Dixmier’s Problem 5 for the Weyl Algebra

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Abstract

In the paper [16], J. Dixmier posed six problems for the first Weyl algebra. In this paper we give a solution to the Dixmier’s Problem 5 from this paper. Problem 3 was solved by Joseph and Stein [21] (using results of McConnell and Robson [25]). Using a (difficult) polarization theorem for the first Weyl algebra Joseph [21] solved problem 6 (a short proof to this problem is given in [7], note that the same result and the proof are true for the ring of differential operators on an arbitrary smooth irreducible algebraic curve [7]). Problems 1, 2, and 4 are still open.

1 Introduction

Let $K$ be a field of characteristic zero. The first Weyl algebra $A_1$ is an associative $K$-algebra generated over $K$ by elements $X$ and $Y$ subject to the defining relation $YX - XY = 1$. The $n$th Weyl algebra $A_n$ is the tensor product $A_1 \otimes \cdots \otimes A_1$ of $n$ copies of the first Weyl algebra. The Weyl algebra $A_n$ is a simple Noetherian domain of Gelfand-Kirillov dimension $2n$ which is canonically isomorphic to the ring of differential operators $K[X_1, \ldots, X_n, \frac{\partial}{\partial X_1}, \ldots, \frac{\partial}{\partial X_n}]$ with polynomial coefficients. The Weyl algebras have been intensively studied during the last fifty years. The Gelfand-Kirillov dimension and the transcendence dimension of the Weyl algebra $A_n$ were computed by Gelfand and Kirillov, [19]. The fact that each derivation of the Weyl algebra $A_n$ is an inner derivation was proved by Dixmier, [18]. The commutativity of the centralizer of an arbitrary nonzero element of the first Weyl algebra was proved by Amitsur, [11]. The structure of maximal commutative subalgebras of the first Weyl algebra was studied and the generators of the group of all algebra automorphisms of $A_1$ were found by Dixmier, [16] (see [8] for a generalization of these results to noncommutative deformations of type-A Kleinian singularities). Simple $A_1$-modules were classified by Block, [13] (see also [4] for an alternative approach and some generalizations). The global dimension of $A_n$ is $n$, this was proved by Rinehart, [28] in the case $n = 1$, and by Roos in the general case. Rentschler and Gabriel proved that the Krull dimension of $A_n$ is $n$, [27]. The finite dimensionality of the vector spaces $\text{Ext}^i_{A_1}$ and $\text{Tor}^i_{A_1}$ for simple $A_1$-modules was established by McConnell and Robson, [25]. The fact that the Gelfand-Kirillov dimension of a nonzero finitely generated $A_n$-module is not less than $n$ (the Bernstein Inequality) was proved by Bernstein, [10]. A finitely generated $A_n$-module of Gelfand-Kirillov dimension $n$ is called a holonomic module. Each simple module over
the first Weyl algebra is holonomic. The situation is completely different for the Weyl algebras \( A_n, \ n \geq 2 \). The first examples of non-holonomic \( A_n \)-modules were constructed by Stafford, \[30\], further progress in this direction was made by Coutinho, \[14\]. Bernstein and Lunts, \[11\] in the case of the second Weyl algebra, and Lunts in the general case, \[23\], showed that "generically" a simple \( A_n \)-module is non-holonomic and has Gelfand-Kirillov dimension \( 2n - 1 \). Simple holonomic \( A_2 \)-modules were classified by the author and van Oystaeyen, \[9\]. Skew subfields of the \( n \)’th Weyl skew field which are invariant under the action of a finite group were studied by Alev and Dumas, \[2\]. Makar-Limanov, \[24\], proved negative answer is given to the 5’th problem.

In his fundamental paper \[16\] Dixmier initiated a systematic study of the structure of the first Weyl algebra \( A_1 \). At the end of his paper he posed 6 problems. In this paper a negative answer is given to the 5’th problem. **Problem 1** concerns the question whether an algebra endomorphism of \( A_1 \) is an algebra automorphism? A positive answer to a similar problem but for the \( n \)’th Weyl algebra implies the Jacobian Conjecture as was shown by Bass, Connel and Wright, \[3\]. For an arbitrary non-scalar element \( u \) of \( A_1 \) one can associate the inner derivation \( ad \ u \) of the Weyl algebra \( A_1 \), \( ad \ u(a) = ua - au, \ a \in A_1 \), and then the \( N \)-filtered algebra \( N(u) = \cup_{i \geq 0} N(u, i) \) where \( N(u, i) := \ker(ad u)^{i+1} \). The zero component of this filtration, \( \ker ad u \), is the centralizer \( C(u) \) of the element \( u \) in \( A_1 \). The algebra \( C(u) \) is a commutative algebra which is a free finitely generated module over its polynomial subalgebra \( K[u] \), \[11\] and \[16\]. Dixmier partitioned all non-scalar elements of the Weyl algebra \( A_1 \) into 5 classes \( \Delta_1, \ldots, \Delta_5 \), and classified up to the action of the group \( \text{Aut}_K(A_1) \) elements from the class \( \Delta_1 \), so-called elements of strongly nilpotent type, and elements from the class \( \Delta_3 \), so-called elements of strongly semi-simple type. Problems 2-6 are concerned with properties and classification of elements from the remaining classes \( \Delta_2, \Delta_4 \) and \( \Delta_5 \). A non-scalar element \( u \in A_1 \) with \( C(u) \neq N(u) \) (resp., with \( N(u) = A_1 \)) is called an element of nilpotent type (resp., of strongly nilpotent type). A non-scalar element \( u \in A_1 \) of nilpotent type belongs to \( \Delta_2 \) iff \( C(u) \neq N(u) \neq A_1 \), and we say that \( u \) is of weakly nilpotent type.

**Dixmier’s Problem 5**, \[16\]: Let \( u \in A_1 \) be an element of nilpotent type. Set \( I_n = (ad \ u)^n N(u, n) \); this an ideal of \( C(u) \). Is \( I_{n+1} = I_1 I_n \) for \( n \) sufficiently large?

The next Theorem shows that the answer in general is negative. This result is proved in Section 4, Corollary \[12\].

For given natural numbers \( n \) and \( m \neq 0 \), there exist and unique natural numbers \( l \) and \( r \) such that \( n = lm + r \) and \( 0 \leq r < m \). The number \( l \) is denoted by \( \lfloor \frac{n}{m} \rfloor \).

**Theorem 1.1** Let \( \alpha(H) \in K[H] \) be a polynomial of degree \( d \geq 1 \) in the variable \( H = XY \). The centralizer of the element \( u = \alpha(YX)X \in A_1 \) is the polynomial ring \( K[u] \), and \( I_k = u^{k-\lfloor \frac{n}{m} \rfloor} K[u] \), for all \( k \geq 1 \). In particular, \( I_1 = uK[u] \) and \( I_{i(d+1)-1} = I_{i(d+1)} = u^{id} K[u] \), for all \( i \geq 1 \). Hence, \( I_1 I_{i(d+1)-1} \neq I_{i(d+1)} \), for all \( i \geq 1 \); and so the Dixmier’s Problem 5 has negative answer.

In order to prove this result we consider the Weyl algebra as the generalized Weyl algebra \( A_1 = K[H](\sigma, H) = \oplus_{i \in \mathbb{Z}} K[H]v_i \) (see Section 2 for details). The localization of
A_1 at the Ore set \( S = K[H]\setminus\{0\} \) is the skew Laurent extension \( B = K(H)[X, X^{-1}; \sigma] = \oplus_{i \in \mathbb{Z}} K(H)X^i \) with the \( K \)-automorphism \( \sigma \) of the field of rational functions \( K(H) \) defined by \( \sigma(H) = H - 1 \). The Weyl algebra \( A_1 \) is a homogeneous subalgebra of the \( \mathbb{Z} \)-graded algebra \( B \). In Section 2, for a homogeneous element \( \alpha X^i, \alpha \in K(H), i \in \mathbb{Z} \), the centralizer \( C(u, B) \) (Proposition 2.1) and the algebra \( N(u, B) \) (Theorem 2.3) are described. In Section 3, using these results, for an arbitrary homogeneous element \( u \) of the Weyl algebra \( A_1 \), the centralizer \( C(u, A_1) \) (Proposition 3.1) and the algebra \( N(u, A_1) \) (Theorem 3.2) are found. Then, for the element \( u = \alpha X \) as in the Theorem 3.3 we can describe the algebra \( N(u, A_1) \) and the ideals \( I_n \). In Section 5, we classify homogeneous elements of the Weyl algebra \( A_1 \) with respect to the Dixmier partition of elements of \( A_1 \) into the classes \( \Delta_i \). We prove (Corollary 5.3) that for an arbitrary homogeneous element \( u \) of the Weyl algebra \( A_1 \) of nilpotent type: (i) the Dixmier’s Problem 4 has positive answer, that is the associated graded algebra \( G(u, A_1) \) of the \( \mathbb{N} \)-algebra \( N(u, A_1) \) is an affine commutative algebra, hence Noetherian, and as a consequence the algebra \( N(u, A_1) \) is affine Noetherian; (ii) the Weyl algebra \( A_1 \) is not a finitely generated (left and right) \( N(u, A_1) \)-module.

