Interactions of fractional $N$-solitons with anomalous dispersions for the integrable combined fractional higher-order mKdV hierarchy

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Abstract: In this paper, we investigate the anomalous dispersive relations, inverse scattering transform with a Riemann-Hilbert (RH) problem, and fractional multi-solitons of the integrable combined fractional higher-order mKdV (fhmKdV) hierarchy, including the fractional mKdV (fmKdV), fractional fifth-order mKdV (f5mKdV), fractional combined third-fifth-order mKdV (f35mKdV) equations, etc., which can be featured via completeness of squared scalar eigenfunctions of the ZS spectral problem. We construct a matrix RH problem to present three types of fractional $N$-solitons illustrating anomalous dispersions of the combined fhmKdV hierarchy for the reflectionless case. As some examples, we analyze the wave velocity of the fractional one-soliton such that we find that the fhmKdV equation predicts a power law relationship between the wave velocity and amplitude, and demonstrates the anomalous dispersion. Furthermore, we illustrate other interesting anomalous dispersive wave phenomena containing the elastic interactions of fractional bright and dark solitons, W-shaped soliton and dark soliton, as well as breather and dark soliton. These obtained fractional multi-solitons will be useful to understand the related nonlinear super-dispersive wave propagations in fractional nonlinear media.

Keywords: Combined fractional mKdV hierarchy; inverse scattering; Riemann-Hilbert problem; fractional $N$-soliton solutions; anomalous dispersive relations

1 Introduction

Fractional (order) calculus (FC) almost has the same long history as the integer order calculus [1,2], however due to the lack of real world applications, the development of fractional calculus became very slowly [3,4]. In fact, many complex phenomena in nature with anomalous dynamics cannot be depicted by he models with only integer order derivatives. In other words, the fractional order models were shown to be more realistic than the usual integer order models to describe some phenomena with some power-law physical quantities, such as time-dependent displacement $t^\alpha, \alpha > 0$ [5,6], power-law potential $\phi^p, p > 0$ for the inflaton field $\phi$ [7]. Particular attention was paid to FC up to 1970’s [3,4]. Since then, it has been applied in many different areas of mathematics and physics, such as random walks [5], random diffusion [8], telomere motion [9], diffusion waves [10–12], turbulence [13–15], control and robotics [16], viscoelastic dynamics [17], and finance [18,19].

Integrable integer-order nonlinear equations are also an important class of nonlinear wave equations in the study of nonlinear dynamics, such as the Korteweg-de Vries (KdV) equation, modified Korteweg-de Vriese (mKdV) equation, Boussinesq equation, Kadomtsev-Petviashvili equation, nonlinear Schrödinger (NLS) equation, Hirota equation, and sine/sinh-Gordon equation [20–22], which can be solved by the inverse scattering transform (IST) [23]. It is a natural idea to extend these integer-order nonlinear equations to the fractional order cases. Many fractional physical models have been paid attention to this extension, such as the fractional NLS equation, and fractional KdV equation, and so on [24–26], but these fractional models cannot usually guarantee the important IST integrability [27]. More recently, based on the definition of Riesz fractional derivative $|\partial_x^\alpha f|, \epsilon \in (0,1)$ [28,29], Ablowitz and his collaborators [30,31] presented some new integrable fractional nonlinear wave equations, such

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as the fractional KdV, fractional NLS, fractional mKdV, and fractional sine/sinh-Gordon equations, and the powerful IST with Gel’fand-Levitan-Marchenko (GLM) integral equation was used to find their fractional one-soliton solutions.

The Riemann-Hilbert (RH) approach \[32\] has been used to solve some integrable integer-order nonlinear wave equations \[33-41\]. However, to the best of our knowledge, the RH approach was not extended to solve the integrable fractional nonlinear wave equations before. In this paper, we would like to give the combined fractional higher-order mKdV (fhmKdV) hierarchy, and use the IST with RH approach to study the interactions of the fractional multi-solitons with anomalous dispersion of the combined fhmKdV hierarchy.

The rest of this paper is organized as follows. In Sec. 2, we study the power-law dispersion relations of the combined fractional higher-order mKdV (fhmKdV) hierarchy, and give its explicit form using the completeness relation of squared scalar eigenfunctions. In Sec. 3, with the aid of the Zakharov–Shabat spectral problem with the real-valued sufficient decay and smoothness potential and the asymptotics depending on time evolutions of eigenfunctions, the matrix RH problem is established, whose solutions are related to the solutions of the the combined fhmKdV hierarchy. By using Plemelj’s formula to solve the RH problem, we can find the fractional solutions of the combined fhmKdV hierarchy. In Sec. 4, for the reflectionless case, we present the explicitly fractional multi-solitons for the combined fhmKdV hierarchy. In particular, we display the wave velocity of the fractional one-soliton such that we find that the fhmKdV equation predicts a power-law relation between the wave velocity and amplitude and displays the anomalous dispersion. Moreover, we illustrate other interesting super-dispersive wave phenomena containing the elastic interactions of bright and dark solitons, W-shaped soliton and dark soliton, breather and dark soliton, as well as bright-dark-dark solitons. Finally, we summarize our work and give some discussions in Sec. 5.

2 Combined fractional \((2n+1)\)th-order mKdV hierarchy with anomalous dispersion

Starting from the \(2 \times 2\) Zakharov–Shabat spectral (eigenvalue) problem \[42\]

\[
\Phi_x = X(x,t;\lambda)\Phi, \quad X(x,t;\lambda) = i\lambda\sigma_3 + U,
\]

where \(\Phi = \Phi(x,t;\lambda)\) is the \(2 \times 2\) matrix-valued eigenfunction, \(\lambda \in \mathbb{C}\) is a spectral parameter, \(\sigma_3\) and the real-valued potential matrix \(U(x,t)\) are defined as

\[
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad U = \begin{pmatrix} 0 & u(x,t) \\ -u(x,t) & 0 \end{pmatrix},
\]

one can, by constructing the proper time part of matrix eigenfunction, deduce the real-valued scalar combined mKdV hierarchy \[43,44\] containing the mKdV equation, combined third-fifth-order mKdV equations, and higher-order equations. The combined \((2n+1)\)th-order mKdV hierarchy is given by \[45,46\]

\[
u_t + \sum_{\ell=1}^{n} a_{2\ell+1} \partial_x K_{2\ell+1}[u(x,t)] = 0, \quad (x,t) \in \mathbb{R}^2, \quad a_{2\ell+1} \in \mathbb{R},
\]

where \(K_{2\ell+1}[u(x,t)]\)'s are defined as

\[
K_3[u(x,t)] = u_{xx} + 3u^3, \\
K_5[u(x,t)] = u_{4x} + 10(u^2u_{xx} + uu_{xx}^2) + 6u^5, \\
K_7[u(x,t)] = u_{6x} + 14(u^2u_{4x} + 4u_{xx}u_{xxx} + 3uu_{xx}^2 + 5u^2u_{xx} + 5u^4u_{xx} + 10u^3u_{x}^2) + 20u^7,
\]
\[ K_n[u(x, t)] = u_8x + 18u^2u_6x + 108uu_uxu_5x + 228uu_uxu_4x + 210u^2u_4x + 126u^4u_xx + 138uu^2u_{xxx} \]
\[ + 756uu_xu_xxu_{xxx} + 1008u^3u_xu_{xxx} + 182u^2_ux + 756u^3u^2_x + 3108u^2u^2_x \]
\[ + 420u^6u_{xxx} + 798uu^4_x + 1260u^5u_x + 70u^9, \ldots \]

