SUPERCUlSPIral RAMIFICATIONS AND TRACES OF ADJOINl LIFTS
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ABSTRACT. We write down the local Brauer classes of the endomorphism algebras of motives attached to non-CM Hecke eigenforms for all supercuspidal primes in terms of traces of adjoint lifts at auxiliary primes. We give an alternative proof of the ramification formulae for odd primes obtained by Bhattacharya-Ghate and write down the ramification formulae for odd unramified supercuspidal primes of level zero also removing a mild hypothesis of them. We also give a complete description of ramifications for \( p = 2 \) where the local Galois representation can be non-dihedral. In the process, we write down the inertial Galois representations even for \( p = 2 \) also generalizing similar description of the same for odd primes by Ghate-Mézard. Some numerical examples using Sage and LMFDB are provided supporting some of our theorems.

1. Introduction

Let \( f(z) = \sum_{n \geq 1} a_n q^n \in S_k(N, \epsilon) \) be a non-CM Hecke eigenform of weight \( k \geq 2 \), level \( N \geq 1 \) and nebentypus \( \epsilon \). One knows that the number field \( E = \mathbb{Q}((a_n)) \) is either a totally real or a CM number field. For \( k = 2 \), let \( M_f \) denote the abelian variety attached to \( f \) by Shimura [27]. For \( k > 2 \), we also denote by \( M_f \) the Grothendieck motive over \( \mathbb{Q} \) with coefficients in \( E \) associated to \( f \) by Scholl [25]. The \( \lambda \)-adic realization of this motive produces a \( \lambda \)-adic Galois representation \( \rho_f \) associated to the modular form \( f \) by a well-known theorem of Deligne [cf. Section 4]. Let \( X_f \) denote the \( \mathbb{Q} \)-algebra of endomorphisms of \( M_f \) defined by \( X = X_f := \text{End}_{\mathbb{Q}}(M_f) \otimes_{\mathbb{Z}} \mathbb{Q} \).

Consider the totally real subfield \( F \) of \( E \) generated by the elements \( a_p^2 \epsilon(p)^{-1} \), for all \( p \nmid N \). The algebra \( X \) has the structure of an explicit crossed product algebra over \( \bar{F} \) due to Momose and Ribet [20] in weight two and Ghate and his collaborators [5, 12, 18] in higher weights. One knows that the class \( [X] \) in \( 2\text{Br}(F) := \text{H}^2(\text{Gal}(\bar{F}/F), \bar{F}^\times) \), the 2-torsion part of the Brauer group of \( F \). In [20], Ribet wondered if it is possible to determine the local Brauer classes by pure thought. We study the Brauer classes of \( X \) locally by the following exact sequence:

\[
0 \to 2\text{Br}(F) \to \bigoplus_v 2\text{Br}(F_v) \to \mathbb{Z}/2 \to 0
\]

where \( v \) runs over all primes in \( F \). It is well-known that \( X_v = X \otimes_F F_v \) is a central simple algebra over \( F_v \) and the class of \( X_v \) is a 2-torsion element in the Brauer group \( \text{Br}(F_v) \) of \( F_v \), that is, the class \( [X_v] \) in \( 2\text{Br}(F_v) \cong \mathbb{Z}/2 \). We say \( X_v \) is unramified if the class of \( X_v \) is trivial and ramified if the class is non-trivial. At the infinite places \( v \), \( X \) is totally indefinite if \( k \) is even, and totally definite if \( k \) is odd [16, Theorem 3.1]. The ramification formula of \( X_v \) is known by a series of papers pioneered by Ghate and his collaborators [2, 3, 5] and [12] for non-supercuspidal primes \( p \) (i.e., the local automorphic factor at \( p \) is not of supercuspidal type). For \( v \mid p \), let \( G_p := \text{Gal}(\bar{\mathbb{Q}}_p|\mathbb{Q}_p) \) and \( G_v := \text{Gal}(\bar{F}_v|F_v) \) be the local Galois groups.

**Definition 1.1.** We call a supercuspidal prime \( p \) to be dihedral for \( f \) if the local Galois representation \( \rho_f|_{G_p} \sim \text{Ind}_{G_K}^{G_p} \chi \) for some quadratic extension \( K|\mathbb{Q}_p \) and some character \( \chi \) of \( G_K := \text{Gal}(\bar{\mathbb{Q}}_p|K) \). Depending on \( K|\mathbb{Q}_p \) is unramified (or ramified), we call the prime \( p \) to be a unramified (or ramified)
supercuspidal prime for \( f \). By the level of an unramified supercuspidal prime \( p \), we mean the level of the corresponding local automorphic representation \( \pi_p \).

We say \( f \) is \( p \)-minimal, if the \( p \)-part of its level is the smallest among all twists \( f \otimes \psi \) of \( f \) by Dirichlet characters \( \psi \). Write \( N = p^{N_p}N' \) with \( p \nmid N' \) and the nebentypus \( \epsilon = \epsilon_p : \epsilon' \) as a product of its \( p \)-part and prime to \( p \)-part with \( \text{cond}(\epsilon_p) = p^{C_p} \). The supercuspidal primes \( p \) for \( f \) can be characterized as follows: \( C_p < N_p \geq 2 \) with \( a_p = 0 \). If \( p \) is a supercuspidal prime, then \( a_p = 0 \) and the corresponding slope is infinity and it is not possible to talk about the parity of slopes. In [4], Bhattacharya-Ghate determined the Brauer classes of \( X_v \) for odd supercuspidal primes (where the local Galois representation is always dihedral) with an extra hypothesis in the level 0 case using a method involving local symbols and \((p, \ell)\)-Galois representation. They determined \( X_v \) except in the following cases:

1. \( p = 2 \),
2. \( p \) is odd, \( K = \mathbb{Q}_p(s) \) is unramified, the \( p \)-minimal twist of \( f \) is of level 0 and \( \chi(s)^{p-1} = -1 \).

We wish to determine the algebra \( X_v \) for all supercuspidal primes (including \( p = 2 \) where the local Galois representation can be non-dihedral and the case (2) above) in terms of the companion adjoint slope \( m_v \) [cf. Definition 3.1] determined by the traces of adjoint lifts at the auxiliary primes. Note that the Theorem 3.3 and a part of Theorem 3.4 proved in this article are just a restatement of the results in [4] for odd primes. However, our method is completely different as it uses local nature of Galois representations associated to modular forms.

In [4], the authors use a formula that determines the local Brauer classes of \( X_v \) as a product of finite number of local Hilbert symbols [12, Theorem 4.1]. In loc. cit., the authors simplify this product and give the ramification formula of \( X_v \) for odd supercuspidal primes. As the formula of local symbols for \( v \mid 2 \) (except \( F = \mathbb{Q} \)) is a bit technical [11, Chapter 7, p. 241], it is not possible to simplify the product in a similar way. Our method circumvents the problem as we are using the local information about modular forms determined by the corresponding filtered \((\phi, N)\)-modules associated to \((p, p)\)-Galois representation. Group cohomology plays a key role in our computation. In Section 5, we write down the inertial Galois representations even for \( p = 2 \). It will be of independent interest similar to that for \( p \) odd.

For supercuspidal primes (specially for \( p = 2 \)), the image of the inertial groups under modular Galois representations are complicated and hence our results are also a bit theoretical in nature. Our results are technical compare to Bhattacharya-Ghate because they used crucially [cf. [4, Lemma 4.2] for instance] the order of the Brauer class is co-prime to the residue degree of the local place. The ingenuity of our results stems from the fact that we manage to identify the cases when the local Brauer classes are not determined by adjoint slopes. Note that for most of the places of \( K \), they are related to adjoint slopes. In an ongoing project, we wish to find out the local Brauer class of the endomorphism algebra of motive attached to a Hilbert modular form [17, Appendix B] using the computation done in the present paper.

**Notation.** Throughout the paper, \( I_p \) and \( I_v \) be the inertia subgroups of \( G_p \) and \( G_v \) respectively. The wild and the tame part of the inertia subgroup \( I_K \) of \( G_K \) will be denoted by \( I_{W}(K) \) and \( I_{T}(K) \) respectively.

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### 2. The Local Brauer Class of X and Its Invariant

Let \( \Gamma = \{ \gamma \in \text{Aut}(E) : \exists \text{ a Dirichlet character } \chi_\gamma \text{ such that } a_p^\gamma = a_p \cdot \chi_\gamma(p) \text{ with } (p, N) = 1 \} \) be the group of extra twists for \( f \). One knows that \( F \) is the fixed field of \( E \) by \( \Gamma \). For \( \gamma, \delta \in \Gamma \), the relation \( \chi_{\gamma \delta} = \chi_{\gamma} \chi_{\delta} \) shows that \( \gamma \mapsto \chi_\gamma(g) \) is a 1-cocycle for a fixed \( g \in G_Q \). Since \( H^1(\Gamma, E^\times) \) is trivial, there
exists $\alpha(g) \in E^\times$ such that
\begin{equation}
\alpha(g)^{\gamma - 1} = \chi_\gamma(g),
\end{equation}
for all $\gamma \in \Gamma$ by Hilbert’s theorem 90. The element $\alpha(g)$ is well defined modulo $F^\times$. The map $\tilde{\alpha} : G_{\mathbb{Q}} \to E^\times/F^\times$, $g \mapsto \alpha(g) \mod F^\times$ is a continuous homomorphism. The map $\alpha : G_{\mathbb{Q}} \to E^\times$ can be thought of as a lift of $\tilde{\alpha}$. Let $\rho_f$ denote the $\lambda$-adic representation attached to $f$ for some prime $\lambda | \ell$ of $E$. We list some properties of any lift $\alpha$ of the homomorphism $\tilde{\alpha}$.

**Proposition 2.1.** [2, Lemma 1] [22, Theorem 5.5] The map $\alpha$ satisfies the following properties:

1. $\alpha^2(g) \equiv \epsilon(g) \mod F^\times$, for all $g \in G_{\mathbb{Q}}$.
2. $\alpha(g) \equiv \mathrm{Tr}(\rho_f(g)) \mod F^\times$, for all $g \in G_{\mathbb{Q}}$, provided that the trace is non-zero.
3. $\alpha(\text{Frob}_p) \equiv a_p \mod F^\times$, for all prime $p \nmid N$ with $a_p \neq 0$.

Comparing (2.1) and the property (1) of the above proposition, we have the following identity: $\chi_\gamma = \epsilon^{\gamma - 1}$, for all $\gamma \in \Gamma$. According to [21], the Brauer class of $X$ in $\text{Br}(F) = H^2(G_{F}, F^\times)$ is given by the $F^\times$-valued 2-cocycle: $c_\alpha(g, h) = \frac{\alpha(g) \alpha(h)}{\alpha(gh)} \forall g, h \in G_{F}$, for any continuous lift $\alpha$ of $\tilde{\alpha}$ and this class is independent of the lift chosen [3, Section 3.2]. The restriction $[c_\alpha|_{G_v}] \in \text{Br}(F_v) = H^2(G_{v}, \tilde{F}_v^\times)$ gives the local Brauer class of $X_v$ for any prime $v$ of $F$. Since $\text{inv}_v(c_\alpha|_{G_v}) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, the invariant map $\text{inv}_v$ completely determines the class $[X_v]$ in $\text{Br}(F_v)$. For the definition of the invariant map, we refer to [26, Ch. XIII, Section 3, p. 193]. Let $S : G_v \to \tilde{F}_v^\times$ be any set map. Recall the following lemma [3, Lemma 9] useful to determine the Brauer class of any local 2-cocycle of the form: $c_S(g, h) = \frac{S(g)S(h)}{S(gh)}$, for all $g, h \in G_v$.

**Lemma 2.2.** Let $S : G_v \to \tilde{F}_v^\times$ be any map and $t : G_v \to \tilde{F}_v^\times$ be an unramified homomorphism with

1. $S(i) \in F_v^\times$, for all $i \in I_v$,
2. $S(g)^2 / t(g) \in F_v^\times$, for all $g \in G_v$.

For any arithmetic Frobenius $\text{Frob}_v$, we then have $\text{inv}_v(c_S) = \frac{1}{2}v\left(\frac{S^2}{t}(\text{Frob}_v)\right) \mod \mathbb{Z} \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Here, $v : \tilde{F}_v^\times \to \mathbb{Z}$ is the surjective valuation.

For $\gamma \in \Gamma$, there is a unique Dirichlet character $\chi_\gamma$ such that $f^\gamma \equiv f \otimes \chi_\gamma$. By restricting to the corresponding decomposition group $G_{\mathbb{Q}}$, we deduce that the $p$-adic Galois representations are similar. In other words, $\rho_{F_v} \sim \rho_{p} \otimes \chi_\gamma$. Using Brauer-Nesbitt theorem, we have $\rho_{F_v} \sim \rho_{F_{p}}$. Hence, the property (2) of Proposition 2.1 is also true even for $p$-adic Galois representation by comparing the traces of the similar $p$-adic Galois representations associated to $f^\gamma$ and $f \otimes \chi_\gamma$.

Since $c_\alpha|_{G_v}$ determines the algebra $X_v$, the most obvious choice for $S$ in the above lemma would be $\alpha$. The main difference of the computation of $X_v$ for $p$ odd and $p = 2$ is as follows: for odd $p$, we will see that $S = \alpha$ except $K_{F_v}[F_v]$ is ramified quadratic or $p$ is a bad prime [cf. Definition 3.2]. When $S \neq \alpha$, we have to divide $\alpha$ by a suitable auxiliary function (i.e., $S = \alpha / f$) to make the above lemma applicable. For $p = 2$, we will always be needed to divide $\alpha$ by (one or more) auxiliary functions unless $N_2 = 2$ and their corresponding cocycles will contribute in the ramification of $X_v$ as error terms.

### 3. Statement of results

If the Brauer class of $X_v$ is determined by the parity of an integer $a$, we write $X_v \sim a$ or $X_v \sim (-1)^a$. Choose a prime $p'$ coprime to $N$, with non-zero Fourier coefficients $a_{p'}$, satisfying the following properties:

\begin{equation}
 p' \equiv 1 \pmod{p^N}, \quad p' \equiv p \pmod{N'}.
\end{equation}

**Definition 3.1.** Let $v$ be a valuation on $F$ such that $v(p) = 1$, We define the “companion adjoint slope” at a place $v$ of $F$ lying above a supercuspidal prime $p$ to be the $v$-adic valuation of the trace of adjoint lift at $p'$. In other words, $m_v := [F_v : \mathbb{Q}_p] \cdot v(a_{p'}^{-1})$ denote the “companion adjoint slope” at $v$. 

We choose the following auxiliary primes with non-zero Fourier coefficients:

- \( p'' \equiv 1 \pmod{N'} \) and \( p'' \) has order \((p-1) \) in \((\mathbb{Z}/pN'\mathbb{Z})^\times \),
- \( p''' \equiv 1 \pmod{N'} \) and \( p''' \) has order 2 in \((\mathbb{Z}/2N\mathbb{Z})^\times \),
- for all \( \gamma \in \Gamma \),

\[
\chi_\gamma(p^l) = \begin{cases} 
-1, & \text{if } \chi_\gamma \text{ is ramified}, \\
1, & \text{if } \chi_\gamma \text{ is unramified}.
\end{cases}
\]

There exist infinitely many such primes since \( f \) is assumed to be non-CM. For an odd supercuspidal prime \( p \), the local Galois representation is always dihedral, i.e., \( \rho_f|_{G_p} \sim \text{Ind}_{G_K}^{G_p} \chi \), with \( K|\mathbb{Q}_p \) quadratic. According to [14], we have \( \chi|_{I_p} = \omega_2^i \chi_1 \chi_2 \) (when \( K \) is unramified) and \( \chi|_{I_K} = \omega^j \chi_1 \chi_2 \) (when \( K \) is ramified) [cf. Section 5]. Here, \( \omega_2 \) is the fundamental character of level 2, \( \omega \) is the Teichmüller character and \( \chi_m \) is the character having some \( p \)-power order for \( m = 1, 2 \).

**Definition 3.2.** We call an odd unramified supercuspidal prime \( p \) of level zero to be “good” if

\[
(\text{H}) \quad l \text{ is not an odd multiple of } (p+1)/2.
\]

If such a prime \( p \) is not “good”, we call it a “bad” level zero unramified supercuspidal prime.

