A New Variable Modified Chaplygin Gas Model Interacting with Scalar Field

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In this letter we present a new form of the well known Chaplygin gas model by introducing inhomogeneity in the EOS. This model explains $\omega = -1$ crossing. Also we have given a graphical representation of the model using $\{r, s\}$ parameters. We have also considered an interaction of this model with the scalar field by introducing a phenomenological coupling function and have shown that the potential decays with time.

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Recent observations reveals [1, 2] that the present Universe is subjected to an accelerated expansion, which can be explained in terms of some new type of matter which violates the strong energy condition $\rho + 3p < 0$. This type of matter is known as dark energy [3-6], which has the cosmological constant to be a strong candidate. However many models have been proposed to play the role of the dark energy, Quintessence [7] or the scalar field being one of the most favoured model because of its decaying potential term dominating the kinetic term so as to generate enough pressure to drive acceleration. Also one can try Chaplygin gas model [8] with equation of state (EOS), $p = -B/\rho$, as it generates negative pressure, where $p$ and $\rho$ are respectively the pressure and energy density and $B$ is a positive constant. Subsequently this fluid has been modified to $p = -B/\rho^\alpha$ with $0 \leq \alpha \leq 1$. and

$$p = A\rho - \frac{B}{\rho^\alpha} \quad \text{with} \quad 0 \leq \alpha \leq 1, \ A, \ B \ \text{are positive constants.} \quad (1)$$

as generalized Chaplygin gas [9, 10] and modified Chaplygin gas [11, 12] respectively. Modified Chaplygin gas can explain the evolution of the Universe from radiation era to $\Lambda$CDM model. Later inhomogeneity has been introduced in the above EOS (1) by considering $B$ to be a function of the scale factor $a(t)$ [13, 14]. This assumption is reasonable since $B(a)$ is related to the scalar potential if we take the Chaplygin gas as a Born-Infeld scalar field [15].

Interaction models where the dark energy weakly interacts with the dark matter have also been studied to explain the evolution of the Universe. This models describe an energy flow between the components. To obtain a suitable evolution of the Universe the decay rate should be proportional to the present value of the Hubble parameter for good fit to the expansion history of the Universe as determined by the Supernovae and CMB data. A variety of interacting dark energy models have been proposed and studied for this purpose [16-19].

In this letter we study a new model by considering both $A$ and $B$ in the EOS (1) to be a function of the scale factor $a(t)$ and thus introducing inhomogeneity in the EOS (1). We solve the EOS to get the energy density and show that the we can explain the evolution of the Universe suitably by choosing different values of the parameters. We then consider an interaction between the fluid and the scalar field by introducing a phenomenological interaction term which describes the energy flow between them, thus showing the effect of interaction in the evolution of the Universe. This kind of interaction term has been studied in ref. [20].

The metric of a spatially flat homogeneous and isotropic universe in FRW model is

$$ds^2 = dt^2 - a^2(t) \left[ dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right] \quad (2)$$
where \(a(t)\) is the scale factor.

The Einstein field equations are

\[
\frac{\dot{a}^2}{a^2} = \frac{1}{3\rho},
\]

and

\[
\frac{\ddot{a}}{a} = \frac{1}{6}(\rho + 3p)
\]

where \(\rho\) and \(p\) are energy density and isotropic pressure respectively (choosing \(8\pi G = c = 1\)).

The energy conservation equation is

\[
\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0
\]

Now, we extend the modified Chaplygin gas with equation of state (1) such that \(A\) and \(B\) are positive function of the cosmological scale factor ‘\(a\)’ (i.e., \(A = A(a), B = B(a)\)). Then equation (3) reduces to,

\[
p = A(a)\rho - \frac{B(a)}{\rho^\alpha}
\]

with \(0 \leq \alpha \leq 1\)

As we can see this is an inhomogeneous EOS [21] where the pressure is a function of the energy density \(\rho\) and the scale factor \(a(t)\). Also if \(\rho = \left(\frac{B(a)}{A(a)}\right)^{-\frac{1}{\alpha}}\), this model reduces to dust model, pressure being zero.