For more information about the Weyl algebras the reader is referred to the following books [12, 22, 26].

2 Centralizer of a Homogeneous Element of the Algebra \( B \)

Let \( D \) be a ring with an automorphism \( \sigma \) and a central element \( a \). The generalized Weyl algebra \( A = D(\sigma, a) \) of degree 1, is the ring generated by \( D \) and two indeterminates \( X \) and \( Y \) subject to the relations:

\[
X\alpha = \sigma(\alpha)X \quad \text{and} \quad Y\alpha = \sigma^{-1}(\alpha)Y, \quad \text{for all} \quad \alpha \in D, \quad YX = a \quad \text{and} \quad XY = \sigma(a).
\]

The algebra \( A = \oplus_{n \in \mathbb{Z}} A_n \) is a \( \mathbb{Z} \)-graded algebra where \( A_n = Dv_n, \ v_n = X^n \ (n > 0), \ v_n = Y^{-n} \ (n < 0), \ v_0 = 1 \). It follows from the above relations that

\[
v_n v_m = (n, m)v_{n+m} = v_{n+m} < n, m >
\]

for some \((n, m) = \sigma^{-n-m}(< n, m >) \in D \). If \( n > 0 \) and \( m > 0 \) then

\[
n \geq m: \ (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \ (-n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),
\]

\[
n \leq m: \ (n, -m) = \sigma^n(a) \cdots \sigma(a), \ (-n, m) = \sigma^{-n+1}(a) \cdots a,
\]

in other cases \((n, m) = 1 \).

Let \( K[H] \) be a polynomial ring in one variable \( H \) over the field \( K \), \( \sigma : H \to H - 1 \) be the \( K \)-automorphism of the algebra \( K[H] \) and \( a = H \). The first Weyl algebra \( A_1 = K < X, Y \mid YX - XY = 1 > \) is isomorphic to the generalized Weyl algebra

\[
A_1 \cong K[H](\sigma, H), \ X \leftrightarrow X, \ Y \leftrightarrow Y, \ YX \leftrightarrow H.
\]
We identify both these algebras via this isomorphism, that is \( A_1 = K[H](\sigma, H) \) and \( H = YX \).

If \( n > 0 \) and \( m > 0 \) then

\[
\begin{align*}
n &\geq m : (n, -m) = (H - n) \cdots (H - n + m - 1), (-n, m) = (H + n - 1) \cdots (H + n - m), \\
n &\leq m : (n, -m) = (H - n) \cdots (H - 1), (-n, m) = (H + n - 1) \cdots H,
\end{align*}
\]
in other cases \((n, m) = 1\).

The localization \( B = S^{-1}A_1 \) of the Weyl algebra \( A_1 \) at the Ore subset \( S = K[H] \setminus \{0\} \) of \( A_1 \) is the skew Laurent polynomial ring \( B = K[H][X, X^{-1}; \sigma] \) with coefficients from the field \( K[H] = S^{-1}K[H] \) of rational functions and \( \sigma \in \text{Aut}_K K(H) \), \( \sigma(H) = H - 1 \). The map \( A_1 \to B, a \to a/1 \), is an algebra monomorphism. We identify the algebra \( A_1 \) with its image in the algebra \( B \), in more detail, via the algebra monomorphism

\[
A_1 \to B, \ X \to X, \ Y \to HX^{-1}.
\]

The subalgebra \( \mathcal{A} := K[H][X, X^{-1}; \sigma] \) of \( B \) contains the Weyl algebra \( A_1 \) (since the algebra generators \( X \) and \( Y = HX^{-1} \) of \( A_1 \) belong to \( \mathcal{A} \)), moreover, \( \mathcal{A} \) is the localization \( \mathcal{A} = A_{1X} \) of \( A_1 \) at the powers of the element \( X \). Clearly, \( B = S^{-1}\mathcal{A} \). The algebras \( B = \bigoplus_{i \in \mathbb{Z}} B_i \) and \( \mathcal{A} = \bigoplus_{i \in \mathbb{Z}} \mathcal{A}_i \) are \( \mathbb{Z} \)-graded algebras where \( B_i = K(H)X^i \) and \( \mathcal{A}_i = K[H]X^i \). The algebras \( A_1 \) and \( \mathcal{A} \) are \( \mathbb{Z} \)-graded subalgebras of \( B \).

A polynomial \( f(H) = \lambda_n H^n + \lambda_{n-1} H^{n-1} + \cdots + \lambda_0 \in K[H] \) of degree \( n \) is called a monic polynomial if the leading coefficient \( \lambda_n \) of \( f(H) \) is 1. A rational function \( h \in K(H) \) is called a monic rational function if \( h = f/g \) for some monic polynomials \( f, g \). A homogeneous element \( u = \alpha X^n \) of \( B \) is called monic iff \( \alpha \) is a monic rational function. We can extend in the obvious way the notion of degree of a polynomial to the field of rational functions setting, \( \deg_H h = \deg_H f - \deg_H g \), for \( h = f/g \in K[H] \). If \( h_1, h_2 \in K(H) \) then \( \deg_{\mathcal{H}} h_1h_2 = \deg_{\mathcal{H}} h_1 + \deg_{\mathcal{H}} h_2 \), and \( \deg_{\mathcal{H}}(h_1 + h_2) \leq \max\{\deg_H h_1, \deg_H h_2\} \). We denote by \( \text{sign}(n) \) and by \( |n| \) the sign and the absolute value of \( n \in \mathbb{Z} \), respectively.

**Proposition 2.1 (Centralizer of a Homogeneous Element of the Algebra \( B \))**

1. Let \( u = \alpha X^n \) be a monic element of \( B_n \) with \( n \neq 0 \). The centralizer \( C(u, B) = K[v, v^{-1}] \) is a Laurent polynomial ring in a uniquely defined variable \( v = \beta X^{\text{sign}(n)s} \) where \( s \) is the minimal positive divisor of \( n \) for which there exists an element \( \beta = \beta_s \in K(H) \), necessarily monic and uniquely defined, such that

\[
\beta \sigma^s(\beta) \sigma^{2s}(\beta) \cdots \sigma^{(n/s-1)s}(\beta) = \alpha, \quad \text{if} \quad n > 0,
\]

\[
\beta \sigma^{-s}(\beta) \sigma^{-2s}(\beta) \cdots \sigma^{-(|n|/s-1)s}(\beta) = \alpha, \quad \text{if} \quad n < 0.
\]

2. Let \( u \in K(H) \setminus K \). Then \( C(u, B) = K(H) \).
Proof. The element $u$ is a homogeneous element of the $\mathbb{Z}$-graded algebra $B$, hence its centralizer
\[ C = C(u, B) = \bigoplus_{i \in \mathbb{Z}} C_i, \quad C_i = C \cap B_i, \]
is a graded subalgebra of $B$. Consider
\[ H \equiv H(u, B) := \{ i \in \mathbb{Z} | C_i \neq 0 \}. \]
The set $H$ is a subgroup of $\mathbb{Z}$: $0 \in H$, since $1 \in C$; $H + H \subseteq H$, since $C_iC_j \subseteq C_{i+j}$ and $B$ is a domain; $-H \subseteq H$, since if $0 \neq v \in C_i$, then $v^{-1} \in C_{-i}$.

