An integral of the Wigner function of a wavefunction $|\psi>$, over some region $S$ in classical phase space is identified as a (quasi) probability measure (QPM) of $S$, and it can be expressed by the $|\psi>$ average of an operator referred to as the region operator (RO). Transformation theory is developed which provides the RO for various phase space regions such as point, line, segment, disk and rectangle, and where all those ROs are shown to be interconnected by completely positive trace increasing maps. The latter are realized by means of unitary operators in Fock space extended by $2D$ vector spaces, physically identified with finite dimensional systems. Bounds on QPMs for regions obtained by tiling with discs and rectangles are obtained by means of majorization theory.

**Key words:** Wigner function, region operators, Quantum Mechanics, quantum tomography, completely positive maps, majorization.

1. Introduction

The Wigner quasidistribution function on phase space $W(p, q)$ [1], provides an important tool in the description of quantum systems from both theoretical and experimental point of view. Of particular interest is for example, its application to the field of quantum tomography [2],[3]. Although the Wigner function shares several properties with classical distribution functions on phase space it is not positive-definite. As a consequence, the value of its integral, (to be called alternatively quasiprobability integral(qpi), mass (qpm), or volume (qpv)), over subregions of phase space, lies in general outside the interval $[0, 1]$, and may admit large positive or negative values. The upper and lower bounds of such integrals define tests e.g on the accuracy of experimental determinations of the
Wigner function in the sense that they give the degree to which the value of the integral lies outside \([0, 1]\), and thus their determination is of particular importance. In [4], [5], [6] and [7] it has been shown that such bounds, for given regions or contours of phase space for 1D quantum systems, are determined by the maximal and minimal eigenvalues of corresponding Hermitian operators \(K_S\), called region operators (RO), whose spectral analysis has been shown to have a rich algebraic structure. Consequently the study of the properties of such operators defined by

\[
K_S = \int_\Gamma \chi_S(\alpha)W(\alpha)da ,
\]

where \(S\) is a subregion in phase space \(\Gamma\), \(\alpha \in \Gamma\), \(\chi_S(\alpha)\) is the characteristic function corresponding to \(S\) and \(W(\alpha)\) are the Wigner operators, is of great importance.

This article presents a detail analysis of transformation theory for RO, and in particular investigates the cases of the point, line, rectangle and disk operators. The development of the theory is mainly based on the notion of completely positive trace increasing map (CPTI) (c.f [8]), which provides the interconnection between various region operators. The application of such maps is also shown to describe tiling processes in phase space. Moreover the eigenvalues of the obtained ROs from step to step during the tiling are shown to respect majorization relations[10]. This allows the determination of upper and lower bounds on qpi over regions in phase space obtained by a tiling process.

An outline of the work is as follows: section two focuses on line and rectangle ROs. The spectral analysis of straight line segment ROs is presented and further it is shown that straight line ROs are projection operators. On the other hand the normally ordered form of rectangle ROs is obtained, and moreover it is shown that all rectangle ROs obtained from an initial one by rigidly shifting its position in the phase plane, are isospectral and thus have the same quasiprobability volume.

In section three an operational construction of quasi-probability measures of canonical polygon ROs is developed by introducing appropriate completely positive trace increasing (CPTI) maps. An explicit example is then given in which a general canonical hexagon RO is constructed from the corresponding RO of an isosceles triangle, by means of CPTI maps.

Finally in section four, appropriate CPTI maps are used to realize a tiling process in phase space acting on rectangle and disk ROs used as building blocks. The step matrix which connects the eigenvalue-vectors of ROs at each step of the tiling process is shown to be simply determined by doubly stochastic matrices. This further allows to determine simple ordering relations between the eigenvalues of successive ROs by means of the majorization theory. This fact provides intervals of extremal values for the eigenvalues i.e the qpm’s, of the sequence of region operators, in a tiling process. Section five briefly summarizes results and outlines some prospects of the theory of region operators.

2. Point, Line and Rectangle Region Operators

**Proposition 1.** Region operators having support on a straight line segment of length \(L\) in angle \(\theta\) with the \(q\)-axis read, \(K^L_\theta = \frac{\sin[(Q\cos \theta + P\sin \theta)L]}{Q\cos \theta + P\sin \theta} \Pi\), and satisfy the eigenvalue
the Hermite polynomials as $\langle \delta \rangle$, the two positive maps $\epsilon$ of the point at the origin along the $q$-axis, respectively the following position and momentum representation:

$$\Psi = L \cos(\theta) \sin(\frac{\pi}{2}) \exp(-i\phi) \Pi.$$ 

In this way we obtain the region operator $K_L^\theta_\parallel$, with support on the straight line segment of length $L$, symmetric with respect to the origin and lying along the position axis, (see figure 1b). Similarly for the segment along the momentum axis, we get $K_L^\theta_\perp = \sin(\frac{\pi}{2}) \exp(-i\phi) \Pi$. The $|q\rangle$ generalized eigenvectors, have respectively the following position and momentum representation: $\psi_\parallel(q) = \langle q | \Psi_\parallel^\theta = \delta(q + q) \pm \delta(q - q)$, and $\psi_\perp(p) = \langle p | \Psi_\perp^\theta = \langle p | \Psi_\parallel^\theta = \frac{1}{\sqrt{2\pi}} \cos(\phi)$. 

If we now rotate by angle $\theta$ the line segment support of region operator $K_L^\theta_\parallel$, by the rotation operator $e^{i\theta N}$, where $N$ the number operator, we end up with the operator

$$K_L^\theta_\parallel = e^{i\theta N} K_L^\theta_\parallel e^{-i\theta N} = \frac{\sin[(Q \cos \theta + P \sin \theta)L]}{Q \cos \theta + P \sin \theta} \Pi.$$ 

As special cases we obtain the region operators for the line segment along the $p-$ axis $K_L^{\theta=0} = K_L^\theta$, and along the $q-$ axis $K_L^{\theta=\pi/2} = K_L^\theta$. 

Remarks: 1) As the trace of a region operator equals the area of the support of the operator itself, the two positive maps $\epsilon^L_\parallel : K_0 \rightarrow K_L^Q, K_0 \rightarrow K_L^P$, introduced previously to get the straight line segment operator from a point operator are trace increasing maps i.e $Tr(K_L^\theta) = Tr(\epsilon^L_\parallel(K_0)) = LTr(K_0)$. 

2) The region operator $K_L^\theta = \sin(\frac{Q(\theta)L}{2}) \Pi$, is written in terms of the rotated operator $Q_\theta = Q \cos \theta + P \sin \theta$. Two copies of the latter with their angles of rotation differing by
\[ Q_0 = Q \cos \theta + P \sin \theta, \quad Q_{\pi / 2} = Q \sin \theta - P \cos \theta, \]  
are canonical i.e \([Q_\theta, Q_{\theta + \pi / 2}] = i \mathbf{1}].

