A limit theorem for moving averages in the $\alpha$–stable domain of attraction

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Abstract

In the early 1990’s, Avram and Taqqu showed that regularly varying moving average processes with all coefficients nonnegative and the tail index $\alpha$ strictly between 0 and 2 satisfy functional limit theorem. They also conjectured that an equivalent statement holds under a certain less restrictive assumption on the coefficients, but in a different topology on the space of càdlàg functions. We give a proof of this result.

Keywords: Functional limit theorem, Regular variation, Stable Lévy process, $M_2$ topology, Moving Average Process

1. Introduction

It is known that the partial sums of i.i.d. regularly varying sequences with the tail index $\alpha \in (0, 2)$ satisfy the functional limit theorem with an $\alpha$–stable Lévy process as a limit. This was first shown by Skorohod in 1950’s using his concept of $J_1$ topology on the space of càdlàg functions. For a nice contemporary presentation of this result we refer to Resnick [5], Chapter 7. Naturally, one wonders whether the same holds for other stationary sequences. It turns out that, by introducing other alternative topologies on the same space, Skorohod gave us the right tools to study this issue. This was first observed by Avram and Taqqu [2] who showed that the functional limit theorem holds for regularly varying moving average processes provided
that they have all coefficients of the same sign. They used Skorohod’s $M_1$ topology to obtain the result and made a further conjecture that a similar theorem holds under a less restrictive assumption on the coefficients of the moving average process, but in somewhat weaker $M_2$ topology. The principle goal of our paper is to show that this is indeed true. We start by stating the problem precisely.

In the sequel, $(Z_i)_{i \in \mathbb{Z}}$ denotes an i.i.d. sequence of regularly varying random variables with index of regular variation $\alpha \in (0, 2)$. In particular, this means that

$$P(|Z_i| > x) = x^{-\alpha} L(x), \quad x > 0,$$

where $L$ is a slowly varying function at $\infty$. Let $(a_n)$ be a sequence of positive real numbers such that

$$n P(|Z_1| > a_n) \to 1,$$

as $n \to \infty$. Regular variation of $Z_i$ can be expressed in terms of vague convergence of measures on $\mathcal{E} = \mathbb{R} \setminus \{0\}$: for $a_n$ as in (1.1) and as $n \to \infty$,

$$n P(a_n^{-1} Z_i \in \cdot) \overset{v}{\to} \mu(\cdot),$$

with the measure $\mu$ on $\mathcal{E}$ given by

$$\mu(dx) = (p 1_{(0,\infty)}(x) + r 1_{(-\infty,0)}(x)) \alpha |x|^{-\alpha-1} dx,$$

where

$$p = \lim_{x \to \infty} \frac{P(Z_i > x)}{P(|Z_i| > x)}$$

and

$$r = \lim_{x \to \infty} \frac{P(Z_i < -x)}{P(|Z_i| > x)}.$$ (1.4)

Under these assumptions on the sequence $(Z_i)$, we study the moving average process of the form

$$X_i = \sum_{j=-\infty}^{\infty} \varphi_j Z_{i-j}, \quad i \in \mathbb{Z},$$

with coefficients satisfying

$$\sum_{j=-\infty}^{\infty} |\varphi_j|^\delta < \infty \quad \text{for some } 0 < \delta < \alpha, \delta \leq 1.$$

Astrauskas [1] and Davis and Resnick [4] showed that the normalized sums of $X_i$’s under these conditions converge in distribution to a stable random
A natural generalization of this result would be a functional limit theorem for the partial sum process of \( X_i \)'s with respect to some natural topology on \( D[0, 1] \). In other words, it is interesting to show

\[
\frac{1}{a_n} \sum_{i=1}^{\lfloor n \cdot \rfloor} (X_i - c_n) \overset{d}{\to} \left( \sum_{j=-\infty}^{\infty} \varphi_j \right) V(\cdot), \tag{1.5}
\]

in \( D[0, 1] \), where \( V(\cdot) \) is an \( \alpha \)-stable Lévy process and \( c_n \) are appropriate centering constants, \( D[0, 1] \) being the space of right continuous functions on \([0, 1]\) with left limits.

If \( X_i \) is a finite order moving average with at least two nonzero coefficients, then the convergence in (1.5) cannot hold in the \( J_1 \) sense, see Avram and Taqqu [2] for instance. However, if all coefficients \( \varphi_i \) are nonnegative, then the convergence in (1.5) holds in the \( M_2 \) topology according to Avram and Taqqu [2], see their Theorem 2 (see also Corollary 1 in Tyran-Kamińska [6]).

In the same article, Avram and Taqqu made the following conjecture: if \( \varphi_j = 0 \) for \( j < 0, \varphi_0, \varphi_1, \ldots \in \mathbb{R} \) and if for every \( K \),

\[
0 \leq \frac{\sum_{j=0}^{K} \varphi_j}{\sum_{j=0}^{\infty} \varphi_j} \leq 1,
\]

then (1.5) holds in \( M_2 \) topology. This topology, is again due to Skorohod (for an extensive discussion of topologies on \( D[0, 1] \) we refer to Whitt [7]). As our main result we give a proof of Avram and Taqqu’s conjecture. In order to do so, we first recall the precise definition of the \( M_2 \) topology. We proceed by proving the conjecture for the finite order moving average processes in Section 2 and then finally in Section 3 we extend this to infinite order moving average processes.

