Linearity and Nonlinearity of groups of polynomial automorphisms of $K^2$

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Abstract

Let $K$ be a field, and let $\text{Aut} K^2$ be the group of polynomial automorphisms of $K^2$. We investigate which subgroups are linear or not. In characteristic zero, there are small nonlinear subgroups and some big linear subgroups. When $K$ has finite characteristic, the whole group $\text{Aut} K^2$ is linear whenever $K$ is finite, and nonlinear otherwise.  \(^1\)

This paper is respectfully dedicated to Jacques Tits.

Introduction

Recall that a group $\Gamma$ is called linear, or linear over a ring in case of ambiguity, if there is an embedding $\Gamma \subset GL(n, R)$ for some integer $n$ and some commutative ring $R$. Moreover $\Gamma$ is called linear over a field if it can be embedded into $GL(n, K)$ for some integer $n$ and some field $K$.

Let $\text{Aut} K^2$ be the group of polynomial automorphisms of the affine plane $K^2$. In this paper, we will investigate the linearity or nonlinearity properties

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for the subgroups of $\text{Aut} K^2$. In particular, we will consider the following subgroups
\[
\text{Aut}_0 K^2 = \{ \phi \in \text{Aut} K^2 | \phi(0) = 0 \}, \quad \text{and} \quad \text{Aut}_1 K^2 = \{ \phi \in \text{Aut}_0 K^2 | d\phi|_0 = \text{id} \}.
\]
The first result of the present paper is

**Theorem A.** *(A.1)* If $K$ is infinite, the group $\text{Aut}_0 K^2$ is not linear, even over a ring.

*(A.2)* Moreover if $\text{ch } K = 0$ the group $\text{Aut}_0 K^2$ contains finitely generated subgroups which are not linear, even over a ring.

It was known that the much larger Cremona group $\text{Cr}_2(\mathbb{Q})$ is not linear over a field, see [4] [11]. More recently, the nonlinearity over a field of the whole group $\text{Aut} \mathbb{Q}^2$ was proved in [5]. In the same paper, Y. Cornulier raises the question (answered by Theorem A) of finding a nonlinear finitely generated (FG in the sequel) subgroup in $\text{Aut} \mathbb{Q}^2$.

Various authors show that automorphism groups share some properties with linear groups, see e.g. [13][1]. More specifically, it was proved in [8] that $\text{Aut} K^2$ satisfies Tits alternative. Theorem A shows that these results are not a consequence of classical results for linear groups.

Roughly speaking, Theorem A.2 means that $\text{Aut}_0 K^2$ contains ”small” subgroups (indeed FG subgroups) which are not linear, at least if $\text{ch } K = 0$.

On the opposite there is

**Theorem B.** For any field $K$, the group $\text{Aut}_1 K^2$ is linear over the field $K(t)$.

Indeed a stronger result is proved: under a mild hypothesis over $K$, the group $\text{Aut}_1 K^2$ is linear over the field $K$ itself, see Corollary D in Section 9.

Following Demazure’s approach [6], the functor $K \mapsto \text{Aut} K^2/\text{Aut}_1 K^2$ is represented by a 6-dimensional variety, namely the affine group of $K^2$. Therefore Theorem B states that $\text{Aut} K^2$ contains some ”big” subgroups (of finite codimension) which are linear over a field.

Assume again that $K$ is infinite. In view of Theorems A and B we can ask which groups between $\text{Aut}_1 K^2$ and $\text{Aut}_0 K^2$ are linear. These groups are defined as
\[
\text{Aut}_S K^2 := \{ \phi \in \text{Aut}_0 K^2 | d\phi|_0 \in S \},
\]
where $S$ is a subgroup of $\text{GL}(K^2)$.

An element $g \in \text{GL}(K^2)$ is called $K$-reducible if its eigenvalues lies in $K$.

The next result provides a partial answer to the previous question
Theorem C. Let $S$ be a subgroup of $GL(K^2)$. Assume that $S$ satisfies one of the following assertions

(i) Any $K$-reducible $g \in S$ is unipotent, or
(ii) the group $S$ is FG, and any $K$-reducible $g \in S$ is quasi-unipotent.

Then the group $Aut_S K^2$ is linear over $K(t)$.

For example, $\pm 1$ are the only two reducible elements of $SO(2, \mathbb{R})$. Therefore $Aut_S \mathbb{R}^2$ is linear over $\mathbb{R}(t)$ for any FG subgroup $S$ of $SO(2, \mathbb{R})$. We believe that the whole group $Aut_{SO(2,\mathbb{R})} \mathbb{R}^2$ is linear and more generally

Question. Let $S$ be a subgroup of $GL(K^2)$. Is the group $Aut_S K^2$ linear if and only if there is an integer $N$ such that $g^N$ is unipotent for any $K$-reducible $g \in S$?

As a consequence of Theorem B we have

Corollary D. (D.1) Let $K$ be a finite field. Then $Aut K^2$ is linear over $K(t)$.

(D.2) Let $p$ be a prime number. Then any FG subgroup of $Aut \mathbb{F}_p$ is linear over $\mathbb{F}_p(t)$.

Note that Corollary D.1 treats the case not covered by Theorem A.1. Moreover, the question of the existence of nonlinear FG subgroups of $Aut K^2$ is solved by Theorem A.2 and Corollary D.2, except for finite characteristic fields of transcendence degree $\geq 1$, therefore we ask the following

Question. Let $K$ be a field containing $\mathbb{F}_p(t)$. Does $Aut K^2$ contain nonlinear FG subgroups?

The paper is organized as follows. In section 1, the general notions concerning $Aut K^2$ are defined, and the classical result of van der Kulk [18] is stated. In the next section, it is proved that some linear groups obtained by amalgamation are indeed linear over a field. Therefore, it is possible to use the theory of algebraic groups to show that some groups are not linear. This is used to prove Theorem A.2 in section 3.

Borel and Tits [3] showed that certain group morphisms of algebraic groups are semi-algebraic. A very simple form of their ideas are used to prove Theorem A.1, see Sections 4 and 5.

Next the proof of Theorem B and C is based on some Ping-Pong ideas. These ideas were originally invented for the dynamic of groups with respect to the euclidean metric topologies [7], but they were used by Tits in the context of the ultrametric topologies [14].
Aknowledgements J.P. Furter and R. Boutonnet informed us that they independently found a FG subgroup of Aut $\mathbb{Q}^2$ which is not linear over a field $[2]$. We also heartily thank S. Lamy for interesting comments and E. Zelmanov for an inspiring talk.

1 The van der Kulk Theorem

In this section, we review the basic facts about the amalgamated products of groups. Then the we recall the classical van der Kulk Theorem.

1.1 Amalgamated products

Let $A$, $G_1$ and $G_2$ be groups. Their free product is denoted by $G_1 \ast G_2$, see [9] ch.4. Now let $A \overset{f_1}{\longrightarrow} G_1$ and $A \overset{f_2}{\longrightarrow} G_2$ be two injective group morphisms and let $K \subset G_1 \ast G_2$ be the invariant subgroup generated by the elements $f_1(a)f_2(a^{-1})$ when $a$ runs over $A$. By definition the group $G_1 \ast A G_2$ is called the amalgamated product of $G$ and $H$ over $A$, and it is denoted by $G_1 \ast_A G_2$, see e.g. [12], ch. I.

In the literature, the amalgamated products are also called free amalgamated products, see [9] ch.8.

Recall that the natural maps $G_1 \rightarrow \Gamma$ and $G_2 \rightarrow \Gamma$ are injective, see the remark after the Theorem 1 of ch. 1 in [12]. Hence, we will use a less formal terminology. The group $A$ will be viewed as a common subgroup of $G_1$ and $G_2$, and $G_1$ and $G_2$ will be viewed as subgroups of $G_1 \ast_A G_2$.

1.2 Reduced words

The usual definition [12] of reduced words is based on the right $A$-cosets. In order to avoid a confusion between the set difference notation $A \setminus X$ and the $A$-orbits notation $A \setminus X$, we will use a definition based on the left $A$-cosets.

Let $G_1$, $G_2$ be two groups sharing a common subgroup $A$, and let $\Gamma = G_1 \ast_A G_2$. Set $G_i^* = G_i \setminus A$, $G_2^* = G_2 \setminus A$ and let $T_i^* \subset G_i^*$ (respectively $T_2^* \subset G_2^*$) be a set of representatives of $G_i^* / A$ (respectively of $G_2^* / A$). Let $S$ be the set of finite alternative sequences of one's and two's, and let $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in S$.

A reduced word is a word $(x_1, \ldots x_n, x_0)$ where $x_0$ is in $A$, and $x_i \in G_i^*$ for $i \geq 1$, where the sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ lies in $S$. Moreover this sequence $\epsilon$ is called the type of the reduced word $(x_1, \ldots x_n, x_0)$. Let $R$ be the set of all reduced words. The next lemma is well-known, see e.g. [12], Theorem 1.
Lemma 1. The map
\[(x_1, \ldots, x_n, x_0) \in \mathcal{R} \mapsto x_1 \ldots x_n x_0 \in G_1 *_A G_2\]
is bijective.

Set \(\Gamma = G_1 *_A G_2\). For \(\gamma \in \Gamma \setminus A\) there is some \(n \geq 1\), some \(\epsilon \in S\) and some \(g_i \in G_{\epsilon_i}^*\), such that \(\gamma = g_1 \ldots g_n\). It follows from the remark of [12] (after Theorem 1, ch. 1) that \(\gamma = x_1 \ldots x_n x_0\) for some reduced word \((x_1 \ldots x_n x_0)\) of type \(\epsilon\). Since \(\epsilon\) and \(n\) are determined by \(\gamma\), the sequence \(\epsilon\) is called the type of \(\gamma\) and the integer \(l(\gamma) := n\) is called its length.

