Photon–number tomography and fidelity
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Abstract
The scheme of photon–number tomography is discussed in the framework of star–product quantization. The connection of dual quantization scheme and observables is reviewed. The quantizer and dequantizer operators and kernels of star–product of tomograms in photon–number tomography scheme and its dual one are presented in explicit form. The fidelity and state purity are discussed in photon–number tomographic scheme, and the expressions for fidelity and purity are obtained in the form of integral of the product of two photon–number tomograms with integral kernel which is presented in explicit form. The properties of quantumness are discussed in terms of inequalities on state photon–number tomograms.

Key words: tomograms, quantizer, dequantizer, photon quadratures, star–product, photon–number tomography
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Introduction
The classical states are associated with the probability density. The quantum states usually are associated either with the wave function or the density operator. So, usually quantum and classical states are described by different objects: probability distribution function or operators. The probability representation of quantum mechanics was suggested in [1], where the quantum states are associated with tomographic probability distribution called symplectic tomogram. The tomographic probability representation of classical mechanics was suggested in [2], where the states of classical systems can be also associated with tomograms. Thus, in the tomographic probability representation, both classical and quantum states are described by the same object – tomogram. In [3–6] the analogous description of quantum spin states by probability distribution was suggested. The photon–number tomography was introduced in [7–9]. Photon–number tomograms of quantum gaussian states in one–mode and multimode cases were obtained in [10 11]. The symplectic and photon–number tomograms of photon–phonon mode in the process of Raman Scattering were considered in [12–14]. The symplectic and photon–number tomograms of even and odd coherent states were considered in [15 16]. The symplectic tomograms of the states of quantum resonant circuit and Josephson junction were considered in [17]. Explicit connection of photon–number tomogram with measurable by homodyne detector optical tomogram [18 19] was obtained in [20 21].
The aim of the star–product approach is to find the description of quantum properties by using classical–like instruments as probability distributions. In the star–product quantization scheme (see e.g. [22] -[25]) functions on phase–space are used to describe physical observables in quantum mechanics instead of operators. The example of such approach is using the Wigner quasidistribution function [26] instead of density operator for describing quantum states. The Wigner function satisfies several conditions which are satisfied by classical probability distributions on phase–space, but the Wigner function can take the negative values and can not be considered as the probability distribution. Due to this it is called the quasiprobability. Another star–product scheme was suggested in [27]–[29] in which the quantum states can be described by the standard positive probability distribution (symplectic tomogram) [1] instead of the description of the quantum states by the Wigner function [26] which is quasidistribution. The spin tomography was discussed in framework of star–product quantization scheme in [30]. The photon–number tomography was considered in the framework of star–product quantization in [11, 31, 32]. The connection of dual quantization scheme and observables was found in [33].

The aim of the paper is to present a review of the photon–number tomography within the framework of star–product quantization, to discuss the connection of dual quantization scheme and observables, to present in explicit form kernels of star–product of symbols of operators in photon–number tomographic scheme and its dual one, and to consider the fidelity and purity of the state within the framework of the photon–number tomographic representation.

The paper is organized as follows. In Sec. 1 we review the general scheme of quantization based on star–product formalism. In Sec. 2 we review the dual quantization scheme and its connection with observables. In Sec. 3 we review photon–number tomography approach and present explicit expressions of kernels of star–product of photon–number tomograms in initial and dual schemes. In Sec. 4 the fidelity and purity of the state within the framework of the photon–number tomographic representation are considered. Conclusions are given in Sec. 5.

1 General star–product quantization scheme

Following [27] [28] [29] [33], let us consider an operator \( \hat{A} \) acting in a Hilbert space and the \( c \)-number function \( f_\hat{A}(\mathbf{x}) \) of vector variables \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \)

\[
f_\hat{A}(\mathbf{x}) = \text{Tr} \left[ \hat{A} \hat{U}(\mathbf{x}) \right].
\]

We suppose that the relation (1) has the inverse

\[
\hat{A} = \int f_\hat{A}(\mathbf{x}) \hat{D}(\mathbf{x}) \, d\mathbf{x}.
\]

