NEWTONIAN LIMIT FOR WEAKLY VISCOELASTIC FLUID FLOWS OF OLROYDS’ TYPE

LUC MOLINET † AND RAAFAT TALHOUK ‡

Key words. viscoelastic fluids, global existence, newtonian limit.

AMS subject classifications. 76D03, 35B05.

Abstract. This paper is concerned with regular flows of incompressible weakly viscoelastic fluids which obey a differential constitutive law of Oldroyd type. We study the newtonian limit for weakly viscoelastic fluid flows in \( \mathbb{R}^N \) or \( \mathbb{T}^N \) for \( N = 2, 3 \), when the Weissenberg number (relaxation time measuring the elasticity effect in the fluid) tends to zero. More precisely, we prove that the velocity field and the extra-stress tensor converge in their existence spaces (we examine the Sobolev-\( H^s \) theory and the Besov-\( B_{s,2}^{1/2} \) theory to reach the critical case \( s = N/2 \)) to the corresponding newtonian quantities. This convergence results are established in the case of “ill-prepared” data. We deduce, in the two-dimensional case, a new result concerning the global existence of weakly viscoelastic fluids flow. Our approach makes use of essentially two ingredients: the stability of the null solution of the viscoelastic fluids flow and the damping effect, on the difference between the extra-stress tensor and the tensor of rate of deformation, induced by the constitutive law of the fluid.

1. Introduction, main results and notations. In this paper we investigate the Newtonian limit of weakly viscoelastic fluid flows of Oldroyd’s type in \( \Omega = \mathbb{R}^N \) or \( \Omega = \mathbb{T}^N \).

The dynamics of an homogeneous, isothermic and incompressible fluid flows, is described by the partial differential derivatives system given by :

\[
\begin{aligned}
\rho \left( u' + (u, \nabla) u \right) &= f + \text{div} \sigma \\
\text{div} u &= 0
\end{aligned}
\]

(1.1)

Here \( \rho > 0 \) is the (constant) density and \( f \) is the external density body forces. \( u = u(t,x) \) is the velocity vector field and \( \sigma = \sigma(t,x) \) is the symmetric stress-tensor, which is split into two parts: \( \sigma = -pI d + \tau \) where \( -pI d \) is the spherical part (\( p = p(t,x) \) the hydrodynamics pressure) and \( \tau \) is the tangential part or the extra-stress tensor. The fluid is called Newtonian if \( \tau \) can be expressed linearly in terms of the rate of strain tensor \( D[u] = \frac{1}{2}(\nabla u + \nabla u^T) \), i.e.

\[
\tau = 2\eta D[u]
\]

(1.2)

where \( \eta \) is the viscosity coefficient of the fluid (in this case (1.1) is the Navier-Stokes system). A fluid for which (1.2) is not valid is called non-Newtonian or complex fluid.

Unfortunately no universal constitutive law exists for non-Newtonian fluids (see for instance [11]). In this paper we consider a class of fluids with memory. For this kind of fluids, the extra-stress tensor at a time \( t \) depends on \( D[u] \) and its history. A model taking into account this properties is the Oldroyd’s one. The constitutive law

---

*This work was partially supported by a "contrat CEDRE" (Université Paris 13 - Université Libanaise).
† L.A.G.A., Institut Galiène, Université Paris 13, 93430 Villetaneuse, France (molinet@math.univ-paris13.fr)
‡ Mathématiques, Faculté des sciences I, Université Libanaise, Beyrouth, Liban (rtalhouk@ul.edu.lb). The author was partially supported by "le programme d’appui des projets de recherche de l’université Libanaise"
of Oldroyd’s type \cite{10} is given by:

\begin{equation}
\tau + \lambda_1 \frac{D_a \tau}{Dt} = 2\eta \left( D + \lambda_2 \frac{D}{Dt} D \right)
\end{equation}

(1.3)

where \(0 \leq \lambda_2 < \lambda_1\), \(\lambda_1\) is the relaxation time and \(\lambda_2\) the retardation time. The symbol \(\frac{D_a \tau}{Dt}\) denotes an objective (frame indifferent) tensor derivative (see \cite{11}). More precisely,

\begin{equation}
\frac{D_a \tau}{Dt} = \tau' + (u \cdot \nabla) \tau + \tau W - W \tau - a(D \tau + \tau D)
\end{equation}

(1.4)

with \(W[u] = \frac{1}{2}(\nabla u - \nabla u^T)\) is the vorticity tensor and \(a\) is a real number verifying 
\(-1 \leq a \leq 1\). The limit case \(\lambda_1 > 0\) and \(\lambda_2 = 0\) corresponds to a purely elastic fluid (which is excluded in our analysis), while the limit case \(\lambda_1 = \lambda_2 = 0\) corresponds to a viscous Newtonian fluid.

The constitutive law (1.3) is not under an evolution form. This equation can be transformed into a transport equation by splitting the extra-stress tensor into two parts \(\tau_s + \tau_p\), where \(\tau_s\) corresponds to a Newtonian part (the solvent) and \(\tau_p\) to the elastic part (the polymer). Setting \(\tau_s = 2\eta(1 - \omega)D[u]\), with \(\omega\) defined by \(0 \leq \omega = 1 - \frac{\lambda_2}{\lambda_1} \leq 1\), it follows from (1.3) that \(\tau_p\) satisfies the following transport equation:

\begin{equation}
\tau_p + \lambda_1 \frac{D_a \tau_p}{Dt} = 2\eta \omega D[u] .
\end{equation}

(1.5)

From now on we shall denote \(\tau_p = \tau\) and rewrite (1.4) and (1.5) by using dimensionless variables, we obtain the following partial differential system:

\begin{equation}
\begin{align}
\text{Re} \left( u' + (u, \nabla) u \right) - (1 - \omega) \Delta u + \nabla p &= f + \text{div} \tau \\
\text{div} u &= 0 \\
\varepsilon \left( \tau' + (u, \nabla) \tau + g(\nabla u, \tau) \right) + \tau &= 2\omega D[u] \\
\end{align}
\end{equation}

(1.5)

where \(g\) is a bilinear tensor valued mapping defined by

\[g(\nabla u, \tau) = \tau W[u] - W[u] \tau - a(D[u] \tau + \tau D[u]),\]

\(\text{Re} = \frac{\rho U L}{\eta}\) and \(\varepsilon = \lambda_1 \frac{U}{L}\) are respectively the well-known Reynolds number and the Weissenberg number (\(U\) and \(L\) represent a typical velocity and typical length of the flow). It is worth noticing that the Weissenberg number is usually denoted by \(\text{We}\). Here, since we will make the Weissenberg number tend to zero, we prefer to denote it by \(\varepsilon\). It is crucial to note that when \(\varepsilon = 0\), (1.5) reduces to the incompressible Navier-Stokes system:

\begin{equation}
\begin{align}
\text{Re} \left( v' + (v, \nabla) v \right) - \Delta v + \nabla p &= f \\
\text{div} v &= 0
\end{align}
\end{equation}

(1.6)

On the other hand, from the definition of the retardation parameter we observe that \(\omega = 1 - \mu/\varepsilon\) where \(0 \leq \mu < \varepsilon\) is given by \(\mu = \frac{\lambda_2}{\lambda_1} \varepsilon\). Therefore the Newtonian limit of (1.5) is actually a limit with two parameters \(\varepsilon\) and \(\mu\). To simplify the study we could drop a parameter by assuming that the rate \(\mu/\varepsilon\) (or equivalently \(\omega\)) is constant as \(\varepsilon\) tends to zero. In this work, instead of doing this, we will only impose an uniform
upper bound on \( \omega ( = 1 - \mu / \varepsilon ) \) with respect to \( \varepsilon \).

System (1.5) is completed by the following initial conditions

\[
(1.7) \quad u|_{t=0} = u_0 \quad \text{and} \quad \tau|_{t=0} = \tau_0
\]

Our approach is quite general and uses the two following ingredients:

- The stability of the null solution of (1.5) for a fixed \( \varepsilon \) (see [2] on \( \mathbb{R}^N \) or \( \mathbb{T}^N \) and [7], [5], [9] for the case of a bounded domain).
- The damping of factor \( 1/\varepsilon \) on the quantity \( \tau - 2\omega D[u] \) induced by equation (1.5).

Our results in the Sobolev spaces are valid for \( \Omega = \mathbb{R}^N \) or \( \mathbb{T}^N \) but to simplify the exposition we will only consider \( \Omega = \mathbb{R}^N \) and give the necessary modification to handle the periodic case.

The main idea is to cut \( u \) and \( \tau \) in low and high frequencies at a level depending on \( 1/\varepsilon \). Roughly speaking, forgetting the nonlinear terms, the high frequency part of \( u - v \) (\( v \) is the Newtonian solution, see (1.6) associated with the initial data \( u_0 \)) will satisfy the homogeneous system linearized around the null solution plus a non-homogeneous part containing a high frequency term of \( v \). But by the Lebesgue monotone convergence theorem, this term will tend to zero in the appropriate norms. The stability of the null solution (cf. [2], [7]) will then force the high frequencies of \( u - v \) and \( \varepsilon^{1/2} \tau \) to remain small (recall that \( (u - v)(0) = 0 \)). On the other hand, the remaining frequencies will tend to zero due to the damping effect on \( \tau - 2\omega D[u] \) which we will use in the same time as a smoothing effect. We will describe the main steps of the proof in Section 1.3.

Note that our analysis is in the spirit of numerous works on the incompressible limit of compressible Navier-Stokes equations (see for instance [3] and references therein). However, our analysis is in some aspects easier since there is a damping effect relating to the small parameter whereas in the incompressible limit it is a dispersive effect.

In our knowledge, no such result exists in the literature concerning our study, i.e. the newtonian limit of non-newtonian fluid flows. Moreover, our global existence result for regular weakly viscoelastic fluids flow in dimension 2 (see Corollary 1.1) is new and, in particular, not contained in the global existence results of [2].

