Dually Vertex-Oblique Graphs

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February 12, 2022

Abstract

A vertex with neighbours of degrees \(d_1 \geq \cdots \geq d_r\) has vertex type \((d_1, \ldots, d_r)\). A graph is vertex-oblique if each vertex has a distinct vertex-type. While no graph can have distinct degrees, Schreyer, Walther and Mel’nikov [Vertex oblique graphs, same proceedings] have constructed infinite classes of super vertex-oblique graphs, where the degree-types of \(G\) are distinct even from the degree types of \(\overline{G}\).

\(G\) is vertex oblique iff \(\overline{G}\) is; but \(G\) and \(\overline{G}\) cannot be isomorphic, since self-complementary graphs always have non-trivial automorphisms. However, we show by construction that there are dually vertex-oblique graphs of order \(n\), where the vertex-type sequence of \(G\) is the same as that of \(\overline{G}\); they exist iff \(n \equiv 0\) or \(1\) (mod 4), \(n \geq 8\), and for \(n \geq 12\) we can require them to be split graphs.

We also show that a dually vertex-oblique graph and its complement are never the unique pair of graphs that have a particular vertex-type sequence; but there are infinitely many super vertex-oblique graphs whose vertex-type sequence is unique.

1 Introduction and basic results

Let \(G\) be a simple graph on \(n\) vertices. A vertex \(v\) of degree \(r\), with neighbours of degrees\(^1\) \(x_1 \geq \cdots \geq x_r\), has vertex type \(t(v) := (x_1, \ldots, x_r)\). \(G\) is vertex-

\(^1\)It is conventional in the literature on degree sequences to list degrees in non-increasing order. We follow this convention here, even though we do not prefer it, because we will
oblique if each vertex has a distinct vertex-type.

The degree of $v$ in $\overline{G}$ (the complement of $G$) is $r := n - 1 - r$. If the degrees of vertices in $G$ are $x_1 \geq \cdots \geq x_r, y_1 \leq \cdots \leq y_r$, then $v$ is non-adjacent to vertices of degrees $y_1 \leq \cdots \leq y_r$, so its vertex-type in $\overline{G}$ is $\overline{t}(v) = (\overline{y}_1, \ldots, \overline{y}_r)$. Thus $G$ is vertex oblique if and only if $\overline{G}$ is.

While no graph can have distinct degrees, Schreyer et al. [10] have constructed infinite classes of vertex-oblique graphs. In fact, their examples are super vertex-oblique, with the degree-types of $G$ being distinct even from degree types of $\overline{G}$.

It is natural to ask whether there are any self-complementary vertex-oblique graphs, but this is impossible because a self-complementary graph always has non-trivial automorphisms [7, 9], obtained by applying twice an(y) isomorphism that maps the graph to its complement. However, in this article we construct infinitely many dually vertex-oblique graphs, where the set of vertex-types of $G$ is the same as that of $\overline{G}$.

Many simple results for self-complementary graphs still hold under the weaker assumption that $G$ and $\overline{G}$ have the same set of vertex-types. In particular, dually vertex-oblique graphs of order $n$ can only exist if $n \equiv 0$ or $1 \pmod{4}$. The main result of this paper is that they exist for all feasible $n$ at least 8:

1. **Theorem.** Dually vertex-oblique graphs of order $n > 1$ exist iff $n$ is congruent to 0 or 1 (mod 4), and $n \geq 8$.

We will make use of the following elementary lemma that is inspired by similar results on self-complementary graphs.

2. **Lemma.** Let $G$ be a graph with the same degree sequence as $\overline{G}$, say $d_1 \geq \cdots \geq d_n$. Then:

   A. $d_i + d_{n-i+1} = n - 1$, for $i = 1, \ldots, n$.

   B. $n \equiv 0$ or 1 (mod 4).

If, moreover, $G$ has the same set of vertex-types as $\overline{G}$, and there are $r_d$ vertices of degree $d$, and $s_{x,y}$ edges joining vertices of degrees $x$ and $y$, then:

discuss degree sequences in Section 5 it is of little importance anyway.
C. \( s_{y,y} + s_{\overline{y} \overline{y}} = \frac{1}{2} \binom{n}{2} \), and if \( x \neq y \), \( s_{x,y} + s_{\overline{x} \overline{y}} = r_x r_y \); in particular, \( s_{d,\overline{d}} = \frac{1}{2} r_d^2 \) for all \( d \), except if \( d = \overline{d} = (n-1)/2 \) in which case \( s_{d,\overline{d}} = \frac{1}{2} \binom{r_d}{2} \).

