Electron-electron interaction in graphene at finite Fermi energy.

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Abstract

The wave equation describing the interaction of two electrons in graphene at arbitrary value of the Fermi energy $E_F$ is derived. For the solutions of this equation, we have found the explicit forms of the density and the current which obey the continuity equation. We have traced the evolution of the wave packet during a scattering process. It is shown that the long-leaving localized quasi-stationary peak may appear at $E_F < 0$. Then this peak decays into a set of wave packets following each other. At $t \to \infty$ a total norm of all outgoing wave packets equals to that of incoming wave packet. At $E_F = 0$ the localized state does not appear. For $E_F < 0$ there is an infinite set of the localized solutions with the finite norms.

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I. INTRODUCTION

Investigation of electron-electron scattering in graphene is a very important task because the results of this investigation may explain a high mobility of the charge carriers. Nowadays there is a set of theoretical publications devoted to this subject, see also reviews. It is well established now that the low-energy single electron dynamics in graphene is described by the massless two-component Dirac equation:

\[ i\hbar \partial_t \psi(t, \mathbf{r}) = v_F \mathbf{\sigma} \cdot \mathbf{\hat{p}} \psi(t, \mathbf{r}) , \]

where \( v_F \) is the Fermi velocity, \( \mathbf{\hat{p}} = -i\hbar \nabla \), and \( \mathbf{\sigma} = (\sigma_x, \sigma_y) \) are the Pauli matrices acting on the pseudospin variables. Below we set \( \hbar = v_F = 1 \). A pair of non-interacting electrons can be described by the equation

\[ i\partial_t \psi(r_1, r_2, t) = \hat{H}_0 \psi(r_1, r_2, t) , \]

\[ \hat{H}_0 = \sigma_1 \cdot \mathbf{\hat{p}}_1 + \sigma_2 \cdot \mathbf{\hat{p}}_2 , \]

where \( \psi(r_1, r_2, t) \) is the wave function depending on the coordinates and pseudospin variables of both electrons. There is a nontrivial problem of generalization of Eq. (1) to the case of interacting electrons. This problem arises because of necessity to take into account the electron-hole interaction, i.e., the interaction of electrons above Fermi surface with electrons below Fermi surface. In quantum electrodynamics this interaction is accounted for by means of the Dyson-Schwinger equation, see, e.g., Refs.\(^ {10,11} \). For massless electrons in graphene, an account for the electron-hole interaction may qualitatively change the properties of the electron-electron interaction. The approach based on the Bethe-Salpeter equation (reduction of the Dyson-Schwinger equation) was used in Refs.\(^ {12,13} \) at the investigation of the electron-hole interaction in graphene.

In Refs.\(^ {2,3} \) the electron-electron interaction has been studied using the equation (1) with the replacement \( \hat{H}_0 \rightarrow \hat{H}_0 + V(r) \), where \( V(r) = V(|r_1 - r_2|) \) is the electron-electron interaction potential. This means that the electron-hole interaction has been neglected. In the frame of zero total momentum the corresponding wave equation reads

\[ i\partial_t \psi(t, \mathbf{r}) = \hat{H} \psi(t, \mathbf{r}) , \]

\[ \hat{H} = (\sigma_1 - \sigma_2) \cdot \mathbf{\hat{p}} + V(r) . \]

(2)
In Ref.\textsuperscript{14} the stationary normalized solutions of Eq. (2) with zero energy are found. In Refs.\textsuperscript{2,3} it is shown that the solutions of the wave equation (2) have unusual properties. In the present paper we investigate the impact of the electron-hole interaction on these properties. We show that the solutions found in Refs.\textsuperscript{2,3} correspond to the Fermi energy \(E_f \to -\infty\). We also show that the electron-electron interaction at \(E_f = 0\) has completely different properties compared to that found in Refs.\textsuperscript{2,3}. A transformation of the solutions with increasing \(E_f\) from \(-\infty\) to zero is traced.