Similarly, we firstly would like to study the combined fractional third-fifth-order mKdV (f35mKdV) equation

\[ u_t + \mathcal{M}(\hat{L})u_x = 0, \quad \mathcal{M}(\hat{L}) = (\alpha_5 - \alpha_3)\hat{L}|\hat{L}|^\epsilon, \quad (4) \]

where \( \epsilon \in (0, 1) \), \( \alpha_3, \alpha_5 \in \mathbb{R} \) and \( \hat{L} = -\partial^2 - 4u^2 - 4u_\alpha \partial^{-1}u \) with \( \partial = \partial / \partial x, \partial^{-1} = \int_{-\infty}^{\infty} dy \) is called the recursion operator. Eq. (4) can be rewritten as

\[ u_t + [\hat{L}]^\epsilon [\alpha_3(\partial^3u_x + 6u^2u_x) + \alpha_5(\partial^5u_x + 10(\partial^2u_x + uu_x^2)x + 30u^4u_x)] = 0. \quad (5) \]

In particular, when \( \alpha_3 = 1, \alpha_5 = 0 \), one can refine the known fmKdV equation

\[ u_t + [\hat{L}]^\epsilon (\partial^3u_x + 6u^2u_x) = 0. \quad (6) \]

As \( \alpha_3 = 0, \alpha_5 = 1 \), one has the fractional fifth-order mKdV (f5mKdV) equation

\[ u_t + [\hat{L}]^\epsilon [\partial^5u_x + 10(\partial^2u_x + uu_x^2)x + 30u^4u_x] = 0. \quad (7) \]

**Anomalous dispersion relation.**—The formal plane wave \( u(x, t) \propto e^{i(\lambda x - w(\lambda)t)} \), where \( \lambda \) is the wave number, and \( w(\lambda) \) angular frequency, is employed to the associated linearization of Eq. (4) to generate the dispersive relation of the linear f35mKdV equation

\[ w(\lambda) = \lambda \mathcal{M}(\lambda^2), \quad (8) \]

where the phase velocity of the wave is \( \mathcal{M}(\lambda^2) \). We further consider the linearization of the f35mKdV equation (6)

\[ u_t + [-\partial^2]^\epsilon (\alpha_5\partial^3u_x + \alpha_3\partial^3u_x) = 0, \]

where \([-\partial^2]^\epsilon \) stands for the Riesz fractional derivative. As a result the anomalous dispersion relation is given by

\[ w(\lambda) = (\alpha_5\lambda^2 - \alpha_3)\lambda^3|\lambda^2|^\epsilon, \quad (9) \]

in which we have the corresponding phase velocity

\[ \mathcal{M}(\lambda^2) = \frac{w(\lambda)}{\lambda} = (\alpha_5\lambda^2 - \alpha_3)\lambda^2|\lambda^2|^\epsilon. \quad (10) \]

**The combined fhmKdV hierarchy and anomalous dispersion relations.**—Similarly to the combined f35mKdV equation (5), one can also consider other combined fractional higher-order mKdV (fhmKdV) equations in the form

\[ u_t + \mathcal{M}_h(\hat{L})u_x = 0, \quad \mathcal{M}_h(\hat{L}) = \left( \sum_{j=1}^{n} a_{2j+1}(-\hat{L})^j \right)|\hat{L}|^\epsilon, \quad a_{2j+1} \in \mathbb{R}, \quad (11) \]

which can be rewritten as

\[ u_t + [\hat{L}]^\epsilon \left( \sum_{j=1}^{n} a_{2j+1}\partial_xk_{2j+1}[u(x, t)] \right) = 0, \quad (12) \]

3
where \( K_{2j+1}[u(x,t)] \)'s are given by Eq. (5). In particular, as \( n = 1, 2, 3, \ldots \), we have the \( f_mKdV \) equation, combined \( f35mKdV \) equation, combined fractional third-fifth-seventh-order \( mKdV \) (\( f357mKdV \)) equation, and etc.

We further use the formal plane wave \( u(x,t) \propto e^{i[kx−\omega_h(\lambda)t]} \) to study the linearization of the combined \( fhmKdV \) equation (12)

\[
u_t + |\partial^2|^c \left( \sum_{i=1}^{n} a_{2i-1} u_{(2i+1)x} \right) = 0,
\]

where \( u_{jx} = \partial^j u/\partial x^j \), such that the anomalous dispersion relation is presented in the form

\[
\omega_h(\lambda) = \sum_{i=1}^{n} a_{2i-1} (-1)^i \lambda^{2i+1} |\lambda^2|^c,
\]

which further leads to the phase velocity

\[
\mathcal{M}_h(\lambda^2) = \frac{\omega_h(\lambda)}{\lambda} = \sum_{i=1}^{n} a_{2i-1} (-\lambda^2)^i |\lambda^2|^c.
\]

To solve the combined \( f35mKdV \) equation (5) and \( fhmKdV \) equation (12) using the IST with the RH problem, we need to consider the Zakharov–Shabat spectral problem (1), and the associated time evolution of eigenfunction \( \Phi \) in the form

\[
\Phi_t = T \Phi, \quad T(x,t;\lambda) = \begin{bmatrix} A(x,t;\lambda) & B(x,t;\lambda) \\ C(x,t;\lambda) & -A(x,t;\lambda) \end{bmatrix},
\]

In general, the expressions of \( A, B, C \) can not be presented explicitly, which is similar to other integrable fractional nonlinear equations [31]. But one can consider the following asymptotic properties:

\[
T_\pm := \lim_{x \to \pm\infty} T(x,t;\lambda) = -i\lambda\mathcal{M}(4\lambda^2)\sigma_3
\]

with \( \mathcal{M}(4\lambda^2) \) given by Eq. (10) for the combined \( f35mKdV \) equation (5), or

\[
T_\pm := \lim_{x \to \pm\infty} T(x,t;\lambda) = -i\lambda\mathcal{M}_h(4\lambda^2)\sigma_3
\]

with \( \mathcal{M}_h(4\lambda^2) \) given by Eq. (14) for the combined \( fhmKdV \) equation (12), such that one has the Jost solutions \( \Phi_\pm(x,t;k) \) satisfying the following boundary conditions

\[
\Phi_\pm(x,t;\lambda) \sim e^{i\lambda[x−\mathcal{M}(4\lambda^2)t]c_3}, \quad x \to \pm\infty
\]

for the combined \( f35mKdV \) equation (5), or

\[
\Phi_\pm(x,t;\lambda) \sim e^{i\lambda[x−\mathcal{M}_h(4\lambda^2)t]c_3}, \quad x \to \pm\infty
\]

for the combined \( fhmKdV \) equation (12).

