On Quantum Deformations of $d = 4$ Conformal Algebra

Andrzej Frydryszak  
*Institute for Theoretical Physics, University of Wroclaw, pl. Maksa Borna 9, 50-204 Wroclaw, Poland*  
and  
*Laboratoire de Physique Théorique, Université Bordeaux I, 33-175 Gradignan, France*  

Jerzy Lukierski  
*Institute for Theoretical Physics, University of Wroclaw, pl. Maksa Borna 9, 50-204 Wroclaw, Poland*  
and  
*Laboratoire de Physique Théorique, Université Bordeaux I, 33-175 Gradignan, France*  

Pierre Minnaert  
*Laboratoire de Physique Théorique, Université Bordeaux I, 33-175 Gradignan, France*  

Marek Mozrzymas  
*Institute for Theoretical Physics, University of Wroclaw, pl. Maksa Borna 9, 50-204 Wroclaw, Poland*  
and  
*Laboratoire de Physique Théorique, Université Bordeaux I, 33-175 Gradignan, France*  

Abstract

Three classes of classical $r$ matrices for $\mathfrak{sl}(4,\mathbb{C})$ algebra are constructed in quasi-Frobenius algebra approach. They satisfy CYBE and are spanned respectively on 8,10,12 generators. The $o(4,2)$ reality condition can be imposed only on the eight dimensional $r$ matrices with dimension-full deformation parameters. Contrary to the Poincaré algebra case, it appears that all deformations with a mass-like deformation parameter ($\kappa$-deformations) are described by classical $r$-matrices satisfying CYBE.

\textsuperscript{1}Partially supported by KBN grant 2P03B130.12  
\textsuperscript{2}To appear in the proceedings of the 7\textsuperscript{th} Colloquium on Quantum Groups and Integrable Systems (Prague, June 18-20, 1998).
1 Introduction.

In four dimensions the deformations of $o(4, 2)$ algebra describing $d = 4$ conformal algebra can be obtained by considering the deformations of complexified $d = 4$ conformal algebra $sl(4, \mathbb{C})$ and then by taking into account the restrictions imposed by the reality conditions (which define the deformed $o(4, 2)$ algebra as a real form of deformed complex $sl(4, \mathbb{C})$). It appears that the reality conditions are quite restrictive. In [1] there were classified all the real forms of Drinfeld-Jimbo deformation $U_q(sl(4, \mathbb{C}))$ of complexified $d = 4$ conformal algebra. It appears that for standard $\ast$-Hopf algebra, with $\ast$-operation being anti-automorphism of algebra and an automorphism of coalgebra there exist only two real forms of the Drinfeld-Jimbo deformations of $U_q(sl(4, \mathbb{C}))$, one for $q$ real and second for $|q| = 1$, providing the $q$-deformations $U_q(o(4, 2))$ of $d = 4$ conformal algebra.

In this paper we describe deformations of $sl(4, \mathbb{C})$ which admit the structure of the real deformed $d = 4$ conformal algebra. To give the classification of quantum deformations, we shall discuss here the classical $sl(4)$ $r$-matrices. We present results which are an essential extension of those obtained in [4] and Stolin [9, 10] and the techniques presented by Alexeevsky, Perelomov and Stolin we describe three classes of classical $r$-matrices for $sl(4, \mathbb{C})$ spanned on 8, 10, 12 dimensional subalgebras. Considering the $o(4, 2)$ real forms we show that only one class (eight dimensional) permits the $o(4, 2)$ reality conditions. It appears that these new classical $r$-matrices are span by the generators of $d = 4$ Weyl algebra (Poincaré generators and dilatations) possibly transformed by Weyl reflections. In Sect. 4 we shall present some remarks and conclusions (in particular, concerning the structure relations of the $\kappa$-deformed $d = 4$ conformal algebra by applying the twist transformation proposed by Kulish, Lyakhovsky and Mudrov [11]).

