QUANTUM AND CLASSICAL FIELDS IN THE
FINITE-DIMENSIONAL FORMALISM

Miguel Navarro

Instituto Carlos I de Física Teórica y Computacional. Granada, Spain

Abstract

The quantization rules recently proposed by M. Navarro [1] (and independently I.V. Kanatchikov [2]) for a finite-dimensional formulation of quantum field theory are applied to the Klein-Gordon and the Dirac fields to obtain the quantum equations of motion of both fields. In doing so several problems arise.

Solving these difficulties leads us to propose a new classical canonical formalism, which, in turn, leads us to new, improved rules of quantization. We show that the new classical equations of motion and rules of quantization overcome several known unsatisfactory features of the previous formalism. We argue that the new formalism is a general improvement with respect to the previous one.

Further we show that the quantum field theory of the Dirac and Klein-Gordon field describes particles with extra, harmonic-oscillator-like degrees of freedom. We argue that these degrees of freedom should give rise to a multi-particle interpretation of the formalism.

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e-mail: nim314159@aol.com

†On leave of absence.
1 Introduction

At present the main goal of Theoretical Physics is to produce an unified quantum theory of all forces of nature, including gravity. Unfortunately, and despite the tremendous success of both these theoretical frameworks, neither General Relativity nor, above all, Quantum Field Theory are theories with firm grounds that would provide confidence for further building. Instead, it seems that both theories will have to be dramatically reformulated before a true unification can be devised.

An approach that may help in this regard is the finite-dimensional formulation of field theory. This approach has a long history, which dates back to the thirties [3]. Much of the work done on the formalism focused on the classical theory or on following routes to the quantum theory which closely mimic the one which, starting from Classical Mechanics, leads to the standard Quantum Mechanics.

Recently, however, a change of strategy has taken place and the focus has shifted towards formulating a quantum theory without paying much attention to ‘deriving’ it from the classical formalism. In particular, inspired by a previous proposal by Good [4] rules of quantization and equations of motion has been postulated by M. Navarro [1] (and independently by I.V. Kanatchikov [2]) that give rise to a promising finite-dimensional formulation of QFT. In this paper that formalism is applied to the scalar field and the Dirac field. In doing so several unsatisfactory aspects of the classical and quantum formalisms become apparent. In order to overcome these problems a modification of the classical as well as the quantum formalism is proposed.

The paper is organised as follows. In section 2 we review the finite-dimensional canonical formalism for the classical fields. In section 3 the formalism put forward by Navarro and Kanatchikov is reviewed. In section 4
we discuss the interpretation of the formalism. In section 5 the formalism is applied to the scalar field and the Dirac field to obtain the quantum equations of motion. In section 6 we present the improved classical canonical formalism. In section 7 improved rules of quantization are presented to be applied to the Dirac and the scalar field. In section 9 we state our conclusions and related comments.

2 The finite-dimensional formulation of the classical field theory

Currently we do not have a quantum theory but classical theories and rules of quantization. The standard way of quantizing a field theory relies on the fact that Classical Field Theory (CFT) can be regarded as a generalization of Classical Mechanics (CM) in which the finite number of degrees of freedom of the latter is replaced with an infinite (continuum) number in the former. In this formulation the fields are considered to be functions $\varphi^a(x)(t) \equiv \varphi^a_x(t)$; that is, the spatial co-ordinates are regarded as labels (the discrete superindex $a$ labels the different fields in the theory). This description is supported primarily by the fact that it is a direct generalization of Quantum Mechanics (QM), which, as a theory with a vast range of successful predictions, is a source of great confidence. The standard framework requires, nonetheless, the use of functionals as well as infinite-dimensional differential calculus, which is plagued with ambiguities. These ambiguities are at the root of the renormalization problem.

