Two-lit trees for lit-only sigma-game

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Abstract

A configuration of the lit-only $\sigma$-game on a finite graph $\Gamma$ is an assignment of one of two states, $on$ or $off$, to all vertices of $\Gamma$. Given a configuration, a move of the lit-only $\sigma$-game on $\Gamma$ allows the player to choose an $on$ vertex $s$ of $\Gamma$ and change the states of all neighbors of $s$. Given any integer $k$, we say that $\Gamma$ is $k$-lit if, for any configuration, the number of $on$ vertices can be reduced to at most $k$ by a finite sequence of moves. Assume that $\Gamma$ is a tree with a perfect matching. We show that $\Gamma$ is 1-lit and any tree obtained from $\Gamma$ by adding a new vertex on an edge of $\Gamma$ is 2-lit.

Keywords: group action, lit-only sigma-game, symplectic forms

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1 Introduction

In 1989, Sutner [9] introduced a one-player game called the $\sigma$-game. The $\sigma$-game is played on a finite directed graph $\Gamma$ without multiple edges. A configuration of the $\sigma$-game on $\Gamma$ is an assignment of one of two states, $on$ or $off$, to all vertices of $\Gamma$. Given a configuration, a move of the $\sigma$-game on $\Gamma$ allows the player to pick any vertex $s$ of $\Gamma$ and change the states of all neighbors of $s$. Given an initial configuration, the goal is to minimize the number of $on$ vertices of $\Gamma$ or to reach an assigned configuration by a finite sequence of moves. If only $on$ vertex can be chosen in each move, we come to the variation: lit-only $\sigma$-game. The goal of the lit-only $\sigma$-game is the same as that of the $\sigma$-game. Given an integer $k$, we say that $\Gamma$ is $k$-lit if, for any configuration, the number of $on$ vertices can be reduced to at most $k$ by a finite sequence of moves of the lit-only $\sigma$-game on $\Gamma$. Motivated by the goal of the lit-only $\sigma$-game, we are interested in the smallest integer $k$, the minimum light number of $\Gamma$ [10], for which $\Gamma$ is $k$-lit.

As far as we know, the notion of the lit-only $\sigma$-game first implicitly occurred in the classification of the equivalence classes of Vogan diagrams, which implies that all simply-laced Dynkin diagrams, the trees shown as below, are 1-lit (cf. [1, 2]). Extending this result, Wang and Wu proved that any tree with $k$ leaves is $\lceil k/2 \rceil$-lit (cf. [10, Theorem 3]). Their recent result gave an insight into the difference between the $\sigma$-game and the lit-only $\sigma$-game on trees with zero or more loops (cf. [11, Theorem 14]). As a consequence, the trees with perfect matchings are 2-lit. The first main result of this paper improves this consequence.
The lit-only $\sigma$-game on a finite simple graph $\Gamma$ can be regarded as a representation of the simply-laced Coxeter group associated with $\Gamma$ (cf. [7]). From this viewpoint, we apply some results from [8] to show that the trees with perfect matchings, except the paths of even order, are 1-lit. Combining this with the result that all paths, namely the trees in class I, are 1-lit, our first result can be simply stated as follows.

**Theorem 1.1.** Any tree with a perfect matching is 1-lit.

Theorem 1.1 gives a large family of 1-lit trees containing the first and third trees in class III. It is natural to ask if there is also a large family of 1-lit trees containing the second tree in class III. This question motivates the discovery of our second result. Assume that $\Gamma$ is a tree with a perfect matching $\mathcal{P}$. An *alternating path* in $\Gamma$ (with respect to $\mathcal{P}$) is a path in which the edges belong alternatively to $\mathcal{P}$ and not to $\mathcal{P}$. For each vertex $s$ of $\Gamma$, we define $a_s$ to be the number of the alternating paths starting from the edge in $\mathcal{P}$ incident to $s$ and ending on some edge in $\mathcal{P}$. An edge of $\Gamma$ is said to be of *odd* (resp. *even*) type if its two endpoints $s, t$ satisfy that $a_s + a_t$ is odd (resp. even). We make use of algebraic and linear algebraic techniques to show that

**Theorem 1.2.** Assume that $\Gamma$ is a tree with a perfect matching. Then the tree obtained from $\Gamma$ by inserting a vertex on an edge of odd (resp. even) type is 1-lit (resp. 2-lit).

Let $\Gamma$ denote the first tree in class III. The edge of $\Gamma$ joining 5 and 6 is of odd type because of $a_5 = 1$ and $a_6 = 2$. Therefore the second tree in class III does be a special case of Theorem 1.2. On the other hand, if we add a vertex on the edge of $\Gamma$ between 1 and 2, the resulting tree is not 1-lit by [3, Proposition 3.2]. Therefore, in general, for any tree $\Gamma$ with a perfect matching, the tree obtained from $\Gamma$ by adding a vertex on an edge is not 1-lit.