The operator \( \hat{U}(\mathbf{x}) \) is called dequantizer [33]. The function \( f_\hat{A}(\mathbf{x}) \) is symbol of the operator \( \hat{A} \) and the operator \( \hat{D}(\mathbf{x}) \) is called quantizer [33]. The formulas (1) and (2) are selfconsistent if one has the following property of the quantizer and dequantizer

\[
\text{Tr} \left[ \hat{U}(\mathbf{x}) \hat{D}(\mathbf{x}') \right] = \delta (\mathbf{x} - \mathbf{x}').
\]
We introduce the product of two functions corresponding to two operators

\[ \hat{A}\hat{B} \rightarrow f_{\hat{A}}(x) * f_{\hat{B}}(x) \]

in the following form

\[ f_{\hat{A}}(x) * f_{\hat{B}}(x) := \text{Tr} \left[ \hat{A}\hat{B}\hat{U}(x) \right]. \]

The map (1) provides the nonlocal product of two functions (star–product)

\[ f_{\hat{A}}(x) * f_{\hat{B}}(x) = \int f_{\hat{A}}(x') f_{\hat{B}}(x'') K(x'', x', x) \, dx' \, dx''. \]

The kernel \( K(x'', x', x) \) of star–product of two symbols is linear with respect to the dequantizer and nonlinear in the quantizer operator

\[ K(x'', x', x) = \text{Tr} \left[ \hat{D}(x'')\hat{D}(x')\hat{U}(x) \right]. \] (4)

The standard product of operators is an associative product

\[ \hat{A}(\hat{B}\hat{C}) = (\hat{A}\hat{B})\hat{C}. \]

The product of functions is associative since the product of the operators is associative

\[ f_{\hat{A}}(x) * \left( f_{\hat{B}}(x) * f_{\hat{C}}(x) \right) = \left( f_{\hat{A}}(x) * f_{\hat{B}}(x) \right) * f_{\hat{C}}(x). \] (5)

The associativity condition for operator symbols means that the kernel (4) satisfies the nonlinear integral equation

\[ \int K(x_1, x_2, y) K(y, x_3, x_4) dy = \int K(x_1, y, x_4) K(x_2, x_3, y) dy. \] (6)

2 The dual star–product scheme and quantum observable

Let us consider the following scheme

\[ f_{\hat{A}}^{(d)}(x) = \text{Tr} \left[ \hat{A}\hat{D}(x) \right], \] (7)

\[ \hat{A} = \int f_{\hat{A}}^{(d)}(x)\hat{U}(x) \, dx. \] (8)

One can see, that we permute the quantizer and the dequantizer because the compatibility condition \([3]\) will be valid in both cases. We consider the new quantizer–dequantizer pair as dual to the initial one

\[ \hat{U}'(x) \mapsto \hat{D}(x), \quad \hat{D}'(x) \mapsto \hat{U}(x). \]
The interchange corresponds to a specific symmetry of the equation (6) for associative star–product kernel. The star–product of dual symbols $f^{(d)}_{\hat{A}}(x), f^{(d)}_{\hat{B}}(x)$ of two operators $\hat{A}$ and $\hat{B}$ is described by dual integral kernel

$$K^{(d)}(x'', x', x) = \text{Tr} \left[ \hat{U}(x'') \hat{U}(x') \hat{D}(x) \right]. \tag{9}$$

The dual kernel (9) is another solution of nonlinear equation (6) \[33\].

Let us consider the mean value of quantum observable $\langle \hat{A} \rangle = \text{Tr}(\hat{\rho} \hat{A}) = \int f(\hat{\rho})(x) \text{Tr}(\hat{D}(x)\hat{A}) \, dx$.

Using the expression for dual symbol of operator (7) we obtain the formula

$$\langle \hat{A} \rangle = \int f(\hat{\rho})(x) f^{(d)}_{\hat{A}}(x) \, dx.$$

So, one can see, that the mean value of an observable $\hat{A}$ is given by the integral of the product of the tomographic symbol of the density operator $f(\hat{\rho})(x)$ in a given quantization scheme and the symbol $f^{(d)}_{\hat{A}}(x)$ of the observable $\hat{A}$ in the dual scheme.

### 3 Photon–number tomography as example of star–product quantization

Photon–number tomography is the method to reconstruct density operator of quantum state using measurable probability distribution function (photon statistics) called tomogram. Photon–number tomography is different from optical tomography method and from symplectic tomography scheme, where the continuous homodyne quadrature are measured for reconstructing quantum state. In photon–number tomography the discrete random variable is measured for reconstructing quantum state. The photon–number tomogram

$$\omega(n, \alpha) = \langle n | \hat{D}(\alpha) \hat{\rho} \hat{D}^{-1}(\alpha) | n \rangle \tag{10}$$

is the function of integer photon number $n$ and complex number $\alpha$, $\hat{\rho}$ is the state density operator. The photon–number tomogram is the photon distribution function (the probability to have $n$ photons) in the state described by the displaced density operator. We take Planck constant $\hbar = 1$.

