OPTIMAL PRICING AND INVENTORY STRATEGIES FOR INTRODUCING A NEW PRODUCT BASED ON DEMAND SUBSTITUTION EFFECTS

ZHJIE SASHA DONG
Ingram School of Engineering, Texas State University
San Marcos, TX 78666, USA

WEI CHEN
School of Management and Engineering, Nanjing University
Nanjing, China 210093

QING ZHAO
Amazon, Seattle, WA 98109, USA

JINGQUAN LI∗
School of Management and Engineering, Nanjing University
Nanjing, China 210093

(Communicated by Jun Fu)

Abstract. This paper studies a single-period inventory-pricing problem with two substitutable products, which is very important in the area of Operations Management but has received little attention. The proposed problem focuses on determining the optimal price of the existing product and the inventory level of the new product. Inspired by practice, the problem considers various pricing strategies for the existing product as well as the cross elasticity of demand between existing and new products. A mathematical model has been developed for different pricing strategies to maximize the expected profit. It has been proven that the objective function is concave and there exists the unique optimal solution. Different sets of computational examples are conducted to show that the optimal pricing and inventory strategy generated by the model can increase profits.

1. Introduction. As people’s living standards improve, the age of the personalized customer experience has arrived with shorter product life cycles and higher product replacement rates. This makes it very common that new products and old products co-exist in the same market. Such coexistence is one of the major features of rapid product development (i.e., upgrade and renew). When a new product enters the market, the existing product continues to be sold usually with a reduced price (e.g., supermarket sales) to attract original customers. However, price reduction may not be the only strategy, and sometimes the sales price of the existing product remains the same. Examples include many items in the supermarket, such as yogurt.
yogurts with different expiration dates have identical prices, and the ones which expire earlier are always moved to the front of the shelf for sale. It is called First Expired-First Out (FEFO) policy, meaning that products which are to expire soon should be served or removed from stock on a priority basis (Sazvar et al. [16]). This is because there are customers who are insensitive about differences between new products and existing products, and retailers could earn higher profits by keeping the price of existing products the same (Sainathan [15]). Whether to decrease the price of old products or not depends on the number of such insensitive customers, the inventory level of existing products and the total demand for these products. At the same time, the price of the existing product will affect the demand for both itself and its alternative (i.e., the new product); that is, there is the cross elasticity of demand between existing and new products.

So, when introducing a new product into the market, there are many decisions need to be made which involves pricing, inventory, and assortment. Traditional research usually assumes that the new product is independent of the existing product. However, customers who originally plan to purchase the existing product may switch to the new product for improved functions and attributes. On the other hand, customers who come in for the new product may be willing to buy the existing one because of its lower price. These are called substitution effects and these two alternative products (i.e., existing and new products) are substitutes. Planning of substitutable products was received attention back in 1970s and there has been an increasing number of studies since 2000s. Shin et al. [17] conducted a thorough literature review on the work published in major Operations Management and Marketing journals from 1974 to 2013. Based on different substitution mechanisms they divided the decision and modeling environment under product substitution into four areas, assortment planning, inventory, capacity planning, and pricing decision. These four areas are not mutually exclusive, and many studies focused on joint problems considering two or more areas of decision. For example, Honhon et al. [3] proposed a dynamic programming algorithm to determine the optimal assortment and inventory levels under stockout-based substitution, and Pan and Honhon [14] developed several algorithms to identify an optimal assortment and price strategy for a category of vertically differentiated products.

Among these different decision areas, pricing and inventory are the most important ones to affect consumers’ product choice because, products with different remaining shelf lives have different attractiveness to customers, and retailers have to make such joint decisions for both new and old stocks (Li et al. [11], Mou et al. [12]). Most previous research concentrated on the conventional pricing-inventory models for a single product. However, these two factors interact with each other and how demand should be distributed among substitutes are determined by the interaction. Research on joint pricing inventory models for substitutable products is still a minor topic. Recent related literature includes Hsieh and Wu [4], Karakul and Chan [7] and Akun et al. [1]. In the modeling structure of joint pricing-inventory models, only two substitutable products in one single period are taken into account for the purpose of simplicity and the assortment is assumed to be fixed. For example, Karakul and Chan [6] dealt with the joint pricing and inventory decisions for a single-period model with two products in which an extant product can be substituted by a newly improved product in case of a shortage. They showed that in the existence of substitution, compared with the extant product, the new product has higher optimal price as well as higher stock level. So this paper considers a
single-period pricing-inventory model with two substitutable products. But unlike
the work of Karakul and Chan [6], the direction of substitutability among products
is two-way; that is, the new product can substitute for the existing product and vice
versa. This bidirectional substitution occurs often among with equally compatible
products regarding their utilities and attributes.

The structure of single-period joint pricing-inventory models with substitution
originates from conventional newsvendor models. It assumes that demand in this
single period is random and all inventories must be ordered before the time period
starts. The objective function is to maximize the expected profitability. Khouja et
al.[8] was one of the earliest work to present a news vendor model with two-way sub-
stitutable products, but without considering the pricing component. Relevant liter-
ature on pricing-inventory models is in general scarce as mentioned earlier. Aydin
and Porteus [2] determined optimal prices and inventory levels of multiples products
for a price-dependent newsvendor model. They showed that the demand model still
has a unique solution regarding both prices and inventory levels when involving
multiplicative uncertainty. Karakul [5] studied the joint pricing and inventory of
fashion products assuming that clearance markets exist, and it has been proved that
the existence of clearance markets will help increase not only price and inventory
quantity but also profits. Karakul and Chan [7] extended their work (Karakul and
Chan [6]) on the single-period joint pricing and inventory of substitutable products
to the case where the noise term in the demand of the low-grade product is contin-
uously distributed. Most relevant work is Nagarajan and Rajagopalan [13], where
they studied the nature of optimal inventory policies for substitutable products in
a single period case and then extended it to a multi-period version with heuristics.