1.3 Amalgamated product of subgroups
Let \(G_1, G_2\) be two groups sharing a common subgroup \(A\). Let \(G_1' \subset G_1\) and \(G_2' \subset G_2\) be subgroups, with the property that \(G_1' \cap A = G_2' \cap A\). Set \(A' := G_1' \cap A = G_2' \cap A\).

Lemma 2. With the previous hypotheses, the natural map
\[G_1' *_{A'} G_2' \hookrightarrow G_1 *_A G_2\]
is injective. Moreover if the the natural maps \(G_1'/A' \rightarrow G/A\) and \(G_2'/A' \rightarrow G_2/A\) are bijective, then we have
\[G_1 *_A G_2/G_1' *_{A'} G_2' \simeq A/A'\].

Proof. For \(i = 1, 2\) set \(G_i^* = G_i \setminus A, G_i'^* = G_i' \setminus A'\). Let \(T_i^* \subset G_i^*\) (respectively \(T_i'^* \subset G_i'^*\)) be a set of representatives of \(G_i^*/A\) (respectively of \(G_i'^*/A'\)).

Since the maps \(G_i'/A' \rightarrow G_i/A\) are injective, it can be assumed that \(T_i'^* \subset T_i^*\). Let \(\mathcal{R}\) (respectively \(\mathcal{R}'\)) be the set of reduced words of \(G_1 *_A G_2\) (respectively of \(G_1' *_{A'} G_2'\)). By definition, we have \(\mathcal{R}' \subset \mathcal{R}\), thus by lemma 2 the map \(G_1' *_{A'} G_2' \hookrightarrow G_1 *_A G_2\) is injective.

Moreover, assume that the maps \(G_1'/A' \rightarrow G/A\) and \(G_2'/A' \rightarrow G_2/A\) are bijective. It follows that \(\mathcal{R}'\) is the set of all reduced words \((x_1, \ldots, x_n, x_0) \in \mathcal{R}\) such that \(x_0 \in A'\). It follows easily that
\[G_1 *_A G_2/G_1' *_{A'} G_2' \simeq A/A'\].

\[\square\]

1.4 The group Aut \(K^2\)
Let \(K\) be a field. Recall that Aut \(K^2\) is the group of polynomial automorphisms of \(K^2\), so an element of Aut \(K^2\) is a polynomial map \(\phi : (x, y) \mapsto (f(x, y), g(x, y))\), where \(f, g \in K[x, y]\) which admits a polynomial inverse. When \(\text{ch } K = 0\), for the existence of a polynomial inverse it is enough that the naive map \(\phi : K^2 \rightarrow K^2\) is bijective. When \(\text{ch } K \neq 0\), this is not enough.
Even when $K$ is infinite, the existence of a polynomial inverse requires some additional separability condition.

1.5 The subgroup $\text{Elem}(K^2)$ of elementary automorphism of $K^2$

By definition, an elementary automorphism of $K^2$ is an automorphism $\phi \in \text{Aut } K^2$ of the form
\[
\phi : (x, y) \mapsto (z_1 x + t, z_2 y + f(x))
\]
for some $z_1, z_2 \in K^*$, some $t \in K$ and some $f \in K[x]$. The group of elementary automorphism is denoted $\text{Elem}(K^2)$. Also set
\[
\text{Elem}_0(K^2) = \text{Elem}(K^2) \cap \text{Aut}_0 K^2, \quad \text{and}
\]
\[
\text{Elem}_1(K^2) = \text{Elem}(K^2) \cap \text{Aut}_1 K^2.
\]
For any $F \in K[t]$, let $\mu(F)$ be the automorphism
\[
\mu(F) : (x, y) \mapsto (x, y + x^2 F(x)).
\]
Clearly $\mu : K[t] \to \text{Elem}_1(K^2)$ is group isomorphism.

1.6 The affine group $\text{Aff}(K^2)$

Let $\text{Aff}(K^2) \subset \text{Aut } K^2$ (respectively $\text{GL}(K^2) \subset \text{Aut } K^2$) be the subgroup of affine automorphisms (respectively linear automorphisms) of $K^2$. Set $B = \text{Aff}(K^2) \cap \text{Elem}(K^2)$. Equivalently, $B$ is the group of affine transformations of the form
\[
(x, y) \in K^2 \mapsto (z_1 x + u, z_2 y + c x + v)
\]
for some $z_1, z_2 \in K^*$ and $c, u, v \in K$. Set $B_0 = B \cap \text{GL}(K^2)$ Indeed $B$ and $B_0$ are Borel subgroups of $\text{Aff}(K^2)$ and $\text{GL}(K^2)$. Moreover we have
\[
\text{Aff}(K^2)/B = \text{GL}(K^2)/B_0 \simeq \mathbb{P}^1_K.
\]

1.7 The van der Kulk Theorem

Recall the classical

van der Kulk Theorem. [18] We have
\[
\text{Aut } K^2 \simeq \text{Aff}(K^2) *_B \text{Elem}(K^2), \quad \text{and}
\]
\[
\text{Aut}_0 K^2 \simeq \text{GL}(K^2) *_{B_0} \text{Elem}_0(K^2).
\]

2 Amalgamated Products and Linearity.

In this section, it is shown that, under a mild assumption, an amalgamated product $G_1 *_A G_2$ which is linear over a ring is also linear over a field, see Lemma 6.

2.1 Linearity Properties
Let $R$ be a ring. For any $n \geq n$, let $GL(n, R)$ be the group of invertible $n \times n$ matrices with entries in $R$. Indeed for a field $K$, the notation $GL(K^2)$ denotes the subgroup of Aut $K^2$ of linear automorphisms of $K^2$ although $GL(2, K)$ is the abstract linear group of degree 2.

A group $\Gamma$ is called linear over $R$ if there is a group embedding $\Gamma \subset GL(n, R)$ for some integer $n$. The group $\Gamma$ is called linear (or linear over a ring) if it is linear over some ring $R$. Similarly $\Gamma$ is called linear over a field if it is linear over some field $K$.

For a group $\Gamma$, the strongest form of linearity is the linearity over a field. On the opposite, there are also groups $\Gamma$ which contain a FG subgroup which is not linear, even over a ring: these groups are nonlinear in the strongest sense.

### 2.2 Minimal embeddings

Let $R$ be a commutative ring, let $n \geq 1$ be an integer and let $\Gamma$ be a subgroup of $GL(n, R)$. For any ideal $J$ of $R$, let $GL(n, J)$ be the kernel of the natural map $GL(n, R) \to GL(n, R/J)$. Any element $g \in GL(n, J)$ can be written as $g = \text{id} + A$, where $A$ is a $n$-by-$n$ matrix with entries in $J$.

The embedding $\Gamma \subset GL(n, R)$ is called minimal if for any ideal $J \neq 0$ we have $\Gamma \cap GL(n, J) \neq \{1\}$.

**Lemma 3.** Let $\Gamma \subset GL(n, R)$. There exist an ideal $J$ with $\Gamma \cap GL(n, J) = \{1\}$ such that the induced embedding $\Gamma \to GL(n, R/J)$ is minimal.

**Proof.** Since $R$ is not necessarily noetherian, the proof requires Zorn Lemma.

Let $S$ be the set of all ideals $J$ of $R$ such that $\Gamma \cap GL(n, J) = \{1\}$. The set $S$ is partially ordered with respect to the inclusion. For any chain $C \subset S$, the ideal $\cup_{I \in C} I$ belongs to $S$. Therefore Zorn Lemma implies that $S$ contains a maximal element $J$. It follows that the induced embedding $\Gamma \to GL(n, R/J)$ is minimal. \[\square\]

### 2.3 Groups with trivial commutants

By definition, a group $\Gamma$ has trivial commutants if the commutant of any nontrivial normal subgroup $K$ of $\Gamma$ is trivial. Equivalently, if $K_1$ and $K_2$ are commuting invariant subgroups of $\Gamma$, then one of them is trivial.

**Lemma 4.** Let $\Gamma$ be a group with trivial commutants.

If $\Gamma$ is linear over a ring, then $\Gamma$ is also linear over a field.
Proof. By hypothesis we have $\Gamma \subset GL(n,R)$ for some commutative ring $R$. By lemma 3, it can be assumed that the embedding $\Gamma \rightarrow GL(n,R)$ is minimal.

Let $I_1, I_2$ be ideals of $R$ with $I_1.I_2 = 0$. Let $A_1$ (respectively $A_2$) be an arbitrary $n$-by-$n$ matrix with entries in $I_1$ (respectively in $I_2$) and set $g_1 = 1 + A_1$, $g_2 = 1 + A_2$. Since we have $A_1.A_2 = A_2.A_1 = 0$, we have $g_1.g_2 = g_2.g_1$, therefore $GL(n,I_1)$ and $GL(n,I_2)$ are commuting invariant subgroups of $GL(n,R)$.

Since $K_1 = \Gamma \cap GL(n,I_1)$ and $K_2 = \Gamma \cap GL(n,I_2)$ are commuting invariant subgroups of $\Gamma$, one of them is trivial. By minimality hypothesis, $I_1$ or $I_2$ is trivial. Thus $R$ is prime.

It follows that $\Gamma \subset GL(n,K)$, where $K$ is the fraction field of $R$. 

\[ \square \]

2.4 The hypothesis $H$

Let $G_1, G_2$ be two groups sharing a common subgroup $A$ and set $\Gamma = G_1 * A G_2$.

