Prime number logarithmic geometry on the plane
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Abstract We found a regularity of the behavior of primes that allows to represent both prime and natural numbers as infinite matrices with a common formation rule of their rows. This regularity determines a new class of infinite cyclic groups that permit the proposition a plane–spiral geometric concept of the arithmetic.

1 Introduction

Counting arithmetic functions for different prime sets can be assigned to the archaic mathematical reality.

Nevertheless, the generated by them prime sequences, named Eratosthenes progressions, became known only in recent years (e.g., [1], [2], [3] and sequences A007097, A063502, A064110 in [4]).

The Eratosthenes progression possesses a common formation law of its elements (an inner prime number distribution law) the realization of which is based on a multiple use of the Eratosthenes sieve [1] (Figure 1).

The derivation of Eratosthenes progressions and their systematic investigation is directed to a learning the nonasymptotic behaviour of primes, i.e., of the function’s behaviour

\[ d(n) = p(n + 1) - p(n), \ n = 1, 2, \ldots, \pi, \]

where \( p(n) \) is the \( n \)th prime and \( \pi \) is a sufficiently large natural number.

The inner prime number distribution law can be applied mostly in mathematics itself, for example, when constructing new geometric concepts in arithmetic.

Following Alain Connes ([5] pp. 208–209), it can be supposed that the specific behaviour of primes will reflect itself in the new geometry sought for understanding quantum gravity.

In biochemistry, the specific behaviour of primes can manifest itself in
the laws of formation and functioning of large molecules, from $10^3$–atomic insulin and hemoglobin up to $3 \cdot 10^5$–atomic proteins and enzymes.

In this paper, the general statement of the problem for derivation of Eratosthenes progressions is given and their basic properties are presented. The general results are applied to the sequence of primes itself

$$P = \{2, 3, 5, 7, 11, \ldots\} = \{p(n)\}_{n=1,2,\ldots},$$

as well as to the following related to $P$ sequences:

$$M = \mathbb{N} \setminus P = \{4, 6, 8, 9, \ldots\} = \{m(n)\}_{n=1,2,\ldots}$$ the set of composite numbers;

$$T = \{t(\nu) = (p(\nu), p(\nu + 1)) : \nu \in \Lambda\}$$ the set of twin pairs, where

$$\Lambda = \{n : p(n + 1) - p(n) = 2, n \in \mathbb{N}\} = \{2, 3, 5, 7, 10, 13, 17, \ldots\};$$

$$T_1 = \{p(\nu) : (p(\nu), p(\nu + 1)) \in T, \nu \in \Lambda\} = \{t_1(\nu)\}_{\nu \in \Lambda}$$ the set of first elements of twins;

$$T_2 = \{p(\nu + 1) : (p(\nu), p(\nu + 1)) \in T, \nu \in \Lambda\} = \{t_2(\nu)\}_{\nu \in \Lambda}$$ the set of second elements of twins;

$$T_3 = T_1 \cup T_2 = \{3, 5, 7, 11, 13, 17, 19, \ldots\}$$ the set of twin elements;

$$S = P \setminus T_3 = \{2, 23, 37, 47, 53, \ldots\}$$ the set of isolated primes\[4\], A007510;

$$D_{6n-1} = \{6n - 1 \in P : n = 1, 2, \ldots\} = \{5, 11, 17, \ldots\}$$ the set of primes of the kind $6n - 1$;

$$D_{6n+1} = \{6n + 1 \in P : n = 1, 2, \ldots\} = \{7, 13, 19, \ldots\}$$ the set of primes of the kind $6n + 1$,

$$T_4 = \{t(n) : (t_1(n) + t_2(n))/2 = 6 \cdot q, q \in P\}$$ the set of twins with minimal average ([2], p. 15).

The sets $T$, $T_1 - T_4$ and $S$ below will be supposed to be infinite.

In this paper some properties of Eratosthenes progressions such as distribution laws of the progression elements, $\zeta$–functions for the progressions and their connection with the Riemann $\zeta$–function are only mentioned.
Multiple Eratosthenes Sieve Machine

Figure 1:
The main result of this paper consists in the proposed plane–spiral geometric concept of arithmetic, compatible with the linear Cartesian concept. The real semiaxis $\mathbb{R}_1^+$ in the new geometric model is isometrically mapped as a logarithmic spline-spiral on the plane $\mathbb{R}^2$ in such a way that the Eratosthenes rays, not intersecting each other, cross the spiral only at the primes.

The spiral arithmetic allows one to interpret in a new way the basic counting function $\pi(x)$, the Littlwood’s $\Omega$–theorem and also gives an arithmetic interpretation of the distribution in natural series of all kinds of clusters of primes (see [6], for example) and twin pairs, in particular.

The basic object in the spiral geometry is a spider–web $W_n$ composed of spiral and Eratosthenes rays intersecting it, in which the number of rotations $n$ infinitely increases.

The web $W_n$ consists of embedded concave–convex trapezoids of primes with a characteristic formation law. This law is a direct consequence of the inner prime number distribution law.

The plane $\mathbb{R}^2$ is considered as a mosaic composed of elementary concave–convex trapezoids.

The web $W_n$ geometrically select (personalizes) primes, and also all kinds of linear and plane configurations of primes.

2 Splitting theorem for infinite sequences of primes

2.1 Basic definitions

Let sets $A \subset \mathbb{N}$ and $B \subset \mathbb{N}$ with the properties

$$ A \cap B = \emptyset, \quad A \cup \overline{B} = \mathbb{N}, $$

where $\overline{B} = \{1\} \cup B$ are given.

Let the arithmetic function $g(n) : \mathbb{N} \to A$

generate (denote) the $n$th element $a(n) \in A$.

Then the counting recurrent law

$$ \varepsilon^+_a(0) : a(n + 1) = g(a(n)), \quad n = 0, 1, 2, \ldots, a(0) \in \mathbb{N} $$

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determines an $A$–counting progression $\varepsilon^+_{a(0)}$ and an $A$–counting ray

$$r_{a(0)} = \{a(n) : a(n + 1) = g(a(n)), n = 0, 1, 2, \ldots, a(0) \in \mathbb{N}\}.$$  

Together with the function $g(n)$, its inverse function, the $n$th number of element $a(n) \in A$, is also uniquely determined (in a purely arithmetical sense it is a counting function)

$$g_1(a) : A \rightarrow \mathbb{N}.$$  

The functions $g(n)$ and $g_1(a)$ are strictly monotonic and satisfy the equalities

$$g(g_1(a)) = a, \quad g_1(g(n)) = n.$$  

By means of $g(n)$ and $g_1(a)$ the compositions

$$g_n(a(0)) = g(\ldots g(a(0)) \ldots) = a(n),$$

$$g_{-n}(a(n_1)) = g_1(\ldots g_{-1}(a(n_1)) \ldots), \quad \text{with } n \leq n_1.$$  

are introduced.

These compositions satisfy the equalities

$$g_{n_1}(g_{n_2}(a(0))) = g_{n_1 + n_2}(a(0)), \quad n_1, n_2 \geq 1,$$

$$g_{-n_1}(g_{n_2}(a(0))) = g_{n_2 - n_1}(a(0)), \quad 1 \leq n_1 \leq n_2.$$  

An extension of the $A$–counting progression $\varepsilon^+_{a(0)}$ with negative numbers $\varepsilon^-_{a(0)} = -\varepsilon^+_{a(0)}$ leads to an infinite cyclic group

$$\varepsilon_{a(0)} = \varepsilon^+_{a(0)} \cup \{a(0)\} \cup \varepsilon^-_{a(0)}, \quad g_{-n}(a(0)) = -g_n(a(0)), \quad n > 0 \quad (4)$$

under composition $g_n(a(0))$, with a depth $n \in \mathbb{Z}$ and a generator $a(0) \in \overline{B}$.

Two elements from $\varepsilon_{a(0)}$ interact under the composition rule

$$g_{n_1}(a(0)) \circ g_{n_2}(a(0)) = g_{n_1}(g_{n_2}(a(0))) = g_{n_1 + n_2}(a(0)), \quad n_1, n_2 \in \mathbb{Z}. \quad (5)$$

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2.2 Basic assertion and its consequences

The following assertion about the splitting of the set $A$ in a denumerable number of denumerable subsets with a common law of formation of its elements is given:

**Theorem 1.** For any sets $A$ and $B$ with properties (1) and (2) the following equalities hold

$$\bigcap_{a(0)\in B} r_{a(0)} = \emptyset, \quad \bigcup_{a(0)\in B} r_{a(0)} = A.$$ 

Theorem 1 leads to a matrix representation of the sequences $A$ and $N$ with peculiar properties of their elements.

**Corollary 1.** There exists an one-to-one mapping

$$\overline{\varphi}(a(0)) : \overline{B} \rightarrow 2^A = \{r_{a(0)}\}_{a(0)\in B} \equiv \{a_{\mu\nu}\}_{\mu,\nu=1,2,...} \quad (6)$$

($2^A$ denotes the matrix representation of the elements of $A$).

From (6) a matrix representation to the natural series

$$2^N = \|\overline{B} \cdot 2^A\|, \quad (7)$$

where $\overline{B} =$ Column$\{a_{\mu0}\}_{\mu=1,2,...}$ also follows.

The matrices $2^A$ and $2^N$ shall be called mesm–matrices.

