EXISTENCE AND SYMMETRY OF PERIODIC NONLOCAL-CMC SURFACES VIA VARIATIONAL METHODS

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Abstract. This paper provides the first variational proof of the existence of periodic nonlocal-CMC surfaces. These are nonlocal analogues of the classical Delaunay cylinders. More precisely, we show the existence of a set in $\mathbb{R}^n$ which is periodic in one direction, has a prescribed (but arbitrary) volume within a slab orthogonal to that direction, has constant nonlocal mean curvature, and minimizes an appropriate periodic version of the fractional perimeter functional under the volume constraint. We show, in addition, that the set is cylindrically symmetric and, more significantly, that it is even as well as nonincreasing on half its period. This monotonicity property solves an open problem and an obstruction which arose in an earlier attempt, by other authors, to show the existence of minimizers.

1. Introduction

For a general set $E \subset \mathbb{R}^n$, the nonlocal (or $s$-fractional, $0 < s < 1$) mean curvature of $\partial E$ (or of $E$) at a boundary point $x \in \partial E$ is defined by

$$H_s[E](x) := PV \int_{\mathbb{R}^n} dy \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+s}} = \lim_{\epsilon \searrow 0} \int_{\{|x-y| > \epsilon\}} dy \frac{\chi_{E^c}(y) - \chi_E(y)}{|x-y|^{n+s}}.$$ 

Here $PV$ denotes the principal value sense, $\chi_E$ is the characteristic function of the set $E$, and $E^c$ is the complement in $\mathbb{R}^n$ of the set $E$. The nonlocal mean curvature arises when computing the first variation of the fractional perimeter, an extension of the classical perimeter that was introduced, in its localized or Dirichlet version, in the pioneering article of Caffarelli, Roquejoffre, and Savin [13]. Their work has allowed for a natural extension of the theory of minimal surfaces to the nonlocal framework.

The current paper concerns the nonlocal analogue of the classical result of Delaunay [17] on the existence of periodic surfaces of revolution with constant mean curvature (CMC). We provide the first variational proof of the existence of periodic surfaces of revolution having constant nonlocal mean curvature and a prescribed (but arbitrary) volume within one period. Throughout the paper, surfaces with constant nonlocal mean curvature will be called nonlocal-CMC surfaces. Before our work, and up to our knowledge, the only papers which had established the existence of some periodic nonlocal-CMC surfaces are [10] [11] [12], by Cabré, Fall, Solà-Morales, and Weth. These works did not use variational methods, but perturbative techniques...
(the implicit function theorem, essentially). Thus, they only obtained nonlocal-CMC surfaces which are very close to certain explicit configurations, as explained next.

In arbitrary dimension, [11] showed the existence of a one parameter family of periodic nonlocal-CMC surfaces bifurcating from a straight cylinder. The paper extended a previous analogous result in \( \mathbb{R}^2 \) from [10]. Later, also by perturbation techniques, [12] proved the existence of a periodic array of disjoint near-balls, the full array having constant nonlocal mean curvature. They bifurcate from infinity, starting from a single round ball of radius 1 —in particular, the near-balls have all the same shape (which is very close to a unit ball) and are very distant from each other. Thus, after rescaling them to have their consecutive centers at distance \( 2\pi \), they become of very small size (and, hence, of small volume within a period). Instead, the \( 2\pi \)-periodic near-cylinders from [11] have a certain positive, bounded below, width.

Prior to the current paper, no result connecting both configurations, through other periodic nonlocal-CMC surfaces, was available. This paper establishes for the first time the existence of a periodic nonlocal-CMC surface in \( \mathbb{R}^n \) for every given volume within a slab. By a variational method we show the existence of a minimizer of the functional

\[
P_s[E] := \int_{E \cap \{-\pi < x_1 < \pi\}} dx \int_{\mathbb{R}^n \setminus E} dy \frac{1}{|x - y|^{n+s}}
\]

(introduced for the first time in [16]), under a volume constraint within the slab \( \{x \in \mathbb{R}^n : -\pi < x_1 < \pi\} \), among subsets \( E \) of \( \mathbb{R}^n = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}\} \) which are \( 2\pi \)-periodic in the coordinate \( x_1 \). Notice that the functional \( P_s \) differs from the fractional \( s \)-perimeter of \( E \) relative to \( \{x \in \mathbb{R}^n : -\pi < x_1 < \pi\} \), a quantity which is given by adding an additional term to \( P_s \), as follows:

\[
P_s[E] + \int_{(\mathbb{R}^n \setminus E) \cap \{-\pi < x_1 < \pi\}} dx \int_{E \cap \{|y_1| > \pi\}} dy \frac{1}{|x - y|^{n+s}}.
\]

Still, and somehow surprisingly, we establish that the first variation of these two functionals —the periodic first variation for (1.1) and the Dirichlet first variation in the slab for (1.2)— lead both to the nonlocal mean curvature.

The following is our first main result. Apart from the existence of a minimizer, it also establishes symmetry and monotonicity properties of any minimizer.

**Theorem 1.1.** Given \( n \geq 2 \), \( s \in (0, 1) \), and \( \mu > 0 \), there exists a minimizer \( E \subset \mathbb{R}^n \) of \( P_s \) among all measurable sets \( F \subset \mathbb{R}^n \) which are \( 2\pi \)-periodic in the \( x_1 \)-variable and satisfy

\[
|F \cap \{-\pi < x_1 < \pi\}| = \mu.
\]

In addition, up to sets of measure zero:

(i) Up to a translation in the \( x' \)-variable, every minimizer \( E \) is of the form

\[
E = \{x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1} : |x'| < u(x_1)\}
\]

for some \( 2\pi \)-periodic nonnegative function \( u \) whose power \( u^{n-1} \) belongs to \( W^{s,1}(-\pi, \pi) \). In particular, we have \( u \in W^{s,1}(-\pi, \pi) \).

\(^3\)Such nonlocal-CMC curves in the plane have no analogue, obviously, for the classical mean curvature.

\(^2\)Recall that, in the local case, the classical Delaunay surfaces vary continuously from a cylinder to an infinite compound of tangent spheres.

\(^3\)The Dirichlet first variation of (1.2) was derived in the seminal paper [13] by Caffarelli, Roquejoffre, and Savin.
(ii) Up to a translation in the $x_1$-direction, $u$ is even, as well as nonincreasing in $(0, \pi)$.

(iii) For every minimizer $E$, there exists a constant $\lambda \in \mathbb{R}$ such that if $x \in \partial E$, $u$ is $C^{1, \alpha}$ in a neighborhood of $x_1$ for some $\alpha > s$, and $u(x_1) > 0$, then $H_s[E](x) = \lambda$.

(iv) For $\mu$ small enough depending only on $n$ (and hence uniformly as $s \uparrow 1$ and as $s \downarrow 0$), every minimizer is not a straight circular cylinder, i.e., the function $u$ is nonconstant.

The symmetry and monotonicity properties of points (i) and (ii) will be discussed in the next Subsection 1.1. Of particular importance is point (ii), since it relies on a deep and not so well-known result on rearrangements in spheres $S^m$. However, we emphasize that our proof of existence of a minimizer only uses simple tools and, in particular, it is independent of this rearrangement result.

The work [16] was the first attempt to show the existence of periodic nonlocal-CMC surfaces with a minimization technique. This paper introduced, for the first time, the functional $P_s$ in (1.1) for periodic sets. However, it did not relate its Euler-Lagrange equation to the nonlocal mean curvature. At the same time, [16] did not prove the existence of a periodic constrained minimizer of $P_s$—it only obtained a minimizer within the class of sets which are even in the $x_1$-variable and nonincreasing in $(0, \pi)$. Note that this restricted minimization class (made of nonincreasing functions) prevents from checking the Euler-Lagrange equation at a minimizer. The result on rearrangements in spheres $S^m$, mentioned above in connection with point (ii) of our theorem, is the additional tool that would have completed the proof in [16]. Instead, as explained later, [16] proved a much stronger result than our statement (iv) (on small volumes) in Theorem 1.1.

The work [16] made apparent, thus, that showing the existence of a periodic constrained minimizer of $P_s$ was a delicate issue. In this article we succeed to find a proof which does not rely, in contrast with the approach in [16], on the nonincreasing property of the functions $u$ in a minimizing sequence. To accomplish this, the main difficulty was finding a suitable, more treatable expression for the periodic nonlocal perimeter. This expression is given in Lemma 2.2, a result that also contains the important lower bound (2.4). With these tools at hand, the existence of minimizer will follow readily from the compact embedding of $W^{s,1}(-\pi, \pi)$ into $L^1(-\pi, \pi)$.

In the course of this work, we first completed our existence proof in the case of the plane, $n = 2$. This was a simpler task, since it relied on an expression for the fractional perimeter which is similar to some for the nonlocal mean curvature contained in our 2D results from [10]. Since the expression for $n = 2$ is more explicit than the one in Lemma 2.2, we will present it in Appendix A.

Our proof of point (iv) in Theorem 1.1—stating that minimizers for small volumes are not straight cylinders, uniformly in $s$—is simple. It is based on an energy-comparison argument and does not require the knowledge of minimizers being nonincreasing in $(0, \pi)$. Instead, Theorem 3 of the above mentioned paper [16] proved a much stronger, quantitative version of this result. Using the monotonicity of the minimizer, it established that for small volume constraints a minimizer must be close, in a measure theoretical sense, to a periodic array of round balls. Notice here that the function $u$ defining the minimizer could be identically zero in a subinterval.

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4At that time, we knew that the first variation of $P_s$ for periodic sets is the nonlocal mean curvature, since we were working on the analogous functional for periodic solutions to fractional semilinear equations (work of us still to appear). When [16] became available, we communicated this fact to its authors, we pointed it out at the beginning of page 35 in the printed version of [11], and was also mentioned in the Master’s Thesis of M. Alvinyà [2] directed within our group.
(a, π) of (0, π) and that this allows for the minimizer to be a periodic array of near-balls. At the same time, the near-balls found in \cite{12} are very close to round balls since they were found through the implicit function theorem. Thus, the result of \cite{16} strongly suggests that for small volumes the minimizers should agree with the near-balls constructed in \cite{12}. However, this is still an open problem.

Regarding point (iii) in Theorem \ref{thm:1.1} we recall that the nonlocal mean curvature is well defined at points \( x \in \partial E \) where \( \partial E \) is of class \( C^{1, \alpha} \), for some \( \alpha > s \), in a neighborhood of \( x \). However, a complete \( C^{1, \alpha} \) regularity theory for volume constrained minimizers is not yet available.

The following subsection addresses the symmetry and monotonicity properties of minimizers.

### 1.1. Symmetry and monotonicity properties of minimizers

Theorem \ref{thm:1.1} contains two results on the symmetry and monotonicity properties of minimizing sets: their cylindrical symmetry, \( (1.3) \), and the fact that \( u \) is an even function which is nonincreasing in \( (0, \pi) \). These properties will follow from two rearrangement inequalities.

The first one concerns the **cylindrical rearrangement**, defined as follows. For a measurable set \( E \subset \mathbb{R}^n \), consider its sections \( E_{x_1} = \{ x' \in \mathbb{R}^{n-1} : (x_1, x') \in E \} \), where \( x_1 \in \mathbb{R} \). Define \( E^{cyl} \) as the set whose sections \( (E^{cyl})_{x_1} \) for every \( x_1 \in \mathbb{R} \) are concentric open balls of \( \mathbb{R}^{n-1} \) centered at \( 0 \) and of the same \((n-1)\)-dimensional Lebesgue measure as \( E_{x_1} \). The fact that the cylindrical rearrangement does not increase the functional \( \mathcal{P}_s \) follows from the Riesz rearrangement inequality, and it was already used in \cite{16} Proposition 13. It will lead to the existence of a cylindrical symmetric minimizer. A stronger fact is that every minimizer is cylindrically symmetric, as stated in Theorem \ref{thm:1.1}. This will follow from a strict version of the Riesz rearrangement inequality which analyzes the case of equality; see Lemma \ref{lem:3.1}.

