Linear response, dynamical friction and the fluctuation-dissipation theorem in stellar dynamics

Robert W. Nelson\textsuperscript{1,3} and Scott Tremaine\textsuperscript{2,3}
\textsuperscript{1}Theoretical Astrophysics, California Institute of Technology 130-33, Pasadena, California 91125, USA
\textsuperscript{2}Canadian Institute for Advanced Research, Program in Cosmology and Gravity
\textsuperscript{3}Canadian Institute for Theoretical Astrophysics, McLennan Labs, University of Toronto, 60 St. George St., Toronto M5S 3H8, Canada

ABSTRACT

We apply linear response theory to a general, inhomogeneous, stationary stellar system, with particular emphasis on dissipative processes analogous to Landau damping. Assuming only that the response is causal, we show that the irreversible work done by an external perturber is described by the anti-Hermitian part of a linear response operator, and damping of collective modes is described by the anti-Hermitian part of a related polarization operator. We derive an exact formal expression for the response operator, which is the classical analog of a well-known result in quantum statistical physics. When the self-gravity of the response can be ignored, and the ensemble-averaged gravitational potential is integrable, the expressions for the mode energy, damping rate, and polarization operator reduce to well-known formulae derived from perturbation theory in action-angle variables. In this approximation, dissipation occurs only via resonant interaction with stellar orbits or collective modes. For stellar systems in thermal equilibrium, the anti-Hermitian part of the response operator is directly related to the correlation function of the fluctuations. Thus dissipative properties of the system are completely determined by the spectrum of density fluctuations—the fluctuation-dissipation theorem. In particular, we express the coefficient of dynamical friction for an orbiting test particle in terms of the fluctuation spectrum; this reduces to the known Chandrasekhar formula in the restrictive case of an infinite homogeneous system with a Maxwellian velocity distribution.
1 Introduction

The aim of this paper is to apply to stellar dynamics two of the powerful tools that have been developed in other branches of statistical physics: linear response theory and the fluctuation-dissipation theorem.

The application of the methods of statistical physics to self-gravitating stellar systems is challenging for several reasons: the interparticle forces are long-range, the systems are intrinsically inhomogeneous, and true thermodynamic equilibrium does not exist (e.g. Lynden-Bell & Wood 1968). Thus, in practice, most of our understanding of relaxation in stellar systems comes from analyses based on the following related approximations (Jeans 1913, 1916; Chandrasekhar 1942; Binney & Tremaine 1987; Spitzer 1987):

1. Local approximation: The stellar system is assumed to be infinite and homogeneous, and the force field from the equilibrium system is neglected; thus the unperturbed stellar orbits are straight lines at constant velocity. The validity of this approximation requires that most of the relaxation arises from close encounters, which have impact parameter $b \ll R$ where $R$ is the size of the system. The failure of the local approximation at large impact parameter is why the Coulomb logarithm is poorly determined in stellar systems.

2. Markov approximation: The interactions between stars are treated as sequential binary encounters of negligible duration, so that the evolution of the one-particle distribution function (hereafter $df$) can be treated as a Markov process. This is a reasonable approximation for short-range forces such as those in neutral gases, but more questionable in a stellar system because of the long range of the gravitational force. The Markov approximation is closely related to the local approximation because only local encounters can plausibly be considered to have negligible duration.

3. Diffusion approximation: Most of the relaxation is assumed to arise from weak encounters, in which the orbital deflection is small; in practice this requires $b \gg b_0 \equiv Gm/\sigma^2$ where $m$ and $\sigma$ are a typical stellar mass and velocity. In this approximation, the evolution of the stellar orbits is a diffusion process that can be described by a Fokker-Planck equation. The diffusion approximation is consistent with the local approximation if $b_0 \ll R$, so that most encounters are weak ($b \gg b_0$) while still local ($b \ll R$); in this case the contribution to the relaxation from encounters with $b < b_0$ is smaller than the contribution from $b > b_0$ by of order the Coulomb logarithm $\ln \Lambda = \ln R/b_0$. With the approximations listed so far, the effects of relaxation on a stellar orbit are completely described by a dynamical friction force, which gives the mean change in velocity per
unit time $\langle \Delta v_i \rangle$, and a diffusion tensor, which gives the mean-square change in velocity per unit time $\langle \Delta v_i \Delta v_j \rangle$.

4. Neglect of self-gravity: The self-gravity of the response of the stellar system is neglected. Thus, for example, dynamical friction arises from the gravitational drag exerted on a body by the wake it creates, but most calculations of dynamical friction neglect the effects of the self-gravity of the wake in determining its amplitude and shape. The error introduced by this approximation can be significant: Weinberg (1989) and Hernquist & Weinberg (1989) find that including the self-gravity of the wake can suppress dynamical friction on an orbiting satellite by a factor of 2–3. Errors of this magnitude are often excused because the precise value of the Coulomb logarithm is not known in any case; however, if the orbital frequency of the perturbing body is close to the frequency of a collective mode of the system, then the effects of self-gravity are increased dramatically (e.g. Weinberg 1993).

The goal of this paper is to summarize some of the insights into fluctuations and dissipation in stellar systems that can be derived without making any of these approximations. Our tools will be linear response theory and the fluctuation-dissipation theorem (e.g. Martin 1968, Forster 1975, Landau & Lifshitz 1980, Reichl 1980, Sitenko 1982, Klimontovich 1986, Kubo et al. 1991).

The intimate relation between fluctuation and dissipation in stellar systems was first recognized by Chandrasekhar (1943), who argued that a star’s random walk in velocity due to stochastic gravitational forces must be balanced by a drag force (dynamical friction) if the stochastic process is to leave a Maxwellian distribution invariant. The most general form of this relation is described by the fluctuation-dissipation theorem. This theorem relates the fluctuations in a dynamical system in thermal equilibrium, described by the correlation function, to the rate at which the system absorbs energy from a weak external field, described by the response operator. In the context of stellar systems, the theorem relates the dynamical friction force to the diffusion tensor (cf. eq. [51] below); however, we shall see that this relation is far more general than the usual derivation of either the dynamical friction force or the diffusion tensor individually.

The conceptual importance of the relation between dynamical friction and stochastic forces was recognized by Bekenstein & Maoz (1992) and Maoz (1993), who concentrated on providing a unified derivation of both effects in a homogeneous system; in contrast, we shall focus on the general properties of drag and fluctuations that can be derived without reference to specific systems.

We begin §2 with a review of linear response theory applied to a stationary inhomoge-
neous system, stressing the constraints that causality places on the analytic properties of the response operator; for example, we derive the Kramers-Kronig relations between the Hermitian and anti-Hermitian parts of this operator and show that dissipation in the system is determined by the anti-Hermitian part of the response operator. We also define the polarization and dielectric operators, and derive their relation to the response operator. We employ these operators to examine collective modes of the system in §3, deriving an expression for the energy of a mode in terms of the Hermitian part of the polarization operator, and show that its (Landau) damping rate is determined by the anti-Hermitian part of the polarization operator. In §4 we derive an exact formal expression for the response operator in a general Hamiltonian system, which has a direct analog in quantum systems. For systems described by integrable potentials, we give an explicit form for the polarization operator in terms of the one-particle $df$. We describe fluctuations in the system using the correlation function in §5, deriving the symmetries that are imposed on the correlation function by the principle of microscopic reversibility. In §6 we prove the fluctuation-dissipation theorem, which relates the response operator and the correlation function for an isothermal system. We describe fluctuations in non-isothermal systems and the dressed-particle approximation in §7. In §8 we apply these results to derive general relations between dynamical friction and relaxation in isothermal stellar systems. We end with a discussion in §9, and a summary of the main formulae and results in §10.

Throughout this paper, we shall apply the term “equilibrium” to a stellar system when the one-particle $df$ is a solution of the time-independent collisionless Boltzmann equation; such a system is stationary except for slow evolution due to relaxation. We reserve the term “thermal equilibrium” for a system whose $N$-particle $df$ is an exponential function of the $N$-particle Hamiltonian. Most of our results apply to any equilibrium stellar system, although some apply only to time-reversible systems (in which the Hamiltonian and the equilibrium one-particle $df$ are invariant under time-reversal), to systems in which the Hamiltonian is integrable, or to systems in thermal equilibrium.

2 Linear response theory

In this section we examine the response of an equilibrium stellar system to small potential perturbations, assuming that the induced density depends linearly on the strength of the perturbation through an—as yet unspecified—response operator. It is often difficult or impossible to find an analytic expression for the response operator. Nevertheless, many of its properties follow directly from causality and linearity.
2.1 The response operator

We consider an equilibrium stellar system that is subjected to a small perturbation $\Phi_e(r, t)$ from an external potential. The density perturbation induced in the system, $\rho_s(r, t)$, can be expressed in terms of a linear response operator $R(r, r', \tau)$, defined by

$$\rho_s(r, t) = \int\! dr' dt' R(r, r', t - t') \Phi_e(r', t'); \quad (1)$$

causality requires $R(r, r', \tau) = 0$ for $\tau < 0$. We stress that $\Phi_e$ does not include the gravitational potential arising from the response density $\rho_s$; in this respect the response operator can be contrasted with the polarization operator defined in §2.2 below.

In this section we consider the general properties of the response operator, deferring the derivation of explicit forms for $R(r, r', \tau)$ until §4. We start by taking Fourier transforms, which we denote by replacing the variable $\tau$ by $\omega$, e.g. $\Phi_e(r, \tau) = \int_{-\infty}^{\infty} d\omega \Phi_e(r, \omega) \exp(-i\omega t)$. Equation (1) simplifies to

$$\rho_s(r, \omega) = 2\pi \int\! dr' R(r, r', \omega) \Phi_e(r', \omega). \quad (2)$$

Since $R(r, r', \tau)$ is real, we must have

$$R^*(r, r', \omega) = R(r, r', -\omega). \quad (3)$$

We can analytically continue the response operator so that it is defined for complex frequencies $z$ (cf. eq. 18 with $\omega \to z$). Causality requires that $R(r, r', z)$ has no poles in the upper half plane.

Equation (2) can be written as an operator equation,

$$\rho_s = 2\pi R(\omega) \Phi_e. \quad (4)$$

If we define the inner product as

$$(\psi, \phi) \equiv \int\! dr \psi^*(r) \phi(r) dr, \quad (5)$$

then the adjoint operator $R^\dagger(\omega)$ satisfies

$$(R^\dagger \psi, \phi) = (\psi, R \phi) \quad \text{for all } \psi(r), \phi(r), \quad (6)$$

and is given by

$$R^\dagger(r, r', \omega) = R^*(r', r, \omega). \quad (7)$$
We can always write the response operator in the form

\[ R(r, r', \omega) \equiv R_H(r, r', \omega) + R_A(r, r', \omega), \]  

(8)

defined by

\[ R_H(r, r', \omega) \equiv \frac{1}{2}R(r, r', \omega) + \frac{1}{2}R^*(r', r, \omega), \]
\[ R_A(r, r', \omega) \equiv \frac{1}{2}R(r, r', \omega) - \frac{1}{2}R^*(r', r, \omega). \]

(9)

The operators \( R_H \) and \( R_A \) satisfy the relations

\[ R^\dagger_H(r, r', \omega) = R^*_H(r', r, \omega) = R_H(r', r', \omega), \quad R^\dagger_A(r, r', \omega) = R^*_A(r', r, \omega) = -R_A(r, r', \omega). \]

(10)

Thus \( R_H \) is Hermitian and \( R_A \) is anti-Hermitian. Using equations (3) and (9) it is easy to show that

\[ R^*_H(r, r', \omega) = R_H(r, r', -\omega), \quad R^*_A(r, r', \omega) = R_A(r, r', -\omega). \]

(11)

In terms of time rather than frequency, we may write

\[ R(r, r', \tau) = R_e(r, r', \tau) + R_o(r, r', \tau), \]

(12)

where

\[ R_e(r, r', \tau) = \int d\omega R_H(r, r', \omega)e^{-i\omega\tau}, \quad R_o(r, r', \tau) = \int d\omega R_A(r, r', \omega)e^{-i\omega\tau}. \]

(13)

It is straightforward to show using (11) that \( R_e \) and \( R_o \) are real. Using (10) we can show that

\[ R_e(r, r', \tau) = R_e(r', r, -\tau), \quad R_o(r, r', \tau) = -R_o(r', r, -\tau). \]

(14)

Moreover, since \( R(r, r', -\tau) = 0 \) for \( \tau > 0 \) we must have

\[ R_e(r, r', \tau) = R_o(r, r', \tau), \quad R_e(r, r', -\tau) = -R_o(r, r', -\tau), \quad \tau > 0. \]

(15)

If \( R(r, r', \tau) \) is well-behaved then \( R(r, r', z) \to 0 \) as \( \text{Im}(z) \to \infty \), where \( z \) is the complex frequency. Since \( R(r, r', z) \) has no poles in the upper half plane, we can write

\[ \int_{-\infty}^{\infty} d\omega' \frac{R(r, r', \omega')}{\omega' - \omega + i\eta} = 0, \]

(16)

where \( \eta > 0 \), since we can close the contour at \( \text{Im}(z) = +\infty \). We next make use of the identity

\[ \lim_{\eta \to 0} \frac{1}{x - y + i\eta} = \mathcal{P} \left( \frac{1}{x - y} \right) - i\pi \text{sgn}(\eta)\delta(x - y), \]

(17)
where $\mathcal{P}$ denotes the Cauchy principal value and $\delta$ denotes the Dirac delta function. Then we have

$$R(r, r', \omega) = -\frac{i}{\pi} \int_{-\infty}^{\infty} d\omega' R(r, r', \omega') \mathcal{P}\left(\frac{1}{\omega' - \omega}\right),$$

(18)

Equating the Hermitian and anti-Hermitian components,

$$R_H(r, r', \omega) = -\frac{i}{\pi} \int_{-\infty}^{\infty} d\omega' R_A(r, r', \omega') \mathcal{P}\left(\frac{1}{\omega' - \omega}\right),$$

$$R_A(r, r', \omega) = -\frac{i}{\pi} \int_{-\infty}^{\infty} d\omega' R_H(r, r', \omega') \mathcal{P}\left(\frac{1}{\omega' - \omega}\right).$$

(19)

These are the Kramers-Kronig relations, which follow from causality and require no other assumptions about the dynamics of the stellar system. They can be analytically continued to complex frequencies $z = \omega + i\eta$.