In the case when $A = P$ and $B = M$ an example of the left upper corner of the matrix $2^N$ (8, pp. 18–22) is given in Appendix 1.

**Corollary 2.** The rows of matrices $2^N$ are isomorphic to the row $r_1$ with respect to the mapping

$$\Psi(g_n(1)) : g_n(1) \rightarrow g_n(g_n(1)) \rightarrow a(0) \rightarrow g_n(a(0)), \quad a(0) \in \overline{B}, \quad a(0) > 1.$$ 

The columns of the matrix $2^A$ are isomorphic to the column $\overline{B}$ with respect to the mapping

$$\varphi(a(0)) : a(0) \rightarrow g_n(a(0)), \quad a(0) \in \overline{B}.$$
In the case $A = P$ and $B = M$ Figure 2 illustrates mentioned isomorphisms. In Figure 2, an one-to-one correspondence between rooted trees and elements of $\mathbb{N}$, proposed by F. Göbel [7] is used (see the 1th row of Figure 2).

Theorem 1 leads also to an important consequence, which reveals the arithmetic nature of the fine structure of the set $A$ elements’ distribution among the natural numbers.

Let $g_{-1}(n', n'')$, $n', n'' \in \mathbb{N}$ denote the number of elements $A$ in the interval $(n', n'')$.

**Corollary 3.** For the matrix $[B \; 2A]$ elements the following equalities hold:

\[
\begin{align*}
&g_{-1}(a_{\mu_0}, 0) = a_{\mu_1} - 1, \quad \mu = 1, 2, \ldots, \\
g_{-1}(a_{\mu_1\nu_1}, a_{\mu_2\nu_2}) = |a_{\mu_1(\nu_1-1)} - a_{\mu_2(\nu_2-1)}| - 1, \\
&\mu_i, \nu_i \geq 1, \quad i = 1, 2.
\end{align*}
\] (8)

3 The Theorem 1 application to special cases of sets $A$ and $B$

3.1 About new $A$–counting progressions

In the case when $A$ and $B$ take usual values the law (3) generates known $A$–counting progressions. So, for example, at $A = \{\text{even}\}$ and $B = \{\text{odd}\}$ a generating function is of kind $g(a(n)) = 2a(n) - 1$ and in this case $\varepsilon_2^+ = \{2, 3, 5, 9, 17, 33, 65, 129\ldots\}$ is a Pisot sequence ([4], A000051).

New $A$–progressions one occur when the behaviour of $A$ elements among natural numbers is unknown and it cannot be considered as a probabilistic. Besides the sequences of primes $P$, all subsequences of $P$, in the formation of which the Eratosthenes sieve combines with an additional deterministic filter $f(n)$ (this is the formation rule of the considered subsequence), should also be considered belonging to this class. The set of these subsequences shall be denoted by $\mathcal{E}_f$.

The Dirichlet theorem about the existence of infinite primes of the kind $\alpha n + \beta$ (an additional filter) for arbitrary coprimes $\alpha$ and $\beta$ shows that $\mathcal{E}_f$ is infinite.

In particular, we have inclusions $T_1, T_2, T_3, S \in \mathcal{E}_f$ and $D_{\alpha n \pm 1} \in \mathcal{E}_f$ at $\alpha = 4, 6$. 

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For all elements $E_f$ there exists a $mesm_f$-process, which is analogous to
the process represented in Figure 1. From the $A$-split theorem it follows
that for every $A_f \in E_f$ and $B_f = \mathbb{N} \setminus A_f$ there exists a $mesm_f$-matrix
$[B_f^2 A_f]$.

As a result of a $mesm$-transition $A_f \rightarrow 2^2 A_f$, the elements of the rows
$2 P, 2 T_1, 2 T_2, 2 T_3, 2 S$ and $2 D_{on+1} (\alpha = 4, 6)$ already will be distributed ac-
cording to the inner law (3), which now should be understood as a specific
self-smoothing (only with respect to the rows $2^2 A_f$) of the irregularities in
the appearance of the elements $A_f$ in the natural series.

3.2 The basic case: $A = P$ and $B = M$.

The upper left corner of the matrix $2 P$ and its extension to the matrix $2 N$
are represented in Appendix 1. The Theorem 1 has been proved inductively
in [2], pp. 4–8.

The first elements of the first rows of the matrix $2 P$ were primarily
determined by hand by means of $MESM$ (Figure 1). In such a way the law
with $g(n) = p(n)$ (Eratosthenes progressions) was discovered [1].

The extension of the matrix $2 P$ rows on negative primes according to
the rules (4), (5) leads to infinite cyclic groups under compo-
sition $p_n(a(0))$, $n \in \mathbb{Z}$ with a generators $a(0) \in M$. An example of such a group is the set

$$\varepsilon_4 = \{\ldots, -p_n(4), \ldots, -59, -17, -7, 4, 7, 17, 59, \ldots, p_n(4), \ldots\}.$$  

A part of $2 P$ represented in Appendix 1 has been computed by means
of Mathematica function NestList[Prime, a(0), n].

The row elements of the matrix $[M^2 P]$ determine new subsets of nat-
ural numbers

$$N_m = \{p_{n_1}^{\alpha_1}(m) \ldots p_{n_k}^{\alpha_k}(m) : \forall n_i, \alpha_i \in \mathbb{N}, i = 1, 2, \ldots, k, \forall k \in \mathbb{N}\}, m \in M :$$

$$N_1 = \{2, 3, 2^2, 5, 2 \cdot 3, 2^3, 3^2, 2 \cdot 5, 11, 2^2 \cdot 3, 3 \cdot 5, 2^4, \ldots\},$$

$$N_4 = \{7, 17, 7^2, 59, 7 \cdot 17, 277, 17^2, 7^3, 7 \cdot 59, 7^2 \cdot 17, 1787, 7^4, \ldots\},$$

$$N_6 = \{13, 41, 13^2, 179, 13 \cdot 41, 1063, 41^2, 13^3, 13^2 \cdot 41, \ldots\},$$

and so on.

According to Corollary 2, the behaviour of composite numbers reflects
on the behaviour of the elements of the columns of the matrix $2 P$. 

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Figure 2: "MESM & F. Göbel" forest of rooted trees
On the other hand, the structure of the set $M$ depends on the structure of the set of primes because $M$ can be represented as a chain of $\alpha_\mu$-element segments (where $\alpha_\mu = d(\mu) - 1$) from consequent composite numbers

$$m_\mu(\alpha_\mu) = \{p(\mu) + 1, p(\mu) + 2, \ldots, p(\mu + 1) - 1\}, \quad \mu = 2, 3, \ldots$$

$$(m_2(1) = \{4\}, m_3(1) = \{6\}, m_4(3) = \{8, 9, 10\}, \ldots).$$

The segments are connected in a whole set $M$ by means of ghost primes $\omega_\mu = \langle p(\mu) \rangle$ ($\omega_2 = \langle 3 \rangle$, $\omega_3 = \langle 5 \rangle$, $\omega_4 = \langle 7 \rangle, \ldots$).

The Eratosthenes progressions $\{\varepsilon_m^+\}_{m \in \overline{M}}$ (i.e., rows of the matrix $^2P$) conform to the inner prime number distribution law

$$a(n + 1) = p(a(n)) = p_{n+1}(a(0)), \quad n = 0, 1, 2, \ldots, a(0) \equiv m \in \overline{M}, \quad (9)$$

but the deviation of the rows $^2P$ between each other (i.e., the distribution of primes in the columns of $^2P$) again persists dependent of the oddish behaviour of primes.

The main information left out of the inner law (9) is reflected in the structure of the first matrix $^2P$ column

$$P_1 = column[p_{11}, p_{21}, \ldots, p_{\mu_1}, \ldots].$$

The following assertion about the $P_1$ structure is valid.

**Theorem 2.** Mapping $\varphi : a(0) \to p_{\mu_1}$ defines a correspondence between segments of composite numbers $m_\mu(\alpha_\mu)$ and clusters of $\alpha_\mu$-successive primes

$$c_\mu(\alpha_\mu) = \{p_1(p(\mu) + 1), p_1(p(\mu) + 2), \ldots, p_1(p(\mu + 1) - 1)\} \subset P$$

in the cases $\alpha_\mu \geq 3$, and separate primes $p_1(p(\mu) + 1)$ in the cases $\alpha_\mu = 1$. At their ends the clusters are complemented by the ghost images up to prime number segments

$$\overline{c}_\mu(\alpha_\mu) = \{p_1(\langle p(\mu) \rangle), c_\mu(\alpha_\mu), p_1(\langle p(\mu + 1) \rangle)\}$$

and the equality $P = \bigcup_{\mu=1}^\infty \overline{c}_\mu(\alpha_\mu)$ is fulfilled.

The next theorem about twin pairs $t(\nu) = (t_1(\nu), t_2(\nu)) \in T$, $\nu = 3, 5, 7, \ldots$ is also justified.
Theorem 3. For each pair $t(\nu)$ (after the pair $(3,5)$) at least one of the elements $t_1(\nu)$ or $t_2(\nu)$ belongs to the first column $P_1$.

The mapping $\varphi^{-1} : p_{\mu_1} \rightarrow m_{\mu}$ defines a correspondence between pairs with both elements on $P_1$ ($u$–twin) and pairs of subsequent elements of some segment $\overline{m_\mu}(\alpha_{\overline{\mu}}) \subset M$ with $\alpha_{\overline{\mu}} \geq 3$.

