ABSTRACT

We show that two knots have matching Vassiliev invariants of order less than \( n \) if and only if they are equivalent modulo the \( n \)th group of the lower central series of some pure braid group, thus characterizing Vassiliev’s knot invariants in terms of the structure of the braid groups. We also prove some results about knots modulo the \( n \)th derived subgroups of the pure braid groups, and about knots modulo braid subgroups in general.

0. INTRODUCTION

(0.1) Vassiliev invariants [25] are a rather recent innovation in low-dimensional topology. The main reason that they have attracted a great deal of interest is that they provide a unified framework in which to consider “quantum invariants” such as the Jones polynomial and its generalizations. The main theorem of this paper characterizes Vassiliev invariants in terms of the structure of the braid groups, thus providing something of a bridge between modern quantum topology and classical topology and group theory:

**Theorem 0.2:** If \( K_1 \) and \( K_2 \) are knots, then \( v(K_1) = v(K_2) \) for every Vassiliev invariant of order < \( n \) if and only if there exist a positive integer \( k \) and braids \( p, b \in B_k \) such that \( K_1 = \overline{b} \), \( K_2 = \overline{pb} \), and \( p \in \text{LCS}_n(P_k) \).

(0.3) By \( \text{LCS}_n(P_k) \) we mean the \( n \)th group of the lower central series of the pure braid group \( P_k \), so \( \text{LCS}_n(P_k) = \left[ \text{LCS}_{n-1}(P_k), P_k \right] \) with \( \text{LCS}_1(P_k) = P_k \). By the closure \( \overline{b} \) of a braid \( b \) we mean the standard notion where the \( i \)th strand on the bottom is looped around and identified with the \( i \)th strand on the top. Two good references on Vassiliev invariants, also called finite-type invariants, are Bar-Natan [1] and Birman [4].

(0.4) For the purposes of this paper, we could define a Vassiliev invariant as follows: Any braid closes up to form a unique link. Therefore, any link invariant pulls back to a braid invariant. If the invariant takes values in an abelian group (often an additive subgroup of the complex numbers) then the invariant may be extended to the group ring \( \mathbb{Z}B_k \) for all \( k > 0 \). Let \( I_k \) be the augmentation ideal of \( \mathbb{Z}P_k \) (so \( I_k \) is generated by all \( p - 1 \) for \( p \in P_k \)). A Vassiliev invariant of order < \( n \) is then an invariant which vanishes on \( I_k^n \) for all \( k \). We formulate this more precisely and relate it to the standard definition of Vassiliev invariant in Proposition 2.17.
We define two knots $K_1$ and $K_2$ to be LCS$_n$-equivalent if there exist $k > 0$ and braids $p, b \in B_k$ such that $K_1 = \overline{b}$, $K_2 = \overline{pb}$, and $p \in$ LCS$_n(P_k)$. We say they are $V_n$-equivalent if $v(K_1) = v(K_2)$ for all Vassiliev invariants $v$ of order $< n$ taking values in any abelian group. Then Theorem 0.2 is reworded in Theorem 2.41 to state that two knots are LCS$_n$-equivalent if and only if they are $V_n$-equivalent.

Crucial inspiration for this paper was provided by Habiro’s paper [11], where a different answer is provided to the question of when two knots have matching invariants of order $< n$. (In particular, the idea to set up the diagram of maps in Figure 2.2 was inspired by Habiro.) We do not, however, rely directly on any of Habiro’s results. On the surface, his approach seems to have little to do with ours. For each positive integer $n$, he defines a class of moves which involve twistings along curves inside of handlebodies, and shows that these moves suffice to go between any two knots with matching invariants up to order $n$. However, Habiro also defines for each $n$ another move, which he calls a “$*$” move”, and this move may be shown to be equivalent to the replacement (unoriented, see (0.9)) in a knot diagram of $1 \in B_{n+1}$ by a particular $p \in$ LCS$_n(P_{n+1})$. The braid $p$ may be written explicitly as a product of iterated commutators, but it is easier to appeal to Theorem 4.2 of [19] to show that it is in LCS$_n(P_{n+1})$.

Another motivator and source of ideas for our work is Gusarov’s paper [10]. Gusarov was the first to construct a group structure on knots dual to finite-type invariants. He also seems to be the first to have realized the importance of invariants which are additive under connected sum, and which are therefore homomorphisms from his group into some other abelian group. It is not hard to see that LCS$_n$-equivalent knots in our sense are $n$-equivalent in the sense of Gusarov and Ohyama [20], and in particular that LCS$_n$-trivial knots are $n$-trivial in their sense. It follows from Theorem 0.2 that both notions of equivalence are the same as $V_n$-equivalence.

The “if” part of Theorem 0.2 was proved in [23], where we used it to construct for any knot $K$ and positive integer $n$, an infinite family of knots whose invariants matched those of $K$ up to order $n$. There are two ways in which Theorem 0.2 is actually a stronger converse to the result in [23] than might have been hoped for. First of all, to go between two knots with the same invariants of order $< n$, one might expect to have to make a number of replacements of $1 \in B_k$ with $p \in$ LCS$_n(P_k)$ (not necessarily the same $k$ or $p$ each time), but in fact what we show here is that the change can be done with a single such replacement. This is a consequence of Proposition 1.22, that LCS$_n$-equivalence is an equivalence relation. Note that we give here no way to control $k$.

In [23], a replacement of $1 \in B_k$ with $p \in$ LCS$_n(P_k)$ could be made without regard for the orientation of strands. That is, the braid orientation did not need to coincide with the knot orientation. We showed that such a replacement did not change invariants of order $< n$. Theorem 0.2 says that if two knots have equal invariants of order $< n$ then they are related by such a replacement where the braid orientation does coincide with the knot orientation. This is the second way in which Theorem 0.2 is a strong converse to Theorem 1 of [23].

The lower central series has shown up elsewhere in the study of finite-type invariants. Milnor defined a set of link invariants in [17] using the lower central series of
link groups. Bar-Natan [2] and Lin [14] showed that these invariants, properly interpreted, are of finite type. Kalfagianni and Lin [12] have given a characterization of knots with trivial finite-type invariants of order $< n$ as knots which admit a certain type of Seifert surface $F$ with certain curves that push off into the $n$th group of the lower central series of the group of the complement $F$ in $S^3$. Garoufalidis and Levine have shown [8] that the lower central series of the Torelli group of a surface plays an important role in the study of finite-type invariants of homology 3-spheres.

(0.11) For groups $G$ whose abelianizations are cyclic, the lower central series stabilizes after one term: $\text{LCS}_n(G) = \text{LCS}_2(G)$ for $n > 1$. Therefore it is not very interesting to study the lower central series of knot groups or braid groups (as opposed to link groups and pure braid groups).

(0.12) A simple but important fact (Proposition 2.13) in the study of Vassiliev invariants via braids is that the ideal of braids with $n$ double points is the $n$th power of the augmentation ideal of $\mathbb{Z}P_k$. It is an almost trivial fact of group theory (Proposition 2.24) that if $g$ is an element of any group $G$ and $g \in \text{LCS}_n(G)$, then $g - 1 \in I^n$, the $n$th power of the augmentation ideal of $\mathbb{Z}G$. If the converse holds in a group $G$ then the group is said to have the “dimension subgroup property.” Not all groups have this property (see Rips [21]), but pure braid groups do. Free groups do too, and in fact one way to show that pure braid groups do is to use the result for free groups, the semidirect product decomposition of $P_k$, and an argument of Sandling [22]. See also Gupta [9] for more discussion of the dimension subgroup problem. For pure braids, therefore, being undetectable by invariants of order $< n$ is exactly the same as being in $\text{LCS}_n(P_k)$ (see also Kohno [13]). Theorem 0.2 may then be viewed as extending this last statement to a corresponding result about knots. The proof of Theorem 0.2 in this paper does not use the dimension subgroup property of $P_k$, however (nor do I know of a proof that does).