Let $S$ be the set of all finite sequences $i = i_1, \ldots, i_n$ of alternating 1 and 2. For $i, j \in \{1, 2\}$, let $S_{i,j}$ be the subset of all $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in S$ starting with $i$ and ending with $j$. Let $\Gamma_{i,j}$ be the set of all $\gamma \in \Gamma$ of type $\epsilon$ for some $\epsilon \in S_{i,j}$. Therefore we have

$$\Gamma = A \sqcup \Gamma_{1,1} \sqcup \Gamma_{2,2} \sqcup \Gamma_{1,2} \sqcup \Gamma_{2,1}.$$

By definition, the amalgamated product $G_1 * A G_2$ is called trivial if $G_1 = A$, or $G_2 = A$ or if both $G_1$ and $G_2$ are isomorphic to $\mathbb{Z}/2\mathbb{Z}$. The latter case is the uninteresting dihedral group $\mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$.

Let consider the following hypothesis

$(H)$ For any $a \in A$ with $a \neq 1$, there is $\gamma \in \Gamma$ such that $a^\gamma \notin A$ where, as usual, $a^\gamma := \gamma a \gamma^{-1}$.

Lemma 5. Let $\Gamma = G_1 * A G_2$ be a nontrivial amalgamated product satisfying the hypothesis $H$. Let $g \in \Gamma$ with $g \neq 1$.

There are $\gamma_1, \gamma_2 \in \Gamma$ such that $g^{\gamma_1} \in \Gamma_{1,1}$ and $g^{\gamma_2} \in \Gamma_{2,2}$.

In particular $\Gamma$ has trivial commutants.

Proof. First it should be noted that $A$ cannot be simultaneously a subgroup of index 2 in $G_1$ and in $G_2$. Otherwise, $A$ would be an invariant subgroup of $G_1$ and $G_2$, the hypothesis $H$ would imply that $A = \{1\}$ and $\Gamma$ would be the
A nonlinear FG subgroup of $\text{Aut}_0\mathbb{Q}^2$.

First we define a certain FG group $\Gamma = G_1 \ast_A G_2$ and we will show that $\Gamma$ is nonlinear, even over a ring. Then we see that $\Gamma$ is a subgroup of $\text{Aut}_0\mathbb{Q}^2$ and therefore $\text{Aut}_0K^2$ contains many nonlinear FG subgroups for any characteristic zero field $K$, what proves Theorem A.2.
To show that $\text{Aut} \mathbb{Q}^2$ is not linear over a field, Cornulier uses a nonFG subgroup $G_{Cor} \subset \text{Aut} \mathbb{Q}^2$ which is locally nilpotent but not nilpotent and therefore not linear over a field [4]. Nevertheless, the group $G_{Cor}$ is linear over a ring, as it is shown at the end of the section.

3.1 Quasi-unipotent endomorphisms

Let $V$ be a finite dimensional vector space over an algebraically closed field $K$. An element $u \in GL(V)$ is called quasi-idempotent if all its eigenvalues are roots of one. The quasi-order of a quasi-idempotent endomorphism $u$ is the smallest positive integer $n$ such that $u^n$ is unipotent.

Lemma 7. Let $u \in GL(V)$. If $u$ and $u^2$ are conjugate, then $u$ is quasi-unipotent and its quasi-order is odd. If moreover $u$ has infinite order, then we have $\text{ch} K = 0$.

Proof. Let $\text{Spec} u$ be the spectrum of $u$. By hypothesis the square map $\text{Spec} u \rightarrow \text{Spec} u, \lambda \mapsto \lambda^2$ is bijective, hence for any $\lambda \in \text{Spec} u$, we have $\lambda^{2n} = \lambda$ for some integer $n$. It follows that all eigenvalues are odd roots of unity, what proves that $u$ is quasi-unipotent of odd quasi-order.

Over any field of finite characteristic, the unipotent endomorphisms have finite order. Hence we have $\text{ch} K = 0$ if $u$ has infinite order.

3.2 The Group $\Gamma = G_1 \ast_A G_2$

Set $G_1 = \mathbb{Z}^2$, and let $\sigma, \sigma'$ be a basis $\mathbb{Z}^2$. Set $\mathbb{Z}(2) = \{ x \in \mathbb{Q} | 2^n x \in \mathbb{Z} \text{ for } n \gg 0 \}$. Indeed $\mathbb{Z}(2)$ is the localization of the ring $\mathbb{Z}$ at 2, but, in what follows, we will only consider its group structure. The element 1 in $\mathbb{Z}(2)$ will be denoted by $\tau$ and the addition in $\mathbb{Z}(2)$ will be denoted multiplicatively.

Let $\sigma$ act on $\mathbb{Z}(2)$ by multiplication by 2, so we can consider the semi-direct product $G_2 = \mathbb{Z} \rtimes \mathbb{Z}(2)$. Let $\Gamma = G_1 \ast_A G_2$, where $A = \mathbb{Z} \sigma$. It is easy to show that $\Gamma$ is generated by $\sigma, \sigma'$ and $\tau$, and it is defined by the following two relations

$$\sigma \sigma' = \sigma' \sigma, \text{ and } \sigma \tau \sigma^{-1} = \tau^2.$$  

Lemma 8. The group $\Gamma$ has trivial commutants.

Proof. First it is proved that the amalgamated product $\Gamma = G_1 \ast_A G_2$ satisfy hypothesis $\mathcal{H}$.

Indeed it is enough to show that, for any nontrivial element $a \in A$, its conjugate $\tau^{-1} a \tau$ is not in $A$. We have $a = \sigma^n$ for some $n \neq 0$, and it can be assumed that $n \geq 1$. 

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We have $\sigma^n\tau\sigma^{-n} = \tau^{2n}$ and therefore $\tau^{-1}a\tau = \tau^{2n-1}a$ what proves that $\tau^{-1}a\tau$ is not in $A$.

Hence the amalgamated product $\Gamma = G_1 \ast_A G_2$ satisfy hypothesis $H$. Therefore by Lemma 6, $\Gamma$ has trivial commutants.

3.3 Nonlinearity of $\Gamma$

**Lemma 9.** The group $\Gamma$ is not linear, even over a ring.

**Proof.** Assume that $\Gamma$ is linear over a ring. By Lemma 8 the group $\Gamma$ has trivial commutants. Thus, by Lemma 4, $\Gamma$ is linear over a field.

Let $\rho : \Gamma \rightarrow GL(V)$ be an embedding, where $V$ is a finite dimensional vector space over a field $K$. Since $\sigma\tau\sigma^{-1} = \tau^2$ it follows from Lemma 7 that $\rho(\tau)$ is quasi-unipotent and its quasi-order $n$ is odd.

Let $\Gamma'$ be the subgroup of $\Gamma$ generated by $\sigma, \sigma'$ and $\tau^n$. Since the morphism $\psi : \Gamma \rightarrow \Gamma'$ defined by $\psi(\sigma) = \sigma, \psi(\sigma') = \sigma'$ and $\psi(\tau) = \tau^n$ is an isomorphism, it can be can assumed that $\rho(\tau)$ is unipotent.

Since $\tau$ has infinite order, $K$ has characteristic zero by Lemma 7. Set $u = \rho(\tau)$, $h = \rho(\sigma)$ and

$$e = \log u = \log 1 - (1 - u) = \sum_{n\geq 1} (1 - u)^n/n,$$

which is defined since $1 - u$ is nilpotent. Since $huh^{-1} = u^2$, we have

$$huh^{-1} = 2e.$$

It can be assumed that $K$ is algebraically closed. Let $V = \oplus_{\lambda \in K} V_{(\lambda)}$ be the decomposition of $V$ into generalized eigenspace relative to $h$. Also, for $\lambda \in K$, set $V_{(\lambda)}^+ = \oplus_{n\geq 0} V_{(2^n\lambda)}$.

Let $\lambda \in K$. Since $\rho(G_1)$ commutes with $h$ we have

$$\rho(G_1)V_{(\lambda)} \subset V_{(\lambda)}.$$

Moreover we have $e.V_{(\lambda)} \subset V_{(2\lambda)}$ and therefore

$$\rho(G_2)V_{(\lambda)}^+ \subset V_{(\lambda)}^+.$$

It follows that $V_{(\lambda)}^+$ is a $\Gamma$-module. Since $\tau$ acts trivially on $V_{(\lambda)}^+/V_{(2\lambda)}^+$, it follows that the image of $\Gamma$ in $GL(V_{(\lambda)}^+/V_{(2\lambda)}^+)$ is commutative. It follows that $V$ has a composition series by one-dimensional $\Gamma$-module, hence $\rho(\Gamma)$ is solvable.

Since it is solvable, $\Gamma$ contains a nontrivial invariant abelian subgroup. This fact contradicts Lemma 8, which states that $\Gamma$ has trivial commutants.

3.4 Proof of Theorem A.2
Theorem A.2. Let $K$ be a field of characteristic zero. Then $\text{Aut}_0 K^2$ contains FG subgroups which are not linear even over a ring.

Proof. Let define three automorphisms $S, S'$ and $T$ of $K^2$ as follows. First $S$ and $S'$ are linear automorphisms where $S = 1/2 \text{id}$ and $S'$ is defined by the matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Next $T$ is the quadratic automorphism $(x,y) \mapsto (x,y + x^2)$.

Set $K_1 = \langle S, S' \rangle$, $K_2 = \langle S, T \rangle$, $C = \langle S \rangle$.

