Quantum Cyclic Code of length dividing $p^t + 1$

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Abstract—In this paper, we study cyclic stabiliser codes over $\mathbb{F}_p$ of length dividing $p^t + 1$ for some positive integer $t$. We call these $t$-Frobenius codes or just Frobenius codes for short. We give methods to construct them and show that they have efficient decoding algorithms.

An important subclass of stabiliser codes are the linear stabiliser codes. For linear Frobenius codes we have stronger results: We completely characterise all linear Frobenius codes. As a consequence, we show that for every integer $n$ that divides $p^t + 1$ for an odd $t$, there are no linear cyclic codes of length $n$. On the other hand for even $t$, we give an explicit method to construct all of them. This gives us a many explicit example of Frobenius codes which include the well studied Laflamme code.

We show that the classical notion of BCH distance can be generalised to all the Frobenius codes that we construct, including the non-linear ones, and show that the algorithm of Berlekamp can be generalised to correct quantum errors within the BCH limit. This gives, for the first time, a family of codes that are neither CSS nor linear for which efficient decoding algorithm exits.

The explicit examples that we construct are summarised in Table I and explained in detail in Tables III (linear case) and IV (non-linear case).

I. INTRODUCTION

Successful implementation of quantum computing requires handling errors that occur while processing, storing and communicating quantum information. Good quantum error correcting codes are therefore a key technology in the eventual building of quantum computing devices, besides, perhaps more importantly, their theory provide some elegant mathematics. An important class of codes are the stabiliser codes [8], which not only captured the isolated examples constructed earlier [13], [14], [4], [11], but built a solid foundation for subsequent works [9, 5, 2].

Constructing stabiliser codes require handling the slightly non-standard symplectic inner product. The CSS construction [7], [15] gives one elegant and natural way, albeit with some loss of generality, to handle this difficulty. For this one needs a self-dual code classical code, or more generally two classical codes one contained in the dual of the other, thereby reusing the intuition built for classical codes. Another approach to the problem, again with some loss of generality, is to look at linear stabiliser codes [6]. Linear stabiliser codes can also be characterised as linear classical codes over a quadratic extension of the base field [6, Theorem 3] [9, Lemma 18] which are Hermitian self-dual.

In this article, we study mainly cyclic stabiliser codes. Cyclic codes, being well studied classically, have recently been studied in detail in [6], [16], [1], [9], mostly from the perspective of either self dual codes or Hermitian self dual codes. We explore another approach to simplify the symplectic condition, namely, we restrict our attention of cyclic codes of length dividing $p^t + 1$ over $\mathbb{F}_p$.

Our contribution: In this article, we focus on cyclic stabiliser codes over the field $\mathbb{F}_p$, whose lengths divide $p^t + 1$, for some positive integer $t$. We call such codes $t$-Frobenius codes, or just Frobenius codes, because of the key role played by the Frobenius automorphism. Restricting to such lengths, while constraining, is not that bad, as there is a healthy, i.e. almost linear, density of such lengths. In bargain, we get a simpler formulation of the isotropy condition, which helps in the analysis of these codes considerably. Furthermore, this simplicity of the isotropic condition allows us to extend the notion of BCH distance for these codes and give efficient decoding algorithms. Since none of the codes that we construct are CSS — all our codes are uniquely cyclic (See Section III for a definition) and by Proposition III.5 are not CSS — and some of them are non-linear, this gives a family of codes for which efficient decoding algorithms were not known before.

We study the subfamily of linear Frobenius codes in detail and completely characterise them (Theorems IV.3 and IV.6). This has two consequence, one negative and another positive. Firstly, over $\mathbb{F}_p$, we show that there are no $t$-Frobenius linear codes when $t$ is odd (Corollary IV.5). This is a somewhat serious limitation of linear cyclic codes as the density of such lengths $n$ seems to be almost linear. Moreover, this impossibility is purely Galois theoretic unlike other known restriction that arise from sphere packing bounds or linear programming bounds.

On the positive side, the characterisation of linear Frobenius codes gives us ways to explicitly construct examples of linear Frobenius codes of lengths $p^{2t} + 1$. Again, since the density of such lengths are also healthy, this technique give sizable number of explicit examples including the well studied Laflamme code. Table II give such examples for $p = 2$ and lengths less than 100.
II. Preliminaries

We give a brief overview of the notation used in this paper. For a prime power $q = p^k$, $\mathbb{F}_q$ denotes the unique finite field of cardinality $q$. The product $\mathbb{F}_p^n$ is a vector space over the finite field $\mathbb{F}_p$ and an element $a = (a_1, \ldots, a_n)^T$ in it is thought of as a column vector. Fix a $p$-dimensional Hilbert space $H$. An orthonormal basis for $H$ is of cardinality $p$. Fix one such basis and denote it by $\{ |a\rangle | a \in \mathbb{F}_p^n \}$. As is standard in quantum computing, for an element $a = (a_1, \ldots, a_n)^T$ in $\mathbb{F}_p^n$, $|a\rangle$ denotes the tensor product $|a_1\rangle \otimes \cdots \otimes |a_n\rangle$. The set $\{ |a\rangle | a \in \mathbb{F}_p^n \}$ forms a basis for the $n$-fold tensor product $H^{\otimes n}$. A quantum code over $\mathbb{F}_p$ of length $n$ is a subspace of the tensor product $H^{\otimes n}$. There is by now a significant literature on quantum codes [10], [8], [9].

Let $\zeta$ denote the primitive $p$-th root of unity $\exp \frac{2\pi i}{p}$. For $a$ and $b$ in $\mathbb{F}_p^n$, define the operators $U_a$ and $V_b$ on $H^{\otimes n}$ as $U_a(x) = |x + a\rangle$ and $V_b(x) = \zeta^{b^T x} |x\rangle$ respectively. The operator $U_a$ can be thought of as a position error and $V_b$ as a phase error. In a quantum channel, both position errors and phase errors can occur simultaneously. These are captured by the Weyl operators $U_a V_b$.

For elements $a$ and $b$ of the vector space $\mathbb{F}_p^n$ the joint weight $w(a, b)$ is the number of positions $i$ such that either $a_i$ or $b_i$ is not zero. The weight of the Weyl operator $U_a V_b$ is the joint weight $w(a, b)$. Occurrence of a quantum error at some $t$ positions is modelled as the channel applying an unknown Weyl operator $U_a V_b$ of weight $t$ on the transmitted message.

An important class of quantum codes are the class of stabiliser codes [8]. One can study stabiliser codes by studying the isotropic sets under the symplectic inner product. For any two vectors $u = (a, b)$ and $v = (c, d)$ of $\mathbb{F}_p^n \times \mathbb{F}_p^n$, define the symplectic inner product $\langle u, v \rangle$ as the scalar $a^T d - b^T c$ of $\mathbb{F}_p$. A subset $S$ of $\mathbb{F}_p^{2n}$ is called totally isotropic [9], or just isotropic, if for any two elements $u$ and $v$ of $S$, $\langle u, v \rangle = 0$.

Isotropic subspaces of $\mathbb{F}_p^{2n}$ are closely related to stabiliser codes. Calderbank et al. [5], [6] were the first to study this relation when the underlying field is $\mathbb{F}_2$. Later, this was generalised to arbitrary fields [3], [2]. We summarise these results in a form convenient for our purposes.

**Theorem II.1** ([5], [3], [2]). Let $S$ be an isotropic subspace of $\mathbb{F}_p^{2n}$ for some positive integer $n$. Let $\omega$ be either the primitive $p$-th root of unity $\exp \frac{2\pi i}{p}$ or $\sqrt{-1}$, depending on whether $p$ is odd or even respectively. Then, the subset $S = \{ \omega^{n a^T b} U_a V_b | (a, b) \in S \}$ of unitary operators forms an Abelian group. Furthermore, the set of vectors invariant under the operators in $S$ forms a quantum stabiliser code and the operator $P = \sum_{U \in S} U$ is the projection to it.

Let $S$ be a subspace of $\mathbb{F}_p^{2n}$. By the centraliser of $S$, denoted by $\mathcal{C}(S)$, we mean the subspace of all $u$ in $\mathbb{F}_p^{2n}$, such that $(u, v) = 0$, for all $v$ in $S$. We have the following theorem on the error correcting properties of the stabiliser codes.