1. In this case, $H = \mathbb{Z}s$ for a uniquely defined positive divisor $s$ of $n$ ($n \in H$, since $u \in C_n$).
   
   **Claim 1:** for every $i \in H$, $C_i = K^* \alpha_i X^i$ for a uniquely defined monic $\alpha_i \in K(H)$. Obviously, $\beta X^i \in C_i$, for some $0 \neq \beta \in K(H)$, iff
   \[ \frac{\sigma^i(\alpha)}{\alpha} = \frac{\sigma^n(\beta)}{\beta}. \] (3)
So, if $0 \neq \beta_1 X^i, \beta_2 X^i \in C_i$, then
\[ \frac{\sigma^n(\beta_1)}{\beta_1} = \frac{\sigma^n(\beta_2)}{\beta_2} \]
or, equivalently, $\sigma^n(\beta_2/\beta_1) = \beta_2/\beta_1 \in K(H)^{\alpha n} = K^*$, this finishes the proof of the claim.

It follows from the claim and from $H = \mathbb{Z}s$, that $C(u, B) = K[v, v^{-1}]$ for some $v$.

By the claim 1, there exists a unique monic element $0 \neq \beta \in K(H)$ such that $v = \beta X^{\text{sign}(n)s}$. If $n > 0$, then
\[ C_n = K^* \alpha X^n \ni v^{n/s} = (\beta X^s)^{n/s} = \beta \sigma^s(\beta) \cdots \sigma^{((n/s)-1)s}(\beta) X^n = \beta_n X^n, \]
hence $\alpha = \beta_n$, since $\beta_n$ is a monic polynomial. If $n < 0$, then
\[ C_n = K^* \alpha X^n \ni (\beta X^{-s})^{n/s} = \beta \sigma^{-s}(\beta) \cdots \sigma^{-(s(s)-1)s}(\beta) X^n = \beta_n X^n, \]
hence $\alpha = \beta_n$, since $\beta_n$ is a monic polynomial.

**Claim 2:** suppose that for some positive divisor $s$ of $n$ and for some $0 \neq \beta \in K(H)$ one of the corresponding equalities, \[1\] or \[2\], holds. Then $\beta X^{\text{sign}(n)s} \in C$. Consider the case $n > 0$. Then
\[ \frac{\sigma^s(\alpha)}{\alpha} = \frac{\sigma^s(\beta) \cdots \sigma^{((n/s)-1)s}(\beta)}{\beta_n} = \frac{\sigma^n(\beta)}{\beta}, \]
hence $\beta X^s \in C$, by \[3\]. Claim 2 proves the minimality of the $s$ (in the Proposition).

2. The centralizer $C$ of $u$ is a homogeneous subalgebra of $B$ which contains $K(H)$. A homogeneous element $\beta X^i$ of $B$ with $i \neq 0$ commutes with $u$ iff $\beta = 0$ since $0 = [\beta X^i, u] = \beta(\sigma^i(u) - u) X^i$ and $u \notin K = K(H)^{\sigma^i}$. This proves that $C = K(H)$. \[\blacksquare\]
**Definition.** The uniquely defined element \( v \) from Proposition 2.2 \((1)\) is called the canonical generator of the algebra \( C(u, B) \).

The set of polynomials

\[
\varphi_0 := 1, \quad \varphi_n := (-1)^n \frac{H(H + 1) \cdots (H + n - 1)}{n!} = (-1)^n \frac{H_\sigma(H) \cdots \sigma^{-n+1}(H)}{n!}, \quad n \geq 1,
\]

is a \( K \)-basis of the polynomial algebra \( K[H] \), \( \deg \varphi_n = n \) and

\[
\sigma(\varphi_n) - \varphi_n = \varphi_{n-1}, \text{ for all } n \geq 0, \quad \varphi_- := 0.
\]

Let \( K \) be a field and let \( K[t] \) be a polynomial ring in an indeterminate \( t \). Let \( M \) be a \( K[t] \)-module. For an element \( p \in K[t] \) we denote by \( \ker p_M \) the kernel of the \( K \)-linear map \( p = p_M : M \to M, m \to pm \). The kernel \( \ker p_M \) is a \( K[t] \)-submodule of \( M \). For \( i \geq 0 \), let \( N_i = N_i(t, M) := \ker t^{i+1} M \). Then

\[
N_0 \subseteq N_1 \subseteq \cdots \subseteq N_i \subseteq \cdots, \quad tN_0 = 0, \quad \text{and } tN_j \subseteq N_{j-1}, \quad \text{for } j \geq 1.
\]

Clearly, \( N = N(t, M) := \cup_{i \geq 0} N_i \) is a \( K[t] \)-submodule of \( M \). We set \( N_{-1} = 0 \). If \( 0 \neq u \in N \) then the unique \( i \) such that \( u \in N_i \setminus N_{i-1} \) is called the nilpotent degree of \( u \), denoted by \( \text{ndeg } u \).

**Definition.** A \( K[t] \)-module \( M \) is called a Jordan \( K[t] \)-module iff \( M = N(t, M) \) and \( tM = M \).

If \( M \) is a nonzero Jordan \( K[t] \)-module then \( N_i \neq N_{i+1} \) for all \( i \geq 0 \) since otherwise \( M = N_j \) for some \( j \), hence \( M = t^{j+1} M = 0 \), a contradiction. From this fact we conclude that each nonzero Jordan \( K[t] \)-module is not a finitely generated module (hence, is not noetherian).

**Example.** A vector space \( \mathcal{J} = \oplus_{i \geq 0} Ke_i \) with the \( K[t] \)-module structure defined by \( te_0 = 0 \) and \( te_i = e_{i-1} \), \( i \geq 1 \), is a Jordan module. The module \( \mathcal{J} \) is isomorphic to the \( K[t] \)-module \( K[t, t^{-1}]/K[t] \), and \( \mathcal{J}_i := \ker t^{i+1} = \oplus_{j=0}^i Ke_j \).

**Lemma 2.2** Let \( M \) be a \( K[t] \)-module. Suppose that \( N(t, M) \) contains a Jordan module \( N' \) such that \( N' \supseteq \ker t \). Then \( N(t, M) = N' \).

**Proof.** Since \( N = \cup_{i \geq 0} N_i \), it suffices to show that each \( N_i \) is contained in \( N' \). We use induction on \( i \). The case \( i = 0 \) is true by the assumption. Suppose that \( N_{i-1} \subseteq N' \) and let \( u \in N_i \). Since \( tu \in N_{i-1} \subseteq N' \) and \( tN' = N' \) \( (N' \text{ is a Jordan module}) \), we have \( tu = tv \) for some \( v \in N' \). Now, \( t(u - v) = 0 \), hence \( u - v = w \in N_0 \subseteq N' \), and finally \( u = v + w \in N' \). It means that \( N_i \subseteq N' \), as required. ■

As we have seen in the Introduction, one can associate with a non-scalar element \( u \) of the Weyl algebra \( A_1 \) the inner derivation \( \text{ad } u \) and the \( \mathbb{N} \)-filtered subspace \( N(u) = N(u, A_1) := \cup_{i \geq 0} N(u, i, A_1) \) where \( N(u, i, A_1) = \ker (\text{ad } u)^{i+1} \). The zero component of this filtration is the centralizer \( C(u, A_1) \) of the element \( u \) in \( A_1 \). The vector space \( N(u, A_1) \) is in fact a \( \mathbb{N} \)-graded algebra as follows from the formula,

\[
(\text{ad } u)^n(ab) = \sum_{i=0}^n \binom{n}{i} (\text{ad } u)^i(a)(\text{ad } u)^{n-i}(b), \quad (5)
\]
where \( a, b \in A_1 \) and \( n \geq 1 \). The algebra \( N(u, A_1) \) is a \( K[\text{ad} u] \)-module, such that

\[
(ad u)^i N(u, j, A_1) \subseteq N(u, j - i, A_1), \quad \text{for all } i, j \geq 0,
\]

where we set \( N(u, -i, A_1) = 0 \) for \( i \geq 1 \). Each \( N(u, i, A_1) \) is a finitely generated \( C(u, A_1) \)-module (Proposition 10.2.(ii), [16]). The associated graded algebra of the \( \mathbb{N} \)-graded algebra \( N(U, A_1) \),

\[
\mathcal{G}(u, A_1) := \oplus_{i \geq 0} N(u, i, A_1)/N(u, i - 1, A_1),
\]

is a commutative domain (Proposition 10.2.(i), [16]).