The eigenvalues of $S'$ are $\frac{1 \pm \sqrt{5}}{2}$. Hence $n \neq 0$ $S'^n$ is not upper diagonal. It follows that $K_1 \cap B_0 = C$. Moreover it is clear that $K_2 \cap B_0 = C$. It follows from Lemma 2 that $K_1 * C K_2$ is a subgroup of $\text{GL}(K^2) * B_0 \text{Elem}_0(K^2)$.

Since $STS^{-1} = T^2$ is is clear that the group morphism $\Gamma : K_1 * C K_2, \sigma \mapsto S, \sigma' \mapsto S', \tau \mapsto T$ is an isomorphism. Therefore $\Gamma$ is a subgroup of $\text{Aut}_0 K^2$.

Hence, by Lemma 9 $\text{Aut}_0 K^2$ contains a FG subgroup which is not linear, even over a ring.

3.5 About Counulier’s subgroup.

The group $G_{Cor}$ considered in [4] is the subgroup of all automorphisms of $\mathbb{Q}^2$ of the form $$(x,y) \mapsto (x + u, y + f(x)).$$

This group, which is not FG, has been used in [4] to prove

**Cornulier Theorem.** The group $G_{Cor}$ is not linear over a field. Consequently, the group $\text{Aut} \mathbb{Q}^2$ is not linear over a field.

Indeed, $G_{Cor}$ is an example of a group which is linear over a ring, but not linear over a field. Set $R = \mathbb{Q}[[x]] \oplus \mathbb{Q}((x))/\mathbb{Q}[[x]]$, where $\mathbb{Q}((x))/\mathbb{Q}[[x]]$ is a square-zero ideal.

**Proposition.** The group $G_{Cor}$ is linear over the ring $R$.

Proof. Let $T \simeq \mathbb{Q}$ be the group whose elements are denoted $\tau^\alpha$ for $\alpha \in \mathbb{Q}$ and the product is given $\tau^\alpha \tau^\beta = \tau^\alpha + \beta$. As a group, $G_{Cor}$ is isomorphic to $T \ltimes \mathbb{Q}[t]$, where the action of $T$ on $\mathbb{Q}[t]$ is defined by $\tau^\alpha f(t) \tau^{-\alpha} = f(t + \alpha)$ for any $\alpha \in \mathbb{Q}$ and $f \in \mathbb{Q}[t]$.

Recall that any $f(t)$ can be written as a finite sum $f(t) = \sum_{n \geq 0} a_n (t^n)$. There are embeddings $\rho_1 : \mathbb{Q}[t] \to \text{GL}(2, \mathbb{R})$ and $\rho_2 : T \to \text{GL}(2, \mathbb{R})$ defined by

\[
\rho_1(a_n (t^n)) = \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix}, \quad \rho_2(\tau^\alpha) = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}.
\]
\[ \rho_1(\sum_{n \geq 0} a_n (t^n)) = \left( \frac{1}{0} \sum_n a_n [x^{-n-1}] \right) \text{ and } \rho_2(\tau^\alpha) = \left( \frac{1}{0} (1 + x)^\alpha \right), \]

where, for \( n < 0 \) the symbol \([x^{-n}]\) is the element \( x^{-n} \mod \mathbb{Q}[[x]]\) in the ideal \( \mathbb{Q}(x)/\mathbb{Q}[[x]]\) of \( R \), and the fractional power \((1 + x)^\alpha\) is the formal series \( \sum_{k \geq 0} (\binom{\alpha}{k}) x^k \).

We claim that
\[ \rho_2(\tau^\alpha) \rho_1(f(t)) \rho_2(\tau^{-\alpha}) = \rho_1(f(t + \alpha)). \]
Indeed for \( \alpha = 1 \), this formula follows from the fact that \( \binom{t+1}{n} - \binom{t}{n} = \binom{t}{n-1} \). It follows that the formula holds for any integer \( \alpha \geq 1 \). Since both sides of the identity are polynomials in \( \alpha \), the formula holds for any \( \alpha \in \mathbb{Q} \).

It follows that \( \rho_1 \) and \( \rho_2 \) can be combined into an embedding \( \rho : G_{Cor} \to GL(2, R) \). \( \square \)

### 4 Semi-algebraic characters.

Let \( K \) be an infinite field of characteristic \( p \).

**Lemma 10.** Let \( n \geq 1 \) be an integer prime to \( p \). Then for any \( x \in K \), there is an integer \( m \) and a collection \( x_1, x_2, \ldots x_m \) of elements of \( K^* \) such that
\[ x = \sum_{1 \leq k \leq m} x_i^n. \]

**Proof.** Let \( K' \) be the additive span of the set \( \{ y^n | y \in K \} \). Since \(-y^n = (p - 1)y^n\), \( K' \) is an additive subgroup of \( K \). Clearly, \( \mathbb{K}' \) is a subring of \( K \). Let \( x \in K' \) with \( x \neq 0 \). It follows from the formula \( x^{-1} = (1/x)^n x^{n-1} \) that \( K' \) is a subfield.

We claim that \( K' \) is infinite. Assume otherwise. Then \( K' = \mathbb{F}_q \) for some power \( q \) of \( p \). Since any element of \( K \) is algebraic over \( K' \) of degree \( \leq n \), it follows that \( K \supseteq \cup_{k \leq n} \mathbb{F}_q^k \), a fact that contradicts that \( K \) is infinite.

It remains to prove that \( K' = K \). For \( x \in K \), let \( P(t) \in K[t] \) be the polynomial \( P(t) = (x + t)^n \). Let \( y_0, \ldots, y_n \) be \( n + 1 \) distinct elements in \( K' \). We have \( P(y_i) \in K' \) for any \( 0 \leq i \leq n \). Using Lagrange’s interpolation polynomials, there exist a polynomial \( Q(t) \in K'[t] \) of degree \( \leq n \) such that \( P(y_i) = Q(y_i) \) for any \( 0 \leq i \leq n \). Since \( P - Q \) has at least \( n + 1 \) roots, we have \( P = Q \) and therefore \( P(t) \in K'[t] \). Since \( P(t) = t^n + nx^n + \ldots \) it follows that \( nx \) belongs to \( K' \), and therefore \( x \in K' \). Hence \( K' = K \). \( \square \)
Lemma 11. Let $K$ be an infinite field of characteristic $p$, let $\mu : K \to K$ be a field automorphism and let $n, m$ be positive integers which are prime to $p$. Assume that

$$x^n = \mu(x)^m, \text{ for all } x \in K^*.$$  

Then we have $m = n$.

Remark: More precisely, the hypotheses of the lemma imply also that $\mu = \text{id}$, but this is not required in what follows.

Proof. Step 1: proof for $K = \mathbb{F}_p(t)$. There are $a, b, c$ and $d \in K$ with $ad - bc \neq 0$ such that $\mu(t) = \frac{at + b}{ct + d}$. The identity

$$t^n = (\frac{at + b}{ct + d})^m$$

clearly implies that $n = m$.

Step 2: proof for $K \subset \mathbb{F}_p$. Since $K$ is infinite, $K$ contains arbitrarily big finite fields. So we have $K \supset \mathbb{F}_{p^N}$ for some positive integer $N$ with $nm < p^N$. Since $K$ contains a unique field of cardinality $p^N$ we have $\mu(\mathbb{F}_{p^N}) = \mathbb{F}_{p^N}$. There is an non negative integer $a < N$ such that $\mu(x) = x^{p^a}$ for any $x \in \mathbb{F}_{p^N}$. Therefore we have

$$x^n = x^{mp^a} \text{ for all } x \in F^*_{p^N}.$$  

It follows that $n \equiv mp^a \mod p^N - 1$. Let $n = \sum_{k \geq 0} n_k p^k$ and $m = \sum_{k \geq 0} m_k p^k$ be the $p$-adic expansions of $n$ and $m$. By definition each digit $n_k, m_k$ is an integer between 0 and $p - 1$. Since $n.m < p^N$, we have $m_k = k = 0$ for $k \geq N$.

For each integer $k$ let $[k]$ be its residue modulo $N$, so we have $0 \leq [k] < N$ by definition. We have

$$n \equiv mp^a \equiv \sum_{0 \leq k < N} m_k p^{[a+k]} \mod p^N - 1.$$  

Since both integers $n$ and $\sum_{0 \leq k < N} m_k p^{[a+k]}$ belong to $[1, p^N - 1]$ and are congruent modulo $p^N - 1$, it follows that

$$n = \sum_{0 \leq k < N} m_k p^{[a+k]}.$$  

Assume $a > 0$. We have $n_a = m_0$ and $n_0 = m_{N-a}$. Since $n$ and $m$ are prime to $p$, the digits $n_0$ and $m_0$ are not zero, therefore we have $n_a \neq 0$ and $m_{N-a} \neq 0$. It follows that

$$n \geq p^a \text{ and } m \geq p^{N-a},$$  

which contradicts that $nm < p^N$.

Therefore, we have $a = 0$, and the equality $n = m$ is obvious.

Step 3: proof for any infinite $K$. In view of the previous step, it can be assumed that $K$ contains a transcendental element $t$. Therefore, $K$ contains the subfield $L := \mathbb{F}_p(t)$.
By Lemma 10, $L$ and $\mu(L)$ are the additive span of the set \( \{ x^n | x \in K^* \} = \{ \mu(x)^m | x \in K^* \} \). Therefore, we have $\mu(L) = L$. Since $\mu$ induces a field automorphism of $L$, the equality $n = m$ follows from the first step.

