}} (\bar{t} - \eta) U(\eta) \, d\eta + c_0 \bar{t} + c_1,$$

which, under conditions (3.10), results in Leontief’s model in the form of a Volterra integral equation of the second kind.

However, in light of the present consideration, the following point is of primary importance. The whole information about the system, i.e., about equation (3.14) and conditions at $t = 0$ and $t = 1$, is reflected by the kernel $k(t, \eta)$ and the free term $q(t)$ of equation (3.16). At the same time, as a result of the repeated integration according to (3.15), the kernel in (3.16) represents, in reality, a rather particular case, i.e., a convolution

$$k(t, \eta) = k(t - \eta),$$

and this convolution is of a very specific type.

In this regard, one may ask a natural question: Based on a priori information, measurements, experiments as well as other factors including heuristics, why not construct the kernel $k(t, \eta)$ in such a way that the function $\varphi(t)$ could possess the property to represent potentially realizable variants of the behavior of the considered system?

Simultaneously, the kernel $k(t, \eta)$ should contain parameters intended for the purposes of adaptation and correction of the model, as well as of forecasting. They can be associated with a part of conditions at $t = 1$ for the function $z(t)$ and its derivatives from (3.14) that are subject to identification. Without any doubt, such a model contains elements of self-training.

However, could a model based on the integral equation (3.16) prove to be more efficient than (3.11), that is, of the model posed in the form of the Cauchy problem that used to be a traditional orientation of the macroeconomic science?

From this point of view, E. Goursat’s constatation is of interest: “The solution of Volterra’s integral equation is a broad generalization of the Cauchy
problem for a linear differential equation” [28] (p. 16). At the same time, however, equations of Volterra’s type represent a particular case of Fredholm equations whose spectrum of solutions is more diverse.

To answer the above-posed question, we consider the representativeness, in a mathematical and, accordingly, objective sense, of the solution of equation (3.16) that depends on the parameters of the model. In this regard, solutions of the exponential type that were present in sections 1 and 2 can be called trivial.

3.3 Representativeness of the integral model

Note that, for every set of the coefficients $a_i$ from (3.14), the parameter $\lambda$ in equation (3.16) is a definite number. At the same time, to study the properties of the function $\varphi(t)$ that satisfies (3.16), it is reasonable to consider $\lambda$ indefinite and to specify it when necessary.

The solution to equation (3.16) in the case of a sufficiently small absolute value of the parameter $\lambda$ can be expressed via the resolvent:

$$
\varphi(t) = q(t) + \lambda \int_{0}^{1} H(t, \eta, \lambda) q(\eta) d\eta,
$$

under the condition $\lambda \neq \lambda_i$, where $\lambda_i$ are the characteristic numbers (alias points of the spectrum) of the kernel $k(t, \eta)$. The above-mentioned spectrum, together with $\lambda_i$ is associated with the eigenfunctions $\varphi_i(t)$ such that

$$
\varphi_i(t) = \lambda_i \int_{0}^{1} k(t, \eta) \varphi_i(\eta) d\eta.
$$

(3.17)

Numerical realization of the resolvent $H(t, \eta, \lambda)$ is a special topic. On the whole, there exist a number of efficient algorithms for the solution of Fredholm integral equations of the second kind. In any case, both the parameter $\lambda$ and the function $H(t, \eta, \lambda)$ are very convenient for analytical purposes. There are publications devoted to the study of the structure of the resolvent in the neighborhood of the characteristic numbers. Analytical estimates of the order of the growth of the characteristic numbers depending on the properties of the kernel are worked out [29] (pp. 61, 62).

The following example clearly demonstrates the influence of the parameter $\lambda$ on the solution. Thus, the equation

$$
\varphi(t) = \lambda \int_{0}^{1} (t + \eta) \varphi(\eta) d\eta + q(t),
$$

for $\lambda \neq -6 \pm 4\sqrt{3}$, has the unique solution [30] (pp. 29, 30)

$$
\varphi(t) = q(t) + \frac{\lambda}{\lambda^2 + 12\lambda - 12} \int_{0}^{1} [6(\lambda - 2)(t + \eta) - 12t\eta - 4\lambda] q(\eta) d\eta.
$$
Equation

\[ \varphi (t) = \lambda \int_0^1 e^{t-\eta} \varphi (\eta) \, d\eta + q(t), \quad (3.18) \]

for \( \lambda \neq 1 \), has the solution

\[ \varphi (t) = q(t) + \frac{\lambda}{1-\lambda} e^t \int e^{-t} q(t) \, dt; \]

if, however, \( \lambda = 1 \) and \( \int e^{-t} q(t) \, dt = 0 \), then

\[ \varphi (t) = q(t) + ce^t, \]

where \( c \) is an arbitrary constant. For \( \lambda = \lambda_1 \), equation (3.18) is, in general, unsolvable, whereas the corresponding homogeneous equation has an infinite number of nontrivial solutions \[27\] (pp. 74-76).

