Hermite polynomials and Fibonacci Oscillators

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Abstract

We compute the $(q_1,q_2)$-deformed Hermite polynomials by replacing the quantum harmonic oscillator problem to Fibonacci oscillators. We do this by applying the Jackson derivative. The deformed energy spectrum is also found in terms of these parameters. We conclude that the deformation is more effective in higher excited states. We conjecture that this achievement may find applications in the inclusion of disorder and impurity in quantum systems. The ordinary quantum mechanics is easily recovered as $q_1 = q_2 = 1$.

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I. INTRODUCTION

It is well known that the resolution of the Schrödinger equation leads to the knowledge of the temporal and spatial evolution of the form of the wave associated with a non-relativistic particle [1, 2]. This is the Schrödinger picture for the non-relativistic quantum mechanics. Several methods and techniques have been developed over the past decades for analytical approximate and exact solutions to improve the understanding of its dynamical behavior.

The insertion of $q$-algebra [3–8] in quantum mechanics as well as the study of $q$-deformed harmonic oscillator have been intensively investigated in the literature [9–15]. F. H. Jackson in his pioneering works introduced the $q$-deformed algebra [16], where several aspects of investigations played an important role for the understanding and development of such an algebra. One of its main ingredients is the presence of a deformation parameter $q$, introduced in the commutation relations that define the Lie algebra of the system with the condition that the original algebra is recovered in the limit $q \to 1$. $q$-Oscillators using so-called Jackson derivative ($JD$) have been considered in order to determine a generalized deformed dynamic in a $q$-commutative phase space [17]. For this purpose one makes use of creation and annihilation operators of $q$-deformed quantum mechanics.

A new proposal for the $q$-calculation is the inclusion of two distinct deformation parameters in some physical applications. Starting with the generalization of $q$-algebra [16], in [18] was generalized the Fibonacci sequence. Here, the numbers are in that sequence of generalized Fibonacci oscillators, where new parameters $(q_1, q_2)$ are introduced [19–21]. They provide a unification of quantum oscillators with quantum groups [22–25], keeping the degeneracy property of the spectrum invariant under the symmetries of the quantum group. The quantum algebra with two deformation parameters may have a greater flexibility when it comes to applications in realistic phenomenological physical models [26, 27].

In this paper we compute the $(q_1, q_2)$-deformed Hermite polynomials by replacing the quantum harmonic oscillator problem to Fibonacci oscillators by changing the ordinary derivative to Jackson derivative. The deformed energy spectrum is also found in terms of these parameters. The ordinary quantum mechanics is easily recovered as $q_1 = q_2 = 1$.

The paper is organized as follows. In Section II we present the $q$-deformed algebra. In Section III we obtain the Hermite polynomials with Fibonacci oscillators and finally, in Section IV we make our final comments.
II. FIBONACCI OSCILLATORS ALGEBRA

The generalization of integers usually is given by a sequence. The two well-known ways to
describe a sequence are the arithmetic and geometric progressions. However, the Fibonacci
sequence encompasses both. By generalizing this sequence, we get the Fibonacci oscillators,
so the spectrum can now be given by the Fibonacci integers. The algebraic symmetry of
the quantum oscillator is defined by the Heisenberg algebra in terms of the annihilation and
creation operators $c, c^\dagger$ respectively, and the number operator $N$, as follows

\[ c_i c_i^\dagger - q_1^2 c_i^\dagger c_i = q_2^{2N_i} \quad \text{and} \quad c_i c_i^\dagger - q_2^2 c_i^\dagger c_i = q_1^{2N_i}, \]

\[ [N, c^\dagger] = c^\dagger, \quad [N, c] = -c. \]  

In addition, the operators obey the relations

\[ c^\dagger c = [N], \quad cc^\dagger = [1 + N], \]

\[ [1 + N_{i,q_1,q_2}] = q_1^2 [N_{i,q_1,q_2}] + q_2^{2N_i}, \quad \text{or} \quad [1 + N_{i,q_1,q_2}] = q_2^2 [N_{i,q_1,q_2}] + q_1^{2N_i}. \]

