Topological Hochschild homology of $X(n)$

Jonathan Beardsley

July 7, 2015

Abstract

We show that Ravenel’s spectrum $X(2)$ is the versal $E_1$-$S$-algebra of characteristic $\eta$. This implies that every $E_1$-$S$-algebra $R$ of characteristic $\eta$ admits an $E_1$-ring map $X(2) \to R$, i.e. an $A_\infty$ complex orientation of degree 2. This implies that $R^*(CP^2) \cong R_*[x]/x^3$. Additionally, if $R$ is an $E_2$-ring Thom spectrum admitting a map (of homotopy ring spectra) from $X(2)$, e.g. $X(n)$, its topological Hochschild homology has a simple description.

Theorem 1. The spectrum $X(n)$, which is the Thom spectrum of the inclusion $\Omega SU(n) \to \Omega SU \simeq BU \to BGL_1(S)$, is of characteristic $\eta$ in the sense of [Szy14] and [ACB14]. In particular $X(2)$ is the versal $E_1$-$S$-algebra of characteristic $\eta$ (as described in Definition 4.3 of [ACB14]).

Proof. We use [ACB14] in a crucial way. Recall that $X(2)$ is the Thom spectrum of the inclusion $i : \Omega S^3 \simeq \Omega SU(2) \to BU \to BGL_1(S)$. Note that this morphism is a two fold loop map, and as such a morphism of $E_2$-algebras in $\text{Top}$. Let $\hat{i}$ be the induced $E_1$-morphism. We have the following equivalences of mapping spaces:

$\text{Map}_{E_1-\text{alg}}(\Omega S^2, BGL_1(S)) \simeq \text{Map}_{T op}(S^2, BGL_1(S)) \simeq \text{Map}_{T op}(S^1, GL_1(S))$.

Since $\pi_1(GL_1(S)) \cong \pi_1(S) \cong \mathbb{Z}/2$ we have that this map is either null homotopic or non-trivial and unique up to homotopy. Since it is not null (i.e. the associated Thom spectrum is not the suspension spectrum of $\Omega S^3$), $i_* : \pi_*(S^1) \to \pi_*(GL_1(S))$ takes $1 \in \pi_1(S^1)$ to $\eta$, the generator of $\pi_1(S) \cong \pi_1(GL_1(S))$. Indeed, the preceding sequence of equivalences implies, by Theorem 4.10 of [ACB14], that $X(2) \simeq S/\eta$, the versal $E_1$-algebra over $S$ of characteristic $\eta$.

Moreover, as $X(n)$ admits a morphism of $E_1$-ring spectra (actually of $E_2$-ring spectra) $X(2) \to X(n)$ for every $n$, we have that the $X(n)$ must be an $E_1$-$S$-algebra of characteristic $\eta$. In particular, the composition $\Sigma S \xrightarrow{\eta} S \to X(n)$ is nullhomotopic.

Remark 1. Of course it’s not necessary to use the machinery of characteristics of structured ring spectra to notice that $\eta$ is trivial in $X(n)_*$, but the identification of $X(2)$ as the versal $E_1$-$S$-algebra of characteristic $\eta$ seems interesting in its own right.
Corollary 1. If $R$ is an $E_1$-ring spectrum of characteristic $\eta$ then $\text{Map}_{E_1}(X(2), R) \simeq \Omega^{\infty+2}R$ and $R^* (\mathbb{CP}^2) \simeq R_4[x]/x^3$.

Proof. The first statement follows immediately from Lemma 4.6 of [ACB14] and the “versality” of $X(2)$. Hence there is at least one $E_1$-morphism from $X(2)$ to $R$. From Proposition 6.5.4 of [Rav86] we obtain the second part of the corollary.

Theorem 2. The topological Hochschild homology of $X(n)$ as an $E_2$-ring spectrum, denoted here by $\text{THH}(X(n))$, is equivalent to $X(n) \wedge SU(n)$. 

Proof. Here we use [BCS10] is a crucial way. In particular, we recall Theorem 2 of that paper which gives $\text{THH}(X(n)) = X(n) \wedge M(\eta \circ Bi)$, where $\eta \circ Bi$ here refers to the morphism $B\Omega SU(n) \simeq SU(n) \xrightarrow{Bi} B^2GL_1(\mathbb{S}) \xrightarrow{\eta} BGL_1(\mathbb{S})$.

Since $X(n)$ is of characteristic $\eta$, we have that the composition $B^2GL_1(\mathbb{S}) \xrightarrow{\eta} BGL_1(\mathbb{S}) \to BGL_1(X(n))$ is nullhomotopic, where $BGL_1(\mathbb{S}) \to BGL_1(X(n))$ is just $BGL_1(-)$ of the unit map of $X(n)$. This implies that $M(\eta \circ Bi)$ is $X(n)$-oriented, so by the associated Thom isomorphism we have $X(n) \wedge M(\eta \circ Bi) \simeq X(n) \wedge SU(n)$. 

Remark 2. By a similar argument, given any Thom spectrum $Mf$, for $f : X \to BGL_1(\mathbb{S})$ a map of double loop spaces, such that the unit map $\mathbb{S} \to Mf$ factors (as maps of $E_2$-rings) $\mathbb{S} \to X(2) \to Mf$, we have that $\text{THH}(Mf) \simeq Mf \wedge \Omega X_+$. 

Conjecture 1. Recall that the morphism of $E_2$-ring spectra $X(n) \to X(n+1)$ is a Hopf-Galois extension with associated spectral Hopf-algebra $\Omega S^{2n+1}$, thought of as the base space of the fibration $\Omega SU(n) \to \Omega SU(n+1) \to \Omega S^{2n+1}$. Then the above results, as well as the results of [BCS10] suggest that one might have relative $\text{THH}$ spectra:

$$\text{THH}_{X(n)}(X(n+1)) \simeq X(n+1) \wedge S^{2n+1}.$$

References

[ACB14] Omar Antolin-Camarena and Tobias Barthel, A simple universal property of Thom ring spectra, 2014, arxiv.org/abs/1411.7988.

[BCS10] Andrew J. Blumberg, Ralph L. Cohen, and Christian Schlichtkrull, Topological Hochschild homology of Thom spectra and the free loop space, Geom. Topol. 14 (2010), no. 2, 1165–1242.

[Rav86] Doug Ravenel, Complex cobordism and the homotopy groups of spheres, Academic Press, 1986.

[Szy14] Markus Szymik, Commutative $\mathbb{S}$-algebras of prime characteristics and applications to unoriented bordism, Algbr. Geom. Topol. 14 (2014), no. 6, 3717–3743.