As is pointed out by V. I. Smirnov \[31\] (p. 130), when considering integral equations whose kernels are analytical functions of the parameter, one can meet with substantial deviations from the regularities present in general theory of Fredholm integral equations of the second kind. From this point of view, of interest is the paper by Z. I. Khalilov \[32\] devoted to investigation into the equation

\[ \varphi(t) = \int_0^1 \left[ k_0(t, \eta) + \mu k_1(t, \eta) \right] \varphi(\eta) \, d\eta + q(t), \quad (3.19) \]

where \( \mu \) is a parameter, under certain restrictions on \( k_0(t, \eta) \), \( k_1(t, \eta) \), and \( q(t) \).

Tamarkin’s theorem is given: Equation (3.19) either has no solutions for all \( q(t) \) and any value of \( \mu \), or it has a unique solution for all \( q(t) \) and any \( \mu \) except for, may be, a countable set of values of \( \mu \) together with a limit point (if it exists) at the infinity. The statement of this theorem for a more general formulation of \( k(t, \eta, \mu) \) is given in \[31\] (p. 132).

In the first case, the author \[32\] has termed the kernel of equation (3.19) exceptional, whereas in the second case, he has termed it non-exceptional. A theorem is proved stating that, in the case of a non-exceptional kernel, equation (3.19) can always be reduced to a usual integral equation. Representations of the kernel of this equation \( r(t, \eta) \) and of the free term via the functions and the parameter from (3.19), including the resolvent, are given.

One more theorem: The spectrum of the kernel in (3.19) either coincides with the whole plane of the complex variable \( \mu \) or coincides with the spectrum of the kernel \( r(t, \eta) \). Conditions of the solvability of equation (3.19) in the case of multiple eigenvalues are also obtained.

In the context of this consideration, the material presented by M. L. Krasnov \[33\] is relevant to the case. Thus, it turns out that the homogeneous integral equation

\[ \varphi(t) = \int_0^1 \left[ \rho(t) \rho(\eta) + \mu \sigma(t) \rho(\eta) \right] \varphi(\eta) \, d\eta, \quad (3.20) \]
where \( \rho(t) \) and \( \sigma(t) \) are continuous functions obeying the relations

\[
\begin{align*}
\int_0^1 \rho^2(\eta) \, d\eta &= 1; \\
\int_0^1 \rho(\eta) \sigma(\eta) \, d\eta &= 0,
\end{align*}
\]

has the nonzero solution

\[
\varphi(t) = \rho(t) + \mu \sigma(t)
\]

for any \( \mu \) \[p. 272\]. In other words, equation (3.20) does not represent a Fredholm integral equation of the second kind.

F. Tricomi summarizes: "In the theory of equations that do not obey Fredholm's theorems, three new phenomena often occur:

- the presence of finite limit points of the spectrum of characteristic numbers or even of a continuous spectrum, i.e., of characteristic numbers that fill the whole interval of the \( \lambda \)-axis and even the whole \( \lambda \)-axis;
- the presence of characteristic numbers of infinite multiplicity, i.e., of characteristic numbers associated with an infinite number of linearly independent functions;
- the presence of bifurcation points (in the real nonlinear case), i.e., of those points of the \( \lambda \)-axis by passing through which the number of solutions of the equation changes while remaining finite" \[p. 207, 208\].

Note that the above-mentioned nonlinearity is not necessary for the appearance of various miracles in the behavior of the solutions. As a matter of fact, we initially have equation (3.16) with the function \( \varphi(t) \) that is explicitly present. Exactly this type of equations is necessary for the study of peculiarities of the solutions, such as branching and bifurcation (they can be interpreted in different ways).

In the general case, to obtain an equation of the second kind, by analogy with (2.16,) one needs to have a nonlinear initial object \[pp. 22, 23, 304-305, 318\]. By definition \[pp. 22, 23, 304-305, 318\], the values of \( \lambda_i \) from (3.17) representing characteristic numbers of the continuous kernel \( k(t, \eta) \) are also points of bifurcation of equation (3.16).

However, only rather special cases of the dependence of the kernel on the parameter have been considered. Thus, Tamarkin's theorem does not envisage the presence, alongside with \( \mu \), of the parameter \( \lambda \). At the same time, if one is motivated by the development of an objective basis of analytical investigation, it is desirable to make the number of the parameters in the kernel sufficiently large. From a formal point of view, there are no obstacles on this way. The main question is whether adequate computational and theoretical means exist.