In what follows, we mainly use the case of the combined \( f35mKdV \) equation (5) as an example to study the fractional multi-solitons. In fact, the case of combined \( fhmKdV \) equation (12) is similar. Hence we introduce a modified matrix function \( J(x,t;\lambda) := \Phi(x,t;\lambda)e^{−i\lambda[x−\mathcal{M}(4\lambda^2)t]c_3} \) such that \( J \) satisfies the boundary conditions \( J_\pm \to \Pi \), as \( x \to \pm\infty \). It is easy to see that \( J \) satisfies the modified spectral problem

\[
J_\epsilon − i\lambda[\sigma, J] = UJ,
\]

which yields

\[
J_\pm(x,t;\lambda) = \Pi + \int_{\pm\infty}^x e^{i\lambda(x−y)c_3} U(y,t)J_\pm(y,t;\lambda)e^{−i\lambda(x−y)c_3} dy.
\]
Let $C^+ = \{ \lambda | \text{Im} \lambda > 0 \}$, $C^- = \{ \lambda | \text{Im} \lambda < 0 \}$, and $\Phi_{\pm}(x,t,\lambda) = (\Phi_{\pm 1}, \Phi_{\pm 2})$ and $J_{\pm}(x,t;\lambda) = (J_{\pm 1}, J_{\pm 2})$. Then for the given $u(x,t) \in L^1(\mathbb{R})$, the matrix-valued functions $\Phi_{\pm}$ and $J_{\pm}$ both have unique solutions in $\mathbb{R}$. Moreover, $J_{\mp 1+2}$, $\Phi_{1+2}$ can be extended analytically to $C^+ (C^-)$, and continuously to $C^+ \cup \mathbb{R} (C^- \cup \mathbb{R})$, since $\Phi_{\pm}(x,t;\lambda)$ are both fundamental solutions of the spectral problem, thus they have the relation

$$
\Phi_{+}(x,t;\lambda) = \Phi_{-}(x,t;\lambda)S(\lambda), \quad \lambda \in \mathbb{R}, \quad S(\lambda) = (s_{ij}(\lambda))_{2 \times 2}, \quad |S(\lambda)| = 1,
$$

which yields

$$
s_{ij}(\lambda) = (-1)^{i+j+1} |\Phi_{-j}(x,t;\lambda), \Phi_{+(3-j)}(x,t;\lambda)|, \quad i, j = 1, 2.
$$

Similarly, the scattering coefficient $s_{11}(\lambda)$ ($s_{22}(\lambda)$) in $\lambda \in \mathbb{R}$ can be extended analytically to $C^- (C^+)$, and continuously to $C^- \cup \mathbb{R} (C^+ \cup \mathbb{R})$, whereas another two scattering coefficients $s_{12}(\lambda)$ and $s_{21}(\lambda)$ cannot be analytically continued away from $\mathbb{R}$.

It follows from the spectral problem (1) that

$$
\hat{L}\phi_{-2} = 4\lambda^2 \phi_{-2}, \quad L\phi_{+1} = 4\lambda^2 \phi_{+1},
$$

where

$$
L = -\partial_x^2 - 4u^2 - 4u\partial_x u_y, \quad \hat{\partial}_x^{-1} = \int_{-\infty}^{\alpha}\partial_x^{-1} dy, \quad \phi_{\pm 1} = \Phi_{\pm 21} \pm \Phi_{\pm 22} \pm \Phi_{\pm 22}. \quad \Phi_{\pm 2} = \Phi_{\pm 21} \pm \Phi_{\pm 22} \pm \Phi_{\pm 22}.
$$

According to the completeness of squared scalar eigenfunctions [31, 47], one can use $M_h(\hat{L})$ to act on a sufficiently smooth and decaying scalar function $g(x)$ to yield

$$
M_h(\hat{L})g(x) = \frac{1}{\pi} \int_{\Gamma_\infty} d\lambda \lambda^2 \int_{\mathbb{R}} \phi_{-2}(x,\lambda)\phi_{+1}(y,\lambda)\phi_{0}(y)g(y)dy,
$$

where $\Gamma_\infty = \lim_{\mathbb{R} \to \infty} \Gamma_R$ with $\Gamma_R$ being the semicircular contour in the upper half plane evaluated from $\lambda = -\mathbb{R}$ to $\lambda = \mathbb{R}$.

For the given $M_h(\hat{L})$ in Eq. (11), one has the combined f35mKdV hierarchy in the form

$$
\begin{align*}
u_t + \frac{1}{\pi} \int_{\Gamma_\infty} d\lambda \lambda^2 \int_{\mathbb{R}} P(x,y,\lambda)u_ydy = 0,
\end{align*}
$$

where $P(x,y,\lambda) = s_{22}^{-2}(\lambda)\phi_{-2}(x,\lambda)\phi_{+1}(y,\lambda)$, that is

$$
\begin{align*}
u_t + \frac{1}{\pi} \int_{\Gamma_\infty} d\lambda |\lambda|^2 \int_{\mathbb{R}} P(x,y,\lambda) \left( \sum_{i=1}^{n} a_{2i+1}(-\hat{L}(y))^i u_y \right)dy = 0,
\end{align*}
$$

where $\hat{L}(y) = -\partial_y^2 - 4u^2(y,t) - 4u_y(y,t)\partial_y^{-1} u(y,t)$ with $\partial_y^{-1} = \int_{-\infty}^{y} ds$, that is

$$
\begin{align*}
u_t + \frac{1}{\pi} \int_{\Gamma_\infty} d\lambda |\lambda|^2 \int_{\mathbb{R}} P(x,y,\lambda) \left( \sum_{i=1}^{n} a_{2i+1} \partial_y K_{2i+1} |u(y,t)| \right)dy = 0.
\end{align*}
$$