2 Cartan-Weyl basis and its real forms.

One can write the complex $sl(4; \mathbb{C})$ algebra in Cartan Weyl basis $e_{AB}$ ($A, B = 1, \ldots, 4$); where the choice of indices $(A, B)$ is taken from the position of non-vanishing entry in the $4 \times 4$ fundamental matrix representation. In particular the diagonal elements $h_1 = e_{11} - e_{22}$, $h_2 = e_{22} - e_{33}$, $h_3 = e_{33} - e_{44}$ describe three commuting Cartan generators, and the simple root generators $e_1 = e_{12}$, $e_2 = e_{23}$ and $e_3 = e_{34}$ describe Cartan-Chevalley basis. The composite roots extending Cartan-Chevalley basis to Cartan-Weyl basis are described by the formulae: $e_4 \equiv e_{13} = [e_{12}, e_{23}]$, $e_5 \equiv e_{24} = [e_{23}, e_{34}]$, $e_6 \equiv e_{14} = [e_{12}, e_{24}] = [e_{13}, e_{34}]$. Classical $sl(4)$ algebra is generated by the relations satisfied by the Cartan-Weyl basis generators.
such that for any $x, y$ \in \text{Lie algebra } \mathfrak{sl}$ relations and the above definitions. In order to describe the real forms of complex Lie algebra $\mathfrak{sl}(4)$ we consider involutive anti-automorphisms $x \rightarrow x^*$ of $U(\mathfrak{sl}(4; C))$ such that for any $x, y \in U(\mathfrak{sl}(4; C))$

$$(xy)^* = y^* x^* \quad (\mu X + \lambda y)^* = \mu^* x^* + \lambda^* y^* \quad \mu, \lambda \in C$$

(2)

There are the following inequivalent real forms describing by means of the reality condition $x = x^*$ the real $o(4, 2)$ algebra $\mathfrak{h}(j = 1, 2, 3)$

$h_j^* = -h_{4-j}, \quad e_{\pm j}^* = e_{\mp (4-j)}$ \quad (3)

$h_j^* = h_j, \quad e_{\pm j}^* = e_{\mp j}$ \quad (4)

with three nonequivalent choices of $(\epsilon_1, \epsilon_2, \epsilon_3): (1, -1, 1), (-1, 1, -1)$ and $(-1, -1, -1)$.

Let us observe that $\mathfrak{sl}(4, C) \cong o(4, 2, C)$ and the generators $M_{RQ} = -M_{QR}$ ($P, Q = 0, 1, 2, 3, 4, 5)$ of $o(4, 2; C)$ $(\eta_{AB} = \text{diag}(-1, 1, 1, 1, -1))$ satisfy the relations

$[M_{PQ}, M_{RS}] = \eta_{PS} M_{QR} - \eta_{PR} M_{QS} + \eta_{QR} M_{PS} - \eta_{QS} M_{PR}$ \quad (5)

We extend the Lorentz generators $M_{\mu \nu} = (M_i = \frac{1}{2} \epsilon_{ijk} M_{jk}, L_i = M_{i0}); \mu, \nu = 0, 1, 2, 3, $ to $d = 4$ conformal algebra generators as follows:

$P_\mu = (M_{4\mu} + M_{5\mu}) \quad K_\mu = (M_{5\mu} - M_{4\mu}) \quad D = M_{45}$ \quad (6)

The reality conditions lead to the following two ways of defining real $d = 4$ conformal algebra generators in terms of Cartan-Weyl basis of $\mathfrak{g} = \mathfrak{sl}(4, C);$ $\mathfrak{g} = B^- \oplus H \oplus B^-$, where $(B^+, H)$, $(B^-, H)$ are two Borel subalgebra and $H = (h_1, h_2, h_3)$ describe Cartan subalgebra.

i) The reality condition $(B^\pm)^* \subset B^\mp$.