D. \( r_d \) is even for all \( d \), except for \( r_{(n-1)/2} \equiv 1 \pmod{4} \).

Furthermore, if \( G \) is dually vertex-oblique, then:

E. \( r_d < 2d, d \neq (n-1)/2; \) in particular, there are no isolated or end-vertices.

F. There must be at least three different degrees in \( G \).

**Proof:**

A. If \( r_d \) vertices have degree \( d \) in \( G \), then \( r_d \) vertices have degree \( d \) in \( \overline{G} \) and, thus, degree \( n - 1 - d \) in \( G \). So the degree sequence is symmetric about \( \frac{1}{2} (n-1) \).

B. The number of edges of \( G \) and \( \overline{G} \) is the same: \( \frac{1}{2} \binom{n}{2} = \frac{1}{2} n(n-1) \), which must be an integer.

C. Since \( s_{x,y} \) is determined by the vertex-types of \( G \), it must remain the same in \( \overline{G} \), that is, \( s_{x,y} = \overline{s}_{x,y} \); similarly, \( s_{x,\overline{y}} = \overline{s}_{x,\overline{y}} \). Now a vertex of degree \( x \) (or \( y \)) in \( G \) has degree \( \overline{x} \) (or \( \overline{y} \)) in \( \overline{G} \). So if there are \( p \) unordered pairs of vertices \( \{v, w\} \) with \( d(v) = x \) and \( d(w) = y \), then \( p - s_{x,y} = \overline{s}_{x,\overline{y}} = \overline{s}_{x,\overline{y}} \). Since \( p = \binom{r_y}{2} \) when \( x = y \), and \( p = r_x r_y \) otherwise, the result follows.

D. By C, \( \frac{1}{2} r_d^2 \) must be an integer, so \( r_d \) is even for \( d \neq \overline{d} \). When there are vertices of degree \( (n-1)/2 \), by B we must have \( n = 4k + 1 \) for some \( k \). By the first part, there are \( 2r \) vertices with degree \( d < (n-1)/2 \), and therefore \( 2r \) vertices with degree \( d > (n-1)/2 \), leaving \( (4k+1) - 2(2r) = 4(k-r) + 1 \) vertices of degree \( (n-1)/2 \).

E. If \( G \) is vertex-oblique, the vertices of degree \( d \) cannot all be adjacent to vertices of degree \( \overline{d} \) only. So \( \frac{1}{2} r_d^2 = s_{d,\overline{d}} < d r_d \).

F. Clearly \( G \) cannot be regular, so there must be at least two different degrees. If \( n \) is odd, the number of different degrees must be odd, by A. Suppose \( n \) is even and there are exactly two degrees, say \( d < (n-1)/2 \) and \( \overline{d} > (n-1)/2 \). Since \( G \) is vertex-oblique, each vertex \( v \) of degree \( d \) must be adjacent to a distinct number \( n_v \) of vertices of degree \( \overline{d} \). Note that \( 0 \leq n_v \leq d \). If \( n_v = 0 \) for some \( v \), then in the complement \( v \) would be a vertex of degree \( \overline{d} \) that is adjacent to all vertices of degree \( d \), so \( \overline{G} \) would have no vertex \( w \) of degree \( d \) with \( n_w = 0 \). Similar reasoning excludes the case \( n_v = d \), so we have \( 0 < n_v < d \) for every \( v \). This means that there are less than \( d < n/2 = r_d \) possible values of \( n_v \), a contradiction. \( \square \)
By Lemma 2.B, the smallest possible orders \( n > 1 \) for a dually vertex-oblique graph are 4 and 5; but by part E these could not have any vertices of degree \( d < (n - 1)/2 \), so no such graphs exist when \( n < 8 \). We now construct graphs for every \( n \equiv 0 \text{ or } 1 \pmod{4}, \ n \geq 8 \).