Let $L$ be another field. A group morphism $\chi : K^* \to L^*$ is called a character of $K^*$. Let $X(K^*)$ be the set of all $L$-valued characters of $K^*$. Let $n$ be an integer prime to $p$. A character $\chi \in X(K^*)$ is called semi-algebraic of degree $n$ if
\[
\chi(x) = \mu(x)^n, \text{ for any } x \in K^*,
\]
for some field embedding $\mu : K \to L$.

Let $X_n(K^*)$ be the set of all semi-algebraic characters of $K^*$ of degree $n$. Of course it can be assumed that $L$ has characteristic $p$, otherwise the set $X_n(K^*)$ is empty.

The next lemma shows that for an integer $n > 0$ prime to $p$, the degree is uniquely determined by the character $\chi$. Indeed this statement is also true for the negative integers $n$ and moreover the field embedding $\mu$ is also determined by $\chi$. For simplicity of the exposition, the lemma is stated and proved in its minimal form.

However, it should be noted that the condition that $n$ is prime to $p$ is essential. Indeed, if $\mu : K \to L$ is a field embedding into a perfect field $L$, then $\mu(x)^n = \mu'(x)^m$, where $\mu'$ is the field embedding defined by $\mu'(x) = \mu(x)^{1/p}$.

**Lemma 12.** Let $n \neq m$ be positive integers which are prime to $p$. Then $X_n(K^*) \cap X_m(K^*) = \emptyset$.

**Proof.** Let $n, m$ be positive integers prime to $p$ and let $\chi \in X_n(K^*) \cap X_m(K^*)$. By definition, there are fields embeddings $\nu, \nu' : K \to L$ such that
\[
\chi(x) = \nu(x)^n = \nu'(x)^m \text{ for any } x \in K^*.
\]

By Lemma 10, $\nu(K)$ and $\nu'(K)$ are the linear span of $\text{Im } \chi$. Therefore we have $\nu(K) = \nu'(K)$ and $\mu := \nu'^{-1} \circ \nu$ is a well defined field automorphism of $K$. We have
\[
\mu(x)^n = \nu'^{-1} \circ \nu(x^n) = \nu'^{-1} \circ \nu'(x^m) = x^m.
\]
Therefore by Lemma 11, we have $n = m$. 

\[\square\]
5 Nonlinearity of $\text{Aut}_0 K^2$ for $K$ infinite of characteristic $p$.

This section provides the proof of Theorem A.1. Since Theorem A.2 has been proved in Section 3, it can be assumed that $K$ is a field of characteristic $p$. Although our setting is different from [3], this section follows the same idea, namely that some abstract morphisms of algebraic groups are, somehow, semi-algebraic.

Recall that an elementary abelian $p$-group is simply a $\mathbb{F}_p$-vector space $E$ viewed as a group. Its $\mathbb{F}_p$-dimension is called the rank of $E$.

**Lemma 13.** Let $L$ be a field and let $E$ be an elementary abelian $p$-group of infinite rank. If $\text{ch} L \neq p$, then $E$ is not linear over $L$.

**Proof.** It can be assumed that $L$ is algebraically closed. Let $V$ be a vector space over $L$ of dimension $n$ and let $F \subset GL(V)$ be an elementary abelian $p$-subgroup. For any character $\chi : K \to L^*$, let $L_\chi$ be the corresponding one-dimensional representation of $F$.

Since $\cap \ker \chi_i$ has $\mathbb{F}_p$-codimension $\leq n$, it follows that the rank of $F$ is $\leq n$.

Therefore no infinite rank elementary abelian $p$-group is linear over $L$.

From now on, $L$ is an algebraically closed field of characteristic $p$.

**Lemma 14.** Let $\chi \in X(K^*)$, let $\mu : K \to L$ be a nonzero additive map and let $n$ be a positive integer prime to $p$.

Assume that

$$\mu(x^n y) = \chi(x) \mu(y)$$

for any $x \in K^*$ and $y \in K$.

Then $\chi$ is a semi-algebraic character of degree $n$.

**Proof.** By Lemma 10, $K$ is the additive span of $(K^*)^n$. Therefore there is some $x \in K^*$ such that $\mu(x^n) \neq 0$. Since $\mu(x^n) = \chi(x) \mu(1)$, it follows that $\mu(1) \neq 0$. After rescaling $\mu$, one can assume that $\mu(1) = 1$.

Let $x, y \in K^*$. We have $\mu(x^n) = \chi(x) \mu(1) = \chi(x)$, and therefore $\mu(x^n y^n) = \chi(x) \mu(y^n) = \mu(x^n) \mu(y^n)$. By lemma 10, $K$ is the additive span of $(K^*)^n$. It follows that
\[ \mu(ab) = \mu(a)\mu(b) \]

for any \( a, b \in K \). Hence \( \mu \) is a field embedding. Moreover, we have \( \chi(x) = \mu(x^n) \) for any \( x \in K^* \). Therefore \( \chi \) is a semi-algebraic character of degree \( n \).

For \( n \geq 1 \), let \( G_n(K) \) be the semi-direct product \( K^* \ltimes K \), where any \( z \in K^* \) acts on \( K \) as \( z^n \). More explicitly, the elements of \( G_n(K) \) are denoted \((z,a)\), with \( z \in K^* \) and \( a \in K \) and the product is defined by

\[
(z, a) \cdot (z', a') = (zz', za^n + a'),
\]

for any \( z, z' \in K^* \) and \( a, a' \in K \).

Let \( V \) be a finite dimensional \( L \)-vector space and let \( \rho : G_n(K) \to GL(V) \) be a group morphism. With respect to the subgroup \((K^*, 0) \) of \( G_n(K) \), there is a decomposition of \( V \) as

\[
V = \bigoplus_{\chi \in \mathcal{X}(K^*)} V_{(\chi)}
\]

where \( V_{(\chi)} := \{ v \in V | \forall x \in K^* : (\rho(x,0) - \chi(x))^nv = 0 \} \) is the generalized eigenspace associated with the character \( \chi \). Here and in the sequel, \( \rho(z,a) \) stands for \( \rho((z,a)) \), for any \((z,a) \in G_n(K) \).

**Lemma 15.** Let \( n \) be a positive integer prime to \( p \). \( \rho : G_n(K) \to GL(V) \) be an injective morphism. Then we have

\[
\operatorname{End}(V_{(\chi)}) \neq 0
\]

for some \( \chi \in \mathcal{X}_n(K^*) \).

**Proof.** Set

\[
V_0 = \{ v \in V | \rho(1,a)v = v, \forall a \in K \}, \quad \text{and} \quad V_1 = \{ v \in V | \rho(1,a)v = v \mod V_0, \forall a \in K \}.
\]

Since \( \rho(1,a)^p = 0 \) for any \( a \in K \), the \( \rho(K) \)-module \( V \) is unipotent. It follows that \( V_1 \not\supseteq V_0 \neq 0 \).

Clearly, \( V_1 \) is a \( G_n(K) \) submodule, and let \( \rho_1 : G_n(K) \to GL(V_1) \) be the restriction of \( \rho \) to \( V_1 \). Let \( \theta : K \to \operatorname{End}(V_1) \) be the map defined by \( \theta(a) = \rho_1(1,a) - 1 \) for \( a \in K \). By definition, we have \( \theta(a)(V_1) \subset V_0 \) and \( \theta(a)(V_0) = \{0\} \). Hence we have \( \theta(a) \circ \theta(b) = 0 \) for any \( a, b \in K \). Since

\[
\rho_1(1,a + b) = 1 + \theta(a + b),
\]

and

\[
\rho_1(1,a) \circ \rho_1(1,b) = (1 + \theta(a))(1 + \theta(b)) = 1 + \theta(a) + \theta(b),
\]

it follows that \( \theta(a + b) = \theta(a) + \theta(b) \).

Let \( W \) be the \( L \)-vector space generated by \( \operatorname{Im} \theta \). Since \( \theta \neq 0 \), we have \( W \neq 0 \). Let \( \rho_W \) the action by conjugacy of \( G_n(K) \) over \( W \). We have \( \rho_W(1,K) = 0 \), therefore there is a \( G_n(K) \)-equivariant map \( g : W \to W' \).
where \( W' \) is a one dimensional quotient of \( W \). Therefore there is some \( \chi \in X(K^*) \) such that
\[
\rho_{W'}(z, 0) = \chi(z),
\]
for any \( z \in K^* \), where \( \rho_{W'} \) is the induced action on \( W' \).

Set \( \mu = g \circ \theta \). It follows from the previous computation that \( \mu \) is additive. Moreover if follows from the identity \( \rho(x, 0)\rho(1, a)\rho(x^{-1}, 0) = \rho(1, x^n a) \) that
\[
\rho_1(x, 0)\theta(a)\rho_1(x^{-1}, 0) = \theta(x^n a)
\]
and therefore
\[
\mu(x^n a) = \chi(x)\mu(a),
\]
for any \( x \in K^* \) and \( a \in K \). By Lemma 14, the character \( \chi \) is algebraic of degree \( n \). Since \( W' \) is a subquotient of \( \text{End}(V) \), it follows that \( \text{End}(V)_{(\chi)} \neq 0 \).

Let \( n \geq 1 \). From now on, we will identify \( G_n(K) \) with the subgroup of \( \text{Aut} K^2 \) of all automorphism of the form
\[
(x, y) \mapsto (zx, zy + ax^{n+1})
\]
for some \( z \in K^* \) and \( a \in K \).

**Lemma 16.** Let \( K \) be an infinite field of characteristic \( p \) and let \( E \) be a subgroup of \( \text{Elem}_0(K^2) \).