The Fibonacci basic number is defined as

\[ [n_{i,q_1,q_2}] = c_i^\dagger c_i = \frac{q_1^{2n_i} - q_2^{2n_i}}{q_1^2 - q_2^2}, \]

where $q_1$ and $q_2$ are parameters of deformation that are real, positive and independent. A
few $(q_1, q_2)$-numbers are given here:

\[ [0] = 0, \]
\[ [1] = 1, \]
\[ [2] = q_1^2 + q_2^2, \]
\[ [3] = q_1^4 + q_2^4 + q_1^2 q_2^2, \]
\[ [4] = q_1^6 + q_2^6 + q_1^2 q_2^4 + q_1^4 q_2^2, \]
\[ [5] = q_1^8 + q_2^8 + q_1^6 q_2^2 + q_1^2 q_2^6 + q_1^4 q_2^4, \]
\[ [6] = q_1^{10} + q_2^{10} + q_1^8 q_2^2 + q_1^2 q_2^8 + q_1^6 q_2^4 + q_1^4 q_2^6, \]

\[ \cdots \cdots \cdots \]
One may transform the $q$-Fock space into the configuration space (the Bargmann holomorphic representation) \[17\] as in the following:

$$c^\dagger = x, \quad c = D_x^{(q_1,q_2)},$$

where $D_x^{(q_1,q_2)}$ is the Jackson derivative (JD) \[16, 21\] defined as

$$D_x^{(q_1,q_2)} f(x) = \frac{q_1^2 - q_2^2}{\ln \left( \frac{q_1^2}{q_2^2} \right)} \partial_x^{(q_1,q_2)} f(x), \quad \text{where} \quad \partial_x^{(q_1,q_2)} f(x) = \frac{f(q_1^2 x) - f(q_2^2 x)}{x(q_1^2 - q_2^2)},$$

such that

$$D_x^{(q_1,q_2)} f(x) = \frac{f(q_1^2 x) - f(q_2^2 x)}{x \ln \left( \frac{q_1^2}{q_2^2} \right)},$$

and

$$D_x^{(q_1,q_2)^2} f(x) = \frac{q_1^2 f(q_1^2 x) + q_2^2 f(q_2^4 x) - (q_1^2 + q_2^2) f\left( q_1^2 q_2^2 x \right)}{q_1^4 q_2^4 x^2 \ln \left( \frac{q_1^2}{q_2^2} \right)^2},$$

where $D_x^{(q_1,q_2)^2} \equiv D_x^{(q_1,q_2)} D_x^{(q_1,q_2)}$ \[16\], and so on.

Using the definition of the $q$-derivative one can easily find several properties of the JD \[28–31\], e.g.,

$$D_x^{q}(\exp_q(ax)) = a \exp_q(ax),$$

and

$$D_x^{q}(ax^n) = a[n]x^{n-1},$$

which will be useful in the calculations that we are going into details shortly.

### III. DEFORMED HERMITE POLYNOMIALS

We start with the Schrödinger equation for the harmonic oscillator and introduce Fibonacci oscillators by replacing the ordinary derivative to Jackson derivative, i.e.,

$$-\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} + \frac{m \omega^2 x^2}{2} \Psi = E \Psi \quad \longrightarrow \quad -\frac{\hbar^2}{2m} D_x^{(q_1,q_2)^2} \Psi + \frac{m \omega^2 x^2}{2} \Psi = E \Psi.$$
We solve this quantum mechanical problem by using the standard power series method (analytical method) found in the literature \cite{1, 2}, and for this let us first introduce the dimensionless variable \( \xi \)

\[
\xi = \sqrt{\frac{m\omega}{\hbar}} x. \tag{15}
\]

Now we can write the ordinary Schrödinger equation (14) in terms of \( \xi \) as in the form

\[
\frac{d^2\Psi}{d\xi^2} = (\xi^2 - K)\Psi, \quad \text{where} \quad K \equiv \frac{2E}{\hbar\omega}. \tag{16}
\]

In the asymptotic limit (\( \xi \) very large) the term \( \xi^2 \) dominates over the constant term \( K \), i.e.,

\[
\frac{d^2\Psi}{d\xi^2} \approx \xi^2\Psi, \tag{17}
\]

which has approximate solution,

\[
\Psi(\xi) \approx A \exp \left( -\frac{\xi^2}{2} \right) + B \exp \left( \frac{\xi^2}{2} \right). \tag{18}
\]