The \( M_2 \) topology on \( D[0, 1] \) is defined using completed graphs. For \( x \in D[0, 1] \) the completed graph of \( x \) is the set

\[
\Gamma_x = \{(t, z) \in [0, 1] \times \mathbb{R} : z = \lambda x(t-) + (1 - \lambda)x(t) \text{ for some } \lambda \in [0, 1]\},
\]

where \( x(t-) \) is the left limit of \( x \) at \( t \). Besides the points of the graph \( \{(t, x(t)) : t \in [0, 1]\} \), the completed graph of \( x \) also contains the vertical line segments joining \((t, x(t))\) and \((t, x(t-))\) for all discontinuity points \( t \) of \( x \). An \( M_2 \) parametric representation of the completed graph \( \Gamma_x \) is a continuous function \((r,u)\) mapping \([0, 1]\) onto \( \Gamma_x \) such that \( r \) is nondecreasing, with \( r \)
being the time component and \( u \) being the spatial component. Let \( \Pi_{s,2}(x) \) denote the set of \( M_2 \) parametric representations of the graph \( \Gamma_x \). For \( x_1, x_2 \in D[0,1] \) define

\[
d_{s,2}(x_1, x_2) = \inf \{ \| r_1 - r_2 \|_{[0,1]} \vee \| u_1 - u_2 \|_{[0,1]} : (r_i, u_i) \in \Pi_{s,2}(x_i), i = 1, 2 \},
\]

where \( \| x \|_{[0,1]} = \sup \{ x(t) : t \in [0,1] \} \) and \( a \vee b = \max\{a, b\} \). Now we say that \( x_n \to x \) in \( D[0,1] \) for a sequence \( (x_n) \) in the Skorohod \( M_2 \) topology if \( d_{s,2}(x_n, x) \to 0 \) as \( n \to \infty \). The \( M_2 \) topology is weaker than the more frequently used \( M_1 \) and \( J_1 \) topologies which are also due to Skorohod. The \( M_2 \) topology can be generated using the Hausdorff metric on the space of graphs. For \( x_1, x_2 \in D[0,1] \) define

\[
d_{M_2}(x_1, x_2) = \left( \sup_{a \in \Gamma_{x_1}} \inf_{b \in \Gamma_{x_2}} d(a, b) \right) \vee \left( \sup_{a \in \Gamma_{x_2}} \inf_{b \in \Gamma_{x_1}} d(a, b) \right),
\]

where \( d \) is the metric on \( \mathbb{R}^2 \) defined by \( d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| \vee |y_1 - y_2| \) for \( (x_i, y_i) \in \mathbb{R}^2, i = 1, 2 \). The metric \( d_{M_2} \) induces the \( M_2 \) topology.

2. Finite order MA processes

Let \( \varphi_0, \varphi_1, \ldots, \varphi_q \) (for some fixed \( q \in \mathbb{N} \)) be real numbers satisfying

\[
0 \leq \sum_{i=0}^{s} \varphi_i / \sum_{i=0}^{q} \varphi_i \leq 1, \quad \text{for every } s = 0, 1, \ldots, q. \tag{2.1}
\]

Put \( \Phi = \Phi(q) = \sum_{i=0}^{q} \varphi_i \). Without loss of generality assume \( \Phi > 0 \). The case \( \Phi < 0 \) is completely equivalent if we multiply the noise sequence \( (Z_i) \) by minus 1, and is therefore omitted. Observe that condition (2.1) implies

\[
\sum_{i=0}^{s} \varphi_i \geq 0 \quad \text{and} \quad \sum_{i=s}^{q} \varphi_i \geq 0, \quad \text{for every } s = 0, 1, \ldots, q.
\]

Let \( (X_t) \) be a moving average process defined by

\[
X_t = \sum_{i=0}^{q} \varphi_i Z_{t-i}, \quad t \in \mathbb{Z}.
\]
Define further the corresponding partial sum process

\[ V_n(t) = \frac{1}{a_n} \left( \sum_{i=1}^{[nt]} X_i - [nt]b_n \right), \quad t \in [0, 1], \quad (2.2) \]

where

\[ b_n = \begin{cases} 
0, & \alpha \in (0, 1] \\
\Phi E(Z_1), & \alpha \in (1, 2). 
\end{cases} \]

**Theorem 2.1.** Let \((Z_i)_{i \in \mathbb{Z}}\) be an i.i.d. sequence of regularly varying random variables with index \(\alpha \in (0, 2)\). When \(\alpha = 1\), suppose further that \(Z_1\) is symmetric. Assume real numbers \(\varphi_0, \varphi_1, \ldots, \varphi_q\) satisfy (2.1). Then

\[ V_n(\cdot) \overset{d}{\to} \Phi V(\cdot), \quad n \to \infty, \]

in \(D[0, 1]\) endowed with the \(M_2\) topology, where \(V\) is an \(\alpha\)-stable Lévy process.