3) Some additional properties of the family of vectors \(\{|q_\theta\rangle, q_\theta \in \mathbb{R}\}\), are: it is a complete set i.e \(\int_{-\infty}^{\infty} dq_\theta |q_\theta\rangle \langle q_\theta| = \mathbf{1}\), its generalized vectors are orthogonal i.e \(\langle q_\theta | q'_\theta \rangle = \delta(q_\theta - q'_\theta)\), and can be constructed from the zero number state as

\[ |q_\theta\rangle = \frac{1}{\pi^{1/4}} \exp \left[ -\frac{1}{2} q_\theta^2 + \sqrt{2} \exp(i\theta) q_\theta a\dagger - \frac{1}{2} \exp(2i\theta) a^{2\dagger} \right] |0\rangle, \tag{5} \]

and so the parity operator flips the sign of its argument i.e \(\Pi |q_\theta\rangle = | - q_\theta\rangle\). These same states are also specified from their overlap with a coherent state i.e

\[ \langle q_\theta | a \rangle = \frac{1}{\pi^{1/4}} \exp \left[ i(Q_{\theta + \pi / 2})q_\theta \right] \exp[-\frac{1}{2}(q_\theta - (Q_\theta))^2] \exp[-\frac{i}{2}(Q_\theta)\langle Q_{\theta + \pi / 2}|], \tag{6} \]

where \(\langle Q_\theta \rangle = \langle \alpha | Q_\theta | \alpha \rangle = \frac{1}{\sqrt{2}} [\alpha \exp(-i\theta) + \alpha^* \exp(i\theta)]\).

4) Region operators with support on straight line segments with proportional lengths, in any direction, are commuting i.e \(K_0 K_L = K_L K_0\), for \(\lambda \geq 1, \) and \(0 \leq \theta < 2\pi\).

**Proposition 2.** Region operators having support on a straight line are projection operators.

**Proof:** Let the region operator \(K_0 = \mathbf{1}\). Its smearing along the whole momentum \(p\)-axis results into the following region operator

\[ K_{Q=0} = \int_{-\infty}^{\infty} dp D(0,p) \Pi D(0,p)^\dagger = \frac{1}{\sqrt{\pi}} : e^{-Q^2} := |q = 0\rangle \langle q = 0|. \tag{7} \]

If we next transform this operator to have support along an axis parallel to \(p\)-axis and crossing the position axis in the point \(q \in \mathbb{R}\) then we will obtain

\[ K_{Q=q} = D(q,0) K_{Q=0} D(q,0)^\dagger = \frac{1}{\sqrt{\pi}} : e^{-(q-Q)^2} := |q\rangle \langle q|, \tag{8} \]

namely the projection operator in position states. Similarly by smearing the point operator along the position \(q\)-axis we obtain

\[ K_{P=0} = \int_{-\infty}^{\infty} dq D(q,0) \Pi D(q,0)^\dagger = \frac{1}{\sqrt{\pi}} : e^{-P^2} := |p = 0\rangle \langle p = 0|. \tag{9} \]

This operator meets the \(p\)-axis at its zero point, so by displacing it arbitrarily by \(p \in \mathbb{R}\) we will obtain

\[ K_{P=p} = D(0,p) K_{P=0} D(0,p)^\dagger = \frac{1}{\sqrt{\pi}} : e^{-(p-P)^2} := |p\rangle \langle p|, \tag{10} \]
they have the same quasiprobability volumes. rigidly shifting its position by respectively on any straight line parallel to the \( q \)-axis at the point \( q \), and respectively on any straight line parallel to the \( q \)-axis crossing the \( p \)-axis at the point \( p \) have been constructed. In order to rotate these support axes by any desired angle \( \theta \) with respect to e.g the \( q \)-axis, we should perform an additional unitary rotation with \( e^{i\theta N} \), where \( N \) is the number operator, the generator of rotations around the origin. Namely, we should consider the transformation \( K_{Q=q} (K_{P=p}) \rightarrow e^{i\theta N} K_{Q=q} e^{-i\theta N} (e^{i\theta N} K_{P=p} e^{-i\theta N}) \). The latter will not change the projective character of \( K_{Q=q} = |q\rangle\langle q| \), and \( K_{P=p} = |p\rangle\langle p| \), operators (see also [9])

**Remarks:** 1) The region operator having support on the whole plane is the unit operator. In order to obtain the region operator on the plane we should "add" the region operators on all lines parallel to any given line. Let e.g \( K_{Q=q} = |q\rangle\langle q| \), and \( K_{P=p} = |p\rangle\langle p| \), the two projective operators that have support on some straight line parallel to \( q \)-axis crossing the \( q \)-axis at the point \( q \), and respectively on some straight line parallel to the \( q \)-axis crossing the \( p \)-axis at the point \( p \). Their integrals \( \int_R K_{Q=q} dq = \int_R |q\rangle\langle q| dq = 1 \), and \( \int_R K_{P=p} dp = \int_R |p\rangle\langle p| dp = 1 \), give the region operator supported on the plane, which actually is the unit operator due to the completeness of position or respectively momentum states. The same result would have been obtained if we would had started with any line in angle \( \theta \) with the \( p, q \) axes.

2) Various region operators can easily be introduced by combining the operators constructed above e.g a region operator with support on a bundle of \( n \) parallel lines crossing the \( q \)-axis at the points \( P_n = \{ q_1, q_2, ..., q_n \} \) is defined as \( K_{P_n} = |q_1\rangle\langle q_1| + |q_2\rangle\langle q_2| + ... + |q_n\rangle\langle q_n| \). The latter can be obtained acting on \( |q_1\rangle\langle q_1| \), the operator with support on the first line of the bundle, with the trace increasing map \( \varepsilon_n \), i.e \( K_{P_n} = \varepsilon_n (|q_1\rangle\langle q_1|) = \sum_{i=1}^{n} V_i |q_1\rangle\langle q_1| V_i^\dagger \), which has an operator sum representation in terms of the unitary operators \( V_i = \exp(i(q_i - q_1) P), \ i = 1, 2, ..., n \).

Let us now show a more special result related to motions that can be done on rectangle operators that leave their spectrum, namely their associated quasiprobability volumes, invariant.

**Proposition 3.** If we denote by \( K_S \equiv K(x_0, k_0; A, B) \) the region operator with support on a rectangle domain \( S \) in phase space with lower left corner at the point \( (x_0, k_0) \) and sides of lengths \( A \) and \( B \) along the axes of position and momentum variables respectively, then i) the normal order form of this operator is given by equation (14) ii) all operators \( K(x_0, k_0; A, B), K(x_0 + s, k_0 + t; A, B) \) for \( s, t \in R \) are isospectral, namely they are region operators obtained from the initial rectangle operator \( K(x_0, k_0; A, B) \) by rigidly shifting its position by \( s, t \), and have all eigenvalues equal, which then implies that they have the same quasiprobability volumes.