Now, assume \(A(a)\) and \(B(a)\) to be of the form

\[
A(a) = A_0 a^{-n}
\]

and

\[
B(a) = B_0 a^{-m}
\]

where \(A_0, B_0, n\) and \(m\) are positive constants. If \(n = m = 0\), we get back the modified Chaplygin gas [12] and if \(n = 0\), we get back variable modified Chaplygin gas model. Using equations (5), (6), (7) and (8), we get the solution of \(\rho\) as,

\[
\rho = a^{-3} e^{\frac{3A_0 a^{-n}}{n}} \left[ C_0 + \frac{B_0}{A_0} \left( \frac{3A_0(1+\alpha)}{n} \right)^{\frac{n(1+\alpha)+n-m}{n}} \Gamma\left( \frac{m-3(1+\alpha)}{n}, \frac{3A_0(1+\alpha)a^{-n}}{n} \right) \right]^{\frac{1}{1+\alpha}}
\]

where \(\Gamma(a, x)\) is the upper incomplete gamma function and \(C_0\) is an integration constant.

Now, considering

\[
\omega_{eff} = \frac{\rho}{\rho}
\]

for this fluid, we have,

\[
\omega_{eff} = A_0 a^{-n} - B_0 a^{-\frac{3A_0(1+\alpha)}{n}} e^{-\frac{3A_0(1+\alpha) a^{-n}}{n}} \left[ C_0 + \left( \frac{3A_0(1+\alpha)}{n} \right)^{\frac{n-\zeta}{n}} \frac{B_0}{A_0} \Gamma\left( \frac{\zeta}{n}, \frac{3A_0(1+\alpha)a^{-n}}{n} \right) \right]^{-1}
\]
Fig. 1 shows the variation of $\omega_{\text{eff}}$ against $a(t)$ for $A_0 = 1, B_0 = 10, \alpha = 1, m = 2, C_0 = 1$ and $n = 3$ (for dotted line), $n = 10$ (for the dark line).

where $\zeta = m - 3(1 + \alpha)$.

For small values of the scale factor $a(t)$, $\rho$ is very large and

$$p = A\rho - \frac{B}{\rho^\alpha} \to A\rho$$

where $A = A_0 a^{-n}$ is a function of $a$, so that for small scale factor we have very large pressure and energy densities. Therefore initially

$$\frac{p}{\rho} = \omega_{\text{eff}} = A^* a^{-n} \leq 1$$

where $A^*$ is a constant,

$$A^* = A_0.$$

If $a = A_0 \frac{1}{A^*}$, the Universe starts from stiff perfect fluid, and if $a = 3A_0 \frac{1}{A^*}$, the Universe starts from radiation era.

Also for large values of the scale factor

$$p = A\rho - \frac{B}{\rho^\alpha} \to -\frac{B}{\rho^\alpha}.$$

If

$$\zeta = m - 3(1 + \alpha) < 0$$

( as we know that upper incomplete Gamma function $\Gamma(a, x)$ exists for $a < 0$ ), the second term dominates and hence $\omega_{\text{eff}} \to -B^* a^{-\zeta}$, where

$$B^* = B_0 \lim_{a \to \infty} e^{-\frac{3A_0(1 + \alpha)a^{-n}}{n}} \left[ C_0 + \left( \frac{3A_0(1 + \alpha)}{n} \right)^{\frac{n - \zeta}{n}} B_0 \frac{\zeta}{n} \frac{3A_0(1 + \alpha)a^{-n}}{n} \right]^{-1}$$

( $\lim_{a \to \infty} e^{-\frac{3A_0(1 + \alpha)a^{-n}}{n}} \to 1$ and $\lim_{a \to \infty} \Gamma(\frac{\zeta}{n}, \frac{3A_0(1 + \alpha)a^{-n}}{n}) \to$ large value, for $\zeta < 0$ ). This will represent dark energy if $a > \left( \frac{1}{A_0} \right)^{\frac{1}{n + 1 + \alpha n - m}}$, $\Lambda$CDM if $a = \left( \frac{1}{A_0} \right)^{\frac{1}{n + 1 + \alpha n - m}}$ and phantom dark energy
Fig. 2 shows the variation of $s$ against $r$ for $A_0 = 1, B_0 = 1, \alpha = \frac{1}{2}, m = 3, n = 2, C_0 = 1$.