$M_+ = e_1 + e_3 \quad M_- = -(e_3 + e_1) \quad M_3 = \frac{i}{2}(h_1 - h_3)

L_+ = i(e_3 - e_1) \quad L_- = -i(e_3 - e_1) \quad L_3 = \frac{i}{2}(h_1 + h_3)

P_1 = -(e_4 + e_5) \quad P_2 = i(e_4 - e_5) \quad P_3 = i(e_2 - e_6)

K_1 = e_4 - e_5 \quad K_2 = i(e_4 + e_5) \quad K_3 = i(e_2 - e_6)

P_0 = -i(e_2 + e_6) \quad K_0 = i(e_2 + e_6) \quad D = \frac{i}{2}(h_1 + 2h_2 + h_3)$

(7)

where $M_\pm = M_1 \pm i M_2, L_\pm = L_1 \pm i L_2$. We see that the Cartan subalgebra $H$ is described by the non compact algebra $(M_3, L_3, D)$, and under the $*$-operation
the operators are real.

ii) The reality condition \((B^\pm)^* \subset B^\mp\). This reality condition can not be applied to the solutions of the CYBE for \(sl(4)\) with a dimension of solution no less than eight. Hence the assignment of the conformal generators for this case will be not needed in further considerations and it is omitted here.

It appears that the Cartan subalgebra \(H\) is described by the compact Abelian subgroup \((M_{12} = M, M_{34} = \frac{1}{2}(P_3 - K_3), M_{50} = \frac{1}{2}(P_0 + K_0))\). The choices of the generators can be modified if we take into consideration the discrete group of Weyl reflections, which preserve the Lie-algebra relations. There are three basic Weyl reflections \(\sigma_1, \sigma_2, \sigma_3\) describing the automorphism of \(sl(4, C)\) Lie algebra.

For example explicit relations defining \(\sigma_1\) are the following:

\[
\begin{align*}
\sigma_1(e_{\pm 1}) &= (a_1)^{\pm 1}e_{\mp 1}, \\
\sigma_1(e_{\pm 2}) &= (a_4)^{\pm 1}e_{\mp 4}, \\
\sigma_1(e_{\pm 3}) &= (a_3)^{\pm 1}e_{\mp 3}, \\
\sigma_1(e_{\pm 4}) &= (a_2)^{\pm 1}e_{\mp 2}, \\
\sigma_1(e_{\pm 5}) &= (a_6)^{\pm 1}e_{\mp 6}, \\
\sigma_1(e_{\pm 6}) &= (a_5)^{\pm 1}e_{\mp 5}
\end{align*}
\]

\((8)\) \(a_4 = a_1a_2\ a_5 = a_2a_3\ a_6 = a_1a_2a_3\). There exists also the isomorphism of Dynkin diagram \((\alpha_1 \leftrightarrow \alpha_3)\) which implies the following isomorphism of \(sl(4; C)\) Lie algebra:

\[
\begin{align*}
\beta(e_{\pm 1}) &= e_{\pm 3}, \\
\beta(e_{\pm 2}) &= e_{\pm 2}, \\
\beta(e_{\pm 4}) &= e_{\pm 5}, \\
\beta(e_{\pm 6}) &= e_{\pm 6}
\end{align*}
\]

\((9)\) Any product of Weyl reflections is again an isomorphism of \(sl(4; C)\), but not all these isomorphisms commute with the \(*\)-operations defining real forms. The condition

\[
\sigma_{i_1...i_k} \cdot * = * \cdot \sigma_{i_1...i_k}
\]

\((10)\) is necessary for defining the restriction of Weyl reflections to \(o(4, 2)\) algebra. We obtain

i) for the \(*\)-operation \((B^\pm)^* \subset B^\pm\) the involutions \(\sigma_2, \sigma_3 = \sigma_3\sigma_1, \beta\) are also isomorphisms of real algebra \(o(4, 2)\) provided that \(b_1^* = b_3, b_2^* = b_2\).

ii) for the \(*\)-operation \((B^\pm)^* \subset B^\mp\) we obtain the following isomorphisms of \(o(4, 2): \sigma_2, \beta\). Provided that \(b_i^*b_i = 1\).