There is however, a different way of looking at CFT as a generalization of Classical Mechanics [3] [5]. In this reading of CFT, all the co-ordinates of the space-time are considered to be on the same footing, no special role is
played by time. The fields are not taken to be an infinite (continuum) set of functions of time but rather a discrete set of functions of all the space-time coordinates: \( \varphi^a = \varphi^a(x) \), with \( x = (x, t) \) and \( a \) a discrete label. Since there is a finite number of functions we refer to this approach as the finite-dimensional formulation of field theory as opposed to the standard or infinite-dimensional formalism.

In this (finite-dimensional) formulation of field theory the canonical formalism is normally assumed to be as follows. Given a Lagrangian \( \mathcal{L} = \mathcal{L}(\varphi^a, \partial_\mu \varphi^a) \), the covariant momenta \( \pi^\mu_a \) are defined by:

\[
\pi^\mu_a = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^a)} \tag{1}
\]

Then the covariant Hamiltonian \( \mathcal{H} \) is obtained by means of the generalized covariant Legendre transform:

\[
\mathcal{H} = \pi^\mu_a \partial_\mu \varphi^a - \mathcal{L} \tag{2}
\]

If we now write the Lagrangian in the following covariant Hamiltonian form

\[
\mathcal{L} = \pi^\mu_a \partial_\mu \varphi^a - \mathcal{H}(\varphi^a, \pi^\mu_a) \tag{3}
\]

its Lagrange equations of motion will also have a covariant Hamiltonian form:

\[
\partial_\mu \varphi^a = \frac{\partial \mathcal{H}}{\partial \pi^\mu_a} \tag{4}
\]
\[
\partial_\mu \pi^\mu_a = -\frac{\partial \mathcal{H}}{\partial \varphi^a} \tag{5}
\]

In this context a mechanical system correspond to a field theory in \( 1 + 0 \) dimensions, i.e. in a spacetime with 1 temporal and 0 spacelike dimensions.
When the system described is mechanical, the finite-dimensional covariant formalism reduces to the ordinary one (which we present here for later use).

In the ordinary canonical formalism the momenta associated to the Lagrangian \( L = L(\dot{q}^i, q^j) \) are defined by

\[
p_i = \frac{\partial L}{\partial \dot{q}^i}
\]

and the Hamiltonian is defined by

\[
H(p_i, q^j) = p_i \dot{q}^i - L
\]

which gives rise to the equation of motion

\[
\dot{q}^i = \frac{\partial H}{\partial p_i}
\]
\[
\dot{p}_i = -\frac{\partial H}{\partial q^i}
\]

3 The finite dimensional formulation of quantum field theory

The challenge is to translate the elegant, finite-dimensional formulation of classical field theory briefly reviewed above to the quantum theory. For doing so suitable rules of quantization and equations of motion are needed.

Let us review the case of ordinary Quantum Mechanics, i.e. the rules of quantization for a mechanical system.

The rules of quantization are

\[
q^i \rightarrow \hat{q}^i = q^i
\]
\[
p_i \rightarrow \hat{p}_i = -i \frac{\partial}{\partial q^i}
\]
The quantum theory is described by a (wave) function $\Psi(q^i, t)$, which obeys the following equation of motion (the Schrödinger equation):

$$i\frac{d}{dt}\Psi = \hat{H}\Psi$$  \hspace{1cm} (11)

The goal is to generalize the formalism described by Eq. (10-11) to the general case described by Eq. (10).

Inspired by previous work by Good [4] (see also [3]) a proposal was recently put forward by M. Navarro [1] (and independently by I.V. Kanatchikov [2]) for quantization rules and evolution equations in the finite-dimensional formalism.

To motivate the proposal, let us consider the ordinary harmonic oscillator and the Dirac field. The respective Lagrangians can be written:

\begin{align*}
\mathcal{L}_{HO} &= a^*(i\dot{a} - a) \hspace{1cm} (12) \\
\mathcal{L}_D &= \bar{\phi}(i\partial_\phi - \phi) \hspace{1cm} (13)
\end{align*}

where $a$ ($a^*$) is the annihilation (creation) operator, $\phi \equiv \gamma^\mu \partial_\mu$, with $\gamma^\mu$ the Dirac’s matrices, and $\bar{\phi} = \phi^\dagger \gamma^0$.

