Para-Sasakian geometry in thermodynamic fluctuation theory

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Abstract
In this work we tie concepts derived from statistical mechanics, information theory and contact Riemannian geometry within a single consistent formalism for thermodynamic fluctuation theory. We derive the concrete relations characterizing the geometry of the thermodynamic phase space stemming from the relative entropy and the Fisher–Rao information matrix. In particular, we show that the thermodynamic phase space is endowed with a natural para-contact pseudo-Riemannian structure derived from a statistical moment expansion which is para-Sasaki and \( \eta \)-Einstein. Moreover, we prove that such manifold is locally isomorphic to the hyperbolic Heisenberg group. In this way we show that the hyperbolic geometry and the Heisenberg commutation relations on the phase space naturally emerge from classical statistical mechanics. Finally, we argue on the possible implications of our results.

Keywords: Sasakian manifolds, fluctuation theory, statistical mechanics, thermodynamics, contact geometry

1. Introduction

Information theory has been widely used in many branches of science, spanning systems from quantum mechanics to biology and from cosmology to statistical inference. In this context, particular attention has been devoted to the notion of the relative entropy—or Kullback–Leibler divergence [1] —which gives an estimation of the information gain (or loss) that one realizes when passing from one probability distribution to another. The relative entropy is a functional whose arguments are pairs of distribution functions. It has been shown in [2] that the first order variation of the relative entropy vanishes. In the case of statistical mechanics, this is the statement of the first law of thermodynamics.

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Another fundamental concept in information theory is that of the Fisher–Rao Information Matrix, which provides us with another measure of the distance between two different probability distributions. Such a measure endows the statistical manifold with a Riemannian structure (see e.g. [3–7]). In fact, while the relative entropy does not define a real distance between distributions (for example, it is not symmetric), it can be shown that the Fisher–Rao Information Matrix arises as the Hessian of the relative entropy over a stationary point. The entries of such a matrix are in correspondence with the components of the metric tensor over the manifold of probability distributions [5–7]. Furthermore, when restricted to Gibbs equilibrium probability distributions, the Fisher–Rao Information Metric has been used to define a thermodynamic length on the space of equilibrium states of thermodynamic systems. This length is equivalent (up to Legendre transformations) to other definitions given in the literature [8, 9], but it has the advantage that it can be extended to the non-equilibrium case [4–6, 10–12]. One of the most important properties of the thermodynamic length is that, for paths out of equilibrium, it gives a bound to the loss of available work—or dissipated availability—during the process [13]. Moreover, measurements of the thermodynamic length can be obtained from non-equilibrium protocols, i.e. this quantity influences also the behavior of systems out of equilibrium and hence can be used to obtain optimal paths (see [11, 12]).

Interestingly, the construction of a Riemannian geometry over the manifold of equilibrium states of a thermodynamic system has been generalized in [3]. In such work, the authors showed that the statistical construction naturally endows both the manifold of equilibrium states and the phase space of equilibrium thermodynamics with a pseudo-Riemannian structure. In particular, the phase space turns out to be equipped with both a contact and a pseudo-Riemannian structures, whose restrictions to Legendre sub-manifolds define the equilibrium manifold itself and the thermodynamic version of the Fisher–Rao Information Metric, respectively.

In this work, we generalize the construction in [3] in order to show that the contact and pseudo-Riemannian structures of the phase space can be defined also for distributions that are different from the Gibbs equilibrium distribution. This is important for extending this geometric picture to systems out of equilibrium. Moreover, in our treatment we will show that these structures can be derived elegantly by means of the variation of the relative entropy. Finally, we will show that the phase space of thermodynamics—as defined in [3]—possesses a number of intriguing geometric properties. In particular, it is a para-Sasakian and \( \eta \)-Einstein manifold whose Ricci scalar of the Levi-Civita connection is constant (see the standard [14–16] for Sasakian manifolds and [17–20] for the para-Sasakian case).

In the context of para-Sasakian geometry there is another connection of geometric significance which is parallel with respect to the metric and the other tensors defining the contact-metric structure. We refer to this as the canonical connection [18, 19]. The main result of this work is a proof that the thermodynamic phase space equipped with such connection has vanishing curvature. This implies that the whole para-Sasakian structure is locally isomorphic to the hyperbolic Heisenberg group. A number of implications will be considered in the conclusions.

This paper is structured as follows. In section 2 we derive the Fisher–Rao Information Metric from two different perspectives. On the one hand, we use a statistical moment expansion of the differential entropy and, on the other hand, by means of the relative entropy. Then, in section 3, we revisit the manner in which the construction of section 2 equips the thermodynamic phase space with a contact-Riemannian structure. In section 4 we present new results regarding the algebraic and geometric structure of this space. In particular we show that this corresponds to the hyperbolic Heisenberg group. Finally, in section 5 we provide some closing remarks and comment on the implications of this construction in a more general setting.
2. Two roads to the Fisher–Rao Information Metric

In this section we will revisit the work of Mrugała et al [3]. In this work, the authors proved that the phase space of thermodynamics—obtained after averaging out the phase space $\Gamma$ of statistical mechanics with respect to the optimal probability distribution $\rho_0$—is naturally endowed with both a contact and a pseudo-Riemannian structure arising from the maximization of the entropy functional. Here, we propose a more general formulation which can be applied to any distribution (not necessarily the Gibbs one). Therefore, our approach is relevant for generalizations of systems out of equilibrium, as we will argue. Moreover, we will derive the Fisher–Rao Information Metric in a natural way and show that, for the case of Gibbs equilibrium distributions, it coincides with the metric introduced in [3].

2.1. Differential entropy moments

Let us consider a system whose macroscopic state is characterized by a set of $n$ observables. Suppose that an experimentalist has measured the values of such observables up to some desirable accuracy. In such case the measurements are identified with the mean values

\[ p_a \equiv \langle F_a \rangle = \frac{\int_{\Gamma} F_a \mu}{\int_{\Gamma} \mu} \quad (1) \]

of the set of stochastic variables $\{F_a\}_{a=1}^n$. Here $\Gamma$ is a sample space together with a well defined measure $\mu$. In the case of a thermodynamic system $\Gamma$ is identified with the phase space of statistical mechanics and $\mu$ is given in terms of an unassigned probability distribution $\rho: \Gamma \to \mathbb{R}^+$ such that

\[ \int_{\Gamma} \mu \rho = \int_{\Gamma} \rho \ dx. \quad (2) \]

For instance, this is the case of the internal energy, which is the average of the microscopic Hamiltonian. The choice of different controllable observables determines the particular statistical ensemble.

In the situation described above, the available information is given solely by the averages (1) and the prescription for assigning the probability distribution (2) is by means of the maximum entropy principle. Thus, let us introduce the microscopic entropy of a generic distribution

\[ s(\rho) = -\ln \rho \quad (3) \]

whose weighted average yields (up to a normalization constant) the entropy functional

\[ S = \langle s(\rho) \rangle = -\int_{\Gamma} \rho \ln \rho \ dx. \quad (4) \]

The maximum entropy principle is expressed as a constrained variational prescription for the functional

\[ \tilde{S} = -\int_{\Gamma} \rho \ln \rho \ dx - w\left( \int_{\Gamma} \rho \ dx - 1 \right) + q^a\left( \int_{\Gamma} F_a(\rho) \ dx - p_a \right), \quad (5) \]

where $w$ and $q^a$ are the corresponding Lagrange multipliers for the constraints given by

\[ \int_{\Gamma} \rho \ dx = 1, \quad (6) \]
\[ \int_{x} F_a(x) \rho(x) \, dx = p_a \quad a = 1, \ldots, n. \quad (7) \]

The result is the well known Gibbs distribution function
\[ \rho_0(\Gamma; w, q^1, \ldots, q^n) = e^{-w + q^a F_a(x)}, \quad (8) \]
where we have used the Einstein summation convention.

