Squeezed field path integral description of second sound in Bose-Einstein condensates

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We propose a generalization of the Feynman path integral using squeezed coherent states. We apply this approach to the dynamics of Bose-Einstein condensates, which gives an effective low energy description that contains both a coherent field and a squeezing field. We derive the classical trajectory of this action, which constitutes a generalization of the Gross Pitaevskii equation, at linear order. We derive the low energy excitations, which provides a description of second sound in weakly interacting condensates as a squeezing oscillation of the order parameter. This interpretation is also supported by a comparison to a numerical c-field method.

The Feynman path integral has been one of the most fruitful concepts of theoretical physics [1]. It provides an alternative view of quantum mechanics by formulating it as a sum of paths. Each of these paths is weighted by a phase given by the classical action, and therefore it recovers the Lagrangian method in the context of quantum mechanics, see also [2]. Path integral formulations have been applied to both dynamic and thermodynamic quantities, and to classical and quantum systems and processes. Numerous analytical and numerical methods have been developed, see e.g. [3–5].

Further down, we will formulate a generalization of the path integral representation of complex $|\phi|^4$ theory, which is naturally realized in condensates of ultra cold atoms. The dynamics of Bose-Einstein condensates continue to be an intriguing and subtle field of research. Phenomena such as superfluidity [6–8] and second sound [9, 10] continue to pose questions, and are under recent and current investigation in ultra cold atom systems. While second sound in helium-II can be described within a hydrodynamic two-fluid approach [11], due to the strong interactions of He, the weakly interacting limit that is realized in atomic condensates is in a qualitatively distinct regime. We note that the second sound velocity is predicted to be $1/\sqrt{3}$ of the first sound velocity within the hydrodynamic approximation. However, as observed in [11], for weak interactions the second sound velocity is above the first sound velocity, and has a different temperature dependence. Theoretical studies on second sound in atomic condensates have been reported in [12–16].

In this paper, we derive a path integral representation of the weakly interacting Bose gas that explicitly captures the squeezed nature of its ordered state. An intuitive motivation is sketched in Fig. 1 (a), which depicts the symmetry broken state of $|\phi|^4$ theory. It has an anisotropic distribution around its expectation value. To include this feature explicitly in the path integral we utilize two-mode squeezed coherent states, instead of the commonly used coherent states. The resulting action that appears in the weight function for each path contains not only the complex field that describes coherent states, but an additional complex field that describes the squeezing of this field. We refer to these as the coherent and the squeezing field. A single path can be visualized in Fig. 1 (b). During the time evolution not only the expectation value of the distribution varies but also the quadratures around it. This also implies that the information in a single path of this path integral, such as the classical path, contains information about higher order fluctuations than the regular coherent state path integral.

We note that the Bogolyubov approximation of the weakly interacting Bose gas uses two mode squeezing of the momenta $k$ and $-k$, in particular

$$b_k = v_k(\eta_k)\beta_k + v_k(\eta_k)\beta_{-k}^\dagger,$$

FIG. 1: (a) Illustration of the ordered state of a weakly interacting condensate, described by $|\phi|^4$ theory. The bosons condense in the minimum of the Mexican hat potential $V(\psi) = -\mu|\psi|^4 + g|\psi|^2/2$, with $\mu, g > 0$. The distribution (shown in red) is squeezed. Panel (b) shows a sketch of a single path in the path integral in which both the expectation value and the squeezing of the distribution vary in time. Panel (c) shows schematically the corresponding fields $\psi(t)$ and $\eta(t)$.
where $b_k$ are the boson operators, and $u_k(\eta_k)$ and $v_k(\eta_k)$ are the Bogolyubov parameters, which both depend on a squeezing parameter $\eta_k$. However, the parameters $\eta_k$ are constant within the Bogolyubov approximation. They are chosen as $\eta_k = \eta_0^k$ to diagonalize the Hamiltonian, resulting in the Bogolyubov modes and their dispersion.

In the path integral that we propose here, the squeezing parameter $\eta_k$ are allowed to evolve in time, and are itself a dynamical field. We derive the equations of motion for the classical path of the resulting Lagrange density for the squeezing field $\xi_k$. The dispersion of the squeezing field $\eta_k$ is the area of the complex plane. We split the time interval into $N$ intervals of length $\Delta t = (t_b - t_a)/N$, and introduce the resolution of the identity $\delta_{k,\eta_k}$ that couples to the time derivative of $\eta_k$: $\eta_k$ is a product of a coherent state with $\psi_0 = \sqrt{N} \eta_0$ for the $k = 0$ mode, which is a fixed throughout the derivation, and a product of coherent squeezed states for all other momentum states. The product operation $\prod_{k \neq 0}^N \delta_{k,\eta_k}$ refers to all momentum states, but excludes double counting [23]. For fixed $\psi_0$, these states resolve the identity in momentum space, excluding $k = 0$, i.e., $\sum_{k \neq 0} \delta_{k,\eta_k} = \prod_{k \neq 0}^N \delta_{k,\eta_k}$.

