ON $\mu$-ZARISKI PAIRS OF LINKS

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Abstract. The notion of Zariski pairs for projective curves in $\mathbb{P}^2$ is known since the pioneer paper of Zariski [25] and for further development, we refer the reference in [3]. In this paper, we introduce a notion of Zariski pair of links in the class of isolated hypersurface singularities. Such a pair is canonically produced from a Zariski (or a weak Zariski) pair of curves $C = \{f(x,y,z) = 0\}$ and $C' = \{g(x,y,z) = 0\}$ of degree $d$ by simply adding a monomial $z^{d+m}$ to $f$ and $g$ so that the corresponding affine hypersurfaces have isolated singularities at the origin. They have the same zeta function and a same Milnor number (16). We give new examples of Zariski pairs which have the same $\mu^*$ sequence and a same zeta function but two functions belong to different connected components of $\mu$-constant strata (Theorem 14). Two link 3-folds are not diffeomorphic and they are distinguished by the first homology which implies the Jordan form of their monodromies are different (Theorem 23). We start from weak Zariski pairs of projective curves to construct new Zariski pairs of surfaces which have non-diffeomorphic link 3-folds. We also prove that hypersurface pair constructed from a Zariski pair give a diffeomorphic links (Theorem 24).

Contents

1. Introduction 2
2. Preliminary 3
2.1. Divisor of rational functions 3
2.2. Zeta function of the Milnor fibration 3
2.3. A’Campo formula 8
2.4. Newton boundary and dual Newton diagram 4
2.5. Toric modification 5
2.6. Varchenko formula 8
2.7. Almost non-degenerate functions 9
2.8. Zeta multiplicity 10
2.9. Lê-Ramunujam result for zeta-functions 10
3. $\mu$-Zariski pairs 11
3.1. Zariski pairs and weak Zariski pairs 11
3.2. Zariski pairs of hypersurfaces 12
3.3. Main theorem 14

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1. Introduction

Consider a germ of an analytic functions $f(z)$ defined in a neighborhood $U_0$ of the origin in $\mathbb{C}^n$ with an isolated singularity at $z = 0$. The $\mu^*$-invariant of $f$ is defined as an $n$-tuple of integers $\mu^* = (\mu^{(n)}, \ldots, \mu^{(1)})$ where $\mu^{(i)}$ is the Milnor number of $f|_{L_i}$ with $L_i$ being a generic $i$-dimensional linear subspace through the origin (16). Consider a Zariski pair (or a weak Zariski pair) of projective curves $C, C'$ of degree $d$ defined by homogeneous polynomials $f_d(x, y, z)$ and $g_d(x, y, z)$. We assume that the singular points of $C$ and $C'$ are Newton non-degenerate. Consider the affine hypersurfaces defined by $f(x, y, z) = f_d(x, y, z) + z^{d+m}$ and $g(x, y, z) = g_d(x, y, z) + z^{d+m}$. Then $f$ and $g$ are almost non-degenerate function in the sense of (16) and two hypersurfaces have a same zeta function. In (16), we called such a pair a Zariski pair of links. It is also easy to observe that $f$ and $g$ have a same $\mu^*$-sequence. Consider the local links at the origin, $K_f = V(f) \cap S^{2n-1}$ and $K_g = V(g) \cap S^{2n-1}$, with a sufficiently small $\varepsilon \ll 1$. We say $\{K_f, K_g\}$ (or $\{f, g\}$) is a $\mu$-Zariski pair of links (or of surfaces), if $f$ and $g$ belong to different connected components of $\mu$-constant strata. For the definition of $\mu$-constant strata, see (4). We say $\{K_f, K_g\}$ are $\mu^*$-Zariski pair if $f$ and $g$ belong to different connected components of $\mu^*$-constant strata. In (4), we gave an example of $\mu^*$-Zariski pair of links. Here $\mu^*$-invariant is introduced by Teissier.

In (23), Teissier proved that if $f$ and $g$ are connected by a complex piece-wise analytic $\mu^*$-constant family $f_t, 0 \leq t \leq 1$ with $f_0 = 0$ and $f_1 = g$, then the local link pairs $(S^{2n-1}_t, K_f)$ and $(S^{2n-1}_t, K_g)$ are diffeomorphic. The same assertion is true for $\mu$-constant $C^\infty$-real family $f_t$ for $n \neq 3$ by Lê and Ramamujam (12).

\[\text{References}\]
The purpose of this paper is to present some examples of $\mu$-Zariski pairs of surfaces in §3 (Theorem [14]). We will show that those pairs of surface links constructed from weak Zariski pairs in §3 are not diffeomorphic. On the other hand, the link pairs constructed from Zariski pairs are always diffeomorphic as 3-manifolds (Theorem [24]) but we do not know if this diffeomorphism can be extended to a diffeomorphism of $S^3$.

2. Preliminary

2.1. Divisor of rational functions. Consider a rational function $\varphi(t) = p(t)/q(t)$ where $p(t), q(t) \in \mathbb{C}[t]$ and $p(0), q(0) \neq 0$ and consider the factorization $p(t) = c \prod_{i=1}^{\ell} (t - \alpha_i)^{\nu_i}, c \neq 0$ and $q(t) = c' \prod_{j=1}^{m} (t - \beta_j)^{\mu_j}, c' \neq 0$. The divisor of $\varphi$ is defined by

$$\text{div}(\varphi) = \sum_{i=1}^{\ell} \nu_i < \alpha_i > - \sum_{j=1}^{m} \mu_j < \beta_j > \in \mathbb{Z}\mathbb{C}^*$$

where $\mathbb{Z}\mathbb{C}^*$ is the group ring of $\mathbb{C}^*$. We denote the divisor of $(t^d - 1)$ by $\Lambda_d$.

The degree of $\varphi$ is defined by $\deg \varphi = \deg p - \deg q = \ell - m$. The following formula (20) is useful in the later discussions:

$$\Lambda_d \cdot \Lambda_{d'} = \gcd(d, d')\Lambda_{\gcd(d, d')}.$$ (1)

2.2. Zeta function of the Milnor fibration. For an analytic function $f(z)$ defined in a neighborhood of the origin, we consider the tubular Milnor fibration $f : E(\varepsilon, \delta)^* \to D_{\delta}^*$ where

$$E(\varepsilon, \delta)^* = \{ z \in \mathbb{C}^n \mid ||z|| \leq \varepsilon, 0 \neq |f(z)| \leq \delta \},$$

$$D_{\delta}^* = \{ \eta \in \mathbb{C} \mid 0 < |\eta| \leq \delta, \delta < \varepsilon \ll 1 \}.$$

Let $F$ be the fiber and let $h : F \to F$ be the monodromy map. Consider the characteristic polynomial $P_j(t) = \det(id - t h_{sj})$ where $h_{sj} : H_j(F; \mathbb{Q}) \to H_j(F; \mathbb{Q})$. The zeta function of the Milnor fibration, denoted as $\zeta_f(t)$ is defined by the alternative product of the characteristic polynomials $\zeta_f(t) = P_0(t)^{-1}P_1(t) \cdots P_{n-1}(t)^{(-1)^n}$. If $f$ has an isolated singularity at the origin, $F$ is (n-2)-connected and

$$\zeta_f(t) = P_{n-1}(t)^{(-1)^n}(1 - t)^{-1}, \quad \deg \zeta_f = -1 + (-1)^n \mu$$

where $\mu$ is the (n-1)-th Betti number of $F$ and $\mu$ is usually called the Milnor number of $f$ at 0 [13].

2.3. A’Campo formula. Consider an analytic function $f(z) = \sum_{\nu} a_{\nu} z^\nu$ of $n$ variables defined in a neighborhood of the origin of $\mathbb{C}^n$. Assume that we are given a good resolution $\hat{\pi} : X \to U_0$ of the function $f$ and let $E_1, \ldots, E_s$ be the exceptional divisors of $\hat{\pi}$, that is $\hat{\pi}^{-1}(V) = \hat{V} \cup_{i=1}^{s} E_j$ where $\hat{V}$ is the strict transform of the hypersurface $V := f^{-1}(0)$ and $U_0$ is a small neighborhood of the origin. The irreducible components of $\hat{V}$ and $E_j, j = 1, \ldots, s$ are non-singular and $\hat{\pi}^{-1}(V)$ has only ordinary normal crossing singularities.
Consider the open subset $E_j' = E_j \setminus (\tilde{V} \cup_{i \neq j} E_i)$ and $E_j'' = E_j' \cap \pi^{-1}(0)$. In particular, if $E_j$ is a compact divisor, $E_j'' = E_j'$. Let $m_j$ be the multiplicity of $\pi^* f$ along $E_j$.

**Lemma 1** (A’Campo [1]). The zeta function of the Milnor monodromy at the origin is given by the formula:

$$
\zeta(t) = \prod_{j=1}^{s} (1 - t^{m_j})^{-\chi(E_j')},
$$

In this formula, the singularities at the origin can be non-isolated. As a simple corollary of the A’Campo formula, we have

**Corollary 2.** The divisor of the zeta function of Milnor monodromy is uniquely expressed as $\text{div} \zeta(t) = \sum_{i=1}^{s} \nu_i \Delta_{d_i}$ with $d_1 < \cdots < d_s$ and $\nu_i \neq 0$ for $i = 1, \ldots, s$.

2.4. **Newton boundary and dual Newton diagram.**

2.4.1. **Newton boundary.** Let $f(z) = \sum_{\nu} a_{\nu} z^\nu$ be a given holomorphic function defined by a convergent power series. Let $M$ be the space of monomials of the fixed coordinate variables $z_1, \ldots, z_n$ of $\mathbb{C}^n$ and let $N$ be the space of weights of the variables $z_1, \ldots, z_n$. We identify the monomial $z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}$ and the integral point $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{R}^n$. A weight $P$ is also identified with the column vector $t(p_1, \ldots, p_n) \in \mathbb{R}^n$ where $p_i = \deg_{p_i}(z_i)$ and we call $P$ a weight vector. The Newton polygon $\Gamma^+(f)$ with respect to the given coordinates $z = (z_1, \ldots, z_n)$ is the convex hull of the union $\cup_{\nu \neq 0} \{ \nu + \mathbb{R}^n \}$ and the Newton boundary $\Gamma(f)$ is defined by the union of compact faces of $\Gamma^+(f)$. For a non-negative weight vector $P = t(p_1, \ldots, p_n)$, we consider the canonical linear function $\ell_P$ on $\Gamma^+(f)$ which is defined by $\ell_P(\nu) = \sum_{i=1}^{n} \nu_i p_i$. This is nothing but the degree mapping $\deg_{p_i} z^\nu = \ell_P(\nu)$. The minimal value of $\ell_P$ is denoted by $d(P; f)$. Put $\Delta(P; f) := \{ \nu \in \Gamma^+(f) \mid \ell_P(\nu) = d(P) \}$. We will use the simplified notations $d(P)$ and $\Delta(P)$ if any ambiguity seems unlikely. In general, $\Delta(P)$ is a face of $\Gamma^+(f)$ and $\Delta(P) \subset \Gamma(f)$ if $P$ is positive (i.e., $p_i > 0, \forall i$). For a maximal dimensional face $\Delta$, i.e. $\Delta \subset \Gamma(f)$ with $\dim \Delta = n - 1$, there is a unique positive primitive integer vector $P$ such that $\Delta(P) = \Delta$. The partial sum $\sum_{\nu \in \Delta} a_{\nu} z^\nu$ is called the face function of the face $\Delta$ and we denote it as $f_\Delta$. For a weight $P$, $f_P$ is defined by $f_\Delta(P)$. Note that $f_P$ is a polynomial if $P$ is positive.

**Remark 3.** In this paper, we used the terminologies for a weight vector positive and non-negative instead of strictly positive and positive weight vectors respectively, terminologies used in [13, 16]. We changed the terminologies so that it is consistent with our paper [4].

2.4.2. **Dual Newton diagram.** Two weight vectors $P, Q$ are equivalent if and only if $\Delta(P) = \Delta(Q)$ and this equivalent relation gives a conical subdivision of the space of the non-negative weight vectors $N^+_\mathbb{R}$, i.e. of $\mathbb{R}^n_{\geq 0}$ (under
the above identification) and we denote it as $\Gamma^*(f)$ and call it the dual Newton diagram of $f$. We say, $f$ is Newton non-degenerate on $\Delta \subset \Gamma(f)$ if $f_\Delta : \mathbb{C}^* \to \mathbb{C}$ has no critical points. We say $f$ is Newton non-degenerate if it is non-degenerate on every face $\Delta \subset \Gamma(f)$ of any dimension. The dimension of a face can be any non-negative integer less than $n$. The closure of an equivalent class can be expressed as

$$\text{Cone}(P_1, \ldots, P_k) := \left\{ \sum_{i=1}^k \lambda_i P_i \mid \lambda_i \geq 0 \right\}$$

where $P_1, \ldots, P_k$ are chosen to be primitive integer vectors. This expression is unique if $k$ is minimal among any possible such expressions. A cone $\sigma = \text{Cone}(P_1, \ldots, P_k)$ is simplicial if dim $\sigma = k$ and $\sigma$ is regular if $P_1, \ldots, P_k$ are primitive integer vectors which can be extended to a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$. Recall that $f$ is convenient if $\Gamma(f)$ touches with every coordinate axis. We say $f$ is pseudo-convenient if $f$ is written as $f(z) = z^w f'(z)$ where $f'$ is a convenient analytic function and $\nu_0$ is a non-negative integer vector. In this case, $\Gamma'(f) = \Gamma^*(f)$.

2.5. Toric modification. A regular simplicial cone subdivision $\Sigma^*$ of the space of non-negative weight vectors $N^+_\mathbb{R} = \mathbb{R}^n_+$ is admissible with the dual Newton diagram $\Gamma^*(f)$ if $\Sigma^*$ is a subdivision of $\Gamma^*(f)$. For such a regular simplicial cone subdivision, we associate a modification $\hat{\pi} : X \to \mathbb{C}^n$ as follows: let $S_n$ be the set of $n$-dimensional cones in $\Sigma^*$. For each $\sigma = \text{Cone}(P_1, \ldots, P_n) \in S_n$, we identify $\sigma$ with the unimodular matrix:

$$\sigma = \begin{pmatrix} P_{11} & \cdots & P_{1n} \\ \vdots & \ddots & \vdots \\ P_{n1} & \cdots & P_{nn} \end{pmatrix}$$

with $P_{ij} = \ell(p_{1j}, \ldots, p_{nj})$. For a unimodular matrix $A = (a_{ij})_{1 \leq i, j \leq n}$, we define an isomorphism of the torus $\psi_A : \mathbb{C}^n \to \mathbb{C}^n$ by $w = \psi_A(z)$, $w_i = z_1^{a_{1i}} \cdots z_n^{a_{ni}}$, $i = 1, \ldots, n$. To each $\sigma \in S_n$, we associate an affine coordinate chart $(\mathbb{C}_\sigma, \mathbf{u}_\sigma)$ with coordinates $\mathbf{u}_\sigma = (u_{\sigma 1}, \ldots, u_{\sigma n})$. The modification $\hat{\pi}$ is defined as follows. For each $\sigma \in S_n$, we associate a birational mapping $\hat{\pi}_\sigma = \psi_\sigma : \mathbb{C}_\sigma^* \to \mathbb{C}^n$ by $z_i = u_{\sigma i 1}^{p_{i1}} \cdots u_{\sigma in}^{p_{in}}$ for $i = 1, \ldots, n$ under the identification of $\sigma$ and the above unimodular matrix. Then a complex manifold $X$ is constructed by gluing $\mathbb{C}_\sigma^*$ and $\mathbb{C}_\tau^*$ by $\hat{\pi}_\tau^{-1} \circ \hat{\pi}_\sigma : \mathbb{C}_\sigma^* \to \mathbb{C}_\tau^*$ where it is well-defined. This defines a proper modification $\hat{\pi} : X \to \mathbb{C}^n$ (for further detail, see Theorem (1.4), Chapter II,[18]).