For example, photon–number tomogram for oscillator ground state is

$$w_0(n, \alpha) = \frac{e^{-|\alpha|^2}}{n!} |\alpha|^{2n}.$$

The photon–number tomograms of excited oscillator states with density operators $\hat{\rho}_m = | m \rangle \langle m |$ are

$$w^{(m)}(n, \alpha) = \frac{n!}{m!} |\alpha|^{2m} e^{-|\alpha|^2} \left( L^{m-n}(|\alpha|^2) \right)^2.$$


The quantizer operator is

\[ D = \frac{m!}{n!} |\alpha|^2 (L_m^{n-m}(|\alpha|^2))^2 \]

where \( L_m^r(x) \) are Laguerre polynomials.

Let us consider photon–number tomogram in the framework of star–product quantization. In the given photon–number tomography quantization scheme the dequantizer is of the form

\[ \hat{U}(x) = \hat{D}(\alpha) |n\rangle \langle n| \hat{D}^{-1}(\alpha), \ x = (n, \alpha). \]  (11)

The quantizer operator is

\[ \hat{D}(x) = \frac{4}{\pi(1-s^2)} \left( \frac{s-1}{s+1} \right) (\hat{a}^\dagger + \alpha^*) (\hat{a} + \alpha)^{-n}, \]  \( n \geq m \),

where \( s \) is ordering parameter, \( \alpha \) is complex number

\[ \alpha = \text{Re} \alpha + i \text{Im} \alpha, \]

\( \hat{D}(\alpha) \) is the Weyl displacement operator

\[ \hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}). \]

The kernel \( K \) of star–product of photon–number tomograms in the given photon–number tomography quantization scheme is

\[ K(n_1, \alpha_1, n_2, \alpha_2, n_3, \alpha_3) = \text{Tr} \{ \hat{D}(n_1, \alpha_1) \hat{D}(n_2, \alpha_2) \hat{U}(n_3, \alpha_3) \}. \]  (13)

Putting in formula (13) the expressions for quantizer (12) and dequantizer (11) operators of photon–number tomography scheme and taking the trace we obtain the kernel of star–product of photon–number tomograms in the explicit form

\[ K(n_1, \alpha_1, n_2, \alpha_2, n_3, \alpha_3) = \left( \frac{4}{\pi(1-s^2)} \right)^2 \exp(it(n_1 + n_2 - 2n_3)) \exp[-|\alpha_3 + \alpha_1 - \alpha_1 e^{-it} \]

\[ + \alpha_2 e^{-it} - \alpha_2 e^{-2it} + \alpha_3 e^{-2it} |^2 + \frac{1}{2}(-\alpha_3 \alpha_1^* + \alpha_3^* \alpha_1 - \alpha_1 \alpha_2 + \alpha_1^* \alpha_2 - \alpha_2 \alpha_3 + \alpha_2^* \alpha_3 + \alpha_3 \alpha_3 e^{-it} + \alpha_3^* \alpha_3 e^{-it} - \alpha_1 \alpha_2 e^{-it} - \alpha_1^* \alpha_2 e^{-it} + \alpha_3 \alpha_3^* e^{2it} + \alpha_2 \alpha_2^* e^{2it} - |\alpha_2| e^{-it + 2it} - |\alpha_3| e^{2it} + \alpha_1 \alpha_3^* e^{2it} - \alpha_1^* \alpha_3 e^{-it + 2it} \]

\[ + \alpha_2 \alpha_3 e^{-it + 2it} - \alpha_3^* \alpha_2 e^{-2it} + \alpha_1^* \alpha_2 e^{-2it} - \alpha_1 \alpha_2^* e^{2it - 2it} - |\alpha_2| e^{-2it - 2it} + |\alpha_3| e^{-2it - 2it} - \alpha_1 \alpha_3 e^{-2it} + \alpha_1^* \alpha_3 e^{-2it - 2it} + \alpha_3 \alpha_3^* e^{-2it - 2it}) |L_n^r(|\alpha_3 + \alpha_1 - \alpha_1 e^{-it} + \alpha_2 e^{-it} - \alpha_2 e^{-2it} + \alpha_3 e^{-2it|^2}]. \]  (14)
where \[
\frac{s-1}{s+1} = e^{it}
\]
and \(L_n(x)\) is Laguerre polynomial.