Another relation is obtained by examining the integral

$$\int \frac{z}{z^2 + s^2} R(r, r', z) dz,$$

(20)

where $s$ is real and positive, and the integral is taken along the real axis and closed in the upper half-plane (e.g., Landau & Lifshitz 1980). Since $R(r, r', z)$ has no poles in the upper half-plane, the only contribution to the integral comes from the pole at $z = is$; thus

$$R(r, r', is) = -\frac{i}{\pi} \int_{-\infty}^{\infty} \frac{\omega R(r, r', \omega)}{\omega^2 + s^2} d\omega.$$ 

(21)

Equating the Hermitian and anti-Hermitian components,

$$R_H(r, r', is) = -\frac{i}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\omega R_A(r, r', \omega)}{\omega^2 + s^2},$$

$$R_A(r, r', is) = -\frac{i}{\pi} \int_{-\infty}^{\infty} d\omega \frac{\omega R_H(r, r', \omega)}{\omega^2 + s^2}.$$ 

(22)

### 2.2 The polarization and dielectric operators

It is sometimes useful to introduce a different measure of the linear response: the polarization operator relates the induced density to the total potential produced by an external perturbation, $\Phi_t = \Phi_e + \Phi_s$,

$$\rho_s(r, t) = \int dr' dt' P(r, r', t - t') \Phi_t(r', t'),$$

(23)
where $\Phi_s$ is associated with the induced density $\rho_s$ by Poisson’s equation, $\nabla^2 \Phi_s = 4\pi G \rho_s$. In frequency space $P(\mathbf{r}, \mathbf{r}', \omega)$ has the same analytic properties described in §2.1 for $R(\mathbf{r}, \mathbf{r}', \omega)$; for example, $P(\mathbf{r}, \mathbf{r}', \omega)$ satisfies Kramers-Kronig relations. In general $P$ is easier to compute explicitly than $R$ because it does not depend on the self-gravity of the response, which is already included in $\Phi_t$.

It is straightforward to show that the two operators are related by the following nonlinear integral equations,

\[
R(\mathbf{r}, \mathbf{r}', \omega) = P(\mathbf{r}, \mathbf{r}', \omega) + 2\pi \int d\mathbf{x} d\mathbf{x}' P(\mathbf{r}, \mathbf{x}, \omega) \Psi(\mathbf{x}, \mathbf{x}') R(\mathbf{x}', \mathbf{r}', \omega),
\]

\[
P(\mathbf{r}, \mathbf{r}', \omega) = P(\mathbf{r}, \mathbf{r}', \omega) + 2\pi \int d\mathbf{x} d\mathbf{x}' R(\mathbf{r}, \mathbf{x}, \omega) \Psi(\mathbf{x}, \mathbf{x}') P(\mathbf{x}', \mathbf{r}', \omega),
\]

(24)

where

\[
\Psi(\mathbf{x}, \mathbf{x}') = -\frac{G}{|\mathbf{x} - \mathbf{x}'|}
\]

is the Coulomb interaction potential. These can be written in operator notation as

\[
R(z) = P(z) + 2\pi P(z) \Psi R(z) = P(z) + 2\pi R(z) \Psi P(z).
\]

(26)

Note that $\Psi$ is self-adjoint, $\Psi^\dagger = \Psi$.

Poisson’s equation may be written

\[
\Phi_s(\mathbf{r}, \omega) = \int d\mathbf{x} \Psi(\mathbf{r}, \mathbf{x}) \rho_s(\mathbf{x}, \omega) = 2\pi \int d\mathbf{x} d\mathbf{r}' \Psi(\mathbf{r}, \mathbf{x}) P(\mathbf{x}, \mathbf{r}', \omega) \Phi_t(\mathbf{r}', \omega).
\]

(27)

Thus the external potential is related to the total potential by

\[
\Phi_e(\mathbf{r}, \omega) = \int d\mathbf{r}' D(\mathbf{r}, \mathbf{r}', \omega) \Phi_t(\mathbf{r}', \omega),
\]

(28)

where

\[
D(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}') - 2\pi \int d\mathbf{x} \Psi(\mathbf{r}, \mathbf{x}) P(\mathbf{x}, \mathbf{r}', \omega)
\]

(29)

is the dielectric response operator. The name arises by analogy to the dielectric constant, which similarly relates the total electric field inside a dielectric to the field generated by external sources. In operator notation

\[
\Phi_e = D(z) \Phi_t \quad \text{where} \quad D(z) = I - 2\pi \Psi P(z).
\]

(30)

The inverse operator satisfies

\[
\Phi_t = D^{-1}(z) \Phi_e, \quad D^{-1}(z) = I + 2\pi \Psi R(z).
\]

(31)
This is easily verified by computing $DD^{-1}$ and using equation (26). The density that generates the external potential is related to the total density by

$$\rho_t = \Lambda(z)\rho_e \quad \text{where} \quad \Lambda(z) = I + 2\pi R(z)\Psi,$$

and $\Phi_e = \Psi \rho_e$. Its inverse is

$$\rho_e = \Lambda^{-1}(z)\rho_t \quad \text{where} \quad \Lambda^{-1}(z) = I - 2\pi P(z)\Psi.$$  

The operators $D^{-1}(z)$ and $\Lambda(z)$ are singular when the response operator $R(z)$ is singular, which occurs at the frequencies of the collective modes of the stellar system (§3).

Two further identities can be derived using (26),

$$P(z) = R(z)D(z),$$

$$R(z) = \Lambda(z)P(z).$$

The operators $R$, $P$, $D$ and $\Lambda$ generally do not commute, and are related through the nonlinear integral equations above. These expressions simplify, however, for an infinite homogeneous stellar system. In this case, the operators are translationally invariant: the spatial dependence of $D(r, r', \omega)$, for example, enters only through $r - r'$. Thus we can define a spatial Fourier transform,

$$D(k, \omega) = \int d(r - r') D(r - r', \omega)e^{-i k \cdot (r - r')},$$

so that operating with $R$, $P$, $D$ or $\Lambda$ reduces to scalar multiplication by a function of the wavelength $k$ and the wavenumber $\omega$. In particular for the gravitational Coulomb potential given by (25),

$$\Psi(k) = -\frac{4\pi G}{k^2}.$$  

Then,

$$D(k, \omega) = 1 + \frac{8\pi^2 G}{k^2} P(k, \omega),$$

$$R(k, \omega) = \frac{P(k, \omega)}{D(k, \omega)},$$

$$\Lambda(k, \omega) = \frac{R(k, \omega)}{P(k, \omega)} = \frac{1}{D(k, \omega)}.$$  

Similar expressions for an infinite homogeneous electron plasma follow with the substitution $Gm^2 \rightarrow -e^2$, where $-e$ is the charge of the electron.
2.3 Work done by an external potential

We gain insight into the response operator by considering the work done on the stellar system by an external potential \( \Phi_e(r, t) \); we assume for simplicity that \( \Phi_e \to 0 \) as \( t \to \pm \infty \). The rate at which the external potential does work on a unit mass is \( -\nabla \Phi_e \cdot v \), where \( v \) is the velocity; thus the rate of doing work on the stellar system is

\[
\dot{E} = -\int dr \nabla \Phi_e \cdot \rho v,
\]

\[
= \int dr \Phi_e \nabla \cdot (\rho v),
\]

\[
= -\int dr \Phi_e(r, t) \frac{\partial \rho(r, t)}{\partial t},
\]

(41)

where the last line follows from the continuity equation. Thus the total energy change is

\[
\Delta E_s = \int \dot{E} dt = -\int dr \int dt \Phi_e(r, t) \frac{\partial \rho(r, t)}{\partial t} = \int dr \int dt \rho(r, t) \frac{\partial \Phi_e(r, t)}{\partial t}.
\]

(42)

Using the relation

\[
\int_{-\infty}^{\infty} A^*(t)B(t)dt = 2\pi \int_{-\infty}^{\infty} A^*(\omega)B(\omega)d\omega
\]

(43)

and the definition of the response operator (2) we have

\[
\Delta E_s = 2\pi i \int dr \omega d\omega \rho(r, \omega) \Phi^*_e(r, \omega)
\]

\[
= (2\pi)^2 i \int dr dr' \omega d\omega R(r, r', \omega) \Phi_e(r', \omega) \Phi^*_e(r, \omega).
\]

(44)

The contribution of the Hermitian component of the response operator to this integral vanishes, so that

\[
\Delta E_s = (2\pi)^2 i \int dr dr' \int_{-\infty}^{\infty} \omega d\omega R_A(r, r', \omega) \Phi_e(r', \omega) \Phi^*_e(r, \omega)
\]

\[
= 8\pi^2 i \int dr dr' \int_{0}^{\infty} \omega d\omega R_A(r, r', \omega) \Phi_e(r', \omega) \Phi^*_e(r, \omega).
\]

(45)

An important special case of this formula occurs when the external potential is nearly monochromatic, with frequency \( \omega_0 \). Then we may write

\[
\Delta E_s = 8\pi^2 i \omega_0 \int dr dr' R_A(r, r', \omega_0) \int_{0}^{\infty} d\omega \Phi_e(r', \omega) \Phi^*_e(r, \omega)
\]

\[
= 4\pi i \omega_0 \int dr dr' R_A(r, r', \omega_0) \int_{-\infty}^{\infty} dt \Phi_e(r', t) \Phi^*_e(r, t).
\]

(46)
For example, suppose that the external potential has the form
\[ \Phi_e(r, t) = g(t) \text{Re} \left[ \phi_e(r) e^{-i\omega_0 t} \right], \tag{47} \]
where the amplitude \( g(t) \) is assumed to be real, vanishingly small in the distant past and future, and varies slowly in the sense that \( |\dot{g}/g| = O(\epsilon) \ll |\omega_0| \). Then the component of \( \Phi_e \) with frequency near \( +\omega_0 \) is \( \frac{1}{2} g(t) \phi_e(r) \exp(-i\omega_0 t) \), and thus
\[ \Delta E_s = \pi i \omega_0 \int dt g^2(t) (\phi_e, R_A \phi_e). \tag{48} \]

It is worthwhile to re-derive this result another way. We begin with the last line of equation (\[\text{11}\]). The induced density may be written \( \rho_s(r, t) = \text{Re}[\rho_s(r, t)e^{-i\omega_0 t}] \), where
\[ \rho_s(r, t) = \iint_0^\infty d\tau R(r, r', t) \phi_e(r') g(t - \tau) e^{i\omega_0 \tau}. \tag{49} \]
Since \( g(t) \) changes slowly, we can expand \( g(t - \tau) = g(t) - \tau \dot{g}(t) + O(\epsilon^2) \) in (\[\text{19}\]), so that
\[ \rho_s(r, t) = 2\pi g(t) \int d\tau' R(r, r', \omega_0) \phi_e(r') + 2\pi i \dot{g}(t) \int d\tau' \frac{\partial R}{\partial \omega}(r, r', \omega_0) \phi_e(r') + O(\epsilon^2). \tag{50} \]
Thus \( \partial \rho/\partial t = \text{Re}[(-i\omega_0 \rho_1 + \dot{\rho}_2)e^{-i\omega_0 t}] + O(\epsilon^2) \), where
\[ \rho_1(r, t) = 2\pi g(t) \int d\tau' R(r, r', \omega_0) \phi_e(r'), \quad \dot{\rho}_2(r, t) = 2\pi i \dot{g}(t) \int d\tau' \frac{\partial \omega R}{\partial \omega}(r, r', \omega_0) \phi_e(r'). \tag{51} \]
We average the rate of doing work over one cycle of the external potential, denoting this average by \( \langle \cdot \rangle \). The time average of the integrand in (\[\text{11}\]) can be written
\[ \langle \Phi_e \frac{\partial \rho}{\partial t} \rangle = -\frac{i\omega_0}{4} (\dot{\phi}_e^* \rho_1 - \phi_e \rho_1^*) + \frac{1}{4} (\dot{\phi}_e^* \dot{\rho}_2 + \phi_e \dot{\rho}_2^*). \tag{52} \]
Finally, writing \( R \) in terms of its Hermitian and anti-Hermitian components, using equations (\[\text{10}\]) and (\[\text{51}\]), we find
\[ \langle \dot{E} \rangle = W + \dot{E}_{\text{int}}, \tag{53} \]
where
\[ W = \pi i \omega_0 \int d\tau d\tau' \phi_e^*(r) R_A(r, r', \omega_0) \phi_e(r') = \pi i \omega_0 (\phi_e, R_A \phi_e), \tag{54} \]
\[ E_{\text{int}} = -\frac{\pi}{2} \int d\tau d\tau' \phi_e^*(r) \frac{\partial \omega R_H}{\partial \omega}(r, r', \omega_0) \phi_e(r') = -\frac{\pi}{2} (\phi_e, \frac{\partial \omega R_H}{\partial \omega} \phi_e), \tag{55} \]
and \( g(t) \) is taken to be unity.