(0.13) Falk and Randell [7] showed that $P_k$ is residually nilpotent. That is, $\cap_n \text{LCS}_n(P_k) = \{1\}$ for all $k$, and combining this with the dimension subgroup property of $P_k$ shows that Vassiliev invariants (of all orders) distinguish braids. See also Bar-Natan [2].

(0.14) Section 1 sets up some notation and basic results for a general idea of $H$-equivalence of links, where $H = \{H_k\}$ is a sequence of normal subgroups of $B_k$ satisfying an inclusion condition which we call “subcoherence” (see Remark 1.6 for the reasoning behind this term). The case of most interest to us is where $H_k = \text{LCS}_n(P_k)$. We conclude the section by showing (Theorem 1.41) that knots modulo $\text{LCS}_n$-equivalence form a group $\mathcal{K}/\mathcal{K}_n$ under connected sum. We also show (Theorem 1.46) that for each $k$ there is a homomorphism of abelian groups from $\text{LCS}_n(P_k)/\text{LCS}_{n+1}(P_k)$ to the group $\mathcal{K}_n/\mathcal{K}_{n+1}$ of $\text{LCS}_n$-trivial knots modulo $\text{LCS}_{n+1}$-trivial knots.

(0.15) The sequence of results in Sections 1 and 2 is mostly self-contained. For basic facts about groups, see Magnus, Karrass, and Solitar [15]. For basic facts about braids and their closures, see Birman [3]. A slightly strengthened version of Markov’s Theorem on braid closures is used to prove Proposition 1.22, that $\text{LCS}_n$-equivalence is an equivalence relation. A generalization of Alexander’s theorem that any knot may be
written as a closed braid is used in Section 2 to show that the definition of Vassiliev invariant given in (0.4) is the same as the standard one.

(0.16) Section 2 is devoted to the proof of Theorem 0.2/2.41 We use the group of LCS\(_n\)-equivalence classes from Section 1. A key idea is to replace formal sums of knots with connected sums. This is done with what we call composite relators, which are of the form \(K_1 \# K_2 - K_1 - K_2\), where \(K_1\) and \(K_2\) are knots. We show that the defining relations for Vassiliev invariants, namely that any knot with \(n\) double points (resolved in the standard way) vanishes, can be generated by composite relators and relations of the form \(pb = b\) for \(p \in \text{LCS}_n(P_k)\), the latter relations being precisely those that define LCS\(_n\)-equivalence. A consequence of this is Theorem 2.43, which says that if two knots have matching additive invariants up to order \(n\) then they have matching invariants up to order \(n\). This was known for rational-valued invariants (see Bar-Natan [1] or Gusarov [10]), since in this case there is a Hopf algebra structure which shows that the space of all Vassiliev invariants is the polynomial algebra generated by the additive ones. It is not clear whether Theorem 2.43 is a nonvacuous extension, however, since all known invariants taking values in \(\mathbb{Z}/q\mathbb{Z}\) lift to integer-valued invariants.

(0.17) Section 3 extends some of the results of Section 1 to the derived series of \(P_k\). The \(n\)th group of the derived series is given by \(\text{DS}_n(P_k) = [\text{DS}_{n-1}(P_k), \text{DS}_{n-1}(P_k)]\) with \(\text{DS}_1(P_k) = P_k\). We show that for all \(n\), the \(\text{DS}_n\)-equivalence classes of knots form a group under connected sum, and we show how to apply an argument from [23] to prove that every \(\text{DS}_n\)-equivalence class of knots contains infinitely many prime, alternating knots.

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1. KNOTS MODULO BRAID SUBGROUPS

(1.1) First we will develop some general ideas about \(H\)-equivalence of links. Then we will consider the main case of interest to us, equivalence modulo the lower central series of the pure braid groups \(P_k\). We will show that the equivalence classes of knots form a group \(K/K_n\) under connected sum, and define homomorphisms from the quotient groups of the lower central series of \(P_k\) into \(K/K_n\).

**Notation 1.2:** Let \(B_k\) be the braid group on \(k\) strands. Let \(\sigma_i\) be the standard generators of \(B_k\), where \(1 \leq i < k\). The defining relations for \(B_k\) are

\[
\sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{for} \quad |i - j| > 1
\]

and

\[
\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}
\]
The symmetric group $S_k$ is the quotient of $B_k$ by the relations $\sigma_i^2 = 1$. If $b$ is a braid then its image under this quotient map to $S_k$ will be called the permutation associated to $b$. Let $P_k \subset B_k$ be the pure braid group. A pure braid is one whose associated permutation is the identity. For $b \in B_k$, denote by $\overline{b}$ the standard closure of $b$ into a knot or link. For $k > 0$, denote by $\iota_k$ the inclusion of $B_k$ into $B_{k+1}$ by adding an unbraided strand on the right end (algebraically, this means adding the generator $\sigma_k$). We will sometimes identify a braid $b \in B_k$ with $\iota_k(b) \in B_{k+1}$, or with a product of such $\iota_k$ into $B_m$ for $m > k$.

**Notation 1.5:** Let $H = \{H_k\}$ be a sequence of groups such that $H_k$ is a normal subgroup of $B_k$ for each $k$, and such that $\iota_k(H_k) \subset H_{k+1}$ for each $k$. We will call such an $H$ a subcoherent sequence of braid subgroups. The subcoherent sequence $P_k \subset B_k$ will be denoted $P$. If $H \subset P$ (that is, $H_k \subset P_k$ for all $k$), then we will call $H$ a subcoherent sequence of pure braid subgroups. If $H$ is a subcoherent sequence of braid subgroups, then two links $L_1$ and $L_2$ will be called $H$-equivalent if there exist $k > 0$, $b \in B_k$, and $h \in H_k$ such that $L_1 = \overline{b}$ and $L_2 = \overline{hb}$.

**Remark 1.6:** Most of the sequences $H$ discussed in this paper, in particular the groups of the lower central and derived series of $P_k$, satisfy a stronger condition, which we might call coherence. A coherent sequence would satisfy $\iota_k(H_k) = H_{k+1} \cap \iota_k(B_k)$. This condition is equivalent to requiring that the map $B_k/H_k \to B_{k+1}/H_{k+1}$ induced by $\iota_k$ be injective. Another reason one might want to consider coherent sequences is that there is a bijective correspondence between coherent sequences of subgroups and normal subgroups of $B_\omega$, the direct limit of the $B_k$ under the $\iota_k$. Given a subcoherent sequence $\{H_k\}$, the direct limit $H_\omega$ is a normal subgroup of $B_\omega$. Different subcoherent sequences may have the same direct limit $H_\omega$, but for coherent sequences this limit is unique. Given the limit $H_\omega$, the coherent sequence may be recovered by $H_k = B_k \cap H_\omega$.

**Example 1.7:** It is not hard to see that two links are $P$-equivalent if and only if they have the same number of components.

(1.8) Birman and Wajnryb [6,26] treated Example 1.7 along with two other interesting examples of subcoherent sequences. In fact, their sequences are coherent, and their point of view is to consider the quotients $B_k/H_k$.