For a pair with one element $t_1(\nu)$ (or $t_2(\nu)$) on $P_1$ ($b$–twin) the mapping $\varphi^{-1} : p_{\mu_1} \rightarrow m(\mu)$ associates $t_1(\nu)$ (or $t_2(\nu)$) with the element $p(n)+1$, or the element $p(n+1)−1$ of some segment $\overline{m_\mu}(\alpha_{\overline{\mu}}) \subset M$ at $\alpha_{\overline{\mu}} \geq 3$, or with the element of some one-element segment $\overline{m_1}(1) \subset M$.

The mapping $\varphi^{-1} : p_{\mu_1 \nu_1} \rightarrow p_{\mu_1(\nu_1-1)}$, $\nu_1 \geq 2$ relates the second element $t_2(\nu)$ (or $t_1(\nu)$) to one of the ghosts $\langle p(\mu) \rangle \equiv p_{\mu_1(\nu_1-1)}$ or $\langle p(\mu + 1) \rangle \equiv p_{\mu_1(\nu_1-1)}$.

The following properties of the matrix $^2P$ rows and columns are briefly viewed:

$q_1$) The difference $d_m(n) = p_{(n+1)}(m) − p_n(m)$, $n = 1, 2, \ldots, m \in \overline{M}$ monotonically increase under the estimate

$$d_m(n) > p_n(m)(\ln p_n(m) − 1)$$

unlike the difference $d(n)$ whose behaviour only on the face of it may seems to be a chaotical one $\mathbb{S}$;

$q_2$) The sequence $\eta(s, m) = \sum_{n=1}^{\infty} \frac{1}{p_n^s(m)}$ converge for all $m \in \overline{M}$ and $s \geq 1$.

Note especially the convergence of the sum $\eta(1, m)$ ($\mathbb{Z}$, p. 10) when the sum $\sum_{n=1}^{\infty} \frac{1}{p(n)}$ diverges;

$q_3$) An analogue of the Euler identity exists

$$\zeta(s, m) \equiv 1 + \sum_{n \in N_m} \frac{1}{n^s} = \prod_{n=1}^{\infty} \left(1 - \frac{1}{p_n^s(m)}\right)^{-1}, m \in \overline{M}, s \geq 1;$$
q4) The Riemann function $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$, $s \in \mathcal{C}$ can be represented by the functions $\zeta(s, m)$

$$\zeta(s) = \prod_{m \in M} \zeta(s, m);$$

q5) The asymptotic law for the primes and the simplified Riemann formula for $\pi(x)$ give an opportunity to find approximately $p_{n+1}(m)$, $m \in M$ by solving the equations with respect to $x$

$$L(x) = p_n(m),$$
$$R(x) = p_n(m),$$

where $L(x) = \int_{0}^{x} \frac{ds}{\ln(s)}$, $R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} L(x^{1/k})$

and $\mu(k)$ is a Möbius function;

q6) There exists an approximate formula

$$n = \int_{\alpha}^{p_n(\beta)} \frac{ds}{s \ln \ln s} + \varepsilon(n, \beta),$$

where $\alpha = 11$, $\beta = 1$, $n > 4$ for $r_1$, $\alpha = 7$, $\beta = 4$ for $r_4$ and $\alpha = \beta = m$ for the other rays $r_m$.

The absolute error $|\varepsilon|$ for the part of the matrix $[M \ 2P]$ in Appendix 1 is not greater than 0.2 when $n$ is small and 0.06 when $n$ is large.

Formula (12) is a prime number distribution law of the rays $2P$.

On Figure 3, the behaviour of the function (12) is presented for the ray $r_9$;

q7) It is obvious that for the number $\mu$ of the element $p_{\mu n}$ in the matrix $2P$ column

$$P_n = \text{column}[p_{1n}, p_{2n}, \ldots, p_{\mu n}, \ldots]$$

there exists an asymptotic formula

$$\mu \sim m - \int_{2}^{m} \frac{ds}{\ln s}.$$
This is the column \( P \) prime number distribution law.

In order to use (12) and (13) it is necessary to know the composite number \( m \).

### 3.3 About other A–counting progressions

Applying the A–split theorem in the cases

\[
A = T_1 \text{ and } B_1 = T_2 \cup M,
\]

\[
A = S \text{ and } B_2 = M \cup T_3,
\]

\[
A = D_{6n-1} \text{ and } B_3 = M \cup D_{6n+1} \cup \{2, 3\},
\]

and

\[
A = D_{6n+1}, \ B_4 = M \cup D_{6n-1} \cup \{2, 3\},
\]

we can obtain the next mesm-matrices of kind (7):

\[
\begin{bmatrix}
1 & 3 & 11 & 137 & 5639 & 641129 & 152921807 \\
2 & 5 & 29 & 641 & 44381 & 7212059 & \ldots \\
4 & 17 & 239 & 12161 & 1583927 & \ldots \\
6 & 41 & 1151 & 93251 & 16989317 & \ldots \\
7 & 59 & 1931 & 176021 & 35263691 & \ldots \\
8 & 71 & 2339 & 221201 & 4749309 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 2 & 23 & 263 & 2917 & 38639 & 603311 & 11093633 \\
3 & 37 & 397 & 4751 & 64403 & 1038629 & 19661749 & \ldots \\
4 & 47 & 491 & 5897 & 81131 & 1328167 & 25467419 & \ldots \\
5 & 53 & 557 & 6709 & 93287 & 1541191 & 29778547 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 5 & 29 & 263 & 3767 & 76253 & 2049263 & 69633521 \\
2 & 11 & 83 & 953 & 16223 & 381221 & 11579489 & \ldots \\
3 & 17 & 137 & 1721 & 31883 & 795803 & 25434641 & \ldots \\
4 & 23 & 197 & 2663 & 51803 & 1348961 & 44635001 & \ldots \\
6 & 41 & 419 & 6329 & 135347 & 3808109 & 134441441 & \ldots \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots
\end{bmatrix}
\]
The matrices $[B_1^2T_1]$, $[B_2^2S]$ and $[P^2M]$ were published in [4] as A063502, A064110 and A025003–A025006, respectively. Matrices $[B_3^2D_{6n-1}]$ and $[B_4^2D_{6n+1}]$ are the new ones.

New mesm–matrices can be obtained also for the Euler primes of the kind $n^2 + n + 41$ ($r_1 = \{41, 1847, 1573316, \ldots\}$), and for the Hardy-Littlwood primes of the kind $H_{n^2+1} = \{n^2 + 1 \in P : n = 1, 2, \ldots\}$ where at $B_5 = \mathbb{N} \setminus H_{n^2+1}$ we have

$$[B_5^2H_{n^2+1}] = \begin{bmatrix}
1 & 2 & 5 & 101 & 746497 & 286961228404901 \\
2 & 17 & 7057 & 11424189457 \ldots \\
3 & 37 & 44101 & 637723627777 \ldots \\
4 & 197 & 3496901 \ldots \\
6 & 257 & 6421157 \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.$$
with the exception of the matrix \([P \ 2M]\) for which an analogue of the law \([13]\) is known. However, the common Corollaries 2 and 3 of the \(A\)-split theorem remain valid for them.

4 Logarithmic geometry of primes on the plane.

4.1 The Prime Number Spider Web (PNSW) Hypothesis

One of the main application of the prime number distribution law \([9]\) consists in constructing the plane spiral geometric concept of arithmetic.

Let

\[
\mathcal{L}_f = \{ \rho(\theta) = (f(\theta))^\theta : f(\theta) \in C^1[0, \infty), f(\theta) \geq 1, 0 \leq \theta < \infty \}
\]

denote a class of logarithmic spirals with an arc length

\[
\lambda(0, \theta) = \int_0^\theta (f(x))^x \left( \left( \ln f(x) + \frac{x f'(x)}{f(x)} \right)^2 + 1 \right)^{1/2} dx.
\]

The plane spiral geometric concept is based on the following PNSW–hypothesis \([11]\).

**Conjecture 1.** On the plane \(\mathbb{R}^2\) there exists a unique spiral \(\rho(\theta) \in \mathcal{L}_f\) and the corresponding to it sets of angles

\[
\{ \theta_{mn} \}_{m \in \overline{M}}, \ n = 1, 2, \ldots, \ \theta_{mn'} < \theta_{mn''} \text{ at } n' < n''
\]

such that the following conditions are fulfilled:

(i) \(\lambda(0, \theta_{mn}) = p_n(m), \ n = 1, 2, \ldots, \ m \in \overline{M};\)

(ii) the primes \(p_n(m), \ n = 1, 2, \ldots\) lie on the same ray \(\ell_m \subset \mathbb{R}^2\) with a positive direction corresponding to increasing \(n;\)

(iii) two arbitrary rays \(\ell_{m_1}\) and \(\ell_{m_2}, \ m_1, m_2 \in \overline{M}\) do not intersect each other and are non–parallel.
4.2 Logarithmic spline–spiral

Under the substitution \( f(\theta) = e^{\cot \varphi} \), \( \mathcal{L}_f \) turns in a one–parametric family of logarithmic spirals

\[
\mathcal{L}_\varphi = \left\{ \rho_\varphi = e^{(\cot \varphi)\theta} : 0 < \varphi < \frac{\pi}{2}, \ 0 \leq \theta < \infty \right\}
\]

with an arc length

\[
\lambda(0, \theta) = \frac{1}{\cos \varphi} \left( e^{(\cot \varphi)\theta} - 1 \right). \quad (14)
\]

Now the required by the PNSW–hypothesis sets of angles with respect to \( m \) and \( n \), according to the condition (i), are given by the formula

\[
\theta_{mn} = \tan \varphi \ln(p_n(m) \cos \varphi + 1)).
\]

For simple logarithmic spirals the conditions (ii) and (iii) of the PNSW–hypothesis are not fulfilled because the equation

\[
S_{n_1n_2}(x) + S_{n_2n_3}(x) + S_{n_3n_1}(x) = 0, \quad (15)
\]

where

\[
S_{\alpha\beta}(x) = (p_\alpha(m)x + 1)(p_\beta(m)x + 1) \sin \left( \sqrt{\frac{1}{x^2} - 1} \ln \frac{(p_\alpha(m)x + 1)}{(p_\beta(m)x + 1)} \right)
\]

cannot be satisfied with the same value \( x = \cos \varphi \) for any triplets \( (p_{n_1}(m), p_{n_2}(m), p_{n_3}(m)) \) from any ray \( r_m, m \in \mathcal{M} \).