The statements of Theorem 1.1 and Theorem 1.2 are combinatorial. It is reasonable to believe that these results can be proved by combinatorial arguments. In addition, motivated by Theorem 1.2, we would like to ask if given a tree $\Gamma$ with a perfect matching, any subdivision of $\Gamma$ is 2-lit. We leave these as open problems.

## 2 Preliminaries

For the rest of this paper, let $\Gamma = (S, R)$ denote a finite simple graph with vertex set $S$ and edge set $R$. The edge set $R$ is a set of some 2-element subsets of $S$. For any distinct $s, t \in S$, we write $st$ or $ts$ to denote the 2-element subset $\{s, t\}$ of $S$. Let $\mathbb{F}_2$ denote the two-element
Let $V$ denote a $\mathbb{F}_2$-vector space that has a basis $\{\alpha_s \mid s \in S\}$ in one-to-one correspondence with $S$. Let $V^*$ denote the dual space of $V$. For each $s \in S$, define $f_s \in V^*$ by

$$f_s(\alpha_t) = \begin{cases} 1 & \text{if } s = t, \\ 0 & \text{if } s \neq t \end{cases}$$

(1)

for all $t \in S$. The set $\{f_s \mid s \in S\}$ is a basis of $V^*$ and called the basis of $V^*$ dual to $\{\alpha_s \mid s \in S\}$. Each configuration $f$ of the lit-only $\sigma$-game on $\Gamma$ is interpreted as the vector

$$\sum_{s \in S} f_s \in V^*,$$

(2)

if all vertices of $\Gamma$ are assigned the off state by $f$, we interpret (2) as the zero vector of $V^*$. For any $s \in S$ and $f \in V^*$, $f(\alpha_s) = 1$ (resp. 0) means that the vertex $s$ is assigned the on (resp. off) state by $f$. For each $s \in S$ define a linear transformation $\kappa_s : V^* \to V^*$ by

$$\kappa_s f = f + f(\alpha_s) \sum_{st \in R} f_t$$

for all $f \in V^*$. (3)

Fix a vertex $s$ of $\Gamma$. Given any $f \in V^*$, if the state of $s$ is on then $\kappa_s f$ is obtained from $f$ by changing the states of all neighbors of $s$; if the state of $s$ is off then $\kappa_s f = f$. Therefore we may view $\kappa_s$ as the move of the lit-only $\sigma$-game on $\Gamma$ for which we choose the vertex $s$ and change the states of all neighbors of $s$ if the state of $s$ is on. In particular $\kappa_s^2 = 1$, the identity map on $V^*$, and so $\kappa_s \in \text{GL}(V^*)$, the general linear group of $V^*$.

The simply-laced Coxeter group $W$ associated with $\Gamma = (S, R)$ is a group generated by the set $S$ subject to the following relations:

$$s^2 = 1,$$

(4)

$$(st)^2 = 1 \quad \text{if } st \not\in R,$$  

(5)

$$(st)^3 = 1 \quad \text{if } st \in R$$

(6)

for all $s, t \in S$. By [7, Theorem 3.2], there is a unique representation $\kappa : W \to \text{GL}(V^*)$ such that $\kappa(s) = \kappa_s$ for all $s \in S$. For any $f, g \in V^*$, observe that $g$ can be obtained from $f$ by a finite sequence of moves of the lit-only $\sigma$-game on $\Gamma$ if and only if there exists $w \in W$ such that $g = \kappa(w)f$. In view of this we define an action of $W$ on $V^*$ by

$$wf = \kappa(w)f \quad \text{for all } w \in W \text{ and } f \in V^*.$$  

In terms of our terminology, given an integer $k$, the simple graph $\Gamma$ is $k$-lit if and only if for any $W$-orbit $O$ of $V^*$, there exists a subset $K$ of $S$ with cardinality at most $k$ such that $\sum_{s \in K} f_s \in O$.

Let $B : V \times V \to \mathbb{F}_2$ denote the symplectic form defined by

$$B(\alpha_s, \alpha_t) = \begin{cases} 1 & \text{if } st \in R, \\ 0 & \text{else} \end{cases}$$

(7)
for all \( s, t \in S \). By (1) and (7), for all \( s \in S \) and \( \alpha \in V \) we have
\[
B(\alpha_s, \alpha) = \sum_{st \in R} f_t(\alpha).
\] (8)

The \textit{radical} of \( V \) (relative to \( B \)), denoted by \( \text{rad} V \), is the subspace of \( V \) consisting of the \( \alpha \in V \) that satisfy \( B(\alpha, \beta) = 0 \) for all \( \beta \in V \). The form \( B \) is said to be \textit{degenerate} if \( \text{rad} V \neq \{0\} \) and \textit{nondegenerate} otherwise. The graph \( \Gamma \) is said to be \textit{degenerate} (resp. \textit{nondegenerate}) if \( B \) is degenerate (resp. nondegenerate). The form \( B \) induces a linear map \( \theta : V \to V^* \) given by
\[
\theta(\alpha)\beta = B(\alpha, \beta) \quad \text{for all} \quad \alpha, \beta \in V.
\] (9)

By (8) and (9), for each \( s \in S \) we have
\[
\theta(\alpha_s) = \sum_{st \in R} f_t.
\] (10)

Let \( A \) denote the adjacency matrix of \( \Gamma \) over \( \mathbb{F}_2 \). Observe that the kernel of \( \theta \) is \( \text{rad} V \) and the matrix representing \( B \) with respect to \( \{\alpha_s \mid s \in S\} \) is exactly \( A \). Therefore we have

\textbf{Lemma 2.1.} The following statements are equivalent:

(i) \( \Gamma \) is a nondegenerate graph.