**Notation 1.9:** Given a group $G$ and two subsets $A, B \subset G$, let $[A, B]$ be the subgroup of $G$ generated by all expressions $[a, b] = aba^{-1}b^{-1}$, where $a \in A$ and $b \in B$. If $H$ and $H'$ are sequences of braid subgroups, then denote by $[H, H']$ the sequence of subgroups $[H_k, H'_k] \subset B_k$.

(1.10) The next proposition will allow us, in particular, to consider the sequences $\text{LCS}_n(P)$ and $\text{DS}_n(P)$ (see Notations 1.40 and 3.8).

**Proposition 1.11:** Let $H$ and $H'$ be two subcoherent sequences of braid subgroups. Then $[H, H']$ is a subcoherent sequence of braid subgroups.

**Proof:** Commutators are preserved by group homomorphisms. □

**Example 1.12:** Consider $H = [P, P]$, so $H_k = [P_k, P_k]$ ($= \text{LCS}_2(P_k) = \text{DS}_2(P_k)$). It turns out that all knots are $[P, P]$-equivalent, since by Theorem 0.2 this corresponds to
having Vassiliev invariants of order < 2 equal, and it is well-known that there are no Vassiliev knot invariants of order 1. For links, \([P, P]\)-equivalence will certainly detect the number of components since \([P_k, P_k] \subset P_k\). Moreover, it is not hard to see that for an \(m\)-component link the unordered \(\binom{m}{2}\)-tuple of linking numbers is an invariant of \([P, P]\)-equivalence. Note that this is a stronger invariant than the single 1st-order Vassiliev invariant of links, which can be obtained by summing up all the pairwise linking numbers. Thus Theorem 0.2 fails for links with more than two (undistinguished) components.

(1.13) We will show below that \(H\)-equivalence is in fact an equivalence relation, but first we need to describe some notation, and introduce a slight strengthening of the Markov Theorem from Birman [3]. For the reader who is interested in keeping track of what gets used where, we note that the sole purpose for the material from here to Proposition 1.22 is to show that \(H\)-equivalence is an equivalence relation. Without Proposition 1.22, we could define \(H\)-equivalence to be generated by \(\bar{b} = \bar{h}\bar{b}\), with \(h\) and \(b\) as in Notation 1.5. This would weaken the result of Theorem 0.2 somewhat (see (0.8)), but its proof would be essentially unchanged.

(1.14) What we need is that a sequence of Markov moves connecting two braids can be replaced by a sequence in which all the stabilization is done first, followed by the destabilization. Consider conjugation, the first Markov move. Conjugate braids in \(B_k\) represent the same link under the closure operation, and, moreover, the \(b \in B_k\) in Notation 1.5 could be replaced by any of its conjugates, since \(h \in H_k\) may also be replaced by any conjugate in \(B_k\) without affecting the definition. For notational convenience in what follows, we do not consider conjugation as a move itself, but rather we consider the second Markov move, stabilization, as a move between conjugacy classes.

**Notation 1.15:** Given a braid \(b \in B_k\), denote by \([b]\) its conjugacy class in \(B_k\). If \(b_1 \in B_k\) and \(b_2 \in B_{k+1}\), then we shall say that \([b_1] < [b_2]\) if \([b_2]\) can be obtained from \([b_1]\) by a stabilization move. This means that there exist \(b'_1 \in [b_1]\) and \(b'_2 \in [b_2]\) such that \(b'_2 = b'_1\sigma_i\) or \(b'_2 = b'_1\sigma_i^{-1}\). This relationship induces a partial ordering on the set of all conjugacy classes in all braid groups \(B_k\). We will denote this partial order by \(<<\).

(1.16) What the standard Markov Theorem says is that the equivalence relation generated by \(<\) is the same as equivalence of the links obtained by closing up braids. Two braid conjugacy classes \([b]\) and \([c]\) will be equivalent if and only if there exists a sequence \([b] = [b_0], [b_1], [b_2], \ldots [b_p] = [c]\) such that for each \(i\) either \([b_i] < [b_{i+1}]\) or \([b_{i+1}] < [b_i]\). It is hard in general to get any control over the sequence \([b_i]\). The relation \(<<\) clearly generates the same equivalence relation as \(<\), and it turns out that with \(<<\) it is never necessary to have more than one intermediate conjugacy class between \(b\) and \(c\). Both \(<\) and \(<<\) behave very well with respect to \(H\)-equivalence, as shown in the next proposition.

**Proposition 1.17:** Let \(x \in B_k, y \in B_l\), and \(h \in H_k\), and suppose \([x] << [y]\). Then there exists \(j \in H_l\) such that \([hx] << [jy]\).

**Proof:** It suffices to show the result for \([x] < [y]\), where \(x \in B_k\) and \(y \in B_{k+1}\). Let \(b^{-1}yb = a^{-1}x\sigma_k^\epsilon\), where \(\epsilon = \pm 1\). Then \(a^{-1}hx\sigma_k^\epsilon = a^{-1}hab^{-1}yb = b^{-1}(ba^{-1}hab^{-1})yb\), so
we may set \( j = ba^{-1}hab^{-1} \). □

(1.18) Here is the strengthened Markov Theorem:

**Lemma 1.19:** Let \( b_1 \in B_k \) and \( b_2 \in B_l \) be such that \( \overline{b_1} = \overline{b_2} \). Then there exists a positive integer \( m \) and a braid \( b_3 \in B_m \) such that \([b_3] >> [b_1]\) and \([b_3] >> [b_2]\).

**Proof:** It suffices, by the standard Markov theorem and induction, to show that if \( b \in B_k \) and \( c_1, c_2 \in B_{k+1} \) with \([c_1] > [b]\) and \([c_2] > [b]\), then there exists \( d \in B_{k+2} \) such that \([d] > [c_1]\) and \([d] > [c_2]\). Choose representatives of conjugacy classes so that \( c_1 = b\sigma_k^{\epsilon_1} \) and \( c_2 = \alpha^{-1}ba\sigma_k^{\epsilon_2} \) where \( \epsilon_1 = \pm 1 \) and \( \epsilon_2 = \pm 1 \). Now let \( d = \alpha^{-1}\sigma_{k+1}\sigma_k^{\epsilon_1}ba\sigma_{k+1}\sigma_k^{\epsilon_2} \).

Using the braid relations (1.3) and (1.4), we obtain

\[
\alpha^{-1}\sigma_{k+1}\sigma_k^{\epsilon_1}ba\sigma_{k+1}\sigma_k^{\epsilon_2} = \alpha^{-1}\sigma_{k+1}\sigma_k^{\epsilon_1}\sigma_{k+1}^{-1}ba\sigma_k^{\epsilon_2} = \alpha^{-1}\sigma_k^{-1}\sigma_k^{\epsilon_1}\sigma_k b\sigma_k^{\epsilon_2} = \alpha^{-1}b\sigma_k^{\epsilon_2}
\]

and, conjugating by \( \sigma_{k+1} \),

\[
\alpha^{-1}\sigma_{k+1}\sigma_k^{\epsilon_1}ba\sigma_{k+1}\sigma_k^{\epsilon_2} = \alpha^{-1}\sigma_k^{\epsilon_1}ba\sigma_{k+1}\sigma_k^{\epsilon_2}\sigma_{k+1}^{-1} = \alpha^{-1}\sigma_k^{\epsilon_1}ba\sigma_k^{\epsilon_2} = \alpha^{-1}b\sigma_k^{\epsilon_1}
\]

□

**Proposition 1.22:** Let \( H \) be a subcoherent sequence of braid subgroups. Then \( H \)-equivalence is an equivalence relation.