In particular, as $n = 2$, we have the combined f35mKdV equation

$$
\begin{align*}
u_t + \frac{1}{\pi} \int_{\Gamma_\infty} d\lambda |\lambda|^2 e \int_{\mathbb{R}} P(x,y,\lambda) \left[ a_3 (u_{yy} + 6u^2 u_y) \
+ a_5 \left( u_{yyyy} + 10(u_y^2 u_y + u_y^2) + 30u^4 u_y \right) \right] dy = 0.
\end{align*}
$$

### 3 Solutions of the Riemann-Hilbert problem

It can be found that $J^{-1}_{\pm}(x,t,\lambda)$ satisfy the the adjoint equation of Eq. (20)

$$
\Psi_x - i\lambda[v_\gamma, \Psi] = -\Psi U.
$$
Let
\[
J^{-1} = \begin{pmatrix} [J^{-1}]_{11} & [J^{-1}]_{12} \\ [J^{-1}]_{21} & [J^{-1}]_{22} \end{pmatrix}^T, \quad J^+ = \begin{pmatrix} [J^+]_{11} & [J^+]_{12} \\ [J^+]_{21} & [J^+]_{22} \end{pmatrix}^T.
\] (31)

Similarly, \([J^{-1}]_{11}, [J^{-1}]_{22}\) are analytic for \(\lambda \in \mathbb{C}^+\), and \([J^+]_{11}, [J^+]_{22}\) are analytic for \(\lambda \in \mathbb{C}^-\). Furthermore, it follows from Eq. (22) that
\[
e^{i|\lambda - M(4\lambda^2)t|x}|J^{-1}| = R(\lambda)e^{-i|\lambda - M(4\lambda^2)t|x}|J^+|, \quad R(\lambda) = S^{-1}(\lambda),
\] (32)
where the inverse scattering matrix is \(R(\lambda) = (r_{ij}(\lambda))_{2 \times 2}\) with \(r_{11}(\lambda) = s_{22}(\lambda), \ r_{22}(\lambda) = s_{11}(\lambda), \ r_{12}(\lambda) = -s_{12}(\lambda), \ r_{21}(\lambda) = -s_{21}(\lambda)\).

Based on the above analysis about \(J_\pm, J_\pm^{-1}\), we define the following new matrices
\[
M_+(x, t; \lambda) := ([J^+]_{11}, [J^+]_{12}) = J_+ L_1 + J_- L_2, \quad M_-(x, t; \lambda) := ([J^+]_{21}, [J^+]_{22}) = L_1 J_+^{-1} + L_2 J_-^{-1},
\] (33) (34)
where \(L_\ell\) is a \(2 \times 2\) matrix, whose element of the \((\ell, \ell)\) position is equal to 1 and others are equal to 0.

Therefore, based on the properties of \(J_\pm(x, t; \lambda)\) and \(J_\pm^{-1}(x, t; \lambda)\), and definitions of \(M_\pm(x, t; \lambda)\), one can construct a Riemann-Hilbert problem:

- **Analytic conditions:** \(M_\pm(x, t; \lambda)\) are analytic for \(\lambda \in \mathbb{C}^\pm\);
- **Jump condition:**
  \[
  M_-(x, t; \lambda)M_+(x, t; \lambda) = G(x, t; \lambda), \quad \lambda \in \mathbb{R},
  \] (35)
  where the jump matrix is
  \[
  G(x, t; \lambda) = e^{i|\lambda - M(4\lambda^2)t|x}|J^{-1}| = e^{i|\lambda - M(4\lambda^2)t|x}|J^+|.
  \]
- \(M_\pm(x, t; \lambda) \rightarrow \mathbb{I}\) as \(\lambda \rightarrow \infty\).

In fact, \(M_+(x, t; \lambda)\) has the following asymptotic expansion at \(\lambda \rightarrow \infty\),
\[
M_+(x, t; \lambda) = \mathbb{I} + \frac{M^{(1)}_+(x, t)}{\lambda} + O(\lambda^{-2}), \quad \lambda \rightarrow \infty,
\] (36)
which is substituted into Eq. (20) to yield the solutions of the combined f3mKdV equation (29)
\[
u(x, t) = -2i(M^{(1)}_+(x, t))_{12},
\] (37)
that is, the solution of Eq. (5) can be converted to solve the above-mentioned Riemann-Hilbert problem.

To present the solutions for the Riemann-Hilbert problem (35), we need to solve the problem under the assumption of irregular, which means that both \(\det M_+\) and \(\det M_-\) should be zero for some \(\lambda\) and all these zeros are simple. The determinants of \(M_+\) and \(M_-\) can be recovered by the elements of \(S(\lambda)\) and \(R(\lambda) := S^{-1}(\lambda)\):
\[
\det M_+(x, t; \lambda) = r_{11}(\lambda) = s_{22}(\lambda), \quad \lambda \in \mathbb{C}^+,
\] (38)
\[
\det M_-(x, t; \lambda) = s_{11}(\lambda), \quad \lambda \in \mathbb{C}^-.
\] (39)
According to \(U^* = -U\), one has \(S^*(\lambda^*) = R(\lambda^*), \ M^*_+((\lambda^*)^*) = M_-(\lambda^*), \ \lambda \in \mathbb{C}^-\). Therefore, one can find that \(\det M_+((\lambda^*)^*) = (\det M_-(\lambda^*))^*, \ \lambda \in \mathbb{C}^+\). Beside, it follows from \(X(\lambda) = X(-\lambda^*)^* \) with \(U^* = U\) that one has \(S(-\lambda^*)^* = S(\lambda)\), which leads to \(s_{11}(-\lambda^*)^* = s_{11}(\lambda)\) and \(\det M_+(\lambda) = (\det M_+(-\lambda^*))^*, \ \det M_-(\lambda) = (\det M_-(-\lambda^*))^*\). Hence, one can assume that if \(\det M_+\) has \(N_1 + 2N_2\) simple zeros \(\lambda_1, \ldots, \lambda_{N_1}\)
in \( i\mathbb{R}^+ \) and \( \lambda_{N_1+1}, \ldots, \lambda_{N_1+N_2}, -\lambda_{N_1+1}, \ldots, -\lambda_{N_1+N_2} \) in \( \mathbb{C}^+ \setminus i\mathbb{R}^+ \), then \( \det M_-\) has \( N_1 + 2N_2 \) simple zeros \( \lambda_1^*, \lambda_2^*, \ldots, \lambda_{N_1}^* \) in \( i\mathbb{R}^- \) and \( \lambda_{N_1+1}^*, \ldots, \lambda_{N_1+N_2}^*, -\lambda_{N_1+1}, \ldots, -\lambda_{N_1+N_2} \) in \( \mathbb{C}^- \setminus i\mathbb{R}^- \).