Nevertheless, the solution (15) for all the denoted prime triplets from all rays of the matrix in Appendix 1 shows that \( x \) remains in a sufficiently narrow interval \( I_x = (0.202, 0.326) \) with an average \( \overline{x} \approx 0.264 \) to which there corresponds a value \( \overline{\varphi} \approx 74.69^\circ \). On Figure 4 a pure–logarithmic web is presented where only the condition (i) is fulfilled.

This result stimulates us to search for a verification of the PNSW–hypothesis in the class of logarithmic spline–spirals (LSS):

\[
\rho_{s_1}(\theta) = e^{s_1(\theta)},
\]

\[
s_1(\theta) = \begin{cases} 
\alpha_{i+1}\theta + \beta_{i+1}, & \theta_i \leq \theta \leq \theta_{i+1}, \quad 0 \leq i \leq k - 1, \\
\alpha_{i-1}\theta_{i-1} + \beta_{i-1} = \alpha_i\theta_{i-1} + \beta_i, & 2 \leq i \leq k,
\end{cases}
\]
Figure 4: Pure logarithmic prime number spider web

where the first degree spline $s_1(\theta)$ is defined on an irregular set

$$\Delta_k : 0 = \theta_0 < \theta_1 < \theta_2 < \ldots < \theta_{k-1} < \theta_k,$$

with a number $k \geq 3$ of subintervals $[\theta_i, \theta_{i+1}], \quad 0 \leq i \leq k - 1$, which increases with the number of rotations $n$ of the web $W_n(P)$.

The unknowns in the spiral $\rho_{s_1}(\theta)$ are both the knots of the set $\Delta_k$ and the spline–spiral coefficients of the elements $e^{\alpha_i \theta + \beta_i}$

$$\{\alpha_i, \theta_i, \beta_i\}_{i=1,2,\ldots,k}.$$

They are determined from the conditions of the PNSW–hypothesis with regard for the initial condition

$$e^{\alpha_1 \theta_0 + \beta_1} = 1 \implies \beta_1 = 0. \quad (16)$$

For arbitrary $x \in \mathbb{R}^1_+$ there exists an unique $p(k_x) \in P$ such that $p(k_x - 1) \leq x < p(k_x)$ and the isometric transformation $x \in \mathbb{R}^1_+$ on $\mathbb{R}^2$, determined by the condition (i) of the PNSW–hypothesis, acquires the explicit form

$$h_{\rho_{s_1}}(x) : \mathbb{R}^1_+ \to \lambda(0, \theta_x) = p(k_x - 1) + \sqrt{1 + \left(\frac{1}{\alpha_{k_x}^2}\right)} e^{\beta_{k_x}} \left(e^{\alpha_{k_x} \theta_x} - e^{\alpha_{k_x} \theta_{k_x-1}}\right), \quad (17)$$
where
\[ 0 \leq x < \infty, \quad \theta_x = \frac{1}{\alpha_k x} \ln E(x), \]
\[ E(x) = \frac{\alpha_k x e^{-\beta_k x (x - p(k_x - 1))} + e^{\alpha_k x \theta_x - 1}}{\sqrt{1 + \alpha_k^2 x}} , \quad p(0) = 0. \]

The points \((u_x, v_x) \in \rho_s(\theta) \subset \mathbb{R}^2\), which correspond to the numbers \(x \in \mathbb{R}_1^+\), have the Euler and Cartesian coordinates respectively:

\[ \rho(\theta_x) = e^{\alpha_k x \theta_x} + \beta_k x = e^{\beta_k x \ln E(x)}, \quad \theta_x = \frac{1}{\alpha_k x} \ln E(x) \]  \hspace{1cm} (18)

and
\[ u_x = \rho(\theta_x) \cos(\theta_x), \quad v_x = \rho(\theta_x) \sin(\theta_x). \]  \hspace{1cm} (19)

The first plane spiral isometric to the semi–axis \(\mathbb{R}_1^+\) was constructed in [9].

4.3 About constructing the webs \(W_n\)

Attempts to construct the spiral \(\rho_s(\theta)\) under the PNSW-hypothesis for a given \(n\) lead to a denial of some number \(k^0\) of starting primes because of the difficulty in fulfilling the condition (ii) around the origin of \(\mathbb{R}^2\) (condition (i) remains valid for the missed primes). In this paper the case \(k^0 = 11\) is considered, i.e., instead of the rays \(r_1, r_4, r_6, r_8, r_9\) and \(r_{10}\), the truncated rays \(\overline{r}_1, \overline{r}_4, \overline{r}_8, \overline{r}_9\) and \(\overline{r}_{10}\) obey the condition (ii), and these rays start with the numbers 127, 59, 41, 87, 83 and 109 respectively.

According to (16), to the first element \(e^{\alpha_1 \theta} (0 \leq \theta \leq \theta_1)\) of the spiral \(\rho_s\) there corresponds the real segment \([0, p(k^0 + 1)]\).

The rotations \(W_n\) are taken in account from the ray 
\[ r_{12} = \{37, 157, 919, 7193, \ldots\} \]
in the direction counter–clockwise.

At first, \(\rho^3_{s_1}\) and \(W_3\) are constructed on the basis of the first 3 elements of the first 25 rays \(\overline{r}_1, \overline{r}_4, \ldots, r_{36}\) plus the fourth element of the ray \(r_{12}\).
The satisfaction of the conditions (i), (ii) of the PNSW–hypothesis with regard to (16) leads to a solution of a nonlinear system ($W_3$–system) of 228 equations and 304 inequalities, caused by the condition (iii), with respect to the 228 unknowns 

$$\{\alpha_i, \theta_i, \beta_i\}_{i=1,2,...,76}.$$ 

All 832 primes up to the number 7193, which remain unused in the construction of the $W_3$–system ($p_{-1}(7193) = 919; \ 11 + (25 \times 3 + 1) + 832 = 919$) are placed by the Cartesian coordinates (19) on the 2nd and 3rd rotation of $\rho^{(3)}_{s_1}$.

The building up of new rotations $n > 3$ on $\rho^{(3)}_{s_1}$ is reduced to the subsequent solution of $3 \times 3$ nonlinear systems of equations.

Both solvability and uniqueness of the mentioned infinite set of nonlinear systems are the analytical interpretation of the content of the PNSW-hypothesis.
A desire to avoid the solution of the $W_3$–system leads to a construction of approximations $\tilde{W}_3$, $\tilde{W}_4$ and $\hat{W}_3$, in which the first two turns are constructed as a simple spiral from $L_\varphi$. The subsequent rotations are constructed as LSS. In these webs $\varphi = 74.18896^\circ$.

The web $\tilde{W}_3$ fairly well illustrates the main properties of the prime number spider webs and the web $\tilde{W}_4$ shows the possibility of continuing the construction of higher rotations. The web $\hat{W}_3$ is created for a generation of initial approximations to a solution of the $W_3$–system. It also illustrates the demerits of the approximated webs.

The web $\tilde{W}_3$ is presented in Figure 5. It is constructed on the basis of 211 primes: 3 numbers from each of the first 20 rays from $\overline{r}_1$ to $r_{30}$, 2 numbers from 5 rays from $r_{32}$ to $r_{36}$ and 2 numbers from each of the 71 rays from $r_{38}$ to $r_{126}$ ($3 \cdot 20 + 2 \cdot 5 + 2 \cdot 71 = 211$); 147 of these primes are placed on the 2nd and the 3rd rotations by means of coordinates (19).

The web $\tilde{W}_3$ does not have a complete 3rd rotation, ending in the number 5381 from the ray $\overline{r}_1$ and not reaching number 7193 from the initial
ray $r_{12}$, marked in Figures 5 and 9 by a thick black line. To simplify the figure, 498 primes remaining until $p_{-1}(5381) = 709$ (709-211=498) are not placed on the 3rd rotation.

The web $\tilde{W}_3$ with a full 3rd rotation is represented in Figure 6. It is built on the basis of 255 primes: 3 from the first 25 rays from $\bar{r}_1$ to $r_{36}$ and 2 from 90 rays from $r_{38}$ to $r_{151}$ ($3 \cdot 25 + 2 \cdot 90 = 255$); 180 of these primes are placed on the 2nd and the 3rd rotations by means of coordinates (19).