**Proof:** The reflexive and symmetric properties are obvious. Suppose that \( A, B, \) and \( C \) are links such that \( A \) is \( H \)-equivalent to \( B \) and \( B \) is \( H \)-equivalent to \( C \). Then there exist \( k \) and \( l \), and braids \( x, h \in B_k \), and braids \( y, j \in B_l \), with \( h \in H_k \) and \( j \in H_l \), such that \( A = \overline{xh}, B = \overline{x} = \overline{y}, \) and \( C = \overline{jj}y \). By Lemma 1.19, there exists a braid \( z \in B_m \) such that \([z] >> [x]\) and \([z] >> [y]\). By Proposition 1.17, there exist \( h', j' \in B_m \) such that \([h'z] >> [hx]\) and \([j'z] >> [jy]\). Since \( h'z = A \) and \( j'z = C \), we get that \( A \) is \( H \)-equivalent to \( C \). □

(1.23) This paper is mostly concerned with pure braids and knots. If we want a notion of \( H \)-equivalence that concerns only knots, then we need to assume that \( H_k \subset P_k \). Since the closure of a pure braid isn’t a knot, we choose a convenient twist \( t_k \) to put on the end of a pure braid so that it will close up to a knot. Once we have chosen such a twist, it is clear that any knot can be represented as a pure braid plus the twist. This twist will serve another useful purpose: If \( x \) is a \( k \)-strand braid which has been inserted into an \( m \)-strand braid \( y \), with \( m > k \), and if we wish to “shift \( x \) to the right” in \( y \), then the precise algebraic way to do it is to replace \( x \) by \( t_m^{-1}xt_m \).
**Notation 1.24:** Denote by $t_k \in B_k$ the braid $\sigma_{k-1}^{-1}\sigma_{k-2}^{-1}\cdots\sigma_1^{-1}$. If $K_1$ and $K_2$ are knots, then we denote their connected sum by $K_1 \# K_2$.

**Proposition 1.25:** Let $x, y \in P_k$. Then

$$(1.26) \quad x t_k^y t_k^x = x t_k^{-y} t_k^{x+1}$$

where in the right-hand side of the equality $x$ and $y$ are identified with their inclusions into $B_{2k}$.

**Proof:** Figure 1.27 shows the case $k = 4$, which clearly generalizes. \(\square\)

**Proposition 1.29:** Let $K_1, K'_1, K_2, \text{ and } K'_2$ be knots, and let $H$ be a subcoherent sequence of pure braid subgroups. If $K_1$ is $H$-equivalent to $K'_1$, and $K_2$ is $H$-equivalent to $K'_2$, then $K_1 \# K_2$ is $H$-equivalent to $K'_1 \# K'_2$.

**Proof:** Without loss of generality we may assume that all four knots are closures of braids $B_k$ for the same $k$. Conjugating if necessary, we may assume that the permutation associated to each of these braids is the same as the permutation associated to $t_k$. So we may write $x_1 t_k = K_1$, $h_1 x_1 t_k = K'_1$, $x_2 t_k = K_2$, and $h_2 x_2 t_k = K'_2$. By Proposition 1.25,

$$(1.30) \quad K_1 \# K_2 = x_1 t_{2k}^{-k} x_2 t_{2k}^{k+1}$$

and

$$(1.31) \quad K'_1 \# K'_2 = h_1 x_1 t_{2k}^{-k} h_2 x_2 t_{2k}^{k+1} = h_1 t_{2k}^{-k} h_2 t_{2k}^{k} x_1 t_{2k}^{-k} x_2 t_{2k}^{k+1}$$

because $x_1$ commutes with $t_{2k}^{-k} h_2 t_{2k}^{k}$. The result follows because $h_1 t_{2k}^{-k} h_2 t_{2k}^{k} \in H_{2k}$. See Figure 1.27. \(\square\)
Proposition 1.33: Let $H$ be a subcoherent sequence of pure braid subgroups. Then knots up to $H$-equivalence form an abelian monoid under the operation of connected sum.

Proof: Follows immediately from the fact that knots form an abelian monoid, and Proposition 1.29. □

(1.34) We are not only interested in a single sequence of subcoherent subgroups $H$, but in various series of such sequences. In this section and the next we are mostly interested in the lower central series of the pure braid group $P_k$. In Section 3 we consider the derived series.

(1.35) Proposition 1.37 allows us to relate the product of two braids with the connected sum of two knots. The proof is illustrated in Figure 1.36. Conjugating the first braid in the figure by $y$ (or, more precisely, by $t^{-3}_{8}yt_{8}^{3}$), we obtain the second, which is then equivalent to the third modulo a commutator involving $x$ and $y$. 

Figure 1.32
Proposition 1.37: Let $H$ be a subcoherent sequence of pure braid subgroups. Let $x \in H_k$ and $y \in P_k$. Then $xt_k \# yt_k$ is $[P,H]$-equivalent to $xyt_k$.

Proof: Let $b_i = xt_k^i yt_k^{i+1}$. For $k = 4$, the first braid in Figure 1.36 is $b_2$, and the third is $b_3$. When $i = 0$ then $b_i = xyt_2k = xyt_k$, and when $i = k$ then $b_i = xt_k \# yt_k$ by Proposition 1.25. We only need show that $b_i$ is $[P,H]$-equivalent to $b_{i+1}$. Conjugating by $t_k^{i-1} yt_k^{i+1}$, we get

\begin{equation}
\overline{b_i} = xt_{2k}^{-i} yt_{2k}^{i+1} = t_{2k}^{-i-1} yt_{2k}^{i+1} xt_{2k} = [t_{2k}^{i-1} yt_{2k}^{i+1}, x] xt_{2k}^{i-1} yt_{2k}^{i+2}
\end{equation}

which is $[P,H]$-equivalent to $b_{i+1}$. \hfill \Box

Proposition 1.39: If a knot $K$ is $H$-trivial, then there exists a knot $K'$ such that $K \# K'$ is $[P,H]$-trivial.

Proof: Let $K = \overline{hut_k}$, where $h \in H_k$ and $ut_k$ is the unknot. Set $K' = \overline{h^{-1}t_k}$. Then by Proposition 1.37, $K_1 \# K'_1$ is $[P,H]$-equivalent to $ut_k$. \hfill \Box

Notation 1.40: We denote by LCS$_n(G)$ the $n$th group of the lower central series of a group $G$. That is, LCS$_1(G) = G$, and LCS$_n(G) = [G, LCS_{n-1}(G)]$ for $n > 1$. If $H_k = LCS_n(P_k)$, then we shall call $H$-equivalent knots LCS$_n$-equivalent and $H$-trivial knots LCS$_n$-trivial. Denote by $\mathcal{K}$, $\mathcal{K}_n$, and $\mathcal{K}/\mathcal{K}_n$ the abelian monoids of all knots, all LCS$_n$-trivial knots, and the LCS$_n$-equivalence classes of knots, respectively.

Theorem 1.41: For every $n > 0$, the set $\mathcal{K}/\mathcal{K}_n$ of LCS$_n$-equivalence classes of knots forms a group under connected sum.