If \( E \) contains \( G_n(K) \) for infinitely many integers \( n \) prime to \( p \), then \( E \) is not linear over a field. In particular, \( \text{Elem}_0(K^2) \) is not linear over a field.

*Proof.* Assume otherwise, and let \( \rho : E \to GL(V) \), where \( V \) is a finite dimensional vector space over an algebraically closed field \( L \). The group \( E \) contains some elementary abelian \( p \)-groups of infinite rank. Therefore, by Lemma 13, the field \( L \) has characteristic \( p \).

The group \( K^* \) is identified with the subgroup of \( \text{Aut} K^2 \) of linear homotheties. Note that \( K^* \) is a common torus of all subgroups \( G_n(K) \) for any \( n \geq 1 \). Let \( \Omega \) be the set of all characters \( \chi \) of \( K^* \), such that \( \text{End}(V)_{(\chi)} \neq 0 \).

By Lemma 15, \( \Omega \) contains a character in \( \chi_n \in \mathcal{X}_n(K^*) \) for infinitely many positive integers \( n \) prime to \( p \). By Lemma 12 these characters are all distincts, which contradicts that \( \Omega \) is a finite set.

**Lemma 17.** The amalgamated product \( \text{GL}(K^2) *_{B_0} \text{Elem}_0(K^2) \) satisfies hypothesis \( \mathcal{H} \).

*Proof.* Let \( g \in B_0 \) with \( g \neq 1 \).
First, if $g$ is not an homothety, there is $\gamma \in GL(K^2)$ such that $g^\gamma$ is not upper triangular and therefore we have $g^\gamma \notin B_0$.

Otherwise $g$ is an homothety with ratio $\lambda \neq 1$. Let $\gamma$ be the automorphism $(x, y) \mapsto (x, y + x^2)$. Then $g^\gamma$ is the automorphism $(x, y) \mapsto (\lambda x, \lambda y + (\lambda^2 - \lambda)x^2)$, so $g^\gamma$ is not in $B_0$.

\[ \square \]

**Theorem A.2.** If $K$ be an infinite field of characteristic $p$, then $\text{Aut}_0 K^2$ is not linear even over a ring.

**Proof.** Assume otherwise. Using van der Kulk Theorem and lemmas 17, 6 and 4 it follows that $\text{Aut}_0 K^2$ is linear over a field. This contradicts the fact that its subgroup $\text{Elem}_0(K^2)$ is not linear over a field by Lemma 16. $\square$

### 6 The Tits Ping-Pong

#### 6.1 The Ping-Pong lemma

Let $(E_p)_{p \in P}$ be a collection of groups indexed by a set $P$. Let $\Gamma := *_{p \in P} E_p$ be the free product of these groups. Let $S$ be the set of all finite sequences $(p_1, \ldots, p_n)$ such that $p_i \neq p_{i+1}$ for any $i < n$. For each $p \in P$, set $E_p^* = E_p \setminus \{1\}$.

Let $\gamma \in \Gamma$. There is a unique $p = (p_1, \ldots, p_n) \in P$ and a unique decomposition of $\gamma$

$$\gamma = \gamma_1 \cdots \gamma_n,$$

where $\gamma_i \in E_{p_i}^*$. The sequence $p$ is called the type of $\gamma$.

The free product $\Gamma := *_{p \in P} E_p$ is called nontrivial if

(i) for any $p \in P$, $E_p \neq \{1\}$,

(ii) Card $P \geq 2$, and

(iii) if Card $P = 2$, $\Gamma$ is not the trivial free product $\mathbb{Z}/2\mathbb{Z} \ast \mathbb{Z}/2\mathbb{Z}$.

**Lemma 18.** Assume that a nontrivial free product $\Gamma = *_{p \in P} E$ acts on some set $\Omega$. Let $(\Omega_p)_{p \in P}$ be a collection of subsets in $\Omega$. Assume

(i) The subsets $\Omega_p$ are nonempty and disjoint, and

(ii) we have $E_p^* \Omega_q \subset \Omega_p$ whenever $p \neq q$.

Then the action of $\Gamma$ on $\Omega$ is faithful.

The hypothesis (ii) is called a Ping-Pong hypothesis.
Proof. Let $\gamma \in \Gamma$ with $\gamma \neq 1$, and let $p = (p_1, \ldots, p_n)$ be its type. We claim that there is $x \in \Omega$ such that $\gamma.x \neq x$.

If Card $P \geq 3$ let $q \in P$ with $q \neq p_1$ and $q \neq p_n$. Let $x \in \Omega_q$. By the Ping-Pong hypothesis, $\gamma.x$ belongs to $\Omega_{p_1}$, therefore $\gamma.x \neq x$.

If Card $P = 2$ it can be assumed that $P = \{1, 2\}$. It follows from Lemma 5 that $\gamma$ has a conjugate $\gamma'$ of type $q = (q_1, \ldots, q_m)$ with $q_1 = q_m = 1$. Let $x' \in \Omega_2$. By the Ping-Pong hypothesis, $\gamma'.x'$ belongs to $\Omega_1$, therefore $\gamma'.x \neq x'$. It follows that there is some $x \in \Omega$ with $\gamma.x \neq x$.

In both cases, any $\gamma \neq 1$ acts nontrivially. Hence the action of $\Gamma$ on $\Omega$ is faithful.

6.2 Mixture of free products, amalgamated products and semi-direct product

This section is devoted to three technical lemmas, for groups with a mix of free products, amalgamated products and semi-direct products.

Let $G$ be a group. A $G$-structure on a group $E$ is a $G$-action on $E$, where $G$ acts by group automorphisms. Equivalently, it means that the semi-direct product $G \ltimes E$ is well defined. For simplicity, a group $E$ with a $G$-structure is called a $G$-group. Two $G$-groups $E$, $E'$ are called $G$-isomorphic if there is an isomorphism from $E$ to $E'$ which commutes with the $G$-structure.

Let $P$ be a set on which $G$ acts, let $E$ be a group and for each $p \in P$ let $E_p$ be a copy of $E$. A compatible $G$-structure on $\ast_{p \in P} E_p$ obviously provides a $A$-structure on $E$, and we have

$$G \ltimes (\ast_{p \in P} E_p) \simeq G \ast_A (A \ltimes E).$$
Proof. Set $\Gamma'_1 = *_{\gamma \in P} E^\gamma$. Clearly $G$ acts over $\Gamma'_1$, so we can consider

$$\Gamma'_0 := G \ltimes \Gamma'_1.$$  

Using the universal properties of amalgamated, free and semi-direct products, one defines morphisms $\phi : \Gamma_0 \rightarrow \Gamma'_0$ and $\psi : \Gamma'_0 \rightarrow \Gamma_0$ which are inverse of each other. It follows that $\Gamma_0$ and $\Gamma'_0$ are isomorphic, and $\phi$ induces an isomorphism from $\Gamma_1$ to $\Gamma'_1$, which proves Lemma 19.

Let 1 be the distinguished point of $G/A$. We have $E^a_1 = E_1$ for any $a \in A$, hence the group $E = E_1$ has an $A$-structure. The rest of the proof of Lemma ?? follows from universal properties, as before.

For the last lemma, let $G$ be a group acting on a set $P$. Let $E$ and $E'$ be two groups. For each $u \in P$, let $G_u$ be the stabilizer in $G$ of $u$ and $E_p$ (respectively $E'_p$) be a copy of $E$ (respectively of $E'$).

Assume given some compatible $G$-structures on $*_{p \in P} E_p$ and $*_{p \in P} E'_p$. Obviously, it provides some $G_p$-structure on $E_p$ and $E'_p$, for any $p \in P$.

**Lemma 21.** Assume that the groups $E_p$ and $E'_p$ are $G_p$-isomorphic for any $p \in P$. Then the groups $G \ltimes (*_{p \in P} E_p)$ and $G \ltimes (*_{p \in P} E'_p)$ are isomorphic.

**Proof.** First, assume that $G$ acts transitively on $P$. Let $u$ be a point of $P$, and let $A := G_u$ be its stabilizer. It follows from Lemma 20 that

$$G \ltimes (*_{p \in P} E_p) \simeq G *_{G_u} (G_u \ltimes E_p),$$

and

$$G \ltimes (*_{p \in P} E'_p) \simeq G *_{G_u} (G_u \ltimes E'_p).$$

Hence the groups $*_{p \in P} E_p$ and $*_{p \in P} E'_p$ are $G$-isomorphic. From this, it follows that the groups $*_{p \in P} E_p$ and $*_{p \in P} E'_p$ are $G$-isomorphic even if $G$ does not act transitively on $P$. Therefore $G \ltimes (*_{p \in P} E_p)$ and $G \ltimes (*_{p \in P} E'_p)$ are isomorphic.

\[ \square \]

### 6.3 The subgroups of $GL_S(2, K[t])$ in $GL(2, K[t])$

For $G(t) \in GL(2, K[t])$, let $G(0)$ its evaluation at 0. For any subgroup $S \subset GL(2, K)$ set

$$GL_S(2, K[t]) := \{ G(t) \in GL(2, K[t]) | G(0) \in S \}.$$  

For $S = \{\}$, the group $GL_S(2, K[t])$ will be denoted by $GL_1(2, K[t])$.

For any $\gamma \in \mathbb{P}_K^1$, let $e_{\gamma} \in \text{End}(K^2)$ such that $e_{\gamma}^2 = 0$ and $\text{Im} e_{\gamma} = \gamma$. Since $e_{\gamma}$ is unique up to a constant multiple, the group

$$E_{\gamma} := \{ \text{id} + t f(t) e_{\gamma} | f \in K[t] \}$$

is a well defined subgroup of $GL_1(2, K[t])$. 