II. GENERAL PROPERTIES OF THE MODEL

We start our consideration with the derivation of the wave equation for two interacting electrons with an account for the electron-hole interaction. The Bethe-Salpeter equation for the two-body function \(\Phi(\varepsilon_1, \mathbf{p}_1|\varepsilon_2, \mathbf{p}_2)\), see, e.g., Ref.\textsuperscript{11}, has the form

\[
\Phi(\varepsilon_1, \mathbf{p}_1|\varepsilon_2, \mathbf{p}_2) = iG(\varepsilon_1, \mathbf{p}_1)G(\varepsilon_2, \mathbf{p}_2) \int \frac{d\omega d\omega}{(2\pi)^3} \tilde{V}(q)\Phi(\varepsilon_1 + \omega, \mathbf{p}_1 + q|\varepsilon_2 - \omega, \mathbf{p}_2 - q),
\]

\[G(\varepsilon_i, \mathbf{p}_i) = \frac{1}{\varepsilon_i - \sigma_i \cdot \mathbf{p}_i + i0 \text{sgn}(\varepsilon_i - E_F)}, \tag{3}\]

where \(\tilde{V}(q)\) is the Fourier transform of the potential \(V(r)\), \(G(\varepsilon, \mathbf{p})\) is the one-particle Green’s function, and \(E_f\) is the Fermi energy. We make the substitution \(\varepsilon_1 = E/2 + \Omega, \varepsilon_2 = E/2 - \Omega\) and take the integral over \(\Omega\) in the both sides of Eq. (3). We have

\[
\Xi(E, \mathbf{p}_1, \mathbf{p}_2) = i \int \frac{d\Omega}{2\pi} G\left(\frac{1}{2}E + \Omega, \mathbf{p}_1\right) G\left(\frac{1}{2}E - \Omega, \mathbf{p}_2\right)
\times \int \frac{d\mathbf{q}}{(2\pi)^2} \tilde{V}(q) \Xi(E, \mathbf{p}_1 + \mathbf{q}, \mathbf{p}_2 - \mathbf{q}), \tag{4}
\]

where

\[
\Xi(E, \mathbf{p}_1, \mathbf{p}_2) = \int \frac{d\Omega}{2\pi} \Phi\left(\frac{1}{2}E + \Omega, \mathbf{p}_1\right|\frac{1}{2}E - \Omega, \mathbf{p}_2\right).
\]

Performing in Eq. (4) the integration over \(\Omega\) we finally obtain the following equation for the wave function \(\Xi(E, \mathbf{p}_1, \mathbf{p}_2)\):

\[
(E - \sigma_1 \cdot \mathbf{p}_1 - \sigma_2 \cdot \mathbf{p}_2)\Xi(E, \mathbf{p}_1, \mathbf{p}_2) = \frac{1}{2} \left\{ \sigma_1 \cdot \mathbf{n}_1 \vartheta(p_1 - |E_F|) + \sigma_2 \cdot \mathbf{n}_2 \vartheta(p_2 - |E_F|) \\
- \text{sgn}(E_F)[\vartheta(|E_F| - p_1) + \vartheta(|E_F| - p_2)] \right\} \int \frac{d\mathbf{q}}{(2\pi)^2} \tilde{V}(q)\Xi(E, \mathbf{p}_1 + \mathbf{q}, \mathbf{p}_2 - \mathbf{q}), \tag{5}
\]

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where \( n_1 = p_1/p_1 \) and \( n_2 = p_2/p_2 \). This equation is valid for any \( p_1, p_2, \) and \( E_f \). Below we consider the most interesting case \( p_1 = -p_2 = p \) and \( \varepsilon_F \equiv -E_F \geq 0 \) (Fermi surface is below or coincides with the Dirac point). Then we have

\[
(E - \Sigma \cdot p) \chi(E, p) = R \int \frac{dq}{(2\pi)^2} \tilde{V}(q) \chi(E, p + q),
\]

\[
R = \frac{1}{2} \Sigma \cdot n \vartheta(p - \varepsilon_F) + \vartheta(\varepsilon_F - p),
\]

(6)

where \( n = p/p, \Sigma = \sigma_1 - \sigma_2, \) and \( \chi(E, p) = \Xi(E, p, -p) \). As should be, in the limit \( \varepsilon_F \to \infty \) \( (E_F \to -\infty) \) the equation (6) transfers in Eq. (2) written in the momentum representation.