To simplify the figure, 664 primes (919-255=664) remaining up to $p_{-1}(7193) = 919$ are not placed on the 3rd rotation.

At the end of the 3rd rotation $\tilde{W}_3$, (at the transition of the logarithmic spiral in LSS) a unessential qualitative defect shows up between numbers 877 from $r_{36}$ and 919 from $r_{12}$. The pointed defect obstructs the exact sewing of the spirals between the numbers 6823 and 7193 from the corresponding rays $r_{36}$ and $r_{12}$. This defect is removable by solving the $W_3$–system.

Figure 7: 4–rotation prime number spider web. In the center of the picture one can see the web $\tilde{W}_3$
The web $\widetilde{W}_4$, represented in Figure 7, is obtained from the web $\widetilde{W}_3$ by adding on the 4th rotation as LSS. In this building, 96 primes are used: 4 elements of the rays from $r_1$ to $r_{30}$, 3 elements of the rays from $r_{32}$ to $r_{36}$ and 4 elements of the rays from $r_{38}$ to $r_{126}$ ($20+5+71=96$). By means of 116 primes the LSS–units $e^{\alpha_i \theta_i + \beta_i}$, $i = 1, 2, \ldots, 116$ are determined by solving a $3 \times 3$ nonlinear system 116 times with respect to 348 unknowns $\alpha_i, \theta_i$ and $\beta_i$: 94 times exactly and 22 times approximately. The cases of inexact solutions result in unessential distortions of the condition (ii) for 22 rays (in the diagrams of $\widetilde{W}_3$ and $\widetilde{W}_4$ those distortions are not seen).

A variety of possibilities for constructing the PNSW–hypothesis is illustrated by web $\widetilde{W}_{\text{deg}}$ (Figure 8), in which the condition (ii) is violated in the following way: the rays lie on straight lines, but the subsequent segments of the rays have the opposite directions. In $\widetilde{W}_{\text{deg}}$, the same primes are used as in $\widetilde{W}_3$. 

Figure 8: Degenerated prime number spider web
4.4 Web formation rules and properties

Satisfying the PNSW–hypothesis, the conditions of $\widetilde{W}_3$ and $\widetilde{W}_4$ fix the individual peculiarities of the behaviour of primes around the origin of $\mathbb{R}^2$ as concrete systems of embedded trapezoids confined between the rays $\ell_{p_{\mu 1}}$ and $\ell_{p_{(\mu+k)1}}$, $k \geq 1$. For example, the trapezoid $t_{19}$ confined between the rays $\ell_{113}$ and $\ell_{127}$ (Figure 9) is one of the nineteen 3-rotation embedded trapezoids (3RET) of the web $\widetilde{W}_3$.

From the Theorem 2 it follows that

$q_8)$ The elements of the clusters $c_{\alpha \mu} (\alpha \mu)$ (they and only they) become the starts of new rays on the spiral $\rho_{s_1}(\theta)$.

Let a cluster $c_\mu(k) = \{ p_{\mu 1}, p_{(\mu+1)1}, \ldots, p_{(\mu+k)1} \}$ lie on the $\nu$th rotation of the spiral $\rho_{s_1}^{(n)}(\theta)$. On the $(\nu + q)$th rotation to it there correspond the primes $\{ p_q(p_{(\mu+i)1}) \}_{i=0,1,\ldots,k}$.

The PNSW–hypothesis shows that

$q_9)$ The geometric figure on $\mathbb{R}^2$, confined between the arcs $(p_{\mu 1}, p_{(\mu+k)1})$ and $(p_q(p_{\mu 1}), p_q(p_{(\mu+k)1}))$ by the spiral $\rho_{s_1}$ and the segments $|p_{\mu 1}, p_q(p_{\mu 1})|$, $|p_{(\mu+k)1}, p_q(p_{(\mu+k)1})|$ of the rays $\ell_{p_{\mu 1}}$ and $\ell_{p_{(\mu+k)1}}$, is the concave–convex trapezoid

$$z(\nu, \mu, k, q) = [(p_{\mu 1}, p_{(\mu+k)1}), (p_q(p_{\mu 1}), p_q(p_{(\mu+k)1}))].$$

The trapezoids of the type $z(\nu, \mu, 1, 1)$ are elementary trapezoids (or holes) to $W_n$. For example

$$z(1, 19, 1, 1) = [(113, 127), (617, 709)]$$

is an elementary $W_2$–trapezoid (Figure 10).

Let the primes $p_{\mu 1}, p_{(\mu+k)1} \in c_\mu(\alpha \mu), \alpha \mu \geq 3$, $0 \leq k \leq \alpha \mu$ lie on the $\nu$th rotation of the spiral $\rho_{s_1}^{(n)}$. On the $(\nu + 1)$th rotation, to them there corresponds the cluster $c_{\alpha \mu_1} (\alpha \mu_1) = \{ p_{\mu 11}, p_{(\mu_1+1)1}, \ldots, p_{(\mu_1+\alpha \mu_1)1} \}$ with a length $\alpha \mu_1 = p_{(\mu+k)1} - p_{\mu 1} - 1$, according to Corollary 3.

From the conditions of the PNSW-hypothesis there stems the following rule for formation of 3RET:

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Figure 9: 3-rotation system of embedded trapezoids $3RET_{19} = z(1, 19, 1, 2) = [(113, 127), (4549, 5381)]$. The thick black line defines the initial ray $r_{12}$. The direction of rotation is counter-clockwise.

Figure 10: The elementary trapezoid $z_{19} = z(1, 19, 1, 1) = [(113, 127), (617, 709)]$
On the plane $\mathbb{R}^2$ the following equalities hold

$$z(\nu + 1, \mu_1, \alpha_{\mu_1} + 1, 1) = \bigcup_{i=0}^{\alpha_{\mu_1}} z(\nu + 1, \mu + i, 1, 1), \quad (20)$$

$$z(\nu, \mu, 1, 2) = z(\nu, \mu, 1, 1) \cup z(\nu + 1, \mu_1, \alpha_{\mu_1} + 1, 1). \quad (21)$$

Equalities (20) and (21) can be regarded as those between the areas (the sign $\cup$ changes to $+$).

Let us apply the rule for formation of 3RET to $3RET_{19}$. Using the matrix from Appendix 1, we obtain

$$3RET_{19} = [(113, 127), (617, 709)] \cup [(617, 619), (4549, 4567)] \cup [(619, 631), (4567, 4663)] \cup \ldots \cup [(701, 709), (5281, 5381)].$$

In this example $\nu = 1$, $\mu = 30$, $\alpha_{\mu_1} = 127 - 113 - 1 = 13$. The starts of the newly–appeared rays on the second rotation are

- $p(114) = 619$, $p(115) = 631$, $p(116) = 641$, $p(117) = 643$,
- $p(118) = 647$, $p(119) = 653$, $p(120) = 659$, $p(121) = 661$,
- $p(122) = 673$, $p(123) = 677$, $p(124) = 683$, $p(125) = 691$, and
- $p(126) = 701$.

The rule for formation of 3RET shows that 3RET is a mosaic of $\alpha_{\mu_1} + 1$ elementary trapezoids, which are grouped in the direction of the ray $\ell_{p_{\mu_1}}$ in the following way:

the first is the elementary trapezoid

$$[(p_{\mu_1}, p_{(\mu+1)1}), (p_1(p_{\mu_1}), p_1(p_{(\mu+1)1}))],$$

followed by the composite trapezoid

$$[(p_1(p_{\mu_1}), p_1(p_{(\mu+\alpha_{\mu_1})1})), (p_2(p_{\mu_1}), p_2(p_{(\mu+\alpha_{\mu_1})1}))].$$

The elements (elementary trapezoids too) of the last one cause in total $\mu_2 = p_1(p_{(\mu+\alpha_{\mu_1})1}) - p_1(p_{\mu_1}) - 1$ new rays on the $(\nu + 2)th$ rotation, to which again the 3RET formation rule is applied for obtaining $\alpha_{\mu_2}$ new composite trapezoids located between the 3rd and 4th rotations of $W_n$.  

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Under the condition of the PNSW-hypothesis the process of growth of the initial 3RET is unlimited and leads to the formation of a class \((\ell_{p_\mu}, \ell_{p_{(\mu+1)}})\)-trapezoids containing an unlimited number of different elementary trapezoids.

Let \(mitos\) denote a closed geometric figure locked between the arc \((p(12), p_1(12))\) of the spiral \(\rho_s\) and the segment \(|p(12), p_1(12)|\) of the ray \(r_{12}\). Then the plane \(\mathbb{R}^2\) can be represented as a mosaic of the initial \(k(k^0)\)-classes of embedded trapezoids (\(k(11)=25\)):  

\[
\mathbb{R}^2 = \bigcup_{i=1}^{k(k^0)} (\ell_{p(k^0+i)}, \ell_{p(k^0+i+1)}) \cup mitos.
\]

Inside the \(mitos\) the starts of the first few rays can intersect each other, i.e., be in a mitosis status.

Properties \(q_{10}\), \(q_{11}\) and \(q_{12}\) are generalized in the property \(q_{13}\).

The plane \(\mathbb{R}^2\) is representable as a mosaic (in the theoretical–set sum sense) of all the prime–numerical elementary trapezoids and \(mitos\).