Proof: In light of Proposition 1.33, it suffices to show that for every equivalence class modulo LCS$_n$ there is an inverse equivalence class. Let $K$ be a knot. By Proposition 1.39, there exists a knot $K_1$ such that $K \# K_1$ is LCS$_2$-trivial. Continuing
inductively, we obtain knots \( K_i \) such that \( K \# K_1 \ldots K_{n-1} \) is LCS\(_n\)-trivial. Then \( K_1 \# K_2 \ldots \# K_{n-1} \) is the desired LCS\(_n\)-inverse for \( K \). □

(1.42) As remarked in (0.7), the group \( K/K_n \) turns out to be the same as Gusarov’s group ([10], see also [19]).

Remark 1.43: From a first glance at the proof of Theorem 1.41, it might appear that the LCS\(_n\)-equivalence class of a knot is related to the number of prime factors it has. This is not the case. First of all, a knot may be connected summed with many multiple copies of some knot in \( K_n \) without affecting its equivalence class. Secondly, an argument from [23] shows that any equivalence class can be represented by a prime knot. The argument is reworked in Section 3 for DS\(_n\)-equivalence, which implies the LCS\(_n\) case.

Notation 1.44: Denote by \( \phi_k : P_k \to K \) the map given by \( \phi_k(p) = pt_k \). Denote also by \( \phi_k \) any quotient or restriction of this map.

(1.45) The following Theorem is a straightforward consequence of Proposition 1.37.

Theorem 1.46: For any \( k, n > 0 \), The map \( \phi_k : \text{LCS}_n(P_k)/\text{LCS}_{n+1}(P_k) \to K_n/K_{n+1} \) is a homomorphism of abelian groups.

Remark 1.47: Note that the proposition implies that for any \( u \in P_k \) such that \( ut_k \) is the unknot, \( \phi(x) = \phi(xu) \). Hence \( \phi_k(x) \) may be evaluated by closing up \( xv \) (where \( v = ut_k \)) for any \( v \in B_k \) such that \( \overline{v} \) is the unknot and \( v \) has the same associated permutation as \( t_k \).

(1.48) The following Theorem is an immediate consequence of Theorem 1.11 in [19]. It says that all LCS\(_n\)-equivalence classes are realized by braids of index \( \leq n \).

Theorem 1.49: For any \( n > 0 \), the homomorphism \( \phi_{n+1} : \text{LCS}_n(P_{n+1})/\text{LCS}_{n+1}(P_{n+1}) \to K_n/K_{n+1} \) is surjective.

Remark 1.50: A good understanding of the kernel of \( \phi_k : \text{LCS}_n(P_k)/\text{LCS}_{n+1}(P_k) \to K_n/K_{n+1} \) could lead to an alternate proof of Kontsevich’s Theorem that any weight system can be integrated to give a Vassiliev invariant. See Bar-Natan [1]. Kontsevich’s Theorem applies only to rational-valued invariants, and perhaps this could be generalized to torsion invariants.

Remark 1.51: Since \( \phi_{k+1} \circ \iota_k = \phi_k \), it is possible to define a unique \( \phi_\omega : P_\omega \to K \), where \( P_\omega \) is the direct limit of the \( P_k \) under the inclusions \( \iota_k \) as in Remark 1.6. For any \( k \), restricting \( \phi_\omega \) to \( P_k \) will yield \( \phi_k \).

2. VASSILIEV INVARIANTS

(2.1) The purpose of this section is to prove that equivalence of knots modulo LCS\(_n\)(\( P \)) is the same as equivalence modulo Vassiliev invariants of order \( < n \). The proof is summarized in the following commutative diagram, which we describe briefly before we begin with the details.
(2.3) All maps in this diagram are considered as \( \mathbb{Z} \)-module homomorphisms (though some have additional structure). All maps except the vertical one in the middle are surjective. The \( \mathbb{Z} \)-modules are finitely-generated, except for \( \mathbb{Z}K \) and \( \mathbb{Z}(\mathcal{K}/\mathcal{K}_n) \). \( \mathbb{Z}K \) is the \( \mathbb{Z} \)-module freely generated by all knot types. The group \( \mathcal{K}/\mathcal{K}_n \) is the group of LCS\(_n\)-equivalence classes of knots, as in Section 1, and \( \mathbb{Z}(\mathcal{K}/\mathcal{K}_n) \) is its integral group ring. The map \( \alpha_n \) is the map which associates to each knot its LCS\(_n\)-equivalence class, extended linearly to \( \mathbb{Z}K \). The map \( a_n \) is the quotient of \( \mathbb{Z}K \) by all knots with \( n \) double points, considered as an alternating sum of \( 2^n \) nonsingular knots in the usual way. Therefore, a Vassiliev knot invariant is a homomorphism from \( V_n \) to some abelian group.

(2.4) The vertical arrow in the middle will not be given a name. It is there to illustrate Proposition 2.29, which states that \( a_n \) factors through \( \alpha_n \), or, in other words, that LCS\(_n\)-equivalent knots have equal Vassiliev invariants of order \( < n \).

Remark 2.5: We note that if \( GK \) is defined to be the Grothendieck group of knots under connected sum, then \( GK \) fits into the diagram above with \( \mathbb{Z}K \) mapping to \( GK \) and \( GK \) mapping to \( \mathcal{K}/\mathcal{K}_n \) and \( W_n \). The point is that the relations that define the maps \( b_n \) and \( \beta_n \) are the images under \( a_n \) and \( \alpha_n \) of the relations that take \( \mathbb{Z}K \) to \( GK \).

(2.6) Now we will develop the modules and maps in Figure 2.2 more formally.

Notation 2.7: We will denote the set of all smooth, tame, and oriented knots in oriented \( S^3 \) by \( \mathcal{K} \). Since this set is an abelian monoid under connected sum \( \# \), we will sometimes use \( 1 \in \mathcal{K} \) to refer to the unknot. The \( \mathbb{Z} \)-module freely generated by \( \mathcal{K} \) will be denoted \( \mathbb{Z}\mathcal{K} \). The connected sum operation \( \# \) extends linearly to \( \mathbb{Z}\mathcal{K} \). A crossing marked with a dot or vertex in a knot diagram will be called a double point, and will be regarded here as simply a shorthand notation for the difference (in \( \mathbb{Z}\mathcal{K} \)) between the knot with the positive crossing and the knot with the negative crossing. So \( K_x = K_+ - K_- \), where the local diagrams are shown in Figure 2.8. A double point in a braid diagram will serve the same purpose, being the difference in \( \mathbb{Z}B_k \) between two braids which are the same except that one has a positive crossing at the double point and the other has a negative crossing. We will denote by \( C_n \) the subgroup of \( \mathbb{Z}\mathcal{K} \) generated by all knot diagrams with
n double points. We will denote by \( a_n : \mathbb{Z}K \to V_n \) the quotient map with kernel \( C_n \). Let \( \alpha_n \) be the map from \( K \) to \( K/K_n \) which sends each knot to its LCS\_n equivalence class. Denote also by \( \alpha_n \) the \( \mathbb{Z} \)-linear extension to \( \mathbb{Z}K \to \mathbb{Z}(K/K_n) \)

![Diagram](image)

**Figure 2.8**

**Proposition 2.9:** Let \( A \) be an abelian group (written multiplicatively) with the usual \( \mathbb{Z} \)-module structure (given by \( z(a) = a^z \) for \( a \in A \) and \( z \in \mathbb{Z} \)), and let \( \mathbb{Z}A \) be its integral group ring with the usual \( \mathbb{Z} \)-module structure (given by \( z \sum z_ia_i = \sum (zz_i)a_i \)). Let \( \beta : \mathbb{Z}A \to A \) be defined by \( \beta(a) = a \) and extended to \( \mathbb{Z}A \) linearly. Then the kernel of \( \beta \) is generated by expressions of the form \( a_1a_2 - a_1 - a_2 \) for \( a_1, a_2 \in A \).