In Lemma 6.3, we prove that level zero unramified supercuspidal primes \( p \equiv 1 \pmod{4} \) with \( C_p = 0 \) and \( p \equiv 3 \pmod{4} \) with \( C_p = 1 \) are always “good”. For \( a, b \in F_v^\times \), we write \( a = \pi_v^{\alpha(a)} \cdot a' \) and \( b = \pi_v^{\alpha(b)} \cdot b' \), where \( \pi_v \) is a uniformizer in \( F_v \). The corresponding local symbol

\[
(a, b)_v = (-1)^{\nu(a)\nu(b)} \frac{\alpha(a)-1}{\nu(a)} \cdot \left( \frac{b'}{v} \right)^{\nu(a)} \cdot \left( \frac{a'}{v} \right)^{\nu(b)}.
\]

Here \( \left( \frac{\cdot}{v} \right) \) is the local quadratic residue symbol in the residue field at \( v \).

We now fix a uniformizer \( \pi \in K \) and let \( g_\pi \in G_K \) be an element which is mapped to \( \pi \in K^\times \) under the reciprocity map. Note that \( g_\pi \) is a Frobenius element in \( G_K \). Assume that

\[
\alpha(g_\pi) \equiv b \mod F_v^\times.
\]

Observe that the element \( b \) depends on the choice of the Frobenius. Any other Frobenius in \( G_K \) has the form \( g_{\pi i} \) for some \( i \in I_K \), and \( \alpha(g_{\pi i}) \equiv \alpha(g_\pi) \alpha(i) \equiv b \cdot \alpha(i) \pmod{F_v^\times} \). The dependence of the result on the choice of uniformizer is not surprising as [4, Theorem 6.1] also depends on a fixed uniformizer \( s \) of the quadratic unramified extension \( K = \mathbb{Q}_p(s) \pmod{\mathbb{Q}_p} \).

Consider the field \( F_v' = F_v(b) \) and let us now define the following error terms:

\[
(-1)^{n_\pi} = \begin{cases} 
(\pi^2, a_\pi^2, v), & \text{if } p \text{ is an odd ramified supercuspidal prime with } KF_v/F_v \text{ ramified quadratic}, \\
(t, c), & \text{if } p \text{ is an odd unramified “bad” supercuspidal prime},
\end{cases}
\]

where \( \pi \) is a uniformizer in \( K \), \( c \in F_v^\times \) is given by \( \alpha(i) \equiv \sqrt{c} \mod F_v^\times \forall i \in I_f(F_v) \) with \( \alpha(i) \notin F_v^\times \) [cf. Equ. 6.6] and \( t \in F_v^\times \) is the quantity given by the quadratic extension \( F_v(\sqrt{i})/F_v \) cut out by the quadratic character \( \psi \) defined as follows: \( \psi(g) = 1 \), if \( \alpha(g) \in (F_v^\times)^k \) and \( \psi(g) = -1 \), if \( \alpha(g) \notin (F_v^\times)^k \).

Note that \( KF_v = F_v \) if and only if \( K \subseteq F_v \). When \( K \not\subseteq F_v \), the extension \( KF_v/F_v \) turns out to be an unramified quadratic extension in the following cases [4]: if \( p \) is an odd unramified supercuspidal prime or \( p \equiv 1 \pmod{4} \) is a ramified supercuspidal prime or \( p \equiv 3 \pmod{4} \) ramified supercuspidal prime with the ramification index \( e(F_v|\mathbb{Q}_p) = e_v \) even. In the remaining case, that is, when \( p \equiv 3 \pmod{4} \) is a ramified supercuspidal prime with \( e_v \) odd, the extension \( KF_v/F_v \) becomes ramified quadratic.

**Theorem 3.3.** Let \( v \) be a place of \( F \) lying above an odd supercuspidal prime \( p \) for \( f \) satisfying one of the following properties:

1. \( p \) is an unramified supercuspidal prime of positive level or it is a “good” level zero unramified supercuspidal prime,
(2) \( p \) is a ramified supercuspidal prime with \( K \subseteq F_v \) or \( KF_v|F_v \) unramified quadratic extension. The local endomorphism algebra \( X_v \) is a matrix algebra if and only if \( m_v \) is even.

We wish to emphasize that the above result is exactly the same as \([4]\). For a “good” level zero unramified supercuspidal prime, the hypothesis (H) here is exactly the same as the condition of \([4, \text{Theorem 6.1}]\) [cf. Lemma 6.2]. Observe that the hypothesis of \([4, \text{Theorem 6.1}]\) is not required for level zero unramified supercuspidal primes \( p \equiv 1 \pmod{4} \) with \( C_p = 0 \) and \( p \equiv 3 \pmod{4} \) with \( C_p = 1 \) [cf. Lemma 6.3]. In the case of odd unramified supercuspidal primes for \( f \) of level zero without the hypothesis, we predict the ramifications of endomorphism algebras using the following theorem:

**Theorem 3.4.** Let \( v \mid p \) be a place of \( F \) with \( p \) a “bad” level zero unramified supercuspidal prime or \( KF_v|F_v \) is a ramified quadratic extension. The ramification of \( X_v \) is determined by the parity of \( m_v + n_v \).

**Remark 3.5.** The result obtained in the present article for \( KF_v|F_v \) ramified quadratic extension is same as that of \([4]\) by Lemma 6.16.

We now consider the case \( p = 2 \). Let \( \omega_2 \) be the fundamental character of level 2, \( \omega \) is a trivial character and \( \chi_m \) is the character on a cyclic group of some 2-power order generated by \( \gamma_m \) for \( m = 1, 2 \). In the dihedral supercuspidal case, we show that the inertia type can be written as follows [cf. Section 5]: \( \chi|_{I_2} = \omega_2 \cdot \chi_1 \cdot \chi_2 \) (when \( K \) is unramified) and \( \chi|_{I_2} = \omega \cdot \chi_1 \cdot \chi_2 \) (when \( K \) is ramified). Assume that \( \chi_1 \) takes \( \gamma_1 \) to \( \zeta_2 \) and \( \chi_2 \) takes \( \gamma_2 \) to \( \zeta_2^* \).

To define the error terms, let \( b \) be as above and consider two fields \( F_v' = F_v(b, \zeta_2 + \zeta_2^{-1}) \) and \( F_v'' = F_v(b, \zeta_2^*) \). We now define two characters \( \psi_1, \psi_2 \) on \( G_v \) as follows:

\[
\psi_1(g) = \begin{cases} 
1 & \text{if } \alpha(g) \in (F_v')^x \\
-1 & \text{if } \alpha(g) \notin (F_v')^x
\end{cases}
\]

\[
\psi_2(g) = \begin{cases} 
1 & \text{if } \alpha(g) \in (F_v'')^x \\
-1 & \text{if } \alpha(g) \notin (F_v'')^x
\end{cases}
\]

Denote by \( F_v(\sqrt{11}), F_v(\sqrt{2}) \), the quadratic extensions of \( F_v \), cut out by the characters \( \psi_1 \) and \( \psi_2 \). Define an integer \( n_v \) modulo 2 as follows:

\[
( -1)^{n_v} = \begin{cases} 
(t_1, \zeta_2^{-1})_v \cdot (t_2, (\zeta_2 + \zeta_2^{-1})^2)_v, & \text{if } p = 2 \text{ and } s \neq 2, \\
(t_2, a^2_{p'})_v, & \text{if } p = 2 \text{ and } s = 2.
\end{cases}
\]

Consider an element \( d_0 \in F_v^x \) given by \( \alpha(i) \equiv \sqrt{d_0} \mod F_v^x, \forall i \in I_T(F_v) \setminus I_T(KF_v) \). An easy check using Lemma 7.1 shows that \( d_0 \) is well-defined. We also define two integers \( n', n'' \) mod 2 by \( (-1)^{n'} = (t_1, \zeta_2^{-1})_v \cdot (t_2, (\zeta_2 + \zeta_2^{-1})^2)_v \cdot (\pi^2, d_0)_v \) and \( (-1)^{n''} = (t_2, a^2_{p'})_v \cdot (\pi^2, d_0)_v \). Note that these error terms can be explicitly computed from the information about a given modular form following \([11]\).

We now state our main theorem for dihedral supercuspidal prime \( p = 2 \).

**Theorem 3.6.** Let \( p = 2 \) be a dihedral supercuspidal prime for \( f \) and \( v \) be a place of \( F \) lying above prime \( p \). The ramification of \( X_v \) is determined by the parity of \( m_v + r_v \).

1. If \( K \subseteq F_v \) or \( KF_v|F_v \) is an unramified quadratic extension, then the error term is \( r_v = n_v \).
2. Assume \( KF_v|F_v \) is a ramified quadratic extension.
   - If \( \zeta_2 + \zeta_2^{-1} \neq 0 \), then the error term \( r_v = n_v \).
   - For \( \zeta_2 + \zeta_2^{-1} = 0 \) the error term is given by \( r_v = n'' \).

In case (2) of the above theorem with \( F = \mathbb{Q} \), we will prove that the quantity \( d_0 \) in the error term is equal to \( a^2_{p'} \) except \( K = \mathbb{Q}_2(\sqrt{d}) \) with \( d = 2, -6 \). The following corollary determines the situation of the above theorem when the local algebra \( X_v \) is determined by the parity of \( m_v \) itself.

**Corollary 3.7.** Let \( p = 2 \) be a dihedral supercuspidal prime for \( f \) with \( N_2 = 2 \). The ramification of the local Brauer class of \( X_v \) is determined by the parity of \( m_v \) for any \( v | 2 \).
Let \( \rho_2(f) \) be the local representation of the Weil-Deligne group of \( \mathbb{Q}_2 \) associated to \( f \) at the prime \( p = 2 \). When inertia acts irreducibly, the projective image of \( \rho_2(f) \) is isomorphic to one of three “exceptional” groups \( A_4, S_4, A_5 \). For any \( v \mid 2 \), let \( D_{K'} \) be the discriminant of the field \( K' \) cut out by the kernel of the homomorphism \( d: G_v \to F_v^x / F_v^{x,2} \) with \( d = \frac{a_v^2}{\ell} \) and \( D = \det(\rho_2(f)) \). In this case, we prove:

**Theorem 3.8.** Let \( p = 2 \) be a non-dihedral supercuspidal prime for a modular form \( f \) and \( v \mid 2 \). The class of \( X_v \) in \( Br(F_v) \) is given by the symbol \( D(-1)^{[F_v: \mathbb{Q}_2]} \cdot (2, D_{K'})_v \).

If \( k \) is odd, we can predict ramification in terms of nebentypus \( \epsilon \). More precisely, we have:

**Corollary 3.9.** If \( p = 2 \) is a non-dihedral supercuspidal prime for a modular form \( f \) of odd weight and \( v \mid 2 \), then we have \( [X_v] \sim \epsilon(-1)^{[F_v: \mathbb{Q}_2]} \cdot (2, D_{K'})_v \).

4. **Galois representation associated to modular forms and local global compatibility**

For all rational prime \( \ell \), we consider a prime \( \lambda \mid \ell \) of \( E \) and let \( E_\lambda \) be the completion of \( E \) at \( \lambda \). For a modular form \( f \) as above, Eichler-Shimura-Deligne constructed a Galois representation \( \rho_f = \rho_{f,\lambda} : G_{\mathbb{Q}} \to GL_2(E_\lambda) \). In this paper, we will be using information about the local Galois representation \( \rho_{f,\lambda} \) with \( \ell = p \) called \( (p, p) \) Galois representation [13].

Let \( A_{\mathbb{Q}} \) denote the adeles of \( \mathbb{Q} \) and \( \pi \) be the automorphic representation of the adele group \( GL_2(A_{\mathbb{Q}}) \) associated to \( f \). This has a decomposition as a restricted tensor product \( \pi = \bigotimes' p \pi_p \) over all primes \( p \) (including the infinite primes). Each local component \( \pi_p \) is an irreducible admissible representation of \( GL_2(Q_p) \). By the local Langlands correspondence for \( n = 2 \), these representations \( \pi_p \) are in a bijection with (isomorphism classes of) complex 2-dimensional Frobenius-semisimple Weil-Deligne representations.

The local global compatibility between these two Galois representations was proved by Carayol in [7] if \( \ell \neq p \). In this paper, we will be using the local global compatibility even for \( \ell = p \) proved by Saito [24] which we describe now. For \( p \neq \ell \) (not necessarily \( p \nmid N \)), the restriction \( \rho_{f,p} := \rho_f|_{G_p} \) induces a representation \( \rho_{f,p} : W_p \to GL_2(\bar{Q}_\ell) \) of the Weil-Deligne group \( W_p \) of \( Q_p \). Let \( \rho_{f,p}^{ss} \) denote its Frobenius semismallification and let the isomorphism class of Frobenius semisimple representation of \( W_p \) associated to \( \pi_p \) be denoted by \( \rho(\pi_p) \). The representation \( \rho(\pi_p) \) of the Weil-Deligne group \( Q_p \) is a pair \( (\rho_p(f), N) \) with a representation \( \rho_p(f) : W_p \to GL_2(\mathbb{C}) \) and a nilpotent endomorphism \( N \) of \( \mathbb{C}^2 \) [10, Section 3]. In this setting, we have the following diagram:

\[
\begin{array}{ccc}
\pi_f = \bigotimes' p \pi_p & \text{Global Langlands} & \rho(\pi) = \rho_f : G_{\mathbb{Q}} \to GL_2(\bar{Q}_\ell) \\
& \downarrow \text{restriction to } G_p \text{ induces} & \\
\rho_{f,p}^{ss} : W_p \to GL_2(\bar{Q}_\ell) & \uparrow (\ast) & \rho(\pi_p) = \rho_p : W_p \to GL_2(\mathbb{C}) \\
& \text{Local Langlands} & \\
\pi_p & \text{de-Rham and by the landmark paper [9] can be studied using filtered } (\phi, N) \text{ modules}.
\end{array}
\]

Recall that for a dihedral supercuspidal prime \( p = 2 \) for \( f \), the local Galois representation \( \rho_{f,G_2} \sim Ind_{G_2}^{G_K} \chi \), with \( K|\mathbb{Q}_2 \) quadratic. In this case, if \( N_p = 2 \), then we show that the extension \( K|\mathbb{Q}_2 \) is always unramified. For the character \( \chi \) of \( G_K \), the usual conductor \( a(\chi) = \min \{ n : \chi(U_n^K) = 1 \} \). Let \( \nu_2 \) be the normalized valuation of \( \mathbb{Q}_2^+ \) and \( \delta(K|\mathbb{Q}_2) \), \( f(K|\mathbb{Q}_2) \) denote the discriminant and the residual degree for \( K|\mathbb{Q}_2 \) respectively. We now recall the formula [8, Proposition 4(b), p. 158] which coincides with the formula for the Artin conductor of a 2-dimensional induced representation of a local Galois group:
\[ a(\text{Ind}_G^G \chi) = v_2(\delta(K|\mathbb{Q}_2)) + f(K|\mathbb{Q}_2)a(\chi). \]

This gives

\[ N_2 = \begin{cases} 
2a(\chi), & \text{if } K|\mathbb{Q}_2 \text{ is unramified,} \\
2 + a(\chi), & \text{if } K|\mathbb{Q}_2 \text{ is ramified with discriminant valuation 2,} \\
3 + a(\chi), & \text{if } K|\mathbb{Q}_2 \text{ is ramified with discriminant valuation 3.} 
\end{cases} \tag{4.1} \]

If \( K|\mathbb{Q}_2 \) is unramified, \( N_2 \) always becomes even. We see that \( N_2 = 2 \) happens in the following cases:

1. \( K|\mathbb{Q}_2 \) is unramified with \( a(\chi) = 1 \) and
2. \( K|\mathbb{Q}_2 \) is ramified with discriminant valuation 2 and \( a(\chi) = 0 \).

The second case cannot occur. Since the algebra \( X_f \) is invariant with respect to twisting by a Dirichlet character \([21, \text{Proposition } 3]\), without loss of generality one can take \( f \) to be minimal in the sense that its level is the smallest among all twists \( f \otimes \psi \) of \( f \) by Dirichlet characters \( \psi \). Then by \([6, \text{§41.4 Lemma}]\), we have \( a(\chi) \geq d = 2 \), a contradiction to \( a(\chi) = 0 \) in the second case.

**Lemma 4.1.** Let \( p = 2 \) be an unramified dihedral supercuspidal prime for \( f \) with \( N_2 = 2 \). For all \( j \in I_W(K) \), we have \( a(j) \in F^\times \).