(ii) \( \theta \) is an isomorphism of vector spaces.

(iii) \( A \) is invertible.

The determinant of \( A \) is 0 (resp. 1) if and only if the number of perfect matchings in \( \Gamma \) is even (resp. odd) (see [4, Section 2.1] for example). Combining this with Lemma 2.1 we have

\textbf{Proposition 2.2.} The following statements are equivalent:

(i) \( \Gamma \) is a nondegenerate graph.

(ii) The number of perfect matchings in \( \Gamma \) is odd.

Since a tree contains at most one perfect matching and by Proposition 2.2, we have

\textbf{Corollary 2.3.} The following statements are equivalent:

(i) \( \Gamma \) is a nondegenerate tree.

(ii) \( \Gamma \) is a tree with a perfect matching.

\textbf{Proposition 2.4.} ([6, Lemma 2.4]). Assume that \( \Gamma \) is a tree of order at least four and with a perfect matching. Then there exist two vertices of \( \Gamma \) with degree two.
3 Proof of Theorem 1.1

The lit-only $\sigma$-game is closely related to another combinatorial game. We call this game the Reeder’s game because as far as we know, this game first appeared in one of Reeder’s papers [8]. The Reeder’s game is a one-player game played on a finite simple graph $\Gamma$. A configuration of the Reeder’s game on $\Gamma$ is an assignment of one of two states, on or off, to each vertex of $\Gamma$. Given a configuration, a move of the Reeder’s game on $\Gamma$ consists of choosing a vertex $s$ and changing the state of $s$ if the number of on neighbors of $s$ is odd. Given an initial configuration, the goal is to minimize the number of on vertices of $\Gamma$ by a finite sequence of moves of the Reeder’s game on $\Gamma$.

We interpret each configuration $\alpha$ of the Reeder’s game on $\Gamma$ as the vector

$$\sum_{\text{on vertices } s} \alpha_s \in V, \tag{11}$$

if all vertices of $\Gamma$ are assigned the off state by $\alpha$, we interpret (11) as the zero vector of $V$. For any $\alpha \in V$, observe that $f_s(\alpha) = 1$ (resp. 0) means that the vertex $s$ is assigned the on (resp. off) state by $\alpha$. For each $s \in S$ define a linear transformation $\tau_s : V \to V$ by

$$\tau_s \alpha = \alpha + B(\alpha_s, \alpha) \alpha_s \quad \text{for all } \alpha \in V. \tag{12}$$

Fix a vertex $s$ of $\Gamma$. By (8), for any $\alpha \in V$, if the number of on neighbors of $s$ is odd then $\tau_s \alpha$ is obtained from $\alpha$ by changing the state of $s$; if the number of on neighbors of $s$ is even then $\tau_s \alpha = \alpha$. Therefore we may view $\tau_s$ as the move of the Reeder’s game on $\Gamma$ for which we choose the vertex $s$ and change the state of $s$ if the number of on neighbors of $s$ is odd. In particular $\tau_s^2 = 1$, the identity map on $V$, and so $\tau_s \in \text{GL}(V)$, the general linear group of $V$.

By [8, Section 5], there exists a unique representation $\tau : W \to \text{GL}(V)$ such that $\tau(s) = \tau_s$ for all $s \in S$. For any $\alpha, \beta \in V$, observe that $\beta$ can be obtained from $\alpha$ by a finite sequence of moves of the Reeder’s game on $\Gamma$ if and only if there exists $w \in W$ such that $\beta = \tau(w)\alpha$. In view of this we define an action $W$ on $V$ by

$$w\alpha = \tau(w)\alpha \quad \text{for all } w \in W \text{ and } \alpha \in V.$$ 

A quadratic form $Q : V \to \mathbb{F}_2$, given in [8, Section 1], is defined by

$$Q(\alpha_s) = 1 \quad \text{for all } s \in S, \tag{13}$$

$$Q(\alpha + \beta) = Q(\alpha) + Q(\beta) + B(\alpha, \beta) \quad \text{for all } \alpha, \beta \in V. \tag{14}$$

Observe that $\tau$ preserves $Q$, namely

$$Q(\tau(w)\alpha) = Q(\alpha) \quad \text{for all } w \in W \text{ and } \alpha \in V. \tag{15}$$

The kernel of $Q$, denoted by $\text{Ker} Q$, is the subspace of $\text{rad} V$ consisting of all $\alpha \in \text{rad} V$ that satisfy $Q(\alpha) = 0$. The orthogonal group $O(V)$ (relative to $Q$) is the subgroup of $\text{GL}(V)$ consisting of the $\sigma \in \text{GL}(V)$ such that $Q(\sigma \alpha) = Q(\alpha)$ for all $\alpha \in V$. 