**Theorem II.2** ([5], [3], [2]). Let $S$ be an isotropic subspace of $\mathbb{F}_p^{2n}$ and let $C$ be the associated stabiliser code. Then the dimension of the subspace $S$ is at most $n$. If $S$ has dimension $n - k$ for some $k > 0$ then the centraliser $\mathcal{C}(S)$ as a vector space over $\mathbb{F}_p$, is of dimension $n + k$ and the code $C$, as a Hilbert space, is of dimension $p^k$. Furthermore, if the minimum weight $\min \{ w(u) | u \in S \}$ is $d$ then $C$ can detect up to $d - 1$ errors and correct up to $\left\lfloor \frac{d - 1}{2} \right\rfloor$ errors.

Let $C$ be a stabiliser code associated with an $n - k$ dimensionally totally isotropic subspace $S$ of $\mathbb{F}_p^{2n}$. By the stabiliser dimension of $C$ we mean the integer $k$. Similarly, we call the weight $\min \{ w(u) | u \in S \}$ the distance of $C$. In this context, recall that the stabiliser code associated to the isotropic set $S$ is called $\delta$-pure, if the minimum of the joint weights of non-zero elements of the centraliser $\mathcal{C}(S)$ is $\delta$. It follows from Theorem II.1 that a $\delta$-pure code is of distance at least $\delta$. A stabiliser code over $\mathbb{F}_p$ of length $n$, stabiliser dimension $k$ and distance $\delta$ is called an $[n, k, \delta]_p$ code.

III. Quantum Cyclic codes

In this section we define quantum cyclic codes and study some of its properties. Fix a prime $p$ and a positive integer $n$ coprime to $p$ for the rest of the section. Let $N$ denote the right shift operator over $\mathbb{F}_p^n$, i.e. the operator that maps $u = (u_1, \ldots, u_n)$ to $(u_n, u_1, \ldots, u_{n-1})$. Consider the unitary operator $\mathcal{N}$ defined as $\mathcal{N}(u) = |\bar{u}\rangle$. Recall that a classical code over $\mathbb{F}_p$ is cyclic if for all code words $u$, its right shift $\mathcal{N} u$ is also a code word. Motivated by this definition, we have the following definition for quantum cyclic codes.

**Definition III.1.** A quantum code $C$ is cyclic if for any vector $|\psi\rangle$ in $C$, the vector $\mathcal{N}|\psi\rangle$ is in $C$.

Let $S$ be a subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$. We say that $S$ is simultaneously cyclic if for all $(a, b)$ in $S$, $(\mathcal{N} a, \mathcal{N} b)$ is also in $S$. Stabiliser codes with simultaneously cyclic isotropic sets were first studied by Calderbank et al. [6] Section 5] and was taken as the definition of cyclic codes in subsequent works [10], [11], [9]. In this context, we show that for stabiliser codes, simultaneous cyclicity and our definition of cyclicity coincide.

**Proposition III.2.** An isotropic subset of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ is simultaneously cyclic if and only if the associated stabiliser code is cyclic.

**Proof:** For a code $C$ with projection operator $P$, it is easy to verify that $C$ is cyclic if and only if $\mathcal{N}^T P \mathcal{N} = P$. Let $S$ be an isotropic subset of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ and let $C$ be the associated stabiliser code.

Assume that $C$ is cyclic. From Theorem II.1 the projection operator to $C$ is given by $P = \sum_{(a,b) \in S} \alpha_{a,b} U_a V_b$, where $\alpha_{a,b} = \omega^{a^T b}$. Notice that $\mathcal{N}^T U_a V_b \mathcal{N} = \mathcal{N} U_a V_b \mathcal{N}$. Therefore, the set $S$ should be simultaneously cyclic, otherwise the support of $\mathcal{N}^T P \mathcal{N}$ will not match with that of $P$.

Conversely, if $S$ is simultaneously cyclic, then we have $(\mathcal{N} a, \mathcal{N} b)$ is in $S$ for all $(a, b)$ in $S$. The inverse of the shift operation $\mathcal{N}$ is just $\mathcal{N}^T$. Therefore, $a^T \mathcal{N}^T \mathcal{N} b = a^T b$ and
hence the scalars $\alpha_{a,b}$ are also preserved. Thus, $N^{t}PN = P$ and as a result, $C$ is cyclic.

Let $R$ denote the cyclotomic ring $\mathbb{F}_p[X]/X^\ell - 1$ of polynomials modulo $X^\ell - 1$. When dealing with cyclic codes, it is often convenient to think of vectors of $\mathbb{F}_p^n$ as polynomials in $R$ by identifying the vector $a = (a_0, \ldots, a_{n-1})$ with the polynomials $a(X) = a_0 + \ldots + a_{n-1}X^{n-1}$. We use the bold face Latin letter, for example $a$, b etc., to denote vectors and the corresponding plain face letter, $a(X)$, $b(X)$ respectively, for the associated polynomial. Recall that, classical cyclic codes are ideals of this ring $R$. In the ring $R$, the polynomial $X$ has a multiplicative inverse namely $X^{-1}$. Often, we write $X^{-1}$ to denote this inverse. Notice that for any two vectors $a$ and $b$ in $\mathbb{F}_p^n$, if $a(X)$ and $b(X)$ denote the corresponding polynomials in $R$, then the coefficient of $X^k$ in the product $a(X)b(X^{-1}) \mod X^n - 1$ is the inner product $a^T X^{-k} b$, where $N$ is the right shift operator. An immediate consequence is the following.

**Proposition III.3.** Let $S$ be a simultaneously cyclic subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$. Then $S$ is isotropic if and only if for any two elements $u = (a, b)$ and $v = (c, d)$, the corresponding polynomials satisfy the condition

$$b(X)c(X^{-1}) - a(X)d(X^{-1}) = 0 \mod X^n - 1.$$

Let $S$ be a simultaneously cyclic subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$. Define $A$ and $B$ to be the projections of $S$ onto the first and last $n$ coordinates respectively, i.e. $A = \{a(a, b) \in S\}$ and $B = \{b(a, b) \in S\}$. Since $S$ is simultaneously cyclic, $A$ and $B$ are cyclic subspaces of $\mathbb{F}_p^n$ and hence are ideals of the ring $R$. Let $g(X)$ be the factor of $X^n - 1$ that generates $A$. Since $g(X)$ is an element of $R$, there exists a polynomial $f(X)$ in $R$ such that $(g, f) \in S$. If this $f$ is unique then we say that $S$ is uniquely cyclic and call the pair $(g(X), f(X))$ of polynomials, a generating pair for $S$. We have the following proposition.

**Proposition III.4.** A simultaneously cyclic subspace $S$ of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ is uniquely cyclic if and only if for every element $(0, a)$ in $S$, $a = 0$. If $S$ is uniquely cyclic generated by the pair $(g, f)$, then every element of $S$ is of the form $(ag, af)$ for some $a(X)$ in $\mathbb{F}_p[X]/X^n - 1$.

For a CSS code, the underlying isotropic set $S$ is a product $C_1 \times C_2$ of two $n$-length classical codes over $\mathbb{F}_p$. In particular, elements $(a, 0)$ and $(0, b)$ for $a$ and $b$ in $C_1$ and $C_2$ respectively belong to $S$. Therefore, we have the following proposition as a consequence of Proposition III.4.

**Proposition III.5.** Any uniquely cyclic stabiliser code is not CSS unless it is of distance 1.

For uniquely cyclic codes the isotropy condition in Proposition III.5 can be simplified as follows.

**Proposition III.6.** Let $S$ be a simultaneously cyclic subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ with generating pair $(g, f)$. Then $S$ is isotropic if and only if $g(X)f(X^{-1}) = g(X^{-1})f(X)$ modulo $X^n - 1$. Moreover, any pair $(a, b)$ belongs to $S$ if and only if $g(X)b(X^{-1}) = a(X^{-1})f(X)$ modulo $X^n - 1$.