1.1. Function spaces and notations. In the sequel \( C \) denotes a positive constant which may differ at each appearance. When writing \( x \preceq y \) (for \( x \) and \( y \) two non negative real numbers), we mean that there exist \( C_1 \) and \( C_2 \) two positive constants (which do not depend of \( x \) and \( y \)) such that \( C_1 x \leq y \leq C_2 x \). When writing \( x \preceq y \) (for \( x \) and \( y \) two non negative real numbers), we mean that there exists \( C_1 \) a positive constant (which does not depend of \( x \) and \( y \)) such that \( x \leq C_1 y \).

\( \mathcal{P} \) will denote the Leray projector on solenoidal vector fields.

For \( 1 \leq p, q \leq \infty \), we denote by \( \| \cdot \|_{L^p} \) the usual Lebesgue norm on \( \Omega = \mathbb{R}^N \),

\[
\| u \|_{L^p} = \left( \int_{\mathbb{R}^N} |u|^p \, dx \right)^{1/p}
\]

and by \( \| \cdot \|_{L_t^q L^r} \) the space-time Lebesgue norm on \( ]0, t[ \times \Omega \),

\[
\| u \|_{L_t^q L^r} = \left[ \int_0^t \| u(\tau) \|_{L^r}^q \, d\tau \right]^{1/q}
\]
Then for any $\delta$, where
\[ (1.8) \quad \|v\|_{H^s} = \left( \int_{\mathbb{R}^N} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 \, d\xi \right)^{1/2} \]

where $\hat{v}$ is the Fourier transform of $v$. The corresponding scalar product will be denoted by $\langle \cdot, \cdot \rangle_{H^s}$. Finally, for any $\varepsilon > 0$ we introduce the following Fourier projectors
\[ (1.9) \quad \tilde{P}_\varepsilon f(\xi) = \chi_{|\xi|<\varepsilon}(|\xi|)\hat{f}(\xi) \quad \text{and} \quad \tilde{Q}_\varepsilon f(\xi) = \chi_{|\xi|>\varepsilon}(|\xi|)\hat{f}(\xi), \]

where $\alpha > 0$ will be specified later.

1.1.1. Homogeneous Besov spaces. Let $\psi$ in $\mathcal{S}(\mathbb{R})$ such that $\hat{\psi}$ is supported by the set $\{z/2^{-1} \leq |z| \leq 2 \}$ and such that
\[ (1.10) \quad \sum_{j \in \mathbb{Z}} \hat{\psi}(2^{-j}z) = 1, \quad z \neq 0. \]

Define $\varphi$ by
\[ (1.11) \quad \varphi = 1 - \sum_{j \geq 1} \hat{\psi}(2^{-j}z), \]

and observe that $\varphi \in \mathcal{D}(\mathbb{R})$, $\hat{\varphi}$ is supported by the ball $\{z/|z| \leq 2 \}$ and $\varphi = 1$ for $|z| \leq 1$. We denote now by $\Delta_j$ and $S_j$ the convolution operators on $\mathbb{R}^N$ whose symbols are respectively given by $\hat{\psi}(2^{-j}|\xi|)$ and $\hat{\varphi}(2^{-j}|\xi|)$. Also we define the operator $\tilde{\Delta}_j$ by
\[ (1.11) \quad \tilde{\Delta}_j = \Delta_{j-1} + \Delta_j + \Delta_{j+1}, \]

which satisfies,
\[ (1.12) \quad \tilde{\Delta}_j \circ \Delta_j = \Delta_j. \]

For $s$ in $\mathbb{R}$, the homogenous Besov space $B^{s,1}_2(\mathbb{R}^N)$ (to simplify the notation we will simply denoted it by $B^s(\mathbb{R}^N)$) is the completion of $\mathcal{S}(\mathbb{R}^N)$ with respect to the semi-norm
\[ (1.13) \quad \|f\|_{B^s} = \|2^{js}\|\Delta_j(f)\|_{L^2}\|_{L^1(\mathbb{R})}. \]

1.2. Main results. Theorem 1.1. Let $N = 2, 3$ and let $(u_0, \tau_0) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ and $f \in L^2_{loc}(\mathbb{R}; H^{s-1})$ with $s > N/2$. Let $v$ be the Newtonian solution satisfying (1.5) with initial data $u_0$ and let $0 < T_0 \leq \infty$ such that $v \in C([0,T_0]; H^s)$. Then for any $\delta \in [0,1]$ there exists
\[ (1.14) \quad \varepsilon_0 = \varepsilon_0(N, \text{Re}, \delta, \|v\|_{L^\infty_{T_0} H^s}, \|\nabla v\|_{L^2_{T_0} H^s}, \|\tau_0\|_{H^s}, \|P f\|_{L^2_{T_0} H^{s-1}}) > 0 \]

such that for any $0 < \varepsilon < \varepsilon_0$ the system (1.5), with
\[ (1.15) \quad 0 < \omega \leq 1 - \delta, \]

with the obvious modification for $p, q = \infty$. For $s \in \mathbb{R}$, we denote by $\| \cdot \|_{H^s}$ the usual Sobolev norms on $\Omega = \mathbb{R}^N$,
admits a unique solution
\[ u_\varepsilon \in C([0, T_0]; H^s), \quad \nabla u_\varepsilon \in L^2(0, T_0; H^s), \quad \tau_\varepsilon \in C([0, T_0]; H^s). \]
Moreover,
\[ u_\varepsilon \xrightarrow{\varepsilon \to 0} v \text{ in } C([0, T_0]; H^s), \]
\[ \tau_\varepsilon - 2\omega D[u_\varepsilon] \xrightarrow{\varepsilon \to 0} 0 \text{ in } L^2(0, T_0; H^s), \]
\[ \varepsilon^{1/2} \tau_\varepsilon \xrightarrow{\varepsilon \to 0} 0 \text{ in } C([0, T_0]; H^s). \]

Recalling that in dimension two, the solution of the Newtonian problem exists for all positive times, we deduce the following result.

**Corollary 1.1.** In dimension 2 there exists
\[ \varepsilon_0 = \varepsilon_0(R, \delta, \|v\|_{L^\infty H^s}, \|\nabla v\|_{L^2 H^s}, \|\tau_0\|_{H^s}, \|P f\|_{L^2 H^{s-1}}) > 0 \]
such that for any \(0 < \varepsilon < \varepsilon_0\) the solution of (1.5) given by Theorem 1.1 exists for all positive times.

**Remark 1.1.** Note that Theorem (1.1) is a convergence result for “ill-prepared” data. Indeed, the quantity \(\tau_0 - 2\omega D[u_0]\) is not assumed to be small with \(\varepsilon\). Moreover, this is a singular limit result since \(\tau\) and \(D[u]\) do not belong to the same function space. In particular, \(D[u_0]\) is not as the same level of Sobolev regularity as \(\tau_0\).

**Remark 1.2.** According to the introduction, the Newtonian limit process is actually a limit process with two parameters \(\varepsilon\) and \(\mu\) tending to zero with \(0 \leq \mu < \varepsilon\).

The assumption (1.2) of Theorem 1.1 means that we impose the following additional conditions on the rate \(\mu/\varepsilon\) as \((\varepsilon, \mu)\) tends to zero \((0, 0)\):
\[ \delta \leq \frac{\mu}{\varepsilon} = \frac{\lambda_2}{\lambda_1} < 1 \]
for some fixed \(1 > \delta > 0\).

As mentioned in the introduction, the use of Besov spaces enables us to reach the critical index \(s = N/2\).

**Theorem 1.2.** Let \(N = 2, 3\) and let \((u_0, \tau_0) \in B^{N/2-1}(\mathbb{R}^N) \times B^{N/2}(\mathbb{R}^N)\) and \(f \in L^1_{\text{loc}}(B^{N/2-1})\). Let \(v\) be the Newtonian solution satisfying (1.1) with initial data \(u_0\) and let \(0 < T_0 \leq \infty\) such that \(v \in C([0, T_0]; B^{N/2-1})\). There exist \(0 < \omega_0 < 1\) and \(\varepsilon_0 = \varepsilon_0(N, Re, \omega_0, \|\tau_0\|_{B^{N/2}}, Pf, u_0) > 0\) such that for any \(0 < \varepsilon < \varepsilon_0\) the system (1.2), with \(0 < \omega \leq \omega_0\), admits a unique solution
\[ u_\varepsilon \in C([0, T_0]; B^{N/2-1}), \quad u_\varepsilon \in L^1(0, T_0; B^{N/2+1}), \quad \tau_\varepsilon \in C([0, T_0]; B^{N/2}). \]
Moreover,
\[ u_\varepsilon \xrightarrow{\varepsilon \to 0} v \text{ in } C([0, T_0]; B^{N/2-1}), \]
\[ \tau_\varepsilon - 2\omega D[u_\varepsilon] \xrightarrow{\varepsilon \to 0} 0 \text{ in } L^1(0, T_0; B^{N/2}). \]
\[ \varepsilon^{1/2} \tau_\varepsilon \to 0 \quad \text{in} \quad C([0, T_0]; B^{N/2}). \]

In dimension two, using the classical global existence result in \( B^0 \) for the Newtonian problem (see for instance [3]), we get the following corollary.

**Corollary 1.2.** In dimension 2 there exists

\[ \varepsilon_0 = \varepsilon_0(\text{Re}, \omega_0, \|v\|_{L^\infty_\varepsilon B^0}, \|\nabla v\|_{L^1_\varepsilon B^2}, \|\tau_0\|_{B^1}, \|P f\|_{L^1_\tau B^0}) \]

such that for any \( 0 < \varepsilon < \varepsilon_0 \) the solution of (1.5) given by Theorem 1.2 exists for all positive times. In the supercritical case, \( s > N/2 \), we get similar results by considering non homogeneous Besov spaces. Note that \( \varepsilon_0 \) depends then explicitly on some norms of \( v \).

