Fixed point theorem on volterra integral equation

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Abstract. This paper discussed the fixed-point theorem in the metric space and its application to the second linear volterra integral equation. This research is done by first studying the concept of metric space and its properties. Then, we studied the application on the second linear volterra integral equation. After obtaining sufficient condition to converge second linear integral volterra equation, we solved the equation by Fixed Point iteration method using Matlab R2013a software. The solution obtained was compared with the exact solution. The results of this study indicate that the numerical solution is quite good.

1. Introduction
Fixed point is one of interesting topic that continues to grow since many years ago. This is because the application of fixed-point theory is found in many fields such as mathematics, physics, biology, chemistry and economics. The \( x \in X \) in metric space \((X, d)\) is called a fixed point for \( T: X \to X \) if \( T(x) = x \). For example, if the mapping \( T: \mathbb{R} \to \mathbb{R} \) is defined by \( T(x) = x^2 \), then 0 and 1 are fixed points of \( T \) since \( T(0) = 0 \) and \( T(1) = 1 \). Not all mappings have fixed points, for example if the mapping \( T: \mathbb{R} \to \mathbb{R} \) is defined by \( T(x) = x + 1 \), then \( T \) has no fixed point. One kind of mapping that ensures the existence of a fixed point is the condition of the contractive mapping. The definition of contractive mapping is given below.

• **Definition 1** Given a metric space \((X, d)\) [1]. Mapping \( T: X \to X \) is said to be contractive in \( X \) if there is a real number \( k \in (0, 1) \), such that it satisfied

\[
d(T(x), T(y)) \leq kd(x, y)
\]

for all \( x, y \in X \) (1)

While the following theorem asserts that contractive mapping guarantees the existence of fixed points in a metric space.

• **Theorem 2** Let \((X, d)\) be a complete metric space and let \( T: X \to X \) be a mapping that satisfies the contractive conditions [2]. Then \( T \) has a fixed point.

• **Proof.** See Nashine & Altun [2].

The use of fixed-point theorems has been developed in many fields specially in determining the specific solution of differential equations and solving the integral equation. Recently, fixed point theorem has been used to show the existence solution of volterra integral equation [3-7]. The application of the volterra integral equation has evolved in the demographic field of viscoelastic
materials and in the mathematical insurance of renewed equations. An approximation solution to volterra integral equation has been also published in many literatures [8-11].

From the above description it is interesting to examine the fixed-point theorem in the metric space and its application to determine the convergence condition of the second linear volterra integral equation. Before discussed the results of this study, first will be reviewed some basic concepts that will support this discussion.

2. Approximation method

General form of volterra integral equation is as follow

\[ x(t) = f(t) + \mu \int_{a}^{t} k(t, s) x(s) \, ds \]  \hspace{1cm} (2)

By choosing \( f(t) \in C[a, b] \) as initial function \( x_0(t) \), we introduce the fixed-point iteration

\[ x_{n+1}(t) = (T x_n)(t) = f(t) + \mu \int_{a}^{t} k(t, s) x_n(s) \, ds, \quad t \in C[a, b], \quad n \geq 0 \]

For \( n \) natural numbers, we find the approximate sequence of \( x_1(t), x_2(t), x_3(t), \ldots, x_n(t), \ldots \). The approximation sequence will be converging to a function that indicates as the completion of the problem above.

The algorithm for working on the above approximations is given in the steps below:

Step (1): specify interval \([a, b]\)
Step (2): input initial condition \( f(t) \) as \( x_0 \)
Step (3): built matrix \( X \)
Step (4): specify the value of \( n \)
Step (5): assume \( x(1) = x_0 \), then substitute to the matrix \( X \)
Step (6): repeat this process to get \( x(i+1), i = 1, 2, 3, \ldots, n - 1 \)

3. Main result

We consider volterra integral equation (2) and assume that \( f \in C[a, b], \mu \in [0, 1] \) and \( k(t, s) \) is a continuous function on \( R = \{(t, s), a < s < t, a < t < b\} \). Now we define operator \( T \) as follows

\[ (T x)(t) = f(t) + \mu \int_{a}^{t} k(t, s) x(s) \, ds \]  \hspace{1cm} (3)

Obviously, the solution of equation (2) is the fixed point of operator \( T \).

3.1. Convergence of the method

The integral equation to be discussed is the integral equation located in space \( C[a, b] \) i.e. the space of all continuous functions defined at interval \( I = [a, b] \) with the metric

\[ d(x, y) = \sup_{t \in I} |x(t) - y(t)| \]  \hspace{1cm} (4)

To apply the Banach fixed point theorem then space \( C[a, b] \) must be completed.

**Theorem 3.1** Let \( T(x) \) in equation (4) be a continuous function in \([a, b]\) and \( k \) is the continuous kernel on the region \( R \) in the \(-xy\) field with \( a \leq y \leq t, a \leq x \leq b \). Then equation (4) has a single solution \( x \) in \([a, b]\) for each \( \mu \).