One could then use the normalization condition (6) for the distribution \( \rho_0 \) (as it is usually the case in statistical mechanics), to set up the functional dependence
\[ w(q^a) = \ln \int_{\Gamma} e^{q^a E(x)} \, dx = \ln Z \quad (9) \]
defining the partition function \( Z \) providing us with an interpretation for \( w \) as the \textit{free entropy} of the thermodynamic system and its derivatives with respect to the \( q^a \) as the equations of state. However, in what follows we want to consider the full phase space of thermodynamics, so we need to have \( w \) independent of the \( q^a \), as we will make clear later. Therefore, we will not consider the normalization condition—equation (6)—but we will keep the form of \( \rho_0 \) to assign our prior probability distribution so that \( w \) will be independent of the \( q^a \).

Now we consider \( \rho \) as a function on the \( n + 1 \) control parameters \( \lambda_i = w, q^1, \ldots, q^n \) and compute the differential of the microscopic entropy \( ds \), to obtain
\[ ds = -\frac{\partial \ln \rho}{\partial \lambda_i} \, d\lambda_i. \quad (10) \]

Now, performing a moment expansion of the differential entropy (10), it follows from (1) that
\[ \langle ds \rangle = -\int_{\Gamma} \left( \frac{\partial \ln \rho}{\partial \lambda_i} \right) \rho(x) \, dx = -\left( \frac{\partial \ln \rho}{\partial \lambda_i} \right) d\lambda_i \quad (11) \]
whilst the second moment yields
\[ \langle (ds)^2 \rangle = \int_{\Gamma} \left( \frac{\partial \ln \rho}{\partial \lambda_i} \right) \frac{\partial \ln \rho}{\partial \lambda_j} d\lambda_i d\lambda_j \quad (12) \]

Computing the averages in (11) and (12) and using (8), one obtains
\[ \langle ds \rangle = -\left( \frac{\partial \ln \rho}{\partial \lambda_i} \right) d\lambda_i = \langle dw - F_a(x) d\lambda^a \rangle = dw - p_a d\lambda^a. \quad (13) \]

Using that the derivative of \( F_a \) with respect to the Lagrange multipliers \( q^j \) (see equation (7)),
\[ \frac{\partial q_b}{\partial q^b} = \langle F_a F_b \rangle - p_b p_b = \left( \langle F_a - p_a \rangle (F_b - p_b) \right), \quad (14) \]
it follows that the variance of \( ds \) is
\[ \text{Var}(ds) = \langle (ds - \langle ds \rangle)^2 \rangle = \langle (F_a - p_a) (F_b - p_b) \rangle d\lambda^a d\lambda^b = dq^a dq^b. \quad (15) \]

where we have used equation (14) to obtain the last identity as it implies that
\[ dp_a = \left( \langle F_a - p_a \rangle (F_b - p_b) \right) dq^b. \quad (16) \]
Finally, using the well known formula for the variance

$$\text{Var}(ds) = \langle (ds)^2 \rangle - \langle ds \rangle^2,$$

one obtains that the second moment of the microscopic entropy change is

$$\langle (ds)^2 \rangle = \text{Var}(ds) + \langle ds \rangle^2 = dp_a dq_a + \langle dw - p_a dq_a \rangle^2.$$

Thus, the first moment of $ds$ defines a 1-form field over an $n + 1$ dimensional control manifold $C^{n+1}$ whose coordinates correspond to the control parameters $\lambda_i$. We will see in the next section that promoting the $p_a$’s to independent variables, such a 1-form is an element of the co-tangent bundle of the thermodynamic phase space introduced by Mrugala [25] and that it defines its contact structure. Note that such a structure is obtained when using a generalized canonical equilibrium distribution of the form (8).

The second moment (12) can be used to define a metric tensor over the control manifold

$$G_{FR} = \sum_{i,j=1}^{n+1} \left[ G_{FR} \right]_{ij} d\lambda_i \otimes d\lambda_j = \left\langle \frac{\partial \ln \rho}{\partial \lambda_i} \frac{\partial \ln \rho}{\partial \lambda_j} \right\rangle d\lambda_i \otimes d\lambda_j,$$

explicitly given by

$$G_{FR}(\rho) = dq \otimes dp + \langle dw - p_a dq_a \rangle \otimes \langle dw - p_b dq_b \rangle,$$

where

$$dq \otimes dp = \frac{1}{2} (dq^a \otimes dp_b + dp_a \otimes dq^b),$$

with $i, j = 1, \ldots, n$ and

$$dp_a = \frac{\partial p_a}{\partial q^b} dq^b,$$

whose components are obtained from equation (14). This result is well known in the literature of statistical estimation theory as it is the Fisher–Rao Information Metric [6, 10]. Note that the position of the indices is conventional and we have adopted lower labels to distinguish the control parameters.

It is an important remark that the Fisher–Rao metric (19) can be defined for any distribution, i.e. independently if $\rho$ corresponds to the equilibrium distribution defined in (8) or not (see [4–6, 10]), and that equation (20) gives its expression for a system in equilibrium, i.e. when $\rho = \rho_0$. It is also worth emphasizing that in this section we have considered the variables $p_a$ to be dependent solely on the Lagrange multipliers $q^b$, as it is clear from equations (14) and (15). In the next section we will assume that the $p_i$ are independent of the $q^j$ and write the metric (20) in a $(2n + 1)$-dimensional space where all the variables $w, q^a, p_b$ are independent, that is, in the thermodynamic phase space.

2.2. Relative entropy

Now, we review the meaning of relative entropy in the framework we are pursuing here (see e.g. [1]). Let us fix the equilibrium distribution $\rho_0$ as the reference distribution and compute the mean value of the microscopic entropy of another distribution $\rho \neq \rho_0$ with respect to $\rho_0$, that is,
\[
\langle s(\rho) \rangle_0 = \int s(\rho) \rho_0 \, dx = S(\rho_0) - S(\rho, \rho_0),
\]

where \( S(\rho, \rho_0) \) is the relative entropy of \( \rho \) with respect to \( \rho_0 \) (also called the Kullback–Leibler divergence of the two distributions \([1]\)), which is defined as

\[
S(\rho, \rho_0) = -\int \rho_0 \ln \left( \frac{\rho_0}{\rho} \right) \, dx
\]

and measures the loss of information that one gets when using the distribution \( \rho \) instead of the proper one \( \rho_0 \) (see e.g. \([4]\)). Therefore, we can obtain a characterization for the relative entropy of a distribution \( \rho \) with respect to the equilibrium distribution \( \rho_0 \) as

\[
S(\rho, \rho_0) = \left\langle s(\rho_0) \right\rangle_0 - \left\langle s(\rho) \right\rangle_0 = \langle 4x \rangle_0,
\]

that is, the relative entropy provides us with the mean difference between the microscopic entropies of the two distributions.