2 Construction on \( 4k \) vertices

A dually vertex-oblique graph on 8 vertices is shown in Figure 1, with the degree and vertex-type displayed next to each vertex. To verify this, one has to check that for every vertex with vertex-type \((x_1, \ldots, x_r)\), there is another vertex that is non-adjacent to vertices of degrees \((7 - x_1, \ldots, 7 - x_r)\).

Figure 1: A dually vertex-oblique graph on 8 vertices.

Given a dually vertex-oblique graph \( G \) on \( n = 4k \) vertices, we now show how to construct \( G' \) on \( n + 4 = 4k' \) vertices, where \( k' := k + 1 \) (see Figure 2). We add vertices \( v_2, w_2 \), that will have degree 2, and \( \overline{v}_2, \overline{w}_2 \) that will have degree \( 2k' - 2 = 2k + 2 \). Moreover, the new vertices induce a \( P_4 \), in a manner reminiscent of Akiyama and Harary's \cite{[1]} method of producing larger self-complementary graphs.

We pick an arbitrary vertex \( x \in V(G) \), and let \( \overline{x} \) be the (unique) vertex such that \( t(\overline{x}) = t(x) \). Note that if \( x \) has degree \( d \), then \( \overline{x} \) has degree \( 4k - 1 - d \neq d \). We make \( v_2 \) adjacent to \( \overline{v}_2 \) and \( x \), \( w_2 \) adjacent to \( \overline{w}_2 \) and \( \overline{x} \). Meanwhile, \( \overline{v}_2 \) is adjacent to \( v_2, \overline{w}_2 \) and \( V(G) \setminus x \); and \( \overline{w}_2 \) is adjacent to \( w_2, \overline{v}_2 \) and \( V(G) \setminus \overline{x} \).

A vertex of degree \( d \) now has degree \( d' := d + 2 \); this also means that a vertex of the complementary degree \( \overline{d} := 4k' - 1 - d \) now still has complementary degree \( \overline{d'} = 4k' - 1 - (d + 2) = \overline{d} + 2 \). The degrees in \( V(G) \) now range between at least 4 and at most \( 4k' \) (by Lemma 2.E), and thus the degrees (and vertex-types) in \( V(G') \) are distinct from those of the new vertices.

If \( u \not\in \{x, \overline{x}\} \) had vertex-type \( t(u) = (d_1, \ldots, d_r) \), in \( G' \) it has type \( t'(u) = (d'_1, \ldots, d'_r, 4k' - 2, 4k' - 2) \). The unique vertex \( \overline{u} \) such that \( t(\overline{u}) = t(u) \) was
Figure 2: Larger dually vertex-oblique graphs from smaller ones.
non-adjacent in $G$ to vertices of degrees $d_1, \ldots, d_r$; so in $G'$ it is non-adjacent to vertices of degrees $d_1', \ldots, d_r'$, as well as two vertices of degree $2 = \frac{4k' - 2}{2}$. Meanwhile $t(x) = \overline{t}(\overline{x}) = (f_1, \ldots, f_d)$ becomes $t'(x) = \overline{t}(\overline{x}) = (2, f_1 + 2, \ldots, f_d + 2, 4k' - 2)$. Distinct vertex-types in $G$ therefore result in distinct vertex-types in $G'$, and complementary vertex-types result in complementary vertex-types. Moreover, $t(v_2) = \overline{t}(\overline{v}_2) = (2, d)$, and $t(w_2) = \overline{t}(\overline{w}_2) = (2, \overline{d})$, so the types of the new vertices are also distinct and complementary, and thus $G$ is dually vertex-oblique.

A graph $G$ is split if its vertex set partitions into $L \cup R$ (the “left” and “right” vertices), where $G[L]$ is edgeless and $G[R]$ is complete. Our constructions are close to being split graphs, with the vertex-set partitioning into vertices of degree less than $2k$ and vertices of degree at least $2k$. With a little more effort we can construct examples (for $n \geq 12$) that are actually split graphs.