**Dixmier’s Problem 4, [16]:** is the algebra \( \mathcal{G}(u, A_1) \) finitely generated? This problem is still open. In the next section a positive answer will be obtained for all homogeneous elements of the Weyl algebra. But first, we need a description of the algebra \( N(u, B) \) for an arbitrary homogeneous element of the algebra \( B \).

**Theorem 2.3 (The Algebra \( N(u, B) \) of a Homogeneous Element of the Algebra \( B \))**

1. Let \( u = \alpha X^n \) where \( 0 \neq n \in \mathbb{Z} \) and \( \alpha \) is a nonzero monic element of \( K(H) \).
   
   (i) The algebra \( N(u, B) \) is generated by the algebra \( C(u, B) \) and the element \( H \). If \( C(u, B) = K[v, v^{-1}] \) where \( v = \beta X^t \) is chosen as in Proposition 2.1 then the algebra \( N(u, B) \) is the skew Laurent extension \( K[H][v, v^{-1}; \sigma^t] \). So, \( N(u, B) \) is an affine Noetherian algebra.
   
   (ii) For \( j \geq 0 \), \( N(u, j, B) = \sum_{i=0}^j C(u, B)H^i \) and \( H^j \in N(u, j, B) \setminus N(u, j - 1, B) \).
   
   (iii) The associated graded algebra \( \mathcal{G}(u, B) \) is the polynomial algebra \( C(u, B)[h] \) with coefficients from \( C(u, B) \) where \( h := H + C(u, B) \in N(u, 1, B) \setminus C(u, B) \). Hence \( \mathcal{G}(u, B) \) is an affine commutative algebra.

2. Let \( u \in K(H) \setminus K \). Then \( N(u, B) = C(u, B) = K(H) \).

**Proof.** 1. Clearly, \( H \in N(u, 1, B) \setminus C(u, B) \) since

\[
[u, H] = (\sigma^n(H) - H)u = -nu
\]

is a nonzero element of \( C(u, B) \). Denote by \( N' \) the subalgebra of \( B \) generated by the algebra \( C(u, B) \) and the element \( H \). The algebra \( N' \) is a homogeneous subalgebra of the \( \mathbb{Z} \)-graded algebra \( B = \oplus_{i \in \mathbb{Z}} K(H)X^i \) since \( N' \) is generated by the homogeneous elements \( v, v^{-1} \) and \( H \) of \( B \). Using this fact we see that

\[
N' = K[H][v, v^{-1}; \sigma^t]
\]

is a skew Laurent polynomial ring with coefficients from the polynomial ring \( K[H] \). We aim to show that \( N' = N(u, B) \). The inclusion \( N' \subseteq N(u, B) \) is obvious since the algebra
generators $v, v^{-1}$ and $H$ of $N'$ belong to $N(u, B)$. By Proposition 2.1, $u = v^k$ where $k = t^{-1}n$ is a natural number. The set of elements

$$\varphi_i(n^{-1}H)v^j, \ i \geq 0, \ j \in \mathbb{Z},$$

is a $K$-basis of $N'$ where the polynomials $\varphi_i = \varphi_i(H)$ are defined in (1), and

$$\text{ad } u(\varphi_i(n^{-1}H)v^j) = \varphi_{i-1}(n^{-1}H)v^{j+k}.$$ 

This means that $N'$ is a Jordan $K[\text{ad } u]$-module which contains $\text{ker ad } u = C(u, B)$ and $N' \subseteq N(u, B)$. By Lemma 2.2, $N' = N(u, B)$.

It follows from (6) that $N(u, j, B) = \sum_{i=0}^j C(u, B)\varphi_i(n^{-1}H) = \sum_{i=0}^j C(u, B)H^i$ and that $H^i \in N(u, j, B) \setminus N(u, j - 1, B)$. So, we have proved the statements (i) and (ii). Statement (iii) is evident.

2. By the assumption, $u \in K(H) \setminus K$, thus, for each nonzero $i \in \mathbb{Z}$, the element $\sigma^i(u) - u$ is nonzero (since $K(H)^{\sigma^i} = K$, the algebra of $\sigma^i$-invariant elements in the field $K(H)$). Let $w = \alpha X^m$ be a nonzero homogeneous element of $B$. For $i \geq 1$,

$$(\text{ad } u)^i w = (u - \sigma^m(u))^i w \neq 0.$$ 

It follows easily from this fact that $N(u, B) = K(H)$.  

3 Centralizer and $N(u, A_1)$ of a Homogeneous Element of the Weyl Algebra

In this section, for an arbitrary homogeneous element $u$ of the Weyl algebra $A_1$, algebra generators are found for the algebras $C(u, A_1)$ (Proposition 3.1) and $N(u, A_1)$ (Theorem 3.2). For certain homogeneous elements of $A_1$, their centralizers were described in Proposition 5.3, [16]. We shall see that the Dixmier’s Problem 4 has positive answer for all homogeneous elements of the Weyl algebra (Theorem 3.2).

Consider an element $u = \alpha n \in A_1$ with $n \neq 0$ and a nonzero monic polynomial $\alpha$ of $K[H]$. If $n > 0$ then $u = \alpha X^n$, and if $n < 0$ then $u = \alpha Y^n$. The element $u$ is a monic homogeneous element of the algebra $B$ since $\alpha$ is a monic polynomial and

$$Y^n = Y^n X^n X^{-n} = (-n, n) X^{-n} = H(H + 1) \cdots (H + n - 1) X^{-n} = (-1)^n n! \varphi_n X^{-n}. \quad (7)$$

By Proposition 2.1, $C(u, B) = K[v, v^{-1}]$ where $v = \beta X^t$ is the canonical generator of the algebra $C(u, B)$, $0 \neq \beta \in K(H)$ and the integer $t$ has the same sign as $n$. Moreover,

$$v^m = u, \ m = t^{-1}n \geq 1. \quad (8)$$

The centralizer $C(u, A_1) = A_1 \cap C(u, B) = \bigoplus_{i \in H} K v^i$ where $H = \{i \in \mathbb{Z} : v^i \in A_1\}$. By 11 or (Theorem 4.2, [16]), the algebra $C(u, A_1)$ is a finitely generated $K[u]$-module. Since $u = v^m$ for some $m \geq 1$, using the graded argument we have $H = \{i \geq 0 : v^i \in A_1\}$.
For \( i = 0, 1, \ldots, m - 1 \), we denote by \( \gamma_i \) a monic polynomial of \( K[H] \) of minimal possible degree, say \( d_i \), such that \( \gamma_i v^i \in A_1 \). We denote by \( \delta \) the inner derivation \( \text{ad} u \) of the Weyl algebra \( A_1 \). Then

\[
A_1 \ni \delta^{d_i}(\gamma_i v^i) = (-n)^{d_i}d_i!v^i u^{d_i} = (-n)^{d_i}d_i!v^{i+d_i}.
\]

Thus, we can define the following non-negative integers,

\[
\mu_i := \min\{ j \geq 0 \mid v^j \in A_1, j \equiv i \, (\text{mod} \, m) \}, \text{ for each } i = 0, 1, \ldots, m - 1.
\]

Then

\[
H = \bigcup_{i=0}^{m-1} \{ \mu_i + m\mathbb{N} \}, \quad \text{(9)}
\]

a disjoint union. The next result describes the centralizer of an arbitrary homogeneous element of the Weyl algebra \( A_1 \).

**Proposition 3.1 (Centralizer of a Homogeneous Element of the Weyl Algebra)**

1. Let \( u = \alpha v_n \in A_1 \) where \( 0 \neq n \in \mathbb{Z} \) and \( \alpha \) is a monic polynomial of \( K[H] \). Then

\[
C(u, A_1) = \bigoplus_{i=0}^{m-1} K[u]v^{\mu_i}.
\]

2. Let \( u \in K[H] \setminus K \). Then \( C(u, A_1) = K[H] \).

*Proof.* 1. By (3), \( C(u, A_1) = \bigoplus_{j \in H} K v^j = \bigoplus \{ K v^j \mid i = 0, \ldots, m-1 \text{ and } j \in \mu_i + m\mathbb{N} \} = \bigoplus_{i=0}^{m-1} K[u]v^{\mu_i} \) since \( v^m = u \).

