RESOLVING IRREDUCIBLE \(\mathbb{C}S_n\)-MODULES BY MODULES RESTRICTED FROM \(GL_n(\mathbb{C})\)

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Abstract. We construct a resolution of irreducible complex representations of the symmetric group \(S_n\) by restrictions of representations of \(GL_n(\mathbb{C})\) (where \(S_n\) is the subgroup of permutation matrices). This categorifies a recent result of Assaf and Speyer. Our construction also gives minimal resolutions of simple \(\mathcal{F}\)-modules (here \(\mathcal{F}\) is the category of finite sets).

1. Introduction

The symmetric group \(S_n\) may be viewed as the subgroup of the general linear group \(GL_n(\mathbb{C})\) consisting of permutation matrices. We may therefore consider the restriction to \(S_n\) of irreducible \(GL_n(\mathbb{C})\) representations.

Let \(S^\lambda\) denote the irreducible representation of \(\mathbb{C}S_n\) indexed by the partition \(\lambda\) (so necessarily \(n\) is the size of \(\lambda\)). Let \(S^\lambda\) denote the Schur functor associated to a partition \(\lambda\), so that \(S^\lambda(C^n)\) is an irreducible representation of \(GL_n(\mathbb{C})\), provided that \(l(\lambda) \leq n\). Let us write \([M]\) for the image of a module in the Grothendieck ring of \(\mathbb{C}S_n\)-modules. Thus, the restriction multiplicities \(a^\lambda_\mu\) are defined via

\[
[\text{Res}_{GL_n}^{\mathbb{C}S_n}(S^\lambda(C^n))] = \sum_{\mu \vdash n} a^\lambda_\mu [S^\mu].
\]

Although a positive combinatorial formula for the restriction multiplicities is not currently known, there is an expression using plethysm of symmetric functions (see [Mac95] Chapter 1 Section 8 for background about plethysm). Let us write \(s_\lambda\) for the Schur functions (indexed by partitions \(\lambda\)). The complete symmetric functions, \(h_n\), are the Schur functions indexed by the one-part partitions \((n)\). We recall the Schur functions \(s_\lambda\) form an orthonormal basis of the ring of symmetric functions with respect to the usual inner product, denoted \(\langle -,-\rangle\) (see [Mac95] Chapter 1 Section 4). Let \(f[g]\) denote the plethysm of a symmetric function \(f\) with another symmetric function \(g\). Then,

\[a^\lambda_\mu = \langle s_\lambda, s_\mu[1+h_1+h_2+\cdots] \rangle,
\]

see [Gay76] or Exercise 7.74 of [SF97]. We will need to consider the Lyndon symmetric function,

\[L_n = \frac{1}{n} \sum_{d | n} \mu(d)p_n^{d/d},
\]

where \(\mu(d)\) is the Möbius function and \(p_d\) is the \(d\)-th power-sum symmetric function. It is important for us that \(L_n\) is the \(GL(V)\) character of the degree \(n\) components of the free Lie algebra on \(V\) (see the first proof of Theorem 8.1 of [Reu93], which proves this to deduce a related result). For convenience we define the total Lyndon symmetric function \(L = L_1 + L_2 + \cdots\); this is the character of the (whole) free Lie algebra on \(V\).

Instead of asking for the restriction coefficients \(a^\lambda_\mu\), we may ask the inverse question: how can one express \([S^\mu]\) in terms of \([\text{Res}_{S_n}^{GL_n}(S^\lambda(C^n))]\)? This question was recently answered by Assaf and Speyer in [AS18]. For a partition \(\mu = (\mu_1, \mu_2, \ldots)\) of any size, let \(\mu[n]\) denote \((n - |\mu|, \mu_1, \mu_2, \ldots)\) (a partition of \(n\) provided that \(n \geq |\mu| + \mu_1\)). Assaf and Speyer showed

\[ [S^{\mu[n]}] = \sum_\lambda b^\mu_\lambda [S^{\lambda}(C^n)],
\]

where

\[b^\mu_\lambda = (-1)^{|\mu|-|\lambda|} \sum_{\mu/\nu \text{ vert. strip}} \langle s_\nu, s_\lambda[L] \rangle.
\]
The notation $\mu/\nu$ vert. strip means that the diagram of $\mu$ may be obtained from the diagram of $\nu$ by adding boxes, no two in the same row, and primes indicate dual partitions.