When the distribution \( \rho \) is infinitesimally close to the equilibrium distribution \( \rho_0 \), we can write equation (25) in differential form. Up to first order terms we have

\[
S(\rho, \rho_0) = \left\langle s(\rho_0) \right\rangle_0 - \left\langle s(\rho) \right\rangle_0 = \left\{ ds(\rho_0) \right\}_0.
\]

However, as \( S(\rho, \rho_0) \) is always positive and has an absolute minimum for \( \rho = \rho_0 \) (with \( S(\rho_0, \rho_0) = 0 \)), expanding up to second order in the control parameters \( \lambda_i \) yields

\[
S(\rho, \rho_0) = \frac{1}{2} \left\{ \frac{\partial^2 s}{\partial \lambda_i \partial \lambda_j} \right\}_0 \, d\lambda_i \, d\lambda_j = \frac{1}{2} \left\{ \frac{\partial \ln \rho}{\partial \lambda_i} \frac{\partial \ln \rho}{\partial \lambda_j} \right\}_0 \, d\lambda_i \, d\lambda_j.
\]

Therefore, the components of Fisher–Rao metric are obtained as the Hessian of the function \( S(\rho, \rho_0) \) at the stationary point \( \rho = \rho_0 \). We will see in the next section that the first order variation of \( S(\rho, \rho_0) \) defines a 1-form while the second order variation yields a metric tensor for the phase space of thermodynamics endowing such manifold with a contact metric structure.

3. The contact and Riemannian structures of the thermodynamic phase space

Let us focus now only on the equilibrium distribution \( \rho_0 \). Motivated by the first moment of the relative entropy (see equation (13)), we promote the mean values \( p_i \) defined in (7) as independent coordinates of a larger manifold which we call the thermodynamic phase space and denote it by \( T^{2n+1} \). Such a space has naturally \( 2n + 1 \) dimensions, \( n + 1 \) corresponding to the control variables and \( n \) to the mean values \( p_i \) corresponding to each Lagrange multiplier \( q^i \). In this manner, equation (13) becomes a contact 1-form for the thermodynamic phase space \( T^{2n+1} \) in a Darboux patch whose local coordinates are given by \( w, q^1, ..., q^n, p_1, ..., p_n \).

Let us clarify our opening statement through a revision of some geometric theories of thermodynamics \([25–29]\). Just as in its symplectic version, Darboux’s theorem states that around each point of a contact manifold, there is a coordinate patch in which the 1-form defining the contact structure of \( T^{2n+1} \) reduces to

\[
\eta = dw - \frac{1}{n} dp_i dq^i, \quad \text{with} \quad D = \ker(\eta),
\]

where \( D \) is the contact distribution of \( T^{2n+1} \) \([24]\). Since the coordinates \( w, q^1, ..., q^n, p_1, ..., p_n \) are all independent in the thermodynamic phase space, the 1-form (28) is non-degenerate and satisfies the maximal non-integrability condition.
Therefore it is a well defined volume form on $T^{2n+1}$.

The spaces of thermodynamic equilibrium states corresponding to particular systems are identified with the Legendre sub-manifolds of the contact distribution $D$ defined by $\eta$. That is, the $n$ dimensional sub-manifolds $E^n$ embedded in $T^{2n+1}$ whose tangent bundle is completely contained in $D$. It is easy to see that if

$$q^n : E^n \longrightarrow T^{2n+1}$$

is one such embedding, then $E^n$ is locally defined through the equation

$$q^n = q^n \left[ dw - p_i dq_i \right] = 0,$$  \hspace{1cm} (31)

where

$$q^n : T^n T^{2n+1} \longrightarrow T^n E^n$$  \hspace{1cm} (32)

is the induced map associated with $q^n$ and (31) provides us with the (local) explicit form of the embedding $q^n: [p_j] \rightarrow [w(q^i), p_j(q^i), q^i]$, that is

$$p_j(q^i) = \frac{\partial w}{\partial q^i}.$$  \hspace{1cm} (33)

where the only freedom rests upon the choice of $w = w(q^i)$.

One can readily see that (31) is a local statement of the first law for a system described by the fundamental relation $w$. This construction has been used as the basis for various programmes in geometric thermodynamics, albeit most of these programmes have focused only on the choice of a metric tensor for the Legendre sub-manifold $E^n$ [8, 9, 21]. In the construction presented here, the metric tensor for the equilibrium space $E^n$ will be obtained by means of the induced map (32) of the metric tensor for $T^{2n+1}$

$$G = \eta \otimes \eta + \frac{1}{2} \left( dq^i \otimes dp_i + dp_i \otimes dq^i \right).$$  \hspace{1cm} (34)

Note that this choice is not arbitrary. It is constructed so that the pull-back of an embedding $\phi$ of the control manifold $C^{n+1}$ into $T^{2n+1}$

$$\phi: (w, q^i) \longrightarrow (w, p_j(q^i), q^i).$$  \hspace{1cm} (35)

coincides with the Fisher–Rao metric on $C^{n+1}$ (see equations (19)–(20)), that is

$$G_{FR} = \phi^*(G).$$  \hspace{1cm} (36)

Moreover, one can directly verify that

$$\phi^*(\eta) = \langle dx \rangle_0.$$  \hspace{1cm} (37)

Thus, at every point of the control manifold, the 1-form $\langle dx \rangle_0 \in T^* S^{n+1}$ is annihilated by vectors lying on the contact distribution $D$ of $T^{2n+1}$. In this sense, the Legendre sub-manifolds of the thermodynamic phase space correspond to equilibrium states maximizing the averaged microscopic entropy as long as the map

$$\left( \phi^{-1} \circ \phi \right)^*(\langle dx \rangle_0) = 0$$  \hspace{1cm} (38)

is well defined. Locally this is always the case provided the embedding $\phi: C^{n+1} \longrightarrow T^{2n+1}$ is a $C^1$ invertible map at any point $P$. That is, if the linear term in the expansion
\[ p_j(q') = p_j(P) + \Delta q' \left( \frac{\partial p_j}{\partial q'} \right)_P + \Phi \left[ (\Delta q')^2 \right] \]  

(39)

is non-vanishing. Thus, equation (38) defines an equivalence, to linear order, between Legendre sub-manifolds and stationary points of the averaged microscopic entropy. In this sense, equation (38) is a re-statement of the first law of thermodynamics. The situation can be summarized by

\[
\begin{array}{c}
\phi \\
\phi^{-1} \circ \phi \\
(\mathcal{E}^n, 0, S_{\text{FR}})
\end{array}
\]

(40)

The local invertibility of \( \phi \) can be interpreted as the existence of local equilibrium, independently of any particular system characterized by a fundamental relation \( w(q') \), defining the embedding \( \varphi \). Moreover, taken all the way down to \( \mathcal{E}^n \), the condition of local invertibility of \( \phi \), equation (39), becomes the condition

\[
\det \left( \frac{\partial^2 w}{\partial q^i \partial q^j} \right) \neq 0, \tag{41}
\]

which is equivalent to demanding that the induced Fisher–Rao metric on \( \mathcal{E}^n \)

\[
g_{\text{FR}} = \phi^n(G) = \frac{\partial^2 w}{\partial q^i \partial q^j} \, dq^i \otimes dq^j, \tag{42}
\]

is non-degenerate.

It is worth noticing here that we have obtained the universality of local equilibrium as a direct consequence of the extremization of the entropy functional \( (\rho_0) \) is the distribution maximizing the entropy) or, alternatively, as the condition of the vanishing of the first order variation of the relative entropy \( S(\rho, \rho_0) \), see equation (26). Moreover, the components for the Fisher–Rao metric correspond to the second moment \( \langle (\Delta s)^2 \rangle_0 \), see equation (20). At equilibrium, the second law implies that the entropy must be a maximum. Therefore, when \( w \) is identified as the entropy, the induced Fisher–Rao metric, equation (42), has to be negative definite. Notice that in such case the metric (42) coincides with the thermodynamic metric introduced by Ruppeiner in the context of thermodynamic fluctuation theory [21–23] up to a sign.