Consider a quadratic extension $\mathbb{F}_{p^2} = \mathbb{F}_p(\eta)$ of $\mathbb{F}_p$ obtained by adjoining a root $\eta$ of some quadratic irreducible polynomial over $\mathbb{F}_p$. Identify the product $\mathbb{F}_p^n \times \mathbb{F}_p^n$ with the the vector space $\mathbb{F}_{p^2}^n$ by mapping a pair of vectors $(a, b)$ to the vector $a + \eta b$. Similarly for the cyclotomic ring $\mathbb{R}$, identify the product ring $\mathbb{R} \times \mathbb{R}$ with the cyclotomic ring $\mathbb{R}(\eta) = \mathbb{F}_{p^2}[X]/X^n - 1$. Let $S$ be any isotropic subspace of $\mathbb{F}_p^n \times \mathbb{F}_p^n$. The associated stabiliser code $C_S$ is said to be linear if $S$ under the above identification is a subspace of $\mathbb{F}_{p^2}^n$. Isotropic subspaces of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ associated to linear stabiliser codes are classical cyclic codes of length $n$ over $\mathbb{F}_p(\eta)$. Thus the following proposition follows.

**Proposition III.7.** Let $S$ be an isotropic simultaneously cyclic subspace of the product $\mathbb{F}_p^n \times \mathbb{F}_p^n$. The associated stabiliser code $C_S$ is linear if and only if $S$ is an ideal of the cyclotomic ring $\mathbb{F}_{p^2}[X]/X^n - 1$. Furthermore, if $C_S$ is linear then the centraliser $S$ is also an ideal of $\mathbb{F}_{p^2}[X]/X^n - 1$.

It follows from the theory of classical codes that both $S$ and $\overline{S}$ are ideals generated by factors of $X^n - 1$ over $\mathbb{F}_{p^2}$. In this context, we make the following definition.

**Definition III.8 (BCH distance).** Let $g$ be a factor of the polynomial $X^n - 1$ over the field $\mathbb{F}_q$, $\eta$ be any linear cyclic stabiliser code of length $n$ over $\mathbb{F}_q$. The BCH distance of the polynomial $g(X)$ is the largest integer $d$ such that the consecutive distinct powers $\beta^t, \beta^{t+1}, \ldots, \beta^{t+d-2}$ are roots of $g$, for some primitive $n$-th root $\beta$.

Recall that, the distance of a classical cyclic code is at least the BCH distance of its generating polynomial. In the setting of stabiliser codes, the distance is related to the minimum joint weight of elements of $\overline{S}$ (Theorem II.2). Motivated by this analogy, we define the BCH distance of linear stabiliser codes as follows.

**Definition III.9.** Let $S$ be an isotropic subset of $\mathbb{F}_p^n \times \mathbb{F}_p^n$ associated to a linear cyclic stabiliser code $C$. The BCH distance of $C$ is the BCH distance of the generator polynomial of the centraliser $\overline{S}$.

We have the following theorem which follows from Theorem II.2.

**Theorem III.10.** Let $C$ be any linear cyclic stabiliser code of BCH distance $d$. Then it is $d$-pure and hence has distance at least $d$.

**IV. LINEAR CYCLIC CODES OF LENGTH DIVIDING $p^t + 1$**

In this section, we study linear cyclic stabiliser codes over $\mathbb{F}_p$ whose length divides $p^t + 1$. The main motivation to restrict our attention to lengths of this form is captured in the following proposition.

**Proposition IV.1.** If the integer $n$ divides $p^t + 1$, for some positive integer $t$ then $X^n - 1$ in the cyclotomic ring $\mathbb{F}_p[X]/X^{p^t + 1} - 1$ is $X^{p^t}$. Therefore, for every polynomial $g(X)$ over any extension of $\mathbb{F}_p$ we have $g(X^{-1})$ is $g(X)^{p^t}$.

The above-mentioned property simplifies the isotropy condition for polynomials considerably and allows us to completely
characterise all linear cyclic codes of such lengths.

Let $\mathbb{F}_p(\eta)/\mathbb{F}_p$ be an extension of degree $d$. When dealing with cyclic quantum codes of length $n$, we use $R$ to denote the cyclotomic ring $\mathbb{F}_p[\eta]/X^n - 1$. The extension ring $R(\eta)$ is then the cyclotomic ring $\mathbb{F}_p(\eta)[X]/X^n - 1$. Linear codes are associated with quadratic extension and identification of the pair of vectors $(a, b)$ with $a + \eta b$ maps its isotropic set to an ideal of $R(\eta)$.

**Lemma IV.2.** Let $S$ be the isotropic ideal associated to a linear cyclic stabiliser code over $\mathbb{F}_p$ of length dividing $p^d + 1$. Then $S$ is uniquely cyclic.

**Proof:** Let $\mathbb{F}_p(\eta)/\mathbb{F}_p$ be the quadratic extension such that $S$ is an ideal of the cyclotomic ring $\mathbb{F}_p(\eta)[X]/(X^n - 1)$. Let $X^2 + c_1 X + c_0 + 0$ be the minimal polynomial of $\eta$ over $\mathbb{F}_p$.

Recall that the projection of $S$ onto the first $n$ co-ordinates forms a classical cyclic code over $\mathbb{F}_p$, and hence is generated by a factor $f(X)$ of $X^n - 1$. Suppose there exist two distinct elements $(g, h)$ and $(g, f')$ in $S$. Let $h = f - f'$ so that $(0, h)$ is also in $S$. We prove that the polynomial $h(X) = 0$ modulo $X^n - 1$. By the Chinese remaindering theorem, it is sufficient to prove separately that all roots of $\frac{X^n - 1}{g}$ and $g$ are roots of $h$.

From Proposition [IV.1], we get, $g(X^{-1}) = g^p(X)$. Applying Proposition [III.3] to the elements $(0, h)$ and $(g, f)$ we have $g^p h = 0 \mod X^n - 1$. Since $g$ is invertible modulo $\frac{X^n - 1}{g}$, every root of $\frac{X^n - 1}{g}$ should also be a root of the polynomial $h(X)$.

We now show that every root of $g$ is also a root of $h$. Since $\eta h$ belongs to $S$, if the code is linear then $\eta h = -c_0 h - \eta c_1 h$ must also belong to $S$ where $\eta$ is a root of the quadratic polynomial $X^2 + c_1 X + c_0$. Any element in $S$ is of the form $ag + \eta h$, where $a(X)$ and $b(X)$ are polynomials in $\mathbb{F}_p[X]/X^n - 1$. Hence, $-c_0 h = ag$ and every root of $g$ is also a root of $h$.

Consider the Frobenius automorphism $\sigma$ on a degree $d$ extension $\mathbb{F}_p(\eta)/\mathbb{F}_p$, which maps any element $x$ in $\mathbb{F}_p(\eta)$ to $x^p$. This can be naturally extended to polynomials over $\mathbb{F}_p(\eta)$ and therefore on $R(\eta)$ as follows: For a polynomial $a(X) = a_0 + \ldots + a_n X^n$ where $a_i$ are $\in \mathbb{F}_p(\eta)$, $\sigma(a)$ is defined as $\sigma(a_0) + \ldots + \sigma(a_n) X^n$. We call this the Frobenius involution.

Constructing linear cyclic codes correspond to constructing generators for the associated isotropic ideal. We make use of the following Galois theoretic lemma to characterise such generators.

**Lemma IV.3.** Let the integer $n$ divide $p^d + 1$ for some positive integer $t$.

1. Any irreducible factor of $X^n - 1$ over $\mathbb{F}_p$ other than the factors $X - 1$ or $X + 1$ has even degree.

2. Let $f(X)$ be any irreducible factor of $X^n - 1$ over $\mathbb{F}_p$ whose degree is divisible by $d$ for some positive integer $d$. Over the extension field $\mathbb{F}_{p^d} = \mathbb{F}_p(\eta)$, $f(X)$ splits into $d$ irreducible factors $f_0(X, \eta), \ldots, f_{d-1}(X, \eta)$ such that $f_i = \sigma^i(f_0)$. 