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Lemma 3.1. ([8, Section 2; Theorem 7.3]). Let \( \Gamma \) denote a tree which is not a path. Assume that \( \ker Q \) is equal to \( \{0\} \). Then \( \tau(W) = O(V) \). Moreover the \( W \)-orbits on \( V \) are

\[
Q^{-1}(1) \setminus \text{rad } V, \quad Q^{-1}(0) \setminus \{0\}, \quad \{\alpha\} \quad \text{for all } \alpha \in \text{rad } V.
\]

As a corollary of Lemma 3.1 we have

Corollary 3.2. Assume that \( \Gamma \) is a nondegenerate tree which is not a path. Then the \( W \)-orbits on \( V \) are

\[
Q^{-1}(1), \quad Q^{-1}(0) \setminus \{0\}, \quad \{0\}.
\]

Recall that the transpose of a linear transformation \( \sigma : V \to V \) is the linear transformation \( t \sigma : V^* \to V^* \) defined by \( (t \sigma f)(\alpha) = f(\sigma \alpha) \) for all \( f \in V^* \) and \( \alpha \in V \).

Lemma 3.3. The representation \( \kappa \) is the dual representation of \( \tau \).

Proof. Let \( s \in S \) be given. Using (3), (8) and (12), we find that \( (\kappa_s f)(\alpha) = (t \tau_s f)(\alpha) \) for all \( f \in V^* \) and \( \alpha \in V \). Therefore \( \kappa_s = t \tau_s \). Since the elements \( s \in S \) generate \( W \) and \( s^{-1} = s \) in \( W \), we have \( \kappa(w) = t \tau(w^{-1}) \) for all \( w \in W \). The result follows.

Lemma 3.4. For all \( w \in W \) and \( \alpha, \beta \in V \) we have

\[
B(\tau(w)\alpha, \tau(w)\beta) = B(\alpha, \beta).
\]

Proof. Fix \( s \in S \). Pick any \( \alpha, \beta \in V \). Using (7), (12) to simplify \( B(\tau_s \alpha, \tau_s \beta) \) we obtain that \( B(\tau_s \alpha, \tau_s \beta) = B(\alpha, \beta) \). The result follows since the elements \( s \in S \) generate \( W \).

We have seen that \( \kappa \) is the dual representation of \( \tau \) and that \( \tau \) preserves the form \( B \). By the principles of representation theory, the following lemma is straightforward. For the convenience of the reader we include the proof.

Lemma 3.5. \( \kappa(w) \circ \theta = \theta \circ \tau(w) \) for all \( w \in W \).

Proof. Let \( w \in W \) be given. Replacing \( \beta \) by \( \tau(w^{-1})\beta \) in Lemma 3.4, we obtain

\[
B(\tau(w)\alpha, \beta) = B(\alpha, \tau(w^{-1})\beta) \quad \text{for all } \alpha, \beta \in V. \tag{16}
\]

Using (9) we can rewrite (16) as

\[
(\theta \circ \tau(w))(\alpha) = (t \tau(w^{-1}) \circ \theta)(\alpha) \quad \text{for all } \alpha \in V. \tag{17}
\]

By Lemma 3.3 the right-hand side of (17) is equal to \( (\kappa(w) \circ \theta)(\alpha) \). The result follows.

As a consequence of Lemma 3.5 we have

Corollary 3.6. Assume that \( \theta \) is an isomorphism of vector spaces. Then the representation \( \tau \) is equivalent to the representation \( \kappa \) via \( \theta \). Moreover the map from the \( W \)-orbits of \( V \) to the \( W \)-orbits of \( V^* \) defined by

\[
O \mapsto \theta(O) \quad \text{for all } W \text{-orbits } O \text{ of } V
\]

is a bijection.
Combining Lemma 2.1, Corollary 3.2 and Corollary 3.6, we have

**Corollary 3.7.** Assume that $\Gamma$ is a nondegenerate tree which is not a path. Then the $W$-orbits of $V^*$ are

\[ \theta(Q^{-1}(1)), \quad \theta(Q^{-1}(0)) \setminus \{0\}, \quad \{0\}. \]

Our last tool for proving Theorem 1.1 is [5, Theorem 6]. Here we offer a short proof of this result.

**Lemma 3.8.** ([5, Theorem 6]). Assume that $\Gamma = (S, R)$ is a nondegenerate graph. Let $s \in S$ and let $f \in V^*$ with $f(\alpha_s) = 0$. Then $f$ and $f + \sum_{st \in R} f_t$ are in distinct $W$-orbits of $V^*$.