**Remark 2.2.** The characteristic Lévy triple of the limiting process \(V\) in the theorem is of the form \((0, \mu, b)\), with \(\mu\) as in (1.3) and

\[ b = \begin{cases} 
0, & \alpha = 1 \\
(p - r) \frac{\alpha}{1 - \alpha}, & \alpha \in (0, 1) \cup (1, 2). 
\end{cases} \]

In the proof of the theorem we are going to use the following simple lemma.

**Lemma 2.3.** (i) For \(k < q\) it holds

\[ \sum_{i=1}^{k} \frac{\Phi Z_i}{a_n} - \sum_{i=1}^{k} \frac{X_i}{a_n} = \sum_{u=0}^{k-1} Z_{k-u} \frac{q}{a_n} \sum_{s=u+1}^{q} \varphi_s - \sum_{u=k-q}^{q-1} Z_{u} \frac{q}{a_n} \sum_{s=u+1}^{q} \varphi_s - \frac{q-k-1}{a_n} \sum_{u=0}^{u+k} \frac{Z_{-u}}{a_n} \sum_{s=u+1}^{u+k} \varphi_s. \]

(ii) For \(k \geq q\) it holds

\[ \sum_{i=1}^{k} \frac{\Phi Z_i}{a_n} - \sum_{i=1}^{k} \frac{X_i}{a_n} = \sum_{u=0}^{q-1} Z_{k-u} \frac{q}{a_n} \sum_{s=u+1}^{q} \varphi_s - \sum_{u=0}^{q-1} Z_{-u} \frac{q}{a_n} \sum_{s=u+1}^{q} \varphi_s =: H_n(k) - G_n. \]
(iii) For $q \leq k \leq n - q$ it holds
\[
\sum_{i=1}^{k} \frac{\Phi Z_i}{a_n} - \sum_{i=1}^{k+q} \frac{X_i}{a_n} = -\sum_{u=0}^{q-1} \frac{Z_{-u}}{a_n} \sum_{s=u+1}^{q} \varphi_s - \sum_{u=1}^{q} \frac{Z_{k+u}}{a_n} \sum_{s=0}^{q-u} \varphi_s
\]
\[= -G_n - T_n(k).\]

**Proof of Lemma 2.3.** We prove only (i), since the other two statements can be proven similarly. Note first that for every $k \in \mathbb{N}$ it holds that
\[
\sum_{i=1}^{k} \sum_{j=0}^{q} \varphi_j Z_{i-j} = \sum_{l=1-q}^{k} Z_l \sum_{i=1 \text{ or } l}^{k \wedge (q+l)} \varphi_{i-l}
\[= \sum_{l=1-q}^{0} Z_l \sum_{s=1-l}^{(k-l)/q} \varphi_s + \sum_{l=1}^{k} Z_l \sum_{s=0}^{(k-l)/q} \varphi_s \tag{2.3}\]

Since $k < q$, by (2.3) we have (recall $\Phi = \sum_{i=0}^{q} \varphi_i$)
\[
\sum_{i=1}^{k} \Phi Z_i - \sum_{i=1}^{k} X_i
\[= \sum_{i=1}^{k} \Phi Z_i - \sum_{l=1-q}^{k-q} Z_l \sum_{s=1-l}^{q} \varphi_s - \sum_{l=1-k-q+1}^{0} Z_l \sum_{s=1-l}^{k-l} \varphi_s - \sum_{l=1}^{k} Z_l \sum_{s=0}^{k-l} \varphi_s
\[= \sum_{l=1}^{k} Z_l \sum_{s=k-l+1}^{q} \varphi_s - \sum_{l=1-q}^{k-q} Z_l \sum_{s=1-l}^{q} \varphi_s - \sum_{l=1-k-q+1}^{0} Z_l \sum_{s=1-l}^{k-l} \varphi_s - \sum_{l=1}^{k} Z_l \sum_{s=0}^{k-l} \varphi_s.
\]

Now, we use the change of variables ($u = k - l$ for the first term on the right hand side in the last equation, and $u = -l$ for the second and third term) and rearrange some sums to arrive at
\[
\sum_{i=1}^{k} \Phi Z_i - \sum_{i=1}^{k} X_i = \sum_{u=0}^{k-1} \frac{Z_{k-u}}{a_n} \sum_{s=u+1}^{q} \varphi_s - \sum_{u=1-k-q+1}^{0} Z_{-u} \sum_{s=u+1}^{q} \varphi_s
\[= \sum_{u=0}^{q-k-1} Z_{-u} \sum_{s=u+1}^{u+k} \varphi_s.
\]

$\square$
Remark 2.4. Note that random variables $H_n(k)$ and $T_n(k)$ are independent.

**Proof of Theorem 2.1**. Case $\alpha \in (0, 1]$. Since the random variables $Z_i$ are i.i.d. and regularly varying, Theorem 7.1 and Corollary 7.1 in Resnick [5] and Karamata’s theorem immediately yield $V_n^Z(\cdot) \xrightarrow{d} \Phi V(\cdot)$, as $n \to \infty$, in $(D[0, 1], d_{M^2})$, where

$$V_n^Z(t) := \sum_{i=1}^{[nt]} \frac{\Phi Z_i}{a_n}, \quad t \in [0, 1]$$

and $V$ is an $\alpha$-stable Lévy process with characteristic triple $(0, \mu, 0)$ if $\alpha = 1$ and $(0, \mu, (p-r)/\alpha/(1-\alpha))$ if $\alpha \in (0, 1)$ with $p$ and $r$ as in (1.4).