To have the physical interpretation of the wave equation (6), it is necessary to derive the continuity equation. For this purpose, let us introduce the projector operators

\[
\Lambda^{\pm \pm}(p) = \frac{1}{4} (1 \pm \sigma_1 \cdot n)(1 \mp \sigma_2 \cdot n),
\]

(7)

and the operators

\[
L_1(p) = \vartheta(p - \varepsilon_F) \Lambda^{++}(p) + \vartheta(\varepsilon_F - p), \quad L_2(p) = \vartheta(p - \varepsilon_F) \Lambda^{--}(p),
\]

\[
L_3(p) = \vartheta(p - \varepsilon_F) \Lambda^{+-}(p), \quad L_4(p) = \vartheta(p - \varepsilon_F) \Lambda^{-+}(p).
\]

(8)

The latter operators are also projectors, i.e., \( L_i L_j = \delta_{ij} L_i \) and \( \sum_{i=1}^{4} L_i = 1 \). We also introduce the functions

\[
\chi_+(E, p) = L_1(p)\chi(E, p), \quad \chi_-(E, p) = L_2(p)\chi(E, p),
\]

\[
\chi_{+-}(E, p) = L_3(p)\chi(E, p), \quad \chi_{-+}(E, p) = L_4(p)\chi(E, p).
\]

(9)

Note that \( \chi = \chi_+ + \chi_- + \chi_{+-} + \chi_{-+} \). It follows from Eq. (10) that \( \chi_{-+}(E, p) = \chi_{+-}(E, p) = 0 \), so that \( \chi = \chi_+ + \chi_- \). The functions \( \chi_+(E, p) \) and \( \chi_-(E, p) \) obey the equations

\[
(E - \Sigma \cdot p) \chi_+(E, p) = L_1(p) \int \frac{dq}{(2\pi)^2} \tilde{V}(q) \chi(E, p + q),
\]

\[
(E - \Sigma \cdot p) \chi_-(E, p) = -L_2(p) \int \frac{dq}{(2\pi)^2} \tilde{V}(q) \chi(E, p + q).
\]

(10)

The time dependent wave functions written in the coordinate space \( (\chi_\pm(E, p) \to \psi_\pm(t, r)) \)
obey the equation:

\[
(i\partial_t - \Sigma \cdot \hat{p}) \psi_\pm(t, \mathbf{r}) = \pm \int d\mathbf{r}' Q_\pm(\mathbf{r} - \mathbf{r}') V(\mathbf{r}') \psi(t, \mathbf{r}'),
\]

where

\[
Q_+(\mathbf{r}) = f_0 + f_1 + f_2, \quad Q_-(\mathbf{r}) = -f_1 + f_2,
\]

\[
f_0 = \frac{yJ_1(y)}{2\pi r^2}, \quad f_1 = i \frac{\Sigma \cdot \mathbf{y}}{8\pi r^2 y} \left[ yJ_0(y) + 1 - \int_0^y J_0(x) \, dx \right]
\]

\[
f_2 = \frac{1}{4} \left[ \delta(\mathbf{r}) - \frac{yJ_1(y)}{2\pi r^2} \right] \left( 1 - \frac{\sigma_1 \cdot \sigma_2}{2} \right) + \frac{2(\sigma_1 \cdot \mathbf{y})(\sigma_2 \cdot \mathbf{y}) - y^2 \sigma_1 \cdot \sigma_2}{16\pi r^2 y^2} [2J_0(y) + yJ_1(y)],
\]

(11)

where \( \mathbf{y} = \varepsilon_F \mathbf{r} \), \( J_n(x) \) are the Bessel functions of the first kind, and \( \psi(t, \mathbf{r}) = \psi_+(t, \mathbf{r}) + \psi_-(t, \mathbf{r}) \). The operators \( Q_+(\mathbf{r}) \) and \( Q_-(\mathbf{r}) \) are the projector operators in the coordinate space corresponding to the projector operators \( L_1(\mathbf{p}) \) and \( L_2(\mathbf{p}) \) in the momentum space. It follows from Eq. (11) that the wave function \( \psi(t, \mathbf{r}) \) obeys the equation

\[
(i\partial_t - \Sigma \cdot \hat{p}) \psi(t, \mathbf{r}) = \int d\mathbf{r}' [Q_+(\mathbf{r} - \mathbf{r}') - Q_-(\mathbf{r} - \mathbf{r}')] V(\mathbf{r}') \psi(t, \mathbf{r}').
\]