**Proof.** Since \( N_2 = 2 \), we have \( a(\chi) = 1 \), that is, \( \chi|_{U_k^1} = 1 \). We know that reciprocity map sends wild inertia group of \( K \) onto the principal unit group of \( K \). Let \( \tau = \tau_k \in I_W(K) \) be an element which is mapped to \( k \in U_k^1 \subset K^\times \) under the reciprocity map. Hence, using property (2) of Proposition 2.1 we obtain \( |\alpha(\tau) = \chi(k) + \chi^\alpha(k) \equiv 1 \mod F^\times \). \( \square \)

We can realize the nebentypus \( \epsilon \) as an idelic character as follows: for \( x \in \mathbb{Q}_p^\times \), let \([x]\) denote the corresponding element \((1, \cdots, x, \cdots, 1)\) in \( A_\mathbb{Q}^\times \). The restriction of \( \epsilon \) to \( \mathbb{Q}_p^\times \) is then given by the formula:

\[ \epsilon([p^m u]) = \epsilon'(p)^m \epsilon_p(u)^{-1}, \tag{4.2} \]

for \( m \in \mathbb{Z} \) and \( u \in \mathbb{Z}_p^\times \). From class field theory, we know that norm residue map sends \( \mathbb{Q}_p^\times \subseteq A_\mathbb{Q}^\times \) onto a dense subset of the decomposition group \( G_p \) at \( p \). The Galois character \( \epsilon|_{G_p} \) can also be determined by this fact using the formula above.

### 5. Inertial Galois representations

In this section, we assume the familiarity of the reader with \([14]\).

**5.1. Odd supercuspidal primes.** For an odd supercuspidal prime, the local Galois representation \( \rho_f|_{G_p} \sim \text{Ind}_{G_K}^G \chi \), with \( K|\mathbb{Q}_p \) quadratic. Assume \( \sigma \) is the generator of \( \text{Gal}(K|\mathbb{Q}_p) \). Choose a finite extension \( L|\mathbb{Q}_p \) with the property that \( \rho_f \) is crystalline over \( L \) (cf. \([14]\) for more details) and \( L|\mathbb{Q}_p \) is Galois.

Let \( \omega_2 \) be the fundamental character of level 2, \( \omega \) is the Teichmüller character and \( \chi_m \) is the character having some \( p \)-power order for \( m = 1, 2 \). The inertia type of \( \chi \) can be written as follows: \( \chi|_{I_p} = \chi|_{I(L|\mathbb{Q}_p)} = \omega_2^\alpha \cdot \chi_1 \cdot \chi_2 \) (when \( K \) is unramified) \([14, \text{Sections 3.3.2}]\) and \( \chi|_K = \chi|_{I(L|K)} = \omega_{\epsilon} \cdot \chi_1 \cdot \chi_2 \) \([14, \text{Sections 3.4.2}]\). The action of \( \sigma \) on these characters are given by the following rule: \( \omega_2^\sigma = \omega_2^\epsilon, \omega_{\epsilon} = \omega_2, \chi_1^\sigma = \chi_1 \) and \( \chi_2^\sigma = \chi_2^{-1} \).

Since \( \chi \) does not extend to \( G_p \), we have \( \chi \not\equiv \chi^\sigma \) on \( G_K \) which is equivalent to that \( \chi \not\equiv \chi^\sigma \) on \( I_K \). The last condition is equivalent to: either \( l \not\equiv 0 \pmod{p+1} \) or \( \chi_2^\sigma \not\equiv \chi_2^{-1} \) (unramified case) and \( \chi_2^\sigma \not\equiv \chi_2^{-1} \) (ramified case).
5.2. Dihedral supercuspidal prime $p = 2$. In this case, we will see that $\chi|_{I_2}$ can be thought of as a character of an inertia subgroup of a finite Galois extension of $\mathbb{Q}_2$. By a computation similar to [14, Sections 3.3.2, Sections 3.4.2] for odd primes $p$, we show that $\chi$ restricted to inertia group can be written as $\chi|_{I_2} = \omega_2^* \cdot \chi_1 \cdot \chi_2$ (in the unramified case) and $\chi|_{I_K} = \omega \cdot \chi_1 \cdot \chi_2$ (in the ramified case). The results in this section are obtained by generalizing the construction of $\chi$ on the inertia group for $p = 2$ following [14].

Let $W_2$ (respectively $W_K$) be the Weil group of $\mathbb{Q}_2$ (respectively $K$) and $\rho_2(f)$ be the local representation associated to the local representation $\pi_2$ [cf. Section 4]. In this case, the inertia group acts reducibly. If it acts irreducibly, then the image of $\rho_2(f)$ becomes an exceptional group. Here, we only concentrate on the dihedral supercuspidal representations. To write down the inertia type $\chi|_{I_2}$ or $\chi|_{I_K}$, we recall the structure of the local Galois representation $\rho_2(f)$ following [14].

5.3. The case $K$ unramified. In this case, $\chi$ is a character of $W_K$ which does not extend to $W_2$ and it is finite on $I_2$. Let $\text{Gal}(K|\mathbb{Q}_2)$ be generated by $\sigma$. Let $K = \mathbb{Q}_2(\omega)$ be the unique unramified quadratic extension of $\mathbb{Q}_2$ with $\omega$ a primitive 3-rd root of unity. We choose a finite extension $L|K$ over which $\rho_2(f)$ becomes crystalline and $L/Q_2$ is Galois. For an integer $m \geq 1$, let $K^m$ be the unique cyclic unramified extension of $K$ of degree $m$. Consider the polynomial $g(X) = \pi X + X^4$, where $\pi$ is a fixed uniformizer of $K$. For a Lubin-Tate module $M$, consider the $\text{O}_K$-module of $\pi^{n+1}$-torsion points

$$W_g^n := \text{ker}(\pi^n|_M)$$

whose module structure is induced by the formal group attached to $g(X)$. Let $K(W_g^n)$ be its field. By local class field theory, it is a totally ramified abelian extension of $K$ and its Galois group

$$\text{Gal}(K(W_g^n)|K) \cong U_K/U_K^{n+1} = F_4^\times \times \text{O}_K/\pi^n,$$

where $U_K$ and $U_K^{n+1}$ denote the units and $(n + 1)$-th principal units of $K$ respectively. Furthermore, since $g(X)$ is defined over $\mathbb{Q}_2$ (if the uniformizer $\pi$ is chosen from $\mathbb{Q}_2$), the extension $K(W_g^n)|\mathbb{Q}_2$ is also Galois.

Consider a finite cyclic extension $F|K$ such that $\chi|_{I_F}$ is trivial. By local class field theory the field $F$ is contained in $K^m K(W_g^n)$, for some $m$ and $n$ and so $\rho_2(f)$ restricted to its inertia subgroup is trivial. For this reason, we take $L = K^m K(W_g^n)$ over which $\rho_2(f)$ becomes crystalline as $\rho_2(f)$ is trivial on $I_L$ and if the fixed uniformizer $\pi$ is chosen to be 2, the extension $L/Q_2$ becomes Galois.

5.3.1. Description of $\text{Gal}(L|\mathbb{Q}_2)$. We now describe $\text{Gal}(L|\mathbb{Q}_2)$ in detail. Let $\alpha$ be a root of $g^{(n+1)}(X)$ but not a root of $g^{(n)}(X)$, where $g^{(n)}(X)$ denote the $n$-th iterate of $g(X)$. We have an identification of fields $K(W_g^n) = K(\alpha)$ with $\mathbb{Q}_2(\alpha)|\mathbb{Q}_2$ a totally ramified extension of degree $(2^2 - 1) \cdot 2^n$ and $L|\mathbb{Q}_2(\alpha)$ is an unramified extension of degree $2m$. Let $\sigma$ be a generator of $\text{Gal}(L|\mathbb{Q}_2(\alpha))$ and its projection to the generator of $\text{Gal}(K|\mathbb{Q}_2)$ is also denoted by $\sigma$.

Let us write the inertia subgroup of $\text{Gal}(L|\mathbb{Q}_2)$ explicitly; i.e., $\text{Gal}(L|K^m) \cong \text{Gal}(K(W_g^n)|K)$. Note that $K(W_g^n) = K(\beta)$ with $\beta$ a root of $X^3 + 2 = 0$. Let $\Delta$ be its Galois group over $K$ which is generated by an element, say $\delta$, of order 3. It is isomorphic to the tame part of the inertia subgroup of $\text{Gal}(L|\mathbb{Q}_2)$. Since the order of $\delta$ and 2 are relatively prime, $\delta$ can be lifted uniquely to an element of order 3 in $\text{Gal}(K(W_g^n)|K)$, again denoted by $\delta$. The wild part of the inertia subgroup of $\text{Gal}(L|\mathbb{Q}_2)$ is isomorphic to $\Gamma = \text{O}_K/2^n \cong \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n = \langle \gamma_1 > \oplus < \gamma_2 >$ with $\gamma_1, \gamma_2$ each have order $2^n$.

The full inertia subgroup of $\text{Gal}(L|\mathbb{Q}_2)$ is $\Delta \times \Gamma$. This is a normal subgroup and it is a direct product of three cyclic groups generated by $\delta, \gamma_1$ and $\gamma_2$ respectively. These generators are characterized by the Equ. (5.4). Since $\sigma^2$ fixes $K(W_g^n)$ the action of $\sigma$ on $\text{Gal}(L|K^m)$ by conjugation is an involution. Indeed, if $h \in \text{Gal}(L|K^m)$ and $x \in K^m$, then we have $\sigma^2 \cdot h(x) = \sigma^2 h \sigma^{-2}(x) = \sigma^2(\sigma^{-2}(x)) = x = h(x)$ and if $x \in K(W_g^n)$, we have $\sigma^2 \cdot h(x) = \sigma^2 h \sigma^{-2}(x) = \sigma^2(h(x)) = h(x)$. This action coincides with the action of $\text{Gal}(K|\mathbb{Q}_2)$ on $\text{Gal}(K(W_g^n)|K)$ by conjugation. The group $\text{Gal}(K|\mathbb{Q}_2)$ acts on $\text{O}_K/\pi^n$ in a natural
way. Note that \( O_K^x/U_{K}^{n+1} = O_K^x/U_1^x \times U_1^x/U_{K}^{n+1} \cong (O_K/2)^x \times O_K/2^n \cong \mathbb{F}_4^x \times O_K/2^n \) and
\[
O_K/2^n = \mathbb{Z}_2[\alpha]/2^n = (\mathbb{Z}_2 \oplus \mathbb{Z}_2 \cdot \alpha)/2^n \cong \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n \cdot \alpha \text{ with } \alpha^2 = -\alpha.
\]
Let \( \rho_K : K^x \rightarrow \text{Gal}(K^{ab}/K) \) be the norm residue map. We again denote the restriction \( \rho_K|_{\sigma_K} \) modulo \( U_{K}^{n+1} \) by \( \rho_K \). This is the isomorphism (5.1). We now consider the following commutative diagram [15, Theorem 6.11]:
\[
\begin{array}{ccc}
O_K^x/U_{K}^{n+1} = \mathbb{F}_4^x \times O_K/2^n & \xrightarrow{\rho_K} & \text{Gal}(F_1|K) \\
\downarrow \sigma & & \downarrow \sigma^* \\
O_K^x/U_{K}^{n+1} = \mathbb{F}_4^x \times O_K/2^n & \xrightarrow{\rho_K} & \text{Gal}(F_1|K),
\end{array}
\]
where \( F_1 = K(W^n_g) \) and the map \( \sigma^* \) is obtained by the conjugated action of \( \sigma \) on \( \text{Gal}(F_1|K) \). Using the commutativity of the above diagram, we have \( \rho_K(\sigma(x)) = \sigma^{-1}\rho_K(x)\sigma \), for all \( x \in O_K^x/U_{K}^{n+1} \). This gives us the following relations:
\[
(5.2) \quad \sigma^{-1}\delta\sigma = \delta^2, \quad \sigma^{-1}\gamma_1\sigma = \gamma_1 \quad \text{and} \quad \sigma^{-1}\gamma_2\sigma = \gamma_2^{-1}.
\]

5.3.2. Action of \( \sigma \). By the action (2.1) of [14], the character \( \chi \) on \( I_2 \) can be thought of a character \( \chi \) on the inertia subgroup of \( \text{Gal}(L|Q_2) \) which is \( \Delta \times \Gamma \) [14, Section 3.3.2]. Write
\[
(5.3) \quad \chi|_{I_2} = \chi|_{I(L|Q_2)} = \omega_2^l \cdot \chi_1 \cdot \chi_2,
\]
where \( \omega_2 \) is the fundamental character of level 2 and \( \chi_m \) is the character taking \( \gamma_m \) to a \( 2^n \)-th root of unity \( \zeta_m \) for \( m = 1, 2 \). Let us assume that \( \chi_1 \) takes \( \gamma_1 \) to \( \zeta_2 \) and \( \chi_2 \) takes \( \gamma_2 \) to \( \zeta_2 \). Here, we denote by \( \zeta_2 \) and \( \zeta_2^s \) a primitive \( 2^n \)-th root of unity and a primitive \( 2^n \)-th root of unity respectively, so \( r, s \leq n \).

Let \( \sigma \) be the non-trivial element of the Galois group of \( K|Q_2 \) and it acts on the above characters in the following way:
\[
(5.4) \quad \omega_2^\sigma = \omega_2^2, \quad \chi_1^\sigma = \chi_1, \quad \chi_2^\sigma = \chi_2^{-1}.
\]
The condition that \( \chi \) does not extend to \( W_2 \), we have \( \chi \neq \chi^\sigma \) on \( W_K \) which is further equivalent to that
\[
l \neq 0 \text{ (mod 3)} \text{ or } \zeta_2 \neq \zeta_2^s \text{.}
\]
Since \( \zeta_2^s = \zeta_2 \) and \( \zeta_2^s = \zeta_2^{-1} \), one can deduce that \( r < s \).

5.4. The case \( K \) ramified. Let us now assume that \( K|Q_2 \) is a ramified quadratic extension with \( \chi \) finite on \( I_K \) such that \( \chi|_{I_K} \) does not extend to \( I_2 \). Let us denote the \( \text{Gal}(K|Q_2) \) by \( < \nu > \). Similar to the unramified case, we find out a Galois extension \( L|Q_2 \) such that \( \rho_2(f)|_{I_L} \) is trivial.

5.4.1. Description of \( \text{Gal}(L|Q_2) \). For an integer \( m \geq 1 \), let \( K^m \) be the unramified extension of \( K \) of degree \( m \). For a uniformizer \( \pi \) of \( K \), let \( g(X) = \pi X + X^2 \) and as before let
\[
W^n_g = \{ \alpha \in \mathcal{O}_g | \pi^{n+1} \cdot \alpha = 0 \}.
\]
Here, \( \mathcal{O}_g \) denote the formal \( \mathcal{O}_K \)-module whose underlying set is the ring of integers of the completion of \( K \) and its module structure is induced by the formal group attached to \( g \). The field \( K^n_g = K(W^n_g) \) is a totally ramified abelian extension of \( K \) with Galois group isomorphic to \( U_K/U_{K}^{n+1} = \{1\} \times \mathcal{O}_K/\pi^n \), where \( U_K \) and \( U_{K}^{n+1} \) denote the units and \( (n+1) \)-th principal units of \( K \) respectively. Note that \( \mathcal{O}_K/\pi^{2n} \cong \mathbb{Z}/2^n \oplus \mathbb{Z}/2^n \).

We now choose a finite cyclic extension \( F|K \) such that \( \chi|_{I_F} = 1 \). As every abelian extension of \( K \) is contained in \( K^m K^n \) for some \( m, n \) by class field theory, we have an inclusion of fields \( F \subset K^m K^n \), for some \( m, n \). We take a uniformizer \( \pi \) of \( K \) such that \( \pi^i = -\pi \) (for any lift of \( i \), again call \( i \)). The polynomial \( g_{-\pi}(X) = -\pi X + X^2 \) gives rise to the Lubin-Tate extension \( K^n_{-\pi} \) with \( K^n_{-\pi} \) which is same as \( K^n_{\pi} \). Indeed, if \( K^n_{\pi} = K(\alpha) \) then \( K^n_{-\pi} = K(-\alpha) \). Thus, the field \( K^n_{\pi} \) is preserved by \( \alpha \) lift of \( \iota \in \text{Gal}(K|Q_2) \) and so \( K^n_{\pi} \) is Galois over \( Q_2 \).
For our convenience, set $L = K^{2m}K_2^{2n}$. Then $L|\mathbb{Q}_2$ is a Galois extension containing $F$. In particular $\rho_2(f)|_{I_K} = 1$ and $\rho_2(f)$ becomes crystalline over $L$. The description of the Galois group of $L|\mathbb{Q}_2$ is given using the following exact sequence:

$$1 \to \text{Gal}(L|K) \to \text{Gal}(L|\mathbb{Q}_2) \to \text{Gal}(K|\mathbb{Q}_2) \to 1$$

where $\text{Gal}(L|K) = \text{Gal}(L|K_2^{2n}) \times \text{Gal}(L|K_2^{2m}) = <\sigma> \times (\Delta \times \Gamma)$, $\text{Gal}(K|\mathbb{Q}_2) = <\iota>$, with $\sigma^{2m} = 1$ and $\Delta = \{1\}$, $\Gamma = \mathcal{O}_K/\mathfrak{p}^{2n} = <\gamma_1 > \times < \gamma_2 >$ with $\gamma_i^{2^n} = 1$ for $i = 1, 2$.