Most previous research on product substitutions (including Nagarajan and Ra-
jagopalan [13]) generally assumes that demand for each substitutable product is
given, or the substitution rate is fixed. However, the change in the price of one
product may have impacts on the demand of the other, and different prices could
lead to varying substitution rates. Either way would influence the expected profit.
Besides this key research gap, our work is inspired by practice (e.g., the yogurt
example in the supermarket) and we also consider the case when the price of the
existing product remains the same, which is ignored by other studies.

To sum it up, this paper discusses a single-period pricing-inventory model for two
products which can substitute each other. This problem is very important in the
area of Operations Management but has not received sufficient attention yet. When
a new product is launched, its price is usually decided already, so our model focuses
on determining the price of the existing product and the inventory level of the new
product. There are three major contributions of this paper. First, it studies the
pricing strategy when the retailer does not reduce the existing product’s sale price.
This strategy is commonly used in many scenarios such as supermarkets. Second,
we consider the cross elasticity of demand between the existing product and the
new product. In other words, the price change of one product will have impacts
on the demand for the other. Third, it is assumed that different substitution rates
are possible and how many consumers are willing to switch from the item they
originally plan to purchase to the alternative is driven by the price of the existing
product.

The rest of the paper is organized as follows. Section 2 formally defines the
single-period pricing-inventory problem proposed in this paper. Section 3 gives
some notations and assumptions about this model. Section 4 discusses necessary
assumptions and constructs mathematical models for different pricing strategies of the existing product. Section 5 conducts three sets of test examples to discuss the models. Section 6 makes conclusions and points out the future research direction.

2. Problem description. Suppose a retailer sells two different generations of the same product simultaneously during time period \([0, T]\) - one is the extant product and the other is the new product. At time 0 the extant product is in stock, and the retailer needs to order a batch of the new product to avoid stockouts of the extant product. When selling both products at the same time in the same market, the retailer has two alternative pricing strategies available. First, keeping the extant product’s price unchanged. It is highly likely that consumers will be appealed to purchase the new product because of improved functions and better attributes, so the existing product would be overstocked. But it is also possible that customers who are indifferent about these two products still buy the existing one, and this usually happens in the supermarket where the old foods are moved to the front of the shelf. The second option is to lower the existing product’s price, which may attract customer demand to switch from the new product to the existing one. This strategy causes a decrease in the profit margin of the extant product but improves its sales, therefore its inventory cost is reduced. It is assumed that the price of the new product is fixed when being introduced into the market, and the retailer only needs determine how much of the existing product to sell and how many of the new product to order. This decision can only be made once and has to be made at the beginning of the sales cycle, i.e., time 0. In other words, once the new product enters the market, the price of the existing product cannot be adjusted and order quantity of the new product cannot be filled even if it is sold out. When the sales cycle ends, remaining inventories of both products will be taken care of based on their salvage values.

It is worth noting that the extant product and the new product are partial substitutes. The level of substitution is dependent of whether consumers are indifferent about two products and price difference between the products. When the new product and the extant product have identical prices, if the extant product is out of stock, customer demand will switch to the new product entirely; and if it is the new product out of stock, only some of the customers who come in for the new product will switch and buy the existing product. When the existing product’s price is reduced, and if it is out of stock only some of the customers are willing to switch from the old product to the new product. Obviously, the substitution percentage is negatively correlated with the price difference for this case. If customers come in for the new product and do not find it in stock, some of them will switch and buy the existing product. This substitution percentage is positively correlated with the price difference.

Therefore, it is assumed that there are only two substitutable products in the market, which are old and new products (i.e., products 1 and 2). The total potential demand, \(D\), is the market size for these two products in one period, and each product takes up the proportion \(p\), and \(1 - p\), respectively. Here, \(p\) is a random variable with a continuous distribution function \(F\). The actual demand for each of these two products may be less than its potential demand. For example, when customers come into a store for a specific product but find it out of stock, they will make either of the following decisions: 1) purchase the other product (since it is substitutable); or 2) just walk away. In the proposed model, we introduce a proportion \(r\) to represent the customers who are willing to switch to the substitute, while there are \((1 - r)\)
proportion of the customers who choose to walk away. The products are substitutes in two respects: their demands are negatively correlated and unsatisfied customers for one product may switch to the other. When the information on the inventory level of the existing product and the price of the new product are given, this paper targets on determining the optimal price of the existing product and order quantity of the new product at time 0 to maximize the seller’s expected profits over time period \([0, T]\).