Suppose the non-zero column vector \( w_j (j = 1, 2, \ldots, N_1 + 2N_2) \) satisfy
\[
M_+(x, t; \lambda_i)w_j = 0.
\]
Then taking the Hermitian of Eq. (40) yields
\[
\overline{w}_j^T M_-(x, t; \lambda^*) = 0.
\]

With the above discussions about the symmetry, the discrete spectra are required to satisfy one of the following three cases, in which \( w_j \) can be found by taking the derivative about \( x \) and \( t \) of Eqs. (40)-(41)
\[
(\theta_{s,j} = i\lambda_j[x - \mathcal{M}(4\lambda_j^2)]);
\]
- **Case a.** \( N = N_1, N_1 \in \mathbb{N}^+, \lambda_j \in i\mathbb{R}^+ \) for \( 1 \leq j \leq N_1 \). In this case, \( w_j = e^{i\sigma_3 x} w_{j0} \), where \( w_{j0} \) are the real constant column vectors.
- **Case b.** \( N = 2N_2, N_2 \in \mathbb{N}^+, \lambda_j + N_2 = -\lambda_j^* \in \mathbb{C}^+ \setminus i\mathbb{R}^+ \) for \( 1 \leq j \leq N_2 \). In this case, \( w_{j+} = e^{i\sigma_3 x} w_{j0}, w_{j+} = w_{j+}^* \) where \( w_{j0} \) are the complex constant column vectors.
- **Case c.** \( N = N_1 + 2N_2, N_1, N_2 \in \mathbb{N}^+, \lambda_j \in i\mathbb{R}^+, \lambda_{N_1+N_2+\ell} = -\lambda_{N_1+\ell}^* \in \mathbb{C}^+ \setminus i\mathbb{R}^+ \) for \( 1 \leq j \leq N_1 \), \( 1 \leq \ell \leq N_2 \). In this case, \( w_j = e^{i\sigma_3 \ell x} w_{j0}, w_{N_1+\ell} = e^{i\sigma_3 \ell x} w_{(N_1+\ell)0} \) and \( w_{N_1+N_2+\ell} = w_{N_1+\ell}^* \) where \( w_{j0}, w_{(N_1+\ell)0} \) are the real and complex constant column vectors, respectively.

**Proposition 1** The non-regular Riemann-Hilbert problem about \( M_\pm \) with zero eigenvalues (40)-(41) can be written as
\[
M_+(x, t; \lambda) = \bar{\mathcal{M}}_+(x, t; \lambda) E(x, t; \lambda),
\]
\[
M_-(x, t; \lambda) = E^{-1}(x, t; \lambda) \bar{\mathcal{M}}_-(x, t; \lambda),
\]
where
\[
E(x, t; \lambda) = \mathbb{I} + \sum_{k,j=1}^{N} \frac{w_k w_j^\dagger (\Omega^{-1})_{kj}}{\lambda - \lambda_j^*}, \quad E^{-1}(x, t; \lambda) = \mathbb{I} - \sum_{k,j=1}^{N} \frac{w_k w_j^\dagger (\Omega^{-1})_{kj}}{\lambda - \lambda_j},
\]
\( \Omega \) is an \( N \times N \) matrix with its \((k,j)\)th element given by
\[
\Omega_{kj} = \frac{w_k^\dagger w_j}{\lambda_k - \lambda_j^*}, \quad 1 \leq k, j \leq N, \quad \det E(\lambda) = \prod_{k=1}^{N} \frac{\lambda - \lambda_k}{\lambda - \lambda_k^*},
\]
and \( \bar{\mathcal{M}}_\pm (x, t; \lambda) \) satisfy the following regular Riemann-Hilbert problem:
- **Analytic conditions:** \( \bar{\mathcal{M}}_\pm (x, t; \lambda) \) are analytic for \( \lambda \in \mathbb{C}^\pm \),
- **Jump condition:** \( \bar{\mathcal{M}}_- (x, t; \lambda) \bar{\mathcal{M}}_+ (x, t; \lambda) = E(\lambda) G(x, t; \lambda) E^{-1}(\lambda), \quad \lambda \in \mathbb{R} \),
- \( \bar{\mathcal{M}}_\pm (x, t; \lambda) \rightarrow \mathbb{I}, \quad \lambda \rightarrow \infty \).

Through the Plemelj’s formula, we have
\[
\bar{\mathcal{M}}_+^{-\dagger} (\lambda) = \mathbb{I} + \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{E(s)[(I - G(s))E^{-1}(s)M_+^{-1}(s)] ds}{s - \lambda},
\]
\[
= \mathbb{I} - \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{E(s)(I - G(s))E^{-1}(s)M_+^{-1}(s) ds + O(\lambda^{-2})}{s - \lambda},
\]

7
Similarly, one has
\[ E(\lambda) = I + \frac{1}{\lambda} \sum_{k,j=1}^{N} w_k w_j^\dagger (\Omega^{-1})_{kj} + O(\lambda^{-2}). \]  

(46)

Therefore, we have
\[ M^{(1)}_+(x,t) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} E(I - G(s)) E^{-1} M^{-1}_+ ds + \sum_{k,j=1}^{N} w_k w_j^\dagger (\Omega^{-1})_{kj}. \]  

(47)

4 Fractional N-solitons with anomalous dispersions

In particular, for the reflectionless case \( G = I \), Eq. (47) becomes
\[ M^{(1)}_+(x,t) = \sum_{k,j=1}^{N} w_k w_j^\dagger (\Omega^{-1})_{kj}. \]  

(48)

Let \( w_j = (a_{1j}, a_{2j})^T \), then one has
\[ w_j = (a_{1j} e^{\theta_{c,j}}, a_{2j} e^{-\theta_{c,j}})^T, \quad \theta_{c,j} = i\lambda_j [x - (16\alpha_5 \lambda_j^4 - 4\alpha_3 \lambda_j^2)|4\lambda_j^2|^T], \quad 1 \leq j \leq N. \]  

(49)

Substituting \( w_j \) into Eq. (48), and using the transform (37), we have the fractional N-soliton solutions of the combined f35mKdV equation (4) in the form
\[ u^{[N]}_+(x,t) = -2i(M^{(1)}_+(x,t))_{12} = -2i \sum_{k,j=1}^{N} a_{1k} a_{2j} e^{\theta_{c,k} - \theta_{c,j}} (\Omega^{-1})_{kj} = 2i \det H \det \Omega, \]  

(50)

where
\[ H = \begin{pmatrix} 0 & a_{11} e^{\theta_{c,1}} & \cdots & a_{1N} e^{\theta_{c,N}} \\ a_{21}^* e^{-\theta_{c,1}} & \Omega_{11} & \cdots & \Omega_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{2N}^* e^{-\theta_{c,N}} & \Omega_{N1} & \cdots & \Omega_{NN} \end{pmatrix}, \]

and \( \Omega = (\Omega_{kj})_{N \times N} \) with
\[ \Omega_{kj} = \frac{a_{1k} a_{1j} e^{\theta_{c,k} + \theta_{c,j}} + a_{2k} a_{2j} e^{-\theta_{c,k} - \theta_{c,j}}}{\lambda_j - \lambda_k}, \quad k, j = 1, 2, \cdots, N. \]  