Using the fact that $J_1$ convergence implies $M_2$ convergence, we obtain

$$V_n^Z(\cdot) \xrightarrow{d} \Phi V(\cdot), \quad n \to \infty,$$

in $(D[0, 1], d_{M^2})$ as well. If one can show that for every $\epsilon > 0$

$$\lim_{n \to \infty} P[d_{M^2}(V_n^Z, V_n) > \epsilon] = 0,$$

an application of Slutsky’s theorem (see for instance Theorem 3.4 in Resnick [5]), will imply $V_n(\cdot) \xrightarrow{d} \Phi V(\cdot)$, as $n \to \infty$, in $(D[0, 1], d_{M^2})$.

Fix $\epsilon > 0$ and let $n \in \mathbb{N}$ be large enough, i.e. $n > \max\{2q, 4q/\epsilon\}$. Then by the definition of the metric $d_{M^2}$, we have

$$d_{M^2}(V_n^Z, V_n) = \left( \sup_{a \in \Gamma_{V_n}^Z} \inf_{b \in \Gamma V_n} d(a, b) \right) \lor \left( \sup_{a \in \Gamma V_n} \inf_{b \in \Gamma_{V_n}^Z} d(a, b) \right)$$

$$=: Y_n \lor T_n.$$

Hence

$$P[d_{M^2}(V_n^Z, V_n) > \epsilon] \leq P(Y_n > \epsilon) + P(T_n > \epsilon) \quad (2.5)$$

Now, we estimate the first term on the right hand side of (2.5). By the definition of $Y_n$, the Hausdorff metric and the choice of number $n$, we see
that

$$\{Y_n > \epsilon\} \subseteq \{\exists a \in \Gamma_{V_n}^Z \text{ such that } d(a, b) > \epsilon \text{ for every } b \in \Gamma_{V_n}\}$$

$$\subseteq \{\exists k \in \{1, \ldots, q - 1\} \text{ such that } |V_n^Z(k/n) - V_n(k/n)| > \epsilon \}$$

$$\cup \{\exists k \in \{q, \ldots, n - q\} \text{ such that } |V_n^Z(k/n) - V_n(k/n)| > \epsilon$$

and

$$|V_n^Z(k/n) - V_n((k + q)/n)| > \epsilon\}$$

$$\cup \{\exists k \in \{n - q + 1, \ldots, n\} \text{ such that } |V_n^Z(k/n) - V_n(k/n)| > \epsilon\}$$

$$=: A_n^Y \cup B_n^Y \cup C_n^Y. \quad (2.6)$$

By Lemma 2.3 (i) and stationarity we obtain

$$P(A_n^Y) \leq \sum_{k=1}^{q-1} P \left| \sum_{i=1}^{k} \frac{\Phi Z_i}{a_n} - \sum_{i=1}^{k} \frac{X_i}{a_n} \right| > \epsilon \right)$$

$$\leq \sum_{k=1}^{q-1} \left[ P \left( \sum_{u=0}^{k-1} \frac{|Z_u - u|}{a_n} \sum_{s=u+1}^{q} |\varphi_s| > \frac{\epsilon}{3} \right) + P \left( \sum_{u=k-q}^{q-1} \frac{|Z_u|}{a_n} \sum_{s=u+1}^{q} |\varphi_s| > \frac{\epsilon}{3} \right) \right]$$

$$+ P \left( \sum_{u=0}^{q-k-1} \frac{|Z_u|}{a_n} \sum_{s=u+1}^{u+k} |\varphi_s| > \frac{\epsilon}{3} \right)$$

$$\leq 3(q - 1)(2q - 1) P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{3(2q - 1)\theta} \right), \quad (2.7)$$

where $$\theta = \sum_{s=0}^{q} |\varphi_s| > 0$$. Hence, by regular variation property we observe

$$\lim_{n \to \infty} P(A_n^Y) = 0. \quad (2.8)$$

Next, using Lemma 2.3 (ii), (iii) together with stationarity and the fact that
$H_n(k)$ and $T_n(k)$ are independent, we obtain

$$P(B_n^Y) = P\left(\exists k \in \{q, \ldots, n-q\} \text{ such that } |H_n(k) - G_n| > \epsilon \right.$$  
and $| - G_n - T_n(k)| > \epsilon$  
$$\leq P\left(|G_n| > \frac{\epsilon}{2}\right) + \sum_{k=q}^{n-q} P\left(|H_n(k)| > \frac{\epsilon}{2} \text{ and } |T_n(k)| > \frac{\epsilon}{2}\right)$$  

$$= P\left(|G_n| > \frac{\epsilon}{2}\right) + \sum_{k=q}^{n-q} P\left(|H_n(k)| > \frac{\epsilon}{2}\right) P\left(|T_n(k)| > \frac{\epsilon}{2}\right)$$  