**Proof of part 1:** Consider any irreducible factor $f(X)$ of $X^n - 1$ over $\mathbb{F}_p$, other than $X - 1$ or $X + 1$. Let $k$ be the degree of $f(X)$. Then, the splitting field of $f(X)$ over $\mathbb{F}_p$ is $\mathbb{F}_{p^k}$. Consider any root $\beta$ of $f(X)$ in $\mathbb{F}_{p^k}$. The Frobenius automorphism $\sigma^k$ is a field automorphism of $\mathbb{F}_{p^k}$ and $\sigma^k(\beta^k) = \beta^p$. Notice that $\beta$ is an $n$-th root of unity and $n$ divides $p^k + 1$. Hence $\sigma^k(\beta) = \beta^{-1}$ and $f(\beta^{-1}) = f^k(\beta) = 0$. Since $f(X)$ is neither $X - 1$ nor $X + 1$, we have $\beta \neq \pm 1$ and hence $\beta \neq \beta^{-1}$. As a result, the roots of $f(X)$ comes in pairs; for every root $\beta$ its inverse $\beta^{-1}$ is also a root. Hence, the degree $k$ of $f(X)$ should be an even number.

**Proof of part 2:** Consider any irreducible factor $f(X)$ of $X^n - 1$ of degree $k = dn$ for some positive integer $m$. Its splitting field $\mathbb{F}_{p^k} = \mathbb{F}_{p^{km}}$ therefore, contains $\mathbb{F}_{p^d}$. Any irreducible factor of $f(X)$ over $\mathbb{F}_{p^d}$ should be of degree equal to the degree of the extension $\mathbb{F}_{p^k}/\mathbb{F}_{p^d}$ which is $m$. The Frobenius $\sigma$ being a field automorphism of $\mathbb{F}_{p^k}$, should map these factors to each other. Further, order of $\sigma$ in $\mathbb{F}_{p^d}$ is $d$. Thus $f(X) = f_0(X, \eta) \cdot \ldots \cdot f_{d-1}(X, \eta)$ over $\mathbb{F}_{p^d}$.

Consider the extension field $\mathbb{F}(\eta) = \mathbb{F}_{p^d}$, and let $S$ be any ideal of $R(\eta)$. The following theorem gives a necessary condition for it to be isotropic and hence give a linear cyclic code.

**Theorem IV.4.** Let $\mathbb{F}_p(\eta)$ be a quadratic extension of $\mathbb{F}_p$. Let $n$ divide $p^d + 1$ and $S$ be an isotropic ideal of $\mathbb{F}_p(\eta)[X]/X^n - 1$. Then $t$ is even and the ideal $S$ is generated by the product polynomial $g(X) \cdot h(X, \eta)$ where $g(X)$ and $h(X, \eta)$ are two coprime factors of $X^n - 1$ satisfying the following condition.

1. $g(X)$ is any factor of $X^n - 1$ over $\mathbb{F}_p$ which contains both $X - 1$ and $X + 1$ as factors.

2. $h(X, \eta)$ is any factor of $\frac{X^n - 1}{g}$ over $\mathbb{F}_{p^d}$, such that for any irreducible factor $r(X, \eta)$ of $\frac{X^n - 1}{g}$ over $\mathbb{F}_{p^d}$, $r(X, \eta)$ divides $h(X, \eta)$ if and only if $\sigma(r) = r(X, \eta)$ does not.

**Proof:** From Lemma [IV.2] it follows that $S$ is uniquely cyclic. Let $(g, f)$ be a generating pair for $S$ where $g(X)$ and $f(X)$ are polynomials over $\mathbb{F}_p$. Then the polynomial $g(X) + \eta f(X)$ is an element of the ideal $S$. It follows from the linearity of $S$ that the polynomial $g(X^t + \eta f(X))$ is also in $S$. However, the set $S$ is uniquely cyclic. Using Proposition [III.3] there is a polynomial $a(X)$ in $\mathbb{F}_p[X]$ such that

$$\eta(g + \eta f) = a(g + \eta f) \mod X^n - 1 \quad (1)$$

Let $c(X) = X^2 + c_1 X + c_0$ be the minimal polynomial of $\eta$ over $\mathbb{F}_p$, where $c_0$ and $c_1$ are elements of $\mathbb{F}_p$. Comparing the coefficients of $\eta$ in Equation (1) we have

$$f = -\frac{a}{c_0} g \mod X^n - 1$$

and

$$c(a(X)) = 0 \mod \frac{X^n - 1}{g}. \quad (2)$$

When $n$ divides $p^d + 1$, for any polynomial $\gamma(X)$ in the cyclotomic ring $\mathbb{F}_{p^d}[X]/X^n - 1, \gamma(X^{-1})$ is just $\gamma^p(X)$. Since
$S$ is isotropic, from Equation 2 and Proposition III 3 it follows that $g^{p^t+1}a^{p^t} = g^{p^t+1}a \mod X^n - 1$. The polynomial $g(X)$ is invertible modulo $\frac{X^n - 1}{g}$. As a result we have,

$$a^{p^t} = a \mod \frac{X^n - 1}{g} \quad (4)$$

Let $r(X)$ be any irreducible factor of $\frac{X^n - 1}{g}$ over $\mathbb{F}_p$ and $\mathbb{K}$ be the extension field $\mathbb{F}_p[X]/r(X)$. From Equation 5 we have, $a \mod r$ is a root of the polynomial $c(Y)$ over the extension field $\mathbb{K}$. If possible, let $t$ be an odd integer $2m + 1$. Since $c(Y)$ divides $Y^{p^{2m}} - Y$, $a^{p^{2m}} = a \mod r$. Using Equation 4 we get $a^p = a$ modulo $r$ and hence $a \mod r$ is an element of the sub field $\mathbb{F}_p$ of $\mathbb{K}$. However, this is a contradiction, since the polynomial $c$ is irreducible over $\mathbb{F}_p$. Therefore, $t$ must be even.

Recall that $a \mod r$ is a root of the polynomial $c(Y)$ over the extension field $\mathbb{K}$. This implies that the extension field $\mathbb{K}$ contains $\mathbb{F}_p[Y]/c(Y) = \mathbb{F}_{p^2}$. Therefore, degree of $r$ must be even and $g$ must have as factors all the odd degree irreducible factors of $X^n - 1$ over $\mathbb{F}_p$. By Lemma IV 3 these odd degree factors are just $X - 1$ and $X + 1$. Thus $g$ satisfy property 1 of the theorem.

Consider the polynomial $h(X, \eta) = \gcd \left( \frac{X^n - 1}{g}, 1 - \frac{X^n}{C(a)} \right)$. Clearly $h$ is coprime to $g(X)$. We claim that $g \cdot h$ generates the ideal $S$. To see this, notice that $S$ as a subspace of $\mathbb{F}_{p^n} \times \mathbb{F}_{p^n}$ is uniquely cyclic and is generated by the pair $(g, f)$, where $f = -\frac{X^n}{C(a)}$ modulo $X^n - 1$ using Equation 2. Therefore, $S$ as an ideal is also generated by the polynomial $\gcd \left( \frac{X^n - 1}{g}, 1 - \frac{X^n}{C(a)} \right)$ which is the product $g \cdot h$. We claim that polynomial $h$ thus constructed satisfies the properties mentioned in the theorem. Any irreducible factor $r(X)$ of $\frac{X^n - 1}{g}$ over the field $\mathbb{F}_p$ is of even degree and hence factorises as $r(X, \eta)\sigma(r(X, \eta))$ over $\mathbb{F}_p(\eta)$. Recall that $a \mod r$ is a root of $c(Y)$. As a result, $a \mod r_1$ is either $\eta$ or $\eta'$. Now, $r_1$ divides $h$ if and only if $1 - \frac{X^n}{C(a)}$ modulo $r_1$ is zero. Therefore, $r_1$ divides $h$ if and only if $a = \eta' \mod r_1$. The polynomial $a$ has coefficients in $\mathbb{F}_p$ and hence $a = \sigma(a)$. As a result by the third property of Proposition IV 1 we have $a = \eta' \mod r_1$ if and only if $a = \sigma(\eta') = \eta \mod \sigma(r_1)$. For each pair $r_1$ and $\sigma(r_1)$, exactly one of them divide $h$ depending one whether $a$ is $\eta$ or $\eta'$ modulo $r_1$. This proves the theorem.