It is more convenient to work with

\[ M_n^\mu = \text{Ind}_{S_n}^{S_n \times S_{n-1}}(S_\mu \boxtimes 1), \]

which decompose into the irreducible $S^{[n]}$ via the Pieri rule:

\[ [M_n^\mu] = \sum_{\mu/\nu \text{ horiz. strip}} [S^{[\nu]}]. \]

Here, $\mu/\nu$ horiz. strip means that the diagram of $\mu$ may be obtained from the diagram of $\nu$ by adding boxes, no two in the same column. The formula for $b^n_\lambda$ is equivalent to the following statement (see Theorem 3 and Proposition 5 of [AS18]):

\[ [M_n^\mu] = \sum_\lambda (-1)^{|\mu|-|\lambda|} \langle s_\mu, s_\lambda \rangle [S_\lambda](s^n \otimes C^n)]. \]

The purpose of this note is to give a categorification of this answer, namely a (minimal) resolution of $M_n^\mu$ by restrictions of $S^\lambda(C^n)$; this is accomplished in Theorem 3.2. Along the way, this explains the presence of the character of the free Lie algebra in the formula, and constructs projective resolutions in the category of $F$-modules (over $\mathbb{Q}$) introduced by Wiltshire-Gordon in [WG14].

Acknowledgements

The author would like to thank Gurbir Dhillon for helpful comments on this paper.

2. The Resolution

We begin by calculating the cohomology of the free Lie algebra on a fixed vector space. Although this result is very well known, it is instrumental in what follows, so we include it for completeness.

Let $L$ be the free Lie algebra on $V = \mathbb{C}^n$. Then $g = L^{\oplus n} = L \otimes \mathbb{C}^n$ is again a Lie algebra. It has an action of $S_n$ by permuting the $L$ summands, coming from an action of $GL_n(\mathbb{C})$ that does not respect the Lie algebra structure. We consider the Lie algebra cohomology of $g$ (with coefficients in the trivial module).

Recall that the Lie algebra cohomology is $\text{Ext}^*_U(g)_{(\mathbb{C}, \mathbb{C})}$. We first consider the case for $t = 1$, so $g = L$. Now $U(L)$ is just the tensor algebra of $V$, which we denote $T(V)$. We therefore have a (graded) free resolution

\[ 0 \rightarrow T(V) \otimes V \xrightarrow{d_1} T(V) \longrightarrow C \rightarrow 0. \]

Here, $d_1(x \otimes v) = xv$ (product in $T(V)$), while $d_0$ simply projects onto the degree zero component. Crucially, $GL(V)$ acts by automorphisms on $L$ (which was the free Lie algebra on $V$), and the above complex is equivariant for this action. The Lie algebra cohomology is given by the cohomology of the complex

\[ 0 \leftarrow \text{hom}_{T(V)}(T(V) \otimes V, \mathbb{C}) \xrightarrow{d_1^*} \text{hom}_{T(V)}(T(V), \mathbb{C}) \leftarrow 0. \]

We easily see the differential $d_1^*$ is zero because any element of $\text{hom}_{T(V)}(T(V), \mathbb{C})$ is zero on a positive degree element of $T(V)$, but the image of $d_1$ is contained in degrees greater than or equal to 1. We thus conclude that $H^0(L, \mathbb{C}) = \mathbb{C}$, and $H^1(L, \mathbb{C}) = V^*$, with all higher cohomology vanishing. Next, we obtain the Lie algebra cohomology of $g = L^{\oplus n}$.

Proposition 2.1. For $0 \leq i \leq n$,

\[ H^i(g, \mathbb{C}) = \text{Ind}_{S_i \times S_{n-1}}^{S_n \times S_{n-1}}((V^*)^\otimes i \otimes \varepsilon_i \otimes \mathbb{C}^\otimes (n-i)), \]

where $\varepsilon_i$ is the sign representation of $S_i$, and $\mathbb{C}^\otimes (n-i)$ is the trivial representation of $S_{n-1}$. Further, for $i > n$, the cohomology $H^i(g, \mathbb{C})$ vanishes.