The second rearrangement that we need to consider is the **symmetric decreasing periodic rearrangement**, which is defined in the following way. First, for a measurable set \( A \subset \mathbb{R}^n \) consider

\[
A^* := \text{Steiner symmetrization of } A \text{ with respect to } \{ x_1 = 0 \}.
\]

This means that \( A^* \) is symmetric with respect to the hyperplane \( \{ x_1 = 0 \} \) and that \( A^* \cap \{ x' = c \} \) is an open interval of the same 1-dimensional Lebesgue measure as \( A \cap \{ x' = c \} \) for every \( c \in \mathbb{R}^{n-1} \). Now, given a set \( E \subset \mathbb{R}^n \) which is \( 2\pi \)-periodic in the \( x_1 \)-variable we define its symmetric decreasing periodic rearrangement by

\[
E^{\text{per}} := 2\pi \text{-periodic extension of } (E \cap \{-\pi < x_1 < \pi\})^*,
\]

where the periodic extension is meant in the coordinate \( x_1 \). It is called “symmetric decreasing” since the characteristic function \( \chi_{E^{\text{per}}} \) is nonincreasing in \( x_1 \) within half of the period, \( (0, \pi) \), for each given \( x' \in \mathbb{R}^{n-1} \). Note also that if \( E \) is cylindrically symmetric, i.e., as in \( (1.3) \), then

\[
E^{\text{per}} = \{ x \in \mathbb{R}^n : |x'| < u^{\text{per}}(x_1) \},
\]

where \( u^{\text{per}} \) is the \( 2\pi \)-periodic function which is even, nonincreasing in \( (0, \pi) \), and equimeasurable with \( u \) (see Section \ref{sec:3} for the precise definition).

\footnote{In the nonperiodic case, the best result (up to our knowledge) on the regularity of volume constrained minimizers of a certain fractional perimeter functional is stated in \cite{23} (which treats sets of revolution in a half-space), claiming their smoothness except for a countable set of points in the rotational axis. Notice that the Euler-Lagrange equation of the functional in \cite{23} is closely related, but not equal to, the nonlocal mean curvature.}
Our second main result establishes that the periodic fractional perimeter $P_s$ in (1.1) does not increase under symmetric decreasing periodic rearrangement. This was left as an open problem in the last paragraph of [16]. Our result also characterizes the case of equality.

**Theorem 1.2.** Let $E \subset \mathbb{R}^n$ be a measurable set which is $2\pi$-periodic in the coordinate $x_1$ and satisfies $P_s[E] < +\infty$.

Then, $P_s[E^{\text{per}}] \leq P_s[E]$. Moreover, if $P_s[E^{\text{per}}] = P_s[E]$ then, up to a set of measure zero, $E = E^{\text{per}} + ce_1$ for some $c \in \mathbb{R}$, where $e_1 := (1, 0, \ldots, 0) \in \mathbb{R}^n$.

A key ingredient in the proof of Theorem 1.2 is a Riesz rearrangement inequality on the circle $S^1$, a result which is not as well-known as the classical Riesz rearrangement inequality on $\mathbb{R}^n$. Such periodic rearrangement inequality is originally due to Baernste in and Taylor [7] and to Friedberg and Luttinger [19], who proved independently different versions of the inequality around the same time; see more details in the comments following Theorem 3.2. The case of equality in the Riesz rearrangement inequality on the circle (and more generally on the sphere $S^n$) was first treated by Burchard and Hajaiej in [8].

### 1.2. On the shape of minimizers: stability of straight cylinders.

Let us first recall what is known about the shape of minimizers in the local case, that is, for the classical isoperimetric problem relative to a slab. It turns out that, for $n \leq 8$, the minimizers are either spheres or straight cylinders, depending on whether the volume constraint is smaller or greater, respectively, than a critical volume; see [4, 27] or [20, Theorem 6] for $n = 3$, and [25] for $n \leq 8$. In particular, the classical Delaunay unduloids are not minimizers for $n \leq 8$. Furthermore, they are known to be unstable, as proven independently in [4, 27] for $n = 3$, and in [25] for $3 \leq n \leq 8$. In fact, when $n \leq 8$ the infinite array of balls and the cylinders are the only stable periodic solutions to the constant mean curvature equation; see [25, Corollary 5.4].

When $n \geq 10$ it has been proven in [25, Proposition 3.4] that there exist unduloids which are minimizers under certain volume constraints (namely for volumes which are close to that of a ball that has the diameter of one period). For large volumes, however, cylinders are the only minimizers in all dimensions, as shown in [26, Theorem 1.1]. For $n = 9$, [21, Section 5] provides numerical evidence that there exist stable unduloids in this dimension (as it is the case for $n \geq 10$).

A simpler fact is to characterize the stability of straight cylinders, still in the local case. A cylinder $\{(x_1, x') \in \mathbb{R}^n : |x'| < R\}$ is stable if and only if

$$R^2 \geq n - 2,$$

as can be verified by a simple calculation (see [4] in the case $n = 3$ and use the Euler-Lagrange equation [21, Equation (4)] for general $n$).

In view of these facts for the local case, and at least in low dimensions, one may expect the minimizers of Theorem 1.1 to be the near-balls found in [12] for small volumes and straight cylinders for large volumes. To sustain this statement for small volumes (which is still an open problem), recall first that the function $u$ defining the minimizer in (1.3) could be identically zero in a subinterval $(a, \pi)$ of $(0, \pi)$ —allowing the minimizer, in this way, to be a periodic array of near-balls. Recall also that [16, Theorem 3] proved that, for small volumes, a minimizer must be close (in a measure theoretical sense) to a periodic array of round balls. On the other hand,
the periodic array of near-spheres found in [12] are obtained as small perturbations of a single round sphere, and hence remains as a stable configuration.

With regard to large volumes, in [9] we will establish the following result on the stability of straight cylinders. Its conclusion supports the conjecture that the only minimizers for large volumes should be straight cylinders. By stability we mean that the second variation of $P_{s}[E_{t}]$ is nonnegative for all $2\pi$-periodic volume preserving variations $\{E_{t}\}_{t \in (-\epsilon, \epsilon)}$ of the set $E$.

**Theorem 1.3** ([9]). Let $n \geq 2$ and $s \in (0, 1)$.

Then, there exists a radius $R_{s} > 0$ (depending only on $n$ and $s$) such that the set $E = \{(x_{1}, x') \in \mathbb{R}^{n} : |x'| < R\}$ is stable (for $P_{s}$ and among sets which are $2\pi$-periodic in the $x_{1}$ direction) if and only if $R \geq R_{s}$. Moreover,

$$\lim_{s \uparrow 1} R_{s}^{2} = n - 2.$$ 

Observe that the last statement fits with (1.4), also in the case $n = 2$ — a dimension for which all above questions are trivial in the local case, but not in the nonlocal case. The proof of Theorem 1.3 in [9] uses tools from [10, 11] mainly, but also the symmetric decreasing periodic rearrangement of the present paper. Indeed, we will use the fact that, thanks to Theorem 1.2, to show stability we will only need to consider even variations of $E$. In this way, we will be able to use the basis $\cos(k \cdot)$, $k \in \mathbb{Z}$, of even eigenfunctions to the linearized operator of $P_{s}$ at a straight cylinder $E$.

1.3. **Plan of the paper.**

— In Section 2 we prove Theorem 1.1, even though we will borrow the results of Section 3 only to prove the symmetry and monotonicity properties of minimizers claimed in the equality (1.3) of (i) and in (ii).

— Section 3 deals with the behaviour of $P_{s}$ under cylindrical rearrangement and under symmetric decreasing periodic rearrangement. In particular, here we prove Theorem 1.2.

— In Appendix A we present a more explicit expression for $P_{s}$ when $n = 2$. This gives an alternative proof of the existence part in Theorem 1.1 for $n = 2$.

— In Appendix B we provide a simple proof of a well-known monotonicity property of the fundamental solution of the one-dimensional heat equation under periodic boundary conditions. The result is crucially used in the proof of Theorem 1.2.

2. **Existence of periodic minimizers**

In this section we establish Theorem 1.1. Before stating its key lemma, we first prove an elementary one which will be used several times in the paper.

**Lemma 2.1.** Let $m$ and $l$ be two positive integers, $u = (u_{1}, \ldots, u_{m})$ and $v = (v_{1}, \ldots, v_{l})$ be $2\pi$-periodic functions in $\mathbb{R}$ with values in $\mathbb{R}^{m}$ and $\mathbb{R}^{l}$, respectively. Suppose that $\Phi : \mathbb{R}^{m} \times \mathbb{R}^{l} \times [0, +\infty) \to \mathbb{R}$ is such that the function $(x, y) \mapsto \Phi(u(x), v(y), |x - y|)$ belongs to $L^{1}((-\pi, \pi) \times \mathbb{R})$.

Then, this last function belongs also to $L^{1}(\mathbb{R} \times (-\pi, \pi))$ and

$$\int_{-\pi}^{\pi} dx \int_{\mathbb{R}} dy \Phi(u(x), v(y), |x - y|) = \int_{\mathbb{R}} dx \int_{-\pi}^{\pi} dy \Phi(u(x), v(y), |x - y|).$$  

(2.1)
Proof. In the integral on the left-hand side of (2.1), both for $|\Phi|$ and for $\Phi$, we can write the domain of integration $\mathbb{R}$ as the union over all $((2k-1)\pi, (2k+1)\pi)$, where $k \in \mathbb{Z}$. Thus the left-hand side is the countable sum of integrals over the domains $(x, y) \in (-\pi, \pi) \times ((2k-1)\pi, (2k+1)\pi)$. For each $k \in \mathbb{Z}$ we carry out the change of variables $x = \bar{x} + 2k\pi$ and $y = \bar{y} + 2k\pi$, where $(\bar{x}, \bar{y}) \in ((-2k-1)\pi, (-2k+1)\pi) \times (-\pi, \pi)$. Now, the periodicity of $u$ and $v$, and the identity $|x - y| = |\bar{x} - \bar{y}|$, yield the claim of the lemma. 

The following lemma will be a key ingredient in the proof of existence of minimizer.

**Lemma 2.2.** Let $n \geq 2$ and $E \subset \mathbb{R}^n$ be a set of the form

$$E = \{x \in \mathbb{R}^n : |x'| < u(x_1)\}$$

with $u : \mathbb{R} \to \mathbb{R}$ measurable, nonnegative, and $2\pi$-periodic in $x_1$.

Then,

$$\mathcal{P}_s[u] := \mathcal{P}_s[E] = \int_{-\pi}^\pi dx_1 \int_{-\infty}^{+\infty} dy_1 \int_{\{x' \in \mathbb{R}^{n-1} : |x'| < u(x_1)\}} dx'$$

$$= \int_{-\pi}^\pi dx_1 \int_{-\infty}^{+\infty} dy_1 \int_{\{x' \in \mathbb{R}^{n-1} : |x'| < u(x_1)\}} dx'$$

$$\quad \int_{\{y' \in \mathbb{R}^{n-1} : |y'| > u(y_1)\}} dy' (|x_1 - y_1|^2 + |x' - y'|^2)^{-\frac{n+s}{2}}$$

$$= \int_{-\pi}^\pi dx_1 \int_{-\infty}^{+\infty} dy_1 \int_{\{x' \in \mathbb{R}^{n-1} : |x'| < u(x_1)\}} dx'$$

$$\quad \int_{\{y' \in \mathbb{R}^{n-1} : |y'| > u(y_1)\}} dy' \left(1 + \left|\frac{x'}{|x_1 - y_1|} - \frac{y'}{|x_1 - y_1|}\right|^2\right)^{-\frac{n+s}{2}}$$

$$= \int_{-\pi}^\pi dx_1 \int_{-\infty}^{+\infty} dy_1 \int_{\{x' \in \mathbb{R}^{n-1} : |x'| < u(x_1)\}} dx'$$

$$\quad \int_{\{y' \in \mathbb{R}^{n-1} : |y'| > u(y_1)\}} dy' (1 + |w' - z'|^2)^{-\frac{n+s}{2}},$$

which gives (2.2)–(2.3).
Next, to prove (2.4) for $0 \leq q \leq p$, we see that
\[
\phi(p, q) = \int_{\{w \in \mathbb{R}^{n-1} : |w'| < p\}} dw' \int_q^{+\infty} dr r^{n-2} \int_{S^{n-2}} d\mathcal{H}^{n-2}(\sigma) \left(1 + |w' - r\sigma|^2\right)^{-\frac{n+4}{2}}
\]
\[
= \int_q^{+\infty} dr r^{n-2} \int_{S^{n-2}} d\mathcal{H}^{n-2}(\sigma) \int_{\{w \in \mathbb{R}^{n-1} : |w'| < p\}} dw' \left(1 + |w' - r\sigma|^2\right)^{-\frac{n+4}{2}}
\]
\[
\geq |S^{n-2}| \int_q^{p} dr r^{n-2} \int_{\{w \in \mathbb{R}^{n-1} : |w'| < p\}} dw' \left(1 + |w' - re'_1|^2\right)^{-\frac{n+4}{2}},
\]
where $e'_1 = (1, 0, \ldots, 0) \in \mathbb{R}^{n-1}$.