**Proof:** i) Let us recall that in general an operator \( K_S \) associated to a phase space
region \( S \) is given in terms of the Wigner operator

\[
W(\alpha) = \frac{1}{2\pi} D(\alpha) (-1)^N D(\alpha)\dagger = \frac{1}{2\pi} e^{i\pi(a^\dagger - \alpha^*) (a - \alpha)} = \frac{1}{2\pi} e^{-2(a^\dagger - \alpha^*) (a - \alpha)}: \quad (11)
\]

The last equation is in normally ordered form and it is easily obtained by means of the formula

\[
e^{\lambda(a^\dagger - \alpha^*) (a - \alpha)} = e^{(e^{\lambda} - 1)(a^\dagger - \alpha^*) (a - \alpha)}: \quad \text{valid for complex parameter } \lambda.
\]

Then by means of the (over)complete basis of coherent state vectors \(|z = q + ip\rangle \equiv |qp\rangle, z \in C\), the region operator becomes

\[
K_S = \frac{1}{2\pi} \int_{z \in C} \int_{\alpha \in \text{supp}(\chi_S)} : e^{-2(a^\dagger - \alpha^*) (a - \alpha)} d^2 \alpha : \quad (12)
\]

At this point we employ the technique of integration within an ordered product (IWOP)\[11\], which applies in our case since the above operator integrand is in normally ordered form. In effect splitting the integration and applying IWOP yields, after using the integral representation of error function \( \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \), the normal ordered expression of the region operator

\[
K_S = \frac{1}{(2\pi)^2} \int_{(q,p) \in \mathbb{R}^2} \left( \int_{x_0}^{x_0 + A} e^{-(q-x)^2} dx \right) \left( \int_{k_0}^{k_0 + B} e^{-(p-k)^2} dk \right)|qp\rangle \langle qp| dq dp dx dk \quad (13)
\]

\[
= \frac{1}{16\pi} \left[ \text{erf}\left( -\frac{1}{\sqrt{2}} (a^\dagger + a) + x_0 + A \right) - \text{erf}\left( -\frac{1}{\sqrt{2}} (a^\dagger + a) + x_0 \right) \right] \times
\]

\[
\left[ \text{erf}\left( -\frac{i}{\sqrt{2}} (a^\dagger - a) + k_0 + B \right) - \text{erf}\left( -\frac{i}{\sqrt{2}} (a^\dagger - a) + k_0 \right) \right] : \quad (14)
\]

ii) If we rewrite the rectangle operator \( K_S \equiv K(x_0, k_0; A, B) \) in terms of position and momentum operators i.e \( Q = \frac{1}{\sqrt{2}} (a^\dagger + a) \), \( P = \frac{1}{\sqrt{2}} (a^\dagger - a) \), use the displacement operator property \( D(x^\dagger, k^\dagger) Q D(x, k) = Q - s \), \( D(x^\dagger, k^\dagger) P D(x, k) = P - t \), and the series expansion of the error function, we obtain that

\[
D\left( \frac{s + it}{\sqrt{2}} \right) K(x_0, k_0; A, B) D\left( \frac{s + it}{\sqrt{2}} \right)\dagger = K(x_0 + s, k_0 + t; A, B) \quad (15)
\]

Interpreting these two formulas we say that the family of region operators resulting from rigidly shifting the vertices of the initial rectangle support i.e \( S(x_0, k_0; A, B) \rightarrow S(x_0 + s, k_0 + t; A, B) \), associates itself with region operators that are unitarily equivalent to the region operator of the original rectangle \( K(x_0, k_0; A, B) \). This implies that all these operators have the same spectrum and in turn means that their respective eigenvalues i.e the quasiprobability volumes are equal.
3. Operational Construction of QPMs: The Case of Canonical Polygons

Let us introduce rotation operators in terms of the number operator $N$, to be the operators

$$R(\frac{2\pi j}{M}) = \exp(\frac{2\pi j}{M} jN),$$

which generate rotations by angles $\frac{2\pi j}{M}$, $j = 0, 1, ..., M−1$. Let us introduce an $M$–sided canonical polygon with radial parameter $a$, defined as the distance between the center of the polygon and the midpoint of a side. Due to its geometric invariance under rotation by angles $\frac{2\pi j}{M}$, $j = 0, 1, ..., M − 1$, the associated polygon region operator denoted by $X_{[a,M]}$ is obtained as the sum of copies of the region operator $X_{[a,\frac{2\pi}{M}]}$ of an isosceles triangle, rotated by all angles $\frac{2\pi j}{M}$, $j = 0, 1, ..., M − 1$, [7], i.e

$$\varepsilon(X_{[a,\frac{2\pi}{M}]}) \equiv X_{[a,M]} = \sum_{j=0}^{M-1} R(\frac{2\pi j}{M}) X_{[a,\frac{2\pi}{M}]} R(\frac{2\pi j}{M})^\dagger. \quad (16)$$

In the above equation the completely positive trace increasing map (CPTI) $\varepsilon$, has been introduced by means of its $M$ generators, given by the rotation operators. Since the trace of a region operator equals the area of its associated region, the CPTI map $\varepsilon$ that produces an $M$–polygon operator from its generating isosceles triangle region operator, should be a trace (area) increasing one. Indeed we have $\text{Tr}\varepsilon(X_{[a,\frac{2\pi}{M}]}) = \text{Tr}X_{[a,M]} = M\text{Tr}X_{[a,\frac{2\pi}{M}]}$.

More explicitly we introduce the linear real function of domain $\rho \in \Gamma$, which applied to CP

$$\text{Tr}_\varepsilon, \rho X_{[a,M]} = \sum_{j=0}^{\infty} \rho R(\frac{2\pi j}{M}) X_{[a,\frac{2\pi}{M}]} R(\frac{2\pi j}{M})^\dagger.$$

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trace preserving maps allows to implement them by means of some unitary operator in a space that is an extension of the original one by an auxiliary (ancilla) space, followed by a projection to the initial space by means of a partial trace. This standard unitary dilation theorem of a CPTP map, will be adapted here to our case for the CPTI dual maps. To this end we introduce an $M$ dimensional ancilla space $H_A = \{ |m\rangle , m = 0, 1, \ldots, M - 1 \}$ and choose some operator $|k\rangle \langle k|$ of it. Then on the total space $\mathcal{H} \otimes H_A$, we define an operator $V$, by means of which we write for the maps $\varepsilon$, $\varepsilon^*$ the respective expressions

$$\varepsilon(X_{[a, \frac{2\pi}{M}]}^r) = X_{[a, M]} = Tr_A V(|k\rangle \langle k| \otimes X_{[a, \frac{2\pi}{M}]}))V^\dagger, \quad \varepsilon^*(\rho) = Tr_A V^\dagger(|k\rangle \langle k| \otimes \rho)V, \quad (19)$$

where we choose $\langle j | V | k \rangle = R(2\pi j/M)$, $j = 0, 1, \ldots, M - 1$.