**Proof:** Note that \( \mathbb{Z}A \) is freely generated by \( a \in A \) as a \( \mathbb{Z} \)-module, so \( \beta \) is a well-defined \( \mathbb{Z} \)-linear map. We have \( \beta(a_1a_2 - a_1 - a_2) = a_1a_2a_1^{-1}a_2^{-1} = 1_A \) for \( a_i \in A \). Conversely, suppose that \( \beta(\sum z_ia_i) = \Pi a_i^{z_i} = 1_A \). We may reduce \( \sum |z_i| \) to 1 or 0 by applying \( a_1 + a_2 = a_1a_2 \) and \( a_1 - a_2 = a_1a_2^{-1} \) repeatedly. Then if there is one remaining element of \( A \) in the sum it must be \( 1_A \). \( \Box \)

**Notation 2.10:** An expression of the form \( K_1 \# K_2 - K_1 - K_2 \), where \( K_1 \) and \( K_2 \) are knots, will be called a composite relator. We will also refer to the image of such an expression under \( a_n : \mathbb{Z}K \to V_n \) or \( \alpha_n : \mathbb{Z}K \to \mathbb{Z}(K/K_n) \) as a composite relator. We will denote by \( b_n : V_n \to W_n \) the quotient map whose kernel is generated by all composite relators in \( V_n \). We will denote by \( \beta_n : \mathbb{Z}(K/K_n) \to K/K_n \) the \( \mathbb{Z} \)-module map which is the identity when restricted to the subset \( K/K_n \) of \( \mathbb{Z}K/K_n \).

**Proposition 2.11:** If \( x, y \in C_1 \subset K \), then \( \beta_n \circ \alpha_n(x\#y) = b_n \circ a_n(x\#y) = 0 \).

**Proof:** First note that the unknot is a composite relator (let \( K_1 = K_2 \) be the unknot in the definition). Both \( x \) and \( y \) are \( \mathbb{Z} \)-sums of expressions of the form \( K - 1 \), where \( K \) is a knot. So the Proposition reduces to the fact that \( (K_1 - 1) \# (K_2 - 1) \) is the sum of the composite relators \( 1 \) and \( K_1 \# K_2 - K_1 - K_2 \). \( \Box \)

**Notation 2.12:** Let \( G \) be a group, and \( \mathbb{Z}G \) its integral group ring. We will denote by \( I \) the augmentation ideal of \( \mathbb{Z}G \), which is the kernel of the map to \( \mathbb{Z} \) which sends each group element to 1. \( I \) is generated (as a \( \mathbb{Z} \)-submodule) by all \( g - 1 \) where \( g \in G \). We will denote by \( I_k \) the augmentation ideal of \( P_k \).

**Proposition 2.13:** \( I_k^n \) is generated by all \( k \)-strand braid diagrams with \( n \) double points.

**Proof:** A diagram with \( n \) double points is of the following form in \( \mathbb{Z}P_k \):

\[
(2.14) \quad w_1(\sigma_{i_1} - \sigma_{i_1}^{-1})w_2(\sigma_{i_2} - \sigma_{i_2}^{-1}) \ldots w_n(\sigma_{i_n} - \sigma_{i_n}^{-1})w_{n+1}
\]
where \( w_j \in B_k \). Set \( v_j = w_1 \sigma_{i_1}^{-1} w_2 \sigma_{i_2}^{-1} \ldots w_n \sigma_{i_n}^{-1} \). Then (2.14) may be rewritten as

\[
(2.15) \quad (v_1 \sigma_{i_1}^2 v_1^{-1} - 1)(v_2 \sigma_{i_2}^2 v_2^{-1} - 1) \ldots (v_n \sigma_{i_n}^2 v_n^{-1} - 1)v_n w_{n+1}
\]

Since \( v_j \sigma_{i_j}^2 v_j^{-1} \in P_k \), (2.15) is an element of \( I_k^n \).

Since \( I_k \) is generated as a \( \mathbb{Z} \)-module by elements of the form \( x - 1 \), it suffices to show that an element of the form \( (x_1 - 1)(x_2 - 1) \ldots (x_n - 1) \), where \( x_i \in P_k \), is a linear combination of diagrams with \( n \) double points. Then, since each \( x_i \) is a pure braid, each may be undone by crossing changes (or, to put it more algebraically, \( P_k \) is normally generated by \( \sigma_i^2 \)). This gives each \( x_i - 1 \) as the \( \mathbb{Z} \)-sum of diagrams with one double point each. Putting them all together with the distributive property gives the desired result. □

**Notation 2.16:** Let \( t_k \in B_k \) be the braid \( \sigma_{n-1}^{-1} \sigma_{n-2}^{-1} \ldots \sigma_1^{-1} \) as in the Section 1. As before, we will denote by \( \overline{\pi} \) the standard closure of a braid into a knot or link. For each \( k \) we may extend the closure operation \( \mathbb{Z} \)-linearly to a map from \( \mathbb{Z}B_k \) to \( \mathbb{Z}K \).

**Proposition 2.17:** \( C_n \) is generated by all elements of the form \( \overline{x t_k} \), where \( k > 0 \) and \( x \in I_k^n \).

**Proof:** Any knot diagram with \( n \) double points may be written as a closed braid diagram with \( n \) double points. This is a generalization of Alexander’s Theorem that every link can be written a closed braid, and is proved in Birman [4]. After conjugating by an element of \( B_k \) if necessary, we may assume that the permutation associated to \( c \) is the same as that associated to \( t_k \), and we may write \( c = q t_k \), where \( q \) is a pure braid diagram with \( n \) double points. Proposition 2.13 completes the proof. □

**Notation 2.18:** A **relator of length \( m \) and order \( \geq n \)** is an expression of the form

\[
(2.19) \quad (x_1 - 1)(x_2 - 1) \ldots (x_m - 1)yt_k
\]

where \( k > 0 \) and \( x_i \in \text{LCS}_{n_i}(P_k) \) for all \( i \), and \( n = n_1 + n_2 + \ldots + n_m \).

(2.20) Since \( \cap_{n} \text{LCS}_{n}(P_k) = \{1\} \subset P_k \), it would in fact be possible to define the exact order of a relator by requiring \( x_i \notin \text{LCS}_{n_i+1}(P_k) \), but this is not necessary for our purposes. These relators are generalizations both of the generators of \( \mathcal{C}_n \) and of \( \text{LCS}_n \)-equivalence, and will be our tool for moving between the two.

**Proposition 2.21:** \( C_n \) is generated by all relators of length \( n \) and order \( \geq n \).

**Proof:** This is just a restatement of Proposition 2.17, using Notation 2.18. □

**Proposition 2.22:** The kernel of \( \alpha_n \) is generated by relators of length 1 and order \( \geq n \).

**Proof:** This is immediate from the definition of \( \text{LCS}_n \)-equivalence. □

(2.23) The next two propositions are well-known facts of group theory.

**Proposition 2.24:** Let \( G \) be a group and let \( I \subset \mathbb{Z}G \) be its augmentation ideal. If \( x \in \text{LCS}_n(G) \), then \( x - 1 \in I^n \).