$$\leq P\left(|G_n| > \frac{\epsilon}{2}\right) + nP\left(|H_n(0)| > \frac{\epsilon}{2}\right) P\left(|T_n(0)| > \frac{\epsilon}{2}\right)$$  

$$\leq q P\left(\frac{|Z_0|}{a_n} > \frac{\epsilon}{2q\theta}\right) + \frac{q^2}{n} \left[n P\left(\frac{|Z_0|}{a_n} > \frac{\epsilon}{2q\theta}\right)\right]^2,$$

whence we conclude

$$\lim_{n \to \infty} P(B_n^Y) = 0. \quad (2.9)$$

In a similar manner as in (2.7), but using (ii) from Lemma 2.3 instead of (i) we get

$$\lim_{n \to \infty} P(C_n^Y) = 0. \quad (2.10)$$

From relations (2.6), (2.8), (2.9) and (2.10) we obtain

$$\lim_{n \to \infty} P(Y_n > \epsilon) = 0. \quad (2.11)$$

It remains to estimate the second term on the right hand side of (2.5). From the definition of $T_n$, the Hausdorff metric and the number $n$ it follows

$$\{T_n > \epsilon\} \subseteq \{\exists a \in \Gamma_{V_n} \text{ such that } d(a, b) > \epsilon \text{ for every } b \in \Gamma_{V_n}\} 
\subseteq \{\exists k \in \{1, \ldots, 2q-1\} \text{ such that } |V_n(k/n) - V_n^Z(k/n)| > \epsilon\}$$  

$$\cup \{\exists k \in \{2q, \ldots, n\} \text{ such that } d((k/n, V_n(k/n)), \Gamma_{V_n}) > \frac{\epsilon}{2}\}$$  

$$= A_n^T \cup B_n^T. \quad (2.12)$$
Using Lemma 2.3 (i) and (ii), one could similarly as before for set $A_n^Y$ obtain
\[
\lim_{n \to \infty} P(A_n^T) = 0. \tag{2.13}
\]

To bound $P(B_n^T)$ we need a new argument. For each $k \geq 2q$, set $V_n^{Z,\min} = \min\{V_n^Z((k-q)/n), V_n^Z(k/n)\}$ and $V_n^{Z,\max} = \max\{V_n^Z((k-q)/n), V_n^Z(k/n)\}$. Since the completed graph $\Gamma_{V_n^Z}$ is connected, if $V_n^Z(k/n) \in (V_n^{Z,\min} - \epsilon/4, V_n^{Z,\max} + \epsilon/4)$
then $d((k/n, V_n^Z(k/n)), \Gamma_{V_n^Z}) < \epsilon/2$ for all $n$ large enough so that $q/n < \epsilon/4$. Therefore, $P(B_n^T)$ is bounded by
\[
P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } \sum_{i=1}^{k} \frac{X_i}{a_n} > V_n^{Z,\max} + \frac{\epsilon}{4} \right) + P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } \sum_{i=1}^{k} \frac{X_i}{a_n} < V_n^{Z,\min} - \frac{\epsilon}{4} \right)
\]

In the sequel we consider only the first of these two probabilities, since the other one can be handled in a similar manner. Note, that the first probability using Lemma 2.3 can be bounded by
\[
P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } G_n - H_n(k) > \frac{\epsilon}{4} \text{ and } G_n + T_n(k-q) > \frac{\epsilon}{4} \right)
\leq P \left( G_n > \frac{\epsilon}{8} \right) + P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } H_n(k) < -\frac{\epsilon}{8} \text{ and } T_n(k-q) > -\frac{\epsilon}{8} \right)
\]

As before $P(G_n > \epsilon/8) \to 0$ as $n \to \infty$. For the second term, note,
\[
H_n(k) = \sum_{u=0}^{q-1} \frac{Z_{k-u}}{a_n} \sum_{s=u+1}^{q} \varphi_s \quad \text{and} \quad T_n(k-q) = \sum_{u=0}^{q-1} \frac{Z_{k-u}}{a_n} \sum_{s=0}^{u} \varphi_s.
\]

Hence, that term is bounded by
\[
nP \left( \sum_{u=0}^{q-1} \frac{Z_{u}}{a_n} \sum_{s=u+1}^{q} \varphi_s < -\frac{\epsilon}{8} \text{ and } \sum_{u=0}^{q-1} \frac{Z_{u}}{a_n} \sum_{s=0}^{u} \varphi_s > \frac{\epsilon}{8} \right)
\]
where we used the stationarity of the sequence \((Z_i)\). Observe now that the sums \(\sum_{s=0}^u \varphi_s\) and \(\sum_{s=u+1}^q \varphi_s\) are both nonnegative and bounded by \(\Phi = \sum_{s=0}^q \varphi_s\), see (2.1). Therefore, the last expression above is bounded by

\[
nP \left( \exists i, j \in \{0, \ldots, q-1\}, i \neq j \text{ such that } \Phi \frac{Z_i - a_n}{a_n} < -\frac{\epsilon}{8q} \text{ and } \Phi \frac{Z_j - a_n}{a_n} > \frac{\epsilon}{8q} \right)
\]

which clearly tends to 0 as \(n \to \infty\), by the regular variation property of the random variables \(Z_i\). Note that the case \(i = j\) above is not possible since then we would have \(Z_{-i} < 0\) and \(Z_{-i} > 0\).