We now claim that, for some constant $c > 0$ depending only on $n$,\footnote{Here and throughout the paper, $B^k_r(y)$ denotes the open ball in $\mathbb{R}^k$ of radius $r$ centered at $y \in \mathbb{R}^k$. We also use the notation $B^k_r$ when $y = 0$, and $B_r$ if in addition $k = n$.}
\[
|B_1^{n-1}(re'_1) \cap B_p^{n-1}| \geq c \quad \text{if } 1 \leq r \leq p.
\]
This is simple. We have $B_p^{n-1} \supset B_r^{n-1} \supset B_1^{n-1}((r-1)e'_1)$. Hence,
\[
|B_1^{n-1}(re'_1) \cap B_p^{n-1}| \geq |B_1^{n-1}(re'_1) \cap B_1^{n-1}((r-1)e'_1)| = |B_1^{n-1} \cap B_1^{n-1}(-e'_1)| =: c.
\]
Therefore, from (2.5) and (2.6) we deduce that, for $1 \leq q \leq p$,
\[
\phi(p, q) \geq |S^{n-2}| \int_q^{p} dr r^{n-2} \int_{B_1^{n-1}(re'_1) \cap B_p^{n-1}} dw' \left(1 + |w' - re'_1|^2\right)^{-\frac{n+4}{2}}
\]
\[
\geq 2^{-\frac{n+4}{2}} |S^{n-2}| \int_q^{p} dr r^{n-2} |B_1^{n-1}(re'_1) \cap B_p^{n-1}|
\]
\[
\geq 2^{-\frac{n+4}{2}} |S^{n-2}| c \int_q^{p} dr r^{n-2} =: 2c_0(p^{n-1} - q^{n-1}).
\]
We conclude that (2.4) holds if $1 \leq q \leq p$.

Next, if $q < 1$ and $p^{n-1} \geq 2$, we see that $\phi(p, q) \geq \phi(p, 1) \geq 2c_0(p^{n-1} - 1)$ by definition of $\phi$ and the previous bound. But $p^{n-1} - 1 \geq (p^{n-1} - q^{n-1})/2$ since we assumed $p^{n-1} \geq 2$. Thus, in this case (2.4) also holds.

It only remains to consider the case $0 \leq q < 1$ and $p^{n-1} < 2$. But then we have
\[
c_0(p^{n-1} - q^{n-1}) - 2c_0 \leq c_0 p^{n-1} - 2c_0 < 0 \leq \phi(p, q),
\]
which finishes the proof. \qed

We now establish Theorem 1.1. The proof of existence of a minimizer relies only on the two previous basic lemmas and on the fact (already used in [16]) that $\mathcal{P}_s$ does not increase under cylindrical rearrangement (as stated in Lemma 3.1). Instead, the proof of the evenness and nonincreasing property of any minimizer uses our new result Theorem 1.2 on the periodic rearrangement.

**Proof of Theorem 1.1** We split the proof into four parts.

*Part 1 (Existence of minimizer)*. Here we prove the existence of a minimizer. Note first that $\mathcal{P}_s \geq 0$ and that $\mathcal{P}_s \not\equiv +\infty$ — this last fact will be easily seen in Part 4 of the proof when we check that $\mathcal{P}_s$ is finite when computed on a periodic array of balls.
Given $\mu > 0$, we take a minimizing sequence of sets with volume $\mu$ within the slab $\{ -\pi < x_1 < \pi \}$. In view of Lemma 3.1, the sets may be assumed to be of the form
\[ E_k = \{ x \in \mathbb{R}^n : |x'| < u_k(x_1) \} \]
for some measurable, nonnegative, and $2\pi$-periodic functions $u_k$.

By Lemma 2.2 we have that, for the constant $c_0$ in (2.4) and some positive constant $C_{n,s}$ depending only on $n$ and $s$,
\[
\int_{-\pi}^\pi dx_1 \int_{-\pi}^\pi dy_1 |x_1 - y_1|^{n-2-s} \left\{ c_0 + \phi \left( \frac{u_k(x_1)}{|x_1 - y_1|}, \frac{u_k(y_1)}{|x_1 - y_1|} \right) \right\} \\
= C_{n,s} c_0 + \int_{-\pi}^\pi dx_1 \int_{-\pi}^\pi dy_1 |x_1 - y_1|^{n-2-s} \phi \left( \frac{u_k(x_1)}{|x_1 - y_1|}, \frac{u_k(y_1)}{|x_1 - y_1|} \right) \\
\leq C_{n,s} c_0 + \mathcal{P}_s[E_k] \leq C
\]
since $\phi \geq 0$, where $C$ is a constant independent of $k$. From this, symmetrizing in $x_1$ and $y_1$, using (2.4) and again, $\phi \geq 0$, we see that
\[
C \geq \int_{-\pi}^\pi dx_1 \int_{-\pi}^\pi dy_1 |x_1 - y_1|^{n-2-s} \left\{ c_0 + \phi \left( \frac{u_k(x_1)}{|x_1 - y_1|}, \frac{u_k(y_1)}{|x_1 - y_1|} \right) \right\} \\
= \frac{1}{2} \int_{-\pi}^\pi dx_1 \int_{-\pi}^\pi dy_1 |x_1 - y_1|^{n-2-s} \left\{ c_0 + \phi \left( \frac{u_k(x_1)}{|x_1 - y_1|}, \frac{u_k(y_1)}{|x_1 - y_1|} \right) \right\} \\
+ \frac{1}{2} \int_{-\pi}^\pi dx_1 \int_{-\pi}^\pi dy_1 |x_1 - y_1|^{n-2-s} \left\{ c_0 + \phi \left( \frac{u_k(y_1)}{|x_1 - y_1|}, \frac{u_k(x_1)}{|x_1 - y_1|} \right) \right\} \\
\geq \frac{1}{2} \int_{-\pi}^\pi dx_1 \int_{-\pi}^\pi dy_1 |x_1 - y_1|^{n-2-s} \\
\quad \times \left\{ 2 c_0 + \phi \left( \frac{\max\{u_k(x_1), u_k(y_1)\}}{|x_1 - y_1|}, \frac{\min\{u_k(x_1), u_k(y_1)\}}{|x_1 - y_1|} \right) \right\} \\
\geq \frac{c_0}{2} \int_{-\pi}^\pi dx_1 \int_{-\pi}^\pi dy_1 |x_1 - y_1|^{n-2-s} \frac{u_k(x_1)^{n-1} - u_k(y_1)^{n-1}}{|x_1 - y_1|^{n-1}} \\
= \frac{c_0}{2} \int_{-\pi}^\pi dx_1 \int_{-\pi}^\pi dy_1 \frac{|u_k(x_1)^{n-1} - u_k(y_1)^{n-1}|}{|x_1 - y_1|^{1+s}}.
\]

This gives a uniform bound on the $W^{s,1}(-\pi, \pi)$ seminorm of the functions $u_k^{n-1}$. At the same time, thanks to our volume constraint, we know that $|B_1^{n-1}| \int_{-\pi}^\pi dx_1 u_k(x_1)^{n-1} = \mu$ for all $k$. Therefore, by the fractional Sobolev compactness theorem (see [18 Theorem 7.1]), a subsequence of $\{u_k^{n-1}\}$ converges strongly in $L^1(-\pi, \pi)$ to a nonnegative function $v \in W^{s,1}(-\pi, \pi)$.

As a consequence, if we set $u := v^{1/(n-1)}$, then $|B_1^{n-1}| \int_{-\pi}^\pi dx_1 u(x_1)^{n-1} = \mu$. Thus, extending $u$ to be $2\pi$-periodic in $\mathbb{R}$, $u$ is an admissible competitor. Finally, since the function $\phi$ in (2.2) is nonnegative, Fatou’s lemma gives that $E := \{ x \in \mathbb{R}^n : |x'| < u(x_1) \}$ is a minimizer.

The fact that $v = u^{n-1} \in W^{s,1}(-\pi, \pi)$ yields $u \in W^{s/(n-1), n/(n-1)}(-\pi, \pi)$ by simply using that $|t - s|^{n-1} \leq |t^{n-1} - s^{n-1}|$ for all nonnegative numbers $t$ and $s$.

**Part 2 (Symmetry of minimizers).** To prove (i) in Theorem 1.1 let $E$ be any minimizer. As in Part 1 above, by Lemma 3.1 we must have $E = \{ x' < u(x_1) \}$ for some nonnegative $2\pi$-periodic function $u$. By (2.7) (with $u_k$ replaced by $u$), we see that $u^{n-1} \in W^{s,1}(-\pi, \pi)$. As before, this leads to $u \in W^{s/(n-1), n/(n-1)}(-\pi, \pi)$.

Next, statement (ii) in the theorem follows immediately from Theorem 1.2.
Part 3 (Euler-Lagrange equation). In this part we address the proof of Theorem 1.1 (iii). Firstly, we will establish that if \( E = \{ x \in \mathbb{R}^n : |x'| < u(x_1) \} \) is a minimizer, \( x_0^1 \in \mathbb{R} \) is such that \( u \) is \( C^{1,\alpha} \) in a neighborhood \( U(x_0^1) \) of \( x_0^1 \) for some \( \alpha > s \), and \( u(x_0^1) > 0 \), then the nonlocal mean curvature of \( \partial E \) is constant in a neighborhood of \( x_0^1 \). Afterwards, we will show that the constant does not depend on the point \( x_0^1 \) in whose neighborhood \( u \) is \( C^{1,\alpha} \).

Note first that we may assume \( x_0^1 \in (-\pi, \pi) \), by the periodicity of \( E \). We take any smooth function \( \xi = \xi(x_1) \) with compact support in the neighborhood \( U(x_0^1) \) of \( x_0^1 \), small enough to be contained in \((-\pi, \pi)\), and we extend \( \xi \) to be \( 2\pi \)-periodic. The first variation of the volume functional —that is, the derivative of \( |B_1^{n-1}| \int_{-\pi}^{\pi} dx_1 (u(x_1) + \epsilon \xi(x_1))^{n-1} \) with respect to \( \epsilon \) evaluated at \( \epsilon = 0 \)— is

\[
(n - 1)|B_1^{n-1}| \int_{-\pi}^{\pi} dx_1 u(x_1)^{n-2} \xi(x_1),
\]

while that of \( \mathcal{P}_s \) is

\[
\frac{d}{d\epsilon} \bigg|_{\epsilon=0} \mathcal{P}_s(u + \epsilon \xi) = \int_{-\pi}^{\pi} dx_1 \int_{-\infty}^{\infty} dy_1 |x_1 - y_1|^{n-2-s} \left\{ \phi_p \left( \frac{u(x_1)}{|x_1 - y_1|}, \frac{u(y_1)}{|x_1 - y_1|} \right) \frac{\xi(x_1)}{|x_1 - y_1|} + \phi_q \left( \frac{u(x_1)}{|x_1 - y_1|}, \frac{u(y_1)}{|x_1 - y_1|} \right) \right\},
\]

We next see that this expression leads to the nonlocal mean curvature of \( \partial E \) at \( x = (x_1, x') \). In what follows, all expressions can be checked to be well defined (as for the nonlocal mean curvature of general, nonperiodic, \( C^{1,\alpha} \) sets) using that \( u \) is \( C^{1,\alpha} \), with \( \alpha > s \), in the neighborhood of \( x_0^1 \) where \( \xi \) has its support.