**Proof.** Suppose on the contrary that there exists $w \in W$ such that $\kappa(w)f = f + \sum_{st \in R} f_t$. (18)

Since $\theta$ is a bijection by Lemma 2.1, there exists a unique $\alpha \in V$ such that $\theta(\alpha) = f$. By (10), we can rewrite (18) as $\kappa(w)(\theta(\alpha)) = \theta(\alpha + \alpha_s)$. By Lemma 3.5 and since $\theta$ is a bijection, we obtain

\[ \tau(w)\alpha = \alpha + \alpha_s. \] (19)

We now apply $Q$ to either side of (19). By (15), the left-hand side is equal to $Q(\alpha)$. By (9) and the assumption on $f$, we have $B(\alpha, \alpha_s) = 0$. By this and using (13) and (14), we find that the right-hand side is equal to $Q(\alpha) + 1$, a contradiction. □

It is now a simple matter to prove Theorem 1.1.

**Proof of Theorem 1.1:** Let $\Gamma$ be a tree with a perfect matching. Recall from Section 1 that all paths are 1-lit. Thus it is enough to treat the case that $\Gamma$ is not a path. Such a $\Gamma$ has order at least four. By Proposition 2.4 there exists a vertex $s$ of $\Gamma$ with degree two. Let $u, v$ denote the neighbors of $s$. By Corollary 2.3 the tree $\Gamma$ is nondegenerate. Applying Lemma 3.8 to $f = f_u$, we obtain $f_u$ and $f_v$ in distinct $W$-orbits of $V^*$. Since there are exactly two nonzero $W$-orbits of $V^*$ by Corollary 3.7, this implies that $\Gamma$ is 1-lit. □

The following example gives a nondegenerate graph which is not 1-lit. Let $\Gamma = (S, R)$ be the graph shown as follows.

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1 2 3 4
5 6 7 8
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The graph $\Gamma$ contains the only perfect matching $\{\{1, 2\}, \{3, 4\}, \{5, 6\}, \{7, 8\}\}$. By Proposition 2.2 the graph $\Gamma$ is nondegenerate. Let $f = f_2 + f_3 + f_6 + f_7$. Let $O$ denote the $W$-orbit of $f$. To see that $\Gamma$ is not 1-lit, we show that $f_s \notin O$ for all $s = 1, 2, \ldots, 8$. Let $\alpha = \alpha_1 + \alpha_4 + \alpha_5 + \alpha_8$, $\alpha_1 = \alpha_2 + \alpha_4 + \alpha_5$ and $\alpha_2 = \alpha_1$. Using (10), we find that $\theta(\alpha) = f$, $\theta(\alpha_1) = f_1$ and $\theta(\alpha_2) = f_2$. Using (13) and (14), we find that $Q(\alpha) = 0$, $Q(\alpha_1) = 1$ and $Q(\alpha_2) = 1$. By (15), neither $\alpha_1$ nor $\alpha_2$ is in the $W$-orbit of $\alpha$. Therefore $f_1 \notin O$ and $f_2 \notin O$ by Corollary 3.6. By symmetry $f_s \notin O$ for $s = 3, 4, \ldots, 8$.
4 Proof of Theorem 1.2

In this section, assume that $\Gamma = (S, R)$ contains at least one edge and fix $x, y \in S$ with $xy \in R$. Define $\hat{\Gamma} = (\hat{S}, \hat{R})$ to be the simple graph obtained from $\Gamma$ by inserting a new vertex $z$ on the edge $xy$. In other words, $z$ is an element not in $S$ and the sets $\hat{S}$ and $\hat{R}$ are $S \cup \{z\}$ and $R \cup \{xz, yz\} \setminus \{xy\}$, respectively. Let $\hat{W}$ denote the simply-laced Coxeter group associated with $\hat{\Gamma}$, namely $\hat{W}$ is the group generated by all elements of $\hat{S}$ subject to the following relations:

\begin{align*}
  s^2 &= 1, \quad (20) \\
  (st)^2 &= 1 \quad \text{if } st \notin \hat{R}, \quad (21) \\
  (st)^3 &= 1 \quad \text{if } st \in \hat{R} \quad (22)
\end{align*}

for all $s, t \in \hat{S}$.

**Lemma 4.1.** For each $u \in \{x, y\}$ there exists a unique homomorphism $\rho_u : W \to \hat{W}$ such that $\rho_u(u) = zuz$ and $\rho_u(s) = s$ for all $s \in S \setminus \{u\}$.

**Proof.** Without loss of generality we assume $u = x$. We first show the existence of $\rho_x$. By (4)–(6) it suffices to verify that for all $s, t \in S \setminus \{x\}$,

\begin{align*}
  s^2 &= 1, \quad (23) \\
  (st)^2 &= 1 \quad \text{if } st \notin R, \quad (24) \\
  (st)^3 &= 1 \quad \text{if } st \in R, \quad (25) \\
  (zxz)^2 &= 1, \quad (26) \\
  (szxz)^2 &= 1 \quad \text{if } sx \notin R, \quad (27) \\
  (szxz)^3 &= 1 \quad \text{if } sx \in R \quad (28)
\end{align*}

hold in $\hat{W}$. It is clear that (23)–(25) are immediate from (20)–(22), respectively. To obtain (26), evaluate the left-hand side of (26) using (20). By (21) and (22), for any $s \in S \setminus \{x, y\}$ we have