**Theorem 1.3.** For \( s > N/2 \), Theorem 1.2 and Corollary 1.2 also hold by replacing the function spaces \( B^{N/2-1} \) by \( B^{-1} \cap B^{N/2-1} \) and \( B^{N/2} \) by \( B^s \cap B^{N/2} \). Moreover, \( \varepsilon_0 \) will depend now explicitly on some norms of \( v \) and \( P f \). More precisely, for \( s > N/2 \), we have

\[ \varepsilon_0 = \varepsilon_0(N, \text{Re}, \omega_0, \|v\|_{L^\infty_\varepsilon B^{N/2-1}}, \|\nabla v\|_{L^1_\varepsilon B^s}, \|\tau_0\|_{B^{N/2}}, \|P f\|_{L^1_\tau B^{N/2-1}}). \]

1.3. Sketch of the proof of Theorem 1.1 In this subsection we want to explain the main steps of the proof of Theorem 1.1. Note that Theorems 1.2 follows from the same arguments. To simplify we drop the nonlinear terms in (1.5). The first step consists in noticing that \( W := u - v \) satisfies the following system:

\[
\begin{cases}
\text{Re} \ W_t - (1 - \omega)Q_\varepsilon \Delta \varepsilon W - P_\varepsilon \Delta W = P_\varepsilon (\text{div} \ \tau - \omega \Delta u) \\
-\omega Q_\varepsilon \Delta v + Q_\varepsilon \text{div} \ \tau \\
\text{div} \ W = 0 \\
\varepsilon \tau + Q_\varepsilon \tau = 2\omega Q_\varepsilon \text{D}[W] + 2\omega Q_\varepsilon \text{D}[v] - P_\varepsilon (\tau - 2\omega \text{D}[u])
\end{cases}
\]

(1.19)

where \( P_\varepsilon \) and \( Q_\varepsilon \) are the projectors on respectively the low and the high frequencies defined in (1.8).

Projecting on the high frequencies with \( Q_\varepsilon \) (see (1.8) for the definition), proceeding as in [2], it is easy to check that we get a differential inequality close to

\[ \frac{d}{dt} \left( \|Q_\varepsilon \varepsilon W\|_{H^{s+\varepsilon}}^2 + \|Q_\varepsilon \varepsilon \tau\|_{H^{s+\varepsilon}}^2 \right) + \|Q_\varepsilon \nabla \varepsilon W\|_{H^{s+\varepsilon}}^2 + \|Q_\varepsilon \varepsilon \tau\|_{H^{s+\varepsilon}}^2 \lesssim \|Q_\varepsilon \nabla \varepsilon v\|_{H^{s+\varepsilon}}^2 \]

where we drop all the constant to clarify the presentation. Therefore, since \( W(0) = 0, \varepsilon \to 0 \) and, by the Lebesgue monotone convergence theorem, \( \|Q_\varepsilon \nabla \varepsilon v\|_{L^\infty_{\varepsilon} H^{s}} \to 0 \), we infer that \( \|Q_\varepsilon \varepsilon W\|_{L^\infty_{\varepsilon} H^{s+\varepsilon}} \) goes to zero with \( \varepsilon \). Now, to treat the low frequency part, we observe that, computing \( P_\varepsilon (1.19)_3 - \frac{2\omega}{\text{Re}} \text{D}[1.19]_4 \) and taking the \( H^{s+1} \)-scalar product of the resulting equation with \( Z := \tau - 2\omega \text{D}[u] \), we obtain something like

\[ \frac{d}{dt} \|P_\varepsilon Z\|_{H^{s-1}}^2 + \frac{1}{\varepsilon} \|P_\varepsilon Z\|_{H^{s-1}}^2 \lesssim \|P_\varepsilon \varepsilon \tau\|_{H^{s+\varepsilon}}^2 + \|P_\varepsilon f\|_{H^{s-1}}^2. \]

(1.20)

On the other hand, \( P_\varepsilon (1.19)_3 \) can be rewritten as

\[ \varepsilon P_\varepsilon \varepsilon t + \varepsilon^3 P_\varepsilon \varepsilon t = 2\omega \varepsilon^3 P_\varepsilon \text{D}[W] + 2\omega \varepsilon^3 P_\varepsilon \text{D}[v] - (1 - \varepsilon^3)P_\varepsilon Z, \]
where 0 < \beta < 1 will be specified later. Therefore, taking the $H^s$-scalar product of this last equality with $r$ and adding with the scalar product of (1.19) with $W$ we get a differential inequality close to

$$
\frac{d}{dt} \left( \| P_\varepsilon W \|^2_{H^s} + \varepsilon \| P_\varepsilon \tau \|^2_{H^s} \right) + \| P_\varepsilon \nabla W \|^2_{H^s} + \varepsilon^\beta \| P_\varepsilon \tau \|^2_{H^s} \lesssim \varepsilon^{-\beta} \| P_\varepsilon Z \|^2_{H^s} + \varepsilon^\beta \| P_\varepsilon \nabla v \|^2_{H^s}.
$$

Adding this last inequality and $\varepsilon^{2\beta}$ (2.20) we finally obtain

$$
\frac{d}{dt} \left( \| P_\varepsilon W \|^2_{H^s} + \varepsilon \| P_\varepsilon \tau \|^2_{H^s} \right) + \| P_\varepsilon \nabla W \|^2_{H^s} + \varepsilon^\beta \| P_\varepsilon \tau \|^2_{H^s} + \varepsilon^{2\beta-1} \| P_\varepsilon Z \|^2_{H^{s-1}} \lesssim \varepsilon^{2\beta} \| P_\varepsilon f \|^2_{H^{s-1}} + \varepsilon^\beta \| P_\varepsilon \nabla v \|^2_{H^s},
$$

since $\varepsilon^{-\beta} \| P_\varepsilon Z \|^2_{H^s} \leq \varepsilon^{-\beta} \varepsilon^{-2\alpha} \| P_\varepsilon Z \|^2_{H^{s-1}} \leq \varepsilon^{2\beta-1} \| Z \|^2_{H^{s-1}}$ as soon as $1 - 3\beta - 2\alpha > 0$. This last inequality enables us to conclude for the low frequency part. Note that we used the damping effect also as a smoothing effect.

2. Proof of Theorem [1.1]. Let us recall the following existence theorem proven by J.-Y. Chemin and N. Masmoudi [2].

**Theorem 2.1.** Let $(u_0, \tau_0) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ with $s > N/2$. Then there exists a unique positive maximal time $T^*$ and a unique solution

$$(u, \tau) \in C([0, T^*]; H^s) \cap L^2_{loc}(0, T^*; H^{s+1}) \times C([0, T^*]; H^s)$$

Moreover, if $T^* < \infty$ then $\forall N/2 < s' \leq s$

$$
\limsup_{t \to T^*} \left( \| u(t) \|_{H^{s'}} + \| \tau(t) \|_{H^{s'}} \right) = +\infty.
$$

**Remark 2.1.** Actually in [2] the following sharper blow up condition is derived

$$
T^* < \infty \implies \int_0^{T^*} \| \nabla u(t) \|_{L^\infty} + \| \tau(t) \|_{L^\infty}^2 dt = +\infty,
$$

but for our purpose the classical blow-up condition (2.1) will be sufficient.

Let us also recall a commutator estimate and classical Leibniz rules for fractional derivatives.

**Lemma 2.2.** Let $\Delta$ be the Laplace operator on $\mathbb{R}^N$, $N \geq 1$. Denote by $J^s$ the operator $(1 - \Delta)^{s/2}$.

- For every $s > N/2$,

$$
\| [J^s, f]g \|_{L^2(\mathbb{R}^N)} \lesssim \| \nabla f \|_{H^s(\mathbb{R}^N)} g \|_{H^{s-1}(\mathbb{R}^N)}.
$$

- For every $s > 0, 1 < q, q' \leq \infty$ and $1 < r, p, p' < \infty$ with $1/p + 1/q = 1/p' + 1/q' = 1/r$,

$$
\| J^s(fg) \|_{L^r(\mathbb{R}^N)} \lesssim \| J^s f \|_{L^p(\mathbb{R}^N)} \| g \|_{L^q(\mathbb{R}^N)} + \| f \|_{L^{p'}(\mathbb{R}^N)} \| J^s g \|_{L^{q'}(\mathbb{R}^N)}.
$$

- For every $p, r, t$ such that $r, p \geq t, r + p \geq 0$ and $r + p - t > N/2$,

$$
\| fg \|_{H^t(\mathbb{R}^N)} \lesssim \| f \|_{H^r(\mathbb{R}^N)} \| g \|_{H^p(\mathbb{R}^N)}.
$$
Proof. (2.3) and (2.4) are classical and can be found in [8] and [9]. (2.2) is a variant of Kato-Ponce’s commutator estimates. It is proven in [13] in dimension 1 but the proof works also in dimension 2 and 3.

To treat some nonlinear terms in dimension 2 we will need moreover the following Gagliardo-Nirenberg type inequality (see for instance [4]).

Lemma 2.3. Let $N \geq 2$, for $u \in H^1(\mathbb{R}^N)$ the following Sobolev type inequality hold for any $2 \leq p < +\infty$ such that $\frac{1}{2} - \frac{1}{N} \leq \frac{1}{p}$:

$$
\|u\|_{L^p(\mathbb{R}^N)} \lesssim \|u\|_{L^2(\mathbb{R}^N)} \left( \|\nabla u\|_{L^2(\mathbb{R}^N)} \right)^{\frac{2}{p} - \frac{1}{2}} \|\nabla u\|_{L^2(\mathbb{R}^N)} \left( \|\nabla u\|_{L^2(\mathbb{R}^N)} \right)^{\frac{1}{p} - \frac{1}{2}}.
$$

2.1. Estimate on $W = u - v$ and $\varepsilon^{1/2} \tau$. We start by deriving a differential inequality for the $H^s$-norms of $W$ and $\varepsilon^{1/2} \tau$. The high frequency part of this inequality is directly inspired by the stability proof of the null solution in [2]. This will enable us to control the very high frequency part $(Q, u, Q, \tau)$ of the solution. The other part $(P, u, P, \tau)$ will be treated by using the damping effect.