The quantity \( W \) is related to the energy change derived earlier through \( \Delta E_s = \int W dt \); \( W \) involves the anti-Hermitian part of the response operator and is present even if the external
potential is maintained at constant amplitude \( (g(t) = \text{constant}) \). It represents the rate of doing work on the stellar system that is required to maintain the periodic potential; thus the anti-Hermitian response \( R_A \) is associated with energy absorption (dissipation) by the stellar system.\(^1\) In collisionless stellar systems with integrable potentials, the work done by the external potential is absorbed by particle resonances or collective modes with frequencies near \( \omega_0 \) (cf. §4.3).

The term \( E_{\text{int}} \) in (53) represents the work required to build up the response to the time-varying field. This work must be done even when the dissipation associated with \( R_A \) is small or zero. We can thus regard \( E_{\text{int}} \) as the total energy associated with the interaction of the external field and the stellar system. If the response operator \( R_H \) is non-singular at the perturbing frequency, this energy will be recovered by the external perturber if the potential is turned off adiabatically.

3 Collective modes

In the previous section we have considered the induced density response of a stellar system that is subjected to an external potential. However, a stellar system may also support a self-induced response even when no external perturbation is present \((\rho_s \neq 0 \text{ even when } \Phi_e = 0)\). Such a response is called a collective mode; equation (28) implies that the potential \( \Phi_s \) associated with a collective mode satisfies the dispersion relation

\[
\int d\mathbf{r}' D(\mathbf{r}, \mathbf{r}', z) \Phi_s(\mathbf{r}', z) = 0 \quad \text{or} \quad D(z) \Phi_s = 0,
\]

where \( z \) is the complex frequency. The response operator \( R \) is singular at the eigenfrequency of a collective mode (cf. eq. 4 and Kalnajs 1971).

For an infinite, homogeneous medium the collective modes are plane waves, \( \Phi_s \propto \exp(ik \cdot \mathbf{r}) \), and equation (56) reduces to an algebraic dispersion relation, \( D(k, z) = 0 \).

3.1 The mode energy

\(^1\)We refer to this process as energy absorption or dissipation; however, in some collisionless systems energy can be emitted rather than absorbed by this process.
We can use the results of §2.3 to determine the energy of a collective mode in the case where the imaginary component of the eigenfrequency is small. We imagine perturbing the stellar system with an external potential of the form (47), where $\omega_0$ is close to the real part of the eigenfrequency of the mode. Averaging equation (41) over one period $2\pi/\omega_0$ and writing the external potential in terms of the total and response potentials, $\Phi_e = \Phi_t - \Phi_s$, we find

$$\langle \dot{E} \rangle = -\int dr \left\langle \Phi_e \frac{\partial \rho}{\partial t} \right\rangle = \int dr \left\langle \Phi_s \frac{\partial \rho}{\partial t} \right\rangle - \int dr \left\langle \Phi_t \frac{\partial \rho}{\partial t} \right\rangle. \quad (57)$$

The first term can be written $dU_m/dt$ where

$$U_m = \frac{1}{2} \int dr \langle \rho \Phi_s \rangle = \frac{1}{8} \left[ (\rho_s, \phi_s) + (\phi_s, \rho_s) \right] = \frac{\pi}{4} [(P \phi_t, \phi_s) + (\phi_s, P \phi_t)] \quad (58)$$

is the potential energy of the induced disturbance. The second term can be evaluated along the lines of equations (49)–(53):

$$-\int dr \left\langle \Phi_t \frac{\partial \rho}{\partial t} \right\rangle = W_m + K_m, \quad (59)$$

where

$$W_m = \pi i \omega_0 \int dr dr' \phi_t^*(r) P_A(r, r', \omega_0) \phi_t(r') = \pi i \omega_0 (\phi_t, P_A \phi_t), \quad (60)$$

$$K_m = -\frac{\pi}{2} \int dr dr' \phi_t^*(r) \frac{\partial \omega}{\partial \omega} P_H(r, r', \omega_0) \phi_t(r') = -\frac{\pi}{2} \left( \phi_t, \frac{\partial \omega}{\partial \omega} \phi_t \right). \quad (61)$$

Since $\omega_0$ is close to the eigenfrequency of the mode, we expect that the response of the system is strong, $|\Phi_s| \gg |\Phi_e|$. Thus to a good approximation we can replace $\phi_t$ by $\phi_s = \phi_t - \phi_e$, and we find that the total energy associated with a collective mode is

$$E_m = K_m + U_m = -\frac{\pi}{2} \left( \phi_s, \frac{\partial \omega}{\partial \omega} \phi_s \right) + \frac{\pi}{2} (\phi_s, P_H \phi_s) = -\frac{\pi}{2} \omega_0 \left( \phi_s, \frac{\partial P_H}{\partial \omega} \phi_s \right), \quad (62)$$

and the energy is dissipated at a rate

$$W_m = \pi i \omega_0 (\phi_s, P_A \phi_s). \quad (63)$$

Since $U_m$ is the potential energy and $E_m$ is the total energy, we may identify $K_m$ as the kinetic energy of the mode.

The quantity $W_m$ represents the rate of absorption of energy from the mode by the stellar system. The amplitude of the mode varies as $\exp(\eta t)$, where $\eta$ is the imaginary part of the eigenfrequency; thus

$$\eta = \frac{1}{2E_m} \frac{dE_m}{dt} = -\frac{W_m}{2E_m} = \frac{i(\phi_s, P_A \phi_s)}{(\phi_s, P_H \phi_s)}, \quad (64)$$
where $P_{H,\omega} = \frac{\partial P_H}{\partial \omega}$. These results are only valid in the limit of weak damping, $|\eta| \ll |\omega_0|$, since we have assumed that the eigenfrequency of the mode is close to the real frequency $\omega_0$.

Equation (64) can also be derived directly from the dispersion relation. Taking the inner product of (56) with the density of the mode gives

$$\langle \rho_s, D(z_0)\phi_s \rangle = 0,$$

(65)

where $z_0 = \omega_0 + i\eta$ is the complex eigenfrequency of the mode. We write $D$ in terms of its Hermitian and anti-Hermitian parts, and assume that $\eta$ is small so that we can expand to first order in $\eta$,

$$\langle \rho_s, D_H(\omega_0)\phi_s \rangle + \langle \rho_s, D_A(\omega_0)\phi_s \rangle + i\eta \langle \rho_s, D_{H,\omega}(\omega_0)\phi_s \rangle = 0.$$  

(66)

It is straightforward to show that the first term is real and the remaining terms are purely imaginary. Equating the imaginary parts to zero and solving for $\eta$, we recover (64).

4 The response operator for a Hamiltonian system

We now examine the dynamics of the stellar system in more detail. We consider a system composed of $N$ stars, with phase-space coordinates $z_i = (r_i, v_i)$, $i = 1, \ldots, N$ (note that we define phase space using velocity, not momentum). For simplicity we assume that all of the stars are identical, with mass $m$, although the results we derive can be generalized to a range of stellar masses. We denote the coordinates and Hamiltonian of the system in 6$N$-dimensional phase space by $Z \equiv (z_1, \ldots, z_N)$ and $mH_0(Z)$, where

$$H_0(Z) = \frac{1}{2} \sum_{i=1}^{N} v_i^2 + \frac{1}{2} m \sum_{i \neq j} \Psi(r_i - r_j);$$

(67)

here $\Psi$ is given by equation (25).

We consider an ensemble of stellar systems, described by a $N$-particle distribution function $df$ $f(Z, t)$, where $f$ is a symmetric function of the $N$ variables $z_1, \ldots, z_N$, normalized so that $\int f(Z, t)dZ = 1$. Thus the ensemble average of any phase function $u(Z)$ may be written $\langle u \rangle = \int u(Z)f(Z, t)dZ$. The evolution of the $df$ is described by Liouville’s equation

$$\frac{\partial f}{\partial t} + [f, H_0] = 0,$$

(68)

where $[ \cdot, \cdot ]$ is a Poisson bracket.
The trajectory of the system $\tilde{Z}(t)$ satisfies Hamilton’s equations

$$\frac{d\tilde{Z}}{dt} = [Z, H_0]_Z. \tag{69}$$

We shall use $Z_\tau$ as shorthand for the image of $Z$ after time $\tau$ under the Hamiltonian flow (39); thus, if $\tilde{Z}(t)$ is a trajectory and $Z = \tilde{Z}(t_0)$, then $Z_\tau = \tilde{Z}(t_0 + \tau)$.

We may derive a generalized response operator for two arbitrary phase functions $u_s(Z)$, $v_{s'}(Z)$, where $s$ is a parameter that labels these functions. Imagine that we apply a small external perturbation $mH_1(Z, t)$ to the stellar system. We suppose that $mH_1$ can be written in the form

$$mH_1(Z, t) = \int d's'v_{s'}(Z)X(s', t). \tag{70}$$

The induced perturbation in the expectation of the phase function $u_s$ may then be written

$$u_{1s}(t) = \int ds'dt'R_{uv}(s, s', t - t')X(s', t'), \tag{71}$$

where $R_{uv}(s, s', t - t')$ is a generalized response operator (cf. eq. 4).

The perturbation to the $df$ induced by $H_1$ is written $f_1(Z, t)$ and satisfies the linearized Liouville equation,

$$\frac{df_1}{dt} = \frac{\partial f_1}{\partial t} + [f_1, H_0] = -[f_0, H_1], \tag{72}$$

where $d/dt$ denotes the Lagrangian derivative along the unperturbed trajectory. The formal solution to this equation is

$$f_1(Z, t) = -\int_0^\infty d\tau[f_0, H_1(t - \tau)]Z_{\tau}. \tag{73}$$

Thus we may write

$$u_{1s}(t) = \int dZf_1(Z, t)u_s(Z)$$

$$= -\int dZu_s(Z)\int_0^\infty d\tau f_0, H_1(t - \tau)]Z_{\tau}$$

$$= -\frac{1}{m}\int dZu_s(Z)\int ds'\int_0^\infty d\tau X(s', t - \tau)[f_0, v_{s'}]Z_{\tau}. \tag{74}$$

Thus the response operator is

$$R_{uv}(s, s', \tau) = -\frac{\Theta(\tau)}{m}\int dZu_s(Z)[f_0, v_{s'}]Z_{\tau}, \tag{75}$$
where $\Theta$ is the step function. Using the identity $A[B, C] = [AB, C] - B[A, C]$ this can be simplified to

$$R_{uv}(s, s', \tau) = \Theta(\tau) \frac{m}{\int dZ f_0(Z)} [u_s(Z), v_{s'}(Z_{-\tau})]$$

$$= \Theta(\tau) \frac{m}{\int dZ f_0(Z)} [u_s(Z), v_{s'}(Z)]$$

$$= \Theta(\tau) \frac{m}{\int dZ f_0(Z)} ([u_s(t+\tau), v_{s'}(t)]). \quad (76)$$

That is, the response operator is just the ensemble-averaged Poisson bracket of $u_s(t+\tau)$ and $v_{s'}(t)$.

The density-density response operator is defined by setting $X(r, t) = \Phi_e(r, t)$, where $\Phi_e$ is the external perturbing potential, and $u_r(t) = v_r(t) = \rho(r, t)$ where

$$\rho(r, t) = m \sum_{i=1}^N \delta[r_i(t) - r], \quad (77)$$

is the exact density distribution for the $N$-body system with coordinates $z_i(t) = (r_i(t), \dot{r}_i(t))$. Thus

$$R(r, r', \tau) = \Theta(\tau) \frac{m}{\int dZ f_0(Z)} ([\rho(r, t+\tau), \rho(r', t)]). \quad (78)$$

The Poisson bracket is taken with respect to the phase-space coordinates $Z = (z_1, \ldots, z_N)$ at time $t$, which determine $r_i(t)$ and $r_i(t+\tau)$ and hence implicitly determine $\rho(r, t+\tau)$ and $\rho(r', t)$ through (77).

