Spectral gaps and discrete magnetic Laplacians

Olaf Post

Mathematik (Fachbereich 4), Universität Trier, Germany

joint work with John Steward Fabila-Carrasco and Fernando Lledó
(Universidad Carlos III de Madrid)

2017-08-01
Analysis and Geometry on Graphs and Manifolds — Potsdam

1 Motivation and main result
2 Methods: Discrete spectral bracketing
Motivation and main result

- **Periodic graphs**: $\Gamma = \mathbb{Z}^r$ acts on infinite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ such that $G = \tilde{G}/\Gamma$ is finite
- **(Combinatorial) Laplacian**: $\Delta \tilde{G} \varphi(v) = \sum_{w \sim v} (\varphi(v) - \varphi(w))$
  $\sigma(\Delta \tilde{G}) \subset [0, 2d_{\infty}]$, $d_{\infty} = \sup_v \deg v$ (< $\infty$) here
- **(Combinatorial) spectral gap**: $S \tilde{G} = [0, 2d_{\infty}] \setminus \sigma(\Delta \tilde{G})$
  $\tilde{G}$ has full (comb.) spectrum iff $S \tilde{G} = \emptyset$

**Theorem (Fabila-Carrasco, Lledó, P (2017))**

Assume $\tilde{G}$ is a $\mathbb{Z}$-periodic tree. Then the following are equivalent:

1. $\tilde{G}$ has full (comb.) spectrum ($S \tilde{G} = \emptyset$)
2. $\tilde{G}$ is the $\mathbb{Z}$-lattice
3. $\tilde{G}$ has no vertex of degree 1

- Clear: (ii)$\Rightarrow$(i) (calculate); (ii)$\Rightarrow$(iii) (obvious); (iii)$\Rightarrow$(ii) (graph th.)
  We will show $\neg$(iii)$\Rightarrow$$\neg$(i)
Motivation and main result

Remarks on main result

- A related result holds for the standard Laplacian given by
  \[(\Delta \tilde{G}, \text{std}) \varphi)(v) = \frac{1}{\deg v} \sum_{w \sim v} (\varphi(v) - \varphi(w))\]
  \(\implies\) \(\sigma(\Delta \tilde{G}, \text{std}) \subset [0, 2],\quad S^{\tilde{G}, \text{std}} := [0, 2] \setminus \sigma(\Delta \tilde{G}, \text{std})\)

- Relation with full spectrum conjecture [HS04] for maximal abelian covering: ("all loops in G are unfolded") if \(\tilde{G}\) (or \(G\)) has no vertices of degree 1 then combinatorial or standard Laplacian has full spectrum (proven if all degrees are even — Euler path) or \(\tilde{G}\) is \((2k + 1)\)-regular with some additional property)
  \(\implies\) we have shown the full spectrum conjecture for trees

- Results and estimates on lengths of bands for periodic discrete gaps (see e.g. [KS14, KS15, KS17])
Motivation and main result

- **Periodic graphs:** $\Gamma = \mathbb{Z}^r$ acts on infinite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ such that $G = \tilde{G}/\Gamma$ is finite
- **(Combinatorial) Laplacian:** $\Delta \tilde{G} \varphi(v) = \sum_{w \sim v} (\varphi(v) - \varphi(w))$
  $\mapsto \sigma(\Delta \tilde{G}) \subset [0, 2d_{\infty}]$, $d_{\infty} = \sup_v \deg v$ ($< \infty$ here)
- **(Combinatorial) spectral gap:** $S \tilde{G} = [0, 2d_{\infty}] \setminus \sigma(\Delta \tilde{G})$
  $\tilde{G}$ has full (comb.) spectrum iff $S \tilde{G} = \emptyset$

**Theorem (Fabila-Carrasco, Lledó, P (2017))**

Assume $\tilde{G}$ is a $\mathbb{Z}$-periodic tree. Then the following are equivalent:

(i) $\tilde{G}$ has full (comb.) spectrum ($S \tilde{G} = \emptyset$)

(ii) $\tilde{G}$ is the $\mathbb{Z}$-lattice

(iii) $\tilde{G}$ has no vertex of degree 1

- Clear: (ii)$\Rightarrow$(i) (calculate); (ii)$\Rightarrow$(iii) (obvious); (iii)$\Rightarrow$(ii) (graph th.)
  