Here, $\Delta$ and $\Gamma$ are the tame and wild parts of the inertia group $I(L|K) = \text{Gal}(L|K_2^{2m})$ respectively. The full inertia subgroup of $\text{Gal}(L|K)$ is a direct product of two cyclic groups each generated by $\gamma_1$ and $\gamma_2$ respectively. These generators are characterized by the Equ. (5.7)

5.4.2. Action of $\iota$. The inertia $I(L|K)$ is a normal subgroup of $I(L|\mathbb{Q}_2)$ and the conjugation action of $\iota$ is given by

$$(5.5) \quad \iota^{-1}\{1\} \iota = \{1\}, \quad \iota^{-1}\gamma_1 \iota = \gamma_1, \quad \iota^{-1}\gamma_2 \iota = \gamma_2^{-1},$$

which can be checked as in the unramified supercuspidal case. As in the previous case we can think of $\chi|_{I_K}$ as a character of $I(L|K) \cong \Delta \times \Gamma$. Write

$$(5.6) \quad \chi|_{I_K} = \chi|_{I(L|K)} = \omega \cdot \chi_1 \cdot \chi_2,$$

where $\omega$ is a trivial character, $\chi_m$ is the character taking $\gamma_m$ to a $2^n$-th root of unity $\zeta_m$ for $m = 1, 2$. Let us assume that $\chi_1$ takes $\gamma_1$ to $\zeta_2$ and $\chi_2$ takes $\gamma_2$ to $\zeta_2^r$. Here, $\zeta_2$ and $\zeta_2^r$ denote a primitive $2^n$-th root of unity and a primitive $2^r$-th root of unity respectively and so $r, s \leq n$. The element $\iota$ acts on the above characters in the following way:

$$(5.7) \quad \omega^\iota = \omega, \quad \chi_1^\iota = \chi_1, \quad \chi_2^\iota = \chi_2^{-1}.$$

The condition $\chi|_{I_K}$ does not extend to $I_2$ is equivalent to $\zeta_2^r \neq \zeta_2^{-1}$ and hence $r < s$. Note that there are seven quadratic extensions $\mathbb{Q}_2(\sqrt{d})$ of $\mathbb{Q}_2$ with $d = -3, -1, 3, 2, -2, 6, -6$. Among them $\mathbb{Q}_2(\sqrt{-3})$ is unramified and rest of them are ramified.

**Remark 5.1.** Note that the above characters $\omega_i^\iota, \omega, \chi_1$ and $\chi_2$ are canonically determined by the modular form $f$ (more precisely, the actions 5.4 and 5.7) as we started with the local representation canonically attached to $f$

**Definition 5.2.** ($\gamma_1$-element and $\gamma_2$-element) An element of $I_W(K)$ (the wild inertia part of $K$) is called a $\gamma_1$-element (resp. $\gamma_2$-element) if its projection to $I_W(L|K)$ is $\gamma_1$ (resp. $\gamma_2$).

6. **Ramifications of Endomorphism Algebras for Odd Supercuspidal Primes**

For an odd supercuspidal prime $p$, the local Galois representation is always dihedral and hence induced by a character $\chi$ of an index two subgroup $G_K$ of the local Galois group $G_p$; namely $\rho_f|_{G_p} \sim \text{Ind}_{G_K}^G \chi$ with $K$ a quadratic extension of $\mathbb{Q}_p$. The structure of $\chi$ on the inertia group is given in the section 5. In this section, we give a proof of the results stated in Section 3 for odd supercuspidal primes. Let $K$ be an unramified quadratic extension. For $i \in I_K$, let $i$ be the projection to $I(L|\mathbb{Q}_p)$ and $\delta$ be as above [cf. Section 5]. We call $\epsilon$ to be tame at $p$ if the order of $\epsilon_p$ divides $p - 1$.

**Lemma 6.1.** If $\epsilon$ is tame at $p$, then $\alpha(j) \in F_\epsilon^\times$ for all $j \in I_W(K)$.

*Proof.* Note that $j$ is an element of a pro $p$-group and $p$ is odd. Since $\epsilon$ is tame at $p$, we must have $\epsilon(j) = 1$ and so $\chi_2^\epsilon(j) = e_{\gamma_1}^{-1}(j) = 1$ for all $\gamma \in \Gamma$. By the nature of $j$ and $p$ is odd, we have $\chi_1(j) = 1$ for all $\gamma \in \Gamma$. This implies that $\alpha(j)^{\gamma_1} = 1$, for all $\gamma \in \Gamma$ [cf. Equ. (2.1)]. Hence, we obtain $\alpha(j) \in F_\epsilon^\times$. □
Let $s$ be a fixed $(p^2 - 1)$-th primitive root of unity as in [4] and $K = \mathbb{Q}_p(s)$ is unramified. Recall that $g_s \in \text{Gal}(\mathbb{Q}_p/K)$ is an element which is mapped to $s \in K^\times$ under the reciprocity map.

The next lemma shows that the hypothesis (H) is same as the condition of [4, Theorem 6.1]. First observe that this condition depends on the choice of $s$. By the structure theorem of the local field $K^\times$, we have $K^\times = < p > \times < s > \times U_K^{(1)}$. Let $L$ be as in the beginning of Section 5. By class field theory, the elements of $< s >$ corresponds to the tame part of the inertia group $I(L|\mathbb{Q}_p)$ under the norm residue map. Let $\delta$ be a $p^2 - 1$-th root of unity as in [14, Equation 3.3] (see also Equation 5.2). Observe that $\delta$ is also a valid choice of $s$.

**Lemma 6.2.** The assumption Tr$(\rho_f(g_s)) \neq 0$ in the [4, Theorem 6.1] is same as (H).

**Proof.** Note that Tr$(\rho_f(g_s)) = \chi(s) + \chi(s)^p$. For the choice of $s = \delta$, this is equivalent to $\chi(\delta) + \chi(\delta)^p \neq 0$.

In other words, $\omega_2(\delta) + \omega_2(\delta)^p \neq 0$ [cf. Section 5]. Since $\omega_2$ takes value in the $(p^2 - 1)$-th roots of unity, the last condition is same as the condition $l$ is not an odd multiple of $(p+1)/2$. \hfill $\square$

**Lemma 6.3.** Let $p$ be an odd unramified supercuspidal prime for $f$ and satisfying one of the following conditions:

(1) $p \equiv 1 \pmod{4}$ with $C_p = 0$

(2) $p \equiv 3 \pmod{4}$ with $C_p = 1$.

Then, the condition (H) is satisfied for $p$.

**Proof.** Note that the condition Tr$(\rho_f(g_s)) = 0$ is equivalent to $\chi(s) + \chi(s)^p = 0$, that is,

$$\chi(s)^{p-1} = -1. \tag{6.1}$$

First consider the case (1). Write $p = 4k + 1$, for some $k \in \mathbb{N}$. Since $s^{p+1} \in \mathbb{Z}_p^\times$, using [4, Equ. (4)] and $C_p = 0$, we have $\chi(s)^{p+1} = \epsilon_p(s^{p+1})^{-1} = 1$. Combining it with (6.1), we get $\chi(s)^2 = -1$. Hence, we obtain $\chi(s) = \pm i$.

On the other hand, using (6.1) we have that $\chi(s)^{4k} = -1$, a contradiction.

We now consider the case (2). By the same equation of [4], we have $\chi(s)^{p+1} = \epsilon_p(s^{p+1})^{-1} = \eta$, where $\eta$ is a $(p-1)$-th root of unity. Combining it with (6.1), we get $\chi(s)^2 = -\eta$.

First assume that $p = 3$. Since $C_3 = 1$ and $s^4 = -1$, we must have $\epsilon_3(s^4) = -1$ and so $\eta = -1$. Thus, we deduce $\chi(s)^4 = -1$, a contradiction to $\chi(s)^2 = -1$.

Now suppose that $p > 3$. Write $p = 4k + 3$, for some $k \in \mathbb{N} \setminus \{0\}$. Since $\chi(s)^2 = -\eta$, we have $\chi(s) = \pm i \cdot \sqrt{\eta}$. Again since $p > 3$ and $\chi(s)$ is a primitive $2(p-1)$-th root of unity, we must have that $\sqrt{\eta}$ is a primitive $2(p-1)$-th root of unity, say $\zeta_{2(p-1)}$. Thus, we get $\chi(s) = \pm i \cdot \zeta_{2(p-1)}$. From the equation (6.1), we have $(\pm i \cdot \zeta_{8k+4})^{4k+2} = -1$. We arrive at a contradiction $\epsilon_{8k+4}^{4k+2} = 1$. \hfill $\square$

Hence, the assumption of [4, Theorem 6.1] is not needed for primes stated in the above lemma.

Note that $\omega_2^{(p-1)}(\delta) = 1$, i.e., $\omega_2^{(p-1)(p+1)/2}(\delta) = -1$. Without (H) we have $\omega_2^{(p-1)}(\delta) = -1$ and it is equivalent to $\omega_2^{(p-1)}(\delta) + \omega_2^{lp} = 0$. Then for $i \in I_F(K)$ with $\bar{i} = \delta$, the last condition is further equivalent to trace$(\rho_f(i)) = (\chi + \chi^p)(i) = \omega_2^{(p-1)}(\delta) + \omega_2^{lp}(\delta) = 0$, i.e., $\omega_2^{(p-1)}(\delta)$ is an primitive $2(p-1)$-th root of unity, say $a$.

**Lemma 6.4.** Let $p$ be an odd unramified supercuspidal prime for $f$ without (H). Suppose that $N_p \geq 3$ and $\epsilon$ is tame at $p$. For all $i \in I_F(K)$, we have:

$$\alpha(i) \equiv \begin{cases} 1 \mod F_v^\times, & \text{if } \bar{i} \text{ is an even power of } \delta, \\ a(\zeta_p - \zeta_p^{-1}) \mod F_v^\times, & \text{otherwise.} \end{cases} \tag{6.2}$$

**Proof.** Let $i \in I_F(K)$ be such that $\bar{i} = \delta$. By above, we deduce that trace$(\rho_f(i)) = \omega_2^{(p-1)}(\delta) + \omega_2^{lp}(\delta) = 0$.

For even $n$, we have $\alpha(i^n) \equiv \omega_2^{(p-1)}(\delta^n) + \omega_2^{lp}(\delta^n) \equiv \text{Tr}_{K|\mathbb{Q}_p}(\delta^n) \equiv 1 \mod F_v^\times$.

We now consider odd $n$. By [4, Lemma 4.1], there exists an element $\tau \in I_W(K)$ such that $\chi(\tau) = \zeta_p$ and $\chi(\tau) = \zeta_p^{-1}$, for some primitive $p$-th root of unity $\zeta_p$ and $\alpha(\tau) \equiv 1 \mod F_v^\times$. Thus, we deduce that $\alpha(i) \equiv \alpha(i\tau) \equiv (\chi + \chi^p)(i\tau) \equiv \omega_2^{(p-1)}(\zeta_p - \zeta_p^{-1}) \mod F_v^\times$. Notice that $\bar{\alpha}$ is a homomorphism and


\[ a^2(\zeta_p - \zeta_p^{-1})^2 \in F_v^\times \] by the same lemma. Hence, we obtain \( \alpha(i^n) \equiv \alpha(i^{n+1}) \equiv \alpha(i) \equiv \alpha(\zeta_p - \zeta_p^{-1}) \mod F_v^\times \) with \( m \) even. \hfill \Box

Consider the field \( F_v' = F_v(h) \) as in Section 3. We have:

**Lemma 6.5.** Let \( p \) be an odd unramified supercuspidal prime for \( f \) with \( N_p \geq 3 \). Assume the hypothesis (H) and \( \epsilon \) is tame at \( p \). If \( g \in G_K \) and \( \alpha(g) \notin (F_v')^\times \), then \( \alpha(g) \equiv \alpha(\zeta_p - \zeta_p^{-1}) \mod (F_v')^\times \).

**Proof.** For an uniformizer \( \pi \) of \( K \), let \( g_\pi \) be the image of \( \pi \) under Norm residue map. Note that every element \( g \in G_K \) can be written as \( g_n^{\frac{1}{2}}i \) for some \( n \in \mathbb{Z} \) and \( i \in I_K \). We use Lemma 6.4 and the homomorphism \( \alpha \) to obtain the result. \hfill \Box

Let \( p \) be an odd unramified supercuspidal prime with \( K \subseteq F_v \). Define a function \( f \) on \( G_v(\subseteq G_K) \) by

\[ f(g) = \begin{cases} 1, & \text{if } \alpha(g) \in (F_v')^\times, \\ \alpha(\zeta_p - \zeta_p^{-1}), & \text{if } \alpha(g) \notin (F_v')^\times. \end{cases} \]

(6.3)

We call an element \( g \) type 1 if \( \alpha(g) \in (F_v')^\times \), otherwise we call it type 2. If \( \epsilon \) is tame at \( p \), then we use the fact \( a^2(\zeta_p - \zeta_p^{-1})^2 \in F_v^\times \) and Lemma 6.5. We see that if \( g \) and \( h \) both are type 1 elements then \( gh \) is also so, but if one of them is of type 1 and the other one is of type 2 then their product is an element of type 2. The product of two type 2 elements is an element of type 1. Thus, we can and do replace the conditions which define the function \( f \) by a quadratic character \( \psi \) in the following way: \( \psi(g) = 1 \), if \( \alpha(g) \in (F_v')^\times \) and \( \psi(g) = -1 \), otherwise. The function \( f \) can be seen alternatively as:

\[ f(g) = \begin{cases} 1, & \text{if } \psi(g) = 1, \\ \alpha(\zeta_p - \zeta_p^{-1}), & \text{if } \psi(g) = -1. \end{cases} \]

The quadratic character \( \psi \) on \( G_v \) cut out a quadratic extension of \( F_v \), namely \( F_v(\sqrt{\ell}) \), for some \( t \in F_v^\times \). To compute \( \text{inv}_\psi(c_f) \), let \( \sigma \) be the non-trivial element of \( \text{Gal}(F_v(\sqrt{\ell})|F_v) \). The cocycle table of the 2-cocycle \( c_f \) is given by:

| \sigma | \begin{bmatrix} 1 & \sigma \\ a^2(\zeta_p - \zeta_p^{-1})^2 \end{bmatrix} |
|-------|----------------------------------|
| 1     | 1                                |
| 1     | 1                                |
| \sigma | 1                                |

which gives the symbol \( (t, a^2(\zeta_p - \zeta_p^{-1})^2) \) \( v \). Note that the element \( t \) has no square root in \( F_v^\times \). For the next lemma, we assume that \( \sqrt{p^n} \notin F_v^\times \). Otherwise we would have the ramified quadratic extension \( \mathbb{Q}_p(\sqrt{p^n}) \subseteq F_v \). As a result, \( \alpha(i) \in F_v^\times \forall i \in I_v \) (cf. Lemma 6.7) and so we do not need any auxiliary function \( f \) in Theorem 6.8.

**Lemma 6.6.** With the above notations, \( (t, a^2(\zeta_p - \zeta_p^{-1})^2) \nu = 1 \), i.e., the cocycle class of \( c_f \) is trivial.

**Proof.** The element \( t \) is unique up to a square in \( F_v^\times \) and it is fixed by the kernel of \( \psi \). Since \( \text{Frob}_v = \text{Frob}_{K(F_v|K)} \), the element \( g_v = g_{\pi}^{f(F_v|K)} \) is a fixed Frobenius in \( G_v \). Hence, we deduce that \( \alpha(g_v) \in (F_v')^\times \).