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equilibrium value is \( \eta_0^0 = -\ln(1 + 2gn/\epsilon_k)/4 \). This value solves \( gn(\eta_0^0 + |\eta_0^0|^2)/2 + (\epsilon_k + gn)\eta_0^0\eta_0 = 0 \), and diagonalizes the Bogolyubov modes. These two expansions give \( \langle \Psi_j | H | \Psi_j \rangle = H_\psi + H_\eta \), with \( H_\psi = \sum_{k \neq 0} \hbar \omega_k |\psi_k|^2 \), where \( \hbar \omega_k = \sqrt{\epsilon_k (\epsilon_k + 2gn)} \) is the standard Bogolyubov dispersion. \( n_0 \) is the condensate density. For the squeezing field we have \( H_\eta = \sum_{k \neq 0} (E_{k,r} \eta_{k,r}^2 + E_{k,i} \eta_{k,i}^2) \), as derived in Apps. B and C, where we also give the full expressions for \( E_{k,r} \) and \( E_{k,i} \). \( \eta_{k,r,i} \) are the real/imaginary part of \( \eta_k \). Combining these results, the propagator takes the form \( iG(\Psi_{t_1}, t; \Psi_{t_2}, t_2) = \int D\Psi \exp(iS/\hbar) \), with the action \( S = \int_{t_1}^{t_2} dt L \). The Lagrangian is \( L = L_\psi + L_\eta \) with

\[
L_\psi = \sum_{k \neq 0} \frac{i\hbar}{2} (\psi_k^* \partial_t \psi_k - \psi_k \partial_t \psi_k^*) - \hbar \omega_k |\psi_k|^2 \tag{7}
\]

\[
L_\eta = \sum_{k \neq 0} i\hbar (a_k^* \partial_t \eta_k - a_k \partial_t \eta_k^*) - E_{k,r} \eta_{k,r}^2 - E_{k,i} \eta_{k,i}^2 \tag{8}
\]

The classical path is given by the Euler-Lagrange extremum, which results in an equation of motion for each of the fields. For the coherent field we have \( i\hbar \partial_t \psi_k = \hbar \omega_k \psi_k \), recovering the standard Bogolyubov result. For the equation of motion of the squeezing field at linear order, we first expand \( a_k \) up to first order, i.e. \( a_k \approx \text{const.} + \epsilon_k \eta_k + \epsilon_k \eta_k^* \), where \( \epsilon_k \eta_k \) and \( \epsilon_k \eta_k^* \) are real-valued expansion coefficients independent of \( \eta_k \), given in App. C. The resulting equation of motion for the squeezing field is

\[
2i\hbar \epsilon_k \partial_t \eta_k = E_r \eta_{k,r} + iE_i \eta_{k,i} \tag{9}
\]

This gives the dispersion \( \hbar \omega_{\eta,k} = \sqrt{E_r E_i/(2\epsilon_k)} \). We expand \( \omega_{\eta,k}^2 \) to first order in \( g \) to obtain

\[
\hbar \omega_{\eta,k} \approx \sqrt{2\epsilon_k (2\epsilon_k + 2gn)} \tag{10}
\]

We note that this has the form of the Bogolyubov dispersion, with the replacement \( \epsilon_k \to 2\epsilon_k \). This replacement derives from the inclusion of two-particle excitations in the squeezing field. The low-frequency limit of this dispersion is

\[
\hbar \omega_{\eta,k} \approx \sqrt{4gn \epsilon_k} = \hbar c_2 |k|, \tag{11}
\]

with \( c_2 = \sqrt{2gn/m} \). This implies that in the weak-coupling limit we have \( c_2 = \sqrt{2c_1} \).