We stress here the fact that this construction on the thermodynamic phase space is completely general and holds for any thermodynamic system. This means that the forms of \( \eta \) and \( G \) do not change from system to system, as well as the first and second law of thermodynamics apply in full generality. The specification of a particular equilibrium system corresponds to a particular choice of the fundamental function \( w(p_j) \). The First Law and the equations of state follow from equation (31). In this way, the induced metric \( g_{\text{FR}} \) on each Legendre sub-manifold \( \mathcal{E}^n \) is specified for each particular system.
4. Geometric properties of the phase space of equilibrium thermodynamics

In this section we study the geometry of the thermodynamics phase space in greater detail. As we have previously discussed, such a space is a metric contact manifold whose contact and metric structures are defined through the mean value and the second moment of the infinitesimal change in the microscopic entropy weighted by the equilibrium Gibbs distribution, respectively (see equations (13) and (18)).

In general, a contact 1-form \( \eta \) is not unique, but it belongs to the class generating the contact structure \( \mathcal{D} \) of \( T^{2n+1} \), see equation (28). Indeed, any other 1-form defining the same \( \mathcal{D} \) is necessarily conformally equivalent to \( \eta \), i.e. for any two 1-forms \( \eta_1 \) and \( \eta_2 \) in the same equivalence class \( \eta \), one has \( \eta_2 = \omega \eta_1 \) for some non-vanishing real function \( \omega \).

For each member in the class generating \( \mathcal{D} \) there is a unique canonical vector field \( \xi \), called the Reeb vector field, satisfying
\[
\eta(\xi) = 1 \quad \text{and} \quad d\eta(\xi, X) = 0,
\]
for any vector field \( X \in T^{2n+1} \). The Reeb vector field generates a splitting of the tangent bundle \( TT^{2n+1} \), that is
\[
TT^{2n+1} = L_\xi \oplus D,
\]
where \( L_\xi \) is the sub-space generated by \( \xi \).

It will be convenient to work in a basis adapted to the splitting (44). Since we have already seen that the Reeb vector field generates the vertical part, it only remains to find a basis for the contact distribution. Choosing Darboux local coordinates, it is easy to see that
\[
\eta \left( \tilde{Q}^b \right) \equiv \left[ dw - p_a dq^a \right] \left( \frac{\partial}{\partial q^b} + p_b \frac{\partial}{\partial w} \right) = p_b - p_a \delta^b_a = 0
\]
and, similarly,
\[
\eta \left( \tilde{P}^a \right) \equiv \left[ dw - p_a dq^a \right] \left( \frac{\partial}{\partial p_b} \right) = 0.
\]

Therefore, the vectors
\[
\left\{ \frac{\partial}{\partial q^b}, \frac{\partial}{\partial q^a} + P_a \frac{\partial}{\partial w} \right\} \subset \ker(\eta)
\]
are linearly independent and generate \( \mathcal{D} \). Thus, the non-coordinate basis
\[
\left\{ \tilde{\xi}, \tilde{Q}^a, \tilde{P}^a \right\} = \left\{ \xi, \frac{\partial}{\partial q^a} + P_a \frac{\partial}{\partial w}, \frac{\partial}{\partial p_b} \right\}
\]
is naturally adapted to the splitting (44) induced by the gauge choice \( \eta \in [\eta] \). Notably, the generators of such basis satisfy the commutation relations
\[
\left[ \tilde{P}^a, \tilde{Q}_b \right] = \delta^a_b \xi, \quad \left[ \tilde{Q}_a, \tilde{P}_b \right] = 0 \quad \text{and} \quad \left[ \xi, \tilde{P}^a \right] = 0,
\]
with respect to the Lie-bracket and where \( \delta^a_b \) represents the \( nxn \)-Kronecker delta. These are the defining relations of the \( n \)th Heisenberg Lie algebra \( \mathcal{H}_n \). For this reason, we call the set (48) the Heisenberg basis of \( TT \). Note that the above commutation relations arose naturally from the definition of the contact structure \( \mathcal{D} \) motivated by the mean value of the microscopic entropy change.
Now, we will show that by taking into account the macroscopic information stemming from the second moment of \( ds \)—equation (19)—the thermodynamic phase space is uniquely defined as the hyperbolic Heisenberg group defined in [19]. That is, we will show that the thermodynamic phase space is a para-Sasakian manifold with a flat canonical connection. To this end, we verify that the metric (34) satisfies some formal definitions following the construction in [18] and [20].

We have already selected a 1-form in the class defining the contact structure of the thermodynamic phase space and equipped this manifold with a metric structure given by (34). That is, we have the quadruple \((T, \eta, \xi, G)\). If, in addition, there is a \((1, 1)\)-tensor field \(\Phi\) satisfying

\[
L_\xi = \ker(\Phi), \quad D = \text{Im}(\Phi) \quad \text{and} \quad \Phi^2 = I - \eta \otimes \xi, \quad (50)
\]
such that

\[
\eta(X) = G(\xi, X), \quad \text{and} \quad \frac{1}{2} \eta(Y, X) = G(X, \eta Y) \quad (51)
\]
for any pair of vector fields \(X\) and \(Y\), we call \(\Phi\) an almost-para-contact structure and \((T, \eta, \xi, \Phi, G)\) a para-contact metric manifold\(^1\).

To determine the form of \(\Phi\) let us consider the Levi-Civita connection associated with \(G\). Then, the covariant derivative of the Reeb vector field satisfies [18]

\[
V_\xi = -\Phi + \Phi h \quad (52)
\]
where

\[
h = \frac{1}{2} L_\xi \Phi. \quad (53)
\]

If \(\xi\) is a killing vector of \(G\), the tensor field \(h\) vanishes identically. Thus, equation (52) directly defines the almost-para-contact structure. Working in Darboux coordinates, the Reeb vector field is simply given by \(\xi = \frac{\partial}{\partial \theta^w}\). Thus, since none of the metric components in (34) is a function of \(\theta^w\), \(\xi\) is indeed a killing vector of \(G\) and \(\Phi = -V_\xi\). Moreover, the vectors in the Heisenberg basis generating the horizontal distribution are all null with respect to the metric \(G\), that is

\[
G(\hat{\theta}^a, \hat{\theta}^a) = G(\hat{\theta}_a, \hat{\theta}_a) = 0. \quad (54)
\]

Thus, the metric has \(n + 1\) ‘space-like’ directions and \(n\) ‘time-like’ directions, i.e. the signature of \(G\) is \((n + 1, n)\). To make explicit the pseudo-Riemannian signature of the metric, let us introduce the orthonormal (dual) basis

\[
\hat{\theta}^{(i)} = \left\{ \hat{\theta}^{(0)}, \hat{\theta}_+^{(a)}, \hat{\theta}_-^{(a)} \right\} \quad i = 0 \ldots 2n, \quad (55)
\]
where

\[
\hat{\theta}^{(0)} = \eta \quad \text{and} \quad \hat{\theta}_\pm^{(a)} = \frac{\sqrt{\eta}}{2p_\pm} \left[p_\pm dq^a \mp dp_\pm \right] \quad (\text{no sum over } a). \quad (56)
\]

\(^1\) Usually in the literature there is a slightly different definition for a para-contact metric manifold (see e.g. [18, 20]), due to the fact that the authors take \(\eta(Y, X) = \frac{1}{2}(\eta(Y) - \eta(X) - \eta(Y(X), Y))\), which in fact differs in our case from the exterior derivative of \(\eta\) by a factor \(1/2\). Taking into account this difference in \(\eta\), one finds that the two definitions coincide.
In this case, the metric (34) can be written as

\[ G = \hat{\theta}^{(0)} \otimes \hat{\theta}^{(0)} + \sum_{a=1}^{n} \left[ \hat{\theta}_\alpha^{(a)} \otimes \hat{\theta}_\alpha^{(a)} - \hat{\theta}_\alpha^{(a)} \otimes \hat{\theta}_\alpha^{(a)} \right]. \]  

(57)

whose matrix representation is

\[ G_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \delta^a_b & 0 \\ 0 & 0 & -\delta^a_b \end{pmatrix}. \]  

(58)

Thus, in this convention, the \( n \) ‘time-like’ directions are given by

\[ \hat{e}^-_{(a)} = -G^{-1} \hat{\theta}_-^{(a)} \]  

(no sum over \( a \)),

(59)

while the \( n + 1 \) ‘space-like’ directions are

\[ \hat{e}^\pm_{(0)} = -G^{-1} \hat{\theta}_\pm^{(0)} \]  

(no sum over \( a \)).