Eqs. (12) and (13) tell us that the Dirac field is a higher-dimensional generalization of the ordinary harmonic oscillator. The generalization is accomplished by replacing the time derivative $d/dt$ with the operator $\hat{\phi} = \gamma^\mu \partial_\mu$.

Mimicking that generalization, the following equation of motion for our finite-dimensional QFT was postulated [1] [2].

**Quantum equation of motion**

$$i\hat{\phi}\Psi = \hat{H}\Psi, \hspace{1cm} \phi \equiv \gamma^\mu \partial_\mu$$  \hspace{1cm} (14)
Here $\gamma_\mu$ are quantities which play a role similar to Dirac’s matrices in the relativistic theory of the electron. In the present letter, and for the sake of clarity, we will identify these quantities with the Dirac matrices. The developments in the present paper support this identification but we should keep in mind, nevertheless, that further developments of the theory may require that identification to be dropped.

The next step is to construct the operator $\hat{H}$. That is, we need quantization rules. The proposal in Ref. [1][2] is the following:

Quantization rules

\[
\begin{align*}
\varphi^a & \quad \rightarrow \quad \hat{\varphi}^a = \varphi^a \\
\pi^\mu_a & \quad \rightarrow \quad \hat{\pi}^\mu_a = -i\gamma^\mu \frac{\partial}{\partial \varphi^a}
\end{align*}
\] (15)

The rules of quantization (15) and the evolution equation (14) fulfill the following properties.

1. Both rules of quantization and equations of motion are explicitly covariant; i.e., space and time co-ordinates are treated on the same footing.

2. Within the limits of mechanical systems these rules of quantization and equations of motion reduce themselves to the familiar canonical rules of quantization and Schrödinger equation of evolution of ordinary Quantum Mechanics.

3. The equations of evolution are second order in derivatives and first order in derivatives of the space-time co-ordinates.

Most notably, and unlike in Good’s proposal, ordinary Quantum Mechanics is contained in this new proposal. Therefore, the vast amount of
experimental predictions of ordinary QM is entirely and automatically incorporated into our proposal. Hence, to rule out the new proposal we would have to look for a test which implied a genuine field system.

4 Interpretation of the quantum theory

The quantum theory is described by a wave function $\Psi(\varphi, x)$. By analogy with mechanical systems the temptation would be to interpret the wave function as the probability density of finding a value $\varphi$ of the field in the point $x$ of the spacetime. With this interpretation, the probability of finding any value of $\varphi$ in the point $x$ would be unity, which would require

$$\int_{-\infty}^{+\infty} d\varphi |\Psi(\varphi, x)|^2 = 1 \quad (16)$$

This interpretation was discussed in Ref. [1] where it was shown that it leads to serious difficulties. These obstacles could perhaps be overcome in ways worth exploring. It turns out however that there is a different and more natural interpretation of the wave function $\Psi(u, x)$.

From eq. (14) it follows that the current

$$j^\mu = \int d\varphi \bar{\Psi} \gamma^\mu \Psi \quad (17)$$

is conserved as long as the Hamiltonian $\hat{H}$ is self-adjoint with respect to $\int d\varphi$ (which seems a reasonable assumption to make).

This leads to an interpretation of the wave function according to which

$$P(\varphi, x) = \bar{\Psi} \gamma^0 \Psi \quad (18)$$

gives the probability density of finding an excitation $\varphi$ of the field in (the vicinity of) the point $x$. This interpretation of the theory is in fact more ap-
pealing and intuitively clear than the interpretation of the standard formalism, which involves configurations of the field over spacelike hypersurfaces.

Moreover, this interpretation clearly maintain the particle interpretation of the theory - but with extra, 'internal' degree of freedom.