Since, by Lemma 2.1 (used with \( m = 1 \) and \( l = 2 \)),

\[
\int_{-\pi}^{\pi} dx_1 \int_{-\infty}^{\infty} dy_1 \frac{\xi(y_1)}{|x_1 - y_1|^{3+s-n}} \phi_q \left( \frac{u(x_1)}{|x_1 - y_1|}, \frac{u(y_1)}{|x_1 - y_1|} \right) = \int_{-\infty}^{\infty} dx_1 \int_{-\pi}^{\pi} dy_1 \frac{\xi(y_1)}{|x_1 - y_1|^{3+s-n}} \phi_q \left( \frac{u(x_1)}{|x_1 - y_1|}, \frac{u(y_1)}{|x_1 - y_1|} \right)
\]

we deduce that \( (2.9) \) is equal to

\[
\int_{-\pi}^{\pi} dx_1 \int_{-\infty}^{\infty} dy_1 \frac{\xi(x_1)}{|x_1 - y_1|^{3+s-n}} \left\{ \phi_p \left( \frac{u(x_1)}{|x_1 - y_1|}, \frac{u(y_1)}{|x_1 - y_1|} \right) + \phi_q \left( \frac{u(y_1)}{|x_1 - y_1|}, \frac{u(x_1)}{|x_1 - y_1|} \right) \right\}.
\]
Thus, by definition of \( \phi \), this last expression becomes

\[
\int_{-\pi}^{\pi} dx_1 \xi(x_1) \int_{-\infty}^{+\infty} dy_1 \left| x_1 - y_1 \right|^{n-3-s} \left\{ \phi_p \left( \frac{u(x_1)}{|x_1 - y_1|}, \frac{u(y_1)}{|x_1 - y_1|} \right) + \phi_q \left( \frac{u(y_1)}{|x_1 - y_1|}, \frac{u(x_1)}{|x_1 - y_1|} \right) \right\} \\
= \int_{-\pi}^{\pi} dx_1 \xi(x_1) \int_{-\infty}^{+\infty} dy_1 \left| x_1 - y_1 \right|^{n-3-s} \times \left\{ \int_{w' \in \mathbb{R}^{n-1}; |w'|=\frac{u(x_1)}{|x_1 - y_1|}} dw' \int_{z' \in \mathbb{R}^{n-1}; |z'|=\frac{u(y_1)}{|x_1 - y_1|}} dz' \left( 1 + \left| w' - z' \right|^2 \right)^{-\frac{n+s}{2}} \right. \\
- \left. \int_{w' \in \mathbb{R}^{n-1}; |w'|<\frac{u(y_1)}{|x_1 - y_1|}} dw' \int_{z' \in \mathbb{R}^{n-1}; |z'|=\frac{u(x_1)}{|x_1 - y_1|}} dz' \left( 1 + \left| w' - z' \right|^2 \right)^{-\frac{n+s}{2}} \right\} \\
= |S^{n-2}| \int_{-\pi}^{\pi} dx_1 \xi(x_1) u(x_1)^{n-2} \int_{-\infty}^{+\infty} dy_1 \left| x_1 - y_1 \right|^{n-3-s} \left( \frac{1}{|x_1 - y_1|} \right)^{n-2} \\
\times \left\{ \int_{z' \in \mathbb{R}^{n-1}; |z'|=\frac{u(y_1)}{|x_1 - y_1|}} dz' \left( 1 + \left| z' - \frac{u(x_1)e'_1}{|x_1 - y_1|} \right|^2 \right)^{-\frac{n+s}{2}} \\
- \int_{w' \in \mathbb{R}^{n-1}; |w'|<\frac{u(y_1)}{|x_1 - y_1|}} dw' \left( 1 + \left| w' - \frac{u(x_1)e'_1}{|x_1 - y_1|} \right|^2 \right)^{-\frac{n+s}{2}} \right\}.
\]

The integral on \( dy_1 \) over \((-\infty, +\infty)\) is equal to

\[
\int_{-\infty}^{+\infty} dy_1 \left| x_1 - y_1 \right|^{-1-s} \\
\times \left\{ \int_{z' \in \mathbb{R}^{n-1}; |z'|=\frac{u(y_1)}{|x_1 - y_1|}} dz' \left| x_1 - y_1 \right|^{1-n} \left( 1 + \left| \frac{z'}{|x_1 - y_1|} - \frac{u(x_1)e'_1}{|x_1 - y_1|} \right|^2 \right)^{-\frac{n+s}{2}} \\
- \int_{w' \in \mathbb{R}^{n-1}; |w'|<\frac{u(y_1)}{|x_1 - y_1|}} dw' \left| x_1 - y_1 \right|^{1-n} \left( 1 + \left| \frac{w'}{|x_1 - y_1|} - \frac{u(x_1)e'_1}{|x_1 - y_1|} \right|^2 \right)^{-\frac{n+s}{2}} \right\} \\
= \int_{-\infty}^{+\infty} dy_1 \left\{ \int_{z' \in \mathbb{R}^{n-1}; |z'|=\frac{u(y_1)}{|x_1 - y_1|}} dz' \left| x_1 - y_1 \right|^{1-n} \left| x_1 - y_1 \right|^2 + \left| u(x_1)e'_1 - z' \right|^2 \right)^{-\frac{n+s}{2}} \\
- \int_{w' \in \mathbb{R}^{n-1}; |w'|<\frac{u(y_1)}{|x_1 - y_1|}} dw' \left| x_1 - y_1 \right|^2 + \left| u(x_1)e'_1 - w' \right|^2 \right)^{-\frac{n+s}{2}} \right\}.
\]

This last expression is, by definition, the nonlocal mean curvature of \( \partial E \) at the point \( x = (x_1, u(x_1)e'_1) = (x_1, u(x_1), 0, \ldots, 0) \), which is defined by

\[
H_s[E](x) = \int_{\mathbb{R}^n} dy \frac{\chi_{E^c}(y) - \chi_E(y)}{|x - y|^{n+s}}.
\]

Thus, by (2.9), we have shown that

\[
\frac{d}{de} \left. \mathcal{P}_s(u + \epsilon \xi) = \right|S^{n-2}| \int_{-\pi}^{\pi} dx_1 \xi(x_1) u(x_1)^{n-2} H_s[E](x_1, u(x_1)e'_1).
\]
From this, (2.8), and the Lagrange multiplier rule\footnote{Note that we are actually using the Lagrange multiplier rule not for the original functional \( P_s \) but for another functional \( \overline{P}_s \) given by the restriction of \( P_s \) to functions which agree with \( u \) outside of the set where \( u \) is regular. The Lagrange multiplier rule applies to \( \overline{P}_s \), since \( u \) is also a minimizer for \( \overline{P}_s \) and \( \overline{P}_s \) is differentiable.} we see that, for some constant \( \lambda \in \mathbb{R} \),

\[
H_s[E](x_1, u(x_1)e_1') = \lambda \quad \text{for all } x_1 \in U(x_1^0).
\]

It remains to show that \( \lambda \) does not depend on \( x_0^1 \). For this purpose, let \( U(x_1^0) \) and \( U(y_1^0) \) be two neighborhoods of two different points \( x_1^0 \) and \( y_1^0 \) where \( u \) is regular. Now, we can simply repeat the previous argument by taking this time a smooth function \( \xi \) with compact support in the union \( U(x_1^0) \cup U(y_1^0) \). Taking such variations \( \xi \) leads to the conclusion that \( H_s[E](x_1, u(x_1)e_1') \) is constant in \( U(x_0^1) \cup U(y_0^1) \). This proves Theorem 1.1 (iii).

**Part 4 (Small volumes).** We finally address the proof of Theorem 1.1 (iv). That is, we prove (uniformly as \( s \uparrow 1 \) and \( s \downarrow 0 \)) that if the volume constraint \( \mu \) is small enough, any minimizer \( E \) of \( P_s \) is not a straight cylinder. To do so, we simply use an energy comparison with the configuration \( \mathcal{B}_r \) given by a periodic array of disjoint balls

\[
\mathcal{B}_r := \{ x \in \mathbb{R}^n : |x - 2k\pi e_1| < r \text{ for some } k \in \mathbb{Z} \},
\]

where \( e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n \) and \( r < \pi \). To satisfy the volume constraint, the radius \( r = r(\mu) \) of the balls must be taken to satisfy \( |\mathcal{B}_1| r(\mu)^n = \mu \).

Now, by expression (1.1) we have, for different constants \( C_n \) depending only on \( n \),

\[
\begin{align*}
\mathcal{P}_s[\mathcal{B}_r] &= \int_{B_r} dx \int_{\mathbb{R}^n \setminus B_r} dy \frac{dy}{|x-y|^{n+s}} \\
&\leq \int_{B_1} dx \int_{\mathbb{R}^n \setminus B_1} dy \frac{dy}{|x-y|^{n+s}} = r^{n-s} \int_{B_1} dx \int_{\mathbb{R}^n \setminus B_1} dy \frac{dy}{|x-y|^{n+s}} \\
&\leq r^{n-s} \int_{B_1} dx \int_{\mathbb{R}^n \setminus B_1, |x|} dy \frac{dy}{|x-y|^{n+s}} \leq C_n r^{n-s} \int_{B_1} dx \int_1^{+\infty} dt t^{-1-s} \\
&= \frac{C_n}{s} r^{n-s} \int_{B_1} dx (1 - |x|)^{-s} = \frac{C_n}{s} s^{n-s} |\mathbb{S}^{n-1}| \int_0^1 \rho^{n-1} (1 - \rho)^{-s} \\
&\leq \frac{C_n}{s(1-s)} r^{n-s} = \frac{C_n}{s(1-s)} \mu^{\frac{n-s}{n}}. 
\end{align*}
\]
Instead, for the straight cylinder satisfying the volume constraint, which is given by \( E_\mu := \{ x \in \mathbb{R}^n : |x'| < c_1 \mu^{1/(n-1)} \} \) for some constant \( c_1 \) depending only on \( n \), we have

\[
\mathcal{P}_s[E_\mu] = \int_{\{ x \in \mathbb{R}^n : |x| < c_1 \mu^{1/(n-1)} \}} dx \int_{\{ y \in \mathbb{R}^n : |y' > c_1 \mu^{1/(n-1)} \}} \frac{dy}{|x - y|^{n+s}}
\]

\[
= c_1^{n-s} \mu^{\frac{n-s}{1-n}} \int_{\{ x \in \mathbb{R}^n : |x| < c_1 \mu^{1/(n-1)} \}} dx \int_{\{ y \in \mathbb{R}^n : |y| > 1 \}} \frac{dy}{|x - y|^{n+s}}
\]

\[
= c_1^{n-s} \mu^{\frac{n-s}{1-n}} \frac{2\pi}{c_1 \mu^{1/(n-1)}} \int_{-\infty}^{+\infty} dt \int_{\{|x| < c_1 \mu^{1/(n-1)}\}} \frac{dx}{|y|^{n+s}} \int_{\{|y| > 1\}} d\gamma (t^2 + |x| - |y|)^{-\frac{n+s}{2}}
\]

\[
\geq c_1^{n-s-1} \mu^{\frac{n-s-1}{1-n}} 2\pi 2^{-\frac{n+s}{2}} \int_{\{|x| < c_1 \mu^{1/(n-1)}\}} \frac{dx}{|y|^{n+s}} \int_{\{|y| > 1\}} d\gamma |x - y|^{-n-s}
\]

\[
\geq c_1^{n-s-1} \mu^{\frac{n-s-1}{1-n}} 2\pi 2^{-\frac{n+s}{2}} \int_{\{|x| < c_1 \mu^{1/(n-1)}\}} \frac{dx}{|y|^{n+s}} \int_{\{|y| > 1\}} d\gamma |x - y|^{1-n-s}.
\]

To bound the last double integral from below, we denote \( \gamma' = (x', z') \in \mathbb{R} \times \mathbb{R}^{n-2} \) to get

\[
\int_{B_1^{-1}} dx \int_{\mathbb{R}^{n-1} \setminus B_1^{-1}} d\gamma |x - \gamma|^{1-n-s}
\]

\[
= |S^{n-2}| \int_0^1 d\rho \rho^{n-2} \int_{\mathbb{R}^{n-1} \setminus B_1^{-1}} dz' |\rho e_1' - z'|^{1-n-s}
\]

\[
\geq |S^{n-2}| \int_0^1 d\rho \rho^{n-2} \int_{\{|z| > 1 \}} dz' |\rho e_1' - z'|^{1-n-s}
\]

\[
= |S^{n-2}| \int_0^1 d\rho \rho^{n-2} \int_{+\infty}^{+\infty} dz'_1 \int_{\mathbb{R}^{n-2}} dz'' ((\rho - z'_1)^2 + |z''|^2)^{\frac{1-n-s}{2}}
\]

\[
\geq |S^{n-2}| \int_0^1 d\rho \rho^{n-2} \int_{+\infty}^{+\infty} dz'_1 \int_{\{|z''| < |\rho - z'_1| \}} dz'' ((\rho - z'_1)^2 + |z''|^2)^{\frac{1-n-s}{2}}
\]

\[
= |S^{n-2}| \int_0^1 d\rho \rho^{n-2} \int_{+\infty}^{+\infty} dz'_1 |\rho - z'_1|^{-1-n-s}
\]

\[
= \frac{1}{s} |S^{n-2}| \int_0^1 d\rho \rho^{n-2} (1 - \rho)^{-n-s}.
\]

Here we implicitly assumed that \( n \geq 3 \). However, when \( n = 2 \) we do not need to introduce the integral over \( z'' \in \mathbb{R}^{n-2} \), and the same bound (with the constant \( \frac{1}{s} |S^{n-2}| \) replaced by \( \frac{2}{s} \)) follows in an easier way. Observe now that

\[
\int_0^1 \rho^{n-2} (1 - \rho)^{-s} d\rho \geq 2^{1-n} \int_{1/2}^1 (1 - \rho)^{-s} d\rho = \frac{2^{1-n+s}}{1-s} \geq \frac{2^{1-n}}{1-s}.
\]

Hence, since \( 2^{-s} \geq 2^{-1} \) and \( c_1^{-s} \geq \min(1, c_1^{-1}) \), we deduce that

\[
\mathcal{P}_s[E_\mu] \geq \frac{c_n}{s(1 - s)} \mu^{\frac{n-1}{n-1}}
\]

\[
\int_{\mathbb{R}^{n-1} \setminus B_1^{-1}} d\gamma |x - \gamma|^{1-n-s}. \]
for some constant $c_n > 0$ depending only on $n$.