2.5.1. Exceptional divisors corresponding to vertices of $\Sigma^*$. Suppose $\sigma = \text{Cone}(P_1, \ldots, P_n)$ and $\tau = \text{Cone}(Q_1, \ldots, Q_n)$ have a same vertex $Q_i = P_i$ for some $i$. Taking a permutation of the vertices, we may always assume that $i = 1$. The hyperplane $\{u_{\sigma 1} = 0\} \subset \mathbb{C}_\sigma^*$ glues canonically with the hyperplane $\{u_{\tau 1} = 0\} \subset \mathbb{C}_\tau^*$. Thus any vertex $P$ of $\Sigma^*$, gluing the hyperplanes on

\[2\text{A vertex is a primitive integral vector which generate a one-dimensional cone of } \Sigma^*.\]
every such toric coordinates with \( P_1 = P \), defines a rational divisor in \( X \), and we denote this divisor by \( \tilde{E}(P) \). We say vertices \( Q_i, i = 1, \ldots, r \) are \textit{adjacent} if \( \text{Cone}(Q_1, \ldots, Q_r) \) is a cone in \( \Sigma^* \). By the assumption that \( \Sigma^* \) is admissible, \( \cap_{i=1}^r \tilde{E}(Q_i) \) is non-empty if and only if \( Q_1, \ldots, Q_r \) are adjacent (Proposition (1.3.2), Chapter II, [LS]). If \( P \) is positive, \( \tilde{E}(P) \) is a compact divisor and \( \pi(\tilde{E}(P)) = \{O\} \). Let \( \mathcal{V}^+ \) be the set of non-negative vertices \( P \) of \( \Sigma^* \) with \( d(P) > 0 \) and \( P \neq e_1, \ldots, e_n \). A germ of a function \( f(z) \in \mathcal{O}_0 \) is called monomial-factor free if the factorization of \( f \) does not have any monomial factor. For a monomial-factor free \( f \), \( \Sigma^* \) is called \textit{small} if \( \mathbb{C}^I \) is a not vanishing coordinate subspace of \( f \), then \( e_I := \text{Cone}\{e_j | j \in I^c \} \) is a cone in \( \Sigma^* \) where \( I^c = \{1 \leq j \leq n | j \notin I \} \). In the case \( f \) being not monomial-factor free, write \( f = Mf' \) where \( M \) is a monomial and \( f' \) is monomial-factor free, and \( \Sigma^* \) is small for \( f \) if it is small for \( f' \). Here we note that \( \Gamma^*(f) = \Gamma^*(f') \). Note that if \( f \) is pseudo-convenient, in a small regular simplicial cone subdivision \( \Sigma^* \), every vertices are positive except for the canonical generators \( \{e_1, \ldots, e_n \} \) of the lattice \( \mathbb{Z}^n \) where \( e_i = t(0, 1, \ldots, 0) \). If \( \Sigma^* \) is \textit{small}, the associated modification \( \hat{\pi} : X \to \mathbb{C}^n \) is called a \textit{small} toric modification hereafter. If \( \hat{\pi} \) is small and \( f \) is pseudo-convenient, \( \hat{E}(e_i)^* \) is surjectively mapped onto \( \{z_i = 0, z_j \neq 0, j \neq i \} \). See \textsection 2.5.3 for the definition of \( \hat{E}(e_i)^* \). In this paper, we consider functions which are either convenient or at most the pseudo convenient and we assume that \( \Sigma^* \) is small. Thus any vertex \( P \in \mathcal{V}^+ \) is, in fact, positive and \( \hat{\pi}^{-1}(O) = \bigcup_{P \in \mathcal{V}^+} \tilde{E}(P) \).

2.5.2. \textit{Pull-back of } \( f \). We consider hypersurface \( V := f^{-1}(0) \) and take a sufficiently small neighborhood \( \mathcal{U}_0 \) where \( f \) is defined. In the following, we restrict \( \hat{\pi} \) over \( \hat{\pi}^{-1}(\mathcal{U}_0) \), whenever we consider the strict transform of \( V \) in the upper space \( X \). The pull-back \( \hat{\pi}^* f \) of \( f \) is expressed in the toric chart \( \mathbb{C}^n_\sigma \) with \( \sigma = \text{Cone}(P_1, \ldots, P_n) \) as follows:

\[
\hat{\pi}^* f(u_\sigma) = \left( \prod_{i=1}^n u_{\sigma,i}^{d(P_i)} \right) \tilde{f}(u_\sigma)
\]

and \( \tilde{f}(u_\sigma) \) is the defining function of the strict transform \( \tilde{V} \) of \( V \). The intersection \( E(P) := \tilde{V} \cap \tilde{E}(P) \) is a divisor in \( \tilde{E}(P) \) and it is defined by \( g(u_{\sigma_2}, \ldots, u_{\sigma_n}) := \tilde{f}(0, u_{\sigma_2}, \ldots, u_{\sigma_n}) = 0 \). (Recall we have assumed \( P = P_1 \).) More explicitly, we have

\[
g(u_{\sigma_2}, \ldots, u_{\sigma_n}) = f_P(\hat{\pi}_\sigma(u_\sigma))/\left( \prod_{i=1}^n u_{\sigma,i}^{d(P_i)} \right)
\]

where \( f_P \) is the face function of \( f \) with respect to \( P = P_1 \). \( E(P) \subset \tilde{V} \) is an exceptional divisor of the restriction \( \pi := \hat{\pi}|_{\tilde{V}} : \tilde{V} \to V \) and \( E(P) \) is non-empty if and only if \( \dim \Delta(P) \geq 1 \).
2.5.3. Toric stratification. Let $P$ be a vertex of $\Sigma^*$ and let $\mathcal{C}(P)$ be the set of cones $\tau = \text{Cone} (P_1, \ldots, P_k) \in \Sigma^*$ with $P_1 = P$. Choose a maximal cone $\sigma = \text{Cone} (P_1, \ldots, P_n)$ which has $\tau$ as a boundary face. Put

$$\hat{E}(\tau)^* := \cap_{i=1}^k \hat{E}(P_i) \cup \bigcup_{j=k+1}^n \hat{E}(P_j)$$

$$= \{ u_\sigma \in \mathbb{C}^n_\sigma \mid u_{\sigma 1} = \cdots = u_{\sigma k} = 0, \ u_{\sigma j} \neq 0, \ j > k + 1 \}$$

$$\cong \mathbb{C}^{*(n-k)}.$$

$\hat{E}(\tau)^*$ does not depend on the choice of $\sigma$ and $\hat{E}(\tau)^*$ is isomorphic to the torus $\mathbb{C}^{*(n-k)}$. Now we see that $\hat{E}(P)$ has a disjoint partition $\Pi_{\tau \in \mathcal{C}(P)} \hat{E}(\tau)^*$ which we call the toric stratification of $\hat{E}(P)$. In particular, we put $\hat{E}(P)^* = \hat{E}(P) \setminus \bigcup_{Q \in \mathcal{V}^+, Q \neq P} \hat{E}(Q)$. This is the maximal dimensional torus in $\hat{E}(P)$. Let $I \subset \{1, \ldots, n\}$ and put $e_i$ the cone generated by $\{e_i \mid i \in I\}$. Let $\hat{E}(P)_I^* := \hat{E}(P) \cap \hat{E}(e_i)^*$, which is empty if vertices $\{P, e_j \mid j \in I^c\}$ are not adjacent and put $\hat{E}(P)'_I = E(P) \cap \hat{E}(P)_I^*$. Here $I^c = \{1, \ldots, n\} \setminus I$. Then we use the toric decomposition of $\hat{E}'(P) := \hat{E}(P) \setminus \bigvee_{Q \neq P} \hat{E}(Q)$ as

$$\hat{E}(P)' = \Pi_I \hat{E}(P)'_I,$$

$$\hat{E}(P)'_I = \hat{E}(P)_I^* \setminus E(P)^*_I.$$

Take a vertex $P \in \mathcal{V}^+$ and we consider the exceptional divisor $\hat{E}(P)$. We compute the Euler characteristic $\chi(\hat{E}(P)^*)$ in the A’Campo formula using the toric stratification. Let $\hat{E}(P)'_I = \hat{E}(P)^*_I \setminus E(P)^*_I$. Note that $\hat{E}(P)'_I$ is non-empty only if $\{P, e_i \mid i \notin I\}$ spans a simplicial cone in $\Sigma^*$. For $I = \{1, \ldots, n\}$, we omit the suffix $I$. By the additivity of Euler characteristics, we have

$$\hat{E}(P)' = \Pi_I (\hat{E}(P)_I^* \setminus E(P)^*_I) = \Pi_I \hat{E}(P)'_I,$$

$$\chi(\hat{E}(P)^*) = - \sum_I \chi(\hat{E}(P)'_I). \quad (2)$$

In the following argument, we use the additivity of Euler characteristics and the decomposition \cite{2} which is also valid for a function which has some Newton degenerate faces. So consider function $f$ which has some degenerate faces like almost Newton non-degenerate functions which we recall in §2.7. First we take an admissible toric modification $\hat{\pi} : X \to \mathbb{C}^n$. The strict transform $\hat{V}$ or $\hat{\pi}^* f^{-1}(0)$ has still some singularities. Take further modification $\omega : Y \to X$ so that $\Pi := \hat{\pi} \circ \omega : Y \to \mathbb{C}^n$ is a good resolution of $f$ when it is restricted over $\mathcal{U}_\emptyset$. Let $D_1, \ldots, D_s$ be the exceptional divisors by $\omega$, let $V$ be the strict transform of $V = V(f)$ to $Y$ and let $d_j$ be the multiplicity of $\Pi^* f$ along $D_j$. The A’Campo formula can be expressed as

$$(AC^*) \quad \zeta(t) = \zeta_\omega(t) \prod_{P \in \mathcal{V}^+} (1 - t^{d(P)})^{-\chi(\hat{E}(P)^*)}$$

$$= \zeta_\omega(t) \prod_{P \in \mathcal{V}^+} \prod_I (1 - t^{d(P)})^{\chi(\hat{E}(P)'_I)}.$$

where $\zeta_\omega(t) = \prod_{j=1}^s (1 - t^{d_j})^{-\chi(D_j)}$. See Lemma \cite{16} below for detail.
2.5.4. **Kouchnirenko formula.** Consider a polynomial \( h(\mathbf{y}) = \sum_{i=1}^{s} a_i \mathbf{y}^m \in \mathbb{C}[y_1, \ldots, y_m] \) of \( m \)-variables \( \mathbf{y} = (y_1, \ldots, y_m) \) where \( a_i \neq 0 \) for \( i = 1, \ldots, s \). The Newton diagram \( \Delta(h) \) of \( h \) is defined by the convex hull of \( \{v_i \mid i = 1, \ldots, s\} \). Note that \( \Delta(h) \) is a compact polyhedron. We say \( h \) is Newton non-degenerate if \( V(h) := h^{-1}(0) \cap \mathbb{C}^m \) has no singular point for every face \( \Delta \) of \( \Delta(h) \) (including \( \Delta(h) \)). A key observation for the calculation is:

**Lemma 4** (Kouchnirenko [7], Oka [19]). Assume that \( h(\mathbf{y}) \in \mathbb{C}[y_1, \ldots, y_m] \) is a Newton non-degenerate polynomial and let \( V(h)^* = \{ \mathbf{y} \in \mathbb{C}^m \mid h(\mathbf{y}) = 0 \} \). Then the Euler characteristic is given as

\[
\chi(V(h)^*) = (-1)^{m-1} m! \text{Vol}_m \Delta(h).
\]

In particular, if \( \dim \Delta(h) < m \), \( \chi(V(h)^*) = 0 \). We use also the following vanishing property.

**Proposition 5.** Assume that \( h(\mathbf{y}) \) is an arbitrary (not necessary Newton non-degenerate) polynomial such that \( \dim \Delta(h) < m \). Then \( \chi(V(h)^*) = 0 \).

**Proof.** Let \( r = \dim \Delta(h) \). We can take a unimodular matrix \( A \) so that after change of variables by \( \mathbf{y} = \pi_A(\mathbf{x}) \), we can write \( h(\pi_A(\mathbf{x})) = M h'(x_1, \ldots, x_r) \) where \( M \) is a monomial of \( x_1, \ldots, x_m \) and \( h' \) is a polynomial of \( r \)-variables \( x_1, \ldots, x_r \). As \( V(h')^* \) is a product \( (V(h')^* \cap \mathbb{C}^r) \times \mathbb{C}^{(m-r)} \), the Euler characteristic is zero. As \( V(h')^* \) is isomorphic to \( V(h)^* \), the assertion follows from this expression. \( \square \)

2.6. **Varchenko formula.** Suppose that \( f(\mathbf{z}) \) is Newton non-degenerate. For each non-vanishing coordinate subspace \( \mathbb{C}^I, I \subset \{1, \ldots, n\} \), let \( \mathcal{P}_I \) be the set of primitive integer weight vectors of \( f^I \) which correspond to the maximal dimensional faces of \( \Gamma(f^I) \). Using a toric modification \( \hat{\pi} : X \rightarrow \mathbb{C}^n \) which is admissible with the dual Newton diagram \( \Gamma^*(f) \), the integer \( \chi(E(P)^*) \) can be computed combinatorially. Namely the zeta function is given as

**Lemma 6** (Varchenko [24]). Suppose that \( f(\mathbf{z}) \) is Newton non-degenerate. Then

\[
\zeta(t) = \prod_{I} \zeta_I(t), \quad \zeta_I(t) = \prod_{Q \in \mathcal{P}_I} (1 - t^{d(Q;f^I)}) \chi(E(Q)^*)
\]

where \( f^I \) is the restriction of \( f \) to the coordinate subspace \( \mathbb{C}^I \).

The number \( \chi(E(Q)^*) \) satisfies the following equality, if \( f \) is Newton non-degenerate.

**Proposition 7.** Suppose that \( f(\mathbf{z}) \) is Newton non-degenerate. Then the above integer satisfies the equality:

\[
\chi(E(Q)^*) = (-1)^{|I|} |I|! \text{Vol}_|I| \text{Cone}(\Delta(Q;f^I))/d(Q;f^I).
\]

**Remark 8.** The vertices in \( \mathcal{P}_I \) are used to compute the zeta function but the vertices of \( \mathcal{P}_I \) are not in \( \mathbb{V}^+ \). Thus they do not appear in A’Campo formula. The correspondence of A’Campo formula and the Varchenko formula is explained by Lemma 4 and the following observation: for any \( Q \in \mathcal{P}_I \),
there is a unique vertex \( P \in \mathcal{V}^+ \) such that \( P \) and \( \{e_i, i \notin I\} \) span a simplex of \( \Sigma^* \) and \( \Delta(P; f) \cap \mathbb{R}^I = \Delta(Q; f^I) \) and \( d(P) = d(Q) \). See §5, Chapter III, [13] for further discussion.

2.7. Almost non-degenerate functions. Consider a convenient analytic function \( f(z) = \sum a_i z^i \) which is expanded in a Taylor series and let \( \Gamma(f) \) be the Newton boundary. Let \( \pi : X \to \mathbb{C}^n \) be a toric modification with respect to \( \Sigma^* \) which is a small simplicial regular subdivision \( \Sigma^* \) of the dual Newton diagram \( \Gamma^*(f) \). Let \( \mathcal{M} \) be the set of maximal dimensional faces of \( \Gamma(f) \) and let \( \mathcal{M}_0 \) be a given subset of \( \mathcal{M} \) so that for \( \Delta \in \mathcal{M}_0 \), \( f_\Delta : \mathbb{C}^n \to \mathbb{C} \) is Newton degenerate. For \( \Delta \in \mathcal{M}_0 \), let \( P_\Delta \) be the unique vertex which corresponds to \( \Delta: \Delta(P_\Delta) = \Delta \). Recall that \( E(P) \) is an exceptional divisor which corresponds to the vertex \( P \). We consider the following conditions on \( f \).

(A1) For any face \( \Delta \) of \( \Gamma(f) \) with either \( \Delta \in \mathcal{M} \setminus \mathcal{M}_0 \) or \( \dim \Delta \leq n - 2 \), \( f \) is Newton non-degenerate on \( \Delta \). For \( \Delta \in \mathcal{M}_0 \), \( f_\Delta : \mathbb{C}^n \to \mathbb{C} \) has a finite number of 1-dimensional critical loci which are \( \Sigma^* \)-orbits through the origin. Recall that \( f_\Delta(z) \) is a weighted homogeneous polynomial with respect to the weight vector \( P_\Delta = (p_1, \ldots, p_n) \) and there is an associated \( \mathbb{C}^* \)-action defined by \( t \circ (z_1, \ldots, z_n) = (t^{p_1} z_1, \ldots, t^{p_n} z_n) \), \( t \in \mathbb{C}^* \).