Let \( i \) denote the elements of \( I_T(F_v) \) such that \( \bar{i} = \delta \). Let \( H \) denote the subgroup of \( G_v \) generated by the elements of \( I_W(F_v) \), even power of \( i \) and \( g_v \). We first show that, \( H = \ker(\psi) \).

Note that \( \ker(\psi) = \{ g \in G_v \mid \alpha(g) \in (F_v')^\times \} \). Since \( \alpha \) is a homomorphism, by Lemmas 6.1 and 6.4 we obtain \( H \subseteq \ker(\psi) \). Using the homomorphism \( \alpha \) again and Lemma 6.4, we have \( \alpha(i^n) \notin (F_v')^\times \), for all \( n \in \mathbb{Z} \) odd and hence it cannot belong to \( \ker(\psi) \). Since every element \( g \in G_v \) has the form \( g = g_v^n i \) for some \( i \in I_v \) and \( n \in \mathbb{Z} \), we have \( \alpha(g) \equiv \alpha(i) \mod (F_v')^\times \). Since \( I_v \) is a product of its tame part and wild part, we have shown \( \ker(\psi) \subseteq H \) and hence \( \ker(\psi) = H \).

We now show that \( \sqrt{p^n} := (\frac{1}{p^n}) \cdot p \) is fixed by all the generators of \( H \). For all \( g \in G_p \), we have \( g(\sqrt{p^n}) = \sqrt{p^n} \) or \( -\sqrt{p^n} \). Let \( j \in I_W(F_v) \) be an element of the wild inertia group of \( F_v \). Since it is an element of a pro-\( p \) group and \( p \) is odd, we must have \( j(\sqrt{p^n}) = \sqrt{p^n} \). For all even \( n \in \mathbb{N} \), the elements
and the definition of the homomorphism.

We now compute \((p^*, a^2(\zeta_p - \zeta_p^{-1})^2)_v\). First consider \(p \equiv 3 \pmod{4}\). Then we have \((Nv - 1)/2 = (p^v - 1)/2 \equiv f_v \pmod{2}\). By [4, Eqns. (15), (16) and (17)], we have \((a^2(\zeta_p - \zeta_p^{-1})^2)^2_v = (a^2)^2 \cdot (\zeta_p - \zeta_p^{-1})^{-1} \cdot (\zeta_p - \zeta_p^{-1})^{-1} = 2e_v/p - 1 \equiv e_v \pmod{2}\). Hence, by (3.2) we get that \((p^*, a^2(\zeta_p - \zeta_p^{-1})^2)_v = (1)^{v_{c_F}} \cdot (\zeta_p - \zeta_p^{-1})^{-1} \cdot (\zeta_p - \zeta_p^{-1})^{-1} = (1)^{v_{c_F}} = e_v = 1\).

Now assume that \(p \equiv 1 \pmod{4}\). Using [4, Lemma 4.1], we have that \(e_v\) is even and \(\sqrt{p} = \sqrt{p^v} \in \mathbb{Q}_p(\zeta_p + \zeta_p^{-1}) \subseteq F_v\). Hence, the symbol \((\sqrt{p^v})^v = 1\). As \(e_v\) is in the exponent, \((p^*, a^2(\zeta_p + \zeta_p^{-1})^2)_v = 1\). □

Suppose that \(K \not\subset F_v\) (i.e., \(G_v \not\subset G_K\)) with \(K|\mathbb{Q}_p\) unramified quadratic. For a fixed Frobenius \(g_v \in G_v\), the element \(g_v \in G_v/G_{KF_v}\) is nontrivial. Thus, every element \(g \in G_v\) can be written as
\[
g = g_v^n h, \quad \text{for some } h \in G_{KF_v} \text{ and } n \in \{0, 1\}.
\]

Note that \(n = 0\) when \(g \in G_{KF_v}(\subseteq G_K)\). Using this decomposition, we extend the function \(f_1\) (defined on \(G_{KF_v} \subseteq G_K\) uniquely to \(G_v\), call it \(F\), as follows: \(F(g) = f(h)\). The inflation map \(\text{Inf}_v : 2\mathbb{H}(G_{KF_v}, (F_v^\times)^{(Gal(KF_v/F_v)}) \rightarrow 2\mathbb{H}(G_v, F_v^\times)\) sends the cocycle \(c_f\) to \(c_F\). Since the inflation map is injective and the class of \(c_F\), the cocycle class of \(c_F\) is trivial.

6.1. The case \(K \subseteq F_v\) or \(KF_v|F_v\) unramified quadratic extension. First we determine the value of \(\alpha\) at the inertia groups.

Lemma 6.7. Let \(p\) be an odd supercuspidal prime with \(K \subseteq F_v\). Assume that \(\epsilon\) is tame at \(p\). When \(p\) is an unramified supercuspidal prime, we also assume (H). For all \(i \in I_v\), we have \(\alpha(i) \in F_v^\times\).

Proof. In this case, we have \(KF_v = F_v\) and \(I_v \subseteq I_K\). Every element \(i \in I_v\) has the form \(i = ij\) for some element \(i\) of the tame part \(T\) and some element \(j\) of the wild part \(W\) of the inertia group \(G_{KF_v}\).

In the unramified case, we deduce that: \(\alpha(i) = \alpha(ij) \equiv \alpha(i) \equiv \chi(i) + \chi'(i) \equiv \omega_j(i) + \omega_j^p(i) \mod F_v^\times\). For ramified supercuspidal primes, we obtain: \(\alpha(i) = \alpha(ij) \equiv \alpha(i) \equiv \omega_0(i) + \omega_0(i) \equiv 2 \omega(i) \mod F_v^\times\). The first congruence relation in both cases follows from Lemma 6.1 and the definition of the homomorphism \(\alpha\), and the second one follows from [Proposition 2.1, property (2)]. Since \(\omega_2(i)\) belongs to \(K = \mathbb{Q}_p\), we obtain \(\omega_2(i) + \omega_2^p(i) = Tr_{K/\mathbb{Q}_p}(\omega_2(i)) \in \mathbb{Q}_p^\times \subseteq F_v^\times\). Again since \(\omega_0\) takes values in the multiplicative group of \((p - 1)\)-th roots of unity, in both cases, we conclude that \(\alpha(i) \in F_v^\times\), for all \(i \in I_v\). □

We now prove Theorem 3.3 when \(K \subseteq F_v\) or \(KF_v|F_v\) is unramified quadratic.

Theorem 6.8. Let \(p\) be an odd supercuspidal prime with \(K \subseteq F_v\) or \(KF_v|F_v\) is unramified quadratic. If \(p\) is an unramified supercuspidal prime, we assume (H) unless \(N_p \geq 3\). Then \(X_v \sim m_v\) for \(v | p\).

Proof. Since the endomorphism algebra is invariant under twisting [21, Proposition 3], without loss of generality one can assume that \(\epsilon\) is tame at \(p\).

(1) Consider \(K \subseteq F_v\) with the hypothesis (H). By the lemma above, \(\alpha(i) \in F_v^\times\) for all \(i \in I_v\). Using [Proposition 2.1, part (1)] and \(\epsilon_p(g) \in \mathbb{Q}_p^\times\) (as \(\epsilon\) is tame at \(p\)), we obtain \(\alpha^2 \epsilon_p(g) \in F_v^\times\), for all \(g \in G_v\). By Lemma 2.2 applied to \(S = \alpha\) and \(t = \epsilon\), we get \(\text{inv}_v(\alpha) = \frac{1}{v_p}(\alpha^2 \epsilon_p(\text{Frob}_v)) \mod \mathbb{Z}\).

(2) Assume \(K \subseteq F_v\) with \(K\) unramified and \(N_p \geq 3\). The previous computation works with (H). Thus, we consider this case without the hypothesis (H).

Note that \(G_v \subseteq G_K\) and \(I_v \subseteq I_K\). Set \(S = \frac{a}{2} \) on \(G_v\) with \(f\) as in (6.3). Since \(\alpha(i) \equiv a(\zeta_p - \zeta_p^{-1}) \mod F_v^\times \forall i \in I_v\) with \(\alpha(i) \notin F_v^\times\) (by Lemmas 6.1 and 6.4), we get \(S(i) \in F_v^\times \forall i \in I_v\). Since \(a^2(\zeta_p - \zeta_p^{-1})^2\) and \(\omega_p^2(g) \in F_v^\times\), we obtain \(\omega_p^2(g) \in F_v^\times \forall g \in G_v\). Then by Lemma 2.2,
\[ \text{inv}_v(c_S) = \frac{1}{2}v\left( \frac{a^2}{c}(\text{Frob}_v) \right) \mod Z = \frac{1}{2}v \left( \frac{a^2}{c} (\text{Frob}_v) \right) \mod Z. \] The cocycle \( c_a \) can be decomposed as \( c_a = c_f \) with \( c_f, c_f \) are the cocycles corresponding to \( S \) and \( f \) respectively. Note that the cocycle class of \( c_f \) is trivial by Lemma 6.6 and hence \( \text{inv}_v(c_a) = \text{inv}_v(c_S) + \text{inv}_v(c_f) = \text{inv}_v(c_S) = \frac{1}{2}v \left( \frac{a^2}{c} (\text{Frob}_v) \right) \mod Z. \]

(3) Next assume that \( K, F \) is unramified quadratic. In this case, we get \( I_v = I_{KF_v} \subseteq I_K \). The same computation in (1) works here with (H). So assume this case without the hypothesis (H).

Define \( S = \frac{2}{p} \) on \( G_v \) with \( F \) as in the previous paragraph of Section 6.1. Since \( I_v = I_{KF_v} \), in the decomposition (6.5) for any element of \( I_v \), we must have \( n = 0 \). By writing the definition of \( F \), we deduce that \( \frac{2}{p} = \frac{2}{p} \) on \( I_v \). By the same argument as in (2), we see that two conditions of Lemma 2.2 are satisfied by \( S \) and \( t = e' \). Hence, we obtain \( \text{inv}_v(c_S) = \frac{1}{2}v \left( \frac{a^2}{c} (\text{Frob}_v) \right) \mod Z \). Since the cocycle class of \( c_f \) is trivial, we deduce that \( \text{inv}_v(c_a) = \text{inv}_v(c_S) + \text{inv}_v(c_f) = \text{inv}_v(c_S) = \frac{1}{2}v \left( \frac{a^2}{c} (\text{Frob}_v) \right) \mod Z. \)

For a prime \( p' \) introduced before, we have that \( \chi_\gamma(\text{Frob}_{p'}) = \chi_\gamma([p]) \equiv (2) \chi_\gamma(p) = \chi_\gamma(p') \), where \( \chi_\gamma \) denote the prime-to-\( p \) part of \( \chi_\gamma \). By a similar computation, we deduce \( \epsilon'(\text{Frob}_{p'}) = \epsilon'(p) \). Thus, using (2.1) we have \( \alpha(\text{Frob}_{p'}) = \alpha(\text{Frob}_{p'}) \equiv a_{p'} \mod F^\times \), where \( \text{Frob}_{p'} \) and \( \text{Frob}_{p} \) denote the Frobenii at the primes \( p \) and \( p' \) respectively. Hence, we deduce that \( \alpha(\text{Frob}_v) = \alpha(\text{Frob}_{p'}) \equiv a_{p'} \mod F_v^\times \). On the other hand, we have \( \epsilon'(p) = \epsilon'(p') = \epsilon(p') \). Hence, in all of the above cases we obtain

\[ \text{inv}_v(c_a) = \frac{1}{2}v \left( \frac{a^2}{\epsilon'} (\text{Frob}_v) \right) = \frac{1}{2} \cdot f_v \cdot v \left( \frac{a^2}{\epsilon'} (\text{Frob}_v) \right) = \frac{1}{2} \cdot f_v \cdot v(a_{p'}^2 \epsilon(p'))^{-1} \mod Z. \]

\[ \square \]

Remark 6.9. Writing multiplicatively the above formula, we obtain the same result as in [4, Theorems 6.1, 6.2 and 7.1]. When \( p \equiv 3 \mod 4 \) is a ramified supercuspidal prime with \( e_v \) even and \( K \not\subseteq F_v \), we have \( [X_v] \sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p'))^{-1}} \). Thus, when \( f_v \) is even, we deduce \( X_v \) is a matrix algebra over \( F_v \) which also follows from the formula (10) of [4] as \( (p^2 - 1)/2 \equiv f_v \mod 2 \). We now consider the case where \( f_v \) is odd. Since \( (p - 1)/2 \) is odd and it divides \( e_v \) [4, Lemma 4.1] which is even, using [4, Lemma 7.2] we get

\[ [X_v] \sim (p, K|Q_p) v(a_{p'}^2 \epsilon(p'))^{-1} \sim (p, K|Q_p) v(a_{p'}^2 \epsilon(p'))^{-1}. \]

Hence, \( X_v \) is unramified, when \( (p, K|Q_p) = 1 \) which we cannot conclude from the result obtained in [4]. When \( (p, K|Q_p) = -1 \), our result matches up with [4, Theorem 7.6].

Remark 6.10. The quantity that determines the algebra \( X_v \) is independent of the choice of \( p' \). For two distinct primes \( p' \) and \( q' \) satisfying (3.1), one has \( \epsilon(p') = \epsilon(q') \). Also, \( \chi_\gamma(p') = \chi_\gamma(q') \mod \Gamma. \) Using (2.1) and [Proposition 2.1, part (3)], we have \( a_{p'}^{-1} = a_{q'}^{-1} \mod \Gamma. \) Thus, we get \( a_{p'} \equiv a_{q'} \mod F^\times \) and so \( a_{p'}^2 \epsilon(p')^{-1} \equiv a_{q'}^2 \epsilon(q')^{-1} \mod (F^\times)^2 \). Hence, they have the same \( v \)-adic valuation modulo 2.

Corollary 6.11. Assume that \( K \subseteq F_v \). If \( p \) is an odd unramified supercuspidal prime (the hypothesis (H) is needed if necessary) or \( p \equiv 3 \mod 4 \) is a ramified supercuspidal prime, then \( X_v \) is a matrix algebra over \( F_v \).

Proof. For such primes \( p \), we have proved that \( X_v \sim (-1)^{f_v \cdot v(a_{p'}^2 \epsilon(p'))^{-1}}, \) for \( v \mid p \). When \( p \) is an odd unramified supercuspidal prime, the containment \( K = \mathbb{Q}_{p,3} \subseteq F_v \) implies that \( f_v \) is even and the result follows.

When \( p \equiv 3 \mod 4 \) is a ramified supercuspidal prime with \( K \subseteq F_v \), we have \( e_v \) is even. On the other hand, \( K \not\subseteq F_v \) if \( e_v \) is odd. If \( (p, K|Q_p) = 1 \), we get the result by using [4, Lemma 7.2] and the fact \( (p - 1)/2 \) is odd and it divides \( e_v \) [4, Lemma 4.1]. If \( (p, K|Q_p) = -1 \), then we have \( \sqrt{p} \in K \subseteq F_v \) [cf.
Lemma 6.15. Let \( g_\pi \) be an element which is mapped to \( \sqrt{\pi} \in K^\times \) and \( g_p \in G_p \) be an element which is mapped to \( p \in \mathbb{Q}_p^\times \) under the reciprocity map. We deduce that \( g_p = g_\sqrt{\pi}^2 \). Thus, using [Proposition 2.1, property (1)] we obtain that \( \alpha^2(g_\pi) \in F_v^\times \) and so \( \alpha^2(g_p) \in (F_v^\times)^2 \). Note that \( g_p \) is one of the Frobenius at \( p \) and \( g_v = g_p^\nu \) is a Frobenius at \( v \). Thus, the valuation \( v\left(\alpha^2(g_v)\right) \) is even. This completes the proof. □

Let \( p \) be an odd unramified supercuspidal prime for \( f \) of level \( 0 \) without the hypothesis (H), i.e. a bad prime. For \( i \in I_T(K) \) with \( i = \delta \), we have \( \omega_2 \) of \( f \) and \( \text{trace}(\rho_f(i)) = 0 \) as before. By Lemma 6.4, we have \( \alpha(i^2) \in F_v^\times \) and we write \( \alpha(i) = \sqrt{f(i)c(i)} \) for some \( t(i), c(i) \in F_v^\times \). Consider two elements \( i, j \in I_T(K) \) with \( i = j = \delta \). Since \( c_\alpha(i, j) \in F_v^\times \) and \( \alpha(ij) \in F_v^\times \) (cf. Lemma 6.4), we must have \( \sqrt{f(i)c(i)} \equiv \sqrt{f(j)c(j)} \mod F_v^\times \). For some fixed \( c \in F_v^\times \), we have \( \sqrt{f(i)} \equiv \sqrt{c} \mod F_v^\times \). By Lemmas 6.1 and 6.4, we have \( \alpha(i) \in F_v^\times \) for all \( i \in I_K \) except those for which \( i \) is an odd power of \( \delta \). For \( i \in I_T(K) \) with \( i = \delta \), let us assume:

\[
\alpha(i) \equiv \sqrt{c} \mod F_v^\times, \quad c \in F_v^\times.
\]

Consider an integer \( n_v \) modulo \( 2 \) as in (3.3) when \( p \) is a bad prime.