(12)

Using (11) we find

\[
\partial_t \rho(t, \mathbf{r}) + \text{div} \mathbf{J}(t, \mathbf{r}) + F(t, \mathbf{r}) = 0,
\]

\[
\rho = \psi_+^* \psi_+ - \psi_-^* \psi_-, \quad \mathbf{J} = \psi_+^* \Sigma \psi_+ - \psi_-^* \Sigma \psi_-,
\]

\[
F = 2 \text{Im} \int d\mathbf{r}' V(\mathbf{r}') \left[ \psi_+^*(t, \mathbf{r}) Q_+(\mathbf{r} - \mathbf{r}') + \psi_-^*(t, \mathbf{r}) Q_-(\mathbf{r} - \mathbf{r}') \right] \psi(t, \mathbf{r}').
\]

(13)

Since \( Q_+ \) and \( Q_- \) are the hermitian projector operators, then

\[
\int d\mathbf{r} F(t, \mathbf{r}) = 2 \text{Im} \int d\mathbf{r}' V(\mathbf{r}') \psi_+^*(t, \mathbf{r}') \psi(t, \mathbf{r}') = 0.
\]

(14)

Therefore \( \int d\mathbf{r} \rho(t, \mathbf{r}) \) is time-independent. The equation (13) may be written in the conventional form

\[
\partial_t \rho(t, \mathbf{r}) + \text{div} \mathbf{J}_{\text{tot}}(t, \mathbf{r}) = 0,
\]

\[
\mathbf{J}_{\text{tot}} = \mathbf{J} + \frac{1}{2\pi} \int d\mathbf{r}' \frac{\mathbf{r} - \mathbf{r}'}{||\mathbf{r} - \mathbf{r}'||^2} F(\mathbf{r}').
\]

(15)

The validity of the continuity equation allows us to treat the function \( \psi(t, \mathbf{r}) \) as a wave function of two electrons and the quantity \( e \rho(t, \mathbf{r}) \) as a local charge density (\( e \) is the electron charge). Note that, generally speaking, \( \rho(t, \mathbf{r}) \) is not positive, but the charge density should not be positive.
III. TIME EVOLUTION OF THE WAVE PACKETS.

To investigate a time evolution of the wave packets, we write the equation for the wave function \( \chi(t, p) \) in the form

\[
(i \partial_t - \Sigma \cdot p) \chi(t, p) = R \int \frac{dq}{(2\pi)^2} \tilde{V}(q) \chi(t, p + q),
\]

where \( R \) is given in Eq. (6). Then we represent the function \( \chi(t, p) \) as

\[
\chi(t, p) = \sum_m c_m \chi_m(t, p, \varphi),
\]

\[
\chi_m(t, p, \varphi) = e^{im\varphi} \left[ a_{00}(t, p)|0,0\rangle + e^{-i\varphi}a_{11}(t, p)|1,1\rangle + e^{i\varphi}a_{1-1}(t, p)|1,-1\rangle + g(t, p)|1,0\rangle \right],
\]

where \( c_m \) are some constants, \( \chi_m(t, p, \varphi) \) are the eigenfunctions of the operator

\[
J^z = T^z - i\partial_\varphi
\]

with the eigenvalue \( m \), where \( T = (\sigma_1 + \sigma_2)/2; |1, k\rangle \) and \(|0, 0\rangle \) are the eigenfunctions of the operator \( T^2 \) and \( T^z \). Let us consider the evolution of the function \( \chi_m(t, p, \varphi) \). It is convenient to pass from the functions \( a_{ij} \) to the functions \( f, h, \) and \( d \):

\[
f = \frac{a_{11} + a_{1-1}}{\sqrt{2}}, \quad h = \frac{a_{11} - a_{1-1}}{\sqrt{2}}, \quad d = a_{00}.
\]

These functions obey the system of integro-differential equations

\[
i\partial_t f = \vartheta(\varepsilon_F - p) \left[ \hat{U}_+ f + \hat{U}_- h \right],
\]

\[
i\partial_t h = -2pd + \vartheta(\varepsilon_F - p) \left[ \hat{U}_- f + \hat{U}_+ h \right] - \vartheta(p - \varepsilon_F)\hat{U}_0 d,
\]

\[
i\partial_t d = -2ph - \vartheta(p - \varepsilon_F) \left[ \hat{U}_- f + \hat{U}_+ h \right] + \vartheta(\varepsilon_F - p)\hat{U}_0 d,
\]

\[
i\partial_t g = \vartheta(\varepsilon_F - p)\hat{U}_0 g.
\]