  We will show $\neg$(iii)$\Rightarrow\neg$(i)
Let $b > 0$, $S^\pm$ self-adjoint in Hilbert space $\mathcal{H}^\pm$, $\dim \mathcal{H}^\pm = n^\pm < \infty$, $\sigma(S^\pm) \subset [0, b]$

**Definition: (spectral ordering)** $S^- \preceq S^+$ iff $\lambda_k(S^-) \leq \lambda_k(S^+)$ for all $k$ ($k$-th eigenvalue) where $\lambda_k(S^\pm) = b$ if $k > n^\pm$ (maximal possible value)

**Magnetic potential:** $\alpha: E \to \mathbb{R}$ with $\alpha(w, v) = -\alpha(v, w)$

($E \subset V \times V$ such that $(v, w) \in E$ iff $(w, v) \in E$)

**(Combinatorial) magnetic Laplacian:**

$\Delta^G_{\alpha} \varphi(v) = \sum_{w \sim v} (\varphi(v) - \alpha(v, w) \varphi(w))$

If $G$ is a tree then $\Delta^G_{\alpha} \cong \Delta^G$

Floquet theory: Let $\tilde{G}$ be $\mathbb{Z}$-periodic tree then $\sigma(\Delta^{\tilde{G}}) = \bigcup_{\alpha} \sigma(\Delta^G_{\alpha})$

($G = \tilde{G}/\mathbb{Z}$) (and $\alpha$ can be supported on one edge only)
A discrete spectral bracketing result:

- **Delete edges:** \( E_0 \subset E \implies G^- := G - E_0 := (V, E^-) \) with \( E^- := E \setminus E_0 \)
- **“Virtualise” vertices:** \( V_0 \subset V, \ G^+ := G - V_0 := (V^+, E) \), \( V^+ := V \setminus V_0 \) (some edges have now vertices not in \( G^+ \) anymore, virtual vertices, \( G^+ \) is a partial subgraph in \( G \))

**Theorem (Fabila-Carrasco, Lled’o, P (2017))**

Choose \( E_0 \subset E, V_0 \subset V \) and magnetic potential \( \alpha : E \rightarrow \mathbb{R} \) such that

- \( G^- = G - E_0 \) is a tree; \( \text{supp} \alpha \subset E_0 \);
- \( E_0 \subset \bigcup_{v \in V_0} E_v \) (edges in \( E_0 \) have at least one end in \( V_0 \))

then \( \Delta G^- \preceq \Delta G^+_\alpha \preceq \Delta G^+ \)

**Corollary**

\( J_k := [\lambda_k(\Delta G^-), \lambda_k(\Delta G^+)], J := \bigcup_k J_k \), then \( \bigcup_\alpha \sigma(\Delta G^+_\alpha) \subset J \ (\ast) \).
Thank you for your attention!

Y. Higuchi and Y. Nomura, *Spectral structure of the Laplacian on a covering graph*, European J. Combin. **30** (2009), 570–585.

Y. Higuchi and T. Shirai, *Some spectral and geometric properties for infinite graphs*, Discrete geometric analysis, Contemp. Math., vol. 347, Amer. Math. Soc., Providence, RI, 2004, pp. 29–56.

E. Korotyaev and N. Saburova, *Schrödinger operators on periodic discrete graphs*, J. Math. Anal. Appl. **420** (2014), 576-611.

E. Korotyaev and N. Saburova, *Spectral band localization for Schrödinger operators on discrete periodic graphs*, Proc. Amer. Math. Soc. **143** (2015), 3951-3967.

E. Korotyaev and N. Saburova, *Magnetic Schrödinger operators on periodic discrete graphs*, J. Funct. Anal. **272** (2017), 1625-1660.

F. Lledó and O. Post, *Eigenvalue bracketing for discrete and metric graphs*, J. Math. Anal. Appl. **348** (2008), 806–833.
Asymptotic Analysis & Spectral Theory

3rd French-German meeting

Trier, Germany
September 25–29, 2017

Invited speakers

Wolfgang Arendt (Ulm) Jussi Behrndt (Graz)
Philippe Briet (Toulon) Pavel Exner (Prague)
Daniel Grieser (Oldenburg) Bernard Helffer (Nantes)
Luc Hillairet (Orleans) Patrick Joly (Palaiseau)
Michael Plum (Karlsruhe) Ivan Veselić (Dortmund)

Organisers

Konstantin Pankrashkin (Orsay)
Olaf Post, Ralf Rückriemen (Trier)

Supported by

Nikolaus Koch Stiftung
International Association of Mathematical Physics

http://math318.uni-trier.de/aspect17/