Let \( \sigma = \text{Cone}(P_1, \ldots, P_n) \) be a simplicial cone in \( \Sigma^* \) such that \( P_1 = P_\Delta \). Let \( u_\sigma = (u_{\sigma 1}, \ldots, u_{\sigma n}) \) be the corresponding toric coordinate chart. The strict transform \( \tilde{V} \) of \( V(f) \) is defined by \( \tilde{f}(u_\sigma) = 0 \) where \( \tilde{f} \) is defined by the equality:

\[
\tilde{f}(u_\sigma) := \hat{\pi}^* f(u_\sigma) = \left( \prod_{i=1}^n u_{\sigma i}^{d(P_1)} \right) \tilde{f}(u_\sigma)
\]

and \( E(P_1) \subset \hat{E}_0 \) is defined by \( \{u_\sigma | u_{\sigma 1} = 0, g_\Delta(u_{\sigma 2}, \ldots, u_{\sigma n}) = 0\} \) where

\[
g_\Delta(u_{\sigma 2}, \ldots, u_{\sigma n}) := \tilde{f}(0, u_{\sigma 2}, \ldots, u_{\sigma n}) = f_\Delta(\pi_\sigma(u))/\prod_{i=1}^n u_{\sigma i}^{d(P_1)}.
\]

For simplicity, we denote \( \hat{\pi}^* f \) by \( \tilde{f} \) hereafter. The assumption (A1) implies that \( E(P_1) \) as a divisor of \( \hat{E}(P_1) \) has a finite number of isolated singular points. In fact, this follows from the isomorphism:

\[
\hat{\pi}_\sigma: V(g_\Delta) \cap \mathbb{C}^n = \mathbb{C}^* \times (V(g_\Delta|_{u_{\sigma 1} = 0}) \cap \mathbb{C}^*^{n-1}) \to V(f_\Delta) \cap \mathbb{C}^n.
\]

Let \( S(\Delta) \) be the set of the singular points of \( E(P_1) \). Take any \( q \in S(\Delta) \). An admissible coordinate chart at \( q \) is an analytic coordinate chart \((U_q, w)\), \( w = (w_1, \ldots, w_n) \) centered at \( q \) where \( U_q \) is an open neighborhood of \( q \) and \((w_2, \ldots, w_n) \) is an analytic coordinate change of \( (u_{\sigma 2}, \ldots, u_{\sigma n}) \) and \( w_1 = u_{\sigma 1} \).

We say that \( f \) is a weakly almost non-degenerate function if it satisfies (A1). As a second condition, we consider
For any \( \Delta \in \mathcal{M}_0 \) and \( q \in S(\Delta) \), there exists an admissible coordinate \((U_q, w)\) centered at \( q \) such that \( \pi^* f(w) \) is Newton non-degenerate and pseudo-convenient with respect to this coordinates \((U_q, w)\).

We say that \( f \) is an almost non-degenerate function if it satisfies \((A1)\) and \((A2)\). For a weakly almost non-degenerate functions, the following theorem holds.

**Theorem 9** \([16]\). Assume that \( f \) is a weakly almost non-degenerate function. Then the zeta function of \( f \) is given by

\[
\zeta(t) = \zeta(t)^{\prime} \prod_{\Delta \in \mathcal{M}_0} \zeta_\Delta(t)
\]

where \( \zeta(t)^{\prime} \) is the zeta function of \( \hat{f} \) outside of the union of \( \varepsilon \) balls centered at \( q \in S(\Delta) \), \( \Delta \in \mathcal{M}_0 \) and \( \zeta(t) = \zeta^{(s)}(t) \zeta^{\epsilon r}(t) \) where \( \zeta^{(s)}(t) \) is the zeta function of the Newton non-degenerate function \( f_s \) with \( \Gamma(f_s) = \Gamma(f) \) and

\[
\zeta^{\epsilon r}(t) = \prod_{\Delta \in \mathcal{M}_0} (1 - t^{d(\Delta)})(-1)^{n-1}\mu_{\Delta}
\]

where \( \mu_{\Delta} \) is the sum of Milnor numbers \( \mu(g_{\Delta}; q) \) for all \( q \in S(\Delta) \). \( \zeta_\Delta(t) \) is the product of the zeta function of \( \hat{f} \) at \( q \in S(\Delta) \).

If \( f \) is almost Newton non-degenerate (so it satisfies \((A2)\)), \( \zeta_\Delta(t) \) can be combinatorially computed by Varchenko formula.

**Remark 10.** In \([16]\), we have assumed the condition \((A2)\) and \( \hat{f} \) is pseudo-convenient at \( q \) but these assumptions are not necessary. If \((A2)\) condition is not satisfied, the assertion is still true but to compute the zeta function \( \zeta_\Delta(t) \), we need an explicit resolution \( \omega : \hat{Y} \to X \) of \( \hat{f} \) and then use the formula of A’Campo instead of Varchenko’s formula.

### 2.8. Zeta multiplicity

By A’Campo formula, the zeta function \( \zeta(t)^{\prime} \) of a germ of analytic function \( f \) is written as \( \prod_{i=1}^s (1 - t^{d_i})^{\nu_i} \) with mutually distinct \( d_1, \ldots, d_s \) and non-zero integers \( \nu_1, \ldots, \nu_s \). Thus we can write \( \text{div}(\zeta) = \sum_{i=1}^s \nu_i \Lambda_{d_i} \). The zeta multiplicity of \( f \) is defined as \( d_{\min} := \min \{ d_i \mid i = 1, \ldots, s \} \) and we denote it as \( m_\zeta(f) \). Suppose \( d_{\min} = d_i \), \( 1 \leq i \leq s \). We call the factor \((1 - t^{d_i})^{\nu_i} \) the zeta multiplicity factor. In general, \( m_\zeta(f) \geq m(f) \) where \( m(f) \) is the multiplicity of \( f \), the lowest degree of the Taylor expansion of \( f \) at \( 0 \). This follows from the following observation.

**Proposition 11.** Assume that \( \hat{\pi} : X \to U_0 \) is a good resolution of an analytic function \( f(z) \) of multiplicity \( m \) and put \( \hat{\pi}^{-1}(V) = \hat{V} \cup_{i=1}^s E_i \) where \( \hat{V} \) is the strict transform of \( V = f^{-1}(0) \). Let \( m_i \) be the multiplicity of \( \hat{\pi}^* f \) along \( E_i \). Then \( m \geq m_i \).

### 2.9. Lé-Ramunujam result for zeta-functions

Consider a piecewise analytic family \( f_s(z) \), \( 0 \leq s \leq 1 \) of functions with isolated singularity at the origin and suppose that the Milnor number \( \mu(f_s) \) of \( f_s \) at \( 0 \in \mathbb{C}^n \) is constant for \( s \). ( \( f_t \) can be piecewise \( C^\infty \).) Then
Lemma 12. The zeta function $\zeta_{f_s}(t)$ of $f_s$ is independent of $s$ and coincides with $\zeta_{f_0}(t)$.

Proof. For $n \neq 3$, the assertion follows from the result of Lê-Ramanujam [12]. For $n = 3$, we consider the family $g_s(x, y, z, w) = f_s(x, y, z) + w^m$. Consider the reduced zeta function $\tilde{\zeta}(t) := \zeta(t)(1 - t)$. For an isolated singularity case, $(-1)^n\tilde{\zeta}(t)$ is equal to the divisor of the characteristic polynomial of the monodromy automorphism $h_s : H_{n-1}(F) \to H_{n-1}(F)$ where $F$ is the Milnor fiber. For a fixed $s$, assume that $\text{div}(\tilde{\zeta}_{f_s}) = \sum_{j=1}^d \mu_j \Lambda_{e_j}$. By Join theorem ([22, 21]), we have the equality $\text{div}(\tilde{\zeta}_{f_s}) = \text{div}(\tilde{\zeta}_{f_0})(-\Lambda_m + 1)$. Taking $m$ to be mutually prime for each $e_j$, we have

$$(J) \quad \text{div}(\tilde{\zeta}_{f_s}) = -\sum_{j=1}^k \mu_j \Lambda_{e_{m,j}} + \sum_{j=1}^k \mu_j \Lambda_{e_j}$$

Note that this divisor does not depend on the parameter $s$ by Lê-Ramanujam ([12]). Assume that $\text{div}(\tilde{\zeta}_{f_0}) = \sum_{j=1}^{k_0} \nu_j \Lambda_{d_j}$. We assume $d_1 > d_2 > \cdots > d_{k_0}$ and $e_1 > \cdots > e_k$ and $m > 1$. By the above equality, we get the equality

$$-\sum_{j=1}^{k_0} \nu_j \Lambda_{d_{m,j}} + \sum_{j=1}^{k_0} \nu_j \Lambda_{d_j} = -\sum_{j=1}^k \mu_j \Lambda_{e_{m,j}} + \sum_{j=1}^k \mu_j \Lambda_{e_j}$$

for any $m$ which is coprime to any $\{d_1, \ldots, d_{k_0}, e_1, \ldots, e_k\}$. We see that $d_1 = e_1$, $\nu_1 = \mu_1$. By an inductive argument, we conclude that $k_0 = k$ and $d_j = e_j$, $\nu_j = \mu_j$ for $j = 1, \ldots, k$. \hfill \Box

3. $\mu$-Zariski pairs

3.1. Zariski pairs and weak Zariski pairs. A pair of projective curves $\{C, C'\}$ in $\mathbb{P}^2$ is called a Zariski pair if they have the same degree and there is a bijective correspondence $\phi : S(C) \to S(C')$ where $S(C)$ and $S(C')$ are the sets of the singular points of $C$ and $C'$ respectively and the local topological type of the singularities of $(C, p)$ and $(C', \phi(p))$ is the same for any $p \in S(C)$ and $\phi$ extends to a homeomorphism $\phi : N(C) \to N(C')$ of a tubular neighborhood $N(C)$ of $C$ to a tubular neighborhood $U(C')$ of $C'$ but this does not extend to a homeomorphism of the ambient spaces $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ for any $\phi$.

We say $\{C, C'\}$ is a weak Zariski pair if they have the same degree and there is a bijective correspondence $\phi : S(C) \to S(C')$ of the singular points of $C$ and $C'$ and the local topological type of the singularities of $(C, p)$ and $(C', \phi(p))$ is the same for any $p \in S(C)$. However there does not exist any homeomorphism $\tilde{\phi} : N(C) \to N(C')$ of their tubular neighborhoods which extends $\phi$. This implies in particular that the pairs $(\mathbb{P}^2, C)$ and $(\mathbb{P}^2, C')$ are not homeomorphic.
3.2. Zariski pairs of hypersurfaces. Assume that we have a pair of hypersurfaces $V(f) = \{ f(z) = 0 \}$ and $V(g) = \{ g(z) = 0 \}$ with isolated singularity at the origin. We say $\{ V(f), V(g) \}$ is a $\mu$-Zariski pair of hypersurfaces (respectively $\mu^*$-Zariski pair of hypersurfaces) if they have a same Milnor number (respectively a same $\mu^*$-invariant) and a same zeta function of the Milnor fibrations but they belong to different connected components of $\mu$-constant strata (resp. of $\mu^*$-constant strata). For the definition of $\mu$-constant strata and $\mu^*$-constant strata, see [4]. They are defined as semi-algebraic sets.

There is a canonical way to produce possible $\mu$-Zariski pairs of surfaces ($n = 3$). Consider a Zariski pair (respectively a weak Zariski pair) of projective curves $C, C'$ of degree $d$ defined by convenient homogeneous polynomials $f_d(x, y, z)$ and $g_d(x, y, z)$. We assume that the singular points of $C$ and $C'$ are Newton non-degenerate with respect to some local coordinates. We assume that $f$ and $g$ are non-degenerate on any face $\Delta$ of their Newton boundary with $\dim \Delta \leq 1$. Consider the affine surfaces defined by $f(x, y, z) = f_d(x, y, z) + z^{d+m}$ and $g(x, y, z) = g_d(x, y, z) + z^{d+m}$. Then $f$ and $g$ are almost non-degenerate functions with isolated singularities at the origin and their zeta functions and Milnor numbers are same. We call $\{ f, g \}$ a Zariski pair (resp. a weak Zariski pair) of surfaces ([16]). In [16], we studied a Zariski pair of surface with $m = 1$ whose links are diffeomorphic. In our paper [4] in preparation, we have shown that the pair $\{ f, g \}$ defined as above starting from a Zariski pair $\{ f_d, g_d \}$ of projective curves is a $\mu^*$-Zariski pair of surfaces. Hereafter in this paper we consider mainly a pair of surfaces constructed from a weak Zariski pairs of curves.

3.2.1. Examples of weak Zariski pairs of surfaces. We consider two weak Zariski pairs of quartics in $\mathbb{P}^2$. Recall that a pair of projective curves $\{ C_1, C_2 \}$ of the same degree is a weak Zariski pair if there is a bijection $\psi : S(C_1) \rightarrow S(C_2)$ of the respective singular points so that the topological singularity type $(C_1, q)$ and $(C_2, \psi(q))$ are equivalent for any $q \in S(C_1)$ but this homeomorphism does not extend to a homeomorphism of any tubular neighborhoods $N(C_1)$ and $N(C_2)$ of $C_1$ and $C_2$ respectively. Examples of weak Zariski pairs which we consider in this paper are:

(a1) $(Q_1, Q_2)$ where $Q_1$ is an irreducible quartic with 3 nodes i.e., 3 $A_1$ singularities. $Q_2$ is union of a smooth cubic and a generic line.

(a2) $(Q_3, Q_4)$ where $Q_3$ is a union of a cubic with one node and a generic line and $Q_4$ is a union of two conics which intersects transversely at 4 points.
More explicitly, as \( Q_1 \) and \( Q_2 \), we can take (see [17]):

\[
Q_1 : \quad q_1(x, y, z) = 0,
q_1 = (x^4 + y^4 + z^4) - 124(xy^2z + x^2yz + x^2yz) + 6(x^2z^2 + y^2z^2 + x^2y^2)
- 4(x^3y + xy^3 + x^3z + x^3z + y^3z + y^3z)
\]

\[
Q_2 : \quad q_2 = 0,
q_2(x, y) = (x^3 + y^3 + z^3)(x + ay + bz), \ a, b \in \mathbb{C}^* : \text{generic}
\]

and \( Q_3 = C_3 \cup L \) and \( C_3 \cap L \) where \( C_3 \) is a cubic with one node. For example, as \( C_3 \) with one node at \((1,1,1)\) we can take

\[
C_3 : \quad c_3^{(1)}(x, y, z) = -(x + y + z)^2 + 27xyz
\]

and adding a generic line component, we get such a quartic \( Q_3 \). For example, we take \( q_3(x, y, z) = c_3^{(1)}(x, y, z) \times (x + 2y + 3z) = 0 \). It has 4 nodes, three of which come from the intersection of \( C_3 \) and the line component. As \( q_4 \) we can take for example, \( q_4 = (x^2 + y^2 + z^2)(x^2 + 2y^2 + 3z^2) \). As affine hypersurface, each quartic \( q_i(x, y, z) = 0 \) has three (or four) singular lines through the origin for \( i = 1, 2 \) (respectively for \( i = 3, 4 \)). In the following, the precise forms of \( q_1, \ldots, q_4 \) are not important. They make no problem for the discussion below.

3.2.2. Isolation of the singularities. We consider the following polynomials which is associated with \( q_i, i = 1, \ldots, 4 \):

\[
f_i(x, y, z) = q_i(x, y, z) + z^{4+m}, \quad i = 1, \ldots, 4
\]

where \( m \) is a fixed positive integer. \( f_1, \ldots, f_4 \) are almost non-degenerate functions and their Milnor numbers are given by \( 27 + 3m \) for \( f_1, f_2 \) and \( 27 + 4m \) for \( f_3, f_4 \). Consider the corresponding hypersurface \( V_i = \{ f_i(x, y, z) = 0 \}, i = 1, \ldots, 4 \). For the calculation of the zeta function, we follow the procedure of Theorem [9]. We first take an ordinary blowing up which is the simplest toric modification \( \hat{\pi} : X \to \mathbb{C}^m \) with one positive weight vector \( P = t(1,1,1) \). Take the toric chart \( \text{Cone}(e_1, e_2, P) \). The exceptional divisor \( E(P) = \hat{E}(P) \cap \hat{V}_i \) contains 3 nodes \( \rho_1, \rho_2, \rho_3 \) for \( V_1, V_2 \) and 4 nodes \( \rho_1, \ldots, \rho_4 \) for \( V_3, V_4 \). Taking the toric coordinate \((u_1, u_2, u_3)\), we have

\[
\hat{f}_i(u) := \hat{\pi}^*f_i(u_1, u_2, u_3) = u_3^{4} \{ q_i(u_1, u_2, 1) + u_3^{m} \}
\]

Recall \((x, y, z) = (u_1 u_3, u_2 u_3, u_3)\). Consider the behavior at a node \( \rho_\alpha \).