Together with relations (2.12) and (2.13) this implies

\[
\lim_{n \to \infty} P(T_n > \epsilon) = 0. \tag{2.14}
\]

Now from (2.5), (2.11) and (2.14) we obtain

\[
\lim_{n \to \infty} P[d_{M_2}(V_n^Z, V_n) > \epsilon] = 0, \tag{2.15}
\]

and finally we conclude that \(V_n(\cdot) \overset{d}{\to} \Phi V(\cdot)\), as \(n \to \infty\), in \((D[0,1], d_{M_2})\).

Case \(\alpha \in (1,2)\). In this case \(E(Z_i) < \infty\). Define

\[
Z'_i = Z_i - E(Z_1), \quad i \in \mathbb{Z}.
\]

Then \(E(Z'_i) = 0\) and \((Z'_i)_i\) is an i.i.d. sequence of regularly varying random variables with index \(\alpha\). Then it is known that, as \(n \to \infty\), the stochastic process

\[
W_n(t) := \sum_{i=1}^{\lfloor nt \rfloor} \frac{Z_i}{a_n} - \lfloor nt \rfloor E \left( \frac{Z_1}{a_n} 1_{\{|Z_1| \leq a_n\}} \right), \quad t \in [0,1],
\]

converges in distribution in \((D[0,1], d_{M_1})\) to an \(\alpha\)-stable Lévy process with characteristic triple \((0, \mu, 0)\) (cf. Theorem 3.4 in Basrak et al. [3]). By Karamata’s theorem, as \(n \to \infty\),

\[
n E \left( \frac{Z_1}{a_n} 1_{\{|Z_1| > a_n\}} \right) \to (p-r) \frac{\alpha}{\alpha-1}.
\]

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with $p$ and $r$ as in (1.4). Thus, as $n \to \infty$,

$$[n \cdot]E\left(\frac{Z_1}{a_n} 1_{\{|Z_1| > a_n\}}\right) \to (\cdot)(p - r)\frac{\alpha}{\alpha - 1}$$

in $(D[0,1], d_{M_1})$. Since the latter function is continuous, an application of Corollary 12.7.1 in Whitt [7] (which gives a sufficient condition for addition to be continuous) and the continuous mapping theorem give that the following stochastic process

$$\sum_{i=1}^{[nt]} \frac{Z_i}{a_n} = \sum_{i=1}^{[nt]} Z_i - [nt]E\left(\frac{Z_1}{a_n}\right) = W_n(t) - [nt]E\left(\frac{Z_1}{a_n} 1_{\{|Z_1| > a_n\}}\right), \quad t \in [0,1],$$

converges in distribution in $(D[0,1], d_{M_1})$ to an $\alpha$–stable Lévy process $V$ with characteristic triple $(0, \mu, (p - r)\alpha/(1 - \alpha))$. Define now

$$X'_i = \sum_{j=0}^q \varphi_j Z'_{i-j}, \quad i \in \mathbb{Z},$$

and

$$V'_n(t) = \sum_{i=1}^{[nt]} \frac{X'_i}{a_n} \quad \text{and} \quad V'^Z_n(t) = \sum_{i=1}^{[nt]} \frac{\Phi Z'_i}{a_n}, \quad t \in [0,1].$$

Now we can repeat all arguments used in the case $\alpha \in (0,1]$ to obtain

$$V'^Z_n(\cdot) \overset{d}{\to} \Phi V(\cdot), \quad n \to \infty, \quad (2.16)$$

in $(D[0,1], d_{M_2})$, and

$$\lim_{n \to \infty} P[d_{M_2}(V'^Z_n, V'_n) > \epsilon] = 0, \quad \epsilon > 0.$$

Note that $V_n = V'_n$ and therefore $V_n(\cdot) \overset{d}{\to} \Phi V(\cdot)$, as $n \to \infty$, in $(D[0,1], d_{M_2})$. This concludes the proof.

3. Infinite order MA processes

Let $\{\varphi_i, i = 0, 1, 2, \ldots\}$ be a sequence of real numbers satisfying

$$\sum_{i=0}^\infty |\varphi_i|^\delta < \infty \quad (3.1)$$
for some $0 < \delta < \min\{1, \alpha\}$, and

$$0 \leq \sum_{i=0}^{s} \varphi_i / \sum_{i=0}^{\infty} \varphi_i \leq 1, \quad \text{for every } s = 0, 1, 2, \ldots \quad (3.2)$$

Let $\Phi = \Phi(\infty) = \sum_{i=0}^{\infty} \varphi_i$. Condition (3.1) implies $\Phi$ is finite. Without loss of generality assume $\Phi > 0$ (as before, the case $\Phi < 0$ can be handled similarly).

Let $(X_t)$ be a moving average process defined by

$$X_t = \sum_{i=0}^{\infty} \varphi_i Z_{t-i}, \quad t \in \mathbb{Z}.$$ 

Condition (3.1) ensures that $X_t$ converges in $L^\delta$ and a.s. Define further the corresponding partial sum stochastic process $V_n$ as in (2.2).