if $a > \left(\frac{1}{B^2}\right)^{\frac{m}{3(1+\alpha)-m}}$. Therefore we can explain the evolution of the Universe till the phantom era depending on the various values of the parameters. We have shown a graphical representation of $\omega_{eff}$ in fig 1 for different values of the parameters. We can see from fig 1 that $\omega_{eff}$ starting from a large values decreases with $a$ crosses $\omega = -1$ for some choices of the parameters.

Since there are various candidates for the dark energy model, we often face with the problem of discriminating between them, which were solved by introducing statefinder parameters [22]. These statefinder diagnostic pair i.e., $\{r, s\}$ parameters are of the following form:

$$r = \frac{\ddot{a}}{aH^3} \quad \text{and} \quad s = \frac{r-1}{3(q-\frac{2}{3})}$$

where $H$ is the Hubble parameter and $q \ (= -\frac{\dot{a}}{aH})$ is the deceleration parameter. These parameters are dimensionless and allow us to characterize the properties of dark energy in a model independent manner. The statefinder is dimensionless and is constructed from the scale factor of the Universe and its time derivatives only. The parameter $r$ forms the next step in the hierarchy of geometrical cosmological parameters after $H$ and $q$.

Now, in our case,

$$H^2 = \frac{\ddot{a}^2}{a^2} = \frac{1}{3} \rho$$

and

$$q = -\frac{\ddot{a}}{aH^2} = \frac{1}{2} + \frac{3}{2} \frac{p}{\rho}$$

So from equation (10) we get

$$r = 1 + \frac{9}{2} \left(1 + \frac{p}{\rho}\right) \frac{\partial p}{\partial \rho} \frac{3}{2} \frac{a \partial p}{\rho \partial a} , \quad s = \frac{2(r-1)}{9 \left(\frac{p}{\rho}\right)}$$

so that, solving we get,

$$r = 1 + \frac{9}{2}(1 + y)(A_0 a^{-n} + \alpha B_0 a^{-m} x) + \frac{3}{2}(nA_0 a^{-n} - mB_0 a^{-m} x) , \quad s = \frac{2(r-1)}{9y}$$
where, \( y = \frac{\xi}{\rho} = A_0a^{-n} - B_0a^{-m}x \) and \( x = \rho^{-(1+\alpha)} \), \( \rho \) is given by equation (9).

We have plotted the \( \{r, s\} \) parameters normalizing the parameters and varying the scale factor \( a(t) \). We can see that the model starts from radiation era. Then we have a discontinuity at the dust era (for radiation era: \( s > 0 \) and \( r > 1 \); dust era: \( r > 1 \) and \( s \to \pm \infty \); \( \Lambda \)CDM: \( r = 1, s = 0 \); phantom: \( r < 1 \)). The model reaches \( \Lambda \)CDM at \( r = 1, s = 0 \) and then crosses \( \Lambda \)CDM to represent phantom dark energy. This model represents the phantom dark energy, whereas, Modified Chaplygin Gas can explain the evolution of the Universe from radiation to \( \Lambda \)CDM and Variable Modified Chaplygin gas describes the evolution of the Universe from radiation to quiessence model.

Now we consider model of interaction between scalar field and the new variable modified Chaplygin Gas model, through a phenomenological interaction term. Keeping into consideration the fact that the Supernovae and CMB data determines that decay rate should be proportional to the present value of the Hubble parameter. This interaction term describes the energy flow between the two fluids. We have considered a scalar field to couple with the New variable modified Chaplygin gas given by EOS (6), (7) and (8).