We will construct an appropriate bipartite graph with partition $L \cup R$ and show that, if we add edges to make $R$ induce a clique, the resulting graph is dually vertex-oblique. If $B$ is bipartite, with partition $L \cup R$, its bipartite complement is the graph $\overline{B}$ with $V(\overline{B}) := V(B)$, $E(\overline{B}) := \{uv \mid u \in L, v \in R, uv \notin E(B)\}$. The vertex-type of $v$ in $\overline{B}$ is $\overline{t}(v)$. A dually semi-vertex-oblique graph is a bipartite graph $B$ with $L = \{\ell_1, \ldots, \ell_{2k}\}$, $R = \{r_1, \ldots, r_{2k}\}$, such that:

(i) $\{t(\ell_1), \ldots, t(\ell_{2k})\}$ contains no repetitions

(ii) $\{t(\ell_1), \ldots, t(\ell_{2k})\} = \{t(r_1), \ldots, t(r_{2k})\}$

(iii) $\{\overline{t}(\ell_1), \ldots, \overline{t}(\ell_{2k})\} = \{t(\ell_1), \ldots, t(\ell_{2k})\}$, and (thus)

$\{\overline{t}(r_1), \ldots, \overline{t}(r_{2k})\} = \{t(r_1), \ldots, t(r_{2k})\}$.

If $\ell_1$, say, had degree $2k$, then in $\overline{B}$ it would have degree 0, so by conditions (ii) and (iii) there must be a vertex of degree 0 in $R$, a contradiction. So the minimum degree in $L$ is at least 1, and by (iii) the maximum degree is at most $2k - 1$, and similarly for $R$.

We now add edges to make $R$ induce a clique, giving us a split graph $G$. The degree of any vertex $r_j$ jumps up by $2k - 1$, so its degree becomes at least $2k$; thus the degrees (and vertex-types) of vertices in $R$ become distinct from those in $L$. If $t(r_i)$ differed from $t(r_j)$ in the number of entries equal to $d$, where $d < 2k$, then in $G$ both types get $2k - 1$ new entries that are all
at least $2k$, but they still differ in the number of entries equal to $d$. If $t(\ell_i)$ differed from $t(\ell_j)$ in the number of entries equal to $d$, in $G$ they will differ in the number of entries equal to $d + 2k - 1$. Thus $G$ is vertex-oblique, and from (iii) we can see that $\overline{G}$ has the same vertex-types as $G$.

Dually semi-vertex-oblique graphs on 12 and 16 vertices are shown in Figure 3, with their degrees and vertex-types. To verify condition (iii), one has to check that for every vertex on the left with vertex-type $(x_1, \ldots, x_r)$, there is another vertex that is non-adjacent to vertices (on the right) of degrees $(2k - x_1, \ldots, 2k - x_r)$.

Given a dually semi-vertex-oblique graph $B$ on $n = 4k$ vertices, we now show how to construct $B'$ on $n + 8 = 4k'$ vertices, where $k' := k + 2$. We add vertices $L_2, L_2', L_2', L_2'$ on the left, and $R_2, R_2', R_2', R_2'$ on the right. The vertices with subscript $2$ will have degree 2, those with subscript $\tilde{2}$ will have degree $2k' - 2 = 2k + 2$. See Figure 4 for a sketch of the new vertices and their adjacencies to each other and to the vertices $\ell, \tilde{\ell}, r, \tilde{r}$ (described below); the vertices with subscript $\tilde{2}$ are also adjacent to all other vertices on the opposite side.
Figure 4: Adding vertices to make larger semi-vertex-oblique graphs.

By (iii) we can find two vertices $r, \tilde{r}$ such that $\tilde{t}(\tilde{r}) = t(r)$; in particular, if $r$ has degree $d$, then $\tilde{r}$ has degree $2k - d$. We make $L_2$ adjacent to $R_2$ and $r$, $L_3$ adjacent to $R_2, R_3, R'_3$ and $\{r, i \neq r\}$. Similarly $L'_2$ is adjacent to $R'_2$ and $\tilde{r}$, $L'_3$ adjacent to $R'_2, R'_3, R'_3$ and $\{r, i \neq \tilde{r}\}$.

By (ii) there are (unique) vertices $\ell, \tilde{\ell}$ with $\tilde{t}(\tilde{\ell}) = t(\ell)$. The adjacencies for $R_2, R'_2, R_2, R'_3$ are defined as above: $N(R_2) := \{L_2, \tilde{\ell}\}, N(R'_2) := \{L_2, L'_2, L'_3\} \cup \{\ell, i \neq \tilde{\ell}\}, N(R'_2) := \{L'_2, \ell\}, N(R'_3) := \{L'_2, L'_3\} \cup \{\ell, i \neq \ell\}$.