3 Classical \(r\)-matrices for \(sl(4)\) and \(o(4, 2)\) reality conditions

We shall consider the antisymmetric solutions of the CYBE i.e.

\[
<< r, r >> = [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0, \quad r \in \hat{g} \wedge \hat{g},
\]

\((11)\) where \(<< r, r >>\) denotes Schouten bracket \((<< r, r >> \in \hat{g} \otimes \hat{g} \otimes \hat{g}\). In order to construct new solutions we shall apply the quasi-Frobenius algebra approach, using the following definitions and results \([13, 12, 9, 10]\):
Lie algebra \( \hat{g} \) is quasi-Frobenius if there exists a skew-symmetric bilinear form \( B : \hat{g} \wedge \hat{g} \to C \) such that for all \( x, y, z \in \hat{g} \)

\[
B([x, y], z) + B([y, z], x) + B([z, x], y) = 0.
\]  
(12)

Let \( b_{ij} = < e_i^* \otimes e_j^*, B > \) then \( r = r_{ij} e_i \wedge e_j \), where \( r_{ij} b_{jk} = \delta^i_k \) satisfies CYBE.

If the bilinear form \( B \) is determined by a functional \( g^*_B \) on \( \hat{g} \) such that \( B(x, y) = < g^*_B, [x, y] > \),

(13)

then \( \hat{g} \) is called Frobenius algebra.

The classification of classical \( r \)-matrices (obtained in this way) can be reduced to the classification of quasi-Frobenius algebras which in turn, are even dimensional and can be identified with a set of parabolic subalgebras.

For \( \hat{g} = sl(4, C) \) we can distinguish three relevant families of the parabolic subalgebras spanning the respective classical \( r \)-matrices. Let \( B_+ = (h_i, e_A); i = 1, 2, 3; A = 1, \ldots, 6 \) denotes a Borel subalgebra, then we have explicitly the following classification of classical \( r \)-matrices:

**d=12.** Parabolic subalgebra \( P_{(-2,-3)} = (B_+, e_{-2}, e_{-3}) \). In this case one obtains the one-parameter generalization of the solution given by Gerstenhaber and Giaquinto.

\[
1 \mathcal{E}_1 = \frac{1}{4}(3h_1 + 2h_2 + h_3) \wedge e_1 + \frac{1}{4}(h_1 + 2h_2 + 3h_3) \wedge e_2 + e_4 \wedge e_{-2} +
\]

\[
- e_6 \wedge e_{-5} + \lambda \left( \frac{1}{2}(h_1 + 2h_2 + h_3) \wedge e_2 + (e_4 + e_5) \wedge e_{-3} \right)
\]

(14)

This solution of the CYBE has the following properties: parameter \( \lambda \) is arbitrary (it has inverse of mass dimension), each part of \( r \) satisfies CYBE separately, it does not permit the restriction of \( sl(4, C) \) to real \( o(4, 2) \).

**d=10.** Parabolic subalgebras \( P_{(j)} = (B_+, e_{-j}), j = 1, 2, 3 \). We have to consider here three separate sets of nonsingular functionals:

i) Parabolic subalgebra \( P_1 \).

\[
g_{1a}^* = e_5^* + e_4^* + e_1^* \\
g_{1b}^* = e_5^* + e_4^* + e_3^* \\
g_{1c}^* = e_6^* + e_5^* + e_3^* \\
g_{1d}^* = e_6^* + e_2^* + e_1^* \\
g_{1e}^* = e_6^* + e_2^* + e_1^* + e_3^* \\
\]

(15)
They yield the following r matrices:

\[ r_{1a}^{(10)} = -e_2 \wedge e_3 + e_6 \wedge e_\pm + \frac{1}{2}(e_1 + e_3) \wedge (h_1 + h_3) \]
\[ + \frac{1}{4}e_4 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{4}e_5 \wedge (h_1 + 2h_2 + 3h_3) \]
\[ r_{1b}^{(10)} = -e_1 \wedge e_2 + e_6 \wedge e_\pm + \frac{1}{2}(e_1 + e_3) \wedge (h_1 + h_3) \]
\[ + \frac{1}{4}e_4 \wedge (3h_1 + 2h_2 - h_3) + \frac{1}{4}e_5 \wedge (-h_1 + 2h_2 + 3h_3) \]
\[ r_{1c}^{(10)} = -e_1 \wedge e_5 + e_4 \wedge e_\pm + \frac{1}{2}(e_3 + e_\mp) \wedge (-h_1 + h_3) \]
\[ + \frac{1}{4}e_2 \wedge (-h_1 + 2h_2 + h_3) + \frac{1}{4}e_6 \wedge (3h_1 + 2h_2 + h_3) \]
\[ r_{1d}^{(10)} = -e_1 \wedge e_5 + e_3 \wedge e_4 + \frac{1}{2}(e_3 + e_\mp) \wedge (-h_1 + h_3) \]
\[ + \frac{1}{4}e_2 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{4}e_6 \wedge (h_1 + 2h_2 + h_3) \]
\[ r_{1e}^{(10)} = -\frac{1}{2}e_1 \wedge e_2 + e_6 \wedge e_\pm + \frac{1}{4}(e_1 + e_3) \wedge (h_1 + h_3) \]
\[ \wedge (h_1 + h_3) - \frac{1}{2}e_2 \wedge (e_3 + e_4 - e_5) \]
\[ + \frac{1}{4}e_4 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{4}e_5 \wedge (h_2 + h_3) \]

(16)

ii) Parabolic subalgebra $P_2$.
We have shown explicitly by considering the most general ansatz that there does not exist a Frobenius algebra structure on $P^*_2$.

iii) Parabolic subalgebra $P_3$.
Nonsingular functionals:

\[ g_{3a} = e_5^* + e_4^* + e_1^* \]
\[ g_{3b} = e_5^* + e_4^* + e_3^* \]
\[ g_{3c} = e_6^* + e_2^* + e_{2+}^* \]

yielding the following r matrices:

\[ r_{3a}^{(10)} = -e_2 \wedge e_3 - e_6 \wedge e_\pm + \frac{1}{2}(e_1 + e_3) \wedge (h_1 + h_3) \]
\[ + \frac{1}{4}e_4 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{4}e_5 \wedge (h_1 + 2h_2 + 3h_3) \]
\[ r_{3b}^{(10)} = -e_1 \wedge e_2 - e_6 \wedge e_\pm + \frac{1}{2}(e_1 + e_3) \wedge (h_1 + h_3) \]
\[ + \frac{1}{2}e_4 \wedge (3h_1 + 2h_2 + h_3) + \frac{1}{2}e_5 \wedge (-h_1 + 2h_2 + h_3) \]
\[ r_{3c}^{(10)} = e_3 \wedge e_4 - e_5 \wedge e_\pm - \frac{1}{2}(e_1 + e_3) \wedge (-h_1 + h_3) \]
\[ + \frac{1}{2}e_2 \wedge (h_1 + 2h_2 - h_3) + \frac{1}{2}e_6 \wedge (h_1 + 2h_2 + 3h_3) \]
\[ r_{3d}^{(10)} = -e_1 \wedge e_5 + e_3 \wedge e_4 - \frac{1}{2}(e_1 + e_\mp) \wedge (-h_1 + h_3) \]
\[ + \frac{1}{2}e_2 \wedge (-h_1 + 2h_2 + h_3) + \frac{1}{2}e_6 \wedge (3h_1 + 2h_2 + h_3) \]
\[ r_{3e}^{(10)} = -\frac{1}{2}e_1 \wedge e_5 - e_6 \wedge e_\pm + \frac{1}{4}(e_1 + e_3) \wedge (-h_1 + h_3) \]
\[ - \frac{1}{2}e_2 \wedge (e_3 + e_4 - e_5) + \frac{1}{2}e_4 \wedge (h_1 + h_2) + \frac{1}{2}e_5 \wedge (h_2 + h_3) \]

(18)

It can be shown that all such generated 10-dimensional classical r-matrices do not permit the $o(4, 2)$ reality conditions. It is that because $\sigma_2$ commutes with the $\ast$ and $(\sigma_2 \otimes \sigma_2) r_3^{(10)} = r_1^{(10)}$, but these r-matrices are not compatible i.e.
\[ < r_1^{(10)}, r_3^{(10)} >> \neq 0. \]