These results are visualized in Fig. 2. In panel (a), we depict the equilibrium state, as in Fig. 1(a). The classical ground state of the squeezed field path integral describes this state with the coherent state amplitude \( \psi_0 \) of the \( b_{k=0} \) mode, as well as the equilibrium state of the squeezing field \( \eta_0 = \{\eta_0^k \} \). The two low energy modes are sketched in Fig. 2(b) and (c). As described above, at linear order the eigenmodes of the coherent field and the squeezing field decouple. In panel (b), we sketch a coherent field mode, which leaves the squeezing field unchanged, and which corresponds to the Bogolyubov modes. In panel (c), we sketch an eigenmode of the squeezing field, which leaves the coherent field unchanged. These modes are breathing modes of the state, and have the dispersion \( \hbar \omega_{\eta,k} \) given above. In Fig. 2(d), we show the dispersion of the squeezing field, Eq. 10, its low energy approximation, Eq. 11, and the Bogolyubov dispersion. We note that higher order terms of the Lagrangian will couple these modes, which renormalize these dispersions and also lead to thermal dependence of them, to be discussed elsewhere.

As a comparison, we consider a numerical implementation of the c-field method, used and described in Refs. 17, 18. We discretize space with discretization length \( \ell \) and approximate the system, Eq. 2, with a Hubbard model

\[
H_l = -J \sum_{\langle ij \rangle} (b_i^\dagger b_j + h.c.) - \sum_i \mu n_i + \frac{U}{2} \sum_i b_i^\dagger b_i^\dagger b_i b_i, \tag{12}
\]

with \( n_i = b_i^\dagger b_i \). The Hubbard parameters are related to the continuous space parameters via \( J = \hbar^2/(2m\ell^2) \) and \( U = g/\ell^3 \). We choose a lattice with the dimensions...
\[ h_{KE}(i) = \sum_{j(i)} (b_i^\dagger b_j + h.c.) \]  

where \( j(i) \) refers to the nearest neighbors of site \( i \). We note that most commonly used correlation functions have a signature of both first and second sound, however, \( h_{KE}(i) \) is a scalar quantity that has naturally a strong overlap with phase dynamics. Within the semi-classical approximation, we calculate the spatial and temporal Fourier transform of its correlation function, i.e.,

\[ g_{KE}(k, \omega) = \langle h_{KE}^\dagger(k, \omega) h_{KE}(k, \omega) \rangle \]  

where \( h_{KE}(k, \omega) \) is related to \( h_{KE}(i, t) \) via

\[ h_{KE}(k, \omega) = \frac{1}{\sqrt{N_t T_s}} \sum_i \int dt e^{-ikr_i - i\omega t} h_{KE}(i, t) \]  

\( N_t \) is the number of lattice sites \( N_t = N_x N_y N_z \), and \( T_s = 1.6s \) is the sampling time for the numerical Fourier transform. In Fig. 4, we show \( g_{KE}(k, \omega) \) for \( k = k_e x \), for a \(^{87}\text{Rb} \) condensate with density \( \rho = 0.7 \times 10^{13}\text{cm}^{-3} \) and temperature \( T/J = 2.5 \). We observe both excitation branches in the numerical result, and compare them with the Bogolyubov dispersion, the dispersion of the squeezing field, Eq. 11, and its low-energy approximation, Eq. 11, and find good agreement.

In conclusion, we have developed a generalized path integral that utilizes squeezed coherent states, and have applied it to the weak coupling limit of Bose-Einstein condensates. We have obtained the corresponding Lagrangian, at linear order, which contains both the standard coherent field as well as an additional squeezing field. We have derived the equations of motion, and determined the low-energy excitations of the condensed state. One of the two excitation branches recovers the Bogolyubov modes, the other one provides an analytical estimate for the second sound dispersion in the weak coupling limit. Furthermore, it provides an interpretation of the phenomenon of second sound as a squeezing oscillation of the order parameter. We note that the method that we have presented here is of broad applicability. It is of conceptual importance, because the same system, described by the same Hamiltonian, gives different generalized Lagrangians in the path integral, depending on the set of states that is used. Furthermore, any analytical approach that is based on a path integral representation can be generalized in the way that we have presented here. Finally, any numerical method that derives from a path integral representation can be generalized by extending the set of states of the path integral, for which we have paved the way in this paper.