On the basis of a research into literature, we can arrive at a conclusion that during the period whose beginning dates back to the publication \[pp. 35\] there has been no more or less significant progress in the field of the theory of integral equations with parameter-dependent kernels. How this fact can be explained?

It seems that the following answer can be given. On the one hand, there are, obviously, considerable difficulties of fundamental character on the way of the development of mathematically rigorous theory. On the other hand, the
theory of integral equations has become part of functional analysis, and, at the modern stage, its problems are considered from a more abstract point of view (the theory of an implicit function). In a certain sense, the factor of nonlinearity may prove to be of minor importance here \[34\] (section 1).

In light of the above, we think that it would be rather interesting to develop the theory of Volterra integral equations of the second kind with the parameter-dependent kernel. On the one hand, the theory of equations of this type is objectively simpler than that of Fredholm. On the other hand, such equations are associated in a natural way with the procedure of the reduction of Leontief’s model to the integral form: see sections 3.1 and 3.2.

Remaining within the framework of the objective orientation of the present consideration, we can draw the following conclusions:
- the solution to an integral equation of the second kind even in a rather particular case of dependence of the kernel on the parameter exhibits a wide range of possibilities of behavior;
- as is illustrated by the simplest examples, the parameter can put the equation outside the class of Fredholm equations of the second class that are characterized by stable dependence of the solution on the data of the problem;
- such properties of the seemingly linear integral equation of the second kind with the parameter-dependent kernel characterize it as a rather efficient tool for modelling of dynamic processes;
- at the same time, the breadth of the class of solutions results in difficulties of an analytical study and of numerical realization of mathematical models even in the case of dependence on the parameter of a particular type.
SUMMARY

Traditional models of macroeconomic dynamics are fundamentally incorrect. The reason lies in a misunderstanding of peculiarities of the analysis of infinitesimal quantities. However, even those types of solutions that are envisaged by the above-mentioned models are nonrepresentative in the sense of the reflection of realities. It became obvious that the techniques of the theory of linear differential equations were insufficient here. Accordingly, the scientists’ attention switched to the theory of nonlinear differential equations.

At the same time, balance and, accordingly, the model with matrix properties are objectively inherent in the economic system. For the reduction of this model to a differential form, there exist rather elementary means that proved to be unclaimed. Macroeconomic rhetoric - the power of the accelerator, a lag on the part of demand, etc. - accompanied by the use of a lot of abstract coefficients prevailed.

Why such an entourage? One cannot but get an impression that the issue, in essence, is political and is deeply rooted. Economics is seemingly a major science because it is more intimately related to the capital than other sciences and it works out wise recommendations for the ruling elite. Therefore, economics is prescribed to use higher mathematics whose peak is differential equations: this has been a mentality since long ago. Nobody has heard anything about other equations, whereas operating the categories of balance is inappropriate because it looks like school arithmetic. The elite, in its turn, acts according to prescriptions of science rather than at will.

However, there is no organic interrelation between matrix and nonlinear differential equations. On the contrary, it can be said that linear theory of integral equations originated in matrix analysis. The Fredholm linear integral equation of the second kind with a parameter-dependent kernel proves to be rather representative with regard to the class of possible solutions. It seems that it can be used for the description of any zigzags of the economy. The price one has to pay for this is the nontriviality of existing theory.
Bibliography

[1] L. V. Kantorovich and A. B. Gorstko, *Optimal Solutions in Macroeconomics* (Mir, Moscow, 1972) (in Russian).

[2] W. Kecs and P.P. Teodorescu, *Introduction to the Theory of Distributions with Applications in Technics* (Mir, Moscow, 1978) (in Russian).

[3] I. M. Gelfand and G. I. Shilov, *Generalized Functions and Operations on them* (Fizmatgiz, Moscow, 1959) (in Russian).

[4] G. A. Baker, Jr. and P. Graves-Morris, *Padé Approximation* (Addison-Wesley, London, 1981) (Russian edition: Mir, Moscow, 1986).

[5] G. E. Shilov, *Mathematical Analysis: The Second Special Course* (MGU, Moscow, 1984) (in Russian).

[6] P. Antosik, J. Mikusiński, and R. Sikorski, *Theory of Distributions: The Sequential Approach* (Elsevier, Amsterdam, 1973) (Russian edition: Mir, Moscow, 1976).

[7] I. G. Petrovsky, *Lectures on the Theory of Ordinary Differential Equations* (Nauka, Moscow, 1970) (in Russian).

[8] R. G. D. Allen, *Mathematical Economics* (Macmillan, London, 1960) (Russian edition: IIIL, Moscow, 1963).

[9] T. Puu, *Nonlinear Economic Dynamics* (Springer, Berlin, 1997) (Russian edition: Reguliarnaya i khaotitcheskaya dinamika, Moscow-Izhevsk, 2000).