(60)

Therefore, let us define the non-coordinate basis

\[ \hat{e}_{(0)} \otimes \hat{e}_{(0)} \]  

(61)

whose structure functions can be read from the only non-vanishing Lie-brackets

\[
\left[ \hat{e}^+_{(a)}, \hat{e}^-_{(a)} \right] = -\frac{1}{2 \sqrt{P_a}} \left( \hat{e}^+_{(a)} + \hat{e}^-_{(a)} \right) + 2 \hat{e}_{(0)} \]  

\( (a = 1 \ldots n). \)

(62)

We call (61) the canonical basis of the thermodynamic phase space. Throughout the rest of the paper, all the calculations will be performed with respect to this basis. Note that in our convention, the indices \( i, j, k \) vary from 0 to \( 2n \) while \( a, b, c \) take values from 1 to \( n \).

First, let us note that the non-vanishing the Levi-Civita connection symbols in this non-coordinate basis are

\[
\Gamma^0_{(a)(b)} = -\Gamma^0_{(b)(a)} = \Gamma^a_{(b)(a)} = \Gamma^a_{(a)(0)} = \Gamma^a_{0(a)} = \Gamma^a_{(a)0} = 1, \]  

\( (a, b = 1 \ldots n). \)

(63)

\[
\Gamma^a_{(a)(a)} = \Gamma^a_{(a)(a)} = -\Gamma^a_{(a)(a)} = -\Gamma^a_{(a)(a)} = \frac{1}{2 \sqrt{P_a}}. \]  

(64)

Now, an expression for the almost-para-contact structure is directly obtained from (52)

whose form in the canonical basis is
Indeed, $\Phi^2$ satisfies

$$\Phi^2 = \left[ \hat{e}^+_{(a)} \otimes \hat{\theta}^-_{(a)} + \hat{e}^-_{(a)} \otimes \hat{\theta}^+_{(a)} \right] \left( - \left[ \hat{e}^+_{(b)} \otimes \hat{\theta}^-_{(b)} + \hat{e}^-_{(b)} \otimes \hat{\theta}^+_{(b)} \right] \right)$$

$$= \left[ \hat{e}^+_{(a)} \otimes \hat{\theta}^-_{(a)} \right] \left( \hat{e}^-_{(b)} \otimes \hat{\theta}^+_{(b)} + \hat{e}^+_{(b)} \otimes \hat{\theta}^-_{(b)} \right) + \left[ \hat{e}^-_{(a)} \otimes \hat{\theta}^+_{(a)} \right] \left( \hat{e}^-_{(b)} \otimes \hat{\theta}^-_{(b)} + \hat{e}^+_{(b)} \otimes \hat{\theta}^+_{(b)} \right)$$

$$= \left[ \delta^a_b \hat{e}^+_{(a)} \otimes \hat{\theta}^+_{(a)} \right] + \left[ \delta^a_b \hat{e}^-_{(a)} \otimes \hat{\theta}^-_{(a)} \right]$$

$$= \left[ \hat{e}^+_{(a)} \otimes \hat{\theta}^+_{(a)} \right] + \left[ \hat{e}^-_{(a)} \otimes \hat{\theta}^-_{(a)} \right].$$

where we have used the Einstein sum convention. Its matrix expression is

$$\left[ \Phi^2 \right]_j^i = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \delta^a_b & 0 \\ 0 & \cdots & \delta^a_b \end{bmatrix}$$

(67)

and, from (65), the action of $\Phi$ on the elements of the basis is trivially given by

$$\Phi \hat{e}_0 = 0,$$

(68)

$$\Phi \hat{e}_{(a)}^- = -\hat{e}_{(a)}^-, \quad (69)$$

$$\Phi \hat{e}_{(a)}^+ = -\hat{e}_{(a)}^+.$$  

(70)

Thus, the defining requirements of (50) are fulfilled.

The metric $G$ is—by construction—compatible and associated to the almost-para-contact structure $\Phi$. Let us revise these concepts explicitly by means of a pair of arbitrary vector fields

$$X = X^i \hat{e}_i = X^0 \hat{e}_0 + \sum_{a=1}^n \left[ X^a_+ \hat{e}_{(a)}^+ + X^a_- \hat{e}_{(a)}^- \right].$$

(71)

and

$$Y = Y^i \hat{e}_i = Y^0 \hat{e}_0 + \sum_{a=1}^n \left[ Y^a_+ \hat{e}_{(a)}^+ + Y^a_- \hat{e}_{(a)}^- \right].$$

(72)
It follows from (68)–(70) that
\[ \Phi X = - \sum_{a=1}^{n} \left[ X^{a} \hat{e}_{(a)}^{+} + X^{-a} \hat{e}_{(a)}^{-} \right], \quad (73) \]
\[ \Phi Y = - \sum_{a=1}^{n} \left[ Y^{a} \hat{e}_{(a)}^{+} + Y^{-a} \hat{e}_{(a)}^{-} \right]. \quad (74) \]

The inner product of the vector fields (73) and (74) induced by the metric (57) is given by
\[ G(\Phi X, \Phi Y) = \sum_{a=1}^{n} \left[ X^{a} Y^{a} - X^{-a} Y^{-a} \right], \quad (75) \]
while that of (71) and (72) is
\[ G(X, Y) = X^{0} Y^{0} - \sum_{a=1}^{n} \left[ X^{a} Y^{a} - X^{-a} Y^{-a} \right]. \quad (76) \]

We say that the metric \( G \) is \textit{compatible with the almost-para-contact structure} \( \Phi \) if the condition
\[ G(\Phi X, \Phi Y) = \frac{1}{2} \eta(X, Y) \quad (77) \]
is satisfied. Thus, it follows from (75) and (76), together with the obvious fact that \( \eta(X) = X^{0} \) and \( \eta(Y) = Y^{0} \), that \( G \)—given by (57)—is compatible with \( \Phi \)—computed in (65). Moreover, the metric \( G \) is an associated metric to the almost-para-contact structure, that is
\[ G(X, \Phi Y) = \frac{1}{2} \eta(X, Y) \quad (78) \]
is satisfied, see definition (51). In the canonical basis, the exterior derivative of the contact 1-form takes the form
\[ d\eta = -2 \sum_{a=1}^{n} \left[ \hat{\theta}_{+}^{(a)} \wedge \hat{\theta}_{-}^{(a)} \right]. \quad (79) \]

Thus, indeed,
\[ G(X, \Phi Y) = - \sum_{a=1}^{n} \left[ X^{a} Y^{a} - X^{-a} Y^{-a} \right] = \frac{1}{2} d\eta(X, Y). \quad (80) \]

A contact manifold endowed with a Riemannian structure such that (78) is satisfied is called a \textit{para-contact metric manifold}. Thus, the thermodynamic phase space \( (T^{2n+1}, \eta, \xi, \Phi, G) \) is a para-contact metric manifold.