For $\varepsilon > 0$ fixed, Theorem (2.4) gives the existence and uniqueness of the solution $(u_\varepsilon, \tau_\varepsilon)$ of (1.5) in $C([0, T^*_\varepsilon); H^s) \cap L^2(0, T^*_\varepsilon; H^{s+1}) \times C([0, T^*_\varepsilon); H^s)$ for some $T^*_\varepsilon > 0$. To simplify the notations, we drop the index $\varepsilon$ on $u$ and $\tau$ in the sequel. Setting $Z = \tau - 2\omega D[u]$ we have the following estimates:

Lemma 2.4. For $\varepsilon > 0$ small enough, the solution $(u, \tau)$ of (1.2) satisfies for all $0 < t < T^*_\varepsilon$ and $0 < \beta < 1$,

$$
\frac{d}{dt} \left( \frac{Re}{2} \|W\|_{H^s}^2 + \frac{\varepsilon}{4\omega} \|\tau\|_{H^s}^2 \right) + \frac{1}{4} \|P\varepsilon \nabla W\|_{H^s}^2 + \frac{(1 - \omega)}{2} \|Q\varepsilon \nabla W\|_{H^s}^2 + \frac{(1 - \omega)}{2} \|Q\varepsilon \nabla W\|_{H^s}^2 + \frac{\varepsilon^2}{4\omega} \|P\varepsilon \tau\|_{H^s}^2
$$

$$
\leq (1 + 4\varepsilon^{-\frac{1}{2}}) \|P\varepsilon Z\|_{H^s}^2 + 4\varepsilon \|Q\varepsilon \nabla v\|_{H^s}^2 + 8\varepsilon \|P\varepsilon \nabla v\|_{H^s}^2 + 8\varepsilon \|P\varepsilon \nabla v\|_{H^s}^2 + 8\varepsilon \|P\varepsilon \nabla v\|_{H^s}^2
$$

whenever $0 < \omega < 1$. Moreover, for $0 < \omega \leq 10^{-2}$, it holds

$$
\frac{d}{dt} \left( \frac{Re}{4} \|W\|_{H^s}^2 + \frac{\varepsilon}{2} \|\tau\|_{H^s}^2 \right) + \frac{1}{4} \|P\varepsilon \nabla W\|_{H^s}^2 + \frac{(1 - \omega)}{4} \|Q\varepsilon \nabla W\|_{H^s}^2 + \frac{\varepsilon^2}{4\omega} \|P\varepsilon \tau\|_{H^s}^2
$$

$$
\leq (1 + 4\varepsilon^{-\frac{1}{2}}) \|P\varepsilon Z\|_{H^s}^2 + 8\varepsilon^2 \|Q\varepsilon \nabla v\|_{H^s}^2 + 8\varepsilon^2 \|P\varepsilon \nabla v\|_{H^s}^2 + 8\varepsilon^2 \|P\varepsilon \nabla v\|_{H^s}^2 + 8\varepsilon \|P\varepsilon \nabla v\|_{H^s}^2
$$

Proof. Notice that $W$ verifies the equation

$$
Re \left( W_t + \mathcal{P}(u, \nabla W) - \Delta W = P \nabla \tau - \omega \Delta u - Re \mathcal{P}(W, \nabla) v \right)
$$

$$
= P \mathcal{P}(\nabla \tau - \omega \Delta u) - \omega Q\varepsilon \Delta u
$$

$$
+ Q\varepsilon P \nabla \tau - \omega Q\varepsilon \Delta W - Re \mathcal{P}(W, \nabla) v.
$$
Therefore, multiplying scalarly (2.8) by $W$ in $H^s(\mathbb{R}^N)$, using Cauchy-Schwarz, Lemma 2.2 and that $u$ is divergence free, we obtain

\[
\frac{1}{2} \operatorname{Re} \frac{d}{dt} \|W\|_{H^s}^2 + \|P_\varepsilon \nabla W\|_{H^s}^2 + (1 - \omega) \|Q_\varepsilon \nabla W\|_{H^s}^2 \\
\leq ((Q_\varepsilon \nabla \tau, W))_{H^s} + \|P_\varepsilon Z\|_{H^s} \|\nabla W\|_{H^s} + \omega \|Q_\varepsilon \nabla v\|_{H^s} \|Q_\varepsilon \nabla W\|_{H^s}
\]  

(2.9)

\[+ C \operatorname{Re} \left| (J^s(u, \nabla) W, J^s W) \right|_{L^2} + \operatorname{Re} \left| (J^s(W, \nabla) v, J^s W) \right|_{L^2} \]

To estimate the second to the last term of (2.9), we rewrite it with the help of a commutator and apply Cauchy-Schwarz to the term containing this commutator to get

\[
\left| (J^s(u, \nabla) W, J^s W) \right|_{L^2} \leq \left| (J^s(u, \nabla) J^s W, J^s W) \right| + \left| (J^s, (u, \nabla) W) \right|_{L^2} \|J^s W\|_{L^2}.
\]

Since $u$ is divergence free, the first term of the right-hand side of this last inequality cancels by integration by parts. Estimating the second term thanks to Lemma 2.2 we then obtain

\[
\left| (J^s(u, \nabla) W, J^s W) \right|_{L^2} \leq C \|\nabla u\|_{H^s} \|\nabla W\|_{H^s} \|W\|_{H^s}.
\]

Now, to estimate the last term of the right-hand side of (2.9) we have to distinguish the cases $N = 2$ and $N = 3$.

- **Case $N = 3$**. Then by Lemma 2.2 Hölder, Sobolev and Young inequalities, we get

\[
\left| (J^s(W, \nabla) v, J^s W) \right|_{L^2} \leq \left| (J^s(W, \nabla) v) \right|_{L^{3/2}} \|J^s W\|_{L^6}
\]

\[
\lesssim \left( \|J^s W\|_{L^2} \|\nabla v\|_{L^3} + \|W\|_{L^3} \|J^s \nabla v\|_{L^2} \right) \|J^s \nabla W\|_{L^2}
\]

\[
\lesssim \|W\|_{H^s} \|\nabla v\|_{H^s} \|\nabla W\|_{H^s}.
\]

- **Case $N = 2$**. In this case, using Hölder and Lemmas 2.2, 2.3 we infer that

\[
\left| (J^s(W, \nabla) v, J^s W) \right|_{L^2} \leq \left| (J^s(W, \nabla) v) \right|_{L^{3/2}} \|J^s W\|_{L^6}
\]

\[
\lesssim \left( \|J^s W\|_{L^6} \|\nabla v\|_{L^2} + \|W\|_{L^6} \|J^s \nabla v\|_{L^2} \right) \|J^s W\|_{L^3}
\]

\[
\lesssim \|W\|_{H^s} \|\nabla v\|_{H^s} \|\nabla W\|_{H^s}.
\]

By Young inequalities it thus follows from (2.9) that

\[
\frac{\operatorname{Re} \frac{d}{dt} \|W\|_{H^s}^2}{2} + \frac{3}{4} \|P_\varepsilon \nabla W\|_{H^s}^2 + \frac{(1 - \omega)}{2} \|Q_\varepsilon \nabla W\|_{H^s}^2 \\
\leq ((Q_\varepsilon \nabla \tau, W))_{H^s} + \|P_\varepsilon Z\|_{H^s} \|\nabla W\|_{H^s} + \frac{\omega^2}{4(1 - \omega)} \|Q_\varepsilon \nabla v\|_{H^s}^2
\]

(2.10)

\[+ C \frac{\operatorname{Re}}{(1 - \omega)^2} \left( \|\nabla u\|_{H^s}^2 + \|\nabla v\|_{H^s}^2 \right) \|W\|_{H^s}^2.
\]

On the other hand, for $0 < \beta < 1$, observing that

\[
\tau - 2\omega D[u] = Q_\varepsilon \tau - 2\omega Q_\varepsilon (D[W] + D[v])
\]

\[+ (1 - \varepsilon \beta) P_\varepsilon Z + \varepsilon \beta \left( P_\varepsilon \tau - 2\omega P_\varepsilon (D[W] + D[v]) \right),
\]
we deduce from \(1.3\) that \(\tau\) satisfies the equation
\[
\varepsilon \left( \tau_t + (u, \nabla) \tau + g(\nabla u, \tau) \right) + Q_\varepsilon \tau + \varepsilon^\beta P_\varepsilon \tau = 2\omega Q_\varepsilon D[W] + 2\omega Q_\varepsilon D[v] \\
+ 2\omega \varepsilon^\beta P_\varepsilon D[W] + 2\omega \varepsilon^\beta P_\varepsilon D[v] - (1 - \varepsilon^\beta) P_\varepsilon Z.
\]

Taking the \(H^s\) scalar product of this equation with \(\tau\), using Lemma 2.2 Cauchy-Schwarz and Young inequalities we get
\[
\varepsilon \frac{d}{dt} \|\tau\|_{H^s}^2 + \frac{1}{2} \|Q_\varepsilon \tau\|_{H^s}^2 + \frac{\varepsilon^\beta}{2} \|P_\varepsilon \tau\|_{H^s}^2 \leq 2\omega((Q_\varepsilon D[W], \tau))_{H^s} + 8\varepsilon^{-\beta} \|P_\varepsilon Z\|_{H^s}^2 \\
+ 8\omega^2 \|Q_\varepsilon \nabla v\|_{H^s}^2 + 8\varepsilon^\beta (\|P_\varepsilon \nabla W\|_{H^s}^2 + \|P_\varepsilon \nabla v\|_{H^s}^2) \\
+ C\varepsilon^{2-\beta} \|\nabla u\|_{H^s}^2 \|\tau\|_{H^s}^2.
\]
\(2.11\)

We now separate the two cases:
- \(\omega \neq 0\). Then, adding \(2.10\) and \(2.11\)/\(2\omega\) we notice that the first term in the right-hand side of \(2.10\) and \(2.11\) cancel each other and \(2.6\) follows. This gives \(2.6\) for \(\varepsilon\) small enough since \(\beta > 0\).
- \(0 < \omega \leq 10^{-2}\). Then adding \(2.10)/2+2.11\), estimating the two remaining \(H^s\)-scalar products by integration by parts, Cauchy-Schwarz inequality and Young inequality, one obtains \(2.7\)

\(2.2.\) **Estimate on** \(Z = \tau - 2\omega D[u]\). We will now take advantage of the damping effect on \(Z = \tau - 2\omega D[u]\).