Taking admissible coordinates \((v_1, v_2, v_3)\) with \( v_3 = u_3 \) in a neighborhood of \( \rho_\alpha \) so that

\[
\hat{q}_{i, \alpha}(v) = v_3^{4}(v_1^{2} + v_2^{2}),
\]

\[
\hat{f}_{i, \alpha}(v) = v_3^{4}(v_1^{2} + v_2^{2} + v_3^{m}).
\]

Thus the zeta function \( \zeta_{\hat{f}_i, \alpha}(t) \) at \( \rho_\alpha \) is given as \( \zeta_{\hat{f}_i, \alpha}(t) = (1 - t^{m + 4})^{-1} \) by Lemma [6]. The geometry at other singular point \( \rho_\beta \) is exactly same with
that of $\rho_\alpha$. Thus using Theorem [9] combining the zeta functions at $\rho_\alpha$, we get

\begin{align}
\zeta_{f_i}(t) &= (1 - t^4)^{-4}(1 - t^{4+m})^{-3}, \quad i = 1, 2 \\
\zeta_{f_i}(t) &= (1 - t^4)^{-3}(1 - t^{4+m})^{-4}, \quad i = 3, 4.
\end{align}

Note that the generic plane sections of $f_1, \ldots, f_4$ have non-degenerate convenient degree 4 components. Therefore the $\mu^*$-invariant of $f_i$ is given as

\begin{equation}
\mu^*(f_i) = \begin{cases}
(27 + 3m, 9, 3), & i = 1, 2 \\
(27 + 4m, 9, 3), & i = 3, 4.
\end{cases}
\end{equation}

We will show that $\{f_1, f_2\}$ and $\{f_3, f_4\}$ are $\mu$-Zariski pairs in the following sections.

3.3. **Main theorem.** We consider the isolation pairs $\{f_1, f_2\}$ and $\{f_3, f_4\}$ of the weak Zariski pairs $\{q_1, q_2\}$ and $\{q_3, q_4\}$ introduced in §3.2.2.

3.3.1. **Non-existence of $\mu^*$-constant path.** First we assert

**Lemma 13.** There are no piecewise analytic $\mu^*$-constant path from $f_1$ to $f_2$ (respectively from $f_3$ to $f_4$).

**Proof.** The assertion is proved in [4] for a pair constructed from Zariski pair and the proof for our case is similar. We give a brief proof for the reader’s convenience. We prove the assertion simultaneously for two pairs. Suppose we have a piecewise analytic family of functions $h_s(x, y, z)$, $0 \leq s \leq 1$ so that $h_0 = f_1$ and $h_1 = f_1$ (respectively $h_0 = f_3$ and $h_1 = f_3$) and $\mu^*$ invariants of $h_s$ are constant. As it is $\mu^*$-constant family, the multiplicity of $h_t$ is 4 for any $s$. Let $h_s = h_{s4} + h_{s5} + \ldots$ be the graduation by the degree. If $h_{s0,4} = 0$ has an non-isolated singularity in $\mathbb{P}^2$ for some $s_0$, the generic plane section $h_{s0} \cap L$ has Milnor number greater than 9 as the tangent cone is a quartic with singularities. Here $L$ is a generic plane through the origin. This contradicts to the $\mu^*$-constancy. Thus $h_{s4} = 0$ has only isolated singularities. Secondly the family of quartics $h_{s4}(x, y, z) = 0$ has the same total Milnor number. In fact, the zeta function of $h_s$, $\zeta_{h_s}(t)$ is constant for $s$ by Lemma [12] and also its zeta multiplicity factor is also constant. This is given by $(1 - t^4)^{-7+\mu_{tot}(s)}$. Here $\mu_{tot}(s)$ is the total Milnor number of the projective curve $D_s := \{h_{s4} = 0\} \subset \mathbb{P}^2$ which is the sum of Milnor numbers at the singular points. Thus the total Milnor numbers of $h_t$ is constant.

We use the following well-known property of the family of curves.

**Bifurcation of singularities.** Consider a continuous family of analytic function $f_t(x, y)$ of plane curves with isolated singularity at the origin. Then there exists a positive number $\varepsilon > 0$ so that for any $t \leq \varepsilon$, $\mu(f_t) \leq \mu(f_0)$ for $|t| \leq \varepsilon$. It is also well-known that if the singularity of $f_0$ at the origin bifurcate into some singularities for $f_t$, the sum of Milnor numbers on the same fiber $f_1 = 0$ is less than $\mu(f_0)$ by Lazzeri [9]. See also [11].
Thus combining the above two observation, the singularities of \{h_{s4} = 0\} has $3A_1$ (respectively $4A_1$) for any $s$ if $h_0 = f_1$ (resp. if $h_0 = f_3$). This implies that the pair $(P^2, D_s)$ is topologically isomorphic to $(P^2, D_0)$ by a result of Lê [10]. However this is a contradiction as $\{Q_1, Q_2\}$ (resp., $\{Q_3, Q_4\}$) is a weak Zariski pair and the pair $(P^2, Q_i), i = 1, 2$ (resp. the pair $(P^2, Q_j), j = 3, 4$) are not homeomorphic. Thus there are no such family of quartics from $Q_1$ to $Q_2$ (resp. from $Q_3$ to $Q_4$). This proves Lemma [13] □

3.3.2. \(\mu\)-Zariski pair. Now we state a stronger result.

**Theorem 14.** The pair \(\{f_1, f_2\}\) and \(\{f_3, f_4\}\) are \(\mu\)-Zariski pairs of hypersurfaces. Namely they belong to different connected components of \(\mu\)-constant strata.

4. **Proof of Theorem 14**

The proof occupies the rest of this section. Assume that we have a \(\mu\)-constant piecewise analytic family \(h_s(x, y, z), s \leq s \leq 1\) so that \(h_0 = f_1, h_1 = f_2\) (respectively \(h_0 = f_3, h_1 = f_4\)). We take an arbitrary \(0 < s < 1\). We will show that zeta function can not be the same as any of \(f_1\) or \(f_3\) if the multiplicity of \(h_s\) is smaller than 4. This part takes the most part of the proof. By Lemma [12] the zeta-function of \(h_s\) is the same as that of \(f_1\) or \(f_3\). We prove the assertion by contradiction.

The argument is to show that the zeta multiplicity factor of \(h_s\) can not be as \((1 - t^4)^{-4}\) for \(f_1\) or \((1 - t^4)^{-3}\) for \(f_3\) if multiplicity of \(h_s\) is less than 4. (There is one exceptional case with \(\zeta\) multiplicity factor is \((1 - t^4)^{-3}\) and the multiplicity is 3. See Lemma [18]) If the multiplicity of \(h_s\) is 4, the singularities of \(h_{s4} = 0\) must be \(3A_1\) (resp. \(4A_1\)). We first show that the multiplicity of \(h_s\) can not be 2 or 3 in (4.1) and (4.2).

4.1. **Case 1.** The multiplicity of \(h_s\) is 2. Assume \(h_s\) has the multiplicity 2 for some \(s\). Fixing \(s\) and apply the generalized Morse Lemma (see for example [2]). Choosing a suitable analytic coordinate \((w_1, w_2, w_3)\), we can write (a) \(h_s(w) = w_1^2 + w_2^2 + w_3^2, \nu \geq 3\) for corank 1 or (b) \(h_s(w) = w_1^2 + j(w)\) for corank 2 where the multiplicity of \(j\) is greater than 2. We show that this is impossible, under the assumption that the zeta function is given as [7] or [8].

For the case (a), it is clearly impossible, as \(\text{div} (\tilde{\zeta}_{h_s}) = \Lambda_\nu - 1\). Assume the case (b). Let \(\Xi_j\) be the divisor of the reduced zeta function of \(j(w)\). By the join theorem ([22], [21]), we need to have

\[
\text{div} (\tilde{\zeta}_{h_s}) = (-\Lambda_2 + 1)\Xi_j,
\]

\[
(-\Lambda_2 + 1)\Xi_j = \text{div} (\tilde{\zeta}_{h_0}) = \begin{cases} 
-4\Lambda_4 - 3\Lambda_{4+m} + 1, & \text{for } f_1, f_2 \\
-3\Lambda_4 - 4\Lambda_{4+m} + 1, & \text{for } f_3, f_4.
\end{cases}
\]

Put \(\Xi_j = \sum_{i=1}^{s} \nu_i \Lambda_{d_i}\) with \(d_1 < d_2 < \cdots < d_s\). First, to obtain 1 in \((-\Lambda_2 + 1)\Xi_j\), we must have \(d_1 = 1\) and \(\nu_1 = 1\). If \(d_2 > 2\), \((-\Lambda_2 + 1)\Xi_j\) gets
vertices of $\Sigma^*$

- $\Lambda_2$ in this summation. This is a contradiction to the above equality. So we need to have $d_2 = 2$ and $\nu_2 = -1$. This implies by Proposition 11 the multiplicity of $j$ is 2 which is also a contradiction to the assumption.

4.2. Case 2. Multiplicity of $h_s$ is 3. Now we show that the multiplicity of $h_s$ can not be 3. Assume that $h_s$ has multiplicity 3 for some $s$ and let $h_s = h_{s3} + h_{s4} + \ldots$ be the graduation by the degree. We consider the cubic curve $C = \{h_{s3} = 0\} \subset \mathbb{P}^2$. In the following, $s$ is fixed as above.

4.2.1. Strategy of the argument. Our argument proceeds as follows. First we take a suitable coordinates, say $(x, y, z)$, and consider the Newton boundary $\Gamma(h_s)$ of $h_s$ with respect to this coordinates. As $h_s = 0$ has an isolated singularity at the origin, we may assume that $h_s$ has a convenient Newton boundary by adding monomials $x^N, y^N, z^N$ where $N$ is a sufficiently large integer. Then we consider the dual Newton diagram $\Gamma^*(h_s)$ and take an admissible regular simplicial subdivision $\Sigma^*$ and consider the associated toric modification $\tilde{\pi}: X \to \mathbb{C}^3$. By the convenience, we may assume that the vertices of $\Sigma^*$ are positive except the canonical ones $\{e_1, \ldots, e_n\}$ and $\tilde{\pi}$ is a small toric modification. Let $\tilde{E}(P), P \in V^+$ be the compact exceptional divisors of $\tilde{\pi}$. The multiplicity of $\hat{h}_s := \tilde{\pi}^*h_s$ along $\tilde{E}(P)$ is $d(P; h_s)$. Let $\tilde{V}_s$ be the strict transform of $V(h_s)$ into $X$. If $\Delta(P)$ is a degenerate face of $\Gamma(h_s)$, $\tilde{V}_s$ can have singularities on $E(P)$. To get a good resolution of $h_s$, we need further blowing ups over singular points of $\tilde{V}_s$ and let $\omega: Y \to X$ is the composition of these blowing ups so that the composition

$$\Pi = \tilde{\pi} \circ \omega: Y \xrightarrow{\omega} X \xrightarrow{\tilde{\pi}} \mathbb{C}^3$$

is a good resolution of $h_s$ and let $D_1, \ldots, D_\ell$ be the exceptional divisors of $\omega$ and let $m_j$ be the multiplicity of $\Pi^*\hat{h}_s$ along $D_j$. Note that $m_j \geq 5$ by Proposition 11 if the multiplicity of the exceptional divisor $\tilde{E}(P)$ of the first modification $\tilde{\pi}: X \to \mathbb{C}^3$ with $\omega(D_j) \subset \tilde{E}(P)$ is at least 4, which implies the multiplicity of $\tilde{\pi}^*f$ is greater than or equal to 5 at a singular point of $V$. Let $V_Y$ be the strict transfrom of $V$ into $Y$ and $D'_j = D_j \setminus \left(\tilde{V}_Y \cup_{P \in V^+} \tilde{E}(P)_Y \cup_{k \neq j} D_j\right)$. We may assume that exceptional divisors are all compact so that its image of the exceptional divisors by $\Pi$ are over the origin. Then the exceptional divisors of $\Pi = \tilde{\pi} \circ \omega$ are $\{\tilde{E}(P)_Y, P \in V^+\} \cup \{D_1, \ldots, D_\ell\}$. Here $\tilde{E}(P)_Y$ is the pull back of $\tilde{E}(P) \subset X$ to $Y$. The contribution of the divisor $\tilde{E}(P)_Y$ in the A’Campo formula is $1 - t_d(P) - \chi(\tilde{E}(P)'_Y)$ where $\tilde{E}(P)'_Y = \tilde{E}(P)_Y \setminus \left(\tilde{V}_Y \cup_{Q \neq P} \tilde{E}(Q)_Y \cup_{k=0} D_k\right)$. Let $\tilde{E}(P)' = \tilde{E}(P) \setminus \left(\tilde{V} \cup_{Q \neq P} \tilde{E}(Q) \subset \tilde{E}(Q)\right)$. As $E(P)'$ is smooth and it does not contain any point of the center of the second blowing-up $\omega$, we have a canonical diffeomorphism $\omega: \tilde{E}(P)'_Y \cong \tilde{E}(P)'$. Thus

**Proposition 15.** We have the equality $\chi(\tilde{E}(P)'_Y) = \chi(\tilde{E}(P)').
Now combining A’Campo formula and the argument of Varchenko formula and Proposition \[5\] we have:

**Lemma 16.** The zeta function of \( f \) is given as

\[
\prod_{P \in \mathcal{V}^+} (1 - t^{d(P)})^{-\chi(\hat{E}(P)')} \times \prod_j (1 - t^{m_j})^{-\chi(D_j')},
\]

The first factor can be written using toric stratification as

\[
\prod_{P \in \mathcal{V}^+} (1 - t^{d(P)})^{-\chi(\hat{E}(P)')} = \prod_I \zeta_I(t)
\]

where

\[
\zeta_I(t) = \prod_{Q \in \mathcal{P}_I} (1 - t^{d(Q)})^{-\chi(\hat{E}(Q)')}
\]

The set \( \mathcal{P}_I \) is the set of weight vectors which correspond to the maximal dimensional faces of \( \Gamma(f') \). If \( f'_P \) is a degenerate face, \( \chi(\hat{E}(P)) \) can not be expressed combinatorially as in the formula \[7\]. The following Lemma is useful to prove Theorem \[14\] Using Lemma \[3\] we have:

**Lemma 17.** Let \( P \) be a positive vertex of \( \Sigma^* \). If \( \hat{E}(P)' \) has non-zero Euler characteristic, there are three possibilities.

1. \( \dim \Delta(P) = 2 \), or
2. \( \dim \Delta(P) = 1 \) and \( P \) is adjacent to one of \( e_1, e_2, e_3 \), or
3. \( \dim \Delta(P) = 0 \) and \( P \) is adjacent to two of \( e_1, e_2, e_3 \).

**Proof.** Assume that \( \dim \Delta(P) = 1 \). Take a toric coordinate chart \( \sigma = \text{Cone}(P, P_2, P_3) \). If \( P \) is not adjacent to any of \( e_1, e_2, e_3 \), \( \hat{E}(P)' = \hat{E}(P; \sigma)^* \backslash E(P; \sigma)^* \) for any toric chart \( \sigma = \text{Cone}(P, P_2, P_3) \) where

\[
\hat{E}(P; \sigma)^* := \{ u_\sigma \in \mathbb{C}^3_\sigma \mid u_{\sigma_1} = 0, u_{\sigma_2}, u_{\sigma_3} \neq 0 \}
\]

\[
E(P; \sigma)^* := \{ (0, u_{\sigma_2}, u_{\sigma_3}) \mid g(u_{\sigma_2}, u_{\sigma_3}) = 0 \}
\]

and \( g \) is the defining polynomial of \( E(P) \) in \( \hat{E}(P) = \{ u_{\sigma_1} = 0 \} \). By the assumption, the Newton polygon of \( g \) is 1-dimensional. Thus by Lemma \[5\] \( \chi(\hat{E}(P)') = -\chi(E(P; \sigma)^*) = 0 \). If \( P \) is adjacent to \( e_1 \), \( \Delta(P) \subset \mathbb{R}^{2,3} \).

The proof of assertion (3) is similar. In this case, \( \hat{E}(\sigma)^* \) is a point (=0-dimensional torus) for \( \sigma \) which is generated by \( P \) and two of \( e_1, \ldots, e_3 \). For example, if \( e_1, e_2 \) is adjacent to \( P \), \( h_3P(x, y, z) = cz^a \) for some \( a > 0 \) and \( c \in \mathbb{C}^* \) and \( P \) take the form \( P = t^1(a, b, 1), a, b > 0 \). In this case, this vertex contributes the zeta function by \( (1 - t^a)^{-1} \). \[\square\]

4.2.2. **Cancellation of Case 2.** Now we are ready to show the impossibility of the multiplicity \( m(h_3) = 3 \). We divide the situation by the geometry of the cubic curve \( C_3 := \{ h_3 = 0 \} \subset \mathbb{P}^2 \). For simplicity, we write hereafter \( h := h_3 \), \( h_3 = h_3 \) etc.