Let us consider the dual photon–number tomography quantization scheme. We replace the quantizer \((12)\) and the dequantizer \((11)\) each the other and consider the dual to the initial one quantizer–dequantizer pair. So, the dequantizer operator in dual photon–number tomography scheme is

\[
\hat{U}'(n, \alpha) = \hat{D}(n, \alpha),
\]
the quantizer operator in dual photon–number tomography scheme has the form

\[
\hat{D}'(n, \alpha) = \hat{U}(n, \alpha).
\]

The kernel \((9)\) of star–product of symbols of operators in the case of dual photon–number tomography quantization scheme is

\[
K^{(d)}(n_1, \alpha_1, n_2, \alpha_2, n_3, \alpha_3) = \text{Tr}[\hat{U}(n_1, \alpha_1)\hat{U}(n_2, \alpha_2)\hat{D}(n_3, \alpha_3)].
\]

Putting in formula \((17)\) the expressions for quantizer \((16)\) and dequantizer \((15)\) operators of photon–number tomography scheme and taking the trace we obtain the dual kernel of star–product of symbols of operators (for example, for the symbols of density operators – photon–number tomograms) in dual photon–number tomography scheme in the explicit form

if \(n_1 \geq n_2\), then

\[
K^{(d)}(n_1, \alpha_1, n_2, \alpha_2, n_3, \alpha_3) = \frac{4n_2!}{\pi(1-s^2)n_1!} \exp(it(n_3 - n_1)) \exp\left[\frac{1}{2}(-\alpha_1\alpha_2^* + \alpha_1^*\alpha_2
\right.

\[\left.-\alpha_2\alpha_3^* + \alpha_2^*\alpha_3 - \alpha_3\alpha_1^* + \alpha_3^*\alpha_1 + \alpha_2\alpha_3 e^{it} - |\alpha_3|^2 e^{it} - \alpha_2\alpha_1 e^{it} + \alpha_3\alpha_1^* e^{it} - \alpha_3\alpha_2^* e^{-it}
\]

\[+|\alpha_3|^2 e^{-it} + \alpha_1\alpha_2 e^{-it} - \alpha_1\alpha_3 e^{-it} - |\alpha_2 - \alpha_1|^2 - |\alpha_3 - \alpha_1 - \alpha_3 e^{-it} + \alpha_1 e^{-it}|^2)]
\]

\[\times[\alpha_2 - \alpha_1 \left(-\alpha_3^* + \alpha_1^* + \alpha_3 e^{it*} - \alpha_1^* e^{it*}\right)]^{(n_1-n_2)} L_{n_2}^{n_1-n_2}(|\alpha_2 - \alpha_1|^2)
\]

\[\times L_{n_2}^{n_1-n_2}(|\alpha_3 - \alpha_1 - \alpha_3 e^{-it} + \alpha_1 e^{-it}|^2),
\]

if \(n_1 \geq n_2\), then

\[
K^{(d)}(n_1, \alpha_1, n_2, \alpha_2, n_3, \alpha_3) = \frac{4n_1!}{\pi(1-s^2)n_2!} \exp(it(n_3 - n_1)) \exp\left[\frac{1}{2}(-\alpha_1\alpha_2^* + \alpha_1^*\alpha_2
\right.

\[\left.-\alpha_2\alpha_3^* + \alpha_2^*\alpha_3 - \alpha_3\alpha_1^* + \alpha_3^*\alpha_1 + \alpha_2\alpha_3 e^{it} - |\alpha_3|^2 e^{it} - \alpha_2\alpha_1 e^{it} + \alpha_3\alpha_1^* e^{it} - \alpha_3\alpha_2^* e^{-it}
\]

\[+|\alpha_3|^2 e^{-it} + \alpha_1\alpha_2 e^{-it} - \alpha_1\alpha_3 e^{-it} - |\alpha_2 - \alpha_1|^2 - |\alpha_3 - \alpha_1 - \alpha_3 e^{-it} + \alpha_1 e^{-it}|^2)]
\]

\[\times[\alpha_2 - \alpha_1 \left(-\alpha_3^* + \alpha_1^* + \alpha_3 e^{it*} - \alpha_1^* e^{it*}\right)]^{(n_1-n_2)} L_{n_2}^{n_1-n_2}(|\alpha_2 - \alpha_1|^2)
\]