A nearly identical result to (76) was first derived by Kubo (1957) in the context of quantum statistical mechanics. In this case the $N$-body $d\mathcal{F}$ is replaced by the quantum mechanical density operator, which satisfies an equation of motion very similar to Liouville’s equation (68); the primary difference being that the Poisson bracket is replaced by a quantum mechanical commutator.

The expression (78) for the response operator is formally exact, but difficult or impossible to evaluate in practice for realistic stellar systems. However, the analogous expression for the polarization operator is simpler. In this case we examine the response of the system to a total potential, and do not have to consider the additional gravitational forces arising from perturbations to the stellar orbits. Thus the Hamiltonian (67) can be written as

$$H_0(Z) = \frac{1}{2} \sum_{i=1}^N v_i^2 + \sum_i \Phi_i(r_i, t); \quad (79)$$

here $\Phi_i(r, t) = m \sum_{j \neq i} j \Psi[r - r_{0j}(t)]$ is the potential from all the stars other than $i$, moving along their unperturbed orbits. This Hamiltonian is separable, $H_0(Z) = \sum_{i=1}^N H_i(z_i)$; the
responses of different stars are independent and hence we can work in 6-dimensional phase space instead of $6N$-dimensional phase space. There is a further simplification in the limit of large $N$, where the potentials $\Phi_i$ can be replaced by the ensemble-averaged potential $\Phi_0$ (the “mean-field” approximation, cf. §6), and the analog to equation (78) is

$$P(r, r', \tau) = \Theta(\tau) \int d\mathbf{x} d\mathbf{v} F_0(\mathbf{x}, \mathbf{v})[\delta(\mathbf{x}_\tau(\mathbf{x}, \mathbf{v}) - r), \delta(\mathbf{x} - \mathbf{r}')] \tag{80}$$

Here $F_0(\mathbf{x}, \mathbf{v})$ is the one-particle df, defined so that $F_0(\mathbf{x}, \mathbf{v}) d\mathbf{x} d\mathbf{v}$ is the (ensemble-averaged) mass in the phase-space volume $d\mathbf{x} d\mathbf{v}$ (in contrast to the $N$-particle df, the integral of $F_0$ over phase space is normalized to the total mass rather than to unity). The Poisson bracket is taken with respect to $\mathbf{x}, \mathbf{v}$; and $\mathbf{x}_\tau(\mathbf{x}, \mathbf{v})$ is the position at time $\tau$ of the particle that was at $(\mathbf{x}, \mathbf{v})$ at time 0.

### 4.1 Symmetries of the response operator

For particular systems, the response and polarization operators may have symmetries in addition to those discussed in §2.1.

A common situation is that the equilibrium stellar system is invariant under time reversal (e.g. non-rotating galaxies). Under time reversal stellar positions remain the same, while velocities are reversed, $(\mathbf{r}_i, \mathbf{v}_i) \rightarrow (\mathbf{r}_i, -\mathbf{v}_i)$. The density transforms as $\rho(\mathbf{r}, t) \rightarrow m \sum_i \delta(\mathbf{r} - \mathbf{r}_i(-t)) = \rho(\mathbf{r}, -t)$, while the Poisson bracket reverses sign. Consequently, under time reversal

$$\langle [\rho(\mathbf{r}, t + \tau), \rho(\mathbf{r}', t)] \rangle \rightarrow -\langle [\rho(\mathbf{r}, -t - \tau), \rho(\mathbf{r}', -t)] \rangle$$

$$= -\langle [\rho(\mathbf{r}, t), \rho(\mathbf{r}', t + \tau)] \rangle$$

$$= \langle [\rho(\mathbf{r}', t + \tau), \rho(\mathbf{r}, t)] \rangle \tag{81}$$

Comparing to equation (78), for systems invariant under time reversal we must have

$$R(\mathbf{r}, \mathbf{r}', \tau) = R(\mathbf{r}', \mathbf{r}, \tau) \tag{82}$$

This result implies in turn that the Hermitian component of the response operator $R_H(\mathbf{r}, \mathbf{r}', \omega)$ is real and even in $\omega$, while the anti-Hermitian component $R_A(\mathbf{r}, \mathbf{r}', \omega)$ is imaginary and odd in $\omega$. Similar relations can be proved for the polarization operator $P(\mathbf{r}, \mathbf{r}', \tau)$, starting from equation (80).

Phase functions generally transform under time reversal as $u[\tilde{Z}(t)] \rightarrow \epsilon_u^T u[\tilde{Z}(-t)]$, where $\epsilon_u^T = +1$ for mass or energy density, while $\epsilon_u^T = -1$ for momentum or angular momentum.
density (e.g. Martin 1968). For systems invariant under time reversal, the general response function satisfies,

\[ R_{uv}(r, r', \tau) = \epsilon_u^T \epsilon_v^T R_{vu}(r', r, \tau). \] (83)

Likewise for stellar systems invariant under parity transformations \((r_i \rightarrow -r_i)\),

\[ R_{uv}(r, r', \tau) = \epsilon_u^P \epsilon_v^P R_{uv}(-r, -r', \tau). \] (84)

where \(\epsilon_u^P\) is the signature of \(u(Z)\) under parity transformation.

### 4.2 Goodman’s stability criterion

The results of the previous section can be used to re-derive an elegant instability test for a time-reversible stellar system (Goodman 1988). Consider the operator

\[ W(z) = P(z) - \frac{1}{2\pi} \Psi^{-1}; \] (85)

where \(\Psi^{-1} = \nabla^2/(4\pi G)\) (cf. eq. 25). A collective mode \(\phi_s\) satisfies \(W(z)\phi_s = 0\). Assume that the frequency \(z = is\) where \(s\) is real and positive. Then \(P(is)\) is real, and hence Hermitian if the stellar system is time-reversible. Similarly \(W\) is real and Hermitian. Therefore the eigenvalues \(\lambda(s)\) of \(W(is)\) are real. Moreover as \(s \rightarrow \infty\), \(P(is) \rightarrow 0\) so \(W \rightarrow -(2\pi)^{-1} \Psi^{-1}\), which is positive-definite for the inner product \((\cdot \mid \cdot)\). Now let \(s\) decrease from infinity. If for any \(s_0 > 0\) there is a function \(\phi\) such that

\[ (\phi, W(is_0)\phi) < 0, \] (86)

then \(W(is_0)\) is no longer positive-definite. Thus some of its eigenvalues \(\lambda(s_0)\) are negative. Therefore there is some \(s_1 > s_0\) for which one of the eigenvalues is zero. Hence there is an unstable mode with growth rate \(s_1\). In other words a necessary condition for stability in time-reversible stellar systems is that

\[ 2\pi (\phi, P(is_0)\phi) < (\phi, \Psi^{-1}\phi) \] (87)

for all \(\phi\) and for all \(s_0 > 0\). This is a slight generalization of Goodman’s result, which was derived only for systems with integrable potentials.
4.3 The polarization operator in action-angle variables

The polarization operator (80) can be evaluated explicitly if the potential $\Phi_0(r)$ is regular, so that phase space can be described by action-angle coordinates $(I, w)$. The orbits are given by $I=\text{constant}$, $w = \Omega t + w_0$ where $\Omega = \partial H_0/\partial I$, and $mH_0$ is the Hamiltonian corresponding to $\Phi_0$. Jeans’s theorem states that the one-particle $df$ depends only on the actions, $F_0 = F_0(I)$. The canonical transformation from $(x, v)$ to $(I, w)$ conserves the volume element in phase space: $dxdv = dwdI$.\footnote{We define the canonical momentum and the actions without the usual factor $m$, so these variables have dimensions (velocity) and (velocity)$\times$(length) respectively.}

Consider a single star of unit mass with action-angle coordinates $(I, w)$; the corresponding spatial coordinate is $x(I, w)$. Formally, the spatial density of the star can be written

$$p(r|I, w) = \delta[r - x(I, w)].$$  \hspace{1cm} (88)

Since $w$ is cyclic, we can expand this in a Fourier series,

$$p(r|I, w) = \sum_l p_l(r|I)e^{il\cdot w} \quad \text{where} \quad p_l(r|I) = \frac{1}{(2\pi)^3} \int dw e^{-il\cdot w}p(r|I, w).$$  \hspace{1cm} (89)

The $p_l$’s are projection operators which give the Fourier components of any function of position $g(r)$ that is expanded in a Fourier series of the form $\sum_l g_l(I) \exp(\text{i}l\cdot w)$:

$$g_l(I) = \frac{1}{(2\pi)^3} \int dw g[x(I, w)]e^{-il\cdot w}$$

$$= \frac{1}{(2\pi)^3} \int dw dr g(r)\delta[r - x(I, w)]e^{-il\cdot w}$$

$$= \frac{1}{(2\pi)^3} \int dw dr g(r)p(r|I, w)e^{-il\cdot w}$$

$$= \int dr p_l(r|I)g(r).$$  \hspace{1cm} (90)

It is straightforward to prove the following useful identity: if $h(I)$ is any function of the actions,

$$(2\pi)^3 \sum_l \int dI h(I)p_l^*(r|I)p_l(r'|I) = \delta(r - r') \int dv h[I(r, v)].$$  \hspace{1cm} (91)

In particular, if $h(I)$ is the one-particle $df$ $F_0(I)$, then

$$(2\pi)^3 \sum_l \int dI F_0(I)p_l^*(r|I)p_l(r'|I) = \delta(r - r') \rho_0(r).$$  \hspace{1cm} (92)
The polarization operator \( \mathbf{P} \) can now be written as
\[
P(\mathbf{r}, \mathbf{r}', \tau) = \Phi(\tau) \int d\mathbf{w} \mathbf{F}_0(\mathbf{I}) \sum_{l,m} \left[ p_l^r(\mathbf{r}|\mathbf{I}) e^{-i(l\cdot\mathbf{w} + \Omega \tau)} \right] p_m(\mathbf{r}'|\mathbf{I}) e^{im\cdot\mathbf{w}}
\]
\[
= -(2\pi)^3 \Phi(\tau)i \int d\mathbf{F}_0(\mathbf{I}) \sum_l 1 \cdot \frac{\partial}{\partial \mathbf{I}} \left[ p_l^r(\mathbf{r}|\mathbf{I}) p_l(\mathbf{r}'|\mathbf{I}) e^{-i l \cdot \Omega \tau} \right]
\]
\[
= (2\pi)^3 \Phi(\tau)i \int d\mathbf{I} \sum_l 1 \cdot \frac{\partial \mathbf{F}_0(\mathbf{I})}{\partial \mathbf{I}} p_l^r(\mathbf{r}|\mathbf{I}) p_l(\mathbf{r}'|\mathbf{I}) e^{-i l \cdot \Omega \tau},
\]
where the last line follows through integration by parts.

The Fourier transform of the polarization operator is
\[
P(\mathbf{r}, \mathbf{r}', \omega) = (2\pi)^2 \sum_l \int d\mathbf{I} p_l^r(\mathbf{r}|\mathbf{I}) p_l(\mathbf{r}'|\mathbf{I}) \frac{1}{1 \cdot \Omega - i\epsilon - \omega} \cdot \frac{\partial \mathbf{F}_0}{\partial \mathbf{I}},
\]
where \( \epsilon \) is a small positive number. This can be split into Hermitian and anti-Hermitian components using the identity \((\ref{eq:identity})\):
\[
P(\mathbf{r}, \mathbf{r}', \omega) \equiv P_H(\mathbf{r}, \mathbf{r}', \omega) + P_A(\mathbf{r}, \mathbf{r}', \omega),
\]
where \((\text{cf. eq. } \ref{eq:identity})\)
\[
P_H(\mathbf{r}, \mathbf{r}', \omega) \equiv (2\pi)^2 \sum_l \int d\mathbf{I} p_l^r(\mathbf{r}|\mathbf{I}) p_l(\mathbf{r}'|\mathbf{I}) \cdot \frac{1}{1 \cdot \Omega - \omega} \cdot \frac{\partial \mathbf{F}_0}{\partial \mathbf{I}} P \left( \frac{1}{1 \cdot \Omega - \omega} \right),
\]
\[
P_A(\mathbf{r}, \mathbf{r}', \omega) \equiv 4\pi^3 i \sum_l \int d\mathbf{I} p_l^r(\mathbf{r}|\mathbf{I}) p_l(\mathbf{r}'|\mathbf{I}) \cdot \frac{\partial \mathbf{F}_0}{\partial \mathbf{I}} \delta(1 \cdot \Omega - \omega).
\]
Thus \( P_A \) is determined entirely by resonant stars satisfying \( 1 \cdot \Omega = \omega \).