At large \( \xi \) this suggest the following Ansatz for the general solution

\[
\Psi(\xi) = h(\xi) \exp \left( -\frac{\xi^2}{2} \right). \tag{19}
\]

Now we can get the first and second Jackson derivatives as follows

\[
\frac{d\Psi}{d\xi} \rightarrow D_{\xi}^{(q_1, q_2)}\Psi = \left( D_{\xi}^{(q_1, q_2)} h - \frac{\xi [2]}{2} \right) \exp \left( -\frac{\xi^2}{2} \right), \tag{20}
\]

\[
\frac{d^2\Psi}{d\xi^2} \rightarrow D_{\xi}^{(q_1, q_2)^2}\Psi = \left[ D_{\xi}^{(q_1, q_2)^2} h - [2] \xi D_{\xi}^{(q_1, q_2)} h + \left( \xi^2 - \frac{[2]}{2} \right) h \right] \exp \left( -\frac{\xi^2}{2} \right). \tag{21}
\]

Replacing this into Eq. (16), we obtain

\[
D_{\xi}^{(q_1, q_2)^2} h - [2] \xi D_{\xi}^{(q_1, q_2)} h + \left( K - \frac{[2]}{2} \right) h = 0. \tag{22}
\]

Many special functions are known as solution to differential equations of the type given in (22). In our particular case, the solution is known as given in terms of Hermite polynomials in \( \xi \). Let us now go into details by proposing a solution in the form of power series in \( \xi \),

\[
h(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + \cdots = \sum_{j=0}^{\infty} a_j \xi^j. \tag{23}
\]
By applying the first and second JD (or the ‘\(q_1, q_2\)-derivative’) to the series we find respectively

\[
D^{(q_1, q_2)} \xi h = a_1 + [2]a_2 \xi + [3]a_3 \xi^2 + \cdots = \sum_{j=0}^{\infty} [j]a_j \xi^{j-1},
\]

and

\[
D^{(q_1, q_2)^2} \xi h = [2]a_2 + [2][3]a_3 \xi + [3][4]a_4 \xi^2 + \cdots = \sum_{j=0}^{\infty} ([j] + 1)([j] + 2)a_j+2 \xi^j.
\]

We can now rewrite Eq. (22) as follows

\[
\sum_{j=0}^{\infty} \left\{ ([j] + 1)([j] + 2)a_{j+2} - [2][j]a_j + \left( K - \frac{[2]}{2} \right) a_j \right\} \xi^j = 0.
\]

Since the coefficient of each power in \(\xi\) should disappear, then

\[
([j] + 1)a_{j+2} - [2][j]a_j + \left( K - \frac{[2]}{2} \right) a_j = 0,
\]

that is

\[
a_{j+2} = \frac{([2][j] - K + \frac{[2]}{2})}{([j] + 1)([j] + 2)} a_j.
\]

For the sake of comparison with the ordinary case, we show that the recursion formula (28) gives explicitly the first three coefficients

\[
a_2 = \frac{([2][0] - K + \frac{[2]}{2})}{([0] + 1)([0] + 2)} a_0 = \frac{([2] - K)}{2} a_0,
\]

\[
a_3 = \frac{([2][1] - K + \frac{[2]}{2})}{([1] + 1)([1] + 2)} a_1 = \frac{([3][2] - K)}{6} a_1,
\]

\[
a_4 = \frac{([2][2] - K + \frac{[2]}{2})}{([2] + 1)([2] + 2)} a_2 = \frac{([2][3][2] - K)}{[2]^2 + 3[2] + 2} a_2.
\]

The complete solution is written as follows:

\[
h(\xi) = h_{even}(\xi) + h_{odd}(\xi) = a_0 + a_1 \xi + a_2 \xi^2 + \cdots.
\]

Thus, Eq. (28) determines \(h(\xi)\) in terms of the arbitrary constants \(a_0\) and \(a_1\), a fact that is expected for a second-order differential equation. However, some obtained solutions are not normalizable. Let us discuss this in details in the following.