The PNSW-hypothesis leads to the geometric interpretation of \(\pi(x)\):

\[
\pi(p_{\nu}(m)) = p_{\nu-1}(m) = \lambda(0, p_{\nu-1}(m)), \quad \nu \geq 2, m \in \overline{M}.
\]  \((22)\)

Equalities \((22)\) allow one to interpret geometrically the \(\Omega\)–theorem of Littlwood and Theorem 1 from [10], where for setting the real values and in particular the values of \(Li(x)\) again the transformation \(h_{\rho_{s_1}}(x)\) and coordinates \((19)\) are used.

The geometric interpretation of the Theorem 3 consists in determination of the laws for appearance of \(u\)– and \(b\)–twins on \(W_n\).

\(u\)–twins appear on \(W_n\) in the following cases:

a) as starts of new rays \(\ell_{p_\mu}\) and \(\ell_{p_{(\mu+1)}}\) on the 1st rotation 3RET, which on the 2nd rotation cause one new ray \(\ell_{p((p_\mu+p_{(\mu+1)})/2)}\). Then 3RET is composed of three elementary trapezoids.

---

### Examples:

- \(m_{8}(3) \rightarrow (\ell_{71}, \ell_{73}) \rightarrow \ell_{359},\)
- \(m_{9}(5) \rightarrow (\ell_{101}, \ell_{103}) \rightarrow \ell_{557},\)

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\[
\bar{m}_{11}(5) \rightarrow \left\{ \begin{array}{l}
(\ell_{137}, \ell_{139}) \rightarrow \ell_{787}, \\
(\ell_{149}, \ell_{151}) \rightarrow \ell_{863};
\end{array} \right.
\]

b) as a pair of subsequent primes on the 2nd rotation of 3RET
Examples on the 2nd turn of the trapezoid \(z(1, 19, 1, 2)\) (Figure 9):

\[
\bar{m}_{30}(13) \rightarrow \left\{ \begin{array}{l}
(\ell_{641}, \ell_{643}) \rightarrow \ell_{4783}, \\
(\ell_{659}, \ell_{661}) \rightarrow \ell_{4937}.
\end{array} \right.
\]

q₁₆) One of the elements of \(b\)-twin \((t₁(\pi), t₂(\pi))\) lies on the existing ray \(\ell_{p_q(p_{\mu_1})}\), \(q \geq 1\), but the other element is a start of a new ray \(\ell_{p_{\mu_2}}\):

\[
t₁(\pi) \equiv p_q(p_{\mu_1}), \ t₂(\pi) \equiv p_{\mu_2} - right \ \text{twin};
\]

\[
t₁(\pi) \equiv p_{\mu_2}, \ t₂(\pi) \equiv p_q(p_{\mu_1}) - left \ \text{twin}
\]

In such a way the \(b\)-elementary trapezoid

\[
[(p_q(p_{\mu_1}), p_{\mu_2}), (p_{q+1}(p_{\mu_1}), p(p_{\mu_2}))]
\]

is sewn together to the right of the ray \(\ell_{p_{\mu_1}}\), and trapezoid

\[
[(p_{\mu_2}, p_q(p_{\mu_1})), (p(p_{\mu_2}), p_{q+1}(p_{\mu_1}))]
\]

to the left of the ray \(\ell_{p_{\mu_1}}\).

Examples: the trapezoid \([(617, 619), (4549, 4567)]\) is sewn to the right of the ray \(\ell_{113} \ ((\ell_{617}, \ell_{619}) \rightarrow \ell_{4561})\);

the trapezoid \([(857, 859), (6653, 6661)]\) is sewn to the left of the ray \(\ell_{149} \ ((\ell_{857}, \ell_{859}) \rightarrow \ell_{6659})\).

The properties q₁₅ and q₁₆) show the following \(W_n\) sewing property

q₁₇) The twin pairs sew uniformly over \(n\) the Eratosthenes rays in an unified plane web \(W_n\), \(n = 1, 2, \ldots \).

5 Conclusion

The study of the inner prime number distribution law remains at the initial stage.

Finally, we shall note how the proof of Conjecture 1 looks like, and we shall find a possibility of its generalization. We should like to indicate
possible applications of the proposed, in this paper, approach to a study of the oddish prime number behaviour.

At the first stage of the proof of Conjecture 1 it is necessary to solve the $W_3$–system and revise the number $k^0$. If the $W_3$–system cannot be solved with $k^0 = 11$, then the numbers $k^0 = 10$ and $k^0 = 12$ should be tried.

In the formulation of the $W_3$–system a function $\tilde{p}(x) \in C^1(0,10000]$, with a property
\[ \tilde{p}(n) = p(n), \quad n = 1, 2, \ldots, 1229 \] (23)
is used.

This function can be built up by the rod spline method [11] using as a rod the approximation ([12], exercise 9.21)
\[ \bar{p}(x) = x \left( \ln x + \ln \ln x + \frac{\ln \ln x - 2}{\ln x} - \frac{(\ln \ln x)^2/2 - 3 \ln \ln x + 5.5}{(\ln x)^2} - 1 \right). \]

The searched function will be of the form $\tilde{p}(x) = s_2(x)\bar{p}(x)$, where $s_2(x)$ is a parabolic spline determined on two nonuniform sets. At the interpolation points equalities [23] are fulfilled.

The $W_3$–system solvability can first be investigated numerically by using, for example, the program LANCELOT [13].

At the second stage of the proof of Conjecture 1, it is necessary to prove inductively the continuability of the basic $k(k^0)$–classes of embedded trapezoids
\[ (\ell_{p(k^0+i)}^{k^0}), \quad i = 1, 2, \ldots, k(k^0), \]
as well as the continuability of the new classes on rotations $n \geq 2$.

Conjecture 1 can be extended to all mesm$^f$–matrices.

**Conjecture 2.** For each matrix $A_f \in \mathcal{E}_f$ there exists LSS-spiral and a web $W_n(A_f)$, satisfying the conditions (i), (ii) and (iii) of the PNSW–hypothesis.

In the infinite set of webs $\mathbf{w} = \{W_n(A_f) : A_f \in \mathcal{E}_f, \ n = 1, 2, \ldots \}$ it is necessary to introduce an operation summing webs @ in such a way that the equalities
\[ W_n(S)@W_n(T_1)@W_n(T_2) = W_n(P) \]
and
\[ W_n(D_{6n-1}) \otimes W_n(D_{6n+1}) = W_n(P), \]
hold, by analogy with the set-theoretical equalities \( P = S \cup T_1 \cup T_2 \) and \( P = D_{6n-1} \cup D_{6n+1} \cup \{2, 3\}, \) where \( \{2, 3\} \subseteq \text{mitos}. \)

On the whole, the algebraic structures of mesm-matrices \([Bf^2A_f]\), webs \( W_n(A_f) \) and the set \( \mathbf{w} \) remain unexplored.

The spiral \( \rho_{s_1}(\theta) \) length from the PNSW–hypothesis can turn out to be an appropriate time coordinate in the description of physical processes taking place in asymmetric and irreversible time.

Indeed, the mapping (17) can be extended also for negative values \( x \in (-\infty, 0] : \)
\[
h_{\rho_{s_1}}(x) : \mathbb{R}_1^+ \to \lambda(0, \theta_x) = \frac{1}{\cos \varphi} \left( e^{(\cot \varphi)\theta_x} - 1 \right),
\]
\[ -\infty < x \leq 0, \varphi = \arccot(\alpha_1). \]

In such a way, in the unit circle (inside the domain \( \text{mitos} \)) there remains a finite negative moustache with a length
\[
\lambda(0, -\infty) = -\frac{1}{\cos \varphi}.
\]

Now, the arc length of the spiral \( \rho_{s_1}(\theta), \) \( -\infty < \theta < \infty \) is split up in three pieces: the length of the negative moustache, the finite positive length spiral in the \( \text{mitosis} \) and the infinite arc length corresponding to the semi-axis \([p(k^0 + 1), \infty)\). 

The solution of equations (10), (11) provides a motivation for the following hypothesis:

**Conjecture 3.** For arbitrary \( m \in \mathbb{M} \) and large \( n \) the inequality
\[
|L(p_{n+1}(m)) - p_n(m)| \leq c_1 \sqrt{p_{n+1}(m) \ln(p_{n+1}(m))} \tag{24}
\]
is fulfilled with a constant \( c_1 \) independent of \( m \).