\begin{align*}
  (sz)^2 &= 1, \quad (29) \\
  (sx)^2 &= 1 \quad \text{if } sx \notin R, \quad (30) \\
  (sx)^3 &= 1 \quad \text{if } sx \in R \quad (31)
\end{align*}

and

\begin{align*}
  (yx)^2 &= 1, \quad (32) \\
  (xz)^3 &= 1, \quad (33) \\
  (yz)^3 &= 1 \quad (34)
\end{align*}

in $\hat{W}$. In what follows, the relation (20) will henceforth be used tacitly in order to keep the argument concise. Concerning (27), let $s \in S \setminus \{x\}$ with $sx \notin R$ be given. By (29), (30) the
element $s$ commutes with $z$ and $x$ in $\hat{W}$, respectively. Therefore the left-hand side of (27) is equal to $(zxx)^3$. Now, by (26) we have (27) in $\hat{W}$. To verify (28) we divide the argument into the two cases: (A) $s \in S \setminus \{x, y\}$ and $sx \in R$; (B) $s = y$ in $S$.

(A) By (29) and (31), we have $zsx = s$ and $xsex = s$ in $\hat{W}$, respectively. In the left-hand side of (28), replace $zsx$ with $s$ twice and then replace $xsex$ with $s$. This yields $(szxx)^3 = (sz)^2$ in $\hat{W}$. By (29) the relation (28) holds.

(B) In this case we need to show that

$$(yzxz)^3 = 1$$

in $\hat{W}$. By (33) we have $zxz = xxz$ in $\hat{W}$. Use this to rewrite (35) as

$$(ygzx)^3 = 1.$$  

(36)

By (32) and (34) we have $xyx = y$ and $zyzyz = y$ in $\hat{W}$, respectively. In the left-hand side of (36), replace $xyx$ with $y$ twice and then replace $zyzyz$ with $y$. This yields $(yzxz)^3 = (yx)^2$ in $\hat{W}$. Now, by (32) we have (36) in $\hat{W}$. Therefore (28) holds.

We have shown the existence of $\rho_x$. Such a homomorphism $\rho_x$ is clearly unique since the elements $s \in S$ generate $W$. \hfill \Box

For the rest of this section, let $\rho_x$ and $\rho_y$ be as in Lemma 4.1. Let $\hat{V}$ denote a $\mathbb{F}_2$-vector space that has a basis $\{\alpha_s \mid s \in \hat{S}\}$ in one-to-one correspondence with $\hat{S}$. Let $\hat{V}^*$ denote the dual space of $\hat{V}$ and let $\{h_s \mid s \in \hat{S}\}$ denote the basis of $\hat{V}^*$ dual to $\{\alpha_s \mid s \in \hat{S}\}$. For each $s \in \hat{S}$ define a linear transformation $\hat{\kappa}_s : \hat{V}^* \to \hat{V}^*$ by

$$\hat{\kappa}_s h = h + h(\alpha_s) \sum_{st \in R} h_t \quad \text{for all } h \in \hat{V}^*. $$

(37)

Let $GL(\hat{V}^\ast)$ denote the general linear group of $\hat{V}^\ast$. Let $\hat{\kappa}$ denote the representation from $\hat{W}$ into $GL(\hat{V}^\ast)$ such that $\hat{\kappa}(s) = \hat{\kappa}_s$ for all $s \in \hat{S}$. Define an action of $\hat{W}$ on $\hat{V}^\ast$ by $wh = \hat{\kappa}(w)h$ for all $w \in \hat{W}$ and $h \in \hat{V}^\ast$. For each $u \in \{x, y\}$, we define a linear transformation $\delta_u : \hat{V}^\ast \to V^\ast$ by

$$\delta_u(h_z) = f_u, \quad \delta_u(h_s) = f_s \quad \text{for all } s \in S. $$

(38)

For each $u \in \{x, y\}$ the linear transformation $\delta_u$ is clearly onto and the kernel of $\delta_u$ is

$$\text{Ker} \delta_u = \{0, h_w + h_z\}. $$

(39)

Using (38), it is routine to verify that for each $u \in \{x, y\}$ and $s \in S$,

$$\sum_{st \in R} \delta_u(h_t) = \begin{cases} f_x + f_y + \sum_{ut \in R} f_t & \text{if } s = u, \\ \sum_{st \in R} f_t & \text{if } s \neq u. \end{cases} $$

(40)

**Lemma 4.2.** Assume that $O$ is a $\hat{W}$-orbit of $\hat{V}^\ast$ with $O \neq \{0\}$. Then $\delta_u(O) \neq \{0\}$ for all $u \in \{x, y\}$.
Proof. Without loss of generality we show that $\delta_x(O) \neq \{0\}$. Suppose on the contrary that $\delta_x(O) = \{0\}$. Since $O \neq \{0\}$ and by (39), this forces that $O = \{h_x + h_z\}$. However $\kappa_z(h_x + h_z) = h_y + h_z \in O$, a contradiction. \qed