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Appendix A: Overlap $\langle \Psi_j | \Psi_{j-1} \rangle$

Because the operators commute for different $k$–modes, it is sufficient to consider a single factor, i.e., a single pair of modes $(k, -k)$. Taking then the product, will lead to a summation in the exponent. We derive the squeezed coherent state in terms of a creation operator acting on the vacuum $|0\rangle$. Our derivation is based on the single mode calculation in reference [19]. Here, it is helpful to consider the squeezed coherent state, i.e., first squeezing the vacuum, then displacing it into the complex plane, instead of the squeezed coherent state. Therefore, we introduce into the state (1) the identity $S_k S_k^\dagger = 1$ and obtain $S_k D_k S_k^\dagger S_k D_{-k} S_k^\dagger S_k^\dagger |0\rangle$. The squeezed coherent state (4) is thus equivalent to the coherent squeezed state, when we squeeze the displacement operators $D_{\pm k}$. By observing that

$$S_k b_{\pm k} S_k^\dagger = u_k b_{\pm k} - v_k b_{\mp k}^\dagger,$$  \hspace{1cm} (A1)

where $u_k = \cosh(|\eta_k|)$ and $v_k = e^{i\phi_k} \sinh(|\eta_k|)$, we can pull the squeezing operators into the exponent of the displacement operator. This will lead us to the standard displacement operator, but with renormalized coherent parameters $\psi'_k = \frac{v_k}{u_k} \psi_{\pm k} + \frac{u_k}{v_k} \psi_{\mp k}^\dagger$, which inherit the two modes coupling from the squeezing operator. Next, we decompose our squeezing and displacement operators into their normal ordered forms

$$D_{\pm k} = e^{\psi_{\mp k} b_{\pm k}^\dagger - \psi_{\pm k}^\dagger b_{\mp k}} e^{-\frac{|\psi_{\pm k}|^2}{2}}, \hspace{1cm} (A2)$$

$$S_k = e^{\frac{v_k}{u_k} b_{\mp k}^\dagger b_{\pm k}} \left( \frac{1}{u_k} \right)^{\eta_k + \eta_{-k} + 1} e^{-\frac{v_k^2}{u_k} b_{\pm k} b_{\pm k}^\dagger}. \hspace{1cm} (A3)$$

After using the Baker-Cambell-Hausdorff-Formula $\exp X \exp Y = \exp \sum_{m=0}^{\infty} \frac{1}{m!} [X, Y]^m$, exp $X$, we find the desired expression for our squeezed state

$$|\eta_k \psi_k \psi_{-k}\rangle = D(\psi'_{\pm k})D(\psi'_{-k})S_k |0\rangle = \frac{1}{u_k} \exp \left( \frac{v_k}{u_k} b_{\pm k}^\dagger - \psi'_{\pm k}^\dagger b_{\mp k} - \psi'_{\pm k} - \psi'_{-k}^\dagger \right)$$

$$+ \psi'_{-k} b_{\pm k} + \psi'_{k} b_{\mp k}^\dagger - \frac{1}{2} (|\psi'_{\pm k}|^2 + |\psi'_{-k}^\dagger|^2) |0\rangle. \hspace{1cm} (A4)$$

By projecting state (A4) onto some two mode coherent state $|\alpha_{k-1} \rangle$ with the closure relation and making use of the definition of coherent states, we can write the $k$–th factor of our scalarproduct as a $c$–number valued Gaussian integral which can be calculated via the relation [20]

$$\int d^2 \alpha e^{-z \bar{\alpha} + \bar{z} \alpha} = \frac{\pi}{w} e^{\frac{\alpha \bar{\alpha}}{w^2}}. \hspace{1cm} (A5)$$

This procedure yields the following overlap Finally, we acquire the corresponding continuous form, i.e., $N \to \infty$, 

In the following, we want to present our detailed derivations and expressions mentioned in the main text.
after expanding all parameters up to first order at time step \( j \) around their values at the previous step \( j - 1 \). The parameters then become continuous functions of time and we obtain the prefactor \( a_k \) of the squeezing part

\[
a_k = \frac{1}{2|\eta_k|^2} \left[ -\eta_k |v_k|^2 (|\psi_k|^2 + |\psi_{-k}|^2 + 1) - (|\eta_k|^2 + u_k \eta_k v_k^* \psi_k \psi_{-k} + (\eta_k^2 - u_k \eta_k v_k^*) \psi_k^* \psi_{-k}^* \right]. \tag{A6}
\]

**Appendix B: Expectation value \( \langle \Psi_j | H | \Psi_j \rangle \)**

In the expectation value, succeeding times do not couple and we can omit the time index \( j \) and consider our parameters to be continuous. Being in the weakly interacting regime at low temperature, we apply the Bogolyubov approximation to our Hamiltonian \( (2) \). The \( k = 0 \) mode is approximated by the \( c- \)number \( \sqrt{N_0} \) and the Hamiltonian is expanded up to second order in \( b_k \). In order to obtain a gapless dispersion, we write the resulting Hamiltonian in terms of the total density \( n_0 = n - n_{ex} \) \[21\] \[22\] and replace the chemical potential by its mean field value \( \mu = g n_0 \). The Hamiltonian under consideration is

\[
H = -\frac{g N^2}{2V} + \sum_{k \neq 0} (\epsilon_k + gn) b_k^\dagger b_k + \frac{gn}{2} \sum_{k \neq 0} (b_k b_{-k}^\dagger + b_{-k} b_k).
\tag{B1}
\]