**Proof.**

Note that equation (4) can be written as \( x = T x \) with \( T: C[a, b] \rightarrow C[a, b] \) is defined as:

\[ T x(t) = f(t) + \mu \int_{a}^{t} k(t, s) x(s) \, ds \]  \hspace{1cm} (5)

Since the \( k \) is continuous on \( R \) and \( R \) is closed and bounded, \( k \) is a finite function on \( R \), for example, \( k(t, s) \leq c \) for each \((t, s) \in R\).

Consider equation (5), we get for all \( x, y \in C[a, b] \),
It will be shown by induction

\[ |T^m x(t) - T^m y(t)| \leq |\mu|^m c(x,y) \frac{(t-a)^m}{m!} d(x,y) \]  

(7)

For \( m = 1 \), an inequality (6) is obtained. Assume the inequality (6) holds for all \( m \), then from inequality (7) we get

\[ |T^{m+1} x(t) - T^{m+1} y(t)| = |T^m(\mu x(t) + \mu \int_a^t k(t,s) x^{m+1}(s) ds) - (\mu y(t) + \mu \int_a^t k(t,s) y^{m+1}(s) ds)| \]

\[ = |\mu \int_a^t \left( k(t,s) x^{m+1}(s) ds - k(t,s) y^{m+1}(s) ds \right)| \]

\[ = |\mu c \int_a^t \left( x^{m+1}(s) - y^{m+1}(s) \right) ds| \]

\[ = |\mu c (x(t) - y(t)) \int_a^t ds| \]

\[ = |\mu c d(x,y)(t-a)| \]

(6)

Thus, it is clearly that inequality (8) applies to every \( m \in \mathbb{N} \).

Using \( t - a \leq b - a \) on the right-hand side of the inequality (7) and the maximum value for \( t \in J \) will be obtained, so we will get

\[ d(T^m x, T^m y) \leq c_m d(x,y) \]

with

\[ c_m = |\mu|^m c(x,y) \frac{(b-a)^m}{m!} \]

(9)

For \( \mu \) fixed and \( m \) large enough it will be obtained \( c_m < 1 \). Means \( T^m \) is a contractive mapping on \( C[a,b] \). The result of the theorem will be obtained with the following entries.
Lemma 3.2 Let $T : X \to X$ be the mapping in the complete metric space $X = (X, d)$, and let $T^m$ is the contractive mapping on $X$ for a positive integer $m$. Then $T$ has a fixed point.

**Proof.**

Assume that $B = T^m$ is the contractive mapping on $X$. Based on the Banach fixed point theorem, the $B$ mapping has a fixed point $\hat{x}$, i.e. $B\hat{x} = \hat{x}$. Means $B^m\hat{x} = \hat{x}$. Banach theorem also results in that $B^m X \to \hat{x}$ if $n \to \infty$ especially $x = T\hat{x}$, because $B^n = T^m$, then we get
\[
\hat{x} = \lim_{n \to \infty} B^n x
= \lim_{n \to \infty} B^n T\hat{x}
= \lim_{n \to \infty} T B^n \hat{x}
= \lim_{n \to \infty} T \hat{x}
= T\hat{x}
\]
This shows that $\hat{x}$ is the fixed point of $T$. Since each fixed point of $T$ is also a fixed point of $B$, consider that $T$ cannot have a fixed point of more than one.

3.2. **Numerical experiment**

The following are examples of how to solve the volterra integral equation problem:

**Problem 1:**

Given the volterra integral equation as follows:
\[
u(x) = \sin(\pi x^2) - \frac{x^2}{\pi} + \int_0^x y^2 \nu(y) \, dy, \quad 0 \leq x \leq 1
\]

Clearly, equation above satisfy the assumption of Theorem 3.1, i.e.:
\[
\sin(\pi x^2) - \frac{x^2}{\pi}
\]
is continuous function on $[a, b]$

Kernel $k(x, y)$ is continuous on $R = [(x, y), 0 < y < x, 0 < x < 1]$

The solution will be determined from the above equation by the fixed-point iteration method
\[
u_{k+1}(x) = \sin(\pi x^2) - \frac{x^2}{\pi} + \int_0^x y^2 \nu_k(y) \, dy
\]

By choosing $\sin(\pi x^2) - \frac{x^2}{\pi}$ as the initial function, we can apply fixed point iteration method to get numerical solution:
\[
u_1(x) = \sin(\pi x^2) - \frac{x^2}{\pi} + \int_0^x y^2 \nu_0(y) \, dy
= \sin(\pi x^2) - \frac{x^2}{\pi} + \int_0^x y^2 \sin(\pi y^2) \, dy
= \sin(\pi x^2) - \frac{x^2}{\pi} + x^2 \frac{1}{\pi^2} \left(1 - \cos(\pi x^2)\right)