Next, we verify that the thermodynamic phase space is, in fact, a \textit{para-Sasakian} manifold. To this end, we need to show that the structure \( (T^{2n+1}, \eta, \xi, \Phi, G) \) is normal, i.e. that the Nijenhuis tensor of the almost-para-contact structure
\[ N_{\phi}(X, Y) \equiv \Phi^{2}[X, Y] + [\Phi X, \Phi Y] - \Phi[\Phi X, Y] - \Phi[X, \Phi Y] \quad (81) \]
satisfies the condition
\[ \left[ N_{\phi} - d\eta \otimes \xi \right](X, Y) = 0. \quad (82) \]
A long but straightforward calculation yields
\[N_{\phi}(X, Y) = -2 \sum_{a=1}^{n} \left( X_a^\alpha Y_a^\beta - X_a^\beta Y_a^\alpha \right) \hat{e}_0(0) \] (83)
and from the second equality of (80), it follows that the normality condition is satisfied. This is a statement of the integrability of the almost-para-complex structure of the horizontal distribution, that is, the restriction of the Nijenhuis tensor to the horizontal space \(D\) vanishes identically, as can be seen from (83). In this case, we say that the structure \((\mathcal{T}^{2n+1}, \eta, \xi, \Phi, G)\) is integrable.

Finally, we use the result of Ivanov, Vassilev and Zamkovoy (IVZ) which states that an integrable para-contact metric manifold of dimension \(2n + 1\) is locally isomorphic to the hyperbolic Heisenberg group exactly when the canonical connection \(\tilde{\nabla}\) has vanishing horizontal curvature, i.e. \(\tilde{R}(X, Y, Z, V) = 0\) for all \(X, Y, Z, V \in D\) (see IVZ theorem in appendix B). Here, the canonical connection refers to the one compatible with all the objects defining the para-Sasakian structure, namely, the one satisfying
\[\tilde{\nabla}_X \eta = \tilde{\nabla}_X \xi = \tilde{\nabla}_X \Phi = \tilde{\nabla}_X G = 0\] (84)
and whose torsion satisfies
\[\tilde{T}(\xi, \Phi Y) = -\Phi \tilde{T}(\xi, Y)\] (85)
and
\[\tilde{T}(X, Y) = 2d\eta(X, Y)\xi.\] (86)

On an integrable para-contact metric manifold such a connection is unique and is defined in terms of the Levi-Civita connection by (see equation (4.44) in [18])
\[\tilde{V}_X Y = V_X Y + \eta(X) \Phi Y - \eta(Y) V_X \xi + (V_X \eta)(Y) \xi.\] (87)
Working out the connection symbols of \(\tilde{\nabla}\) with respect to the canonical basis,
\[\tilde{V}_{\hat{e}_i} \hat{e}_j = \tilde{\Gamma}_{ij}^k \hat{e}_k\] (88)
it follows that
\[\tilde{\Gamma}_{ij}^0 = -\delta^0_i \tilde{\Gamma}_{ij}^0 - \delta^0_j \tilde{\Gamma}_{ij}^0 = 0\] (89)
and
\[\tilde{\Gamma}_{ij}^a = \Gamma_{ij}^a - \delta^0_i \Gamma_{ij}^a - \delta^0_j \Gamma_{ij}^a \] (90)
where Levi-Civita connection symbols are given by (63) and (64). Thus, the only non-vanishing connection symbols are
\[-\tilde{\Gamma}_{(a+a)}^a = \Gamma_{(a+a)}^{a+a} = \tilde{\Gamma}_{a(a+a)}^{a+a} = \frac{1}{2} \sqrt{\tilde{E}_a} (a = 1 \ldots n).\] (91)
Finally, the components of the curvature tensor of the canonical connection are
\[\tilde{R}_{jkl}^i = \hat{e}_i(\tilde{\Gamma}_{jkl}) - \hat{e}_i(\tilde{\Gamma}_{jkl}) + \tilde{\Gamma}_{jml}^i \tilde{\Gamma}_{mk}^l - \tilde{\Gamma}_{jml}^i \tilde{\Gamma}_{mk}^l + \gamma_{ijkl} \tilde{\Gamma}_{jm}.\] (92)
Using the definition of the canonical basis (61) together with the the structure functions \(\gamma_{ijkl}\) obtained from (62) and the expression for the non-vanishing connection symbols (91) it can be directly verified that all the components of the Riemann tensor (92) are identically zero (see appendix A).
It is an interesting fact that the Levi–Civita and the canonical connections play a dual geometric role in the following sense: the former is the unique torsion-free and metric compatible connection whose curvature is non-trivial whereas the latter is the unique curvature-free and fully adapted connection with non-trivial torsion providing us with two geometrically independent pictures of the same object. Indeed, the Ricci curvature of the Levi–Civita connection

\[ \text{Ric} = -2n\theta^{(0)} \otimes \theta^{(0)} + 2 \sum_{a=1}^{n} \left[ \theta_{+}^{(a)} \otimes \theta_{+}^{(a)} - \theta_{-}^{(a)} \otimes \theta_{-}^{(a)} \right], \tag{93} \]

satisfies the property

\[ \text{Ric}(X, Y) = \lambda\eta(X)\eta(Y) + \nu G(X, Y), \tag{94} \]

where \( \lambda = -(2n + 2) \) and \( \nu = 2 \). Thus, the phase space is, additionally, an \( \eta \)-Einstein para-Sasakian structure.

We conclude that the thermodynamic phase space—(\( T^{2n+1}, \eta, \xi, \Phi, G \))—is a canonically flat \( \eta \)-Einstein para-Sasakian manifold. Hence, it follows from the IVZ theorem (see appendix B) that the thermodynamic phase space is locally isomorphic to the hyperbolic Heisenberg group. However, the metrics on the hyperbolic Heisenberg group and on the thermodynamic phase space are not exactly equivalent. In fact, although the Ricci part of the curvature is the same, the Weyl part is different in the two cases. Furthermore, it can be verified that these metrics satisfy the field equations for the vacua of Einstein–Gauss–Bonnet gravity. This will be the subject of further investigation [30].

5. Closing remarks

In this work, we have presented a geometric formulation of the emergence of the macroscopic phase space of thermodynamics based on the maximum entropy principle. This construction unifies various aspects arising from statistical mechanics, information theory and metric contact geometry. One important aspect of our formulation is that it can be generalized to the case of non-equilibrium systems by considering probability distributions different from the Gibbs one as reference in the relative entropy functional.

We derived a number of useful properties of the phase space geometry. In particular, we showed that it is a para-Sasakian manifold defined by a metric of signature \((n + 1, n)\) whose associated Levi–Civita connection satisfies the defining property of an \( \eta \)-Einstein manifold. Moreover, introducing another connection which is parallel with respect to the tensors defining the para-Sasakian structure and using the IVZ theorem (see appendix B) we have shown that this manifold is locally isomorphic to the hyperbolic Heisenberg group.

The relationship with the Heisenberg group is not surprising since, by construction, the observables we considered correspond to mean values of random variables defined on a suitable microscopic space of events. Such a set up is sufficiently general in the sense that the data measured in an experiment can never be known with full accuracy; we can never achieve complete knowledge of the defining variables of a system. This is the starting point for the general problem of model selection in statistical inference theory. In this setting, the maximum entropy principle is a criterion for selecting the ‘optimal’ probability distribution consistent with the given data: the one which is least informative about anything else not contained in the data set. Therefore, there is an intrinsic uncertainty encoded in the very definition of the thermodynamic phase space which manifests itself by restricting the geometry to be the hyperbolic Heisenberg group.
Let us close this work by pointing some future directions. Firstly, the signature of the metric of the thermodynamic phase space allows for the existence of null directions providing us with a cone structure similar to the one present in relativity. The existence of such a cone structure is a signal that there may be a way of characterizing correlations between states (points in the thermodynamic phase space) in a geometric way analogous to that of points in a space-time. Secondly, the hyperbolic nature of the thermodynamic phase space remains to be studied, a possible direction can be directly related to chaotic motion, which is a necessary ingredient in order to obtain correct statistical mechanical calculations. Finally, noting that the thermodynamic phase space has constant Levi–Civita scalar curvature suggests that it is highly symmetric. Indeed, the Levi–Civita connection satisfies the defining condition of an $\eta$-Einstein manifold, a slight modification of an Einstein manifold. In gravitational theories, an important example are the dS and AdS solutions for the vacuum Einstein field equations with a cosmological constant. It can be directly verified that the metric presented in this work is a solution for the vacuum of Einstein-Gauss-Bonnet theory of gravity analogous to the de Sitter solution. This is a remarkable connection between information geometry and gravity which deserves further exploration [30].