A corollary of the above theorem is the following impossibility result.

**Corollary IV 5.** Let $n$ be any integer that divides $p^t + 1$, where $t$ is odd. Then there does not exist any linear cyclic stabiliser codes of length $n$ over $\mathbb{F}_p$.

For example, 9, 11, 19, 27, 33, 43, 57, 59, 67, 81, 83, 99 are the numbers less then hundred that divide $2^t + 1$ for some odd $t$. Hence there is no binary linear cyclic code of such lengths.

The next theorem shows that the conditions in Theorem IV 4 are also sufficient to construct isotropic ideals of $\mathcal{R}(\eta)$. This gives us a way of constructing linear cyclic stabiliser of length dividing $p^{2m} + 1$. This theorem directly follows from a more generalised construction given in Theorem VI 1 and Theorem V 5.

**Theorem IV 6.** Let $n$ divide $p^{2m} + 1$ and $\mathbb{F}_p(\eta)$ be a quadratic extension of $\mathbb{F}_p$. Let $g(X)$ and $h(X, \eta)$ be factors of $X^n - 1$ satisfying the properties 1 and 2 of Theorem IV 4. Then the ideal $S$ of $\mathbb{F}_p(\eta)[X]/X^n - 1$ generated by the product $g \cdot h$ is isotropic as a subset of $\mathbb{F}_p \times \mathbb{F}_p$ and the associated stabiliser code is linear and cyclic.

In the rest of the article, we refer to cyclic stabiliser codes whose length divide $p^t + 1$ as $t$-Frobenius codes. For linear $2m$-Frobenius codes, we call the factorisation $g(X) \cdot h(X, \eta)$ characterised above as the canonical factorisation associated to the code.

**Theorem IV 7.** Let $C$ be a linear $2m$-Frobenius code over $\mathbb{F}_p$ with canonical factorisation $g \cdot h$. The stabiliser dimension of the code $C$ is $\deg(g)$. The centraliser $\mathcal{S}$ of $S$ is the ideal generated by $h(X, \eta)$ and hence the BCH distance of $C$ is $\text{BCH distance of } h$.

Again the proof follows from the more general theorem VI 7 and V 5.

**V. Generalisation to nonlinear codes**

We have already shown that if $n$ divides $p^t + 1$ for some odd integer $t$ then no linear code of length $n$ exists. In this section we show how to construct nonlinear codes of such length. The construction is a generalisation of Theorem IV 6. The major difference is that the extension of $\mathbb{F}_p$ is no longer restricted to be quadratic.

**Theorem V 1.** Let $n$ divide $p^{2m} + 1$ and $\mathbb{F}_p(\eta)$ be a degree $d$ extension of $\mathbb{F}_p$. Let $g(X)$ and $h(X, \eta)$ be co-prime factors of $X^n - 1$ satisfying the following properties.

1) $g(X)$ is any factor of $X^n - 1$ over $\mathbb{F}_p$ which contains all the irreducible factor of $X^n - 1$ over $\mathbb{F}_p$ whose degree is not divisible by $d$.

2) $h(X, \eta)$ is any factor of $\frac{X^n - 1}{g}$ over $\mathbb{F}_p(\eta)$ such that for any irreducible factor $r(X, \eta)$ of $\frac{X^n - 1}{g}$ over $\mathbb{F}_p(\eta)$, $r(X, \eta)$ divides $h(X, \eta)$ if and only if none of the factors $\sigma(r), \ldots, \sigma^{d-1}(r)$ divide $h$ i.e. $\frac{X^n - 1}{\sigma(\eta)} = \prod_{i=0}^{d-1} \sigma^i(h)$.

Fix any nonzero $\alpha$ in $\mathbb{F}_p$ and let $a(X, \eta)$ be the polynomial, uniquely defined by Chinese remaindering, as follows.

$$a = \begin{cases} 1 \mod g \\ \sigma^i(a\eta) \mod \sigma^i(h) \text{ for all } 0 \leq i < d \end{cases}$$

Then $a(X, \eta)$ is a polynomial in $\mathbb{F}_p[X]$ and the uniquely cyclic subspace generated by $(g, ag)$ is isotropic.

The proof of this theorem, involves verifying certain equations modulo $X^n - 1$. Almost always we do this in by verifying the said equation separately modulo $g$ and each irreducible factor $r(X, \eta)$ of $\frac{X^n - 1}{g}$. Then by Chinese remaingdering, we have the said equation modulo $X^n - 1$. We call this the Chinese remainder verification.
Let us call the polynomial $\frac{X^n - 1}{g(X)}$ as $f(X)$. From the definition of the factor $h(X, \eta)$, it follows that over $\mathbb{F}_p[\eta]$, the polynomial $f(X)$ splits as $h \cdot \sigma(h) \cdots \sigma^{d-1}(h)$.

**Claim V.2.** The polynomial $a(X, \eta)$ is a polynomial in $\mathbb{F}_p[X]$.

**Proof:** It is sufficient to prove that $\sigma(a) = a$. Notice that since $a = 1 \mod g$, $\sigma(a) = a \mod g$. Using Proposition [IV.1] it is sufficient to show that $\sigma(a) = a \mod f$. Since $\sigma^d(h) = h$ and $a = \sigma^i(\alpha \eta)$, $\sigma^i(h)$ for all $i$, applying $\sigma$ on both side we get $\sigma(a) = \sigma^i(\alpha \eta) \mod \sigma^i(h)$ for all $i$. By Chinese remainder verification the claim follows.

**Claim V.3.** $a(X) = a(X^{-1}) \mod f(X)$

**Proof:** From Proposition [IV.1] we have, $a(X^{-1}) = \sigma^d(a(X)) \mod X^n - 1$. We know, $a = \sigma^i(\alpha \eta) \mod \sigma^i(h)$.

Since $\sigma^i(\alpha \eta) = \alpha \eta$, applying $\sigma^i$ on $a$ we get $\sigma^i(a) = \sigma^i(\alpha \eta) \mod \sigma^i(h)$. Hence by Chinese remaindering, $\sigma^i(a) = a \mod f$.

**Claim V.4.** The uniquely cyclic subspace generated by the pair $(g, \eta)$ is isotropic.

**Proof:** Using Proposition [III.6] it is sufficient to prove that $g(X)a(X^{-1})g(X^{-1}) - g(X^{-1})a(X)g(X) = 0 \mod X^n - 1$. (5)

Clearly Equation (5) holds modulo $g(X)$ as both the terms are divisible by $g$. From Proposition [IV.1] and Equation (5) we have, $a(X^{-1}) = a \mod \frac{X^n - 1}{g}$. We then apply Chinese remaindering and conclude that the subspace is isotropic.

The following theorem shows that the linear codes obtained from Theorem [V.6] are indeed a subclass of the codes generated from Theorem [V.1]

**Theorem V.5.** Let $c(X) = X^2 + c_1 X + c_0$ be an irreducible polynomial over $\mathbb{F}_p$ and $\eta, \eta'$ be roots of $c(X)$. Fix $d = 2$, $\mathbb{F}_p[\eta']/\mathbb{F}_p$ to be the extension and $a = -c_0^{-1}$ in Theorem [V.7] and let $S$ be the corresponding isotropic subspace. Then the image of $S$ under the map $(u, v) \rightarrow u + \eta v$ is an ideal of the cyclotomic ring $\mathbb{F}_p[\eta]/(X^n - 1)$ and its generator is given by the polynomial $g(X)h(X, \eta)$ where $g, h$ satisfies the properties in Theorem [IV.4]. Moreover the centraliser $S$ also maps to the ideal generated by $h$.

**Proof:** We know from Theorem [V.1] that the polynomial $a(X, \eta)$ defined by the following

$$a = \sigma^i(-c_0^{-1} \eta') \mod \sigma^i(h) \text{ where } i \in \{0, 1\} \quad (6)$$

belongs to the ring $\mathbb{F}_p[X]/g(X)$ and the uniquely cyclic subspace $S$ generated by the pair $(g, \eta)$ is isotropic.