Here the following notations are used

\[
\hat{U}_\pm H = \frac{1}{2} \int_0^\infty dk \, k[V_{m-1}(p, k) \pm V_{m+1}(p, k)]H(k), \quad \hat{U}_0 H = \int_0^\infty dk \, kV_m(p, k)H(k),
\]

\[
V_m(p, k) = \int_0^\infty dr \, rV(r) J_m(pr)J_m(kr).
\]
where $H(k)$ is an arbitrary function. In the limiting case $\varepsilon_F \to \infty$, we have

$$
\begin{align*}
    i\partial_t f &= \hat{U}_+ f + \hat{U}_- h, \\
    i\partial_t h &= -2pd + \hat{U}_- f + \hat{U}_+ h, \\
    i\partial_t d &= -2ph + \hat{U}_0 d, \\
    i\partial_t g &= \hat{U}_0 g. 
\end{align*}
$$

This system of equations have been investigated in detail in Ref.\textsuperscript{2} in coordinate space. It turns out that the solutions at $m \neq 0$ have very unusual properties. Namely, the time evolution of the wave packet, corresponding to the scattering problem setup, leads to the appearance of the localized state at large time. This is because the first equation in (21), written in a coordinate space, does not contain derivatives over a spatial variable $r$ and reduces to the equation of constraint. Below we trace whether the localized states survive at a finite value of $\varepsilon_F$.

Note that the system (19) at $m = 0$ does not reveal the unusual properties at any $\varepsilon_F$. This statement can be explained as follows. For $m = 0$ we have $U_- = 0$ and the system (19) reduces to

$$
\begin{align*}
    i\partial_t f &= \vartheta(\varepsilon_F - p)\hat{U}_+ f, \\
    i\partial_t h &= -2pd + \vartheta(\varepsilon_F - p)\hat{U}_+ h - \vartheta(p - \varepsilon_F)\hat{U}_0 d, \\
    i\partial_t d &= -2ph - \vartheta(p - \varepsilon_F)\hat{U}_+ h + \vartheta(\varepsilon_F - p)\hat{U}_0 d, \\
    i\partial_t g &= \vartheta(\varepsilon_F - p)\hat{U}_0 g. 
\end{align*}
$$

(22)

It is possible to show that the functions $f$ and $g$ at $m = 0$ tend to zero at $r \to \infty$ faster than $1/\sqrt{r}$, so that they are irrelevant to the scattering problem. Then we have a coupled system of equation for the functions $h$ and $d$ without any constraints and without any unusual properties. This is why for the scattering problem only the cases $m \neq 0$ is investigated below.

Let us consider the convergent at $t \to -\infty$ wave packet, corresponding to $m \neq 0$, with the average energy $E$ and some width $\delta l$ (the energy spread is $\delta E \sim 1/\delta l \ll E$) scattered on the potential

$$
V(r) = u_0 \exp[-r^2/a^2]
$$

(23)

with $u_0 > 0$. We also assume that $\delta E \ll u_0$. A goal of this section is to trace the evolution of the density $\rho(t,r)$. To calculate this function, we firstly find the solutions of Eq. (19) and then pass to the coordinate space making the Fourier transform. As a result we come
to the expression for the density $\rho(t, r) = \rho_+(t, r) - \rho_-(t, r)$ corresponding to the function $\chi_m(t, p, \phi)$:

$$\rho_{\pm}(t, r) = |a_{\pm}|^2 + |b_{\pm}|^2 + 2|c_{\pm}|^2,$$

$$\begin{pmatrix} a_+ \\ b_+ \\ c_+ \end{pmatrix} = \int_{\varepsilon_F}^{\infty} dp\, (h - d) \begin{pmatrix} J_{m-1} \\ J_{m+1} \\ J_m \end{pmatrix}, \quad \int_{\varepsilon_F}^{\infty} dp\, (h - f) J_{m+1} \begin{pmatrix} (h + f) J_{m-1} \\ (h - f) J_{m+1} \\ -dJ_m \end{pmatrix}, \quad \int_{\varepsilon_F}^{\infty} dp\, (h + d) \begin{pmatrix} J_{m-1} \\ J_{m+1} \\ J_m \end{pmatrix}, \quad (24)$$

where the arguments of all Bessel functions are $pr$ and the functions $f$, $h$, and $d$ are the solutions of Eq. (19). Note that the function $g(t, r)$ is irrelevant to the scattering problem at any $m$, and we have omitted it in Eq. (24).