5 The quantum fields

A The scalar field

The Lagrangian for the scalar field is:

\[ \mathcal{L}_S = \frac{1}{2}(\partial_{\mu}u\partial^\mu u - m^2 u^2) \]  \hspace{1cm} (19)

The momenta are given by \((u_\mu \equiv \partial_\mu u)\)

\[ \pi^\mu = \frac{\partial \mathcal{L}}{\partial u_{\mu}} = \partial^\mu u \]  \hspace{1cm} (20)

The classical covariant Hamiltonian is given by

\[ \mathcal{H} = \frac{1}{2}(\pi_\mu \pi^\mu + m^2 u^2) \]  \hspace{1cm} (21)

The rules of quantization (13) yield the following operator

\[ \hat{\mathcal{H}} = \frac{1}{2}(-4 \frac{\partial^2}{\partial u^2} + m^2 u^2) \]  \hspace{1cm} (22)

and quantum equation of motion

\[ i\hbar \Psi = \frac{1}{2}(-4 \frac{\partial^2}{\partial u^2} + m^2 u^2) \Psi \]  \hspace{1cm} (23)

Eq. (23) can be interpreted as describing a Dirac field with a mass that is not constant but has the dynamics of a harmonic-oscillator. Clearly it is
expected that the equally-spaced levels of this harmonic oscillator correspond to different particle numbers in the system.

B The Dirac field

The Lagrangian for the Dirac field is

\[ \mathcal{L}_D = i \bar{\psi} \gamma^\mu \partial_\mu \psi - m \bar{\psi} \psi \] (24)

The canonical momenta are given by

\[ \pi^\mu = i \bar{\psi} \gamma^\mu \] (25)

This equality means that the Lagrangian (24) is already written in the canonical form (3) and that the covariant Hamiltonian is

\[ \mathcal{H} = m \bar{\psi} \psi \] (26)

Now, we can use the rules of quantization (15) to produce the operator

\[ \hat{\mathcal{H}} = m \frac{\partial}{\partial v} \psi \] (27)

and the quantum equation of motion

\[ i \gamma^\mu \partial_\mu \Psi = m \frac{\partial}{\partial v} \psi \Psi \] (28)

Clearly here there can be ordering issues that should be taken into consideration when further developing the theory.

Eq. (26) closely mimics the mechanical harmonic oscillator (13), which after all inspired our finite-dimensional, covariant formalism.
6 Improved classical formalism

There is a serious problem, however, with the Dirac field as discussed above: the covariant Hamiltonian (26) together with the classical canonical equations of motion (5) do not yield the correct classical equation of motion (Dirac equation) for the Dirac field. Moreover, trying to find a (different) covariant Hamiltonian by using the requirement that it reproduces the correct classical equations of motions leads to nowhere because the equations of motion (5) tend to produce second-order equations of motion.

This difficulty highlights the fact that there are actually a number of aspects of the classical canonical formalism (1-5) that are unsatisfactory. Most notable among them is the lack of a symmetry fields-momenta. This symmetry is one of the principal features of the ordinary canonical formalism and one of the reasons why the canonical formalism is used at all. In fact, in the ordinary canonical formalism no distinction should be made between fields and momenta since these quantities should be interpreted as coordinates of the phase space and there is no intrinsic reason to identify some of the coordinates of the phase space with fields and the others with momenta. Moreover, this symmetry is carried through towards ordinary quantum theory where it plays an important role.

There is therefore a strong case for modifying the (classical) formalism. Our proposal goes as follows.