3. Notations and assumptions. Index \(i\) is used to index two products: \(i = 1\) if it is the extant product, and \(i = 2\) if it is the new product. For simplicity's sake, in this section and Section 4, they are called product 1 and product 2, respectively. The variables and parameters are defined below.

\[D: \text{the total potential demand (i.e., market size) for products 1 and 2 over the period } [0, T];\]
\[s_i: \text{product } i\text{'s price; }\]
\[h_i: \text{holding cost per unit of unsold inventory of product } i;\]
\[c: \text{unit cost of product 2; }\]
\[Q_i: \text{order quantity of product } i;\]
\[d_i: \text{demand for product } i;\]
\[r_i: \text{a fraction of customers who plan to buy product } i \text{ but do not find it available will switch to the other product if it is in stock, i.e., substitution rate; }\]
\[p: \text{a probability that customers are indifferent about two products when these two products have identical prices. It is a random variable between 0 and 1. The cumulative distribution and probability density functions for are given by } F(\cdot) \text{ and } f(\cdot);\]
\[\lambda: \text{the increased fraction of total demand for product 1 when its price drops. }\]
\[\lambda = \Delta d_1/D. \text{ This is because when the price drops, some consumers originally for product 2 will switch for product 1. } \lambda \text{ is related to product 1’s price;}\]

Assumption 1.

\(A. s_i > c_i > h_i \text{ for } \forall i. \text{ For each product, its cost is lower than the price but greater than the unit salvage value. It is worth mentioning that product 1’s unit salvage value could be negative.}\)

\(B. s_2 \geq s_1, h_2 \geq h_1. \text{ Product 2’s price is at least as high as product 1’s, and so is the salvage value.}\)

Assumption 2.

When product 1’s price remains unchanged, the initial demand for these two products is given by \(p*D\) and \((1-p)*D\).

When these two products have identical prices, to reduce possible unsold inventory of product 1 the retailer always puts product 1 in the front of shelves so that customers who are indifferent about two products will be more likely to buy product 1 if it is available. Therefore, such customers are assumed to always buy product 1 when no price difference exists.

Assumption 3.

When product 1’s price remains unchanged, \(r_1 = 1, r_2 = 0. \text{ In other words, all the customers who come in for product 1 and find it unavailable would like to switch for product 2, but none of the customers who come in for product 2 wants to buy product 1 instead. When the price of product 1 drops, fractions } r_1 \text{ and } r_2 \text{ depend on how much the price drops.}\)
Assumption 4. When product 1’s price decreases, \( r_1(s_1), r_2(s_1) \in [0,1] \), and \( r'_1(s_1) > 0, r''_1(s_1) < 0 \), \( r'_2(s_1) < 0, r''_2(s_1) < 0 \).

\( r'_1(s_1) > 0 \) indicates that the lower product 1’s price is, the fewer consumers who come in for product 1 but find it unavailable will switch for product 2. \( r'_2(s_1) < 0 \) indicates that the lower product 1’s price is, the more consumers for product 2 will substitute to product 1 if product 2 is out of stock. \( r''_1(s_1) < 0 \) and \( r''_2(s_1) < 0 \) demonstrate that the lower product 1’s price is, the greater the change in substitution fraction will be.

Assumption 5. As we mentioned earlier, the price of product 1 will affect the order quantity of product 2; also, the price of product 2 will affect the inventory level of product 1. In other words, there is the cross elasticity of demand between product 1 and product 2. It means that when product 1’s price drops, some consumers will switch their demand from product 2 to product 1. Such cross elasticity of demand is \( e = \frac{\lambda/D}{(1−p)/D} \).

Assumption 6. The difference between product \( i \)'s price and salvage value is no smaller than the product of the substitution rate and the difference between the other product’s price and the salvage value, \( s_i − h_i \geq r_i(s_j − h_j) \), where \( i = 1, 2 \) and \( j = 1, 2 \) and \( i \neq j \).

This assumption ensures that when consumers come in for product \( i \) they may switch to product \( j \) as a substitute only when \( i \) is out of stock; otherwise they will purchase product \( i \). In other words, a product can substitute for the other only after stockout.

4. Mathematical models. As discussed above, the retailer has two pricing strategies for the existing product when introducing the new product into the market - keeping its price unchanged or reducing its price. Therefore, the following models are developed for each case.

4.1. Case I. Product 1’s price remains unchanged. Since products 1 and 2 have identical prices, \( s, s_1 = s_2 = s \). Order quantities of two products are \( Q_1 = \alpha * D \) and \( Q_2 = \beta * D \), where \( 0 \leq \alpha, \beta \leq 1 \). Without loss of generality, we let \( D = 1 \) here. Denote the expected profit of the retailer by \( \Pi(\beta) \). The formulation \( \Pi(\beta) \) is split into two scenarios depending on whether available inventory is sufficient to satisfy stockout generated substitution demand (i.e., \( \alpha + \beta \geq 1 \)) or not (i.e., \( \alpha + \beta < 1 \)).

\[
\Pi(\beta) = \begin{cases} 
\Pi_1(\beta), & \alpha + \beta \geq 1, \\
\Pi_2(\beta), & \alpha + \beta < 1.
\end{cases}
\]

(1)

For \( \Pi_1(\beta) \), we consider the following three sub-scenarios by possible values of \( p \).
1. \( 0 < p \leq 1 - \beta \), inventory exceeds demand for product 1 but not product 2, and remaining inventory of 1 can fully satisfy substitution demand of 2.
2. \( 1 - \beta < p \leq \alpha \), inventory is sufficient to satisfy demands for both products.
3. \( \alpha < p \leq 1 \), inventory exceeds demand for product 2 but not product 1, and remaining inventory of 2 can fully satisfy substitution demand of 1.

So the derivation of \( \Pi_1(\beta) \) can be written as follows:

\[
\Pi_1(\beta) = \int_{0}^{1-\beta} [s(p + \beta) + h_1(\alpha - p)] dF(p)
\]
For $\Pi_2(\beta)$, we consider the following three sub-scenarios by possible values of $p$.