To prove that $S$ maps to an ideal it is sufficient to show that for any element $(ug, vag)$ in $S$ there exists another element $(vg, vag)$ in $S$ such that $\eta(ug + vag) = v + vag \mod X^n - 1$. We claim that we can always choose $v = c_0 a u$ to satisfy this condition.

**Claim V.6.** $\eta(g + vag) = c_0 a(g + vag) \mod X^n - 1$

**Proof:** The cyclotomic polynomial $X^n - 1$ is product of $g, h$ and $\sigma(h)$. Using equation (6) and the fact that $c_0 = \eta \eta'$, $-\eta' = c_1 \eta + c_0$ it is straight-forward to verify the claim separately modulo $g(X)$, $h$ and $\sigma(h)$. Then by Chinese remaindering conclude that it is true modulo $X^n - 1$. Equations (6) implies that for any irreducible factor $r(X, \eta)$ of $\frac{X^n - 1}{g}$ over $\mathbb{F}_p[\eta]$

$$a \mod r = \begin{cases} -\eta' - 1 \quad \text{if } r|h \\ -c_0^{-1} \eta \quad \text{if } r|\sigma(h) \end{cases} \quad (7)$$

Now $\tilde{g}(X, \eta) = (g + \eta a g)$ is a generator of $S$ as an ideal and equation (2) implies that $g$ $\mod r$ is zero and if $r|\sigma(h)$ then $\tilde{g}$ mod $r$ is $\frac{\tilde{g}}{c_0} + 1$ which is nonzero. Therefore $\gcd(X^n - 1, \tilde{g}) = gh$ and $gh$ is also a generator of $S$.

Let $I$ be the ideal generated by $1 + \eta a$. Since $\gcd(X^n - 1, 1 + \eta a) = h$, the ideal $I$ is also generated by $h$. We know that $a(X) = a(X^{-1}) \mod \frac{X^n - 1}{g}$ (see proof of Claim V.2).

**Proposition V.3.** It can be verified that any element of the form $(u, a u)$ belongs to the centraliser. Since $S$ itself is an ideal, $I$ is a subideal of $S$. To show that $I$ is actually $S$ we show that they have same cardinality. The cardinality of $I$ is $(p^d)^n - \deg(g)$ which is equal to $p^{n + \deg(g)}$, since $\deg(g) + 2 \deg(h) = n$. On the other hand cardinality of $S$ is $p^n - \deg(g)$. Hence by Theorem [IV.2] cardinality of $S$ is $p^n - \deg(g)$.

As before, we call $g \cdot h$ as the canonical factorisation associated with the above mentioned t-Frobenius codes. We also call the BCH distance of $h$ to be the BCH distance of $C$.

**Theorem V.7.** Let $g(X) \cdot h(X, \eta)$ be the canonical factorisation associated with a t-Frobenius code $C$ as in Theorem [V.7]. The stabiliser dimension of $C$ is $\deg(g)$. If the BCH distance of $h$ is $\delta$ then $C$ is $\delta$-pure and hence has distance at least $\delta$.

**Proof:** Let $a(X)$ be the polynomial corresponding to $C$ such that the isotropic subspace $S$ of $C$ is generated by the pair $(g, \eta g)$ as in Theorem [V.1]. Since any element in $S$ is of the form $(ug, uag)$, the number of distinct values $u$ can take is the cardinality of the ring $\mathbb{F}_p[X]/g(X)$ which is $p^{n - \deg(g)}$. Hence by Theorem [IV.2] we conclude that the stabiliser dimension of $C$ is $\deg(g)$.

To prove the lower bound on distance we first need the following result.

**Claim V.8.** Any element in the centraliser $S$ is of the form $(u, au + u \frac{X^n - 1}{g})$ for some polynomials $u(X)$ and $v(X)$ over $\mathbb{F}_p$ such that $g(X)$ and $v(X)$ are coprime.

**Proof:** Let $A$ be the set of all pairs $(u, au + u \frac{X^n - 1}{g})$ where $u(X)$ and $v(X)$ are polynomials over $\mathbb{F}_p$ such that $v(X)$ is coprime to $g(X)$. It follows from Proposition [III.6] that the set $A$ is contained in $S$. However, the cardinality of $A$ is the product of the cardinalities of the rings $\mathbb{F}_p[X]/(X^n - 1)$ and $\mathbb{F}_p[X]/g(X)$ which is $p^n - \deg(g)$. By Theorem [IV.2] cardinality of $S$ itself is $p^n - \deg(g)$. Hence $S$ is equal to the set $A$.

Notice that the joint weight of a pair $(u'(X), v'(X))$ is equal to the weight of $uv' - u' v$ as a polynomial over $\mathbb{F}_p(\eta)$. We know, for any factor $r(X, \eta)$ of $X^n - 1$, weight of any
polynomial over \( \mathbb{F}_p(\eta) \) which is a multiple of \( r \), is at least the BCH distance of \( r \). Therefore, to prove that \( \mathcal{C} \) is \( \delta \)-pure it is sufficient to show that for any element \((u', v')\) in the centraliser \( \mathcal{S} \) the polynomial \( h(X, \eta) \) is a factor of \( \alpha u' - v' \).

By Claim \( \square \) we know that there exists a polynomials \( v \) such that \( v' = a u' + v = \alpha^{-1} u' \). Since \( h \) divide \( \alpha^{-1} u' \), it follows that \( \alpha u' - v' \mod h \) if 0.

As a demonstration of our construction we list (Table I) some explicit examples of codes where the characteristic \( p \) of the underlying finite field is 2. The distance given in this table is the BCH distance. The actual distance can be larger. Canonical factors and their roots are given in the appendix. We have both linear and non-linear codes for parameters with } dagger whereas star denotes only nonlinear codes.

| Length | Parameters |
|--------|------------|
| 5      | [5,1,3]*   |
| 9      | [9,3,2]*   |
| 13     | [13,1,3]   |
| 17     | [17,1,1],[17,9,3] |
| 19     | [19,1,3]*   |
| 25     | [25,5,1],[25,5,3] |
| 27     | [27,21,2],[27,9,3]* |
| 29     | [29,1,3]   |
| 37     | [37,1,2]   |
| 41     | [41,1,4],[41,21,4] |
| 53     | [53,1,3]   |
| 57     | [57,21,3],[57,39,3]* |
| 61     | [61,1,7]*   |
| 65     | [65,5,13],[65,13,8],[65,17,9],[65,17,11]* , [65,29,7],[65,41,5],[65,53,3]* |
| 67     | [67,1,7]*   |
| 81     | [81,21,4],[81,75,2]* |
| 97     | [97,1,9],[97,49,3] |
| 99     | [99,99,3]   |

**Table I**

**Explicit examples of Frobenius codes over \( \mathbb{F}_2 \)**

Let \( \mathcal{C} \) be a \( t \)-Frobenius code based on a degree \( d \) extension \( \mathbb{F}_p(\eta) \) as in Theorem \( \square \). Let the code \( \mathcal{C} \) have length \( n \) and BCH distance \( \delta = 2t + 1 \). Much like in the classical case, we show that there is an \( \mathbb{F}_2 \) time quantum algorithm to correct any quantum error of weight at most \( \tau \). We use two key algorithms: (1) Kitaev’s phase estimation \( \square \) algorithm and (2) The Berlekamp decoding algorithm \( \square \) for classical BCH codes.

**Theorem VI.1** (Berlekamp). Let \( h(X) \) be a factor of \( X^n - 1 \) of BCH distance \( \delta = 2t + 1 \) over a finite field \( \mathbb{F}_q \), and \( n \) coprime. Let \( e(X) \) be any polynomial of weight at most \( \tau \) over \( \mathbb{F}_q \). Given a polynomial \( r(X) = e(X) \mod h(X) \), there is a polynomial time algorithm to find \( e(X) \).