Lemma 4.3. For all $u \in \{x, y\}$ and $w \in W$, we have

$$\kappa(w) \circ \delta_u = \delta_u \circ \kappa(\rho_u(w)).$$

Proof. Let $u \in \{x, y\}$ be given. By Lemma 4.1 and since the elements $s \in S$ generate $W$, it suffices to show that

$$\kappa_u \circ \delta_u = \delta_u \circ \kappa_u \circ \kappa_z \circ \kappa_u \circ \kappa_z,$$

$$\kappa_s \circ \delta_u = \delta_u \circ \kappa_s \quad \text{for all} \ s \in S \setminus \{u\}. \quad (41) \quad \text{(42)}$$

To verify (41), we show that

$$(\kappa_u \circ \delta_u)(h_s) = (\delta_u \circ \kappa_z \circ \kappa_u \circ \kappa_z)(h_s) \quad \text{for all} \ s \in \hat{S}. \quad (43)$$

The argument is divided into the two cases: (A) $s \in \{u, z\}$; (B) $s \in \hat{S} \setminus \{u, z\}$.

(A) Using (37) we find that $(\kappa_z \circ \kappa_u \circ \kappa_z)(h_s)$ is equal to

$$h_s + h_x + h_y + \sum_{u \in \hat{R}} h_t.$$  

By this and using (38) and (40), the right-hand side of (43) is equal to

$$f_u + \sum_{u \in \hat{R}} f_t. \quad (44)$$

Using (3) and (38), the left-hand side of (43) is equal to (44). Therefore (43) holds.

(B) By (3), (37) and (38), we have $\kappa_u(f_s) = f_s$, $\kappa'_u(h_s) = \kappa'_z(h_s) = h_s$ and $\delta_u(h_s) = f_s$, respectively. Using these we find either side of (43) is equal to $f_s$. We have shown (41).

To verify (42), we fix $s \in S \setminus \{u\}$ and show that

$$(\kappa_s \circ \delta_u)(h_t) = (\delta_u \circ \kappa_s)(h_t) \quad \text{for all} \ t \in \hat{S}. \quad (45)$$

The argument is divided into the two cases: (C) $t \in \{u, z\}$; (D) $t \in \hat{S} \setminus \{u, z\}$.

(C) By (3), (37) and (38), we have $\kappa_s(f_u) = f_u$, $\kappa'_s(h_t) = h_t$ and $\delta_u(h_t) = f_u$, respectively. Using these we find either side of (45) is equal to $f_u$. Therefore (45) holds.

(D) Using (37) and (40), the right-hand side of (45) is equal to

$$f_t + h_t(\alpha_s) \sum_{sv \in R} f_v. \quad (46)$$

Using (3) and (38), the left-hand side of (45) is equal to

$$f_t + f_t(\alpha_s) \sum_{sv \in R} f_v. \quad (47)$$

Clearly (46) and (47) are equal since $f_t(\alpha_s) = h_t(\alpha_s)$. We have shown (42). The result follows. \qed
From now on, assume that $\Gamma = (S, R)$ is a tree with a perfect matching $\mathcal{P}$. For each $s \in S$, define $A_s$ to be the subset of $S$ consisting of all elements $t \in S \setminus \{s\}$ for which the path in $\Gamma$ joining $s$ and $t$ is an alternating path which starts from and ends on edges in $\mathcal{P}$. Clearly, for each $s \in S$ the number $a_s$ is equal to the size of $A_s$. For each $s \in S$ we define

$$\alpha_s^\vee = \sum_{t \in A_s} \alpha_t. \quad (48)$$

Using (13), (14) and (48), the following lemma is straightforward.

**Lemma 4.4.** For each $s \in S$ we have $Q(\alpha_s^\vee) \equiv a_s \pmod{2}$.

**Lemma 4.5.** For each $s \in S$ we have $\theta(\alpha_s^\vee) = f_s$.

*Proof.* Suppose on the contrary that the set $\{A_s \mid s \in S \text{ and } \theta(\alpha_s^\vee) \neq f_s\}$ is nonempty. From this set, we choose a minimal element $A_s$ under inclusion. Let $t \in S$ with $st \in \mathcal{P}$. Observe that $A_s$ is equal to the disjoint union of $\{t\}$ and the sets $A_u$ for all $u \in S \setminus \{s\}$ with $ut \in R$. By this observation and (48), we deduce that

$$\alpha_t = \sum_{u \in R} \alpha_u^\vee. \quad (49)$$

By the choice of $A_s$, for each $u \in S \setminus \{s\}$ with $ut \in R$ we have $\theta(\alpha_u^\vee) = f_u$. Apply $\theta$ to either side of (49) and then apply (10) to the left-hand side. Simplifying the resulting equation, we obtain that $\theta(\alpha_s^\vee) = f_s$, a contradiction. \[\Box\]

We are now ready to prove Theorem 1.2.