**Theorem 6.12.** Let \( v | p \) be a place of \( F \) with \( p \) a “bad” level zero unramified supercuspidal prime. The ramification of \( X_v \) is determined by the parity of \( m_v + n_v \).

**Proof.** We will proceed the same way as before. When \( K \subset F_v \), consider the function on \( G_v \) defined by: \( f(g) = 1 \), if \( \alpha(g) \in (F_v^\times)^2 \) and \( f(g) = \sqrt{\pi} \), if \( \alpha(g) \notin (F_v^\times)^2 \). Consider the extension \( F_v(\sqrt{l})|F_v \) cut out by the quadratic character obtained from the conditions that define \( f \). By a computation as in the previous cases, the cocycle class of \( c_f \) is determined by the symbol \( (t, c)_v \).

If \( K \subsetneq F_v \), we extend this function \( f \) uniquely to \( G_v \), call it \( F \). As above, observe that both \( c_f \) and \( c_{KF} \) have the same Brauer class. Define a function \( \alpha' \) on \( G_v \) as follows:

\[
\alpha' = \begin{cases} \sqrt{\pi}, & \text{if } K \subsetneq F_v, \\ \sqrt{c}, & \text{if } K \subset F_v. \end{cases}
\]

Then the assumptions of Lemma 2.2 will be satisfied by \( S = \alpha' \) and \( t = \epsilon' \) and we get the result. □

6.2. The case \( K F_v | F_v \) is ramified. This case will happen only if \( p \) is an odd ramified supercuspidal prime with \( p \equiv 3 \pmod{4} \) and \( \nu_p \) odd. For any quadratic extension \( L_1|L_2 \) and \( x \in L_2^\times \), the symbol \( (x, L_1|L_2) = 1 \) or \( -1 \) according as \( x \) is a norm of an element of \( L_1 \) or not.

In the ramified case, the possibilities for \( K \) are \( \mathbb{Q}_p(\sqrt{-p}) \) and \( \mathbb{Q}_p(\sqrt{-p^{p-1}}) \) depending on (\( p, K|\mathbb{Q}_p \)) = 1 or \(-1\) respectively. We can choose \( \pi = \sqrt{-p} \) or \( \sqrt{-p^{p-1}} \) as a unramified \( K \) and write \( K = \mathbb{Q}_p(\pi) \). For any lift \( \sigma \) of the generator of \( \text{Gal}(K|\mathbb{Q}_p) \) to \( G_p \), we have \( \pi^\sigma = -p \) and \( N_{K|\mathbb{Q}_p}(\pi) = -p^2 \).

For a field \( L \), let \( O_L^\times \) be the ring of units inside \( O \). Since \( K F_v | F_v \) is a ramified quadratic extension, \( N_{K F_v F_v}(\mathcal{O}_{K F_v}^\times) = \mathcal{O}_{F_v}^\times \). Let \( \pi \) be a fixed uniformizer in \( N_{K F_v F_v}(\mathcal{O}^\times_{K F_v}) \subseteq F_v^\times \). Writing \( a = \pi^{(a)} \cdot \alpha' \in F_v^\times \), we have \( (\pi^2)^\nu = (a, K F_v F_v) \).

Note that \( f_v \) is odd in our case. For these primes, we have

\[
(-1, K F_v F_v) = \left(\frac{-1}{v}\right) = \left(\frac{-1}{p}\right)^f = (-1)^{f_v} = -1.
\]

Otherwise, \( X_v \) is a matrix algebra over \( F_v \) [cf. Remark 6.17]. Since \( N_{K F_v F_v}(\sqrt{-p^{p-1}}) = p^{p-1} \) and \( N_{K F_v F_v}(\sqrt{-p}) = p \), we deduce that

\[
\left(\frac{\pi^2}{v}\right) = \begin{cases} \left(\frac{-p^{p-1}}{v}\right) = \left(\frac{-1}{v}\right)(p, K F_v F_v) = \left(\frac{-1}{v}\right), & \text{if } (p, K|\mathbb{Q}_p) = 1, \\ \left(\frac{-p^{p-1}}{v}\right) = \left(\frac{-1}{v}\right)(p^{p-1}, K F_v F_v) = \left(\frac{-1}{v}\right), & \text{if } (p, K|\mathbb{Q}_p) = -1. \end{cases}
\]
Lemma 6.13. For all \( i \in I_v \setminus I_{KF_v} \), and \( \alpha(i) \notin F_v^\times \), we have \( \alpha^2(i) \equiv d \mod F_v^{\times^2} \), for some fixed \( d \in F_v^\times \).

Proof. Let us consider an element \( i \in I_v \setminus I_{KF_v} \), which we fix now. Since \( i^2 \in I_{KF_v} \subseteq I_K \), we have \( \alpha^2(i) \in F_v^\times \) [cf. Lemma 6.7]. Hence, \( \alpha(i) \equiv \sqrt{d} \mod F_v^\times \) for some \( d \in F_v^\times \). Any element \( i \in I_v \setminus I_{KF_v} \) can be written as \( i = ij \) for some \( j \in I_{KF_v} \). Using Lemma 6.7 and the homomorphism \( \tilde{\alpha} \), we have \( \alpha(i) \equiv \alpha(i)\alpha(j) \equiv \alpha(i) \equiv \sqrt{d} \mod F_v^\times \).

We show that the value of the constant \( d \) is \( a^2 \). Let \( [\_]_v : F_v^\times \to \mathbb{G}_a^\text{ab} \) be the usual norm residue map.

Lemma 6.14. As an element of the Galois group, we have \( i = [-1]_v \in G_v \setminus G_{KF_v} \). Moreover, the value of the map \( \alpha \) at \( i \) is given by: \( \alpha(i) \equiv a^\omega \mod (F_v^\times) \).

Proof. As the norm residue map is surjective, we need to show that \( [-1]_v \neq [x]_{KF_v} \) for any \( x \in (KF_v)^\times \). Suppose towards a contradiction that \( [-1]_v = [x]_{KF_v} \), for some \( x \in (KF_v)^\times \). Let \( \rho_{KF_v} = [\_]_{KF_v} \) and \( \rho_o = [\_]_v \) be the norm reciprocity maps. Recall, the following commutative diagram from the class field theory:

\[
\begin{array}{ccc}
(KF_v)^\times & \longrightarrow & \mathbb{G}_a^\text{ab}_{KF_v} \\
\downarrow N_{KF_v/F_v} & & \downarrow \text{incl} \\
F_v^\times & \longrightarrow & \mathbb{G}_a^\text{ab}.
\end{array}
\]

From the above diagram, we have \([x]_{KF_v} = [N(x)]_v \) and so \([-1]_v = [N(x)]_v \). We write \(-1 = N(x)y \) for some \( y \in N_{F_v/F_v}(F_v^\times) = \bigcap_{F_v \subseteq L} \text{finite } N_{K/L}(K^\times) \). As \( KF_v \) is a finite extension of \( F_v \), we deduce \(-1 \) is a norm of some element of \((KF_v)^\times \), a contradiction to (6.8). Since the norm residue map sends the unit group of \( F_v \) onto the inertia subgroup of \( G_v \), the element \( i = [-1]_v \in I_v \setminus I_{KF_v} \).

Note that \( i = [-1]_v \in G_v(\subseteq G_p) \) is one of the several elements that maps to \(-1 \in \mathbb{Q}_p^\times \). For all \( \gamma \in \Gamma \), we obtain

\[
\alpha(i)\gamma^{-1} = \gamma\gamma(i) = \gamma([-1]) = \gamma(\frac{a^\omega}{a^\omega}) = \gamma(p^\omega)(-1) = \gamma(p^\omega)(-1) = \gamma(p^\omega)^{\frac{p-1}{2}} = (\alpha(Frob_p^\omega))^{-1} \cdot \frac{p-1}{2}.
\]

We deduce that \( \alpha(i) \equiv a^\omega \mod F_v^\times \). Using the property (1) of Proposition 2.1 and \( \alpha(Frob_p^\omega) \equiv a^\omega \mod F_v^\times \), we have \( \epsilon(p^\omega) \equiv \epsilon(p) \mod F_v^\times \). Since \( p^\omega \) has order \((p-1)\) in \((\mathbb{Z}/p\mathbb{Z})^\times \), we have \( a^\omega \equiv a^\omega \mod F_v^\times \). As \( p \equiv 3 \) (mod 4) and \((p-1)/2 \) is odd, we deduce that \( \alpha(i) \equiv a^\omega \mod F_v^\times \).

Lemma 6.15. If \( p \equiv 3 \) (mod 4) and \( (p, K) = 1 \), then we have \( a^2 \equiv a \mod F_v^\times \), for some unit \( u \in \mathbb{O}_v^\times \).

Proof. For an odd prime \( p \), the two ramified quadratic extensions of \( \mathbb{Q}_p \) are \( \mathbb{Q}_p(\sqrt{-p}) \) and \( \mathbb{Q}_p(\sqrt{-p^{s_p}-1}) \) up to an isomorphism. Note that \( \mathbb{Q}_p(\sqrt{p}) \) is always a ramified quadratic extension of \( \mathbb{Q}_p \) and \(-1 \) has no square root modulo \( p \) for primes \( p \equiv 3 \) (mod 4). Since when \((p, K) = 1 \), the only possibility for \( K \) is \( Q_p(\sqrt{p}) = Q_p(\sqrt{-p^{s_p}-1}) \).

We obtain \( \omega' = \sqrt{-p^{s_p}-1} \in K \) and let \( g_{\omega'} \in G_K \) be an element which is mapped to \( s' \in K^\times \) under the reciprocity map. We have a following equality:

\[
\chi_{\gamma}(g_{\omega'}) = \chi_{\gamma}(N_{KQ_p(s')}([\_])) = \chi_{\gamma}([\_](\sqrt{p^{s_p}-1})) = \chi_{\gamma}(\sqrt{p^{s_p}-1} = \chi_{\gamma}(p^{s_p}-1).
\]

Using (2.1), we deduce that \( \alpha(g_{\omega'}) \alpha(Frob_p^\omega) \in F_v^\times \) and hence \( \alpha(g_{\omega'}) \cdot a^\omega \equiv F_v^\times \). We now claim that \( \alpha(g_{\omega'}) \equiv u \mod F_v^\times \), for some unit \( u \in \mathbb{Q}_v^\times \). Since \( s' \) is a root of unity, the element \( g_{\omega'} \in I_T(K) \) by class field theory. For an element \( i \in I_T(K) \), we know that \( \alpha(i) \in F_v^\times \) and \( \alpha(i) \equiv \omega(i) \mod F_v^\times \) [cf. Lemma 6.7]. Since \( \omega \) takes values in the \((p-1)\)-th roots of unity, we get the result.

We now prove Theorem 3.4 when \( KF_v/F_v \) is a ramified quadratic extension.
Proof. Define a function \( f \) on \( G_v \) by

\[
(6.11) \quad f(g) = 1, \text{ if } g \in G_{KF_v} \text{ and } f(g) = a_p^\nu, \text{ if } g \in G_v \setminus G_{KF_v}.
\]

Note that \( KF_v = F_v(\pi) \). Denote the image of \( g \in G_v \) under the projection in \( G_v / G_{KF_v} = \text{Gal}(F_v(\pi) | F_v) \) by \( \bar{g} \). We now consider the function \( F \) on \( \text{Gal}(KF_v | F_v) \):

\[
(6.12) \quad F(g) = 1, \text{ if } \overline{\gamma} = 1 \text{ and } F(g) = a_p^\nu, \text{ if } \overline{\gamma} \neq 1.
\]

Using equations (6.11) and (6.12), one can check that \( c_F(\bar{g}, \bar{h}) = c_F(g, h) \). In other words, we deduce that the inverse of the inflation map \( \text{Inf} : H^2(F_v(\pi) | F_v) \to H^2(F_v | F_v) \) sends \( c_F \) to \( c_F \). Let \( \sigma \) be the non-trivial element of \( \text{Gal}(F_v(\pi) | F_v) \). The cocycle table of \( C_F \) is given by:

\[
\begin{array}{c|c|c|c}
\sigma & 1 & \sigma \\
1 & 1 & 1 \\
\sigma & 1 & a_p^\nu \\
\end{array}
\]

which gives the symbol \( (\pi^2, a_p^{2\nu})_v \). Using the above inflation map we get both \( c_F \) and \( c_F \) have same class in their respective Brauer groups. Define an integer \( n_v \) mod 2 by \( (-1)^{n_v} = (\pi^2, a_p^{2\nu})_v \).

Let \( \alpha'(g) = \frac{\alpha(g)}{f(g)} \) on \( G_v \). Then the cocycle \( c_\alpha \) can be decomposed as \( = c_{\alpha'}c_F \). The two conditions of Lemma 2.2 are satisfied by \( S = \alpha' \) and \( t = \epsilon' \). Thus, we obtain: \( \text{inv}_v(c_{\alpha'}) = \frac{1}{2} \cdot v\left( \frac{a_p^2}{\epsilon} \cdot (\text{Frob}_v) \right) = \frac{1}{2} \cdot v\left( \frac{a_p^2}{\epsilon} \cdot \epsilon(p')^{-1} \right) \mod \mathbb{Z} \), as before. Hence, we deduce that: \( \text{inv}_v(c_\alpha) = \text{inv}_v(c_{\alpha'}) + \text{inv}_v(c_F) = \frac{1}{2} \cdot \left( f_v \cdot v(a_p^2 \cdot \epsilon(p')^{-1}) + n_v \right) \mod \mathbb{Z}. \)

Multiplicatively, we can write the above as \( [X_v] \sim (-1)^{f_v (a_p^2 \epsilon(p')^{-1})} (\pi^2, a_p^{2\nu})_v \). The following lemma will complete the proof which is a simplification of this product depending upon the value of \( (p, K|\mathbb{Q}_p) \).