Proof. We apply the Künneth theorem in an $S_n$-equivariant way. The sign representation $\varepsilon_i$ arises because of the Koszul sign rule (cohomology is only graded commutative).
Now let us compute the Lie algebra cohomology of \( g \) using the Chevalley-Eilenberg complex [Wei95]. Recall that the \( i \)-th cochain group is
\[
\text{hom}_C(\bigwedge^i (g), \mathbb{C})
\]
and the differential \( d \) is given by the formula
\[
d(f)(x_1, \ldots, x_{k+1}) = \sum_{i<j} (-1)^{i-j} f([x_i, x_j], x_1, \ldots, \hat{x}_i, \ldots, \hat{x}_j, \ldots, x_{k+1})
\]
where hats indicate omitted arguments. This differential is \( GL(V) \times S_n \)-equivariant and homogeneous in terms of the grading on \( g \) (the grading corresponds to the action of \( \mathbb{C}^\times = Z(GL(V)) \)).

Note that \( g \) is graded in strictly positive degrees. As an algebraic representation of \( GL(V) \), the \( i \)-th cochain group,
\[
\text{hom}_R(\bigwedge^i (g), \mathbb{C})
\]
is contained in degrees \( \leq -i \). This means that if we are interested only in the degree \( -i \) component of the cohomology, we may truncate the Chevalley-Eilenberg complex after \( i \) steps. Thus, if we write a subscript \( -i \) to indicate the degree \( -i \) component of an \( GL(V) \) representation, we obtain the following.

**Proposition 2.2.** The complex (with differential inherited from the Chevalley-Eilenberg complex)
\[
0 \leftarrow \text{hom}_C(\bigwedge^i (g), \mathbb{C})_{-i} \leftarrow \text{hom}_C(\bigwedge^{i-1} (g), \mathbb{C})_{-i} \leftarrow \cdots \leftarrow \text{hom}_C(\bigwedge^1 (g), \mathbb{C})_{-i} \leftarrow \text{hom}_C(\bigwedge^0 (g), \mathbb{C})_{-i} \leftarrow 0
\]
has cohomology \( H^i(g, \mathbb{C}) \) on the far left, and zero elsewhere.

3. **Resolving the \( CS_n \)-modules \( M_n^\mu \)**

Let us take the multiplicity space of the \( GL(V) \)-irreducible \( S^\mu(V^*) \).

**Proposition 3.1.** The \( S^\mu(V^*) \) multiplicity space in the cohomology \( H^i(g, \mathbb{C}) \) is
\[
M_n^\mu = \text{Ind}_{S_n \times S_{n-i}}(S^\mu \boxtimes 1)
\]

**Proof.** We apply Schur-Weyl duality to Proposition 2.1 noting that \( S^\lambda \otimes \epsilon_i = S^\lambda' \):
\[
H^i(g, \mathbb{C}) = \text{Ind}_{S_n \times S_{n-i}}((V^*)^\otimes i \otimes \epsilon_i \boxtimes 1) = \text{Ind}_{S_n \times S_{n-i}}(\bigoplus_{\lambda \vdash i} S^\lambda(V^*) \otimes S^{\lambda'} \boxtimes 1).
\]

Hence, the \( S^\mu(V^*) \) multiplicity space is \( \text{Ind}_{S_n \times S_{n-i}}(S^\mu \boxtimes 1) \). \( \square \)

Because the complex we constructed in Proposition 2.2 is \( GL(V) \) equivariant, taking cohomology commutes with taking the \( S^\mu(V^*) \) multiplicity space. We immediately obtain the following.

**Theorem 3.2.** Consider the complex of \( S_n \) representations
\[
\text{hom}_{GL(V)}(S^\mu(V^*), \text{hom}_C(\bigwedge^i (g), \mathbb{C}))
\]
for \( |\mu| \geq i \geq 0 \) with maps induced by the differential of the Chevalley-Eilenberg complex. This is a resolution of \( M_n^\mu \) by representations restricted from \( GL_n(\mathbb{C}) \).

**Proof.** This is immediate from Proposition 3.1 and Proposition 2.2 \( \square \)

Should we wish to resolve the irreducible \( S^\mu \), rather than \( M_n^\mu \), we simply take \( n = |\mu| \) so that \( M_n^\mu = S^\mu \).