Let the canonical factorisation of the \( \mathcal{C} \) be \( g \cdot h \) so that its isotropic subspace is generated by the pair \((g, a g)\) where \( a = \sigma^t(a \eta) \mod \sigma^{\gamma}(h) \). Assume that we transmitted a quantum message \( |\varphi\rangle \) over the quantum channel and received the corrupted state \( |\psi\rangle = U_u V_v |\varphi\rangle \), where the vectors \( u \) and \( v \) are unknown but fixed for the rest of the section. We show that using quantum phase finding we can recover the polynomial \( \alpha u(X^{-1}) - v(X^{-1}) \mod h \) without disturbing \( |\psi\rangle \). Provided the joint weight \( w(u, v) \leq \tau \) we can now find \( u \) and \( v \) using Berlekamp algorithm. The sent message is recovered by applying the inverse map \( V_v^T U_u^T \) on \( |\psi\rangle \). Hence we have the following theorem about decoding linear \( t \)-Frobenius codes.

**Theorem VI.2.** Let \( \mathcal{C} \) be a \( t \)-Frobenius code, as in Theorem \( \square \) of length \( n \) and BCH distance \( \delta = 2t + 1 \). There is an efficient quantum algorithm which, given the state \( |\psi\rangle \) and any element \((a, b)\) in \( \mathcal{C} \), computes the polynomial \( b(X) u(X^{-1}) - a(X) v(X^{-1}) \mod X^n - 1 \) without destroying \( |\psi\rangle \).

**Proof:** Let \( S \) be the isotropic subspace corresponding to \( \mathcal{C} \). We need the following lemma to prove the theorem.

**Lemma VI.3.** Let \( |\varphi\rangle \) be a codeword in \( \mathcal{C} \) and \( |\psi\rangle = U_u V_v |\varphi\rangle \), where \( u \) and \( v \) are any vectors in \( \mathbb{F}_p^n \). There is an efficient quantum algorithm which, given the state \( |\psi\rangle \) and any element \((a, b)\) in \( \mathcal{C} \), computes the polynomial \( b(X) u(X^{-1}) - a(X) v(X^{-1}) \mod X^n - 1 \) without destroying \( |\psi\rangle \).

**Proof:** Recall that for every \((a, b)\) in \( \mathcal{C} \) and \( |\varphi\rangle \) in \( \mathcal{C} \), the operator \( W_{a,b} = \omega^a b^T U_u V_v \) stabilises \( |\varphi\rangle \) (Theorem \( \square \)). Suppose the received vector is \( |\psi\rangle = U_u V_v |\varphi\rangle \). It is easy to verify that the vector \( |\psi\rangle \) is an eigen vector of the operator \( W_{a,b} \), and the associated eigenvalue is \( \zeta^b u - a^T v \) where \( \zeta \) is the primitive \( p \)-th root of unity. One can recover this phase without disturbing \( |\psi\rangle \) using quantum phase finding. Repeating the algorithm with \((N^2 a, N^2 b)\), all the inner products \( b^T N^2 k u - a^T N^2 v \) can be recovered. These are precisely the coefficients the polynomial \( b(X) u(X^{-1}) - b(X) v(X^{-1}) \mod X^n - 1 \). Hence proved.

Let \( \mathcal{C} \) be based on a degree \( d \) extension \( \mathbb{F}_p(\eta) \) as in Theorem \( \square \). Let the canonical factorisation associated with \( \mathcal{C} \) be \( g(X) \cdot h(X, \eta) \). The isotropic subspace \( S \) of \( \mathcal{C} \) is generated by the pair \((g, a g)\) where \( a = \sigma^t(a \eta) \mod \sigma^{\gamma}(h) \). Using Lemma \( \square \), we can compute the polynomial \( e'(X) = a(X) g(X) u(X^{-1}) - g(X) v(X^{-1}) \mod X^n - 1 \). The factor \( g(X) \) has an inverse \( g^{-1}(X) \) in the ring \( \mathbb{F}_p[X]/X^n - 1 \) i.e. \( g g^{-1} = 1 \mod \frac{X^n - 1}{\gamma} \). Hence the joint weight \( |\psi\rangle \) is same as the weight of the polynomial \( e(X, \eta) \). Since the BCH distance of \( h \) is \( 2t + 1 \), if \( w \) is at most \( \tau \) then using Berlekamp algorithm (Theorem \( \square \)) we can compute the polynomial \( e \) and therefore the vectors \( u \) and \( v \). Applying \( V_v^T U_u^T \) on \( |\psi\rangle \) we recover the original codeword.

**VII. Density of numbers that divide \( p^t + 1 \)**

Let \( f_p(x) \) denote the number of positive integers less than \( x \) such that \( n - 1 = p^t \mod n \) for some \( t \), i.e. \( f_p(x) = \# \{ n \leq x \mid \exists t \mod n \} \).
For \( p = 2 \) we have \( f_2(100,000) = 12741 \) of which 6641 lengths divide \( 2^t+1 \) for an even exponent \( t \) and the rest 66100 for an odd exponent \( t \). Figure 1 gives plot of \( f_2(x) \) for \( x \) in the range \([0,10^5]\) and Table II gives some explicit values.

| \( x \)   | \( f_2(x) \) | \( f_2^*(x) \) | \( f_2^*(x) \) |
|----------|--------------|----------------|----------------|
| 10       | 2            | 1              | 1              |
| 100      | 23           | 11             | 12             |
| 1,000    | 189          | 101            | 88             |
| 10,000   | 1521         | 790            | 731            |
| 100,000  | 12741        | 6641           | 6100           |

Table II
VALUES OF \( f_2(x) \)

We believe that the density is of \( f_p(x) \) the form \( \frac{x}{\log x} \), although the plots look linear. We show a weaker lower bound.

**Claim VII.1.** For any prime \( p \), \( f_p(x) \geq c_p \frac{x}{\log x} \), where \( c_p \) is a constant that depends only on \( p \).

**Proof:** Fix a characteristic \( p \) of the base field. Let us estimate only the prime lengths \( n \) that are good, i.e., primes \( n \) such that \( p^t \equiv -1 \mod n \). When \( n \) is also a prime, if \( p \) has an even order say \( 2\ell \) in the group \( \mathbb{Z}/n\mathbb{Z}^* \) then \( p^\ell = -1 \) and hence \( n \) is good. In particular, if \( p \) is a quadratic non-residue modulo \( n \) then \( n \) is good. First consider the case when \( p = 2 \). Using quadratic reciprocity, we have 2 is a quadratic non-residue if and only if \( n \equiv 3 \mod 8 \). Therefore, the density \( f_2(X) \) is at least the density of primes in the arithmetic progression \( 3 \mod 8 \) (or for that matter \( 5 \mod 8 \)). We can now use the density version of Dirichlet’s theorem on prime numbers in AP.

On the other hand if \( p \) is odd then again using quadratic reciprocity we have

\[
\left( \frac{2}{n} \right) = (-1)^{\frac{n-1}{2} \frac{p-1}{2}} = \left( \frac{n}{p} \right). \tag{8}
\]

Pick any quadratic non-residue \( 1 \leq \beta \leq p-1 \mod p \). Then, from Equation (8) \( n \) is good if it simultaneously satisfies the equations.

\[
\begin{align*}
\beta & \equiv 1 \mod p \\
1 & \equiv \beta \mod 4
\end{align*}
\tag{9}
\]

Using Chinese reminder theorem, we can find an element \( n_0 \) satisfying Equation (9) and coprime to \( 4p \) such that \( 1 \leq n_0 \leq 4p \). Therefore, all \( n \) that is \( n_0 \mod 4p \) are good. The result then follows by using the density version of Dirichlet’s theorem.

**VIII. Conclusion**

In this paper, we studied cyclic stabiliser codes of length dividing \( p^t+1 \) over \( \mathbb{F}_p \). It is natural to ask whether the construction can be generalised for arbitrary code length. For higher degree extensions the gap between actual and BCH distance could be significant. Therefore, it would be interesting to find a better lower bound and in particular to know whether Berlekamp-like algorithms can be used to decode up to that bound. Unlike previous definition of cyclicity, our definition is applicable to non-stabiliser codes as well. An open problem is to construct cyclic non-stabiliser codes.

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Appendix

Proof of Proposition III.2

Proof of Lemma IV.2

Frobenius Involution

The Frobenius involution satisfies few properties which are crucial in many of our proofs.