**Theorem 3.1.** Let $(Z_i)_{i \in \mathbb{Z}}$ be an i.i.d. sequence of regularly varying random variables with index $\alpha \in (0, 2)$. When $\alpha = 1$, suppose further that $Z_1$ is symmetric. Let $\{\varphi_i, i = 0, 1, 2, \ldots\}$ be a sequence of real numbers satisfying (3.1) and (3.2). Then

$$V_n(\cdot) \xrightarrow{d} \Phi V(\cdot), \quad n \to \infty,$$

in $D[0, 1]$ endowed with the $M_2$ topology, where $V$ is an $\alpha$-stable Lévy process.

**Remark 3.2.** The characteristic triple of the limiting process $V$ in Theorem 3.1 is of the same form as in Remark 2.2.

**Proof of Theorem 3.1.** Case $\alpha \in (0, 1]$. Fix $q \in \mathbb{N}$ and define

$$X_i^q = \sum_{j=0}^{q-1} \varphi_j Z_{i-j} + \varphi'_i Z_{i-q}, \quad i \in \mathbb{Z},$$

where $\varphi'_i = \sum_{i=q}^{\infty} \varphi_i$, and

$$V_{n,q}(t) = \sum_{i=1}^{\lfloor nt \rfloor} \frac{X_i^q}{a_n}, \quad t \in [0, 1].$$
Since the coefficients $\varphi_0, \ldots, \varphi_{q-1}, \varphi'_q$ satisfy condition (2.1), an application of Theorem 2.1 to a finite order moving average process $(X^q_i)$, yields that

$$V_{n,q} \left( \cdot \right) \overset{d}{\to} \Phi V \left( \cdot \right), \quad n \to \infty,$$

in $(D[0,1], d_{M_2})$. If we show that for every $\epsilon > 0$

$$\lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P}[d_{M_2}(V_{n,q}, V_n) > \epsilon] = 0,$$

then by a generalization of Slutsky’s theorem (see for instance Theorem 3.5 in Resnick [5]) it will follow $V_n \left( \cdot \right) \overset{d}{\to} \Phi V \left( \cdot \right)$, as $n \to \infty$, in $(D[0,1], d_{M_2})$.

Since the Skorohod $M_2$ metric on $D[0,1]$ is bounded above by the uniform metric on $D[0,1]$, it suffices to show that

$$\lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq 1} \left| V_{n,q}(t) - V_n(t) \right| > \epsilon \right) = 0.$$

Recalling the definitions, we have

$$\lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq 1} \left| V_{n,q}(t) - V_n(t) \right| > \epsilon \right) \leq \lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sum_{i=1}^{n} \left| \frac{X^q_i - X_i}{a_n} \right| > \epsilon \right).$$

Put $\varphi''_q = \varphi'_q - \varphi_q = \sum_{j=q+1}^{\infty} \varphi_j$ and observe

$$\sum_{i=1}^{n} \left| X^q_i - X_i \right| = \sum_{i=1}^{n} \left| \sum_{j=0}^{q-1} \varphi_j Z_{i-j} + \varphi'_q Z_{i-q} - \sum_{j=0}^{\infty} \varphi_j Z_{i-j} \right|$$

$$= \sum_{i=1}^{n} \left| \varphi''_q Z_{i-q} - \sum_{j=q+1}^{\infty} \varphi_j Z_{i-j} \right|$$

$$\leq \sum_{i=1}^{n} \left[ |\varphi''_q| |Z_{i-q}| + \sum_{j=q+1}^{\infty} |\varphi_j| |Z_{i-j}| \right]$$

$$\leq \left( 2 \sum_{j=q+1}^{\infty} |\varphi_j| \right) \sum_{i=1}^{n} |Z_{i-q}| + \sum_{i=-\infty}^{0} |Z_{i-q}| \sum_{j=1}^{n} |\varphi_{q-i+j}|$$

$$=: O_n(q).$$
Therefore we have to show
\[ \lim_{q \to \infty} \limsup_{n \to \infty} P \left( \frac{O_n(q)}{a_n} > \epsilon \right) = 0. \]  
(3.4)

By Lemma 2 in Avram and Taqqu [2], we obtain, for large \( q \),
\[
P \left( \frac{O_n(q)}{a_n} > \epsilon \right) \leq M \epsilon^{-(\alpha+\eta)} \frac{n}{n} \left( n \left( 2 \sum_{j=q+1}^{\infty} |\varphi_j| \right)^{\alpha-\eta} + \sum_{i=-\infty}^{0} \left( \sum_{j=1}^{n} |\varphi_{q-i+j}| \right)^{\alpha-\eta} \right),
\]
where \( \eta \) is some positive real number satisfying \( \alpha - \eta > \delta \) and \( M \) is a positive constant. Since \( \alpha - \eta < \alpha \leq 1 \), an application of the inequality \(|\sum_{i=1}^{n} a_i|^\gamma \leq \sum_{i=1}^{n} |a_i|^\gamma\) with \( a_i, \ldots, a_n \) real numbers and \( \gamma \in (0, 1] \), yields
\[
\left( \sum_{j=1}^{n} |\varphi_{q-i+j}| \right)^{\alpha-\eta} \leq \sum_{j=1}^{n} |\varphi_{q-i+j}|^{\alpha-\eta}.
\]