**Lemma 2.5.** The solution \((u, \tau)\) of \(1.3\) satisfies for all \(\varepsilon\) small enough and \(0 < t < T^*_\varepsilon\),
\[
\frac{1}{2} \frac{d}{dt} \|Z\|^2_{H^{s-1}} + \frac{1}{2\varepsilon} \|\nabla Z\|^2_{H^{s-1}} \leq \frac{4\omega}{Re(1-\omega)} \|Pf\|^2_{H^{s-1}} + \frac{(1+\omega)^2}{Re(1-\omega)} \|\tau\|^2_{H^s} \\
+ \frac{4}{1-\omega} \left( Re \|\nabla u\|^2_{H^s} + \|\tau\|^2_{H^s} \right) \|u\|^2_{H^s}.
\]
\(2.12\)

**Proof.** We apply \(\frac{2\varepsilon^\beta}{Re} D[\cdot]\) to \(1.5\) and substract the resulting equation from \(1.5\) to obtain
\[
Z_t - \frac{(1-\omega)}{Re} \Delta Z + \frac{1}{\varepsilon} Z = -f_1 - f_2
\]
where
\[
f_1 = \frac{2\omega}{Re} D[\mathcal{P} \text{div} \tau] - \frac{(1-\omega)}{Re} \Delta \tau + \frac{2\omega}{Re} D[Pf] - 2\omega D[\mathcal{P}(u, \nabla)u]
\]
and
\[
f_2 = \mathcal{P}(u, \nabla)\tau + g(\nabla u, \tau) .
\]
Taking the \(H^{s-1}\)-scalar product of \(2.13\) with \(Z\) we get
\[
\frac{1}{2} \frac{d}{dt} \|Z\|^2_{H^{s-1}} + \frac{(1-\omega)}{4 Re} \|\nabla Z\|^2_{H^{s-1}} + \frac{1}{\varepsilon} \|Z\|^2_{H^{s-1}} \leq C \left( \frac{(1+\omega)^2}{Re(1-\omega)} \|\tau\|^2_{H^s} + \frac{4\omega}{Re(1-\omega)} \|Pf\|^2_{H^{s-1}} \\
+ \frac{4\omega}{1-\omega} \|(u, \nabla)u\|^2_{H^{s-1}} + 4\|(u, \nabla)\tau\|^2_{H^{s-1}} + \|g(\nabla u, \tau)\|^2_{H^{s-1}} \right),
\]
\(2.14\)
where we used that
\[
\frac{2\omega}{\text{Re}} \left| (D\{P\text{div }\tau \}, Z) \right|_{H^s} \leq C \frac{2\omega}{\text{Re}} \|\text{div }\tau\|_{H^{s-1}} \|\nabla Z\|_{H^{s-1}} \\
\leq \frac{1 - \omega}{8 \text{ Re}} \|\nabla Z\|_{H^{s-1}}^2 + C \frac{\omega^2}{\text{Re} (1 - \omega)} \|\tau\|_{H^s}^2.
\]

Finally to control the nonlinear terms we notice that thanks to (2.4),
\[
\|a, \nabla b\|_{H^{s-1}} \lesssim \|a\|_{H^s} \|\nabla b\|_{H^{s-1}}
\]
which concludes the proof of (2.12).

**2.3. Convergence to the Newtonian flow.** We give here the proof in the case \(10^{-2} \leq \omega \leq 1 - \delta\). The case \(0 < \omega \leq 10^{-2}\) is simpler and can be handled in the same way by using (2.7) instead of (2.6). Adding (2.6) and \(\varepsilon^{2\beta} (2.12)\), we obtain for \(\varepsilon\) small enough
\[
\frac{d}{dt} \left( \frac{\text{Re}}{2} \|W\|_{H^s}^2 + \frac{\varepsilon}{4\omega} \|\tau\|_{H^s}^2 + \frac{\varepsilon^{2\beta}}{2} \|Z\|_{H^{s-1}}^2 \right) \\
+ \frac{(1 - \omega)}{4} \|\nabla W\|_{H^s}^2 + \frac{1}{8\omega} \|Q \varepsilon \tau\|_{H^s}^2 + \varepsilon^\beta \|P \varepsilon \tau\|_{H^s}^2 \right) + \frac{\varepsilon^{2\beta-1}}{4} \|Z\|_{H^{s-1}}^2 \\
\leq \frac{8\omega \|Q \varepsilon \nabla v\|_{H^s}^2}{\varepsilon} + \frac{8\omega^2}{\varepsilon} \|P \varepsilon \nabla v\|_{H^s}^2 + C \varepsilon^{1-\beta} \|\nabla u\|_{H^s}^2 \frac{\varepsilon}{4\omega} \|\tau\|_{H^s}^2 \\
+ C \frac{\text{Re}}{(1 - \omega)^2} \left( \|\nabla v\|_{H^s}^2 + \|\nabla u\|_{H^s}^2 \right) \|W\|_{H^s}^2
\]
(2.15)
\[+ C \varepsilon^{2\beta} \left( \frac{1}{\text{Re}} \right) \|P f\|_{H^{s-1}}^2 + (\text{Re} \|\nabla u\|_{H^s}^2 + \|\tau\|_{H^s}^2) \|u\|_{H^s}^2 \]

Here, we used that for \(\varepsilon\) small enough,
\[
\frac{(1 + \omega)^2}{\text{Re} (1 - \omega)} \varepsilon^{2\beta} \|\tau\|_{H^s}^2 \leq \frac{\varepsilon^\beta}{8\omega} \|\tau\|_{H^s}^2
\]
and
\[
(1 + \frac{4\varepsilon^{-\beta}}{\omega}) \|P \varepsilon Z\|_{H^s}^2 \leq (1 + 20^2 \varepsilon^{-\beta}) \varepsilon^{-2\alpha} \|P \varepsilon Z\|_{H^{s-1}}^2 \leq \frac{\varepsilon^{2\beta-1}}{4} \|Z\|_{H^{s-1}}^2
\]
as soon as \(\beta > 0\) and \(1 - 3\beta - 2\alpha > 0\).
From now on we thus take to simplify \((\alpha, \beta) = (1/8, 1/8)\). Setting
\[
X_s(t) = \frac{\text{Re}}{2} \|W(t)\|_{H^s}^2 + \frac{\varepsilon}{4\omega} \|\tau(t)\|_{H^s}^2 + \frac{\varepsilon^{2\beta}}{2} \|Z\|_{H^{s-1}}^2 \\
+ \frac{(1 - \omega)}{4} \|\nabla W\|_{H^s}^2 + \frac{1}{8\omega} \|Q \varepsilon \tau\|_{H^s}^2 + \frac{\varepsilon^\beta}{8} \|P \varepsilon \tau\|_{H^s}^2 + \frac{\varepsilon^{2\beta-1}}{4} \|Z\|_{H^{s-1}}^2 \right. ds
\]
we infer that \(X_s\) satisfies the following differential inequality
\[
\frac{d}{dt} X_s \leq \frac{8\omega^2 \|Q \varepsilon \nabla v\|_{H^s}^2}{\varepsilon} + \frac{8\omega^2}{\varepsilon} \|P \varepsilon \nabla v\|_{H^s}^2 + C \frac{\varepsilon^{2\beta}}{\text{Re} (1 - \omega)} \|P f\|_{H^{s-1}}^2 \\
+ C(\text{Re}, \delta) \left[ \varepsilon^{2\beta} \|\tau\|_{H^s}^2 + \|\nabla u\|_{H^s}^2 + \|\nabla v\|_{H^s}^2 \right] X_s
\]
(2.16)
\[+ C \varepsilon^{2\beta} \left( \frac{1}{\text{Re}} \right) \|\nabla u\|_{H^s}^2 + \varepsilon^{\beta} \|\tau\|_{H^s}^2 \|v\|_{H^s}^2\]
where we rewrite $u$ as $W + v$ and use the triangle inequality when necessary. Hence, Gronwall inequality leads to

$$X_s(t) \leq \exp \left[ C(\Re, \delta) \left( \varepsilon^{2\beta} \|\nabla v\|_{L^2_{\tau; H^s}}^2 + \|\nabla u\|_{L^2_{\tau; H^s}}^2 + \|\nabla v\|_{L^2_{\tau; H^s}}^2 \right) \right]$$

$$\left[ X_s(0) + 8\varepsilon^2 \|Q\varepsilon \nabla v\|_{L^2_{\tau; H^s}}^2 + 8\varepsilon^2 \|\nabla v\|_{L^2_{\tau; H^s}}^2 + C\varepsilon^2 \|Pf\|_{L^2_{\tau; H^{s-1}}}^2 \right]$$

$$+ \frac{\varepsilon^2 \Re}{\delta} \|\nabla v\|_{L^2_{\tau; H^s}}^2 \|\nabla v\|_{L^2_{\tau; H^s}}^2 .$$

Rewriting $u$ as $v + W$, we finally obtain

$$X_s(t) \leq \exp \left[ C(\Re, \delta) \left( \|\nabla v\|_{L^2_{\tau; H^s}}^2 + X_s(t) \right) \right]$$

$$\left[ X_s(0) + 8\varepsilon^2 \|Q\varepsilon \nabla v\|_{L^2_{\tau; H^s}}^2 + 8\varepsilon^2 \|\nabla v\|_{L^2_{\tau; H^s}}^2 + C\varepsilon^2 \|Pf\|_{L^2_{\tau; H^{s-1}}}^2 \right]$$

$$+ C \frac{(1 + \Re)\varepsilon^2}{\delta} \|\nabla v\|_{L^2_{\tau; H^s}}^2 \|\nabla v\|_{L^2_{\tau; H^s}}^2 X_s(t) + C \frac{\Re \varepsilon^2}{\delta} \|\nabla v\|_{L^2_{\tau; H^s}}^2 \|\nabla v\|_{L^2_{\tau; H^s}}^2$$

where

$$X_s(0) = \frac{\varepsilon}{4\omega} \|\tau_0\|_{H^s}^2 + \frac{\varepsilon^2}{2} \|\tau_0 - 2\omega D\|_{H^{s-2}} .$$

Let us now assume that $T^*_r \leq T_0$. Since (2.18) holds for any $N/2 < s' < s$, noticing that

$$\|Q\varepsilon \nabla v\|_{L^2_{\tau; H^{s'}}} \leq \varepsilon^{\alpha(s-s')} \|\nabla v\|_{L^2_{\tau; H^s}} ,$$

we deduce from (2.18), (2.19) and the continuity of $t \mapsto X_s(t)$ that there exists $\varepsilon_0(s, \|\tau_0\|_{H^s}, \|u_0\|_{H^s}, \|\nabla v\|_{L^2_{\tau_0; H^s}}, \|v\|_{L^2_{\tau_0; H^s}} > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $0 < t < T^*_r$

$$X_s(t) \leq C\varepsilon^{\min(\beta, 2\alpha(s-s'))} \leq C\varepsilon^{\min(1, s-s')/8}$$

which contradicts (2.2) of Theorem 2.1. This ensures that $T^*_r > T_0$. Now, since by Lebesgue monotone convergence theorem

$$\lim_{\varepsilon \to 0} \|Q\varepsilon \nabla v\|_{L^2_{\tau_0; H^s}} = 0 ,$$

it follows from (2.18)-(2.19) that $X_s(T_0) \to 0$ as $\varepsilon \to 0$. This proves (1.13) and (1.15). To prove (1.14) we observe that from this last limit and (2.21), $\|Q(\tau - 2\omega D\|_{L^2_{\tau_0; H^s}} \to 0$ and $\varepsilon^{2\beta-1}\|P(\tau - 2\omega D\|_{L^2_{\tau_0; H^{s-1}}} \to 0$. This yields the result by Bernstein inequality since $2\beta - 1 + 2\alpha < 0$.