1. $0 < p \leq \alpha$, inventory exceeds demand for product 1 but not product 2, and remaining inventory of 1 can fully satisfy substitution demand of 2.
2. $\alpha < p \leq 1 - \beta$, inventory cannot afford demand for neither product 1 nor product 2.
3. $1 - \beta < p \leq 1$, inventory exceeds demand for product 2 but not product 1, and remaining inventory of 2 cannot fully satisfy substitution demand of 2.

So the derivation of $\Pi_2(\beta)$ can be written as follows:

$$\Pi_2(\beta) = \int_0^\alpha [s(\alpha + \beta) + h_1(\alpha - p)] dF(p) + \int_\alpha^1 [s(\alpha + \beta)] dF(p) - c \beta$$

(3)

Theorem 4.1. The expected profit function, $\Pi(\beta)$, is concave and continuously differentiable. There exists an optimal solution, $\beta^*$, to maximize this expected profit.

Proof. The first derivatives of $\Pi_1(\beta)$ and $\Pi_2(\beta)$ are

$$\begin{cases} \frac{d\Pi_1(\beta)}{d\beta} = \int_0^{1-\beta} (s - h_2) dF(p) + h_2 - c \\ \frac{d\Pi_2(\beta)}{d\beta} = s - c, \end{cases}$$

(4)

respectively.

The second derivatives of $\Pi_1(\beta)$ and $\Pi_2(\beta)$ are

$$\begin{cases} \frac{d^2\Pi_1(\beta)}{d\beta^2} = -(s - h_2)f(1 - \beta) \leq 0 \\ \frac{d^2\Pi_2(\beta)}{d\beta^2} = 0, \end{cases}$$

(5)

respectively.

Since $\Pi_1(1 - \alpha) = \Pi_2(1 - \alpha)$,

$$\Pi_2(1 - \alpha) - \Pi_1'(1 - \alpha) = (s - c) - [(s - h_2)F(\alpha) + h_2 - c]$$

$$= s - h_2 - (s - h_2)F(\alpha)$$

$$= (s - h_2)[1 - F(\alpha)]$$

$$\geq 0.$$

(6)

Given that the expected profit function is $\Pi(\beta) = \begin{cases} \Pi_1(\beta), \alpha + \beta \geq 1 \\ \Pi_2(\beta), \alpha + \beta < 1 \end{cases}$.

When $\alpha + \beta = 1$,

$$\Pi_1(\beta) = \Pi_1(1 - \alpha)$$

$$= \int_0^\alpha [s(\alpha + 1 - \alpha) + h_1(\alpha - p)] dF(p) + \int_\alpha^1 s dF(p) - c(1 - \alpha)$$

$$= \int_0^\alpha (s - h)p dF(p) + \int_\alpha^1 (s - h)p dF(p) + \int_\alpha^1 s dF(p) - c(1 - \alpha)$$

$$= s[1 - F(\alpha)] - c(1 - \alpha).$$

(7)
When \( \alpha + \beta \to 1 \),
\[
\lim_{\alpha+\beta\to 1} \Pi_2(\beta) = \int_0^{\alpha} [s(p + 1 - \alpha) + h_1(\alpha - p)] dF(p) + \int_{\alpha}^{1} s dF(p) - c(1 - \alpha) \\
= s[1 - F(\alpha)] - c(1 - \alpha) \\
= \Pi_1(1 - \alpha).
\]

Thus \( \Pi(\beta) \) is a concave and continuously differentiable function on the interval \([0,1]\). By the definition of an optimal solution, we know that there exists an optimally policy so that the maximum of \( \Pi(\beta) \) is attained. \( \square \)

Let \( \frac{d\Pi_1(\beta)}{d\beta} = \int_0^{1-\beta} (s - h_2) dF(p) + h_2 - c = 0 \), we have \( F(1 - \beta^*) = \frac{c-h_2}{s-h_2} \) when \( \alpha + \beta^* \geq 1 \). It is worth mentioning that when \( \alpha + \beta^* < 1 \), the optimal inventory level \( \beta^* = 1 - \alpha \).

### 4.2. Case II. Product 1’s price decreases
Since \( s_1 < s_2 \), customers who are indifferent about two products will choose to buy product 1 when it is in stock. Let \( p_t = p + \lambda \). So demand for product 1 is \( d_1 = p_t * D \) and demand for product 2 is \( d_2 = (1 - p_t) * D \).

When product 1 is out of stock, only some of the customers who come in for 1 will switch for 2 while others will walk away because 2 is more expensive. When product 2 is out of stock, some of the customers who come in for 2 will substitute to 1. So \( r_1 \leq 1 \) and \( r_2 \geq 0 \). The expected profit function of the retailer is:
\[
\Pi(\beta_p) = \begin{cases} 
\Pi_1(\beta, p_t), & \alpha + \beta \geq 1, \\
\Pi_2(\beta, p_t), & \alpha + \beta < 1.
\end{cases}
\]

Similar to \( \Pi_1(\beta) \), for \( \Pi_1(\beta, p_t) \) we consider the following three sub-scenarios by possible values of \( p_t \):

1. \( 0 < p_t \leq 1 - \beta \), inventory exceeds demand for product 1 but not product 2, and remaining inventory of 1 can fully satisfy substitution demand of 2.
2. \( 1 - \beta < p_t \leq \alpha \), inventory is sufficient to satisfy demands for both products.
3. \( \alpha < p_t \leq 1 \), inventory exceeds demand for product 2 but not product 1, and remaining inventory of 2 can fully satisfy substitution demand of 1.