2. By Proposition 2.1.(2), \( C(u, B) = K(H) \), thus \( C(u, A_1) = A_1 \cap C(u, B) = K[H] \). \( \square \)

Observe that \( X^t \) is equal to \( v_1 \) if \( t > 0 \), and to \( (t, -t)^{-1} v_t = (H(H+1) \cdots (H-t-1))^{-1} v_t \) if \( t < 0 \) (by (7)). Thus the canonical generator \( v \) can be written in the form \( \gamma v_t \) where \( \gamma = \beta \) if \( t > 0 \), and \( \gamma = \beta(t, -t)^{-1} \) if \( t < 0 \). The element \( \gamma \) is a monic element of \( K(H) \). Set \( \mu := \max\{ \mu_0, \ldots, \mu_{m-1} \} \). Then

\[
v^i = \gamma^{\sigma^t(\gamma)} \cdots \sigma^{(i-1)t}(\gamma) v_{it} \in A_1, \text{ for all } i \geq \mu,
\]

hence,

\[
\gamma^{\sigma^t(\gamma)} \cdots \sigma^{(i-1)t}(\gamma) \in K[H], \text{ for all } i \geq \mu.
\]

For each \( i = 1, \ldots, \mu - 1 \), there exists a unique monic polynomial \( g_i \in K[H] \) of minimal possible degree such that \( g_i v^i \in A_1 \). The polynomial \( g_i \) is the *denominator* of the rational function \( \gamma^{\sigma^t(\gamma)} \cdots \sigma^{(i-1)t}(\gamma) \) multiplied by a proper nonzero scalar. By definition, the denominator of a rational function \( \alpha = pq^{-1} \) (\( p, q \in K[H] \)) is \( q \) provided \( \gcd(p, q) = 1 \). Clearly,

\[
K[H]v^i \cap K[H]v_t = K[H]g_i v^i, \quad i = 1, \ldots, \mu - 1.
\]

It follows from the equality \( v_{-k} v_k = (-k, k) \in K[H], k \in \mathbb{Z} \), that

\[
v^{-1}_k = (-k, k)^{-1} v_{-k}.
\]
Now, by (10),
\[ v^{-i} = v_{it}^{-1}(\gamma\sigma^t(\gamma)\cdots\sigma^{(i-1)t}(\gamma))^{-1} = \{(it)^{-i}(\gamma\sigma^t(\gamma)\cdots\sigma^{(i-1)t}(\gamma))\}^{-1}v_{-it}. \]
By (11),
\[ v_{-it} = (it)^{-i}(\gamma\sigma^t(\gamma)\cdots\sigma^{(i-1)t}(\gamma))v^{-i} \in N(u, A), \text{ for all } i \geq \mu. \quad (12) \]
For each \( i \geq 1 \), there exists a unique monic polynomial \( f_i \in K[H] \) such that
\[ K[H]v^{-i} \cap K[H]v_{-it} = (f_i)v_{-it}, \]
where \( (f_i) = f_iK[H] \). By (12), \( f_i = 1 \) for all \( i \geq \mu \). For each \( i = 1, \ldots, \mu - 1 \), the polynomial \( f_i \) is the denominator of the rational function \( (it)^{-i}(\gamma\sigma^t(\gamma)\cdots\sigma^{(i-1)t}(\gamma)) \) multiplied by a proper nonzero scalar.

Let \( R = \bigcup_{i \in \mathbb{N}} R_i \) be an \( \mathbb{N} \)-graded algebra and \( gr R = \oplus_{i \in \mathbb{N}} R_i/R_{i-1} \) be its associated graded algebra. Denote by \( \pi : R \to gr R \) the principal symbol map defined \( \pi(r) = r + R_{i-1} \) where \( r \in R_i \setminus R_{i-1} \).

**Definition.** A basis \( E = \{e_j, j \in J\} \) of the algebra \( R \) is called a principal basis iff the set \( \pi(E) = \{\pi(e_j), j \in J\} \) is a basis of the associated graded algebra \( gr R \).

Suppose that \( F = \{f_l, l \in L\} \) is a basis of the algebra \( gr R \) such that each element \( f_l \) is a homogeneous element of the algebra \( gr R \). For each \( f_l \) we fix its preimage \( e_l \) under the principal symbol map \( \pi \), that is \( \pi(e_l) = f_l \). Then the set \( E = \{e_l, l \in L\} \) is a principal basis of the algebra \( R \).

The next theorem describes the algebra \( N(u, A_1) \) for an arbitrary homogeneous element of the Weyl algebra \( A_1 \), and gives a positive answer to the Dixmier’s Problem 4 for all such elements. In the next section, this result will lead us to a solution of the Dixmier’s Problem 5.

**Theorem 3.2 (The Algebra \( N(u, A_1) \) of a Homogeneous Element of the Weyl Algebra)**

1. Let \( u = \alpha v_n \in A_1 \) where \( 0 \neq n \in \mathbb{Z} \) and \( \alpha \) is a monic polynomial of \( K[H] \). We keep the notation above, then
   (i) \( N(u, A_1) = \oplus_{i \geq \mu} K[H]v_{-it} \oplus (\oplus_{i = 1}^{\mu-1} K[H]f_i v_{-it}) \oplus K[H] \oplus (\oplus_{i = 1}^{\mu-1} K[H]g_i v^i) \oplus (\oplus_{i \geq \mu} K[H]v^i) \).
   (ii) The set \( S = \{H^kv_{-it}, H^kf_jv_{-jt}, H^kg_jv^j, H^kg_jv^j \mid k \geq 0, i \geq \mu, j = 1, \ldots, \mu - 1\} \) is a principal basis of the algebra \( N(u, A_1) \), with \( ndeg v_{-it} = i(|t| + \deg_H \gamma) \) for \( i \geq \mu \), \( ndeg f_jv_{-jt} = \deg f_j + j(|t| + \deg_H \gamma) \) and \( ndeg g_jv^j = \deg_H g_j \) for \( j = 1, \ldots, \mu - 1 \).
   (iii) The algebra \( G(u, A_1) \) is an affine (commutative) algebra, hence Noetherian.
   (iv) The algebra \( N(u, A_1) \) is an affine Noetherian algebra.

2. Let \( u \in K[H] \setminus K \). Then \( N(u, A_1) = K[H] = C(u, A_1) \).
Proof. 1.(i). By Theorem 2.3 (1), the algebra \( N(u, B) = \oplus_{i \in \mathbb{Z}} K[H]v^i \) and the Weyl algebra \( A_1 = \oplus_{j \in \mathbb{Z}} K[H]v_j \) are homogeneous subalgebras of the algebra \( B = \oplus_{j \in \mathbb{Z}} K[H]X^j \). So, the intersection

\[
N(u, A_1) = A_1 \cap N(u, B) = \oplus_{i \in \mathbb{Z}} (K[H]v^i \cap K[H]v_i),
\]

is a homogeneous subalgebra of the Weyl algebra \( A_1 \). If we recall the definition of the polynomials \( f_i \) and \( g_i \) then the result follows immediately from the fact above and [10], [12].

(ii) Since \( N(u, A_1) \) is a homogeneous subalgebra of \( A_1 \), it is easy to see that the set \( S \) is a principal basis of \( N(u, A_1) \). Since \( \text{ndeg } H = 1 = \deg_H H \) and

\[
v_{-it} = (-it, it)\sigma^{-it}(\gamma\sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma))v^{-i},
\]

for all \( i \geq \mu \), we have \( \text{ndeg } v_{-it} = \deg_H (-it, it)\sigma^{-it}(\gamma\sigma^t(\gamma) \cdots \sigma^{(i-1)t}(\gamma)) = i(|t| + \deg_H \gamma) \). For each

\[
f_jv_{-jt} = f_j(-jt, jt)\sigma^{-jt}(\gamma\sigma^t(\gamma) \cdots \sigma^{(j-1)t}(\gamma))v^{-j},
\]

hence, \( \text{ndeg } f_jv_{-jt} = \deg_H f_j(-jt, jt)\sigma^{-jt}(\gamma\sigma^t(\gamma) \cdots \sigma^{(j-1)t}(\gamma)) = \deg f_j + j(|t| + \deg_H \gamma) \). The rest is obvious.