### 2.4. The periodic setting.

Let us give here the modifications needed to handle with the case $\Omega = \mathbb{T}^N$, $N = 2, 3$. It is worth noticing that Lemma 2.2 holds also with $\Omega = \mathbb{T}^N$. On the other hand, the Sobolev inequality (2.3) does not hold for general functions in $\mathbb{T}^N$ but holds, for instance, for zero-mean value functions. Note that if $f(t)$ has mean value zero for all time $t \geq 0$ then using the invariance by Galilean transformations, $u \mapsto u(t, x - z t) + z$ with $z \in \mathbb{R}^3$, we can assume that $u$ has zero mean-value for all time and we are done. Otherwise, we have only to care about the
treatment of the nonlinear term \((W_\nabla)v\) in (2.8). Denoting by \(W\) the \(L^2\)-projection of \(W\) on zero mean-value functions, we rewrite \((W_\nabla)v\) as

\[
(W_\nabla)v = (W_\nabla)v + \left( \int_\Omega W \right) \nabla v.
\]

We take the \(H^s\)-scalar product of (2.8) with \(W\) and add with the \(L^2\)-scalar product of (2.8) with \(W\). The \(H^s\)-scalar product coming from the first term of the right-hand side of (2.22) can be treated as in \(R^N\). For the second term, we observe that

\[
\left| \left( \left( \int_\Omega W \nabla v, W \right) \right) \right|_{H^s} = \left| \int_\Omega W \left| \left( \nabla v, W \right) \right|_{H^s} \right| \lesssim \|W\|_{L^2} \|\nabla v\|_{H^s} \|\nabla W\|_{H^{s-1}}.
\]

On the other hand, concerning the \(L^2\)-scalar product we notice that

\[
\left| \left( \left( (W_\nabla)v, W \right) \right) \right|_{L^2} = \left| \left( \left( (W_\nabla)v, W \right) \right) \right|_{L^2} + \left( \int_\Omega W \right) \left| \left( \nabla v, W \right) \right|_{L^2} \lesssim \|W\|_{L^2} \|\nabla v\|_{L^2} \|W\|_{L^2} + \left| \int_\Omega W \right| \|\nabla v\|_{L^2} \|W\|_{L^2} \lesssim \|W\|_{H^s} \|\nabla W\|_{L^2} \|\nabla v\|_{L_2}
\]

We thus obtain exactly as in (2.10),

\[
\frac{\text{Re}}{2} \frac{d}{dt} \left( \|W\|_{H^s}^2 + \|W\|_{L^2}^2 \right) + \frac{3}{4} \|P_\varepsilon \nabla W\|_{H^s}^2 + \frac{(1 - \omega)}{2} \|Q_{\varepsilon} \nabla W\|_{H^s}^2 \leq \left( \left( \nabla v, W \right) \right)_{H^s} + \left| \left( \nabla v, W \right) \right|_{H^s}^2 + \frac{\omega^2}{4(1 - \omega)} \|Q_{\varepsilon} \nabla v\|_{H^s}^2 + \frac{C \text{Re}}{(1 - \omega)} \left( \left( \|\nabla u\|_{H^s}^2 + \|\nabla v\|_{H^s}^2 \right) \right) \|W\|_{H^s}^2.
\]

The remainder of the analysis is now exactly the same as in \(R^N\).

3. Proof of Theorems 1.2 and 1.3 In this section we prove a convergence result in the Besov spaces \(B_2^{s-1,1}, s \geq N/2\). It will require a smallness assumption on the retardation parameter \(\omega\) but on the other hand will enable us to reach the critical regularity space for (1.5). Note that our smallness assumption on the retardation parameter is the same as the one in [2] to get the stability of the null solution in such function spaces.

Let us recall the following well-posedness result derived in [2].

**Theorem 3.1.** Let \((u_0, \tau_0) \in B_2^{s-1}(R^N) \cap B^{N/2-1}(R^N) \times B_2^s(R^N) \cap B^{N/2}(R^N)\) with \(s \geq N/2\). Then there exists a unique positive maximal time \(T^*\) and a unique solution

\[
(u, \tau) \in C([0, T^*]; B_2^{s-1} \cap B^{N/2-1}) \cap L^1_{loc}(0, T^*; B_2^{s+1} \cap B^{N/2+1}) \times C([0, T^*]; B_2^s \cap B^{N/2})
\]

Moreover, if \(T^* < \infty\) then

\[
\limsup_{t \nearrow T^*} \left( \|u(t)\|_{B^{N/2-1}} + \|\tau(t)\|_{B^{N/2}} \right) = +\infty
\]
We will make use of the following classical commutator and product estimates 
(see for instance [2, 3] and [12]):

Lemma 3.2. For all \( s \in ]1 - N/2, 1 + N/2[ \) we have

\[
(3.2) \quad \| \tilde{\Delta}_j [(a, \nabla), \Delta_j] b \|_2 \lesssim 2^{\frac{1}{2}(s-1)} \gamma_j \| \nabla a \|_{B^{N/2+1}} \| b \|_{B^{s-1}},
\]

with \( \| \gamma_j \|_{L^4(Z)} \lesssim 1 \).

For all \( s_1, s_2 \leq N/2 \) with \( s_1 + s_2 > 0 \) it holds

\[
(3.3) \quad \| ab \|_{B^{s_1+s_2-N/2}} \lesssim \| a \|_{B^{s_1}} \| b \|_{B^{s_2}}.
\]

For any \( \varepsilon > 0 \) we divide \( Z \) into the three following subsets

\( I := Z^\varepsilon = \{ j \in Z, 0 < 2^j < 1 \}, \quad J_\varepsilon := \{ j \in Z, 1 \leq 2^j \leq \varepsilon^{-\alpha} \} \) and \( K_\varepsilon := \{ j \in Z, 2^j > \varepsilon^{-\alpha} \} \)

and for any subset \( N \subset Z \) we denote by \( \| \cdot \|_{B_N} \) the semi-norm

\[
\| u \|_{B_N} = \sum_{j \in N} 2^{s_j} \| \Delta_j u \|_{L^2}.
\]

3.1. Estimate on \( W \) and \( \varepsilon \tau \). Lemma 3.3. The solution \( (u, \varepsilon \tau) \) of (3.5) satisfies for all \( 0 < t < T^* \)

\[
\frac{d}{dt} \left( Re \| W \|_{B^{s-1}} + 4 \varepsilon \| \tau \|_{B^s} \right)
+ \frac{1}{2} \| (1 - \omega) \Delta_j \Delta W \|_{B^{s+1}_K} + \| W \|_{B^{s+1}_{i,j_e}} + 2 \| \varepsilon \|_{B^{s}_{i,j_e}} + 2 \| \tau \|_{B^{s}_{i,j_e}} + 2 \| \varepsilon^\beta \|_{B^{s}_{i,j_e}}
\leq 5 \| Z \|_{B^{s}_{i,j_e}} + 16 \omega \| v \|_{B^{s+1}_{i,j_e}} + 16 \omega \| \varepsilon^\beta \|_{B^{s+1}_{i,j_e}} + \frac{1}{2} \| \varepsilon^\beta \|_{B^{s+1}_{i,j_e}}

(3.4) + C \varepsilon \mu_1 \| u \|_{B^{N/2+1}} \| \tau \|_{B^s} + C (\| u \|_{B^{N/2+1}} + \| v \|_{B^{N/2+1}}) \| W \|_{B^{s-1}}.
\]

Proof. Applying \( \Delta_j \) to (3.5) we have for \( j \in J_\varepsilon \),

\[
Re \left( \partial_t \Delta_j W + P(u, \nabla) \Delta_j W \right) - (1 - \omega) \Delta_j \Delta W = -\omega \Delta_j \Delta v + \Delta_j P \text{div} \tau
+ \text{Re} \Delta_j P[(u, \nabla), \Delta_j] W + \text{Re} \Delta_j P(\nabla) W v
\]

and for \( j \in I \),

\[
Re \left( \partial_t \Delta_j W + P(u, \nabla) \Delta_j W \right) - \Delta_j \Delta W = \Delta_j Z
+ \text{Re} \Delta_j P[(u, \nabla), \Delta_j] W + \text{Re} \Delta_j P(\nabla) W v.
\]

Taking the scalar product in \( L^2(\mathbb{R}^N) \) of (3.5) with \( \Delta_j W \), using that \( W \) is divergence free and Cauchy-Schwarz inequality a we get

\[
\frac{1}{2} \text{Re} \frac{d}{dt} \| \Delta_j W \|_2^2 + (1 - \omega) \| \nabla \Delta_j W \|_2^2 \leq \| \Delta_j W \|_2 \left( \omega \| \Delta_j \Delta v \|_2 + \| \Delta_j \text{div} \tau \|_2 \right)
+ \text{Re} \| \Delta_j P[(u, \nabla), \Delta_j] W \|_2 + \text{Re} \| \Delta_j P(\nabla) v \|_2.
\]

(3.7)