In physical terms, this realization describes the coupling of the given quantum density matrix $\rho$ or the quantum observable $X_{[a, \frac{2\pi}{M}]}$, with an ancilla, ”atomic” $M$–dimensional system, their dynamical interaction by means of the operator $V$, and their subsequent decoupling as a result of partially tracing the ancilla system. The physical resources of such an operational realization is however very demanding since they require the interaction with a high dimensional ”atomic” system. More feasible realizations which are also motivated geometrically by the various ways of covering a polygon by rotated and reflected triangles as desired. In fact one would like to have realization that use only two dimensional ancilla systems i.e qubits. Although there is no explicit proof detail search shows that there are no realizations that would use only qubits.

In fact alternative realizations would involve multiple CP maps and Kraus generators[8]. Those generators would be the identity map $1$, a rotation operator $R(\varphi)$ of angle $\varphi$, or a reflection operator $A(r)$ with respect to some axis e.g going through the origin and making an angle $\varphi$, with the x-axis. Two types of maps operating on quantum observables are then introduced, one for rotations and one for reflections, which read respectively

$$\varepsilon_\varphi(X) = X + R(\varphi)XR(\varphi)^\dagger, \quad \varepsilon_r(X) = X + A(r)XA(r)^\dagger. \quad (20)$$

For observables that are region operators corresponding to a phase space domain with parameters specified by $D$, we denote the region operator by $X_D$. Then the linearity of CP map yields for the transformed, by rotation and reflection maps, region operators the respective relations $\varepsilon_\varphi(X_D) = X_{D \cup R(\varphi), D}$, and $\varepsilon_r(X_D) = X_{D \cup A(r), D}$. The notation $D \cup R(\varphi), D \cup A(r), D$, denotes that the domain $D \cup R(\varphi), D \cup A(r), D$ of transformed, by the rotation (reflection) CP map, region operator, is the union of the domain of the untransformed one by the rotated $R(\varphi), D$ (reflected $A(r), D$) image of it. This geometric property common to all CPTI maps, would probably justify the name ”copy-paste transformation” for them. Dually we consider next the CP map $\varepsilon_k^\varphi$, $k = \varphi, r$, with respective Kraus generators $S = R(\varphi), A(r)$, acting on some density operator. The introduction of an ancilla Hilbert space $H_A = span\{|\Phi\rangle, |\Phi^\perp\rangle\}$, with orthogonal basis vectors i.e $\langle \Phi | \Phi^\perp \rangle = 0$, where $|\Phi\rangle = a |0\rangle + b |1\rangle$, allows one to implement those CP maps unitarily as

$$\varepsilon_k^\varphi(\rho) = \rho + S^\dagger \rho S = Tr_A(V_k |\Phi\rangle \langle \Phi| \otimes \rho V_k^\dagger). \quad (21)$$
The Kraus generators are obtained by the relations $1 = \langle \Phi | V_k | \Phi \rangle$, and $S^\dagger = \langle \Phi^\perp | V_k | \Phi \rangle$. The unitary operator $V_k$, determines given CP map up to a unitary local (operating non trivially only in ancilla space ) operator i.e for $W$ a unitary operator in $H_A$, the transformation $V_k \rightarrow W \otimes V_k$, generates the same CP map. In our case such a $V_k$ operator in the $\{|\Phi\rangle, |\Phi^\perp\rangle\}$ basis is

$$V_k = \langle \Phi | \Phi \rangle \otimes 1 + \langle \Phi | \Phi^\perp \rangle \otimes (-S) + \langle \Phi^\perp | \Phi \rangle \langle \Phi^\perp | \Phi^\perp \rangle \otimes 1$$

$$= \sum_{i,j=1,2} |\Phi^i\rangle \langle \Phi^j | \otimes V_k^{ij} = \begin{pmatrix} 1 & -S \\ S^\dagger & 1 \end{pmatrix}$$  \hspace{1cm} (22)

where $\{|\Phi^1\rangle, |\Phi^2\rangle\} = \{|\Phi\rangle, |\Phi^\perp\rangle\}$, and $V_k^{11} = V_k^{22} = 1$, $V_k^{12} = V_k^{21} = -S = S$.

Next introducing and combining the next two maps $\varepsilon_1, \varepsilon_1$, we can construct as an example, the hexagon region operator with radial parameter $a = \frac{\sqrt{3}}{2}$. This realization reads

$$\varepsilon_2(\varepsilon_1(X_{[a, \frac{\sqrt{3}}{2}]}) = X_{[a, 6]} \rangle$$  \hspace{1cm} (23)

where, using the equality $R(\frac{\pi}{3}) = R(\frac{\pi}{3})^\dagger = R(-\frac{\pi}{3})$ (see fig.2 also), we have the explicit maps

$$\varepsilon_1(X_{[a, \frac{\sqrt{3}}{2}]}) = X_{[a, \frac{\sqrt{3}}{2}]} + R(\frac{\pi}{3})X_{[a, \frac{\sqrt{3}}{2}]}R(\frac{\pi}{3})^\dagger + R(\frac{5\pi}{3})X_{[a, \frac{\sqrt{3}}{2}]}R(\frac{5\pi}{3})^\dagger, \hspace{1cm} (24)$$

$$\varepsilon_2(X_{[a, \frac{\sqrt{3}}{2}]}) = X_{[a, \frac{\sqrt{3}}{2}]} + A(\frac{\pi}{2})X_{[a, \frac{\sqrt{3}}{2}]}A(\frac{\pi}{2})^\dagger. \hspace{1cm} (25)$$

The domain of the initial isosceles triangle $[a, \frac{\sqrt{3}}{2}]$ increases by successive ”copy-paste” induced by the two maps until reaching the hexagon domain $[a, 6]$ as the following equations show,

$$Dn[\varepsilon_1(X_{[a, \frac{\sqrt{3}}{2}]})] = [a, \frac{2\pi}{6}] \cup R(\frac{\pi}{3}), [a, \frac{2\pi}{6}] \cup R(-\frac{\pi}{3}), [a, \frac{2\pi}{6}]$$

$$Dn[\varepsilon_2(\varepsilon_1(X_{[a, \frac{\sqrt{3}}{2}]})] = Dn[\varepsilon_1(X_{[a, \frac{\sqrt{3}}{2}]})] \cup A(\frac{\pi}{2}), Dn[\varepsilon_1(X_{[a, \frac{\sqrt{3}}{2}]})] = Dn[X_{[a, 6]}] = [a, 6]. \hspace{1cm} (26)$$