14
**Proof:** Let \( x \in \text{LCS}_n(G) \) and \( y \in G \). Assume inductively that therefore \( x - 1 \in I^n \). We have \([x, y] - 1 = (xy - yx)x^{-1}y^{-1} = ((x - 1)(y - 1) - (y - 1)(x - 1))x^{-1}y^{-1} \in I^{n+1} \). Also, if \( x - 1 \in I^n \) and \( y - 1 \in I^n \), then \( xy - 1 = x(y - 1) + (x - 1) \in I^n \); and if \( x - 1 \in I^n \) then \( x^{-1} - 1 = -x^{-1}(x - 1) \in I^n \). \( \square \)

**Proposition 2.25:** Let \( G \) be a group. Let \( x \in \text{LCS}_m(G) \) and \( y \in \text{LCS}_n(G) \). Then \([x, y] \in \text{LCS}_{m+n}(G)\).

**Proof:** It is not hard to find commutator identities, which hold in a free group and hence in any group, and which allow one to break up and reorder commutators of products and commutators of commutators. For example,

\[
\text{(2.26)} \quad [xy, z] = (x[y, z]x^{-1})[x, z]
\]

and

\[
\text{(2.27)} \quad [[x, y], z] = (xyx^{-1}z[[z^{-1}, y^{-1}], x]z^{-1}xy^{-1}x^{-1})(x[y, [x^{-1}, z]])x^{-1}
\]

\( \square \)

\[
\text{(2.28)} \quad \text{The next proposition was proved (with different notation) in [23].}
\]

**Proposition 2.29:** The map \( a_n \) factors through the map \( \alpha_n \).

**Proof:** By Propositions 2.21 and 2.22, this is the same as saying that a relator of length 1 and order \( n \) can be written as a sum of relators of length \( n \) and order \( n \). This follows from Proposition 2.24. \( \square \)

**Proposition 2.30:** A relator of length \( m \) and order \( \geq mn \) is a linear combination of relators of length 1 and order \( n \).

**Proof:** In such a relator, there must exist \( i \) such that \( x_i \in \text{LCS}_n(P_k) \). Using the distributive property of multiplication, we may write our given relator as a sum of expressions of the form \( w(x_i - 1)yt_k = (wx_iw^{-1} - 1)wy_{t_k} \), where \( w, y \in P_k \). \( \square \)

\[
\text{(2.31)} \quad \text{The following proposition is the key to the theorems of this section. Its proof depends on the same idea as Proposition 1.37. There a connected sum was made into a braid product by sliding a braid \( y \) around and around modulo the next term down in the lower central series. Here we do the opposite. Given a product of expressions \( (x_i - 1) \) in \( \mathbb{Z}P_k \), we can slide \( x_1 - 1 \) around and around, modulo relations closer to what we want, until it forms a connected sum which is zero because of the composite relators.}
\]

**Proposition 2.32:** Any relator of order \( \geq n \) can be written as the \( \mathbb{Z} \)-sum of composite relators and relators of length 1 and order \( \geq n \).

**Proof:** Suppose on the contrary that there is some relator of order \( \geq n \) which cannot be so written. Choose such a relator with minimal length \( m \), and among those with minimal length, one with maximal order. The order must be less than \( mn \) by Proposition 2.30. Let us write our relator \((x_1 - 1)(x_2 - 1)\ldots(x_m - 1)yt_{2k}\), where initially \( y \in P_k \) and
\( x_i \in \text{LCS}_{n_i}(P_k) \), included into \( B_{2k} \) in the standard way. As the argument progresses, the \( x_i \) and \( y \) will shift and combine to occupy more than \( k \) strands in \( P_{2k} \).

First of all, we may interchange \((x_i - 1)\) with \((x_{i+1} - 1)\) modulo relators of shorter length or greater order:

\[
(x_i - 1)(x_{i+1} - 1) - (x_{i+1} - 1)(x_i - 1) = x_ix_{i+1} - x_{i+1}x_i
\]

where the relation containing \([x_i, x_{i+1}] - 1\) has the same order but shorter length than the original, and the relation containing \([x_i, x_{i+1}] - 1)(x_{i+1}x_i - 1\) has the same length but greater order. We are applying Proposition 2.25 here. Secondly, \(x_m - 1\) may be exchanged with \(y\) modulo relators of greater order:

\[
(x_m - 1)y - y(x_m - 1) = ([x_m, y] - 1)y x_m
\]

Thirdly, by conjugation we have

\[
\frac{(x_1 - 1)(x_2 - 1) \ldots y(x_m - 1)t_{2k}}{t_{2k}^{-1}x_m t_{2k} - 1}(x_1 - 1)(x_2 - 1) \ldots (x_{m-1} - 1)y t_{2k}
\]

Finally, applying (2.33), (2.34), and (2.35) repeatedly, we obtain

\[
\frac{(x_1 - 1)(x_2 - 1) \ldots (x_m - 1)y t_{2k}}{t_{2k}^{-k}(x_1 - 1)t_{2k}^k(x_2 - 1)(x_3 - 1) \ldots (x_{m-1} - 1)y t_{2k}}
\]

modulo relators of shorter length and relators of greater order. This last relator is equal to

\[
\frac{(x_2 - 1)(x_3 - 1) \ldots (x_m - 1)y t_k \# (x_1 - 1)t_k}{(x_2 - 1)(x_3 - 1) \ldots (x_m - 1)y t_k \# (x_1 - 1)t_k}
\]

by Proposition 1.25, and is therefore a \( \mathbb{Z} \)-sum of composite relators by Proposition 2.11.

\[
\boxed{	ext{Proposition 2.38:}}\]

The kernel of \( \beta_n \circ \alpha_n : \mathbb{Z}K \rightarrow K/K_n \) is equal to the kernel of \( b_n \circ a_n : \mathbb{Z}K \rightarrow W_n \).

\[
\text{Proof:} \quad \text{By the preceding propositions, both kernels are generated by relators of length 1 and order } \geq n \text{ and by composite relators.}\]

\[
\boxed{\text{Notation 2.39:}}\]

We say that two knots are \( W_n \)-equivalent if they are equivalent in \( W_n \). We say that they are \( V_n \)-equivalent if they are equivalent in \( V_n \).

\[
\boxed{\text{Proposition 2.40:}}\]

If two knots are \( W_n \)-equivalent, then they are LCS\(_n\)-equivalent.

\[
\text{Proof:} \quad \text{In both cases, this just says that the difference of the two knots is in the kernel of } b_n \circ a_n.\]

16
Theorem 2.41: Two knots are $V_n$-equivalent if and only if they are LCS$_n$-equivalent.

Proof: If two knots are equivalent modulo $K_n$ then they are $V_n$-equivalent by Proposition 2.29. If two knots are $V_n$-equivalent then they are $W_n$-equivalent, and therefore they are LCS$_n$-equivalent. $\blacksquare$

(2.42) As noted in (0.16), it is not known whether the following theorem actually applies to anything new.

Theorem 2.43: If $K_1$ and $K_2$ are knots, then $v(K_1) = v(K_2)$ for all Vassiliev invariants of order $< n$ if and only if $w(K_1) = w(K_2)$ for all additive Vassiliev invariants of order $< n$.

3. THE DERIVED SERIES

(3.1) If instead of the set $\mathcal{K}$ of all knots we restrict ourselves to $H$-trivial knots for some $H$, we get the following immediate generalization of Theorem 1.41:

Theorem 3.2: Let $H = \{H_k\}$ be a subcoherent sequence of pure braid subgroups. Let $H_k^{(n)} = [P_k, H_k^{(n-1)}]$, with $H_k^{(1)} = H_k$. Then for any $n > 0$, the set of $H^{(n)}$ equivalence classes of $H$-trivial knots forms a group under connected sum.