(iii) Denote by \( R \) the subalgebra of \( G = G(u, A_1) \) generated by the principal symbols of the elements \( v_{-\mu t}, H \) and \( v^\mu \). The set \( S \) is a principal basis of the algebra \( N(u, A_1) \), thus the set \( \pi(S) = \{\pi(s), s \in S\} \) is a basis of the algebra \( G \). The algebra \( G \) is affine since it is a finitely generated \( R \)-module with generators which are the principal symbols of the elements

\[
v_{-it}, f_jv_{-jt}, 1, g_jv^j, \text{ and } v^it, \text{ where } i = \mu + 1, \ldots, 2\mu - 1; j = 1, \ldots, \mu - 1.
\]

(iv) The algebra \( N(u, A_1) \) is a Noetherian affine algebra since the algebra \( G \) is so.

2. By Theorem 2.3 (2), \( N(u, A_1) = A_1 \cap N(u, B) = A_1 \cap K(H) = K[H] = C(u, A_1) \).

Let \( u = \alpha v_n \in A_1 \) be as in Theorem 3.2 (1). The algebra \( N(u, A_1) \) is a \( \mathbb{Z} \)-graded algebra with zero graded component \( K[H] \), hence the (left and right) Krull dimension (in the sense of Rentschler and Gabriel, [27]),

\[
\text{K.dim } N(u, A_1) \geq \text{K.dim } K[H] = 1.
\]

(13)

The homogeneous subalgebra \( A \) of \( N(u, A_1) \) generated by the homogeneous elements \( y := v_{-\mu t}, H \) and \( x := v^\mu \) is the generalized Weyl algebra

\[
A = K[H](\sigma^\mu, a := (-\mu t, \mu t)\sigma^{-\mu t}(\gamma\sigma^t(\gamma) \cdots \sigma^{(\mu-1)t}(\gamma))),
\]

since, by [12],

\[
xH = \sigma^\mu(H)x, \quad yH = \sigma^{-\mu t}(H)y, \quad yx = a, \quad \text{and } xy = \sigma^\mu(a).
\]

In the terminology of [20], [15], the algebra \( A \) is called a noncommutative deformation of type-A Kleinian singularity. The algebra \( A \) is a (left and right) Noetherian algebra, [1], [20].
Corollary 3.3 Let $u = \alpha v_n \in A_1$ be as in Theorem 3.2 (1).

1. The algebra $N(u, A_1)$ is a finitely generated $A$-module.

2. The (left and right) Krull dimension of the algebra $N(u, A_1)$ is 1.

3. If $\deg_H \alpha > 0$ then the Weyl algebra $A_1$ is not a finitely generated (left and right) $N(u, A_1)$-module.

Proof. 1 and 2. By Theorem 3.2 (1), the algebra $N(u, A_1)$ is a finitely generated (left and right) $A$-module, hence $K \dim N(u, A_1) \leq K \dim A$. Since $a \neq 0$ and char $K = 0$, the Krull dimension of the generalized Weyl algebra $A$ is 1 (see [4] or [20]), hence $K \dim N(u, A_1) = 1$, by (13).

Since $N(u, A_1)$ is a finitely generated $A$-module, the Weyl algebra $A_1$ is not a finitely generated $N(u, A_1)$-module iff it is not a finitely generated $A$-module. So, it suffices to prove that $A_1$ is not a finitely generated $A$-module. By (8) and (10), $\alpha v_n = u = \gamma \sigma^t(\gamma) \cdots \sigma^{(m-1)t}(\gamma)v_n$, hence $0 < \deg_H \alpha = \deg_H \gamma \sigma^t(\gamma) \cdots \sigma^{(m-1)t}(\gamma) = m \deg_H \gamma_1$, thus $\deg_H \gamma > 0$ and $\deg_H a > 0$. It suffices to show that the factor module $A_1/A$ is not a finitely generated $A$-module. Observe that $A$ is a homogeneous subalgebra of $A_1$, and that

$$M := (\oplus_{i \in \mathbb{Z}} K[H]v_{i\mu})/A = \oplus_{i \geq 1} (K[H]v_{i\mu}/K[H]v^{i\mu})$$

is an $\mathbb{N}$-graded $A$-submodule of $A_1/A$. The $i$'th component of $M$, $M_i := K[H]v_{i\mu}/K[H]v^{i\mu}$, as a $K[H]$-module, is canonically isomorphic to

$$K[H]v_{i\mu}/K[H]v_{i\mu} \gamma_{i\mu} \simeq K[H]/K[H]v_{i\mu},$$

where $\gamma_{i\mu} := \gamma \sigma^t(\gamma) \cdots \sigma^{(i\mu-1)t}(\gamma)$. Since $\gamma_{i\mu} M_i = 0$, for all $i \geq 1$, the $K[H]$-module $M_i$ is a $K[H]$-torsion module. Each finitely generated $K[H]$-torsion module over the generalized Weyl algebra $A$ has finite length and Gelfand-Kirillov dimension $\leq 1$ (H 6).

Suppose that $M$ is a finitely generated $A$-module, then $\operatorname{GK} (M) = 1$ since $\dim_K M = \infty$. The algebra $A$ is a somewhat commutative algebra, [20], hence there exists a natural number $c$ such that

$$\sum_{i=1}^n \dim M_i \leq cn \text{ for all } n > 0,$$

which contradicts $\sum_{i=1}^n \dim M_i = \sum_{i=1}^n \deg_H \gamma_{i\mu} = \sum_{i=1}^n i \mu \deg_H \gamma = \mu \deg_H(\gamma) \frac{n(n+1)}{2}$. Thus $M$ is not a finitely generated $A$-module. \qed

4 Solution to the Dixmier’s Problem 5

In this section we apply the results from the previous sections to show that the Dixmier’s Problem 5 has negative solution.

Lemma 4.1 Let $\alpha$ be a monic polynomial of $K[H]$ of degree $d \geq 1$ and $u = \alpha X \in A_1$. Then
1. $C(u, A_1) = K[u]$.

2. $N(u, A_1) = \bigoplus_{i \geq 1} K[H]Y^i \oplus \bigoplus_{i \geq 0} K[H]u^i$ and the set $\{\varphi_i Y^{j+1}, \varphi_i u^j \mid i, j \geq 0\}$ is a basis of the algebra $N(u, A_1)$.

3. $Y \in N(u, d + 1, A_1) \setminus N(u, d, A_1)$ and, for $k \geq 1$, $N(u, k, A_1) = \bigoplus_{i,j \geq 0} \{K\varphi_i Y^j \mid i + (d+1)j \leq k\} \oplus \bigoplus_{i=0}^k K[u] \varphi_i$.

**Proof.** 1. By Proposition 2.1(1), the algebra $C(u, B) = K[u, u^{-1}]$, hence, by Proposition 3.1(1), we have $C(u, A_1) = K[u]$.

2. The element $u$ is the canonical generator of the algebra $C(u, B) = K[u, u^{-1}]$. Since $\alpha \in K[H]$, by Theorem 3.2(1), $N(u, A_1) = \bigoplus_{i \geq 1} K[H]Y^i \oplus \bigoplus_{i \geq 0} K[H]u^i$. The rest is evident.

3. By Theorem 2.8, we have $H \in N(u, 1, A_1) \setminus N(u, 0, A_1)$, hence

$$Y = HX^{-1} = H\sigma^{-1}(\alpha)(\alpha X)^{-1} = H\sigma^{-1}(\alpha)u^{-1} \in N(u, d + 1, A_1) \setminus N(u, d, A_1),$$

and

$$\varphi_i(H)Y^j \in N(u, i + (d+1)j, A_1) \setminus N(u, i + (d+1)j - 1, A_1), \text{ for all } i, j \geq 0.$$

Now, the result follows from Statement 2. ■

Let $u = \alpha X$ be as in Lemma 4.1. We denote by $\delta$ the inner derivation $ad^\Delta u$ of the Weyl algebra $A_1$.