\textbf{d=8.} Here we have the Borel subalgebra $B_+$.

\[ r_1^{(8)} = e_4 \wedge e_3 - e_5 \wedge e_1 + ah_2 \wedge e_6 + h_6 \wedge e_6 \]

(19)

Taking into account that above classical r-matrix is real under the $\ast$-operation (3) and using the Weyl automorphism commuting with this $\ast$-involution we obtain
another form of the $d=8$ solution:

$$r_2^{(8)} = (\sigma_2 \otimes \sigma_2) \circ r_1^{(8)} = [(\sigma_1 \sigma_3) \otimes (\sigma_1 \sigma_3)] \circ r_1^{(8)} =$$

$$e_5 \wedge e_{-3} - e_4 \wedge e_{-1} + h_2 \wedge e_2 + a h_6 \wedge e_2$$  \hspace{1cm} (20)

Let us note that in the physical basis above $r$-matrices are spanned on generators of the $d = 4$ Weyl subalgebra ($M_i, L_i, P_\mu, D$).

4 Final remarks.

In this paper we have considered $r$-matrices satisfying CYBE. From their scaling properties in the physical basis and the fact that invariant three form for $o(4,2)$; $I = e_{ij} \wedge e_{jk} \wedge e_{ki} \sim M^B_A \wedge M^C_B \wedge M^D_C$ is scale invariant it follows that every $r$-matrix giving dimension-full deformation ($\kappa$-deformation) satisfies CYBE.

We do not present here the description of the complete $\kappa$-deformed algebra. Using the results of the work [11] one can obtain the coproduct applying to the $\Delta_0$ the twist $F$ of the form: $\Delta_F(x) = F \circ \Delta_0(x) \circ F^{-1}$, where $F = \exp (h_6 \otimes \sigma(e_6)) \cdot \exp (2\lambda e_1 \otimes e_5 \cdot e^{-2\sigma(e_6)}) \cdot \exp (2\lambda e_4 \otimes e_3 e^{-2\sigma(e_6)})$ and $\sigma(e_6) = -\frac{1}{2\lambda} \ln (1 - 2\lambda e_6) \sim \ln (1 - \frac{2}{\kappa}(P_0 + P_3))$. It will be given in a forthcoming paper of the present authors.

References

[1] J. Lukierski, A. Nowicki, J. Sobczyk, J. Phys. A 26 (1993), 4047.
[2] J. Lukierski, A. Nowicki, H. Ruegg, Phys. Lett. B 271 (1991), 321
[3] M. Schneuert, Journ. Math. Phys.
[4] J. Lukierski, P. Minnaert, M. Mozrzymas, Phys. Lett. B 371(1996), 251.
[5] A. Ballesteros, F.J. Herranz, A.A. del Olmo, M. Santander, Phys. Lett. B 351 (1995), 137.
[6] P. Kosiński, P. Maślanka, in ”From Field Theory to Quantum Groups”, ed. B. Jancewicz and J. Sobczyk, World Scientific, Singapore, 1996, p. 41.
[7] D.V. Alekseevsky and A.M. Perelomov, ”Poisson brackets on simple Lie algebra and symplectic Lie algebra”, preprint CERN TH 6983/93 unpublished.
[8] M. Gerstenhaber, A. Giaquinto, Lett. Math. Phys. 40, 337 (1997), 337.
[9] A. Stolin, ” On rational solutions of the classical Yang Baxter equation”, Ph. D. Thesis, Stockholm 1991, unpublished.
[10] A. Stolin, Math. Scand. 69 (1991), 81.

[11] P. P. Kulish, V. D. Lyakhovsky, A. I. Mudrov "Extended jordanian twists for Lie algebras", q-alg/9806014.

[12] V. G. Drinfeld, Soviet. Math. Dokl. 273 (1983), 667.

[13] A. G. Elashvili, Funct. Anal. Dppl. 16 (1982), 94.