We divide the case 2 in three subcases.

2-1 \( C_3 : h_3 = 0 \) a union of 3 lines.

2-2 \( h_3 = 0 \) is a union of conic and a line.
2-3 \( h_3 = 0 \) is an irreducible cubic.

First we consider Case 2-1.

**Case 2-1.** \( C_3 : h_3 = 0 \) a union of 3 lines.

We further divide this case into four cases depending the geometry of the lines \( h_3 = 0 \):

(a) \( C_3 \) is a union of three lines which are generic in \( \mathbb{P}^2 \), or
(b) \( C_3 \) is a union of three lines which are intersecting at one point in \( \mathbb{P}^2 \).
(c) \( C_3 \) is union of two lines where one line has multiplicity 2.
(d) \( C_3 \) is one line with multiplicity 3.

Subcase 2-1-a. We first consider the case that \( h_3 = 0 \) is three generic lines.

Taking a new linear coordinate if necessary and putting this coordinates as \( (x, y, z) \), we may assume that \( h_3(x, y, z) = xyz \). Put \( A = (1, 1, 1) \in \Gamma(h) \).

Let \( P = \Gamma(a_1, a_2, a_3) \) be a positive vertex of \( \Sigma^* \) with \( d(P) \leq 4 \). If \( d(P) = 3 \), then \( P = \Gamma(1, 1, 1) \) and \( \Delta(P) = \{A\} \). This vertex does not give any factor in zeta function and we do not need to consider this case. Suppose \( d(P) = 4 \). Then \( \Delta(P) \) contains only degree 4 monomials and possibly \( xyz \).

Here we used the trivial inequality \( \deg x^ay^bz^c \geq a + b + c \). If \( E(P)^\ell \) has non-zero Euler characteristic, the possibility is (a) \( \dim \Delta(P) = 2 \), or (b) \( \dim \Delta(P) \geq 1 \) and \( P \) is adjacent to one of \( e_1, e_2, e_3 \), or (c) \( \dim \Delta(P) \geq 0 \) and \( P \) is adjacent to two of \( e_1, e_2, e_3 \). For (a) or (b), the possible weights are \( \Gamma(2, 1, 1), \Gamma(1, 2, 1), \Gamma(1, 1, 2) \). Thus we may assume, for example, that \( P = \Gamma(1, 1, 2) \) and any degree 4 monomial except \( xyz \) must be a monomial \( x^iy^4-i \) of degree 4 in \( x, y \).

Thus we assume that \( I = \{1, 2\} \) and suppose \( h^\ell(x, y) \) have 1-dimensional support. We first assume that \( \sigma = \mathrm{Cone}(P, e_2, e_3) \) is a simplex in \( \Sigma^* \), assuming \( x^4 \) is in \( h^\ell \). (Note \( \Delta(P) \cap \Delta(E_2) \cap \Delta(e_3) = \{(4,0,0)\} \) and we can consider \( \sigma \in \Sigma^* \).) As a unimodular matrix, \( \sigma \) takes the form:

\[
\sigma = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
2 & 0 & 1 
\end{pmatrix}.
\]

Then \( \hat{\pi}_\sigma(u_\sigma) = (u_{\sigma 1}, u_{\sigma 1}u_{\sigma 2}, u_{\sigma 1}u_{\sigma 3}) \) and

\[
\hat{h}(u_\sigma) \equiv \hat{h}_P(u_\sigma) \text{ modulo}(u_{\sigma 1}^5),
\]

\[
\hat{h}_P(u_\sigma) = u_{\sigma 1}^4 (h^\ell_1(1, u_{\sigma 2}) + u_{\sigma 2}u_{\sigma 3}).
\]

Let \( \nu_\ell \) be the number of non-zero distinct roots of \( h^\ell(1, u_{\sigma 2}) = 0 \) and let \( \delta_\ell \) be the number of monomials of \( \{x^4, y^4\} \) in \( h^\ell \). Then

\[
E(P)^* = \{(u_{\sigma 2}, u_{\sigma 3}) | h^\ell_1(1, u_{\sigma 2}) + u_{\sigma 2}u_{\sigma 3} = 0, u_{\sigma 2}, u_{\sigma 3} \neq 0\}
\]

and it is easy to see that \( E(P)^* \) is homeomorphic to \( \mathbb{C}^* \setminus \{\nu_\ell \text{ points}\} \) by the projection \( (u_{\sigma 2}, u_{\sigma 3}) \mapsto u_{\sigma 2} \). Thus \( \chi(E(P)^*) = -\nu_\ell \). On the other hand, \( E(P)^\ell_\ell = \{u_{\sigma 2} \in \mathbb{C}^* | h^\ell_2(1, u_{\sigma 2}) = 0\} \) and \( \chi(E(P)^\ell_\ell) = \nu_\ell \). Thus those two terms are cancelled out. Thus the contribution of the stratum \( E(P)^* \) and \( E(P)^\ell_\ell \) to the zeta multiplicity factor is \( (1-t^4)^{\delta_\ell} \) where \( \delta_\ell \) is the number of monomials in \( \{x^4, y^4\} \) which are in \( h^\ell_1 \) and \( \delta_\ell \in \{0,1,2\} \). If \( y^4 \)
appears in $h^I_4$ and $x^4$ does not appear in $h^I_4$, we do the same argument by
$\sigma' = \text{Cone}(P, e_1, e_3)$. If $x^4$ and $y^4$ are not in $h^I_4$ and assume that $h^I_4(x, y) =
y^a j_{4-\alpha}(x, y), 1 \leq \alpha \leq 2$ where $j_{4-\alpha}$ is a polynomial of degree $4 - \alpha$ with
$j_{4-\alpha}(x, 0) := c \neq 0$. Take a vector $Q = (a - 1, a, b)$ with $a$ sufficiently large
and $b \gg a$. We can see $h_Q = cy^a x^{4-\alpha}$ and $\Delta(P) \supset \Delta(e_3) \supset \Delta(Q)$. (Recall
Cone $(P, Q, R)$ is an admissible cone if $\Delta(P) \cap \Delta(Q) \cap \Delta(R) \neq \emptyset$.) This means

$$\tau = \text{Cone}(P, Q, e_3) \iff \tau = \begin{pmatrix} 1 & a - 1 & 0 \\ 1 & a & 0 \\ 2 & b & 1 \end{pmatrix}$$

is an admissible regular simplicial cone. We may assume that $\tau$ is a simplicial cone
of a regular simplicial cone subdivision $\Sigma^*$ of $\Gamma^*(h)$. However this choice
of $\tau$ and an explicit construction of $\Sigma^*$ is not necessary and this particular
choice of $\tau$ does not make any difference in the calculation of $\chi(E(P^*))$ which
is clear from the following calculation. In fact, in this coordinate chart, $\pi^* h$
is defined by

$$\tilde{h}(u_{r_1}) = u_{r_2}^4 u_{r_3}^{4a-4+\alpha} \left( j_{4-\alpha}(1, u_{r_2}) + u_{r_2}^{b-2a+3-\alpha} u_{r_3} \right) \mod (u_{r_1}^5)$$

and $E(P)$ is defined by

$$g(u_{r_2}, u_{r_3}) = j_{4-\alpha}(1, u_{r_2}) + u_{r_2}^{b-2a+3-\alpha} u_{r_3} = 0.$$ 

Thus we see $E(P)^*$ is isomorphic to $\mathbb{C}^* \setminus \{\nu_i \text{ points}\}$ (isomorphism is given
by the projection $(u_{r_2}, u_{r_3}) \mapsto u_{r_2}$) and $E(P)^*_\nu$ is $\nu_i$ points
which are roots of $j_{4-\alpha}(1, u_{r_2}) = 0$. Note that $\nu_i$ does not depend
on the choice of $b \gg a \gg 1$.

We do the same discussion for $J = \{2, 3\}$ and $K = \{1, 3\}$ and
we conclude the zeta-multiplicity factor is given as $(1 - t^4)^{-\delta}$ where $\delta$
is the number of monomials in $\{x^4, y^4, z^4\}$ in $h_4$. Thus if $\delta < 3$, the zeta
multiplicity factor is $(1 - t^4)^{-\delta}$ and it can not be same with that of $f_i, i = 1, 3, (1 - t^4)^{-4+\epsilon}, \epsilon = 0, 1$.
The case $\delta = 3$ is a bit different, as the zeta-multiplicity factor
coincides with that of $f_3, f_4$. In this case, $h_4$ contains three monomials $x^4, y^4, z^4$ and
$h_4$ is convenient. We assert

**Lemma 18.** Assume that $h_3 = xyz$ and $h_4$ is convenient. Then the zeta
function of $h$ is given as $(1 - t^4)^{-3}$ and the Milnor number
is 11.

Thus assuming this lemma, $h$ can not have the same zeta-function as $f_1$
or $f_3$ which are $(1 - t^4)^{-4+\epsilon}(1 - t^{4+m})^{-3-\epsilon}, \epsilon = 0, 1$.

**Proof.** We choose another linear coordinate $(x', y', z')$ so that $x = \ell_1 :=
(a_1 x' + a_2 y' + a_3 z'), y = \ell_2 := (b_1 x' + b_2 y' + b_3 z'), z = \ell_3 := (c_1 x' + c_2 y' + c_3 z')$
where $a_i, b_i, c_i \neq 0, i = 1, 2, 3$ are generic non-zero complex numbers and they
satisfy $h_4(a_i, b_i, c_i) \neq 0$ for $i = 1, 2, 3$. In this coordinate, we have $h_3 = \ell_1 \ell_2 \ell_3$ and
consider the homogeneous polynomial $H_3(x', y', z') := h_3(\ell_1, \ell_2, \ell_3)$. The
intersection points in $\mathbb{P}^2$ of three lines $x = 0, y = 0, z = 0$ are $\rho_1 = (1, 0, 0),
\rho_2 = (0, 1, 0)$ and $\rho_3 = (0, 0, 1)$. In the new coordinates $(x', y', z')$, we have:
Assertion 19. In the coordinates \((x', y', z')\), \(\rho_i, i = 1, 2, 3\) are not on the coordinate lines \(\{x'y'z' = 0\}\) and \(H_3(x', y', z')\) is convenient.

Proof. By solving explicitly respective linear equations \(\ell_1 - 1 = \ell_2 = \ell_3 = 0, \ell_1 = \ell_2 - 1 = \ell_3 = 0\) and \(\ell_1 = \ell_2 = \ell_3 - 1 = 0\) in \((x', y', z')\), we can easily see that \(\rho_1, \rho_2, \rho_3\) are outside of the lines \(x'y'z' = 0\) as long as \(a_i, b_i, c_i, i = 1, 2, 3\) are generically chosen. (To see the intersection of \(\ell_2 = \ell_3 = 0\) in \((x', y', z')\) coordinates of \(\mathbb{P}^2\), we may assume that \(\ell_1 = 1\) on that point.) As \(H_3(0, 0, 1) = h_3(a_3, b_3, c_3) \neq 0\). Similarly \(H_3(1, 0, 0)\) and \(H_3(0, 1, 0) \neq 0\). This means \(H_3\) is a convenient polynomial.

Put \(\alpha_1, \alpha_2, \alpha_3\) be the coefficients of \(x'^4, y'^4, z'^4\) in \(H_4(x', y', z') := h_4(\ell_1, \ell_2, \ell_3)\). Now we consider the toric modification \(\hat{\pi} : X \to \mathbb{C}^3\) with respect to \(\Sigma^*\) with vertices \(\{e_1, e_2, e_3, P\}\) where \(P = t^i(1, 1, 1)\) and the coordinates are \((x', y', z')\). The exceptional divisor \(E(P)\) has three \(A_1\) singularities at \(\rho_i, i = 1, 2, 3\). Take the toric chart \(\sigma = \text{Cone}(P, e_2, e_3)\). Let \(w = (w_1, w_2, w_3)\), \(w_1 = u_{\sigma_1}\) be an admissible coordinate at \(\rho_i\) so that

\[
\hat{H} = H(u_{\sigma_1}, u_{\sigma_2}, u_{\sigma_3}) \\
= u_{\sigma_1}^3(H_3(1, u_{\sigma_2}, u_{\sigma_3}) + \alpha_1 u_{\sigma_1} + R) \\
= u_{\sigma_1}^3(w_2^2 + w_3^2 + \alpha_1 w_1 + R)
\]

where \(R \in (w_1^2)\). Note that zeta function of \(w_1^3(w_2^2 + w_3^2 + \alpha_1 w_1 + R)\) is determined by \(w_1^3(w_2^2 + w_3^2 + \alpha_1 w_1)\) whose zeta function is \((1 - t^4)^{-1}\). Thus \(H\) is an almost non-degenerate function in the coordinates \((x', y', z')\). This function is the same as the function considered in Example 2, §3.4 [16]. We apply Theorem [5, 16] to get \(\zeta_H(t) = (1 - t^4)^{-3}\) which proves the assertion. \(\square\)

Subcase 2-1-b. Suppose \(h_3 = 0\) is a 3 lines which intersect at a point in \(\mathbb{P}^2\). We may assume that \(h_3 = xy(x + ay), a \neq 0\) (after a linear change of coordinate). Take a toric modification with an admissible regular simplicial subdivision \(\Sigma^*\). It has a vertex \(P = t^i(1, 1, \alpha)\), \(\alpha \geq 1\) in \(\Sigma^*\) which is adjacent to \(e_3\) and \(h_P = h_3\). This vertex gives \(d(P) = 3\) and it gives factor \((1 - t^3)\) in the zeta function by A’Campo formula. (In the Varchenko formula, this corresponds to \(\zeta_i(t)\) with \(I = \{1, 2\}\).) We can see no other vertices \(\Sigma^*\) contribute the factor \((1 - t^3)\). After further blowing ups, no exceptional divisors appears with multiplicity 3. This is a contradiction to the assumption.

Subcase 2-1-c. \(h_3 = 0\) are two lines where one line is doubled. Then we may assume that \(h_3 = x^2y\). If \(\Gamma(h)\) has a face of dimension 2 of degree 4, the only possibility is \(h_4(x, 0, z)\) has 1 dimensional support and with \(x^2y\), it generate a face of dimension 2 with weight vector \(P = t^i(1, 2, 1)\). 1-dimensional faces can be on \(h_4(x, 0, z)\) and \(h_4(0, y, z)\). We consider again degree 4 component \(h_4\). Let \(I = \{1, 3\}\). If \(h_4^I\) is 0 or a single monomial \(x^a z^{4-a}\) with \(1 \leq a \leq 3\), there are no possible degree 4 face of dimension 2. If \(h_4^I(x, z)\) is not a monomial, assuming \(x^4\) appears in \(I\), we consider \(\sigma = \text{Cone}(P, e_2, e_3)\) where \(P = t^i(1, 2, 1)\). Let \(\nu_I\) be the number of non-zero distinct roots of
If \( f_4'(x, z) = 0 \) as before. Then \( x = u_{31}, y = u_{21}^2 u_{32}, z = u_{31} u_{32} \) and \( E(P)^* \) is defined by \( u_{32} + f_4'(1, u_{33}) = 0 \). Thus \( \chi(\tilde{E}(P)^*) = -\nu_I \) and \( \chi(E(P)^*) = \nu_I \). If \( z^4 \) appears in \( h_4^I \), we consider in the chart \( \text{Cone}(P, e_1, e_3) \). If neither \( x^4 \) nor \( z^4 \) appears in \( h_4^I \), we take an admissible simplical cone \( \tau = \text{Cone}(P, Q, e_2) \) where \( Q = t(a - 1, b, a) \) with \( a \gg 1 \) and \( b \gg a \) and the similar argument works as in Case 2-1-2. Anyway the contribution from \( h_4 \) is cancelled with contribution from \( E(P)^* \). However for \( J = \{2, 3\} \), \( h^J(y, z) \) contribute to the zeta function by \( (1 - t^4)^\nu_J \). On \( \{x, y\} \) planes, there are no degree 4 edges.

Let \( \delta \) be the number of monomials among \( \{x^4, y^4, z^4\} \) which appears in \( h_4(x, y, z) \). Each monomial contribute by the factor \( (1 - t^4)^{-1} \) in the zeta function of \( h \) and altogether we get \( (1 - t^4)^{-\delta} \). Altogether the contribution to the zeta function on the factor is \( (1 - t^4)^{-\delta + \nu_J} \). As \(-\delta + \nu_J > -3\), we get a contradiction to the assumption.