(51)

Similarly, we can find the fractional multi-solitons of the combined fractional higher-order mKdV equation (12) in the form (50) with
\[ \theta_{c,j} \rightarrow \theta^{(h)}_{c,j} = i\lambda_j \left[ x - \left( \sum_{\ell=1}^{n} a_{2\ell+1} (-4\lambda_j^2)^\ell |4\lambda_j^2|^T \right) \right], \quad 1 \leq \ell \leq n. \]  

(52)

In what follows, we will illustrate the fractional multi-solitons (50) of the f35mKdV equations for \( N = 1, 2, 3 \). In fact, one can also exhibit the fractional multi-solitons (50) with (52) of the combined fhmKdV equation.

Case 1. As \( N = N_1 = 1, \lambda_1 = i\eta (\eta \in \mathbb{R}^+), \) and \( a_{11}, a_{21} \in \mathbb{R}\setminus\{0\} \), it follows from Eq. (50) that the fractional one-soliton solution of Eq. (5) can be written as
\[ u^{[1]}_+(x,t) = -2i a_{11} a_{21} (\lambda_1 - \lambda_1) e^{\theta_{c,1} - \theta_{c,1}} \frac{a_{11} e^{\theta_{c,1} + \theta_{c,1}} + a_{21} e^{-\theta_{c,1} - \theta_{c,1}}}{a_{11} e^{\theta_{c,1} + \theta_{c,1}} + a_{21} e^{-\theta_{c,1} - \theta_{c,1}}}, \]

(53)

\[ = -2\eta \text{sgn}(a_{11}a_{21}) \text{sech}\left\{ 2\eta \left[ x - (4\alpha_5 \eta^2 + a_3) (2\eta)^{2+2\ell} \right] + \ln |a_{21}/a_{11}| \right\}. \]
Figure 1: Wave velocities (54) vs $\epsilon$ of fractional solitons for the fmKdV equation ($\alpha_3 = 1, \alpha_5 = 0$, red dashed line), f5mKdV equation ($\alpha_3 = 0, \alpha_5 = 1$, blue dash-dotted line) and f35mKdV equation ($\alpha_3 = \alpha_5 = 1$, green solid line) equations. (a) $\lambda_1 = i\eta = 0.4i$ (i.e., $\eta = 0.4$); (b) $\lambda_1 = i\eta = 0.6i$ (i.e., $\eta = 0.6$).

Figure 2: $N = N_1 = 1$. Fractional one-soliton solutions (53) for $\lambda_1 = i\eta = 0.6i, a_{11} = 1, a_{21} = -1$. (a1)-(a4) $\alpha_3 = 1, \alpha_5 = 0$ (fmKdV equation), (b1)-(b4): $\alpha_3 = 0, \alpha_5 = 1$ (f5mKdV equation), (c1)-(c4): $\alpha_3 = 1, \alpha_5 = 1$ (f35mKdV equation). (a4)-(c4) the fractional one-soliton curves with $\epsilon = 0.5$ at different times $t = -2$ (dashed line), $t = 0$ (solid line) and $t = 2$ (dash-dotted line).

In particular, when $\alpha_3 = 1, \alpha_5 = 0$ and $\text{sgn}(a_{11}a_{21}) = -1$, we refind the fractional one-soliton solution of the fmKdV equation [31]. As $\alpha_3 = 0, \alpha_5 = 1$, we have the fractional one-soliton solutions of the f5mKdV equation. When $\alpha_3\alpha_5 \neq 0$, we find the fractional one-soliton solutions of the f35mKdV equation.

The wave velocity of the fractional one-soliton solution is

$$v_\epsilon(\eta) = (4\alpha_5\eta^2 + \alpha_3)(2\eta)^{2+2\epsilon}, \quad (54)$$

which depends on the parameter $\epsilon$ for the given $\alpha_3, \alpha_5$ and $\eta$, and implies that the f35mKdV equation predicts a power law relationship between the wave velocity $v_\epsilon(\eta)$ depending on $\epsilon \in (0,1)$ and wave amplitude $\eta$, and displays the anomalous dispersion. The absolute value of the wave velocity, $|v_\epsilon(\eta)|$ is $|4\alpha_5\eta^2 + \alpha_3|(2\eta)^{1+2\epsilon}$ times more than the amplitude $2\eta$. For the given parameter $\epsilon$, the taller solitons travel more quickly than shorter ones. Furthermore, it follows from Eq. (54) that

- As $\alpha_3, \alpha_5 \geq 0$ with $\alpha_3^2 + \alpha_5^2 \neq 0$, one has $v_\epsilon(\eta) > 0$, in which the wave is a right-going travelling wave;
Figure 3: \(N = 2N_2 = 2\). Fractional one-breather (or W-shaped soliton) solutions for \(\lambda_1 = -\lambda_2^* = 0.5 + 0.5i, a_{11} = a_{22} = a_{31} = a_{32} = 1\). (a1)-(a4) \(a_3 = 1, a_5 = 0\) (fmKdV equation), (b1)-(b4) \(a_3 = 0, a_5 = 1\) (W-shaped soliton of the f5mKdV equation), (c1)-(c4): \(a_3 = 1, a_5 = 1\) (f35mKdV equation). (a4)-(c4) the fractional one-soliton curves with \(\epsilon = 0.5\) at different times \(t = -2\) (dashed line), \(t = 0\) (solid line) and \(t = 2\) (dash-dotted line).

- As \(a_3, a_5 \leq 0\) with \(a_3^2 + a_5^2 \neq 0\), one has \(v_\epsilon(\eta) < 0\), in which the wave is a left-going travelling wave;

- As \(a_3a_5 < 0\) and \(4a_5\eta^2 + a_3 > 0\), one has \(v_\epsilon(\eta) > 0\), in which the wave is a right-going travelling wave;

- As \(a_3a_5 < 0\) and \(4a_5\eta^2 + a_3 < 0\), one has \(v_\epsilon(\eta) < 0\), in which the wave is a left-going travelling wave;

- As \(a_3a_5 < 0\) and \(4a_5\eta^2 + a_3 = 0\), one has \(v_\epsilon(\eta) = 0\), in which the wave is a stationary one.