For a Lagrangian \( \mathcal{L} = \mathcal{L}(\varphi^a, \partial_\mu \varphi^a) \) we define the momenta

\[
\pi_a = \gamma_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^a)}
\]

(29)

The Hamiltonian \( \mathcal{H} \) is defined by

\[
\mathcal{H}(\varphi^a, \pi_a) = \pi_a \gamma_\mu \partial_\mu \varphi^a - \mathcal{L}
\]

(30)
If the Lagrangian is written in the form

\[
\mathcal{L} = \pi_a \gamma^\mu \partial_\mu \varphi^a - \mathcal{H}(\varphi^a, \pi_a)
\]

\[
= \pi_a \hat{\partial} \varphi^a - \mathcal{H}(\varphi^a, \pi_a)
\]  

(31)

it leads to the following classical canonical equations of motion:

\[
\hat{\partial} \varphi^a = \frac{\partial \mathcal{H}}{\partial \pi_a}
\]

\[
\hat{\partial} \pi_a = -\frac{\partial \mathcal{H}}{\partial \varphi^a}
\]  

(32)

The new classical canonical formalism have several advantages when compared with the old one. Most notably it is symmetric under the change fields ↔ momenta and it reproduced the correct equations of motion for the Dirac field with the covariant Hamiltonian \( \mathcal{H} = m\bar{v}v \).

Aesthetically the new formalism is also more like the ordinary formalism. Like in classical mechanics we have a single momenta (albeit it is a matrix-valued one) for each field. The generalization from one to the other is accomplished by replacing everywhere the temporal derivative operator \( \frac{d}{dt} \) with the Dirac operator \( \mathcal{D} \).

7 Improved rules of quantization

The new classical canonical formalism leads naturally to new, improved rules of quantization (the quantum equation of motion (14) remains unchanged though). We proposed the new, improved rules of quantization:

\[
\varphi^a \longrightarrow \hat{\varphi}^a = \varphi^a
\]
\[ \pi_a \rightarrow \hat{\pi}_a = -i \frac{\partial}{\partial \varphi^a} \] (33)

8 Improved quantum fields

For the Dirac field the improved rules of quantization reproduce the same quantum theory as the old one (with the important difference that now the classical theory is consistent). For the Klein-Gordon field the change of rules of quantization gives rise to minor but important differences.

The Lagrangian in Eq. (19) yields the momenta

\[ \pi = \partial u \] (34)

and the Hamiltonian

\[ H_S = \frac{1}{2} (\pi^2 + u^2) \] (35)

Now, the rules of quantization (33) yield the quantum operator

\[ \hat{H}_S = \frac{1}{2} (-\frac{\partial^2}{\partial u^2} + u^2) \] (36)

The quantum covariant Hamiltonian has been altered in that the numerical factor multiplying the first term in the operator has changed. The 4 (which would become D for a D-dimensional space-time) has disappeared.

That such a numerical factor (a D) appeared in the quantum operator for the scalar field has already been noted in the literature as an unsatisfactory feature of the previous formalism [6]. That it does not appear when the improved rules of quantization are used is another factor in their favor.
9 Conclusions and other comments

After a detailed discussion of the finite-dimensional formalism in field theory we have applied the formulation proposed by M. Navarro [I] (and independently by I.V. Kanatchikov [2]) to the scalar field and the Dirac field. In doing so several unsatisfactory aspects of the classical and quantum formalism have become apparent. In order to overcome these problems a modification of the classical as well as the quantum formalism have been proposed.

In summary, our proposal goes as follows:

**Classical finite-dimensional canonical formalism**

Lagrangian: \( \mathcal{L} = \mathcal{L}(\varphi^a, \partial_\mu \varphi^a) \)

Momenta: \( \pi_a = \gamma^\mu \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial (\partial_\mu \varphi^a)} \right) \)

Hamiltonian: \( \mathcal{H}(\pi_a, \varphi^a) = \pi_a \hat{\varphi}^a - \mathcal{L} \)

Canonical Equations of motion:

\[
\begin{align*}
\hat{\varphi}^a &= \frac{\partial \mathcal{H}}{\partial \pi_a} \\
\hat{\pi}_a &= -i \frac{\partial \mathcal{H}}{\partial \varphi^a}
\end{align*}
\]  

(37)

**Rules of quantization**

\( \varphi^a \rightarrow \hat{\varphi} = \varphi^a \)

\( \pi_a \rightarrow \hat{\pi}_a = -i \frac{\partial}{\partial \varphi^a} \)  

(38)

**Quantum (Schrödinger like) equation of motion**

\( i\hat{\varphi} \Psi = \hat{\mathcal{H}} \Psi \)  

(39)

The new formalism solves three known problems of the previous formalism:
• It restores the symmetry fields-momenta of the ordinary canonical formalism.