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Lemma 22. The group $GL_1(2, K[t])$ is generated by its subgroups $E_\gamma$, where $\gamma$ runs over $\mathbb{P}^1_K$.

Proof. For $A, B \in \text{End}(K^2)$, set

$$< A | B >= \det (A + B) - \det A - \det B.$$ 

Any element $G(t) \in GL_1(2, K[t])$ can be written as a polynomial

$$G(t) = \sum_{n \geq 0} A_n t^n$$

where $A_n$ belong to $\text{End}(K^2)$ and $A_0 = \text{id}$.

The proof runs by induction on the degree $N$ of $G$. It can be assumed that $N \geq 1$. For any $n \geq 0$, let $I_n$ be the set of pairs of integers $(i, j)$ with $0 \leq i < j$ and $i + j = n$. We have $\det G(t) = \sum_{n \geq 0} c_n t^n$, where the scalars $c_n$ are given by

$$c_n = \det A_n/2 + \sum_{(i, j) \in I_n} < A_i|A_j >,$$

where it is understood that $\det A_n/2 = 0$ if $n$ is odd.

Since $\det G(t)$ is an invertible polynomial, we have $c_n = 0$ for any $n > 0$. The identity $c_{2N} = 0$ implies that $\det A_N = 0$ hence $A_N$ has rank one. Let $\delta$ be the image of $A_N$ and set $E = \{a \in \text{End}(K^2) | \text{Im} a \subset \delta\}$

There exist an integer $n$ such that

$$A_n \notin E$$

but $A_m \in E$ for any $m > n$.

We have

$$c_{N+n} = \det A_{N+n} + \sum_{(i, j) \in I_{N+n}} < A_i|A_j >.$$ 

Since $E$ consists of rank one endomorphisms, we have $\det a = 0$ for any $a \in E$ and $< a, b >= 0$ for any $a, b \in E$. Therefore we have $\det A_{N+n} = 0$ and $< A_i|A_j >= 0$, whenever $(i, j)$ lies in $I_{N+n}$ and $i \neq n$. Thus it follows that $< A_n|A_N >= 0$.

Set $\delta' = \text{Ker} A_N$. An easy computation shows that the previous relation $< A_N|A_n >= 0$ implies that $A_n(\delta') \subset \delta$. Set $B = e_{\delta} \circ A_n$. Clearly we have $\text{Ker} B \supset \delta'$ and $\text{Im} B \subset \delta$, therefore $B$ is proportional to $A_N$. Since $B$ is not in $E$, we have $B \neq 0$ and therefore $ce_{\delta} \circ A_n = A_N$ for some $c \in K$. Set

$$H(t) = (1 - ct^{N-n}e_{\delta}).G(t).$$

We have $e_{\delta} \circ a = 0$ for any $a \in E$, therefore we have $e_{\delta} \circ A_m = 0$ for any $m > n$. It follows that $H(t)$ has degree $\leq N$. Moreover, its degree $N$ component is $A_N - c e_{\gamma} A_n = 0$. Therefore $H(t)$ has degree $< N$, et the proof runs by induction.

6.4 Free products in $GL_1(K[t])$

Let $K$ be a field.
Lemma 23. We have
\[ \text{GL}_1(2, K[t]) = *_{\delta \in \mathbb{P}^1_K} E_\delta. \]

Proof. Set \( \Omega := K[t]^2 \setminus \{0\} \). Any \( v \in \Omega \) can be written as a finite sum
\[ v = \sum_{0 \leq k} v_k \otimes t^k, \]
where \( v_k \) lies in \( K^2 \). The biggest integer \( n \) with \( v_n \neq 0 \)
is the degree of \( v \) and \( hc(v) := v_n \) is called its highest component. For any \( \delta \in \mathbb{P}^1_K \), set
\[ E_\delta^* = E_\delta \setminus \{1\} \]
and
\[ \Omega_\delta = \{v \in \Omega \mid hc(v) \in \delta\}. \]

Let \( H(t) \in E_\delta^* \). Since \( H(t) \neq 1 \), its degree \( n \) is positive and its degree \( n \)
component is \( c.e_\delta \) for some \( c \neq 0 \). Let \( \delta' \in \mathbb{P}^1_K \) be a line distinct from \( \delta \) and
let \( v \in \Omega_{\delta'} \). Since \( e_\delta, u \neq 0 \) for any non-zero \( u \in \delta' \), we have
\[ hc(H(t).v) = ce_\delta hc(v), \]
for any \( v \in \Omega_{\delta'} \). It follows that \( E_\delta^* \Omega_{\delta'} \subset \Omega_\delta \) for any \( \delta \neq \delta' \).

Therefore by Lemma 18, the free product \( *_{\delta \in \mathbb{P}^1_K} E_\delta \) embeds in \( \text{GL}_1(2, K[t]) \).
Hence by Lemma 22, we have \( *_{\delta \in \mathbb{P}^1_K} E_\delta = \text{GL}_1(2, K[t]) \).

Remark. Set
\[ U^+ = \left\{ \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} \mid f \in K[t] \right\}, \]
\[ U^- = \left\{ \begin{pmatrix} 1 & 0 \\ f & 1 \end{pmatrix} \mid f \in K[t] \right\}, \]
and
\[ U = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in K \right\}. \]

It is easy to show that Lemma 23 is equivalent to the fact that \( U^+ * U^- = \text{GL}_U(2, K) \). This results in stated in the context of Kac-Moody groups in
Tits notes [15]. Since these notes are not widely distributed, let mention that
an equivalent result is stated in [17], Section 3.2 and 3.2. For Tits original
proofs, see [16].

7 The Linear Representation of \( \text{Aut}_1 K^2 \)

In this section, we prove Theorem B.

7.1 The subgroups \( F_\delta \) in \( \text{Aut}_1 K^2 \).

Let \( \delta \in \mathbb{P}^1_K \) and let \( (a, b) \in \delta \) be nonzero. For any \( f \in K[t] \), let \( \tau_\delta(f) \) be the
automorphism
\[ \tau_\delta(f) : (x, y) \rightarrow (x + a f(bx - ay), y + b f(bx - ay)). \]
We have $\tau_\delta(f) \circ \tau_\delta(g) = \tau_\delta(f + g)$. Let $F_\delta = \{\delta(f) | f \in t^2 K[t]\}$. Indeed for $\delta_0 = K(0, 1)$, $\tau_{\delta_0}(f)$ is the elementary automorphism 

$$(x, y) \mapsto (x, y + f(x)),$$

and $F_{\delta_0}$ is the group $\text{Elem}_1(K^2)$ of Section 1.5. In general, we have $F_\delta = \text{Elem}_1(K^2)^g$, where $g \in GL(K^2)$ satisfies $g.\delta_0 = \delta$.

**Lemma 24.** We have 

$$\text{Aut}_1(K^2) = \ast_{\delta \in \mathcal{P}_1^1 K} F_\delta.$$

**Proof.** First we check that $\text{Aut}_0 K^2$ satisfies the hypotheses of Lemma 19. By van der Kulk Theorem, we have 

$$\text{Aut}_0 K^2 = G_1 \ast_A G_2 = \Gamma_0$$

where $G_1 = GL(K^2)$, $G_2 = \text{Elem}_0(K^2)$ and $A = B_0$ is the Borel subgroup of $GL(K^2)$. We have $G_2 = A \times E$ where $E = \text{Elem}_1(K^2)$. Clearly the map $\chi : G_1 \ast_A G_2 \rightarrow G_1$ is simply the map 

$$\phi \in \text{Aut}_0(K^2) \rightarrow d\phi_0,$$

and its kernel is $\Gamma_1 = \text{Aut}_1(K^2)$. Since $G_1/A = GL(2)/B_0 = \mathbb{P}_K^1$, and $E^\gamma = F_\delta$ for any $\gamma \in GL(K^2)$ with $\gamma.\delta_0 = \delta$, it follows from Lemma 19 that 

$$\text{Aut}_1(K^2) = \ast_{\gamma \in \mathbb{P}_K^1} F_\delta.$$ 

\begin{proof}

7.2 Proof of Theorem B

For any $\delta \in \mathbb{P}_K^1$, let $\psi_\delta : F_\delta \rightarrow E_\delta$ be the isomorphism defined by 

$$\psi_\delta(\tau_\delta(f)) = \text{id} + f/t \otimes e_\delta.$$

**Theorem B.** The collection of isomorphisms $(\psi_\delta)_{\delta \in \mathbb{P}_K^1}$ induces an isomorphism 

$$\psi : \text{Aut}_1 K^2 \simeq GL_1(2, K[t]).$$

In particular, $\text{Aut}_1 K^2$ is linear over $K(t)$

**Proof.** The first statement is a consequence of Lemmas 23 and 24. It follows that $\text{Aut}_1 K^2 \subset SL(2, K(t))$, therefore $\text{Aut}_1 K^2$ is linear over $K(t)$. 

\begin{remark}

The isomorphism $\text{Aut}_1 K^2 \simeq GL_1(2, K[t])$ is not canonical, since it depends upon a choice, for each $\delta \in \mathbb{P}_K^1$, of a basis of $\delta$.

Moreover the formula defining $\psi$ is complicated. Indeed if an automorphism $\sigma \in \text{Aut}_1(K^2)$ is written as a free product $\sigma = \sigma_1 \ldots \sigma_m$, where $\sigma_i \in F_{\delta_i}$ has degree $m_i$, then $\sigma$ has degree $m_1 \ldots m_n$ although $\psi(\sigma)$ has degree $m_1 + \ldots + m_n - m$.