Figs. [a1] and (b) display the wave velocities of the fractional one-soliton solutions for equation parameters \(a_3, a_5 \geq 0\) with \(a_3^2 + a_5^2 \neq 0\) and two spectral parameters \(\lambda_1 = i\eta = 0.4i, 0.6i\), respectively. It follows from Fig. [a1] that for the given \(\eta = 0.4 \in (0, 0.5)\), all wave velocities are decreasing functions in \(\epsilon \in [0, 1]\), however, as \(\eta = 0.6 \in (0.5, 1]\), all wave velocities are increasing functions in \(\epsilon \in [0, 1]\) (see Fig. [b]). In particular, as \(\eta = 0.5\), one has \(v_\epsilon = 4\eta^2(4a_5\eta^2 + a_3)\), which is independent of \(\epsilon\). It follows from Fig. [a1-b] that for the fixed \(\epsilon\), their wave velocities have the relation: \(v_{f35mKdV} > v_{f5mKdV} > v_{fmKdV}\) for \(a_3, a_5 \geq 0\).

Figures [2(a1)-(c4)] illustrate the fractional one-soliton solutions of the fmKdV, f5mKdV and f35mKdV equations for \(\lambda_1 = 0.6i\) and different parameter \(\epsilon = 0, 0.5, 0.9\), respectively, which are all right-going travelling-wave solitons without dissipating or spreading out.

Similarly, as \(N = N_1 = 1\), that is, \(\lambda_1 = i\eta\) (\(\eta \in \mathbb{R}^+\)), we can also find the fractional one-soliton solutions of the combined fractional higher-order mKdV equation [12] in the form

\[
u^{[1]}(x, t) = -2\eta \text{sgn}(a_{11}a_{21}) \text{sech}\left\{2\eta \left[x - \left(\sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell} \right)(2\eta)^{2\ell} t\right] + \ln |a_{21}/a_{11}|\right\}, \tag{55}
\]
whose wave velocity is

$$v_{h,e}(\eta) = \left( \sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell} \right) (2\eta)^{2e}. \quad (56)$$

The absolute value of the wave velocity, \( |v_{h,e}(\eta)| \) is \( \left| \sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell-1} \right| (2\eta)^{2e} \) times more than the amplitude \( 2\eta \). For the given parameter \( e \), the taller fractional soliton propagates more quickly than the shorter one.

We have the following conclusions about the fractional soliton for the parameters \( \eta, a_{2\ell+1}, \ell = 1, 2, \cdots, n, \) and \( e \):

- When \( \sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell} > 0 \), the fractional one-soliton solution (55) is a right-going travelling wave;
- When \( \sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell} < 0 \), the fractional one-soliton solution (55) is a left-going travelling wave;
- When \( \sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell} = 0 \), the fractional one-soliton solution (55) is a stationary wave.
- As \( \sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell} > (<) 0 \) and \( 0 < \eta < 0.5 \), the wave velocity \( v_{h,e}(\eta) > (<) 0 \) and it is a decreasing (increasing) function of \( e \);
- As \( \sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell} > (<) 0 \) and \( \eta > 0.5 \), the wave velocity \( v_{h,e}(\eta) > (<) 0 \) and it is a increasing (decreasing) function of \( e \);
- As \( \eta = 0.5 \), the wave velocity a constant function of \( e \), i.e. \( v_{h,e}(\eta) = \sum_{\ell=1}^{n} a_{2\ell+1}(2\eta)^{2\ell} \).

Notice that since there are the more parameters \( a_{2\ell+1}, \ell = 1, 2, \cdots, n \) in the combined fhmKdV equation (12), thus the fractional one-soliton solution can generate more wave structures than the fmKdV equation (6).

Figure 4: \( N = N_1 = 2 \). Elastic interactions of the fractional two-soliton (bright-dark solitons) solutions for \( \lambda_1 = 0.4i, \lambda_2 = 0.6i, a_{11} = a_{12} = a_{21} = a_{22} = 1 \). (a1)-(a4) \( a_3 = 1, a_5 = 0 \) (fmKdV equation), (b1)-(b4): \( a_3 = 0, a_5 = 1 \) (f5mKdV equation), (c1)-(c4): \( a_3 = 1, a_5 = 1 \) (f35mKdV equation). (a4)-(c4) the fractional one-soliton curves with \( \epsilon = 0.5 \) at different times \( t = -2 \) (dashed line), \( t = 0 \) (solid line) and \( t = 2 \) (dash-dotted line).
Figure 5: \( N = N_1 + 2N_2 = 3, N_1 = 1, N_2 = 1 \) and \( a_{ij} = 1, i = 1, 2; j = 1, 2, 3 \). Elastic interactions of fractional one-breather solution (or W-shaped soliton) and one-dark-soliton solution for (a1)-(a4) \( \lambda_1 = 0.6i, \lambda_2 = -\lambda_3^* = 0.6 + 0.4i, \alpha_3 = 1, \alpha_5 = 0 \) (f3mKdV equation), (b1)-(b4) \( \lambda_1 = 0.7i, \lambda_2 = -\lambda_3^* = 0.8 + 0.2i, \alpha_3 = 1, \alpha_5 = 1 \) (f35mKdV equation); (c1)-(c4) \( \lambda_1 = 0.7i, \lambda_2 = -\lambda_3^* = 0.8 + 0.2i, \alpha_3 = 0, \alpha_5 = 1 \) (f5mKdV equation); (d1)-(d4) \( \lambda_1 = 0.3i, \lambda_2 = -\lambda_3^* = 0.5 + 0.5i, \alpha_3 = 1, \alpha_5 = 1 \) (f5mKdV). (a4)-(d4) the fractional two-soliton curves with \( \epsilon = 0.5 \) at different times \( t = -2 \) (dashed line), \( t = 0 \) (solid line) and \( t = 2 \) (dash-dotted line).

**Case 2a.** As \( N = 2N_2 = 2 \), that is, a pair of anti conjugate spectral parameters \( \lambda_1 \in \mathbb{C}^+ \backslash \mathbb{R}^+ \) and \( \lambda_2 = -\lambda_1^* \) is considered, the fractional one-breather (or W-shaped soliton) solutions of Eq. (5) are given by

\[
\left(u^{[2]}(x,t) = \frac{2i(a_{11}a_{21}e^{\theta_{11}^*} - \theta_{12}^* \Omega_{22} - a_{11}a_{22}e^{\theta_{11}^* - \theta_{21}^*} \Omega_{12} - a_{12}a_{21}e^{\theta_{12}^*} - \theta_{12}^* \Omega_{21} + a_{12}a_{22}e^{\theta_{12}^*} - \theta_{22}^* \Omega_{11})}{\Omega_{12} \Omega_{21} - \Omega_{11} \Omega_{22}}, \right) (57)
\]

where \( \Omega_{ij}, k, j = 1, 2 \) are given by Eq. (51).