• It produces the right equations of motions for the Dirac with the natural covariant Hamiltonian $\mathcal{H} = m\bar{\psi}\psi$.

• It eliminates a numerical factor $D$ (the spacetime dimension) from the quantum Hamiltonian operator of the scalar field.

Further we have shown that the quantum equations of motion correspond to fields with harmonic-oscillator-like internal degrees of freedoms. It seems natural to predict that these extra degrees of freedom will lead to a multi-particle interpretation of the theory.

It has been shown that the formalism behaves consistently in addition to being very elegant. The next step should be to produce falsifiable predictions.

In this regard a satisfactory description of the electromagnetic field should be produced. The difficulty here, of course, is how to covariantly describe in a canonical way a theory that has reduced degrees of freedom because of the gauge invariance $A_\mu \rightarrow A_\mu + \partial_\mu \phi$. Of course, similar but surely even more difficult problems are faced when applying the formalism to gravity theories.

We hope to be able to report progress in these areas in the near future.

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References
[1] M. Navarro, Toward a finitedimensional formulation of quantum field theory *Found. Phys. Lett.* **11** (1998) 585-593, Imperial-TP/95-96, [hep-th/9805010](https://arxiv.org/abs/hep-th/9805010).

[2] I.V. Kanatchikov, “Toward the Born-Weyl Quantization of Fields”, *Int. J. Th. Phys.* **37**, (1998) 333, [quant-ph/9712058](https://arxiv.org/abs/quant-ph/9712058).

[3] M. Born, *Proc. Roy. Soc. London A* **143**, 410 (1934). H. Weyl, *Phys. Rev.* **46**, 505 (1934). Th. de Donder, *Theorie invariente du calcul des variations* (Gauthier–Villars, Paris, 1935). C. Carathodory, *Acta Sci. Math.* (Szeged) **4** (1929) 193. R. H. Good, *Phys. Rev.* **93**, 239 (1954). R. S. Liotta, *Nuovo Cimento* **3**, 438 (1956). V. Tapia, *Nuovo Cimento B* **102**, 123 (1988).

[4] R. H. Good, *J. Math. Phys* **35**, 3333 (1994); *ibid* **36**, 707 (1995). Good’s rules of quantisation are: $\varphi^a \rightarrow \hat{\varphi}^a = \varphi^a$; $\pi^a_\mu \rightarrow \hat{\pi}^a_\mu = -\frac{\partial^2}{\partial \varphi^a \partial x^\mu}$. Good’s quantum equation of motion is: $-\frac{\partial^2}{\partial x^\nu \partial x^\mu} \Psi(\varphi^a, x) = \hat{\mathcal{H}} \Psi(\varphi^a, x)$. This proposal, along with many attractive features, involves a number of undesired properties which prevent it from being a good starting point for a finite-dimensional formulation of QFT. These drawbacks are serious enough to rule out this proposal as a good candidate for a finite-dimensional QFT. Moreover, it was shown in ref. [5] that this theory leads to (measurable) predictions which do not agree with standard Quantum Mechanics. Hence, this proposal should be discarded.

[5] M. Navarro, *J. Math. Phys.* **36** (1995)6665.

[6] I.V. Kanatchikov, *Int. J. Th. Phys.* **40**, (2001) 1121-1149, [gr-qc/0012074](https://arxiv.org/abs/gr-qc/0012074).