(3.3) Given $H$, we would like to consider $H$-trivial knots modulo LCS$_n(H)$, but what stops us is that we have no control over the braid $u$ in the proof of Proposition 1.39. So we will consider a stronger notion of $H$-triviality, somewhat artificial but useful nonetheless.

Notation 3.4: Let $H$ be a subcoherent sequence of pure braid subgroups. We shall say that a knot $K$ is strongly $H$-trivial if there exist $k > 0$ and $h \in H_k$ such that $K = ht_k$.

Proposition 3.5: Let $H$ be a subcoherent sequence of pure braid subgroups. Then the connected sum of two strongly $H$-trivial knots is strongly $H$-trivial.

Proof: Use Proposition 1.25, with $x, y \in H_k$. $\blacksquare$

(3.6) Recall that Proposition 1.11 tells us that we can form the commutator between two subcoherent sequences of braid subgroups and obtain a new subcoherent sequence.

Proposition 3.7: Let $K$ be a strongly $H$-trivial knot. Then there exists a strongly $H$-trivial knot $K'$ such that $K \neq K'$ is strongly $[H,H]$-trivial.

Proof: Follow the proofs of Propositions 1.37 and 1.39. The difference is that there is no $u$ in this case, so that $x$ and $y$ may be assumed by induction to be in DS$_n(P_k)$. $\blacksquare$

Notation 3.8: The $n$th group of the derived series of a group $G$ will be denoted $\text{DS}_n(G)$. Set $\text{DS}_1(G) = G$, and $\text{DS}_{n+1}(G) = [\text{DS}_n(G), \text{DS}_n(G)]$. If $H = \{H_k\}$ is a sequence of groups, then $\text{DS}_n(H)$ will denote the sequence $\{\text{DS}_n(H_k)\}$. If two knots are $\text{DS}_n(P)$-equivalent, then we shall call them $\text{DS}_n$-equivalent, and a $\text{DS}_n(P)$-trivial knot will be called $\text{DS}_n$-trivial.
Theorem 3.9: Let $H$ be a subcoherent sequence of pure braid subgroups. Then for any $n > 0$, the set of $DS_n(H)$-equivalence classes of strongly $H$-trivial knots forms a group under connected sum.

Proof: Same as Theorem 1.41. □

Corollary 3.10: For any $n > 0$, the $DS_n$-equivalence classes of knots form a group under connected sum.

Proof: If $H_k = P_k$, then all knots are strongly $H$-trivial. □

(3.11) Corollary 3.10 implies Theorem 1.41, since $DS_n(G) \subset LCS_n(G)$ for any $n$ and any group $G$. More generally, we obtain

Corollary 3.12: Let $H$ be a subcoherent sequence of pure braid subgroups. Then for any $n > 0$, the set of $LCS_n(H)$-equivalence classes of strongly $H$-trivial knots forms a group under connected sum.

Remark 3.13: It follows from Proposition 2.25 that for any group $G$, $DS_n(G) \subset LCS_{2n-1}(G)$. At the very least, then, the $DS_n$-equivalence classes of knots pick up Vassiliev invariants much faster than the $LCS_n$-equivalence classes. It would be interesting to know whether they pick up anything else. Note that the quotients $LCS_n(P_k)/LCS_{n+1}(P_k)$ are all finitely-generated abelian groups, whereas the quotients $DS_n(P_k)/DS_{n+1}(P_k)$ are abelian but are not finitely-generated. So on the one hand it seems very possible that $DS_n$-equivalence is more than just a faster way to get at Vassiliev invariants, and on the other hand it seems that it will be very difficult to analyze $DS_n$-equivalence.

(3.14) In [23] we showed that for any $n > 0$ and any $LCS_n$-equivalence class of knots, it is possible to choose a representative of that class which is prime and alternating. In fact, it is possible to choose an infinite number of such representatives, and moreover, the same trick works for $DS_n$-equivalence classes, as we shall show in Theorem 3.23 below. The idea for the following approach to $B_3$ came from Birman and Menasco [5].

Notation 3.15: In $B_3$, let $a = \sigma_1$ and $b = \sigma_2$. Let $A = a^{-1}$ and $B = b^{-1}$. Let $d = bab = aba$ and $D = ABA = BAB$. Let $w$ stand for a word in $a$ and $B$ (no $A$ or $b$), not necessarily the same word each time it is used. Let $W$ stand for a word in $A$ and $b$, not necessarily the same each time.

Proposition 3.16: $DaD = dad = b$, $DbD = dbd = a$, $DAD = dAd = B$, $DbD = DBD = A$, $DWD = dWd = w$, $DwD = dwd = W$.

Proof: Follows from the braid relations (1.4). □

Proposition 3.17: For any $n > 0$, there exist in $DS_n(P_3)$ words of each of the following forms: $awa, awB, Bwa, BwB, awaD, awBD, BwaD, dawa, dawB, dBwa, dBwb$.

Proof: Clearly one can find such words in $P_3$. So assume by induction that we have $awa, awB, Bwa, BwB \in DS_n(P_3)$. Then
\[(awa, awB) = awaawBAWAbWA \]
\[= awaawa(ABA)WAbWA \]
\[= awaawawBawB(ABA) \]
\[= awBD \]

and similarly
\[(3.19) \quad [awa, BwB] = awaBwa(ABA)WAbWb = awaD \]
\[(3.20) \quad [Bwa, awB] = Bwaawa(ABA)WbbWA = BwBD \]
\[(3.21) \quad [Bwa, BwB] = BwaBwa(ABA)WbbWb = BwaD \]

The above words are all in DS\(_{n+1}(P_3)\). Since DS\(_{n+1}(P_3)\) is a fully invariant subgroup of P\(_3\), we may conjugate the inverses of these words by \(d\) to get \(dawa, dawB, dBwa, dBwB\). For example, \((awaD)^{-1} = dAWA\), and \(ddAWAD = dBwB\). The words that end in \(D\) may now be combined with the words that begin with \(d\) to make \(awa, awB, Bwa, and BwB\), and these may then be used for the next induction step. \(\blacksquare\)

**Proposition 3.22:** Let \(x \in B_3\), and let \(n > 0\). Then there exists a \(x' \in B_3\), which is a word in \(a = \sigma_1\) and \(B = \sigma_2^{-1}\) only, such that \(x\) is congruent to \(x'\) modulo DS\(_{n}(P_3)\).

**Proof:** Write \(x\) as a word in \(a, b, A, B\). Replace every occurrence of \(A\) by \(wDAdw = wBw\), where \(wD \in DS_n(P_3)\) is one of the words from Proposition 3.17. Similarly, replace \(b\) by \(wDbdw = waw\). \(\blacksquare\)

**Theorem 3.23:** Let \(K\) be a knot and let \(n > 0\). Then there exist an infinite number of prime, alternating knots in the DS\(_{n}\)-equivalence class of \(K\).

**Proof:** The ideas are the same as in the proof of Theorem 2 in [23]. Take a braid word in \(B_k\) and insert elements in DS\(_n(P_k)\) of the forms from Proposition 3.17. so that the braid word becomes alternating. Insert more such elements to make the knot prime, using Menasco’s theorem [16] that an alternating prime diagram represents a prime knot. Then insert still more such elements to create an infinite family of knots, using the fact that an alternating diagram has minimal crossing number (see Murasugi [18] or Thistlethwaite [24]). It suffices to work with three consecutive strands at a time, and to use the elements of DS\(_n(P_3)\) from Proposition 3.17 (shifted over appropriately) as in the proof of Proposition 3.22. \(\blacksquare\)
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