See fig.2 for the geometric effect of the combined CPTI maps to construct the hexagon region operator. An implementation of the maps based on unitary operators is then given by the relation

$$\varepsilon_2(\varepsilon_1(X_{[a, \frac{\sqrt{3}}{2}]}) = \text{Tr}_A \otimes \text{Tr}_A \left(V_{r=\frac{\pi}{6}, \varphi=\pm \frac{\pi}{2}} |\Theta^1\rangle \langle \Theta^1 | \otimes |\Phi^1\rangle \langle \Phi^1 | X_{[a, \frac{\sqrt{3}}{2}]}V^\dagger_{r=\frac{\pi}{6}, \varphi=\pm \frac{\pi}{2}} \right)$$

$$\equiv \sum_{i,j=1,2} \sum_k l=1,2 V^{ij}_{r=\frac{\pi}{6}, \varphi=\pm \frac{\pi}{2}} V^k_{l=\frac{\pi}{6}, \varphi=\pm \frac{\pi}{2}} X_{[a, \frac{\sqrt{3}}{2}]}V^{kl}_{r=\frac{\pi}{6}, \varphi=\pm \frac{\pi}{2}} \hspace{1cm} (27)$$

where two ancilla Hilbert spaces, one three dimensional i.e $H^1_A = \text{span}\{|\Theta^1\rangle, |\Theta^2\rangle, |\Theta^3\rangle\}$, and one two dimensional $H^2_A = \text{span}\{|\Phi^1\rangle, |\Phi^2\rangle\}$ have been used. The operators involved are also of two kinds: the first one $V_{r=\frac{\pi}{6}, \varphi=\pm \frac{\pi}{2}}$ is a three dimensional unitary dilation of the Kraus generators of the CPTI ”rotation” map $\varepsilon_1$ i.e $1 = \langle 0 | V_{r=\frac{\pi}{6}, \varphi=\pm \frac{\pi}{2}} |\Theta^1\rangle$, etc.
The CPTI displacement map $\varepsilon_2$ defined in the canonical basis as $V_{\pi/2} = \begin{pmatrix} 1 & -A(\pi/2) \varepsilon \\ 0 & 1 \end{pmatrix}$, and is such that $V_{\pi/2} V_{\pi/2}^\dagger = 21$. It should be noticed that the reflection with respect to the y-axis i.e. $(x, y) \rightarrow (-x, y)$, is not a canonical transformation, but since the domain $D_n\varepsilon_1(X_{q,p})$, is the semicircle we actually need the reflection with respect to the origin i.e. $(x, y) \rightarrow (-x, -y)$, which in fact is a canonical transformation, and it is realized by the parity operator $\Pi = e^{i\pi N} = (-1)^N$, so we will in effect use the operator $V_{\pi/2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Indeed the parity operator with properties $\Pi^1 = \Pi$, and $\Pi^2 = 1$, operates on the canonical generators as $\Pi a \Pi = -a$, $\Pi a^\dagger \Pi = -a^\dagger$, and on the displacement operator as $\Pi D_n \Pi = D_{-\alpha}$.

4. Bounds, Tiling and Majorization for QPMs

Let us start by pointing out the covariance property of the displaced parity kernel operator $\Delta(q, p) = D(q, p)\Pi D^\dagger(q, p)$ of the region operator under the displacement operator, which reads

$$D(q', p')\Delta(q, p)D^\dagger(q', p') = \Delta(q + q', p + p'). \hspace{1cm} (28)$$

We now introduce a CPTI displacement map by the equation

$$\varepsilon_{(q', p')} (X) = X + D(q', p') XD^\dagger(q', p'). \hspace{1cm} (29)$$

Two particular such maps are the $q'$-west and the $p'$-north displacement maps defined correspondingly as

$$\varepsilon_N^W = \varepsilon_{(-q', 0)} \hspace{0.5cm} \varepsilon_N^N = \varepsilon_{(0, p')} \hspace{1cm} (30)$$

For a region operator $X_{[q,p]}$ or $X_{[c,c;d]}$ with support $[q, p]$ a rectangle with sides $q, p$ along $x, y$ axes, or $[c, c; d]$ a disk of diameter $d$ and center at the point $(c, c)$, the previously introduced domain operator is respectively

$$Dn[X_{[q,p]}] = [q, p], \hspace{1cm} Dn[X_{[c,d]}] = [c; d]. \hspace{1cm} (31)$$

Also we will use the following symbols to denote $q'$-west and the $p'$-north displacements, of respective domains of phase space

$$W_{q'}[q, p] = [q' + q, p], \hspace{1cm} N_{p'}[q, p] = [q, p' + p], \hspace{1cm} (32)$$

$$W_{d}[c, c; d] = [c + d, c; d], \hspace{1cm} N_{d}[c, c; d] = [c, c + d; d]. \hspace{1cm} (33)$$

The CPTI maps that can induce the respective displacement in the domains of rectangle region operators are

$$\varepsilon_N^N(X_{[q,p]}) = X_{[q,p]} + D(0, p')X_{[q,p]}D^\dagger(0, p'), \hspace{1cm} (34)$$

$$\varepsilon_N^W(X_{[q,p]}) = X_{[q,p]} + D(q', 0)X_{[q,p]}D^\dagger(q', 0). \hspace{1cm} (35)$$
Indeed we have for the rectangle that
\[ \varepsilon_{\rho}^N(X_{c,c;\rho}) = X_{c,c;\rho} + D(0,p')X_{c,c;\rho}D^\dagger(0,p'), \tag{36} \]
\[ \varepsilon_{q}^W(X_{c,c;\rho}) = X_{c,c;\rho} + D(q',0)X_{c,c;\rho}D^\dagger(q',0), \tag{37} \]

and similarly for the respective displacements of a \([c,c;d]\) disk,

\[ \varepsilon_{\rho}^N(X_{c,c;d}) = X_{c,c;d} + D(0,p')X_{c,c;d}D^\dagger(0,p'), \tag{36} \]
\[ \varepsilon_{q}^W(X_{c,c;d}) = X_{c,c;d} + D(q',0)X_{c,c;d}D^\dagger(q',0), \tag{37} \]

Indeed we have for the rectangle that
\[ D_n[\varepsilon_{\rho}^N(X_{q,p})] = [q,p] \cup N_{p'}[q,p] = [q,p] \cup [q,p' + p], \tag{38} \]
\[ D_n[\varepsilon_{q}^W(X_{q,p})] = [q,p] \cup W_{q'}[q,p] = [q,p] \cup [q' + q,p]. \tag{39} \]

and for the disk
\[ D_n[\varepsilon_{\rho}^N(X_{c,c;\rho})] = [c,c;d] \cup N_{p'}[c,c;d] = [c,c;d] \cup [c,c + p';d], \tag{40} \]
\[ D_n[\varepsilon_{q}^W(X_{c,c;\rho})] = [c,c;d] \cup W_{q'}[c,c;d] = [c,c;d] \cup [c + q',c;d]. \tag{41} \]