Subcase 2-1-d. \( h_3 = 0 \) is a line with multiplicity 3. We assume that \( h_3 = x^3 \). It is easy to see that the only possible vertex \( P \) with \( d(P) = 4 \) which satisfies one of the conditions in Lemma 17 is \( P := t(1, 1, 1) \), \( a > 1 \) and \( h_3(x, y, z) = h_4(0, y, z) \). Let \( \delta \) be the number of monomials \( \{y^4, z^4\} \) in \( h_4(0, y, z) \) and let \( \nu \) be the number of non-zero roots of \( h_4(0, 1, z) = 0 \). Then the contribution to zeta-multiplicity factor is \( (1 - t^4)^{\nu - \delta} \). As \(-1 \leq \nu - \delta \leq 2\), this is a contradiction to the assumption. Alternatively we can also have a contradiction by showing that \( x^3 \) gives a factor \( (1 - t^3)^{-1} \).

Subcase 2-2. Suppose \( h_3 = 0 \) is a union of a conic \( C \) and a line \( L \). Subcase 2-2-1. The conic \( C \) and the line \( L \) are transversal in \( \mathbb{P}^2 \). We may assume that \( h_3 \) is convenient and two \( A_1 \) is not on \( xyz = 0 \). After one toric blowing up with vertices \( \{e_1, e_2, e_3, P\} \) with \( P := t(1, 1, 1) \), we see that the exceptional divisor \( E(P) \subset \tilde{E}(P) \) has 2\( A_1 \) singularities at the intersection of the conic and the line component and we see that the divisor \( \tilde{E}(P) \) gives zeta-multiplicity factor \( (1 - t^3)^{-1} \). This is a contradiction to the assumption.

Subcase 2-2-2. The conic \( C \) and the line \( L \) are tangent in \( \mathbb{P}^2 \). Put \( p = C \cap L \). Then the singularity of \( (C \cup L, p) \) is isomorphic to \( A_3 \). Using the same toric modification, we may assume that \( E(P) \) has one \( A_3 \) singularity. This divisor does not gives any zeta factor as we see below. Note that the zeta function of a non-singular cubic is \( (1 - t^3)^{-3} \) and by Theorem 14 in the zeta function of \( h \), the exponent of zeta factor becomes \(-3 + \mu(A_3) = 0\). Consider the toric chart \( \sigma = \text{Cone}(e_1, e_2, P) \). Then the pull-back of \( h \) is written as

\[
\hat{h}(u_\sigma) = u_{33}^2 (h_3(u_{31}, u_{32}, 1) + u_{33} R(u_\sigma))
\]

where \( u_{33} R \) is coming from higher terms of \( h \). Taking an admissible coordinates \( (w_1, w_2, w_3) \), \( w_3 = u_{33} \) at the singular point \( \rho = (\alpha_1, \alpha_2, 0) \), we can write

\[
\hat{h}(w) = w_3^3 (w_1^2 + w_2^4 + w_3 R(w))
\]

If the multiplicity of \( w_3 R(w) \) is greater than or equal to 2, the multiplicity of \( \hat{h} \) at the singular point is greater than or equal to 5 and no factor \( (1 - t^4) \)
appears in the zeta function. Only possible case is when \( w_3 R(w) \equiv aw_3, a \neq 0 \) modulo higher terms. Then \( \hat{h}(w) \) is non-degenerate and by Theorem 9, \( \zeta_h(t) = (1 - t^{16})^{-1}(1 - t^8)(1 - t^4)^{-1} \). (For example, we can take as \( h, (x^2 + y^2 - 2z^2)(x + y - 2z) + z^4 \).) This is also contradiction to the assumption.

Subcase 2-3. Suppose that \( h_3 = 0 \) is an irreducible cubic. We can choose a generic linear coordinates so that \( h_3(x, y, z) \) is convenient and a possible singularity is one \( A_1 \) or one \( A_2 \). By Theorem 9, the zeta-multiplicity factor is one of \((1 - t^3)^{-3}, (1 - t^3)^{-2}, (1 - t^3)^{-1}\) according to the cubic is non-singular, or one \( A_1 \) or one \( A_2 \). This is also a contradiction.

4.3. Case 3. Multiplicity 4. We consider the last case. Assume that \( h = h_4 \) has multiplicity 4. We divide this case in two cases by the geometry of the curve \( h_4 = 0 \).

(3-1) \( h_4 = 0 \) has non-isolated singularity for some \( s \).

(3-2) \( h_4 = 0 \) has only isolated singularity for any \( s \).

We will show that the only possible case is (3.2).

Subcase (3-1). Suppose it has non isolated singularity for some \( s \). Then the possibility of \( h_4 = 0 \) are
- (a) one line with multiplicity 4, or
- (b) two lines where one line has multiplicity 3, or
- (c) two lines of multiplicity 2, or
- (d) one line with multiplicity 2 and two other lines, or
- (e) one line with multiplicity 2 and an irreducible conic, or
- (f) one conic with multiplicity 2.

Subcase (3-1-a) We assume the case (a). Choose a linear coordinates so that \( h_4 = x^4 \). Then it is easy to see that there are no faces of dimension 2 or 1 with degree 4. The only possible effective vertex with multiplicity 4 is of the form \( P = t(1, a, b) \) corresponding to the monomial \( x^4 \) and the contribution is \((1 - t^4)^{-1}\). This is a contradiction to the assumption.

Subcase (3-1-b) Consider the case (b). We assume that \( h_4 = x^3y \). In this case, it is impossible to have an effective exceptional divisor of multiplicity 4. See Lemma 17.

Subcase (3-1-c) Assume that \( h_4 = x^2y^2 \). The same reason as the case (b).

Subcase (3-1-d) Assume that \( h_4 = 0 \) has three lines \( L_1, L_2, L_3 \) where \( L_1 \) has multiplicity 2. After one point blowing-up \( \tilde{\pi} : X \to \mathbb{C}^3, E(P) \cong \mathbb{P}^2 \) and \( E(P) \) is a union of three lines \( L_1, L_2, L_3 \) where \( L_1 \) has multiplicity 2. We can assume that \( h_4 = xyz^2 \) or \( x^2y(x + ay), a \neq 0 \). In the case \( h_4 = xyz^2 \), there are no possibility of vertex \( P \) with \( d(P) = 4 \) and which contribute to the zeta function. Assume that \( h_4 = x^2y(x + ay) \), the only possible vertex take the form \( P = t(1, 1, b), b \geq 1 \) which is adjacent to \( e_3 \) and \( h_P = h_4 \).

If this is the case, its contribution is \((1 - t^4)\). This is also a contradiction to the assumption. (This case, we can also do the same discussion as Case (3-1-c) below.)
Subcase (3-1-e) We can assume that $h_4(x, y) = x^2j_2(x, y, z)$ with $j_2$ is a smooth conic. We take a toric modification $\hat{\pi} : X \to C^3$ which respect to $\Sigma^* = \{e_1, e_2, e_3, P\}$, $P = t(1, 1, 1)$. $E(P)$ is a union of smooth conic $C$ and a line of multiplicity 2. Using the toric chart $\sigma := \text{Cone}(e_1, e_2, P)$, the pull-back of $h$ takes the form

$$h(u_\sigma) = h_4(u_\sigma) + h_\ell(u_\sigma) + (\text{higher degree terms}), \, \ell \geq 5$$

$$\hat{h}(u_{\sigma 1}, u_{\sigma 2}, u_{\sigma 3}) = u_{\sigma 3}^2 \left( u_{\sigma 1}^2 j_2(u_{\sigma 1}, u_{\sigma 2}, 1) + u_{\sigma 3}^{\ell-4} h_\ell(u_{\sigma 1}, u_{\sigma 2}, 1) + \ldots \right)$$

$C \cap L$ is either 2 points or one point. Put them $\rho_1$ and $\rho_2$ be the intersection of $C$ and $L$ in $E(P)$. In the latter case, $L$ is tangent to $C$ and $\rho_2 = \rho_1$. On $L$ and $C$, there are finite points such that the function $\hat{h}$ is not equi-singular along $L$ or $C$. These exceptional points on $L$ are $\rho_1, \rho_2$ and the points in the intersection $u_{\sigma 1} = h_\ell(u_{\sigma 1}, u_{\sigma 2}, 1) = 0$ on $L$ and $j_2(u_{\sigma 1}, u_{\sigma 2}, 1) = h_\ell(u_{\sigma 1}, u_{\sigma 2}, 1) = 0$ on $C$ respectively. Put them $\rho_3, \ldots, \rho_k$. Take a small $\varepsilon$ ball $B_\varepsilon(\rho_1)$ centered at $\rho_1$ and put $B = \bigcup_{i=1}^k B_\varepsilon(\rho_i)$. Let $N(\tilde{E}(P))$, $N(L)$ and $N(C)$ be the sufficiently small controlled tubular neighborhoods of $\tilde{E}(P) \setminus (L \cup C)$, $L \setminus (L \cap B)$ and $C \setminus (C \cap B)$ respectively. Put $N = N(L) \cup N(C)$. Let $N(\tilde{E}(P))' = N(\tilde{E}(P)) \setminus (B \cup N)$. We divide the Milnor fibration of $\hat{h}$ into fibrations on $N(\tilde{E}(P))', N(L), N(C)$ and $B_\varepsilon(\rho_i), i = 1, \ldots, k$ and carry out the exact same argument as that in [16]. Note that $\chi(N(\tilde{E}(P))') = 1$ or 0, according $\rho_1 \neq \rho_2$ or $\rho_1 = \rho_2$ and $\chi(L \cup C) = 2$ or 3. The zeta function of $\hat{h}|_{N(\tilde{E}(P))'}$ is $(1-t^4)^{-1}$ or $(1-t^4)^0 = 1$. The contribution of the zeta function $\hat{h}|_{N(L)}$ to the zeta-multiplicity factor $(1-t^4)$ is trivial as the normal zeta function is described by the $u_{\sigma 3}^4(u_{\sigma 1}^2 + u_{\sigma 3}^{\ell-4})$ (Sublemme 4.16). Similarly the restriction of the Milnor fibration $\hat{h}$ on $N(C)$ does not contribute to the zeta-multiplicity factor as the normal zeta function corresponds to $u_{\sigma 3}^4(j_2 + u_{\sigma 3}^{\ell-4})$ where $(u_{\sigma 3}, j_2)$, $j_2 = j_2(u_{\sigma 1}, u_{\sigma 2}, 1)$ is coordinates of the normal slice. That is, there is a local coordinates $(u_{\sigma 3}, j_2, \exists v_1)$ locally where $v_1$ is a local coordinate of $C$. To get the zeta function of the restriction of Milnor fibration on $B \setminus \hat{h}^{-1}(0)$, we have to take further resolution. However over $\rho_i$, we get exceptional divisors of multiplicity greater than or equal to 5, as the multiplicity of $\hat{h}$ at $\rho_i$ is greater than or equal to 5. Over $N(L)$ and $N(C)$, the multiplicity of $\hat{h}$ is 6 and 5 respectively. Combining these data, the possible zeta-multiplicity factor in $\zeta_\hat{h}(t)$ is either $(1-t^4)^{-1}$ or 1, a contradiction to the assumption.

Subcase (3-1-f) Assume that $h_4 = 0$ is non-reduced conic $C$ of multiplicity 2. We assume that $h_4 = j_2^2$ and $h = j_2^2 + h_\ell + (\text{higher terms})$ as above. After one point blowing-up $\hat{\pi} : X \to C^3$, the exceptional divisor is $\tilde{E}(P) = \mathbb{P}^2$ and $E(P)$ is a non-reduced conic. Use again the toric chart $\sigma := \text{Cone}(e_1, e_2, P)$ as above. Then

$$\hat{h}(u_\sigma) = u_{\sigma 3}^4 \left( j_2(u_{\sigma 1}, u_{\sigma 2}, 1)^2 + u_{\sigma 3}^{\ell-4} h_\ell(u_{\sigma 1}, u_{\sigma 2}, 1) + (\text{higher terms}) \right).$$
Let \( \rho_1, \ldots, \rho_k \) be the intersection of \( j_2(u_{\sigma_1}, u_{\sigma_2}, 1) = h_\ell(u_{\sigma_1}, u_{\sigma_2}, 1) = 0 \) and take a small disk \( B_\varepsilon(\rho_i) \) centered at \( \rho_i \) for each \( i = 1, \ldots, k \) and put \( B = \bigcup_{i=1}^k B_\varepsilon(\rho_i) \). Take a controlled tubular neighborhoods \( N(\hat{E}(P) \setminus C), N(C) \) of \( C = E(P) \setminus B \) and \( N(\hat{E}(P)) \) be a tubular neighborhood of \( E(P) \setminus (B \cup N(C)) \). We divide Milnor fibration into the following parts. The complement \( N(\hat{E}(P))' := N(\hat{E}(P)) \setminus (N(C) \cup B) \) and \( N(C) \setminus (N(C) \cap B) \) and \( B \). We do the same discussion as in Subcase (3-1-e). Thus \( N(\hat{E}(P))' \) contribute to the zeta function by \( (1 - t^4)^{-3+2} = (1 - t^4)^{-1} \). On \( N(C) \), the normal zeta function is described by \( u_{\rho_1}^3(j_2(u_{\sigma_1}, u_{\sigma_2}, 1)^2 + u_{\rho_3}^{-4}(u_{\sigma_1}, u_{\sigma_2}, 1) + \text{(higher terms)} \) which is equivalent to \( u_{\rho_3}^4(j_2^2 + u_{\rho_3}^{-4}) \) with \( (u_{\rho_3}, j_2) \) are coordinates of the normal slice. Thus it contribute for the factor \( (1 - t^4) \) trivially. On \( \rho_i, \hat{h} \) has multiplicity \( 7 \) and also this part also gives nothing for the zeta-multiplicity factor. Thus we conclude that the case \( (3-1-f) \) trivially and the proof of Theorem 9 is completed.

\textbf{Subcase 3-2 (Last case).} Assume that the family \( h_s = 0 \) has only isolated singularity and multiplicity is \( 4 \) and the total and local Milnor numbers of \( h_{s_4} \) must be constant for any \( 0 \leq s \leq 1 \) by Theorem \( [9] \). This implies \( h_s \) is a \( \mu^* \)-constant family from \( f_1 \) to \( f_2 \) or from \( f_3 \) to \( f_4 \). However by Lemma \( [13] \) this is impossible and the proof of Theorem \( [9] \) is completed.

5. \textbf{Geometric structure of the links } \( K_{f_1} \text{ and } K_{f_2} \)

In this section, we study further geometric structure of the link 3-manifolds \( K_{f_i}, i = 1, 2 \). As second result, we will show that they are not diffeomorphic, though their zeta functions are equal. Recall that

\[
\begin{align*}
    f_1(x, y, z) &= q_1(x, y, z) + z^{m+4}, \\
    f_2(x, y, z) &= q_2(x, y, z) + z^{m+4}.
\end{align*}
\]

where \( q_1 \) is a quadric with 3 \( A_1 \) singularities and \( q_2 \) is a union of a smooth cubic and a generic line. We assume that all of these polynomials are convenient. Let \( f(z) \) be one of \( f_1 \) or \( f_2 \). First we take a toric modification \( \pi : X \to \mathbb{C}^3 \) with respect to the vertices \( \{e_1, e_2, e_3, P\} \) with \( P = \{1, 1, 1\} \). Take the coordinate chart \( \xi = \text{Cone}(e_1, e_2, P) \) with coordinates \( u_\xi = \langle u_{\xi 1}, u_{\xi 2}, u_{\xi 3} \rangle \). The pull back of \( f \) is given as

\[
\begin{align*}
    \hat{f}_i(u_\xi) := \hat{\pi}^* f_i(u_\xi) &= u_{\xi 3}^4 \left( q_i(u_{\xi 1}, u_{\xi 2}, 1) + u_{\xi 3}^{m} \right), \quad i = 1, 2
\end{align*}
\]

as \( (x, y, z) = \langle u_{\xi 1} u_{\xi 2}, u_{\xi 2} u_{\xi 3}, u_{\xi 3} \rangle \).