So the derivation of \( \Pi_1(\beta, p_t) \) can be written as follows:
\[
\Pi_1(\beta, s_1) = \int_0^{1-\beta} \{s_1[p_t + r_2(1 - p_t - \beta)] + s_2\beta + h_1[\alpha - p_t - r_2(1 - p_t - \beta)]\}
\]
\[
dF(p_t) + \int_{1-\beta}^{\alpha} \{s_1p_t + s_2(1 - p_t) + h_1(\alpha - p_t) + h_2[\beta - (1 - p_t)]\}
\]
\[
dF(p_t) + \int_{\alpha}^{1} \{s_1\alpha + s_2[1 - p_t + r_1(p_t - \alpha)] + h_2[\beta - (1 - p_t)] - r_1(p_t - \alpha)\} dF(p_t) - c\beta.
\]

To facilitate the derivation of \( \Pi_2(\beta, p_t) \), we define \( \tilde{\alpha} = [\alpha - r_2(1 - \beta)]/[1 - r_2] \) and \( \tilde{\beta} = [1 - \beta - r_1\alpha]/[1 - r_1] \). It is obvious that when \( p_t = \tilde{\alpha} \), remaining inventory of product 1 is exactly equal to the substitution demand from product 2. Similarly, when \( p_t = \tilde{\beta} \), remaining inventory of product 2 is exactly equal to the substitution demand from product 1.

According to possible values of \( p_t \), there are five different sub scenarios as follows:

1. \( 0 < p_t \leq \tilde{\alpha} \), inventory exceeds demand for product 1 but not product 2, and excess inventory of 1 can fully satisfy substitution demand of 2.
2. $\alpha < p_t \leq \beta$, inventory exceeds demand for product 1 but not product 2, and excess inventory of 1 cannot fully satisfy substitution demand of 2.

3. $\alpha < p_t \leq 1 - \beta$, inventory is not sufficient to satisfy demand for either product 1 or product 2.

4. $1 - \beta < p_t \leq \hat{\beta}$, inventory exceeds demand for product 2 but not product 1, and excess inventory of 2 can fully satisfy substitution demand of 1.

5. $\beta < p_t \leq 1$, inventory exceeds demand for product 2 but not product 1, and excess inventory of 1 cannot completely satisfy substitution demand of 1.

So we can write $\Pi_2(\beta, s_1)$ as

$$\Pi_2(\beta, s_1) = \int_0^\alpha \{s_1[p_t + r_2(1 - p_t - \beta)] + s_2 \beta + h_1[\alpha - p_t - r_2(1 - p_t - \beta)]\}
+ \int_0^\beta (s_1 \alpha + s_2 \beta) dF(p_t) + \int_0^1 \{s_1 \alpha + s_2[1 - p_t + r_1(p_t - \alpha)]\} (11)
+ h_2[\beta - (1 - p_t) - r_1(p_t - \alpha)] \ dF(p_t) - c\beta. $$

**Theorem 4.2.** For any price of product 1, $s_1$, the expected profit function is a concave function of $\beta$.

**Proof.** The first derivatives of $\Pi_1(\beta, s_1)$ and $\Pi_2(\beta, s_1)$ are

$$\frac{d\Pi_1}{d\beta} = \int_0^{1-\beta} (s_2 - s_1 r_2 + h_1 r_2) dF(p_t) + \int_0^1 h_2 dF(p_t) - c$$

$$\frac{d\Pi_2}{d\beta} = \int_0^\beta (s_2 - s_1 r_2 + h_1 r_2) dF(p_t) + \int_\beta^1 s_2 dF(p_t) + \int_0^1 h_2 dF(p_t) - c, (12)$$

respectively.

The second derivatives of $\Pi_1(\beta, s_1)$ and $\Pi_2(\beta, s_1)$ are

$$\frac{d^2\Pi_1}{d\beta^2} = -[s_2 - h_2 - r_2(s_1 - h_1)] f(1 - \beta) \leq 0$$

$$\frac{d^2\Pi_2}{d\beta^2} = -r_2(s_1 - h_1) f(\hat{\alpha}) \frac{r_2}{1 - r_2} - (s_2 - h_2) f(\hat{\beta}) \frac{1}{1 - r_2} \leq 0, (13)$$

respectively.

Since $\Pi_1(1 - \alpha) = \Pi_2(1 - \alpha)$ we have

$$\Pi_2'(1 - \alpha) = -\Pi_1'(1 - \alpha)$$

$$\Pi_2'(1 - \alpha) = [(s_2 - h_2)f(\hat{\beta}) - r_2(s_1 - h_1)f(\hat{\alpha})] - [s_2 - h_2 - r_2(s_1 - h_1)] f(\alpha)$$

$$\Pi_1'(1 - \alpha) = [(s_2 - h_2)f(\alpha) - r_2(s_1 - h_1)f(\alpha)] - [s_2 - h_2 - r_2(s_1 - h_1)] f(\alpha)$$

$$= 0. (14)$$

So the expected profit function is a concave and continuously differentiable function of $\beta$ with given $s_1$. Therefore there exists a unique solution $\beta^*(s_1)$ to maximize the expected profit.