From this and (2.10), we conclude that $\mathcal{P}_s[E_\mu] > \mathcal{P}_s[\mathcal{B}_r(\mu)]$ if $\mu$ is small enough, since clearly $(n - 1 - s)/(n - 1) < (n - s)/n$. \hfill $\Box$

3. CYLINDRICAL AND PERIODIC SYMMETRIC DECREASING REARRANGEMENTS

In this section we analyze the behavior of $\mathcal{P}_s$ under the two rearrangements defined in Subsection 1.1, namely, the cylindrical rearrangement and the symmetric decreasing periodic rearrangement.

In the previous section we have shown the existence of a cylindrically symmetric minimizer using the Riesz rearrangement inequality (the key ingredient in the proof of Lemma 3.1) \footnote{The Riesz rearrangement inequality was already used in \cite[Proposition 13]{10} for the functional $\mathcal{P}_s$ acting on periodic sets, to conclude that the cylindrical rearrangement does not increase the functional $\mathcal{P}_s$.} In the next lemma we will see that characterizing the case of equality in the Riesz rearrangement using the Riesz rearrangement inequality will actually yield that every minimizer is cylindrically symmetric. The same argument has been used in \cite[Lemma 4.2]{23} to show the cylindrical symmetry of minimizers of a different functional.

Before stating the lemma we recall the notation introduced in Subsection 1.1. For a measurable set $E \subset \mathbb{R}^n$, given $x_1 \in \mathbb{R}$ consider

$$E_{x_1} := \{x' \in \mathbb{R}^{n-1} : (x_1, x') \in E\}.$$  

Then, denoting balls in $\mathbb{R}^{n-1}$ by $B_r^{n-1}$ and the $(n - 1)$-dimensional Lebesgue measure by $\mathcal{L}^{n-1}$, we define

$$E^{\text{cyl}} := \left\{(x_1, x') \in \mathbb{R}^n : x' \in B_r^{n-1}(x_1), \text{ where } \mathcal{L}^{n-1}(B_{r(x_1)}^{n-1}) = \mathcal{L}^{n-1}(E_{x_1}), \right\},$$

with the understanding that $B_{r(x_1)}^{n-1} = \mathbb{R}^{n-1}$ if $\mathcal{L}^{n-1}(E_{x_1}) = +\infty$. Notice that by definition

$$(E^{\text{cyl}})_{x_1} = (E_{x_1})^{s(n-1)}, \quad (3.1)$$

where $A^{s(n-1)}$ denotes the Schwarz rearrangement of a set $A \subset \mathbb{R}^{n-1}$, i.e., the ball in $\mathbb{R}^{n-1}$ centered at 0 and of the same $(n - 1)$-dimensional Lebesgue measure as $A$ — in case that the measure of $A$ is infinite, we set $A^{s(n-1)} := \mathbb{R}^{n-1}$.

The following lemma concerns our periodic fractional perimeter, but its statement does not require the periodicity of the set $E$. Observe that $\mathcal{P}_s[E]$ makes sense also for sets which are not periodic. Its proof will use the classical Riesz rearrangement inequality, as well as its strict version needed to discuss the case of equality.

**Lemma 3.1.** Let $E \subset \mathbb{R}^n$ be a measurable set such that $|E \cap ((-\pi, \pi) \times \mathbb{R}^{n-1})| < +\infty$.

Then, $\mathcal{P}_s[E^{\text{cyl}}] \leq \mathcal{P}_s[E]$. Moreover, if in addition $|E \cap ((-\pi, \pi) \times \mathbb{R}^{n-1})| > 0$ and $\mathcal{P}_s[E] < +\infty$, the equality $\mathcal{P}_s[E^{\text{cyl}}] = \mathcal{P}_s[E]$ holds if and only if $E = E^{\text{cyl}} + (0, c)$ for some $c \in \mathbb{R}^{n-1}$ up to a set of measure zero.

**Proof.** For simplicity of notation, given $x_1, y_1 \in \mathbb{R}$ with $x_1 \neq y_1$, define $g_{x_1,y_1} : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$g_{x_1,y_1}(z') := \left(|x_1 - y_1|^2 + |z'|^2\right)^{-\frac{n+4}{2}}.$$  

Note that $g_{x_1,y_1} \in L^1(\mathbb{R}^{n-1})$ is a positive and radially symmetric decreasing function.
To address the proof of the inequality $\mathcal{P}_s[\mathcal{L}^{\text{cyl}}] \leq \mathcal{P}_s[E]$, let us first make some preliminary observations. Let $x_1, y_1 \in \mathbb{R}$ with $x_1 \neq y_1$, and assume that $\mathcal{L}^{n-1}(E_{x_1}) < +\infty$. On the one hand, if $\mathcal{L}^{n-1}(E_{y_1}) < +\infty$, the Riesz rearrangement inequality [22, Theorem 3.7] yields that

$$\int_{E_{y_1}} dx' \int_{E_{y_1}} dy' g_{x_1,y_1}(x' - y') \leq \int_{(E_{x_1})^{(n-1)}} dx' \int_{(E_{y_1})^{(n-1)}} dy' g_{x_1,y_1}(x' - y'). \tag{3.2}$$

In addition, the strict rearrangement inequality [6, Corollary 2.19] or [22, Theorem 3.9] shows that the equality in (3.2) holds (when $E_{x_1}$ and $E_{y_1}$ have finite measure) if and only if either $\mathcal{L}^{n-1}(E_{x_1}) \mathcal{L}^{n-1}(E_{y_1}) = 0$ or

$$E_{x_1} = B^{n-1}_{r(x_1)} + c(x_1,y_1) \quad \text{and} \quad E_{y_1} = B^{n-1}_{r(y_1)} + c(x_1,y_1) \tag{3.3}$$

for some $r(x_1)$ and $r(y_1)$ belonging to $(0, +\infty)$ and for some $c(x_1,y_1) \in \mathbb{R}^{n-1}$.

On the other hand, if $\mathcal{L}^{n-1}(E_{y_1}) = +\infty$ then $(E_{y_1})^{(n-1)} = \mathbb{R}^{n-1}$ and, therefore,

$$\int_{E_{x_1}} dx' \int_{E_{y_1}} dy' g_{x_1,y_1}(x' - y') \leq \int_{E_{x_1}} dx' \int_{\mathbb{R}^{n-1}} dy' g_{x_1,y_1}(x' - y')
= \mathcal{L}^{n-1}(E_{x_1}) \|g_{x_1,y_1}\|_{L^1(\mathbb{R}^{n-1})} = \mathcal{L}^{n-1}((E_{x_1})^{(n-1)}) \|g_{x_1,y_1}\|_{L^1(\mathbb{R}^{n-1})}
= \int_{(E_{x_1})^{(n-1)}} dx' \int_{\mathbb{R}^{n-1}} dy' g_{x_1,y_1}(x' - y')
= \int_{(E_{y_1})^{(n-1)}} dx' \int_{(E_{y_1})^{(n-1)}} dy' g_{x_1,y_1}(x' - y'). \tag{3.4}$$

In addition, since $g_{x_1,y_1}$ is positive and integrable in $\mathbb{R}^{n-1}$, equality in (3.4) holds if and only if either $\mathcal{L}^{n-1}(E_{x_1}) = 0$ or $E_{y_1} = \mathbb{R}^{n-1}$ up to a set of measure zero. In particular, if $E_{x_1}$ has finite measure, $E_{y_1}$ has infinite measure, and equality in (3.4) holds, then either $\mathcal{L}^{n-1}(E_{x_1}) = 0$ or

$$E_{y_1} = (E_{y_1})^{(n-1)} + c \quad \text{for every } c \in \mathbb{R}^{n-1}. \tag{3.5}$$

Having these observations in mind, we can now proceed to the proof of the first part of the lemma.

First of all, the assumption $|E \cap ((-\pi, \pi) \times \mathbb{R}^{n-1})| < +\infty$ leads to $\mathcal{L}^{n-1}(E_{x_1}) < +\infty$ for a.e. $x_1 \in (-\pi, \pi)$. In particular, for a.e. $x_1 \in (-\pi, \pi)$ and all $y_1 \in \mathbb{R}$ with $x_1 \neq y_1$, we have

$$\int_{E_{x_1}} dx' \int_A dy' g_{x_1,y_1}(x' - y') \leq \|E_{x_1}\| \|g_{x_1,y_1}\|_{L^1(\mathbb{R}^{n-1})} < +\infty$$

for every measurable set $A \subset \mathbb{R}^{n-1}$. This justifies the forthcoming computations. Using (3.2) and (3.4) (depending on whether $\mathcal{L}^{n-1}(E_{y_1}) < +\infty$ or $\mathcal{L}^{n-1}(E_{y_1}) = +\infty$), and recalling (3.1),

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we get
\[
\mathcal{P}_s[E] = \int_{-\pi}^{\pi} dx_1 \int_{\mathbb{R}} dy_1 \int_{E_{x_1}} dx' \int_{E_{y_1}} dy' g_{x_1,y_1}(x' - y')
\]
\[
= \int_{-\pi}^{\pi} dx_1 \int_{\mathbb{R}} dy_1 \left( \int_{E_{x_1}} dx' \int_{E_{y_1}} dy' g_{x_1,y_1}(x' - y') - \int_{E_{x_1}} dx' \int_{E_{y_1}} dy' g_{x_1,y_1}(x' - y') \right)
\]
\[
= \int_{-\pi}^{\pi} dx_1 \int_{\mathbb{R}} dy_1 \left( \mathcal{L}^{-1}(E_{x_1}) \int_{\mathbb{R}} dz' g_{x_1,y_1}(z') - \int_{E_{x_1}} dx' \int_{E_{y_1}} dy' g_{x_1,y_1}(x' - y') \right)
\]
\[
\geq \int_{-\pi}^{\pi} dx_1 \int_{\mathbb{R}} dy_1 \left( \mathcal{L}^{-1}(E_{x_1}) \int_{\mathbb{R}} dz' g_{x_1,y_1}(z') - \int_{(E^{*cyl})_{x_1}} dx' \int_{(E^{*cyl})_{y_1}} dy' g_{x_1,y_1}(x' - y') \right)
\]
\[
= \mathcal{P}_s[E^{*cyl}],
\]
which proves the first part of the lemma.