\end{remark}
8 Linearity of $\text{Aut}_S K^2$ for some $S \subset GL(2, K)$

In this section, Theorem C1 and C2 are proved.

For a subgroup $S$ of $GL(2, K)$, the following hypotheses $\mathcal{U}$ and $\mathcal{Q}\mathcal{U}$ will be considered

(\mathcal{U}) any $K$-reducible element $g \in S$ is unipotent,

(\mathcal{Q}\mathcal{U}) any $K$-reducible element $g \in S$ is quasi-unipotent.

8.1 Proof of Theorem C1

Theorem C.1. Assume that the subgroup $S$ satisfies the hypothesis $\mathcal{U}$. Then we have

$$\text{Aut}_S K^2 \simeq GL_S(2, K[t]).$$

In particular, $\text{Aut}_S K^2$ is linear over $K(t)$.

Proof. It follows from Lemmas 23 and 24 that

$$GL_1(2, K[t]) = \ast_{\delta \in \mathbb{P}^1_K} E_\delta \text{ and } \text{Aut}_1 K^2 = \ast_{\delta \in \mathbb{P}^1_K} F_\delta,$$

where the groups $E_\delta$ and $F_\delta$ are obviously isomorphic. For $\delta \in \mathbb{P}^1_K$, let $S_\delta$ be the stabilizer in $S$ of $\delta$. Any element in $S_\delta$ is $K$-reducible, so by hypothesis (\mathcal{U}) all elements in $S_\delta$ are unipotent. It follows that $S_\delta$ acts trivially on $E_\delta$ and $F_\delta$, therefore the $S_\delta$ groups $E_\delta$ and $F_\delta$ are $S_\delta$-isomorphic. It follows from Lemma 21 that the $S$ groups $\ast_{\delta \in \mathbb{P}^1_K} E_\delta$ and $\ast_{\delta \in \mathbb{P}^1_K} F_\delta$ are $S$-isomorphic. Therefore $GL_S(2, K[t]) = S \ltimes (\ast_{\delta \in \mathbb{P}^1_K} E_\delta)$ and $\text{Aut}_1 K^2 = S \ltimes (\ast_{\delta \in \mathbb{P}^1_K} F_\delta)$ are isomorphic. \[\square\]

8.2 FG subgroups of $GL(2, K)$

Lemma 25. Let $S$ be a FG subgroup of $GL(2, K)$. There is an integer $N > 0$ such that any quasi-unipotent $g \in S$ has quasi-order divisible by $N$.

Proof. Let $F$ be the prime field of $K$, namely $F = \mathbb{F}_p$ if $K$ has characteristic $p$ and $F = \mathbb{Q}$ otherwise. There is a FG ring $R \subset K$ such that $S$ is a subgroup of $GL(2, R)$. It can be assumed that $K$ is the fraction field of $R$.

Let $X$ be the set of all roots of unity which are eigenvalues of some element $g \in S$. We claim that the set $X$ is finite.

Set $E = \mathbb{F} \cap K$. Since $K$ is finitely generated, $E$ is a finite extension of $F$. Any $\zeta \in X$ lies in $E$ or in a quadratic extension of $E$. We will now consider separately the case where $K$ has zero characteristic or finite characteristic.
First assume that $\text{ch} \ K = 0$. Let $m$ be the order of $\zeta$. Since $[\mathbb{Q}(z) : \mathbb{Q}] = \phi(m)$, we have $\phi(m) \leq 2[E : \mathbb{Q}]$. Thus $m$ is bounded, and therefore $X$ is finite.

Next assume that $\text{ch} \ K = p$. Then $E$ is a finite field and $\zeta$ lies in the unique quadratic extension $E'$ of $E$. Therefore $X \subset E'$ is finite.

Since $X$ is finite, we have $X \subset \mu_N$ for some $N$, where $\mu_N$ is the set of $N^{th}$-root of unity. Therefore any quasi-unipotent element $g \in S$ has eigenvalues in $\mu_N$, and therefore $g^N$ is unipotent.

\begin{lemma}
Let $S$ be a FG subgroup of $GL(2, K)$ satisfying the hypothesis \QU. Then $S$ contains a finite index subgroup $S'$ satisfying the hypothesis $\U$.

Proof. By Lemma 25, there exists an integer $N \geq 1$ such that any $K$-reducible element in $S$ is quasi-idempotent of quasi-order divisible by $N$.

Let $C$ be the set of pairs $(s, p)$ in $K^2$ such that $s = \zeta_1 + \zeta_2$ and $p = \zeta_1\zeta_2$ for some $\zeta_1, \zeta_2 \in \mu_N$.

There is a finitely generated subring $R$ of $K$ such that $S \subset GL(2, R)$. Since the intersection of all cofinite ideal $m$ in $R$ is zero, there is a cofinite ideal $m \subset R$ such that $(s, p) \not\equiv (2, 1) \mod m$ for any $(s, p) \in C \cap R^2$ with $(s, p) \not\equiv (2, 1)$.

Set $S' = S \cap GL(2, m)$. Since $GL(2, m)$ has finite index in $GL(2, R)$, $S'$ has finite index in $S$. We claim that $S'$ satisfies the hypothesis $\U$.

Let $g \in S'$ be $K$-reducible. Since $g$ is quasi-idempotent of quasi-order divisible by $N$, the couple $(\text{tr} \ g, \text{det} \ g)$ belongs to $C$. Since $g \equiv 1 \mod m$, we have $(\text{tr} \ g, \text{det} \ g) \equiv (2, 1) \mod m$. By the construction of $m$, we have $(\text{tr} \ g, \text{det} \ g) = (2, 1)$, and therefore $g$ is unipotent. Hence $S'$ satisfies the hypothesis $\U$.
\end{lemma}

8.3 Proof of Theorem C.2

**Theorem C.2.** Let $S$ be a FG subgroup of $GL(K^2)$ satisfying hypothesis \QU. Then $\text{Aut}_S K^2$ is linear over $K(t)$.

**Proof.** By Lemma 26, there is a subgroup $S'$ of $S$ of finite index satisfying the hypothesis $\U$. By Theorem C.1, the group $\text{Aut}_{S'} K^2$ is linear over $K(t)$. Since it is a finite index subgroup, the group $\text{Aut}_S K^2$ is also linear over $K(t)$.
9 Some Corollaries

9.1 Linearity of Aut $K^2$ for a finite field $K$

**Corollary D.1.** Let $K$ be a finite field of characteristic $p$. The group $\text{Aut} K^2$ is linear over $\mathbb{F}_p(t)$.

*Proof.* By Theorem B, the group $\text{Aut}_1 K^2$ is Since $[\text{Aut} K^2 : \text{Aut}_1 K^2]$ is finite, the group $\text{Aut} K^2$ is also linear over $\mathbb{F}_p(t)$. \qed

9.2 Linearity of FG subgroups of Aut $K^2$ for a quasi-finite field $K$

**Corollary D.2.** Let $K$ be an infinite subfield of $\mathbb{F}_p$.

The group $\text{Aut} K^2$ is not linear, even over a ring. However any FG subgroup is linear over $\mathbb{F}_p(t)$.

*Proof.* Indeed $\text{Aut} K^2$ is not linear by Theorem B. Any FG subgroup $\Gamma$ of $\text{Aut} K^2$ is a subgroup of $\text{Aut} L^2$ for some finite subfield $L \subset K$, so it is linear by Corollary D.1. \qed

9.3 Linearity over $K$ of Aut $K^2$ for a big enough field $K$

**Lemma 27.** Let $K$, $L$ be fields. If $\text{Card} K = \text{Card} L$ and $\text{ch} K = \text{ch} L$, then $\text{Aut}_1 K^2$ and $\text{Aut}_1 L^2$ are isomorphic.

*Proof.* It can be assumed that $K$ is infinite. Let $F$ be its prime field, let $P$ be a set and let $E$ be a $F$-vector space with

$$\text{Card} P = \text{Card} K$$

and $\text{dim}_F E = \text{Card} K$.

By lemma 24, we have

$$\text{Aut}_1 K^2 \cong \bigast_{p \in P} E_p,$$

where each $E_p$ is a copy of $E$. It follows that $\text{Aut}_1 K^2$ and $\text{Aut}_1 L^2$ are isomorphic if $\text{Card} K = \text{Card} L$ and $\text{ch} K = \text{ch} L$. \qed

A field $K$ is called *big enough* if its absolute transcendence degree is $\geq 1$ and if $K$ is not a finite extension of $\mathbb{F}_p(t)$ for some prime number $p$. For example, $\mathbb{Q}(t)$, $\mathbb{F}_p(t_1, t_2)$ and $L(t)$, where $L$ is an infinite subfield of $\mathbb{F}_p$, are big enough.

**Corollary D.** Let $K$ be a big enough field. There is an embedding

$$\text{Aut}_1 K^2 \subset SL(2, K).$$
Proof. Since $K$ is big enough, there is an embedding $L(t) \subset K$, where $L$ is a field with $\text{Card } L = \text{Card } K$.

It follows from Lemma 27 that $\text{Aut}_1 K^2$ is isomorphic to $\text{Aut}_1 L^2$. Moreover by Theorem B, $\text{Aut}_1 L^2$ is isomorphic to $GL_1(L[t])$. Therefore we have $\text{Aut}_1 K^2 \simeq GL_1(L[t]) \subset SL_2(L(t)) \subset SL(2, K)$.

\[ \square \]

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