Proposition A.1. Let \( \sigma \) denote the Frobenius involution on the ring of polynomials over the quadratic extension \( \mathbb{F}_p(\eta) / \mathbb{F}_p \). Then

1) Let \( \eta' \) be the conjugate \( \eta^p \) of \( \eta \) then for polynomials \( a(X) \) and \( b(X) \) over \( \mathbb{F}_p \) we have \( \sigma(a + \eta b) = a + \eta' b \).
2) Any \( a(X, \eta) \) in \( \mathbb{F}_p(\eta) \) is a polynomial over \( \mathbb{F}_p \) if and only if \( \sigma(a) = a \).
3) Let \( a, b \) and \( g \) be polynomials over \( \mathbb{F}_p(\eta) \) then \( a = b \mod g \) if and only if \( \sigma(a) = \sigma(b) \mod \sigma(g) \).

Explicit Examples of Linear Frobenius Codes

We now demonstrate our construction for the case when the characteristic \( p \) of the underlying finite field is 2. Tables III and IV give some explicit codes for all lengths \( n \) less than 100 which is a factor of \( 2^{dm} + 1 \) for some \( m \) where \( d \) is either 2 or 3. When \( d = 2 \) the codes are all linear and for \( d = 3 \) they are non-linear. Recall that for a \( t \)-Frobenius code with canonical factorisation \( g \cdot h \), it is necessary for both \( X + 1 \) and \( X - 1 \) to divide \( g \). However, since 1 and \(-1\) are the same in \( \mathbb{F}_2 \), the polynomial \( g \) needs to have only the factor \( X - 1 \). The notation used in the tables are the following: Let \( \beta \) denote any fixed primitive \( n \)-th root of unity. Roots of any degree \( l \) irreducible factor of \( X^n - 1 \) over \( \mathbb{F}_q \) are exactly \( \beta^{kq^l} \ldots \beta^{kq^{l-1}} \) for some \( k \), where \( l \) is the smallest positive integer such that \( kq^l = k \mod n \). Call this factor \( f_{q,k} \). Let the polynomials \( g_k \) and \( h_k \) in the tables denote \( f_{2,k} \) and \( f_{2^3,k} \) respectively. Notice that \( g_0 = X - 1 \) and in case of \( d = 2 \), for any \( k \neq 0 \), \( g_k = h_k h_{2k} \) where \( \sigma(h_k) = h_{2k} \). Similarly in case of \( d = 3 \) if degree of \( g_k \) is divisible by 3 then \( g_k = h_k h_{2k} h_{3k} \). The distance given in these tables is the BCH distance. The actual distance can be larger.
| $m$ | Canonical factors | Roots of $h$ | Code       |
|-----|------------------|--------------|------------|
| 1   | $g_0$            | $h_2$        | $\beta^2, \ldots, \beta^3$ | [5, 1, 3] |
| 2   | $g_0$            | $h_2 h_6$    | $\beta^6, \ldots, \beta^{11}$ | [17, 1, 7] |
|     | $g_0 g_1$        | $h_6$        | $\beta^6, \ldots, \beta^7$  | [17, 9, 3] |
| 3   | $g_0$            | $h_1 h_5$    | $\beta^4, \ldots, \beta^5$  | [25, 1, 4] |
|     | $g_0 g_5$        | $h_2$        | $\beta^2, \ldots, \beta^3$  | [25, 5, 3] |
| 4   | $g_0$            | $h_1$        | $\beta^4, \ldots, \beta^7$  | [29, 1, 5] |
| 5   | $g_0$            | $h_1$        | $\beta^9, \ldots, \beta^{12}$ | [37, 1, 5] |
| 6   | $g_0 g_1$        | $h_2 h_6$    | $\beta^{14}, \ldots, \beta^{19}$ | [41, 1, 7] |
|     | $g_0 g_1 h_6$    | $h_2$        | $\beta^{14}, \ldots, \beta^{19}$ | [41, 1, 7] |
| 7   | $g_0$            | $h_2$        | $\beta^{28}, \ldots, \beta^{31}$ | [61, 1, 7] |
| 8   | $g_0 g_1 h_6$    | $h_2 h_6 h_10$ | $\beta^{22}, \ldots, \beta^{28}$ | [65, 13, 8] |
|     | $g_0 g_9 g_1$    | $h_2 h_6 h_10$ | $\beta^{29}, \ldots, \beta^{36}$ | [65, 17, 9] |
|     | $g_0 g_9 g_1 h_6$ | $h_2 h_6 h_10$ | $\beta^{30}, \ldots, \beta^{35}$ | [65, 29, 7] |
|     | $g_0 g_9 g_1 h_6$ | $h_2 h_6 h_10$ | $\beta^{31}, \ldots, \beta^{34}$ | [65, 41, 5] |
|     | $g_0 g_9 g_1 h_6$ | $h_2 h_6 h_10$ | $\beta^{32}, \ldots, \beta^{33}$ | [65, 53, 3] |
| 12  | $g_0 g_1 h_7$    | $h_7$        | $\beta^{33}, \ldots, \beta^{40}$ | [97, 1, 9] |
|     | $g_0 g_1 h_7$    | $h_7$        | $\beta^{37}, \ldots, \beta^{40}$ | [97, 49, 5] |

**TABLE III**

LINEAR CYCLIC STABILISER CODES OF LENGTH DIVIDING $4^m + 1$ OVER $\mathbb{F}_2$

| $m$ | Canonical factors | Roots of $h$ | Code       |
|-----|------------------|--------------|------------|
| 1   | $g_0 g_3$        | $h_4$        | $\beta^1, \ldots, \beta^5$  | [9, 3, 3] |
| 2   | $g_0$            | $h_4$        | $\beta^6, \ldots, \beta^7$  | [13, 1, 3] |
| 3   | $g_0$            | $h_4$        | $\beta^9, \ldots, \beta^{12}$ | [16, 1, 5] |
| 4   | $g_0 g_9$        | $h_4$        | $\beta^{12}, \ldots, \beta^{15}$ | [27, 3, 5] |
| 5   | $g_0 g_9 g_9$    | $h_4$        | $\beta^{15}, \ldots, \beta^{18}$ | [27, 21, 2] |
| 6   | $g_0$            | $h_4$        | $\beta^{22}, \ldots, \beta^{25}$ | [27, 29, 9] |
| 7   | $g_0 g_9$        | $h_4$        | $\beta^{25}, \ldots, \beta^{28}$ | [57, 1, 4] |
| 8   | $g_0 g_9 g_9$    | $h_4$        | $\beta^{28}, \ldots, \beta^{30}$ | [57, 21, 5] |
| 9   | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{29}, \ldots, \beta^{32}$ | [57, 39, 3] |
| 10  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{31}, \ldots, \beta^{36}$ | [61, 1, 5] |
| 11  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{32}, \ldots, \beta^{38}$ | [65, 5, 13] |
|     | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{37}, \ldots, \beta^{38}$ | [65, 17, 11] |
| 12  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{39}, \ldots, \beta^{42}$ | [65, 29, 7] |
|     | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{43}, \ldots, \beta^{46}$ | [65, 41, 5] |
|     | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{47}, \ldots, \beta^{50}$ | [65, 53, 3] |
| 13  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{50}, \ldots, \beta^{53}$ | [67, 1, 7] |
| 14  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{54}, \ldots, \beta^{56}$ | [81, 21, 4] |
| 15  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{57}, \ldots, \beta^{60}$ | [81, 75, 2] |
| 16  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{60}, \ldots, \beta^{69}$ | [81, 9, 5] |
| 17  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{66}, \ldots, \beta^{68}$ | [81, 27, 3] |
| 18  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{77}, \ldots, \beta^{79}$ | [97, 49, 3] |
| 19  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{80}, \ldots, \beta^{83}$ | [97, 94, 9] |
| 20  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{84}, \ldots, \beta^{86}$ | [97, 94, 9] |
| 21  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{87}, \ldots, \beta^{90}$ | [99, 93, 2] |
| 22  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{91}, \ldots, \beta^{93}$ | [99, 69, 3] |
| 23  | $g_0 g_9 g_9 g_9$ | $h_4$        | $\beta^{94}, \ldots, \beta^{97}$ | [99, 69, 3] |

**TABLE IV**

NON-LINEAR CYCLIC STABILISER CODES OF LENGTH DIVIDING $8^m + 1$ OVER $\mathbb{F}_2$