For very large \(j\), the recursion formula will be given approximately by

\[
a_{j+2} \approx \frac{[2]}{[j]} a_j, \quad a_j \approx \frac{C}{([j]/[2])!},
\]
where $C$ is a constant, and with large values of $\xi$, we have

$$h(\xi) \approx C \sum \frac{1}{(j/[2])!} \xi^j \approx C \sum \frac{1}{[j]!} \xi^j \approx C \exp_{q_1,q_2}(\xi^2),$$

where the $q$-exponential function is defined as

$$\exp_q(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]!} x^n, \quad \text{or} \quad \exp_{q_1,q_2}(ax) = \sum_{n=0}^{\infty} \frac{a^n}{[n]_{q_1,q_2}!} x^n.$$ 

Returning to Eq.(19), where we have the asymptotic behavior and using Eq.(34) where we have that $h$ behaves as $\exp_{q_1,q_2}(\xi^2)$, so $\Psi$ behaves as $\exp(\xi^2/2)$, for instance when $q_1 = q_2 = 1$, which is precisely the solution we disregarded since the very beginning. These are types of non-normalizable solutions.

In order to obtain normalizable solutions, the power series must terminate. This must happens in the highest $j$ that we call $n$, so that Eq.(28) produces

$$a_{n+2} = 0, \quad a_1 = 0 \text{ for } n \text{ even and } a_0 = 0 \text{ for } n \text{ odd}. \quad (36)$$

Physically acceptable solutions require that Eq.(28) gives

$$K = [2][n] + \frac{[2]}{2} \quad \text{or} \quad K = [n_{q_1,q_2}] + [n_{q_1,q_2} + 1],$$

for some non-negative number $n_{q_1,q_2}$, i.e., the energy must be

$$E_{n_{q_1,q_2}} = \frac{\hbar \omega}{2} \left([n_{q_1,q_2}] + [n_{q_1,q_2} + 1]\right) = \frac{\hbar \omega}{2} + \frac{\hbar \omega (2 \ln(q_2) - 2 \ln(q_1)) n}{q_2 - q_1^2},$$

and when $q_1 = q_2 = 1$, we have

$$E_n = \frac{\hbar \omega}{2} \left(2n + 1\right).$$

We now focus on determining the $(q_1,q_2)$-deformed Hermite polynomials. The Hermite polynomials are a sequence of orthogonal polynomials that arise in probability theory and physics [32]. As we know, they give rise to the eigenstates of the ordinary (undeformed) quantum harmonic oscillator.

For the allowed values of $K$, we have the recursion formula

$$a_{j+2} = \frac{[2]}{(j-[n])} \frac{[j]-[n]}{(j+1)(j+2)} a_j.$$

Let us now obtain the first three terms of the series (32). If $n = 0$ we have only one term, and we must choose $a_1 = 0$ to neutralize $h_{odd}$, and for $j = 0$ in Eq.(40), we obtain $a_2 = 0$, such that

$$h_0(\xi) = a_0, \quad \text{and} \quad \Psi_0(\xi) = a_0 \exp \left(-\frac{\xi^2}{2}\right).$$

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For $n = 1$, $a_0 = 0$ and $j = 1$, one finds $a_3 = 0$, then
\[ h_1(\xi) = a_1 \xi, \quad \text{and} \quad \Psi_1(\xi) = a_1 \xi \exp\left(-\frac{\xi^2}{2}\right). \] (42)

For $n = 2$ and $j = 0$, we have $a_2 = -\frac{[2]^2}{2}a_0$, and $j = 2$ yields $a_4 = 0$, thus
\[ h_2(\xi) = a_0 \left(1 - \frac{[2]^2}{2} \xi^2\right), \quad \text{and} \quad \Psi_2(\xi) = a_0 \left(1 - \frac{[2]^2}{2} \xi^2\right) \exp\left(-\frac{\xi^2}{2}\right). \] (43)

In general, $h_n(\xi)$ is a polynomial of degree $n$ in $\xi$, for $n$ being either even or odd. Up to the general factor ($a_0$ or $a_1$) we will call them ($q_1, q_2$)-Hermite polynomials, $H_n^{(q_1, q_2)}(\xi)$. Since the Eq.(22) is homogeneous, the Hermite polynomials are defined up to a multiplicative constant. Adopting the same usual convention of the ordinary (undeformed) case, we choose the constants $a_0$ or $a_1$ so that the coefficient of the highest term $\xi^n$ in $h_n(\xi)$ is $[2]^n$. This completely defines the other coefficients from the recursion relation (40) by using the allowed values of $K$.