The adjacencies of the new vertices are well-defined, and the construction is symmetric (as far as degrees and vertex-types go) with respect to $L$ and $R$, so (ii) holds. The degrees of every $\ell, i$ increase by 2 (so they now range between at least 3 and at most $2k + 1$). If $t(\ell, i)$ differed from $t(\ell, j)$ in the number of entries equal to $d$, in $B'$ they will differ in the number of entries equal to $d + 2$. By construction, $t(L_2), t(L'_2), t(L_3)$ and $t(L'_3)$ are distinct from each other (and from the $t(\ell, i)$’s, because of their degrees). Thus (i) holds.

In the bipartite complement, $N(L_2) = \{R_2, R'_2, R'_3\} \cup \{r, i \neq r\}, N(L'_2) = \{R_2, R'_2, R'_3\} \cup \{r, i \neq \tilde{r}\}$ and $N(L'_3) = \{R_2, \tilde{r}\}$. The neighbourhoods of $R_2, R'_2, R_3$ and $R'_3$ are changed similarly. Recall also that $\tilde{t}(\tilde{\ell}) = t(\ell)$ and $\tilde{t}(\tilde{r}) = t(r)$. Thus $B'$ is obtained from $B$ in the same way as we obtained $B'$ from $B$ (with the roles of $L_2$ and $L'_2$ interchanged, and similarly for $L'_2$ and $L'_2$, $\ell$ and $\tilde{\ell}$, and so on). Since $B$ satisfied (iii), $B'$ does too.
3 Construction on $4k + 1$ vertices

Take a dually vertex-oblique graph $G$ on $4k$ vertices, and introduce a new vertex $u_0$ that is adjacent to the $2k$ vertices with degree $d \geq 2k$. We claim that the resulting graph $G'$ of order $n' = 4k + 1$ is again dually vertex-oblique. Note that if $G$ was a split graph, then the high-degree vertices must have formed a clique, and thus $G'$ will also be split.

A vertex of degree $d := d_G(v)$ in $G$ has degree $d' := d_{G'}(v)$ in $G'$. So $d' = d$ if $d < 2k$, and $d' = d + 1$ if $d \geq 2k$. If $v$ and $w$ had complementary degrees in $G$, that is, $d_G(v) + d_G(w) = n - 1 = 4k - 1$, then $d_{G'}(v) + d_{G'}(w) = n' - 1 = 4k$; so $v$ and $w$ still have complementary degrees in $G'$; this means that if $\vec{d} = f$ in $G$, then $\vec{d'} = f'$ in $G'$. Also, $d_{G'}(u_0) + d_{G'}(u_0) = 2k + 2k = n' - 1$.

In $G'$, $u_0$ will be the unique vertex of degree $2k$; and if $v, w$, were adjacent in $G$ to different numbers of vertices of degree $d$, then in $G'$ they are adjacent to different numbers of vertices of degree $d'$; so $G'$ is vertex-oblique.

Since $G$ is dually vertex-oblique, for every vertex $v$ there is a unique vertex $v'$ with $\vec{t}(v') = t(v)$. If $t(v) = (x_1, \ldots, x_r)$, then $v'$ has non-neighbours in $G$ of degrees $x_1, \ldots, x_r$. If $r < 2k$, then $u_0$ is adjacent to $v'$ but not to $v$, so in $G'$ $v$ has vertex-type $t'(v) = (x_1', \ldots, x_r')$, and $v'$ has non-neighbours of degrees $x_1', \ldots, x_r'$; thus $\vec{t}(v') = t'(v)$. If $r > 2k$ then $u_0$ is adjacent to $v$ in $G'$, and to $v'$ in $\overline{G'}$; thus $t'(v) = (x_1', \ldots, 2k, \ldots, x_t') = \vec{t}(v')$. Finally $t'(u_0) = \vec{t}(u_0)$, so $G'$ is dually vertex-oblique.

4 Vertex-type sequences:
uniqueness and non-uniqueness

The degree sequence of a graph on $n$ vertices is the sequence $d_1 \geq \cdots \geq d_n$ of its degrees (see footnote 1 p. 2). The vertex-type sequence is the sequence $t_1 \geq \cdots \geq t_n$ of vertex-types, where $t_i > t_j$ if $t_i$ is longer than $t_j$, or if $t_i$ and $t_j$ have the same length and $t_i$ is lexicographically larger than $t_j$. $G_d$ is the subgraph of $G$ induced by vertices of degree $d$, and (for $p \neq q$) $G_{p,q}$ is the bipartite subgraph induced by edges joining a vertex of degree $p$ to a vertex of degree $q$.