Below we analyze the process at a few values of $\varepsilon_F$. We start with the case $\varepsilon_F = 0$ when the system (19) reduces to

$$i\partial_t f = 0, \quad i\partial_t g = 0,$$

$$i\partial_t h = -2pd - \tilde{U}_0 d, \quad i\partial_t d = -2ph - \tilde{U}_- f - \tilde{U}_+ h. \quad (25)$$

It is seen that the functions $f$ and $g$ are time-independent, so that they are irrelevant to the scattering process, and we can set them to be equal to zero. Thus, we have a coupled system of equations

$$i\partial_t h = -2pd - \tilde{U}_0 d, \quad i\partial_t d = -2ph - \tilde{U}_+ h. \quad (26)$$

In contrast to Eqs. (21), the system (26) does not contain any constraints, so that the localized states do not appear during a scattering process. To illustrate this statement, we show in Fig. 1 the time evolution of the function $r\rho(t, r)$, see Eq. (13), for scattering of the wave packet with the parameters $m = 1$, $\delta E = 0.1$, $E = 4$ (left picture) and $E = 2$ (right picture) on the potential with the parameters $u_0 = 3$ and $a = 2$. It is seen from Fig. 1 that localized states have not appeared, the shape of the outgoing wave is the same as that of the incoming one, and $\int dr\, r\, \rho(-\infty, r) = \int dr\, r\, \rho(\infty, r)$. This statement is valid for both cases $E > u_0$ and $E < u_0$. 

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Then we pass to the case \( \varepsilon_F = 6 \) at the same parameters of the incoming wave packet and the potential. The evolution of the function \( r\rho(t,r) \) is shown in Fig. 2. This picture is typical for the evolution of the wave packet at non-zero but finite \( \varepsilon_F \). For \( E > u_0 \) (left picture) we have only one outgoing wave with the same shape as the incoming wave and with the same norm. For \( E < u_0 \) (right picture) the situation is completely different. When the initial wave came to the nonzero potential region, the long-leaving localized quasi-stationary peak appeared, decaying then into a set of wave packets following each other. At \( t \to \infty \) a total norm of all outgoing wave packets equals to that of incoming wave packet. At \( \varepsilon_F \to \infty \) the lifetime of the localized quasi-stationary peak tends to infinity, as it was shown in Ref. 3. It is possible to estimate the lifetime of this peak at finite \( \varepsilon_F \) as \( \tau \sim \varepsilon_F/|V'(r_0)| \), where \( r_0 \) is defined via the equation \( E = V(r_0) \).
IV. LOCALIZED STATES IN STATIONARY PROBLEM.

As it was mentioned in a previous section, the functions \( f \) and \( g \) at \( m = 0 \) are irrelevant to the scattering problem. Therefore, it is interesting to investigate whether these functions correspond to any localized states at some energies. The answer is positive. To illustrate this statement, let us consider the equations for \( f \) and \( g \) at \( m = 0 \) and fixed energy \( E \):

\[
Ef = \vartheta(\varepsilon_F - p)\hat{U}_+ f, \quad Eg = \vartheta(\varepsilon_F - p)\hat{U}_0 g.
\]

(27)

In the particular case of the potential (23), these equations reduce to

\[
Ef(p) = \frac{u_0 a^2}{2} \vartheta(\varepsilon_F - p) \int_0^\infty dk k \exp[-a^2(p^2 + k^2)/4] I_1(a^2pk/2)f(k),
\]

\[
Eg(p) = \frac{u_0 a^2}{2} \vartheta(\varepsilon_F - p) \int_0^\infty dk k \exp[-a^2(p^2 + k^2)/4] I_0(a^2pk/2)g(k),
\]