Therefore now the conservation equation becomes

\[
\dot{\rho}_{\text{tot}} + 3\frac{\dot{a}}{a}(\rho_{\text{tot}} + p_{\text{tot}}) = 0
\]

so that the equations of motion of the the new fluid and scalar field read,

\[
\dot{\rho} + 3H(\rho + p) = -3H\rho\delta
\]

and

\[
\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 3H\rho\delta
\]

( \( \delta \) is a constant ).

Where the total energy density and pressure of the universe are given by,

\[
\rho_{\text{tot}} = \rho + \rho_\phi
\]

and

\[
p_{\text{tot}} = p + p_\phi
\]

where \( \rho \) and \( p \) are the energy density and pressure of the extended modified Chaplygin gas model given by equations (6), (7), (8), (9) and \( \rho_\phi \) and \( p_\phi \) are the energy density and pressure due to the scalar field given by,

\[
\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi)
\]

and

\[
p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi)
\]

where, \( V(\phi) \) is the relevant potential for the scalar field \( \phi \).

Thus the field equations become

\[
\frac{\dot{a}^2}{a^2} = \frac{1}{3}\rho_{\text{tot}}
\]

and

\[
\frac{\ddot{a}}{a} = -\frac{1}{6}(\rho_{\text{tot}} + 3p_{\text{tot}})
\]
Solving the equations we get the solution for $\rho$ as

$$\rho = a^{-3(1+\delta)} e^{\frac{3A_0 a^{-n}}{n}} \left[ C_0 + \frac{B_0}{A_0} \left( \frac{3A_0 (1+\alpha)}{n} \right)^{\frac{3(1+\alpha)(1+\delta)+n-m}{n}} \Gamma\left( \frac{m-3(1+\alpha)(1+\delta)}{n}, \frac{3A_0 (1+\alpha)}{n} a^{-n} \right) \right]$$

where $C_0$ is an integration constant.

Further substitution in the above equations give,

$$V(\phi) = 3H^2 + \dot{H} + \frac{p - \rho}{2}$$

To get an explicit form of the energy density and the potential corresponding to the scalar field we consider a power law expansion of the scale factor $a(t)$ as,

$$a = t^\beta$$

so that, for $\beta > 1$ we get accelerated expansion of the Universe thus satisfying the observational constrains. If $\beta = 1$ or $\beta < 1$ we get constant and decelerated expansion respectively.

Using equations (18), (22) and (26), we get,

$$\rho_\phi = \frac{3\beta^2}{t^2} - \rho$$

where $\rho$ is given by equation (25) along with (27). Since $\rho_\phi$ is always positive, so we may have at least for some range of the values of the free parameters.

Also the potential takes the form,

$$V = \frac{3\beta^2 - \beta}{t^2} + \frac{p - \rho}{2}$$

The graphical representation of $V$ against time is shown in figure 3 normalizing the parameters. We see that the potential decays with time.

Here we present a new variable modified Chaplygin gas model which is an unified version of the dark matter and the dark energy of the Universe. It behaves like dark matter at the initial stage and later it explains the dark energy of the Universe. Unlike the Generalized or Modified Chaplygin gas model, it
can explain the evolution of the Universe at phantom era depending on the parameters. Also we have calculated the \( \{r, s\} \) parameters corresponding to this model. Normalizing the parameters such that \( m - 3(1 + \alpha) < 0 \), show the diagrammatical representation of \( \{r, s\} \) for our model (in Fig.2), varying the scale factor. We see that starting from the radiation era it crosses \( \omega = -1 \) and extends till phantom era. Also we can see that the deceleration parameter starting from a positive point becomes negative, indicating deceleration initially and acceleration at later times. Again we have considered an interaction of this fluid with that of scalar field by introducing a phenomenological coupling term, so that there is a flow of energy between the field and the fluid which decays with time, as in the initial stage the fluid behaves more like dark matter and the field that of dark energy, whereas in the later stage both explain the dark energy present in the Universe. In Fig.3, we have shown the nature of the potential by considering a power law expansion of the Universe to keep the recent observational support of cosmic acceleration, and we see that the potential decays with time.

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