Some graphs, such as complete graphs, edgeless graphs and matchings, have unique degree sequence (that is, no other graph has the same degree sequence) and, thus, unique vertex-type sequence. If $G$ is dually vertex-oblique, then by definition its complement shares the same vertex-type sequence, and
is not isomorphic to $G$ because self-complementary graphs have non-trivial automorphisms. But could this complementary pair be the unique graphs with that vertex-type sequence? We show here that the answer is always ‘No’, but that there are infinitely many super vertex-oblique graphs with unique vertex-type sequence.

The key to the proofs is a restricted switching operation. A switch is the replacement of edges $v_0w_0, v_1w_1$, with new edges $v_0w_1, v_1w_0$ (that is, $v_0w_1$ and $v_1w_0$ did not appear in the original graph); this does not change the degree of any vertex, but may change the vertex-types. A $(d, d')$-switch (or just ‘restricted switch’, when $d$ and $d'$ are not specified) is a switch where $v_0$ and $v_1$ both have degree $d$, and $w_0$ and $w_1$ both have degree $d'$ (possibly equal to $d$); such a switch does not change the type of any vertex. In a bipartite graph, a switch respects the bipartition if $v_0, v_1$, are in the same part, and (thus) $w_0, w_1$, are in the opposite part.

3. **Theorem.** For any dually vertex-oblique graph $G$, there is a graph $H \not\in \{G, \overline{G}\}$ with the same vertex-type sequence as $G$.

**Proof:** We will establish:

**Claim.** For any degree $d \neq (n-1)/2$, $G$ has distinct vertices $v_0, v_1$, of degree $d$, and $w_0, w_1$, of degree $\overline{d}$, such that $v_0w_0, v_1w_1 \in E(G)$, $v_0w_1, v_1w_0 \not\in E(G)$.

The result follows from the claim since we can then perform a $(d, \overline{d})$-switch which gives us another graph $H$ without changing the type of any vertex. Since $G$ has trivial automorphism group, $H \not\cong G$, and we will show that $H \not\cong \overline{G}$.

Let $x$ be any vertex not in $\{v_0, v_1, w_0, w_1\}$. Let $\overline{x}$ be the unique vertex that has the same vertex-type in $\overline{G}$ as $x$ has in $G$. If $\overline{x} \neq x$ (possibly $\overline{x} \in \{v_0, v_1, w_0, w_1\}$), note that $x$ and $\overline{x}$ are adjacent in $H$ iff they are adjacent in $G$ iff they are not adjacent in $\overline{G}$. If $x = \overline{x}$, take another vertex $y \not\in \{x, v_0, v_1, w_0, w_1\}$; note that $x$ and $y$ exist by the remark after Lemma 2. If $y \neq \overline{y}$ we are done, otherwise note that $x$ and $y$ are adjacent in $H$ iff they are adjacent in $G$ iff $\overline{x} = x$ and $\overline{y} = y$ are not adjacent in $\overline{G}$.

We now turn to proving the Claim, which is equivalent to saying that the be the bipartite graph $G_{d, \overline{d}}$ has an induced $2K_2$. If there is any vertex
z of degree d that is adjacent to no (or all) vertices of degree $\overline{d}$, then in $\overline{G}$ z would be a vertex of degree $\overline{d}$ adjacent to all (or no) vertices of degree $d$, contradicting the fact that $G$ and $\overline{G}$ have the same vertex-type sequence. So in $G_{d,\overline{d}}$ every vertex has at least one neighbour and one non-neighbour from the opposite part.