\nu_2(x) = \sin(\pi x^2) - \frac{x^2}{\pi} + \int_0^x y^2 \nu_1(y) \, dy
= \sin(\pi x^2) - \frac{x^2}{\pi} + \int_0^x y^2 \left(\sin(\pi y^2) - \frac{x^2}{\pi} + y^2 \frac{1}{\pi^2} \left(1 - \cos(\pi y^2)\right)\right) \, dy
= \sin(\pi x^2) - \frac{x^2}{\pi} + \frac{x^2}{\pi^2} \left(-4\pi^2 - 2 + 4\pi^2 \cos(\pi x^2) + x^4 \pi^2 + 2x^2 \pi^2 \sin(\pi x^2) + 2\cos(\pi x^2)\right)

\nu_3(x) = \sin(\pi x^2) - \frac{x^2}{\pi} + \int_0^x y^2 \nu_2(y) \, dy
= \sin(\pi x^2) - \frac{x^2}{\pi} + \frac{x^2}{\pi^2} \left(-4\pi^2 - 2 + 4\pi^2 \cos(\pi y^2) + y^4 \pi^2 + 2y^2 \pi^2 \sin(\pi y^2) + 2\cos(\pi y^2)\right) \, dy
Consider that for $|x| \leq 1$, sequence $\{u_n(x)\}$ will converge to $u(x) = \sin(\pi x^2) - \frac{x^2}{\pi}$.

Below is the comparison of numerical results with analytic results:

$$
\sin(\pi x^2) - \frac{x^2}{\pi} = x^2 \left\{ -\frac{1}{54\pi^4} \left( -32\pi^5 - 16\pi^3 - 24\pi + 32\pi^5 \cos(\pi x^2) + 3x^4\pi^5 - 4x^4\pi^3 + 16x^2\pi^4 \left( \sin(\pi x^2) + 16\pi^3 \cos(\pi x^2) + x^4\pi^3 \cos(\pi x^2) + 24x^2\pi^2 \left( \sin(\pi x^2) \right) \right) \right) \right\}
$$

![Figure 1. Graph of approximation solution compare to exact solution for problem 1.](image)

Error calculation of approximation solution compare to exact solution is given below.

**Table 1.** Approximation solution compare to exact solution for problem 1.

| $x$  | approximation solution | exact solution | error    |
|------|------------------------|----------------|----------|
| 0,000| 0,000                  | 0,000          | 0,000    |
| 0,100| 0,023                  | 0,028          | 0,005    |
| 0,200| 0,102                  | 0,113          | 0,011    |
| 0,300| 0,234                  | 0,250          | 0,016    |
| 0,400| 0,412                  | 0,431          | 0,019    |
| 0,500| 0,609                  | 0,628          | 0,018    |
| 0,600| 0,779                  | 0,790          | 0,011    |
| 0,700| 0,848                  | 0,844          | 0,005    |
| 0,800| 0,730                  | 0,701          | 0,029    |
| 0,900| 0,358                  | 0,304          | 0,054    |
| 1,000| -0,251                 | -0,318         | 0,067    |

Table 1 shows that error of the approximation solution compare to exact solution is relatively small.
Problem 2:
Given the volterra integral equation is as follows:
\[ u(x) = 1 - x - \frac{x^2}{2} + \int_0^x (x - t) u(t) \, dt, \quad 0 \leq x \leq 1 \]

Solution:
Below is the comparison of numerical results with analytic results

![Figure 2. Graph of approximation solution compare to exact solution with \( h = 0.01 \) for problem 2.](image)

Error calculation of approximation solution compare to exact solution for example 2 is given below.

| \( x_i \) | approximation solution | exact solution | error |
|-----------|------------------------|----------------|-------|
| 0.000     | 1.000                  | 1.000          | 0.000 |
| 0.100     | 0.906                  | 0.900          | 0.006 |
| 0.200     | 0.792                  | 0.799          | 0.007 |
| 0.300     | 0.668                  | 0.695          | -0.028|
| 0.400     | 0.534                  | 0.589          | -0.055|
| 0.500     | 0.390                  | 0.479          | -0.089|
| 0.600     | 0.236                  | 0.363          | -0.127|
| 0.700     | 0.072                  | 0.241          | -0.169|
| 0.800     | -0.102                 | 0.112          | -0.214|
| 0.900     | -0.286                 | -0.027         | -0.260|
| 1.000     | -0.480                 | -0.175         | -0.305|

Table 2 shows that error of the approximation solution compare to exact solution is also relatively small.
4. Conclusion
The result of this study indicates that volterra integral equations that meet contractive condition have a fixed point. The numerical solution with fixed point approximation compares to exact solution give quite good solution.

Acknowledgements
The author thanks the Kemenristekdikti for sponsored this research through Penelitian Dosen Pemula (PDP) Scheme and Universitas PGRI Palembang for facilitate author in doing this research.

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