If the one-particle \( \Phi \) depends only on the energy per unit mass \( E, \mathbf{F}_0 = F_0(E) \), then these expressions are simplified; for example,
\[
P_A(\mathbf{r}, \mathbf{r}', \omega) = 4\pi^3 i \omega \sum_l \int d\mathbf{I} p_l^r(\mathbf{r}|\mathbf{I}) p_l(\mathbf{r}'|\mathbf{I}) \frac{d\mathbf{F}_0}{dE} \delta(1 \cdot \Omega - \omega).
\]

Equation \((\ref{eq:energy_dissipation})\) allows us to find an explicit expression for the rate of energy dissipation in a collective mode \((\text{eq. } \ref{eq:energy_dissipation})\),
\[
W_s = -4\pi^4 \omega \sum_l \int d\mathbf{I} |\phi_l(\mathbf{I})|^2 \cdot \frac{\partial \mathbf{F}_0}{\partial \mathbf{I}} \delta(1 \cdot \Omega - \omega),
\]
where the potential of the collective mode has been expanded as \( \phi_s(\mathbf{r}) = \sum_l \phi_l(\mathbf{I}) \exp(i l \cdot \mathbf{w}) \), and we have used \((\ref{eq:energy_dissipation})\). This result is closely related to formulae originally given by Lynden-Bell & Kalnajs \((1972)\) and can also be derived using second-order Lagrangian perturbation theory \((Nelson & Tremaine 1995)\). Similar expressions to those in this subsection are also given by Tremaine & Weinberg \((1984)\), Goodman \((1988)\), and Palmer \((1994)\).
5 Fluctuations

The density and potential of any stellar system fluctuate about their mean local values due to the finite number of stars. These fluctuations are described by the density-density correlation function and its Fourier transform. For an isothermal stellar system, it turns out that the correlation function is directly related to the response function. Thus, dissipation processes described by $R_A(\omega)$ are determined by the fluctuation spectrum of the stellar system—the fluctuation-dissipation theorem.

5.1 The correlation function

We examine the density fluctuations $\delta \rho(r, t)$ in the stellar system,

$$\delta \rho(r, t) = m \sum_i \delta[r_i(t) - r] - \rho_0(r),$$

(99)

where $\rho_0(r) = mN\langle \delta(r_i - r) \rangle$ is the mean density at $r$ and as usual $\langle \cdot \rangle$ denotes an ensemble average. The density fluctuations are characterized by the correlation function

$$C(r, r', \tau) = \langle \delta \rho(r, t + \tau)\delta \rho(r', t) \rangle.$$  

(100)

Since the equilibrium system is stationary, the correlation function is independent of time $t$. If we replace $t$ by $t - \tau$ we derive the symmetry relation

$$C(r, r', \tau) = C(r', r, -\tau).$$

(101)

The Fourier transform of the correlation function is called the dynamic form factor:

$$S(r, r', \omega) = \frac{1}{2\pi} \int d\tau e^{i\omega \tau} C(r, r', \tau) = \frac{1}{2\pi} \int d\tau e^{i\omega \tau} \langle \delta \rho(r, t + \tau)\delta \rho(r', t) \rangle,$$

(102)

Alternatively, we can define $S(r, r', \omega)$ by expressing the correlation function in terms of the frequency transform of the density fluctuations,

$$C(r, r', \tau) = \int d\omega e^{-i\omega \tau} \int d\omega' \langle \delta \rho(r, \omega)\delta \rho(r', \omega') \rangle e^{-i(\omega + \omega')t}.$$  

(103)

In order that the integral on the right side be independent of time $t$, we must have

$$\langle \delta \rho(r, \omega)\delta \rho(r', -\omega') \rangle = S(r, r', \omega)\delta(\omega - \omega').$$

(104)
The dynamic form factor satisfies the symmetry relations

\[ S^*(r, r', \omega) = S(r, r', -\omega), \quad S^*(r, r', \omega) = S(r', r, \omega); \]  

(105)

the first of these follows because the correlation function is real, and the second follows from (101) and implies that the dynamic form factor is Hermitian. Finally, the correlation function at zero time difference is given by integrating \( S(r, r', \omega) \) over all frequencies,

\[ C(r, r', 0) = \int d\omega S(r, r', \omega) = \langle \delta \rho(r, t) \delta \rho(r', t) \rangle; \]  

(106)

\( C(r, r', 0) \) is sometimes referred to as the static form factor.

If the \( ef \) of the stellar system is invariant under time-reversal (cf. §4.2), and the Hamiltonian has the form (67), the correlation function satisfies the principle of microscopic reversibility,

\[ C(r, r', \tau) = C(r', r, \tau) \quad \text{or} \quad C(r, r', \tau) = C(r, r', -\tau). \]  

(107)

This implies in turn that the dynamic form factor \( S(r, r', \omega) \) is real, symmetric in \( r \) and \( r' \), and an even function of \( \omega \).

A simple proof of (107) relies on notation and results from §4. We consider a generalized correlation function associated with any two phase functions \( u_s(Z) \), \( v_s(Z) \), labeled by the parameter \( s \). This may be written

\[ C_{uu}(s, s', \tau) = \langle \delta u_s(t + \tau) \delta v_{s'}(t) \rangle = \int dZ f_0(Z) u_s(Z) v_{s'}(Z) - \langle u_s \rangle \langle v_{s'} \rangle. \]

(108)

The density-density correlation function may be written in the form

\[ C(r, r', \tau) = \int dZ f_0(Z) v_r(Z) v_{r'}(Z) - \rho_0(r) \rho_0(r'), \]

(109)

where \( v_r(Z) = \rho(r, t) \) is defined by equation (77), and \( \rho_0(r) \equiv \langle v_r(Z) \rangle \) is the ensemble-averaged density at \( r \). Now change the dummy variable from \( Z \) to \( \bar{Z} \), where \( \bar{Z} \) is obtained from \( Z \) by reversing all of the velocities. Since \( f_0 \) is time-reversible, we can replace \( f_0(Z) \) by \( f_0(\bar{Z}) \). We can replace the volume element \( d\bar{Z} \) by \( dZ \) because the Jacobian \( |\partial(\bar{Z})/\partial(Z)| = 1 \). Moreover, \( v_r(\bar{Z}) = v_{r'}(\bar{Z}) \) because \( v_r \) depends only on coordinates, not momenta. Finally, \( \bar{(\bar{Z})}_\tau = \bar{Z}_{-\tau} \) because reversing the velocities yields the time-reversed orbit in the Hamiltonian (67); thus \( v_r[(\bar{Z})_\tau] = v_r(Z_{-\tau}) \). Equation (109) becomes

\[ C(r, r', \tau) = \int dZ f_0(Z) v_r(Z_{-\tau}) v_{r'}(Z) - \rho_0(r) \rho_0(r') = C(r, r', -\tau), \]

(110)

and equation (107) then follows from (101).
For some purposes it is useful to define a modified correlation function that removes long-term correlations:

\[
\tilde{C}(\mathbf{r}, \mathbf{r}', \tau) = C(\mathbf{r}, \mathbf{r}', \tau) - C(\mathbf{r}, \mathbf{r}', \infty).
\] (111)

In contrast to most systems examined in statistical mechanics, the correlation function does not vanish at large times in most stellar systems. This is simple to understand in the context of spherical systems. The time-averaged density of each star is an annulus with fixed orientation and fixed inner and outer radii. The time-averaged total density is composed of the sum of the densities from \(N\) such annuli, and hence contains permanent irregularities that are not present in the ensemble-averaged density. We expect a non-zero correlation function at large times unless the system is ergodic.

Correlations between particles are present for two conceptually distinct reasons: random fluctuations in the number of particles in a given small volume, which are present even if the particles move on their unperturbed trajectories; and gravitational interactions between particles, which deflect their mutual orbits. It is sometimes useful to isolate the correlation function arising only from random fluctuations in particle number, which we denote

\[
C^{(0)}(\mathbf{r}, \mathbf{r}', \tau) = \langle \delta \rho(\mathbf{r}, t + \tau) \delta \rho(\mathbf{r}', t) \rangle^{(0)}.
\] (112)

As in equation (80), in the limit of large \(N\) we can replace the exact potential \(\Phi_i\) by the ensemble-averaged potential \(\Phi_0\), and write the correlation function as

\[
C^{(0)}(\mathbf{r}, \mathbf{r}', \tau) = m \int d\mathbf{x} d\mathbf{v} F_0(\mathbf{x}, \mathbf{v}) \delta(\mathbf{x}_r(\mathbf{x}, \mathbf{v}) - \mathbf{r}) \delta(\mathbf{x} - \mathbf{r}') - \frac{1}{N}\rho_0(\mathbf{r}) \rho_0(\mathbf{r}').
\] (113)

The factor proportional to \(1/N\) derives from a near-cancellation of terms analogous to the derivation of equation (118) below.

Because the particle orbits are independent in this approximation, \(\delta \rho(\mathbf{r}, t)\) is a Gaussian random field, and hence its properties are completely described by the correlation function \(C^{(0)}(\mathbf{r}, \mathbf{r}', \tau)\).

5.2 The correlation function in action-angle variables

We can find an explicit expression for the density-density correlation function \(C^{(0)}(\mathbf{r}, \mathbf{r}', \tau)\) if the ensemble-averaged potential \(\Phi_0(\mathbf{r})\) is regular, using the same approximations and notation as in §4.3.
Assuming that the stars move on their unperturbed orbits in the potential \( \Phi_0 \), we may write
\[
\rho(r, t) = m \sum_{i=1}^{N} p[r|I_i, \mathbf{w}_i(t)] = m \sum_{i=1}^{N} p_l(r|I_i)e^{i\Omega_l(t+w_{l0})}. \tag{114}
\]

Since the initial phases \( \mathbf{w}_{i0} \) are uniformly distributed, only the terms with \( l = 0 \) survive in the ensemble average; thus the ensemble-averaged density is
\[
\rho_0(r) = \langle \rho(r, t) \rangle = (2\pi)^3 \int d\mathbf{I}_0 \rho_0(r|\mathbf{I}). \tag{115}
\]

We may now write
\[
\delta \rho(r, t + \tau) \delta \rho(r', t) = m^2 \sum_{i,j=1}^{N} \sum_{l,l'} p^*_l(r|I_i)p_l(r'|I_j)e^{-i[l\cdot(\mathbf{I}_i(t+\tau)+\mathbf{w}_{l0})+l'\cdot(\mathbf{I}_j+t+w_{l'0})]}
- \rho(r, t)\rho_0(r') - \rho_0(r)\rho_0(r') + \rho_0(r)\rho_0(r'). \tag{116}
\]

Now take an ensemble average; since the phases \( \mathbf{w}_{i0} \) are randomly distributed, the average over \( \exp[i(l' \cdot \mathbf{w}_{j0} - l \cdot \mathbf{w}_{i0})] \) will vanish unless either (i) \( i \neq j \) and \( l = l' = 0 \), or (ii) \( i = j \) and \( l = l' \). In case (i) the sum over \( i \neq j \) yields \( N(N-1) \) terms, each of which is equal to \( \rho_0(r)\rho_0(r')/N^2 \). Thus
\[
\langle \delta \rho(r, t + \tau) \delta \rho(r', t) \rangle = m^2 \left( \sum_{i=1}^{N} \sum_{l} p^*_l(r|I_i)p_l(r'|I_i) \exp(-i\Omega_l \cdot \tau) \right) - \frac{1}{N}\rho_0(r)\rho_0(r'). \tag{117}
\]

We can replace the sum over stars by the integral over the one-particle \( df \) to get
\[
C^{(0)}(r, r', \tau) = \langle \delta \rho(r, t + \tau) \delta \rho(r', t) \rangle
= (2\pi)^3 m \sum_{l} \int d\mathbf{I}_0 |p^*_l(r|I)|p_l(r'|I) \exp(-i\Omega \cdot \tau) - \frac{1}{N}\rho_0(r)\rho_0(r'); \tag{118}
\]
the superscript \( "0" \) is a reminder that we have neglected the self-gravity of the density fluctuations by using the unperturbed orbits of the stars. This result can also be derived from equations (88), (89) and (113).

The dynamic form factor (102) becomes
\[
S^{(0)}(r, r', \omega) = (2\pi)^3 m \sum_{l} \int d\mathbf{I}_0 |p^*_l(r|I)|p_l(r'|I)\delta(\mathbf{l} \cdot \mathbf{\Omega} - \omega) - \frac{1}{N}\rho_0(r)\rho_0(r')\delta(\omega). \tag{119}
\]
Thus the dynamic form factor is determined entirely by resonant stars.
It is straightforward to calculate the static form factor \( C^{(0)} \), starting from equation (113):

\[
C^{(0)}(\mathbf{r}, \mathbf{r}', 0) = \int d\omega S^{(0)}(\mathbf{r}, \mathbf{r}', \omega) = m \int d\mathbf{x} d\mathbf{v} F_0(\mathbf{x}, \mathbf{v}) \delta(\mathbf{x} - \mathbf{r}) \delta(\mathbf{x} - \mathbf{r}') - \frac{1}{N} \rho_0(\mathbf{r}) \rho_0(\mathbf{r}')
\]

\[
= m \int d\mathbf{x} \rho_0(\mathbf{x}) \delta(\mathbf{x} - \mathbf{r}) \delta(\mathbf{x} - \mathbf{r}') - \frac{1}{N} \rho_0(\mathbf{r}) \rho_0(\mathbf{r}')
\]

\[
= m \rho_0(\mathbf{r}) \delta(\mathbf{r} - \mathbf{r}') - \frac{1}{N} \rho_0(\mathbf{r}) \rho_0(\mathbf{r}').
\]

(120)

This result also follows immediately from equations (12) and (118).