Let $G_{d,\overline{d}}$ have bipartition $D \cup \overline{D}$. In what follows, $x_i$ will be a vertex in $D$, $N_i \subseteq \overline{D}$ the set of its neighbours, and $N'_i := \overline{D} \setminus N_i$; $N_i \not= \emptyset \not= N'_i$ by the previous argument. Take an arbitrary vertex $x_0 \in D$. Pick a vertex $y_1 \in N'_0$, and let $y_1$ be adjacent to some vertex $x_1$; clearly $x_1 \not= x_0$. If there is a vertex $\tilde{y}_0 \in N_0$ such that $\{x_0, x_1, \tilde{y}_0, y_1\}$ induce a $2K_2$, we are done; otherwise, $x_1$ is adjacent to all of $N_0$, as well as $y_1$, so $N_1 \supseteq N_1$, and $N'_1 \subseteq N'_0$. Pick a vertex $y_2 \in N_1$, and let $y_2$ be adjacent to $x_2$; as before, $x_2 \not= x_1$, and either there is $\tilde{y}_1 \in N_1$ such that $\{x_1, x_2, \tilde{y}_1, y_2\}$ induce a $2K_2$, or $N'_2 \subseteq N'_1$. Repeating this procedure we must eventually find an induced $2K_2$, since $N'_i$ can never be empty. 

A graph $G$ can be transformed by switches into any other graph $H$ with the same degree sequence. If $H$ even has the same vertex-type sequence as $G$, then we will show how to achieve this using only restricted switches.

Suppose the vertices of a graph $G$ are labeled $v_1, \ldots, v_n$, with $\Delta = d(v_1) \geq \cdots \geq d(v_n)$. By switching, we can transform $G$ into a canonical labeled graph $G_0$ that is determined completely by the degree sequence (the first step in this recursive process is to use switches to make $v_1$ adjacent to $v_2, \ldots, v_{\Delta+1}$); any other labeled graph $H$ with the same vertex-set and the same degree sequence (i.e. $d_G(v_i) = d_H(v_i)$ for all $i$) can also be transformed into $G_0$. These ideas, and analogous ones for bipartite graphs, give us:

**Theorem [3, 5].** If $G$, $H$, are two labeled graphs with the same degree sequence, then $G$ can be obtained from $H$ by a sequence of switches. Moreover, if $G$ and $H$ are bipartite, the switches respect the bipartition. 

We use this to prove the next result, that has probably also appeared in [11]:

**4. Theorem.** If $G$, $H$, are two labeled graphs with the same vertex-type sequence, then $G$ can be obtained from $H$ by a sequence of restricted switches.
Proof: The vertex-type sequence clearly determines the degree sequence. Moreover, for every degree \( d \), the subgraphs \( G_d \) and \( H_d \) have the same vertex-set and the same degree sequence, since this is also determined by the vertex-types; we can therefore transform \( G_d \) into \( H_d \) by a sequence of switches; note that in \( G \) these are just \((d, d)\)-switches. Similarly, for every \( p \neq q \) in the degree sequence, the bipartite graphs \( G_{p, q} \) and \( H_{p, q} \) have the same vertex-set, the same bipartition, and the same degrees, so we can transform \( G_{p, q} \) into \( H_{p, q} \) by switches; moreover, we can use switches that respect the bipartition, and these will be valid \((p, q)\)-switches in \( G \) even though \( G_{p, q} \) is not a vertex-induced subgraph of \( G \).

\[ \square \]

5. Corollary. If no \( G_d \) and no \( G_{p, q} \) contains an induced \( 2K_2 \), then \( G \) has unique vertex-type sequence. In particular, if every degree appears at most once in \( G \), except for some degree that appears at most three times, then \( G \) has unique vertex-type sequence. \[ \square \]

The converse of the corollary is not true (the matchings are a counterexample), because a restricted switch may give us a graph \( G' \) isomorphic to \( G \). But it can be used to show, for example, that the super vertex-oblique graphs \( G_6^0, G_1^1, G_2^2, G_2^0 \), in \[10\] have unique vertex-type sequence. In particular, every degree appears exactly once in \( G_2^8 \), except for five vertices of the same degree that induce a graph with only one edge; it can be checked that applying Construction 1 of \[10\] with \( k = 1 \) preserves these properties, and it is shown in that paper that the result is again connected and super vertex-oblique. We thus have:

6. Corollary. There are infinitely many connected super vertex-oblique graphs with a unique vertex-type sequence. \[ \square \]

5 Recognising degree and vertex-type sequences

A graph \( G \) realises its degree sequence, and its vertex-type sequence. Erdős and Gallai\(^2\)\[11\] showed that a sequence \( d_1 \geq \cdots \geq d_n \) is realised by some