(28)

where \( I_n(x) \) is the modified Bessel function of the first kind. It is seen that \( E \) can be written as \( E = u_0 \mathcal{E} \), where \( \mathcal{E} \) depends on \( \varepsilon_F a \) and is independent of \( u_0 \). Passing from the functions \( f(p) \) and \( g(p) \) to the functions \( \sqrt{p} f(p) \) and \( \sqrt{p} g(p) \), we obtain the integral equations with the symmetric kernels. Thus, it follows from the theory of such equations that there are infinite set of normalized orthogonal solutions with the energies \( |E_1| > |E_2| > |E_3|... \)

Let us consider two limiting cases, \( \varepsilon_F a \ll 1 \) and \( \varepsilon_F a \gg 1 \). In the first case we have \( |E_{n+1}/E_n| \ll 1 \), and the asymptotic forms of the solutions for the functions \( f \) and \( g \) with the maximal energies read

\[
f(p) \propto p \vartheta(\varepsilon_F - p), \quad f(r) = \int_0^\infty dppJ_1(pr)f(p) = \frac{1}{r}J_2(\varepsilon_F r), \quad E_1 = u_0(\varepsilon_F a)^4/32;
\]

\[
g(p) \propto \vartheta(\varepsilon_F - p), \quad g(r) = \int_0^\infty dppJ_0(pr)g(p) = \frac{1}{r}J_1(\varepsilon_F r), \quad E_1 = u_0(\varepsilon_F a)^2/4.
\]

(29)

In the second case we have \( |1 - E_{n+1}/E_n| \ll 1 \), and the asymptotic forms of the solutions for the functions \( f \) and \( g \) with the maximal energies read

\[
f(p) \propto \vartheta(\varepsilon_F - p), \quad f(r) = \frac{1}{\varepsilon_F r^2} \int_{\varepsilon_F r}^{\infty} dx xJ_1(x), \quad E_1 = u_0;
\]

\[
g(p) \propto \vartheta(\varepsilon_F - p), \quad g(r) = \frac{1}{r}J_1(\varepsilon_F r), \quad E_1 = u_0.
\]

(30)

In both cases a typical size of the wave functions is \( r \sim 1/\varepsilon_F \).
For $m \neq 0$ the localized solutions with some energies exist only for the function $g$. We have

$$g(p) \propto p^m \vartheta(\varepsilon_F - p), \quad g(r) = \frac{1}{r} J_{m+1}(\varepsilon_F r), \quad E_1 = u_0(\varepsilon_F a/2)^{2m+2}/(m + 1)!$$

(31)

for $\varepsilon_F a \ll 1$ and

$$g(p) \propto \vartheta(\varepsilon_F - p), \quad g(r) = \frac{1}{\varepsilon_F r^2} \int_0^{\varepsilon_F r} dx x J_m(x), \quad E_1 = u_0$$

(32)

for $\varepsilon_F a \gg 1$.

V. CONCLUSION

Using the Bethe-Salpeter equation with the kernel calculated in the leading approximation we have derived the wave equation (5) describing the interaction of two electrons in graphene at arbitrary value of the Fermi energy $E_F$ and the equation (11) for the case $E_F \leq 0$. We have found the explicit forms of the density $\rho(t, r)$ and the current $J_{tot}(t, r)$ which obey the continuity equation (13). We have traced how the picture of the wave packet scattering depends on $E_F$. At $m \neq 0$, $E_F < 0$, and $E < u_0$, the initial wave comes to the nonzero potential region and the long-leaving localized quasi-stationary peak appears. Then this peak decays into a set of wave packets following each other. At $t \to \infty$ a total norm of all outgoing wave packets equals to that of the incoming wave packet. At $E_F \to -\infty$ the lifetime of the localized quasi-stationary peak tends to infinity, which is in agreement with the results of Ref.2. At $E > u_0$ there is only one outgoing wave with the same shape as the incoming wave and with the same norm. At $m = 0$ and any $E_F$, the localized state does not appear in a scattering process. At $E_F = 0$ the localized state does not appear for any $m$.

For $E_F < 0$ there is an infinite set of the localized solutions with the discrete energies and the finite norms. A typical size of the localization is $1/|E_F|$. These solutions are irrelevant to the scattering problem. The experimental observation of these states would be a very interesting task.

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