**Theorem 4.3.** The expected profit function is concave in $(\beta, s_1)$.

**Proof.** Theorem 2 shows that for any given $s_1$, $d\Pi/d\beta = 0$ has a unique solution $\beta^*(s_1)$. The expected profit function can be re-written as follows:

$$\Pi_1(\beta, s_1) = \{s_1 \alpha + s_2 r_1(1 - \alpha) + h_2[\beta - r_1(1 - \alpha)]\}$$

$$- \int_0^{1-\beta} (s_1 - h_1)(1 - r_2) dF(p_t)$$
and

\[
\Pi_2(\beta, s_1) = \{s_1[\hat{\alpha} + r_2(1 - \hat{\alpha} - \beta)] + h_1[\alpha - \hat{\alpha} - r_2(1 - \hat{\alpha} - \beta)]\} F(\hat{\alpha})
\]
\[
- \int_0^{\hat{\alpha}} (s_1 - h_1)(1 - r_2) dF(p_t)
\]
\[
+ (s_2 - h_2)[\beta - (1 - \hat{\beta}) - r_1(\hat{\beta} - \alpha)] F(\hat{\beta})
\]
\[
- \int_\beta^1 (h_2 - s_2)(1 - r_1) dF(p_t)
\]
\[
+ \{s_1 \alpha + s_2 r_1(1 - \alpha) + h_2[\beta - r_1(1 - \alpha)]\} - c_\beta.
\]

The first derivatives of \(\Pi_1(\beta)\) and \(\Pi_2(\beta)\) are

\[
\frac{\partial \Pi_1(\beta, s_1)}{\partial s_1} = [\alpha + (s_2 - h_2)(1 - \alpha)r_1'] - \int_0^{1-\beta} [1 - r_2 - (s_1 - h_1)r_2'] dF(p_t)
\]
\[
- \int_0^\alpha dF(p_t) - \int_\alpha^1 (s_2 - h_2)r_1' dF(p_t)
\]

and

\[
\frac{\partial \Pi_2(\beta, s_1)}{\partial s_1} = [\hat{\alpha} - \alpha + (r_2 + s_1 r_2' - h_1 r_2')(1 - \hat{\alpha} - \beta)] F(\hat{\alpha})
\]
\[
- \int_0^{\hat{\alpha}} [1 - r_2 - (s_1 - h_1)r_2'] dF(p_t) - [(s_2 - h_2)(\hat{\beta} - \alpha)r_1'] F(\hat{\beta})
\]
\[
- \int_\beta^1 (s_2 - h_2)r_1' dF(p_t) + [\alpha + (s_2 - h_2)(1 - \alpha)r_1'],
\]

respectively.

The second derivatives of \(\Pi_1(\beta)\) and \(\Pi_2(\beta)\) are

\[
\frac{\partial^2 \Pi_1(\beta, s_1)}{\partial s_1^2} = [2r_2' + (s_1 - h_1)r_2''] \int_0^{1-\beta} dF(p_t)
\]
\[
+ (s_2 - h_2)r_1'' \int_\alpha^1 [1 - F(p_t)] dp_t
\]
\[
\leq 0
\]

and

\[
\frac{\partial^2 \Pi_2(\beta, s_1)}{\partial s_1^2} = [2r_2' + (s_1 - h_1)r_2''](1 - \hat{\alpha} - \beta) F(\hat{\alpha})
\]
\[
+ [2r_2' + (s_1 - h_1)r_2''] \int_0^{\hat{\alpha}} F(p_t) dp_t
\]
\[
+ (s_2 - h_2)r_1'' [1 - \alpha - (\hat{\beta} - \alpha)] F(\hat{\beta})
\]
\[
- \int_\beta^1 F(p_t) dp_t \leq 0,
\]

respectively.
Since $\Pi_1(1 - \alpha, s_1) = \Pi_2(1 - \alpha, s_1)$, we have $\frac{\partial \Pi_2(1 - \alpha, s_1)}{\partial s_1} - \frac{\partial \Pi_1(1 - \alpha, s_1)}{\partial s_1} = 0$. Therefore the expected profit function is concave of $s_1$ and it is continuously differentiable. This shows that for any given $\beta$, there exists a unique optimal solution $s_1^*$ to maximize the expected profit. 

5. Computational study. In this section, we use three sets of computational instances to discuss the models proposed in the previous section. When $n$ is large enough, the demand for product 1, $d_1$, is approximately normally distributed with a mean of $np$, where $n$ is the number of consumers and $p$ is the probability that consumers are indifferent about products (i.e., product 1 and 2) with identical prices. Let $n = 1,000$ here, then the total demand for both products, $D$, is 1,000. We also assume that when product 1’s price drops the substitution rate is a linear function of its price, i.e., $r_1 = s_1/s_2$ and $r_2 = (s_2 - s_1)/s_2$.

The first set of computational examples is to analyze the effect of the existing product’s decreased price on the inventory level of the new product and the expected profit. The parameters used are $Q_1 = 250, c = 8, s_2 = 12, e = 0.5, p = 0.2, h_1 = 0, and h_2 = 4$. Computational results are presented in Figures 1 and 2. Figure 1 shows that the order quantity of the new product decreases when the existing product’s price drops. It is because customers who come in originally for the new product will be attracted to switch for the extant product by its lower price. The lower the extant product’s price is, the more customers for the new product will be willing to buy the extant product instead. This also leads to a decreasing substitution rate $r_1$ and an increasing substitution rate $r_2$. Figure 2 demonstrates that when the existing product’s price drops, the expected profit increases first and then decreases. There is a unique turning point where the expected profit achieves maximum, and this is the optimal solution.