The correlation function at large times is easily derived from equation (118):

\[
C^{(0)}(\mathbf{r}, \mathbf{r}', \infty) = (2\pi)^3 m \int d\mathbf{I} F_0(\mathbf{I}) p^*_0(\mathbf{r}|\mathbf{I}) p_0(\mathbf{r}'|\mathbf{I}) - \frac{1}{N} \rho_0(\mathbf{r}) \rho_0(\mathbf{r}');
\]

(121)

additional terms would appear if the potential had global resonances such that \( \mathbf{l} \cdot \Omega = 0 \) for non-zero \( \mathbf{l} \) (cf. Rauch & Tremaine 1996). Thus the modified correlation function defined in equation (111) takes on the simpler form

\[
\tilde{C}^{(0)}(\mathbf{r}, \mathbf{r}', \tau) = (2\pi)^3 m \sum_{\mathbf{l} \neq 0} \int d\mathbf{I} F_0(\mathbf{I}) p^*_0(\mathbf{r}|\mathbf{I}) p_0(\mathbf{r}'|\mathbf{I}) \exp(-i\mathbf{l} \cdot \Omega \tau).
\]

(122)

6 Thermal equilibrium and the fluctuation-dissipation theorem

A system with Hamiltonian \( mH_0(\mathbf{Z}) \) is in thermal equilibrium if the \( N \)-particle \( \mathcal{D}\mathcal{F} \) has the form \( f_0(\mathbf{Z}) \propto \exp[-\beta H_0(\mathbf{Z})] \) (in contrast the term “equilibrium” simply denotes that the one-particle \( \mathcal{D}\mathcal{F} \) is a solution of the time-independent collisionless Boltzmann equation). For the usual gravitational force, the interaction potential \( m^2 \Psi(\mathbf{r}) = -Gm^2/|\mathbf{r}| \); in this case \( f_0(\mathbf{Z}) \) diverges exponentially as \( |\mathbf{r}_i - \mathbf{r}_j| \to 0 \) and hence a thermal equilibrium state does not exist. Nevertheless, the useful concept of a thermal equilibrium can be retained for stellar systems, by modifying the interaction potential in one of two ways:

- We can eliminate the interaction potential, setting \( \Psi(\mathbf{r}) = 0 \), so that the gravitational potential is determined entirely by the external potential \( \Phi_{\text{ext}}(\mathbf{r}) \); we then augment \( \Phi_{\text{ext}}(\mathbf{r}) \) by adding the mean potential field of the unperturbed system. This “mean-field” approximation is equivalent to including the self-gravity of the equilibrium stellar
system but neglecting the self-gravity of fluctuations, which is the same as approximation (4) of §1. In this case the thermal equilibrium $df$ is the product of one-particle isothermal $df$s,

$$f_0(Z) \propto \Pi_{i=1}^N \exp[-\beta H(z_i)], \quad \text{where} \quad H(z) = \frac{1}{2}v^2 + \Phi_{\text{ext}}(r); \quad (123)$$

in other words the two-particle correlation function vanishes. This is a plausible model for galaxies, since (i) the one-particle $df$ in galaxies is often approximately isothermal, and (ii) the relaxation times in galaxies are generally much larger than their age, so significant correlations between stars have not had time to develop. However, the mean-field approximation neglects the dynamical effects of the self-gravity of the perturbed density; thus, for example, there is no distinction between the polarization and response operators and no collective modes.

- A more realistic approach is to soften the interaction potential to $m^2\Psi(r) = -Gm^2/(r^2 + b^2)^{1/2}$. The softening length $b$ should be much smaller than the size of the system, so the large-scale equilibrium structure is unaffected; in fact it should also be much less than the typical interstellar separation so that softening does not affect most encounters between stars. On the other hand the softening length should be large enough to suppress the strong correlations between stars that are present if the gravitational force is unsoftened. To see what is required, consider an infinite homogeneous stellar system, for which the equilibrium two-particle $df$ is

$$p^{(2)}(z_1, z_2) \propto \exp\left\{ -\beta \left[ \frac{1}{2}v_1^2 + \frac{1}{2}v_2^2 + m\Psi(r_1 - r_2) \right] \right\} = p^{(1)}(z_1)p^{(1)}(z_2)[1 + g(r_1 - r_2)], \quad (124)$$

where

$$p^{(1)}(z) \propto \exp(-\frac{1}{2}\beta v^2), \quad g(\Delta r) = \exp[-\beta m\Psi(\Delta r)] \quad (125)$$

are the one-particle $df$ and the two-particle correlation function. We would like the two-particle (and higher order) correlation functions to be small, so that the $N$-particle $df$ is the product of one-particle $df$s as in equation (123). This requires that $\max|\beta m\Psi|$ is small, which in turn requires $b \gg b_0 = Gm/\sigma^2$, where $\sigma^2 = 1/\beta$ is the mean-square velocity in one dimension. Fortunately, it is easy to satisfy these conditions in most stellar systems; thus, in the solar neighborhood we might (for example) choose $b = 0.001$ pc, which is much less than the typical interstellar separation of 1 pc but much greater than $b_0 \sim 10^{-5}$ pc.

### 6.1 The fluctuation-dissipation theorem
We examine the relation between the response operator (75) and the correlation function (108) when the stellar system is in thermal equilibrium. If \( f_0(Z) \propto \exp[-\beta H_0(Z)] \),

\[
[f_0, v_{s'}]_{Z_\tau} = -\beta f_0(Z_\tau)[H_0, v_{s'}]_{Z_\tau} = -\beta f_0(Z) \frac{d}{d\tau} v_{s'}(Z_\tau);
\]

(126)
in the second line, the replacement of the Poisson bracket with a time derivative follows from Hamilton’s equations (69), and the argument of \( f_0 \) can be changed from \( Z_\tau \) to \( Z \) because \( H_0 \) and therefore \( f_0 \) is invariant along a trajectory. The response operator may now be written

\[
R_{uv}(s, s', \tau) = \frac{\beta \Theta(\tau)}{m} \frac{\partial}{\partial \tau} \int dZ u_s(Z) f_0(Z) v_{s'}(Z_\tau) = \frac{\beta \Theta(\tau)}{m} \frac{d}{d\tau} \int dZ f_0(Z) u_s(Z_\tau) v_{s'}(Z); \]

(127)
in the second equality we have relabeled the variables \( Z \) and \( Z_\tau \) as \( Z_\tau \) and \( Z \) respectively, and then changed the dummy variable from \( Z_\tau \) to \( Z \); the Jacobian of the transformation is unity by Liouville’s theorem. Finally, using equation (108),

\[
R_{uv}(s, s', \tau) = \frac{\beta \Theta(\tau)}{m} \frac{\partial}{\partial \tau} C_{uv}(s, s', \tau). \]

(128)
This is the fluctuation-dissipation theorem (Callen & Welton 1951), which relates the response operator to the correlation function for any system in thermal equilibrium.

If we set \( u_s = v_s = \rho(s, t) \), where \( \rho(s, t) \) is defined by equation (77), then the generalized correlation function \( C_{uv} \) and response operator \( R_{uv} \) of equations (108) and (71) reduce to their analogs \( C \) and \( R \) defined earlier in equations (100) and (1). Thus the fluctuation-dissipation theorem implies that

\[
R(r, r', \tau) = \frac{\beta \Theta(\tau)}{m} \frac{\partial}{\partial \tau} C(r, r', \tau), \]

(129)
or

\[
R_o(r, r', \tau) = \frac{\beta}{2m} \frac{\partial}{\partial \tau} C(r, r', \tau) = \frac{\beta}{2m} \frac{\partial}{\partial \tau} \tilde{C}(r, r', \tau), \]

(130)
where \( R_o \) is defined in equation (13) and \( \tilde{C} \) is defined in equation (111). The formal analog of this result in Fourier space is

\[
R_A(r, r', \omega) = -\frac{i\omega}{2m} S(r, r', \omega); \]

(131)
the equation is equally valid if \( \omega S(r, r', \omega) \) is replaced by \( \omega \tilde{S}(r, r', \omega) \) since the difference between the two functions is proportional to \( \omega \delta(\omega) \) which is zero.
If we suppress the interparticle gravitational interaction (the mean-field approximation), then the response operator $R(r, r', \tau)$ is replaced by the polarization operator $P(r, r', \tau)$, and the correlation function $C(r, r', \tau)$ is replaced by $C^{(0)}(r, r', \tau)$; thus

$$P(r, r', \tau) = \frac{\beta \Theta(\tau)}{m} \frac{\partial}{\partial \tau} C^{(0)}(r, r', \tau). \quad (132)$$

It is straightforward to verify this result when the ensemble-averaged potential is integrable, using the explicit expressions for $P(r, r', \tau)$ and $C^{(0)}(r, r', \tau)$ in equations (93) and (118). The analogous equation in Fourier space is

$$P_A(r, r', \omega) = -\frac{i \omega \beta}{2m} S^{(0)}(r, r', \omega); \quad (133)$$

once again, the equation is equally valid if $S^{(0)}$ is replaced by $\tilde{S}^{(0)}$.

### 7 Correlations in nonisothermal systems and the dressed-particle approximation

Gilbert (1968) has described collisional relaxation in stellar systems that are not necessarily in thermal equilibrium, by expanding the exact Liouville equation in powers of $N^{-1}$. His treatment is based on similar results derived by Rosenbluth et al. (1957) and Rostoker (1961) for a Coulomb plasma, and accounts fully for the self-gravity of the medium.

These calculations show that fluctuations in the system can be properly accounted for—to $O(N^{-1})$—by considering an individual test star and the polarization cloud it induces in the background medium together, as a “dressed particle”; for a test star with trajectory $r_*(t)$ the density of the bare particle is $\rho_0(r, t) = m\delta[r - r_*(t)]$, and the density of the dressed particle is given by

$$\rho_d(r, \omega) = [I + 2\pi R(\omega) \Psi] \rho_0(r, \omega) = \Lambda(\omega) \rho_0(r, \omega) \quad (134)$$

(cf. eq. 32). The test star itself is drawn from the one-particle df and moves on its unperturbed orbit; thus in this approximation the only fluctuations in the density of dressed particles are statistical fluctuations due to the finite number of particles. The papers described above effectively show that induced correlations between dressed particles are $O(N^{-2})$ or higher.
In the dressed-particle approximation, the dynamic form factor $S$ can be computed from $S^{(0)}$, simply by replacing each particle by a dressed particle: starting from equation (104)

$$
S(\omega)\delta(\omega - \omega') = \langle \delta\rho_d(r, \omega)\delta\rho_d(r', -\omega') \rangle = \langle \Lambda(\omega)\delta\rho_0(\omega)\Lambda(-\omega')\delta\rho_0(-\omega') \rangle = \Lambda(\omega)S^{(0)}(\omega)\Lambda^\dagger(\omega)\delta(\omega - \omega').
$$

The advantage of this replacement is that $S^{(0)}$ can be computed directly from the unperturbed orbits of the particles (e.g. eq. 119, if the ensemble-averaged potential is regular).

These results are consistent with the two forms of the fluctuation-dissipation theorem involving the response and polarization operators (eqs. 131 and 133), in the following sense. The latter equation can be written as $\omega S^{(0)} = (2mi/\beta)P_A$; substituting this into equation (135) gives

$$
\omega S = \frac{2mi}{\beta}AP_A\Lambda^\dagger.
$$

Using equation (26) we can show that $AP_A\Lambda^\dagger = R + 2\pi R\Psi R^\dagger$, so that $AP_A\Lambda^\dagger = (AP_A\Lambda^\dagger)_A = R_A$; hence equation (136) simplifies to

$$
\omega S = \frac{2mi}{\beta}R_A,
$$

which is the same as equation (131).

Ichimaru (1965) argues that the result (135) should hold for any stationary plasma, whether or not in thermal equilibrium, so long as the relaxation time is much longer than the correlation time; in the context of stellar systems, the analogous constraint is that the relaxation time should be much longer than the crossing time, which is equivalent to the modest requirement that the number of stars $N \gg 1$ and entirely compatible with the results of Gilbert (1968).

### 7.1 Fluctuations and collective modes

One consequence of equation (135) is that the density fluctuations described by $S(\omega)$ become very large for frequencies near singularities of $\Lambda(z)$ or $R(z)$, i.e. near the frequencies of collective modes. To see this more explicitly, we can expand $R(z)$ in a Laurent series near a collective mode with complex eigenfrequency $z_0 = \omega_0 + i\eta$. The resonant part is then

$$
R_{\text{res}}(z) = \frac{R_{-1}}{z - z_0}.
$$
where $R_{-1}$ is the residual of $R$ at $z_0$. Consequently, the dynamic form factor at real frequencies near a weakly damped collective mode is

$$S_{\text{res}}(\omega) = \frac{\Lambda_{-1}S^{(0)}\Lambda_{-1}^\dagger}{(\omega - \omega_0)^2 + \eta^2} \rightarrow \frac{\pi \Lambda_{-1}S^{(0)}\Lambda_{-1}^\dagger}{|\eta|} \delta(\omega - \omega_0)$$

as $\eta \to 0$, \hspace{1cm} (139)

where $\Lambda_{-1} = 2\pi R_{-1}\Psi$, and we have used equation (17).