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Appendix A. Calculation of the canonical curvature

In this appendix we compute all the non-trivial components of the Riemann tensor of the canonical connection, i.e.

$$\tilde{R}^i_{jkl} = \tilde{\epsilon}_{(i)} \tilde{F}^j_{jk} - \tilde{\epsilon}_{(k)} \tilde{F}^j_{ij} + \tilde{F}^m_{jk} \tilde{F}^i_{ml} - \tilde{F}^m_{ij} \tilde{F}^i_{mk} + \gamma^m_{ij} \tilde{F}^i_{jm}. \quad (A.1)$$

Firstly, the non-vanishing structure functions of the canonical basis (see equation (62)) are given by

$$\gamma_{a}^{0} = 2 \quad \text{and} \quad \gamma_{a}^{a} = \gamma_{a}^{\alpha + a} = \frac{1}{2\sqrt{\bar{B}_a}}. \quad (A.2)$$

Note that these coefficients are anti-symmetric in their lower indices. Notice as well that—from (91)—the curvature tensor does not have $i, j, k, l = 0$. Thus, let us split the calculation in its eight different non-trivial possibilities. In the following, we do not use Einstein sum convention except for the label $m$. 

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1. \( i = a, j = a \) and \( k = a \) case.

\[
\begin{align*}
\tilde{R}_{lad}^a &= \hat{\epsilon}_{(i)} f_{ia}^a - \hat{\epsilon}_{(a)} f_{al}^a + \gamma_{aal} f_{ma}^a - \gamma_{amb} f_{al}^a f_{ma}^a + \gamma_{alm} f_{al}^a f_{am}^a \\
&= f_{a(l+a)}^{a(l+a)} - f_{a(l+n+a)}^{a(l+n+a)} \\
&= \begin{cases} 
-\frac{1}{\sqrt{R_{a}}} & -\frac{1}{\sqrt{R_{a}}}, \\
-\frac{1}{\sqrt{R_{a}}} & \frac{1}{\sqrt{R_{a}}}, \\
\frac{1}{\sqrt{R_{a}}} & \frac{1}{\sqrt{R_{a}}}, \\
0 & 0 \end{cases} \quad l = a \\
&= \begin{cases} 
-\frac{1}{\sqrt{R_{a}}} & \frac{1}{\sqrt{R_{a}}}, \\
\frac{1}{\sqrt{R_{a}}} & \frac{1}{\sqrt{R_{a}}}, \\
0 & 0 \end{cases} \quad l = n + a \\
&= 0. \quad (A.3)
\end{align*}
\]

2. \( i = a, j = a \) and \( k = n + a \) case.

\[
\begin{align*}
\tilde{R}_{a(l+n+a)}^a &= \hat{\epsilon}_{(i)} f_{a(l+n+a)}^a - \hat{\epsilon}_{(a)} f_{a(l+n+a)}^a + \gamma_{a(l+n+a)l} f_{m(l+n+a)}^a - \gamma_{a(l+n+a)m} f_{a(l+n+a)m} + \gamma_{a(l+n+a)l} f_{a(l+n+a)m} \\
&= f_{a(l+n+a)}^{a(l+n+a)} - \left[ f_{a(l+n+a)}^{a(l+n+a)} f_{m(l+n+a)}^{a(l+n+a)} \right] \\
&= \begin{cases} 
-\frac{1}{\sqrt{R_{a}}} & -\frac{1}{\sqrt{R_{a}}}, \\
-\frac{1}{\sqrt{R_{a}}} & \frac{1}{\sqrt{R_{a}}}, \\
\frac{1}{\sqrt{R_{a}}} & \frac{1}{\sqrt{R_{a}}}, \\
0 & 0 \end{cases} \quad l = a \\
&= \begin{cases} 
-\frac{1}{\sqrt{R_{a}}} & \frac{1}{\sqrt{R_{a}}}, \\
\frac{1}{\sqrt{R_{a}}} & \frac{1}{\sqrt{R_{a}}}, \\
0 & 0 \end{cases} \quad l = n + a \\
&= 0. \quad (A.4)
\end{align*}
\]

3. \( i = a, j = n + a \) and \( k = a \) case.

\[
\begin{align*}
\tilde{R}_{a(n+a)l}^a &= \hat{\epsilon}_{(i)} f_{a(n+a)l}^a - \hat{\epsilon}_{(n+a)l} f_{a(n+a)l}^a + \gamma_{a(n+a)l} f_{m(n+a)l}^a - \gamma_{a(n+a)l,m} f_{a(n+a)l}^a f_{m(n+a)l}^a + \gamma_{a(n+a)l} f_{a(n+a)l}^a f_{a(n+a)l}^a \\
&= \hat{\epsilon}_{(i)} - \frac{1}{2\sqrt{R_{a}}} - \hat{\epsilon}_{(n+a)l} f_{a(n+a)l} \\
&= \frac{\partial}{\partial \eta_{l}} \left( -\frac{1}{2\sqrt{R_{a}}} \right) + \frac{\partial}{\partial \eta_{l}} \left( \frac{1}{2\sqrt{R_{a}}} \right) \quad l = a \\
&= \frac{\partial}{\partial \eta_{l}} \left( -\frac{1}{2\sqrt{R_{a}}} \right) + \frac{\partial}{\partial \eta_{l}} \left( \frac{1}{2\sqrt{R_{a}}} \right) \quad l = n + a \\
&= 0. \quad (A.5)
\end{align*}
\]
4. $i = a$, $j = n + a$ and $k = n + a$ case.

\[
\mathbf{\tilde{R}}_a^{(i)(j)(k)} = \mathbf{\tilde{e}}_a^{(i)} \tilde{F}_{(j)(k)}^a - \mathbf{\tilde{e}}_a^{(i)} \tilde{F}_{(j)(k)}^a + \tilde{F}_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} - \tilde{F}_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} - \gamma_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} + \gamma_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m}
\]

\[
= \mathbf{\tilde{e}}_a^{(i)} \left( \frac{1}{2 \sqrt{P_a^2}} \right) + \frac{\partial}{\partial \eta_a} \left( \tilde{F}_{a}^{(i)} \right) + \gamma_{a} \mathbf{\tilde{F}}_{a}^{m} + \gamma_{a} \mathbf{\tilde{F}}_{a}^{m} \]

\[
= \begin{cases} 
\frac{\partial}{\partial \eta_a} \left( \frac{1}{2 \sqrt{P_a^2}} \right) & l = a \\
\frac{\partial}{\partial \eta_a} \left( \frac{1}{2 \sqrt{P_a^2}} \right) & l = n + a
\end{cases}
\]

\[
= 0. \quad (A.6)
\]

5. $i = n + a$, $j = a$ and $k = a$ case.