To prove its second part, we need to show that if \( E \) has positive finite measure in \( (-\pi, \pi) \times \mathbb{R}^{n-1} \) and \( \mathcal{P}_s[E] = \mathcal{P}_s[E^{*cyl}] < +\infty \), then there exists \( c \in \mathbb{R}^{n-1} \) such that \( E_{y_1} = (E_{y_1})^{*n-1} + c \) for a.e. \( y_1 \in \mathbb{R} \). For this, a key point is to use (3.6), already proven, as follows. Since \( \mathcal{L}^{-1}(E_{x_1}) = \mathcal{L}^{-1}(E^{*cyl}_{x_1}) < +\infty \) for a.e. \( x_1 \in (-\pi, \pi) \) and \( \mathcal{P}_s[E] = \mathcal{P}_s[E^{*cyl}] < +\infty \), (3.6) shows that we must have
\[
\int_{-\pi}^{\pi} dx_1 \int_{\mathbb{R}} dy_1 \left( \int_{E_{x_1}} dx' \int_{E_{y_1}} dy' g_{x_1,y_1}(x' - y') \right)
\]
\[
= \int_{-\pi}^{\pi} dx_1 \int_{\mathbb{R}} dy_1 \left( \int_{(E^{*cyl})_{x_1}} dx' \int_{(E^{*cyl})_{y_1}} dy' g_{x_1,y_1}(x' - y') \right).
\]
Using now (3.2) in case \( \mathcal{L}^{-1}(E_{y_1}) < +\infty \), or (3.4) in case \( \mathcal{L}^{-1}(E_{y_1}) = +\infty \), we deduce that for a.e. \( (x_1, y_1) \in (-\pi, \pi) \times \mathbb{R} \) it holds
\[
\int_{E_{x_1}} dx' \int_{E_{y_1}} dy' g_{x_1,y_1}(x' - y') = \int_{(E_{x_1})^{*n-1}} dx' \int_{(E_{y_1})^{*n-1}} dy' g_{x_1,y_1}(x' - y').
\]
Now, note that the assumption \( 0 < |E \cap ((-\pi, \pi) \times \mathbb{R}^{n-1})| < +\infty \) yields that
\[
\Lambda := \{ x_1 \in (-\pi, \pi) : 0 < \mathcal{L}^{-1}(E_{x_1}) < +\infty \}
\]
has positive measure. Recall that we do not assume \( E \) to be periodic, and therefore there might exist a set \( I \subset (-\infty, -\pi) \cup (\pi, +\infty) \) of positive measure such that \( \mathcal{L}^{-1}(E_{y_1}) = +\infty \) for all \( y_1 \in I \). In view of this fact we define
\[
\Sigma := \{ y_1 \in \mathbb{R} : 0 < \mathcal{L}^{-1}(E_{y_1}) < +\infty \},
\]
which also has positive measure because \( \Lambda \subset \Sigma \).

Let now \((x_1, y_1) \in \Lambda \times \Sigma\) be such that (3.7) holds. Then, by the statement concluding (3.3) we have
\[
E_{x_1} = B_{r(x_1)}^{n-1} + c(x_1, y_1) \quad \text{and} \quad E_{y_1} = B_{r(y_1)}^{n-1} + c(x_1, y_1).
\]
Repeating this argument for any other \((\tilde{x}_1, y_1) \in \Lambda \times \Sigma\) we also obtain \( E_{y_1} = B_{r(y_1)}^{n-1} + c(\tilde{x}_1, y_1) \), and hence \( c \) is independent of \( x_1 \). Analogously, the same argument on any other \((x_1, \tilde{y}_1) \in \Lambda \times \Sigma\)
yields that \( E_{y_1} = B_{n-1}^r + c(x_1, y_1) \). Hence, \( c \) is also independent of \( y_1 \). Thus, there exists \( c \in \mathbb{R}^{n-1} \) such that \( E_{y_1} = B_{n-1}^r + c \) for a.e. \( y_1 \in \mathbb{R} \) satisfying \( 0 < \mathcal{L}^{n-1}(E_{y_1}) < +\infty \). 

For this constant \( c \in \mathbb{R}^{n-1} \), we now claim that 
\[
E_{y_1} = (E_{y_1})^{(n-1)} + c \tag{3.8}
\]
for a.e. \( y_1 \in \mathbb{R} \). On the one hand, we have already proven (3.8) for a.e. \( y_1 \in \Sigma \). On the other hand, if \( y_1 \in \mathbb{R} \setminus \Sigma \), then we must have that either \( \mathcal{L}^{n-1}(E_{y_1}) = 0 \) or \( \mathcal{L}^{n-1}(E_{y_1}) = +\infty \). But if \( \mathcal{L}^{n-1}(E_{y_1}) = 0 \), then \( E_{y_1} = (E_{y_1})^{(n-1)} + d \) for every \( d \in \mathbb{R}^{n-1} \) up to a set of \( \mathcal{L}^{n-1} \)-measure zero, since \( (E_{y_1})^{(n-1)} \) is the empty set in \( \mathbb{R}^{n-1} \). Thus, (3.8) holds in this case too. It only remains to show (3.8) for a.e. \( y_1 \in \mathbb{R} \) such that \( \mathcal{L}^{n-1}(E_{y_1}) = +\infty \). Recall that (3.7) holds for a.e. \((x_1, y_1) \in (-\pi, \pi) \times \mathbb{R} \). Hence, since \(|\Lambda| > 0\), for a.e. \( y_1 \in \mathbb{R} \) with \( \mathcal{L}^{n-1}(E_{y_1}) = +\infty \) there exists a point \( x_1 \in \Lambda \) such that (3.7) holds. Then the statement concluding (3.5) yields that 
\[
E_{y_1} = (E_{y_1})^{(n-1)} + d \quad \text{for every } d \in \mathbb{R}^{n-1}.
\]
Thus, (3.8) also follows in this last case.

In conclusion, there exists \( c \in \mathbb{R}^{n-1} \) such that \( E_{y_1} = (E_{y_1})^{(n-1)} + c \) for a.e. \( y_1 \in \mathbb{R} \). Equivalently, \( E = E^{* \text{cyl}} + (0, c) \) up to a set of measure zero. \( \Box \)

We now address the proof of Theorem 1.2. Let us first recall some definitions to facilitate the reading. For a measurable set \( A \subset \mathbb{R}^n \) we denote its Steiner symmetrization with respect to \( \{ x_1 = 0 \} \) by \( A^* \). If \( A \subset \mathbb{R}^n \) is measurable and of finite measure, then \( A^* \) is an interval centered at 0 with the same measure as \( A \). The Steiner symmetrization is well defined whenever the 1-dimensional set \( A \cap \{ x' = c \} \) has finite length for all \( c \in \mathbb{R}^{n-1} \). Thus, the periodic rearrangement 
\[
E^{* \text{per}} := \bigcup_{k \in \mathbb{Z}} [(E \cap \{ -\pi < x_1 < \pi \})^* + 2k\pi e_1]
\]
is well defined for every measurable set \( E \subset \mathbb{R}^n \) which is 2\pi-periodic in the \( x_1 \) direction.

By periodicity, note that \( E^{* \text{per}} = (E + (c, 0))^{* \text{per}} \) for all \( c \in \mathbb{R} \).

We denote the symmetric decreasing rearrangement of a real-valued function \( f \) on \( \mathbb{R} \) by \( f^* \). It is defined by the following properties: \( f^* \) is nonnegative, equiregular with \(|f|\), and its superlevel sets are open intervals centered at 0. Two explicit expressions for \( f^* \) are 
\[
f^*(x) = \int_0^{+\infty} dt \chi_{\{|f| > t\}^*}(x) = \sup \{ t : x \in \{|f| > t\}^* \}.
\]
For a 2\pi-periodic function \( u : \mathbb{R} \to \mathbb{R} \), the periodic symmetric decreasing rearrangement of \( u \), denoted by \( u^{* \text{per}} \), is the unique 2\pi-periodic function such that \( u^{* \text{per}} \chi_{(-\pi, \pi)} = (u \chi_{(-\pi, \pi)})^* \).

It follows from these definitions that if \( E \) is as in (1.3), then 
\[
E^{* \text{per}} = \{ x \in \mathbb{R}^n : |x'| < u^{* \text{per}}(x_1) \}.
\]

The main ingredient of our proof of Theorem 1.2 will be the following Riesz rearrangement inequality on the circle.

**Theorem 3.2.** ([7, Theorem 2], [19, Theorem 1], [8, Theorem 2], [6, Theorem 7.3]) Let \( f, h, g : \mathbb{R} \to \mathbb{R} \) be three nonnegative 2\pi-periodic measurable functions. Assume that \( g \) is even, as well as nonincreasing in \((0, \pi)\).

Then, 
\[
\int_{-\pi}^\pi dx \int_{-\pi}^\pi dy f(x)g(x - y)h(y) \leq \int_{-\pi}^\pi dx \int_{-\pi}^\pi dy f^*(x)g(x - y)h^*(y). \tag{3.9}
\]
In addition, if \( g \) is decreasing in \((0, \pi)\) and the left-hand side of (3.9) is finite, then equality holds in (3.9) if and only if at least one of the following conditions holds:
(i) either \( f \) or \( h \) is constant almost everywhere.
(ii) there exists \( z \in \mathbb{R} \) such that \( f(x) = f^{\text{per}}(x + z) = h^{\text{per}}(x + z) \) for almost every \( x \in \mathbb{R} \).

The inequality in Theorem 3.2 was first discovered, independently, in [7] and [19]. Both references contain more general inequalities: [7] deals with the sphere \( S^n \), whereas [19] deals with a product of more than three functions. The result [6, Theorem 7.3] is also more general than Theorem 3.2, as it deals with a Riesz rearrangement inequality on the sphere \( S^n \) and not only on the circle. Moreover, [6] treats more general functions of \( f(x) \) and \( h(y) \) than simply the product \( f(x)h(y) \). The inequality (3.9) can also be found in [5] in a more general form where \( g \) is also rearranged.

Instead, the statement in Theorem 3.2 concerning equality in (3.9) follows from Burchard and Hajaiej [8, Theorem 2], who treated the case of equality in \( S^n \) for the first time. For this result, we also cite [6, Theorem 7.3] since, being less general than [8], fits precisely with our setting.

We find reference [6] to be the simplest one for looking up all statements of Theorem 3.2.

Proof of Theorem 7.2 Let us first show that \( \mathcal{P}_s[E^{\text{per}}] \leq \mathcal{P}_s[E] \). In contrast to the proof of Lemma 3.1, we now write \( \mathcal{P}_s[E] \) as

\[
\mathcal{P}_s[E] = \int_{\mathbb{R}^{n-1}} dx' \int_{E_{x'} \cap (-\pi, \pi)} dE^{\text{per}}(x') dx_1 \int_{\mathbb{R}^{n-1}} dy' \int_{\mathbb{R} \setminus E'_{y'}} \frac{1}{|x - y|^{n+s}},
\]

where we have set \( E_{x'} := \{ x_1 \in \mathbb{R} : (x_1, x') \in E \} \) (in analogy with the sections \( E_{x_1} \) defined above). Writing \( \mathcal{P}_s[E^{\text{per}}] \) in the same way (simply replacing \( E \) by \( E^{\text{per}} \)) and interchanging the order of integration \( dx_1 dy' \), we see that it is sufficient to prove that

\[
\int_{E_{x'} \cap (-\pi, \pi)} dx_1 \int_{\mathbb{R} \setminus E'_{y'}} \frac{dy_1}{|x - y|^{n+s}} \geq \int_{(E^{\text{per}})_{x'} \cap (-\pi, \pi)} dx_1 \int_{\mathbb{R} \setminus (E^{\text{per}})_{y'}} \frac{dy_1}{|x - y|^{n+s}} \tag{3.10}
\]

for almost every \( (x', y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \).

For this, set

\[
a = |x' - y'| \quad \text{and} \quad \gamma = \frac{n + s}{2} \quad \text{(and thus } |x - y|^{n+s} = ((x_1 - y_1)^2 + a^2)^\gamma \text{)},
\]

observe that the following integrals are all finite when \( a \neq 0 \), and that the condition \( a \neq 0 \) holds for almost every \( (x', y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1} \). We have

\[
\int_{\mathbb{R} \setminus E'_{y'}} \frac{dy_1}{|x - y|^{n+s}} = \int_{\mathbb{R}} \frac{dy_1}{|x - y|^{n+s}} - \int_{E'_{y'}} \frac{dy_1}{|x - y|^{n+s}}
\]

and

\[
\int_{\mathbb{R} \setminus (E^{\text{per}})_{y'}} \frac{dy_1}{|x - y|^{n+s}} = \int_{\mathbb{R}} \frac{dy_1}{|x - y|^{n+s}} - \int_{(E^{\text{per}})_{y'}} \frac{dy_1}{|x - y|^{n+s}}.
\]

From the equality \( |E_{x'} \cap (-\pi, \pi)| = |(E^{\text{per}})_{x'} \cap (-\pi, \pi)| \), we infer that

\[
\int_{E_{x'} \cap (-\pi, \pi)} dx_1 \int_{\mathbb{R}} \frac{dy_1}{|x - y|^{n+s}} = \int_{(E^{\text{per}})_{x'} \cap (-\pi, \pi)} dx_1 \int_{\mathbb{R}} \frac{dy_1}{|x - y|^{n+s}}.
\]

Therefore, to prove (3.10) it is enough to show that

\[
\int_{E_{x'} \cap (-\pi, \pi)} dx_1 \int_{E'_{y'}} \frac{dy_1}{((x_1 - y_1)^2 + a^2)^\gamma} \leq \int_{(E^{\text{per}})_{x'} \cap (-\pi, \pi)} dx_1 \int_{(E^{\text{per}})_{y'}} \frac{dy_1}{((x_1 - y_1)^2 + a^2)^\gamma} \tag{3.11}
\]
for all \(a \neq 0\).