\(^2\)Several authors have given different characterisations of degree sequences of graphs.
graph if and only if, for \( r = 1, \ldots, n - 1 \), we have

\[
\sum_{i=1}^{r} d_i \leq r(r - 1) + \sum_{j=r+1}^{n} \min(r, d_j).
\]

If \( G \) is a graph with the same degree sequence as \( \overline{G} \), and \( r_d \) is the number of vertices of degree \( d \), then:

\begin{align*}
(*) & \quad d_i + d_{n-i+1} = n - 1, \text{ for } i = 1, \ldots, n; \\
(**) & \quad r_d \text{ is even for all } d, \text{ except for } r_{(n-1)/2} \equiv 1 \pmod{4}.
\end{align*}

Clapham and Kleitman [2] showed by construction that every sequence that satisfies (*) and the Erdős-Gallai conditions, is realised by a self-complementary graph. However (Lemma 2.F), not all such sequences are realised by a dually vertex-oblique graph. It would be interesting to characterise the degree sequences of dually vertex-oblique graphs. One might also ask similar questions about vertex-type sequences:

**Problem.** A. When is a sequence (of sequences of positive integers) the vertex-type sequence of some graph?
B. Characterise the degree sequences and vertex-type sequences of:

- vertex-oblique graphs,
- super vertex-oblique graphs, and
- dually vertex-oblique graphs.

The Erdős-Gallai results on degree-sequences, together with the Gale-Ryser conditions explained below, lead to an efficient algorithm to solve the vertex-type sequence problems; if the sequence is realised by some graph, the algorithm can also be made to construct an example. However, we would like a more succinct characterisation similar to that of Erdős-Gallai, Clapham-Kleitman or Gale-Ryser, especially as this might shed light on the degree sequence problems.

Gale [6] and Ryser [8] showed that sequences \( p_1 \geq \cdots \geq p_m \) and \( q_1 \geq \cdots \geq q_n \) are the degrees of a bipartite graph \( B \) (with the \( p_i \)'s being degrees
on one side, and the $q_j$’s the degrees on the other side) if and only if, for $r = 1, \ldots, n - 1$:

$$\sum_{i=1}^{m} \min\{r, p_i\} \geq \sum_{j=1}^{r} q_j.$$  

Given the vertex-type sequence of a graph $G$, we can recover the degree sequence, and compute the vertex-types of $\overline{G}$ (as noted at the beginning of the introduction); it is then straightforward to check whether $G$ is (super or dually) vertex-oblique. So we turn our attention to Problem A.

If we want to check whether a given sequence is actually the vertex-type sequence of some graph $G$, we recover the degree-sequences of the $G_d$’s and $G_{p,q}$’s (for all $d, p, q$, in the degree-sequence of $G$), and check the Erdős-Gallai and Gale-Ryser conditions, respectively. If the conditions are not all satisfied, we have a contradiction; otherwise, we can construct $G_d$’s and $G_{p,q}$’s that together give us a graph with the given vertex-type sequence.

6 Other open problems

In a self-complementary graph of order $4k + 1$, one can always remove an appropriate vertex to get a self-complementary graph of order $4k$. It is not clear whether an analogous claim is true for dually vertex-oblique graphs.

**Problem.** Is there a dually vertex oblique graph on $4k + 1$ vertices, such that removing any vertex of degree $2k$ leaves a subgraph $H$ such that (a) $H$ does not have the same vertex-types as its complement, or (b) $H$ is not vertex-oblique, or both (a) and (b)?

For any fixed $k$, Schreyer et al. [10] constructed super vertex-oblique graphs that were $k$-connected, with $k$-connected complements. Our examples of dually vertex-oblique graphs have vertices of degree 2, and thus connectivity at most 2.

**Problem.** Are there (complementary pairs of) dually vertex-oblique graphs of arbitrarily high connectivity?
7 Acknowledgements

My studies in Canada are fully funded by the Canadian government through a Canadian Commonwealth Scholarship.
I would like to thank Jens Schreyer and Hansjoachim Walther for introducing me to the concept of vertex-oblique graphs at the Cycles and Colourings workshop in Stará Lesná (Slovakia, 2002), and the Graph Theory conference at Czorstyn (Poland, 2002). Participation in these conferences was funded by the University of Waterloo.

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