Lemma 6.16. If \( KF_v | F_v \) is a ramified quadratic extension, then the ramification formula is given by:

\[
[X_v] \sim ((-1)^ka_p^{2\nu} \epsilon(p')^{-1}, KF_v | F_v).
\]

Proof. Since \( p \equiv 3 \pmod{4} \), we get \( (Nv - 1)/2 = (p^{f_v} - 1)/2 \equiv f_v \pmod{2} \). Recall that both \( v(\pi^2) = e_v \) and \( (p - 1)/2 \) is odd that divides \( e_v \) [4, Lemma 4.1]. We have an equality of symbols: \( (\pi^2, a_p^{2\nu})_v \equiv (-1)^{v(\pi^2) + (a_p^{2\nu})_v(p^{f_v} - 1)/2} \left( \frac{(a_p^{2\nu})_v}{\pi^2} \right)^{v(\pi^2)} \left( \frac{(a_p^{2\nu})_v}{\pi^2} \right)^{v(a_p^{2\nu})} \left( \frac{1}{\pi^2} \right)^{v(a_p^{2\nu})} = \left( \frac{(a_p^{2\nu})_v}{\pi^2} \right)^{v(a_p^{2\nu})}. \)
When \( (p, K|\mathbb{Q}_p) = 1 \), using [4, Lemma 7.2] we deduce that:

\[
[X_v] \sim (-1)^{f_v\cdot v(\alpha^{e(v)(p')^{-1}})} \cdot (\pi^2, a^{2}_{p'})_v
\]

\[
= (-1)^{e_v f_v (k-1) - \epsilon_p(1) 2 e_v f_v / (p-1)} \cdot \left( \frac{(a^{2}_{p'})^\prime}{v} \right)
\]

\[
= (-1)^{k f_v (-\epsilon_p(1)) f_v \left( \frac{(a^{2}_{p'})^\prime}{v} \right)}
\]

\[
= (-1)^k f_v \left( \epsilon(p'')^{(p'-1)/2} \right) f_v \cdot \left( \frac{(a^{2}_{p'})^\prime}{v} \right)
\]

\[
= (-1)^k f_v \left( \epsilon(p')^{p''} \right) f_v \cdot \left( \frac{(a^{2}_{p'})^\prime}{v} \right)
\]

\[
= (-1)^k f_v \left( \epsilon(p'')^{p''-1} F_v | F_v \right).
\]

On the other hand for \( (p, K|\mathbb{Q}_p) = -1 \), using [4, Lemma 7.2] we have:

\[
[X_v] \sim (-1)^{f_v\cdot v(\alpha^{e(v)(p')^{-1}})} \cdot (\pi^2, a^{2}_{p'})_v
\]

\[
= (-1)^{e_v f_v (k-1) - \epsilon_p(1) 2 e_v f_v / (p-1)} \cdot (-1)^{f_v\cdot v(\alpha^{e(v)(p')^{-1}})} \cdot \left( \frac{(a^{2}_{p'})^\prime}{v} \right)
\]

\[
= ((-1)^k a^{2}_{p'} \epsilon(p'')^{-1} F_v | F_v) \cdot (-1)^{f_v\cdot v(\alpha^{e(v)(p')^{-1}})}
\]

\[
(6.15)
\]

\[
= ((-1)^k a^{2}_{p'} \epsilon(p'')^{-1} F_v | F_v).
\]

\[
\square
\]

**Remark 6.17.** In general, when \(KF_v | F_v \) is ramified quadratic, we have \([X_v] \sim (-1)^{f_v\cdot v(\alpha^{e(v)(p')^{-1}})} \cdot (\pi^2, d)_v\) with \(d\) as in Lemma 6.13. If \(f_v\) is even, then by (3.2), the symbol \((\pi^2, d)_v = 1\) as \((p^{f_v} - 1)/2 \equiv f_v \pmod{2}\). Hence, \(X_v\) is unramified.

### 7. Ramifications for Primes Lying Above Dihedral Supercuspidal Prime \(p = 2\)

For \(i \in I_K\), let \(i\) denote the projection to the inertia subgroup \(I(L/K)\)[cf. Section 5]. The following lemma will give information about \(\alpha\) on the inertia group \(I_K\).

**Lemma 7.1.** Let \(p = 2\) be a dihedral supercuspidal prime for \(f\). For all \(i \in I_K\), we have

\[
\alpha(i) \equiv \begin{cases} 1 \mod F_v^\times, & \text{if } i \in I_T(K), \\ \zeta_2^x \mod F_v^\times, & \text{if } \bar{i} \text{ is an odd power of } \gamma_1, \\ \zeta_2 + \zeta_2^{-1} \mod F_v^\times, & \text{if } \bar{i} \text{ is an odd power of } \gamma_2 \text{ and } \zeta_2 + \zeta_2^{-1} \neq 0. \end{cases}
\]

Furthermore, \(\alpha(i^{2k}) \equiv 1 \mod F_v^\times\) for all \(k \in \mathbb{Z}\).

**Proof.** Consider the case \(K\) is unramified. By the part (2) of Proposition 2.1 and using (5.3), (5.4) we have \(\alpha(i) \equiv \omega^2(\bar{i}) + \omega^2(\bar{i}) \mod F_v^\times\) for all \(i \in I_T(K)\). Since \(\omega_2\) takes values in the third roots of unity, we have \(\alpha(i) \equiv 1 \mod F_v^\times\).
In the ramified case, we obtain $\alpha(i) \equiv \omega(i) + \omega(i) \equiv 1 \pmod{F_v^\times}$. Let $j_1$ and $j_2$ be a $\gamma_1$-element and a $\gamma_2$-element respectively [cf. Definition 5.2]. Then $\alpha(j_1) \equiv \chi_1(\gamma_1) + \chi^7(\gamma_1) \equiv \zeta_{2^r}$ mod $F_v^\times$ and $\alpha(j_2) \equiv \chi_2(\gamma_2) + \chi^7(\gamma_2) \equiv \zeta_{2^s} + \zeta_{2^s}^{-1}$ mod $F_v^\times$.

Since $\alpha(j_2) \equiv \zeta_{2^r} + \zeta_{2^s}^{-1}$ mod $F_v^\times$, using [Proposition 2.1, property (1)] we have that $\epsilon(j_2) = a(\zeta_{2^r} + \zeta_{2^s}^{-1})^2$ for some $a \in F_v^\times$. Recall that $\epsilon(j_2)$ is a root of unity. Thus, we obtain $\epsilon(j_2) \in \{\pm 1\}$ and so $(\zeta_{2^r} + \zeta_{2^s}^{-1})^2 \in F_v^\times$. In turn, this implies $\zeta_{2^r-1} + \zeta_{2^s-1} \in F_v^\times$. As $r < s$, we must have $\zeta_{2^r} + \zeta_{2^s}^{-1} \in F_v^\times$. Note that the field $F(\zeta_{2^r})$ inside $E$ has degree 2 over both the fields $F = F(\zeta_{2^r} + \zeta_{2^s}^{-1})$ and $F(\zeta_{2^s-1})$. Since $F \subseteq F(\zeta_{2^r-1})$, we conclude that $\zeta_{2^r-1} \in F_v^\times$. We get the desired result using the homomorphism $\tilde{\alpha}$. The last statement follows from the observation $\zeta_{2^r-1}, (\zeta_{2^r} + \zeta_{2^s}^{-1})^2 \in F_v^\times$. \hfill $\square$

**Lemma 7.2.** If $p = 2$ is a dihedral supercuspidal prime for $f$, then $\epsilon$ is $F_v^\times$-valued on $I_W(K)$.

**Proof.** For an extra twist $(\gamma, \chi_\gamma)$, we have $\rho_{f^\gamma} \sim \rho_f \otimes \chi_\gamma$. By restricting the representation to $G_2$, we obtain:

$$
\left(\begin{array}{cc}
\chi_\gamma^3 & 0 \\
0 & \chi_\gamma^3
\end{array}\right) \sim \left(\begin{array}{cc}
\chi_\gamma & 0 \\
0 & \chi_\gamma^r\chi_\gamma^{-1}
\end{array}\right).
$$

Equating determinants on both sides, we get $(\chi_\gamma^r)^\gamma = \chi_\gamma^r\chi_\gamma^{-2}$ for all $\gamma \in \Gamma$. Since $\chi_\gamma^2 = e^{-1}$, the quantity $\Delta_2 \in F_v^\times$ and so $\frac{\Delta_2}{2} \in F_v^\times$ on the wild inertia group $I_W(K)$. We get the lemma as $\chi_\gamma^2 \in F_v^\times$. \hfill $\square$

**Lemma 7.3.** Assume that $K \subseteq F_v$ with $\zeta_{2^r} + \zeta_{2^s}^{-1} = 0$. Let $j_2$ be a $\gamma_2$-element. Then $\chi_\gamma$ becomes unramified (or ramified) depending on $\gamma \in \{j_2\} = 1$ (or $-1$), for all $\gamma \in \Gamma$.

**Proof.** By assumption, we have $s = 2$. Since $e$ is $F_v^\times$-valued on $I_W(K)$ by Lemma 7.2, we have $\chi_\gamma^2(j_2) = e(j_2)\gamma^{-1} = 1$ for all $\gamma \in \Gamma$. Thus, we obtain $\chi(\gamma, j_2) = \pm 1$ for all $\gamma \in \Gamma$. Depending on the value of the Dirichlet character $\chi$, $\chi_\gamma$ is unramified or ramified. For elements $i \in I_T(F_v)$, we have $\chi(i) = 1$ as $\alpha(i) \in F_v^\times$. This is also the case for the elements of $\Gamma_1$, one part of the wild inertia group, as $r = 0, 1(r < s)$ [cf. Lemma 7.1]. \hfill $\square$

Note that the above is true for any $\gamma_2$-element. Since $f$ is non-CM, we can and do choose an auxiliary prime $p^i$ stated in the introduction imitating a similar construction of [3, Section 6.2.3]. Since $e^{-1}$ is an extra twist, $e(p^i) = -1$ and we obtain $a_{p^i}^2 = -a_{p^i}^2 e(p^i)^{-1} \in F_v^\times$. We deduce that $\alpha(j_2) \equiv \alpha(\text{Frob}_{p^i}) = a_{p^i}^2 \equiv a_{p^i} \mod F_v^\times$. Consider the Frobenius element $g_{\tau}$ in $G_K$ and two fields $F_v = F_v(b, \zeta_{2^r} + \zeta_{2^s}^{-1})$ and $F_v' = F_v(b, \zeta_{2^s})$ as in Section 3.

**Lemma 7.4.** Let $p = 2$ be a dihedral supercuspidal prime for $f$ and $\zeta_{2^r} + \zeta_{2^s}^{-1} \neq 0$.

- If $g \in G_K$ and $\alpha(g) \notin (F_v')^\times$, then $\alpha(g) \equiv \zeta_{2^r} \pmod{(F_v')^\times}$.
- If $\alpha(g) \notin (F_v')^\times$, then we have $\alpha(g) \equiv (\zeta_{2^r} + \zeta_{2^s}^{-1}) \pmod{(F_v')^\times}$.

**Proof.** Recall that every element $g \in G_K$ can be written as $g_{\tau}^n i$, for some $n \in \mathbb{Z}$ and $i \in I_K$. Using the homomorphism $\tilde{\alpha}$ and Lemma 7.1, we get the result. \hfill $\square$

**Lemma 7.5.** Let $p = 2$ be a dihedral supercuspidal prime for $f$ and $\zeta_{2^r} + \zeta_{2^s}^{-1} = 0$. If $g \in G_K$ and $\alpha(g) \notin (F_v')^\times$, then we have $\alpha(g) \equiv a_{p^i} \mod (F_v')^\times$.

**Proof.** Since $r \in \{0, 1\}$ (as $r < s = 2$), we must have $\alpha(i) \in F_v^\times$, for all $i \in I_K$ with $\tilde{i} \in \gamma_1 > [\text{cf. Lemma 7.1}]$. Then as in lemma above, we get the result. \hfill $\square$

### 7.1. Error terms for $p = 2$.

First assume that $K \subseteq F_v$ with $\zeta_{2^r} + \zeta_{2^s}^{-1} \neq 0$. Define two functions on $G_v$ by

$$
f_1(g) = \begin{cases} 
1 & \text{ if } \alpha(g) \in (F_v')^\times \\
\zeta_{2^r} & \text{ if } \alpha(g) \notin (F_v')^\times 
\end{cases}
$$

and

$$
f_2(g) = \begin{cases} 
1 & \text{ if } \alpha(g) \in (F_v')^\times \\
\zeta_{2^r} + \zeta_{2^s}^{-1} & \text{ if } \alpha(g) \notin (F_v')^\times 
\end{cases}
$$
Recall that we defined two quadratic characters $\psi_1, \psi_2$ on $G_v$ and $t_1, t_2 \in F_v^\times$ in Section 3. We replace the conditions that define $f_1$ and $f_2$ by the quadratic characters $\psi_1$ and $\psi_2$ respectively defined in that section.

The functions $f_1$ and $f_2$ will induce 2-cocycles $c_{f_1}$ and $c_{f_2}$ in $\text{Br}(F_v)$. To compute $\text{inv}_v(c_{f_1})$, consider the non-trivial element $\sigma$ of the Galois group $\text{Gal}(F_v(\sqrt{v_1})/F_v)$. The cocycle table of the 2-cocycle $c_{f_1}$ is given by

| $\sigma$ | 1 | $\zeta_{2r-1}$ |
|----------|---|----------------|
| 1        | 1 | 1              |
| $\sigma$ | 1 | $\zeta_{2r-1}$ |

which gives the symbol $(t_1, \zeta_{2r-1})$. Similarly, the cocycle table of $c_{f_2}$ gives the symbol $(t_2, (\zeta_{2r} + \zeta_{2r}^{-1})^2)_v$. Define two integers $n_{1,v}, n_{2,v}$ mod 2 by $(-1)^{n_{1,v}} = (t_1, \zeta_{2r-1})_v$ and $(-1)^{n_{2,v}} = (t_2, (\zeta_{2r} + \zeta_{2r}^{-1})^2)_v$. Let us consider an integer $n_v$ mod 2 defined by $(-1)^{n_v} = (-1)^{n_{1,v} + n_{2,v}} = (t_1, \zeta_{2r-1})_v \cdot (t_2, (\zeta_{2r} + \zeta_{2r}^{-1})^2)_v$.

We now assume the case $K \nsubseteq F_v$ with $\zeta_{2r} + \zeta_{2r}^{-1} \neq 0$. Consider a non-trivial element $\delta_v \in G_v/G_{K,F_v}$ for some $\sigma_v \in G_v$. Then any element $g \in G_v$ can be written as:

$$g = \sigma_v^n h$$

for some $h \in G_{K,F_v}$ and $n \in \{0,1\}$.

Note that $n = 0$ when $g \in G_{K,F_v}$. Using this decomposition one can extend $f_1$ and $f_2$ uniquely to $G_v$, call it $F_1$ and $F_2$, by defining $F_1(g) = f_1(h)$ and $F_2(g) = f_2(h)$. The inflation map $\text{Inf} : 2\mathbb{H}^2(G_{K,F_v}, \tilde{F}_v^\times) \rightarrow 2\mathbb{H}^2(G_v, \tilde{F}_v^\times)$ sends $c_{f_1}$ and $c_{f_2}$ to $c_{F_1}$ and $c_{F_2}$ respectively and hence their classes are same in their respective Brauer groups.

**Definition 7.6.** (Auxiliary functions for $s = 2$) Assume that $\zeta_{2r} + \zeta_{2r}^{-1} = 0$. If $K \subseteq F_v$, we define a function $f$ on $G_v(\subseteq G_K)$ by

$$f(g) = \begin{cases} 1, & \text{if } \psi_2(g) = 1, \\ a_{p^t}, & \text{if } \psi_2(g) = -1. \end{cases}$$

When $K_{F_v}|F_v$ is quadratic, we use the above decomposition to extend $f$ uniquely to $G_v$, call it $F$.

As before, both $c_f$ and $c_F$ have the same cocycle class in their respective Brauer groups. The cocycle table of $c_f$ is determined by the symbol $(t_2, a_{p^t})_v$, with $t_2$ as above. In order to keep same notation, we write $(-1)^{n_v} = (t_1, \zeta_{2r-1})_v \cdot (t_2, (\zeta_{2r} + \zeta_{2r}^{-1})^2)_v$, if $s \neq 2$, or $(t_2, a_{p^t})_v$, if $s = 2$.

### 7.2. The case $K \subseteq F_v$ or $K_{F_v}|F_v$ unramified quadratic

Let $f_1, f_2, F_1, F_2, f, F$ and $n_v$ be as in the previous section. We now prove Theorem 3.6 with this assumption.

**Proof.** First assume that $\zeta_{2r} + \zeta_{2r}^{-1} \neq 0$. We now consider a function $\alpha'$ on $G_v$ defined as follows:

$$\alpha' = \frac{\alpha}{f_{F_2}}$$

if $K \subseteq F_v$ and $\alpha' = \frac{\alpha}{f_{F_1}}$, if $K_{F_v}|F_v$ unramified quadratic. With the assumption as above, we have $I_v = I_{K,F_v} \subseteq I_K$. We can apply Lemma 7.1 to determine $\alpha(i) \forall i \in I_v$. If $K_{F_v}|F_v$ is unramified quadratic, we have $I_v = I_{K,F_v}$ and hence, in the decomposition (7.2) for any element of $I_v$, we must have

$$n_v = 0. \text{ As a map on } I_v, \text{ we have } \frac{\alpha}{f_{F_1}} = \frac{\alpha}{f_{F_2}}. \text{ By unravelling the definition of } \alpha', \text{ we see that the two conditions of Lemma 2.2 are satisfied by } S = \alpha' \text{ and } t = \epsilon'. \text{ Hence, we obtain}$$

$$\text{inv}_v(c_{\alpha'}) = \frac{1}{2} \cdot v\left(\frac{\alpha'^2}{\epsilon'}(\text{Frob}_v)\right) = \frac{1}{2} \cdot v\left(\alpha'^2(\text{Frob}_v)\right) = \frac{1}{2} \cdot f_v \cdot v(a_{p^t}^2 \epsilon(p')^{-1}) \mod \mathbb{Z}.$$ 

We conclude that, $\text{inv}_v(c_{\alpha}) = \frac{1}{2} \cdot (f_v \cdot v(a_{p^t}^2 \epsilon(p')^{-1}) + n_v) \mod \mathbb{Z}$.