In order to do this, recall that \((E^{\text{per}})_{y'} = (E_{y'})^{\text{per}}\) and that both \(E_{y'}\) and \((E^{\text{per}})_{y'}\) are \(2\pi\)-periodic. Taking these facts in consideration, (3.11) will follow if we show that, for every two measurable subsets \(A \) and \(B \) of \((-\pi, \pi)\), it holds

\[
\int_A dx_1 \int_{\bigcup_{k \in \mathbb{Z}} (B + 2k\pi)} dy_1 \frac{dy_1}{((x_1 - y_1)^2 + a^2)\gamma} \leq \int_A dx_1 \int_{\bigcup_{k \in \mathbb{Z}} (B + 2k\pi)} dy_1 \frac{dy_1}{((x_1 - y_1)^2 + a^2)\gamma}.
\]

By the change of variables \(y_1 = y_1 - 2k\pi\), this can be equivalently written as

\[
\int_A dx_1 \int_B dy_1 g(x_1 - y_1) \leq \int_A dx_1 \int_B dy_1 g(x_1 - y_1) \quad \text{for all subsets} \ A \text{ and } B \text{ of } (-\pi, \pi),
\]

(3.12)

where we have set \(g(z) := \sum_{k \in \mathbb{Z}} \frac{1}{((z + 2k\pi)^2 + a^2)\gamma} \).

Clearly, the function \(g\) is positive, \(2\pi\)-periodic, and even. We will next show that \(g\) is decreasing in \((0, \pi)\), and hence (3.12) will follow from Theorem 3.2 applied to the characteristic functions \(f = \chi_A\) and \(h = \chi_B\).

In order to show that \(g\) is decreasing in \((0, \pi)\), we introduce the Laplace transform of the function \(t \mapsto t^{\gamma-1}\), which amounts to the equality

\[
w^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} dt \, t^{\gamma-1} e^{-wt} \quad \text{for all } w > 0,
\]

(3.14)

where \(\Gamma(\gamma) := \int_0^{+\infty} dr \, r^{\gamma-1} e^{-r}\) is the Gamma function. The identity (3.14) follows simply by a change of variables \(r = wt\) in the definition of \(\Gamma(\gamma)\). Using (3.14) in the definition of \(g\) yields

\[
g(z) = \frac{1}{\Gamma(\gamma)} \sum_{k \in \mathbb{Z}} \int_0^{+\infty} dt \, t^{\gamma-1} e^{-((z + 2k\pi)^2 + a^2)t} = \frac{1}{\Gamma(\gamma)} \int_0^{+\infty} dt \, t^{\gamma-1} e^{-a^2 t} G_t(z),
\]

(3.15)

where

\[
G_t(z) := \sum_{k \in \mathbb{Z}} e^{-((z + 2k\pi)^2 + a^2)t}.
\]

We will now show that \(G_t\) is decreasing in \((0, \pi)\) for every \(t > 0\), yielding, in view of (3.15), that \(g\) is also decreasing in \((0, \pi)\). The claim on the strict monotonicity of \(G_t\) follows directly from the fact that the fundamental solution of the heat equation in \((-\pi, \pi)\) with periodic boundary conditions (or, in other words, of the heat equation on the circle), which is

\[
\Lambda_t(z) := \frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}} e^{-((z + 2k\pi)^2 + a^2)t},
\]

is decreasing in \((0, \pi)\) for all \(t > 0\). For the sake of completeness, in Theorem B.1 we give a simple proof of this well-known fact, based solely on maximum principles for the heat equation.

We now prove that if \(\mathcal{P}_h[E^{\text{per}}] = \mathcal{P}_s[E] < +\infty\), then \(E = E^{\text{per}} + c\mathcal{E}_1\) for some \(c \in \mathbb{R}\), up to a set of measure zero. Indeed, in view of (3.10) and the equalities following it, if \(\mathcal{P}_s[E^{\text{per}}] = \mathcal{P}_s[E]\) then there must be equality in (3.11) for almost every \((x', y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}\). Equivalently, using the formulation of (3.12),

\[
\int_{E_x \cap (-\pi, \pi)} dx_1 \int_{E_y \cap (-\pi, \pi)} dy_1 g(x_1 - y_1) = \int_{(E^{\text{per}})_x \cap (-\pi, \pi)} dx_1 \int_{(E^{\text{per}})_y \cap (-\pi, \pi)} dy_1 g(x_1 - y_1)
\]

(3.16)
for almost every \((x', y') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}\).

Now, let \(J \subset \mathbb{R}^{n-1}\) be the set of points \(x' \in \mathbb{R}^{n-1}\) for which \(E_{x'} \cap (-\pi, \pi)\) is neither \((-\pi, \pi)\) nor \(\emptyset\) (up to zero measure sets). We may assume that \(J\) has positive \((n - 1)\)-dimensional Lebesgue measure —if \(J\) has measure zero then \(E = E^{\ast\text{per}}\) up to a set of measure zero in \(\mathbb{R}^n\), and we would be done. Then, \(J \times J\) has positive measure in \(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}\), and (3.16) holds for almost every \((x', y') \in J \times J\). Note that, by definition of \(J\), we don’t fall in case \((i)\) of Theorem 3.2 when using (3.16) on points \(x'\) and \(y'\) in \(J\). Hence, by (3.16) and Theorem 3.2 \((ii)\) we deduce that, for almost every \((x', y') \in J \times J\), we have

\[
E_{x'} = (E^{\ast\text{per}})_{x'} + c(x', y') \quad \text{and} \quad E_{y'} = (E^{\ast\text{per}})_{y'} + c(x', y')
\]

for some constant \(c(x', y') \in \mathbb{R}\) depending, a priori, on \(x'\) and \(y'\). This easily yields that \(c(x', y') = c\) is actually independent of \(x'\) and \(y'\). Indeed, considering two different values \(x' \in J\) and \(\overline{x} \in J\) for the same \(y'\), we have

\[
(E^{\ast\text{per}})_{y'} + c(x', y') = E_{y'} = (E^{\ast\text{per}})_{y'} + c(\overline{x}', y').
\]

Since \(y' \in J\), \((E^{\ast\text{per}})_{y'}\) is a \(2\pi\)-periodic set for which \((E^{\ast\text{per}})_{y'} \cap (-\pi, \pi)\) is a nonempty open interval different from \((-\pi, \pi)\). Therefore (3.17) yields that \(c(x', y') = c(\overline{x}', y')\) modulo \(2\pi\). By symmetry, \(c(x', y')\) is also independent of \(y'\). Therefore, \(E_{y'} = (E^{\ast\text{per}})_{y'} + c\) for some \(c \in \mathbb{R}\) and all \(y' \in J\). By the definition of \(J\), we conclude that \(E = E^{\ast\text{per}} + ce_1\) up to a set of measure zero.

\[\square\]

**Remark 3.3.** From the last part of the proof of Theorem 1.2 in particular in what regards (3.14), we see that Theorem 1.2 easily extends to any kernel which is the Laplace transform of a nonnegative function. More precisely, consider a periodic nonlocal perimeter functional of the form

\[
\mathcal{P}_K[E] := \int_{\mathbb{R}^n} \int_{E \cap (-\pi < x_1 < \pi)} dx \int_{\mathbb{R}^n \setminus E} dy K(|x - y|^2),
\]

where \(K : (0, +\infty) \to (0, +\infty)\) is a kernel given by

\[
K(w) := \int_0^{+\infty} dt \kappa(t)e^{-wt}, \quad w > 0,
\]

for some measurable function \(\kappa : (0, +\infty) \to [0, +\infty)\) which is not identically zero. Then, Theorem 1.2 holds also for \(\mathcal{P}_K\).

Indeed, arguing as in the proof of Theorem 1.2 up to (3.12), one sees that it is enough to show that

\[
\int_A dx_1 \int_B dy_1 g_K(x_1 - y_1) \leq \int_{A^*} dx_1 \int_{B^*} dy_1 g_K(x_1 - y_1)
\]

for all subsets \(A\) and \(B\) of \((-\pi, \pi)\), where

\[
g_K(z) := \sum_{k \in \mathbb{Z}} K((z + 2k\pi)^2 + a^2), \quad z \in \mathbb{R}.
\]

But

\[
g_K(z) = \sum_{k \in \mathbb{Z}} \int_0^{+\infty} dt \kappa(t)e^{-(z+2k\pi)^2+a^2)t} = \int_0^{+\infty} dt \kappa(t)e^{-a^2t}G_1(z)
\]

is a decreasing function in \((0, \pi)\) by Theorem 3.1 and by the fact that \(\kappa \geq 0\) is not identically zero in \((0, +\infty)\). Hence, Theorem 3.2 still applies in this framework and, thus, we deduce that \(\mathcal{P}_K\) is nonincreasing under periodic decreasing rearrangement. Moreover, as before, if \(\mathcal{P}_K[E^{\ast\text{per}}] = \mathcal{P}_K[E]\) then \(E = E^{\ast\text{per}} + ce_1\) for some \(c \in \mathbb{R}\), up to a set of measure zero.
APPENDIX A. PERIODIC MINIMIZERS IN $\mathbb{R}^2$

In this appendix we present, when $n = 2$, an expression for the functional $\mathcal{P}_s$ which is simpler than that of Lemma 2.2 in $\mathbb{R}^n$. As mentioned in the Introduction, this expression allowed us to prove the existence of minimizer when $n = 2$ in a simple way. We later found the existence proof of Theorem 1.1 in all dimensions.

The formula for the periodic version of the fractional perimeter when $n = 2$ reads as follows. If $E = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < u(x_1)\}$, for some nonnegative $2\pi$-periodic function $u$, then $\mathcal{P}_s[E]$ can be written in terms of $u$ as

$$\mathcal{P}_s[E] = \int_{-\pi}^{\pi} dx_1 \int_{-\infty}^{+\infty} dy_1 \frac{2}{|x_1 - y_1|^s} \left\{ G \left( \frac{u(x_1) - u(y_1)}{|x_1 - y_1|} \right) + H \left( \frac{u(x_1) + u(y_1)}{|x_1 - y_1|} \right) \right\}, \quad (A.1)$$

where

$$G(t) := \int_0^t d\tau (t - \tau) (1 + \tau^2)^{-\frac{2+s}{2}} \quad \text{and} \quad H(t) := G'(+\infty) t - G(t) \quad \text{for} \ t \in \mathbb{R}. \quad \text{(A.1)}$$

Note that $G(0) = 0$,

$$0 < G'(t) = \int_0^t d\tau (1 + \tau^2)^{-\frac{2+s}{2}} < G'(+\infty) < +\infty,$$

and $G''(t) = (1 + t^2)^{-\frac{2+s}{2}} > 0$ for all $t \in \mathbb{R}$. Therefore, $G$ is a nonnegative even function which is increasing in $(0, +\infty)$ and strictly convex on $\mathbb{R}$. Furthermore, it behaves as a quadratic function near the origin and approaches a linear one as $t \to \pm \infty$. On the other hand, $H(0) = 0$, $H' = G'(+\infty) - G' > 0$, and $H'' = -G'' < 0$ in $[0, +\infty)$. Thus, $H$ is positive, increasing, and strictly concave in $[0, +\infty)$. Finally, by Fubini’s theorem,

$$H(+\infty) = \int_0^{+\infty} dt H'(t) = \int_0^{+\infty} dt \left( G'(+\infty) - G'(t) \right) = \int_{-\infty}^{+\infty} dt \int_t^{+\infty} d\tau (1 + \tau^2)^{-\frac{2+s}{2}}$$

$$= \int_{-\infty}^{+\infty} d\tau (1 + \tau^2)^{-\frac{2+s}{2}} < +\infty.$$ 

For the sake of completeness we will give the proof of (A.1), even though this expression was not used to prove the results of this paper. But before, let us explain how to use (A.1) to give a simple proof of the existence of a constrained minimizer of $\mathcal{P}_s$ when $n = 2$. The argument goes as follows. Since $G(t) + C_1 \geq C_2 |t|$ for some constants $C_1, C_2 > 0$, the term in (A.1) involving $G$ bounds the $W^{s,1}(-\pi, \pi)$ norm of $u$ up to an additive constant. In this way, since $H \geq 0$, a minimizing sequence for $\mathcal{P}_s$ is a bounded sequence in $W^{s,1}(-\pi, \pi)$. By compactness, this yields the existence of the desired constrained minimizer. We refer to [2] for the details on this argument; see also the end of Part 1 in our proof of Theorem 1.1.