Note that the level of fluctuations becomes very large as $\eta \to 0$—that is, as the mode approaches neutral stability. This phenomenon is analogous to opalescence in the vicinity of a critical point. Large fluctuations have been observed in numerical simulations of systems near marginal stability (Ivanov 1992, Weinberg 1993).

8 Applications of the fluctuation-dissipation theorem

A useful preliminary calculation is the response of a stellar system in thermal equilibrium to a static or slowly growing external potential $\Phi_e(r)$. Using equations (1), (129) and (111) we find that the density perturbation induced by $\Phi_e$ is

$$\rho_s(r) = \int dr' \int_0^\infty d\tau R(r, r', \tau)\Phi_e(r')$$

$$= \frac{\beta}{m} \int dr' \Phi_e(r') \int_0^\infty d\tau \frac{\partial}{\partial \tau} \bar{C}(r, r', \tau)$$

$$= -\frac{\beta}{m} \int dr' \Phi_e(r') \bar{C}(r, r', 0).$$

(140)

If, in addition, we ignore induced fluctuations and approximate $\bar{C}(r, r', \tau)$ by $\bar{C}^{(0)}(r, r', \tau)$, and the ensemble-averaged potential is integrable, we find using equations (122) and (10) that

$$\rho_s^{(0)}(r) = -(2\pi)^3 \beta \sum_{l \neq 0} \int dI F_0(I) p_l^*(r|I) \Phi_e(l)$$

$$= -\beta \int d\nu F_0(r, \nu) [\Phi_e(r) - \langle \Phi_e \rangle];$$

(141)

here $\langle \Phi_e \rangle \equiv \Phi_{e,0}[I(r, \nu)]$ is the orbit-averaged potential experienced by the particle passing through the phase-space point $(r, \nu)$. The last line is a special case of the general result (e.g. Lynden-Bell 1969) that the linear response of a stellar system whose $df$ depends only on energy $E$ to a slowly varying potential is

$$f = \frac{dF_0(E)}{dE} (\Phi - \langle \Phi \rangle).$$

(142)
8.1 Dynamical friction on an orbiting body

Consider a body whose center of mass travels on an orbit \( \mathbf{r}_s(t) \) through a stellar system, and whose gravitational potential is \( \Phi_*(\mathbf{r} - \mathbf{r}_s) \) (normally \( \Phi_* \) is spherically symmetric but this is not necessary for the derivation; we do, however, neglect changes in the orientation of the body). We wish to compute the mean force \( \mathbf{F}_1(t) \) exerted on the body by the response it induces in the stellar system.

The density in the wake, \( \rho_s(\mathbf{r}, t) \), is given by equation (141), and \( \mathbf{F}_1 \) is equal and opposite to the force exerted on the wake by the body, that is,

\[
\mathbf{F}_1(t) = \int d\mathbf{r} \nabla \Phi_*(\mathbf{r} - \mathbf{r}_s(t)) \rho_s(\mathbf{r}, t) = \int d\mathbf{r} d\mathbf{r}' dt' \nabla \Phi_*(\mathbf{r} - \mathbf{r}_s(t)) R(\mathbf{r}, \mathbf{r}', t - t') \Phi_s[\mathbf{r}' - \mathbf{r}_s(t')]
\]

\[
= \int d\mathbf{r} d\mathbf{r}' \nabla \Phi_*(\mathbf{r} - \mathbf{r}_s(t)) \int_0^\infty d\tau R(\mathbf{r}, \mathbf{r}', \tau) \Phi_s[\mathbf{r}' - \mathbf{r}_s(t - \tau)]. \tag{143}
\]

For completeness we point out that there is another component of the mean force on the body, which is not caused by the response it induces in the system. The fluctuating force on the body is given by

\[
\delta \mathbf{F}(t) = \int d\mathbf{r} \nabla \Phi_*(\mathbf{r} - \mathbf{r}_s(t)) \delta \rho(\mathbf{r}, t). \tag{144}
\]

Although \( \langle \delta \rho(\mathbf{r}, t) \rangle = 0 \), the mean fluctuating force does not vanish because the orbit \( \mathbf{r}_s(t) \) is correlated with the fluctuations \( \delta \rho(\mathbf{r}, t) \); thus there is an additional contribution to the mean force \( \mathbf{F}_2 = \langle \delta \mathbf{F} \rangle \).

To compute \( \mathbf{F}_2 \), we first compute the perturbation to the orbit caused by the fluctuating field, \( \mathbf{r}_s(t) = \mathbf{r}_0(t) + \delta \mathbf{r}_s(t) \) where

\[
\delta \mathbf{r}_s(t) = \int_{-\infty}^t dt' \delta \mathbf{v}_s(t') = \frac{1}{M} \int_{-\infty}^t dt' \int_{-\infty}^{t'} dt'' \delta \mathbf{F}(t'') = \frac{1}{M} \int_0^\infty d\tau \delta \mathbf{F}(t - \tau)
\]

\[
= \frac{1}{M} \int d\mathbf{r}' \int_0^\infty d\tau \nabla \cdot \Phi_s[\mathbf{r}' - \mathbf{r}_s(t - \tau)] \delta \rho(\mathbf{r}', t - \tau) \tag{145}
\]

plus higher-order terms.

Expanding equation (144) and integrating by parts gives,

\[
\mathbf{F}_2(t) = \langle \delta \mathbf{F}(t) \rangle = \int d\mathbf{r} \nabla \Phi_*(\mathbf{r} - \mathbf{r}_s(t)) \langle \delta \mathbf{r}_s(t) \cdot \nabla \delta \rho(\mathbf{r}, t) \rangle \tag{146}
\]

where

\[
\langle \delta \mathbf{r}_s(t) \cdot \nabla \rho(\mathbf{r}, t) \rangle = \frac{1}{M} \int d\mathbf{r}' \int_0^\infty d\tau \nabla \cdot \Phi_s[\mathbf{r}' - \mathbf{r}_s(t - \tau)] \cdot \nabla \langle \delta \rho(\mathbf{r}, t) \delta \rho(\mathbf{r}', t - \tau) \rangle
\]

\[
= \frac{1}{M} \int d\mathbf{r}' \int_0^\infty d\tau \nabla \cdot \Phi_s[\mathbf{r}' - \mathbf{r}_s(t - \tau)] \cdot \nabla C(\mathbf{r}, \mathbf{r}', \tau) \tag{147}
\]
Thus,
\[
\mathbf{F}_2(t) = \frac{1}{M} \int d\mathbf{r} d\mathbf{r}' \nabla \Phi_\ast[\mathbf{r} - \mathbf{r}_\ast(t)] \int_0^\infty d\tau \nabla C(\mathbf{r}, \mathbf{r}', \tau) \cdot \nabla' \Phi_\ast[\mathbf{r}' - \mathbf{r}_\ast(t - \tau)].
\] (148)

Note that \( \mathbf{F}_2 \) can be formally divergent if \( C(\mathbf{r}, \mathbf{r}', \infty) \) is non-zero. The reason is that in this case the time-averaged density is not the same as the ensemble-averaged density, so that the “fluctuating” force can contain a constant component that leads to a secular change in the orbit.

The two components of the mean force, \( \mathbf{F}_1 \) and \( \mathbf{F}_2 \), have different dependences on the mass of the orbiting body; in particular, if this mass is much greater than the mass of the stars in the system, \( M \gg m \), then the dominant force is \( \mathbf{F}_1 \), which is greater than \( \mathbf{F}_2 \) by \( O(M/m) \). In the opposite limit, \( M \rightarrow 0 \) and \( \Phi_\ast \propto M \), the mean acceleration \( \bar{a}_1 = \mathbf{F}_1/M \) goes to zero as \( M \) while the acceleration \( \bar{a}_2 \) remains finite.

If the stellar system is in thermal equilibrium, we can use the fluctuation-dissipation theorem in the form (129) to write equation (143) as
\[
\mathbf{F}_1(t) = \frac{\beta}{m} \int d\mathbf{r} d\mathbf{r}' \nabla \Phi_\ast[\mathbf{r} - \mathbf{r}_\ast(t)] \int_0^\infty d\tau \frac{\partial C(\mathbf{r}, \mathbf{r}', \tau)}{\partial \tau} \Phi_\ast[\mathbf{r}' - \mathbf{r}_\ast(t - \tau)].
\] (149)

Now replace \( C(\mathbf{r}, \mathbf{r}', \tau) \) by \( \tilde{C}(\mathbf{r}, \mathbf{r}', \tau) = C(\mathbf{r}, \mathbf{r}', \tau) - C(\mathbf{r}, \mathbf{r}', \infty) \) and integrate by parts with respect to \( \tau \):
\[
\mathbf{F}_1(t) = -\frac{\beta}{m} \int d\mathbf{r} d\mathbf{r}' \nabla \Phi_\ast[\mathbf{r} - \mathbf{r}_\ast(t)] \tilde{C}(\mathbf{r}, \mathbf{r}', 0) \Phi_\ast[\mathbf{r}' - \mathbf{r}_\ast(t)]
- \frac{\beta}{m} \int d\mathbf{r} d\mathbf{r}' \nabla \Phi_\ast[\mathbf{r} - \mathbf{r}_\ast(t)] \int_0^\infty d\tau \tilde{C}(\mathbf{r}, \mathbf{r}', \tau) \mathbf{v}_\ast(t - \tau) \cdot \nabla \Phi_\ast[\mathbf{r}' - \mathbf{r}_\ast(t - \tau)]
\equiv \mathbf{F}_s + \mathbf{F}_d.
\] (150)

where \( \mathbf{v}_\ast = d\mathbf{r}_\ast/dt \). Using equations (140) and (144) it is easy to show that the first term, \( \mathbf{F}_s \), is simply the force on the body due to the static response of the stellar system; in other words this is the force that would result if the body were fixed in its present position. The second term, \( \mathbf{F}_d \), vanishes if \( \mathbf{v}_\ast = 0 \) and corresponds to dynamical friction. The dynamical friction force can be rewritten as
\[
\mathbf{F}_{d1}(t) = -\frac{\beta}{m} \int_0^\infty d\tau \mathbf{v}_\ast(t - \tau) \langle \delta F_i(t) \delta F_j(t - \tau) \rangle,
\] (151)

where the fluctuating forces \( \delta F_i(t) \) on the body are computed using the modified correlation function \( \tilde{C}(\mathbf{r}, \mathbf{r}', \tau) \). The result (151) holds for any stellar system in thermal equilibrium.

In most calculations of dynamical friction the self-gravity of the wake is neglected. In this approximation the derivation of equation (151) remains the same, except that the
fluctuation-dissipation theorem is used in the form (132) and the fluctuating quantities $\delta \rho$ and $\delta \mathbf{F}$ are computed using the correlation function $\tilde{C}^{(0)}$, which neglects interactions between stars.

8.2 Infinite homogeneous medium

It is instructive to work out the mean force and fluctuating force on a point mass $M$ traveling through an infinite homogeneous system of stars of mass $m$. We neglect the self-gravity of the equilibrium system, so that all objects travel on straight-line orbits at constant velocity; we also neglect the self-gravity of the response, since otherwise the equilibrium system is Jeans unstable. Thus there is no distinction between the polarization and response functions $P$ and $R$, or between the correlation functions $C^{(0)}$ and $C$.

Following equation (78), the response function may be written

$$
R(r, r', \tau) = \Theta(\tau) \frac{\Theta(\tau)}{m} \sum_{ij} \langle [\delta(r - r_{i0} - v_{i0}\tau), \delta(r' - r_{j0})] \rangle,
$$

(152)

where the orbit of particle $i$ is $r_i(t + \tau) = r_{i0} + v_{i0}\tau$. The Poisson bracket is taken with respect to the phase-space variables $(r_{i0}, v_{i0})$. Thus

$$
[\delta(r - r_{i0} - v_{i0}\tau), \delta(r' - r_{j0})] = -\delta_{ij} \tau \nabla \delta[r - r_i(t + \tau)] \cdot \nabla' \delta[r' - r_j(t)].
$$

(153)

The response function is then

$$
R(r, r', \tau) = -m\Theta(\tau) \tau \sum_i \langle \nabla \delta[r - r_i(t + \tau)] \cdot \nabla' \delta[r - r_j(t)] \rangle = -\frac{\Theta(\tau)}{m} \tau \nabla \cdot \nabla' C(r, r', \tau).
$$

(154)

Integrating equation (148) by parts with respect to $r'$, substituting equation (154), and comparing to (143) shows that in an infinite homogeneous medium there is a simple relation between the components of the mean force,

$$
\mathbf{F}_2 = \left(\frac{m}{M}\right) \mathbf{F}_1.
$$

(155)

Thus

$$
\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \left(1 + \frac{m}{M}\right) \int dr dr' \int_0^\infty d\tau \nabla \Phi [r - r_*(t)] R(r, r', \tau) \Phi [r' - r_*(t - \tau)].
$$

(156)
To proceed further, we shall assume that the stellar system has a uniform number density $n$ and a Maxwellian velocity distribution with one-dimensional dispersion $\sigma$—which therefore is isothermal, with $\beta = 1/\sigma^2$. In this case the force correlation function can be shown to be (cf. Cohen 1975)

$$\langle F_i(\mathbf{r}, t + \tau) F_j(\mathbf{r}', t) \rangle = \pi n (GmM)^2 \times \left\{ \frac{a_i a_j}{a^3} \left[ \left( \frac{3}{u^2} - 2 \right) \text{erf}(u) - \frac{6}{\pi^{1/2} u} e^{-u^2} \right] + \frac{\delta_{ij}}{a} \left[ \left( 2 - \frac{1}{u^2} \right) \text{erf}(u) + \frac{2}{\pi^{1/2} u} e^{-u^2} \right] \right\},$$  \hspace{.5cm} (157)

where $\mathbf{a} = \mathbf{r}' - \mathbf{r}$, $u = a/(2^{1/2}\sigma|\tau|)$, and $\text{erf}(x) = 2\pi^{-1/2} \int_0^x \exp(-y^2) dy$ is the usual error function. The trace of (157) is simpler,

$$\langle \mathbf{F}(\mathbf{r}, t + \tau) \cdot \mathbf{F}(\mathbf{r}', t) \rangle = 4\pi n (GmM)^2 \frac{\mathbf{a} \text{erf}(u)}{a}, \hspace{.5cm} (158)$$

an expression given by Cohen (1975).