In view of fig.3, we now introduce a rectangle operator \( X_{[\alpha,\beta]} \), (fig.3a), by \( \varepsilon_{\beta}^W \) along the \( x \) axis to west, (fig.3b), and one displacement of the resulting rectangle by \( \varepsilon_{\alpha}^N \) along the \( y \) axis to north, (fig.3c). The combined displacements define the map \( \varepsilon_{[\alpha,\beta]} \) realizing a simple step in the tiling by operating on the square and disk region operators respectively as follows

\[ X_{[2\alpha,2\beta]} = \varepsilon_{(\alpha,\beta)}(X_{[\alpha,\beta]}) \equiv \varepsilon_{\alpha}^N \circ \varepsilon_{\beta}^W(X_{[\alpha,\beta]}), \tag{42} \]
\[ X_{[c+d,c+d;\rho]} = \varepsilon_{(d,d)}(X_{[c,c;\rho]}) \equiv \varepsilon_{\rho}^N \circ \varepsilon_{\rho}^W(X_{[c,c;\rho]}), \tag{43} \]

where the notation \( X_{[c+d,c+d;\rho]} \) means the cluster of disks of diameter \( d \), with center at the points \( \{(c,c);(c,c+d);(c+d,d);(c+d,c+d)\} \). This notation extended in the form \( X_{[c+md,c+md;\rho]} \), where \( m = 0,1,2,... \), describes the cluster of all disks of diameter \( d \), "under" the disk with center at the point \( (c+md,c+md) \).

Referring to fig.3 (fig.4) we introduce a west-north tiling of an initial rectangle (disk) region operator \( X_{[\alpha,\beta]} \), \( X_{[c,c;\rho]} \), by successively operating on it with the CPTI maps \( \{\varepsilon_{(\alpha,\beta)};\varepsilon_{(2\alpha,2\beta)};\varepsilon_{(4\alpha,4\beta)};...\} \), \( \{(\varepsilon_{(d,d)};\varepsilon_{(2d,2d)};\varepsilon_{(4d,4d)};...\} \), which results into the sequence of region operators \( \{\epsilon_{[\alpha,\beta]};\epsilon_{[2\alpha,2\beta]};\epsilon_{[3\alpha,3\beta]};...\} \), \( \{\epsilon_{[c,c;\rho]};\epsilon_{[c+2d,c+2d;\rho]};\epsilon_{[c+2d,c+2d;\rho]};...\} \), the sequence of domains of which corresponds to a rectangle (disk) shaped west-north tiling of the upper left part of the plane. In other words we have the iteration

\[ X_{[2\alpha,2\beta]} = \varepsilon_{(\mu,\nu)}(X_{[\mu,\nu]}), \quad \mu = m\alpha, \quad \nu = m\beta, \quad m \in \{2,4,6,...\}, \tag{44} \]
\[ X_{[c+md,c+md;\rho]} = \varepsilon_{(m-1)d,(m-1)d}(X_{[c+(m-1)d,c+(m-1)d;\rho]}), \quad m \in \{1,2,3,...\}, \tag{45} \]

for the two types of tiling of the original rectangle and disk region operators.

Before proceeding let’s us mention that the CPTI maps can be unitarily extended by utilizing an auxiliary two dimensional Hilbert space and a partial tracing as follows

\[ \varepsilon_{\rho}^W(X_{[\rho]}) = Tr_A(U_p^W|0\rangle \otimes X_{[\rho]}U_p^W), \quad \varepsilon_{\rho}^N(X_{[\rho]}) = Tr_A(U_p^N|0\rangle \otimes X_{[\rho]}U_p^N). \tag{46} \]
The unormalized unitary operators chosen for the extension of the west and north displacement maps are respectively the following two

\[
U^W_{q'} = \begin{pmatrix}
\frac{1}{D(q',0)} & -D^\dagger(q',0) \\
D(q',0) & 1
\end{pmatrix}, \quad U^N_{p'} = \begin{pmatrix}
\frac{1}{D(0,p')} & -D^\dagger(0,p') \\
D(0,p') & 1
\end{pmatrix}.
\] (47)

Let us turn now to the question of determining the quasi probability masses (qpm) identified with the eigenvalues of rectangle (disk) region operator \(X_{[\mu, 2\nu]}(X_{[c+m, c+md, d]}\) at some time \([2\mu, 2\nu](m)\), of the tiling iteration process, from its previous one \([\mu, \nu](m-1)\). This would be an important relation for determining qpm’s, and their bounds i.e the maximal and minimal ones among them, in the course of rectangle (disk) tiling, once the qpm’s of the initial rectangle (disk) region operator is given or can be measured or estimated. To this end let us take the canonical decomposition of the matrices of a region operator \(X = V^\dagger X^d V\), before and after a general tiling CPTI map i.e of the operator \(X = \varepsilon_{(e', e)}(X) = W^\dagger X^d' W\). Here \(V, W\) and \(X^d, X^d'\), are the respective diagonalizing unitary matrices and the diagonal matrices of the eigenvalues appearing in the canonical decomposition. The latter two are related by the equation

\[
X^d' = W[V^\dagger X^d V + D(q',p')V^\dagger X^d D^\dagger(q',p')]W^\dagger
\] (48)

Let \(\lambda' = (\lambda'_{jj})_{j=1}^\infty = (X^d'_{jj})_{j=1}^\infty\), the eigenvalue column vector, and similarly from the eigenvalues of the \(X^d\) the vector \(\lambda\). In view of the last equation these two real column vectors are connected by the following matrix \(\Sigma\)

\[
\lambda' = \left(WV^\dagger \circ \overline{WD(q',p')}V^\dagger \circ \overline{WD(q',p')}V^\dagger\right) \lambda \equiv \Sigma \lambda.
\] (49)

Above, the overbar denotes complex conjugation and, the elementwise or Hadamard product between matrices defined as \((A \circ B)_{ij} = A_{ij} B_{ij}\), has been used. Since for a given unitary matrix \(U\) the Hadamard product \(U \circ \overline{U}\) is a column and row stochastic matrix (bistochastic), i.e for the unit column vector \(c = (1, 1, \ldots)^T\), we have that \(U \circ \overline{U}c = c\), and \(e^T U \circ \overline{U} = e^T\), then our present sigma matrix has column and row sums equal to two i.e \(\Sigma c = 2c\), and \(e^T \Sigma = 2e^T\).