We can now write the ($q_1, q_2$)-deformed stationary states for the Fibonacci oscillators as follows
\[ \Psi_n^{(q_1, q_2)}(\xi) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n[n_{q_1, q_2}]!}} H_n^{(q_1, q_2)}(\xi) \exp\left(-\frac{\xi^2}{2}\right), \] (44)
where the first ($q_1, q_2$)-Hermite polynomials are given by
\[ H_0 = 1, \]
\[ H_1 = [2] \xi, \]
\[ H_2 = ([2] \xi)^2 - 2, \]
\[ H_3 = ([2] \xi)^3 - 3[2]^2 \xi, \]
\[ H_4 = 3[2]^2 - ([2] \xi)^4 - 12([2] \xi)^2, \]
\[ H_5 = 15[2]^3 - 10[2]^4 \xi^3 + ([2] \xi)^5. \] (45)

We can also write these polynomials through the ($q_1, q_2$)-deformed version of the well-known Rodrigues formula \[1, 2\]
\[ H_n^{(q_1, q_2)}(\xi) = (-1)^n \exp_{q_1, q_2}(\xi^2)D_{\xi}^{n(q_1, q_2)} \exp_{q_1, q_2}(-\xi^2). \] (46)

Below, in Figs.1-3 we depict the first three wave functions $\Psi_n^{(q_1, q_2)}$. Notice the presence of the deformation in the Fibonacci oscillators is evident in the curves for different ($q_1, q_2$) parameters.
Figure 1: Stationary state $\Psi_1$ for several $(q_1, q_2)$ parameters. The undeformed case is $q_1 = q_2 = 1$ (black curve) and the most deformed case is $q_1 = q_2 = 0.1$ (blue curve).

Figure 2: Stationary state $\Psi_2$ for several $(q_1, q_2)$ parameters. The undeformed case is $q_1 = q_2 = 1$ (black curve) and the most deformed case is $q_1 = q_2 = 0.1$ (blue curve).

We can observe in the Figs. 1-3 that the behavior of the curves is altered with the presence of the $(q_1, q_2)$-deformation. In all cases the black curves do not present deformation, since they are in the limit $q_1 = q_2 = 1$ and the most deformed case is $q_1 = q_2 = 0.1$ depicted by the blue curves.

In Fig. 1 we can observe that the behaviors are similar. The deformation acts simply varying the amplitude of the curves.

In Figs. 2 and 3 we note that the presence of the deformation develops a greater role in the
Figure 3: Stationary state $\Psi_3$ for several $(q_1, q_2)$ parameters. The undeformed case is $q_1 = q_2 = 1$ (black curve) and the most deformed case is $q_1 = q_2 = 0.1$ (blue curve).

next excited states. The behavior of the other curves become very different from the black curve. On the right panel we have a zoom in three curves to carry out a better analysis of the deformation.

IV. CONCLUSIONS

As expected from Eq.(44), the Fibonacci oscillators have modified behavior in the stationary states. It is clear that as the deformation parameters decrease in relation to the undeformed case $q_1 = q_2 = 1$, the differentiated behavior of the curves becomes more evident. This also becomes clear as we look at the Hermite polynomials (45) where $H_3$ feels a greater presence of $q_1$ and $q_2$ than $H_1$. We can conclude that the more the states are excited the more strong is the deformation on them. This may find interesting applications in quantum mechanics such as inclusion of disorders and impurities in the quantum system. For instance, in [21] several studies were put forward uncovering the fact that the $q$-deformation affects the oscillator frequency which may be associated with the changing in the strength of the ‘spring constant’ associated with such an oscillator as a consequence of introduction of impurities or disorders in the system. This should be further addressed elsewhere.
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