For each $i \geq 1$, $\delta(\varphi_i) = (\sigma(\varphi_i) - \varphi_i)u = \varphi_{i-1}u$, hence

$$\delta^i(\varphi_i) = u^i. \quad (14)$$

Clearly,

$$Y^i = (HX^{-1})^i = H(H + 1) \cdots (H + i - 1)X^{-i}$$

$$= H(H + 1) \cdots (H + i - 1)\sigma^{-1}(\alpha)\sigma^{-2}(\alpha) \cdots \sigma^{-i}(\alpha)u^{-i}$$

$$= (H^{i(d+1)} + \cdots)u^{-i} = (-1)^{i(d+1)}[(d+1)! \varphi_{i(d+1)}u^{-i} + \cdots],$$

where by three dots we denote, as usually, elements of smaller nilpotent degree. So,

$$\delta^{(d+1)i}(Y^i) = (-1)^{(d+1)i}[(d+1)!u^{id}. \quad (15)$$

Using (15), we have

$$\delta^{i + (d+1)j}(\varphi_i Y^j) = \binom{i + (d+1)j}{i} \delta^i(\varphi_i)\delta^{(d+1)j}(Y^j) = (-1)^{j(d+1)}\binom{i + (d+1)j}{i}[(d+1)j!]u^{i+ji}, \quad (16)$$

for all $i, j \geq 0$. 

13
Corollary 4.2 (Solution to the Dixmier’s Problem 5) Let \( u = \alpha X \) be as in Lemma 4.1. Then \( I_k = u^{k-\left\lfloor \frac{k}{d+1} \right\rfloor} K[u] \), for all \( k \geq 1 \). In particular, \( I_1 = uK[u] \) and \( I_{i(d+1)-1} = I_{i(d+1)} = u^{id}K[u] \), for all \( i \geq 1 \). Hence, \( I_1I_{i(d+1)-1} \neq I_{i(d+1)} \), for all \( i \geq 1 \), and the Dixmier’s Problem 5 has negative solution.

Proof. By Lemma 4.1.(3) and (16), \( I_k = u^{k-\left\lfloor \frac{k}{d+1} \right\rfloor} K[u] \), for all \( k \geq 1 \). The rest is obvious.

It turns out that, for the element \( u \) as above, the algebra \( N(u, A_1) \) is a generalized Weyl algebra of a special sort. So, applying the results of the papers [4]–[6], [20], where these algebras were studied, we can say a lot about them. We collect some of the results in the following corollary.

Corollary 4.3 Let \( u \in A_1 \) be as in Lemma 4.1. Then

1. The algebra \( N(u, A_1) \) is a generalized Weyl algebra \( K[H](\sigma, H\sigma^{-1}(\alpha)) \), a so-called noncommutative deformation of type A-Kleinian singularity in the terminology of [20], [13].

2. The algebra \( N(u, A_1) \) is simple iff, for any two distinct monic irreducible factors \( p \) and \( q \) from \( K[H] \) of the polynomial \( H\sigma^{-1}(\alpha) \), there is no an integer \( i \) such that \( \sigma^i(p) = q \).

3. The algebra \( N(u, A_1) \) has only finitely many (two-sided) ideals, they are classified in [5]. Each nonzero ideal has finite codimension in \( N(u, A_1) \).

4. The Krull dimension of the algebra \( N(u, A_1) \) is 1.

5. Let \( H\sigma^{-1}(\alpha) = p_1^{n_1} \cdots p_s^{n_s} \) be a product of distinct monic irreducible polynomials. The global dimension

\[
\text{gl.dim } N(u, A_1) = \begin{cases} 
\infty, & \text{if there exists } n_i \geq 2; \\
2, & \text{if } n_1 = \cdots = n_s = 1 \text{ and } \sigma^i(p_j) = p_k \\
1, & \text{for some } j \neq k \text{ and some integer } i; \\
1, & \text{otherwise.}
\end{cases}
\]

Proof. 1. By Lemma 4.1.(2), the algebra \( N(u, A_1) \) is generated by the elements \( Y, H \) and \( X' = \alpha X \). Since

\[
X'H = \sigma(H)X', \ YH = \sigma^{-1}(H)Y, \ YX' = H\sigma^{-1}(\alpha) \text{ and } X'Y = \sigma(H\sigma^{-1}(\alpha)),
\]

the algebra \( N(u, A_1) \) is isomorphic to the generalized Weyl algebra \( K[H](\sigma, H\sigma^{-1}(\alpha)) \) in a view of the decomposition from Lemma 4.1.(2).

2 and 3. These results were proved in [4, 5, 6].
4 and 5. These results were proved in [4, 6, 20].
Corollary 4.4 Let $u \in A_1$ be as in Lemma 4.1. Then the left $N(u, A_1)$-module $M = A_1/N(u, A_1)$ is a $K[H]$-torsion, not finitely generated left $N(u, A_1)$-module of Gelfand-Kirillov dimension 1. Each finitely generated $N(u, A_1)$-submodule of $M$ has finite length. The set of isomorphism classes of simple subfactors of all finitely generated $N(u, A_1)$-submodules of $M$ is a finite set.

Proof. The algebra $N(u, A_1)$ is a homogeneous subalgebra of the Weyl algebra $A_1$, thus the $N(u, A_1)$-module $M = \oplus_{i \geq 1} M_i$ is an $\mathbb{N}$-graded $N(u, A_1)$-module where the $i$'th component $M_i$, as a $K[H]$-module, is canonically isomorphic to

$$K[H]/u^iK[H] = K[H]/\alpha_i X^i K[H] \simeq K[H]/\alpha_i K[H],$$

where $\alpha_i := \alpha \sigma(\alpha) \cdots \sigma^{i-1}(\alpha)$. Since $\alpha_i M_i = 0$, for all $i \geq 1$, the module $M$ is a $K[H]$-torsion module. Each finitely generated $K[H]$-torsion module over a generalized Weyl algebra of the type $K[H]/(\sigma, a \neq 0)$, for example $N(u, A_1)$, has finite length and Gelfand-Kirillov dimension $\leq 1$, thus $\text{GK}(M) \leq 1$. Observe that the $N(u, A_1)$-submodule $L$ of $M$ generated by the element $X = X + N(u, A_1)$ is not finite dimensional since

$$u^i \bar{X} = \alpha_i X^i \bar{X} = X^{-1}(\alpha_i) X^i + N(u, A_1) = X^{-1}(\alpha_i - \alpha_i) X^i + N(u, A_1) = (\alpha_i - \sigma(\alpha_i)) \bar{X} \neq 0$$

since $0 < \deg(\alpha_i - \sigma(\alpha_i)) < \deg \alpha_i$. So, $1 \leq \text{GK}(L) \leq \text{GK}(M) \leq 1$, hence $\text{GK}(M) = 1$.

A finitely generated $N(u, A_1)$-submodule, say $V$, of $M$ is a submodule of the module $U_s$ generated by $\oplus_{i=1}^s M_i$ for some $s$. The $N(u, A_1)$-module $U_s$ is an epimorphic image of the $N(u, A_1)$-module $\oplus_{i=1}^s N(u, A_1)/N(u, A_1)\alpha_i$. Each $N(u, A_1)$-module $N(u, A_1)/N(u, A_1)\alpha_i$ has finite length, and the set of all isomorphic classes of all simple subfactors of all modules $N/Na_\alpha$ is a finite set since $\alpha_i := \alpha \sigma(\alpha) \cdots \sigma^{i-1}(\alpha)$ (see [1, 6] for details). Now the result follows.