In order to apply these facts to two successive region operators in the rectangle tiling, we introduce first their canonical decompositions

\[
X_{[2\mu, 2\nu]} = W^\dagger X^d_{[2\mu, 2\nu]} W = W^\dagger diag(\lambda_{[2\mu, 2\nu]}) W,
\] (50)

\[
X_{[\mu, \nu]} = V^\dagger X^d_{[\mu, \nu]} V = V^\dagger diag(\lambda_{[\mu, \nu]}) V.
\] (51)

Then we obtain the following relation among their diagonal components

\[
X^d_{[2\mu, 2\nu]} = W\{V^\dagger X^d_{[\mu, \nu]} V + D(\mu,0)V^\dagger X^d_{[\mu, \nu]} V D(\mu,0)^\dagger + D(0,\nu)V^\dagger X^d_{[\mu, \nu]} V D(0,\nu)^\dagger + D(0,\nu)D(\mu,0)V^\dagger X^d_{[\mu, \nu]} V D(\mu,0)^\dagger D(0,\nu)^\dagger\}W^\dagger.
\] (52)
The diagonal elements of these matrices are the column vectors $\lambda_{[\mu,\nu]}$, $\lambda_{[2\mu,2\nu]}$, and they are identified with qpm’s before and after the tiling map. They are explicitly related as follows

$$
\lambda_{[2\mu,2\nu]} = \left( WV^\dagger \circ W V^\dagger + WD(\mu,0)V^\dagger \circ WD(\mu,0)V^\dagger \\
+ WD(0,\nu)V^\dagger \circ WD(0,\nu)V^\dagger + WD(\mu,\nu)V^\dagger \circ WD(\mu,\nu)V^\dagger \right) \lambda_{[\mu,\nu]}
$$

$$
\equiv \Gamma \lambda_{[\mu,\nu]}.
$$

(53)

The matrix gamma introduced above updates the vector of qpm’s in the iteration of rectangle tiling, and it has column and row sum equal to four i.e $\Gamma \epsilon = 4 \epsilon$, $\epsilon^T \Gamma = 4 \epsilon^T$. This last fact signifies the quadrupling of a initial rectangle at each tiling step. As to the disk tiling, the analogous relation results after simplifying by setting $W = 1$, since the disk range operator $X_{[c,c,d]}$ is diagonal in the number state basis i.e $X_{[c,c,d]} = X_{[c,c,d]}^d = \text{diag}(\lambda_{[c,c,d]})$. Then the relation connecting the column vectors $\lambda_{[c,c,d]}$ and $\lambda_{[c+d,c+d,d]}$ of the diagonal elements of the respective disk range operators $X_{[c+d,c+d,d]} = \varepsilon_{[d,d]}(X_{[c,c,d]}) = V^\dagger \text{diag}(\lambda_{[c+d,c+d,d]})V$, and $X_{[c,c,d]}$, i.e the operators before and after a disk tiling map, reads

$$
\lambda_{[c+d,c+d,d]} = \left( V^\dagger \circ V^\dagger + D(0,0)V^\dagger \circ D(0,0)V^\dagger \\
+ D(0,d)V^\dagger \circ D(0,d)V^\dagger + D(d,d)V^\dagger \circ D(d,d)V^\dagger \right) \lambda_{[c,c,d]} \equiv E \lambda_{[c,c,d]}.
$$

(54)

As with the matrix $\Gamma$, the matrix $E$ has column and row sums equal to four.

Remarks: 1) The eigenvalues $\lambda_{[2\mu,2\nu]}$ i.e the qpm’s, will result if we evaluate the matrix elements of some square region operators in some of their eigenvectors. These eigenvectors however may not be known and not even been relevant vectors for some problems, in such cases the above recurrent relations may not be useful. Instead the determination of matrix elements in the number states which are physically relevant states, is desirable and this will be pursued next.

2) The same remark is in fact true for the determination of eigenvalues $\lambda_{[c+m,d,c+m,d,d]}$, i.e the qpm’s in the case of disk tiling. So in the following we evaluate the diagonal matrix elements of the respective region operators in the number states.

Lemma 4. Let the transformation $D \rightarrow \varepsilon(D) \equiv X = \int_{x \in \Delta} V(x)DV^\dagger(x)du(x)$, of the operator $D = \sum_{k=0,1,\ldots} d_k |k\rangle \langle k|$, diagonal in the number state basis, and $V = V(x)$ a unitary operator depending on some variable $x$ with values in some domain $\Delta$, on which a integration measure $du(x)$ is defined. Let the vector $|\Psi\rangle = \sum_{k=0,1,\ldots} d_k |k\rangle$, then the diagonal elements of operator $X$, are determined by $\langle k|X|k\rangle = \langle k| \left( \int_{x \in \Delta} V(x) \circ V(x)du(x) \right) |\Psi\rangle$.

The proof is straightforward, and the lemma is also true for the case of transformations defined by discrete sums. All the above can now be summarized in the following proposition:
**Proposition 5.** Let the rectangle and disk tiling with iteration maps \( \varepsilon_{(\mu, \nu)} \) and \( \varepsilon_{(md, md)} \) respectively, acting on rectangle and disk region operator as \( \varepsilon_{(\mu, \nu)}(X_{[\mu, \nu]}) = X_{[2\mu, 2\nu]} \) and \( \varepsilon_{(md, md)}(X_{[c+md, c+md,d,d]})) = X_{[c+(m+1)d,c+(m+1)d,d,d]} \). Let the canonical decomposition of the rectangle tiling range operator \( X_{[\mu, \nu]} = V^\dagger X_{[\mu, \nu]} V = V^\dagger \text{diag}(\lambda_{[\mu, \nu]}) V \), with \( \lambda_{[\mu, \nu]} = \{\lambda^k_{[\mu, \nu]}\}_{k=0}^\infty \), the column vector of its eigenvalues. Let \( X_{[c+md, c+md]} = V^\dagger X_{[c+md, c+md]}^d V = V^\dagger \text{diag}(\lambda_{[c+md, c+md]}) V \) been also the canonical decomposition of the disk tiling range operator with \( \lambda_{[c+md, c+md]} = \{\lambda^k_{[c+md, c+md]}\}_{k=0}^\infty \) the column vector of its eigenvalues. Given that the eigenvalue-vector problem has been solved for these operators after some given tiling step, then their diagonal elements in the number state basis, (which physically signify the quasiprobability mass of the Wigner function in the respective number states), are given by the two corresponding expressions,

\[
\langle k | X_{[2\mu, 2\nu]} | k \rangle = \langle k | V^\dagger \circ \nabla + D(\mu, 0)V^\dagger \circ \overline{V} D(\mu, 0)^\dagger + D(0, \nu)V^\dagger \circ \overline{V} D(0, \nu)^\dagger + D(0, 0)V^\dagger \circ \overline{V} D(0, 0)^\dagger \rangle | \Phi_{[\mu, \nu]} \rangle,
\]

and

\[
\langle k | X_{[c+(m+1)d,c+(m+1)d,d,d]} | k \rangle = \langle k | V^\dagger \circ \nabla + D(md, 0)V^\dagger \circ \overline{V} D(md, 0)^\dagger + D(0, md)V^\dagger \circ \overline{V} D(0, md)^\dagger + D(md, md)V^\dagger \circ \overline{V} D(md, md)^\dagger \rangle | \Psi_{[c+md, c+md]} \rangle.
\]

where \( | \Phi_{[\mu, \nu]} \rangle = \sum_{k=0,1,\ldots} \lambda^k_{[\mu, \nu]} | k \rangle \) and \( | \Psi_{[c+md, c+md]} \rangle = \sum_{k=0,1,\ldots} \lambda^k_{[c+md, c+md]} | k \rangle \).