The unitarity of the squeezing operator is manifest in the relation \( S(\eta_k)^\dagger = S(-\eta_k) \). Hence the transformation \[21\] is modified to \( S_k b_{\pm, k} \bar{S}_k = u_k b_{\pm, k} + v_k b_{\mp, k} \). With this, we can now calculate the squeezed Hamiltonian and employ the definition of the coherent state. The resulting expectation value reads as follows

\[
\langle \Psi | H | \Psi \rangle = -\frac{g N^2}{2V} + \sum_{k \neq 0} \left( \epsilon_k (u_k^2 + |v_k|^2) + gn |u_k + v_k|^2 \right) |\psi_k|^2 +
\left( \frac{gn}{2} u_k^2 + v_k^2 \right) + (\epsilon_k + gn) u_k v_k \psi_k^* \psi_{-k} +
\left( \frac{gn}{2} u_k^2 + v_k^2 \right) + (\epsilon_k + gn) u_k v_k \psi_k \psi_{-k} +
\epsilon_k |v_k|^2 + \frac{gn}{2} (|u_k + v_k|^2 - 1).
\tag{B2}
\]

This \( c- \) number valued Hamiltonian is diagonalized for the equilibrium value \( \eta_k^0 \) given in the main text.

**Appendix C: Expansion around equilibrium**

Since we are in the weak interaction and low temperature regime, we want to analyze the system around the equilibrium parameter values up to second order.

The prefactor \( a_k \) in the overlap is expanded up to first order since it is multiplied by the time derivatives

\[
a_k = a_{k,1} \eta_k + \alpha_{k,1} \eta_k + \frac{(\omega_k - \epsilon_k)^2}{8 \omega_k \epsilon_k \eta_k^0}, \tag{C1}
\]

\[
a_{k,1} = \frac{\omega_k^2 - \epsilon_k^2}{8 \omega_k \epsilon_k \eta_k^0}, \tag{C2}
\]

where \( \omega_k \) is the Bogolyubov dispersion.

We get the corresponding Hamiltonian by expanding equation \( (B2) \)

\[
H = H_0 + \sum_{k \neq 0} \omega_k |\psi_k|^2 + \sum_{k \neq 0} \left( E_{k,r} \eta_k^2 + E_{k,i} \eta_k^2 \right),
\tag{C3}
\]

\[
H_0 = -\frac{g N^2}{2V} - \sum_{k \neq 0} \left( \frac{(\omega_k - \epsilon_k)^2}{4 \epsilon_k} \right),
\tag{C4}
\]

\[
E_{k,r} = \frac{1}{8 \epsilon_k (\eta_k^0)^2} \left( 2 gn_0 \omega_k - \left( h^2 \omega_k^2 - \omega_k \epsilon_k \right) \left( 2 - \frac{3 \epsilon_k}{\omega_k} + \frac{3 \epsilon_k^2}{h^2 \omega_k^2} \right) \right). \tag{C5}
\]

The overlap can be interpreted as the kinetic part and the Hamiltonian as the potential part of the Lagrangian. After applying the Euler-Lagrange equation, we obtain the two linear differential equation for the coherent parameter \( \psi_k \) and for the squeezing parameter \( \eta_k \) mentioned in the main text. The dispersion of the coherent field can be read out directly. However, in case of the squeezing field, we transform the differential equation into its matrix form

\[
\partial_t \begin{bmatrix} \eta_{k,r} \\ \eta_{k,i} \end{bmatrix} = \frac{1}{2\hbar} \begin{bmatrix} 0 & E_{k,r} \\ -E_{k,r} & 0 \end{bmatrix} \begin{bmatrix} \eta_{k,r} \\ \eta_{k,i} \end{bmatrix}. \tag{C6}
\]

Now, we can read out the dispersion immediately

\[
\hbar^2 \omega_{k,j}^2 = \frac{E_j E_i}{4 \alpha_{k,1}} = \frac{2 \omega_k^2 + 3 \omega_k \epsilon_k}{\omega_k + \epsilon_k}. \tag{C7}
\]

The expansion of this dispersion in \( g \), up to first order, reveals the two mode structure of the squeezing field, as is given in the main text.