We now take the Euler characteristic of our complex, viewed as an element of the Grothendieck ring of \( \mathbb{C}S_n \)-modules tensored with the Grothendieck ring of \( GL(V) \)-modules; we view the latter as the ring of symmetric functions. In the language of symmetric functions, the Schur function \( s_\lambda \) corresponds to the irreducible representation \( S^\lambda(V) \) (strictly speaking, we must quotient out \( s_\lambda \) for \( \lambda \) with more parts than
of Equation 1. In particular, such a module with fixed $\lambda$ and passing to Grothendieck rings, this becomes $\sum_{\mu} s_{\mu'} (x^{-1}) [M_{\mu}^n]$, where a Schur function indicates a representation of $GL(V)$ (the inverted variables account for the dualised space $V^*$). Calculating the Euler characteristic directly from the cochain groups, we consider the $i$-th exterior power of $g = L \otimes C^n$,

\[
\bigwedge^i (g) = \bigoplus_{\lambda \vdash i} S^\lambda (L) \otimes S^\lambda (C^n)
\]

which gives $\sum_{\lambda, i} [S^\lambda (C^n)] s_{\lambda'} [L] (x)$.

Thus the coefficient of $[S^\lambda (C^n)]$ in $[M_{\mu}^n]$ is the coefficient of $s_{\mu'} (x^{-1})$ in $\sum_{\lambda} (\sum_{\mu} [-1] [\lambda] [S^\lambda (C^n)] s_{\lambda'} [L] (x^{-1})$, which gives us

\[
[M_{\mu}^n] = \sum_{\lambda} (\sum_{\mu} [-1] [\lambda] [S^\lambda (C^n)] s_{\lambda'} [L], s_{\mu'}).
\]

This provides an alternative proof the formula from ASIS for expressing the irreducible representation $S^\mu [n]$ of $S_n$ in terms of restrictions $Res_{S_n}^{GL_n} (S^\lambda (C^n))$. This construction addresses a remark of Assaf and Speyer by examining the presence of the character of the free Lie algebra (namely, $L$) in the formula.

4. Application to $\mathcal{F}$-modules

Let $\mathcal{F}$ denote the category of finite sets. An $\mathcal{F}$-module is a functor from $\mathcal{F}$ to vector spaces over a fixed field. These were introduced in [WGL4], and their homological algebra was studied over $\mathbb{Q}$. An $\mathcal{F}$-module consists of a $S_n$-module for each $n$ together with suitably compatible maps between them. (This is because the image of an $n$-element set carries an action of $Aut(\{1, 2, \ldots, n\}) = S_n$.) When $\mu$ is a partition different from $(1^k)$ (i.e. not a single column), $M_{\mu}^n$ (considered for fixed $\mu$ but varying $n$) defines an irreducible $\mathcal{F}$-module, by demanding that an $n$-element set in $\mathcal{F}$ map to $M_{\mu}^n$ (see Theorem 5.5 of [WGL4]). Furthermore, in this category, objects obtained by restricting $S^\lambda (\mathbb{Q}^n)$ to $S_n$ are projective (see Definition 4.8 and Proposition 4.12 of [WGL4]). Our resolution (provided we replace all instances of $\mathbb{C}$ with $\mathbb{Q}$) therefore gives a projective resolution of these simple $\mathcal{F}$-modules $M_{\mu}^n$. This resolution is in fact minimal (in the sense that each step in the projective resolution is as small as possible). This follows from the following two facts. Firstly, the $r$-th term in the resolution of $M_{\mu}^n$ is a sum of $Res_{S_n}^{GL_n} (S^\lambda (\mathbb{Q}^n))$ with $|\lambda| = |\mu| - r$, which is a consequence of Equation 1. In particular, such a module with fixed $\lambda$ can only appear in one step of the resolution. Secondly, a theorem of Littlewood (Theorem XI of [Lit58]), states that the restriction multiplicity $a_{\mu, \lambda}^r$ is equal to $\delta_{\mu, \lambda}$ if $|\mu| \geq |\lambda|$. Thus, $[Res_{S_n}^{GL_n} (S^\lambda (\mathbb{Q}^n))$ are linearly independent elements of the Grothendieck ring of $S_n$-modules, provided $n$ is sufficiently large. Furthermore, the $[Res_{S_n}^{GL_n} (S^\lambda (\mathbb{Q}^n))$ should only occur in the resolution in order of decreasing $|\lambda|$ (as in our resolution). Together with Observation 4.25 of [WGL4], which provides a projective resolution of certain $\mathcal{F}$-modules $D_k$ (which can be thought of as substitutes for $M_{\mu}^n$ when $\mu = (1^k)$), we obtain minimal projective resolutions of all finitely-generated $\mathcal{F}$-modules over $\mathbb{Q}$.

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