\[
\mathbf{\tilde{R}}_{a}^{(i)(j)(k)} = \mathbf{\tilde{e}}_a^{(i)} \tilde{F}_{(j)(k)}^a - \mathbf{\tilde{e}}_a^{(i)} \tilde{F}_{(j)(k)}^a + \tilde{F}_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} - \tilde{F}_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} + \gamma_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} - \gamma_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m}
\]

\[
= \mathbf{\tilde{e}}_a^{(i)} \left( -\frac{1}{2 \sqrt{P_a^2}} \right) + \mathbf{\tilde{e}}_a^{(i)} \tilde{F}_{a}^{(j)(k)}
\]

\[
= \begin{cases} 
\frac{\partial}{\partial \eta_a} \left( -\frac{1}{2 \sqrt{P_a^2}} \right) & l = a \\
\frac{\partial}{\partial \eta_a} \left( -\frac{1}{2 \sqrt{P_a^2}} \right) & l = n + a
\end{cases}
\]

\[
= 0. \quad (A.7)
\]

6. $i = n + a$, $j = a$ and $k = n + a$ case.

\[
\mathbf{\tilde{R}}_{a}^{(i)(j)(k)} = \mathbf{\tilde{e}}_a^{(i)} \tilde{F}_{(j)(k)}^a - \mathbf{\tilde{e}}_a^{(i)} \tilde{F}_{(j)(k)}^a + \tilde{F}_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} - \tilde{F}_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} + \gamma_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m} - \gamma_{(j)(k)}^a \mathbf{\tilde{F}}_{a}^{m}
\]

\[
= \mathbf{\tilde{e}}_a^{(i)} \left( \frac{1}{2 \sqrt{P_a^2}} \right) + \frac{\partial}{\partial \eta_a} \tilde{F}_{a}^{(j)(k)}
\]

\[
= \begin{cases} 
\frac{\partial}{\partial \eta_a} \left( \frac{1}{2 \sqrt{P_a^2}} \right) & l = a \\
\frac{\partial}{\partial \eta_a} \left( \frac{1}{2 \sqrt{P_a^2}} \right) & l = n + a
\end{cases}
\]

\[
= 0. \quad (A.8)
\]
7. $i = n + a, j = n + a$ and $k = a$ case.

\[
\mathcal{R}^{(n+a)}_{(n+a)(n+a)} = \mathcal{R}^{(n+a)}_{(n+a)(n+a)} \mathcal{R}^m_{(n+a)m} - \mathcal{R}^{(n+a)}_{(n+a)(n+a)} \mathcal{R}^m_{(n+a)m} + \mathcal{R}^{(n+a)}_{(n+a)(n+a)} \mathcal{R}^m_{(n+a)m} + \mathcal{R}^{(n+a)}_{(n+a)(n+a)} \mathcal{R}^m_{(n+a)m}.
\]

\[
= \left\{ \begin{array}{l}
\left( \frac{1}{2 \sqrt{b_i}} \right) \left( \frac{1}{2 \sqrt{b_i}} \right) - \left( \frac{1}{2 \sqrt{b_i}} \right) \left( \frac{1}{2 \sqrt{b_i}} \right), \\
\left( \frac{1}{2 \sqrt{b_i}} \right) \left( \frac{1}{2 \sqrt{b_i}} \right) - \left( \frac{1}{2 \sqrt{b_i}} \right) \left( \frac{1}{2 \sqrt{b_i}} \right), \\
\end{array} \right. \\
\end{array}
\]

\[
= 0.
\]

Therefore, the Riemann tensor of the canonical connection is identically zero.

\[
(A.9)
\]

8. $i = n + a, j = n + a$ and $k = n + a$ case.

\[
\mathcal{R}^{(n+a)}_{(n+a)(n+a)} = \mathcal{R}^{(n+a)}_{(n+a)(n+a)} \mathcal{R}^m_{(n+a)m} - \mathcal{R}^{(n+a)}_{(n+a)(n+a)} \mathcal{R}^m_{(n+a)m} + \mathcal{R}^{(n+a)}_{(n+a)(n+a)} \mathcal{R}^m_{(n+a)m}.
\]

\[
= \left\{ \begin{array}{l}
\left( \frac{1}{2 \sqrt{b_i}} \right) \left( \frac{1}{2 \sqrt{b_i}} \right) - \left( \frac{1}{2 \sqrt{b_i}} \right) \left( \frac{1}{2 \sqrt{b_i}} \right), \\
\left( \frac{1}{2 \sqrt{b_i}} \right) \left( \frac{1}{2 \sqrt{b_i}} \right) - \left( \frac{1}{2 \sqrt{b_i}} \right) \left( \frac{1}{2 \sqrt{b_i}} \right), \\
\end{array} \right. \\
\end{array}
\]

\[
= 0.
\]

\[
(A.10)
\]

Therefore, the Riemann tensor of the canonical connection is identically zero.

**Appendix B. The hyperbolic Heinseberg group**

In this appendix we give the definition of the hyperbolic Heisenberg group as in [19] and we state the theorem 4.2 in the same reference.

Let us introduce a manifold $G(\mathbb{P}) = \mathbb{R}^{2n} \times \mathbb{R}$ with the group law

\[
(p^*, t^*) = (p', t') \cdot (p, t) = \left( p' + p, t' + t - \sum_{k=1}^{n} (u'_k v_k - v'_k u_k) \right),
\]

with $p = (u_1, v_1, ..., u_n, v_n)$, $p' = (u'_1, v'_1, ..., u'_n, v'_n)$ and $t, t' \in \mathbb{R}$. The contact structure is given by the 1-form

\[
\tilde{\Theta} = -\frac{1}{2} dt - \sum_{k=1}^{n} (u_k dv_k - v_k du_k)
\]

and therefore the vector fields

\[
\xi = \frac{2}{\sqrt{b_i}} \frac{\partial}{\partial t}, \quad U_k = \frac{\partial}{\partial u_k} - 2v_k \frac{\partial}{\partial t}, \quad V_k = \frac{\partial}{\partial v_k} + 2u_k \frac{\partial}{\partial t}
\]

\[
(B.3)
\]
form the canonical basis for the tangent space, with the vectors $U_k$ and $V_k$ spanning the horizontal distribution $\mathcal{D} = \ker \Theta$ and $\xi$ spanning the vertical direction, as in (44). One can give such manifold a para-contact structure $\Phi$ defined by the following rules

$$\Phi^\xi \xi = 0, \quad \Phi U_k = V_k, \quad \Phi V_k = U_k.$$  \hfill (B.4)

Finally, one can define on $G(\mathcal{P})$ a metric such that

$$\tilde{G}(\xi, \xi) = 1, \quad \tilde{G}(U_k, U_k) = 1, \quad \tilde{G}(V_k, V_k) = -1,$$

so that the canonical basis (B.3) turns out to be the orthonormal basis for this metric. Note that this metric differs from the standard Sasaki metric on the Heisenberg group in the signature, while the group laws are the same. The structure $(G(\mathcal{P}), \tilde{\Theta}, \xi, \Phi, G)$ is the hyperbolic Heisenberg group. It is an example of an integrable para-contact Hermitian structure with flat canonical connection. Moreover, the following theorem from [19] states that locally it is the only such example.

**Theorem 1**  IVZ theorem. Let $(M, \eta, \xi, \Phi, G)$ be an integrable para-contact hermitian manifold of dimension $2n + 1$.

(i) If $n > 1$, then $(M, \eta, \xi, \Phi, G)$ is locally isomorphic to the hyperbolic Heisenberg group if and only if the canonical connection (87) has vanishing horizontal curvature.

(ii) If $n = 1$, then $(M, \eta, \xi, \Phi, G)$ is locally isomorphic to the three-dimensional hyperbolic Heisenberg group if and only if the canonical connection (87) has vanishing horizontal curvature and zero torsion.

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