**Lemma A.1.** Let $u : \mathbb{R} \to \mathbb{R}$ be nonnegative, $2\pi$-periodic, and such that $u \in W^{s,1}(-\pi - \epsilon, \pi + \epsilon)$ for some (and hence for all) $\epsilon > 0$. Let $E = \{(x_1, x_2) \in \mathbb{R}^2 : |x_2| < u(x_1)\}$. Then, (A.1) holds.

**Proof.** We prove (A.1) using (2.2). Let us first rewrite $\phi$ in (2.3) for $n = 2$ in a more convenient way by introducing the function $G'$. For $p \geq 0$ and $q \geq 0$ we have (using the change of variables
Note that \( G \) is odd, \( 0 < -G'(-\infty) = G'(\infty) < +\infty \), and \( G'(-q - \cdot) - G'(q - \cdot) \) is an even function. Therefore,

\[
\phi(p, q) = 2 \int_0^p dw \left\{ G'(-q - w) - G'(q - w) + 2G'(\infty) \right\} = 4G'(\infty)p + 2 \int_0^p dw G'(-q - w) - 2 \int_0^p dw G'(q - w)
\]

where we also used in the last equality that \( G \) is even. Now, from (2.2) and (A.2) we deduce that

\[\mathcal{P}_s[E] = 2 \int_{-\pi}^{\pi} dx_1 \int_{-\infty}^{+\infty} dy_1 |x_1 - y_1|^{-s} \left\{ G\left( \frac{u(x_1) - u(y_1)}{|x_1 - y_1|} \right) - G\left( \frac{u(x_1) + u(y_1)}{|x_1 - y_1|} \right) + 2G'(\infty) \frac{u(x_1)}{|x_1 - y_1|} \right\}.\]  

(A.3)

Our goal is to show that in the term involving \( 2G'(\infty)u(x_1)|x_1 - y_1|^{-1} \) we can replace \( u(x_1) \) by \( u(y_1) \) and the identity (A.3) remains unchanged. Once this is proved, we get that in this last term one can replace \( 2u(x_1) \) by \( u(x_1) + u(y_1) \). Hence, using that \( H(t) = G'(\infty)t - G(t) \), this gives (A.1).

Therefore, it only remains to show that we can replace \( u(x_1) \) by \( u(y_1) \) in the last term on the right-hand side of (A.3). This follows immediately from Lemma 2.1 (used with \( m = l = 1 \)) applied to the integral

\[
\int_{-\pi}^{\pi} dx \int_{-\infty}^{+\infty} dy \frac{f(x, y)}{|x - y|^s},
\]

where

\[f(x, y) := -G\left( \frac{u(x) + u(y)}{|x - y|} \right) + 2G'(\infty) \frac{u(x)}{|x - y|}.\]

To check the hypothesis of the lemma, we need to verify that

\[\int_{-\pi}^{\pi} dx \int_{-\infty}^{+\infty} dy \left| \frac{f(x, y)}{|x - y|^s} \right| < +\infty.\]  

(A.4)

For this purpose we write \( f(x, y) = A + B \), where

\[A := -G\left( \frac{u(x) + u(y)}{|x - y|} \right) + G\left( \frac{2u(x)}{|x - y|} \right)\]

and

\[B := -G\left( \frac{2u(x)}{|x - y|} \right) + 2G'(\infty) \frac{u(x)}{|x - y|} = H\left( \frac{2u(x)}{|x - y|} \right).\]
Concerning the quantity $A$, observe that

$$|A| = \left| \int_{-\pi}^{\pi} \frac{du(x)}{|x-y|} \, dt \right| \leq \sup_{t \in \mathbb{R}} |G'(t)| \frac{|u(x) - u(y)|}{|x-y|} = G'(\infty) \frac{|u(x) - u(y)|}{|x-y|}. \quad (A.6)$$

Recall that we are assuming $u \in W^{s,1}(\pi)$ for some $\epsilon > 0$. According to this, to prove (A.4) we split the integral and we use (A.5) and (A.6) to get

$$\int_{-\pi}^{\pi} dx \int_{|x-y| > \epsilon} dy |f(x, y)| \leq G'(\infty) \int_{-\pi}^{\pi} dx \int_{|x-y| > \epsilon} dy |A| + G'(\infty) \int_{-\pi}^{\pi} dx \int_{|x-y| > \epsilon} dy \left| \frac{3u(x)}{|x-y|^{1+s}} \right|$$

$$+ G'(\pi) \int_{-\pi}^{\pi} dx \int_{|x-y| > \epsilon} dy \left| \frac{u(y)}{|x-y|^{1+s}} \right|. \quad (A.8)$$

The first two terms on the right-hand side of (A.7) are finite. To estimate the third one, we use that $0 \leq G(t) \leq G'(\infty) t$ for all $t \geq 0$, and hence

$$\int_{-\pi}^{\pi} dx \int_{|x-y| > \epsilon} dy \left| \frac{3u(x)}{|x-y|^{1+s}} \right| \leq G'(\pi) \int_{-\pi}^{\pi} dx \int_{|x-y| > \epsilon} \frac{dy}{|x-y|^{1+s}}.$$
one in [24] is a straightforward application of the sharp estimates for the heat kernel on the sphere \( S^n \) found there.

The fundamental solution of the heat equation in \((-\pi, \pi)\) with periodic boundary conditions is given by

\[
\Gamma(z, t) := \frac{1}{\sqrt{4\pi t}} \sum_{k \in \mathbb{Z}} e^{-\frac{(x-2k\pi)^2}{4t}} \quad \text{for } z \in \mathbb{R} \text{ and } t > 0.
\]

Given a \(2\pi\)-periodic initial data \(g\), the function

\[
u(x, t) = \int_{-\pi}^{\pi} dy \Gamma(x - y, t)g(y)
\]

satisfies \(\partial_t u - \partial_{xx} u = 0\) in \(\mathbb{R} \times (0, +\infty)\), that \(u(\cdot, t)\) is \(2\pi\)-periodic for all \(t > 0\), and that \(u(\cdot, 0) = g\) in \(\mathbb{R}\). This follows immediately from the properties of the standard (nonperiodic) heat kernel and the fact that

\[
\sum_{k \in \mathbb{Z}} \int_{-\pi}^{\pi} dy e^{-\frac{(x-y+2k\pi)^2}{4t}} g(y) = \sum_{k \in \mathbb{Z}} \int_{-\pi}^{\pi} dy e^{-\frac{(x-y+2k\pi)^2}{4t}} g(y - 2k\pi)
\]

\[
= \sum_{k \in \mathbb{Z}} \int_{-\pi-2k\pi}^{\pi-2k\pi} dy e^{-\frac{(x-y)^2}{4t}} g(y) = \int_{\mathbb{R}} dy e^{-\frac{(x-y)^2}{4t}} g(y).
\]

We now state and prove the result on the periodic heat kernel that we used in the proof of Theorem 1.2.

**Theorem B.1.** For every \(t > 0\), the function

\[z \mapsto \Gamma(z, t) \text{ is decreasing in } (0, \pi).\]

**Proof.** Let us first show that \(\Gamma(\cdot, t)\) is nonincreasing in \((0, \pi)\). For this, we take functions \(g_\epsilon \in C^\infty_c(-\epsilon, \epsilon)\), where \(0 < \epsilon < \pi/2\), approximating the Dirac delta as \(\epsilon \downarrow 0\), and we consider their \(2\pi\)-periodic extensions from \([-\pi, \pi]\) to \(\mathbb{R}\) (which we also denote by \(g_\epsilon\)). In particular, we assume that

\[
\int_{-\epsilon}^{\epsilon} g_\epsilon = 1 \quad \text{for all } 0 < \epsilon < \pi/2.
\]

We additionally assume that the \(g_\epsilon\) are nonnegative, even, and nonincreasing in \((0, \pi)\).

Let \(u_\epsilon\) be the solution to

\[
\begin{align*}
\partial_t u_\epsilon - \partial_{xx} u_\epsilon &= 0 \quad \text{in } \mathbb{R} \times (0, +\infty), \\
u_\epsilon(\cdot, 0) &= g_\epsilon \quad \text{in } \mathbb{R}.
\end{align*}
\]

Thus, \(u_\epsilon(x, t) = \int_{-\pi}^{\pi} dy \Gamma(x - y, t)g_\epsilon(y)\) and \(u_\epsilon(\cdot, t)\) is \(2\pi\)-periodic for all \(t > 0\). Notice that \(u_\epsilon(\cdot, t)\) is even with respect to \(x = 0\) and \(x = \pi\) (by uniqueness, since so is \(g_\epsilon\)). We deduce that the derivative \(v_\epsilon := \partial_x u_\epsilon\) solves

\[
\begin{align*}
\partial_t v_\epsilon - \partial_{xx} v_\epsilon &= 0 \quad \text{in } (0, \pi) \times (0, +\infty), \\
v_\epsilon(0, t) &= v_\epsilon(\pi, t) = 0 \quad \text{for all } t > 0.
\end{align*}
\]

Moreover, by the assumptions on \(g_\epsilon\), we have that \(v_\epsilon(x, 0) = \partial_x u_\epsilon(x, 0) = g_\epsilon'(x) \leq 0\) for all \(x \in [0, \pi]\). Hence, the maximum principle yields

\[v_\epsilon \leq 0 \quad \text{in } [0, \pi] \times [0, +\infty). \quad (B.1)
\]

We now claim that, for every given \(t > 0\) and \(m = 0, 1, 2, \ldots\),

\[
u_\epsilon(\cdot, t) \text{ converges to } \Gamma(\cdot, t) \text{ in } C^m([0, \pi]) \text{ as } \epsilon \downarrow 0. \quad (B.2)
\]
This follows by combining the uniform continuity in $[0, \pi]$ of $\Gamma(\cdot, t)$ and of all its derivatives with the fact that, for every $\delta > 0$, we have

$$\left| \partial_x^m u_\epsilon(x, t) - \partial_x^m \Gamma(x, t) \right| \leq \int_{-\epsilon}^{\epsilon} dy \left| \partial_x^m \Gamma(x - y, t) - \partial_x^m \Gamma(x, t) \right| |g_\epsilon(y)| \leq \delta$$

if $\epsilon$ is chosen small enough.

Now, from (B.1) and (B.2) we conclude that

$$\partial_x \Gamma(x, t) = \lim_{\epsilon \downarrow 0} \partial_x u_\epsilon(x, t) = \lim_{\epsilon \downarrow 0} v_\epsilon(x, t) \leq 0$$

for all $x \in [0, \pi]$ and $t > 0$.

Finally, we show that $\Gamma(\cdot, t)$ is decreasing in $(0, \pi)$. For this, notice that we already proved that, for all $t > 0$, $\Gamma(\cdot, t)$ is even with respect to $x = 0$ and $x = \pi$. Therefore, given any $t_0 > 0$, we see that $v := \partial_x \Gamma$ solves

$$\begin{cases}
\partial_t v - \partial_{xx} v = 0 & \text{in } (0, \pi) \times (t_0, +\infty), \\
v(0, t) = v(\pi, t) = 0 & \text{for all } t > t_0, \\
v(\cdot, t_0) = \partial_x \Gamma(\cdot, t_0) \leq 0 & \text{in } [0, \pi].
\end{cases}$$

Now, the strong maximum principle yields that either $v < 0$ in $(0, \pi) \times (t_0, +\infty)$ or $v \equiv 0$ in $[0, \pi] \times [t_0, +\infty)$. The proof is now complete, since $v \equiv 0$ is absurd—it would give that $\Gamma(\cdot, t)$ is constant in $x$, clearly contradicting the fact that $\Gamma$ is the fundamental solution of the heat equation with periodic boundary conditions, as shown in the beginning of this appendix. □

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