We use now that, according to Bernstein inequality, \( \| \nabla \Delta_j W \|_2 \geq 2^{j-1} \| \Delta_j W \|_2 \) and divide (3.7) by \( \| \Delta_j W \|_2 \). Then, estimating the commutator term thanks to (3.2)
and the last term thanks to (3.3) with \( s_1 = s - 1 \) and \( s_2 = N/2 \), using Bernstein inequalities, it follows that

\[
\text{Re} \frac{d}{dt} \| \Delta_j W \|_2 + \frac{(1 - \omega)}{2} 2^2 \| \Delta_j W \|_2 \leq 2 \| \Delta \Delta_j v \|_{L^2} + 2 \| \Delta_j \text{div} \tau \|_2
\]

\[
(3.8)
\]

\[
+ \gamma_j 2^{-j(s-1)} (\| u \|_{B^{N/2+1}} + \| v \|_{B^{N/2+1}}) \| W \|_{B^{s-1}},
\]

with \( \| (\gamma_j) \|_{\mathcal{H}(Z)} \lesssim 1 \). Multiplying by \( 2^{j(s-1)} \) and summing in \( j \in K_\varepsilon \), it follows that

\[
\text{Re} \frac{d}{dt} \| W \|_{B^{s-1}_{K_\varepsilon}} + \frac{(1 - \omega)}{2} \| W \|_{B^{s+1}_{K_\varepsilon}} - 2 \| \tau \|_{B^s_{K_\varepsilon}} \leq (\| u \|_{B^{N/2+1}} + \| v \|_{B^{N/2+1}}) \| W \|_{B^{s-1}}.
\]

(3.9)

Proceeding in the same way with (3.6) but summing in \( j \in I \cup J_\varepsilon \), we obtain

\[
\text{Re} \frac{d}{dt} \| W \|_{B^{s+1}_{I \cup J_\varepsilon}} + \frac{1}{2} \| W \|_{B^{s+1}_{I \cup J_\varepsilon}} - \| Z \|_{B^{s+1}_{I \cup J_\varepsilon}} \leq (\| u \|_{B^{N/2+1}} + \| v \|_{B^{N/2+1}}) \| W \|_{B^{s-1}}.
\]

(3.10)

Now, for \( j \in Z \), we infer from (1.2) that

\[
\varepsilon \partial_t \Delta_j \tau + \varepsilon (u, \nabla) \Delta_j \tau + \Delta_j \tau = 2 \omega \Delta_j D[u] \\
- \varepsilon [(u, \nabla), \Delta_j] \tau + \varepsilon \Delta_j g(\nabla u, \tau).
\]

(3.11)

Rewriting \( \Delta_j (\tau - 2 \omega D[u]) \) as \( \Delta_j (\tau - 2 \omega D[W] - 2 \omega D[v]) \) for \( j \in K_\varepsilon \) and as

\[
\varepsilon^\beta \Delta_j \tau - 2 \omega \varepsilon^\beta \Delta_j (D[W] + D[v]) + (1 - \varepsilon^\beta) \Delta_j Z
\]

for \( j \in I \cup J_\varepsilon \), similar considerations as above lead to the two following inequalities

\[
\varepsilon \frac{d}{dt} \| \tau \|_{B^{s+1}_{K_\varepsilon}} + \| \tau \|_{B^{s+1}_{K_\varepsilon}} \leq 4 \omega \| W \|_{B^{s+1}_{K_\varepsilon}} \\
+ 4 \omega \| v \|_{B^{s+1}_{K_\varepsilon}} + C \varepsilon \| u \|_{B^{N/2+1}} \| \tau \|_{B^s},
\]

(3.12)

and

\[
\varepsilon \frac{d}{dt} \| \tau \|_{B^{s+1}_{I \cup J_\varepsilon}} + \varepsilon^\beta \| \tau \|_{B^{s+1}_{I \cup J_\varepsilon}} \leq \| Z \|_{B^{s+1}_{I \cup J_\varepsilon}} + 4 \omega \varepsilon^\beta \| W \|_{B^{s+1}_{I \cup J_\varepsilon}} \\
+ 4 \omega \varepsilon^\beta \| v \|_{B^{s+1}_{I \cup J_\varepsilon}} + C \varepsilon \| u \|_{B^{N/2+1}} \| \tau \|_{B^s}.
\]

(3.13)

Adding \((3.9) + (3.10) + 4(3.12) + (3.13)\), \((3.4)\) follows.

**3.2. Estimate on \( \tau - 2 \omega D[u] \).** *Lemma 3.4.*

\[
\frac{d}{dt} \| Z \|_{B^{s+1}_{K_\varepsilon}} + \frac{1}{\varepsilon} \| Z \|_{B^{s+1}_{K_\varepsilon}} \leq \frac{(1 + \omega)}{\text{Re}} \| \tau \|_{B^{s+1}_{K_\varepsilon}} + \| Pf \|_{B^{s+1}_{K_\varepsilon}} \\
+ C \alpha \ln(\varepsilon^{-1}) \left( \| u \|_{B^{N/2+1}} + \| \tau \|_{B^{N/2}} \right) \| u \|_{B^{s+1}}.
\]

(3.14)

\[
\frac{d}{dt} \| Z \|_{B^{s+1}_{I \cup J_\varepsilon}} + \frac{1}{\varepsilon} \| Z \|_{B^{s+1}_{I \cup J_\varepsilon}} \leq \frac{(1 + \omega)}{\text{Re}} \| \tau \|_{B^{s+1}_{I \cup J_\varepsilon}} + \| Pf \|_{B^{s+1}_{I \cup J_\varepsilon}} \\
+ C \left( \| u \|_{B^{N/2+1}} + \| \tau \|_{B^{N/2}} \right) \| u \|_{B^{s+1}}.
\]

(3.15)
Proof. Applying $\Delta_j$ to (2.13) and taking the $L^2$-scalar product with $\Delta_jZ$ we get
\[
\frac{d}{dt}\|\Delta_jZ\|_{L^2} + \frac{(1-\omega)}{2\text{Re}} \|\Delta_jZ\|_{L^2} + \frac{1}{\varepsilon}\|\Delta_jZ\|_{L^2} \lesssim \|\Delta_jf_1\|_{L^2} + \|\Delta_jf_2\|_{L^2},
\]
where
\[
f_1 = \frac{2\omega}{\text{Re}} D[\mathcal{P}\text{div}\tau] - \frac{(1-\omega)}{\text{Re}} \Delta\tau + \frac{2\omega}{\text{Re}} D[\mathcal{P}f] - 2\omega D[\mathcal{P}(u\nabla)u]
\]
and
\[
f_2 = \mathcal{P}(u\nabla)\tau + g(\nabla u, \tau).
\]
Multiplying this inequality by $2\varepsilon^{(s-2)}$, summing in $j \in J_\varepsilon$ we infer that
\[
\frac{d}{dt}\|Z\|_{B_{\varepsilon}^{-2}} + \frac{(1-\omega)}{2\text{Re}} \|Z\|_{B_{\varepsilon}^{-2}} + \frac{1}{\varepsilon}\|Z\|_{B_{\varepsilon}^{-2}} \leq \frac{(1+\omega)}{\text{Re}} \|\tau\|_{B_{\varepsilon}^{-2}} + \|\mathcal{P}f\|_{B_{\varepsilon}^{-2}} + \|g(\nabla u, \tau)\|_{B_{\varepsilon}^{-2}}.
\]
(3.16)
For $s > 1$ we estimate the nonlinear term thanks to (3.3) with respectively $(s_1, s_2) = (s-1, N/2)$, $(s-1, N/2 - 1)$ and $(s-2, N/2)$. For $s = 1$ (of course $N = 2$) we estimate the first nonlinear term in the same way and use the following lemma to estimate the two last ones. This lemma follows directly from the definitions of $I$ and $J_\varepsilon$ and the fact that, for $|s| \leq N/2$, the usual product maps continuously $B^{-s,1}_2 \times B^{s,1}_2$ into $B^{-N/2,\infty}_2$ (see for instance [12]). Note, in particular, that $|J_\varepsilon| \lesssim a\ln(\varepsilon^{-1})$.

Lemma 3.5. For all $s_1, s_2 \leq N/2$ with $s_1 + s_2 = 0$ it holds
\[
\|ab\|_{B_{\varepsilon}^{-N/2}} \lesssim a\ln(\varepsilon^{-1})\|ab\|_{B_{\varepsilon}^{-1}} \|b\|_{B_{\varepsilon}^{2}}.
\]
(3.17)
and
\[
\|ab\|_{B_{\varepsilon}^{-N/2+2}} \lesssim \|a\|_{B_{\varepsilon}^{-1}} \|b\|_{B_{\varepsilon}^{2}}.
\]
(3.18)
We apply this lemma with $(s_1, s_2) = (0, 0)$ and $(-1, 1)$ for respectively the second and the third nonlinear term of (3.16) to complete the proof of (3.14). Finally (3.15) can be easily obtained in the same way by using that $\|a\|_{B_{\varepsilon}^{s}} \leq \|a\|_{B_{\varepsilon}^{s'}}$ for $s' \leq s$ and (3.18).