The description of the tiling process that has been given above with rectangle and disc tiles is based respectively on the one step CPTI maps \( \varepsilon_{(\mu, \nu)}(X_{[\mu, \nu]}) = X_{[2\mu, 2\nu]} \) and \( \varepsilon_{(md, md)}(X_{[c+md, c+md,d,d]}) = X_{[c+(m+1)d,c+(m+1)d,d,d]} \). These maps induce in the eigenvalue vectors of the region operators the corresponding transformations \( \lambda_{[c+d,c+d]} = E \lambda_{[c,c]} \) and \( \lambda_{[2\mu, 2\nu]} = \Gamma \lambda_{[\mu, \nu]} \). The matrices \( E \) and \( \Gamma \) involved, are almost bistochastic, since as we have shown they have column and raw sums equal to four. This property allows to estimate the behavior of upper and lower bounds of the eigenvalues of the region operators after a one step operation of the tiling process. Specifically in the subsequent corollary we will determine the relation before and after the performance of one step tiling, between the extremal values of the eigenvalues of rectangle and disk region operators. These values of course stand for the extremal values of the quasiprobability mass of the respective region operators in their associated eigenstates. These extremal values provide useful information by means of which we can estimate the interval of values of the qpm's in the course of tiling process.

To this end let us introduce the following notation: let the matrix \( S \) which stands for the matrices \( E \) and \( \Gamma \) used above, sharing the same properties. This matrix will be used to write generically \( \lambda' = S \lambda \) for a transformation of the eigenvalues in a one-step tiling. Let us introduce the non-increasing (non-decreasing) ordering of the eigenvalue
vectors i.e respectively $\lambda^↓ = (\lambda^↓_1, \lambda^↓_2, ..., \lambda^↓_i, \lambda^↓_{i+1}, ...)$, with $\lambda^↓_i \geq \lambda^↓_{i+1}, i = 1, 2, ...$, and $\lambda^↑ = (\lambda^↑_1, \lambda^↑_2, ..., \lambda^↑_i, \lambda^↑_{i+1}, ...)$, with $\lambda^↑_i \leq \lambda^↑_{i+1}, i = 1, 2, ...$

**Corollary 6.** Let the transformation $\lambda' = S\lambda$, of the column vector of rectangle/disk region operator eigenvalues in a one-step tiling map. For the non-increasing and the non-decreasing eigenvalue vectors the following respective majorization relations are valid, $4\lambda^↓ ≻ \lambda'^↓$ and $4\lambda^↑ ≻ 4\lambda'^↑$. Also for the maximum and minimum values of the eigenvectors before and after the transformation the “squeezing” property is valid $4\lambda'^{\min} \leq \lambda'^{\min} < \lambda'^{\max} \leq 4\lambda^{\max}$.

**Proof:** We start with the equation $\lambda' = S\lambda$, but it suffices to assume $\lambda'^↓ = S\lambda^↓$, and $\lambda'^↑ = S\lambda^↑$, for we could always consider two suitable permutations $P, Q$ such that $\lambda'^↓ = P\lambda^↓$, $\lambda^↓ = P\lambda$. The equation for the non-increasing ordered vectors $\lambda'^↓ = PSW^T\lambda^↓$, involves the matrix $PSW^T$ with the same property as the $S$, matrix i.e $PSW^Te = PSe = 4Pe = 4e$ and $e^TPSW^T = e^TSW^T = 4e^TW^T = 4e^T$. Similar result are obtained for the non-decreasingly ordered vectors. We start now with the equation $\lambda'^↑ = S\lambda^↑$ and write $\lambda'^↑ = 4S\lambda^↑ = 4H\lambda^↑$, where $H$ is a bistochastic matrix. The transformation $\bar{\lambda}^↓ \equiv H\lambda^↓$, leads to the majorization $\lambda^↓ \succ \bar{\lambda}^↓$ [10]. Then the relation $4\lambda'^↓ = \bar{\lambda}^↓$, leads to the majorization $4\lambda^↓ \succ \lambda'^↓$. The proof of the second majorization relation is similar. Finally, taking the maximum and minimum values of qpm’s before i.e $\lambda^{\max} = \lambda^↓_1$, $\lambda^{\min} = \lambda^↓_1$, and after i.e $\lambda^{\max} = \lambda^↑_1$, $\lambda^{\min} = \lambda^↑_1$, of the tiling map, and using the inequalities ensuing from the majorization relations we obtain the “squeezing” property $4\lambda'^{\min} \leq \lambda'^{\min} < \lambda'^{\max} \leq 4\lambda^{\max}$.

5. **Conclusions**

Region operators (RO) associated with integrals of Wigner quasiprobability function over domains of phase space provide an example of new type of quantum mechanical observables. These operators can be considered as a quantum version of characteristic functions of classical regions, or as generalized operator valued probability measures (OVM). The theory connecting geometric transformations of regions to spectral characteristics of ROs, has been initiated in this work. Applications of ROs and their associated qpm’s with their respected bounds, to the field of tomographic reconstruction of Wigner function over certain regions [2, 3], should be a testing ground for the theory. Also the application of region operators in the study of coupled bipartite quantum systems, could provide a framework for investigation of the geometric manifestation and quantification of quantum entanglement, cast in the language of generalized operator valued measures on phase space. We aim to return to these matters elsewhere.
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Figure captions.

Fig.1 Point and line region operators $K_0 = \Pi$ and $K_0^P = \varepsilon^P_L(\Pi)$ respectively.

Fig.2 Construction of hexagon region operators $X_{[a,6]}(2c)$, by implementation of two CPTI maps on the isosceles triangle region operator $X_{[a,\pi]}(2a)$.

Fig.3 Construction of west-north tiling by successive application of CPTI maps on the rectangle region operators $X_{[a,\beta]}(3a)$.

Fig.4 Construction (i) of west-north (4b)and (ii) of reflection-rotation (4d, 4e) tiling by successive application of CPTI maps on region operators of clusters of disks (4a) and rectangles (4c) respectively.