Figures 3(a1)-(a4) and (c1)-(c4) illustrate, respectively, the fractional one-breather solutions of the f3mKdV and f35mKdV equations for two spectral parameters \( \lambda_1 = -\lambda_2^* = 0.5 + 0.5i \) and different parameters \( \epsilon = 0, 0.5, 0.9 \), which are all left-going travelling-wave solitons without dissipating or spreading out. However, Figures 3(b1)-(b4) exhibit the fractional W-shaped one-soliton solutions of the f5mKdV equation for two spectral parameters \( \lambda_1 = -\lambda_2^* = 0.5 + 0.5i \) and different parameters \( \epsilon = 0, 0.5, 0.9 \).

**Case 2b.** As \( N = N_1 = 2 \), that is, two spectral parameters \( \lambda_1, \lambda_2 \in \mathbb{R}^+ \) are considered, the fractional two-soliton (bright-dark) solutions of Eq. (5) are given by

\[
\left(u^{[2]}(x,t) = \frac{2i(a_{11}a_{21}e^{\theta_{11}^*} - \theta_{12}^* \Omega_{22} - a_{11}a_{22}e^{\theta_{11}^* - \theta_{21}^*} \Omega_{12} - a_{12}a_{21}e^{\theta_{12}^*} - \theta_{12}^* \Omega_{21} + a_{12}a_{22}e^{\theta_{12}^*} - \theta_{22}^* \Omega_{11})}{\Omega_{12} \Omega_{21} - \Omega_{11} \Omega_{22}}, \right) (58)
\]
Figure 6: $N = N_1 = 3$. Elastic interactions of fractional two-dark-soliton and one-bright-soliton solution for $\lambda_1 = 0.3i$, $\lambda_2 = 0.5i$, $\lambda_3 = 0.7i$, $a_{ij} = 1$, $i = 1, 2; j = 1, 2, 3$. (a1)-(a4) $\alpha_3 = 1$, $\alpha_5 = 0$ (fKdV equation), (b1)-(b4) $\alpha_3 = 0$, $\alpha_5 = 1$ (one W-shaped and one bright solitons of the f5mKdV equation), (c1)-(c4): $\alpha_3 = 1$, $\alpha_5 = 1$ (f35mKdV equation). (a4)-(c4) the fractional one-soliton curves with $\epsilon = 0.5$ at different times $t = -2$ (dashed line), $t = 0$ (solid line) and $t = 2$ (dash-dotted line).

where $\Omega_{kj}$, $k, j = 1, 2$ are given by

$$\Omega_{kj} = \frac{a_{1k}a_{1j}e^{\theta_k^*+\theta_j^*}+a_{2k}a_{2j}e^{-\theta_k^*+\theta_j^*}}{\lambda_k^* - \lambda_j^*}.$$ 

Figures 4(a1)-(c4) illustrate, respectively, the elastic interactions of the fractional two-soliton (bright-dark) solutions of the fmKdV, f5mKdV and f35mKdV equations for two spectral parameters $\lambda_1 = 0.4i$, $\lambda_2 = 0.6i$ and different parameter $\epsilon = 0, 0.5, 0.9$, which are all interactions of one left-going travelling-wave bright soliton and another right-going travelling-wave dark soliton without dissipating or spreading out (see Figs. 4(c1)-(c4)).

Case 3a. As $N = N_1 + 2N_2 = 3$, $N_1 = N_2 = 1$, that is, three spectral parameters $\lambda_1 \in iR^+$, $\lambda_2 \in C^+\setminus iR^+$ and $\lambda_3 = -\lambda_2^*$ are considered, the fractional two-soliton (one dark soliton and one breather/one W-shaped soliton) solutions of Eq. (5) are given in the form (50).

Figures 5(a1)-(a4), (b1)-(b4), and (c1)-(c4) illustrate the elastic interactions of the fractional one-breather and one-dark-soliton solutions of the fmKdV equation for three spectral parameters $\lambda_3 = 0.6i$, $\lambda_2 = -\lambda_3^* = 0.6 + 0.4i$, the f5mKdV and f35mKdV equations for $\lambda_1 = 0.7i$, $\lambda_2 = -\lambda_3^* = 0.8 + 0.2i$, and different parameters $\epsilon = 0, 0.5, 0.9$, respectively, which are all the interactions of one left-going travelling-wave breather and another right-going travelling-wave dark soliton without dissipating or spreading out. However, Figures 5(d1)-(d4) exhibit the elastic interaction of the fractional one W-shaped-soliton and one-dark-soliton solutions of the f5mKdV equation for three spectral parameters $\lambda_1 = 0.3i$, $\lambda_2 = -\lambda_3^* = 0.5 + 0.5i$ and different parameters $\epsilon = 0, 0.5, 0.9$, which are all the interactions of one left-going travelling-wave W-shaped soliton and another right-going travelling-wave dark soliton without dissipating or spreading out.
Case 3b. As $N = N_1 = 3$, that is, three spectral parameters $\lambda_1$, $\lambda_2$, $\lambda_3 \in i\mathbb{R}^+$ (i.e., pure imaginary spectral parameters) are considered, the fractional three-soliton solutions of Eq. (5) are given in the form (50).

Figures (a1)-(c4) illustrate, respectively, the elastic interactions of the fractional three-soliton (two-dark-one-bright) solutions of the fmKdV, f5mKdV and f35mKdV equations for three pure imaginary spectral parameters $\lambda_1 = 0.3i$, $\lambda_2 = 0.5i$, $\lambda_1 = 0.7i$ and different parameter $\epsilon = 0, 0.5, 0.9$, which are all interactions of one right-going travelling-wave bright soliton, one left-going travelling-wave bright soliton, and another right-going travelling-wave dark soliton without dissipating or spreading out (see Figs. (c1)-(c4)).

In fact, we can also display the more abundant wave structures of the fractional $N$-solitons as $N > 3$.

5 Conclusions and discussions

In conclusion, we have studied the integrable combined fractional higher-order mKdV hierarchy with the aid of the Riesz fractional derivative, and their corresponding anomalous dispersion relations. Moreover, the combined fhmKdV hierarchy can be characterized via completeness of squared scalar eigenfunctions. Moreover, the Riemann-Hilbert approach is used to explore the fractional multi-soliton solutions of combined fhmKdV hierarchy. In particular, we display the fractional one-soliton (one-breather) solutions, interactions of fractional two-soliton (e.g., one breather and one dark soliton, one bright and one dark solitons, one dark and one W-shaped solitons, one bright soliton and two dark solitons) of the fmKdV, f5mKdV, and f35mKdV equations. These found fractional multi-soliton solutions will be useful to understand the related super-dispersion transports of nonlinear waves in fractional nonlinear media. Moreover, the RH approach can also be extended to other integrable fractional nonlinear wave equations.

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