**Proof of Theorem 1.2:** By Corollary 2.3 the tree $\Gamma$ is nondegenerate. Since all paths are 1-lit, we may assume that $\Gamma$ is not a path. Let $O$ denote any nonzero $\hat{W}$-orbit of $\hat{V}^*$. By Lemma 4.5 we have $\theta(\alpha_s^\vee) = f_x$ and $\theta(\alpha_y^\vee) = f_y$. We first suppose that the edge $xy$ is of odd type. To see that $\hat{V}$ is 1-lit, it suffices to show that there exists $s \in \hat{S}$ such that $h_s \in O$. By Lemma 4.2, there exists $h \in O$ such that $\delta_x(h) \neq 0$. By Lemma 4.4 one of $Q(\alpha_x^\vee)$ and $Q(\alpha_y^\vee)$ is 1 and the other is 0. By Corollary 3.7 there exists $w \in W$ such that $\kappa(w)\delta_x(h)$ is equal to $f_x$ or $f_y$. By Lemma 4.3 we have $\delta_x(\hat{\kappa}(\rho_x(w))h)$ is equal to $f_x$ or $f_y$. Using (39), we deduce that one of $h_x, h_y, h_z, h_x + h_y + h_z$ is in $O$. By this and since $\hat{\kappa}(h_x + h_y + h_z) = h_z$, one of $h_x, h_y, h_z$ is in $O$, as desired.

We now suppose that $xy$ is of even type. By Lemma 4.4 we have $Q(\alpha_x^\vee) = Q(\alpha_y^\vee)$. By Corollary 3.7, the two vectors $f_x$ and $f_y$ are in the same nonzero $W$-orbit of $V^*$ and there exists $u \in S \setminus \{x, y\}$ such that $f_u$ is in the other nonzero $W$-orbit of $V^*$ by Theorem 1.1. Without loss of generality, we assume that $u$ and $x$ lie in the same component of the graph $(S, R \setminus \{xy\})$. By Lemma 4.2, there exists $h \in O$ such that $\delta_x(h) \neq 0$. By the above comments, there exists $w \in W$ such that $\kappa(w)\delta_x(h)$ is equal to $f_u$ or $f_x$. By Lemma 4.3, we have $\delta_x(\hat{\kappa}(\rho_x(w))h)$ is equal to $f_u$ or $f_x$. Using (39), we find that $\hat{\kappa}(\rho_x(w))h$ is equal to one of $h_u, h_x, h_z, h_u + h_x + h_z$. In particular $(\hat{\kappa}(\rho_x(w))h)(\alpha_s) = 0$ for all $s \in \hat{S} \setminus \{u, x, z\}$. If $xy \in \mathcal{P}$ (resp. $xy \notin \mathcal{P}$), we let $\hat{\Gamma}_x$ denote the component of the graph $(\hat{S}, \hat{R} \setminus \{yz\})$ (resp. $(\hat{S}, \hat{R} \setminus \{xz\})$) containing $x$. Clearly $\hat{\Gamma}_x$ is a tree with a perfect matching and contains $u$. \[11\]
Applying Theorem 1.1 to $\hat{\Gamma}_x$, there exists a finite sequence of moves for which we only choose the vertices of $\hat{\Gamma}_x$ such that $\hat{\kappa}(\rho_x(w))h$ is transferred to $h'$, where $h'(\alpha_s) = 0$ for all $s \in \hat{S}$ except some vertex in $\hat{\Gamma}_x$ and the vertex of $\hat{\Gamma}$ that is adjacent to $\hat{\Gamma}_x$ and not in $\hat{\Gamma}_x$. Therefore $\hat{\Gamma}$ is 2-lit.

\[\square\]

References

[1] A. Borel, J. de Siebenthal. Les sous-groupes fermés de rang maximum des groupes de Lie clos. Commentarii Mathematici Helvetici 23 (1949) 200–221.

[2] M. Chuah, C. Hu. Equivalence classes of Vogan diagrams. Journal of Algebra 279 (2004) 22–37.

[3] M. Chuah, C. Hu. Extended Vogan diagrams. Journal of Algebra 301 (2006) 112–147.

[4] C. Godsil. Algebraic Combinatorics. Chapman and Hall, New York, 1993.

[5] J. Goldwasser, X. Wang, Y. Wu. Does the lit-only restriction make any difference for the $\sigma$-game and $\sigma^+$-game? European Journal of Combinatorics 30 (2009) 774–787.

[6] Y. Hou, J. Li. Bounds on the largest eigenvalues of trees with a given size of matching. Linear Algebra and its Applications 342 (2002) 203–217.

[7] H. Huang, C. Weng. Combinatorial representations of Coxeter groups over a field of two elements. arXiv:0804.2150v2.

[8] M. Reeder. Level-two structure of simply-laced Coxeter groups. Journal of Algebra 285 (2005) 29–57.

[9] K. Sutner. Linear cellular automata and the Garden-of-Eden. Intelligencer 11 (1989) 40–53.

[10] X. Wang, Y. Wu. Minimum light number of lit-only $\sigma$-game on a tree. Theoretical Computer Science 381 (2007) 292–300.

[11] X. Wang, Y. Wu. Lit-only sigma-game on pseudo-trees. Discrete Applied Mathematics 158 (2010) 1945–1952.

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