The second set of computational examples is to study how the inventory level of the existing product affects the retailer’s optimal solution and the expected profit. The parameters used are $c = 8, s_2 = 12, e = 0.5, p = 0.2, h_1 = 0, and h_2 = 4$. Computational results are summarized in Table 1 with two pricing policies: either keeping the price of the extant product as what it is or reducing the price. $Q_1$ is
Figure 2. Effect of the extant product’s price on the expected profit.

Table 1. Effect of the inventory level of the existing product on the retailer’s optimal policy and the expected profit.

| Strategy | Variables | Values | $Q_1$ | $Q_2$ | $s_1$ ($) | $E_P$ ($) | $R_P$ ($) |
|----------|-----------|--------|-------|-------|-----------|-----------|-----------|
| Unchanged |           |        | 160   | 840   | 12       | 5280      | 100.0     |
|           |           |        | 170   | 830   | 12       | 5360      | 100.0     |
|           |           |        | 180   | 820   | 12       | 5440      | 100.0     |
|           |           |        | 190   | 810   | 12       | 5520      | 100.0     |
|           |           |        | 200   | 800   | 12       | 5600      | 100.0     |
|           |           |        | 210   | 800   | 12       | 5600      | 100.0     |
|           |           |        | 220   | 800   | 12       | 5600      | 100.0     |
|           |           |        | 230   | 800   | 12       | 5600      | 100.0     |
|           |           |        | 240   | 800   | 12       | 5600      | 100.0     |
|           |           |        | 250   | 800   | 12       | 5600      | 100.0     |
| Decreased |           |        |       | 840   | 12       | 5280      | 100.0     |
|           |           |        |       | 830   | 12       | 5360      | 100.0     |
|           |           |        |       | 820   | 12       | 5440      | 100.0     |
|           |           |        |       | 810   | 12       | 5520      | 100.0     |
|           |           |        |       | 800   | 12       | 5600      | 100.0     |
|           |           |        |       | 790   | 11.7     | 5617      | 99.7      |
|           |           |        |       | 780   | 11.4     | 5628      | 99.7      |
|           |           |        |       | 770   | 11.1     | 5633      | 99.7      |
|           |           |        |       | 760   | 11.0     | 5632      | 99.7      |

quantity in stock of the extant product, $Q_2$ is order quantity of the new product, $s_1$ is the price of the extant product, and $E_P$ is the expected profits. $FR$ stands for fill rate, which is an indicator showing the percentage of demands that are met at the time they are placed.

As this table shows, if the new product and the existing product have identical prices, with an increase in the inventory of the existing product, the optimal order quantity of the new product will decrease first and then remain unchanged. This is because when inventory of the existing product is low, it cannot fully afford customer’s demand and some customers may purchase the new product instead; while the inventory level is going up, more and more demand for the existing product can be satisfied and fewer customers will switch to the new product. After the inventory of the existing product can fully afford its demand, there is no substitution happened and the optimal order quantity of the new product is its demand.

Here we assume that for customers who are indifferent about these two products with the same price will always buy the extant product when it is available. If the retailer reduces the extant product’s price, when the inventory increases there is a continuous drop in the optimal order quantity of the new product. At the same time, the optimal pricing strategy for the extant product is to stay unchanged first and then decrease with the increasing inventory. This is because at earlier times since the extant product’s inventory cannot fully satisfy its demand, there is no need to reduce its price. The higher the inventory of the existing product is, the
Table 2. Effect of the salvage value of the existing product on the retailer’s optimal policy and the expected profit

| Strategy | Variables | $h_1$ | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|----------|-----------|-------|----|----|----|----|----|---|---|---|---|---|
| Unchanged | $Q_2$ | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 | 800 |
|          | $s_1$ | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 | 12 |
|          | $E(P)$ | 5350 | 5400 | 5450 | 5500 | 5550 | 5600 | 5650 | 5700 | 5750 | 5800 | 5800 |
| Decreased | $Q_2$ | 750 | 750 | 750 | 750 | 750 | 750 | 750 | 750 | 750 | 750 | 750 |
|          | $s_1$ | 10.5 | 10.5 | 10.5 | 10.5 | 10.5 | 10.5 | 10.5 | 10.5 | 10.5 | 10.5 | 10.5 |
|          | $E(P)$ | 5625 | 5625 | 5625 | 5625 | 5625 | 5625 | 5625 | 5625 | 5625 | 5625 | 5625 |

fewer people would like to switch for the new product and the smaller quantity of the new product will be ordered. When there is excess inventory of the existing product, the retailer will decide whether to lower its price or not based on how the pricing strategy will affect the expected profit. If reducing the price of the extant product will increase the expected profit, how much to reduce the price is positively correlated with how much the inventory excesses. In this case, more customers who come in for the new product will switch to the existing one, and it causes substitution rate $r_2$ to increase and substitution rate $r_1$ to decrease leading to a fall in the optimal inventory of the new product.

Table 1 also demonstrates that when the existing product’s inventory is in short, there is no difference between these two pricing strategies regarding the total expected profit. However, when there is excess inventory of the existing product, reducing its price will bring a greater expected profit.

It is worth mentioning that when the existing product’s inventory is 250 units, the optimal order quantity of the new product is 800 units if the existing product’s price stays the same while it will be only 746.6 units if the price drops. If we calculate fill rates, they are 100% and 99.7%, respectively. This means that although lowering the price of the extant product can increase the expected profit, the fill rate may decrease.