[10] R. F. Harrod, *Towards a Dynamic Economics* (Macmillan, London, 1948) (Russian edition: Ekonomika, Moscow, 1997; vol. 1).

[11] A. V. Lusse, *Macroeconomics: Key Issues: A Handbook* (Piter, St. Petersburg, 1999) (in Russian).

[12] L. Stoleru, *L’équilibre et la croissance économique* (Dunod, Paris, 1968) (Russian edition: Statistika, Moscow, 1974).

[13] A. R. Bergstrom, *The Construction and Use of Economic Models* (The English Universities Press, London, 1967) (Russian edition: Progress, Moscow, 1970).
[14] J. Casti, *Connectivity, Complexity, and Catastrophe in Large-Scale Systems* (Wiley, New York, 1979) (Russian edition: Mir, Moscow, 1982).

[15] S. Yu. Glaziev, G. I. Mikerin, P. N. Teslya et al., *Long Waves: Scientific-Technological Progress and Social-Economic Development* (Nauka, Novosibirsk, 1991) (in Russian).

[16] Ya. G. Panovko, *Introduction to the Theory of Mechanical Oscillations* (Nauka, Moscow, 1970) (in Russian).

[17] M. Morishima, *Equilibrium, Stability, and Growth: A Multi-sectoral Analysis* (Clarendon Press, Oxford, 1964) (Russian edition: Nauka, Moscow, 1972).

[18] V. V. Druzhinin and D. S. Kontorov, *Systems Engineering* (Radio i Sviaz, Moscow, 1985) (in Russian).

[19] S. A. Lomov, *Introduction to General Theory of Singular Perturbations* (Nauka, Moscow, 1970) (in Russian).

[20] Yu. V. Rakitskii, S. M. Ustinov, and I. G. Chernorutskii, *Numerical Methods of the Solution of Rigid Problems* (Nauka, Moscow, 1979) (in Russian).

[21] E. P. Doolan, J. J. Miller, and W. H. A. Schilders, *Uniform Numerical Methods for Problems with Initial and boundary Layers* (Boole Press, Dublin, 1980) (Russian edition: Mir, Moscow, 1983).

[22] V. N. Fadeeva, *Computational Methods of Linear Algebra* (Gostekhteorizdat, Moscow, 1950) (in Russian).

[23] J. M. Ortega and W. G. Poole, Jr., *An Introduction to Nonlinear Methods for Differential Equations* (Pitman, 1981) (Russian edition: Nauka, Moscow, 1986).

[24] R. Bellman, *Introduction to Matrix Analysis* (McGraw-Hill, New York, 1960) (Russian edition: Nauka, Moscow, 1969).

[25] A. J. van der Schaft, "On Realization of Nonlinear Systems Described by Higher-Order Differential Equations", Mathematical Systems Theory, v. 19, 239-275 (1987) [Russian edition: System Theory. Mathematical Methods of Modeling (Mir, Moscow, 1989), pp. 192-237].

[26] A. V. Voronin and S. I. Chernyshov, "Cycles in Nonlinear Macroeconomics", arXiv: 0706.1013.

[27] F. Tricomi, *Integral Equations*, (Interscience, New York, 1957) (Russian edition: IL, Moscow, 1960).

[28] É. Goursat, *Course d’Analyse Mathématique*, Bd.3 (Gauthier-Villard, Paris, 1923) (Russian edition: Gostekhteorizdat, Moscow, 1934, Vol. 3, Part 2).
[29] P. P. Zabreiko, A. I. Koshelev, M. A. Krasnoselskii et al, *Integral Equations* (Nauka, Moscow, 1958) (in Russian).

[30] S. G. Mikhlin, *Integral Equations and their Applications to Some Problems of Mechanics, Mathematical Physics and Engineering* (Gostekhteorizdat, Moscow, 1949) (in Russian).

[31] V. I. Smirnov, *A Course of Higher Mathematics*, Vol. 4, Part 1 (Nauka, Moscow, 1974) (in Russian).

[32] Z. I. Khalilov, "On a Fredholm Integral Equation whose Kernel is Linear in a Parameter", DAN USSR, v. 54, No 7, 571-574 (1946)

[33] M. L. Krasnov, *Integral Equations: An Introduction to the Theory* (Nauka, Moscow, 1975) (in Russian).

[34] M. A. Krasnoselskii, G. M. Vinikko, P. P. Zabreiko et al., *Approximate Solution of Operator Equations* (Nauka, Moscow, 1969) (in Russian).

[35] J. D. Tamarkin, "On Fredholm's Integral Equations, whose Kernels are Analytic in a Parameter", Ann. of Math., v. 28, No 2, 1217-153 (1927)