Inequalities (24) result in the common estimate
\[
|L(x) - \pi(x)| < c_2 \sqrt{x \ln x}, \ c_2 = \text{const.} \tag{25}
\]
The proof of inequality (24) is easier than the proof of inequality (25). If from estimation (25) follows the truth of the Riemann hypothesis that complex solutions of the equations \( \zeta(s) = 0, \ s \in \mathbb{C} \) have a form \( s = 1/2 + i\gamma_n, \ \gamma_n \in \mathbb{R}, \ n = 1, 2, \ldots \), then the inner prime number distribution law (9) will prove to be a new useful tool of the analytical number theory.
### Appendix 1

**Matrix $[\mathbf{M}^{\,2\mathbf{P}}]$**

|   |  1  |  2  |  3  |  5  |  11 |  31  |
|---|-----|-----|-----|-----|-----|------|
| 1 | 127 | 709 | 5381| 52711| 3657500101|
| 2 | 648391 | 973733 | 174440041| 3657500101|
| 3 | 88362852307 | 2428095424619 | 5063692618249| ...|
| 4 | 17 | 59 | 277 | 1787 | 37139213 |
| 5 | 15299 | 167449 | 2269733 | 37139213 |
| 6 | 718064159 | 16123689073 | 414507281407 | ...|
| 7 | 12055296811267 | ... | ... | ... |
| 8 | 41 | 179 | 1063 | 8527 | 326851121 |
| 9 | 87803 | 1128889 | 17624813 | 326851121 |
| 10 | 7069067389 | 175650481151 | 495201938323 | 326851121 |
| 11 | 8 | 67 | 331 | 2221 | 19577 |
| 12 | 219613 | 3042161 | 50728129 | 997525853 |
| 13 | 22742734291 | 592821132889 | 1746120452132 | ...|
| 14 | 93 | 431 | 3001 | 27457 | 1559861749 |
| 15 | 318211 | 4535189 | 77557187 | 1559861749 |
| 16 | 36294260117 | 963726515729 | 28871271685163 | ...|
| 17 | 109 | 599 | 4397 | 42043 | 274711961 |
| 18 | 506683 | 7474967 | 131807699 | 274711961 |
| 19 | 64988430769 | 1765037224331 | 53982894593057 | ...|
| 20 | 157 | 919 | 7193 | 72727 | 5545806481 |
| 21 | 919913 | 14161729 | 259336153 | 5545806481 |
| 22 | 136395369829 | 3809491708961 | ... | ... |
| 23 | 191 | 1153 | 9319 | 96797 | 8012791231 |
| 24 | 1254739 | 19734581 | 368345293 | 8012791231 |
| 25 | 200147986693 | 5669795882633 | ... | ... |
| 26 | 211 | 1297 | 10631 | 112129 | 9672485827 |
| 27 | 1471343 | 23391799 | 440817757 | 9672485827 |
| 28 | 243504973489 | 6947574946087 | ... | ... |
| 29 | 241 | 1523 | 12763 | 137077 | 12501968177 |
| 30 | 1828669 | 29499439 | 563167303 | 12501968177 |
| 31 | 318083817907 | 9163611272327 | ... | ... |
| 32 | 283 | 1847 | 15823 | 173867 | 16917026909 |
| 33 | 2364361 | 38790341 | 751783477 | 16917026909 |
| 34 | 435748987787 | 12695664159413 | ... | ... |
| 35 | 2381 | 21179 | 239489 | 239489 |
| 36 | 3338989 | 56011909 | 1107276647 | 25366202179 |
| 37 | 664090238153 | 19638537755027 | ... | ... |
| 38 | 367 | 2477 | 22093 | 250751 | 26887732891 |
| 39 | 3509299 | 59053067 | 1170710369 | 26887732891 |
| 40 | 70555301183 | 20909033866927 | ... | ... |
Appendix 1

Matrix \[\mathbf{M}^2 \mathbf{P}\]... Continuation 1.

\[
\begin{array}{cccc}
22 & 79 & 401 & 2749 \\
4030889 & 68425619 & 24894639811901 & 285191 \\
835122557939 & 1367161723 & 31621854169 & 24859 \\
8479 & 401 & 2749 & 24894639811901 \\
8479 & 401 & 2749 & 24894639811901 \\
24 & 89 & 461 & 3259 \\
5054303 & 87019979 & 1760768239 & 41192432219 \\
1099216100167 & 33080040753131 & 685085753131 & 401519 \\
25 & 97 & 509 & 3637 \\
5823667 & 101146501 & 2062666783 & 48596930311 \\
1305164025929 & 3510004035659 & 685085753131 & 401519 \\
26 & 101 & 547 & 3943 \\
6478961 & 113256643 & 232114841 & 55022031709 \\
1484830174901 & 39510004035659 & 685085753131 & 401519 \\
27 & 103 & 563 & 4091 \\
6816631 & 119535373 & 2458721501 & 58379844161 \\
1579041544637 & 48112275898789 & 685085753131 & 401519 \\
28 & 107 & 587 & 4273 \\
7220981 & 127065427 & 2621760397 & 62427213623 \\
1692866362237 & 51702420222709 & 685085753131 & 401519 \\
30 & 113 & 617 & 4549 \\
7807321 & 138034009 & 2860139341 & 68363711327 \\
1860306318433 & 56997887937671 & 685085753131 & 401519 \\
32 & 131 & 739 & 5623 \\
10311439 & 185350441 & 3898093877 & 94434956839 \\
2606906998739 & 80783250929599 & 685085753131 & 401519 \\
33 & 137 & 773 & 5869 \\
10875143 & 196100297 & 4135824247 & 100450108949 \\
2773622459039 & 86127342906779 & 685085753131 & 401519 \\
34 & 139 & 797 & 6113 \\
11469013 & 207460717 & 4387715993 & 106839327589 \\
2956887579073 & 83600528759703 & 685085753131 & 401519 \\
35 & 149 & 859 & 6661 \\
12838937 & 233784751 & 4973864561 & 121763369327 \\
3386468161121 & 864835951121 & 685085753131 & 401519 \\
36 & 151 & 877 & 6823 \\
13243033 & 241568891 & 5147813641 & 126206581463 \\
3514741569337 & 83108741569337 & 685085753131 & 401519 \\
38 & 163 & 967 & 7607 \\
15239333 & 280256489 & 6016014239 & 148471002899 \\
4159843299587 & & & 4159843299587 \\
\end{array}
\]
### Appendix 1

**Matrix $[\hat{M} ^{2P}]$**

Continuation 2.

|   | 39   | 40   | 41   | 42   | 43   | 44   | 45   | 46   | 47   | 48   | 49   | 50   | 51   | 52   | 53   | 54   | 55   | 56   |
|---|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
|   | 167  | 173  | 181  | 193  | 197  | 199  | 223  | 227  | 229  | 233  | 239  | 251  | 257  | 263  |
|   | 991  | 1031 | 1087 | 1171 | 1201 | 1217 | 1409 | 1433 | 1447 | 1471 | 1499 | 1597 | 1621 | 1669 | 33991309 | 10735307868743 |
|   | 7841 | 8221 | 8719 | 9461 | 9739 | 9859 | 11743| 11953 | 12097 | 12301 | 12547 | 13469 | 13709 | 14177 | 647927381 | 370600719481 |
|   | 80071| 84347| 90023| 98519| 101701|103069|125113|127643|129229|131707|134597|145547|148439|153877|1589901|2071583 |
|   | 1021271|1080923|1159901|1278779|1323503|1342907|1656649|1693031|1715761|1751411|1793237|1850629|1950629|1993039|2071583|370600719481|... |
|   | 155231019913|166500464477|181639026043|204869160779|21736685679|281736685679|289357897711|294145810687|301656862553|301656862553|301656862553|301656862553|301656862553|301656862553|301656862553|301656862553|301656862553|301656862553 |
|   | 4356423418499|...|...|...|...|...|...|...|...|...|...|...|...|...|...|...|... |

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Appendix 1

Matrix $[\mathbf{M} \ 2\mathbf{P}]$… Continuation 3.

|   |   |   |   |   |
|---|---|---|---|---|
| 57 | 269 | 1723 | 14723 | 160483 | 2167937 |
|   |   | 35368547 | 682005953 | 15277169617 | 391886115431 |
| 58 | 271 | 1741 | 14867 | 162257 | 2193689 |
|   |   | 35815873 | 691097513 | 15490445177 | 397580778799 |
| 60 | 281 | 1823 | 15641 | 171697 | 2332537 |
|   |   | 38235377 | 740436923 | 16649917331 | 428592846379 |
| 62 | 293 | 1913 | 16519 | 182261 | 2487943 |
|   |   | 40951019 | 796004027 | 17959785803 | 463728180431 |
| 63 | 307 | 2027 | 17627 | 195677 | 2685911 |
|   |   | 44432569 | 867503173 | 19651365719 | 509248998611 |
| 64 | 311 | 2063 | 17987 | 200017 | 2750357 |
|   |   | 45564719 | 890830471 | 20205837393 | 524169678691 |
| 65 | 313 | 2081 | 18149 | 202001 | 2779781 |
|   |   | 46089287 | 901517753 | 2045825581 | 531016168117 |
| 66 | 317 | 2099 | 18311 | 204067 | 2810191 |
|   |   | 46620709 | 912598217 | 20721384791 | 538121923037 |
| 68 | 337 | 2269 | 20063 | 225503 | 3129913 |
|   |   | 52286593 | 1029838717 | 23513901553 | 613739626127 |
| 69 | 347 | 2341 | 20773 | 234293 | 3260657 |
|   |   | 54615469 | 1078227191 | 24670634249 | 645165616243 |
| 70 | 349 | 2351 | 20899 | 235891 | 3284657 |
|   |   | 55043683 | 1087126459 | 24883634693 | 650958710863 |
| 72 | 359 | 2417 | 21529 | 243781 | 3403457 |
|   |   | 57160969 | 1131224411 | 25940205719 | 679722101701 |
| 74 | 373 | 2549 | 22811 | 259657 | 3643579 |
|   |   | 61460533 | 1221036307 | 28097383163 | 73858245417 |
| 75 | 379 | 2609 | 23431 | 267439 | 3760921 |
|   |   | 63567289 | 1265161649 | 29159843309 | 767640499331 |
Appendix 1