5 Classification of Homogeneous Elements of the Weyl Algebra and the Dixmier’s Problem 4

Let $u$ be a non-scalar element of the Weyl algebra $A_1$. The corresponding inner derivation $\text{ad} u$ of $A_1$ is denoted by $\delta$. Denote by $\text{Ev}(\text{ad} u, A_1) = \text{Ev}(u, A_1)$ the set of all eigenvalues of the linear map $\delta$ acting in the vector space $A_1$. For an eigenvalue $\lambda$ of $\delta$ we denote by $D(u, \lambda, A_1)$ the set of all eigenvectors of $\delta$ with the eigenvalue $\lambda$. The map $\delta$ is a derivation of the Weyl algebra $A_1$, so the set $\text{Ev}(u, A_1)$ is an additive submonoid of the field $K$, and the vector space

$$D(u)^{\text{ev}} = D(u, A_1)^{\text{ev}} := \oplus_{\lambda \in \text{Ev}(u, A_1)} D(u, \lambda, A_1)$$

is a $\text{Ev}(u, A_1)$-graded algebra, that is,

$$D(u, \lambda, A_1)D(u, \mu, A_1) \subseteq D(u, \lambda + \mu, A_1), \text{ for all } \lambda, \mu \in \text{Ev}(u, A_1).$$
Let a field $\bar{K}$ be an algebraic closure of the field $K$. The tensor product of algebras $A_1 = K \otimes A_1$ over the field $\bar{K}$ is the Weyl algebra over the field $\bar{K}$ which contains the Weyl $K$-algebra $A_1$. Then

$$D(u) = D(u, A_1) := A_1 \cap D(u, A_1)^{ev}$$

is a $K$-subalgebra of $A_1$ which contains the algebra $D(u)^{ev}$ but does not necessarily coincide with this algebra. The next result, obtained by Dixmier, \[16\], classifies non-scalar elements of the Weyl algebra $A_1$ with respect to the properties of the corresponding inner derivations of elements.

**Theorem 5.1 (The Dixmier’s Classification of Non-scalar Elements of the Weyl Algebra)** The set of non-scalar elements of the Weyl algebra $A_1$ is a disjoint union of the following subsets:

1. $\Delta_1 \equiv \{ x \in A_1 \mid K : N(x) = A_1, \ D(x) = C(x) \}$.
2. $\Delta_2 \equiv \{ x \in A_1 \mid K : N(x) \neq A_1, \ N(x) \neq C(x), \ D(x) = C(x) \}$.
3. $\Delta_3 \equiv \{ x \in A_1 \mid K : D(x) = A_1, \ N(x) = C(x) \}$.
4. $\Delta_4 \equiv \{ x \in A_1 \mid K : D(x) \neq A_1, \ D(x) \neq C(x), \ N(x) = C(x) \}$.
5. $\Delta_5 \equiv \{ x \in A_1 \mid K : D(x) = N(x) = C(x) \}$. 

Each subset $\Delta_i$ of $A_1$ is a non-empty set.

**Definition.** Elements of $\Delta_3 \cup \Delta_4$ (resp. of $\Delta_3$) are called elements of *semi-simple type* (resp. of strongly semi-simple type). Dixmier classified elements of strongly semi-simple type, Theorem 9.2, \[16\]: if there exists an automorphism $\tau \in \text{Aut}_K A_1$ such that $\tau(x) = \lambda Y^2 + \mu X^2 + \nu$ for some scalars $\lambda, \mu$ and $\nu$ such that $\lambda \neq 0$ and $\mu \neq 0$. It can be easily seen that, if the polynomial $\lambda t^2 + \mu$ has a root in the field $K$ then there exists an automorphism $\tau_1 \in \text{Aut}_K A_1$ such that $\tau_1(x) = \alpha H + \beta$ for some scalars $0 \neq \alpha$ and $\beta$ (see Corollary 9.3, \[16\]).

**Theorem 5.2 (Classification of Homogeneous Elements of the Weyl Algebra)** Let $u = \alpha v_i, \ \alpha \in K[H], \ i \in \mathbb{Z}$, be a homogeneous non-scalar element of $A_1$. Then $u \in \Delta_1 \cup \Delta_2 \cup \Delta_3 \cup \Delta_5$. In more detail,

1. $u \in \Delta_1 \iff \alpha \in K^* \text{ and } i \neq 0$.
2. $u \in \Delta_2 \iff \alpha \not\in K \text{ and } i \neq 0$.
3. $u \in \Delta_3 \iff \deg_H \alpha = 1 \text{ and } i = 0$.
4. $u \in \Delta_5 \iff \deg_H \alpha > 1 \text{ and } i = 0$.

**Proof.** If $\alpha \in K^*$ and $i \neq 0$ then by \[16\], Proposition 10.3, $N(u, A_1) = N(v_i, A_1) = N(v_{i+1}, A_1) = A_1$, hence $u \in \Delta_1$. If $\alpha \not\in K$ and $i \neq 0$ then $u \in \Delta_2$, by Theorem \[3\,2\](1). If $\deg_H \alpha = 1$ and $i = 0$, i.e. $u = \lambda H + \mu$ for some scalars $\lambda \neq 0$ and $\mu$, then $u \in \Delta_3$. If $\deg_H \alpha > 1$ and $i = 0$ then, by Theorem \[3\,2\](2), $N(u, A_1) = C(u, A_1) = K[H]$. The element $u = \alpha$ is a homogeneous element of the algebra $A_1$, thus the algebra
$D(u, A_1)$ is a homogeneous subalgebra of $A_1$. Suppose that $D(u, A_1) \neq C(u, A_1)$ then there exists a homogeneous element $\beta v_m$ ($\beta \in K[H]$) of $A_1$ and a nonzero scalar $\lambda$ such that $\lambda \beta v_m \in D(u, \lambda, A_1)$, hence $\lambda \beta v_m = [\alpha, \beta v_m] = (\alpha - \sigma^m(\alpha))\beta v_m$. So, $\lambda = \alpha - \sigma^m(\alpha)$, hence $\deg_H \alpha = 1$, a contradiction. This means that $D(u, A_1) = C(u, A_1)$ and $u \in \Delta_5$. This finishes the proof of the theorem.

**Corollary 5.3** Let $u$ be a homogeneous element of weakly nilpotent type of the Weyl algebra $A_1$, i.e. $u \in \Delta_2$. Then

1. The associated graded algebra $G(u, A_1)$ is an affine commutative algebra (thus, the Dixmier’s Problem 4 has a positive answer for homogeneous elements of nilpotent type) hence the algebra $N(u, A_1)$ is an affine Noetherian algebra.

2. The algebra $A_1$ is not a finitely generated (left and right) $N(u, A_1)$-module.

**Proof.** 1. By Theorem 5.2 (2), each homogeneous element of nilpotent type of the Weyl algebra $A_1$ has the form as in Theorem 2.3 (1). Now observe that the result is already was proved in Theorem 5.2 (2) (iii) and (iv).

2. This follows from Corollary 3.3 (3) and Theorem 5.2 (2).

Let $a$ and $p$ be nonzero elements of the Weyl algebra $A_1$ satisfying $[a, p] = \lambda p$ for some $0 \neq \lambda \in K$. Then the element $p$ is of nilpotent type, and $[a, cp] = \lambda cp$ for all nonzero elements $c \in C(a, A_1)$. The next result shows how the type of the element $p$ changes when $p$ is multiplied by an element from $C(a, A_1)$.

**Corollary 5.4** Let $a \in \Delta_3(A_1)$ and $[a, p] = \lambda p$ for some $0 \neq \lambda \in K$ and $p \in A_1$. Then $C(a, A_1) = K[a]$.

1. Suppose that $p \in \Delta_1(A_1)$ and $0 \neq \alpha(t) \in K[t]$. Then
   
   (i) $\alpha(a)p \in \Delta_1(A_1)$ if and only if $\alpha \in K^*$.

   (ii) $\alpha(a)p \in \Delta_2(A_1)$ if and only if $\alpha \not\in K$.

2. Suppose that $p \in \Delta_2(A_1)$. Then $\alpha(a)p \in \Delta_2(A_1)$ for all nonzero polynomials $\alpha(t) \in K[t]$.

**Proof.** The fact that $C(a, A_1) = K[a]$ easily follows from Theorem 9.2 and Corollary 9.4. [16]

We may assume that $K$ is an algebraically closed field. Then, by Theorem 9.2 and Corollary 9.4, [16], there exists an automorphism $\nu \in \text{Aut}_K(A_1)$ such that $\nu(a) = \mu H + \gamma$ for some scalars $\mu \neq 0$ and $\gamma$. So, multiplying the element $a$ by $\mu^{-1}$ and adding an appropriate scalar to $H$, without loss of generality we may assume that $a = H$. Now, $p$ is a homogeneous nonscalar element of the algebra $A_1$, and the result follows from Theorem 3.2.

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