The last set explores how the salvage value of the extant product and $p$ affect the retailer’s optimal solution and his expected profit. The parameters are $Q_1 = 250$, $c = 8$, $s_2 = 12$, $e = 0.5$, $p = 0.2$, $h_1 = 0$, and $h_2 = 4$. Computational results are summarized in Table 2, where $h_1$ is the salvage value of the existing product.

Table 2 shows that with an increase in the salvage value of the existing product, regardless of which pricing strategy the optimal order quantity of the new product stays the same, and so does the optimal price for the existing product. This demonstrates that the existing product’s salvage value does not have a significant impact on the retailer’s optimal policy. Table 2 also presents that if the extant product and the new product have identical prices, when $p$ increases, the optimal order quantity of the new product will decrease first and then remain unchanged. When $p$ is large enough, customer demand for the existing product cannot be fully satisfied, and the optimal order quantity of the new product is $\beta^* = 1 - \alpha$. When $p$ is relatively small the retailer has to lower the existing product’s price to attract more customers from the new product.

6. Conclusions and future research. There is a lack of research on the joint inventory-pricing models under product substitution, especially with different pricing strategies. This paper studies a single-period inventory problem with two substitutable products, which determines the optimal price of the extant product and
order quantity of the new product. Motivated by practice (e.g., yogurts in the supermarkets), it is assumed that the new product is introduced into the market with the same price of the existing product, and the seller can choose either lower the price of the existing or keep it unchanged. Moreover, the proposed problem considers the cross elasticity of demand between these two products as well as varying substitution rates. We have formulated this problem to maximize the expected profit and proved that the objective function is concave with a unique optimal solution. The numerical study shows that implementing the optimal pricing strategy for the existing product (i.e., lowering the price) will help improve the expected profit but may fail to satisfy the total consumers’ demand. It also shows that salvage values of these products have little impact on the optimal strategies.

There are several potential extensions of our current work, such as a more general case with $N$ substitutable products and more decision makers (e.g., retailers). We want to explore possible distributions of the probability random variable ($p$) to catch up the consumers’ behavior so that we can gain more interesting managerial insights. We are also interested in considering incomplete information (Lan et al.[9][10]) as well as applying the proposed models into practice.

REFERENCES

[1] M. Akan, B. Ata and R. C. Savaskan-Ebert, Dynamic pricing of remanufacturable products under demand substitution: a product life cycle model, Annals of Operations Research, 211 (2013), 1–25.
[2] G. Aydin and E. L. Porteus, Joint inventory and pricing decisions for an assortment, Operations Research, 56 (2008), 1247–1255.
[3] D. Honhon, V. Gaur and S. Seshadri, Assortment planning and inventory decisions under stockout-based substitution, Operations Research, 58 (2010), 1364–1379.
[4] C. C. Hsieh and C. H. Wu, Coordinated decisions for substitutable products in a common retailer supply chain, European Journal of Operational Research, 196 (2009), 273–288.
[5] M. Karakul, Joint pricing and procurement of fashion products in the existence of clearance markets, International Journal of Production Economics, 114 (2008), 487–506.
[6] M. Karakul and L. M. A. Chan, Analytical and managerial implications of integrating product substitutability in the joint pricing and procurement problem, European Journal of Operational Research, 190 (2008), 179–204.
[7] M. Karakul and L. M. A. Chan, Joint pricing and procurement of substitutable products with random demands - A technical note, European Journal of Operational Research, 201 (2010), 324–328.
[8] M. Khouja, A. Mehrez and G. Rabinowitz, A two-item newsboy problem with substitutability, International Journal of Production Economics, 44 (1996), 267–275.
[9] Y. Lan, Z. Liu and B. Niu, Pricing and design of after-sales service contract: The value of mining asymmetric sales cost information, Asia-Pacific Journal of Operational Research, 34 (2017), 1740002.
[10] Y. Lan, R. Zhao and W. Tang, A fuzzy supply chain contract problem with pricing and warranty, Journal of Intelligent & Fuzzy Systems, 26 (2014), 1527–1538.
[11] X. Li, G. Sun and Y. Li, A multi-period ordering and clearance pricing model considering the competition between new and out-of-season products, Annals of Operations Research, 242 (2016), 207–221.
[12] S. Mou, D. J. Robb and N. DeHoratius, Retail store operations: Literature review and research directions, European Journal of Operational Research, 265 (2018), 399–422.
[13] M. Nagarajan and S. Rajagopalan, Inventory models for substitutable products: Optimal policies and heuristics, Management Science, 54 (2008), 1453–1466.
[14] X. A. Pan and D. Honhon, Assortment planning for vertically differentiated products, Production and Operations Management, 21 (2012), 253–275.
[15] A. Sainathan, Pricing and replenishment of competing perishable product variants under dynamic demand substitution, Production and Operations Management, 22 (2013), 1157–1181.
[16] Z. Sazvar, S. M. J. Mirzapour Al-e-hashem, K. Govindan and B. Bahli, A novel mathematical model for a multi-period, multi-product optimal ordering problem considering expiry dates in a FEFO system, *Transportation Research Part E: Logistics and Transportation Review*, 93 (2016), 232–261.

[17] H. Shin, S. Park, E. Lee and W. C. Benton, A classification of the literature on the planning of substitutable products, *European Journal of Operational Research*, 246 (2015), 686–699.

Received July 2017; 1st revision January 2018; final revision August 2018.

E-mail address: sasha.dong@txstate.edu
E-mail address: chenveiy@163.com
E-mail address: qz74@cornell.edu
E-mail address: ljq@nju.edu.cn