Let \( \rho_\alpha, \alpha = 1, 2, 3 \) be the singular points of \( E(P) \). Take \( \langle v_{\alpha 1}, v_{\alpha 2}, v_{\alpha 3} \rangle \) admissible coordinates in the neighborhood \( U_\alpha \) centered at \( \rho_\alpha \) so that \( v_{\alpha 3} = u_{\xi 3} \) and

\[
\begin{align*}
    \hat{f}_i(v) &= u_3^4(v_{\alpha 1}^2 + v_{\alpha 2}^2 + v_{\alpha 3}^m).
\end{align*}
\]

To distinguish from the local coordinates, we write \( v_{\alpha 3} = u_{\xi 3} \) as \( u_{\xi 3} \). For \( f_2 \), we consider also another coordinates. Let \( q_{2, 3}, q_{2, 1} \) be the defining polynomial of the cubic and linear component of \( q_2 = 0 \). Thus \( q_2 = q_{2, 3} q_{2, 1} \). We
define \( w_1 := q_{2,3}(u_{\xi}, u_{\xi}, 1) \) and \( w_2 := q_{2,1}(u_{\xi}, u_{\xi}, 1) \). Note that 
\[
q_{2,3}(u_{\xi}u_{\xi}, u_{\xi}u_{\xi}, u_{\xi}) = u_{\xi}^3 q_{2,3}(u_{\xi}, u_{\xi}, 1) = u_{\xi}^3 w_1,
\]
\[
q_{2,1}(u_{\xi}u_{\xi}, u_{\xi}u_{\xi}, u_{\xi}) = u_{\xi} q_{2,1}(u_{\xi}, u_{\xi}, 1) = u_{\xi} w_2
\]
and \( q_{2,i}(u_{\xi}, u_{\xi}, 1) = 0 \) with \( i = 3, 1 \) are the defining polynomials of the strict transform of the cubic and the line component respectively. As the cubic and the line intersect transversely, \((w_1, w_2)\) is a local coordinate of \( E(P) \) and \((w_1, w_2, u_3)\) is a local coordinate of \( X \) in the neighborhood of \( \rho_\alpha, \alpha = 1, 2, 3 \) for \( f_2 \) so that the pull-back of \( \hat{f}_2(u_3) \) is now given as 
\[
\hat{f}_2(w_1, w_2, u_3) = u_3^{1/2}(w_1 w_2 + u_3^{1/2}).
\]

Thus the local coordinates \((v_\alpha, v_\alpha)\) for \( f_2 \) are chosen so that they satisfy
\[
(14) \quad v_\alpha + \sqrt{-1} v_\alpha = w_1, \quad v_\alpha - \sqrt{-1} v_\alpha = w_2.
\]

\( w_1, w_2 \) is also globally defined on the toric chart \( U_\xi \) (and also on \( X \) as a meromorphic function). As is obvious from the expression, \( \hat{f}_i \) is weighted homogeneous in \( v_\alpha \) and the dual Newton diagram \( \Gamma^*(\hat{f}_i; v_\alpha) \) at \( \rho_\alpha \) has only one positive vertex \( R_\alpha = \ell(m, m, 2) \) or \( S_{m_0} = \ell(m_0, m_0, 1) \) for \( m = 2m_0 \) odd or even \( (m = 2m_0) \) respectively. \( R_\alpha \) or \( S_{m_0} \) is the weight vector of \( \hat{f}_i(v_\alpha) \).

5.1. Regular simplicial subdivision \( \Sigma^*_\alpha \). Suppose \( m \) is an odd integer and put \( m = 2m_0 + 1 \). The regular simplicial cone subdivision \( \Sigma^*_\alpha \) is given as the left \( \Sigma^*_\alpha \) of Figure 1. Here \( R_\alpha = \ell(m, m, 2) \), \( T_\alpha = \ell(m + 1, m + 1, 1) \) for \( m = 2m_0 + 1 \) and \( m_0 \) vertices \( S_{0,1}, \ldots, S_{0,m_0} \) are added where \( S_{0,i} = \ell(i, i, 1), i = 0, \ldots, m_0 \). For an even \( m = 2m_0 \), we do not need \( T_\alpha \) and \( R_\alpha = \ell(m, m, 1) \). See the right subdivision \( \Sigma^*_\alpha \) of Figure 1. In the following, we consider the case \( m = 2m_0 + 1 \). The case \( m = 2m_0 \) is similar.

To see the manifold structure, we consider a toric modification \( \omega_\alpha : Y_\alpha \rightarrow X, \alpha = 1, 2, 3 \) with respect to \( \Sigma^*_\alpha \); \( \Sigma^*_\alpha \). (\( \Sigma^*_\alpha \); \( \Sigma^*_\alpha \) is the same for every \( \rho_\alpha \)). Three modification can be canonically glued together to get a final resolution \( \omega : Y \rightarrow X \) and by taking composition with \( \hat{x} : X \rightarrow \mathbb{C}^3 \), we get a good resolution of \( f_\alpha \) restricting \( \Pi : Y \rightarrow \mathbb{C}^3 \) to an open neighborhood \( U_0 \) of the origin.

The exceptional divisors of \( \Pi \) are all compact and they are \( E(P) \) (from \( \hat{x} \)) and \( E(R_\alpha), E(S_{\alpha,i}), i = 0, \ldots, m_0 \) from \( \omega_\alpha, \alpha = 1, 2, 3 \), and \( E(T_\alpha) \) for an odd \( m = 2m_0 + 1 \). (If \( m = 2m_0 \), the exceptional divisors are \( E(S_i), i = 0, \ldots, m_0 \).) Let \( \hat{V}_\beta \) be the strict transform of \( V_\beta = \hat{f}_\beta^{-1}(0) \) to \( Y \). Hereafter \( \alpha = 1, 2, 3 \) are the indices of the singular points and \( \beta = 1, 2 \) are the choice of functions. Recall that for a vertex \( K \in \Sigma^* \), the restriction of the exceptional divisors \( E(K) \) to \( \hat{V}_\beta \) are non-empty if the supporting face \( \Delta(K) \) has dimension greater than or equal to \( 1 \). Thus \( E(T_\alpha) \) is empty as \( \hat{f}_\beta T_\alpha = u_3^{m_0 + 4} \).

Remark 20. To be precise, the resolution space \( X \) and \( Y \) for \( f_1 \) and for \( f_2 \) are different complex spaces and it is better to be distinguished and to be written as \( \hat{x}_1 : X_1 \rightarrow \mathbb{C}^3, \hat{x}_2 : X_2 \rightarrow \mathbb{C}^3 \) and \( \Pi_1 : Y_1 \rightarrow \mathbb{C}^3 \) and \( \Pi_2 : Y_2 \rightarrow \mathbb{C}^3 \).
Figure 1. \( \Sigma_o^*: m = 2m_0 + 1 \): odd, \( \Sigma_e^*: m = 2m_0 \): even

Also the exceptional divisors \( S_{\alpha,i}, i = 1, \ldots, m_0 \) and \( R_\alpha \) are sitting in the different space \( Y_1 \) and \( Y_2 \). Thus it is more precise to write them as \( S_{\alpha,i,j} \) and \( R_{\alpha,j} \) for \( j = 1, 2 \). However except the central divisor \( P \), they are the same Riemann surfaces and the resolution graphs are isomorphic and the link 3-manifolds are determined by the dual resolution graphs. Therefore we ignore this too precise notations (with too many suffixes) and we use the same letter for the both cases, unless any confusion is likely.

We denote the strict transform of the exceptional divisor of \( \bar{\pi}_\beta : X_\beta \to \mathbb{C}^3 \), \( E(P) \subset X_\beta \), to \( Y \) by \( P_{\beta, \beta} = 1, 2 \) respectively, as they are topologically different.

**Proposition 21.** For \( f_1 \), \( P_1 \) is a smooth rational curve. For \( f_2 \), \( P_2 \) is a union of smooth cubic and a line \( P_2 = P_{2,3} + P_{2,1} \), where \( P_{2,3} \) and \( P_{2,1} \) are the strict transforms of the cubic and the line respectively.

**Proof.** By the discussion of Euler characteristics, we know that \( \chi(E(P)) = -4 + 3 = -1 \) for \( f_1 \). Recall \( E(P) \) is a quartic with \( 3A_1 \), \( \chi(E(P)) = -1 \). \( P_1 \) is the normalization of \( E(P) \) at three points \( \rho_\alpha, \alpha = 1, 2, 3 \). In \( Y_\alpha \), each singular point \( \rho_\alpha \) is separated in two points. Thus \( \chi(P_1) = -1 + 3 = 2 \). The second assertion is obvious from the assumption on \( q_2 \). \( \square \)

Note that \( P_{2,3} \cap P_{2,1} = \emptyset \) (after the modification \( \omega \)). By abuse of notation, we also denote the exceptional divisors \( E(S_{\alpha,i}) \) and \( E(R) \) by the same letter \( S_{\alpha,i} \) and \( R_\alpha \) for simplicity of the notations.
Proposition 22. (1) $S_{o_1}$ has two components which are $\mathbb{P}^1$ for $1 \leq i \leq m_0$ if $m = 2m_0 + 1$ and for $1 \leq i \leq m_0 - 1$ for $m = 2m_0$. We can call each component as $S^+_{o_1}$, $S^-_{o_1}$ so that $S^+_i, S^-_i = 1$ and $S^+_i, S^-_i = 0$.

(2) For $m = 2m_0 + 1$, $R_{o}$ is a rational sphere and $R_{o} \cdot S^+_{o,m_0} = 1$ and $S_{o,m_0}$ is a union of two rational spheres.

(3) $P_3 \cdot S_{o_1} = 2$. More precisely, $P_1 \cdot S^+_{o_1} = 1$ for $f_1$ and for $f_2$, $P_2 = P_{2,3} + P_{2,1}$ and we have that $P_{2,3} \cdot S^+_{o_1} = 1, 0$ and $S^+_{o,1} \cdot P_{2,1} = 0, 1$ respectively.

Proof. As we are working on $Y_{o}$ for $\alpha = 1, 2$ simultaneously, we skip the suffix $\alpha$. First consider the toric chart $\sigma_i := \text{Cone}(S_{i}, S_{i+1}, e_1)$ for $0 \leq i \leq m_0 - 1$. which corresponds to the unimodular matrix

$$
\begin{pmatrix}
  i & i + 1 & 1 \\
  i & i + 1 & 0 \\
  1 & 1 & 0
\end{pmatrix}
$$

We understand $S_0 = e_3$ in the above notation. Denote the coordinates of $\mathbb{C}^3_{\alpha,i}$ as $(u_{i,1}, u_{i,2}, u_{i,3})$ and

$$
v_{o_1} = u_{i,1} u_{i,2} u_{i,3}, \quad v_{o_2} = u_{i,1} u_{i,2}^2, \quad v_3 = u_{i,1} u_{i,2}
$$

and pull back of $\hat{f}$ by $w_0$ for $m = 2m_0 + 1$ is given as

$$
\omega^* \hat{f} = u^{2i+4}_{i,1} u^{2i+6}_{i,2} \bigg( u^2_{i,3} + 1 + u_{i,1}^{-m} u_{i,2}^{-2m-2i} \bigg), \quad i < m_0 - 1.
$$

The divisor $S_i$ and $S_{i+1}$ are defined in this chart by $u_{i,1} = 0$ and $u_{i,2} = 0$ respectively and their two components $S^+_i$, $S^-_{i+1}$ correspond to $u_{i,3} = \pm \sqrt{-1}$ respectively. In the chart $\sigma_{m_0} = \text{Cone}(S_{m_0}, R, e_1)$,

$$
\omega^* \hat{f} = u^{2m_0+4}_{m_0,1} u^{2m_0+8}_{m_0,2} \big\{ u^2_{m_0,3} + 1 + u_{m_0,1} \big\}.
$$

We can see $R$ is defined by $u_{m_0,2} = 0$ and $R \cdot S_{m_0} = 2$. This is described as $u_{m_0,1} = u_{m_0,3}^2 + 1 = 0$. For an even $m$ with $m = 2m_0$, $\omega^* \hat{f}$ is as above (15) for $i < m_0 - 1$ and for $i = m_0 - 1$, the above equation takes the form:

$$
\omega^* \hat{f} = u^{2m_0+2}_{m_0-1,1} u^{2m_0+4}_{m_0-1,2} \bigg( u^2_{m_0-1,3} + 1 + u^2_{m_0-1,1} \bigg) \quad \text{if} \quad m = 2m_0.
$$

As $E(S_{m_0}) = \{u_{m_0-2,1} = u_{m_0-1,1} = 0\}$ if $m = 2m_0$, we see that it is connected and a rational curve. Other part, the argument for $m$ odd or even is exactly the same, and we do the argument for $m = 2m_0 + 1$ hereafter.

In the chart $\mathbb{C}^2_{\alpha,i}, \hat{E}(S_{i})$ and $\hat{E}(S_{i+1})$ is defined by $u_{i,1} = 0$ and $u_{i,2} = 0$ respectively and two components are $u_{i,1} = u_{i,3} \pm \sqrt{-1} = 0$. We define $S^+_i = \{u_{i,1} = u_{i,3} + \sqrt{-1} = 0\}$ and $S^-_i = \{u_{i,1} = u_{i,3} - \sqrt{-1} = 0\}$. In the next chart $\sigma_{i+1} = \text{Cone}(S_{i+1}, S_{i+2}, e_1)$ with coordinate $(u_{i+1,1}, u_{i+1,2}, u_{i+1,3})$, they are related by

$$
u_{i+1,3} = u_{i,3}, \quad u_{i,1} = u_{i+1,1}^{-1}, \quad u_{i,2} = u_{i+1,1} u_{i+1,2}.$$
Thus we see $S_i^+ \cdot S_{i+1}^+ = 1$. For $f_2$, in the chart $\sigma_0$, we have
\[ w_1 = v_1 + \sqrt{-1}v_2 = u_{0,2}(u_{0,3} + \sqrt{-1}), \quad w_2 = v_1 - \sqrt{-1}v_2 = u_{0,2}(u_{0,3} - \sqrt{-1}) \]
where $\sigma_0 = \text{Cone}(e_3, S_1, e_1)$ and $v_1 = u_{0,2}u_{0,3}, v_2 = u_{0,2}, u_3 = u_{0,1}u_{0,2}$.
(Recall $w_1 = 0$ is the defining function of $P_{2,3}$ and $w_2 = 0$ defines $P_{2,1}$.)
This implies $S_0^+ = P_{2,3}$ and $S_0^- = P_{2,1}$ as is desired. Now we consider the last chart $\tau = \text{Cone}(S_{m_0}, R, e_1)$ with coordinates $(u_{r_1}, u_{r_2}, u_{r_3})$ (here $m = 2m_0 + 1$) and
\[
\omega^* \hat{f} = u_{r_1}u_{r_2}^2(u_{r_3}^2 + 1 + u_{r_1})
\]
We see $\chi(E(R)^*) = -2$ and $\chi(E(R) \cap E(S_{m_0}^\pm)) = 1$ and $\chi(E(R) \cap E(e_1)) = \chi(\hat{E}(R) \cap \hat{E}(e_2)) = 1$. Thus $\chi(E(R)) = 2$. \hfill \Box

5.2. Resolution graph of $V_1, V_2$. Now we come to the crucial part. Let $\Gamma_\beta$ be the dual resolution graph of $\Pi_{\tilde{V}_\beta} \tilde{V}_\beta \to V_\beta, \beta = 1, 2$. Here $V_\beta = V(f_\beta), \beta = 1, 2$.
First we consider $V_1$. $\Gamma_1$ has $6m_0 + 4$ vertices corresponding to $P_1, S_{\alpha_1}^+, \ldots, S_{\alpha_m_0}^+, R_\alpha, \alpha = 1, 2, 3$ and $\Gamma_1$ is three cycle graph centered at $P_1$. For each $\alpha$, one cycle centered at $P_1$ is this:

$P_1 \rightarrow S_{\alpha_1}^+ \rightarrow \ldots \rightarrow S_{\alpha_m_0}^+ \rightarrow R_\alpha \rightarrow S_{\alpha_m_0}^- \rightarrow \ldots \rightarrow S_{\alpha_1}^- \rightarrow P_1$

Figure 2 show the graph $\Gamma_1$ for $m = 3$.
Now we consider the case $f_2$. The central divisor $E(P)$ split into two vertices corresponding to $P_{2,3}$ and $P_{2,1}$ and $\Gamma_2$ has $6m_0 + 5$ vertices. There are three trees from $P_{2,3}$ to $P_{2,1}$. See Figure 3.

$P_{2,3} \rightarrow S_{\alpha_1}^+ \rightarrow \ldots \rightarrow S_{\alpha_m_0}^+ \rightarrow R_\alpha \rightarrow S_{\alpha_m_0}^- \rightarrow \ldots \rightarrow S_{\alpha_1}^- \rightarrow P_{2,1}$
5.3. **Two links are not diffeomorphic.** The calculation in the previous subsection shows the following important theorem. Let $K_{f_i} := V_{f_i} \cap S^5_\varepsilon$ for
5.3.1. Wang sequence and Jordan block. As far as we know, this is a first example of a pair of links with the same zeta function and non-diffeomorphic links. Recall the Wang sequence of the Milnor fibration \cite{13}:

\[ 0 \to H_3(S^5_x \setminus K_{f_i}) \to H_2(F_i)^{h_* - \text{id}} \to H_2(F_i) \to H_2(S^5_x \setminus K_{f_i}) \to 0. \]

Here \( F_i \) is the Milnor fiber of \( f_i \). By Alexander duality, \( H^1(K_{f_i}) \cong H_3(S^5_x \setminus K_{f_i}) \cong \text{Ker}(h_* - \text{id}) \). Theorem \cite{23} says that the monodromy mappings for \( f_1 \) and \( f_2 \) have different Jordan blocks on the second homology of the Milnor fiber, though their characteristic polynomials are given by \((1 - t)^3(1 - t^{4+m})^4(1 - t)^{-1}\) and therefore the multiplicity of eigenvalue 1 for \( h_* : H_2(F) \to H_2(F) \) is 6 for both of \( f_1, f_2 \). However the number of Jordan blocks of eigenvalue 1 is 3 and 4 respectively for \( f_1 \) and \( f_2 \) by Theorem \cite{23}.