We now assume that $\zeta_{2r} + \zeta_{2r}^{-1} = 0$. Define a map $\alpha'$ on $G_v$ by: $\alpha' = \frac{\alpha}{\alpha}$, if $K \subseteq F_v$, and $\alpha' = \frac{\alpha}{\alpha}$, if $K_{F_v}|F_v$ unramified quadratic. In this case, we have $I_v = I_{K,F_v} \subseteq I_K$. As before since $\frac{\alpha}{\alpha} = \frac{\alpha}{\alpha}$ on $I_v$, using Lemma 7.5 the two conditions of Lemma 2.2 are satisfied by $S = \alpha'$ and $t = \epsilon'$. Thus, we get the required result as before.

We now prove Corollary 3.7.
Proof. If $N_2 = 2$, then the extension $K|\mathbb{Q}_2$ is unramified [cf. Section 4]. By the Lemmas 4.1 and 7.1, we have $\alpha(t) \in F_v^\times$ for all $t \in I_K$. Since $I_v = I_{K_F} \subseteq I_K$, this is true for all $t \in I_v$. We choose $S = \alpha$ and use Lemma 2.2 to complete the proof. 

7.3. The case $KF_v|F_v$ ramified. We now prove Theorem 3.6 with the present assumption.

Proof. As $KF_v|F_v$ is a ramified quadratic extension, we choose an element $a_0 \in \mathcal{O}_v^\times$ which is not a norm of the extension. Let $i_0 \in G_v$ be an element which is mapped to $a_0$ under the reciprocity map. Assume that $i_0$ is an element of the tame inertia part. Since $I_T(F_v)^2 = I_T(KF_v)$, we have that $i_0 \in I_T(F_v) \setminus I_T(KF_v)$. Since $i_0^2$ belongs to the tame inertia part of $K$, we deduce that $\alpha(i_0) \equiv \sqrt{d_0} \pmod{F_v^\times}$ for some $d_0 \in F_v^\times$ [cf. Lemma 7.1].

First assume that $s \neq 2$. Define a function $f$ on $G_v$ by

$$f(g) = 1, \text{ if } g \in G_{KF_v} \text{ and } f(g) = \sqrt{d_0}, \text{ if } g \in G_v \setminus G_{KF_v}$$

and define the function $F$ on $\text{Gal}(KF_v|F_v)$ which is 1 for the identity element and $\sqrt{d_0}$ else. Consider the inflation map $\text{Inf}: H^2(F_v(\pi)|F_v) \rightarrow H^2(F_v|F_v)$ as before and it is easy to check that $\text{Inf}^{-1}(c_F) = c_F$ and the cocycle table of $c_F$ is given by:

| 1 | 1 | $\sigma$ |
| --- | --- | --- |
| 1 | 1 | 1 |
| $\sigma$ | 1 | $d_0$ |

which gives the symbol $(\pi^2, d_0)_v$. We now define an integer $n_v'$ mod 2 by $(-1)^{n_v'} = (-1)^{n_v} \cdot (\pi^2, d_0)_v$. Let $F_1, F_2$ be the functions as in the previous theorem. We apply Lemma 2.2 to compute the invariant of the cocycle $[c_s]$ with $G_v = \frac{F(g)F_2(g)F_3(g)}{F(g)F_2(g)f(g)}$ on $G_v$ and $t = \epsilon'$.

We now deduce that $S(i) \in F_v^\times$, for all $i \in I_v$. Consider the element $i_0 \in G_v \setminus G_{KF_v}$ and hence $\tilde{i}_0$ is a non-trivial element in $G_v/G_{KF_v}$. We will consider the decomposition (7.2) with respect to the element $i_0$ instead of $\sigma_v$. For $i \in I_v \setminus I_{K_{F_v}}$, we have $i = i_0\tilde{i}$, for some $i \in I_{K_{F_v}}(\subseteq I_K)$. We obtain

$$S(i) = \frac{\alpha(i)}{F_1(i)F_2(i)f(i)} = \frac{\alpha(i_0)\alpha(\tilde{i})}{F_1(i_0)F_2(i_0)f(i)} = \frac{\alpha(\tilde{i})}{f_1(\tilde{i})f_2(\tilde{i})} \equiv 1 \pmod{F_v^\times}.$$ 

The last equality follows from the fact that $\alpha(i) \equiv f_1(i)f_2(i) (\text{ mod } F_v^\times)$ for all $i \in I_{K_{F_v}}$. As usual, the cocycle $c_s$ can be decomposed as $c_s = c_{\alpha}c_{F_1}c_{F_2}c_F$. By Lemma 2.2, we deduce that $\text{inv}_v(c_s) = \frac{1}{2} \cdot v\left(\frac{\alpha'}{F(z)}(\text{Frob}_v)\right) = \frac{1}{2} \cdot v\left(\frac{a_{F_v}'}{F(z)}(\text{Frob}_v)\right) = \frac{1}{2} \cdot f_v \cdot v(a_{F_v}^2\epsilon(p')^{-1}) \pmod{Z}$. Hence, we conclude that:

$$\text{inv}_v(c_\alpha) = \text{inv}_v(c_s) + \text{inv}_v(c_{F_1}) + \text{inv}_v(c_{F_2}) + \text{inv}_v(c_F) = \frac{1}{2} \cdot (f_v \cdot v(a_{F_v}^2\epsilon(p')^{-1}) + n_v') \pmod{Z}.$$ 

We now assume that $s = 2$. Let $\alpha'(g) = \frac{\alpha(g)}{F(g)}$ on $G_v$, where $F$ be the function as in Definition 7.6 and the function $f$ be as in (7.4). Then the cocycle $c_{\alpha}$ can be decomposed as $c_{\alpha} = c_{\alpha'}c_{F_1}c_{F_2}$. Define an integer $n_v''$ mod 2 by $(-1)^{n_v''} = (t_2, a_{p'}^2)_v \cdot (\pi^2, d_0)_v$. Two conditions of Lemma 2.2 are satisfied by $S = \alpha'$ and $t = \epsilon'$ as before. Hence, we have $\text{inv}_v(c_{\alpha'}) = \frac{1}{2} \cdot f_v \cdot v(a_{p'}^2\epsilon(p')^{-1}) \pmod{Z}$. Thus, $\text{inv}_v(c_{\alpha'}) = \text{inv}_v(c_{\alpha'}) + \text{inv}_v(c_{F_1}) + \text{inv}_v(c_{F_2}) + \text{inv}_v(c_F) = \frac{1}{2} \cdot (f_v \cdot v(a_{p'}^2\epsilon(p')^{-1}) + n_v'') \pmod{Z}$.

The next lemma will determine the value of $d_0$ in some special cases.

**Lemma 7.7.** Assume $F = \mathbb{Q}$ and the supercuspidal prime $p$ satisfies the second condition of Theorem 3.6. The quantity $d_0$ involved in the error term is equal to $a_{p''}^2$, except $K = \mathbb{Q}_2(\sqrt{2})$ and $\mathbb{Q}_2(\sqrt{-6})$. 

Proof. When $F = \mathbb{Q}$, the ramified quadratic extension becomes $K|\mathbb{Q}_2$. We have that $\alpha_0 = -1$ is not a norm of the extension $K|\mathbb{Q}_2$ except $K = \mathbb{Q}_2(\sqrt{d})$ with $d = 2, -6$, see [28, p. 34]. In this case, we find out the value of $d_0$. For a prime $p^{'''}$ chosen before, $\alpha(i_0)\gamma^{-1} = \chi_\gamma(i_0) = (1, 2)\gamma_2(-1)^{-1} = \chi_\gamma(p^{''}) = \alpha(Frob_{p^{'''}})\gamma^{-1}$, for all $\gamma \in \Gamma$ and so $\alpha(i_0) \equiv \alpha_{p^{'''}} \mod F_v^\times$.

\section{Ramifications for primes lying above non-di
dehedral supercuspidal prime $p = 2$}

Let $\rho_2(f)$ be the local representation of the Weil-Deligne group of $\mathbb{Q}_2$ associated to $f$ at the prime $p = 2$ and denote by $\hat{\rho}_2$ the projective image of $\rho_2 := \rho_2(f)$. When the inertia group $I_2$ acts irreducibly, the image of $\hat{\rho}_2$ is one of three exceptional groups $A_4, S_4, A_5$. The $A_5$ case cannot occur since the Galois group $G_2 := \text{Gal}(\bar{\mathbb{Q}}_2|\mathbb{Q}_2)$ is solvable. Weil proved in [29] that over $\mathbb{Q}$, the $A_4$ case also does not occur, so the only exceptional case has image $S_4$. For $D = \text{det}(\hat{\rho}_2(f))$ and $d = \frac{a^2}{D}$, we have a cocycle class decomposition $[X] = [c_D] \cdot [c_d]$, where the cocycles $c_D$ and $c_d$ are given by $c_D(g, h) = \frac{\sqrt{d(g)\sqrt{D(h)}}}{\sqrt{D(gh)}}$, $c_d(g, h) = \frac{\sqrt{d(g)\sqrt{D(h)}}}{\sqrt{d(gh)}}$ respectively. In this section, we find the local Brauer class $[X_v] = [c_D]_v \cdot [c_d]_v$ for any $v | 2$.

Let $D_{K'}$ be the discriminant of the field $K'$ cut out by $\ker(d)$.

The following Lemma is a straightforward adaptation in our setting of Lemma 7.3.17 of [1].

\begin{lemma}
The 2-cocycle $[c_d]_v = 1$ if and only if $D(1 - 1) = 1$.
\end{lemma}

Recall that $f^\gamma \equiv f \otimes \chi_\gamma$ implies that $\rho_2(f)^\gamma \sim \rho_2(f) \otimes \chi_\gamma$ and taking determinant gives $\text{det}(\rho_2(f))^\gamma = \text{det}(\rho_2(f))(\chi_\gamma)^2$. Thus, we obtain $\text{det}(g)^\gamma = 1$ and so $d(g) \in F_v^\times$. Consider the topological space $F_v^\times/F_v^{\times 2}$ with the discrete topology. We deduce that $d : G_v \rightarrow F_v^\times/F_v^{\times 2}$ is a continuous homomorphism and hence $G_v/\ker(d) \cong \text{Gal}(K'|F_v) \cong (\mathbb{Z}/2\mathbb{Z})^m$ for some elementary 2-extension $K'$ of $F_v$. For each $i = 1, \ldots, m$, let $\sigma_i \in \text{Gal}(K'|F_v)$ be the element corresponding to $(0, \cdots, 1, \cdots, 0) \in (\mathbb{Z}/2\mathbb{Z})^m$ (1 in the $i$-th position). Define $t_j \in F_v^\times (1 \leq j \leq m)$ as follows:

$$
\sigma_i(\sqrt{t_j}) = \delta_{ij} \sqrt{t_j}.
$$

Lift $\sigma_i$ to an element of $G_v$ denoted also by $\sigma_i$ and set $d_i := d(\sigma_i) \in F_v^\times/F_v^{\times 2}$. As in [18], the class of $c_d$ is given by $[c_d]_v = (t_1, d_1)_v \cdots (t_m, d_m)_v$.

\begin{proposition}
$[c_d]_v = (2, D_{K'})_v$.
\end{proposition}

\begin{proof}
We first prove that $\ker(\hat{\rho}_2) \subset \ker(d)$. Suppose $g \in \ker(\hat{\rho}_2)$, then $\rho_2(g)$ is a scalar matrix $\rho_2(g) = (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix})$. Thus, the quantity $\text{Tr}(\rho_2(g)^2)/\text{det}(\rho_2(g))$ is 4. As trace is non-zero, using the part (2) of Proposition 2.1, this quantity is equal to $d(g)$ up to an element of $F_v^{\times 2}$ and so $d(g) \equiv 1 \mod F_v^{\times 2}$. Hence, we get the inclusion.

Thus, there is an onto map $S_4 = G_v/\ker(\hat{\rho}_2) \rightarrow G_v/\ker(d)$. Only 2-subgroup that can be quotient of $S_4$ is the trivial group or $\mathbb{Z}/2\mathbb{Z}$. Thus $m$ is either 0 or 1. Since every element of the projective image of $\rho_2$ (which is $S_4$) has order 1, 2, 3 or 4, we have $\text{Tr}(\rho_2(g)^2)/\text{det}(\rho_2(g)) \in \{4, 0, 1, 2\}$ for every $g \in G_v$. For all $g \in G_v$, observe that $d(g) \in \{4, 0, 1, 2\}$ up to $F_v^{\times 2}$. We conclude that $d(g) \in F_v^{\times 2}$ or $d(g) \equiv \sqrt{2} \mod F_v^{\times 2}$, for all $g \in G_v$. The value of $m \in \{0, 1\}$ depends on this. Since the projective image of $\rho_2$ is $S_4$, there is an element $g \in G_v$ whose projective image in $S_4$ is a 4-cycle. For such an element $g$, we have $d(g) \equiv 2 \mod F_v^{\times 2}$ and so we conclude that $m = 1$. Thus, the field $K'$ cut out by the kernel of the homomorphism $d$ must be a quadratic field. As $t_1 = D_{K'}$ (the discriminant of $K'$) and $d_1 = 2$ up to an element of $F_v^{\times 2}$, the class $[c_d]_v$ is determined by the symbol $(2, D_{K'})_v$. We obtain the result.
\end{proof}

We now prove Theorem 3.8 and Corollary 3.9.
Proof. Since \([X_v] \sim [c_d]_v \cdot [c_D]_v\), we obtain the Theorem 3.8 for a non-dihedral prime \(p = 2\).

By the isomorphism \((\ast)\) [cf. Section 4] we can replace \(D\) by \(\det(\rho_{f,2}) = \chi_2^{k-1} \epsilon\), where \(\chi_2\) is the 2-adic cyclotomic character. When \(k\) is odd, the cocycle class of \(c_D\) is same as the cocycle class of \(c_e\), that is, \(c_D \sim c_e\), where the 2-cocycle \(c_e\) is defined as follows: \(c_e(g,h) = \frac{\sqrt[2]{\epsilon(g)\sqrt[2]{\epsilon(h)}}}{\sqrt[2]{\epsilon(gh)}}\). Apply Lemma 8.1 and observe that \([c_D] = [c_e]\) for odd \(k\), we obtain Corollary 3.9. \(\Box\)

9. Numerical Examples

For an odd prime \(p\), our results are concurrent with the theorems proved in [4]. However, the example (5) of loc. cit. shows that \(X_v\) is not determined by \(m_v\), if \(p\) is an unramified “bad” level zero supercuspidal prime. This example corroborates our Theorem 3.4.

To support our results, we give numerical examples about local ramifications at supercuspidal prime \(p = 2\). The examples are provided in the table of [12]. Using Sage and L-function and modular forms database (LMFDB), we determine the \(v\)-adic valuation of the trace of adjoint lift at the prime \(p'\).

(1) \(f \in S_5(20,[0,3])\) with \(E = \mathbb{Q}(\sqrt{-1})\) and \(F = \mathbb{Q}\). Since \(N_2 = 2\) the prime \(p = 2\) is an unramified dihedral supercuspidal prime. We choose \(p' = 17\). Using Sage we check that \(a_{p'} = 1 - i\) and hence \(v_2(a_{p'}^2\epsilon(p')^{-1}) = 1\), so \(X_v\) is ramified.

(2) \(f \in S_5(36,[0,3])\), \(E = \mathbb{Q}(\sqrt{-2})\) and \(F = \mathbb{Q}\). Here \(p = 2\) is an unramified dihedral supercuspidal prime as \(N_2 = 2\). We choose \(p' = 29\) and compute that \(a_{p'}^2 = a_{29}^2 = -421362 = -2 \cdot 459^2\). Hence \(v_2(a_{p'}^229^{-1}) = 1\), so \(X_v\) is ramified.

(3) \(f \in S_5(24,[0,0,1])\), \(E = \mathbb{Q}(\sqrt{-2})\), \(F = \mathbb{Q}\). Here \(p = 2\) is a non-dihedral supercuspidal prime for \(f\) [23, Section 6]. We have \(D_K = 64\) [23, Section 6] and \(\epsilon(-1) = -1\). Hence \(X_v\) is ramified.

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