Matrix $[\overline{M} \ 2P]$... Continuation 4.

| 76  | 383 | 2647 | 23801 | 271939 | 3829223 |
|-----|-----|------|-------|--------|----------|
|     |     |      | 64795981 | 1290918281 | 29780778613 | 784640376427 |
|     |     |      | 23339094889519... |          |          |          |
| 77  | 389 | 2683 | 24107 | 275837 | 3888551 |
|     |     |      | 65864459 | 1313343397 | 30321784529 | 799462887341 |
|     |     |      | 23795492951147... |          |          |          |
| 78  | 397 | 2719 | 24509 | 280913 | 3965483 |
|     |     |      | 67247771 | 1342401539 | 31023447269 | 818701472243 |
|     |     |      | 24388288001989... |          |          |          |
| 80  | 409 | 2803 | 25423 | 292489 | 4142053 |
|     |     |      | 70432519 | 1409422013 | 32644249103 | 863205467819 |
|     |     |      | 25761357737977... |          |          |          |
| 81  | 419 | 2897 | 26371 | 304553 | 4326473 |
|     |     |      | 73768631 | 1479780677 | 34349423377 | 910115902141 |
|     |     |      | 27211243680073... |          |          |          |
| 82  | 421 | 2909 | 26489 | 305999 | 4348681 |
|     |     |      | 74172503 | 1488302867 | 34556157661 | 915809403721 |
|     |     |      | 27387388206553... |          |          |          |
| 84  | 433 | 3019 | 27689 | 321017 | 4578163 |
|     |     |      | 78339559 | 1576442723 | 36697520357 | 974856473813 |
|     |     |      | 29216297536511... |          |          |          |
| 85  | 439 | 3067 | 28109 | 326203 | 4658099 |
|     |     |      | 79794157 | 1607252663 | 37447368857 | 995564440951 |
|     |     |      | 29858589333061... |          |          |          |
| 86  | 443 | 3109 | 28573 | 332099 | 4748047 |
|     |     |      | 81428323 | 1641908027 | 38291437141 | 1018893116299 |
|     |     |      | 30582699050611... |          |          |          |
| 87  | 449 | 3169 | 29153 | 339601 | 4863959 |
|     |     |      | 83543071 | 1686826109 | 39386748617 | 1049194449883 |
|     |     |      | 31524064728311... |          |          |          |
| 88  | 457 | 3229 | 29803 | 347849 | 4989697 |
|     |     |      | 85839547 | 1735649329 | 40578571003 | 1082201297941 |
|     |     |      | 32550506359429... |          |          |          |
| 90  | 463 | 3299 | 30557 | 357473 | 5138719 |
|     |     |      | 88565483 | 1793681753 | 41997140089 | 1121535591721 |
|     |     |      | 33775078562347... |          |          |          |
| 91  | 467 | 3319 | 30781 | 360293 | 5182717 |
|     |     |      | 89369047 | 1810798861 | 42415879469 | 1133155938589 |
|     |     |      | 34137123380603... |          |          |          |
| 92  | 479 | 3407 | 31667 | 371981 | 5363167 |
|     |     |      | 92678347 | 1881482157 | 44145738083 | 1181205761389 |
|     |     |      | 35635464099689... |          |          |          |
### Appendix 1

**Matrix $\overline{M^2P}$**... Continuation 5.

|   |   |   |   |   |   |
|---|---|---|---|---|---|
| 93 | 487 | 3469 | 32341 | 380557 | 5496349 |
| 95121911 | 1933651711 | 45426482839 | 1216826411041 |
| 3674753244474... |  |  |  |  |  |
| 94 | 491 | 3517 | 32797 | 386401 | 5587537 |
| 96797411 | 1969496239 | 46306458839 | 1241322670799 |
| 37512927359291... |  |  |  |  |  |
| 95 | 499 | 3559 | 33203 | 391711 | 5670851 |
| 98330021 | 2002298621 | 47112340151 | 1263771327193 |
| 38214783465337... |  |  |  |  |  |
| 96 | 503 | 3593 | 33569 | 396269 | 5741453 |
| 99630571 | 2030158657 | 47797243919 | 1282861540019 |
| 38811965770483... |  |  |  |  |  |
| 98 | 521 | 3733 | 35023 | 415253 | 6037513 |
| 105089261 | 2147305243 | 50681376121 | 1363360331743 |
| 41333311232987... |  |  |  |  |  |
| 99 | 523 | 3761 | 35311 | 418961 | 6095731 |
| 106166089 | 2170447637 | 51251887327 | 1379303865481 |
| 41833278300773... |  |  |  |  |  |
| 100 | 541 | 3911 | 36887 | 439357 | 6415081 |
| 112073683 | 2297602183 | 54391267121 | 1467155677657 |
| 44591599921641... |  |  |  |  |  |
| 102 | 557 | 4027 | 38153 | 455849 | 6673993 |
| 116881321 | 2401362767 | 56958606937 | 1539140110927 |
| 46855727983837... |  |  |  |  |  |
| 104 | 569 | 4133 | 39239 | 470207 | 6898807 |
| 121064467 | 2491797367 | 59200082443 | 1602086508713 |
| 48838469899327... |  |  |  |  |  |
| 105 | 571 | 4153 | 39451 | 472837 | 6940103 |
| 121834483 | 2508461203 | 59613478459 | 1613705610163 |
| 49204743622123... |  |  |  |  |  |
| 106 | 577 | 4217 | 40151 | 481847 | 7081709 |
| 124469621 | 2565499711 | 61029312569 | 1653521623993 |
| 50460527025823... |  |  |  |  |  |
| 108 | 593 | 4339 | 41491 | 499403 | 7359427 |
| 129647857 | 2677808011 | 63821022049 | 1732128413677 |
| 52942646093899... |  |  |  |  |  |
| 110 | 601 | 4421 | 42293 | 510031 | 7528669 |
| 132814411 | 2746597487 | 65533394977 | 1780407360517 |
| 54468962620717... |  |  |  |  |  |
| 111 | 607 | 4463 | 42697 | 515401 | 7612799 |
| 134389627 | 2780844971 | 66386576369 | 1804479121591 |
| 55230488801623... |  |  |  |  |  |
### Appendix 1

Matrix $[\overline{M} 2\overline{P}]$... Continuation 6.

|   |   |   |   |   |   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 112 | 613 | 4517 | 43283 | 522829 | 7730539 |
|     | 136593931 | 2828789699 | 67581794939 | 1838220650251 |
| 114 | 619 | 4567 | 43889 | 530773 | 7856939... |
| 115 | 631 | 4663 | 44879 | 543967 | 8066533... |
| 116 | 641 | 4759 | 45971 | 558643 | 8300687... |
| 117 | 643 | 4787 | 46279 | 562711 | 8365481... |
| 118 | 647 | 4801 | 46451 | 565069 | 8402833... |
| 119 | 653 | 4877 | 47297 | 576203 | 8580151... |
| 120 | 659 | 4933 | 47857 | 583523 | 8696917... |
| 121 | 661 | 4943 | 47963 | 584999 | 8720227... |
| 122 | 673 | 5021 | 48821 | 596243 | 8900383... |
| 123 | 677 | 5059 | 49207 | 601397 | 8982923... |
| 124 | 683 | 5107 | 49739 | 608459 | 9096533... |
| 125 | 691 | 5189 | 50591 | 619739 | 9276991... |
| 126 | 701 | 5281 | 51599 | 633467 | 9498161... |
| 127 | 709 | 5381 | 52711 | 648391 | 9737333... |
| 128 | 719 | 5441 | 53353 | 657121 | 9878657... |
| 129 | 727 | 5503 | 54013 | 665843 | 10020343... |
| 130 | 733 | 5557 | 54601 | 673793 | 10147877... |
| 132 | 743 | 5651 | 55681 | 688249 | 10382033... |
| 133 | 751 | 5701 | 56197 | 695239 | 10493953... |
| 134 | 757 | 5749 | 56701 | 702173 | 10606223... |
| 135 | 761 | 5801 | 57193 | 708479 | 10707449... |
| 136 | 769 | 5851 | 57751 | 715969 | 10829519... |
| 138 | 787 | 6037 | 59723 | 742681 | 11261903... |
| 140 | 809 | 6217 | 61819 | 771079 | 11723507... |
| 141 | 811 | 6229 | 61979 | 773317 | 11760029... |
| 142 | 821 | 6311 | 62921 | 786053 | 11967047... |
| 143 | 823 | 6323 | 63059 | 788009 | 11999111... |
| 144 | 827 | 6353 | 63391 | 792413 | 12071197... |
| 145 | 829 | 6361 | 63467 | 793511 | 12089177... |
| 146 | 839 | 6469 | 64679 | 809917 | 12356863... |
| 147 | 853 | 6599 | 66089 | 828923 | 12667463... |
| 148 | 857 | 6653 | 66749 | 838091 | 12816389... |
| 150 | 863 | 6691 | 67157 | 843613 | 12907091... |
| 152 | 881 | 6841 | 68821 | 866329 | 13280819... |
| 153 | 883 | 6863 | 69109 | 870161 | 13343881... |
| 154 | 887 | 6899 | 69491 | 875519 | 13431967... |
| 155 | 907 | 7057 | 71287 | 900157 | 13836751... |

...
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