Using this and the fact that every \(|\varphi_i|^{\alpha-\eta}\), for \( i = q + 1, q + 2, \ldots \), appears in the sum \( \sum_{i=-\infty}^{0} \sum_{j=1}^{n} |\varphi_{q-i+j}|^{\alpha-\eta} \) at most \( n \) times, we obtain
\[
P \left( \frac{O_n(q)}{a_n} > \epsilon \right) \leq M \epsilon^{-(\alpha+\eta)} \left( \left( 2 \sum_{j=q+1}^{\infty} |\varphi_j| \right)^{\alpha-\eta} + \sum_{j=q+1}^{\infty} |\varphi_j|^{\alpha-\eta} \right)
= M \epsilon^{-(\alpha+\eta)} \left( \left( 2 \sum_{j=q+1}^{\infty} |\varphi_j| \right)^{\alpha-\eta} + \sum_{j=q+1}^{\infty} |\varphi_j|^{\alpha-\eta} \right). \quad (3.5)
\]

Since for large \( q \) it holds that \(|\varphi_j| \leq |\varphi_j|^\delta\) and \(|\varphi_j|^{\alpha-\eta} \leq |\varphi_j|^\delta\) for all \( j \geq q + 1 \), from condition (3.1) we immediately obtain, as \( q \to \infty \),
\[
\sum_{j=q+1}^{\infty} |\varphi_j| \to 0 \quad \text{and} \quad \sum_{j=q+1}^{\infty} |\varphi_j|^{\alpha-\eta} \to 0.
\]

Therefore from (3.5) letting \( q \to \infty \), follows (3.4), which means that \( V_n(\cdot) \xrightarrow{d} \Phi V(\cdot) \), as \( n \to \infty \), in \( (D[0,1], d_{M_2}) \).

Case \( \alpha \in (1, 2) \). Define \( Z_i' = Z_i - E(Z_1), i \in \mathbb{Z} \). Fix \( q \in \mathbb{N} \) and define
\[
X_i'^q = \sum_{j=0}^{q-1} \varphi_j Z_{i-j}' + \varphi_q' Z_{i-q}', \quad i \in \mathbb{Z},
\]
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and

\[ V'_{n,q}(t) = \sum_{i=1}^{\lfloor nt \rfloor} X'_{q,i} \cdot \frac{a_n}{a_n}, \quad t \in [0,1]. \]

Then

\[ V'_{n,q}(t) = \frac{1}{a_n} \left( \sum_{i=1}^{\lfloor nt \rfloor} X'_{q,i} \right) - \lfloor nt \rfloor \Phi E(Z_1). \]

Since the coefficients \( \varphi_0, \ldots, \varphi_{q-1}, \varphi' \) satisfy condition (2.1), Theorem 2.1, applied to a finite order moving average process \( (X'_{q,i}) \), yields that

\[ V'_{n,q}(\cdot) \xrightarrow{d} \Phi V(\cdot), \quad n \to \infty, \quad (3.6) \]

in \((D[0,1], d_{M_2})\). In order to obtain \( V_n(\cdot) \xrightarrow{d} \Phi V(\cdot)\), as in the previous case, it remains to show that for every \( \epsilon > 0 \)

\[ \lim_{q \to \infty} \limsup_{n \to \infty} P[d_{M_2}(V_{n,q}', V_n) > \epsilon] = 0, \]

i.e.

\[ \lim_{q \to \infty} \limsup_{n \to \infty} P \left( \frac{O_n(q)}{a_n} > \epsilon \right) = 0. \]

As before, by Lemma 2 in Avram and Taqqu [2], for large \( q \),

\[ P \left( \frac{O_n(q)}{a_n} > \epsilon \right) \leq \frac{M^{\epsilon-(\alpha+\eta)}}{n} \left( \sum_{j=q+1}^{\infty} |\varphi_j| \right)^{\alpha-\eta} \sum_{i=-\infty}^{0} \left( \sum_{j=1}^{n} |\varphi_{q-i+j}| \right)^{\alpha-\eta}, \]

where \( \eta \) is some positive real number satisfying \( \alpha - \eta > 1 \). Now using the inequality \( |\sum_{i=1}^{n} a_i|^\gamma \leq \sum_{i=1}^{n} |a_i| \) with \( a_i, \ldots, a_n \) real numbers such that \( |a_1 + \ldots + a_n| < 1 \) and \( \gamma \in (1,2) \), similarly as before we obtain

\[ P \left( \frac{O_n(q)}{a_n} > \epsilon \right) \leq \frac{M^{\epsilon-(\alpha+\eta)}}{n} \left( \sum_{j=q+1}^{\infty} |\varphi_j| \right)^{\alpha-\eta} + \sum_{i=-\infty}^{\infty} \left( \sum_{j=1}^{\infty} |\varphi_{q-i+j}| \right)^{\alpha-\eta}, \]

and again letting \( \lim_{q \to \infty} \limsup_{n \to \infty} \), the desired result follows. This completes the proof. \( \square \)
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