For a homogeneous system the static induced force $\mathbf{F}_s$ (eq. 150) vanishes, and we have

$$\mathbf{F}_j = \left( 1 + \frac{m}{M} \right) \mathbf{F}_{1j} = - \left( 1 + \frac{m}{M} \right) \frac{v_\ast i}{m\sigma^2} \int_{\tau_1}^{\tau_2} d\tau \langle \delta F_i(\mathbf{r}, t + \tau) \delta F_j(\mathbf{r} - \mathbf{v}_\ast \tau, t) \rangle; \hspace{.5cm} (159)$$

we have changed the limits of integration to $\tau_1 > 0$ and $\tau_2 < \infty$ to avoid divergences. The integral is easily evaluated to yield

$$\mathbf{f} = -\mathbf{v}_\ast \frac{4\pi G^2 M(m + m) n \ln \Lambda}{v_\ast^3} [\text{erf}(u) - u \text{erf}'(u)], \hspace{.5cm} (160)$$

where $u = v_\ast/(2^{1/2}\sigma)$ and $\Lambda = \tau_2/\tau_1$. This is the standard formula for dynamical friction in an infinite medium with Maxwellian velocity dispersion (Chandrasekhar 1943, Binney & Tremaine 1987), except that the Coulomb logarithm $\ln \Lambda$ is determined by the limits on the time lag in the correlation function rather than by limits on the impact parameter. Note also that the frictional force induced by the body—which is what we defined to be dynamical friction in the previous subsection—is actually $\mathbf{F}_{11}$, not $\mathbf{f}$, which is smaller by a factor $M/(M + m)$.

The effects of relaxation in an infinite homogeneous medium are often expressed in terms of diffusion coefficients, $\langle \Delta v_j \rangle$ and $\langle \Delta v_i \Delta v_j \rangle$, representing the mean and mean-square changes in velocity per unit time $\Delta t$. These changes are related to the forces on a point mass $M$ by

$$M\langle \Delta v_j \rangle = \int_0^{\Delta t} \mathbf{F}_j(t) dt, \hspace{.5cm} M^2 \langle \Delta v_i \Delta v_j \rangle = \int_0^{\Delta t} dt \int_0^{\Delta t} dt' \langle \delta F_i(t) \delta F_j(t') \rangle.$$  \hspace{.5cm} (161)
If we assume that the correlation function is negligible for time lag $\tau > \tau_{\text{max}}$, where $\tau_{\text{max}} \ll \Delta t$—this is the Markov approximation of §1—then the second of these equations may be written

$$M^2 \langle \Delta v_i \Delta v_j \rangle = \int_0^{\Delta t} dt \int_{-\tau_{\text{max}}}^{\tau_{\text{max}}} \langle \delta F_i(t) \delta F_j(t - \tau) \rangle = 2 \int_0^{\Delta t} dt \int_{0}^{\tau_{\text{max}}} \langle \delta F_i(t) \delta F_j(t - \tau) \rangle,$$

(162)

where the second equation follows from (107). Equation (159) then yields

$$\langle \Delta v_j \rangle = -\frac{M + m}{2m \sigma^2} v_{*i} \langle \Delta v_i \Delta v_j \rangle,$$

(163)

a result derived already by Chandrasekhar (1943).

9 Discussion

We have described stellar dynamics using the language of statistical physics: linear response operators, correlation functions, and the fluctuation-dissipation theorem. It is fair to ask what we have gained from this rather formal approach, especially since stellar systems do not satisfy many of the simplifying assumptions commonly used in many-body theory, such as homogeneity, local forces, and thermodynamic equilibrium. We believe that there are two main reasons why pursuing this approach is worthwhile. First, these techniques have proven to be extremely powerful in other branches of physics, so it is important to understand to what extent they can be applied to gravitating $N$-body systems. Second, it is useful to know which features in stellar systems result from general properties such as causality and time-reversal symmetry, and which depend on specific approximations used to make the dynamics tractable.

An occasional controversy, for example, is the appropriate maximum impact parameter that should be used in the Coulomb logarithm that enters the diffusion tensor computed using the local and Markov approximations of §1. Chandrasekhar (1942) and Kandrup (1981) advocate terminating the integration over impact parameters at the typical interstellar separation, whereas Cohen et al. (1950), followed by most modern authors, argue that the integration should include all impact parameters up to the characteristic size of the stellar system (or the Debye length in a plasma). In an $N$-star system with $N \gg 1$, the polarization and response operators depend only on the one-body $df$, which is unchanged if the number of stars is changed so long as the system mass $Nm$ is conserved. The response operator determines dynamical friction, and the diffusion tensor is related to dynamical friction through (151). Thus the diffusion tensor is unchanged—except for small terms
whose fractional contribution is $O(m/M)$—if the number of stars is changed, so long as $Nm$ is conserved. However, this change affects the interstellar separation; hence the diffusion tensor cannot depend on the interstellar separation and the effective maximum impact parameter must be of order the system size.

A central result of this paper is equation (151), which relates the dynamical friction force to the force-force correlation function in any stellar system in thermal equilibrium. This quite general result illuminates the relation between stochastic and dissipative gravitational forces in stellar systems. For example, Rauch & Tremaine (1996) have argued that the rate of angular momentum relaxation is strongly enhanced in nearly Keplerian star clusters, such as those found around massive black holes (“resonant relaxation”). Equation (151) immediately implies that dynamical friction is similarly enhanced in such clusters, an effect analyzed by Rauch and Tremaine and termed “resonant friction”.

We have not discussed more general issues related to the long-term relaxation of the one-particle $df$, as described by the full collisional Boltzmann equation or its generalizations. For isothermal systems, the fluctuation-dissipation theorem already suggests that the Boltzmann collision integral describing this relaxation must be related to the power spectrum of potential fluctuations in the background medium. Indeed, in plasma physics, the corresponding Balescu-Lenard collision term can be expressed in terms of the fluctuation spectrum determined by the collisionless plasma (e.g. Lifshitz & Pitaevskii 1981). Weinberg (1993) has derived this collision integral for a model periodic system, and argues that fluctuations associated with nearly unstable collective modes can strongly enhance the relaxation process.

The concepts in this paper can equally be applied to study artificial fluctuations and dissipation in numerical methods that approximate stellar systems, such as self-consistent field codes (Hernquist & Ostriker 1992), which approximate the gravitational field by a truncated multipole expansion.

The fluctuation-dissipation theorem in statistical physics also relates transport coefficients (e.g. electrical and thermal conductivity, diffusion coefficients) to frequency moments of the fluctuation spectrum. In many cases, these transport coefficients can be calculated without full knowledge of the correlation function. Moreover, conjugate transport coefficients are related by the Onsager relations. Similar relationships may exist for stellar dynamical systems, although their usefulness remains unclear, especially for non-isothermal systems. To study the dynamics of a stellar system described by a general one-particle $dr$, it may be possible, for example, to consider phenomenological response operators that are consistent with the lower-order moments of the exact response function. These models may offer analytic insight into the linearized dynamics of the stellar system without computing the full
perturbed distribution function.

10 Summary

The main goal of this paper has been to assemble and discuss the general properties of linear response, dissipation and fluctuations in stationary stellar systems. Many of the results are not new, having already been derived in other arenas of statistical mechanics—although usually in the context of spatially homogeneous systems with short-range forces. Here we summarize our main results.

We have expressed the dynamical response of a stellar system in terms of a linear response operator \( R(\mathbf{r}, \mathbf{r}', \tau) \), which determines the density perturbation induced by an external potential,

\[
\rho_s(\mathbf{r}, t) = \int d\mathbf{r}' dt' R(\mathbf{r}, \mathbf{r}', t - t') \Phi_e(\mathbf{r}', t').
\]

The response operator is analogous to the conductivity tensor used in plasma physics; in statistical physics it is sometimes called the generalized susceptibility (Landau & Lifshitz 1980). Many of its analytic properties in the frequency domain follow from the requirement that the response must be causal. For example, its Hermitian and anti-Hermitian parts \( R_H(\omega) \) and \( R_A(\omega) \) are related through the Kramers-Kronig relations (19).

The anti-Hermitian part of the response operator is associated with dissipation and dynamical friction; it determines, for example, the irreversible work done on the system by an external time-dependent potential (§2.3). If the perturbation is periodic, \( \Phi_e(\mathbf{r}, t) = \text{Re} [\phi_e(\mathbf{r}) e^{-i\omega_0 t}] \) the rate of irreversible work done is

\[
W = \pi i \omega_0 \int d\mathbf{r} d\mathbf{r}' \phi_e^*(\mathbf{r}) R_A(\mathbf{r}, \mathbf{r}', \omega_0) \phi_e(\mathbf{r}') = \pi i \omega_0 (\phi_e, R_A \phi_e).
\]

For a Hamiltonian system, an exact formal expression for the response operator is given by the ensemble average of a Poisson bracket in 6\( N \)-dimensional phase space,

\[
R(\mathbf{r}, \mathbf{r}', \tau) = \Theta(\tau) \frac{\Theta(\tau)}{m} \langle [\rho(\mathbf{r}, t + \tau), \rho(\mathbf{r}', t)] \rangle,
\]

where \( \rho(\mathbf{r}, t) = m \sum_i \delta[\mathbf{r} - \mathbf{r}_i(t)] \) is the exact density distribution of the system.

Equation (78) generally cannot be explicitly evaluated for realistic stellar systems (although short-time expansions and sum rules can be calculated). However, a closely related
polarization operator \( P(r, r', \tau) \) is easier to evaluate. The polarization operator relates the induced density perturbation to the total (external plus induced) potential,

\[
\rho_s(r, t) = \int dr'dt' P(r, r', t - t') \Phi_t(r', t').
\]

If the ensemble-averaged potential of the stellar system is integrable—so that individual stellar orbits have well-defined actions \( I \)—the polarization operator is given by

\[
P(r, r', \omega) = (2\pi)^2 \sum_l \int dI_p^l (r|I) p_l^* (r'|I) \frac{\partial F_0}{\partial I},
\]

where \( F_0(I) \) is the one-particle \( df \), and the functions \( p_l^l (r|I) \) are projection operators onto action space (eq. 89).

The operators \( R \) and \( P \) are related by a nonlinear equation (26). The operator \( P \) determines the dispersion relation for collective modes of the stellar system (56). The rate of energy loss of a mode to Landau damping is determined by its anti-Hermitian part,

\[
W_m = \pi i \omega (\phi_s, P_A \phi_s) = 2\eta E_m,
\]

where \( E_m \) and \( \eta \) are the energy and damping rate of the mode. Dissipation occurs through resonant interaction with stellar orbits commensurate with the mode frequency.

Density fluctuations in a stellar system are characterized by the correlation function

\[
C(r, r', \tau) = \langle \delta \rho(r, t + \tau) \delta \rho(r', t) \rangle.
\]

For a stellar system in thermal equilibrium, described by an \( N \)-particle distribution function \( f_0(Z) \propto \exp[-\beta H_0(Z)] \), the response function is directly related to the correlation function by the fluctuation-dissipation theorem, which states that

\[
R(r, r', \tau) = \frac{\beta \Theta(\tau)}{m} \frac{\partial}{\partial \tau} C(r, r', \tau).
\]

The equivalent expression in the frequency domain reads

\[
R_A(r, r', \omega) = -\frac{i \omega \beta}{2m} S(r, r', \omega),
\]

where \( S \) is the Fourier transform of the correlation function (eq. 102). Thus dissipational processes in the medium, determined by \( R_A(\omega) \), are directly related to the power spectrum of density fluctuations.

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