3.3. Convergence to the Newtonian flow. From now on we set $\gamma(\omega) = (1-\omega)/2 - 16\omega$ and assume that $0 \leq \omega \leq \omega_0$ with $\gamma(\omega_0) > 0$. We proceed as in Section 2.3. For $0 < \beta < 1$, we add (3.14) and $\varepsilon^{2\beta}(3.14) + 3.15$ to get
\[
\frac{d}{dt} \left( \text{Re}\|W\|_{B^{\beta-1}_2} + 4\varepsilon\|\tau\|_{B^{\beta}_2} + \varepsilon^{2\beta}(\|Z\|_{B_{\varepsilon}^{-2}} + \|Z\|_{B_{\varepsilon}^{1}}) \right)
\]
\[
\leq 16\omega\|v\|_{B^{\beta+1}_2} + 16\omega\varepsilon^{\beta}\|v\|_{B^{\beta+1}_{1,\varepsilon}} + \varepsilon^{2\beta}\|\mathcal{P}f\|_{B_{\varepsilon}^{-1}}
\]
\[
+C\varepsilon\|u\|_{B^{N/2+1}}\|\tau\|_{B^{\beta}_2} + C\|u\|_{B^{N/2+1}}\|v\|_{B^{N/2+1}}\|W\|_{B^{1-1}_2}
\]
(3.19)
\[
\varepsilon^{2\beta}\ln(\varepsilon^{-1})(\|u\|_{B^{N/2+1}} + \|\tau\|_{B^{N/2}_2})\|v\|_{B^{1-1}_2}.
\]
\footnote{For $1 \leq p \leq \infty$, $\|f\|_{p,p} = \|\{2^s\|\Delta_j(f)\|_{L^2}\}\|_p(Z)$.}
Here we used that for $\varepsilon$ small enough, $\varepsilon^\beta \leq \min(16\gamma(\omega_0), \frac{\text{Re}}{4})$, $\varepsilon^{2\beta - 1}/2 \geq 5$ and

$$5\|Z\|_{B^1_{t\varepsilon}} \leq \varepsilon^{-2\alpha}\|Z\|_{B^{\varepsilon}_{t\varepsilon}} \leq \frac{\varepsilon^{2\beta - 1}}{2}\|Z\|_{B^{\varepsilon}_{t\varepsilon}}$$

as soon as

$$0 < \alpha < 1/2 \quad \text{and} \quad 0 < 2\beta < 1 - 2\alpha.$$

From now on we set $(\alpha, \beta) = (1/8, 1/8)$ so that (3.20) is satisfied. Setting

$$X_s(t) = \text{Re} \|W(t)\|_{B^{\varepsilon}_{t\varepsilon}} + 4\varepsilon\|\tau\|_{B^{\varepsilon}_{t\varepsilon}} + \varepsilon^{2\beta}\|Z\|_{B^{\varepsilon}_{t\varepsilon}}$$

$$+ \int_0^t \frac{\gamma(\omega_0)}{2}\|W\|_{B^{\varepsilon}_{t\varepsilon}} + \left(\|\tau\|_{B^{\varepsilon}_{t\varepsilon}} + \varepsilon^\beta\|\tau\|_{B^{\varepsilon}_{t\varepsilon}}\right) + \frac{\varepsilon^{2\beta - 1}}{2}\left(\|Z\|_{B^{\varepsilon}_{t\varepsilon}} + \|Z\|_{B^1_{t\varepsilon}}\right) ds,$$

we infer that

$$\frac{d}{dt} X_s(t) \leq 16\omega\|v\|_{B^{\varepsilon}_{t\varepsilon}} + 16\varepsilon^{2\beta}\|v\|_{B^{\varepsilon}_{t\varepsilon}} + \varepsilon^{2\beta}\|PF\|_{B^{\varepsilon}_{t\varepsilon}}$$

$$+ C(\text{Re}, \omega) \left[\|W\|_{B^{N/2+1}_{t\varepsilon}} + \|v\|_{B^{N/2+1}_{t\varepsilon}} + \varepsilon^{2\beta}\ln(\varepsilon^{-1})\|\tau\|_{B^{N/2}_{t\varepsilon}}\right] X_s$$

$$+ C\varepsilon^{2\beta}\ln(\varepsilon^{-1})\left(\|v\|_{B^{N/2+1}_{t\varepsilon}} + \|W\|_{B^{N/2+1}_{t\varepsilon}} + \|\tau\|_{B^{N/2}_{t\varepsilon}}\|v\|_{B^{\varepsilon}_{t\varepsilon}}\right).$$

By Gronwall lemma we infer that

$$X_s(t) \leq \exp \left(C(\omega, \text{Re}) \left(\|v\|_{L^1_t B^{N/2+1}_{t\varepsilon}} + X_{N/2}(t)\right)\right)$$

$$\left[X_s(0) + 16\omega\|v\|_{L^1_t B^{N/2+1}_{t\varepsilon}} + 16\varepsilon^{2\beta}\|v\|_{L^1_t B^{N/2+1}_{t\varepsilon}} + \varepsilon^{2\beta}\|PF\|_{L^1_t B^{\varepsilon}_{t\varepsilon}}\right.$$

$$+ C\varepsilon^{2\beta}\ln(\varepsilon^{-1})\left(X_{N/2}(0)\|v\|_{L^\infty_t B^{N/2}_{t\varepsilon}} + \varepsilon^\beta\|v\|_{L^1_t B^{\varepsilon}_{t\varepsilon}}\|v\|_{L^\infty_t B^{\varepsilon}_{t\varepsilon}}\right)\left.$$

(3.21)

$$+ C\varepsilon^{2\beta}\ln(\varepsilon^{-1})\left(\|v\|_{L^1_t B^{N/2+1}_{t\varepsilon}} + \|W\|_{B^{N/2+1}_{t\varepsilon}} + \|\tau\|_{B^{N/2}_{t\varepsilon}}\|v\|_{B^{\varepsilon}_{t\varepsilon}}\right).$$

where

$$X_s(0) = 4\varepsilon\|\tau_0\|_{B^{\varepsilon}_{t\varepsilon}} + \varepsilon^{2\beta}\left(\|\tau_0 - 2\omega D[u_0]\|_{B^{\varepsilon}_{t\varepsilon}} + \|\tau_0 - 2\omega D[u_0]\|_{B^{\varepsilon}_{t\varepsilon}}\right).$$

Assuming that $T^* \leq T_0$ and noticing that

$$\|v\|_{L^1_t B^{N/2+1}_{t\varepsilon}} \rightarrow 0 \quad \text{as} \quad \varepsilon \rightarrow 0,$$

we deduce from (3.21) and (3.22) and the continuity of $t \mapsto X_{N/2}(t)$ that there exists $\varepsilon_0 = \varepsilon_0(N, \|\tau_0\|_{B^{N/2}_{t\varepsilon}}, \mathcal{P}f, u_0)$ such that for any $0 < \varepsilon < \varepsilon_0$ and any $0 < t < T^*_\varepsilon$,

$$X_{N/2}(t) \leq \Lambda(\varepsilon)$$

with $\Lambda(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. This contradicts (3.1) of Theorem 3.1 and thus ensures that $T^*_\varepsilon > T_0$. (1.16) and (1.18) follow as well. To prove (1.18) we notice that from last limit and (3.23), $\|\tau - 2\omega D[u]\|_{L^1_t B^{N/2}_{t\varepsilon}} \rightarrow 0$, $\varepsilon^{2\beta - 1}\|\tau - 2\omega D[u]\|_{L^1_t B^{N/2}_{t\varepsilon}} \rightarrow 0$ and $\varepsilon^{2\beta - 1}\|\tau - 2\omega D[u]\|_{L^1_t B^{N/2-2}_{t\varepsilon}} \rightarrow 0$. This gives the result since $2\beta - 1 + 2\alpha \leq 0$ and thus

$$\|\tau - 2\omega D[u]\|_{L^1_t B^{N/2-2}_{t\varepsilon}} \leq \varepsilon^{-2\alpha}\|\tau - 2\omega D[u]\|_{L^1_t B^{N/2}_{t\varepsilon}} \leq \varepsilon^{2\beta - 1}\|\tau - 2\omega D[u]\|_{L^1_t B^{N/2-2}_{t\varepsilon}}.$$
Finally, for $s > N/2$, the proof follows the same lines using that
\begin{equation}
\|v\|_{L_{t_0}^{1}B_{K_{r}}^{N/2+1}} \leq \varepsilon^{\alpha(s-N/2)}\|v\|_{L_{t_0}^{1}B_{K_{r}}^{s+1}}
\end{equation}
and thus with
\[\varepsilon_0 = \varepsilon_0(N, \|\tau_0\|_{B^{N/2}}, \|u_0\|_{B^{N/2-1}}, \|v\|_{L_{t_0}^{1}B_{K_{r}}^{s+1}}, \|Pf\|_{L_{t_0}^{1}B^{N/2-1}}).\]
This completes the proof of Theorems 1.2 and 1.3.

Acknowledgments. The authors are grateful to the Referees for useful remarks.

REFERENCES

[1] R. A. Adams, Sobolev spaces, Academic Press, 1975.
[2] J.-Y. Chemin and N. Masmoudi, About lifespan of regular solutions of equations related to viscoelastic fluids, SIAM J. Math. Anal., 33 (2001), pp. 84–112.
[3] R. Danchin, Zero Mach number limit in critical spaces for compressible Navier-Stokes equations, Annales Scientifique de l’École Normale Supérieure, 35 (2002), pp. 27-75.
[4] L. C. Evans Partial differential equations, Graduate Studies in Mathematics, 19. A.M.S., Providence, RI, 1998.
[5] E. Fernandez-Cara, F. Guillen, R.R. Ortega, Some theoretical results concerning non-Newtonian fluids of Oldroyd kind, Ann. Scuola Norm. Sup. Pisa Cl. Sci., 26 (1998), 1-29.
[6] P. Grisvard, Elliptic problems in non-smooth domains. Monographs and Studies in Mathematics, 24. Pitman (Advanced Publishing Program), Boston, MA, 1985.
[7] C. Guillopé and J.C. Saut, Existence results for flow of viscoelastic fluids with a differential constitutive law, Nonlinear Analysis TMA, 15 (1990), pp. 849–869.
[8] C.E. Kenig, G. Ponce and L. Vega, Well-posedness and scattering results for the generalized Korteweg-de Vries equation via contraction principle, Comm. and Pure and Appl. Math., 46 (1993), pp. 527-620.
[9] L. Molinet and R. Talhouk On the global and periodic regular flows of viscoelastic fluids with a differential constitutive law, NoDEA, 11 (2004), 349–359.
[10] J.G. Oldroyd, On the formation of rheological equations of state, Proc. R. Soc. Lond. A, 200 (1950), pp. 523–541.
[11] M. Renardy, W.J. Hrusa and J.A. Nohel, Mathematical Problems in viscoelasticity, Longman, London.
[12] M. E. Taylor, Tools for PDE. Pseudodifferential operators, paradifferential operators, and layer potentials. Mathematical Surveys and Monographs, 81. A.M.S., Providence, RI, 2000.
[13] M. Toma, Smoothing properties of some weak solutions of the Benjamin-Ono equation, Differ. Int. Equ., 3 (1990), pp. 683–694.