5.4. Intersection numbers and dual resolution graphs. To compute the self-intersection numbers, we consider the divisor of pull-back function \( \Pi^* x \). For simplicity, we consider the case \( m = 2m_0 + 1 \) and \( m_0 \geq 1 \). By \cite{20} and the center of \( \omega_i \) does not intersect with the coordinate plane \( u_{\xi_1} = 0 \), we get

\begin{align}
(18) \quad (\Pi^* x) &= (u_{\xi_1}) + P + \sum_{\alpha=1}^{3} \sum_{i=1}^{m_0} S_{\alpha i} + 2R_{\alpha}, \quad \text{for } f_1, f_2 \\
(19) &= (u_{\xi_1}) + P + \sum_{\alpha=1}^{4} \sum_{i=1}^{m_0} S_{\alpha i} + 2R_{\alpha}, \quad \text{for } f_3, f_4
\end{align}

Here \( S_{\alpha i} = S_{\alpha i}^+ + S_{\alpha i}^- \), and \( P \) it is equal to \( P_1 \) for \( f_1 \) and \( P_{2,3} + P_{2,1} \) for \( f_2 \) and \( P = P_{3,3} + P_{3,1} \) for \( f_3 \), and \( P = P_{4,2} + P'_{4,2} \) for \( f_4 \). Note that the genus of \( P_{2,3} \) is 1 for \( f_2 \) but \( P_{3,3} \) is a normalization of a nodal cubic \( c_3^{(1)} = 0 \) and it is rational. \( P_{4,2}, P'_{4,2} \) are smooth conics. Using the property that \( (\Pi^* x) \cdot C = 0 \) for a compact exceptional divisor \( C \subset Y \) (see for example Theorem 2.6, \cite{8}), we get for \( \Gamma_1 \) and \( \Gamma_2 \):

\[
\begin{align*}
P_1^2 &= -10, \quad S_{\alpha i}^\pm = -2, \quad R_{\alpha}^2 = -1, \quad \text{for } f_1 \\
P_{2,3}^2 &= -6, P_{2,1}^2 = -4, \quad S_{\alpha i}^\pm = -2, \quad R_{\alpha}^2 = -1, \quad \text{for } f_2
\end{align*}
\]
and for the resolution of \( f_3, f_4 \), exceptional divisors are all rational and:

\[
\begin{align*}
P^2_{3,3} &= -8, & P^2_{4,1} &= -4, & S^{\pm 2}_{\alpha i} &= -2, & R^2_{\alpha} &= -1, & \text{for } f_3 \\
P^2_{4,2} &= -6, & P'^2_{4,2} &= -6, & S^{\pm 2}_{\alpha i} &= -2, & R^2_{\alpha} &= -1, & \text{for } f_4
\end{align*}
\]

We have used the following equality for the calculation:

\[(u_{\xi 1}) \cdot E(P) = 4\]

as \( E(P) \cap (u_{\xi 1}) \) corresponds to the roots of \( q_i(0, u_{\xi 2}, 1) = 0 \) for each \( i = 1, \ldots, 4 \).

5.4.1. Remarks on the pair \( \{f_3, f_4\} \). The calculation of the links \( K_{f_3} \) and \( K_{f_4} \) are similar. We only give few remarks and leave the detail to the reader. Put \( V_j = f_j^{-1}(0) \) for \( j = 3, 4 \). Let \( \Gamma_3, \Gamma_4 \) be the dual resolution graphs for \( V_3 \) and \( V_4 \) respectively. We put for \( V_3, P_{3,3} \cap P_{3,1} = \{\rho_1, \rho_2, \rho_3\} \) and \( \rho_4 \) is the inner singularity of \( P_{3,3} \). \( P_{3,3} \) is the normalization of the nodal cubic component and \( P_{3,1} \) is the line component. For \( V_4, P_{4,2} \cap P'_{4,2} = \{\rho_1, \ldots, \rho_4\} \) where \( P_{4,2}, P'_{4,2} \) are conics.

1. After the first blowing up \( \hat{\pi} : X \to \mathbb{C}^3 \), the exceptional divisor \( E(P) \) for \( V_3 \) is a union of cubic \( P_{3,3} \) with one node and a line \( P_{3,1} \) where \( P_{3,3} \) and \( P_{3,1} \) intersects transversely. For \( V_4, E(P) \) is two smooth conics \( P_{4,2} \) and \( P'_{4,2} \). They intersect transversely. The modifications at \( 4A_1 \) singularities are exactly same as those in the previous section using the same regular simplicial cone subdivision \( \Sigma^* \). For simplicity, we explain the outline assuming \( m = 2m_0 + 1 \). We use the regular simplicial cone subdivision \( \Sigma^*_0 \) as before for the toric modification at each singular points.

2. For three \( A_1 \) singularities \( \rho_1, \rho_2, \rho_3 \) of \( E(P) \) are located at the intersection of two components \( P_{3,3} \cap P_{3,1} \) for \( V_3 \) or \( P_{4,2} \cap P'_{4,2} \) for \( V_4 \), we take the toric modification \( \omega_{\alpha} : Y_{\alpha} \to X \). Two divisors \( P_{3,3} \) and \( P_{3,1} \) or \( P_{4,2} \) and \( P'_{4,2} \) are separated by \( \omega \) and it gives a tree of exceptional divisors from \( P_{3,3} \) to \( P_{3,1} \) or from \( P_{4,2} \) to \( P'_{4,2} \) respectively:

\[
\begin{array}{cccccccc}
P_{3,3} & S^+_{\alpha 1} & \cdots & S^+_{\alpha m_0} & R_{\alpha} & S^-_{\alpha m_0} & \cdots & S^-_{\alpha 1} & P_{h,1} \\
P_{4,2} & \bullet & \bullet & \cdots & \bullet & \bullet & \cdots & \bullet & P'_{h,1}
\end{array}
\]

3. The last \( A_1 \) singularity \( \rho_4 \) is an inner singularity of \( P_{3,3} \) for \( V_3 \) and the fourth intersection of two conics for \( V_4 \). Thus after a toric modification \( \omega_{\alpha} : Y_{\alpha} \to X, \Gamma_3 \) get a closed chain at \( P_{3,3} \). For \( \Gamma_4 \), it is a same pass as in (2). The resolution graphs are given in Figure 4. Two graphs have same number of independent cycles, 3. All exceptional divisors are \( \mathbb{P}^1 \) and the number of independent cycles is 3 for \( \Gamma_3 \) and \( \Gamma_4 \). (\( P_{3,3} \) is rational as it has one node.) Thus \( K_{f_3} \) and \( K_{f_4} \) has the first Betti number 3. However their graphs are not isomorphic (even as weighted graphs). To show that two links \( K_{f_3} \) and \( K_{f_4} \) are not diffeomorphic, we can use Theorem 3.2, [15].
5.5. **Link pairs constructed from Zariski pairs.** Consider a Zariski pair of projective curves $C : f_d(x, y, z) = 0$ and $C' : g_d(x, y, z) = 0$ of degree $d$ with simple singularities. We assume that $C, C'$ are irreducible for simplicity. Consider the isolation

$$f(x, y, z) := f_d(x, y, z) + z^{d+m}$$

$$g(x, y, z) := g_d(x, y, z) + z^{d+m}.$$  

We assume $f$ and $g$ are convenient polynomial as before. Take the simplest toric modification $\hat{\pi} : X → C^3$ and $\hat{\pi}' : X' → C^3$ with $\Sigma^*$ with 4 vertices \{e_1, e_2, e_3, P\} with $P = t(1, 1, 1)$ as before. In the toric coordinates $σ = \text{Cone}(e_1, e_2, P)$, $\hat{f}$ and $\hat{g}$ are written as

$$\hat{f}(u_σ) = u_{σ3}^d(f_d(u_{σ1}, u_{σ2}, 1) + u_{σ3}^m),$$

$$\hat{g}(u_σ) = u_{σ3}^d(g_d(u_{σ1}, u_{σ2}, 1) + u_{σ3}^m)$$

Let $ρ_1, \ldots, ρ_s$ be the singular points of $E(P)$. Then choose an admissible coordinates system $w_i = (w_{i,1}, w_{i,2}, w_{i,3})$ for each $i = 1, \ldots, s$ with $w_{i,3} = u_{σ3}$. As $E(P)$ is projective space $\mathbb{P}^2$ and $f_d(u_{σ1}, u_{σ2}, 1) = 0$ and $g_d(u_{σ1}, u_{σ2}, 1) = 0$ is the affine equation of the projective curve $C, C'$ respectively, we may assume that $\hat{f}$ and $\hat{g}$ take the exact same polynomial expression at each $ρ_i$. That is, $\hat{f}, \hat{g}$ take the form

$$\hat{f}(w), \hat{g}(w) = u_{σ3}^d(ψ_i(w_1, w_2) + u_{σ3}^m)$$

where $ψ_i(w_1, w_2)$ is a fixed normal form of the simple singularity $(E(P), ρ_i) = (C, ρ_i)$ at $ρ_i$. We proceed the further toric modifications at each $ρ_i, \omega_i : Y_i → \mathbb{P}^2.$
X or $\omega'_i : Y'_i \to X'$ with respect to the same regular simplicial cone subdivision $\Sigma'_i$. Let $\Pi : Y \to \mathbb{C}^3$ and $\Pi' : Y' \to \mathbb{C}^3$ be the resolution of $f$ and $g$ obtained by composing these toric modification with $\pi$ as before. In this way two surface singularities get the exact same configuration of the exceptional divisors $E_i, E_1, \ldots, E_i, r_i$. Here we are abusing notations $E_i, j$'s which are exceptional divisors of either $\Pi$ or of $\Pi'$. Thus two hypersurface $V(f)$ and $V(g)$ have the exact same dual resolution graph. The assumption that $\{C, C'\}$ is a Zariski pair of irreducible curves implies after the resolution $\Pi$ and $\Pi'$, the central divisor $E(P)$ has the same genus for $f$ and $g$. (This was not the case for a weak Zariski pair.) We can compute intersection numbers of exceptional divisors using the divisor $(\Pi^* x)$ as in §5.4 and use the property that $(\Pi^* x) \cdot E = 0$ for any compact divisor $E$ in $Y$. It is easy to see that $(\Pi^* x)$ has the exactly same expression for $V(f)$ and $V(g)$. Thus as the link 3-manifolds $K_f$ and $K_g$ can be considered as the graph manifolds, we have

**Theorem 24.** Assume that $\{C, C'\}$ is a Zariski pair of irreducible curves with simple singularities. The two links $K_f$ and $K_g$ are diffeomorphic.

In [16], we gave an example of such links. Though two links are diffeomorphic, we do not know if the diffeomorphism can be extended to a diffeomorphism of $S^5_s$ or not.

**Remark 25.** Theorem 24 also valid for non-irreducible Zariski pairs. The argument is exactly same. The simple singularities assumption can be replaced by Newton no-degeneracy of singularities. The assumption that $\{C, C'\}$ is a Zariski pair is crucial, because otherwise, the geometry of the central divisor $E(P)$ in $Y$ and $Y'$ are different as we have seen in the case of weak Zariski pair.

5.6. **Appendix.** Starting from weak Zariski pairs, we can construct many other examples with non-diffeomorphic links. We give two examples. We leave the detail for the reader. More interesting problem is: Are they $\mu$-Zariski pairs of links?

Consider irreducible projective curve of degree $d$ with $k A_1$'s singularities and note it as $C_d^{(k)}$. Note that the genus of the normalization of $C_d^{(k)}$ is $\frac{(d-1)(d-2)}{2} - k$. We denote by $r$ the number of independent cycles in the resolution graph and by $g_{tot}$ the sum of the genus of the exceptional divisors. Put $b_1 = r + 2g_{tot}$, which is the Betti number of the link. The calculation can be done in exact same way as our examples $f_1, f_2, f_3, f_4$.

Example 1. Consider the pair of sextic curves $\{C_{6,1}, C_{6,2}\}$ with $9 A_1$ singularities where $C_{6,1} = C_6^{(9)}$ (9 nodal sextic) and $C_{6,2} = C_3^{(0)} + C_3^{(0)}$, two generic cubics. We assume that irreducible components are intersecting transversely. Let $f_{6,i}(x, y, z)$ be the defining homogeneous polynomials. We always assume that $f_{6,i}$ is convenient. We consider the isolation surfaces

$$V_i : g_i(x, y, z) := f_{6,i}(x, y, z) + z^{6+m} = 0, \quad i = 1, 2, m \geq 1.$$
The resolution is given exactly as \[5.2\]. After one point blowing up \(\tilde{\pi} : X \to \mathbb{C}^3\), \(E(P)\) has 9 nodes and they are resolved by 9 toric modifications omega: \(Y \to X\). At each node of \(E(P)\), the second toric resolution is exactly as in \[5.2\]. In \(V_1\), \(E(P)\) is normalized by \(\omega\) into a genus 1 curve. For \(V_2\), \(E(P)\) has two torus (=surface of genus 1) and separated by \(\omega\). Then the corresponding links \(K_i\) are not diffeomorphic and the invariants of the dual resolution graphs are given as follows. Their zeta function is given by

\[
(1 - t^6)^{-12}(1 - t^{6+m})^{-9} \text{ and } \mu^* = (134, 25, 5).
\]

Example 2. Consider triple of projective curve \(\{C_{6,3}, C_{6,4}, C_{6,5}\}\) of sex-tics with 10 \(A_1\) singularities where \(C_{6,3} = C^{(10)}_6\) (10 nodal sextic), \(C_{6,4} = C^{(1)}_3 + C^{(0)}_3\) and \(C_{6,5} = C^{(1)}_4 + C_1 + C'_1\) with \(C_1, C'_1\) being lines. Irreducible components are intersecting transversally. Let \(f_{6,j}(x, y, z)\) be the defining convenient homogeneous polynomials and let \(g_{6,j}(x, y, z) = f_{6,j}(x, y, z) + z^{6+m}, j = 3, 4, 5\) respectively. Let \(K_j, j = 3, 4, 5\) be the corresponding link 3-manifolds. Among the exceptional divisors, \(C^{(0)}_3\) has genus 1 and the normalization of \(C^{(1)}_3\) has genus 2 and the corresponding invariants of the resolution graphs are given as follows.

| Table 2. Invariants |
|---------------------|
| \(\text{link} \) | \(r\) | \(g_{tot}\) | \(b_1\) |
| \(K_3\) | 10 | 0 | 10 |
| \(K_4\) | 9 | 1 | 11 |
| \(K_5\) | 8 | 2 | 12 |

Their zeta-function is \((1-t^6)^{-11}(1-t^{6+m})^{-10}\). The \(\mu^*\)-invariant is \((135, 25, 5)\).

A related problem. 1. In the proof of Theorem \[14\], we have proved that the multiplicity of \(\mu\)-constant family is constant in our example \(\{f_1, f_2\}\) or \(\{f_3, f_4\}\). We ask if this is true for other \(\mu\)-constant family. If this is not true, give an explicit counter example.

2. Suppose that \(\{g_1, g_2\}\) is a weak Zariski pair (respectively Zariski pair) of degree \(d\) and consider \(g_i := f_i + z^{d+m}, i = 1, 2\).

(1) If \(\{f_1, f_2\}\) is a weak Zariski pair, is \(\{f_1, f_2\}\) a \(\mu\)-Zariski pair of hypersurface?

(2) If they are a weak Zariski pair, but not a Zariski pair, are their links \(K_{g_i}, i = 1, 2\) not diffeomorphic?
(3) Suppose \( \{f_1, f_2\} \) is a Zariski pair. Is \( \{g_1, g_2\} \) a \( \mu \)-Zariski pair? (We have shown that they are \( \mu^* \)-Zariski pair in [4].

3. Are the examples in this appendix \( \mu \)-Zariski pairs (resp. \( \mu \)-Zariski triple)?

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