\[\times L_{n_2}^{n_1-n_2}(|\alpha_3 - \alpha_1 - \alpha_3 e^{-it} + \alpha_1 e^{-it}|^2),
\]
where the kernel $K_{\text{photon quadratures}}$ the generalized fidelity which equals to trace of product of the form of integrals where the tomographic probability distributions are involved. For continuous purity. In fact, we have to present the known quantities given in terms of density operators in terms of photon–number tomograms such physical quantities as fidelity and purity were calculated by using expression of the density operators in terms of the symplectic tomograms in [35]. In the framework of photon–number tomography scheme we get for the fidelity the expression

$$F = \text{Tr}(\hat{\rho}_1 \hat{\rho}_2) = \sum_{n_1, n_2=0}^{\infty} \int w_1(n_1, \alpha_1) w_2(n_2, \alpha_2) \mathcal{K}(n_1, n_2, \alpha_1, \alpha_2) \, d\alpha_1 \, d\alpha_2 \tag{19}$$

and for the state purity the analogous expression

$$P = \text{Tr}(\hat{\rho}_2) = \sum_{n_1, n_2=0}^{\infty} \int w(n_1, \alpha_1) w(n_2, \alpha_2) \mathcal{K}(n_1, n_2, \alpha_1, \alpha_2) \, d\alpha_1 \, d\alpha_2, \tag{20}$$

where the kernel $\mathcal{K}(n_1, n_2, \alpha_1, \alpha_2)$ is of the form

$$\mathcal{K}(n_1, n_2, \alpha_1, \alpha_2) = e^{it(n_1+n_2)} \exp\left[\frac{1}{2}(-|\alpha_1|^2 e^{-it} + |\alpha_1|^2 e^{-it} + \alpha_1 \alpha_2 (1 - e^{-it}) (e^{-it} - e^{-2it})
- \alpha_1^* \alpha_2 (1 - e^{-it}) (e^{-it} - e^{-2it})) - |\alpha_1 - \alpha_2 e^{-it} + \alpha_2 e^{-2it}|^2 - e^{-2it}\right]
\times J_0(2e^{-it}|\alpha_1 - \alpha_2 e^{-it} + \alpha_2 e^{-2it}|) \tag{21}$$

and $J_0(x)$ is Bessel function. So, we obtain the expressions for fidelity [19] and purity [20] in the form of integral of the product of two photon–number tomograms with integral kernel $\mathcal{K}(n_1, n_2, \alpha_1, \alpha_2)$, which we obtain in explicit form [21].

We have following inequalities for fidelity [19] and purity [20] of quantum states of real physical system written in terms of photon–number tomograms

$$0 \leq \sum_{n_1, n_2=0}^{\infty} \int w_1(n_1, \alpha_1) w_2(n_2, \alpha_2) \mathcal{K}(n_1, n_2, \alpha_1, \alpha_2) \, d\alpha_1 \, d\alpha_2 \leq 1, \tag{22}$$

4 Fidelity and purity in photon–number tomography scheme

Both kernels (14) and (18) are solutions of equation (6).
\[ 0 \leq \sum_{n_1, n_2=0}^{\infty} \int w(n_1, \alpha_1)w(n_2, \alpha_2)K(n_1, n_2, \alpha_1, \alpha_2) \, d\alpha_1 \, d\alpha_2 \leq 1. \]  
\tag{23}

Also photon–number tomograms associated with quantum states of real physical system must satisfy inequalities which are conditions of density operators nonnegativity

\[ \frac{4}{\pi(1 - s^2)} \sum_{n=0}^{\infty} \int \left( \frac{s - 1}{s + 1} \right)^{(\hat{a}^\dagger + \hat{a}^\star)(\hat{a} + \alpha) - n} w(n, \alpha) \, d\alpha \geq 0. \]  
\tag{24}

For the photon states we obtained expressions for fidelities (22) and purities (23) given in terms of photon–number tomograms which are probability distributions. The expressions (22), (23) and (24) can be used for checking quantumness of the states analogous to [36], where the expressions written in terms of measurable optical tomograms [35] were used for checking quantumness of the states in experiments with homodyne detection. The inequality (24) can be violated for the classical electromagnetic field.

**Conclusion**

We review the notion of quantum state in photon–number tomography approach. The scheme of photon–number tomography is discussed in the framework of star–product quantization. As new results presented in the paper we want to mention the explicit expressions of the kernels of star–product of photon–number tomograms: expression (14) in given and expression (18) in dual quantization schemes. The fidelity and state purity are discussed in the framework of the photon–number tomography scheme and the explicit expressions for them in the form of the product of two photon–number tomograms with integral kernel (21), which is obtained in explicit form, are presented. The properties of quantumness and classicality are discussed in terms of inequalities on state photon–number tomograms.

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